ON THE FREE LIE ALGEBRA WITH MULTIPLE BRACKETS

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Abstract. It is a classical result that the multilinear component of the free Lie algebra is isomorphic (as a representation of the symmetric group) to the top (co)homology of the proper part of the poset of partitions $\Pi_n$ tensored with the sign representation. We generalize this result in order to study the multilinear component of the free Lie algebra with multiple compatible Lie brackets. We introduce a new poset of weighted partitions $\Pi^k_n$ that allows us to generalize the result. The new poset is a generalization of $\Pi_n$ and of the poset of weighted partitions $\Pi^w_n$ introduced by Dotsenko and Khoroshkin and studied by the author and Wachs for the case of two compatible brackets. We prove that the poset $\Pi^k_n$ with a top element added is EL-shellable and hence Cohen-Macaulay. This and other properties of $\Pi^k_n$ enable us to answer questions posed by Liu on free multibracketed Lie algebras. In particular, we obtain various dimension formulas and multicolored generalizations of the classical Lyndon and comb bases for the multilinear component of the free Lie algebra. We also obtain a plethystic formula for the Frobenius characteristic of the representation of the symmetric group on the multilinear component of the free multibracketed Lie algebra.

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1. Introduction

Recall that a Lie bracket on a vector space $V$ is a bilinear binary product $[\cdot, \cdot] : V \times V \to V$ such that for all $x, y, z \in V$,

\begin{align}
[&x, y] + [y, x] = 0 \quad \text{(Antisymmetry)}, \\
[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \quad \text{(Jacobi Identity)}.
\end{align}

Throughout this paper let $k$ denote an arbitrary field. The free Lie algebra on $\{1, 2, \ldots, n\}$ is the $k$-vector space generated by the elements of $\{1, 2, \ldots, n\}$ and all the possible bracketings involving these elements subject only to the relations (1.1) and (1.2). Let $\text{Lie}(n)$ denote the multilinear component of the free Lie algebra on $\{1, 2, \ldots, n\}$, i.e., the subspace generated by bracketings that contain each element of $\{1, 2, \ldots, n\}$ exactly once. We call these bracketings bracketed permutations. For example $[[2, 3], 1]$ is a bracketed permutation in $\text{Lie}(3)$, while $[[2, 3], 2]$ is not. For any set $S$, the symmetric group $S_S$ is the group of permutations of $S$. In particular we denote by $S_n := S_{\{1, 2, \ldots, n\}}$ the group of permutations of the set $\{1, 2, \ldots, n\}$. The symmetric group $S_n$ acts naturally on $\text{Lie}(n)$ making it into an $S_n$-module. A permutation $\tau \in S_n$ acts on the bracketed permutations by replacing each letter $i$ by $\tau(i)$. For example $(1, 2) [[3, 5], [2, 4], 1] = [[[3, 5], [1, 4]], 2]$. Since this action respects the relations (1.1) and (1.2), it induces a representation of $S_n$ on $\text{Lie}(n)$. It is a classical result that

$$\dim \text{Lie}(n) = (n - 1)!.$$ 

Although the $S_n$-module $\text{Lie}(n)$ is an algebraic object it turns out that the information needed to completely describe this object is of a combinatorial nature. Let $P$ denote a partially ordered set (or poset for short). To every poset $P$ one can associate a simplicial complex $\Delta(P)$ (called the order complex) whose faces are the chains (totally ordered subsets) of $P$. Consider now the poset $\Pi_n$ of set partitions of $\{1, 2, \ldots, n\}$ ordered by refinement. The symmetric group $S_n$ acts naturally on $\Pi_n$ and this action induces isomorphic representations of $S_n$ on the unique nonvanishing reduced simplicial homology $\tilde{H}_{n-3}(\Pi_n)$ and cohomology $\tilde{H}^{n-3}(\Pi_n)$ of the order complex $\Delta(\Pi_n)$ of the proper part $\bar{\Pi}_n := \Pi_n \setminus \{\hat{0}, \hat{1}\}$ of $\Pi_n$. It is a classical result that

$$\text{Lie}(n) \cong_{S_n} \tilde{H}_{n-3}(\Pi_n) \otimes \text{sgn}_n,$$

where $\text{sgn}_n$ is the sign representation of $S_n$.

Equation (1.3) was observed by Joyal [28] by comparing a computation of the character of $\tilde{H}_{n-3}(\Pi_n)$ by Hanlon and Stanley (see [36]), to
an earlier formula of Brandt [10] for the character of $\mathfrak{Lie}(n)$. Joyal [28] gave a proof of the isomorphism using his theory of species. The first purely combinatorial proof was obtained by Barcelo [2] who provided a bijection between known bases for the two $S_n$-modules (Björner’s NBC basis for $\tilde{\mathfrak{H}}_{n-3}(\Pi_n)$ and the Lyndon basis for $\mathfrak{Lie}(n)$). Later Wachs [41] gave a more general combinatorial proof by providing a natural bijection between generating sets of $\tilde{\mathfrak{H}}_{n-3}(\Pi_n)$ and $\mathfrak{Lie}(n)$, which revealed the strong connection between the two $S_n$-modules. Connections between Lie type structures and various types of partition posets have been studied in other places in the literature, see for example [3], [4], [25], [23], [17], [40], [11], [30].

The moral of equation (1.3) is that we can describe $\mathfrak{Lie}(n)$ and understand its algebraic properties by studying and applying poset theoretic techniques to the combinatorial object $\Pi_n$. This observation will play a central role throughout this paper.

1.1. **Doubly bracketed Lie algebra.** Two Lie brackets $[\cdot, \cdot]_1$ and $[\cdot, \cdot]_2$ on a vector space $V$ are said to be **compatible** if every linear combination of the brackets is also a Lie bracket on $V$, that is, satisfies relations (1.1) and (1.2). As pointed out in [13, 29], this kind of compatibility is equivalent to the **mixed Jacobi** condition: for all $x, y, z \in V$,

$$[x, [y, z]_1]_2 + [z, [x, y]_1]_2 + [y, [z, x]_1]_2 + \text{(Mixed Jacobi)}$$

$$= 0.$$ (1.4)

Let $\mathfrak{Lie}_2(n)$ be the multilinear component of the free Lie algebra on $[n]$ with two compatible brackets, that is, the multilinear component of the $k$-vector space generated by (mixed) bracketings of elements of $[n]$ subject only to the five relations given by (1.1) and (1.2), for each bracket, and (1.4). For each $i$, let $\mathfrak{Lie}_2(n, i)$ be the subspace of $\mathfrak{Lie}_2(n)$ generated by bracketed permutations with exactly $i$ brackets of the first type and $n - 1 - i$ brackets of the second type. The symmetric group $S_n$ acts naturally on $\mathfrak{Lie}_2(n)$ and since this action preserves the number of brackets of each type, we have the following decomposition into $S_n$-submodules:

$$\mathfrak{Lie}_2(n) = \bigoplus_{i=0}^{n-1} \mathfrak{Lie}_2(n, i).$$

Note that interchanging the roles of the two brackets makes evident the $S_n$-module isomorphism
\[ \text{Lie}_2(n, i) \cong_{\mathfrak{S}_n} \text{Lie}_2(n, n - 1 - i) \]

for every \( i \). Also note that in particular \( \text{Lie}(n) \) is isomorphic to the submodules \( \text{Lie}_2(n, i) \) when \( i = 0 \) or \( i = n - 1 \).

It was conjectured by Feigin and proved independently by Dotsenko-Khoroshkin [13] and Liu [29] that

\[ \dim \text{Lie}_2(n) = n^{n-1}. \]

In [29] Liu proves the conjecture by constructing a combinatorial basis for \( \text{Lie}_2(n) \) indexed by rooted trees giving as a byproduct the refinement

\[ \dim \text{Lie}_2(n, i) = |T_{n,i}|, \]

where \( T_{n,i} \) is the set of rooted trees on vertex set \( [n] \) with \( i \) descending edges (a parent with a greater label than its child).

The Dotsenko-Khoroshkin proof [13, 14] of Feigin’s conjecture was operad-theoretic; they used a pair of functional equations that apply to Koszul operads to compute the \( SL_2 \times \mathfrak{S}_n \)-character of \( \text{Lie}_2(n) \). They also proved that the dimension generating polynomial has a nice factorization:

\[ \sum_{i=0}^{n-1} \dim \text{Lie}_2(n, i)t^i = \prod_{j=1}^{n-1} ((n - j) + jt). \]

Since, as can be obtained from the results in [16] (see also [19] and [15]), the right hand side of (1.7) is equal to the generating function for rooted trees on node set \( [n] \) according to the number of descents of the tree, it follows that for each \( i \), the dimension of \( \text{Lie}_2(n, i) \) equals the number of rooted trees on node set \( [n] \) with \( i \) descents. (This product formula is a refinement of the well-known result that the number of trees on node set \( [n] \) is \( n^{n-1} \).)

Although Dotsenko and Khoroshkin [13] did not use poset theoretic techniques in their ultimate proof of (1.5), they introduced the poset of weighted partitions \( \Pi_w^n \) as a possible approach to establishing Koszulness of the operad associated with \( \text{Lie}_2(n) \), a key step in their proof. In [22] Wachs and the author applied poset theoretic techniques to the poset of weighted partitions to give an alternative proof of (1.5) and (1.6) and to obtain further results on \( \text{Lie}_2(n) \).

The poset \( \Pi_w^n \) has a minimum element \( \hat{0} := \{\{1\}^0, \{2\}^0, \ldots, \{n\}^0\} \) and \( n \) maximal elements \( \{\{n\}^0\}, \{\{n\}^1\}, \ldots, \{\{n\}^{n-1}\} \). For each \( i \), the maximal intervals \([\hat{0}, [n]^i]\) and \([\hat{0}, [n]^{n-1-i}]\) are isomorphic to each other,
and the two maximal intervals $[\hat{0}, [n]^0]$ and $[\hat{0}, [n]^{n-1}]$ are isomorphic to $\Pi_n$.

In [22] the authors found a nice EL-labeling of $\Pi_n^w \cup \{\hat{1}\}$ that generalized a classical EL-labeling of $\Pi_n$ due to Björner and Stanley (see [5]). An EL-labeling of a poset (defined in Section 3.2) is a labeling of the edges of the Hasse diagram of the poset that satisfies certain requirements. Such a labeling has important topological and algebraic consequences, such as the determination of the homotopy type of each open interval of the poset. The so called ascent-free maximal chains give a basis for cohomology of the open intervals. A poset that admits an EL-labeling is said to be EL-shellable. See [5], [7] and [43] for further information.

**Theorem 1.1** (Theorem 3.2, Corollary 3.5 and Theorem 3.6 [22]). The poset $\hat{\Pi}_n^w := \Pi_n^w \cup \{\hat{1}\}$ is EL-shellable and hence Cohen-Macaulay. Consequently, for each $i = 0, \ldots, n-1$, the order complex $\Delta((\hat{0}, [n]^i))$ has the homotopy type of a wedge of $|T_{n,i}|$ spheres.

Theorem 1.1 implies that the unique nonvanishing cohomology of $\Delta((\hat{0}, [n]^i))$ is the top cohomology $\tilde{H}^{n-3}((\hat{0}, [n]^i))$ and that its dimension is $|T_{n,i}|$, the same dimension as $\mathcal{L}ie_2(n, i)$ by equation (1.6). The symmetric group acts naturally on each $\mathcal{L}ie_2(n, i)$ and on each open interval $(\hat{0}, [n]^i)$. In [22] González D’León and Wachs give an explicit $S_n$-module isomorphism which establishes

$$\mathcal{L}ie_2(n, i) \simeq_{S_n} \tilde{H}^{n-3}((\hat{0}, [n]^i)) \otimes \text{sgn}_n. \quad (1.8)$$

For $i = 0$ or $i = n-1$, equation (1.8) reduces to equation (1.3) and the isomorphism reduces to the one in [41]. In [22] bases for $\tilde{H}^{n-3}((\hat{0}, [n]^i))$ and for $\mathcal{L}ie_2(n, i)$ are constructed generalizing the classical Lyndon tree basis and the comb basis for $\tilde{H}^{n-3}(\Pi_n)$ and $\mathcal{L}ie(n)$. In particular, the general Lyndon basis is obtained from the ascent-free maximal chains of the EL-labeling of Theorem 1.1. The authors of [22] also define a basis for $\tilde{H}_{n-3}((\hat{0}, [n]^i))$ in terms of labeled rooted trees that generalizes the Björner NBC basis for homology of $\Pi_n$ (see [6] Proposition 2.2).

1.2. **Multibracketed Lie algebras.** Liu posed the following natural question.

**Question 1.2** (Liu [29], Question 11.7). Is it possible to define an $S_n$-module $\mathcal{L}ie_k(n)$ for any $k \geq 1$ so that it has nice dimension formulas like those for $\mathcal{L}ie(n)$ and $\mathcal{L}ie_2(n)$? What are the right combinatorial objects to describe bases and compute dimensions for such $\mathcal{L}ie_k(n)$?
The results developed in this paper provide an answer to this question.

Let \( \mathbb{N} \) denote the set of nonnegative integers and \( \mathbb{P} \) the set of positive integers. We say that a set \( B \) of Lie brackets on a vector space is *compatible* if every linear combination of the brackets in \( B \) is a Lie bracket (see Proposition 2.1 for an equivalent definition). We now consider compatible Lie brackets \([\cdot, \cdot]_j\) indexed by positive integers \( j \in \mathbb{P} \) and define \( \mathcal{L}ie_{\mathbb{P}}(n) \) to be the multilinear component of the free multi-bracketed Lie algebra on \([n]\); that is, the \( k \)-vector space generated by \([n]\) subject only to the relations given by \([1.1]\) and \([1.2]\), for each bracket, and the compatibility relations for any set of brackets. For example, \([[2, 5], 3]_{1}, [1, 4]_{1} \) is a generator of \( \mathcal{L}ie_{\mathbb{P}}(5) \).

A *weak composition* \( \mu \) of \( n \) is a sequence of nonnegative integers \((\mu(1), \mu(2), \ldots)\) such that \(|\mu| := \sum_{i \geq 1} \mu(i) = n\). Let \( wcomp \) be the set of weak compositions and \( wcomp_n \) the set of weak compositions of \( n \). For \( \mu \in wcomp_{n-1} \), define \( \mathcal{L}ie(\mu) \) to be the subspace of \( \mathcal{L}ie_{\mathbb{P}}(n) \) generated by bracketed permutations of \([n]\) with \( \mu(j) \) brackets of type \( j \) for each \( j \). For example \( \mathcal{L}ie(0, 1, 2, 0, 1) \) is generated by bracketed permutations of \([5]\) that contain one bracket of type 2, two brackets of type 3, one bracket of type 5 and no brackets of any other type.

As before, \( \mathfrak{S}_n \) acts naturally on \( \mathcal{L}ie(\mu) \) by replacing the letters of a bracketed permutation. Interchanging the roles of the brackets reveals that for every \( \nu, \mu \in wcomp_{n-1} \), such that \( \nu \) is a rearrangement of \( \mu \), we have that \( \mathcal{L}ie(\nu) \simeq_{\mathfrak{S}_n} \mathcal{L}ie(\mu) \). In particular, if \( \mu \) has a single nonzero component, \( \mathcal{L}ie(\mu) \) is isomorphic to \( \mathcal{L}ie(n) \). If \( \mu \) has at most two nonzero components then \( \mathcal{L}ie(\mu) \) is isomorphic to \( \mathcal{L}ie(n, i) \) for some \( 0 \leq i \leq n - 1 \).

For \( \mu \in wcomp_n \) define its *support* \( \text{supp}(\mu) = \{ j \in \mathbb{P} \mid \mu(j) \neq 0 \} \) and for a subset \( S \subseteq \mathbb{P} \) let

\[
\mathcal{L}ie_{S}(n) := \bigoplus_{\mu \in wcomp_{n-1} \atop \text{supp}(\mu) \subseteq S} \mathcal{L}ie(\mu).
\]

Note that \( \mathcal{L}ie_{k}(n) := \mathcal{L}ie_{[k]}(n) \) generalizes \( \mathcal{L}ie(n) = \mathcal{L}ie_{1}(n) \) and \( \mathcal{L}ie_{2}(n) \).

The isomorphisms \([1.3]\) and \([1.8]\) provide a way to study the algebraic objects \( \mathcal{L}ie(n) \) and \( \mathcal{L}ie_{2}(n) \) by applying poset topology techniques to \( \Pi_n \) and \( \Pi_{n}^{w} \). In particular the dimensions of the modules can be read from the structure of the posets and the bases for the cohomology of the posets can be directly translated into bases of \( \mathcal{L}ie(n) \) and \( \mathcal{L}ie_{2}(n) \).

It is then natural to look for a poset whose cohomology allows us to analyze \( \mathcal{L}ie_{k}(n) \).
1.3. The poset of weighted partitions. We introduce a more general poset of weighted partitions $\Pi_n^k$ where the weights are given by weak compositions supported in $[k]$. A (composition)-weighted partition of $[n]$ is a set $\{B_1^\mu, B_2^\mu, \ldots, B_i^\mu\}$ where $\{B_1, B_2, \ldots, B_t\}$ is a set partition of $[n]$ and $\mu_i \in \text{wcomp}_{|B_i|-1}$ with $\text{supp}(\mu_i) \subseteq [k]$. For $\nu, \mu \in \text{wcomp}$, we say that $\mu \leq \nu$ if $\mu(i) \leq \nu(i)$ for every $i$. Since weak compositions are infinite vectors we can use component-wise addition and subtraction, for instance, we denote by $\nu + \mu$, the weak composition defined by $(\nu + \mu)(i) := \nu(i) + \mu(i)$.

The poset of weighted partitions $\Pi_n^k$ is the set of weighted partitions of $[n]$ with order relation given by $\{A_1^\mu, A_2^\mu, \ldots, A_s^\mu\} \leq \{B_1^\nu, B_2^\nu, \ldots, B_t^\nu\}$ if the following conditions hold:

- $\{A_1, A_2, \ldots, A_s\} \subseteq \{B_1, B_2, \ldots, B_t\}$ in $\Pi_n$
- If $B_j = A_i \cup A_i \cup \ldots \cup A_i$ then $\nu_j \geq (\mu_1 + \mu_2 + \ldots + \mu_t)$ and $|\nu_j - (\mu_1 + \mu_2 + \ldots + \mu_t)| = l - 1$

Equivalently, we can define the covering relation $\{A_1^\mu, A_2^\mu, \ldots, A_s^\mu\} \preceq \{B_1^\nu, B_2^\nu, \ldots, B_t^\nu\}$ by:

- $\{A_1, A_2, \ldots, A_s\} \prec \{B_1, B_2, \ldots, B_t\}$ in $\Pi_n$
- if $B_j = A_i \cup A_i$ then $\nu_j - (\mu_1 + \mu_2) = e_r$ for some $r \in [k]$, where $e_r$ is the weak composition with a 1 in the $r$-th component and 0 in all other entries.
- if $B_k = A_i$ then $\nu_k = \mu_i$.

In Figure 1 below the set brackets and commas have been omitted.

![Figure 1](image-url)

**Figure 1.** Weighted partition poset for $n = 3$ and $k = 3$

The poset $\Pi_n^k$ has a minimum element

$$\hat{0} := \{1\}^{(0,\ldots,0)}, \{2\}^{(0,\ldots,0)}, \ldots, \{n\}^{(0,\ldots,0)}$$

and $\binom{k+n-2}{n-1}$ maximal elements

$$\{[n]^\mu\} \text{ for } \mu \in \text{wcomp}_{n-1} \text{ and } \text{supp}(\mu) \subseteq [k].$$
We write each maximal element \( \{[n]^{\mu}\} \) as \([n]^{\mu}\) for simplicity. Note that for every \( \nu, \mu \in w\text{comp}_{n-1} \) with \( \text{supp}(\nu), \text{supp}(\mu) \subseteq [k] \), such that \( \nu \) is a rearrangement of \( \mu \), the maximal intervals \( [0, [n]^{\nu}] \) and \( [0, [n]^{\mu}] \) are isomorphic to each other. In particular, if \( \mu \) has a single nonzero component, these intervals are isomorphic to \( \Pi_n \). Indeed, we can think of a composition \((i, n-1-i)\) as being the weight \( i \) in the poset \( \Pi_k \) in \( [22] \). Hence \( \Pi_1 \approx \Pi_n \) and \( \Pi_2 \approx \Pi_2 \).

1.4. Main results. The symmetric group acts naturally on each open interval \((\hat{0}, [n]^{\mu})\). Using Wachs’ technique in Section 2 we give an explicit isomorphism that proves the following theorem.

**Theorem 1.3.** For \( \mu \in \text{wcomp}_{n-1} \),

\[
\mathcal{L} \text{ie}(\mu) \simeq_{\text{sgn}_n} \tilde{H}^{n-3}(\hat{0}, [n]^{\mu}) \otimes \text{sgn}_n.
\]

Theorem 1.3 is a generalization of equations (1.3) and (1.8). It reduces to equation (1.3) when \( \text{supp}(\mu) \subseteq [1] \) and to equation (1.8) when \( \text{supp}(\mu) \subseteq [2] \). We use Theorem 1.3 to give information about \( \mathcal{L} \text{ie}(\mu) \) by studying the algebraic and combinatorial properties of the poset \( \Pi_k \).

In [14] Dotsenko and Khoroshkin prove using operad-theoretic techniques that the operad related to \( \mathcal{L} \text{ie}_k(n) \) is Koszul. This implies using Vallette’s theory of operadic partition posets [40] that the maximal intervals \([\hat{0}, [n]^{\mu}]\) of \( \Pi_k \) are Cohen-Macaulay. In Section 3.2 we prove a stronger property.

**Theorem 1.4.** The poset \( \hat{\Pi}_n^k := \Pi_n^k \cup \{\hat{1}\} \) is EL-shellable and hence Cohen-Macaulay. Consequently, for each \( \mu \in \text{wcomp}_{n-1} \), the order complex \( \Delta((\hat{0}, [n]^{\mu})) \) has the homotopy type of a wedge of \((n-3)\)-spheres.

Using Vallette’s theory, Theorem 1.4 gives a new proof of the fact that the operads \( \mathcal{L} \text{ie}_k \) and \( k \text{Com} \) considered in [14] are Koszul.

The set of ascent-free maximal chains of this EL-labeling provides a basis for \( \tilde{H}^{n-3}((\hat{0}, [n]^{\mu})) \) and hence, by the isomorphism of Theorem 1.3, also a basis for \( \mathcal{L} \text{ie}(\mu) \). This basis is a multicolored generalization of the classical Lyndon basis for \( \mathcal{L} \text{ie}(n) \). We also construct a multicolored generalization of the classical comb basis for \( \mathcal{L} \text{ie}(n) \) and use our multicolored Lyndon basis to show that our construction does indeed yield a basis for \( \mathcal{L} \text{ie}(\mu) \).

