Anomalous quantum correlations in the motion of a trapped ion

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Abstract
In a previous paper (Krumm and Vogel 2018 Phys. Rev. A 97 043806) we presented a method to solve the nonlinear Jaynes–Cummings dynamics, describing the quantized motion of a trapped ion exactly, including detuning. Here we investigate this model with respect to nonclassical effects, such as squeezing and sub-Poisson statistics. We show that for the versatile model under study there exist quantum phenomena beyond squeezing and sub-Poisson statistics, such as anomalous quantum correlations of two non-commuting observables. In particular, it is shown that for the excitation of the zeroth sideband neither squeezing nor sub-Poisson statistics occur, but anomalous correlations can be verified. Furthermore, it is shown how these anomalous correlation functions can be derived from measured data.

Keywords: quantum physics, quantum optics, trapped ions

1. Introduction
In the wide-ranging field of quantum optics, vital areas of interest are the identification, characterization and quantification of nonclassical effects—i.e. effects that cannot be explained within Maxwell’s theory of classical electrodynamics. During the last decades significant efforts were made to develop techniques that allow not only for the theoretical description but also for the experimental verification of nonclassical states. Prominent examples are photon antibunching [1–3], squeezing [4–8], sub-Poisson statistics [9–11], and entanglement [12–17].

On a general basis, nonclassicality can be subdivided into two sets, namely single-time and multi-time nonclassicality. This means, there exist effects that can be characterized by using a single point in time and effects which need, for its description, two or more points in time. One example of the latter is photon antibunching as two points in time are required for its general analysis.

A general treatment of quantum correlations of radiation fields was introduced in [18]. Based on normal- and time-ordered correlation functions, it was shown that a plethora of multi-time nonclassicality criteria can be derived. They verify nonclassical effects beyond photon antibunching and, in the special single-time scenario, nonclassicality beyond squeezing and sub-Poisson light. Those phenomena include so-called anomalous correlations of non-commuting observables [19]. Such effects have recently been demonstrated to occur for squeezed coherent light, even for phase values when squeezing does not occur [20].

The question arises whether or not a similar behavior can be found in other, more sophisticated physical systems. An encouraging approach can be based on the Jaynes–Cummings model [21, 22], which was widely applied in cavity QED, see e.g. [23]. Using a vibrational rotating-wave approximation, this model also applies to describe the quantized center-of-mass motion of a trapped ion in a Paul trap [24, 25], for related experiments, see [26]. During the years it became feasible to study many nonclassical motional states of the ion [27–36]. Under more general conditions, the Hamiltonian describing the dynamics of a trapped ion attains the form of a nonlinear Jaynes–Cummings model [37]. Recently, the latter was extended to include some frequency mismatch, leading to an explicitly time-dependent dynamics [38]. It is noteworthy that related approaches can include the counter-rotating terms of the Hamiltonian, which are neglected within the vibrational
is a coherent input state of a cavity mode. In the rotating-wave approximation. The corresponding framework is referred to as Quantum Rabi Model, which is for example treated in [39–44].

In the present paper, we study the nonlinear Jaynes–Cummings dynamics to analyze quantum effects in the atomic center-of-mass motion of a trapped ion, with particular emphasis on anomalous quantum correlations. As those correlations are normal-ordered ones, they are not hindered by vacuum fluctuations which typically occur in the presence of losses. In addition, as it was demonstrated in a recent quantum-optics experiment, the anomalous quantum correlations are capable of certifying nonclassicality beyond squeezing; see [20]. The reduction of quantum noise effects by the use of squeezed states is limited to narrow phase intervals, in particular for strong squeezing. As strongly squeezed states can be easily prepared in the center-of-mass motion of trapped ions, see [28], this opens new applications of squeezing for phase-noise tolerant applications of trapped ions in quantum technology, with particular emphasis on anomalous quantum correlations. As those correlations can be even prepared when squeezing or sub-Poisson statistics do not persist.

The paper is structured as follows. In section 2 we briefly recapitulate the model under study. Afterwards, in section 3 we analyze in some detail nonclassical phenomena. A detailed consideration of the measurement of the correlation functions under study is provided in section 4. Finally, we give a summary and some conclusions in section 5.

