Strong stability of Nash equilibria in load balancing games

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Abstract We study strong stability of Nash equilibria in load balancing games of \(m\) (\(m \geq 2\)) identical servers, in which every job chooses one of the \(m\) servers and each job wishes to minimize its cost, given by the workload of the server it chooses. A Nash equilibrium (NE) is a strategy profile that is resilient to unilateral deviations. Finding an NE in such a game is simple. However, an NE assignment is not stable against coordinated deviations of several jobs, while a strong Nash equilibrium (SNE) is. We study how well an NE approximates an SNE. Given any job assignment in a load balancing game, the improvement ratio (IR) of a deviation of a job is defined as the ratio between the pre- and post-deviation costs. An NE is said to be a \(\rho\)-approximate SNE (\(\rho \geq 1\)) if there is no coalition of jobs such that each job of the coalition will have an IR more than \(\rho\) from coordinated deviations of the coalition. While it is already known that NEs are the same as SNEs in the 2-server load balancing game, we prove that, in the \(m\)-server load balancing game for any given \(m \geq 3\), any NE is a \((5/4)\)-approximate SNE, which together with the lower bound already established in the literature yields a tight approximation bound. This closes the final gap in the literature on the study of approximation of general NEs to SNEs in load balancing games. To establish our upper bound, we make a novel use of a graph-theoretic tool.

Keywords load balancing game, Nash equilibrium, strong Nash equilibrium, approximate strong Nash equilibrium

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1 Introduction

In game theory, a fundamental notion is Nash equilibrium (NE), which is a state that is stable against deviations of any individual participants (known as agents) of the game in the sense that any such deviation will not bring about additional benefit to the deviator. Much stronger stability is exhibited by a strong Nash equilibrium (SNE), a notion introduced by Aumann [3], at which no coalition of agents exists such that each member of the coalition can benefit from coordinated deviations by the members of the coalition.

Evidently selfish individual agents stand to benefit from cooperation and hence SNEs are much more preferred to NEs for stability. However, SNEs do not necessarily exist [2] and, even if they do, they are much more difficult to identify and to compute [4, 7]. It is therefore very much desirable to have

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the advantages of both computational efficiency and strong stability, which motivates our study in this paper. We establish that, for general NE job assignments in load balancing games, which exist and are easy to compute, their loss of strong stability possessed by SNEs is at most 25%.

In a load balancing game, there are $n$ selfish agents, each representing one of a set $J = \{J_1, \ldots, J_n\}$ of $n$ jobs. In the absence of a coordinating authority, each agent must choose one of $m$ identical servers, $M = \{1, \ldots, m\}$, to assign his job to in order to complete the job as soon as possible. All jobs assigned to the same server will finish at the same time, which is determined by the workload of the server, defined to be the total processing time of the jobs assigned to the server. Let job $J_j$ have a processing time $p_j$ ($1 \leq j \leq n$) and let $S_i$ denote the set of jobs assigned to server $i$ ($1 \leq i \leq m$). For convenience, we will use “agent” and “job” interchangeably, and consider job processing times also as their “lengths”. The completion time $c_j$ of job $J_j \in S_i$ is the workload of its server: $L_i = \sum_{J_j \in S_i} p_j$.

The notions of NE and SNE can be stated more specifically for the load balancing game. A job assignment $S = (S_1, \ldots, S_m)$ is said to be an NE if no individual job $J_j \in S_i$ can reduce its completion time $c_j$ by unilaterally migrating from server $i$ to another server. A job assignment $S = (S_1, \ldots, S_m)$ is said to be an SNE if no subset $\Gamma \subseteq J$ of jobs can each reduce their job completion times by forming a coalition and making coordinated migrations from their own current servers.

NEs in the load balancing game have been widely studied (see, e.g., [5, 6, 8, 10, 11]) with the main focus of quantifying their loss of global optimality in terms of the price of anarchy, a term coined by Koutsoupias and Papadimitriou [11], as largely summarized in [12]. In this paper, we study NEs in load balancing games from a different perspective by quantifying their loss of strong stability.

We focus on pure NEs, those corresponding to deterministic job assignments in load balancing games. While high-quality NEs are easily computed, identification of an SNE is strongly NP-hard [4]. Given any job assignment in a load balancing game, the improvement ratio (IR) of a deviation of a job is defined as the ratio between the pre- and post-deviation costs. An NE is said to be a $\rho$-approximate SNE ($\rho \geq 1$) (which is called $\rho$-SE in [1]) if there is no coalition of jobs such that each job of the coalition will have an IR more than $\rho$ from coordinated deviations of the coalition. Clearly, the stability of NE improves with a decreasing value of $\rho$ and a 1-approximate SNE is in fact an SNE itself.

For the load balancing game of two servers, one can easily verify that every NE is also an SNE [2]. If there are three or four servers in the game, then it is proved in [4,7], respectively, that any NE assignment is a $(5/4)$-approximate SNE, and the bound is tight. Furthermore, it is a $(2 - 2/(m + 1))$-approximate SNE if the game has $m$ servers for $m \geq 5$ (see [7]).

We establish in this paper that, in the $m$-server load balancing game ($m \geq 3$), any NE is a $(5/4)$-approximate SNE, which is tight and hence closes the final gap in the literature on the study of NE approximation of SNE in load balancing games. To establish our approximation bound, we make a novel use of a powerful graph-theoretic tool.

2 Definitions and preliminaries

2.1 A lower bound

We start with an example to help the reader get some intuition of the problem under consideration. The example also provides a lower bound of $5/4$ for any NE assignment to approximate SNE. The left panel of Figure 1 shows an NE assignment of six jobs to three identical servers with job completion time of 5, 5 and 10, respectively, for the three pairs of jobs. If the four jobs of lengths 2 and 5 form a coalition and make a coordinated deviation as shown in the figure, then in the resulting assignment, each of the four jobs in the coalition achieves an improvement ratio of $5/4$.

