Addendum to “The VIT Transform Approach to Discrete-Time Signals and Linear Time-Varying Systems”

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Abstract: This addendum contains clarifications and a sharpening of some of the results on the VIT transform framework developed in [1]. The focus is on the right-coefficient and left-coefficient forms of the transform, the extraction of a first-order term from a left polynomial fraction, and the application to linear time-varying systems.

Keywords: VIT transform, discrete-time signals, linear time-varying systems

1. Definition of the VIT transform

Given a real or complex-valued discrete-time signal \( x(n) \), where \( n \) is the integer-valued time variable, in [1] the VIT transform \( X(z, k) \) of \( x(n) \) is defined to be the formal power series in \( z^{-1} \) given by the right-coefficient form

\[
X(z, k) = \sum_{i=0}^{\infty} z^{-i} x(k + i),
\]

where \( k \) is the integer-valued initial time variable. Here \( z \) is a symbol or indeterminate. The transform \( X(z, k) \) is an element of the set \( A[[z^{-1}]] \) consisting of all formal power series in \( z^{-1} \) with coefficients in \( A \), where \( A \) is the ring of all functions from the integers into the field of complex numbers with the usual pointwise operations.

It follows from Equation (1) that the VIT transform \( X(z, k) \) depends only on the values of the signal \( x(n) \) for \( n \geq k \). Hence the VIT transform is a one-sided transform. Note that the VIT transform of \( x(n) \) is equal to the VIT transform of \( x(n) \mathcal{H}(n - k) \), where \( \mathcal{H}(n - k) \) is the Heaviside step function defined by \( \mathcal{H}(n - k) = 1 \) for \( n \geq k \) and \( \mathcal{H}(n - k) = 0 \) for \( n < k \).

In [1], the set \( A[[z^{-1}]] \) is given the structure of a noncommutative ring by defining the usual addition of power series and with multiplication defined by \( z^{-(i+j)} = z^{-i}z^{-j} \) and

\[
a(k)z^{-i} = z^{-i}a(k + i), \quad a(k) \in A,
\]

Noncommutativity of the ring \( A[[z^{-1}]] \) is a consequence of the noncommutative multiplication given by Equation (2). Due to this noncommutative multiplication, the elements of \( A[[z^{-1}]] \) are
sometimes referred to as skew power series, and polynomials in \( z \) with coefficients in \( A \) and with the multiplication \( a(k)z^i = z^i a(k - i) \) are referred to as skew polynomials.

Given a signal \( x(n) \), in [2] the generalized \( z \)-transform \( \hat{x}(z, k) \) of \( x(n) \) is defined to be the skew power series \( \hat{x}(z, k) = \sum_{i=0}^{\infty} z^{-i} x(i) \delta(k) \), where \( \delta(k) \) is the unit pulse located at \( k = 0 \). The transform \( \hat{x}(z, k) \) can be written in the form \( \hat{x}(z, k) = \sum_{i=0}^{\infty} z^{-i} x(i + k) \delta(k) \), and thus \( \hat{x}(z, k) = X(z, k) \delta(k) \), where \( X(z, k) \) is the VIT transform of \( x(n) \). In the generalized \( z \)-transform, the initial time \( k \) is equal to zero; whereas in the VIT transform, the initial time \( k \) is a variable ranging over the set of integers.

2. Right-Coefficient and Left-Coefficient Forms of the VIT Transform

Now consider the signal \( x(n) = f(n)\mathcal{H}(n - k) \), where \( f(n) \) is a real or complex-valued function of \( n \), and the values of \( f(n) \) do not depend on the initial time variable \( k \). Then using the definition of multiplication (2), we can write the transform \( X(z, k) \) in the left-coefficient form

\[
X(z, k) = \sum_{i=0}^{\infty} z^{-i} f(k + i) \mathcal{H}(i) = \sum_{i=0}^{\infty} f(k) \mathcal{H}(i) z^{-i} = \sum_{i=0}^{\infty} f(k) z^{-i}.
\]  

