On the directional asymptotic approach in optimization theory  
Part B: constraint qualifications

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During the last years, asymptotic (or sequential) constraint qualifications, which postulate upper semicontinuity of certain set-valued mappings and provide a natural companion of asymptotic stationarity conditions, have been shown to be comparatively mild, on the one hand, while possessing inherent practical relevance from the viewpoint of numerical solution methods, on the other one. Based on recent developments, the theory in this paper enriches asymptotic constraint qualifications for very general nonsmooth optimization problems over inverse images of set-valued mappings by incorporating directional data. We compare these new directional asymptotic regularity conditions with standard constraint qualifications from nonsmooth optimization. Further, we introduce directional concepts of pseudo- and quasi-normality which apply to set-valued mappings. It is shown that these properties provide sufficient conditions for the validity of directional asymptotic regularity. Finally, a novel coderivative-like variational tool is introduced which allows to study the presence of directional asymptotic regularity. For geometric constraints, it is illustrated that all appearing objects can be calculated in terms of initial problem data.

Keywords:  Asymptotic regularity, Constraint qualifications, Pseudo-normality, Quasi-normality, Super-coderivative

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1 Introduction

In recent years, sequential concepts of stationarity and regularity received much attention not only in standard nonlinear optimization, see Andreani et al. (2010, 2011, 2016, 2018), but also in complementarity-, cardinality-, and switching-constrained programming, see Andreani et al. (2019); Kanzow et al. (2021a); Liang and Ye (2021); Ramos (2021), conic optimization, see Andreani et al. (2021), nonsmooth optimization, see Helou et al. (2020); Mehlitz (2020, 2023), or even infinite-dimensional optimization, see Börgens et al. (2020); Kanzow et al. (2018); Kruger and Mehlitz (2022). The interest in sequential stationarity conditions is based on the observation that they hold at local minimizers in the absence of constraint qualifications, and that different types of solution algorithms like multiplier-penalty- and some SQP-methods naturally compute such points. Sequential constraint qualifications provide conditions which guarantee that a sequentially stationary point is already stationary in classical sense, e.g., a Karush–Kuhn–Tucker-point in standard nonlinear programming or an Mordukhovich-stationary (M-stationary for short) point in nonsmooth optimization. Naturally, this amounts to upper semicontinuity of certain problem-tailored set-valued mappings. It has been reported, e.g., in Andreani et al. (2016); Liang and Ye (2021); Mehlitz (2020); Ramos (2021) that sequential constraint qualifications are comparatively mild. Inherently from their construction, sequential constraint qualifications simplify the convergence analysis of some numerical solution procedures.

In Benko and Mehlitz (2022b), we have shown how sequential stationarity for very general nonsmooth problems can be enriched by directional information in terms of critical directions at local minimizers and limiting normals in such directions, see Benko et al. (2019) for an introduction to as well as an overview of the directional limiting calculus. We then used so-called pseudo-coderivatives, see Gfrerer (2014) as well, in order to come up with mixed-order and even M-stationarity conditions under suitable qualification conditions at local minimizers. In the present paper, we strike a different path to benefit from these novel findings regarding directional sequential stationarity.

The particular sequential stationarity conditions from Benko and Mehlitz (2022b) allow us to introduce directional sequential (or asymptotic) qualification conditions whose validity directly yields M-stationarity of local minimizers, see Section 3. Roughly speaking, these conditions demand certain control of unbounded input sequences associated with the regular coderivative of the underlying set-valued mapping in a neighborhood of the reference point. The directional approach reveals that asymptotic regularity is only necessary in critical directions and with respect to (w.r.t.) sequences satisfying some additional conditions. This way, we can relate our new constraint qualifications with already existing ones from the literature. Exemplary, as in Mehlitz (2020), we observe that the concept is independent of both, (directional) metric subregularity, see Gfrerer (2013), and the celebrated First-Order Sufficient Condition for Metric Subregularity, see Gfrerer and Klatte (2016).

In Section 4, we introduce directional versions of pseudo- and quasi-normality for abstract set-valued mappings. It is illustrated that these conditions generalize the ones from Bai et al. (2019); Benko et al. (2022) where the authors merely consider so-called

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geometric and, in particular, disjunctive constraint systems. We show that directional pseudo- and quasi-normality are sufficient for directional metric subregularity as well as directional asymptotic regularity. Furthermore, we discuss how directional pseudo- and quasi-normality can be specified for equilibrium-constrained programs which cover bilevel optimization problems and models with (quasi-) variational inequality constraints, see e.g. Dempe (2002); Dempe et al. (2015); Facchinei and Pang (2003); Luo et al. (1996); Outrata et al. (1998).

Finally, a new directional coderivative-like tool, the directional super-coderivative, is introduced in Section 5 which can be applied beneficially for checking validity of directional asymptotic regularity. In the presence of so-called metric pseudo-regularity, see Gfrerer (2014) again, this leads to conditions in terms of the aforementioned pseudo-coderivatives. Noting that these generalized derivatives can be computed in terms of initial problem data for so-called feasibility mappings associated with geometric constraints, we can specify our findings for such constraint systems. As it turns out, the approach recovers our findings from Benko and Mehlitz (2022b) in different way. Furthermore, we show that the explicit sufficient conditions for directional asymptotic regularity provide constraint qualifications for M-stationarity which are not stronger than the First- and Second-Order Sufficient Condition for Metric Subregularity from Gfrerer and Klatte (2016).

The remainder of the paper is organized as follows. In Section 2, we comment on the notation in this paper and recall some fundamental tools from variational analysis and generalized differentiation which we are going to exploit for our analysis. Furthermore, the underlying model problem from nonsmooth optimization is introduced and the associated sequential stationarity condition from Benko and Mehlitz (2022b) is presented. Section 3 is dedicated to the introduction of directional notions of asymptotic regularity. Additionally, we comment on elementary relations to other constraint qualifications from nonsmooth optimization. Some examples are used to visualize our findings. In Section 4, directional notions of pseudo- and quasi-normality are suggested which address arbitrary set-valued mappings with a closed graph. It is shown that these conditions serve as sufficient conditions for directional metric subregularity and directional asymptotic regularity. Furthermore, we demonstrate that these conditions provide suitable generalizations of already available concepts in the literature which address geometric constraint systems. Finally, we specify directional pseudo- and quasi-normality for constraint systems associated with equilibrium conditions since these are modeled with the aid of set-valued mappings in general. Yet another way to check the presence of directional asymptotic regularity is presented in Section 5. Therein, we define the super-coderivative of a set-valued mapping, which is closely related to pseudo-coderivatives, and comment on its relationship to the validity of directional asymptotic regularity. These findings are made precise for geometric constraint systems by exploiting the calculus rules for the pseudo-coderivative we already obtained in Benko and Mehlitz (2022b). Some concluding remarks close the paper in Section 6.
2 Notation and preliminaries

In this paper, we mainly exploit standard notation coined in Aubin and Frankowska (2009); Bonnans and Shapiro (2000); Rockafellar and Wets (1998); Mordukhovich (2018).

2.1 Basic notation

In this paper, $\mathbb{X}$ and $\mathbb{Y}$ are Euclidean spaces, i.e., finite-dimensional Hilbert spaces, with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$ (the associated space will always be clear from the context). The unit sphere of $\mathbb{X}$ will be denoted by $S_\mathbb{X}$. For a given set $Q \subset \mathbb{X}$ and some point $\bar{x}$, we use $\bar{x} + Q := \{x + \bar{x} \in \mathbb{X} \mid x \in Q\}$ for simplicity. The adjoint of a given linear operator $A: \mathbb{X} \to \mathbb{Y}$ will be denoted by $A^*: \mathbb{Y} \to \mathbb{X}$.

For a continuously differentiable mapping $g: \mathbb{X} \to \mathbb{Y}$, we use $\nabla g(\bar{x}): \mathbb{X} \to \mathbb{Y}$ in order to denote the derivative of $g$ at $\bar{x} \in \mathbb{X}$ which is a linear mapping between $\mathbb{X}$ and $\mathbb{Y}$. For twice continuously differentiable $g$ and a vector $\lambda \in \mathbb{Y}$, $\langle \lambda, g(x) \rangle := \langle \lambda, g(x) \rangle$ for each $x \in \mathbb{X}$ defines the associated scalarization mapping $\langle \lambda, g \rangle: \mathbb{X} \to \mathbb{R}$. By $\nabla \langle \lambda, g \rangle(\bar{x})$ and $\nabla^2 \langle \lambda, g \rangle(\bar{x})$ we denote the first- and second-order derivatives of this map w.r.t. the variable which enters $g$ at $\bar{x}$. Furthermore, for each $u \in \mathbb{X}$, we set

$$\nabla^2 g(\bar{x})[u, u] := \sum_{i=1}^m \langle u, \nabla^2 \langle e_i, g \rangle(\bar{x})(u) \rangle e_i$$

where $\{e_1, \ldots, e_m\} \subset \mathbb{Y}$ is the canonical basis of $\mathbb{Y}$.

2.2 Variational analysis and generalized differentiation

Let us fix a closed set $Q \subset \mathbb{X}$ and some point $\bar{x} \in Q$. The set

$$T_Q(\bar{x}) := \left\{ d \in \mathbb{X} \mid \exists\{d_k\}_{k \in \mathbb{N}} \subset \mathbb{X}, \exists\{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_+: d_k \to d, t_k \searrow 0, \bar{x} + t_k d_k \in Q \forall k \in \mathbb{N} \right\}$$

is called the tangent (or Bouligand) cone to $Q$ at $\bar{x}$. Furthermore, we exploit

$$\hat{N}_Q(\bar{x}) := \{ \eta \in \mathbb{X} \mid \forall x \in Q: \langle \eta, x - \bar{x} \rangle \leq o(\|x - \bar{x}\|) \},$$

$$N_Q(\bar{x}) := \left\{ \eta \in \mathbb{X} \mid \exists\{x_k\}_{k \in \mathbb{N}} \subset Q, \exists\{\eta_k\}_{k \in \mathbb{N}} \subset \mathbb{X}: x_k \to \bar{x}, \eta_k \to \eta, \eta_k \in N_Q(x_k) \forall k \in \mathbb{N} \right\}$$

which are referred to as the regular (or Fréchet) and limiting (or Mordukhovich) normal cone to $Q$ at $\bar{x}$ in the literature. It is well known that these cones coincide with the normal cone in the sense of convex analysis whenever $Q$ is a convex set. For the purpose of completeness, for each $\bar{x} \notin Q$, we put $T_Q(\bar{x}) := \emptyset$ and $\hat{N}_Q(\bar{x}) = N_Q(\bar{x}) := \emptyset$.

For some direction $u \in \mathbb{X}$, we make use of

$$N_Q(\bar{x}; u) := \left\{ \eta \in \mathbb{X} \mid \exists\{u_k\}_{k \in \mathbb{N}} \subset \mathbb{X}, \exists\{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_+, \exists\{\eta_k\}_{k \in \mathbb{N}} \subset \mathbb{X}: u_k \to u, t_k \searrow 0, \eta_k \to \eta, \eta_k \in N_Q(\bar{x} + t_k u_k) \forall k \in \mathbb{N} \right\}$$
which is called the limiting normal cone to $Q$ at $\bar{x}$ in direction $u$. Note that this set is empty when $\bar{x} \notin Q$ or $u \notin T_Q(\bar{x})$. In case where $Q$ is convex, we obtain $\mathcal{N}_Q(\bar{x}; u) = \mathcal{N}_{\partial Q}(\bar{x}+tu)$ where $[u]^- := \{\eta \in \mathbb{R} | \langle \eta, u \rangle = 0\}$ is the annihilator of $u$.

The limiting normal cone to a set is well known for its robustness, i.e., it is outer semicontinuous as a set-valued mapping. In the course of the paper, we exploit an analogous property of the directional limiting normal cone which has been validated in (Gfrerer et al., 2022, Proposition 2).

**Lemma 2.1.** Let $Q \subset \mathbb{R}^n$ be closed and fix $\bar{x} \in Q$. Then, for each $u \in \mathbb{R}^n$, we have

$$\mathcal{N}_Q(\bar{x}; u) = \left\{ \eta \in \mathbb{R}^n \left| \exists \{u_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n, \exists \{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^+, \exists \{\eta_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n : u_k \to u, t_k \searrow 0, \eta_k \to \eta, \eta_k \in \mathcal{N}_Q(\bar{x}+t_ku_k) \forall k \in \mathbb{N} \right. \right\}.$$ 

Next, we recall some fundamental notions of generalized differentiation. Let us start with a locally Lipschitz continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$ and fix $\bar{x} \in \mathbb{R}^n$. The sets

$$\partial \varphi(\bar{x}) := \left\{ \eta \in \mathbb{R}^n | (\eta, -1) \in \mathcal{N}_{\text{epi} \varphi}(\bar{x}, \varphi(\bar{x})) \right\},$$

$$\hat{\partial} \varphi(\bar{x}) := \left\{ \eta \in \mathbb{R}^n | (\eta, -1) \in \mathcal{N}_{\text{epi} \varphi}(\bar{x}, \varphi(\bar{x})) \right\}$$

are referred to as the regular and limiting subdifferential of $\varphi$ at $\bar{x}$. Here, $\text{epi} \varphi := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} | \varphi(x) \leq \alpha\}$ denotes the epigraph of $\varphi$. Furthermore, for some direction $u \in \mathbb{R}^n$,

$$\partial \varphi(\bar{x}; u) := \left\{ \eta \in \mathbb{R}^n \left| \exists \{u_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n, \exists \{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^+, \exists \{\eta_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n : u_k \to u, t_k \searrow 0, \eta_k \to \eta, \eta_k \in \partial \varphi(\bar{x}+t_ku_k) \forall k \in \mathbb{N} \right. \right\}$$

is referred to as the limiting subdifferential of $\varphi$ at $\bar{x}$ in direction $u$.

Let $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping. The sets $\text{dom} \Phi := \{x \in \mathbb{R}^n | \Phi(x) \neq \emptyset\}$, $\text{gph} \Phi := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | y \in \Phi(x)\}$, $\text{ker} \Phi := \{x \in \mathbb{R}^n | 0 \in \Phi(x)\}$, and $\text{Im} \Phi := \bigcup_{x \in \mathbb{R}^n} \Phi(x)$ are called the domain, graph, kernel, and image of $\Phi$, respectively.

We fix some point $(\bar{x}, \bar{y}) \in \text{gph} \Phi$. The set-valued mapping $D\Phi(\bar{x}, \bar{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is called graphical derivative of $\Phi$ at $(\bar{x}, \bar{y})$. In case where $\Phi$ is single-valued at $\bar{x}$, we exploit $D\Phi(\bar{x}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ for brevity of notation. In (Benko and Mehlitz, 2022), Definition 2.4), we introduced the so-called graphical subderivative of $\Phi$ at $(\bar{x}, \bar{y})$ to be the set-valued mapping $D_{\text{sub}}\Phi(\bar{x}, \bar{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ which assigns to every $u \in \mathbb{R}^n$ the set of vectors $v \in \mathbb{R}^n$ such that there are sequences $\{u_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$, $\{v_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$, and $\{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^+$, such that $u_k \to u$, $v_k \to v$, $t_k \searrow 0$, $t_k \to \infty$, and $(\bar{x}+t_ku_k, \bar{y}+t_kv_k) \in \text{gph} \Phi$ for all $k \in \mathbb{N}$. Some fundamental calculus rules for the graphical subderivative, particularly, regarding so-called normal cone mappings, can be found in (Benko and Mehlitz, 2022, Section 2.3).

Let us now turn our attention to dual concepts of generalized differentiation which will be of essential importance in this paper. We refer to $D^*\Phi(\bar{x}, \bar{y})$, $D^*\Phi(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ given by

$$D^*\Phi(\bar{x}, \bar{y})(y^*) := \left\{ x^* \in \mathbb{R}^n | (x^*, -y^*) \in \mathcal{N}_{\text{gph} \Phi}(\bar{x}, \bar{y}) \right\},$$
for each $y^* \in \mathcal{Y}$ as regular and limiting coderivative of $\Phi$ at $(\bar{x}, \bar{y})$. For a pair of directions $(u, v) \in \mathcal{X} \times \mathcal{Y}$, the set-valued mapping $D^* \Phi((\bar{x}, \bar{y}); (u, v))$ is called the limiting coderivative of $\Phi$ at $(\bar{x}, \bar{y})$ in direction $(u, v)$. We note that the latter is only reasonable if $v \in D\Phi((\bar{x}, \bar{y}))(u)$, and that

$$D^* \Phi((\bar{x}, \bar{y}); (u, v))(y^*) = \{ x^* \in \mathcal{X} \mid (x^*, -y^*) \in \mathcal{N}_{\text{gph}} \Phi((\bar{x}, \bar{y})); (u, v)) \}$$

holds for all $y^* \in \mathcal{Y}$. Again, we exploit $\hat{D}^* \Phi((\bar{x}, \bar{y}))(x^*), D^* \Phi((\bar{x}, \bar{y}); (u, v))$ for brevity if $\Phi$ is single-valued at $\bar{x}$.

