Topological Susceptibility and Zero Mode Size in Lattice QCD

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We use the overlap formalism to define a topological index on the lattice. We study the spectral flow of the hermitian Wilson-Dirac operator and identify zero crossings with topological objects. We determine the topological susceptibility and zero mode size distribution, and we comment on the stability of our results.

1. INTRODUCTION

In this proceedings, we summarize our work on the determination of the topological susceptibility and zero mode size distribution of SU(2) and SU(3) gauge fields using the overlap formalism. Most of this work has appeared in print [1]. We will focus here on new results which are extensions of the published work, namely the topological susceptibility for SU(2) pure gauge theory, and the zero mode size distribution for both SU(2) and SU(3), and the density of zero eigenvalues \( \rho(0) \) for SU(2) and SU(3).

A more detailed discussion of the role of the zero eigenvalue density can be found in the proceedings of the plenary talk given by Narayanan [2].

2. SPECTRUM OF THE HERMITIAN WILSON–DIRAC OPERATOR

The overlap formalism for constructing a chiral gauge theory on the lattice [3] provides a natural definition of the index, \( I \), of the associated chiral Dirac operator. The index is equal to half the difference of negative and positive eigenvalues of the hermitian Wilson-Dirac operator \( H_L(m) = \gamma_5 W(-m) \) where \( W(m) \) is the usual Wilson–Dirac operator (we will use negative sign for the mass term throughout).

On the lattice, because of the additive mass renormalization, the crossings of zero occur at positive \( m \) and spread out in \( m \). It is easy to see, that no eigenvalues of \( H_L(m) \) can be zero for \( m < 0 \). Since in the free case the first doublers become massless at \( m = 2 \) we restrict ourselves to \( m < 2 \). A simple way to compute the index \( I \) is to compute the lowest eigenvalues of \( H_L(m) \) at some suitably small \( m \) before any crossings of zero occurred. Then \( m \) is slowly varied and the number and direction of zero crossings are tracked. The net number at some \( m_t \) is the index of the overlap chiral Dirac operator.

We have applied this procedure to compute the index, which we take as the definition of topological charge on the lattice, of various gauge field ensembles with gauge group SU(3) including pure gauge, Symanzik improved pure gauge, and two dynamical flavor generated gauge fields [1], and recently pure gauge SU(2). We found that the zero crossings start occurring at some, ensemble dependent, \( m_1 > 0 \) and continue occurring for all \( m \) in \( m_1 < m < 2 \) in sufficiently large lattice volumes. Hence, the spectral gap is closed in the entire range.

To further investigate the zero level crossings, we compute the density of zero eigenvalues \( \rho(0; m) \) for \( H_L(m) \) by fitting the integrated density \( \int_0^\lambda \rho(x') \, dx' \) to a line through the origin for some cutoff \( \lambda \). By studying the scaling of \( \rho(0; m) \) for quenched SU(3), we conclude that the density of zero eigenvalues falls exponentially in the inverse lattice spacing and only vanishes in the continuum limit (see Figure 6 of Narayanan’s talk [2]). We will observe latter that these zeros are due to small localized modes. More discussion can be found in the talk of Narayanan [2].
3. TOPOLOGY AND SMALL ZERO MODES

Using the index of the chiral Dirac operator as our (fermionic) definition of the topological charge of the gauge field background, we obtain the topological susceptibility $\chi$ as a function of $m_t$. All of the gauge ensembles studied have the general characteristic that the susceptibility rises sharply in the region where $\rho(0; m)$ is peaked, and then it essentially flattens out in the region where $\rho(0; m)$ is small [1].

We define a size of the eigenvector associated with the level crossing zero mode motivated by the 'tHooft zero mode $\rho^2_z/2(t^2 + \rho^2_z)^{3/2}$ where $\phi_k$ are the eigenvectors of $H_L$.

We show in Figure 1 a detailed study of the $\beta = 6.0, 16^3 \times 32$ pure gauge ensemble. On the first line is shown the density of zero eigenvalues $\rho(0; m)$ and the number of crossings in each mass bin. Since there are a nonzero number of crossings, we see that $\rho(0; m)$ does indeed measure zero eigenvalues, and not just small eigenvalues near
We also see that $\rho(0; m)$ rises sharply in $m$, then falls to a nonzero value where there is a small number of zero level crossings.

On the second line of Figure 1, we show the size of the zero modes $\rho_z(m)$. The modes are large near $m_1$ where $\rho(0; m)$ is large, then $\rho_z$ drops to about 1 or 2 lattice spacings up to $m = 2$. We see that the corresponding $\chi$ rises sharply when $\rho_z$ is large for $m$ near $m_1$, then is quite stable when $\rho_z$ is small. This result show that while the index, $I$, of the field is $m$ dependent, $\chi$ (a physical quantity) is independent of the contribution from the small modes for $m > 1$.

To further clarify the relative contribution of the zero modes, in the last line of Figure 1 the zero mode size distribution is shown which peaks for $\rho_z \leq 2$. In the adjacent graph, $\chi$, here defined by the contribution of zero modes of size $\rho_z$ and larger, is stable when $\rho_z \leq 2$. Hence, the small modes do not affect the estimate of $\chi$. Our estimates of $\chi$ are shown in Table 1 where we use the string tension value $\sqrt{\sigma} = 440$(MeV) to set the scale. Our results are in rough agreement with other groups \cite{4, 5}.

In Figure 2 we plot the zero mode size distribution for several $\beta$'s for $SU(3)$ and $SU(2)$ which is of interest for the instanton liquid model. We find the distributions always peaks for sizes about 1 - 2 lattice spacings. For $SU(3)$, we see small secondary peaks at $\sim 0.65$fm for $\beta = 6.0$ and 5.7. However, there is only a shoulder for 5.85 at 0.3fm. For $SU(2)$ which are all the same lattice size, we find the secondary peaks shift to smaller $\rho_z$ for increasing $\beta$. These secondary peaks do not occur at any general physical radius and are consistent with a finite volume effect.

### Table 1

| $\beta$ | $\rho_z$ (fm) | $N_{cont}$ | $\chi^{1/2}$(MeV) |
|---------|---------------|------------|-------------------|
| $SU(3)$ | 6.0           | $16^3 \times 32$ | 75 | 194(10) |
|         | 5.85          | $8^3 \times 16$ | 200 | 108(05) |
|         | 5.7           | $8^3 \times 16$ | 50  | 193(10) |
| $SU(2)$ | 2.6           | $16^3$       | 400 | 229(05) |
|         | 2.5           | $16^4$       | 100 | 232(10) |
|         | 2.4           | $16^4$       | 200 | 220(06) |

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