A tree expansion formula of a homology intersection numbers on the configuration space $\mathcal{M}_{0,n}$

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Abstract

In [23], Sebastian Mizera discovered a tree expansion formula of a homology intersection number on the configuration space $\mathcal{M}_{0,n}$. The formula originates in a study of Kawai-Lewellen-Tye relation in string theory. In this paper, we give an elementary proof of the formula. The basic ingredients are the combinatorics of the real moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ and a combinatorial identity related to the face number of the associahedron.

1 Introduction

One of the distinguished properties of hypergeometric functions is that they enjoy integral representations whose integrands are elementary functions. These integrals are called hypergeometric integrals. The standard machinery of studying the hypergeometric integral is to regard it as a period integral, a pairing between a de Rham cohomology group and a homology group with local system coefficient. The latter object is also called twisted homology group in the context of special functions. Combining this viewpoint with Poincaré duality, the intersection theory of twisted homology groups naturally comes into play ([8], [9], [17], [18], [20], [21], [22], [23], [25], [26]).

Though a period integral can be defined for a more general class of integrals, a remarkable feature of the intersection theory of hypergeometric integral is that intersection numbers of specific homology classes have a combinatorial expression. In this paper, we are interested in the intersection number associated to the following integral:

$$\int_{\Delta} \prod_{1 \leq i < j \leq n} (t_i - t_j)^{s_{ij}} \omega. \quad (1)$$

Here, the ambient space of integration is the configuration space $\mathcal{M}_{0,n} := \text{Conf}_n(\mathbb{P}^1)/\text{GL}(2; \mathbb{C})$ of $n$-points on the Riemann sphere $\mathbb{P}^1$ with $\text{Conf}_n(\mathbb{P}^1) := \{(t_1, \ldots, t_n) \in (\mathbb{P}^1)^n \mid t_i \neq t_j \text{ for any } 1 \leq i < j \leq n\}$, $s_{ij}$ are complex parameters subject to a linear constraint, $\Delta$ is an integration cycle and $\omega$ is a meromorphic differential form on $\mathcal{M}_{0,n}$. Note that the action of $\text{GL}(2; \mathbb{C})$ on $\text{Conf}_n(\mathbb{P}^1)$ is given by möbius transform of each coordinate $t_i$.

If we normalize $t_1, t_{n-1}$ and $t_n$ to 0, 1 and $\infty$, $\mathcal{M}_{0,n}$ is the complement of the Selberg arrangement.

The real part $\mathcal{M}_{0,n}(\mathbb{R})$ of the configuration space $\mathcal{M}_{0,n}$ consists of finitely many connected components $\{\Delta(\alpha)\}_\alpha$ labeled by the quotient $\mathfrak{S}_n/D_n$ of the permutation group $\mathfrak{S}_n$. Here, $D_n$ is a subgroup of $\mathfrak{S}_n$ isomorphic to the dihedral group of order $2n$. We can

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regard $\Delta(\alpha)$ as an element of the twisted (Borel-Moore) homology group\(^1\) by specifying the determination of the multivalued function $\prod_{1 \leq i < j \leq n} |t_i - t_j|^{|\alpha|}$ and we write $[C^+(\alpha)]$ for the homology class determined by $\Delta(\alpha)$ in this sense. The set $\{[C^+(\alpha)]\}_\alpha$ generates the twisted homology group. If we write $[C^-(\alpha)]$ for the homology class obtained by replacing $s_{ij}$ by $-s_{ij}$ in the definition of $[C^+(\alpha)]$, it is natural to expect that the homology intersection number $\langle [C^+(\alpha)], [C^-(\beta)] \rangle_h$ has a combinatorial formula. The study of the homology intersection number $\langle [C^+(\alpha)], [C^-(\beta)] \rangle_h$ is not new. For example, the authors of \cite{25} describes the recursive structure of the intersection number with respect to the natural fibration $\mathcal{M}_{0,n} \to \mathcal{M}_{0,n-1}$. The usual Selberg integral \cite{23} appears if we specialize the parameters $s_{ij}$ of $\{1\}$ in a specific manner. In this case, a symmetric group acts on the twisted homology group and the invariant part of the (dual) twisted homology group is a one-dimensional vector space spanned by the class $[C^\pm] := \sum_{\alpha \in \mathfrak{S}_{n-3}} [C^\pm(\alpha)]$. Here, the subset $\mathfrak{S}_{n-3} \subset \mathfrak{S}_n/D_n$ of permutations of $\{1, \ldots, n\}$ whose element fixes $1, n-1$ and $n$ corresponds to the bounded chambers of the Selberg arrangement. The authors of \cite{19} evaluated the self intersection number $\langle [C^+], [C^-] \rangle_h$ in terms of sine functions of the parameters.

In \cite{23}, Sebastian Mizera discovered yet another formula of the intersection number $\langle [C^+(\alpha)], [C^-(\beta)] \rangle_h$: a tree expansion formula. Tree diagrams naturally appear from the fact that the closure of $\mathcal{M}_{0,n}$ is the Deligne-Knudsen-Mumford compactification $\overline{\mathcal{M}}_{0,n}^2$ \cite{15} is the associahedron \cite{20}, whose faces are in one-to-one correspondence to a set of trees. What is remarkable in his formula is the fact that only a few tree diagrams actually contribute to the homology intersection number $(C^+(\alpha), C^-(\beta))_h$. Namely, we only need to focus on the trees of which the valency at any internal vertex is odd. We call such a tree diagram an admissible tree. Following \cite{23}, we set $\langle C^+(\alpha), C^-(\beta) \rangle_h = \left( \sqrt{\frac{1}{2}} \right)^{n-3} m(\alpha|\beta)^3$. Let us explain how the formula looks like when both $\alpha$ and $\beta$ are taken to be the identity permutation $\underline{n} := 12\cdots n$. If $T$ is an admissible tree, we assign a Catalan number $C_{|e|\leq 3}$ to each internal vertex $v$ and assign a linear combination $s_v$ of the parameters $s_{ij}$ to each internal edge $e$. Then, the formula of \cite{23} Lemma 4.1 computes the number $m(\underline{n}|\underline{n})$ as a sum of terms of the form

$$\prod_{\text{winternal vertex of } T} \prod_{\text{cinternal edge of } T} C_{|e|\leq 3} \cot(\pi s_v)$$

for all admissible trees $T$. A precise formulation is given in \cite{14} of this paper.