2.2 Graph-theoretic tool

As a tool of our analysis, we start with the minimal deviation graph introduced by Chen [4]. For convenience we collect into this subsection some basic results on minimal deviation graphs from [4]. Given an NE job assignment $S = (S_1, \ldots, S_m)$, as an NE-based coalitional deviation or simply coalitional
deviation $\Delta$, we refer to a collective action of a subset $\Gamma \subseteq J$ of jobs in which each job of $\Gamma$ migrates from its server in the assignment $S$ so that its completion time is decreased after the migration. Accordingly, $\Gamma = \Gamma(\Delta)$ is called the corresponding coalition. We introduce deviation graphs to characterize coalitional deviations. In a coalitional deviation, a server $i$ is said to be participating or involved if its job set changes after the deviation. Given a coalitional deviation $\Delta$ with the corresponding coalition $\Gamma = \Gamma(\Delta)$, we define the corresponding (directed) deviation graph $G(\Delta) = (V, A)$ as follows:

$$V = V(G) := \{i : \text{server } i \text{ is a participating server}\},$$

$$A = A(G) := \{(u, v) : \text{a job } J_j \in \Gamma \text{ migrates from } S_u \text{ to } S_v\}.$$ 

In what follows, without loss of generality we consider coalitional deviations with $V(G) = M$. Given a coalitional deviation $\Delta$, we denote by $L'_i(\Delta)$ the workload of server $i$ after deviation $\Delta$, and by $IR(\Delta)$ the minimum of the improvement ratios of all jobs taking part in $\Delta$. Then we have the following definition and lemmas from [4]:

**Lemma 1.** The out-degree $\delta^+(i)$ of any node $i$ of a deviation graph is at least 1, and hence $|S_i| \geq 2$.

**Lemma 2.** If all $m$ servers are involved in a coalitional deviation, then the deviation graph does not contain a set of node-disjoint directed cycles such that each node of the graph is in one of the directed cycles.

**Definition 1.** Let $\Delta$ be a coalitional deviation and $\Gamma = \Gamma(\Delta)$ be the corresponding coalition. Deviation graph $G = G(\Delta)$ is said to be minimal if $IR(\Delta') < IR(\Delta)$ for any coalitional deviation $\Delta'$ such that the corresponding coalition $\Gamma' = \Gamma'(\Delta')$ is a proper subset of $\Gamma$.

**Lemma 3.** The in-degree $\delta^-(i)$ of any node $i$ of a minimal deviation graph is at least 1.

**Lemma 4.** A minimal deviation graph is strongly connected.

### 2.3 Some observations

In our study of bounding NE approximation of SNE, we can apparently focus on those coalitional deviations that correspond to minimal deviation graphs. We start with several observations on any NE-based coalitional deviation $\Delta$ involving $m$ servers for $m \geq 3$. Let $G(\Delta)$ denote the corresponding minimal deviation graph.

If two jobs assigned to server $i \in M$ in the NE assignment migrate to server $j \in M$ ($j \neq i$) together, or both stay on the server, then we can treat them as one single job without loss of generality in our study of the minimal deviation graph. With this understanding, if we let $a_i$ ($i \in M$) denote the number of jobs assigned to server $i$ in the NE assignment, then the following is immediate.

**Observation 1.** For any $i \in M$, we have $2 \leq a_i \leq m$. $\delta^+(i) = a_i$ or $\delta^+(i) = a_i - 1$.

As a result of the above observation, the node set $M$ can be partitioned into two, $M'$ and $M''$, as follows:

$$M' := \{i \in M : a_i = \delta^+(i)\},$$

$$M'' := M \setminus M' = \{i \in M : a_i = \delta^+(i) + 1\}.$$
By applying a data scaling if necessary, we assume without loss of generality that
\[
\min_{i \in M} L_i = 1. \tag{1}
\]

**Observation 2.** For any \( i \in M \), we have \( L_i \leq a_i/(a_i - 1) \).

**Proof.** Suppose to the contrary that \( L_i > a_i/(a_i - 1) \), which implies that \( a_i > L_i/(L_i - 1) \).

Let \( x_i \) denote the length of the shortest job assigned to server \( i \) in the NE assignment. We have \( L_i \geq a_i x_i \), which leads to \( L_i > L_i x_i/(L_i - 1) \), i.e., \( L_i > x_i + 1 \), which implies that the shortest job assigned to server \( i \) in the NE assignment can have the benefit of reducing its job completion time by unilaterally migrating to the server of which the workload is 1, contradicting the NE property.

The following observation states that, if all jobs on a server participate in the migration, then none of the servers they migrate to will have all its jobs migrate out.

**Observation 3.** If \((i,j) \in A\) and \( i \in M'\), then \( j \in M''\).

**Proof.** Suppose to the contrary that \( a_j \neq \delta^+(j) + 1 \). According to Observation 1, we have \( a_j = \delta^+(j) \), which implies that all the jobs assigned to server \( i \) and server \( j \) in the NE assignment belong to coalition \( \Gamma \).

Since \((i,j) \in A\), there is a job \( J_k \in \Gamma \) that migrates from server \( i \) to server \( j \). Consider the new coalition \( \Gamma' \) formed by all members of \( \Gamma \) except \( J_k \). Then we have \( 0 \neq \Gamma' \subset \Gamma \). Let \( \Delta' \) be such a coalesional deviation of \( \Gamma' \) that is the same as \( \Delta \) except without the involvement of \( J_k \) and the job(s) that migrate(s) to \( i \) (resp. \( j \)) in \( \Delta \) will migrate to \( j \) (resp. \( i \)) in \( \Delta' \). Then we have \( IR(\Delta') = IR(\Delta) \), contradicting the minimality of the deviation graph \( G \) according to Definition 1.

The following observation is a direct consequence of Observation 3:

**Observation 4.** Assume \( i,j \in M' \). Hence \((i,j),(j,i) \notin A\) according to Observation 3. Let \( \Delta' \) be the same as \( \Delta \) except that any job that migrates to \( i \) (resp. \( j \)) in \( \Delta \) will migrate to \( j \) (resp. \( i \)) in \( \Delta' \). Then \( IR(\Delta') = IR(\Delta) \), and \( G(\Delta') \) is also minimal.

### 3 A key inequality

To help our analysis, we will introduce in this section a special arc set \( \tilde{A} \subseteq A \) in the minimal deviation graph \( G(\Delta) \).

**3.1 Auxiliary arc set \( \tilde{A} \)**

For any node \( i \in M \), denote \( Q^+(i) := \{ j \in M : (i,j) \in A \} \) and \( Q^-(i) := \{ j \in M : (j,i) \in A \} \). For notational convenience, for any node set \( S \subseteq M \), we denote
\[
Q^+(S) := \bigcup_{i \in S} Q^+(i) \quad \text{and} \quad Q^-(S) := \bigcup_{i \in S} Q^-(i).
\]

With \( A \) replaced by \( \tilde{A} \) above, we similarly define \( \tilde{Q}^+(i), \tilde{Q}^-(i), \tilde{Q}^+(S) \) and \( \tilde{Q}^-(S) \).

