Integer Quantization of the Chern-Simons Coefficient in a Broken Phase

Lusheng Chen, Gerald Dunne, Kurt Haller and Edwin Lim-Lombridas
Department of Physics
University of Connecticut
Storrs, CT 06269 USA

October 28, 1994

Abstract

We consider a spontaneously broken nonabelian topologically massive gauge theory in a broken phase possessing a residual nonabelian symmetry. Recently there has been some question concerning the renormalization of the Chern-Simons coefficient in such a broken phase. We show that, in this broken vacuum, the renormalized ratio of the Chern-Simons coupling to the gauge coupling is shifted by $1/4\pi$ times an integer, preserving the usual integer quantization condition on the bare parameters.

1 Introduction

It is well known that the addition of a Chern-Simons term to the conventional Yang-Mills gauge field action in $2 + 1$ dimensions leads to a massive transverse gauge mode, with mass equal to the Chern-Simons coupling constant, $m$. In a nonabelian theory, consistency of the quantum theory under large gauge transformations requires that the dimensionless ratio $\frac{4\pi m}{g^2}$ be quantized as an integer, where $g^2$ is the gauge coupling. This is most easily understood as a field theoretical analogue of Dirac’s monopole quantization condition which requires $\exp(i\text{action})$ to be gauge invariant [2]. Pisarski and Rao have shown [3] that in a 1-loop perturbative analysis of topologically massive $SU(N)$ Chern-Simons-Yang-Mills gauge theory, the bare ratio $q \equiv \frac{4\pi m}{g^2}$ receives a finite renormalization shift equal to the integer $N$. It has since been shown that there is no further correction at two loops [4], and there exist general arguments that this should be true to all orders [3, 4]. On the other hand, one can
also generate a gauge field mass by the conventional Higgs mechanism, in a vacuum in which
the gauge symmetry is spontaneously broken. In this Letter we consider the perturbative fate of the integer quantization condition on the Chern-Simons coupling in the broken phase of a spontaneously broken nonabelian topologically massive theory. We show that if there is still a nonabelian symmetry present in the broken phase, then the one-loop renormalized ratio \[
\left(\frac{4\pi m}{g^2}\right)_{ren}
\] is shifted from its bare value by an integer.

The abelian spontaneously broken Chern-Simons theory was investigated in [5, 6], where
it was shown that in the broken vacuum \[q \equiv \frac{4\pi m}{g^2}\] receives a finite renormalization shift
which is a complicated function of the various mass scales: the Chern-Simons mass \(m\), the
gauge coupling \(g^2\), and the symmetry breaking mass scale. A similar situation exists for a
completely broken nonabelian theory [7]. However, here there is no apparent contradiction
with the Chern-Simons quantization condition since there is no residual nonabelian symmetry
in the broken vacuum. To probe this question more deeply, the authors of Ref. [8] introduced
an ingenious model in which a nonabelian topologically massive theory is partially broken,
leaving a nonabelian gauge symmetry in the broken phase. One’s immediate expectation
is that once again the renormalized ratio \[
\left(\frac{4\pi m}{g^2}\right)_{ren}
\] should be quantized, but with an integer renormalization shift characteristic of the smaller unbroken residual gauge symmetry. We have reconsidered this model both in a detailed canonical treatment and, as reported in this Letter, in a direct perturbative analysis. We disagree with the result reported in Ref. [8]. We
find that the integer quantization condition is indeed preserved to one loop. The mechanism
by which this quantization arises in a perturbative analysis is considerably more involved
than the Pisarski and Rao case [3] where there is no symmetry breaking. The cancellations
required to produce the integer shift can be viewed as new ‘topological Ward identities’ for
the spontaneously broken theory, generalizing the ‘topological Ward identity’ found in [1].

For simplicity, we consider an \(SU(3)\) topologically massive gauge theory, coupled to a
triplet of charged scalar fields with a potential possessing a vacuum in which the \(SU(3)\)
symmetry is spontaneously broken to \(SU(2)\). We later show that our result applies also to
the case of \(SU(N)\) broken to \(SU(N-1)\), for \(N \geq 3\).

