Analytical solutions of a Dirac bound state equation and their field interpretation

L. Micu
Horia Hulubei Institute for Physics and Nuclear Engineering
Bucharest POB MG-6, 76900 Romania

Institute for Theoretical Physics of the University of Bern
CH-3012 Bern, Sidlerstrasse 5

We solve the single particle Dirac equation with a particular confining potential and comment its significance from the point of view of the quantum field theory. We show that the solutions describe a complex physical system made of independent constituents: a free particle and an effective field representing the confining potential.

I. THE DIRAC EQUATION

In the description of low energy hadronic processes the so called ”QCD inspired” models based on phenomenological notions of constituent quarks, confining potentials and hyperfine interaction, remain very useful tools. To understand their connection with QCD a first requirement is to express in field language the information acquired from these models. In a recent paper [1] we have presented a way to do this. We have shown that in the momentum space representation a bound system of particles can be treated like a gas of free particles and a collective excitation of a background field which must be seen as the stationary time averaged result of a continuous series of quantum fluctuations.

The main features of the method are now discussed on a simple example which has analytical solutions.

We consider the case of a single particle with spin 1/2 confined to a certain region of space by an external field represented by a particular scalar-vector combination of linear rising and Coulomb-like potentials frequently used in quark models [2]. The bound state function, \( \psi \), is an eigenfunction of the single particle Dirac Hamiltonian and satisfies the equation

\[
(\vec{\alpha} \cdot \vec{p} + \beta m + V(\vec{r})) \psi(\vec{r}) = \varepsilon \psi(\vec{r})
\]

where

\[
V(\vec{r}) = V(\vec{r}) = \frac{1}{2} \beta (V_1(\vec{r}) + V_2(\vec{r})) \pm \frac{1}{2} (V_1(\vec{r}) - V_2(\vec{r}))
\]

and

\[
V_1(\vec{r}) = \zeta |\vec{r}| \quad \text{and} \quad V_2(\vec{r}) = \frac{\zeta}{|\vec{r}|} - 2m_i + 2 \sqrt{\frac{\zeta}{\xi}} \left( \vec{\sigma} \cdot \vec{L} + \frac{1}{2} \right)
\]

We mention that the potential confinement has been studied in connection with the problem of a light quark bound to a heavy antiquark [3] and exact solutions of the single particle problem have been found also for scalar-vector oscillator potential [4] and for other combinations of scalar-vector Coulomb-like and linear rising potentials [5]. Our solutions are however particularly simple and have some nice features which deserve a detailed examination.

Following the usual treatment of the Dirac equation with a central potential [6] we write \( \psi \) as:

---

*E-mail address: lmicu@theory.nipne.ro
where \( l, l' \) are either \( l = J - \frac{1}{2}, \ l' = J + \frac{1}{2} \), or \( l = J + \frac{1}{2}, \ l' = J - \frac{1}{2} \) and \( \sum_{m,s} C_{m,s}^l \frac{1}{2} M Y_{l,m}^M(\vec{r}) \chi^s \). Taking now \( \mathcal{V}(i) = \mathcal{V}_+ \) in (4) and recalling the relations

\[
\vec{\alpha} \cdot \vec{p} = \alpha_r \left[ p_r + \frac{i}{r} (J^2 - \vec{L}^2 + \frac{1}{4}) \right]
\]

and

\[
\frac{\vec{J} \cdot \vec{r}}{r} \gamma^M_{(J + \frac{1}{2})} J = -\gamma^M_{(J + \frac{1}{2})} J
\]

where \( \alpha_r = \bar{\varepsilon} \vec{\alpha}, \ p_r = -i \frac{\partial}{\partial r}, \ \vec{J} \) is the total angular momentum, \( \vec{L} \) is the orbital momentum we replace ec.(1) by two coupled differential equations

\[
-G'(r) + \frac{J(J + 1) - l'(l' + 1) + \frac{1}{4}}{r} G(r) = (\varepsilon - m - \zeta r) F(r)
\]

\[
F'(r) - \frac{J(J + 1) - l(l + 1) + \frac{1}{4}}{r} F(r) = \left[ \varepsilon - m + \frac{\xi}{r} + 2 \sqrt{\frac{\xi}{J}} \left( J(J + 1) - l(l' + 1) - \frac{1}{4} \right) \right] G(r).
\]

