Parameterization of irreducible characters for \( p \)-solvable groups

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1. Introduction

1.1. Let \( p \) be a prime number, \( k \) an algebraically closed field of characteristic \( p \) and \( G \) a \( p \)-solvable finite group. In [16], up to the choice of a polarization \( \omega \), we give a natural parameterization of the set of isomorphism classes of the simple \( kG \)-modules, in terms of the set of \( G \)-conjugacy classes of the weights of \( G \) with respect to \( p \), introduced by Jon Alperin in [1] in order to formulate his celebrated conjecture.

1.2. In [4] Everett Dade formulates a refinement of Alperin’s conjecture involving ordinary irreducible characters — with their defect — instead of simple \( kG \)-modules, and in [17] Geoffrey Robinson proves that the new conjecture holds for \( p \)-solvable groups. But this refinement is formulated in terms of a vanishing alternating sum of cardinals of suitable sets of irreducible characters, without giving any possible refinement for the weights.

1.3. Our purpose in this note is to show that, in the case of the \( p \)-solvable finite groups, the method developed in [16] can be suitably refined to provide, up to the choice of a polarization \( \omega \), a natural bijection — namely compatible with the action of the group of outer automorphisms of \( G \) — between the sets of absolutely irreducible characters of \( G \) and of \( G \)-conjugacy classes of suitable inductive weights of \( G \) defined in §8 below. Moreover, any inductive weight of \( G \) admits an associated block of \( G \) and a natural definition of a defect in such a way that then our bijection preserves the associated blocks and the defects.

1.4. Let \( \mathcal{O} \) be a complete discrete valuation ring of characteristic zero admitting \( k \) as the residue field and denote by \( K \) the quotient field of \( \mathcal{O} \). As in [16], the inductive arguments lead to consider a more general situation including central extensions of the involved finite groups; but, as we will see below, in the present case the central \( k^* \)-extensions considered in [16] have to be replaced by the central \( \mathcal{O}^* \)-extensions, analogously called \( \mathcal{O}^* \)-groups.

1.5. Contrarily to the \( k^* \)-group case, a block of an \( \mathcal{O}^* \)-group need not be isomorphic to a block of a finite group; although the arguments are very similar, this forces us to develop to some extend the basic results on \( \mathcal{O}^* \)-groups; in particular, our results on \( p \)-solvable finite \( k^* \)-groups in [15] have to be extended to the \( p \)-solvable finite \( \mathcal{O}^* \)-groups.
2. Generalities on $O^*$-groups

2.1. As above, let $p$ be a prime number and $O$ a complete discrete valuation ring with a quotient field $K$ of characteristic zero and an algebraically closed residue field $k$ of characteristic $p$. An $O^*$-group is a group $\hat{G}$ endowed with an injective group homomorphism $\varphi: O^* \to Z(\hat{G})$ where $Z(\hat{G})$ denotes the center of $\hat{G}$. As in the $k^*$-group case [6, §5], we call $O^*$-quotient of $\hat{G}$ the quotient $G = \hat{G}/\text{Im}(\varphi)$ and write $\lambda.x$ instead of $\varphi(\lambda)x$ for any $\lambda \in O^*$ and any $x \in \hat{G}$; we say that $\hat{G}$ is split whenever $\hat{G} \cong O^* \times G$. Note that, the central product

$$\hat{G}^k = k^* \times_{O^*} \hat{G} \cong \hat{G}/\varphi(1 + J(O))$$ 2.1.1

is a $k^*$-group of $k^*$-quotient $G$. Similarly, if $K'$ is a finite field extension of $K$ and $O'$ is the ring of integers of $K'$, the central product

$$\hat{G}^{O'} = O'^* \times_{O^*} \hat{G}$$ 2.1.2

has an evident structure of $O'^*$-group with the same $O'^*$-quotient $G$; moreover, since the residue field of $O'$ coincides with $k$, the degree of $K'$ over $K$ coincides with the corresponding ramification index of $O'$ over $O$.

2.2. The $O^*$-group algebra of an $O^*$-group $\hat{G}$ is the $O$-algebra

$$O_\bullet \hat{G} = O \otimes_{O^*} O\hat{G}$$ 2.2.1

where $O \otimes O^*$ and $O \hat{G}$ denote the respective ordinary group algebras of $O^*$ and $\hat{G}$ over $O$; we also set $K_\bullet \hat{G} = K \otimes O_\bullet \hat{G}$. The introduction of $O^*$-groups mainly depends on the well-known situation exhibited by the next proposition where $\hat{G}$ need not be split even if $\hat{H}$ is so. We use terminology introduced in 2.6 below.

**Proposition 2.3.** Let $G$ be a finite group, $H$ a normal subgroup of $G$, $\hat{H}$ an $O^*$-group with $O^*$-quotient $H$. Assume that the action of $G$ on $\hat{H}$ can be lifted to an action of $G$ on $\hat{H}$ and that $G$ stabilizes an absolutely irreducible character $\zeta$ of $\hat{H}$. Denote by $e_\zeta$ the primitive idempotent of $Z(\hat{K}_\bullet \hat{H})$ associated with $\zeta$. If $K'$ is a finite field extension of $K$ of degree divisible by $[G/\langle G, G \rangle]$, denoting by $O'$ the ring of integers of $K'$ there are an $O'^*$-group $\hat{G}$ with $O'^*$-quotient $G$ containing and normalizing $\hat{H}^{O'}$, and an $O'^*$-group homomorphism

$$\hat{G} \longrightarrow (K'_\bullet \hat{H}^{O'} e_\zeta)^*$$ 2.3.1

lifting the action of $G$ on $K'_\bullet \hat{H}^{O'} e_\zeta$ and extending the structural $O'^*$-group homomorphism from $\hat{H}^{O'}$, and they are unique up to a unique $O'^*$-group isomorphism inducing the identity on $G$. 
Proof: We are assuming that $G$ acts on the $\mathcal{K}$-algebra $\mathcal{K}_* \hat{H} e_\xi$ which is a full matrix algebra over $\mathcal{K}$ and therefore the pull-back

$$
\begin{align*}
G & \longrightarrow \text{Aut}(\mathcal{K}_* \hat{H} e_\xi) \\
\uparrow & \\
\hat{G}^\mathcal{K} & \longrightarrow (\mathcal{K}_* \hat{H} e_\xi)^*
\end{align*}
$$

2.3.2

determines a central $\mathcal{K}^*$-extension $\hat{G}^\mathcal{K}$ of $G$; note that the inclusion $H \subset G$ and the structural $\mathcal{O}^*$-group homomorphism $\hat{H} \to (\mathcal{K}_* \hat{H} e_\xi)^*$ determine a $G$-stable “inclusion” $\hat{H} \subset \hat{G}^\mathcal{K}$.

On the other hand, the valuation maps $\vartheta: \mathcal{K}^* \to \mathbb{Z}$ and $\vartheta': \mathcal{K}'^* \to \mathbb{Z}$ determine a commutative diagram of exact sequences

$$
\begin{align*}
1 & \to \mathcal{O}^* \longrightarrow \mathcal{K}^* \vartheta \longrightarrow \mathbb{Z} \to 1 \\
& \cap \quad \cap \quad \quad \downarrow \\
1 & \to \mathcal{O}'^* \longrightarrow \mathcal{K}'^* \vartheta' \longrightarrow \mathbb{Z} \to 1
\end{align*}
$$

2.3.3

where the vertical right-hand arrow is the multiplication by the degree $[\mathcal{K} : \mathcal{K}']$ of the extension; hence, since $\mathbb{H}^1(G, \mathbb{Z}) = \{0\}$, they determine the following commutative diagram of exact sequences [3, Chap. XII, §2]

$$
\begin{align*}
0 & \to \mathbb{H}^2(G, \mathcal{O}^*) \to \mathbb{H}^2(G, \mathcal{K}^*) \to \mathbb{H}^2(G, \mathbb{Z}) \\
0 & \to \mathbb{H}^2(G, \mathcal{O}'^*) \to \mathbb{H}^2(G, \mathcal{K}'^*) \to \mathbb{H}^2(G, \mathbb{Z})
\end{align*}
$$

2.3.4

where the vertical right-hand arrow is the multiplication by $[\mathcal{K} : \mathcal{K}']$; hence, since we have $[9, 2.6.2]$

$$
\mathbb{H}^2(G, \mathbb{Z}) \cong \text{Ext}_G^1(G/[G,G], \mathbb{Z})
$$

2.3.5

it follows from our hypothesis that this vertical arrow is zero.

Consequently, the element of $\mathbb{H}^2(G, \mathcal{K}'^*)$ determined by the central product $\mathcal{K}'^* \times_{\mathcal{K}^*} \hat{G}^\mathcal{K}$ has a trivial image in $\mathbb{H}^2(G, \mathbb{Z})$ or, equivalently, there is a surjective group homomorphism

$$
\theta: \mathcal{K}'^* \times_{\mathcal{K}^*} \hat{G}^\mathcal{K} \longrightarrow \mathbb{Z}
$$

2.3.6

fulfilling $\theta((\mathcal{O}'^*,1)) = \{0\}$; moreover, this homomorphism is unique since we still have $\mathbb{H}^1(G, \mathbb{Z}) = \{0\}$. Thus, $\hat{G} = \text{Ker}(\theta)$ is an $\mathcal{O}'^*$-group with $\mathcal{O}'^*$-quotient $G$, $\hat{G}$ contains and normalizes $\hat{H}^\mathcal{O}'$ since $\hat{G}^\mathcal{K}$ contains and normalizes $\hat{H}$, and it is easily checked that the bottom arrow in diagram 2.3.2 induces an $\mathcal{O}'^*$-group homomorphism $\hat{G} \longrightarrow (\mathcal{K}_* \hat{H} e_\xi)^*$ which fulfills all the requirements.
2.4. Let $\hat{G}$ be an $O^*$-group with a finite $O^*$-quotient $G$; whereas any finite $k^*$-group contains a finite subgroup covering its own $k^*$-quotient [6, Lemma 5.5], $\hat{G}$ need not contain a finite subgroup covering $G$; but, if $G'$ and $G''$ are two finite subgroups of $\hat{G}$ covering $G$, they normalize each other and therefore $G'.G''$ is also a finite subgroup of $\hat{G}$ covering $G$; in particular, if $\hat{G}$ contains finite subgroups covering $G$, $\text{Aut}(\hat{G})$ stabilizes one of them.

**Proposition 2.5.** For some finite extension $K'$ of $K$, denoting by $O'$ the ring of integers of $K'$, the $O'^*$-group $\hat{G}^{O'}$ contains a finite subgroup covering $G$.

**Proof:** Denote by $M$ the group of Schur multipliers of $G$ and by $A$ the maximal Abelian quotient of $G$; since we have the exact sequence [9, 2.6.2]

$$0 \to \text{Ext}_1^Z(A, O^*) \to H^2(G, O^*) \to \text{Hom}(M, O^*) \to 0$$

the element $h$ of $H^2(G, O^*)$ corresponding to $\hat{G}$ induces a group homomorphism $\sigma: M \to O^*$ which then determines an evident commutative diagram of exact sequences

$$0 \to \text{Ext}_1^Z(A, M) \to H^2(G, M) \to \text{Hom}(M, M) \to 0$$

$$0 \to \text{Ext}_1^Z(A, O^*) \to H^2(G, O^*) \to \text{Hom}(M, O^*) \to 0$$

Then, an element of $H^2(G, M)$ lifting $\text{id}_M$ determines an $M$-extension $E$ of $G$ (which is finite!) and the central product $O^* \times_M E$ determines $h' \in H^2(G, O^*)$ lifting $\sigma$, so that the difference $h' - h$ comes from $\text{Ext}_1^Z(A, O^*)$.

That is to say, $h' - h$ determines an Abelian $O^*$-extension $Z$ of $A$ and, choosing a finite subset $X$ of $Z$ such that

$$A = \prod_{x \in X} < x >$$

where $\bar{x}$ is the image of $x \in X$ in $A$, we consider the field extension $K'$ of $K$ generated by $\mid < \bar{x} > \mid$-th roots $\lambda_x$ of the elements $x^{\mid < \bar{x} > \mid}$ of $O^*$ when $x$ runs over $X$; then, denoting by $O'$ the ring of integers of $K'$, the $O'^*$-group $Z^{O'}$ is split since the subgroup

$$A' = \prod_{x \in X} < \lambda_x^{-1}.x >$$

of $Z$ provides an $O'^*$-section. Consequently, denoting by $\iota : O^* \to O'^*$ the inclusion map, we get the equality $H^2(G, \iota)(h' - h) = 0$ or, equivalently, an $O'^*$-group isomorphism

$$O'^* \times_M E \cong \hat{G}^{O'}$$

inducing the identity on $G$; now, the image of $E$ in $\hat{G}^{O'}$ is the announced finite subgroup.
2.6. If \( \hat{G} \) and \( \hat{G}' \) are two \( O^* \)-groups, we denote by \( \hat{G} \times \hat{G}' \) the quotient of the direct product \( \hat{G} \times \hat{G}' \) by the image in \( \hat{G} \times \hat{G}' \) of the inverse diagonal of \( O^* \times O^* \), which has an obvious structure of \( O^* \)-group with \( O^* \)-quotient \( G \times G' \); moreover, if \( G = G' \) then we denote by \( \hat{G} \ast \hat{G}' \) the \( O^* \)-group obtained from the inverse image of \( \Delta(G) \subset G \times G \) in \( \hat{G} \times \hat{G}' \), which is nothing but the so-called sum of both central \( O^* \)-extensions of \( G \); in particular, we have a canonical \( k^* \)-group isomorphism

\[
\hat{G} \ast \hat{G}' \cong O^* \times G
\]

Moreover, an \( O^* \)-group homomorphism \( \hat{f} : \hat{G} \to \hat{G}' \) is a group homomorphism such that \( \hat{f}(\lambda x) = \lambda \hat{f}(x) \) for any \( \lambda \in O^* \) and \( x \in \hat{G} \); it is clear that \( \hat{f} \) determines an ordinary group homomorphism \( f : G \to G' \) between the corresponding \( O^* \)-quotients.

2.7. Note that for any \( O \)-free \( O \)-algebra \( A \) of finite \( O \)-rank — just called \( O \)-algebra in the sequel — the group \( A^* \) of invertible elements has a canonical \( O^* \)-group structure; we call point of \( A \) any \( A^* \)-conjugacy class \( \alpha \) of primitive idempotents of \( A \) and denote by \( A(\alpha) \) the simple quotient of \( A \) determined by \( \alpha \), and by \( \mathcal{P}(A) \) the set of points of \( A \). If \( S \) is a full matrix algebra over \( O \) then \( Aut_O(S) \) coincides with the \( O^* \)-quotient of \( S^* \); in particular, any finite group \( G \) acting on \( S \) determines — by pull-back — an \( O^* \)-group \( \hat{G} \) of \( k^* \)-quotient \( G \), together with an \( O^* \)-group homomorphism

\[
\rho : \hat{G} \to S^*
\]

2.8. Let \( \hat{G} \) be an \( O^* \)-group with finite \( O^* \)-quotient \( G \); a \( \hat{G} \)-interior algebra is an \( O \)-algebra \( A \) endowed with an \( O^* \)-group homomorphism \( \hat{G} \to A^* \); we say that \( A \) is primitive whenever the unity element is primitive in \( A^G \). Note that the corresponding \( k \)-algebra \( k \otimes O A \) is only concerned by the \( k^* \)-group \( \hat{G}^k \) (cf. 2.1.1). A \( \hat{G} \)-interior algebra homomorphism from \( A \) to another \( \hat{G} \)-interior algebra \( A' \) is a not necessarily unitary algebra homomorphism \( f : A \to A' \) fulfilling \( f(\hat{x} a) = \hat{x} f(a) \) and \( f(a \hat{x}) = f(a) \hat{x} \); we say that \( f \) is an embedding whenever \( \ker(f) = \{0\} \) and \( \im(f) = f(1) A' f(1) \).

Occasionally, it is handy to consider the \( (A^G)^* \)-conjugacy class of \( f \) that we denote by \( \hat{f} \) and call exterior homomorphism from \( A \) to \( A' \); note that the exterior homomorphisms can be composed [7, Definition 3.1]. Moreover, \( A \) is a \( G \)-algebra and all the pointed group terminology applies. If \( H \) is a pointed group on the \( G \)-algebra \( A \) and \( \hat{H} \) is the converse image of \( H \) in \( \hat{G} \), we call \( \hat{H} \) a pointed \( O^* \)-group on \( A \) and we set \( A(\hat{H}) = A(H) \) denoting by \( s_\alpha : A^H \to A(\hat{H}) \) the canonical map. It is clear that \( A_\alpha \) becomes an \( \hat{H} \)-interior algebra and that we get a structural \( \hat{H} \)-interior algebra exterior embedding

\[
\hat{f}_\alpha : A_\alpha \to \operatorname{Res}_{\hat{H}}^\hat{G}(A)
\]
2.9. We similarly proceed with inclusions and localness but, contrarily to the \( k^* \)-group case, if \( \hat{P}_\gamma \) is a local pointed \( O^* \)-group on \( A \) — namely, \( P_\gamma \) is a local pointed group on the \( G \)-algebra \( A \) and \( \hat{P} \) is the converse image of \( P \) in \( \hat{G} \) — then \( \hat{P} \) need not be split. Recall that all the maximal local pointed \( O^* \)-groups \( \hat{P}_\gamma \) on \( A \) contained in \( \hat{H}_\alpha \) — called defect pointed \( O^* \)-groups of \( \hat{H}_\alpha \) — are mutually \( H \)-conjugate \cite{7, Theorem 1.2}, and that the \( O \)-algebras \( A_\alpha \) and \( A_\gamma \) are mutually Morita equivalent \cite{7, Corollary 3.5}. As usual, we consider the Brauer quotient and the Brauer algebra homomorphism

\[
\Br^A_P : A^P \rightarrow A(P) = A^P / \bigoplus_Q A^P_Q
\]

2.10. It is clear that \( \hat{G} \) acts on the set of pointed \( O^* \)-groups on \( A \) and, if \( \hat{H}_\alpha \) and \( \hat{K}_\beta \) are two of them, we denote by \( E_G(\hat{K}_\beta, \hat{H}_\alpha) \) the set of \( H \)-conjugacy classes of \( O^* \)-group homomorphisms from \( \hat{K} \) to \( \hat{H} \) induced by the elements \( x \in G \) such that \( (\hat{K}_\beta)^x \subset \hat{H}_\alpha \); as usual, we set

\[
E_G(\hat{H}_\alpha) = E_G(\hat{H}_\alpha, \hat{H}_\alpha) = N_G(\hat{H}_\alpha)/\hat{H}C_G(\hat{H})
\]

If \( \hat{P}_\gamma \) is a local pointed \( O^* \)-group on \( A \), as in \cite{9, 6.2} we denote by \( \hat{N}_G(\hat{P}_\gamma) \) the \( k^* \)-group obtained from the pull-back

\[
\begin{align*}
N_G(\hat{P}_\gamma) & \rightarrow \text{Aut}(A(\hat{P}_\gamma)) \\
\uparrow & \\
\hat{N}_G(\hat{P}_\gamma) & \rightarrow A(P_\gamma)^*
\end{align*}
\]

and then, since the structural maps from \( \hat{P}.C_G(\hat{P}) \) to \( N_G(\hat{P}_\gamma) \) and to \( A(\hat{P}_\gamma)^* \) determine a \( N_G(\hat{P}_\gamma) \)-stable \( k^* \)-group homomorphism \( \hat{P}^X.C_G(\hat{P}) \rightarrow \hat{N}_G(\hat{P}_\gamma) \), we still can define the \( k^* \)-group

\[
\hat{E}_G(\hat{P}_\gamma) = (\hat{N}_G(\hat{P}_\gamma)^* N_G(\hat{P}_\gamma)^*)/P.C_G(\hat{P})
\]

2.11. As in \cite{5, 2.5}, we say that an injective \( O^* \)-group homomorphism \( \hat{\varphi} : \hat{K} \rightarrow \hat{H} \) is an \( A \)-fusion from \( \hat{K}_\beta \) to \( \hat{H}_\alpha \) whenever there is a \( \hat{K} \)-interior algebra embedding

\[
f_{\hat{\varphi}} : A_\beta \rightarrow \text{Res}_{\hat{\varphi}}(A_\alpha)
\]

such that the inclusion \( A_\beta \subset A \) and the composition of \( f_{\hat{\varphi}} \) with the inclusion \( A_\alpha \subset A \) are \( A^* \)-conjugate; we denote by \( F_A(\hat{K}_\beta, \hat{H}_\alpha) \) the set of \( H \)-conjugacy classes of \( A \)-fusions from \( \hat{K}_\beta \) to \( \hat{H}_\alpha \) and, as usual, we write \( F_A(\hat{H}_\alpha) \) instead of \( F_A(\hat{H}_\alpha, \hat{H}_\alpha) \). If \( A_\alpha = iA_i \) for \( i \in \alpha \), as in \cite[Corollary 2.13]{5} we have a group homomorphism

\[
F_A(\hat{H}_\alpha) \rightarrow N_{A_\alpha^*}(\hat{H}.i)/\hat{H}(A_\alpha^H)^*
\]

and it is clear that

\[
E_G(\hat{K}_\beta, \hat{H}_\alpha) \subset F_A(\hat{K}_\beta, \hat{H}_\alpha)
\]
Note that if $B$ is another $\hat{G}$-interior algebra and $h : A \to B$ a $\hat{G}$-interior algebra embedding, we have

$$F_B(\hat{K}_\beta, \hat{H}_\alpha) = F_A(\hat{K}_\beta, \hat{H}_\alpha)$$  \hspace{1cm} 2.11.4.

2.12. If $\hat{P}_{\gamma}$ is a local pointed $O^*$-group on $A$, we have a canonical isomorphism $\hat{P}^k \cong k^* \times P$; then, choosing $j \in \gamma$, setting $A_\gamma = jA_j$ and denoting by $\hat{P}'$ the converse image of $1 \times P$ in $\hat{P}$, $F_A(\hat{P}_\gamma)$ denotes the $k^*$-group obtained from the pull-back \[9, \text{Proposition 6.12}\]. Thus, the inclusion $E_{\gamma}(\hat{P}_\gamma) \subset F_A(\hat{P}_\gamma)$ can be lifted to a canonical $k^*$-group homomorphism

$$E_{\gamma}(\hat{P}_\gamma)^o \to F_A(\hat{P}_\gamma)$$  \hspace{1cm} 2.12.2

as we show in the next proposition.

