HOMOGENIZED DYNAMICS OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH DYNAMICAL BOUNDARY CONDITIONS

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Abstract. A microscopic heterogeneous system under random influence is considered. The randomness enters the system at physical boundary of small scale obstacles as well as at the interior of the physical medium. This system is modeled by a stochastic partial differential equation defined on a domain perforated with small holes (obstacles or heterogeneities), together with random dynamical boundary conditions on the boundaries of these small holes.

A homogenized macroscopic model for this microscopic heterogeneous stochastic system is derived. This homogenized effective model is a new stochastic partial differential equation defined on a unified domain without small holes, with static boundary condition only. In fact, the random dynamical boundary conditions are homogenized out, but the impact of random forces on the small holes’ boundaries is quantified as an extra stochastic term in the homogenized stochastic partial differential equation. Moreover, the validity of the homogenized model is justified by showing that the solutions of the microscopic model converge to those of the effective macroscopic model in probability distribution, as the size of small holes diminishes to zero.

Dedicated to Giuseppe Da Prato on the occasion of his 70th birthday

Date: September 13, 2006 submitted; March 2, 2007 accepted.

2000 Mathematics Subject Classification. Primary 60H15; Secondary 86A05, 34D35.

Key words and phrases. Stochastic PDEs, random dynamical boundary condition, two-scale convergence, effective macroscopic model, stochastic homogenization, convergence in probability distribution.

A part of this work was done while J. Duan was visiting the Ennio De Giorgi Center of Mathematical Research (www.crm.sns.it), Pisa, Italy. J. Duan would like to thank Giuseppe Da Prato and Franco Flandoli for their financial support and hospitality. This work was partly supported by the NSF Grants DMS-0209326 & OCE-0620539 and the Outstanding Overseas Chinese Scholars Fund of the Chinese Academy of Sciences.
1. Introduction

Stochastic effects in the multiscale modeling of complex phenomena have drawn more and more attention in many areas such as material science [10], climate dynamics [27], chemistry and biology [21, 51]. Stochastic partial differential equations (SPDEs or stochastic PDEs) arise naturally as mathematical models for multiscale systems under random influences. The need to include stochastic effects in mathematical modeling of some realistic complex behaviors has become widely recognized in science and engineering. But implementing this approach poses some challenges both in mathematical theory and computation [44, 16, 26, 27, 51, 43]. The addition of stochastic terms to mathematical models has led to interesting new mathematical problems at the interface of dynamical systems, partial differential equations, scientific computing, and probability theory.

Sometimes, noise affects a complex system not only inside the physical medium but also at the physical boundary. Such random boundary conditions arise in the modeling of, for example, the air-sea interactions on the ocean surface [42], heat transfer in a solid in contact with a fluid [31], chemical reactor theory [32], and colloid and interface chemistry [56]. Randomness in such boundary conditions are often due to various fluctuations.

In this paper we consider a microscopic heterogeneous system, modeled by a SPDE with random dynamical boundary condition, in a medium which exhibits small-scale spatial heterogeneities or obstacles. One example of such microscopic systems of interest is composite materials containing microscopic holes (i.e., cavities), under the impact of random fluctuations in the domain and on the surface of the holes [28, 35]. A motivation for such a model is based on the consideration that the interaction between the atoms of the different compositions in a composite material causes the thermal noise when the scale of the heterogeneity scale is small. A similar consideration appears also in a microscopic stochastic lattice model [6] for a composite material. Here the microscopic structure is perturbed by random effect and the complicated interactions on the boundary of the holes is dynamically and randomly evolving. The heterogeneity scale is assumed to be much smaller than the macroscopic scale, i.e., we assume that the heterogeneities are evenly distributed. From a mathematical point of view, one can assume that microscopic heterogeneities (holes) are periodically placed in the media. This spatial periodicity with small period can be represented by a small positive parameter $\epsilon$ (i.e., the period). In
fact we work on the spatial domain \( D_\epsilon \), obtained by removing \( S_\epsilon \), a collection of small holes of size \( \epsilon \), periodically distributed in a fixed domain \( D \). When taking \( \epsilon \to 0 \), the holes inside domain \( D \) are smaller and smaller and their numbers goes to \( \infty \). This signifies that the heterogeneities are finer and finer.

In other words, we consider a spatially extended system with state variable \( u_\epsilon \), where stochastic effects are taken into account both in the model equation and in the boundary conditions, defined on a domain perforated with small scale holes. Specifically, we study a class of stochastic partial differential equations driven by white noise on a perforated domain with random dynamical boundary conditions:

\[
\begin{align*}
du_\epsilon(t,x) &= \left[ \Delta u_\epsilon(t,x) + f(t,x,u_\epsilon,\nabla u_\epsilon) \right] dt + g_1(t,x)dW_1(t,x) \\
in \qquad D_\epsilon \times (0,T),
\end{align*}
\]

\[
\begin{align*}
\epsilon^2 du_\epsilon(t,x) &= \left[ - \frac{\partial u_\epsilon(t,x)}{\partial \nu_\epsilon} - \epsilon b u_\epsilon(t,x) \right] dt + \epsilon g_2(t,x)dW_2(t,x) \\
on \qquad \partial S_\epsilon \times (0,T).
\end{align*}
\]

This model will be described in more detail in the next section.

The goal is to derive a homogenized effective equation, which is a new stochastic partial differential equation (see Theorems 5.1, 6.1, 6.2 and 6.3), for the above microscopic heterogenous system, by homogenization techniques in the sense of probability. Homogenization theory has been developed for deterministic systems, and compactness discussion for the solutions \( \{u_\epsilon\}_\epsilon \) in some function space is a key step in various homogenization approaches [12]. However, due to the appearance of the stochastic terms in the above microscopic system considered in this paper, such compactness result does not hold for this stochastic system. Fortunately the compactness in the sense of probability, that is, the tightness of the distributions for \( \{u_\epsilon\}_\epsilon \), still holds. So one appropriate way is to homogenize the stochastic system in the sense of probability. It is shown that the solution \( u_\epsilon \) of the microscopic or heterogeneous system converges to that of the macroscopic or homogenized system as \( \epsilon \downarrow 0 \) in probability distribution. This means that the distribution of \( \{u_\epsilon\}_\epsilon \) weakly converges, in some appropriate space, to the distribution of a stochastic process which solves the macroscopic effective equation.

It is interesting to note that, for the above system with random dynamical boundary conditions, the random force on the boundary of microscopic scale holes leads, in the homogenization limit, to a random force distributed all over the physical domain \( D \), even when the model equation itself contains no stochastic influence in the domain; see Remark 5.2 in §5. We could also say that the impact of small scale random dynamical boundary conditions
is quantified or carried over to the homogenized model as an extra random forcing. Therefore, the homogenized effective model is a new stochastic partial differential equation, defined on a unified domain without holes.

In the present paper, the two-scale convergence techniques are employed in our approach. Two-scale convergence method is an important method in homogenization theory which is a formal mathematic procedure for deriving macroscopic models from microscopic systems. Two-scale convergence method contains more information than the usual weak convergence method; see [2] or §. Moreover by use of the two-scale convergence, we do not need the extension operator as introduced in [13].

Partial differential equations (PDEs) with dynamical boundary conditions have been studied recently in, for example, [1, 20, 22, 23, 25, 47] and reference therein. The parabolic SPDEs with noise in the static Neumann boundary conditions have also been considered in [16, 17, 36]. In [11], the authors have studied well-posedness of the SPDEs with random dynamical boundary conditions. One of the present authors, with collaborators, has considered [18, 57] dynamical issues of SPDEs with random dynamical boundary conditions.

The homogenization problem for the deterministic systems defined in perforated domains or in other heterogeneous media has been investigated in, for example, [8, 39, 40, 46, 48] for heat transfer in a composite material, [8, 13, 15] for the wave propagation in a composite material and [34, 38] for the fluid flow in a porous media. For a systematic introduction in homogenization in the deterministic context, see [12, 28, 45, 35]. In [47], the effective macroscopic dynamics of a deterministic partial differential equation with deterministic dynamical boundary condition on the microscopic heterogeneity boundary is studied.

Recently there are also works on homogenization of partial differential equations (PDEs) in the random context; see [29, 37, 41, 28] for PDEs with random coefficients, and [7, 58, 59, 28] for PDEs in randomly perforated domains. A basic assumption in these works is the ergodic hypotheses on the random coefficients, for the passing of the limit as $\epsilon \rightarrow 0$. Note that the microscopic models in these works are partial differential equations with random coefficients, so-called random partial differential equations (random PDEs)[9, 30, 41, 44, 29, 53], instead of stochastic PDEs — PDEs with noises — in the present paper; see also [52]. Another novelty in the present paper is that the microscopic system is under the influence of random dynamical boundary conditions.

We first consider the linear system and then present results about nonlinear systems with special nonlinear terms. This paper is organized as follows. The problem formulation is stated in §. Section 3 is devoted to basic properties of
the microscopic heterogeneous system, and some knowledge to be used in our approach is introduced in §4. The homogenized effective macroscopic model for the linear system is derived in §5. In the last section, homogenized effective macroscopic models are obtained for three types of nonlinear systems.

2. Problem formulation

Let the physical medium $D$ be an open bounded domain in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial D$, and let $\epsilon > 0$ be a small parameter. Let $Y = [0, l_1] \times [0, l_2] \times \cdots \times [0, l_n]$ be a representative elementary cell in $\mathbb{R}^n$ and $S$ an open subset of $Y$ with smooth boundary $\partial S$, such that $\overline{S} \subset Y$. The elementary cell $Y$ and the small cavity or hole $S$ inside it are used to model small scale obstacles or heterogeneities in a physical medium $D$. Write $l = (l_1, l_2, \ldots, l_n)$.

