COUNTABLY COMPACT GROUP TOPOLOGIES ON ARBITRARILY LARGE FREE ABELIAN GROUPS

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Abstract. We prove that if there are $c$ incomparable selective ultrafilters then, for every infinite cardinal $\kappa$ such that $\kappa^\omega = \kappa$, there exists a group topology on the free Abelian group of cardinality $\kappa$ without nontrivial convergent sequences and such that every finite power is countably compact. In particular, there are arbitrarily large countably compact groups. This answers a 1992 question of D. Dikranjan and D. Shakhmatov.

1. Introduction

1.1. Some history. It is well known that a non-trivial free Abelian group does not admit a compact Hausdorff group topology. Tomita [22] showed that it does not admit even a group topology whose countable power is countably compact.

Tkachenko [20] showed in 1990 that the free Abelian group generated by $c$ elements can be endowed with a countably compact Hausdorff group topology under CH. Tomita [22], Koszmider, Tomita and Watson [15], and Madariaga-Garcia and Tomita [17] obtained such examples using weaker assumptions. Boero, Castro Pereira and Tomita obtained such an example using a single selective ultrafilter [2]. Using $2^c$ selective ultrafilters, the example in [17] showed the consistency of a countably compact group topology on the free Abelian group of cardinality $2^c$. All forcing examples obtained so far had their cardinalities bounded by $2^c$.

Boero and Tomita [4] showed from the existence of $c$ selective ultrafilters that there exists a free Abelian group of cardinality $c$ whose square is countably compact. Tomita [26] showed that there exists a group topology on the free Abelian group of cardinality $c$ that makes all its finite powers countably compact.

E. van Douwen showed in [8] that the cardinality of a countably compact group cannot be a strong limit of countable cofinality.

Using the result in the abstract, we obtain the following:

Theorem 1.1. Assume GCH. Then a free Abelian group of infinite cardinality $\kappa$ can be endowed with a countably compact group topology (without non-trivial convergent sequences) if and only if $\kappa = \kappa^\omega$.

The result above answers a question of Dikranjan and Shakhmatov that was posed in the survey by Comfort, Hoffman and Remus [6].
Because of the way our examples are constructed we can raise their weights in the same way as in the papers \[23\] or \[5\] and obtain the following result – the examples in these references are Boolean but the trick is similar.

**Theorem 1.2.** It is consistent that there is a proper class of cardinals of countable cofinality that can occur as the weight of a countably compact free Abelian group.

1.2. Basic results, notation and terminology. We recall that a topological space is *countably compact* if, and only if, every countable open cover of it has a finite subcover.

**Definition 1.3.** Let \( \mathcal{U} \) be a filter on \( \omega \) and let \((x_n : n \in \omega)\) be a sequence in a topological space \( X \). We say that \( x \in X \) is a *\( \mathcal{U} \)-limit point of \((x_n : n \in \omega)\) if, for every neighborhood \( U \) of \( x \), the set \( \{ n \in \omega : x_n \in U \} \) belongs to \( \mathcal{U} \).

If \( X \) is Hausdorff, every sequence has at most one \( \mathcal{U} \)-limit and we write \( x = \mathcal{U}\text{-}\lim(x_n : n \in \omega) \) in that case.

The set of all free ultrafilters on \( \omega \) is denoted by \( \omega^* \). The following proposition is a well known result on ultrafilter limits.

**Proposition 1.4.** A topological space is countably compact if and only if each sequence in it has a \( \mathcal{U} \)-limit point for some \( \mathcal{U} \in \omega^* \).

The concept of almost disjoint families will be useful in our construction.

**Definition 1.5.** An almost disjoint family is an infinite family \( \mathcal{A} \) of infinite subsets of \( \omega \) such that distinct elements of \( \mathcal{A} \) have a finite intersection.

It is well known that there exists an almost disjoint family of size continuum (see \[10\]).

**Definition 1.6.** The unit circle group \( \mathbb{T} \) will be the metric group \((\mathbb{R}/\mathbb{Z}, \delta)\) where the metric \( \delta \) is given by
\[
\delta(x + \mathbb{Z}, y + \mathbb{Z}) = \min \{|x - y + a| : a \in \mathbb{Z}\}
\]
for every \( x, y \in \mathbb{R} \).

Given an open interval \((a, b)\) of \( \mathbb{R} \) with \( a < b \), we let \( \delta((a, b)) = b - a \).

An arc of \( \mathbb{T} \) is a set of the form \( I + \mathbb{Z} = \{a + \mathbb{Z} : a \in I\} \), where \( I \) is an open interval of \( \mathbb{R} \).

An arc is said to be proper if it is distinct from \( \mathbb{T} \).

If \( U \) is a proper arc and \( U = \{a + \mathbb{Z} : a \in I\} = \{b + \mathbb{Z} : a \in J\} \), then the Euclidean length of \( I \) equals the Euclidean length of \( J \), and we define the length of \( U \) as \( \delta(U) = \delta(I) \). We also let \( \delta(\mathbb{T}) = 1 \).

Given an arc \( U \) such that \( \delta(U) \leq \frac{1}{2} \), it follows that \( \text{diam}_{\mathbb{R}} U = \delta(U) \).

Our free Abelian groups will all be represented as direct sums of copies of the group of integers \( \mathbb{Z} \); we fix some notation. The additive group of rationals will also be used, so in the following definition one should read \( \mathbb{Z} \) or \( \mathbb{Q} \) for \( G \).

**Definition 1.7.** If \( f \) is a map from a set \( X \) to a group \( G \) then the *support* of \( f \), denotes \( \text{supp} f \) is defined to be the set \( \{x \in X : f(x) \neq 0\} \).

We define \( G^{(X)} = \{f \in G^X : |\text{supp} f| < \omega\} \).

If \( Y \) is a subset of \( X \) then, as an abuse of notation, we often write \( G^{(Y)} = \{x \in G^{(X)} : \text{supp} x \subseteq Y\} \).

Given \( x \in X \), we denote by \( \chi_x \) the characteristic function of \( \{x\} \), whose support is \( \{x\} \) and which value \( \chi_x(x) = 1 \).

For a sequence \( \zeta : \omega \to X \) in \( X \) we define \( \chi_{\zeta} : \omega \to G^X \) by \( \chi_{\zeta}(n) = \chi_{\zeta(n)} \).

Finally, for \( x \in X \), we let \( \bar{x} : \omega \to X \) be the constant sequence with value \( x \).
Note that then $\chi_{f}$ is also constant, with value $\chi_{x}$.

**Definition 1.8.** Let $U$ be a filter on $\omega$ and $X$ a set. We say that the sequences $f, g \in X^{\omega}$ are $U$-equivalent and write $f \equiv_{U} g$ if $\{ n \in \omega : f(n) = g(n) \} \in U$.

It is easy to verify that $\equiv_{U}$ is an equivalence relation. We denote the equivalence class of $f \in X^{\omega}$ by $[f]_{U}$. We also denote the set of all equivalence classes by $X^{\omega}/U$.

If $R$ is a ring and $X$ is an $R$-module, then $X^{\omega}/U$ has a natural $R$-module structure given by $[f]_{U} + [g]_{U} = [f + g]_{U}$, $-[f]_{U} = -[f]_{U}$, $r \cdot [f]_{U} = [r \cdot f]_{U}$ and the class of the zero function as its zero element.

If $p$ is a free ultrafilter, then the ultrapower of the $R$-module $X$ by $p$ is the $R$-module $X^{\omega}/p$. For the rest of this paper we will fix a cardinal number $\kappa$ that satisfies $\kappa^{\omega} = \kappa$.

Throughout this article, we will work inside ultrapowers of $\mathbb{Q}^{(\kappa)}$. These ultrapowers contain copies of ultrapowers of $\mathbb{Z}^{(\kappa)}$, which will be useful for the construction. So it is useful to define some notation.

**Definition 1.9.** Let $p$ be a free ultrafilter on $\omega$. We define $\text{Ult}(\mathbb{Q}, p)$ as the $\mathbb{Q}$-vector space $(\mathbb{Q}^{(\kappa)})^{\omega}/p$ and $\text{Ult}(\mathbb{Z}, p) = \{ [g]_{p} : g \in \mathbb{Z}^{\omega} \}$ with the subgroup structure.

