ON EQUIVARIANTLY FORMAL 2-TORUS MANIFOLDS

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Abstract. A 2-torus manifold is a closed connected smooth \(n\)-manifold with a non-free effective smooth \(\mathbb{Z}_2^n\)-action. In this paper, we prove that a 2-torus manifold is equivariantly formal if and only if the \(\mathbb{Z}_2^n\)-action is locally standard and every face of its orbit space (including the whole orbit space) is mod 2 acyclic. Our study is parallel to the study of torus manifolds with vanishing odd-degree cohomology by M. Masuda and T. Panov in [27]. As an application, we determine when such kind of 2-torus manifolds can have regular \(m\)-involutions (i.e. involutions with only isolated fixed points of the maximum possible number).

1. Introduction

Let \(G\) be a compact Lie group and \(BG\) be the classifying space of \(G\). For a \(G\)-space \(X\), the \(G\)-equivariant cohomology of \(X\) with coefficients in a field \(k\) is the singular cohomology of the Borel construction \(X_G\) (see [6])

\[ H^*_G(X; k) := H^*(X_G; k). \]

There is a natural fibration \(X \to X_G \to BG\) associated to \(X_G\) called the Borel fibration. If the inclusion of the fiber \(i_X : X \to X_G\) induces a surjection on cohomology \(i_X : H^*_G(X; k) \to H^*(X; k)\), \(X\) is called called (cohomologically) equivariantly formal over \(k\). This term was coined in 1998 by Goresky, Kottwitz, and MacPherson in [18]. But this condition had already been studied by A. Borel in [5, §4] and [6, Ch. XII] where \(X\) is called totally non-homologous to zero in \(X_G\) (also see [7, Ch. VII]).

For some special groups \(G\) shown below, the equivariant formality of a \(G\)-action can be interpreted in some other ways (see [5, §4], [11 Ch. 3] and [2, Sec. 4]).

- When \(BG\) is simply connected (e.g. \(G\) is a torus \(T^r = (S^1)^r\)), \(X\) is equivariantly formal if and only if the Serre spectral sequence of the Borel fibration of \(X\) degenerates at the \(E_2\) stage.
• When $G$ is the $p$-torus $\mathbb{Z}_p^r$ ($p$ is prime), $X$ being equivariantly formal is equivalent to either one of the following conditions.

  (i) The Serre spectral sequence with $\mathbb{Z}_p$-coefficients of the Borel fibration of $X$ degenerates at the $E_2$ stage and the induced action of $\mathbb{Z}_p^r$ on $H^*(X; \mathbb{Z}_p)$ is trivial.

  (ii) $H^*_\mathbb{Z}_p(X; \mathbb{Z}_p) \cong H^*(X; \mathbb{Z}_p) \otimes H^*(B\mathbb{Z}_p^r; \mathbb{Z}_p)$ is a free $H^*(B\mathbb{Z}_p^r; \mathbb{Z}_p)$-module.

Due to the above fact, we call a $\mathbb{Z}_p^r$-action on $X$ weakly equivariantly formal if we only assume that the Serre spectral sequence (with $\mathbb{Z}_p$-coefficients) of the Borel fibration of $X$ degenerates at the $E_2$ stage. So an equivariantly formal $\mathbb{Z}_p^r$-action is always weakly equivariantly formal.

When $G = T^r$ or $\mathbb{Z}_2^r$ and $k = \mathbb{Q}$ or $\mathbb{Z}_2$ respectively, there is another equivalent description of equivariantly formal $G$-actions given by the so called “Atiyah-Bredon sequence” (see Bredon \cite{8} and Franz-Puppe \cite{16} for the $T^r$ case, and Allday-Franz-Puppe \cite{2} for the $\mathbb{Z}_2^r$ case). In addition, there are many sufficient conditions for a $T^r$-action to be equivariantly formal (for example: vanishing of odd-degree cohomology, all homology classes being representable by $T^r$-invariant cycles, etc.).

Equivariantly formal $G$-spaces provide many nice examples in geometry and topology. Some of them are:

• Smooth compact toric varieties.

• Hamiltonian $G$-actions on symplectic manifolds which have moment maps (see Atiyah-Bott \cite{3} and Jeffrey-Kirwan \cite{22}).

• Quasitoric manifolds and small covers defined in Davis-Januszkiewicz \cite{14}.

• Torus manifolds with vanishing odd degree cohomology (see Masuda-Panov \cite{27}).

In addition, when $G = T^r$ or $(\mathbb{Z}_p)^r$, the following theorem gives us an easy way to recognize equivariantly formal $G$-actions.

**Theorem 1.1** (see Theorem (3.10.4) in Allday-Puppe \cite{1}). Let $G = T^r$ or $(\mathbb{Z}_p)^r$ where $p$ is a prime and $k = \mathbb{Q}$ or $\mathbb{Z}_p$ respectively. Let $X$ be a paracompact $G$-space with only finitely many orbit types and $\dim_k H^*(X; k) < \infty$. Then the fixed point set $X^G$ always satisfies

$$\dim_k H^*(X^G; k) \leq \dim_k H^*(X; k)$$

where the equality holds if and only if $X$ is equivariantly formal over $k$. Here $\dim_k H^*(X; k)$ denotes the sum of the rank of the cohomology groups of $X$ in all dimensions over $k$. 
A very special case is when $G = \mathbb{Z}_2$ and $X^{\mathbb{Z}_2}$ consists only of isolated points. By Theorem 1.1, we have

$$|X^{\mathbb{Z}_2}| = \dim_{\mathbb{Z}_2} H^*(X^{\mathbb{Z}_2}; \mathbb{Z}_2) \leq \dim_{\mathbb{Z}_2} H^*(X; \mathbb{Z}_2)$$

Such a $\mathbb{Z}_2$-action on $X$ is equivariantly formal if and only if the number of the fixed points reaches the maximum, i.e. $|X^{\mathbb{Z}_2}| = \dim_{\mathbb{Z}_2} H^*(X; \mathbb{Z}_2)$. In this case, the involution determined by the $\mathbb{Z}_2$-action is called an $m$-involution on $X$ (this term was named by Puppe [28]).

There is an interesting relation between $m$-involutions on closed manifolds and binary codes. It was shown in [28] that one can obtain a self-dual binary code from any $m$-involution on an odd-dimensional closed manifold. This motivates the study in Chen-Lü-Yu [12] on the $m$-involutions on a special kind of closed manifolds called small covers (see [14]). In this paper, we want to study a more general type of closed manifolds with $2$-torus actions defined below.

**Definition 1.2** (see Lü-Masuda [25]). A $2$-torus manifold is a closed connected smooth $n$-manifold $W$ with a non-free effective smooth action of $\mathbb{Z}_2^n$. For such a manifold $W$, since $\dim(W) = n = \text{rank}(\mathbb{Z}_2^n)$ and the $\mathbb{Z}_2^n$-action is effective, the fixed point set $W^{\mathbb{Z}_2^n}$ must be discrete. Then since $W$ is compact, $W^{\mathbb{Z}_2^n}$ is a finite set of isolated points (if not empty). Note that we require all $2$-torus manifolds to be connected in this paper.

- For brevity, we call a $2$-torus manifold $W$ equivariantly formal or weakly equivariantly formal if the $\mathbb{Z}_2^n$-action on $W$ is so, respectively.
- We call $W$ locally standard if for every point $x \in W$, there is a $\mathbb{Z}_2^n$-invariant neighborhood $V_x$ of $x$ such that $V_x$ is equivariantly homeomorphic to an invariant open subset of a real $n$-dimensional faithful linear representation space of $\mathbb{Z}_2^n$. An equivalently way to describe such a neighborhood $V_x$ is: $V_x$ is weakly equivariantly homeomorphic to an invariant open subset of $\mathbb{R}^n$ under the standard $\mathbb{Z}_2^n$-action defined by: for any $(x_1, \cdots, x_n) \in \mathbb{R}^n$ and $(g_1, \cdots, g_n) \in \mathbb{Z}_2^n$,

$$(g_1, \cdots, g_n) \cdot (x_1, \cdots, x_n) \mapsto ((-1)^{g_1} x_1, \cdots, (-1)^{g_n} x_n).$$

- Every non-zero element $g \in \mathbb{Z}_2^n$ determines a nontrivial involution $\tau_g$ on $W$, called a regular involution on $W$.

We will prove in Theorem 3.3 that if a 2-torus manifold is equivariantly formal, then it must be locally standard.

For an $n$-dimensional locally standard 2-torus manifold $W$, the orbit space $Q = W/\mathbb{Z}_2^n$ naturally becomes a connected smooth nice $n$-manifold with corners and with non-empty boundary (since the $\mathbb{Z}_2^n$-action is non-free). Moreover,
The $\mathbb{Z}_2$-action on $W$ determines a characteristic function

$$\lambda_W : \{F_1, \cdots, F_m\} \to \mathbb{Z}_2^n$$

where $F_1, \cdots, F_m$ are all the facets (codimension-one faces) of $Q$.

The free part of the $\mathbb{Z}_2^n$-action on $W$ determines a principal $\mathbb{Z}_2^n$-bundle $\xi_W$ over $Q$.

It is shown in Lü-Masuda [25, Lemma 3.1] that $W$ can be recovered from the data $(Q, \lambda_W, \xi_W)$ up to equivariant homeomorphism. In addition, let $\pi : W \to Q$ denote the orbit map. If $f$ is a codimension-$k$ face of $Q$, then $W_f := \pi^{-1}(f)$ is a codimension-$k$ submanifold of $W$ called a facial submanifold of $W$. Let $G_f$ denote the isotropy subgroup of $W_f$. Then $W_f$ is also a 2-torus manifold with respect to the induced action of $\mathbb{Z}_2^n / G_f$. In the following, when we say $W_f$ is equivariantly formal, we always consider $W_f$ being equipped with the induced $\mathbb{Z}_2^n / G_f$-action from $W$.

The main purpose of this paper is to answer the following two questions.

**Question-1:** What kind of 2-torus manifolds are equivariantly formal?

**Question-2:** What kind of locally standard 2-torus manifolds have regular m-involutions?

Generally speaking, it is very hard to compute the equivariant cohomology of a locally standard 2-torus manifold $W$ directly from its orbit space $Q$ and the data $(\lambda_W, \xi_W)$. So it is difficult to judge whether $W$ is equivariantly formal by directly verifying the condition in the definition. Meanwhile, it was proved by Masuda-Panov [27] that a smooth $T^n$-action on a connected smooth 2n-manifold with non-empty fixed points is equivariantly formal if and only if the $T^n$-action is locally standard and every face of its orbit space is acyclic (also see Goertsches-Töben [19, Theorem 10.19] for a reformulation of this result). This result is also implied by Franz [15, Theorem 1.3]. The arguments in [27] inspire us to prove the following parallel result for 2-torus manifolds.

**Theorem 1.3.** Let $W$ be a 2-torus manifold with orbit space $Q$.

(i) $W$ is equivariantly formal if and only if $W$ is locally standard and $Q$ is mod 2 face-acyclic.

(ii) $W$ is equivariantly formal and $H^*(W; \mathbb{Z}_2)$ is generated by its degree-one part as a ring if and only if $W$ is locally standard and $Q$ is a mod 2 homology polytope.

The definitions of “mod 2 face-acyclic” and “mod 2 homology polytope” are given in Definition 2.1.

The main strategy in our proof of Theorem 1.3 is very similar to the strategy used in [27] for equivariantly formal torus manifolds. Besides, our proof uses the
mod 2 GKM theory introduced in Biss-Guillemin-Holm [4] which allows us to observe the equivariant cohomology of an equivariantly formal 2-torus manifold by restricting to its fixed point set (see Section 2.3).

**Remark 1.4.** If a 2-torus manifold $W$ is assumed to be locally standard in the first place, Theorem 1.3(i) can also be derived from Chaves [11, Theorem 1.1] whose proof uses the theory of syzygies in the mod 2 equivariant cohomology (see Allday-Franz-Puppe [2, Theorem 10.2]) and the mod 2 “Atiyah-Bredon sequence”. But we will use a completely different approach in our proof here.

Using Theorem 1.3 we can easily derive the following theorem which gives an answer to Question-2.

