Quantum cooling by unitary transformations

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Abstract. We study the unitary time evolution of a simple quantum Hamiltonian describing a heat engine coupled to two heat baths. The engine is modeled as a three-level system. Each heat bath consists of a single harmonic oscillator. The engine is operated via time-dependent external fields. The $S$-matrix of the thermodynamic cycle is obtained in analytic form. We conjecture that the spectrum of this $S$-matrix contains a continuous part, and that this is a requirement for the operation as a heat engine. Energy currents flow in both directions through the engine. The balance of these currents determines whether the engine performs work or whether its operation requires the application of external forces.

1. Introduction

Quantum heat engines produce work or pump energy by repeated execution of a thermodynamic cycle, much in analogy with the original Carnot engine or the Otto engine [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. The theoretical modelling of these engines usually relies on the following ingredients. The state of the engine is described by a density matrix. During phases 1 and 3 of the cycle the engine is in contact with one of the heat baths. The time evolution of the density matrix is then described by a master equation of the Lindblad form [14, 15]. During phases 2 and 4 the state of the engine is modified by external forces working upon it. The adiabatic theorem is invoked to obtain that energy levels are modified without changing their occupational probability.

More detailed models going beyond this standard approach are needed to understand some fundamental questions about topics such as the entanglement between the quantum engine and the heat baths or the role of processes of decoherence in the heat baths. An early attempt [16] in this direction, using the exact solution of a model describing a system in interaction with a heat bath, reveals important deviations when the decoherence processes are failing.

It is rather straightforward to improve the modelling of the heat engine during the operation of the engine (phases 2 and 4). One of the techniques of quantum cooling is known as the STIRAP technique [17, 18, 19]. It involves population inversion of
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three level systems. Detailed theoretical studies are available on how the three level system behaves when two nearly simultaneous light pulses are applied [20, 21]. Also analytical results are available [22] describing the behaviour of three level systems in time-dependent external fields.

It is less easy to improve the description of the interactions between the engine and the heat baths. A full quantum mechanical treatment of a large heat bath seems not to be feasible. Never the less some results have been obtained by explicitly treating the heat bath [23, 24, 25]. We follow this line of research and limit each of the two heat baths to a single harmonic oscillator.

An additional motivation for our study comes from the investigation of thermodynamics in small closed quantum systems [26, 23, 27]. It is obvious to say that one harmonic oscillator is warmer than the other if the initial conditions are such that the engine performs work while energy flows from warm to cold. This can then be used to define the temperature of each of the two heat baths. Our present results only form a first step of such a rather challenging programme.

We use units in which \( \hbar = 1 \).

The model is introduced in the next Section. The \( S \)-matrix approach is explained in Section 3. The analytic expression for the \( S \)-matrix corresponding with one thermodynamic cycle is obtained. In Section 4 we analyse our results. Final conclusions and an outlook follow in Section 5. The details of our calculations are explained in the Appendices.

2. The model

The model Hamiltonian \( H \) consists of an unperturbed part \( H_0 \) describing the cold and warm heat baths and the heat engine, to which are added time-dependent external fields operating the heat engine and time-dependent interactions between the cold and warm baths and the engine.

It is quite common to model a heat bath with a collection of harmonic oscillators — see for instance [16]. To keep the model simple the cold and the warm heat bath are each made up by a single harmonic oscillator. All together, the unperturbed Hamiltonian reads

\[
H_0 = \omega_1 a^\dagger a + H_{gef} + \omega_3 c^\dagger c
\]

The operators \( a \) and \( c \) are the annihilation operators of the cold and of the warm harmonic oscillator, respectively. The hamiltonian of the heat engine \( H_2 \) is discussed below.

The use of a three level system as a heat engine has been proposed half a century ago by Scovil and Schulz-Dubois [1, 2]. In particular, the Stimulated Raman Adiabatic Passage (STIRAP) technique [18, 20, 21] has become a very efficient tool [19] to change the level populations of molecular systems. In the present study we limit our selves to a rather primitive sequence of two square pulses, although more realistic pulses can be treated analytically as well [22].
The Hamiltonian of the three level system is given by

$$H_{gef} = \begin{pmatrix} -\mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu + 2\delta \end{pmatrix}. \quad (2)$$

The three levels are labelled $g$, $e$, and $f$, and have energies $-\mu$, $\mu$, and $\mu + 2\delta$, respectively.

To this three level system we add two time-dependent external fields which are used to pump the energy levels of the heat engine. Their contribution is

$$I_{gef} = -\epsilon_a(t)\lambda_1 - \epsilon_b(t)\lambda_6, \quad (3)$$

where $\lambda_1$ and $\lambda_6$ are the Gell-Mann matrices — see the Appendix A.

The interaction between the three level system and each of the harmonic oscillators is inspired by the Jaynes-Cummings model. The choice for this model is in the first place justified by the consideration that its eigenvalues and eigenvectors can be calculated analytically.

The coupling at the cold side is given by\(^\dagger\)

$$H_{12} = \kappa_{12}(t)(a^\dagger E_+ + aE_-) \quad (4)$$

with

$$E_+ = \frac{1}{2}(\lambda_1 + i\lambda_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$E_- = \frac{1}{2}(\lambda_1 - i\lambda_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (5)$$

It couples the $g$ and $e$ levels of the three level system. At the warm side the interaction Hamiltonian is given by

$$H_{23} = \kappa_{23}(t)(F_+c^\dagger + F_-c) \quad (6)$$

with

$$F_+ = \frac{1}{2}(\lambda_6 + i\lambda_7) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$F_- = \frac{1}{2}(\lambda_6 - i\lambda_7) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (7)$$

It couples the $e$ and $f$ levels of the three level system with the hot harmonic oscillator.

The total time-dependent Hamiltonian is now

$$H = H_0 + H_{12} + I_{gef} + H_{23}. \quad (8)$$

\(^\dagger\) In the Jaynes-Cummings model $a^\dagger$ is multiplied with $\sigma_+$ instead of $\sigma_-$. The change made here is needed because the ground state of our three level system corresponds with the excited state in the Jaynes-Cummings model.
3. Cycles

The external field strengths $\epsilon_a(t)$ and $\epsilon_b(t)$ and the coupling parameters $\kappa_{12}(t)$ and $\kappa_{23}(t)$ all depend on time $t$. They are pulsed one after another in such a way that a thermodynamic cycle is traversed. See the Figure 1.

![Figure 1](image1.png)

**Figure 1.** The 4 phases of the thermodynamic cycle. On the horizontal axis is the energy of the heat engine. On the vertical axis is the total energy of the system.

The cycle starts by coupling the heat engine to the cold bath. The switching on and off changes the total energy of the system (this contribution is omitted in the figure). But during the first phase of the cycle the total energy is constant. In the second phase the energy of the heat engine is pumped up by applying a sequence of two pulses. Work is performed by doing so. In phase 3 the heat engine releases energy to the warm bath. In phase 4 the engine delivers work to the environment. This is again modelled by two externally applied pulses which pump down the internal energy of the heat engine.