Notice that each $[g]_{p}$ in $\text{Ult}(\mathbb{Z}, p)$ is formally an element of $(\mathbb{Q}^{(\kappa)})^{\omega}/p$, not of $(\mathbb{Z}^{(\kappa)})^{\omega}/p$. Nevertheless it is clear that $(\mathbb{Z}^{(\kappa)})^{\omega}/p$ is isomorphic to $\text{Ult}(\mathbb{Z}, \kappa)$ via the obvious isomorphism that carries the equivalence class of a sequence $g \in (\mathbb{Z}^{(\kappa)})^{\omega}$ in $(\mathbb{Z}^{(\kappa)})^{\omega}/p$ to its class in $(\mathbb{Q}^{(\kappa)})^{\omega}/p$.

### 2. Selective Ultrafilters

In this section we review some basic facts about selective ultrafilters, the Rudin-Keisler order and some lemmas we will use in the next sections.

**Definition 2.1.** A selective ultrafilter (on $\omega$), also called Ramsey ultrafilter, is a free ultrafilter $p$ on $\omega$ with the property that for every partition $(A_{n} : n \in \omega)$ of $\omega$, either there exists $n$ such that $A_{n} \in p$ or there exists $B \in p$ such that $|B \cap A_{n}| = 1$ for every $n \in \omega$.

The following proposition is well known. We provide [14] as a reference.

**Proposition 2.2.** Let $p$ be a free ultrafilter on $\omega$. Then the following are equivalent:

a) $p$ is a selective ultrafilter,

b) for every $f \in \omega^{\kappa}$, there exists $A \in p$ such that $f$ is either constant or one-to-one on $A$,

c) for every function $f : [\omega]^{2} \to 2$ there exists $A \in p$ such that $f$ is constant on $[A]^{2}$.

The Rudin-Keisler order is defined as follows:

**Definition 2.3.** Let $U$ be a filter on $\omega$ and $f : \omega \to \omega$. We define $f_{*}(U) = \{ A \subseteq \omega : f^{-1}[A] \in U \}$.

It is easy to verify that $f_{*}(U)$ is a filter; if $U$ is an ultrafilter then so is $f_{*}(U)$; if $f, g : \omega \to \omega$, then $(f \circ g)_{*} = f_{*} \circ g_{*}$; and $(\text{id}_{\omega})_{*}$ is the identity over the set of all filters. This implies that if $f$ is bijective, then $(f^{-1})_{*} = (f_{*})^{-1}$.

**Definition 2.4.** Let $U$ and $V$ be filters. We say that $U \leq V$ (or $U \leq_{\text{RK}} V$, if we need to be clear) iff there exists $f \in \omega$ such that $f_{*}(U) = V$.

The *Rudin-Keisler order* is the set of all free ultrafilters over $\omega$ ordered by $\leq_{\text{RK}}$. We say that two ultrafilters $p$ and $q$ are equivalent iff $p \leq q$ and $q \leq p$.

It is easy to verify that $\leq$ is a preorder and that the equivalence defined above is indeed an equivalence relation. Moreover, the equivalence class of a fixed ultrafilter is the set of all fixed ultrafilters, so the relation restricts to $\omega^{*}$ without modifying the equivalence classes. We refer to [14] for the following proposition:
Proposition 2.5. The following are true:

1. If \( p \) and \( q \) are ultrafilters, then \( p \leq q \) and \( q \leq p \) is equivalent to the existence of a bijection \( f : \omega \to \omega \) such that \( f_*(p) = q \).

2. The selective ultrafilters are exactly the minimal elements of the Rudin-Keisler order.

This implies that if \( f : \omega \to \omega \) and \( p \) is a selective ultrafilter, then \( f_*(p) \) is either a fixed ultrafilter or a selective ultrafilter. If \( f_*(p) \) is the ultrafilter generated by \( n \), then \( f^{-1}([n]) \in p \), so, in particular, if \( f \) is finite to one and \( p \) is selective, then \( f_*(p) \) is a selective ultrafilter equivalent to \( p \).

The existence of selective ultrafilters is independent from ZFC. Martin’s Axiom for countable orders implies the existence of \( 2^\omega \) pairwise incomparable selective ultrafilters in the Rudin-Keisler order.

The lemma below appears in [24].

Lemma 2.6. Let \( (p_k : k \in \omega) \) be a family of pairwise incomparable selective ultrafilters. For each \( k \) let \( (a_{k,i} : i \in \omega) \) be a strictly increasing sequence in \( \omega \) such that \( \{a_{k,i} : i \in \omega\} \in p_k \) and \( i < a_{k,i} \) for all \( i \in \omega \). Then there exists \( \{I_k : k \in \omega\} \) such that:

a) \( \{a_{k,i} : i \in I_k\} \in p_k \) for each \( k \in \omega \).

b) \( I_j \cap I_j = \emptyset \) whenever \( i, j \in \omega \) and \( i \neq j \), and

c) \( \{i, a_{k,i} : i \in I_k \text{ and } k \in \omega\} \) is a pairwise disjoint family.

In the course of the construction we will often use families of ultrafilters indexed by \( \omega \) and finite sequences of infinite subsets of \( \omega \). The following definition fixes some convenient notation.

Definition 2.7. A finite tower in \( \omega \) is a finite sequence \((A_0, \ldots, A_{k-1})\) of infinite subsets of \( \omega \) such that \( A_{t+1} \subseteq A_t \) for every \( t < k - 1 \). The set of all finite towers in \( \omega \) is called \( \mathcal{T} \). If \( T = (A_0, \ldots, A_{k-1}) \) then \( l(T) = A_{k-1} \), the last term of the sequence \( T \). For the empty sequence we write \( l(\emptyset) = \omega \).

Lemma 2.8. Assume there are \( \varepsilon \) incomparable selective ultrafilters. Then there is a family of incomparable selective ultrafilters \( (p_{T,n} : T \in \mathcal{T}, n \in \omega) \) such that \( l(T) \in p_{T,n} \) whenever \( T \in \mathcal{T} \) and \( n \in \omega \).

Proof. Index the \( \varepsilon \) incomparable selective ultrafilters faithfully as \( \{q_{T,n} : T \in \mathcal{T}, n \in \omega\} \). For each \( T \), let \( f_T : \omega \to l(T) \) be a bijection and define \( p_{T,n} = f_T(q_{T,n}) \). Since \( f \) is one-to-one, it follows that \( p_{T,n} \) is a selective ultrafilter equivalent to \( q_{T,n} \). The family \( (p_{T,n} : T \in \mathcal{T}, n \in \omega) \) is as required. \( \square \)

3. Main Ideas

From now on we fix a family \( (p_{T,n} : n \in \omega, T \in \mathcal{T}) \) of selective ultrafilters as provided by Lemma 2.8.

The idea will be to use these ultrafilters to assign \( p \)-limits to enough injective sequences in \( Z^\omega \) to ensure countable compactness of the resulting topology. We take some inspiration from [2] where a large independent family was used such that, up to a permutation every injective sequence in \( Z^\omega \) was part of this family. Since this group has cardinality \( \varepsilon \), there were indeed enough permutations to accomplish this. For an arbitrarily large group, we shall consider large linearly independent pieces to make sure every sequence has an accumulation point.

The following definition will be used to construct a witness for linearly independence in an ultraproduct that does not depend on the free ultrafilter.

Definition 3.1. Let \( \mathcal{F} \) be a subset of \( (Z^\omega)^\omega \) and \( A \in [\omega]^\omega \). We shall call \( \mathcal{F} \) linearly independent mod \( A^* \) iff for every free ultrafilter \( p \) with \( A \in p \) the set

\[ ([f]_p : f \in \mathcal{F}) \cup (\langle \chi^\omega_{\Xi} : \xi < \kappa \rangle) \]
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Lemma 3.2. A

We will abbreviate “linearly independent mod $A^*$” to l.i. mod $A^*$.

An application of Zorn’s Lemma will establish the following lemma.

**Lemma 3.2.** Every set of sequences that is l.i. mod $A^*$ can be extended to a maximal linearly independent set mod $A^*$. □

It should be clear that $A \subseteq B \subseteq \omega$ and $A$ and $B$ are infinite, then a set that is l.i. mod $B^*$ is also l.i. mod $A^*$. Then by using recursion, this easily implies the following corollary:

**Corollary 3.3.** There exists a family $(E_T : T \in \mathcal{T})$ such that:

1. For every $T \in \mathcal{T}$ the set $E_T$ is maximal l.i. mod $l(T)^*$, and
2. For every $T \in \mathcal{T}$, if $n \leq |T|$ then $E_{T \cap n} \subseteq E_T$.

