RANK-CRANK TYPE PDE’S FOR HIGHER LEVEL APPELL FUNCTIONS

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Abstract. In this paper we consider level \( l \) Appell functions, and find a partial differential equation for all odd \( l \). For \( l = 3 \) this recovers the Rank-Crank PDE, found by Atkin and Garvan, and for \( l = 5 \) we get a similar PDE found by Garvan.

1. Introduction and statement of results

Dyson in [5] introduced the rank of a partition, to explain the first two of the three Ramanujan-congruences

\[
\begin{align*}
p(5n + 4) &\equiv 0 \pmod{5}, \\
p(7n + 5) &\equiv 0 \pmod{7}, \\
p(11n + 6) &\equiv 0 \pmod{11}.
\end{align*}
\]

Here \( p(n) \) denotes the number of partitions of \( n \). He defined the rank of a partition as the largest part minus the number of its parts and conjectured that the partitions of \( 5n + 4 \) (resp. \( 7n + 5 \)) form \( 5 \) (resp. \( 7 \)) groups of equal size when sorted by their ranks modulo \( 5 \) (resp. \( 7 \)). This was later proven by Atkin and Swinnerton-Dyer in [3]. We are interested here in the generating function

\[
R(w; q) := \sum_{\lambda} w^{\text{rank}(\lambda)} q^{||\lambda||} = \frac{(1 - w)}{(q)_{\infty}} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\frac{3n^2 + 1}{2}}}{1 - wq^n},
\]

where \((q)_{\infty} := \prod_{n=1}^{\infty}(1 - q^n)\). In the first sum the \( \lambda \) run over all partitions, \( \text{rank}(\lambda) \) denotes the rank of \( \lambda \) and \( ||\lambda|| \) denotes the size of the partition (the sum of all the parts).

Another partition statistic is the so called crank of a partition. For the generating function we have

\[
C(w; q) := \sum_{\lambda} w^{\text{crank}(\lambda)} q^{||\lambda||} = \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - wq^n)(1 - w^{-1}q^n)}
= \frac{(1 - w)}{\sum_{n \in \mathbb{Z}}(-1)^n q^{\frac{2n(n-1)}{2}}w^n}.
\]

The crank was introduced by Andrews and Garvan in [1] to explain the Ramanujan congruence (1.1) with modulus 11.

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In the setting of Jacobi forms it is more natural to consider the following modified rank and crank generating functions

\[ R(z; \tau) := \frac{w^{1/2} q^{-1/24}}{1 - w} R(w, q), \]
\[ C(z; \tau) := \frac{w^{1/2} q^{-1/24}}{1 - w} C(w, q). \]

Here we use \( w = \exp(2\pi i z) \) and \( q = \exp(2\pi i \tau) \), with \( z \in \mathbb{C} \) and \( \tau \) in the complex upper half plane \( \mathbb{H} \).

**Remark 1.1.** \( C \) is a meromorphic Jacobi form of weight \( 1/2 \) and index \( -1/2 \) and in [7] it is shown that \( R \) is mock Jacobi form of weight \( 1/2 \) and index \( -3/2 \).

The two (modified) generating functions are related by a partial differential equation, which we will refer to as the Rank-Crank PDE.

**Theorem 1.2** (see [2]). If we define the heat operator \( H \) by

\[ H := \frac{3}{\pi i} \frac{\partial}{\partial \tau} + \frac{1}{(2\pi i)^2} \frac{\partial^2}{\partial z^2}, \]

then

\[ H R = 2\eta^2 C^3, \]

where \( \eta \) is the Dedekind \( \eta \)-function, given by \( \eta(\tau) = q^{1/24}(q)_{\infty} \).

Note that the identity found in [2] is slightly different, because they use a different normalization. However, the two are easily seen to be equivalent. In [4] it is explained how the Rank-Crank PDE arises naturally in the setting of certain non-holomorphic Jacobi forms and a generalization is given to partial differential equations for an infinite family of related functions.

