Derivative corrections to the symmetry energy and the isovector dipole-resonance structure in nuclei

J P Blocki\(^1\), A G Magner\(^2\) and P Ring\(^3\)

\(^1\)National Centre of Nuclear Research, PL-00681 Warsaw, Poland
\(^2\)Institute for Nuclear Research, Kyiv 03680, Ukraine
\(^3\)Technical Munich University, D-85747 Garching, Germany

E-mail: magner@kinr.kiev.ua

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Abstract

The effective surface approximation is extended accounting for derivatives of the symmetry energy density per particle. The new expressions for the isovector surface energy constants are used for calculations of improved energies and sum rules of the isovector dipole-resonance strength structure within the Fermi-liquid droplet model. Our results are in reasonable agreement with experimental data and with other theoretical approaches.

Keywords: nuclear binding energy, liquid drop model, extended Thomas–Fermi approach, surface symmetry energy, neutron skin, isovector stiffness

1. Introduction

The symmetry energy is a key quantity for the study of fundamental properties of exotic nuclei with a large neutron excess in nuclear and astro-physics [1–9]. In spite of very intensive studies of these properties, the derivatives of the symmetry energy and its surface coefficient are still rather underestimated in the calculations by the liquid droplet model (LDM) [1], or within more general local density approaches (LDA) [10, 11]. In particular, this applies for the extended Thomas–Fermi (ETF) approximation [12], and for models based on the Hartree–Fock (HF) method [13] using Skyrme forces [14–17], where, in contrast to the main volume symmetry-energy constant, the derivatives are still not well determined. Within the nuclear LDA, one can use the variational condition, minimizing the nuclear energy under the restrictions of a fixed particle number and a neutron excess. The corresponding Lagrange equations are significantly simplified by using the leptodermous expansion for heavy nuclei within the effective surface (ES) approximation (ESA) [18–22]. A separation of the nuclear energy into the volume and surface (and curvature) components of the LDM and ETF makes obviously sense for a small leptodermous parameter defined as the edge diffuseness with respect to the mean curvature radius of the nuclear surface. The accuracy of the ESA in the ETF approach [12] was checked in [20] taking into account the spin–orbit (SO) and the asymmetry terms [21, 22]. The ESA calculations were compared with the results obtained within the HF as well with other versions of the ETF theory for some Skyrme forces.

Expressions for isoscalar and isovector particle densities within the ETF ESA were used in calculations of the fundamental surface symmetry-energy coefficients, in particular the neutron skin in leading order of the leptodermous parameter [21, 22]. Our results are compared with the previous ones [1–4] in the LDM and recent works [6–9, 23–27].

The structure of the isovector dipole-resonance (IVDR) strength in terms of the main and satellite (pygmy) modes [24–33] depending on the isovector surface energy coefficients in the ESA can be described by using the Fermi-liquid droplet (FLD) model [34, 35]. Within the FLD model [33] the surface symmetry energy constants correspond to reasonable energies and sum rules of the IVDR.

In the present work, we shall extend the variational ES method [22, 33] taking into account also derivatives of the non-gradient terms in the symmetry energy density along with its gradient ones. The fundamental isovector derivative and surface-tension constants are not fixed yet well enough by using the experimental data for the neutron skin thickness [9]. We suggest to use also empirical data for the splitting of the
2. Symmetry energy and particle densities

The nuclear energy $E$ is a functional of the isoscalar ($\rho_s$) and the isovector ($\rho_v$) particle densities, $\rho_s = \rho_i \pm \rho_p$, in the LDA [12, 14]:

$$E = \int \! dr \, \rho_s \mathcal{E}(\rho_s, \rho_v),$$

(1)

with

$$\mathcal{E}(\rho_s, \rho_v) \approx -b_v + JI^2 + \varepsilon_+ + \varepsilon_- + \left(\frac{\rho_s}{\rho_v}\right)^2,$$

(2)

being the energy density per particle and

$$\varepsilon_+(\rho_s, \rho_v, \varepsilon) = \mathcal{S}(\varepsilon) / \rho_v^2 - JI^2,$$

$$\mathcal{S}(\varepsilon) = J - L - K_+ \varepsilon^2 / 2.$$

(3)

Following suggestions of Myers and Swiatecki [1] we also introduced a small parameter $\varepsilon = (\rho_v - \rho_s) / \rho_v$, where $\rho_v = 3/4 \pi r_0^3 \approx 0.16 \text{ fm}^{-3}$ is the density of infinite nuclear matter and $r_0$ is the commonly accepted constant in the $A^{1/3}$ dependence of a mean radius. In (2), $b_v \approx 16 \text{ MeV}$ is the separation energy per particle. The isoscalar component of the surface energy density (2) (the zero-order terms in the expansion over $\rho_v / \rho_s$, denoted as $\delta$ [1]) are explicitly independent of the density gradient terms. They are determined by a function $e_+$. We use here a representation of $e_+$ by the dimensionless quantity $\varepsilon$:

$$e_+ = \varepsilon_+ / \varepsilon,$$

(4)

where $K_+$ is the isoscalar in-compressibility modulus of symmetric nuclear matter, $K_+ \approx 220 - 245 \text{ MeV}$,

$$\varepsilon_+ = 9\varepsilon^2 + \frac{1}{2} \left[ \mathcal{S}(\varepsilon) - J \right] / K_+. $$