Interestingly, the tree expansion formula was discovered in physical context. The Kawai-Lewellen-Tye relation in string theory is a manifestation of open/closed string duality and it expresses the closed string amplitude in terms of a quadratic combination of open string amplitudes. We can arrange the coefficients of the quadratic combination into a square matrix which we call the KLT kernel. It turns out that, at tree-level, KLT relation can be reformulated as the twisted period relation \cite{14} THEOREM 2, \cite{10} (5.1), \cite{19} Theorem 6.2), a relation among period integrals, twisted homology and cohomology intersection numbers (\cite{23} (3.19)). Thus, the inverse of the KLT kernel can be seen as the twisted homology intersection matrix. The tree expansion formula of $m(\alpha|\beta)$ in \cite{23} Lemma 4.1, Theorem 4.1 is then an $\alpha'$-correction of the formula of the field theory limit of the inverse of the KLT kernel obtained in \cite{5}. More precisely, if we replace the parameters

\footnote{In this paper, we assume that $s_{ij}$ are generic parameters in a sense that will be clarified later (see \cite{14}). Under this condition, the twisted homology group is canonically isomorphic to its Borel-Moore counterpart.}

\footnote{In \cite{23}, the Deligne-Knudsen-Mumford compactification $\overline{\mathcal{M}}_{0,n}$ is denoted by $\mathcal{M}_{0,n}$.}

\footnote{More precisely, $m(\alpha|\beta)$ should be denoted by $m_1(\alpha|\beta)$ (\cite{23}, \cite{24}).}
by rescaled ones $\alpha'^{n-3}$ computes the field theory inverse KLT kernel \cite[(3.2)]{24}. At the limit, we only have the contribution from all trivalent tree diagrams which amounts to focusing on the vertices of the associahedron.

Unfortunately (for mathematicians), the argument in \cite{23} makes use of physical intuition developed in \cite{24} which deals with the $\alpha'$-correction of the bi-adjoint scalar amplitude discussed in \cite{5}. The aim of this paper is to provide a math-friendly proof of the tree expansion formula of \cite[Lemma 4.1, Theorem 4.1]{23}. Once we recall the well-known cell decomposition of $M_{0,n}(\mathbb{R})$ \cite{6}, it is easy to see that the homology intersection number $\langle C^+(\alpha), C^-(\beta) \rangle_h$ has a tree expansion formula. Indeed, it has already appeared in \cite[Lemma 1]{22} in a slightly different form. However, the sum is taken over all tree diagrams and we have sine-like functions in the summand. Therefore, it is important to see why many terms cancel each other in the cotangent expansion. The answer is a simple, but non-trivial combinatorial identity (Theorem 7.1). Combining Theorem 7.1 with the combinatorics of the real moduli space $M_{0,n}(\mathbb{R})$, we obtain the desired formula. Since the proof turned out to be short and concise, we expect an analogous formula of homology intersection numbers of Coxeter arrangements \cite{11}. This aspect will be discussed in a forthcoming paper.

The author thanks Sebastian Mizera for letting me know his formula \cite[Theorem 4.1]{23}, asking me if there is a short mathematical proof of the result and many other valuable comments. The author also thanks Genki Shibukawa for reporting to me a simpler proof of Theorem 7.1 than the one we originally obtained. With his permission, we include his proof as the second proof of Theorem 7.1. This work is supported by JSPS KAKENHI Grant Number 19K14554 and JST AIP-PRISM Grant.

2 Convention for faces of associahedron

In this section, we introduce some basic notation related to the associahedron. Let $n \geq 3$ be an integer and let us consider a sequence of letters $12 \cdots n-1$. A bracket $a$ is a consecutive digits $a = i \cdots j$ ($1 \leq i < j \leq n - 1$) which is not $12 \cdots n-1$. We write $|a|$ for the length $j - i + 1$ of $a$. A bracketing $F$ of letters $12 \cdots n - 1$ is a collection of brackets such that for any pair of elements $a, a' \in F$ either $a \cap a' = \emptyset$, $a \subset a'$, or $a' \subset a$ is true.

**Definition 2.1** (associahedron). The associahedron (or Stasheff polytope) $K_{n-1}$ is a convex polytope of dimension $n - 3$ whose face poset is isomorphic to that of bracketings of $n - 1$ letters $12 \cdots n - 1$, ordered so that $F \prec F'$ if $F$ is obtained from $F'$ by adding new brackets.

If $F$ is a face of associahedron $K_{n-1}$, we write $F < K_{n-1}$. A face $F < K_{n-1}$ corresponds to a polyhedral subdivision of a planer convex $n$-gon whose edges are labeled by $1, 2, \ldots, n$ in a clockwise order on a unit circle which we regard as the boundary of a unit disk. We define codim $F$ as the number of brackets appearing in $F$. We set dim $F := n - 3 - \text{codim} F$. Another interpretation of a face $F < K_{n-1}$ is a rooted tree embedded in a unit disk of which the external vertices are labeled by $1, 2, \ldots, n$ in a clockwise manner. The label $n$ corresponds to the root vertex. The rule is as follows:

1. Write a planer $n$-gon whose edges are labeled by $1, \ldots, n$ in a clockwise way.

2. To each polygon $\Delta_i$ in $F$, we associate a vertex $v_i$ located at the barycenter of $\Delta_i$. Draw external edges from $v_i$ to edges of $\Delta_i$. Finally, if $\Delta_i$ and $\Delta_j$ share an edge, we connect $v_i$ and $v_j$.
Figure 1: The tree and the polyhedral subdivision corresponding to $1((23)4)$

By abuse of notation, the resulting graph is still denoted by $F$. An unlabeled vertex $v$ of $F$ is called an internal vertex and the symbol $V(F)_{\text{int}}$ denotes the set of internal vertices. An edge $e = (v_1, v_2)$ of $F$ is called an internal edge if both $v_1$ and $v_2$ are internal vertices. The set of internal edges of $F$ is denoted by $E(F)_{\text{int}}$. The valency (or degree) $|v|$ of a vertex $v$ is the number of edges containing $v$. If we take any internal edge $e$ from $F$, $F$ is decomposed into a pair of connected components. The component which does not contain the external vertex $n$ defines a subset of labels $a \subset \{1, \ldots, n-1\}$ consisting of consecutive numbers. This $a$ corresponds to a bracket in $1 \cdots (n-1)$. Conversely, any bracket $a \in F$ is obtained from an internal edge in this fashion.

Trees are identified with each other under the dihedral symmetry. With this in mind, we can also view the tree diagram $F$ as a bracketing of $(i+1) \cdots n1 \cdots (i-1)$ for any $i = 1, \ldots, n-1$. For example, the tree in Figure 1 can be identified with $1((23)4) = (5(23)4)5 = 3(4(51)) = (23)4 = (51)(23)$. The connected component of the complement of an internal edge $e$ which does not contain the external vertex $i$ corresponds to a bracket in $(i+1) \cdots n1 \cdots (i-1)$. If we regard $F$ as a bracketing of $(i+1) \cdots n1 \cdots (i-1)$, any bracket is obtained from an internal edge in this fashion. In the following, we regard a face $F < K_{n-1}$ as a bracketing of the digits $1 \cdots (n-1)$ unless otherwise stated.