**Proposition 2.13.** Let $\hat{P}_\gamma$ be a local pointed $O^*$-group on $A$, choose $j \in \gamma$ and set $A_\gamma = jA_j$. For any $\hat{x} \in N_j(\hat{P}_\gamma)$ and any $a \in (A^j_p)^*$ having the same action on the simple $k$-algebra $A(\hat{P}_\gamma)$, the element $j(\hat{x}^{-1}a)j$ is invertible in $A_\gamma$ and normalizes $\hat{P} :\gamma$. Moreover, denoting by $x$, $\hat{x}^k$ and $\hat{x}$ the respective images of $\hat{x}$ in $G$, $\hat{G}$ and $E_{\gamma}(\hat{P}_\gamma)$, and setting

$$c_{\hat{x}, a} = j(\hat{x}^{-1} a)j \cdot \hat{P}'(j + J(A^P_{\gamma}))$$  \hspace{1cm} 2.13.1

there is a $k^*$-group homomorphism

$$E_{\gamma}(\hat{P}_\gamma)^o \to F_A(\hat{P}_\gamma)$$  \hspace{1cm} 2.13.2

which maps the image of $(x, s_{\gamma}(a)) \otimes (\hat{x}^k)^{-1}$ in $E_{\gamma}(\hat{P}_\gamma)$ on the element $(\hat{x}^{-1}, c_{\hat{x}, a})$ of $F_A(\hat{P}_\gamma)$.

**Proof:** Since $a^{-1} \hat{x}$ acts trivially on $A(\hat{P}_\gamma)$, the idempotents $j$ and $j a^{-1} \hat{x}$ have the same image in $A^p / J(A^p)$ and thus the image in this quotient of

$$b = j j a^{-1} \hat{x} + (1 - j)(1 - j) a^{-1} \hat{x}$$  \hspace{1cm} 2.13.3

is the unity element, so that $b$ is invertible in $A^p$; hence, since we have

$$b = (j(\hat{x}^{-1} a)j + (1 - j)(\hat{x}^{-1} a)(1 - j))(a^{-1} \hat{x})$$  \hspace{1cm} 2.13.4,
\( j(\hat{x}^{-1}.a)j \) is invertible in \( A_\gamma \); moreover, it is clear that, for any \( \hat{u} \in \hat{P} \), we have
\[
j(\hat{x}^{-1}.a)j \cdot \hat{u} = j(\hat{x}^{-1}\hat{u}.a)j = \hat{u}\hat{x}.j(\hat{x}^{-1}.a)j \tag{2.13.5}
\]
so that \( j(\hat{x}^{-1}.a)j \) normalizes \( \hat{P}.j \).

If \( a' \in (A^P)^* \) has the same image as \( a \) in \( A(\hat{P}_\gamma) \), and moreover we have \( j^{a^{-1}.\hat{x}} = j \), we get
\[
(j(\hat{x}^{-1}.a)j)(j(a'^{-1}.\hat{x})j) = j(\hat{x}^{-1}.a.a'^{-1}.\hat{x})j = j(a.a'^{-1}.x)j \tag{2.13.6}
\]
and this element belongs to \( j + J(A^P) \) since \( s_\gamma(aa'^{-1}) = 1 \); hence, the class \( c_{\hat{x},a} \) does not depend on our choice of \( a \) and we may assume that \( j^{a^{-1}.\hat{x}} = j \);
in particular, we get the announced map from \( \hat{E}_G(\hat{P}_\gamma)^o \) to \( \hat{F}_A(\hat{P}_\gamma) \). Finally, if \( \hat{x}' \in N_G(\hat{P}_\gamma) \) and \( a' \in (A^P)^* \) have the same action on \( A(\hat{P}_\gamma) \), we obtain
\[
(j(\hat{x}^{-1}.a)j)(j(\hat{x}'^{-1}.a')j) = j(\hat{x}^{-1}.a.\hat{x}'^{-1}.a')j \tag{2.13.7}
\]
and therefore, since \( s_\gamma(a'a'^{-1}.\hat{x}') = s_\gamma(aa') \), it is easily checked that this map is a \( k^* \)-group homomorphism.

2.14. If \( A' \) is another \( \hat{G} \)-interior algebra and \( f:A \to A' \) a \( \hat{G} \)-interior algebra embedding, it follows from Proposition 3.5 below that, as in [16, 2.9], denoting by \( \gamma' \) the point of \( P \) on \( A' \) containing \( f(\gamma) \), we have a canonical \( O^* \)-group isomorphism
\[
\hat{F}_f(P_\gamma) : \hat{F}_A(P_\gamma) \cong \hat{F}_{A'}(P_{\gamma'}) \tag{2.14.1}
\]
More precisely, let \( Q_\delta \) be another local pointed group on \( A \) and denote by \( \delta' \) the point of \( Q \) on \( A' \) containing \( f(\delta) \) and by
\[
\tilde{f}_\gamma' : A_\gamma \to A'_{\gamma'} \quad \text{and} \quad f_\delta' : A_\delta \to A'_{\delta'} \tag{2.14.2}
\]
the induced embeddings; if there is an \( A \)-fusion \( \varphi : Q \cong P \) from \( Q_\delta \) to \( P_\gamma \) then, according to equality 2.11.4 above, \( \varphi \) is also an \( A' \)-fusion from \( Q_\delta' \) to \( P_{\gamma'} \), so that we have two \( Q \)-interior algebra isomorphisms
\[
f_\varphi : A_\delta \cong \text{Res}_\varphi(A_\gamma) \quad \text{and} \quad f_\varphi' : A'_{\delta'} \cong \text{Res}_\varphi(A'_{\gamma'}) \tag{2.14.3}
\]
and the uniqueness of the exterior isomorphisms \( \tilde{f}_\varphi \) and \( f_\varphi' \) forces the equality
\[
\tilde{f}_\varphi' \circ f_\delta' = \text{Res}_\varphi(\tilde{f}_\gamma') \circ \tilde{f}_\varphi \tag{2.14.4}
\]
In particular, since by our very definition we have
\[ \hat{F}_{\text{Res}}(A_\gamma)(Q_\delta) = \hat{F}_A(P_\gamma) \quad \text{and} \quad \hat{F}_{\text{Res}}(A'_{\gamma'}) = \hat{F}_{A'}(P_{\gamma'}) \]
we get the following commutative diagram of $O^*$-group isomorphisms
\[ \begin{array}{ccc}
\hat{F}_A(Q_\delta) & \cong & \hat{F}_A(P_\gamma) \\
\hat{F}_f(Q_\delta) \ & || & \ & || \\
\hat{F}_{A'}(Q_{\delta'}) & \cong & \hat{F}_{A'}(P_{\gamma'})
\end{array} \]

2.15. If $\hat{H}$ is an $O^*$-subgroup of $\hat{G}$ and $B$ an $\hat{H}$-interior algebra, we call
induced $\hat{G}$-interior algebra of $B$ the $O^*$-$\hat{G}$-bimodule
\[ \text{Ind}^{\hat{G}}_{\hat{H}}(B) = O_\ast \hat{G} \otimes_{O_\ast \hat{H}} B \otimes_{O_\ast \hat{H}} O_\ast \hat{G} \]
edowed with the distributive multiplication defined by
\[
(x \otimes b \otimes y)(x' \otimes b' \otimes y') = \begin{cases} 
  x \otimes b \cdot yx' \cdot b' \otimes y' & \text{if } yx' \in \hat{H} \\
  0 & \text{otherwise}
\end{cases}
\]

2.16. If $G'$ is another $O^*$-group with $O^*$-quotient $G$ and $A'$ is a $G'$-interior algebra, setting $\hat{G} = \hat{G} \ast G'$, the tensor product $A'' = A \otimes O A'$ has an evident structure of $G$-interior algebra. Then, for any pointed $O^*$-group $H_{a''}$ on $A''$, there are points $\alpha$ of $\hat{H}$ on $A$ and $\alpha'$ of $\hat{H}'$ on $A'$, and an $\hat{H}$-interior algebra embedding $A''_{a''} \rightarrow A_\alpha \otimes_O A'_{\alpha'}$ such that we have a commutative diagram
\[ \begin{array}{ccc}
\text{Res}_{\hat{H}}^{\hat{G}}(A'') & \rightarrow & A_{a''} \\
\hat{A}'_{a''} & \rightarrow & A_\alpha \otimes_O A'_{\alpha'}
\end{array} \]
Moreover, if $P_{a''}$ is a local pointed $O^*$-group on $A''$, the corresponding points $\gamma$ of $\hat{P}$ on $A$ and $\gamma'$ of $\hat{P}'$ on $A'$ have to be local.

2.17. As a matter of fact, we are mainly interested in the case where a Sylow $p$-subgroup of $G$ stabilizes an $O$-basis of $A$; then, $\gamma''$ is the unique local point of $P_{a''}$ on $A_{\gamma} \otimes_O A'_{\gamma'}$ [7, Proposition 5.6] and therefore, for any $\tilde{\phi} \in F_A(\hat{P}_\gamma)$ and any $\tilde{\phi}' \in F_{A'}(\hat{P}'_{\gamma'})$ having the same image in $\hat{\text{Aut}}(P)$, the corresponding
$\mathcal{O}^*$-group outer automorphism $\varphi'' = \tilde{\varphi} \ast \varphi'$ of $\hat{P}$ belongs to $F_{A''}(P_{\gamma''})$. In particular, if $F$ is a finite group and we have group homomorphisms

$$\theta : F \to F_A(\hat{P}_\gamma) \quad \text{and} \quad \theta' : F \to F_A'(P_{\gamma'})$$

2.17.1

inducing the same group homomorphism $F \to \hat{\text{Aut}}(P)$, we get a group homomorphism $\theta'' : F \to F_{A''}(P_{\gamma''})$ and, setting

$$\hat{F}^{\gamma} = \text{Res}_\theta(\hat{F}_A(\hat{P}_\gamma))$$

2.17.2

$$\hat{F}^{\gamma'} = \text{Res}_{\theta'}(\hat{F}_A(\hat{P}_{\gamma'})) \quad \text{and} \quad \hat{F}^{\gamma''} = \text{Res}_{\theta''}(\hat{F}_A(\hat{P}_{\gamma''}))$$

the corresponding embedding in the bottom of diagram 2.16.1 induces a $k^*$-group isomorphism [7, Proposition 5.11]

$$\hat{F}^{\gamma} \ast \hat{F}^{\gamma'} \cong \hat{F}^{\gamma''}$$

2.17.3

3. The $\mathcal{O}^*$-group algebra

3.1. Let $\hat{G}$ be an $\mathcal{O}^*$-group with finite $\mathcal{O}^*$-quotient $G$, and $\mathcal{O}'$ the ring of integers of a finite field extension $K'$ of $K$ such that the $\mathcal{O}^*$-group $\hat{G}^{\mathcal{O}'}$ contains a finite subgroup $G'$ covering $G$ (cf. Proposition 2.5). It is clear that the product $\mathcal{O}'^* \otimes \hat{G}$ in $\mathcal{O}' \otimes \mathcal{O} \hat{G}$ can be identified to $\hat{G}^{\mathcal{O}'}$, so that we have $\mathcal{O}'_{\lambda} \hat{G}^{\mathcal{O}'} = \mathcal{O}' \otimes \mathcal{O}_{\lambda} \hat{G}$, and we will analyse the surjective $G$-algebra homomorphism $\mathcal{O}'G' \to \mathcal{O}'_{\lambda} \hat{G}^{\mathcal{O}'}$ induced by the inclusion $G' \subset \hat{G}^{\mathcal{O}'}$. We respectively denote by $Z'$ the Sylow $p$-subgroup of $(\mathcal{O}'^*).1 \cap G'$ and by $\Lambda'$ the converse image in $\mathcal{O}'^*$ of the Hall $p'$-subgroup of this intersection, and in the group algebra $\mathcal{O}'G'$ we set

$$e_{\Lambda'} = \frac{1}{|\Lambda'|} \sum_{\lambda' \in \Lambda'} \lambda^{-1}(\lambda'.1)$$

3.1.1

then, we still get a surjective $G$-algebra homomorphism over $\mathcal{O}'$

$$s_{Z'} : \mathcal{O}'G' e_{\Lambda'} \to \mathcal{O}'_{\lambda} \hat{G}^{\mathcal{O}'}$$

3.1.2

which can be actually considered as a homomorphism of $G'$-interior algebras; more precisely, considering the obvious group homomorphism $Z' \to \mathcal{O}'^*$, $s_{Z'}$ induces a $G'$-interior algebra isomorphism

$$\mathcal{O}' \otimes_{\mathcal{O}'Z'} \mathcal{O}'G' e_{\Lambda'} \cong \mathcal{O}'_{\lambda} \hat{G}^{\mathcal{O}'}$$

3.1.3

Note that, if we were working over $k$, only the $k^*$-group $\hat{G}^k$ would be concerned since we have

$$k \otimes \mathcal{O}, \hat{G} \cong k^* \hat{G}^k \cong kG' e_{\Lambda'}$$

3.1.4

where we set $G' = G'/Z'$ and $e_{\Lambda'}$ is the image of $e_{\Lambda'}$ in $kG'$. 
Proposition 3.2. The radical $J(O'G'e')$ contains $\text{Ker}(s_{Z'})$, and for any subgroup $H$ of $G$ we have

$$s_{Z'}((O'G'e_{\Lambda'})^H) = (O'_G\hat{G}^{O'})^H$$

3.2.1.

In particular, $s_{Z'}$ is a strict covering homomorphism of $G$-algebras. Moreover, for any $p$-subgroup $P$ of $G$, $Br_{P,\hat{G}}^{O'}$ induces an $N_G(P)$-algebra isomorphism

$$k_\ast C_{G^s}(\hat{P}) \cong (O_\ast\hat{G})(P)$$

3.2.2.

where $\hat{P}$ denotes the converse image of $P$ in $\hat{G}$.

Proof: Since $Z'$ is a finite $p$-group, from isomorphism 3.1.3 we get

$$\text{Ker}(s_{Z'}) \subset J(O'Z'), (O'G'e_{\Lambda'}) \subset J(O'G'^{e_{\Lambda'}})$$

3.2.3;

moreover, since any subgroup $H$ of $G$ stabilizes the following direct sum decomposition

$$O'_G\hat{G}^{O'} = \bigoplus_{x \in G} O'x'$$

3.2.4

where $x' \in G'$ is a lifting of $x \in G$, $(O'_G\hat{G}^{O'})^H$ is linearly generated by the elements $T_{\hat{G}^H}(x')(x')$ where $x'$ runs over the set of elements of $G'$ such that the stabilizer of $O'x'$ in $H$ centralizes $x'$, and this implies the equality 3.2.1.

Consequently, $s_{Z'}$ is a strict covering homomorphism of $G$-algebras [10, §4]. In particular, if $H = P$ is a $p$-group and $\hat{P}$ is the converse image of $P$ in $G$, it is easily checked that

$$(O_\ast\hat{G})(P) \cong (O'_G\hat{G}^{O'})(P)$$

$$= \bigoplus_{x \in C_G(\hat{P})} kBr_P(x') \cong k_\ast C_{G^s}(\hat{P})$$

3.2.5

where again $x' \subset C_{G'}(\hat{P})$ lifts $x \in C_G(\hat{P})$.

Corollary 3.3. The inclusion $O_\ast\hat{G} \subset O'_G\hat{G}^{O'}$ and the homomorphism $s_{Z'}$ induce a bijection, preserving inclusions and localness, between the set of pointed $O'^*$-groups $\hat{H}_\beta$ on $O_\ast\hat{G}$ and the set of pointed groups $H'_\beta'$ on $O'G'e_{\Lambda'}$ such that $H'$ contains $\Lambda'Z'$. Moreover, if $H_\beta$ and $H'_\beta'$ correspond to each other, then we have $\hat{H}^{O'} \cap G' = H'$ and this equality together with $s_{Z'}$ determine the following group isomorphism and $H'$-interior algebra isomorphism

$$E_{G'}(H'_\beta') \cong E_{G}(\hat{H}_\beta)$$

3.3.1.

$$O' \otimes_{O'Z'} (O'G'e_{\Lambda'})_{\beta'} \cong O' \otimes_{O'} (O_\ast\hat{G})_{\beta}$$

3.3.1.
Proof: Since $O'_\ast\hat{G}' \cong O' \otimes_O O_*\hat{G}$, for any subgroup $H$ of $G$ we have

$$(O'_\ast\hat{G}'_H)^H \cong O' \otimes_O (O_*\hat{G})^H$$

and therefore, since $k$ is algebraically closed, the inclusion $O_*\hat{G} \subset O'_\ast\hat{G}'_H$ induces a bijection, preserving inclusions and localness, between the sets of corresponding pointed $O'$- and $O'_\ast$-groups, which maps any pointed $O'$-group $H_\beta$ on $O_*\hat{G}$ to a pointed $O'_\ast$-group $(\hat{H}'_H_\beta)'_\beta$ on $O'_\ast\hat{G}'_H$ fulfilling $\beta \subset \beta'$; moreover, since $s_Z$ is a strict covering homomorphism of $G$-algebras, it does the same between the sets of pointed groups on $O'_\ast\hat{G}'_eA'$ and $O'_\ast\hat{G}'_H$ [10, Propositions 4.4 and 4.18], namely $H_{\beta_\beta'}$ or, equivalently, $(\hat{H}'_H_\beta)'_\beta$ comes from the pointed group $H_{\beta_\beta'}$ on $O'_\ast\hat{G}'_eA'$ fulfills $s_Z(\beta') \subset \beta'$, where $H' = \hat{H}'_H \cap G'$ is the converse image of $H$ in $G$; now, it is clear that

$$N_G(H_{\beta_\beta'}) = N_G(\hat{H}_\beta) \quad \text{and} \quad C_G(H') = C_G(\hat{H})$$

and the bottom isomorphism in 3.3.1 follows from isomorphism 3.1.3.

3.4. More generally, for any $\hat{G}$-interior algebra $A$, it is clear that the extension $A' = O' \otimes_O A$ becomes a $G'$-interior algebra and, as in the corollary above, the inclusion $A \subset A'$ induces a bijection, preserving inclusions and localness, between the sets of pointed $O'$-groups $H_\beta$ on $A$ and of pointed groups $H_{\beta'}$ on $A'$ fulfilling $\Lambda' \cap Z' \subset H'$, which maps $H_\beta$ on $H_{\beta'}$, if and only if we have

$$H' = \hat{H}' \cap G' \quad \text{and} \quad \beta \subset \beta'$$

Indeed, from the left-hand equality and from any $O$-basis of $O'$ we easily get $A' \cap H' \cong O' \otimes_O A'H'$ and therefore, since $k$ is also the residue field of $O'$, we obtain an obvious bijection between the sets of points of $H'$ on $A'$ and of $H$ on $A$. Actually, the bijection above also preserves the fusions as it shows the next result.

Proposition 3.5. With the notation above, for any pair of pointed $O'$-groups $H_\beta$ and $K_\gamma$ on $A$, we have

$$F_A(K_\gamma, \hat{H}_\beta) = F_{A'}(K'_\gamma, H'_{\beta'})$$

where we are setting $H' = \hat{H}' \cap G'$ and $K' = \hat{K}' \cap G'$ and the points $\beta'$ of $H'$ and $\gamma'$ of $K'$ respectively contain $\beta$ and $\gamma$.

Proof: Since $A'_{\beta'} \cong O' \otimes_O A_{\beta}$ and $A'_{\gamma'} \cong O' \otimes_O A_{\gamma}$, if $\phi : \hat{K} \to \hat{H}$ is an $A$-fusion from $K_{\gamma}$ to $H_{\beta}$, the corresponding $K$-interior algebra embedding

$$f_{\hat{\phi}} : A_{\gamma} \to \text{Res}_{\beta}(A_{\beta})$$

determines a $K'$-interior algebra embedding

$$f'_{\hat{\phi}} : A'_{\gamma'} \to \text{Res}_{\beta'}(A'_{\beta'})$$
where \( \varphi': K' \to H' \) is the group homomorphism determined by \( \hat{\varphi} \), and it is clear that the inclusion \( A'_b \subset A' \) and the composition of \( f'_{\varphi'} \) with the inclusion \( A'_\beta \subset A' \) are also \( A^*-\)conjugate.

Conversely, since any \( A'\text{-fusion} \) from \( K'_\gamma \) to \( H'_\gamma \) can be decomposed as a composition of an isomorphism with an inclusion \([8, 2.11]\), we may assume that \( \varphi' \) is an isomorphism; in this case, choosing \( j \in \beta \subset \beta' \) and \( \ell \in \gamma \subset \gamma' \), it follows from \([8, 2.11]\) that the right-hand multiplication by a suitable invertible element \( a' \) of \( A^* \) determines an isomorphism from \( A' \) considered as an \( A' \otimes \mathcal{O}' K' \)-module by left- and right-hand multiplication, onto \( \text{Res}_{id_A \otimes \varphi'}(A'j) \) where \( A'j \) is similarly considered as an \( A' \otimes \mathcal{O}' \mathcal{O}' H' \)-module by left- and right-hand multiplication.

But, considering an \( \mathcal{O}\)-basis \( \Delta' \) of \( \mathcal{O}' \), it is quite clear that \( A'\ell \) and \( A'j \) are respectively isomorphic to \( (A\ell)^{\Delta'} \) and \( (Aj)^{\Delta'} \) considered as \( A \otimes \mathcal{O} \hat{K} \)- and \( A \otimes \mathcal{O} \hat{\mathcal{H}} \)-modules. Consequently, \( A\ell \) and \( \text{Res}_{id_A \otimes \varphi}(A\ell) \) are isomorphic considered as \( A \otimes \mathcal{O} \hat{K} \)-modules; then, the quotients
\[
A/A\ell \cong A(1 - \ell) \quad \text{and} \quad \text{Res}_{id_A \otimes \varphi}(A/A\ell) \cong \text{Res}_{id_A \otimes \varphi}(A(1 - j))
\]
are also isomorphic considered as \( A \otimes \mathcal{O} \hat{K} \)-modules and therefore the right-hand multiplication by a suitable invertible element \( a \) of \( A^* \) still determines an isomorphism
\[
A\ell \cong \text{Res}_{id_A \otimes \varphi}(Aj)
\]
so that it follows again from \([8, 2.11]\) that \( \hat{\varphi} \) an \( A\text{-fusion} \) from \( \hat{K}_\gamma \) to \( \hat{H}_\beta \). We are done.