(5)

The parameter of the nuclear asymmetry $I = (N - Z) / A$ is defined in the usual way through a difference between the neutron, $N = \int \! dr \, \rho_n (r)$, and proton, $Z = \int \! dr \, \rho_p (r)$, numbers, $A = N + Z$. The second term, $\alpha \varepsilon^2$, appears due to the derivative corrections to the volume symmetry energy with the nuclear matter constant $J \approx 30 \text{ MeV}$. In contrast to the simplest approximation [22], the terms with $L \approx 20 \div 120 \text{ MeV}$, and even less known $K_-$ [5, 6, 9], defined by the first and second derivatives of the symmetry energy expansion with respect to the variable $\varepsilon$, were taken into account in equations (3) and (5). Equation (2) is general for applications to the semiclassical approximation for realistic Skyrme forces [15–17]. In particular, one can neglect both higher $\varepsilon^2$ corrections related to the ETF kinetic energy [12] and Coulomb terms [20–22]. All of them can be easily taken into account [18, 22] neglecting relatively small Coulomb exchange terms. In this case, such a correction can be calculated numerically. For the constants $C_+$ in equation (2) one has the following functions of Skyrme force parameters [12, 14–17].

$$C_+ = \frac{1}{12} \left( \frac{t_1}{t_2} - \frac{5}{12} - \frac{3}{t_2} x_3 \right),$$

$$C_- = -\frac{1}{48} t_1 \left( 1 + \frac{5}{2} x_1 - \frac{1}{36} t_2 \left( 1 + \frac{19}{8} x_3 \right) \right).$$

(6)

The isoscalar SO gradient terms in (2) are defined through a constant: $D_s = -9 m W_{0}^2 / 16 \hbar^2$ with $W_0 \approx 100 - 130 \text{ MeV fm}^{-5}$ and the nucleon mass $m$ [12, 14–16]. The isovector constant $D_v$ is usually relatively small and will be neglected here for simplicity. We emphasize that (2) has a general form [19-21] for any heavy dense system with a sharp edge while it is not applied for sharpness of the surfaces in more dilute systems like very exotic nuclei. In particular, it can be derived from the comparison with one of the energy density functionals of Skyrme type in order to find the relations (6) for $C_-$ and others in terms of its parameters.

The energy density per particle in equation (2) contains in the first line two volume terms and surface components including the new $L$ and $K_-$ derivative corrections $e_-$ (in contrast to [22]) and in the second and third lines density gradients. Both are important for finite nuclear systems. These gradient terms together with the other surface components in the energy density within the ESA are responsible for the surface tension in finite nuclei.

As usual, we minimize the energy $E (1)$ under the constraints of fixed particle number $A = \int \! dr \, \rho_s (r)$ and neutron excess $N - Z = \int \! dr \, \rho_n (r)$. For other constraints, for instance of deformation and nuclear angular momentum, see [18–20, 22]. The results are the Lagrange equations with the isoscalar and isovector chemical potentials as corresponding multipliers. In order to ensure these constraints for calculations of the particle densities $\rho_s$, one needs the leading order terms in the leptoderous parameter $a/R \sim A^{-1/3}$, $a$ is of the order of the diffuse edge thickness of the nucleus and $R$ is the mean curvature radius of the ES. Using these densities for the accurate calculations of the nuclear energy we account for higher order surface corrections in $a/R$ with respect to the leading order terms [22].

3. Extended densities and energies

Up to the leading terms in the parameter $a/R$, one obtains the usual first-order differential Lagrange equation for the isoscalar particle density, $w = w_0 = \rho_s / \rho_v$. This equation can be easily integrated as shown in [20–22]. Thus, we find the following analytical expression depending however on the
derivatives \( L \) and \( K_\gamma \) through \( e_\gamma \) in equations (4) and (5),
\[
x = \int_{w_0}^{w} \frac{1 + \beta y}{y [e(y)]}, \quad x = \frac{\xi}{a}, \quad e(w) = \frac{1 - w}{3},
\]
for \( x < x(w = 0) \) and \( w = 0 \) for \( x \geq x(w = 0) \), where \( x(w = 0) \) is the turning point. Here, we introduced a local coordinate system \((\xi, \eta)\), where \( \xi \) is the distance from a given spatial point to the ES \((\xi = r - R)\) for the spherically-symmetric case) and \( \eta \) is the tangent-to-ES coordinate \([19]\).

