DISCRETIZED SUM-PRODUCT FOR LARGE SETS

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Abstract. Let $A \subset [1, 2]$ be a $(\delta, \sigma)$-set with measure $|A| = \delta^{1-\sigma}$ in the sense of Katz and Tao. For $\sigma \in (1/2, 1)$ we show that

$$|A + A| + |AA| \geq \delta^{-c}|A|,$$

for $c = \frac{(1-\sigma)(2\sigma-1)}{6\sigma+1}$. This improves the bound of Guth, Katz, and Zahl for large $\sigma$.

1. Introductions

Erdős-Volkmann [4] showed that for any $\sigma \in [0, 1]$ there exists a subgroup of reals with Hausdorff dimension $\sigma$, and they conjectured that this property does not hold for subring of reals. Precisely the Erdős-Volkmann ring conjecture claims that there does not exist a subring of reals with Hausdorff dimension strictly between zero and one. Edgar and Miller [3] first proved this ring conjecture via the orthogonal projections of fractal sets. A slightly later Bourgain [1] independently proved the Erdős-Volkmann ring conjecture via the discretized ring conjecture (discretized sum-product) of Katz and Tao [10].

Discretized sum-product also has many other applications. For instance it is closely related to Falconer distance sets problem and the dimension of Furstenburg sets, see Katz and Tao [10] for more details. Bourgain [2] showed a different approach for the discretized sum-product, and given applications in the projections of fractal sets and Fourier analysis. For the applications of the discretized sets to projections of fractal sets see He [9], Orponen [12] and reference therein. For the applications of discretized sum-product to the Fourier decay of measures see Li [11].

We note that Bourgain [1, 2] does not produce an explicit bound. Recently Guth, Katz, and Zahl [7] given a short proof of the discretized sum-product theorem, and they showed an explicit bound for the discretized sum-product theorem. They [7] used some ideas from Garaev [6] and applied some discretized version of arguments from additive combinatorics.

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We show some notation of Katz and Tao [10] first. Let \( \varepsilon, \delta \) be small and positive parameters. We use \( f \preceq g \) to denote \( |f| \leq C\varepsilon^{-C\varepsilon}|g| \), \( f \asymp g \) if \( g \preceq f \) and \( f \approx g \) if \( f \preceq g \) and \( g \preceq f \). We say that a subset \( A \subset \mathbb{R} \) is called \( \delta \)-discretized if \( A \) is the union of intervals of lengths \( \approx \delta \). For a positive constant \( M \) we say that a subset \( A \subset (-M, M) \) is a \((\delta, \sigma)\)-set if it is \( \delta \)-discretized, and one has

\[
|A \cap B(x, r)| \preceq \delta(r/\delta)^{\sigma}
\]

for all \( \delta \leq r < 1 \) and \( x \in \mathbb{R} \).

We note that a \((\delta, \sigma)\)-set here may have much smaller measure than \( \delta^{1-\delta} \). However we ask that any \((\delta, \sigma)\)-set has positive measure to avoid to be an empty set.

We remark that the \((\delta, \sigma)\)-set can be considered as the discrete approximation of set in \( \mathbb{R} \) at scale \( \delta \). Suppose that \( F \) is a compact subset of \( \mathbb{R} \). For each \( k \in \mathbb{Z} \) let

\[
I_{k,\delta} = [k\delta, (k+1)\delta],
\]

and \( F_\delta \) be the union of the interval \( I_{k,\delta} \) which intersects \( F \), that is

\[
F_\delta = \bigcup_{k \in \mathbb{Z}, F \cap I_{k,\delta} \neq \emptyset} I_{k,\delta}.
\]

In many cases, the set \( F_\delta \) is a \((\delta, \sigma)\)-set for some \( 0 \leq \sigma \leq 1 \). Here the parameter \( \sigma \) is often related to the “fractal dimension” of \( F \). For instance if \( F \) is the classical Cantor ternary set, then for any \( 0 < \delta < 1 \) the set \( F_\delta \) is a \((\delta, \log 2/\log 3)\)-set. Indeed this follows from Falconer [5, Chapter 3], which shows that the box dimension of the Cantor set is \( \log 2/\log 3 \). We remark that there are various dimensions in fractal geometry, we refer to [5] for more details.

