STABILITY FOR THE MULTI-DIMENSIONAL BORG–LEVINSON THEOREM OF THE BIHARMONIC OPERATOR

PEIJUN LI, XIAOHUA YAO, AND YUE ZHAO

Abstract. We prove for the first time a conditional Hölder stability related to the multi-dimensional Borg–Levinson theorem, which is concerned with determining a potential from spectral data for the biharmonic operator. The proof depends on the theory of scattering resonances to obtain the resolvent estimate and a Weyl-type law for the biharmonic operator.

1. Introduction

The multi-dimensional version of the classical one-dimensional Borg–Levinson theorem [5,19] for the Schrödinger operator \(-\Delta + V\) concerns the unique determination of the potential \(V \in L^\infty\) from the Dirichlet eigenvalues and the boundary values of the normal derivatives of the eigenfunctions. The uniqueness result was established in [23], where the proof relies on connecting the spectral data to the Dirichlet-to-Neumann (DtN) map. Moreover, the uniqueness with incomplete data was proved in [14]. The result implies that it still suffices to determine the potential \(V\) for the multi-dimensional Schrödinger operator from the knowledge of all large Dirichlet eigenvalues and the boundary values of the normal derivatives of the corresponding eigenfunctions.

The first conditional Hölder stability estimate for the multi-dimensional Borg–Levinson theorem was obtained in [1]. The idea is to transform the stability estimate for the spectral problem into a stability estimate for the inverse problem of determining the potential in a wave equation from the corresponding hyperbolic DtN map. A stronger conditional Hölder stability estimate was proved in [8]. The proof is based on high frequency asymptotic techniques instead of relating the spectral problem to a hyperbolic DtN map for the wave equation. Some relevant stability estimates for the problem can be found [4,7]. We refer to [3,15,17] for related inverse spectral problems of the second order elliptic operators on compact Riemannian manifolds and in a periodic waveguide. We also refer to [2,25] for recent developments on numerical algorithms for the one-dimensional inverse spectral problems.

Since there is already a vast amount of literature on the multi-dimensional Borg–Levinson theorem for the Schrödinger operator, we wish to extend the results to higher order elliptic operators. The inverse problems for biharmonic operators have important applications in elasticity and the theory of vibration of beams, e.g., the beam equation [11], the hinged

2000 Mathematics Subject Classification. 31B20, 35R30, 58J50.

Key words and phrases. The biharmonic operator, Borg–Levinson theorem, stability, spectral data, theory of scattering resonances.

The first author was supported in part by the NSF grant DMS-1912704. The second author was supported partially by NSFC (No. 11771165 and 12171182). The third author was supported in part by NSFC (No. 12001222).
plate configurations [11], and the scattering by grating stacks [22]. We refer to [12, 13] for uniqueness of the inverse problem of higher order elliptic operators. In [18], the uniqueness with full or incomplete spectral data were studied for the elliptic operators of higher order with constant coefficients. However, to the best of our knowledge, there is no stability estimate so far for the multi-dimensional Borg–Levinson theorem of the elliptic operators of higher order.

This work concerns the resolvent and Weyl’s law for the biharmonic operator. We prove a conditional Hölder stability for the biharmonic operator even with incomplete data. The proof combines an Isozaki’s representation formula (cf. Lemma 4) and the Weyl-type law (cf. Lemma 6). Compared with the second order equations, the analysis of the biharmonic operator is more sophisticated. For instance, it is necessary to investigate two sets of the DtN maps and use more spectral data in order to study the inverse problems of the biharmonic operator.

Since the work relies on the analysis of the resolvent for the biharmonic operator, it is closely related to the theory of scattering resonances. We refer to the monograph [9] for a comprehensive account of this subject. Specifically, we study the meromorphic continuation of the resolvent for the biharmonic operator and derive its upper bound. Recently, the theory of scattering resonances has found successful applications in the study of inverse scattering problems (e.g., [6, 20, 21]). In [21], the authors showed for the first time the stability of the inverse source problem for the three-dimensional Helmholtz equation in an inhomogeneous background medium. The analysis employs the theory of scattering resonances to obtain the holomorphic domain and an upper bound for the resolvent of the elliptic operator. Later, the results are extended to the biharmonic operator [20], which inspires us to explore the current work. As another application, we refer to [6] for a duality between scattering poles and transmission eigenvalues in scattering theory. In this paper we employ the theory of scattering resonances again to study the inverse boundary spectral problem for the biharmonic operator.

Next we introduce some notations and state the main result of this paper.

Let $B_R = \{x \in \mathbb{R}^n : |x| < R\}$, where $n \geq 3$ is odd and $R > 0$ is a constant. Denote by $\partial B_R$ the boundary of $B_R$. Consider the biharmonic operator

$$H(V) := \Delta^2 + V(x), \quad x \in \mathbb{R}^n,$$

where $\Delta$ is the Laplacian and $V(x) \in L^\infty(B_R)$ is the real-valued potential. The domain of $H$ is

$$D(H) = H_0^2(B_R) \cap H^4(B_R).$$

Consider the spectrum of the operator $H(V)$ with the Navier boundary condition

$$\begin{cases}
(\Delta^2 + V)\phi_k(V) = \lambda_k(V)\phi_k(V) & \text{in } B_R, \\
\Delta\phi_k(V) = \phi_k(V) = 0 & \text{on } \partial B_R.
\end{cases}$$

where $\{\lambda_k(V), \phi_k(V)\}_{k=1}^\infty$ are the increasingly ordered eigenvalues and normalized orthonormal eigenfunctions of $H$ in $B_R$ satisfying

$$-\infty < \lambda_1(V) \leq \lambda_2(V) \leq \cdots \leq \lambda_k(V) \leq \cdots + \infty$$
Hereafter, the notation $a \preceq b$ stands for $a \leq Cb$, where $C > 0$ is a generic constant which may change step by step in the proofs. The following Weyl-type law provided in Lemma 6 is crucial in the proof of the stability:

$$|\lambda_k(V)| \sim k^{4/n}, \quad \|\partial_\nu \phi_k(V)\|_{L^2(\partial B_R)} \preceq k^{2/n}, \quad \|\partial_\nu (\Delta \phi_k(V))\|_{L^2(\partial B_R)} \preceq k^{4/n},$$

where $\nu$ is the unit outward normal to $\partial B_R$. Let $m > n/4 + 1$ be a fixed integer. It follows from (1.1) that both the series

$$\sum_{k \geq 1} k^{-4m/n} \|\partial_\nu \phi_k(V)\|_{L^2(\partial B_R)}$$

and

$$\sum_{k \geq 1} k^{-4m/n} \|\partial_\nu (\Delta \phi_k(V))\|_{L^2(\partial B_R)}$$

converge absolutely in $L^2(\partial B_R)$.

Let $N \geq 0$ be a fixed integer and define

$$\delta_0(V_1, V_2) = \max_{k \geq 1} |\lambda_{k+N}(V_1) - \lambda_{k+N}(V_2)|.$$ 

Let

$$\delta_1(V_1, V_2) = \sum_{k \geq 1} k^{-4m/n} \|\partial_\nu \phi_{k+N}(V_1) - \partial_\nu \phi_{k+N}(V_2)\|_{L^2(\partial B_R)},$$

$$\delta_2(V_1, V_2) = \sum_{k \geq 1} k^{-4m/n} \|\partial_\nu (\Delta \phi_{k+N}(V_1)) - \partial_\nu (\Delta \phi_{k+N}(V_2))\|_{L^2(\partial B_R)}.$$ 

The following theorem concerns the stability of the inverse problem and is the main result of the paper.

**Theorem 1.** Let $V_1, V_2 \in L^\infty(B_R)\text{ such that } V := V_1 - V_2 \in H_0^1(B_R)$ and

$$\|V_1\|_{L^\infty(B_R)} + \|V_2\|_{L^\infty(B_R)} + \|V\|_{H_0^1(B_R)} \leq M.$$ 

Then there exist two constants $C = C(m, M)$ and $0 < \gamma < 1$ such that

$$\|V_1 - V_2\|_{L^2(B_R)} \leq C\delta^\gamma,$$

where $\delta = \delta_0(V_1, V_2) + \delta_1(V_1, V_2) + \delta_2(V_1, V_2)$.