\( e[y] \) in equation (5) is the dimensionless surface non-gradient term \( e_\gamma \) of the energy density per particle \((2)\) and \( \beta = D_{d\beta c}/C_\gamma \) is the dimensionless SO parameter. For the ES \( w_0 = w(x = 0) \) one has, up to \( \beta \) corrections, the boundary equation, \( d^2 w(x)/dx^2 = 0 \) at the ES defined as \( x = 0 \) (maximum of the particle density gradient), i.e.

\[
e[y(w_0)] + w_0 [1 + \beta w_0] [d[e(w)]/dw]_{w = w_0} = 0.
\]

In equation (7), \( a \approx 0.5\)–0.6 fm is the diffuseness parameter \([22]\),
\[
a = \frac{c_s \rho_{c} K_\gamma}{3 b_c^2},
\]
found from the asymptotic behavior of the particle density, \( w \sim \exp(-\xi/a) \) for large \( \xi \) as the solution of equation (7).

Within the same leading precision in \( a/R \), after simple transformations, one finally arrives at the isovector Lagrange equations and the boundary conditions for the isovector density \( w_\gamma(w) = \rho_\gamma/\rho_s \), as a function of \( w(x) \),
\[
dw_\gamma = \frac{c_s}{J} \frac{S_{\text{sym}}(e)[1 + \beta w]}{J e[e(w)]} \sqrt{1 - \frac{w^2}{w_0^2}},
\]
and \( w_\gamma(w = 1) = 1 \). Here \( S_{\text{sym}}(e) \) is given by (3) and
\[
c_{\text{sym}} = \frac{a}{J} \left[ J/[\rho_s C_\gamma] \right]^{1/2},
\]
see \([22]\) for the particular case of constant \( S_{\text{sym}} = J \). For the derivation of an analytical solution \( w_\gamma = w \cos[\psi(w)] \), one expands \( \psi \) in powers of
\[
\gamma = 3 \varepsilon(w) c_{\gamma},
\]
Up to the second order in \( \gamma \), one finds \([36]\)
\[
w_\gamma = w \cos[\psi(w)] \approx w \left[ 1 - \psi^2(w)/2 \right],
\]
where
\[
\psi(w) = \gamma \left[ 1 + \tilde{c} \gamma \left( \gamma + \cdots \right) \right]^{1/2},
\]
with
\[
\tilde{c} = \left[ \beta c_\gamma^2 + 2 + c_\gamma^2 L(1 + \beta) \right]/(3[1 + \beta]).
\]
The \( L \) dependence of \( w_\gamma \) appears at second order in \( \gamma \) but \( w_\gamma \) is independent of \( K_\gamma \) at this order. Note that \( K_\gamma \) is the coefficient in the expansions (3) and (5) at order \( e_\gamma^2 \), and therefore, it shows up at third order in the expansions (13) and (14) in \( \gamma \).

In figure 1, the \( L \) dependence of the function \( w_\gamma(x) \) is shown within a rather large interval \( L = 0\)–100 MeV \([7]\) and it is compared to that of the density \( w_\gamma(x) \) for the SLy5* force as a typical example \((L = 45.885 \approx 50 \text{ MeV} \) \([37]\) and table 1). As shown in figure 2 in a logarithmic scale, one observes a big difference in the isovector densities \( w_\gamma \) derived from several Skyrme forces \([15–17]\) within the diffuse edge. All these calculations have been done with a finite value of the slope parameter \( L \) taken approximately from \([16, 37]\) (table 1). This is important for calculations of the neutron skins of nuclei. As mentioned above, the isovector particle density \( w_\gamma \) \((\gamma) \) does not depend on the symmetry energy incompressibility \( K_\gamma \) in the second order of the small parameter \( \gamma \). Therefore, it is possible to study first the main slope effects of \( L \) neglecting small \( \beta \) corrections to the isoscalar particle density \( w_\gamma \) \([22]\) through \( e(\epsilon) \) in equation (5). Afterwards,
taking into account higher order terms, we may deal more precisely with the effect of the second derivatives \( K_n \). We point out that the densities \( w(x) \) (7) and \( w_n(x) \) (13) (figures 1 and 2) were obtained in the dominating ESA \((a/R \ll 1)\). They depend on the parameters of the Skyrme force like \( b \) and \( c_\text{sym} \) (11) but taking now into account the \( L \)-dependence. These densities are at the leading order in the lepton-densities parameter \( a/R \) approximately universal functions independent of the peculiarities of the specific nucleus. It yields largely the local density distributions in the normal-to-ES direction \( \xi \) with the correct asymptotic behavior outside of the deformed ES layer at \( a/R \ll 1 \), as it is the case for semi-infinite nuclear matter. Therefore, within the dominating ESA the particle densities \( w_n \) are universal distribution approximations independent of the specific properties of nuclei while high order corrections to the densities \( w_n \) depend on the specific macroscopic properties of nuclei like the surface tension coefficient.