The method in [4] works only in certain special cases and no results are found for the level \( l \) Appell functions

\[ A_l(z; \tau) := w^{l/2} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{\frac{l}{2}(n+1)}}{1 - wq^n}, \quad l \in \mathbb{Z}_{>0}, \quad \tag{1.2} \]

for values of \( l \) higher than 3.

Garvan ([6]), however, found the following PDE for a level 5 Appell function

**Theorem 1.3** (Garvan). Let

\[ G_5(z; \tau) := \frac{A_5(z; \tau)}{\eta(\tau)^5}, \]

and define the heat operator

\[ H := \frac{5}{\pi i} \frac{\partial}{\partial \tau} + \frac{1}{(2\pi i)^2} \frac{\partial^2}{\partial z^2}, \]

then

\[ (H^2 - E_4) G_5 = 24\eta^2 C^5, \]

where \( E_4 \) is the usual Eisenstein series

\[ E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \left( \sum_{d|n} d^3 \right) q^n. \]
Note that the identity found by Garvan is slightly different, because he uses a different normalization. However, the two are easily seen to be equivalent.

This theorem is a special case of the following

**Theorem 1.4.** Let \( l \) be an odd positive integer. Define

\[
\mathcal{H}_k := \frac{l}{\pi i} \frac{\partial}{\partial \tau} + \frac{1}{(2\pi i)^2} \frac{\partial^2}{\partial z^2} - \frac{l(2k - 1)}{12} E_2,
\]

\[
\mathcal{H}^k := \mathcal{H}_{2k-1} \mathcal{H}_{2k-3} \cdots \mathcal{H}_3 \mathcal{H}_1,
\]

where \( E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \left( \sum_{d|n} d \right) q^n \) is the usual Eisenstein series in weight 2. Then there exist holomorphic modular forms \( f_j \) (\( j = 0, 2, 4, \ldots, l-1 \)) on \( \text{SL}_2(\mathbb{Z}) \) of weight \( j \), such that

\[
(l-1)^j \sum_{k=0}^{(l-1)/2} f_{l-2k-1} \mathcal{H}^k A_l = (l-1)! f_0 \eta^l C^l.
\]

**Remark 1.5.** In the proof of the theorem we will see an explicit construction for the \( f_j \)'s for given \( l \).

In the next section we will proof Theorem 1.4 and in section 3 we will look at the first few cases and in particular we’ll see that the theorem for \( l = 5 \) is equivalent to Theorem 1.3.

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2. Proof of Theorem 1.4

Throughout we assume that \( l \) is an odd positive integer. We (trivially) have

\[
A_l(z + 1; \tau) = -A_l(z; \tau),
\]

and if we replace \( z \) by \( z + \tau \) and \( n \) by \( n - 1 \) in (1.2) we find

\[
e^{-2\pi i lz - \pi il \tau} A_l(z + \tau; \tau) = -w^{-l/2} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2} n(n-1)} \frac{1}{1 - w q^n},
\]

and so

\[
A_l(z; \tau) + e^{-2\pi i lz - \pi il \tau} A_l(z + \tau; \tau) = -w^{-l/2} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2} n(n-1)} \left( 1 - w^{1/2} q^n \right)
\]

\[
= -w^{-l/2} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2} n(n-1)} \sum_{r=0}^{l-1} w^r q^{nr}
\]

\[
= - \sum_{r=0}^{l-1} w^{r-1/2} q^{\frac{1}{4} (r-1/2)^2} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2} (n-1/2 + r/l)^2}
\]

\[
= - \sum_{r=0}^{l-1} e^{2\pi i (r-1/2)z - \frac{\pi i}{4} (r-1/2)^2} \vartheta_{1,r}(\tau),
\]
with

$\vartheta_{l,r}(\tau) := \sum_{n \in \mathbb{Z}} (-1)^n q^{2(n-1/2+r/l)^2}$.