The theorem shows that even if a finite number of spectral data are missing, we can still determine the potential $V$ with a H"older stability. An explicit value of $\gamma$ will be given as a function of $n$ at the end of the proof. Clearly, the stability estimate (1.2) implies the uniqueness of the inverse spectral problem.

The paper is organized as follows. Some necessary preliminaries are provided in Section 2. Section 3 is devoted to the proof of the stability. In Appendix, we present the estimate of the resolvent and a Weyl-type law for the biharmonic operator.
2. Preliminaries

We consider two families of DtN maps which are parameterized by the spectral parameter. Let \( V \in L^\infty(B_R) \) and \( \lambda \in \rho(H(V)) \) which is the resolvent set of \( H(V) \). Given any \( f \in H^{3/2}(\partial B_R) \) and \( g \in H^{-1/2}(\partial B_R) \), consider the boundary value problem

\[
\begin{align*}
H(V)u - \lambda u &= 0 \quad \text{in } B_R, \\
u &= f \quad \text{on } \partial B_R, \\
\Delta u &= g \quad \text{on } \partial B_R.
\end{align*}
\]

(2.1)

Clearly, it has a unique weak solution \( u \in H^2(B_R) \). We introduce two DtN maps

\[
\Lambda_1(V, \lambda) : f \to \partial_\nu u|_{\partial B_R},
\]

\[
\Lambda_2(V, \lambda) : g \to \partial_\nu (\Delta u)|_{\partial B_R},
\]

where \( \Lambda_1(V, \lambda) \) and \( \Lambda_2(V, \lambda) \) define bounded operators from \( H^{3/2}(\partial B_R) \) to \( H^{1/2}(\partial B_R) \) and from \( H^{-1/2}(\partial B_R) \) to \( H^{-3/2}(\partial B_R) \), respectively.

Next, we derive formal representations of \( \Lambda_1(V, \lambda) \) and \( \Lambda_2(V, \lambda) \) by using the spectral data. Multiplying both sides of (2.1) by \( \phi_k \) and using the integration by parts, we have

\[
\int_{B_R} u \phi_k \, dx = \frac{1}{\lambda_k(V)} - \frac{1}{\lambda_k(V)} \left( \int_{\partial B_R} \partial_\nu \phi_k(V) f \, ds(y) + \int_{\partial B_R} \partial_\nu (\Delta \phi_k(V)) g \, ds(y) \right),
\]

which formally gives

\[
u(x, \lambda) = \sum_{k=1}^{\infty} \phi_k(V) \frac{1}{\lambda_k(V)} - \frac{1}{\lambda_k(V)} \left( \int_{\partial B_R} \partial_\nu \phi_k(V) f \, ds(y) + \int_{\partial B_R} \partial_\nu (\Delta \phi_k(V)) g \, ds(y) \right), \quad x \in B_R.
\]

Thus, for \( \lambda \notin \{\lambda_k(V)\}_{k=1}^{\infty} \), the DtN maps can be represented by

\[
\Lambda_1(V, \lambda)(f) = \sum_{k=1}^{\infty} \partial_\nu \phi_k(V) \bigg|_{\partial B_R} \frac{1}{\lambda_k(V)} - \frac{1}{\lambda_k(V)} \left( \int_{\partial B_R} \partial_\nu \phi_k(V) f \, ds(y) + \int_{\partial B_R} \partial_\nu (\Delta \phi_k(V)) g \, ds(y) \right)
\]

and

\[
\Lambda_2(V, \lambda)(g) = \sum_{k=1}^{\infty} \partial_\nu (\Delta \phi_k(V)) \bigg|_{\partial B_R} \frac{1}{\lambda_k(V)} - \frac{1}{\lambda_k(V)} \left( \int_{\partial B_R} \partial_\nu \phi_k(V) f \, ds(y) + \int_{\partial B_R} \partial_\nu (\Delta \phi_k(V)) g \, ds(y) \right).
\]

However, the series on the right hand side may not converge absolutely. As suggested in [17], it can be shown that some higher order formal derivatives converge absolutely. Let

\[
\Lambda_1^{(m)}(V, \Lambda) := \frac{d^m}{d \lambda^m} \Lambda_1(V, \lambda), \quad \Lambda_2^{(m)}(V, \Lambda) := \frac{d^m}{d \lambda^m} \Lambda_2(V, \lambda).
\]

By the Weyl-type law [11], for \( m \gg 1 \), the series converge absolutely. Precisely, we have the following lemma.
Lemma 2. For $m > n/4 + 1$ and $\lambda \notin \{\lambda_k\}_{k=1}^{\infty}$, the series
\[
\Lambda_1^{(m)}(V, \lambda)(f) = -m! \sum_{k=1}^{\infty} \partial_\nu \phi_k(V) \mid_{\partial B_R} \left( \frac{1}{(\lambda_k(V) - \lambda)^{m+1}} \int_{\partial B_R} \partial_\nu \phi_k(V) f ds(y) \right) 
+ \int_{\partial B_R} \partial_\nu (\Delta \phi_k(V)) g ds(y),
\]
and
\[
\Lambda_2^{(m)}(V, \lambda)(f) = -m! \sum_{k=1}^{\infty} \partial_\nu (\Delta \phi_k(V)) \mid_{\partial B_R} \left( \frac{1}{(\lambda_k(V) - \lambda)^{m+1}} \int_{\partial B_R} \partial_\nu \phi_k(V) f ds(y) \right) 
+ \int_{\partial B_R} \partial_\nu (\Delta \phi_k(V)) g ds(y),
\]
converge absolutely in $H^{1/2}(\partial B_R)$ and $H^{-3/2}(\partial B_R)$, respectively. Moreover, $\Lambda_1^{(m)}(V, \lambda)$ and $\Lambda_2^{(m)}(V, \lambda)$ can be extended to meromorphic families with poles at the eigenvalues.

The following lemma gives the mapping properties of the derivatives of the DtN maps.

Lemma 3. Let $l$ be any positive integer. The following estimates hold:
\[
\|\Lambda_1^{(j)}(V_1, \lambda) - \Lambda_1^{(j)}(V_2, \lambda)\|_{L(H^{1/2}(\Gamma), H^{1/2}(\Gamma))} \lesssim \frac{1}{|\Re \lambda|^{j+\sigma_1}},
\]
\[
\|\Lambda_2^{(j)}(V_1, \lambda) - \Lambda_2^{(j)}(V_2, \lambda)\|_{L(H^{-1/2}(\Gamma), H^{-1/2}(\Gamma))} \lesssim \frac{1}{|\Re \lambda|^{j+\sigma_2}},
\]
where $0 \leq j \leq l$, $\Re \lambda \leq -2M$, and
\[
\sigma_1 = \frac{1 - 2t_1}{4}, \quad -\frac{3}{2} \leq t_1 \leq \frac{1}{2}, \quad \sigma_2 = \frac{-3 - 2t_2}{4}, \quad -\frac{7}{2} \leq t_2 \leq -\frac{3}{2}.
\]

Proof. The proof is adapted from [7, Lemma 2.32] for the Laplacian operator to the complex case for the biharmonic operator.