In this improved ESA, the energy \( E \) of the nucleus can be represented as a sum of volume \( (E_V) \) and surface \( (E_S) \) terms,

\[
E \approx E_V + E_S, \quad E_V = -b_V A + J(N - Z) \gamma / A, \\
E_S = E_S^{(+)} + E_S^{(-)}. \\
\] (16)

For the isoscalar \((+)\) and isovector \((-)\) surface energy components \( E_S^{(\pm)} \), one obtains \( E_S^{(\pm)} = b_S^{(\pm)} S / (4 \pi \rho_\xi^2) \), where \( S \) measures the ES area, \( b_S^{(\pm)} \) are the isoscalar \((+\) and isovector \((-)\) surface energy constants,

\[
b_S^{(\pm)} \approx 8 \pi \rho_\xi^2 C_S \int_{-\infty}^{0} d\xi \left(1 + \frac{D_{\text{sym}}}{C_S}(\partial \rho_\xi / \partial \xi)^2\right). \quad (17)
\]

These constants are proportional to the corresponding surface tension coefficients \( \sigma_\xi = b_S^{(\pm)} / (4 \pi \rho_\xi^2) \) through solutions for \( \rho_\xi (\xi) \) (see (7) and (13)) which can be taken into account in the main order of \( a/R \). In the derivations, one accounts for the equivalence of contributions coming from the surface non-gradient and gradient terms, according to the equations (7) and (10) for \( w_n \) (see [22] for the case \( L = K_n = 0 \)). These coefficients \( \sigma_\xi \) coincide with the expressions for the capillary pressures of the macroscopic boundary conditions (see [22] extended now to new values of \( \xi \) (3) and (5) modified by \( L \) and \( K_n \) derivative corrections.

To obtain isoscalar and isovector particle densities with higher precision, one may take into account in the Lagrange equations higher order corrections in the parameter \( a/R \) within an improved ESA. We multiply these equations by \( \partial \rho_\xi / \partial \xi \) terms and integrate them through the ES in the normal \( (\xi) \) direction. Applying then the solutions for \( w_n(x) \) up to the dominating orders (see (7) and (13)), one arrives at the ES equations in the form of the macroscopic boundary conditions [22, 35] which however depend now on derivatives of the symmetry energy. They ensure the macroscopic equilibrium of the volume and surface (capillary) pressure (isoscalar or isovector) forces acting on the ES. The capillary pressures are proportional to the corresponding surface tension coefficients \( \sigma_\xi \). For the coefficients \( b_S^{(\pm)} \) (17), one finds [36]

\[
b_S^{(+)} = 6C \rho_\xi \frac{J_+}{\eta_0}, \quad J_+ = \int_0^1 dw \left[ w(1 + \beta w) \, e[w(\xi)] \right]^{1/2},
\] (18)

and

\[
b_S^{(-)} = k_S I_2, \quad k_S = 6C \frac{\rho_\xi J_-}{\eta_0},
\] (19)

where

\[
J_+ = \int_0^1 dw \left[ \frac{w \, e[w(\xi)]}{(1 + \beta w)} \right]^{1/2} \left[ \cos \psi + w \sin \psi \left( \frac{c_\text{sym} \sqrt{1 + \beta}}{1 + 2\gamma \beta \tau(w)} \right) \right]_1^2.
\] (20)

For \( \gamma \) and \( \tau \), see (12) and (15). For \( J_- \) one can use the following approximation:

\[
J_- \approx \int_0^1 \left[ 1 - w \right] dw \left( \frac{w}{1 + \beta w} \right) \left[ 1 + \frac{2\gamma \beta \tau(w)}{c_\text{sym}(1 + \beta)} \right] \left[ \frac{1}{c_\text{sym}} + 6(1 + \beta) \left( \frac{\xi}{c_\text{sym}} - \frac{1}{2} \right) \right].
\] (21)

Simple expressions for constants \( b_S^{(\pm)} \) can be easily derived in terms of algebraic and trigonometric functions for the quadratic form of \( e(\xi) \) (5). In these derivations we neglected here curvature corrections of the same order as well as shell corrections which were discarded from the very beginning. The isovector energy terms were calculated within the ESA with high accuracy up to a small \( (\xi)^2 \).

