Active plasma resonance spectroscopy: eigenfunction solutions in spherical geometry

J Oberrath and R P Brinkmann

Institute for Theoretical Electrical Engineering, Ruhr University Bochum, D-44780 Bochum, Germany

Received 9 April 2014, revised 7 July 2014
Accepted for publication 4 August 2014
Published 30 September 2014

Abstract

The term active plasma resonance spectroscopy denotes a class of related techniques which utilize, for diagnostic purposes, the natural ability of plasmas to resonate on or near the electron plasma frequency $\omega_{pe}$: a radio frequent signal (in the GHz range) is coupled into the plasma via an antenna or probe, the spectral response is recorded, and a mathematical model is used to determine plasma parameters like the electron density. The mathematical model of an arbitrarily shaped probe–plasma system can be written in an abstract but very compact equation. It contains an appropriate operator, which describes the dynamical behavior and can be split into a conservative and a dissipative part. Based on the cold plasma model, this manuscript provides a solution strategy to determine the electrical admittance of a specific probe–plasma system derived from the abstract dynamical equation. Focusing on probes with a spherical-shaped probe tip the general admittance can be derived analytically. Therefore, the matrix representation of the resolvent of the dynamical operator is determined. This matrix representation is derived by means of the eigenfunctions and eigenvalues of the conservative operator. It can be shown that these eigenvalues represent the resonance frequencies of the probe–plasma system which are simply connected to the electron density. As an example, the result is applied to established probe designs: the spherical impedance probe and the multipole resonance probe.

Keywords: active plasma resonance spectroscopy, multipole resonance probe, impedance probe, functional analytic, eigenvalues, resonance frequencies, eigenfunctions

(Some figures may appear in colour only in the online journal)

1. Introduction

The term active plasma resonance spectroscopy (APRS) denotes a number of similar plasma diagnostic methods, which exploit the natural ability of plasmas to resonate at or close to the plasma frequency of electrons. The principle is quite simple: an electrical signal in the GHz range is coupled to the plasma via an electrical probe. The spectrum of the response of the plasma is recorded, and then evaluated based on a specific mathematical model. From the structure of the spectrum one is able to calculate the electron density.

An overview and a classification of the different proposed APRS designs is given in [1]. One particular class of this method is that of electrostatic concepts [2–9]. (The term electrostatic refers to the fact that the mathematical model of the resonance behavior is based on the electrostatic approximation of Maxwell’s equations.) Within this class the coupling of a surface wave to the plasma is utilized, which excites resonance modes of frequencies below the electron plasma frequency. A number of different approaches has been reported to provide a deeper understanding of the resonance behavior of the system [10–25]. All of them are based on a fluid dynamical approach and restricted to specific probe designs.

However, a more general description has been provided by members of our own group [26]. There, the whole class of electrostatic probes is analyzed by means of functional analytic (Hilbert space) methods. These methods are particular suited because explicit information on the probe geometry or the electrical operation do not enter in the description. The most important result is given by the interpretation of a complex term...
as the electrical admittance of the probe–plasma system. The complex term refers to the resolvent of a dynamical operator that describes the system.

In this manuscript we provide a solution strategy to determine the admittance of a specific probe design derived from the general electrical admittance. Focusing on probes with a spherical shaped probe tip the admittance can be derived analytically. Therefore, the matrix representation of the resolvent of the dynamical operator—the operator can be split into a conservative and dissipative part—is determined. This matrix representation is derived by means of the eigenfunctions and eigenvalues of the conservative operator. We show that these eigenvalues represent the resonance frequencies of the probe–plasma system which are simply connected to the electron density. As an example, the result is applied to (a) the spherical impedance probe (IP) [6] and (b) the multipole resonance probe (MRP) [9].

2. Model of an electrostatic probe of arbitrary shape

In a recent paper a general model of an electrostatic probe is derived and analyzed [26]. For details we refer this work. Here, we summarize the most important aspects and results: the plasma chamber (see figure 1) is given as a simply connected, spatially bounded domain $\mathcal{V}$, most of which is plasma (a simply connected subdomain $\mathcal{P}$). Other subdomains of $\mathcal{V}$ are the plasma boundary sheath $\mathcal{S}$, which shields the plasma from all material objects, and possibly dielectric domains $\mathcal{D}$. The boundary $\partial\mathcal{V}$ of the domain $\mathcal{V}$ is either grounded (G) or ideally insulating (I) with vanishing conductivity and permittivity.

Into this idealized plasma chamber an arbitrarily shaped probe is immersed. The probe contains a finite number of powered electrodes $\mathcal{E}_n$, $n = 1 \ldots N$, which are insulated from each other and from ground. The electrodes are driven by rf voltages $U_n$. Grounded surfaces can be treated as another electrode $\mathcal{E}_0$. A possible dielectric shielding of the probe is represented as a part of the subdomain $\mathcal{D}$ within the plasma chamber $\mathcal{V}$.

Within the subdomain $\mathcal{P}$, the dynamical behavior of the plasma, given by the dynamics of the charge density $\rho_e$ and the current density $j_e$, is appropriately described by the cold plasma model in electrostatic approximation. Assuming a complete electron depletions within the sheath $\mathcal{S}$, a surface charge density $\sigma_e$ at the sheath edge $\mathcal{K}$ has to be taken into account. The corresponding equations, including the constant plasma frequency $\omega_{pe}$ and the collision frequency for electron–neutral-collisions $\nu$ is given by

$$\frac{\partial \sigma_e}{\partial t} = -n \cdot j_e \bigg|_{r \in \mathcal{K}},$$

$$\frac{\partial \rho_e}{\partial t} = -\nabla \cdot j_e.$$

(1)

$\phi$ is the inner electrostatic potential and is governed by Poisson’s equation in the domain $\mathcal{V}$ subject to homogeneous boundary conditions

$$-\nabla \cdot (\varepsilon_0 \varepsilon_r \nabla \phi) = \begin{cases} 0 & r \in \mathcal{D} \cup \mathcal{S} \\ \sigma_e & r \in \mathcal{K} \\ \rho_e & r \in \mathcal{P}. \end{cases}$$

(2)

The functions $\psi_n$, representing the vacuum coupling between the electrodes, are solutions to the homogeneous Poisson equation,

$$-\nabla \cdot (\varepsilon_0 \varepsilon_r \nabla \psi_n) = 0.$$  

(3)

They contain information about the geometry and satisfy the boundary conditions $\psi_n \big|_{r \in \mathcal{E}_n} = \delta_{nm}$ at the electrodes $\mathcal{E}_n$. ($\delta_{nm}$ is Kronecker’s delta.) The respective permittivity $\varepsilon_r$ is given as 1 within $\mathcal{S}$ and $\mathcal{P}$ and with $\varepsilon_r = \text{const}$ within $\mathcal{D}$.