Assume that $\lambda \in \rho(H(V_1)) \cap \rho(H(V_2))$. Let $f \in H^{3/2}(\partial B_R)$, $g \in H^{-1/2}(\partial B_R)$ and $u_j, j = 1, 2$ be the solution to the boundary value problem
\[
\begin{cases}
\Delta^2 u_j + V_j u_j - \lambda u_j = 0 & \text{in } B_R, \\
u_j = f & \text{on } \partial B_R, \\
\Delta u_j = g & \text{on } \partial B_R.
\end{cases}
\]
Let $u := u_1 - u_2$. A simple calculation yields
\[
\begin{cases}
\Delta^2 u + V_1 u - \lambda u = (V_2 - V_1)u_2 & \text{in } B_R, \\
u = 0 & \text{on } \partial B_R, \\
\Delta u = 0 & \text{on } \partial B_R.
\end{cases}
\]

For sufficiently large $|\lambda|$, multiplying both sides of the above equation by $u$ and integrating by parts, we obtain
\[
\|u\|_{L^2(B_R)} \lesssim \frac{1}{|\lambda|} \|u_2\|_{L^2(B_R)}. \tag{2.2}
\]
Combining (2.3) and (2.4) leads to
\[ \|u_2\|_{L^2(B_R)} \leq \|f\|_{H^{3/2}(\partial B_R)} + \|g\|_{H^{-1/2}(\partial B_R)}, \]
which gives
\[ \|u\|_{L^2(B_R)} \leq \frac{1}{\lambda}\left(\|f\|_{H^{3/2}(\partial B_R)} + \|g\|_{H^{-1/2}(\partial B_R)}\right). \tag{2.3} \]

Denote \(u'(\lambda)\) and \(u'_1(\lambda)\) the derivatives with respect to \(\lambda\). It can be verified that \(u'_2(\lambda)\) satisfies
\[
\begin{aligned}
\Delta^2 u'_2(\lambda) + V_2 u'_2(\lambda) - \lambda u'_2(\lambda) &= u_2 \quad \text{in } B_R, \\
u'_2(\lambda) &= 0 \quad \text{on } \partial B_R, \\
\Delta u'_2(\lambda) &= 0 \quad \text{on } \partial B_R.
\end{aligned}
\]
Using similar arguments as (2.2), we get
\[ \|u'_2(\lambda)\|_{L^2(B_R)} \leq \frac{1}{\lambda}\|u(\lambda) + (V_2 - V_1) u'_2(\lambda)\|_{L^2(B_R)}. \tag{2.4} \]
Since \(u'(\lambda)\) satisfies
\[
\begin{aligned}
\Delta^2 u'(\lambda) + V_1 u'(\lambda) - \lambda u'(\lambda) &= u(\lambda) + (V_2 - V_1) u'_2(\lambda) \quad \text{in } B_R, \\
u'(\lambda) &= 0 \quad \text{on } \partial B_R, \\
\Delta u'(\lambda) &= 0 \quad \text{on } \partial B_R,
\end{aligned}
\]
we have
\[ \|u'(\lambda)\|_{L^2(B_R)} \leq \frac{1}{\lambda}\|u(\lambda) + (V_2 - V_1) u'_2(\lambda)\|_{L^2(B_R)}. \]
Combining (2.3) and (2.4) leads to
\[ \|u'(\lambda)\|_{L^2(B_R)} \leq \frac{1}{\lambda^2}\left(\|f\|_{H^{3/2}(\partial B_R)} + \|g\|_{H^{-1/2}(\partial B_R)}\right). \tag{2.5} \]
On the other hand, it follows from the standard regularity results of elliptic equations that
\[ \|u'(\lambda)\|_{H^4(B_R)} \leq \lambda \|u'(\lambda)\|_{L^2(B_R)} + \|u(\lambda)\|_{L^2(B_R)} + \|u'_2(\lambda)\|_{L^2(B_R)}, \]
which gives
\[ \|u'(\lambda)\|_{H^2(B_R)} \leq \frac{1}{\lambda^2}\left(\|f\|_{H^{3/2}(\partial B_R)} + \|g\|_{H^{-1/2}(\partial B_R)}\right). \tag{2.6} \]
Recalling the interpolation inequality
\[ \|w\|_{H^s(B_R)} \leq \|w\|_{L^2(B_R)}^{1-s/2} \|w\|_{H^{s/2}(B_R)}, \quad 0 \leq s \leq 2, \quad w \in H^2_0(B_R), \]
we obtain from (2.5)–(2.6) that
\[ \|u'(\lambda)\|_{H^s(B_R)} \leq \frac{1}{\lambda^2-s/2}\left(\|f\|_{H^{3/2}(\partial B_R)} + \|g\|_{H^{-1/2}(\partial B_R)}\right), \quad 0 \leq s \leq 2. \]
Therefore, we have
\[ \|\partial_\nu u'(\lambda)\|_{H^{s-3/2}(B_R)} \leq \frac{1}{\lambda^2-s/2}\left(\|f\|_{H^{3/2}(\partial B_R)} + \|g\|_{H^{-1/2}(\partial B_R)}\right), \quad 0 \leq s \leq 2, \]
Multiplying both sides of the above equation by $\varphi$ which has a unique trivial solution $u$, which completes the proof.

Proof. Consider the boundary value problem

$$S(V)(\omega, \theta) = -\sqrt{\Lambda} \int_{\partial B_R} \Lambda_1(V, \lambda)(\varphi_\omega) \varphi_{-\theta} + \Lambda_2(V, \lambda)(\varphi_\omega) \varphi_{-\theta} ds(x).$$

Lemma 4. For $\lambda \in \rho(H(V)) \setminus (-\infty, 0], \omega, \theta \in \mathbb{S}^2$, it holds that

$$S(V, \omega, \theta) = -\int_{B_R} V e^{i \sqrt{\Lambda} (\omega-\theta) \cdot x} dx - \int_{B_R} R(V, \lambda)(-V \varphi_{\omega}) V \varphi_{-\theta} dx - 2\sqrt{\Lambda} \int_{\partial B_R} \varphi_{\omega} \varphi_{-\theta} ds(x).$$

Proof. Consider the boundary value problem

$$\begin{cases}
H(V)u - \lambda u = 0 & \text{in } B_R, \\
u = 0 & \text{on } \partial B_R, \\
\Delta u = 0 & \text{on } \partial B_R,
\end{cases}$$

which has a unique trivial solution $u = 0$. Decompose $u$ as $u = \tilde{u} + \varphi_\omega$. Then we have $\tilde{u} = R(V, \lambda)(-V \varphi_{\omega})$. Moreover, $\tilde{u}$ satisfies the boundary value problem

$$\begin{cases}
H(V)\tilde{u} - \lambda \tilde{u} = -V \varphi_{\omega} & \text{in } B_R, \\
\tilde{u} = -\varphi_{\omega} & \text{on } \partial B_R, \\
\Delta \tilde{u} = -\Delta \varphi_{\omega} & \text{on } \partial B_R.
\end{cases}$$

Multiplying both sides of the above equation by $\varphi_{-\theta}$ and integrating by parts yield

$$\int_{\partial B_R} \varphi_{-\theta} (\varphi_{-\theta}) ds(x) + \int_{\partial B_R} \varphi_{-\theta} \Delta \varphi_{-\theta} ds(x)$$

$$= -\int_{B_R} (V \tilde{u} \varphi_{-\theta} + V \varphi_{\omega} \varphi_{-\theta}) dx + \int_{\partial B_R} \Delta \tilde{u} \varphi_{-\theta} ds(x),$$

which completes the proof. \qed

Let

$$\theta_{\tau} = c_{\tau} \eta + \frac{1}{2\tau} \xi, \quad \omega_{\tau} = c_{\tau} \eta - \frac{1}{2\tau} \xi, \quad \sqrt{\Lambda_{\tau}} = \tau + i,$$

where we choose $\tau > 1, \xi \in \mathbb{R}^3$, $\eta \in \mathbb{S}^2$, and $\eta \parallel \xi$. For simplicity, we drop the subscript $\tau$ in $\lambda_{\tau}, \omega_{\tau}$, and $\theta_{\tau}$.

Using the resolvent estimate in Theorem\cite{7} gives

$$\|R(V, \lambda)\|_{L^2(B_R)} \lesssim \frac{1}{\sqrt{\|\Lambda\|}}, \quad \exists \lambda > 0.$$ \hfill (2.7)
We obtain from Lemma 4 and (2.7) that
\[ |(\hat{V}_1 - \hat{V}_2)(\xi + \frac{i}{\tau} \xi)| \lesssim \frac{1}{\tau^2} + \sup_{0 \leq s \leq 1} |\nabla \hat{V}(\xi + \frac{is}{\tau})| \].
(2.8)

Hence, it is required to estimate \(|S(V_1, \omega, \theta) - S(V_2, \omega, \theta)|\) in terms of the spectral data in order to prove the stability.

