Generalized $K$-Shift Forbidden Substrings in Permutations

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In this note we continue the analysis started in [2] and generalize propositions regarding permutations that avoid substrings $12, 23, \ldots, (n-1)n$ (and others) to permutations that for fixed $k$, $k < n$, avoid substrings $j(j+k)$, $1 \leq j \leq n-k$, as well as substrings $j(j+k) \pmod{n}$, $1 \leq j \leq n$, (ie. $k$-shifts in general, as defined in Section 2).

Keywords: $k$-shifts $k$-successions, permutations, linear arrangements, forbidden patterns substrings (mod $n$), fixed points, bijections.

1. Introduction and Previous Results

In this section we summarize some results obtained in [2] and we recall the following definitions.

$$d_n := \text{the number of permutations on } [n] \text{ that avoid substrings } 12, 23, \ldots, (n-1)n.$$  

$$D_n := \text{the number of permutations on } [n] \text{ that avoid substrings } 12, 23, \ldots, (n-1)n, n1.$$  

$$\text{Der}_n := \text{the } n\text{th derangement number, i.e.}$$

$$\text{Der}_n = n! \sum_{k=0}^{n} \frac{(-1)^n}{k!}.$$  

(1.1)

In [2] we discussed the existing result

$$d_n = \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} (n-j)!.$$  

(1.2)

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2Note: In [2], the term “linear arrangement” was used instead of “permutation”, and “pattern” instead of “substring”. Here we use the more conventional terminology. Permutations are meant to be in one-line notation.
We also proved (Equation 2.1)

\[ D_n = n! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}, \]  

which is equivalent to

\[ D_n = \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} (n-j)!. \]  

Finally in Proposition 2.4, we proved that \( D_n = Der_n + (-1)^{n-1}, n \geq 1 \), which we called the “alternating derangement sequence” since these numbers alternate plus or minus one from the derangement sequence itself. This is sequence A000240 in OEIS [3].

Now we extend the results to forbidden substrings that are not one space apart but \( k \) spacings apart (what we call “\( k \)-shifts” in the following section).

We define a minimal forbidden substring as two consecutive elements \( jk \). We assign to this minimal substring a length equal to one. Hence the length of forbidden substrings will be one less than the number of elements.

2. Main Lemmas and Propositions

2.1 Results for \( \{d_n\} \) and \( \{d_n^k\} \)

For the sake of compactness, we define \( \{d_n\} \) as the set of permutations on \([n]\) that avoid substrings 12, 23, \ldots, \((n-1)n\), with \( d_n \) being the number of such permutations.

We generalize to \( \{d_n^k\}, k < n \), defined as the set of permutations on \([n]\) that for fixed \( k \), avoid substrings \( j(j+k) \), \( 1 \leq j \leq n-k \). We will refer to these substrings that are \( k \) spacings apart as “\( k \)-successions”, or “\( k \)-successions”\). We let \( d_n^k \) be the number of such permutations (the reason for power notation will become apparent in the next section).

The forbidden substrings in these permutations can be pictured as a diagonal running \( k \) places to the right of the main diagonal of an \( n \times n \) chessboard (hence the term “\( k \)-shifts”) as can be seen in Figure 1 below.
Figure 1: Forbidden positions in \( \{d^6_n\} \).

Figure 1 shows the forbidden positions on a 6 × 6 chessboard that correspond to forbidden substrings of permutations in \( \{d^6_n\} \). The forbidden substrings are \{13, 24, 35, 46\}. Note that there are \( n-k \) forbidden substrings in \( \{d^6_n\} \).

The permutations that avoid these substrings are not too difficult to handle, and in fact we can count them for any \( k \), as we show in the following proposition.

**Proposition 2.1.** For any fixed \( k \), \( 0 < k < n \), if \( d^k_n \) denotes the number of permutations that avoid substrings \( j(j+k) \), \( 1 \leq j \leq n-k \), then

\[
d^k_n = \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} (n-j)!.
\]

**Proof.** For any fixed \( n \) and \( k \), there are a total of

\[
\binom{n-k}{j} (n-j)!
\]

forbidden substrings of length \( j \) since the combinatorial term counts the number of ways to get such substrings while the term \( (n-j)! \) counts the permutations of the substrings and the remaining elements. Using inclusion-exclusion we get the result. \( \square \)

We note that the case \( k = 1 \) is just the result we had for \( d_n \) in Equation 1.2.

