Development of Frequency Weighted Model Order Reduction Techniques for Discrete-Time One-Dimensional and Two-Dimensional Linear Systems with Error Bounds

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ABSTRACT Enns’ frequency weighted model reduction method yields an unstable reduced model. Many stability-preserving techniques for one-dimensional and two-dimensional reduced-order systems have been demonstrated; however, these methods produce significant truncation errors. This article presents a frequency weighted stability preserving framework, which addresses Enns’ main problem concerning reduced-order model instability. Unlike other stability-preserving techniques, the offered frameworks provide an easily computable a priori error-bound expression. The simulation results show that the proposed frameworks outperform existing stability-preserving approaches, demonstrating effectiveness.

INDEX TERMS Model reduction, minimal realization, Hankel-Singular values, optimal Hankel norm approximation, frequency response error, error bound.

ACRONYM/ABBREVIATION AND ELEMENTARY OPERATORS

In this article, following acronyms/abbreviations are used:

MOR Model order reduction
ROM Reduced order model
ODEs Ordinary differential equations
PDEs Partial differential equations
m-D Multi-dimensional
1-D One-dimensional
2-D Two-dimensional
BT Balanced Truncation
HNA Hankel norm approximation
CRSD Causal recursive separable denominator
GA Gugercin & Antoulas

The Table 1 provides some basic terminologies and their corresponding operators used in this article.

I. INTRODUCTION

A. MOTIVATION AND INCITEMENT

THE MOR challenge aims to develop an alternate model for the original large-scale stable model that is simple to measure and has the same responses as the original. In ROMs, the MOR attempts to retain the key characteristics
Table 1. Elementary Operators and Terminologies.

| Terminology                        | Description |
|------------------------------------|-------------|
| $F^*$                               | $A$ is reducible to $A = BC$ realization $\{A, B, C, D\}$ |
| $F \circ [\cdot]$                  | $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ |
| $\mathcal{H}_\infty$               | $\mathcal{H}_\infty$ operator |

The researcher focused on fundamental problems such as decomposition, factorization, stability, and model reduction, etc. As decomposition, factorization, stability, and model reduction are not straightforward extensions for 1-D models, the fundamental theorem of the algebra does not apply to m-D systems directly [24–27].

### B. LITERATURE REVIEW

The most often used MOR approach is BT [28], while using BT [28] methods, it is necessary to balance the system, which is equivalent to determining the system’s controllability and observability Gramians in a unique diagonal form. The Cholesky factors of these Gramians can be efficiently computed as dual Lyapunov equation solutions for systems with few inputs and outputs. BT [28] provides ROMs for the 1-D LTI continuous and the discrete-time systems that guarantee stability and yield error bounds. However, an entire frequency interim is used to execute MOR operations, while the particular frequency band is concerned only in practical applications, i.e., the controller reduction case. Similarly, Glover [29] used an optimal HNA to perform the MOR operation. The HNA is a model reduction method that offers the best Hankel semi-norm approximation. These promote the usage of frequency weights in MOR. Therefore, Enns [30], [31] provided the frequency weighted MOR approach for the 1-D LTI continuous and the discrete-time systems by inducing frequency weights (i.e., input, output and double-sided) in BT [28] approach. However, this approach [30] generates unstable ROMs in the case of double-sided weightings [32]. Similarly, the limited frequency interval [33] is of concern for some applications (i.e., controller and filter reduction). The frequency-limited intervals Gramians based MOR approach for the 1-D LTI continuous and the discrete-time systems were implemented by the GJ [34] and WZ [35], respectively; however, it does often result in unstable ROMs at certain frequency-intervals [36], [37] and there exist no a priori error bound expressions for these techniques [30], [34], [35].

Recently, a significant amount of research has been conducted on the MOR of large-scale systems, and a number of different MOR approaches have been developed. [7–12]. In [7], second-order dynamical systems using structure-preserving balanced truncation approaches are provided, which deals with first-order constrained balanced truncation approaches and apply them to second-order systems utilizing various second-order balanced truncation formulas. The
work presented in [8] is based on a balanced truncation model order reduction for discrete-time systems that preserves stability after reduction. However, due to the iterative nature of this method, it becomes more complicated when the order of the original system increases. Correspondingly, [9] presents MOR based on cross Gramians; this method [9] uses Sylvester equations rather than Lyapunov equations as described by BT [28], Enns [30], GJ [34], and WZ [35]. Furthermore, this method is only applicable to bilinear systems that employ a truncated cross Gramian projection approach. A similar work based on interpolation is presented in [10]. It proposes adaptive techniques for computing time delay systems’ reduced-order model. The algorithms use greedy iterations to choose expansion locations and interpolate the transfer function. Similarly, another interpolation-based approach is presented in [11]. It focuses on dominated and temporal moment retention. It condenses the large-scale complete order model into a lower order system, allowing approximate computation denominator by employing generalized pole clustering. The factors division procedure yields the approximate numerator, which results in the ROM. In [12], the MOR for 2-D discrete-time system MOR is presented. This method ensures the stability of the filtering error system and $H_{\infty}$ performance when the noise frequency ranges are known beforehand. Using the gKYP lemma, Finsler’s lemma, and some independent matrices yield fewer conservative findings. The research briefly discussed above are based on cross Gramians, interpolation, and Kalman filtering. Furthermore, to overcome the shortcomings as appeared in Enns [30], GJ [34], and WZ [35] substantial amount of research have been conducted over the couple of decades [36]–[46], which are briefly discussed as follows with their drawbacks.

To overcome the main drawback as appeared in [30], the Lin & Chiu [38] introduced strictly proper two-sided weights to ensure the stability of ROMs; however, this method cannot be used in controller reduction applications due to no pole-zero cancellation assumption required in the method. Later on, VA [39] introduced an alternative approach to ensure the stability of the ROM for the continuous-time frequency weighted systems. Since the main weakness of Lin and Chiu’s [38] technique is the requirement that no pole-zero cancellation occurs when forming the augmented systems (input augmented and output augmented). This prevents the applicability of this method when solving controller reduction problems involving weights; however, this technique [39] is only valid for strictly proper original systems.

The instability problem in [30] is related to the indefiniteness of the corresponding input and output matrices; CB [40] provided the stability-preserving frequency weighted MOR method by ensuring the input and output matrices are positive/semi-positive definite. As a result, some eigenvalues have significant variations while others have slight variations. Dissimilar effects on each eigenvalue of the input and output matrices result in a significant approximation error in the ROM. The GS method [41] combines unweighted balanced and partial-fraction-based frequency weighted balanced reduction techniques, ensuring ROM stability but being parameterized. The GS [42] also proposed a MOR technique for 2-D discrete-time weighted systems. However, truncating negative eigenvalues causes a significant approximation error in 2-D ROM. The stability-preserving frequency-weighted MOR approach introduced by IG [43] involves varying the input and output matrices, but subtracting all eigenvalues from minor eigenvalues results in zeroing the last eigenvalue, resulting in an unequal effect to eigenvalues and a significant approximation error in the ROM.

Together with the use of positive/semi-positive definiteness of input and output matrices, GA [36] established stability preserving frequency weighted limited Gramians based MOR approach. However, the asymmetrical impacts on all eigenvalues cause significant approximation error [36]. By using frequency-limited intervals, GS [41] developed ROM stability. GS’s approach [41] produces a large approximation error due to the significant variation in the original system. In later work, IG [44] adjusted the eigenvalues matrix by subtracting the least dependent negative eigenvalue from all the eigenvalues; nonetheless, the modified eigenvalues cause significant changes to the original systems and large approximation error. Similarly, [45] offers three techniques to maintain ROM stability; however, [45] is iterative, which is inefficient when the original system’s order rises.

Similarly, to overcome the main drawback as appeared in [35], GS [37] ensures the stability of the ROM by improving the eigenvalues matrix; however, due to the truncation of negative eigenvalues and absolute of all the eigenvalues, it increases a distance from the eigenvalues matrix of the original systems, which leads to a large approximation in the ROM. Similarly, IG [46] also introduced frequency limited MOR approach for the discrete-time systems; however, this approach results in significant truncation errors in the desired discrete frequency intervals due to the significant variance from the original system and zeroing the effect of the last eigenvalue.

Recently, a significant amount of research has been conducted on the MOR of large-scale systems based on balanced approach [47]–[51]. In [47], weighted and limited interval discrete-time 1-D systems are provided. The frequency limited intervals for 1-D and 2-D systems are given in [48]. Similarly, frequency weighted and limited MOR approaches for power systems are given in [49]–[51].

The BT [28], Enns [30], GJ [34], and WZ [35] yield unstable ROM and do not provide a priori error-bound expressions. Further, their successive stability preserving approaches [36]–[46] ensure stability in some conditions and generate significant truncation error due to the substantial variation to the original systems (i.e., pole-zero cancellation, absolute of negative eigenvalues, truncation of all negative eigenvalues, zeroing the effect of the last eigenvalue, etc.).

C. MAIN CONTRIBUTION AND PAPER ORGANIZATION
A novel method for 1-D and 2-D discrete-time systems is proposed. For 1-D and 2-D discrete-time systems, the
suggested method offers a new discrete frequency weighted strategy exhibiting small truncation error. The square root of all eigenvalues with similar effects prevents the zeroing of the last eigenvalues, provides an equal impact on all eigenvalues, and preserves the eigenvalues’ structure of some input and output matrices. Compared to other stability-preserving model reduction frameworks based on frequency-weighted Gramians, the proposed method provides small variation to the original system.

The main contributions of this paper are as follows:

- Decomposition of the discrete-time 2-D CRSD model based on frequency weightings into two decomposed 1-D sub-models is attained by using the minimal rank-decomposition method.

- Modifications to associated input and output matrices are performed for 1-D models and corresponding decomposed 1-D sub-models to assure positive and semi-positive definiteness of associated input and output matrices.

- The controllability and observability Gramians for 1-D and 2-D systems is given for frequency weights are computed, corresponding to modified input and output matrices.

- Stability of ROMs are ensured incase of 1-D and 2-D weighted systems.

- Frequency weighted a priori error bound formula for the 1-D and 2-D systems are derived based on balance truncation.

- Frequency weighted a priori error bound formula for the 1-D and 2-D systems are derived based on an optimal HNA.

- Comparison among different existing frequency weighted MOR techniques (including 1-D and 2-D) with proposed techniques are presented.

The MOR framework based on frequency weighted for linear time-invariant discrete-time 1-D and 2-D systems is presented in this paper. The 1-D and 2-D un-weighted and weighted models are discussed in Section II, and the 2-D model decomposition via minimal rank-decomposition conditions. The balance truncation approach, as well as frequency weighted MOR approaches, are discussed in Section III. The existing stability-preserving frequency weighted balancing related techniques for 1-D and 2-D discrete-time systems are also discussed in this part. Section IV lays out the proposed work for 1-D and 2-D discrete-time systems and the a priori error-bound expressions for 1-D and 2-D cases. In addition, the numerical simulation results are presented in section V, where a comparison is made between existing 1-D, and 2-D frequency weighted MOR techniques and proposed techniques, demonstrating the proposed techniques’ efficacy.

II. PRELIMINARIES

This section presents the corresponding un-weighted and frequency weighted 1-D and 2-D state space systems.

A. 1-D STATE SPACE SYSTEM

Here we provide a brief overview of un-weighted, and frequency weighted 1-D state-space discrete-time systems.

1) Un-Weighted 1-D State-Space System

Consider a 1-D discrete time system be given as:

\[
x[k+1] = A_x x[k] + B_x u[k], \\
y[k] = C_x x[k] + D_x u[k], \\
F_r[z] = D_s + C_s [z I - A_s]^{-1} B_s,
\]

where \( \{A_s \in \mathbb{R}^{n \times n}, B_s \in \mathbb{R}^{n \times m}, C_s \in \mathbb{R}^{p \times n}, D_s \in \mathbb{R}^{p \times m}\} \) is its \( n^{th} \) order minimal realization with \( m \) number of inputs and \( p \) number of outputs. The ROM is obtained as:

\[
x_r[k+1] = A_r x_r[k] + B_r u[k], \\
y_r[k] = C_r x_r[k] + D_r u[k], \\
F_r[z] = D_r + C_r [z I - A_r]^{-1} B_r,
\]

is achieved by truncating the large-scale stable original system \cite{28} (i.e., in the entire-frequency intervals \( \omega_1, \omega_2 \in [-\pi, \pi] \)), where \( \{A_r, B_r, C_r, D_r \in \mathbb{R}^{p \times r}, D_r \in \mathbb{R}^{p \times m}\} \) with \( r \ll n \).

2) Frequency Weighted 1-D State-Space System

Consider a transfer function form of a stable discrete-time input-weighting model be given as:

\[
x_i[k+1] = A_{iw} x_i[k] + B_{iw} u_i[k], \\
y_i[k] = C_{iw} x_i[k] + D_{iw} u_i[k], \\
G_i[z] = D_{iw} + C_{iw} [z I - A_{iw}]^{-1} B_{iw},
\]

where \( A_{iw} \in \mathbb{R}^{(n_i \times n_i)}, B_{iw} \in \mathbb{R}^{(n_i \times m_i)}, C_{iw} \in \mathbb{R}^{(p_i \times m_i)}, D_{iw} \in \mathbb{R}^{(p_i \times n_i)} \) is its \( n_i^{th} \) order minimal realization. Similarly, consider a transfer function form of a stable discrete-time output-weighting model

\[
x_o[k+1] = A_{ow} x_o[k] + B_{ow} u_o[k], \\
y_o[k] = C_{ow} x_o[k] + D_{ow} u_o[k], \\
H_o[z] = D_{ow} + C_{ow} [z I - A_{ow}]^{-1} B_{ow},
\]

where \( A_{ow} \in \mathbb{R}^{(n_o \times n_o)}, B_{ow} \in \mathbb{R}^{(n_o \times m_o)}, C_{ow} \in \mathbb{R}^{(p_o \times m_o)}, D_{ow} \in \mathbb{R}^{(p_o \times n_o)} \) is its \( n_o^{th} \) order minimal realization. The input-augmented and the output-augmented systems are given by:

\[
F[z]G_i[z] = C_{ai}[z I - A_{ai}]^{-1} B_{ai} + D_{ai}, \\
H_o[z]F[z] = C_{ao}[z I - A_{ao}]^{-1} B_{ao} + D_{ao},
\]

where

\[
\begin{bmatrix}
A_{ai} & B_{ai} \\
C_{ai} & D_{ai}
\end{bmatrix} = 
\begin{bmatrix}
A_s & B_s C_{iw} & B_s D_{iw} \\
0 & A_{iw} & D_{iw}
\end{bmatrix},
\]

\[
\begin{bmatrix}
A_{ao} & B_{ao} \\
C_{ao} & D_{ao}
\end{bmatrix} = 
\begin{bmatrix}
A_{ow} & B_{ow} C_{iw} & B_{ow} D_{iw} \\
0 & A_s & D_{ow}
\end{bmatrix}.
\]
B. 2-D STATE SPACE SYSTEMS

Here we provide a brief overview of un-weighted and frequency weighted 2-D systems with its decomposition based on minimal rank-decomposition criteria and weighted 2-D state-space discrete-time systems.