(3)

Since \( \sum_{i=0}^{\infty} z^{-i} z = (z - 1)^{-1}z \), the left-coefficient form of \( X(z, k) \) reduces to \( f(k) \) multiplied on the right by the fraction \( (z - 1)^{-1}z \); that is,

\[
X(z, k) = f(k)(z - 1)^{-1}z.
\]  

(4)

To check this result, first observe that the inverse VIT transform of \( (z - 1)^{-1}z \) is equal to \( \mathcal{H}(n - k) \). Then by the multiplication by a time function property given in [1], the inverse VIT transform of \( f(k)(z - 1)^{-1}z \) is equal to \( f(n)\mathcal{H}(n - k) \). This verifies the form \( f(k)(z - 1)^{-1}z \) for the VIT transform of the signal \( x(n) = f(n)\mathcal{H}(n - k) \).

Note that setting \( k = 0 \) in the right-coefficient form in Equation (3) results in \( \sum_{i=0}^{\infty} z^{-i} f(i) \), which is equal to the formal \( z \)-transform of the function \( f(n) \). However, setting \( k = 0 \) in the left-coefficient form in Equation (4) results in \( f(0)(z - 1)^{-1}z \), which is not equal to the \( z \)-transform of \( f(n) \). To eliminate this inconsistency, we define the evaluation of the VIT transform \( X(z, k) \) at a particular value of \( k \) to be the evaluation of the right-coefficient form at that value of \( k \).

It is important to note that if the values of \( f(n) \) depend on the initial time variable \( k \), then the form \( f(k)(z - 1)^{-1}z \) for the VIT transform of \( f(n)\mathcal{H}(n - k) \) is not valid. If \( f(n) \) does depend on \( k \), then \( f(n) = f(n, k) \), and the left-coefficient form of the VIT transform of \( x(n) = f(n, k)\mathcal{H}(n - k) \) is

\[
X(z, k) = \sum_{i=0}^{\infty} z^{-i} f(k + i, k) \mathcal{H}(i) = \sum_{i=0}^{\infty} f(k, k - i) z^{-i}.
\]  

(5)

Now since \( f(n, k) \) varies as \( k \) is varied, \( f(k, k - i) \neq f(k, k) \), and thus the VIT transform cannot be written in the form \( f(k)(z - 1)^{-1}z \), where \( f(k) = f(k, k) \).
For an example, let $a(k)$ and $b(k)$ be functions from the integers into the real or complex numbers, and let $a_0(k) = 1, a_i(k) = a(k)a(k + 1) \cdots a(k + i - 1), i \geq 1$. Consider the signal

$$x(n, k) = a_{n-k}(k)b(k), \quad n \geq k,$$

(6)

with initial value $x(k, k) = b(k)$. Note that if $a(k)$ is the constant function $a(k) = c$ for all $k$, where $c$ is a real or complex number, then $a_{n-k} = c^{n-k}$ and $x(n, k) = c^{n-k}b(k), \quad n \geq k$. The right-coefficient form of the VIT transform of the signal $x(k, n)$ given by Equation (6) is

$$X(z, k) = \sum_{i=0}^{\infty} z^{-i} a_i(k)b(k).$$

This transform can be written in the left polynomial fraction form

$$X(z, k) = (z - a(k))^{-1} z b(k) = (z - a(k))^{-1} [b(k + 1)z].$$

(7)

To verify Equation (7), divide $z - a(k)$ into $z$ using left long division.