We consider the generating function

\[
L_n(x) := \sum_{\mu \in \text{wcomp}_n} \dim \mathcal{L} \text{ie}(\mu) x^{\mu},
\]

(1.9)
where $x^\mu = x_1^{\mu_1}(x_2^{\mu_2}) \cdots$. Since for any rearrangement $\nu$ of $\mu$ it happens that $\text{Lie}(\nu) \cong_{\mathbb{C}}\text{Lie}(\mu)$ it follows that (1.9) belongs to the ring of symmetric functions $\Lambda_{\mathbb{Z}}$. The following theorem gives a characterization of this symmetric function.

**Theorem 1.5.** We have

$$\sum_{n \geq 1} \sum_{\mu \in \text{wcomp}_{n-1}} \dim \text{Lie}(\mu) x^\mu y^n \frac{n!}{n!} = \left[ \sum_{n \geq 1} (-1)^{n-1} h_{n-1}(x) \frac{y^n}{n!} \right]^{-1},$$

where $h_n$ is the complete homogeneous symmetric function and $(\cdot)^{-1}$ denotes the compositional inverse of a formal power series.

It follows from our construction of the multicolored Lyndon basis for $\text{Lie}(\mu)$ that the symmetric function $L_n(x)$ is $e$-positive; i.e., the coefficients of the expansion of $L_n(x)$ in the basis of elementary symmetric functions are all nonnegative. We give various combinatorial interpretations of these coefficients in this paper. Two of the interpretations involve binary trees and two involve the Stirling permutations introduced by Gessel and Stanley in [18]. We will now give one of the binary tree interpretations (Theorem 1.6). The others are given in Theorems 4.3 and 4.9.

We say that a planar labeled binary tree with label set $[n]$ is normalized if the leftmost leaf of each subtree has the smallest label in the subtree. See Figure 2 for an example of a normalized tree and Section 3.3 for the proper definitions. We denote the set of normalized binary trees with label set $[n]$ by $\text{Nor}_n$.

![Figure 2. Example of a normalized tree](image)

We associate a type (or integer partition) to each $\Upsilon \in \text{Nor}_n$ in the following way: Let $\pi_{\text{Comb}}(\Upsilon)$ be the finest (set) partition of the set of internal nodes of $\Upsilon$ satisfying

- for every pair of internal nodes $x$ and $y$ such that $y$ is a right child of $x$, $x$ and $y$ belong to the same block of $\pi_{\text{Comb}}(\Upsilon)$. 


We define the **comb type** \( \lambda^{\text{Comb}}(\Upsilon) \) of \( \Upsilon \) to be the (integer) partition whose parts are the sizes of the blocks of \( \pi^{\text{Comb}}(\Upsilon) \). In Figure 2 the associated partition is \( \lambda^{\text{Comb}}(\Upsilon) = (3, 2, 1, 1) \). The following theorem gives a direct method, alternative to Theorem 1.5, for computing the dimensions of \( \mathcal{L} \text{ie}(\mu) \).

**Theorem 1.6.** For all \( n \),

\[
\sum_{\mu \in \text{wcomp}_{n-1}} \dim \mathcal{L} \text{ie}(\mu) x^\mu = \sum_{\Upsilon \in \text{Nor}_n} e_{\lambda^{\text{Comb}}(\Upsilon)}(x),
\]

where \( e_\lambda \) is the elementary symmetric function associated with the partition \( \lambda \).

To prove Theorem 1.6 we use another normalized tree type \( \lambda^{\text{Lyn}} \), related to our colored Lyndon basis for \( \mathcal{L} \text{ie}(\mu) \), which come from the EL-labeling of \([\hat{0}, [n]]\). We use the colored Lyndon basis to show that Theorem 1.6 holds with \( \lambda^{\text{Comb}} \) replaced by \( \lambda^{\text{Lyn}} \). We then construct a bijection on \( \text{Nor}_n \) which takes \( \lambda^{\text{Lyn}} \) to \( \lambda^{\text{Comb}} \). This bijection makes use of Stirling permutations and leads to two versions of Theorem 1.6 involving Stirling permutations.

In terms of these combinatorial objects, the dimension of \( \mathcal{L} \text{ie}_k \) has a simple description as an evaluation of the symmetric function (1.9).

**Corollary 1.7.** For all \( n \) and \( k \),

\[
\dim \mathcal{L} \text{ie}_k(n) = \sum_{\Upsilon \in \text{Nor}_n} e_{\lambda^{\text{Comb}}(\Upsilon)}(1, \ldots, 1, 0, 0, \ldots, k \, \text{times}).
\]

From equation (1.7), it follows that the polynomial \( \sum_{i=0}^{n-1} \dim \mathcal{L} \text{ie}_2(n, i) t^i \) has only negative real roots and hence it has a property known as \( \gamma \)-**positivity**, i.e., when written in the basis \( t^i(1 + t)^{n-1-2i} \) it has positive coefficients. Note that this polynomial is actually \( L_{n-1}(t, 1, 0, 0, \ldots) \). The property of \( \gamma \)-positivity of this polynomial is a consequence of the \( e \)-positivity of \( L_n(x) \).

A more general question is to understand the representation of \( \mathfrak{S}_n \) on \( \mathcal{L} \text{ie}(\mu) \). The characters of the representation of \( \mathfrak{S}_n \) on \( \mathcal{L} \text{ie}(n) \) and \( \mathcal{L} \text{ie}_2(n) \) were computed in ([10], [36], and [13]). Here we consider

(1.10)

\[
\sum_{\mu \in \text{wcomp}_{n-1}} \text{ch} \mathcal{L} \text{ie}(\mu) x^\mu,
\]

where \( \text{ch} \mathcal{L} \text{ie}(\mu) \) denotes the Frobenius characteristic in variables \( y = (y_1, y_2, \ldots) \) of the representation \( \mathcal{L} \text{ie}(\mu) \). The generating function of
(1.10) belongs to the ring $\Lambda_R$ of symmetric functions in $y$ with coefficients in the ring of symmetric functions $R = \Lambda_Q$ in $x$. The following result generalizes Theorem 1.5.

**Theorem 1.8.** We have that

$$\sum_{n \geq 1} \sum_{\mu \in \text{wcomp}_{n-1}} \text{ch} \text{Lie}(\mu) x^{\mu} = -\left( - \sum_{n \geq 1} h_{n-1}(x) h_n(y) \right)^{[-1]},$$

where $(\cdot)^{[-1]}$ denotes the plethystic inverse in the ring of symmetric power series in $y$ with coefficients in the ring $\Lambda_Q$ of symmetric functions in $x$.

To prove Theorem 1.8 we use Theorem 1.3 and the Whitney (co)homology technique developed by Sundaram in [39], and further developed by Wachs in [42].

The paper is organized as follows: In Section 2 we describe generating sets of $\text{Lie}(\mu)$ and $\tilde{H}^{n-3}((\hat{0}, [n]^\mu))$ in terms of labeled binary trees with colored internal nodes. The description makes transparent the isomorphism of Theorem 1.3 which we prove using Wachs’ technique, as in [41] and [22]. In Section 3 we use the recursive definition of the Möbius invariant of $\Pi_n^k$ to prove an analogue of Theorem 1.5 for the poset $\Pi_n^k$, that together with the results of Section 2 implies Theorem 1.5. We also prove Theorem 1.4 and we give a description of the ascent-free maximal chains of the EL-labeling. Theorem 1.5 and the version of Theorem 1.6 in which $\lambda^{\text{Comb}}$ is replaced by $\lambda^{\text{Lyn}}$, are presented in Section 4 as corollaries of results in the previous chapters. We prove Theorem 1.6 and we use the language of Stirling permutations to give two additional combinatorial descriptions of the dimension of $\text{Lie}(\mu)$. In Section 5 we summarize some of the results of Sections 3 and 4 on the colored Lyndon basis and we present the colored comb basis for $\text{Lie}(\mu)$ and $\tilde{H}^{n-3}((\hat{0}, [n]^\mu))$. We also discuss bases for $\tilde{H}^{n-2}(\Pi_n^k \setminus \{\hat{0}\})$ in terms of the two families of colored binary trees. We present in Section 6 results on Whitney numbers of the first and second kind and on Whitney cohomology. In section 7 we prove Theorem 1.8.

Some of the results in this work are generalizations of results in [22] and we refer the reader to that article for the context and some of the proofs.

2. **The isomorphism** $\text{Lie}(\mu) \simeq_{\mathfrak{S}_n} \tilde{H}^{n-3}((\hat{0}, [n]^\mu)) \otimes \text{sgn}_n$

In this section we establish the isomorphism of Theorem 1.3. We will use this isomorphism to study $\text{Lie}(\mu)$ by understanding the algebraic and combinatorial properties of the maximal intervals $[\hat{0}, [n]^\mu]$ of $\Pi_n^k$. 
In [22] Wachs and the author gave a proof of the isomorphism for the case \( k = 2 \), analogous to a proof for the case \( k = 1 \) in [41]. For the sake of completeness in the discussion, we reproduce some of the steps but omit the proofs that are similar to the ones in [22] and [41].

2.1. A combinatorial description of \( \text{Lie}(\mu) \). We give a description of the generators and relations of \( \text{Lie}(\mu) \). A tree is a simple connected graph that is free of cycles. A tree is said to be rooted if it has a distinguished node or root. For an edge \( \{x, y\} \) in a tree \( T \) we say that \( x \) is the parent of \( y \), or \( y \) is the child of \( x \), if \( x \) is in the unique path from \( y \) to the root. A node that has children is said to be internal, otherwise we call a node without children a leaf. A rooted tree is said to be planar if for every internal node its set of children has been totally ordered. In the following we will be only considering trees that are rooted and planar and so when using the word tree we mean a planar rooted tree.

A binary tree is a tree for which every internal node has two children that we call the left (or first) child and the right (or second) child (according to their total order). A colored binary tree is a binary tree for which each internal node \( x \) has been assigned an element \( \text{color}(x) \in \mathbb{P} \).

For a colored binary tree \( T \) with \( n \) leaves (and \( n - 1 \) internal nodes) and \( \sigma \in \mathfrak{S}_n \), we define the labeled colored binary tree \((T, \sigma)\) to be the colored tree \( T \) whose \( j \)th leaf from left to right has been labeled \( \sigma(j) \).

For \( \mu \in \text{wcomp}_{n-1} \) we denote by \( \mathcal{BT}_\mu \) the set of labeled colored binary trees with \( n \) leaves and \( \mu \) internal nodes with color \( j \) for each \( j \). We call these trees \( \mu \)-colored binary trees. We will often denote a labeled binary tree by \( \Upsilon = (T, \sigma) \). If \( \Upsilon \) is a colored labeled binary tree, we use \( \tilde{\Upsilon} \) to denote its underlying uncolored labeled binary tree. It will also be convenient to consider trees whose label set is more general than \([n]\). For a finite subset \( A \) of positive integers with \( |A| = |\mu| + 1 \), let \( \mathcal{BT}_{A,\mu} \) be the set of \( \mu \)-colored binary trees whose leaves are labeled by a permutation of \( A \). If \( (S, \alpha) \in \mathcal{BT}_{A,\mu} \) and \( (T, \beta) \in \mathcal{BT}_{B,\nu} \), where \( A \) and \( B \) are disjoint finite sets, and \( j \in \mathbb{P} \) then \( (S, \alpha) \langle_j (T, \beta) \) denotes the tree in \( \mathcal{BT}_{A \cup B, \mu + \nu + e_j} \) with root colored \( j \), whose left subtree is \( (S, \alpha) \) and whose right subtree is \( (T, \beta) \).

We can represent the bracketed permutations that generate \( \text{Lie}(\mu) \) with labeled colored binary trees. More precisely, let \( (T_1, \sigma_1) \) and \( (T_2, \sigma_2) \) be the left and right labeled subtrees of the root \( r \) of \( (T, \sigma) \in \mathcal{BT}_\mu \). Then define recursively

\[
[T, \sigma] = \begin{cases} 
[T_1, \sigma_1], [T_2, \sigma_2] & \text{if color}(r) = j \text{ and } n > 1 \\
\sigma & \text{if } n = 1.
\end{cases}
\]
Clearly $[T, \sigma]$ is a bracketed permutation of $\text{Lie} (\mu)$. See Figure 3.

Recall that we call a set $B$ of Lie brackets on a vector space *compatible* if every linear combination of the brackets in $B$ is a Lie bracket. As it turns out the description of the relations in $\text{Lie} (\mu)$ are simplified by the following proposition.

**Proposition 2.1.** A set of Lie brackets is compatible if and only if the brackets in the set are pairwise compatible.

**Proof.** Assume that the brackets $\{ [\cdot, \cdot]_j \mid j \in S \}$ are pairwise compatible. Hence for every $i, j \in S$ we have that the relation (1.4) holds.

Now for scalars $\alpha_j \in k$ and a finite subset $\{i_1, \ldots, i_k\} \subseteq S$ define

$$\langle \cdot, \cdot \rangle = \sum_{j=1}^{k} \alpha_j [\cdot, \cdot]_{i_j}.$$ 

By relations (1.2) and (1.4) and bilinearity of the brackets, we have

$$0 = \sum_{j=1}^{k} \alpha_j^2 ([x, [y, z]_j]_j + [z, [x, y]_j]_j + [y, [z, x]_j]_j)$$

$$+ \sum_{l<j} \alpha_l \alpha_j ([x, [y, z]_{i_l}]_{i_l} + [z, [x, y]_{i_l}]_{i_l} + [y, [z, x]_{i_l}]_{i_l})$$

$$+ [x, [y, z]_{i_l}]_{i_l} + [z, [x, y]_{i_l}]_{i_l} + [y, [z, x]_{i_l}]_{i_l})$$

$$= \sum_{l,j=1}^{k} \alpha_l \alpha_j [x, [y, z]_{i_l}]_{i_l} + \alpha_l \alpha_j [z, [x, y]_{i_l}]_{i_l} + \alpha_l \alpha_j [y, [z, x]_{i_l}]_{i_l}$$

$$= \langle x, \langle y, z \rangle \rangle + \langle z, \langle x, y \rangle \rangle + \langle y, \langle z, x \rangle \rangle.$$
This implies that \( \langle \cdot, \cdot \rangle \) satisfies relation (1.2). It follows from the definition that \( \langle \cdot, \cdot \rangle \) also satisfies the relation (1.1) and hence it is a Lie bracket.

For the converse note, from the definition of compatibility, that all the brackets in a compatible set of Lie brackets are pairwise compatible.

Thus we see that \( \mathcal{L}(\mu) \) is subject only to the relations (1.1) and (1.2), for each bracket \( j \), and (1.4) for every pair of brackets \( i, j \in [k] \).

If the characteristic of \( k \) is not 2 we can even say that \( \mathcal{L}(\mu) \) is subject only to relations (1.1) and (1.4) for every pair of brackets \( i, j \in [k] \) (including \( i = j \)).

We denote by \( \Upsilon_1 \wedge \Upsilon_2 \), the labeled colored binary tree whose left subtree is \( \Upsilon_1 \), right subtree is \( \Upsilon_2 \) and root color is \( j \), with \( j \in P \).

Using this notation we are able to represent any labeled colored binary tree as a properly parenthesized word with letters from the set \([n] \cup \{ i \mid j \in P \}\). As an example, the tree of Figure 3 can be represented as \(((3 \wedge 4) \wedge 6) \wedge (1 \wedge 5)) ((2 \wedge 7) \wedge (9 \wedge 8)).\) If \( \Upsilon \) is a labeled colored binary tree then \( \alpha(\Upsilon) \beta \) denotes an arbitrary labeled colored binary tree with \( \Upsilon \) as a subtree, where \( \alpha \) and \( \beta \) are respectively the prefix and suffix in the word representation of such tree. The following result is an easy consequence of relations (1.1) and (1.2) for each \( j \), and (1.4) for each pair \( i \neq j \).

**Proposition 2.2.** The set \{ \( [T, \sigma] \mid (T, \sigma) \in \mathcal{B}T_\mu \) \} is a generating set for \( \mathcal{L}(\mu) \), subject only to the relations for \( i \neq j \in \text{supp}(\mu) \)

\[
\begin{align*}
(2.1) \quad [\alpha(\Upsilon_1 \wedge \Upsilon_2) \beta] + [\alpha(\Upsilon_2 \wedge \Upsilon_1) \beta] = 0 \\
(2.2) \quad [\alpha(\Upsilon_1 \wedge (\Upsilon_2 \wedge \Upsilon_3)) \beta] - [\alpha((\Upsilon_1 \wedge \Upsilon_2) \wedge \Upsilon_3) \beta] \\
- [\alpha((\Upsilon_1 \wedge \Upsilon_2) \wedge \Upsilon_3) \beta] = 0 \\
(2.3) \quad [\alpha(\Upsilon_1 \wedge (\Upsilon_2 \wedge \Upsilon_3)) \beta] + [\alpha(\Upsilon_1 \wedge (\Upsilon_2 \wedge \Upsilon_3)) \beta] \\
- [\alpha((\Upsilon_1 \wedge \Upsilon_2) \wedge \Upsilon_3) \beta] - [\alpha((\Upsilon_1 \wedge \Upsilon_2) \wedge \Upsilon_3) \beta] \\
- [\alpha(\Upsilon_2 \wedge (\Upsilon_1 \wedge \Upsilon_3)) \beta] - [\alpha(\Upsilon_2 \wedge (\Upsilon_1 \wedge \Upsilon_3)) \beta] = 0.
\end{align*}
\]
2.2. A generating set for $\bar{H}^{n-3}(\hat{0}, [n]^{\mu})$. The top dimensional cohomology of a pure poset $P$, say of length $\ell$, has a particularly simple description. Let $\mathcal{M}(P)$ denote the set of maximal chains of $P$ and let $\mathcal{M}'(P)$ denote the set of chains of length $\ell - 1$. We view the coboundary map $\delta$ as a map from the chain space of $P$ to itself, which takes chains of length $d$ to chains of length $d + 1$ for all $d$. Since the image of $\delta$ on the top chain space (i.e. the space spanned by $\mathcal{M}(P)$) is 0, the kernel is the entire top chain space. Hence top cohomology is the quotient of the space spanned by $\mathcal{M}(P)$ by the image of the space spanned by $\mathcal{M}'(P)$. The image of $\mathcal{M}'(P)$ is what we call the coboundary relations. We thus have the following presentation of the top cohomology

$$\bar{H}^{\ell}(P) = \langle \mathcal{M}(P) | \text{coboundary relations} \rangle.$$ 

Recall that the postorder listing of the internal nodes of a binary tree $T$ is defined recursively as follows: first list the internal nodes of the left subtree in postorder, then list the internal nodes of the right subtree in postorder, and finally list the root. The postorder listing of the internal nodes of the binary tree of Figure 3 is illustrated in Figure 4a.

Let $A^{\mu_1}_1, A^{\mu_2}_2, \ldots, A^{\mu_s}_s$ be $s$ different blocks in a weighted partition $\alpha$ and let $\nu \in w\text{comp}_{s-1}$. We say that we $\nu$-merge these blocks if we remove them from $\alpha$ and replace them with the single block $(\bigcup A_i)^{\sum \mu_i + \nu}$ to obtain a new weighted partition. For $(T, \sigma) \in BT_{\mathcal{A}, \mu}$, let $\pi(T, \sigma) = A^\mu$.

**Definition 2.3.** For $(T, \sigma) \in BT_{\mu}$ and $t \in [n - 1]$, let $T_t = L_t \wedge R_t$ be the subtree of $(T, \sigma)$ rooted at the $t$th node listed in postorder. The chain $c(T, \sigma) \in \mathcal{M}(\hat{0}, [n]^{\mu})$ is the one whose rank $t$ weighted partition is obtained from the rank $t - 1$ weighted partition by $e_j$-merging the blocks $\pi(L_t)$ and $\pi(R_t)$. See Figure 4b.

Not all maximal chains in $\mathcal{M}(\hat{0}, [n]^{\mu})$ can be described as $c(T, \sigma)$. For some maximal chains postordering of the internal nodes is not enough to describe the process of merging the blocks. We need a more flexible construction in terms of linear extensions (cf. [41]). Let $v_1, \ldots, v_{n-1}$ be the postorder listing of the internal nodes of $T$. A listing $v_{\tau(1)}, v_{\tau(2)}, \ldots, v_{\tau(n-1)}$ of the internal nodes such that each node precedes its parent is said to be a linear extension of $T$. We will say that the permutation $\tau$ induces the linear extension. In particular, the identity permutation $\varepsilon$ induces postorder which is a linear extension. Denote by $e(T)$ the set of permutations that induce linear extensions of the internal nodes of $T$. We extend the construction of $c(T, \sigma)$ by letting $c(T, \sigma, \tau)$ be the chain in $\mathcal{M}(\hat{0}, [n]^{\mu})$ whose rank $t$ weighted partition
In particular, $c(T, \sigma, \tau)$ is obtained from the rank $t - 1$ weighted partition by $e_{j_{\tau(t)}}$-merging the blocks $\pi(L_{\tau(t)})$ and $\pi(R_{\tau(t)})$, where $L_v R_v$ is the subtree rooted at $v_i$. In particular, $c(T, \sigma) = c(T, \sigma, \varepsilon)$. From each maximal chain we can easily construct a binary tree and a linear extension that encodes the merging instructions along the chain. Thus, any maximal chain can be obtained in this form.

For any colored labeled binary tree $(T, \sigma)$, the chains obtained with any two different linear extensions are cohomologous in the sense of Lemma 2.4 below.

The number of inversions of a permutation $\tau \in \mathfrak{S}_n$ is defined by $\text{inv}(\tau) := |\{(i, j) \mid 1 \leq i < j \leq n, \tau(i) > \tau(j)\}|$ and the sign of $\tau$ is defined by $\text{sgn}(\tau) := (-1)^{\text{inv}(\tau)}$. For $T \in \mathcal{B}T_{n, \mu}$, $\sigma \in \mathfrak{S}_n$, and $\tau \in e(T)$, write $\bar{c}(T, \sigma, \tau)$ for $c(T, \sigma, \tau) := c(T, \sigma, \tau) \setminus \{0, [n]^\mu\}$ and $c(T, \sigma)$ for $c(T, \sigma) := c(T, \sigma) \setminus \{0, [n]^\mu\}$.

**Lemma 2.4 (cf. [41] Lemma 5.2).** Let $(T, \sigma) \in \mathcal{B}T_{\mu}$, $\tau \in e(T)$. Then in $\tilde{H}^{n-3}((0, [n]^\mu))$

$$\bar{c}(T, \sigma, \tau) = \text{sgn}(\tau)\bar{c}(T, \sigma).$$

The proof of Lemma 2.4 is essentially the same as [41] Lemma 5.2.

For $\Upsilon = (T, \sigma)$, let $I(\Upsilon)$ denote the set of internal nodes of $\Upsilon$. The following result generalizes [22] Theorem 4.4.