The above argument shows that the \((\delta, \sigma)\)-set in \( \mathbb{R}^d \) is also important for understanding the structure of set in \( \mathbb{R}^d \). However in this project we consider \((\delta, \sigma)\)-set on the line \( \mathbb{R} \) only.

Let \( A, B \subset \mathbb{R} \). The sum sets of \( A \) and \( B \) is defined as

\[
A + B = \{ a + b : a \in A, b \in B \},
\]

and similarly the product sets of \( A \) and \( B \) is defined as

\[
AB = \{ ab : a \in A, b \in B \}.
\]

Using the above notation, Bourgain’s discretized sum product theorem claims that for a \((\delta, \sigma)\)-set \( A \subset [1, 2] \) with \( 0 < \sigma < 1 \) and the measure \( |A| = \delta^{1-\sigma} \), there exists a constant \( c = c(\sigma) > 0 \) such that

\[
|A + A| + |AA| \geq \delta^{-c}|A|.
\]
Under the same condition, Guth, Katz, and Zahl [7] proved that for any $c < \frac{\sigma(1-\sigma)}{4(7+3\sigma)}$ one has

$$|A + A| + |AA| \gtrsim \delta^{-c}|A|.$$  

By adapting the arguments of Garaev [6], Guth, Katz, and Zahl [7], and the bilinear bound of Bourgain [2, Theorem 7], we obtain the following.

**Theorem 1.1.** Let $\sigma \in (1/2, 1)$ and $A \subset [1, 2]$ be a $(\delta, \sigma)$-set with measure $|A| = \delta^{1-\sigma}$. Then

$$|A + A| + |AA| \gtrsim \delta^{-\frac{1-\sigma}{6\sigma+4}}|A|.$$  

Note that for any $1/2 < \sigma < 1$, Theorem 1.1 gives a non-trivial lower bound. Furthermore for $\sigma > \frac{\sqrt{225-10}}{9} = 0.5703\ldots$, Theorem 1.1 improves the bound of Guth, Katz, and Zahl [7].

2. Preliminaries

We use $\#S$ to denote the cardinality of set $S$. Let $X, Y, Z \subset \mathbb{R}$ be finite sets with $Z \neq \emptyset$, then the Ruzsa triangle inequality claims that

$$\#(X + Y) \leq \frac{\#(X + Z)\#(Z + Y)}{\#Z}.$$  

See [13, Chapter 2] for a proof and many other useful sum sets estimates.

We need the following well known discretized version of Ruzsa triangle inequality. Our proof is based on Guth, Katz, and Zahl [7, Proof of Corollary 2.3], Orponen [12, Remark 4.40]. For many other discretized version of sum sets estimates see He [8], Tao [14].

We show a geometric observation first. Let $S \subset \mathbb{R}$ be the union of disjoint intervals with length $\gtrsim \delta$. Then for all $0 < c < 10$ we have

$$|S + B(0, c\delta)| \lesssim |S|.$$  

Here we can change the parameter 10 to any other fixed positive constant.

**Lemma 2.1.** Let $A, B, C \subset \mathbb{R}$ be $\delta$-discretized sets. Then

$$|A + B| \lesssim \frac{|A + C||C + B|}{|C|}.$$  

**Proof.** Without losing general we may assume that each interval of $A, B$ and $C$ has length at least $\delta$. In the end we change the estimate $\ll$ to $\lesssim$, and this does not change our result.
For any set $S \subset \mathbb{R}$ let $S_\delta = (\delta/3)\mathbb{Z} \cap S$. For any $a \in A, b \in B$ there exists $a' \in A_\delta, b' \in B_\delta$ such that

$$|a + b - a' - b'| \leq \delta.$$ 

It follows that

$$A + B \subset A_\delta + B_\delta + B(0, \delta) \subset A + B(0, 2\delta).$$

Combining with (1) we obtain

$$|A + B| \ll \#(A_\delta + B_\delta)\delta \ll |A + B|.$$ 

Applying Ruzsa triangle inequality to sets $A_\delta, B_\delta, C_\delta$ and applying (2) to $A + C$ and $C + B$ we obtain the result. 