Define $\epsilon S = \{\epsilon y : y \in S\}$. Denote by $S_{\epsilon, k}$ the translated image of $\epsilon S$ by $kl$, $k \in \mathbb{Z}^n$, $kl = (k_1l_1, k_2l_2, \ldots, k_n l_n)$. And let $S_{\epsilon}$ be the set all the holes contained in $D$ and $D_{\epsilon} = D \setminus S_{\epsilon}$. Then $D_{\epsilon}$ is a periodically perforated domain with holes of the same size as period $\epsilon$. We remark that the holes are assumed to have no intersection with the boundary $\partial D$, which implies that $\partial D_{\epsilon} = \partial D \cup \partial S_{\epsilon}$. See Fig. 1 for the case $n = 2$. This assumption is only needed to avoid technicalities and the results of our paper will remain valid without this assumption.

In the sequel we use the notations $Y^* = Y \setminus \overline{S}$, $\vartheta = \frac{|Y^*|}{|Y|}$ with $|Y|$ and $|Y^*|$ the Lebesgue measure of $Y$ and $Y^*$ respectively. Denote by $\chi$ the indicator function, which takes value 1 on $Y^*$ and value 0 on $Y \setminus Y^*$. In particular, let $\chi_A$ be the indicator function of $A \subset \mathbb{R}^n$. Also denote by $\tilde{v}$ the
zero extension to the whole $D$ for any function $v$ defined on $D_\epsilon$:

$$ \tilde{v} = \begin{cases} v & \text{on } D_\epsilon, \\ 0 & \text{on } S_\epsilon. \end{cases} $$

Now for $T > 0$ fixed final time, we consider the following Itô type nonautonomous stochastic partial differential equation defined on the perforated domain $D_\epsilon$ in $\mathbb{R}^n$

$$ du_\epsilon(t, x) = \left[ \Delta u_\epsilon(t, x) + f(t, x, u_\epsilon, \nabla u_\epsilon) \right] dt + g_1(t, x) dW_1(t, x) \quad (2.1) $$

$$ \epsilon^2 du_\epsilon(t, x) = \left[ -\frac{\partial u_\epsilon(t, x)}{\partial \nu_\epsilon} - \epsilon b u_\epsilon(t, x) \right] dt + \epsilon g_2(t, x) dW_2(t, x) \quad (2.2) $$

on $\partial S_\epsilon \times (0, T)$,

$$ u_\epsilon(t, x) = 0 \quad \text{on } \partial D \times (0, T), \quad (2.3) $$

$$ u_\epsilon(0, x) = u_0(x) \quad \text{in } D_\epsilon, \quad (2.4) $$

where $b$ is a real constant, $f : [0, T] \times D \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies some property which will be described later and $\nu_\epsilon$ is the exterior unit normal vector on the boundary $\partial S_\epsilon$, $v_0 \in L^2(\partial S_\epsilon)$ and $u_0 \in L^2(D)$. Moreover, $W_1(t, x)$ and $W_2(t, x)$ are mutually independent $L^2(D)$ valued Wiener processes on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a canonical filtration $(\mathcal{F}_t)_{t \geq 0}$. Denote by $Q_1$ and $Q_2$ the covariance operators of $W_1$ and $W_2$ respectively. Here we assume that $g_i(t, x) \in \mathcal{L}(L^2(D))$, $i = 1, 2$ and that there is a positive constant $C_T$ independent of $\epsilon$ such that

$$ \|g_i(t, \cdot)\|_{L^2_{Q_i}}^2 := \sum_{j=1}^{\infty} \|g_i e_j\|_{L^2_{Q_i}(D)}^2 \leq C_T, \quad i = 1, 2, \quad t \in [0, T] \quad (2.5) $$

where $\{e_j\}_{j=1}^{\infty}$ are eigenvectors of operator $-\Delta$ on $D$ with Dirichlet boundary condition and they form an orthonormal basis of $L^2(D)$. Here $\mathcal{L}(L^2(D))$ denotes the space of bounded linear operators on $L^2(D)$ and $\mathcal{L}_{Q_i}^{Q_i} = \mathcal{L}_{Q_i}^{Q_i}(H)$ denotes the space of Hilbert-Schmidt operators related to the trace operator $Q_i$. We also denote by $\mathbf{E}$ the expectation operator with respect to $\mathbb{P}$.

Let $S$ be a Banach space and $S'$ be the strong dual space of $S$. We recall the definitions and some properties of weak convergence and weak* convergence.

**Definition 2.1.** A sequence $\{s_n\}$ in $S$ is said to converge weakly to $s \in S$ if $\forall s' \in S'$,

$$ \lim_{n \to \infty} (s', s_n)_{S', S} = (s', s)_{S', S} $$
which is written as \( s_n \to s \) weakly in \( S \). Note that \((s', s)\) denotes the value of the continuous linear functional \( s' \) at the point \( s \).

**Lemma 2.2.** (Eberlein-Shmulyan) Assume that \( S \) is reflexive and let \( \{s_n\} \) be a bounded sequence in \( S \). Then there exists a subsequence \( \{s_{n_k}\} \) and \( s \in S \) such that \( s_{n_k} \to s \) weakly in \( S \) as \( k \to \infty \). If all the weak convergent subsequence of \( \{s_n\} \) has the same limit \( s \), then the whole sequence \( \{s_n\} \) weakly converges to \( s \).

**Definition 2.3.** A sequence \( \{s'_n\} \) in \( S' \) is said to converge weakly* to \( s' \in S' \) if \( \forall s \in S \),
\[
\lim_{n \to \infty} (s'_n, s)_{S', S} = (s', s)_{S', S}
\]
which is written as \( s'_n \to s' \) weakly* in \( S' \).

**Lemma 2.4.** Assume that the dual space \( S' \) is reflexive and let \( \{s'_n\} \) be a bounded sequence in \( S' \). Then there exists a subsequence \( \{s'_{n_k}\} \) and \( s' \in S' \) such that \( s'_{n_k} \to s' \) weakly* in \( S' \) as \( k \to \infty \). If all the weakly* convergent subsequence of \( \{s'_n\} \) has the same limit \( s' \), then the whole sequence \( \{s'_n\} \) weakly* converges to \( s' \).

In the following, for a fixed \( T > 0 \), we always denote by \( C_T \) a constant independent of \( \epsilon \). And denote by \( D_T \) the set \([0, T] \times D\).

### 3. Basic properties of the microscopic model

In this section we will present some estimates for solutions of the microscopic model (2.1), and then discuss the tightness of the distributions of the solution processes in some appropriate space. We focus our argument in the case of linear microscopic systems, where the term \( f \) is independent of \( u_\epsilon \) and \( \nabla u_\epsilon \) and \( f(\cdot, \cdot) \in L^2(0, T; L^2(D)) \). Then we briefly extend this to the case of nonlinear microscopic systems with Lipschitz nonlinearities.

Define by \( H^1_\epsilon(D) \) the space of elements of \( H^1(D) \) which vanish on \( \partial D \). Denote by \( H^{-1}_\epsilon(D) \) the dual space of \( H^1_\epsilon(D) \) with the usual norm and let \( \gamma_\epsilon : H^1(D) \to L^2(\partial S_\epsilon) \) be the trace operator with respect to \( \partial S_\epsilon \) which is continuous \([49]\). We also denote that \( H^{\frac{1}{2}}(\partial S_\epsilon) = \gamma_\epsilon(H^1(D)) \) and let \( H^{-\frac{1}{2}}_\epsilon(D) \) be the dual space of \( H^{\frac{1}{2}}_\epsilon(D) \).

Introduce the following function spaces
\[
X^1_\epsilon = \{(u, v) \in H^1_\epsilon(D) \times H^{\frac{1}{2}}_\epsilon(\partial S_\epsilon) : v = \epsilon \gamma_\epsilon u\}
\]
and
\[
X^0_\epsilon = \{L^2(D) \times L^2_\epsilon(\partial S_\epsilon)\}
\]
with the usual product and norm. Define an operator $B_\epsilon$ on the space $H^1_\epsilon(D_\epsilon)$ as
\[ B_\epsilon u = \frac{\partial u}{\partial n} + \epsilon bu, \quad u \in H^1_\epsilon(D_\epsilon). \] (3.1)
Now we define the operator $A_\epsilon$ on $\mathcal{D}(A_\epsilon) = \{(u, v) \in X_\epsilon^1 : (-\Delta u, R_\epsilon B_\epsilon u) \in X_\epsilon^0\}$, where $R_\epsilon$ is the restriction to $\partial S_\epsilon$, as
\[ A_\epsilon z = (-\Delta u, \frac{1}{\epsilon} R_\epsilon B_\epsilon u), \quad z = (u, v) \in \mathcal{D}(A_\epsilon). \] (3.2)

Associated with the operator $A_\epsilon$, we introduce the bilinear form on $X_\epsilon^1$
\[ a_\epsilon(z, \tilde{z}) = \int_{D_\epsilon} \nabla u \nabla \tilde{u} dx + \epsilon b \int_{S_\epsilon} \gamma_\epsilon(u) \gamma_\epsilon(\tilde{u}) ds \] (3.3)
with $z = (u, v)$, $\tilde{z} = (\tilde{u}, \tilde{v}) \in X_\epsilon^1$. Notice that $|\gamma_\epsilon(u)|_{L^2(S_\epsilon)}^2 \leq C(S_\epsilon)|u|_{H^1_\epsilon(D_\epsilon)}^2$, we see that there is $M > 0$, independent of $\epsilon$, such that
\[ a_\epsilon(z, \tilde{z}) \leq M|u|_{H^1_\epsilon(D_\epsilon)}|\tilde{u}|_{H^1_\epsilon(D_\epsilon)} \]
and the following coercive property of $a$ holds
\[ a_\epsilon(z, z) \geq \tilde{\alpha}|z|_{X_\epsilon^1}^2 - \tilde{\beta}|z|_{X_\epsilon^0}^2, \quad z \in X_\epsilon^1 \] (3.4)
for some constants $\tilde{\alpha}$, $\tilde{\beta} > 0$ which are also independent of $\epsilon$. Write the $C_0$-semigroup generated by operator $-A_\epsilon$ as $S_\epsilon(t)$.

