Emergence of maximal acceleration from non-commutativity of space-time

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Abstract

In this work, we show that the causally connected 4-dimensional line element of the $\kappa$-deformed Minkowski space-time induces an upper cut-off on the proper acceleration and derive this maximal acceleration, valid up to first order in the deformation parameter. We also construct the $\kappa$-deformed geodesic equation and obtain its $\kappa$-deformed Newtonian limit, valid up to first order in deformation parameter. Using this, we constrain certain non-commutative parameters present in the expression for maximal acceleration. We analyse different limits of the maximal acceleration and also discuss its implication to maximal temperature.

Keywords: $\kappa$-space time, maximal acceleration, geodesic equation

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1 Introduction

One of the major stumbling blocks in our understanding of the universe is the lack of a consistent quantum theory of gravity. Irrespective of numerous approaches developed over the last several decades and substantial progress made, an entirely satisfactory microscopic theory of gravity still eludes us. Different frameworks to understand the quantum theory of gravity have pointed to certain characteristic features expected of this quantum gravity theory. Existence of a fundamental length scale, below which quantum gravity effects become important, is one of them [1].

The existence of a minimal length scale in some quantum gravity models has been shown to be associated with the emergence of an upper limit on the proper acceleration, known as maximal acceleration [2]. In [3] it has been shown that covariant loop quantum gravity exhibits the existence of maximal acceleration, compatible with the local Lorentz symmetry. String theory has also predicted an upper bound on the string acceleration [4]. The notion of an upper bound on the value of allowed acceleration has a long history. The maximal acceleration in commutative space-time has been obtained in [5], from the 8-dimensional phase-space element, by imposing the events in the corresponding phase-space to be time-like. Alternatively, the form of the maximal acceleration has been obtained in [6] using Heisenberg’s uncertainty principle. This maximal acceleration has been used to smoothen the UV divergences in local QFT [7]. Various aspects of kinematics and dynamics of a relativistic particle possessing maximal acceleration in 8-dimensional space-time tangent bundle have been studied extensively [8]. Several authors have studied different phenomenological consequences of maximal acceleration models in [9]. The existence of a minimal time scale in physical phenomena leads to an upper cut-off on the allowed value of acceleration [10], and using this argument, a model of an electron with maximal acceleration...
has been constructed and analysed [11]. It has been shown that the maximal acceleration modifies the standard model of cosmology by introducing an inflationary expansion in the beginning stage [12].

Investigations of accelerating observers have resulted in unravelling curious results such as Unruh temperature [13] and Hawking radiation [14]. The implication of acceleration of the observer to the locality in general relativity had studied in detail [15]. The study of physics of accelerated frames/observers has brought out a plethora of insights regarding different aspects of gravity. Thus it is of intrinsic interest to study possible limits placed on the acceleration of observers by any approach to quantum gravity, such as non-commutative geometry.

Models with maximal acceleration have been studied by various authors. In [5] maximal acceleration was shown to arise as a consequence of geometrising quantum mechanics by introducing a metric in the 8-dimensional phase-space. This was followed by analysing line element in the 8-dimensional phase-space which depends not only on the coordinates $x^\mu$ but also on $E$ and $\vec{p}$, and using energy-momentum dispersion relation, a limit on the maximal acceleration of massive particle was derived. This maximal acceleration depends on the mass of the particle. Various aspects of this approach have been studied. In all these studies, the starting line element in 8-dimensions was not general covariant. In [16], this issue was solved by constructing a covariant line element by introducing non-linear connection in time-derivatives appearing in the velocity-dependent part of the 8-dimensional metric. In [16], the existence of maximal acceleration argued and calculated by combining Unruh effect [13] and Sakharov's result of the existence of a maximal temperature [17]. Another approach based on a pseudo-complex generalisation of Minkowski metric has been also shown to give a maximal cut-off value for acceleration. The line element used in this method [18] was also covariant, like the one used in [16].

In all these approaches described above, one start with the assumption of form a metric/ line element in 8-dimension space which has been taken as a direct product of Minkowski space-time and corresponding tangent space. This 8-dimensional metric depends on the uniquely defined metric of the underlying Minkowski space-time. The basic entity of these calculations is 8-dimensional metric and hence the connection of the 4-dimensional Minkowski metric with experimental results is ambiguous. This shortcoming has been circumvented in an approach based on the effective theory of metrics [19] and different geometric structures associated with maximal acceleration was studied in [20].

