Kaluza–Klein-type models of de Sitter and Poincaré gauge theories of gravity

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Abstract

We construct Kaluza–Klein-type models with a de Sitter or Minkowski bundle in the de Sitter or Poincaré gauge theory of gravity, respectively. A manifestly gauge-invariant formalism has been given. The gravitational dynamics is constructed by the geometry of the de Sitter or Minkowski bundle and a global section which plays an important role in the gauge-invariant formalism. Unlike the old Kaluza–Klein-type models of gauge theory of gravity, a suitable cosmological term can be obtained in the Lagrangian of our models and the models in the spin-current-free and torsion-free limit will come back to general relativity with a corresponding cosmological term. We also generalize the results to the case with a variable cosmological term.

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1. Introduction

The first Kaluza–Klein-type model of gauge theory of gravity is presented by Mansouri and Chang [1], which is similar to the Kaluza–Klein-type unifications of non-Abelian gauge theories with gravitation (for example, see [2]). A Kaluza–Klein-type model of gauge theory of gravity is a gravitational model with the Lagrangian constructed from the scalar curvature of a fiber bundle, in which the structure group is the gauge group for gravity and the Ehresmann connection is related to the geometry of the spacetime as the base manifold. In the model of [1], the fiber bundle is the principal bundle with both the structure group and fiber being the Lorentz or Poincaré group, and is assumed to be torsion free. The parallel transport of vector fields in the spacetime is used to uniquely relate the gauge potential in the fiber bundle to the connection of the spacetime. The scalar curvature of the bundle is equal to the sum of the scalar curvature of the spacetime, a Yang–Mills Lagrangian and the scalar curvature of the group space. The action of the model is the integration of the Lagrangian over the bundle. The
model has been generalized to the torsional case [3], with the help of Guo’s definition of the torsion field of the fiber bundle in terms of the torsion field of the spacetime [4].

In these models, however, even if the gauge group is chosen to be the Poincaré group, a cosmological term still appears from the scalar curvature of the group space. It will prevent the Minkowski space from being a vacuum solution, or the cosmological term should be canceled out by hand. How to deal with this problem consistently? Guo and Chang [3] have proposed the idea of using the associated Minkowski bundle to solve the problem. In this case, the cosmological term is replaced by the scalar curvature of the Minkowski space, which is equal to zero.

In the gauge theories of gravity, the gauge group is usually chosen to be the Poincaré, de Sitter (dS) or anti-de Sitter (AdS) group. There are several methods to obtain the gauge-invariant expressions of the metric and torsion fields. For the dS and AdS gauge theories of gravity established on an umbilical manifold [5–7], Guo [8] writes down a dS/AdS-invariant metric field by making use of the normal vector field of the manifold. An AdS-invariant metric field is also given in another geometric framework [9], in which a global section of the AdS bundle is used. Locally, the global section of the AdS bundle corresponds to a 5-vector valued, non-dynamical scalar field. The scalar field has further been generalized to the dynamical case [9], which is equivalent to using the AdS bundle where the radius of the AdS fiber is variable. Following [9], a Poincaré-invariant metric field has been implicitly used in [10]. The explicit expressions for the Poincaré-invariant metric and torsion fields can be found in [11]. But, to our knowledge, a gauge-invariant expression of the torsion field for the case with the dynamical scalar field is still absent in the literatures.

The gauge-invariant expressions of metric and torsion fields give a relation between the Ehresmann connection of the bundle and the geometry of the spacetime. The formalism with such gauge-invariant expressions will be called the manifestly gauge-invariant formalism. In this formalism, the configuration variables are the Ehresmann connection of the principal bundle and a global section of the associated bundle. The Ehresmann connection is different from the nonlinear connection used in the nonlinear realization [12]. Generally, the nonlinear connection is related to the Ehresmann connection in a nonlinear way. When the Ehresmann connection performs a Poincaré, dS or AdS transformation, the nonlinear connection only performs a Lorentz transformation.

One of the purposes of this paper is to construct new Kaluza–Klein-type models with a dS or Minkowski bundle in the dS or Poincaré gauge theory of gravity, respectively. We will define both the metric and torsion fields on the dS or Minkowski bundle and calculate the corresponding Riemann–Cartan scalar curvatures. In the torsion-free case, the Riemann–Cartan scalar curvatures reduce to the Riemann scalar curvatures in [13], where the scalar curvatures of fiber bundles with generic homogenous fibers in the framework of Kaluza–Klein theory have been systematically computed, but the torsional case and the relation between the gauge potential and the geometry of the spacetime are not taken into account. The Lagrangian is constructed in such a way that a suitable cosmological term can be obtained in the model, so that the dS or Minkowski fiber is one of the vacuum solutions of the corresponding theory.

Concretely, in the dS case, the gravitational Lagrangian consists of two parts: one is the pullback of the scalar curvature of the dS bundle by a global section and the other is the Lagrangian for that global section itself. The coupling constant between the two parts is proportional to the cosmological constant. In the Poincaré case, the gravitational Lagrangian is simply chosen to be the pullback of the scalar curvature of the Minkowski bundle by a global section. The global section is just the one used in the gauge-invariant expressions for the metric and torsion fields. The gravitational action is the integration of the Lagrangian over some spacetime region rather than the bundle as in [1]. It will be shown that the scalar curvature
of the fiber bundle is a sum of the scalar curvature of the spacetime, the scalar curvature of the dS or Minkowski fiber and a Yang–Mills-like term. The pullback of the Yang–Mills-like term is merely a quadratic torsion term. These models will come back to general relativity (GR) with or without a cosmological term in the spin-current-free and torsion-free case.

The second purpose of this paper is to generalize the above results to the case with a variable cosmological term. The gauge-invariant expressions for both metric and torsion fields will be given. For this case, the global section used in the gauge-invariant expressions becomes dynamic and the introduction of its Lagrangian would become more natural. The variable cosmological term may be of significance as many variable cosmological constant models could solve the coincidence problem of the cosmological constant (for example, see [14]).

This paper is organized as follows. In section 2, we introduce a manifestly gauge-invariant formalism for the dS and Poincaré gauge theories of gravity. In section 3, we construct concrete Kaluza–Klein-type models with the dS or Minkowski bundle in the dS or Poincaré gauge theory of gravity, respectively. A suitable cosmological term will be obtained in these models. The results of sections 2 and 3 are generalized to the case with a variable cosmological term in section 4. Finally, we end with some remarks in the last section.

2. de Sitter and Poincaré gauge theories of gravity

We will first introduce the geometric framework for the dS gauge theory of gravity and then turn to the Poincaré case. In the latter part of the section, we will briefly discuss two specific models of gauge theory of gravity.

2.1. de Sitter gauge theory of gravity

To introduce a dS bundle, let \( \mathcal{P} \) be a principal fiber bundle with the dS group \( SO(1, 4) \) as its structure group and with the spacetime manifold \( \mathcal{M} \) as the base space. As \( SO(1, 4) \) may be realized at a five-dimensional (5D) Minkowski space with a fixed origin, we may set up a 5D Minkowski bundle \( \mathcal{Q}_{5M} \) with a zero section and associate them with \( \mathcal{P} \). A local section \( \sigma \) of \( \mathcal{P} \) presents a local trivialization of \( \mathcal{P} \), which induces a local trivialization on the associated bundle \( \mathcal{Q}_{dS} \). The local coordinates in the corresponding region of \( \mathcal{M} \) and Minkowski coordinates in the typical fiber define the local coordinates on \( \mathcal{Q}_{dS} \):

\[
\eta_{AB} \xi^A \xi^B = l^2,
\]

where \( l \) is a constant with the dimension of length and the signature is chosen so that \( \eta_{AB} = \text{diag}(-1, 1, 1, 1, 1) \). The vertical coordinate basis vector fields \( \bar{\partial}_A = \partial / \partial \xi^A \) of \( \mathcal{Q}_{5M} \) define by projecting the vector fields which are tangent to the dS bundle \( \mathcal{Q}_{dS} \),

\[
\bar{\partial}_A = \partial_A - l^{-2} \xi^B \partial_B.
\]

The horizontal basis vector field of the dS bundle can be chosen as follows [13]:

\[
E_\mu = \partial_\mu - \Omega^A_{\mu B} \xi^B \bar{\partial}_A,
\]

where \( \partial_\mu = \partial / \partial x^\mu \) and \( \Omega^A_{\mu B} \) is the Ehresmann connection of the principal bundle in the local section \( \sigma \) (cf the appendix).