3.6. As in the ordinary case, we call block of \( \hat{G} \) any primitive idempotent \( b \) of \( Z(\mathcal{O}\hat{G}) \); thus, \( \alpha = \{b\} \) is a point of \( G \) (or of \( \hat{G} \) ) on \( \mathcal{O}\hat{G} \) and, if \( P_\gamma \) is a defect pointed group of \( G_\alpha \), we call defect of \( b \) the integer \( d = d(b) \) such that \( p^d = |P| \). Then, according to the corollary above, \( G_\alpha \) and \( P_\gamma \) respectively determine pointed groups \( G'_\alpha \) and \( P'_\gamma \) of \( \mathcal{O}' \hat{G}' e_{\alpha'} \) where we denote by \( P' \) the Sylow \( p \)-subgroup of \( \hat{\mathcal{O}}' \cap \mathcal{O}' \), so that \( P'_\gamma \) is a defect pointed group of \( G'_\alpha \); once again, we have \( \alpha' = \{b'\} \) for a block \( b' \) of \( G' \) such that \( b' e_{\alpha'} = b' \), but note that this block has a defect \( d(b') \) such that \( p^d(b') = p^d|Z'| \); in particular, since isomorphism 3.1.3 implies that
\[
\text{rank}_\mathcal{O}(\langle (\mathcal{O}' G' e')_\gamma \rangle) = \text{rank}_\mathcal{O}(\langle (\mathcal{O}\hat{G})_\gamma \rangle[Z])
\]
and we have \( E_{G'}(P'_\gamma) \cong E_{\hat{G}}(\hat{P}_\gamma) \) (cf. isomorphisms 3.3.1), as in the ordinary case \( E_{\hat{G}}(\hat{P}_\gamma) \) is a \( p \)-group, \( p^d \) divides \( \text{rank}_\mathcal{O}(\langle (\mathcal{O}\hat{G})_\gamma \rangle) \) and we get
\[
p^{-d} \text{rank}_\mathcal{O}(\langle (\mathcal{O}\hat{G})_\gamma \rangle) \equiv |E_{\hat{G}}(\hat{P}_\gamma)| \pmod{p}
\]
Moreover, note that the image \( b' \) of \( b \) in \( \mathcal{O}\hat{G} \) is a block of \( \mathcal{G} \) and corresponds to the image \( b' \) of \( b \) in \( k\mathcal{G}^k \) by isomorphism 3.1.4; thus, \( \overline{b} \) is a \( k \)-block of \( \mathcal{G}^k \) and this correspondence clearly determines a bijection between the sets of blocks of \( \mathcal{G} \) and \( \mathcal{G}^k \) which preserves the defects.
3.7. It is clear that the \( K \)-algebra \( K_\ast \hat{G} b \) is semisimple and let us assume that \( K \) is a splitting field for \( \hat{G} \) or, equivalently, that \( K_\ast \hat{G} b \) is a direct product of full matrix algebras over \( K \). Since the \( O \)-algebra \( O_\ast \hat{G} b \) is symmetric, the \( O \)-module of the symmetric \( O \)-linear forms \( Z^O(O_\ast \hat{G} b) \) is a free \( Z(O_\ast \hat{G} b) \)-module of rank one [12, Proposition 2.2] and therefore, denoting by \( \text{Irr}_K(\hat{G}, b) \) the set of characters of the simple \( K_\ast \hat{G} b \)-modules — simply called irreducible characters of \( \hat{G} \) in the sequel — the \( Z(O_\ast \hat{G} b) \)-submodule

\[
Z^O_{\text{ch}}(O_\ast \hat{G} b) = \bigoplus_{\chi \in \text{Irr}_K(\hat{G}, b)} O \cdot \chi
\]

3.7.1,
determines an ideal \( Z_{\text{ch}}(O_\ast \hat{G} b) \) of \( Z(O_\ast \hat{G} b) \) [12, Proposition 2.9]. Note that, if \( \hat{x} \in \hat{G} \) and \( C_G(\hat{x}) \neq C_G(x) \) where \( x \) denotes the image of \( \hat{x} \) in \( G \), the elements \( \hat{x} \) and \( \lambda \cdot \hat{x} \) are \( G \)-conjugate for some \( \lambda \in O^* - \{1\} \) and therefore any \( \mu \in Z^O(O_\ast \hat{G}) \) vanishes on \( \hat{x} \).

3.8. Moreover, for any \( \chi \in \text{Irr}_K(\hat{G}, b) \), recall that the image of the restriction of \( \chi \) to \( Z(O_\ast \hat{G} b) \) is contained in \( \chi(1)O \) and that \( \chi(1)^{-1} \chi \) defines an \( O \)-algebra homomorphism from \( Z(O_\ast \hat{G} b) \) to \( O \); we call defect of \( \chi \) the integer \( d(\chi) \) such that

\[
\frac{\chi}{\chi(1)}(Z_{\text{ch}}(O_\ast \hat{G} b)) = p^{d(\chi)} O
\]

3.8.1.

More explicitly, according to Corollary 2.4, there is a finite field extension \( K' \) of \( K \) such that, denoting by \( O' \) the ring of integers of \( K' \), the \( O'^* \)-group \( G^{O'} \) contains a finite subgroup \( G' \) covering \( G \); then, \( \chi \) can be extended to an irreducible character of \( G^{O'} \) which, by restriction, determines an irreducible character \( \chi' \) of \( G' \) fulfilling \( \chi'((\lambda')^{-1} \chi') = \lambda' \chi(1) \) for any \( \lambda' \in \Lambda' \), where \( b' \) is the corresponding block of \( G' \), and therefore we get

\[
\frac{\chi'}{\chi'(1)}(Z_{\text{ch}}(O'G'b')) = \frac{|G'|}{\chi'(1)} O'
\]

3.8.2.

But, it follows from isomorphisms 3.1.3 and 3.1.4 that

\[
s_{Z'}(Z_{\text{ch}}(O'G'b')) = |Z'| O' \otimes_O Z_{\text{ch}}(O_\ast \hat{G} b)
\]

3.8.3.

Consequently, we actually get

\[
p^{d(\chi')} = p^{d(\chi)} |Z'| \quad \text{and} \quad d(\chi) = \vartheta_p \left( \frac{|G'|}{\chi(1)} \right)
\]

3.8.4.

where \( \vartheta_p : \mathbb{Z} \to \mathbb{Z} \) denotes the \( p \)-adic valuation; in particular, since \( d(b') \) coincides with the maximal value of \( d(\chi') \) when \( \chi' \) runs over the set of irreducible characters of \( KG'b' \), it is not difficult to prove that \( d(b) \) coincides with the maximal value of \( d(\chi) \) when \( \chi \) runs over \( \text{Irr}_K(\hat{G}, b) \).
Proposition 3.9. Let $b$ be a block of $\hat{G}$ and $\hat{P}_\gamma$ a defect pointed $\mathcal{O}^*$-group of $\hat{G}(b)$. Then, the following statements are equivalent:

3.9.1 The block $b$ has defect zero.

3.9.2 The source algebra $(\mathcal{O}_*\hat{G})_\gamma$ has $\mathcal{O}$-rank one.

3.9.3 The block algebra $\mathcal{O}_*\hat{G}b$ is a full matrix algebra over $\mathcal{O}$.

3.9.4 The block $\mathcal{K}$-algebra $\mathcal{K}_*\hat{G}b$ is a direct product of full matrix algebras over $\mathcal{K}$ and there is $\chi \in \text{Irr}_\mathcal{K}(\hat{G}, b)$ with defect zero.

In this case, we have $\mathcal{O}_p(G) = \{1\}$.

Proof: With the notation above, if $d(b) = 0$ then $Z'$ is the unique defect group of the corresponding block $b'$ of $G'$ and therefore the $Z'$-interior source algebra of $b'$ is isomorphic to $\mathcal{O}'Z'$ [9, Proposition 14.6], so that statement 3.9.2 follows from isomorphism 3.1.3. Since the block algebra is always Morita equivalent to the source algebra [7, Corollary 3.5], statement 3.9.3 follows from statement 3.9.2.

If statement 3.9.3 holds then $\text{Irr}_\mathcal{K}(\hat{G}, b)$ has a unique element $\chi$ and, since an indecomposable direct summand of $\mathcal{O}_*\hat{G}b$ as $\mathcal{O}_*\hat{G}b$-module has $\mathcal{O}$-rank equal to $\chi(1)$, $|G|_p$ divides $\chi(1)$ and therefore we have $d(\chi) = 0$. Finally, statement 3.9.4 implies that the sum

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{x \in G} \chi(\hat{x}) \cdot \hat{x}^{-1}$$  

3.9.5

where $\hat{x} \in \hat{G}$ is a lifting of $x \in G$, belongs to $Z(\mathcal{O}_*\hat{G})$; but, $e_\chi$ is an idempotent and therefore we get $e_\chi = b$, so that $\chi$ is the unique element in $\text{Irr}_\mathcal{K}(\hat{G}, b)$, which forces $d(b) = d(\chi) = 0$; moreover, in this case the image $\bar{b}$ of $b$ in $\mathcal{K}_*\hat{G}^k$ is a block of defect zero of $\hat{G}^k$, so that there is a projective simple $\mathcal{K}_*\hat{G}^k$-module which forces $\mathcal{O}_p(G) = \{1\}$.

Proposition 3.10. For any pair of local pointed $\mathcal{O}^*$-groups $\hat{P}_\gamma$ and $\hat{Q}_\delta$ on $\mathcal{O}_*\hat{G}$, we have

$$E_G(\hat{Q}_\delta, \hat{P}_\gamma) = F_{\mathcal{O}_*\hat{G}}(\hat{Q}_\delta, \hat{P}_\gamma)$$  

3.10.1

Proof: We already know that the left-hand member is contained in the right-hand one (cf. 2.11.3). For any $\hat{\varphi} \in F_{\mathcal{O}_*\hat{G}}(\hat{Q}_\delta, \hat{P}_\gamma)$ and any representative $\varphi$ of $\hat{\varphi}$, it is quite clear that there is a local point $\varepsilon$ of $\hat{R} = \varphi(\hat{Q})$ on $\mathcal{O}_*\hat{G}$ such that $\hat{R}_\varepsilon \subset \hat{P}_\gamma$ and that $\varphi$ determines an element of $F_{\mathcal{O}_*\hat{G}}(\hat{Q}_\delta, \hat{R}_\varepsilon)$; thus, we may assume that the $\mathcal{O}^*$-quotients $P$ of $\hat{P}$ and $Q$ of $\hat{Q}$ have the same order; in this case, according to 2.11 and choosing $i \in \gamma$ and $j \in \delta$, there is $a \in (\mathcal{O}_*\hat{G})^*$ fulfilling $(u,j)^a = \varphi(u).i$ for any $u \in \hat{Q}$.
With the notation above, let \( P'_\gamma \) and \( Q'_\delta \) be the corresponding local pointed groups on \( O'G'e' \) (cf. Corollary 3.3), and choose \( i' \in \gamma' \) and \( j' \in \delta' \) respectively lifting \( i \) and \( j \); then, \( O'Q'j' \) is an indecomposable direct summand of \( j'(O'G')j' \) as \( O'(Q' \times Q') \)-modules and therefore the quotient
\[
(O'_j \dot{Q} \sigma' j) \cong O' \otimes_{O(Z)} O'Q'j'
\]
is a direct summand of the quotient
\[
j(O'_j \dot{Q} \sigma' j) \equiv O' \otimes_{O(Z)} j'(O'G')j'
\]
always as \( O'(Q' \times Q') \)-modules. The \( O'(Q' \times Q') \)-module \( j(O'_j \dot{Q} \sigma' j) \) is still indecomposable since, denoting by \( \Delta(Q') \) the diagonal subgroup of \( Q' \times Q' \) and considering \( O' \) as an \( O'((Z' \times Z').\Delta(Q')) \)-module where \( \Delta(Q') \) acts trivially and \( Z' \times Z' \) throughout the canonical homomorphism \( Z' \to O'^* \), we have
\[
(O'_j \dot{Q} \sigma' j) \cong \text{Ind}^{Q' \times Q'}_{(Z' \times Z').\Delta(Q')}(O')
\]
Consequently, since we clearly have
\[
j(O'_j \dot{Q} \sigma' j)ja = j(O'_j \dot{Q} \sigma' j)i
\]
\( j(O'_j \dot{Q} \sigma' j)ja \) becomes an indecomposable direct summand of \( j(O'_j \dot{Q} \sigma' j)i \) as \( O'(Q' \times P') \)-modules and therefore, since we have the \( O'(Q' \times P') \)-module isomorphism
\[
j(O'_j \dot{Q} \sigma' j)i \equiv O' \otimes_{O(Z)} j'(O'G')i'
\]
the \( O'(Q' \times P') \)-module \( j'(O'G')i' \) admits a direct summand of \( O' \)-rank equal to \( |P'|Z'| = |P'| \); thus, the \( O'(Q' \times P') \)-module \( O'G'e' \) admits such a direct summand, which forces the existence of \( x' \in G' \) fulfilling \( O'Q'x' = x'O'P' \), together with an \( O'(Q' \times P') \)-module isomorphism
\[
(O'_j \dot{Q} \sigma' j)ja \equiv O' \otimes_{O(Z)} O'Q'x'
\]
But, the left-hand member of this isomorphism admits a vertex \( V' \) equal to \( \{ (z'u', \varphi(u')) \}_{z' \in Z', u' \in Q'} \) and a \( V' \)-source of \( O' \)-rank one where the subgroup \( \{ (u', \varphi(u')) \}_{u' \in Q'} \subset V' \) is the kernel, whereas the right-hand member admits a vertex equal to \( \{ (z'u', u'tx') \}_{z' \in Z', u' \in Q'} \) and the corresponding kernel is equal to \( \{ (u', u'tx') \}_{w' \in Q'} \); hence, up to modifying our choice of \( x' \), we may assume that \( \varphi(u') = u'tx' \) for any \( u' \in Q' \). In conclusion, \( x' \) induces \( \varphi : \dot{Q} \to P \) and therefore \( \hat{\varphi} \) belongs to \( E_G(\dot{Q}_\delta, \dot{P} \sigma') \); in particular, since \( E_G(\dot{Q}_\delta, \dot{P} \sigma') \subset F_{O' \sigma'}(\dot{Q}_\delta, \dot{P} \sigma') \) and \( \varphi \) is an isomorphism, \( \text{id}_{\dot{P}} \) belongs to \( F_{O' \sigma'}(\dot{P} \sigma', \dot{P} \gamma) \), which forces \( \delta x' = \gamma \) (cf. 2.11) and \( \hat{\varphi} \in E_G(\dot{Q}_\delta, \dot{P} \gamma) \). We are done.
3.11. Let \( \hat{Q}_\delta \) be a local pointed \( \mathcal{O}^\ast \)-group on \( \mathcal{O}_s \hat{G} \); if \( \hat{H} \) is an \( \mathcal{O}^\ast \)-subgroup of \( \hat{G} \) containing \( \hat{Q}.C_{\hat{G}}(\hat{Q}) \) then from isomorphism 3.2.2 we get (cf. 2.8)

\[
s_\delta((\mathcal{O}_s \hat{G})^H) \subset s_\delta((\mathcal{O}_s \hat{G})Q.C_{\hat{G}}(\hat{Q})) = (\mathcal{O}_s \hat{G})(Q)C_{\hat{G}}(\hat{Q}) \cong k
\]

and therefore there is a unique point \( \beta \) of \( \hat{H} \) on \( \mathcal{O}_s \hat{G} \) such that \( \hat{Q}_\delta \subset \hat{H}_\beta \); in particular, there are unique points \( \zeta \) of \( \hat{Q}.C_{\hat{G}}(\hat{Q}) \) and \( \nu \) of \( N_{\hat{G}}(\hat{Q}_\delta) \) on \( \mathcal{O}_s \hat{G} \) fulfilling

\[
\hat{Q}_\delta \subset \hat{Q}.C_{\hat{G}}(\hat{Q})\zeta \subset N_{\hat{G}}(\hat{Q}_\delta)\nu
\]

and, according to Corollary 3.3, \( \hat{Q}_\delta \) is a defect pointed \( \mathcal{O}^\ast \)-group of \( N_{\hat{G}}(\hat{Q}_\delta) \) if and only if it is a maximal local pointed \( \mathcal{O}^\ast \)-group on \( \mathcal{O}_s \hat{G} \) [7, Corollary 1.4]. We denote by \( b(\delta) \) the block of \( \hat{Q}.C_{\hat{G}}(\hat{Q}) \) fulfilling \( b(\delta)\cdot\zeta = \zeta \), by \( \hat{C}_{\hat{G}}(\hat{Q}) \) the \( k^\ast \)-group \( C_{\hat{G}}(\hat{Q})/C_Q(\hat{Q}) \) and by \( \hat{b}(\delta) \) the image of \( b(\delta) \) in \( k^\ast \hat{C}_{\hat{G}}(\hat{Q}) \), which is a block of \( \hat{C}_{\hat{G}}(\hat{Q}) \) over \( k \). We say that \( \hat{Q}_\delta \) is selfcentralizing, or that it is a selfcentralizing pointed \( \mathcal{O}^\ast \)-group on \( \mathcal{O}_s \hat{G} \); if \( \hat{Q} \) is a defect pointed \( \mathcal{O}^\ast \)-group of \( \hat{Q}.C_{\hat{G}}(\hat{Q}) \); moreover, as in [16, 3.1], we say that \( \hat{Q}_\delta \) is radical if it is selfcentralizing and we have

\[
\mathcal{O}_p(E_G(\hat{Q}_\delta)) = \{1\}
\]

**Proposition 3.12.** For any local pointed \( \mathcal{O}^\ast \)-group \( \hat{Q}_\delta \) on \( \mathcal{O}_s \hat{G} \), the following statements are equivalent:

1. \( \hat{Q}_\delta \) is selfcentralizing.
2. The structural homomorphism induces \( kC_Q(\hat{Q}) \cong (\mathcal{O}_s \hat{G})_{\delta}(Q) \).
3. The block \( \hat{b}(\delta) \) of \( \hat{C}_{\hat{G}}(\hat{Q}) \) has defect zero.
4. For any local pointed \( \mathcal{O}^\ast \)-group \( \hat{P}_\gamma \) on \( \mathcal{O}_s \hat{G} \) containing \( \hat{Q}_\delta \), \( \hat{Q} \) contains \( C_{\hat{P}}(\hat{Q}) \).

**Proof:** With the notation above, it follows from Proposition 3.2 that \( s_Z \) determines a surjective \( k \)-algebra homomorphism

\[
k \otimes_{\mathcal{O}'(Q')} e_{\Lambda'}(Q') \longrightarrow (\mathcal{O}'(\hat{G})^{G'})\zeta(Q')
\]

but, setting \( Q' = \hat{Q}^{G'} \cap G' \) and denoting by \( Q'_s \) and by \( Q'.C_{G'}(Q')_\zeta \) the respective pointed groups on \( \mathcal{O}'G' e_{\Lambda'}(Q')_\zeta \), the statement 3.12.1 implies that \( Q'_s \) is a defect pointed group of \( Q'.C_{G'}(Q')_\zeta \); in this case, it follows from [9, statements 14.5.1 and 2.9.2] that the structural homomorphism induces an isomorphism

\[
kZ(Q') \cong (\mathcal{O}'G' e_{\Lambda'})(Q')
\]

consequently, the structural homomorphism (cf. isomorphism 3.2.2)

\[
kC_Q(\hat{Q}) \longrightarrow (\mathcal{O}'G^{G'})(Q) \cong (\mathcal{O}_s \hat{G})_{\delta}(Q)
\]

is surjective, which proves statement 3.12.2.
Now, if we have the $k$-algebra isomorphism
\[ kC_Q(\hat{Q}) \cong (O_\ast \hat{G})_\delta(Q) \] 3.12.8,
then we have a $k$-algebra embedding $k \to k_*\tilde{C}_{\hat{G}^k}(\hat{Q})\tilde{b}(\delta)$ (cf. isomorphism 3.2.2) which determines an indecomposable projective $k_*\tilde{C}_{\hat{G}^k}(\hat{Q})\tilde{b}(\delta)$-module isomorphic to its own socle; thus, the block algebra $k_*\tilde{C}_{\hat{G}^k}(\hat{Q})\tilde{b}(\delta)$ admits a simple projective module and therefore the block $\tilde{b}(\delta)$ has defect zero.

Assume that statement 3.12.3 holds and let $\hat{P}_\gamma$ be a local pointed $O^\ast$-group on $O_\ast \hat{G}$ containing $\hat{Q}_\delta$; then, there are successively local points $\varepsilon$ of $\hat{Q}, C_{\hat{P}}(\hat{Q})$ and $\varphi$ of $\hat{Q}$ on $O_\ast \hat{G}$ fulfilling
\[ \hat{Q}_\varphi \subset \hat{Q}.C_{\hat{P}}(\hat{Q})_\varepsilon \subset \hat{P}_\gamma \] 3.12.9
which forces $b(\varphi) = b(\delta)$ [2, Theorem 1.8] and therefore, since we have (cf. isomorphism 3.2.2)
\[ k_*\tilde{C}_{\hat{G}^k}(\hat{Q})\tilde{b}(\delta) \cong (O_\ast \hat{G})(\hat{Q}_\delta) \] 3.12.10,
we get $\varphi = \delta$; in particular, since we have [2, Proposition 1.5]
\[ ((O_\ast \hat{G})(Q))(C_{P'}(\hat{Q})) \cong (O_\ast \hat{G})(Q.C_{P'}(\hat{Q})) \] 3.12.11,
$\text{Br}_{Q}(\varepsilon)$ is a local point of $C_{P'}(\hat{Q})$ on $(O_\ast \hat{G})(Q)$; but, it follows from [13, Theorem 2.9] that the canonical map
\[ k_*\tilde{C}_{\hat{G}^k}(\hat{Q}) \rightarrow k_*\tilde{C}_{\hat{G}^k}(\hat{Q}) \] 3.12.12
is a semicovering $C_{\hat{G}^k}(\hat{Q})$-interior algebra homomorphism; hence, according to [6, Proposition 3.15] and to isomorphism 3.2.2 above, $\text{Br}_{Q}(\varepsilon)$ determines a local point of $C_{P'}(\hat{Q})/C_Q(\hat{Q})$ on $k_*\tilde{C}_{\hat{G}^k}(\hat{Q})\tilde{b}(\delta)$, which forces this $p$-group to be trivial, so that $C_{P'}(\hat{Q}) = Z(\hat{Q})$.