Now we consider the case \( l = J - \frac{1}{2}, \ l' = J + \frac{1}{2} \) and observe that a first condition for \( \psi \) to be finite at the origin is \( F(r) \sim r^{l+k} \), \( k \geq 0 \). From (7) it follows \( G(r) \sim r^{2-k} \) and from (8) one gets \( l = l' \). We also observe from (8) that \( G'(r) \) behaves like \( rF(r) \) at infinity, so we take

\[
F(r) = r^{l+1}e^{-\alpha r} \sum_{i=0} a_i r^i
\]

\[
G(r) = r^{l+2}e^{-\alpha r} \sum_{i=0} b_i r^i.
\]

Introducing the expressions (9) and (10) in the equations (7) and (8) we get the following relations connecting the coefficients \( a_i, b_i \):

\[
-(2l + n + 3)b_n + \alpha b_{n-1} = (\varepsilon - m)a_n - \zeta a_{n-1}
\]

\[
n a_n - \alpha a_{n-1} = \left[ \varepsilon - m - 2 \sqrt{\frac{\xi}{J}} \left( l + \frac{3}{2} \right) \right] b_{n-2} + \xi b_{n-1}.
\]

We cut the series in (1) at \( n = N \) by requiring \( a_{N+k} = 0 \) for any \( k \geq 1 \) and obtain from (12) the following restrictions for \( b_{N+k} \):

\[
0 = \left[ \varepsilon - m - 2 \sqrt{\frac{\xi}{J}} \left( l + \frac{3}{2} \right) \right] b_{N+k-1} + \xi b_{N+k}.
\]

Equation (11) gives for any \( k \geq 1 \):

\[
-(2l + N + k + 1)b_{N+k} + \alpha b_{N+k-1} = -\delta_{k1} \zeta a_{N+k-1}.
\]

Supposing the series in \( G(r) \) is cut at \( i = M \) where \( M \geq N + 1 \), it follows from (14) that \( b_i = 0 \) for any \( N < i \leq M \). On the other side, the condition \( b_N \neq 0 \) implies
\[ \varepsilon - m - 2 \sqrt{\frac{\xi}{\zeta}} \left( l + \frac{3}{2} \right) = 0 \]  

which is the quantization condition for the energy levels. Introducing the condition (15) in the equations (11) and (12) we write them as follows

\[- (2l + n + 3)b_n + \alpha b_{n-1} = 2 \sqrt{\frac{\xi}{\zeta}} \left( l + \frac{3}{2} \right) a_n - \zeta a_{n-1} \]  

\[(n+1)a_{n+1} - \alpha a_n = \xi b_n. \]  

For \( n = 0, 1, 2, ..., N \) we get from (16) and (17)

\[-(2l+3)b_0 = 2 \sqrt{\frac{\xi}{\zeta}} \left( l + \frac{3}{2} \right) a_0 \]  

\[a_1 - \alpha a_0 = \xi b_0 \]  

\[-2(l+2)b_1 + \alpha b_0 = 2 \sqrt{\frac{\xi}{\zeta}} \left( l + \frac{3}{2} \right) a_1 - \zeta a_0 \]  

\[2a_2 - \alpha a_1 = \xi b_1 \]  

\[-\alpha a_N = \xi b_N \]  

\[\alpha b_N = -\zeta a_N. \]  

From the last two equalities it follows that

\[ \alpha = \sqrt{\xi \zeta}; \quad \zeta \xi > 0. \]  

Introducing these relations into the first equalities of the set (18) we have successively: \( a_1 = 0, \ b_1 = 0, \ a_2 = 0, \ldots \) which means that the solution of the eigenvalue equation (1) reads

\[ \psi^M_J(\vec{r}) = \begin{pmatrix} r^{J+\frac{1}{2}} e^{-\sqrt{\xi \zeta} r} Y^M_{J+\frac{1}{2}J} \\ -i \sqrt{\xi} r^{J+\frac{1}{2}} e^{-\sqrt{\xi \zeta} r} Y^M_{J-\frac{1}{2}J} \end{pmatrix} \]  

and \( \varepsilon_J = m + 2 \sqrt{\frac{\xi}{\zeta}} (J + 1). \)

Proceeding in a similar manner it can be shown that there is no solution in the case \( l = J + \frac{1}{2}, \ l' = J - \frac{1}{2}, \) so (20) is the single one and \( \varepsilon_J \) is \((2J+1)\) fold degenerate.

