Algebraic intersection for translation surfaces in the stratum $\mathcal{H}(2)$

Intersection algébrique dans la strate $\mathcal{H}(2)$

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Abstract

Nous étudions la quantité KV ol définie par l’équation (1) sur la strate $\mathcal{H}(2)$ des surfaces de translation de genre 2, avec une singularité conique. Nous donnons une suite explicite de surfaces $L(n, n)$ telles que $\text{KV ol}(L(n, n)) \rightarrow 2$ quand $n$ tend vers l’infini, 2 étant l’infimum-conjectural-de KV ol sur $\mathcal{H}(2)$.

We study the quantity KV ol defined in Equation (1) on the stratum $\mathcal{H}(2)$ of translation surfaces of genus 2, with one conical point. We provide an explicit sequence $L(n, n)$ of surfaces such that $\text{KV ol}(L(n, n)) \rightarrow 2$ when $n$ goes to infinity, 2 being the conjectured infimum for KV ol over $\mathcal{H}(2)$.

1 Introduction

Let $X$ be a closed surface, that is, a compact, connected manifold of dimension 2, without boundary. Let us assume that $X$ is oriented. Then the algebraic intersection of closed curves in $X$ endows the first homology $H_1(X, \mathbb{R})$ with an antisymmetric, non degenerate, bilinear form, which we denote $\text{Int}(\cdot, \cdot)$.

Now let us assume $X$ is endowed with a Riemannian metric $g$. We denote $\text{Vol}(X, g)$ the Riemannian volume of $X$ with respect to the metric $g$, and for any piecewise smooth closed curve $\alpha$ in $X$, we denote $l_g(\alpha)$ the length of $\alpha$ with respect to $g$. When there is no ambiguity we omit the reference to $g$.

We are interested in the quantity

$$\text{KVol}(X, g) = \text{Vol}(X, g) \sup_{\alpha, \beta} \frac{\text{Int}(\alpha, \beta)}{l_g(\alpha)l_g(\beta)}$$

where the supremum ranges over all piecewise smooth closed curves $\alpha$ and $\beta$ in $X$. The $\text{Vol}(X, g)$ factor is there to make KVol invariant to re-scaling of the metric $g$. See [6] as to why KVol is finite. It is easy to make KVol go to infinity, you just need to pinch a non-separating closed curve $\alpha$ to make its length go to zero. The interesting surfaces are those $(X, g)$ for which KVol is small.
When $X$ is the torus, we have $\KV_{ol}(X, g) \geq 1$, with equality if and only if the metric $g$ is flat (see [6]). Furthermore, when $g$ is flat, the supremum in $\eqref{eq:KVol}$ is not attained, but for a negligible subset of the set of all flat metrics. In [6] $\KV_{ol}$ is studied as a function of $g$, on the moduli space of hyperbolic (that is, the curvature of $g$ is $-1$) surfaces of fixed genus. It is proved that $\KV_{ol}$ goes to infinity when $g$ degenerates by pinching a non-separating closed curve, while $\KV_{ol}$ remains bounded when $g$ degenerates by pinching a separating closed curve.

This leaves open the question whether $\KV_{ol}$ has a minimum over the moduli space of hyperbolic surfaces of genus $n$, for $n \geq 2$. It is conjectured in [6] that for almost every $(X, g)$ in the moduli space of hyperbolic surfaces of genus $n$, the supremum in $\eqref{eq:KVol}$ is attained (that is, it is actually a maximum).

In this paper we consider a different class of surfaces: translation surfaces of genus 2, with one conical point. The set (or stratum) of such surfaces is denoted $\mathcal{H}(2)$ (see [3]). By [7], any surface $X$ in the stratum $\mathcal{H}(2)$ may be unfolded as shown in Figure 1, with complex parameters $z_1, z_2, z_3, z_4$. The surface is obtained from the plane template by identifying parallel sides of equal length.

It is proved in [4] (see also [2]) that the systolic volume has a minimum in $\mathcal{H}(2)$, and it is achieved by a translation surface tiled by six equilateral triangles. Since the systolic volume is a close relative of $\KV_{ol}$, it is interesting to keep the results of [4] and [2] in mind.

We have reasons to believe that $\KV_{ol}$ behaves differently in $\mathcal{H}(2)$, both from the systolic volume in $\mathcal{H}(2)$, and from $\KV_{ol}$ itself in the moduli space of hyperbolic surfaces of genus 2; that is, $\KV_{ol}$ does not have a minimum over $\mathcal{H}(2)$.