Finally, it follows from Corollary 3.3 that statement 3.12.4 implies that, for any local pointed groups $P'_\gamma$ containing $Q'_{\delta'}$, we have $C_{P'_\gamma}(Q') = Z(Q')$; then, according to [14, 4.8 and Corollary 7.3], $Q'_{\delta'}$ is selfcentralizing and therefore it is a defect pointed group of $Q'.C_{G'}(Q'_{\delta'})$; hence, once again by Corollary 3.3, $\hat{Q}_\delta$ is selfcentralizing too.

**Corollary 3.13.** If $\hat{P}_\gamma$ and $\hat{Q}_\delta$ are local pointed $O^\ast$-groups on $O_\ast \hat{G}$ such that $Q_\delta \subset \hat{P}_\gamma$ and $\hat{Q}_\delta$ is selfcentralizing then $\hat{P}_\gamma$ is selfcentralizing too.

**Proof:** In this situation, it is clear that $\hat{P}_\gamma$ fulfills also statement 3.12.4.
Proposition 3.14. For any local pointed \( O^* \)-group \( \hat{P}_\gamma \) on \( O_* \hat{G} \) there is a unique local point \( \hat{\gamma}^k \) of \( P \) on \( k_* \hat{G}^k \) containing the image of \( \gamma \), the canonical group homomorphism

\[
E_G(\hat{P}_\gamma) \rightarrow E_G(P_{\gamma^k}) \tag{3.14.1}
\]

is surjective and its kernel is an abelian \( p \)-group. In particular, \( P_{\gamma^k} \) is self-centralizing if and only if \( E_G(\hat{P}_\gamma) \) is isomorphic to \( E_G(P_{\gamma^k}) \) and \( \hat{P}_\gamma \) is self-centralizing too. Moreover, for any local pointed group \( Q_{\delta^k} \) on \( k_* \hat{G}^k \), there is a local point \( \delta \) of \( O_* \hat{G} \) of the converse image \( \hat{Q} \) of \( Q \) in \( \hat{G} \) such that \( \delta^k \) contains the image of \( \delta \) and that the canonical map

\[
E_G(\hat{Q}_\delta, \hat{P}_\gamma) \rightarrow E_G(Q_{\delta^k}, P_{\gamma^k}) \tag{3.14.2}
\]

is surjective.

**Proof:** With the notation above, it follows from Corollary 3.3 that \( \hat{P}_\gamma \) determines a local pointed group \( P'_{\gamma'} \) on \( O' G' e_{A'} \); but, it is clear that the image \( \gamma'^k \) of \( \gamma' \) in \( k G' e_{A'} \) is a local point of \( P \) on this \( k \)-algebra; moreover, it follows from [13, Theorem 2.9] that the canonical map

\[
k G' e_{A'} \rightarrow k_* \hat{G}^k \tag{3.14.3}
\]

is a semicovering \( G \)-algebra homomorphism; hence, according to [6, Proposition 3.15], the image of \( \gamma'^k \) is contained in a local point \( \gamma^k \) of \( P \) on \( k_* \hat{G}^k \), which then contains the image of \( \gamma \); moreover, if follows from [13, Corollary 2.13] that \( P_{\gamma^k} \) is selfcentralizing if and only if \( E_G(\hat{P}_\gamma) \) is isomorphic to \( E_G(P_{\gamma^k}) \) and \( \hat{P}_\gamma \) is self-centralizing too.

The existence of the local point \( \delta \) of \( \hat{Q} \) in the converse image of \( \delta^k \) follows from [6, Proposition 3.15] and from the fact that homomorphism 3.14.3 is a semicovering; moreover, if \( E_G(Q_{\delta^k}, P_{\gamma^k}) \) is not empty then it is quite clear that we can choose \( \delta \) in such a way that \( E_G(\hat{Q}_\delta, \hat{P}_\gamma) \) is also not empty; in this case, the surjectivity of the canonical map 3.14.2 follows from [13, Theorem 2.9]† and from Proposition 3.10 above.

4. Fong reduction for interior algebras over an \( O^* \)-group

4.1. As a matter of fact, all our arguments in [15, §3] on interior algebras over finite \( k^* \)-groups can be translated to interior algebras over finite \( O^* \)-groups; we will explicit the translation of the statements since they demand some modifications, and some indications on the proofs if necessary; but, it seems useless to repeat the identical part of the proofs.

† In [13, Theorem 2.9] the necessary assumption that there is at least an \( A \)-fusion from \( Q_\delta \) to \( P_\gamma \) has been forgotten.
4.2. The first modification concerns, for any finite $p$-group $P$, the involved Dade $P$-algebras; in our present situation, we call Dade $P$-algebra a full matrix algebra $S$ over $\mathcal{O}$ endowed with an action of $P$ which stabilizes an $\mathcal{O}$-basis of $S$ containing $1_S$; recall that two Dade $P$-algebras $S$ and $S'$ are similar if $S$ can be embedded (cf. 2.8) in the tensor product $\text{End}(N) \otimes k S'$ for a suitable $kP$-module $N$ with a $P$-stable $\mathcal{O}$-basis [11, 1.5 and 2.5.1]. Moreover, recall that $S(P)$ is a simple $k$-algebra [11, 1.8]; in particular, if $S$ is primitive then we have

$$\text{rank}_{\mathcal{O}}(S) \equiv 1 \mod p$$

4.2.1;

hence, in all the cases, it follows from [11, 3.13] that the action of $P$ on $S$ can be lifted to a group homomorphism $P \to S^*$ and we will consider $S$ as a $P$-interior algebra. Since $S(P)$ is simple, $P$ has a unique local point $\pi$ on $S$ [11, 1.8] that very often we omit, respectively writing $F_S(P)$ and $F_S(P)$ instead of $F_S(P_\pi)$ and $F_S(P_\pi)$ (cf. 2.11 and 2.12). Moreover, since we have an evident $k^*$-group homomorphism

$$F_S(P) \to F_{k \otimes \mathcal{O}}(P)$$

4.2.2,

any polarization $\omega$ considered in [16, 2.15] still supplies a $k^*$-group homomorphism

$$\omega(P, S) : F_S(P) \to k^*$$

4.2.3.

4.3. Let $\hat{P}$ be an $\mathcal{O}^*$-group with $\mathcal{O}^*$-quotient $P$; we call monomial any $\mathcal{O}_*\hat{P}$-module $M$ such that the sources of any indecomposable direct summand of $M$ have $\mathcal{O}$-rang one or, equivalently, it admits an $\mathcal{O}$-module decomposition $M = \bigoplus_{i \in I} L_i$ such that $\text{rank}_{\mathcal{O}}(L_i) = 1$ and that $\hat{P}$ stabilizes the family $\{L_i\}_{i \in I}$; then, $k \otimes \mathcal{O} M$ is a permutation $kP$-module. More precisely, we call twisted diagonal $\mathcal{O}_*(\hat{P} \times \hat{P})$-module any monomial $\mathcal{O}_*(\hat{P} \times \hat{P})$-module $M$ such that the intersections of any vertex of any indecomposable direct summand of $M$ with the images in $\hat{P} \times \hat{P}$ of $\hat{P}$ and $\hat{P}$ coincide with the image of $\mathcal{O}^*$.

4.4. Let $\hat{G}$ be an $\mathcal{O}^*$-group with finite $\mathcal{O}^*$-quotient $G$ and $A$ a $\hat{G}$-interior algebra; as in [13, 2.8], we denote by $\mathcal{L}_A$ the local category of $A$ where the objects are the local pointed groups on $A$ and the morphisms are the $A$-fusions (cf. 2.11) between them with the usual composition [8, Definition 2.15]. Let $S$ be a $G$-stable unitary subalgebra of $A$ isomorphic to a direct product of full matrix algebras over $\mathcal{O}$ and assume that $G$ acts transitively on the set $I$ of primitive idempotents of the center $Z(S)$ of $S$; let $i$ be an element of $I$ and denote by $\hat{H}$ the stabilizer of $i$ in $\hat{G}$. Thus, the $\mathcal{O}^*$-quotient $H$ of $\hat{H}$ acts on the full $\mathcal{O}$-matrix algebra $S_i$ determining an $\mathcal{O}^*$-group $\hat{H}$, together with an $\mathcal{O}^*$-group homomorphism $\rho : H \to (S_i)^*$ (cf. 2.7) and we set (cf. 2.6)

$$\hat{H} = \hat{H} \ast (\hat{H})^\circ$$

4.4.1;
moreover, if \( P \) is a \( p \)-subgroup of \( H \) and \( \text{Res}_\text{H}^H(Si) \) is a Dade \( P \)-algebra, according to 4.2 the converse image \( \hat{P} \) of \( P \) in \( \hat{H} \) is split and therefore, up to the choice of a splitting \( \hat{P} \cong \mathcal{O}^* \times \hat{P} \), we can identify to each other the converse images in \( \hat{H} \) and \( \hat{H} \) of any subgroup \( Q \) of \( P \); if \( p \) does not divide \( \text{rank}_\mathcal{O}(Si) \), we fix our choice assuming that \( \rho(1 \times P) \subset \text{Ker}(\text{det}_\mathcal{O}(Si)) \).

**Proposition 4.5.** With the notation and the hypothesis above, there exists an \( H \)-interior algebra \( B \), unique up to isomorphisms, such that we have a \( \hat{G} \)-interior algebra isomorphism

\[
A \cong \text{Ind}_{\hat{H}}^\hat{G}(Si \otimes_k B)
\]

mapping \( s \in Si \) on \( 1 \otimes (s \otimes 1_B) \otimes 1 \). In particular, \( A \) and \( B \) are Morita equivalent.

**Proof:** Since [7, Proposition 2.1] holds over \( \mathcal{O} \), the proof of [15, Proposition 3.2] applies.

**Corollary 4.6.** With the notation and the hypothesis above, assume that \( B \) has a unique \( H \)-conjugacy class of maximal local pointed \( \mathcal{O}^* \)-groups \( \hat{P}_\gamma \), that \( \text{Res}_\text{H}^H(Si) \) is a Dade \( P \)-algebra and that \( A \) and \( B \) are twisted diagonal \( \mathcal{O}_s(\hat{P} \times \hat{P}^*) \)-modules by left- and right-hand multiplication. Choosing a splitting \( \hat{P} \cong \mathcal{O}^* \times \hat{P} \), for any local pointed \( \mathcal{O}^* \)-group \( \hat{Q}_\delta \) on \( B \) we have a local point \( \iota(\delta) \) of \( \hat{Q} \) on \( A \) such that isomorphism 4.5.1 induces a \( \hat{Q} \)-interior algebra embedding

\[
A_{\iota(\delta)} \longrightarrow \text{Res}_\text{Q}^\text{H}(Si) \otimes_\mathcal{O} B_{\delta}
\]

and this correspondence determines an equivalence of categories \( \iota: \mathcal{L}_B \rightarrow \mathcal{L}_A \) between the local categories of \( B \) and \( A \). In particular, \( A \) has a unique \( G \)-conjugacy class of maximal local pointed \( \mathcal{O}^* \)-groups.

**Proof:** The proof of [15, Corollary 3.3] shows the existence of a map \( \iota \) between the sets of local pointed \( \mathcal{O}^* \)-groups on \( B \) and on \( A \). Then, in order to prove the equality

\[
F_B(\hat{R}_\varepsilon, \hat{Q}_\delta) = F_A(\hat{R}_{\iota(\varepsilon)}, \hat{Q}_{\iota(\delta)})
\]

for any pair of local pointed \( \mathcal{O}^* \)-groups \( \hat{Q}_\delta \) and \( \hat{R}_\varepsilon \) on \( B \), note that it follows from Proposition 3.5 above that [8, Proposition 2.14] and [10, Theorem 5.3] apply to our present situation; thus, we still have

\[
F_{iAi}(\hat{R}_{\iota(\varepsilon)}, \hat{Q}_{\iota(\delta)}) = F_A(\hat{R}_{\iota(\varepsilon)}, \hat{Q}_{\iota(\delta)})
\]

and, since we have \( iAi \cong Si \otimes_\mathcal{O} B \), it suffices to prove that \( F_{Si}(R, Q) \) contains both \( F_{iAi}(\hat{R}_{\iota(\varepsilon)}, \hat{Q}_{\iota(\delta)}) \) and \( F_B(\hat{R}_\varepsilon, \hat{Q}_\delta) \).
Let \( \hat{\varphi} : \hat{R} \to \hat{Q} \) be an \( O^* \)-group homomorphism which belongs either to 
\( F_A(\hat{R}_{\epsilon(\xi)}, \hat{Q}_{\epsilon(\xi)}) \) or to \( F_B(\hat{R}_{\epsilon}, \hat{Q}_{\epsilon}) \); once again, according to Proposition 3.5 above, [8, Proposition 2.18] apply to our present situation and therefore, since 
\( \text{Res}^H_P(Si) \) is a Dade \( P \)-algebra, if follows from [11, statement 2.5.1] that it 
suffices to prove that the Dade \( R \)-algebras \( \text{Res}^H_R(k \otimes O Si) \) and \( \text{Res}_\varphi(\text{Res}^H_Q(k \otimes O Si)) \) 
are similar (cf. 4.2). But, it follows from [11, statement 1.5.2] that these 
Dade \( R \)-algebras are similar if and only if the corresponding Dade \( R \)-algebras 
over \( k \) are so; moreover, according to our hypothesis, the actions of \( P \times P \) on 
\( k \otimes O A \) and \( k \otimes O B \) by left and right multiplication stabilize bases where 
\( P \times \{1\} \) and \( \{1\} \times P \) act freely, so that the hypothesis of [15, Corollary 3.3] 
are fulfilled.

Consequently, the proof of [15, Corollary 3.3] applies, proving that the 
corresponding Dade \( P \)-algebras over \( k \)

\[
\text{Res}^H_R(k \otimes O Si) \quad \text{and} \quad \text{Res}_\varphi(\text{Res}^H_Q(k \otimes O Si))
\]

are indeed similar to each other which shows that the functor \( \iota : \mathcal{L}_B \to \mathcal{L}_A \) is 
fully faithful; once again, the proof of [15, Corollary 3.3] applies showing that 
this functor is essentially surjective, so that it is an equivalence of categories.
We are done.

4.7. The main point in Fong reduction is that if \( A \) is a block algebra 
\( O_* \hat{G} b \) for a block \( b \) of \( \hat{G} \) then \( i \) is a block of \( \hat{H} \) and if moreover \( p \) does 
not divide \( \text{rank} O(Si) \) then \( B \) is also a block algebra. Denote by \( V \) a simple 
k \otimes O Si-module, which becomes an \( k_* \hat{H}^k \)-module through \( \rho \) (cf. 4.4).

**Proposition 4.8.** With the notation and the hypothesis above, if \( A \cong O_* \hat{G} b \) 
for a block \( b \) of \( \hat{G} \) then \( i \) is a block of \( \hat{H} \) which belongs to a point \( \beta \) of \( \hat{H} \) 
on \( A \) and we have \( i(O_* \hat{G}) i = O_* \hat{H} i \). In particular, we have an equivalence 
of categories \( \mathcal{L}_{O_* \hat{H} i} \cong \mathcal{L}_{O_* \hat{G} b} \).

**Proof:** Replacing [15, Corollary 3.3] by Corollary 4.6 above, the proof of [15, Proposition 3.5] applies and then the last statement follows from Corollary 4.6 above.

**Theorem 4.9.** With the notation and the hypothesis above, assume that we 
have \( A \cong O_* \hat{G} b \) for a block \( b \) of \( \hat{G} \) and that \( p \) does not divide \( \text{rank} O(Si) \).
Then, we have \( B \cong O_* \hat{H} c \) for a block \( c \) of \( \hat{H} \) and \( V \) is a simple \( k_* \hat{H}^k \)-mo-
dule. Moreover, if \( \text{Res}^H_P(Si) \) is a Dade \( P \)-algebra, we have an equivalence of 
categories \( \mathcal{L}_{O_* \hat{H} c} \cong \mathcal{L}_{O_* \hat{G} b} \).

**Proof:** It follows from Proposition 3.5 above that [10, §4] apply to our present 
situation; thus, replacing [15, Corollary 3.3] by Corollary 4.6 above, the proof of [15, Theorem 3.6] applies.
Theorem 4.10. With the notation and the hypothesis above, assume that $A \cong O_∗G b$ for a block $b$ of $\hat{G}$ and that $S = O_∗K b$ for a normal $O^*$-subgroup $\hat{K}$ of $\hat{G}$ having a block $d$ of defect zero such that $d b \neq 0$. Then, $K$ is a normal subgroup of $H$ and we have $B \cong O_∗(H/K)\bar{c}$ for a block $\bar{c}$ of $H/K$.

Proof: The corresponding part of the proof of [15, Theorem 3.7] applies.

4.11. It is well-known that, up to replacing $O$ by $K$, the so-called Clifford’s reduction can be viewed as a particular case of Fong’s reduction. Explicitly, let $\hat{N}$ be a normal $O^*$-subgroup of $\hat{G}$ and $\nu$ an absolutely irreducible character of $\hat{N}$; denote by $\text{Irr}_K(\hat{G}, \nu)$ the set of $\chi \in \text{Irr}_K(\hat{G})$ such that $\nu$ is involved in $\text{Res}_K(\chi)$, by $e_\nu$ the primitive idempotent of $Z(K_{\hat{N}})$ associated with $\nu$ and by $H$ and $\hat{H}$ the respective stabilizers of $\nu$ in $G$ and $\hat{G}$. If $\hat{K}$ is an extension of $K$ of degree divisible by $|\hat{G}|$, it follows from Proposition 2.3 that, denoting by $\hat{O}$ the ring of integers of $\hat{K}$ and setting

$$\hat{G} = \hat{G}^{\hat{O}}, \quad \hat{H} = \hat{H}^{\hat{O}} \quad \text{and} \quad \hat{N} = \hat{N}^{\hat{O}}$$

4.11.1, there are an $\hat{O}$-group $\hat{H}$ with $\hat{O}$-quotient $H$ containing and normalizing $\hat{N}$, and an $\hat{O}$-group homomorphism

$$\hat{H} \rightarrow (\hat{K}, \hat{N} e_\zeta)^*$$

4.11.2

lifting the action of $H$ on $\hat{K}, \hat{N} e_\zeta$ and extending the structural $\hat{O}$-group homomorphism from $\hat{N}$. It is clear that this homomorphism determines an irreducible character $\text{Ext}(\nu)$ of $\hat{H}$; moreover, we set $\hat{H}^\circ = \hat{H} * (\hat{H})^\circ$ and identify $N \subset \hat{N} * \hat{N}^\circ$ with its canonical image in $H^\circ$.

Proposition 4.12. With the notation above, we have a $\hat{G}$-interior algebra isomorphism

$$\hat{K}, \hat{G}, \text{Tr}_{\hat{G}}(e_\nu) \cong \text{Ind}_H^\hat{G}(\hat{K}, \hat{N} e_\nu \otimes \hat{K}, (H^*/N))$$

4.12.1

mapping $y e_\nu$ on $1 \otimes (y e_\nu \otimes 1) \otimes 1$ for any $y \in \hat{N}$. In particular, if $K$ is a splitting field for $\hat{G}$ then $\hat{K}$ is a splitting field for $H^*/N$ and the map sending any $\zeta \in \text{Irr}_K(H^*/N)$ to

$$\chi = \text{Ind}_H^\hat{G}(\text{Ext}(\nu) \otimes \text{Res}(\zeta))$$

4.12.2,

where $\text{Res}(\zeta)$ denotes the corresponding irreducible character of $H^\circ$, determines a bijection

$$\text{Irr}_K(H^*/N) \cong \text{Irr}_K(\hat{G}, \nu)$$

4.12.3

fulfilling $d(\chi) = d(\nu) + d(\zeta)$. 
**Proof:** Considering a set \( X \subset G \) of representatives for \( G/H \) and the \( G \)-stable pairwise orthogonal set of idempotents \( \{ (e_\nu)^{-1} \}_{\nu \in X} \), from [9, 2.14.2] we get a \( \hat{G} \)-interior algebra isomorphism

\[
\hat{\mathcal{K}}_* \hat{G} \operatorname{Tr}_{\hat{H}}^G (e_\nu) \cong \operatorname{Ind}_{\hat{H}}^G (e_\nu (\hat{\mathcal{K}}_* \hat{G}) e_\nu)
\]

and, as in the proof of [15, Proposition 3.5], we actually get

\[
e_\nu (\hat{\mathcal{K}}_* \hat{G}) e_\nu = \hat{\mathcal{K}}_* \hat{H} e_\nu
\]

moreover, since \( \hat{\mathcal{K}}_* \hat{N} e_\nu \) is a full matrix algebra over \( \hat{\mathcal{K}} \) and a unitary subalgebra of \( \hat{\mathcal{K}}_* \hat{H} e_\nu \), the multiplication in this \( \hat{\mathcal{K}} \)-algebra induces a \( \hat{\mathcal{K}} \)-algebra isomorphism

\[
\hat{\mathcal{K}}_* \hat{N} e_\nu \otimes_{\hat{\mathcal{K}}} C \cong \hat{\mathcal{K}}_* \hat{H} e_\nu
\]

where \( C \) is the centralizer of \( \hat{\mathcal{K}}_* \hat{N} e_\nu \) in \( \hat{\mathcal{K}}_* \hat{H} e_\nu \) [7, Proposition 2.1]; then, as in the proof of [15, Proposition 3.2], \( C \) becomes an \( \hat{H}/\hat{N} \)-interior algebra and it is easily checked that the structural \( \hat{\mathcal{K}} \)-algebra homomorphism

\[
\hat{\mathcal{K}}_* (\hat{H}/\hat{N}) \rightarrow C
\]

is an isomorphism.

In particular, if \( \mathcal{K} \) is a splitting field for \( \hat{G} \) then \( \mathcal{K}_* \hat{G} \operatorname{Tr}_{\hat{H}}^G (e_\nu) \) is a direct product of full matrix algebras over \( \mathcal{K} \) and therefore \( \mathcal{K}_* \hat{G} \operatorname{Tr}_{\hat{H}}^G (e_\nu) \) is a direct product of full matrix algebras over \( \mathcal{K} \); since \( \mathcal{K}_* \hat{N} e_\nu \) is a full matrix algebra over \( \mathcal{K} \), this forces \( \mathcal{K}_* (\hat{H}/\hat{N}) \) to be also a full matrix algebra over \( \mathcal{K} \) and induces bijection 4.12.3. Finally, setting \( \xi = \operatorname{Ext}(\nu) \otimes \operatorname{Res}(\zeta) \), we have

\[
d(\chi) = \vartheta_p \left( \frac{|G|}{\chi(1)} \right) = \vartheta_p \left( \frac{|H|}{\xi(1)} \right) = \vartheta_p \left( \frac{|N||H/N|}{\nu(1)\zeta(1)} \right) = d(\nu) + d(\zeta)
\]

We are done.