1) Un-Weighted 2-D State-Space System

Consider a stable LTI MIMO, minimal separable denominator 2-D discrete-time Roesser’s state-space model be given as [52]:

\[
\begin{align*}
x[i, j] &= Ax[i, j] + Bu[i, j], \quad (7) 
y[i, j] &= Cx[i, j] + Du[i, j], \quad (8) 
F[z_1, z_2] &= D + C[z_1 I_n \oplus z_2 I_m - A]^{-1} B. \quad (9)
\end{align*}
\]

where

\[
A = \begin{bmatrix} A_1 & A_2 \\
A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\
B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix},
\]

and \( i, j \) are vertical and horizontal coordinates respectively, \( x_h(i, j) \in \mathbb{R}^n \) and \( x_v(i, j) \in \mathbb{R}^m \) are the horizontal and vertical state vectors that convey horizontal and vertical information, respectively, \( u(i, j) \in \mathbb{R}^p \) and \( v(i, j) \in \mathbb{R}^q \) and \( A \in \mathbb{R}^{(n+m)\times(n+m)}, B \in \mathbb{R}^{(n+m)\times(p)}, C \in \mathbb{R}^{(q)\times(n+m)}, D \in \mathbb{R}^{q \times p} \) is its \((n+m)\)th order minimal realization with \( p \) number of inputs and \( q \) number of outputs.

The MOR challenge is to determine

\[
F_r[z_1, z_2] = D_r + C_r[z_1 I_{r_1} \oplus z_2 I_{r_2} - A_r]^{-1} B_r, \quad (10)
\]

where \( A_r \in \mathbb{R}^{(r_1 r_2)\times(r_1 r_2)}, B_r \in \mathbb{R}^{(r_1 r_2)\times(p)}, C_r \in \mathbb{R}^{(q)\times(r_1 r_2)}, D_r \in \mathbb{R}^{q \times p} \) with \( r_1 << n_1 \) and \( r_2 << n_2 \).

Let the minimal rank-decomposition of Roesser’s state-space realization subject to \( A_3 = 0 \) be written as:

\[
\begin{bmatrix} A_2 & B_1 \\
C_2 & D \end{bmatrix} = \begin{bmatrix} B_1 \\
D_2 \end{bmatrix} \begin{bmatrix} C_2 & D_2 \end{bmatrix}, \quad (11)
\]

consequently, 2-D separable denominator state-space can be given as:

\[
\begin{align*}
A &= \begin{bmatrix} A_1 & B_1 C_2 \\
0 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\
B_2 \end{bmatrix}, \quad (12) 
C &= \begin{bmatrix} C_1 & D_1 C_2 \end{bmatrix}, \quad D = \begin{bmatrix} D_1 \\
D_2 \end{bmatrix}, \quad (13)
\end{align*}
\]

that results \( F[z_1, z_2] = \hat{F}[z_1, z_2] = \hat{F}_1[z_1] \hat{F}_2[z_2] \), where

\[
\begin{align*}
\hat{F}_1[z_1] &= \hat{D}_1 + C_1[z_1 I - A_1]^{-1} B_1, \quad (14) 
\hat{F}_2[z_2] &= \hat{D}_2 + C_2[zz_2 I - A_2]^{-1} B_2. \quad (15)
\end{align*}
\]

The decomposed 1-D system \( \hat{F}_1[z_1] \) is a \( p \)-input/\( p_1 \)-output system, and the decomposed 1-D system \( \hat{F}_2[z_2] \) is a \( p_2 \)-input/q-output system [52].

Similarly, the minimal rank-decomposition of Roesser’s state-space realization subject to \( A_2 = 0 \) can be written as:

\[
\begin{bmatrix} A_3 & B_2 \\
C_1 & D \end{bmatrix} = \begin{bmatrix} B_2 \end{bmatrix} \begin{bmatrix} C_1 & D \end{bmatrix}, \quad (16)
\]

consequently, 2-D separable denominator state-space can be written as:

\[
\begin{align*}
A &= \begin{bmatrix} A_1 & 0 \\
B_2 C_1 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\
B_2 \end{bmatrix}, \quad (17) 
C &= \begin{bmatrix} C_2 \\
D_2 C_1 \end{bmatrix}, \quad D = \begin{bmatrix} D_2 \end{bmatrix}, \quad (18)
\end{align*}
\]

that results \( F[z_1, z_2] = \tilde{F}[z_1, z_2] = \tilde{F}_1[z_1] \tilde{F}_2[z_2] \), where

\[
\begin{align*}
\tilde{F}_1[z_1] &= \tilde{D}_1 + C_1[z_1 I - A_1]^{-1} B_1, \quad (19) 
\tilde{F}_2[z_2] &= \tilde{D}_2 + C_2[zz_2 I - A_2]^{-1} B_2. \quad (20)
\end{align*}
\]

Remak 1: The 2-D models, as in (9), generally don’t exist in CRSD form; however, existing 1-D MOR schemes are only applicable to the 2-D systems when it exists in 2-D CRSD form. In addition, we need minimal rank-decomposition criteria to obtain decomposed 1-D sub-models as in (14)-(15) and (19)-(20).

Lemma 1: Let the ROM for 2-D discrete-time systems be \( F_r[z_1, z_2] = F_{1r}[z_1] F_{2r}[z_2] \) obtained by using 1-D BT [28], then the 2-D discrete-time ROM \( F_r[z_1, z_2] \) is asymptotically stable. Moreover, the frequency response truncation error is bounded by:

\[
\|F[z_1, z_2] - F_r[z_1, z_2]\|_\infty \leq 2(\|D_1\|_2 + \sum_{i=1}^{n} \hat{\rho}_i \times \sum_{i=m+1}^{m} \hat{\varphi}_i ) + (\|D_2\|_2 + \sum_{i=1}^{n} \hat{\varphi}_i \times \sum_{i=n+1}^{m} \hat{\rho}_i )
\]

Alternatively,

\[
\|F[z_1, z_2] - F_r[z_1, z_2]\|_\infty \leq 2(\|D_1\|_2 + \sum_{i=1}^{n} \hat{\rho}_i \times \sum_{i=m+1}^{m} \hat{\varphi}_i ) + (\|D_2\|_2 + \sum_{i=1}^{n} \hat{\varphi}_i \times \sum_{i=n+1}^{m} \hat{\rho}_i )
\]

where \( \hat{\rho}_i \) and \( \hat{\varphi}_i \) are the Hankel Singular-values of the decomposed sub-systems \( \hat{F}_1[z_1] \) and \( \hat{F}_2[z_2] \), respectively.

Lemma 2: Let the ROM for 2-D discrete-time systems be \( F_{rh}[z_1, z_2] = F_{1rh}[z_1] F_{2rh}[z_2] \) obtained by using 1-D an optimal Hankel norm approximation [29], then the 2-D discrete-time ROM \( F_{rh}[z_1, z_2] \) is asymptotically stable.
Moreover, the frequency response truncation error is bounded by:

$$
\|F[z_1, z_2] - F_{rh}[z_1, z_2]\| \leq \langle \|\tilde{D}_1\| + 2 \sum_{i=1}^{m} \tilde{\rho}_i \rangle \times 2 \sum_{i=n+1}^{m} \bar{\phi}_i \\
+ \langle \|\tilde{D}_2\| + 2 \sum_{i=1}^{m} \bar{\phi}_i + 3 \sum_{i=n+1}^{m} \bar{\phi}_i \rangle \times 2 \sum_{i=n+1}^{m} \tilde{\rho}_i,
$$

Alternatively,

$$
\|F[z_1, z_2] - F_{rh}[z_1, z_2]\| \leq \langle \|\tilde{D}_1\| + 2 \sum_{i=1}^{n} \tilde{\rho}_i \rangle \times 2 \sum_{i=n+1}^{m} \bar{\phi}_i \\
+ \langle \|\tilde{D}_1\| + 2 \sum_{i=1}^{n} \bar{\phi}_i + 3 \sum_{i=n+1}^{m} \bar{\phi}_i \rangle \times 2 \sum_{i=n+1}^{m} \tilde{\rho}_i,
$$

where $\tilde{\rho}_i$ and $\bar{\phi}_i$ are the optimal Hankel Singular-values of the decomposed sub-systems $F_1[z_1]$ and $F_2[z_2]$, respectively.

2) **Frequency Weighted 2-D State-Space System**

The 2-D weighted discrete-time systems arrangement is shown in Figure 1. Consider a transfer function stable 2-D linear time-invariant discrete-time input weighted system [42] be given as:

$$
G_i[z_1, z_2] = D_i + C_i [z_1 I_{n_1} + z_2 I_{n_2} - A_i]^{-1} B_i, \quad (21)
$$

where

$$
A_i = \begin{bmatrix} A_{11} & A_{21} \\ A_{31} & A_{41} \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}, \quad (22)
$$

and augmented-output (i.e., $H_o[z_1, z_2]F[z_1, z_2]$, respectively, (see Figure 2) [42].

$$
H_o[z_1, z_2] = D_o + C_o [z_1 I_{n_1} + z_2 I_{n_2} - A_o]^{-1} B_o, \quad (23)
$$

where

$$
A_o = \begin{bmatrix} A_{1o} & A_{2o} \\ A_{3o} & A_{4o} \end{bmatrix}, \quad B_o = \begin{bmatrix} B_{1o} \\ B_{2o} \end{bmatrix}, \quad (24)
$$

and augmented-output (i.e., $H_o[z_1, z_2]F[z_1, z_2]$).

Figure 1. Input and output weighted 2-D discrete-time system.

Figure 2. Auxiliary Input and output weighted 2-D discrete-time system.
III. 1-D MODEL REDUCTION TECHNIQUES

Here we provide a brief overview of un-weighted [28] and frequency weighted [30] model reduction techniques for the discrete-time 1-D systems.

A. UN-WEIGHTED 1-D MODEL REDUCTION TECHNIQUE

Let the controllability Gramians $P_{a_1}$ and the observability Gramians $Q_{a_2}$ for the entire frequency interim be given as [28]:

$$P_{a_1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{j\psi} I - A_1)^{-1} B_{1} B_{1}^T (e^{-j\psi} I - A_1^T)^{-1} d\psi,$$

$$Q_{a_2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{j\psi} I - A_2)^{-1} C_{1} C_{1}^T (e^{-j\psi} I - A_2^T)^{-1} d\psi,$$

that are the solution of the following Lyapunov equations:

$$A_1 P_{a_1} A_1^T - P_{a_1} + B_1 B_1^T = 0, \quad (27)$$

$$A_2^T Q_{a_2} A_2 - Q_{a_2} + C_{1} C_{1}^T = 0, \quad (28)$$

Let a similarity transformation matrix $T_1$ be given as:

$$T_1^T Q_{1_2} T_1 = T_1^T P_{1_2} T_1^{-T} = \Sigma_{co} = \begin{bmatrix} \xi_1 & 0 & \cdots & 0 \\ 0 & \xi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_n \end{bmatrix},$$

where $\Sigma_{co} = \text{diag}(\Sigma_{c_1}, \Sigma_{c_2}), \quad \xi_i \geq \xi_{i+1}, \quad i = 1, 2, 3, \ldots, n - 1, \quad \xi_n > \xi_{r+1}$. The ROM is attained as [28], [29]:

$$T_1^{-1} A_1 T_1 = \begin{bmatrix} A_{r_1} & A_{1_2} \\ A_{2_1} & A_{2_2} \end{bmatrix}, \quad T_1^{-1} B_1 = \begin{bmatrix} B_{r_1} \\ B_{2_1} \end{bmatrix}, \quad (29)$$

$$C T_1 = \begin{bmatrix} C_{r_1} & C_{2_1} \end{bmatrix}, \quad D = D_{r_1}. \quad (30)$$

Lemma 3 ([28]): The ROM (i.e., $F_{r_1}[z]$) obtained by using BT [28] is stable and the truncation error is bounded by:

$$\| F_{r_1}[z] - F_{r_1}[z] \|_{\infty} = 2 \sum_{i=r+1}^{n} \xi_i. \quad (31)$$