For a given order $\gamma > 1$ and $(u, v) \in S_{\mathcal{X}} \times \mathcal{Y}$, let the pseudo-coderivative of $\Phi$ of order $\gamma$ at $(\bar{x}, \bar{y})$ in direction $(u, v)$ be the set-valued mapping $D^\gamma \Phi((\bar{x}, \bar{y}); (u, v))$: $\mathcal{Y} \rightrightarrows \mathcal{X}$ which assigns to $y^* \in \mathcal{Y}$ the set of all vectors $x^* \in \mathcal{X}$ such that there exist sequences $\{ u_k \}_{k \in \mathbb{N}}, \{ x_k^* \}_{k \in \mathbb{N}} \subset \mathcal{X}, \{ v_k \}_{k \in \mathbb{N}}, \{ y_k^* \}_{k \in \mathbb{N}} \subset \mathcal{Y}$, and $\{ t_k \}_{k \in \mathbb{N}} \subset \mathbb{R}_+$ such that $u_k \rightarrow u$, $v_k \rightarrow v$, $x_k^* \rightarrow x^*$, $y_k^* \rightarrow y^*$, $x_k^* \in \hat{D}^* \Phi((\bar{x} + t_k u_k, \bar{y} + t_k v_k))(y_k^*) \forall k \in \mathbb{N}$, and $\forall k \in \mathbb{N}$:

$$t_k \| u_k \|^\gamma \rightarrow \| x_k^* \| \gamma \rightarrow (t_k \| u_k \|^\gamma)(y_k^*).$$

This notion of a pseudo-coderivative originates from Benko and Mehlitz (2022b) where it has been used to derive mixed-order stationarity conditions for nonsmooth optimization problems. However, the concept of pseudo-coderivative is a little older and dates back to Gfrerer (2014). Therein, a set-valued mapping $\hat{D}^\gamma \Phi((\bar{x}, \bar{y}); (u, v))$: $\mathcal{Y} \rightrightarrows \mathcal{X}$ is called pseudo-coderivative of $\Phi$ of order $\gamma$ at $(\bar{x}, \bar{y})$ in direction $(u, v)$ when there exist sequences $\{ u_k \}_{k \in \mathbb{N}}, \{ x_k^* \}_{k \in \mathbb{N}} \subset \mathcal{X}, \{ v_k \}_{k \in \mathbb{N}}, \{ y_k^* \}_{k \in \mathbb{N}} \subset \mathcal{Y}$, and $\{ t_k \}_{k \in \mathbb{N}} \subset \mathbb{R}_+$ such that $u_k \rightarrow u$, $v_k \rightarrow v$, $x_k^* \rightarrow x^*$, $y_k^* \rightarrow y^*$, $t_k \rightarrow 0$, and $\forall k \in \mathbb{N}$:

$$t_k \| u_k \|^\gamma \rightarrow \| x_k^* \| \gamma \rightarrow (t_k \| u_k \|^\gamma)(y_k^*).$$

In order to distinguish both concepts, we use the slightly different notation from above and refer to $\hat{D}^\gamma \Phi((\bar{x}, \bar{y}); (u, v))$ as Gfrerer’s directional pseudo-coderivative. By definition of these tools, we have the trivial estimate

$$D^\gamma \Phi((\bar{x}, \bar{y}); (u, v))(y^*) \subset \hat{D}^\gamma \Phi((\bar{x}, \bar{y}); (u, 0))(y^*)$$

for all $y^* \in \mathcal{Y}$.

Let us recall that $\Phi$ is said to be metrically subregular at $(\bar{x}, \bar{y})$ in direction $u \in \mathcal{X}$ whenever there are constants $\varepsilon > 0$, $\delta > 0$, and $\kappa > 0$ such that

$$\forall x \in \bar{x} + B_{\varepsilon, \delta}(u): \text{dist}(x, \Phi^{-1}(\bar{y})) \leq \kappa \text{dist}(\bar{y}, \Phi(x)),$$
where $B_{\epsilon,\delta}(u) := \{ v \in \mathbb{X} \mid \| v \| \leq \delta \| u \|, \| v \| \leq \epsilon \}$ is a so-called directional neighborhood of $u$ and $\Phi^{-1}(\bar{y}) := \{ x \in \mathbb{X} \mid \bar{y} \in \Phi(x) \}$ is the inverse image of $\bar{y}$ under $\Phi$. In case where this is fulfilled for $u := 0$, $\Phi$ is said to be metrically subregular at $(\bar{x}, \bar{y})$.

Coderivatives have turned out to be suitable tools in order to characterize local Lipschitz or regularity properties of set-valued mappings. Exemplary, let us mention that the so-called Mordukhovich criterion

$$\ker D^*\Phi(\bar{x}, \bar{y}) = \{0\}$$

is equivalent to $\Phi$ being metrically regular at $(\bar{x}, \bar{y})$, see e.g. (Mordukhovich, 2018, Section 3.1) for a definition and this result. Furthermore, the condition

$$\forall u \in \mathbb{S}_X:\ \ker D^*\Phi((\bar{x}, \bar{y}); (u, 0)) = \{0\}$$

is sufficient for $\Phi$ to be metrically subregular at $(\bar{x}, \bar{y})$, see Gfrerer (2013), which is why it is called First-Order Sufficient Condition For Metric Subregularity (FOSCMS for short) in the literature. We also note that, for some fixed $u \in \mathbb{S}_X$, $\ker D^\gamma\Phi((\bar{x}, \bar{y}); (u, 0)) = \{0\}$ implies that $\Phi$ is metrically subregular at $(\bar{x}, \bar{y})$ in direction $u$. An analogous sufficient condition for metric pseudo-subregularity of order $\gamma > 1$ in terms of Gfrerer’s directional pseudo-coderivative was derived in Gfrerer (2014), and in (Benko and Mehlitz, 2022b, Lemma 2.9) it was slightly modified to

$$\forall u \in \mathbb{S}_X:\ \ker D^\gamma\Phi((\bar{x}, \bar{y}); (u, 0)) = \{0\}.$$

Finally, we would like to provide some basic calculus rules for the coderivative of so-called constraint mappings. Therefore, recall that some single-valued function $g: \mathbb{X} \to \mathbb{Y}$ is called calm at $x \in \mathbb{X}$ whenever there are a neighborhood $U \subset \mathbb{X}$ of $x$ and a constant $L > 0$ such that

$$\forall x' \in U:\ \| g(x') - g(x) \| \leq L \| x' - x \|.$$

Furthermore, for some direction $u$, $g$ is referred to as calm in direction $u$ at $x$ if there are constants $\epsilon > 0$, $\delta > 0$, and $L > 0$ such that

$$\forall x' \in \bar{x} + B_{\epsilon,\delta}(u):\ \| g(x') - g(x) \| \leq L \| x' - x \|.$$

Lemma 2.2. Let $g: \mathbb{X} \to \mathbb{Y}$ be continuous, and let $D \subset \mathbb{Y}$ be nonempty as well as closed. We consider the constraint map $\Phi: \mathbb{X} \rightrightarrows \mathbb{Y}$ given by $\Phi(x) := g(x) - D$ for all $x \in \mathbb{X}$. Fix $(x, y) \in \text{gph} \Phi$. Then the following statements hold.

(a) For each $y^* \in \mathbb{Y}$, we have

$$\hat{D}^*\Phi(x, y)(y^*) \subset \begin{cases} \hat{D}^*g(x)(y^*) & y^* \in \hat{N}_D(g(x) - y), \\ \emptyset & \text{otherwise,} \end{cases}$$

and the opposite inclusion holds if $g$ is calm at $x$. 

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(b) For each \( y^* \in \mathcal{Y} \), we have
\[
D^*\Phi(x, y)(y^*) \subset \begin{cases} 
D^*g(x)(y^*) & \text{if } y^* \in \mathcal{N}_D(g(x) - y), \\
\emptyset & \text{otherwise}.
\end{cases}
\]

(c) For each pair of directions \((u, v) \in X \times Y\) and each \( y^* \in \mathcal{Y} \), we have
\[
D^*\Phi((x, y); (u, v))(y^*) \subset \bigcup_{w \in Dg(x)(u)} \begin{cases} 
D^*g(x; (u, w))(y^*) & \text{if } y^* \in \mathcal{N}_D(g(x) - y; w - v), \\
\emptyset & \text{otherwise}
\end{cases}
\]
provided \( g \) is calm at \( x \).

Proof. (a) For the proof of the statement, we observe that \( \text{gph } \Phi = \text{gph } g - (\{0\} \times D) \) is valid. Now, we exploit the sum rule from Benko and Mehlitz (2022a). Therefore, let us introduce the surrogate mapping \( M : X \times Y \Rightarrow (X \times Y) \times (X \times Y) \) given by
\[
M(x, y) := \{(\tilde{x}, g(\tilde{x})), (0, \tilde{y}) \mid \tilde{x} = x, \tilde{y} \in -D, y = g(\tilde{x}) + \tilde{y}\}
\]
for all \((x, y) \in X \times Y\), and observe that \( \text{gph } \Phi = \text{dom } M \) holds while \( M \) is single-valued and continuous on \( \text{gph } \Phi \). Now, we find
\[
\hat{N}_{\text{gph } \Phi}(x, y) \subset \hat{D}^*M((x, y), ((x, g(x)), (0, y - g(x))))((0, 0), (0, 0))
\]
for all \((x, y) \in \text{gph } \Phi\) from (Benko and Mehlitz, 2022a, Theorem 3.1), and the converse inclusion holds if \( g \) is calm at \( x \) since this ensures that \( M \) is so-called isolatedly calm at the point of interest, see (Benko and Mehlitz, 2022a, Corollary 4.4, Section 5.1.1). Now, computing the regular coderivative of \( M \) via (Benko and Mehlitz, 2022a, Lemmas 2.1, 2.2) yields the claim.

(b) The proof is similar as the one of the first statement. Again, we exploit the mapping \( M \) given in (2.1) and apply (Benko and Mehlitz, 2022a, Theorem 3.1) while observing that \( M \) is so-called inner semicompact w.r.t. its domain at each point \((x, y) \in \text{gph } \Phi \) by continuity of \( g \).

(c) This assertion can be shown in similar way as the second one.

Let us note that the upper estimate in (a) was also shown in (Bai et al., 2019, Lemma 3.2), but it actually follows directly from (Rockafellar and Wets, 1998, Exercise 6.44) upon realizing \( \text{gph } \Phi = \text{gph } g - (\{0\} \times D) \). In case where \( g \) is not calm at the reference point, one can still obtain an upper estimate for the directional limiting coderivative from (Benko and Mehlitz, 2022a, Theorem 3.1) which is slightly more technical since it comprises another union over \( w \in Dg(x)(0) \cap \mathcal{S}_Y \).
2.3 The model problem

Let \( \varphi : X \to \mathbb{R} \) be a locally Lipschitz continuous mapping, assume that \( \Phi : X \rightrightarrows Y \) has a closed graph, and fix \( \bar{y} \in \text{Im } \Phi \). In this paper, we investigate the rather general nonsmooth optimization problem

\[
\min \{ \varphi(x) | \bar{y} \in \Phi(x) \}.
\]

The feasible set of (P) will be denoted by \( \mathcal{F} \subset X \) and is, by \( \bar{y} \in \text{Im } \Phi \), nonempty. Let us remark that the model (P) covers diverse classes of optimization problems from the literature including standard nonlinear problems, problems with geometric (particularly, disjunctive or conic) constraints, problems with (quasi-) variational inequality constraints, and bilevel optimization problems. Optimality conditions and constraint qualifications for problems of this type can be found, e.g., in Gfrerer (2013); Mehlitz (2020); Mordukhovich (2006); Ye and Ye (1997). A standard notion of stationarity, which applies to (P) and is based on the tools of limiting variational analysis, is the one of M-stationarity.

**Definition 2.3.** A feasible point \( \bar{x} \in \mathcal{F} \) of (P) is called M-stationary whenever there is a multiplier \( \lambda \in Y \) such that

\[
0 \in \partial \varphi(\bar{x}) + D^* \Phi(\bar{x}, \bar{y})(\lambda).
\]

In the following lemma, whose proof is analogous to the one of (Bai and Ye, 2022, Lemma 3.1), we point out that directional metric subregularity of \( \Phi \) implies that penalizing the constraint in (P) with the aid of the distance function yields a directionally exact penalty function.

**Lemma 2.4.** Let \( \bar{x} \in \mathcal{F} \) be a local minimizer of (P), and assume that \( \Phi \) is metrically subregular at \((\bar{x}, \bar{y})\) in direction \( u \in S_X \). Then there are constants \( \varepsilon > 0, \delta > 0, \) and \( C > 0 \) such that \( \bar{x} \) is a local minimizer of

\[
\min \{ \varphi(x) + C \text{dist}(\bar{y}, \Phi(x)) | x \in \bar{x} + \mathcal{B}_{\varepsilon, \delta}(u) \}.
\]

Let us note that this result refines well-known theory about classical exact penalization in the presence of metric subregularity, see e.g. Burke (1991); Clarke (1983); Klatte and Kummer (2002).

In order to state one of the essential findings of Benko and Mehlitz (2022b) which provides the basis of our investigations, we need to recall the notion of critical directions associated with (P).

**Definition 2.5.** For some feasible point \( \bar{x} \in \mathcal{F} \) of (P), a direction \( u \in S_X \) is called critical for (P) at \( \bar{x} \) whenever there are sequences \( \{u_k\}_{k \in \mathbb{N}} \subset X, \{v_k\}_{k \in \mathbb{N}} \subset Y, \) and \( \{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_+ \) such that \( u_k \to u, v_k \to 0, t_k \searrow 0, \) and \( (\bar{x} + t_k u_k, \bar{y} + t_k v_k) \in \text{gph } \Phi \) for all \( k \in \mathbb{N} \) as well as

\[
\limsup_{k \to \infty} \frac{\varphi(\bar{x} + t_k u_k) - \varphi(\bar{x})}{t_k} \leq 0.
\]
Let us note that whenever \( \varphi \) is directionally differentiable at \( \bar{x} \in \mathcal{F} \), then \( u \in \mathbb{S}_X \) is a critical for \((P)\) at \( \bar{x} \) if and only if \( \varphi'(\bar{x}; u) \leq 0 \) and \( 0 \in D\Phi(\bar{x}, \bar{y})(u) \).

A directionally refined concept of M-stationarity has been shown to serve as a necessary optimality condition under validity of directional metric subregularity in (Gfrerer, 2013, Theorem 7).

**Lemma 2.6.** Let \( \bar{x} \in \mathcal{F} \) be a local minimizer of \((P)\), let \( u \in \mathbb{S}_X \) be a critical direction for \((P)\) at \( \bar{x} \), and let \( \Phi \) be metrically subregular at \((\bar{x}, \bar{y})\) in direction \( u \). Then there is a multiplier \( \lambda \in \mathcal{Y} \) such that

\[
0 \in \partial \varphi(\bar{x}; u) + D^* \Phi((\bar{x}, \bar{y}); (u, 0))(\lambda).
\]

Particularly, \( \bar{x} \) is M-stationary.

Let us note that the above result can also be distilled from **Lemma 2.4** by following ideas used to prove (Bai and Ye, 2022, Theorem 3.1).

The following result is taken from (Benko and Mehlitz, 2022b, Corollary 4.4) and sharpens the information provided by (Mehlitz, 2020, Theorem 3.2) or (Kruger and Mehlitz, 2022, Theorem 4.1).

**Theorem 2.7.** Let \( \bar{x} \in \mathcal{F} \) be a local minimizer of \((P)\). Then \( \bar{x} \) is M-stationary or there exist a critical direction \( u \in \mathbb{S}_X \) for \((P)\) at \( \bar{x} \) and \( y^* \in \mathcal{Y} \) as well as sequences \( \{x_k\}_{k \in \mathbb{N}}, \{x'_k\}_{k \in \mathbb{N}}, \{\eta_k\}_{k \in \mathbb{N}} \subseteq X \) and \( \{y_k\}_{k \in \mathbb{N}}, \{y^*_k\}_{k \in \mathbb{N}} \subseteq \mathcal{Y} \) such that \( x_k, x'_k \notin \Phi^{-1}(\bar{y}) \), \( y_k \neq \bar{y} \), and \( y^*_k \neq 0 \) for all \( k \in \mathbb{N} \),

\[
\begin{align*}
x_k, x'_k &\rightarrow \bar{x}, & y_k &\rightarrow \bar{y}, & \eta_k &\rightarrow 0, \quad (2.3a) \\
\|x_k - \bar{x}\| &\rightarrow u, & \|x'_k - \bar{x}\| &\rightarrow u, & \|y_k - \bar{y}\| &\rightarrow 0, \quad (2.3b) \\
y^*_k &\rightarrow y^*, & \|y_k - \bar{y}\| / \|y^*_k\| &\rightarrow \infty, & \|y_k - \bar{y}\| - \|y^*_k\| &\rightarrow 0, \quad (2.3c)
\end{align*}
\]

and

\[
\forall k \in \mathbb{N}: \quad \eta_k \in \hat{\partial} \varphi(x'_k) + D^* \Phi(x_k, y_k) \left( \frac{\|x_k - \bar{x}\|}{\|y_k - \bar{y}\|} y^*_k \right). \quad (2.4)
\]

Observe that **Theorem 2.7** yields a necessary optimality condition for \((P)\) which holds in the absence of any constraint qualification. Basically, **Theorem 2.7** says that either a local minimizer of \((P)\) is M-stationary or the so-called approximate (or asymptotic) stationarity condition \((2.4)\) holds along certain sequences such that the involved sequence of multiplier estimates given by

\[
\forall k \in \mathbb{N}: \quad \lambda_k := y^*_k \|x_k - \bar{x}\| / \|y_k - \bar{y}\|
\]

is unbounded. Note that in case where \( \{\lambda_k\}_{k \in \mathbb{N}} \) would be bounded, one could simply take the limit in \((2.4)\) along a suitable subsequence and, respecting the convergences from \((2.3)\), would end up with M-stationarity again. Thus, divergence of the multiplier
estimates is natural in Theorem 2.7 since not all local minimizers of \((P)\) are M-stationary in general, see (Mehlitz, 2020, Lemma 3.4) as well.

The sequential information from (2.3) describes in great detail what must “go wrong” if M-stationarity fails. We will refer to (2.3a)-(2.3c) as basic, directional, and multiplier (sequential) information, respectively. Clearly, one can secure M-stationarity of a local minimizer by ruling out the second alternative in Theorem 2.7 and, as we will show, various known constraint qualifications for M-stationarity indeed do precisely that. Let us mention here two such conditions. Rescaling (2.4) by \(\|\lambda_k\|\) and taking the limit \(k \to \infty\) leads to a contradiction with the Mordukhovich criterion/metric regularity of \(\Phi\) at \((\bar{x}, \bar{y})\). Respecting also the directional information (2.3b) yields a contradiction with FOSCMS at \((\bar{x}, \bar{y})\). Thus, we obtain a result related to Lemma 2.6, see (Benko and Mehlitz, 2022b, Theorem 4.3) as well. The advantage of these two conditions lies in their simplicity, since they can be expressed via suitable derivatives, but they are a bit more restrictive.

In both cases, we have essentially discarded the multiplier information (2.3c) which deserves some remarks. We have used \(\|\lambda_k\| \to \infty\), but this information is not really very important, since as we already explained, if the multipliers remain bounded, we end up with M-stationarity anyway. The fact that \(\{y^*_k\}_{k \in \mathbb{N}}\) converges tells us how fast the multipliers \(\{\lambda_k\}_{k \in \mathbb{N}}\) grow since we have \(y^*_k = \lambda_k \|y_k - \bar{y}\|/\|x_k - \bar{x}\|\) for each \(k \in \mathbb{N}\). In Section 5, we introduce the so-called super-coderivative which incorporates this information.