5. The \( p \)-solvable \( \mathcal{O}^* \)-group case

5.1. As above, \( \hat{G} \) is an \( \mathcal{O}^* \)-group with finite \( \mathcal{O}^* \)-quotient \( G \) and in this section we assume that \( G \) is \( p \)-solvable. Let \( b \) be a block of \( \hat{G} \) and \( S \) a \( G \)-stable unitary subalgebra of \( A \) isomorphic to a direct product of full matrix algebras over \( \mathcal{O} \), maximal such that \( p \) does not divide the \( \mathcal{O} \)-rank of its indecomposable factors. Since \( b \) is primitive in \( Z(\mathcal{O}, \hat{G}b) \), \( G \) acts transitively on the set \( I \) of primitive idempotents of \( Z(S) \) and we borrow the notations \( i, \hat{H}, H \) and \( H' \) from 4.4. According to Propositions 4.5 and 4.8, and to Theorem 4.9,
i is a block of \( \hat{H} \) which belongs to a point \( \beta \) of \( \hat{H} \) on \( \mathcal{O}_s \hat{G} \) and, for a suitable block \( c \) of \( H \), we have \( \mathcal{O}_s \hat{G} \)- and \( \hat{H} \)-interior algebra isomorphisms

\[
\mathcal{O}_s \hat{G} b \cong \text{Ind}^{\hat{H}}_H(\mathcal{O}_s \hat{H} i) \quad \text{and} \quad (\mathcal{O}_s \hat{G})_\beta \cong \mathcal{O}_s \hat{H} i \cong \mathcal{O} \otimes \mathcal{O}_s H c \quad 5.1.1
\]

and an equivalence of categories \( \iota : \mathcal{L}_{\mathcal{O}_s H c} \to \mathcal{L}_{\mathcal{O}_s \hat{G} b} \); in particular, there is a defect pointed \( \mathcal{O}^* \)-group \( \hat{P}_\gamma \) of \( b \) contained in \( \hat{H} \). Moreover, we denote by \( \mathcal{O}_p'(\hat{H}) \), \( \mathcal{O}_p'(H) \) and \( \mathcal{O}_p'(H') \) the respective inverse images in \( \hat{H} \), \( H \) and \( H' \) of \( \mathcal{O}_p'(H) \).

**Proposition 5.2.** With the notation and the hypothesis above, \( P \) is a Sylow \( p \)-subgroup of \( H \), \( S_i \) coincides with \( \mathcal{O}_s \mathcal{O}_p'(\hat{H}) \iota \) and is a Dade \( P \)-algebra, and the inclusion of \( \mathcal{O}_p'(H) \) in \( (Si)^* \) induces an \( H \)-stable \( \mathcal{O}^* \)-group isomorphism \( \mathcal{O}^* \times \mathcal{O}_p'(H) \cong \mathcal{O}_p'(H) \) such that, identifying \( \mathcal{O}_p'(H) \) with its image, we have

\[
c = \frac{1}{|\mathcal{O}_p'(H)|} \sum_{p \in \mathcal{O}_p'(H)} y \quad \text{and} \quad \mathcal{O}_s H c \cong \mathcal{O}_s(H/\mathcal{O}_p'(H)) \quad 5.2.1.
\]

In particular, we have an equivalence of categories \( \mathcal{L}_{\mathcal{O}_s \hat{G} b} \cong \mathcal{L}_{\mathcal{O}_s (H/\mathcal{O}_p'(H))} \) and the \( \mathcal{O} \)-algebras \( \mathcal{O}_s \hat{G} b \) and \( \mathcal{O}_s(H/\mathcal{O}_p'(H)) \) are Morita equivalent. Moreover, setting \( Q = P \cap \mathcal{O}_p'(H) \), \( c \) is primitive in \( (\mathcal{O}_s H c)^Q \).

**Proof:** Note that the Brauer First Main Theorem still holds for blocks of \( \mathcal{O}^* \)-groups (cf. 2.7) and that it follows from Proposition 3.5 above that [10, Proposition 5.6 and Corollary 5.8] apply to our present situation; thus, the proof of [15, Proposition 4.2] applies. Moreover, since \( S_i \) is generated by a finite \( p' \)-group [9, Lemma 5.5], \( P \) stabilizes an \( \mathcal{O} \)-basis of \( S_i \) and, since \( p \) does not divide \( \text{rank}_\mathcal{O}(S_i) \), \( P \) fixes some element in any \( P \)-stable \( \mathcal{O} \)-basis; then, the last statement follows from Proposition 4.5 and Theorem 4.9.

**Corollary 5.3.** With the notation and the hypothesis above, denote by \( Q \) the intersection \( P \cap \mathcal{O}_p'(H) \), by \( \hat{Q} \) the converse image of \( Q \) in \( \hat{P} \) and by \( \delta \) a local point of \( \hat{Q} \) on \( \mathcal{O}_s \hat{G} \) such that \( Q_\delta \subset \hat{P}_\gamma \subset \hat{H}_\beta \). Then, \( Q_\delta \) is the unique local pointed \( \mathcal{O}^* \)-group on \( \mathcal{O}_s \hat{G} \) fulfilling the following conditions

5.3.1 We have \( \hat{Q}_\delta \trianglelefteq \hat{P}_\gamma \), \( C_P(Q) = Z(Q) \) and \( \mathcal{O}_p(E_{\mathcal{O}_s(Q_\delta)}(\hat{Q}_\delta \hat{P}_\gamma)) = \{1\} \)

5.3.2 We have \( E_{\mathcal{O}_s}(\hat{P}_\gamma \hat{Q}_\delta) = E_{\mathcal{O}_s(Q_\delta)}(\hat{Q}_\delta \hat{P}_\gamma) \) for any local pointed \( \mathcal{O}^* \)-group \( \hat{Q}_\delta \) on \( \mathcal{O}_s \hat{G} \) contained in \( \hat{P}_\gamma \).

Moreover, denoting by \( b \) and \( f \) the respective blocks of \( \hat{G} \) and \( N_G(Q_\delta) \) determined by \( \hat{P}_\gamma \), the \( \mathcal{O} \)-algebras \( \mathcal{O}_s \hat{G} b \) and \( \mathcal{O}_s N_G(Q_\delta) f \) are Morita equivalent.

**Proof:** It follows from Proposition 5.2 that \( S_i \) is a Dade \( P \)-algebra; hence, Corollary 4.6 and Theorem 4.9 above apply, and therefore the proof of [15,
Corollary 4.3] applies too. Moreover, it follows from isomorphisms 5.1.1 and Proposition 5.2 above that 
\( \mathcal{O}_* (\tilde{G} b) \) is Morita equivalent to 
\( \mathcal{O}_* (H / \mathcal{O}_p' (H)) \); 
similarly, setting \( \tilde{N} = N_{\tilde{H}} (\tilde{Q}_\delta) \) and \( N = N_H (Q_\delta) \), 
\( \mathcal{O}_* N_{\tilde{C}} (\tilde{Q}_\delta) f \) is Morita equivalent to 
\( \mathcal{O}_* (N / \mathcal{O}_p' (N)) \); but, the uniqueness of \( \delta \) forces \( N = N_H (\tilde{Q}) \) 
and thus, by the Frattini argument, we get \( H = \mathcal{O}_p' (H) N \), so that we still get 
\( H / \mathcal{O}_p' (H) \cong N / \mathcal{O}_p' (N) \)  
5.3.3.

Finally, we claim that we have an \( \mathcal{O}^* \)-group isomorphism 
\( H / \mathcal{O}_p' (H) \cong N / \mathcal{O}_p' (N) \)  
5.3.4; 
indeed, it suffices to prove that \( \tilde{N} \) is an \( \mathcal{O}^* \)-subgroup of \( H \); but, these 
\( \mathcal{O}^* \)-groups come from the respective actions of \( N \) and \( H \) on \( \mathcal{O}_p' (\tilde{N}) \) and 
\( \mathcal{O}_p' (\tilde{H}) \) and therefore it suffices to prove that \( N^k \) is an \( \mathcal{O}^* \)-subgroup of \( H^k \); 
moreover, it is clear that 
\( \mathcal{O}_p' (\tilde{N}) = C_{\mathcal{O}_p' (\tilde{N})} (\tilde{Q}) \)  
5.3.5.

Consequently, the corresponding block \( j \) of \( \tilde{N} \) fulfills \( \text{Br}_{\tilde{Q}} (j) = \text{Br}_{\tilde{Q}} (i) \) and 
therefore we have to consider the respective actions of \( N \) and \( H \) on \( \mathcal{O}_p' (\tilde{N}) \) and 
\( \mathcal{O}_p' (\tilde{H}) \) and thus, our claim follows from the existence of a splitting (cf. 4.2) 
\( \omega (q, k \otimes \sigma Si) : \hat{F}_k \otimes \sigma Si (Q) \rightarrow k^* \)  
5.3.6.

5.4. Similarly as in [16], we call Fitting pointed \( \mathcal{O}^* \)-group of \( \tilde{G} \) any local 
pointed \( \mathcal{O}^* \)-group \( \tilde{O}_\eta \) on \( \mathcal{O}_* \tilde{G} \) fulfilling conditions 5.3.1 and 5.3.2 above with 
respect to some maximal local pointed \( \mathcal{O}^* \)-group \( \tilde{P}_\gamma \) on \( \mathcal{O}_* \tilde{G} \).

**Theorem 5.5.** With the notation and the hypothesis above, there are a 
p-solvable finite \( \mathcal{O}^* \)-group \( \mathcal{L} \) containing \( \tilde{P} \) as an \( \mathcal{O}^* \)-subgroup and a primitive 
Dade \( \mathcal{P} \)-algebra \( T \), both unique up to isomorphisms, fulfilling the following 
conditions 
5.5.1 \( C_\mathcal{L} (\mathcal{O}_p (\mathcal{L})) = Z (\mathcal{O}_p (\mathcal{L})) \) where \( \mathcal{L} \) is the \( \mathcal{O}^* \)-quotient of \( \mathcal{L} \). 
5.5.2 There is a \( \mathcal{P} \)-interior algebra embedding \( e_\gamma : (\mathcal{O}_* \tilde{G})_\gamma \rightarrow T \otimes \mathcal{O}_* \tilde{L} \). 
In particular, \( e_\gamma \) induces an equivalence of categories \( \mathcal{L}_{\mathcal{O}_* \tilde{G}} \cong \mathcal{L}_{\mathcal{O}_* \tilde{L}} \) and a 
Morita equivalence between the \( \mathcal{O} \)-algebras \( \mathcal{O}_* \tilde{G} b \) and \( \mathcal{O}_* \tilde{L} \). Moreover, for 
any local pointed \( \mathcal{O}^* \)-group \( \hat{Q}_\delta \) on \( \mathcal{O}_* \tilde{G} \) contained in \( \tilde{P}_\gamma \), we have 
\( F_{\mathcal{O}_* \tilde{G}} (\hat{Q}_\delta, \tilde{P}_\gamma) \subset F_T (Q, P) \)  
5.5.3

and, denoting by \( \hat{Q}_\delta \) the corresponding local pointed \( \mathcal{O}^* \)-group on \( \mathcal{O}_* \tilde{L} \), any 
polarization \( \omega \) determines a \( k^* \)-group isomorphism 
\( \hat{F}_{\mathcal{O}_* \tilde{G}} (\hat{Q}_\delta) \cong \hat{F}_{\mathcal{O}_* \tilde{L}} (\hat{Q}_\delta) \)  
5.5.4.
Proof: As in [15, 4.4], the existence of \( \hat{L} \), \( T \) and \( e_\gamma \) follows from the results above; moreover, it follows from [11, statement 1.5.2] that, in order to prove the uniqueness of the Dade \( P \)-algebra \( T \), it suffices to prove the uniqueness of the \( Dade P \)-algebra \( k \otimes_\mathbb{O} T \) over \( k \) and this uniqueness follows from [15, Lemma 4.5 and Theorem 4.6].

Moreover, from Corollary 4.6, any local pointed \( \mathcal{O}^* \)-group \( \hat{Q}_\delta \) on \( \mathcal{O}, \hat{G} \) contained in \( \hat{P}_\gamma \) determines a local pointed \( \mathcal{O}^* \)-group \( \hat{Q}_\delta \) on \( \mathcal{O}, \hat{L} \), and this correspondence is bijective; then, since \( (\mathcal{O}, \hat{G})_\gamma \) and \( \mathcal{O}, \hat{L} \) are twisted diagonal \( \mathcal{O}^*(\hat{P} \times \hat{P}) \)-modules by left- and right-hand multiplication, it follows again from Corollary 4.6 above and from the uniqueness of the \( \hat{P}_\gamma \)-source pair \( (T, \hat{L}) \) that we have

\[
F_{\mathcal{O},\hat{G}}(\hat{Q}_\delta, \hat{P}_\gamma) = F_{\mathcal{O},\hat{L}}(\hat{Q}_\delta, \hat{P}_\gamma) \subset F_T(Q, P)
\]

5.5.5.

At this point, according to [10, Proposition 5.11], any polarization \( \omega \) determines a \( k^* \)-group isomorphism

\[
\hat{F}_{\mathcal{O},\hat{G}}(\hat{Q}_\delta) \cong \hat{F}_{\mathcal{O},\hat{L}}(\hat{Q}_\delta)
\]

5.5.6.

Finally, isomorphism 5.6.4 below together with Proposition 4.5 implies the Morita equivalence between \( \mathcal{O}, \hat{G} b \) and \( \mathcal{O}, \hat{L} \). We are done.

5.6. Let \( b \) be a block of \( \hat{G} \) and \( \hat{P}_\gamma \) a maximal local pointed \( \mathcal{O}^* \)-group on \( \mathcal{O}, \hat{G} b \); we call \( \hat{P}_\gamma \)-source of \( b \) any pair \( (T, \hat{L}) \) formed by a primitive Dade \( P \)-algebra \( T \) — considered as a \( P \)-interior algebra fulfilling

\[
\text{det}_T(P \cdot \text{id}_T) = \{1\}
\]

5.6.1

— and by a \( p \)-solvable finite \( \mathcal{O}^* \)-group \( \hat{L} \) containing \( \hat{P} \) as an \( \mathcal{O}^* \)-subgroup, which fulfills conditions 5.5.1 and 5.5.2 above. Then, since we have \( \hat{P} \)-interior algebra embeddings

\[
\mathcal{O}, \hat{L} \rightarrow T^\circ \otimes_\mathcal{O} T \otimes_\mathcal{O} \mathcal{O}, \hat{L} \rightarrow T^\circ \otimes_\mathcal{O} (\mathcal{O}, \hat{G})_\gamma
\]

5.6.2

and \( \hat{P} \) has a unique local point \( (T^\circ \otimes_\mathcal{O} T) \times \hat{\gamma} \) on \( T^\circ \otimes_\mathcal{O} T \otimes_\mathcal{O} \mathcal{O}, \hat{L} \) [10, Theorem 5.3], we still have a \( \hat{P} \)-interior algebra embedding

\[
e_\gamma : \mathcal{O}, \hat{L} \rightarrow T^\circ \otimes_\mathcal{O} (\mathcal{O}, \hat{G})_\gamma
\]

5.6.3

Note that, according to Corollary 4.6 and Proposition 5.2 above, the uniqueness of the \( \hat{P}_\gamma \)-source pair \( (T, \hat{L}) \) forces the existence of an \( \mathcal{O}^* \)-group isomorphism

\[
\hat{L} \cong H/\mathcal{O}_P(H)
\]

5.6.4

inducing the canonical \( \mathcal{O}^* \)-group homomorphism \( \hat{P} \rightarrow H/\mathcal{O}_P(H) \), and of a \( P \)-algebra embedding \( T \rightarrow \text{Res}^H_P(S_l) \).

5.7. Let \( \hat{Q}_\delta \) be a local pointed \( \mathcal{O}^* \)-subgroup on \( \mathcal{O}, \hat{G} \) contained in \( \hat{P}_\gamma \) and containing the Fitting pointed \( \mathcal{O}^* \)-group \( \mathcal{O}_n \) of \( \hat{G} \) contained in \( \hat{P}_\gamma \);
we know that there is a unique point $\beta$ of $N_G(\hat{Q}_\delta)$ on $\mathcal{O}_* \hat{G}$ fulfilling (cf. 3.11)

$$\hat{Q}_\delta \subset N_G(\hat{Q}_\delta)_\beta$$  \hspace{1cm} 5.7.1

and that, up to replacing $\hat{P}_\gamma$ by a $G$-conjugate, we may assume that $N_{\hat{P}}(\hat{Q}_\delta)_\rho$ is a defect pointed $\mathcal{O}^*$-group of $N_G(\hat{Q}_\delta)_\beta$ for a suitable local point $\rho$ of $N_{\hat{P}}(\hat{Q}_\delta)$ on $\mathcal{O}_* \hat{G}$; then, denoting by $f$ the block of $C_{\hat{G}}(\hat{Q})$ determined by $\delta$, note that $f$ is still a block of $N_{\hat{G}}(\hat{Q}_\delta)$ and that $N_{\hat{P}}(\hat{Q}_\delta)_\rho$ determines a maximal local pointed $\mathcal{O}^*$-group $N_{\hat{P}}(\hat{Q}_\delta)_\beta$ on $\mathcal{O}_* N_G(\hat{Q}_\delta) f$

**Proposition 5.8.** With the notation and the hypothesis above, if $(T^s, \hat{L}^s)$ is a $N_{\hat{P}}(\hat{Q}_\delta)_\rho$-source of the block $f$ of $N_G(\hat{Q}_\delta)$, we have an $N_{\hat{P}}(\hat{Q}_\delta)$-algebra embedding and an $\mathcal{O}^*$-group isomorphism inducing the identity on $N_{\hat{P}}(\hat{Q}_\delta)$

$$k \otimes_{\mathcal{O}} T^s \longrightarrow T(Q) \quad \text{and} \quad \hat{L}^s \cong N_{\hat{P}}^*(\hat{Q})$$  \hspace{1cm} 5.8.1.

**Proof:** It follows from Proposition 5.2 that $Si = \mathcal{O}_* \mathcal{O}_{\hat{P}}'(\hat{H})i$ is a Dade $P$-algebra and therefore the $k$-algebra (cf. isomorphism 3.2.2)

$$(Si)(Q) \cong k_* \mathcal{O}_{\hat{P}}'(C_{\hat{H}^s}(\hat{Q})) Br_Q(i)$$  \hspace{1cm} 5.8.2

is simple [11, 1.8]; thus, denoting by $f^k$ the image of $f$ in $k_* N_{\hat{G}^s}(\hat{Q}_\delta)$, it is clear that $N_G(\hat{Q}_\delta)$ stabilizes the semisimple $k$-subalgebra

$$\prod_x k_* \mathcal{O}_{\hat{P}}'(C_{\hat{H}^s}(\hat{Q})) Br_Q(i^x)$$  \hspace{1cm} 5.8.3

of $k_* N_{\hat{G}^s}(\hat{Q}_\delta)^f$, where $x \in N_G(\hat{Q}_\delta)$ runs over a set of representatives for $N_G(\hat{Q}_\delta)/N_H(\hat{Q}_\delta)$.

Hence, it follows from [15, Proposition 3.5] that $Br_Q(i)$ is a block of $N_{\hat{H}^s}(\hat{Q}_\delta)$ over $k$ and therefore it comes from a block $j$ of $N_{\hat{H}}(\hat{Q}_\delta)$ (cf. 3.6); actually, it is easily checked that $j$ belongs to $\mathcal{O}_* \mathcal{O}_{\hat{P}}'(C_{\hat{H}}(\hat{Q})) i \subset Si$; in particular, $N_G(\hat{Q}_\delta)$ stabilizes in $\mathcal{O}_* N_G(\hat{Q}_\delta) f$ the following direct product of full matrix algebras over $\mathcal{O}$

$$S^s = \prod_x \mathcal{O}_* \mathcal{O}_{\hat{P}}'(C_{\hat{H}}(\hat{Q})) j^x$$  \hspace{1cm} 5.8.4,

where $x \in N_G(\hat{Q}_\delta)$ runs over a set of representatives for $N_G(\hat{Q}_\delta)/N_H(\hat{Q}_\delta)$, it acts transitively over the set $J$ of primitive idempotents of $Z(S^s)$, and $N_H(\hat{Q}_\delta)$ is the stabilizer of $j \in J$.

Consequently, setting $\hat{K} = \mathcal{O}_{\hat{P}}'(C_{\hat{H}}(\hat{Q}))$ and considering the corresponding $\mathcal{O}^*$-groups $N_H(\hat{Q}_\delta)$ and $N_{\hat{P}}(\hat{Q}_\delta)$, it follows from Proposition 5.2
that we have a \(N_{\hat{G}}(\hat{Q}_\delta)\)-interior algebra isomorphism
\[
\mathcal{O}_*N_{\hat{G}}(\hat{Q}_\delta)f \cong \text{Ind}_{N_{\hat{P}}(\hat{Q}_\delta)}^{N_{\hat{H}}(\hat{Q}_\delta)}(\mathcal{O}_*\hat{K}_j \otimes \mathcal{O}_*(N_{\hat{H}}(\hat{Q}_\delta)))/K
\]
5.8.5;
thus, it follows from 5.6 above that we have an \(N_{\hat{P}}(\hat{Q}_\delta)\)-algebra embedding \(T^\delta \to \mathcal{O}_*\hat{K}_j\) and therefore, since we have a \(P\)-algebra embedding \(T \to Si\), from isomorphism 5.8.2 we get \(N_{\hat{P}}(\hat{Q}_\delta)\)-algebra embeddings
\[
k \otimes \mathcal{O}_*T^\delta \longrightarrow (Si)(Q) \longleftarrow T(Q)
\]
5.8.6
which induces the \(N_{\hat{P}}(\hat{Q}_\delta)\)-algebra embedding in 5.8.1 since the \(N_{\hat{P}}(\hat{Q}_\delta)\)-algebra \(T^\delta\) is primitive.