We note explicitly that even though $E_T$ is only demanded to be maximal l.i. mod $l(T)^*$ it will, because of item (2), depend on all of $T$, not just on $l(T)$.

**Lemma 3.4.** Let $g$ be an element of $(\mathbb{Z}^\omega)^\omega$ and let $E \subseteq (\mathbb{Z}^\omega)^\omega$ be maximal l.i. mod $B^*$. Then there exist an infinite subset $A$ of $B$, a finite subset $E$ of $E$, a finite subset $D$ of $\kappa$, and sets $\{r_f : f \in E\}$ and $\{s_\nu : \nu \in D\}$ of rational numbers such that

$$g|_\kappa = \sum_{f \in E} r_f \cdot f|_\kappa + \sum_{\nu \in D} s_\nu \cdot \chi_\nu|_\kappa.$$

**Proof.** If $g \in E$ or $g = \chi_\kappa$ for some $\nu < \kappa$, then we are done. Otherwise, by the maximality of $E$, there exists a free ultrafilter $p$ with $B \in p$ such that the set

$$\{[g]_p \cup \{[h]_p : h \in E\} \cup \{[\chi_\xi]_p : \xi < \kappa\}$$



is not linearly independent.

This means that we can find finite subsets $E$ and $D$ of $E$ and $\kappa$ respectively and finite sets $\{r_f : f \in E\}$ and $\{s_\nu : \nu \in D\}$ of rational numbers such that

$$[g]_p = \sum_{f \in E} r_f \cdot [f]_p + \sum_{\nu \in D} s_\nu \cdot [\chi_\nu]_p.$$

Now choose $A \in p$ with $A \subseteq B$ that witnesses this equality. □

**Corollary 3.5.** If $E \subseteq (\mathbb{Z}^\omega)^\omega$ is maximal l.i. mod $B^*$, then $|E| = \kappa$.

**Proof.** First notice that $|E| \leq |(\mathbb{Z}^\omega)^\omega| = \kappa^\omega = \kappa$. Assume $|E| < \kappa$. Then the set $C = \bigcup\{\text{supp } f(n) : n \in \omega, f \in E\}$ has cardinality less than $\kappa$.

Take some injective sequence $\{\xi_n : n \in \omega\}$ in $\kappa \setminus C$ and define $g : \omega \to \mathbb{Z}^\omega$ by $g(n) = \chi_{\xi_n}$ for all $n$. Clearly then $\bigcup\{\text{supp } g(n) : n \in \omega\}$ is disjoint from $C$, all values of $g$ are non-zero and the values have disjoint supports.

Apply Lemma 3.4 to obtain sets $A, E, D, \{r_f : f \in E\}$, and $\{s_\nu : \nu \in D\}$ such that

$$g|_\kappa = \sum_{f \in E} r_f \cdot f|_\kappa + \sum_{\nu \in D} s_\nu \cdot \chi_\nu|_\kappa.$$ (*)&

Since $A$ is infinite and $D$ is finite, there is a $k \in A$ such that $\xi_k \notin D$. Now $f(k)(\xi_k) = 0$ when $f \in E$ because $\xi_k \notin C$, and $\chi_\nu(k)(\xi_k) = 0$ when $\nu \in D$ because $\xi_k \notin D$, and also $g(k)(\xi_k) = 1$, which contradicts (*).

Henceforth we fix a family $(E_T : T \in \mathcal{T})$ as in Corollary 3.3 and enumerate each $E_T$ faithfully as $E_T = \{f_\xi^T : \kappa \leq \xi < \kappa + \kappa\}$.
Definition 3.6. For each $T \in \mathcal{T}$ and $n \in \omega$, we denote by $G_{T,n}$ the intersection of Ult($\mathbb{Z}_{\mathcal{P}_T,n}$) and the free Abelian group generated by $\{\frac{1}{\mathcal{P}_T}[f]_{\mathcal{P}_T} : \kappa \leq \xi < \kappa + \kappa}\} \cup \{\frac{1}{\mathcal{P}_T}[\chi_{\xi}]_{\mathcal{P}_T} : \xi < \kappa\}$.

For the next lemma, we are going to use the following proposition:

Proposition 3.7. If $G$ is an abelian group and $H$ is a subgroup of $G$ such that $G/H$ is an infinite cyclic group, then there exists $a \in G$ such that $G = H \oplus \langle a \rangle$.

A proof may be found in [9, 14.4]. This is not the last statement of the theorem but it is exactly what is proved by the author.

The main idea of the proof of the following lemma is to mimic the well known proof of the fact that every subgroup of a free abelian group is free.

Lemma 3.8. The group $G_{T,n}$ has a basis of the form $\{[\chi_{\xi}]_{\mathcal{P}_T} : \xi < \kappa\} \cup \{f]_{\mathcal{P}_T} : f \in \mathcal{F}_{T,n}\}$ for some subset $\mathcal{F}_{T,n}$ of $(\mathbb{Z}^{(\kappa)})^\omega$.

Proof. Let $H_\mu$ the the group generated by $\{[\chi_{\xi}]_{\mathcal{P}_T} : \xi < \mu\}$ if $\mu \leq \kappa$ and by the union of $\{[\chi_{\xi}]_{\mathcal{P}_T} : \xi < \kappa\}$ and $\{\frac{1}{\mathcal{P}_T}[f]_{\mathcal{P}_T} : \kappa \leq \xi < \mu\}$ when $\kappa < \mu \leq \kappa + \kappa$.

Let $G_\mu = H_\mu \cap \text{Ult}(\mathbb{Z}_{\mathcal{P}_T,n})$ for all $\mu$.

For every $\mu < \kappa + \kappa$ we shall find $h_\mu$ so that $G_{\mu+1} = G_\mu \oplus \langle [h_\mu]_{\mathcal{P}_T,n} \rangle$, as follows.

For $\mu < \kappa$ the group $G_\mu$ is generated by $\{[\chi_{\xi}]_{\mathcal{P}_T} : \xi < \mu\}$, so $G_{\mu+1} = G_\mu \oplus \langle [\chi_{\xi}] \rangle$ and we have $h_\mu = \chi_{\xi}$.

For $\mu \geq \kappa$ observe that $G_{\mu+1} \cap H_\mu = G_\mu$, so:

$$\frac{G_{\mu+1}}{G_\mu} = \frac{G_{\mu+1} \cap H_\mu}{H_\mu} \approx \frac{G_{\mu+1} + H_\mu}{H_\mu} \leq \frac{H_{\mu+1}}{H_\mu}.$$ 

The group $\frac{H_{\mu+1}}{H_\mu}$ is cyclic infinite, so either $\frac{G_{\mu+1}}{G_\mu}$ is infinite and cyclic or $G_{\mu+1} = G_\mu$. By Proposition 3.7 there exists $a_\mu \in G_{\mu+1}$ such that $G_{\mu+1} = G_\mu \oplus \langle \{a_\mu\} \rangle$ (and $a_\mu = 0$ in case $G_{\mu+1} = G_\mu$). Take $h_\mu$ such that $[h_\mu]_{\mathcal{P}_T,n} = a_\mu$.

For every $\mu < \kappa + \kappa$, it follows that $G_{\mu+1} = G_\mu \oplus \langle [h_\mu]_{\mathcal{P}_T,n} \rangle$. Since $G_{T,n} = \bigcup_{\mu < \kappa + \kappa} G_\mu$, it follows that $G_{T,n} = \bigoplus_{\mu < \kappa + \kappa} \langle [h_\mu]_{\mathcal{P}_T,n} \rangle$.

The set $\mathcal{F}_{T,n} = \{h_\mu : \kappa \leq \mu < \kappa + \kappa, [h_\mu]_{\mathcal{P}_T,n} \neq 0\}$ is as required. \hfill \square

For the rest of this article we fix such a set $\mathcal{F}_{T,n}$ as above for each pair $(T,n)$ in $\mathcal{T} \times \omega$.

The next lemma makes good on the promise from the beginning of this section as it shows how to make our topology countably compact.