Let us define \( \tilde{A} \) as follows. According to Lemma 3, \( |Q^-(i)| \geq 1 \) for any \( i \in M \). For each \( i \in M \), we pick up an arc from the non-empty set \( Q^-(i) \) to form an \( m \)-element subset \( \tilde{A} \subseteq A \). Then \( \tilde{A} \) possesses the following property:
\[
|\tilde{Q}^-(i)| = 1 \quad \text{for any} \ i \in M. \tag{2}
\]

Denote \( b_i := |\tilde{Q}^+(i)| \) for any \( i \in M \). Then it is clear that
\[
\sum_{i=1}^{m} b_i = |\tilde{A}| = m. \tag{3}
\]

If node set \( S \) is a singleton, then we will also use \( S \) to denote the singleton if no confusion can arise. Hence, due to (2) we will also use \( \tilde{Q}^-(i) \) to denote the single element of the corresponding set. Any arc set \( \tilde{A} \subseteq A \) that possesses property (2) is said to be *tilde-valid*. 
3.2 Main result

Our main result is stated in the following theorem:

**Theorem 1.** For any minimal deviation graph $G(\Delta_m)$ involving $m$ servers, its improvement ratio $IR(\Delta_m) \leq 5/4$.

Let us perform some initial investigation to see what we need to do to prove the theorem. Recall that, for any $i \in M$, $a_i$ is the number of jobs assigned to server $i$ in the NE assignment and $b_i = |Q_+^i|$ for a fixed arc set $\bar{A}$ defined in Subsection 3.1 for the minimal deviation graph $G(\Delta_m)$. For a pair of integers $a$ and $b$ with $2 \leq a \leq m$ and $0 \leq b \leq a$, let

$$M_{ab}^b := \{i \in M : a_i = a, b_i = b\}.$$  

Then it is clear that

$$\bigcup_{2 \leq a \leq m} \bigcup_{0 \leq b \leq a} M_{ab}^b = M. \tag{4}$$

Denote $m_{ab}^b = |M_{ab}^b|$ for all possible pairs $a$ and $b : 2 \leq a \leq m$ and $0 \leq b \leq a$. Let $r = IR(\Delta_m)$. Then according to (2) and (4), we have

$$\sum_{a=2}^{m} \sum_{b=0}^{a} m_{ab}^b = m a \quad \text{and} \quad \sum_{a=2}^{m} \sum_{b=0}^{a} bm_{ab}^b = m. \tag{5}$$

According to the definition of IR, we have $r L'_{ij} \leq L_i$ for $(i, j) \in A$. Summing up these inequalities over all $m$ arcs in $\bar{A}$ leads to

$$\sum_{j=1}^{m} r L'_{ij} \leq \sum_{i=1}^{m} b_i L_i,$$

which implies that

$$r \leq \frac{\sum_{i=1}^{m} b_i L_i}{\sum_{i=1}^{m} L_i}. \tag{6}$$

According to Observation 2, we have $L_i \leq a_i/(a_i - 1) \leq 2$, which implies that the right-hand side of (6), which we denote by $R$, is at most 2, since $R$ is a convex combination of $L_1, \ldots, L_m$ and 0 with the corresponding combination coefficients

$$\lambda_i = b_i/\sum_{k=1}^{m} L_k, \quad i = 1, \ldots, m$$

and

$$\lambda_{m+1} = 1 - m/\sum_{k=1}^{m} L_k \geq 0$$

due to (1) and (3). On the other hand, since $r \geq 1$ according to the definition, we conclude that $1 \leq R \leq 2$, which implies that $R$ is a decreasing function of $L_i$ for which $b_i = 0$ or $b_i = 1$, and an increasing function of $L_i$ for which $b_i \geq 2$. Therefore, we increase $R$ by increasing $L_i$ to $a_i/(a_i - 1)$ for $i$ such that $b_i \geq 2$, and by decreasing $L_i$ to 1 for $i$ such that $b_i = 0$ or $b_i = 1$. Noticing that $2 \leq a_i \leq m$ according to Observation 1, we obtain

$$r \leq \frac{\sum_{a=2}^{m} \sum_{b=2}^{a} \frac{ab}{a-1}m_{ab}^b + \sum_{a=2}^{m} m_{aa}^1}{\sum_{a=2}^{m} \sum_{b=2}^{a} \frac{1}{a-1}m_{ab}^b + \sum_{a=2}^{m} m_{aa}^0},$$

which together with (5) implies that

$$r \leq \frac{\sum_{a=2}^{m} \sum_{b=2}^{a} \frac{b}{a-1}m_{ab}^b + m}{\sum_{a=2}^{m} \sum_{b=2}^{a} \frac{1}{a-1}m_{ab}^b + m}.$$
In order to prove Theorem 1, we need to show \( r \leq 5/4 \). Then it suffices to show
\[
\frac{\sum_{a=2}^{m} \sum_{b=2}^{a} \frac{b}{a-1} m_a^b + m}{\sum_{a=2}^{m} \sum_{b=2}^{a} \frac{1}{a-1} m_a^b + m} \leq \frac{5}{4},
\]
which is equivalent to
\[
\sum_{a=2}^{m} \sum_{b=2}^{a} \frac{4b - 5}{a-1} m_a^b \leq m.
\]
By replacing the right-hand side \( m \) of the above inequality with the left-hand side of the second equality in (5), we have
\[
0 \leq \sum_{a=2}^{m} m_a^1 + \sum_{a=2}^{m} \sum_{b=2}^{a} \left( b - \frac{4b - 5}{a-1} \right) m_a^b,
\]
i.e.,
\[
m_2 + \frac{1}{2} m_3^2 \leq \sum_{a=2}^{m} m_a^1 + \frac{1}{2} m_3^2 + \sum_{a=4}^{m} \sum_{b=2}^{a} \left( b - \frac{4b - 5}{a-1} \right) m_a^b. \tag{7}
\]

In what follows, we are to prove (7) and thereby Theorem 1 through a series of lower bounds established in Section 5 on different terms of the right-hand side of (7).

4 Preparations

We introduce an auxiliary node set \( W \) in addition to the auxiliary arc set \( \tilde{A} \) introduced earlier.

4.1 Auxiliary node set \( W \)

Let \( W_0 := \{ i \in M : b_i = 0 \} \). Then we immediately have the following lemma.