3. Proof of the main result

Let \( V = V_1 - V_2 \). We have
\[ |\hat{V}(\xi)| \lesssim |\hat{V}(\xi + \frac{i}{\tau})| + \frac{|\xi|}{\tau} e^{\frac{c|\xi|}{\tau}} \|V\|_{L^\infty(B_R)}, \]
where \( c \) is some positive constant. Thus,
\[ |\hat{V}(\xi)| \lesssim |\hat{V}(\xi + \frac{i}{\tau})| + \frac{|\xi|}{\tau} e^{\frac{c|\xi|}{\tau}} \|V\|_{L^\infty(B_R)}. \]
(3.1)

Let \( \sqrt{\tau} = \tau + i, \tau \geq 1 \). Combining (2.8) and (3.1), we have
\[ |\hat{V}(\xi)| \lesssim \frac{1}{\tau^2} + \frac{|\xi|}{\tau} e^{\frac{c|\xi|}{\tau}} + |S(V_1, \omega, \theta) - S(V_2, \omega, \theta)|. \]

Integrating the above inequality in the ball \(|\xi| \leq \tau^\alpha|\) gives
\[ \int_{|\xi| \leq \tau^\alpha} |\hat{V}(\xi)|^2 d\xi \lesssim \frac{\tau^{\alpha n}}{\tau^{2}} + \frac{\tau^{\alpha(2+n)}}{\tau^{2}} e^{c\tau^{(\alpha-1)}} + \tau^{\alpha n} |S(V_1, \omega, \theta) - S(V_2, \omega, \theta)|^2. \]

Choose \( \alpha = 1/(2+n) \) to get
\[ \int_{|\xi| \leq \tau^{1/(2+n)}} |\hat{V}(\xi)|^2 d\xi \lesssim \frac{1}{\tau^3} + \tau^{n/(n+2)} |S(V_1, \omega, \theta) - S(V_2, \omega, \theta)|^2. \]
(3.2)

Since \( V \in H^1(\mathbb{R}^3) \), we have
\[ \|V\|^2_{L^2(B_R)} = \|\hat{V}\|^2_{L^2(\mathbb{R}^3)} = \int_{|\xi| \leq \tau^{1/(2+n)}} |\hat{V}(\xi)|^2 d\xi + \int_{|\xi| > \tau^{1/(2+n)}} |\hat{V}(\xi)|^2 d\xi \lesssim \int_{|\xi| \leq \tau^{1/(2+n)}} |\hat{V}(\xi)|^2 d\xi + \frac{1}{\tau^{2(n+2)}} \int_{|\xi| > \tau^{1/(2+n)}} |\xi|^2 |\hat{V}(\xi)|^2 d\xi \lesssim \int_{|\xi| \leq \tau^{1/(2+n)}} |\hat{V}(\xi)|^2 d\xi + \frac{1}{\tau^{2(n+2)}} \|V\|^2_{H^1(\mathbb{R}^3)}.
\]

It follows from (3.2) that
\[ \|V\|^2_{L^2(B_R)} \lesssim \frac{1}{\tau^{n+2}} + \tau^{n/(n+2)} |S(V_1, \omega, \theta) - S(V_1, \omega, \theta)|^2. \]
Next we estimate $|S(V_1, \omega, \theta) - S(V_2, \omega, \theta)|^2$ in terms of $\|\Lambda_1(V_1, \lambda) - \Lambda_1(V_2, \lambda)\|_1$ and $\|\Lambda_2(V_1, \lambda) - \Lambda_2(V_2, \lambda)\|_2$. We have from Lemma 3 that
\[
|S(V_1, \omega, \theta) - S(V_2, \omega, \theta)|
\leq \left| \sqrt{\frac{\tau}{2}} \int_{\partial B_R} \Lambda_1(V_1, \lambda)(\varphi_\omega)\varphi_{-\theta} - \Lambda_1(V_2, \lambda)(\varphi_\omega)\varphi_{-\theta} \text{d}s(x) \right|
+ \left| \sqrt{\frac{\tau}{2}} \int_{\partial B_R} \Lambda_2(V_1, \lambda)(\varphi_\omega)\varphi_{-\theta} - \Lambda_2(V_2, \lambda)(\varphi_\omega)\varphi_{-\theta} \text{d}s(x) \right|
\leq \tau^2 \|\Lambda_1(V_1, \lambda) - \Lambda_1(V_2, \lambda)\|_1 \|\varphi_\omega\|_{H^{3/2}(\partial B_R)} \|\varphi_{-\theta}\|_{L^2(\partial B_R)}
+ \tau^2 \|\Lambda_2(V_1, \lambda) - \Lambda_2(V_2, \lambda)\|_2 \|\varphi_\omega\|_{H^{1/2}(\partial B_R)} \|\varphi_{-\theta}\|_{H^{3/2}(\partial B_R)},
\]
where $\|\cdot\|_1$ and $\|\cdot\|_2$ stand for the norms in $L(H^{3/2}(\partial B_R), L^2(\partial B_R))$ and $L(H^{-1/2}(\partial B_R), H^{-3/2}(\partial B_R))$, respectively, by choosing $t_1 = 0$ and $t_2 = -\frac{3}{2}$ in Lemma 3. Since
\[
\|\varphi_\omega\|_{H^{3/2}(\partial B_R)} \leq \tau^{3/2}, \quad \|\varphi_{-\theta}\|_{H^{1/2}(\partial B_R)} \leq C, \quad \|\varphi_{-\theta}\|_{H^{3/2}(\partial B_R)} \leq \tau^{3/2},
\]
we have
\[
|S(V_1, \omega, \theta) - S(V_2, \omega, \theta)| \leq \tau^{7/2} \left( \|\Lambda_1(V_1, \lambda) - \Lambda_1(V_2, \lambda)\|_1^2 + \|\Lambda_2(V_1, \lambda) - \Lambda_2(V_2, \lambda)\|_2^2 \right).
\]
Then
\[
\|V\|_{L^2(\partial B_R)}^2 \leq \frac{1}{\tau^{-n+2}} + \tau^{-n/2} \left( \|\Lambda_1(V_1, \lambda) - \Lambda_1(V_2, \lambda)\|_1^2 + \|\Lambda_2(V_1, \lambda) - \Lambda_2(V_2, \lambda)\|_2^2 \right). \tag{3.3}
\]

In what follows we study $\Lambda_1(V_i, \lambda), i = 1, 2$. Decompose $\Lambda_1(V_i, \lambda)$ and $\Lambda_2(V_i, \lambda)$ in the following forms:
\[
\hat{\Lambda}_1(V_i, \lambda) = \hat{\Lambda}_1(V_i, \lambda) + \hat{\Lambda}_1(V_i, \lambda),
\]
\[
\hat{\Lambda}_2(V_i, \lambda) = \hat{\Lambda}_2(V_i, \lambda) + \hat{\Lambda}_2(V_i, \lambda),
\]
where
\[
\hat{\Lambda}_1(V_i, \lambda) = \sum_{k > N} \partial_\nu \phi_k(V_i) \bigg|_{\partial B_R} \frac{1}{\lambda_k^{(V_i)}} - \lambda \left( \int_{\partial B_R} \partial_\nu \phi_k(V_i) f \text{d}s(y) + \int_{\partial B_R} \partial_\nu (\Delta \phi_k(V_i)) g \text{d}s(y) \right),
\]
\[
\hat{\Lambda}_2(V_i, \lambda) = \sum_{k < N} \partial_\nu \phi_k(V_i) \bigg|_{\partial B_R} \frac{1}{\lambda_k^{(V_i)}} - \lambda \left( \int_{\partial B_R} \partial_\nu \phi_k(V_i) f \text{d}s(y) + \int_{\partial B_R} \partial_\nu (\Delta \phi_k(V_i)) g \text{d}s(y) \right),
\]
and
\[
\hat{\Lambda}_2(V_i, \lambda) = \sum_{k > N} \partial_\nu (\Delta \phi_k(V_i)) \bigg|_{\partial B_R} \frac{1}{\lambda_k^{(V_i)}} - \lambda \left( \int_{\partial B_R} \partial_\nu \phi_k(V_i) f \text{d}s(y) + \int_{\partial B_R} \partial_\nu (\Delta \phi_k(V_i)) g \text{d}s(y) \right),
\]
\[
\hat{\Lambda}_2(V_i, \lambda) = \sum_{k < N} \partial_\nu (\Delta \phi_k(V_i)) \bigg|_{\partial B_R} \frac{1}{\lambda_k^{(V_i)}} - \lambda \left( \int_{\partial B_R} \partial_\nu \phi_k(V_i) f \text{d}s(y) + \int_{\partial B_R} \partial_\nu (\Delta \phi_k(V_i)) g \text{d}s(y) \right).
\]