**Theorem 1.5.** Let $W$ be an $n$-dimensional locally standard 2-torus manifold with orbit space $Q$. Then there exists a regular $m$-involution on $W$ if and only if $Q$ is mod 2 face-acyclic (or equivalently $W$ is equivariantly formal) and the values of the characteristic function $\lambda_W$ on all the facets of $Q$ consist exactly of a linear basis of $\mathbb{Z}_2^n$.

A nice manifold with corners $Q$ is called $k$-colorable if we can assign $k$ different colors to all the facets of $Q$ so that no two adjacent facets are of the same color. Clearly, there exists a 2-torus manifold over $Q$ whose characteristic function takes value in a linear basis of $\mathbb{Z}_2^n$ if and only if $Q$ is $n$-colorable.

**Remark 1.6.** By Theorem 1.5 and the construction in Puppe [28], we can obtain a self-dual binary code $C_Q$ from an $n$-colorable mod 2 face-acyclic nice smooth $n$-manifold with corners $Q$ when $n$ is odd. This generalizes the self-dual binary codes from $n$-colorable simple convex $n$-polytopes in Chen-Lü-Yu [12]. Moreover, we can write down $C_Q$ explicitly in the same way as the self-dual binary code obtained in [12, Corollary 4.5].

The paper is organized as follows. In Section 2 we review the definitions and some basic facts of locally standard 2-torus manifolds and quote some well known results that are useful for our proof. In Section 3 we study various properties of equivariantly formal 2-torus manifolds. Since the philosophy of our study is very similar to the study of torus manifolds with vanishing odd degree cohomology in Masuda-Panov [27], many lemmas in this paper are parallel to those in [27]. In Section 4 we prove some special properties of equivariantly formal 2-torus manifolds whose mod 2 cohomology rings are generated by their degree-one part. Then finally in Section 5 we prove Theorem 1.3 and Theorem 1.5.

**2. Preliminaries**

2.1. Manifolds with corners and locally standard 2-torus manifolds.
Recall a (smooth) $n$-dimensional manifold with corners $Q$ is a Hausdorff space together with a maximal atlas of local charts onto open subsets of $\mathbb{R}^n_{\geq 0}$ such that the transition functions are (diffeomorphisms) homeomorphisms which preserve the codimension of each point. Here the codimension $c(x)$ of a point $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n_{\geq 0}$ is the number of $x_i$ that is 0. So we have a well defined map $c : Q \to \mathbb{Z}_{\geq 0}$ where $c(q)$ is the codimension of a point $q \in Q$. An open face of $Q$ of codimension $k$ is a connected component of $c^{-1}(k)$. A (closed) face is the closure of an open face. A face of codimension one is called a facet of $Q$. When $Q$ is connected, we also consider $Q$ itself as a face (of codimension zero).

- For any $k \in \mathbb{Z}_{\geq 0}$, the $k$-skeleton of $Q$ is the union of all the faces of $Q$ with dimension $\leq k$.
- The face poset of $Q$, denoted by $\mathcal{P}_Q$, is the set of faces of $Q$ ordered by reversed inclusion (so $Q$ is the initial element).

A manifold with corners $Q$ is said to be nice if either its boundary $\partial Q$ is empty or $\partial Q$ is non-empty and any codimension-$k$ face of $Q$ is a component of the intersection of $k$ different facets in $Q$. If $Q$ is nice, $\mathcal{P}_Q$ is a simplicial poset. But in general $\mathcal{P}_Q$ may not be the face poset of a simplicial complex. Indeed, $\mathcal{P}_Q$ is the face poset of a simplicial complex if and only if all non-empty multiple intersections of facets of $Q$ are connected (see [27, Sec. 5.2]).

**Definition 2.1.** Let $Q$ be a nice manifold with corners.

- We call $Q$ mod 2 face-acyclic if every face of $Q$ (including $Q$ itself) is a mod 2 acyclic space.
- We call $Q$ a mod 2 homology polytope if $Q$ is mod 2 face-acyclic and $\mathcal{P}_Q$ is the face poset of a simplicial complex.

A topological space $B$ is called mod 2 acyclic if $H^*(B; \mathbb{Z}_2) \cong H^*(\text{pt}; \mathbb{Z}_2)$.

It is not difficult to prove the following lemma (see [27, p.743 Remark] for a short argument).

**Lemma 2.2.** If $Q$ is mod 2 face-acyclic, then every face of $Q$ has a vertex and the 1-skeleton of $Q$ is connected.

In the following, let $W$ be an $n$-dimensional locally standard 2-torus manifold with orbit space $Q$. Then $Q$ is a smooth nice manifold with corners with $\partial Q \neq \emptyset$. Let $\pi : W \to Q$ denote the projection and let the set of facets of $Q$ be

$$\mathcal{F}(Q) = \{F_1, \cdots, F_m\}.$$ 

Then $\pi^{-1}(F_1), \cdots, \pi^{-1}(F_m)$ are embedded codimension-one closed connected submanifolds of $W$, called the characteristic submanifolds of $W$. Moreover, the $\mathbb{Z}_2^n$-action on $W$ determines a characteristic function on $Q$ which is a map

$$\lambda_W : \mathcal{F}(Q) \to \mathbb{Z}_2^n$$

(2)
where $\lambda_W(F_i) \in \mathbb{Z}_2^n$ is the generator of the $\mathbb{Z}_2$ subgroup that pointwise fixes the submanifold $\pi^{-1}(F_i)$, $1 \leq i \leq m$. Since the $\mathbb{Z}_2^n$ action is locally standard, the function $\lambda_W$ satisfies the following linear independence condition:

whenever the intersection of $k$ different facets $F_{i_1}, \ldots, F_{i_k}$ is non-empty, the elements $\lambda_W(F_{i_1}), \ldots, \lambda_W(F_{i_k})$ are linearly independent when viewed as vectors of $\mathbb{Z}_2^n$ over the field $\mathbb{Z}_2$.

For a codimension-$k$ face $f$ of $Q$, let $F_{i_1}, \ldots, F_{i_k}$ be all the facets containing $f$. Then the isotropy subgroup of the facial submanifold $W_f$ is

$$G_f = \text{the subgroup generated by } \{\lambda_W(F_{i_1}), \ldots, \lambda_W(F_{i_k})\} \subseteq \mathbb{Z}_2^n.$$

By the linear independence condition of $\lambda_W$, $G_f \cong \mathbb{Z}_2^k$. Hence $W_f$ is also a $2$-torus manifold with respect to the induced action of $\mathbb{Z}_2^n/G_f \cong \mathbb{Z}_2^{n-k}$.

In addition, $W$ determines a principal $\mathbb{Z}_2^n$-bundle over $Q$ as follows. We take a small invariant open tubular neighborhood for each characteristic submanifold of $W$ and remove their union from $W$. Then the $\mathbb{Z}_2^n$-action on the resulting space is free and its orbit space can naturally be identified with $Q$, which gives a principal $\mathbb{Z}_2^n$-bundle over $Q$, denoted by $\xi_W$. It is shown in Lü-Masuda [25] that $W$ can be recovered (up to equivariant homeomorphism) from $(Q, \xi_W, \lambda_W)$. For example, when $\xi_W$ is a trivial $\mathbb{Z}_2^n$-bundle, $W$ is equivariantly homeomorphic to the following “canonical model” determined by $(Q, \lambda_W)$.

$$M_Q(\lambda_W) := Q \times \mathbb{Z}_2^n / \sim$$

where $(q, g) \sim (q', g')$ if and only if $q = q'$ and $g - g' \in G_{f(q)}$ where $f(q)$ is the unique face of $Q$ that contains $q$ in its relative interior. This canonical model is a generalization of a result of Davis-Januszkiewicz [14, Prop. 1.8]. We will see that the canonical model plays an important role in our proof of Theorem 1.3 in Section 5.

2.2. Borel construction and equivariant cohomology.

For a topological group $G$, there exists a contractible free right $G$-space $EG$ called the universal $G$-space. The quotient $BG = EG/G$ is called the classifying space for free $G$-actions. For example when $G = \mathbb{Z}_2^n$, we can choose

$$EZ_2^n = (EZ_2)^n = (S^\infty)^n, \quad BZ_2^n = (B\mathbb{Z}_2)^n = (\mathbb{R}P^\infty)^n.$$

Let $X$ be a topological space with a left $G$-action (we call $X$ a $G$-space for brevity). The Borel construction of $X$ is denoted by

$$EG \times_G X = EG \times X / \sim$$

where $(e, x) \sim (eg, g^{-1}x)$ for any $e \in EG$, $x \in X$ and $g \in G$. 

The equivariant cohomology of $X$ with coefficients in a field $k$ is defined as 
\[ H^*_G(X; k) := H^*(EG \times_G X; k). \]

**Convention:** The term “cohomology” of a space in this paper, always mean singular cohomology if not specified otherwise.

The Borel construction determines a canonical fibration called *Borel fibration*:
\[ X \to EG \times_G X \to BG. \]

The map $\rho$ collapsing $X$ to a point induces a homomorphism
\[ \rho^*: H^*_G(pt; k) = H^*(BG; k) \to H^*_G(X; k) \]
which defines a canonical $H^*(BG; k)$-module structure on $H^*_G(X; k)$. A useful fact is: when $X$ is a paracompact space with finite cohomology dimension, and $G = T^r$ or $(\mathbb{Z}_p)^r$ where $p$ is a prime and $k = \mathbb{Q}$ or $\mathbb{Z}_p$ respectively, $\rho^*$ is injective if and only if the fixed point set $X^G$ is non-empty (see [21, Ch. IV]).

In general, $H^*_G(X; k)$ may not be a free $H^*(BG; k)$-module. The following *localization theorem* due to A. Borel (see [21, p. 45]) says that we can compute the free $H^*(BG; k)$-module part of $H^*_G(X; k)$ by restricting to the fixed point set.

**Theorem 2.3** (Localization Theorem). Let $G = T^r$ or $(\mathbb{Z}_p)^r$ where $p$ is a prime and $k = \mathbb{Q}$ or $\mathbb{Z}_p$ respectively. For a paracompact $G$-space $X$ with finite cohomology dimension, the following localized restriction homomorphism is an isomorphism:
\[ S^{-1}H^*_G(X; k) \to S^{-1}H^*_G(X^G; k) = H^*(X^G; k) \otimes_k (S^{-1}H^*(BG; k)) \]
where $S = R - \{0\}$ and $R$ is the polynomial subring of $H^*(BG; k)$. So the kernel of the restriction $H^*_G(X; k) \to H^*_G(X^G; k)$ lies in the $H^*(BG; k)$-torsion of $H^*_G(X; k)$. In particular if $X$ is equivariantly formal, $H^*_G(X; k) \to H^*_G(X^G; k)$ is injective.

The Borel construction can also be applied to a $G$-vector bundle $\pi: E \to X$ (i.e. both $E$ and $X$ are $G$-spaces and the projection $\pi$ is $G$-equivariant). In this case, the Borel construction $E_G$ of $E$ is a vector bundle over $X_G$ whose mod 2 Euler class, denoted by $e^G(E)$, lies in $H^*_G(X; \mathbb{Z}_2)$. Note that using $\mathbb{Z}_2$-coefficients allows us to ignore the orientation of a vector bundle.

**2.3. Mod 2 GKM-theory.**

Let $W$ be an $n$-dimensional equivariantly formal 2-torus manifold. Then the fixed point set $W^{\mathbb{Z}_2}$ is a finite non-empty set (by Theorem 1.1), and $H^*_G(W; \mathbb{Z}_2)$ is a free module over $H^*(B\mathbb{Z}_2^n; \mathbb{Z}_2)$. Moreover, $H^*_G(W; \mathbb{Z}_2)$ can be computed by the so called Mod 2 GKM-theory (see Biss-Guillemin-Holm [1]) which is an extension of the GKM-theory in [13] to 2-torus actions. In this section, we briefly review
some results related to our study. The reader is referred to [4] and [24] for more details.