Note that the cycle does not necessarily close. It is obvious that in the energy transfer mode the engine will consume more work than it can deliver. Because the heat baths are finite this implies that the total energy of the system goes up after every cycle of the process. The time inversion symmetry of the purely mechanical system is broken by the sequence in which external pulses are given and couplings are activated. See the Figure 2.

![Figure 2](image2.png)

**Figure 2.** Overview of the activation of the time-dependent terms in the Hamiltonian.
The time evolution of the system with Hamiltonian (8) is studied without making any approximation. This is best done in the interaction picture. Then the wave function of the total system — heat engine plus baths — is time-independent in the periods when none of the time-dependent terms is active. The effect of activating one of the interaction terms or one of the external fields is then to transform the wave function $\psi$ by means of an S-matrix into a new wave function $S\psi$.

**Step 1: Absorbing energy from the cold system**

In the first phase of the cycle the three level system is connected to the cold harmonic oscillator during a time $\tau_1$. The corresponding S-matrix is denoted $S_1$. It is not very difficult to calculate it exactly. See the Appendix B. The result is of the form

$$S_1 = e^{i\tau_1 H_0} e^{-i\tau_1 H}$$

$$= e^{\frac{i}{\tau_1} (\omega_1 - 2\mu)} \left[ a^\dagger (A - iC)a E_1 + ia^\dagger B E_+ \right]$$
$$+ e^{-\frac{i}{\tau_1} (\omega_1 - 2\mu)} \left[ iB a E_- + aa^\dagger (A + iC) E_2 \right]$$
$$+ G_1 E_1 + E_3,$$

with

$$E_1 = E_+ E_- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = E_- E_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$A = \sum_n \frac{1}{n+1} \cos(\tau_1 \lambda_n) |n\rangle \langle n|,$$

$$B = \sum_n \frac{1}{\sqrt{n+1}} \sin(\tau_1 \lambda_n) \sin(2\theta_n) |n\rangle \langle n|,$$

$$C = \sum_n \frac{1}{n+1} \sin(\tau_1 \lambda_n) \cos(2\theta_n) |n\rangle \langle n|.$$  \hfill (11)

The coefficients $\lambda_n$ and the angles $\theta_n$ are given by

$$\lambda_n = \frac{1}{2} \sqrt{4\kappa_{12}^2 (n+1) + (\omega_1 - 2\mu)^2}$$
$$\tan(\theta_n) = \frac{2\kappa_{12} \sqrt{n+1}}{2\lambda_n + \omega_1 - 2\mu}.$$  \hfill (12)

The operator $G_1$ is the orthogonal projection $|0\rangle \langle 0|$ onto the ground state of the cold harmonic oscillator.
Step 2: Pumping up

We apply a sequence of two pulses of the on/off type. The first pulse results by giving \( \epsilon_a(t) \) a constant non-zero value during a time \( \tau_a \). It tries to invert the population of the levels \( e \) and \( f \). The change of the population as a consequence of this pulse is given by the S-matrix \( S_{2a} \) which is now calculated.

\[
S_{2a} = e^{i \tau_a H_0} e^{-i \tau_a H} = e^{i \tau_a H_{ge}} e^{-i \tau_a (H_{ge} - \epsilon_a \lambda_e)} = e^{-i \tau_a \delta \sigma_3} e^{i \tau_a \delta \sigma - \epsilon_a \sigma_1}.
\] (13)

Note that we switched notations, using two-dimensional Pauli matrices instead of the Gell-Mann matrices, omitting one dimension for a moment. Introduce the constant \( T_a = \frac{1}{\sqrt{\delta^2 + \epsilon_a^2}} \). There follows

\[
S_{2a} = \left[ \cos(\tau_a \delta) - i \sin(\tau_a \delta)\sigma_3 \right] \times \left[ \cos(\tau_a/T_a) + iT_a \sin(\tau_a/T_a)(\delta \sigma_3 - \epsilon_a \sigma_1) \right].
\] (14)

Let us now make an appropriate choice of the pulse duration \( \tau_a \). The goal is to minimize the population of the \( e \)-level after the pulse. Since one can expect that before the pulse the \( e \)-level is more populated than the \( f \)-level the best one can do is to require that the \( e \) matrix element of \( S_{2a} \) is as small as possible in modulus. Let therefore \( \tau_a = \frac{1}{2} \pi T_a \). Then the S-matrix becomes

\[
S_{2a} = iT_a \left[ \cos(\tau_a \delta) - i \sin(\tau_a \delta)\sigma_3 \right] \sin(\tau_a/T_a)(\delta \sigma_3 - \epsilon_a \sigma_1).
\] (15)

Restoring the third dimension this becomes

\[
S_{2a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -i & 0 \\ \end{pmatrix} + iT_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta e^{-i \tau_a \delta} & -\epsilon_a e^{-i \tau_a \delta} \\ 0 & -\epsilon_a e^{i \tau_a \delta} & -\delta e^{i \tau_a \delta} \\ \end{pmatrix}.
\] (16)

In the limit of a strong short pulse this becomes

\[
S_{2a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \\ \end{pmatrix}.
\] (17)

The first pulse of the second phase of the thermodynamic cycle is followed by a pulse of duration \( \tau_b \), intended to invert the population of levels \( e \) and \( g \). The corresponding S-matrix reads, using the notation \( T_b = 1/\sqrt{\mu^2 + \epsilon_b^2} \),

\[
S_{2b} = e^{i \tau_b H_0} e^{-i \tau_b H} = e^{i \tau_b H_{ge}} e^{-i \tau_b (H_{ge} - \epsilon_b \lambda_1)} = \left[ \cos(\tau_b \mu) - i \sin(\tau_b \mu)\sigma_3 \right] \times \left[ \cos(\tau_b/T_b) + iT_b \sin(\tau_b/T_b)(\mu \sigma_3 + \epsilon_b \sigma_1) \right]
\] (18)
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With similar arguments as before let us choose \( \tau_b = \frac{1}{2} \pi T_b \). Then the S-matrix becomes

\[
S_{2b} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} + iT_b \begin{pmatrix}
\mu e^{-i \tau_b \mu} & \epsilon_b e^{-i \tau_b \mu} & 0 \\
\epsilon_b e^{i \tau_b \mu} & -\mu e^{i \tau_b \mu} & 0 \\
0 & 0 & 0 \\
\end{pmatrix}.
\] (19)

In the limit of a strong short pulse this becomes

\[
S_{2b} = \begin{pmatrix}
0 & i & 0 \\
i & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}.
\] (20)