For example, consider permutations in \( \{d^3_5\} \). The \( n-k \) forbidden substrings are \{14, 25\}. For \( j = 0 \) we get the 5! permutations in \( S_5 \). For \( j = 1 \) there are \( \binom{3}{1} \) ways to choose one forbidden substring, and we can permute it with the remaining elements in 4! ways (for example, select the forbidden substring 14 and permute the 4 blocks 14, 2, 3, 5). For \( j = 2 \) there are \( \binom{3}{2} \) ways to choose two forbidden substrings and we can permute them with the remaining elements in 3! ways (that is, permute the 3 blocks 14, 25, 3). Hence \( d^3_5 = 78 \), so there are 78 permutations in \( S_5 \) that avoid substrings \{14, 25\}.

**Corollary 2.2.** The following relation holds for \( d^k_n \):

\[
d^{k+1}_n = d^k_n + d^k_{n-1}.
\]

\[3\]
Proof. By Equation 2.1 and elementary methods. □

Now we define \( d_0^n := \text{Der}_n \), which makes sense since in a chessboard of forbidden positions, a derangement is represented by an \( X \) in the position \((j, j)\), i.e., a 0-shift.

Note that Equation 2.2 generalizes the relation in Lemma 2.3 in [2], and we have the following equations starting at \( n = 1 \):

\[
\begin{align*}
d_n &= d_1^n = \text{Der}_n + \text{Der}_{n-1} \\
d_2^n &= d_n + d_{n-1} \\
d_3^n &= d_2^n + d_{n-1} \\
&\quad \vdots
\end{align*}
\]

Using the initial condition \( d_1^n = \text{Der}_2 \), Equation 2.2 defines a binomial-type relation, which, upon iteration, gives us the triangle in Table 1 in the Appendix. Note in particular that for \( k = n - 1 \), \( d_k^n = n! - k! \).

Note from the triangle that we may get \( d_k^n \) starting only from derangement numbers. For example, to get \( d_5^8 \), i.e., the number of permutations of length 8 with forbidden substrings \( \{16, 27, 38\} \), we can start from the upper-left corner of the table and by successive addition along the triangle we can reach cell \( d_5^8 = 27,240 \) (or we can obviously use Equation 2.1). Hence there are 27,240 permutations in \( S_8 \) that avoid substrings \( \{16, 27, 38\} \).

For further references, the sequence \( \langle d_k^n \rangle \) is available in OEIS. For example, for \( k = 3 \), the sequence is now A277609 [4].

2.2 Results for \( \{D_n\} \) and \( \{D^k_n\} \)

Now we define \( \{D_n\} \) as the set of permutations on \([n]\) that avoid substrings 12, 23, \ldots, \((n-1)n, n1\), with \( D_n \) being the number of such permutations.

We generalize to define permutations without \( k \)-shifts (or \( k \)-successions) (mod \( n \)), \( \{D^k_n\} \), as the set of permutations on \([n]\) that for fixed \( k \), \( k < n \), avoid substrings \( j(j+k) \) for \( 1 \leq j \leq n-k \), and avoid substrings \( j(j+k) \) (mod \( n \)) for \( n-k < j \leq n \). Note that we can summarize in the single definition “avoid substrings \( j(j+k) \) (mod \( n \)) for all \( j, 1 \leq j \leq n \)” if we agree to write \( n \) instead of 0 when doing addition (mod \( n \)).

We let \( D^k_n \) be the number of such permutations. These forbidden substrings are easily seen along an \( n \times n \) chessboard, where for \( j > n-k \), the forbidden

\[\text{This triangle follows the same recurrence as the so-called Euler’s Difference Table, which originally had no combinatorial interpretation. Euler’s Table also has a } k! \text{ term in each column, terms that don’t apply in our context of } k\text{-shifts.}\]
positions start again from the first column along a diagonal \((n-k)\) places below the main diagonal as in Figure 2 below.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & & & & \times & \\
3 & & & \times & &
\end{array}
\]

Figure 2: Forbidden positions in \(\{D^2_n\}\).

Figure 2 shows the forbidden positions on a \(6 \times 6\) chessboard that correspond to forbidden substrings of permutations in \(\{D^2_n\}\). These forbidden substrings are \{13, 24, 35, 46; 51, 62\}. The forbidden substrings below the diagonal are separated by a semicolon; these are the forbidden substrings \(j(j+k)\) \((\text{mod } n)\) for \(n-k < j \leq n\). Note that while there are only \(n-k\) forbidden substrings in \(\{d^k_n\}\), there are \(n\) forbidden substrings in \(\{D^k_n\}\).

It turns out that the numbers \(D^k_n\) are more difficult to get. They depend on whether \(n\) is prime, and more generally, on whether \(n\) and \(k\) are relatively prime.