According to the macroscopic theory [1, 18–22], one may derive the isovector stiffness \( Q \) in the collective variable \( R_n - R_p \), where \( R_n \) and \( R_p \) are the neutron and proton effective radii defined as maxima of the density distribution gradients, respectively,

\[
Q = k_S I_2^2 / \xi^2.
\] (22)
with \( \tau \) being the neutron skin in units of \( r_0 \),
\[
\tau = \frac{R_n - R_p}{r_0} \approx \frac{8aI g(w_\nu)}{2n c_{\text{sym}}}, \quad (23)
\]
The function \( g(w) \),
\[
g(w) = \frac{w^{3/2}(1 + \beta w)^{3/2}}{(1 + \beta)(3w + 1 + 4\beta w)} \left\{ w(1 + 2\bar{\epsilon} \gamma)^2 + 2\gamma(1 + \bar{\epsilon} \gamma) \left[ \bar{\epsilon} w - c_{\text{sym}}(1 + 2\bar{\epsilon} \gamma) \right] \right\}, \quad (24)
\]
is taken at the ES value \( w = w_\nu \) \( [w_\nu(w_\nu) = 0 \ (8)] \). Accounting also for (19) and (21), one finally results in
\[
Q = -\nu J^2/k_S, \quad \nu = 9J^2/\left[16g^2(w_\nu)\right], \quad (25)
\]
where \( J \) and \( g(w) \) are given by (19), (21) and (24), respectively. This \( Q \) has been predicted earlier \([1] \) with \( \nu = 9/4 \). Note that in our derivations \( \nu \) is proportional to the expression \( J^2/g^2(w_\nu) \) and therefore, it deviates significantly from the number 9/4. The functions \( \nu \) and \( Q \) depend especially on the value of the SO interaction parameter \( \beta \) which is basically the same for most of the Skyrme forces \([22, 33] \). In addition, the stiffness coefficient \( Q \) (25) depends essentially on the constants \( C_\nu \) and \( L \) (also \( K \)) through \( \nu \) (25) and \( k_S \) (19) \( \) (and equations (5) and (3)).

Within the dominating order of the ESA, the universal functions \( w(x) \) (7) and \( w_{\nu}(x) \) (13) (in the sense explained above) can be used analytically in the quadratic approximation for \( e(\nu) \) for calculations of the surface energy constants \( b_{\text{sym}}^{(n)} \) (17) and the neutron skin \( \tau \approx I \). Note that only the particle density distributions \( w(x) \) and \( w_{\nu}(x) \) within the surface layer are necessary to be used through their derivatives.

Therefore, the surface symmetry-energy coefficient \( k_S \) (19), the neutron skin \( \tau \) (23) and the neutron skin stiffness \( Q \) (25) can be approximated in terms of functions of the critical surface combinations of the Skyrme parameters \( a, \beta, b_{\text{sym}}^{(n)}, c_{\text{sym}} \), as well as the volume ones \( k_0, K, J \) and derivatives of the symmetry energy \( L \) and \( K \). Thus, in the ESA, the functional dependence of these constants, for instance \( b_{\text{sym}}^{(n)} \) (17), on the Skyrme parameters does not depend mainly on the specific properties of the nucleus.

4. Discussion of the results

The surface symmetry energy constant \( k_S \) (19), the neutron skin stiffness \( Q \) (25), its constant \( \nu \) and the neutron skin \( \tau \) (23) are shown in table 1. They were obtained for a few Skyrme forces \([15–17] \) with different values of \( L \) within the ESA using the quadratic approximation for \( e(\nu) \) (5), and neglecting the \( \tilde{p} \) slope corrections. We also show the quantities \( k_S(0), \nu_0, Q_0 \) and \( r_0 \) where the slope corrections are neglected \( (L = 0) \). In contrast to a fairly good agreement for the isoscalar energy constant \( b_{\text{sym}}^{(n)} \) \([22] \) with that of \([15–17, 22] \), the neutron-skin energy coefficients \( Q \) (or \( k_S = J^2/Q \)) are more sensitive to the choice of the Skyrme forces than the isoscalar ones. The modula of \( Q \) are significantly smaller for most of the Skyrme forces SLy \([15, 17] \) and SV \([16] \) than for the other ones, in contrast to the symmetry energy constant \( k_S \) for which one has the opposite behavior. However, the \( L \) dependence of \( k_S \) is not more pronounced than that of \( Q \) (tables 1 and 2). For SLy and SV forces the stiffnesses \( Q \) are correspondingly significantly smaller in absolute value being more close to the well-known empirical values of \( Q \) \([4] \) and semi-microscopic HF calculations \([7, 12] \), especially with increasing \( L \). Note that \( |Q| \) decreases more with \( L \) than \( k_S \), sometimes by a factor two as for the SLy and SV forces. Thus, the neutron-skin stiffness \( Q \) is even much more sensitive to the constants of the Skyrme force and to the slope parameter \( L \) than \( k_S \).