We also define \( \psi^c_J = (\psi^M_J)^+ \gamma^0 \) which satisfies the same equation as \( \psi^M_J \) and the charge conjugate solution \( \psi^c_J \)

\[ \psi^c_J(\vec{r}) = i \gamma^2 \gamma^0 (\psi^M_J)^T(\vec{r}) = \begin{pmatrix} i \sqrt{\xi} r^{J+\frac{1}{2}} e^{-\sqrt{\xi \zeta} r} Y^M_{J+\frac{1}{2}J} \\ r^{J-\frac{1}{2}} e^{-\sqrt{\xi \zeta} r} Y^M_{J-\frac{1}{2}J} \end{pmatrix} \]  

which satisfies the equation (11) with the potential \( V_+ \) and corresponds to the eigenvalue \( \varepsilon^c_J = -\varepsilon_J = -m - 2 \sqrt{\xi} (J + 1). \)

**II. THE FIELD INTERPRETATION OF THE BOUND STATE FUNCTION**

Our purpose now is to find the relativistic meaning of the bound state functions \( \psi \) and \( \psi^c \) in the hope to create a link with the quantum field theory.

To this end we have first to expand \( \psi \) and \( \psi^c \) in terms of the free Dirac solutions, the only ones having a real relativistic character and then to interpret in a Lorentz covariant manner the contribution of the confining potential to the bound state energy.
For reasons of simplicity we shall work with the lowest $J$ functions

\[ \psi_{\frac{J}{2}}(\vec{r}) = \begin{pmatrix} e^{-\sqrt{\zeta \xi} \chi^\rho} \\ -i \sqrt{\frac{2}{\zeta}} \Delta r e^{-\sqrt{\zeta \xi} \phi^\rho} \end{pmatrix} \]  

(22)

and

\[ \psi_{\frac{J}{2}}(\vec{r}) = \begin{pmatrix} i \sqrt{\frac{2}{\zeta}} \Delta r e^{-\sqrt{\zeta \xi} \phi^\rho} \\ e^{-\sqrt{\zeta \xi} \chi^\rho} \end{pmatrix} \]  

(23)

where $\chi^\rho$ and $\phi^\rho$ are two component spinors and project them on the free Dirac spinors.

In the first case the projection $\phi_{\rho s}^{(+)}(\vec{k})$ on a positive energy free state has the following expression

\[ \phi_{\rho s}^{(+)}(\vec{k}) = \int d^3r e^{-i\vec{k}\cdot\vec{r}} \bar{u}_s(\vec{k}) \psi_{\frac{J}{2}}(\vec{r}) = \]  

\[ = 4\pi \sqrt{\frac{\zeta \xi}{\zeta + \vec{k}^2}} \frac{\sqrt{e + m}}{2m(e + m)} \left( 1 - 4 \sqrt{\frac{\zeta}{\zeta + \vec{k}^2}} \right) \delta_{\rho s} \]  

(24)

where $e = \sqrt{\vec{k}^2 + m^2}$ and, according to the general principles of the quantum mechanics, it represents the probability amplitude to find a free particle with momentum $\vec{k}$ and spin $s$ in the bound state $\psi_{\frac{J}{2}}^\rho$.