Finally, \((y_k - \bar{y})/\|y_k - \bar{y}\| - \lambda_k/\|\lambda_k\| \to 0\), which equals \((y_k - \bar{y})/\|y_k - \bar{y}\| - y^*_k/\|y^*_k\| \to 0\), means that the multipliers precisely capture the direction from which \(\{y_k\}_{k \in \mathbb{N}}\) converges to \(\bar{y}\). Equivalently, it can be expressed via \(\lambda_k/\|\lambda_k\|, (y_k - \bar{y})/\|y_k - \bar{y}\| \to 1\), which was used in the sufficient condition for metric subregularity in (Gfrerer, 2014, Corollary 1). This information is behind the notions of pseudo- and quasi-normality and we discuss it in detail in Section 4 and also utilize it in Section 5 to some extent.

Let us also mention that if the regular subdifferential and coderivative are replaced by the limiting ones in (2.4), \(y_k\) and \(y^*_k\) can be chosen such that \((y_k - \bar{y})/\|y_k - \bar{y}\| = y^*_k/\|y^*_k\|\) holds for all \(k \in \mathbb{N}\). This owes to the fact that the fuzzy calculus used in the proof of (Benko and Mehlitz, 2022b, Theorem 4.3) can be replaced by the exact calculus for limiting subdifferentials.

3 Directional asymptotic regularity in nonsmooth optimization

Based on Theorem 2.7, the following definition introduces concepts which may serve as (directional) qualification conditions for \((P)\).

**Definition 3.1.** Let \((\bar{x}, \bar{y}) \in \text{gph} \Phi\) be fixed.

(a) The map \(\Phi\) is said to be asymptotically regular at \((\bar{x}, \bar{y})\) whenever the following condition holds: for every sequences \(\{(x_k, y_k)\}_{k \in \mathbb{N}} \subset \text{gph} \Phi\), \(\{x^*_k\}_{k \in \mathbb{N}} \subset \mathbb{X}\), and \(\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{Y}\) as well as \(x^* \in \mathbb{X}\) satisfying \(x_k \to \bar{x}, y_k \to \bar{y}, x^*_k \to x^*\), and \(x^*_k \in D^* \Phi(x_k, y_k)(\lambda_k)\) for all \(k \in \mathbb{N}\), we find \(x^* \in \text{Im} D^* \Phi(\bar{x}, \bar{y})\).
(b) For the fixed direction \( u \in S_X \), \( \Phi \) is said to be asymptotically regular at \((\bar{x}, \bar{y})\) in direction \( u \) whenever the following condition holds: for every sequences \( \{(x_k, y_k)\}_{k \in \mathbb{N}} \subset \text{gph} \, \Phi \), \( \{x^*_k\}_{k \in \mathbb{N}} \subset X \), and \( \{\lambda_k\}_{k \in \mathbb{N}} \subset Y \) as well as \( x^* \in X \) and \( y^* \in Y \) satisfying \( x_k \notin \Phi^{-1}(\bar{y}) \), \( y_k \neq \bar{y} \), and \( x^*_k \in D^*\Phi(x_k, y_k)(\lambda_k) \) for each \( k \in \mathbb{N} \) as well as the convergences

\[
\begin{align*}
x_k &\to \bar{x}, & y_k &\to \bar{y}, & x^*_k &\to x^*, \\
\frac{x_k - \bar{x}}{\|x_k - \bar{x}\|} &\to u, & \frac{y_k - \bar{y}}{\|y_k - \bar{y}\|} &\to 0, & \|\lambda_k\| &\to \infty, \quad (3.1)
\end{align*}
\]

we find \( x^* \in \text{Im} \, D^*\Phi(\bar{x}, \bar{y}) \).

(c) For the fixed direction \( u \in S_X \), \( \Phi \) is said to be strongly asymptotically regular at \((\bar{x}, \bar{y})\) in direction \( u \) whenever the following condition holds: for every sequences \( \{(x_k, y_k)\}_{k \in \mathbb{N}} \subset \text{gph} \, \Phi \), \( \{x^*_k\}_{k \in \mathbb{N}} \subset X \), and \( \{\lambda_k\}_{k \in \mathbb{N}} \subset Y \) as well as \( x^* \in X \) and \( y^* \in Y \) satisfying \( x_k \notin \Phi^{-1}(\bar{y}) \), \( y_k \neq \bar{y} \), and \( x^*_k \in D^*\Phi(x_k, y_k)(\lambda_k) \) for each \( k \in \mathbb{N} \) as well as the convergences \( (3.1) \), we have \( x^* \in \text{Im} \, D^*\Phi(\bar{x}, \bar{y}; (u, 0)) \).

Let us briefly note that asymptotic regularity of a set-valued mapping \( \Phi : X \rightrightarrows Y \) at some point \((\bar{x}, 0)\) in gph \( \Phi \) in the sense of Definition 3.1 equals AM-regularity of the set \( \Phi^{-1}(0) \) at \( \bar{x} \) mentioned in (Mehlitz, 2020, Remark 3.17), see Proposition 3.5 as well. The concepts of directional asymptotic regularity from Definition 3.1 are new.

In the subsequent remark, we summarize some obvious relations between the different concepts from Definition 3.1.

**Remark 3.2.** Let \((\bar{x}, \bar{y}) \in \text{gph} \, \Phi \) be fixed. Then the following assertions hold.

(a) Let \( u \in S_X \) be arbitrarily chosen. If \( \Phi \) is strongly asymptotically regular at \((\bar{x}, \bar{y})\) in direction \( u \), it is asymptotically regular at \((\bar{x}, \bar{y})\) in direction \( u \).

(b) If \( \Phi \) is asymptotically regular at \((\bar{x}, \bar{y})\), then it is asymptotically regular at \((\bar{x}, \bar{y})\) in each direction from \( S_X \).

We note that strong asymptotic regularity in each unit direction is indeed not related to asymptotic regularity. On the one hand, the subsequently stated example, taken from (Mehlitz, 2020, Example 3.15), shows that asymptotic regularity does not imply strong asymptotic regularity in each unit direction. On the other hand, Example 3.6 illustrates that strong asymptotic regularity in each unit direction does not yield asymptotic regularity.

**Example 3.3.** We consider \( \Phi : \mathbb{R} \rightrightarrows \mathbb{R} \) given by

\[
\forall x \in \mathbb{R} : \quad \Phi(x) := \begin{cases} 
\mathbb{R} & \text{if } x \leq 0, \\
[x^2, \infty) & \text{if } x > 0
\end{cases}
\]
at \((\bar{x}, \bar{y}) := (0,0)\). It is demonstrated in (Mehlitz, 2020, Example 3.15) that \(\Phi\) is asymptotically regular at \((\bar{x}, \bar{y})\). We find \(\mathcal{T}_{\partial \Phi}(\bar{x}, \bar{y}) = \{(u, v) \in \mathbb{R}^2 \mid u \leq 0 \lor v \geq 0\}\) so \((\pm 1, 0) \in \mathcal{T}_{\partial \Phi}(\bar{x}, \bar{y})\). Let us consider \(u := 1\). Then we find \(\text{Im} \ D^*\Phi((\bar{x}, \bar{y}); (u, 0)) = \{0\}\). Taking \(x^* := 1, y^* := 1/2,\) as well as

\[
\forall k \in \mathbb{N}: \quad x_k := \frac{1}{k}, \quad y_k := \frac{1}{k^2}, \quad x_k^* := 1, \quad \lambda_k := \frac{k}{2},
\]

we have \(x_k^* \in \hat{D}^*\Phi(x_k, y_k)(\lambda_k)\) for all \(k \in \mathbb{N}\) as well as the convergences \((3.1)\). However, due to \(x_k^* \to x^* \notin \text{Im} \ D^*\Phi((\bar{x}, \bar{y}); (u, 0))\), \(\Phi\) is not strongly asymptotically regular at \((\bar{x}, \bar{y})\) in direction \(u\).

Combining Theorem 2.7 with the concepts from Definition 3.1, we immediately obtain the following result due to local boundedness of the regular subdifferential of Lipschitzian functions, see e.g. (Mordukhovich, 2018, Theorem 1.22).

**Corollary 3.4.** Let \(\bar{x} \in \mathcal{F}\) be a local minimizer of \((P)\) such that, for each critical direction \(u \in \mathcal{S}_\mathcal{F}\) for \((P)\) at \(\bar{x}\), \(\Phi\) is asymptotically regular at \((\bar{x}, \bar{y})\) in direction \(u\). Then \(\bar{x}\) is \(M\)-stationary.

In the light of Remark 3.2 (b), our result from Corollary 3.4 improves (Mehlitz, 2020, Theorem 3.9) by a directional refinement of the constraint qualification since it suffices to check asymptotic regularity with respect to particular directions.

We point out that, unlike typical constraint qualifications, (directional) asymptotic regularity allows the existence of sequences satisfying \((3.1)\) as long as the limit \(x^*\) is included in \(\text{Im} \ D^*\Phi(\bar{x}, \bar{y})\) which is enough for \(M\)-stationarity.

For the purpose of completeness, we show that the notions from Definition 3.1 can be stated in terms of the limiting coderivative completely.

**Proposition 3.5.** Definition 3.1 can be equivalently formulated with \(x_k^* \in \hat{D}^*\Phi(x_k, y_k)(\lambda_k)\) replaced by \(x_k^* \in D^*\Phi(x_k, y_k)(\lambda_k)\).

**Proof.** For non-directional asymptotic regularity the proof is standard and follows from a diagonal sequence argument. The proof for strong directional asymptotic regularity parallels the one for directional asymptotic regularity which is presented below.

Since one implication is clear by definition of the regular and limiting coderivative, we only show the other one. Thus, let us fix sequences \(\{(x_k, y_k)\}_{k \in \mathbb{N}} \subset \partial \Phi, \{x_k^*\}_{k \in \mathbb{N}} \subset X, \) and \(\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{Y}\) as well as \(x^* \in X\) and \(y^* \in \mathbb{Y}\) satisfying \(x_k \notin \Phi^{-1}(y), y_k \neq y,\) and \(x_k^* \in \hat{D}^*\Phi(x_k, y_k)(\lambda_k)\) for each \(k \in \mathbb{N}\) as well as the convergences \((3.1)\). For each \(k \in \mathbb{N}\), we find sequences \(\{(x_{k, \ell}, y_{k, \ell})\}_{\ell \in \mathbb{N}} \subset \partial \Phi, \{x_{k, \ell}^*\}_{\ell \in \mathbb{N}} \subset X, \) and \(\{\lambda_{k, \ell}\}_{\ell \in \mathbb{N}} \subset \mathbb{Y}\) with \(x_{k, \ell} \to x_k, x_{k, \ell}^* \to x_k^*, y_{k, \ell} \to y_k,\) and \(\lambda_{k, \ell} \to \lambda_k\) as \(\ell \to \infty\) as well as \(x_{k, \ell}^* \in \hat{D}^*\Phi(x_{k, \ell}, y_{k, \ell})(\lambda_{k, \ell})\) for each \(\ell \in \mathbb{N}\). Observing that \(\Phi^{-1}(\bar{y})\) is closed, its complement is open so that \(x_{k, \ell} \notin \Phi^{-1}(\bar{y})\) holds for sufficiently large \(\ell \in \mathbb{N}\). Furthermore, since \(\|x_k - \bar{x}\| > 0\) and \(\|y_k - \bar{y}\| > 0\) are valid, we can choose an index \(\ell(k) \in \mathbb{N}\) so large such that the estimates

\[
\|x_{k, \ell(k)} - x_k\| < \frac{1}{k}\|x_k - \bar{x}\|, \quad \|x_{k, \ell(k)}^* - x_k^*\| < \frac{1}{k},
\]

\[
\|y_{k, \ell(k)} - y_k\| < \frac{1}{k}\|y_k - \bar{y}\|, \quad \|\lambda_{k, \ell(k)} - \lambda_k\| < \frac{1}{k}
\]
and \( x_{k,\ell(k)} \notin \Phi^{-1}(\tilde{y}) \) as well as \( y_{k,\ell(k)} \neq \tilde{y} \) are valid. For each \( k \in \mathbb{N} \), we set \( \bar{x}_k := x_{k,\ell(k)} \), \( \bar{x}_k^* := x_{k,\ell(k)}^* \), \( \bar{y}_k := y_{k,\ell(k)} \), and \( \bar{\lambda}_k := \lambda_{k,\ell(k)} \). Clearly, we have \( \bar{x}_k \to \bar{x}, \bar{y}_k \to \tilde{y}, \) \( \bar{x}_k^* \to x^*, \|\bar{\lambda}_k\| \to \infty \), \( \{(\bar{x}_k, \bar{y}_k)\}_{k \in \mathbb{N}} \subset \text{gph} \Phi \), and \( \bar{x}_k \notin \Phi^{-1}(\tilde{y}) \), \( \bar{y}_k \neq \tilde{y} \), as well as \( \bar{x}_k^* \in D^*\Phi(\bar{x}_k, \bar{y}_k)(\bar{\lambda}_k) \) for each \( k \in \mathbb{N} \) by construction. Furthermore, we find

\[
\|\bar{x}_k - \bar{x}\| \geq \|x_k - \bar{x}\| - \|\bar{x}_k - x_k\| \geq \frac{k-1}{k}\|x_k - \bar{x}\|
\]

for each \( k \in \mathbb{N} \). With the above estimates at hand, we obtain

\[
\left\| \frac{x_k - \bar{x}}{\|x_k - \bar{x}\|} - \frac{\bar{x}_k - \bar{x}}{\|\bar{x}_k - \bar{x}\|} \right\| = \left\| \frac{x_k - \bar{x}_k}{\|x_k - \bar{x}\|} + (\bar{x}_k - \bar{x}) \left( \frac{1}{\|x_k - \bar{x}\|} - \frac{1}{\|\bar{x}_k - \bar{x}\|} \right) \right\| \\
\leq \frac{\|x_k - \bar{x}_k\|}{\|x_k - \bar{x}\|} + \frac{\|\bar{x}_k - \bar{x}\|}{\|\bar{x}_k - \bar{x}\|} \leq \frac{2}{k}
\]

and

\[
\left\| \frac{y_k - \tilde{y}}{\|x_k - \bar{x}\|} - \frac{\bar{y}_k - \tilde{y}}{\|\bar{x}_k - \bar{x}\|} \right\| = \left\| \frac{y_k - \bar{y}_k}{\|x_k - \bar{x}\|} + (\bar{y}_k - \tilde{y}) \left( \frac{1}{\|x_k - \bar{x}\|} - \frac{1}{\|\bar{x}_k - \bar{x}\|} \right) \right\| \\
\leq \frac{\|y_k - \bar{y}_k\|}{\|x_k - \bar{x}\|} + \frac{\|\bar{y}_k - \tilde{y}\|}{\|\bar{x}_k - \bar{x}\|} \frac{\|x_k - \bar{x}\|}{\|\bar{x}_k - \bar{x}\|} \leq \frac{1}{k}\frac{\|y_k - \bar{y}_k\|}{\|x_k - \bar{x}\|} + \frac{1}{k-1}\frac{\|y_k - \bar{y}_k\|}{\|x_k - \bar{x}\|} \\
\leq \frac{1}{k} + \frac{1}{k(k-1)} \leq \frac{2}{k-1} \frac{\|y_k - \tilde{y}\|}{\|x_k - \bar{x}\|}
\]

so that, with the aid of (3.1), we find \( (\bar{x}_k - \bar{x})/\|\bar{x}_k - \bar{x}\| \to u \) and \( (\bar{y}_k - \tilde{y})/\|\bar{x}_k - \bar{x}\| \to 0 \). With the aid of (3.2),

\[
\left\| \frac{\bar{y}_k - \tilde{y}}{\|\bar{x}_k - \bar{x}\|} \right\| \lambda_k - \frac{\|y_k - \tilde{y}\|}{\|x_k - \bar{x}\|} \lambda_k \leq \left\| \frac{\bar{y}_k - \tilde{y}}{\|\bar{x}_k - \bar{x}\|} \right\| \lambda_k - \frac{\|y_k - \tilde{y}\|}{\|x_k - \bar{x}\|} \lambda_k \leq \frac{1}{k}\left\| \frac{\bar{y}_k - \tilde{y}}{\|\bar{x}_k - \bar{x}\|} \right\| \lambda_k \\
\leq \frac{2}{k-1} \frac{\|y_k - \tilde{y}\|}{\|x_k - \bar{x}\|} \lambda_k
\]

is obtained, which gives \( \bar{\lambda}_k\|\tilde{y}_k - \tilde{y}\|/\|\bar{x}_k - \bar{x}\| \to y^* \). Similar as above, we find

\[
\left\| \frac{\bar{y}_k - \tilde{y}}{\|\bar{y}_k - \tilde{y}\|} - \frac{y_k - \tilde{y}}{\|y_k - \tilde{y}\|} \right\| \leq \frac{2}{k}
\]

and

\[
\left\| \frac{\bar{\lambda}_k}{\|\bar{\lambda}_k\|} - \frac{\lambda_k}{\|\lambda_k\|} \right\| \leq 2\|\lambda_k - \bar{\lambda}_k\|/\|\lambda_k\| \leq 2/(k \|\lambda_k\|),
\]

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so that (3.1) gives us
\[
\lim_{k \to \infty} \left( \frac{\tilde{y}_k - \bar{y}}{\|\tilde{y}_k - \bar{y}\|} - \frac{\lambda_k}{\|\lambda_k\|} \right) = \lim_{k \to \infty} \left( \frac{y_k - \bar{y}}{\|y_k - \bar{y}\|} - \frac{\lambda_k}{\|\lambda_k\|} \right) = 0.
\]

Now, since \( \Phi \) is asymptotically regular at \((\bar{x}, \bar{y})\) in direction \( u \), we obtain \( x^* \in \text{Im} D^* \Phi(\bar{x}, \bar{y}) \). \( \square \)

Since (directional) asymptotic regularity (w.r.t. all critical unit directions) yields M-stationarity of a local minimizer by Corollary 3.4, in the remaining part of the paper, we put it into context of other common assumptions that work as a constraint qualification for M-stationarity associated with problem \((P)\). Let us clarify here some rather simple or known connections.

(a) A polyhedral mapping is asymptotically regular at each point of its graph.

(b) Metric regularity implies asymptotic regularity.

(c) Strong metric subregularity implies asymptotic regularity.