**Definition 2.2.** A face $F$ is said to be admissible if for each internal vertex $v$, the valency $|v|$ is odd.

We conclude this section by recalling a well-known

**Proposition 2.3** (§2 of [29]). Any face $F < K_{n-1}$ is isomorphic to a product of associahedra. To be more precise, one has an isomorphism

$$F \simeq \prod_{v \in V(F)_{\text{int}}} K_{|v|-1}. \quad (3)$$

For readers’ convenience, let us explain the meaning of the isomorphism (3). If $F' < F$ is a face, $F'$ defines a polyhedral subdivision of a planer convex $n$-gon which is a refinement of $F$. Therefore, $F'$ induces a polyhedral subdivision of each polygon appearing in $F$. It is easy to see that the valency $|v|$ at a vertex $v \in V(F)_{\text{int}}$ is equal to the number of edges of the polygon containing $v$. Thus, the proposition follows.
3 Real moduli space $\overline{M}_{0,n}(\mathbb{R})$ as a patchwork of associahedra ([6], [11], [30])

In this section, we briefly recall the combinatorics of the real part $\overline{M}_{0,n}(\mathbb{R})$ of the moduli space $\overline{M}_{0,n}$ of stable pointed curves of genus 0. The readers can refer to [6], [11] or [30] for proofs and more explanations.

Let $\mathbb{P}^1$ denote the complex projective line. We set $\text{Conf}_n(\mathbb{P}^1) := \{(t_1, \ldots, t_n) \in (\mathbb{P}^1)^n \mid t_i \neq t_j (1 \leq i < j \leq n)\}$. The group $\text{GL}(2; \mathbb{C})$ acts on $\text{Conf}_n(\mathbb{P}^1)$ through möbius transform of each coordinate $t_i$. The quotient $\text{Conf}_n(\mathbb{P}^1)/\text{GL}(2; \mathbb{C})$ is denoted by $\overline{M}_{0,n}$. Moving $t_1, t_{n-1}$ and $t_n$ to 0, 1 and $\infty$, we have an identification $\overline{M}_{0,n} \cong \{(t_2, \ldots, t_{n-2}) \in \mathbb{C}^{n-3} \setminus \bigcup_{i=2}^{n-2} \{t_i(t_i-1) = 0\} \cup \bigcup_{2 \leq i < j \leq n-2} \{t_i = t_j\}\}$.

We set $\Delta := \{0 < t_2 < \cdots < t_{n-2} < 1\} \subset \overline{M}_{0,n}$. The permutation group $\mathfrak{S}_n$ of $\{1, \ldots, n\}$ naturally acts on $\overline{M}_{0,n}$ by $\alpha \cdot (t_1, \ldots, t_n) := (\alpha^{-1}(1), \ldots, \alpha^{-1}(n))$. For any element $\alpha \in \mathfrak{S}_n$, we set $\Delta(\alpha) := \alpha \cdot \Delta$. These chambers $\Delta(\alpha)$ cellulate the real part of $\overline{M}_{0,n}$.

The Deligne-Knudsen-Mumford compactification $\overline{M}_{0,n}$ of $M_{0,n}$ is a smooth projective variety defined over $\mathbb{Q}$ ([15]). In this paper, we simply regard it as a complex variety. The complex structure determines the set of real points $\overline{M}_{0,n}(\mathbb{R})$, which was investigated in detail in [6]. For our purpose, it is important to recall the cell decomposition of $\overline{M}_{0,n}(\mathbb{R})$ as a patchwork of associahedra. Any element $\alpha \in \mathfrak{S}_n$ is a bijection of the set $\{1, \ldots, n\}$ and we identify $\alpha$ with the number sequence $\alpha(1)\alpha(2)\cdots\alpha(n)$. In this sense, let $D_n$ be a subgroup of $\mathfrak{S}_n$ generated by two elements $23\cdots n1$ and $n(n-1)\cdots 1$. It is easy to see that $D_n$ is isomorphic to the dihedral group of order $2n$. Moreover, the natural inclusion $\mathfrak{S}_{n-1} \hookrightarrow \mathfrak{S}_n$ induces an isomorphism $\mathfrak{S}_{n-1}/(n-1)(n-2)\cdots 1 \cong \mathfrak{S}_n/D_n$. The closure of the cell $\Delta(\alpha)$ in $\overline{M}_{0,n}(\mathbb{R})$ gives rise to an associahedron for which we write $K(\alpha)$. Let us choose a representative $[\alpha] \in \mathfrak{S}_n/D_n$ so that $\alpha(n) = n$. Then, the set of brackets on $\alpha(1)\cdots\alpha(n-1)$ forms an associahedron which we identify with $K(\alpha)$. $K(\alpha)$ and $K(\beta)$ are identified in $\overline{M}_{0,n}(\mathbb{R})$ precisely when the equivalence classes $[\alpha]$ and $[\beta]$ are identical in the quotient $\mathfrak{S}_n/D_n$. On the other hand, any face $F_1 < K(\alpha)$ is identified with a face $F_2 < K(\beta)$ in $\overline{M}_{0,n}(\mathbb{R})$ precisely when the corresponding polyhedral subdivisions of a planar convex $n$-gon are related to each other by a sequence of twists along diagonals in the sense of [6] §3.1. If we regard $F_1$ and $F_2$ as labeled trees, they are identified in $\overline{M}_{0,n}(\mathbb{R})$ precisely when one is obtained by the other by a sequence of twists around an internal edge (Figure 2). In particular, twists do not change the set of internal/external vertices nor the set of internal/external edges.

![Figure 2: A twist along a diagonal amounts to taking an isomer](image)

Here, it is an important question how we can compute the intersection $K(\alpha) \cap K(\beta)$
in $\mathcal{M}_{0,n}(\mathbb{R})$. The answer is quite simple. One can describe the intersection $K(\alpha) \cap K(\beta)$ in terms of a tree diagram. The graphical rule is simple and effective. We call this rule CHY rule since the same figure appeared in the paper [3] by Cachazo, He and Yuan. CHY rule was further investigated in [24]. Let the symbol $I_n$ denote the identity permutation $12\cdots n$. Since we have an identity $K(\alpha) \cap K(\beta) = \alpha \cdot (K(I_n) \cap K(\alpha^{-1}\beta))$, we are reduced to the case of $K(I_n) \cap K(\alpha)$. CHY rule is defined as follows:

1. Draw a circle with $n$ marked points $1, 2, \ldots, n$ arranged in a clockwise order. Connect the labeled points $\alpha(i)$ and $\alpha(i+1)$ for any $i = 1, \ldots, n$ by a segment. Here, we put $\alpha(n+1) := \alpha(1)$.

2. We have several polygons inside the circle. For any pair of polygons $\Delta$ and $\Delta'$, we write $\Delta \sim \Delta'$ if there is a sequence of polygons $\Delta = \Delta_0, \Delta_1, \ldots, \Delta_k = \Delta'$ such that $\Delta_i$ and $\Delta_{i+1}$ share a vertex and are in the diagonal position for $i = 0, \ldots, k-1$. Pick any polygon $\Delta$ with at least one marked point as a vertex. We write $\Delta \sim \Delta'$ for the set of polygons $\Delta'$ such that $\Delta \sim \Delta'$.

