Length of separable states and symmetrical informationally complete (SIC) POVM

Lin Chen$^{1,2}$

$^1$Department of Pure Mathematics and Institute for Quantum Computing, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada
$^2$Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 11754

(Dated: February 5, 2013)

This short note reviews the notion and fundamental properties of SIC-POVM and its connection with the length of separable states. We also review the t-design.

PACS numbers: 03.67.Mn, 03.65.Ud

I. DEFINITION AND BACKGROUND OF SIC-POVM

1. [3, 12, 17] In the $d$-dimensional Hilbert space, a SIC-POVM consists of $d^2$ outcomes that are subnormalized projectors onto pure states $\Pi_j = \frac{1}{d}\left| \psi_j \right\rangle \left\langle \psi_j \right|$ for $j, k = 1, \ldots, d^2$, such that

$$\left| \langle \psi_j | \psi_k \rangle \right|^2 = \frac{1 + d\delta_{jk}}{d + 1}. \quad (1)$$

2. [12, Theorem 2] Using Eq. (1) we can show that any SIC-POVM forms a 2-design:

$$\sum_{i=1}^{d^2} \left| \psi_i, \psi_i \right\rangle \left\langle \psi_i, \psi_i \right| = \frac{2d}{d+1} S_d. \quad (2)$$

Here, the operator $S_d$ denotes the $d \times d$ symmetrizer operator, i.e.,

$$S_d := \sum_{i=1}^{d} \left| ii \right\rangle \left\langle ii \right| + \sum_{j>i=1}^{d} \frac{|ij| + |ji|}{\sqrt{2}} \frac{|ij| + |ji|}{\sqrt{2}}. \quad (3)$$

3. Eq. (2) implies that $\sum_{j=1}^{d^2} \Pi_j = I$, so SIC-POVM is a complete measurement in physics.

4. Three basic papers on SIC-POVMs are [3, 12, 17].

- (1) G. Zauner, "Quantendesigns - Grundzüge einer nicht kommutativen Designtheorie," PhD thesis (University of Vienna, 1999).
- (2) J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, J. Math. Phys. 45, 2171 (2004). (provide analytical $d = 2, 3, 4$, numerical $d \leq 45$)
- (3) D. M. Appleby, J. Math. Phys. 46, 052107 (2005). It provides the analytical solutions of SIC-POVM for $d = 2, \ldots, 7, 19$.

*Electronic address: cqtcl@nus.edu.sg (Corresponding Author)
5. Example 1 SIC-POVM for $d=2$. Let

$$|\psi_0\rangle = \frac{\sqrt{3+\sqrt{3}}}{6}|0\rangle + e^{\pi i/4}\frac{\sqrt{3-\sqrt{3}}}{6}|1\rangle,$$

$$|\psi_1\rangle = \frac{\sqrt{3+\sqrt{3}}}{6}|0\rangle - e^{\pi i/4}\frac{\sqrt{3-\sqrt{3}}}{6}|1\rangle,$$

$$|\psi_2\rangle = \frac{\sqrt{3+\sqrt{3}}}{6}|1\rangle + e^{\pi i/4}\frac{\sqrt{3-\sqrt{3}}}{6}|0\rangle,$$

$$|\psi_3\rangle = \frac{\sqrt{3+\sqrt{3}}}{6}|1\rangle - e^{\pi i/4}\frac{\sqrt{3-\sqrt{3}}}{6}|0\rangle.$$  

Then one can verify

$$\sum_{i=0}^{3} |\psi_i\rangle \langle \psi_i| = \frac{4}{3}S_2.$$  

The four states $|\psi_i\rangle$, $i=1,2,3,4$ form a regular tetrahedron when represented on the Bloch sphere.

6. Analytical SIC-POVMs have been constructed for dimension $d=2,\cdots, 16, 19, 24, 28, 31, 35, 37, 43, 48$, see [14]. Numerical SIC-POVMs have been constructed for $d \leq 67$, see the details in [20]. This is achieved by the popular method of Weyl-Heisenberg group in quantum information community. However the construction becomes hard for higher dimensions. So it is unknown, though widely believed, that whether SIC-POVM exists for any dimension $d$.

7. Constructing SIC-POVM is one of the most important questions in quantum information. It is related to quantum tomography [18], Mutually unbiased bases (MUBs) [4, 16], entanglement theory [7, 19], Lie Algebra [2], Galois field [1], foundations of quantum mechanics [9] and so on.