In \([4] \) the neutron-skin stiffness values \( Q \equiv 30–80 \) MeV were suggested accounting for the additional experimental data. A more precise \( A \)-dependence of the averaged isovector giant dipole resonance (IVGDR) energy constant \( D_{\beta L D} \) for the values of \( Q \) are obviously beyond the accuracy of both the hydrodynamical and the FLD model calculations. As shown in \([7, 12] \), more realistic self-consistent HF calculations with the Coulomb interaction, surface-curvature and quantum-shell effects lead to larger values \( Q \equiv 30–80 \) MeV.

The IVGDR energies and the energy weight sum rules (EWSR) obtained within the FLD approach based on the Landau–Vlasov equation \([33, 35] \) with the macroscopic boundary conditions \([22] \) are also basically insensitive to the surface energy constants \( k_S \) or \( Q \). They are similarly in good agreement with the experimental data, and basically independent of the Skyrme forces, also when the finite slope parameter \( L \) of the symmetry energy is included \( \) (the two last rows in table 1 and the seventh column of table 2).

Assuming a macroscopic nature, the IVDR splitting into a dominating peak which exhausts mainly the model-independent EWSR and a much broader satellite \( \) (pygmy-like resonances with a much smaller contribution to the EWSR \([24–27, 33] \) \) is extended now within the FLD approach by taking into account the \( L \)-dependence of the symmetry energy density. This macroscopic picture should be valid for skin-like collective excitation modes in heavy neutron-rich nuclei. Note that the relative strength of the satellite mode with respect to that of the main peak disappears as \( I \) in the limit of symmetric nuclei \([34, 35] \).

The total IVDR strength function being the sum of the two \( (\) out-of-phase\( ) ) \( n = 1 \) and \( \) in-phase\( ) ) \( n = 2 \) modes for the isovector- and isoscalar-like volume particle-density vibrations, respectively \( \) (figure 3) \) have a shape asymmetry \([33] \) \( \) (the SLy5* forces are taken again as a typical example). The \( L \)-dependence of the total IVDR strength is shown in figure 4 for SLy5* and in figure 5 for SVsym32. The characteristic values of \( L \) used in these calculations are \( L \equiv 50 \) MeV for SLy5* \([37] \) and \( L \equiv 60 \) MeV for SVsym32 \([16] \). In figures 3 and 4 for the SLy5* and in figure 5 for the SVsym32 forces, one has the \( \) in-phase\( ) satellite to the right of the main \( \) out-of-phase\( ) peak. An enhancement on its left part for SLy5* is due to the increasing \( \) out-of-phase\( ) strength \( \) (dotted curve) at small energies as compared to the IVGDR \( \) \( \) (figures 3 and 4), in contrast to the SVsym32 case shown in figure 5. The IVDR energies of the two modes in the nucleus \( ^{132} \)Sn do not change significantly with \( L \) in both cases \( \) (table 2).

However, as seen
from table 2 and figures 4 and 5, the \( L \) dependence of the EWSRs of these modes for \( \text{SVsym32} \) is larger than for \( \text{SLy5}\). Notice, the essential redistribution of the EWSR contributions among them due to a significant enhancement of the main ‘out-of-phase’ peak with increasing \( L \) for \( \text{SLy5}\) (figures 3 and 4) and the more pronounced EWSR redistribution with \( L \) for \( \text{SVsym32} \) (figure 5) in the same nucleus (table 2). The slopes of the \( L \) dependence of \( \tau/T \) which is almost linear are approximately the same for both considered Skyrme forces but smaller than those found in [9] probably due to another definition of the neutron skin thickness \( \tau \) [22].