Suppose that \( \eta_{\mathcal{M}}(t) \) is a curve on \( \mathcal{M} \) with \( \eta_{\mathcal{M}}(0) = x_0 \in \mathcal{M} \). Its tangent vector field is designated by \( v^\mu(t) \partial_\mu \). Denote the tangent vector at \( x_0 \) by \( v_0 \). Let \( \eta(t) \) be the horizontal lift of \( \eta_{\mathcal{M}}(t) \), passing through the point \( p_0 \in \mathcal{Q}_{dS} \) which lies in the fiber over \( x_0 \). [\( x^\mu(\eta(t)), \xi^A(\eta(t)) \)]
are the coordinates of $\eta(t)$ in the bundle. The tangent vector field of $\eta(t)$ is required to be $v^\mu(t)E_\mu$. It can be realized by the following definition of the horizontal lift:

$$
\frac{dv^\mu(\eta(t))}{dt} = v^\mu(t), \quad \frac{d\xi^A(\eta(t))}{dt} = -v^\mu(t)\Omega^A_{\beta\mu}(\eta_M(t))\xi^\beta(\eta(t)).
$$

Then, the gauge-covariant derivative of a cross section locally represented by $\xi^A(x)$ ($x \in M$) at $x_0$ with respect to $v_0$ can be defined by

$$
D_{v_0}\xi^A(x_0) = \lim_{t \to 0} \frac{\xi^A(\eta_M(t)) - \xi^A(\eta(t))}{t},
$$

where $\xi^A(\eta_M(t))$ is the value of the cross section at $\eta_M(t)$. It can be observed that

$$
D_{v_0}\xi^A(x_0) = [\partial_\mu \xi^A(x_0) + \Omega^A_{\beta\mu}(x_0)\xi^\beta(x_0)](v_0)^\mu.
$$

Remarkably, equation (6) is easy to be generalized to the gauge-covariant derivative of a cross section with respect to any vector field $v$ on $M$:

$$
D_v\xi^A(x) = [\partial_\mu \xi^A(x) + \Omega^A_{\beta\mu}(x)\xi^\beta(x)]v^\mu.
$$

This derivative will be used in the gauge-invariant expressions of the metric and torsion fields of the spacetime later.

Now, we will show how to tie the bundle structure and the spacetime structure together. Let $P_H$ be the right-handed orthonormal frame bundle of a Riemann–Cartan spacetime manifold $M$, where $H = SO(1, 3)$ stands for the Lorentz group. Identify $H$ to a subgroup of $G = SO(1, 4)$. As $H$ acts on $G$ by the group multiplication, we may set up an associated bundle $P$ of $P_H$ with $G$ as the typical fiber. Actually, $P$ turns out to be a principal bundle with $G$ as the structure group [15]. Any element of $P$ can be expressed by

$$
p = p_H \cdot g = \{(p_H h^{-1}, hg)| h \in H\},
$$

where $p_H \in P_H$ and $g \in G$. Suppose that $M$ could be covered by finite charts of right-handed orthonormal frame fields. They correspond to finite charts of local sections $[\sigma_H(x)]$ of $P_H$ and therefore finite charts of local sections $[\sigma(x) = \sigma_H(x) \cdot l]$ of $P$, where $i$ denotes the $i$th local section and $l$ stands for the identity element of $G$. Let $\xi = (0, 0, 0, 0, 1)^T$; then $[\sigma(x) \cdot \xi]$ forms a global section $\phi$ of the dS bundle $Q_{dS}$, where

$$
\sigma_i(x) \cdot \xi = \{[\sigma_i(x)g^{-1}, g\xi]| g \in G\}
$$

is a local section of $Q_M$ as well as $Q_{dS}$. In the local section $\sigma_i(x)$, the connection 1-form of the principal bundle $P$ can be defined as follows:

$$
\Omega^A_{\beta\mu} = \begin{pmatrix}
\Gamma^\alpha_{\beta\alpha} & I^{-1}\xi^\alpha \\
-I\xi^\beta & 0
\end{pmatrix},
$$

where $\alpha$ is an abstract index [16], $\alpha, \beta = 0 \sim 3$, $[\epsilon^\alpha_\beta]$ is the dual frame field of the orthonormal frame field $[e^\alpha_\beta]$, which corresponds to the local section $\sigma_H(x)$, and $\Gamma^\alpha_{\beta\gamma}$ is the metric-compatible connection 1-form of $M$ in $[e^\alpha_\beta]$. The dual frame fields $\epsilon^\alpha_\beta$ and the connection 1-forms $\Omega^A_{\beta\mu}$, $\Gamma^\alpha_{\beta\gamma}$ may also be denoted by $e^\alpha$ and $\Omega^A_B$, $\Gamma^\alpha_{\beta\gamma}$, respectively. The curvature 2-form of $\Omega^A_{\beta\mu}$, denoted by $F^A_{B\mu}$ or $F^A_B$, is

$$
F^A_{B\mu} = (d\Omega^A_B)_{\mu\nu} + \Omega^A_C \wedge \Gamma^C_{B\mu},
$$

(d$\Omega^A_B$ is a 2-form and thus may be denoted as $(d\Omega^A_B)_{ab}$ in terms of abstract indices.) It can be shown that relation (10) is equivalent to the following manifestly gauge-invariant form:

$$
g_{ab} = \eta_{AB}(D_a\xi^A(D_b\xi^B)),
$$

$$
S_{cab} = F_{AB\mu}(D_c\xi^A)\xi^B.
$$
They are the metric and torsion field of \( \mathcal{M} \), respectively. Equation (12) has been given in [8, 9] and the earlier references therein. Here, \( \xi^A = \xi^A(x) \) is the local representation of the global section \( \phi \), \( D_\phi \xi^A \) is defined by

\[
u^\alpha(D_\phi \xi^A) = D_\phi \xi^A
\]

(14)

for any vector field \( \nu^\alpha \) on \( \mathcal{M} \) and can be interpreted as the local representation of the gauge covariant derivative of \( \phi \). The metric-compatible connection 1-form \( \Gamma^a_{\beta\alpha} \) corresponds to a metric-compatible derivative operator \( \nabla_a \) of \( \mathcal{M} \), such that

\[
\Gamma^a_{\beta\alpha} = e^a_b \nabla_\alpha e^b_b.
\]

(15)

Then, the torsion and curvature tensors of \( \mathcal{M} \) are defined as usual:

\[
(\nabla_a \nabla_b - \nabla_b \nabla_a)f = -S^c_{ab} \nabla_c f
\]

(16)

\[
(\nabla_a \nabla_b - \nabla_b \nabla_a)\omega_d = -R^c_{dab} \omega_c - S^c_{ab} \nabla_c \omega_d
\]

(17)

for any function \( f \) and 1-form \( \omega_a \) on \( \mathcal{M} \). Let \( S^a_{ab} = S^a_{ab} e^c_c, R^a_{\beta\alpha} = R^a_{dab} e^e_c e^b_d \); then equations (16) and (17) are equivalent to

\[
S^a_{ab} = (de^a)_{ab} + \Gamma^a_{\beta\alpha} \wedge e^b_b.
\]

(18)

\[
R^a_{\beta\alpha} = (d\Gamma^a_{\beta\alpha})_{ab} + \Gamma^a_{\gamma\alpha} \wedge \Gamma^\gamma_{\beta\beta}.
\]

(19)