The (Euclidean) Lagrange density for this model is
\[
\mathcal{L} = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + i \frac{m}{2} \epsilon_{\mu\nu\rho} \left( \partial_\mu A_\nu^a A_\rho^a + \frac{1}{3} g f^{abc} A_\mu^a A_\nu^b A_\rho^c \right) + (D_\mu \Phi)^\dagger (D_\mu \Phi) - \mu^2 \left( \Phi^\dagger \Phi \right) + \lambda \left( \Phi^\dagger \Phi \right)^2 \tag{1}
\]
The gauge fields, \(A_\mu = A_\mu^a T^a\), take values in the defining representation. We use the hermitean generators \(T^a = \frac{1}{2} \lambda^a\), satisfying \([T^a, T^b] = if^{abc} T^c\), where the \(\lambda^a\) are the \(SU(3)\) Gell-Mann matrices. We present our analysis in Euclidean space-time to permit direct comparison with the results of Ref. [3]. The covariant derivative is \(D_\mu \Phi = (\partial_\mu - ig A_\mu) \Phi\), and the
gauge curvature is \(F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c\). Note that both the Chern-Simons coupling \(m\) and the gauge coupling \(g^2\) have dimensions of mass. For definiteness we choose \(m\) positive. The Chern-Simons term in (1) is not invariant under large gauge transformations
with nonzero winding number, but the (Euclidean) action changes by an integer multiple of

$2\pi i$

provided the following quantization condition holds [1, 3]:

$$q \equiv \frac{4\pi m}{g^2} = \text{integer}$$

(2)

The Higgs potential in (1) has a nontrivial vacuum in which $\Phi$ has the (tree approximation)

vacuum expectation value

$$\Phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}, \quad v \equiv \sqrt{\frac{\mu^2}{\lambda}}$$

(3)

In this nontrivial vacuum we express $\Phi$ as $\Phi = \Phi' + \Phi_0$, where

$$\Phi' = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_1 + i\chi_1 \\ \psi_2 + i\chi_2 \\ \psi_3 + i\chi_3 \end{pmatrix}$$

(4)

The Lagrange density (1) is supplemented with the 't Hooft gauge fixing and ghost terms:

$$L_g = -\frac{1}{2\xi} \left( \partial_\mu A_\mu^a - ig\xi \left( \Phi_0^a T^a \Phi - \Phi_0^a T^a \Phi_0 \right) \right)^2 + \partial_\mu \bar{\eta}^a \partial_\mu \eta^a - ig f^{abc} \partial_\mu \bar{\eta}^a A_\mu^b \eta^c$$

(5)

A simple calculation leads to the following quadratic gauge field portion of the Lagrange
density

$$L_{\text{quad}} = \frac{1}{2} \partial_\mu A_\nu (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) + \frac{m}{2} \epsilon_{\mu\nu\rho} \partial_\mu A_\rho^a A_\nu^a - \frac{1}{2\xi} (\partial_\mu A_\mu^a)^2 + \frac{g^2}{2} \left( \Phi_0^a \{T^a, T^b\} \Phi_0 \right) A_\mu^a A_\mu^b$$

(6)

and the anticommutation relations in (5) are conveniently summarized as: $\{T^a, T^b\} = \frac{1}{3} \delta^{ab} + d^{abc} T^c$. We deduce the following gauge field propagators (in the Landau gauge: $\xi = 0$), diagonal in color indices. In the unbroken sector (corresponding to gauge field components $a = 1, 2, 3$) the gauge field has the standard topologically massive propagator:

$$\Delta_{\text{unbroken}}^{\mu\nu} = \frac{(k^2 \delta_{\mu\nu} - k_\mu k_\nu) - m \epsilon_{\mu\nu\rho} k_\rho}{k^2 (k^2 + m^2)}$$

(7)