□

**Lemma 2.2.** Let $A \subset [0, 2]$ be a $\delta$-discretized set, and the union of the intervals of $A$ are pairwise disjoint. Then for any $t \in [0.5, 2.1]$ and any $x \in [-\delta, \delta]$ we have

$$|A + tA| \lesssim |A + (t + x)A| \lesssim |A + tA|.$$ 

**Proof.** For any $t \in [0.5, 2], x \in [-\delta, \delta]$ and $a, b \in A$ we have

$$|(a + tb) - (a + (t + x)b)| \leq 2\delta,$$

and hence

$$A + tA \subset A + (t + x)A + B(0, 2\delta) \subset A + tA + B(0, 4\delta).$$

Combining with (1) we finish the proof. 

□

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The Fourier transform of the function $f$ at $\xi \in \mathbb{R}$ is defined as

$$\hat{f}(\xi) = \int_\mathbb{R} e^{-x\xi}f(x)dx,$$

where throughout the paper we denote $e(x) = e^{2\pi ix}$. Let $\mu$ be a measure on $\mathbb{R}$. The Fourier transform of the measure $\mu$ at $\xi \in \mathbb{R}$ is defined as

$$\hat{\mu}(\xi) = \int e(-x\xi)d\mu(x).$$

For a subset $S \subset \mathbb{R}$ we will also use $S$ to denote the characteristic function of $S$. Let $A, B \subset \mathbb{R}$ be two bounded sets. The convolution of $A$ and $B$ is defined as

$$(A * B)(x) = \int_\mathbb{R} A(x - y)B(y)dy.$$ 

The (additive) energy of $A, B$ is defined as

$$(3) \quad E(A, B) = \int_\mathbb{R} (A * B)(x)^2dx = \int_\mathbb{R} |\hat{A}(\xi)|^2|\hat{B}(\xi)|^2d\xi.$$
The second equality holds by applying the Plancherel identity and convolution theorem. Clearly we have
\[ \int_{\mathbb{R}} (A \ast B)(x) \, dx = |A||B|. \]
By Cauchy-Schwarz inequality we obtain
\[ (|A||B|)^2 \leq E(A, B)|A + B|. \]
We will frequently use the Plancherel identity for a set. Precisely for a bounded subset \( S \subset \mathbb{R} \) we have
\[ \int_{\mathbb{R}} |\mathcal{S}(\xi)|^2 \, d\xi = |S|. \]
We formulate the following version of Bourgain [2, Theorem 7]. Note that the interval \([0, 4]\) is not essential, in fact Lemma 2.3 holds for any bounded interval.

**Lemma 2.3.** Let \( \mu, \nu \) be probability measures on \([0, 4]\) such that for all \( \delta < r \leq 1 \) and all \( x \in \mathbb{R} \),
\[ \mu(B(x, r)) \leq K_1 r^\alpha \text{ and } \nu(B(x, r)) \leq K_2 r^\beta. \]
Then for \( 1 \leq |\xi| \leq \delta^{-1} \) we have
\[ \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e(-xy\xi) \, d\mu(x) \right| \, d\nu(y) \lesssim \sqrt{K_1 K_2} \, |\xi|^{-\frac{\alpha+\beta-1}{2}}. \]
We remark that the statement of [2, Theorem 7] gives a bound for
\[ \left| \int_{\mathbb{R}} \int_{\mathbb{R}} e(-xy\xi) \, d\mu(x) \, d\nu(y) \right|. \]
However the proof of [2, Theorem 7] (see [2, (8.3)]) indeed works for the bound
\[ \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e(-xy\xi) \, d\mu(x) \right| \, d\nu(y), \]
which is used for bounding (22) below. Moreover the term \( \sqrt{K_1 K_2} \) appears, since the using of Cauchy-Schwarz inequality in the proof of [2, Theorem 7].
In particularly we have the following version for \((\delta, \sigma)\)-sets which is easier for our using.