(d) FOSCMS does not imply asymptotic regularity, but it implies strong asymptotic regularity in each unit direction.

(e) Metric subregularity does not imply asymptotic regularity in each unit direction. However, if the map of interest is metrically subregular at every point of its graph near the reference point with a uniform constant, then strong asymptotic regularity in each unit direction follows.

(f) Neither asymptotic regularity nor strong directional asymptotic regularity yields the directional exact penalty property of Lemma 2.4.

Statements (a) and (b) were shown in (Mehlitz, 2020, Theorems 3.10 and 3.12). Let us now argue that strong metric subregularity (the “inverse” property associated with isolated calmness), see Dontchev and Rockafellar (2014), also implies asymptotic regularity at the point. This follows easily from the discussion above (Benko and Mehlitz, 2022a, Corollary 4.6), which yields that the domain of the limiting coderivative, at the point where the mapping is isolatedly calm, is the whole space. Equivalently, the range of the limiting coderivative, at the point where the mapping is strongly metrically subregular, is the whole space and asymptotic regularity thus follows trivially. Thus, statement (c) follows.

Regarding (d), the fact that FOSCMS implies strong asymptotic regularity in each unit direction easily follows by similar arguments that show that metric regularity implies asymptotic regularity, see (Mehlitz, 2020, Lemma 3.11, Theorem 3.12). Indeed, let us fix \((\bar{x}, \bar{y}) \in \text{gph} \Phi\) and \( u \in \mathbb{S}_X \) such that \( \ker D^* \Phi((\bar{x}, \bar{y}); (u, 0)) = \{0\} \). Furthermore, choose sequences \( \{(x_k, y_k)\}_{k \in \mathbb{N}} \subset \text{gph} \Phi, \{x_k^*\}_{k \in \mathbb{N}} \subset X, \) and \( \{\lambda_k\}_{k \in \mathbb{N}} \subset Y \) as well as \( x^* \in X \) and \( y^* \in Y \) satisfying \( x_k \notin \Phi^{-1}(\bar{y}) \), \( y_k \neq \bar{y} \), and \( x_k^* \in D^* \Phi(x_k, y_k)(\lambda_k) \) for all \( k \in \mathbb{N} \) as well as the convergences (3.1). Then we also have \( x_k^*/\|\lambda_k\| \in D^* \Phi(x_k, y_k)(\lambda_k/\|\lambda_k\|) \) for
each $k \in \mathbb{N}$, and taking the limit $k \to \infty$ along a suitably chosen subsequence, we end up with $0 \in D^*\Phi((\bar{x}, \bar{y}); (u, 0))$ for some $\lambda \in \mathbb{S}_Y$ which is a contradiction. Hence, such sequences cannot exist and $\Phi$ is strongly asymptotically regular at $(\bar{x}, \bar{y})$ in direction $u$.

The following example shows that FOSCMS does not imply asymptotic regularity.

**Example 3.6.** Let $\Phi: \mathbb{R} \to \mathbb{R}$ be given by

$$
\forall x \in \mathbb{R}: \quad \Phi(x) := \begin{cases} 
[x, \infty) & \text{if } x \leq 0, \\
\frac{1}{k} - \frac{1}{k} (x - \frac{1}{k}) , \infty) & \text{if } x \in \left(\frac{1}{k+1}, \frac{1}{k}\right) \text{ for some } k \in \mathbb{N}, \\
\emptyset & \text{otherwise.}
\end{cases}
$$

Then $\{(1/k, 1/k)\}_{k \in \mathbb{N}} \subset \text{gph } \Phi$ converges to $(\bar{x}, \bar{y}) := (0, 0)$ and

$$
\mathcal{N}_{\text{gph } \Phi}(1/k, 1/k) = \{(x^*, y^*) \in \mathbb{R}^2 | y^* \leq 0, y^* \leq kx^*\}
$$

is valid showing that $\text{Im } D^*\Phi(1/k, 1/k) = \mathbb{R}$ is valid for all $k \in \mathbb{N}$. On the other hand, we have

$$
\mathcal{N}_{\text{gph } \Phi}(0, 0) = \{(x^*, y^*) \in \mathbb{R}^2 | x^* \geq 0, y^* \leq 0\},
$$

and thus $\text{Im } D^*\Phi(0, 0) = \mathbb{R}_+$. This means that $\Phi$ is not asymptotically regular at $(\bar{x}, \bar{y})$.

On the other hand, we find

$$
\mathcal{T}_{\text{gph } \Phi}(\bar{x}, \bar{y}) = \{(u, v) \in \mathbb{R}^2 | u \leq v\}.
$$

Each pair $(u, 0) \in \mathcal{T}_{\text{gph } \Phi}(\bar{x}, \bar{y})$ with $u \neq 0$ satisfies $u < 0$, i.e., the direction $(u, 0)$ points into the interior of $\text{gph } \Phi$. Thus, we have $\mathcal{N}_{\text{gph } \Phi}(\bar{x}, \bar{y}), (u, 0)) = \{(0, 0)\}$ which shows that FOSCMS is valid.

Regarding (c), let us fix $(\bar{x}, \bar{y}) \in \text{gph } \Phi$ and note that metric subregularity of $\Phi$ on a neighborhood of $(\bar{x}, \bar{y})$ (restricted to $\text{gph } \Phi$) with a uniform constant $\kappa > 0$ is clearly milder than metric regularity at $(\bar{x}, \bar{y})$ since it is automatically satisfied e.g. by polyhedral mappings. To see that it implies asymptotic regularity, consider sequences $\{(x_k, y_k)\}_{k \in \mathbb{N}} \subset \text{gph } \Phi$, $\{x_k^*\}_{k \in \mathbb{N}} \subset \mathbb{X}$, and $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{Y}$ as well as $x^* \in \mathbb{X}$ and $y^* \in \mathbb{Y}$ satisfying $x_k^* \in D^*\Phi(x_k, y_k)(\lambda_k)$ for each $k \in \mathbb{N}$ and the convergences (3.1) for some unit direction $u \in \mathbb{S}_X$. Due to (Benko and Mehlitz, 2022a, Theorem 3.2) and $-x_k^* \in \text{dom } D^*\Phi^{-1}(y_k, x_k)$, we find $x_k^* \in D_{\Phi^{-1}(y_k)}(x_k) \subset \mathcal{N}_{\Phi^{-1}(y_k)}(x_k)$ for each $k \in \mathbb{N}$. Furthermore, (Benko and Mehlitz, 2022a, Theorem 3.2) also gives the existence of $\lambda_k \in \mathbb{Y}$ with $\|\lambda_k\| \leq \kappa \|x_k^*\|$ and $x_k^* \in D^*\Phi(x_k, y_k)(\lambda_k)$. Noting that $\{x_k^*\}_{k \in \mathbb{N}}$ converges, this shows that there is a limit point $\lambda \in \mathbb{Y}$ of $\{\lambda_k\}_{k \in \mathbb{N}}$ which satisfies $x^* \in D^*\Phi((\bar{x}, \bar{y}); (u, 0))(\lambda)$ by robustness of the directional limiting coderivative which can be distilled from Lemma 2.1. Hence, $\Phi$ is strongly asymptotically regular at $(\bar{x}, \bar{y})$ in direction $u$. Note that for the above arguments to work, we only need uniform metric subregularity along all sequences $\{(x_k, y_k)\}_{k \in \mathbb{N}} \subset \text{gph } \Phi$ converging to $(\bar{x}, \bar{y})$ from direction $(u, 0)$.

On the other hand, the following example shows that metric subregularity in the neighborhood of the point of interest does not imply asymptotic regularity in each unit direction.
Example 3.7. We consider the mapping $\Phi: \mathbb{R} \rightharpoonup \mathbb{R}$ given by
\[
\forall x \in \mathbb{R}: \quad \Phi(x) := \{0, x^2\}.
\]
Due to $\Phi^{-1}(0) = \mathbb{R}$, $\Phi$ is metrically subregular at all points $(x, 0)$ where $x \in \mathbb{R}$ is arbitrary. Furthermore, at all points $(x, x^2)$ where $x \neq 0$, the Mordukhovich criterion shows that $\Phi$ is metrically regular. Thus, $\Phi$ is metrically subregular at each point of its graph. Note that the moduli of metric subregularity tend to $\infty$ along the points $(t, t^2)$ and $(-t, t^2)$ as $t \searrow 0$.

Let us consider the point $(\bar{x}, \bar{y}) := (0, 0)$ where we have $N_{gph \Phi}(\bar{x}, \bar{y}) = \{0\} \times \mathbb{R}$ and, thus, $\text{Im} D^*\Phi(\bar{x}, \bar{y}) = \{0\}$. Choosing $x^* := 1$, $y^* := 1/2$, as well as
\[
\forall k \in \mathbb{N}: \quad x_k := \frac{1}{k}, \quad y_k := \frac{1}{k^2}, \quad x_k^* := 1, \quad \lambda_k := \frac{k}{2},
\]
we have $x_k^* \in \text{Im} D^*\Phi(x_k, y_k) (\lambda_k)$ for all $k \in \mathbb{N}$ as well as the convergences (3.1) for $u := 1$. Due to $x_k^* \to x^* \notin \text{Im} D^*\Phi(\bar{x}, \bar{y})$, $\Phi$ is not asymptotically regular at $(\bar{x}, \bar{y})$ in direction $u$.

Finally, let us address item (f) with the aid of an example.

Example 3.8. Let us define $\varphi: \mathbb{R} \to \mathbb{R}$ and $\Phi: \mathbb{R} \rightharpoonup \mathbb{R}$ by means of
\[
\forall x \in \mathbb{R}: \quad \varphi(x) := -x, \quad \Phi(x) := \begin{cases} \mathbb{R} & \text{if } x \leq 0, \\
[x^2, \infty) & \text{if } x = \frac{1}{k} \text{ for some } k \in \mathbb{N}, \\
\emptyset & \text{otherwise}.
\end{cases}
\]
Furthermore, we fix $\bar{y} := 0$. One can easily check that $\bar{x} := 0$ is the uniquely determined global minimizer of the associated problem (P). Furthermore, we have $\text{Im} D^*\Phi(\bar{x}, \bar{y}) = \text{Im} D^*\Phi((\bar{x}, \bar{y}); (1, 0)) = \mathbb{R}$ which shows that $\Phi$ is asymptotically regular at $(\bar{x}, \bar{y})$ as well as strongly asymptotically regular at $(\bar{x}, \bar{y})$ in direction 1. Furthermore, it is obvious that $\Phi$ is strongly asymptotically regular at $(\bar{x}, \bar{y})$ in direction $-1$. Finally, let us mention that $\Phi$ fails to be metrically subregular at $(\bar{x}, \bar{y})$ in direction 1.

Now, define $x_k := 1/k$ for each $k \in \mathbb{N}$ and observe that for each constant $C > 0$ and sufficiently large $k \in \mathbb{N}$, we have $\varphi(x_k) + C \text{ dist}(\bar{y}, \Phi(x_k)) = -1/k + C/k^2 < 0 = \varphi(\bar{x})$, i.e., $\bar{x}$ is not a local minimizer of (2.2) for any choice of $C > 0$, $\varepsilon > 0$, $\delta > 0$, and $u := 1$.

4 Directional pseudo- and quasi-normality

In this section, we connect asymptotic regularity with the notions of pseudo- and quasi-normality. Note that the latter concepts have been introduced for standard nonlinear programs in Bertsekas and Ozdaglar (2002); Hestenes (1975), and reasonable generalizations to more general geometric constraints have been established in Guo et al. (2013). Furthermore, problem-tailored notions of these conditions have been coined e.g. for so-called cardinality-, complementarity-, and switching-constrained optimization problems, see Kanzow et al. (2021b); Kanzow and Schwartz (2010); Liang and Ye (2021). Let us
point out that these conditions are comparatively mild constraint qualifications and sufficient for the presence of metric subregularity of the underlying feasibility mapping which equals the so-called error bound property, see e.g. (Guo et al., 2013, Theorem 5.2). Here, we extend pseudo- and quasi-normality from the common setting of geometric constraint systems to arbitrary set-valued mappings and comment on the qualitative properties of these conditions. Naturally, we aim for directional versions of these concepts, which, in the setting of geometric constraints, were recently introduced in Bai et al. (2019) and further explored in Benko et al. (2022). Furthermore, we briefly discuss directional pseudo- and quasi-normality in the context of equilibrium-constrained optimization.

4.1 Pseudo- and quasi-normality for set-valued mappings

The definition below introduces the notions of our interest.

Definition 4.1. Fix \((\bar{x}, \bar{y}) \in \text{gph } \Phi\) and a direction \(u \in \mathbb{S}_X\).

(a) We say that pseudo-normality in direction \(u\) holds at \((\bar{x}, \bar{y})\) if there does not exist a nonzero vector \(\lambda \in \ker D^*\Phi((\bar{x}, \bar{y}); (u, 0))\) satisfying the following condition: there are sequences \(\{(x_k, y_k)\}_{k \in \mathbb{N}} \subset \text{gph } \Phi\) with \(x_k \neq \bar{x}\) for all \(k \in \mathbb{N}\) and \(\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{Y}\), \(\{\eta_k\}_{k \in \mathbb{N}} \subset \mathbb{X}\), such that

\[
\begin{align*}
x_k & \to \bar{x}, \\
y_k & \to \bar{y}, \\
\lambda_k & \to \lambda, \\
\eta_k & \to 0,
\end{align*}
\]

and \(\eta_k \in \hat{D}^*\Phi(x_k, y_k)(\lambda_k)\) as well as \(\langle \lambda, y_k - \bar{y} \rangle > 0\) for all \(k \in \mathbb{N}\).

(b) Let \(\mathcal{E} := \{e_1, \ldots, e_m\} \subset \mathbb{Y}\) be an orthonormal basis of \(\mathbb{Y}\). We say that quasi-normality in direction \(u\) holds at \((\bar{x}, \bar{y})\) w.r.t. \(\mathcal{E}\) if there does not exist a nonzero vector \(\lambda \in \ker D^*\Phi((\bar{x}, \bar{y}); (u, 0))\) satisfying the following condition: there are sequences \(\{(x_k, y_k)\}_{k \in \mathbb{N}} \subset \text{gph } \Phi\) with \(x_k \neq \bar{x}\) for all \(k \in \mathbb{N}\) and \(\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{Y}\), \(\{\eta_k\}_{k \in \mathbb{N}} \subset \mathbb{X}\), such that we have the convergences from (4.1) and, for all \(k \in \mathbb{N}\) and \(i \in \{1, \ldots, m\}\), \(\eta_k \in \hat{D}^*\Phi(x_k, y_k)(\lambda_k)\) as well as \(\langle \lambda, e_i \rangle \langle y_k - \bar{y}, e_i \rangle > 0\) if \(\langle \lambda, e_i \rangle \neq 0\).

In case where the canonical basis is chosen in \(\mathbb{Y} := \mathbb{R}^m\), the above concept of quasi-normality is a direct generalization of the original notion from Bertsekas and Ozdaglar (2002) which was coined for standard nonlinear problems and neglected directional information. Let us just mention that a reasonable, basis-independent definition of quasi-normality would require that there exists some basis w.r.t. which the mapping of interest is quasi-normal, see also Theorem 4.3.

Note that the sequence \(\{y_k\}_{k \in \mathbb{N}}\) in the definition of directional pseudo- and quasi-normality needs to satisfy \(y_k \neq \bar{y}\) for all \(k \in \mathbb{N}\). In the definition of directional pseudo-normality, this is clear from \(\langle \lambda, y_k - \bar{y} \rangle > 0\) for all \(k \in \mathbb{N}\). Furthermore, in the definition of directional quasi-normality, observe that \(\lambda \neq 0\) implies the existence of \(j \in \{1, \ldots, m\}\) such that \(\langle \lambda, e_j \rangle \neq 0\) holds, so that \(\langle y_k - \bar{y}, e_j \rangle \neq 0\) is necessary for each \(k \in \mathbb{N}\).

In the following lemma, we show the precise relation between directional pseudo- and quasi-normality.
Lemma 4.2. Fix $(\bar{x}, \bar{y}) \in \text{gph} \Phi$ and some direction $u \in S_X$. Then $\Phi$ is pseudo-normal at $(\bar{x}, \bar{y})$ in direction $u$ if and only if $\Phi$ is quasi-normal at $(\bar{x}, \bar{y})$ in direction $u$ w.r.t. each orthonormal basis of $Y$.

Proof. $[\implies]$ Let $\Phi$ be pseudo-normal at $(\bar{x}, \bar{y})$ in direction $u$, let $\mathcal{E} := \{e_1, \ldots, e_m\} \subset Y$ be an orthonormal basis of $Y$, and pick $\lambda \in \ker D^*\Phi((\bar{x}, \bar{y}); (u, 0))$ as well as sequences $\{(x_k, y_k)\}_{k \in \mathbb{N}} \subset \text{gph} \Phi$ with $x_k \neq \bar{x}$ for all $k \in \mathbb{N}$ and $\{\lambda_k\}_{k \in \mathbb{N}} \subset Y$, $\{\eta_k\}_{k \in \mathbb{N}} \subset X$, satisfying the convergences (4.1) and, for all $k \in \mathbb{N}$ and $i \in \{1, \ldots, m\}$, $\eta_k \in D^* \Phi(x_k, y_k)(\lambda_k)$ as well as $\langle \lambda, e_i \rangle \langle y_k - \bar{y}, e_i \rangle > 0$ if $\langle \lambda, e_i \rangle \neq 0$. Observing that we have

$$
\langle \lambda, y_k - \bar{y} \rangle = \left( \sum_{i=1}^m \langle \lambda, e_i \rangle e_i, \sum_{j=1}^m \langle y_k - \bar{y}, e_j \rangle e_j \right) = \sum_{i=1}^m \sum_{j=1}^m \langle \lambda, e_i \rangle \langle y_k - \bar{y}, e_j \rangle \langle e_i, e_j \rangle = \sum_{i=1}^m \langle \lambda, e_i \rangle \langle y_k - \bar{y}, e_i \rangle,
$$

validity of pseudo-normality at $(\bar{x}, \bar{y})$ in direction $u$ gives $\lambda = 0$, i.e., $\Phi$ is quasi-normal at $(\bar{x}, \bar{y})$ in direction $u$ w.r.t. $\mathcal{E}$.