2. The nonlinear Jaynes–Cummings model including detuning

The time-independent version of the nonlinear Jaynes–Cummings model was introduced in [37]. In order to study the influence of time ordering and a time-dependent Hamiltonian in general, we extended the model such that a detuning between the monochromatic driving laser and the electronic transitions can be included [38]. To solve the resulting time-dependent Hamiltonian analytically, the driving laser-field was quantized. That is, the amplitude of the laser, \( \beta_0 \), was replaced by the corresponding Hilbert-space operator \( \hat{b} \), which obeys the eigenvalue equation

\[
\hat{b}|\beta_0\rangle = \beta_0|\beta_0\rangle,
\]

where \( |\beta_0\rangle \) is a coherent input state of a cavity mode. In the limit of a strong coherent amplitude, \( \beta_0 \gg 1 \), the classical solutions of the dynamics are recovered [45]. In this section we recapitulate the basic equations and the obtained analytic solutions of the interaction problem. The total Hamiltonian to be studied in the Schrödinger picture, including the quantized pump field (cavity field), reads as

\[
\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}},
\]

\[
\hat{H}_0 = \hbar \omega_0 \hat{a}^\dagger \hat{a} + \hbar \omega_2 \hat{b}^\dagger \hat{b} + \hbar \omega_{21} \hat{a} \hat{b}^\dagger,
\]

\[
\hat{H}_{\text{int}} = \hbar \omega_{21} \hat{b}^\dagger \hat{a} \hat{b} + \hbar \omega_{12} \hat{a} \hat{b} + \h.c.
\]

(2)

The first term of \( \hat{H}_0 \) describes the free evolution of the vibrational center-of-mass motion, with the vibrational frequency \( \nu \). The second term represents the free evolution of the cavity field, with the laser frequency \( \omega_L = \omega_{21} - k \nu + \Delta \omega \). The free evolution of the electronic degrees of freedom of the two-level ion, with the electronic transition frequency \( \omega_{21} = \omega_2 - \omega_1 \), is given by the third term of \( \hat{H}_0 \). The operators \( \hat{a}^\dagger (\hat{a}) \) create (annihilate) the quanta of the vibrational mode whose energy levels are equidistantly separated by \( \nu \). The atomic \( |j\rangle \rightarrow |j+1\rangle \) transitions are described by the atomic flip operators \( \hat{A}_j = |j\rangle \langle j| (i, j = 1, 2) \). In \( \hat{H}_{\text{int}} \), \( \kappa \) is the coupling of the vibronic system and

\[
\hat{\mathcal{F}}(\hat{\alpha}; \eta) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(i\eta)^n}{(n+k)!} \hat{a}^k \hat{a}^\dagger + \text{H.c.}
\]

(3)

describes the mode structure of the driving laser (standing wave) at the operator-valued position of the ion. \( L_n^{(2)} \) denotes the generalized Laguerre polynomials, \( |n\rangle \) the motional number states, \( \hat{\alpha} = \hat{a} \hat{a}^\dagger \) the corresponding number operator, \( \Delta \phi \) defines the position of the trap potential relative to the laser wave, and \( \eta \) is the Lamp–Dicke parameter.

The physical interpretation of the Hamiltonian in equation (2) is as follows: a cavity photon is absorbed (\( \hat{b} \)) and the trapped ion is excited (\( \hat{a} \)). The transitions of the vibrational states \( \hat{\mathcal{F}}(\hat{\alpha}; \eta) \) occur according to the chosen laser frequency such that only the \( |n\rangle \leftrightarrow |n-k\rangle \) transitions are driven. That is, we assume we operate in a limit that the vibrational sidebands can be resolved very well (resolved sideband regime). Note that the electronic de-excitation process is described by the H.c. term.

The dynamics of the Hamiltonian in equation (2) is described by the time-evolution operator [38]

\[
\hat{U}(t, t_0) = \sum_{\sigma = \pm} \sum_{m, n = 0} e^{-i\omega_{\sigma}^0(t-t_0)|\psi^{\sigma}_{mn}\rangle\langle \psi^{\sigma}_{mn}|}
\]

\[
+ \sum_{n=0}^{\infty} e^{-i\omega_{21}(t-t_0)|1, 0, n\rangle\langle 1, 0, n|}
\]

\[
+ \sum_{m=0}^{\infty} \sum_{q=0}^{m-1} e^{-i[\nu + \omega_{12}(m+1)](t-t_0)}|1, m+1, q\rangle\langle 1, m+1, q|.
\]

(4)