The dynamical equations (1) can be written in matrix form. This allows to interpret $\sigma_e$, $\rho_e$, and $j_e$ as variables of the state vector $|z\rangle$, given by

$$|z\rangle = \begin{pmatrix} \sigma_e \\ \rho_e \\ j_e \end{pmatrix}.$$  

(4)

The state vector is an element of the linear vector space $\mathcal{H}$. For two different state vectors $|z\rangle$ and $|z'\rangle$ a scalar product, which
is motivated by the inner energy, can be defined by
\[
\langle z | z \rangle = \int V \varepsilon_0 \phi \nabla \phi^* \cdot \nabla \phi \, d^3r + \int \frac{1}{\varepsilon_0 \omega_{pe}^2} \mathbf{j}_e \cdot \mathbf{j}_e \, d^3r. \tag{5}
\]
It is compatible with the dynamical equations and induces the corresponding norm \( ||z|| = \sqrt{\langle z | z \rangle} \). By means of \( ||z|| \) it is possible to show that \( \mathcal{H} \) is complete for all square integrable state vectors including singular functions like the surface charge density. Hence, \( \mathcal{H} \) is a Hilbert space.

Another important vector is the excitation vector
\[
|e_n\rangle = \begin{pmatrix} 0 \\ 0 \\ - \varepsilon_0 \omega_{pe}^2 \nabla \psi_n \end{pmatrix}. \tag{6}
\]
Computing the scalar product between the excitation vector and the state vector, it turns out that this scalar product is equal to the inner current \( i_n \) at the electrode \( \psi_n \). The inner current represents the observable response of the dynamic system and is given by
\[
i_n = \langle e_n | z \rangle = - \int V \nabla \psi_n \cdot \mathbf{j}_e \, d^3r. \tag{7}
\]
This result shows, that the excitation vector acts also as observation vector.

Furthermore, two operators can be identified: the conservative operator \( T_C \) which is anti-Hermitian and the dissipative operator \( T_D \) which is Hermitian and positive definite. These operators contain information about the frequency and collisional damping behavior of the system, respectively. One can find
\[
T_C |z\rangle = \begin{pmatrix} -n \cdot \mathbf{j}_e |z\rangle \in \mathcal{K} \\ -\nabla \cdot \mathbf{j}_e \\ - \varepsilon_0 \omega_{pe}^2 \nabla \phi \end{pmatrix}, \tag{8}
\]
\[
T_D |z\rangle = \begin{pmatrix} 0 \\ 0 \\ - \nu \mathbf{j}_e \end{pmatrix}. \tag{9}
\]
By means of these definitions it is possible to describe the dynamical behavior of the probe–plasma system in an abstract, but very compact form
\[
\frac{\partial}{\partial t} |z\rangle = T_C |z\rangle + T_D |z\rangle + \sum_{n=1}^N U_n |e_n\rangle. \tag{10}
\]
Concerning measurements, the stationary solutions lie on the focus of interest. Therefore, a harmonic ansatz with the frequency \( \omega_{RF} \) > 0 is adequate to solve the dynamic equation for the state vector,
\[
|z\rangle = \sum_{n=1}^N U_n (i \omega_{RF} - T_C - T_D)^{-1} |e_n\rangle. \tag{11}
\]
Entering the general solution of the state vector (11) into the expression (7), one finds that the current \( i_n \) is given by the resolvent of the complete dynamical operator \( T_C + T_D \)
\[
i_n = \langle e_n | z \rangle = \sum_{n=1}^N \langle e_n | (i \omega_{RF} - T_C - T_D)^{-1} |e_n\rangle U_n = \sum_{n=1}^N Y_{nn} U_n. \tag{12}
\]
Hence, the scalar product between two excitation vectors and the resolvent can be interpreted as the admittance \( Y_{nn'} \) between two electrodes,
\[
Y_{nn'} = \langle e_n | (i \omega_{RF} - T_C - T_D)^{-1} |e_{n'}\rangle. \tag{13}
\]
The interpretation of equation (13) is the main result of the analysis of the general model in [26]. Based on functional analytic methods, this result can be used to determine an approximated or analytic expression for the admittance between two arbitrary electrodes, which is not derived yet. For this purpose, a complete orthonormal basis \( \{|k\rangle\} \) of the Hilbert space is needed. Two of these basis vectors are orthonormal to each other and they satisfy the completeness relation
\[
\langle k' | k \rangle = \delta_{kk'}, \quad \sum_k |k\rangle \langle k| = 1. \tag{14}
\]
Inserting these expressions into (13) allows to expand the admittance via the orthonormal basis. It yields an expression for \( Y_{nn'} \) which is determined by a vector–matrix–vector product
\[
Y_{nn'} = \sum_{k'} |e_n | k' \rangle \sum_k \langle k | (i \omega_{RF} - T_C - T_D)^{-1} |k\rangle |e_{n'}\rangle. \tag{15}
\]
The matrix in this product is the algebraic representation of the resolvent. Its elements are given by the scalar product \( \langle k' | (i \omega_{RF} - T_C - T_D)^{-1} |k\rangle \). Based on (14) it is possible to show, that the matrix of the resolvent is equal to the inverse matrix of the operator \( i \omega_{RF} - T_C - T_D \)
\[
\sum_k \langle k' | (i \omega_{RF} - T_C - T_D)^{-1} |k\rangle = 1. \tag{16}
\]
This result provides the opportunity to first determine the matrix of the operator and then to calculate its inverse to find the matrix of the resolvent.