In the last decades, non-commutative geometry has been proposed as a paradigm to study the microscopic structure of space-time [21,22]. One of the non-commutative space-times that has been attracting wide attention is the $\kappa$-deformed space-time [23,24]. This space-time appears in the low energy limit of certain loop gravity models. It is also the space-time associated with doubly(deformed) special relativity (DSR), which incorporate an additional constant having length dimensions, apart from the velocity of light, in a consistent manner with the principle of relativity [1,25,26]. Different aspects of this space-time have been investigated in recent times [27,31]. Its implication to various phenomena were analysed in [32,33,40,41]. Recently, generalising the approach of [5], the maximal acceleration has been derived from the 8-dimensional phase-space metric defined in $\kappa$-deformed phase-space [42]. As in the commutative space-time, this study also used the 4-dimensional ($\kappa$-deformed) Minkowski metric to construct the 8-dimensional line element. We now initiate a construction which avoids the 8-dimensional metric to obtain the maximal acceleration in the $\kappa$-deformed space-time. Since the symmetry algebra associated with $\kappa$-deformed space-time has two fundamental constants, viz; $a$ having dimensions of the length and $c$, the velocity of light, it is possible to associate a natural time scale $\frac{a}{c}$ with phenomena in this space-time. This will have implications to the locality of accelerated observers [15] as well as on the allowed values of acceleration. Thus it is interesting to see whether one can get any limit on the acceleration within the $\kappa$-deformed Minkowski space-time, without resorting to the 8-dimensional space obtained by taking the direct product of this space with its tangent space. In this paper, we address this and show that an upper cut-off on allowed value of acceleration naturally emerges from the 4-dimensional, $\kappa$-deformed line element and derive the expression of maximum acceleration valid up to first order in the deformation parameter $a$. This expression of maximal acceleration depends on the mass of the particle, background energy $E$ and dimensionless parameters entering through the realisation of the non-commutative coordinates we adopt. We then derive the geodesic equation in the $\kappa$-deformed Minkowski space-time, valid up to first order in $a$ and obtain its Newtonian limit. Comparing this result with the same limit obtained in [42], we fix the dimensionless parameters appearing in the expression...
for maximal acceleration. We also analyse various limits of the maximal acceleration and discuss the implication to maximal temperature.

The organisation of this paper is as follows. In sec.2 we construct the $\kappa$-deformed metric, valid up to first in the deformation parameter $a$, by using the generalised commutation relation between $\kappa$-deformed phase-space coordinates. In sec.3 we show the emergence of the maximal acceleration in the $\kappa$-deformed space-time from the causally connected, 4-dimensional, $\kappa$-deformed Minkowskian space-time. In sec.4 we derive the expression for the geodesic equation of a test particle in the $\kappa$-deformed space-time, valid up to first order in the deformation parameter, using the $\kappa$-deformed metric. In sub-sec.4.1, we obtain the $\kappa$-deformed Newtonian force equation from the $\kappa$-deformed geodesic equation. We use this to constrain the non-commutative parameters present in the deformed maximal acceleration expression. We then plot the deformed maximal acceleration of electron against the deformation parameter. Finally, in sec. 5 we summarise our results and give the concluding remarks. Here we use $\eta_{\mu\nu} = diag(1, -1, -1, -1)$.

2 $\kappa$-deformed metric

In this section, we begin with a summary of the construction of metric in the $\kappa$-deformed space-time [39]. The $\kappa$-deformed space-time coordinates can be realised in various ways; one can directly work with $\kappa$-deformed coordinates, or alternately one can represent the $\kappa$-deformed space-time coordinates as functions of commutative coordinate and their derivatives. We use the second method in our analysis. Using the generalised commutation between $\kappa$-deformed phase-space coordinates, we show how to derive the $\kappa$-deformed metric, valid up to first order in $a$.

The $\kappa$-deformed space-time coordinates satisfy a Lie-algebra type commutation relations given by

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu) \quad (2.1)$$

where $a_\mu = (a, \vec{0})$. Here $a$ is the $\kappa$-deformation parameter having the dimension of $[L]$ and we choose a specific realisation for the $\kappa$-deformed space-time coordinate as [39]

$$\hat{x}_\mu = x_\alpha \varphi^\alpha_\mu(p). \quad (2.2)$$

The consistency of Eq.(2.2) and Eq.(2.1) gives, up to order $a$ term, we get

$$\varphi^\alpha_\mu(p) = \delta^\alpha_\mu \left(1 + \alpha \frac{a \cdot p}{\hbar}\right) + \frac{1}{\hbar} \left[a \cdot p + (\alpha + 1)(a \cdot p)a_\mu\right]. \quad (2.3)$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are dimensionless parameters and they are constrained by the relation $\gamma = \alpha + 1$. Note that $\varphi^\alpha_\mu(p)$ reduces $\delta^\alpha_\mu$ in the limit $a \to 0$ where as $\varphi^\alpha_\mu(p)$ diverges as $\hbar \to 0$. Hence we recover the commutative limit (where there is no $\hbar$ dependence) by taking $a \to 0$. The limit, $\frac{a}{\hbar} \to 0$ of $\varphi^\alpha_\mu(p)$ is also well defined and in this case also we recover the commutative coordinates. Thus using $\varphi^\alpha_\mu(p)$, Eq.(2.2) is re-written as

