Discrepancy Analysis of a New Randomized Diffusion Algorithm

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Abstract

For an arbitrary initial configuration of discrete loads over vertices of a distributed graph, we consider the problem of minimizing the discrepancy between the maximum and minimum loads among all vertices. For this problem, this paper is concerned with the ability of natural diffusion-based iterative algorithms: at each discrete and synchronous time step on an algorithm, each vertex is allowed to distribute its loads to each neighbor (including itself) without occurring negative loads or using the information of previous time steps.

In this setting, this paper presents a new randomized diffusion algorithm like multiple random walks. Our algorithm archives $O(\sqrt{d \log N})$ discrepancy for any $d$-regular graph with $N$ vertices with high probability, while deterministic diffusion algorithms have $\Omega(d)$ lower bound. Furthermore, we succeed in generalizing our algorithm to any symmetric round matrix. This yields that $O(\sqrt{d_{\text{max}} \log N})$ discrepancy for arbitrary graphs without using the information of maximum degree $d_{\text{max}}$.

Key words: load balancing algorithms, diffusion, Markov chains

1 Introduction

This paper is concerned with the load balancing problem on distributed networks. Let $G = (V, E)$ be a connected graph with $N = |V|$ vertices, and let $X_v(0)$ denote an initial amount of loads on each $v \in V$. Then, from an arbitrary initial configuration of loads $X(0)$, we consider iterative algorithms which update the configuration of loads from $X(t)$ to $X(t+1)$ on each time step $t$ with the goal to minimize the discrepancy between the maximum and minimum loads among all vertices as well as possible. Especially, this paper focuses on the diffusion algorithms: in an update, each vertex distributes its loads to each neighbor synchronously. Because of not only its locality and simplicity, but also deep connection with the theory of multiple random walks and mixing time of Markov chains, diffusion algorithms have been well studied recently.

1.1 Previous works

Continuous loads: If the loads are continuous ($X_v(t) \in \mathbb{R}$), i.e. divisible, techniques to estimate the discrepancy corresponding to the theory of Markov chains have been well studied (See e.g. [13]). For example, on $d$-regular graphs, a natural diffusion algorithm such that each vertex $v$ sends $X_v(t)/(d+1)$ loads to each neighbor and keeps the same amount of loads for itself achieves a constant discrepancy after appropriate time steps. Strictly speaking, for any vector $\xi \in \mathbb{R}^N$, let

$$\text{Disc}(\xi) := \max_{v,u \in V} |\xi_v - \xi_u|,$$

(1)

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which represents the discrepancy between the maximum and minimum values of $\xi$ among all vertices. Then, $\text{Disc}(\mathcal{X}^{(T)}) = O(1)$ after $T := O(\log(\text{Disc}(\mathcal{X}^{(0)}) N) / (1 - \lambda_2))$ steps, where $\lambda_2$ is the second largest eigenvalue of the transition matrix of the graph. The convergence time $T$ is highly related to the mixing time of Markov chains (we will give precise discussions in Section 2).

**Discrete loads:** On the other hand, if the loads are discrete ($\mathcal{X}_v^{(t)} \in \mathbb{Z}$), i.e. indivisible, it is not easy to estimate the discrepancy compared with the continuous case although the analytic techniques are also related to the theory of Markov chains. Considering the discrete loads is important as practical settings and there are many previous works.

For example, it is easy to consider a natural discretization diffusion algorithm such that each vertex $v$ partitions $\mathcal{X}_v^{(t)} \in \mathbb{Z}$ into $d + 1$ integers $\left\lceil \frac{\mathcal{X}_v^{(t)}}{d+1} \right\rceil, \ldots, \left\lceil \frac{\mathcal{X}_v^{(t)}}{d+1} \right\rceil$, and sends each of them to each neighbor (including itself). Rabani et al. [11] gave a framework of the analysis to deal with the discrepancy of discrete load balancing algorithms including this SEND($\left\lceil \mathcal{X}_v^{(t)}/(d+1) \right\rceil$ or $\left\lceil \mathcal{X}_v^{(t)}/(d+1) \right\rceil$) algorithm. From their result, it was shown that $\text{Disc}(\mathcal{X}^{(T)}) = O(d \log N / (1 - \lambda_2))$ for any $d$ regular graph. Almost same but slightly refined upper bound on the discrepancy was given later by Shiraga et al. [14]. They showed $\text{Disc}(\mathcal{X}^{(T)}) = O(dt_{\text{mix}})$, where $t_{\text{mix}}$ is a mixing time of the graph. It is well known that $t_{\text{mix}} = O(\log N / (1 - \lambda_2)).$

Berenbrink et al. [4] studied a randomized diffusion algorithm to get smaller discrepancy with high probability, i.e. with probability larger than $1 - 1/N^c$ for some constant $c$. In their algorithm, each vertex $v$ sends $\left\lceil \mathcal{X}_v^{(t)}/(d+1) \right\rceil$ loads to each neighbor (including itself) firstly. Then the remaining loads are randomly sent one by one without replacing to neighbors (including itself). For this RS algorithm on $d$ regular graphs, they showed that $\text{Disc}(\mathcal{X}^{(T)})$ is bounded by $O \left( \left( d + \sqrt{\frac{\log d}{1 - \lambda_2}} \right) \sqrt{\log N} \right)$ and $O((d \log \log N) / (1 - \lambda_2))$ with high probability. Later, Sauerwald and Sun [12] gave an $O(d^2 \sqrt{\log N})$ discrepancy. Note that this bound is independent to the expansion of graphs, i.e. independent of the second largest eigenvalue $\lambda_2$. As an other randomized diffusion algorithm, Akbari and Berenbrink [1] dealt with a randomized ordering version of so called rotor-router model, and showed that $\text{Disc}(\mathcal{X}^{(T)}) = O((d \log \log N) / (1 - \lambda_2)).$

A recent progress on diffusion algorithms was given by Berenbrink et al. [5]. They gave a strong framework of the deterministic diffusion algorithms and analyzed the discrepancy. For example, for a proposed algorithm such that each vertex $v$ sends $\left\lceil \mathcal{X}_v^{(t)}/2d \right\rceil$ loads to each neighbor and keeps the remaining loads for itself, they showed that $\text{Disc}(\mathcal{X}^{(T)}) = O(d \sqrt{\log N / (1 - \lambda_2)})$ on $d$ regular graphs. It means that this SEND($\left\lceil \mathcal{X}_v^{(t)}/2d \right\rceil$) algorithm improves the upper bound of the discrepancy of the SEND($\left\lceil \mathcal{X}_v^{(t)}/(d+1) \right\rceil$ or $\left\lceil \mathcal{X}_v^{(t)}/(d+1) \right\rceil$). In a sense, SEND($\left\lceil \mathcal{X}_v^{(t)}/2d \right\rceil$) is a kind of lazy version of SEND($\left\lceil \mathcal{X}_v^{(t)}/(d+1) \right\rceil$ or $\left\lceil \mathcal{X}_v^{(t)}/(d+1) \right\rceil$). Laziness is a famous property used in the field of random walks (lazy random walk stays current vertex with probability larger than 1/2). Furthermore, they gave an algorithm which achieves $O(d)$ the discrepancy within $O(T + (\log N) / (1 - \lambda_2))$ step. In this algorithm, each vertex $v$ sends $\left\lceil \mathcal{X}_v^{(t)}/3d \right\rceil$ loads to each neighbor and keeps the remaining loads itself, where $\lfloor : \rfloor$ denotes rounding to the nearest integer. They also showed that the lower bound of the discrepancy is $\Omega(d)$ for the deterministic (stateless) discrete diffusion algorithms, hence the SEND($\left\lceil \mathcal{X}_v^{(t)}/3d \right\rceil$) gives a tight upper bound.

**On arbitrary graphs:** Since real computer networks often have the scale free property nowadays, the demand of studying load balancing algorithms on irregular graph is increasing.

For the continuous case, it is not too difficult to discuss the discrepancy since the theory of symmetric (or reversible) Markov chains has been established in the framework including irregular graphs. Strictly speaking, let $P \in [0, 1]^{V \times V}$ be a transition matrix (round matrix) on $V$. Then, for the algorithm such that each vertex $v$ sends $\mathcal{X}_v^{(t)} P_{v,u}$ loads to a neighbor $u$, the discrepancy converges to a constant within $O(T)$ step if $P$ is symmetric. For example, on an arbitrary graph, the algorithm such that each vertex $v$ sends
$\chi_v^{(t)}/d_{\text{max}}$ loads to each neighbor and keeps the remaining loads itself achieves this property, where $d_{\text{max}}$ is the maximum degree of the graph. If one would rather not use $d_{\text{max}}$ since this is a global variable, then the algorithm with $P_{v,u} = 1/\min\{d_v, d_u\}$ for any $\{v,u\} \in E$ (and $P_{v,v} = 1 - \sum_{u: \{v,u\} \in E} P_{v,u}$) also converges to a constant discrepancy since $P$ is symmetric. This chain is called Metropolis chain. Note that this algorithm only require each vertex the knowledge of the degree of each neighbor (cf. [10]).