II. RELATING SIC-POVM TO LENGTH

For a bipartite state $\rho$ acting on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the partial transpose computed in the standard orthonormal (o.n.) basis $\{|i\rangle\}$ of system $A$, is defined by $\rho^\Gamma_B = \sum_{ij} |j\rangle \langle i| \otimes \langle i| \rho^\Gamma|j\rangle$. One can similarly define the partial transpose $\Gamma_A$ on the system $A$. Let $r(\rho)$ denote the rank of $\rho$. We call the integer pair $(r(\rho), r(\rho^\Gamma))$ the birank of $\rho$, and the two integers may be different. The length, $L(\rho)$, of a separable state $\rho$ is the minimal number of pure product states over all such decompositions of $\rho$ [3]. It is known that $L(\rho) \geq \max\{r(\rho), r(\rho^\Gamma)\}$.

One can verify that the partial transpose of the state $\rho_2 = \frac{1}{d^2+d}I + |\Psi_d\rangle \langle \Psi_d|$ is

$$\rho_2^\Gamma = \frac{1}{d^2+d}(I + |\Psi_d\rangle \langle \Psi_d|),$$  

where $|\Psi_d\rangle = \sum_{i=1}^{d} |ii\rangle$ is the non-normalized $d$-level maximally entangled state. So the separable state $\rho_2$ has birank $(\frac{d^2+d}{d^2+d}, d^2)$. Therefore we have $L(\rho_2) \geq d^2$. The equality holds for $d=2$ by Example 1. It also holds for $d=2, \cdots, 16, 19, 24, 28, 31, 35, 37, 43, 48$ [14]. However the question is whether

Conjecture 2 $L(\rho_2) = d^2$ for any $d \geq 2$.

The positive answer of this conjecture would imply that the SIC-POVM exists for any integer $d \geq 2$. This argument has been proved by using the notion of weighted 2-design in [13, Theorem 4]. On the other hand if Conjecture 2 turned out to fail for some $d$, i.e., $L(\rho_2) > d^2$, then SIC-POVM would not exist for this $d$. This argument has been proved by Eq. (2) and [12, Theorem 2].

To conclude, either the positive or negative answer to Conjecture 2 will solve the SIC-POVM problem.
III. MORE GENERAL BACKGROUND: T-DESIGN

Let $t \geq 1$ be an integer. The t-design of dimension $d$ is defined as a set $S$ of pure product states $|a_i⟩ ∈ C^d$ if

$$\frac{1}{|S|} \sum_i |a_i⟩⟨a_i|^{⊗t} = ρ_t = \left(\frac{d + t - 1}{t}\right)^{-1} S_{d,t}, \quad (10)$$

where $S_{d,t}$ is the t-partite symmetrizer operator in the space $(C^d)^{⊗t}$. For example, $S_{d,t} = S_d$ for $t = 2$ in Eq. (3). It is known \cite{13} that the number of design points satisfies

$$|S| \geq \left(\frac{d + \lceil t/2 \rceil - 1}{\lfloor t/2 \rfloor}\right) \left(\frac{d + \lceil t/2 \rceil - 1}{\lfloor t/2 \rfloor}\right). \quad (11)$$

A design which achieves this lower bound is called tight. For example, the bound is equal to $d$, $d^2$ and $d^2(d + 1)/2$ for $t = 1, 2, 3$, respectively. The t-designs exist for any $d$ \cite{13}. In the language of quantum information, it means that any t-partite symmetrizer operator is a non-normalized separable state. However it is unknown that whether tight t-designs exist, i.e., whether the length of t-partite symmetrizer operator reaches the lower bound in Eq. (11).

Here are a few known results from the field of t-designs. For $d = 2$, tight t-designs exist for $t = 1, 2, 3, 5$ \cite{11}. For a few $d > 2$, tight t-designs exist for $t = 1, 2, 3, 5 \cite{12}$. Here is the detail. It is trivial that tight 1-designs exist for any $d$. The existence of tight 2-designs is equivalent to the positive answer for Conjecture [2] in terms of Eq. (10). So far this is true for $d = 2, \cdots, 16, 19, 24, 28, 31, 35, 37, 43, 48$, see \cite{14}. Third, the tight 3-designs are known only for $d = 2, 4, 6 \cite{10}$. In particular for $d = 2$, the six states from an MUB in $C^2$ form a tight 3-design \cite{20}. It can also be directly verified by computing the frame potential.