$$\hat{x}_\mu = \hat{x}_\mu \left(1 + \alpha \frac{a \cdot p}{\hbar}\right) + \frac{1}{\hbar} \left[a \cdot p + (\alpha + 1)(a \cdot p)a_\mu\right]. \quad (2.4)$$

The $\kappa$-deformed metric, $\hat{g}_{\mu\nu}$, is constructed by defining the generalised commutation relations between deformed the phase space coordinates as

$$[\hat{x}_\mu, \hat{P}_\nu] = i\hbar \hat{g}_{\mu\nu}. \quad (2.5)$$

Here $\hat{P}_\mu$ is realised as, $\hat{P}_\mu = g_{\alpha\beta}(\hat{y})p^\beta \varphi^\alpha_\mu(p)$, where $\hat{y}$ is an auxiliary $\kappa$-deformed space-time coordinate satisfying $[\hat{y}_\mu, \hat{x}_\mu] = 0$ and hence any function of $\hat{y}_\mu$ also commutes with $\hat{x}_\mu$, i.e, $[f(\hat{y}), \hat{x}_\mu] = 0$ [39]. Thus using Eq.(2.2) and Eq.(2.3), we find $\hat{y}_\mu$ and $f(\hat{y})$ as

$$\hat{y}_\mu = x_\mu + \alpha x_\mu \frac{a \cdot p}{\hbar} + \frac{1}{\hbar} \left[a \cdot p + (\alpha + 1)x_\mu a \cdot p\right], \quad (2.6)$$

$$f(\hat{y}) = f(x) + \alpha \left(\frac{a}{\hbar} \frac{\partial f}{\partial x} \cdot p\right) + \left(\frac{a}{\hbar} \cdot x\right) \left(\frac{\partial f}{\partial x} \cdot p\right) + (\alpha + 1) \left(x \cdot \frac{\partial f}{\partial x} \cdot \frac{a}{\hbar} \cdot p\right). \quad (2.7)$$
Now we expand the LHS of Eq. (2.5) using the above defined realisation for \( \hat{x}_\mu \) and \( \hat{P}_\nu \), and we obtain
\[
\hat{g}_{\mu\nu} = g_{\alpha\beta}(\hat{y}) \left( p^\beta \frac{\partial \hat{\varphi}_\alpha}{\partial \hat{p}_\nu} + \varphi_\alpha^\beta \varphi_\nu^\beta \right).
\]

Using Eq. (2.7) we re-express \( g_{\mu\nu}(\hat{y}) \) in terms of \( x \) as
\[
g_{\mu\nu}(\hat{y}) = g_{\mu\nu}(x) + \alpha \left( \frac{a}{\hbar} \cdot \frac{\partial g_{\mu\nu}}{\partial x} \right) (x \cdot p) + \beta \left( \frac{a}{\hbar} \cdot \frac{\partial g_{\mu\nu}}{\partial x} \right) (x \cdot p) + (\alpha + 1) \left( x \cdot \frac{\partial g_{\mu\nu}}{\partial x} \right) (\frac{a}{\hbar} \cdot p).
\]

Substituting Eq. (2.3) and Eq. (2.9) in Eq. (2.8), we get the expression for the \( \kappa \)-deformed metric, valid up to first order in \( a \), as
\[
\hat{g}_{\mu\nu} = g_{\mu\nu} + \alpha \left( \frac{p^\beta \varphi_{\alpha\beta}}{\hbar} + 2g_{\mu\nu} \left( \frac{a}{\hbar} \cdot \frac{\partial g_{\mu\nu}}{\partial x} \right) x \cdot p \right) + \beta \left( \frac{a^\beta}{\hbar} \eta_{\alpha\beta} \eta_{\mu\nu} + \frac{a^\beta}{\hbar} p_\nu g_{\mu\beta} + \frac{a^\beta}{\hbar} p_\mu g_{\nu\beta} \right)
+ \left( \frac{a}{\hbar} \cdot x \right) \left( \frac{\partial g_{\mu\nu}}{\partial x} \right) (x \cdot p) + (\alpha + 1) \left( 2g_{\mu\beta} \varphi_{\alpha\beta} + p_\beta \frac{a^\mu}{\hbar} g_{\alpha\nu} + \left( x \cdot \frac{\partial g_{\mu\nu}}{\partial x} \right) (\frac{a}{\hbar} \cdot p) \right).
\]