The left-coefficient form of the VIT transform of the signal given by Equation (6) is

$$X(z, k) = \sum_{i=0}^{\infty} a_i(k-i)b(k-i)z^{-i}$$

(8)

The right side of Equation (8) can be written in the right polynomial fraction form

$$X(z, k) = [b(k)z] \left[ z - \frac{b(k-1)}{b(k)} a(k-1) \right]^{-1},$$

(9)

where $\frac{1}{b(k)}$ is viewed as an element of the quotient field $Q(A)$ of $A$ in the case when $b(k)$ has values that are equal to zero. The right polynomial fraction form in Equation (9) can be verified by dividing $z - \frac{b(k-1)}{b(k)} a(k-1)$ into $z$ using right long division. Then combining (7) and (9), we have

$$(z - a(k))^{-1} [b(k + 1)z] = [b(k)z] \left[ z - \frac{b(k-1)}{b(k)} a(k-1) \right]^{-1}.$$ 

(10)

From Equation (10), it is seen that moving $b(k + 1)z$ to the left through $(z - a(k))^{-1}$ changes the coefficient $a(k)$ in the denominator polynomial to $\frac{b(k-1)}{b(k)} a(k-1)$. This noncommutativity is a fundamental aspect of the VIT transform framework.

In this example, it can be shown that the right polynomial fraction form of the transform $X(z, k)$ can be derived directly from the left polynomial fraction form. This turns out to be true in the general case when $X(z, k) = \mu(z, k)^{-1} v(z, k)$, where $\mu(z, k)$ and $v(z, k)$ are skew polynomials in $z$ with coefficients in $A$ . Similar to the discussion of the extended right Euclidean algorithm given in [1], the extended left Euclidean algorithm can be used to determine polynomials $\alpha(z, k)$ and $\beta(z, k)$ such that $\mu(z, k)\alpha(z, k) = v(z, k)\beta(z, k)$. In general, the coefficients of $\alpha(z, k)$ and $\beta(z, k)$ belong to the quotient field $Q(A)$ of $A$. Then $\mu(z, k)^{-1} v(z, k) = \alpha(z, k)[\beta(z, k)]^{-1}$, and therefore, $\alpha(z, k)[\beta(z, k)]^{-1}$ is a right polynomial fraction form of $X(z, k)$. 
3. Extraction of a First-Order Term

Given a skew polynomial $\xi(z,k)$ with coefficients in $A$ and a function $a(k) \in A$, consider the polynomial fraction $\left([z - a(k)]\xi(z,k)\right)^{-1}$. The extended right Euclidean algorithm can be used to determine a polynomial $\eta(z,k)$ with coefficients in $Q(A)$ and $d(k) \in Q(A)$ such that $\eta(z,k)(z - a(k)) + d(k)\xi(z,k) = 1$. Then

$$
\left([z - a(k)]\xi(z,k)\right)^{-1} = [\eta(z,k)(z - a(k)) + d(k)\xi(z,k)] [\xi(z,k)]^{-1} [z - a(k)]^{-1} = [\eta(z,k)(z - a(k))][\xi(z,k)]^{-1} [z - a(k)]^{-1} + d(k)[z - a(k)]^{-1}. \tag{11}
$$

If $a(k)$ and the coefficients of $\xi(z,k)$ are constant functions, then $z - a(k)$ commutes with $\xi(z,k)$, that is, $(z - a(k))\xi(z,k) = \xi(z,k)(z - a(k))$, and $[\xi(z,k)]^{-1} [z - a(k)]^{-1} = [z - a(k)]^{-1} [\xi(z,k)]^{-1}$. In this case, Equation (11) reduces to

$$
\left([z - a(k)]\xi(z,k)\right)^{-1} = [\eta(z,k)][\xi(z,k)]^{-1} + d(k)[z - a(k)]^{-1}. \tag{12}
$$

Hence the first-order term $d(k)[z - a(k)]^{-1}$ is extracted from the fraction $\left([z - a(k)]\xi(z,k)\right)^{-1}$.