3. Associate a vertex $v_i$ to the barycenter of each $\Delta_i$. Connect each $v_i$ to the marked points in $\Delta_i$. Connect a pair of vertices $v_i$ and $v_j$ if $\Delta_i$ and $\Delta_j$ share a vertex and are in a diagonal position.

Here, we assumed that the marked points $1, 2, \ldots, n$ are in a general position. Namely, they are arranged so that any triplet of segments connecting $\alpha(i)$ and $\alpha(i+1)$ does not have an intersection. The rule above produces a graph $G$. One may notice that there is an ambiguity in the third step and $G$ is not uniquely determined. Nonetheless, we have a

**Proposition 3.1.** Let $G$ be the graph produced by CHY rule. The intersection $F = K(I_n) \cap K(\alpha)$ is non-empty if and only if $G$ is a tree. If $G$ is a tree, $G$ is uniquely determined and it is the tree diagram corresponding to the face $F$ of $K(I_n)$.

**Proof.** Recall that the equivalence class $[\alpha] \in \mathfrak{S}_n/D_n$ is uniquely determined by the intersection $F = K(I_n) \cap K(\alpha)$ if it is non-empty. Therefore, it is enough to prove that if the intersection $F$ is non-empty, the graph produced by CHY rule is a tree and it coincides with the tree diagram corresponding to $F$.

We regard $F$ as a collection of brackets in $12\cdots (n-1)$. We choose a maximal (with respect to inclusion) element $a \in F$. Let us consider a planer convex $n$-gon whose vertices are labeled by $1, 2, \ldots, n$ in a clockwise order. Since $a$ is a set of consecutive digits $a = i \cdots j$, we can flip the vertices $i \cdots j$ to obtain an hourglass. We replace $F$ by $F \setminus \{a\}$ and repeat this process. Each time we pick a maximal element $a \in F$, we flip the vertices contained in $a$ to obtain an hourglass with several sections. If we regard this hourglass as a tree diagram, it is the tree diagram corresponding to $F$. An example of $F = 12(34567)$ is illustrated in Figure 3. We can also recover the permutation $\alpha$ such that $K(I_n) \cap K(\alpha) = F$ as follows. We begin with the consecutive digits $\alpha = 12\cdots n$. Each time we take a maximal element $a = i \cdots j \in F$, we revert the digits $i \cdots j$ or $j \cdots i$ in $\alpha$. In the end, we arrive at a number sequence $\alpha(1) \cdots \alpha(n-1)\alpha(n)$. For example, if we take $F = 12(34567)$, the process is $12345678 \rightarrow 12765438 \rightarrow 12745638$. By construction, CHY rule applied to the permutation $\alpha$ produces the tree diagram corresponding to $F$.

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4One can also regard the hourglass as a bubble (§).
Remark 3.2. We call $\alpha \in S_n$ a standard representative if $\alpha$ is obtained by the following process: If $n = 3$, $\alpha = 123$ is the unique standard representative. Suppose $n \geq 4$. We begin with a consecutive digits $12 \cdots (n - 1)$. First, we choose a consecutive digits $i(i + 1)$ and revert them. We write $\alpha'(1)\alpha'(2)\cdots\alpha'(n - 1)$ for the resulting sequence. Then, we choose a consecutive digits $\alpha'(i')\alpha'(i' + 1)$ and revert it to $\alpha'(i' + 1)\alpha'(i')$ unless $(\alpha'(i'), \alpha'(i' + 1)) = (1, n - 1)$. We repeat this process and we arrive at a sequence $\alpha(1)\cdots\alpha(n - 1)$. A standard representative $\alpha \in S_n$ is then given by $\alpha(1)\cdots\alpha(n - 1)n$. For example, if $n = 4$, standard representatives are $1234$, $1324$, $2134$. Clearly, standard representatives give rise to a complete set of representatives of the quotient $S_n/D_n$. Under the identification $\mathcal{M}_{0,n} \simeq \{(t_2, \ldots, t_{n - 2}) \in \mathbb{C}^{n - 3}\} \cup \bigcup_{i=2}^{n-2} \{t_i(t_i - 1) = 0\} \cup \bigcup_{2 \leq i < j \leq n - 2} \{t_i = t_j\}$, each standard representative $\alpha \in S_n$ corresponds to a chamber $\Delta(\alpha) = \{t_{\alpha(1)} < \cdots < t_{\alpha(n-1)}\}$ where we have set $t_1 = 0$ and $t_{n-1} = 1$.

Example 3.3. We take $\alpha = 134256 \in S_6$. CHY rule produces the graph as in Figure 4.

Let us take $\alpha = 146325 \in S_6$. In this case, the graph produced by the CHY rule depends on the configuration of the marked points 1, 2, \ldots, 6 (Figure 5). However, it has a cycle in any case. This means that the intersection $K(\mathbb{I}_6) \cap K(146325)$ is empty in $\mathcal{M}_{0,6}(\mathbb{R})$. 

Figure 3: Producing an hourglass with several sections

Figure 4: The tree corresponding to the intersection $K(\mathbb{I}_6) \cap K(134256)$.
Let us recall the definition of the relative winding number \( w(\alpha|\beta) \) \((23)\). The rule is to first draw the permutation \( \alpha \) on a circle in a clockwise order, and then follow the points according to the other permutation \( \beta \) by always going clockwise. The relative winding number \( w(\alpha|\beta) \), is then given by the total number of cycles completed. As an example, we have \( w(I_5|31425) = 3 \) as in Figure 6.

4 A tree expansion formula

We consider complex parameters \( s_{ij} \) for \( 1 \leq i < j \leq n \). For any pair \( i < j \), we set \( s_{ji} := s_{ij} \). We assume the following constraint on the complex parameters \( s_{ij} \):

\[
\sum_{1 \leq j \leq n, j \neq i} s_{ij} = 0 \quad (i = 1, \ldots, n).
\]

We take \( p \geq 3 \) numbers \( 0 \leq \alpha_1 < \cdots < \alpha_p < n \). We define consecutive digits \( a_1, \ldots, a_{p+1} \) by \( a_1 := \alpha_1 + 1 \cdots \alpha_2, a_2 := \alpha_2 + 1 \cdots \alpha_3, \ldots, a_p := \alpha_p + 1 \cdots n 1 \cdots \alpha_1 \). We regard the associahedron \( K_{p-1} \) as a set of brackets of digits \( a_1 a_2 \cdots a_{p-1} \). In this sense, \( K_{p-1} \) is also denoted by \( K_{a_1 \cdots a_{p-1}} \). For any bracket \( e = (a_1 \cdots a_j) \), we set \( s_e := s_{a_i a_{i+1} a_{j+1} a_{j+2} \cdots a_j} = \sum_{\alpha_i < k < \ell \leq \alpha_j} s_{kl} \). Since any face \( F < K_{a_1 \cdots a_{p-1}} \) can be reinterpreted as a tree diagram, we set

\[
m(a_1, \ldots, a_p) := \sum_{F < K_{a_1 \cdots a_{p-1}}: \text{admissible}} \prod_{v \in V(F)_{\text{int}}} C_{\frac{|v|-3}{2}} \prod_{e \in E(F)_{\text{int}}} \cot \pi s_{e}.
\]