Lemma 3.9. Assume that for every pair $(T,n)$ in $\mathcal{T} \times \omega$ every sequence $f$ in $\mathcal{F}_{T,n}$ has a $\mathcal{P}_T,n$-limit in $\mathbb{Z}^{(\kappa)}$. Then every finite power of $\mathbb{Z}^{(\kappa)}$ is countably compact.

Proof. A sequence in some finite power of $\mathbb{Z}^{(\kappa)}$ is represented by finitely members of $(\mathbb{Z}^{(\kappa)})^\omega$, say $g_0, \ldots, g_m$. We show that there is one ultrafilter $\mathcal{G}$ such that $\mathcal{G}$-lim $g_i$ exists for all $i$, namely $\mathcal{P}_T,n$ for a suitable $T$ and $n$.

Recursively, we define a tower $T = (A_0, \ldots, A_m)$ and for $i \leq m$ finite subsets $E_i$ and $D_i$ of $\mathcal{E}_{T,i}$ and $\kappa$ respectively together with finite sets $(r^i_j : f \in E_i)$ and $(s^i_\nu : \nu \in D_i)$ of rational numbers such that

$$g_i|_{A_i} = \sum_{f \in E_i} r^i_j \cdot f|_{A_i} + \sum_{\nu \in D_i} s^i_\nu \cdot \chi_{\nu}|_{A_i} \quad (*)$$

For $i = 0$, use Lemma 3.3 applied to $\mathcal{E}_0'$ to obtain $A_0, E_0, D_0, (r^0_f : f \in E_0)$ and $(s^0_\nu : \nu \in D_0)$ such that $(*)$ holds with $i = 0$.

To go from $i$ to $i + 1$ apply Lemma 3.3 to $\mathcal{E}(A_0, \ldots, A_i)$ to obtain $A_{i+1}, E_{i+1}, D_{i+1}, (r^{i+1}_f : f \in E_{i+1})$, and $(s^{i+1}_\nu : \nu \in D_{i+1})$ so that $(*)$ holds for $i + 1$. 


Let $A = A_m$ and let $n$ be sufficiently large so that $n! r_i^j$ and $n! s_i^j$ are integers, for all $i \leq m$, $f \in E_i$, and $\nu \in D_i$. Then $g_i | A = \sum_{f \in E_i} n! r_i^j \cdot \left( \frac{1}{T_i^j} \right) \cdot (\frac{T_i^j}{m} \cdot f) | A + \sum_{\nu \in D_i} n! s_i^j \cdot \left( \frac{1}{T_i^j} \right) \cdot (\frac{T_i^j}{m} \cdot \nu) | A$ for all $i$.

As $l(T) = A \in \mathcal{P}_T$, and for each $E_i$, it is a subset of $\mathcal{E}_T$. Therefore, each $[g_i]_{\mathcal{P}_T}$ is an integer combination of $\{(f)_{\mathcal{P}_T} : f \in F_{T,n}\} \cup \{[\chi_{\mathcal{P}_T}]_{\mathcal{P}_T} : \chi < \kappa\}$. Then, by hypothesis, it follows that each $g_i$ has a $p_{T,n}$-limit. This completes the proof. \hfill $\Box$

4. Constructing Homomorphisms

Through this section, we let $G = \mathbb{Z}^{(\kappa)}$ and we let $\{h_\xi : \omega \leq \xi < \kappa\}$ be an enumeration of $G^\omega$ such that $\text{supp} h_\xi(n) \subseteq \xi$ whenever $n \in \omega$ and $\omega \leq \xi < \kappa$, and so that each element of $G^\omega$ appears at least $\varepsilon$ many times.

**Lemma 4.1.** There exists a family $(J_{T,n} : T \in T, n \in \omega)$ of pairwise disjoint subsets of $\kappa$ such that $\{h_\xi : \xi \in J_{T,n}\} = F_{T,n}$.

**Proof.** For each $f \in G^\omega$ there is an injective map $\phi_f : T \times \omega \to \{\xi \in \kappa : f = h_\xi\}$. Let $J_{T,n} = \{\phi_f(T,n) : f \in F_{T,n}\}$ and we are done. \hfill $\Box$

For the rest of this section, we fix a family $(J_{T,n} : T \in T, n \in \omega)$ as above.

The following lemma is the key to the main result.

**Lemma 4.2.** Assume we have a non-zero element $d \in G$, an injective sequence $r$ in $G$, and a countably infinite subset $D$ of $\kappa$ such that

1. $\omega \cup \text{supp} d \cup \bigcup_{n \in \omega} \text{supp} r(n) \subseteq D$,
2. $D \cap J_{T,n} \neq \emptyset$ for infinitely many $(T,n)$'s and,
3. $\text{supp} h_\xi(n) \subseteq D$ for all $n \in \omega$ and $\xi \in D \setminus \omega$

Then there exists a homomorphism $\phi : \mathbb{Z}^{(D)} \to \mathbb{T}$ such that:

1. $\phi(d) \neq 0$
2. $\text{pr}_{T,n} (\bigcup_{\xi \in D} \phi(h_\xi(k))) = \phi(\chi_\xi)$, whenever $T \in T$, $n \in \omega$, and $\xi \in D \cap J_{T,n}$.
3. $\phi \circ r$ does not converge.

Before proving this lemma, we show how to use it to prove the main result. First, we use it to prove another lemma:

**Lemma 4.3.** Assume $d$ is a non-zero element of $G$ and $r$ is an injective sequence in $G$. Then there exists a homomorphism $\phi : \mathbb{Z}^{(\kappa)} \to \mathbb{T}$ such that

1. $\phi(d) \neq 0$
2. $\text{pr}_{T,n} (\bigcup_{\xi \in D} \phi(h_\xi(k))) = \phi(\chi_\xi)$, whenever $T \in T$, $n \in \omega$ and $\xi \in J_{T,n}$.
3. $\phi \circ r$ does not converge.

**Proof.** Using a closing-off argument construct a countable subset $D$ of $\kappa$ that intersects infinitely many sets $J_{T,n}$, and that contains $\omega$, $\text{supp} d$, $\text{supp} r(n)$ for all $n$ as well as $\text{supp} h_\xi(n)$ whenever $\xi \in D \setminus \omega$ and $n \in \omega$.

By the previous Lemma, there exists a homomorphism $\phi_0 : \mathbb{Z}^{(D)} \to \mathbb{T}$ such that $\phi_0(d) \neq 0$, $\phi_0 \circ r$ does not converge, and $\text{pr}_{T,n} \cdot \text{lim}_{\kappa} \phi_0(h_\xi(k)) = \phi_0(\chi_\xi)$ whenever $T \in T$, $n \in \omega$ and $\xi \in D \cap J_{T,n}$.

We let $\{a_\delta : \delta < \kappa\}$ be the monotone enumeration of $\kappa \setminus D$. For $\gamma \leq \kappa$, let $D_\gamma = D \cup \{a_\delta : \delta < \kappa\}$. So $D_0 = D$ and $D_\kappa = \kappa$.

Recursively, we construct, for $\gamma \leq \kappa$, an increasing sequence of homomorphisms $\phi_\gamma : \mathbb{Z}^{(D_\gamma)} \to \mathbb{T}$ such that $\text{pr}_{T,n} \cdot \text{lim}_{\kappa} \phi_\gamma(h_\xi(k)) = \phi_\gamma(\chi_\xi)$ whenever $T \in T$, $n \in \omega$ and $\xi \in D_\gamma \cap J_{T,n}$. Our homomorphism $\phi$ will be $\phi_\kappa$. The basis step is already done, and for limit steps, we just unite all previous homomorphisms.

To define $\phi_{\gamma+1}$ given $\phi_\gamma$ it suffices to specify the value $\phi_{\gamma+1}(\chi_\alpha)$. 

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The page continues with more mathematical content, but the initial content provided above outlines the structure and foundational steps for constructing homomorphisms in the context of countably compact free abelian groups.
If \( \alpha \gamma \in J_{T,n} \) for some \( T \in \mathcal{T} \) and \( n \in \omega \) then we put \( \phi_{\gamma+1}(\chi_{\alpha}) = pr_{n} n \lim_{n} \phi_{\gamma}(h_{\gamma}(n)) \). This is well defined because \( \text{supp} h_{\gamma}(n) \subseteq \gamma \subseteq D_{\gamma} \) for all \( n \) and because \( \mathbb{T} \) is compact. In the other case let \( \phi_{\gamma+1}(\chi_{\alpha}) = 0 \). \( \square \)

We can now prove our main result.