In fact, similar to the torsion tensor, the curvature tensor also has the following manifestly invariant form:

\[
R_{cdab} = (2/l^2) g_{ab} (\xi^A) (D_\xi^B) = F^A_{\beta\alpha} (D_{\xi^A} \xi^B).
\]

(20)

Similar to \( \Omega^A_{\beta\alpha} \), in the local section \( \sigma(x) \), \( F^A_{\beta\alpha} \) has the following expression:

\[
F^A_{\beta\alpha} = \begin{pmatrix} R^a_{\beta\alpha} - l^{-2} e^a_a \wedge e^b_b & l^{-1} S^a_{ab} \\ -l^{-1} S^a_{ab} & 0 \end{pmatrix}.
\]

(21)

According to equations (12) and (13), the gravitational action \( S_G \) which is a functional of \( g_{ab} \) and \( S^a_{ab} \) could also be viewed as a functional of \( \Omega^A_{\beta\alpha} \) and \( \xi^A \). It is invariant under the gauge transformation:

\[
\xi^A \rightarrow k^A_B \xi^B,
\]

\[
\Omega^A_{\beta\alpha} \rightarrow k^A_C \Omega^C_{Da} (g^{-1})^D_B + k^A_C \partial_a (g^{-1})^C_B.
\]

(22)

where \( g^A_B = g^A_B(x) \) is the matrix representation of a \( G \)-valued local function of \( \mathcal{M} \). A \( G \)-valued local function of \( \mathcal{M} \) is a smooth map from a region of \( \mathcal{M} \) to the Lie group \( G \). For the \( \delta S \) gauge theory of gravity, the group \( G \) is \( SO(1, 4) \).

By equation (1), there exists \( A \in \{0, 1, 2, 3, 4\} \), such that \( \xi^A \neq 0 \) locally. Without loss of generality, it can be assumed that \( \xi^A \neq 0 \) locally; then \( \xi^A \) can be viewed as a function of \( \xi^a \), according to condition (1). The gravitational equations can be given by

\[
\delta S_T / \delta \Omega = 0,
\]

(23)

\[
\delta S_T / \delta \xi^a + (\delta S_T / \delta \xi^4)(\partial \xi^4 / \partial \xi^a) = 0,
\]

(24)

where \( S_T \) is the total action including the gravitational action and the action for matter fields. Substituting \( \partial \xi^4 / \partial \xi^a = -\xi_a / \xi^4 \) into equation (24), there will be

\[
\delta S_T / \delta \xi^A = \lambda \xi_A,
\]

(25)

where \( \lambda \) is a local function of spacetime, determined by the detailed information of the total action.
Actually, equation (25) can be deduced from equation (23) and the fact that the action $S_T$ is gauge invariant. By equation (23),

$$\delta S_T = \int (\delta S_T/\delta \xi^A) \delta \xi^A.$$  

(26)

We may let

$$\delta \xi^A = \delta [g^A_B(x, \lambda) \xi^B(x)] = (\delta g^A_B(x)) \xi^B,$$

(27)

where $g^A_B(x, \lambda)$ is a family of $SO(1, 4)$-valued local functions, and thus $\delta g^A_B(x) \equiv \frac{d}{d\lambda} |_{\lambda=0} [g^A_B(x, \lambda)]$ is an $so(1, 4)$-valued local function. Since $S_T$ is invariant under the gauge transformation (22), $\delta S_T$ given by equations (26) and (27) should be equal to zero. Therefore,

$$\delta S_T/\delta \xi^A \xi^B - (\delta S_T/\delta \xi^B) \xi_A = 0,$$

(28)

which results in equation (25).

### 2.2. Poincaré gauge theory of gravity

Now, we turn to the Poincaré case. Most of the above formalism is valid and the differences are as follows. The structure group for the principal bundle $\mathcal{P}$ is now the Poincaré group $ISO(1, 3)$. The meanings of $\Omega^A_{Ba}, F^A_{Bab}, D_\alpha$, etc change to those of the corresponding objects of the principal Poincaré bundle or associated 4D Minkowski bundle $\mathcal{Q}_{M4}$. As a definition of the 4D Minkowski bundle $\mathcal{Q}_{M4}$ from $\mathcal{Q}_{M5}$, equation (1) should be replaced by

$$\xi^4 = l.$$  

(29)

and equation (2) should be replaced by

$$\tilde{\alpha} = \alpha, \quad \tilde{d}_4 = 0.$$  

(30)

They are tangent to $\mathcal{Q}_{M4}$, equation (10) should be replaced by

$$\Omega^A_{Ba} = \begin{pmatrix} \Gamma^a_{\beta a} & l^{-1}e_{0}^a \\ 0 & 0 \end{pmatrix}.$$  

(31)

The gauge-invariant expression for the curvature tensor (20) should be replaced by

$$R_{cdab} = F_{dab}(D_c \xi^A)(D_d \xi^B),$$  

(32)

and equation (21) replaced by

$$F^A_{Bab} = \begin{pmatrix} R^a_{\beta ab} & l^{-1}S^a_{ab} \\ 0 & 0 \end{pmatrix}.$$  

(33)

g^A_B = g^A_B(x) in equation (22) is now the matrix representation of an $ISO(1, 3)$-valued local function of $\mathcal{M}$.

The gravitational field equations are given by equation (23) and

$$\delta S_T/\delta \xi^a = 0.$$  

(34)

Similar to the dS case, equation (34) can be deduced from equation (23) and the gauge-invariant property of the action $S_T$. By equations (23) and (29),

$$\delta S_T = \int (\delta S_T/\delta \xi^A) \delta \xi^A = \int (\delta S_T/\delta \xi^a) \delta \xi^a.$$  

(35)

We may let $\delta \xi^A = (\delta g^A_B(x)) \xi^B$, then $\delta S_T$ given by the above equation will be equal to zero, as $S_T$ is invariant under the gauge transformation (22). Now $g^A_B = g^A_B(x, \lambda)$ is a family of $ISO(1, 3)$-valued local functions, $\delta g^A_B$ is an $iso(1, 3)$-valued local function and, therefore,

$$\delta S_T/\delta \xi^a \xi^B - (\delta S_T/\delta \xi^B) \xi_a = 0, \quad (\delta S_T/\delta \xi^a) \xi_A = 0,$$

(36)

which results in equation (34).
2.3. Example 1: a model of de Sitter gauge theory of gravity

Naively, a gauge theory of gravity with a manifest gauge invariance should have a Yang–Mills-like action for gravitation

\[ S_{GYM} = \kappa \int \mathcal{F}^{AB}_{\ ab} \mathcal{F}_{AB}^{\ ab}, \]  

(37)

plus the gauge-invariant action for matter fields, where \( \kappa \) is the dimensionless coupling constant between the matter and the gravitational field. For the dS case, equation (37) gives [5, 17, 7]

\[ S_{GYM} = \kappa \int \left[ R_{\abcd} R^{\abcd} - \frac{4}{l^2} \left( R - \frac{6}{l^2} \right) + \frac{2}{l^2} S_{abcd} S_{abcd} \right], \]  

(38)

where equation (21) is used. The field equations of the dS gravity model are

\[ \nabla_c S_{ab}^{\ c} + \frac{1}{2} S_{cd} T_b^{\ cd} + G_{ba} + \frac{3}{l^2} g_{ab} = - \frac{l^2}{8\kappa} \Sigma_{ab} + \frac{l^2}{2} \left( R_{cdea} R^{cdea} - \frac{1}{4} R_{cdef} R^{cdef} g_{ab} \right) + S_{cda} S^{\ cda}_b - \frac{1}{4} S_{abcd} S^{abcd} g_{ab}, \]  

(39)

\[ \nabla_d R_{bc}^{\ da} - \frac{1}{2} T_{bc}^{\ da} R_{de}^{\ bc} + \frac{1}{l^2} T^{\ a}_{bc} + \frac{2}{l^2} S_{[bc]}^{\ a} = \frac{1}{4\kappa} \tau_{bc}^{\ a}, \]  

(40)

where \( \Sigma_{ab} \) and \( \tau_{bc}^{\ a} \) are the stress–energy tensor and spin current of matter fields, respectively.

\[ G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}, \quad T_{ab}^{\ c} = S_{ab}^{\ c} + 2 \delta_{(a}^{\ c} \left[ a S^{d)}_{b]d} \right]. \]

In the torsion-free and vacuum case, the field equations reduce to

\[ G_{ab} + \frac{3}{l^2} g_{ab} = \frac{l^2}{2} \left[ C_{abcd} R^{\abcd} + \frac{R}{6} \left( R_{ab} - \frac{1}{4} R g_{ab} \right) \right], \]

(41)

\[ \nabla_d R^{\ da}_{bc} = 0 \quad \Leftrightarrow \quad \nabla_b R^{\ da}_{c} = \nabla_d R^{\ da}_{b}, \]  

(42)

where \( C_{abcd} \) is the Weyl curvature tensor. It has been shown that all the torsion-free vacuum solutions of the dS gravity model are the vacuum solutions of GR with the same cosmological constant, and vice versa [18, 19].