Lemma 4 ([29]): The ROM (i.e., $F_{r_1}[z]$) obtained by using an optimal Hankel norm approximation [29] is stable and the truncation error is bounded by:

$$\| F_{r_1}[z] - F_{r_1}[z] \|_{\infty} = \sum_{i=r+1}^{n} \xi_i. \quad (32)$$

B. FREQUENCY WEIGHTED 1-D MODEL REDUCTION TECHNIQUE

Let the controllability Gramians $P_{a_1}$ and the observability Gramians $Q_{a_2}$ for the corresponding input-augmented (5) and the output-augmented (6) realization respectively, that satisfy the following Lyapunov equations:

$$P_{a_1} = \begin{bmatrix} P_E & P_{1_2} \\ P_{1_2}^T & P_G \end{bmatrix}, \quad Q_{a_2} = \begin{bmatrix} Q_H & Q_{1_2}^T \\ Q_{1_2} & Q_E \end{bmatrix},$$

that satisfy the following Lyapunov equations:

$$A_{r_1} P_{a_1} A_{r_1}^T - P_{a_1} + B_{r_1} B_{r_1}^T = 0, \quad (33)$$

$$A_{2_1}^T Q_{a_2} A_{2_1} - Q_{a_2} + C_{1} C_{1}^T = 0, \quad (34)$$

Truncating $1^{st}$ and $4^{th}$ block of (33) and (34), respectively, we have the following Lyapunov equations:

$$A_1 P_{E_1} A_1^T - P_{E_1} + X_E = 0, \quad (35)$$

$$A_*^T Q_{E_1} A_* - Q_{E_1} + Y_E = 0. \quad (36)$$

By using the eigenvalues decomposition of $X_E$ and $Y_E$ we have the following:

$$X_E = U_E \begin{bmatrix} S_{E_1} & 0 \\ 0 & S_{E_2} \end{bmatrix} U_E^T, \quad (39)$$

$$B_E = U_E \begin{bmatrix} S_{E_1}^{1/2} & 0 \\ 0 & S_{E_2}^{1/2} \end{bmatrix} U_E S_{E_1}^{1/2}, \quad (40)$$

$$Y_E = V_E \begin{bmatrix} R_{E_1} & 0 \\ 0 & R_{E_2} \end{bmatrix} V_E^T, \quad (41)$$

$$C_E = \begin{bmatrix} R_{E_1}^{1/2} & 0 \\ 0 & R_{E_2}^{1/2} \end{bmatrix} V_E R_{E_1}^{1/2}, \quad (42)$$

where $S_{E_1}, S_{E_2}, R_{E_1}$, and $R_{E_2}$ have $(l - 1)$ and $(k - 1)$ numbers of positive eigenvalues respectively; similarly, $S_{E_1}$ and $R_{E_2}$ have $(n - l)$ and $(n - k)$ numbers of negative eigenvalues respectively. Let $T_2$ be the transformation matrix obtained as:

$$T_2^T Q_E T_2 = T_2^{-1} P_E T_2^{-T} = \text{diag}(\Xi_1, \Xi_2) = \begin{bmatrix} \xi_1 & 0 & \cdots & 0 \\ 0 & \xi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_n \end{bmatrix},$$

where $\Xi_1 = \text{diag}(\xi_1, \xi_2, \cdots, \xi_r), \quad \Xi_2 = \text{diag}(\xi_{r+1}, \xi_{r+2}, \cdots, \xi_{n+1}), \quad \xi_j \geq \xi_{j+1}, \quad j = 1, 2, \ldots, n - 1, \quad \xi_r > \xi_{r+1}$. The transformation matrix $T_2$ transforms the original stable large-scale system realization into a balanced realization. The ROM $F_r[z] = D_{r_1} + \hat{C}_{r_1} [z I - A_{r_1}]^{-1} B_{r_1}$ is acquired by truncating the transformed balanced realization.

$$T_2^{-1} A_1 T_2 = \hat{A}_{r_1} = \begin{bmatrix} \hat{A}_{r_1} & \hat{A}_{2_1} \\ \hat{A}_{2_1} & \hat{A}_{2_2} \end{bmatrix}, \quad T_2^{-1} B_1 = \hat{B}_{r_1} = \begin{bmatrix} \hat{B}_{r_1} \\ \hat{B}_{2_1} \end{bmatrix}, \quad (43)$$

$$C_{1} T_2 = \hat{C}_{r_1} = \begin{bmatrix} \hat{C}_{r_1} & \hat{C}_{2_1} \end{bmatrix}, \quad D_* = \hat{D}_{r_1}. \quad (44)$$
Remark 2: This technique [30] provide unstable ROMs because input/output associated matrices $X_E$ and $Y_E$ respectively are indefinite (i.e., $X_E \leq 0$ and $Y_E \leq 0$) [32] when both-sided weights are used.

IV. EXISTING STABILITY PRESERVING FREQUENCY WEIGHTED MOR TECHNIQUES

Here we provide a brief overview of existing frequency weighted model reduction techniques for the 1-D [39], [40], [43] and 2-D [42] systems.

A. EXISTING 1-D STABILITY PRESERVING FREQUENCY WEIGHTED MOR TECHNIQUES

CB [40], VA [39], and IG [43] improvised Enns’s [30] input and output associated matrices $X_E$ and $Y_E$, respectively, to yield positive and positive-semi definiteness of these matrices, which consequently yield stability of the ROM. These techniques also offer an error bounds formula. The controllability and observability Gramians $P_{ex}$ and $Q_{ex}$, respectively, satisfying the following Lyapunov equations:

$$A_s P_{ex} A_s^T - P_{ex} + B_{ex} B_{ex}^T = 0,$$  \hspace{1cm} (45)

$$A_s^T Q_{ex} A_s - Q_{ex} + C_{ex}^T C_{ex} = 0.$$ \hspace{1cm} (46)

The improvisation by CB [40], VA [39], and IG [43] introduced fictitious input and output associated matrices $B_{ex} \in \{B_{ex1} [40], B_{ex2}, [39], B_{ex3} [43]\}$ and $C_{ex} \in \{C_{ex1} [40], C_{ex2} [39], C_{ex3} [43]\}$, respectively, can be computed as:

$$B_{ex1} = U_{ex1} S_{ex1} U_{ex1}^T,$$

$$B_{ex2} = U_{ex2} S_{ex2} U_{ex2}^T,$$

$$B_{ex3} = U_{ex3} S_{ex3} U_{ex3}^T.$$

The matrices $S_{ex} \in \{S_{ex1}, S_{ex2}, S_{ex3}\}$ and $S_{ex} \in \{S_{ex1}, S_{ex2}, S_{ex3}\}$ are composed two 1-D systems and also yield error bounds. For the output related matrix $C_{ex} \in \{C_{ex1} [40], C_{ex2}, [39], C_{ex3} [43]\}$ and $Q_{ex} \in \{Q_{ex1}, [40], Q_{ex2}, [39], Q_{ex3} [43]\}$ in a unique way. This leads to the existence of the different transformation matrices $T_{ex} \in \{T_{ex1}, [40], T_{ex2}, [39], T_{ex3} [43]\}$. As a consequence, three existing stability-preserving model order reduction techniques are established.

B. EXISTING 2-D STABILITY PRESERVING FREQUENCY WEIGHTED MOR TECHNIQUE

GS [42] modified Enns’s [30] matrices $X_E$ and $Y_E$ and applied these matrices for 2-D MOR case (by using minimal rank-decomposition conditions) to grant positive and positive-semi definite of these input and output associated matrices, which consequently grant stable ROMs for the decomposed two 1-D systems and also yield error bounds. For decomposed systems $\tilde{F}_{1}[z] = \tilde{D}_{1} + C_{1}[zI - A_{1}]^{-1}B_{1}$, and $\tilde{F}_{2}[z] = \tilde{D}_{2} + C_{2}[zI - A_{2}]^{-1}B_{2}$ the controllability and observability Gramians $\tilde{P}_{ex} \in \{\tilde{P}_{ex1} [42], \tilde{P}_{ex2} [42]\}$ and $\tilde{Q}_{ex} \in \{\tilde{Q}_{ex1}, [42], \tilde{Q}_{ex2}, [42]\}$ respectively, satisfying following Lyapunov equations:

$$\tilde{A}_{ex}^T \tilde{P}_{ex} \tilde{A}_{ex} - \tilde{P}_{ex} + \tilde{B}_{ex} \tilde{B}_{ex}^T = 0,$$

$$\tilde{A}_{ex}^T \tilde{Q}_{ex} \tilde{A}_{ex} - \tilde{Q}_{ex} + \tilde{C}_{ex}^T \tilde{C}_{ex} = 0.$$

where $\tilde{\xi}_j \geq \tilde{\xi}_{j+1}$, $j = 1, 2, \ldots, n - 1$, $\tilde{\xi}_r > \tilde{\xi}_{r+1}$ where $r$ is the order of the ROM. The ROM $F_{ex}[z] = D_{ex} + C_{ex}[zI - A_{ex}]^{-1}B_{ex}$ is acquired as:

$$T_{ex}^{-1} A_{ex} T_{ex} = \tilde{A}_{ex}, \quad T_{ex}^{-1} B_{ex} = \tilde{B}_{ex},$$

Remark 3: Since $X_E \leq B_{ex} B_{ex}^T$, $Y_E \leq C_{ex}^T C_{ex}$, $\{A_{ex}, B_{ex}, C_{ex}\}$; consequently, yield minimal and stable ROMs. These techniques offer formula for the error bounds.

Remark 4: The following error-bound expression exists [40]:

$$\|F_{ex}[z] - F_{ex}[z]\| \leq 2\|L\|\|K\| \sum_{j=r+1}^n \tilde{\xi}_j,$$

with the existence of the rank conditions $\text{rank}(B_{ex} B_{ex}) = \text{rank}(B_{ex} B_{ex}) = \text{rank}(C_{ex} C_{ex})$, respectively; which results into the positive and positive-semi definite of the original system's input and the original system's output associated matrices, respectively; which results into the positive and positive-semi definite of the controllability matrices $P_{ex} \in \{P_{ex1} [40], P_{ex2} [39], P_{ex3} [43]\}$ and the observability matrices $Q_{ex} \in \{Q_{ex1}, [40], Q_{ex2}, [39], Q_{ex3} [43]\}$ in a unique way. This leads to the existence of the different transformation matrices $T_{ex} \in \{T_{ex1}, [40], T_{ex2}, [39], T_{ex3} [43]\}$. As a consequence, three existing stability-preserving model order reduction techniques are established.
For the systems $A \in \{A_1, A_4\}$ the input and output related matrices $B_{ex} \in \{B_{ex_1}, B_{ex_2}\}$ and $C_{ex} \in \{C_{ex_1}, C_{ex_2}\}$, respectively, can be computed as:

$$
\hat{B}_{ex_1} = \hat{U}_{\hat{e}x_1}\hat{S}_{\hat{e}x_1}\hat{U}_{\hat{e}x_1}^T = \left[ \begin{array}{c}
\hat{U}_{\hat{e}x_1} \\
\hat{U}_{\hat{e}x_2}
\end{array} \right],
$$

$$
\hat{B}_{ex_2} = \hat{U}_{\hat{e}x_2}\hat{S}_{\hat{e}x_2}\hat{U}_{\hat{e}x_2}^T = \left[ \begin{array}{c}
\hat{V}_{\hat{e}x_1} \\
\hat{V}_{\hat{e}x_2}
\end{array} \right],
$$

$$
\hat{C}_{ex_1} = \hat{V}_{\hat{e}x_1}\hat{R}_{\hat{e}x_1}\hat{V}_{\hat{e}x_1}^T = \left[ \begin{array}{c}
\hat{V}_{\hat{e}x_1} \\
\hat{V}_{\hat{e}x_2}
\end{array} \right],
$$

$$
\hat{C}_{ex_2} = \hat{V}_{\hat{e}x_2}\hat{R}_{\hat{e}x_2}\hat{V}_{\hat{e}x_2}^T = \left[ \begin{array}{c}
\hat{V}_{\hat{e}x_1} \\
\hat{V}_{\hat{e}x_2}
\end{array} \right].
$$

Let $\hat{T}_{ex_1} \in \{\hat{T}_{ex_1}, \hat{T}_{ex_2}\}$ a transformation matrix be obtained as:

$$
\hat{T}_{ex_1}^T\hat{Q}_{ex_1}\hat{T}_{ex_1} = \hat{T}_{ex_1}^{-1}\hat{P}_{ex}\hat{T}_{ex_1}^{-T} = \left[ \begin{array}{cccc}
\hat{\xi}_1 & 0 & \cdots & 0 \\
0 & \hat{\xi}_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \hat{\xi}_n
\end{array} \right],
$$

where $\hat{\xi}_{i} \geq \hat{\xi}_{i+1}$, $j = 1, 2, 3, \ldots, n - 1$, $\hat{\xi}_{r} > \hat{\xi}_{r+1}$ where $r$ is the order of the ROM. The ROM $F_{1}[z_{1}] = C_{1r}\left[z_{1} - A_{11}\right]^{-1}B_{1r} + D_{1r}$ and $F_{2}[z_{2}] = C_{2r}\left[z_{2} - A_{4r}\right]^{-1}B_{2r} + D_{2r}$ can be acquired as:

$$
\hat{T}_{ex_1}^{-1}A_{\hat{T}_{ex_1}} = \left[ \begin{array}{cc}
A_{11} & A_{12} \\
A_{12} & A_{22}
\end{array} \right],
$$

$$
\hat{T}_{ex_1}^{-1}\hat{B}_{1r} = \left[ \begin{array}{c}
\hat{B}_{1r} \\
\hat{B}_{2r}
\end{array} \right],
$$

$$
\hat{C}_{1r}\hat{T}_{ex_1}^{-1} = \left[ \begin{array}{cc}
0 & 0 \\
0 & 0
\end{array} \right],
$$

$$
\hat{D}_{1r} = \hat{D}_{1r},
$$

$$
\hat{D}_{2r} = \hat{D}_{2r},
$$

Similarly,

$$
\hat{T}_{ex_2}^{-1}A_{\hat{T}_{ex_2}} = \left[ \begin{array}{cc}
A_{41} & A_{42} \\
A_{41} & A_{42}
\end{array} \right],
$$

$$
\hat{T}_{ex_2}^{-1}\hat{B}_{1r} = \left[ \begin{array}{c}
\hat{B}_{1r} \\
\hat{B}_{2r}
\end{array} \right],
$$

$$
\hat{C}_{2r}\hat{T}_{ex_2}^{-1} = \left[ \begin{array}{cc}
0 & 0 \\
0 & 0
\end{array} \right],
$$

$$
\hat{D}_{2r} = \hat{D}_{2r},
$$

For each decomposed original systems $F_{1}[z_{1}] = D_{1r} + C_{1r}\left[z_{1} - A_{11}\right]^{-1}B_{1r} + D_{1r}$ and $F_{2}[z_{2}] = D_{2r} + C_{2r}\left[z_{2} - A_{4r}\right]^{-1}B_{2r} + D_{2r}$ the ROMs obtained $\{A_{1r}, B_{1r}, C_{1r}, D_{1r}\}$ and $\{A_{4r}, B_{2r}, C_{2r}, D_{2r}\}$ respectively are minimal and stable.

Remark 6: Since $X_{E} \leq B_{ex}B_{ex}^T$ and $Y_{E} \leq C_{ex}^TC_{ex}$; consequently, the realizations $A_{T}, B_{ex}, C_{ex}$; and $A_{T}, B_{ex}, C_{ex}$ are minimal and stable respectively, moreover, yield minimal and stable ROMs. These techniques offer a formula for the error bounds.