Projecting $\psi$ on a negative energy free Dirac state one obtains

\[ \phi_{\rho s}^{(-)}(\vec{k}) = \int d^3r e^{i\vec{k}\cdot\vec{r}} \bar{v}_s(\vec{k}) \psi_{\frac{J}{2}}(\vec{r}) = \]  

\[ = \frac{4\pi \sqrt{\frac{\zeta \xi}{\zeta + \vec{k}^2}}}{(\zeta + \vec{k}^2)^2} \frac{1}{\sqrt{2m(e + m)}} \left( 1 + 4 \sqrt{\frac{\zeta}{\zeta + \vec{k}^2}} \right) \bar{v}_s(\vec{k}) \gamma^0 u^\rho(\vec{k}) \]  

(25)

which shows that the existence of a negative energy free particle is accompanied by the dissolution into the vacuum of a particle-antiparticle pair. We recall that the projections of negative energy free states on the free states with positive energy are always zero and hence the existence of the nonvanishing result (24) has to be considered an effect of a classical potential. This demonstrates that $\psi$ is not a single particle state in the sense of the classical quantum mechanics and that its physical content is more complex than that. This means also that the part missing from the complete description of the bound state is related to the confining potential which is now indissoluble connected to the bound state function.

In the attempt to identify this part we notice that the position vector with respect to the center of forces, $\vec{r}$ can be written as $\vec{R} - \vec{R}_0$, where $\vec{R}, \vec{R}_0$ are the position vectors of the particle and of the center of forces with respect to the origin of the coordinate frame and therefore $\psi$ appears to describe a system made of two components: a free particle having the momentum $\vec{k}$ and an additional component, denoted $\Phi$ in the following, carrying the momentum $\vec{Q}$ defined as:

\[ \vec{Q} = -\vec{k} \]  

(26)

which is the recoil momentum of the center of forces due to the motion of the free particle.

Besides, recalling that the bound state function $\psi$ describes a stationary system with the energy $\varepsilon$ we conclude that the additional component must carry the energy $Q^0$ which represents the potential energy of the bound system and is defined as follows:

\[ Q^0 = \varepsilon \mp \sqrt{\vec{k}^2 + m_1^2}. \]  

(27)

It is important to mention that by introducing $\Phi$ as an independent component of the bound state and by defining its 4-momentum as a linear combination of free 4-momenta (see 24)
and \((27)\) the contribution of the binding potential to the bound state energy acquires a well defined, Lorentz covariant significance. The major advantage of this fact is that one can write immediately a Lorentz covariant representation of the bound state without worrying about the transformation properties of the binding potential at boosts. The single change to be made is to write the functions \(\phi^{(\pm)}\) in a Lorentz covariant form which is obtained by replacing \(\vec{k}\) by \(k_\mu' = k\mu - (\pi \cdot k)\pi^\mu\) where the vector \(\pi^\mu = \left(\frac{1}{\sqrt{1 - \omega^2}}, \frac{1}{\sqrt{1 - \omega^2}}\omega\right)\) is used to write the Lorentz transformation from the initial frame where the confining potential is defined to a frame moving with the velocity \(\omega\) with respect to it.

We have:

\[
\begin{align*}
\phi^{(+)}_{\rho s}(k) &= \frac{4\pi\sqrt{\xi\zeta}}{(\xi + (\pi \cdot k)^2 - m^2)^2} \sqrt{(\pi \cdot k) - m} \delta_{\rho s}, \\
\phi^{(-)}_{\rho s}(k) &= \frac{4\pi\sqrt{\xi\zeta}}{(\xi + (\pi \cdot k)^2 - m^2)^2} \left(\frac{1}{\sqrt{2m((\pi \cdot k) + m)}} \right) \left(1 + 4 \sqrt{\frac{\zeta}{\xi} \xi + (\pi \cdot k)^2 - m^2} \right).
\end{align*}
\]

\((28)\)

Then, in field notation, the stationary Lorentz covariant expression of the bound state function \(\psi\) can be written:

\[
\begin{align*}
\psi^R(\vec{R}, \vec{R}_0, t) &= \sum_{\rho} \int d^3k \frac{m}{e} \int dtQ \left( \delta^{(4)}(Q + k - P) \phi^{(+)}_{\rho s}(k) a_s(k) \Phi(Q) u^s(k)e^{-i(\xi + Q^0)t + i\vec{k}\vec{R} + i\vec{q}\vec{R}_0} \\
&- \delta^{(4)}(Q - k - P) \phi^{(-)}_{\rho s}(k) b_s^\dagger(k) \Phi(Q) v^s(k)e^{i(\xi - Q^0)t - i\vec{k}\vec{R} + i\vec{q}\vec{R}_0} \right)
\end{align*}
\]