All together the S-matrix for the second phase of the cycle equals

\[
S_2 = S_{2b} S_{2a} = \begin{pmatrix}
i T_b \mu e^{-i \tau_b \mu} & -T_a T_b \epsilon_b \delta e^{-i \tau_a \delta - i \tau_b \mu} & T_a T_b \epsilon_a \epsilon_b e^{-i \tau_a \delta - i \tau_b \mu} \\
i T_b \epsilon_b e^{i \tau_b \mu} & T_a T_b \mu \delta e^{-i \tau_a \delta + i \tau_b \mu} & -T_a T_b \mu \epsilon_a e^{-i \tau_a \delta + i \tau_b \mu} \\
0 & -iT_a \epsilon_a e^{i \tau_a \delta} & -iT_a \delta e^{-i \tau_a \delta} \\
\end{pmatrix}.
\] (21)

In the limit of strong short pulses it becomes

\[
S_2 = \begin{pmatrix}
0 & 0 & 1 \\
i & 0 & 0 \\
0 & -i & 0 \\
\end{pmatrix}.
\] (22)

**Step 3: Exchanging energy with the warm bath**

In the third phase of the cycle the three level system is connected to the warm harmonic oscillator during a time \( \tau_3 \). The corresponding S-matrix is denoted \( S_3 \). The calculation is similar to that in Step 1. The result is of the form

\[
S_3 = e^{i \tau_3 \bar{H}} e^{-i \tau_3 \bar{H}} = E_1 + E_2 G_3 + e^{i \tau_3 (\omega_3 - 2 \delta)} \left[ c^\dagger (Z - i V) c E_2 + i c^\dagger Y F_+ \right] + e^{-i \tau_3 (\omega_3 - 2 \delta)} \left[ i Y c F_- + c c^\dagger (Z + i V) E_3 \right]
\] (23)

with

\[
Z = \sum_n \frac{1}{n + 1} \cos(\tau_3 \xi_n) |n \rangle \langle n |
\]

\[
Y = \sum_n \frac{1}{\sqrt{n + 1}} \sin(\tau_3 \xi_n) \sin(2 \phi_n) |n \rangle \langle n |
\]

\[
V = \sum_n \frac{1}{n + 1} \sin(\tau_3 \xi_n) \cos(2 \phi_n) |n \rangle \langle n |
\]

(24)

The coefficients \( \xi_n \) and the angles \( \phi_n \) are given by

\[
\xi_n = \frac{1}{2} \sqrt{4 \kappa_{23}^2 (n + 1) + (\omega_3 - 2 \delta)^2}
\]
The full S-matrix reads
\[
\tan(\phi_n) = \frac{2\kappa_3 \sqrt{n + 1}}{2\xi_n + \omega_3 - 2\delta}.
\]

The operator \( G_3 \) is the orthogonal projection \(|0\rangle\langle 0|\) onto the ground state of the warm harmonic oscillator.

**Step 4: Pumping down**

The operation in the fourth phase is the inverse of that in the second phase. We thus have \( S_4 = S_2^\dagger \).

**4. Analysis**

In the previous Section the contribution to the S-matrix from each of the four phases of the cycle has been obtained. The composite matrix \( S = S_4 S_3 S_2 S_1 \) is now calculated. The result is a rather complicated. Therefore a tensor notation is appropriate. Remember that the Hilbert space of wave functions of the total system is the tensor product
\[
\mathcal{H} = \mathcal{H}_{\text{cold}} \otimes \mathbb{C}^3 \otimes \mathcal{H}_{\text{warm}}.
\]

The first and the last factor are the Hilbert space of the cold and of the warm harmonic oscillator, respectively. The middle factor is the space of vectors with three complex components.

**4.1. The composed S-matrix**

The full S-matrix reads
\[
S = \mathbb{I} \otimes S_2^\dagger \otimes \mathbb{I} \\
\times \left\{ \begin{array}{l}
\mathbb{I} \otimes E_1 \otimes \mathbb{I} + \mathbb{I} \otimes E_2 \otimes G_3 \\
+ e^{\frac{i}{2} \tau_1(\omega_3 - 2\delta)} \left[ \mathbb{I} \otimes E_2 \otimes c^\dagger(Z - iV)c + i \mathbb{I} \otimes F_+ \otimes c^\dagger Y \right] \\
+ e^{-\frac{i}{2} \tau_1(\omega_3 - 2\delta)} \left[ i \mathbb{I} \otimes F_- \otimes Yc + \mathbb{I} \otimes E_3 \otimes cc^\dagger(Z + iV) \right] \end{array} \right\} \\
\times \mathbb{I} \otimes S_2 \otimes \mathbb{I} \\
\times \left\{ e^{\frac{i}{2} \tau_1(\omega_1 - 2\mu)} \left[ a^\dagger(A - iC)a \otimes E_1 \otimes \mathbb{I} + ia^\dagger B \otimes E_+ \otimes \mathbb{I} \right] \\
+ e^{-\frac{i}{2} \tau_1(\omega_1 - 2\mu)} \left[ iB a \otimes E_- \otimes \mathbb{I} + aa^\dagger(A + iC) \otimes E_2 \otimes \mathbb{I} \right] \\
+ G_1 \otimes E_1 \otimes \mathbb{I} + \mathbb{I} \otimes E_3 \otimes \mathbb{I} \right\} \\
= \left\{ \begin{array}{l}
\mathbb{I} \otimes S_2^\dagger E_1 S_2 \otimes \mathbb{I} + \mathbb{I} \otimes S_2^\dagger E_2 S_2 \otimes G_3 \\
+ e^{\frac{i}{2} \tau_1(\omega_3 - 2\delta)} \left[ \mathbb{I} \otimes S_2^\dagger E_2 S_2 \otimes c^\dagger(Z - iV)c + i \mathbb{I} \otimes S_2^\dagger F_+ S_2 \otimes c^\dagger Y \right] \\
+ e^{-\frac{i}{2} \tau_1(\omega_3 - 2\delta)} \left[ i \mathbb{I} \otimes S_2^\dagger F_- S_2 \otimes Yc + \mathbb{I} \otimes S_2^\dagger E_3 S_2 \otimes cc^\dagger(Z + iV) \right] \end{array} \right\} \\
\times \left\{ e^{\frac{i}{2} \tau_1(\omega_1 - 2\mu)} \left[ a^\dagger(A - iC)a \otimes E_1 \otimes \mathbb{I} + ia^\dagger B \otimes E_+ \otimes \mathbb{I} \right] \\
+ e^{-\frac{i}{2} \tau_1(\omega_1 - 2\mu)} \left[ iB a \otimes E_- \otimes \mathbb{I} + aa^\dagger(A + iC) \otimes E_2 \otimes \mathbb{I} \right] \\
+ G_1 \otimes E_1 \otimes \mathbb{I} + \mathbb{I} \otimes E_3 \otimes \mathbb{I} \right\}
\]
Table 1. Interpretation of the terms appearing in (28).
The arrows indicate the direction of the energy flow.