**Theorem 2.5.** The set $\{\bar{c}(T, \sigma) \mid (T, \sigma) \in \mathcal{B}T_{\mu}\}$ is a generating set for $\tilde{H}^{n-3}((0, [n]^\mu))$, subject only to the relations for $i \neq j \in \text{supp}(\mu)$.
The proof is analogous to [22, Theorem 4.3]. The “only” part follows from Proposition 5.2.

2.3. The isomorphism. The symmetric group \( \mathfrak{S}_n \) acts naturally on \( \Pi_n^k \). Indeed, let \( \sigma \in \mathfrak{S}_n \) act on the weighted blocks of \( \Pi_n^k \) by replacing each element \( x \) of each weighted block of \( \pi \) with \( \sigma(x) \). Since the maximal elements of \( \Pi_n^k \) are fixed by each \( \sigma \in \mathfrak{S}_n \) and the order is preserved, each open interval \((0, [n]^\mu)\) is an \( \mathfrak{S}_n \)-poset. Hence (see [43]) we have that \( ^nH^{|\pi|}((0, [n]^\mu)) \) is an \( \mathfrak{S}_n \)-module. The symmetric group \( \mathfrak{S}_n \) also acts naturally on \( \text{Lie}(\mu) \). Indeed, let \( \sigma \in \mathfrak{S}_n \) act by replacing letter \( x \) of a bracketed permutation with \( \sigma(x) \). Since this action preserves the number of brackets of each type, \( \text{Lie}(\mu) \) is an \( \mathfrak{S}_n \)-module for each \( \mu \in \text{wcomp}_{n-1} \). In this section we obtain an explicit sign-twisted isomorphism between the \( \mathfrak{S}_n \)-modules \( ^{n-3}H((0, [n]^\mu)) \) and \( \text{Lie}(\mu) \).

Define the sign of a binary tree \( T \) recursively by

\[
\text{sgn}(T) = \begin{cases} 
1 & \text{if } I(T) = \emptyset \\
(-1)^{|I(T)|} \text{sgn}(T_1) \text{sgn}(T_2) & \text{if } T = T_1 \wedge T_2,
\end{cases}
\]

where \( I(T) \) is the set of internal nodes of the binary tree \( T \). The sign of a colored (labeled or unlabeled) binary tree is defined to be the sign of the binary tree obtained by removing the colors and leaf labels.

**Theorem 2.6.** For each \( \mu \in \text{wcomp}_{n-1} \), there is an \( \mathfrak{S}_n \)-module isomorphism \( \varphi : \text{Lie}(\mu) \to ^{n-3}H((0, [n]^\mu)) \otimes \text{sgn}_n \) determined by

\[
\varphi([T, \sigma]) = \text{sgn}(\sigma) \text{sgn}(T) \tilde{c}(T, \sigma),
\]
for all \((T, \sigma) \in BT_\mu\).

**Proof.** The proof of this result is almost identical to the one in [22], so we omit the details. The map \(\varphi\) maps the generators and relations of Proposition 2.2 onto the generators and relations of Proposition 2.5 and clearly respects the \(\mathfrak{S}_n\) action. \(\square\)

3. Homotopy type of the intervals of \(\Pi^k_n\)

We assume familiarity with basic terminology and results in poset topology. The reader is referred to [22, Section 3 and Appendix] for a review of poset (co)homology. For further poset topology terminology not defined here the reader could also visit [38] and [43].

For \(u \leq v\) in a poset \(P\), the open interval \(\{w \in P \mid u < w < v\}\) is denoted by \((u, v)\) and the closed interval \(\{w \in P \mid u \leq w \leq v\}\) by \([u, v]\). A poset is said to be bounded if it has a minimum element \(\hat{0}\) and a maximum element \(\hat{1}\). For a bounded poset \(P\), we define the proper part of \(P\) as \(\hat{P} := P \setminus \{\hat{0}, \hat{1}\}\). A poset is said to be pure (or ranked) if all its maximal chains have the same length, where the length of a chain \(s_0 < s_1 < \cdots < s_\ell\) is \(\ell\). If \(P\) is pure and has a minimal element \(\hat{0}\), we can define a rank function \(\rho\) by requiring that \(\rho(\hat{0}) = 0\) and \(\rho(\beta) = \rho(\alpha) + 1\) whenever \(\alpha \prec \beta\) in \(P\).

The length \(\ell(P)\) of a poset \(P\) is the length of its longest chain. For a bounded poset \(P\), let \(\mu_P\) denote its Möbius function. The reason for the nonstandard notation \(\mu_P\) is that we have been using the symbol \(\mu\) to denote a weak composition.

3.1. Möbius invariant. For \(\alpha = \{A^{\mu_1}_1, \ldots, A^{\mu_s}_s\} \in \hat{\Pi}^k_n\), \(\alpha \neq \hat{1}\), and \(\nu \in \text{wcomp}_{n-1}\) such that \(\nu - \mu(\alpha) \in \text{wcomp}_{|\alpha|-1}\),

\[
\begin{align*}
(1) & \ [\alpha, \hat{1}] \text{ and } \hat{\Pi}^k_n \text{ are isomorphic posets,} \\
(2) & \ [\alpha, [n]^s] \text{ and } [\hat{0}, [\alpha]|^{\nu-\mu(\alpha)}] \text{ are isomorphic posets,} \\
(3) & \ [\hat{0}, \alpha] \text{ and } [\hat{0}, [A_1]|^{\mu_1}] \times \cdots \times [\hat{0}, [A_s]|^{\mu_s}] \text{ are isomorphic posets.}
\end{align*}
\]

Proposition 3.1 is a general statement that is satisfied by any partition poset associated to a basic set operad (see [40]) replacing the composition \(\mu\) by an element of the given operad (see also [32]).

Recall that \(x^\mu = x_1^{\mu(1)} \cdots x_k^{\mu(k)}\) and \((\cdot)^{-1}\) denotes compositional inverse. We use the recursive definition of the Möbius function \(\mu_P\) and the compositional formula to derive the following theorem.
**Theorem 3.2.** We have that
\[
\sum_{n \geq 1} \sum_{\mu \in \text{wcomp}_{n-1}} \bar{\mu}_{\Pi_n}(\hat{0}, [n]^{\mu}) x^{\mu} y^n n! = \left[ \sum_{n \geq 1} h_{n-1}(x_1, \ldots, x_k) y^n n! \right]^{-1},
\]
where $h_n$ is the complete homogeneous symmetric polynomial.

**Proof.** By the recursive definition of the Möbius function we have that
\[
\delta_{n,1} = \sum_{\mu \in \text{wcomp}_{n-1}} x^{\mu} \sum_{\hat{0} \leq \alpha \leq [n]^{\mu}} \bar{\mu}_{\Pi_n}(\alpha, [n]^{\mu})
\]
\[
= \sum_{\alpha \in \Pi_n} x^{\mu(\alpha)} \sum_{\nu \in \text{wcomp}_{|\alpha|-1}} \bar{\mu}_{\Pi_n}(\alpha, [|\alpha|]^{\nu}) x^{\mu-\mu(\alpha)}.
\]
Now using Proposition 3.1
\[
\delta_{n,1} = \sum_{\alpha \in \Pi_n} x^{\mu(\alpha)} \sum_{\nu \in \text{wcomp}_{|\alpha|-1}} \bar{\mu}_{\Pi_n}(\alpha, [|\alpha|]^{\nu}) x^{\mu-\mu(\alpha)}
\]
\[
= \sum_{\alpha \in \Pi_n} \prod_{i=1}^{|\alpha|} h_{|\alpha|-1}(x_1, \ldots, x_k) \sum_{\nu \in \text{wcomp}_{|\alpha|-1}} \bar{\mu}_{\Pi_n}(\alpha, [|\alpha|]^{\nu}) x^{\mu-\mu(\alpha)}.
\]
The last statement implies using the compositional formula see ([37, Theorem 5.1.4]) that the two power series are compositional inverses. \qed

### 3.2. EL-labeling.
Let $P$ be a bounded poset. An *edge labeling* is a map $\tilde{\lambda} : \mathcal{E}(P) \rightarrow \Lambda$, where $\mathcal{E}(P)$ is the set of edges of the Hasse diagram of a poset $P$ and $\Lambda$ is a fixed poset. We denote by

\[
\tilde{\lambda}(c) = \tilde{\lambda}(x_0, x_1) \tilde{\lambda}(x_1, x_2) \cdots \tilde{\lambda}(x_{t-1}, x_t),
\]
the word of labels corresponding to a maximal chain $c = (\hat{0} = x_0 \prec x_1 < \cdots < x_{t-1} < x_t = \hat{1})$. We say that $c$ is *increasing* if its word of labels $\tilde{\lambda}(c)$ is strictly increasing, that is, $c$ is increasing if $\tilde{\lambda}(x_0, x_1) < \tilde{\lambda}(x_1, x_2) < \cdots < \tilde{\lambda}(x_{t-1}, x_t)$.

We say that $c$ is *ascent-free* (or decreasing, or falling) if its word of labels $\tilde{\lambda}(c)$ has no ascents, i.e. $\tilde{\lambda}(x_i, x_{i+1}) \neq \tilde{\lambda}(x_{i+1}, x_{i+2})$, for all $i = 0, \ldots, t-2$. An *edge-lexicographical labeling* (EL-labeling, for short) of $P$ is an edge labeling such that in each closed interval $[x, y]$ of $P$, there
is a unique increasing maximal chain, and this chain lexicographically precedes all other maximal chains of \([x, y]\).

A classical EL-labeling for the partition lattice \(\Pi_n\) is obtained as follows. Let \(\Lambda = \{(i, j) \in [n-1] \times [n] \mid i < j\}\) with lexicographic order as the order relation on \(\Lambda\). If \(x < y\) in \(\Pi_n\) then \(y\) is obtained from \(x\) by merging two blocks \(A\) and \(B\), where \(\min A < \min B\). Let \(\bar{\lambda}(x, y) = (\min A, \min B)\). This defines a map \(\bar{\lambda} : E(\Pi_n) \to \Lambda\) (Note that \(\bar{\lambda}\) in this section is an edge labeling and not an integer partition).

By viewing \(\Lambda\) as the set of atoms of \(\Pi_n\), one sees that this labeling is a special case of an edge labeling for geometric lattices, which first appeared in Stanley \([35]\) and was one of Björner’s \([5]\) initial examples of an EL-labeling. A generalization of the Björner-Stanley EL-labeling was given in \([22]\) for the poset \(\Pi^w_n\). We generalize further this labeling by providing one for \(\Pi^k_n\) that reduces to the one in \([22]\) for \(k = 2\) and to the Björner-Stanley EL-labeling when \(k = 1\).

**Definition 3.3** (Poset of labels). For each \(a \in [n]\), let \(\Gamma_a := \{(a, b)^u \mid a < b \leq a + 1, u \in [k]\}\). We partially order \(\Gamma_a\) by letting \((a, b)^u \leq (a, c)^v\) if \(b \leq c\) and \(u \leq v\). Note that \(\Gamma_a\) is isomorphic to the direct product of the chain \(a + 1 < a + 2 < \cdots < n + 1\) and the chain \(1 < 2 < \cdots < k\). Now define \(\Lambda^k_n\) to be the ordinal sum \(\Lambda^k_n := \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_n\) (see Figure 5).

**Definition 3.4** (EL-labeling). If \(x < y\) in \(\Pi^k_n\) then \(y\) is obtained from \(x\) by \(e_r\)-merging two blocks \(A\) and \(B\) for some \(r \in [k]\), where \(\min A < \min B\).
Let
\[ \bar{\lambda}(x < y) = (\min A, \min B)' \]
This defines a map \( \bar{\lambda} : \mathcal{E}(\Pi_n^k) \to \Lambda_n^k \). We extend this map to \( \bar{\lambda} : \mathcal{E}(\Pi_n^k) \to \Lambda_n \) by letting \( \bar{\lambda}([n]^\mu, 1) = (1, n + 1)^1 \), for all \( \mu \in \text{wcomp}_n \) with \( \text{supp}(\mu) \subseteq [k] \) (See Figure 6).

**Remark 3.5.** Recall that when \( \mu \) has a single nonzero entry (equal to \( n - 1 \)), the interval \([0, [n]^\mu]\) is isomorphic to \( \Pi_n \). Note that \( \bar{\lambda} \) reduces to the Björner-Stanley EL-labeling on those intervals.

\[ \text{Figure 6. Labeling } \bar{\lambda} \text{ on } \Pi_3^3 \]

The proof of the following theorem follows the same ideas of [22, Theorem 3.2].

**Theorem 3.6.** The labeling \( \bar{\lambda} : \mathcal{E}(\Pi_n^k) \to \Lambda_n \) defined above is an EL-labeling of \( \Pi_n^k \).

Theorem 1.4 is then a corollary of Theorem 3.6 and the following theorem linking lexicographic shellability and topology.

**Theorem 3.7** (Björner and Wachs [8]). Let \( \bar{\lambda} \) be an EL-labeling of a bounded poset \( P \). Then for all \( x < y \) in \( P \),

1. the open interval \( (x, y) \) is homotopy equivalent to a wedge of spheres, where for each \( r \in \mathbb{N} \) the number of spheres of dimension \( r \) is the number of ascent-free maximal chains of the closed interval \( [x, y] \) of length \( r + 2 \).
2. the set \( \{ \bar{c} \mid c \text{ is an ascent-free maximal chain of } [x, y] \text{ of length } r + 2 \} \) forms a basis for cohomology \( \tilde{H}^r((x, y)) \), for all \( r \).
Since the Möbius invariant of a bounded poset $P$ equals the reduced Euler characteristic of the order complex of $P$ (see [38]), Theorem 3.7 and the Euler-Poincaré formula imply the following corollary.

**Corollary 3.8.** Let $P$ be a pure EL-shellable poset of length $n$. Then

1. $P$ has the homotopy type of a wedge of spheres all of dimension $n - 2$, where the number of spheres is $|\mu_P(\hat{0}, \hat{1})|$.
2. $P$ is Cohen-Macaulay, which means that $\tilde{H}_i((x, y)) = 0$ for all $x < y$ in $P$ and $i < l([x, y]) - 2$.

**Theorem 3.9** (Theorem 1.4). The poset $\hat{\Pi}_n^k := \Pi_n^k \cup \{\hat{1}\}$ is EL-shellable and hence Cohen-Macaulay. Consequently, for each $\mu \in wcomp_{n-1}$, the order complex $\Delta((\hat{0}, [n]^\mu))$ has the homotopy type of a wedge of $(n-3)$-spheres.

In [14] Dotsenko and Khoroshkin use operad theory to prove that the operad related to $\mathbb{L}ie_k(n)$ is Koszul, which implies using Vallette’s theory ([40]) that all intervals of $\Pi_n^k$ are Cohen-Macaulay. Theorem 1.4 is an extension of their result.

3.3. **The ascent-free maximal chains from the EL-labeling.** We will describe the ascent-free maximal chains of the maximal intervals $[\hat{0}, [n]^\mu]$ given by the EL-labeling of Theorem 3.6. A Lyndon tree is a labeled binary tree $(T, \sigma)$ such that for each internal node $x$ of $T$, the smallest leaf label of the subtree $T_x$ rooted at $x$ is in the left subtree of $T_x$ and the second smallest label is in the right subtree of $T_x$. An alternative characterization of a Lyndon tree is given in Proposition 3.10 below.

For each internal node $x$ of a labeled binary tree, let $L(x)$ denote the left child of $x$ and $R(x)$ denote its right child. For each node $x$ of a labeled binary tree $(T, \sigma)$ define its valency $v(x)$ to be the smallest leaf label of the subtree rooted at $x$. A Lyndon tree is depicted in Figure 7 illustrating the valencies of the internal nodes.

We say that a labeled binary tree is normalized if the leftmost leaf of each subtree has the smallest label in the subtree. This is equivalent to requiring that for every internal node $x$,

$$v(x) = v(L(x)).$$

Note that a normalized tree can be thought of simply as a labeled nonplanar binary tree (or a phylogenetic tree) that has been drawn in the plane following the convention above. We denote the set of normalized labeled binary trees on label set $[n]$ by $\text{Nor}_n$ and the set of normalized binary trees on some arbitrary finite subset $A$ of $\mathbb{P}$ by
It is well-known that there are \((2n - 3)!! := 1 \cdot 3 \cdots (2n - 3)\) phylogenetic trees on \([n]\) and so \(|\text{Nor}_n| = (2n - 3)!!\).

**Proposition 3.10** ([22, Proposition 5.6]). Let \((T, \sigma)\) be a labeled binary tree. Then \((T, \sigma)\) is a Lyndon tree if and only if it is normalized and for every internal node \(x\) of \(T\) we have

\[
(3.1) \quad v(R(L(x))) > v(R(x)).
\]

**Figure 7.** Example of a Lyndon tree. The numbers above the lines correspond to the valencies of the internal nodes.

We will say that an internal node \(x\) of a labeled binary tree \((T, \sigma)\) is a **Lyndon node** if (3.1) holds. Hence Proposition 3.10 says that \((T, \sigma)\) is a Lyndon tree if and only if it is normalized and all its internal nodes are Lyndon nodes.

A **colored Lyndon tree** is a normalized binary tree such that for any node \(x\) that is not a Lyndon node it must happen that

\[
(3.2) \quad \text{color}(L(x)) > \text{color}(x).
\]

For \(\mu \in \text{wcomp}_{n-1}\), let \(\text{Lyn}_\mu\) be the set of colored Lyndon trees in \(\mathcal{BT}_\mu\) and \(\text{Lyn}_n = \bigcup_{\mu \in \text{wcomp}_{n-1}} \text{Lyn}_\mu\). Note that equation (3.2) implies that the monochromatic Lyndon trees are just the classical Lyndon trees.

The set of bicolored Lyndon trees for \(n = 3\) is depicted in Figure 8.

We will show that the ascent-free maximal chains of the EL-labeling of \([\hat{0}, [n]]\) given in Theorem 3.6 are of the form \(c(T, \sigma, \tau)\), where \((T, \sigma) \in \text{Lyn}_\mu\) and \(\tau\) is the linear extension of the internal nodes of \(T\), which we now describe: It is easy to see that there is a unique linear extension of the internal notes of \((T, \sigma) \in \mathcal{BT}_\mu\) in which the valencies of the nodes weakly decrease. Let \(\tau_{T, \sigma}\) denote the permutation that induces this linear extension.
Theorem 3.11. The set \( \{ c(T, \sigma, \tau_{T, \sigma}) \mid (T, \sigma) \in \text{Lyn}_\mu \} \) is the set of ascent-free maximal chains of the EL-labeling of \([0, [n]^\mu]\) given in Theorem 3.6.

Proof. We begin by showing that \( c := c(T, \sigma, \tau) \) is ascent-free whenever \( (T, \sigma) \in \text{Lyn}_\mu \) and \( \tau = \tau_{T, \sigma} \). Let \( x_i \) be the \( i \)th internal node of \( T \) in postorder. Then by the definition of \( \tau_{T, \sigma} \),

\[
v(x_{\tau(1)}) \geq v(x_{\tau(2)}) \geq \cdots \geq v(x_{\tau(n-1)}),
\]

where \( v \) is the valency. For each \( i \), the \( i \)th letter of the label word \( \bar{\lambda}(c) \) is given by

\[
\bar{\lambda}_i(c) = (v(L(x_{\tau(i)})), v(R(x_{\tau(i)})))^{u_i} = (v(x_{\tau(i)}), v(R(x_{\tau(i)})))^{u_i},
\]

where \( u_i = \text{color}(x_{\tau(i)}) \). Note that since \( (T, \sigma) \) is normalized, \( v(R(x_{\tau(i)})) \neq v(R(x_{\tau(i+1)})) \) for all \( i \in [n-1] \). Now suppose the word \( \bar{\lambda}(c) \) has an ascent at \( i \). Then it follows from (3.3) that

\[
v(x_{\tau(i)}) = v(x_{\tau(i+1)}), \quad v(R(x_{\tau(i)})) < v(R(x_{\tau(i+1)})), \quad \text{and} \quad u_i \leq u_{i+1}.
\]

(3.4)

The equality of valencies implies that \( x_{\tau(i)} = L(x_{\tau(i+1)}) \) since \( (T, \sigma) \) is normalized and \( \tau \) is a linear extension. Hence by (3.4),

\[
v(R(L(x_{\tau(i+1)}))) < v(R(x_{\tau(i+1)})).
\]

It follows that \( x_{\tau(i+1)} \) is not a Lyndon node. So by the coloring restriction on colored Lyndon trees

\[
u_i = \text{color}(x_{\tau(i)}) = \text{color}(L(x_{\tau(i+1)})) > \text{color}(x_{\tau(i+1)}) = u_{i+1},
\]

which contradicts (3.4). Hence the chain \( c \) is ascent-free.
Conversely, assume \( c \) is an ascent-free maximal chain of \([\hat{0}, [n]^\mu]\). Then \( c = c(T, \sigma, \tau) \) for some bicolored labeled tree \((T, \sigma)\) and some permutation \( \tau \in \mathfrak{S}_{n-1} \). We can assume without loss of generality that \((T, \sigma)\) is normalized. Since \( c \) is ascent-free, \([3.3]\) holds. This implies that \( \tau \) is the unique permutation that induces the valency-decreasing linear extension, namely \( \tau_{T, \sigma} \).

If all internal nodes of \((T, \sigma)\) are Lyndon nodes we are done. So let \( i \in [n-1] \) be such that \( x_{\tau(i)} \) is not a Lyndon node. That is
\[
\text{color}(x_{\tau(i-1)}) > \text{color}(x_{\tau(i)})
\]
which is precisely what we need to conclude that \((T, \sigma)\) is a colored Lyndon tree. \(\square\)

From Theorem 3.7, Theorem 3.11 and Corollary 3.8, we have the following corollary.

**Corollary 3.12.** For all \( n \geq 1 \) and for all \( \mu \in \text{wcomp}_{n-1} \) with \( \text{supp}(\mu) \subseteq [k] \), the order complex \( \Delta((\hat{0}, [n]^\mu)) \) has the homotopy type of a wedge of \( |\text{Lyn}_\mu| \) spheres of dimension \( n-3 \). Consequently,
\[
\dim \tilde{H}^{n-3}((\hat{0}, [n]^\mu)) = |\text{Lyn}_\mu|
\]
and
\[
\bar{\mu}^{\Pi_k}(\hat{0}, [n]^\mu) = (-1)^{n-1}|\text{Lyn}_\mu|.
\]

4. THE DIMENSION OF \( \text{Lie}(\mu) \)

In this section we present various formulas for the dimension of \( \text{Lie}(\mu) \). We begin by using the isomorphism between \( \text{Lie}(\mu) \) and \( \tilde{H}^{n-3}((\hat{0}, [n]^\mu)) \) of Theorem 2.6 to transfer information on \( \tilde{H}^{n-3}((\hat{0}, [n]^\mu)) \) obtained in the previous section to \( \text{Lie}(\mu) \).