It is easy to check that

$$\left( \frac{l}{\pi i} \frac{\partial}{\partial \tau} + \frac{1}{(2\pi i)^2} \frac{\partial^2}{\partial z^2} \right) e^{2\pi i (r-l/2)z - \frac{\pi i}{4}(r-l/2)^2 \tau} = 0,$$

and that for functions $F : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$

$$\mathcal{H}_k \left(F(z+1; \tau)\right) = \left(\mathcal{H}_k F\right)(z+1; \tau),$$

$$\mathcal{H}_k \left(e^{-2\pi i l z - \pi i l \tau} F(z+\tau; \tau)\right) = e^{-2\pi i l z - \pi i l \tau} \left(\mathcal{H}_k F\right)(z+\tau; \tau),$$

with $\mathcal{H}_k$ as in the theorem. Hence we get from applying $\mathcal{H}_1$ to equations (2.1) and (2.2)

$$(\mathcal{H}_1 A_l)(z+1; \tau) = -(\mathcal{H}_1 A_l)(z; \tau),$$

and

$$(\mathcal{H}_1 A_l)(z; \tau) + e^{-2\pi i l z - \pi i l \tau} (\mathcal{H}_1 A_l)(z+\tau; \tau)
= -2l \sum_{r=0}^{l-1} e^{2\pi i (r-l/2)z - \frac{\pi i}{4}(r-l/2)^2 \tau} \left(D_{l/2} \vartheta_{l,r}\right)(\tau),$$

with the operator $D_k$ defined by

$$D_k := \frac{1}{2\pi i} \frac{\partial}{\partial \tau} - \frac{k}{12} E_2.$$

If we now apply $\mathcal{H}_3, \mathcal{H}_5, \ldots,$ upto $\mathcal{H}_{2k-1}$ we find

$$(\mathcal{H}_k A_l)(z+1; \tau) = -(\mathcal{H}_k A_l)(z; \tau),$$

and

$$(\mathcal{H}_k A_l)(z; \tau) + e^{-2\pi i l z - \pi i l \tau} (\mathcal{H}_k A_l)(z+\tau; \tau)
= -(2l)^k \sum_{r=0}^{l-1} e^{2\pi i (r-l/2)z - \frac{\pi i}{4}(r-l/2)^2 \tau} \left(D_k \vartheta_{l,r}\right)(\tau),$$

with

$$D^k := D_{2k-3/2} D_{2k-7/2} \cdots D_{5/2} D_{1/2}.$$

We need the following

**Lemma 2.1.** Let $l$ be an odd positive integer, then there exist holomorphic modular forms $F_j$ ($j = 0, 2, 4, \ldots, l-1$) on $SL_2(\mathbb{Z})$ of weight $j$, such that

$$\sum_{k=0}^{(l-1)/2} F_{l-2k-1} D^k \vartheta_{l,r} = 0$$

for all $r \in \mathbb{Z}$. 

If we now define 
\[ P = \sum_{k=0}^{(l-1)/2} f_{l-2k-1} \mathcal{H}^k A_l, \]
with \( f_{l-2k-1} = (2l)^{-k} F_{l-2k-1} \) and \( F_j \) as in the lemma, then we see from equation (2.3) and (2.4) 
\[ P(z + 1; \tau) = e^{-2\pi i l z - \pi i l \tau} P(z + \tau; \tau) = -P(z; \tau). \] (2.6)

Now consider the Jacobi theta function
\[ \vartheta(z; \tau) := \sum_{n \in \mathbb{Z}} (-1)^n w^{n+1/2} q^{1/2 (n+1)^2} = w^{1/2} q^{1/8} \prod_{n=1}^{\infty} (1 - q^n)(1 - w q^n)(1 - w^{-1} q^{n-1}) = -\frac{\eta(\tau)^2}{\mathcal{C}(z; \tau)}. \]

This function satisfies
\[ \vartheta(z + 1; \tau) = e^{2\pi i z + \pi i \tau} \vartheta(z + \tau; \tau) = -\vartheta(z; \tau), \] (2.7)
\( z \mapsto \vartheta(z; \tau) \) has simple zeros in \( \mathbb{Z} \tau + \mathbb{Z} \) and
\[ \frac{1}{2\pi i} \frac{\partial}{\partial z} \vartheta(z; \tau) = \eta(\tau)^3. \] (2.8)