If $a(k)$ and/or the coefficients of $\xi(z,k)$ vary as functions of $k$, then $z - a(k)$ and $\xi(z,k)$ do not commute. In this case, in [1] the extraction of a first-order term from the fraction $\left([z - a(k)]\xi(z,k)\right)^{-1}$ is approached by first computing $\beta(k) \in Q(A)$ and a polynomial $\varphi(z,k)$ with coefficients in $Q(A)$ such that

$$
(z - a(k))\xi(z,k) = \varphi(z,k)(z - \beta(k)). \tag{13}
$$

However, it is not necessary to express $(z - a(k))\xi(z,k)$ in the form given by Equation (13) in order to extract a first-order term. A sufficient condition for extracting a first-order term is that there exist a polynomial $\rho(z,k)$ with coefficients in $Q(A)$ and $b(k) \in Q(A)$ such that

$$
\rho(z,k)(z - a(k)) + \xi(z,k)b(k) = 1. \tag{14}
$$

To prove sufficiency, multiple both sides of Equation (14) on the left by $\xi(z,k)^{-1}$ and on the right by $[z - a(k)]^{-1}$. This results in the decomposition

$$
\xi(z,k)^{-1}[z - a(k)]^{-1} = [\xi(z,k)\xi(z,k)]^{-1} = [\xi(z,k)]^{-1} \rho(z,k) + b(k)[z - a(k)]^{-1}, \tag{15}
$$

and thus, the first-order term $b(k)[z - a(k)]^{-1}$ is extracted from the fraction. Also note that the denominators of the terms in the decomposition (15) are equal to the factors $\xi(z,k)$ and $z - a(k)$ comprising the denominator of the fraction $\left([z - a(k)]\xi(z,k)\right)^{-1}$.

The computation of $b(k)$ that satisfies Equation (14) can be carried out by evaluating both sides of Equation (14) at $z^i = a_i(k)$, where $a_0(k) = 1, a_i(k) = a(k)a(k + 1)\ldots a(k + i - 1), i \geq 1$. First, it follows from the results in [3] that the evaluation at $z^i = a_i(k)$ of a skew polynomial $\gamma(z,k)$, with coefficients written on the left, is equal to the remainder after dividing $\gamma(z,k)$ on the right by $z - a(k)$. If $\gamma(z,k)$ has $z - a(k)$ as a right factor, the remainder after dividing by $z - a(k)$ on the right is equal to zero. Hence, in this case, the evaluation of $\gamma(z,k)$ at $z^i = a_i(k)$ is
equal to zero. Finally, evaluation is an additive operation; that is, the evaluation of the sum of two skew polynomials is equal to the sum of the evaluations of the two polynomials.

Now suppose that \( \xi(z,k) = z^N + \sum_{i=0}^{N-1} \xi_i(k) z^i \), \( \xi_i(k) \in A \). Then

\[
\xi(z,k) b(k) = b(k+N) z^N + \sum_{i=0}^{N-1} \xi_i(k) b(k+i) z^i ,
\]

and evaluating both sides of Equation (14) at \( z^i = a_i(k) \) gives

\[
b(k+N) a_N(k) + \sum_{i=0}^{N-1} \xi_i(k) b(k+i) a_i(k) = 1 \tag{16}
\]

Thus, the solution \( b(k) \) to Equation (14) satisfies the \( N \)th-order linear time-varying difference equation (16). Note that if \( a(k) = a \) and \( \xi_i(k) = \xi_i \) for all integers \( k \), then \( b(k) \) is also a constant and is equal to \([a^N + \sum_{i=0}^{N-1} \xi_i a_i]^{-1}\).

Equation (16) specifies \( b(k) \) for all \( k \) ranging over the set of integers. Given a fixed integer \( k_0 \), \( b(k) \) can be computed iteratively for \( k \geq k_0 \) by solving Equation (16) with initial values \( b(k_0−i),i = 1,2, ..., N \). Since the values of \( b(k) \) for \( k \geq k_0 \) depend on the initial values, there is no unique solution for \( b(k) \) for \( k \geq k_0 \). If \( b(k) \) is approximately constant over every \((N+1)\)-step interval \( k, k + 1, ..., k + N \), then \( b(k) \) can be approximated by

\[
b(k) = [a_N(k) + \sum_{i=0}^{N-1} \xi_i(k) a_i(k)]^{-1} .
\]

Once \( b(k) \) has been computed for some desired range of \( k \), the polynomial \( \rho(z,k) \) in Equation (14) can be determined by equating the coefficients of like powers of \( z \) in the left and right sides of Equation (14), with the coefficients of polynomials written on the left of the \( z^i \). The details are omitted.