**Lemma 5.** \( W_0 \neq \emptyset \).

**Proof.** Suppose to the contrary that \( W_0 = \emptyset \). Then any \( b_i \geq 1 \) in (3), which implies that \( b_i = 1 \) for any \( i \in M \), so that \( \tilde{A} \) forms some node-disjoint directed cycles that span all nodes, contradicting Lemma 2. \( \square \)

Note that, from the formation of arc set \( \tilde{A} \), it is clear that \( \tilde{A} \) as a tilde-valid arc set may not be unique. However, among all possible choices of a tilde-valid arc set \( \tilde{A} \subseteq A \), we choose one that has some additional properties in terms of minimum cardinalities of some combinatorial structures, which we shall define in due course. These additional properties will be presented in a sequence of three assumptions, which are made without loss of generality due to the finiteness of the total number of tilde-valid arc sets. With the same reason, we assume that our coalitional deviation \( \Delta \) is chosen in such a way that it has a certain property (see Assumption 4).

**Assumption 1.** Arc set \( \tilde{A} \) is tilde-valid and it minimizes \( |W_0(\tilde{A})| \).

Let \( \tilde{W}_0 = Q^+(W_0) \). Then \( \tilde{W}_0 \neq \emptyset \) according to Lemmas 1 and 5. A node \( i \in M \) is said to be **associated** with \( W_0 \) if it is **linked** to an element of \( \tilde{W}_0 \) through a sequence of arcs (but not a directed path) in \( \tilde{A} \) and \( A \) in alternation (see Figure 2 for an illustration). More formally, \( i \in M \) is associated with \( W_0 \) if and only if, for some integer \( k \geq 0 \), there are nodes \( \{ i_0, \ldots, i_k, j_0, \ldots, j_k \} \subseteq M \) with \( i = i_k \) and \( j_0 \in \tilde{W}_0 \), such that
\[
(i_0, j_0), \ldots, (i_k, j_k) \in \tilde{A} \quad \text{and} \quad (i_0, j_1), \ldots, (i_{k-1}, j_k) \in A. \tag{8}
\]

Note that in the above definition, if \( i = i_k \) is associated with \( W_0 \), then \( i_0, \ldots, i_{k-1} \) used in (8) are each associated with \( W_0 \). Define
\[
W_1 := \{ i \in M : \text{node } i \text{ is associated with } W_0 \},
\]
\[
\tilde{W}_1 := Q^+(W_1).
\]
Immediately we have $\tilde{Q}^- (\tilde{W}_0) \subseteq W_1$, which implies that
\[
\tilde{W}_0 \subseteq \tilde{W}_1. \tag{9}
\]
On the other hand, since $\tilde{Q}^- (\tilde{W}_0) \neq \emptyset$ according to (2), we have $W_1 \neq \emptyset$.

**Lemma 6.** For any $i \in W_1$, $b_i = 1$. Furthermore, $Q^+ (W_0 \cup W_1) = \tilde{W}_1$.

**Proof.** It is clear from the definition that $W_1 \cap W_0 = \emptyset$. Hence $b_i \geq 1$ for any $i \in W_1$. Assume for contradiction that $b_i \geq 2$ for some $i \in W_1$. Since $i$ is associated with $W_0$, in addition to nodes
\[
\{i_0, \ldots, i_k, j_0, \ldots, j_k\} \subseteq M
\]
satisfying (8), we have a node $h \in W_0$ (hence $h \notin \{i_0, \ldots, i_k\}$) such that $(h, j_0) \in A$ according to the definition of $\tilde{W}_0$. Now we remove $k + 1$ arcs $(i_0, j_0), \ldots, (i_k, j_k)$ from $\tilde{A}$ and add $k + 1$ new arcs $(h, j_0), (i_0, j_1), \ldots, (i_k-1, j_k)$ to $\tilde{A}$. It is easy to see that the new set $\tilde{A}$ still has property (2). Additionally, under the new $\tilde{A}$, all $\{b_k\}$ remain the same except two of them: $b_h$ and $b_i$, with the former increased by 1 and the latter decreased by 1. Since $b_i \geq 2$ under the original $\tilde{A}$ and $i \notin W_0$ under the new $\tilde{A}$, then $i \notin W_0$ under the new $\tilde{A}$. Consequently, the new $W_0$ determined by the new $\tilde{A}$ contains a smaller number of elements, contradicting Assumption 1 about the original $\tilde{A}$.

To prove the second part of the lemma, let us first prove $Q^+ (W_1) \subseteq \tilde{W}_1$. Let $i \in W_1$ and $(i, j) \in A$. We show that $j \in \tilde{W}_1$. In fact, since $|\tilde{Q}^- (j)| = 1$ according to (2), we have a node $h \in M$ such that $(h, j) \in \tilde{A}$. Now since $i$ is associated with $W_0$, we conclude that $h$ is also associated with $W_0$, which implies that $j \in \tilde{W}_1$. Therefore, with (9) we have proved that
\[
Q^+ (W_0 \cup W_1) \subseteq \tilde{W}_1.
\]
The other direction of the inclusion is apparent.

It follows from Lemma 6 and (2) that the mapping $\tilde{Q}^+ (\cdot)$ from $W_1$ onto $\tilde{W}_1$ is a one-to-one correspondence and hence
\[
|W_1| = |\tilde{W}_1| > 0. \tag{10}
\]
Let
\[
W := W_0 \cup W_1 \cup \tilde{W}_1.
\]

### 4.2 Notation

As we can see from (7), bounding the sizes $m_2^3$ and $m_3^3$ of the respective sets $M_2^3$ and $M_3^3$ is vital in our establishment of the desired bound. We therefore take a close look at the two sets by partitioning
\[
X := M_2^3 \cup M_3^3
\]
into a number of subsets, so that different bounding arguments can be applied to different subsets.

We assemble our notation here in one place for easy reference and the reader is advised to conceptualize each *only* when it is needed in an analysis at a later point.