The remaining gauge field components acquire mass through the Chern-Simons-Higgs mecha

nism [3], leading to a propagator with two mass poles:

$$\Delta_{\text{broken}}^{\mu\nu} = \frac{(k^2 \delta_{\mu\nu} - k_\mu k_\nu) (k^2 + m_+^2) / k^2 - m \epsilon_{\mu\nu\rho} k_\rho}{(k^2 + m_+^2)(k^2 + m^2)}$$

(8)
where the two mass poles are related to the Chern-Simons mass and the “W-boson” mass $m_W$ by

$$m_\pm = \sqrt{m_{W}^2 + \frac{m_\pm^2}{4}} \pm \frac{m_\pm}{2}$$

(9)

For the iso-doublet massive gauge fields (with $a = 4, 5, 6, 7$) the W-boson mass is $m_W = v g/2 \equiv m_D$, while for the iso-singlet massive gauge field (with $a = 8$) the W-boson mass is $m_W = v g/\sqrt{3} \equiv m_S$. The most direct way [3] to derive the broken propagators (8) is to invert the self-energy in (6), but a more physical approach is to note that the two masses $m_\pm$ correspond precisely to the two characteristic frequencies $\omega_\pm$ of a planar quantum mechanical system in the presence of a perpendicular magnetic field (corresponding to the Chern-Simons interaction) and a harmonic potential (corresponding to the Higgs potential) [9].

The 3-gluon vertex differs from the standard QCD vertex by the inclusion of a parity-odd Chern-Simons contribution, giving the combined vertex :

$$V^{abc}_{\mu\nu\rho}(p,q,r) = -igf^{abc}[(r-q)_{\mu}\delta_{\nu\rho} + (p-r)_{\nu}\delta_{\mu\rho} + (q-p)_{\rho}\delta_{\mu\nu} - m\epsilon_{\mu\nu\rho}]$$

(10)

In addition there are the conventional vertices of spontaneously broken gauge theory.

The gauge field self-energy can be expressed as

$$\Pi_{\mu\nu} = (p^2 \delta_{\mu\nu} - p_\mu p_\nu)\Pi_{\text{even}}(p^2) + m\epsilon_{\mu\nu\rho}p_\rho\Pi_{\text{odd}}(p^2)$$

(11)

In perturbation theory, the bare (integer) ratio $q = \frac{4\pi m}{g^2}$ appearing in (4) acquires a multiplicative renormalization [3]

$$q_{\text{ren}} = 4\pi \left[\frac{m}{g^2}\right]_{\text{ren}} = \frac{4\pi m}{g^2} Z_m \bar{Z}^2$$

(12)

The multiplicative renormalization factor $Z_m$ is defined in terms of the odd part of the gauge self-energy evaluated at zero momentum:

$$Z_m = 1 + \Pi_{\text{odd}}(0)$$

(13)

The renormalization factor $\bar{Z}$ is similarly defined in terms of the ghost self energy $\bar{\Pi}$ [3].

There are three types of contribution to the computation of $\Pi_{\text{odd}}$, as shown in Fig. 1. The first contribution, corresponding to the diagram in Fig. 1(a), arises from the external unbroken gluons coupling to internal unbroken gluons, and so is exactly the same as that computed by Pisarski and Rao - so we find

$$\Pi_{\text{odd}}^{\text{Fig.1(a)}}(p^2) = 2g^2 \int \frac{d^3k}{(2\pi)^3} \left[ \frac{p^2 k^2 - (p \cdot k)^2}{p^2} \right] \frac{5k^2 + 5k \cdot p + 4p^2 + 2m^2}{k^2(k^2 + m^2)(k + p)^2((k + p)^2 + m^2)}$$

(14)
where the overall factor of 2 in front corresponds to the fact that this is essentially an $SU(2)$ computation. At $p^2 = 0$ this reduces to

$$
\Pi_{\text{odd}}^{\text{Fig.1(a)}}(0) = 2g^2 \left( \frac{2}{3} \right) \int \frac{d^3k}{(2\pi)^3} \frac{5k^2 + 2m^2}{k^2(k^2 + m^2)^2} = 2 \frac{g^2}{m} \frac{7}{12\pi}
$$  \hspace{1cm} (15)