We notice that in the limit \( a \to 0 \) (also in the limit \( \hbar \to 0 \)), we recover the commutative metric, i.e., \( \hat{g}_{\mu\nu} \) becomes \( g_{\mu\nu} \). Further we note that for \( a \neq 0 \), in the limit \( \hbar \to 0 \), the metric \( \hat{g}_{\mu\nu} \) diverges.

3 \( \kappa \)-deformed maximal acceleration

In this section, we start with the general expression for the line-element in \( \kappa \)-deformed space-time. Using this we then obtain the line-element for the \( \kappa \)-deformed Minkowski space-time, valid up to first order in \( a \). We further impose the condition that the events in the 4-dimensional \( \kappa \)-deformed Minkowski space-time to be causally connected and derive the explicit form of the maximal acceleration emerging from the \( \kappa \)-deformed space-time geometry, valid up to first order in \( a \).

The \( \kappa \)-deformed line element is defined in terms of the deformed metric and the differential of deformed space-time coordinate as
\[
d\hat{s}^2 = \hat{\eta}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu.
\]

We evaluate \( d\hat{x}^\mu \) by taking the differential of Eq. (2.4) and replacing \( g_{\mu\nu} \) with \( \hat{g}_{\mu\nu} \) in Eq. (2.10) we find \( \hat{\eta}_{\mu\nu} \) and \( d\hat{x}^\mu \) in Eq. (3.1) we obtain the \( \kappa \)-deformed Minkowskian line element, valid up to first order in \( a \), as
\[
\hat{d}s^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \alpha \left( 2p_\mu \frac{a_\nu}{\hbar} + 4\eta_{\mu\nu} \frac{E}{\hbar c} + 2p_\mu \frac{a_\nu}{\hbar} \right) dx^\mu dx^\nu + \frac{a}{\hbar} \eta_{\mu\nu} \frac{dE}{c} dx^\mu x^\nu + (x \cdot dx) \frac{a}{\hbar} \frac{dE}{c}
+ 2\frac{a}{\hbar} \frac{cdt (dx \cdot p) + 2(x \cdot dp) c dt} + \beta \left( \eta_{\mu\nu} \frac{E}{\hbar c} + \frac{a_\mu}{\hbar} p_\nu + \frac{a_\nu}{\hbar} p_\mu \right) dx^\mu dx^\nu + 2\frac{a}{\hbar} (dx \cdot p) c dt
+ 2\frac{a}{\hbar} (dx \cdot dp) (x \cdot dx) \frac{a}{\hbar} \frac{dE}{c} dx^\mu x^\nu + 2\frac{a}{\hbar} c dt (dx \cdot p) + 2\frac{a}{\hbar} c dt (dx \cdot dp) \right).
\]

Note that the deformed line element depends on \( E, \vec{p} \). For the remaining analysis, without loss of generality, we work with 1 + 1 dimension \( \kappa \)-deformed space-time function here onwards. The deformed line element in 1 + 1 dimension as
\[
ds^2 = c^2 dt^2 - dz^2 + \alpha \left( 6\frac{a}{\hbar} c dt (Edt - p dz) + 4\frac{a}{\hbar} \frac{E}{c} \left( c^2 dt^2 - dz^2 \right) + 2\frac{a}{\hbar} dE \left( c^2 dt - dz \right) \right)
+ 2\frac{a}{\hbar} c dt \left( tdE - zd\right) + \beta \left( \frac{a}{\hbar} \frac{E}{c} \left( c^2 dt^2 - dz^2 \right) + 4\frac{a}{\hbar} c dt (Edt - pdz) + 2\frac{a}{\hbar} c dt \left( dE dt - pdz \right) \right)
+ 5\frac{a}{\hbar} c dt (Edt - pdz) + 2\frac{a}{\hbar} c dt (tdE - zd). \]
Now let us consider a time-like event, so that $ds^2 \geq 0$ and we divide the above equation by $dt^2$ and denote $\frac{dz}{dt}$ as $v$, which is the velocity and $\frac{dp}{dt} = \frac{ma}{1-c^2/v^2}^{\frac{1}{2}}$, where $m$ is the rest mass and $A$ is the proper acceleration, respectively. Using these definitions, above expression takes the form