| Term | Interpretation |
|------|----------------|
| $a^\dagger(A-iC)a \otimes E_1 \otimes c^\dagger(Z-iV)c$ | $\rightarrow$ |
| $a^\dagger B \otimes E_+ \otimes c^\dagger(Z-iV)c$ | $\leftarrow$ |
| $Ba \otimes E_1 \otimes c^\dagger Y$ | $\rightarrow$ |
| $aa^\dagger(A+iC) \otimes E_\pm \otimes c^\dagger Y$ | $\rightarrow$ |
| $a^\dagger(A-iC)a \otimes E_- \otimes Yc$ | $\leftarrow$ |
| $a^\dagger B \otimes E_2 \otimes Yc$ | $\leftarrow$ |
| $Ba \otimes E_- \otimes cc^\dagger(Z+iV)$ | $\rightarrow$ |
| $aa^\dagger(A+iC) \otimes E_2 \otimes cc^\dagger(Z+iV)$ | $\rightarrow$ |

For simplicity, we use the value (22) of $S_2$ in the limit of strong short pulses. In this limit one has $S_2^\dagger E_1 S_2 = E_3$, $S_2^\dagger E_2 S_2 = E_1$, $S_2^\dagger E_3 S_2 = E_2$, $S_2^\dagger F_+ S_2 = -E_+$, $S_2^\dagger F_- S_2 = -E_-$. Hence, the above expression for $S$ simplifies to

$$S = \left\{ \mathbb{I} \otimes E_3 \otimes \mathbb{I} + \mathbb{I} \otimes E_1 \otimes G_3 + e^{\frac{i}{\hbar} \tau_2(\omega_3-2\delta)} \left\{ \left[ \mathbb{I} \otimes E_1 \otimes c^\dagger(Z-iV)c - i\mathbb{I} \otimes E_+ \otimes c^\dagger Y \right] + e^{-\frac{i}{\hbar} \tau_1(\omega_3-2\delta)} \left[ \left[ -i\mathbb{I} \otimes E_- \otimes Yc + \mathbb{I} \otimes E_2 \otimes cc^\dagger(Z+iV) \right] + e^{\frac{i}{\hbar} \tau_1(\omega_1-2\mu)} \right\} \right\}$$

$$= \mathbb{I} \otimes E_3 \otimes \mathbb{I} + G_1 \otimes E_1 \otimes G_3 + e^{\frac{i}{\hbar} \tau_1(\omega_1-2\mu)} \left[ a^\dagger(A-iC)a \otimes E_1 \otimes G_3 + ia^\dagger B \otimes E_+ \otimes G_3 \right]$$

$$+ e^{\frac{i}{\hbar} \tau_1(\omega_3-2\delta)} G_1 \otimes E_1 \otimes c^\dagger(Z-iV)c - ie^{\frac{i}{\hbar} \tau_1(\omega_3-2\delta)} G_1 \otimes E_- \otimes Yc$$

$$+ e^{\frac{i}{\hbar} \tau_1(\omega_3-2\delta)} e^{\frac{i}{\hbar} \tau_1(\omega_1-2\mu)} \left[ [Ba \otimes E_1 - iaa^\dagger(A+iC) \otimes E_+] \otimes c^\dagger Y \right]$$

$$+ e^{-\frac{i}{\hbar} \tau_1(\omega_3-2\delta)} e^{\frac{i}{\hbar} \tau_1(\omega_1-2\mu)} \left[ -ia^\dagger(A-iC)a \otimes E_- + a^\dagger B \otimes E_2 \otimes Yc \right]$$

$$+ e^{-\frac{i}{\hbar} \tau_1(\omega_3-2\delta)} e^{-\frac{i}{\hbar} \tau_1(\omega_1-2\mu)} \left[ iBa \otimes E_- + aa^\dagger(A+iC) \otimes E_2 \right] \otimes cc^\dagger(Z+iV).$$

Note that the operators $A$, $B$, $C$, $Y$, $Z$, $V$, commute with the counting operators of the two harmonic oscillators. Hence the two terms which directly transfer energy between the two oscillators are those proportional to $Ba \otimes E_1 \otimes c^\dagger Y$ and $a^\dagger B \otimes E_2 \otimes Yc$ respectively. They act in opposite directions. Other terms do not transfer energy or they exchange energy between the heat engine and one of the oscillators. See the Table 1.

4.2. Eigenvectors of the $S$-matrix

The above $S$-matrix describes the effect in the interaction picture of performing one thermodynamic cycle. It is immediately clear that the ground state $|0, g, 0\rangle$ of the system...
is an eigenstate of this S-matrix with eigenvalue 1. This is an immediate consequence of the fact that the ground state of the Jaynes-Cummings model is not affected by the interactions of the model. An important question is whether the S-matrix has other eigenvectors. Indeed, such eigenvectors describe situations in which the action of the heat engine has no effect at all. Of course, on a superposition of eigenvectors the engine can have effect. But the result is an almost periodic function which always returns arbitrary close to its starting point.

On the other hand, if part of the spectrum of $S$ is continuous then a genuine energy transfer is possible by which the system approaches a stationary regime. We did not yet succeed to prove the existence of such a continuous spectrum. But note that several terms in the expression (28) are linear in the creation and annihilation operators of the cold and warm oscillators. These operators are linear combinations of position and momentum operators which are known to have a purely continuous spectrum. Hence it would not be strange when this continuous spectrum survives when adding other terms.

4.3. Energy transfer

The result (28) seems hopelessly complicated but can never the less be used to derive some unexpected properties of the heat engine. The change in the state of the cold oscillator before and after one cycle is defined by

$$D = S^\dagger a^\dagger aS - a^\dagger a.$$  \hfill (29)

One finds (see the Appendix C)

$$D = \left[aa^\dagger B^2 \otimes E_2 - a^\dagger B^2 a \otimes E_1 + ia^\dagger aa^\dagger (A + iC)B \otimes E_+ - i(A - iC)Baa^\dagger a \otimes E_- \right] \otimes \mathbb{I}. \hfill (30)$$

The eigenvectors of $D$ are of the form

$$\psi = u|n + 1\rangle \otimes |g\rangle + v|n\rangle \otimes |e\rangle$$  \hfill (31)

(we neglect the Hilbert space of the warm oscillator for a moment). The condition $D\psi = \rho\psi$ then yields

$$0 = u \left( \rho + \sin^2(\tau_1 \lambda_n) \sin^2(2\theta_n) \right)$$
$$+ v \sin(\tau_1 \lambda_n) \sin(2\theta_n) \left[ \sin(\tau_1 \lambda_n) \cos(2\theta_n) - i \cos(\tau_1 \lambda_n) \right]$$

$$0 = u \sin(\tau_1 \lambda_n) \sin(2\theta_n) \left[ \sin(\tau_1 \lambda_n) \cos(2\theta_n) + i \cos(\tau_1 \lambda_n) \right]$$
$$+ v \left( \rho - \sin^2(\tau_1 \lambda_n) \sin^2(2\theta_n) \right). \hfill (32)$$

This set of equations has a non-trivial solution when

$$\rho = \pm \sin(\tau_1 \lambda_n) \sin(2\theta_n). \hfill (33)$$

Corresponding eigenvectors are then

$$u = \sin(\tau_1 \lambda_n) \cos(2\theta_n) - i \cos(\tau_1 \lambda_n),$$
$$v = \mp 1 - \sin(\tau_1 \lambda_n) \sin(2\theta_n). \hfill (34)$$
Note that $D|0\rangle \otimes |g\rangle = 0$. Hence, the spectrum of $D$ is completely known. For each strictly positive eigenvalue $\rho > 0$ also $-\rho$ is an eigenvalue. $\rho > 0$ corresponds with warming up of the left oscillator, $\rho < 0$ with cooling.