The second contribution to $\Pi_{\text{odd}}$ comes from the diagram in Fig.1(b) in which the external unbroken gluons couple to internal broken gluons, whose propagator is now of the form (8), and so the Pisarski-Rao computation must be modified. A lengthy, but straightforward computation, leads to the expression

$$
\Pi_{\text{odd}}^{\text{Fig.1(b)}}(p^2) = \frac{1}{2} g^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{Q} \left[ \frac{p^2k^2 - (p \cdot k)^2}{p^2} \right] \left\{ 6m^2 - m^2 \left( ((p + k)^2 + m_D^2)/(p + k)^2 + (k^2 + m_D^2)/k^2 \right) - \left( 6k^2 + 4p^2 + 6p \cdot k \right) \left( k^2 + m_D^2 \right) \left( (p + k)^2 + m_D^2 \right)/(k^2(p + k)^2) + (4p^2 + 4p \cdot k + 8k^2) k^2 + m_D^2/k^2 \right\}
$$  \hspace{1cm} (16)

where

$$
Q \equiv (k^2 + m_+^2)(k^2 + m_-^2)((p + k)^2 + m_+^2)((p + k)^2 + m_-^2)
$$  \hspace{1cm} (17)

At $p^2 = 0$ this reduces to

$$
\Pi_{\text{odd}}^{\text{Fig.1(b)}}(0) = \frac{1}{3} g^2 \int \frac{d^3k}{(2\pi)^3} \frac{6m^2k^2 - 2m^2(k^2 + m_D^2) - 6(k^2 + m_D^2)^2 + 16k^2(k^2 + m_D^2)}{(k^2 + m_+^2)^2(k^2 + m_-^2)^2} = g^2 \frac{2}{2\pi(m_+ + m_-)}
$$  \hspace{1cm} (18)

Expressions for this diagram Fig. 1(b) (at $p^2 = 0$) have been published previously in [7, 8]. However these two previous expressions disagree with one another, as well as with our result (18). There is however, a check that when $m_D^2$ is set to zero inside the integral appearing in the expression (16) (i.e. no symmetry breaking), this should reduce (apart from the overall group theoretical factor of 2) to the Pisarski-Rao expression in (14) for all $p^2$. This is true of our expression (16), but it is not true if the coefficients are modified to agree with the expressions in either [7] or [8]. It is a remarkably unfortunate coincidence of numerology that if one tries this $m_D^2 \to 0$ reduction for the diagram evaluated at $p^2 = 0$, then all three expressions reproduce the $\frac{7}{12\pi}$ factor in the Pisarski and Rao result (15), after the $k$ integration has been performed.
The third contribution to \( \Pi_{\text{odd}} \) comes from the diagram in Fig. 1(c) in which the external unbroken gluons couple to an internal broken gluon and to the internal unphysical scalar fields \( \psi_1, \psi_2, \chi_1 \) or \( \chi_2 \). This diagram contributes

\[
\Pi_{\text{odd}}^{\text{Fig.1(c)}}(p^2) = g^2 m_D^2 \int \frac{d^3k}{(2\pi)^3} \frac{p \cdot k}{p^2} \frac{1}{(k^2 + m_+^2)(k^2 + m_-^2)(p + k)^2}
\]

which reduces at \( p^2 = 0 \) to

\[
\Pi_{\text{odd}}^{\text{Fig.1(c)}}(0) = g^2 \frac{1}{6\pi(m_+ + m_-)}
\]

Combining these three results we find that the total one-loop mass renormalization factor is

\[
Z_m = 1 + g^2 \left[ 2 \frac{7}{12\pi m} + \frac{2}{3\pi(m_+ + m_-)} \right]
\]