\[ c^2 - v^2 + \alpha \left( 6a \frac{c}{h} (E - pv) + 4a \frac{E}{h} c (c^2 - v^2) + 2a \frac{c}{h} \frac{dE}{dt} \left( c^2 t - zv \right) + 2a \frac{c}{h} (t \frac{dE}{dt} - z \frac{mA}{1-v^2/c^2}^{3/2}) \right) + \beta \left( \frac{a}{h} c \left( c^2 - v^2 \right) + 4a c \frac{E}{h} (E - pv) + 2a \frac{c}{h} ct \left( \frac{pc^2}{E} - v \frac{mA}{1-v^2/c^2}^{3/2} \right) + \left( 5a \frac{c}{h} \frac{E}{1-v^2/c^2}^{3/2} \frac{dt}{dE} - z \frac{mA}{1-v^2/c^2}^{3/2} \right) \right) \geq 0 \]

(3.4)

We divide the Eq.[3.4] throughout by $c^2 - v^2$ and use the commutative dispersion relation to re-write $\frac{dE}{dt}$ as $\frac{pc^2}{E} \frac{dp}{dt} = \frac{pc^2}{E} \frac{mA}{(1-v^2/c^2)^{3/2}}$. Note that since we are working to the order $a$ in our analysis, we need to use only commutative expression for $\frac{dE}{dt}$ as $\frac{dE}{dt}$ term in Eq.[3.4] are already of order $a$. Thus we re-express the above Eq.[3.4] as

\[ 1 + \alpha \left( 6a \frac{c}{h} (E - pv) + 4a \frac{E}{h} c (c^2 - v^2) + 2a \frac{pc}{h} \frac{mA}{1-v^2/c^2}^{3/2} \left( c^2 t - zv \right) + 2a \frac{c}{h} \frac{pc^2}{E} - z \frac{mA}{1-v^2/c^2}^{3/2} \right) + \beta \left( \frac{a}{h} c \left( c^2 - v^2 \right) + 4a \frac{c}{h} \frac{E}{c^2 - v^2} \left( \frac{pc^2}{E} - v \frac{mA}{1-v^2/c^2}^{3/2} \right) + \left( 5a \frac{c}{h} \frac{E}{1-v^2/c^2}^{3/2} \frac{pc^2}{E} - z \frac{mA}{1-v^2/c^2}^{3/2} \right) \right) \geq 0, \]

(3.5)

which is valid up to first order in $a$. As in the Minkowski space-time, maximum allowed velocity for any particle in the $\kappa$-deformed space-time is $c$. Hence acceleration of a particle can reach the maximum value, if it starts with zero velocity. Thus, to calculate the maximum acceleration, we choose instantaneous rest frame of the particle in which $v = 0$ [5][12]. Thus from Eq.(3.5) we derive the expression for magnitude of the maximal acceleration $A$, valid up to first order in $a$, as

\[ A \leq \frac{c}{2(\frac{a}{h})zm} \frac{1}{1+\alpha} \left[ 1 + 5a \frac{E}{h} c (1 + \beta + 2a) \right]. \]

(3.6)

Here $E$ represents the back ground energy scale of non-commutativity which entered through of the deformed metric. Note that in the limit $a \to 0$, $A \to \infty$ as expected in the commutative limit. The appearance of $z$ term in $A$ is due to the presence of $x_\mu$ in the realisation, i.e Eq.(2.4) (Can also be traced back to $p_0$ appearing in Eq. (2.4), which defines deformed coordinates.). Thus we find the maximal acceleration induced by the non-commutativity depends inversly on the spatial coordinate and hence the value of maximal acceleration blows to infinity when $z = 0$. But note that for a particle moving under the influence of a force exerted by another object, this $z$ has to be the minimum separation between them and thus the position dependance of $A$ will replaced with a fixed length. Also note that in the limit $h \to 0$, $A \leq \frac{5}{2zm} \frac{E(1+\beta+2a)}{(1+\alpha)}$.

We note that the expression for the maximal acceleration in Eq.(3.6) in the $\kappa$-space-time is derived from the 4-dimensional line element unlike [5] and other earlier works where the maximal acceleration in commutative space-time was obtained from line element defined in 8-dimensional phase-space. Thus we find here that the maximal acceleration arises due to the geometry of the non-commutative space-time itself.

