We construct consistent brane-world Kaluza-Klein reductions involving the radion mode that measures the separation of the domain-wall branes. In these new examples, we can obtain matter supermultiplets coupled to supergravity on the brane, starting from pure gauged supergravity in the higher dimension. This contrasts with previously-known examples of consistent brane-world reductions involving the radion, where either pure supergravity reduced to pure supergravity, or else supergravity plus matter reduced to supergravity plus matter. As well as considering supersymmetric reductions, we also show that there exist broader classes of consistent reductions of bosonic systems. These include examples where the lower-dimensional theory has non-abelian Yang-Mills fields and yet the scalar sector has a potential that admits Minkowski spacetime as a solution. Combined with a sphere reduction to obtain the starting point for the brane-world reduction, this provides a Kaluza-Klein mechanism for obtaining non-abelian gauge symmetries from the geometry of the reduction, whilst still permitting a Minkowski vacuum in the lower dimension.

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1 Introduction

In conventional Kaluza-Klein reductions, the internal space is taken to be compact with isometries, leading to lower-dimensional theories that comprise a finite number of massless modes, together with infinite towers of massive modes. In certain circumstances, it is possible to perform a consistent truncation of all the massive modes (meaning that setting the massive modes to zero is consistent with their equations of motion). In some cases, such as Kaluza-Klein reduction on a circle, torus, or other group manifold, there is a clear-cut group-theoretic reason for the consistency of the truncation, in that all the fields that are group singlets are retained, whilst all those that are non-singlets are discarded. Clearly, then, the retained fields cannot act as sources for those that are set to zero. In other cases, there are much more remarkable consistent reductions for which there is no fully-understood group-theoretic explanation. Examples of this kind include the reductions of \( D = 11 \) supergravity on \( S^4 \) or \( S^7 \), and the reduction of type IIB supergravity on \( S^5 \).

In the usual circle reduction from \((D+1)\) to \(D\) dimensions, retaining just the massless sector, the internal space (and the reduction ansatz) is assumed to have a \( U(1) \) isometry. A rather different kind of reduction has been considered recently, in which the \( D \)-dimensional world is viewed as the world-volume of a \((D-1)\)-brane (i.e. a domain wall) embedded in \((D+1)\) dimensions. Clearly, the one-dimensional transverse space no longer has a \( U(1) \) isometry; it is broken by the location of the domain wall, and by the warp-factor of the domain-wall metric. Thus the conventional Kaluza-Klein technique has to be modified for these new situations.

Firstly, it is necessary to examine whether gravity would indeed localise on the brane, so that one has a genuinely lower-dimensional theory. One way to achieve the localisation is by considering two branes, one located at each end of a finite interval. In such a scenario, the localisation of gravity is guaranteed, since the internal space is finite, and so the massive Kaluza-Klein tower will have a discrete mass spectrum. An alternative to this compactification is to make use of the fact that with the domain-wall warp factor, it is possible to trap gravity even when there is only a single brane, with an extra dimension of infinite extent \([1]\). This is because although the extra dimension is infinite, its volume is finite owing to the warp factor. Using either the single-brane or double-brane scenario, it is possible to arrive at a lower-dimensional gravity theory on the brane. In fact, the equations of motion for the domain wall solution require only that the world-volume metric have vanishing Ricci-tensor, rather than the more stringent condition of vanishing Riemann tensor and hence Minkowskian spacetime. Furthermore, the domain-wall solution preserves
a certain fraction of supersymmetry (typically $\frac{1}{2}$). It follows that one would expect to find a supergravity theory with lesser supersymmetry trapped on the domain-wall world-volume. It has been shown in [2, 3, 4] that this can indeed be the case. The associated consistent reduction procedure is known as a brane-world Kaluza-Klein reduction.

In these examples, a pure supergravity theory was obtained on the brane, by using a modified, but nonetheless consistent, Kaluza-Klein procedure. It is of considerable interest to see whether matter supermultiplets can also arise through such brane-world Kaluza-Klein reductions. Generating matter using Kaluza-Klein is not always guaranteed. For example, in the Horava-Witten model, the $E_8 \times E_8$ Yang-Mills fields of the heterotic string are not expected to come from a Kaluza-Klein reduction; rather, their existence is argued on the grounds of anomaly cancellation [5]. In the present paper, we shall consider examples where we can obtain matter supermultiplets from consistent brane-world Kaluza-Klein reductions.

In the reductions that we shall obtain in this paper, the breathing mode (i.e. the scalar that measures the “size” of the extra dimension) plays an important role. It can also be thought of as a “radion mode,” since in the double-brane picture it measures the relative separation of the two branes in the transverse dimension. In fact consistent brane-world Kaluza-Klein reductions involving the radion mode were first introduced in [3]. In those examples, which include the reduction of the massive IIA theory to $D = 9$, and the reduction of gauged $D = 8$ pure supergravity to $D = 7$, the radion mode becomes the dilaton of the pure supergravity multiplet in the lower dimension. A further example of a consistent brane-world reduction involving the radion mode was then obtained in [6]; in that case the starting point was gauged five-dimensional $\mathcal{N} = 2$ supergravity coupled to a hypermultiplet, and the radion became the scalar member of a chiral matter multiplet in four dimensional $\mathcal{N} = 1$ supergravity.

In this paper, we shall obtain various examples of consistent brane-world reductions involving the radion, which give rise to lower-dimensional supergravities (with a halving of supersymmetry) coupled to matter multiplets. We begin in section 2 with a general discussion of the circumstances under which we can obtain a consistent brane-world reduction of a bosonic theory comprising gravity, a dilaton with an exponential potential, and a $p$-form field strength. We find that a consistent reduction is possible if there is a specific relation between the dilaton coupling to the $p$-form and the dilaton coupling in the exponential potential. We make extensive use of these general results in the subsequent sections, when we consider consistent reductions of gauged supergravities. Our first supergravity example, in section 3, starts from gauged $\mathcal{N} = 2$ supergravity in $D = 7$. We show that a consistent
brane-world reduction is possible in which we obtain ungauged $\mathcal{N} = (1, 0)$ chiral supergravity in $D = 6$, coupled to a chiral tensor multiplet. The radion mode in this reduction forms the scalar member of the chiral tensor multiplet. (The brane-world reduction to give pure $\mathcal{N} = (1, 0)$ chiral six-dimensional supergravity was obtained in [2].) Solutions in the six-dimensional theory can then be lifted back to $D = 7$. We consider the BPS dyonic string, and demonstrate that in $D = 7$ it leads to a bending of the domain walls.

In section 4, we obtain further supersymmetric consistent brane-world reductions to ungauged supergravities plus matter. In one of these, we obtain five-dimensional supergravity with a vector multiplet, starting from six-dimensional gauged $\mathcal{N} = (1, 1)$ supergravity. In fact this, and the above reduction from $D = 7$, are the first examples where supermatter as well as supergravity is obtained in consistent brane-world reductions of pure supergravity theories. In a further example, we obtain four-dimensional $\mathcal{N} = 1$ supergravity with a chiral multiplet, starting from gauged $D = 5$ supergravity with a single vector multiplet.

In section 5 we consider some extended bosonic systems for which we can obtain consistent brane-world reductions. Of particular interest are cases where the starting point is the bosonic sector of a gauged supergravity in which we now augment the previous brane-world reductions by including $SU(2)$ Yang-Mills fields. We find that consistent brane-world reductions are possible in which we end up with these Yang-Mills fields in the lower dimension, but still in a theory where there is no cosmological term or scalar potential. The higher-dimensional gauged theories can themselves be obtained by $S^3$ reduction of ungauged supergravity in a yet higher dimension. We therefore have the intriguing situation that we can view the $S^3$ plus brane-world reduction as a (3 + 1)-dimensional reduction scheme in which non-abelian Yang-Mills emerges from Kaluza-Klein reduction, without any cosmological term or scalar potential being generated. The (3 + 1)-dimensional reduction can viewed as a reduction on a singular cone over $S^3$.

In section 6 we consider the brane-world reductions of bosonic theories with additional scalars as well as the dilaton in the higher dimension. Such theories typically arise as subsectors of gauged supergravities. We find that under appropriate circumstances we can obtain consistent brane-world reductions in which all the extra scalars are retained. The scalar potential in the lower dimension is related to that in the higher dimension, but with a modification which means, in particular, that it admits a Minkowski spacetime vacuum.
2 Consistent Reduction of a $p$-form and Radion

We begin by deriving a general result for a consistent Kaluza-Klein brane-world reduction of a $(D + 1)$-dimensional theory comprising a metric, a dilatonic scalar and a $p$-form field strength. In the class of theory we shall be considering, the dilaton has a scalar potential which is a single exponential function:

$$
\hat{L} = R \hat{s} \mathbb{I} - \frac{1}{2} \hat{s} d\hat{\phi} \wedge d\hat{\phi} - \frac{1}{2} e^{\gamma \hat{\phi}} \hat{s} \hat{F} \wedge \hat{F} + g^2 e^{a \hat{\phi}} \hat{s} \mathbb{I}.
$$

(1)

It will frequently turn out to be convenient to parameterise the constant $a$ in the scalar potential in terms of a quantity $\Delta$, where

$$
a^2 \equiv \Delta + \frac{2D}{D-1} = \Delta + 2 + \frac{2}{D-1}. \tag{2}
$$

The equations of motion for this theory,

$$
\hat{R}_{AB} = \frac{1}{2} \partial_A \hat{\phi} \partial_B \hat{\phi} + \frac{1}{2} \left( \hat{F}^2_{AB} - \frac{(p-1)}{p(D-2)} \hat{F}^2 \eta_{AB} \right) - \frac{g^2}{D-1} e^{a \hat{\phi}} \eta_{AB},
$$

$$
\hat{\Box} \hat{\phi} = \frac{\gamma}{2p!} e^{\gamma \hat{\phi}} \hat{F}^2 - a g^2 e^{a \hat{\phi}},
$$

$$
d(e^{a \hat{\phi}} \hat{s} \hat{F}) = 0,
$$

(3)

do not admit an AdS$_{D+1}$ “vacuum” solution, but they do allow a domain wall, given by

$$
ds^2 = W \frac{4}{(D-1) \Delta} dx^\mu dx_\mu + W \frac{4D}{(D-1) \Delta} dy^2,
$$

$$
e^{\hat{\phi}} = W^{\frac{2a}{\Delta}}, \quad \hat{F} = 0,
$$

(4)

where $W$ is a linear function of the transverse coordinate $y$;

$$
W = 1 + my, \quad m^2 = -\frac{1}{2} \Delta g^2. \tag{5}
$$

(For an actual domain wall one would replace $y$ by $|y|$. Since our focus here is on the Kaluza-Klein reductions rather than the properties of the wall itself, it is preferable for our present purposes to omit the modulus sign, which would lead to additional delta-function contributions in the curvature.)

The domain-wall “vacuum” can be thought of as a background solution around which a Kaluza-Klein brane-world reduction can be performed. In fact, we can do much better than merely describing linearised fluctuations; in appropriate circumstances we can obtain a fully

$^1\Delta$ is preserved under toroidal dimensional reduction, as discussed in [7].
consistent Kaluza-Klein embedding that is exact to all non-linear orders. Specifically, we find that we can obtain a consistent reduction if the constants $\gamma$ and $a$ in (1) are related by
\begin{equation}
\gamma = -\frac{2(p-1)}{(D-1)a}.
\end{equation}

We find that the reduction ansatz is given by\footnote{Here, and throughout the paper, we shall place hats on the higher-dimensional fields, which depend, \textit{a priori}, on the lower-dimensional coordinates $x^\mu$ and the extra coordinate $y$. Unhatted fields live in the lower dimension, and depend only on the $x^\mu$ coordinates.}
\begin{equation}
\begin{aligned}
ds^2 &= W^{\frac{4}{D-1}\Delta} e^{2\alpha \varphi} ds^2 + W^\frac{4D}{(D-1)\Delta} e^{-2(D-2)\alpha \varphi} dy^2, \\
e^{a\hat{\phi}} &= W^{-\frac{2a^2}{\Delta}} e^{2(D-2)\alpha \varphi}, \quad \hat{F} = F. 
\end{aligned}
\end{equation}

It is useful to record that in the obvious vielbein basis $\hat{e}^a = W^{2/(D-1)\Delta} e^{\alpha \varphi} e^a$, $\hat{e}^0 = W^{2D/((D-1)\Delta)} e^{-(D-2)\alpha \varphi} dy$, the torsion-free spin connection and the components of the Ricci tensor turn out to be given by
\begin{equation}
\begin{aligned}
\hat{\omega}_{0a} &= -(D-2)\alpha W^{-\frac{2}{(D-1)\Delta}} e^{-\alpha \varphi} \hat{e}^0 - \frac{2m}{(D-1)\Delta} W^{\frac{2D+(D-1)\Delta}{(D-1)\Delta}} e^{(D-2)\alpha \varphi} \hat{e}^a, \\
\hat{\omega}_{ab} &= \omega_{ab} + \alpha W^{-\frac{2}{(D-1)\Delta}} e^{-\alpha \varphi} (\partial_b \varphi \hat{e}^a - \partial_a \varphi \hat{e}^b), \\
\hat{R}_{00} &= (D-2)\alpha e^{-2\alpha \varphi} W^{\frac{2}{D-1}} \Box \varphi, \\
\hat{R}_{0a} &= m \alpha (D-2) W^{\frac{2}{D-1}} e^{(D-3)\alpha \varphi} \partial_a \varphi, \\
\hat{R}_{ab} &= W^{\frac{2}{D-1}} e^{-2\alpha \varphi} \left( R_{ab} - (D-1)(D-2)\alpha^2 \partial_a \varphi \partial_b \varphi - \alpha \Box \varphi \eta_{ab} \right) \\
&\quad - \frac{m^2}{D-1} W^{\frac{2}{D-1}} e^{2(D-2)\alpha \varphi} \eta_{ab}. 
\end{aligned}
\end{equation}