$[\impliedby]$ Assume that $\Phi$ is quasi-normal at $(\bar{x}, \bar{y})$ in direction $u$ w.r.t. each orthonormal basis of $Y$. Suppose that $\Phi$ is not pseudo-normal at $(\bar{x}, \bar{y})$ in direction $u$. Then we find some nonzero $\lambda \in \ker D^*\Phi((\bar{x}, \bar{y}); (u, 0))$ as well as sequences $\{(x_k, y_k)\}_{k \in \mathbb{N}} \subset \text{gph} \Phi$ with $x_k \neq \bar{x}$ for all $k \in \mathbb{N}$ and $\{\lambda_k\}_{k \in \mathbb{N}} \subset Y$, $\{\eta_k\}_{k \in \mathbb{N}} \subset X$, satisfying the convergences (4.1) and $\eta_k \in D^* \Phi(x_k, y_k)(\lambda_k)$ as well as $\langle \lambda, y_k - \bar{y} \rangle > 0$ for all $k \in \mathbb{N}$. Noting that $\lambda$ does not vanish, we can construct an orthonormal basis $\mathcal{E}_\lambda := \{e_1^\lambda, \ldots, e_m^\lambda\}$ of $Y$ with $e_1^\lambda := \lambda/\|\lambda\|$. Note that, for $i \in \{1, \ldots, m\}$, we have $\langle \lambda, e_i^\lambda \rangle \neq 0$ if and only if $i = 1$ by construction of $\mathcal{E}_\lambda$. Furthermore, we find

$$
\langle \lambda, e_1^\lambda \rangle \langle y_k - \bar{y}, e_1^\lambda \rangle = \|\lambda\| \langle \lambda, y_k - \bar{y} \rangle = \langle \lambda, y_k - \bar{y} \rangle > 0.
$$

This, however, contradicts quasi-normality of $\Phi$ at $(\bar{x}, \bar{y})$ in direction $u$ w.r.t. $\mathcal{E}_\lambda$. $\square$

Let us note that (Bertsekas and Ozdaglar, 2002, Example 1) shows in the non-directional situation of standard nonlinear programming that pseudo-normality might be more restrictive than quasi-normality w.r.t. the canonical basis in $\mathbb{R}^m$. On the other hand, due to Lemma 4.2, there must exist another basis such that quasi-normality w.r.t. this basis fails since pseudo-normality fails. This depicts that validity of quasi-normality indeed may depend on the chosen basis. In Bai et al. (2019), the authors define directional quasi-normality for geometric constraints in Euclidean spaces in componentwise fashion although this is somehow unclear in situations where the image space is different from $\mathbb{R}^m$. Exemplary, in the $\frac{1}{2}m(m + 1)$-dimensional space $S_m$ of all real symmetric $m \times m$-matrices, the canonical basis, which seems to be associated with a componentwise calculus, comprises $\frac{1}{2}(m - 1)m$ matrices with precisely two nonzero entries. Our definition of quasi-normality from Definition 4.1 gives some more freedom since the choice of the underlying basis allows to rotate the coordinate system.

Following the arguments in (Benko et al., 2022, Section 3.2), it also might be reasonable to define intermediate conditions bridging pseudo- and quasi-normality. In the light of this paper, however, the concepts from Definition 4.1 are sufficient for our purposes.
As the following theorem shows, directional quasi- and, thus, pseudo-normality also serve as sufficient conditions for strong directional asymptotic regularity and directional metric subregularity which explains our interest in these conditions. Both statements follow once we clarify that pseudo- and quasi-normality are in fact specifications of the multiplier sequential information in (3.1), namely \( (y_k - \tilde{y})/\|y_k - \tilde{y}\| - \lambda_k/\|\lambda_k\| \to 0 \).

**Theorem 4.3.** If \( \Phi : X \rightrightarrows Y \) is quasi-normal in direction \( u \in S_X \) at \( (\bar{x}, \tilde{y}) \in gph \Phi \) w.r.t. some orthonormal basis \( E := \{e_1, \ldots, e_m\} \subset Y \) of \( Y \), then it is also strongly asymptotically regular as well as metrically subregular in direction \( u \) at \( (\bar{x}, \tilde{y}) \).

**Proof.** Fix arbitrary sequences \( \{(x_k, y_k)\}_{k \in \mathbb{N}} \subset gph \Phi, \{x_k^*\}_{k \in \mathbb{N}} \subset X, \) and \( \{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{Y} \) as well as \( x^* \in X \) and \( y^* \in \mathbb{Y} \) satisfying \( x_k \notin \Phi^{-1}(\tilde{y}), \) \( y_k \neq \tilde{y}, \) and \( x_k^* \in D^*\Phi(x_k, y_k)(\lambda_k) \) for each \( k \in \mathbb{N} \) as well as the convergences (3.1). Let us define \( w_k := (y_k - \tilde{y})/\|y_k - \tilde{y}\| \) and \( \bar{\lambda}_k := \lambda_k/\|\lambda_k\| \) for each \( k \in \mathbb{N} \). The requirements from (3.1) imply that \( \{w_k\}_{k \in \mathbb{N}} \) and \( \{\bar{\lambda}_k\}_{k \in \mathbb{N}} \) converge, along a subsequence (without relabeling), to the same nonvanishing limit which we will call \( \lambda \in S_Y \). Moreover, given \( i \in \{1, \ldots, m\} \) with \( \langle \lambda, e_i \rangle \neq 0 \), for sufficiently large \( k \in \mathbb{N} \), we get \( \langle w_k, e_i \rangle \neq 0 \) and

\[
0 < \langle \lambda, e_i \rangle \langle w_k, e_i \rangle = \langle \lambda, e_i \rangle \langle y_k - \tilde{y}, e_i \rangle/\|y_k - \tilde{y}\|.
\]

Observing that we have \( x_k^*/\|\lambda_k\| \to 0 \) from (3.1), we find \( \lambda \in \ker D^*\Phi((\bar{x}, \tilde{y}); (u, 0)) \) by definition of the directional limiting coderivative. This contradicts validity of quasi-normality of \( \Phi \) at \( (\bar{x}, \tilde{y}) \) in direction \( u \) w.r.t. \( E \). Particularly, such sequences \( \{(x_k, y_k)\}_{k \in \mathbb{N}}, \{x_k^*\}_{k \in \mathbb{N}}, \) and \( \{\lambda_k\}_{k \in \mathbb{N}} \) cannot exist which means that \( \Phi \) is strongly asymptotically regular in direction \( u \) at \( (\bar{x}, \tilde{y}) \).

The claim about metric subregularity now follows from (Gfrerer, 2014, Corollary 1), since the only difference from quasi-normality is the requirement

\[
\langle \lambda_k/\|\lambda_k\|, (y_k - \tilde{y})/\|y_k - \tilde{y}\| \rangle \to 1
\]

which is the same as \( (y_k - \tilde{y})/\|y_k - \tilde{y}\| - \lambda_k/\|\lambda_k\| \to 0 \) as mentioned in the comments at the end of Section 2.3. \( \square \)

Relying on this result, Lemma 2.6 yields that directional pseudo- and quasi-normality provide constraint qualifications for (P) which ensure validity of directional M-stationarity at local minimizers.

We would like to point the reader’s attention to the fact that non-directional versions of pseudo- and quasi-normality are not comparable with the non-directional version of asymptotic regularity. This has been observed in the context of standard nonlinear programming, see (Andreani et al., 2016, Sections 4.3, 4.4). The reason is that the standard version of asymptotic regularity makes no use of the multiplier information (2.3c).

Let us now also justify the terminology by showing that the new notions from Definition 4.1 coincide with directional pseudo- and quasi-normality in case of standard constraint mappings from Benko et al. (2022).
We begin by a general result relying on calmness of the constraint function. Note that we consider the particular situation $\bar{y} := 0$ for simplicity of notation. This is not restrictive since $\Phi$ can be shifted appropriately if $\bar{y}$ does not vanish to achieve this setting. Furthermore, we only focus on the concept of directional quasi-normality in our subsequently stated analysis. Analogous results can be obtained for directional pseudonormality.

**Proposition 4.4.** A constraint mapping $\Phi: \mathbb{X} \rightharpoonup \mathbb{Y}$ given by $\Phi(x) := g(x) - D$ for all $x \in \mathbb{X}$, where $g: \mathbb{X} \to \mathbb{Y}$ is calm in direction $u \in \mathbb{S}_\mathbb{X}$ at $\bar{x} \in \mathbb{X}$ such that $(\bar{x}, 0) \in \text{gph} \Phi$ and $D \subset \mathbb{Y}$ is closed, is quasi-normal in direction $u$ at $(\bar{x}, 0)$ w.r.t. some orthonormal basis $E := \{e_1, \ldots, e_m\} \subset \mathbb{Y}$ of $\mathbb{Y}$ provided there do not exist a direction $v \in \mathbb{Y}$ and a nonzero vector $\lambda \in \mathcal{N}_D(g(\bar{x}); v)$ with $0 \in D^*g(\bar{x}; (u, v))(\lambda)$ satisfying the following condition: there are sequences $\{x_k\}_{k \in \mathbb{N}} \subset \mathbb{X}$ with $x_k \neq \bar{x}$ for all $k \in \mathbb{N}$, $\{z_k\}_{k \in \mathbb{N}} \subset D$, $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{Y}$, and $\{\eta_k\}_{k \in \mathbb{N}} \subset \mathbb{X}$ satisfying $x_k \to \bar{x}$, $z_k \to g(\bar{x})$, $\lambda_k \to \lambda$, $\eta_k \to 0$,

$$\frac{x_k - \bar{x}}{\|x_k - \bar{x}\|} \to u, \quad \frac{z_k - g(\bar{x})}{\|x_k - \bar{x}\|} \to v, \quad \frac{g(x_k) - g(\bar{x})}{\|x_k - \bar{x}\|} \to v, \quad (4.2)$$

and, for all $k \in \mathbb{N}$ and $i \in \{1, \ldots, m\}$, $\eta_k \in \tilde{D}^*g(x_k)(\lambda_k)$, $\lambda_k \in \tilde{N}_D(z_k)$, as well as $(\lambda, e_i) \langle g(x_k) - z_k, e_i \rangle > 0$ if $(\lambda, e_i) \neq 0$.

Moreover, if $g$ is even calm (particularly Lipschitz continuous) near $\bar{x}$, the two conditions are equivalent.

**Proof.** $[\iff]$ Choose $\lambda \in \ker D^*\Phi((\bar{x}, 0); (u, 0))$ and sequences $\{(x_k, y_k)\}_{k \in \mathbb{N}} \subset \text{gph} \Phi$ with $x_k \neq \bar{x}$ for all $k \in \mathbb{N}$ and $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{Y}$, $\{\eta_k\}_{k \in \mathbb{N}} \subset \mathbb{X}$ satisfying (4.1) with $\bar{y} := 0$ and, for all $k \in \mathbb{N}$ and $i \in \{1, \ldots, m\}$, $\eta_k \in \tilde{D}^*\Phi(x_k, y_k)(\lambda_k)$ as well as $(\lambda, e_i) \langle g(x_k) - z_k, e_i \rangle > 0$ if $(\lambda, e_i) \neq 0$. Applying Lemma 2.2(a) yields $\eta_k \in \tilde{D}^*g(x_k)(\lambda_k)$ and $\lambda_k \in \tilde{N}_D(g(x_k) - y_k)$ for each $k \in \mathbb{N}$. The assumed calmness of $g$ at $\bar{x}$ in direction $u$ yields boundedness of the sequence $\{(g(x_k) - g(\bar{x}))/\|x_k - \bar{x}\|\}_{k \in \mathbb{N}}$, i.e., along a subsequence (without relabeling) it converges to some $v \in \mathbb{V}$. Note also that $(u, v) \in T_{\text{gph} g}(\bar{x}, g(\bar{x}))$, i.e., $v \in Dg(\bar{x})(u)$, and that $\{(x_k, g(x_k))\}_{k \in \mathbb{N}}$ converges to $(\bar{x}, g(\bar{x}))$ from direction $(u, v)$. Setting $z_k := g(x_k) - y_k$ for each $k \in \mathbb{N}$, we get $z_k \to g(\bar{x})$ by continuity of $g$ as well as $\lambda_k \in \tilde{N}_D(g(z_k))$ and $(\lambda, e_i) \langle g(x_k) - z_k, e_i \rangle > 0$ if $(\lambda, e_i) \neq 0$ for each $k \in \mathbb{N}$ and $i \in \{1, \ldots, m\}$. Moreover, we have

$$\frac{z_k - g(\bar{x})}{\|x_k - \bar{x}\|} = \frac{g(x_k) - g(\bar{x})}{\|x_k - \bar{x}\|} - \frac{y_k}{\|x_k - \bar{x}\|} \to v - 0 = v$$

and $v \in T_D(g(\bar{x}))$ follows as well. Finally, taking the limit yields $\lambda \in \mathcal{N}_D(g(\bar{x}); v)$ and $0 \in D^*g(\bar{x}; (u, v))(\lambda)$, so that the assumptions of the proposition imply $\lambda = 0$. Consequently, $\Phi$ is quasi-normal in direction $u$ at $(\bar{x}, 0)$ w.r.t. $E$.

$[\implies]$ Assume that quasi-normality in direction $u$ holds at $(\bar{x}, 0)$ w.r.t. $E$ and that $g$ is calm around $\bar{x}$. Suppose that there are some $v \in \mathbb{Y}$, $\lambda \in \mathcal{N}_D(g(\bar{x}); v)$ with $0 \in D^*g(\bar{x}; (u, v))(\lambda)$, and sequences $\{x_k\}_{k \in \mathbb{N}} \subset \mathbb{X}$ with $x_k \neq \bar{x}$ for all $k \in \mathbb{N}$ and $\{z_k\}_{k \in \mathbb{N}} \subset D$, $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{Y}$, $\{\eta_k\}_{k \in \mathbb{N}} \subset \mathbb{X}$ with $x_k \to \bar{x}$, $z_k \to g(\bar{x})$, $\lambda_k \to \lambda$, $\eta_k \to 0$, (4.2), and, for all $k \in \mathbb{N}$ and $i \in \{1, \ldots, m\}$, $\eta_k \in \tilde{D}^*g(x_k)(\lambda_k)$, $\lambda_k \in \tilde{N}_D(z_k)$, as well as
Lemma 2.2 (a). The directional, Bai et al. (Benko et al. Liang and Ye Benko et al. Hestenes (2022) Kanzow et al. Corollary 4.5 Proposition 4.4 Kanzow and Schwartz Bai et al. Bertsekas and Ozdaglar Corollary 4.5. Let us consider the setting discussed in Remark 4.6. We believe that it is useful to know how to verify if a mapping is subregular in a specific direction. Calmness of \( g \) is needed precisely for preservation of directional information. We believe that it is useful to know how to verify if a mapping is subregular in a specific direction.

If \( g \) is continuously differentiable, the situation becomes a bit simpler and we precisely recover the notion of directional quasi-normality for geometric constraint systems as discussed in (Benko et al., 2022, Definition 3.4).

Corollary 4.5. A constraint mapping \( \Phi : \mathbb{X} \rightrightarrows \mathbb{Y} \) given by \( \Phi(x) = g(x) - D \) for all \( x \in \mathbb{X} \), where \( g : \mathbb{X} \to \mathbb{Y} \) is continuously differentiable and \( D \subset \mathbb{Y} \) is closed, is quasi-normal in direction \( u \) w.r.t. \( \mathcal{E} \) if and only if \( g \) yields \( \mathcal{E} \) yields \( \lambda = 0 \) and the claim follows.

\[
\langle \lambda, e_i \rangle (g(x_k) - z_k, e_i) > 0 \text{ as soon as } \langle \lambda, e_i \rangle \neq 0. \text{ Set } y_k := g(x_k) - z_k \text{ for each } k \in \mathbb{N}. \text{ Then we have } y_k \to 0,
\]

\[
\frac{y_k}{\|x_k - \bar{x}\|} = \frac{g(x_k) - z_k}{\|x_k - \bar{x}\|} = \frac{g(x_k) - g(x)}{\|x_k - x\|} - \frac{z_k - g(x)}{\|x_k - x\|} \to v - v = 0,
\]

and, for all \( k \in \mathbb{N} \) and \( i \in \{1, \ldots, m\} \), \( \lambda_k \in \hat{N}_D(g(x_k) - y_k) \) as well as \( \langle \lambda, e_i \rangle \langle y_k, e_i \rangle > 0 \) if \( \langle \lambda, e_i \rangle \neq 0. \text{ Since } \eta_k \in \hat{D}^* g(x_k)(\lambda_k) \text{, calmness of } g \text{ at } x_k \text{ implies } \eta_k \in \hat{D}^* \Phi(x_k, y_k)(\lambda_k) \text{ due to Lemma 2.2 (a), and taking the limit yields } \lambda \in \ker \hat{D}^* \Phi((\bar{x}, \hat{y}); (u, 0)). \text{ Thus, the assumed quasi-normality of } \Phi \text{ at } (\bar{x}, 0) \text{ in direction } u \text{ w.r.t. } \mathcal{E} \text{ yields } \lambda = 0 \text{ and the claim follows.}
\]

In (Benko et al., 2022, Section 3.3), it has been reported that under additional conditions on the set \( D \), we can drop the sequences \( \{z_k\}_{k \in \mathbb{N}} \) and \( \{\lambda_k\}_{k \in \mathbb{N}} \) from the characterization of directional quasi-normality in Corollary 4.5. Particularly, this can be done for so-called ortho-disjunctive programs which cover e.g. standard nonlinear, complementarity-, cardinality-, or switching-constrained optimization problems. In this regard, Corollary 4.5 reveals that some results from Bertsekas and Ozdaglar (2002); Hestenes (1975); Kanzow et al. (2021b); Kanzow and Schwartz (2010); Liang and Ye (2021) are covered by our general concept from Definition 4.1.

Let us briefly compare our results with the approach from Bai et al. (2019).

Remark 4.6. Let us consider the setting discussed in Proposition 4.4. The directional versions of quasi- and pseudo-normality from Bai et al. (2019) operate with all nonzero pairs of directions \((u, v)\), rather than just a fixed \( u \). The advantage is that calmness of \( g \) plays no role. The reason is, however, that the authors in Bai et al. (2019) only derive statements regarding metric subregularity, but not metric subregularity in some fixed direction. Calmness of \( g \) is needed precisely for preservation of directional information. We believe that it is useful to know how to verify if a mapping is subregular in a specific direction.
direction since only some directions play a role in many situations. We could drop the calmness assumption from Proposition 4.4, but, similarly as in (Benko et al., 2019, Theorem 3.1), additional directions of the type \((0, v)\) for a nonzero \(v\) would appear. Clearly, such directions are included among all nonzero pairs \((u, v)\), but the connection to the original direction \(u\) would have been lost.