Given \( \eta(z,k) = \sum_{i=0}^{M} \eta_i(k) z^i \), \( \eta_i(k) \in A, M \leq N \), we shall now extract a first-order term from the left polynomial fraction \( [(z − a(k))\xi(z,k)]^{-1} \eta(z,k) \). First, define \( \hat{\eta}(z,k) = \sum_{i=0}^{M} \eta_i(k−i) z^i \). Then from the results in [3], the remainder \( r(k) \) after dividing \( \eta(z,k) \) on the left by \( z − a(k) \) is equal to \( \hat{\eta}(z,k) \) evaluated at \( z^i = \hat{a}_i(k) \), where \( \hat{a}_i(k) = a(a(k)a(k−1) \cdots a(k−i+1), i \geq 1, \hat{a}_0(k) = 1 \). Thus,

\[
(z − a(k))^{-1} \eta(z,k) = q(z,k) + (z − a(k))^{-1} r(k), \tag{17}
\]

where \( q(z,k) \) is a polynomial in \( z \) with coefficients in \( Q(A) \) and

\[
r(k) = \sum_{i=0}^{M} \eta_i(k−i) \hat{a}_i(k) . \tag{18}
\]

Multiplying both sides of Equation (15) on the right by \( \eta(z,k) \) gives

\[
[(z − a(k))\xi(z,k)]^{-1} \eta(z,k) = \xi(z,k)^{-1} \rho(z,k) \eta(z,k) + b(k)[z − a(k)]^{-1} \eta(z,k) . \tag{19}
\]

Inserting Equation (17) into Equation (19) yields
\[
\left[(z - a(k))\xi(z,k)\right]^{-1} \eta(z,k) = \xi(z,k)^{-1} \rho(z,k) \eta(z,k) + b(k)q(z,k) + b(k)[z - a(k)]^{-1}r(k).
\]  

(20)

Finally, since \( M \leq N \), the fraction \( \left[(z - a(k))\xi(z,k)\right]^{-1} \eta(z,k) \) is strictly proper, and since \( b(k)[z - a(k)]^{-1}r(k) \) is also strictly proper, there exists a polynomial \( \pi(z,k) \) such that

\[
\xi(z,k)^{-1} \rho(z,k) \eta(z,k) + b(k)q(z,k) = \xi(z,k)^{-1} \pi(z,k).
\]

(21)

Inserting Equation (21) into Equation (20) completes the proof of the following result:

**Theorem 1.** Given \( a(k) \in A, \xi(z,k) = z^N + \sum_{i=0}^{N-1} \xi_i(k)z^i, \xi_i(k) \in A, \) and \( \eta(z,k) = \sum_{i=0}^{M} \eta_i(k)z^i, \eta_i(k) \in A, M \leq N, \) the left polynomial fraction \( \left[(z - a(k))\xi(z,k)\right]^{-1} \eta(z,k) \) has the decomposition

\[
\left[(z - a(k))\xi(z,k)\right]^{-1} \eta(z,k) = \xi(z,k)^{-1} \pi(z,k) + b(k)[z - a(k)]^{-1}r(k),
\]

(22)

where \( b(k) \) is the solution to the \( N \)th-order linear time-varying difference equation (16), and \( r(k) \) is given by Equation (18).