### Table 2.

| Skyrme   | \( L \) | \( E_1 \) MeV | \( E_2 \) MeV | \( S_1 \) % | \( S_2 \) % | \( D \) | \( k_S \) | \( Q \) | \( \tau/T \) |
|----------|--------|---------------|---------------|------------|------------|------|--------|--------|----------|
| \( \text{SLy5}\) \( ^{132}\text{Sn} \) | 0      | 17.15         | 19.85         | 87.9       | 12.1       | 88.9 | -13.1  | 72     | 0.43     |
|          | 50     | 17.18         | 19.86         | 88.8       | 11.2       | 88.8 | -14.2  | 48     | 0.54     |
|          | 60     | 17.18         | 19.86         | 89.0       | 11.0       | 88.8 | -14.5  | 45     | 0.57     |
|          | 70     | 17.18         | 19.87         | 89.2       | 10.8       | 88.7 | -14.8  | 42     | 0.59     |
|          | 80     | 17.19         | 19.87         | 89.5       | 10.5       | 88.8 | -15.1  | 40     | 0.62     |
|          | 90     | 17.19         | 19.87         | 89.7       | 10.3       | 88.8 | -15.5  | 38     | 0.64     |
|          | 100    | 17.20         | 19.87         | 89.9       | 10.1       | 88.8 | -15.8  | 36     | 0.67     |
| \( \text{SVsym32} \) \( ^{132}\text{Sn} \) | 50     | 14.65         | 17.14         | 72.9       | 27.1       | 78.8 | 15.6   | -55    | 0.53     |
|          | 50     | 14.69         | 17.99         | 75.1       | 24.9       | 78.4 | 17.2   | -39    | 0.66     |
|          | 60     | 14.69         | 17.98         | 75.5       | 24.5       | 78.3 | 17.6   | -37    | 0.69     |
|          | 70     | 14.69         | 17.98         | 76.0       | 24.0       | 78.2 | 18.0   | -35    | 0.71     |
|          | 80     | 14.69         | 17.98         | 76.4       | 23.6       | 78.2 | 18.4   | -33    | 0.74     |
|          | 90     | 14.68         | 17.97         | 76.9       | 23.1       | 78.3 | 18.9   | -32    | 0.77     |
|          | 100    | 14.68         | 17.97         | 77.4       | 22.6       | 78.0 | 19.3   | -30    | 0.80     |
| \( ^{208}\text{Pb} \) | 50     | 14.70         | 18.00         | 72.9       | 27.1       | 78.8 | 15.6   | -55    | 0.53     |

**Figure 3.** IVDR strength functions \( S(\omega) \) versus the excitation energy \( \hbar \omega \) are shown for vibrations of the nucleus \( ^{132}\text{Sn} \) for the Skyrme force \( \text{SLy5}\) by a solid line at \( L = 50 \) MeV; dotted (‘out-of-phase’), and dashed (‘in-phase’) curves show separately the main and satellite excitation modes, respectively [33]; the collision relaxation time \( T = 4.3 \times 10^{-21} \) s in agreement with the IVGDR widths [40].

**Figure 4.** The same total strengths as in figure 3 are shown for different \( L = 0, 50, 60 \) and \( 100 \) MeV by dashed, thick and thin solid and dotted lines.

which is related to the value of \( \delta_S = \rho_1/\rho_2 \) at the nuclear ES [1]. Note also that these more precise calculations (see with those of [22, 33]) take into account higher (fourth order) terms of the power expansion in the small parameter \( \gamma \) for any reasonable \( L \) change in (13) and (14). This is essentially important for the IVDR strength distribution for SV forces because of the smaller values of \( \delta_S \) as compared to those for the other Skyrme interactions. The constants \( \tilde{c} \) for the iso-vector solutions \( \tilde{w}_- \), (13), are modified essentially (besides of the \( L \) dependence) by higher order terms due to the nonlinear equation (10) (re-written for \( \psi(w) \) through the left equation (13)) which was solved by using the power series. Decreasing the relaxation time \( T \) in the factor of about 1.5 with respect to the value (figures 3–5) evaluated from the data.
on the widths of the IVGDRs at their energies almost does not change the IVDR strength structure [40]. However, we found a strong dependence on \( T \) in a wider region (a factor of about 2–3). The ‘in-phase’ strength component with a rather wide maximum is weakly dependent on the choice of the Skyrme forces [15–17] and on the slope parameter \( L \), as well as on the relaxation time \( T \).

The essential parameter of the Skyrme HF approach leading to the significant differences in the \( k_S \) and \( Q \) values is the constant \( C_\tau \) (6) in the gradient terms of the energy density (2). The key quantity in the expression for \( Q \) (25) and the isovector surface energy constant \( k_S \) (19), is really \( C_\tau \) because one approximately has \( k_S \propto C_\tau \) [22], and \( Q \propto 1/k_S \propto 1/C_\tau \). Concerning \( k_S \) and the IVDR strength structure, this is even more important than the \( L \) dependence though the latter changes significantly the neutron-skin stiffness \( Q \) and the neutron skin \( \tau \). The constant \( C_\tau \) changes considerably for different Skyrme forces in the absolute value and even in sign, \( C_\tau \approx -23 \) to \( 26 \) MeV \( \text{fm}^{-5} \) (\( k_S \approx -15 \) to \( 18 \) MeV). Contrary to the isoscalar parameters (\( b^{(+)}_S \)), there are so far no clear experiments which would fix the value of \( k_S \) well enough because the mean energies of the IVGDR (main peaks) do not depend very much on \( k_S \) for different Skyrme forces (last row in table 1 and seventh column of table 2). Another difference in \( k_S \) and \( Q \) values might be traced back to the difficulties in deducing \( k_S \) directly from the HF calculations due to the curvature and quantum effects in contrast to \( b^{(+)}_S \). It is worthwhile to study semi-infinite nuclear matter within our approach to avoid such effects and to compare with semi-microscopic models. According to (19), we have to go far away from the nuclear stability line to subtract uniquely the coefficient \( k_S \) in the dependence of \( b^{(+)}_S \) on \( T^2 \). More problems in connection of \( k_S \) to the experimental data is related to a sufficient precision for exotic nuclei. The \( L \) dependence of the neutron skin \( \tau \) is essential but not dramatic in the case of SLy and SV forces (table 1). The precision of such a description depends more on the specific nuclear models [7]. Our results for the neutron skin \( R_n-R_p \approx 0.10–0.13 \) fm in \(^{208}\text{Pb} \) (table 2) in reasonable agreement with the experimental data (figure 3 of [9]), \( (R_n-R_p)_{\text{exp}} = 0.12–0.14 \) fm (the coefficient (3/5)\(^{1/2} \) of the square-mean neutron and proton radii was taken into account to adopt the definitions, see also very precise experimental data of [41]). For \(^{132}\text{Sn} \) our analytical evaluation predicts the neutron skin thickness \( R_n-R_p \approx 0.12–0.15 \) fm (0.08–0.10 fm for a more known nucleus \(^{12}\text{Sn} \) in agreement with the experimental data collected in [9]). The neutron skin thickness \( \tau \), like the stiffness \( Q \), is interesting in many aspects for studying exotic nuclei, for instance, in nuclear astrophysics.