One concludes that raising or lowering the energy of the cold oscillator after one cycle of the engine depends completely on the choice of the initial wave function. The important question is of course what happens after one cycle with a wave function originally chosen as an eigenvector $\psi$ of $D$ with negative eigenvalue. Will $S\psi$ be a superposition of eigenvectors all with negative eigenvalues? Or will part of them have a positive eigenvalue? We expect, but did not succeed to prove, that the latter is the case and that in this way the cooling efficiency of the engine gradually goes down so that after a small number of cycles a stationary regime is approached.

A similar calculation for the warm oscillator is possible. But note that an easy result only follows when starting the cycle with coupling the heat bath to the warm oscillator instead of the cold one, as used in the above calculations.

4.4. Performing work

The previous subsection gives a partial answer to the question whether the engine is capable of transferring energy between the two oscillators. Now follows a discussion of the work needed to operate the engine.

In phases 1 and 3 of the cycle some work is needed to operate the valves connecting the engine with the cold respectively the warm oscillator. Indeed, switching on and off the interaction terms (4, 6) changes the total energy of the system. Since the wave function of the system evolves in time between the switching on and switching off the involved energy changes to not necessarily cancel. Hence we expect that a tiny amount of work is needed to operate these valves.

It is now indicated to consider a cycle starting with phase 2 instead of phase 1. Then the energy changes during the respective phases are given by

$$
\Delta E_1 = H_0 - S_1 H_0 S_1^\dagger
$$
$$
\Delta E_2 = S_2^\dagger H_0 S_2 - H_0,
$$
$$
\Delta E_3 = S_2^\dagger (S_4^\dagger H_0 S_3 - H_0) S_2,
$$
$$
\Delta E_4 = S_2^\dagger S_3^\dagger (S_2 H_0 S_2^\dagger - H_0) S_3 S_2. \quad (35)
$$

Using the simplified expression (17) for $S_2$ one obtains

$$
\Delta E_1 = (\omega_1 - 2\mu) \left( B^2 a a^\dagger E_2 - a^\dagger B^2 a E_1 \right)
+ i(\omega_1 - 2\mu) a^\dagger Baa^\dagger (A + iC) E_+ 
- i(\omega_1 - 2\mu) Baa^\dagger (A - iC) a E_- \quad (36)
$$

and

$$
\Delta E_2 = 2[\mu, \delta, -\mu - \delta] \quad (37)
$$

and

$$
\Delta E_3 = (\omega_3 - 2\delta) \left[ E_2 c c^\dagger Y^2 - E_1 c^\dagger Y^2 c \right]
$$
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\[-i(\omega_3 - 2\delta)E_+c^\dagger(Z + iV)cY + i(\omega_3 - 2\delta)E_-(Z - iV)cYc\]  \hspace{1cm} (38)

and

\[\Delta E_4 = - \Delta E_2 + 2(\mu - \delta) \left( E_1c^\dagger Y^2 c - E_2 Y^2 c c^\dagger \right) + 2i(\mu - \delta) \left[ E_+c^\dagger(Z + iV)cY - E_-Y c c^\dagger(Z - iV)c \right], \hspace{1cm} (39)\]

where \([a, b, c]\) denotes the diagonal matrix with eigenvalues \(a, b, c\). See the Appendix D.

Several features can be observed. The contributions \(\Delta E_3\) and \(\Delta E_1\) represent the energy needed to switch on and off the interactions with the harmonic oscillators. They vanish when the coupling between the heat engine and the oscillators is at resonance.

The work performed by the engine equals the quantum expectation of the operator \(-\Delta E_2 - \Delta E_4\). When \(\mu = \delta\) then the operation of the engine is meaningless and no net energy is used and no net work is performed during the phases 2 and 4. In the general case the eigen values of \(\Delta E_2 + \Delta E_4\) can be calculated analytically. One obtains

\[\lambda = \pm \sin(\tau_3 \xi_n) \sin(2\phi_n). \hspace{1cm} (40)\]

The corresponding eigen vectors are linear combinations of \(|g, n+1\rangle\) and \(|e, n\rangle\) (neglecting the state of the cold oscillator). Hence also the spectrum of this operator is symmetric under a change of sign. This means that the initial conditions determine whether operating the engine consumes energy or whether it performs work.

5. Conclusions

It is feasible to obtain analytic results for a closed quantum system consisting of a heat engine operating between two small heat baths. The engine is operated by switching external fields on and off. The state of the system is at any moment determined by its wave function. The time evolution follows from solving the Schrödinger equation using a time-dependent Hamiltonian. In the traditional approach the engine is the system and the cold and warm heat baths belong to the environment which is taken into account in a phenomenological way. In the present approach the boundary between the system and the environment is shifted so that the heat baths belong to the system while the operation of the engine is still implemented in a phenomenological way. The main advantages of this approach are that the time evolution is unitary and that the quantum entanglement between the engine and the heat baths is treated rigorously.

From our toy model we have learned a number of points.

- The use of the interaction picture for doing the calculations improves the transparency of the calculations.
- We do not make use of the adiabatic theorem. The change in the population of the energy levels of the engine results from the time evolution. As a consequence all results depend only on intra-level distances and not on the positioning of oscillator levels w.r.t. levels of the engine.
At each of the two interfaces the energy flows in both directions. Energy leaks away in the direction opposite to the intended one. Eight different energy contributions have been distinguished in the Table 1. In the usual approach these are replaced by two phenomenological terms.

We conjecture that the $S$-matrix of the thermodynamic cycle has a continuous spectrum and that this is an essential feature of a realistic heat engine. The argument for the latter is that eigenstates of the $S$-matrix are unchanged after one cycle. Hence they correspond with states of the system in which the engine is inactive.

To our surprise the operator $D = S^\dagger a^\dagger aS - a^\dagger a$ which measures the change in energy of the cold bath during one cycle has a fully discrete spectrum with explicitly known eigenvectors and eigenvalues. This is probably an artefact caused by the Jaynes-Cummings mechanism used by the toy model for implementing the interaction between heat bath and engine. However, the spectrum of this operator $D$ is symmetric under the change of sign. This could be a more general feature being a consequence of time inversion symmetry.