The eigenstates of the Hamiltonian read as

\[
|\psi^{\pm}_{mn}\rangle = \alpha^{\pm}_{mn}|1, m, n\rangle + \beta^{\pm}_{mn}|1, m+1, n+k\rangle.
\]

(5)

In \( |i, m, n\rangle \), the \( i = 1, 2 \) refer to the electronic excitations, \( m \) and \( n \) are the photon number of the cavity field and the motional excitation of the ion, respectively. Furthermore, one
finds
\[
\begin{align*}
C_{\text{sq}}^+ & = \frac{1}{\sqrt{1 + |\alpha_{\text{sq}}|^2}}, \\
\alpha_{\text{sq}}^+ & = \Delta \omega \pm \sqrt{\Delta \omega^2 + |\Omega_{\text{sq}}|^2}, \\
\omega_{\text{sq}}^\pm & = \frac{1}{2} (\Delta \omega (2m + 1) + \nu (2n - 2km) + \omega_{21} (2m + 2) \\
& \pm \sqrt{\Delta \omega^2 + |\Omega_{\text{sq}}|^2}), \\
\Omega_{\text{sq}} & = 2\kappa \sqrt{m + 1} f_k (n; \eta) \sqrt{(n + k)!/n!}, \\
f_k (n; \eta) &= \langle n | \hat{f}_k (\hat{n}; \eta) | n \rangle.
\end{align*}
\]

Based on these solutions, general properties of the center-of-mass motion can be described.

3. Nonclassicality

During the last decades various criteria to identify nonclassicality of different types were derived. The most elementary conditions are those for squeezing [4–8] and sub-Poisson statistics (Mandel \(Q\) parameter) [9–11]. In this section we investigate nonclassical properties and their temporal evolutions in the explicitly time-dependent nonlinear Jaynes–Cummings model. In [38], it was already shown that the vibrational states are clearly nonclassical for the times under study. Here, we discuss this behavior in more detail, especially with respect to the anomalous quantum correlation effects.

3.1. Special nonclassical effects

In the following we denote the nonclassicality criteria by \(C\). Squeezing is defined through the negativity of the normal-ordered variance of the quadrature operator, \(\hat{\chi}(\varphi; \tau) = e^{i\varphi} \hat{\alpha}(\tau) + e^{-i\varphi} \hat{\alpha}^\dagger(\tau)\), see [46]. Thus, if
\[
\langle [\Delta \hat{\chi}(\varphi; \tau)]^2 \rangle < 0
\]
for some \(\varphi\)-interval, with \(\Delta \hat{\chi} = \hat{\chi} - \langle \hat{\chi} \rangle\), the state is referred to as a quadrature squeezed state. The normal-ordering prescription orders the operators \(\hat{\alpha}\) and \(\hat{\alpha}^\dagger\) such that all creation operators \(\hat{\alpha}^\dagger\) are placed to the left of the annihilation operators \(\hat{\alpha}\). Consequently, we define the criterion for squeezing as
\[
C_{\text{sq}} := \min_{\varphi \in [0, 2\pi]} \{ \langle [\Delta \hat{\chi}(\varphi; \tau)]^2 \rangle \} < 0.
\]

That is, if \(C_{\text{sq}} < 0\) then squeezing occurs at time point \(\tau\). Beside the condition in equation (8), we consider the Mandel \(Q\) parameter [9–11]:
\[
C_{\text{SP}} := Q(\tau) = \frac{\langle [\Delta \hat{n}(\tau)]^2 \rangle}{\langle \hat{n}(\tau) \rangle} < 0.
\]

In terms of radiation fields a negative Mandel \(Q\) parameter certifies a photocounting statistics of sub-Poisson type. Such a statistics does not possess a classical analog and, hence, \(C_{\text{SP}} < 0\) verifies nonclassicality. We also consider an anomalous quantum-correlation condition, which cannot be fulfilled by classical states [18, 19],
\[
C_{\text{AC}} := \min_{\varphi \in [0, 2\pi]} \{ \langle [\Delta \hat{n}(\tau)]^2 \rangle \} \langle [\Delta \hat{\chi}(\varphi; \tau)]^2 \rangle - \langle [\Delta \hat{\chi}(\varphi; \tau) \Delta \hat{n}(\tau)]^2 \rangle < 0.
\]

The squeezing condition [equation (8)] depends solely on the normal-ordered variance of the quadrature operator \(\hat{\chi}\). The sub-Poisson condition [equation (9)] depends on the normal-ordered version of the variance of the number operator \(\hat{n}\). The anomalous quantum-correlation condition in equation (10) contains contributions of both quantities together with their quantum correlations in the last term. Recently, it was shown via homodyne cross-correlation measurements [47] that this condition certifies nonclassicality for radiation fields beyond squeezing [20].