Then the system (2.1)-(2.4) can be rewritten as the following abstract stochastic evolutionary equation
\[ dz_\epsilon(t, x) = [-A_\epsilon z_\epsilon(t, x) + F_\epsilon(t, x)] dt + G_\epsilon(t, x) dW(t, x), \quad z_\epsilon(0) = z_0 \] (3.5)
where
\[ F_\epsilon(t, x) = (f(t, x), 0)^t, \quad G_\epsilon(t, x) dW(t) = (g_1(t, x) dW_1(t, x), g_2(t, x) dW_2(t, x))^t \]
and $z_0 = (u_0, v_0)$. And the solution of (3.5) can be written in the mild sense
\[ z_\epsilon(t) = S_\epsilon(t) z_0 + \int_0^t S_\epsilon(t - s) F_\epsilon(s) ds + \int_0^t S_\epsilon(t - s) G_\epsilon(s) dW(s). \] (3.6)

Moreover, the variational formulation is
\[ \int_0^T \int_{D_\epsilon} \dot{u}_\epsilon \varphi dx dt + \epsilon^2 \int_0^T \int_{\partial S_\epsilon} \nu_\epsilon \varphi dx dt + \epsilon b \int_0^T \int_{S_\epsilon} u_\epsilon \varphi dx dt = \] (3.7)
\[ - \int_0^T \int_{D_\epsilon} \nabla u_\epsilon \nabla \varphi dx dt + \int_0^T \int_{D_\epsilon} f \varphi dx dt + \int_0^T \int_{D_\epsilon} g_1 \varphi \dot{W}_1 dx dt + \epsilon \int_0^T \int_{\partial S_\epsilon} g_2 \varphi \dot{W}_2 dx dt \]
for $\varphi(t, x) \in C_0^\infty([0, T] \times D_\epsilon)$. Here $\dot{\epsilon}$ denotes $\epsilon^{\frac{d}{dt}}$.

For the well-posedness of system (3.5) we have the following result.
Theorem 3.1. (Global well-posedness of microscopic model) Assume that (2.5) holds for $T > 0$. If $z_0 = (u_0, v_0)$ is a $(\mathcal{F}_0, \mathcal{B}(X_0^0))$-measurable random variable, then the system (3.3) has a unique solution $z_\epsilon \in L^2(\Omega, C(0, T; X_\epsilon^0) \cap L^2(0, T; X_\epsilon^1))$, which is also a weak solution in the following sense.

\[
(z_\epsilon(t), \phi)_{X_\epsilon^1} = (z_0, \phi)_{X_\epsilon^1} + \int_0^t (-A_\epsilon z_\epsilon(s), \phi)_{X_\epsilon^1} ds + \int_0^t (F_\epsilon, \phi)_{X_\epsilon^1} ds + \int_0^t (G_\epsilon dW, \phi)_{X_\epsilon^1}
\]

for $t \in [0, T)$ and $\phi \in X_\epsilon^1$. Moreover if $z_0$ is independent of $W(t)$ with $\mathbb{E}|z_0|_{X_\epsilon^0}^2 < \infty$, then

\[
\mathbb{E}|z_\epsilon(t)|_{X_\epsilon^0}^2 + \mathbb{E} \int_0^t |z_\epsilon(s)|_{X_\epsilon^1}^2 ds \leq (1 + \mathbb{E}|z_0|_{X_\epsilon^0}^2)CT, \quad \text{for } t \in [0, T]
\]

and

\[
\mathbb{E}\left\{ \sup_{t \in [0, T]} |z_\epsilon(t)|_{X_\epsilon^0}^2 \right\} \leq (1 + \mathbb{E}|z_0|_{X_\epsilon^0}^2 + \mathbb{E} \int_0^T |z_\epsilon(s)|_{X_\epsilon^1}^2 ds)CT.
\]

Proof. By the assumption (2.5), we have

\[
\|G_\epsilon(t, x)\|_{L_2}^2 = \|g_1(t, x)\|_{L_2}^2 + \|g_2(t, x)\|_{L_2}^2 < \infty.
\]

Then the stochastic Fubini theorem \cite{16} yields the local existence of $z_\epsilon$. By applying the classical result \cite{16}, it can be verified that the local mild solution is also a weak solution.

Now we give the following a priori estimates which yields the existence of weak solution on $[0, T]$ for any $T > 0$.

Applying Itô formula to $|z_\epsilon|_{X_\epsilon^0}^2$, we derive

\[
d|z_\epsilon(t)|_{X_\epsilon^0}^2 + 2(A_\epsilon z_\epsilon, z_\epsilon)_{X_\epsilon^0} dt = 2(F_\epsilon(t, x), z_\epsilon)_{X_\epsilon^0} dt + 2(G_\epsilon(t, x)dW(t), z_\epsilon)_{X_\epsilon^0} + |G_\epsilon(t, x)|_{L_2}^2 dt.
\]

By the coercivity (3.4) of $a_\epsilon(\cdot, \cdot)$, integrating (3.11) with respect to $t$ yields

\[
|z_\epsilon(t)|_{X_\epsilon^0}^2 + 2\bar{\alpha} \int_0^t |z_\epsilon(s)|_{X_\epsilon^1}^2 ds \leq |z_0|_{X_\epsilon^0}^2 + |F_\epsilon|_{L_2([0, T]; X_\epsilon^0)}^2 + (2\bar{\alpha} + 1) \int_0^t |z_\epsilon(s)|_{X_\epsilon^0}^2 ds + 2 \int_0^t (G_\epsilon(s)dW(s), z_\epsilon(s))_{X_\epsilon^0} + \int_0^t |G_\epsilon(s)|_{L_2}^2 ds.
\]
Taking expectation on both sides of the above inequality yields
\[
\mathbb{E}|z_{\epsilon}(t)|_{X^0}^2 + 2\alpha \mathbb{E} \int_0^t |z_{\epsilon}(s)|_{X^1}^2 ds
\]
\[
\leq \mathbb{E}|z_0|_{X^0}^2 + |F_\epsilon|_{L^2(0,T;X^0)}^2 + (2\beta + 1) \int_0^t \mathbb{E}|z_{\epsilon}(s)|_{X^0}^2 ds + \int_0^t |G_{\epsilon}(s)|_{L^2_\sigma}^2 ds.
\]
Then the Gronwall lemma gives the estimate (3.9). Notice that, by Lemma 7.2 in [16],
\[
\mathbb{E}\sup_{t \in [0,T]} \int_0^t S_{\epsilon}(t-s)G_{\epsilon}(s,x)ds\leq C_T \int_0^T |G_{\epsilon}(s)|_{L^2_\sigma}^2 ds.
\]
Therefore by the assumption on f and (3.6) we have the estimate (3.10). The proof is hence complete. \(\square\)

By the above result and the definition of \(z_{\epsilon}\) we have the following corollary.

**Corollary 3.2.** Assume the conditions in Theorem 3.1. Then for \(t \in [0,T]\), we have
\[
\mathbb{E}\left(|u_{\epsilon}(t)|_{L^2(D_{\epsilon})}^2 + \epsilon^2 |\gamma u_{\epsilon}(t)|_{L^2(\partial D_{\epsilon})}^2\right) + \int_0^t \mathbb{E}\left(|u_{\epsilon}(s)|_{H^1(D_{\epsilon})}^2 + \epsilon^2 |\gamma u_{\epsilon}(s)|_{H^1(\partial D_{\epsilon})}^2\right)ds \leq (1 + \mathbb{E}|z_0|_{X^0}^2)C_T (3.12)
\]
and
\[
\mathbb{E}\left\{\sup_{t \in [0,T]} |u_{\epsilon}(t)|_{L^2(D_{\epsilon})}^2 + \epsilon^2 |\gamma u_{\epsilon}(t)|_{L^2(\partial D_{\epsilon})}^2\right\} \leq (1 + \mathbb{E}|z_0|_{X^0}^2)C_T. (3.13)
\]

We recall a probability concept. Let \(z\) be a random variable taking values in a Banach space \(S\), namely, \(z : \Omega \to S\). Denote by \(\mathcal{L}(z)\) the distribution (or law) of \(z\). In fact, \(\mathcal{L}(z)\) is a Borel probability measure on \(S\) defined as [16]
\[
\mathcal{L}(z)(A) = \mathbb{P}\{\omega : z(\omega) \in A\},
\]
for every event (i.e., a Borel set) \(A\) in the Borel \(\sigma\)-algebra \(\mathcal{B}(S)\), which is the smallest \(\sigma\)-algebra containing all open balls in \(S\).

As stated in [11] for the SPDE (2.1), we aim at deriving an effective equation in the sense of probability. A solution \(u_{\epsilon}\) may be regarded as a random variable taking values in \(L^2(0,T;L^2(D_{\epsilon}))\). So for a solution \(u_{\epsilon}\) of (2.1), defined on \([0,T]\), we focus on the behavior of distribution of \(u_{\epsilon}\) in \(L^2(0,T;L^2(D_{\epsilon}))\) as \(\epsilon \to 0\). For this purpose, the tightness [19] of distributions is necessary. Note that the function space changes with \(\epsilon\), which is a difficulty for obtaining the
tightness of distributions. Thus we will treat \( \{ \mathcal{L}(u_\epsilon) \}_{\epsilon > 0} \) as a family of distributions on \( L^2(0, T; L^2(D)) \) by extending \( u_\epsilon \) to the whole domain \( D \). Recall that the distribution (or law) of \( \tilde{u}_\epsilon \) is defined as:

\[
\mathcal{L}(\tilde{u}_\epsilon)(A) = \mathbb{P}\{ \omega : \tilde{u}_\epsilon(\cdot, \cdot, \omega) \in A \}
\]

for Borel set \( A \) in \( L^2(0, T; L^2(D)) \).