Now, the solution strategy is obvious: one has to choose a set of orthonormal basis functions, after that the matrix elements of the operator \( i \omega_{RF} - T_C - T_D \) can be determined to find the matrix of the resolvent, then the scalar products between the basis vectors and the excitation vectors have to be computed, and finally the admittance is given by a vector–matrix–vector multiplication.

In a complex geometry an appropriate set of orthonormal basis functions has to be found to determine an approximated matrix representation of the operators. This leads to an efficient calculation of an approximated admittance instead of a simulation. It can be used to determine the spectral response, e.g., of the plasma absorption probe [7]. However, in this manuscript we focus on probes with a spherical probe tip. This allows, as we will show in the following sections, to derive an analytic solution of the admittance in a spherical probe–plasma system.
sheath thickness $\delta$ can be written with a pure imaginary eigenvalue $i$.

As shown in the end of the last section, an orthonormal basis is needed to calculate the matrix representation of the resolvent. The ideal basis would be the eigenfunction set of the complete dynamic operator $T_C + T_D$. However, it is worth noting, that $T_C$ and $T_D$ do not commute, which means that they do not have the same set of eigenvectors. Therefore, we follow the perturbation approach for operators since the collision frequency $\nu$ in a low pressure plasma is much smaller than the frequency range of interest.

For this purpose we have to determine the eigenvectors of the conservative operator $T_C$. Here, we focus on spherical geometry because the idealized spherical impedance probe and the idealized multipole resonance probe have a perfectly spherical geometry. They are depicted in Figure 2 with the probe radius $R$, the thickness of the dielectric $d$, and the sheath thickness $\delta$.

$T_C$ is anti-Hermitian. Due to that the eigenvalue equation can be written with a pure imaginary eigenvalue $io|z| = T_C|z|$. To solve the eigenvalue problem in spherical geometry we expand all scalar functions in spherical harmonics

\[
\sigma_{\ell}(\vartheta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sigma_{lm} Y_{lm}(\vartheta, \phi),
\]

\[
\rho_{\ell}(r, \vartheta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \rho_{lm}(r) Y_{lm}(\vartheta, \phi),
\]

(17)

\[
\phi(r, \vartheta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi_{lm}(r) Y_{lm}(\vartheta, \phi),
\]

and the current density in vectorial spherical harmonics

\[
j_e(r, \vartheta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} j_{lm}^{(Y)}(r) Y_{lm}(\vartheta, \phi) + j_{lm}^{(Z)}(r) Z_{lm}(\vartheta, \phi).
\]

This expansion leads to the following set of equations:

\[
\text{i} \omega \sigma_{lm} = -j_{lm}^{(Y)}|_{r=R+d},
\]

(19)

\[
\text{i} \omega \rho_{lm} = \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 j_{lm}^{(Y)} \right) - \frac{\text{i}}{r} \sqrt{l(l+1)} j_{lm}^{(Z)} \right],
\]

(20)

\[
\text{i} \omega j_{lm}^{(X)} = 0,
\]

(21)

\[
\text{i} \omega j_{lm}^{(Y)} = -\varepsilon_0 \omega^2 \varepsilon D \frac{\partial}{\partial r} \sigma_{lm},
\]

(22)

\[
\text{i} \omega j_{lm}^{(Z)} = \varepsilon_0 \omega^2 \varepsilon D \frac{\text{i}}{r} \sqrt{l(l+1)} \sigma_{lm}.
\]

(23)

Consequently, Poisson’s equation reads

\[
-\varepsilon_0 \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \rho_{lm}}{\partial r} \right) - \frac{l(l+1)}{r^2} \rho_{lm} \right] = \begin{cases} 0 & r \in [R-d, R+d) \\ \rho_{lm} & r \in [R+d, \infty) \end{cases}
\]

(24)

with the corresponding boundary conditions

\[
\phi_{lm}^{(D)}(R-d) = 0 \quad \text{and} \quad \lim_{r \to \infty} \phi_{lm}^{(P)}(r) = 0,
\]

(25)

and transition conditions

\[
\phi_{lm}^{(P)}(R-d) = \phi_{lm}^{(S)}(R+d) = 0,
\]

(26)

within the subdomain $\mathcal{P}$ we are able to combine the dynamical equations (20), (22), and (23) with Poisson’s equation (24) to obtain

\[
\left( 1 - \frac{\omega^2 \varepsilon D}{\omega^2} \right) \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi_{lm}^{(P)}}{\partial r} \right) - \frac{l(l+1)}{r^2} \phi_{lm}^{(P)} = 0.
\]

(27)

From (27) it is obvious that two cases can be distinguished: $\omega = \pm \omega_{pe}$ and $\omega \neq \pm \omega_{pe}$.

Since for $\omega \neq \pm \omega_{pe}$, the term in the right brackets of equation (27) has to equal zero. This term is identical to the radial component of the expanded Laplace equation. It has to
be satisfied by the potential in the plasma. The same holds of course for the potential in $D$ and $S$. Hence, all potentials are governed by the same equation because the permittivity is constant in the different regions

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi_{lm}^{(D,S,p)}}{\partial r} \right) - \frac{l(l+1)}{r^2} \phi_{lm}^{(D,S,p)} = 0. \quad (28)$$

Its general solution is given by

$$\phi_{lm}^{(D,S,p)}(r) = A^{(D,S,p)}_l r^l + B^{(D,S,p)}_l r^{-l-1}. \quad (29)$$