Here, \( C_k \) denotes the \( k \)-th Catalan number. We also write \( m(a_1, \ldots, a_p, a_1, \ldots, a_{i-1}) \) for \( m(a_1, \ldots, a_p) \). Note that \( m(a_1, a_2, a_3) = 1 \) if \( a_1a_2a_3 = 12 \cdots n \).
Now suppose that a face \( F < K_{n-1} = K_{12\ldots n-1} \) is given. For any internal vertex \( v \) of \( F \), we define the consecutive digits \( a_1, \ldots, a_{|v|} \) as follows: we consider all the edges \( e_1, \ldots, e_{|v|} \) containing \( v \). If we remove an edge \( e \) from \( F \), \( F \setminus e \) is decomposed into two connected components. We consider the component \( C \) which does not contain this vertex \( v \). The component \( C \) has labeled vertices \( i \cdots j \) which is a sequence of consecutive numbers. Here, we regard a sequence of the form \( aa \cdots b \) \((b < a)\) as consecutive numbers. We set \( a_e := i \cdots j \). Without loss of generality, we may assume that \( a_{e_1} \cdots a_{e_{|v|}} = 1 \cdots n \) in a circular sense, namely, we identify \( 1 \cdots n \) and so on. We set

\[
m_v(F) := m(a_{e_1}, \ldots, a_{e_{|v|}}).
\]

(6)

If we use a sequence \( \alpha(1) \cdots \alpha(n) \) instead of a sequence \( 12 \cdots n \) for some \( \alpha \in \mathcal{S}_n \) and \( F < K(\alpha) \simeq K(\alpha(1) \cdots \alpha(n-1)) \), we write \( m^\alpha_v(F) \). Note that if \( e \) is an internal edge of \( F < K(\alpha) \) and if \( I \) is a connected component of the complement \( F \setminus e \), we have

\[
s_e = \sum_{i,j \in I, i<j} s_{ij}.
\]

(7)

Note that the formula (7) does not depend on the choice of the connected component \( I \) in view of (4). We set

\[
\Phi(t_1, \ldots, t_n) := \prod_{1 \leq i < j \leq n} (t_j - t_i)^{s_{ij}}.
\]

(8)

We write \( \mathbb{C}\Phi \) for the local system on \( \text{Conf}_n(\mathbb{P}^1) \) whose local section is a determination of the multi-valued function \( \Phi \). In view of (4), \( \mathbb{C}\Phi \) induces a local system on \( \mathcal{M}_{0,n} \) which is still denoted by the same symbol. We write \( \mathbb{C}\Phi^{-1} \) for the dual local system of \( \mathbb{C}\Phi \). We are interested in computing the twisted homology intersection form at the middle dimension

\[
(\bullet, \bullet)_h : H_{n-3}(\mathcal{M}_{0,n}; \mathbb{C}\Phi) \times H_{n-3}^l(\mathcal{M}_{0,n}; \mathbb{C}\Phi^{-1}) \to \mathbb{C}.
\]

(9)

Here, the superscript \( l \) stands for the word “locally finite” and \( H_{n-3}^l(\mathcal{M}_{0,n}; \mathbb{C}\Phi) \) denotes the locally finite (or Borel-Moore) homology group. We say that the regularization condition is satisfied if the canonical morphism

\[
H_k(\mathcal{M}_{0,n}; \mathbb{C}\Phi) \to H_k^l(\mathcal{M}_{0,n}; \mathbb{C}\Phi)
\]

(10)

is an isomorphism for any \( k \). If the regularization condition is satisfied, both the homology group \( H_k(\mathcal{M}_{0,n}; \mathbb{C}\Phi) \) and the locally finite homology group \( H_k^l(\mathcal{M}_{0,n}; \mathbb{C}\Phi) \) vanish unless \( k = n-3 \). The inverse map of the canonical morphism (10) is denoted by \( \text{reg} \). In order to justify the regularization condition, let us recall the following result.

**Proposition 4.1 (3).** Let \( \mathcal{A} = \{H\} \) be a hyperplane arrangement in \( \mathbb{C}^n \) and let \( \mathcal{A}_\infty := \{H\} \cup H_\infty \) be its associated projective arrangement where \( H_\infty \) is the hyperplane at infinity. Let \( l_H \) be linear forms defining \( H \). We consider a local system \( \mathcal{L} = \mathbb{C}\prod_{H \in \mathcal{A}} l_H^{s_H} \) on the complement \( X := \mathbb{C}^n \setminus \mathcal{A} \) for some \( \alpha_H \in \mathbb{C} \). We set \( \alpha_{H_\infty} := -\sum_{H \in \mathcal{A}} \alpha_H \).

If for any dense edge \( E \in D(\mathcal{A}_\infty) \), the condition \( \alpha_E := \sum_{E \subseteq H \in \mathcal{A}_\infty} \alpha_H \notin \mathbb{Z} \) holds, then one has a canonical isomorphism

\[
H_k(X; \mathcal{L}) \simeq H_k^l(X; \mathcal{L})
\]

(11)

for any integer \( k \).
To be more precise, Proposition 4.1 follows from [3] lemma 3] combined with the composition theorem of derived functors. We assume the following condition:

Condition (\ast)

For any element \([\alpha] \in \mathcal{S}_n/D_n\), any face \(F < K(\alpha)\) and for any internal edge \(e\) of \(F\), the complex number \(s_e\) is not an integer.

In view of the fact that \(\overline{M}_{0,n}\) can be realized as an iterated blowing-up of \(\mathbb{P}^{n-3}\) along dense edges of a hyperplane arrangement ([6] §4, [11] §4, [12] Chapter 4) and each edge \(e\) corresponds to a dense edge at which the sum of relevant exponents \(s_{ij}\) is precisely \(s_e\), the regularization condition is satisfied under the condition (\ast). Namely, both \(H_k(M_{0,n}; \mathbb{C} \Phi^\pm)\) and \(H^f_{n-3}(M_{0,n}; \mathbb{C} \Phi^\pm)\) are zero unless \(k = n - 3\) and we have a natural isomorphism \(H_{n-3}(M_{0,n}; \mathbb{C} \Phi^\pm) \simeq H^f_{n-3}(M_{0,n}; \mathbb{C} \Phi^\pm)\).