First we define the following spaces which will be used in our approach. For Banach space \( U \) and \( p > 1 \), define \( W^{1,p}(0, T; U) \) as the space of functions \( h \in L^p(0, T; U) \) such that

\[
|h|^p_{W^{1,p}(0, T; U)} = |h|^p_{L^p(0, T; U)} + \left| \frac{dh}{dt} \right|^p_{L^p(0, T; U)} < \infty.
\]

And for any \( \alpha \in (0, 1) \), define \( W^{\alpha,p}(0, T; U) \) as the space of function \( h \in L^p(0, T; U) \) such that

\[
|h|^p_{W^{\alpha,p}(0, T; U)} = |h|^p_{L^p(0, T; U)} + \int_0^T \int_0^T \frac{|h(t) - h(s)|^p_U}{|t - s|^{1+\alpha p}} ds dt < \infty.
\]

For \( \rho \in (0, 1) \), we denote by \( C^\rho(0, T; U) \) the space of functions \( h : [0, T] \to X \) that are Hölder continuous with exponent \( \rho \).

**Theorem 3.3. (Tightness of distributions)** Assume that \( z_0 = (u_0, v_0) \) be a \( (\mathcal{F}_0, \mathcal{B}(X_0^d)) \)-measurable random variable which is independent of \( W(t) \) with \( \mathbb{E}|z_0|^2_{X_0^d} < \infty. \) Then for any \( T > 0, (\mathcal{L}(\tilde{u}_\epsilon))_\epsilon \), the distributions of \((\tilde{u}_\epsilon)_\epsilon\), is tight in \( L^2(0, T; L^2(D)) \cap C(0, T; H^{-1}(D)) \).

**Proof.** Denote the projection \((u, v) \to u \) by \( P \). By the result of Corollary 3.2

\[
\mathbb{E}|u_\epsilon|^2_{L^2(0, T; H^1(D))} \leq C_T.
\]

Write \( z_\epsilon(t) \) as

\[
z_\epsilon(t) = z_\epsilon(0) - \int_0^t A_\epsilon z_\epsilon(s) ds + \int_0^t F_\epsilon(s, x) ds + \int_0^t G_\epsilon(s, x) dW(s).
\]

Then by (3.3) and (3.8), when \((h, 0) \in X_1^d\), we have the following estimate, for some positive constant \( C > 0 \) independent of \( \epsilon \)

\[
\left| \left( - P \int_0^t A_\epsilon z_\epsilon(s) ds + P \int_0^t F_\epsilon(s, x) ds, h \right)_{L^2(D_\epsilon)} \right| \\
\leq \left| \int_0^t a(Pz_\epsilon(s), h) ds \right| + \left| \int_0^t (f(s, x), h)_{L^2(D_\epsilon)} ds \right| \\
\leq C \left( \int_0^t |u_\epsilon(s)|_{H^1(D_\epsilon)} ds + \int_0^t |f(s)|_{L^2(D)} ds \right) |h|_{H^0_0(D_\epsilon)}.
\]
Thus we have
\[ E \left| -P \int_0^t A_\epsilon z_\epsilon(s)ds + P \int_0^t F_\epsilon(s, x)ds \right|_{W^{1,2}(0, T; H^{-1}(D_\epsilon))}^2 \leq C_T. \]  
(3.15)

Let \( M_\epsilon(s, t) = \int_s^t G_\epsilon(s, x)dW(s) \). By Lemma 7.2 of [16] and the Hölder inequality, we have
\[ E|PM_\epsilon(s, t)|_{L^2(D_\epsilon)}^4 \leq E|PM_\epsilon(s, t)|_{L^2(D)}^4 \leq cE\left( \int_s^t |g_1(\tau)|_{L^2_\epsilon}^4 d\tau \right)^2 \]
\[ \leq K(t - s) \int_s^t E|g_1(\tau)|_{L^2_\epsilon}^4 d\tau \]
\[ \leq K'(t - s)^2 \]
for \( t \in [s, T] \), and for positive constants \( K \) and \( K' \) independent of \( \epsilon, s \) and \( t \). Therefore
\[ E \int_0^T |PM_\epsilon(0, t)|_{L^2(D_\epsilon)}^4 dt \leq C_T \]
(3.16)
and for \( \alpha \in \left( \frac{1}{4}, \frac{1}{2} \right) \)
\[ E \int_0^T \int_0^T \frac{|PM_\epsilon(0, t) - PM_\epsilon(0, s)|_{L^2(D_\epsilon)}^4}{|t - s|^{1+4\alpha}} dsdt \leq C_T. \]  
(3.17)

Combining the estimates (3.14)-(3.17) with the Chebyshev inequality [16, 19], it is clear that for any \( \delta > 0 \) there is a bounded set
\[ K_\delta \subset \mathcal{X} \]
with \( \mathcal{X} = L^2(0, T; H^1_\epsilon(D)) \cap (W^{1,2}(0, T; H^{-1}(D)) + W^{\alpha,4}(0, T; L^2(D))) \), such that
\[ \mathbb{P}\{\bar{u}_\epsilon \in K_\delta\} > 1 - \delta. \]

Moreover by the compact embedding
\[ L^2(0, T; H^1(D)) \cap W^{1,2}(0, T; H^{-1}(D)) \subset L^2(0, T; L^2(D)) \cap C(0, T; H^{-1}(D)) \]
and
\[ L^2(0, T; H^1(D)) \cap W^{\alpha,4}(0, T; L^2(D)) \subset L^2(0, T; L^2(D)) \cap C(0, T; H^{-1}(D)), \]
we conclude that \( K_\delta \) is compact in \( L^2(0, T; L^2(D)) \cap C(0, T; H^{-1}(D)) \). Thus \( \{L(\bar{u}_\epsilon)\}_\epsilon \) is tight in \( L^2(0, T; L^2(D)) \cap C(0, T; H^{-1}(D)) \).

The proof is complete. \( \square \)

**Remark 3.4.** When \( f = f(t, x, u_\epsilon) \) is nonlinear (i.e., it depends on \( u_\epsilon \)) but is also globally Lipschitz in \( u_\epsilon \), the results in Theorem 3.1 and Corollary 3.2 still hold. For example, see [11] for such SPDEs with stochastic dynamical boundary conditions. Moreover, by the Lipschitz property, we have \( |f(t, x, u_\epsilon)|_{L^2(D)} \leq \)
Hence a similar analysis as in the proof of Theorem 3.3 yields the tightness of the distribution for $u_\epsilon$ in this globally Lipschitz nonlinear case. This fact will be used in the beginning of §6 to get the homogenized effective model when $f = f(t, x, u_\epsilon)$ is globally Lipschitz nonlinear.

In fact, in §6, we will also derive homogenized effective models for three types of nonlinearities $f = f(t, x, u_\epsilon)$ that are not globally Lipschitz in $u_\epsilon$.

4. Two-scale convergence and some preliminary results

In this section we present some basic results about the two-scale convergence. In the following we denote by $C^\infty_{\text{per}}(Y)$ the space of infinitely differentiable functions in $\mathbb{R}^n$ that are periodic in $Y$. We also denote by $L^2_{\text{per}}(Y)$ or $H^1_{\text{per}}(Y)$ the completion of $C^\infty_{\text{per}}(Y)$, in the usual norm of $L^2(Y)$ or $H^1(Y)$, respectively. We also introduce the space $H^1_{\text{per}}(Y)/\mathbb{R}$, which is the space of the equivalent classes of $u \in H^1_{\text{per}}(Y)$ under the following equivalent relation

$$u \sim v \Leftrightarrow u = v + \text{constant}.$$ 

Definition 4.1. A sequence of functions $u_\epsilon(t, x)$ in $L^2(D_T)$ is said to be two-scale convergent to a limit $u(t, x, y) \in L^2(D_T \times Y)$, if for any function $\varphi(t, x, y) \in C^\infty_0(D_T, C^\infty_{\text{per}}(Y))$,

$$\lim_{\epsilon \to 0} \int_{D_T} u_\epsilon(t, x) \varphi(t, x, \frac{x}{\epsilon}) dx dt = \frac{1}{|Y|} \int_D \int_Y u_0(t, x, y) \varphi(t, x, y) dy dx dt.$$ 

This two-scale convergence is written as $u_\epsilon \overset{2-s}{\to} u$.

The following result ensures the existence of two-scale limit and for the proof see [2,12].

Lemma 4.2. Let $u_\epsilon$ be a bounded sequence in $L^2(D_T)$. Then there exist a function $u \in L^2(D_T \times Y)$ and a subsequence $u_{\epsilon_k}$ with $\epsilon_k \to 0$ as $k \to \infty$ such that $u_{\epsilon_k}$ two-scale converges to $u$.

Remark 4.3. Taking $\varphi$ independent of $y$ in the definition of two-scale convergence, then $u_\epsilon \overset{2-s}{\to} u$ implies that $u_\epsilon$ weakly converges to its spatial average:

$$u_\epsilon(t, x) \rightharpoonup \bar{u}(t, x) = \frac{1}{|Y|} \int_Y u(t, x, y) dy.$$ 

So, we see that, for a given bounded sequence $L^2(D_T)$, the two-scale limit $u(t, x, y)$ contains more information than the weak limit $u(t, x)$: $u$ gives some knowledge on the periodic oscillations of $u_\epsilon$, while $\bar{u}$ is just the average with respect to $y$. Another advantage of the usage of two-scale convergence is that...
we do not need an extending operator such as in \[13, 15\] in the homogenization procedure. For more properties of two-scale convergence we refer to \[2\].

The following result is useful when considering two-scale convergence of the product of two convergent sequences, see \[2, 12\].

**Lemma 4.4.** Let \(v_\epsilon\) be a sequence in \(L^2(D_T)\) that two-scale converges to a limit \(v(x, y) \in L^2(D_T \times Y)\). Further assume that

\[
\lim_{\epsilon \to 0} \int_{D_T} |v_\epsilon(t, x)|^2 dx dt = \frac{1}{|Y|} \int_{D_T} \int_Y |v(t, x, y)|^2 dy dx dt. \tag{4.1}
\]

Then, for any sequence \(u_\epsilon \in L^2(D_T)\), which two-scale converges to a limit \(u \in L^2(D_T \times Y)\), we have the weak convergence of the product \(u_\epsilon v_\epsilon\):

\[
u_\epsilon v_\epsilon \rightharpoonup \frac{1}{|Y|} \int_Y u(\cdot, \cdot, y)v(\cdot, \cdot, y)dy, \quad \text{as} \ \epsilon \to 0 \ \text{in} \ L^2(D_T).
\]

**Remark 4.5.** Condition (4.1) always holds for a sequence of functions \(\varphi(t, x, x/\epsilon)\), with \(\varphi(t, x, y) \in L^2(D_T; C_{per}(Y))\). Such functions \(v_\epsilon\) are called admissible test functions. With the additional condition (4.1), the two-scale convergence of \(v_\epsilon\) is also called strong two-scale convergence \[2\].