For each \(1 \leq i \leq n\), let \(\rho_i \in \text{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)\) be the homomorphism defined by

\[
\rho_i((g_1, \ldots, g_n)) = g_i, \forall (g_1, \ldots, g_n) \in \mathbb{Z}_2^n.
\]

By a canonical isomorphism \(\text{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2) \cong H^1(B\mathbb{Z}_2^n; \mathbb{Z}_2)\), we can identify \(H^*(B\mathbb{Z}_2^n; \mathbb{Z}_2)\) with the graded polynomial ring \(\mathbb{Z}_2[\rho_1, \ldots, \rho_n]\) where \(\deg(\rho_i) = 1, 1 \leq i \leq n\).

Let \(Q = W/\mathbb{Z}_2^n\) be the orbit space of \(W\). By our Theorem 3.3 proved later, a 2-torus manifold \(W\) being equivariantly formal implies that it is locally standard. Hence \(Q\) is a nice manifold with corners. Then the 1-skeleton of \(Q\), consisting of vertices (0-faces) and edges (1-faces) of \(Q\), is an \(n\)-valent graph denoted by \(\Gamma(Q)\).

Let \(V(Q)\) and \(E(Q)\) denote the set of vertices and edges of \(Q\), respectively.

**Convention:** We will not distinguish a vertex of \(Q\) and the corresponding fixed point in \(W/\mathbb{Z}_2^n\) in the rest of the paper.

- Let \(\pi : W \to Q\) be the quotient map.
- For each edge \(e \in E(Q)\), \(\pi^{-1}(e)\) is a circle whose isotropy subgroup \(G_e\) is a rank \(n - 1\) subgroup of \(\mathbb{Z}_2^n\). Then we obtain a map

\[
\alpha : E(Q) \to \text{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2) \cong H^1(B\mathbb{Z}_2^n; \mathbb{Z}_2)
\]

where for each edge \(e \in E(Q)\), \(\ker(\alpha(e)) = G_e\).
- For each vertex \(p \in V(Q)\), let \(\alpha_p = \{\alpha(e) \mid p \in e\} \subset \text{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)\).

Such a map \(\alpha\) is called an axial function which has the following properties:

(i) For every vertex \(p \in V(Q)\), \(\alpha_p\) is a linear basis of \(\text{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)\).
(ii) For every edge \(e \in E(Q)\), \(\alpha_p \equiv \alpha_{p'} \mod \alpha(e)\) where \(p, p'\) are the two vertices of \(e\).

By [4] Theorem C and [4] Remark 5.9, we have the following theorem which is a consequence of the \(\mathbb{Z}_2\)-version Chang–Skjelbred theorem (see [4] Theorem 4.1 and [10]).

**Theorem 2.4** (see [4]). Let \(W\) be an \(n\)-dimensional equivariantly formal 2-torus manifold. If we choose an element \(\eta_p \in H^*_\mathbb{Z}_2(W/\mathbb{Z}_2^n; \mathbb{Z}_2)\) for each \(p \in W/\mathbb{Z}_2^n\), then

\[
(\eta_p) \in \bigoplus_{p \in W/\mathbb{Z}_2^n} H^*(B\mathbb{Z}_2^n; \mathbb{Z}_2) \cong H^*_\mathbb{Z}_2(W/\mathbb{Z}_2^n; \mathbb{Z}_2)
\]

is in the image of the restriction homomorphism \(r : H^*_\mathbb{Z}_2(W; \mathbb{Z}_2) \to H^*_\mathbb{Z}_2(W/\mathbb{Z}_2^n; \mathbb{Z}_2)\) if and only if for every edge \(e \in E(Q)\) with vertices \(p\) and \(p'\), \(\eta_p - \eta_{p'}\) is divisible by \(\alpha(e)\).
Moreover, we can understand the above axial function $\alpha$ in the following way. For brevity, we use the following notations for an $n$-dimensional locally standard 2-torus manifold $W$ in the rest of this section.

- Let $G = \mathbb{Z}_2^n$.
- Let $W_i := W_{F_i} = \pi^{-1}(F_i)$, $1 \leq i \leq m$, be all the characteristic submanifolds of $W$ where $F_1, \ldots, F_m$ are all the facets of $Q$.
- Let $G_i := \langle \lambda_W(F_i) \rangle \cong \mathbb{Z}_2$ be subgroup of $G$ that fixes $W_i$ pointwise.
- Let $\nu_i$ be the (equivariant) normal bundle of $W_i$ in $W$. So we have the equivariant Euler class of $\nu_i$, denoted by $e^G(\nu_i) \in H^1_G(W_i; \mathbb{Z}_2)$.
- For any fixed point $p \in W_{\mathbb{Z}_2^n}$, let $I(p) := \{ i \mid p \in W_i \}$. We have the decomposition of tangent space $T_p W$ as

$$T_p W = \bigoplus_{i \in I(p)} \nu_i|_p,$$

where $\nu_i|_p$ denotes the restriction of $\nu_i$ to $p$. So $\nu_i|_p$ is a 1-dimensional linear representation of $G$ whose equivariant Euler class

$$e^G(\nu_i|_p) = e^G(\nu_i)|_p \in H^1(B\mathbb{Z}_2^n; \mathbb{Z}_2).$$

The inclusion map $\psi_i : W_i \hookrightarrow W$ defines an equivariant Gysin homomorphism $\psi_i : H^*_G(W_i; \mathbb{Z}_2) \rightarrow H^*_G(W; \mathbb{Z}_2)$ (see [1, §5.3] for example). For brevity, let

$$\tau_i = \tau_{F_i} = \psi_i(1) \in H^*_G(W; \mathbb{Z}_2)$$

be the image of the identity $1 \in H^0_G(W_i; \mathbb{Z}_2)$. The element $\tau_i$ can be thought of as the Poncaré dual of the Borel construction of $W_i$ in $H^*_G(W; \mathbb{Z}_2)$ and is called the equivariant Thom class of $\nu_i$. A standard fact is

$$\tau_i|_p$$

agrees with the equivariant Euler class of $\nu_i|_p$.

Note that the elements of $\text{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$ are in one-to-one correspondence with all the 1-dimensional linear representations of $\mathbb{Z}_2^n$. So the canonical isomorphism between $\text{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$ and $H^1(B\mathbb{Z}_2^n; \mathbb{Z}_2)$ is given by the equivariant Euler class of a 1-dimensional representations of $\mathbb{Z}_2^n$. Then we have the following identification:

$$\alpha_p = \{ \alpha(e) \mid p \in e \} \leftrightarrow \{ e^G(\nu_i)|_p = \tau_i|_p : i \in I(p) \}.$$

where an edge $e$ containing $p$ corresponds to the unique index $i \in I(p)$ so that the facet $F_i$ intersects $e$ transversely (or equivalently $e \not\subseteq F_i$).

- For a codimension-$k$ face $f$ of $Q$, let $\nu_f$ denote the (equivariant) normal bundle of $W_f$ in $W$. Denote by $\tau_f \in H^*_{\mathbb{Z}_2}(W; \mathbb{Z}_2)$ the equivariant Thom class of $\nu_f$. Then the restriction of $\tau_f$ to $H^*_G(W_f; \mathbb{Z}_2)$ is the equivariant Euler class of $\nu_f$, denoted by $e^G(\nu_f)$. In particular, if $f = Q$, $W_f = W$ and so $\tau_f$ is the identity element of $H^0_G(W_f; \mathbb{Z}_2)$. 
Let \( r_p : H^*_G(W; \mathbb{Z}_2) \to H^*_G(p; \mathbb{Z}_2) \cong H^*(BG; \mathbb{Z}_2) \) denote the restriction map at a fixed point \( p \in W^G \). Then
\[
(8) \quad r = \bigoplus_{p \in W^G} r_p : H^*_G(W; \mathbb{Z}_2) \to H^*_G(W^G; \mathbb{Z}_2) = \bigoplus_{p \in W^G} H^*(BG; \mathbb{Z}_2).
\]
By Theorem 2.3, the kernel of \( r \) is the \( H^*(BG; \mathbb{Z}_2) \)-torsion subgroup of \( H^*_G(W; \mathbb{Z}_2) \).

Clearly, \( r_p(\tau_f) = 0 \) unless \( p \in (W_f)^G \) (i.e. \( p \) is a vertex of \( f \)). It follows from (7) that for any \( p \in W^G \),
\[
(9) \quad r_p(\tau_f) = \begin{cases} 
\prod_{p \in e, e \not\in f} \alpha(e), & \text{if } p \in f; \\
0, & \text{otherwise}.
\end{cases}
\]
In addition, define
\[
(10) \quad \hat{H}_G^*(W; \mathbb{Z}_2) := \frac{H^*_G(W; \mathbb{Z}_2)}{H^*(BG; \mathbb{Z}_2) \text{-torsion}}.
\]
By the localization theorem (Theorem 2.3), the restriction homomorphism \( r \) induces a monomorphism \( \hat{H}_G^*(W; \mathbb{Z}_2) \to H^*_G(W^G; \mathbb{Z}_2) \), still denoted by \( r \).

The following proposition is parallel to [27, Proposition 3.3].

**Proposition 2.5.** Let \( W \) be an \( n \)-dimensional locally standard 2-torus manifold.

(i) For each characteristic submanifold \( W_i \) with \( (W_i)^G \not= \emptyset \) where \( G = \mathbb{Z}_2^n \), there is a unique element \( a_i \in H_1(BG; \mathbb{Z}_2) \) such that
\[
(6) \quad \rho^*(t) = \sum_i \langle t, a_i \rangle \tau_i \mod \text{H}^*(BG; \mathbb{Z}_2) \text{-torsion}
\]
for any element \( t \in H^1(BG; \mathbb{Z}_2) \). Here the sum is taken over all the characteristic submanifolds \( W_i \) with \( (W_i)^G \not= \emptyset \) and \( \rho^* \) is defined in (6).

(ii) For each \( W_i \) with \( (W_i)^G \not= \emptyset \), the subgroup \( G_i \) fixing \( W_i \) coincides with the subgroup determined by \( a_i \in H_1(BG; \mathbb{Z}_2) \) through the identification \( H_1(BG; \mathbb{Z}_2) \cong \text{Hom}(\mathbb{Z}_2, G) \).

(iii) If \( n \) different characteristic submanifolds \( W_i, \ldots, W_i \) have a \( G \)-fixed point in their intersection, then the elements \( a_i, \ldots, a_i \) form a linear basis of \( H_1(BG; \mathbb{Z}_2) \) over \( \mathbb{Z}_2 \).

**Proof.** The argument is completely parallel to the arguments for torus manifolds in the proof of [26, Lemma 1.3, Lemma 1.5, Lemma 1.7]. Indeed, we can just replace the torus manifold \( M \) in [26] by our 2-torus manifold \( W \) and, replace \( T^n \) by \( \mathbb{Z}_2^n \) and \( H^2(M; \mathbb{Z}) \) by \( H^1(W; \mathbb{Z}_2) \) to obtain our proof here. The details of the proof are left to the reader. \( \square \)

In addition, the following lemma is completely parallel to the torus manifold case [27, Lemma 6.2]. Its proof is also parallel to [27], hence omitted.
Lemma 2.6. Let $W$ be a locally standard 2-torus manifold with orbit space $Q$. For any $\eta \in H^*_G(W; \mathbb{Z}_2)$ and any edge $e \in E(Q)$, $r_p(\eta) - r_{p'}(\eta)$ is divisible by $\alpha(e)$ where $p$ and $p'$ are the endpoints of $e$.

2.4. Face ring.

A poset (partially ordered set) $P$ is called simplicial if it has an initial element $\hat{0}$ and for each $x \in P$ the lower segment $[\hat{0}, x]$ is a boolean lattice (the face lattice of a simplex).

Let $P$ be a simplicial poset. For each $x \in P := P - \{\hat{0}\}$, we assign a geometrical simplex whose face poset is $[\hat{0}, x]$ and glue these geometrical simplices together according to the order relation in $P$. The cell complex we obtained is called the geometrical realization of $P$, denoted by $|P|$. We may also say that $|P|$ is a simplicial cell complex.

For any two elements $x, x' \in P$, denote by $x \vee x'$ the set of their least common upper bounds, and by $x \wedge x'$ their greatest common lower bounds. Since $P$ is simplicial, $x \wedge x'$ consists of a single element if $x \vee x'$ is non-empty.