Remark 7: Since for each input related matrix $\hat{B}_{ex} \in \{\hat{B}_{ex_1}, \hat{B}_{ex_2}\}$ and for each output related matrix $C_{ex} \in \{C_{ex_1}, C_{ex_2}\}$ grant positive and positive-semi definite of decomposed original system’s input and decomposed original system’s output related matrices respectively; which results into the positive and positive-semi definite of the controllability matrices $\hat{P}_{ex} \in \{\hat{P}_{ex_1}, \hat{P}_{ex_2}\}$ and the observability matrices $\hat{Q}_{ex} \in \{\hat{Q}_{ex_1}, \hat{Q}_{ex_2}\}$ in a unique way. This leads to the existence of the different transformation matrices $\hat{T}_{ex_1} \in \{\hat{T}_{ex_1}, \hat{T}_{ex_2}\}$ which subsequently results in ROMs $\{A_{1r}, B_{1r}, C_{1r}, D_{1r}\}$ and $\{A_{4r}, B_{2r}, C_{2r}, D_{2r}\}$ for the given decomposed systems $F_{1}[z_{1}] = D_{1r} + C_{1r}\left[z_{1} - A_{11}\right]^{-1}B_{1r}$ and $F_{2}[z_{2}] = D_{2r} + C_{2r}\left[z_{2} - A_{4r}\right]^{-1}B_{2r}$ respectively. As a consequence, ROMs obtained are stable, and these techniques yield error bound formula.

Remark 8: Similarly, ROMs for decomposed systems $F_{1}[z_{1}], F_{2}[z_{2}] = F_{2}[z_{2}]F_{1}[z_{1}]$ as in (19) and (20) are obtained in similar way as in (49-50) and (51-52) respectively. Moreover, ROMs obtained are stable and also yield error bound formula.

V. MAIN RESULTS

The stability preserving strategies for 1-D discrete-time systems proposed by CB [40], GS [37], and IG [46] modified $X_{E}$ and $Y_{E}$ to ensure the stability of the ROM by making positive and semi-positive definite of the associated input and the associated output matrices. However, these methods induce significant truncation errors in some distinct frequency weights due to significant variance form the original systems.

This paper presents a stability preserving frequency-weighted MOR technique for discrete-time 1-D and 2-D systems. For the 1-D and 2-D systems, the ROM’s stability is ensured by inserting some fictitious input and output matrices. The fictitious matrices are created by square-rooting eigenvalues that have identical effects on each eigenvalue of 1-D and 2-D discrete-time input and output matrices to construct stable ROMs with low truncation errors at specified frequency weights. Decomposition is performed first for the discrete-time 2-D weighted system using the minimal rank-decomposition condition as illustrated in (11,16); then, the controllability and the observability Gramians are computed based on modified associated input and output matrices for decomposed 1-D sub-systems. The proposed scheme also provides an a priori error bound expressions by using the BT and an optimal Hankel norm approximation approaches, respectively, for the 1-D and 2-D discrete-time frequency weighted systems. A comparison among different existing frequency weighted MOR techniques (including 1-D and 2-D systems) with proposed techniques are presented, which show the efficacy of proposed methods.

A. 1-D FREQUENCY WEIGHTED MODEL REDUCTION TECHNIQUE FOR DISCRETE-TIME SYSTEMS

Let a new fictitious controllability Gramians matrix $\bar{P}_{m}$ and the observability Gramians matrix $\bar{Q}_{m}$ for 1-D discrete-time systems are computed as

$$
A_{*}\bar{P}_{m}A_{*}^T - \bar{P}_{m} + \bar{X}_{m} = 0,
$$

$$
A_{*}^T\bar{Q}_{m}A_{*} - \bar{Q}_{m} + \bar{Y}_{m} = 0,
$$

where $\bar{X}_{m} = \bar{U}_{m}\bar{S}_{m}\bar{U}_{m}^T$, and $\bar{Y}_{m} = \bar{V}_{m}\hat{R}_{m}\bar{V}_{m}^T$. By eigenvalues decomposition of $\bar{X}_{m}$ and $\bar{Y}_{m}$ we have the following:

$$
\bar{X}_{m} = \bar{U}_{m}\bar{S}_{m}\bar{U}_{m}^T,
$$

$$
\bar{Y}_{m} = \bar{V}_{m}\hat{R}_{m}\bar{V}_{m}^T.
$$
The new fictitious \( \tilde{B}_m \) and \( \tilde{C}_m \) are given as input and output associated matrices respectively, where

\[
\tilde{B}_m = \begin{cases} \sqrt{U_m \left( \frac{(E_k - s)^{1/2}}{s} \right) U_m} & \text{for } s_n < 0 \\ U_m \left( \frac{(E_k - s)^{1/2}}{s} \right) U_m \tilde{s}_m^{1/2} & \text{for } s_n \geq 0 \end{cases}
\]

\[
\tilde{C}_m = \begin{cases} \sqrt{V_m \left( \frac{(E_k - s)^{1/2}}{s} \right) V_m} & \text{for } r_n < 0 \\ V_m \left( \frac{(E_k - s)^{1/2}}{s} \right) V_m \tilde{s}_m^{1/2} & \text{for } r_n \geq 0 \end{cases}
\]

Let the similarity transformation matrix \( \bar{T}_m \) is calculated as:

\[
\Sigma_m = \bar{T}_m^T Q_m T_m = T_m^{-1} P_m T_m^{-T} = \begin{bmatrix} \bar{p}_1 & 0 & \ldots & 0 \\ 0 & \bar{p}_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \bar{p}_n \end{bmatrix},
\]

where \( \Sigma_m = \text{diag}\{\Sigma_m_1, \Sigma_m_2\} \geq \bar{p}_j \geq \bar{p}_{j+1} \) and \( \bar{p}_r \geq \bar{p}_{r+1} \).

The above MOR procedure can be viewed in the context of non-minimum phase systems.

**Lemma 5 ([54]):** If the \( n \)th order square discrete-time 1-D minimal realization be given as:

\[
F_s \equiv \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} = \begin{bmatrix} A_k & A_2 \\ A_3 & A_4 \\ B_1 & B_2 \\ C_1 & C_2 \\ D_1 & D_2 \end{bmatrix}
\]

and let \( H_0[z] = F_s^{-1}[z] \); then, \( A_l = A_l - B_l D_l^{-1} C_l \) has \( k \) eigenvalues outside the unit circle. Let \( \lambda_l[A_l] \lambda_j[A_j] \neq 1 \) for \( l, j \) where, there exist a unique controllability and the observability matrices, \( P_{c_s} \) and \( Q_{o_s} \), respectively, which are the solution to the Lyapunov equation as in (27) and \( (A_k^T Q_{o_s} A_l - Q_{o_s} + C_k^T (D_1^{-1})^T D_1 C_l) = 0 \), respectively.\n
**Remark 9:** The realization \( F_s[z] \) can be decomposed into two sub-systems as:

\[
F_s[z] = F_k[z] + F_{n-k}[z]
\]

where

\[
F_k[z] = \begin{bmatrix} A_k & B_1 \\ C_1 & D_1 \end{bmatrix},
\]

\[
F_{n-k}[z] = \begin{bmatrix} A_{n-k} & B_2 \\ C_2 & D_2 \end{bmatrix}.
\]

The realization \( F_k[z] \) has exactly \( k \) zeros outside of the unit disk; whereas, the rest of the zeros are inside the unit disc. Similarly, the above MOR procedure can be viewed in the context of unstable minimum phase systems.

**Lemma 6 ([54]):** If the \( n \)th order square discrete-time 1-D realization be given as:

\[
F_s \equiv \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} = \begin{bmatrix} A_k & A_2 \\ A_3 & A_4 \\ B_1 & B_2 \\ C_1 & C_2 \\ D_1 & D_2 \end{bmatrix}
\]

with no eigenvalues on the unit circle and let \( P_{c_s} = P_{c_s}^T \) be the solution to the Lyapunov equation as in (53) with \( P_{c_s} = \text{diag}\{\Sigma_{c_1}, \Sigma_{c_2}\} \), where \( \Sigma_{c_1} \) is non-singular matrix and \( \Sigma_{c_2} > 0 \) is a diagonal matrix. Then, \( A_k \) and \( A_n \) have \( k \) unstable poles (eigenvalues) outside the unit circle; also, \( A_k \) has no eigenvalues inside the unit circle. Assume that \( F_s[n] = D_s \) is a nonsingular and \( \lambda_l[A_k] \lambda_j[A_s] \neq 1 \) for \( l, j \). Further, \( P_{c_s} \) contain \( k \) and \( n - k \) negative and positive eigenvalues, respectively.

**Remark 10:** The realization \( F_{c_s}[z] \) can be decomposed into two sub-systems as:

\[
F_{c_s}[z] = F_k[z] + F_{n-k}[z]
\]

where

\[
F_k[z] \leftrightarrow \begin{bmatrix} A_k & B_1 \\ C_1 & D_1 \end{bmatrix},
\]

\[
F_{n-k}[z] \leftrightarrow \begin{bmatrix} A_{n-k} & B_2 \\ C_2 & D_2 \end{bmatrix}.
\]

The realization \( F_k[z] \) has exactly \( k \) poles (eigenvalues) outside of the unit disk; whereas, the rest of the poles are inside the unit disc.

Furthermore, the proposed MOR procedure can be employed for the marginally stable systems by decomposing the original systems into sub-systems (i.e., asymptotically stable + marginally stable).

**Lemma 7 ([55]):** There exists a similarity transformation matrix \( T_{sm} \) that satisfies:

\[
A_s = T_{sm} \begin{bmatrix} A_s & 0 \\ 0 & A_m \end{bmatrix} T_{sm}^{-1},
\]

such that the full-order-model as in (1) is marginally stable and matrix \( A_s \) has a full rank.

**Remark 11:** The decomposition as in Lemma 7 enables each sub-system to be reduced in a manner that preserves its particular notion of stability. Further, the MOR for each sub-systems are obtained in a similar way as in (59)-(60).

**Remark 12:** Since \( X_E \leq B_m B_m^T \geq 0 \), \( Y_E \leq C_m C_m^T \geq 0 \), \( P_m > 0 \) and \( Q_m > 0 \). Therefore, the transformed-realization \( \{A_s, \tilde{B}_m, \tilde{C}_m\} \) is minimal and the stability of the ROM is guaranteed.

**Lemma 8:** The fictitious input associated matrices \( X_E \leq B_m B_m^T \geq 0 \) and the fictitious output associated matrices \( Y_E \leq C_m C_m^T \geq 0 \), likewise, the controllability matrices \( P_E \leq \tilde{P}_m > 0 \) and the observability matrices \( Q_E < Q_m > 0 \). Therefore, the transformed-realization \( \{A_s, \tilde{B}_m, \tilde{C}_m\} \) obtained is minimal and stable which also guaranteed the ROM’s stability in the desired frequency-intervals.

**Proof of Lemma 8:** we will demonstrate that the realization \( \{A_s, \tilde{B}_m, \tilde{C}_m\} \) is minimal (i.e., controllable and observable). Since the controllability Gramians matrix \( P_m \) and the observability Gramians
matrix $Q_m$ are solution of Lyapunov equations as in (53) and (54) respectively, so
\[
\tilde{B}_m \tilde{B}_m^T X_E \geq 0
\]

\[
P_m - P_m \geq \frac{2\pi}{\nu \eta} \int_{0}^{\eta} (e^{i\omega T} I - A_m) - 1 \tilde{B}_m \tilde{B}_m^T (e^{-i\omega T} I - A_m^T)^{-1} d\omega
\]
\[
- \frac{1}{2\pi} \int_{0}^{\eta} (e^{i\omega T} I - A_m) - 1 \tilde{B}_m \tilde{B}_m^T (e^{-i\omega T} I - A_m^T)^{-1} d\omega
\]
\[
= \frac{1}{2\pi} \int_{0}^{\eta} (e^{i\omega T} I - A_m) - 1 (\tilde{B}_m \tilde{B}_m^T X_E) (e^{-i\omega T} I - A_m^T)^{-1} d\omega \geq 0
\]
Since for $P_E \geq 0$; consequently, $\tilde{P}_m \geq 0$ [37]. Similarly, for $Q_E \geq 0$; consequently, $Q_m \geq 0$. As a consequence, the original-system matrix $A_s$ is stable. Resultantly, the pair $(A_s, B_m)$ is controllable and the pair $(A_s, C_m)$ is observable (i.e., $(A_s, B_m, C_m)$ is minimal).

**Remark 9** [56] Since the pair $(A, B_m)$ satisfy the following Lyapunov equation (53),
\[
A_s P_m A_s^T - P_m = -B_m B_m^T,
\]
for $\tilde{P}_m \geq 0$; then, the original large-scale system is asymptotically stable iff it is controllable. Suppose the original system is not asymptotically stable. In that case, eigenvalues of the original large-scale system (i.e., $e^{i\lambda t} A_s$) are outside of the unit circle, not on the inside of the unit circle.

**Proof of Lemma 9:**
The first part is obvious. To proof the second part, let $A_s$ and $\nu$ have eigenvalue $\lambda$ and corresponding left eigenvector $v^*$ respectively; then, $v^* A_s = \nu v^* \lambda$ and $A_s^T \nu = \lambda v$. Appropriately pre- multiplying and post-multiplying the Lyapunov equation (53) by $v^*$ and $\nu$ respectively; consequently, gives
\[
\nu^* A_s P_m A_s^T \nu - \nu^* P_m \nu = -v^* \tilde{B}_m \tilde{B}_m^T v = (\nu \lambda - 1) \nu^* P_m \nu.
\]
Since the matrix $\nu^* \tilde{P}_m \nu \geq 0$ and the matrix $\nu^* \tilde{B}_m \tilde{B}_m^T \nu \geq 0$, this results $|\lambda \lambda| \leq 1$. Furthermore, if $Re|\lambda| \neq 0$; then, $\nu^* \tilde{B}_m \tilde{B}_m^T \nu \neq 0$; hence, $\nu^* B_m \neq 0$ which results the transformed-realization $(A_s, B_m, C_m)$ is controllable and stable.