**Proposition 2.3.** In the set of permutations \(\{D^k_n\}\) with \(0 < k < n\), \(k\) relative prime to \(n\), \(n \geq 2\), we can form a forbidden substring of length \(j = n-1\).

**Proof.** Start with forbidden substrings \(12, 23, \ldots, (n-1)n\) in \(\{D_n\}\) and form the permutation \((12 \ldots n)\) in cycle notation. Since \(k\)-powers of the permutation produce forbidden \(k\)-shifts (or \(k\)-successions), we see that the longest cycle of forbidden substrings will have length \(n/(n,k)\), where \((n,k)\) stands for the greatest common divisor. Hence the longest cycle length of forbidden substrings will be achieved for \((n,k) = 1\), and in this case we will have a cycle of length \(n\), which represents a forbidden substring of length \(n-1\). \(\square\)

Note that the proof of the proposition justifies the power notation in \(\{D^k_n\}\) (and in \(k\)-shifts in general). Note also that for \(n \geq 2\), Proposition 2.3 implies that we will always have the longest forbidden substrings of length \(n-1\) in \(\{D_n\}\), \((i.e. k = 1)\), as well as in some other \(\{D^k_n\}\) whenever \((n,k) = 1\). Moreover, Proposition 2.3 will show that the number of permutations in these sets are equal.

As an example, Figure 3 shows the maximum cycle lengths achieved for the case \(n = 6\) and all its possible \(k\)-shifts (or \(k\)-successions), \(k = 1, \ldots, 5\). In this case there exist maximum length substrings in \(\{D_6\}\) and \(\{D^5_6\}\).

Note that Proposition 2.3 is not true if \(k\) is not relative prime to \(n\), for example in \(\{D^2_6\}\). In this case, the forbidden substrings are \{13, 24, 35, 46; 51, 62\}, and the maximum cycle length is 3, which means we can have forbidden substrings
Forbidden Substrings

| k    | Forbidden Substrings | Permutation | kth-power | max cycle length |
|------|---------------------|-------------|-----------|------------------|
| k = 1 | 12, 23, 34, 45, 56; 61 | (123456)    | (123456)  | 6                |
| k = 2 | 13, 24, 35, 46; 51, 62 | (123456)    | (135)(246)| 3                |
| k = 3 | 14, 25, 36; 41, 52, 63 | (123456)    | (14)(25)(36)| 2             |
| k = 4 | 15, 26; 31, 42, 53, 64 | (123456)    | (153)(264)| 3                |
| k = 5 | 16; 21, 32, 43, 54, 65 | (123456)    | (165432)  | 6                |

Figure 3: $k$-shifts and maximum cycle lengths for $n = 6$.

of length at most 2 (such as 135 or 246). (Recall that our minimal forbidden substring consists of two elements $jk$, with length equal to one).

**Corollary 2.4.** In the set of permutations $\{D_n^k\}$ with $k$ relative prime to $n$, $n \geq 2$, there exist forbidden substrings of any length $j$, $j = 1, 2, \ldots, n - 1$.

**Proof.** By the previous proposition, for $(n, k) = 1$ we can get the longest forbidden substring of length $j = n - 1$. Note that it can be considered either a single substring of length $n - 1$ or $n - 1$ substrings of length 1. Hence once this substring is obtained, we can split it to get forbidden substrings of any length $j$, $j = 1, \ldots, n - 1$. □

**Proposition 2.5.** The number of permutations in $\{D_n^k\}$ with $0 < k < n$, $k$ relative prime to $n$, $n \geq 2$, is the same as the number of permutations in $\{D_n\}$.

**Proof.** By the previous Corollary and Proposition, since for $k$’s such that $(n, k) = 1$, we can have forbidden substrings of any length, $j = 1, \ldots, n - 1$. It is easy to count that there are exactly $\binom{n}{j}$ ways to get $j$ forbidden substrings (either disjoint or overlapping), and $(n - j)!$ permutations of these substrings and the remaining elements. Then by inclusion-exclusion we have that

$$D_n^k = \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} (n - j)!.$$  \hfill (2.3)

But this is the same as Equation 1.4 which counts the number of permutations in $\{D_n\}$. □