Definition 2.7 (see Stanley [29]). The face ring of a simplicial poset $P$ over a field $k$ is the quotient

$$k[P] := k[v_x : x \in P]/I_P$$

where $I_P$ is the ideal generated by all the elements of the form

$$v_x v_{x'} - v_{x \wedge x'} \cdot \sum_{x'' \in x \vee x'} v_{x''}.$$ 

Let $Q$ be a nice manifold with corners. It is easy to see that the face poset of $Q$ is a simplicial poset, denoted by $P_Q$. We call $|P_Q|$ the simplicial cell complex dual to $Q$.

We define the face ring of $Q$ to be the face ring of $P_Q$. Equivalently, we can write the face ring of $Q$ as

$$k[Q] := k[v_f : f \text{ a face of } Q]/I_Q,$$

where $I_Q$ is the ideal generated by all the elements of the form

$$v_f v_{f'} - v_{f \vee f'} \cdot \sum_{f'' \in f \cap f'} v_{f''}.$$ 

where $f \vee f'$ denotes the unique minimal face of $Q$ containing both $f$ and $f'$.

Convention: For any face $f$ of $Q$, define the degree of $v_f$ to be the codimension of $f$. Then $k[Q] = k[P_Q]$ becomes a graded ring. Note that in the discussion of torus manifolds in [27], the degree of $v_f$ is defined to be twice the codimension of $f$ to fit the study there.
The \textit{f-vector} of \( Q \) is defined as \( f(Q) = (f_0, \ldots, f_{n-1}) \) where \( n = \dim(Q) \) and \( f_i \) is the number of faces of codimension \( i + 1 \). The equivalent information is contained in the \textit{h-vector} \( h(Q) = (h_0, \ldots, h_n) \) determined by the equation:

\begin{equation}
    h_0 t^n + \cdots + h_{n-1} t + h_n = (t-1)^n + f_0 (t-1)^{n-1} + \cdots + f_{n-1}.
\end{equation}

The \textit{Hilbert series} of \( k[Q] \) is \( F(k[Q]) := \sum_i \dim_k k[Q]_i t^i \) where \( k[Q]_i \) denotes the homogeneous degree \( i \) part of \( k[Q] \). By [29, Proposition 3.8],

\begin{equation}
    F(k[Q]; t) = \frac{h_0 + h_1 t + \cdots + h_n t^n}{(1-t)^n}.
\end{equation}

The following construction is taken from [27, Section 5]. For any vertex \((0\text{-face}) p \in Q \), we define a map

\begin{equation}
    s_p : k[Q] \to k[Q]/(v_f : p \notin f).
\end{equation}

If \( p \) is the intersection of \( n \) different facets \( F_1, \ldots, F_n \), then \( k[Q]/(v_f : p \notin f) \) can be identified with the polynomial ring \( k[v_{F_1}, \ldots, v_{F_n}] \).

\begin{lemma}[Lemma 5.6 in [27]]. If every face of \( Q \) has a vertex, then the direct sum \( s = \bigoplus_p s_p \) over all vertices \( p \in Q \) is a monomorphism from \( k[Q] \) to the sum of polynomial rings \( k[Q]/(v_f : p \notin f) \).
\end{lemma}

A finitely generated graded commutative ring \( R \) over \( k \) is called \textit{Cohen-Macaulay} if there exists an \textit{h.s.o.p} (homogeneous system of parameters) \( \theta_1, \ldots, \theta_n \) such that \( R \) is a free \( k[\theta_1, \ldots, \theta_n] \)-module. Clearly, if \( k[Q] = k[\mathcal{P}_Q] \) is Cohen-Macaulay, then it has a \textit{l.s.o.p} (linear system of parameters).

A simplicial complex \( K \) is called a \textit{Gorenstein* complex} over \( k \) if its face ring \( k[K] \) is Cohen-Macaulay and \( H^*(K; k) \cong H^*(S^d; k) \) where \( d = \dim(K) \). The reader is referred to Bruns-Herzog [9] and Stanley [30] for more information of Cohen-Macaulay rings and Gorenstein* complexes.

The following proposition is parallel to [27, Lemma 8.2 (1)].

\begin{proposition}
If \( Q \) is an \( n \)-dimensional mod 2 homology polytope, then the geometrical realization \( |\mathcal{P}_Q| \) of \( \mathcal{P}_Q \) is a Gorenstein* simplicial complex over \( \mathbb{Z}_2 \). In particular, \( \mathbb{Z}_2[\mathcal{P}_Q] \) is Cohen-Macaulay and \( H^*(|\mathcal{P}_Q|; \mathbb{Z}_2) \cong H^*(S^{n-1}; \mathbb{Z}_2) \).
\end{proposition}

\begin{proof}
The proof is almost identical to the proof in [27, Lemma 8.2] except that we use \( \mathbb{Z}_2 \)-coefficients instead of \( \mathbb{Z} \)-coefficients when applying [30, II 5.1] in the argument.
\end{proof}

3. Equivariantly formal 2-torus manifolds

In this section, we study various properties of equivariantly formal 2-torus manifolds. One may find that many discussions on 2-torus manifolds here are parallel to the discussions in [27] on torus manifolds. The condition “vanishing
of odd degree cohomology” on a torus manifold in [27] is now replaced by the equivariant formality condition on a 2-torus manifold and, the coefficients \( \mathbb{Z} \) is replaced by \( \mathbb{Z}_2 \). Many arguments in [27] are transplanted into our proof here while some of them actually become simpler.

In Section 3.1, we prove some general results of equivariantly formal \( \mathbb{Z}_2 \)-actions on compact manifolds. In particular, we prove that any equivariantly formal 2-torus manifold is locally standard, and the equivariant formality of a 2-torus manifold is inherited by all its facial submanifolds.

In Section 3.2, we explore the relations between the equivariant cohomology of a locally standard 2-torus manifold and the face ring of its orbit space.

In Section 3.3, we prove that the equivariant formality of a 2-torus manifold is preserved under real blow-ups along its facial submanifolds. Our proof uses a result from Gitler [17].

### 3.1. Equivariantly formal \( \Rightarrow \) locally standard.

**Lemma 3.1.** Suppose \( M \) is a compact manifold whose connected components are \( M_1, \ldots, M_k \). A \( \mathbb{Z}_2 \)-action on \( M \) is equivariantly formal if and only if each \( M_i \) is \( \mathbb{Z}_2 \)-invariant and the restricted \( \mathbb{Z}_2 \)-action on \( M_i \) is equivariantly formal.

**Proof.** The “if” part is obvious. For the “only if” part, assume that \( M_1, \ldots, M_s, s \leq k \), are all the components each of which is preserved under the \( \mathbb{Z}_2 \)-action.

Since the \( \mathbb{Z}_2 \)-action on \( M \) is equivariantly formal, by Theorem [14] we have

\[
\dim_{\mathbb{Z}_2} H^*(M; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(M_i; \mathbb{Z}_2).
\]

So in particular, \( M^{\mathbb{Z}_2} \) is not empty. Clearly, \( M^{\mathbb{Z}_2} \) must lie in \( M_1 \cup \cdots \cup M_s \), so \( s > 0 \) and \( M^{\mathbb{Z}_2} \) is the disjoint union of \( M_1^{\mathbb{Z}_2}, \ldots, M_s^{\mathbb{Z}_2} \). Then by Theorem [14]

\[
\sum_{i=1}^{s} \dim_{\mathbb{Z}_2} H^*(M_i^{\mathbb{Z}_2}; \mathbb{Z}_2) \leq \sum_{i=1}^{s} \dim_{\mathbb{Z}_2} H^*(M_i; \mathbb{Z}_2) \leq \dim_{\mathbb{Z}_2} H^*(M; \mathbb{Z}_2).
\]

By comparing this inequality with the previous equation, we can deduce that \( s = k \) and on every component \( M_i \), \( \dim_{\mathbb{Z}_2} H^*(M_i^{\mathbb{Z}_2}; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(M_i; \mathbb{Z}_2) \). So by Theorem [14] again, the \( \mathbb{Z}_2 \)-action on \( M_i \) is equivariantly formal. \( \square \)

**Lemma 3.2.** If a \( \mathbb{Z}_2 \)-action on a compact manifold \( M \) is equivariantly formal, then for every subgroup \( H \) of \( \mathbb{Z}_2 \),

(i) The action of \( H \) on \( M \) is equivariantly formal.

(ii) The induced action of \( \mathbb{Z}_2/H \) on \( M/H \) and \( \mathbb{Z}_2 \) on \( M/H \) are both equivariantly formal.

(iii) The induced action of \( \mathbb{Z}_2 \) (or \( \mathbb{Z}_2/H \)) on every connected component \( N \) of \( M/H \) is equivariantly formal, hence \( N \) has a \( \mathbb{Z}_2 \)-fixed point.


Proof. (i) By Theorem 1.1, it is equivalent to prove

\[(14) \dim_{\mathbb{Z}_2} H^*(M^H; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(M; \mathbb{Z}_2).\]

Otherwise assume \(\dim_{\mathbb{Z}_2} H^*(M^H; \mathbb{Z}_2) < \dim_{\mathbb{Z}_2} H^*(M; \mathbb{Z}_2)\). Observe that the \(\mathbb{Z}_2^r\)-action on \(M\) induces an action of \(\mathbb{Z}_2^r/H\) on \(M^H\) and we have

\[(15) M^{\mathbb{Z}_2^r} = (M^H)^{\mathbb{Z}_2^r/H}.\]

So by Theorem 1.1, \(\dim_{\mathbb{Z}_2} H^*(M^{\mathbb{Z}_2^r}; \mathbb{Z}_2) \leq \dim_{\mathbb{Z}_2} H^*(M^H; \mathbb{Z}_2) < \dim_{\mathbb{Z}_2} H^*(M; \mathbb{Z}_2)\), which contradicts the assumption that the \(\mathbb{Z}_2^r\)-action on \(M\) is equivariantly formal.

This proves (i).

(ii) By (15) and the assumption that the \(\mathbb{Z}_2^r\)-action is equivariantly formal,

\[
\dim_{\mathbb{Z}_2} H^*((M^H)^{\mathbb{Z}_2^r/H}; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(M^{\mathbb{Z}_2^r}; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(M; \mathbb{Z}_2) \leq \dim_{\mathbb{Z}_2} H^*(M^H; \mathbb{Z}_2).
\]

Then by Theorem 1.1 the action of \(\mathbb{Z}_2^r/H\) on \(M^H\) is equivariantly formal, so is the action of \(\mathbb{Z}_2^r\) on \(M\).

(iii) By the conclusion in (ii) and Lemma 3.2 (iii), the induced action of \(\mathbb{Z}_2^r\) (or \(\mathbb{Z}_2^r/H\)) on every connected component \(N\) of \(M^H\) is equivariantly formal. So by Theorem 1.1, \(N\) must have a \(\mathbb{Z}_2^r\)-fixed point. \(\square\)

Next, we prove a theorem that is parallel to [27, Theorem 4.1].

**Theorem 3.3.** If a 2-torus manifold \(W\) is equivariantly formal, then \(W\) must be locally standard.

**Proof.** Suppose \(\dim(W) = n\). For a point \(x \in W\), denote by \(G_x\) the isotropy group of \(x\).

- If \(G_x\) is trivial, then \(x\) is in a free orbit of the \(\mathbb{Z}_2^n\)-action. So \(W\) is locally standard near \(x\).

- Otherwise, let \(N\) be the connected component of \(W^{G_x}\) containing \(x\). By Lemma 3.2 (iii), the induced \(\mathbb{Z}_2^n\)-action on \(N\) has a fixed point, say \(x_0\).

  Since \(W^{\mathbb{Z}_2^n}\) is discrete, the tangential \(\mathbb{Z}_2^n\)-representation \(T_{x_0}W\) is faithful.

  Then since \(x\) and \(x_0\) are in the same connected component fixed pointwise by \(G_x\), the \(G_x\)-representation on \(T_{x_0}W\) agrees with the restriction of the tangential \(\mathbb{Z}_2^n\)-representation \(T_{x_0}W\) to \(G_x\). This implies that \(W\) is locally standard near \(x\).