Substituting (7) into the higher-dimensional equations of motion (3), we obtain $D$-dimensional equations of motion for the metric, the $p$-form $F$ and the radion $\varphi$, which can be derived from the Lagrangian
\begin{equation}
\mathcal{L} = R * 1 - \frac{1}{2} * d\varphi \wedge d\varphi - \frac{1}{2} e^{-\frac{2(p-1)(\Delta+4)}{2(D-2)(D-1)(\Delta+4)}} F * F,
\end{equation}
where $F = dA$ and we have chosen
\begin{equation}
\alpha^2 = \frac{a^2}{2(D-2)(D-1)(\Delta+4)},
\end{equation}
so that the radion is canonically normalised. Note that the relation (6) between $\gamma$ and $a$ is essential in order that the $y$-dependence in the various higher-dimensional equations of motion balances properly, giving rise to consistent lower-dimensional equations of motion.
In the above consistent reduction, a $p$-form in $(D + 1)$ dimensions is reduced only to a $p$-form in $D$ dimensions. Since a $p$-form is dual to a $(D + 1 - p)$ form in the original $(D + 1)$-dimensional theory, there is dual description in terms of a $\tilde{p} = D + 1 - p$ form field $\tilde{G} \equiv e^{\gamma \phi} \tilde{F}$, for which the reduction ansatz is

$$\tilde{G} = W^4 \Delta G \wedge dy,$$

where the lower-dimensional field strength $G$ is a $(\tilde{p} - 1)$-form. This field is related by dualisation to the previous $p$-form field $F$ in $D$ dimensions in the usual way, namely

$$G = e^{-\frac{2(p-1)(\Delta + 4) \alpha}{a^2}} \tilde{F}.$$  \hspace{1cm} (12)

Thus an equivalent statement about the circumstances under which a consistent braneworld reduction of (1) can be performed is that $\gamma$ must be related to $a$ by

$$\gamma = -\frac{2(p-1)}{(D-1)a}, \quad \text{or} \quad \gamma = \frac{2(D-p)}{(D-1)a}.$$  \hspace{1cm} (13)

In the first case, the $p$-form field strength is reduced according to $\tilde{F} = F$, while in the second case the reduction of the $p$-form $\tilde{F}$ is instead performed using

$$\tilde{F} = W^4 \Delta F \wedge dy.$$  \hspace{1cm} (14)

In this case the resulting $D$-dimensional Lagrangian is given by

$$\mathcal{L} = R \ast 1 - \frac{1}{2} d\varphi \wedge d\varphi - \frac{1}{2} e^{\frac{2(D-p)(\Delta + 4) \alpha}{a^2}} \varphi \ast F \wedge F,$$  \hspace{1cm} (15)

where $\alpha$ is again given by (10), and now $F = dA$ is a $(p-1)$-form.

The consistent brane-world reduction that we have derived here makes essential use of the radion field $\varphi$ that characterises the scale in the $y$ direction transverse to the lower-dimensional spacetime. Such braneworld reductions involving the radion mode were first introduced in some of the consistent brane-world reductions obtained [3]. One of these was a reduction of massive type IIA supergravity to give $\mathcal{N} = 1$ ungauged supergravity in $D = 9$, and the other was a reduction of gauged supergravity in $D = 8$ to give ungauged $\mathcal{N} = 2$ supergravity in $D = 7$. A further example of a consistent brane-world reduction involving the radion mode was obtained in [6], where the five-dimensional theory resulting from a generalised Calabi-Yau reduction from $D = 11$ was further reduced to give $\mathcal{N} = 1$ supergravity plus a chiral scalar multiplet in $D = 4$.

In subsequent sections, we shall make use of the results that we have obtained here in order to construct consistent brane-world reductions of various gauged supergravity theories.
It turns out that for all the examples we shall consider, the scalar potential in the higher-dimensional gauged theory is of the single exponential form in (1), with the constant $a$ given by (2) with $\Delta = -2$. In these cases, it follows from (2) and (6) that we have a consistent reduction if either

$$a^2 = \frac{2}{D-1}, \quad \gamma = -(p-1) a, \quad \text{and} \quad \hat{A} = A,$$

yielding the $D$-dimensional Lagrangian

$$\mathcal{L} = R \ast 1 - \frac{1}{2} d\varphi \wedge d\varphi - \frac{1}{2} e^{-2(p-1)(D-1)\alpha \varphi} * F \wedge F,$$

where $F = dA$ is a $p$-form, or else if

$$a^2 = \frac{2}{D-1}, \quad \gamma = (D-p) a, \quad \text{and} \quad \hat{A} = W^{-2} A \wedge dy,$$

yielding the $D$-dimensional Lagrangian

$$\mathcal{L} = R \ast 1 - \frac{1}{2} d\varphi \wedge d\varphi - \frac{1}{2} e^{2(D-p)(D-1)\alpha \varphi} * F \wedge F,$$

where $F = dA$ is a $(p-1)$-form. In each case the metric and dilaton reduction ansatz is

$$ds^2 = W^{-\frac{2}{D-1}} e^{2\alpha \varphi} ds^2 + W^{-\frac{2D}{D-1}} e^{-2(D-2)\alpha \varphi} dy^2,$$

$$e^{\alpha \hat{\varphi}} = W^{-\frac{2}{D-1}} e^{2(D-2)\alpha \varphi},$$

and

$$\alpha^2 = \frac{1}{2(D-2)(D-1)^2}.$$  

3 $D = 7$ Reduced to $D = 6$, $\mathcal{N} = (1,0)$ supergravity with matter

In this section, we shall carry out in detail the consistent brane-world reduction of a gauged $\mathcal{N} = 2$ seven-dimensional supergravity.\(^3\) Gauged $\mathcal{N} = 4$ supergravity can be obtained via a consistent $S^4$ reduction from $D = 11$ [8, 9], and the $\mathcal{N} = 2$ theory can be obtained as a truncation of this. In the process, the $SO(5)$ Yang-Mills fields of the $\mathcal{N} = 4$ theory are truncated to $SU(2)$. Explicit expressions for the $S^4$ reduction were obtained in [10]. The bosonic field content comprises the metric, a dilaton $\hat{\varphi}$, a 4-form field strength $\hat{F}_{(4)}$, and the

\(^3\)We use the convention where the allowed supersymmetries in $D = 7$ are $\mathcal{N} = 2$ and $\mathcal{N} = 4$. Thus, the $\mathcal{N} = 2$ theory has half of maximal supersymmetry.
SU(2) Yang-Mills fields $\hat{F}_{(2)}^i$. There is a scalar potential which is the sum of three different exponentials of the dilaton $\hat{\phi}$, of the form

$$V = 2g_1^2 e^{-\frac{2}{\sqrt{10}} \hat{\phi}} + 2g_1 g_2 e^{\frac{3}{\sqrt{10}} \hat{\phi}} - \frac{1}{4} g_2^2 e^{\frac{8}{\sqrt{10}} \hat{\phi}}. \quad (22)$$

This has a stationary point, and hence the theory admits an AdS$_7$ “vacuum” solution. Note that the two constants $g_1$ and $g_2$ have interpretations as the SU(2) gauge coupling and a topological mass term respectively.

A consistent brane-world reduction that yielded just ungauged chiral $\mathcal{N} = (1, 0)$ supergravity in six dimensions was constructed in [2]. It could be viewed as a fully non-linear generalisation of a linearised Kaluza-Klein reduction around the AdS$_7$ vacuum. In the bosonic sector, the resulting six-dimensional theory comprised just the metric and a self-dual 3-form. In the present paper, we wish to extend the scope of the brane-world reduction, so that we obtain $\mathcal{N} = (1, 0)$ supergravity coupled to an $\mathcal{N} = (1, 0)$ matter multiplet. Specifically, we shall show how we can obtain the tensor matter multiplet comprising an anti-self-dual 3-form plus a scalar field. In order to do this, we shall employ the reduction scheme derived in section 2. This reduction requires that there be only a single exponential in the seven-dimensional theory, and that it be related to the dilaton coupling for the 4-form $\hat{F}_{(4)}$ in the specific way discussed in section 2. In fact we find that this can be achieved by setting the topological mass term $g_2$ to zero.\textsuperscript{4}

3.1 The bosonic sector

After setting $g_2 = 0$ and relabeling $g_1 = g$, the seven-dimensional bosonic Lagrangian becomes

$$\hat{L}_7 = \hat{R} \ast \mathbf{1} - \frac{1}{2} d\hat{\phi} \wedge d\hat{\phi} - \frac{1}{2} e^{-\frac{2}{\sqrt{10}} \hat{\phi}} \ast \hat{F}_{(4)} \wedge \hat{F}_{(4)} - \frac{1}{4} e^{\frac{4}{\sqrt{10}} \hat{\phi}} \ast \hat{F}_{(2)}^i \wedge \hat{F}_{(2)}^i + \frac{1}{4} \hat{F}_{(2)}^i \wedge \hat{F}_{(2)}^i \wedge \hat{A}_{(3)} + 2g^2 e^{-\frac{2}{\sqrt{10}} \hat{\phi}} \ast \mathbf{1}, \quad (23)$$

where $\hat{F}_{(4)} = d\hat{A}_{(3)}$ and $\hat{F}_{(2)} = d\hat{A}_{(1)} + \frac{1}{2} g \epsilon_{ijk} \hat{A}_{(1)}^j \wedge \hat{A}_{(1)}^k$. It is evident that the 4-form dilaton coupling with $\gamma = -\frac{4}{\sqrt{10}}$ and the scalar potential with $a = -\frac{2}{\sqrt{10}}$ satisfy the second of the two criteria in (13), implying that we can obtain a consistent reduction of the 4-form $\hat{F}_{(4)}$ to give a 3-form in six dimensions.\textsuperscript{5} If we dualise the 4-form field strength to a 3-form, the

\textsuperscript{4}The theory with $g_2 = 0$ also arises as the Scherk-Schwarz group-manifold reduction of ten-dimensional type I supergravity on $S^3$, truncated to the pure $\mathcal{N} = 2$ supergravity sector [11]. By contrast, the theory where $g_1$ is instead set to zero arises from the generalised reduction of $D = 11$ supergravity on $T^4$ [12].

\textsuperscript{5}Note that the constant $a$ is given by $\Delta = -2$ in (2). In fact in all our examples, the strength of dilaton coupling in the scalar potential is characterised by $\Delta = -2$. 

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resulting Lagrangian can be obtained from the $SU(2)$ Scherk-Schwarz reduction of $N = 1$ supergravity in $D = 10$ [11].

From the formulae in section 2, we are therefore led to the following brane-world reduction ansatz for the seven-dimensional theory:

\[
\begin{align*}
    d\hat{s}_7^2 &= W^{-\frac{2}{3}} e^{2\alpha \varphi} ds_6^2 + W^{-\frac{12}{5}} e^{-8\alpha \varphi} dy^2, \\
    e^{-\frac{1}{\sqrt{10}} \hat{\phi}} &= W^\frac{2}{5} e^{8\alpha \varphi}, \\
    \hat{A}_{(3)} &= W^{-2} A_{(2)} \wedge dy, \quad \hat{A}_{(1)} = 0.
\end{align*}
\]  

(24)

where $\alpha = -1/(10\sqrt{2})$, $W = 1 + m y$, and $m^2 = 2g^2$. Substituting this into the equations of motion following from (23), we find that we obtain a consistent Kaluza-Klein reduction, resulting in six-dimensional equations that can be derived from the Lagrangian

\[
\mathcal{L}_6 = R \ast 1 - \frac{1}{2} d\varphi \wedge d\varphi - \frac{1}{2} e^{-\sqrt{10} \varphi} F_{(3)} \wedge F_{(3)},
\]

(25)

where $F_{(3)} = dA_{(2)}$. This is precisely the bosonic sector of six-dimensional $N = (1,0)$ supergravity coupled to an $N = (1,0)$ tensor matter multiplet. The supergravity multiplet comprises the metric and the self-dual part of $F_{(3)}$, and the tensor multiplet comprises the “radion” $\varphi$ and the anti-self-dual part of $F_{(3)}$.

In the next subsection, we shall show that the consistent reduction we have performed here can be extended to include the fermionic sector, and thus that we can obtain the full $N = (1,0)$ supergravity coupled to the tensor multiplet, via the brane-world reduction.