First let us consider $\hat{\Lambda}_d^{(j)}(V_i, z)$ for $d = 1, 2$. Since $\lambda_k(V) \leq k^{4/n}$ for all $k \geq 1$, we have
\[
\|\hat{\Lambda}_d^{(j)}(V_i, z)\|_d \leq \frac{1}{|\Re z|^{d+1}}, \quad N^{4/n} \leq \Re z \text{ or } \Re z \leq 0. \tag{3.4}
\]
In particular, for some sufficiently large \( \tau_0 \geq 1 \) depending on \( N \), we obtain from (3.4) that
\[
\| \tilde{A}_d(V_i, \lambda) \|_d \lesssim \frac{1}{\tau^d}, \quad \tau \geq \tau_0.
\] (3.5)

Combining (3.3) and (3.5) gives
\[
\| V \|_{L^2(B_R)}^2 \lesssim \frac{1}{\tau^{n+2}} + \frac{1}{\tau^{n+2}} + \tau^{n/(n+2)+\tau} \left( \| \tilde{A}_1(V_1, \lambda) - \tilde{A}_1(V_2, \lambda) \|_1^2 + \| \tilde{A}_2(V_1, \lambda) - \tilde{A}_2(V_2, \lambda) \|_2^2 \right)
\lesssim \frac{1}{\tau^{n+2}} + \tau^{n/(n+2)+\tau} \left( \| \tilde{A}_1(V_1, \lambda) - \tilde{A}_1(V_2, \lambda) \|_1^2 + \| \tilde{A}_2(V_1, \lambda) - \tilde{A}_2(V_2, \lambda) \|_2^2 \right). \tag{3.6}
\]

Using Lemma 3 with \( t_1 = 0, t_2 = -2 \) and (3.4), we have for \( d = 1, 2 \) that
\[
\| \tilde{A}_d^{(j)}(V_1, z) - \tilde{A}_d^{(j)}(V_2, z) \|_d \lesssim \frac{1}{|\Re z|^{\rho + \sigma}}, \tag{3.7}
\]
where \( z \in \mathbb{C}, \Re z \leq -2M, 0 \leq j \leq m \) and \( \sigma = \min\{\sigma_1, \sigma_2\} \). Here the constants \( \sigma_1 \) and \( \sigma_2 \) are introduced in Lemma 3.

Hereafter we assume \( \tau \gg 1 \) such that \( \Re \lambda \geq 2M \). For \( \rho \geq 2\Re \lambda \), we set \( \tilde{\lambda} = -\rho + \lambda \). By Taylor’s formula, we have for \( d = 1, 2 \) that
\[
\tilde{A}_d(V_i, \lambda) = \sum_{k=0}^{m-1} \frac{(\lambda - \tilde{\lambda})^k}{k!} \tilde{A}_d^{(k)}(V_i, \tilde{\lambda}) + \int_0^1 \frac{(1-s)^m(\lambda - \tilde{\lambda})^m}{(m-1)!} \tilde{A}_d^{(m)}(V_i, \tilde{\lambda} + s(\lambda - \tilde{\lambda}))ds
\]
\[
:= I_d(V_i, \lambda) + R_d(V_i, \lambda). \tag{3.8}
\]

Since \( \Re \tilde{\lambda} \leq -2M \), an application of (3.7) leads to
\[
\| I_d(V_i, \lambda) - I_d(V_2, \lambda) \|_d \lesssim \frac{1}{\rho^\sigma}. \tag{3.9}
\]

Next we study \( R_1(V_i, \lambda), \ i = 1, 2 \). We start with \( \tilde{A}_1^{(m)}(V_i, z) \) appearing in the integral of \( R_1(V_i, \lambda) \). We know from Lemma 2 that
\[
\tilde{A}_1^{(m)}(V_i, z) f = \sum_{k > N} \partial_\nu (\lambda_k)(V_i) \bigg|_{\partial B_R} \frac{1}{(\lambda_k(V_i) - z)^m}
\]
\[
\times \left( \int_{\partial B_R} \partial_\nu (\lambda_k)(V_i) f ds(y) + \int_{\partial B_R} \partial_\nu (\lambda_k)(V_i) g ds(y) \right).
\]

Denote \( \mu = \mu(s) = \tilde{\lambda} + s(\lambda - \tilde{\lambda}) = \lambda - (1-s)\rho \) and
\[
N_i(\lambda) = \min\{k \geq N; \lambda_{k+1}(V_i) \geq 2\Re \lambda\}.
\]

Let \( N = \max\{N_1, N_2\} \). For \( \Re \lambda \gg 1 \), we decompose \( \tilde{A}_1^{(m)}(V_i, \mu) f \) as follows
\[
\tilde{A}_1^{(m)}(V_i, \mu) f = \tilde{A}_{1,1}^{(m)}(V_i, \mu) f + \tilde{A}_{1,2}^{(m)}(V_i, \mu) f,
\]
where

\[ \tilde{\Lambda}_{1,1}^{(m)}(V_i, \mu) = \sum_{k=N+1}^{N(\lambda)} \partial_{\nu} \phi_k(V_i) \left|_{\partial B_R} \frac{1}{(\lambda_k(V_i) - \mu)^{m+1}} \right. \]

\times \left( \int_{\partial B_R} \partial_{\nu} \phi_k(V_i) f ds(y) + \int_{\partial B_R} \partial_{\nu} (\Delta \phi_k(V_i)) g ds(y) \right),

and

\[ \tilde{\Lambda}_{1,2}^{(m)}(V_i, \mu) = \sum_{k>N(\lambda)} \partial_{\nu} \phi_i(V_i) \left|_{\partial B_R} \frac{1}{(\lambda_k(V_i) - \mu)^{m+1}} \right. \]

\times \left( \int_{\partial B_R} \partial_{\nu} \phi_k(V_i) f ds(y) + \int_{\partial B_R} \partial_{\nu} (\Delta \phi_k(V_i)) g ds(y) \right).

We have

\[ \tilde{\Lambda}_{1,1}^{(m)}(V_1, \mu) - \tilde{\Lambda}_{1,1}^{(m)}(V_2, \mu) = L_1 + L_2 + L_3, \]

where

\[ L_1 f = \sum_{k=N+1}^{N(\lambda)} \partial_{\nu} \phi_k(V_1) \left( \frac{1}{(\lambda_k(V_1) - \mu)^{m+1}} - \frac{1}{(\lambda_k(V_2) - \mu)^{m+1}} \right) \]

\times \left( \int_{\partial B_R} \partial_{\nu} \phi_k(V_1) f ds(y) + \int_{\partial B_R} \partial_{\nu} (\Delta \phi_k(V_1)) g ds(y) \right),

\[ L_2 f = \sum_{k=N+1}^{N(\lambda)} \frac{\partial_{\nu} \phi_k(V_1)}{(\lambda_k(V_2) - \mu)^{m+1}} \left( \int_{\partial B_R} (\partial_{\nu} \phi_k(V_1) - \partial_{\nu} \phi_k(V_2)) f ds(y) \right. \]

\[ + \int_{\partial B_R} (\partial_{\nu} (\Delta \phi_k(V_1)) - \partial_{\nu} (\Delta \phi_k(V_2))) g ds(y) \right), \]

\[ L_3 f = \sum_{k=N+1}^{N(\lambda)} \frac{1}{(\lambda_k(V_2) - \mu)^{m+1}} \left( \int_{\partial B_R} \partial_{\nu} \phi_k(V_2) f ds(y) \right. \]

\[ + \int_{\partial B_R} \partial_{\nu} (\Delta \phi_k(V_2)) g ds(y) \right) (\partial_{\nu} \phi_k(V_1) - \partial_{\nu} \phi_k(V_2)). \]