**Theorem 1:** The following error bound expression exists:
\[
\|H_0[z](F_s[z] - F_r[z]) G_t[z]\|_\infty \leq \frac{1}{2}\|H_0[z]\|_\infty \|L_m\|_\infty \|K_m\|_\infty \|G_t[z]\|_\infty \sum_{j=r+1}^{n} \bar{\rho}_j,
\]
with the existence of the rank conditions $\text{rank } [\tilde{B}_m B_s] = \text{rank } [\tilde{C}_m C_s] = \text{rank } [\tilde{C}_m]$,
\[
\bar{L}_m = \left\{ \begin{array}{ll}
C_s \tilde{V}_m \hat{\tilde{R}}_{mh}^{-1/2} & \text{if } r_n < 0 \text{ exists} \\
C_s V E R E^{-1/2} & \text{otherwise}
\end{array} \right.
\]
\[
\bar{K}_m = \left\{ \begin{array}{ll}
\hat{\tilde{S}}_{mh}^{-1/2} U_{mh}^T B_s & \text{if } s_n < 0 \text{ exists} \\
S_{E}^{-1/2} U_E^T B_s & \text{otherwise}
\end{array} \right.
\]

**Proof of Theorem 1:**
Since $\text{rank } [\tilde{B}_m B_s] = \text{rank } [\tilde{B}_m]$ and $\text{rank } [\tilde{C}_m C_s] = \text{rank } [\tilde{C}_m]$, respectively. In that case, eigenvalues $A_s$ of the relationships $B_s = \tilde{B}_m \hat{K}_m$ and $C = \bar{L}_m \tilde{C}_m$ holds. By partitioning $\tilde{B}_m = [\tilde{B}_{m1} \tilde{B}_{m2}], \tilde{C}_m = [\tilde{C}_{m1} \tilde{C}_{m2}]$ and substituting $\bar{B}_r = B_m K_m$, $\bar{C}_r = \tilde{L}_m \tilde{C}_m$, respectively, yields:
\[
\|H_0[z](F_s[z] - F_r[z]) G_t[z]\|_\infty \leq \frac{1}{2}\|H_0[z]\|_\infty \|L_m\|_\infty \|K_m\|_\infty \|G_t[z]\|_\infty \sum_{j=r+1}^{n} \bar{\rho}_j.
\]

**Theorem 2:** The following error-bound expression exists:
\[
\|H_0[z](F_r[z] - F_{r_{mh}}[z]) G_t[z]\|_\infty \leq \frac{1}{2}\|H_0[z]\|_\infty \|L_{mh}\|_\infty \|K_{mh}\|_\infty \|G_t[z]\|_\infty \sum_{j=r+1}^{n} \bar{\rho}_j,
\]
with the existence of the rank conditions $\text{rank } [\bar{B}_{mh} B_s] = \text{rank } [\tilde{C}_{mh}]$ and $\text{rank } [\tilde{C}_{mh} C_s] = \text{rank } [\bar{C}_{mh}]$, where
\[
\bar{L}_{mh} = \left\{ \begin{array}{ll}
C_s \tilde{V}_m \hat{\tilde{R}}_{mh}^{-1/2} & \text{if } r_n < 0 \text{ exists} \\
C_s V E R E^{-1/2} & \text{otherwise}
\end{array} \right.
\]
\[
\bar{K}_{mh} = \left\{ \begin{array}{ll}
\hat{\tilde{S}}_{mh}^{-1/2} U_{mh}^T B_s & \text{if } s_n < 0 \text{ exists} \\
S_{E}^{-1/2} U_E^T B_s & \text{otherwise}
\end{array} \right.
\]

**Proof of Theorem 2:**
The proof of above-mentioned Theorem 2 is similar to the proof of Theorem 1; hence, omitted for the brevity.

**Corollary 1:** Theorem 1 holds true subject to the following rank conditions: $\text{rank } [\tilde{B}_m B_s] = \text{rank } [\tilde{B}_m]$ and $\text{rank } [\tilde{C}_m C_s] = \text{rank } [\tilde{C}_m]$ (which follows from [57]) are satisfied.

**Remark 13:** When $X_E \geq 0$ and $Y_E \geq 0$; then, $P_E = P_{ex} = \tilde{P}_m$ and $Q_E = Q_{ex} = \tilde{Q}_m$, consequently, ROMs obtained by using [30], [40], [39], [43], and suggested technique are the equivalent. Otherwise $P_E < \tilde{P}_m$ and $Q_E < \tilde{Q}_m$. This work is licensed under a Creative Commons Attribution 4.0 License. For more information, see https://creativecommons.org/licenses/by/4.0/
Furthermore, the frequency-weighted Hankel singular-values satisfy: \( \lambda_j [P_E Q_E]^{1/2} \leq (\lambda_j [P_m Q_m])^{1/2} \).

**Remark 14:** When \( X_E \geq 0 \) and \( Y_E \geq 0 \); then, ROMs obtained using Enns [30] and suggested framework are the equivalent.

**Remark 15:** For the fictitious-input matrix \( \tilde{B}_m \) and the fictitious-output matrix \( \tilde{C}_m \) grant positive and positive-semi definite of the input associated matrix \( B_e \) and the output associated matrix \( C_e \) respectively; consequently, positive and positive-semi definite of the controllability Gramians matrix \( \tilde{P}_m \) and the observability Gramians matrix \( \tilde{Q}_m \). This corresponds to transformation matrix \( \tilde{T}_m \), resulting in the stability retention MOR algorithm. In addition, constants (i.e., \( \tilde{L}_m \) and \( \tilde{K}_m \)) provides the relationship between the systems matrices (i.e., \( B_e \) and \( C_e \)) with the fictitious matrices (i.e., \( \tilde{B}_m \) and \( \tilde{C}_m \)), resulting in the error-bound expression for the suggested framework.

**Remark 16 ([58]):** The ill-conditioning of the relevant discrete-time Lyapunov equations as in (53)-(54) causes difficulty in computing the ROM based on Gramians of sampled-data models for smaller sampling periods. The numerical results are distorted by errors up to a particular limit for the sampling step. To get over this limitation, an "approximately" balanced realization of the sampled-data system is obtained straight from its continuous-time counterpart’s balanced realization. When the sample time is reduced to zero, this realization comes “near” to be exactly balanced for "extremely small" (i.e., \( \delta[T] = T_2 - T_1 = \iota \)) sample steps (i.e., considerably less than the systems’ time constants), where \( T \) is sampling time, and \( \iota \) is a very small number. Similarly, the error based on the Hankel singular values (i.e., \( \tilde{\rho}_j \)) and frequency response error will be the same. It’s also worth noting that the bilinear mapping (i.e., \( z \rightarrow (1 + s)/(1 - s) \)) produces a balanced continuous-time equivalent system if the original discrete-time approach was similarly balanced [29].

**Theorem 3:** The following Lyapunov equation for the suggested framework holds:

\[
\begin{align*}
A_x \tilde{P}_{(ext)} A_x^T - \tilde{P}_{(ext)} + \tilde{B}_{(ext)} \tilde{B}_{(ext)}^T &= 0, \\
A_x^T \tilde{Q}_{(ext)} A_x - \tilde{Q}_{(ext)} + \tilde{C}_{(ext)} \tilde{C}_{(ext)}^T &= 0.
\end{align*}
\]  

**Proof of Theorem 3:**

Using (39), (41), (57) and (58) we have the following:

\[
\begin{align*}
S_E &= \text{diag}[S_{E_1}, S_{E_2}] = \text{diag}[(s_1, \ldots, s_{1-1}), (s_1, \ldots, s_n)], \\
\tilde{S}_m &= \text{diag}[\tilde{S}_{m_1}, \tilde{S}_{m_2}] = \text{diag}[(\tilde{s}_1, \ldots, \tilde{s}_{1-1}), (\tilde{s}_1, \ldots, \tilde{s}_n)], \\
R_E &= \text{diag}[R_{E_1}, R_{E_2}] = \text{diag}[(r_1, \ldots, r_{p-1}), (r_1, \ldots, r_{n-p})], \\
\tilde{R}_m &= \text{diag}[\tilde{R}_{m_1}, \tilde{R}_{m_2}] = \text{diag}[(\tilde{r}_1, \ldots, \tilde{r}_{p-1}), (\tilde{r}_1, \ldots, \tilde{r}_{n-p})],
\end{align*}
\]

\( \tilde{S}_{(ext)} \) and \( \tilde{R}_{(ext)} \) are obtained by \( \tilde{S}_m - S_E \) and \( \tilde{R}_m - R_E \), respectively.

\[
\tilde{S}_{(ext)} = \begin{bmatrix} S_{(ext)}^1 & 0 \\ 0 & S_{(ext)}^2 \end{bmatrix}, \quad \tilde{R}_{(ext)} = \begin{bmatrix} R_{(ext)}^1 & 0 \\ 0 & R_{(ext)}^2 \end{bmatrix},
\]

where matrices \( \tilde{B}_{(ext)} \) and \( \tilde{C}_{(ext)} \) are obtained by (57 - 40) and (58 - 42), respectively.

**Corollary 2:** Theorem 3 holds true subject to the balanced-realization \( \{A_x, \tilde{B}_{(ext)}, \tilde{C}_{(ext)}\} \) is minimal (i.e., controllable and observable) and stable.

**Remark 17:** For the balanced-realization \( \{A_x, \tilde{B}_{(ext)}, \tilde{C}_{(ext)}\} \) to the following Lyapunov equation:

\[
\begin{align*}
A_x \tilde{P}_{(ext)} A_x^T - \tilde{P}_{(ext)} + \tilde{B}_{(ext)} \tilde{B}_{(ext)}^T &= 0, \\
A_x^T \tilde{Q}_{(ext)} A_x - \tilde{Q}_{(ext)} + \tilde{C}_{(ext)} \tilde{C}_{(ext)}^T &= 0,
\end{align*}
\]

where the matrix \( \tilde{B}_{(ext)} \geq 0 \) and the matrix \( \tilde{C}_{(ext)} \geq 0 \) grant positive and positive-semi definite of the matrix \( \tilde{B}_m \) and the matrix \( \tilde{C}_m \) respectively; consequently, positive and positive-semi definite of the controllability Gramians matrix \( \tilde{P}_{(ext)} \) and the observability Gramians matrix \( \tilde{Q}_{(ext)} \) in a way leads to the positive and positive-semi definite of the matrix \( \tilde{P}_m \) and the matrix \( \tilde{Q}_m \).

**Remark 18:** Note that by applying stability robustness theorem [59] to the frequency weighted model reduction problem, the combine weighted systems is stable if the following inequalities hold (see chapter 3 of [31] for more detail)

\[
\begin{align*}
(i) \quad \|H_{x} z [F_{x} z - F_{x} z]\|_{\infty} &\leq 1. \\
(ii) \quad \|(F_{x} z - F_{x} z)G_{x z}\|_{\infty} &\leq 1. \\
(iii) \quad \|H_{x} z [F_{x} z - F_{x} z]G_{x z}\|_{\infty} &\leq 1.
\end{align*}
\]
The above inequalities also provide the criteria for the choice of weightings (i.e., input weightings and output weightings).

### B. 2-D FREQUENCY WEIGHTED MODEL REDUCTION TECHNIQUE FOR DISCRETE-TIME SYSTEMS

Let the controllability Gramians $P_{ia}$ and the observability Gramians $Q_{oa}$ for the corresponding input-augmented (25) and the output-augmented (26) realizations, respectively, be given as:

$$P_{ia} = \begin{bmatrix} P_{ia1 \bar{1}} & P_{ia2 \bar{1}} \\ P_{ia3 \bar{1}} & P_{ia4 \bar{1}} \end{bmatrix},$$

$$Q_{oa} = \begin{bmatrix} Q_{oa1 \bar{1}} & Q_{oa2 \bar{1}} \\ Q_{oa3 \bar{1}} & Q_{oa4 \bar{1}} \end{bmatrix},$$

that are the solution of the following Lyapunov equations:

$$A_{ia}^T P_{ia} A_{ia} - P_{ia} + B_{ia} B_{ia}^T = 0,$$

$$A_{oa}^T Q_{oa} A_{oa} - Q_{oa} + C_{oa}^T C_{oa} = 0.$$  

Truncating (3, 3) and (1, 1) block of (65) and (66), respectively, we have the following Lyapunov equations:

$$A_{ia}^T P_{ia} A_{ia} - P_{ia} + X_{ia} = 0,$$

$$A_{oa}^T Q_{oa} A_{oa} - Q_{oa} + Y_{oa} = 0,$$

where

$$Y_{ia} = A_{ia}^T Q_{oa} A_{ia},$$

$$X_{oa} = A_{oa}^T Q_{oa} A_{oa}.$$  

The stability is ensured for 2-D discrete-time system by making the input $X_{ia} = U_{e_{ia}} S_{e_{ia}}^T V_{e_{ia}}$ and the output $Y_{oa} = C_{oa}^T C_{oa} = V_{o_{oa}} R_{o_{oa}} V_{o_{oa}}^T$ (69) associated matrices positive and positive semi definite. The fictitious matrices $\hat{B}_{m_{e_{2}}} \bar{C}_{m_{e_{2}}}$ are obtained by improving $B_{e_{2}} = U_{e_{2}} S_{e_{2}} V_{e_{2}}^T$ and $C_{e_{2}} = R_{e_{2}} V_{e_{2}}^T$, respectively.

$$\hat{B}_{m_e_2} = \begin{bmatrix} U_{m_e} \sqrt{(s_n - s_{n-1})^{1/2}} \\ U_{e_2} S_{e_2}^{1/2} \end{bmatrix}$$

$$\bar{C}_{m_e_2} = \begin{bmatrix} V_{m_e} \sqrt{(r_n - r_{n-1})^{1/2}} \\ R_{e_2} V_{e_2} \end{bmatrix}$$

Remark 19: When the following rank conditions holds:

$$\text{rank} [\hat{B}_{m_{e_{2}}} B_{2}] = \text{rank} [\hat{B}_{m_{e_{2}}}],$$

$$\text{rank} [\bar{C}_{m_{e_{2}}} C_{1}] = \text{rank} [\bar{C}_{m_{e_{2}}};]$$

then, the following relationship holds for the fictitious input and the fictitious output matrices.

$$B_{2} = \hat{B}_{m_{e_{2}}} \bar{K}_{m_{e_{2}}},$$

$$C_{1} = \bar{L}_{m_{e_{2}}} \bar{C}_{m_{e_{2}}},$$

where

$$\bar{K}_{m_{e_{2}}} = \begin{cases} \bar{S}_{m_{e_{2}}}^{-1/2} \bar{U}_{m_{e_{2}}} B_{2} & \text{if } \bar{s}_{n} < 0 \text{ exists} \\ \bar{S}_{m_{e_{2}}}^{-1/2} U_{e_{2}} B_{2} & \text{otherwise} \end{cases}$$

$$\bar{L}_{m_{e_{2}}} = \begin{cases} C_{1} V_{m_{e_{2}}} \bar{R}_{m_{e_{2}}}^{-1/2} & \text{if } \bar{r}_{n} < 0 \text{ exists} \\ C_{1} V_{e_{2}} R_{e_{2}}^{-1/2} & \text{otherwise} \end{cases}$$

Remark 20: It can be seen in [40] that (73) is always true. It can be seen that in (70), each term is expressed as $B_{2} [\star]$ or $\star B_{2}^T$ or $B_{2} [\star] B_{2}^T$, that is same as in [40], here terms $[\star]$ are some matrices which doesn’t affect our analysis. So (73) is always true. Similarly, it can also be seen in [40] that (74) is always true. It can also be seen that in (69), each terms are expressed as $C_{1} [\star]$ or $[\star] C_{1}^T$ or $C_{1} [\star] C_{1}^T$, that is same as in [40], here terms $[\star]$ are some matrices which doesn’t affect our analysis. So (74) is always true.