**Theorem 4.1** (Theorem 1.5). We have
\[
\sum_{n \geq 1} \sum_{\mu \in \text{wcomp}_{n-1}} \dim \text{Lie}(\mu) \frac{x^n y^n}{n!} = \left[ \sum_{n \geq 1} (-1)^{n-1} h_{n-1}(x) \frac{y^n}{n!} \right]^{-1}.
\]

**Proof.** From Corollary 3.12 we have that
\[
\bar{\mu}^{\Pi_k}(\hat{0}, [n]^\mu) = (-1)^{n-1} \dim \tilde{H}^{n-3}((\hat{0}, [n]^\mu)).
\]
The theorem now follows from Theorems 2.6 and 3.2 when we let $k$ get large.

We have a combinatorial description for the dimension of $\mathcal{L}ie(\mu)$.

**Theorem 4.2.** For all $n \geq 1$ and $\mu \in \text{wcomp}_{n-1}$,

$$\dim \mathcal{L}ie(\mu) = |\text{Lyn}_{\mu}|.$$  

**Proof.** We know from Corollary 3.12 that $\dim \tilde{H}^{n-3}((0,[n]^\mu)) = |\text{Lyn}_{\mu}|$. Hence, the isomorphism of Theorem 2.6 proves the theorem.

4.1. **Lyndon type of a normalized tree.** With a normalized tree $\Upsilon \in \text{Nor}_n$ we can associate a (set) partition $\pi^{\text{Lyn}}(\Upsilon)$ of the set of internal nodes of $\Upsilon$, defined to be the finest partition satisfying the condition:

- for every internal node $x$ that is not Lyndon, $x$ and $L(x)$ belong to the same block of $\pi^{\text{Lyn}}(\Upsilon)$.

For the tree in Figure 9 the shaded rectangles indicate the blocks of $\pi^{\text{Lyn}}(\Upsilon)$.

Note that the coloring condition (3.2) implies that in a colored Lyndon tree $\Upsilon$ there are no repeated colors in each block $B$ of the partition $\pi^{\text{Lyn}}(\Upsilon)$ associated with $\Upsilon$. Hence after choosing a set of $|B|$ colors for the internal nodes in $B$ there is a unique way to assign the different colors such that the colored tree $\Upsilon$ is a colored Lyndon tree (the colors must decrease towards the root in each block of $\pi^{\text{Lyn}}(\Upsilon)$).

Define the **Lyndon type** $\lambda^{\text{Lyn}}(\Upsilon)$ of a normalized tree (colored or uncolored) $\Upsilon$ to be the (integer) partition whose parts are the block sizes of the partition $\pi^{\text{Lyn}}(\Upsilon)$. For the tree $\Upsilon$ in Figure 9 we have $\lambda^{\text{Lyn}}(\Upsilon) = (3,2,2,1)$.

![Figure 9. Example of a colored Lyndon tree of type (3,2,2,1). The numbers above the lines correspond to the valencies of the internal nodes](image)
Let $e_\lambda(x)$ be the elementary symmetric function associated with the partition $\lambda$.

**Theorem 4.3.** For all $n$,

$$
\sum_{\mu \in \text{wcomp}_{n-1}} \dim \mathcal{L}ie(\mu) x^\mu = \sum_{\Upsilon \in \text{Nor}_n} e_{\lambda_{\text{Lyn}}(\Upsilon)}(x).
$$

**Proof.** For a colored labeled binary tree $\Psi$ we define the *content* $\mu(\Psi)$ of $G$ as the weak composition $\mu$ where $\mu(i)$ is the number of internal nodes of $\Psi$ that have color $i$. Recall that $\widetilde{\Psi}$ denotes the underlying uncolored labeled binary tree of $\Psi$. Note that the comments above imply that for $\Upsilon \in \text{Nor}_n$, the generating function of colored Lyndon trees associated with $\Upsilon$ is

$$
\sum_{\Psi \in \text{Lyn}_n, \Psi = \Upsilon} x^{\mu(\Psi)} = e_{\lambda_{\text{Lyn}}(\Upsilon)}(x).
$$

(4.1)

Indeed the internal nodes in a block of size $i$ in the partition $\pi_{\text{Lyn}}(\Upsilon)$ can be colored uniquely with any set of $i$ different colors and so the contribution from this block of $\pi_{\text{Lyn}}(\Upsilon)$ to the generating function in (4.1) is $e_i(x)$.

By Theorem 4.2

$$
\sum_{\mu \in \text{wcomp}_{n-1}} \dim \mathcal{L}ie(\mu) x^\mu = \sum_{\mu \in \text{wcomp}_{n-1}} |\text{Lyn}_\mu| x^\mu
$$

$$
= \sum_{\Psi \in \text{Lyn}_n} x^{\mu(\Psi)}
$$

$$
= \sum_{\Upsilon \in \text{Nor}_n} \sum_{\Psi \in \text{Lyn}_n, \Psi = \Upsilon} x^{\mu(\Psi)}
$$

$$
= \sum_{\Upsilon \in \text{Nor}_n} e_{\lambda_{\text{Lyn}}(\Upsilon)}(x),
$$

with the last equation following from (4.1). □

### 4.2. Stirling permutations.

A *Stirling permutation* on the set $[n]$ is a permutation of the multiset $\{1,1,2,2,\ldots,n,n\}$ such that for all $m \in [n]$, all numbers between the two occurrences of $m$ are larger than $m$. The set of Stirling permutations on $[n]$ will be denoted by $\mathcal{Q}_n$. For example, the permutation 12332144 is in $\mathcal{Q}_n$ but 43341122 is not since 3 is between the two occurrences of 4. Stirling permutations were introduced by Stanley and Gessel in [18] and have been also studied by Bóna, Park, Janson, Kuba, Panholzer and others (see [9, 33, 27, 24]).
For an arbitrary subset \( A := \{a_1, a_2, \ldots, a_n\} \) of positive integers, we denote by \( Q_A \), the set of Stirling permutations of \( A \); that is, permutations of the multiset \( \{a_1, a_1, a_2, a_2, \ldots, a_n, a_n\} \), satisfying the condition above.

It is known that \(|Q_{n-1}| = (2n - 3)!!\). So this set of Stirling permutations is equinumerous with the set \( \text{Nor}_n \) of normalized binary trees with label set \([n]\). We will present an explicit bijection between these two sets. Moreover, this bijection has some nice properties that allow us to translate the previous results in this section to the language of Stirling permutations and to ultimately prove Theorem 1.6.

### 4.3. Type of a Stirling permutation.

A segment \( u \) of a Stirling permutation \( \theta = \theta_1 \theta_2 \cdots \theta_{2n} \) is a subword of \( \theta \) of the form \( u = \theta_i \theta_{i+1} \cdots \theta_{i+\ell} \), i.e., all the letters of \( u \) are adjacent in \( \theta \). A block in a Stirling permutation \( \theta \) is a segment of \( \theta \) that starts and ends with the same letter. For example, \( 455774 \) is a block of \( 12245577413366 \). We define \( B_\theta(a) \) to be the block of \( \theta \) that starts and ends with the letter \( a \), and define \( \hat{B}_\theta(a) \) to be the segment obtained from \( B_\theta(a) \) after removing the two occurrences of the letter \( a \). For example, \( B_\theta(1) = 1224557741 \) in \( \theta = 12245577413366 \) and \( \hat{B}_\theta(1) = 22455774 \).

We call \((a, b)\) an ascending adjacent pair if \( a < b \) and the blocks \( B_\theta(a) \) and \( B_\theta(b) \) are adjacent in \( \theta \), i.e., \( \theta = \theta' B_\theta(a) B_\theta(b) \theta'' \). An ascending adjacent sequence of \( \theta \) of length \( k \) is a subsequence \( a_1 < a_2 < \cdots < a_k \) such that \((a_j, a_{j+1})\) is an ascending adjacent pair for \( j = 1, \ldots, k-1 \). Similarly, for a Stirling permutation \( \theta \in Q_n \) we call \((a, b)\) a terminally nested pair if \( a < b \) and the block \( B_\theta(b) \) is the last block in \( \hat{B}_\theta(a) \), i.e., \( \hat{B}_\theta(a) = \theta' B_\theta(b) \) for some Stirling permutation \( \theta' \). A terminally nested sequence of \( \theta \) of length \( k \) is a subsequence \( a_1 < a_2 < \cdots < a_k \) such that \((a_j, a_{j+1})\) is a terminally nested pair for \( j = 1, \ldots, k-1 \).

We can associate a type to a Stirling permutation \( \theta \in Q_n \) in two ways. We define the ascending adjacent type \( \lambda^{\text{AA}}(\theta) \), to be the partition whose parts are the lengths of maximal ascending adjacent sequences; and we define the terminally nested type \( \lambda^{\text{TN}}(\theta) \), to be the partition whose parts are the lengths of maximal terminally nested sequences. We will show that these two types are equinumerous in \( Q_n \).

**Example 4.4.** If \( \theta = 158851244667723399 \), then the maximal ascending adjacent sequences are \( 1239, 467, 5 \) and \( 8 \); then \( \lambda^{\text{AA}}(\theta) = (4, 3, 1, 1) \), which is a partition of \( n = 9 \). Also the maximal terminally nested sequences are \( 158, 27, 3, 4, 6 \) and \( 9 \); then \( \lambda^{\text{TN}}(\theta) = (3, 2, 1, 1, 1, 1) \), which is also a partition of \( n = 9 \).
It is easy to see that every Stirling permutation has a unique factorization \( \theta = B_\theta(a_1)B_\theta(a_2) \cdots B_\theta(a_k) \) into adjacent blocks. We call this factorization the block factorization of \( \theta \). For example, 12245577413366 has a block factorization 1224557741 – 33 – 66. A Stirling factorization of a Stirling permutation \( \theta \) is a decomposition \( \theta = \theta^1 \theta^2 \cdots \theta^\ell \), such that \( \theta^i \) is a Stirling permutation for all \( i \). Note that the block factorization of \( \theta \) is the finest Stirling factorization.

Denote by \( \kappa(a) := a_k \), the largest letter of the maximal terminally nested sequence \( a = a_1 < a_2 < \cdots < a_k \) of \( B_\theta(a) \) that contains \( a \). In \( \theta = 15885124467723399 \), we have for example that \( \kappa(1) = 8 \), \( \kappa(2) = 7 \) and \( \kappa(7) = 7 \). We define the following two types of restricted Stirling factorizations:

- The ascending adjacent factorization of \( \theta \) is the Stirling factorization \( \theta = \theta^1 \theta^2 \) in which \( \theta^1 \) is the shortest nonempty prefix of \( \theta \) such that if \( \theta^1 = \alpha B_\theta(a) \) and \( \theta^2 = B_\theta(b) \beta \) then \( a > b \).

  For example if \( \theta = 133155442662 \), then the ascending adjacent factorization of \( \theta \) is 133155 – 442662.

- The terminally nested factorization of \( \theta \) is the Stirling factorization \( \theta = \theta^1 \theta^2 \) in which \( \theta^1 \) is the shortest nonempty prefix of \( \theta \) such that if \( \theta^1 = B_\theta(a) \alpha \) and \( \theta^2 = B_\theta(b) \beta \) then \( \kappa(a) > b \).

  In the case of \( \theta = 133155442662 \), the terminally nested factorization of \( \theta \) is 13315544 – 2662.

An irreducible AA-word is a Stirling permutation that has no nontrivial ascending adjacent factorization. It is not difficult to see that an irreducible AA-word is a Stirling permutation of the form

\[
B_\theta(a_1)B_\theta(a_2) \cdots B_\theta(a_k) = a_1 \tau_1 a_1 a_2 \tau_2 a_2 \cdots a_{k-1} \tau_{k-1} a_{k-1} a_k \tau_k a_k,
\]

where \( a_1 < a_2 < \cdots < a_k \) and \( \tau_i \) are Stirling permutations for each \( i \).

An irreducible TN-word is a Stirling permutation that has no nontrivial terminally nested factorization. It is not difficult to see that an irreducible TN-word is a Stirling permutation of the form

\[
B_\theta(a) \alpha
\]

where \( \kappa(a) < a' \) for any letter \( a' \) in \( \alpha \). Equivalently, an irreducible TN-word is a Stirling permutation of the form

\[
a_1 \tau_1 a_2 \tau_2 a_{k-1} \tau_{k-1} a_k a_k a_k \cdots a_2 a_1 \tau_k
\]

where \( a_1 < a_2 < \cdots < a_k \) and \( \tau_i \) are Stirling permutations for each \( i \) with \( a_k < a' \) for any letter \( a' \) in \( \tau_k \).

The complete ascending adjacent (terminally nested) factorization of \( \theta \) is the factorization \( \theta = \theta^1 \theta^2 \cdots \theta^\ell \) that we obtain by factoring \( \theta \) into
\( \theta^1 \theta^2 \) by the ascending adjacent (resp., terminally nested) factorization and then recursively applying the same procedure to \( \theta^2 \).

Let \( A \) be a subset of the positive integers. We define a map \( \xi : \mathcal{Q}_A \rightarrow \mathcal{Q}_A \) recursively as follows:

1. If \( \theta = mm \) then \( \xi(\theta) = mm \).
2. If \( \theta \) is an irreducible AA-word \( a_1 \tau_1 a_1 a_2 \tau_2 a_2 \cdots a_k \tau_k a_k \) then
   \[ \xi(\theta) = a_1 \xi(\tau_1) a_2 \xi(\tau_2) \cdots a_k \xi(\tau_k-1) a_k a_k-1 \cdots a_2 a_1 \xi(\tau_k). \]
3. If \( \theta = \theta^1 \theta^2 \cdots \theta^l \) is the complete ascending adjacent factorization of \( \theta \) then
   \[ \xi(\theta) = \xi(\theta^1) \xi(\theta^2) \cdots \xi(\theta^l). \]

Step (2) guarantees that \( \xi \) is well-defined. Indeed, in an irreducible AA-word of the form given in (2), we have \( a_s < a_{s+1} < \cdots < a_k \) for any \( s \). Hence, we are inserting only letters that are greater than \( a_s \) between the two occurrences of \( a_s \).

The map \( \xi \) is in fact a bijection and it is not difficult to check that its inverse \( \xi^{-1} : \mathcal{Q}_A \rightarrow \mathcal{Q}_A \) is defined by:

1. If \( \theta = mm \) then \( \xi^{-1}(\theta) = mm \).
2. If \( \theta \) is an irreducible TN-word \( a_1 \tau_1 \cdots a_k \tau_k a_k \cdots \cdots a_2 a_1 \tau_k \) then
   \[ \xi^{-1}(\theta) = a_1 \xi^{-1}(\tau_1) a_2 \xi^{-1}(\tau_2) \cdots a_k \xi^{-1}(\tau_k-1) a_k-1 \cdots a_2 a_1 \xi^{-1}(\tau_k). \]
3. If \( \theta = \theta^1 \theta^2 \cdots \theta^l \) is the complete terminally nested factorization of \( \theta \) then
   \[ \xi^{-1}(\theta) = \xi^{-1}(\theta^1) \xi^{-1}(\theta^2) \cdots \xi^{-1}(\theta^l). \]

Step (2) guarantees that \( \xi^{-1} \) is well-defined since in an irreducible TN-word, \( a_k < b \) for any letter \( b \) in \( \tau_k \).

**Example 4.5.** Consider \( \theta = 237372499468861551 \). Its complete ascending factorization is \( 23737249946886 - 1551 \); then

\[
\xi(\theta) = \xi(23737249946886 - 1551) \\
= \xi(23737249946886) - \xi(1551) \\
= 2\xi(3377)4\xi(99)6642\xi(88) - 11\xi(55) \\
= 237734996642881155.
\]

Note that the maximal ascending adjacent sequences of \( \theta \) are \((246, 37, 1, 5, 8, 9)\) which are also the maximal terminally nested sequences of \( \xi(\theta) \). These observations hold in general.
Proposition 4.6. The map $\xi : Q_A \to Q_A$ is a well-defined bijection that satisfies:

1. $(i, j)$ is an ascending adjacent pair in $\theta$ if and only if $(i, j)$ is a terminally nested pair in $\xi(\theta)$,
2. $\lambda^{TN}(\xi(\theta)) = \lambda^{AA}(\theta)$.

Proof. Note that if $\theta = \theta^1\theta^2 \cdots \theta^l$ is the complete ascending adjacent factorization of a Stirling permutation $\theta$, then an ascending adjacent pair can only occur within one of the factors $\theta^i$. Similarly, if $\theta = \theta^1\theta^2 \cdots \theta^l$ is the complete terminally nested factorization of a Stirling permutation $\theta$, then a terminally nested pair can only occur within one of the factors $\theta^i$. Hence, without loss of generality, as a consequence of step (3) in the definitions of $\xi$ and $\xi^{-1}$, we can assume that the word $\theta$ is an irreducible AA-word or an irreducible TN-word. Then the first assertion follows directly from step (2) in the definitions of $\xi$ and $\xi^{-1}$ and induction on the length of $\theta$. The second assertion is an immediate consequence of the first. \hfill $\square$

From Proposition 4.6 we see that $\lambda^{AA}$ and $\lambda^{TN}$ are equidistributed on $Q_n$.

4.4. A bijection between normalized trees and Stirling permutations. Let $\hat{Q}_n$ be the set of permutations $\theta \in Q_n$ where $\theta_1 = \theta_{2n} = 1$. There is a natural bijection $\text{red} : \hat{Q}_n \to Q_{n-1}$ obtained by removing the leading and trailing 1 from $\theta = 1\theta'1$ and then reducing the word $\theta'$ by decreasing every letter in $\{2, \ldots, n\}$ by one. For example, $\text{red}(12332441) = 122133$. In greater generality, for $A$ a subset of the positive integers, let $\hat{Q}_A$ be the set of Stirling permutations of $A$ such that both the first and last letter of the permutation is $\min A$. Define the map $\tilde{\gamma} : \text{Nor}_A \to \hat{Q}_A$ recursively by:

1. If $\Upsilon = (\bullet, m)$ then $\tilde{\gamma}(\Upsilon) = mm$.
2. If $\Upsilon$ is of the form

```
       \gamma_j
     / \        \\
    /   \       \\
   /     \      \\
 m ------ \G_j-1 --- \gamma_j
     \\
   /     \      \\
    /   \       \\
       \gamma_1
```

then $\tilde{\gamma}(\Upsilon) = m\tilde{\gamma}(\gamma_1)\tilde{\gamma}(\gamma_2) \cdots \tilde{\gamma}(\gamma_{j-1})\tilde{\gamma}(\gamma_j)m$.

The function $\tilde{\gamma}$ is well-defined since the tree is normalized. Indeed, $m$ is the minimal letter and we always obtain a word with values greater than 1.

\[1] The same map has appeared before in \[12\].
than \( m \) between the two occurrences of \( m \). Proceeding by induction on the number of internal nodes of \( \Upsilon \), we have that the words \( \gamma(\Upsilon_i) \) are Stirling permutations for each \( i \) and so it is \( \gamma(\Upsilon) \).

It is not difficult to check that the inverse \( \tilde{\gamma}^{-1} : \hat{Q}_A \to \text{Nor}_A \) can also be defined recursively by

1. If \( \theta = mm \) then \( \tilde{\gamma}^{-1}(mm) = (\bullet, m) \).
2. If \( \theta = B_\theta(m) \) and \( \tilde{B}_\theta(m) = B_\theta(a_1)B_\theta(a_2) \cdots B_\theta(a_{j-1})B_\theta(a_j) \), then

\[
\tilde{\gamma}^{-1}(\theta) = \tilde{\gamma}^{-1}(B_\theta(a_j)) \cdot \tilde{\gamma}^{-1}(B_\theta(a_{j-1})) \cdots \tilde{\gamma}^{-1}(B_\theta(a_2)) \cdot \tilde{\gamma}^{-1}(B_\theta(a_1))
\]

The tree defined in the step above is clearly normalized. So we can encode any normalized binary tree with a permutation in \( \hat{Q}_n \). See Figure 10 for an example of the bijection.

We give an alternative description of \( \tilde{\gamma} \). First we extend the leaf labeling of \( \Upsilon \in \text{Nor}_n \) to a labeling \( \theta \) that includes the internal nodes. For each internal node \( x \), let \( \theta(x) \) be the smallest leaf label in the right subtree of the subtree of \( \Upsilon \) rooted at \( x \); for each leaf \( x \), let \( \theta(x) \) be the leaf label of \( x \) (See Figure 10). Let \( x_1, \ldots, x_{2n-1} \) be the listing of all the nodes of \( \Upsilon \) (internal and leaves) in postorder and let \( \theta(\Upsilon) := \theta(x_1)\theta(x_2) \cdots \theta(x_{2n-1}) \).

**Proposition 4.7.** For all \( \Upsilon \in \text{Nor}_n \),

\[
\tilde{\gamma}(\Upsilon) = \theta(\Upsilon)\theta(x_1)
\]

where \( x_1 \) is the leftmost leaf of \( \Upsilon \).
Proof. If $\Upsilon = (\bullet, m)$ is a single node then $\tilde{\gamma}(\Upsilon) = mm = \theta(\Upsilon)\theta(x_1)$. If $\Upsilon$ has internal nodes, it can be expressed as
\[
\Upsilon = (\ldots (((x_1, v(x_1)) \land \Upsilon_1) \land \Upsilon_2) \land \cdots \land \Upsilon_j)
\]
(as in step (2) of the definition of $\tilde{\gamma}$).

Let $y_i$ denote the parent of the root of $\Upsilon_i$ for each $i$. As a consequence of the definition of $\theta$, we have that $\theta(y_i) = \theta(z_i)$, where $z_i$ is the smallest leaf of $\Upsilon_i$. By induction, using the definition of $\tilde{\gamma}$,
\[
\tilde{\gamma}(\Upsilon) = v(x_1)\tilde{\gamma}(\Upsilon_1) \ldots \tilde{\gamma}(\Upsilon_j)v(x_1)
= \theta(x_1)\theta(y_1) \theta(y_2) \theta(y_3) \cdots \theta(y_j)\theta(x_1)
= \theta(\Upsilon)\theta(x_1).
\]
The last step holds since the postorder traversal of $\Upsilon$ lists first $x_1$, followed by postorder traversal of $\Upsilon_1$ followed by $y_1$, followed by postorder traversal of $\Upsilon_2$ followed by $y_2$, and so on. $\square$

To remove the unnecessary leading and trailing ones in $\tilde{\gamma}(\Upsilon)$, we consider instead the map $\gamma: \text{Nor}_n \to \mathbb{Q}_{n-1}$ defined by $\gamma(\Upsilon) := \text{red}(\tilde{\gamma}(\Upsilon))$ for each $\Upsilon \in \text{Nor}_n$.