**Lemma 2.4.** Let \( 0 < \alpha, \beta < 1 \). Let \( A \subset [0, 4] \) and \( B \subset [0, 4] \) be \((\delta, \alpha)\)-set and \((\delta, \beta)\)-set respectively. Then for any \( 1 \leq |\xi| \leq \delta^{-1} \) we have
\[ \int_{A} \left| \int_{B} e(-xy\xi) \, dy \right| \, dx \lesssim |\xi|^{\frac{\alpha+\beta-1}{2}} \delta^{\frac{\sigma-\beta}{2}} \sqrt{|A||B|}. \]
Proof. For $A$ we define a measure $\mu$ by letting
$$\mu(X) = \frac{|X \cap A|}{|A|} \text{ for } X \subset \mathbb{R}.$$  

By the condition that $A$ is a $(\delta, \alpha)$-set we obtain
$$\mu(B(x, r)) \lesssim r^{\alpha \delta^{-1} - \alpha} |A|^{-1} \text{ for } \delta \leq r \leq 1. \tag{6}$$

Similarly for $B$ we define a measure $\nu$ by letting
$$\nu(X) = \frac{|X \cap A|}{|A|} \text{ for } X \subset \mathbb{R},$$

and we have
$$\nu(B(x, r)) \lesssim r^{\beta \delta^{-1} - \beta} |B|^{-1} \text{ for } \delta \leq r \leq 1. \tag{7}$$

Clearly we have
$$\int_A \int_B e(-xy\xi) dx \, dy = |A||B| \int_A \int_B e(-xy\xi) d\mu(x) \, d\nu(y).$$

Then Lemma 2.4 and estimates (6), (7) give the result. \qed

3. Proof of Theorem 1.1

By adapting the arguments in Guth, Katz, and Zahl [7] and especially in Garaev [6] we obtain the following.

Lemma 3.1. Let $0 < \sigma < 1$. Let $A \subset [1, 2]$ be a $(\delta, \sigma)$-set with measure $|A| = \delta^{1-\sigma}$. Then there exists a $(\delta, \sigma)$-set $T \subset [0.4, 2.1]$ with measure
$$|T| \gtrsim \frac{|A|^2}{|AA|}$$

such that for any $t \in T$ we have
$$|A + tA| \lesssim \frac{|A + A|^2 |AA|}{|A|^2}.$$  

Proof. Let $D = \{a_1, \ldots, a_N\}$ be a maximal $\delta$-separated subset of $A$, i.e., any two distinct elements of $D$ has distance at least $\delta$, furthermore for any $a \in A$ there exists $a_i \in D$ such that $|a - a_i| \leq \delta$. Note that
$$N \approx \delta^{-\sigma}, \tag{8}$$

and for all $\delta \leq r \leq 1$ and $x \in \mathbb{R},$

$$\# (D \cap B(x, r)) \lesssim \left( \frac{r}{\delta} \right)^{\sigma}. \tag{9}$$

Let
$$f(x) = \sum_{i=1}^{N} 1_{a_i A}(x).$$
Then we have
\[(10) \quad \int_{\mathbb{R}} f(x)dx = \sum_{i=1}^{N} |a_iA| \geq N|A|, \]
and
\[(11) \quad \int_{\mathbb{R}} f(x)^2dx = \sum_{1 \leq i, j \leq N} |a_iA \cap a_jA|. \]

By Cauchy-Schwarz inequality we arrive
\[(12) \quad \left(\int_{\mathbb{R}} f(x)dx \right)^2 \leq \int_{\mathbb{R}} f(x)^2dx \|\{x \in \mathbb{R} : f(x) > 0\}\|. \]