The change of energy during one cycle can be obtained analytically as well. The operation of the valves connecting the engine with the heat baths costs energy except when the interaction is at resonance. The pumping up and down of the occupational probabilities of the engine levels can cost energy or can perform work depending on the initial state of the system, this is, depending on its wave function. This shows that the engine can be used either to transfer energy from the cold to the warm oscillator or to perform work produced by the energy flow from warm to cold.

The present paper shows that our approach of studying a heat engine by means of unitary time evolutions can produce new insights. Future work goes into two directions. Many questions about the model are still unresolved. We have studied only one cycle of the engine. The repeated operation of the engine as an energy pump will lead to a steady increase of the total energy. But the reduced density operator of the cold oscillator might converge. Alternatively one could try to analyse how the effect of a single cycle depends on the initial wave function.

It is tempting to analyse our results in terms of thermodynamic concepts. This is not easy because in a closed system it is not clear at all what the distinction is between energy and heat. But one could for instance analyse the change of entropy of the reduced density matrices.

Another line of research could try to study some features of the toy model in a more general context. Particularly intriguing is the question whether the continuous spectrum of the $S$-matrix is a general feature of heat engines. Also the effects of the time inversion symmetry of the equations of motion require further investigation.
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Appendix A. The Gell-Mann matrices

Conventionally, the Gell-Mann matrices are defined as follows.

\[
\begin{align*}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\
\lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\
\lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{align*}
\] (A.1)

Appendix B. The S-matrix of phase 1 of the cycle

Here we calculate the S-matrix of a Jaynes-Cummings Hamiltonian in which the interaction is switched on during a finite time. The coupling is constant with strength \(\kappa_{12}\) during a time interval of length \(\tau_1\). The relevant Hamiltonian is

\[
H = \omega_1 a^\dagger a - \mu \sigma_z - \kappa_{12} (a^\dagger \sigma_+ + a \sigma_-).
\] (B.1)

Let

\[
|g\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |e\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\] (B.2)

The eigenstates of the harmonic oscillator are denoted \(n\rangle\), with \(n = 0, 1, \cdots\).

The ground state of \(H\) is

\[
|0, -\rangle \equiv |0\rangle \otimes |g\rangle.
\] (B.3)

It satisfies \(H|0, -\rangle = -\mu|0, -\rangle\). The pairs of excited states are denoted \(|n, \pm\rangle\), with \(n = 0, 1, \cdots\). They are of the form

\[
|n, -\rangle = \cos(\theta_n) |n\rangle \otimes |e\rangle + \sin(\theta_n) |n + 1\rangle \otimes |g\rangle,
\]

\[
|n, +\rangle = -\sin(\theta_n) |n\rangle \otimes |e\rangle + \cos(\theta_n) |n + 1\rangle \otimes |g\rangle.
\] (B.4)

From

\[
H|n\rangle \otimes |e\rangle = (n\omega_1 + \mu) |n\rangle \otimes |e\rangle - \kappa_{12} \sqrt{n + 1} |n + 1\rangle \otimes |g\rangle
\]

\[
H|n\rangle \otimes |g\rangle = (n\omega_1 - \mu) |n\rangle \otimes |g\rangle - \kappa_{12} \sqrt{n} |n - 1\rangle \otimes |e\rangle
\] (B.5)

follows

\[
H|n, -\rangle = \cos(\theta_n) \left[ (n\omega_1 + \mu) |n\rangle \otimes |e\rangle - \kappa_{12} \sqrt{n + 1} |n + 1\rangle \otimes |g\rangle \right]
\]
The solution is
\[
+ \sin(\theta_n) \left[ ((n + 1)\omega_1 - \mu)|n + 1\rangle \otimes |g\rangle - \kappa_{12}\sqrt{n + 1}|n\rangle \otimes |e\rangle \right],
\]
\[
H|n, +\rangle = -\sin(\theta_n) \left[ (n\omega_1 + \mu)|n\rangle \otimes |e\rangle - \kappa_{12}\sqrt{n + 1}|n + 1\rangle \otimes |g\rangle \right] + \cos(\theta_n) \left[ ((n + 1)\omega_1 - \mu)|n + 1\rangle \otimes |g\rangle - \kappa_{12}\sqrt{n + 1}|n\rangle \otimes |e\rangle \right].
\]

The requirement that \( H|n, \pm\rangle = E_n^\pm |n, \pm\rangle \) then yields the set of equations
\[
(n\omega_1 + \mu)\cos(\theta_n) - \kappa_{12}\sqrt{n + 1}\sin(\theta_n) = E_n^\pm \cos(\theta_n)
\]
\[
-\kappa_{12}\sqrt{n + 1}\cos(\theta_n) + (n + 1)\omega_1 - \mu \sin(\theta_n) = E_n^\pm \sin(\theta_n)
\]
\[
-\kappa_{12}\sqrt{n + 1}\cos(\theta_n) - (n\omega_1 + \mu)\sin(\theta_n) = -E_n^+ \sin(\theta_n)
\]
\[
((n + 1)\omega_1 - \mu)\cos(\theta_n) + \kappa_{12}\sqrt{n + 1}\sin(\theta_n) = E_n^+ \cos(\theta_n)
\]

The solution is
\[
E_n^\pm = \left(n + \frac{1}{2}\right)\omega_1 \pm \lambda_n
\]
\[
\tan(\theta_n) = \frac{2\lambda_n + 2\mu - \omega_1}{2\kappa_{12}\sqrt{n + 1}}
\]
with
\[
\lambda_n = \sqrt{\kappa_{12}^2(n + 1) + \left(\mu - \frac{1}{2}\omega_1\right)^2}.
\]

A short calculation now gives
\[
S_1|n\rangle \otimes |e\rangle = e^{itH_0} e^{-itH} |n\rangle \otimes |e\rangle
\]
\[
= e^{itH_0} \left[ \cos(\theta_n) e^{-itE_n^\pm} |n, -\rangle - \sin(\theta_n) e^{-itE_n^\pm} |n, +\rangle \right]
\]
\[
= e^{it(\mu - \omega_1/2)} \left[ \cos(t\lambda_n) + i \cos(2\theta_n) \sin(t\lambda_n) \right] |n\rangle \otimes |e\rangle
\]
\[
+ ie^{-it(\mu - \omega_1/2)} \sin(2\theta_n) \sin(t\lambda_n) |n + 1\rangle \otimes |g\rangle
\]
\[
S_1|n + 1\rangle \otimes |g\rangle = e^{itH_0} e^{-itH} |n + 1\rangle \otimes |g\rangle
\]
\[
= e^{itH_0} \left[ \cos(\theta_n) e^{-itE_n^\pm} |n, +\rangle + \sin(\theta_n) e^{-itE_n^\pm} |n, -\rangle \right]
\]
\[
= ie^{it(\mu - \omega_1/2)} \sin(2\theta_n) \sin(t\lambda_n) |n\rangle \otimes |e\rangle
\]
\[
+ e^{-it(\mu - \omega_1/2)} \left[ \cos(t\lambda_n) - i \cos(2\theta_n) \sin(t\lambda_n) \right] \times |n + 1\rangle \otimes |g\rangle.
\]

These expressions can be written as (9).

