NON-RIEMANNIAN EINSTEIN-RANDERS METRICS ON $E_6/A_4$ AND $E_6/A_1$

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Abstract. In this paper, we first prove that homogeneous spaces $E_6/A_4$ and $E_6/A_1$ admit Einstein metrics which are $\text{Ad}(T \times A_1 \times A_4)$-invariant, and then show that they admit Non-Riemannian Einstein-Randers metrics.

1. Introduction

Randers metrics were introduced by Randers in the context of general relativity, and named after him by Ingarden. It shows the importance of Randers metrics in physics. Moreover, Randers metrics are useful in other fields; see Ingarden’s account on [10] for their application in the study on the Lagrangian of relativistic electrons.

Just as in the Riemannian case, it is a fundamental problem to classify homogeneous Einstein-Randers spaces. In particular, it is very important to know if a homogeneous manifold admits invariant Einstein-Randers metrics. There are a lot of studies on Einstein-Randers metrics on homogeneous manifolds, see [4] [6] [11] [12] [13] [14] [15] [16] [17].

In this paper, we will discuss Einstein metrics and Einstein-Randers metrics on homogeneous spaces $E_6/A_4$ and $E_6/A_1$. In Section 2, we prove that there are only four Einstein metrics on $E_6/A_4$ and two Einstein metrics on $E_6/A_1$ which are $\text{Ad}(T \times A_1 \times A_4)$-invariant. Furthermore, in Section 3, we prove that there are at least four and two families of $E_6$-invariant non-Riemannian Einstein-Randers metrics on $E_6/A_4$ and $E_6/A_1$ respectively.

2. Einstein metrics

Consider the symmetric spaces $E_6/A_1 \times A_5$ and $A_5/T \times A_4$. Let $E_6 = A_1 \oplus A_5 \oplus m_1$ and $A_5 = h_0 \oplus A_4 \oplus m_2$ be the corresponding decompositions of the Lie algebras. Here $h_0$ is the Lie algebra of $T$. Let $H$ be the Lie group $T \times A_4 \times A_1$ with the Lie algebra $h_0 \oplus A_4 \oplus A_1$. Then we have the following decomposition of the Lie algebra $E_6$:

$$E_6 = h_0 \oplus A_4 \oplus A_1 \oplus m_1 \oplus m_2.$$  \hfill (2.1)

Here the $\text{Ad}(H)$-modules $m_i$, $i = 1, 2$ are irreducible and mutually non-equivalent and $\dim h_0 = 1$. For the structure of such decomposition, see [2] [7]. Clearly $\dim A_4 = 24$, $\dim A_1 = 3$, $\dim m_1 = 40$ and $\dim m_2 = 10$. The left-invariant metric on $E_6$ which is $\text{Ad}(H)$-invariant must be of the form

$$\langle \cdot, \cdot \rangle = u_0 \cdot B|_{h_0} + u_1 \cdot B|_{A_4} + u_2 \cdot B|_{A_1} + x_1 \cdot B|_{m_1} + x_2 \cdot B|_{m_2},$$  \hfill (2.2)

where $u_0, u_1, u_2, x_1, x_2 \in \mathbb{R}^+$. The space of left-invariant symmetric covariant 2-tensors on $E_6$ which are $\text{Ad}(H)$-invariant is given by

$$v_0 \cdot B|_{h_0} + v_1 \cdot B|_{A_4} + v_2 \cdot B|_{A_1} + v_3 \cdot B|_{m_1} + v_4 \cdot B|_{m_2},$$  \hfill (2.3)

where $v_0, v_1, v_2, v_3, v_4 \in \mathbb{R}$. In particular, the Ricci tensor $r$ of a left-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ on $G$ is a left invariant symmetric covariant 2-tensor on $G$ which is $\text{Ad}(H)$-invariant. Thus $r$ is of the
The authors give in [2] the formulae of the Ricci tensor corresponding to the metric (2.2) on $E_6$. In fact, the case for $E_6$ is just one case of what the authors study in [2].