Observe that
\[\{x \in \mathbb{R} : f(x) > 0\} \subset AA.\]

Thus together with (10), (11), and (12), we obtain
\[\sum_{1 \leq i, j \leq N} |a_iA \cap a_jA| \geq \frac{N^2|A|^2}{|AA|}. \]

Thus there exists \(1 \leq j_0 \leq N\) such that
\[\sum_{1 \leq i \leq N} |a_iA \cap a_{j_0}A| \geq \frac{N|A|^2}{|AA|}. \]

Let
\[P = \{1 \leq i \leq N : |a_iA \cap a_{j_0}A| \geq \frac{|A|^2}{2|AA|}\}. \]

Then
\[\sum_{i \in P} |a_iA \cap a_{j_0}A| \geq \frac{N|A|^2}{2|AA|}. \]

Taking dyadic decomposition for \(|a_iA \cap a_{j_0}A|\) with \(i \in P\), we obtain
\[\sum_{k=1}^{K} 2^{-k} \#P_k \geq \sum_{i \in P} |a_iA \cap a_{j_0}A| \geq \frac{N|A|^2}{2|AA|}, \]

where
\[P_k = \{i \in P : 2^{-k-1} < |a_iA \cap a_{j_0}A| \leq 2^{-k}\}, \]

and \(K\) is an integer parameter such that
\[2^{-K-1} < \frac{|A|^2}{2|AA|} \leq 2^{-K}. \]

Since the product set \(AA\) is a subset of \([1, 4]\), we have \(|AA| \ll 1\). It follows that
\[K \ll \log \left(\frac{1}{\delta}\right). \]
Thus there exist $\tau$ and $D_\tau \subset P$ such that

$$\tau \# D_\tau \gtrapprox \sum_{k=1}^{K} 2^{-k} \# P_k \gtrapprox \frac{N|A|^2}{|AA|},$$

and for any $i \in D_\tau$ we have

$$\tau \leq |a_i A \cap a_{j_0} A| \leq 2\tau. \quad (13)$$

Since $\tau \leq |A|$ and $\# D_\tau \leq N$, we obtain

$$\tau \gtrapprox \frac{|A|^2}{|AA|}, \quad (14)$$

and

$$\# D_\tau \gtrapprox \frac{N|A|}{|AA|}. \quad (15)$$

Now we intend to bound the measure of the set $a_i A + a_{j_0} A$ for each $i \in D_\tau$. For this purpose we introduce some notation first.

For each $k \in \mathbb{Z}$ let $J_{k,\delta} = [k\delta, (k+1)\delta)$. For each $1 \leq i \leq N$ let

$$U_i = \bigcup_{k \in \mathbb{Z}, a_i A \cap J_{k,\delta} \neq \emptyset} J_{k,\delta}.$$

Note that

$$a_i A \subseteq U_i \subseteq a_i A + B(0,\delta), \quad (16)$$

and the intersection $U_i \cap U_j$ is a $\delta$-discretized set for $1 \leq i, j \leq N$. Moreover for each $i \in D_\tau$ by $(13)$, $(14)$ we have

$$|U_i \cap U_{j_0}| \gtrapprox |a_i A \cap a_{j_0} A| \gtrapprox \frac{|A|^2}{|AA|}. \quad (17)$$

For any $i \in D_\tau$ applying Ruzsa triangle inequality Lemma 2.1 for the sets $a_i A, a_{j_0} A$ and $U_i \cap U_{j_0}$, we derive

$$|a_i A + a_{j_0} A| \lesssim \frac{|a_i A + U_i \cap U_{j_0}| |U_i \cap U_{j_0} + a_{j_0} A|}{|U_i \cap U_{j_0}|}. \quad (18)$$