Let
\[
\tilde{M}_2^3 := \{ \ell \in M_2^3 : \tilde{Q}^+ (\ell) \notin W \}.
\]
For convenience, we reserve letter $\ell$ to exclusively index elements of $\tilde{M}^2_2$ and let $\tilde{Q}^+(\ell) = \{\ell_1, \ell_2\}$ with the understanding that it is always the case that $\ell_1 \notin W$. For any $\ell \in \tilde{M}^2_2$, $\ell_1 \notin W$ implies $b_{\ell_1} \geq 1$ since $W_0 \subseteq W$. On the other hand, since arc $(\ell, \ell_1) \in \tilde{A} \subseteq \tilde{A}$, we have

$$a_{\ell_1} = \delta^+(\ell_1) + 1 \geq b_{\ell_1} + 1$$

according to Observation 3, which implies that $\ell_1$ must belong to one of the following three mutually disjoint node sets:

$$Z_1 := \{i \in M \setminus W : b_i = 1\},$$
$$Z_2 := \{i \in M \setminus W : b_i > 1, a_i > b_i + 1\},$$
$$Z := \{i \in M \setminus W : b_i > 1, a_i = b_i + 1\}.$$

Therefore, if we define

$$\begin{align*}
X_2 & := \{i \in M^3_2 : \tilde{Q}^+(i) \not\subseteq W\}, \\
X_3 & := \{\ell \in \tilde{M}^2_2 : \tilde{Q}^+(\ell) \cap W = \emptyset\}, \\
X_4 & := \{\ell \in \tilde{M}^2_2 \setminus X_3 : \ell_1 \in Z_1 \cup Z_2\}, \\
X_5 & := \{\ell \in \tilde{M}^2_2 \setminus X_3 : \ell_1 \in Z, Q^+(\ell_1) \not\subseteq W, \delta^-(\ell_1) > 1\}, \\
X_6 & := \{\ell \in \tilde{M}^2_2 \setminus X_3 : \ell_1 \in Z, Q^+(\ell_1) \not\subseteq W, \delta^-(\ell_1) = 1\},
\end{align*}$$

then we have

$$X_{11} := \tilde{M}^2_2 \setminus \bigcup_{k=3}^6 X_k = \{\ell \in \tilde{M}^2_2 \setminus X_3 : \ell_1 \in Z, Q^+(\ell_1) \subseteq W\},$$

and

$$|\tilde{Q}^+(\ell) \cap W| = |\tilde{Q}^+(\ell) \setminus W| = 1,$$

for any

$$\ell \in \tilde{M}^2_2 \setminus X_3 = X_4 \cup X_5 \cup X_6 \cup X_{11}.$$

In other words, for any element

$$\ell \in \tilde{M}^2_2 \setminus X_3,$$

the two-element set $\tilde{Q}^+(\ell)$ has exactly one element in $W$. Now let

$$X_1 := \{i \in X : \tilde{Q}^+(i) \subseteq W\} \cup X_{11}.$$

Clearly, $X_i \cap X_j = \emptyset$ ($1 \leq i \neq j \leq 6$) and $X = \bigcup_{k=1}^6 X_k$.

## 5 Proving upper bounds

To bound from below the right-hand side of the key inequality (7), or equivalently, to bound from above the left-hand side of (7), we establish through a series of five lemmas and a corollary that the number of nodes in $X_i$ is at most $c_t|Y_t|$, where $c_t \in \{1, \frac{1}{2}\}$ and $Y_t$ are defined below for $t = 1, \ldots, 6$ with $Y_1, \ldots, Y_6 \subseteq M \setminus X$ mutually disjoint node sets. We divide our proofs into two parts with the second part on bounding $|X_6|$.

### 5.1 Part 1

Let us start with some straightforward upper bounds. Since $\tilde{Q}^+(i) \setminus W \neq \emptyset$ for any $i \in X_2$ according to the definition of $X_2$, we immediately have the following lemma thanks to Observation 3.

**Lemma 7.** Let $Y_2 := \bigcup_{i \in X_2} \tilde{Q}^+(i) \setminus W$. Then $Y_2 \subseteq M'' \setminus W$ and $|X_2| \leq |Y_2|$. 
Note that \( |\tilde{Q}^+(\ell)| = 2 \) for any \( \ell \in X_3 \) and \( \tilde{Q}^+(i) \cap \tilde{Q}^+(j) = \emptyset \) \( (i \neq j) \) due to (2), which lead to the following lemma.

**Lemma 8.** Let \( Y_3 := \bigcup_{\ell \in X_3} \tilde{Q}^+(\ell) \). Then \( Y_3 \subseteq M' \setminus W \) and \( 2|X_3| \leq |Y_3| \).

The following lemma follows directly from the definition of \( X_4 \):

**Lemma 9.** Let \( Y_4 := \bigcup_{\ell \in X_4} \tilde{Q}^+(\ell) \setminus W \). Then \( Y_4 \subseteq M' \setminus W \) and \( |Y_4| \leq |X_4| \). For any \( j \in Y_4 \), \( b_j > 1 \) and \( a_j > b_j + 1 \), unless \( b_j = 1 \).

At this point, we introduce our second additional assumption about \( \tilde{A} \) without loss of generality.

**Assumption 2.** Arc set \( \tilde{A} \) is such that it first satisfies Assumption 1 and then minimizes \( |M_2^2(\tilde{A})| \).

For any \( \ell \in X_5 \), since \( \delta^-(\ell_1) > 1 \) according to the definition of \( X_5 \), there is \( j \in Q^-(\ell_1) \setminus \{\ell\} \). Then \( j \notin W_0 \cup W_1 \) (otherwise we would have \( \ell_1 \in W \) according to Lemma 6). In fact, node \( j \) has the following property:

\[
j \in M_2^1 \cap M' \setminus W. \tag{12}
\]

To see this, consider replacing \((\ell, \ell_1)\) with \((j, \ell_1)\) in \( \tilde{A} \) to form a new tilde-valid arc set \( \tilde{A}' \). It is easy to see that \( \tilde{A}' \) satisfies Assumption 1. However, with the new arc set \( \tilde{A}' \), \( \ell \) is no longer a node in the new \( M_2^2(\tilde{A}') \), which implies that \( j \) has to become a node in \( M_2^2(\tilde{A}') \) in order not to contradict Assumption 2 with the original choice of \( \tilde{A} \), which in turn implies properties (12). Furthermore, since \( j \in M_2^1 \) and \((j, \ell_1) \in A \setminus \tilde{A} \), there is no \( k \neq \ell_1 \) such that \((j, k) \in A \setminus \tilde{A} \), which implies that

\[
j \notin Q^{-1}(\tilde{Q}^+(\ell')) \setminus \{\ell'\}.
\]

Consequently, we have the following lemma.