When the dS gravity model applies to the evolving universe and the matter fields are composed of spin-current-free, pressureless ideal gas, the model may explain the accelerating expansion of the universe and supply a natural transit from a decelerating expansion to an accelerating expansion without the help of dark energy [20, 21]. The torsion together with curvature makes the universe transit from the decelerating expansion to the accelerating expansion. Besides, the attractors in the dS gravity model are analyzed [22, 23].

It is remarkable that the dS gravity model becomes highly non-trivial when the exterior (vacuum) solutions are required to join some interior solutions [24–26]. For example, the spin-current-free matter in torsionless interior solutions must be distributed uniformly [24] because equation (42) sets a new constraint. In the torsional case, the exterior solutions can join with the interior solutions with the non-uniformly distributed matter field, satisfying Newton’s law in the weak field approximation, and supply an alternative way to explain the galactic rotation curves without involving the dark matter [25, 26]. But unfortunately, the model may be inconsistent with the solar-system-scale observations, since the Schwarzschild–dS solutions, which play an important role in the explanation of the solar-system-scale observations, could not be smoothly connected to regular internal solutions, in the weak field approximation [26].
2.4. Example 2: a model of Poincaré gauge theory of gravity

For the Poincaré case, equation (37) gives the Stephenson–Kilmister–Yang action [27]

$$S_{GYM} = \kappa \int R_{abcd} R^{abcd},$$

(43)

where equation (33) has been used. The Einstein term, i.e., the scalar curvature term, is absent, and the resulting field equations are underdetermined at least in the weak field approximation [25]. In addition, it has been shown that there is no physical degree in the Stephenson–Kilmister–Yang action [28], for the number of constraints is greater than the number of degrees of freedom in Dirac’s prescription for constrained Hamiltonian systems.

In the following sections, we will construct several other viable models, by utilizing the configuration variables, i.e., the Ehresmann connection $\Omega_{\mu}^{A}$ and the global section $\phi$, in the manifestly gauge-invariant formalism.

3. Kaluza–Klein-type models

In this section, we will construct new Kaluza–Klein-type models of dS and Poincaré gauge theories of gravity with a fixed parameter $l$. Again, we will discuss the dS case first and turn to the Poincaré case later.

3.1. New Kaluza–Klein-type model of dS gauge theory of gravity

In the dS bundle $Q_{\text{dS}}$, we may locally define the following 1-form fields:

$$\theta^{A} = d\xi^{A} + \Omega^{A}_{\mu} \xi^{\mu} \, dx^{\mu},$$

(44)

which satisfy $\theta^{A}(E_{\mu}) = 0$. Note that the vector field $\tilde{\partial}_{A}$ and the fiber coordinates $\xi^{A}$ have their corresponding meanings in the vertical dS spacetime. In the dS spacetime, there exist local functions $\tilde{E}^{A}_{\mu}(\xi)$ and $E^{A}_{\mu}(\xi)$ such that $\{\tilde{E}^{A}_{\mu}(\xi) \, \tilde{\partial}_{A}\}$ is an orthonormal frame field with $\{E^{A}_{\mu}(\xi) \, d\xi^{A}\}$ as its dual frame field. In the dS bundle $Q_{\text{dS}}$, we may define $\tilde{E}^{A}_{\mu}(x, \xi) = \tilde{E}^{A}_{\mu}(\xi)$, $E^{A}_{\mu}(x, \xi) = E^{A}_{\mu}(\xi)$ in a special gauge first and then let them transform by

$$\tilde{E}^{A}_{\mu} \rightarrow \tilde{g}^{A}_{B} E^{B}_{\mu}, \quad E^{A}_{\mu} \rightarrow E^{A}_{B} (\tilde{g}^{-1})^{B}_{\mu},$$

(45)

under the local gauge transformation (22). Let

$$E_{\mu} = \tilde{E}^{A}_{\mu}(x, \xi) \tilde{\partial}_{A}, \quad E^{A}_{\mu} = E^{A}_{\mu}(x, \xi) \theta^{A}.$$  

(46)

Then, $\{E_{A}\} = \{E_{\mu}, E_{a}\}$ ($A = \mu, 4 + \alpha$) becomes a local frame field for $Q_{\text{dS}}$ with $\{E^{A}\} = \{d\sigma^{\nu}, E_{a}\}$ as its dual frame field. Moreover, we may define new functions on the bundle, $E^{A}_{\mu}$ and $\tilde{E}^{A}_{\mu}$, by

$$E^{A}_{\mu} = E_{a}(\xi^{A}), \quad \tilde{\partial}_{A} = \tilde{E}^{A}_{\mu} E_{\mu}.$$  

(47)

They will be used later.

The metric field for $Q_{\text{dS}}$ can be defined as follows:

$$\mathcal{G} = g_{\mu\nu} \, dx^{\mu} \otimes dx^{\nu} + \eta_{AB} \theta^{A} \otimes \theta^{B},$$

(48)

with its inverse

$$\mathcal{G}^{-1} = g^{\mu\nu} E_{\mu} \otimes E_{\nu} + \eta^{AB} \tilde{\partial}_{A} \otimes \tilde{\partial}_{B},$$

(49)

where $g_{\mu\nu}$ is the metric field of the spacetime manifold, with $g^{\mu\nu}$ as its inverse. Recall that there is a global section $\phi$ on the dS bundle, defined by $\{\sigma_{i}(x) \cdot \xi^{A}\}$ and locally represented by $\xi^{A}(x)$. In fact, by equations (12) and (44), the pullback

$$\phi^{*}(g_{\mu\nu} \, dx^{\mu} \otimes dx^{\nu}) = \phi^{*}(\eta_{AB} \theta^{A} \otimes \theta^{B})$$

(50)

is just the metric field of the spacetime.
The bundle torsion can be defined by the following way: the only nonzero components of S in \([E_A]\) can be calculated by the following formulas [2]:

\[
\Gamma^C_{AB} = \frac{1}{2} \mathcal{G}^{CD} \left[ E_A(\mathcal{G}_{BD}) + E_B(\mathcal{G}_{AD}) - E_D(\mathcal{G}_{AB}) \right] - \mathcal{K}^C_{AB} - \frac{1}{2} (C^C_{AB} + C_{AB}^C + C_{BA}^C),
\]

(51)

\[
\mathcal{R}^A_{BCD} = 2E_C(\Gamma^A_{[B|D]}) + 2\Gamma^A_{E[C} \mathcal{R}^{E}_{|D]} - \Gamma^A_{BD} C^E_{CD},
\]

(52)

where \(\mathcal{K}^C_{AB}\) is the contorsion tensor related to the torsion tensor \(\mathcal{S}^C_{AB}\) by

\[
\mathcal{K}^C_{AB} = \frac{1}{4} (\mathcal{S}^C_{AB} + \mathcal{S}_{AB}^C + \mathcal{S}_{BA}^C)
\]