### 3.2 The fermionic sector

The bosonic Lagrangian (23) of the previous subsection has a supersymmetric completion [13], given up to quartic terms in the fermions by

\[
\begin{align*}
    \hat{L}_7^{\text{fermion}} &= -\frac{1}{2} \hat{\gamma}_i \hat{\gamma}_M \hat{\gamma}^{MNP} \hat{D}_N \hat{\psi}_{P_i} - \frac{1}{2} \hat{\lambda}_i \hat{\gamma}_M \hat{D}_M \hat{\lambda}_i \\
    &\quad - \frac{1}{12} \hat{\gamma}_i \hat{\gamma}_M \hat{\gamma}^{MNP} \hat{\psi}_{N_i} + \frac{1}{2} \hat{\gamma}_i \hat{\gamma}_M \hat{D}_M \hat{\psi}_{P_i} e^{-\frac{2}{\sqrt{10}} \hat{\phi}} \hat{F}_{ABCD} \\
    &\quad - \frac{i}{\sqrt{10}} \hat{\gamma}_i \hat{\gamma}_M \hat{\gamma}^{MNP} \hat{\psi}_{N_i} e^{-\frac{1}{\sqrt{10}} \hat{\phi}} \hat{F}_{ABCD} \\
    &\quad + \frac{1}{48\sqrt{5}} (\hat{\lambda}_i \hat{\gamma}_M \hat{\gamma}^{MNP} \hat{\psi}_{P_i} e^{-\frac{2}{\sqrt{10}} \hat{\phi}} \hat{F}_{ABCD} \\
    &\quad - \frac{i}{4\sqrt{10}} (\hat{\lambda}_i \hat{\gamma}_M \hat{\gamma}^{MNP} \hat{\psi}_{M} e^{-\frac{1}{\sqrt{10}} \hat{\phi}} \hat{F}_{ABCD} - \frac{1}{2\sqrt{2}} (\hat{\lambda}_i \hat{\gamma}_M \hat{\gamma}^{A} \hat{\psi}_{M_i} \partial_\phi \hat{\phi}) \\
    &\quad + \frac{1}{320} (\hat{\lambda}_i \hat{\gamma}_{ABC} \hat{\lambda}_i) e^{-\frac{1}{\sqrt{10}} \hat{\phi}} \hat{F}_{ABCD} - \frac{3i}{480} (\hat{\lambda}_i \hat{\gamma}_{AB} \hat{\lambda}_i) e^{-\frac{1}{\sqrt{10}} \hat{\phi}} \hat{F}_{ABCD} \\
    &\quad - \frac{1}{160} g e^{-\frac{1}{\sqrt{10}} \hat{\phi}} \hat{\gamma}_i \hat{\gamma}_M \hat{\gamma}_\mu \hat{\psi}_{\nu i} - \frac{1}{2\sqrt{10}} g e^{-\frac{1}{\sqrt{10}} \hat{\phi}} \hat{\gamma}_i \hat{\psi}_{\mu} \hat{\psi}_{\nu i} - \frac{3}{20} g e^{-\frac{1}{\sqrt{10}} \hat{\phi}} \hat{\gamma}_i \hat{\psi}_{\mu} \hat{\psi}_{\nu i}.
\end{align*}
\]

(26)
The fully gauge-covariant derivative $\hat{D}_\mu \hat{\epsilon}_i$ is defined as, e.g.

$$\hat{D}_\mu \hat{\epsilon}_i = \nabla_M \hat{\epsilon}_i + \frac{i}{2} g \hat{A}_M \hat{\epsilon}_i$$  \hspace{1cm} (27)

where $\hat{A}_M \hat{\epsilon}_i$ is given by $\hat{A}_M \hat{\epsilon}_i \equiv \hat{A}^k \epsilon^j (-\sigma^k)_{ij}$. This results in a field strength given by $\hat{F}_{MN} \hat{\epsilon}_i = \partial_M \hat{A}_N \hat{\epsilon}_i + \frac{i}{2} g \hat{A}_N \hat{A}_M \hat{\epsilon}_i - \delta (M \leftrightarrow N)$. The supersymmetry transformations are given by [13]

$$\delta \hat{\psi}_{Mi} = [\hat{D}_M + \frac{1}{160}(\gamma^M)_{PQR} \gamma^P \gamma^R e^{-\frac{\sqrt{2}}{5} \delta \phi} \hat{F}_{NPQR} - \frac{1}{5 \sqrt{2}} g \gamma_M e^{-\frac{\sqrt{2}}{5} \delta \phi}] \hat{\epsilon}_i$$

$$+ \frac{1}{20 \sqrt{2}} (\gamma^M)_{NP} \gamma^P e^{-\frac{\sqrt{2}}{5} \delta \phi} \hat{F}_{NP} \hat{\epsilon}_i \hat{\epsilon}_j,$$

$$\delta \lambda_i = [-\frac{1}{2 \sqrt{2}} \gamma^M \hat{D}_M \hat{\phi} + \frac{1}{48 \sqrt{2}} e^{-\frac{\sqrt{2}}{5} \delta \phi} \hat{F}_{MNPQ} \gamma^{MNPQ} + \frac{1}{\sqrt{2}} \hat{\phi} \hat{F}_{MN} \gamma^{MN} \hat{\epsilon}_i]$$

$$+ \left[ -\frac{1}{4 \sqrt{2}} e^{-\frac{\sqrt{2}}{5} \delta \phi} \hat{F}_{MN} \hat{\epsilon}_i \hat{\epsilon}_j \right],$$  \hspace{1cm} (28)

for the fermions, and

$$\delta \hat{\phi} = -\frac{1}{2 \sqrt{2}} \hat{\psi}_{Mi},$$

$$\delta \hat{\epsilon}_A = \frac{1}{4} \hat{\psi}_{Mi},$$

$$\delta \hat{A}_{MN} = e^{-\frac{1}{\sqrt{2}} \delta \phi} \left[ (\hat{\psi}_{Mi} \hat{\epsilon}_i) + \frac{1}{\sqrt{2}} \hat{\lambda}_i \gamma_{MNP} \hat{\epsilon}_i \right],$$

$$\delta \hat{A}_M \hat{\epsilon}_i = \frac{i}{2} e^{-\frac{1}{\sqrt{2}} \delta \phi} \left[ (\hat{\psi}_{Mi} \hat{\epsilon}_i - \frac{1}{\sqrt{2}} \hat{\lambda}_i \gamma_M \hat{\epsilon}_i) - \frac{1}{2 \sqrt{2}} \delta \phi \gamma^j \hat{\psi}_{M} \hat{\epsilon}_j \right],$$  \hspace{1cm} (29)

for the bosons. Here, the supersymmetry transformation parameter $\hat{\epsilon}_i$ is normalized according to

$$[\delta_1, \delta_2] \Xi = \frac{1}{4} (\hat{\psi}_{Mi} \hat{\epsilon}_i) \hat{\phi} + \text{(general coordinate)} + \text{(local Lorentz)} + \text{(gauge)},$$  \hspace{1cm} (30)

where $\Xi$ represents any of the fields in the theory.

The $D = 7$ spinors are sympletic-Majorana, with $i,j = 1,2$ being an $Sp(1) \equiv SU(2)$ index. We take a convenient basis where all $D = 7$ Dirac matrices are antisymmetric, obeying $\{\gamma^A, \gamma^B\} = 2 \eta^{AB}$. The Majorana condition is simply $\tilde{\lambda}^i = e^{ij} \chi^T_j$, and the Majorana flip relation reads

$$\tilde{\chi}^i \gamma_{M_1 M_2 \ldots M_n} \psi_i = (-)^n \tilde{\psi}_j \gamma_{M_n M_{n-1} \ldots M_1} \tilde{\lambda}_i$$  \hspace{1cm} (31)

(the triplet combination picks up an additional sign). This spinor convention is most convenient for reduction to $D = 6$, as the $D = 7$ sympletic-Majorana spinors reduce trivially to their six-dimensional counterparts. Furthermore, an additional $D = 6$ Weyl condition may be imposed consistent with this Majorana condition, as will be seen below.
To reduce the fermions, we first examine the Killing spinors of the domain wall background, given by (24) with \( \varphi \) and \( A_{(2)} \) set to zero. Inserting this solution into (28), we find

\[
\begin{align*}
\delta \hat{\lambda}_i &= \frac{1}{\sqrt{10}} g W \frac{1}{2} (1 + \gamma^7) \hat{\epsilon}_i, \\
\delta \hat{\psi}_{\mu i} &= -\frac{1}{5\sqrt{2}} g W^{-1} \gamma^7 (1 - 10 g^{-1} W \gamma^7 \partial_y) \hat{\epsilon}_i, \\
\delta \hat{\psi}_{\mu i} &= -\frac{1}{5\sqrt{2}} g \gamma_{\mu} (1 + \gamma^7) \hat{\epsilon}_i,
\end{align*}
\]

which leads to a half-BPS solution with Killing spinors given by \( \hat{\epsilon}_i = W^{-\frac{1}{10}} \epsilon^{(-)}_0 \). Here, \( \epsilon^{(-)}_0 \) is a constant six-dimensional symplectic-Majorana-Weyl spinor with the chiral components defined by

\[
\epsilon^{(\pm)} = P^{(\pm)} \epsilon \equiv \frac{1}{2} (1 \pm \gamma^7) \epsilon.
\]

Hence \( \gamma^7 \), the Dirac matrix in the \( y \) direction, provides the chirality operator on the brane. This generation of chirality from a non-chiral theory is a novel feature of this class of braneworld reductions, and was previously investigated in Refs. [2, 3, 4]. Note that, were there to be a modulus sign in \( W \), the projection would instead be \( P^{(\pm)} = \frac{1}{2} (1 \pm \gamma^7 \text{sgn} y) \).

This provides an obstruction to having globally well-defined Killing spinors, unless the gauge coupling constant \( g \) changes sign as well, thus compensating for the sign change in \( \partial_y W \) [14, 15].

Using the Killing spinors of the background as a guideline, it is then straightforward to reduce the \( D = 7 \) fermions. There is one important feature of the consistent reduction that needs mention, however. Given that the bosonic sector is that of a \((1,0)\) supergravity multiplet coupled to a \((1,0)\) tensor multiplet, one must identify two chiral spinors, \( \psi_{\mu i}^{(-)} \) and \( \lambda_i^{(+)} \), in six dimensions. However a straightforward reduction of \( \hat{\psi}_{M i} \) would suggest the presence of an additional unwanted spin-1/2 field \( \psi_{y i}^{(+)} \). The resolution of this puzzle is that both \( \hat{\lambda} \) and \( \psi_{y i} \) transform identically (up to factors), given the bosonic ansatz (24). Thus they may be consistently set equal to one another. As a result, we obtain the reduction ansatz on the fermions

\[
\begin{align*}
\hat{\psi}_{\mu i} &= W^{-\frac{1}{10}} e^{\frac{1}{2} \alpha \varphi} \left[ \psi_{\mu i}^{(-)} + \frac{1}{10} \gamma_{\mu} \lambda_i^{(+)} \right], \\
\hat{\psi}_{y i} &= -\frac{2}{5} W^{-\frac{1}{10}} e^{-\frac{1}{2} \alpha \varphi} \gamma^7 \lambda_i^{(+)}, \\
\hat{\lambda}_i &= \frac{2}{\sqrt{5}} W^{-\frac{1}{10}} e^{-\frac{1}{2} \alpha \varphi} \lambda_i^{(+)}, \\
\hat{\epsilon}_i &= W^{-\frac{1}{10}} e^{\frac{1}{2} \alpha \varphi} \epsilon_i^{(-)}.
\end{align*}
\]

Although \( \gamma^7 \) has a definite eigenvalue when acting on definite chirality spinors, we nevertheless retain it here and in the equations below to avoid ambiguity in our choice of sign.
conventions.

Substitution of this ansatz into the $D = 7$ fermion transformations, (28), yields the $D = 6$ transformations

$$
\delta \psi_{\mu i}^{(-)} = [\nabla_{\mu} - \frac{1}{18} e^{-\frac{1}{\sqrt{2}} \varphi} F_{\nu\rho\sigma} \gamma^{\nu\rho\sigma} \gamma_7] e_{i}^{(-)},
$$

$$
\delta \lambda_{i}^{(+)} = [-\frac{1}{2\sqrt{2}} \gamma^{\mu} \partial_{\mu} \varphi + \frac{1}{24} e^{-\frac{1}{\sqrt{2}} \varphi} F_{\mu\nu\rho} \gamma^{\mu\nu\rho} \gamma_7] e_{i}^{(-)},
$$

while substitution into the bosonic transformations, (29), yields

$$
\delta \varphi = -\frac{1}{2\sqrt{2}} \varepsilon^{(-)} \bar{\lambda}_{i}^{(+)} ,
$$

$$
\delta e_{\mu}^{(\alpha)} = \frac{1}{4} \varepsilon^{(-)} \gamma^{\alpha} \psi_{\mu i}^{(-)} ,
$$

$$
\delta B_{\mu\nu} = \frac{1}{2} e^{\frac{1}{\sqrt{2}} \varphi} (\varepsilon^{(-)} \gamma_{[\mu} \gamma_{\nu]} \psi_{i}^{(-)} + \frac{1}{2} \varepsilon^{(-)} \gamma_{\mu\nu} \gamma_7 \lambda_{i}^{(+)} ).
$$

Furthermore, consistency of the bosonic ansatz, (24), is maintained under supersymmetry, as we have verified that fields initially set to zero remain so under their variations. We see that these transformations are simply those of ungauged $N = (1, 0)$ supergravity\footnote{One also sees that these transformations match the appropriate truncation of the $N = (1, 1)$ transformations given below in Eqs. (42) and (43).} in six dimensions [16] with supergravity multiplet $(e_{\mu}^{(\alpha)} , \psi_{\mu i}^{(-)} , B_{\mu\nu}^{(+)} )$ and tensor multiplet $(B_{\mu\nu}^{(-)} , \lambda_{i}^{(+)} , \varphi )$.