Consider rank $[B_{1} B_{2}] = \text{rank}[B_{2}]$ and rank $\begin{bmatrix} C_{1} \\ C_{2} \end{bmatrix} = \text{rank}[C_{1}]$; then, there exists some constant matrices $\bar{K}_{m_{e_{1}}}$ and $\bar{L}_{m_{e_{2}}}$, such that

$$B_{1} = \bar{K}_{m_{e_{1}}} B_{2},$$

$$C_{2} = C_{1} \bar{L}_{m_{e_{2}}},$$

Remark 21: Assumptions rank $[B_{1} B_{2}] = \text{rank}[B_{2}]$ and rank $\begin{bmatrix} C_{1} \\ C_{2} \end{bmatrix} = \text{rank}[C_{1}]$ will always be satisfied for $B_{2}$ and $C_{1}$ be full column rank and row rank, respectively. Using (75), (76), (79) and (80), we can derive new matrices $\hat{B}_{m_{e_{1}}} \bar{C}_{m_{e_{2}}}$ as follows:

$$\hat{B}_{m_{e_{1}}} = \bar{K}_{m_{e_{1}}} \hat{B}_{m_{e_{2}}},$$

$$\bar{C}_{m_{e_{2}}} = \bar{L}_{m_{e_{2}}} \bar{C}_{m_{e_{2}}},$$

then,

$$\begin{bmatrix} B_{1} \\ B_{2} \end{bmatrix} = \begin{bmatrix} \hat{B}_{m_{e_{1}}} \\ \bar{B}_{m_{e_{2}}} \end{bmatrix} \bar{K}_{m_{e_{1}}} := \hat{B}_{m} \bar{K}_{m_{e_{1}}},$$

$$\begin{bmatrix} C_{1} \\ C_{2} \end{bmatrix} = \bar{L}_{m_{e_{1}}} \bar{C}_{m_{e_{2}}} := \bar{L}_{m} \bar{C}_{m_{e_{2}}}.$$
Theorem 5: The realization \( \{A, B_{m_r}, C_{m_r}, D\} \) is minimal, stable, and separable denominator.

Proof of Theorem 5:
The proof of above Theorem 5 follows from the minimality, stability, and separability of the 2-D discrete-time system realization \( \{A, B, C, D\} \).
The minimal rank-decomposition of new realization \( \{A, B_{m_r}, C_{m_r}, D\} \) subject to \( A_3 = 0 \) can be written as:

\[
\begin{bmatrix}
A_2 & B_{m_2} \\
C_{m_2} & D_{m_2}
\end{bmatrix} = \begin{bmatrix}
B_{m_2} \\
D_{m_2}
\end{bmatrix} \begin{bmatrix}
C_{m_2} & D_{m_2}
\end{bmatrix} \tag{85}
\]

that results \( \hat{F}_{m_r}[z_1, z_2] = \hat{F}_{m_r}[z_1] \hat{F}_{m_2}[z_2] \), where

\[
\begin{align*}
\hat{F}_{m_r}[z_1, z_2] &= D_{m_r} + C_{m_r} [z_1 I_n \oplus z_2 I_m - A]^{-1} B_{m_r}, \\
\hat{F}_{m_r}[z_1] &= D_{m_r} + C_{m_r} [z_1 I - A]^{-1} B_{m_r}, \\
\hat{F}_{m_2}[z_2] &= D_{m_2} + C_{m_2} [z_2 I - A_4]^{-1} B_{m_2}, \\
D &= L_{m_r} D_{m_2} K_{m_2} \tag{86}
\end{align*}
\]

Remark 22: The equation (86) can be solvable for \( D_m \) if and only one of the following equivalent conditions holds [60]:
1. rank \( [L_{m_1}] = \text{rank} [L_{m_1} D] \) and rank \( [K_{m_2}] = [K_{m_2} D] \).
2. There exist some matrices \( Y_e \) and \( Z_e \) such that \( D = L_{m_1} Y_e \) and \( D = Z_e K_{m_2} \).

Remark 23: The requirements for the existence of (86) for strictly proper original systems is immediately met. This requirement will be met when the full row rank \( L_{m_1} \) and the full column rank \( K_{m_2} \) is exist. We notice that even by setting \( D_{m_2} = 0 \) we can get rid of this condition.

Remark 24: The realizations \( \{A_{1 r}, B_{m_{r 1}}, C_{m_{r 1}}, D_{m_{r 1}}\} \) and \( \{A_{4 r}, B_{m_{r 2}}, C_{m_{r 2}}, D_{m_{r 2}}\} \) are minimal and stable.

The new controllability (\( \hat{P}_{m_{r 1}}, \hat{P}_{m_{r 2}} \)) and observability (\( \hat{Q}_{m_{r 1}}, \hat{Q}_{m_{r 2}} \)) Gramians correspond to the decomposed sub-system \( \hat{F}_{m_r}[z_1, \hat{F}_{m_2}[z_2]] \), respectively, these Gramians satisfy the following corresponding Lyapunov equations i.e., for sub-system \( \hat{F}_{m_r}[z_1] = D_{m_{r 1}} + C_{m_{r 1}} [z_1 I - A_1]^{-1} B_{m_{r 1}} \):

\[
\begin{align*}
A_1 \hat{P}_{m_{r 1}} A_1^T - \hat{P}_{m_{r 1}} + \hat{X}_{m_{r 1}} &= 0, \\
A_1^T \hat{Q}_{m_{r 1}} A_1 - \hat{Q}_{m_{r 1}} + \hat{Y}_{m_{r 1}} &= 0, \tag{88}
\end{align*}
\]

where \( \hat{X}_{m_{r 1}} = B_{m_{r 1}} \hat{B}_{m_{r 1}}^T \) and \( \hat{Y}_{m_{r 1}} = \hat{C}_{m_{r 1}}^T \hat{C}_{m_{r 1}} \). Let the similarity transformation matrix \( \hat{T}_{m_{r 1}} \) is calculated as:

\[
\hat{T}_{m_{r 1}}^T \hat{Q}_{m_{r 1}} \hat{T}_{m_{r 1}} = \hat{T}_{m_{r 1}}^{-1} \hat{P}_{m_{r 1}} \hat{T}_{m_{r 1}}^{-T} = \begin{bmatrix}
\hat{\rho}_{e_1} & 0 & \cdots & 0 \\
0 & \hat{\rho}_{e_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \hat{\rho}_{e_1}
\end{bmatrix}, \tag{89}
\]

where \( \hat{\rho}_{e_j} \geq \hat{\rho}_{e_{j+1}} \) and \( \hat{\rho}_{e_r} \geq \hat{\rho}_{e_{r+1}} \). The ROM \( \hat{F}_{m_{r 1}}[z_1] = D_{m_{r 1} r} + C_{m_{r 1}} [z_1 I - A_{1 r}]^{-1} \hat{B}_{m_{r 1}} \) is obtained as:

\[
\begin{align*}
\tilde{T}_{m_{r 1}}^{-1} A_1 \tilde{T}_{m_{r 1}} &= \hat{A}_{1 r}, \\
\tilde{T}_{m_{r 1}}^{-1} \hat{B}_{m_{r 1}} &= \hat{B}_{m_{r 1}}, \\
\tilde{C}_{m_{r 1}} \tilde{T}_{m_{r 1}} &= \hat{C}_{m_{r 1}}, \\
\tilde{D}_{m_{r 1}} &= \hat{D}_{m_{r 1}}, \tag{90}
\end{align*}
\]

i.e., for sub-system \( \hat{F}_{m_2}[z_2] = D_{m_{r 2}} + C_{m_{r 2}} [z_2 I - A_{4 2}]^{-1} B_{m_{r 2}} \):

\[
\begin{align*}
A_4 \hat{P}_{m_{r 2}} A_4^T - \hat{P}_{m_{r 2}} + \hat{X}_{m_{r 2}} &= 0, \\
A_4^T \hat{Q}_{m_{r 2}} A_4 - \hat{Q}_{m_{r 2}} + \hat{Y}_{m_{r 2}} &= 0, \tag{91}
\end{align*}
\]

where \( \hat{X}_{m_{r 2}} = \hat{B}_{m_{r 2}} \hat{B}_{m_{r 2}}^T \) and \( \hat{Y}_{m_{r 2}} = \hat{C}_{m_{r 2}}^T \hat{C}_{m_{r 2}} \). Let the similarity transformation matrix \( \hat{T}_{m_{r 2}} \) is calculated as:

\[
\hat{T}_{m_{r 2}}^T \hat{Q}_{m_{r 2}} \hat{T}_{m_{r 2}} = \hat{T}_{m_{r 2}}^{-1} \hat{P}_{m_{r 2}} \hat{T}_{m_{r 2}}^{-T} = \begin{bmatrix}
\hat{\rho}_{t_1} & 0 & \cdots & 0 \\
0 & \hat{\rho}_{t_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \hat{\rho}_{t_1}
\end{bmatrix}, \tag{92}
\]

Remark 25: The realizations \( \{A_{1 r}, B_{m_{r 1}}, C_{m_{r 1}}, D_{m_{r 1}}\} \) and \( \{A_{4 2 r}, B_{m_{r 2}}, C_{m_{r 2}}, D_{m_{r 2}}\} \) are stable and minimal. Furthermore, the 2-D discrete-time weighted ROM \( \hat{F}_{m_r}[z_1, z_2] = D_{m_{r 2}} + C_{m_{r 2}} [z_1 I_n \oplus z_2 I_m - A_{r 4}]^{-1} B_{m_{r 2}} \) of an original 2-D discrete-time \( F[z_1, z_2] = D + C [z_1 I_n \oplus z_2 I_m - A]^{-1} B \), where

\[
\begin{align*}
A_{m_{r 1}} &= \begin{bmatrix}
A_{1 r} \\
\hat{B}_{m_{r 1}} \\
\hat{C}_{m_{r 1}} \\
D_{m_{r 1}} = L_{m_{r 1}} D_{m_{r 1}} \hat{D}_{m_{r 1}} K_{m_{r 2}} \tag{100}
\end{align*}
\]

Algoirthm 1: Given a discrete time 2-D system \( F[z_1, z_2] \) with input and output frequency weights \( G_{i}[z_1, z_2] \) and \( H_{o}[z_1, z_2] \).
The ROM $\tilde{F}_{m_r}[z_1, z_2]$ for 2-D discrete-time systems are obtained by using the following steps:

1. Compute the controllability Gramians $P_{oa}$ and the observability Gramians $Q_{oo}$ by using (65) and (66), respectively.
2. Compute $Y_{e_1}$ and $X_{e_4}$ by using (69) and (70), respectively.
3. Decompose $Y_{e_1}$ and $X_{e_4}$ by using singular-values decomposition: $C_1 C_1^H = V_r R_r V_r^T$ and $B_2 B_2^H = U_r S_r U_r^T$, respectively, to compute $\tilde{C}_{m_1} = \tilde{R}^{-1/2} \tilde{V}_r$, and $\tilde{B}_{m_2} = \tilde{U}_r S_r^{-1/2}$ by using (72) and (71), respectively.
4. Compute constants $K_{m_2}, \tilde{L}_{m_1}, \tilde{K}_{m_1}$, and $\tilde{L}_{m_2}$ by using (77), (78), (79) and (80), respectively.
5. Compute $\tilde{B}_{m_1}$ and $\tilde{C}_{m_2}$ by using (81) and (82), respectively.
6. Compute $\tilde{P}_{m_1}, \tilde{Q}_{m_1}, \tilde{P}_{m_2}$, and $\tilde{Q}_{m_2}$ by using (87), (88), (94) and (95), respectively.
7. Compute the transformation matrices $\tilde{T}_{m_1}$ and $\tilde{T}_{m_2}$ to satisfy (89) and (96), respectively.
8. Compute the realizations $\{\tilde{A}_1, \tilde{B}_{m_1}, \tilde{C}_{m_1}, \tilde{D}_{m_1}\}$ and $\{\tilde{A}_4, \tilde{B}_{m_2}, \tilde{C}_{m_2}, \tilde{D}_{m_2}\}$ by using (90-93) and (97-100), respectively, to obtain corresponding ROMs $\tilde{F}_{1m_1}[z_1]$ and $\tilde{F}_{2m_2}[z_2]$.
9. Compute 2-D discrete-time systems ROMs by using (101-104):

$$\begin{bmatrix}
A_{m_r} & B_{m_r} \\
C_{m_r} & D_{m_r}
\end{bmatrix} =
\begin{bmatrix}
\tilde{A}_1 & \tilde{B}_{m_2} \tilde{C}_{m_2} & \tilde{D}_{m_2} \tilde{K}_{m_2} \\
0 & \tilde{A}_4 & \tilde{B}_{m_1} \tilde{K}_{m_1}
\end{bmatrix}
\begin{bmatrix}
L_{m_1} C_{m_1} & D_{m_1} & \tilde{K}_{m_1} \\
L_{m_2} D_{m_2} & C_{m_2} & \tilde{K}_{m_2}
\end{bmatrix},
$$

where

$A_{m_r} \in \mathbb{R}^{(n_r+m_r) \times (n_r+m_r)}, B_{m_r} \in \mathbb{R}^{(n_r+m_r) \times p}, C_{m_r} \in \mathbb{R}^{p \times (n_r+m_r)}, D_{m_r} \in \mathbb{R}^{q \times p}$, and $n_r < n, m_r < m$.