On the other hand, for the discrete case, it is difficult to analyze on arbitrary graphs since it is not clear that if one can generalize the analytic techniques for the discrete diffusion of regular graphs or not. For the algorithm such that each vertex $v$ sends $\lceil \chi_v^{(t)} P_{v,u} \rceil$ or $\lfloor \chi_v^{(t)} P_{v,u} \rfloor$ loads to a neighbor $u$, the analysis of Rabani et. al [11] showed that $\text{Disc}(\chi^{(T)}) = O(\Psi_1(P))$, where $\Psi_1(P)$ is called the local-1 divergence (See (1.4) for the precise definition). They also showed that $\Psi_1(P) = O(d_{\text{max}} \log N/(1 - \lambda_2))$ for any symmetric $P$. Shiraga et al. [14] showed that $\text{Disc}(\chi^{(T)}) = O(d_{\text{max}} t_{\text{mix}})$ for this algorithm.

1.2 Related works

**Algorithms with the state:** Note that all above algorithms are stateless, i.e. each vertex does not use any information of previous time steps, while some previous works concerned with diffusion algorithms with the state. The Rotor-router model is a typical one, which is a well studied deterministic process analogous to random walks. In this algorithm, each vertex sends loads one by one to neighboring vertices in the round robin fashion. Rotor-router model has the same upper bound as $\text{SEND}[\lceil \chi_v^{(t)}/(d + 1) \rceil]$ or $\lceil \chi_v^{(t)}/(d + 1) \rceil$ (cf. [11]). Berenbrink et. al [4] gave a lazy version of the rotor-router model L.Rotor-router model, i.e. each vertex has self loops more than $d/2$, and showed that $O(d)$ discrepancy within $O(T + (d \log^2 N)/(1 - \lambda_2))$ steps.

**Algorithms occurring negative loads:** We also note that all above algorithms satisfy the property such that each tokens distributes its own loads to each neighbor (and itself). There are several previous works corresponding to the algorithms occurring negative loads, i.e. each vertex possibly sends more than its own loads to each neighbor (and itself). For example, consider the algorithm such that each vertex sends randomly rounded $\lfloor \chi_v^{(t)}/(d+1) \rfloor$ or $\lceil \chi_v^{(t)}/(d+1) \rceil$ to each neighbor and itself. This algorithm possibly occurs negative loads, i.e. total amount of the sent loads possibly becomes $\lfloor \chi_v^{(t)}/(d+1) \rfloor (d+1) > \chi_v^{(t)}$. Sauerwald and Sun showed that $O(\sqrt{d \log N})$ discrepancy within $O(T)$ with high probability for this algorithm. They also showed that on arbitrary graphs, the algorithm rounding $\lceil \chi_v^{(t)}/2d_{\text{max}} \rceil$ or $\lceil \chi_v^{(t)}/d_{\text{max}} \rceil$ with appropriate probability archives $\text{Disc}(\chi^{(T)}) = O(\Psi_2(P) \sqrt{\log N})$ with high probability, where $\Psi_2(P)$ denotes the local 2-divergence (See [14] for the precise definition), and showed that $\Psi_2(P) = O(d_{\text{max}})$ for the round matrix $P_{v,u} = 1/cd_{\text{max}}$ for any $\{v,u\} \in E$ (and the remaining is the self loop), where $c$ is some constant. The algorithm in [2] corresponds to a diffusion algorithm with the state and negative loads. This algorithm achieves $O(\sqrt{d_{\text{max}} \log N})$ discrepancy for arbitrary graphs.

**Matching algorithms:** Matching based algorithms have been well studied as well as diffusion algorithms. Matching-based algorithms generate a matching of the graph in a distributed way at each round, and the endpoints of each matching edge balances loads as evenly as possible. Friedrich and Sauerwald [7] studied randomized version of matching models on regular graphs. They showed that $\text{Disc}(\chi^{(T)}) = O(\Psi_2(P) \sqrt{\log N})$ with high probability. They also showed that $\Psi_2(P) = O(\sqrt{d/(1 - \lambda_2)})$. The results of Sauerwald and Sun [12] is the best result of the discrepancy so far. They showed that a constant discrepancy within $O(T)$ step for a randomized matching model on regular graphs.

1.3 This work

**Motivation:** The strength of diffusion algorithms is its strong locality. The ability required to each vertex is only to count its own loads (degree) and to send its own loads to each neighbor. There is no need to
communicate and check the amount of neighbor’s loads like matching algorithms.

Our main concern of this paper is to investigate the discrepancy of the discrete diffusion algorithms with simplest assumptions, i.e. stateless and non-negative loads diffusion algorithms. We call these algorithms natural diffusion algorithms. This framework contains Markov chains (multiple random walks).

In previous works, the $\Omega(d)$ lower bound of the discrepancy for any deterministic natural diffusion algorithms on $d$ regular graphs has been shown. Similarly, one can guess a $\Omega(d)$ bound for any randomized natural diffusion algorithm since the discrepancies of all previous upper bounds of randomized natural diffusion algorithms depend on the polynomial of $d$. However, a lower bound of randomized natural diffusion algorithms has not been known, i.e. no one knows that whether there is a randomized natural diffusion algorithm with the discrepancy $o(d)$ or not.

**Results:** For this question, this paper proposes a new randomized natural diffusion algorithm which archives $O(\sqrt{d\log N})$ discrepancy within $O(T)$ steps with high probability on $d$ regular graphs. This result gives a positive answer to the question, i.e. breaks $\Omega(d)$ barrier, since the result guarantees that the discrepancy is $o(d)$ for $d = \omega(\log N)$ regular graphs. Surprisingly, even though we compared with the best upper bound of the diffusion algorithm allowing negative loads or with the state, this upper bound for our natural diffusion algorithm is the same magnitude.

Furthermore, we succeed in generalizing the proposed algorithm for arbitrary symmetric round matrices. This allows us to construct a randomized natural diffusion algorithms on arbitrary graphs with $O(\sqrt{d_{\text{max}} \log N})$ discrepancy within $O(T)$ steps with high probability.

### 1.3.1 Result on regular graphs

First, we introduce a new randomized natural diffusion algorithm (Algorithm 1). The main idea of this algorithm is to add the laziness to RSEND ($[X_v^{(t)}/(d+1)]$ or $[X_v^{(t)}/(d+1)]$). Let $G = (V, E)$ be an arbitrary $d$-regular graph with $|V| = N$ vertices. For each $v \in V$, let $v_0, v_1, \ldots, v_{d-1}$ denote the $d$ neighbors of $v$.

**Definition of Algorithm 1:** Let $X^{(0)} \in \mathbb{Z}_{\geq 0}^N$ be a initial configuration of $K$ loads over $V$, and let $X^{(t)} \in \mathbb{Z}_{\geq 0}^N$ denote the configuration of $K$ loads over $V$ at time $t \in \mathbb{Z}_{\geq 0}$ in our algorithm. In an update from $X^{(t)}$ to $X^{(t+1)}$ in our algorithm, at each vertex $v$, each load $k$ ($k \in \{0, 1, \ldots, X_v^{(t)} - 1\}$) randomly samples a number $r_v^{(t)}(k)$ from the interval $[\frac{k}{X_v^{(t)}}, \frac{k+1}{X_v^{(t)}}]$. Then, each load $k$ moves to a neighbor $v_i$ if sampled random number is in the interval of $v_i$, i.e. each load $k$ moves to $v_i$ if $r_v(k) \in \left[\frac{i}{2d}, \frac{i+1}{2d}\right]$ (if $r_v(k) \in [1/2, 1)$, load $k$ stays at $v$).

Obviously, Algorithm 1 is a natural diffusion algorithm. If each load randomly samples a number from $[0, 1)$, then this is multiple lazy random walks. Laziness is a important property to estimate the discrepancy (cf. [5]). For this algorithm, we showed the following upper bound of the discrepancy.

**Theorem 1.1 (Result on regular graphs).** Suppose that $G = (V, E)$ is an arbitrary connected $d$-regular graph. Then, for any $X^{(0)}$ and for each $T \geq \frac{\log(4 \text{Disc}(X^{(0)})N)}{1 - \lambda_2}$, $X^{(T)}$ of Algorithm 1 satisfies that

$$\Pr \left[\text{Disc}(X^{(T)}) \leq 18\sqrt{d \log N}\right] \geq 1 - \frac{2}{N}.$$

Similar to natural deterministic diffusion algorithms, using laziness improves the previous upper bounds of randomized natural diffusion algorithms and the state. Table 1 summarizes the discrepancies of previous results and this work on regular graphs.