To calculate all needed quantities, we express the inequalities in terms of the creation and annihilation operators. The condition for squeezing [equation (8)] reads as
\[
\langle [\Delta \hat{\chi}(\varphi; \tau)]^2 \rangle = 2 \text{Re} \{ e^{2i\varphi} \langle \hat{\alpha}(\tau)^2 \rangle \} - 4 \text{Re} \{ e^{i\varphi} \langle \hat{\alpha}(\tau) \rangle \}^2 + 2 \langle \hat{n}(\tau) \rangle.
\]

Re \([z]\) denotes the real part of the variable \(z\). The anomalous correlation function in equation (10) may be rewritten as
\[
\langle [\Delta \hat{n}(\varphi; \tau)]^2 \rangle = 2 \text{Re} \{ e^{i\varphi} \langle \hat{n}(\tau) \rangle \} \langle \hat{\alpha}(\tau) \rangle - 2 \text{Re} \{ e^{i\varphi} \langle \hat{\alpha}(\tau) \rangle \} \langle \hat{n}(\tau) \rangle,
\]
and
\[
\langle [\Delta \hat{n}(\tau)]^2 \rangle = \langle \hat{n}(\tau)^2 \rangle - \langle \hat{n}(\tau) \rangle^2,
\]
with \(\langle \hat{n}(\tau)^2 \rangle = \langle \hat{n}(\tau) \rangle^2 - \langle \hat{n}(\tau) \rangle\). Thus, the sub-Poisson-condition [equation (9)] can be rewritten as
\[
C_{\text{SP}} = \frac{\langle \hat{n}(\tau)^2 \rangle - \langle \hat{n}(\tau) \rangle^2 - 1}{\langle \hat{n}(\tau) \rangle},
\]
which equals the commonly used form of the Mandel \(Q\) parameter.

Since we consider only single-time expectation values of operators that are initially attributed to the motion of the ion, in general denotes by \(\hat{A}(0)\), the expectation values can be calculated via
\[
\langle \hat{A}(t) \rangle = \text{Tr} \{ \hat{\rho}_{\text{mot}}(t) \hat{A}(0) \},
\]
with the reduced motional density matrix \(\hat{\rho}_{\text{mot}}(t)\). The reduced density matrix of the motional subsystem is obtained by the trace over electronic degrees of freedom and the cavity field,
\[
\hat{\rho}_{\text{mot}}(t, t_0) = \sum_{i=1,2} \sum_{m=0}^{\infty} \langle i, m | \hat{\rho}(t, t_0) | i, m \rangle
\]
\[
= \sum_{m=0}^{\infty} |\beta_0|^2 m e^{-|\beta_0|^2} \sum_{n,m} \rho_{n,m} \times \sum_{\sigma, \sigma' = \pm} e^{i e_{\sigma} e_{\sigma'} (t-t_0)} |c_{\sigma,\sigma'}^{m} c_{\sigma,\sigma'}^{m'}|^2 \times \{ |n| \langle n | + \alpha_{mn}^* (\alpha_{mn})^n |n + k \rangle \langle n + k | \}.
\]
electronic degree of freedom of the ion is prepared in the excited state as $|2\rangle$.

### 3.2. Analytical results

We recapitulate: nonclassicality at time $\tau$ is certified via $C_x < 0$ for $x = \{ \text{Sq, SP, AC} \}$ (squeezing, sub-Poisson statistics, anomalous quantum correlations), according to equations (8)–(10). Let us first consider the case where the ion is driven quasi-resonantly to the zeroth sideband. That is, in our model we choose $k = 0$, which means that we consider the $|1, n\rangle \leftrightarrow |2, n\rangle$ transitions.

According to [48], where the authors considered the semiclassical, $|\beta_0| \gg 1$, case with exact resonance, $\Delta \omega = 0$, the quantum nondemolition measurement of the motional energy of the trapped ion was proposed. This case, extended by a detuning and a quantized pump, is studied in the following with respect to its nonclassical properties. The pump is chosen to be strong such that the dynamics is close to the semiclassical one, as it was treated in [48] for the resonant case.