To compute \( \tilde{\Pi} \) we need to compute the ghost self-energy, which has two contributions, as shown in Figure 2. The contribution of Fig 2(a) corresponds to an internal unbroken gluon, and so this is once again identical to the Pisarski-Rao computation. This yields

\[
\tilde{\Pi}_{\text{Fig.2(a)}}(p^2) = -2g^2 \int \frac{d^3k}{(2\pi)^3} \left[ \frac{p^2k^2 - (p \cdot k)^2}{p^2} \right] \frac{1}{k^2(p + k)^2(k^2 + m^2)}
\]

At \( p^2 = 0 \) this reduces to

\[
\tilde{\Pi}_{\text{Fig.2(a)}}(0) = -2g^2 \frac{1}{6\pi m}
\]

The second contribution comes from Fig. 2(b) in which the internal gluon propagator is of the broken form, giving

\[
\tilde{\Pi}_{\text{Fig.2(b)}}(p^2) = -g^2 \int \frac{d^3k}{(2\pi)^3} \left[ \frac{p^2k^2 - (p \cdot k)^2}{p^2} \right] \frac{1}{k^2(p + k)^2(k^2 + m_+^2)(k^2 + m_-^2)}
\]

which reduces at \( p^2 = 0 \) to

\[
\tilde{\Pi}_{\text{Fig.2(b)}}(0) = -g^2 \frac{1}{3\pi(m_+ + m_-)}
\]

Thus the renormalization factor \( \tilde{Z} \) is given by

\[
\tilde{Z} = 1 - g^2 \left[ 2 \frac{1}{6\pi m} + \frac{1}{3\pi(m_+ + m_-)} \right]
\]
We are now able to compute the renormalized ratio \( q_{\text{ren}} \) appearing in (12):

\[
q_{\text{ren}} = \frac{4\pi m}{g^2} \left\{ 1 + g^2 \left[ \frac{7}{12\pi m} - \frac{4}{6\pi m} + \frac{2}{3\pi(m_+ + m_-)} - \frac{2}{3\pi(m_+ + m_-)} \right] \right\} = \frac{4\pi m}{g^2} \left\{ 1 + 2\frac{g^2}{4\pi m} \right\}
\]

\( = q_{\text{bare}} + 2 \) \hspace{1cm} (27)

This shift by 2 corresponds precisely to the Pisarski-Rao shift for the residual \( SU(2) \) symmetry in the broken vacuum. For an \( SU(N) \) theory spontaneously broken to \( SU(N - M) \), with \( N - M \geq 2 \), this computation goes through almost entirely unchanged. For example, for \( SU(N) \) broken to \( SU(N - 1) \), the ‘additional diagrams’ computed in \( \text{(16,19,24)} \) are completely unchanged except for the values of \( m_\pm \). However, these cancel in the end and the \( SU(N) \to SU(N - 1) \) generalization of the result (27) is

\[
q_{\text{ren}} = \frac{4\pi m}{g^2} \left\{ 1 + g^2 \left[ (N - 1) \frac{7}{12\pi m} - 2(N - 1) \frac{1}{6\pi m} + \frac{2}{3\pi(m_+ + m_-)} - \frac{2}{3\pi(m_+ + m_-)} \right] \right\} = q_{\text{bare}} + (N - 1) \hspace{1cm} (28)
\]

To conclude, we comment that this result fills a gap in our previous understanding of the interplay between gauge invariance and the spontaneous breaking of parity. The general picture in the abelian theories is well understood - in the broken phase there is a radiative correction to the odd part of the photon propagator \( [4,5] \), but this should be understood not as a renormalization of the topological mass, but as due to the appearance of gauge invariant terms in the effective action which reduce to a Chern-Simons-like term when the scalar field takes its vacuum expectation value \( [10] \). This then extends the Coleman-Hill theorem \( [11] \), concerning the absence of corrections to the topological mass in certain abelian theories, to include the case of spontaneous symmetry breaking \( [10] \). Presumably a similar mechanism operates in the completely broken nonabelian theories \( [7] \), although this has not yet been demonstrated explicitly. When a nonabelian symmetry is only partially broken, leaving a residual nonabelian symmetry in the broken vacuum, this mechanism no longer works - there is no obvious way to construct gauge invariant terms in the effective action which could reduce to the nonabelian Chern-Simons term when the scalar fields take their vacuum expectation value. However, the result described here implies that no such extra terms are required. Our result should also be relevant for the nonabelian generalizations of the abelian pure Chern-Simons-matter theories considered in Ref. \( [12] \).