By $(16)$ we have

$$a_i A + U_i \cap U_{j_0} \subseteq a_i A + a_i A + B(0,\delta).$$

Since $1 \leq a_i \leq 2$ and the simper fact (1) we obtain

$$|a_i A + U_i \cap U_{j_0}| \ll |A + A|.$$

Similarly, we have

$$|a_{j_0} A + U_i \cap U_{j_0}| \ll |A + A|.$$
Combining with (17) and (18) we arrive
\[ |a_i A + a_j A| \leq \frac{|A + A|^2|AA|}{|A|^2}. \]

Applying Lemma 2.2 we obtain that for any \( i \in D_r \) and \( x \in (-\delta, \delta) \) we have
\[ |A + (a_i/a_j + x)A| \leq \frac{|A + A|^2|AA|}{|A|^2}. \]

Let
\[ T = \bigcup_{i \in D_r} B(a_i/a_j, \delta/2). \]

We ask that \( \delta \) is a small positive parameter, and hence \( T \subset [0.4, 2.1] \). Furthermore, the estimates (8), (15) imply
\[ |T| \gg \#D_r \delta \gg \frac{|A|^2}{|AA|}. \]

By (9) we obtain that \( T \) is a \((\delta, \sigma)\)-set which finishes the proof. \( \square \)

Applying Lemma 2.4 we obtain the following upper bound of the mean value of energies \( E(A, tA) \).

**Lemma 3.2.** Fix \( 1/2 < \sigma < 1 \). Let \( T \subset [0.4, 2.1] \) be a \((\delta, \sigma)\)-set with the measure \( |T| \geq \delta \). Let \( A \subset [1, 2] \) be a \((\delta, \sigma)\)-set with measure \( |A| = \delta^{1-\sigma} \). Then
\[ \int_T E(A, tA)dt \leq |A|^3|T| \left( |A|^\frac{2\sigma}{1+\sigma} |T|^{-\frac{1}{1+\sigma}} + \delta(|A||T|)^{-1} \right). \]

**Proof.** For each \( t \in \mathbb{R} \) and \( \xi \in \mathbb{R} \) we have
\[ \hat{tA} (\xi) = t \hat{A} (t\xi). \]

Thus by (3) and the condition \( T \subset [0.4, 2.1] \) we conclude that
\[ \int_T E(A, tA)dt \ll \int_T \int_{\mathbb{R}} |\hat{A}(\xi)|^2|\hat{A}(t\xi)|^2 dt d\xi. \]

Let \( 0 < L < 1/\delta \) be a parameter which will be determined later. We decompose \( \mathbb{R} \) into three parts, and then bound (19) by three corresponding parts. Precisely,
\[ (19) \ll I_0 + I_1 + I_2 \]

where
\[ I_0 = \int_T \int_{|\xi| \leq L} |\hat{A}(\xi)|^2|\hat{A}(t\xi)|^2 dt d\xi, \]
\[ I_1 = \int_T \int_{L \leq |\xi| \leq 1/\delta} |\hat{A}(\xi)|^2|\hat{A}(t\xi)|^2 dt d\xi, \]
and
\[ I_2 = \int_T \int_{|\xi| \geq \delta^{-1}} |\hat{A}(\xi)|^2 |\hat{A}(t\xi)|^2 dt d\xi. \]

For \( I_0 \), we use the trivial bound \( |\hat{A}(\xi)| \leq |A| \), and we obtain
\[ (21) \quad I_0 \leq |A|^4 |T| L. \]

For \( I_1 \), clearly the trivial bound \( |\hat{A}(\xi)| \leq |A| \) gives
\[ (22) \quad \int_T |\hat{A}(\xi t)|^2 dt \leq |A| \int_T |\hat{A}(\xi t)| dt. \]

Applying Lemma 2.4, and the condition \( |A| = \delta^{1-\sigma} \), we obtain
\[ \int_T |\hat{A}(\xi t)| dt \leq |A|^{3/2} |T|^{1/2} |\xi|^{(2\sigma-1)/2}. \]

Thus we arrive
\[ (22) \leq |A|^{5/2} |T|^{1/2} |\xi|^{-(2\sigma-1)/2}. \]