### 1.3.2 Generalized algorithm and the result on arbitrary graph

Furthermore, we succeeded in generalizing Algorithm 1 for any round matrix $P$ (Algorithm 2).
Table 1: Previous works and our result on \(d\)-regular graphs. D/R represents a deterministic/randomized algorithm. NL represents an algorithm which does not occur negative loads. SL represents state less algorithms.

| Algorithm                  | D/R | Disc(\(\chi^{(T)}\)) | NL | SL | Ref. |
|----------------------------|-----|----------------------|----|----|------|
| SEND \(\chi^{(t)}_{d+1}\) or \(\chi^{(t)}_{d+1}\) | D   | \(O (d \log N \sqrt{\frac{1}{1-\lambda_2}})\) | ✓  | ✓  | [11] |
| SEND \(\chi^{(t)}_{d}\) | D   | \(O (d \sqrt{\frac{\log N}{1-\lambda_2}})\) | ✓  | ✓  | [5]  |
| SEND \(\chi^{(t)}_{d}\) | D   | \(O(d)\) | ✓  | ✓  | [5]  |
| RSEND \(\chi^{(t)}_{d+1}\) or \(\chi^{(t)}_{d+1}\) | R   | \(O \left(\left(d + \sqrt{\frac{d \log d}{1-\lambda_2}}\right) \sqrt{\log N}\right)\) | ✓  | ✓  | [4]  |
|                          |     | \(O \left(\frac{d \log \log N}{1-\lambda_2}\right)\) |     |     | [4]  |
|                          |     | \(O \left(d^2 \sqrt{\log N}\right)\) |     |     | [12] |
| RRotor-router            | R   | \(O \left(d \log \log N\right)\) | ✓  | ✓  | [11] |
| Algorithm 1 : Rotor-router | D   | \(O (d \log \log N)\) | ✓  | ✓  | Theorem 1.1 |
| LRotor-router            | D   | \(O(d)\) | ✓  | ✓  | [5]  |
| Algorithm in [2]       | R   | \(O (d \log \log N)\) | ✓  | ✓  | [2]  |
| Neg.RSEND               | R   | \(O (d \log \log N)\) | ✓  | ✓  | [12] |

Figure 1: In this example, there are 5 loads at \(v\). Then, load 0 moves \(v_0 / v_1 / v_2\) w.p. \(\frac{5}{80}, \frac{5}{80}, \frac{5}{80}, \frac{6}{80}, \frac{5}{80}\). Load 1 moves \(v_2 / v_3\) w.p. \(\frac{1}{20}, \frac{3}{20}\). Load 2 moves to \(v_3 / v_4\) w.p. \(\frac{1}{2} / \frac{1}{2}\). Load 3 / 4 moves to \(v_4\) w.p. 1.

Notations: Let \(V\) be a vertex set, and let \(N = |V|\). Let \(P \in \mathbb{R}_{\geq 0}^{N \times N}\) be a round (transition) matrix on \(V\), i.e. \(\sum_{u \in V} P_{v,u} = 1\) holds for any \(v \in V\), where \(P_{v,u}\) denotes \((v,u)\) entry of \(P\). For a round matrix \(P\) on \(V\), let \(N_v^P\) be the set of neighbors of \(v \in V\), i.e. \(N_v^P := \{u \in V \mid P_{v,u} > 0\}\). In this paper, we assume an arbitrary ordering on \(N_v^P\), i.e. we denote \(N_v^P = \{v_0, v_1, \ldots, v_{d_v^P-1}\}\), where \(d_v^P = |N_v^P|\).

Definition of Algorithm 2: Let \(\chi^{(0)} \in \mathbb{Z}_{\geq 0}^N\) be an initial configuration of \(K\) loads over \(V\), and let \(\chi^{(t)} \in \mathbb{Z}_{\geq 0}^N\) denote the configuration of \(K\) loads over \(V\) at time \(t \in \mathbb{Z}_{\geq 0}\) in our algorithm. In an update from \(\chi^{(t)}\) to \(\chi^{(t+1)}\) in our algorithm, at each vertex \(v\), each load \(k (k \in \{0, 1, \ldots, \chi_v^{(t)} - 1\})\) randomly samples a random number \(r_v(k)\) from the interval \([k \chi_v^{(t)}, k+1 \chi_v^{(t)}]\). Then, each load \(k\) moves to its corresponding neighbor \(v_i\), i.e. load \(k\) moves to \(v_i\) if \(r_v(k) \in \left[\sum_{j=0}^{k-1} P_{v,v_j}, \sum_{j=0}^{k} P_{v,v_j}\right]\) (let \(\sum_{j=0}^{k-1} P_{v,v_j} = 0\)).

This is a generalization of Algorithm 1. We put an example on Figure 1. Only different point compared with multiple random walks according to \(P\) is that each load \(k\) randomly samples a number from

\(^a\)Discrepancy after \(O(T + (\log N)/(1 - \lambda_2))\) steps.
\(^b\)Discrepancy after \(O(T + (d \log^2 N)/(1 - \lambda_2))\) steps.
[\textcolor{red}{[k/\mathcal{X}_{v}^{(t)}, (k + 1)/\mathcal{X}_{v}^{(t)}]}], while each load randomly samples a number from \([0, 1]\) in multiple random walks.

One of the main reasons introducing the generalized algorithm is to use the (lazy) Metropolis chain \(P_M\) on arbitrary graphs. This chain is defined by as follows: \((P_M)_{v,u} := 1/(2 \max\{d_v, d_u\})\) for any \(\{v, u\} \in E\), \((P_M)_{v,v} := 1 - \sum_{\{v,u\} \in E} (P_M)_{v,u}\) for any \(v \in V\) and \((P_M)_{v,u} := 0\) for any \(\{v, u\} \notin E\). The main strength of this chain is that there is no need to require each vertex to know the maximum degree over all vertices.

Then, we show the following upper bound of Algorithm 2 according to \(P_M\) on arbitrary graphs.

**Theorem 1.2** (Results on arbitrary graphs). Suppose that \(G = (V, E)\) is an arbitrary connected graph. Then, for any \(\mathcal{X}^{(0)}\) and for each \(T \geq \log(4\text{Disc}(\mathcal{X}^{(0)}))\), \(\mathcal{X}^{(T)}\) of Algorithm 2 according to \(P_M\) satisfies that

\[
\mathbb{P} \left[ \text{Disc}(\mathcal{X}^{(T)}) \leq 16\max\log N \right] \geq 1 - \frac{2}{N}.
\]

This upper bound dramatically improves the previous works on the framework of the natural diffusion adaptive to the metropolis chain: \(O(d_{\max}\log N/(1 - \lambda_2))\) for DSMS\((\mathcal{X}_{v}^{(t)}(P_M)_{v,u}\) or \(\mathcal{X}_{u}^{(t)}(P_M)_{v,u}\)).

Even though we compared with the best upper bound of the diffusion algorithm allowing negative loads and the information of the maximum degree [12], the upper bound of Theorem 1.2 for Algorithm 2 satisfies the same magnitude.

### 1.3.3 Idea of the proof and technical contribution

Above our main theorems are shown by a load (token)-based analysis. The load configuration of Algorithm 2 is determined by random variables of the destinations of each load at each vertex (Observations 3.1). The properties of the destinations of each load are described in Observations 3.2 (Expectation of the destinations) and 3.3 (Independency of the destinations). Especially, Observations 3.3 plays a key role to prove the key lemma (Lemma 3.8) stating the Martingale difference. This and the well-known concentration inequality (AzumaHoeffding inequality) allow us to get the following good upper bound of the Discrepancy between discrete and continuous diffusions in general form.

**Theorem 1.3** (Discrepancy between discrete and continuous diffusions). For any \(\mathcal{X}^{(0)}\), for any round matrix \(P\) and for each time \(T \in \mathbb{Z}_{\geq 0}\), \(\mathcal{X}^{(T)}\) of Algorithm 2 according to \(P\) satisfies that

\[
\mathbb{P} \left[ \max_{w \in V} |\mathcal{X}_{w}^{(T)} - (\mathcal{X}^{(0)}P^{T})_{w}| \leq 4\Psi(P)\sqrt{\log N} \right] \geq 1 - \frac{2}{N}.
\]

\(\mathcal{X}^{(0)}P^{T}\) is the configuration of loads on the continuous diffusion according to \(P\) (See Section 2 for the detail). Note that this value also represents the expected configuration of multiple random walks.

As an technical contribution, we obtain a general upper bound of the \(\Psi(P)\). The definition of the local \(p\)-divergence is as follows.

**Definition 1.4** (local \(p\)-divergence, [11][7]). For any \(p \in \mathbb{Z}_{\geq 0}\), the local-\(p\) divergence of \(P\) is defined by

\[
\Psi_{P}(P) := \max_{w \in V} \left( \sum_{t=0}^{\infty} \sum_{(v,u) \in V \times V: P_{v,u} > 0} |P_{v,w}^{t} - P_{u,w}^{t}|^{p} \right)^{1/p}.
\]

Although \(\Psi_{P}(P) = O(\sqrt{d_{\max}})\) for the transition matrix on \(G = (V, E)\) such that \(P_{v,u} = 1/cd_{\max}\) for any \(\{v, u\} \in E\) and \(P_{v,v}\) is the remaining probability for any \(v \in V\) [12], this proof is not simple and it was not clear that if we can extend this proof to any round matrix like Metropolis chain.

For this problem, we realized that the equation corresponding to the local-2 divergence can be transformed into a simple equation by the idea of the Dirichlet form of the reversible transition matrix (Lemma 4.1). Then, adding the monotonicity of the lazy transition matrix \((P_{v,w}^{t+1} \geq P_{v,w}^{t})\), we succeeded in extending the previous work of the upper bound of the local-2 divergence as follows.
Theorem 1.5 (Upper bound of the local 2 divergence). Suppose that $P$ is reversible and lazy, and let $\pi$ be the stationary distribution of $P$. Then, it holds that

$$\Psi_2(P) \leq \sqrt{\frac{2 \max_{v \in V} \pi_v}{\min_{(v,u) \in E} P_{v,u} > 0}}.$$

Note that if $P$ is symmetric, Theorem 1.5 becomes simpler since $\pi_v = 1/N$ for any $v \in V$.