Additionally, consistent reduction of the fermion equations of motion following from (26) results in six-dimensional equations of motion that may be derived from the Lagrangian

$$
L_{6}^{\text{fermion}} = -\frac{1}{2} \bar{\psi}_{\mu i}^{(-)} \gamma^{\mu\nu} \nabla_{\nu} \psi_{\mu i}^{(-)} - \frac{1}{8} \bar{\lambda}_{i} \gamma^{\mu} \nabla_{\mu} \lambda_{i} - \frac{1}{8} [\bar{\psi}_{\mu i}^{(-)} \gamma^{\mu\nu\rho\sigma} \gamma_7 \psi_{\nu i}^{(-)} - \bar{\psi}_{\mu i}^{(-)} \gamma^{\mu\nu\rho\sigma} \gamma_7 \psi_{\nu i}^{(-)}] e^{-\frac{1}{\sqrt{2}} \varphi} F_{\alpha\beta\gamma} 
$$

$$
- \frac{1}{8} [\bar{\psi}_{\mu i}^{(-)} \gamma^{\mu\alpha\beta\gamma} \gamma_7 \lambda_{i} - \bar{\psi}_{\mu i}^{(-)} \gamma^{\mu\alpha\beta\gamma} \gamma_7 \lambda_{i}] e^{-\frac{1}{\sqrt{2}} \varphi} F_{\alpha\beta\gamma} - \frac{1}{2\sqrt{2}} (\bar{\psi}_{\mu i}^{(-)} \gamma^{\alpha\beta\gamma} \lambda_{i}) \partial_{\alpha} \varphi + \frac{1}{48} (\bar{\lambda}_{i} \gamma^{\alpha\beta\gamma} \gamma_7 \lambda_{i}) e^{-\frac{1}{\sqrt{2}} \varphi} F_{\alpha\beta\gamma}.
$$

Along with the bosonic Lagrangian, (25), this reproduces the ungauged six-dimensional $(1, 0)$ model [16], up to four-fermion terms.

### 3.3 Lifting the dyonic string

The six-dimensional supergravity with tensor matter multiplet that we have obtained via brane-world reduction admits a BPS dyonic string solution. By reversing the steps of the brane-world reduction we can therefore obtain a BPS solution in the seven-dimensional gauged supergravity. Of course, this can be further lifted to ten-dimensions, since the seven-dimensional gauged supergravity arises from an $S^{3}$ Scherk-Schwarz reduction of type I supergravity.
The dyonic string solution in six dimensions is given by

\[ ds^2_6 = (H_e H_m)^{-\frac{1}{2}} (-dt^2 + dx^2) + (H_e H_m)^{\frac{1}{2}} (dr^2 + r^2 d\Omega_3^2), \]

\[ e^{\sqrt{2} \varphi} = \frac{H_m}{H_e}, \quad H_e = 1 + \frac{Q}{r^2}, \quad H_m = 1 + \frac{P}{r^2}, \quad (38) \]

\[ F_{(3)} = 2P \Omega_{(3)} - dt \wedge dx \wedge dH_e^{-1}, \]

where \( \Omega_{(3)} \) is the volume form of the 3-sphere in the transverse space. Lifting to \( D = 7 \), we therefore obtain the solution

\[ ds^2_7 = W^{-\frac{4}{5}} \left( H_e^{-\frac{8}{5}} H_m^{-\frac{4}{5}} (-dt^2 + dx^2) + H_e^{\frac{4}{5}} H_m^{\frac{8}{5}} (dr^2 + r^2 d\Omega_3^2) \right) + W^{-\frac{24}{25}} H_e^{-\frac{16}{25}} H_m^{\frac{12}{25}} dy^2, \]

\[ e^{-\phi} = \left( \frac{W H_e}{H_m} \right)^{\frac{1}{5}}, \quad (39) \]

\[ \hat{F}_{(4)} = W^{-2} (2P \Omega_{(3)} - dt \wedge dx \wedge dH_e^{-1}) \wedge dy. \]

This seven-dimensional solution can be interpreted as the intersection of a membrane and a string, living in the world-volume of a 5-brane (domain wall). The corresponding harmonic functions are \( H_e, H_m \) and \( W \) respectively. In the standard Randall-Sundrum I scenario, there are two distinct domain walls, with a separation \( L \) in the vacuum state. Turning on the radion mode will bend the walls locally, giving them a space-time position dependent separation. In this particular BPS solution corresponding to the six-dimensional dyonic string, the separation length is given by

\[ L \sim \left( \frac{H_m}{H_e} \right)^{\frac{1}{5}} = \left( \frac{r^2 + P}{r^2 + Q} \right)^{\frac{1}{5}}. \quad (40) \]

In the case \( P = Q \), corresponding to the self-dual string, the radion mode decouples and the separation is a constant. For \( P > Q \), the two domain walls are convex, being closest at large \( r \), with their greatest separation occurring at \( r = 0 \). In the limit \( Q \rightarrow 0 \), which is a purely magnetic string, the separation at \( r = 0 \) becomes infinite. Conversely, if \( P < Q \) the domain walls are concave, being closest at \( r = 0 \). The separation at \( r = 0 \) becomes zero in the limit of a purely electric string, when \( P \rightarrow 0 \). This demonstrates that a BPS configuration can connect the visible world and the hidden brane, in the Randall-Sundrum I scenario.

It is worth remarking that a scalar potential characterised by \( \Delta = -2 \), such as we have here, is capable of trapping gravity in a Randall-Sundrum II model [17, 18, 19].
4 Further Supersymmetric Examples

4.1 \( D = 6 \) reduced to \( D = 5, \mathcal{N} = 2 \) with vector multiplet

Our starting point is the six-dimensional gauged \( \mathcal{N} = (1, 1) \) supergravity. The bosonic fields in this theory comprise the metric, a dilaton \( \phi \), a 2-form potential \( A_2 \), and a 1-form potential \( B_1 \), together with the gauge potentials \( A_1 \) of \( SU(2) \) Yang-Mills. The theory can be obtained from a consistent local \( S^4 \) reduction of massive type IIA supergravity [20]. The bosonic Lagrangian [21], converted to the language of differential forms, is [20]

\[
\hat{L}_6 = \hat{R} \ast 1 - \frac{1}{2} \ast \hat{d} \hat{\phi} \wedge d \hat{\phi} - \frac{2}{9} g_2^2 X^{-6} + \frac{8}{9} g_1 g_2 X^{-2} + 2 g_1^2 X^2 \ast 1
\]

\[
- \frac{1}{2} X^4 \hat{F}_{(3)} \wedge \hat{F}_{(3)} - \frac{1}{2} X^{-2} \left( \hat{\ast} \hat{G}_{(2)} \wedge \hat{G}_{(2)} + \hat{\ast} \hat{F}_{(2)} \wedge \hat{F}_{(2)} \right) \tag{41}
\]

\[
- \hat{A}_{(2)} \wedge (\frac{1}{2} \hat{d} \hat{B}(1) \wedge \hat{d} \hat{B}(1) + \frac{1}{2} g_2 \hat{A}_{(2)} \wedge d \hat{B}_1 + \frac{2}{27} g_2^2 \hat{A}_{(2)} \wedge \hat{A}_{(2)} + \frac{1}{2} \hat{F}_{(2)} \wedge \hat{F}_{(2)}),
\]

where \( X \equiv e^{-\hat{\phi}/(2\sqrt{2})} \), \( \hat{F}_{(3)} = d \hat{A}_{(2)} \), \( \hat{G}_{(2)} = d \hat{B}_1 + \frac{2}{3} g_2 \hat{A}_{(2)} \), \( \hat{F}_{(2)} = d \hat{A}_{(1)} + \frac{1}{2} g_1 \epsilon_{ijk} \hat{A}_{(1)} \wedge \hat{A}_{(1)} \), and here \( \ast \) denotes the six-dimensional Hodge dual. (We have rescaled fields and coupling constants relative to the expression in [20], to make explicit the gauge coupling \( g_1 \) and the mass parameter \( g_2 \).

In addition, the fermions comprise a symplectic-Majorana gravitino \( \hat{\psi}_{M i} \) and spinor \( \hat{\lambda}_i \). The \((1, 1)\) supersymmetry transformations are

\[
\delta \hat{\psi}_{M i} = [\hat{D}_M - \frac{1}{32} X^2 \hat{F}_{ABC} \gamma^{ABC} \gamma^7 - \frac{1}{4\sqrt{2}} (g_1 X + \frac{1}{2} g_2 X^{-3}) \hat{\gamma}_M] \hat{\epsilon}_i
\]

\[
- \frac{1}{16 \sqrt{2}} (\hat{\gamma}_M \hat{A}_B - 6 \hat{\delta} \hat{A}_M \hat{\gamma}_B) X^{-1} (\hat{G}_{AB} \hat{\delta}_i^j + \frac{i}{2} \hat{\gamma}_i \hat{F}_{AB} \hat{\gamma}^j \hat{\epsilon}_j),
\]

\[
\delta \hat{\lambda}_i = \left[-\frac{1}{2 \sqrt{2}} \sigma^M \partial_M \hat{\phi} + \frac{1}{2 \sqrt{2}} X^2 \hat{F}_{MNP} \hat{\gamma}^{MNP} \hat{\gamma}^7 + \frac{1}{2 \sqrt{2}} (g_1 X - g_2 X^{-3}) \hat{\epsilon}_i \right]
\]

\[
+ \frac{1}{8 \sqrt{2}} X^{-1} \left( \hat{G}_{MN} \hat{\delta}_i^j + \frac{i}{2} \hat{\gamma}_i \hat{F}_{MN} \hat{\gamma}^j \hat{\epsilon}_j \right),
\]

for the fermions, and

\[
\delta \hat{e}_M^A = \frac{1}{4} \hat{e}^A \gamma^M \hat{\psi}_{M i},
\]

\[
\delta \hat{\phi} = - \frac{1}{2 \sqrt{2}} \hat{\gamma}_i \hat{\lambda}_i,
\]

\[
\delta \hat{B}_M = \frac{1}{2 \sqrt{2}} X (\hat{e}^A \gamma^M \hat{\psi}_{M i} - \frac{1}{2} \hat{e}^A \hat{\gamma}_M \hat{\gamma}^7 \hat{\lambda}_i),
\]

\[
\delta \hat{A}_{MN} = \frac{1}{2} \hat{X}^{-2} (\hat{e}^A \hat{\gamma}_{[M} \hat{\gamma}^7 \hat{\psi}_{N]}, + \frac{1}{2} \hat{e}^A \hat{\gamma}_{MN} \hat{\gamma}^7 \hat{\lambda}_i),
\]

\[
\delta \hat{A}_{M i j} = \frac{i}{\sqrt{2}} X \left( \frac{1}{2} \hat{e}^j \hat{\psi}_{M i} + \frac{1}{2} \hat{e}^j \hat{\gamma}_M \hat{\lambda}_i \right) - \frac{1}{2} \hat{\delta}_i^j \left( \hat{e}^k \hat{\psi}_{M k} + \frac{1}{2} \hat{e}^k \hat{\gamma}_M \hat{\lambda}_k \right)
\]

for the bosons. Our convention for symplectic-Majorana spinors parallels that given above for the \( D = 7 \) case. In particular, the Majorana flip relation is identical to (31). Additionally, we follow the same normalization as (30). Here, as in seven dimensions, the gauge
covariant derivative acting on an $Sp(1)$ spinor is defined as $\hat{D}_M \hat{\psi}_i = \hat{\nabla}_M \hat{\psi}_i + \frac{i}{2} g_1 \hat{A}_M \hat{\psi}_i$, where $\hat{A}_M \hat{\psi}_i \equiv \hat{A}^k_M (-\sigma^k)_j$, so that $\hat{F}_{MN} \hat{\psi}_i = \partial_M \hat{A}_N \hat{\psi}_i + \frac{1}{2} g \hat{A}_M \hat{A}_N \hat{\psi}_i - (m \leftrightarrow n)$.

For our brane-world reduction, we shall take the mass parameter $g_2$ to zero, and relabel $g_1$ as $g$, giving

$$\hat{\mathcal{L}}_0 = \hat{R} \hat{\Phi} - \frac{1}{2} e \hat{d} \hat{\phi} \wedge e \hat{d} \hat{\phi} + 2 g X^2 \hat{d} \hat{\phi}$$
$$- \frac{1}{2} X^4 \hat{d} \hat{F}(3) \wedge \hat{F}(3) - \frac{1}{2} X^{-2} \left( \hat{d} \hat{G}(2) \wedge \hat{G}(2) + \frac{1}{2} \hat{d} \hat{F}(2) \wedge \hat{F}(2) \right)$$

(44)

with $\hat{G}(2) = d\hat{B}(1), \hat{F}(2) = d\hat{A}(1) + \frac{1}{2} g \epsilon_{ijk} \hat{\lambda}(1) \wedge \hat{A}^k(1)$. From section 2, we are then led to make the reduction ansatz

$$d\hat{s}_6^2 = W^{-\frac{1}{2}} e^{2\alpha \phi} d\hat{s}_5^2 + W^{-\frac{5}{2}} e^{-6\alpha \phi} dy^2$$
$$e^{-\frac{1}{2} \alpha \phi} = W^\frac{1}{2} e^{6\alpha \phi}$$
$$\hat{\lambda}(2) = W^{-2} A_1 \wedge dy, \quad \hat{B}_1 = B_1, \quad \hat{A}_i(1) = 0,$$

(45)

with $\alpha = -1/(4\sqrt{3})$ and $W = 1 + \sqrt{2} y$. Substituting into the equations of motion that follow from (44), we find consistent five-dimensional equations of motion that can be derived from the Lagrangian

$$\mathcal{L}_5 = R \hat{\Phi} - \frac{1}{2} e \hat{d} \hat{\phi} \wedge e \hat{d} \hat{\phi} - \frac{1}{2} e^{\frac{1}{2} \alpha \phi} \hat{F}(2) \wedge \hat{F}(2) - \frac{1}{2} e^{\frac{5}{2} \alpha \phi} \hat{G}(2) \wedge \hat{G}(2) - \frac{1}{2} e^{\frac{1}{2} \alpha \phi} \hat{d} \hat{A}(1) \wedge d\hat{B}(1) \wedge d\hat{B}(1),$$

(46)

where $F(2) = dA(1)$ and $G_2 = dB(1)$. This is the bosonic sector of ungauged $\mathcal{N} = 2$ five-dimensional supergravity, coupled to a vector multiplet whose bosonic fields are $\phi$ and $B_1$.