Let us mention that some of the comments from Bai et al. (2019) about improving (Benko et al., 2019, Proposition 2.2) are not accurate since these results are actually not comparable. Moreover, e.g. (Bai et al., 2019, Corollary 3.1) can be easily derived on the basis of (Benko et al., 2019, Theorem 3.1).

### 4.2 Pseudo- and quasi-normality for problems with equilibrium constraints

In mathematical optimization, problems with so-called equilibrium constraints are used to model situations where some variables need to solve a given variational problem. Exemplary, this covers optimization problems with variational inequality constraints, see e.g. Facchinei and Pang (2003); Luo et al. (1996); Outrata et al. (1998), or bilevel optimization problems, see e.g. Dempe (2002); Dempe et al. (2015). In order to model such problems, we need to split the decision space into two parts, i.e., we assume that \(X = X_1 \times X_2\) for Euclidean spaces \(X_1, X_2\) and exploit \(x := (x_1, x_2)\) for \(x \in X, x_1 \in X_1, \) and \(x_2 \in X_2\). Furthermore, let \(S: X_1 \Rightarrow X_2\) be the solution mapping of the underlying variational problem and assume that \(\text{gph} S\) is closed. For some locally Lipschitz continuous function \(\varphi: X \rightarrow \mathbb{R}\) and some closed set \(\Omega \subset X_1\), the problem of interest is given by

\[
\min\{\varphi(x) \mid x_1 \in \Omega, x_2 \in S(x_1)\} \quad \text{(MPEC)}
\]

Introducing \(\Phi: X \Rightarrow X\) by means of

\[
\forall x \in X: \quad \Phi(x) := (\Omega - x_1, S(x_1) - x_2),
\]

we easily see that (MPEC) is a special instance of (P) with \(\tilde{y} := 0\).

In order to apply our new notions of directional pseudo- and quasi-normality from Definition 4.1 to (MPEC), we need to compute the regular and directional limiting coderivative of \(\Phi\) from (4.4). For \((x, y) \in \text{gph} \Phi\) and \((\lambda, \mu) \in X\), (Benko and Mehlitz, 2022a, Lemma 5.7(ii)) yields

\[
\hat{D}^* \Phi((x, 0); (\lambda, 0)) = \begin{cases} (\hat{D}^* S(x_1, x_2 + y^2)(\mu) - \lambda, -\mu) & -\lambda \in \hat{N}_{\Omega}(x_1 + y^1), \\ \emptyset & \text{otherwise.} \end{cases}
\]

Furthermore, for \((x, 0) \in \text{gph} \Phi, (\lambda, \mu) \in X,\) and some direction \(u \in X\), (Benko and Mehlitz, 2022a, Lemma 5.6(ii)) gives us the upper estimate

\[
D^* \Phi((x, 0); (u, 0))(\lambda, \mu) \subset \begin{cases} (D^* S(x; u)(\mu) - \lambda, -\mu) & -\lambda \in N_{\Omega}(x_1; u_1), \\ \emptyset & \text{otherwise.} \end{cases}
\]

Particularly, we obtain

\[
\ker D^* \Phi((x, 0); (u, 0)) \subset (D^* S(x; u)(0) \cap (-N_{\Omega}(x_1; u_1))) \times \{0\}.
\]

With this in mind, we have the following result.
Lemma 4.7. Fix \((\bar{x}, 0) \in gph \Phi\) where \(\Phi\) is given in (4.4) as well as a direction \(u \in S_X\). Then the following statements hold.

(a) Suppose that there does not exist a nonzero \(\lambda \in D^* S(x; u)(0) \cap (-\mathcal{N}_\Omega(x^1; u^1))\) satisfying the following condition: there are sequences \(\{(x_k, y_k)\}_{k \in \mathbb{N}} \subset gph \Phi\) with \(x_k \neq \bar{x}\) for all \(k \in \mathbb{N}\) and \(\{\lambda_k\}_{k \in \mathbb{N}}, \{\eta_k\}_{k \in \mathbb{N}} \subset \mathbb{X}^1\), \{\mu_k\}_{k \in \mathbb{N}} \subset \mathbb{X}^2\), such that (4.1) and \(\mu_k \to 0\) as well as \(\eta_k + \lambda_k \in D^* S(x^1_k, x^2_k + y^2_k)(\mu_k), -\lambda_k \in \mathcal{N}_\Omega(x^1_k + y^1_k),\) and \(\langle \lambda, y^1_k \rangle > 0\) for all \(k \in \mathbb{N}\). Then \(\Phi\) is pseudo-normal at \((\bar{x}, 0)\) in direction \(u\).

(b) Let \(\mathcal{E}^j := \{e^j_1, \ldots, e^j_{m_j}\}\) be an orthonormal basis of \(\mathbb{X}^j\) for \(j = 1, 2\). Suppose that there does not exist a nonzero \(\lambda \in D^* S(x; u)(0) \cap (-\mathcal{N}_\Omega(x^1; u^1))\) satisfying the following condition: there are sequences \(\{(x_k, y_k)\}_{k \in \mathbb{N}} \subset gph \Phi\) with \(x_k \neq \bar{x}\) for all \(k \in \mathbb{N}\) and \(\{\lambda_k\}_{k \in \mathbb{N}}, \{\eta_k\}_{k \in \mathbb{N}} \subset \mathbb{X}^1\), \{\mu_k\}_{k \in \mathbb{N}} \subset \mathbb{X}^2\), such that (4.1) and \(\mu_k \to 0\) as well as, for all \(k \in \mathbb{N}\) and \(i \in \{1, \ldots, m_1\}\), \(\eta_k + \lambda_k \in D^* S(x^1_k, x^2_k + y^2_k)(\mu_k), -\lambda_k \in \mathcal{N}_\Omega(x^1_k + y^1_k),\) and \(\langle \lambda, e^1_i \rangle \langle y^1_k, e^1_i \rangle > 0\) if \(\langle \lambda, e^1_i \rangle \neq 0\). Then \(\Phi\) is quasi-normal at \((\bar{x}, 0)\) in direction \(u\) w.r.t. the orthonormal basis \(\mathcal{E}^1 \times \mathcal{E}^2\) of \(\mathbb{X}\).

We note that, depending on the structure of the underlying variational problem, the appearing coderivatives of \(S\) can be specified or at least estimated from above in terms of initial problem data under mild assumptions, see e.g. Dontchev and Rockafellar (2014); Gfrerer and Outrata (2016a,b); Mordukhovich (2006) and the references therein. Observe that working with upper estimates of these derivatives in Lemma 4.7 still yields sufficient conditions for directional pseudo- and quasi-normality of \(\Phi\). Clearly, when applied to bilevel optimization problems, Lemma 4.7 provides suitable notions of directional pseudo- and quasi-normality for this problem class in hierarchical form. Combined with Lemma 2.6 and Theorem 4.3, validity of the conditions from Lemma 4.7 guarantees directional M-stationarity of local minimizers associated with (MPEC). Let us briefly mention Bai and Ye (2022) where another notion of directional quasi-normality has been introduced for bilevel optimization problems which is based on the so-called value function reformulation. The latter is a single-level optimization problem with highly irregular nonsmooth inequality constraints and essentially different from (MPEC). However, a common feature of both approaches is that the considered problems are only implicitly given. While in (MPEC), \(S\) is an implicit object, the same holds true for the value function. Using tools of generalized differentiation allows us to characterize the derivatives of these objects and, thus, end up with explicit conditions in terms of problem data.

Let us investigate a small example to illustrate the conditions from Lemma 4.7.

Example 4.8. Let \(\mathbb{X}^1 := \mathbb{X}^2 := \mathbb{R}\) and \(S: \mathbb{R} \Rightarrow \mathbb{R}\) be given by

\[
\forall t \in \mathbb{R}: \quad S(t) := \begin{cases} \{t^2\} & t \leq 0, \\ \{\sqrt{t}\} & t > 0. \end{cases}
\]

We consider the feasible region

\[
\{x \in \mathbb{R}^2 | x^1 \in \mathbb{R}^+, x^2 \in S(x^1)\}
\]

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at \( \bar{x} := (0, 0) \).

One obtains \( T_{\text{gph}} S(\bar{x}) = (\mathbb{R}^- \times \{0\}) \cup (\{0\} \times \mathbb{R}^+) \) and \( T_{\mathbb{R} \times \mathbb{R}^+}(\bar{x}^1) = \mathbb{R}^+ \), so that the only interesting direction from \( S_{\mathbb{R}^2} \) is \( u := (0,1) \). Note that we have \( D^* S(\bar{x}; u)(0) \cap (-\mathbb{N}_{\mathbb{R}^+}(\bar{x}^1, u^1)) = \mathbb{R}^+ \). Suppose that there are \( \lambda > 0 \) and sequences \( \{(x_k, y_k)\}_{k \in \mathbb{N}} \subset \text{gph} \Phi \), \( \{\lambda_k\}_{k \in \mathbb{N}} \), \( \{\eta_k\}_{k \in \mathbb{N}} \), \( \{\mu_k\}_{k \in \mathbb{N}} \subset \mathbb{R} \) as in statement (a) of Lemma 4.7. Due to \( \lambda_k \to \lambda \), \(-\lambda_k \in \mathbb{N}_{\mathbb{R}^+}(x_k^1 + y_k^1)\), and \( \lambda y_k^1 > 0 \), we find \( x_k^1 = -y_k^1 < 0 \) for each \( k \in \mathbb{N} \). Thus, we find \( S(x_k^1) = \{-(x_k^1)^2\} \) for each \( k \in \mathbb{N} \), and an evaluation of the coderivative condition regarding \( S \) yields \( \eta_k + \lambda_k = -2x_k^1\mu_k \) for each \( k \in \mathbb{N} \). Taking the limit \( k \to \infty \) yields \( \lambda = 0 \) which is a contradiction. Thus, due to Lemma 4.7(a), the associated mapping \( \Phi \) from (4.4) is pseudo-normal w.r.t. all directions from \( S_{\mathbb{R}^2} \).

5 Sufficient conditions for asymptotic regularity via pseudo-coderivatives

5.1 On the super-coderivative

As we will observe in this section, (strong) directional asymptotic regularity can be suitably investigated via the following novel concept of generalized differentiation, which generalizes the notion of pseudo-coderivatives.

Definition 5.1. Let \( \Phi : X \rightrightarrows Y \) be a set-valued mapping with a closed graph and fix \((\bar{x}, \bar{y}) \in \text{gph} \Phi \) and \((u, v) \in S_X \times S_Y \). The super-coderivative of \( \Phi \) at \((\bar{x}, \bar{y})\) in direction \((u, v)\) is the mapping \( D^*_\text{sup} \Phi((\bar{x}, \bar{y}); (u, v)) : Y \rightrightarrows X \), which assigns to every \( y^* \in Y \) the set of all \( x^* \in X \) for which there exist sequences \( \{u_k\}_{k \in \mathbb{N}}, \{x_k^*\}_{k \in \mathbb{N}} \subset X, \{v_k\}_{k \in \mathbb{N}}, \{y_k^*\}_{k \in \mathbb{N}} \subset Y \), \( \{t_k\}_{k \in \mathbb{N}}, \{\tau_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_+ \) which satisfy \( u_k \to u, v_k \to v, x_k^* \to x^*, y_k^* \to y^*, t_k \downarrow 0, \tau_k \downarrow 0 \), and \( \tau_k/t_k \to 0 \) such that

\[
(\tau_k \|u_k\||x_k^* \in D^*_\text{sup} \Phi(\bar{x} + t_k u_k, \bar{y} + \tau_k v_k)(((\tau_k \|u_k\||)/(\tau_k \|v_k\|))y_k^*)
\]

or, equivalently,

\[
x_k^* \in D^*_\text{sup} \Phi(\bar{x} + t_k u_k, \bar{y} + \tau_k v_k)(((\tau_k \|u_k\||)/(\tau_k \|v_k\|))y_k^*)
\]  

holds for all \( k \in \mathbb{N} \).

We start with some remarks regarding Definition 5.1. First, observe that we only exploit the super-coderivative w.r.t. unit directions \((u, v) \in S_X \times S_Y \) which also means that \( \{u_k\}_{k \in \mathbb{N}} \subset X \) and \( \{v_k\}_{k \in \mathbb{N}} \subset Y \) can be chosen such that \( u_k \neq 0 \) and \( v_k \neq 0 \) hold for all \( k \in \mathbb{N} \). Particularly, condition (5.1) is reasonable.

Second, we would like to note that \( x^* \in D^*_\text{sup} \Phi((\bar{x}, \bar{y}); (u, v))(y^*) \) implies the existence of sequences \( \{u_k\}_{k \in \mathbb{N}} \subset X, \{v_k\}_{k \in \mathbb{N}} \subset Y \), and \( \{t_k\}_{k \in \mathbb{N}}, \{\tau_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_+ \) which satisfy \( u_k \to u, v_k \to v, t_k \downarrow 0, \tau_k \downarrow 0 \), and \( \tau_k/t_k \to 0 \) as well as \((\bar{x} + t_k u_k, \bar{y} + \tau_k v_k) \in \text{gph} \Phi \) for all \( k \in \mathbb{N} \). Thus, in the light of the definition of the graphical subderivative, one might be tempted to say that the pair \((u, v)\) belongs to the graph of the graphical super-derivative of \( \Phi \) at \((\bar{x}, \bar{y})\). This justifies the terminology in Definition 5.1.
Let us briefly discuss the relation between pseudo-coderivatives and the novel supercoderivative from Definition 5.1. Consider \( \gamma > 1 \) and \( x^* \in \partial^* \Phi((\bar{x}, \bar{y}); (u, v)) \) for \((u, v) \in S_X \times S_Y\), and \( y^* \in Y^*\). Setting \( \tau_k := (t_k \|u_k\|)^{\gamma} \) for each \( k \in \mathbb{N} \), where \( \{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_+ \) and \( \{u_k\}_{k \in \mathbb{N}} \subset X \) are the sequences from the definition of the pseudo-coderivative, we get \( x^* \in D^*_{\sup} \Phi((\bar{x}, \bar{y}); (u, v))(y^*) \) since \( t_k^{\gamma - 1} \|u_k\|^{\gamma} \to 0 \).

In the subsequent lemma, we comment on the converse inclusion which, to some extent, holds in the presence of a qualification condition.

**Lemma 5.2.** Let \((\bar{x}, \bar{y}) \in \text{gph} \Phi, (u, v) \in S_X \times S_Y, y^* \in Y^*\), and \( \gamma > 1 \) be fixed. Furthermore, assume that \( \text{ker} D^* \Phi((\bar{x}, \bar{y}); (u, 0)) = \{0\} \) holds. Then there exists \( \alpha > 0 \) such that

\[
D^*_{\sup} \Phi((\bar{x}, \bar{y}); (u, v))(y^*) \subset D^*_\gamma \Phi((\bar{x}, \bar{y}); (u, 0))(0) \cup D^*_\gamma \Phi((\bar{x}, \bar{y}); (u, \alpha v))(y^*/\alpha)
\]

\[
\cup \text{Im} D^*_\gamma \Phi((\bar{x}, \bar{y}); (u, 0))
\]

Proof. Let \( x^* \in D^*_{\sup} \Phi((\bar{x}, \bar{y}); (u, v))(y^*) \) be arbitrarily chosen. Then we find sequences \( \{u_k\}_{k \in \mathbb{N}}, \{x_k^*\}_{k \in \mathbb{N}} \subset X, \{v_k\}_{k \in \mathbb{N}}, \{y_k^*\}_{k \in \mathbb{N}} \subset Y^*\), and \( \{t_k\}_{k \in \mathbb{N}}, \{\tau_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_+ \) which satisfy \( u_k \to u, v_k \to v, x_k^* \to x^*, y_k^* \to y^*, t_k \to 0, \tau_k \to 0, \) and \( \tau_k/t_k \to 0 \) as well as (5.1) for all \( k \in \mathbb{N} \). This also gives us

\[
x_k^* \in \tilde{D}^*_\gamma \Phi \left( \bar{x} + t_k u_k, \bar{y} + (t_k \|u_k\|)^\gamma \frac{\tau_k v_k}{(t_k \|u_k\|)^\gamma} \right) \left( (t_k \|u_k\|)^{1-\gamma} \frac{\tau_k v_k}{(t_k \|u_k\|)^\gamma} y_k^* \right)
\]

(5.2)

for all \( k \in \mathbb{N} \). Set \( \tilde{y}_k^* := (t_k \|u_k\|)^\gamma/\tau_k v_k y_k^* \) for each \( k \in \mathbb{N} \). In case where \( \{\tilde{y}_k^*\}_{k \in \mathbb{N}} \) is not bounded, we have \( (\tau_k \|v_k\|)/(t_k \|u_k\|)^\gamma \to 0 \) along a subsequence (without relabeling), and taking the limit in

\[
x_k^* \|\tilde{y}_k^*\| \in \tilde{D}^*_\gamma \Phi \left( \bar{x} + t_k u_k, \bar{y} + (t_k \|u_k\|)^\gamma \frac{\tau_k v_k}{(t_k \|u_k\|)^\gamma} \right) \left( (t_k \|u_k\|)^{1-\gamma} \tilde{y}_k^*/\|\tilde{y}_k^*\| \right)
\]

yields \( \text{ker} D^*_\gamma \Phi((\bar{x}, \bar{y}); (u, 0)) \neq \{0\} \) which is a contradiction. Hence, \( \{\tilde{y}_k^*\}_{k \in \mathbb{N}} \) is bounded.