Since the poles of \( z \mapsto A_l(z; \tau) \) are simple poles in \( \mathbb{Z} \tau + \mathbb{Z} \), the function \( z \mapsto P(z; \tau) \) has poles of order \( l \) in \( \mathbb{Z} \tau + \mathbb{Z} \), and so the function
\[ p(z; \tau) := \frac{\partial}{\partial z} P(z; \tau), \]
is a holomorphic function as a function of \( z \). Using (2.6) and (2.7) we find that
\[ p(z + 1; \tau) = p(z + \tau; \tau) = p(z; \tau), \]
from which we get that \( p \) is constant (as a function of \( z \)). To determine the constant, we consider the behaviour for \( z \to 0 \). From (1.2) we easily see that for \( z \to 0 \)
\[ A_l(z; \tau) = -\frac{1}{2\pi i z} + \mathcal{O}(1), \]
and so
\[ P(z; \tau) = -f_0(\tau) \frac{(l-1)!}{(2\pi i)^l} \frac{1}{z^l} + \mathcal{O}\left(\frac{1}{z^{l-1}}\right). \]

Combining this with (2.8) we see
\[ p(z; \tau) = -f_0(\tau) (l-1)! \eta(\tau)^3, \]
and so
\[ P(z; \tau) = -f_0(\tau) (l-1)! \frac{\eta(\tau)^3}{\vartheta(z; \tau)} = (l-1)! f_0(\tau) \eta(\tau)^3 \mathcal{C}(z; \tau)^l, \]
which finishes the proof.
Proof of Lemma 2.7. Throughout, let \( l \) be an odd integer. Because of the trivial relations
\[
\begin{align*}
\vartheta_{l,r+l} &= -\vartheta_{l,r} \\
\vartheta_{l,-r} &= -\vartheta_{l,r}
\end{align*}
\]
it suffices to consider \( \vartheta_{l,r} \) for \( r = 1, 2, \ldots, (l-1)/2 \). Define
\[
\Theta_l = \begin{pmatrix}
\vartheta_{l,1} \\
\vartheta_{l,2} \\
\vdots \\
\vartheta_{l,(l-1)/2}
\end{pmatrix},
\]
then \( \Theta_l \) transforms as a (vector-valued) modular form of weight 1/2 on the full modular group \( \text{SL}_2(\mathbb{Z}) \):
\[
\begin{align*}
\Theta_l(\tau+1) &= \text{diag} \left( \zeta_{8l}^{j(l-2j)^2} \right)_{1 \leq j \leq (l-1)/2} \Theta_l(\tau), \\
\Theta_l(-1/\tau) &= (-1)^{(l+1)/2} \sqrt{\tau/\ell} \left( 2\sin 2\pi r k / l \right)_{1 \leq r,k \leq (l-1)/2} \Theta_l(\tau).
\end{align*}
\]
Using
\[
E_2 \left( \frac{a \tau + b}{c \tau + d} \right) = (c \tau + d)^2 E_2(\tau) + \frac{6}{\pi i} c (c \tau + d) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),
\]
we can easily verify that
\[
D_k \left( (c \tau + d)^{-k} f \left( \frac{a \tau + b}{c \tau + d} \right) \right) = (c \tau + d)^{-k-2} \left( D_k f \right) \left( \frac{a \tau + b}{c \tau + d} \right),
\]
and so
\[
D^k \left( (c \tau + d)^{-1/2} \Theta_l \left( \frac{a \tau + b}{c \tau + d} \right) \right) = (c \tau + d)^{-2k-1/2} \left( D^k \Theta_l \right) \left( \frac{a \tau + b}{c \tau + d} \right).
\]
Now define the \( (l-1)/2 \times (l-1)/2 \)-matrix
\[
T_l = \begin{pmatrix}
\Theta_l & D^1 \Theta_l & D^2 \Theta_l & \cdots & D^{(l-3)/2} \Theta_l
\end{pmatrix},
\]
then \( T_l \) transforms as a (matrix-valued) modular form on the full modular group \( \text{SL}_2(\mathbb{Z}) \):
\[
\begin{align*}
T_l(\tau+1) &= \text{diag} \left( \zeta_{8l}^{j(l-2j)^2} \right)_{1 \leq j \leq (l-1)/2} T_l(\tau), \\
T_l(-1/\tau) &= (-1)^{(l+1)/2} \sqrt{\tau/\ell} \left( 2\sin 2\pi r k / l \right)_{1 \leq r,k \leq (l-1)/2} T_l(\tau) \text{diag} \left( \tau^{2j-2} \right)_{1 \leq j \leq (l-1)/2}.
\end{align*}
\]
From this we see that
\[
\begin{align*}
\det(T_l(\tau+1)) &= \zeta_{24}^{(l-1)(l-2)/2} \det(T_l(\tau)), \\
\det(T_l(-1/\tau)) &= (-i\tau)^{(l-1)(l-2)/4} \det(T_l(\tau)),
\end{align*}
\]
and so \( \det(T_l) \) is a multiple of \( \eta^{(l-1)(l-2)/2} \). We determine what that multiple is by looking at the lowest order terms:
First observe that by doing elementary column operations we get
\[
\det(T_l(\tau)) = \det \left( \Theta_l \quad \partial_\tau \Theta_l \quad \partial_\tau^2 \Theta_l \quad \cdots \quad \partial_\tau^{(l-3)/2} \Theta_l \right),
\]
with \( \partial_\tau := \frac{1}{2\pi i} \frac{\partial}{\partial \tau} \).
For \( 1 \leq r \leq (l-1)/2 \) we have