This identification of the resulting theory with five-dimensional $\mathcal{N} = 2$ supergravity may be confirmed by reducing the fermionic sector. Applying the procedure outlined in section 3 to the present fermions, we find the appropriate reduction to be

$$\hat{\psi}_{\mu i} = W^{-\frac{1}{2}} e^{\frac{1}{2} \alpha \phi} \left[ \psi_{\mu}^{(-)} + \frac{1}{2\sqrt{6}} \gamma_{\mu} \lambda^{(+)} \right],$$
$$\hat{\psi}_{y i} = -\frac{3}{2\sqrt{6}} W^{-\frac{1}{2}} e^{-\frac{1}{2} \alpha \phi} \gamma^6 \lambda^{(+)}$$
$$\hat{\lambda}_i = \frac{3}{\sqrt{6}} W^\frac{1}{2} e^{-\frac{1}{2} \alpha \phi} \lambda^{(+)}$$
$$\hat{e}_i = W^{-\frac{1}{2}} e^{\frac{1}{2} \alpha \phi} e^{(-)},$$

(47)

where the ‘chirality’ is determined by the projection $P^{(\pm)} = \frac{1}{2} (1 \pm \gamma^6)$. For the moment, we retain the six-dimensional form of the spinors and Dirac matrices. The reduction of the fermion transformations, (42) yields

$$\delta \psi_{\mu i}^{(-)} = \left[ \nabla_{\mu} - \frac{1}{2\sqrt{6}} (\gamma_{\mu} \nu \lambda - 4 \delta^{\nu}_{\mu} \gamma^\lambda) (\sqrt{2} e^{\frac{1}{2} \alpha \phi} G_{\nu \lambda} - e^{-\frac{1}{2} \alpha \phi} F_{\nu \lambda} \gamma^6) \gamma^7 \right] \epsilon_i,$$
$$\delta \lambda^{(+)}_i = \left[ -\frac{1}{2} \gamma^{\mu} \partial_{\mu} \phi + \frac{1}{8\sqrt{3}} \left( e^{\frac{1}{2} \alpha \phi} G_{\mu \nu} + \sqrt{2} e^{-\frac{1}{2} \alpha \phi} F_{\mu \nu} \gamma^6 \right) \gamma^{\mu \nu} \gamma^7 \right] \epsilon_i,$$

(48)
while the reduction of the boson transformations, (43) yields

\[
\begin{align*}
\delta e_\mu^\alpha &= \frac{1}{4} \bar{e}^\lambda \gamma^\alpha \psi_\mu^i, \\
\delta \phi &= -\frac{1}{2} \bar{e} \lambda_i, \\
\delta A_\mu &= \frac{1}{2} e \sqrt{6} \phi \left( \frac{2}{\sqrt{6}} \bar{e}^i \gamma_\mu \gamma^6 \gamma^7 \lambda_i - \frac{1}{2} \bar{e}^i \gamma^6 \gamma^7 \psi_\mu^i \right), \\
\delta B_\mu &= -\frac{1}{\sqrt{2}} e^{-\sqrt{3} \phi} \left( \frac{1}{\sqrt{6}} \bar{e}^i \gamma_\mu \gamma^7 \lambda_i - \frac{1}{2} \bar{e}^i \gamma^7 \psi_\mu^i \right). 
\end{align*}
\]

(49)

Again, we find that this reduction is consistent.

Finally, to make connection to the $D = 5$, $N = 2$ supergravity, we rewrite the fermions in terms of natural five-dimensional spinors. To do so, we first note that the expressions in (48) and (49) are trivial under $Sp(1)$, and hence the symplectic-Majorana spinors may be combined into ordinary six-dimensional Dirac spinors. Then we may choose a decomposition of the Dirac matrices as, e.g.

\[
\begin{align*}
\gamma_\mu &= \tilde{\gamma}_\mu \times \sigma^1, \quad \mu = 0, 1, \ldots, 4, \\
\gamma^6 &= 1 \times \sigma^3, \\
\gamma^7 &= \gamma_{01234} \times \sigma^2.
\end{align*}
\]

(50)

Here, $\tilde{\gamma}_\mu$ are a set of $4 \times 4$ Dirac matrices for $D = 5$. Note that ‘chiral’ spinors under $P(\pm)$ may be written as

\[
\chi^{(+)} = \begin{pmatrix} \chi \\ 0 \end{pmatrix}, \quad \chi^{(-)} = \begin{pmatrix} 0 \\ \chi \end{pmatrix}.
\]

(51)

Each five-dimensional Dirac matrix flips $P(\pm)$ chirality, and furthermore, we have $\gamma_7 \chi^{(\pm)} = \pm i \chi^{(\mp)}$. As a result, Eqs. (48) and (49) take on the five-dimensional form

\[
\begin{align*}
\delta \psi_\mu &= \left[ \nabla_\mu + \frac{i}{24} \left( \gamma_\mu \gamma^\lambda \gamma_\lambda - 4 \delta_\mu^\nu \gamma^\lambda \right) \left( \sqrt{2} e \sqrt{6} \phi \tilde{G}_{\mu \lambda} - e^{-\sqrt{3} \phi} F_{\mu \lambda} \right) \right] e, \\
\delta \lambda &= -\frac{1}{4} \gamma_\mu \partial_\mu \phi - \frac{i}{8 \sqrt{3}} \left( e \sqrt{6} \phi \tilde{G}_\mu \gamma^\nu + \sqrt{2} e^{-\sqrt{3} \phi} F_\mu \gamma^\nu \right) \gamma_\nu e, \\
\delta e_\mu^\alpha &= \frac{1}{4} \bar{e} \gamma_\mu \psi_\mu^i, \\
\delta \phi &= -\frac{1}{2} \bar{e} \lambda, \\
\delta A_\mu &= -\frac{1}{2} e \sqrt{6} \phi \left( \frac{2}{\sqrt{6}} \bar{e}^i \gamma_\mu \lambda - \frac{1}{2} \bar{e} \psi_\mu \right), \\
\delta B_\mu &= -\frac{1}{2} e^{-\sqrt{3} \phi} \left( \frac{1}{\sqrt{6}} \bar{e}^i \gamma_\mu \lambda - \frac{1}{2} \bar{e} \psi_\mu \right). 
\end{align*}
\]

(52)

where we have now dropped the tildes on the five-dimensional Dirac matrices. After transforming to a Dirac normalization and taking $\lambda \rightarrow -i \lambda$, these transformations agree with those of $\mathcal{N} = 2$ supergravity coupled to a single vector multiplet, as given below in (63).
One can construct BPS solutions (strings or black holes) in five dimensions, and then lift them back to \( D = 6 \). Again, the bending of the domain walls will be convex or concave according to the relative sizes of the two charges carried by \( F_{(2)} \) and \( G_{(2)} \).

### 4.2 \( D = 5 \) reduced to \( D = 4 \), \( \mathcal{N} = 1 \) with a chiral multiplet

In this example, we shall take as the starting point the five-dimensional \( \mathcal{N} = 2 \) gauged supergravity coupled to a vector multiplet. In general, this \( \mathcal{N} = 2 \) theory is described by very special geometry \([22, 23]\), and we begin with a brief outline of some relevant facts.

For the coupling of supergravity to \( n \) vector multiplets, in addition to the graviton \( \hat{g}_{MN} \) and gravitino \( \hat{\psi}_M \), one introduces \( n + 1 \) vector potentials \( \hat{A}_I^{(1)} \), as well as \( n \) scalars \( \hat{\phi}^i \) and gauginos \( \hat{\lambda}^i \). Of the \( n + 1 \) vectors, the \( \mathcal{N} = 2 \) graviphoton is given by the linear combination

\[
\hat{A}_I^{(1)} = V_I A_I^{(1)},
\]

where the \( V_I \) are a set of constants related to the gauging.

The bosonic Lagrangian takes the form

\[
\hat{L}_5 = \hat{R} \hat{\epsilon} - G_{IJ} \hat{\psi}^I \wedge \hat{\psi}^J + 2g^2 V \hat{\epsilon} - G_{IJ} \hat{F}^I_{(2)} \wedge \hat{F}^J_{(2)} - \frac{1}{6} C_{IJK} \hat{A}_I^{(1)} \wedge d\hat{B}_J^{(1)} \wedge d\hat{B}_K^{(1)},
\]

where the constants \( C_{IJK} \) specify a homogeneous cubic polynomial

\[
\mathcal{V} = \frac{1}{6} C_{IJK} X^I X^J X^K.
\]

Here, the \( n + 1 \) quantities \( X^I \) are functions of the \( n \) scalar fields \( \hat{\phi}^i \), and are required to satisfy the condition \( \mathcal{V} = 1 \). The quantities \( G_{IJ} \) and \( G_{ij} \) in the Lagrangian are given by

\[
G_{IJ} = -\frac{1}{2} \partial_I \partial_J \log \mathcal{V} \bigg|_{\mathcal{V} = 1},
\]

\[
G_{ij} = \partial_i X^I \partial_j X^J G_{IJ} \bigg|_{\mathcal{V} = 1},
\]

and the potential, which arises from the gauging has the form

\[
V = V_I V_J (6X^I X^J - \frac{9}{2} G^{ij} \partial_i X^I \partial_j X^J).
\]

For a more complete treatment, see \([22, 23]\).

In terms of a single five-dimensional Dirac spinor, the \( \mathcal{N} = 2 \) supersymmetry transformations are

\[
\delta \hat{\psi}_M = [\hat{D}_M + \frac{i}{3} (\gamma_M N_P - 4\delta_M N_P) X_I \hat{F}^I_{NP} + \frac{1}{2} g \gamma_M X^I V_I] \hat{\epsilon},
\]

\[
\delta \hat{\lambda}_i = \partial_i X^I [-\frac{1}{4} G_{IJ} \gamma^{MN} \hat{F}^I_{MN} + \frac{3i}{4} \gamma^M \partial_M X_I + \frac{3i}{2} g V_I] \hat{\epsilon},
\]

\[
\delta \hat{e}_M^A = \frac{1}{2} \hat{\gamma}^A \hat{\psi}_M ,
\]

\[
\delta \hat{A}_M^I = \frac{i}{2} \hat{g}^{ij} \partial_i X^I \hat{\gamma}_M \hat{\lambda} - \frac{i}{2} X^I \hat{\gamma} \hat{\psi}_M ,
\]

\[
\delta \hat{\phi}^i = \frac{i}{2} \hat{g}^{ij} \hat{\epsilon} \hat{\lambda}_j .
\]
The supersymmetry transformation parameter $\hat{\epsilon}$ is normalized according to

$$[\delta_1, \delta_2] \Xi = \frac{1}{2} (\hat{\epsilon}_2 \gamma^M \hat{\epsilon}_1) \partial_M \Xi + \cdots ,$$

and the gauge covariant derivative acting on a charged spinor is given by

$$\hat{D}_M = \hat{\nabla}_M - \frac{3i}{2} g \hat{A}_M = \hat{\nabla}_M - \frac{3i}{2} g V_I \hat{A}^I_M .$$

With these preliminaries out of the way, we now focus on the model on hand, namely gauged supergravity coupled to a single vector multiplet (i.e. $n = 1$). This model is obtained by taking $C_{112} = C_{121} = C_{211} = 1$, and by specifying the gauging according to

$$g V_1 = \frac{\sqrt{2}}{6} g_1 , \quad g V_2 = \frac{1}{6} g_2 .$$

A convenient scalar parametrization preserving $V = 1$ is then

$$X^1 = \sqrt{2} e^{-\frac{4}{\sqrt{6}} \hat{\phi}} , \quad X^2 = e^{\frac{4}{\sqrt{6}} \hat{\phi}} .$$