Let \( x(n,k) \) denote the signal \( x(n,k) = a_{n-k-1}(k), n > k, \) with initial value \( x(k,k) = 0. \) Then the inverse VIT transform of the first-order term \( b(k)[z - a(k)]^{-1}r(k) \) in the decomposition (22) is equal to \( b(n)x(n,k)r(k). \) Hence, the inverse transform of the first-order term is a scaling of \( x(n,k) \) by \( b(n) \) in the time variable \( n, \) and a scaling of \( x(n,k) \) by \( r(k) \) in the initial time variable \( k. \) Also note that if the skew polynomial \( \xi(z,k) \) has the factorization

\[ \xi(z,k) = (z - e(k))\theta(z,k), \] where \( e(k) \in A, \) and \( \theta(k,z) \) is a polynomial with coefficients in \( A, \) the above procedure can be repeated to extract a first-order term from \( \left[(z - e(k))\theta(z,k)\right]^{-1}. \)

Continuing this process will yield a decomposition of the fraction \( \left[(z - a(k))\xi(z,k)\right]^{-1} \eta(z,k) \) given in terms of a sum of first-order terms.

**4. Application to Linear Time-Varying Systems**

Consider a causal linear time-varying discrete-time system with input \( u(n) \) and resulting output response \( y(n) \). It is assumed that the input is applied beginning at time \( k, \) and is zero before time \( k. \) Here \( k \) is the initial time which is allowed to vary over the set of integers. Then the input can be expressed in the form \( u(n) = u(n) \mathcal{H}(n - k), \) which shows that \( u(n) \) depends on \( k, \) so we shall write the input as \( u(n,k). \) For example, let \( \delta(n - k) \) denote the unit pulse defined by \( \delta(n - k) = 1, n = k, \delta(n - k) = 0, n \neq k. \) Then the input \( u(n,k) = \delta(n - k) \) is the unit pulse applied at the initial time \( k. \) The output response of the system to the unit-pulse \( \delta(n - k) \) with zero initial conditions is the unit-pulse response function \( h(n,k). \) By casualty, \( h(n,k) \) is equal to zero when \( n < k. \)

Now consider the input \( u(n,k) \mathcal{H}(n - k) \) which can be written in the form
\[ u(n, k) \mathcal{H}(n - k) = \sum_{r=k}^{\infty} u(r, k) \delta(n - r). \]

By definition of \( h(n, k) \), the response to \( \delta(n - r) \) is \( h(n, r) \). Then by linearity, the output response to the input \( u(n, k) \mathcal{H}(n - k) \) with zero initial energy at time \( k \) is given by

\[ y(n, k) = \sum_{r=k}^{\infty} h(n, r)u(r,k). \]  

This is the input/output relationship of the system when the input \( u(n, k) \mathcal{H}(n - k) \) depends on both the current time \( n \) and the initial time \( k \).

In [2], the transfer function \( H(z, k) \) of the system given by Equation (23) is defined to be the skew power series

\[ H(z, k) = \sum_{i=0}^{\infty} z^{-i}h(k + i, k), \]  

and in the case when \( u(n, k) = u(n) \mathcal{H}(n) \), it is proved that \( \hat{y}(z, k) = H(z, k)\hat{u}(z, k) \), where \( \hat{u}(z, k) \) and \( \hat{y}(z, k) \) are the generalized z-transforms of the input and output, respectively. Here the output \( y(n) \) is the response of the system to the input \( u(n) \) applied beginning at the initial time \( n = 0 \).

The transfer function \( H(z, k) \) defined by Equation (24) is equal to the VIT transform of the unit-pulse response function \( h(n, k) \), and as proved in [1], the VIT transform \( Y(z, k) \) of the output \( y(n, k) \) resulting from the input \( u(n, k) \) is given by the product

\[ Y(z, k) = H(z, k)U(z, k), \]  

where \( U(z, k) \) is the VIT transform of the input \( u(n, k) \). Here the output \( y(n, k) \) is the response of the system to the input \( u(n, k) \) applied beginning at the initial time \( n = k \), where \( k \) varies over the set of integers. Hence, the VIT transform framework captures the dependency of the output response on the time when the input is applied, which is a key aspect of time-varying systems.