The coefficients $A^{(S,p)}_l$ and $B^{(D,S,p)}_l$ are determined by five of the boundary and transition conditions given in $\text{(25)}$ and $\text{(26)}$

$$B^{(D)} = \frac{A^{(D)}}{(R - d)^{2l+1}} \quad \text{and} \quad B^{(S)} = \frac{A^{(S)}_l}{2l+1} \quad \text{and} \quad B^{(P)} = \frac{A^{(P)}_l}{2l+1},$$

$$A^{(S)} = \frac{A^{(D)}}{2l+1} \left[ l + (\varepsilon_D + 1) + (l + 1)(\varepsilon_D - 1) \left( 1 - \frac{d}{R} \right)^{2l+1} \right] = A^{(S)}_l A^{(S)}_l,$$

$$A^{(S)} = \frac{A^{(D)}}{2l+1} \left[ l + (\varepsilon_D - 1) \left( 1 - \frac{d}{R} \right)^{2l+1} \right] + A^{(S)}_l A^{(S)}_l.$$

Each of these coefficients depends on $A^{(D)}_l = A^{(S)}_l$ and they allow for the complete inner potential to be written as

$$\phi(r) = \sum_{l,m} A^{(S)}_l Y_{lm} l^{(D)} l^{(S)} l^{(P)} r^{-l-1}, \quad r \in D \quad \text{and} \quad r \in S \quad \text{and} \quad r \in P. \quad (31)$$

The sixth condition given in $\text{(25)}$ and $\text{(26)}$ determines the eigenvalues of $T_S$ in spherical coordinates

$$\omega_{lm} = \pm \omega_{pe} \sqrt{\frac{l + 1}{2l + 1} \left[ 1 - b_l \left( 1 + \frac{\delta}{R} \right)^{2l+1} \right]} = \pm \omega_{pe} \eta_l,$$

$$b_l = \frac{l(\varepsilon_D - 1) + (\varepsilon_D + 1)(\varepsilon_D - 1) \left( 1 - \frac{d}{R} \right)^{2l+1}}{1 + (l + 1)\varepsilon_D + (l + 1)(\varepsilon_D - 1) \left( 1 - \frac{d}{R} \right)^{2l+1}}.$$

They are proportional to the electron plasma frequency and thus, simply connected to the electron density. It is important to note that the eigenvalues are independent of the index $m$. This shows that the resonance modes of a spherical probe–plasma system described by the cold plasma model are always symmetric referred to a rotation around an arbitrary rotation axis. The corresponding charge, surface charge, and current density depend on the inner potential defined by the equations $\text{(19)}$ to $\text{(23)}$.

In the case $\omega = \pm \omega_{pe}$ the left brackets of equation (27) vanish. Hence, an arbitrary potential fulfills Laplace’s equation in the plasma region

$$\phi_{lm}(r) = \text{arbitrary }, \forall r \in P \text{ with } \lim_{r \to \infty} \phi_{lm}(r) = 0. \quad (33)$$

The potential in $S$ and $D$ vanishes due to the rewritten boundary and transition conditions in this case and the complete inner potential is given by

$$\phi(r) = \sum_{l,m} Y_{lm} \left\{ 0, \quad \phi_{lm}^{(P)}(r), \quad r \in S \cup D, \quad r \in P. \quad (34)$$

Again, the corresponding charge, surface charge, and current density depend on the inner potential.

In summary, we find the following two different eigenvectors of the conservative operator $T_C$, which build a complete orthogonal set in the Hilbert space

$$\omega = \pm \omega_{pe} : \quad \frac{\omega_{pe}^2}{\omega_{pe}^2} = \frac{l + 1}{(R + \delta)^{2l+2}} Y_{lm}$$

$$\delta \left| \phi_{lm}^{(1+1)} \right| = \frac{\omega_{pe}^2}{\omega_{pe}^2} \left| \phi_{lm}^{(P)}(r) \right| _{r \to \infty}$$

$$\left| \phi_{lm}^{(2+1)} \right| = \frac{\omega_{pe}^2}{\omega_{pe}^2} \left| \phi_{lm}^{(P)}(r) \right| _{r \to \infty}.$$

A scalar product naturally induces a norm, whereby the eigenvectors can be normalized. In spherical geometry it is possible to simplify the scalar product $\langle \rangle$ via the expansion in the orthogonal spherical harmonics

$$\langle \phi | z \rangle = \sum_{l,m} \int_{r=d}^{r=\infty} \varepsilon_{0} e_{r} \left( |\phi_{lm}|^2 + \frac{l(l+1)}{r^2} \right)^{1/2} \phi_{lm}^{(1)} \phi_{lm}^{(2)} r^2 dr + \frac{1}{\varepsilon_{0} \omega_{pe}^2} \sum_{l,m} \int_{r=d}^{r=\infty} \left( |\phi_{lm}^{(1)}|^2 + |\phi_{lm}^{(2)}|^2 \right) r^2 dr.$$

By means of the simplified scalar product (37) each norm of the different eigenvectors can be determined. Introducing the components of the inner potential and the current density into the scalar product, the squared norm of the first eigenvector can explicitly be evaluated

$$\left| \phi_{lm}^{(1)} \right|^2 = \left| \phi_{lm}^{(2)} \right|^2 = \frac{2 \varepsilon_{0} (l + 1) B_{lm}^{(P)} r^2}{(R + \delta)^{2l+1} \eta_{l}}. \quad (38)$$

The elements of the second eigenvector depend on the inner potential $\phi_{lm}^{(P)}$ in the plasma region $P$. It turns out that the
remaining integrals in the scalar product are equal and the norm results in
\[ \| z^{(2)}_{lm} \| = \left\| z^{(2)}_{lm} \right\|_2 = 2 \varepsilon_0 \int_{R+d}^{\infty} \left[ \frac{\partial \phi^{(2)}_{lm}}{\partial r} \right]^2 + \frac{l(l+1)}{r^2} \left| \phi^{(2)}_{lm} \right|^2 r^2 \, dr. \] (39)

The integral remains undetermined due to the arbitrary inner potential in \( P \).