For each \( k \in \mathbb{N} \), we set \( \alpha_k := \tau_k \|v_k\|/(t_k \|u_k\|)^\gamma \). First, suppose that \( \{\alpha_k\}_{k \in \mathbb{N}} \) is not bounded. Then, along a subsequence (without relabeling), we may assume \( \alpha_k \to \infty \). By boundedness of \( \{y_k^*\}_{k \in \mathbb{N}}\), \( \tilde{y}_k^* \to 0 \) follows. Rewriting (5.2) yields

\[
x_k^* \in \tilde{D}^*_\gamma \Phi \left( \bar{x} + t_k u_k, \bar{y} + t_k \frac{\tau_k v_k}{t_k} \right) \left( (t_k \|u_k\|)^{1-\gamma} \tilde{y}_k^* \right)
\]

for each \( k \in \mathbb{N} \), and taking the limit \( k \to \infty \) while respecting \( \tau_k/t_k \to 0 \), thus, gives \( x^* \in \tilde{D}^*_\gamma \Phi((\bar{x}, \bar{y}); (u, 0))(0) \). In case where \( \{\alpha_k\}_{k \in \mathbb{N}} \) converges to some \( \alpha > 0 \) (along a subsequence without relabeling), we can simply take the limit \( k \to \infty \) in (5.2) in order to find \( x^* \in \tilde{D}^*_\gamma \Phi((\bar{x}, \bar{y}); (u, \alpha v))(y^*/\alpha) \). Finally, let us consider the case \( \alpha_k \to 0 \) (along a subsequence without relabeling). Then, by boundedness of \( \{\tilde{y}_k^*\}_{k \in \mathbb{N}}\), taking the limit \( k \to \infty \) in (5.2) gives \( x^* \in \text{Im} D^*_\gamma \Phi((\bar{x}, \bar{y}); (u, 0)) \). Thus, we have shown the first inclusion.

The second inclusion follows by the trivial upper estimate for the pseudo-coderivative. \( \square \)
Let us now interrelate the concepts of super-coderivatives and asymptotic regularity. Choose sequences \( \{ (x_k, y_k) \}_{k \in \mathbb{N}} \subseteq \text{gph} \Phi \), \( \{ x_k^\ast \}_{k \in \mathbb{N}} \subseteq \mathcal{X} \), and \( \{ \lambda_k \}_{k \in \mathbb{N}} \subseteq \mathcal{Y} \) as well as \( x^\ast \in \mathcal{X} \) and \( y^\ast \in \mathcal{Y} \) satisfying \( x_k \notin \Phi^{-1}(y_k) \), \( y_k \neq y^\ast \), and \( x_k^\ast \in \mathcal{D}^\ast \Phi(x_k, y_k)(\lambda_k) \) for all \( k \in \mathbb{N} \) as well as the convergences (3.1). For each \( k \in \mathbb{N} \), we set \( t_k := \| x_k - x^\ast \| \), \( \tau_k := \| y_k - y^\ast \| \),
\[
  u_k := \frac{x_k - x^\ast}{\| x_k - x^\ast \|}, \quad v_k := \frac{y_k - y^\ast}{\| y_k - y^\ast \|}, \quad y_k^\ast := \frac{\| y_k - y^\ast \|}{\| x_k - x^\ast \|} \lambda_k
\]
and find
\[
  \forall k \in \mathbb{N}: \quad x_k^\ast \in \mathcal{D}^\ast \Phi(x + t_k u_k, y + \tau_k v_k)(t_k/\tau_k) y_k^\ast.
\]
Along a subsequence (without relabeling), \( v_k \to v \) holds for some \( v \in \mathcal{S}_Y \). Thus, taking the limit \( k \to \infty \), we have \( x^\ast \in \mathcal{D}^\ast \Phi((\bar{x}, \bar{y}); (u, v))(y^\ast) \) by definition of the super-coderivative. Moreover, from (3.1) we also know that \( y^\ast = \| y^\ast \| v \). Consequently, we come up with the following lemma.

**Lemma 5.3.** Let \((\bar{x}, \bar{y}) \in \text{gph} \Phi \) and \( u \in \mathcal{S}_X \) be fixed. If
\[
  \bigcup_{v \in \mathcal{S}_Y} \mathcal{D}^\ast \Phi((\bar{x}, \bar{y}); (u, v))(\beta v) \subseteq \text{Im} \mathcal{D}^\ast \Phi(\bar{x}, \bar{y})
\]
holds for all \( \beta \geq 0 \), then \( \Phi \) is asymptotically regular at \((\bar{x}, \bar{y})\) in direction \( u \). If the above estimate holds for all \( \beta \geq 0 \) with \( \text{Im} \mathcal{D}^\ast \Phi(\bar{x}, \bar{y}) \) replaced by \( \text{Im} \mathcal{D}^\ast \Phi((\bar{x}, \bar{y}); (u, 0)) \), then \( \Phi \) is strongly asymptotically regular at \((\bar{x}, \bar{y})\) in direction \( u \).

The next result follows as a corollary of Lemmas 5.2 and 5.3, and gives new sufficient conditions for directional asymptotic regularity. Note that strong directional asymptotic regularity can be handled analogously.

**Theorem 5.4.** Let \((\bar{x}, \bar{y}) \in \text{gph} \Phi \), \( u \in \mathcal{S}_X \), and \( \gamma > 1 \) be fixed. Furthermore, assume that \( \ker \mathcal{D}^\ast \Phi((\bar{x}, \bar{y}); (u, 0)) = \{ 0 \} \) holds. If
\[
  \mathcal{D}^\ast \Phi((\bar{x}, \bar{y}); (u, 0))(0) \cup \bigcup_{v \in \mathcal{S}_Y} \mathcal{D}^\ast \Phi((\bar{x}, \bar{y}); (u, \alpha v))(\beta v) \subseteq \text{Im} \mathcal{D}^\ast \Phi(\bar{x}, \bar{y})
\]
holds for all \( \alpha, \beta \geq 0 \), particularly, if
\[
  \text{Im} \mathcal{D}^\ast \Phi((\bar{x}, \bar{y}); (u, 0)) \subseteq \text{Im} \mathcal{D}^\ast \Phi(\bar{x}, \bar{y})
\]
holds, then \( \Phi \) is asymptotically regular at \((\bar{x}, \bar{y})\) in direction \( u \).

In case where the pseudo-coderivatives involved in the statement of Theorem 5.4 can be computed or estimated from above, new applicable sufficient conditions for (strong) directional asymptotic regularity are at hand. Particularly, in situations where \( \Phi \) is given in form of a constraint mapping and \( \gamma := 2 \) is fixed, we can rely on the results obtained in (Benko and Mehlitz, 2022b, Section 3) in order to make the findings of Theorem 5.4 more specific. This will be done in the next subsection.
5.2 Constraint mappings

Throughout the section, we assume that $\Phi: X \Rightarrow Y$ is given by $\Phi(x) := g(x) - D$, $x \in X$, where $g: X \to Y$ is a single-valued, twice continuously differentiable function and $D \subset Y$ is a closed set. Furthermore, for simplicity of notation, we fix $\bar{y} := 0$ which is not restrictive as already mentioned earlier.

We start with the proof of the first statement. From (Proof. Implies we find $\tilde{\Phi} \subset D$.

**Theorem 5.5.** Let $(\bar{x}, 0) \in \text{gph} \Phi$ as well as $u \in S_X$ be fixed. Assume that the condition

$$\nabla g(\bar{x})^* y^* = 0, \nabla^2 (y^*, g) (\bar{x})(u) + \nabla g(\bar{x})^* z^* = 0, \quad y^* \in N_D(g(\bar{x}); \nabla g(\bar{x})u), \ z^* \in D\nabla D(g(\bar{x}), y^*)(\nabla g(\bar{x})u) \quad \implies \quad y^* = 0 \quad (5.5)$$

is valid. Furthermore, let

$$\nabla g(\bar{x})^* y^* = 0, \nabla g(\bar{x})^* z^* = 0, \quad y^* \in N_D(g(\bar{x}); \nabla g(\bar{x})u), \ z^* \in D\nabla D(g(\bar{x}), y^*)(0) \quad \implies \quad z^* = 0 \quad (5.6)$$

or, in case $\nabla g(\bar{x})u \neq 0$,

$$\nabla g(\bar{x})^* y^* = 0, \nabla g(\bar{x})^* z^* = 0, \quad y^* \in N_D(g(\bar{x}); \nabla g(\bar{x})u) \quad \implies \quad z^* \notin D_{\text{sub}}\nabla D(g(\bar{x}), y^*) \left( \frac{\nabla g(\bar{x})u}{\|\nabla g(\bar{x})u\|} \right) \quad (5.7)$$

hold.

(a) If, for each $x^* \in X$ and $y^*, z^* \in Y$ satisfying

$$x^* = \nabla^2 (y^*, g) (\bar{x})(u) + \nabla g(\bar{x})^* z^*, \quad (5.8a)$$

$$y^* \in N_D(g(\bar{x}); \nabla g(\bar{x})u) \cap \ker \nabla g(\bar{x})^* \quad (5.8b)$$

$$z^* \in D\nabla D(g(\bar{x}), y^*)(\nabla g(\bar{x})u) \quad (5.8c)$$

there is some $\lambda \in N_D(g(\bar{x}))$ such that $x^* = \nabla g(\bar{x})^* \lambda$, then $\Phi$ is asymptotically regular at $(\bar{x}, 0)$ in direction $u$.

(b) If, for each $x^* \in X$ and $y^*, z^* \in Y$ satisfying (5.8), there is some $\lambda \in N_D(g(\bar{x}); \nabla g(\bar{x})u)$ such that $x^* = \nabla g(\bar{x})^* \lambda$, then $\Phi$ is strongly asymptotically regular at $(\bar{x}, 0)$ in direction $u$.

**Proof.** Let us start with the proof of the first statement. From (Benko and Mehltitz, 2022b, Theorem 3.2(b)), (5.5) together with (5.6) or, in case $\nabla g(\bar{x})u \neq 0$, (5.7) imply $\ker D^2 \Phi((\bar{x}, 0); (u, 0)) = \{0\}$ which also implies $\ker D^3 \Phi((\bar{x}, 0); (u, 0)) = \{0\}$. Now, pick $x^* \in \text{Im} D^2 \Phi((\bar{x}, 0); (u, 0))$. Then, due to (Benko and Mehltitz, 2022b, Theorem 3.2(b)), we find $y^*, z^* \in Y$ satisfying (5.8). The assumptions guarantee that we can find $\lambda \in N_D(g(\bar{x}))$ such that $x^* = \nabla g(\bar{x})^* \lambda \in \text{Im} D^* \Phi(\bar{x}, 0)$ where we used Lemma 2.2(b). It follows $\text{Im} D^2 \Phi((\bar{x}, 0); (u, 0)) \subset \text{Im} D^* \Phi(\bar{x}, 0)$. Thus, Theorem 5.4 shows that $\Phi$ is asymptotically regular at $(\bar{x}, 0)$ in direction $u$. The second statement follows in analogous way while respecting Lemma 2.2(c).

\[ \square \]
We note that (5.6) is stronger than (5.7) when \( \nabla g(\bar{x})u \neq 0 \) holds, see (Benko and Mehlitz, 2022b, formula (2.2)). Naturally, this means that it is sufficient to check (5.6) regardless whether \( \nabla g(\bar{x})u \) vanishes or not. In case \( \nabla g(\bar{x})u \neq 0 \), however, it is already sufficient to check the milder condition (5.7). This will be important later on, see Proposition 5.8 and Remark 5.9 below.

The subsequently stated results address the particular case where \( D \) is a polyhedral set, i.e., it is the union of finitely many convex polyhedral sets. Similarly, \( D \) is referred to as locally polyhedral around \( y \in D \) whenever \( D \cap V \) is polyhedral for some neighborhood \( V \) of \( y \).

**Theorem 5.6.** Let \((\bar{x}, 0) \in \text{gph} \Phi \) as well as \( u \in S_\mathcal{X} \) be fixed. Let \( \mathcal{Y} := \mathbb{R}^m \) and let \( D \) be polyhedral locally around \( g(\bar{x}) \). Assume that the following condition holds:

\[
\begin{align*}
\nabla g(\bar{x})^* y^* & = 0, \quad \nabla^2 (y^*, g)(\bar{x})(u) + \nabla g(\bar{x})^* z^* = 0, \\
y^* \in \mathcal{N}_{TD(g(\bar{x}))}(\nabla g(\bar{x})u), \quad z^* \in \mathcal{N}_{TD(g(\bar{x}))}(\nabla g(\bar{x})u)(y^*)
\end{align*}
\]

\( \implies \quad y^* = 0. \) (5.9)

(a) If, for each \( x^* \in \mathcal{X} \) and \( y^*, z^* \in \mathbb{R}^m \) satisfying (5.8a) and

\[
\begin{align*}
y^* & \in \mathcal{N}_{TD(g(\bar{x}))}(\nabla g(\bar{x})u) \cap \ker \nabla g(\bar{x})^*, \\
z^* & \in \mathcal{N}_{TD(g(\bar{x}))}(\nabla g(\bar{x})u)(y^*)
\end{align*}
\]

there is some \( \lambda \in \mathcal{N}_{D}(g(\bar{x})) \) such that \( x^* = \nabla g(\bar{x})^* \lambda \), then \( \Phi \) is asymptotically regular at \((\bar{x}, 0)\) in direction \( u \).

(b) If, for each \( x^* \in \mathcal{X} \) and \( y^*, z^* \in \mathbb{R}^m \) satisfying (5.8a) and (5.10), there is some \( \lambda \in \mathcal{N}_{TD(g(\bar{x}))}(\nabla g(\bar{x})u) \) such that \( x^* = \nabla g(\bar{x})^* \lambda \), then \( \Phi \) is strongly asymptotically regular at \((\bar{x}, 0)\) in direction \( u \).

**Proof.** The proof is analogous to the one of Theorem 5.5 and exploits (Benko and Mehlitz, 2022b, Theorem 3.2 (c)). \( \square \)

In Theorems 5.5 and 5.6, we relied on the more restrictive assumption (5.4) from Theorem 5.4. In the general case, we were not able to utilize the more refined condition (5.3), but in the polyhedral case, we obtain the following improved result. We would like to point out that, based on Theorem 5.4, one can state even finer but more technical sufficient conditions for directional asymptotic regularity in the polyhedral case.

**Theorem 5.7.** Let \((\bar{x}, 0) \in \text{gph} \Phi \) as well as \( u \in S_\mathcal{X} \) be fixed. Let \( \mathcal{Y} := \mathbb{R}^m \) and let \( D \) be polyhedral locally around \( g(\bar{x}) \). Furthermore, we set \( \mathcal{T}(u) := \mathcal{T}_{TD(g(\bar{x}))}(\nabla g(\bar{x})u) \) and, for arbitrary \( s \in \mathcal{X} \) and \( v \in \mathbb{R}^m \), \( w_s(u, v) := \nabla g(\bar{x})s + 1/2\nabla^2 g(\bar{x})[u, u] - v \). Assume that the following condition holds for each \( s \in \mathcal{X} \):

\[
\begin{align*}
\nabla g(\bar{x})^* y^* & = 0, \quad \nabla^2 (y^*, g)(\bar{x})(u) + \nabla g(\bar{x})^* z^* = 0, \\
y^* & \in \mathcal{N}_{\mathcal{T}(u)}(w_s(u, 0)), \quad z^* \in \mathcal{N}_{\mathcal{T}(u)}(w_s(u, 0))(y^*)
\end{align*}
\]

\( \implies \quad y^* = 0. \) (5.11)
(a) If, for each $x^*, s \in \mathbb{X}$ and $y^*, z^*, v \in \mathbb{R}^m$ satisfying $\langle y^*, v \rangle \geq 0$, (5.8a), and
\[
y^* \in N_{T(u)}(w_s(u, v)) \cap \ker g(\bar{x})^*,
z^* \in T_{N_{T(u)}(w_s(u, v))}(y^*),
\]
there is some $\lambda \in N_D(g(\bar{x}))$ such that $x^* = \nabla g(\bar{x})^* \lambda$, then $\Phi$ is asymptotically regular at $(\bar{x}, 0)$ in direction $u$.

(b) If, for each $x^*, s \in \mathbb{X}$ and $y^*, z^*, v \in \mathbb{R}^m$ satisfying $\langle y^*, v \rangle \geq 0$, (5.8a), and (5.12), there is some $\lambda \in N_{T_D(g(\bar{x}))}(\nabla g(\bar{x})u)$ such that $x^* = \nabla g(\bar{x})^* \lambda$, then $\Phi$ is strongly asymptotically regular at $(\bar{x}, 0)$ in direction $u$.

Proof. We start with the proof of the first assertion. With the aid of (Benko and Mehlitz, 2022b, Theorem 3.2 (d)), we easily see that (5.11) yields $\ker D^2_\Phi((\bar{x}, 0); (u, 0)) = \{0\}$ in the present situation. Now, fix $x^* \in D^2_\Phi((\bar{x}, 0); (u, 0))(0)$. Then (Benko and Mehlitz, 2022b, Theorem 3.2 (c)) shows the existence of $z^* \in N_{T_D(g(\bar{x}))}(\nabla g(\bar{x})u)$ such that $x^* = \nabla g(\bar{x})^* z^*$. In case where $x^* \in D^2_\Phi((\bar{x}, 0); (u, \alpha w))(\beta w)$ holds for some $w \in \mathbb{S}_{\mathbb{R}^m}$ and $\alpha, \beta \geq 0$, (Benko and Mehlitz, 2022b, Theorem 3.2 (d)) implies the existence of $s \in \mathbb{X}$ such that (5.8a) and (5.12) hold with $v := \alpha w$ and $y^* := \beta w$, and this gives $\langle y^*, v \rangle = \alpha \beta \|w\|^2 \geq 0$. Now, the postulated assumptions guarantee the existence of $\lambda \in N_{T_{D_D}(g(\bar{x}))}(\nabla g(\bar{x})u)$ such that $x^* = \nabla g(\bar{x})^* \lambda$. Respecting Lemma 2.2 (b), this shows (5.3) with $y := 0$ and $\gamma := 2$. Thus, Theorem 5.4 yields that $\Phi$ is asymptotically regular at $(\bar{x}, 0)$ in direction $u$.

The second statement follows in analogous fashion while exploiting Lemma 2.2 (c).

We note that the sufficient conditions for directional asymptotic regularity stated in Theorems 5.5 to 5.7 recover the constraint qualifications for M-stationarity we obtained in (Benko and Mehlitz, 2022b, Section 4.2) by a different approach. In the remaining part of the paper, we prove that the assumptions of Theorem 5.5 are not stronger than FOSCMS while the assumptions of Theorems 5.6 and 5.7 are weaker than the so-called Second-Order Sufficient Condition for Metric Subregularity (SOSCMS).