(53)

and the structural coefficients \(C^C_{AB}\) are defined by

\[
[E_A, E_B] = C^C_{AB} E_C.
\]

(54)

After some calculations, the explicit expressions of \(C^C_{AB}\) can be attained as follows:

\[
C^\sigma_{\mu\nu} = 0, C^\alpha_{\mu\nu} = -\mathcal{F}^A_{\beta\mu\nu} \xi^B_{\alpha} \tilde{E}_A^\alpha,
\]

(55)

\[
C^\nu_{\mu\alpha} = 0, C^\beta_{\mu\alpha} = [E_\mu(\tilde{E}_A^\alpha) + \Omega^A_{\beta\mu} E_\alpha] \tilde{E}_A^\beta,
\]

(56)

\[
C^\mu_{\nu\beta} = 0
\]

(57)

and \(C^\nu_{\mu\alpha}\) is the same as the structural coefficients of the orthonormal frame field \([\tilde{E}_A^\alpha(\xi)\tilde{d}_A]\) in the dS spacetime. Components of the bundle metric field in \([E_A]\) are as follows:

\[
\mathcal{G}_{\mu\nu} = g_{\mu\nu}, \quad \mathcal{G}_{\mu\alpha} = 0, \quad \mathcal{G}_{\alpha\beta} = \eta_{\alpha\beta}.
\]

(58)

The bundle torsion can be defined by the following way: the only nonzero components of \(\mathcal{S}^C_{AB}\) are \(S^\sigma_{\mu\nu} = S^\alpha_{\mu\nu}\) [4]. Then, the connection coefficients have the following expressions:

\[
\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\mu\nu}, \quad \Gamma^\nu_{\alpha\beta} = \Gamma^\nu_{\alpha\beta},
\]

(59)

\[
\Gamma^\mu_{\alpha\nu} = \frac{1}{2} \mathcal{F}^A_{\beta\mu\nu} \xi^B_{\alpha} \tilde{E}_A^\alpha,
\]

(60)

\[
\Gamma^\nu_{\mu\alpha} = \Gamma^\mu_{\alpha\nu} = \frac{1}{2} \mathcal{F}^B_{\alpha\beta\nu} \xi^B_{\mu} E_\alpha^\beta
\]

(61)

\[
\Gamma^\mu_{\nu\beta} = 0, \quad \Gamma^\beta_{\mu\alpha} = 0, \quad \Gamma^\beta_{\alpha\mu} = -C^\beta_{\alpha\mu}
\]

(62)

where \(\Gamma^\nu_{\alpha\beta}\) is the connection coefficient of the dS metric in the orthonormal frame field \([\tilde{E}_A^\alpha(\xi)\tilde{d}_A]\). To obtain the above results, some of these formulas are useful:

\[
E^A_{E^B^A} = \delta^A_B, \quad \tilde{E}_A^\alpha \tilde{E}_B^\beta = \delta^\alpha_\beta,
\]

(63)

\[
E^A_{\tilde{E}^B} = \delta^A_B - \xi^A B/[B^2],
\]

(64)

\[
E^A_\xi = \tilde{E}^A_\xi - \xi^A \xi_\xi / [B^2],
\]

(65)

\[
C_{\mu\alpha\nu} = [E_\mu(\tilde{E}_A^\alpha) + \Omega^A_{\beta\mu} E_\alpha] \eta_{\beta\nu} E_\beta^C.
\]

(67)

Moreover, it can be shown that the contracted curvature components are as follows:

\[
\mathcal{R}^{\mu\nu}_{\mu\nu} = R_{\mu\nu} - \frac{3}{2} \mathcal{F}^{\mu\nu}_{AB} \mathcal{F}^A_{\mu\nu} \xi^B_\xi \xi^C_\xi,
\]

(68)

\[
\mathcal{R}^{\mu\beta}_{\alpha\beta} = R_{\alpha\beta},
\]

(69)

\[
\mathcal{R}^{\mu\alpha}_{\mu\alpha} = \frac{1}{2} \mathcal{F}^{\mu\alpha}_{AB} \mathcal{F}^A_{\mu\alpha} \xi^B_\xi \xi^C_\xi.
\]

(70)
where $R_M$ is the scalar curvature of the spacetime, $R_F$ is the scalar curvature of the typical fiber, i.e., the scalar curvature of a dS spacetime. Therefore, the scalar curvature of the dS bundle is

$$
R = \bar{R}^{uv}_{\mu\nu} + \bar{R}^{a\beta}_{\alpha\beta} + 2\bar{R}^{\mu\mu}_{\mu\mu} = R_M + R_F - (1/4)F_{A\mu}^{\alpha}F^{A}_{\mu\nu}\xi^{\mu}\xi^{\nu}.
$$

(71)

The gravitational Lagrangian may be chosen as

$$
\mathcal{L}_G = \chi \left( \phi^{-1}R - \frac{a}{\Lambda^2} \mathcal{L}_\phi \right),
$$

(72)

where

$$
\mathcal{L}_\phi = \frac{1}{2} (D_a \xi^\Lambda) D^a \xi^\Lambda + \frac{b}{\Lambda^2} \xi^\Lambda \xi^\Lambda = 2 + b,
$$

(73)

$\xi^\Lambda = \xi^\Lambda (x)$ is the local representation of $\phi$, $\chi$ is the gravitational coupling constant, and $a$ and $b$ are two new dimensionless coupling constants and $0 < a < \infty$. In order to guarantee that the dS space is one of the torsion-free vacuum solutions of the model, the following condition should hold:

$$
a = \frac{18}{2 + b}.
$$

(74)

By equations (21) and (71),

$$
\phi^{-1}R = R_M + R_F - \frac{1}{4}S_{abc}S^{abc},
$$

(75)

where $R_F = 4\Lambda = 12/\ell^2$. Therefore, with condition (74), the Lagrangian (72) is equal to

$$
\mathcal{L}_G = \chi (R_M - 2\Lambda - \frac{1}{4}S_{abc}S^{abc}).
$$

(76)

The gravitational action is the following integration:

$$
S_G = \int_U \mathcal{L}_G \epsilon,
$$

(77)

where $U$ is some spacetime region and $\epsilon$ is the metric-compatible volume form. This action is dS gauge invariant under the gauge transformation (22). The field equations for the theory given by equations (76), (77) and (23) are

$$
\frac{1}{2} \nabla_c S_{ab}^\epsilon + \frac{1}{4} S_{ac\tau} T_{d}^{\epsilon\tau} + G_{bc} + \frac{3}{\ell^2} g_{ab} = \frac{1}{2\chi} \Sigma_{ab} + \frac{1}{2} S_{cd\epsilon} S^{cd\epsilon} g_{ab},
$$

(78)

$$
T_{bc}^a + S_{[bc]}^a = - \frac{1}{\chi} \tau_{bc}^a.
$$

(79)

Comparison with the dS gravity model in section 2.3, the most important features of the theory are that the covariant derivative of the curvature does not enter equation (79) and that equation (79) is an algebraic equation, having the solution,

$$
S_{abc} = - \frac{1}{2\chi} (3\tau_{bca} - \tau_{cab} - \tau_{abc}) - \frac{8}{5\chi} g_{[ab\tau_c]},
$$

(80)

where $\tau_a = \tau_{a\epsilon}^\epsilon$. Therefore, for the gravity coupled to the matter without spin current, equation (78) reduces to the Einstein field equation with the same cosmological constant. It is remarkable that although the Lagrangian (76) falls into a special case of the general quadratic models of previous literature [29], it is deduced from the first principle here.
3.2. New Kaluza–Klein-type models of Poincaré gauge theory of gravity