Note that the left-coefficient form of \( H(z, k) \) is given by

\[ H(z, k) = \sum_{i=0}^{\infty} h(k, k - i)z^{-i}. \]  

When \( z \) is viewed as a complex variable, \( H(z, k) \) defined by Equation (26) is equal to the ordinary z-transform of \( h(k, k - n) \), which is the definition of the transfer function given in [4]. Thus, the transfer function has the same form in both the z-transform approach developed in [4] and the VIT transform approach developed in [1]. However, the two approaches differ significantly since the skew ring framework in [1] is based on the noncommutative multiplication \( a(k)z^i = z^{-i}a(k - i), \ a(k) \in A \), whereas there is no noncommutative multiplication in the z-transform framework. It is a consequence of the noncommutative multiplication in the ring framework that the VIT transform of the output is equal to the product of \( H(z, k) \) with the VIT transform of the input; whereas the z-transform of the output is not equal in general to the product of \( H(z, k) \) with the z-transform of the input.
From the results in [1] and [2], when the system is given by the input/output difference equation
\[ y(n + N, k) + \sum_{i=0}^{N-1} \xi_i(n)y(n + i, k) = \sum_{i=0}^{M} v_i(n)u(n + i, k), \quad \xi_i(n), v_i(n) \in A, \]
where \( M \leq N \), the transfer function \( H(z, k) \) in the skew ring framework has the left polynomial fraction form given by
\[ H(z, k) = \left[z^N + \sum_{i=0}^{N-1} \xi_i(k)z^i\right]^{-1} \left[\sum_{i=0}^{M} v_i(k)z^i\right]. \]
Let \( \xi(z, k) = z^N + \sum_{i=0}^{N-1} \xi_i(k)z^i \) and \( v(z, k) = \sum_{i=0}^{M} v_i(k)z^i \). Then the VIT transform \( Y(z, k) \) of the output of the system defined by the difference equation (27) is equal to \( \xi(z, k)^{-1}v(z, k)U(z, k) \).

One consequence of the form \( \xi(z, k)^{-1}v(z, k)U(z, k) \) for \( Y(z, k) \) is that it leads directly to the inverse system. To show this, multiply both sides of \( Y(z, k) = \xi(z, k)^{-1}v(z, k)U(z, k) \) on the left by \( \xi(z, k) \), and then multiply the result on the left again by \( v(z, k)^{-1} \). This yields
\[ v(z, k)^{-1}\xi(z, k)Y(z, k) = U(z, k). \]

From Equation (29), it is seen that \( v(z, k)^{-1}[\xi(z, k)] \) is the transfer function of the inverse system; that is, this system reproduces the input \( u(n, k) \) from the output \( y(n, k) \). However, if \( M < N \), then dividing \( v(z, k) \) into \( \xi(z, k) \) using left long division will result in a term of the form \( w(k)z^{N-M} \), where \( w(k) \in A \) if \( v_M(k) \neq 0 \) for all integers \( k \). As a result, the inverse system defined by Equation (29) is not causal. To achieve causality, multiply both sides of Equation (29) on the left by \( z^{-(N-M)} \) which gives
\[ z^{-(N-M)}[v(z, k)]^{-1}\xi(z, k)Y(z, k) = [v(z, k)z^{N-M}]^{-1}\xi(z, k)Y(z, k) = z^{-(N-M)}U(z, k). \]

From the shifting property of the VIT transform given in [1], the inverse transform of \( z^{-(N-M)}U(z, k) \) is equal to \( u(n - (N - M), k) \). Hence, the causal inverse system with transfer function \( [v(z, k)z^{N-M}]^{-1}\xi(z, k) \) reproduces a time delayed version of \( u(n, k) \) from the output \( y(n, k) \).