The theorem is proved. \(\square\)

**Proposition 3.4.** Let \(W\) be an equivariantly formal 2-torus manifold with orbit space \(Q\). For any face \(f\) of \(Q\), the facial submanifold \(W_f\) is also an equivariantly formal 2-torus manifold.
Suppose \( \dim(W) = n \) and \( f \) is a codimension-\( k \) face of \( Q \). By Theorem \ref{thm:3.3} \( W \) is locally standard. Then \( W_f \) is a connected \((n - k)\)-dimensional embedded submanifold of \( W \) fixed pointwise by \( G_f \cong \mathbb{Z}_2^k \) (see \ref{thm:3.3}). By Lemma \ref{lem:3.2}(iii), the induced action of \( \mathbb{Z}_2^n / G_f \cong \mathbb{Z}_2^{n-k} \) on \( W_f \) is equivariantly formal. \( \square \)

### 3.2. Equivariant cohomology of locally standard 2-torus manifolds.

Let \( W \) be an \( n \)-dimensional locally standard 2-torus manifold with orbit space \( Q \). We explore the relation between \( H^*_G(W; \mathbb{Z}_2) \) where \( G = \mathbb{Z}_2^n \) and the face ring \( \mathbb{Z}_2[Q] \) under some conditions on \( Q \). In the following, we use the notations from Section \ref{sec:2.3}.

First of all, we have a lemma that is parallel to \cite[Lemma 6.3]{27}.

**Lemma 3.5.** For any faces \( f \) and \( f' \) of \( Q \), the relation below holds in \( \hat{H}^*_G(W; \mathbb{Z}_2) \):

\[
\tau_f \tau_{f'} = \tau_{f \cap f'} \sum_{f'' \in f \cap f'} \tau_{f''}.
\]

Here we define \( \tau_{\varnothing} = 0 \).

**Proof.** The proof is parallel to the proof of \cite[Lemma 6.3]{27}. The idea is to use the monomorphism \( r : \hat{H}^*_G(W; \mathbb{Z}_2) \rightarrow H^*_G(W^G; \mathbb{Z}_2) \) to map both sides of the identity to the fixed points and then use the formula \ref{eq:9} to check that they are equal. \( \square \)

By Lemma \ref{lem:3.5} we obtain a well-defined homomorphism

\[
\varphi : \mathbb{Z}_2[Q] \rightarrow \hat{H}^*_G(W; \mathbb{Z}_2).
\]

\[
v_f \mapsto \tau_f
\]

The following lemma and its proof are parallel to \cite[Lemma 6.4]{27}.

**Lemma 3.6.** The homomorphism \( \varphi \) is injective if every face \( Q \) has a vertex.

**Proof.** According to the definitions of \( r \) and \( s \) (see \ref{eq:8} and \ref{eq:13}), we have \( s = r \circ \varphi \) by identifying \( H^*_G(p; \mathbb{Z}_2) \) with \( \mathbb{Z}_2[Q]/(v_f : p \notin f) \) for every vertex \( p \) of \( Q \). Then by Lemma \ref{lem:2.8} \( s \) is injective if every face of \( Q \) has a vertex, so is \( \varphi \). \( \square \)

The following lemma is parallel to \cite[Proposition 7.4]{27}.

**Lemma 3.7.** If the 1-skeleton of every face of \( Q \) (including \( Q \) itself) is connected, then \( \hat{H}^*_G(W; \mathbb{Z}_2) \) is generated by the elements \( \tau_{F_1}, \ldots, \tau_{F_m} \in H^1_G(W; \mathbb{Z}_2) \) as an \( H^*(BG; \mathbb{Z}_2) \)-module, where \( F_1, \ldots, F_m \) are all the facets of \( Q \).

**Proof.** The argument is a bit technical, but it is completely parallel to the proof of \cite[Proposition 7.4]{27}. The main idea of the proof is to consider the restriction of an element \( \eta \in H^*_G(W; \mathbb{Z}_2) \) to the fixed point set \( W^G \) via \( r : H^*_G(W; \mathbb{Z}_2) \rightarrow H^*_G(W^G; \mathbb{Z}_2) \), and then use \( \tau_{F_1}, \ldots, \tau_{F_m} \) and elements in \( H^*(BG; \mathbb{Z}_2) \) to spell out
The following theorem is parallel to [27, Theorem 7.5].

**Theorem 3.8.** Let $W$ be a locally standard 2-torus manifold with orbit space $Q$. If every face $f$ of $Q$ has a vertex and the 1-skeleton of $f$ is connected, then the map $\varphi : \mathbb{Z}_2[Q] \to \check{H}^*_G(W; \mathbb{Z}_2)$ is an isomorphism of graded rings.

**Proof.** By Lemma 3.6 $\varphi$ is injective and, by Lemma 3.7 $\varphi$ is surjective. □

**Lemma 3.9.** Let $W$ be an equivariantly formal 2-torus manifold with orbit space $Q$. Then the 1-skeleton of every face of $Q$ (including $Q$ itself) is connected.

**Proof.** Since $W$ is equivariantly formal, the localization theorem (Theorem 2.3) implies that the restriction homomorphism $r : H^*_G(W; \mathbb{Z}_2) \to H^*_G(W^G; \mathbb{Z}_2)$ is injective. In addition, since $W$ is connected, the image of $H^0(G(W; \mathbb{Z}_2))$ under the restriction homomorphism is isomorphic to $\mathbb{Z}_2$. So the “if” part of Theorem 2.4 implies that the 1-skeleton of $Q$ must be connected.

For any proper face $f$ of $Q$, the facial submanifold $W_f$ is also an equivariantly formal 2-torus manifold by Proposition 3.4. Then by applying the above argument to $W_f$, we obtain that the 1-skeleton of $f$ is also connected. □

**Corollary 3.10.** If $W$ is an equivariantly formal 2-torus manifold, then the map $\varphi : \mathbb{Z}_2[Q] \to \check{H}^*_G(W; \mathbb{Z}_2)$ is an isomorphism of graded rings.

**Proof.** Since $W$ is equivariantly formal, its equivariant cohomology $H^*_G(W; \mathbb{Z}_2)$ is a free module over $H^*(BG; \mathbb{Z}_2)$. So by definition, $\check{H}^*_G(W; \mathbb{Z}_2) = H^*_G(W; \mathbb{Z}_2)$. For any face $f$ of $Q$, the facial submanifold $W_f$ is also an equivariantly formal 2-torus manifold by Proposition 3.4. This implies that $f$ has a vertex. Moreover, the 1-skeleton of $f$ is connected by Lemma 3.9. Then the corollary follows from Theorem 3.8. □

When a 2-torus manifold $W$ is equivariantly formal, Corollary 3.10 tells us that the equivariant cohomology ring of $W$ is completely determined by the face poset of its orbit space (so independent on the characteristic function $\lambda_W$ or the principal bundle $\xi_W$). This suggests that the orbit space of $W$ should be rather special.

The following corollary is parallel to [27, Corollary 7.8]. It generalizes the calculation of the mod 2 cohomology ring of a small cover in [14].

**Corollary 3.11.** If a 2-torus manifold $W$ is equivariantly formal, then

$$H^*(W; \mathbb{Z}_2) \cong \mathbb{Z}_2[v_f : f \text{ a face of } Q]/I$$

where $I$ is the ideal generated by the following two types of elements:
Figure 1. Cutting off a face from a nice manifold with corners

\begin{align*}
(a) \quad v_f v'_f - v_{f \cap f'} \sum_{f'' \in f \cap f'} v_{f''}, \quad (b) \quad \sum_{i=1}^m (t, a_i) v_{F_i}, \quad t \in H^1(BG; \mathbb{Z}_2).
\end{align*}

Here $F_1, \ldots, F_m$ are all the facets of $Q$ and the elements $a_i \in H_1(BG; \mathbb{Z}_2)$ are defined in Proposition \ref{prop:facets}.

Proof. Since $W$ is equivariantly formal, $\iota^*_W : H^*_G(W; \mathbb{Z}_2) \to H^*(W; \mathbb{Z}_2)$ is surjective and its kernel is generated by all $\rho^*(t)$ with $t \in H^1(BG; \mathbb{Z}_2)$ (see (6)). Then the statement follows from Corollary \ref{cor:formalness} and Proposition \ref{prop:kernel}.

3.3. Real blow-up of a locally standard 2-torus manifold along a facial submanifold.

Let $W$ be a locally standard 2-torus manifold with orbit space $Q$. For a codimension-$k$ face $f$ of $Q$, the facial submanifold $W_f$ is an embedded connected codimension-$k$ submanifold of $W$. So the equivariant normal bundle $\nu_f$ of $W_f$ in $W$ is a real vector bundle of rank $k$. If we replace $W_f \subset W$ by the real projective bundle $P(\nu_f)$, we obtain a new 2-torus manifold denoted by $\tilde{W}_f$ called the real blow-up of $W$ along $W_f$. This is analogous to the blow-up of a torus manifold in [27, Sec. 9] (also see [20, p. 605] and [15, Sec. 4]).

The orbit space of $\tilde{W}_f$, denoted by $Q_f$, is the result of “cutting off” the face $f$ from $Q$ (see Figure 1). So $\tilde{W}_f$ is also locally standard. Correspondingly, the simplicial cell complex $|P_{Q_f}|$ is obtained from $|P_Q|$ by a stellar subdivision of the face dual to $f$.

Proposition 3.12. Let $W$ be a locally standard 2-torus manifold with orbit space $Q$ and $f$ be a proper face of $Q$ with codimension-$k$. Then $\tilde{W}_f$ is equivariantly formal if and only if so is $W$.

Proof. (a) Let $\tilde{\nu}_f$ denote the equivariant normal bundle of $P(\nu_f)$ in $\tilde{W}_f$. Besides, let $\text{Th}(\nu_f)$ and $\text{Th}(\tilde{\nu}_f)$ be the Thom space of $\nu_f$ and $\tilde{\nu}_f$, respectively. Then we
have a natural commutative diagram of continuous maps:

$$
\begin{array}{ccc}
P(\nu_f) & \xrightarrow{i} & \tilde{W}^f \\
\downarrow{p_0} & & \downarrow{p} \\
W_f & \xrightarrow{i} & W \\
\end{array}
\xrightarrow{t} \text{Th}(\nu_f)
$$

where $i$ and $\tilde{i}$ are the inclusions, $t$ and $\tilde{t}$ are the Thom-Pontryagin maps; and $p : \tilde{W}^f \to W$ is the blow-down map, $p_0$ is the restriction of $p$ to $P(\nu_f)$ and $q$ is the induced map by $p$ in the Thom spaces.