We furthermore define $\hat{F}^1_{(2)} = \hat{G}_{(2)} = d\hat{B}_{(1)}$, and $\hat{F}^2_{(2)} = \hat{F}_{(2)} = d\hat{A}_{(1)}$. The Lagrangian of the bosonic sector is then given by

$$\hat{\mathcal{L}}_5 = \hat{R} \hat{1} - \frac{1}{2} \hat{d}\hat{\phi} \wedge \hat{d}\hat{\phi} + (2 g_1 g_2 e^{\frac{4}{\sqrt{6}} \hat{\phi}} + g_2^2 e^{-\frac{4}{\sqrt{6}} \hat{\phi}}) \hat{1}
- \frac{1}{2} e^{-\frac{4}{\sqrt{6}} \hat{\phi}} \hat{F}_{(2)} \wedge \hat{F}_{(2)} - \frac{1}{2} e^{\frac{4}{\sqrt{6}} \hat{\phi}} \hat{G}_{(2)} \wedge \hat{G}_{(2)} + \frac{1}{2} \hat{A}_{(1)} \wedge d\hat{B}_{(1)} \wedge d\hat{B}_{(1)} ,$$

with supersymmetry transformations

$$\delta \hat{\psi}_M = [\hat{D}_M + \frac{1}{4 \sqrt{3}} (\hat{\gamma}_M^{NP} - 4 \delta M^{N} \hat{\gamma}^P)] (\sqrt{2} e^{\frac{4}{\sqrt{6}} \hat{\phi}} \hat{G}_{NP} + e^{-\frac{4}{\sqrt{6}} \hat{\phi}} \hat{F}_{NP})
+ \frac{12}{12} (2 g_1 e^{-\frac{4}{\sqrt{6}} \hat{\phi}} + g_2 e^{\frac{4}{\sqrt{6}} \hat{\phi}}) \hat{\epsilon} ,$$
$$\delta \hat{\lambda} = [-\frac{i}{4} \hat{\gamma}^M \partial_M \hat{\phi} + \frac{1}{8 \sqrt{3}} \hat{\gamma}^{MN} (e^{\frac{4}{\sqrt{6}} \hat{\phi}} \hat{G}_{MN} - \sqrt{2} e^{-\frac{4}{\sqrt{6}} \hat{\phi}} \hat{F}_{MN})]
- \frac{i}{2 \sqrt{6} \sqrt{6}} (g_1 e^{-\frac{4}{\sqrt{6}} \hat{\phi}} - g_2 e^{\frac{4}{\sqrt{6}} \hat{\phi}}) \hat{\epsilon} ,$$
$$\delta \hat{e}_M^A = \frac{1}{2} \hat{\gamma}^A \hat{\psi}_M ,$$
$$\delta \hat{\phi} = \sqrt{6} \hat{\lambda} ,$$
$$\delta \hat{A}_M = e^{\frac{4}{\sqrt{6}} \hat{\phi}} (\frac{2}{\sqrt{6}} \hat{\psi}_M \hat{\lambda} - \frac{i}{2} \hat{\psi}_M) ,$$
$$\delta \hat{B}_M = \sqrt{2} e^{-\frac{4}{\sqrt{6}} \hat{\phi}} (\frac{1}{\sqrt{6}} \hat{\psi}_M \hat{\lambda} - \frac{i}{2} \hat{\psi}_M) .$$

The explicit embedding in eleven dimensional supergravity of this AdS$_5$ supergravity coupled to a vector multiplet was given in [24].

For our present purposes, we shall set $g_2 = 0$ and relabel $g_1$ as $g$. The resulting theory has a domain wall as its vacuum solution instead of AdS$_5$. As shown in appendix B, this
domain wall supergravity can be obtained from the $S^2$ reduction of the seven-dimensional
domain-wall supergravity (23) discussed in section 3. From the results in section 2, we are
led to make the following Kaluza-Klein reduction ansatz:

\[
\begin{align*}
\hat{s}^2_5 & = W^{-\frac{2}{3}} e^{2\alpha \varphi} ds^2_6 + W^{-\frac{2}{3}} e^{-4\alpha \varphi} dy^2, \\
e^{-\frac{2}{\sqrt{6}} \hat{\phi}} & = W^{\frac{2}{3}} e^{4\alpha \varphi}, \\
\hat{A}_1 & = \chi W^{-2} dy, \\
\hat{B}_1 & = 0, \\
\end{align*}
\]

where $\alpha = \frac{1}{6}$ and $m^2 = g^2$ (so that $W = 1 + gy$). Substituting into the equations of
motion following from (62) (with $g_1 = g$, $g_2 = 0$), we obtain a consistent reduction of the
bosonic fields to a four-dimensional system whose equations of motion can be derived from
the Lagrangian

\[
L_4 = R \ast 1 - \frac{1}{2} d\varphi \wedge d\varphi - \frac{1}{2} e^{2\varphi} \ast d\chi \wedge d\chi.
\]

This is precisely the bosonic sector of four-dimensional $\mathcal{N} = 1$ supergravity coupled to a
chiral scalar multiplet.

The fermionic reduction may similarly be obtained following the method developed in
section 3. Defining here the four-dimensional chirality projection $P^{(\pm)} = \frac{1}{2} (1 \pm \gamma^5)$, the
appropriate reduction on the fermions is given by

\[
\begin{align*}
\hat{\psi}_\mu & = W^{-1} e^{\frac{1}{6} \varphi} [\psi^{(+)}_\mu - \frac{i}{6} \gamma_\mu \lambda^{(-)}], \\
\hat{\psi}_y & = \frac{i}{3} W^{-\frac{2}{3}} e^{-\frac{2}{3} \varphi} \gamma^5 \lambda^{(-)}, \\
\hat{\lambda} & = \frac{1}{\sqrt{6}} W e^{\frac{2}{3} \varphi} \lambda^{(-)}, \\
\hat{\epsilon} & = W^{-\frac{2}{3}} e^{-\frac{2}{3} \varphi} \epsilon^{(+)}.
\end{align*}
\]

The reduction of the five-dimensional supersymmetry transformations, (63), yields

\[
\delta \psi^{(+)}_\mu = [\nabla_\mu - \frac{1}{2} e^{\varphi} \partial_\mu \chi \gamma^5] \epsilon^{(+)} + \delta e^{\mu}_\mu = \frac{1}{2} \epsilon^{(+)} \gamma^\alpha \psi^{(+)}_\mu,
\]

for the four-dimensional supergravity multiplet, and

\[
\begin{align*}
\delta \lambda^{(-)} & = \frac{1}{2} \gamma^\mu [i \partial_\mu \varphi - e^{\varphi} \partial_\mu \chi \gamma^5] \epsilon^{(+)}, \\
\delta \varphi & = -\frac{1}{2} \epsilon^{(+)} \chi^{(-)}, \\
\delta \chi & = \frac{1}{2} e^{-\varphi} \epsilon^{(+)} \gamma^5 \chi^{(-)}.
\end{align*}
\]

for the matter fields. Again, we have verified that this is a consistent reduction on the
fermions. This confirms our identification of the reduced theory as $\mathcal{N} = 1$ supergravity cou-
pled to a chiral multiplet. Standard techniques may be used to rewrite the four-dimensional
Weyl spinors in terms of Majorana ones.
5 Extended Bosonic Examples

In this section, we show that it is possible to enlarge the bosonic ansätze of the previous consistent reductions in sections 3 and 4, to obtain bosonic reductions that yield larger numbers of fields from the same higher-dimensional starting points. Although these larger bosonic reductions are still fully consistent, we find that they are no longer the bosonic sectors of supergravities in the lower dimensions. In other words, although the bosonic sectors of the higher-dimensional supergravities admit enlarged consistent reductions that retain more lower-dimensional fields, it is not possible to make corresponding enlarged consistent reductions in the fermionic sectors. Nonetheless, the fact that the bosonic sectors admit enlarged consistent reductions is of interest in its own right.

5.1 $D = 7$ reduced with $SU(2)$ Yang-Mills in $D = 6$

In this enlarged consistent reduction, we begin with the same seven-dimensional Lagrangian (23) that we used in section 3, but we now include the $SU(2)$ Yang-Mills fields as well, which were previously set to zero in (24). From the results in section 2, we see that a consistent reduction for the $\hat{A}^i_{(1)}$ potentials should be possible, with the reduction ansatz given by

$$\hat{A}_i = A_i.$$  \hfill (69)

It is easy to verify that within the bosonic sector, the entire reduction, with ansatz given by (24) except that $\hat{A}^i_{(1)} = A^i_{(1)}$ instead of $\hat{A}^i_{(1)} = 0$, is consistent, and the resulting bosonic Lagrangian in $D = 6$ is given by

$$\mathcal{L}_6 = R \ast 1 - \frac{1}{2} d\varphi \wedge d\varphi - \frac{1}{2} e^{-\sqrt{2}\varphi} \ast F_3 \wedge F_3 - \frac{1}{2} e^{\sqrt{2}\varphi} \ast F_3 \wedge F_3 + \frac{1}{2} F^i_{(2)} \wedge F^i_{(2)} + \frac{1}{2} F^i_{(2)} \wedge A^i_{(2)},$$  \hfill (70)

where $F_{(3)} = dA_{(2)}$ and $F^i_{(2)} = dA^i_{(1)} + \frac{1}{2} g \epsilon_{ijk} A^j_{(1)} \wedge A^k_{(1)}$.

We have obtained a lower-dimensional theory that includes $SU(2)$ Yang-Mills fields, but where nevertheless there is no scalar potential. This result is rather surprising. The $D = 7$ gauged supergravity itself can be obtained from $N = 1, D = 10$ supergravity, which is ungauged, by reduction on $S^3$. Thus by combining a standard Scherk-Schwarz group manifold reduction with an additional stage of brane-world reduction, we can obtain a theory with non-abelian Yang-Mills fields coming from the geometry of the internal space, and yet this lower-dimensional theory has no cosmological term.

At first sight the Lagrangian (70) appears to be precisely the bosonic sector of $D = 6 (1, 0)$ supergravity coupled to a tensor multiplet together with an $SU(2)$ adjoint vector multiplet. However, this is in fact not the case. One way to see this is to note that, while

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the lower-dimensional theory has no potential, the original $D = 7$ gauged supergravity has gravitini charged under this same $SU(2)$ (which is in fact the $Sp(1)$ symmetry of the symplectic Majorana spinors). Dimensional reduction along the lines of (69) can never remove the charge of the gravitini. Thus, were the resulting theory to be supersymmetric, one would end up with a $D = 6$, $\mathcal{N} = (1, 0)$ gauged supergravity without a scalar potential. The fact that there are no known theories of this form immediately provides a hint that the reduction cannot be supersymmetric.

Of course, one may entertain the possibility that such a class of gauged (or ‘partially gauged’) supergravities might in fact exist. Thus it is worth examining the supersymmetry of the extended reduction in some detail. Since the domain wall vacuum preserves $\mathcal{N} = (1, 0)$, we are still concerned with only a chiral supersymmetry, parameterized by $\epsilon_i^{(-)}$. However, now additional chiralities show up in the fermion reduction. Relaxing (34) to include both chiralities of the $D = 6$ spinors (but retaining the connection between $\hat{\psi}_{y i}$ and $\hat{\lambda}_i$, which is still consistent), we obtain, in addition to (35) with $\nabla_{\mu}$ replaced by $D_{\mu}$ and (36), the transformations

$$
\delta \psi^{(+)}_{\mu i} = \left[ \frac{i}{16} e^{\frac{3}{2} \Phi} (\gamma_{\mu}^{\alpha \beta} - 6 \delta_{\mu}^{\alpha} \gamma^{\beta}) F_{\alpha \beta i j} \right] \epsilon_j^{(-)};
\delta \lambda_{-}^{(-)} = \left[ -\frac{i}{8} e^{\frac{3}{2} \Phi} F_{\mu \nu \lambda}^{\lambda \mu \nu} \right] \epsilon_j^{(-)};
\delta A_{\mu i j} = \left[ \frac{1}{2} e^{-\frac{3}{2} \Phi} \left( \psi_{\mu i}^{(+)} \epsilon_j^{(-)} - \frac{1}{2} \lambda_{-}^{(-)} \gamma_{\mu} \epsilon_j^{(-)} \right) - \frac{1}{2} \delta_j^{\nu}(\psi_{\mu}^{(+)} \epsilon_k^{(-)} - \frac{1}{2} \lambda_{-}^{(-)} \gamma_{\mu} \epsilon_k^{(-)}) \right].
$$

(71)

At first, this is quite intriguing, as this suggests that the $SU(2)$ vectors are in fact grouped into a spin-$\frac{3}{2}$ multiplet. Essentially, since the gauge fields were superpartners of the gravitini in $D = 7$, they remain superpartners of spin-$\frac{3}{2}$ matter in the reduced $D = 6$ theory. However, this identification presupposes the existence of an abelian vector which is lacking in the reduction, as the spin-$\frac{3}{2}$ multiplet consists of the fields $(\psi_{\mu i}^{(+)}, A_{\mu i j}, A_{\mu}, \lambda_{-}^{(-)})$.

Another way to see that this reduction cannot be supersymmetric is to note that the presence of $\psi_{\mu i}^{(+)}$ and $\lambda_{-}^{(-)}$ yields the $D = 7$ transformations

$$
\delta \hat{g}_{\mu y} = \frac{1}{4} W^{-\frac{7}{8}} e^{\frac{3}{2} \Phi} e^{c^{(-)}(\gamma_{\mu}^{\alpha \beta} + \frac{1}{2} \gamma_{\mu} \lambda_{-}^{(-)})},
\delta \hat{A}_{\mu \nu \rho} = -\frac{3}{2} W^{-1} e^{\frac{3}{2} \Phi} e^{c^{(-)}(\gamma_{\mu \nu} \psi_{\mu i}^{(+)} - \frac{1}{6} \gamma_{\mu \nu \rho} \lambda_{-}^{(-)})}.
$$

(72)

So although both $\hat{g}_{\mu y}$ and $\hat{A}_{\mu \nu \rho}$ are set to zero in the reduction ansatz (24), this structure cannot be maintained under supersymmetry transformations in the presence of these additional fields. Consistency of the reduction under supersymmetry then requires the vanishing of $A_{\mu i j}$, $\psi_{\mu i}^{(+)}$, and $\lambda_{-}^{(-)}$. Nevertheless, the form of (72) suggests a possible further generalization of the reduction ansatz to include a vector arising from an appropriate linear
combination of the dual of $\hat{A}_{\mu\nu\rho}$ and an off-diagonal metric component $\hat{g}_{\mu y}$. This possibility is currently under investigation.