4 $\kappa$-deformed geodesic equation

In this section, we derive the $\kappa$-deformed Christoffel symbol by replacing the commutative metric with the $\kappa$-deformed metric in the definition of Christoffel symbol. We then use this $\kappa$-deformed Christoffel
symbol and derive the geodesic equation of a test particle in the \( \kappa \)-deformed space-time, valid up to first order in \( a \). It is to be noted that one can also obtain the \( \kappa \)-deformed geodesic equation in an alternate way by generalising the Feynman’s approach to \( \kappa \)-deformed space-time. Using this approach, the curvature effects in \( \kappa \)-deformed space-time were studied and the geodesic equation was derived earlier in [39]. Such non-vanishing curvature like contributions are also observed in Moyal space-time [43].

We analyse the special case of Eq. (2.10) by setting \( \beta = 0 \) and hence the expression for the deformed metric becomes

\[
\hat{g}_{\mu\nu} = g_{\mu\nu} + a\left(p^\beta g_{\nu\beta} \frac{a\mu}{\hbar} + 2g_{\mu\nu} \left(\frac{a}{\hbar} \cdot p\right) + \left(\frac{a}{\hbar} \cdot \frac{\partial}{\partial x} p\right) x \cdot p\right) + (\alpha + 1) \left(2g_{\mu\beta} p^\beta \frac{a}{\hbar} + p^\alpha \frac{a}{\hbar} g_{\alpha\nu} + \left(x \cdot \frac{\partial g_{\mu\nu}}{\partial x}\right) \left(\frac{a}{\hbar} \cdot p\right)\right).
\]  

(4.1)

The Christoffel symbol in \( \kappa \)-deformed space-time is defined as

\[
\hat{\Gamma}^\mu_{\nu\lambda} = \frac{1}{2} \hat{g}^{\mu\rho} \left(\partial_\nu \hat{g}_{\rho\lambda} + \partial_\lambda \hat{g}_{\nu\rho} - \partial_\rho \hat{g}_{\nu\lambda}\right)
\]  

(4.2)

Substituting the deformed metric given in Eq. (4.1) in the above equation, we calculate the deformed Christoffel symbol, valid up to first order in \( a \), as

\[
\hat{\Gamma}^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda} + \frac{1}{2} \frac{a}{\hbar} \left(mB^\mu_{\nu\lambda\sigma} \frac{dx^\sigma}{d\tau} + E\frac{c}{E} E^\mu_{\nu\lambda}\right),
\]  

(4.3)

where,

\[
B^\mu_{\nu\lambda\sigma} = \alpha g^{\mu\rho} \left(2\partial_\nu g_{\rho\sigma} [\delta^\lambda_0 + 2\partial_\lambda g_{\sigma\rho} [\delta^\nu_0 - 2\partial_\rho g_{\sigma\rho} [\delta^\nu_0 + \frac{1}{c} \partial_\nu (\frac{\partial g_{\rho\lambda}}{\partial t} x^\sigma) + \frac{1}{c} \partial_\lambda (\frac{\partial g_{\nu\rho}}{\partial t} x^\sigma)\right) - \frac{1}{c} \partial_\rho (\frac{\partial g_{\nu\lambda}}{\partial t} x^\sigma)\right) + \frac{3}{2} g^{\mu\rho} \left(\partial_\nu g_{\rho\lambda} [\delta^\sigma_0 + \partial_\sigma g_{\lambda\rho} [\delta^\nu_0 - \partial_\rho g_{\lambda\rho} [\delta^\nu_0 + 3 \left(\partial_\lambda g_{\nu\rho} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\nu\lambda}\right) g^{\alpha\beta} \delta^\mu_0 [\sigma^\nu_0 [\sigma^\nu_0 + \frac{1}{c} \partial_\rho (\frac{\partial g_{\nu\lambda}}{\partial t} x^\sigma)\right).
\]  

(4.4)

and

\[
C^\mu_{\nu\lambda} = \left(4\alpha g^{\mu\rho} + (\alpha + 1) \left(x \cdot \frac{\partial g^{\mu\rho}}{\partial x}\right)\right) \left(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda}\right) + (\alpha + 1) g^{\mu\rho} \left(\partial_\nu \left(x \cdot \frac{\partial g_{\rho\lambda}}{\partial x}\right) + \partial_\lambda \left(x \cdot \frac{\partial g_{\nu\rho}}{\partial x}\right) - \partial_\rho \left(x \cdot \frac{\partial g_{\nu\lambda}}{\partial x}\right)\right).
\]  

(4.5)

Thus we see that the deformed Christoffel symbol, valid up to first order in \( a \), contains two non-commutative correction terms, in which first one depends on the velocity of the test particle whereas the second correction term is independent of \( E \), the non-commutative energy scale.