We consider moderate detuning, $\Delta \omega/|\kappa| = 20$, and a Lamb–Dicke parameter of $\eta = 0.3$. The results are depicted in figure 1(a). Obviously, since $C_{\text{SP}} \geq 0$ and $C_{\text{Sq}} \geq 0$, neither sub-Poisson statistics nor quadrature squeezing are observed. Nonclassicality is only revealed by the anomalous correlations as defined in equation (10). The same results are found if other values of $\eta$ or $\Delta \omega$ are considered. On the investigated time scales we can, using the considered nonclassicality criteria, certify nonclassicality criteria only through anomalous correlations.

For the same choice of $\eta = 0.3$ and $\Delta \omega/|\kappa| = 20$, we now consider the excitation to the second vibrational sideband, $k = 2$. The results are given in figure 1(b). Naively, one would expect a significant squeezing contribution as the squeezing operator consists of quadratic contributions of the creation and annihilation operators. Unexpectedly, we only find small regions where the system is nonclassical regarding the squeezing condition [equation (8)] and the sub-Poisson condition [equation (9)]. Again, the anomalous correlations certify nonclassicality in a very pronounced manner, over nearly the whole considered time range. Additionally, for large times we see that the criteria develop in different directions. This counterintuitive behavior is caused by the nonlinearities, occurring beyond the Lamb–Dicke regime, which have a significant impact on the dynamics. Note that a larger detuning leads to a decrease of the overall strength of the effects.

For convenience, let us visualize the motional state in the corresponding phase-space picture. That is, we need to choose an appropriate phase-space distribution. Here, we use the regularized version of the Glauber–Sudarshan $P$ function. The $P$ function [49, 50] itself can be used to express the density operator of an arbitrary state as a pseudo-mixture of coherent states, namely

$$\hat{\rho}(t) = \int d^2 \alpha P(\alpha; t)|\alpha\rangle \langle \alpha|,$$

where $P(\alpha; t)$ can become negative and even strongly singular. A state is referred to as classical state if the corresponding $P$ function has the properties of a classical probability density—i.e. it is non-negative [51, 52]. However, for the most states the $P$ function is not experimentally accessible due to its singularities. Hence, a regularization procedure was introduced in [53] to transform the ordinary $P$ function into a well behaved phase-space representation, $P_\Omega$, of the state under study. This procedure works as follows: since the singularities of $P$ are caused by an unbounded characteristic function $\Psi$, one introduces a suitable filter function $\Omega_w$, with a width $w$. It is constructed such that the filtered function $\Psi_\Omega = \Omega_w \Psi$ is square-integrable for any quantum state. Since we are mainly interested in the negativities of $P$, it is important that the filter function must not introduce additional negativities in the filtered $P$ function denoted by $P_\Omega$. Thus, the Fourier transform of $\Omega_w$ must be non-negative.

Using the procedure outlined in [38], one may calculate $P_\Omega$ directly out of the reduced density matrix in equation (16). A plot of $P_\Omega$ is given in figure 2, where we considered the same situation as in figure 1(a) for $|\kappa|t = 0.2$. The depicted state does neither reveal squeezing nor sub-Poisson statistics.
but only anomalous quantum correlations [see figure 1(a)]. The nonclassicality is uncovered by the negative values of \( P_{12} \).

Altogether, we see that for many situations the most commonly used definitions of nonclassicality, such as the negative Mandel Q parameter (sub-Poisson statistics) [equation (9)] and quadrature squeezing [equation (8)], fail to certify nonclassicality in the detuned nonlinear Jaynes–Cummings model. In the scenario \( k = 0 \), where the zeroth sideband is only excited, the anomalous quantum-correlation condition reveals the nonclassical character of the dynamics. In the \( k = 2 \) case, the applicability of the criteria, for the purpose to uncover nonclassicality, depends on the choice of \( \eta \) and \( \Delta \omega \). However, the anomalous quantum-correlation condition (10) is a powerful tool to certify nonclassicality for nearly the full timescale under study. This underlines the strength of this condition and it encourages one to investigate quantum effects beyond the mostly considered criteria. Especially, the excitation to the zeroth sideband, which only reveals its nonclassical character in terms of anomalous correlations, is a promising scenario to further analyze the physical relevance of such quantum signatures.