Corollary 1.6. Suppose that $P$ is symmetric and lazy. Then, it holds that

$$\Psi_2(P) \leq \sqrt{\frac{2}{\min_{(v,u) \in E} P_{v,u} > 0}}.$$

Corollary 1.6 allows us to estimate an upper bound of the (lazy) Metropolis chain $\Psi_2(P_M) = O(\sqrt{t_{\text{max}}})$. We believe that our simple transformation of the local-2 divergence (Lemma 4.1) has applications in other problems.

2 Continuous diffusion and Markov chains

Before describing our algorithms, we give a precise discussion of the continuous diffusion algorithm according to the round matrix $P$. Let $X(0) \in \mathbb{R}_{\geq 0}^n$ be a initial configuration of $K (= \sum_{v \in V} X_v(0))$ loads over a vertex set $V$, and let $X(t) \in \mathbb{R}_{\geq 0}^n$ denotes the configuration of the loads at time $t$ on the diffusion algorithm according to the round matrix $P$. Then, for each $t$ and $v$, $X_v(t)$ is defined by $X(t)$ and the round matrix $P$, $(X(t+1))_v := (X(t)P)_v = \sum_{u \in V} X_u(t)P_{u,v}$. From this definition, the configuration of loads at time $t$ is described by the initial configuration of loads and the round matrix as follows.

$$X(t) = X(0)P^t. \quad (2)$$

Our main concern is the discrepancy $\text{Disc}(X(t))$ defined in [1]. The limit behavior of the discrepancy is characterized by the rich theory of the convergence of the Markov chains. To discuss clearly, we introduce some terminologies. A $P$ is called irreducible if for any $v,u \in V$ there exists a $t$ such that $P_{v,u} > 0$. Note that $P$ is irreducible if and only if the transition diagram $\mathcal{G} = (V,E)$ is connected, where $E := \{(v,u) \in V \times V \mid P_{v,u} > 0\}$. A $P$ is called symmetric if $P_{v,u} = P_{u,v}$ holds for any $v,u \in V$. Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be the eigenvalues of $P$. We assume $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_N|$. Then, it is easy to derive the following proposition stating the convergence time of the diffusion algorithm according to $P$. We put the proof in Appendix A.

Proposition 2.1 (The discrepancy of the continuous diffusion algorithm [13, 11]). Suppose that $P$ is irreducible and symmetric. Then, for any $X(0) \in \mathbb{R}_{\geq 0}^n$, $w \in V$, $\varepsilon \in (0, 1)$ and $T \geq \frac{1}{1-\lambda_2} \log \left( \frac{4\text{Disc}(X(0))v}{\varepsilon} \right)$, $\text{Disc}(X(T)) = \text{Disc}(X(0)P^T) \leq \varepsilon$ holds.

Note that combining Proposition 2.1, Theorem 1.3 (Discrepancy between discrete and continuous diffusions) and Theorem 1.5 (Upper bound of the local 2 divergence), it is easy to obtain our main Theorems 1.1 (Result on regular graphs) and 1.2 (Results on arbitrary graphs). Thus, we start proving Theorem 1.3 (Section 3) and Theorem 1.5 (Section 4).
3 Proof of Theorem 1.3

3.1 Key properties of our model

In this section, we observe key properties of our model (Algorithm 2). The properties are described by Observations 3.1, 3.2 and 3.3. Each of them is concerned with the random variable \( D_v^{(t)}(k) \), which denotes the destination of \((k+1)\)-th token on \( v \in V \) at \( t \geq 0 \). For the notational convenience, we define \( \bar{P}_{v,u} \), which denotes the interval of \( P_{v,u} \) on \([0, 1)\). For any \( v \in V \) and \( u = v_i \in N_v^P \), let

\[
\bar{P}_{v,u} := \left[ \sum_{j=0}^{i-1} P_{v,v_j}, \sum_{j=0}^{i} P_{v,v_j} \right].
\]  

(3)

We assume \( \sum_{j=0}^{-1} P_{v,v_j} := 0 \). Note that the length of \( \bar{P}_{v,u} \) is equal to \( P_{v,u} \).

Recall that \( r_v^{(t)}(k) \) is a random number sampled from \( [k/X_v^{(t)}, (k+1)/X_v^{(t)}) \), where \( X_v^{(t)} \) denotes the configuration of loads at time \( t \) in Algorithm 2 (see the definition of Algorithm 2 in Section 1.3.2). Then, \( D_v^{(t)}(k) \) is defined as follows.

\[
D_v^{(t)}(k) = u \text{ if } r_v^{(t)}(k) \in \bar{P}_{v,u}.
\]  

(4)

Now, let us observe the following 3 properties of \( D_v^{(t)}(k) \). First one is concerned with the connection between the configuration of loads and \( D_v^{(t)}(k) \). Note that \( \sum_{k=0}^{X_v^{(t)}-1} 1\{D_v^{(t)}(k) = u\} \) denotes the number of loads sent from \( v \) to \( u \) at time \( t \) in Algorithm 2.

**Observation 3.1** (Relation between the configuration and destinations). *For each time step \( t \geq 0 \), \( X_v^{(t+1)} \) of Algorithm 2 is determined by \( X_v^{(t)} \) and \( \{D_v^{(t)}(0), D_v^{(t)}(1), \ldots, D_v^{(t)}(X_v^{(t)}-1)\} \) \( v \in V \), i.e. for each time \( t \geq 0 \) and \( u \in V \),

\[
X_u^{(t+1)} = \sum_{v \in V} \sum_{k=0}^{X_v^{(t)}-1} 1\{D_v^{(t)}(k) = u\}.
\]  

(5)

Next one describe the expected value of \( D_v^{(t)}(k) \) conditioned on \( X_v^{(t)} \).

**Observation 3.2** (Expectation of the destination). *For any \( t \geq 0 \), \( v \in V \), \( k \in \{0, 1, \ldots, X_v^{(t)} - 1\} \) and \( u \in N_v^P \), it holds that

\[
E\left[ 1\{D_v^{(t)}(k) = u\} \mid X_v^{(t)} \right] = \left[ k \left[ X_v^{(t)}, k+1 \right] \right] \cap \bar{P}_{v,u} \right] \cdot X_v^{(t)}.
\]

\[
E\left[ 1\{D_v^{(t)}(k) = u\} \mid X_v^{(t)} \right] = \Pr \left[ D_v^{(t)}(k) = u \mid X_v^{(t)} \right] = \Pr \left[ r_v^{(t)}(k) \in \bar{P}_{v,u} \mid X_v^{(t)} \right]
\]

holds and we obtain the claim since \( r_v^{(t)}(k) \) is randomly sampled from \( \left[ k \left[ X_v^{(t)}, k+1 \right] \right] \).

Proof.

The last one shows that the conditional independency of each destination in Algorithm 2.
Observation 3.3 (Independency of the destinations). Suppose that $X^{(t)}$ is fixed. Then, for any $v, v' \in V$, $k \in \{0, 1, \ldots, X^{(t)} - 1\}$ and $k' \in \{0, 1, \ldots, X^{(t)} - 1\}$, $D^{(t)}(k)$ and $D^{(t)}(k')$ are independent if $v \neq v'$ or $k \neq k'$.

Proof. For any intervals $[a, b] \subseteq [0, 1)$, $[a', b') \subseteq [0, 1)$ and $v, v', k, k'$ s.t. $v \neq v'$ or $k \neq k'$,

$$\Pr \left[ \left( r^{(t)}(k) \in [a, b) \right) \text{ and } \left( r^{(t)}(k') \in [a', b') \right) \mid X^{(t)} \right] = \Pr \left[ r^{(t)}(k) \in [a, b) \mid X^{(t)} \right] \cdot \Pr \left[ r^{(t)}(k') \in [a', b') \mid X^{(t)} \right]$$

(6)

since $r^{(t)}(k)$ and $r^{(t)}(k')$ are randomly sampled from each fixed interval independently. □

The analysis of this paper is derived from the properties described in Observations 3.1, 3.2. For example, the expected value of the configuration of loads is derived from Observations 3.1 and Observations 3.2.

Lemma 3.4 (Expectation of the configuration). For any $T \geq 0$ and initial configuration $X^{(0)}$, it holds that

$$\mathbb{E} \left[ X^{(T)} \right] = X^{(0)} P^T.$$ 

Proof. Since

$$\mathbb{E} \left[ X^{(t+1)} \mid X^{(t)} \right] = \sum_{v \in V} \sum_{k=0}^{X^{(t)} - 1} \mathbb{E} \left[ 1 \{ D^{(t)}(k) = u \} \mid X^{(t)} \right]$$

$$= \sum_{v \in V} \sum_{k=0}^{X^{(t)} - 1} \left[ \frac{k}{X^{(t)}-1} \right] \cap P_{v,u} \cdot X^{(t)} = \sum_{v \in V} X^{(t)} P_{v,u} = (X^{(t)} P)_{v,u}$$

holds from Observations 3.1 and 3.2, we have

$$\mathbb{E} \left[ X^{(t+1)} \right] = \mathbb{E} \left[ \mathbb{E} \left[ X^{(t+1)} \mid X^{(t)} \right] \right] = \mathbb{E} \left[ X^{(t)} \right] P$$

(7)

and we obtain the claim by iterating the equation (7). □

At the end of this section, we introduce the following 2 lemmas, each of which guarantees an upper bound of the discrepancy between $1 \{ D^{(t)}(k) = u \}$ and its expectation.