5.2 $D = 6$ reduced with $SU(2)$ Yang-Mills in $D = 5$

Analogously, the $SU(2)$ Yang-Mills fields in the $D = 6$ Lagrangian (44) can also be consistently reduced. The complete reduction ansatz is given by (45), except that now $\hat{A}^i_{(1)} = A^i_{(1)}$, instead of being set to zero. The resulting Lagrangian in $D = 5$ is given by

$$L_5 = R \ast 1 - \frac{1}{8} d\varphi \wedge d\varphi - \frac{1}{2} e^{-2\varphi} \ast F_{(2)} \wedge F_{(2)} - \frac{1}{2} e^{-2\varphi} \left( \ast G_{(2)} \wedge G_{(2)} + \ast F^i_{(2)} \wedge F^i_{(2)} \right)$$

$$- \frac{1}{2} A_{(1)} \wedge \left( dB_{(1)} \wedge dB_{(1)} + F^i_{(2)} \wedge F^i_{(2)} \right),$$

(73)

where $F_{(2)} = dA_{(1)}$, $G_{(2)} = dB_{(1)}$ and $F^i_{(2)} = dA^i_{(1)} + \frac{1}{2} g \epsilon_{ijk} A^j_{(1)} \wedge A^k_{(1)}$.

5.3 $D = 5$ reduced with an additional vector

The vector $B_{(1)}$ in section 4.2 can also be included in a consistent reduction, with the ansatz given by (64) except that $\hat{B}_{(1)} = B_{(1)}$. The resulting Lagrangian is now given by

$$L_4 = R \ast 1 - \frac{1}{8} d\varphi \wedge d\varphi - \frac{1}{2} e^{2\varphi} \ast d\chi \wedge d\chi - \frac{1}{2} e^{-\varphi} \ast G_{(2)} \wedge G_{(2)} + \frac{1}{2} \chi G_{(2)} \wedge G_{(2)},$$

(74)

where $G_{(2)} = dB_{(1)}$.

At first sight, this is exactly the bosonic Lagrangian of $\mathcal{N} = 1$ supergravity coupled to a vector and a chiral multiplet. On the other hand, this runs into a similar difficulty with supersymmetry as found above in section 5.1. Were supersymmetry to be valid, somehow, one again runs into a problem identifying the superpartner to the vector as either spin-$\frac{3}{2}$ or spin-$\frac{5}{2}$. In this case, the generalization of (66) yields the additional transformations

$$\delta \psi^{(-)}_{\mu} = \frac{i}{8\sqrt{2}} e^{-\frac{1}{2} \varphi} G_{\alpha\beta} \gamma^{\alpha\beta} \gamma_{\mu} \epsilon^{(+)}$$

$$\delta \lambda^{(+)} = \frac{1}{8\sqrt{3}} e^{-\frac{1}{2} \varphi} G_{\alpha\beta} \gamma^{\alpha\beta} \epsilon^{(+)},$$

(75)

which is suggestive of both spin-$\frac{3}{2}$ and spin-$\frac{5}{2}$ simultaneously. Note here that the possibility of consistently setting $\lambda^{(+)}$ to be proportional to $\gamma^{\mu} \psi^{(-)}_{\mu}$ will not work, as the latter is kinematically vanishing. Similarly, one finds by supersymmetry that both $\hat{g}_{\mu y}$ and $\hat{A}_{\mu}$ cannot be consistently set to zero unless both $\psi^{(-)}_{\mu}$ and $\lambda^{(+)}$ are absent. So we again conclude that the extend reduction ansatz here is inconsistent with the inclusion of the fermions.

Unlike for the extended $D = 7$ to $D = 6$ case, however, (which lacks even the requisite bosonic fields for supersymmetry) here the bosonic sector has a natural supersymmetric
fermionic completion. It just so happens that the extended domain wall reduction does not yield this natural completion, and instead gives rise to the inconsistent set of fermions given by (75). The fields that we have identified, namely $\psi^{(-)}_\mu$, $\lambda^{(+)}$ and $B_\mu$, is suggestive of a vector multiplet coupled to a spin-$3/2$ multiplet (which is indeed the case for the $\mathcal{N}=2$ theory in five dimensions). So from this point of view, there is in fact a missing second vector that would complete this coupled set of multiplets. Presumably this missing vector may be identified as an appropriate linear combination of $\hat{g}_{\mu y}$ and $\hat{A}_\mu$. Denoting this vector field strength as $K_{\mu\nu}$, we speculate that it would correct the transformations, (75), so that $G_{\mu\nu}$ would be replaced by, for example, $G_{\mu\nu} + K_{\mu\nu}$ and $G_{\mu\nu} - K_{\mu\nu}$ in the spin-$3/2$ and spin-$1/2$ equations, respectively. If this were the case, the additional matter would separate cleanly into independent supermultiplets.

6 Consistent Reduction of Scalar Potential and the Radion

So far we have considered brane-world reductions involving $p$-forms, and a dilaton $\hat{\phi}$ with a running potential. The consistency of these reductions requires turning on the radion, and equating the dilatonic and radionic degrees of freedom. In this section, we show that we can also obtain consistent brane-world reductions for systems comprising a set of additional scalars as well as the dilaton. Specifically, our starting point is the $(D+1)$-dimensional theory described by the Lagrangian

$$\hat{L} = \hat{R} \ast 1 - \frac{1}{2} \hat{s} \hat{d} \hat{d} \hat{\phi} - \frac{1}{2} \hat{d} \hat{d} \Phi_i + g^2 e^{\alpha \hat{\phi}} V(\Phi_i),$$

(76)

where $\Phi_i$ denotes an additional set of scalars, with a potential $V(\Phi_i)$ that has a stationary point $V_0$. Clearly, the solution admits a domain wall solution (4) with $m^2 = -\frac{1}{2} \Delta g^2 V_0$. We find that the scalars $\Phi_i$ can be consistently reduced into this domain wall world-volume spacetime, provided that $a = \sqrt{2/(D-1)}$, corresponding again to $\Delta = -2$. The reduction ansatz is given by

$$ds^2 = W^{-\frac{2}{D-1}} e^{2\alpha \phi} ds^2 + W^{\frac{2D}{D-1}} e^{-2(D-2)\alpha \phi} dy^2,$$

$$e^{\alpha \hat{\phi}} = W^{-\frac{2a^2}{\Delta}} e^{2(D-2)\alpha \phi}, \quad \hat{\Phi}_i = \Phi_i.$$

(77)

We find that the resulting Lagrangian of the lower-dimensional theory is given by

$$\mathcal{L} = R \ast 1 - \frac{1}{2} * d\varphi \land d\varphi - \frac{1}{2} * d\Phi_i \land d\Phi_i + g^2 e^{2(D-1)\alpha \varphi} (V(\Phi_i) - V_0),$$

(78)

where $\alpha^2 = 1/(2(D-2)(D-1)^2)$. Thus we see that the coupling of the dilaton $\phi$ in the lower dimensional scalar potential is of the form $\sqrt{2/(D-2)}$, again corresponding to $\Delta = -2$. 

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Scalar potentials of the type appearing in (76), with a dilaton coupling $\Delta = -2$ in the scalar potential, typically arise in gauged supergravities, such as those that come from $S^3$ reductions of ungauged supergravities, where the higher-dimensional 3-form field strength is taken to be proportional to the volume of the $S^3$.

General $S^3$ reductions of this kind were derived in [25]. Starting in $D$ dimensions from the Lagrangian

$$L_D = \hat{R} \ast \mathbf{1} - \frac{1}{2} \ast d\phi \wedge d\phi - \frac{1}{2} e^{-b \hat{\phi}} \ast \hat{F}(3) \wedge \hat{F}(3),$$

(79)

where $b^2 = 8/(D - 2)$, it is shown in [25] that the following Kaluza-Klein ansatz gives a consistent $S^3$ reduction:

$$d\hat{s}_D^2 = Y^{-1/2} \left( \Omega^{-2} ds_{D-3}^2 + g^{-2} \Omega^{-D-4} T_{ij} \mathcal{D} \mu^i \mathcal{D} \mu^j \right),$$

$$e^{\sqrt{(D-2)/2} \hat{\phi}} = \Omega^{-1} Y^{(D-4)/4},$$

$$\hat{F}(3) = F(3) + \frac{1}{6} \epsilon_{ijk\ell \kappa \lambda} \left( g^{-2} U \Omega^{i} \mathcal{D} \mu^{j} \wedge \mathcal{D} \mu^{k} \wedge \mathcal{D} \mu^{i} + 3g^{-2} \Omega^{-2} \mathcal{D} \mu^{i} \wedge \mathcal{D} \mu^{j} \wedge \mathcal{D} \mu^{k} + 3g^{-1} \Omega^{-1} F^{ij} \wedge \mathcal{D} \mu^{j} T_{ij} \mu^j \right),$$

(80)

where

$$\mu^i \mu^i = 1, \quad \Omega = T_{ij} \mu^i \mu^j, \quad U = 2T_{ik} T_{jk} \mu^i \mu^j - \Omega T_{ii}, \quad Y = \det(T_{ij}),$$

(81)

and the indices $i, j, \ldots$ range of 4 values. Here, a summation over repeated $SO(n+1)$ indices is understood. The gauge-covariant exterior derivative $D$ is defined so that

$$\mathcal{D} \mu^i = d\mu^i + g A^{ij}_{(1)} \mu^j, \quad \mathcal{D} T_{ij} = dT_{ij} + g A^{ik}_{(1)} T_{kj} + g A^{jk}_{(1)} T_{ik},$$

(82)

where $A^{ij}_{(1)}$ denotes the $SO(4)$ gauge potentials coming from the isometry group of the 3-sphere, and

$$F^{ij} = dA^{ij}_{(1)} + g A^{ik}_{(1)} \wedge A^{kj}_{(1)}.$$

(83)

Thus the lower-dimensional fields appearing in the Kaluza-Klein Ansatz comprise the metric $ds_{D-3}^2$, the six gauge potentials $A^{ij}_{(1)}$ of $SO(4)$, the ten scalar fields described by the symmetric tensor $T_{ij}$, and the 2-form potential $A_{(2)}$, whose (Chern-Simons modified) field strength is $F(3)$. The resulting $(D - 3)$-dimensional equations of motion can be derived from the Lagrangian [25]

$$L_{D-3} = R \ast \mathbf{1} - \frac{1}{2} \ast d\phi \wedge d\phi - \frac{1}{2} T_{ij} \ast D T_{jk} \wedge T_{ik} \mathcal{D} T_{li}$$

$$- \frac{1}{2} T^{-1} \ast F(3) \wedge F(3) - \frac{1}{2} \ast Y^{-1} T_{ik} \ast T_{j\ell} \ast T_{i\ell} \ast F^{ij} \wedge F^{k\ell} - V \ast \mathbf{1},$$

(84)
where $Y \equiv \exp(-\sqrt{8/(D-5)} \phi)$, $\tilde{T}_{ij} \equiv Y^{-1/4} T_{ij}$ (implying that $\det(\tilde{T}_{ij}) = 1$), and the potential $V$ is given by

$$V = \frac{1}{2} g^2 Y^2 \left( 2 \tilde{T}_{ij} \tilde{T}_{ij} - (\tilde{T}_{ii})^2 \right). \quad (85)$$

The 3-form field strength $F_3$ is given by

$$F_3 = d\tilde{A}_3 + \frac{1}{8} \epsilon_{ijk\ell} (F_{(2)}^{ij} \wedge A^{k\ell}_{(1)} - \frac{1}{3} g A^{ij}_{(1)} \wedge A^{km}_{(1)} \wedge A^{m\ell}_{(1)}). \quad (86)$$

The scalars $\tilde{T}_{ij}$ parameterise the coset $SL(4, \mathbb{R})/SO(4)$.