Now suppose that the system defined by the difference equation (27) is asymptotically stable as defined in [1]. Using the VIT transform framework, we shall determine the steady-state output response of the system to the input \( u(n, k) = a_{n-k}(k) = a(k)a(k+1) \cdots a(n-1), n > k, \) with initial value \( u(k, k) = a_0(k) = 1 \). Here \( a(k) \) is an element of \( A \) with the condition that \( a_{n-k}(k) \) does not converge to zero as \( n \to \infty \). Note that \( u(n+1, k) = a(n)u(n, k) \), and the VIT transform of \( u(n, k) \) is equal to \( (z - a(k))^{-1}z \).

Then the VIT transform \( Y(z, k) \) of the output response is
\[ Y(z, k) = \xi(z, k)^{-1}v(z, k)(z - a(k))^{-1}z. \]

Dividing \( v(z, k) \) on the right by \( z - a(k) \) gives
\[ v(z, k)(z - a(k))^{-1} = \theta(z, k) + s(k)(z - a(k))^{-1}. \]
where \( \theta(z, k) \) is a polynomial in \( z \) with coefficients in \( Q(A) \), and the remainder \( s(k) \) is equal to the evaluation of \( v(z, k) \) at \( z^i = a_i(k) \); that is, \( s(k) = \sum_{i=0}^{M} v_i(k) a_i(k) \). Then inserting (31) into (30) results in

\[
Y(z, k) = \xi(z, k)^{-1} \theta(z, k)z + \xi(z, k)^{-1} s(k)(z - a(k))^{-1} z. \tag{32}
\]

The inverse VIT transform of the first term on the right side of Equation (32) decays to zero as \( n \to \infty \) since the system is asymptotically stable.

As in the above constructions leading to the proof of Theorem 1, it is possible to determine a polynomial \( \tau(z, k) \) with coefficients in \( Q(A) \) and \( p(k) \in Q(A) \) such that

\[
\tau(z, k)(z - a(k)) + \xi(z, k)p(k) = s(k), \tag{33}
\]

Evaluating Equation (33) at \( z^i = a_i(k) \) results in the following difference equation for \( p(k) \):

\[
p(k + N) a_N(k) + \sum_{i=0}^{N-1} \xi_i(k) a_i(k) p(k + i) = s(k) = \sum_{i=0}^{M} v_i(k) a_i(k). \tag{34}
\]

Then multiplying both sides of Equation (33) on the left by \( \xi(z, k)^{-1} \) and on the right by \( (z - a(k))^{-1} \), we have that the second term on the right side of Equation (32) has the decomposition

\[
\xi(z, k)^{-1} s(k)(z - a(k))^{-1} z = \xi(z, k)^{-1} \tau(z, k)z + p(k)(z - a(k))^{-1} z. \tag{35}
\]

The inverse VIT transform of the first term on the right side of Equation (35) decays to zero as \( n \to \infty \). Hence, the steady-state response to the input \( u(n, k) = a_{n-k}(k), k \geq n \), is equal to the inverse VIT transform of \( p(k)(z - a(k))^{-1} z \), which is equal to \( p(n) a_{n-k}(k), k \geq n \). This proves the following result.

**Theorem 2.** Suppose that the system defined by the input/output difference equation (27) is asymptotically stable. Then the steady-state response to the input \( u(n, k) = a_{n-k}(k), k \geq n \), \( a(n) \in A \), is equal to \( p(n) u(n, k) \), where \( p(n) \) is the solution to the difference equation (34) with \( k = n \).

By Theorem 2, the steady-state output response of an asymptotically stable system to the input \( u(n, k) = a_{n-k}(k), k \geq n \), is equal to a scaling of the input by the time function \( p(n) \). It is interesting to note that the expression for \( p(n) \) given by Equation (34) can be generated directly from the input/output difference equation by inserting \( u(n, k) = a_{n-k}(k) \) and \( y(n, k) = p(n) a_{n-k}(k) \) into Equation (27) and solving for \( p(n) \). The VIT transform framework as utilized here verifies that this solution to the input/output difference equation is in fact the steady-state response in the case when the system is asymptotically stable.
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