Note that for specific Skyrme forces there exist an abnormal dependence of the surface constants \( k_S \) and \( Q \) on the symmetry energy. This is related mainly to the fundamental constant \( C_\tau \) of the energy density (2) but not much to the derivative corrections of the symmetry-energy density. The coefficients \( \tau \) (25) are almost independent on \( C_\tau \) for SLy and SV Skyrme forces (table 1). Their magnitudes are significantly smaller than the number 9/4 of [1] for most of the Skyrme forces, besides SkM* (\( \approx 2.3 \)).

In contrast to all other Skyrme forces, for RATP [15] and SV [16] (like for SkI) forces, the neutron-skin stiffness \( Q \) is even negative as \( C_\tau > 0 \) (\( k_S > 0 \)), that would correspond to the unstable vibration of the neutron skin. In relativistic investigations [38, 39] of the pygmy modes and the structure of the IVDR distributions, the dependence of these quantities on the derivative terms has not been investigated so far. It therefore remains an interesting task for the future to apply similar semiclassical methods such as the ESA used in this manuscript also in relativistic models.

5. Conclusions

Neutron and proton particle densities and the surface symmetry energy were obtained analytically in the ESA as functions of the slope parameter \( L \) and other well-known parameters of the Skyrme forces. Our expressions for the surface energy constant \( k_S \), the neutron skin thickness \( \tau \) and the stiffness \( Q \) depend essentially on the choice of the parameters of the Skyrme energy density (2), mostly on its gradient terms through the parameter \( C_\tau \) (6). The values of these constants \( k_S \), \( \tau \), and especially, \( Q \) depend also essentially on the slope parameter \( L \), and the SO interaction constant \( \beta \). The neutron-skin stiffness constants \( Q \) become more close to the empirical data accounting for their \( L \)-dependence. The mean IVGDR energies and sum rules calculated in the macroscopic models like the FLD model [33, 35] are in a fairly good agreement with the experimental data. We found a reasonable two-mode main and satellite structure of the IVDR strength within the FLD model as compared to the experimental data [31] and recent other theoretical works [24–27]. We may interpret semiclassically the IVDR satellites as some kind of the pygmy resonances. Their energies and sum rules obtained analytically within the semiclassical FLD approximation are
sensitive to the surface symmetry energy constant $k_S$. The energies $E_1$ and $E_2$ (table 2) are independent of the slope parameter $L$ but the EWSRs $S_1$ and $S_2$ do depend on $L$, in a more pronounced way for SVsym32. It seems helpful to describe them in terms of the only few critical constants, like $k_S$ and $L$. Therefore, their comparison with the experimental data on the IVDR strength splitting can be used for the evaluation of $k_S$ and $L$, in addition to the experimental data for the neutron skin. Our ESA evaluations of the neutron skins for several neutron-rich nuclei are in a reasonable agreement with their experimental data.

As for perspectives, it would be worth to apply our results to calculations of pygmy resonances in the IVDR strength within the FLDR model [35] in a more systematic way. More general problems of the classical and quantum chaos in terms of the level statistics [42] and Poincaré and Lyapunov exponents [43, 44] might lead to a progress in studying the fundamental properties of the collective dynamics like nuclear fission within the Swiatecki and Strutinsky macroscopic-microscopic model. Our approach is helpful also for further investigation of the effects in the surface symmetry energy because it gives the approximate analytical universal expressions for the constants $k_S$, $r$, and $Q$ as functions of the symmetry slope parameter $L$ which are basically independent of the specific properties of the nucleus.

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