4. Measurement

In the following we consider the possible measurement of the correlations studied for the quantized motion of the trapped ion. For radiation fields, the anomalous correlations were measured recently [20]. However, the reconstruction of the motional state of a trapped ion is a sophisticated problem by itself [54, 55]. In the following, we are interested in the measurement of the full vibronic quantum state by the technique introduced in [56], for the purpose to detect entanglement of the vibronic degrees of freedom, see the scheme in figure 3.

The strategy is as follows: the weak \( \{1\} \rightarrow \{2\} \) transition is the one whose joint quantum state we are interested in. The electronic state is tested by a strong \( \{1\} \rightarrow \{3\} \) transition via the appearance of resonance fluorescence [57–59]. If the latter is detected, the ion is in the state \( \{1\} \), otherwise in the state \( \{2\} \). The incident laser is tuned to the zeroth sideband, which, in the resolved sideband limit, leads to the interaction Hamiltonian, see [48],

\[
\hat{H}_{\text{int}} = \hbar \kappa' \hat{n}(\alpha) \hat{A}_1 \hat{A}_2 + \text{H.c.} \tag{18}
\]

Here we use the notation \( \hat{H}_{\text{int}} \) to distinguish this Hamiltonian from the previously discussed one in equation (2). The operator-valued function \( \hat{f}_\kappa(\alpha, \eta) \) is defined in equation (3). The corresponding time-evolution operator is obtained via \( \hat{U}_{\text{int}}(\tau) = \exp( -\frac{i}{\hbar} \hat{H}_{\text{int}} \tau) \).

Usually, one focuses on the no-fluorescence events since the motional state is then not disturbed due to recoil effects. The initial probe cycle is performed as follows: first, the motional state is coherently displaced by the amplitude \( \alpha \), which can be accomplished via the application of a radiofrequency field. Second, the driving laser with frequency \( \omega_d \) (figure 3) is switched on for a certain interaction time \( \tau \). Afterwards, the electronic state is measured via probing for resonance fluorescence. After such a probe cycle, if no fluorescence is detected, the unnormalized density operator of the ion reads as

\[
\hat{\rho}^{(1)}(\tau) = \{2\} \{2\} \otimes \hat{\rho}_{\text{red}}^{(1)}(\tau), \tag{19}
\]

with \( \hat{\rho}_{\text{red}}^{(1)}(\tau) = \langle 2 | \hat{U}_{\text{int}}(\tau) \hat{\rho}(\alpha) \hat{U}_{\text{int}}^\dagger(\tau) | 2 \rangle \) and \( \hat{\rho}(\alpha) = D^\dagger(\alpha) \hat{\rho}(0) D(\alpha) \), where \( D(\beta) = \exp(\beta \hat{a}^\dagger - \beta^* \hat{a}) \) is the coherent displacement operator. Here, \( \hat{\rho}(0) \) denotes the density operator of the vibronic degrees of freedom. This means that the cavity mode is traced out. As long as the coherent amplitude \( \beta_0 \) is sufficiently large—i.e. the dynamics is close to the semiclassical case—\( \hat{\rho}(0) \) contains the complete information of the nonclassical properties.
Applying $K$ of those probe cycles, with interaction times $\tau_1, \ldots, \tau_K$, yields the diagonal elements (still unnormalized)

$$\langle n | \hat{p}^{(K)}(\tau_n) | n \rangle = \prod_{q=2}^{K} \cos^2 (|\kappa|^2 L_n (\gamma^2) e^{-\gamma^2 / 2} \gamma_n) \langle n | \hat{p}^{(1)}(\gamma_n) | n \rangle.$$ 

(20)

The probability to obtain such a sequence of cycles, equals the trace of the latter expression. For appropriately chosen interaction times, see the end of section III of [56], this probability can directly be related to the displaced density operator elements $\rho^{nn}_{ij}(\alpha) \equiv \langle i, n | \hat{p}(\alpha) | j, n \rangle$ for $i, j = 1, 2$.