As an example of the previous proposition, consider again $n = 8$, $k = 5$. In this case, forbidden substrings in $\{D_8^5\}$ are $\{16, 27, 38; 41, 52, 63, 74, 85\}$. It is easy to count, for instance, that there are $\binom{8}{4}$ forbidden substrings of length $j = 4$ and $(8 - 4)!$ permutations of these substrings and the remaining elements. For example, a disjoint substring of length 4 (alternatively, four substrings of length 1) is given by 1638, 74 and we count $(8 - 4)! = 4!$ permutations of the four blocks 1638, 74, 2, 5. Another substring of length 4 is given by 16385 and we count 4! permutations of the blocks 16385, 2, 4, 7. Note that since $(8, 5) = 1$, there are $\binom{8}{7}$ substrings of maximum length $j = n - 1 = 7$ (for example, 16385274),
and we can permute these single blocks in \((8 - 7)! = 1\) way since there are no remaining elements to permute them with. Then, by Equation 2.3, we see that \(D_8^5 = 14,832\) so there are this number of permutations in \(S_8\) that avoid substrings \(\{16, 27, 38; 41, 52, 63, 74, 85\}\). Note that \(D_8^5 < d_8^5\) since there are more forbidden substrings in \(D_8^5\) than in \(d_8^5\).

**Corollary 2.6.** In the set of permutations \(\{D_k^p\}\) with \(p\) prime, we can form a substring of length \(j = p - 1\) for any \(k\)-shift, \(k = 1, 2, \ldots, p - 1\).

\[\text{Proof. } (p, k) = 1, k = 1, 2, \ldots, p - 1. \]

For example, for \(p = 5\), we can get longest forbidden substrings of length 4 for all \(k\)-shifts, \(k = 1, 2, 3, 4\) by taking powers of the permutation \((12345)\), as done in Figure 3 above. For \(k = 3\), for instance, a longest forbidden substring is given by \((12345)^3 = (14253)\) \(\rightarrow\) 14253. This substring in \(\{D_5^5\}\) corresponds to forbidden positions along a 5 \(\times\) 5 chessboard for the \(k\)-shift \(k = 3\), as can be seen in Figure 4.

![Figure 4: Forbidden positions in \(\{D_5^5\}\).](image)

Proposition 2.5 and Corollary 2.6 in turn imply:

**Corollary 2.7.** The number of permutations in \(\{D_k^p\}\) for any \(k\)-shift, \(k = 1, 2, \ldots, p - 1\), is the same as the number of permutations in \(\{D_p\}\), \(p\) prime.

One can see from Table 2 in the Appendix, for instance, that for \(p = 5\) there are 45 permutations in \(\{D_5^5\}\) for any \(k\)-shift \(k = 1, 2, \ldots, 4\).

The maximum cycle length achieved for a particular \(n\) and \(k\) is a very important statistic. In fact, for any fixed \(n\), \(k\)-shifts that have the same maximum cycle length will produce the same number of permutations, as can be seen in Table 2 in the Appendix. This table shows that there will be the same number of permutations in \(\{D_n^{k_1}\}\) and \(\{D_n^{k_2}\}\) whenever \((n, k_1) = (n, k_2)\), since then the maximum cycle lengths will be equal.

\[\text{No similar table appears in other references to our knowledge.}\]
References

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APPENDIX

\[
\begin{array}{ccccccc}
 n & D_{\text{er}} & d_1 & d_2 & d_3 & d_4 & d_5 \\
 1 & 0 & & & & & \\
 2 & 1 & 1 & & & & \\
 3 & 2 & 3 & 4 & & & \\
 4 & 9 & 11 & 14 & 18 & & \\
 5 & 44 & 53 & 64 & 78 & 96 & \\
 6 & 265 & 309 & 362 & 426 & 504 & 600 \\
 7 & 1,854 & 2,119 & 2,428 & 2,790 & 3,216 & 3,720 \\
 8 & 14,833 & 16,687 & 18,806 & 21,234 & 24,024 & 27,240 \\
\end{array}
\]

Table 1: Some values for \(d_n^k\).

\[
\begin{array}{cccccccc}
 n & k = 1 & k = 2 & k = 3 & k = 4 & k = 5 & k = 6 \\
 2 & 0 & & & & & \\
 3 & & & & & & \\
 4 & 3 & 3 & 8 & & & \\
 5 & 8 & 8 & 8 & & & \\
 6 & 45 & 45 & 45 & 45 & & \\
 7 & 264 & 270 & 240 & 270 & 264 & \\
 8 & 1,855 & 1,855 & 1,855 & 1,855 & 1,855 & 1,855 \\
 9 & 14,832 & 14,816 & 14,832 & 13,824 & 14,832 & 14,816 \\
 10 & 133,497 & 133,497 & 134,298 & 133,497 & 133,497 & 134,298 \\
\end{array}
\]

Table 2: Some values for \(D_n^k\).