For the Poincaré case, we do not need the concepts of $\tilde{\varepsilon}_\alpha^A$ and $\tilde{E}_\alpha^A$. Instead, we may define

$$E_\alpha = \partial_\alpha, \quad E^a = \theta^a$$  \hspace{1cm} (81)

in a special gauge and then define

$$E_\alpha^A = E_\alpha (\xi^A), \quad E^a_A = E^a (\tilde{\theta}_A)$$  \hspace{1cm} (82)

in an arbitrary gauge, where the Poincaré version of $\tilde{\theta}_A$ is given by equation (30). They satisfy the following properties:

$$\tilde{\theta}_A = E^a_A E_\alpha^a, \quad \eta_{\alpha\beta} E^A_\alpha \eta^{AB} E^a_B = E^a_B,$$  \hspace{1cm} (83)

$$E^a_A E^a_B = \delta^A_B, \quad E^a_A E^a_B = \begin{cases} \delta^A_B & A \neq 4 \\ 0 & A = 4 \end{cases}.$$  \hspace{1cm} (84)

For the structural coefficients, $C'_{\alpha\beta} = 0$, the second formula of equation (55) should be replaced by

$$C''_{\mu\nu} = -F_{B\mu\nu} \xi^B E^a_A,$$  \hspace{1cm} (85)

and the second formula of equation (56) should be replaced by

$$C''_{\mu\alpha} = \left[ E_{\mu} (E^A_\alpha) + \Omega^A_{B\mu} E^B_\nu E^a_A \right].$$  \hspace{1cm} (86)

In the special gauge with respect to equation (81), the first term on the right-hand side of the above equation is equal to zero. For the connection coefficients, equation (60) should be replaced by

$$\Gamma''^a_{\mu\nu} = \frac{1}{2} F_{B\mu\nu} \xi^B E^a_A.$$  \hspace{1cm} (87)

The gravitational Lagrangian (72) should be replaced by

$$L_G = \chi \phi^* \vec{R} = \chi (R_{\mathcal{M}} - (1/4) S_{abc} S^{abc}).$$  \hspace{1cm} (88)

Apparently, the Lagrangian and thus the field equations are different from the dS case (76) only by a cosmological term $\Lambda = 3/l^2$. Actually, they are Poincaré gauge invariant because $g_{\mu\nu}$ and $S_{abc}$ defined by equations (12) and (13) with the gauge group $G = ISO(1, 3)$ are now viewed as the Poincaré gauge invariants.

4. Kaluza–Klein-type models with a variable cosmological term

As was pointed out by [9], we may view $l$ in equation (1) as a positive function of the spacetime. Then, $\partial_\mu$ in equation (3) is no longer tangent to the dS bundle and should be replaced by

$$\tilde{\partial}_\mu = \partial_\mu + (\partial_\mu l)(\xi^A/l) \partial_A,$$  \hspace{1cm} (89)

and thus $E_{\mu}$ should be replaced by

$$\tilde{E}_\mu = \tilde{\partial}_\mu - \Omega^A_{B\mu} \xi^B \tilde{\theta}_A.$$  \hspace{1cm} (90)

Accordingly, the second formula of equation (4) for the horizontal lift of $\eta_{\mathcal{M}}(l)$ should be replaced by

$$\frac{d\xi^A}{dt} = v^\mu (l) [(\partial_\mu l)(\xi^A/l) - \Omega^A_{B\mu} \xi^B],$$  \hspace{1cm} (91)

and the gauge-covariant derivative (7) should be replaced by

$$\tilde{D}_\mu \xi^A(x) = [\partial_\mu \xi^A(x) - (\partial_\mu l)(\xi^A/l) + \Omega^A_{B\mu} \xi^B(x)] v^\mu.$$  \hspace{1cm} (92)
The metric field is given by equations (92), (12) and (14) with the replacement of $D$ by $\tilde{D}$, which is similar to but slightly different from equation (6.6) of [9] and has a different explanation. In addition, the gauge-invariant expression for the torsion, equation (13), should be replaced by

$$S_{\alpha\beta\gamma} = F_{\alpha\beta\mu}(\tilde{D}_\mu \xi^\alpha)\xi^\beta - 2(g_{\mu\nu}\partial_\mu l)/l,$$

(93)

but the gauge-invariant expression for the curvature tensor is still given by equation (20) with the replacement of $D$ by $\tilde{D}$. The curvature 2-form is still defined by equation (11), but in the local section $\sigma(x)$, $F_{\alpha\beta}$ in equation (21) should be replaced by

$$F_{\alpha\beta} = \left(\begin{array}{ccc} R^{\alpha\beta}_{\gamma\delta} & -l^{-2}e^\alpha_{\gamma\delta} & \quad l^{-1}S_{\alpha\beta} + 2l^{-2}e^\alpha_{\gamma\delta}\partial_\delta l \\ 0 & 0 & 0 \end{array}\right).$$

(94)

The gravitational equation will be given by equation (23) and

$$\delta S_T/\delta \xi^\alpha = 0.$$  

(95)

Equation (44) is now modified to be

$$\tilde{\theta}^\alpha = d\xi^\alpha - (\xi^\alpha/l)(\partial_\alpha l)\,dx^\mu + \Omega^B_{\beta\mu}\xi^\beta\,dx^\mu,$$

(96)

so that $\tilde{\theta}^\alpha(\tilde{E}_\alpha) = 0$. The new definitions with respect to $\tilde{E}_\alpha^A$ and $E^A_A$ are given as follows. In the dS spacetime with radius $l$, there exist local functions $\tilde{E}_\alpha^A(\xi, l)$ and $E^A_A(\xi, l)$ such that $\{\tilde{E}_\alpha^A(\xi, l)\partial_\alpha\}$ is an orthonormal frame field with $\{E^A_A(\xi, l)\,d\xi^A\}$ as its dual frame field. In the dS bundle $\tilde{Q}_{dS}$, we may define

$$\tilde{E}_\alpha^A(x, \xi) = \tilde{E}_\alpha^A(\xi, l(x)),$$

(97)

$$\tilde{E}^A_A(\xi, l) = E^A_A(\xi, l(x))$$

in a special gauge and then let them transform by equation (45) under the gauge transformation (22). The vertical orthonormal frame and dual frame fields are

$$\tilde{E}_\alpha = \tilde{E}_\alpha^A(\xi, \xi)\tilde{\partial}_\alpha,$$

$$\tilde{E}^A_A = \tilde{E}^A_A(\xi, \xi)\tilde{\theta}^A.$$

(98)

The structural coefficients are now defined by

$$[\tilde{E}_A, \tilde{E}_B] = \tilde{C}^C_{AB}\tilde{E}_C$$

(99)

with $\tilde{E}_A = \{\tilde{E}_\mu, \tilde{E}_\alpha\}$. A straightforward calculation shows that the second formula of equations (56) and (67) should be replaced by

$$\tilde{C}^\beta_{\mu\alpha} = [\tilde{E}_\mu(\tilde{E}_\alpha^A) - (\partial_\mu l)(1/l)\tilde{E}_\alpha^A + \Omega^B_{\beta\mu}\tilde{E}_\alpha^B] \tilde{E}^\beta_A$$

(100)

and

$$\tilde{C}_{\beta\mu\alpha} = [\tilde{E}_\mu(\tilde{E}_\alpha^A) - (\partial_\mu l)(1/l)\tilde{E}_\alpha^A + \Omega^B_{\beta\mu}\tilde{E}_\alpha^B] \eta_{\alpha\beta}\tilde{E}^\nu_C,$$

(101)

respectively, with $\tilde{E}_\alpha^A = \tilde{E}_\alpha^A(\xi, l)$ and $\tilde{E}^A_A$ defined by $\tilde{\partial}_A = \tilde{E}^\alpha_A\tilde{E}_\alpha$. The connection coefficients, defined by the formula similar to equation (51) with the replacement of the symbols with $\tilde{\cdot}$, take the similar results except equation (62) which should be replaced by

$$\tilde{\Gamma}^\mu_{\nu\beta} = -\tilde{\Gamma}^\mu_{\beta\nu} = (\partial_\nu l)(1/l)\eta_{\mu\beta},$$

(102)