Let \(u_\epsilon\) be a sequence of functions defined on \([0, T] \times D_\epsilon\) which is bounded in \(L^2(0, T; H^1_\epsilon(D_\epsilon))\). Then we have the following result concerning the two-scale limit of the bounded sequences \(\tilde{u}_\epsilon\) and \(\tilde{\nabla} u_\epsilon\); for the proof see \[2\].

**Lemma 4.6.** There exist \(u(t, x) \in H^1_0(D_T), u_1(t, x, y) \in L^2(D_T; H^1_{per}(Y))\) and a subsequence \(u_{\epsilon_k}\) with \(\epsilon_k \to 0\) as \(k \to \infty\), such that

\[
\tilde{u}_{\epsilon_k}(t, x) \overset{2-\delta}{\rightharpoonup} \chi(y)u(t, x), \quad k \to \infty
\]

and

\[
\tilde{\nabla} u_{\epsilon_k} \overset{2-\delta}{\rightharpoonup} \chi(y)[\nabla u(t, x) + \nabla u_1(t, x, y)], \quad k \to \infty
\]

where \(\chi(y)\) is the indicator function of \(Y^*\) (which takes value 1 on \(Y^*\) and value 0 on \(Y \setminus Y^*\)).

Since we consider the dynamical boundary condition, the technique of transforming the surface integrals into the volume integrals is useful in our approach. For this we follow the method of \[55\] (see also \[14\]) for the nonhomogeneous Neumann boundary problem for an elliptic equation.

For \(h \in H^{-1/2}(\partial S)\) and \(Y\)-periodic, define

\[
\Lambda_h = \frac{1}{|Y^*|} \int_{\partial S} h(x)dx.
\]
Also define
\[
\lambda_h = \frac{1}{|Y|} \langle h, 1 \rangle_{H^{-1/2}, H^{1/2}} = \partial \Lambda_h.
\]
Thus, in particular \( \Lambda_1 = \frac{|\partial S|}{|Y|} \) and
\[
\lambda := \lambda_1 = \frac{|\partial S|}{|Y|},
\]
(4.2)
where \( | \cdot | \) denotes Lebesgue measure.

For \( h \in L^2(\partial S) \) and \( Y \)-periodic, define \( \lambda^\varepsilon_h \in H^{-1}(D) \) as
\[
\langle \lambda^\varepsilon_h, \varphi \rangle = \varepsilon \int_{\partial S} h(\frac{x}{\varepsilon})\varphi(x)dx, \quad \text{for} \quad \varphi \in H^1_0(D).
\]
Then we have the following result about the convergence of the integral on the boundary.

**Lemma 4.7.** Let \( \varphi_\varepsilon \) be a sequence in \( H^1_0(D) \) such that \( \varphi_\varepsilon \rightharpoonup \varphi \) weakly in \( H^1_0(D) \) as \( \varepsilon \to 0 \). Then
\[
\langle \lambda^\varepsilon_h, \varphi_\varepsilon \rangle_{| D_\varepsilon } \to \lambda_h \int_{D} \varphi dx, \quad \text{as} \quad \varepsilon \to 0.
\]

For the proof we refer to [14].

5. **Homogenized macroscopic model**

In this section we derive the effective macroscopic model for the original model (2.1), by the two-scale convergence approach. We first obtain a two-scale limiting model. Then the homogenized macroscopic model is obtained by exploiting the relation between weak limit and the two-scale limit.

By the proof of Theorem 3.3 for any \( \delta > 0 \) there is a bounded closed set \( K_\delta \subset X \) which is compact in \( L^2(0, T; L^2(D)) \) such that
\[
P\{ \tilde{u}_\varepsilon \in K_\delta \} > 1 - \delta.
\]
Then the Prohorov theorem and the Skorohod embedding theorem ([16]) assure that for any sequence \( \{ \varepsilon_j \}_j \) with \( \varepsilon_j \to 0 \) as \( j \to \infty \), there exist subsequence \( \{ \varepsilon_{j(k)} \}_k \), random variables \( \{ u^*_j(\varepsilon_{j(k)}) \}_k \subset L^2(0, T; H_{\varepsilon_{j(k)}}) \) and \( u^* \in L^2(0, T; H) \) defined on a new probability space \( (\Omega^*, \mathcal{F}^*, \mathbb{P}^*) \), such that
\[
\mathcal{L}(\tilde{u}^*_{\varepsilon_{j(k)}}) = \mathcal{L}(\tilde{u}^*_{\varepsilon_{j(k)}})
\]
and
\[
\tilde{u}^*_{\varepsilon_{j(k)}} \to u^* \text{ in } L^2(0, T; H) \text{ as } k \to \infty,
\]
for almost all \( \omega \in \hat{\Omega} \). Moreover \( u^*_{\varepsilon_{j(k)}} \) solves system (2.1)-(2.4) with \( W \) replaced by Wiener process \( W^*_k \) defined on probability space \( (\Omega^*, \mathcal{F}^*, \mathbb{P}^*) \) with same
distribution as \( W \). In the following, we will determine the limiting equation (homogenized effective equation) that \( u^* \) satisfies and the limiting equation is independent of \( \epsilon \). After this is done we see that \( L(\tilde{u}_\epsilon) \) weakly converges to \( L(u^*) \) as \( \epsilon \downarrow 0 \).

For \( u_\epsilon \) in set \( K_\delta \), by Lemma 4.6 there is \( u(t, x) \in H^1_0(D_T) \) and \( u_1(t, x, y) \in L^2(D_T; H^1_{\text{per}}(Y)) \) such that

\[
\tilde{u}_{\epsilon, j}(t, x) \xrightarrow{2-s} \chi(y)u(t, x)
\]

and

\[
\nabla_x \tilde{u}_{\epsilon, j} \xrightarrow{2-s} \chi(y)[\nabla_x u(t, x) + \nabla_y u_1(t, x, y)].
\]

Then by Remark 4.3

\[
\tilde{u}_{\epsilon, j}(t, x) \rightharpoonup 1 \left| Y \right| \int_Y \chi(y)u(t, x)dy = \vartheta u(t, x), \text{ weakly in } L^2(D_T).
\]

In fact by the compactness of \( K_\delta \), the above convergence is strong in \( L^2(D_T) \). In the following, we will determine the limiting equation, which is a two-scale system that \( u \) and \( u_1 \) satisfy. Then the limiting equation (homogenized effective equation) that \( u_0 \) satisfies can be easily obtained by the relation between weak limit and the two-scale limit.

Define a new probability space \((\Omega_\delta, \mathcal{F}_\delta, \mathbb{P}_\delta)\) as

\[
\Omega_\delta = \{ \omega \in \Omega : \tilde{u}_\epsilon(\omega) \in K_\delta \},
\]

\[
\mathcal{F}_\delta = \{ F \cap \Omega_\delta : F \in \mathcal{F} \}
\]

and

\[
\mathbb{P}_\delta(F) = \frac{\mathbb{P}(F \cap \Omega_\delta)}{\mathbb{P}(\Omega_\delta)}, \text{ for } F \in \mathcal{F}_\delta.
\]

Denote by \( \mathbb{E}_\delta \) the expectation operator with respect to \( \mathbb{P}_\delta \). Now we restrict the system on the probability space \((\Omega_\delta, \mathcal{F}_\delta, \mathbb{P}_\delta)\).

Replace the test function \( \varphi \) in (3.7) by \( \varphi_\epsilon(t, x) = \phi(t, x) + \epsilon \Phi(t, x, x/\epsilon) \) with \( \phi(t, x) \in C^\infty_0(D_T) \) and \( \Phi(t, x, y) \in C^\infty_0(D_T; C^\infty_{\text{per}}(Y)) \). We will consider the terms in (3.7) respectively.

By the choice of \( \varphi_\epsilon \) and noticing that \( \chi_{D_\epsilon} \rightharpoonup \vartheta \), weakly* in \( L^\infty(D) \), we have

\[
\int_0^T \int_{D_\epsilon} f(t, x) \varphi_\epsilon(t, x)dxdt = \int_0^T \int_D \chi_{D_\epsilon} f(t, x) \varphi_\epsilon(t, x)dxdt
\]

\[
\rightarrow \vartheta \int_0^T \int_D f(t, x) \phi(t, x)dxdt, \text{ } \epsilon \rightarrow 0. \tag{5.1}
\]
And by the condition (2.5)

\[
\int_0^T \int_D g_1(t,x) \varphi_\epsilon(t,x) dxdW_1(t) = \int_0^T \int_D \chi_D g_1(t,x) \varphi_\epsilon(t,x) dxdW_1(t)
\]

\[
\rightarrow \vartheta \int_0^T \int_D g_1(t,x) \phi(t,x) dxdW_1(t), \quad \epsilon \to 0 \quad \text{in} \quad L^2(\Omega).
\]

(5.2)

Integrating by parts and noticing that \(\tilde{u}_\epsilon\) converges strongly to \(\vartheta u(t,x)\) in \(L^2(D_T)\),

\[
\int_0^T \int_D \dot{u}_\epsilon(t,x) \varphi_\epsilon(t,x) dxdt = - \int_0^T \int_D u_\epsilon(t,x) \dot{\varphi}_\epsilon(t,x) dxdt
\]

\[
= - \int_0^T \int_D \tilde{u}_\epsilon(t,x) \dot{\phi}(t,x) dxdt - \epsilon \int_0^T \int_D \tilde{u}_\epsilon(t,x) \dot{\Phi}(t,x, x/\epsilon) dxdt
\]

\[
\rightarrow - \int_0^T \int_D \vartheta u(t,x) \dot{\phi}(t,x) dxdt = \int_0^T \int_D \vartheta u(t,x) \dot{\phi}(t,x) dxdt. \quad (5.3)
\]