Once the norms are determined, we are able to define the following completeness relation because the normalized eigenvectors build a complete orthonormal set in the Hilbert space
\[ \sum_{lm} \left( \left| z_{lm}^{(1+)} \right|^2 + \left| z_{lm}^{(1-)} \right|^2 + \left| z_{lm}^{(2+)} \right|^2 + \left| z_{lm}^{(2-)} \right|^2 \right) = 1. \] (40)

As seen in equation (15) the completeness relation is needed to expand and simplify the admittance given in equation (13). Additionally, it allows the expansion of an arbitrary state vector in the eigenvectors of the conservative operator
\[ |z\rangle = \sum_{l,m} A_{lm}^{(1+)} \left| z_{lm}^{(1+)} \right\rangle + A_{lm}^{(1-)} \left| z_{lm}^{(1-)} \right\rangle + A_{lm}^{(2+)} \left| z_{lm}^{(2+)} \right\rangle + A_{lm}^{(2-)} \left| z_{lm}^{(2-)} \right\rangle. \] (41)

4. General excitation vector

After the orthonormal basis is derived, we have to compute the excitation vector to determine the admittance (15). Based on the specified geometry we are able to calculate the general excitation vector \( |e_n\rangle \). It contains the characteristic functions \( \psi_n \), which follow Laplace’s equation
\[ \nabla \cdot (\varepsilon_0 \varepsilon_r \nabla \psi_n) = 0 \quad \text{with} \quad \lim_{r \to \infty} \psi_n = 0 \quad \text{and} \quad \psi_n |_{E_n'} = \delta_{nn'} \]. (42)

Similar to the eigenvector calculation we expand the characteristic functions in spherical harmonics and determine the solution in \( r \)-direction
\[ \psi_{lm}(r) = \begin{cases} \alpha^{(D)}_{lm} r^l + \beta^{(D)}_{lm} r^{-(l+1)}, & r \in [R - d, R) \\ \beta^{(vac)}_{lm} r^{-(l+1)}, & r \in [R, \infty). \end{cases} \] (43)

Due to the continuity of the vacuum potential and the electric flux density at the surface of the dielectric we find the following transition conditions
\[ \psi_{lm}^{(D)}(R) = \psi_{lm}^{(vac)}(R), \quad \varepsilon_D \frac{\partial \psi_{lm}^{(D)}}{\partial r} \bigg|_R = \frac{\partial \psi_{lm}^{(vac)}}{\partial r} \bigg|_R. \] (44)

They allow for \( \alpha^{(D)} \) and \( \beta^{(D)} \) to be determined dependent on \( \beta^{(vac)} = \beta^{(n)}_{lm} \)
\[ \alpha^{(D)} = \frac{(l+1)(\varepsilon_D - 1)}{(2l+1)\varepsilon_D R^{2l+1}} \beta^{(vac)}, \quad \beta^{(D)} = \frac{1 + l(l + \varepsilon_D)}{(2l+1)\varepsilon_D} \beta^{(vac)}. \] (45)

By means of these coefficients the general characteristic functions are defined as
\[ \psi_{n}(r) = \sum_{l,m} \beta^{(n)}_{lm} \psi_{nm}^{(n)}(r) Y_{lm}(\vartheta, \varphi) \] (46)
and determine the general excitation vector to
\[ |e_n\rangle = \sum_{l,m} \left[ 0, 0, -\varepsilon_0 \varepsilon_r^2 \beta^{(n)}_{lm} \right] \left( \frac{\partial \psi_{lm}^{(n)}}{\partial r} Y_{lm} - \frac{i}{r} \sqrt{l(l+1)} \psi_{lm}^{(n)} Z_{lm} \right) \]. (47)

The remaining coefficient \( \beta^{(n)}_{lm} \) can be evaluated by the boundary condition \( \psi_{lm}^{(vac)}(R - d) = \delta_{lm'} \) at the electrodes \( E_n' \). Utilizing the orthogonality of the spherical harmonics we find
\[ \beta^{(n)}_{lm} = \frac{1}{\gamma_l} \int_{E_n} Y_{lm}^\ast(\vartheta, \varphi) \, d\Omega \quad \text{with} \quad \gamma_l = \frac{(l+1)(\varepsilon_D - 1) \left( 1 - \frac{d}{R} \right)^{2l+1} + l(l + \varepsilon_D)}{(2l+1)\varepsilon_D R^{2l+1} \left( 1 - \frac{d}{R} \right)^l}. \] (48)

The integral in \( \beta^{(n)}_{lm} \) contains the information about the electrode configuration within the probe tip and has to remain undetermined until the configuration is defined.

5. General admittance in spherical geometry

Now, we are equipped with all the necessary elements to expand the admittance in equation (13) via the completeness relation (40). Based on this expansion we derive the general admittance in a spherical geometry in this section. Introducing the completeness relation (40) twice into (13)—between the excitation vectors and the resolvent, respectively—yields a long expression. Within the long expression scalar products between the eigenvectors and the excitation vectors appear. The scalar products between the excitation vector and second eigenvector becomes zero and simplify the admittance
The remaining scalar products between $|e_{n}\rangle$ and $|z_{l/m}^{(1)}\rangle$ can explicitly be evaluated. They differ only in their sign and the expansion index. Hence, they are given by

$$
\langle e_{n}|z_{l/m}^{(1)}\rangle = \frac{i\varepsilon_{0}\omega_{pe}(l+1)\beta_{lm}^{(m)} B_{l}^{(p)} }{\eta_{l}(R+\delta)^{2i+1}} |z_{l/m}^{(1)}\rangle = -\langle e_{n}|z_{l/m}^{(-)}\rangle
$$

(50)

$$
\langle z_{l/m}^{(1)}|e_{n}\rangle = \frac{i\varepsilon_{0}\omega_{pe}(l+1)\beta_{lm}^{(m)} B_{l}^{(p)} }{\eta_{l}(R+\delta)^{2i+1}} |z_{l/m}^{(1)}\rangle = -\langle z_{l/m}^{(1)}|e_{n}\rangle
$$

(51)

Finally, we have to evaluate the matrix elements of the resolvent. Equation (16) shows that the matrix of the resolvent is equal the inverse of the matrix of the operator $i\omega_{RF} - T_{C} - T_{D}$. As an example, we compute the matrix element concerning to the first term in equation (49), whereas the eigenvalue representation of the conservative operator $T_{C}$ is used

$$
\langle z_{l/m}^{(1)}|i\omega_{RF} - T_{C} - T_{D}|z_{l/m}^{(1)}\rangle = i(\omega_{RF} - \omega_{pe}\eta_{l})\delta_{ll}\delta_{mm} + \langle z_{l/m}^{(1)}|T_{D}|z_{l/m}^{(1)}\rangle
$$