Now let us recall the orientable double cover \(\overline{M}_{0,n}(\mathbb{R})\) of the real moduli space \(M_{0,n}(\mathbb{R})\) constructed in [7]. Let \(\pi : \overline{M}_{0,n}(\mathbb{R}) \to M_{0,n}(\mathbb{R})\) be the covering map of loc. cit. and let us fix an orientation of \(M_{0,n}(\mathbb{R})\). In loc. cit., the authors fix one point \(t_n = \infty\) which amounts to the natural bijection \(\mathcal{S}_{n-1}/((n-1)(n-2)\cdots1) \simeq \mathcal{S}_n/D_n\). Note that \(\mathcal{S}_{n-1}\) can be identified with a subgroup of \(\mathcal{S}_n\) consisting of permutations \(\alpha\) such that \(\alpha(n) = n\). By construction, the preimage \(\pi^{-1}(M_{0,n}(\mathbb{R}))\) is a disjoint union of \((n-1)!\) copies of the associahedron \(K_{n-1}\) labeled by the elements of \(\mathcal{S}_{n-1}\). Let the symbol \(C(\alpha)\) denote the associahedron in \(\pi^{-1}(M_{0,n}(\mathbb{R}))\) labeled by an element \(\alpha \in \mathcal{S}_{n-1} \simeq \mathcal{S}_n/\langle23\cdots n1\rangle\). We have \(\pi(C(\alpha)) = \Delta(\alpha)\). The orientation of \(C(\alpha)\) is naturally induced from that of \(\overline{M}_{0,n}(\mathbb{R})\). By abuse of notation, we write \(C(\alpha)\) for the image of \(C(\alpha)\) through the morphism \(\pi\) in \(M_{0,n}(\mathbb{R})\). We choose the standard loading of the multivalued function \(\Phi\) on \(C(\alpha)\), that is, we choose the branch of \(\Phi\) so that we have \(\Phi > 0\) on \(C(\alpha)\) when all the parameters \(s_{ij}\) are real. With this choice of a branch of \(\Phi\), \(C(\alpha)\) defines an element of the locally finite homology group \(H^f_{n-3}(M_{0,n}; \mathbb{C} \Phi)\) which is denoted by \([C^+(\alpha)]\). The same argument defines a homology class \([C^- (\alpha)]\) of \(H^f_{n-3}(M_{0,n}; \mathbb{C} \Phi^-)\) for any \(\alpha \in \mathcal{S}_{n-1} \simeq \mathcal{S}_n/\langle23\cdots n1\rangle\). In view of [10], we see that the twisted homology group \(H^f_{n-3}(M_{0,n}; \mathbb{C} \Phi^\pm)\) is generated by \([\{C^\pm(\alpha)\}]_{\alpha \in \mathcal{S}_{n-1}}\). As a basis of \(H^f_{n-3}(M_{0,n}; \mathbb{C} \Phi^\pm)\), one can take, for example, \([\{C^\pm(\alpha)\}]_{\alpha \in \mathcal{S}_{n-3}}\) where \(\mathcal{S}_{n-3}\) is identified with the set of permutations which fix \(1, n-1\) and \(n\). This is a basis consisting of bounded chambers. For the cycles \([C^\pm(U_n)] \in H^f_{n-3}(M_{0,n}; \mathbb{C} \Phi^\pm)\), we have the following formula.

**Theorem 4.2.** We assume the condition (\ast). Then, one has a formula

\[
\langle \text{reg}(C^+(U_n)), [C^- (U_n)] \rangle_h = \left(\frac{\sqrt{-1}}{2}\right)^{n-3} \prod_{e \in V(F)_{\text{int}}} C_{\frac{|e|}{2}} \prod_{e \in E(F)_{\text{int}}} \cot(\pi s_e). \quad (13)
\]

More generally, we have a

**Theorem 4.3.** We assume the condition (\ast). For any \([\alpha], [\beta] \in \mathcal{S}_n \langle23\cdots n1\rangle\) with a
non-empty intersection \( K(\alpha) \cap K(\beta) = \emptyset \) in \( \mathcal{M}_{0,n}(\mathbb{R}) \), one has a formula

\[
\langle \text{reg}[C^+(\alpha)], [C^-(\beta)] \rangle_h = (-1)^{w(\alpha | \beta)+1} \left( \frac{\sqrt{-1}}{2} \right)^{n-3} \prod_{e \in E(F)_{\text{int}}} \csc(\pi s_e) \prod_{v \in V(F)_{\text{int}}} m_v^\alpha(F).
\]

(14)

Note that when \( \alpha = \beta \), \( F \) is equal to \( K(\alpha) \) and therefore, \( E(F)_{\text{int}} = \emptyset \) and \( V(F)_{\text{int}} \) is a single point. This case is reduced to Theorem \[1.2\]. In the following, we simply write \( \langle [C^+(\alpha)], [C^-(\beta)] \rangle_h \) for the homology intersection number \( \langle \text{reg}[C^+(\alpha)], [C^-(\beta)] \rangle_h \).

5 Proof of Theorem 4.2

Proof. By an induction on the natural number \( n \), we can prove an identity

\[
\frac{1}{(T_1 - 1) \cdots (T_n - 1)} = \frac{1}{2^n} \sum_{k=0}^{n} (-1)^{n-k} \sum_{I \subset \{1, \ldots, n\}} \prod_{i \in I} T_i + 1 \prod_{i \in I} T_i - 1.
\]

(15)

We set \( e(\alpha) := \sqrt{-1} e^{2\pi \sqrt{-1} \alpha} \) and \( t(\alpha) := \frac{e(\alpha)+1}{e(\alpha)-1} \). For a vector \( s = (s_1, s_2, \ldots) \), we set \( t(s) := t(s_1)t(s_2) \cdots \). For any bracket \( a := (i \cdots j) \), we set \( s_a := s_{i-j} := \sum_{i \leq k \leq j} s_k \). Since any face \( F < K_{n-1} \) is a set of brackets, we set \( s_F := (s_a)_{a \in F} \). Let us recall the construction of \( \mathcal{M}_{0,n} \) as an iterated blowing-up of \( \mathbb{P}^{n-3} \) [18 §4, [21] §4, [22] Chapter 4]. Any bracket \( a = i \cdots j \) in \( 12 \cdots n-1 \) corresponds to the proper transform of the linear subvariety \( \{ t_i = \cdots = t_j \} \) in \( \mathcal{M}_{0,n} \) around which the local system \( \mathbb{C} \Phi^\pm \) has the eigenvalue of local monodromy \( e(\pm s_a) \). Therefore, as in [22] Lemma 1], we obtain

\[
\langle [C^+(I_n)], [C^-(I_n)] \rangle_h = (-1)^{n-3} \sum_{F < K_{n-1}} \prod_{a \in F} \frac{1}{e(s_a) - 1},
\]