**Theorem 4.4.** Assume the existence of pairwise incompatible \( \epsilon \) selective ultrafilters and that \( \kappa \) is an infinite cardinal such that \( \kappa^\omega \). Then the free abelian group of cardinality \( \kappa \) has a Hausdorff group topology without nontrivial converging sequences such that all of its finite powers are countably compact.

**Proof.** Following the notation of the rest of the article, given \( d \in G \setminus \{0\} \) and an injective sequence \( r \) in \( G \), Lemma 4.3 provides a homomorphism \( \phi_{d,r} : G \to \mathbb{T} \) such that \( \phi_{d}(d) \neq 0 \), such that \( \phi_{d,r} \circ r \) does not converge, and such that \( pr_{n} n \lim_{n} \phi_{d,r}(h_{\xi}(k)) = \phi_{d,r}(\chi_{\xi}) \) whenever \( T \in \mathcal{T} \), \( n \in \omega \) and \( \xi \in J_{T,n} \). We give \( G \) the initial topology generated by the collection of homomorphisms \( \{ \phi_{d,r} : d \in G \setminus \{0\} \} \), \( r \in G^\omega \) is injective) thus obtained and the natural topology of \( T \).

Since the initial topology generated by any collection of group homomorphisms is a group topology we do indeed obtain a group topology. Since \( T \) is Hausdorff and for every \( d \neq 0 \) there are many \( \phi_{d,r} \) with \( \phi_{d,r}(d) \neq 0 \) it follows at once that our topology is Hausdorff.

To see that every finite power of \( G \) is countably compact we use Lemma 3.2.

Given \( T \in \mathcal{T} \), \( n \in \omega \) and \( f \in \mathcal{F}_{T,n} \), there exist \( \xi \in J_{T,n} \) such that \( h_{\xi} = f \). For every \( d \in G \setminus \{0\} \) and injective \( r \in G^\omega \), we have \( pr_{n} n \lim_{n} \phi_{d,r}(h_{\xi}(n)) = \phi_{d,r}(\chi_{\xi}) \). So \( pr_{n} n \lim_{n} f(n) = \chi_{\xi} \) and we are done.

Since for a given injective sequence \( r \) and any \( d \in G^\omega \) the sequence \( \phi_{d,r} \circ r \) does not converge and \( \phi_{d,r} \) is continuous, it follows that \( r \) does not converge. So \( G \) has no nontrivial convergent sequences. \( \square \)

Towards the proof of Lemma 4.2 we formulate a definition and a (very) technical lemma.

**Definition 4.5.** Let \( \epsilon > 0 \). An \( \epsilon \)-arc function is a function \( \psi \) from \( \kappa \) into the set of open arcs of \( \mathcal{T} \) (including \( \mathcal{T} \) itself) such that for all \( \alpha \) either \( \psi(\alpha) = \mathcal{T} \) or the length of \( \psi(\alpha) \) is equal to \( \epsilon \), and the set \( \{ \alpha \in \kappa : \psi(\alpha) \neq \mathcal{T} \} \) is finite. We will call this finite set the support of \( \psi \) and denote it by \( \text{supp} \psi \).

Given two arc functions \( \psi \) and \( \varphi \) we write \( \psi \leq \varphi \) if \( \overline{\text{supp} \psi} \subseteq \text{supp} \varphi \) or \( \psi(\alpha) = \varphi(\alpha) \) for each \( \alpha \in \kappa \).

We shall obtain our homomorphisms using limits of such arc functions. The following lemmas are instrumental in its construction.

The following result follows from an argument implicit in the construction of [2], but it may be difficult to extract it from that paper. We postpone its rather technical proof to the next section.

**Lemma 4.6.** Let \( p \) be a selective ultrafilter and \( \mathcal{F} \) a finite subset of \( G^\omega \) such that the set \( \{ [f]_p : f \in \mathcal{F} \} \cup \{ [\chi_\alpha]_p : \alpha < \kappa \} \) is linearly independent.

Then for a given \( \epsilon > 0 \) and a finite subset \( E \) of \( \kappa \) there exist \( A \in p \) and a sequence \( \{ \delta_n : n \in A \} \) of positive real numbers such that

\( \star \) whenever \( \{ U_f : f \in \mathcal{F} \} \) is a family of arcs of length \( \epsilon \) and \( \varphi \) is an arc function of length at least \( \epsilon \) with \( \text{supp} \varphi \subseteq E \) there exist for each \( n \in A \) a \( \delta_n \)-arc function \( \psi_n \leq \varphi \) such that \( \text{supp} \psi_n = \bigcup_{f \in \mathcal{F}} \text{supp} f(n) \cup E \), and \( \sum_{n \in \text{supp} f} f(n)(\{\mu\} \cdot \psi_n)(\mu) \subseteq U_f \) for each \( f \in \mathcal{F} \).

Now we proceed to prove Lemma 4.2. We will use the following lemma:

**Lemma 4.7.** Let \( \mathcal{F}^k : k \in \omega \) be a sequence of countable subsets of \( G^\omega \) and let \( \{ p_k : k \in \omega \} \) is a sequence of pairwise incomparable selective ultrafilters such that for each \( k \in \omega \) the set
Suppose we have defined recursion.

\[ \omega \text{ contains } \{p \mid F \text{ and take a similar sequence (in such that } \supp \text{ disjoint. Without loss of generality we can assume that } \delta \text{ that } \setminus \text{ set } t_i \in k \text{ whenever } k \in \omega \text{ and } f \in F^k.\]

**Proof.** Write \( D \) as the union of an increasing sequence \( (D_n : n \in \omega) \) of finite nonempty subsets, and take a similar sequence \( (F_n : n \in \omega) \) for each \( F^k. \)

Take a sufficiently small positive number \( \epsilon_0 \) and an \( \epsilon_0 \)-arc function \( g_* \) such that \( \supp \subseteq \supp d \) and \( 0 \notin \sum_{\mu \in \supp d} d(\mu)g_*(\mu) \cup \sum_{\mu \in \supp d'} d'(\mu)g_*(\mu). \)

Let \( E_0 = \supp g_* \cup D_0 \) and \( B^{k}_0 = \omega \) for each \( k \in \omega. \)

We will define, by recursion, for \( m \in \omega \): finite sequences \( B^{k}_m : 0 \leq k \leq m \) of finite sets \( E_m \subseteq \omega, \) and real numbers \( \epsilon_m > 0 \) satisfying:

1. For all \( k \) and \( m \) in \( \omega \): we have \( B^{k}_m \in p_k. \)
2. For each \( m \geq 1 \) and \( k \leq m, \) we have a sequence \( (\delta^{k}_m,n : n \in \omega) \) of positive real numbers such that: if \( (U_f : f \in F^k_{m+1}) \) is a family of arcs of length \( \epsilon_{m-1} \) and \( g \) is an arc function of length \( \epsilon_{m-1} \) and \( \supp \subseteq \supp d \) then for each \( n \in \omega \) there exists a \( \delta^{k}_m,n \text{-arc function } \psi \) with \( \psi \leq g \), and \( \supp \psi = \bigcup_{f \in F^k_{m+1}} \supp f(n) \cup E_{m-1}, \) and \( \sum_{\mu \in \supp f} f(n)(\mu)\psi(\mu) \leq U_f \) for each \( f \in F^k_{m+1}. \)
3. For all \( k \) and \( m \) we have \( B^{k}_{m+1} \subseteq B^{k}_m. \)
4. \( \epsilon_{m+1} = \frac{1}{2} \min \{ \delta^{k}_{m,n} : k \leq m + 1 \} \).

Suppose we have defined \( B^{k}_l \) for all \( k \) as well as \( E_t \) and \( \epsilon_t \) for all \( t \leq m. \) As will be clear from the step below the set \( B^{k}_m \) is only non-trivial whenever \( k \leq m. \) Therefore we let \( B^{k}_{m+1} = B^{k}_m = \omega \) for \( k > m + 1 \) and we concentrate on the case \( k \leq m + 1. \)

Let \( k \leq m + 1. \) By Lemma 4.6 there exist \( B^{k}_{m+1} \in p_k \) and \( (\delta^{k}_{m+1,n} : n \in \omega) \) that satisfy (2) for \( m + 1. \) Without loss of generality we can assume that \( B^{k}_{m+1} \subseteq B^{k}_m. \)

Condition (4) now specifies \( E_{m+1}. \)

Setting \( E_{m+1} = E_m \cup \bigcup \{ \supp f(k) : k \leq m, f \in \bigcup_{k=m+1} F^k_{m+1} \} \cup D_{m+1} \) completes the recursion.