We invite the reader to recall the definition of comb type $\lambda^{\text{Comb}}(\Upsilon)$ of a normalized tree $\Upsilon$ given in Section 4.1 before Theorem 4.6 and the definition of Lyndon type $\lambda^{\text{Lyn}}(\Upsilon)$ given in Section 4.1. Recall also the definition of ascending adjacent and terminally nested pairs of a Stirling permutation $\theta \in \mathbb{Q}_n$, and the associated types $\lambda^{\text{AA}}(\theta)$ and $\lambda^{\text{TN}}(\theta)$, given in the first part of this section. We give an equivalent characterization of these pairs. An ascending adjacent pair of $\theta \in \mathbb{Q}_n$ is a pair $(a, b)$ such that $a < b$ and in $\theta$ the second occurrence of $a$ is the immediate predecessor of the first occurrence of $b$. A terminally nested pair of $\theta \in \mathbb{Q}_n$ is a pair $(a, b)$ such that $a < b$ and in $\theta$ the second occurrence of $a$ is the immediate successor of the second occurrence of $b$.

For any node (internal or leaf) $x$ of $\Upsilon$ we define the (reduced valency) $v^\circ(x) := v(x) - 1$.

**Proposition 4.8.** The map $\gamma: \text{Nor}_n \to \mathbb{Q}_{n-1}$ is a well-defined bijection satisfying for each $\Upsilon \in \text{Nor}_n$ the following properties:

1. There is a bijection between the set of non-Lyndon nodes of $\Upsilon$ and the set of ascending adjacent pairs of $\gamma(\Upsilon)$. This bijection associates a non-Lyndon node $x$ with the ascending adjacent pair $(v^\circ(R(L(x))), v^\circ(R(x)))$.
2. There is a bijection between the set of internal nodes of $\Upsilon$ that have a right child that is also an internal node and the set of terminally nested pairs of $\gamma(\Upsilon)$. This bijection associates a node...
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that has a right child \(R(x)\) that is also internal with the terminally nested pair \((v^f(R(x)), v^f(R(R(x))))\).

\[(3) \lambda^{AA}(\gamma(\Upsilon)) = \lambda^{Lyn}(\Upsilon).\]

\[(4) \lambda^{TN}(\gamma(\Upsilon)) = \lambda^{Comb}(\Upsilon).\]

Proof. Let \(\Upsilon \in \text{Nor}_n\) and let \(x_i\) be the \(i\)th node of \(\Upsilon\) listed in postorder. We use the alternative characterization of \(\tilde{\gamma}\) given in Proposition 4.7.

We claim that:

Claim 1: The pair \((\theta(x_i), \theta(x_{i+1}))\) is an ascending adjacent pair of \(\tilde{\gamma}(\Upsilon)\) if and only if \(x_i\) is a left child that is not a leaf and its parent \(p(x_i)\) satisfies \(\theta(p(x_i)) > \theta(x_i)\). (The latter condition is equivalent to \(p(x_i)\) being a non-Lyndon node.)

Claim 2: \((\theta(x_{i+1}), \theta(x_i))\) is a terminally nested pair of \(\tilde{\gamma}(\Upsilon)\) if and only if \(x_i\) is a right child that is not a leaf.

We say that \(\theta \in Q_n\) has a first occurrence of the letter \(\theta_i\) at position \(i\) if \(\theta_j \neq \theta_i\) for all \(j < i\). We say that \(\theta \in Q_n\) has a second occurrence of the letter \(\theta_i\) at position \(i\) if there is a \(j < i\) such that \(\theta_j = \theta_i\).

Before proving these claims we first observe that in the word \(\tilde{\gamma}(\Upsilon) = \theta(\Upsilon)\theta(x_1)(\text{Proposition 4.7})\), there is a first occurrence of a letter at position \(i\) if \(x_i\) is a leaf and a second occurrence of a letter if \(x_i\) is an internal node.

The proof of the two claims follows from the following four cases that in turn are consequences of this observation.

Case 1: Let \(x_i\) be a left child that is not a leaf. Then \(x_{i+1}\) is the leftmost leaf of the right subtree of the subtree of \(\Upsilon\) rooted at \(p(x_i)\). By the observation above, the position \(i\) of \(\tilde{\gamma}(\Upsilon)\) contains the second occurrence of a letter while the position \(i + 1\) contains the first occurrence of a letter. Note that \(\theta(p(x_i)) = \theta(x_{i+1})\).

Case 2: Let \(x_i\) be a left child that is a leaf. Then \(x_{i+1}\) is the smallest leaf of the right subtree of the subtree of \(\Upsilon\) rooted at \(p(x_i)\). Hence, positions \(i\) and \(i + 1\) contain first occurrences of letters in \(\tilde{\gamma}(\Upsilon)\) and \(\theta(x_i) < \theta(x_{i+1}) = \theta(p(x_i))\).

Case 3: Let \(x_i\) be a right child that is not a leaf. Then by postorder \(x_{i+1} = p(x_i)\) and positions \(i\) and \(i + 1\) contain second occurrences of letters in \(\tilde{\gamma}(\Upsilon)\). Note that \(\theta(x_i) > \theta(x_{i+1})\).

Case 4: Let \(x_i\) be a right child that is a leaf. Then by postorder \(x_{i+1} = p(x_i)\) and by the definition of \(\theta\) we have that \(\theta(x_i) = \theta(p(x_i)) = \theta(x_{i+1})\).

It is not difficult to see that the two claims imply (1) and (2) after applying the definitions of \(\text{red}\) and \(v^f\). Parts (3) and (4) are immediate consequences of parts (1) and (2), respectively.

We have now four different combinatorial interpretations of the coefficients of the symmetric function \(L_n(x) := \sum_{\mu \in \text{wcomp}_n} \dim Lie(\mu) x^\mu\) in
the elementary symmetric function basis. Theorem 4.9 below includes Theorem 1.6.

**Theorem 4.9.** For all $n$,

$$
\sum_{\mu \in \text{wcomp}_{n-1}} \dim \mathcal{L}(\mu) x^\mu = \sum_{T \in \text{Nor}_n} e_{\lambda_{\text{sym}}(T)}(x)
$$

$$
= \sum_{\theta \in Q_{n-1}} e_{\lambda_{\AA}(\theta)}(x)
$$

$$
= \sum_{\theta \in Q_{n-1}} e_{\lambda_{\NN}(\theta)}(x)
$$

$$
= \sum_{\Upsilon \in \text{Nor}_n} e_{\lambda_{\text{Comb}}(\Upsilon)}(x).
$$

**Proof.** The first equality comes from Theorem 4.3, the third equality is a consequence of Proposition 4.6, and the second and fourth equality are consequences of Proposition 4.8. □

For a permutation $\theta \in Q_n$ we define the *initial permutation* $\text{init}(\theta) \in S_n$ to be the subword of $\theta$ formed by the first occurrence of each of the letters in $\theta$. For example, $\text{init}(23772499468861551) = 237496815$.

**Proposition 4.10.** For any $\theta \in Q_n$ and $\Upsilon = (T, \sigma) \in \text{Nor}_n$,

1. $\text{init}(\xi(\theta)) = \text{init}(\theta)$
2. $\sigma = \text{init}(\tilde{\gamma}(\Upsilon))$.

**Proof.** In the definition of $\xi$, the relative order of the initial occurrence of the letters is not changed; which proves (1). We consider the alternative characterization of $\tilde{\gamma}$ of Proposition 4.7. Recall that $\theta(x_i)$ is a first occurrence of a letter in $\theta(\Upsilon)$ if and only if $x_i$ is a leaf of $\Upsilon$. Hence, part (2) follows from the fact that postorder of the nodes of $\Upsilon$ restricted to the leaves is just left to right reading of the leaves. □

We have the following diagram of bijections:

$\text{Nor}_n \xrightarrow{\gamma} Q_{n-1} \xrightarrow{\xi} Q_{n-1} \xrightarrow{\gamma^{-1}} \text{Nor}_n$.

The following theorem is a generalization of the classical bijections between Lyndon trees, combs and permutations in $\mathfrak{S}_{n-1}$. See Figure 11 for a complete example of the bijections.

**Corollary 4.11.** The map $\gamma^{-1} \xi \gamma$ is a bijection on $\text{Nor}_n$ that translates between the Lyndon type and comb type. Moreover, the bijection preserves the permutation of leaf labels for each tree.
Proof. By Propositions 4.6 and 4.8,

$$\lambda_{\text{Comb}}(\gamma^{-1}\xi\gamma(\Upsilon)) = \lambda^{\text{TN}}(\xi\gamma(\Upsilon))$$

$$= \lambda^{\text{AA}}(\gamma(\Upsilon))$$

$$= \lambda^{\text{Lyn}}(\Upsilon).$$

Proposition 4.10 implies that the order of the leaf labels is preserved. □

We can combine Theorem 1.5 with Theorem 4.9 and conclude the following e-positivity result.

**Theorem 4.12.** We have

$$\left[\sum_{n \geq 1} (-1)^{n-1} h_{n-1}(x) \frac{y^n}{n!}\right]^{<-1>} = \sum_{n \geq 1} \sum_{\Upsilon \in \text{Nor}_n} e_{\lambda(\Upsilon)}(x) \frac{y^n}{n!}$$

$$= \sum_{n \geq 1} \sum_{\theta \in \mathcal{Q}_{n-1}} e_{\lambda(\theta)}(x) \frac{y^n}{n!},$$

where $\lambda(\Upsilon)$ is either the Lyndon type or the comb type of the normalized tree $\Upsilon$ and $\lambda(\theta)$ is either the AA type or the TN type of the Stirling permutation $\theta$. 

![Figure 11. Example of the bijections $\tilde{\gamma}$, red and $\xi$](image-url)
In [21] the author gives another proof of Theorem 4.12 that does not involve poset topology and instead involves a nice interpretation of the compositional inverse of exponential generating functions given by B. Drake in [15].

4.5. A remark about colored Stirling permutations. We can also define colored Stirling permutations in analogy with the case of colored normalized binary trees. An AA colored Stirling permutation $\Theta = (\theta, c)$ is a Stirling permutation $\theta \in Q_n$ together with a map $c : [n] \to \mathbb{P}$ such that for every occurrence of an ascending adjacent pair $(a, b)$ in $\theta$, $c$ satisfies the condition $c(a) > c(b)$.

**Example 4.13.** If $\theta = 23772499468861551$, the map $c : [9] \to \mathbb{P}$ defined by the pairs $(i, c(i))$: 

$$
\{(1, 1), (2, 3), (3, 3), (4, 2), (5, 2), (6, 1), (7, 1), (8, 2), (9, 1)\}
$$

is an AA coloring, but 

$$
\{(1, 1), (2, 2), (3, 3), (4, 3), (5, 2), (6, 1), (7, 1), (8, 2), (9, 1)\}
$$

is not since 24 is an adjacent ascending pair but $c(2) = 2 < 3 = c(4)$.

In the same manner we define a TN colored Stirling permutation to the pair $\Theta = (\theta, c)$, where $c$ satisfies $c(a) > c(b)$ whenever $(a, b)$ is a terminally nested pair. For $\mu \in \text{wcomp}_n$, we say that a colored Stirling permutation $(\theta, c)$ is $\mu$-colored if $\mu(i) = |c^{-1}(i)|$ for all $i$. We denote by $Q^{AA}_\mu$ the set of AA $\mu$-colored Stirling permutations of $[n]$ and $Q^{TN}_\mu$ the set of TN $\mu$-colored Stirling permutations of $[n]$.

**Corollary 4.14** (of Corollary 3.12). For all $n \geq 1$ and $\mu \in \text{wcomp}_{n-1}$

$$
\mu_{\Pi_n^k}(\hat{0}, [n]^\mu) = (-1)^{n-1}|Q^{AA}_\mu| = (-1)^{n-1}|Q^{TN}_\mu|.
$$

Consequently,

$$
\text{dim } \tilde{H}^{n-3}((\hat{0}, [n]^\mu)) = |Q^{AA}_\mu| = |Q^{TN}_\mu|.
$$

**Proof.** Note that the bijection $\gamma : \text{Nor}_n \to Q_{n-1}$ extends naturally to a bijection $\text{Lyn}_\mu \cong Q^{AA}_\mu$ and the bijection $\xi : Q_{n-1} \to Q_{n-1}$ extends naturally to a bijection $Q^{AA}_\mu \cong Q^{TN}_\mu$. Thus the result is a corollary of Corollary 3.12. \qed

By Theorem 2.6

**Corollary 4.15.** For all $n \geq 1$ and $\mu \in \text{wcomp}_{n-1}$

$$
\text{dim } \mathcal{L}ie(\mu) = |Q^{AA}_\mu| = |Q^{TN}_\mu|.
$$
5. Combinatorial bases

In this section we discuss various bases for \( \text{Lie} (\mu), \tilde{H}^{n-3}(\hat{0}, [n]^\mu) \) and \( \tilde{H}^{n-2}(\Pi_k^\mu \setminus \{0\}) \).

5.1. Colored Lyndon basis. Recall from Theorem 3.7 that the ascent-free maximal chains of the EL-labeling of \([\hat{0}, [n]^\mu]\) yield a basis for the cohomology \( \tilde{H}^{n-3}(\hat{0}, [n]^\mu) \). Hence, Theorem 3.11 gives a description of this basis in terms of colored Lyndon trees. The following result gives a basis closely related (equal up to signs). By applying the isomorphism of Theorem 2.6, one gets a corresponding basis for \( \text{Lie} (\mu) \), which reduces to the classical Lyndon basis for \( \text{Lie} (n) \) when \( \mu \) has a single nonzero component.

Theorem 5.1. The set \( \{\bar{c}(T, \sigma) \mid (T, \sigma) \in \text{Lyn}_\mu\} \) is a basis of \( \tilde{H}^{n-3}(\hat{0}, [n]^\mu) \) and the set \( \{[T, \sigma] \mid (T, \sigma) \in \text{Lyn}_\mu\} \) is a basis for \( \text{Lie} (\mu) \).

Proof. By Theorem 3.11 and Theorem 3.7, the set \( \{\bar{c}(T, \sigma, \tau_{T, \sigma}) \mid (T, \sigma) \in \text{Lyn}_\mu\} \) is a basis of \( \tilde{H}^{n-3}(\hat{0}, [n]^\mu) \). Lemma 2.4 implies that we can replace \( \tau_{T, \sigma} \) by any other linear extension and still obtain a basis for \( \tilde{H}^{n-3}(\hat{0}, [n]^\mu) \). In particular, we can replace it by postorder. □

Theorem 5.1 already implies that the set of maximal chains coming from colored Lyndon trees spans \( \tilde{H}^{n-3}(\hat{0}, [n]^\mu) \). To complete the proof of Theorem 2.5 and conclude that the relations in the theorem generate all the cohomology relations, we will show that we can represent any \( \bar{c} \in \tilde{H}^{n-3}(\hat{0}, [n]^\mu) \) as a linear combination of chains in \( \{\bar{c}(T, \sigma) \mid (T, \sigma) \in \text{Lyn}_\mu\} \) using only the relations in Proposition 2.5.

Proposition 5.2. The relations in Proposition 2.5 generate all the cohomology relations in \( \tilde{H}^{n-3}(\hat{0}, [n]^\mu) \).

Proof. We use a "straightening" strategy using the relations of Theorem 2.5 in order to prove the result. Recall that for an internal node \( x \) of a normalized binary tree \( \Upsilon \in \text{Nor}_n \), we define the valency \( v(x) \) to be the smallest of the labels in the subtree rooted at \( x \). We define a valency inversion in \( \Upsilon \in \text{Nor}_n \) to be a pair of internal nodes \( (x, y) \) such that:

- \( x \) is in the subtree rooted at the left child of \( y \),
- \( v(R(x)) < v(R(y)) \).

Let \( \text{valinv}(\Upsilon) \) denote the number of valency inversions in \( \Upsilon \). Note for example that a Lyndon tree is a normalized binary tree such that \( \text{valinv}(\Upsilon) = 0 \).

A coloring inversion is a pair of internal nodes \( (x, y) \) in \( \Upsilon \) such that...
\begin{itemize}
\item $v(x) = v(y)$,
\item $x$ is in the subtree rooted at the left child of $y$,
\item $\text{color}(x) < \text{color}(y)$.
\end{itemize}

We denote by $\text{colinv}(Y)$, the number of coloring inversions in $Y$.

Define the \textit{inversion pair} of $Y$ to be $(\text{valinv}(Y), \text{colinv}(Y))$. We order these pairs lexicographically; that is, we say

$$(\text{valinv}(Y), \text{colinv}(Y)) < (\text{valinv}(Y'), \text{colinv}(Y')),$$

if either $\text{valinv}(Y) < \text{valinv}(Y')$ or $\text{valinv}(Y) = \text{valinv}(Y')$ and $\text{colinv}(Y) < \text{colinv}(Y')$. Note that if the inversion pair of $Y$ is $(0, 0)$ then $Y$ is a colored Lyndon tree since in particular its underlying uncolored tree is a Lyndon tree.

Now let $Y \in B_{T_\mu}$ be a colored normalized binary tree that is not a colored Lyndon tree. Then $Y$ must have a subtree of the form: $(\gamma_1^i \gamma_2^j \gamma_3^k)$, with $v(\gamma_2) < v(\gamma_3)$ and $i \leq j$. We will show that $\overline{c}(Y)$ can be expressed as a linear combination of chains associated with colored normalized binary trees with smaller inversion pairs.

\textbf{Case $i = j$:} Using relation (2.5) (and relation (2.4)) we have that

$$\overline{c}(\alpha((\gamma_1^i \gamma_2^j \gamma_3^k) \beta)) = \pm \overline{c}(\alpha((\gamma_1^i \gamma_2^j \gamma_3^k) \beta)) \pm \overline{c}(\alpha((\gamma_1^i \gamma_2^j \gamma_3^k) \beta)).$$

(The signs in the relations of Theorem 2.5 are not relevant here and have therefore been suppressed.)

Let $p(\gamma_j)$ denote the parent of the root of the subtree $\gamma_j$ in $Y$. We then have that

$$\text{valinv}(\alpha((\gamma_1^i \gamma_2^j \gamma_3^k) \beta)) - \text{valinv}(\alpha((\gamma_1^i \gamma_2^j \gamma_3^k) \beta)) \geq 1,$$

since the pair $(p(\gamma_2), p(\gamma_3))$ and any other valency inversion between an internal node of $\gamma_1$ and $p(\gamma_3)$ are valency inversions in the former tree but not in the latter and no other change occurs to the set of valency inversions. We also have that

$$\text{valinv}(\alpha((\gamma_1^i \gamma_2^j \gamma_3^k) \beta)) - \text{valinv}(\alpha((\gamma_1^i \gamma_2^j \gamma_3^k) \beta)) \geq 1,$$

since the pair $(p(\gamma_2), p(\gamma_3))$ and any other valency inversion between an internal node of $\gamma_2$ and $p(\gamma_3)$ are valency inversions in the former tree but not in the latter and no other change occurs to the set of valency inversions.

\textbf{Case $i < j$:} Using relation (2.6) (and relation (2.4)) we have that

$$\overline{c}(\alpha((\gamma_1^i \gamma_2^j \gamma_3^k) \beta)) = \pm \overline{c}(\alpha((\gamma_1^i \gamma_2^j \gamma_3^k) \beta)) \pm \overline{c}(\alpha((\gamma_1^i \gamma_2^j \gamma_3^k) \beta)) \pm \overline{c}(\alpha((\gamma_1^i \gamma_2^j \gamma_3^k) \beta)).$$
$$\pm \tilde{c}(\alpha((Y_1,Y_3)\langle\langle Y_2)\beta)$$
$$\pm \tilde{c}(\alpha((Y_1\langle\langle Y_3)\langle\langle Y_2)\beta).$$

Just as in the previous case, all the labeled colored trees on the right hand side of the equation, except for the first, have fewer valency inversions than the tree in the left hand side. The first labeled colored tree $\tilde{c}(\alpha((Y_1j\langle\langle Y_3)\langle\langle Y_2)\beta)$ has the same number of valency inversions as that of $c(\alpha((Y_1\langle\langle Y_2)\langle\langle Y_3)\beta).$ However the coloring inversion number is reduced by one and so the inversion pair is reduced.

From the two cases above we conclude that if $Y \in BT_\mu$ is a colored normalized binary tree that is not a colored Lyndon tree then $\tilde{c}(Y)$ can be expressed as a linear combination of chains, associated to colored normalized binary trees, of smaller inversion pair. Hence by induction on the inversion pair, $\tilde{c}(Y)$ can be expressed as a linear combination of chains of the form $\tilde{c}(Y')$ where $Y' \in Lyn_\mu.$ Also since by relation (2.4) any $Y \in BT_\mu$ is of the form $\pm \tilde{c}(Y')$, where $Y'$ is a colored normalized binary tree, the same is true for any $Y \in BT_\mu.$

Since the set $\{\tilde{c}(Y) \mid Y \in BT_\mu\}$ is a spanning set for $\tilde{H}^{n-3}((0, [n])\mu)$, we have shown using only the relations in Theorem 2.5 that $\{\tilde{c}(Y) \mid Y \in Lyn_\mu\}$ is also a spanning set for $\tilde{H}^{n-3}((0, [n])\mu)$. The fact that $\{\tilde{c}(Y) \mid Y \in Lyn_\mu\}$ is a basis (Theorem 5.1), proves the result. □

5.2. Colored comb basis. A colored comb is a normalized colored binary tree that satisfies the following coloring restriction: for each internal node $x$ whose right child $R(x)$ is not a leaf,

(5.1) $\text{color}(x) > \text{color}(R(x)).$

Let $\text{Comb}_n$ be the set of colored combs in $BT_n$ and $\text{Comb}_\mu$ be the set of the $\mu$-colored ones. Note that in a monochromatic comb every right child has to be a leaf and hence they are the classical left combs that yield a basis for $Lie(n)$ (see [41, Proposition 2.3]). The colored combs generalize also the bicolored combs that yield a basis of $Lie(n,i)$ in [22]. Figure 12 illustrates the bicolored combs for $n = 3.$

Remark 5.3. Note that the coloring condition (5.1) is closely related to the comb type of a normalized tree defined in Section 1 before Theorem 1.6. The coloring condition implies that in a colored comb $Y$ there are no repeated colors in each block $B$ of the partition $\pi^{\text{Comb}}(Y)$ associated to $Y$. So after choosing $|B|$ different colors for the internal nodes of $Y$ in $B$, there is a unique way to assign the colors such that $Y$ is a colored comb (the colors must decrease towards the right in each block of $\pi^{\text{Comb}}(Y)$). In Figure 13 this relation is illustrated.
Theorem 5.4. There is a bijection
\[ \text{Lyn}_\mu \cong \text{Comb}_\mu. \]

Proof. This is a consequence of Theorem 4.11. Indeed, the bijection \( \gamma^{-1} \xi \gamma \) that translates between the Lyndon type and comb type on \( \text{Nor}_n \) extends naturally to a bijection \( \text{Lyn}_\mu \cong \text{Comb}_\mu. \) \( \square \)

We obtain the following corollary from Corollary 3.12.