(16)

which can be deduced from [14] p175, Proposition]. On the other hand, we obtain a formula

\[
\prod_{a \in F} \frac{1}{e(s_a) - 1} = \frac{1}{2 \text{codim} F} \sum_{F' \subset G} (-1)^{\text{dim} F - \text{dim} F'} t(s_{F'})
\]

(17)

in view of \[15\]. Comparing \[16\] and \[17\], we can expand the intersection number \( \langle [C^+(I_n)], [C^-(I_n)] \rangle_h \) into a sum of \( t(s_F)'s \). The coefficient of \( t(s_F) \) is given by

\[
(-1)^{n-3} \sum_{F' < F} \frac{1}{2 \text{codim} F'} (-1)^{\text{dim} F' - \text{dim} F} = \frac{(-1)^{\text{codim} F}}{2^{n-3}} \sum_{F' < F} (-2)^{\text{dim} F'}.
\]

(18)

We set \( C_F := \sum_{F' < F} (-2)^{\text{dim} F'} \). In view of Proposition \[2.3\] any face \( F' \) of \( F \) is decomposed into a product of faces \( f_v \) of \( K_{|v|-1} \) as \( F' = \prod_{v \in V(F)_{\text{int}}} f_v \). Thus, we obtain

\[
C_F = \prod_{v} \sum_{f_v < K_{|v|-1}} (-2)^{\sum_{v} \text{dim} f_v} = \prod_{v \in V(F)_{\text{int}}} \left( \sum_{f_v < K_{|v|-1}} (-2)^{\text{dim} f_v} \right) = \prod_{v \in V(F)_{\text{int}}} C_{K_{|v|-1}}.
\]

(19)
Therefore, we are reduced to computing $C_{K_{n-1}}$. Since the number of $k$-codimensional faces of $K_{n-1}$ is \( \frac{1}{n-1} \binom{n-3}{k} \binom{n+k-1}{k+1} \) (3 LEMMA 3.2.1), we have a formula

\[
C_{K_{n-1}} = \frac{1}{n-1} \sum_{k=0}^{n-3} (-2)^{n-k} \binom{n-3}{k} \binom{n+k-1}{k+1}.
\]

(20)

In view of Theorem [7,4], we see that $C_F$ is zero unless $F$ is admissible. When $F$ is admissible, we obtain Theorem [1,2] again from Theorem [7,4].

6 Proof of Theorem [4.3]

Before going into the proof, let us first discuss the signature effect of blowing-up on twisted homology intersection numbers. Let $l_1, \ldots, l_q$ be non-constant linear polynomials in $r$-variables $x = (x_1, \ldots, x_r)$ with real coefficients. We assume that $l_1, \ldots, l_p$ do not have a constant term and $\bigcup_{i=1}^{p} \{ l_i = 0 \}$ is normal crossing. We consider domains $D_1, D_2 \subset \mathbb{R}^r$ specified by the following relations:

\[
D_1 : l_1 \geq 0, \ldots, l_p \geq 0; l_{p+1} \geq 0, \ldots, l_q \geq 0 \\
D_2 : l_1 \leq 0, \ldots, l_p \leq 0; l_{p+1} \geq 0, \ldots, l_q \geq 0 \\
\bar{D}_1 \cap \bar{D}_2 : l_1 = 0, \ldots, l_p = 0; l_{p+1} \geq 0, \ldots, l_q \geq 0
\]

for some integer $1 \leq p \leq r$. Here, $\bar{D}_i$ denotes the closure of $D_i$. It is important to observe that we can equip $D_1$ and $D_2$ with an orientation induced from that of $\mathbb{R}^r$. After blowing-up, the orientation may differ. Let us see how the orientation changes. Since it is a local problem, we may assume that $l_1 = x_1, \ldots, l_p = x_p, D_1 = \{ x_1 \geq 0, \ldots, x_p \geq 0 \}$ and $D_2 = \{ x_1 \leq 0, \ldots, x_p \leq 0 \}$. We consider the blowing-up $\pi : X \to \mathbb{C}^r$ of $\mathbb{C}^r$ along $\{ x_1 = \cdots = x_p = 0 \}$. Let us take $(w_1 := x_1, w_2 := \frac{x_2}{x_1}, \ldots, w_p := \frac{x_p}{x_1}, x_{p+1}, \ldots, x_r)$ as a local coordinate of $X$. On $\mathbb{R}^r$, $D_1$ and $D_2$ are oriented in such a way that the $r$-form $\omega := dx_1 \land \cdots \land dx_r$ is positive. Let us observe that $\pi^* \omega = w_1^{p-1} dw_1 \land \cdots \land dw_p \land dx_{p+1} \land \cdots \land dx_r$. The proper transforms of $D_1$ and $D_2$ are locally given by the equations

\[
\pi^{-1}(D_1) = \{ w_1 > 0, \ldots, w_p > 0 \} \\
\pi^{-1}(D_2) = \{ w_1 < 0, w_2 > 0, \ldots, w_p > 0 \}.
\]

(21)

(22)

We can equip $\pi^{-1}(D_i)$ ($i = 1, 2$) with an orientation $dw_1 \land \cdots \land dw_p \land dx_{p+1} \land \cdots \land dx_r > 0$. Then, $\pi : \pi^{-1}(D_1) \to D_1$ is orientation-preserving for any $p$. However, $\pi : \pi^{-1}(D_1) \to D_1$ is orientation-preserving if and only if $p$ is odd. Therefore, even-codimensional blowing-up add a signature ($-$) to the formula [14] p175, Proposition 1] of homology intersection numbers.

Now, we prove Theorem [4.3]. Observe that it is enough to evaluate $\langle [C^+(\mathbb{I}_n)], [C^-(\alpha)] \rangle_h$. Let us suppose that the intersection $F = K(\mathbb{I}_n) \cap K(\alpha)$ in $\mathcal{M}_{0, n}(\mathbb{R})$ is non-empty and $\alpha$ is a standard representative in the sense of Remark [3.2]. The formula [14] p175, Proposition 6 in our setting reads

\[
\langle [C^+(\mathbb{I}_n)], [C^-(\alpha)] \rangle_h = \prod_{a \in F} (-1)^{|a|-2} \left( \frac{1}{2\sqrt{-1}} \right)^{|F|} \prod_{a \in F} \csc(\pi s_a) \times (-1)^{\dim F} \sum_{F' \subset F} \prod_{a \in F'} \frac{1}{e(s_a) - 1}.
\]

(23)

\textsuperscript{6}This is a formula of a homology intersection number on a complement of a hyperplane arrangement in a projective space. However, since the computation of intersection is a local problem, we can apply the formula even after blowing-up the projective space. The signature effect of blowing-up must be taken into account.
Note that each bracket $a = (i \cdots j) \in F$ corresponds to a $(|a| - 1)$-codimensional blowing-up along $\{t_i = \cdots = t_j\}$. We have $\prod_{a \in F} (-1)^{|a| - 2} \times (-1)^{|F|} = \prod_{a \in F} (-1)^{|a| - 1} = (-1)^{w(l_n[\alpha] + 1)}$. In view of Proposition 2.3 and the proof of Theorem 4.2, we have an equality

$$(1 - \frac{\dim F}{2}) \sum_{F' \leq F, a \in F'} \frac{1}{e(s_a) - 1} = \left( \frac{\sqrt{-1}}{2} \right)^{\dim F} \prod_{w \in V(F)_{int}} m_w(F). \tag{24}$$

Combining (23) and (24), we obtain the desired formula (14). Lastly, if $\alpha$ is not a standard representative, we see that $\alpha' := \alpha \circ (n(n-1) \cdots 1) \circ (23 \cdots n1)^l$ for some integer $l$ is standard. Combining this observation with $(-1)^{w(l_n[\alpha'])} = (-1)^{w(l_n[\alpha] + n)}$ and $[C^-(\alpha')] = (-1)^n [C^{-}(\alpha)]$, we obtain (14).