We also believe that the infimum of $\KV_{ol}$ over $\mathcal{H}(2)$ is 2. This paper is a first step towards the proof: we find an explicit sequence $L(n, n)$ of surfaces in $\mathcal{H}(2)$, whose $\KV_{ol}$ tends to 2 (see Proposition 2.5). These surfaces are obtained from very thin, symmetrical, L-shaped templates (see Figure 2).
In the companion paper [1] we study KVol as a function on the Teichmüller disk (the $SL_2(\mathbb{R})$-orbit) of surfaces in $\mathcal{H}(2)$ which are tiled by three identical parallelograms (for instance $L(2,2)$), and prove that KVol does have a minimum there, but is not bounded from above. Therefore KVol is not bounded from above as a function on $\mathcal{H}(2)$. In [1] we also compute KVol for the translation surface tiled by six equilateral triangles, and find it equals 3, so it does not minimize KVol, neither in $\mathcal{H}(2)$, nor even in its own Teichmüller disk.

2 $L(n, n)$

2.1 Preliminaries

Following [8], for any $n \in \mathbb{N}$, $n \geq 2$, we call $L(n+1, n+1)$ the $(2n+1)$-square translation surface of genus two, with one conical point, depicted in Figure 2, where the upper and rightmost rectangles are made up with $n$ unit squares. We call $A$ (resp. $B$) the region in $L(n+1, n+1)$ obtained, after identifications, from the uppermost (resp. rightmost) rectangle, and $C$ the region in $L(n+1, n+1)$ obtained, after identifications, from the bottom left square. Both $A$ and $B$ are annuli with a pair of points identified on the boundary, while $C$ is a square with all four corners identified. We call $e_1, e_2$, (resp. $f_1, f_2$) the closed curves in $L(n+1, n+1)$ obtained by gluing the endpoints of the horizontal (resp. vertical) sides of $A$ and $B$. The closed curve which sits on the opposite side of $C$ from $e_1$ (resp. $f_1$) is called $e'_1$ (resp. $f'_1$), it is homotopic to $e_1$ (resp. $f_1$) in $L(n+1, n+1)$. The closed curves in $L(n+1, n+1)$ which correspond to the diagonals of the square $C$ are called $g$ and $h$.

Figure 3 shows a local picture of $L(n+1, n+1)$ around the singular (conical) point $S$, with angles rescaled so the $6\pi$ fit into $2\pi$.

Since $e_1, e_2, f_1, f_2$ do not meet anywhere but at $S$, the local picture yields the algebraic intersections between any two of $e_1, e_2, f_1, f_2$, summed up in the following matrix:

\[
\begin{array}{cccc}
\text{Int} & e_2 & f_1 & e_1 & f_2 \\
e_2 & 0 & 1 & 0 & -1 \\
f_1 & -1 & 0 & 0 & 0 \\
e_1 & 0 & 0 & 0 & 1 \\
f_2 & 1 & 0 & -1 & 0 \\
\end{array}
\]

We call $T_A$ (resp. $T_B$) the flat torus obtained by gluing the opposite sides of the rectangle made with the $n+1$ leftmost squares (resp. with the $n+1$ bottom squares), so the homology of $T_A$ (resp. $T_B$) is generated by $e_1$ and the concatenation of $f_1$ and $f_2$ (resp. $f_1$ and the concatenation of $e_1$ and $e_2$).

Lemma 2.1. The only closed geodesics in $L(n+1, n+1)$ which do not intersect $e_1$ nor $f_1$ are, up to homotopy, $e_1$, $f_1$, $g$, and $h$.

Proof. Let $\gamma$ be such a closed geodesic. It cannot enter, nor leave, $A$, $B$, nor $C$. If it is contained in $A$, and does not intersect $e_1$, then it must be homotopic to $e_1$, which is the
Figure 2: $L(n+1, n+1)$

Figure 3: Local picture around the conical point
soul of the annulus from which \( A \) is obtained by identifying two points on the boundary. Likewise, if it is contained in \( B \), and does not intersect \( f_1 \), then it must be homotopic to \( f_1 \). Finally, if \( \gamma \) is not contained in \( A \) nor in \( B \), it must be contained in \( C \). The only closed geodesics contained in \( C \) are the sides and diagonals of the square from which \( C \) is obtained, which are \( e_1, e'_1, f_1, f'_1, g, \) and \( h \).