For this decomposition of $E_6$, the left-invariant metric on $E_6/A_4 \times A_1$ which is $Ad(H)$-invariant must be of the form

$$
\langle \cdot, \cdot \rangle = u_0 \cdot B|_{b_0} + x_1 \cdot B|_{m_1} + x_2 \cdot B|_{m_2},
$$

where $u_0, x_1, x_2 \in \mathbb{R}^+$. Based on the formulae given in [2], the authors classify in [6] Einstein metrics on $E_6/A_4 \times A_1$ which are $Ad(H)$-invariant.

The following is to discuss the left-invariant metrics on $E_6/A_4$ and $E_6/A_1$ which are $Ad(H)$-invariant corresponding to the decomposition (2.1) of $E_6$.

### 2.1. The case of $E_6/A_4$

The left-invariant metric on $E_6/A_4$ which is $Ad(H)$-invariant must be of the form

$$
\langle \cdot, \cdot \rangle = u_0 \cdot B|_{b_0} + u_2 \cdot B|_{A_1} + x_1 \cdot B|_{m_1} + x_2 \cdot B|_{m_2},
$$

where $u_0, u_2, x_1, x_2 \in \mathbb{R}^+$. Based on the formulae given in [2], we have the components of the Ricci tensor $\bar{r}$ of the metric (2.5) on $E_6/A_4$:

$$
\begin{aligned}
\bar{r}_{b_0} &= \frac{u_0}{8x_1^2} + \frac{u_0}{8x_2^2}, \\
\bar{r}_{A_1} &= \frac{1}{24u_2} + \frac{5u_2}{24x_1^2}, \\
\bar{r}_{m_1} &= \frac{1}{2x_1} - \frac{x_2}{16x_1^2} - \frac{u_0}{160x_1^2} - \frac{u_2}{32x_1^2}, \\
\bar{r}_{m_2} &= \frac{1}{4x_2} + \frac{x_2}{8x_1^2} - \frac{u_0}{40x_2^2}.
\end{aligned}
$$

Furthermore, the metric is Einstein if and only if there exists a positive solution $\{u_0, u_2, x_1, x_2\}$ of the system of equations

$$
\bar{r}_{b_0} = \bar{r}_{A_1} = \bar{r}_{m_1} = \bar{r}_{m_2}.
$$

The following is to solve the equations by the theory of Gröbner basis. Putting $u_0 = 1$ and by $\bar{r}_{b_0} = \bar{r}_{A_1}, \bar{r}_{b_0} = \bar{r}_{m_1}, \bar{r}_{b_0} = \bar{r}_{m_2}$, we have

$$
\begin{aligned}
f_1 &= -x_1^2x_2^3 - 5x_1^3u_2^2 + 3x_1^2u_2 + 3x_1^2u_2 = 0, \\
f_2 &= -80x_1x_2^2 + 10x_2^3 + 20x_1^2 + 21x_1^2 + 5x_2^2u_2 = 0, \\
f_3 &= -10x_1^2x_2 + 5x_2^3 + 6x_1^2 + 5x_2^2 = 0.
\end{aligned}
$$

Consider the polynomial ring $R = \mathbb{Q}[z, x_1, x_2, u_2]$ and an ideal $I$ generated by $\{f_1, f_2, f_3, zx_1x_2u_2 - 1\}$ to find non-zero solutions of (2.6). Take a lexicographic order $> \text{with } z > u_2 > x_1 > x_2$ for a monomial ordering on $R$. By the help of computer, we have the polynomial of $x_2$ containing in the Gröbner basis of the ideal $I$:

$$
f(x_2) = 27263765625x_2^6 - 82709987500x_2^5 + 94104102500x_2^4 - 48116787500x_2^3 + 9948352750x_2^2 - 491681700x_2 + 74376100x_2^2 - 1183780x_2 + 142129.
$$