**Lemma 10.** Let 

\[
Y_5 := \bigcup_{\ell \in X_5} Q^-(\tilde{Q}^+(\ell) \setminus W) \setminus \{\ell\}.
\]

Then \( Y_5 \subseteq M_2^1 \cap M' \setminus W \) and \( |X_5| \leq |Y_5| \).

Now, let us establish an upper bound on \( |X_1| \) in the following lemma with the minimality of our deviation graph \( G(\Delta) \).

**Lemma 11.** Let \( Y_1 := W_1 \). Then \( |X_1| \leq |Y_1| \).

**Proof.** Suppose to the contrary that \( |X_1| > |W_1| \), i.e., \( |X_1| > |W_1| = |\tilde{W}_1| \) according to (10). Let \( X'_1 \subsetneq X_1 \) be a proper subset of \( |\tilde{W}_1| \) > 0 elements. Define

\[
K := W \setminus \tilde{W}_1 \subseteq W_0 \cup W_1,
\]

\[
K' := Q^+(X'_1) \setminus W,
\]

\[
\Gamma' := \{ J_j \in \Gamma : J_j \in \bigcup_{i \in \tilde{M}} S_i \},
\]

where \( \tilde{M} := X'_1 \cup K \cup K' \) (see Figure 3 for an illustration with explanations to follow).

![Figure 3](proof_of_lemma_11.png)
Then $\Gamma' \neq \emptyset$ since $X'_i \neq \emptyset$. We claim $\Gamma'$ is a proper subset of $\Gamma$. To see this, let $i \in X_1 \setminus X'_1 \neq \emptyset$. Since $X \cap (W_0 \cup W_1) = \emptyset$ (see Lemma 6), we have $i \notin K$. Observation 3 implies $i \notin K'$. Therefore, we have $i \notin \tilde{M}$, i.e., $S_i \cap \Gamma' = \emptyset$, but $S_i \subseteq \Gamma$. With the same arguments we note that the three constituent subsets of $\tilde{M}$ are mutually disjoint. In Figure 3, the set $X_{11}'$ is a subset of $X_{11}$ according to the definition of $X_1$ and the mapping between $X_{11}'$ and $K'$ is a one-to-one correspondence due to equation (11).

Since $|X'_1| = |\tilde{W}_1|$, we can assume there is a one-to-one correspondence $\phi$ between the nodes (i.e., servers) of the two sets $X'_1$ and $\tilde{W}_1$. Now let us define a new coalitional deviation $\Delta'$ with $\Gamma' = \Gamma'(\Delta')$, which is the same as $\Delta$ restricted on $\Gamma'$ except that, if $J_j \in \Gamma'$ migrates in $\Delta$ to a server of $\tilde{W}_1$, then let $J_j$ migrate in $\Delta'$ to the corresponding (under $\phi$) server of $X'_1$.

We show that the improvement ratio of any job deviation in $\Delta'$ is at least the same as that in $\Delta$, which then implies that $\text{IR}(\Delta') \geq \text{IR}(\Delta)$, contradicting the minimality of $G = G(\Delta)$ according to Definition 1. To this end, we only need to show that the new coalitional deviation $\Delta'$ takes place among the servers assigned with jobs of the coalition $\Gamma'$, i.e.,

$$Q^+(\tilde{M}) \subseteq W \cup K' = \tilde{W}_1 \cup K \cup K',$$

(13)

so that benefit of any job deviation will not decrease due to the fact that all jobs on servers of $X'_1$ migrate out in $\Delta$ and hence in $\Delta'$ as well, leaving empty space for deviational jobs under $\Delta'$, which originally migrate to servers of $\tilde{W}_1$ under $\Delta$.

First, we have $Q^+(K) \subseteq \tilde{W}_1$ according to Lemma 6. On the other hand, it can be easily verified that

$$Q^+(X'_1) \subseteq W \cup K'$$

according to the definition of $K'$. Now we show $Q^+(K') \subseteq W$, which then implies (13). In fact, for any $k \in K'$, noticing that $X'_1 \subseteq X_1$, according to the definitions of $K'$ and $X_1$, we have $k \in Q^+(X_{11}) \setminus W$, which implies that $Q^+(k) \in W$ according to the definition of $X_{11}$.

5.2 Part 2

To prove our final upper bound, we need to introduce the following two structures in graph $G(\Delta)$ with tilde-valid arc set $\tilde{A}$:

$$\Omega(\tilde{A}) := \{i \in M' : i_1 = \tilde{Q}^{-1}(i) \in \tilde{G}^+(i), \delta^-(i_1) = 1, \delta^+(i_1) = b_{i_1} > 1\},$$

$$\Pi(\tilde{A}) := \{(i, i_1, j) : i \in \tilde{M}_2^1 \setminus X_3, i_1 \in \tilde{Q}^+(\ell) \cap \tilde{Q}^-(j) \setminus W, i \neq j, \delta^-(i_1) = 1, \delta^+(i_1) = b_{i_1} > 1, j \in M'\}.$$

Note that each element in $\Omega(\tilde{A})$ represents a directed 2-cycles of both arcs in $\tilde{A}$ and each element in $\Pi(\tilde{A})$ is a directed 2-path of both arcs in $\tilde{A}$. In both cases of $\Omega(\tilde{A})$ and $\Pi(\tilde{A})$, the interior node $i_1$ has an in-degree $\delta^-(i_1) = 1$ and all its out-arcs are in $\tilde{A}$. Our next result is based on the following further refinement of the tilde-valid arc set $\tilde{A}$.

**Lemma 12.** If $\Omega(\tilde{A}) \neq \emptyset$ for some arc set $\tilde{A}$ satisfying Assumption 2, then there exists an arc set $\tilde{A}'$ such that, while it also satisfies Assumption 2, additionally, $\Omega(\tilde{A}')$ is a proper subset of $\Omega(\tilde{A})$.