Using the geodesic equation in the \( \kappa \)-deformed space-time given by

\[
\frac{d^2 x^\mu}{d\tau^2} + \hat{\Gamma}^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0,
\]  

(4.6)

and \( \frac{dx^\mu}{d\tau} = \frac{d^2 x^\mu}{d\tau^2} \) calculated from Eq. (2.4) and Eq. (4.3), the \( \kappa \)-deformed geodesic equation, valid up to first order in \( a \) is derived and it is given by

\[
\frac{d^2 x^\mu}{d\tau^2} + \hat{\Gamma}^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} + \frac{a}{\hbar} D^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} + \frac{a}{\hbar} E^\mu_{\nu\lambda\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau} + \frac{a}{\hbar} F^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} \frac{dx^\mu}{d\tau^2} = 0,
\]  

(4.7)

where,

\[
D^\mu_{\nu\lambda} = \alpha \frac{E}{c} \Gamma^\mu_{\nu\lambda} + \frac{1}{2} \alpha \frac{E}{c} C^\mu_{\nu\lambda},
\]  

\[
E^\mu_{\nu\lambda\sigma} = m(\alpha + 1) \Gamma^\mu_{\rho\gamma} \left(\delta^\rho_{\sigma} \delta^\gamma_0 + \delta^\gamma_{\sigma} \delta^\rho_0\right) \eta_{\nu\lambda} + \frac{1}{2} mB^\mu_{\nu\lambda\sigma},
\]  

(4.8)

\[
F^\mu_{\nu\lambda} = 3m(\alpha + 1) \eta_{\nu\lambda} \delta^\rho_0 + m(\alpha + 1) x_{\nu} \left(\delta^\rho_{\lambda} \delta^\gamma_0 + \delta^\gamma_{\lambda} \delta^\rho_0\right) \Gamma^\mu_{\gamma\rho}.
\]
4.1 $\kappa$-deformed Newtonian limit

In this subsection, we derive the $\kappa$-deformed Newton’s force equation, valid up to first order in $a$, by using the same approach used in the commutative case. We constrain the $\alpha$ parameter present in the maximal acceleration expression, by comparing this $\kappa$-deformed Newton’s equation with the one obtained in [42], and then analyse the change in maximal acceleration with respect to $\frac{a}{\hbar}$ by plotting the same.

For deriving the Newtonian limit, we keep only the lowest order derivative terms in the equations of motion and hence we neglect the terms in $\frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} \frac{dx^\rho}{d\tau}$ and thus the $\kappa$-deformed geodesic equation obtained in Eq.(4.7) becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} + \frac{a}{\hbar} \hat{D}^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0. \tag{4.9}$$

Here $\Gamma^\mu_{\nu\lambda}$ is the Christoffel symbol in the commutative space-time. The $\kappa$-deformed geodesic equation is written in terms of commutative space-time coordinate and the deformation parameter, so we use the same approach as that in the commutative case to obtain the $\kappa$-deformed Newtonian limit of Eq.(4.9).

In this approach, one assumes

- the test particles are moving slowly, i.e, $\frac{dx_i}{d\tau} << \frac{dx_0}{d\tau}$.
- the metric is static, i.e, $\frac{\partial g_{\mu\nu}}{\partial t} = 0$.
- gravitational field is weak and the metric is linearised as, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \ |h_{\mu\nu}| << 1$.

Using these conditions in Eq.(4.9), we get

$$\frac{d^2x^0}{d\tau^2} = 0,$$

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2} \partial_i h_{00} \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} + \frac{1}{2} \frac{a}{\hbar} E \left( 5\alpha \partial_i \partial_0 h_{00} - 3\partial_i \left( \frac{\partial h_{00}}{\partial x} \right) \right) \frac{dx^0}{d\tau} \frac{dx^0}{d\tau}. \tag{4.10}$$

The Newtonian potential is defined as, $h_{00} = \frac{2GM}{r}$. We note that, $\frac{d^2x^i}{d\tau^2}$ expression in Eq.(4.10) is reparametrisation invariant and hence we use the definition $\dot{x}^0 = \frac{c}{\hbar}$ and obtain the $\kappa$-deformed Newton’s equation, valid up to first order in $a$, as

$$\hat{F}^i = F^i \left( 1 - \frac{a}{\hbar} \frac{E}{c} \left( 5\alpha + 3 \right) \right), \tag{4.11}$$

where $F^i = -\frac{mMG}{r^2}$. Here we find that the Newton’s force equation, defined in the $\kappa$-deformed space-time, valid up to first order in $a$, picks up an $\frac{a}{\hbar}$ dependent correction term and it has only radial force component as in the commutative case.