In the Gröbner basis of the ideal $I$, $x_1$ and $u_2$ can be written into polynomials of $x_2$. The equation $f(x_2) = 0$ has four solutions:

$$
x_2 \approx 0.6513015810, \quad x_2 \approx 0.6770950751, \quad x_2 \approx 0.8288266917, \quad x_2 \approx 0.8641265950.
$$
In fact, we have the following solutions of (2.6):
\[
\{ u_2 \approx 0.1141930615, \quad x_1 \approx 1.200678505, \quad x_2 \approx 0.6513015810 \},
\{ u_2 \approx 1.746579387, \quad x_1 \approx 0.9798479028, \quad x_2 \approx 0.6770950751 \},
\{ u_2 \approx 0.7564861893, \quad x_1 \approx 0.5068895851, \quad x_2 \approx 0.8288266917 \},
\{ u_2 \approx 0.0549236976, \quad x_1 \approx 0.4382514353, \quad x_2 \approx 0.8641265950 \}.
\]

In summary, we have:

**Theorem 2.1.** Let the notations be as above. Then every left invariant metric on $E_6/A_4$ which is $Ad(H)$-invariant is of the form (2.5). Up to scaling, there are four Einstein metrics on $E_6/A_4$ which are $Ad(H)$-invariant.

**2.2. The case of $E_6/A_1$.** The left-invariant metric on $E_6/A_1$ which is $Ad(H)$-invariant must be of the form
\[
(\cdot, \cdot) = u_0 \cdot B|_{h_0} + u_1 \cdot B|_{A_4} + x_1 \cdot B|_{m_1} + x_2 \cdot B|_{m_2},
\]
where $u_0, u_1, x_1, x_2 \in \mathbb{R}^+$. Based on the formulae given in [2], we have the components of the Ricci tensor $\tilde{\omega}$ of the metric (2.7) on $E_6/A_1$:
\[
\begin{align*}
\tilde{\omega}_0 &= \frac{u_0}{8x_1^2} + x_1, \\
\tilde{\omega}_A &= \frac{5}{48u_1} + \frac{u_0}{8x_2^2} + \frac{u_1}{48x_2^2}, \\
\tilde{\omega}_m &= \frac{1}{2x_1} - \frac{x_2}{16x_1^2} - \frac{u_0}{160x_1^2} - \frac{3u_1}{20x_1^2}, \\
\tilde{\omega}_m &= \frac{1}{4x_2} - \frac{x_2}{8x_2^2} - \frac{u_0}{40x_2^2} - \frac{u_1}{10x_2^2}
\end{align*}
\]

Furthermore, the metric is Einstein if and only if there exists a positive solution $\{ u_0, u_1, x_1, x_2 \}$ of the system of equations
\[
\tilde{\omega}_0 = \tilde{\omega}_A = \tilde{\omega}_m = \tilde{\omega}_m.
\]

Similar to the discussion on $E_6/A_4$. Putting $u_0 = 1$ and by $\tilde{\omega}_0 = \tilde{\omega}_A$, $\tilde{\omega}_b = \tilde{\omega}_m$, $\tilde{\omega}_b = \tilde{\omega}_m$, we have
\[
\begin{align*}
f_1 &= -5x_1^2x_2^2 - 6x_2^2u_1 - x_1^2u_1^2 + 6x_2^2u_1 + 6x_2^2u_1 = 0, \\
f_2 &= -80x_1^2x_2 + 10x_3^2 + 20x_2^2 + 21x_3^2 + 24x_2^2u_1 = 0, \\
f_3 &= -10x_3^2x_2 - 5x_3^2 + 6x_2^2 + 5x_2^2 + 4x_3^2u_1 = 0.
\end{align*}
\]

Consider the polynomial ring $R = \mathbb{Q}[z, x_1, x_2, u_1]$ and an ideal $I$ generated by $\{ f_1, f_2, f_3, zx_1x_2u_1 - 1 \}$ to find non-zero solutions of (2.8). Take a lexicographic order $\gg$ with $z > u_1 > x_1 > x_2$ for a monomial ordering on $R$. By the help of computer, we have the polynomial of $x_2$ containing in the Gröbner basis of the ideal $I$:
\[
f(x_2) = 40733269776x_2^8 - 95717471616x_2^7 + 80248108328x_2^6 - 31589680504x_2^5 + 7669961625x_2^4 - 1207801950x_2^3 + 120201725x_2^2 - 5089500x_2 + 422500.
\]