**Proof.** Assume $i \in \Omega(\tilde{A})$ and let $i_1 = \tilde{Q}^{-1}(i)$ be as in the definition of $\Omega(\tilde{A})$. Then there must be a node $h \in Q^-(i)$ with $h \neq i_1$, since otherwise $\delta^-(i) = \delta^-(i_1) = 1$, which implies that there would be no directed path from any other nodes in $G(\Delta)$ to nodes $i$ or $i_1$, contradicting Lemma 4. Therefore, the following set is not empty:

$$H_i := \{h \in M : (h, i) \in A, \text{ either } \delta^-(h) > 1 \text{ or } (h, i) \notin \tilde{A}\}.$$

(14)

Let $h \in H_i \neq \emptyset$. We define a new tilde-valid arc set

$$\tilde{A}' := (\tilde{A} \setminus \{(i_1, i)\}) \cup \{(h, i)\}.$$

(15)
Assumption 4. Coalitional deviation assumption. For generality, our coalitional deviation $\Delta$ is considered to have been chosen so that it satisfies the following assumption:

Any arc set $A$ of $\ell \in X_\ell$ that maximizes $\ell$ and $\ell_j \in Z$ is such that the arc $(\ell, \ell_j)$ is lexicographically smaller than $\Delta$. This can be obtained by contracting Corollary 13 with our Assumption 3 due to $b_{\ell_j} > 1$, and hence also satisfies Assumption 2 since $\delta^+(h) < a_{\ell_j}$ (which implies that $h \notin \Omega(\tilde{A}) \cup \Omega(\tilde{A}')$) according to Observation 3 (as no other node not in $M^2(\tilde{A})$ can possibly become a member of $M^2(\tilde{A}')$).

As a result of Lemma 12, we can further refine our initial choice of $\tilde{A}$ so that it satisfies the following assumption, where the benefit of minimizing $|\Pi(\tilde{A})|$ will be seen in the proof of Lemma 14 (see (16)).

Assumption 3. Any arc set $\tilde{A}$ is such that it first satisfies Assumption 2 and then lexicographically minimizes $(|\Omega(\tilde{A})|, |\Pi(\tilde{A})|)$.

Corollary 13. Any arc set $\tilde{A}$ satisfying Assumption 3 must satisfy $\Omega(\tilde{A}) = \emptyset$.

An arc set $\tilde{A}$ in graph $G(\Delta)$ that satisfies Assumption 3 is said to be derived from $\Delta$. Without loss of generality, our coalitional deviation $\Delta$ is considered to have been chosen so that it satisfies the following assumption.

Assumption 4. Coalitional deviation $\Delta$ defining minimal deviation graph $G(\Delta)$ is such that the arc set $\tilde{A}$ derived from $\Delta$ gives lexicographical minimum

$$V(\Delta) := (|W_0(\tilde{A})|, |M^2(\tilde{A})|, |\Omega(\tilde{A})|, |\Pi(\tilde{A})|).$$

Lemma 14. Let minimal deviation graph $G(\Delta)$ with $\Delta$ satisfying Assumption 4 be given. For any $\ell \in X_\ell$, there is $j \neq \ell$, such that $Q^{-1}(j) = Q^+(\ell) \setminus W$ (note (11)) and $j \in M'' \setminus W$. 

Proof. Given $\ell \in X_\ell$ and $\ell_1 \in Z$, we have $a_{\ell_1} = b_{\ell_1} + 1$ according to the definition of $X_\ell$, and hence $\delta^+(\ell_1) = Q^+(\ell_1)$ since $a_{\ell_1} = \delta^+(\ell_1) + 1$ according to Observation 3. Since $b_{\ell_1} > 1$ and $Q^+(\ell_1) = Q^+(\ell_1) \notin W$ (again according to the definition of $X_\ell$), we let $j \in Q^+(\ell_1) \setminus W$. Then $j \neq \ell$ since otherwise we would have $\ell \in \Omega(\tilde{A})$, contracting Corollary 13 with our Assumption 3 (see Figure 4 for an illustration with more explanations to follow).

We claim $j \in M''$ and hence are done. Let us assume for a contradiction that $j \in M'$. Note that with $\{\ell, \ell_1, j\}$ replacing $\{i, i_1, j\}$ in the definition of $\Pi(\tilde{A})$, we conclude that $\ell \in \Pi(\tilde{A})$. Now let us define a new coalitional deviation $\Delta'$ so that its derived arc set $\tilde{A}'$ gives a

$$\mu(\Delta') := (|W_0(\tilde{A}')|, |M^2(\tilde{A}')|, |\Omega(\tilde{A}')|, |\Pi(\tilde{A}')|)$$

that is lexicographically smaller than

$$\mu(\Delta) := (|W_0(\tilde{A})|, |M^2(\tilde{A})|, |\Omega(\tilde{A})|, |\Pi(\tilde{A})|),$$

a desired contraction to Assumption 4.

In fact, let $\Delta'$ be defined as in Observation 4 after node $i$ has been replaced by $\ell$ in the statement of Observation 4. Denote $A'$ as the arc set of the resulting minimal deviation graph $G(\Delta')$. Let $\tilde{A}'$ be the natural result of $\tilde{A}$ after the re-orientation from $\Delta$ and $\Delta'$, i.e., an arc in $\tilde{A}$ pointing to $\ell$ (resp. $j$) will become an arc in $\tilde{A}'$ pointing to $j$ (resp. $\ell$). Other arcs are the same for $\tilde{A}$ and $\tilde{A}'$. Apparently,

$$|W_0(\tilde{A}')| = |W_0(\tilde{A})|, |M^2(\tilde{A}')| = |M^2(\tilde{A})|.$$
On the other hand, if the value of $|Ω(\bar{A}')|$ has increased from $|Ω(\bar{A})|$, then clearly it must be the result of $\ell$ and/or $j$ becoming element(s) of $Ω(\bar{A}')$. In any such case (say, the former case for the sake of argument), based on the definition of $Ω(\bar{A}')$, we can use the approach in Lemma 12 to find $h \in H_\ell$ as defined in (14) and perform an arc-swap as in (15) with $i$ and $i_1$ replaced by $\ell$ and $\ell_1$, respectively, to reduce $|Ω(\bar{A}')|$ while maintaining the values of $|W_{0}(\bar{A})|$ and $|M_{2}^{\ell}(\bar{A}')|$. For convenience, we still use $\bar{A}'$ to denote the tilde-valid arc set after such arc-swap(s) if needed. Consequently, we have

$$Ω(\bar{A}') = Ω(\bar{A}) = \emptyset.$$  

However, we claim

$$|Π(\bar{A}')| < |Π(\bar{A})|,$$  

(16) a desired contradiction. To see (16), we first note that (i) any 2-path in $Π(\bar{A})$ starting at $i \neq \ell$, $j$ is also a 2-path in $Π(\bar{A}')$, and vice versa, and (ii) any 2-path in $Π(\bar{A})$ (resp. $Π(\bar{A}')$) starting at $\ell$ (resp. $j$) must have the first arc $(\ell, \ell_1)$ (resp. $(j, Q^{+}(j)\backslash W)$), since $|Q^{+}(j)\backslash W| = 1$ due to $j \in M_{2}^\ell \backslash X_3$ according to (11)).