According to [17, §5] and [17, Theorem 3.7], there is a short exact sequence:

$$0 \to H^*(\text{Th}(\nu_f); \mathbb{Z}_2) \xrightarrow{\alpha} H^*(W; \mathbb{Z}_2) \oplus H^*(\text{Th}(\nu_f); \mathbb{Z}_2) \xrightarrow{\beta} H^*(\tilde{W}^f; \mathbb{Z}_2) \to 0.
$$

where $\alpha = (t^*, q^*)$ and $\beta = p^* - \tilde{t}^*$. This implies:

$$\dim_{\mathbb{Z}_2} H^*(\tilde{W}^f; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(W; \mathbb{Z}_2) + \dim_{\mathbb{Z}_2} H^*(\text{Th}(\nu_f); \mathbb{Z}_2) - \dim_{\mathbb{Z}_2} H^*(\text{Th}(\nu_f); \mathbb{Z}_2).
$$

By the Thom isomorphism, we have

$$\dim_{\mathbb{Z}_2} H^*(\text{Th}(\nu_f); \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(W_f; \mathbb{Z}_2),$$

$$\dim_{\mathbb{Z}_2} H^*(\text{Th}(\nu_f); \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(P(\nu_f); \mathbb{Z}_2).
$$

By Leray-Hirsch theorem, $H^*(P(\nu_f); \mathbb{Z}_2) \cong H^*(W_f; \mathbb{Z}_2) \otimes H^*(\mathbb{R}P^{k-1}; \mathbb{Z}_2)$ (as $\mathbb{Z}_2$-vector spaces), which implies $\dim_{\mathbb{Z}_2} H^*(P(\nu_f); \mathbb{Z}_2) = k \cdot \dim_{\mathbb{Z}_2} H^*(W_f; \mathbb{Z}_2)$. So

$$\dim_{\mathbb{Z}_2} H^*(\tilde{W}^f; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(W; \mathbb{Z}_2) + (k - 1) \cdot \dim_{\mathbb{Z}_2} H^*(W_f; \mathbb{Z}_2).
$$

If $W$ is equivariantly formal, then $W$ is locally standard and so $Q$ is a nice manifold with corners. It is easy to see

$$\#\text{vertices of } Q^f = \#\text{vertices of } Q + (k - 1) \cdot \#\text{vertices of } f.$$

Since the fixed point set $W^G$ ($G = \mathbb{Z}_2^a$) corresponds to the vertex set of $Q$ which is discrete, the number of fixed points of the $G$-action satisfies

$$\left|\left(\tilde{W}^f\right)^G\right| = \left|W^G\right| + (k - 1) \cdot \left|(W_f)^G\right|.
$$

By Proposition [3.4], $W_f$ is also equivariantly formal. So by Theorem [11]

$$\dim_{\mathbb{Z}_2} H^*(W; \mathbb{Z}_2) = \left|W^G\right|, \quad \dim_{\mathbb{Z}_2} H^*(W_f; \mathbb{Z}_2) = \left|(W_f)^G\right|.
$$

It follows from [17] and [18] that $\left|\left(\tilde{W}^f\right)^G\right| = \dim_{\mathbb{Z}_2} H^*(\tilde{W}^f; \mathbb{Z}_2)$. So we deduce from Theorem [11] that $\tilde{W}^f$ is equivariantly formal.
Conversely, if $\tilde{W}^f$ is equivariantly formal, we have
\[
\dim_{\mathbb{Z}_2} H^*(W; \mathbb{Z}_2) \overset{\text{(17)}}{=} \dim_{\mathbb{Z}_2} H^*(\tilde{W}^f; \mathbb{Z}_2) - (k - 1) \cdot \dim_{\mathbb{Z}_2} H^*(W_f; \mathbb{Z}_2)
\]
(by Theorem 1.1) \leq |(\tilde{W}^f)^G| - (k - 1) \cdot |(W_f)^G| \overset{\text{(18)}}{=} |W^G| = \dim_{\mathbb{Z}_2} H^*(W^G; \mathbb{Z}_2).

But by Theorem 1.1, $\dim_{\mathbb{Z}_2} H^*(W^G; \mathbb{Z}_2) \leq \dim_{\mathbb{Z}_2} H^*(W; \mathbb{Z}_2)$. So we must have $\dim_{\mathbb{Z}_2} H^*(W; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(W^G; \mathbb{Z}_2)$, which implies that $W$ is equivariantly formal. The proposition is proved. \(\square\)

The following lemma is parallel to [27, Lemma 9.1]. Its proof is almost identical to the proof in [27], hence omitted.

**Lemma 3.13.** Let $Q$ be a nice manifold with corners and $f$ be a proper face of $Q$. Then $Q^f$ is mod 2 face-acyclic if and only if so is $Q$.

\[\begin{align*}
4. \text{Equivariantly formal 2-torus manifolds with mod 2 cohomology generated by degree-one part}
\end{align*}\]

In our study of equivariantly formal 2-torus manifolds, those manifolds whose mod 2 cohomology rings are generated by their degree-one part are of special importance. We will see in Section 5 that the study of general equivariantly formal 2-torus manifolds can be reduced to the study of these special 2-torus manifolds by a sequence of real blow-ups along facial submanifolds.

The following lemma is parallel to [27, Lemma 2.3].

**Lemma 4.1.** Suppose there is an equivariantly formal $\mathbb{Z}_2^r$-action on a compact manifold $M$ where the cohomology ring $H^*(M; \mathbb{Z}_2)$ is generated by its degree-one part. Then for any subgroup $H$ of $\mathbb{Z}_2^r$ and every connected component $N$ of $M^H$, the homomorphism $i^* : H^*(M; \mathbb{Z}_2) \to H^*(N; \mathbb{Z}_2)$ is surjective where $i : N \hookrightarrow M$ is the inclusion. In particular, $H^*(N; \mathbb{Z}_2)$ is also generated by its degree-one part.

**Proof.** First we assume $H \cong \mathbb{Z}_2$. We have a commutative diagram as follows:
\[
\begin{array}{ccc}
H^*_H(M; \mathbb{Z}_2) & \overset{\tilde{i}_H^*}{\longrightarrow} & H^*_H(N; \mathbb{Z}_2) \\
\varepsilon^*_M & & \varepsilon^*_N \\
H^*(M; \mathbb{Z}_2) & \overset{i^*}{\longrightarrow} & H^*(N; \mathbb{Z}_2)
\end{array}
\]

where $H^*_H(N; \mathbb{Z}_2) \cong H^*(N; \mathbb{Z}_2) \otimes H^*(BH; \mathbb{Z}_2)$ and $\tilde{i}_H^*$ is the homomorphism on equivariant cohomology induced by $i$. By our assumption, both $\varepsilon^*_M$ and $\varepsilon^*_N$ are surjective. The following argument is parallel to the proof of [27, Lemma 2.3].
By [7 Theorem VII.1.5], the inclusion \( M^H \hookrightarrow M \) induces an isomorphism 
\( H^k_H(M; \mathbb{Z}_2) \rightarrow H^k_H(M^H; \mathbb{Z}_2) \) for sufficiently large \( k \), which implies that
\[
\hat{\imath}^*_H : H^k_H(M; \mathbb{Z}_2) \rightarrow H^k_H(N; \mathbb{Z}_2)
\]
is surjective if \( k \) is sufficiently large.

Let \( v_1, \cdots, v_d \in H^1(M; \mathbb{Z}_2) \) be a set of multiplicative generators of \( H^*(M; \mathbb{Z}_2) \). For each \( 1 \leq l \leq d \), let \( \hat{v}_l \) be a lift of \( v_l \) in \( H^*_H(M; \mathbb{Z}_2) \) and \( w_l := \hat{\imath}^*(v_l) \in H^1(N; \mathbb{Z}_2) \). Let \( t \) be a generator of \( H^1(BH; \mathbb{Z}_2) \cong \mathbb{Z}_2 \). By the commutativity of the above diagram (19),
\[
\hat{\imath}^*(\hat{v}_l) = bt + w_l \quad \text{for some} \quad b_t \in \mathbb{Z}_2.
\]

Then for an arbitrary element \( \zeta \in H^*(N; \mathbb{Z}_2) \), there exists a large enough integer \( q \in \mathbb{Z} \) and a polynomial \( P(x_1, \cdots, x_d) \) such that
\[
\hat{\imath}^*(P(\hat{v}_1, \cdots, \hat{v}_d)) = \zeta \otimes t^q.
\]

On the other hand, we have
\[
\hat{\imath}^*(P(\hat{v}_1, \cdots, \hat{v}_d)) = P(b_1 t + w_1, \cdots, b_d t + w_d) = \sum_{k \geq 0} P_k(w_1, \cdots, w_d) \otimes t^k
\]
for some polynomials \( P_k, k \geq 0 \). Hence \( \zeta = P_s(w_1, \cdots, w_d) = \hat{\imath}^*(P(v_1, \cdots, v_d)) \). Therefore, \( \hat{\imath}^* \) is surjective and \( H^*(N; \mathbb{Z}_2) \) is generated by \( w_1, \cdots, w_d \in H^1(N; \mathbb{Z}_2) \).

For the general case, suppose \( H \cong \mathbb{Z}_2^r, 1 \leq s \leq r \). Then we have a sequence:
\[
\{0\} = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_s = H
\]
where \( H_l \cong \mathbb{Z}_2^r \) for each \( 0 \leq l \leq s \). Moreover, we have
\[
M^H = ((M^H_{H_1 \cap H_1}) \cdots)^{H_s/H_{s-1}}, \quad H_l/H_{l-1} \cong \mathbb{Z}_2, \quad l = 1, \cdots, s.
\]
Repeating the above argument for each \( H_l/H_{l-1} \) proves the lemma. \( \square \)

The following lemma is parallel to [27 Lemma 3.4].

**Lemma 4.2.** Let \( W \) be an equivariantly formal 2-torus manifold whose cohomology ring \( H^*(W; \mathbb{Z}_2) \) is generated by its degree-one part. Then all non-empty multiple intersections of the characteristic submanifolds of \( W \) are equivariantly formal 2-torus manifolds whose mod 2 cohomology rings are generated by their degree-one part as well.

**Proof.** Let \( F_1, \cdots, F_m \) be all the facets of \( Q \) and \( G = \mathbb{Z}_2^n \) where \( n = \dim(W) \). In the following, we use the notations defined in Section 2.3. First of all, since the characteristic submanifold \( W_i \) is a connected component of the fixed point set \( X^{G_i} \), Lemma 4.1 implies that the restriction \( H^*(W; \mathbb{Z}_2) \rightarrow H^*(W_i; \mathbb{Z}_2) \) is
surjective. So the $G$-action on $W_i$ is equivariantly formal (by Proposition ??). Then we have
\[
H^*_G(W; \mathbb{Z}_2) \cong H^*(W; \mathbb{Z}_2) \otimes H^*(BG; \mathbb{Z}_2), \\
H^*_G(W_i; \mathbb{Z}_2) \cong H^*(W_i; \mathbb{Z}_2) \otimes H^*(BG; \mathbb{Z}_2).
\]
It follows that the restriction $H^*_G(W; \mathbb{Z}_2) \to H^*_G(W_i; \mathbb{Z}_2)$ is also surjective. In addition, by using Proposition 2.5(i) and a completely parallel argument to the proof of [26] Prop. 3.4(2), we can prove the following claim.

**Claim:** $H^*_G(W; \mathbb{Z}_2)$ is generated as a ring by all the equivariant Thom classes $\tau_1, \ldots, \tau_m$ of the normal bundles of the characteristic submanifolds $W_1, \ldots, W_m$. $\square$

When $W_{j_1} \cap \cdots \cap W_{j_s} = \emptyset$, $\tau_{j_1} \cdots \tau_{j_s}$ clearly vanish. So the above claim implies that for any $k \geq 0$, $H^*_G(W; \mathbb{Z}_2)$ is additively generated by the monomials $\tau_{j_1}^{k_1} \cdots \tau_{j_s}^{k_s}$ such that $W_{j_1} \cap \cdots \cap W_{j_s} \neq \emptyset$ and $k_1 + \cdots + k_s = k$.

Let $N$ be a connected component of $W_{i_1} \cap \cdots \cap W_{i_k}$, $1 \leq k \leq n$. Then $N$ is the facial submanifold $W_f$ over some codimension-$k$ face $f$ of $Q$. So by Lemma 4.2, $N$ is an equivariantly formal 2-torus manifold whose cohomology ring $H^*(N; \mathbb{Z}_2)$ is generated by its degree-one part. Moreover, by a completely parallel argument to the proof of [27] Lemma 3.4, we can show that $N$ is the only connected component of $W_{i_1} \cap \cdots \cap W_{i_k}$ from the above discussion of $H^*_G(W; \mathbb{Z}_2)$. The lemma is proved.

The following proposition is parallel to [27] Lemma 8.2(2)].

**Proposition 4.3.** Suppose $W$ is an $n$-dimensional equivariantly formal 2-torus manifold with orbit space $Q$ and the cohomology ring $H^*(W; \mathbb{Z}_2)$ is generated by its degree-one part. Then the geometrical realization $\vert \mathcal{P}_Q \vert$ of the face poset $\mathcal{P}_Q$ of $Q$ is a Gorenstein* simplicial complex over $\mathbb{Z}_2$. In particular, $\mathbb{Z}_2[\mathcal{P}_Q] = \mathbb{Z}[Q]$ is Cohen-Macaulay and $H^*(\mathcal{P}_Q; \mathbb{Z}_2) \cong H^*(S^{n-1}; \mathbb{Z}_2)$.

