Numerical studies of Anderson transition
P. Markoš
Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 842 28 Bratislava, Slovakia

Abstract
We present numerical results for the statistics of \( z \)'s (\( z \)'s are defined as logarithm of eigenvalues of the transfermatrix \( T^\dagger T \)) at the critical points of Anderson transition in 3D and 4D. The change of the density of \( z \) due to the crossover from the metallic to the localized regime is described. Linear behavior \( \rho(z) = z \) at the critical point in 3D is proven and discussed. In the insulating regime, the universal form of \( \rho \) has been found.

While the finite-size scaling analysis of the disorder-induced metal-insulator transition enables us to find critical disorder \( W_c \) and critical exponent \( \nu \) from the knowledge of the smallest positive Lyapunov exponent, \( z_1 \) in the quasi-one dimensional limit \( [1] \), the knowledge of the higher \( z \)'s are necessary to understand the statistical properties of transport in cubic samples. Here, \( z_i \) is defined as logarithm of the \( i \)-th eigenvalue of the matrix \( T^\dagger T \) (\( T \) is the transfer matrix).

Statistics of \( z \)'s is well known understood in the limit of the small disorder. Its properties, derived in the analogy with the random matrix theory, \( [2] \), enables us to explain universal features of the transport in weakly disordered mesoscopic systems \( [3] \). The key role in this explanation plays the density \( \rho(z) \), defined as

\[
\rho(z) = \langle \sum_i \delta(z - z_i) \rangle, \tag{1}
\]

where the summation covers all channels and brackets mean averaging over statistical ensemble. In the weak disorder limit, \( \rho(z) = \text{const} \). Another typical characteristics of the distribution \( \langle \delta \rangle \) of (normalized) differences between \( z \)'s. In the weak disorder limit, \( P(\delta) \) equals to the Wigner surmise \( P(s) = \frac{\pi}{2} s \exp -\frac{\pi}{4} s^2 \) \( [2] \).

Our aim in this paper is to shown how the statistics of \( z \)'s changes when system undergoes the metal-insulator transition. We believe that the understanding of the statistical properties of \( z \)'s at the critical region would provide us with serious basis for the more general understanding of the Anderson transition, including the description of the system size and disorder dependence of conductance and its statistical moments in the critical region.

We consider Anderson model:

\[
\mathcal{H} = W \sum_n \varepsilon_n |n\rangle \langle n| + \sum_{n,n'} |n\rangle \langle n'|. \tag{2}
\]

In \( [2] \), \( n \) numerates sites on the \( d \)-dimensional cubic lattice \( L^d \), and \( W \) measures the disorder. For the box distributed random energies \( \varepsilon, |\varepsilon| < 1/2 \), model exhibits the metal-insulator transition at the critical point \( W_c \approx 16.5 \) (34.5) for \( d = 3 \) (4), respectively.
Recently, some speculative models has been proposed, in which weak disorder statistics of \( z \)'s has been generalized to models which reflect some special features of the metal-insulator transition. Unfortunately, no one of them succeeded to describe the metal-insulator transition completely.

The analysis of the statistics of \( z \)'s has its counterpart in the level statistics. Here, the most important quantity is the distribution \( P(s) \) of the (normalized) differences between energy levels. It has been shown, that there are three typical form of \( P(s) \): Wigner surmisises (WS) for the metallic, Poison for the insulating and the third, universal distribution at the critical point. The same scenario has been found for the statistics \( P(\delta) \) of the (normalized) differences between \( z \)'s.

We present in Fig. 1 \( P(\delta) \) in the critical point. Data show that (i) \( P(\delta) \) is system size invariant, (ii) it depends on the dimension, and that (iii) it follow neither the Wigner surmises nor the Semipoisson distribution \( P(s) = 4s \exp(-2s) \).

The first two results correspond to that of the level statistics; the second one confirms that the statistics of \( z \)'s is more similar to the localized one in higher dimension. However, the third one is in the contradiction with numerical observation reported in Ref. , where the Semipoisson distribution of the level statistics has been found as the result of the sensitivity of the level statistics to the boundary conditions at the critical point. Although we consider only periodic boundary condition in this work, we do not believe that the use of different boundaries will remove this disagreement. The reason is that the typical values of \( z \)'s are rather large at the critical point: The distribution of higher \( z \)'s is Gaussian with mean value \( \geq 5 \) and variance \( \sim 1 \). It is therefore highly unprobale that the change of boundaries could influence the statistics of higher
Figure 2: The density $\rho(z)$ for cubic system with $L = 12$.

$\rho(z)$ changes due to the growth of disorder. Fig. 2. shows how the density $\rho(z)$ changes due to the growth of disorder. The typical semicircle form of the $\rho(z)$ could be found only for extremely small disorder ($W = 6$) (we remove the contribution of closed channels in this case). The growth of the disorder deforms $\rho(z)$: the last decreases for small $z$ in favor of the maximum, which moves towards the higher values of $z$. At the critical point, $\rho(z)$ becomes linear (quadratic) in $z$ in 3D (4D), respectively. This agrees with our previous result, obtained for the quasi-one dimensional samples [5].

The critical density deserves the special Figure. In Fig. 3a we present $\rho(z)$ for different size of the sample. It is evident, that only small part of $\rho(z)$ is system size invariant. The interval of the size invariance equals approximately to that at which $\rho(z)$ remains linear. When comparing $\rho(z)$ for different system sizes, $> L_{\text{min}}$, then the interval, at which all $\rho$’s coincides, is $z \leq L_{\text{min}}/2$. This restriction must be taken into account in analysis of the scaling properties of higher $z$’s [6].

In Fig. 3b,c we present the normalized critical density for 3D and 4D Anderson models. For 4D, $\rho(z) \sim z^2$ in the limit of small $z$.

The form of $\rho(z)$ remain approximately the same also when $W$ exceeds critical point. The whole distribution only shifts towards the higher values of $z$. In the limit of large disorder we find that

$$\rho_{W_1}(z - \langle z_1(W_1) \rangle) = \rho_{W_2}(z - \langle z_1(W_2) \rangle)$$

(Fig. 4). The most remarkable differences could be found only in the tail of the
Figure 3: (a) Critical $\rho(z)$ for different system size. Only small (linear) part of $\rho(z)$ is system-size invariant. (b) normalized $\rho(z)$ for 3D and (c) 4D system.

density for small values of $z$.

In conclusion, we present numerical data for Lyapunov exponents for 3D and 4D Anderson models in the neighbor of the critical point. We show how two the most famous characteristics of the statistics of $z$'s change when the strength of the disorder exceeds the weak disorder limit. In difference to the level statistics, we found no Semipoisson distribution of the differences of $z$’s. Our data support belief that the statistics of $z$’s and, consequently, of the conductance has a simple form similar that developed for the metallic regime.

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Figure 4: Density $\rho(z)$ for the 3D system with $W = 32$ ($\circ$), 45 ($\diamond$) and 60 (full symbols) for two system size ($L = 8$ and 12). We shifted the densities for $W = 32$ and 45 in $\langle z_1(W = 60) \rangle - \langle z_1(W) \rangle$ to show their the universal form (3). $\langle z_1 \rangle$ for $W = 60$ is 20 (3) for $L = 8$ (12), respectively.

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