Using these elements, $\rho^{nn}_{ij}(\alpha)$, one can derive the Wigner-function matrix straightforwardly [56]:

$$W_{ij}(\alpha) = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \rho^{nn}_{ij}(\alpha).$$ 

(21)

The latter is a unification of the ordinary Wigner function [60], including the electronic degrees of freedom. Hence it was shown that using $W_{ij}(\alpha)$ one can uncover entanglement between the motional and electronic states of the ion which would not be verified by using only the reduced density matrix. However, nowadays we have experimental access to the regularized $P$ function which possesses several advantages over other quasiprobabilities [61, 62]. Remarkably, in [63] a regularized hybrid version of the $P$ function was introduced, utilizing the description of continuous- and discrete-variable systems. The definition applies here as well and reads as

$$P_{ij}(\alpha) = \langle \hat{A}_{ij} \otimes \delta(\hat{a} - \alpha) \rangle,$$ 

(22)

with $\hat{A}_{ij} = |j\rangle \langle i|$. Using this definition, one can define the overall density operator of the system as

$$\hat{\rho} = \sum_{ij} \int d^2 \alpha P_{ij}(\alpha) \langle i | \otimes | \alpha \rangle \langle \alpha | \otimes | j \rangle.$$ 

(23)

The questions arise how the anomalous moments can be obtained and how $P_{ij}(\alpha)$ can be reconstructed by using the scheme in figure 3.

Let us start with the definition given in equation (21). Applying the inverse Fourier transform yields the characteristic-function matrix of the Wigner-function matrix (indicated by the index $W$),

$$\Phi_{ij,W}(\beta) = \int d^2 \alpha W_{ij}(\alpha) e^{i \beta a^\dagger - \beta^* a} = \langle \hat{A}_{ij} \otimes \hat{D}(\beta) \rangle.$$ 

(24)

To transform the symmetric ordered function $\Phi_{ij,W}(\beta)$ into normal order, one may apply the Baker–Campbell–Hausdorff formula to obtain the Fourier transform of the $P$-function matrix $P_{ij}(\alpha)$,

$$\Phi_{ij}(\beta) = \langle \hat{A}_{ij} \otimes \hat{D}(\beta) \rangle = \langle \hat{A}_{ij} \otimes \hat{D}(\beta) \rangle e^{i \beta^2 / 2}.$$ 

(25)

As soon as the latter function is derived from $W_{ij}(\alpha)$, one may calculate its trace over the electronic degrees of freedom to obtain the characteristic function of the motional subsystem:

$$\sum_{i=1}^{2} \Phi_{ij}(\beta) = \langle \hat{D}(\beta) \rangle.$$ 

(26)

Differentiation yields the expectation values needed in equation (10). In principle, all possible combinations of normal-ordered moments can be obtained in this way. For example, in

$$\langle \hat{\epsilon}(\varphi) \Delta \hat{n}(\alpha) \rangle = \langle \hat{\epsilon}(\varphi) \rangle - \langle \hat{\epsilon}(\varphi) \rangle \langle \hat{n} \rangle.$$ 

(27)

one may derive all terms as follows, defining $\beta = [\beta e^{i \varphi}]$:

$$\langle \hat{\epsilon}(\varphi) \rangle = \frac{1}{i} \int \frac{d\beta}{|\beta|} \langle \hat{D}(\beta) \rangle,$$

$$\langle \hat{n} \rangle = - \frac{\partial}{\partial \beta} \frac{\partial}{\partial \beta^*} \langle \hat{D}(\beta) \rangle,$$

$$\langle \hat{\epsilon}(\varphi) \hat{n} \rangle = - \frac{1}{i} \frac{\partial}{\partial \beta} \frac{\partial}{\partial \beta^*} \frac{\partial}{\partial \beta } \langle \hat{D}(\beta) \rangle.$$ 

(28)

Via $\frac{\partial}{\partial \beta} \frac{\partial}{\partial \beta^*} = \left( e^{i \varphi \beta} \frac{\partial}{\partial \beta} + e^{-i \varphi \beta^*} \frac{\partial}{\partial \beta^*} \right)$ all moments can be derived with respect to the derivatives of $\beta$ and $\beta^*$.

Furthermore, we may calculate the regularized $P$-function matrix out of equation (25) via multiplication of an appropriate filter function $\Omega_{\alpha}(\beta)$ [53, 64] (with a width $\omega$). A subsequent Fourier transformation and the usage of equation (24) yields

$$P_{\omega,ij}(\alpha) = \frac{1}{\pi^2} \int d^2 \beta \Omega_{\alpha}(\beta) e^{i \beta a^\dagger - \beta^* a} \Phi_{ij}(\beta)$$

$$= \int d^2 \alpha' W_{ij}(\alpha') \int d^2 \beta \Omega_{\alpha}(\beta) e^{i \beta a^\dagger - \beta^* a} e^{i \beta a - \beta^* a'},$$

$$= \lambda_{ij}(\alpha', \alpha)$$ 

(29)