By the choice of \(\varphi_\epsilon\),

\[
\nabla_x \phi(t,x) + \nabla_y \Phi(t,x, x/\epsilon) \xrightarrow{\epsilon \to 0} \nabla_x \phi(t,x) + \nabla_y \Phi(t,x,y),
\]

and

\[
\lim_{\epsilon \to 0} \| \nabla_x \phi(t,x) + \nabla_y \Phi(t,x, x/\epsilon) \|_{L^2(D_T)^n} = \frac{1}{|Y|} \int_{D_T \times Y} \left| \nabla_x \phi(t,x) + \nabla_y \Phi(t,x,y) \right|^2 dydxdt.
\]

Hence by Theorem [4.4] we have

\[
\int_0^T \int_D \nabla u_\epsilon(t,x) \nabla \varphi_\epsilon(t,x) dxdt = \int_0^T \int_D \nabla u_\epsilon(t,x) \left( \nabla \phi(t,x) + \nabla_y \Phi(t,x, x/\epsilon) \right) dxdt
\]

\[
= \int_0^T \int_D \nabla \tilde{u}_\epsilon(t,x) \left( \nabla \phi + \nabla_y \Phi(t,x, x/\epsilon) \right) dxdt
\]

\[
\rightarrow \frac{1}{|Y|} \int_0^T \int_D \int_Y \chi(y) \left| \nabla_x u(x) + \nabla_y u_1(x,y) \right| \left| \nabla_x \phi(t,x) + \nabla_y \Phi(t,x,y) \right| dydxdt
\]

\[
= \frac{1}{|Y|} \int_0^T \int_D \int_Y \left[ \nabla_x u(x) + \nabla_y u_1(x,y) \right] \left[ \nabla_x \phi(t,x) + \nabla_y \Phi(t,x,y) \right] dydxdt. \quad (5.4)
\]
Now we consider the integrals on the boundary. First for a fixed $T > 0$, it is easy to see that
\[
\epsilon^2 \int_0^T \int_{\partial S} \dot{u}_\epsilon(t,x) \psi_\epsilon(t,x) dx dt = -\epsilon^2 \int_{\partial S} \int_0^T u_\epsilon(t,x) \dot{\psi}_\epsilon(t,x) dtdx = -\epsilon \left\langle \lambda_1^\epsilon, \int_0^T \dot{u}_\epsilon(t,x) \psi_\epsilon(t,x) dt \right\rangle \bigg|_{D_\epsilon} \rightarrow 0, \ \epsilon \rightarrow 0. \tag{5.5}
\]
And then
\[
eb \int_0^T \int_{\partial S} u_\epsilon(t,x) \psi_\epsilon(t,x) dx dt = \left\langle \lambda_1^\epsilon, \int_0^T \dot{u}_\epsilon(t,x) \psi_\epsilon(t,x) dt \right\rangle_{D_\epsilon} \rightarrow b\partial \lambda \int_0^T \int_D u(t,x) \phi(t,x) dx dt, \ \epsilon \rightarrow 0. \tag{5.6}
\]
By the same method as above and the condition (2.5) we have the limit of the stochastic integral on the boundary
\[
\epsilon \int_0^T \int_{\partial S} g_2(t,x) \psi_\epsilon(t,x) dx dW_2(t) \rightarrow \lambda \int_0^T \int_D g_2(t,x) \phi(t,x) dx dW_2(t), \ \epsilon \rightarrow 0, \ \text{in} \ L^2(\Omega). \tag{5.7}
\]
Combining the above analysis in (5.1)-(5.7) and by the density argument we have
\[
\partial \int_0^T \int_D \dot{u}(t,x) \phi(t,x) dx dt = -\frac{1}{|Y|} \int_0^T \int_D \int_{Y^*} \left[ \nabla_x u(x) + \nabla_y u_1(x,y) \right] \left[ \nabla_x \phi(t,x) + \nabla_y \Phi(t,x,y) \right] dx dt
\]
\[
- b\partial \lambda \int_0^T \int_D u(t,x) \phi(t,x) dx dt + \partial \int_0^T \int_D f(t,x) \phi(t,x) dx dt
\]
\[
+ \partial \int_0^T \int_D g_1(t,x) \phi(t,x) dx dW_1(t) + \lambda \int_0^T \int_D g_2(t,x) \phi(t,x) dx dW_2(t) \tag{5.8}
\]
for any $\phi \in H^1_0(D_T)$ and $\Phi \in L^2(D_T; H^1_{per}(Y)/\mathbb{R})$. Integrating by parts, we see that (5.8) is the variational problem of the following two-scale homogenized
\[
\dot{u} = -\text{div}_x A(\nabla_x u) - b\partial \lambda_1 u + \vartheta f \ dt + \vartheta g_1 dW_1(t) + \lambda g_2 dW_2(t), \tag{5.9}
\]

\[
[\nabla_x u + \nabla u_1] \cdot \boldsymbol{\nu} = 0, \quad \text{on } \partial Y^* - \partial Y \quad \tag{5.10}
\]

where \(\boldsymbol{\nu}\) is the unit exterior norm vector on \(\partial Y^* - \partial Y\) and

\[
A(\nabla_x u) = \frac{1}{|Y|} \int_{Y^*} [\nabla_x u(t, x, y) + \nabla_y u_1(t, x, y)] dy, \tag{5.11}
\]

with \(u_1\) satisfying the following integral equation

\[
\int_{Y^*} [\nabla_x u + \nabla_y u_1] \nabla y \Phi dy = 0, \quad u_1 \text{ is } Y - \text{periodic}, \tag{5.12}
\]

for any \(\Phi \in H^1_0(D_T; H^1_{per}(Y))\). The problem (5.12) has a unique solution for any fixed \(u\), and so \(A(\nabla_x u)\) is well-defined. Furthermore \(A(\nabla_x u)\) satisfies

\[
\langle A(\xi_1) - A(\xi_2), \xi_1 - \xi_2 \rangle_{L^2(D),L^2(D)} \geq \alpha \|\xi_1 - \xi_2\|^2_{L^2(D)} \tag{5.13}
\]

and

\[
|\langle A(\xi), \xi \rangle_{L^2(D),L^2(D)}| \leq \beta \|\xi\|^2_{L^2(D)} \tag{5.14}
\]

with some \(\alpha, \beta > 0\) and any \(\xi, \xi_1, \xi_2 \in H^1_0(D)\). For more detailed properties of \(A(\nabla u)\) and (5.12) we refer to [24]. Then by the classical theory of the SPDEs, \(\text{(5.9)-(5.10)}\) is well-posed.

In fact \(A(\nabla u)\) can be transformed to the classical homogenized matrix by

\[
u_1(t, x, y) = \sum_{i=1}^{n} \frac{\partial u(t, x)}{\partial x_i}(w_i(y) - \mathbf{e}_iy), \tag{5.15}
\]

where \(\{\mathbf{e}_i\}_{i=1}^{n}\) is the canonical basis of \(\mathbb{R}^n\) and \(w_i\) is the solution of the following cell problem (problem defined on the spatial elementary cell)

\[
\Delta_y w_i(y) = 0 \quad \text{in } Y^* \tag{5.16}
\]

\[
w_i - \mathbf{e}_iy \quad \text{is } Y - \text{periodic} \tag{5.17}
\]

\[
\frac{\partial w_i}{\partial \nu} = 0 \quad \text{on } \partial S. \tag{5.18}
\]

Then a simple calculation yields

\[
A(\nabla u) = A^* \nabla u
\]

with \(A^* = (A^*_{ij})\) being the classical homogenized matrix defined as

\[
A^*_{ij} = \frac{1}{|Y|} \int_{Y^*} w_i(y)w_j(y)dy. \tag{5.19}
\]
Then the above two-scale system (5.9) is equivalent to the following homogenized system,

\[
\vartheta du = \left[-\operatorname{div}_x(A^*\nabla_x u) - b\vartheta\lambda u + \vartheta f\right]dt + \vartheta g_1 dW_1(t) + \lambda g_2 dW_2(t). \tag{5.20}
\]

Let \( U(t,x) = \vartheta u(t,x) \). We thus have the limiting homogenized equation

\[
dU = \left[-\vartheta^{-1}\operatorname{div}_x(A^*\nabla_x U) - b\lambda U + \vartheta f\right]dt + \vartheta g_1 dW_1(t) + \lambda g_2 dW_2(t). \tag{5.21}
\]

And then \( u^* \), we have mentioned at the beginning of this section, satisfies (5.21) with \( W = (W_1,W_2) \) replaced by a Wiener process \( W^* \) with the same distribution as \( W \). By the classical existence result [16], the homogenized model (5.21) is well-posed. We formulate the main result of this section as follows.

**Theorem 5.1. (Homogenized macroscopic model)**

Assume that (2.5) holds. Let \( u_\epsilon \) be the solution of (2.1)-(2.4). Then for any fixed \( T > 0 \), the distribution \( \mathcal{L}(\tilde{u}_\epsilon) \) converges weakly to \( \mu \) in \( L^2(0,T;H) \) as \( \epsilon \downarrow 0 \), with \( \mu \) being the distribution of \( U \), which is the solution of the following homogenized effective equation

\[
dU = \left[-\vartheta^{-1}\operatorname{div}_x(A^*\nabla_x U) - b\lambda U + \vartheta f\right]dt + \vartheta g_1 dW_1(t) + \lambda g_2 dW_2(t), \tag{5.22}
\]

with the boundary condition \( U = 0 \) on \( \partial D \), the initial condition \( U(0) = u_0/\vartheta \) and the effective matrix \( A^* = (A_{ij}^*) \) being determined by (5.19). Moreover, the constant coefficient \( \vartheta = \frac{|Y^*|}{|Y|} \) is defined in the beginning of §2 and \( \lambda = \frac{|\partial S|}{|Y|} \) is defined in (4.2).