Lemma 3.5 (Discrepancy around one load). For any $t \geq 0$, $v \in V$ and $k \in \{0, 1, \ldots, X^{(t)} - 1\}$, it holds that

$$\sum_{u \in N^P_v} \left| 1 \{ D^{(t)}(k) = u \} - \mathbb{E} \left[ 1 \{ D^{(t)}(k) = u \} \mid X^{(t)} \right] \right| \leq 2.$$

Proof. Since

$$\sum_{u \in N^P_v} 1 \{ D^{(t)}(k) = u \} = 1$$

and

$$\sum_{u \in N^P_v} \mathbb{E} \left[ 1 \{ D^{(t)}(k) = u \} \mid X^{(t)} \right] = \sum_{u \in N^P_v} \Pr \left[ D^{(t)}(k) = u \right] = 1$$

hold from the definition (4), we obtain the claim. Note that

$$\left| 1 \{ D^{(t)}(k) = u \} - \mathbb{E} \left[ 1 \{ D^{(t)}(k) = u \} \mid X^{(t)} \right] \right| \leq \left| 1 \{ D^{(t)}(k) = u \} \right| + \mathbb{E} \left[ 1 \{ D^{(t)}(k) = u \} \mid X^{(t)} \right].$$

□
Lemma 3.6 (Discrepancy around one neighbor). For any $t \geq 0, v \in V$ and $u \in N_v$, it holds that

$$
\sum_{k=0}^{\mathcal{X}^{(t)} - 1} \left| 1\{D_v^{(t)}(k) = u \} - \mathbb{E}\left[ 1\{D_v^{(t)}(k) = u \} \mid \mathcal{X}_v^{(t)} \right] \right| \leq 2.
$$

Proof. Let $\overline{P}_{v,u} = [a_u, b_u]$, and let

$$
A_v^{(t)}(u) := \left\{ k \in \{0, \ldots, \mathcal{X}^{(t)} - 1 \} \mid \frac{k}{\mathcal{X}^{(t)}} < a_u < \frac{k + 1}{\mathcal{X}^{(t)}} \right\} \quad \text{and}
$$

$$
B_v^{(t)}(u) := \left\{ k \in \{0, \ldots, \mathcal{X}^{(t)} - 1 \} \mid \frac{k}{\mathcal{X}^{(t)}} < b_u < \frac{k + 1}{\mathcal{X}^{(t)}} \right\}.
$$

Note that both $|A_v^{(t)}(u)| \leq 1$ and $|B_v^{(t)}(u)| \leq 1$ hold since $\left[ \frac{k}{\mathcal{X}^{(t)}}, \frac{k + 1}{\mathcal{X}^{(t)}} \right]$ and $\left[ \frac{k'}{\mathcal{X}^{(t)}}, \frac{k' + 1}{\mathcal{X}^{(t)}} \right]$ are disjoint for any $k \neq k'$.

Now, we observe that if $k \not\in (A_v^{(t)}(u) \cup B_v^{(t)}(u))$, then $1. \left[ \frac{k}{\mathcal{X}^{(t)}}, \frac{k + 1}{\mathcal{X}^{(t)}} \right] \subseteq \overline{P}_{v,u}$ or $2. \left[ \frac{k}{\mathcal{X}^{(t)}}, \frac{k + 1}{\mathcal{X}^{(t)}} \right] \cap \overline{P}_{v,u} = 0$. For the case 1., $\mathbb{P}[D_v^{(t)}(k) = u \mid \mathcal{X}_v^{(t)}] = 1$ from the Observation 3.2. Then, $1\{D_v^{(t)}(k) = u \} = 1$ since $D_v^{(t)}(k)$ must be $u$. For the case 2., $\mathbb{P}[D_v^{(t)}(k) = u \mid \mathcal{X}_v^{(t)}] = 0$ from the Proposition 3.2. Then, $1\{D_v^{(t)}(k) = u \} = 0$ since $D_v^{(t)}(k)$ must not be $u$. Hence $1\{D_v^{(t)}(k) = u \} - \mathbb{E}[1\{D_v^{(t)}(k) = u \} \mid \mathcal{X}_v^{(t)}] = 0$ for both cases. Thus

$$
\sum_{k=0}^{\mathcal{X}^{(t)} - 1} \left| 1\{D_v^{(t)}(k) = u \} - \mathbb{E}\left[ 1\{D_v^{(t)}(k) = u \} \mid \mathcal{X}_v^{(t)} \right] \right| = \sum_{k \in A_v^{(t)}(u) \cup B_v^{(t)}(u)} \left| 1\{D_v^{(t)}(k) = u \} - \mathbb{E}\left[ 1\{D_v^{(t)}(k) = u \} \mid \mathcal{X}_v^{(t)} \right] \right| \leq |A_v^{(t)}(u) \cup B_v^{(t)}(u)| \leq 2.
$$

$\square$

3.2 Framework of the proof

Now, we estimate the discrepancy between $\mathcal{X}_v^{(t)}$ and $(\mathcal{X}_v^{(0)} P^T)_w$ for the Theorem 1.3. For the convenience, we introduce an useful notation. Let $V = \{0, 1, \ldots, N - 1\}$. Then, for any $\tau \in \{0, 1, \ldots, T - 1\}, v \in V$, and $\kappa \in \{0, 1, \ldots, K - 1\}$, let

$$
D_{KN\tau + K\nu + \kappa} := \begin{cases} D_v^{(\tau)}(\kappa) & \text{if } \kappa \in \{0, 1, \ldots, K - 1\} \\ -1 & \text{(otherwise)} \end{cases}.
$$

(8)

This definition means that $1\{D_{KN\tau + K\nu + \kappa} = u \} = 0$ for any $\mathcal{X}_v^{(t)} \leq \kappa \leq K - 1$ and $u \in V$. Then, the following is easily observed from Observation 3.1 and the definition (8).

Observation 3.7. For any $t \in \mathbb{Z}_{\geq 0}$, $\mathcal{X}_v^{(t)}$ is determined by $\mathcal{X}_v^{(0)}$ and $D_0, D_1, \ldots, D_{KN\tau - 1}$.

Note that $D_{KN\tau - 1} = D_{KN(t - 1) + K(N - 1) + (K - 1)}$, which is $D_{N - 1}^{(t - 1)}(K - 1)$ or $-1$.

The main idea to estimate $\mathcal{X}_w^{(T)} - (\mathcal{X}_v^{(0)} P^T)_w$ is applying the Azuma-Hoeffding inequality (See Appendix B) to the martingale $\mathcal{Y}_0, \mathcal{Y}_1, \ldots, \mathcal{Y}_{KN\tau}$ respect to $D_0, D_1, \ldots, D_{KN\tau - 1}$, where

$$
\mathcal{Y}_\ell := \mathbb{E}\left[ \mathcal{X}_w^{(T)} - (\mathcal{X}_v^{(0)} P^T)_w \mid D_0, D_1, \ldots, D_{\ell - 1} \right],
$$

(9)
We assume that \( \mathcal{Y}_0 := \mathbb{E}[\mathcal{X}_w^{(T)} - (\mathcal{X}^{(0)}P^T)_w] \). Since \( \mathcal{Y}_{KNT} = \mathcal{X}_w^{(T)} - (\mathcal{X}^{(0)}P^T)_w \) from Observation 3.7 and \( \mathcal{Y}_0 = \mathbb{E}[\mathcal{X}_w^{(T)} - (\mathcal{X}^{(0)}P^T)_w] = 0 \) from Lemma 3.4,

\[
\Pr\left[|\mathcal{Y}_{KNT} - \mathcal{Y}_0| \geq \eta\right] \leq 2 \exp\left[-\eta^2 / 2 \sum_{\ell=0}^{KNT} (c\ell)^2\right] \tag{10}
\]

for any \( \eta > 0 \) from Azuma-Hoeffding inequality, where \( c\ell \) is a value satisfies \( \mathcal{Y}_{\ell+1} - \mathcal{Y}_\ell \leq c\ell \). Hence

\[
\Pr\left[\max_{w \in V} |\mathcal{X}_w^{(T)} - (\mathcal{X}^{(0)}P^T)_w| \geq 2 \sqrt{\sum_{\ell=0}^{KNT-1} \log N} \right] \leq \frac{2}{N} \tag{11}
\]

by taking \( \eta = \sqrt{2 \sum_{\ell=0}^{KNT-1} \log N} \) and using the union bound.

Thus, our main concern to obtain Theorem 1.3 is the upper bound of the difference \( \mathcal{Y}_{\ell+1} - \mathcal{Y}_{\ell} = \mathbb{E}[\mathcal{X}_w^{(T)} - (\mathcal{X}^{(0)}P^T)_w \mid D_0, \ldots, D_{\ell}] - \mathbb{E}[\mathcal{X}_w^{(T)} - (\mathcal{X}^{(0)}P^T)_w \mid D_0, D_1, \ldots, D_{\ell-1}] \). For this key value, we showed the following Lemma.