In the Gröbner basis of the ideal $I$, $x_1$ and $u_1$ can be written into polynomials of $x_2$. The equation $f(x_2) = 0$ has two solutions:
\[
x_2 \approx 0.8651778712, \quad x_2 \approx 0.9203114422.
\]

In fact, we have the following solutions of (2.8):
\[
\begin{align*}
\{ u_1 \approx 0.1945580092, \quad x_1 \approx 0.5189654864, \quad x_2 \approx 0.8651778712 \}, \\
\{ u_1 \approx 0.7881276805, \quad x_1 \approx 2.582407960, \quad x_2 \approx 0.9203114422 \}.
\end{align*}
\]
In summary, we have:

**Theorem 2.2.** Let the notations be as above. Then every left invariant metric on $E_6/A_1$ which is $\text{Ad}(H)$-invariant is of the form (2.7). Up to scaling, there are two Einstein metrics on $E_6/A_1$ which are $\text{Ad}(H)$-invariant.

### 3. Einstein-Randers metrics

A Randers metric $F$ on $M$ is built from a Riemannian metric and a 1-form, i.e.,

$$F = \alpha + \beta,$$

where $\alpha$ is a Riemannian metric and $\beta$ is a 1-form whose length with respect to the Riemannian metric $\alpha$ is less than 1 everywhere. Obviously, a Randers metric is Riemannian if and only if it is reversible, i.e., $F(x, y) = F(x, -y)$ for any $x \in M$ and $y \in T_x(M)$. Sometimes it is convenient to use the following presentation of a Randers metric in [3], i.e.,

$$F(x, y) = \sqrt{h(W, y)^2 + h(y, y)\lambda} - \frac{h(W, y)}{\lambda}$$

(3.1)

here $\lambda = 1 - h(W, W) > 0$. The pair $(h, W)$ is called the navigation data of the corresponding Randers metric $F$.

The Ricci scalar $\mathfrak{Ric}(x, y)$ of a Finsler metric is defined to be the sum of those $n - 1$ flag curvatures $K(x, y, e_v)$, where \{e_v : v = 1, 2, \ldots, n - 1\} is any collection of $n - 1$ orthonormal transverse edges perpendicular to the flagpole, i.e.,

$$\mathfrak{Ric}(x, y) = \sum_{v=1}^{n-1} R_{vv}.$$  

(3.2)

The Ricci tensor is defined by

$$\text{Ric}_{ij} = (\frac{1}{2} F^2 \mathfrak{Ric})_{yy'}.$$  

(3.3)

Obviously, the Ricci scalar depends on the position $x$ and the flagpole $y$, but does not depend on the specific $n - 1$ flags with transverse edges orthogonal to $y$ (see [3, 4]). In the Riemannian case, it is a well known fact that the Ricci scalar depends only on $x$. Thus it is quite interesting to study a Finsler manifold whose Ricci scalar does not depend on the flagpole $y$. Generally, a Finsler metric with such a property is called an Einstein metric, i.e.,

$$\mathfrak{Ric}(x, y) = (n - 1)K(x)$$  

(3.4)

for some function $K(x)$ on $M$. In particular, for a Randers manifold $(M, F)$ with $\text{dim} M \geq 3$, $F$ is an Einstein metric if and only if there is a constant $K$ such that (3.4) holds (see [3]). The following lemma is an important result on Einstein-Randers metrics.