(52)

The matrix elements of $T_{D}$ have to be evaluated explicitly. Applying the operator $T_{D}$ to $|z_{l/m}^{(1)}\rangle$ yields a state vector with a vanishing charge contribution. Hence, the scalar product with $(|z_{l/m}^{(1)}\rangle)$ is reduced to the integral over the current components and is determined by

$$
\langle z_{l/m}^{(1)}|T_{D}|z_{l/m}^{(1)}\rangle = \frac{\nu\varepsilon_{0}[((l+1)(l'+1) + (R+\delta)^{2i+1})]}{(l+1)\eta_{R}} |z_{l/m}^{(1)}\rangle = |z_{l/m}^{(1)}\rangle
$$

(53)

Entering (53) in (52) yields the complete first matrix element

$$
\langle z_{l/m}^{(1)}|i\omega_{RF} - T_{C} - T_{D}|z_{l/m}^{(1)}\rangle = \left[\frac{\nu\varepsilon_{0}(\omega_{RF} - \omega_{pe}\eta_{l}) + \nu\eta_{l}}{\eta_{R}}\right] \delta_{ll}\delta_{mm}
$$

(54)

The fact that $T_{D}$ is represented by pure diagonal elements shows that $T_{C}$ and $T_{D}$ can be projected on the same subdomain. (The corresponding basis functions are the spherical harmonics.) Therefore, the perturbation approach becomes dispensable. Following the same calculation, we find the other matrix elements of the dynamical operator

$$
\langle z_{l/m}^{(-)}|T_{D}|z_{l/m}^{(-)}\rangle = -\left[\frac{\nu\varepsilon_{0}(\omega_{RF} + \omega_{pe}\eta_{l}) + \nu\eta_{l}}{\eta_{R}}\right] \delta_{ll}\delta_{mm}
$$

(55)

$$
\langle z_{l/m}^{(1)}|T_{D}|z_{l/m}^{(-)}\rangle = -\left[\frac{\nu\varepsilon_{0}(\omega_{RF} + \omega_{pe}\eta_{l}) + \nu\eta_{l}}{\eta_{R}}\right] \delta_{ll}\delta_{mm}
$$

(56)

This result shows that the operator $i\omega_{RF} - T_{C} - T_{D}$ is represented by a diagonal block matrix, where the block elements on the main diagonal are given by $(2 \times 2)$-matrices. The inverse of such a matrix is also a diagonal block matrix with $(2 \times 2)$-matrices on the main diagonal. Each block element on the diagonal is determined by the inverse of the $(2 \times 2)$-matrix of the original block element.

Now, all terms in the admittance (49) are evaluated and can be introduced. Finally, we exploit the Kronecker deltas to determine the general admittance in a spherical geometry

$$
Y_{mv} = \sum_{l=0}^{\infty} \sum_{m=1}^{l} \frac{l+1}{(R+\delta)^{2i+1}} i\omega\varepsilon_{0}\omega_{pe}^{2} \beta_{lm}^{(m)} \beta_{lm}^{(n)} - \omega_{RF}
$$

(57)

The coefficients $\beta_{lm}^{(m)}$ and $\beta_{lm}^{(n)}$ are still not determined. They contain the information about the electrode configuration. Therefore, we have to distinguish between different probe designs and focus on the IP and the MRP in the next two sections.

6. Admittance of the impedance probe

In figure 2 (left) the idealized impedance probe is depicted. It contains one spherical powered electrode $E_{1}$ and is, in our case, surrounded by a dielectric. The general current (12) is reduced to the current flowing to $E_{1}$, where the voltage $U_{1}$ is applied (For a shorter notation we substitute $\omega_{RF} = \omega$ in the rest of the manuscript.)

$$
i_{1} = Y_{11}U_{1} = \sum_{l=0}^{\infty} \sum_{m=1}^{l} \frac{l+1}{(R+\delta)^{2i+1}} i\omega\varepsilon_{0}\omega_{pe}^{2} \beta_{lm}^{(1)} \beta_{lm}^{(1)} U_{1}
$$

(58)

The specified electrode configuration allows for $\beta_{lm}^{(1)}$ to be evaluated explicitly by its definition given in (48)

$$
\beta_{lm}^{(1)} = \frac{\sqrt{4\pi}}{\gamma} \delta_{ll}\delta_{mm}
$$

(59)

Entering (59) into the current (58) and exploiting the Kronecker deltas yields the admittance of the spherical impedance probe

$$
Y_{IP} = \frac{4\pi \varepsilon_{0}\omega_{pe}^{2}}{\sqrt{R}} i\omega
$$

$$
\times \left( \frac{1}{\varepsilon_{D}\varepsilon_{R} + \varepsilon_{D}(\rho + \varepsilon_{D} - 1)(1 + \frac{\delta}{R})} \right)
$$

(60)

It describes the coupling between the electrode and ground, which is in infinite distance to the probe. Equation (60) shows that the impedance probe provides just one resonance mode with the resonance frequency $\omega_{0}$. This resonance frequency is given by the eigenvalue of the conservative operator $T_{C}$

$$
\omega_{0} = \pm \omega_{pe} \sqrt{1 - \frac{\varepsilon_{D}(1 - \frac{\delta}{R})}{1 + (\varepsilon_{D} - 1)(1 + \frac{\delta}{R})}}
$$

(61)

Neglecting the dielectric ($\varepsilon_{D} = 0$ and $\varepsilon_{D} = 1$), the resonance frequency (61) reduces to the well known sheath resonance in spherical geometry [11]. It can also be called ‘monopole resonance’ referring to the one electrode system.