**Lemma 3.8** (Martingale difference). For any \( \tau \in \{0, 1, \ldots, T-1\} \), \( \nu \in \{0, 1, \ldots, N-1\} \) and \( \kappa \in \{0, 1, \ldots, K-1\} \), let \( \ell = KN\tau + K\nu + \kappa \). Then, it holds that

\[
\mathbb{E} \left[ \mathcal{X}_w^{(T)} - (\mathcal{X}^{(0)}P^T)_w \mid D_0, D_1, \ldots, D_{\ell} \right] - \mathbb{E} \left[ \mathcal{X}_w^{(T)} - (\mathcal{X}^{(0)}P^T)_w \mid D_0, D_1, \ldots, D_{\ell-1} \right] = \sum_{u \in N^w_{\nu}} \left(1 \{D_{\ell} = u\} - \mathbb{E} \left[1 \{D_{\ell} = u\} \mid \mathcal{X}^{(\tau)}_{\nu}\right]\right) \left(P_{u,w}^{T-\tau-1} - P_{\nu,w}^{T-\tau-1}\right).
\]

Combining Lemma 3.8, (11), Lemma 3.5 and Lemma 3.6, we can show Theorem 1.3.

**Proof of Theorem 1.3** From the Lemma 3.8, Cauchy-schwarz inequality and Lemma 3.6,

\[
(\mathcal{Y}_{\ell+1} - \mathcal{Y}_{\ell})^2 \leq \sum_{u \in N^w_{\nu}} \left(1 \{D_{\ell} = u\} - \mathbb{E} \left[1 \{D_{\ell} = u\} \mid \mathcal{X}^{(\tau)}_{\nu}\right]\right)^2 \tag{12}
\]

Thus, using Lemma 3.6 and note the definition of \( D_{\ell} \),

\[
\sum_{\ell=0}^{KNT-1} (\mathcal{Y}_{\ell+1} - \mathcal{Y}_{\ell})^2 = \sum_{\tau=0}^{T-1} \sum_{\nu=0}^{N-1} \sum_{\kappa=0}^{K-1} \left(\mathcal{Y}_{KN\tau + K\nu + \kappa + 1} - \mathcal{Y}_{KN\tau + K\nu + \kappa}\right)^2 \leq \sum_{\tau=0}^{T-1} \sum_{\nu=0}^{N-1} \sum_{\kappa=0}^{K-1} \left(2 \mathbb{E} \left[1 \{D^{(\tau)}_{\nu}(\kappa) = u\} \mid \mathcal{X}^{(\tau)}_{\nu}\right] \left(P_{u,w}^{T-\tau-1} - P_{\nu,w}^{T-\tau-1}\right)^2\right) \leq 4 \sum_{\tau=0}^{T-1} \sum_{\nu=0}^{N-1} \sum_{u \in N^w_{\nu}} \left(P_{u,w}^{T-\tau-1} - P_{\nu,w}^{T-\tau-1}\right)^2 \leq (2\Psi_2(P))^2. \tag{13}
\]

Combining (11) and (13), we obtain the claim.
To complete the proof, we prove Lemma 3.8 in the following subsection.

3.3 Proof of Lemma 3.8

Lemma 3.8 is shown by carefully discussions of the conditional expectations characterized by the following Lemmas derived from Observations 3.1-3.3. First, we introduce the following lemma, which characterizes the difference between $\mathcal{X}_w^{(T)}$ and $(\mathcal{X}^{(0)} P^T)_w$ by the summation of the geometric series of the round matrix and $1\{D_v^{(t)}(k) = u\} - E[1\{D_v^{(t)}(k) = u\} | \mathcal{X}_v^{(t)}]$. 

**Lemma 3.9 (Relation between the discrete diffusion and the continuous diffusion).**

\[
\mathcal{X}_w^{(T)} - (\mathcal{X}^{(0)} P^T)_w = \sum_{t=0}^{T-1} \sum_{v \in V} \sum_{k=0}^{X_v^{(t)}-1} \left(1\{D_v^{(t)}(k) = u\} - E\left[1\{D_v^{(t)}(k) = u\} | \mathcal{X}_v^{(t)}\right]\right) (P_{P_{u,w}}^t - P_{P_{w,w}}^t)
\]

holds for any $\mathcal{X}^{(0)} \in \mathbb{Z}_+^N$, $T \geq 0$, $\mathcal{X}^{(T)}$ of Algorithm 2 and $w \in V$.

**Proof.** Since

\[
\sum_{t=0}^{T-1} \left(\mathcal{X}^{(t+1)} - \mathcal{X}^{(t)} P\right) P^{t-1} = \sum_{t=0}^{T-1} \left(\mathcal{X}^{(t+1)} P^{T-t-1} - \mathcal{X}^{(t)} P^{T-t}\right) = \mathcal{X}^{(T)} P^0 - \mathcal{X}^{(0)} P^T = \mathcal{X}^{(T)} - \mathcal{X}^{(0)} P^T,
\]

\[
\mathcal{X}_w^{(T)} - (\mathcal{X}^{(0)} P^T)_w = \sum_{t=0}^{T-1} \sum_{u \in V} \sum_{v \in V} \left(\mathcal{X}_u^{(t)} - (\mathcal{X}^{(t)} P)_v\right) P_{P_{u,w}}^{T-t-1}
\]

holds. Then, from Observation 3.1

\[
\mathcal{X}_u^{(t+1)} = \sum_{v \in V} \mathcal{X}_v^{(t)} - \sum_{k=0}^{X_v^{(t)}-1} 1\{D_v^{(t)}(k) = u\}
\]

holds and from Observation 3.2

\[
(\mathcal{X}^{(t)} P)_u = \sum_{v \in V} \mathcal{X}_u^{(t)} P_{v,u} = \sum_{v \in V} \mathcal{X}_v^{(t)} \sum_{k=0}^{X_v^{(t)}-1} \left\lfloor \frac{k}{\mathcal{X}_v^{(t)}} \right\rfloor \left\lfloor \frac{k + 1}{\mathcal{X}_v^{(t)}} \right\rfloor \cap \widetilde{P}_{v,u}
\]

\[
= \sum_{v \in V} \sum_{k=0}^{X_v^{(t)}-1} E\left[1\{D_v^{(t)}(k) = u\} | \mathcal{X}_v^{(t)}\right].
\]

holds. Thus, from (14)-(16),

\[
\mathcal{X}_w^{(T)} - (\mathcal{X}^{(0)} P^T)_w = \sum_{t=0}^{T-1} \sum_{u \in V} \sum_{v \in V} \sum_{k=0}^{X_v^{(t)}-1} \left(1\{D_v^{(t)}(k) = u\} - E\left[1\{D_v^{(t)}(k) = u\} | \mathcal{X}_v^{(t)}\right]\right) P_{P_{u,w}}^{T-t-1}
\]

\[
= \sum_{t=0}^{T-1} \sum_{v \in V} \sum_{k=0}^{X_v^{(t)}-1} \sum_{u \in V} \sum_{k=0}^{X_v^{(t)}-1} \left(1\{D_v^{(t)}(k) = u\} - E\left[1\{D_v^{(t)}(k) = u\} | \mathcal{X}_v^{(t)}\right]\right) P_{P_{u,w}}^{T-t-1}.
\]
Then, since
\[
\sum_{u \in N_P} \left( \mathbf{1}\{D_v^{(t)}(k) = u\} - \mathbb{E}\left[ \mathbf{1}\{D_v^{(t)}(k) = u\} \mid X_v^{(t)} \right] \right) P_{v,w}^{T-t-1} = 0, \tag{18}
\]
we obtain the claim by subtracting (18) from (17). Note that we have \(\sum_{u \in N_P} \mathbf{1}\{D_v^{(t)}(k) = u\} = 1\) and \(\sum_{u \in N_P} \mathbb{E}[\mathbf{1}\{D_v^{(t)}(k) = u\} \mid X_v^{(t)}] = 1\).

Next, we introduce the following three lemmas which are concerned with the conditional expectation of \(\mathbf{1}\{D_t = u\}\) and \(\mathbb{E}[\mathbf{1}\{D_t = u\} \mid X_v^{(t)}]\). Each of them is derived from our key Observations 3.1 and 3.3.

**Lemma 3.10.** For any \(t \in \{0, 1, \ldots, T-1\}\), \(v \in \{0, 1, \ldots, N-1\}\) and \(k \in \{0, 1, \ldots, K-1\}\), let \(l = KN_t + Kv + k\). Then, it holds that
\[
\mathbb{E}[\mathbf{1}\{D_t = u\} \mid D_0, D_1, \ldots, D_{t-1}] = \mathbb{E}\left[\mathbf{1}\{D_t = u\} \mid X_v^{(t)} \right].
\]

**Proof.** From the chain rule of conditional expectations, we have
\[
\mathbb{E}[\mathbf{1}\{D_t = u\} \mid D_0, \ldots, D_{t-1}] = \mathbb{E}[\mathbf{1}\{D_t = u\} \mid D_0, \ldots, D_{KN_t-1}, D_{KN_t}, \ldots, D_{t-1}] = \mathbb{E}[\mathbf{1}\{D_t = u\} \mid D_0, \ldots, D_{KN_t-1}] = \mathbb{E}\left[\mathbf{1}\{D_t = u\} \mid X_v^{(t)} \right].
\]
The second equality holds from the independency (Observation 3.3). The last equality holds from Observation 3.7, i.e. \(D_0, \ldots, D_{KN_t-1}\) determines \(X_v^{(t)}\).