\[
\partial_{l,r}(\tau) = q^{(l-2r)^2/8l} (1 + \mathcal{O}(q)),
\]

so

\[
\begin{pmatrix}
\Theta_l & \partial_\tau \Theta_l & \partial_{\tau}^2 \Theta_l & \cdots & \partial_{\tau}^{(l-3)/2} \Theta_l
\end{pmatrix}
= \text{diag} \left( q^{(l-2i)^2/8l} \right)_{1 \leq i \leq (l-1)/2} \cdot \left( \frac{(l-2i)^2}{8l} \right)^{j-1} + \mathcal{O}(q) \right)_{1 \leq i,j \leq (l-1)/2},
\]

\[
\det \begin{pmatrix}
\Theta_l & \partial_\tau \Theta_l & \partial_{\tau}^2 \Theta_l & \cdots & \partial_{\tau}^{(l-3)/2} \Theta_l
\end{pmatrix}
= q^{(l-1)(l-2)/48} (\det(B) + \mathcal{O}(q)),
\]

and hence

\[
\det(T_l(\tau)) = \det(B) \eta(\tau)^{(l-1)(l-2)/2},
\tag{2.10}
\]

with

\[
B_{ij} = \left( \frac{(l-2i)^2}{8l} \right)^{j-1} \quad \text{for} \quad 1 \leq i, j \leq (l-1)/2.
\]

\( B \) is a Vandermonde matrix: an \( m \times n \) matrix \( V \), such that \( V_{ij} = \alpha_i^{j-1} \) with \( \alpha_i \in \mathbb{R} \). Since a square Vandermonde matrix is invertible if and only if the \( \alpha_i \) are distinct, we see that \( B \) is invertible. From (2.10) and the fact that \( \eta \) has no zeros on \( \mathbb{H} \) we then get that \( T_l(\tau) \) is invertible for all \( \tau \in \mathbb{H} \). We can rewrite the condition that (2.5) holds for \( 1 \leq r \leq (l-1)/2 \) as

\[
T_l \begin{pmatrix} F_{l-1} \\ F_{l-3} \\ \vdots \\ F_2 \end{pmatrix} + F_0 D^{(l-1)/2} \Theta_l = 0.
\]

If we take \( F_0 = 1 \) we get the other \( F_j \)'s by inverting \( T_l \)

\[
\begin{pmatrix} F_{l-1} \\ F_{l-3} \\ \vdots \\ F_2 \end{pmatrix} = -T_l^{-1} D^{(l-1)/2} \Theta_l.
\tag{2.11}
\]