On the other hand, the following can be easily observed:

1. If $(\ell, \ell_1, j') \in Π(\bar{A})(j' \neq j)$, then $(\ell, \ell_1, j') \in Π(\bar{A}')$, and vice versa.
2. If $(j, j_1, j') \in Π(\bar{A})(j' \neq j)$, then $(j, j_1, j') \in Π(\bar{A}')$, and vice versa.
3. $(\ell, \ell_1, j) \in Π(\bar{A}) \cap Π(\bar{A}')$, since $(\ell, \ell_1, j) \in Π(\bar{A}')$ would imply $(\ell_1, j) \in \bar{A}' \subseteq A'$ by definition of $Π(\bar{A}')$ and hence $(\ell_1, \ell) \in A$ by definition of $A'$, which in turn implies that $(\ell_1, \ell) \in \bar{A}$ since $b_{\ell_1} = \delta^{+}(\ell_1)$ under $\bar{A}$. Consequently, we obtain $\ell \in Ω(\bar{A})$, contradicting Corollary 13.
4. With similar reasons for $(\ell, \ell_1, j) \notin Π(\bar{A}')$, we have $(j, j_1, \ell) \notin Π(\bar{A}')$.

Therefore, overall $Π(\bar{A}')$ contains at least one element less than $Π(\bar{A})$ as indicated in points 3 and 4 above.

We call $j \in M'\backslash W$ identified in the above lemma a company of $\ell \in X_6$. Clearly, any $j \in M'\backslash W$ cannot be a company of two different elements of $X_6$ according to the statement of the lemma, which leads us to the following corollary.

**Corollary 15.** Denote $X[\ell] := \{j \in M\backslash X: node j is a company of \ell\}$ for any $\ell \in X_6$ and let $Y_\ell := \bigcup_{\ell \in X_6}((\bar{Q}^{+}(\ell)\backslash W) \cup X[\ell])$. Then $Y_\ell \subseteq M'\backslash W$ and $2|X_6| \leq |Y_\ell|$.  

We have used the cardinalities of the six sets $Y_1, \ldots, Y_6$ to bound $|X_1|, |X_2|, 2|X_3|, |X_4|, |X_5|$ and $2|X_6|$, respectively. Let us make sure these sets do not overlap with $X$ and are mutually disjoint. According to Lemmas 7–10 and Corollary 15, we have

$$Y_5 \subseteq M_{3}^{\ell} \cap M'\backslash W \subseteq M\backslash X,$$

$$Y_2, Y_3, Y_4, Y_6 \subseteq M'\backslash W \subseteq M\backslash X,$$

and

$$Y_t \subseteq \bar{Q}^{+}(X_t), \quad t = 2, 3, 4,$$

$$Y_6 \subseteq \bar{Q}^{+}(X_6) \cup \bar{Q}^{+}(M\backslash X).$$

Hence, $W \cap X = \emptyset$ and $W \cap Y_t = \emptyset (t = 2, \ldots, 6)$. Since $\bar{Q}^{+}(i) \cap \bar{Q}^{+}(j) = \emptyset$ for any $i \neq j$ (definition of $\bar{A}$) and $\bar{Q}^{+}(i) \cap M' = \emptyset$ for any $i$ (see Observation 3), noticing that $Y_1 = W_1 \subseteq W\backslash X$ (see Lemma 6), we conclude that

$$Y_t \cap X = \emptyset \quad \text{and} \quad Y_t \cap Y_s = \emptyset, \quad s \neq t, \quad s, t \in \{1, \ldots, 6\}. \quad (17)$$

### 6 Establishment of strong stability

Now we are ready to go back to proving (7) and hence Theorem 1. Since $X_2 \subseteq M_{3}^{\ell}$, the left-hand side of (7) is at most

$$|X| - \frac{1}{2}|M_{3}^{\ell}| \leq \sum_{t=1}^{6} |X_t| - \frac{1}{2}|X_2| \quad (18)$$
On the other hand, if we let 

\[ Y'_t := \{ i \in Y_t : b_i = 1 \} \quad \text{and} \quad Y''_t := Y_t \setminus Y'_t, \quad \text{for } t = 2, 3, 4, 6, \]

which imply

\[
Y''_4 = \{ i \in Y_4 : b_i \geq 2, a_i \geq b_i + 2 \} \subseteq \bigcup_{2 \leq b \leq a-2} M^b_a,
\]

\[
\bigcup_{t \in \{2,3,4,6\}} Y'_t \cup Y_1 \cup Y_5 \subseteq \bigcup_{a \geq 2} M^1_a,
\]

\[
\bigcup_{t \in \{2,3,6\}} Y''_t \subseteq \bigcup_{2 \leq b < a} M^b_a,
\]

then noticing the properties (17) and that

\[
b - \frac{4b - 5}{a - 1} \geq \begin{cases} 
\frac{1}{2}, & \text{if } 2 \leq b \leq a - 2, \\
1, & \text{if } 2 \leq b < a,
\end{cases}
\]

we see that the right-hand side of (7) is at least

\[
|Y_1| + |Y_5| + \sum_{t \in \{2,3,4,6\}} |Y'_t| + \frac{1}{2} \sum_{t \in \{2,3,6\}} |Y''_t| + |Y''_4|
\]

\[
\geq |Y_1| + \frac{1}{2} |Y_2| + \frac{1}{2} |Y_3| + |Y_4| + |Y_5| + \frac{1}{2} |Y_6|.
\]

(19)

According to Lemmas 7–11 and Corollary 15, the right-hand side of (18) is at most that of (19), which in turn ultimately leads to (7). Consequently, Theorem 1 is established.

From Theorem 1 and the lower bound demonstrated in Subsection 2.1, the following theorem follows.

**Theorem 2.** In the m-server load balancing game \((m \geq 3)\), any NE is a \((5/4)\)-approximate SNE and the bound is tight.

### 7 Concluding remarks

By establishing a tight bound of \(5/4\) for the approximation of general NEs to SNEs in the \(m\)-server load balancing game for \(m \geq 3\), we have closed the final gap for the study of approximation of general NEs to SNEs. However, as demonstrated by Feldman and Tamir [7], and by Chen [4], a special subset of NEs known as LPT assignments, which can be easily identified as NEs [9], do approximate SNEs better than general NEs. It is still a challenge to provide a tight approximation bound for this subset of NEs.

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