For fixed \( \beta > \frac{8}{n} + 1 \), we have from a simple calculation that

\[ \| L_1 \| \leq \frac{N(\lambda)^{\beta}}{|3\lambda|^{m+2}} \delta_0(V_1, V_2) \left( \sum_{k=N+1}^{N(\lambda)} k^{-\beta} \| \partial_{\nu} \phi_k(V_1) \|_{L^2(\partial B_R)}^2 \right. \]

\[ + \sum_{k=N+1}^{N(\lambda)} k^{-\beta} \| \partial_{\nu} \phi_k(V_1) \|_{L^2(\partial B_R)} \| \partial_{\nu} (\Delta \phi_k(V_1)) \|_{L^2(\partial B_R)} \right). \]

Since

\[ \| \partial_{\nu} \phi_k(V_i) \|_{L^2(\partial B_R)} \leq k^{2/n}, \quad \| \partial_{\nu} (\Delta \phi_k(V_i)) \|_{L^2(\partial B_R)} \leq k^{4/n}, \]
we get
\[ N(\lambda) \sum_{k=N+1}^{N} k^{-\beta} \| \partial_{\nu} \phi_k(V_1) \|_{L^2(\partial B_R)}^2 + k^{-\beta} \| \partial_{\nu} \phi_k(V_1) \|_{L^2(\partial B_R)} \| \partial_{\nu} (\Delta \phi_k(V_1)) \|_{L^2(\partial B_R)} \leq \sum_{k \geq 1} k^{-\beta+6/n}. \]

Thus,
\[ \| L_1 \| \leq \frac{N(\lambda)^{\beta}}{3\lambda|m+2}\delta_0(V_1, V_2). \]

Let
\[ \delta_1(V_1, V_2) = \sum_{k \geq 1} k^{-4m/n} \| \partial_{\nu} \phi_k + N(V_1) - \partial_{\nu} \phi_k + N(V_2) \|_{L^2(\partial B_R)}, \]
\[ \delta_2(V_1, V_2) = \sum_{k \geq 1} k^{-4m/n} \| \partial_{\nu} (\Delta \phi_k + N(V_1)) - \partial_{\nu} (\Delta \phi_k + N(V_2)) \|_{L^2(\partial B_R)}. \]

Similarly, we obtain
\[ \| L_2 \| \leq \frac{N(\lambda)^{4m/n+2/n}}{|3\lambda|m+1}(\delta_1(V_1, V_2) + \delta_2(V_1, V_2)), \]
\[ \| L_2 \| \leq \frac{N(\lambda)^{4m/n+4/n}}{|3\lambda|m+1}\delta_1(V_1, V_2). \]

Letting \( \delta = \delta_0(V_1, V_2) + \delta_1(V_1, V_2) + \delta_2(V_1, V_2) \), we have
\[ \| L_1 \| + \| L_2 \| + \| L_3 \| \leq \frac{N(\lambda)^{\beta} + N(\lambda)^{4(m+1)/n}}{|3\lambda|m+1}\delta. \]

By choosing \( \beta = 4(m+1)/n \) we have \( \beta > \frac{8}{n} + 1 \) from our choice of \( m \), which gives
\[ \| \tilde{\Lambda}_{1,1}^{(m)}(V_1, \mu) - \tilde{\Lambda}_{1,1}^{(m)}(V_2, \mu) \|_1 \leq \frac{N(\lambda)^{4(m+1)/n}}{|3\lambda|m+1}\delta. \tag{3.10} \]

Since
\[ |\lambda_k(V_i) - \mu| \geq \lambda_f(V_i) - \Re \lambda + (1-s)\rho \geq \lambda_k(V_i) - \Re \lambda \geq \frac{\lambda_k(V_i)}{2}, \]
which gives
\[ (\lambda(V_i) - \lambda_k)^{m+1} \leq \frac{1}{\lambda_k^{m+1}(V_i)} \leq \frac{1}{(k^n)^{m+1}} \leq k^{-4m/n}. \]

Therefore, using similar arguments to decompose \( \tilde{\Lambda}_{1,1}^{(m)}(V_1, \lambda) - \tilde{\Lambda}_{1,1}^{(m)}(V_2, \lambda) \) into three parts, we may obtain
\[ \| \tilde{\Lambda}_{1,1}^{(m)}(V_1, \lambda) - \tilde{\Lambda}_{1,1}^{(m)}(V_2, \lambda) \|_1 \leq \delta, \]
which, together with (3.10), implies
\[ \| \tilde{\Lambda}_{1}^{(m)}(V_1, \lambda) - \tilde{\Lambda}_{1}^{(m)}(V_2, \lambda) \|_1 \leq N(\lambda)^{4(m+1)/n}\delta. \]

From the definition of \( N(\lambda) \), we have
\[ N^{4/n} \leq \lambda_N(V_i) \leq 2\Re \lambda. \]
Hence,
\[ \| \tilde{\Lambda}_1^{(m)}(V_1, \lambda) - \tilde{\Lambda}_1^{(m)}(V_2, \lambda) \|_1 \lesssim |\Re \lambda|^{m+1} \delta, \]
which, together with $|\Re \lambda| \leq \tau^4$, gives
\[ \| \tilde{\Lambda}_1^{(m)}(V_1, \lambda) - \tilde{\Lambda}_1^{(m)}(V_2, \lambda) \|_1 \lesssim \tau^{4(m+1)} \delta. \]

Then
\[ \| R_1(V_1, \lambda) - R_1(V_2, \lambda) \|_1 \lesssim \rho^m \tau^{4(m+1)} \delta. \]  \hfill (3.11)

It follows from (3.8), (3.9) and (3.11) that
\[ \| \tilde{\Lambda}_1(V_1, \mu) - \tilde{\Lambda}_1(V_2, \mu) \|_1 \lesssim \frac{1}{\rho^\sigma} + \rho^m \tau^{4(m+1)} \delta. \]

Using similar arguments, we obtain
\[ \| R_2(V_1, \lambda) - R_2(V_2, \lambda) \|_2 \lesssim \rho^m \tau^{4(m+1)} \delta, \]
which gives
\[ \| \tilde{\Lambda}_2(V_1, \mu) - \tilde{\Lambda}_2(V_2, \mu) \|_2 \lesssim \frac{1}{\rho^\sigma} + \rho^m \tau^{4(m+1)} \delta. \]

Substituting the above estimates into (3.6) yields
\[ \| V \|_{L^2(B_R)}^2 \lesssim \frac{1}{\tau^{n+2}} + \tau^{n/(n+2)+7} \left( \frac{1}{\rho^\sigma} + \rho^2 \tau^{8(m+1)} \right)^2. \]

Taking $\rho = (2 \Re \lambda)^{\kappa}$ with $\kappa = 1/\sigma$ gives
\[ \| V \|_{L^2(B_R)}^2 \lesssim \frac{1}{\tau^{n+2}} + \tau^{n/(n+2)+7+8\kappa m+8(m+1)} \delta^2. \]

Using the standard minimization with respect to $\tau$, we obtain the stability estimate
\[ \| V \|_{L^2(B_R)}^2 \lesssim \delta^7, \]
where
\[ \gamma = \frac{1}{4(n+2)(2+\kappa m+m)}. \]

**Appendix A. Useful estimates**

**Theorem 5.** Let $u \in H^2(\Omega)$ be a weak solution of the following boundary value problem:

\[
\begin{align*}
H(V)u &= F_1 \quad \text{in } \Omega, \\
u &= f \quad \text{on } \partial \Omega, \\\n\Delta u &= g \quad \text{on } \partial \Omega,
\end{align*}
\]
where $H(V) = \Delta^2 + V$ and $0$ is not an eigenvalue of $H(V)$ with the Navier boundary condition. Then
\[ \| u \|_{H^2(\Omega)} \lesssim \| F \|_{L^2(\Omega)} + \| f \|_{H^{\frac{4}{3}}(\partial \Omega)} + \| g \|_{H^{-\frac{1}{2}}(\partial \Omega)}. \]

The following lemma gives an estimate for the normal derivatives of the eigenfunctions on $\partial B_R$ and a Weyl-type inequality for the Dirichlet eigenvalues.
Lemma 6. The following estimate holds in \(\mathbb{R}^n\):
\[
\|\partial_\nu \phi_k\|_{L^2(\partial B_R)} \leq C \lambda_k^{\frac{1}{2}}, \quad \|\partial_\nu (\Delta \phi_k)\|_{L^2(\partial B_R)} \leq C \lambda_k,
\]
where the positive constant \(C\) is independent of \(k\). Moreover, the following Weyl-type inequality holds for the Dirichlet eigenvalues \(\{\mu_k\}_{k=1}^\infty\):
\[
E_1 k^{4/n} \leq \lambda_k \leq E_2 k^{4/n},
\]
where \(E_1\) and \(E_2\) are two positive constants independent of \(k\).