Different spectra of the impedance probe will be depicted and discussed within section 8. We will compare them to the spectra of the multipole resonance probe and discuss advantages and disadvantages. The admittance of the MRP is determined in the next section.
The idealized multipole resonance probe is shown in figure 2 (right). The probe consists of an upper electrode \( E_1 \) and a lower electrode \( E_2 \), where the voltages \( U_1 \) and \( U_2 \) are applied, respectively. We calculate the current flowing to the electrode \( E_1 \)

\[
i_1 = \sum_{n' = 1}^{2} Y_{1n'} U_{n'}
\]

\[
= \sum_{n' = 1}^{2} \frac{1}{\gamma_l} \sum_{l = 0}^{\infty} \frac{l + 1}{(R + \delta)^{2l+1}} \frac{i \omega \varepsilon_0 \varepsilon_{pc}^2}{\omega_{pc}^2 \eta_l^2 + i \omega \nu - \omega^2} \beta_{lm}^{(n')} U_{n'}.
\]

(62)

Owing to the two different electrodes, we have to determine two coefficients \( \beta_{lim}^{(1)} \) and \( \beta_{lim}^{(2)} \)

\[
\beta_{lim}^{(1)} = \frac{\sqrt{2(2l + 1)\pi}}{\gamma_l} \frac{\sqrt{\pi}}{2(1 - l + \frac{1}{2})} \frac{\Gamma\left(l + \frac{1}{2}\right)}{\Gamma\left(l + 1\right)} \delta_{m0},
\]

(63)

\[
\beta_{lim}^{(2)} = \frac{\sqrt{2(2l + 1)\pi}}{\gamma_l} \frac{\sin\left(l\pi\right)}{l\pi(1 + l)} \left( \frac{1}{2} \Gamma\left(1 - l + \frac{1}{2}\right) \right) \delta_{m0}.
\]

(64)

The gamma function \( \Gamma\left(1 - \frac{1}{2}\right) \), which is present in both coefficients, becomes infinity for all even \( l > 0 \). Furthermore, the sine vanishes for all \( l > 0 \). Hence, the coefficients vanish also for all even \( l > 0 \) and the remaining coefficients for odd \( l = 2l' - 1 \) can be combined

\[
\beta_{lim}^{(1/2)} = \frac{\sqrt{4l' - 1}}{2(2l' - 1)^{3/2}} \frac{\pi \delta_{l0} \delta_{m0}}{2\Gamma\left(l' + 1\right)} \delta_{m0}, \quad l' \in \mathbb{N}.
\]

(65)

The positive sign belongs to \( \beta_{lim}^{(1)} \) and the negative to \( \beta_{lim}^{(2)} \). (65) shows the influence of the symmetric geometry to the resonance modes of the probe–plasma system. The even modes, instead of \( l = 0 \), vanish in the calculation. In the limit \( l \to 0 \) the coefficients become equal

\[
\beta_{l0}^{(1)} = \sqrt{\pi} \frac{\sqrt{\pi}}{\gamma_l} = \beta_{l0}^{(2)}.
\]

(66)

Introducing (65) and (66) into the current (62), the current can be simplified

\[
i_1 = \sum_{l' = 1}^{\infty} \frac{2l'}{(R + \delta)^{2l'-1}} \frac{i \omega \varepsilon_0 \varepsilon_{pc}^2}{\omega_{pc}^2 \eta_{l'-1}^2 + i \omega \nu - \omega^2} \left( (U_1 - U_2) \right) + \frac{1}{R + \delta} \frac{i \omega \varepsilon_0 \varepsilon_{pc}^2}{\omega_{pc}^2 \eta_0^2 + i \omega \nu - \omega^2} \left( U_1 + U_2 \right).
\]

(67)

8. Comparison between IP and MRP

Within the last two sections the admittances of the impedance probe and multipole resonance probe are derived. Now, we compare their spectra based on the parameters of the MRP prototype: probe tip radius \( R = 4 \) mm, thickness of the dielectric \( d = 1 \) mm, and permittivity \( \varepsilon_D = 4.6 \) [1]. Additionally, we assume a sheath thickness of \( \delta = 0.2 \) mm,

\[
\text{Figure 3. Spectra of the impedance probe (dashed) and multipole resonance probe (solid) with the prototype parameter of the MRP } R = 4 \text{ mm, } d = 1 \text{ mm, } \varepsilon_D = 4.6 \text{ and assumed plasma parameter } \delta = 0.2 \text{ mm, } \omega_{pc} = 2\pi \times 10^5 \text{ s}^{-1}, \nu = 0.015 \omega_{pc}.
\]
a plasma frequency of $\omega_{pe} = 2\pi \times 10^9 \text{s}^{-1}$, and a collision frequency of $\nu = 0.015\omega_{pe}$. Figure 3 shows the corresponding spectra of both probes (IP dashed and MRP solid). Obviously, the resonance frequency of the IP is smaller than the ones of the MRP, as expected from the derived eigenvalues. Furthermore, one can see that the mode of the IP is less damped than the ones of the MRP.

The spectra depict in figure 4 are computed with a thickness of the dielectric that is equal to the sheath thickness $d = \delta = 0.2 \text{ mm}$. This leads to a shift of the resonance frequencies to smaller values and to less damping of the higher modes. Due to that, the contribution of the higher modes to the spectrum is observable. Increasing the thickness of the dielectric to $d = 2 \text{ mm}$ yields the spectra in figure 5, where the resonance frequencies are shifted to higher frequencies. The higher modes of the multipole resonance probe are not observable anymore.

Both probes provide spectra with a dominant resonance peak. The monopole mode of the impedance probe is unique and just slightly damped by an increasing thickness of the dielectric. This allows a measurement also in a plasma with higher collision frequencies. However, the fact that the current couples to ground at infinity means that the interpretation of a measurement—based on the excitation of the resonance mode—has to be understood as an average reaction of the whole plasma. Therefore, the measurement is not local, which can be seen as a disadvantage of the impedance probe. Another advantage is the simple design.