For each \( k \in \omega, \) apply the selectivity of \( p_k, \) to choose an increasing sequence \( (a_{k,i} : i \in \omega) \) with \( \{a_{k,i} : i \in \omega \} \subseteq p_k \) and such that \( a_{k,i+1} \in B^{k}_{m+1} \) and \( a_{k,i+1} > i \) for all \( i. \)

Next apply Lemma 2.6 and let \( (I_k : k \in \omega) \) be a sequence of pairwise disjoint subsets of \( \omega \) such that \( \{i_k : i \in I_k \} \subseteq p_k \) and the family of intervals \( \{[i, a_{k,i}] = k \in \omega, i \in I_k \} \) is pairwise disjoint. Without loss of generality we can assume that \( k < \min I_k. \)

Enumerate \( \bigcup_{k \in \omega} I_k \) in increasing order as \( (i_t : t \in \omega) \). For each \( t \in \omega, \) let \( k_t \) be such that \( i_t \in I_{k_t}. \) Thus, for each \( t \) we have \( i_t \in I_{k_t}, \) and hence \( i_t \geq \min I_{k_t} > k_t \) and \( a_{k_t,i_t} > i_t. \)

By recursion we define a sequence of arc functions, \( (g_i : t \in \omega, \) such that \( g_0 \leq g_* \) and \( g_{i+1} \leq g_i. \)

We start with \( t = 0. \) Then we have \( k_0 < i_0 < a_{k_0,i_0}, \) and \( a_{k_0,i_0} \in B^{k_0}_{m} \) and \( \epsilon_{i_0} \leq \epsilon_0. \)

Since \( g_* \) has length at least \( \epsilon_{i_0-1}, \) there exists an arc function \( g_{i_0} \) of length \( \delta^{k_0}_{i_0,a_{k_0,i_0}} \) such that \( \sum_{\mu \in \supp f} f(a_{k_0,i_0})(\mu)g_{i_0}(\mu) \leq g_*(\xi_{i_0}) \) for each \( f \in F^{k_0}_{i_0}. \) We have by the definition that \( \delta^{k_0}_{i_0,a_{k_0,i_0}} \geq \epsilon_{i_0-1}. \)

Suppose \( t > 0 \) and that \( g_{i_t-1} \) has been defined with length at least \( \epsilon_{i_t-1}. \)

Apply item (2) to the arc function \( g_{i_t-1}, \) the finite set \( F = F^{k_{i_t}}_{i_t}, \) the number \( \epsilon_{i_t-1}, \) the finite set \( E_{i_t-1}, \) the arcs \( U_f = g_{i_t-1}(\xi_f) \) for \( f \in F^{k_{i_t}}_{i_t}, \) and \( n = a_{k_t,i_t} \in B^{k_{i_t}}_{i_t} \) to obtain an arc function
\( \varrho_t \leq \varrho_{t-1} \) such that \( \sum_{\mu \in \supp f} f(\alpha_{k_i,i})(\mu) \varrho_i(\mu) \subseteq \varrho_{t-1}(\xi_f) \) for all \( f \in F^k_t \), and \( \varrho_i \) has length \( \delta_{k_t,i}^{k_{i_t},i_{i_t}} \).

Because \( k_t < i_t < a_{k_t,i_t} \leq i_{t+1} - 1 \) and \( a_{k_t,i_t} \in B_{k_t}^{k_t} \) we get \( \delta_{k_t,i}^{k_{i_t},i_{i_t}} > \epsilon_{i_{t+1}-1} \).

If \( \xi \in D_t \) then \( \xi \in \supp \varrho_i \) and the length of \( \varrho_i(\xi) \) is not greater than \( \epsilon_{i_t-1} \) which in turn is not larger than \( \frac{1}{2^{t-1}} \).

It follows that for all \( \xi \in D \) the intersection \( \bigcap_{t \in \omega} \varrho_t(\xi) \) consists of a unique element; we define \( \phi(\chi_\xi) \) to be that element and extend \( \phi \) to a group homomorphism.

By construction \( \phi(f(\alpha_{k,i}))(\mu) \) is in \( \sum_{\mu \in \supp f} f(\alpha_{k,i})(\mu) \varrho_i(\mu) \) which is a subset of \( \varrho_{t-1}(\xi_f) \) whenever \( f \in F^k_t \). Therefore, the sequence \( \phi(f(\alpha_{k,i}))_{i \in I_k} \) converges to \( \phi(\chi_{\xi_f}) \), for each \( k \in \omega \) and \( f \in F^k \).

Furthermore \( \phi(d(\mu)) \in \sum_{\mu \in \supp d(\mu)} \varrho_i(\mu) \), therefore, \( \phi(d) \neq 0 \); and likewise \( \phi(d') \neq 0 \).

It is clear that this implies the conclusion of Lemma 4.7.

Now we are ready to prove Lemma 4.6.

**Proof of Lemma 4.6.** There are only a countably many of pairs \( (T,n) \in T \times \omega \) such that \( J_{T,n} \cap D \neq \emptyset \). We enumerate them as \( ((T_m,n_m) : m \geq 2) \).

For \( m \geq 2 \) let \( F^m = \{ \xi : \xi \in D \cap J_{T_m,n_m} \} \) and \( p_m = p_{T_m,n_m} \). Let \( p_0 \) and \( p_1 \) be two ultrafilters that were not listed and let \( F^0 = \{ r \} \). For each \( m \geq 2 \) and \( \xi \in J_{T_m,n_m} \cap D \), let \( \xi_{m,n} = \xi \). Let \( h_{r,0} = \chi_k \) and \( h_{r,1} = \chi_{k'} \), where \( k,k' \in \omega \) are not in \( \supp d \). Then, by applying Lemma 4.7, with \( d' = \chi_k - \chi_{k'} \), there exist \( \phi : \mathbb{Z}^[T] \to T \) satisfying (1) and (2). To see it also satisfies (3), notice that \( p_0 \text{-lim} \phi \circ r = p_1 \text{-lim} \phi \circ r \).

5. **Proof of Lemma 4.6**

In this section we present a proof of Lemma 4.6. We will need the notion of integer stack, which was defined in [26].

The integer stacks are collections of sequences in \( \mathbb{Z}^{[\xi]} \) that are usually associated to a selective ultrafilter. Given an finite set of sequences \( F \) it is possible to associate it to a integer stack which generates the same \( \mathbb{Q} \) vector space as \( F \). The sequences in the stack have some nice properties that help us to construct well behaved arcs when constructing homomorphisms, and the linear relations between \( \mathcal{F} \) and the sequences of the stack helps us to transform these arcs into arcs that work for the functions of \( \mathcal{F} \). Below, we give the definition of integer stack.

**Definition 5.1.** An integer stack \( \mathcal{S} \) on \( A \) consists of

(i) an infinite subset \( A \) of \( \omega \);
(ii) natural numbers \( i, t, \) and \( M \); positive integers \( r_i \) for \( 0 \leq i < s \) and positive integers \( r_{i,j} \) for \( 0 \leq i < s \) and \( 0 \leq j < r_i \);
(iii) functions \( f_{i,j,k} \in (\mathbb{Z}^{[\xi]} \mathbb{A}) \) for \( 0 \leq i < s, 0 \leq j < r_i, \) and \( 0 \leq k < r_{i,j} \) and elements \( g_{i} \in (\mathbb{Z}^{[\xi]} \mathbb{A}) \) for \( 0 \leq l < t \);
(iv) sequences \( \xi_i \in \mathbb{A} \) for \( 0 \leq i < s \) and \( \mu_i \in \mathbb{A} \) for \( 0 \leq l < t \) and \( 0 \); and
(v) real numbers \( \theta_{i,j,k} \) for \( 0 \leq i < s, 0 \leq j < r_i, \) and \( 0 \leq k < r_{i,j} \)

These are required to satisfy the following conditions:

1. \( \mu_{i}(n) \in \supp g_{i}(n) \) for each \( n \in A \);
2. \( \mu_{i}(n) \notin \supp g_{i}(n) \) for each \( n \in A \) and \( 0 \leq l < t \);
3. the elements of \( \{ \mu_{i}(n) : 0 \leq l < t \} \) are pairwise distinct;
4. \( |g_{i}(n)| \leq M \) for each \( n \in A \) and \( 0 \leq l < t \);
5. \( \{ \theta_{i,j,k} : 0 \leq k < r_{i,j} \} \) is a linearly independent subset of \( \mathbb{R} \) as a \( \mathbb{Q} \)-vector space for each \( 0 \leq i < s \) and \( 0 \leq j < r_i \).
functions that are linear combinations of functions the
followed by Lemma 6.2 (depending on the case), it follows that the stack constructed consists of
the proof of 7.1 the stack is constructed by applying Lemma 5.5 or Lemma 6.2 or Lemma 5.5
h combinations of the
g ten
integer
Lemma 5.3. Thus a function
functions
Sketch in this paper, for the sake of completeness, a proof for the second part by indicating which
Given an integer stack
space
Q
p
(\[Z\]
0
Then, following the proof of Lemma 7.1, using the
ated by it does not contain nonzero constant classes, then it satisfies the conclusion of Lemma 4.1.
First, notice that if \{\[h_0\]|U, \ldots, \[h_{m-1}\]|U\} is a \(Q\)-linearly independent set and the group generated by it does
not contain nonzero constant classes, then each restriction \[h_i]|_A\ is an integer combination of the stack \[1|S\] on \(\[\bigcap\]|U\). On the other hand, each element of the integer stack \(\[1|S\] is an integer combination of \{\[h_0|, \ldots, \[h_{m-1}|\} restricted to \(A\).

Proof. We prove (#). All numbered references in this proof are to the paper \[26\].

First, notice that if \{\[h_0|\], \ldots, \[h_{m-1}|\] is a \(Q\)-linearly independent set and the group generated by it does not contain nonzero constant classes, then it satisfies the conclusion of Lemma 4.1.

Then, following the proof of Lemma 7.1, using the \(f\)’s as the \(h\)’s themselves, we see that the functions \(h_0, \ldots, h_{m-1}\) are integer combinations of the stack \[1|S\] that was constructed.

It remains to see that the functions of \(\[\bigcap\]|U\) are integer combinations of the functions \(h_i\) restricted to \(A\). First, notice that in the statement of Lemma 5.4, by x), xi), xii) and xiv) the functions \(f_i^0\) and \(g_q^0\) are integer combinations of the \(h_i\). This Lemma is used in the proof of Lemma 5.5, where the functions \(f_i^j\) become the functions \(f_i,j,k\), so there are integer combinations of the \(h_i\)’s.

Now notice that in Lemma 6.1, by g), c) and finite induction, the functions \(g_q^j\) are integer combinations of the \(h_i\), and some of these become the \(q\)’s in the proof of Lemma 6.2. As in the proof of 7.1 the stack is constructed by applying Lemma 5.5 or Lemma 6.2 or Lemma 5.5 followed by Lemma 6.2 (depending on the case), it follows that the stack constructed consists of functions that are linear combinations of functions the \(h_i\)’s (restricted to \(A\)).

\[\square\]
Now we define some integers related to Kronecker’s Theorem that will be useful in our proof. The existence of these integers follows from Lemma 4.3. of [26]. These integers were also defined and used in that paper.

**Definition 5.4.** If \( \{\theta_0, \ldots, \theta_r-1\} \) is a linearly independent subset of the \( \mathbb{Q} \)-vector space \( \mathbb{R} \) and \( \epsilon > 0 \) then \( L(\theta_0, \ldots, \theta_r-1, \epsilon) \) denotes a positive integer, \( L \), such that \( \{(\theta_0 x + z, \ldots, \theta_r x + z) : x \in I\} \) is \( \epsilon \)-dense in \( \mathbb{T}^r \) in the usual Euclidean metric product topology, for any interval \( I \) of length at least \( L \).

The last lemma we are going to need is Lemma 8.3 from [26], stated below.

**Lemma 5.5.** Let \( \epsilon, \gamma \) and \( \rho \) be positive reals, \( N \) a positive integer and \( \psi \) be an arc function. Let \( S \) be an integer stack on \( A \in [\omega]^\omega \) and \( s, t, r_i, r_{i,j}, M, f_{i,j,k}, g_l, \xi_l, \mu_j \) and \( \theta_{i,j,k} \) be as in Definition [7, 4].

Let \( L \) be an integer greater or equal to \( \max\{L(\theta_{i,j,0}, \ldots, \theta_{i,j,r_{i,j}-1}, \frac{24}{L}) : 0 \leq i < s \) and \( 0 \leq j < r_i \} \) and let \( r = \max\{r_{i,j} : 0 \leq i < s \) and \( 0 \leq j < r_i \} \).

Suppose that \( n \in A \) is such that

\begin{enumerate}
  \item \( \{V_{i,j,k} : 0 \leq i < s, 0 \leq j < r_i, 0 \leq k < r_{i,j}\} \cup \{W_l : 0 \leq l < t\} \) is a family of open arcs of length \( \epsilon \);
  \item \( \delta(\psi(\beta)) \geq \epsilon \) for each \( \beta \) in \( \text{supp} \psi \);
  \item \( \epsilon > 3\sqrt{N} \cdot \max\{\|g_l(n)\| : 0 \leq l < t\} \cup \{\|f_{i,j,k}(n)\| : 0 \leq i < s, 0 \leq j < r_i, 0 \leq k < r_{i,j}\} \); \label{eq1}
  \item \( 3MN\gamma \leq \epsilon \);
  \item \( |f_{i,j,k}(n)(\xi_l(n))| \cdot \gamma > 3L \) for each \( 0 \leq i < s \);
  \item \( |f_{i,j,k}(n)(\xi_l(n))| \cdot \gamma \geq 3L \) for each \( 0 \leq i < s \) and \( 0 \leq j < r_i \);
  \item \( \theta_{i,j,k} = \frac{f_{i,j,k}(n)(\xi_l(n))}{f_{i,j,k}(n)(\xi_l(n))} > 3L \) for each \( 0 \leq i < s \) and \( 0 \leq j < r_i \);
  \item \( |f_{i,j,k}(n)(\xi_l(n))| \cdot \gamma > 3L \) for each \( 0 \leq i < s \) and \( 0 \leq j < r_i \);
  \item \( \sup \psi \cap \{\mu_0(n), \ldots, \mu_{\gamma-1}(n)\} = \emptyset \).
\end{enumerate}

Then there exists an arc function \( \phi \) such that

\begin{enumerate}
  \item \( N \cdot \phi(\beta) \subseteq N \cdot \psi(\beta) \) for each \( \beta \) in \( \text{supp} \psi \);
  \item \( \sum_{\beta \in \text{supp} \psi} g_l(n)(\beta) \phi(\beta) \subseteq W_l \) for each \( 0 \leq l < t \);
  \item \( \sum_{\beta \in \text{supp} \psi} f_{i,j,k}(n)(\beta) \phi(\beta) \subseteq V_{i,j,k} \) for each \( i < s, j < r_i \) and \( k < r_{i,j} \);
  \item \( \delta(\phi(\beta)) = \rho \) for each \( \beta \) in \( \text{supp} \phi \) and \( \phi \) can be chosen to be any finite set containing

\[
\sup \psi \cup \bigcup_{0 \leq i < s, 0 \leq j < r_i, 0 \leq k < r_{i,j}} \text{supp} f_{i,j,k}(n) \cup \bigcup_{0 \leq l < t} \text{supp} g_l(n). \quad \square
\]
\end{enumerate}

Now we are ready to prove Lemma 4.6.

**Proof of Lemma 5.6.** Write \( F = \{u_0, \ldots, u_{q-1}\} \) with no repetition. Let \( S \) be an integer stack on \( A' \in p \) and let \( N \) be a positive integer such that \( \left( \frac{1}{N}S, A', p \right) \) is associated to \( F \).

As in Definition [5, 4], the components of \( S \) will be denoted \( s, t, M, (r_i : i < s), (r_{i,j} : i < s, j < r_i), (f_{i,j,k} : i < s, j < r_i, k < r_{i,j}), (g_l : 0 \leq l < t), (\xi_i : i < s), (\mu^p : i < t) \) and \( (\theta_{i,j,k} : 0 \leq i < s, 0 \leq j < r_i, k < r_{i,j}) \).