What remains to be shown is that the \( F_j \) found this way are holomorphic modular forms of weight \( j \) on \( \text{SL}_2(\mathbb{Z}) \). The modular transformation properties follow easy from (2.9) and those of \( D^{(l-1)/2} \Theta_l \)

\[
D^{(l-1)/2} \Theta_l(\tau + 1) = \text{diag} \left( \left( \frac{(l-2j)^2}{8l} \right) \right)_{1 \leq j \leq (l-1)/2} D^{(l-1)/2} \Theta_l(\tau),
\]

\[
D^{(l-1)/2} \Theta_l(-1/\tau) = (-1)^{(l+1)/2} \sqrt{\tau/l} \tau^{(l-1)/2} (2\sin 2\pi r k/l)_{1 \leq r,k \leq (l-1)/2} D^{(l-1)/2} \Theta_l(\tau).
\]
Since \( \det T_l \) has no zeros on \( \mathbb{H} \) we get that \( F_j \) is a holomorphic function on \( \mathbb{H} \). That it also doesn’t have a pole at infinity follows from

\[
D^{(l-1)/2} \Theta_l = \text{diag} \left( q^{(l-2i)^2/8l} \right)_{1 \leq i \leq (l-1)/2} \cdot \begin{pmatrix} \mathcal{O}(1) \\ \mathcal{O}(1) \\ \vdots \\ \mathcal{O}(1) \end{pmatrix},
\]

\[
T_l = \text{diag} \left( q^{(l-2i)^2/8l} \right)_{1 \leq i \leq (l-1)/2} \cdot (C_{ij} + \mathcal{O}(q))_{1 \leq i,j \leq (l-1)/2},
\]

for some \((l-1)/2 \times (l-1)/2\)-matrix \( C \), with

\[
\det(C) = \det(B) \neq 0.
\]

\[\square\]

3. Some examples

From (2.11) we can calculate the first few coefficients in the Fourier expansion of the \( F_j \)’s and since they are holomorphic modular forms on \( \text{SL}_2(\mathbb{Z}) \), that means that we can easily identify them.

For \( l = 3 \), we have

\[
\Theta_l = (\vartheta_{3,1}) = (\eta).
\]

Using

\[
D_{k/2} \left( \eta^k \right) = 0,
\]

which follows from

\[
E_2 = \frac{12 \eta'}{\pi i \eta},
\]

we see

\[
D_{1/2} \Theta_l = 0,
\]

and so we find

\[
F_0(\tau) = 1 \quad \text{and} \quad F_2(\tau) = 0,
\]

and

\[
f_0(\tau) = 1/6 \quad \text{and} \quad f_2(\tau) = 0.
\]

If we put this into Theorem 1.4 and multiply by 6 we get

\[
\mathcal{H}_1 A_3 = 2\eta^3 C^3.
\]

Using

\[
\mathcal{R}(z; \tau) = \frac{A_3(z; \tau)}{\eta(\tau)} + e^{\pi iz - \pi i\tau/12},
\]

we see

\[
\mathcal{H}_{1/2} \mathcal{R} = \mathcal{H}_{1/2} \left( \frac{A_3}{\eta} \right) = \frac{\mathcal{H}_1 A_3}{\eta} + 6 A_3 D_{-1/2} \left( \frac{1}{\eta} \right) = 2\eta^2 C^3,
\]

which is the Rank-Crank PDE.