For the contracted curvature components, equations (69) and (70) should be replaced by

$$\tilde{R}^{\alpha\beta}_{\mu\nu} = R_{\alpha\beta} - (12/l^2)(\nabla_\alpha l)\nabla^\beta l$$

(103)

and

$$\tilde{R}^{\alpha\beta}_{\mu\nu} = \frac{1}{4} F_{B\mu}^{\alpha\nu} F_{C\lambda\nu} \xi^B \xi^C - (4/l^2)(\nabla_\alpha l)\nabla^\beta l - \tilde{\nabla}^\alpha [4/(l^2)\nabla_\beta l] + S_{\alpha\beta}^\mu (4/l)\nabla_\mu l,$$

(104)
where $\hat{\nabla}_a$ is the torsion-free derivative operator compatible with $g_{ab}$. As a result, the scalar curvature (71) should be modified to be

$$\hat{R} = R_M + R_F - (1/4)F_{AB}^{\mu\nu}F_{CA}^{\rho\sigma}g_{\rho\sigma}g^{BC} - (20/1^2)(\nabla_a l)\nabla^a l + (8/1)S^{ab}a \nabla b l - 2\hat{\nabla}^a [ (4/1)\nabla a l].$$  \hspace{1cm} (105)

The gravitational Lagrangian is chosen to be the same as equation (72), with $\hat{R}$ replaced by $\hat{R}$, and $l$ replaced by $l_0$, where $l_0 = \text{const}$ is a fixed value of $l$-function at some $x$ or a limit value of $l$-function on the spacetime manifold. By equations (94) and (105),

$$\phi^* \hat{R} = R_M + R_F - (1/4)S_{abc}S^{abc} - (43/2l^2)(\nabla a l)\nabla^a l + (9/1)S^{ab}a \nabla b l - 2\hat{\nabla}^a [ (4/1)\nabla a l],$$

(106)

where $R_F = 4A = 12/l^2$ and the last term only contributes to a boundary term. In the case with $l = l_0$, equation (106) will come back to equation (75). In order to guarantee that the dS space with radius $l_0$ is one of the torsion-free vacuum solutions of the model in the case with $l = l_0$ and $\Lambda_0 = 3/l_0^2$, the following conditions should hold:

$$\frac{a}{l_0^2} (2 + b) = 6\Lambda_0, \quad ab = -12,$$  \hspace{1cm} (107)

which result in

$$a = 15, \quad b = -\frac{4}{5},$$  \hspace{1cm} (108)

$$\mathcal{L}_G = \frac{1}{2} (D_a \xi A \nabla^a \xi A - \frac{4}{2} l_0^{-2} \xi A = 2 - (4/5)(l/l_0)^{-2},$$  \hspace{1cm} (109)

$$\mathcal{L}_G = (\phi^* \hat{R} - 5\Lambda_0) \mathcal{L}_\phi) = \chi\left( R_M + 6l^{-2}(2 - 5(l/l_0)^2 + 2(l/l_0)^4) - \frac{1}{4}S_{abc}S^{abc} - (43/2l^2)(\nabla a l)\nabla^a l + (9/1)S^{ab}a \nabla b l - 2\hat{\nabla}^a [ (4/1)\nabla a l].

(110)

The field equations for the theory are

$$\frac{1}{2} \nabla c S_a^{\cdots c} + \frac{1}{4} S_a^{\cdots d} T_{bcd} + G_{ba} - M^{-2}(2 - 5(l/l_0)^2 + 2(l/l_0)^4)g_{ab}$$

$$= \frac{1}{2X} S_{ab} + \frac{1}{2} \left( S^{\cdots cd} S_{ab} - \frac{1}{4} S^{\cdots de} S_{de} g_{ab} \right)$$

$$+ \frac{9}{2l^2} \left[ \nabla c \nabla d - g_{ab} \nabla c \nabla d \right] S_{bc}$$

$$+ \frac{1}{2l^2} \left[ 34 (\nabla c l) (\nabla d l) - \frac{25}{2} g_{ab} (\nabla c l) (\nabla d l) \right].$$

(111)

$$S_{[bc]}^a + T^a_{bc} = -\frac{1}{X} \tau_{bc}^a + \frac{9}{l} S^a_{[bc]} \nabla c l,$$

(112)

$$\frac{43}{l} [\hat{\nabla}_a \nabla^a l - l^{-4} (\nabla a l) (\nabla^a l)] + 24l^2 (l_0^{-4} - l^{-4}) = 9\hat{\nabla}_b S^b_{c a}.$$  \hspace{1cm} (113)

The last line of equation (111) can be regarded as the stress–energy tensor for the scalar field $l$. Again, equation (112) is an algebraic equation and has the solution

$$S_{abc} = -\frac{1}{2X} (3\tau_{bca} - \tau_{cab} - \tau_{abc}) - \frac{8}{5X} g_{a[b} \tau_{c]} - \frac{18}{5l} g_{a[b} \nabla c l].$$

(114)
It means that the variable $l$ also serves as the source of torsion. Therefore, among the following three conditions, if two are satisfied, then the third one must be also satisfied, and the gravity reduces to GR with the cosmological constant $\Lambda_0$:

\begin{align}
S_{abc} &= 0, \\
\tau_{bca} &= 0, \\
l &= l_0.
\end{align}

For the Poincaré case, equation (94) should be replaced by

\[ F_{A Bab} = \left( R^\alpha_{\beta ab} l^{-1} S^\alpha_{ab} + 2l^{-2} e^\alpha a [a b] l \right). \tag{118} \]

We do not need the concepts of $\tilde{\tilde{E}}_A^\alpha$ and $\tilde{E}_A^\alpha$. Instead, we may define $\tilde{E}_A^\alpha$ and $\tilde{E}_A^\alpha$ by equations (81) and (82) with the replacement of the symbols with $\tilde{\tilde{E}}$. Equation (100) should be replaced by

\[ \tilde{C}_{\mu a} = [\tilde{E}_\mu (\tilde{E}_a^A) - (\partial_\mu l)(1/l)\tilde{E}_a^A + \Omega^A_{B\mu} \tilde{E}_a^B] \tilde{E}_A^\beta \tag{119} \]

and equation (110) should be replaced by

\[ \mathcal{L}_G = \chi \phi^2 \tilde{R}, \tag{120} \]

where $\phi^2 \tilde{R}$ is given by equation (106) with $R_F = 0$. The second field equation is identical to equation (112), and the other two equations are now

\begin{align}
\frac{1}{2} \nabla_c S_{ab}^c + \frac{1}{4} S_{a}^d T_{bcd} + G_{ba} &= \frac{1}{2 \chi} \Sigma_{ab} + \frac{1}{2} \left( S_{a}^d S_{c}^d - \frac{1}{4} S_{abc} S_{abc} g_{ab} \right) \\
&+ \frac{9}{2l} \left( \nabla_a \nabla_b l - g_{ab} \nabla_c \nabla^c l - (\nabla_a l) \nabla^a l \right) \\
&+ \frac{1}{2l^2} \left[ 34 (\nabla_a l) \nabla_b l - \frac{25}{2} g_{ab} (\nabla_a l) (\nabla^a l) \right], \tag{121} \\
l \tilde{\nabla}_a \nabla^a l - (\nabla_a l) (\nabla^a l) &= \frac{9l^2}{43} \tilde{\nabla}_a S_{ab}^a. \tag{122} 
\end{align}

The local Poincaré gauge-invariance model can reduce to GR under the same conditions as the above local dS gauge-invariance model, with $l = l_0$ replaced by $l =$ const. An important difference is that the Minkowski space is a vacuum solution of the model in those cases.