\((29)\)

where the annihilation operator of a negative energy free state \(a_-\) has been replaced by the creation operator of an antiparticle, \(b^\dagger\). The four components of the momentum \(P^\mu\) are \(E = \sqrt{1 - \omega^2}\) and \(\vec{P} = -\vec{\omega}E\). \(\Phi\) designates the additional component representing the confining potential and, as it can be seen from \((29)\), it must be seen as a reservoir of particles and energy in the sense the vacuum is for the free Dirac equation.

Similar results are obtained in the case of the charge conjugate function where one writes

\[
\begin{align*}
\psi^{cR}(\vec{R}, \vec{R}_0, t) &= \sum_{\rho} \int d^3k \frac{m}{e} \int dtQ \left( \delta^{(4)}(k + Q - P) \phi^{(-)}_{\rho s}(k) a_s(k) \Phi(Q) u^s(k)e^{-i(\xi + Q^0)t + i\vec{k}\vec{R} + i\vec{q}\vec{R}_0} \\
&- \delta^{(4)}(Q - P - k) \phi^{(+)}_{\rho s}(k) b_s^\dagger(k) \Phi(Q) v^s(k)e^{i(\xi - Q^0)t - i\vec{k}\vec{R} + i\vec{q}\vec{R}_0} \right)
\end{align*}
\]

\((30)\)

where \(\phi^{(+)}\) and \(\phi^{(-)}\) can be obtained from \(\phi^{(+)}\) and \(\phi^{(-)}\) respectively by performing the replacement \(u^r \leftrightarrow -v^r\).

Concluding this paper we notice that from the point of view of the field theory a particle in a bound state can be seen as a system made of two components: a free particle and an effective field \(\Phi\) which has a double face: in momentum space it is a reservoir of particles and energy while in coordinate representation it represents a kind of a box where the particle is confined.

This image is similar to that of bag models \([7]\). It is a time averaged image, not an instantaneous one and it is expected to hold whenever the observation time is longer or at least equal to the time giving a stable average.

We also notice that, as suggested by the relativistic interpretation of the bound state functions \([24]\) and \((30)\), the Dirac equation with a confining potential is a field equation and its solutions represent a complex physical structure where the free positive and negative energy states are mixed.

Acknowledgments The author thanks Fl. Stancu for valuable suggestions and continuous encouragement.
The work has been completed during author’s visit at ITP of the University of Bern under the SCOPE Programme. The warm hospitality at ITP and the financial support from the Swiss National Science Foundation are gratefully acknowledged.

[1] L. Micu, hep-ph/0202244.
[2] W. Lucha, F. Schöberl and D. Gromes, Phys. Rep. 200 (1991) 127.
[3] M. G. Olsson, S. Veseli and K. Williams, Phys. Rev D 51 (1995) 5079; J. Sucher, Phys. Rev. D 51 (1995) 5965.
[4] R. Tegen, R. Brockmann and W. Weise, Z. Phys. A - Atoms and Nuclei 307 (1982) 339.
[5] J. Franklin, Mod. Phys. Lett. A14 (1999) 2409; A. S. De Castro and J. Franklin, hep-ph/0011137; A. S. De Castro and J. Franklin, J. Mod. Phys. A15: (2000) 4355.
[6] M. E. Rose, Relativistic electron theory, John Wiley & Sons Inc. New York-London, 1960.
[7] A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn and V. F. Weisskopf, Phys. Rev. D 9, 3471 (1974); A. Chodos, R. L. Jaffe, K. Johnson and C. B. Thorn, Phys. Rev. D 10, 2599 (1974); T. DeGrand, R. L. Jaffe, K. Johnson and J. Kiskis, Phys. Rev. D 12, 2060 (1975).