**Lemma 3.1 ([3]).** Suppose $(M, F)$ is a Randers space with the navigation data $(h, W)$. Then $(M, F)$ is Einstein with Ricci scalar $\mathfrak{Ric}(x) = (n - 1)K(x)$ if and only if there exists a constant $\sigma$ satisfying the following conditions:

1. $h$ is Einstein with Ricci scalar $(n - 1)K(x) + \frac{1}{16}\sigma^2$, and
2. $W$ is an infinitesimal homothety of $h$, i.e., $\mathcal{L}_W h = -\sigma h$.

Furthermore, $\sigma$ must be zero whenever $h$ is not Ricci-flat.

It is well known that $K(x)$ is a constant if $(M, F)$ is a homogeneous Einstein Finsler manifold. Here a Finsler manifold $(M, F)$ is called homogeneous if its full group of isometries acts transitively on $M$. Based on Lemma 3.1 Deng-Hou obtained a characterization of homogeneous Einstein-Randers metrics.
Lemma 3.2 ([9]). Let G be a connected Lie group and H a closed subgroup of G such that G/H is a reductive homogeneous space with a decomposition \( \mathfrak{g} = \mathfrak{h} + \mathfrak{m} \). Suppose \( h \) is a G-invariant Riemannian metric on G/H and W \( \in \mathfrak{m} \) is invariant under H with \( h(W, W) < 1 \). Let \( \tilde{W} \) be the corresponding G-invariant vector field on G/H with \( \tilde{W}|_o = W \). Then the Randers metric \( F \) with the navigation data \((h, \tilde{W})\) is Einstein with Ricci constant \( K \) if and only if \( h \) is Einstein with Ricci constant \( K \) and \( W \) satisfies
\[
\langle [W, X]_\mathfrak{m}, Y \rangle + \langle X, [W, Y]_\mathfrak{m} \rangle = 0, \quad \forall X, Y \in \mathfrak{m},
\]
where \( \langle , \rangle \) is the restriction of \( h \) on \( T_o(G/H) \simeq \mathfrak{m} \). In this case, \( \tilde{W} \) is a Killing vector field with respect to the Riemannian metric \( h \).

For simple, denote \( H_i = A_4, \mathfrak{h}_j = A_1, \) or \( H_i = A_1, \mathfrak{h}_j = A_4 \). By the equivalence of the adjoint representation and the isotropy representation of \( H_i \) on \( \mathfrak{h}_0 \oplus \mathfrak{h}_j \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \), the vector field
\[
\tilde{W}|_{\mathfrak{h}H} = d(\tau(g))|_o(W), \forall g \in G, W \in \mathfrak{h}_0
\]
is well-defined, and it is G-invariant (see [8]). For every metric given in Theorem 2.1 and Theorem 2.2, one can easily verify the equation
\[
\langle [W, X]_{\mathfrak{h}_0 \oplus \mathfrak{h}_j \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2}, Y \rangle_{G/H_i} + \langle X, [W, Y]_{\mathfrak{h}_0 \oplus \mathfrak{h}_j \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2} \rangle_{G/H_i} = 0
\]
holds for any \( W \in \mathfrak{h}_0 \) and \( X, Y \in \mathfrak{h}_0 \oplus \mathfrak{h}_j \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \), using the facts that \( \mathfrak{h}_0 \subset \mathfrak{h} \) and that the metric is \( Ad(H) \)-invariant. Then by Lemma 3.2 the homogeneous metric
\[
F(x, y) = \frac{\sqrt{\langle [W, y]_{G/H_1}, x \rangle^2 + \langle y, y \rangle_{G/H_1}} - \langle W, y \rangle_{G/H_1}}{\lambda}
\]
is a G-invariant Einstein-Randers metric on \( G/H_1 \) when \( W \) satisfies \( \langle W, W \rangle_{G/H_1} < 1 \), and \( F \) is Riemannian if and only if \( W = 0 \). That is, we have the following theorem.

Theorem 3.3. There are at least four families of \( E_6 \)-invariant non-Riemannian Einstein-Randers metrics on \( E_6/A_4 \), and two families of \( E_6 \)-invariant non-Riemannian Einstein-Randers metrics on \( E_6/A_1 \).

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