**Lemma 2.2.** For any closed geodesic \( \gamma \) in \( L(n+1, n+1) \), we have \( l(\gamma) \geq n|\text{Int}(\gamma, e_1)| \).

*Proof.* For each intersection with \( e_1, \gamma \) must go through \( A \), from boundary to boundary. Obviously a similar lemma holds with \( f_1 \) instead of \( e_1 \). For \( g \) and \( h \) the proof is a bit different:

**Lemma 2.3.** For any closed geodesic \( \gamma \) in \( L(n+1, n+1) \), we have \( l(\gamma) \geq n|\text{Int}(\gamma, g)| \).

*Proof.* First, observe that between two consecutive intersections with \( g \), \( \gamma \) must go through either \( A \) or \( B \), unless \( \gamma \) is \( g \) itself, or \( h \) : indeed, the only geodesic segments contained in \( C \) with endpoints on \( g \) are segments of \( g \), or \( h \). Obviously \( \text{Int}(g, g) = 0 \), and from the intersection matrix (2), knowing that \( [g] = [e_1] - [f_1], [h] = [e_1] + [f_1] \), we see that \( \text{Int}(g, h) = 0 \).

Thus, either \( \text{Int}(\gamma, g) = 0 \), or each intersection must be paid for with a trek through \( A \) or \( B \), of length at least \( n \).

Obviously a similar lemma holds with \( h \) instead of \( g \). Note that Lemmata 2.1, 2.2, 2.3 imply that the only geodesics in \( L(n+1, n+1) \) which are shorter than \( n \) are \( e_1, f_1, g, h \), and closed geodesics homotopic to \( e_1 \) or \( f_1 \).

**Lemma 2.4.** Let \( I, J \) be positive integers, take \( a_{ij}, i = 1, \ldots, I, j = 1, \ldots, J \) in \( \mathbb{R}_+ \), and \( b_1, \ldots, b_I, c_1, \ldots, c_J \) in \( \mathbb{R}_+^* \). Then we have

\[
\frac{\sum_{i,j} a_{ij}}{(\sum_{i=1}^I b_i)(\sum_{j=1}^J c_j)} \leq \max_{i,j} \frac{a_{ij}}{b_i c_j}.
\]

*Proof.* Re-ordering, if needed, the \( a_{ij}, b_i, c_j \), we may assume

\[
\frac{a_{ij}}{b_i c_j} \leq \frac{a_{11}}{b_1 c_1} \quad \forall i = 1, \ldots, I, j = 1, \ldots, J.
\]

Then \( a_{ij} b_1 c_1 \leq a_{11} b_i c_j \quad \forall i = 1, \ldots, I, j = 1, \ldots, J \), so

\[
b_1 c_1 \sum_{i,j} a_{ij} \leq a_{11} \sum_{i,j} b_i c_j = a_{11} \left( \sum_{i=1}^I b_i \right) \left( \sum_{j=1}^J c_j \right).
\]
2.2 Estimation of $K\text{Vol}(L(n, n))$

Proposition 2.5.

\[
\lim_{n \to +\infty} K\text{Vol}(L(n + 1, n + 1)) = 2.
\]

Proof. First observe that $\text{Vol}(L(n+1, n+1)) = 2n+1$, $l(e_1) = 1$, $l(f_2) = n$, $\text{Int}(e_1, f_2) = 1$, so

\[
K\text{Vol}(L(n + 1, n + 1)) \geq 2 + \frac{1}{n}.
\]

To bound $K\text{Vol}(L(n + 1, n + 1))$ from above, we take two closed geodesics $\alpha$ and $\beta$; by Lemmata 2.2, 2.3 if either $\alpha$ or $\beta$ is homotopic to $e_1$, $f_1$, $g$, or $h$, then

\[
\frac{\text{Int}(\alpha, \beta)}{l(\alpha)l(\beta)} \leq \frac{1}{n},
\]

so from now on we assume that neither $\alpha$ or $\beta$ is homotopic to $e_1$, $f_1$, $g$, $h$. We cut $\alpha$ and $\beta$ into pieces using the following procedure: we consider the sequence of intersections $\beta_1$, $\ldots$, $\beta_6$, so $1$, so $l(\alpha)$, $l(\beta) = \sum_{i=1}^{J} l(\alpha_i)$, and $l(\beta) = \sum_{j=1}^{I} l(\beta_j)$, and

\[
|\text{Int}(\alpha, \beta)| \leq \sum_{i,j} |\text{Int}(\alpha_i, \beta_j)|,
\]

so Lemma 2.4 says that

\[
\frac{|\text{Int}(\alpha, \beta)|}{l(\alpha)l(\beta)} \leq \max_{i,j} \frac{|\text{Int}(\alpha_i, \beta_j)|}{l(\alpha_i)l(\beta_j)}.
\]