7 A combinatorial identity

**Theorem 7.1.** For any non-negative integer $p$, we have

$$\frac{1}{p+2} \sum_{k=0}^{p} (-2)^{p-k} \binom{p}{k} \binom{p + k + 2}{k + 1} = \begin{cases} \left(-1\right)^\frac{p}{2} \frac{1}{2} C_{\frac{p}{2}} \quad (p : \text{even}) \\ 0 \quad (p : \text{odd}) \end{cases}. \tag{25}$$

We give two different proofs of Theorem 7.1. One is based on the Wilf-Zeilberger method ([27], [32]), the other is based on the method of generating function. We thank Genki Shibukawa for sharing the second proof.

(The first proof) We set $F(p, k) := \frac{1}{p+2} (-2)^{-k(p)} \binom{p + k + 2}{k + 1}$. We set $\Delta_p \cdot F(p, k) := F(p + 1, k)$, $R(p, k) := -\frac{8(p+1)(2p+5)}{(p+3)(p-k+1)(p-k+2)} G(p, k) := R(p, k) F(p, k)$. By a direct computation, we obtain

$$[(p + 4)\Delta^2_p + 4(p + 1)F(p, k)] = G(p, k + 1) - G(p, k). \tag{26}$$

We set $f(p) := \sum_{k=0}^{p} F(p, k) = \sum_{k=0}^{\infty} F(p, k)$. Since $F(p, k) = G(p, k) = 0$ if $k > p$, taking a summation of (26) gives rise to a difference equation

$$(p + 4) f(p + 2) + 4(p + 1) f(p) = 0. \tag{27}$$

On the other hand, we set

$$g(p) := \begin{cases} \left(-1\right)^\frac{p}{2} \frac{1}{2} C_{\frac{p}{2}} \quad (p : \text{even}) \\ 0 \quad (p : \text{odd}) \end{cases}. \tag{28}$$

It is easy to check that $g(p)$ is also a solution of (27). Since $f(0) = g(0)$ and $f(1) = g(1)$, we obtain $f(p) = g(p)$ for any positive integer $p$.

(The second proof) We first observe that the left-hand side of (25) is same as the sum

$$\frac{1}{p+1} \sum_{k=0}^{p} (-2)^{p-k} \binom{p + 1}{k + 1} \binom{p + k + 2}{k}. \tag{29}$$

which in turn equals to $\frac{1}{p+1} [u^p]_1 (1 - u)^{-2} \left( \frac{1 - 2u}{1-u} \right)^{p+1}$. Here, the symbol $[u^p] f(u)$ denotes the coefficient of $u^p$ in the Taylor series expansion of $f(u)$. We apply the so-called Lagrange-Bürmann formula ([31] p.129)]. In our setting, it is convenient to state it as follows: Let
and \( \phi \) be holomorphic functions defined at the origin. We assume that \( \phi(0) \neq 0 \). Let \( g(z) \) be the inverse function of \( \frac{u}{\phi(u)} \). Then, one has an equality
\[
[z^{p+1}]H(g(z)) = \frac{1}{p+1} [u^p]H'(u)\phi(u)^{p+1}
\]
for any non-negative integer \( p \). In our case, we set \( \phi(u) = \frac{1-2u}{1-u} \), \( H(u) = \frac{1}{1-u} \) and \( g(z) = \frac{1+2z-\sqrt{1+4z^2}}{2} \). The left-hand side of (30) is nothing but the sum (29). A simple computation shows that the right-hand side of (30) is that of (25). Note that the generating function \( \sum_{k=0}^{\infty} C_k z^k \) of Catalan numbers is given by \( \frac{1-\sqrt{1-4z^2}}{2z} \).

**Remark 7.2.** Let \( M_p(x; \beta, \gamma) \) be the \( p \)-th Meixner polynomial (\cite[p.346]{p}). (29) is nothing but the special value \( M_{p+1}(p; 2, \frac{1}{2}) \).

## 8 An illustrative example

Let us compute the intersection number \( \langle [C^+(I_6)], [C^-(134256)] \rangle_h \). The intersection \( K(I_6) \cap K(134256) \) is computed in Example 3.3. We have \( w(I_6|134256) = 2 \). Therefore, formula (14) reads
\[
\langle [C^+(I_6)], [C^-(134256)] \rangle_h = -\left(\frac{\sqrt{-1}}{2}\right)^3 \csc(\pi s_{34}) \csc(\pi s_{234}) \times
\]
\[
\begin{array}{c}
6 \\
1 \\
3 \\
5 \\
234 \\
\end{array}
\times
\begin{array}{c}
561 \\
34 \\
2 \\
4 \\
3 \\
\end{array}
\times
\begin{array}{c}
5612 \\
34 \\
2 \\
4 \\
3 \\
\end{array}
\]
\[
= -\left(\frac{\sqrt{-1}}{2}\right)^3 \csc(\pi s_{34}) \csc(\pi s_{234}) m(1, 234, 5, 6) m(2, 34, 561) m(3, 4, 5612)
\]
\[
= \frac{1}{16} \cdot \frac{1}{5} \cdot \frac{1}{234} \cdot \frac{1}{6} \cdot \frac{1}{234} \times \frac{1}{234} \cdot \frac{1}{6} \cdot \frac{1}{234} \cdot \frac{1}{234} 
\]
\[
= \cot(\pi s_{2345}) + \cot(\pi s_{1234}).
\]

Both \( m(2, 34, 561) \) and \( m(3, 4, 5612) \) are a summation over one point \( K_2 \) and they are both equal to 1. On the other hand, admissible trees in \( K_3 = K_{1,234,5} \) are precisely vertices and we have
\[
m(1, 234, 5, 6) = \frac{1}{6} \times \frac{1}{234} \times \frac{1}{234} \times \frac{1}{234} 
\]
\[
= \cot(\pi s_{2345}) + \cot(\pi s_{1234}).
\]
The interested readers can find more examples in [23] and [24].

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16
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