**Proof.** By Lemma 4.2, all non-empty multiple intersections of the characteristic submanifolds of $W$ are connected. This implies that $\vert \mathcal{P}_Q \vert$ is a simplicial complex. Moreover, by [30] II 5.1(d)], it is enough to verify the following three conditions to prove that $\vert \mathcal{P}_Q \vert$ is Gorenstein* over $\mathbb{Z}_2$:

(a) $\mathbb{Z}_2[\mathcal{P}_Q]$ is Cohen-Macaulay;
(b) Every $(n-2)$-simplex in $\mathcal{P}_Q$ is contained in exactly two $(n-1)$-simplices;
(c) $\chi(\mathcal{P}_Q) = \chi(S^{n-1}).$

Since $W$ is equivariantly formal, $H^*_G(W; \mathbb{Z}_2)$ is a free $H^*(BG; \mathbb{Z}_2)$-module and $\mathbb{Z}_2[\mathcal{P}_Q] = \mathbb{Z}_2[Q]$ is isomorphic to $H^*_G(W; \mathbb{Z}_2)$ (by Corollary 3.10) where $G = \mathbb{Z}_2^n$. This implies (a).

Note that each $(n-2)$-simplex of $\mathcal{P}_Q$ corresponds to a non-empty intersection of $n-1$ characteristic submanifolds of $W$. The latter intersection is an equivariantly
formal 1-manifold by Lemma 4.2 so it is a circle with exactly two $G$-fixed points. This implies (b).

The proof of (c) is completely parallel to [27, Lemma 8.2 (2)], so we leave it to the reader. The proposition is proved. □

Using the above proposition and the lemmas from Section 3, we obtain the following theorem that is parallel to [27, Theorem 7.7].

**Theorem 4.4.** Let $W$ be a 2-torus manifold whose orbit space is $Q$. Then $W$ is equivariantly formal and the cohomology ring $H^*(W; \mathbb{Z}_2)$ is generated by its degree-one part if and only if the following three conditions are satisfied:

(a) $H^*_G(W; \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2[Q] = \mathbb{Z}_2[\mathcal{P}_Q]$ as a graded ring.

(b) $\mathbb{Z}_2[Q]$ is Cohen-Macaulay.

(c) $|\mathcal{P}_Q|$ is a simplicial complex.

**Proof.** The argument is completely parallel to the proof of [27, Theorem 7.7]. We only need to replace $T^n$ by $\mathbb{Z}_2^n$ and $\mathbb{Q}$-coefficients by $\mathbb{Z}_2$-coefficients to obtain our proof here. □

5. Proof of Theorem 1.3

In this section, we give a proof of Theorem 1.3. Our proof follows the proof of [27, Theorem 8.3, Theorem 9.3] almost step by step, while some arguments for 2-torus manifolds here are simpler than those for torus manifolds in [27].

5.1. Equivariant cohomology of the canonical model.

Let $Q$ be a connected compact smooth nice $n$-manifold with corners. We call any function $\lambda : \mathcal{F}(Q) \to \mathbb{Z}_2^n$ that satisfies the linear independence relation in Section 2.1 a characteristic function on $Q$. By the same gluing rule in (4), we can obtain a space $M_Q(\lambda)$ from any characteristic function $\lambda$ on $Q$, called the canonical model determined by $(Q, \lambda)$. It is easy to see that $M_Q(\lambda)$ is a 2-torus manifold of dimension $n$.

Let $Q^\vee$ denote the cone of the geometrical realization of the order complex $\text{ord}(\overline{\mathcal{P}}_Q) = \mathcal{P}_Q - \{0\}$. So topologically, $Q^\vee$ is homeomorphic to $\text{Cone}(|\mathcal{P}_Q|)$. Moreover, $Q^\vee$ is a “space with faces” (see Davis [13, Sec. 6]) where each proper face $f$ of $Q$ determines a unique “face” $f^\vee$ of $Q^\vee$ that is the geometrical realization of the order complex of the poset $\{f' \mid f' \subseteq f\}$. More precisely, $f^\vee$ consists of all simplices of the form $f'_k \subseteq \cdots \subseteq f'_1 \subseteq f'_0 = f$ in $\text{ord}(\overline{\mathcal{P}}_Q)$. The “boundary” of $Q^\vee$, denoted by $\partial Q^\vee$, is $\text{ord}(\overline{\mathcal{P}}_Q)$ which is homeomorphic to $|\mathcal{P}_Q|$. So we have homeomorphisms:

$$\partial Q^\vee \cong |\mathcal{P}_Q|, \quad Q^\vee \cong \text{Cone}(|\mathcal{P}_Q|).$$
Remark 5.1. When $|\mathcal{P}_Q|$ is a simplicial complex, the space $Q^\vee$ with the face decomposition was called in [14, p. 428] a simple polyhedral complex.

Suppose $F_1, \ldots, F_m$ are all the facets of $Q$. Let $\mathcal{F}(Q^\vee) = \{F^\vee_1, \ldots, F^\vee_m\}$. Then any characteristic function $\lambda : \mathcal{F}(Q) \to \mathbb{Z}_2^n$ induces a map $\lambda^\vee : \mathcal{F}(Q^\vee) \to \mathbb{Z}_2^n$ where $\lambda^\vee(F^\vee_i) = \lambda(F_i)$, $1 \leq i \leq m$. Then by the same gluing rule in [14], we obtain a space $M_{Q^\vee}(\lambda^\vee)$ with a canonical $\mathbb{Z}_2^n$-action. By the same argument as in the proof of [27, Proposition 5.14], we can prove the following.

**Proposition 5.2.** There exists a continuous map $\phi : Q \to Q^\vee$ which preserves the face structure and induces an equivariant continuous map $\Phi : M_Q(\lambda) \to M_{Q^\vee}(\lambda^\vee)$.

Here $\phi : Q \to Q^\vee$ is constructed inductively, starting from an identification of vertices and extending the map on each higher-dimensional face by a degree-one map. Since every face $f^\vee$ of $Q^\vee$ is a cone, there are no obstructions to such extensions.

In addition, by a similar argument to that in [14, Theorem 4.8], we can obtain the following result.

**Proposition 5.3.** $H^*_G(M_{Q^\vee}(\lambda^\vee); \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2[Q]$ where $G = \mathbb{Z}_2^n$.

On the other hand, $H^*_G(M_Q(\lambda); \mathbb{Z}_2)$ could be much more complicated. Indeed, it is shown in [31, Theorem 1.7] that $H^*_G(M_Q(\lambda); \mathbb{Z}_2)$ isomorphic to the so called topological face ring of $Q$ over $\mathbb{Z}_2$ which involves the mod 2 cohomology rings of all the faces of $Q$.

5.2. Proof of Theorem 1.3(ii).

**Proof.** We first prove the “if” part. Let $Q$ be an $n$-dimensional mod 2 homology polytope and $G = \mathbb{Z}_2^n$. Since $H^1(Q; \mathbb{Z}_2) = 0$ and $W$ is locally standard, the principal $G$-bundle $\xi_W$ determined by $W$ is a trivial $G$-bundle over $Q$. Then by [25, Lemma 3.1], $W$ is equivariantly homeomorphic to the canonical model $M_Q(\lambda_W)$ (see (1)). So by Proposition 5.3 there exists an equivariant continuous map $\Phi : W = M_Q(\lambda_W) \to M_{Q^\vee}(\lambda^\vee_W) := W^\vee$.

Let $\pi : W \to Q$ and $\pi^\vee : W^\vee \to Q^\vee$ be the projections, respectively. Let $F_1, \ldots, F_m$ be all the facets of $Q$. Since $Q$ is a mod 2 homology polytope, so are $F_1, \ldots, F_m$. For brevity, let $W_i = \pi^{-1}(F_i)$, $W^\vee_i = (\pi^\vee)^{-1}(F^\vee_i)$, $1 \leq i \leq m$. 


It is easy to see that the $\mathbb{Z}_2$-actions on $W \setminus \bigcup_i W_i$ and $W^\vee \setminus \bigcup_i W_i^\vee$ are both free. Then we have
$$H_G^*(W, \bigcup W_i; \mathbb{Z}_2) \cong H^*(Q, \partial Q; \mathbb{Z}_2), \quad H_G^*(W^\vee, \bigcup_i W_i^\vee; \mathbb{Z}_2) \cong H^*(Q^\vee, \partial Q^\vee; \mathbb{Z}_2).$$

So $\Phi : W \to W^\vee$ induces a map between the following two exact sequences:
\[
\begin{array}{c}
\xrightarrow{\phi^*} H^*(Q^\vee, \partial Q^\vee; \mathbb{Z}_2) \longrightarrow H_G^*(W^\vee; \mathbb{Z}_2) \longrightarrow H_G^*(\bigcup_i W_i^\vee; \mathbb{Z}_2) \longrightarrow \cdots
\end{array}
\]
\[
\begin{array}{c}
\xrightarrow{\phi^*} H^*(Q, \partial Q; \mathbb{Z}_2) \longrightarrow H_G^*(W; \mathbb{Z}_2) \longrightarrow H_G^*(\bigcup_i W_i; \mathbb{Z}_2) \longrightarrow \cdots
\end{array}
\]

Each $W_i$ is a 2-torus manifold over the homology polytope $F_i$. So using induction and a Mayer-Vietoris argument, we may assume that in the diagram (21), $\Phi^* : H_G^*(\bigcup_i W_i^\vee; \mathbb{Z}_2) \to H_G^*(\bigcup_i W_i; \mathbb{Z}_2)$ is an isomorphism.

By Proposition 2.9, $H^*(|\mathcal{P}_Q|; \mathbb{Z}_2) \cong H^*(S^{n-1}; \mathbb{Z}_2)$. Then by (20), we obtain
$$H^*(Q^\vee, \partial Q^\vee; \mathbb{Z}_2) \cong H^*(D^n, S^{n-1}; \mathbb{Z}_2).$$

We also have $H^*(Q, \partial Q; \mathbb{Z}_2) \cong H^*(D^n, S^{n-1}; \mathbb{Z}_2)$ since $Q$ is an $n$-dimensional mod 2 homology polytope. By the construction of $\phi$, it is easy to see that the homomorphism $\phi^* : H^*(Q^\vee, \partial Q^\vee; \mathbb{Z}_2) \to H^*(Q, \partial Q; \mathbb{Z}_2)$ is an isomorphism. Then by applying the five-lemma to the diagram (21), we can deduce that $\Phi^* : H_G^*(W^\vee; \mathbb{Z}_2) \to H_G^*(W; \mathbb{Z}_2)$ is also an isomorphism. So by Proposition 5.3, $H_G^*(W; \mathbb{Z}_2) \cong \mathbb{Z}_2[Q]$. 

Besides, we also know that $\mathbb{Z}_2[Q]$ is Cohen-Macaulay by Proposition 2.9. Then since $|\mathcal{P}_Q|$ is a simplicial complex, all the three conditions in Theorem 4.4 are satisfied. Hence $W$ is equivariantly formal and $H^*(W; \mathbb{Z}_2)$ is generated by its degree-one part as a ring. The “if” part is proved.

Next, we prove the “only if” part. By the assumption on $W$ and Lemma 4.2, all non-empty multiple intersections of characteristic submanifolds of $W$ are connected and their cohomology rings are generated by their degree-one elements. So we may assume by induction that all the proper faces of $Q$ are mod 2 homology polytopes. In particular, the proper faces of $Q$ are all mod 2 acyclic. From these assumptions, we need to prove that $Q$ itself is mod 2 acyclic.

By Proposition 5.3, $|\mathcal{P}_Q|$ is a simplicial complex. So $|\mathcal{P}_Q|$ is the nerve simplicial complex of the cover of $\partial Q$ by the facets of $Q$. By a Mayer–Vietoris sequence argument, we can deduce that $H^*(\partial Q; \mathbb{Z}_2) \cong H^*(|\mathcal{P}_Q|; \mathbb{Z}_2)$. This together with Proposition 4.3 shows that
\[
H^*(\partial Q; \mathbb{Z}_2) \cong H^*(S^{n-1}; \mathbb{Z}_2).
\]

Claim: $H^1(Q; \mathbb{Z}_2) = 0$. 