Corollary 5.5. For all \( n \geq 1 \) and \( \mu \in \text{wcomp}_{n-1}, \)
\[ \mu_{H_n}(\hat{0}, [n]^\mu) = (-1)^{n-1}|\text{Comb}_\mu|. \]

Hence,
\[ \dim \tilde{H}^{n-3}(\hat{0}, [n]^\mu) = |\text{Comb}_\mu|. \]

By Theorem 4.2 or Theorem 2.6 we have,
Corollary 5.6. For all $n \geq 1$ and $\mu \in \text{wcomp}_{n-1}$

$$\dim \text{Lie}(\mu) = |\text{Comb}_{\mu}|.$$ 

Theorem 5.7. $\{\bar{c}(T, \sigma) \mid (T, \sigma) \in \text{Comb}_{\mu}\}$ is a basis for $\tilde{H}^{n-3}((\hat{0}, [n]^\mu))$ and $\{[T, \sigma] \mid (T, \sigma) \in \text{Comb}_{\mu}\}$ is a basis for $\tilde{\text{Lie}}(\mu)$.

Proof. The “spanning” part of the proof for $\tilde{H}^{n-3}((\hat{0}, [n]^\mu))$ follows the same idea as in the proof of Proposition 5.2. Instead of using $\text{valinv}(T)$ we use $\sum_{x \in I(T)} r(x)$, where $I(T)$ is the set of internal nodes of $T$ and $r(x)$ is the number of internal nodes in the right subtree of $x$. And instead of using $\text{colinv}(T)$ we use the number of pairs $(x, y)$ of internal nodes of $T$ such that $y$ is a descendant of $x$ that can be reached from $x$ along a path of right edges and $\text{color}(x) < \text{color}(y)$ (see [22, Theorem 5.1] for the special case in which $\text{supp}(\mu) \subseteq [2]$).

Using Corollary 5.5 we conclude that the set $\{\bar{c}(T, \sigma) \mid (T, \sigma) \in \text{Comb}_{\mu}\}$ is a basis for $\tilde{H}^{n-3}((\hat{0}, [n]^\mu))$ and using Theorem 2.6 that $\{[T, \sigma] \mid (T, \sigma) \in \text{Comb}_{\mu}\}$ is a basis for $\tilde{\text{Lie}}(\mu)$. \hfill \Box

5.3. Bases for cohomology of the full weighted partition poset.

In [22] bicolored combs and bicolored Lyndon trees are used to construct bases for $\tilde{H}^{n-2}(\Pi^w_n \setminus \{\hat{0}\})$.

Denote the root of a colored binary tree $T$ by $\text{root}(T)$, and define

$$\mathcal{BT}^k_n := \bigcup_{\mu \in \text{wcomp}_{n-1}^k, \text{supp}(\mu) \subseteq [k]} \mathcal{BT}_\mu,$$

$$\text{Comb}^k_n := \bigcup_{\mu \in \text{wcomp}_{n-1}^k, \text{supp}(\mu) \subseteq [k]} \text{Comb}_\mu,$$

$$\text{Lyn}^k_n := \bigcup_{\mu \in \text{wcomp}_{n-1}^k, \text{supp}(\mu) \subseteq [k]} \text{Lyn}_\mu.$$

For a chain $c$ in $\Pi^k_n$, let

$$\bar{c} := c \setminus \{\hat{0}\}.$$ 

Theorem 5.8. The set

$$\{\bar{c}(T, \sigma) \mid (T, \sigma) \in \text{Lyn}^k_n, \text{color}(\text{root}(T)) \neq 1\}$$

is a basis for $\tilde{H}^{n-2}(\Pi^k_n \setminus \{\hat{0}\})$.

Proof. From the EL-labeling of Theorem 3.6 we have that all the maximal chains of $\hat{\Pi}^k_n$ have last label $(1, n + 1)^t$. Then for a maximal chain to be ascent-free it must have a second to last label of the form $(1, a)^j$...
for $a \in [n]$ and $j \in [k] \setminus \{1\}$. By Theorem 3.11, we see that the ascent-free chains correspond to colored Lyndon trees such that the color of the root is different from 1. It therefore follows from Theorem 3.7 and Lemma 2.4 (with $\bar{c}$ replaced by $\hat{c}$) that the set is a basis for $\tilde{H}^{n-2}(\Pi^n \setminus \{\hat{0}\})$. □

Using an extended version of the straightening algorithm of [22, Theorem 5.15] with an additional cohomology relation that is more general than the one in [22, Theorem 5.15], the reader can verify the following theorem (or see [20, Proposition 6.3.2] for the proof).

**Proposition 5.9.** The set
\[
\{ \hat{c}(T, \sigma) \mid (T, \sigma) \in \text{Comb}^k_n, \text{color}(\text{root}(T)) \neq k \}
\]
spans $\tilde{H}^{n-2}(\Pi^n \setminus \{\hat{0}\})$.

**Theorem 5.10** ([22, Theorem 5.15]). The set
\[
\{ \hat{c}(T, \sigma) \mid (T, \sigma) \in \text{Comb}^2_n, \text{color}(\text{root}(T)) \neq 2 \}
\]
is a basis of $\tilde{H}^{n-2}(\Pi^n \setminus \{\hat{0}\})$.

We propose the following conjecture.

**Conjecture 5.11.** The set
\[
\{ \hat{c}(T, \sigma) \mid (T, \sigma) \in \text{Comb}^k_n, \text{color}(\text{root}(T)) \neq k \}
\]
is a basis of $\tilde{H}^{n-2}(\Pi^n \setminus \{\hat{0}\})$.

We can combine Theorem 5.8 and the exact same idea of the proof of Proposition 5.2 to show that the set $B = \{ \hat{c}(T, \sigma) \mid (T, \sigma) \in \text{Lyn}^k_n, \text{color}(\text{root}(T)) \neq k \}$ spans $\tilde{H}^{n-2}(\Pi^n \setminus \{\hat{0}\})$ by using only the relations of Theorem 2.5 and the additional relation
\[
(5.2) \quad \hat{c}(\Upsilon_1^1 \Upsilon_2) + \hat{c}(\Upsilon_1^2 \Upsilon_2) + \cdots + \hat{c}(\Upsilon_1^k \Upsilon_2) = 0.
\]
for all $A \subseteq [n]$ and for all $\Upsilon_1 \in B\mathcal{T}_A$ and $\Upsilon_2 \in B\mathcal{T}_{[n] \setminus A}$. We conclude that these are the only relations in a presentation of $\tilde{H}^{n-3}(\Pi^n \setminus \{\hat{0}\})$ since $B$ is a basis. We summarize with the following result.

**Theorem 5.12.** The set $\{ \hat{c}(\Upsilon) : \Upsilon \in B\mathcal{T}^k_n \}$ is a generating set for $\tilde{H}^{n-3}(\Pi^n \setminus \{\hat{0}\})$, subject only to the relations of Theorem 2.5 (with $\bar{c}$ replaced by $\hat{c}$) and relation (5.2).

6. Whitney numbers and Whitney (co)homology

In this section we discuss weighted Whitney numbers and Whitney (co)homology of $\Pi^n$.
6.1. **Whitney numbers and weighted uniformity.** Let $P$ denote a pure poset with a minimum element $\hat{0}$. Denote by $Int(P)$ the set of closed intervals $[x, y]$ in the poset $P$. For some unitary commutative ring $R$ (for example $k[x]$ or $k[x_1, \ldots, x_k]$) we say that a weight function $\varpi_P : Int(P) \to R$ is $P$-compatible if

- for any $\alpha \in P$, $\varpi_P(\alpha, \alpha) = 1$ and,
- $\theta \leq \alpha \leq \beta$ in $P$ implies $\varpi_P(\theta, \beta) = \varpi_P(\theta, \alpha)\varpi_P(\alpha, \beta)$.

Equivalently, let $k[\text{Int}(P)]$ be the unitary commutative algebra over $k$ generated by intervals $[x, y]$ in $\text{Int}(P)$ subject to the relations:

- $[\alpha, \alpha] = 1$ for any $\alpha \in P$, and
- $[\theta, \beta] = [\theta, \alpha][\alpha, \beta]$ for all $\theta \leq \alpha \leq \beta$ in $P$.

Then a $P$-compatible weight function is just an algebra homomorphism $\varpi_P : k[\text{Int}(P)] \to R$. The poset $\Pi_n^k$ has a natural $\Pi_n^k$-compatible weight function $\varpi_{\Pi_n^k}$. Indeed, we define the map $\varpi_{\Pi_n^k} : k[\text{Int}(P)] \to k[x_1, \ldots, x_k]$ by letting $\varpi_{\Pi_n^k}(\hat{0}, \hat{0}) = 1$ and $\varpi_{\Pi_n^k}(\hat{0}, \alpha) = x_1^{w(1)} \cdots x_k^{w(k)}$ for any $\alpha = \{A_1^{\mu_1}, \ldots, A_s^{\mu_s}\} \in \Pi_n^k$, with $w = \mu(\alpha) = \sum_{i=1}^s \mu_i$. This extends to any interval $[\alpha, \beta]$, by setting $\varpi_{\Pi_n^k}(\alpha, \beta) = \frac{\varpi_{\Pi_n^k}(\emptyset, \beta)}{\varpi_{\Pi_n^k}(\emptyset, \alpha)}$ (clearly a monomial in $k[x_1, \ldots, x_k]$), and to $k[\text{Int}(P)]$ by linearity.

The weighted Whitney numbers $w_j(P, \varpi_P)$ and $W_j(P, \varpi_P)$ of the first and second kind are defined as:

$$w_j(P, \varpi_P) = \sum_{\alpha \in P, \rho(\alpha) = j} \overline{\mu}_P(\hat{0}, \alpha)\varpi_P(\hat{0}, \alpha),$$

$$W_j(P, \varpi_P) = \sum_{\alpha \in P, \rho(\alpha) = j} \varpi_P(\hat{0}, \alpha).$$

Note that if $\varpi_P$ is the trivial $P$-compatible function defined by $\varpi_P(\alpha, \alpha') = 1$ for all $\alpha \leq \alpha' \in P$, then $w_j(P) := w_j(P, \varpi_P)$ and $W_j(P) := W_j(P, \varpi_P)$ are the classical Whitney numbers of the first and second kind respectively.

Recall that for each $\alpha \in \Pi_n^k$, we have $\rho(\alpha) = n - |\alpha|$. For a partition $\lambda \vdash n$, with $\ell(\lambda) = r$ and where a part of size $i$ occurs $m_i(\lambda)$ times, let $\lambda \setminus (1^r)$ denote the partition obtained from $\lambda$ by decreasing each of its parts by 1. Recall the symmetric function

$$(6.1) \quad L_n(x) := \sum_{\mu \in \text{comp}_{\Pi_n}^k} \dim \mathcal{L}(\mu) \cdot x^\mu,$$

and for a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, define

$$L_\lambda(x) := L_{\lambda_1}(x) \cdots L_{\lambda_r}(x).$$
Note that $L_\lambda(x)$ is a homogeneous symmetric function of degree $|\lambda|$. Define $m(\lambda)! := \prod_{s=1}^n m_s(\lambda)!$.

**Proposition 6.1.** For all $n \geq 1$, the weighted Whitney numbers are given by

\begin{equation}
(6.2) \quad w_r(\Pi_n^k, \varpi_{\Pi_n^k}) = (-1)^r \sum_{\lambda \vdash n, \ell(\lambda) = n - r} \binom{n}{\lambda} \frac{1}{m(\lambda)!} L_{\lambda \setminus (1^{n-r})}(x_1, \ldots, x_k),
\end{equation}

\begin{equation}
(6.3) \quad W_r(\Pi_n^k, \varpi_{\Pi_n^k}) = \sum_{\lambda \vdash n, \ell(\lambda) = n - r} \binom{n}{\lambda} \frac{1}{m(\lambda)!} h_{\lambda \setminus (1^{n-r})}(x_1, \ldots, x_k),
\end{equation}

where $h_{\lambda}$ denotes the complete homogeneous symmetric function associated with the partition $\lambda$.

**Proof.** We want to construct a weighted partition $\alpha$ that has underlying (unweighted) set partition $\pi \in \Pi_n$. For a block of size $s$ in $\pi$, any monomial $x_1^{\mu(1)} \cdots x_k^{\mu(k)}$ with $|\mu| = s - 1$ is a valid weight, so the contribution of this block corresponds to the complete homogeneous symmetric polynomial $h_{s-1}(x_1, \ldots, x_k)$. Then $\pi$ has a contribution of $h_{\lambda(\pi) \setminus (1^{|\pi|})}$, where $\lambda(\pi)$ denotes the integer partition whose parts are equal to the block sizes of $\pi$, proving equation (6.3).

By Proposition 3.1, intervals of the form $[0, \alpha]$ are isomorphic to products of maximal intervals of smaller copies of $\Pi_n^k$. Following a similar argument as in (6.3), using the fact that the Möbius function is multiplicative and Theorem 2.6, equation (6.2) follows. \hfill \square

**Definition 6.2.** A pure poset $P$ of length $\ell$ with minimum element $\hat{0}$ and with rank function $\rho$, is said to be uniform if there is a family of posets $\{P_i \mid 0 \leq i \leq \ell\}$ such that for all $x \in P$, the upper order ideal $I_x := \{y \in P \mid x \leq y\}$ is isomorphic to $P_i$, where $i = \ell - \rho(x)$.

We refer to $(P_0, \ldots, P_\ell)$ as the associated uniform sequence. It follows from Proposition 3.1 that $P = \Pi_n$ is uniform with $P_i = \Pi_{n-i+1}$ for $i = 0, \ldots, n-1$.

Note that a $P$-compatible weight function $\varpi_P$ induces, for any $x \in P$, an $I_x$-compatible weight function $\varpi_{I_x}$, the restriction of $\varpi_P$ to $k[\text{Int}(I_x)]$. For a uniform poset $P$, we say that a $P$-compatible weight function $\varpi_P$ is uniform if for any two elements $x, y \in P$ such that $\rho(x) = \rho(y)$ there is a poset isomorphism $f : I_x \to I_y$, such that the induced weight functions $\varpi_{I_x}$ and $\varpi_{I_y}$ satisfy $\varpi_{I_x}(z, z') = \varpi_{I_y}(f(z), f(z'))$ for all $z \leq z' \in I_x$. For example, the $\Pi_n^k$-compatible weight function $\varpi_{\Pi_n^k}$ defined before is uniform. It is clear that for a uniform poset $P$ with associated uniform sequence $(P_0, \ldots, P_\ell)$ and uniform
\( P \)-compatible weight function \( \varpi_P \) there is a well-defined induced \( P_i \)-compatible weight function \( \varpi_{P_i} \) for each \( i \). The following proposition is a weighted version of a variant of \cite{38} Exercise 3.130 (a).

**Proposition 6.3.** Let \( P \) be a uniform poset of length \( \ell \), with associated uniform sequence \( (P_0, \ldots, P_\ell) \) and a uniform \( P \)-compatible weight function \( \varpi_P \). Then the matrices \( [w_{i-j}(P_i, \varpi_{P_i})]_{0 \leq i,j \leq \ell} \) and \( [W_{i-j}(P_s, \varpi_P)]_{0 \leq i,j \leq \ell} \) are inverses of each other.

**Proof.** For a fixed \( \alpha \in P \) with \( \rho(\alpha) = \ell - i \) we have by the recursive definition of the Möbius function and the uniformity of \( P \)

\[
\delta_{i,j} = \sum_{\beta \in P} \varpi_P(\alpha, \beta) \sum_{x \in [\alpha, \beta]} \mu_P(\alpha, x)
\]

\[
= \sum_{s=0}^\ell \sum_{\beta \in P} \mu_P(\alpha, x) \varpi_P(\alpha, x) \sum_{\beta \geq x} \varpi_P(x, \beta)
\]

\[
= \sum_{s=0}^\ell \sum_{\beta \in P} \mu_P(\alpha, x) \varpi_P(\alpha, x) \sum_{\beta \geq x} \varpi_P(\alpha, \beta)
\]

\[
= \sum_{s=0}^\ell w_{i-s}(P_i, \varpi_{P_i}) W_{i-j}(P_s, \varpi_P).
\]

From the uniformity of the pair \( (\Pi_k^n, \varpi_{\Pi_k^n}) \) and Proposition 6.1, we have the following consequence of Proposition 6.3.

**Corollary 6.4.** The matrices \( A = \left( (-1)^{i-j} \sum_{\lambda(\lambda) = j} \binom{i}{\lambda} \frac{1}{m(\lambda)!} L_{\lambda \setminus \{1\}}(x) \right)_{0 \leq i,j \leq n-1} \)

\( B = \left( \sum_{\lambda(\lambda) = j} \binom{i}{\lambda} \frac{1}{m(\lambda)!} h_{\lambda \setminus \{1\}}(x) \right)_{0 \leq i,j \leq n-1} \)

are inverses of each other.

When \( x_1 = x_2 = 1 \) and \( x_i = 0 \) for \( i \geq 3 \), these matrices have the simpler form given in the following result.

**Theorem 6.5** (\cite{22} Corollary 2.11). The matrices \( A = \left( (-1)^{i-j} \binom{i-1}{j-1} j^{i-j} \right)_{1 \leq i,j \leq n} \)

\( B = \left( \binom{i}{j} j^{i-j} \right)_{1 \leq i,j \leq n} \)

are inverses of each other.

It can be shown that when \( x_1 = 1 \) and \( x_i = 0 \) for \( i \geq 2 \), Corollary 6.4 reduces to the following classical result since \( \Pi_1^n = \Pi_n \).
Theorem 6.6 (see [38]). Let $s(i,j)$ and $S(i,j)$ denote respectively, the Stirling numbers of the first and of the second kind. The matrices $A = [s(i,j)]_{1 \leq i,j \leq n}$ and $B = [S(i,j)]_{1 \leq i,j \leq n}$ are inverses of each other.

6.2. Whitney (co)homology. Whitney homology was introduced by Baclawski in [1] giving an affirmative answer to a question of Rota about the existence of a homology theory on the category of posets where the Betti numbers for geometric lattices are given by the Whitney numbers of the first kind. Whitney homology was later used to compute group representations on the homology of Cohen-Macaulay posets by Sundaram [39] and generalized to the non-pure case by Wachs [42] (see also [43]).

Whitney cohomology (over the field $k$) of a poset $P$ with a minimum element $\hat{0}$ can be defined for each integer $r$ as follows:

$$WH^r(P) := \bigoplus_{x \in P} \tilde{H}^{r-2}((\hat{0}, x); k).$$

In the case of a Cohen-Macaulay poset this formula becomes

$$WH^r(P) := \bigoplus_{x \in P \atop \rho(x) = r} \tilde{H}^{r-2}((\hat{0}, x); k).$$

(6.4)

Note that

$$\dim WH^r(P) = |w_r(P)|.$$  

(6.5)

where $w_r(P)$ is the classical $r$th Whitney number of the first kind.

Define $\wedge^r Lie_k(n)$ to be the multilinear component of the $r$th exterior power of the free Lie algebra on $[n]$ with $k$ compatible brackets. From the definition of $\wedge^r Lie_k(n)$ and equation (6.1) we can derive the following proposition.

Proposition 6.7. For $0 \leq r \leq n-1$ and $k \geq 1$,

$$\dim \wedge^r Lie_k(n) = \sum_{\lambda \vdash n \atop \ell(\lambda) = r} \binom{n}{\lambda} \frac{1}{m(\lambda)!} L_{\lambda \setminus (1^r)}(1^k).$$

Consequently, if $\wedge Lie_k(n)$ is the multilinear component of the exterior algebra of the free Lie algebra with $k$ compatible brackets on $n$ generators then

$$\dim \wedge Lie_k(n) = \sum_{\lambda \vdash n} \binom{n}{\lambda} \frac{1}{m(\lambda)!} L_{\lambda \setminus (1^{\ell(\lambda)})}(1^k).$$
Equivalently,
\[ \dim \wedge \mathcal{L} \text{ie}_k(n) = n! [x^n] \exp \left( \sum_{i \geq 1} \frac{L_{i-1}(1^k)^x}{i!} \right), \]
where \([x^n]F(x)\) denotes the coefficient of \(x^n\) in the formal power series \(F(x)\) and \(\exp(x) = \sum_{n \geq 1} \frac{x^n}{n!}\) (see [37, Theorem 5.1.4]).

Note that since, by equation (6.5), \(\dim WH_r(\Pi^k_n)\) equals the signless \(r\)th Whitney number of the first kind \(|w_r(\Pi^k_n)|\), Propositions 6.1 and 6.7 imply that the dimensions of \(\dim WH^{n-r}(\Pi^k_n)\) and \(\wedge^r \mathcal{L} \text{ie}_k(n)\) are equal.

**Corollary 6.8.** For \(0 \leq r \leq n - 1\) and \(k \geq 1\),
\[ \dim \wedge^r \mathcal{L} \text{ie}_k(n) = \dim WH^{n-r}(\Pi^k_n), \]
\[ \dim \wedge^r \mathcal{L} \text{ie}_k(n) = \dim WH(\Pi^k_n), \]
where \(WH(\Pi^k_n) := \oplus_{r \geq 0} WH^r(\Pi^k_n)\).

If a group \(G\) of automorphisms acts on a poset \(P\), this action induces a representation of \(G\) on \(WH^r(P)\) for every \(r\). Thus the action of \(\mathfrak{S}_n\) on \(\Pi^k_n\) turns \(WH^r(\Pi^k_n)\) into an \(\mathfrak{S}_n\)-module for each \(r\). Moreover, the symmetric group \(\mathfrak{S}_n\) acts naturally on \(\wedge^r \mathcal{L} \text{ie}_k(n)\) giving it the structure of an \(\mathfrak{S}_n\)-module. We will present an equivariant version of Corollary 6.8 below.

In [3] Barcelo and Bergeron proved the following \(\mathfrak{S}_n\)-module isomorphism for the poset of partitions:
\[ WH^{n-r}(\Pi_n) \cong_{\mathfrak{S}_n} \wedge^r \mathcal{L} \text{ie}_1(n) \otimes \text{sgn}_n. \]
In [41] Wachs shows that an extension of her correspondence between generating sets of \(\widetilde{H}^{n-3}(\Pi_n)\) and \(\mathcal{L} \text{ie}(n) \otimes \text{sgn}_n\) can be used to prove this result. The same technique in [22] proves:
\[ WH^{n-r}(\Pi^w_n) \cong_{\mathfrak{S}_n} \wedge^r \mathcal{L} \text{ie}_2(n) \otimes \text{sgn}_n. \]

We use the same technique to prove that in general
\[ WH^{n-r}(\Pi^k_n) \cong_{\mathfrak{S}_n} \wedge^r \mathcal{L} \text{ie}_k(n) \otimes \text{sgn}_n. \]

A colored binary forest is a sequence of colored binary trees. Given a colored binary forest \(F\) with \(n\) leaves and \(\sigma \in \mathfrak{S}_n\), let \((F, \sigma)\) denote the labeled colored binary forest whose \(i\)th leaf from left to right has label \(\sigma(i)\). Let \(\mathcal{B} \mathcal{F}_{n,r}\) be the set of labeled colored binary forests with
n leaves and r trees. If the jth labeled colored binary tree of \((F, \sigma)\) is 
\((T_j, \sigma_j)\) for each \(j = 1, \ldots, r\) then define 
\[
[F, \sigma] := [T_1, \sigma_1] \wedge \cdots \wedge [T_r, \sigma_r],
\]
where now \(\wedge\) denotes the wedge product operation in the exterior algebra. The set \(\{(F, \sigma) : (F, \sigma) \in BF_{n,r}\}\) is a generating set for \(\wedge^r \text{Lie}_k(n)\).