5. Remarks

In these new Kaluza–Klein-type models, we use the Riemann–Cartan scalar curvature of the associated bundle and a global section to construct the gravitational dynamics. The action is the integration of the Lagrangian over some spacetime region as usual. A suitable cosmological term can be obtained, so that the dS or Minkowski space as the typical fiber is one of the vacuum solutions of the theory. In the spin-current-free and torsion-free limit, the models reduce to GR with the same cosmological term. It should be mentioned that these models are different from the original Kaluza–Klein-type models [1, 3] of gauge theory of gravity, which use the Riemann or Riemann–Cartan scalar curvature of a principal bundle to serve as the gravitational Lagrangian and use the integration of the Lagrangian over the principal bundle to serve as the action. In addition, the old models cannot reduce to GR in the spin-current-free and torsion-free limit and have not provided a rule to obtain a suitable cosmological term.
It should be emphasized that both dS and Poincaré gauge theories of gravity presented in this paper are manifestly gauge invariant. In the formalism, the configuration variables are the Ehresmann connection $\Omega^A_{Ba}$ and vector-valued scalar field $\xi^A$ which are covariant under the gauge transformations (22). The geometric variables, such as metric $g_{ab}$ and torsion $S'_{ab}$, are expressed as the functions of $\Omega^A_{Ba}$ and $\xi^A$ and are invariant under the gauge transformations. The actions are the functionals of $\Omega^A_{Ba}$ and $\xi^A$, which are invariant under the gauge transformations. The manifestly gauge-invariant formalism is motivated by the principle of localization [7, 30], which states that gravity should be based on the localization of the full symmetry of the corresponding special relativity (SR) as well as dynamics. It should be noted that the manifestly gauge-invariant formalism may also be applied to SR and GR. For example, in Einstein’s SR, the flat metric field can be expressed in a local Poincaré invariant form via equations (7), (12) and (31), while the Ehresmann connection is constrained by the flat condition $F^A_{Bab} = 0$, and the vector-valued scalar field is constrained by the condition $\xi^4 = \text{const} \neq 0$. In GR, the metric field can be expressed in a local Poincaré gauge-invariant form via the above-mentioned equations, while the Ehresmann connection and the vector-valued scalar field are constrained by the torsion-free condition $F^A_{Bab}(Dc)\xi^B = 0$ and the condition $\xi^4 = \text{const} \neq 0$. In contrast, in the Poincaré gauge theory of gravity discussed in section 3 of this paper, the Ehresmann connection is completely determined by the variation principle, while the vector-valued scalar field is still constrained by the condition $\xi^4 = \text{const} \neq 0$. In the gauge with $\xi^A = (0, 0, 0, 0, l)^T$, it is easy to see that the actions which are functionals of $\Omega^A_{Ba}$ and $\xi^A$ can also be viewed as the functionals of $\Gamma^a_{\beta\alpha}$ and $e^a$ which can be identified with the nonlinear connection in the nonlinear realization [12]. The formalism with the nonlinear connection as variable emphasizes on the Lorentz invariance of the theory, which confirms the remaining symmetry in the gauge with $\xi^A = (0, 0, 0, 0, l)^T$.

The actions for both dS and Poincaré gauge theories of gravity can be written as the functionals of the metric and torsion fields. It is remarkable that once the actions in different gauge theories of gravity are written in these forms, it is difficult to say that they are dS gauge invariant or Poincaré gauge invariant or even only Lorentz gauge invariant. The information of different gauge groups has been hidden in the concrete expressions of the metric and torsion fields in terms of $\Omega^A_{Ba}$ and $\xi^A$. What can be confirmed is that such actions are diffeomorphism invariant and Lorentz invariant through the tetrad formalism. One may try to determine the gauge group by observing whether the dS spacetime or Minkowski spacetime is one of the vacuum solutions of the corresponding theory. It is interesting to see whether there are any observational effects to distinguish the internal symmetries.

We have also considered the case where the Ehresmann connection and the vector-valued scalar field are completely determined by the variation principle. As a result, the global section becomes dynamic, and a variable cosmological term appears in the Kaluza–Klein-type Lagrangian. Actually, many variable cosmological constant models may solve the coincidence problem of the cosmological constant and numerous works have been done to search for the theoretical foundation of such models [14]. If our models turn out to be able to consistently explain the observational data of the universe while solving the coincidence problem, they would serve as elegant explanations for the accelerating expanding universe.

Although only the dS and Poincaré cases have been discussed in this paper, it is an easy thing to modify the results of the dS case to obtain the corresponding results of the AdS case.

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Appendix. Ehresmann connection

Consider a generic principal fiber bundle $P(M, G)$ and its tangent bundle $TP$. $TP$ may be decomposed as

$$TP = V \oplus H,$$

(A.1)

where $V$ and $H$ are the smooth vertical and horizontal subbundles, respectively, and $\oplus$ is a direct sum. The vertical subbundle is defined canonically by the fundamental vector fields, which are generated by elements in the Lie algebra $g$ of the bundle $P(M, G)$ and are tangent to $G_p$ at every point $p \in P$. The horizontal subbundle is an Ehresmann connection in terms of distributions [31].

If and only if,

$$H_{pg} = d(R_g)_p(H_p), \quad p \in P, \ g \in G,$$

(A.2)

the Ehresmann connection defines a Lie algebra valued 1-form $\nu \in g \otimes \Omega^1(P)$:

$$\nu : T_P \rightarrow V \simeq g, \quad p \in P$$

(A.3)

such that $H = \ker(\nu)$,

$$\nu(\xi) = \xi, \quad \xi \in g;$$

(A.4)

$$R_g^*\nu = \text{ad}_g^{-1}\nu = g^{-1}\nu g, \quad g \in G,$$

(A.5)

where $R_g$ denotes right multiplication by $g$, $\xi$ is the fundamental vector field on $P$ associated with $\xi$ by differentiating the $G$ action on $P$, and $\text{ad}_g(X) := \frac{d}{dt}g \exp(tX)g^{-1}|_{t=0}$ is the adjoint action.

Locally, on a given coordinate patch $U_i$ of $M$ and for a given local sections $\sigma_i$, the pullback $\sigma_i^*$ and $\nu$ may define a local connection $A_i \in g \otimes \Omega^1(U_i)$ by

$$A_i \equiv \sigma_i^* \nu.$$  

(A.6)

It is the local form of the Ehresmann connection $\nu$ and may be identified with the gauge potential up to some Lie algebra factor. Remember that $\nu$ is globally defined. In the intersection $U_i \cap U_j$ of two coordinate patches $U_i$ and $U_j$, the local connection should transform as

$$A_j = \Lambda_{ji} A_i \Lambda_{ji}^{-1} + \Lambda_{ji} d(\Lambda_{ji}^{-1}),$$

(A.7)

where $\Lambda_{ij}$ is the transition function from $U_i$ to $U_j$. (The repeated indices are not summed up here.) Similarly, for the local sections $\sigma_i$ and $\tilde{\sigma}_i$ in two given gauges, where $\tilde{\sigma}_i = \sigma_i(g_i)^{-1}$ with $g_i \in G$, the local connections $A_i$ and $\tilde{A}_i$ for the two sections transform as

$$\tilde{A}_i = g_i A_i g_i^{-1} + g_i d g_i^{-1},$$

(A.8)

where $i$ is not summed up.

Expanding $A$ in terms of Lie algebra generators $\tau_a$ and the dual coordinate bases $dx^\mu$, we have

$$A = A^a \tau_a = A^a_{\mu} \tau_a dx^\mu.$$  

(A.9)
where the index for the $i$th coordinate patch has been omitted. When a matrix representation of the Lie algebra $g$ is used, we have the matrix form of $A$:

$$
A^A_B = A^A_{B\mu} \, dx^\mu.
$$

(A.10)

Then, equation (A.8) can be written as

$$
\tilde{A}^A_B = g^{AB} C^C_D (g^{-1})^B_D + g^{AC} d (g^{-1})^C_B.
$$

(A.11)

In the notation of [16], $A^A_B$ is also written as $A^A_{B\nu}$. The connection $\Omega^\mu_{AB}$ in equation (3) is defined in the same way as $A^A_{B\mu}$. We do not use the notation $\Omega^\mu_{1AB}$ in this appendix for the reason that it is similar to that of the 1-form space $\Omega^1$, which appears in the line above equation (A.3).

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