**Proof.** Noticing the arbitrariness of the choice of \( \delta \), this is direct result of the analysis of the first part in this section by the Skorohod theorem and the \( L^2(\Omega_\delta) \)-convergence of \( \tilde{u}_\epsilon \) on \( (\Omega_\delta,\mathcal{F}_\delta;\mathbb{P}_\delta) \). \( \square \)

**Remark 5.2.** It is interesting to note the following fact. Even when the original microscopic model equation (2.1) is a deterministic PDE (i.e., \( g_1 = 0 \)), the homogenized macroscopic model (5.22) is still a stochastic PDE, due to the impact of random dynamical interactions on the boundary of small scale heterogeneities.

**Remark 5.3.** For the macroscopic system (5.22), we see that the fast scale random fluctuations on the boundary is recognized or quantified in the homogenized equation, through the \( \mu_1 g_2 dW_2(t) \) term. The effect of random boundary evolution is thus felt by the homogenized system on the whole domain.
6. Homogenized macroscopic dynamics for nonlinear microscopic systems

In this section, we derive homogenized macroscopic model for the microscopic system (2.1)-(2.4), when the nonlinearity $f$ is either globally Lipschitz, or non-globally Lipschitz.

As Remark 3.4 has pointed out that if $f$ is a globally Lipschitz nonlinear function of $u$, all the estimates in §3 hold. In fact, similar results in §5 on homogenized model also hold. In fact for $f$ satisfying $f(t,x,u_1) = 0$ and

$$|f(t,x,u_1) - f(t,x,u_2)| \leq L|u_1 - u_2|$$

for any $t \in \mathbb{R}$, $x \in D$ and $u_1, u_2 \in \mathbb{R}$ with some positive constant $L$. Since $\tilde{u}_\epsilon \to \varphi u$ strongly in $L^2(0,T;L^2(D))$ and by the Lipschitz property of $f(t,x,u)$ with respect to $u$, $f(t,x,\tilde{u}_\epsilon(t,x)) \to f(t,x,u(t,x))$ strongly in $L^2(0,T;L^2(D))$. (5.1) still hold for $f(t,x,u_\epsilon)$. Then we can obtain the same effective macroscopic system as (5.22) with nonlinearity $f = f(t,x,U)$:

$$dU = [-\varphi^{-1}div_x(A^*\nabla_x U) - b\lambda U + \varphi f(t,x,U)]dt + \varphi g_1dW_1(t) + \lambda g_2dW_2(t).$$

(6.1)

For the rest of this section, we consider three types of nonlinear systems with $f$ being non-global-Lipschitz nonlinear function in $u_\epsilon$. The difficulty is at passing the limit $\epsilon \to 0$ in the nonlinear term. These three types of nonlinearity include: Polynomial nonlinearity; nonlinear term that is sublinear; and nonlinearity that contains a gradient term $\nabla u_\epsilon$. We look at these nonlinearities case by case, and only highlight the difference with the analysis in §5.

**Case 1: Polynomial nonlinearity**

First we suppose $f$ is in the following form

$$f(t,x,u) = -a(t,x)|u|^pu$$

(6.2)

with $0 < a_0 \leq a(t,x) \leq a_1$ for $t \in [0,\infty)$, $x \in D$. And $p$ satisfies the following condition

$$p \leq \frac{2}{n-2}, \text{ if } n \geq 3; \quad p \in \mathbb{R}, \text{ if } n = 2.$$  

(6.3)

For this case we need the following Weak convergence lemma from Lions [33].

Let $Q$ be a bounded region in $\mathbb{R} \times \mathbb{R}^n$. For any given functions $g_\epsilon$ and $g$ in $L^p(Q)$ ($1 < p < \infty$), if

$$|g_\epsilon|_{L^p(Q)} \leq C, \quad g_\epsilon \to g \text{ in } Q \text{ almost everywhere}$$

for some positive constant $C$, then $g_\epsilon \rightharpoonup g$ weakly in $L^p(Q)$. 

Notice that \( F_\epsilon(t, x, z_\epsilon) = (f(t, x, u_\epsilon), 0) \) and \( (F_\epsilon(t, x, z_\epsilon), z_\epsilon) \chi_\theta \leq 0 \), the results in Theorem 3.1 can be obtained by the same method as in the proof of Theorem 3.1. Moreover by the assumption \((6.3)\), \(|f(t, x, u_\epsilon)|_{L^2(D_T)} \leq CT\), which by the analysis of Theorem 3.3 yields the tightness of the distribution of \( \tilde{u}_\epsilon \).

Now we pass the limit \( \epsilon \to 0 \) in \( f(t, x, \tilde{u}_\epsilon) \). In fact, noticing that \( \tilde{u}_\epsilon \) converges strongly to \( \vartheta u \) in \( L^2(0, T; L^2(D)) \), by the above weak convergence lemma with \( g_\epsilon = f(t, x, \tilde{u}_\epsilon) \) and \( p = 2\), \( f(t, x, \tilde{u}_\epsilon) \) converges weakly to \( f(t, x, \vartheta u) \) in \( L^2(D_T) \).

Therefore by the analysis for linear system in \( \textsection 5 \) we have the following result.

**Theorem 6.1.** Assume that \((2.5)\) holds. Let \( u_\epsilon \) be the solution of \((2.1)-(2.4)\) with nonlinear term \( f \) being \((6.2)\). Then for any fixed \( T > 0 \), the distribution \( \mathcal{L}(\tilde{u}_\epsilon) \) converges weakly to \( \mu \) in \( L^2(0, T; H) \) as \( \epsilon \downarrow 0 \), with \( \mu \) being the distribution of \( U \), which is the solution of the following homogenized effective equation

\[
d U = \left[ -\partial^{-1} \text{div} \left(A^v \nabla U\right) - b\lambda U + \partial f(t, x, U)\right] dt + \partial g_1 dW_1(t) + \lambda g_2 dW_2(t),
\]

with the boundary condition \( U = 0 \) on \( \partial D \), the initial condition \( U(0) = u_0/\vartheta \) and the effective matrix \( A^v = (A^v)^t \) being determined by \((5.19)\). Moreover, the constant coefficient \( \vartheta = \frac{|V^*|}{|V|} \) is defined in the beginning of \( \textsection 2 \) and \( \lambda = \frac{|\partial S|}{|V|} \) is defined in \((4.2)\).

**Case 2: Nonlinear term that is sublinear**

More generally, we consider \( f : [0, T] \times D \times \mathbb{R} \to \mathbb{R} \) a measurable function which is continuous in \((x, \xi) \in D \times \mathbb{R} \) for almost all \( t \in [0, T] \) and satisfies

\[
[f(t, x, \xi_1) - f(t, x, \xi_2)][\xi_1 - \xi_2] \geq 0
\]

for \( t \geq 0, x \in D \) and \( \xi_1, \xi_2 \in \mathbb{R} \). Moreover, we assume that \( f \) is sublinear,

\[
|f(t, x, \xi)| \leq g(t)(1 + |\xi|), \quad \xi \in \mathbb{R}, \quad t \geq 0,
\]

where \( g \in L^\infty_{\text{loc}}(0, \infty) \). Notice that under the assumption \((6.5)\) and \((6.6)\), \( f \) may not be a Lipschitz function.

By the assumption \((6.6)\) we can also have the tightness of the distributions of \( \tilde{u}_\epsilon \) and also conclude that \( \chi_D f(t, x, \tilde{u}_\epsilon) \) two-scale converges to a function denoted by \( f_0(t, x, y) \in L^2(D_T \times Y) \). In the following we need to identity \( f_0(t, x, y) \).

Let \( \phi \in C^\infty_0(D_T) \) and \( \psi \in C^\infty_0(D_T; C^\infty_{\text{per}}(Y)) \). And for \( \kappa > 0 \) let

\[
\xi_\epsilon(t, x) = \phi(t, x) + \kappa \psi(t, x, \frac{x}{\epsilon}).
\]
Then by the assumption (6.5) we have

\[
0 \leq \int_0^T \int_{D_\varepsilon} \left[ f(t, x, u_\varepsilon) - f(t, x, \xi_\varepsilon) \right] [u_\varepsilon - \xi_\varepsilon] \, dxdt
\]

\[
= \int_{D_T} \chi\left( \frac{x}{\varepsilon} \right) \left[ f(t, x, \tilde{u}_\varepsilon) - f(t, x, \xi_\varepsilon) \right] [\tilde{u}_\varepsilon - \xi_\varepsilon] \, dxdt
\]

\[
\triangleq I_\varepsilon = I_{1,\varepsilon} - I_{2,\varepsilon} - I_{3,\varepsilon} + I_{4,\varepsilon}
\]

with

\[
I_{1,\varepsilon} = \int_{D_T} \chi\left( \frac{x}{\varepsilon} \right) f(t, x, \tilde{u}_\varepsilon) \tilde{u}_\varepsilon \, dxdt
\]

\[
\xrightarrow{\varepsilon \to 0} \frac{1}{|Y|} \int_{D_T} \int_Y f_0(t, x, y) \vartheta u(t, x) \, dydxt, \quad (6.8)
\]

\[
I_{2,\varepsilon} = \int_{D_T} \chi\left( \frac{x}{\varepsilon} \right) f(t, x, \tilde{u}_\varepsilon) \xi_\varepsilon \, dxdt
\]

\[
\xrightarrow{\varepsilon \to 0} \frac{1}{|Y|} \int_{D_T} \int_Y f_0(t, x, y) \left[ \phi(t, x) + \kappa \psi(t, x, y) \right] \vartheta u(t, x) \, dydxt, \quad (6.9)
\]

\[
I_{3,\varepsilon} = \int_{D_T} \chi\left( \frac{x}{\varepsilon} \right) f(t, x, \xi_\varepsilon) \tilde{u}_\varepsilon \, dxdt
\]

\[
\xrightarrow{\varepsilon \to 0} \frac{1}{|Y|} \int_{D_T} \int_Y \chi(y) f(t, x, \phi(t, x) + \kappa \psi(t, x, y)) \vartheta u(t, x) \, dydxt,
\]

\[
\quad \times \left[ \phi(t, x) + \kappa \psi(t, x, y) \right] \, dydxt, \quad (6.10)
\]

and

\[
I_{4,\varepsilon} = \int_{D_T} \chi\left( \frac{x}{\varepsilon} \right) f(t, x, \xi_\varepsilon) \xi_\varepsilon \, dxdt
\]

\[
\xrightarrow{\varepsilon \to 0} \frac{1}{|Y|} \int_{D_T} \int_Y \chi(y) f(t, x, \phi(t, x) + \kappa \psi(t, x, y)) \times
\]

\[
\left[ \phi(t, x) + \kappa \psi(t, x, y) \right] \, dydxt. \quad (6.11)
\]

In (6.8)-(6.11) we have used the fact of strong two-scale convergence of \( \chi\left( \frac{x}{\varepsilon} \right) \) and \( f(t, x, \xi_\varepsilon) \), and the strong convergence of \( u_\varepsilon \) to \( \vartheta u \).