Finally, since the \(k^*\)-group \(N_{\hat{H}}(\hat{Q}_\delta)^k\) comes from the action of \(N_{\hat{H}}(\hat{Q}_\delta)\) on the simple \(k\)-algebra \((Si)(Q)\), the existence of a splitting (cf. 4.2)
\[
\omega_{(Q,k \otimes \sigma Si)}: \hat{F}_{k \otimes \sigma Si}(Q) \longrightarrow k^*
\]
5.8.7
shows that \(N_{\hat{H}}(\hat{Q}_\delta)^k\) can be identified to the \(k^*\)-group \(N_{\hat{H}^k}(\hat{Q}_\delta)\) and therefore we get a \(k^*\)-group isomorphism
\[
(N_{\hat{H}}(\hat{Q}_\delta)^k \cong N_{(\hat{H})^k}(\hat{Q}_\delta)
\]
5.8.8;
moreover, since both \(O^*\)-groups \(N_{\hat{H}}(\hat{Q}_\delta)\) and \(N_{\hat{H}}(\hat{Q}_\delta)\) contain \(N_{\hat{P}}(\hat{Q}_\delta)\), and \(N_{\hat{P}}(\hat{Q}_\delta)\) is a Sylow \(p\)-subgroup of \(N_{\hat{H}}(\hat{Q}_\delta)\), it follows from [3, Ch. XII, Theorem 10.1] that these \(O^*\)-groups are isomorphic, so that we get \(O^*\)-group isomorphisms
\[
\hat{\mathcal{L}}^\delta \cong N_{\hat{H}}(\hat{Q}_\delta)/K \cong N_{\hat{H}}(\hat{Q}_\delta)/K \cong N_{\mathcal{L}}(\hat{Q})
\]
5.8.9.
We are done.

5.9. Let us denote by \(\text{Out}_{\hat{P}}((\mathcal{O}_*\hat{G})_\gamma)\) the group of exterior automorphisms (cf. 2.8) of the \(\hat{P}\)-interior algebra \((\mathcal{O}_*\hat{G})_\gamma\), which is Abelian according to [9, Proposition 14.9].

**Proposition 5.10.** With the notation above, there are group isomorphisms
\[
\text{Out}_{\hat{P}}((\mathcal{O}_*\hat{G})_\gamma) \cong \text{Out}_{\hat{P}}(\mathcal{O}_*\hat{L}) \cong \text{Hom}(L, k^*)
\]
5.10.1
mapping \(\tilde{\sigma} \in \text{Out}_{\hat{P}}((\mathcal{O}_*\hat{G})_\gamma)\) on an element \(\tilde{\sigma} \in \text{Out}_{\hat{P}}(\mathcal{O}_*\hat{L})\) such that, for any \(\hat{P}\)-interior algebra embedding \(e_{\gamma}:(\mathcal{O}_*\hat{G})_\gamma \to T \otimes \mathcal{O}_*\hat{L}\) we have
\[
\hat{e}_{\gamma} \circ \tilde{\sigma} = (\hat{\mu}_T \otimes \tilde{\sigma}) \circ \hat{e}_{\gamma}
\]
5.10.2
and mapping \(\zeta \in \text{Hom}(L, k^*)\) on the exterior class of the \(\hat{P}\)-interior algebra automorphism of \(\mathcal{O}_*\hat{L}\) sending \(\hat{y} \in \hat{L}\) to \(\zeta(y)\cdot \hat{y}\) where \(y\) is the image of \(\hat{y}\) in \(L\).
Moreover, \(\text{Out}_{\hat{P}}((\mathcal{O}_*\hat{G})_\gamma)\) acts regularly over the set of exterior embeddings from \((\mathcal{O}_*\hat{G})_\gamma\) to \(T \otimes \mathcal{O}_*\hat{L}\).

**Proof:** The proof of [16, Proposition 5.2] applies.
6. Charactered pointed $O^*$-groups

6.1. Let $\hat{G}$ be an $O^*$-group with a finite $p$-solvable $O^*$-quotient $G$ and assume that $\mathcal{K}$ is a splitting field for all the $O^*$-subgroups of $\hat{G}$. The main difference from [16] is that, in our present situation, any $O^*$-subgroup $\hat{Q}$ of $\hat{G}$ with a finite $p$-group $O^*$-quotient $Q$ has to be always “accompanied” with an irreducible character.

6.2. Thus, let us call charactered $O^*$-subgroup of $\hat{G}$ any pair $(\hat{Q}, \mu)$ formed by an $O^*$-subgroup $\hat{Q}$ of $\hat{G}$ with a finite $p$-group $O^*$-quotient $Q$ and an irreducible $K$-character $\mu$ of $\hat{Q}$, and call defect of $(\hat{Q}, \mu)$ the defect of $\mu$ (cf. 3.8); denote by $e_\mu$ the central idempotent of $\mathbb{K}_e \hat{Q}$ determined by $\mu$ and by $N_G(\hat{Q}, \mu)$ the stabilizer of $\mu$ in $N_G(\hat{Q})$.

6.3. Now, as in 4.11 above, if $\hat{K}$ is an extension of $\mathcal{K}$ of degree divisible by $|G|$ and $\hat{O}$ the ring of integers of $\hat{\mathcal{K}}$, it follows from Proposition 2.3 that there are an $\hat{O}^*$-group $\hat{N}_G(\hat{Q}, \mu)$ with $\hat{O}^*$-quotient $N_G(\hat{Q}, \mu)$ containing and normalizing $\hat{Q} = \hat{Q}^\hat{O}$, and an $\hat{O}^*$-group homomorphism

$$\hat{N}_G(\hat{Q}, \mu) \rightarrow (\hat{\mathbb{K}}_e \hat{Q} e_\mu)^*$$

lifting the action of $N_G(\hat{Q}, \mu)$ on $\hat{\mathbb{K}}_e \hat{Q} e_\mu$ and extending the structural $\hat{O}^*$-group homomorphism from $\hat{Q}$; then, setting $\hat{G} = \hat{G}^\hat{O}$, the group $Q \subset \hat{Q} * \hat{Q}^*$ becomes a normal subgroup of $N_G(\hat{Q}, \mu) * \hat{N}_G(\hat{Q}, \mu)^*$ and we set

$$\hat{N}_G(\hat{Q}, \mu) = (N_G(\hat{Q}, \mu) * \hat{N}_G(\hat{Q}, \mu)^*)/Q$$

which is an $\hat{O}^*$-group with $\hat{O}^*$-quotient $N_G(\hat{Q}, \mu) = N_G(\hat{Q}, \mu)/\hat{Q}$; moreover, according to Proposition 4.12, we have a bijection

$$\text{Irr}_{\mathcal{K}}(N_G(\hat{Q}, \mu)) \cong \text{Irr}_{\mathcal{K}}(\hat{N}_G(\hat{Q}, \mu))$$

fulfilling $d(\zeta) = d(\mu) + d(\bar{\zeta})$ if $\zeta \in \text{Irr}_{\mathcal{K}}(N_G(\hat{Q}, \mu))$ maps on $\bar{\zeta}$.

6.4. We say that a charactered $O^*$-subgroup $(\hat{R}, \nu)$ of $\hat{G}$ is normal in $(\hat{Q}, \mu)$ if $\hat{R}$ is normal in $\hat{Q}$ and $\nu$ stabilizes $\nu$, and $\nu$ is involved in $\text{Res}^{\hat{Q}}_R(\mu)$ or, equivalently, we have $\frac{\mu(1)}{\nu(1)} \nu = \text{Res}^{\hat{Q}}_R(\mu)$; then, setting $\hat{R} = \hat{R}^\hat{O}$, it follows from Propositions 3.2 and 4.12 that the action of $Q$ on $\hat{\mathbb{K}}_e \hat{R} e_\nu$ determines an $\hat{O}^*$-group $\hat{Q}^\nu$ of $\hat{O}^*$-quotient $Q$, that $R$ can be identified to a normal subgroup of $\hat{Q} * (\hat{Q}^\nu)^*$ and that, setting

$$\hat{Q} = Q/R \quad \text{and} \quad \hat{Q} = (\hat{Q} * (\hat{Q}^\nu)^*)/R$$

4.1
and denoting by $\mu^\diamond$ the irreducible character of $\hat{Q}$ determined by $\mu$, there is a unique irreducible character $\hat{\mu}$ of $\hat{Q}$ fulfilling

$$\mu^\diamond = \text{Ext}(\nu) \otimes \text{Res}(\hat{\mu}) \quad \text{and} \quad d(\mu) = d(\nu) + d(\hat{\mu}) \quad 6.4.2$$

where $\text{Ext}(\nu)$ and $\text{Res}(\hat{\mu})$ are the corresponding irreducible characters of $\hat{Q}^\vee$ and $\hat{Q} \ast (\hat{Q}^\vee)^\circ$.

**Proposition 6.5.** With the notation and the hypothesis above, let $(\hat{Q}, \mu)$ and $(\hat{R}, \nu)$ be characterized $\mathcal{O}^*$-subgroups of $G$ such that $(\hat{R}, \nu)$ is normal in $(\hat{Q}, \mu)$.

Set $N = N_G(\hat{R}, \nu)$ and $\bar{N} = N/R$. Denote by $\hat{K}$ an extension of $\hat{K}$ of degree divisible by $\bar{N}$ and by $\hat{O}$ the ring of integers of $\hat{K}$. Then, $(\hat{Q}, \hat{\mu})$ is a characterized $\mathcal{O}^*$-subgroup of $\bar{N}_G(\hat{R}, \nu)$ and the natural group isomorphism $\bar{N}_N(\hat{Q}, \mu) \cong \bar{N}_\bar{N}(\hat{Q}, \hat{\mu})$ can be canonically lifted to an $\hat{O}$-group isomorphism

$$\bar{N}_N(\hat{Q}, \mu)^\diamond \cong \bar{N}_\bar{N}(\hat{Q}, \hat{\mu}) \quad 6.5.1.$$

**Proof:** Clearly $\hat{Q}$ is contained in $\hat{N} = \bar{N}_G(\hat{R}, \nu)$ (cf. 6.3.2); thus, $(\hat{Q}, \hat{\mu})$ is a characterized $\mathcal{O}^*$-subgroup of $\hat{N}$ and, according to our definition, we have

$$\bar{N}_N(\hat{Q}, \mu) = (N_N(\hat{Q}, \hat{\mu}) \ast \bar{N}_N(\hat{Q}, \hat{\mu})^\circ)/\hat{Q} \quad 6.5.2$$

where we set $\hat{N} = \bar{N}^\diamond$. But, it follows from Proposition 4.12 that we have

$$\hat{\mathcal{K}} \ast \hat{Q}e_{\hat{\mu}} \cong \hat{\mathcal{K}} \ast \hat{R}e_{\nu} \otimes _{\hat{K}} \hat{\mathcal{K}} \ast \hat{Q}e_{\hat{\mu}} \quad 6.5.3$$

and that $N_N(\hat{Q}, \mu)$ stabilizes each factor of this tensor product. Consequently, we get a canonical $\hat{O}^*$-group isomorphism

$$\bar{N}_N(\hat{Q}, \mu)^\diamond \cong N_{N_G(\hat{R}, \nu)}(\hat{Q}, \mu)^\diamond \ast \bar{N}_N(\hat{Q}, \hat{\mu}) \quad 6.5.4,$$

where $\bar{N}_N(\hat{Q}, \hat{\mu})$ denotes the corresponding pull-back from $\bar{N}_\bar{N}(\hat{Q}, \hat{\mu})$, and therefore we still get

$$\bar{N}_N(\hat{Q}, \hat{\mu}) \cong (\bar{N}_N(\hat{Q}, \mu) \ast N_{N_G(\hat{R}, \nu)}(\hat{Q}, \mu)^\circ)^\diamond /\hat{Q} \quad 6.5.5.$$

Moreover, always according to our definition, we have

$$\hat{N} \cong (\bar{N}_G(\hat{R}, \nu) \ast \bar{N}_G(\hat{R}, \nu)^\circ)/\hat{R} \quad 6.5.6$$

and therefore we still have

$$N_{\hat{N}}(\hat{Q}, \hat{\mu}) \cong (N_{N_G(\hat{R}, \nu)}(\hat{R}, \nu) \ast N_{N_G(\hat{R}, \nu)}(\hat{R}, \nu)^\circ)^\diamond /\hat{R} \quad 6.5.7.$$
Hence, setting \( \hat{N} = \hat{N}^\delta \), we finally obtain

\[
\hat{N}_N(\hat{Q}, \mu) \equiv \left( N_{\hat{N}_G(\hat{Q}, \mu)}(\hat{R}, \nu) * \hat{N}_N(\hat{Q}, \mu)^\delta \right)^\delta / Q
\]

\[
= \left( N_{\bar{N}}(\hat{Q}, \mu) * \hat{N}_N(\hat{Q}, \mu)^\delta \right)^\delta / Q = \hat{N}_N(\hat{Q}, \mu)^\delta
\]

6.5.8.

We are done.

6.6. Coherently, let us call charactered weight of \( \hat{G} \) any triple \((\hat{Q}, \mu, \zeta)\) formed by a charactered \( \mathcal{O}^\ast \)-subgroup \((\hat{Q}, \mu)\) of \( \hat{G} \) and, considering the ring of integers \( \mathcal{O} \) of an extension \( \mathcal{K} \) of \( K \) of degree divisible by \(|G|\), by an irreducible character \( \zeta \) of defect zero of the \( \mathcal{O}^\ast \)-group \( \hat{N}_G(\hat{Q}, \mu) \), and we call defect of \((\hat{Q}, \mu, \zeta)\) the defect of \( \mu \) (cf. 3.8). Some of them will form our set of parameters and, for a charactered weight \((\hat{Q}, \mu, \zeta)\) of \( \hat{G} \), this depends on the following local points of \( \hat{Q} \) on \( \mathcal{O}_s \hat{G} \) determined by \( \zeta \).

6.7. First of all, denoting by \( e_\zeta \) the idempotent of \( Z(\hat{K}_s \hat{N}_G(\hat{Q}, \mu)) \) determined by \( \zeta \), it follows from Proposition 3.9 that \( e_\zeta \) is actually a block of \( \hat{O}_s \hat{N}_G(\hat{Q}, \mu) \) and that \( \hat{O}_s \hat{N}_G(\hat{Q}, \mu)e_\zeta \) is a full matrix algebra over \( \hat{O} \). Moreover, \( C_G(\hat{Q}) \) is contained in \( N_G(\hat{Q}, \mu) \) and acts trivially on \( \hat{K}_s \hat{Q} e_\mu \), so that the converse image of \( C_G(\hat{Q}) \) in \( \hat{N}_G(\hat{Q}, \mu) \) is split, and we denote by \( \bar{C}_G(\hat{Q}) \) the converse image of \( C_G(\hat{Q}) \cong C_G(\hat{Q})/C_Q(\hat{Q}) \) in \( \hat{N}_G(\hat{Q}, \mu) \). Then, it follows from Proposition 4.12 that all the irreducible characters of \( \bar{C}_G(\hat{Q}) \) involved in the restriction of \( \zeta \) to this normal \( \hat{O} \)-subgroup still have defect zero; hence, it follows again from Proposition 3.9 that the \( \mathcal{O} \)-subalgebra \( \hat{O}_s \bar{C}_G(\hat{Q})e_\zeta \) of \( \hat{O}_s \hat{N}_G(\hat{Q}, \mu)e_\zeta \) is isomorphic to direct product of full matrix algebras over \( \hat{O} \) and, in particular, there is a block of defect zero \( \bar{f} \) of \( \bar{C}_G(\hat{Q}) \) such that \( \bar{f}e_\zeta \neq 0 \).

6.8. Consequently, since the image \( \bar{f}^k \) of \( \bar{f} \) in \( k_s \bar{C}_G(\hat{Q}) \) is a block of defect zero of \( \bar{C}_G(\hat{Q}) \), according to isomorphism 3.2.2, we get a local point \( \delta \) of \( \hat{Q} \) on \( \mathcal{O}_s \hat{G} \) fulfilling

\[
(\mathcal{O}_s \hat{G})(\hat{Q}_\delta) \cong k_s \bar{C}_G(\hat{Q})\bar{f}^k
\]

6.8.1, and, according to Proposition 3.9, the local pointed \( \mathcal{O}^\ast \)-group \( \hat{Q}_\delta \) is self-centralizing. Moreover, it follows from Proposition 4.10 that there are an \( \mathcal{O}^\ast \)-group \( \hat{F}_{\mathcal{O}, G}(\hat{Q}_\delta, \mu) \) with an \( \mathcal{O}^\ast \)-quotient equal to the stabilizer of \( \mu \) in \( F_{\mathcal{O}, G}(\hat{Q}_\delta) \), and an irreducible character of defect zero \( \zeta \) of this \( \mathcal{O}^\ast \)-group
such that we have an $\tilde{N}_G(\hat{Q}, \mu)$-interior algebra isomorphism

$$\hat{O}_s \tilde{N}_G(\hat{Q}, \mu)e_\xi \cong \text{Ind}_{\tilde{N}_G(\hat{Q}, \mu)}^{\hat{N}_G(\hat{Q}, \mu)} \left( \hat{C}_G(\hat{Q}) \hat{f} \otimes \hat{O}_s \hat{F}_{\hat{O}_s \hat{G}}(\hat{Q}_\delta, \mu)e_\xi \right)$$ 6.8.2

where $\tilde{N}_G(\hat{Q}, \mu)$ denotes the stabilizer of $\tilde{f}$ or, equivalently, of $\delta$ in $\tilde{N}_G(\hat{Q}, \mu)$.

6.9. More generally, we call characterized pointed $\mathcal{O}^*$-group on $\mathcal{O}_s \hat{G}$ any pair $(\hat{Q}_\delta, \mu)$ formed by a selfcentralizing pointed $\mathcal{O}^*$-group $\hat{Q}_\delta$ (cf. 3.11) on $\mathcal{O}_s \hat{G}$ and an irreducible $\mathcal{K}$-character $\mu$ of $\hat{Q}$, and call defect of $(\hat{Q}_\delta, \mu)$ the defect of $\mu$ (cf. 3.8); denote by $N_G(\hat{Q}_\delta, \mu)$ and $F_{\mathcal{O}_s \hat{G}}(\hat{Q}_\delta, \mu)$ the respective stabilizers of $\mu$ in $N_G(\hat{Q}_\delta)$ and $F_{\mathcal{O}_s \hat{G}}(\hat{Q}_\delta)$, and by $f$ the block of $C_G(\hat{Q})$ determined by $\delta$. Since $\hat{Q}_\delta$ is selfcentralizing, it follows from Proposition 3.9 that the image $\tilde{f}$ of $f$ in $\hat{O}_s \hat{C}_G(\hat{Q})$ is a block of defect zero and therefore, considering the blocks of $\hat{N}_G(\hat{Q}, \mu)$ involved in $\text{Tr}_{\hat{C}_G(\hat{Q})}^{\hat{N}_G(\hat{Q}, \mu)}(\tilde{f})$, we can claim as above Proposition 4.10 in order to exhibit an $\hat{O}$-group $\hat{F}_{\mathcal{O}_s \hat{G}}(\hat{Q}_\delta, \mu)$; let us give an alternative definition of this $\hat{O}$-group.

6.10. Let $\hat{P}_s$ be a maximal local pointed $\mathcal{O}^*$-group on $\mathcal{O}_s \hat{G}$ containing $\hat{Q}_\delta$; as in 5.7 above, we may assume that $N_{\hat{P}_s}(\hat{Q}_\delta)$ is a maximal local pointed $\mathcal{O}^*$-group on $\mathcal{O}_s N_G(\hat{Q}_\delta)f$ and then consider a $N_{\hat{P}_s}(\hat{Q}_\delta)$-source $(S^\delta, \hat{L}^\delta)$ of the block $f$ of $N_G(\hat{Q}_\delta)$; thus, $\hat{L}^\delta$ contains $N_{\hat{P}_s}(\hat{Q}_\delta)$ which contains $\hat{Q}$, and it actually follows from isomorphism 5.6.4 that $\hat{L}^\delta$ normalizes $\hat{Q}$ and that $\mathcal{K}$ is also a splitting field for all the $\mathcal{O}^*$-subgroups of $\hat{L}^\delta$. On the one hand, Proposition 2.3 applies to $N_{\hat{L}^\delta}(\hat{Q}, \mu)$, $\hat{Q}$, $\mu$ and $\hat{\mathcal{K}}$, and as in 6.3.3 above we get an $\mathcal{O}^*$-group $\tilde{N}_{\hat{L}^\delta}(\hat{Q}, \mu)$ with $\mathcal{O}^*$-quotient $\hat{N}_{\hat{L}^\delta}(\hat{Q}, \mu)$ where $\hat{L}^\delta$ denotes the $\mathcal{O}^*$-quotient of $\hat{L}^\delta$. On the other hand, since we have

$$Q \subseteq \mathcal{O}_{\hat{P}_s}(\hat{L}^\delta) \subseteq N_{\hat{P}_s}(\hat{Q}_\delta) \quad \text{and} \quad C_{\hat{P}}(Q) \subseteq Q$$ 6.10.1,

any $p'$-subgroup of $L^\delta$ which centralizes $Q$ still centralizes $\mathcal{O}_{\hat{P}_s}(\hat{L}^\delta)$ [5, Ch. 5, Theorem 3.4] and therefore, according to condition 5.5.1, it is trivial; hence, since $N_{\hat{P}_s}(\hat{Q}_\delta)$ is a Sylow $p'$-subgroup of $L^\delta$, we still have

$$C_L(Q) = Z(Q)$$ 6.10.2.

In particular, $\hat{Q}$ has a unique local point $\hat{\delta}$ on $\mathcal{O}_s \hat{L}^\delta$ (cf. isomorphism 3.2.2) and actually we have $\hat{\delta} = \{1\}$ [14, 1.19].