Given a point $\bar{x} \in \mathbb{X}$ with $(\bar{x}, 0) \in \text{graph } \Phi$, Lemma 2.2 (c) shows that the condition
\[
u \in S_{\mathbb{X}}, \nabla g(\bar{x})u \in T_D(g(\bar{x})), \nabla g(\bar{x})^* y^* = 0, y^* \in N_D(g(\bar{x})); \nabla g(\bar{x})u \implies y^* = 0
\]
equals FOSCMS in the current setting. In case where $D$ is locally polyhedral around $g(\bar{x})$, the refined condition
\[
u \in S_{\mathbb{X}}, \nabla g(\bar{x})u \in T_D(g(\bar{x})), \nabla g(\bar{x})^* y^* = 0, \\
\nabla^2 \langle y^*, g \rangle (\bar{x})|u, u| \geq 0, y^* \in N_D(g(\bar{x})); \nabla g(\bar{x})u \implies y^* = 0,
\]
is referred to as SOSCMS in the literature. As these names suggest, both conditions are sufficient for metric subregularity of $\Phi$ at $(\bar{x}, 0)$, see (Gfrerer and Klatte, 2016, Corollary 1). Particularly, they provide constraint qualifications for M-stationarity of local minimizers. Moreover, validity of these conditions for a fixed direction $u \in S_{\mathbb{X}}$, denoted by FOSCMS($u$) and SOSCMS($u$), respectively, is sufficient for metric subregularity of $\Phi$ at $(\bar{x}, 0)$ in direction $u$.

We split our remaining considerations into the general and the polyhedral case.
Proposition 5.8. Consider \((\bar{x}, 0) \in \text{gph} \Phi\) and \(u \in S_X\). Under FOSCMS\((u)\) all assumptions of Theorem 5.5 are satisfied.

Proof. Let \(y^* \in N_D(g(\bar{x}) u)\) be such that \(\nabla g(\bar{x})^* y^* = 0\). Then FOSCMS\((u)\) yields \(y^* = 0\) and so (5.5) is satisfied. Moreover, we only need to show the remaining assertions for \(y^* = 0\).

First, we claim that

\[
D_N(g(\bar{x}), 0)(q) \left( D_{sub} N_D(g(\bar{x}), 0)(q) \right) \subset N_D(g(\bar{x}); q) \tag{5.13}
\]

holds for any \(q \in Y\) \((q \in S_Y)\). Indeed, let \(\hat{z}^* \in D_N(g(\bar{x}), 0)(q)\). The definition yields that there are sequences \(\{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_+\) and \(\{q_k\}_{k \in \mathbb{N}} \subset Y\) with \(t_k \searrow 0\), \(q_k \to q\), \(\hat{z}^*_k \to \hat{z}^*\), and \(t_k \hat{z}^*_k \in N_D(g(\bar{x}) + t_kq_k)\) for each \(k \in \mathbb{N}\). Since, for each \(k \in \mathbb{N}\), \(N_D(g(\bar{x}) + t_kq_k)\) is a cone, however, we get \(\hat{z}^*_k \in N_D(g(\bar{x}) + t_kq_k)\), and \(\hat{z}^* \in N_D(g(\bar{x}); q)\) follows by robustness of the directional limiting normal cone, see Lemma 2.1. The case \(D_{sub} N_D(g(\bar{x}), 0)(q)\) is almost identical.

Next, assume that \(\nabla g(\bar{x})u \neq 0\) holds. Suppose now that (5.7) is violated, i.e., there exists \(\hat{z}^* \in D_{sub} N_D(g(\bar{x}), 0)(q)\) for \(q := \nabla g(\bar{x})u / \|\nabla g(\bar{x})u\|\) with \(\nabla g(\bar{x})^* \hat{z}^* = 0\). By (5.13) and FOSCMS\((u)\) we thus get \(\hat{z}^* = 0\) which is a contradiction since \(\hat{z}^* \in S_Y\) by definition. Similarly, in case \(\nabla g(\bar{x})u = 0\), we can verify (5.6) which reduces to

\[
\nabla g(\bar{x})^* \hat{z}^* = 0, \quad \hat{z}^* \in D_N(g(\bar{x}), 0)(0) \implies \hat{z}^* = 0.
\]

Applying (5.13) with \(q := 0\), we get \(\hat{z}^* \in N_D(g(\bar{x}))\) which again implies \(\hat{z}^* = 0\) since FOSCMS\((u)\) corresponds to the Mordukhovich criterium due to \(\nabla g(\bar{x})u = 0\). Thus, we have shown that (5.6) or, in case \(\nabla g(\bar{x})u \neq 0\), (5.7) holds.

Validity of the last assumption follows immediately since \(z^* \in N_D(g(\bar{x}); \nabla g(\bar{x})u)\) follows from (5.13), and so we can just take \(\lambda := z^*\) due to \(y^* = 0\). \(\square\)

Remark 5.9. Note that for \(u \in S_X\) satisfying \(\nabla g(\bar{x})u \neq 0\), we have the trivial upper estimate \(D_{sub} N_D(g(\bar{x}), y^*)(\nabla g(\bar{x})u / \|\nabla g(\bar{x})u\|) \subset D_N(g(\bar{x}), y^*)(0)\), but keeping only this non-directional information, i.e., relying only on (5.6), we cannot show that FOSCMS\((u)\) is sufficient for it to hold.

Proposition 5.10. Let \((\bar{x}, 0) \in \text{gph} \Phi\) as well as \(u \in S_X\) be fixed and assume that SOSCMS\((u)\) is valid. Furthermore, let \(Y := \mathbb{R}^m\) and let \(D\) be polyhedral locally around \(g(\bar{x})\). Then the following statements hold.

(a) The assumptions of Theorem 5.7 are satisfied.

(b) The assumptions of Theorem 5.6 hold for \(x^* \in X\) satisfying \((x^*, u) \geq 0\).

Proof. For the proof of the first statement, let \(y^* \in N_{T(u)}(w_s(u, 0))\) for some \(s \in X\) be such that \(\nabla g(\bar{x})^* y^* = 0\). We get

\[
N_{T(u)}(w_s(u, 0)) = N_{T_D(g(\bar{x}))}(\nabla g(\bar{x})u; w_s(u, 0)) \subset N_{T_D(g(\bar{x}))}(\nabla g(\bar{x})u) \cap [w_s(u, 0)]^\perp = N_D(g(\bar{x}); \nabla g(\bar{x})u) \cap [w_s(u, 0)]^\perp
\]

due to (local) polyhedrality of $\mathcal{T}_D(g(\bar{x}))$ and $D$ from (Benko and Mehlitz, 2022b, Lemma 2.1). From $\nabla g(\bar{x})^*y^* = 0$ we thus obtain

$$\frac{1}{2} \nabla^2(y^*, g)(\bar{x})[u, u] = \frac{1}{2} \langle \nabla^2 g(\bar{x})[u, u], y^* \rangle = \langle w_s(u, 0), y^* \rangle = 0$$

and SOSCMS$(u)$ yields $y^* = 0$ which shows validity of (5.11).

Next, for arbitrary $y^* \in \mathcal{N}_{\mathcal{T}(u)}(w_s(u, v)) \cap \ker \nabla g(\bar{x})^*$ with $s \in \mathcal{X}$ and $v \in \mathbb{R}^m$ satisfying $\langle y^*, v \rangle \geq 0$, we get $y^* \in \mathcal{N}_D(g(\bar{x})\nabla g(\bar{x})u) \cap [w_s(u, v)]^\perp$ and

$$\frac{1}{2} \nabla^2(y^*, g)(\bar{x})[u, u] = \langle w_s(u, v), y^* \rangle + \langle v, y^* \rangle = \langle v, y^* \rangle \geq 0,$$

so SOSCMS$(u)$ can still be applied to give $y^* = 0$. Now, for $z^* \in \mathcal{T}_{\mathcal{N}(u)}(w_s(u, v))(0)$, we get $z^* \in \mathcal{N}_{\mathcal{T}(u)}(w_s(u, v)) \subset \mathcal{N}_D(g(\bar{x})\nabla g(\bar{x})u)$ and so under SOSCMS$(u)$ we can always take $\lambda := z^*$ since $y^* = 0$.

In order to prove the second claim, let $y^*, z^* \in \mathbb{R}^m$ satisfy the requirements of condition (5.9). By (local) polyhedrality of $\mathcal{N}_{\mathcal{T}(g(\bar{x}))}(\nabla g(\bar{x})u)$ and $D$, for sufficiently small $\alpha > 0$, we obtain

$$y^* + \alpha z^* \in \mathcal{N}_{\mathcal{T}(g(\bar{x}))}(\nabla g(\bar{x})u) \subset \mathcal{N}_D(g(\bar{x})) \cap [\nabla g(\bar{x})u]^\perp,$$

see (Benko and Mehlitz, 2022b, Lemma 2.1) again. Due to $\nabla g(\bar{x})^*y^* = 0$, we also get

$$0 = \langle u, \nabla g(\bar{x})^*(y^* + \alpha z^*) \rangle = \alpha \langle u, \nabla g(\bar{x})^* z^* \rangle,$$

i.e., $\langle u, \nabla g(\bar{x})^* z^* \rangle = 0$. Taking into account the equation $\nabla^2(y^*, g)(\bar{x})(u) + \nabla g(\bar{x})^* z^* = 0$, we get

$$\nabla^2(y^*, g)(\bar{x})[u, u] = \langle u, \nabla^2(y^*, g)(\bar{x})(u) \rangle = -\langle u, \nabla g(\bar{x})^* z^* \rangle = 0$$

and SOSCMS$(u)$ yields $y^* = 0$ since $y^* \in \mathcal{N}_D(g(\bar{x})\nabla g(\bar{x})u)$.

Finally, assume that there are $x^* \in \mathcal{X}$, $y^* \in \mathcal{N}_{\mathcal{T}(g(\bar{x}))}(\nabla g(\bar{x})u) \cap \ker \nabla g(\bar{x})^*$, and $z^* \in \mathcal{T}_{\mathcal{N}(g(\bar{x}))}(\nabla g(\bar{x})u)(y^*)$ such that $\langle x^*, u \rangle \geq 0$ and $x^* = \nabla^2(y^*, g)(\bar{x})(u) + \nabla g(\bar{x})^* z^*$.

Exploiting the above arguments, we find

$$\nabla^2(y^*, g)(\bar{x})[u, u] = \langle u, \nabla^2(y^*, g)(\bar{x})(u) \rangle = -\langle u, \nabla g(\bar{x})^* z^* \rangle + \langle u, x^* \rangle = \langle u, x^* \rangle \geq 0$$

and due to $y^* \in \mathcal{N}_D(g(\bar{x})\nabla g(\bar{x})u)$, SOSCMS$(u)$ yields $y^* = 0$. Thus, we also have $z^* \in \mathcal{N}_D(g(\bar{x})\nabla g(\bar{x})u)$ and we can choose $\lambda := z^*$ again.

We immediately arrive at the following corollary.

**Corollary 5.11.** The constraint mapping $\Phi$ is strongly asymptotically regular at $(\bar{x}, 0) \in \text{gph} \Phi$ in direction $u \in \mathcal{S}_\mathcal{X}$ if FOSCMS$(u)$ holds or if $\mathcal{Y} := \mathbb{R}^m$, $D$ is locally polyhedral around $g(\bar{x})$, and SOSCMS$(u)$ holds.

**Remark 5.12.** Note that showing that SOSCMS implies strong directional asymptotic regularity was only possible via the refined conditions from Theorem 5.7. In the context of $M$-stationarity, however, also the simpler, more restrictive conditions from Theorem 5.6
can be useful. Indeed, let $\bar{x} \in F$ be a local minimizer of (P). In order to justify M-stationarity of $\bar{x}$, it is sufficient to verify the assumptions from Theorem 5.6 for $x^* \in -\partial \varphi(\bar{x})$ and, taking into account (Benko and Mehltz, 2022b, Remark 4.7), we only need to consider

$$-x^* \in \partial \varphi(\bar{x}; (u, \mu)) := \{x^* \in X \mid (x^*, -1) \in N_{\text{epi} \varphi}(\bar{x}, \varphi(\bar{x})); (u, \mu)\}$$

for some $\mu \leq 0$, where $\partial \varphi(\bar{x}; (u, \mu))$ denotes the (geometric) subdifferential of $\varphi$ at $\bar{x}$ in direction $(u, \mu)$, see Benko et al. (2019). Then, whenever $\text{epi} \varphi$ is so-called semismooth* at $(\bar{x}, \varphi(\bar{x}))$, we directly get $\langle x^*, u \rangle = -\mu \geq 0$ and Proposition 5.10 (b) can be applied.

Note that, as recently introduced in Gfrerer and Outrata (2021), a closed set $Q \subset X$ is called semismooth* at $x \in Q$ if for all $w \in T_Q(x)$ and $\eta \in N_Q(x; w)$, we have $\langle \eta, w \rangle = 0$. Let us briefly mention that broad classes of sets enjoy the semismoothness* property, see (Gfrerer and Outrata, 2021, Section 3). Exemplary, each union of finitely many closed, convex sets and graphs, epigraphs, as well as hypographs of continuously differentiable mappings are semismooth* everywhere.

The following example shows that our new conditions are in fact strictly milder than SOSCMS. We conjecture that they are also strictly milder than FOSCMS but, unfortunately, have no example available which shows this.

**Example 5.13.** Let $g : \mathbb{R} \to \mathbb{R}^2$ and $D \subset \mathbb{R}^2$ be given by $g(x) := (x, -x^2)$ for all $x \in \mathbb{R}$ and $D := (\mathbb{R}_+ \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}_+)$. Observe that $D$ is a polyhedral set. We consider the constraint map $\Phi : \mathbb{R} \mapsto \mathbb{R}^2$ given by $\Phi(x) := g(x) - D$ for all $x \in \mathbb{R}$. We note that $\Phi^{-1}(0) = [0, \infty)$ holds. Hence, fixing $\bar{x} := 0$, we can easily check that $\Phi$ is metrically subregular at $(\bar{x}, 0)$ in direction 1 but not in direction $-1$. Hence, FOSCMS and SOSCMS must be violated.

First, we claim that all the assumptions from Theorem 5.6 (and, hence, also Theorem 5.7) are satisfied for $u = \pm 1$. Thus, let us fix $u = \pm 1$, $y^* \in N_{T_D(g(\bar{x}))}(\nabla g(\bar{x})u) \cap \ker \nabla g(\bar{x})^*$, and $z^* \in T_N_{T_D(g(\bar{x}))}(\nabla g(\bar{x})u)(y^*)$ such that $\nabla^2(y^*, g)(\bar{x})(u) + \nabla g(\bar{x})^*z^* = 0$. From $y^* \in \ker \nabla g(\bar{x})^*$ we have $y_1^* = 0$. Furthermore, we have $\nabla g(\bar{x})u = (u, 0)$, $\nabla^2(y^*, g)(\bar{x})(u) = -2y_2^*u$, and

$$N_{T_D(g(\bar{x}))}(\nabla g(\bar{x})u) = \begin{cases} \{0\} \times \mathbb{R}_- & u = -1, \\ \{(0, 0)\} & u = 1. \end{cases}$$

Thus, for $u = 1$, condition (5.9) holds trivially. For $u = -1$, we fix $y_2^* \leq 0$ and, thus, $T_N_{T_D(g(\bar{x}))}(\nabla g(\bar{x})u)(y^*) \subset \{0\} \times \mathbb{R}$, i.e., $z_1^* = 0$. Thus, from $-2y_2^*u + z_1^* = 0$, we deduce $y_2^* = 0$, and (5.9) follows. In order to check the second assumption of Theorem 5.6, we fix $x^* \in \mathbb{R}$, $u = \pm 1$, $y^* \in N_{T_D(g(\bar{x}))}(\nabla g(\bar{x})u) \cap \ker \nabla g(\bar{x})^*$, and $z^* \in T_N_{T_D(g(\bar{x}))}(\nabla g(\bar{x})u)(y^*)$ such that $x^* = \nabla^2(y^*, g)(\bar{x})(u) + \nabla g(\bar{x})^*z^*$. In case $u = 1$, we have $y^* = 0$ from above. This yields $z^* \in N_{T_D(g(\bar{x}))}(\nabla g(\bar{x})u)$, and we can choose $\lambda := z^*$ to find $x^* = \nabla g(\bar{x})^*\lambda$ as well as $\lambda \in N_{T_D(g(\bar{x}))}(\nabla g(\bar{x})u)$. Thus, let us consider $u = -1$. Then we find $y_2^* \leq 0$ and $z_1^* = 0$ from above. Next from $x^* = -2y_2^*u + z_1^* = 2y_2^* \leq 0$ we can choose $\lambda := (x^*, 0) \in N_{T_D(g(\bar{x}))}$ to get $\nabla g(\bar{x})^*\lambda = x^*$.

Note, however, that $\lambda = (x^*, 0) \notin N_{T_D(g(\bar{x}))}(\nabla g(\bar{x})u) = \{0\} \times \mathbb{R}_-$ unless $x^* = 0$. 

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Regarding the assumptions of Theorem 5.5, let us just mention, without providing all the details, that (5.6) and (5.7) fail since the graphical (sub-) derivative is too large. Particularly, this clarifies that the first assumption is not necessary e.g. in the polyhedral setting, but not because it would be automatically satisfied.

6 Concluding remarks

In this paper, we introduced directional notions of asymptotic regularity for set-valued mappings. These conditions have been shown to serve as constraint qualifications guaranteeing M-stationarity of local minimizers in nonsmooth optimization. These new qualification conditions have been embedded into the landscape of constraint qualifications which are already known from the literature, and we came up with the impression that these conditions are comparatively mild. Noting that directional asymptotic regularity might be difficult to check in practice, we then focused on the derivation of applicable sufficient conditions for its validity. First, we suggested directional notions of pseudo- and quasi-normality for that purpose which have been shown to generalize related concepts for geometric constraint systems to arbitrary set-valued mappings. Second, with the aid of so-called super- and pseudo-coderivatives, sufficient conditions for the presence of directional asymptotic regularity for geometric constraint systems in terms of first- and second-order derivatives of the associated mapping as well as standard variational objects associated with the underlying set were derived. These sufficient conditions turned out to recover some of our findings from Benko and Mehlitz (2022b), and we showed that they are not stronger than FOSCMS and SOSCMS. In this paper, we completely neglected to study the potential value of directional asymptotic regularity in numerical optimization which might be a promising topic of future research. Furthermore, it has been shown in Mehlitz (2020) that non-directional asymptotic regularity can be applied nicely as a qualification condition in the limiting variational calculus. Most likely, directional asymptotic regularity may play a similar role in the directional limiting calculus.

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