Proof. We begin with the estimate (A.1) for the eigenfunctions on the boundary. Let \(u\) be an eigenfunction with eigenvalue \(\mu\) such that
\[
\begin{aligned}
Hu &= \lambda u & \text{in } B_R, \\
u &= \Delta u = 0 & \text{on } \partial B_R.
\end{aligned}
\]
Define a differential operator
\[
A = \frac{1}{2} (x \cdot \nabla + \nabla \cdot x) = x \cdot \nabla + \frac{n}{2} = |x| \partial_\nu + \frac{n}{2},
\]
Denote the commutator of two differential operators by \([\cdot, \cdot]\) such that \([O_1, O_2] = O_1 O_2 - O_2 O_1\) for two differential operators \(O_1\) and \(O_2\). Then we have
\[
[\Delta^k, A] = 2k \Delta^k, \quad k \in \mathbb{N}^+.
\]
Denote \(B = A \Delta\). A simple calculation gives
\[
\int_{\partial B_R} u[H, B]u \, dx = \int_{B_R} \left( (u(\Delta^2 + V)(Bu) - uB(\Delta^2 + V)u) \right) \, dx
\]
\[
= \int_{B_R} (\Delta^2 u + V u - \lambda u) Bu \, dx + \int_{\partial B_R} (u \partial_\nu (\Delta Bu) - \partial_\nu (\Delta u Bu)) \, ds
\]
\[
+ \int_{\partial B_R} (\Delta u \partial_\nu (Bu) - \partial_\nu (\Delta u) Bu) \, ds
\]
\[
= -\int_{\partial B_R} (\partial_\nu u \Delta (Bu) + \partial_\nu (\Delta u) Bu) \, ds
\]
\[
= -\int_{\partial B_R} (\partial_\nu u \Delta (Bu) + R |\partial_\nu (\Delta u)|^2) \, ds,
\]
where we have used \(u = \Delta u = 0\) on \(\partial B_R\) and Green’s formula. By (A.3), we have
\[
\Delta Bu = \Delta A \Delta = (A \Delta + 2 \Delta) \Delta = A \Delta^2 + 2 \Delta^2.
\]
It holds that
\[
\int_{\partial B_R} \partial_\nu u \Delta (Bu) \, ds = \int_{\partial B_R} \partial_\nu u (A \Delta^2 + 2 \Delta^2) \, ds
\]
\[
= \int_{\partial B_R} \left( \partial_\nu u \left( (R \partial_\nu + \frac{n}{2}) \Delta^2 u \right) + 2 \Delta^2 u \right) \, ds
\]
\[
= R \int_{\partial B_R} \partial_\nu u \partial_\nu (\Delta^2 u) \, ds = R \int_{\partial B_R} \partial_\nu u \partial_\nu (\mu u - Vu) \, ds,
\]
where we have used $\Delta^2 u = -Vu + \lambda u = 0$ and $u = 0$ on $\partial B_R$. Hence
\[
\left| \int_{\partial B_R} \partial_{\nu} u \Delta(Bu) ds \right| \geqslant (\lambda - \|V\|_{L^\infty(B_R)}) \int_{\partial B_R} |\partial_{\nu} u|^2 ds. \tag{A.4}
\]

On the other hand we have
\[
\int_{\partial B_R} \partial_{\nu} (\Delta u) Bu ds = \int_{\partial B_R} \partial_{\nu} (\Delta u) Bu ds = R \int_{\partial B_R} |\partial_{\nu} (\Delta u)|^2 ds. \tag{A.5}
\]
Moreover, it follows from \[A.3\] that $[H, B] = 4\Delta^3 + [V, A\Delta]$, which gives
\[
\left| \int_{B_R} u[H, B]udx \right| = \left| \int_{B_R} (4u\Delta^3 u + [V, A\Delta] u) dx \right|
= \left| \int_{B_R} (4u\Delta(-Vu + \lambda u) + [V, A\Delta] u) dx \right|
\leqslant C\lambda \|u\|^2_{H^2(B_R)} \leqslant C\lambda^2. \tag{A.6}
\]
Here we have used the fact that the commutator $[V, A\Delta]$ has order of 2 at most. Using \[A.4\]–\[A.6\] we obtain
\[
\|\partial_{\nu} u\|^2_{L^2(\partial B_R)} \leqslant \lambda, \quad \|\partial_{\nu} (\Delta u)\|^2_{L^2(\partial B_R)} \leqslant \lambda^2,
\]
which completes the proof of \[A.1\].

Next, we prove the Weyl-type inequality \[A.2\]. Assume $\lambda_1 < \lambda_2 < \cdots$ are the eigenvalues of the operator $H$. Denote the functional space
\[
H^2_0(B_R) = \{\psi \in H^2(B_R); \Delta \psi = \psi = 0 \text{ on } \partial B_R\},
\]
Then we have following min-max principle:
\[
\lambda_k = \max_{\psi_1, \cdots, \psi_{k-1}} \min_{\psi \in H^2_0(B_R)} \frac{\int_{B_R} |\Delta \psi|^2 + V|\psi|^2 dx}{\int_{B_R} \psi^2 dx}.
\]
Assume that $\lambda_1^{(1)} < \lambda_2^{(1)} < \cdots$ are the eigenvalues for the operator $\Delta^2$. By the min-max principle, we have
\[
C_1 \lambda_k^{(1)} < \lambda_k < C_2 \lambda_k^{(1)}, \quad k = 1, 2, \ldots,
\]
where $C_1$ and $C_2$ are two positive constants depending on $\|V\|_{L^\infty(B_R)}$. We have from Weyl’s law \[24\] for $\Delta^2$ that
\[
\lim_{k \to +\infty} \frac{\lambda_k^{(1)}}{k^{4/n}} = D,
\]
where $D$ is a constant. Therefore there exist two constants $E_1$ and $E_2$ such that
\[
E_1 k^{4/n} \leqslant \lambda_k \leqslant E_2 k^{4/n},
\]
which completes the proof. □

Denote the resolvent by $R(V, \lambda) = (-\Delta + V - \lambda)^{-1}, \lambda \in \mathbb{C}$. The following theorem gives a resonance-free region and a resolvent estimate of $\rho R(V, \lambda) \rho : L^2(\mathbb{R}^n) \to H^4(\mathbb{R}^n)$ for a given $\rho \in C_0^\infty(\mathbb{R}^n)$ when $n \geqslant 3$ is odd.
Theorem 7. Let $V(x) \in L^\infty_{\text{comp}}(\mathbb{R}^n, \mathbb{C})$ and $n \geq 3$ be odd. Then for any given $\rho \in C_0^\infty(\mathbb{R}^n)$ satisfying $\rho V = V$, i.e., $\text{supp}(V) \subset \text{supp}(\rho) \subset B_R$, there exists a positive constant $C$ depending on $\rho$ and $V$ such that

$$
\|\rho \mathcal{R}_V(\lambda)\rho\|_{L^2(B_R) \to H^1(B_R)} \leq C|\lambda|^{\frac{2n}{4}} \left( e^{2R(3\sqrt[n]{\lambda})_-} + e^{2R(3\sqrt[n]{\lambda})_-} \right), \quad j = 0, 1, 2, 3, 4, \quad (A.7)
$$

where $\lambda \in \Omega_\delta$. Here $\Omega_\delta$ denotes the resonance-free region defined as

$$
\Omega_\delta := \left\{ \lambda : \Re \sqrt[n]{\lambda} \geq -A - \delta \log(1 + |\lambda|^{1/4}), \Re \sqrt[n]{\lambda} \geq -A - \delta \log(1 + |\lambda|^{1/4}), |\lambda|^{1/4} \geq C_0 \right\},
$$

where $A$ and $C_0$ are two positive constants and $\delta$ satisfies $0 < \delta < \frac{1}{4R}$.