Appendix C. Change in the state of the cold oscillator

Here we calculate (30).

Note that \([a^\dagger a, G_1] = 0\) and \([a^\dagger a, a^\dagger Aa] = 0\) and \([a^\dagger a, a^\dagger B] = a^\dagger B\) and \([a^\dagger a, a^\dagger Ca] = 0\). Using these relations one obtains
\[
[a^\dagger a, S] = ie^{\tau_1(\omega_1 - 2\mu)}a^\dagger B \otimes E_+ \otimes G_3
\]
\[
+ ie^{\tau_3(\omega_3 - 2\delta)} e^{\tau_1(\omega_1 - 2\mu)}a^\dagger B \otimes E_+ \otimes e^i(Z - iV)c
\]
Therefore we start with calculating $c^\dagger$. One calculates using
\[ cc^\dagger \text{ so that } \]
\[ E = \text{ } \]
It is then straightforward to obtain
\[ S_2^\dagger(S_3^\dagger H_0 S_3 - H_0)S_2 = (\omega_3 - 2\delta) E_2 Y cc^\dagger Y - (\omega_3 - 2\delta) E_1 c^\dagger Y^2 c - i(\omega_3 - 2\delta) E_+ c^\dagger (Z + iV) cc^\dagger Y + i(\omega_3 - 2\delta) E_- (Z - iV) cc^\dagger Y c. \] (D.5)
This yields $\Delta E_3$.

Finally calculate $\Delta E_4$. One has using the simplified expression (17)
\[ S_2 H_0 S_2^\dagger - H_0 = 2[\mu + \delta, -\mu, -\delta]. \] (D.6)

Note that
\[ X \equiv cc^\dagger (Z^2 + V^2) + Y^2 = \sum_n \frac{1}{n+1} |n\rangle \langle n| \] (C.2)
so that $cc^\dagger X = G_3 + c^\dagger X c = I$. Hence, from (C.1) one obtains
\[ D = S^\dagger[a^\dagger a, S] = + [ia^\dagger a a^\dagger (A + iC) B \otimes E_+ + a a^\dagger B^2 \otimes E_2] \otimes (G_3 + c^\dagger X c) - [a^\dagger B^2 a \otimes E_1 + i(A - iC) a a^\dagger B a \otimes E_-] \otimes cc^\dagger X = \left[ a a^\dagger B^2 \otimes E_2 - a^\dagger B^2 a \otimes E_1 \right.
\[ + ia^\dagger a a^\dagger (A + iC) B \otimes E_+ - i(A - iC) B a a^\dagger a \otimes E_- ] \otimes I. \] (C.3)
This is (30).

**Appendix D. Work performed during phases 1, 3, 4**

We first calculate $\Delta E_1 = H_0 - S_1 H_0 S_1^\dagger$. Note that one can write $\Delta E_1 = S_1 [S_1^\dagger, H_0]$. Therefore we start with calculating
\[ [S_1^\dagger, H_0] = - i(\omega_1 - 2\mu) e^{-\frac{i}{2}r_1 (\omega_1 - 2\mu)} B a \otimes E_-
\[ + i(\omega_1 - 2\mu) e^{\frac{i}{2}r_1 (\omega_1 - 2\mu)} a^\dagger B \otimes E_+. \] (D.1)
Now multiplying from the left with $S_1$ yields (36).

Next calculate $\Delta E_3$ using
\[ \Delta E_3 = S_3^\dagger H_0 S_3 - H_0 = S_3^\dagger [H_0, S_3]. \] (D.2)
One calculates using $[c^\dagger c, c^\dagger (Z - iV)c] = [c^\dagger c, cc^\dagger (Z + iV)] = [c^\dagger c, Y] = 0$
\[ [H_0, S_3] = \delta[E_3 - E_2, S_3] + \omega_3 [c^\dagger c, S_3]
\[ = i(\omega_3 - 2\delta) e^{\frac{i}{2}r_3 (\omega_3 - 2\delta)} F_+ c^\dagger Y - i(\omega_3 - 2\delta) e^{-\frac{i}{2}r_3 (\omega_3 - 2\delta)} F_- Y c. \] (D.3)
This gives using $F_- F_+ = E_3, E_2 F_+ = F_+, F_+ F_- = E_2, E_3 F_- = F_-$
\[ S_3^\dagger [H_0, S_3] = (\omega_3 - 2\delta) E_3 Y cc^\dagger Y + i(\omega_3 - 2\delta) F_+ c^\dagger (Z + iV) cc^\dagger Y
\[ - (\omega_3 - 2\delta) E_2 c^\dagger Y^2 c - i(\omega_3 - 2\delta) F_- (Z - iV) cc^\dagger Y c. \] (D.4)
It is then straightforward to obtain
\[ S_2^\dagger (S_3^\dagger H_0 S_3 - H_0)S_2 = (\omega_3 - 2\delta) E_2 Y cc^\dagger Y - (\omega_3 - 2\delta) E_1 c^\dagger Y^2 c - i(\omega_3 - 2\delta) E_+ c^\dagger (Z + iV) cc^\dagger Y
\[ + i(\omega_3 - 2\delta) E_- (Z - iV) cc^\dagger Y c. \] (D.5)
Note that (using $E_2 S_2 = i E_-, F_+ S_2 = -i E_2$, and $E_3 S_2 = -i F_-)$
\[
S_3 S_2 = \mathbb{I} \otimes E_1 S_2 \otimes \mathbb{I} + \mathbb{I} \otimes E_\perp \otimes G_3 \\
+ e^{\mp \gamma_{32}(\omega - 2\delta)} \mathbb{I} \otimes [i E_- \otimes c^\dagger (Z - i V) c + E_2 \otimes c^\dagger Y] \\
+ e^{-\mp \gamma_{32}(\omega - 2\delta)} \mathbb{I} \otimes [i F_- S_2 \otimes Y c - i F_- \otimes cc^\dagger (Z + i V)]. \quad (D.7)
\]
This gives (using $S_4^1 E_1 S_2 = E_3$ and $S_2^1 E_2 S_2 = E_1$)
\[
S_2^1 S_3^1 E_1 S_3 S_2 = E_3 \\
S_2^1 S_3^1 E_2 S_3 S_2 = \mathbb{I} \otimes E_1 \otimes G_3 + \mathbb{I} \otimes E_1 \otimes c^\dagger (Z^2 + V^2) cc^\dagger c + \mathbb{I} \otimes E_2 \otimes Y^2 cc^\dagger \\
- i \mathbb{I} \otimes E_+ \otimes c^\dagger (Z + i V) cc^\dagger Y + i \mathbb{I} \otimes Y cc^\dagger (Z - i V) c \\
S_2^1 S_3^1 E_3 S_3 S_2 = \mathbb{I} \otimes E_1 \otimes Y^2 c + \mathbb{I} \otimes E_2 \otimes (Z^2 + V^2) \{cc^\dagger\}^2 \\
+ i \mathbb{I} \otimes E_+ \otimes c^\dagger (Z + i V) cc^\dagger Y - i \mathbb{I} \otimes E_- \otimes Y cc^\dagger (Z - i V) c. \quad (D.8)
\]
The result is
\[
\Delta E_4 = 2(\mu + \delta) E_3 \\
- 2 \mu \mathbb{I} \otimes \left[ E_1 \otimes G_3 + E_1 \otimes c^\dagger (Z^2 + V^2) cc^\dagger c + E_2 \otimes Y^2 cc^\dagger \right] \\
- 2 \delta \mathbb{I} \otimes \left[ E_1 \otimes c^\dagger Y^2 c + E_2 \otimes (Z^2 + V^2) \{cc^\dagger\}^2 \right] \\
+ 2 i (\mu - \delta) \mathbb{I} \otimes \left[ E_+ \otimes c^\dagger (Z + i V) cc^\dagger Y - E_- \otimes Y cc^\dagger (Z - i V) c \right]. \quad (D.9)
\]
Using $G_3 + c^\dagger X c = \mathbb{I}$ and $cc^\dagger X = \mathbb{I}$ this can be written as
\[
\Delta E_4 = - \Delta_2 + 2(\mu - \delta) \mathbb{I} \otimes E_1 \otimes c^\dagger Y^2 c - 2(\mu - \delta) \mathbb{I} \otimes E_2 \otimes y^2 cc^\dagger \\
+ 2 i (\mu - \delta) \mathbb{I} \otimes \left[ E_+ \otimes c^\dagger (Z + i V) cc^\dagger Y - E_- \otimes Y cc^\dagger (Z - i V) c \right]. \quad (D.10)
\]
This is (39).