**Remark 26:** For the only input weighting, the realization based on frequency weighted becomes $\{A, \tilde{B}_{m_1}, C, D\}$; consequently, $C_2$ replaces $\tilde{C}_{m_1}$ in (85).

**Remark 27:** For the only output weighting, the realization based on frequency weighted becomes $\{A, B, \tilde{C}_{m_2}, D\}$; consequently, $B_1$ replaces $\tilde{B}_{m_2}$ in (85).

**Remark 28:** Notice that also in Remark (23) we can get rid of this assumption by setting $D_m = 0$, then setting $D_{m_r} = D$ into (6) the appropriate 2-D dimensions. However, this comment might not be helpful if we use 1-D singular perturbation approximation for the method of 2-D MOR.

**Remark 29:** While it is expressly indicated for balanced truncation, the above algorithms can be easily expanded/defined for almost all 1-D reduction schemes, such as the Hankel norm approximation and singular perturbation approximation, etc.

**Theorem 6:** The 2-D ROM obtained with this procedure is stable.

**Proof of Theorem 6:** The proof follows directly from the stability of the un-weighted approximation and is thus excluded.

**Remark 30:** For the 2-D discrete-time 2-D system $F[z_1, z_2] = F_1[z_1] F_2[z_2]$, the corresponding decomposed sub-systems $\tilde{F}_{m_1}[z_1, z_2] = F_{1m_1}[z_1] F_{2m_2}[z_2]$ are formed, the matrices $X_{e_2} = B_2 B_2^H < B_{m_2} B_{m_2}^H$, and $Y_{e_1} = C_1^H C_1 < C_{m_1}^H C_{m_1}$; therefore, $B_{m_1} B_{m_1}^H \geq 0$, $C_{m_1}^H C_{m_1} \geq 0$; resultantly, $P_{m_1} > 0, \bar{P}_{m_2} > 0$ and $Q_{m_1} > 0, \bar{Q}_{m_2} > 0$. Realization $\{\tilde{A}_1, \tilde{B}_{m_1}, \tilde{C}_{m_1}\}$ and $\{\tilde{A}_4, \tilde{B}_{m_2}, \tilde{C}_{m_2}\}$ are minimal and the obtained ROMs are stable.

**Remark 31:** Similar to Remark (30), the 2-D discrete-time 2-D system $F[z_1, z_2] = F_2[z_2] F_1[z_1]$, the corresponding decomposed sub-systems $\tilde{F}_{m_1}[z_1, z_2] = F_{2m_2}[z_2] F_{1m_1}[z_1]$ can also be formed; consequently, their stability of ROMs are also ensured by making corresponding inputs (i.e., $B_1$ and $B_2$) and corresponding outputs (i.e., $C_1$ and $C_2$) matrices positive and positive semi definite (i.e., results in fictitious input and output matrices $B_{m_1}, B_{m_2}, C_{m_1}$ and $C_{m_2}$) in a similar way as given in equations (71,72), respectively, such that their corresponding controllability Gramians matrices (i.e., $P_{m_1}$ and $P_{m_2}$) and the observability Gramians matrices (i.e., $Q_{m_1}$ and $Q_{m_2}$) are positive and positive semi definite, which leads to two different transformation matrices (i.e., $\tilde{T}_{m_1}$ and $\tilde{T}_{m_2}$) for their corresponding sub-systems $\tilde{F}_{1m_1}[z_1]$ and $\tilde{F}_{2m_2}[z_2]$ respectively; subsequently, transformed-realization correspond to $\tilde{F}_{1m_1}[z_1]$ and $\tilde{F}_{2m_2}[z_2]$ are minimal and their ROMs are stable.

**Theorem 7:** Let ROMs be attained by using balanced truncation, then the frequency response approximation error is bounded by:

$$\|H_0[z_1, z_2] (F[z_1, z_2] - F_r[z_1, z_2]) G_1[z_1, z_2]\|_{\infty} \leq 2\kappa (\|\tilde{D}_{m_1}\|_{\infty} + 2 \sum_{i=m+1}^{n} \tilde{\nu}_i) \sum_{i=m+1}^{n} \tilde{\rho}_i$$

where $\kappa = H_0[z_1, z_2] L_{m_1} K_{m_2} G_1[z_1, z_2], \tilde{\nu}_i$ and $\tilde{\rho}_i$ are the Hankel singular-values of the realizations $F_{1m_1}[z_1]$ and $F_{2m_2}[z_2]$, respectively.

**Proof of Theorem 7:**

$$\|H_0[z_1, z_2] (F[z_1, z_2] - F_r[z_1, z_2]) G_1[z_1, z_2]\|_{\infty} = \|H_0[z_1, z_2] (C[z_1 I_n \oplus z_2 I_m - A^{-1} B] - C_r [z_1 I_n \oplus z_2 I_m - A_r^{-1} B_r] G_1[z_1, z_2])\|_{\infty}$$

$$= \|H_0[z_1, z_2] (L_{m_1} C_{m_1} [z_1 I_n \oplus z_2 I_m - A^{-1} B_m] - \tilde{L}_{m_1} C_{m_2} [z_1 I_n \oplus z_2 I_m - A_{m_2}]^{-1} B_{m_2} \tilde{K}_{m_2} G_1[z_1, z_2])\|_{\infty}$$

$$= \|H_0[z_1, z_2] (\tilde{L}_{m_1}) (\tilde{C}_{m_1} [z_1 I_n \oplus z_2 I_m - A^{-1} B_m] - \tilde{L}_{m_1} C_{m_2} [z_1 I_n \oplus z_2 I_m - A_{m_2}]^{-1} B_{m_2} \tilde{K}_{m_2} G_1[z_1, z_2])\|_{\infty}$$
\[ \leq \| H_o[z_1, z_2] \|_\infty \| (C_m z_1 I_n \oplus z_2 I_m - A)^{-1} B_{m_2} - C_{m_2} z_1 I_n \oplus z_2 I_m - A_{m_2}^{-1} B_{m_2} \|_\infty \| \hat{K}_{m_2} G_i[z_1, z_2] \|_\infty \]

Since \( \{ A, B_{m_2}, C_{m_2} \} \) is the balanced realization and \( \{ A_{m_2}, B_{m_2}, C_{m_2} \} \) is its ROM, using Lemma (2) we have the following:

\[ \| (C_m z_1 I_n \oplus z_2 I_m - A)^{-1} B_{m_2} - C_{m_2} z_1 I_n \oplus z_2 I_m - A_{m_2}^{-1} B_{m_2} \|_\infty \leq 2(\| \hat{D}_{m_2} \| + 2 \sum_{i=1}^{n} \bar{\varphi}_i) 2 \sum_{i=m+1}^{n} \bar{\varphi}_i, \]

Therefore,

\[ \| H_o[z_1, z_2] (F[z_1, z_2] - F_{rh}[z_1, z_2]) G_i[z_1, z_2] \|_\infty \leq 2\kappa(\| \hat{D}_{m_2} \| + 2 \sum_{i=1}^{n} \bar{\varphi}_i) 2 \sum_{i=m+1}^{n} \bar{\varphi}_i, \]

\[ +2\kappa(\| \hat{D}_{m_1} \| + 2 \sum_{i=m+1}^{n} \bar{\varphi}_i) 2 \sum_{i=m+1}^{n} \bar{\varphi}_i. \]

Theorem 8: Let ROMs be attained by using optimal Hankel norm approximation, then the frequency response approximation error is bounded by:

\[ \| H_o[z_1, z_2] (F[z_1, z_2] - F_{rh}[z_1, z_2]) G_i[z_1, z_2] \|_\infty \leq 2\kappa(\| \hat{D}_{m_2} \| + 2 \sum_{i=1}^{n} \bar{\varphi}_i + 3 \sum_{i=m+1}^{n} \bar{\varphi}_i) 2 \sum_{i=m+1}^{n} \bar{\varphi}_i, \]

\[ +2\kappa(\| \hat{D}_{m_1} \| + 2 \sum_{i=m+1}^{n} \bar{\varphi}_i) 2 \sum_{i=m+1}^{n} \bar{\varphi}_i, \]

where \( \kappa = H_o[z_1, z_2] L_{m_{rh}}, \hat{K}_{m_{rh}} G_i[z_1, z_2], \bar{\varphi}_i, \) and \( \bar{\varphi}_i \) are the optimal Hankel singular-values of the realizations \( F_{1_{m_{rh}}}[z_1] \) and \( F_{2_{m_{rh}}}[z_2] \), respectively.

Proof of Theorem 8:
The proof of above-mentioned Theorem 8 is similar to the proof of Theorem 7; hence, omitted for the brevity.

Corollary 3: When the only input-weighting or the only output-weighting is present, then \( \kappa \) becomes \( \| \hat{K}_{m_2} G_i[z_1, z_2] \|_\infty \) or \( \| H_o[z_1, z_2] L_{m_{rh}} \|_\infty \), respectively. Furthermore, when there is no weighting (i.e., input and output) present, \( \kappa = 1 \).

Remark 32: For the decomposed sub-systems (i.e., \( F_{1_{m_1}}[z_1] \) and \( F_{2_{m_2}}[z_2] \)), the fictitious-input matrices (i.e., \( B_{m_1}, \) and \( B_{m_2} \)) and the fictitious-output matrices (i.e., \( C_{m_1}, \) and \( C_{m_2} \)) grant positive and positive-semi definite of the input associated matrix (i.e., \( B_1 \) and \( B_2 \)) and the output associated matrix (i.e., \( C_1, \) and \( C_2 \)), respectively; consequently, positive and positive-semi definite of the controllability Gramians matrices (i.e., \( P_{m_1} \) and \( P_{m_2} \)) and the observability Gramians matrices (i.e., \( Q_{m_1} \) and \( Q_{m_2} \)). This corresponds to transformation matrices (i.e., \( T_{m_1} \) and \( T_{m_2} \)), resulting in stability retention MOR algorithm. In addition, constants (i.e., \( K_{m_1}, L_{m_1}, \hat{K}_{m_2}, \) and \( L_{m_2} \)) provide the relationship between the systems matrices (i.e., \( B_1, B_2, C_1, \) and \( C_2 \)) and the fictitious matrices (i.e., \( \hat{B}_{m_1}, \hat{B}_{m_2}, \hat{C}_{m_1}, \) and \( \hat{C}_{m_2} \)), resulting in the error-bound expression for the suggested algorithm.

Remark 33: Similarly, for the decomposed sub-systems (i.e., \( F_{1_{m_1}}[z_1] \) and \( F_{2_{m_2}}[z_2] \)), the fictitious-input matrices (i.e., \( B_{m_1}, \) and \( B_{m_2} \)) and the fictitious-output matrices (i.e., \( C_{m_1}, \) and \( C_{m_2} \)) grant positive and positive-semi definite of the input associated matrices (i.e., \( B_1 \) and \( B_2 \)) and the output associated matrix (i.e., \( C_1 \) and \( C_2 \)), respectively; consequently, positive and positive-semi definite of the controllability Gramians matrices (i.e., \( P_{m_1} \) and \( P_{m_2} \)) and the observability Gramians matrices (i.e., \( Q_{m_1} \) and \( Q_{m_2} \)). This corresponds to transformation matrices (i.e., \( \hat{T}_{m_1} \) and \( \hat{T}_{m_2} \)), resulting in stability retention MOR algorithm. In addition, constants (i.e., \( K_{m_1}, L_{m_1}, \hat{K}_{m_2}, \) and \( L_{m_2} \)) provide the relationship between the systems matrices (i.e., \( B_1, B_2, C_1, \) and \( C_2 \)) and the fictitious matrices (i.e., \( \hat{B}_{m_1}, \hat{B}_{m_2}, \hat{C}_{m_1}, \) and \( \hat{C}_{m_2} \)), resulting in the error-bound expression for the suggested algorithm.

Remark 34: Similar to the Remark 18, by applying the stability robustness theorem [59] to the frequency weighted model reduction problem for the discrete-time 2-D systems, the combined weighted systems is stable if the following inequalities hold:

\[ (i) \| H_o[z_1, z_2] (F[z_1, z_2] - F_{rh}[z_1, z_2]) G_i[z_1, z_2] \|_\infty \leq 1. \]

\[ (ii) \| (F[z_1, z_2] - F_{rh}[z_1, z_2]) G_i[z_1, z_2] \|_\infty \leq 1. \]

\[ (iii) \| H_o[z_1, z_2] (F[z_1, z_2] - F_{rh}[z_1, z_2]) G_i[z_1, z_2] \|_\infty \leq 1. \]

VI. NUMERICAL SIMULATIONS

To highlight the comparison of existing frequency weighted models ( [30], [37], [46] ), a numerical example of a multi-input multi-output doubly-fed induction generator (DFIG) based variable-speed wind turbine (double-cage induction generator) for the power system (current model) is presented in Example-1. Furthermore, the 2-D discrete-time system is demonstrated in Example-2. Figs. 3, 4, and 7 depicted the frequency response error for the entire frequency-weights of the approximated model obtained by using existing ( [30], [37], [46] ) and suggested frameworks. In addition, Figs. 5 and 6 depict the original 2-D model, and ROMs acquired using the existing and suggested methods, in the specified frequency-weights, of the ROMs acquired through the use of different existing ( [30], [42] ) and suggested techniques.

1. Induction Generator Parameters: Base voltage = 690V, Base power = 2MW, Angular velocity = 2πf_m, f_m = 50Hz, Stator resistance = 0.00488 p.u., Double-cage reactance = 0.0453 p.u., Stator leakage reactance = 0.09241 p.u., Rotor resistance = 0.00549 p.u., Rotor leakage reactance = 0.09955 p.u., Rotor to double-cage mutual reactance = 0.02 p.u., Magnetizing reactance
2. **DFIG Control Parameters:** Speed limit = 1800 r/min, Cut-in speed = 1000 r/min, Shutdown Speed = 2000 r/min. 