The set \(BF_{n,r}\) also provides a natural generating set for \(WH^{n-r}(\Pi^k_n)\).

For \((F, \sigma) \in BF_{n,r}\), let \(c(F, \sigma)\) be the unrefinable chain of \(\Pi^k_n\) whose rank \(i\) partition is obtained from its rank \(i-1\) partition by col\(_i\)-merging the blocks \(L_i\) and \(R_i\), where col\(_i\) is the color of the \(i\)th postorder internal node \(v_i\) of \(F\), and \(L_i\) and \(R_i\) are the respective sets of leaf labels in the left and right subtrees of \(v_i\).

We have the following generalization of Theorem 2.6 and [41, Theorem 7.2]. The proof is similar to that of Theorem 2.6 and is left to the reader.

**Theorem 6.9.** For each \(r\), there is an \(S_n\)-module isomorphism 
\[
\phi : \wedge^r \text{Lie}_k(n) \to WH^{n-r}(\Pi^k_n) \otimes \text{sgn}_n
\]
determined by 
\[
\phi([F, \sigma]) = \text{sgn}(\sigma) \text{sgn}(F) \bar{c}(F, \sigma), \quad (F, \sigma) \in BF_{n,r},
\]
where if \(F\) is the sequence \(T_1, \ldots, T_r\) of colored binary trees then 
\[
\text{sgn}(F) := (-1)^{I(T_2) + I(T_4) + \cdots + I(T_{2 \lfloor r/2 \rfloor})} \text{sgn}(T_1) \text{sgn}(T_2) \cdots \text{sgn}(T_r).
\]

**Remark 6.10.** When \(k = 1\) it is proved in [3] that 
\[
\dim \wedge \text{Lie}(n) = \dim WH(\Pi_n) = n!,
\]
and when \(k = 2\) it is proved in [22] that 
\[
\dim \wedge \text{Lie}_2(n) = \dim WH(\Pi^w_n) = (n + 1)^{n-1}.
\]

**7. The Frobenius Characteristic of \(\text{Lie}(\mu)\)**

In this section we prove Theorem 1.8. We use a technique developed by Sundaram [39], and further developed by Wachs [42], to compute group representations on the (co)homology of Cohen-Macaulay posets using Whitney (co)homology. We introduce and develop first the concepts and results necessary to prove Theorem 1.8 in Sections 7.1 and 7.2 we give a proof of the theorem in Section 7.3. For information not presented here about symmetric functions, plethysm and the representation theory of the symmetric group see [31], [34], [26] and [37, Chapter 7].
7.1. Wreath product modules and plethysm. In the following we follow closely the exposition and the results in [42].

Let \( R \) be a commutative ring containing \( \mathbb{Q} \) and let \( \Lambda_R \) denote the ring of symmetric functions with coefficients in \( R \) with variables \((y_1, y_2, \ldots)\). The power-sum symmetric functions \( p_k \) are defined by \( p_0 = 1 \) and

\[
p_k = y_1^k + y_2^k + \cdots \quad \text{for } k \in \mathbb{P}.
\]

For a partition \( \lambda \vdash n \), \( p_\lambda \) denotes the power-sum symmetric function associated to \( \lambda \), i.e., \( p_\lambda = p_{\lambda_1}p_{\lambda_2} \cdots p_{\lambda_{\ell(\lambda)}} \), where \( \ell(\lambda) \) is the number of nonzero parts of \( \lambda \). It is well-known that the set \( \{ p_\lambda \mid \lambda \vdash n \} \) is a basis for the component \( \Lambda^n_R \) of \( \Lambda_R \) consisting of homogeneous symmetric functions of degree \( n \).

Let \( \mathbb{Q}[[z_1, z_2, \ldots]] \) be the ring of formal power series in variables \((z_1, z_2, \ldots)\). If \( g \in \mathbb{Q}[[z_1, z_2, \ldots]] \) then plethysm \( p_k[g] \) of \( p_k \) and \( g \) is defined as:

\[
(7.1) \quad p_k[g] = g(z_1^k, z_2^k, \ldots).
\]

The definition of plethysm is then extended to \( p_\lambda \) multiplicatively and then to all of \( \Lambda_R \) linearly with respect to \( R \).

It follows from (7.1), that if \( f \in \Lambda^n_R \) and \( g \in \mathbb{Q}[[z_1, z_2, \ldots]] \), the following identity holds:

\[
(7.2) \quad f[-g] = (-1)^n \omega(f)[g].
\]

where \( \omega(\cdot) \) is the involution in \( \Lambda_R \) that maps \( p_i(y) \) to \((-1)^{i-1}p_i(y)\).

For (perhaps empty) integer partitions \( \nu \) and \( \lambda \) such that \( \nu \subseteq \lambda \) (that is \( \nu(i) \leq \lambda(i) \) for all \( i \)), let \( S^{\lambda/\nu} \) denote the Specht module of shape \( \lambda/\nu \) and \( s^{\lambda/\nu} \) the Schur function of shape \( \lambda/\nu \). Recall that \( s^{\lambda/\nu} \) is the image in the ring of symmetric functions of the Specht module \( S^{\lambda/\nu} \) under the Frobenius characteristic map \( \text{ch} \), i.e., \( \text{ch} S^{\lambda/\nu} = s^{\lambda/\nu} \).

We will use the following standard results in the theory of symmetric functions and the representation theory of the symmetric group, respectively.

**Proposition 7.1** (cf. [31] and [42]). Let \( \nu \) be a non empty integer partition and let \( \{f_i\}_{i \geq 1} \) be a sequence of formal power series \( f_i \in \mathbb{Z}[z_1, z_2, \ldots] \) such that the sum \( \sum_{i \geq 1} f_i \) exists as a formal power series. Then

\[
s_\lambda \left[ \sum_{i \geq 1} f_i \right] = \sum_{\emptyset = \nu_0 \subseteq \nu_1 \subseteq \cdots \subseteq \nu_j = \lambda} \prod_{i=1}^j s_{\nu_i/\nu_{i-1}}[f_i].
\]
Proposition 7.2 (cf. [26] and [42]). Let $\lambda \vdash \ell$ and let $(m_1, m_2, \ldots, m_t)$ be a sequence of nonnegative integers whose sum is $\ell$. Then the restriction of the $S_\ell$-module $S^\lambda$ to the Young subgroup $\times_{i=1}^t S_{m_i}$ decomposes into a direct sum of outer tensor products of $S_{m_i}$-modules as follows,

$$S^\lambda \downarrow_{\times_{i=1}^t S_{m_i}} = \bigoplus_{\emptyset = \nu_0 \subseteq \nu_1 \subseteq \cdots \subseteq \nu_t = \lambda} \bigotimes_{i=1}^t S^\nu_{\nu_i - \nu_{i-1}}.$$ 

Recall that the wreath product of the symmetric groups $S_m$ and $S_n$, denoted $S_m[S_n]$, is the normalizer of the Young subgroup $S_m \times \cdots \times S_n$ of $S_{mn}$. Each element of $S_m[S_n]$ can be represented as an $(m+1)$-tuple $(\sigma_1, \ldots, \sigma_m; \tau)$ with $\tau \in S_m$ and $\sigma_i \in S_n$ for all $i \in [m]$.

From an $S_n$-module $W$ we can construct a representation $\tilde{W}$ of $S_m[S_n]$ on the vector space $W^\otimes m := W \otimes \cdots \otimes W$ with action given by

$$(\sigma_1, \ldots, \sigma_m; \tau)(w_1 \otimes \cdots \otimes w_m) := \sigma_1 w_{\tau^{-1}(1)} \otimes \cdots \otimes \sigma_m w_{\tau^{-1}(m)},$$

and from an $S_m$-module $V$ we can construct a representation $\hat{V}$ of $S_m[S_n]$ with action given by

$$(\sigma_1, \ldots, \sigma_m; \tau)(v) := \tau v,$$

called the pullback of $V$ from $S_m$ to $S_m[S_n]$. The wreath product module $V[W]$ of the $S_m$-module $V$ and the $S_n$-module $W$ is the $S_m[S_n]$-module

$$V[W] := \tilde{W} \otimes \hat{V},$$

where $\otimes$ denotes inner tensor product.

Proposition 7.3 ([31]). Let $V$ be an $S_m$-module and $W$ an $S_n$-module. Then

$$\text{ch} \left( (V \otimes W) \uparrow_{S_m \times S_n}^{S_{m+n}} \right) = \text{ch} V \text{ch} W,$$

$$\text{ch} \left( V[W] \uparrow_{S_m[S_n]}^{S_{mn}} \right) = \text{ch} [\text{ch} W],$$

where $\uparrow^*$ denotes induction.

7.2. Weighted integer partitions. Now let $\Phi$ be a finitary totally ordered set and let $|| \cdot || : \Phi \to \mathbb{P}$ be a map. We call a finite multiset $\tilde{\lambda} = (\tilde{\lambda}_1 \geq \Phi \tilde{\lambda}_2 \geq \Phi \cdots \geq \Phi \tilde{\lambda}_j)$ with elements from $\Phi$ a $\Phi$-partition of length $\ell(\tilde{\lambda}) := j$. We also define $|\tilde{\lambda}| := \sum_j ||\tilde{\lambda}_j||$ and say that $\tilde{\lambda}$ is a $\Phi$-partition of $n$ if $|\tilde{\lambda}| = n$. Denote the set of $\Phi$-partitions by $\text{Par}(\Phi)$.
and the set of $\Phi$-partitions of length $\ell$ by $\text{Par}_\ell(\Phi)$. For $\phi \in \Phi$, we denote by $m_\phi(\tilde{\lambda})$, the number of times $\phi$ appears in $\tilde{\lambda}$.

Let $V$ be an $\mathfrak{S}_\ell$-module, $W_\phi$ be an $\mathfrak{S}_{|\phi||}$-module for each $\phi \in \Phi$ and $\tilde{\lambda}$ a $\Phi$-partition with $\ell$ parts. Note that $\times_{\phi \in \Phi} \mathfrak{S}_{m_\phi(\tilde{\lambda})}[\mathfrak{S}_{|\phi||}]$ is a finite product since $\tilde{\lambda}$ is a finite multiset. The module

$$\bigotimes_{\phi \in \Phi} W_\phi^{\otimes m_\phi(\tilde{\lambda})} \otimes \tilde{V}^{\tilde{\lambda}},$$

is the inner tensor product of two $\times_{\phi \in \Phi} \mathfrak{S}_{m_\phi(\tilde{\lambda})}[\mathfrak{S}_{|\phi||}]$-modules. The first module is the outer tensor product $\bigotimes_{\phi \in \Phi} W_\phi^{\otimes m_\phi(\tilde{\lambda})}$ of the $\tilde{\mathfrak{S}}_{m_\phi(\tilde{\lambda})}[\mathfrak{S}_{|\phi||}]$-modules $W_\phi^{\otimes m_\phi(\tilde{\lambda})}$ (cf. Section 7.1) and the second module is the pull-back $\tilde{V}^{\tilde{\lambda}}$ of the restricted representation $V^{\mathfrak{S}_t} \times_{\phi \in \Phi} \mathfrak{S}_{m_\phi(\tilde{\lambda})}[\mathfrak{S}_{|\phi||}]$ through the product of the natural homomorphisms $\tilde{\mathfrak{S}}_{m_\phi(\tilde{\lambda})}[\mathfrak{S}_{|\phi||}] \rightarrow \mathfrak{S}_{m_\phi(\tilde{\lambda})}$ given by $(\sigma_1, \ldots, \sigma_{m_\phi(\tilde{\lambda})}; \tau) \mapsto \tau$.

The following theorem generalizes [42, Theorem 5.5] and the proof follows the same idea.

**Theorem 7.4.** Let $V$ be an $\mathfrak{S}_\ell$-module and $W_\phi$ be an $\mathfrak{S}_{|\phi||}$-module for each $\phi \in \Phi$. Then

$$\sum_{\tilde{\lambda} \in \text{Par}_\ell(\Phi)} \text{ch} \left( \bigotimes_{\phi \in \Phi} W_\phi^{\otimes m_\phi(\tilde{\lambda})} \otimes \tilde{V}^{\tilde{\lambda}} \right) |_{\times_{\phi \in \Phi} \mathfrak{S}_{m_\phi(\tilde{\lambda})}[\mathfrak{S}_{|\phi||}]} z_{\tilde{\lambda}} = \text{ch}(V) \left[ \sum_{\phi \in \Phi} \text{ch}(W_\phi) z_\phi \right],$$

where $z_\phi$ are indeterminates with $z_{\tilde{\lambda}} := z_{\tilde{\lambda}_1} \cdots z_{\tilde{\lambda}_\ell}$.

**Proof.** Note that restriction, induction, pullback, ch and plethysm in the outer component are all linear and inner tensor product is bilinear. Thus it is enough to prove the theorem for $V$ equal to an irreducible $\mathfrak{S}_t$-module $S^\eta$ (the Specht module associated to the partition $\eta \vdash \ell$). Since the set $\Phi$ is a finitary totally ordered set, we can denote by $\phi^i$, the $i$th element in the total order of $\Phi$. Consider $\tilde{\lambda} \in \text{Par}_\ell(\Phi)$ and let $t := \max\{i \mid \phi^i \in \tilde{\lambda}\}$. Now using Proposition 7.2 and the definition of a wreath product module in equation (7.3) yields

$$\bigotimes_{i=1}^t W_{\phi^i}^{\otimes m_{\phi^i}(\tilde{\lambda})} \otimes S^\eta = \bigotimes_{i=1}^t W_{\phi^i}^{\otimes m_{\phi^i}(\tilde{\lambda})} \otimes \bigoplus_{\emptyset = \nu_0 \subset \nu_1 \subset \cdots \subset \nu_t = \eta} \bigotimes_{i=1}^t S^{\nu_i/\nu_{i-1}},$$
\[
\begin{align*}
&= \bigoplus_{0=\nu_0 \leq \nu_1 \leq \cdots \leq \nu_t = \eta} \bigoplus_{|\nu_i|-|\nu_{i-1}|=m_\phi(\lambda)} \left( \bigotimes_{i=1}^{t} W^\otimes_{m_\phi(\lambda)}(\lambda) \otimes \bigotimes_{i=1}^{t} S^\nu_i/\nu_{i-1} \right) \\
&= \bigoplus_{0=\nu_0 \leq \nu_1 \leq \cdots \leq \nu_t = \eta} \bigoplus_{|\nu_i|-|\nu_{i-1}|=m_\phi(\lambda)} \left( \bigotimes_{i=1}^{t} W^\otimes_{m_\phi(\lambda)}(\lambda) \otimes S^\nu_i/\nu_{i-1} \right) \\
&= \bigoplus_{0=\nu_0 \leq \nu_1 \leq \cdots \leq \nu_t = \eta} \bigotimes_{|\nu_i|-|\nu_{i-1}|=m_\phi(\lambda)} S^\nu_i/\nu_{i-1}[W^\phi].
\end{align*}
\]

We induce and then apply the Frobenius characteristic map \( \text{ch} \). Then using Proposition 7.3 and the transitivity property of induction of representations, we have that

\[
\text{ch} \left( \left( \bigotimes_{i=1}^{t} W^\otimes_{m_\phi(\lambda)}(\lambda) \otimes S^\lambda \right)^\dagger \right) = \prod_{i=1}^{t} \mathfrak{S}_{\lambda} \]

\[
= \bigoplus_{0=\nu_0 \leq \nu_1 \leq \cdots \leq \nu_t = \eta} \bigoplus_{|\nu_i|-|\nu_{i-1}|=m_\phi(\lambda)} \left( \bigotimes_{i=1}^{t} S^\nu_i/\nu_{i-1}[W^\phi] \right)^\dagger \]

\[
= \sum_{0=\nu_0 \leq \nu_1 \leq \cdots \leq \nu_t = \eta} \text{ch} \left( \left( \bigotimes_{i=1}^{t} S^\nu_i/\nu_{i-1}[W^\phi] \right)^\dagger \right) \]

\[
= \sum_{0=\nu_0 \leq \nu_1 \leq \cdots \leq \nu_t = \eta} \text{ch} \left( \left( \bigotimes_{i=1}^{t} S^\nu_i/\nu_{i-1}[W^\phi] \right)^\dagger \right) \]

\[
= \sum_{0=\nu_0 \leq \nu_1 \leq \cdots \leq \nu_t = \eta} \text{ch} \left( \left( \bigotimes_{i=1}^{t} S^\nu_i/\nu_{i-1}[W^\phi] \right)^\dagger \right) \]

\[
= \sum_{0=\nu_0 \leq \nu_1 \leq \cdots \leq \nu_t = \eta} \prod_{i=1}^{t} \mathfrak{S}_{\lambda} \]

\[
= \prod_{i=1}^{t} \mathfrak{S}_{\lambda}.
\]
\[ \sum_{i=1}^{t} \prod_{\nu_i / \nu_{i-1}} \nu_{i-1} \left[ \text{ch } W_{\phi^i} \right] \]

Now note that \( s_{\nu_i / \nu_{i-1}} [\text{ch } W_{\phi^i}] = s_{\nu_i / \nu_{i-1}} [\text{ch } W_{\phi^i}] \sum_{i=1}^{t} \prod_{\nu_i / \nu_{i-1}} \nu_{i-1} \) and that \( s_{\nu_i / \nu_{i-1}} = s_{\emptyset} = 1 \) if \( \nu_i = \nu_{i-1} \). And using Proposition 7.1, we obtain

\[ \sum_{\lambda \in \text{Par}_{\ell}(\Phi)} \left( \bigotimes_{i \geq 1} \left( W_{\phi^i} \otimes \tilde{S}_{\emptyset} \right) \right)^{\uparrow G} \left( \bigotimes_{i=1}^{t} \prod_{\nu_i / \nu_{i-1}} \nu_{i-1} \right) \]

7.3. Using Whitney (co)homology to compute (co)homology.

The technique of Sundaram [39] to compute characters of \( G \)-representations on the (co)homology of pure \( G \)-posets is based on the following result:

**Lemma 7.5** ([39] Lemma 1.1). Let \( P \) be a bounded poset of length \( \ell \geq 1 \) and let \( G \) be a group of automorphisms of \( P \). Then the following isomorphism of virtual \( G \)-modules holds

\[ \bigoplus_{i=0}^{\ell} (-1)^i WH^i(P) \cong_G 0. \]

Recall that if a group \( G \) of automorphisms acts on the poset \( P \), this action induces a representation of \( G \) on \( WH^r(P) \) for every \( r \). From equation (6.4), when \( P \) is Cohen-Macaulay, \( WH^r(P) \) breaks into the direct sum of \( G \)-modules

\[ WH^r(P) \cong_p \bigoplus_{x \in P/\sim} \tilde{H}^{r-2}((0, x); k)^{\uparrow G}_{\rho(x) = r}. \]
where $P/\sim$ is a set of orbit representatives and $G_x$ the stabilizer of $x$.

Let $\mu \in \text{wcomp}_n$. We want to apply Lemma 7.3 to the dual poset $[\hat{0}, [n]^{\mu}]^*$ of the maximal interval $[\hat{0}, [n]^{\mu}]$, which by Theorem 1.4 is Cohen-Macaulay. In order to compute $WH^r([\hat{0}, [n]^{\mu}]^*)$, by equation (7.4), we need to specify a set of orbit representatives for the action of $\mathcal{S}_n$ on $[\hat{0}, [n]^{\mu}]^*$. For this we consider the set

$$\Phi = \{ \phi \in \text{wcomp} \mid \text{supp}(\phi) \subseteq [k] \}$$

and the map $||\phi|| := |\phi| + 1$ for $\phi \in \text{wcomp}$ (cf. Section 7.2). We fix any finitary total order on $\Phi$. For any $\Phi$-partition $\lambda$ of $n$ of length $\ell$, we denote by $\alpha_{\lambda}$ the weighted partition $\{A_{\lambda^1}, \ldots, A_{\lambda^\ell}\}$ of $[n]$ whose blocks are of the form

$$A_i = \left[ \sum_{j=1}^i ||\lambda_j|| \right] \setminus \left[ \sum_{j=1}^{i-1} ||\lambda_j|| \right].$$

Recall that for $\nu, \mu \in \text{wcomp}$, we say that $\mu \leq \nu$ if $\mu(i) \leq \nu(i)$ for every $i$ and we denote by $\nu + \mu$, the weak composition defined by $(\nu + \mu)(i) := \nu(i) + \mu(i)$. Let

$$\text{Par}^{\mu}(\Phi) := \{ \lambda \in \text{Par}(\Phi) \mid ||\lambda|| = ||\mu||, \sum_i \lambda_i \leq \mu \}.$$ 

It is not difficult to see that $\{\alpha_{\lambda} \mid \lambda \in \text{Par}^{\mu}(\Phi)\}$ is a set of orbit representatives for the action of $\mathcal{S}_n$ on $[\hat{0}, [n]^{\mu}]^*$. Indeed, any weighted partition $\beta \in [\hat{0}, [n]^{\mu}]^*$ can be obtained as $\beta = \sigma\alpha_{\lambda}$ for suitable $\lambda \in \text{Par}^{\mu}(\Phi)$ and $\sigma \in \mathcal{S}_n$. It is also clear that $\alpha_{\lambda} \neq \sigma\alpha_{\lambda'}$ for $\lambda \neq \lambda'$ in $\text{Par}^{\mu}(\Phi)$ and for every $\sigma \in \mathcal{S}_n$. Observe that the partition $\alpha_{\lambda}$ has stabilizer $\times_{\phi \in \mathcal{S}} \mathcal{G}_{m_{\phi}(\lambda)}[\mathcal{S}|_{\phi||}]$. By equation (7.4) applied to $[\hat{0}, [n]^{\mu}]^*$,

$$WH^r([\hat{0}, [n]^{\mu}]^*) \cong \mathcal{S}_n \bigoplus_{\lambda \in \text{Par}^{\mu}(\Phi)} \tilde{H}^{r-3}((\alpha_{\lambda}, [n]^{\mu})) \cap \mathcal{S}_n$$

Note that if $r = 2$ then the open interval $(\alpha_{\lambda}, [n]^{\mu})$ is the empty poset. Hence $\tilde{H}^{r-3}((\alpha_{\lambda}, [n]^{\mu}))$ is isomorphic to the trivial representation of $\times_{\phi \in \mathcal{S}} \mathcal{G}_{m_{\phi}(\lambda)}[\mathcal{S}|_{\phi||}]$. If $r = 1$ then $\alpha_{\lambda} = [n]^{\mu}$. In this case we use the convention that $\tilde{H}^{r-3}((\alpha_{\lambda}, [n]^{\mu}))$ is isomorphic to the trivial representation of $\times_{\phi \in \mathcal{S}} \mathcal{G}_{m_{\phi}(\lambda)}[\mathcal{S}|_{\phi||}]$.