We view each piece $\alpha_i$ (resp. $\beta_j$) as a geodesic arc in the torus $T_A$ (resp. $T_B$), with endpoints on the image in $T_A$ (or $T_B$) of $f_1$ or $f'_1$ (resp. $e_1$ or $e'_1$), which is a geodesic arc of length 1, so we can close each $\alpha_i$ (resp. $\beta_j$) with a piece of $f_1$ or $f'_1$ (resp. $e_1$ or $e'_1$), of length $\leq 1$. We choose a closed geodesic $\hat{\alpha}_i$ (resp. $\hat{\beta}_j$) in $T_A$ (resp. $T_B$) which is homotopic to the closed curve thus obtained. We have $l(\hat{\alpha}_i) \leq l(\alpha_i) + 1$, $l(\hat{\beta}_j) \leq l(\beta_j) + 1$, so

\[
\frac{1}{l(\hat{\alpha}_i)l(\hat{\beta}_j)} \geq \frac{1}{(l(\alpha_i) + 1)(l(\beta_j) + 1)}.
\]

Now recall that $l(\alpha_i), l(\beta_j) \geq n$, so $l(\alpha_i) + 1 \leq (1 + \frac{1}{n}) l(\alpha_i)$, whence

\[
\frac{1}{l(\hat{\alpha}_i)l(\hat{\beta}_j)} \geq \frac{1}{l(\alpha_i)l(\beta_j)} \left( \frac{n}{n+1} \right)^2.
\]

Next, observe that $|\text{Int}(\alpha_i, \beta_j)| \leq |\text{Int}(\hat{\alpha}_i, \hat{\beta}_j)| + 1$, because $\hat{\alpha}_i$ (resp. $\hat{\beta}_j$) is homologous to a closed curve which contains $\alpha_i$ (resp. $\beta_j$) as a subarc, and the extra arcs cause at
most one extra intersection, depending on whether or not the endpoints of \( \alpha_i \) and \( \beta_j \) are intertwined. So,

\[
\frac{|\text{Int}(\alpha_i, \beta_j)|}{l(\alpha_i)l(\beta_j)} \leq \frac{|\text{Int}(\hat{\alpha}_i, \hat{\beta}_j)| + 1}{l(\hat{\alpha}_i)l(\hat{\beta}_j)} \left( \frac{n + 1}{n} \right)^2 \leq \left( \frac{|\text{Int}(\hat{\alpha}_i, \hat{\beta}_j)|}{l(\hat{\alpha}_i)l(\hat{\beta}_j)} + \frac{1}{n^2} \right) \left( \frac{n + 1}{n} \right)^2,
\]

where the last inequality stands because \( l(\hat{\alpha}_i) \geq n \), \( l(\hat{\beta}_j) \geq n \), since \( \hat{\alpha}_i \) and \( \hat{\beta}_j \) both have to go through a cylinder \( A \) or \( B \) at least once. Finally, since \( \hat{\alpha}_i \) and \( \hat{\beta}_j \) are closed geodesics on a flat torus of volume \( n + 1 \), we have (see [6])

\[
\frac{|\text{Int}(\hat{\alpha}_i, \hat{\beta}_j)|}{l(\hat{\alpha}_i)l(\hat{\beta}_j)} \leq \frac{1}{n + 1},
\]

so

\[
\frac{|\text{Int}(\alpha_i, \beta_j)|}{l(\alpha_i)l(\beta_j)} \leq \left( \frac{1}{n + 1} + \frac{1}{n^2} \right) \left( \frac{n + 1}{n} \right)^2 = \frac{1}{n} + o\left( \frac{1}{n} \right),
\]

which yields the result, recalling that \( \text{Vol}(L(n + 1, n + 1)) = 2n + 1 \). \( \square \)

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