Combining with Fubini’s theorem and the condition \( \sigma > 1/2 \), we have
\[ I_1 \leq |A|^{5/2} |T|^{1/2} \int_{L \leq |\xi| \leq \delta^{-1}} |\hat{A}(\xi)|^2 |\xi|^{-(2\sigma-1)/2} d\xi \]
\[ \leq |A|^{5/2} |T|^{1/2} L^{-(2\sigma-1)/2} \int_{L \leq |\xi| \leq \delta^{-1}} |\hat{A}(\xi)|^2 d\xi. \]

Plancherel identity (5) implies
\[ \int_{L \leq |\xi| \leq \delta^{-1}} |\hat{A}(\xi)|^2 d\xi \leq |A|. \]

Thus we arrive
\[ (23) \quad I_1 \leq |A|^{7/2} |T|^{1/2} L^{-(2\sigma-1)/2}. \]

Now we optimize the choice of the parameter \( L \) to find the smallest upper bound for the parts \( I_0, I_1 \). Recalling that we ask \( 0 < L < 1/\delta \). In the end, the parameter \( L_0 \), which makes the right hand sides of (21), (23) “comparable”, satisfies our need. Thus we derive
\[ (24) \quad L_0 = (|A||T|)^{-1/(1+2\sigma)}. \]

Indeed the conditions \( |A| = \delta^{1-\sigma}, |T| \geq \delta \), and \( 1/2 < \sigma < 1 \), imply that \( L_0 \leq \delta^{-1} \). It follows that
\[ (25) \quad I_0, I_1 \leq |A|^4 |T| L_0 \leq |A|^3 |T| (|A|^{2\sigma/|T|} |T|^{-(1+2\sigma)}). \]

Now we turn to the estimate for \( I_2 \). By changing variables and applying the Plancherel identity we obtain
\[ \int_{\mathbb{R}} |\hat{A} (t\xi)|^2 dt \leq |A||\xi|^{-1}. \]
Again by applying the Plancherel identity we have
\[ I_2 \leq \int_{|\xi| \geq \delta^{-1}} |\hat{A}(\xi)|^2 |A| |\xi|^{-1} d\xi \ll |A|^2 \delta. \]

Combining with (20), (21), (24), (25), we obtain the desired bound. □

Now we turn to the proof of Theorem 1.1. Suppose that
\[ \max\{|A + A|, |AA|\} = K|A|. \]

By Lemma 3.1 there exists a \((\delta, \sigma)\)-set \(T \subset [0, 2]\) such that
\[ |T| \geq |A|/K, \]

and for each \(t \in T\) we have
\[ |A + tA| \leq K^3 |A|. \]

Applying Lemma 3.2 to \(A\) and \(T\), we conclude that there exists a \(t_0 \in T\) such that
\[ E(A, t_0 A) \leq |A|^3 \left(|A|^{2\sigma/1 + 2\sigma} |T|^{-\frac{1}{1 + 2\sigma}} + \delta(|A||T|)^{-1}\right). \]

By (4) and estimates (26), (27), we obtain that
\[
|A| \leq \left(|A|^{2\sigma/1 + 2\sigma} |T|^{-\frac{1}{1 + 2\sigma}} + \delta(|A||T|)^{-1}\right) |A + t_0 A|
\]
\[ \leq |A|^{2\sigma/2\sigma + 1} |A| K^{\frac{6\sigma + 4}{2\sigma + 1}} + |A|^{-1} \delta K^4. \]

It follows that
\[ K \geq \min\{\delta^{-(1 - \sigma)(2\sigma - 1)/6\sigma + 4}, \delta^{-2\sigma - 1/4}\}. \]

Note that for \(1/2 < \sigma < 1\) we have
\[ \frac{(1 - \sigma)(2\sigma - 1)}{6\sigma + 4} \leq \frac{2\sigma - 1}{4}, \]

and hence
\[ K \geq \delta^{-(1 - \sigma)(2\sigma - 1)/6\sigma + 4}, \]

which gives the result.

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