Note that $ρ_t$ is a t-partite separable state. We have

**Lemma 3** The tight t-design of dimension $d$ exists if and only if $L(ρ_t) = \left(\frac{d + \lceil t/2 \rceil - 1}{\lfloor t/2 \rfloor}\right) \left(\frac{d + \lceil t/2 \rceil - 1}{\lfloor t/2 \rfloor}\right)$.

The proof is based on Ref. [41,42] of \cite{12}. Nevertheless, it is known that the tight t-design does not exist for $d ≥ 3, t ≥ 5 \cite{13}$.

Acknowledgments

I thank Dr. Huangjun Zhu for careful reading this note and pointing out a few errors in an early version of this note.

\[\text{References}\]

[1] D. M. Appleby, Hulya Yadsan-Appleby, Gerhard Zauner, *Galois Automorphisms of a Symmetric Measurement*, quant-ph/1209.1813 (2012).
[2] D. M. Appleby, S. T. Flammia, and C. A. Fuchs, *The Lie algebraic significance of symmetric informationally complete measurements*, J. Math. Phys. 52, 022202 (2011).
[3] D. M. Appleby, J. Math. Phys. 46, 052107 (2005).
[4] D. M. Appleby. *SIC-POVMs and MUBs: Geometrical relationships in prime dimension*, AIP Conf. Proc. 1101, 223 (2009).
[5] Bannai E and Hoggar S G, *On tight t-designs in compact symmetric spaces of rank one*, Proc. Japan Acad. 61, 78 (1985).
[6] Bannai E and Hoggar S G *Tight t-designs and squarefree integers* Eur. J. Comb. 10, 113 (1989).
[7] Lin Chen, Huangjun Zhu, and Tzu-Chieh Wei, *Connections of geometric measure of entanglement of pure symmetric states to quantum state estimation*, Phys. Rev. A 83, 012305 (2010).
[8] D.P. DiVincenzo, B.M. Terhal, and A.V. Thapliyal, *Optimal decomposition of barely separable states*, J. Mod. Opt. 47 (2000), 377-385.
[9] Christopher A. Fuchs and Ruediger Schack, *Quantum-Bayesian Coherence: The No-Nonsense Version*, quant-ph/1301.3274 (2013).
[10] Hoggar S G, *t-designs in projective spaces*, Eur. J. Comb. 3, 233 (1982).
[11] Hardin R H and Sloane N J A, *McLaren’s improved snub cube and other new spherical designs in three dimensions Discrete Comput. Geom. 15, 429 (1996).*
[12] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, J. Math. Phys. 45, 2171 (2004).
[13] A. J. Scott, *Tight informationally complete quantum measurements*. J. Phys. A -Mathematical and General, 2006. 39(43): p. 13507-13530.
[14] A. J. Scott and M. Grassl, *SIC-POVMs: A new computer study*, J. Math. Phys. 51, 042203 (2010).
[15] Seymour P D and Zaslavsky T, *Averaging sets: a generalization of mean values and spherical designs*, Adv. Math. 52, 213 (1984).

[16] W. K. Wootters. *Quantum measurements and finite geometry*, Found. Phys., 36, 112, (2006).

[17] G. Zauner, Ph.D. thesis, University of Vienna, 1999; available online at [http://www.gerhardzauner.at/qdmye.html](http://www.gerhardzauner.at/qdmye.html). See also the English version: *Quantum designs: foundations of a noncommutative design theory*, International Journal of Quantum Information (IJQI) 9(1): 445 (2011).

[18] H. Zhu and B.-G. Englert, *Quantum state tomography with fully symmetric measurements and product measurements*, Phys. Rev. A 84, 022327 (2011).

[19] H. Zhu, Y. S. Teo, and B.-G. Englert. *Two-qubit symmetric informationally complete positive-operator-valued measures*, Phys. Rev. A, 82, 042308 (2010).

[20] H. Zhu, PhD Thesis, [http://scholarbank.nus.edu.sg/handle/10635/35247](http://scholarbank.nus.edu.sg/handle/10635/35247)