We write \( \{f_{i,j,k} : i < s, j < r_i, k < r_{i,j}\} \cup \{g_l : 0 \leq l < t\} \) as \( \{v_0, \ldots, v_{q-1}\} \).

Let \( M \) be the \( q \times q \) matrix of integer numbers such that \( Nv_n = \sum_{j < q} M_{i,j}v_j \) for all \( n \in A \) and \( i < q \).

By (5.3) each \( v_j \) is an integer combination of the \( u_i \)'s, therefore the inverse matrix of \( \frac{1}{N}M \), which we denote by \( N' \), has integer entries.

Let \( e' = \epsilon \cdot (\sum_{j < q} |M_{i,j}|)^{-1} \) and \( \gamma < e'/(3MN) \). Let \( L \) be larger than \( e \) and equal to the maximum of the set \( \{L(\theta_{i,j,0}, \ldots, \theta_{i,j,r_{i,j}-1}, e'/24) : i < s, j < r_i\} \).
For each $n \in A'$, let $\delta_n < \frac{1}{2}$ be such that:

$$
\epsilon' > 3N \cdot \max\{\|g_l(n)\| : 0 \leq l < t\} \cup \bigcup\{\|f_{i,j,k}(n)\| : 0 \leq i < s, 0 \leq j < r, 0 \leq k < r_{i,j}\} \cdot \frac{\delta_n}{N}
$$

We note that both $N$’s above cancel but we write this way as we will use $\delta_n/N$ in the place of $\rho$ in item c) of Lemma 5.5.

Let $r = \max\{r_{i,j} : 0 \leq i < s, 0 \leq j < r_i\}$. Let $A$ be the set of $n$’s in $A'$ such that:

- $|f_{i,r-1,0}(n)(\xi_i(n))| > 3L$ for each $0 \leq i < s$,
- $|f_{i,j-1,0}(n)(\xi_i(n))| \cdot \frac{\epsilon'}{6\sqrt{r_{i,j}}f_{i,j,0}(n)} > 3L$ for each $0 \leq i < s$ and $0 < j < r_i$,
- $|\theta_{i,j,k} - f_{i,j,k}(n)(\xi_i(n))| < \frac{\epsilon'}{24\sqrt{rL}}$ for each $i < s, j < r_i$ and $k < r_{i,j}$, and
- $E \cap \{\mu_0(n), \ldots, \mu_{t-1}(n)\} = \emptyset$.

Notice that $A$ is cofinite in $A'$, therefore $A \in p$.

We claim this $A$ and this sequence $(\delta_n : n \in A)$ work.

Fix $n \in A$.

Let $(U_f : f \in F)$ be a family of arcs of length $\epsilon$ and let $\varrho$ be an arc function of length at least $\epsilon$ with $\supp \varrho \subseteq E$. We rewrite the family of arcs as $(U_i : i < q)$, where $U_i = U_{f_i}$ for each $i < q$. For each $i < q$ let $y_i$ be a real such that $y_i + Z$ is the center of $U_i$. Let $z_j = \sum_{i < q} N_{j,i} \frac{y_i}{N}$ and, for each $j$ let $R_j$ be the arc of center $z_j$ and length $\epsilon'$. Since $N$ is a matrix of integers, $z_j + Z = \sum_{i < q} M_{i,j} R_j$ is a subset of $U_i$ for each $i < q$.

Now we aim to apply Lemma 5.5. Set $\psi = \varrho$, $\rho = \delta_n/N$ and $\epsilon'$ in the place of $\epsilon$. For $i < s$, $j < r_i$, $k < r_{i,j}$ we put $V_{i,j,k} = R_x$ if $f_{i,j,k} = v_x$ for some $x < q$, and for $j < t$ we put $W_j = R_x$ if $g_j = v_x$ for some $x < q$.

Then there exists an arc function $\tilde{\psi}_n$ such that

(A) $N\tilde{\psi}_n \subseteq N\psi_n \subseteq \varrho(\beta)$ for each $\beta \in \supp \psi$;
(B) $\sum_{\beta \in \supp g_l(n)} g_l(n)(\beta)\tilde{\psi}_n(\beta) \subseteq W_l$ for each $l < t$;
(C) $\sum_{\beta \in \supp f_{i,j,k}(n)} f_{i,j,k}(n)(\beta)\tilde{\psi}_n(\beta) \subseteq V_{i,j,k}$ for each $i < s, j < r_i$ and $k < r_{i,j}$;
(D) $\delta(\psi_n(\beta)) = \delta_n/N$ for each $\beta \in \supp \psi_n$ and
(E) $\supp \tilde{\psi}_n$ is equal to

$$
\bigcup_{0 \leq i < s, 0 \leq j < r_i, 0 \leq k < r_{i,j}} \supp f_{i,j,k}(n) \cup \bigcup_{0 \leq l < t} \supp g_l(n) \cup E = \bigcup_{f \in F} \supp f(n) \cup E.
$$

Let $\psi_n = N\tilde{\psi}_n$. By (A), $\psi_n \leq \varrho$. By (E) and (D), $\supp \psi_n = \bigcup_{f \in F} \supp f(n) \cup E$ and for each $\beta \in \supp \psi_n$, we have $\delta(\psi_n(\beta)) = \delta_n$. Let $S = \supp \psi_n$. Now notice that given $u_i \in F$ we
have:

\[
\sum_{\mu \in \text{supp } u_i} u_i(n)(\mu)\psi_n(\mu) = \sum_{\mu \in S} u_i(n)(\mu)N\tilde{\psi}_n(\mu)
\]

\[
= \sum_{\mu \in S} \left( \sum_{j<q} M_{i,j}v_j(n)(\mu) \right) \tilde{\psi}_n(\mu)
\]

\[
= \sum_{j<q} M_{i,j} \left( \sum_{\mu \in S} v_j(n)(\mu)\tilde{\psi}_n(\mu) \right)
\]

Then by (B), (C) and the definitions of the $W_l$'s and $V_{i,j,k}$'s:

\[
\sum_{\mu \in \text{supp } u_i} u_i(n)(\mu)\psi_n(\mu) = \sum_{\mu \in S} u_i(n)(\mu)N\tilde{\psi}_n(\mu) \subseteq \sum_{j<q} M_{i,j}R_j \subseteq U_i.
\]

As intended. \qed

6. Final comments

The method to construct countably compact free Abelian groups came from the technique to construct countably compact groups without non-trivial convergent sequences. It is not known if there is an easier method to produce countably compact group topologies on free Abelian groups if we do not care if the resulting topology has convergent sequences.

In fact, even to produce a countably compact group topology with convergent sequences in non-torsion groups it is used a modification of the technique to construct countably compact groups without non-trivial convergent sequences, see [1] and [2].

The first examples of countably compact groups without non-trivial convergent sequences were obtained by Hajnal and Juhász [11] under CH. E. van Douwen [7] obtained an example from MA and asked for a ZFC example. Other examples were obtained using MAcountable [15], a selective ultrafilter [10] and in the Random real model [19]. Only recently, Hrušák, van Mill, Shelah and Ramos obtained an example in ZFC [13].

This motivates the following questions in ZFC:

**Question 6.1.** Are there large countably compact groups without non-trivial convergent sequences in ZFC? Is there an example of cardinality $2^\omega$?

The example of Hrušák et al has size continuum and it is not clear if their construction could yield larger examples.

**Question 6.2.** Is there a countably compact free Abelian group in ZFC? A countably compact free Abelian group without non-trivial convergent sequences in ZFC?

It is still open if there exists a torsion-free group in ZFC that admits a countably compact group topology without non-trivial convergent sequences. If such example exists then there is a countably compact group topology without non-trivial convergent sequences in the free Abelian group of cardinality $\mathfrak{c}$ (see [25] or [27]).

**Question 6.3.** Is there a both-sided cancellative semigroup that is not a group that admits a countably compact semigroup topology (a Wallace semigroup) in ZFC?

The known examples were obtained in [18] under CH, in [21] under MAcountable, in [17] from $\mathfrak{c}$ incomparable selective ultrafilters and in [2] from one selective ultrafilter. The last two use the known fact that a free Abelian group without non-trivial convergent sequences contains a Wallace semigroup, which was used in [18]. The example in [21] was a modification of [12].
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