Focusing first on the scalar sector of this theory, we see that there is a dilaton $\phi$ with exponential coupling in the scalar potential that is exactly of the strength $\Delta = -2$ that we required for our consistent brane-world reduction in this section. The nine scalars described by the unimodular symmetric tensor $\tilde{T}_{ij}$ correspond to the scalars $\Phi_i$ in our earlier general discussion. In fact we can see that the strengths of the dilaton coupling to $F_3$ and $F_{(2)}^i$ in (84) are also exactly what we found to be necessary in section 2 in order to obtain consistent brane-world reductions of these fields too. Thus we can start from the theory (79) in $D$ dimensions, perform a consistent $S^3$ reduction to obtain (84) in $(D-3)$ dimensions, and then perform a further brane-world consistent reduction, ending up with a theory in $(D-4)$ dimensions that comprises the metric, a radion $\varphi$, nine scalars $\tilde{T}_{ij}$ parameterising $SL(4, \mathbb{R})/SO(4)$, a 3-form $F_3$, and the six $SO(4)$ Yang-Mills fields $F_{(2)}^i$. The lower dimensional Lagrangian is given by

$$\mathcal{L}_{D-4} = R \ast 1 - \frac{1}{2} d\tilde{\varphi} \wedge d\varphi - \frac{1}{4} \tilde{T}_{ij}^{-1} \ast D \tilde{T}_{jk} \wedge \tilde{T}_{kl}^{-1} D \tilde{T}_{li} - \frac{1}{2} e^{-\sqrt{\frac{8}{D-6}} \varphi} \ast F_3 \wedge F_3 - \frac{1}{4} e^{-\sqrt{\frac{2}{D-6}} \varphi} \tilde{T}_{ij}^{-1} \ast F_{(2)}^{ij} \wedge F_{(2)}^{kl} - V \ast 1, \quad (87)$$

where the scalar potential $V$ is given by

$$V = \frac{1}{2} g^2 e^{-\sqrt{\frac{2}{D-6}} \varphi} \left( 2 \tilde{T}_{ij} \tilde{T}_{ij} - (\tilde{T}_{ii})^2 + 8 \right). \quad (88)$$

It is interesting to note that we have started from the theory in $D$ dimensions described by the Lagrangian (79), where there is no gauging and no Yang-Mills fields, and we have ended up in $(D-4)$ dimensions with the theory described by (87), where we have $SO(4)$ Yang-Mills fields, and a set of scalars with a non-trivial scalar potential. However, remarkably, the scalar potential in the $(D-4)$-dimensional gauged theory admits Minkowski spacetime as a solution. It is quite unusual in Kaluza-Klein reductions that one can end up with non-abelian gauge theories coming from a reduction on a compact space with isometries, and yet still have a Minkowski vacuum solution.
It is of interest therefore to examine the geometry of the 4-dimensional internal spaces that combine the $S^3$ reduction and the brane-world reduction. This can be seen most clearly by just looking at the “vacuum” solution, where the $S^3$ is undistorted and the solution in $(D-3)$-dimensional gauged theory is taken to be the domain wall. Retracing the steps described above, we see therefore that this domain-wall solution lifts back to give

$$d\hat{s}_D^2 = W^{-\frac{2}{D-2}} dx^\mu dx_\mu + W^{-\frac{2(D-1)}{D-2}} dy^2 + g^{-2} W^{-\frac{2}{D-2}} d\Omega_3^2. \tag{89}$$

If we define a new coordinate $r$ in place of $y$, by setting $W = g^2 r^2$, the $D$-dimensional metric becomes

$$d\hat{s}_D^2 = (g r)^{\frac{4}{D-2}} dx^\mu dx_\mu + (g r)^{-\frac{2(D-4)}{D-2}} (dr^2 + r^2 d\Omega_3^2). \tag{90}$$

(Here we have used the relation $m^2 = 4g^2$, which follows from the relation given below (76).) The metric (90) can be recognised as the near-horizon limit of the $(D-5)$-brane in $D$ dimensions:

$$d\hat{s}_D^2 = \left(1 + \frac{Q}{r^2}\right)^{-\frac{2}{D-2}} dx^\mu dx_\mu + \left(1 + \frac{Q}{r^2}\right)^{\frac{D-4}{D-2}} (dr^2 + r^2 d\Omega_3^2), \tag{91}$$

where $Q = g^{-2}$.

The “internal” four-dimensional metric in the vacuum solution (90) can be seen to be singular at $r = 0$, on account of the conformal factor $(g r)^{-2(D-4)/(D-2)}$ that multiplies the flat 4-metric $dr^2 + r^2 d\Omega_3^2$. (An exception, of course, arises if $D = 4$, since then the entire metric is “internal,” and is merely the Euclidean metric itself.) Another way to view the internal geometry is by introducing a new radial coordinate $\rho$, related to $r$ by $g \rho = (g r)^{2/(D-2)}$. The metric (90) then becomes

$$d\hat{s}_D^2 = (g \rho)^2 dx^\mu dx_\mu + \frac{1}{4}(D-2)^2 d\rho^2 + \rho^2 d\Omega_3^2. \tag{92}$$

Again, we see that the internal 4-metric is in general singular, describing a cone over $S^3$, with $D = 4$ being the exceptional case where the metric becomes non-singular.

Finally, we remark that the “anomaly term” term $L_{\text{anom}} = -(D-26) m^2 e^{\sqrt{2/(D-2)}} \phi$ in the $D$-dimensional non-critical string effective action also has a $\Delta = -2$ coupling strength. It is easy to verify that one can perform a consistent brane-world reduction on the theory described by (79) together with the additional contribution $L_{\text{anom}}$. A particularly interesting case is the 27-dimensional non-critical bosonic string, which can then be reduced to give a critical string in 26 dimensions by means of this brane-world reduction.
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APPENDICES

A Conventions for differential forms

Our conventions for differential forms are as follows. A $p$-form $\omega$ has components $\omega_{\mu_1 \cdots \mu_p}$ such that

$$\omega = \frac{1}{p!} \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}.$$  \hspace{1cm} (93)

The Hodge dual in $n$ dimensions is defined by

$$*(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}) = \frac{1}{(n-p)!} \epsilon_{\nu_1 \cdots \nu_{n-p}}^{\mu_1 \cdots \mu_p} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-p}},$$  \hspace{1cm} (94)

which implies that the components of the dual $\ast \omega$, defined by

$$(*\omega)_{\nu_1 \cdots \nu_{n-p}} = \frac{1}{p!} \epsilon_{\nu_1 \cdots \nu_{n-p}}^{\mu_1 \cdots \mu_p} \omega_{\mu_1 \cdots \mu_p}.$$  \hspace{1cm} (95)

are given by

$$(*\omega)_{\nu_1 \cdots \nu_{n-p}} = \frac{1}{p!} \epsilon_{\nu_1 \cdots \nu_{n-p}}^{\mu_1 \cdots \mu_p} \omega_{\mu_1 \cdots \mu_p}.$$  \hspace{1cm} (96)

We therefore have that

$$\ast \omega \wedge \omega = \frac{1}{p!} \omega_{\mu_1 \cdots \mu_p} \omega^{\mu_1 \cdots \mu_p} \ast \mathbb{1},$$  \hspace{1cm} (97)

where

$$\ast \mathbb{1} = \sqrt{-g} dx^0 \wedge dx^1 \wedge \cdots dx^{n-1}$$  \hspace{1cm} (98)

is the volume form, and we are taking $\epsilon_{012\cdots} = +\sqrt{-g}$. Thus a Lagrangian with fields normalised so that

$$e^{-1} L = R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2p!} F_{\mu_1 \cdots \mu_p} F^{\mu_1 \cdots \mu_p},$$  \hspace{1cm} (99)

can be written instead in terms of the $n$-form

$$\mathcal{L} = R \ast \mathbb{1} - \frac{1}{2} \ast d\phi \wedge d\phi - \frac{1}{2} \ast F \wedge F.$$  \hspace{1cm} (100)
B  $S^2$ reduction of gauged $D = 7$ supergravity

In this appendix, we address the question of whether the $D = 5, \mathcal{N} = 2$ gauged supergravity coupled to a vector multiplet discussed in section 4.2 can also be obtained from an $S^2$ reduction of the $D = 7, \mathcal{N} = 2$ gauged pure supergravity discussed in 3.1. At the level of the bosonic sector, the following ansatz, which was considered in general in [26], gives a consistent reduction of the seven-dimensional gauged supergravity (23) to $D = 5$:

$$
ds^2_7 = e^{-2\sqrt{\frac{2}{5}} \phi} ds^2_5 + \frac{2}{\lambda^2} e^{3\sqrt{\frac{2}{5}} \phi} d\Omega^2_2, \hspace{1cm} \hat{\phi} = \sqrt{\frac{3}{5}} \phi$$

$$
\hat{F}_{(4)} = \frac{2}{\lambda^2} F_{(2)} \wedge \Omega_{(2)}, \hspace{1cm} \hat{F}^3_{(2)} = \frac{2}{\lambda} \Omega_{(2)} + G_{(2)}, \hspace{1cm} \hat{F}^1_{(2)} = 0 = \hat{F}^0_{(2)}. \quad (101)
$$

The resulting five-dimensional equations of motion are those of the bosonic sector of five-dimensional gauged supergravity coupled to a vector multiplet, described by (62) with $g_2 = 0$ and

$$
g_1^2 = 2g^2 + \frac{1}{2} \lambda^2. \quad (102)
$$

Note that the lower-dimensional “cosmological constant” $g_1^2$ is the sum of contributions from both $\lambda$ and $g$. This implies that there is a 1-parameter family of ways of obtaining the same five-dimensional bosonic theory, with different proportions of the contributions to the lower-dimensional cosmological term coming from the curvature of the reduction 2-sphere versus the already-present cosmological term in the seven-dimensional gauged theory. In fact in one extreme, we can take $\lambda = 0$, which amounts to making a $T^2$ reduction of the $D = 7$ gauged theory. At the other extreme, we can take $g = 0$, which amounts to an $S^2$ reduction of the ungauged $D = 7$ theory.

The situation changes somewhat when we include the fermionic sector in the reduction. We find that for the general case with $g$ and $\lambda$ both non-zero, we cannot obtain a consistent five-dimensional supersymmetric theory. In fact this can be illustrated by considering a simple known supersymmetric solution of the five-dimensional gauged theory, namely the domain wall that preserves half of the $D = 5$ supersymmetry. In $D = 5$ this is given by

$$
ds^2_5 = W^{-\frac{2}{3}} dx^\mu dx_\mu + W^{-\frac{8}{3}} dy^2, \hspace{1cm} e^\phi = W^{-\sqrt{\frac{2}{3}}}, \quad (103)
$$

where $W = 1 + m y$ and $m = g_1$. Lifting this to $D = 7$ using (101), we find

$$
ds^2_7 = W^{-\frac{2}{5}} (dx^\mu dx_\mu + \frac{2}{\lambda^2} d\Omega^2_2) + W^{-\frac{12}{5}} dy^2, \hspace{1cm} e^{-\frac{1}{\sqrt{10}} \hat{\phi}} = W_5^{\frac{1}{5}}, \hspace{1cm} \hat{F}^3_{(2)} = \frac{2}{\lambda} \Omega_{(2)}. \quad (104)$$

29
To illustrate the issue of supersymmetry, it suffices to consider just the dilatino transformation rule given in (28). In the background (104), the dilatino variation gives
\[ \delta \hat{\lambda}_i = \frac{1}{\sqrt{10}} W^{\frac{1}{2}} \left[ \frac{1}{\sqrt{2}} m \hat{\gamma}^5 \epsilon_i + g \epsilon_i + \frac{i}{2} \lambda (\sigma_3)_i \hat{\gamma}^{67} \epsilon_j \right], \]
where the \( y \) direction is denoted by “5” and the two directions on \( S^2 \) are denoted by “6” and “7.” We see that to get preserved supersymmetry, we must have
\[ m = \sqrt{2} (\pm g \pm \frac{1}{2} \lambda) \]
(106).

This is compatible with (102) only if \( g = 0 \) or \( \lambda = 0 \).

In general, one may search for a fermion reduction by substituting the bosonic ansatz, (101), into the supersymmetry transformations (28). For the dilatino, we find
\[ \delta \hat{\lambda}_i = -i \sqrt{\frac{6}{5}} e^{\frac{1}{2} \sqrt{2} \phi} \left[ \frac{i}{4} \gamma^\mu \partial_\mu \phi \delta_i \hat{\gamma}^j + \frac{1}{8} \sqrt{3} \gamma^{\mu \nu} \left( e^{\frac{1}{2} \sqrt{2} \phi} G_{\mu \nu} (\sigma_3)_i \hat{\gamma}^j + \sqrt{2} e^{-\sqrt{2} \phi} F_{\mu \nu} (i \gamma^{67}) \delta_i \hat{\gamma}^j \right) \right] \]
\[ + \frac{i}{\sqrt{6}} \left( \sqrt{2} g + \frac{1}{\sqrt{2}} \lambda (i \gamma^6 \gamma^7) (\sigma^3)_i \right) e^{-\frac{1}{2} \sqrt{2} \phi} \hat{\gamma}^j, \]
(107)
which may be compared with the corresponding \( D = 5 \) dilatino transformation of (63). A more direct comparison may be obtained by converting the spinors into their natural five-dimensional counterparts, following (50). However, even at present, it is clear that for the reduced dilatino to transform properly, one requires (up to an overall sign)
\[ m = \sqrt{2} [g + \frac{1}{2} \lambda (i \gamma^6 \gamma^7) (\sigma^3)] \]
(108)
(where this notation refers to the eigenvalues of the Dirac matrices on the reduced spinor parameter \( \epsilon \)). This confirms (106), and furthermore indicates that the sign is dependent on the orientation of the \( S^2 \).

The supersymmetry of several breathing mode sphere reductions was considered in Ref. [27]. There it was conjectured that breathing mode reductions could be consistent, at least in the case of reduced supersymmetries. In the examples of Ref. [27], contributions to the lower-dimensional potential were similarly quadratic, as in (102), while contributions to the “superpotential” were linear, as in (108). However, unlike the present case, in those examples, the potential and superpotential comprised more than a single exponential, thus allowing the functional relationship between potential and superpotential to be satisfied for arbitrary values of the parameters. Furthermore, the earlier examples were all for sphere reduction of ungauged supergravities. From this point of view, the obstruction to having a consistent reduction on the fermions is similar to that discussed for the bosonic examples of section 5. Namely, turning on the graviphoton \( \hat{F}^3_{(3)} \) (even when restricted to the sphere direction) gives rise to inconsistencies in the fermion sector.
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