Acknowledgement: We are grateful to Manu Paranjape for discussions and correspondence. This work has been supported by the DOE grant DE-FG02-92ER40716.00.
References

[1] S. Deser, R. Jackiw and S. Templeton, “Topologically Massive Gauge Theory”, *Ann. Phys.* **140** (1982) 372.

[2] P. Dirac, “Quantized Singularities in the Electromagnetic Field”, *Proc. Roy. Soc.* **A133** (1931) 60.

[3] R. Pisarski and S. Rao, “Topologically Massive Chromodynamics in the Perturbative Regime”, *Phys. Rev.* **D32** (1985) 2081.

[4] G. Giavarini, C. P. Martin and F. Ruiz Ruiz, “Chern-Simons Theory as the Large Mass Limit of Topologically Massive Yang-Mills Theory”, *Nucl. Phys.* **B381** (1992) 222.

[5] S. Yu. Khlebnikov, “Spontaneous Parity Violation in Three-Dimensional Scalar Electrodynamics”, *JETP Lett.* **51** (1990) 81; V. Spiridonov, “Higher-Order Corrections to the Chern-Simons Term in Scalar Electrodynamics”, *JETP Lett.* **52** (1990) 513.

[6] V. Spiridonov, “Quantum Dynamics of the D=3 Abelian Higgs Model and Spontaneous Breaking of Parity”, *Phys. Lett.* **B247** (1990) 337; V. Spiridonov and F. Tkachov, “Two-Loop Contribution of Massive and Massless Fields to the Abelian Chern-Simons Term”, *Phys. Lett.* **B260** (1991) 109.

[7] S. Yu. Khlebnikov and M. Shaposhnikov, “Spontaneous Symmetry Breaking Versus Spontaneous Parity Violation”, *Phys. Lett.* **B254** (1991) 148.

[8] A. Khare, R. MacKenzie, P. Panigrahi and M. Paranjape, “Spontaneous Symmetry Breaking and the Renormalization of the Chern-Simons Term”, Montréal Preprint, UdeM-LPS-TH-93-150, hep-th/9306027.

[9] G. Dunne, “Symmetry Breaking in the Schrödinger Representation for Chern-Simons Theories”, *Phys. Rev.* **D50** (1994) 5321.

[10] A. Khare, R. MacKenzie and M. Paranjape, “On the Coleman-Hill Theorem”, Montréal Preprint (August 1994), hep-th/9408091.

[11] S. Coleman and B. Hill, “No More Corrections to the Topological Mass Term in QED3”, *Phys. Lett.* **B159** (1985) 184.

[12] H-C. Kao, K. Lee, C. Lee and T. Lee, “The Chern-Simons Coefficient in the Higgs Phase”, Columbia-Seoul Preprint CU-TP-647 and SNUTP94-71, hep-th/9408079.
Figure 1: These diagrams represent the one-loop contributions to the odd part of the (unbroken) gluon propagator $\Pi_{\text{odd}}$. Figure 1(a) involves internal unbroken gluon propagators, as in (7), while Figure 1(b) involves internal broken propagators, as in (8). Figure 1(c) involves an internal broken gluon propagator, and an internal unphysical scalar propagator.

Figure 2: These diagrams represent the one-loop contributions to $\tilde{\Pi}$. Figure 2(a) involves an internal unbroken gluon propagator, while Figure 2(b) involves an internal broken propagator.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9411062v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9411062v1