**Lemma 3.11.** For any \(t \in \{0, 1, \ldots, T-1\}\), \(v \in \{0, 1, \ldots, N-1\}\) and \(k \in \{0, 1, \ldots, K-1\}\), let \(l = KN_t + Kv + k\). Then, for any \(\ell \in \{0, 1, \ldots, KNT-1\}\), it holds that
\[
\mathbb{E}[\mathbf{1}\{D_t = u\} \mid D_0, D_1, \ldots, D_{\ell}] = \begin{cases} \mathbf{1}\{D_t = u\} & \text{if } l \leq \ell \\ \mathbb{E}\left[\mathbb{E}[\mathbf{1}\{D_t = u\} \mid X_v^{(t)}] \mid D_0, D_1, \ldots, D_{\ell}\right] & \text{if } l \geq \ell + 1 \end{cases}
\]

**Proof.** For the case \(l \leq \ell\), the claim is true obviously. If \(l \geq \ell + 1\), it holds that \(l - 1 \geq \ell\) and from the chain rule of the conditional expectation, we have
\[
\mathbb{E}[\mathbf{1}\{D_t = u\} \mid D_0, \ldots, D_{\ell}] = \mathbb{E}\left[\mathbb{E}[\mathbf{1}\{D_t = u\} \mid X_v^{(t)}] \mid D_0, \ldots, D_{t-1}\right] \mid D_0, \ldots, D_{\ell}].
\]
Thus we obtain the claim from Lemma 3.10.

**Lemma 3.12.** For any \(t \in \{0, 1, \ldots, T-1\}\), \(v \in \{0, 1, \ldots, N-1\}\) and \(k \in \{0, 1, \ldots, K-1\}\), let \(l = KN_t + Kv + k\). Then, for any \(\ell \geq KNT - 1\), it holds that
\[
\mathbb{E}\left[\mathbb{E}[\mathbf{1}\{D_t = u\} \mid X_v^{(t)}] \mid D_0, D_1, \ldots, D_{\ell}\right] = \mathbb{E}\left[\mathbf{1}\{D_t = u\} \mid X_v^{(t)}\right].
\]

**Proof.** Since \(\mathbb{E}[\mathbf{1}\{D_t = u\} \mid X_v^{(t)}]\) is a function of \(X_v^{(t)}\) (Observation 3.2) and \(X_v^{(t)}\) is determined by \(D_0, \ldots, D_{KN_t-1}\) (Observation 3.7), we obtain the claim.
Proof of Lemma 3.8. For any \( t \in \{0,1,\ldots,T-1\} \), \( v \in \{0,1,\ldots,N-1\} \) and \( k \in \{0,1,\ldots,K-1\} \), let \( l = KNt + Kv + k \). Then, for any \( u \in N_v^P \) and \( \ell \in \{0,1,\ldots,KNT-1\} \), let

\[
\Delta_u(l, \ell) := \left(\mathbb{E} \left[ \mathbf{1}\{D_l = u\} \mid D_0, \ldots, D_{\ell}\right] - \mathbb{E} \left[ \mathbf{1}\{D_l = u\} \mid \mathcal{X}_v^{(\ell)} \mid D_0, \ldots, D_{\ell}\right]\right)
- \left(\mathbb{E} \left[ \mathbf{1}\{D_l = u\} \mid D_0, \ldots, D_{\ell-1}\right] - \mathbb{E} \left[ \mathbf{1}\{D_l = u\} \mid \mathcal{X}_v^{(\ell)} \mid D_0, \ldots, D_{\ell-1}\right]\right). \tag{19}
\]

From Lemma 3.9 and the definition of \( D_v \), it holds that

\[
\mathbb{E} \left[\mathcal{X}_u^{(T)} - (\mathcal{X}_v^{(0)} P^{T})_w \mid D_0, \ldots, D_{\ell}\right] - \mathbb{E} \left[\mathcal{X}_v^{(T)} - (\mathcal{X}_v^{(0)} P^{T})_w \mid D_0, \ldots, D_{\ell-1}\right] = \sum_{t=0}^{T-1} \sum_{v=0}^{N-1} \sum_{k=0}^{K-1} \sum_{u \in N_v^P} \Delta_u(KNt + Kv + k, \ell) \left(P^{T-t-1}_{u,v,w} - P^{T-t-1}_{v,w}\right). \tag{20}
\]

To obtain the claim, we show that \( \Delta_u(KNt + Kv + k, \ell) = \Delta_u(l, \ell) = 0 \) for any \( l \neq \ell \) firstly. We consider this by the following case 1. \((l \leq \ell - 1)\) and case 2. \((l \geq \ell + 1)\).

**case 1.** \( l \leq \ell - 1 \): In this case, it holds that both \( l \leq \ell \) and \( l \leq \ell - 1 \). From Lemma 3.11, we have

\[
\mathbb{E} \left[ \mathbf{1}\{D_l = u\} \mid D_0, \ldots, D_{\ell}\right] = \mathbf{1}\{D_l = u\} \quad \text{and} \quad \mathbb{E} \left[ \mathbf{1}\{D_l = u\} \mid D_0, \ldots, D_{\ell-1}\right] = \mathbf{1}\{D_l = u\}. \tag{21}
\]

Thus \( \Delta_u(l, \ell) = 0 \) from (21) in this case.

**case 2.** \( l \geq \ell + 1 \): In this case, it holds that both \( l \geq \ell + 1 \) and \( l \geq \ell - 1 \). Using Lemma 3.11, we have

\[
\mathbb{E} \left[ \mathbf{1}\{D_l = u\} \mid D_0, \ldots, D_{\ell}\right] = \mathbb{E} \left[ \mathbb{E} \left[ \mathbf{1}\{D_l = u\} \mid \mathcal{X}_v^{(\ell)} \right] \mid D_0, \ldots, D_{\ell}\right] \quad \text{and} \tag{25}
\]

\[
\mathbb{E} \left[ \mathbf{1}\{D_l = u\} \mid D_0, \ldots, D_{\ell-1}\right] = \mathbb{E} \left[ \mathbb{E} \left[ \mathbf{1}\{D_l = u\} \mid \mathcal{X}_v^{(\ell)} \right] \mid D_0, \ldots, D_{\ell-1}\right]. \tag{26}
\]

Thus \( \Delta_u(l, \ell) = 0 \) from (25), (26) and (19) in this case.

From these discussion, all term such that \( l \neq \ell \) in (20) is 0, hence we have

\[
\Delta_u(l, \ell) = \sum_{u \in N_v^P} \Delta_u(l, \ell) \left(P^{T-t-1}_{u,v,w} - P^{T-t-1}_{v,w}\right). \tag{27}
\]

Note that \( \ell = KN\tau + Kn' + \kappa \) from the assumption of the Lemma 3.8. We conclude the proof by showing that \( \Delta_u(l, \ell) = \mathbf{1}\{D_\ell = u\} - \mathbb{E}[\mathbf{1}\{D_\ell = u\} \mid \mathcal{X}_v^{(\tau)}]\). We have

\[
\mathbb{E} \left[ \mathbf{1}\{D_\ell = u\} \mid D_0, \ldots, D_{\ell}\right] = \mathbf{1}\{D_\ell = u\} \quad \text{and} \tag{28}
\]

\[
\mathbb{E} \left[ \mathbf{1}\{D_\ell = u\} \mid D_0, \ldots, D_{\ell-1}\right] = \mathbb{E} \left[ \mathbf{1}\{D_\ell = u\} \mid \mathcal{X}_v^{(\tau)} \right], \tag{29}
\]

where the second equality holds from Lemma 3.10. Note that both \( \ell \geq KN\tau - 1 \) and \( \ell - 1 \geq KN\tau - 1 \) hold since \( l - 1 = KN\tau + Kn' + \kappa - 1 \geq K\bar{N}\tau - 1 \). From these facts and Lemma 3.12, we have

\[
\mathbb{E} \left[ \mathbb{E} \left[ \mathbf{1}\{D_\ell = u\} \mid \mathcal{X}_v^{(\tau)} \right] \mid D_0, \ldots, D_{\ell}\right] = \mathbb{E} \left[ \mathbf{1}\{D_\ell = u\} \mid \mathcal{X}_v^{(\tau)} \right] \quad \text{and} \tag{30}
\]

\[
\mathbb{E} \left[ \mathbb{E} \left[ \mathbf{1}\{D_\ell = u\} \mid \mathcal{X}_v^{(\tau)} \right] \mid D_0, \ldots, D_{\ell-1}\right] = \mathbb{E} \left[ \mathbf{1}\{D_\ell = u\} \mid \mathcal{X}_v^{(\tau)} \right]. \tag{31}
\]

Thus \( \Delta_u(l, \ell) = \mathbf{1}\{D_\ell = u\} - \mathbb{E}[\mathbf{1}\{D_\ell = u\} \mid \mathcal{X}_v^{(\tau)}] \) from (28), (31) and (19), and we obtain the claim. \( \square \)
4 General upper bound of the local 2-divergence

This section gives a general upper bound of $\Psi_2(P)$ for any irreducible, reversible and lazy chain. Let $\pi \in \mathbb{R}_{\geq 0}$ denotes the stationary distribution of the round matrix $P$, i.e. the probability distribution such that $\pi P = \pi$ holds. A $P$ is called reversible if $\pi_v P_{v,u} = \pi_u P_{u,v}$ holds for any $v, u \in V$. A $P$ is called lazy if $P_{v,v} \geq 1/2$ holds for any $v \in V$.