For \( l = 5 \), we find from (2.11) \((F_0 = 1)\)

\[
F_4(\tau) = \frac{-11}{3600} - \frac{11}{15}q + \mathcal{O}(q^2),
\]

\[
F_2(\tau) = \mathcal{O}(q^2),
\]
and hence we can identify them as

\[ F_4 = -\frac{11}{3600} E_4 \quad \text{and} \quad F_2 = 0. \]

So

\[ f_0 = \frac{1}{100}, \quad f_2 = 0 \quad \text{and} \quad f_4 = -\frac{11}{3600} E_4. \]

If we put this into Theorem 1.4 and multiply by 100 we get

\[ \left( \mathcal{H}_3 \mathcal{H}_1 - \frac{11}{36} E_4 \right) A_5 = 24 \eta^5 c^5. \]

We now rewrite this in terms of \( G_5 \):

\[ \mathcal{H}_1 A_5 = \mathcal{H}_1 (\eta^3 G_5) \]

\[ = 10 \left( D_{3/2} \eta^3 \right) G_5 + \eta^3 \mathcal{H}_{-1/2} G_5 = \eta^3 \mathcal{H}_{-1/2} G_5, \]

\[ \mathcal{H}_3 \mathcal{H}_1 A_5 = \mathcal{H}_3 (\eta^3 \mathcal{H}_{-1/2} G_5) \]

\[ = 10 \left( D_{3/2} \eta^3 \right) \mathcal{H}_{-1/2} G_5 + \eta^3 \mathcal{H}_{3/2} \mathcal{H}_{-1/2} G_5 = \eta^3 \mathcal{H}_{3/2} \mathcal{H}_{-1/2} G_5, \]

and so we get

\[ \left( \mathcal{H}_{3/2} \mathcal{H}_{-1/2} - \frac{11}{36} E_4 \right) G_5 = 24 \eta^2 c^5. \]

Using

\[ \mathcal{H}_{3/2} \mathcal{H}_{-1/2} = \mathcal{H}^2 + \frac{25}{3} \left( \frac{1}{2\pi i} E'_2 - \frac{1}{12} E_2^2 \right) \]

and

\[ \frac{1}{2\pi i} E'_2 - \frac{1}{12} E_2^2 = -\frac{1}{12} E_4 \]

we see that is equivalent to the statement of Theorem 1.3.

For \( l = 7 \) we find

\[ F_6 = \frac{85}{74088} E_6, \quad F_4 = -\frac{5}{252} E_4, \quad F_2 = 0 \quad \text{and} \quad F_0 = 1. \]

For \( l = 9 \)

\[ F_8 = -\frac{253}{559872} E_8, \quad F_6 = \frac{53}{5832} E_6, \quad F_4 = -\frac{13}{216} E_4, \quad F_2 = 0, \quad F_0 = 1. \]

For \( l = 11 \)

\[ F_{10} = -\frac{7888}{39135393} E_{10}, \quad F_8 = -\frac{6151}{1724976} E_8, \quad F_6 = \frac{295}{8712} E_6, \]

\[ F_4 = -\frac{53}{396} E_4, \quad F_2 = 0, \quad F_0 = 1. \]

And for \( l = 13 \)

\[ F_{12} = -\frac{1462986875}{14412774445056} E_{12} + \frac{170060275}{5683867488} \Delta, \quad F_{10} = \frac{377735}{296120448} E_{10}, \]

\[ F_8 = -\frac{62165}{45556992} E_8, \quad F_6 = \frac{3281}{36504} E_6, \quad F_4 = -\frac{459}{1872} E_4, \quad F_2 = 0, \quad F_0 = 1. \]
References

[1] G.E. Andrews and F.G. Garvan, *Dyson’s crank of a partition*, Bull. Amer. Math. Soc. (N. S.) 18 No. 2 (1988), pages 167–171.

[2] A.O.L. Atkin and F.G. Garvan, *Relations between the ranks and the cranks of partitions*, Ramanujan Journal 7, pages 137–152.

[3] A.O.L. Atkin and H.P.F. Swinnerton-Dyer, *Some properties of partitions*, Proc. London Math. Soc. 4 (1954), pages 84–106.

[4] K. Bringmann and S.P. Zwegers, *Rank-crank type PDE’s and non-holomorphic Jacobi forms*, Math. Res. Lett., accepted for publication.

[5] F. Dyson, *Some guesses in the theory of partitions*, Eureka (Cambridge) 8 (1944), pages 10–15.

[6] F.G. Garvan, personal communication.

[7] S.P. Zwegers, *Mock theta functions I: Appell functions and the Mordell integral*, in preparation.

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