Proof. Denote the free resolvent by $R_0(\lambda) = (-\Delta - \lambda)^{-1}$, $\lambda \in \mathbb{C}$. Using the following identity

$$
R_0(\lambda) = (-\Delta^2 - \lambda)^{-1} = \frac{1}{2\sqrt{\lambda}} \left[ (-\Delta - \sqrt[n]{\lambda})^{-1} - (-\Delta + \sqrt[n]{\lambda})^{-1} \right] \quad (A.8)
$$

and [9, Theorem 3.1], we can prove that when $n \geq 3$ is odd, for each $\rho \in C_0^\infty(\mathbb{R}^n)$ with supp($\rho$) $\subset B_R$ and $\lambda \neq 0$

$$
\|\rho R_0(\lambda)\rho\|_{L^2(B_R) \to L^2(B_R)} \leq \frac{1}{\sqrt{\lambda}} \left( e^{2R(3\sqrt[n]{\lambda})_-} + e^{2R(3\sqrt[n]{\lambda})_-} \right), \quad (A.9)
$$

where $t_- := \max\{-t, 0\}$. Consequently, using similar arguments as in the proofs of [20, Theorem 2.1] and [20, Theorem 3.3], we can prove the estimate (A.7). \hfill \Box

Remark 8. We discuss the resolvent estimates in even dimensions $n \geq 2$. Since the free resolvent $G_0(\lambda) = (-\Delta - \lambda)^{-1}$ in even dimensions is a convolution operator with the kernel (see e.g. [10])

$$
G_0(\lambda) = \frac{c_n e^{i\sqrt[n]{\lambda}|x|}}{|x|^{n-2}} \int_0^\infty e^{-\frac{t}{2}} \left( \frac{t}{2} - i\sqrt[n]{\lambda}|x| \right)^{\frac{n-3}{2}} dt,
$$

where $c_n$ is a positive constant depending on the dimension $n$. Then by (A.8) and a direct calculation we have

$$
|G_0(\lambda)| \leq \frac{|\lambda|^{\frac{n-3}{4}}}{|\lambda|} \left( e^{(3\sqrt[n]{\lambda})_-|x|} + e^{(R\sqrt[n]{\lambda})_-|x|} \right) \leq \frac{1}{|\lambda|^{1-\frac{n-3}{4}}} \left( e^{(3\sqrt[n]{\lambda})_-|x|} \right),
$$

which implies that only when $1 - \frac{n-3}{4} > 0$, by repeating the above arguments we may obtain similar resolvent estimates for even dimensional cases.

References

[1] G. Alessandrini and J. Sylvester, Stability for a multidimensional inverse spectral theorem, Comm. Part. Diff. Eqs., 15 (1990), 711–736.
[2] G. Bao, X. Xu, and J. Zhai, An inverse spectral problem for a damped wave operator, SIAM J. Appl. Math., to appear.
[3] M. Belishev and Y. Kurylev, To the reconstruction of a Riemannian manifold via its spectral data (BC-method), Commun. Part. Diff. Eq., 17 (1992), 767–804.
[4] M. Bellassoued and D. Dos Santos Ferreira, Stability estimates for the anisotropic wave equation from the Dirichlet-to-Neumann map, Inverse Problems, 27 (2011), 745–773.
[5] G. Borg, Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte [A reversal of the Sturm-Liouville eigenvalue problem. Determination of the differential equation by the eigenvalues], Acta Math., 78 (1946), 1–96.
[6] F. Cakoni, D. Colton, and H. Haddar, A duality between scattering poles and transmission eigenvalues in scattering theory, Proc. R. Soc. A, 476 (2020), 20200612.

[7] M. Choulli, Une Introduction aux Problemes Inverses Elliptiques et Paraboliques [An Introduction to Elliptic and Parabolic Inverse Problems]. Mathématiques and Applications (Berlin), vol. 65 (2009), Berlin: Springer-Verlag.

[8] M. Choulli and P. Stefanov, Stability for the multi-dimensional Borg–Levinson theorem with partial spectral data, Commun. Part. Diff. Eq., 38 (2013), 455–476.

[9] S. Dyatlov and M. Zworski, Mathematical Theory of Scattering Resonances, vol. 200, American Mathematical Soc., 2019.

[10] D. Finco and K. Yajima, The $L^p$ boundedness of wave operators for Schrödinger operators with threshold singularities. II. Even dimensional case, J. Math. Sci. Univ. Tokyo, 13 (2006), 277–346.

[11] F. Gazzola, H.-C. Grunau, and G. Sweers, Polyharmonic Boundary Value Problems, Positivity Preserving and Nonlinear Higher Order Elliptic Equations in Bounded Domains, Lecture Notes in Mathematics, Springer-Verlag, Berlin, Heidelberg, 2010.

[12] M. Ikehata, A special Green’s function for the biharmonic operator and its application to an inverse boundary value problem, Multidimensional inverse problems, Comput. Math. Appl., 22 (1991), 53–66.

[13] V. Isakov, Completeness of products of solutions and some inverse problems for PDE, J. Differential Equations, 92 (1991), 305–316.

[14] H. Isozaki, Some remarks on the multi-dimensional Borg–Levinson theorem, J. Math. Kyoto Univ., 31 (1991), 743–753.

[15] A. Katchalov and Y. Kurylev, Multidimensional inverse problem with incomplete boundary spectral data, Comm. Partial Differential Equations, 23 (1998), 55–95.

[16] A. Kachalov, Y. Kurylev, and M. Lassas, Inverse Boundary Spectral Problems, in: CRC Res. Notes in Math., 2001.

[17] O. Kavian, Y. Kian, and E. Soccorsi, Uniqueness and stability results for an inverse spectral problem in a periodic waveguide, J. Math. Pures Appl., 104 (2015), 1160–1189.

[18] K. Krupchyk and P. Päivärinta, Borg–Levinson theorem for higher order elliptic operators, Int. Math. Res. Notices, 6 (2012), 1321–1351.

[19] N. Levinson, The inverse Sturm–Liouville problem, Mat. Tidsskr. B., 1949, 25–30.

[20] P. Li, X. Yao, and Y. Zhao, Stability for an inverse source problem of the biharmonic operator, SIAM J. Appl. Math., to appear.

[21] P. Li, J. Zhai, and Y. Zhao, Stability for the acoustic inverse source problem in inhomogeneous media, SIAM J. Appl. Math., 80 (2020), 2547–2559.

[22] N.V. Movchan, R.C. McPhedran, A.B. Movchan, and C.G. Poulton, Wave scattering by platonic grating stacks, Proc. R. Soc. A, 465 (2009), 3383–3400.

[23] A. Nachman, J. Sylvester, and G. Uhlmann, An n-dimensional Borg–Levinson theorem, Comm. Math. Phys., 115 (1988), 595–605.

[24] H. Weyl, Das asymptotische verteilungsgesetz der eigenwerte linear partieller differentialgleichungen, Math. Ann., 71 (1911), 441–479.

[25] X. Xu and J. Zhai, Inversion of trace formulas for a Sturm-Liouville operator, J. Comput. Math., to appear.

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907, USA

Email address: lipeijun@math.purdue.edu

School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, China

Email address: yaoxiaohua@mail.ccnu.edu.cn

School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, China

Email address: zhaoyueccnu@163.com