Now we have

\[
\lim_{\varepsilon \to 0} I_\varepsilon
\]

\[
= \int_{D_T} \int_Y \left[ f_0(t, x, y) - \chi(y) f(t, x, \phi + \lambda \psi) \right] [\vartheta u(t, x) - \phi(t, x) - \kappa \psi] \, dydxt \geq 0,
\]
for any $\phi \in C_0^\infty(D_T)$ and $\psi \in C_0^\infty(D_T; C_{per}(Y))$. Letting $\phi \to \vartheta u$, dividing the above formula by $\kappa$ on both sides of the above formula and letting $\kappa \to 0$ yields
\[
\int_{D_T} \int_Y \left[ f_0(t, x, y) - \chi(y) f(t, x, \vartheta u) \right] \psi dy dx dt \leq 0
\]
for any $\psi \in C_0^\infty(D_T; C_{per}(Y))$, which means
\[
f_0(t, x, y) = \chi(y) f(t, x, \vartheta u).
\]

Then by the similar analysis for linear systems in §5 we have the following homogenized model.

**Theorem 6.2.** Assume that (2.5) holds. Let $u_\epsilon$ be the solution of (2.1)-(2.4) with nonlinear term $f$ satisfying (6.5) and (6.6). Then for any fixed $\epsilon > 0$, the distribution $L(u_\epsilon)$ converges weakly to $\mu$ in $L^2(0, T; H)$ as $\epsilon \downarrow 0$, with $\mu$ being the distribution of $U$, which is the solution of the following homogenized effective equation
\[
dU = \left[ -\vartheta^{-1} \text{div}_x \left( A^* \nabla_x U \right) - b\lambda U + \vartheta f(t, x, U) \right] dt
\]
\[+ \vartheta g_1 dW_1(t) + \lambda g_2 dW_2(t), \quad (6.12)
\]
with the boundary condition $U = 0$ on $\partial D$, the initial condition $U(0) = u_0 / \vartheta$ and the effective matrix $A^* = (A^*_{ij})$ being determined by (5.19). Moreover, the constant coefficient $\vartheta = \frac{|Y^*|}{|Y|}$ is defined in the beginning of §2 and $\lambda = \frac{|\partial S|}{|V|}$ is defined in (4.2).

**Case 3: Nonlinearity that contains a gradient term**

We next consider $f$ in the following form containing a gradient term,
\[
f(t, x, u, \nabla u) = h(t, x, u) \cdot \nabla u \quad (6.13)
\]
where $h(t, x, u) = (h_1(t, x, u), \cdots, h_n(t, x, u))$ and each $h_i : [0, T] \times D \times \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, n$, is continuous with respect to $u$ and $h(\cdot, \cdot, u(\cdot, \cdot)) \in L^2(0, T; L^2(D))$ for $u \in L^2(0, T; L^2(D))$. Moreover assume that $h$ satisfies
\[
(1) \ |(h(t, x, u) \cdot \nabla v, v)|_{L^2(D)} \leq C_0 |\nabla u|_{L^2(D)} |v|_{L^2(D)} \quad \text{with some positive constant } C_0.
\]
(2) $|h_i(t, x, \xi_1) - h_i(t, x, \xi_2)| \leq k|\xi_1 - \xi_2|$ for $\xi_1, \xi_2 \in \mathbb{R}$, $i = 1, \cdots, n$ and $k$ is a positive constant.

Now we have
\[
|(F_\epsilon(t, x, z_\epsilon), z_\epsilon)|_{X^0} = |(h(t, x, u_\epsilon) \cdot \nabla u_\epsilon, u_\epsilon)|_{L^2(D)} \leq C_0 |z_\epsilon|_{X^0} |z_\epsilon|_{X^1}. \quad (6.14)
\]

By applying Itô formula to $|z_\epsilon|^2_{X^0}$, we obtain
\[
d|z_\epsilon(t)|^2_{X^0} + 2(A_\epsilon z_\epsilon, z_\epsilon)|_{X^0} dt = 2(F_\epsilon(t, x, z_\epsilon), z_\epsilon)|_{X^0} dt + 2(G_\epsilon(t, x) dW(t), z_\epsilon)|_{X^0} + |G_\epsilon(t, x)|^2_{L^2(Q)} dt. \quad (6.15)
\]
By (6.14), coercivity (3.4) of $a_\epsilon(\cdot, \cdot)$ and the Cauchy inequality, integrating (6.15) with respect to $t$ yields

$$
|z_\epsilon(t)|_{X_0}^2 + \bar{\alpha} \int_0^t |z_\epsilon(s)|_{X_1}^2 \, ds
\leq |z_0|_{X_0}^2 + 2 \beta + \Lambda_1(\bar{\alpha}) \int_0^t |z_\epsilon(s)|_{X_0}^2 \, ds + 2 \int_0^t (G_\epsilon(s)dW(s), z_\epsilon(s))_{X_0^2} + \int_0^t |G_\epsilon(s)|_{L^2_{\epsilon}}^2 \, ds
$$

where $\Lambda_1$ is a positive constant depending on $\bar{\alpha}$. Then by the Gronwall lemma we see that (3.9) and (3.10) hold. Moreover, the fact $|h(t, x, u_\epsilon) \cdot \nabla u|_{L^2} \leq C_0|z_\epsilon|_{X_1}$, together with the Hölder inequality yields

$$
E \left| - P \int_0^t A_\epsilon z_\epsilon(s) \, ds + P \int_0^t F_\epsilon(s, x, z_\epsilon) \, ds \right|_{W^{1,2}(0,T;H^{-1}(D_\epsilon))}^2 \leq C_T \quad (6.16)
$$

where $P$ is defined in Theorem 3.3. Then by the same discussion of Theorem 3.3, we have the tightness of the distributions of $\tilde{u}_\epsilon$.

Now we pass the limit $\epsilon \to 0$ in the nonlinear term $f(t, x, u_\epsilon, \nabla u_\epsilon)$. In fact we restrict the system on $(\Omega_\delta, F_\delta, P_\delta)$. By the assumption (2) on $h$ and the fact that $\tilde{u}_\epsilon$ strong converges to $\tilde{u}$ in $L^2(D_T)$, we have

$$
\lim_{\epsilon \to 0} \int_{D_T} \left[ h(t, x, \tilde{u}_\epsilon(t, x)) - h(t, x, \tilde{u}(t, x)) \right]^2 \, dxdt = 0.
$$

For any $\psi \in C_0^\infty(D_T),$

$$
\int_{D_T} h(t, x, \tilde{u}_\epsilon) \cdot \nabla \tilde{u}_\epsilon \psi \, dxdt = \int_{D_T} \left[ h(t, x, \tilde{u}_\epsilon) - h(t, x, \tilde{u}) \right] \cdot \nabla \tilde{u}_\epsilon \psi \, dxdt + \int_{D_T} h(t, x, \tilde{u}) \cdot \nabla \tilde{u}_\epsilon \psi \, dxdt \xrightarrow{\epsilon \to 0} \frac{1}{|Y|} \int_{D_T} \int_Y h(t, x, \tilde{u}) \cdot \chi(y) \left[ \nabla_x u + \nabla_y u_1 \right] \psi \, dy \, dxdt. \quad (6.17)
$$

Combining with the analysis for linear system in §5, we have the following result.

**Theorem 6.3.** Assume that (2.5) holds. Let $u_\epsilon$ be the solution of (2.1)-(2.4) with nonlinear term (6.13). Then for any fixed $T > 0$, the distribution $\mathcal{L}(\tilde{u}_\epsilon)$
converges weakly to $\mu$ in $L^2(0,T;H)$ as $\epsilon \downarrow 0$, with $\mu$ being the distribution of $U = \vartheta u$ which satisfies the following homogenized effective equation

$$dU = \left[-\vartheta^{-1}\text{div}_x \left(A^* \nabla_x U\right) - b\lambda U + f^*(t,x,U,\nabla_x U)\right]dt + \vartheta g_1 dW_1(t) + \lambda g_2 dW_2(t), \quad (6.18)$$

where the boundary condition $U = 0$ on $\partial D$, the initial condition $U(0) = u_0/\vartheta$, the effective matrix $A^* = (A^*_{ij})$ is determined by (5.19) and $f^*$ is the following spatial average

$$f^*(t,x,U,\nabla_x U) = \frac{1}{|Y|} \int_Y h(t,x,U) \cdot \chi(y) \left[\vartheta^{-1} \nabla_x U + \nabla_y u_1\right]dy$$

with $u_1$ being given by (5.12) and $\chi(y)$ the indicator function of $Y^*$. Moreover, the constant coefficient $\tilde{\vartheta} = \frac{|Y^*|}{|Y|}$ is defined in the beginning of §2 and $\lambda = \frac{|\partial S|}{|Y|}$ is defined in (4.2).

**Remark 6.4.** All the results in this paper hold when $\Delta$ is replaced by a more general strong elliptic operator $\text{div}(A\nabla u)$, where $A_\epsilon$ is $Y-$periodic and satisfies the strong ellipticity condition.

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