6.11. Consequently, from Proposition 3.10 we get

$$\tilde{N}_{\hat{L}^\delta}(\hat{Q}, \mu) = N_{\hat{L}^\delta}(\hat{Q}, \mu)/Q = E_{\hat{L}^\delta}(\hat{Q}_\delta, \mu) = F_{\mathcal{O}_s \hat{L}^\delta}(\hat{Q}_\delta, \mu)$$ 6.11.1;
but, it follows from Theorem 5.5 that the choice of a $\hat{P}$-interior algebra embedding
\[
e_\gamma : (\mathcal{O}_* \mathcal{N}_G(\hat{Q}_\delta))_\gamma \rightarrow S^\delta \otimes_{\mathcal{O}_*} \mathcal{O}_* \hat{L}^\delta
\]
induces an equivalence of categories $\mathcal{L}_{\mathcal{O}_*, \mathcal{N}_G(\hat{Q}_\delta)} \cong \mathcal{L}_{\mathcal{O}_*, \hat{L}^\delta}$: consequently, denoting by $\hat{Q}_\delta$ the local pointed $\mathcal{O}^*$-group on $\mathcal{O}_*, \mathcal{N}_G(\hat{Q}_\delta)$ determined by $\hat{Q}_\delta$, we easily get the equalities (cf. Proposition 3.10)
\[
F_{\mathcal{O}, \hat{L}^\delta}(\hat{Q}_\delta, \mu) = F_{\mathcal{O}_*, \mathcal{N}_G(\hat{Q}_\delta)}(\hat{Q}_\delta, \mu) = E_G(\hat{Q}_\delta, \mu) = F_{\mathcal{O}_*, \hat{G}}(\hat{Q}_\delta, \mu)
\]
Thus, the $\mathcal{O}^*$-group $\hat{N}_{\mathcal{L}^\delta}(\hat{Q}, \mu)$ has $\mathcal{O}^*$-quotient $F_{\mathcal{O}_*, \hat{G}}(\hat{Q}_\delta, \mu)$ and, since this $\mathcal{O}^*$-group contains $N_{\hat{P}}(\hat{Q}_\delta)$, up to the choice of a polarization $\omega$ it easily follows from isomorphism 5.5.4, from Proposition 5.10 and from the uniqueness of the $N_{\hat{P}}(\hat{Q}_\delta)_\mu$-source $(S^\delta, \hat{L}^\delta)$ that this $\mathcal{O}^*$-group is independent of our choices up to unique $\mathcal{O}^*$-group exterior isomorphisms; hence, it makes sense to define
\[
\hat{F}_{\mathcal{O}_*, \hat{G}}(\hat{Q}_\delta, \mu) = \hat{N}_{\mathcal{L}^\delta}(\hat{Q}, \mu)
\]

note that the $\mathcal{O}^*$-group isomorphism 5.6.4 shows that this definition agrees with the definition above.

6.12. Moreover, it still follows from Theorem 5.5 that the choice of $e_\gamma$ induces a Morita equivalence between the $\mathcal{O}$-algebras $\mathcal{O}_* N_{\hat{G}}(\hat{Q}_\delta)$ and $\mathcal{O}_* \hat{L}^\delta$, and therefore we clearly get a bijection between the sets $\text{Irr}_{\mathcal{K}}(N_{\hat{G}}(\hat{Q}_\delta), \mu)$ and $\text{Irr}_{\mathcal{K}}(\hat{L}^\delta, \mu)$ preserving the defect (cf. 3.8.1); but, in the present situation bijection 6.3.3 yields a new bijection
\[
\text{Irr}_{\mathcal{K}}(\hat{L}^\delta, \mu) \cong \text{Irr}_{\mathcal{K}}(\hat{N}_{\mathcal{L}^\delta}(\hat{Q}, \mu))
\]
finally, with the definition above, we get a bijection
\[
\Gamma_{(\hat{Q}_\delta, \mu)} : \text{Irr}_{\mathcal{K}}(N_{\hat{G}}(\hat{Q}_\delta), \mu) \cong \text{Irr}_{\mathcal{K}}(\hat{F}_{\mathcal{O}_*, \hat{G}}(\hat{Q}_\delta, \mu))
\]
fulfilling
\[
d(\xi) = d(\mu) + d(\Gamma_{(\hat{Q}_\delta, \mu)}(\xi))
\]
for any $\xi \in \text{Irr}_{\mathcal{K}}(N_{\hat{G}}(\hat{Q}_\delta), \mu)$; once again, up to the choice of a polarization $\omega$ it easily follows from isomorphism 5.5.4, from Proposition 5.10 and from the uniqueness of the $N_{\hat{P}}(\hat{Q}_\delta)_\mu$-source $(S^\delta, \hat{L}^\delta)$ that this bijection is independent of our choices.

6.13. More precisely, denote by $b$ the block of $\hat{G}$ determined by $\hat{Q}_\delta$; in the case that $Q_\delta$ is a Fitting pointed $\mathcal{O}^*$-group (cf. 5.4), it follows from Corollary 5.3 that the $\mathcal{O}$-algebras $\mathcal{O}_* \hat{G} b$ and $\mathcal{O}_* N_{\hat{G}}(\hat{Q}_\delta)$ are Morita equivalent and therefore, choosing a set of representatives $\text{Fit}(\hat{G})$ for the set of
for a suitable block $b$ isomorphism 6.8.2 determines an irreducible $\hat{\chi}$ inversely, for any block $b$ zero $\bar{\chi}$ ( $\hat{\chi}$ irreducible ( $\hat{\chi}$ weight $\hat{\chi}$ which allows us to identify to each other both sets. and therefore it induces the same bijection $\hat{x}$-group exterior isomorphism

$$\hat{F}_{\hat{\chi}, G}(\hat{\chi}, \nu) \cong \hat{F}_{\hat{\chi}, G}(\hat{\chi}', \nu')$$

6.13.3

and therefore it induces the same bijection

$$\text{Irr}_{\hat{\chi}}(\hat{F}_{\hat{\chi}, G}(\hat{\chi}', \nu')) \cong \text{Irr}_{\hat{\chi}}(\hat{F}_{\hat{\chi}, G}(\hat{\chi}, \nu'))$$

6.13.4

which allows us to identify to each other both sets.

6.14. On the other hand, we have proved that any characterized weight $(\hat{\chi}, \nu, \zeta)$ of $\hat{G}$ determines an $N_G(\hat{\chi}, \nu)$-orbit of local points $\delta$ of $\hat{G}$ on $\hat{G}$ $b$ for a suitable block $b$ of $\hat{G}$ — in such a way that $Q_3$ is selfcentralizing — and an irreducible $K$-character of defect zero $\zeta$ of $\hat{F}_{\hat{\chi}, G}(\hat{\chi}, \nu)$. Conversely, for any block $b$ of $\hat{G}$, let us call characterized $b$-weight of $\hat{G}$ any triple $(\hat{Q}_3, \mu, \zeta)$ formed by a selfcentralizing pointed $\hat{G}$-group on $\hat{G}$ $b$, by an irreducible $K$-character of defect zero $\zeta$ of $\hat{F}_{\hat{\chi}, G}(\hat{\chi}, \nu)$; then, it follows from Propositions 4.10 and 4.12 that isomorphism 6.8.2 determines an irreducible $K$-character of defect zero $\zeta$ of the $\hat{G}$-group $\hat{N}_G(\hat{\chi}, \nu)$, so that the triple $(\hat{Q}, \mu, \zeta)$ becomes a characterized weight of $\hat{G}$.

6.15. Actually, these correspondences define a bijection between the set of $G$-conjugacy classes of characterized $G^*$-groups of $\hat{G}$, and the union of sets of $G^*$-conjugacy classes of characterized $b$-weights of $\hat{G}$ when $b$ runs over the set of blocks of $\hat{G}$. Let $b$ be a block of $\hat{G}$, $\hat{\chi}$ a Fitting pointed $\hat{G}$-group on $\hat{G}$ $b$ and $\nu$ an irreducible character of $\hat{\chi}$; let us denote by $\text{Chw}_{\hat{\chi}}(\hat{G}, \hat{\chi}, \nu)$ the subset of $\text{Chw}_{\hat{\chi}}(\hat{G}, b)$ determined by the characterized $b$-weights $(\hat{Q}_3, \mu, \zeta)$ of $\hat{G}$ fulfilling

$$\hat{Q}_3 \subset \hat{Q}_3 \quad \text{and} \quad \text{Res}_{\hat{\chi}}^\hat{\chi}(\mu) = \frac{\mu(1)}{\nu(1)} \nu$$

6.15.1

then, $(\hat{\chi}, \hat{\nu})$ is normal in $(\hat{Q}, \mu)$ and in 6.4 above we have defined a characterized $\hat{G}$-group $(\hat{Q}, \hat{\mu})$. 
Proposition 6.16. With the notation and the hypothesis above, \((\hat{Q}, \hat{\mu})\) is a charactered \(\hat{O}^*\)-subgroup of \(\hat{F}_{O,G}(\hat{O}_\eta, \nu)\), and any polarization \(\omega\) determines an \(\hat{O}^*\)-group isomorphism

\[
\hat{F}_{O,G}(\hat{Q}_\delta, \mu)\hat{\delta} \cong \hat{N}_{F_{O,G}(O_n, \nu)}(\hat{Q}, \hat{\mu})
\]

6.16.1.

In particular, denoting by \(\tilde{\zeta}\) the character of the right-hand member determined by \(\zeta\), the triple \((\hat{Q}, \hat{\mu}, \tilde{\zeta})\) is a charactered weight of \(\hat{F}_{O,G}(\hat{O}_\eta, \nu)\). Moreover, the correspondence mapping \((\hat{Q}_\delta, \mu, \zeta)\) on \((\hat{Q}, \hat{\mu}, \tilde{\zeta})\) induces a bijection

\[
\Delta^\omega_{(\hat{O}_\eta, \nu)} : \text{Chw}_{\hat{K}}(\hat{G}, \hat{O}_\eta, \nu) \cong \text{Chw}_{\hat{K}}(\hat{F}_{O,G}(\hat{O}_\eta, \nu))
\]

6.16.2.

**Proof:** Let \(\hat{P}_\gamma\) be a maximal local pointed \(O^*\)-group on \(O, \hat{G}\) b containing \(\hat{Q}_\delta\) and \((T, \hat{L})\) a \(\hat{P}_\gamma\)-source of \(b\); according to Proposition 6.5, in particular \((\hat{Q}, \hat{\mu})\) is a charactered \(\hat{O}^*\)-subgroup of \(\hat{N}_L(\hat{O}, \nu)\). But, denoting by \(g\) the block of \(N_G(\hat{O}_\eta, \nu)\), by \(\hat{P}_\gamma\) the maximal local pointed \(O^*\)-group on \(O, N_G(\hat{O}_\eta, \nu)g\) determined by \(\hat{P}_\gamma\) and by \((T^n, \hat{L}^n)\) a \(\hat{P}_\gamma\)-source of \(g\), it follows from Proposition 5.8 that we may assume that \(\hat{L}^n = \hat{L}\) and then, by its very definition (cf. 6.11.4), we get

\[
\hat{F}_{O,G}(\hat{O}_\eta, \nu) = \hat{N}_L(\hat{O}, \nu)
\]

6.16.3.

Consequently, \((\hat{Q}, \hat{\mu})\) is a charactered \(\hat{O}^*\)-subgroup of \(\hat{F}_{O,G}(\hat{O}_\eta, \nu)\) and from Proposition 6.5 we obtain an \(\hat{O}^*\)-group isomorphism

\[
\hat{N}_{N_L(\hat{O}, \nu)}(\hat{Q}, \mu)\hat{\delta} \cong \hat{N}_{N_L(\hat{O}, \nu)}(\hat{Q}, \hat{\mu})
\]

6.16.4;

moreover, with the notation and the hypothesis in 5.7 above, it follows from Proposition 5.8 that we may assume that \(\hat{L}' = N_L(\hat{Q})\) and then, by its very definition (cf. 6.11.4), we get

\[
\hat{F}_{O,G}(\hat{Q}_\delta, \mu) = \hat{N}_L(\hat{Q}, \mu)
\]

6.16.5;

finally, since \(\mu\) determines \(\nu\), we still have \(\hat{N}_L(\hat{Q}, \mu) = \hat{N}_{N_L(\hat{O}, \nu)}(\hat{Q}, \mu)\) and isomorphism 6.16.1 follows from isomorphism 6.16.4 above.

Moreover, it is easily checked that this correspondence is compatible with \(G\)- and \(F_{O,G}(\hat{O}_\eta, \nu)\)-conjugation and therefore it induces a map

\[
\Delta^\omega_{(\hat{O}_\eta, \nu)} : \text{Chw}_{\hat{K}}(\hat{G}, \hat{O}_\eta, \nu) \rightarrow \text{Chw}_{\hat{K}}(\hat{F}_{O,G}(\hat{O}_\eta, \nu))
\]

6.16.6;

in order to prove that it is bijective, let us define the inverse map. If \((\hat{Q}', \hat{\mu}', \tilde{\zeta}')\) is a charactered weight of \(\hat{F}_{O,G}(\hat{O}_\eta, \nu) = \hat{N}_L(\hat{O}, \nu)\) then the converse image
$Q'$ of $\tilde{Q}'$ in $L$ fulfills $O \subset Q' \subset P$ and determines an $O^*$-group $\tilde{Q}'$ such that $\tilde{O} \subset \tilde{Q}' \subset \tilde{P}$; since $\tilde{O}_n$ is selfcentralizing and contained in $\tilde{P}_\gamma$, it is easily checked that we still have $\tilde{O}_\eta \subset \tilde{Q}'_{\delta'} \subset \tilde{P}_\gamma$ for a unique local point $\delta'$ of $\tilde{Q}'$ on $\tilde{O}_*, \tilde{G}$.

On the other hand, since $\mathcal{K}$ is a splitting field for $\tilde{Q}'$, it follows from Proposition 4.12 that there is an irreducible $\mathcal{K}$-character $\mu'$ of $\tilde{Q}'$ fulfilling (cf. 6.16.2)

$$\mu'^\gamma = \text{Ext}(\nu) \otimes \text{Res}(\bar{\mu}')$$

and then, since $\mathcal{K}$ is a splitting field for $\hat{F}_{\mathcal{O}, \mathcal{C}}(\tilde{Q}'_{\delta'}, \mu')$, there is an irreducible $\mathcal{K}$-character $\bar{\zeta}'$ determining $\hat{\zeta}'$ via the $\hat{O}^*$-group isomorphism 6.16.1. Thus, $(\tilde{Q}', \bar{\nu}', \bar{\zeta}')$ determines the characterized $b$-weight $(\tilde{Q}'_{\delta'}, \mu', \bar{\zeta}')$ of $\hat{G}$ and it is quite clear that this correspondence induces the announced inverse map. We are done.

7. The characterized Fitting sequences

7.1 Let $\hat{G}$ be an $O^*$-group with a finite $p$-solvable $O^*$-quotient $G$ and assume that $\mathcal{K}$ is a splitting field for all the $O^*$-subgroups of $\hat{G}$. In order to exhibit bijections between the sets of irreducible $\mathcal{K}$-characters of $\hat{G}$ and of $G$-conjugacy classes of the inductive weights of $\hat{G}$ defined below, we need a third set — namely the set of $G$-conjugacy classes of characterized Fitting sequences of $\hat{G}$ — which depends on the choice of a polarization $\omega$. We call characterized Fitting $\omega$-sequence of $\hat{G}$ any sequence

$$\mathcal{B} = \{(\mathcal{K}_n, \hat{G}_n, \hat{O}_n^{\mathcal{K}_n}, \nu_n)\}_{n \in \mathbb{N}}$$

of quadruples formed by a field extension $\mathcal{K}_n$ of $\mathcal{K}$, by an $O_n^*$-group $\hat{G}_n$ where $O_n$ denotes the ring of integers of $\mathcal{K}_n$, by a Fitting pointed $O_n^*$-group of $\hat{G}_n$ (cf. 5.4), and by an irreducible $\mathcal{K}_n$-character $\nu_n$ of $\hat{O}_n$, in such a way that $\mathcal{K}_0 = \mathcal{K}$, that $\hat{G}_0 = \hat{G}$ and that, for any $n \in \mathbb{N}$, $\mathcal{K}_{n+1}$ is the field extension of degree $|G_n|$ of $\mathcal{K}_n$ and we have (cf. definition 6.11.4)

$$\hat{G}_{n+1} = \hat{F}_{\mathcal{O}_n, \mathcal{G}_n}(O_n, \nu_n)$$

Note that $|G_{n+1}| \leq |G_n|$ and actually we have $|G_{n+1}| = |G_n|$ if and only if $O^n = \{1\}$ (cf. condition 5.3.1). Moreover, for any $h \in \mathbb{N}$, the sequence

$$\mathcal{B}_h = \{(\mathcal{K}_{h+n}, \hat{G}_{h+n}, \hat{O}_{h+n}^{\mathcal{K}_{h+n}}, \nu_{h+n})\}_{n \in \mathbb{N}}$$

is clearly a characterized Fitting $\omega$-sequence of $\hat{G}_h$. 
7.2. If \( \hat{G}' \) is an \( \mathcal{O}'^* \)-group isomorphic to \( \hat{G} \) and \( \theta : \hat{G} \cong \hat{G}' \) an \( \mathcal{O}'^* \)-group isomorphism, it is quite clear that, from any characterized Fitting \( \omega \)-sequence
\[
\mathcal{B} = \{(K_n, \hat{G}_n, \hat{O}^n_{\eta_n}, \nu_n)\}_{n \in \mathbb{N}} \quad 7.2.1
\]
of \( \hat{G} \), we are able to construct a characterized Fitting \( \omega \)-sequence
\[
\mathcal{B}' = \{(K_n, \hat{G}'_n, \hat{O}'^n_{\eta'_n}, \nu'_n)\}_{n \in \mathbb{N}} \quad 7.2.2
\]
of \( \hat{G}' \) inductively defining a sequence of \( \mathcal{O}'^*_n \)-group isomorphisms \( \theta_n : \hat{G}_n \cong \hat{G}'_n \) for any \( n \in \mathbb{N} \), by \( \theta_0 = \theta \) and, for any \( n \in \mathbb{N} \), by \( (cf. \ 2.14) \)
\[
\hat{F}_{\theta_n}(\hat{O}^n_{\eta_n}, \nu_n) : \hat{F}_{\mathcal{O}_n, \hat{G}_n}(\hat{O}^n_{\eta_n}, \nu_n) \cong \hat{F}_{\mathcal{O}_n, \hat{G}'_n}(\hat{O}'^n_{\eta'_n}, \nu'_n) \quad \theta_{n+1} \quad \hat{G}_{n+1} \quad 7.2.3
\]
where we still denote by \( \theta_n : \mathcal{O}_n \hat{G}_n \cong \mathcal{O}_n \hat{G}'_n \) the corresponding \( \mathcal{O} \)-algebra isomorphism and set
\[
\hat{O}'^n = \theta_n(\hat{O}^n) \quad \eta'_n = \theta_n(\eta_n) \quad \text{and} \quad \nu'_n = \text{res}_{\theta_n}(\nu_n) \quad 7.2.4
\]
In particular, the group of inner automorphisms of \( G \) acts on the set of characterized Fitting \( \omega \)-sequences of \( \hat{G} \) and we denote by \( \text{Chs}_x(\hat{G}) \) the set of “\( G \)-conjugacy classes” of the characterized Fitting \( \omega \)-sequences of \( \hat{G} \).

7.3. If \( \mathcal{B} = \{(K_n, \hat{G}_n, \hat{O}^n_{\eta_n}, \nu_n)\}_{n \in \mathbb{N}} \) is a characterized Fitting \( \omega \)-sequence of \( \hat{G} \), it is clear that \( \hat{O}^n_{\eta_n} \) determines a block \( b_n \) of \( \hat{G}_n \) for any \( n \in \mathbb{N} \) and we call defect of \( \mathcal{B} \) the sum
\[
d(\mathcal{B}) = \sum_{n \in \mathbb{N}} d(\nu_n) \quad 7.3.1
\]
which makes sense since \( d(\nu_m) = 0 \) for \( m \) big enough; as in \( [16, \ 6.4] \), let us call irreducible character \( \omega \)-sequence associated to \( \mathcal{B} \) any sequence \( \{\chi_n\}_{n \in \mathbb{N}} \) where \( \chi_n \) belongs to \( \text{Irr}_{\mathcal{O}'_n}(\hat{G}_n, b_n) \) in such a way that, up to identifications, we have \( (cf. \ 6.13.1) \)
\[
\Gamma_{\hat{G}_n}(\chi_n) = \chi_{n+1} \quad 7.3.2
\]
for any \( n \in \mathbb{N} \); in this case, it easily follows from equality 6.13.2 that
\[
d(\chi_0) = d(\mathcal{B}) \quad 7.3.3.
\]

**Theorem 7.4.** With the notation and the choice above, any characterized Fitting \( \omega \)-sequence \( \mathcal{B} = \{(K_n, \hat{G}_n, \hat{O}^n_{\eta_n}, \nu_n)\}_{n \in \mathbb{N}} \) of \( \hat{G} \) admits a unique irreducible character \( \omega \)-sequence \( \{\chi_n\}_{n \in \mathbb{N}} \) associated to \( \mathcal{B} \). Moreover, the correspondence mapping \( \mathcal{B} \) to \( \chi_0 \) induces a natural bijection
\[
\text{Chs}_x(\hat{G}) \cong \text{Irr}_x(\hat{G}) \quad 7.4.1
\]
which preserves the defect.
Proof: Since the sequence \( \{ \hat{G}_n \}_{n \in \mathbb{N}} \) “stabilizes”, we can argue by induction on the minimal \( n \in \mathbb{N} \) fulfilling \( \hat{G}_{n+1} = \hat{G}_n \). If \( n = 0 \) then \( \hat{O}^0 = \mathcal{O}^* \), \( \nu_0 = id_{\mathcal{O}^*} \) and the block \( b_0 \) of \( \hat{G}_0 \) has defect zero, so that \( \text{Irr}_k(\hat{G}_0, b_0) \) has a unique element \( \chi_0 \) (cf. Proposition 3.9) and, setting \( \chi_n = \chi_0 \) for any \( n \in \mathbb{N} \), we get an irreducible character \( \omega \)-sequence associated to \( \mathcal{B} \).

If \( n \neq 0 \) then, according to the induction hypothesis, the characterized Fitting \( \omega \)-sequence \( \mathcal{B}_1 = \{(\hat{\mathcal{K}}_{1+n}, \hat{\mathcal{G}}_{1+n}, \hat{\mathcal{O}}_{\eta_{1+n}}, \nu_{1+n})\}_{n \in \mathbb{N}} \) of \( \hat{G}_1 \) already admits an irreducible character \( \omega \)-sequence \( \{\chi_{1+n}\}_{n \in \mathbb{N}} \); thus, in order to get an irreducible character \( \omega \)-sequence associated to \( \mathcal{B} \), up to identifications it suffices to define (cf. 7.3.2)

\[
\chi_0 = (\Gamma_{\hat{G}_0}^\omega)^{-1}(\chi_1)
\]

7.4.2.

On the other hand, since the maps \( \Gamma_{\hat{G}_n}^\omega \) are bijective, equality 7.3.2 shows that an irreducible character \( \omega \)-sequence \( \{\chi_n\}_{n \in \mathbb{N}} \) associated to \( \mathcal{B} \) is uniquely determined by one of their terms; but, for \( n \) big enough, we know that \( b_n \) is a block of defect zero of \( \hat{G}_n \) and then \( \chi_n \) is uniquely determined; consequently, \( \{\chi_n\}_{n \in \mathbb{N}} \) is uniquely determined and it is quite clear that, up to identifications, it only depends on the \( \mathcal{G} \)-conjugacy class of \( \mathcal{B} \); thus, we have obtained a natural map

\[
\text{Chs}_{\mathcal{K}}^\omega(\hat{G}) \rightarrow \text{Irr}_K(\hat{G})
\]

7.4.3.

which preserves the defect (cf. equality 7.3.3).