References

[1] H. E. D. Scovil and E. O. Schulz-DuBois, Three-Level Masers as Heat Engines, Phys. Rev. Lett. 2, 262–263 (1959).
[2] J. E. Geusic, E. O. Schulz-DuBois, and H. E. D. Scovil, Quantum Equivalent of the Carnot Cycle, Phys. Rev. 156, 343–351 (1967).
[3] C.M. Bender, D.C. Brody and B.K. Meister, Quantum mechanical Carnot engine, J. Phys. A33, 4427–4436 (2000).
[4] M. O. Scully, M. S. Zubairy, G.S. Agarwal, and H. Walther, Extracting Work from a Single Heat Bath via Vanishing Quantum Coherence, Science 299, 862–864 (2003).
[5] T. D. Kieu, The Second Law, Maxwell’s Demon, and Work Derivable from Quantum Heat Engines, Phys. Rev. Lett. 93, 140403 (2004).
[6] T. D. Kieu, Quantum heat engines, the second law and Maxwell’s daemon, Eur. Phys. J. D39, 115–128 (2006).
[7] Y. Rezek and R. Kosloff, Irreversible performance of a quantum harmonic heat engine, New J. Phys. 8, 83 (2006).
[8] H. T. Quan, Yu-xi Liu, C. P. Sun, and F. Nori, Quantum thermodynamic cycles and quantum heat engines, Phys. Rev. E 76, 031105 (2007)
Quantum cooling by unitary transformations

[9] H. T. Quan, *Quantum thermodynamic cycles and quantum heat engines. II*, Phys. Rev. E**79**, 041129 (2009).

[10] P. Salamon, K. H. Hoffmann, Y. Rezek and R. Kosloff, *Maximum work in minimum time from a conservative quantum system*, Phys. Chem. Chem. Phys. **11**, 1027–1032 (2009).

[11] M. Esposito, R. Kawai, K. Lindenberg, C. Van den Broeck, *Efficiency at Maximum Power of Low-Dissipation Carnot Engines*, Phys. Rev. Lett. **105**, 150603 (2010).

[12] M. Esposito, R. Kawai, K. Lindenberg, C. Van den Broeck, *Quantum-dot Carnot engine at maximum power*, Phys. Rev. E**81**, 041106 (2010).

[13] J. Naudts and W. O’Kelly de Galway, *On the BCH formula of Rezek and Kosloff*, Physica A**390**, 3317–3319 (2011).

[14] A. Kossakowski, *On quantum statistical mechanics of non-Hamiltonian systems*, Rep. Math. Phys. **3**, 247–274 (1972).

[15] G. Lindblad, *On the generators of quantum dynamical semigroups*, Commun. Math. Phys. **48**, 119–130 (1976).

[16] N. G. van Kampen, *A Soluble Model for Quantum Mechanical Dissipation*, J. Stat. Phys. **78**, 299–310 (1995).

[17] K. Winkler, F. Lang, G. Thalhammer, P. van der Straten, R. Grimm, J. H. Denschlag, *Coherent Optical Transfer of Feshbach Molecules to a Lower Vibrational State*, Phys. Rev. Lett. **98**, 043201 (2007).

[18] K. Bergmann, H. Theuer, B. W. Shore, *Coherent population transfer among quantum states of atoms and molecules*, Rev. Mod. Phys. **70**, 1003–1025 (1998).

[19] E. Kuznetsova, M. Gacesa, Ph. Pellegrini, S. F. Yelin, R. Côté *Efficient formation of ground-state ultracold molecules via STIRAP from the continuum at a Feshbach resonance*, New J. Phys. **11**, 055028 (2009).

[20] V.N. Vitanov, *Analytic model of a three-state system driven by two laser pulses on two-photon resonance*, J. Phys. B**31**, 709–725 (1998).

[21] K. Na, L.E. Reichl, *Nonlinear dynamics of ladder and lambda STIRAP*, Chaos, Solitons & Fractals **25**, 185–196 (2005).

[22] J. Naudts and W. O’Kelly de Galway, *Analytic solutions for a three-level system in a time-dependent field*, Physica D**240**, 542–545 (2011).

[23] M. Campisi, P. Talkner, and P. Hänggi, *Finite bath fluctuation theorem*, Phys. Rev. E**80**, 031145 (2009).

[24] M. Campisi, P. Talkner, and P. Hänggi, *Thermodynamics and fluctuation theorems for a strongly coupled open quantum system: an exactly solvable case*, J. Phys. A**42**, 392002 (2009).

[25] M. Esposito, K. Lindenberg, C. Van den Broeck, *Entropy production as correlation between system and reservoir*, New J. Phys. **12**, 013013 (2010).

[26] P. Talkner, P. Hänggi, and M. Morillo, *Microcanonical quantum fluctuation theorems*, Phys. Rev. E**77**, 051131 (2008).

[27] M. Esposito, R. Kawai, K. Lindenberg, C. Van den Broeck, *Finite-time thermodynamics for a single-level quantum dot*, Eur. Phys. Lett. **89**, 20003 (2010).