Since \( W \) is equivariantly formal, \( H^*_G(W; \mathbb{Z}_2) \) is a free \( H^*(BG; \mathbb{Z}_2) \)-module. On the other hand, \( H^*(Q, \partial Q; \mathbb{Z}_2) \) is finitely generated over \( \mathbb{Z}_2 \) since \( Q \) is compact. So \( H^*(Q, \partial Q; \mathbb{Z}_2) \) is a torsion \( H^*(BG; \mathbb{Z}_2) \)-module. It follows that the whole bottom row in the diagram (21) splits into short exact sequences:

\[
0 \to H^k_G(W; \mathbb{Z}_2) \to H^k_G\left( \bigcup_i W_i; \mathbb{Z}_2 \right) \to H^{k+1}(Q, \partial Q; \mathbb{Z}_2) \to 0, \ k \geq 0.
\]

Take \( k = 0 \) above, we clearly have \( H^0_G(W; \mathbb{Z}_2) \cong H^0_G\left( \bigcup_i W_i; \mathbb{Z}_2 \right) \cong \mathbb{Z}_2 \). This implies \( H^1(Q, \partial Q; \mathbb{Z}_2) = 0 \). So in the following exact sequence,

\[
\cdots \to H^1(Q, \partial Q; \mathbb{Z}_2) \to H^1(Q; \mathbb{Z}_2) \to H^1(\partial Q; \mathbb{Z}_2) \to \cdots,
\]

\( H^1(Q; \mathbb{Z}_2) \) is mapped injectively into \( H^1(\partial Q; \mathbb{Z}_2) \cong H^1(S^{n-1}; \mathbb{Z}_2) \). Note that if \( n = 1 \), the claim is trivial. When \( n = 2 \), we have \( \partial Q = S^1 \) and \( H^1(Q; \mathbb{Z}_2) = 0 \) or \( \mathbb{Z}_2 \). But by the classification of compact surfaces, the latter case is impossible. When \( n \geq 3 \), we have \( H^1(\partial Q; \mathbb{Z}_2) = 0 \), so \( H^1(Q; \mathbb{Z}_2) = 0 \). The claim is proved.

Now since \( H^1(Q; \mathbb{Z}_2) = 0 \), by the above proof of the “if” part, there exists an equivariant homeomorphism \( \Phi \) from \( W \) to the canonical model \( M_Q(\lambda_W) \). In addition, by (20) and Proposition 4.3, we have

\[
H^*(\partial Q^\vee; \mathbb{Z}_2) \cong H^*([\mathcal{P}_Q]; \mathbb{Z}_2) \cong H^*(S^{n-1}; \mathbb{Z}_2).
\]

So we have an isomorphism

\[
H^*(Q^\vee, \partial Q^\vee; \mathbb{Z}_2) \cong H^*(D^n, S^{n-1}; \mathbb{Z}_2).
\]

Then by the construction of \( \phi \), the map \( \phi^*: H^*(Q^\vee, \partial Q^\vee; \mathbb{Z}_2) \to H^*(Q, \partial Q; \mathbb{Z}_2) \) is an isomorphism in degree \( n \) (since \( Q \) is connected) and thus is injective in all degrees. So by an extended version of the 5-lemma, we can deduce that in the diagram (21) the map \( \Phi^*: H^*_G(W^\vee; \mathbb{Z}_2) \to H^*_G(W; \mathbb{Z}_2) \) is injective. Moreover,

\begin{itemize}
  \item \( H^*_G(W^\vee; \mathbb{Z}_2) = H^*_G(M_{Q^\vee}(\lambda_W); \mathbb{Z}_2) \cong \mathbb{Z}_2[Q] \) by Proposition 5.3 and
  \item \( \mathbb{Z}_2[Q] \cong H^*_G(W; \mathbb{Z}_2) \) by Corollary 3.10
\end{itemize}

So \( H^*_G(W^\vee; \mathbb{Z}_2) \) and \( H^*_G(W; \mathbb{Z}_2) \) have the same dimension over \( \mathbb{Z}_2 \) in each degree. Therefore, the monomorphism \( \Phi^*: H^*_G(W^\vee; \mathbb{Z}_2) \to H^*_G(W; \mathbb{Z}_2) \) is actually an isomorphism. Then by the 5-lemma again, we can deduce from the diagram (21) that \( \phi^*: H^*(Q^\vee, \partial Q^\vee; \mathbb{Z}_2) \to H^*(Q, \partial Q; \mathbb{Z}_2) \) is an isomorphism. So by (21),

\[
H^*(Q, \partial Q; \mathbb{Z}_2) \cong H^*(D^n, S^{n-1}; \mathbb{Z}_2)
\]

which implies that \( Q \) is mod 2 acyclic by Poincaré-Lefschetz duality. This finishes the proof. \( \square \)
Figure 2. Cutting a vertex and an edge

5.3. Proof of Theorem 1.3(i).

Proof. We can reduce Theorem 1.3(i) to Theorem 1.3(ii) by real blow-ups of $W$ along sufficient many facial submanifolds, which corresponds to doing some barycentric subdivisions of the face poset $\mathcal{P}_Q$ of $Q$ (see Figure 2). Indeed, after doing enough barycentric subdivisions to $\mathcal{P}_Q$, we can turn $|\mathcal{P}_Q|$ into a simplicial complex. Let $\hat{W}$ be the 2-torus manifold obtained after these real blow-ups on $W$ and $\hat{Q}$ be its orbit space (with $|\mathcal{P}_\hat{Q}|$ being a simplicial complex).

Fact-1: $\hat{W}$ is equivariantly formal if and only if so is $W$ (by Proposition 3.12).
Fact-2: $\hat{Q}$ is mod 2 face-acyclic if and only if so is $Q$ (by Lemma 3.13).

We first prove the “if” part. Suppose $W$ is locally standard and $Q$ is mod 2 face-acyclic. Then $\hat{W}$ is also locally standard and $\hat{Q}$ is a mod 2 homology polytope by Fact-2. So by Theorem 1.3(ii), $\hat{W}$ is equivariantly formal, then so is $W$.

Next, we prove the “only if” part. If $W$ is equivariantly formal, then so is $\hat{W}$, and $W$ is locally standard by Theorem 3.3. So by Corollary 3.10 we have a graded ring isomorphism $H^*_G(\hat{W}; \mathbb{Z}_2) \cong \mathbb{Z}_2[\hat{Q}]$. Moreover, since $|\mathcal{P}_\hat{Q}|$ is a simplicial complex, $\mathbb{Z}_2[\hat{Q}]$ is generated by its degree-one elements, then so is $H^*_G(\hat{W}; \mathbb{Z}_2)$. In addition, since $i^*_\hat{W} : H^*_W(\hat{W}; \mathbb{Z}_2) \to H^*(\hat{W}; \mathbb{Z}_2)$ is surjective, $H^*(\hat{W}; \mathbb{Z}_2)$ is also generated by its degree-one elements. Then by Theorem 1.3(ii), $\hat{Q}$ is a mod 2 homology polytope. So by Fact-2, $Q$ is mod 2 face-acyclic. □

5.4. Proof of Theorem 1.5

Proof. We first prove the “if” part. Assume that there exists a regular $m$-involution $\tau$ on $W$. By definition the fixed point set $W^\tau$ of $\tau$ is discrete, then so is $W^{\mathbb{Z}_2} \subseteq W^\tau$. This implies that $Q$ must have vertices. Let $p$ be a vertex of $Q$
and let $F_1, \cdots, F_n$ be all the facets containing $p$. By the property of $\lambda_W$, 
$$e_1 = \lambda_W(F_1), \cdots, e_n = \lambda_W(F_n)$$
form a linear basis of $\mathbb{Z}_2^n$ over $\mathbb{Z}_2$. Then since the $\mathbb{Z}_2^n$-action on $W$ is locally standard, it is easy to see that only when $g = e_1 + \cdots + e_n$ could the fixed point set $W^g$ be discrete. So we must have $\tau = \tau_{e_1+\cdots+e_n}$, and in particular 
$$W^\tau = W^{\tau_{e_1+\cdots+e_n}} = W^{\mathbb{Z}_2^n}.$$ 
Hence 
$$\dim_{\mathbb{Z}_2} H^*\left(\mathbb{Z}^{\mathbb{Z}_2^n}; \mathbb{Z}_2\right) = \dim_{\mathbb{Z}_2} H^*(W^\tau; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(W; \mathbb{Z}_2)$$
where the second “=” is due to the assumption that $\tau$ is an $m$-involution. So by Theorem 1.1, $W$ is equivariantly formal. Then $Q$ is mod 2 face-acyclic by Theorem 1.3. In particular, every face of $Q$ has a vertex and the 1-skeleton of $Q$ is connected (by Lemma 2.2).

It remains to prove that the image of $\lambda_W : \mathcal{F}(Q) \to \mathbb{Z}_2^n$ is exactly $\{e_1, \cdots, e_n\}$. Indeed, take an edge $e$ of $Q$ whose vertices are $p$ and $p'$. So the $n$ facets of $Q$ that contain $p'$ are $F_1, \cdots, F_{i-1}, F'_i, F_{i+1}, \cdots, F_n$ for some $1 \leq i \leq n$. Then since $\tau_{e_1+\cdots+e_n}$ is an $m$-involution, we must have 
$$\lambda_W(F_1) + \cdots + \lambda_W(F_{i-1}) + \lambda_W(F'_i) + \lambda_W(F_{i+1}) + \cdots + \lambda_W(F_n) = e_1 + \cdots + e_n.$$ 
This implies $\lambda_W(F'_i) = e_i$. Then since the 1-skeleton of $Q$ is connected and every facet $F$ of $Q$ contains a vertex, we can iterate the above argument to prove that every $\lambda_W(F)$ must take value in $\{e_1, \cdots, e_n\}$.

Next, we prove the “only if” part. Suppose $Q$ is mod 2 face-acyclic and the values of the characteristic function $\lambda_W$ of $Q$ consist exactly of a linear basis $e_1, \cdots, e_n$ of $\mathbb{Z}_2^n$. By Theorem 1.3(i), $W$ is equivariantly formal. So we have 
$$\dim_{\mathbb{Z}_2} H^*\left(\mathbb{Z}^{\mathbb{Z}_2^n}; \mathbb{Z}_2\right) = \dim_{\mathbb{Z}_2} H^*(W; \mathbb{Z}_2)$$ (by Theorem 1.1).

On the other hand, our assumption on $\lambda_W$ implies that the regular involution $\tau = \tau_{e_1+\cdots+e_n}$ satisfies $W^\tau = \mathbb{Z}_2^n$ which is a discrete set. Then we have 
$$\dim_{\mathbb{Z}_2} H^*(W^\tau; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(\mathbb{Z}^{\mathbb{Z}_2^n}; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^*(W; \mathbb{Z}_2).$$
So $\tau$ is a regular $m$-involution on $W$ by definition. The theorem is proved. 

**Remark 5.4.** If we do not assume a 2-torus manifold $W$ to be locally standard, even if $W$ admits a regular $m$-involution, $W$ may not be equivariantly formal or locally standard. For example: let 
$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$ 
Define two involutions $\sigma$ and $\sigma'$ on $S^2$ by 
$$\sigma(x_1, x_2, x_3) = (-x_1, -x_2, x_3), \quad \sigma'(x_1, x_2, x_3) = (x_1, x_2, -x_3).$$
It is easy to see that $\sigma$ is an $m$-involution on $S^2$ with two isolated fixed points $(0,0,1)$ and $(0,0,-1)$. But since the $\mathbb{Z}_2^2$-action on $S^2$ determined by $\sigma$ and $\sigma'$ has no global fixed point, it is not equivariantly formal. We can also directly check that this $\mathbb{Z}_2^2$-action on $S^2$ is not locally standard.

Finally, we propose some questions on weakly equivariantly formal 2-torus manifolds.

**Question-3:** Does there exist a weakly equivariantly formal 2-torus manifold which is not equivariantly formal?

**Question-4:** If a 2-torus manifold is weakly equivariantly formal, are there any restrictions on the topology and combinatorial structure of its orbit space?

**Question-5:** Whether or not a 2-torus manifold being weakly equivariantly formal is determined only by the topology and combinatorial structure of its orbit space?

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**Conflict of interest**

The author declares that he has no conflict of interest.

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