For a reversible $P$, its stationary distribution $\pi$ and a vector $f \in \mathbb{R}^N$, the Dirichlet form $E(f)$ is defined as follows.

$$E(f) := \frac{1}{2} \sum_{(v,u) \in V \times V} (f_v - f_u)^2 \pi_v P_{v,u}. \quad (32)$$

Dirichlet form is concerned with many techniques to estimating the key measures of Markov chains such as eigenvalues (cf. [3][8]). Now, we consider the connection between the Dirichlet form and the local 2-divergence. Since there exists a $w \in V$ such that $(\Psi_2(P))^2 = \sum_{t=0}^{\infty} \sum_{(v,u) \in V \times V : P_{v,u} > 0} (P^t_{v,w} - P^t_{u,w})^2$ from Definition 1.4.

$$\left( \Psi_2(P) \right)^2 \leq \sum_{t=0}^{\infty} \sum_{(v,u) \in V \times V} (P^t_{v,w} - P^t_{u,w})^2 \frac{\pi_v P_{v,u}}{\min_{(v,u) \in V \times V : P_{v,u} > 0} \pi_v P_{v,u}} \frac{2}{\min_{(v,u) \in V \times V : P_{v,u} > 0} \pi_v P_{v,u}} \sum_{t=0}^{\infty} E(P^t_{v,u}). \quad (33)$$

holds by taking $f_x = P^t_{x,w}$. Now, we introduce the following main lemma to give an upper bound of the local 2-divergence.

**Lemma 4.1 (Dirichlet form of $P^t_{v,w}$).** For any reversible $P$, $w \in V$ and $t \geq 0$, it holds that

$$E(P^t_{v,u}) = \pi_w (P^t_{w,w} - P^{2t+1}_{w,w}).$$

**Proof.** From the definition of the Dirichlet form, it holds that

$$2E(P^t_{v,u}) = \sum_{v \in V} \sum_{u \in V} \pi_v P_{v,u} (P^t_{v,w} - P^t_{u,w})^2$$

$$= \sum_{v \in V} \sum_{u \in V} \pi_v P_{v,u} (P^t_{v,w})^2 + \sum_{v \in V} \sum_{u \in V} \pi_v P_{v,u} (P^t_{u,w})^2 - 2 \sum_{v \in V} \sum_{u \in V} \pi_v P_{v,u} P^t_{v,w} P^t_{u,w}. \quad (34)$$

Then, from the reversibility of $P$,

$$\sum_{v \in V} \sum_{u \in V} \pi_v P_{v,u} (P^t_{v,w})^2 + \sum_{v \in V} \sum_{u \in V} \pi_v P_{v,u} (P^t_{u,w})^2$$

$$= \sum_{v \in V} \sum_{u \in V} \pi_v P_{v,u} (P^t_{v,w})^2 + \sum_{v \in V} \sum_{u \in V} \pi_u P_{u,v} (P^t_{u,w})^2$$

$$= \sum_{v \in V} \pi_w P^t_{w,v} P^t_{v,w} + \sum_{u \in V} \pi_w P^t_{w,u} P^t_{u,w} = 2\pi_w P^{2t}_{w,w} \quad \text{and} \quad (35)$$

and

$$2 \sum_{v \in V} \sum_{u \in V} \pi_v P_{v,u} P^t_{v,w} P^t_{u,w} = 2\pi_w \sum_{v \in V} \sum_{u \in V} P^t_{w,v} P_{v,u} P^t_{u,w} = 2\pi_w P^{2t+1}_{w,w} \quad (36)$$

hold. Combining (34)-(36), we obtain the claim. \qed
Proof of Theorem 1.5 Combining (33), Lemma 4.1 and the assumption of laziness of $P$, 
\[
\sum_{t=0}^{\infty} \mathcal{E}(P^t_{w,v}) = \pi_w \sum_{t=0}^{\infty} (P^{2t}_{w,v} - P^{2t+1}_{w,v}) \leq \pi_w \sum_{t=0}^{\infty} (P^{2t}_{w,v} - P^{2t+2}_{w,v}) \leq \pi_w
\]
holds, thus we obtain the Theorem 1.5. Note that $P^t_{w,v} \geq P^{t+1}_{w,v}$ holds for any lazy $P$ (cf. Proposition 10.25 of [8]).

5 Upper bound on the discrepancy for symmetric round matrices

Now, we conclude this paper stating the proofs of Theorems 1.1 and 1.2. First, we show the following general theorem according to the discrepancy for any symmetric round matrix.

Theorem 5.1 (Result for symmetric matrices). Suppose that $P$ is irreducible and symmetric. Then, for any $X(0)$ and for each $T \geq \log(4\text{Disc}(X(0))N) / (1-\lambda_2^2)$, $X(T)$ of Algorithm 2 satisfies that
\[
\Pr\left[\text{Disc}(X(T)) \leq 9\Psi_2(P) \sqrt{\log N}\right] \geq 1 - \frac{2}{N}.
\]

Proof. Since $X_v^{(T)} - X_u^{(T)} = (X_v^{(T)} - (X(0)P^T)_v) + ((X(0)P^T)_v - (X(0)P^T)_u) + ((X(0)P^T)_u - X_u^{(T)})$, we have
\[
\left|X_v^{(T)} - X_u^{(T)}\right| \leq \left|X_v^{(T)} - (X(0)P^T)_v\right| + \left|(X(0)P^T)_v - (X(0)P^T)_u\right| + \left|(X(0)P^T)_u - X_u^{(T)}\right|
\]
\[
\leq 2\text{Disc}(X^{(T)}) + \left|(X(0)P^T)_v - (X(0)P^T)_u\right|
\]
Combining Proposition 2.1 and Theorem 1.3 we obtain the claim.

Proof of Theorem 1.1 Since Algorithm 1 is Algorithm 2 according to the transition matrix of the lazy random walk on $G = (V, E)$, i.e. $P$ such that $P_{v,u} = 1/(2d)$ for any $\{v,u\} \in E$, $P_{v,v} = 1/2$ for any $v \in V$, and $P_{v,u} = 0$ for any $\{v,u\} \notin E$. For this $P$, since $\min_{\{v,u\} \in V \times V: P_{v,u} > 0} P_{v,u} = 1/(2d)$, $\Psi_2(P) \leq 2\sqrt{d}$ holds from Corollary 1.6 Thus we obtain the claim by Theorem 5.1.

Proof of Theorem 1.2 From the definition of $P_M$ in Section 1.3.2 this chain is lazy and symmetric. Since $\min_{\{v,u\} \in V \times V: P_{v,u} > 0} P_{v,u} = 1/(2d_{\text{max}})$, $\Psi_2(P) \leq 2\sqrt{d_{\text{max}}}$ holds from Corollary 1.6 Thus we obtain the claim by Theorem 5.1.

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References

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A Proof of Proposition 2.1

Proof. We have

\[
\left| \left( X(0) P^T \right)_w - \frac{K}{N} \right| = \left| \sum_{v \in V} X_v(0) \left( P^T_{w,v} - \frac{1}{N} \right) \right| = \left| \sum_{v \in V} X_v(0) \left( P^T_{w,v} - \frac{1}{N} \right) \right| \\
= \left| \sum_{v \in V} (X_v(0) - X_v(0)) \left( P^T_{w,v} - \frac{1}{N} \right) \right| \leq 2\text{Disc}(X(0)) \cdot \frac{1}{2} \sum_{v \in V} \left| P^T_{w,v} - \frac{1}{N} \right|.
\]

Note that we used the assumption of the symmetry of \( P \) and \( \sum_{v \in V} X_v(0) \left( P^T_{w,v} - \frac{1}{N} \right) = 0 \) for any \( x \in V \). Using Theorem 12.4 in [8], \( \frac{1}{2} \sum_{v \in V} \left| P^T_{w,v} - \frac{1}{N} \right| \leq \varepsilon' \) holds for any \( T \geq \frac{1}{1 - \lambda_2} \log \left( \frac{N}{\varepsilon'} \right) \). Thus, with the assumption of \( T \), we have \( \frac{1}{2} \sum_{v \in V} \left| P^T_{w,v} - \frac{1}{N} \right| \leq \frac{\varepsilon}{4\text{Disc}(X(0))} \). Hence we have

\[
\left| \left( X(0) P^T \right)_v - \left( X(0) P^T \right)_u \right| = \left| \left( X(0) P^T \right)_v - \frac{K}{N} \right| - \left( \frac{K}{n} - \left( X(0) P^T \right)_u \right) \leq \left| X(0) P^T \right)_v - \frac{K}{N} \right| + \left| \left( X(0) P^T \right)_u - \frac{K}{N} \right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]
for any \( v, u \in V \), and we obtain the claim.

\[ \Box \]

## B Concentration inequality

**Theorem B.1** (Asuma-Hoeffding Inequality, [9]). Let \( X_0, \ldots, X_n \) be a martingale such that

\[
|X_k - X_{k-1}| \leq c_k.
\]

Then, for all \( t \geq 1 \) and any \( \lambda > 0 \),

\[
\Pr[|X_t - X_0| \geq \lambda] \leq 2 \exp \left[ -\frac{\lambda^2}{2 \sum_{k=1}^{t} (c_k)^2} \right].
\]