We apply Lemma 7.5 together with equation (7.5) to obtain the following result.
Lemma 7.6. For \( n \geq 1 \) and \( \mu \in \text{wcomp}_{n-1} \) we have the following \( \mathfrak{S}_n \)-module isomorphism

\[
\begin{align*}
\ell_{\mathfrak{S}_n} \delta_{n,1} \cong & \mathfrak{S}_n \bigoplus_{\lambda \in \text{Par}(\Phi)} (-1)^{\ell(\lambda)-1} \tilde{H}^{\ell(\lambda)-3}((\alpha_{\lambda}, [n]^\mu)) \uparrow_{\mathfrak{S}_n}^{\mathfrak{S}_{\ell(\lambda)}} \times_{\varphi \in \Phi} \mathfrak{S}_{m_{\phi}(\lambda)}[\mathfrak{S}_{\|\phi\|}]
\end{align*}
\]

where \( \ell_{\mathfrak{S}_n} \) denotes the trivial representation of \( \mathfrak{S}_n \).

Lemma 7.7. For all \( \tilde{\lambda} \in \text{Par}(\Phi) \) with \( |\tilde{\lambda}| = n \) and \( \nu \in \text{wcomp}_{\ell(\tilde{\lambda})-1} \), the following \( \times_{\varphi \in \Phi} \mathfrak{S}_{m_{\phi}(\lambda)}[\mathfrak{S}_{\|\phi\|}] \)-module isomorphism holds:

\[
\tilde{H}^{\ell(\tilde{\lambda})-3}((\alpha_{\tilde{\lambda}}, [n]^\nu \Sigma \lambda_j)) \cong \left( \bigotimes_{\varphi \in \Phi} (1_{\mathfrak{S}_{\|\phi\|}})^{\otimes m_{\phi}(\tilde{\lambda})} \right) \otimes \tilde{H}^{\ell(\tilde{\lambda})-3}((\tilde{0}, [\ell(\tilde{\lambda})]^\nu)) \tilde{\lambda}.
\]

Proof. The poset \( [\tilde{0}, [\ell(\tilde{\lambda})]^\nu] \) is a \( \times_{\varphi \in \Phi} \mathfrak{S}_{m_{\phi}(\lambda)}[\mathfrak{S}_{\|\phi\|}] \)-poset with the action given by the pullback through the product of the natural homomorphisms \( \mathfrak{S}_{m_{\phi}(\lambda)}[\mathfrak{S}_{\|\phi\|}] \to \mathfrak{S}_{m_{\phi}(\tilde{\lambda})} \). There is a natural poset isomorphism between \( [\alpha_{\tilde{\lambda}}, [n]^\nu \Sigma \lambda_j] \) and \( [\tilde{0}, [\ell(\tilde{\lambda})]^\nu] \). Indeed, for a weighted partition \( \{B_1^\mu, \ldots, B_t^\mu\} \geq \alpha_{\tilde{\lambda}} = \{A_1^\lambda_1, \ldots, A_s^\lambda_s\} \), each weighted block \( B_j^\mu \) is of the form \( B_j = A_{i_1} \cup A_{i_2} \cup \ldots \cup A_{i_s} \) and \( \mu_j = u_j + \sum_k \lambda_{i_k} \), where \( |u_j| = s - 1 \) and \( \sum_j u_j \leq \nu \). Let

\[
\Gamma : [\alpha_{\tilde{\lambda}}, [n]^\nu \Sigma \lambda_j] \to [\tilde{0}, [\ell(\tilde{\lambda})]^\nu]
\]

be the map such that \( \Gamma([B_1^\mu, \ldots, B_t^\mu]) \) is the weighted partition in which each weighted block \( B_j^\mu \) is replaced by \( \{i_1, i_2, \ldots, i_s\}^\mu \). The map \( \Gamma \) is an isomorphism of posets that commutes with the action of \( \times_{\varphi \in \Phi} \mathfrak{S}_{m_{\phi}(\lambda)}[\mathfrak{S}_{\|\phi\|}] \). The isomorphism of \( \times_{\varphi \in \Phi} \mathfrak{S}_{m_{\phi}(\lambda)}[\mathfrak{S}_{\|\phi\|}] \)-posets induces an isomorphism of the \( \times_{\varphi \in \Phi} \mathfrak{S}_{m_{\phi}(\lambda)}[\mathfrak{S}_{\|\phi\|}] \)-modules \( \tilde{H}^{\ell(\tilde{\lambda})-3}((\alpha_{\tilde{\lambda}}, [n]^\nu \Sigma \lambda_j)) \) and \( \tilde{H}^{\ell(\tilde{\lambda})-3}((\tilde{0}, [\ell(\tilde{\lambda})]^\nu)) \tilde{\lambda} \). The result follows since \( \bigotimes_{\varphi \in \Phi} (1_{\mathfrak{S}_{\|\phi\|}})^{\otimes m_{\phi}(\tilde{\lambda})} \) is the trivial representation of \( \times_{\varphi \in \Phi} \mathfrak{S}_{m_{\phi}(\lambda)}[\mathfrak{S}_{\|\phi\|}] \). \( \square \)

Let \( R \) be the ring of symmetric functions \( \Lambda_{Q} \) in variables \( x = (x_1, x_2, \ldots) \). There is a natural inner product in \( \Lambda_{R} \) defined for arbitrary partitions \( \lambda \) and \( \nu \), by

\[
\langle p_{\lambda}, p_{\nu} \rangle = z_{\lambda} \delta_{\lambda, \nu}
\]

and then extended linearly to \( \Lambda_{R} \), where \( z_{\lambda} \) is an integer that depends on \( \lambda \). This inner product defines a notion of convergence. Indeed, for a sequence of symmetric functions \( f_n \in \Lambda_{R}, n \geq 1 \), we say that \( \{f_n\}_{n \geq 1} \) converges if for every partition \( \nu \) there is a number \( N \) such
that \( \langle f_n, p_m \rangle = \langle f_n, p_m \rangle \) whenever \( n, m \geq N \). We use \( \Lambda_R \) to denote the completion of the ring of \( \Lambda_R \) with respect to this topology. It is not difficult to verify that \( \Lambda_R \) consists of the class of formal power series in two sets of variables, \( x = (x_1, x_2, \ldots) \) and \( y = (y_1, y_2, \ldots) \), that can be expressed as \( \sum_{\lambda} c_{\lambda}(x)p_{\lambda}(y) \), where \( c_{\lambda}(x) \in \Lambda_Q \). Given a formal power series \( F(y) = \sum c_{\lambda}(x)p_{\lambda}(y) \) in \( \Lambda_R \) and a formal power series \( g \in \mathbb{Q}[[z_1, z_2, \ldots]] \) satisfying \( g(0) = 0 \), we can extend the definition of plethysm from symmetric functions to formal power series in \( \Lambda_R \) by

\[
F[g] := \sum_{\lambda} c_{\lambda}(x)p_{\lambda}[g].
\]

The reader can check that \( \Lambda_R \), together with plethysm and the plethystic unit \( p_1(y) \), has the structure of a monoid.

Let \( G(y) \) and \( F(y) \) be in \( \Lambda_R \) such that \( G(0) = 0 \). The power series \( G(y) \) is said to be a plethystic inverse of \( F(y) \) with respect to \( y \), if \( F(y)[G(y)] = p_1(y) \). It can be shown that if this is the case, then \( F(y) \) is unique and also \( G(y)[F(y)] = p_1(y) \). Thus \( G(y) \) and \( F(y) \) are said to be plethystic inverses of each other with respect to \( y \), and we write \( G(y) = F[-1](y) \). Note that

\[
\sum_{\mu \in \text{wcomp}_{n-1}} \text{ch} \tilde{H}^{n-3}((\hat{0}, [n]^{\mu})) x^{\mu} = \sum_{\lambda \vdash n-1} \text{ch} \tilde{H}^{n-3}((\hat{0}, [n]^{\lambda})) m_\lambda(x) \in \Lambda_R,
\]

where \( m_{\lambda} \) is the monomial symmetric function associated to the partition \( \lambda \). Hence the left hand side of equation (7.7) below is in \( \Lambda_R \).

**Theorem 7.8.** We have

\[
(7.7) \quad \sum_{n \geq 1} (-1)^{n-1} \sum_{\mu \in \text{wcomp}_{n-1}} \text{ch} \tilde{H}^{n-3}((\hat{0}, [n]^{\mu})) x^{\mu} = \left( \sum_{n \geq 1} h_{n-1}(x)h_n(y) \right)^{-1}.
\]

**Proof.** For presentation purposes let us use temporarily the notation \( G_\lambda := \times_{\phi \in \Phi} \mathfrak{G}_{m_{\phi}(\lambda)}|\mathfrak{G}_{|\phi||} \). We also use the convention that

\[
\tilde{H}^{\ell(\lambda)-3}((\alpha_\lambda, [n]^{\mu})) = 0
\]

whenever \( \alpha_\lambda \not\in [n]^{\mu} \). Applying the Frobenius characteristic map \( \text{ch} \) (in \( y \) variables) to both sides of equation (7.6) multiplying by \( x^{\mu} \) and summing over all \( \mu \in \text{wcomp}_{n-1} \) with \( \text{supp}(\mu) \subseteq [k] \) yields

\[
h_1(y)\delta_{n,1} = \sum_{\mu \in \text{wcomp}_{n-1} \subseteq [k]} x^{\mu} \text{ch} \left( \bigoplus_{\lambda \in \Par^{\mu}(\Phi)} (-1)^{\ell(\lambda)-1} \tilde{H}^{\ell(\lambda)-3}((\alpha_\lambda, [n]^{\mu}))^{\mathfrak{G}_n}_{G_\lambda} \right)
\]
Lemma 7.7 and summing over all \( n \geq 1 \) we have

\[
\begin{align*}
    h_1(y) &= \sum_{\ell \geq 1} \sum_{\nu \in \text{wcomp}_{\ell-1}} \sum_{\tilde{\lambda} \in \text{Par}_r(\Phi)} x^{\nu + \sum \lambda_r} ch \left( \left( \bigotimes_{\phi \in \Phi} (1_{\mathcal{E}_{||\phi||}} \otimes m_\phi(\tilde{\lambda}) \otimes \tilde{H}_\ell^{\tilde{\lambda}}) \right) \bigg| G_{\tilde{\lambda}} \right) \\
    &= \sum_{\ell \geq 1} \sum_{\nu \in \text{wcomp}_{\ell-1}} \sum_{\tilde{\lambda} \in \text{Par}_r(\Phi)} x^{\nu} \sum_{\lambda_r} x^{\sum \lambda_r} ch \left( \left( \bigotimes_{\phi \in \Phi} (1_{\mathcal{E}_{||\phi||}} \otimes m_\phi(\tilde{\lambda}) \otimes \tilde{H}_\ell^{\tilde{\lambda}}) \right) \bigg| G_{\tilde{\lambda}} \right) \\
    &= \sum_{\ell \geq 1} \sum_{\nu \in \text{wcomp}_{\ell-1}} \sum_{\tilde{\lambda} \in \text{Par}_r(\Phi)} x^{\nu} \sum_{\lambda_r} x^{\sum \lambda_r} ch \left( \left( \bigotimes_{\phi \in \Phi} (1_{\mathcal{E}_{||\phi||}} \otimes m_\phi(\tilde{\lambda}) \otimes \tilde{H}_\ell^{\tilde{\lambda}}) \right) \bigg| G_{\tilde{\lambda}} \right).
\end{align*}
\]

Now we use Theorem 7.4 with \( z_\phi = x^\phi \),

\[
\begin{align*}
    h_1(y) &= \sum_{\ell \geq 1} \sum_{\nu \in \text{wcomp}_{\ell-1}} \sum_{\phi \in \Phi} ch(H_\ell^\nu) \left[ \sum_{\phi \in \text{wcomp}_{\ell-1}} \sum_{\phi \in \Phi} h_{||\phi||}(y)x^\phi \right] \\
    &= \left( \sum_{\ell \geq 1} \sum_{\nu \in \text{wcomp}_{\ell-1}} \sum_{\phi \in \Phi} ch(H_\ell^\nu)x^\nu \right) \left[ \sum_{j \geq 1} h_j(y)h_{j-1}(x_1, \ldots, x_k) \right].
\end{align*}
\]

The last step uses the definition of the complete homogeneous symmetric polynomial \( h_{j-1}(x_1, \ldots, x_k) \). To complete the proof we let \( k \) get arbitrarily large. \( \square \)
Theorem 7.9 (Theorem 1.8). We have that
\[
\sum_{n \geq 1} \sum_{\mu \in \text{wcomp}_{n-1}} \text{ch Lie}(\mu) x^\mu = -\left( - \sum_{n \geq 1} h_{n-1}(x) h_n(y) \right)^{-1},
\]

Proof. We have
\[
p_1(y) = \left( \sum_{n \geq 1} (-1)^{n-1} \sum_{\mu \in \text{wcomp}_{n-1}} \text{ch}\tilde{H}^{n-3}((\hat{0}, [n]^\mu)) x^\mu \right) \left[ \sum_{n \geq 1} h_{n-1}(x) h_n(y) \right]
\]
\[
= \left( - \sum_{n \geq 1} \sum_{\mu \in \text{wcomp}_{n-1}} \omega \left( \text{ch}\tilde{H}^{n-3}((\hat{0}, [n]^\mu)) \right) x^\mu \right) \left[ - \sum_{n \geq 1} h_{n-1}(x) h_n(y) \right]
\]
\[
= \left( - \sum_{n \geq 1} \sum_{\mu \in \text{wcomp}_{n-1}} \text{ch Lie}(\mu) x^\mu \right) \left[ - \sum_{n \geq 1} h_{n-1}(x) h_n(y) \right].
\]
The first two equalities above follow from Theorem 7.8 and from equation (7.2), respectively. Recall that for an \( \mathfrak{S}_n \)-module \( V \) we have that \( \text{ch}(V \otimes_{\mathfrak{S}_n} \text{sgn}_n) = \omega(\text{ch} V) \), which proves the third equality. Finally the last equality makes use of Theorem 2.6. \qed

In the case \( k = 2, \) Theorem 1.8 specializes to the following result when \( x_1 = t, \) \( x_2 = 1 \) and \( x_i = 0 \) for \( i \geq 3. \)

Theorem 7.10. For \( n \geq 1, \)
\[
\sum_{n \geq 1} \sum_{i=0}^{n-1} \text{ch Lie}_2(n, i) t^i = -\left( - \sum_{n \geq 1} \frac{t^n - 1}{t - 1} h_n(y) \right)^{-1}.
\]

For a closely related result obtained using operad theoretic arguments, see [13]. A proof of Theorem 1.8 could also be obtained via operad theory using Vallette’s results in [40].

And we obtain a well-known classical result when \( x_1 = 1, \) \( x_i = 0 \) for \( i \geq 2. \)

Theorem 7.11. For \( n \geq 1, \)
\[
\sum_{n \geq 1} \text{ch Lie}(n) = -\left( - \sum_{n \geq 1} h_n(y) \right)^{-1}.
\]
We show that Theorem 1.8 reduces to Theorem 1.5 after applying an appropriate specialization. Recall that $R = \Lambda_{\mathbb{Q}}$ and consider the map $E : \hat{\Lambda}_R \to R[[y]]$ defined by:

$$E(p_i(y)) = y^\delta_{i,1}$$

for $i \geq 1$ and extended multiplicatively, linearly and taking the corresponding limits to all of $\hat{\Lambda}_R$. It is not difficult to check that $E$ is a ring homomorphism since $E$ is defined on generators. Moreover, we show in the following proposition that the specialization $E$ maps plethysm in $\hat{\Lambda}_R$ to composition in $R[[y]]$.

**Proposition 7.12.** For all $F, G \in \hat{\Lambda}_R$ such that $G(0) = 0$,

$$E(F[G]) = E(F)(E(G)).$$

**Proof.** Using the definition of plethysm and using the convention $x^k = (x_1^k, x_2^k, \ldots)$,

$$p_\nu(y) \left[ \sum_\lambda c_\lambda(x)p_\lambda(y) \right] = \prod_{i=1}^{\ell(\nu)} p_{\nu_i}(y) \left[ \sum_\lambda c_\lambda(x)p_\lambda(y) \right]$$

$$= \prod_{i=1}^{\ell(\nu)} \sum_\lambda c_\lambda(x^{\nu_i})p_\lambda(y^{\nu_i})$$

$$= \prod_{i=1}^{\ell(\nu)} \sum_\lambda c_\lambda(x^{\nu_i}) \prod_{j=1}^{\ell(\lambda)} p_{\lambda_j}(y^{\nu_i})$$

$$= \prod_{i=1}^{\ell(\nu)} \sum_\lambda c_\lambda(x^{\nu_i}) \prod_{j=1}^{\ell(\lambda)} \sum_\nu c_{\lambda\nu_i}(x^{\nu_i}) p_{\lambda_j}(y^{\nu_i}).$$

(7.8)

Note that $E(p_{\lambda_{i,1}}(y)) = y^\delta_{\lambda_{i,1}} = y^\delta_{\lambda_{i,1}}$. Then if $\nu$ has at least one part $\nu_i$ of size greater than 1, equation (7.8) implies

$$E \left( p_\nu(y) \left[ \sum_\lambda c_\lambda(x)p_\lambda(y) \right] \right) = 0 = E(p_\nu(y)) \left( E \left( \sum_\lambda c_\lambda(x)p_\lambda(y) \right) \right),$$

since $E(p_\nu(y)) = 0$. If $\nu = (1^m)$, then

$$E \left( p_\nu(y) \left[ \sum_\lambda c_\lambda(x)p_\lambda(y) \right] \right) = E \left( \prod_{i=1}^{\ell(\nu)} \prod_\lambda c_\lambda(x^{\nu_i}) \prod_{j=1}^{\ell(\lambda)} p_{\lambda_j}(y) \right)$$
$$\begin{align*}
= & \ E \left( \prod_{i=1}^{m} \sum_{\lambda} c_{\lambda}(x) \prod_{j=1}^{\ell(\lambda)} p_{\lambda_j}(y) \right) \\
= & \ E \left( \sum_{\lambda} c_{\lambda}(x) \prod_{j=1}^{\ell(\lambda)} p_{\lambda_j}(y) \right)^m \\
= & \ E(p_{(1^m)}(y)) \left( E \left( \sum_{\lambda} c_{\lambda}(x) \prod_{j=1}^{\ell(\lambda)} p_{\lambda_j}(y) \right) \right)^m.
\end{align*}$$

We just proved that $E(p_{\nu}(y)[G]) = E(p_{\nu}(y))[E(G)]$ for any $G \in \widehat{\Lambda}_R$. The proof of the proposition follows by extending this result to all of $\widehat{\Lambda}_R$ by linearity and taking limits. 

Since $E(p_1(y)) = y$, we conclude that $E$ is a monoid homomorphism between the monoid of power series $G(y) \in \widehat{\Lambda}_R$ such that $G(0) = 0$, under plethysm and with unit $p_1$, and the set of power series $g(y) \in R[[y]]$ such that $g(0) = 0$, under composition and unit $y$.

The specialization $E$ can be better understood under the definition of the Frobenius characteristic map. Let $V$ be a representation of $\mathfrak{S}_n$ and $\chi^V$ its character, then

$$\text{ch}(V) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi^V(\sigma)p_{\lambda(\sigma)}(y),$$

where $\lambda(\sigma)$ is the cycle type of the permutation $\sigma \in \mathfrak{S}_n$.

We have that

$$E(\text{ch } V) = \frac{1}{n!} E \left( \sum_{\sigma \in \mathfrak{S}_n} \chi^V(\sigma)p_{\lambda(\sigma)}(y) \right) = \chi^{V}(\text{id}) \frac{y^n}{n!} = \dim V \frac{y^n}{n!}.$$

In particular since $h_n(y) = \text{ch}(1_{\mathfrak{S}_n})$, the Frobenius characteristic of the trivial representation of $\mathfrak{S}_n$, we have that $E(h_n(y)) = \frac{y^n}{n!}$. Therefore Theorem 1.8 reduces to Theorem 1.5 after we apply the specialization $E$.

Theorem 1.8 gives an implicit description of the character for the representation of $\mathfrak{S}_n$ on $\text{Lie}(\mu)$; Theorem 7.10 gives a description of the character of $\text{Lie}_2(n, i)$. Dotsenko and Khoroshkin in [13] computed an explicit product formula for the $\text{SL}_2 \times \mathfrak{S}_n$-character of $\text{Lie}_2(n)$. From this one can get the coefficients (as polynomials in $t$) of $p_\lambda$ in the symmetric function $\sum_{i=0}^{n-1} \text{ch } \text{Lie}_2(n, i) t^i$. 

Question 7.13. Can we find explicit character formulas for the representation of $\mathfrak{S}_n$ on $\text{Lie}(\mu)$ for general $\mu \in \text{wcomp}_{n-1}$? What are the multiplicities of the irreducibles?

Since $\sum_{\mu \in \text{wcomp}_{n-1}} \text{ch} \text{Lie}(\mu) x^{\mu}$ is a symmetric function in $x$ with coefficients that are symmetric functions in $y$, we can write

$$\sum_{\mu \in \text{wcomp}_{n-1}} \text{ch} \text{Lie}(\mu) x^{\mu} = \sum_{\lambda \vdash n-1} C_{\lambda}(y) e_{\lambda}(x),$$

where $C_{\lambda}(y)$ is a homogeneous symmetric function of degree $n$ with coefficients in $\mathbb{Z}$.

By Theorem 4.9, $E(C_{\lambda}(y))$ equals the number $c_{n,\lambda}$ of trees $\Upsilon \in \text{Nor}_n$ of comb type (or Lyndon type) $\lambda(\Upsilon) = \lambda$. We propose the following conjecture.

Conjecture 7.14. The coefficients $C_{\lambda}(y)$ are Schur positive.

The conjecture basically asserts that $C_{\lambda}(y)$ is the Frobenius characteristic of a representation of dimension $c_{n,\lambda}$. An approach to proving the conjecture is to find such a representation.

8. Related work

In a subsequent paper we generalize the results presented here further. We consider a more general family of weighted partition posets with different restrictions on the sizes of the blocks. Then the corresponding generalization of the isomorphism of Theorem 1.3 allows us to study generalizations of the free multibracketed Lie algebras. These generalizations include for example multibracketed versions of free Lie $k$-algebras closely related to the ones studied by Hanlon and Wachs in [25].

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