**Example 1:** Consider a stable LTI 6th order DFIG model (current model) as given in [61], the discretized sampling time is $T_s = 0.001$ sec, with the following input weights and the output weights:

$$A_{iw} = \begin{bmatrix} -0.75 & 0 & 0 & 0 & 0 & 0 \\ -0.75 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.75 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.75 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.75 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.75 & 0 \end{bmatrix},$$

$$B_{iw} = \begin{bmatrix} -0.33281 & -0.34895 \\ -0.15539 & -0.07846 \\ -0.02101 & -0.22836 \\ -0.30336 & -0.21175 \\ -0.22092 & -0.13554 \\ -0.14248 & -0.04977 \end{bmatrix},$$

$$C_{iw} = \begin{bmatrix} -0.00377 & 0.05555 & 0.02653 & 0.03357 & 0.02709 & 0.09909 \\ 0.06317 & 0.12623 & 0.14361 & 0.05603 & 0.00676 & 0.05758 \\ 0.02762 & 0.11013 & 0.03980 & 0.01313 & 0.10840 & 0.09410 \\ 0.10887 & 0.08565 & 0.13869 & 0.09602 & 0.05212 & 0.06325 \end{bmatrix},$$

$$D_{iw} = \begin{bmatrix} 0.09106 & 0.03833 \\ 0.08006 & 0.06173 \\ 0.07458 & 0.05755 \\ 0.08131 & 0.05301 \end{bmatrix},$$

$$A_{ow} = \begin{bmatrix} -0.95 & 0 & 0 & 0 & 0 & 0 \\ -0.95 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.95 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.95 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.95 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.95 & 0 \end{bmatrix},$$

$$B_{o} = \begin{bmatrix} -0.28072 & -0.22311 \\ -0.13737 & -0.03178 \\ -0.07214 & -0.20447 \\ -0.22917 & -0.13898 \\ -0.22780 & -0.06365 \\ -0.22219 & -0.02956 \end{bmatrix},$$

$$C_{ow} = \begin{bmatrix} 0.12354 & 0.10541 & 0.12138 & 0.05983 & 0.09805 & 0.06601 \\ 0.02625 & 0.02304 & 0.11229 & 0.06226 & 0.13989 & 0.03864 \\ 0.02454 & 0.14302 & 0.01803 & 0.02711 & 0.02453 & 0.11279 \\ 0.09990 & 0.08113 & 0.07876 & 0.03831 & 0.13816 & 0.03430 \\ 0.13416 & 0.10196 & 0.04888 & 0.00308 & 0.11920 & 0.00963 \\ 0.07748 & 0.04548 & 0.08197 & 0.13855 & 0.08661 & 0.11510 \end{bmatrix},$$

$$D_{ow} = \begin{bmatrix} 0.06712 & 0.03174 \\ 0.07152 & 0.08145 \\ 0.06421 & 0.07891 \\ 0.04190 & 0.08523 \\ 0.03908 & 0.05056 \\ 0.08161 & 0.06357 \end{bmatrix}. $$

The frequency-response error comparison is given in Fig. 3 and 4 of 2nd and 3rd order ROMs, respectively. The pole locations of existing ([30], [40], [39], [43]) and proposed techniques are provided in Table 2. It can also be observed that [30] produces unstable 2nd and 3rd order ROMs along with the pole locations at $z_1 = -1.12469 \pm 1.5327i$ and $z = 1.12133, 1.001579 \pm 1.002044i$, respectively. However, in the given frequency-weights, proposed techniques produce low frequency-response truncation error with stable ROMs comparable to existing stability-preserving algorithms ([40], [39], [43]).

**Example 2:** Consider a 6th order stable 2-D discrete-time system [37]: with the desired frequency-weights as given in [42]. Fig. 5 and 6 show the stable 2-D original and ROMs obtained using the existing [42] and proposed techniques, respectively. The frequency-response error comparison in the desired frequency-weights is given in Fig. 7. The pole locations of [30] and proposed techniques are provided in Table 2. It can also be observed that [30] produce unstable 3rd dimension ROMs along with the pole locations at $z_1 = 1.00889,$
1.14789 ± 0.00479i and \( z_2 = 1.45781, -0.12147 \pm 0.12471i \), respectively. However, in the given frequency-weights, proposed techniques produce low frequency-response truncation error with stable ROMs comparable to existing stability-preserving algorithms ([42]).

**ANALYSIS & DISCUSSION**

Figures 3, 4 and 7 indicate that ROMs attained with the Enns technique [30] provide a low-frequency response truncation error as compared to the other methods; however, this also yields unstable ROMs as seen in Table 2. Proposed techniques, however, generates a low-frequency response truncation error with stable ROMs as compared with the existing stability preserving algorithms.

**VII. CONCLUSION**

In this work, the frequency-weighted model order reduction framework for the discrete-time one-dimensional and two-dimensional models is proposed by using balance truncation (proposed technique 1) and an optimal Hankel norm approximation (proposed technique 2), respectively. The suggested approach guarantees that some associated input matrices and associated output matrices for one-dimensional and two-dimensional discrete-time systems, which produce stable reduced-order models, are positive and semi-positive definite. The proposed algorithm also provides frequency-response error bound expression by using balance truncation and an optimal Hankel norm approximation, respectively, for the one-dimensional and two-dimensional discrete-time weighted systems. There are comparisons between existing frequency-weighted model order reduction methods with the proposed framework, indicate that the low-frequency response truncation errors with stable reduced-order models are obtained comparable to existing stability-preserving algorithms, which show the efficacy of the proposed framework. However, the proposed methodology only applies to linear time-invariant one-dimensional and causal recursive separable denominator two-dimensional discrete-time systems. Furthermore, more research is required to extend the pro-

**TABLE 2.** Poles Locations of Reduced Order Models.

| Examples | ROMs | Enns [30] | Proposed Technique 1 | Proposed Technique 2 |
|----------|------|-----------|-----------------------|----------------------|
| Example-1 | 2\(^{nd}\) order | \( z = -1.12469 \pm 1.5327i \) | \( z = 0.12119 \pm 0.5514i \) | \( z = 0.12166 \pm 0.5514i \) |
| Example-2 | 3\(^{rd}\) order | \( z = 1.12131, 1.001579 \pm 1.0000144i \) | \( z = 0.1157, 0.3347 \pm 0.1254i \) | \( z = 0.1255, 0.551119 \pm 0.1255i \) |
| Example-2 | \( F_{\text{freq}}[z] = 3\^{rd}\) order | \( z_1 = 1.008895, 1.14789 \pm 0.008895 \) | \( z_1 = 0.14551, 0.00551i \) | \( z_1 = 0.16512, 0.00117 \) |
| Example-2 | \( F_{\text{freq}}[z] = 3\^{rd}\) order | \( z_2 = 1.45781, -0.12147 \pm 0.12471i \) | \( z_2 = 0.14789, 0.14789 \pm 0.12471i \) | \( z_2 = 0.3745, 0.127512i \) |
posed method to linear time-variant, descriptor, and bilinear systems. This method can also be used to analyze continuous systems in the time domain. Moreover, the proposed methodology may be expended for other variants of 2-D systems, such as positive 2-D continuous delayed systems based on $L_1$-gain control design.

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**REFERENCES**

[1] X. Cheng, Z. Guan, and P. Zhu, “Nearest neighbor transformation of quantum circuits in 2d architecture,” *IEEE Access*, vol. 8, pp. 222466–222475, 2020.

[2] B. J. Wolf, P. Pirih, M. Kruusmaa, and S. M. Van Netten, “Shape classification using hydrodynamic detection via a large-scale large-scale 2d-sensitive artificial lateral line,” *IEEE Access*, vol. 8, pp. 11393–11404, 2020.

[3] S. Birogul, G. Temür, and U. Kose, “Yolo object recognition algorithm and “buy-sell decision” model over 2d candlestick charts,” *IEEE Access*, vol. 8, pp. 91894–91915, 2020.

[4] M. Nakao, K. Imanishi, N. Ueda, Y. Imai, T. Kiriti, and T. Matsuda, “Regularized three-dimensional generative adversarial nets for supervised metal artifact reduction in head and neck ct images,” *IEEE Access*, vol. 8, pp. 109453–109465, 2020.

[5] R. M. Terol, A. R. Reina, S. Ziaei, and D. Gil, “A machine learning approach to reduce dimensional space in large datasets,” *IEEE Access*, vol. 8, pp. 148181–148192, 2020.

[6] P. Benner and S. W. Werner, “Frequency-and time-limited balanced truncation for large-scale second-order systems,” *Linear Algebra and its Applications*, vol. 623, pp. 68–103, 2021.

[7] A. Satapathi and D. Kumar, “A new stability preserving model reduction technique for discrete-time systems using frequency-limited gramians,” in *2019 IEEE Students Conference on Engineering and Systems (SCES)*, IEEE, 2019, pp. 1–5.

[8] M. Tahvori, “Model reduction via truncated cross-granular for bilinear systems,” in *2021 International Conference on Recent Advances in Mathematics and Informatics (ICRAMI)*, IEEE, 2021, pp. 1–5.

[9] D. Alfke, L. Feng, L. Lombardi, G. Antonini, and P. Benner, “Model order reduction with balanced realizations: An error bound,” in *Proceedings of the 39th IEEE Conference on Decision and Control* (CDC), IEEE, 2000, pp. 4209–4214.

[10] C. Wang and L. Jia, “$h_{\infty}$ model reduction for robust uncertain 2-d continuous systems with time-varying delays,” in *3rd International Conference on Systems and Control*. IEEE, 2013, pp. 545–550.

[11] C. Wang and L. Jia, “$h_{\infty}$ model reduction for uncertain 2-d continuous systems in roesser model,” in *Proceedings of the 10th World Congress on Intelligent Control and Automation*. IEEE, 2012, pp. 1733–1738.

[12] W. Laming, W. Weiqin, C. Weinim, and Z. Guangchen, “Frequency finite fault detection observer design for 2-d continuous-discrete systems in roesser model,” in *2015 34th Chinese Control Conference (CCC)*. IEEE, 2015, pp. 6147–6152.

[13] D. Meng, Y. Jia, J. Du, and F. Yu, “Data-driven control for relative degree systems via iterative learning,” *IEEE Transactions on neural networks*, vol. 22, no. 12, pp. 2213–2225, 2011.

[14] P. Dabkowski, K. Galkowski, E. Rogers, Z. Cai, C. T. Freeman, P. L. Lewin, Z. Hurak, and A. Kummert, “Experimentally verified iterative learning control based on repetitive process stability theory,” in *2012 American Control Conference (ACC)*. IEEE, 2012, pp. 604–609.

[15] Y. Yanling, B. Xuhui, Y. Shuake, and L. Jian, “Robust quantized ilc design for linear systems using a 2-d model,” in *2016 Chinese Control and Decision Conference (CCDC)*. IEEE, 2016, pp. 4209–4214.

[16] M. Imran and A. Ghafoor, “Transformation of 2d roesser into causal recursive separable denominator model and decomposition into 1d systems,” *Circuits, Systems, and Signal Processing*, pp. 1–12, 2021.

[17] N. K. Bose, *Multidimensional systems theory and applications*. Springer Science & Business Media, 2003.

[18] S. K. Tadepalli and V. J. Leite, “Robust stabilization of uncertain 2-d discrete delayed systems,” *Journal of Control, Automation and Electrical Systems*, vol. 29, no. 3, pp. 280–291, 2018.

[19] Z. Duan, Y. Sun, C. K. Ahn, Z. Xiang, and I. Ghous, “L1-gain control design for positive 2d continuous delayed systems,” *IEEE Transactions on Circuits and Systems II: Express Briefs*, 2021.

[20] B. Moore, “Principal component analysis in linear systems: Controllability, observability, and model reduction,” *IEEE transactions on automatic control*, vol. 26, no. 1, pp. 17–32, 1981.

[21] Z. Glover, “All optimal Hankel-norm approximations of linear multivariable systems and their $\infty$-error bounds,” *International Journal of control*, vol. 39, no. 6, pp. 1115–1193, 1984.

[22] D. F. Enns, “Model reduction with balanced realizations: An error bound and a frequency weighted generalization,” in *The 23rd IEEE conference on decision and control*. IEEE, 1984, pp. 127–132.

[23] D. Enns, *Model reduction for control system design*. PhD Dissertation, University of Stanford, 1984.

[24] V. Sreeram, B. Anderson, and A. Madiyeksi, “New results on frequency-weighted bounded reduction technique,” in *Proceedings of 1995 American Control Conference-ACC* ’95, vol. 6, IEEE, 1995, pp. 4004–4009.

[25] A. V. Oppenheim, *Discrete-time signal processing*. Pearson Education India, 1999.

[26] W. Gawronski and J.-N. Jiang, “Model reduction in limited time and frequency intervals,” *International Journal of Systems Science*, vol. 21, no. 2, pp. 349–376, 1990.

[27] D. Wang and A. Zilouchian, “Model reduction of discrete linear systems via frequency-domain balanced structure,” *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 47, no. 6, pp. 830–837, 2000.

[28] S. Gugercin and A. C. Antoulas, “A survey of model reduction by balanced truncation and some new results,” *International Journal of Control*, vol. 77, no. 8, pp. 748–766, 2004.

[29] A. Ghafoor and V. Sreeram, “Model reduction via limited frequency interval gramians,” *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 55, no. 9, pp. 2806–2812, 2008.

[30] C.-A. Lin and T.-Y. Chiu, “Model reduction via frequency weighted balanced realization,” in *1990 American Control Conference*. IEEE, 1990, pp. 2069–2070.

[31] A. Varga and B. D. Anderson, “Accuracy enhancing methods for the frequency-weighted balanced related model reduction,” in *Proceedings of the 40th IEEE Conference on Decision and Control (Cat. No. 01CH37228)*, vol. 4. IEEE, 2001, pp. 3659–3664.

[32] K. Campbell, V. Sreeram, and G. Wang, “A frequency-weighted discrete system balanced truncation method and an error bound,” in *Proceedings of the 2000 American Control Conference*. IEEE, 2000, pp. 2403–2404.

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IMRAN ET AL.: “Preparation of Papers for IEEE TRANSACTIONS and JOURNALS”

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