Hence, using equation (21) we finally arrive at

$$P_{\omega,ij}(\alpha) = \frac{2}{\pi^3} \sum_{n=0}^{\infty} (-1)^n \int d^2 \alpha' \lambda_{ij}(\alpha', \alpha) \rho_{ij}^{nn}(\alpha'),$$ 

(30)

where $\rho_{ij}^{nn}(\alpha) \equiv \langle i, n | \hat{p}(\alpha) | j, n \rangle$ for $i, j = 1, 2$. Thus, out of the Wigner-function matrix we derive the moments in equation (28) and furthermore, out of the $\rho_{ij}^{nn}(\alpha)$, we may obtain the regularized $P$-function matrix. The trace $\sum_{i=1}^{2} P_{\omega,ij}(\alpha)$ would yield the regularized $P$ representation including merely the motional subsystem, which we discussed for a special case in figure 2.

Note that in equation (30) one needs to evaluate an integral over the whole $\alpha'$-plane. One can avoid this integration by using an alternative approach related to the ideas presented in [65, 66]. The nonclassicality witnesses for harmonic oscillators, which also apply here, lead to the expression

$$P_{\omega,ij}(\alpha) = \frac{w^2}{16} \sum_{m=0}^{\infty} \frac{(-w^2 / 4)^m}{(m + 1)!} \left( \frac{2m + 2}{m} \right) \langle \hat{A}_{ij} \otimes \hat{n}(\alpha)^{m} \rangle,$$ 

(31)

where $\hat{n}(\alpha) = \hat{D}(\alpha) \hat{n} \hat{D}(\alpha)^\dagger$. This result, expressed in terms of normal-ordered displaced-number moments, is obtained via the application of a particular disc-function filter. Inserting the expression

$$\langle \hat{A}_{ij} \otimes \hat{n}(\alpha)^{m} \rangle = \sum_{n=m}^{\infty} \rho_{ij}^{nn}(\alpha) \frac{n!}{(n - m)!}$$ 

(32)
in equation (31), we directly relate the regularized $P$-function matrix, $P_{ijkl}(\alpha)$, to the elements $\rho_{ij}^{\text{eff}}(\alpha)$. Especially, we in this formulation we do not have to evaluate an integral over the complex plane, as in equation (30). We can choose a certain value $\alpha$ and calculate $P_{ijkl}$ at this point in phase-space.

5. Summary and conclusions

In this work we studied nonclassical properties of the recently introduced generalization of the nonlinear Jaynes–Cummings model for the vibronic dynamics of a trapped ion—including a quantized pump field and a small detuning with respect to the vibronic excitation in the resolved sideband regime. We showed that for the excitation of the zeroth and second sideband the so-called anomalous quantum correlations of non-commuting observables certify nonclassicality when established criteria, verifying sub-Poisson number statistics or quadrature squeezing, fail. In particular, in the case of driving the zeroth sideband, the anomalous quantum-correlation condition is the favored one that uncover the nonclassicality. In addition, we studied the influence of the nonlinearities occurring beyond the Lamb–Dicke regime as well as the detuning from resonance. The great importance of the anomalous quantum correlations in the dynamics under study raises the question whether these phenomena may be useful for practical applications in quantum technologies. In any case, the verification of the nonclassical nature of the system under study through anomalous quantum correlations is of fundamental interest—in particular if standard quantum signatures (e.g. squeezing and sub-Poisson statistics) are negligibly small.

To access the studied quantum signatures in experiments, we studied the possibilities to determine the needed correlations from measured data. For this aim, a measurement technique is suited which was originally proposed for the purpose to verify entanglement within the vibronic quantum system of the trapped ion. We show in detail how the needed moments and correlation functions, including those characterizing the anomalous quantum correlations, are obtained from measured quantities. In the underlying measurement scenario, the Wigner-function matrix was considered as the quantity to be determined. In the present paper we demonstrated how the regularized version of the Glauber–Sudarshan $P$-function matrix can be obtained from the Wigner-function matrix. This is needed as the desired correlation functions for analyzing the quantum effects of interest are normal-ordered ones. The advantage of normal ordering consists in the fact that these correlations are robust against losses and they are not washed out by vacuum fluctuations which are caused by losses. Based on these techniques, very general quantum effects in the vibronic degrees of freedom of trapped ions may be studied.

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