In case of the MRP the thickness of the dielectric influences strongly the resonance behavior. A thickness of $d = 1 \text{ mm}$ is a reasonable compromise to observe a unique dipole mode, which is also detectable in a measurement of several Pa. This thickness of the dielectric was chosen by the prototype of the MRP. The electric field of the dipole mode decreases rapidly and provides a local measurement of the electron density. A small disadvantage is the complex design, which is necessary to ensure the geometrical and electrical symmetry.
9. Summary and conclusion

Based on the result that the admittance of an electrostatic probe for APRS in arbitrary geometry is given by the resolvent of the dynamical operator $T_C + T_D$, we derived the general admittance in spherical geometry. Therefore, we determined the matrix representation of the resolvent by the eigenvalues and eigenfunctions of the conservative operator $T_C$. In general, the operators $T_C$ and $T_D$ do not commute. However, in spherical geometry they can be projected on the same subdomain with spherical harmonics as basisfunctions. This allows for the exact analytical representation of the general admittance in spherical geometry.

Two different probe designs were chosen to compare the general admittance with established results: The spherical impedance probe and the multipole resonance probe. (The practical importance and the verification between theory and experiment of these probes are shown in the references [1, 6, 20, 27–31].) In both cases we showed that the admittances are simplified expressions of the general admittance. The corresponding resonance frequencies are given by the eigenvalues of the conservative operator $T_C$, respectively. Concerning the impedance probe, the resonance frequency is equal to the sheath resonance and might also be called monopole resonance. The admittance of the multipole resonance probe is identical to the admittance, which is derived directly for the specific design [9].

Both probe designs provide spectra with a dominant resonance peak which is clearly detectable in a measurement. The impedance probe has a simple design and due to that always a unique resonance, but the measurement is not local. The multipole resonance probe has a more complex design to ensure the geometrical and electrical symmetry. Due to that symmetry the MRP acts like a dipole with a rapidly decreasing field, that the measurement is local.

The analytic solution presented here is restricted to spherical geometry. However, the solution strategy can also be performed in an arbitrary geometry. Therefore, an appropriate set of orthonormal basis functions has to be found to determine an approximated matrix representation of the operators. This will lead to an efficient calculation of the approximated spectral response instead of a simulation. Possibly, a perturbation approach is useful to determine the admittance by the matrix representation of the resolvent.

Acknowledgments

The authors acknowledge the support by the Federal Ministry of Education and Research (BMBF) in frame of the project PluTO, and also the support by the Deutsche Forschungsgemeinschaft (DFG) via Graduiertenkolleg GK 1051, Collaborative Research Center TRR 87, and the Ruhr University Research School. Gratitude is expressed to M Lapke, C Schulz, R Storch, T Stynroll, P Awakowicz, T Musch, T Mussenbrock, and I Rolles, who are or were part of the MRP-Team at the Ruhr University Bochum.

References

[1] Lapke M et al 2011 Plasma Sources Sci. Technol. 20 042001
[2] Takayama K, Ikegami H and Miyazaki S 1960 Phys. Rev. Lett. 5 238
[3] Messiaen A M and Vandenplas P E 1966 J. Appl. Phys. 37 1718
[4] Waletzko J A and Beketh G 1967 Radio Sci. 2 489
[5] Vernet N, Manning R and Steinberg J L 1975 Radio Sci. 10 517
[6] Blackwell D D, Walker D N and Amatucci W E 2005 Rev. Sci. Instrum. 76 023503
[7] Kokura H, Nakamura K, Ghanashev I P and Sugai H 1999 Japan. J. Appl. Phys. 38 5262
[8] Scharwitz C, Böke M, Winter J, Lapke M, Mussenbrock T and Brinkmann R P 2009 Appl. Phys. Lett. 94 011502
[9] Lapke M, Mussenbrock T and Brinkmann R P 2008 Appl. Phys. Lett. 93 051502
[10] Fejer J A 1964 Radio Sci. 68D 1171
[11] Harp R S 1964 Appl. Phys. Lett. 4 186
[12] Harp R S and Crawford F W 1964 J. Appl. Phys. 35 3436
[13] Dote T and Ichimiyu T 1965 J. Appl. Phys. 36 1866
[14] Kostelnick R J 1968 Radio Sci. 3 319
[15] Cohen A J and Beketh G 1971 Phys. Fluids 14 1512
[16] Tarstropp J and Heikikila W J 1972 Radio Sci. 4 493
[17] Aso T 1973 Radio Sci. 8 139
[18] Bantin C C and Balmain K G 1974 Can. J. Phys. 52 291
[19] Dine S, Booth J P, Curley G A, Corr C S, Jolly J and Guillen J 2005 Plasma Sources Sci. Technol. 14 777
[20] Walker D N, Fernsler R F, Blackwell D D, Amatucci W E and Messer S J 2006 Phys. Plasmas 13 032108
[21] Lapke M, Mussenbrock T, Brinkmann R P, Scharwitz C, Böke M and Winter J 2007 Appl. Phys. Lett. 90 121502
[22] Xu J, Nakamura K, Zhang Q and Sugai H 2009 Plasma Sources Sci. Technol. 18 045009
[23] Xu J, Shi J, Zhang J, Zhang Q, Nakamura K and Sugai H 2010 Chin. Phys. B 19 075206
[24] Li B, Li H, Chen Z, Xie J, Feng G and Liu W 2010 Plasma Sci. Technol. 12 513
[25] Linag I, Nakamura K and Sugai H 2011 Appl. Phys. Express 4 066101
[26] Lapke M, Oberrath J, Mussenbrock T and Brinkmann R P 2013 Plasma Sources Sci. Technol. 22 025005
[27] Blackwell D D, Walker D N, Messer S J and Amatucci W E 2005 Phys. Plasmas 12 093510
[28] Walker D N, Fernsler R F, Blackwell D D and Amatucci W E 2008 Phys. Plasmas 15 123506
[29] Walker D N, Fernsler R F, Blackwell D D and Amatucci W E 2010 Phys. Plasmas 17 113503
[30] Stynroll T, Harhausen J, Lapke M, Storch R, Brinkmann R P and Awakowicz P 2013 Plasma Sources Sci. Technol. 22 045008
[31] Stynroll T, Bielholz S, Lapke M and Awakowicz P 2014 Plasma Sources Sci. Technol. 23 025013