In [42], we have obtained the $\kappa$-deformed Newton force equation as $\hat{F}^i = F^i \left( 1 - \frac{a}{\hbar} \frac{E}{c} \right)$ for a choice of realisation $\varphi^0 = \delta^0 \mu e^{-\frac{aE}{\hbar c}}$. On comparing this with Eq.(4.11), we found that $\alpha = -\frac{2}{5}$. Thus for $\beta = 0$ and $\alpha = -\frac{2}{5}$, the expression for the maximal acceleration given in Eq.(3.6) becomes

$$A \leq \frac{5}{6} \left( \frac{c}{\hbar} \right) \frac{E}{m} \frac{r_s}{m} \left( 1 + \frac{aE}{\hbar c} \right). \tag{4.12}$$

We see that in the limit $a \rightarrow 0$, $A \rightarrow \infty$ and in the limit $h \rightarrow 0$, $A \leq \frac{5E}{6r_s m}$. Note that we have re-expressed $z$ as $r_s$ in the above (see Eq.(3.6)). Here $r_s$ is the shortest distance of approach of the particle of mass $m$ to the gravitating body exerting Newtonian force. For a test particle falling radially to a Schwarzschild black hole, $r_s$ would be Schwarzschild radius; for an electron moving under gravitational force of another particle, $2r_s = \lambda_c$, the Compton wave length of electron. Below, we plot variation of $A$ against $\frac{a}{\hbar}$, for fixed value of $E$ and $2r_s = \lambda_c$. Compton wave length of electron and $m$ is mass of electron. Note that in the commutative limit, $a \rightarrow 0$, maximum acceleration diverges to infinity. Here in plotting the graph, we have taken $E = E_{Planck} = 1.956 \times 10^9 J$. 

7
Conclusions

In this paper, we have shown that in the $\kappa$-deformed space-time, for every massive particle, an upper cut-off value for the allowed acceleration exists. This is obtained by constructing the 4-dimensional line element in the $\kappa$-deformed Minkowski space-time, valid up to first order in the deformation parameter, $a$ and analysing the condition for a causal connection between any two events. This should be contrasted with earlier works where maximal acceleration is derived from an 8-dimensional line element defined in the phase space. The maximal acceleration we obtained depends on the non-commutative parameter(s), $\hbar$, the mass of the particle, a minimum distance of approach and the non-commutative energy scale.

We have seen that the maximal acceleration blows up as the non-commutative parameter $a$ goes to zero. But unlike the earlier results in the commutative space-time, in the limit, $\hbar \to 0$, the maximal acceleration goes to a finite value (when $a \neq 0$). This limiting value depends on the mass of the particle, distance of shortest approach and non-commutative energy scale $E$. Note that this limit is different from the corresponding limit obtained in the commutative space-time. In [20], the implication of the existence of maximum allowed acceleration was studied using the approach of the jet bundle, which also avoids the use of 8-dimensional line element. In this case, the usual Minkowski space-time is obtained in the limit where the maximum acceleration diverges to infinity as in the present case.

Note that the Unruh temperature $T = \frac{\hbar A}{2 \pi k c}$, where $k$ is the Boltzmann constant is shown to be unaffected by the $\kappa$-deformation of space-time [42]. Using the value of $A$ obtained in eqn. (4.12), we see that the maximum temperature gets a non-commutative correction, which depends on the deformation parameter $a$ and back ground energy scale $E$. After re-expressing the bound given in Eq. (4.12) by substituting the Compton wave length of the particle of mass $m$, that is $\lambda_C = \hbar/mc$ for $2r_s$, we find

$$T_{\text{max}} = \frac{5}{6\pi k} c \left( 1 + \frac{aE}{\hbar c} \right).$$

(5.1)

In the commutative limit, $a \to 0$, we find that $T_{\text{max}} \to \infty$. Also for $a \neq 0$, $T_{\text{max}}$ diverges to infinity in the vanishing limit of $\hbar$. Thus in the classical limit (as opposed to quantum, i.e., as $\hbar \to 0$), $T_{\text{max}}$ blows up in the $\kappa$-deformed space-time. This should be contrasted with the $\hbar \to 0$ limit of maximal temperature obtained in [17][44].

We note that the maximal acceleration in $\kappa$-space-time, derived using an 8-dimensional line element constructed in the phase-space [42] do reduce to the commutative value obtained in [5] when $a \to 0$. Thus the corresponding maximal temperature will also reduce to the one obtained in [17][44]. Thus we see a clear difference in the commutative limit of maximal acceleration obtained here and the corresponding...
limit of the $\kappa$-deformed maximal acceleration obtained by generalising the approach of [5] to $\kappa$-deformed space-time in [12].

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