We claim that it is bijective; actually, we will define the inverse map. For any \( \chi \in \text{Irr}_K(\hat{G}) \), we inductively define a sequence \( \{\chi_n\}_{n \in \mathbb{N}} \) in the following way: we set \( \mathcal{K}_0 = \mathcal{K} \), \( \hat{G}_0 = \hat{G} \) and \( \chi_0 = \chi \); we denote by \( b_0 \) the block of \( \chi \), and we choose a Fitting pointed \( \mathcal{O}^* \)-group \( \hat{\mathcal{O}}_n^{0} \) on \( \mathcal{O}_0 \hat{\mathcal{G}} b \) and an irreducible character \( \nu_0 \) of \( \hat{\mathcal{O}}_n^{0} \) involved in the irreducible character of \( \mathcal{N}_{\mathcal{G}}(\hat{\mathcal{O}}_n^{0}) \) determined by \( \chi \) (cf. Corollary 5.3); moreover, for any \( n \in \mathbb{N} \), we denote by \( \mathcal{K}_{n+1} \) the field extension of degree \( |\mathcal{G}_n| \) of \( \mathcal{K}_n \), we set

\[
\hat{G}_{n+1} = \hat{F}_{\mathcal{K}_n, \mathcal{G}_n}(\mathcal{O}_{\eta_{n+1}}) \quad \text{and} \quad \chi_{n+1} = \Gamma_{\hat{G}_n}^\omega(\chi_n)
\]

7.4.4.

we denote by \( b_{n+1} \) the block of \( \chi_{n+1} \), and we choose a Fitting pointed \( \mathcal{O}^* \)-group \( \hat{\mathcal{O}}_{\eta_{n+1}}^{n+1} \) on \( \mathcal{O}_n \hat{\mathcal{G}} b_n \) and an irreducible character \( \nu_{n+1} \) of \( \hat{\mathcal{O}}_{\eta_{n+1}}^{n+1} \) involved in the irreducible character of \( \mathcal{N}_{\mathcal{G}_{n+1}}(\hat{\mathcal{O}}_{\eta_{n+1}}^{n+1}) \) determined by \( \chi_{n+1} \) (cf. Corollary 5.3). Then, it is clear that \( \mathcal{B} = \{(\mathcal{K}_n, \hat{\mathcal{G}}_n, \mathcal{O}_{\eta_{n+1}}^{n+1}, \nu_{n+1})\}_{n \in \mathbb{N}} \) is a characterized Fitting \( \omega \)-sequence of \( \hat{G} \) and that \( \{\chi_n\}_{n \in \mathbb{N}} \) becomes the irreducible character \( \omega \)-sequence associated to \( \mathcal{B} \).

Our construction only depends on the choices of a Fitting pointed \( \mathcal{O}^* \)-group \( \hat{\mathcal{O}}_{\eta_{n+1}}^{n+1} \) on \( \mathcal{O}_n \hat{\mathcal{G}} b_n \), and of an irreducible character \( \nu_n \) of \( \hat{\mathcal{O}}_n^{0} \) involved
in the irreducible character of $N_{G_n}(\hat{O}^n_{\eta_n})$ determined by $\chi_n$, for a finite set of values of $n$. Moreover, since all the Fitting pointed $O^*_n$-groups on $(O_n, G_n) b_n$ are mutually $G_n$-conjugate, $\chi$ determines a unique $G$-conjugacy class of characterized Fitting $\omega$-sequence of $G$. That is to say, we have obtained a map

$$\text{Irr}_K(G) \longrightarrow \text{Chs}_K^\omega(\hat{G})$$

and it is easily checked that this map is the inverse of the map 7.4.3. We are done.

8. The inductive weights

8.1. Let $\hat{G}$ be an $O^*$-group with a finite $p$-solvable $O^*$-quotient $G$, assume that $K$ is a splitting field for all the $O^*$-subgroups of $\hat{G}$, choose a polarization $\omega$, and consider a characterized Fitting $\omega$-sequence of $\hat{G}$

$$\mathcal{B} = \{(K_n, \hat{G}_n, \hat{O}^n_{\eta_n}, \nu_n)\}_{n \in \mathbb{N}}$$

we call characterized weight $\omega$-sequence associated to $\mathcal{B}$ any sequence

$$\mathcal{W} = \{\hat{O}^n_n, \mu_n, \zeta_n\}_{n \in \mathbb{N}}$$

where $(\hat{Q}^n, \mu_n, \zeta_n)$ is a characterized weight of $\hat{G}_n$ such that, denoting by $b_n$ and by $(\hat{Q}^n_{b_n}, \mu_n, \zeta_n)$ the block and a characterized $b_n$-weight corresponding to $(\hat{Q}^n, \mu_n, \zeta_n)$ (cf. 6.14), we have

$$\hat{O}^n_{\eta_n} \subset \hat{Q}^n_{b_n} \quad \text{and} \quad \text{Res}_{\hat{Q}^n_{b_n}}^\hat{Q}^n_n(\mu_n) = \frac{\mu_n(1)}{\nu_n(1)} \nu_n$$

and the corresponding characterized weight $(\pi^1, \bar{b}_n, \zeta_n)$ (cf. Proposition 6.16) coincides with $(\pi^1, \bar{b}_n+1, \zeta_n+1)$ for any $n \in \mathbb{N}$

8.2. It is clear that, for any $h \in \mathbb{N}$, the sequence

$$\mathcal{W}_h = \{(\hat{Q}^{h+n}, \mu_{h+n}, \zeta_{h+n})\}_{n \in \mathbb{N}}$$

is a characterized weight $\omega$-sequence associated to $\mathcal{B}_h$. Moreover, calling characterized weight $\omega$-sequence of $\hat{G}$ any sequence $\mathcal{W} = \{(\hat{Q}^n, \mu_n, \zeta_n)\}_{n \in \mathbb{N}}$ associated to some characterized Fitting $\omega$-sequence $\mathcal{B}$ of $\hat{G}$, as in 7.2 above the group of inner automorphisms of $G$ acts on this set; let us denote by $\text{Cws}_K(\hat{G})$ the corresponding set of $G$-conjugacy classes.

**Theorem 8.3.** With the notation and the choice above, any characterized Fitting $\omega$-sequence $\mathcal{B} = \{(K_n, \hat{G}_n, O^n_{\eta_n}, \nu_n)\}_{n \in \mathbb{N}}$ of $\hat{G}$ admits an associated characterized weight $\omega$-sequence $\mathcal{W} = \{(\hat{Q}^n, \mu_n, \zeta_n)\}_{n \in \mathbb{N}}$, unique up to identifications, and then we have $d(\mathcal{B}) = d(\mu_0)$. Moreover, the correspondence sending $\mathcal{B}$ to $\mathcal{W}$ induces a natural bijection

$$\text{Chs}_K^\omega(\hat{G}) \cong \text{Cws}_K(\hat{G})$$
Proof: Since the sequence \( \{ \hat{G}_n \}_{n \in \mathbb{N}} \) “stabilizes”, we can argue by induction on the minimal \( n \in \mathbb{N} \) fulfilling \( \hat{G}_{n+1} = \hat{G}_n \). If \( n = 0 \) then \( \hat{G}^0 = O^* \), \( \nu_0 = id_{O^*} \) and the block \( b_0 \) of \( \hat{G}_0 \) has defect zero, so that \( b_0 \) only admits the trivial charactered \( b_0 \)-weight \( (O^*_c, id_{O^*}, id_{O^*}) \) (cf. Proposition 3.9) and, setting

\[
(\hat{Q}^n, \mu_n, \zeta_n) = (O^*, id_{O^*}, id_{O^*})
\]

for any \( n \in \mathbb{N} \), we get a charactered weight \( \omega \)-sequence associated to \( B \); moreover, it is clear that \( d(B) = 0 = d(id_{O^*}) \).

If \( n \neq 0 \) then, according to our induction hypothesis, the charactered Fitting \( \omega \)-sequence \( B_1 = \{ (K_{1+n}, \hat{G}_{1+n}, \hat{O}^{1+n}_{n+1}, \nu_{1+n}) \}_{n \in \mathbb{N}} \) of \( \hat{G}_1 \) already admits an associated charactered weight \( \omega \)-sequence

\[
\mathcal{W}_1 = \{ (\hat{Q}^{1+n}, \mu_{1+n}, \zeta_{1+n}) \}_{n \in \mathbb{N}}
\]

fulfilling \( d(B_1) = d(\mu_1) \); but, it follows from Proposition 16.6 that there is a charactered \( b_0 \)-weight \( (\hat{Q}^0_{b_0}, \mu_0, \zeta_0) \) of \( \hat{G} \) fulfilling condition 8.1.3 and determining a charactered weight \( (\hat{Q}^0, \mu_0, \zeta_0) \) of \( \hat{F}_{O^*, G}(\hat{O}^0, \nu_0) = \hat{G}_1 \) which coincides with \( (\hat{Q}^1, \mu_1, \zeta_1) \); then, it suffices to choose the charactered weight \( (\hat{Q}^0, \mu_0, \zeta_0) \) of \( \hat{G} \) determined by \( (\hat{Q}^0_{b_0}, \mu_0, \zeta_0) \) (cf. 6.14) to get a charactered weight \( \omega \)-sequence \( \mathcal{W} \) associated to \( B \); moreover, we have (cf. equalities 6.4.2 and Proposition 6.16)

\[
d(B) = d(\nu_0) + d(B_1) = d(\nu_0) + d(\mu_1) = d(\mu_0)
\]

On the other hand, since the maps \( \Delta_\omega \) are bijective, it is quite clear that a charactered weight \( \omega \)-sequence \( \mathcal{W} = \{ (\hat{Q}^n, \mu_n, \zeta_n) \}_{n \in \mathbb{N}} \) associated to \( B \) is, up to identifications, uniquely determined by one of their terms; but, for \( n \) big enough, we know that \( b_n \) is a block of defect zero of \( \hat{G}_n \) and then \( (\hat{Q}^0, \mu_n, \zeta_n) \) is uniquely determined (cf. 8.3.2); consequently, \( \mathcal{W} \) is, up to identifications, uniquely determined by \( B \). Moreover, this correspondence is clearly compatible with \( G \)-conjugation and therefore it defines a natural map

\[
\text{Chs}_K^G(\hat{G}) \longrightarrow \text{Cws}_K^G(\hat{G})
\]

Conversely, the elements of \( \text{Cws}_K(\hat{G}) \) are the \( G \)-conjugacy classes of charactered weight \( \omega \)-sequences \( \mathcal{W} = \{ (\hat{Q}^n, \mu_n, \zeta_n) \}_{n \in \mathbb{N}} \), namely associated to some charactered Fitting \( \omega \)-sequence \( B = \{ (K_n, \hat{G}_n, \hat{O}^n_{n_n}, \nu_n) \}_{n \in \mathbb{N}} \) of \( \hat{G} \), so that the map 8.3.5 sends the \( G \)-conjugacy class of \( B \) to the \( G \)-conjugacy class of \( \mathcal{W} \); moreover, condition 8.1.3 shows that \( \mathcal{W} \) determines \( B \). We are done.

8.4. Finally, we inductively define the announced inductive weights. We call inductive weight of \( G \) any charactered weight \( (\hat{Q}, \mu, \zeta) \) of \( \hat{G} \) which is either...
trivial or, denoting by $b$ the block determined by $\hat{Q}_\delta$ and by $(\hat{Q}_\delta, \mu, \zeta)$ some charactered $b$-weight associated to $(\hat{Q}, \mu, \zeta)$ (cf. 6.14), $\hat{Q}_\delta$ contains a Fitting pointed $\mathcal{O}^*$-group $\hat{O}_n$ of $\hat{G}$, we have
\[
\text{Res}_{\hat{O}}^G(\mu) = \frac{\mu(1)}{\nu(1)} \nu
\]
for a suitable irreducible character of $\hat{O}$, and the corresponding charactered weight $(\hat{Q}^n, \bar{\mu}_n, \bar{\zeta}_n)$ is an inductive weight of $\hat{F}_{\mathcal{O}, \hat{G}}(\hat{O}_n, \nu)$ (cf. Proposition 6.16). Let us denote by $\text{Inw}_\mathcal{K}^\omega(\hat{G})$ the set of $G$-conjugacy classes of inductive weights of $\hat{G}$.

**Proposition 8.5.** A charactered weight $(\hat{Q}, \mu, \zeta)$ of $\hat{G}$ is an inductive weight if and only if there is a charactered weight $\omega$-sequence
\[
W = \{(\hat{Q}^n, \mu_n, \zeta_n)\}_{n \in \mathbb{N}}
\]
associated to a charactered Fitting $\omega$-sequence $B = \{(\mathcal{K}_n, \hat{G}_n, \hat{O}^n_{\eta_n}, \nu_n)\}_{n \in \mathbb{N}}$ of $\hat{G}$ such that $(\hat{Q}, \mu, \zeta) = (\hat{Q}^0, \mu_0, \zeta_0)$.

**Proof:** We argue by induction on $|G|$ and may assume that $(\hat{Q}, \mu, \zeta)$ is not trivial. The existence of $W$ and $B$ yields the existence of the Fitting pointed $\mathcal{O}^*$-group $\hat{O}^0_{\eta_0}$ contained in $\hat{Q}^0_{\delta_0} = \hat{Q}_\delta$ and of the irreducible character $\nu_0$ of $\hat{O}^0$ fulfilling equality 8.4.1 for $\mu_0 = \mu$; moreover, the charactered weight $(\hat{Q}, \bar{\mu}, \bar{\zeta})$ coincides with $(\hat{Q}^1, \mu_1, \zeta_1)$; then, since the charactered weight $\omega$-sequence
\[
W_1 = \{(\hat{Q}^{1+n}, \mu_{1+n}, \zeta_{1+n})\}_{n \in \mathbb{N}}
\]
is associated to $B_1$, our induction hypothesis guarantees that $(\hat{Q}, \bar{\mu}, \bar{\zeta})$ is already an inductive weight of $\hat{F}_{\mathcal{O}, \hat{G}}(\hat{O}^0_{\eta_0}, \nu_0)$ and therefore $(\hat{Q}, \mu, \zeta)$ is an inductive weight too.

Conversely, if $(\hat{Q}, \bar{\mu}, \bar{\zeta})$ is an inductive weight of $\hat{F}_{\mathcal{O}, \hat{G}}(\hat{O}_\eta, \nu)$ for suitable $\hat{O}_\eta$ and $\nu$ fulfilling the conditions above, it follows from our induction hypothesis that there is a charactered weight $\omega$-sequence
\[
W_1 = \{(\hat{Q}^{1+n}, \mu_{1+n}, \zeta_{1+n})\}_{n \in \mathbb{N}}
\]
associated to a charactered Fitting $\omega$-sequence
\[
B_1 = \{(\mathcal{K}_{1+n}, \hat{G}_{1+n}, \hat{O}^{1+n}_{\eta_{1+n}}, \nu_{1+n})\}_{n \in \mathbb{N}}
\]
of $\hat{F}_{\mathcal{O}, \hat{G}}(\hat{O}_\eta, \nu)$ such that $(\hat{Q}, \bar{\mu}, \bar{\zeta}) = (\hat{Q}^1, \mu_1, \zeta_1)$, and we may assume that $\mathcal{K}_1$ is the field extension of $\mathcal{K}$ of degree $|G|$. Then, setting $\mathcal{K}_0 = \mathcal{K}$, $G_0 = \hat{G}$, $\hat{O}^0 = \hat{O}$ and $\nu_0 = \nu$, it is clear that
\[
B = \{(\mathcal{K}_n, \hat{G}_n, \hat{O}^n_{\eta_n}, \nu_n)\}_{n \in \mathbb{N}}
\]
is a charactered Fitting $\omega$-sequence of $\hat{G}$, and setting $\hat{Q}^0 = \hat{Q}$, $\mu_0 = \mu$ and $\zeta^0 = \zeta$, the sequence $\mathcal{W} = \{(\hat{Q}^n, \mu_n, \zeta_n)\}_{n \in \mathbb{N}}$ is a charactered weight $\omega$-sequence associated to $B$. We are done.

**Corollary 8.6.** With the notation and the choice above, the correspondence mapping any charactered Fitting $\omega$-sequence $\mathcal{B} = \{(\mathcal{K}_n, \hat{G}_n, \hat{O}_{\eta_n}, \nu_n)\}_{n \in \mathbb{N}}$ of $\hat{G}$ on the inductive weight $(\hat{Q}^0, \mu_0, \zeta_0)$, where $\mathcal{W} = \{(\hat{Q}^n, \mu_n, \zeta_n)\}_{n \in \mathbb{N}}$ is the charactered weight $\omega$-sequence of $\hat{G}$ associated to $\mathcal{B}$, induces a natural bijection

$$\text{Chs}^\omega_{\mathcal{K}}(\hat{G}) \cong \text{Inw}^\omega_{\mathcal{K}}(\hat{G})$$

8.6.1.

In particular, we have $d(\mathcal{B}) = d(\mu_0)$.

**Proof:** It follows from Theorem 8.3 that there is such a correspondence mapping $\mathcal{B}$ on $(\hat{Q}^0, \mu_0, \zeta_0)$ and that it fulfills $d(\mathcal{B}) = d(\mu_0)$; moreover, it is clear that this correspondence is compatible with the $G$-conjugation, and therefore we get a natural map

$$\text{Chs}^\omega_{\mathcal{K}}(\hat{G}) \rightarrow \text{Inw}^\omega_{\mathcal{K}}(\hat{G})$$

8.6.2;

furthermore, since $(\hat{Q}^0, \mu_0, \zeta_0)$ clearly determines $\mathcal{W}$, this map is injective.

In order to prove the surjectivity, we argue by induction on $|G|$; let $(\hat{Q}, \mu, \zeta)$ be an inductive weight that we may assume not trivial; denoting by $b$ and by $(\hat{Q}_b, \mu, \zeta)$ the block and a characterized $b$-weight corresponding to $(\hat{Q}, \mu, \zeta)$ (cf. 6.14), it follows from the very definition that $\hat{Q}_b$ contains a Fitting pointed $O^n$-group $\hat{O}_b$ of $\hat{G}$, that we have

$$\text{Res}_{\hat{Q}}^\mathcal{O}(\mu) = \frac{\mu(1)}{\nu(1)} \nu$$

8.6.3

for a suitable irreducible character of $\hat{O}$, and that the corresponding characterized weight $(\hat{Q}, \hat{\mu}, \hat{\zeta})$ is an inductive weight of $\hat{F}_{\mathcal{O},\hat{C}}(\hat{O}_b, \nu)$.

Thus, by our induction hypothesis, there exists a characterized Fitting $\omega$-sequence

$$\mathcal{B}_1 = \{(\mathcal{K}_{1+n}, \hat{G}_{1+n}, \hat{O}_{\eta_{1+n}}, \nu_{1+n})\}_{n \in \mathbb{N}}$$

8.6.4

of $\hat{G}_1 = \hat{F}_{\mathcal{O},\hat{C}}(\hat{O}_b, \nu)$ such that the unique characterized weight $\omega$-sequence $\mathcal{W}_1 = \{(\hat{Q}^{1+n}, \mu_{1+n}, \zeta_{1+n})\}_{n \in \mathbb{N}}$ of $\hat{G}_1$ associated to $\mathcal{B}_1$ fulfills

$$(\hat{Q}^1, \mu_1, \zeta_1) = (\hat{Q}, \hat{\mu}, \hat{\zeta})$$

8.6.5;

moreover, we may assume that $\mathcal{K}_1$ is the field extension of $\mathcal{K}$ of degree $|G|$.

Once again, setting $\mathcal{K}_0 = \mathcal{K}$, $\hat{G}_0 = \hat{G}$, $\hat{O}^0 = \hat{O}$ and $\nu_0 = \nu$, it is clear that

$$\mathcal{B} = \{(\mathcal{K}_n, \hat{G}_n, \hat{O}_{\eta_n}, \nu_n)\}_{n \in \mathbb{N}}$$

8.6.6
is a charactered Fitting \( \omega \)-sequence of \( \hat{G} \), and setting \( \hat{Q}^0 = \hat{Q}, \mu_0 = \mu \) and \( \zeta^0 = \zeta \), the sequence \( W = \{(\hat{Q}^n, \mu_n, \zeta_n)\}_{n \in \mathbb{N}} \) is a charactered weight \( \omega \)-sequence associated to \( B \); thus, the map 8.6.2 sends \( B \) to \( (\hat{Q}, \mu, \zeta) \). We are done.

**Corollary 8.7.** With the notation and the choice above, we have a natural bijection
\[
\text{Irr}_K(\hat{G}) \cong \text{Inw}_K(\hat{G}) \tag{8.7.1}
\]

preserving the defects, which maps \( \chi \in \text{Irr}_K(\hat{G}) \) on the \( G \)-conjugacy class of an inductive weight \( (\hat{Q}, \mu, \zeta) \) provided there is a charactered Fitting \( \omega \)-sequence \( B \) of \( \hat{G} \) such that the irreducible character \( \omega \)-sequence \( \{\chi_n\}_{n \in \mathbb{N}} \) and the charactered weight \( \omega \)-sequence \( W = \{(\hat{Q}^n, \mu_n, \zeta_n)\}_{n \in \mathbb{N}} \) associated to \( B \) fulfill
\[
\chi_0 = \chi \quad \text{and} \quad (\hat{Q}^0, \mu_0, \zeta_0) = (\hat{Q}, \mu, \zeta) \tag{8.7.2}
\]

**Proof:** It suffices to compose the inverse of isomorphism 7.4.1 with isomorphism 8.6.1.

**References**

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Abstract. The weights for a finite group $G$ with respect to a prime number $p$ where introduced by Jon Alperin, in order to formulate his celebrated conjecture. In 1992, Everett Dade formulates a refinement of Alperin’s conjecture involving ordinary irreducible characters — with their defect — and, in 2000, Geoffrey Robinson proves that the new conjecture holds for $p$-solvable groups. But this refinement is formulated in terms of a vanishing alternating sum, without giving any possible refinement for the weights. In this note we show that, in the case of the $p$-solvable finite groups, the method developed in a previous paper can be suitably refined to provide, up to the choice of a polarization $\omega$, a natural bijection — namely compatible with the action of the group of outer automorphisms of $G$ — between the sets of absolutely irreducible characters of $G$ and of $G$-conjugacy classes of suitable inductive weights, preserving blocks and defects.