DUALITY AND NORMALIZATION,
VARIATIONS ON A THEME OF SERRE AND REID

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WITH AN APPENDIX BY HAILONG DAO

Abstract. We discuss the naive duality theory of coherent, torsion free, \( S_2 \)
sheaves on schemes.

On a normal, algebraic variety the most important coherent sheaf is its canonical sheaf, also called dualizing sheaf. One can define a dualizing sheaf on more general schemes, but frequently the definitions show neither that the object is canonical nor that it has anything to do with duality; see for example [BH93 3.3.1] or [Sta15, Tag 0A7B]. I started to contemplate this while re-reading [Ser59, Chap.IV] and [Rei94 Secs.2–3]. These notes are the result of a subsequent attempt to generalize the naive duality of reflexive sheaves from normal varieties to schemes. As the reader will see, the essential ideas are in the works of my predecessors, especially [Ser59, Rei94, Har07]. However, I hope that the formulation and the generality of some of the results may be new and of interest.

On a normal scheme \( X \) the reflexive hull of a coherent sheaf \( F \) is given by the formula

\[ F^{**} := \text{Hom}_X (\text{Hom}_X (F, \mathcal{O}_X), \mathcal{O}_X). \]

While this definition makes sense over any integral scheme (see [Sta15 Tag 0AUY] and [Sta15 Tag 0AVT]), it does not seem to have many of the good properties of the normal case.

My claim is that, on non-normal schemes, the correct analogs of reflexive sheaves and reflexive hulls are

- torsion free, \( S_2 \) sheaves, abbreviated as \( \text{TfS}_2 \) sheaves, and
- torsion free, \( S_2 \)-hulls, abbreviated as \( \text{TfS}_2 \)-hulls.

(I call a coherent sheaf \( F \) torsion free if every associated point of \( F \) is a generic point of \( X \). See [13] for the precise definition of the \( \text{TfS}_2 \)-hull.)

Our first result says that the theory of torsion free and \( S_2 \) sheaves on noetherian schemes can be reduced to the study of \( S_2 \) schemes.

**Theorem 1.** Let \( X \) be a noetherian scheme. Then there is a unique noetherian, \( S_2 \) scheme \( X^H \) and a finite morphism \( \pi : X^H \to X \) such that \( F \mapsto \pi_* F \) establishes an equivalence between the categories of coherent, torsion free, \( S_2 \) sheaves on \( X^H \) and coherent, torsion free, \( S_2 \) sheaves on \( X \).

This \( X^H \) is called the **torsion free, \( S_2 \)-hull** or \( \text{TfS}_2 \)-hull of \( X \).

Note that \( \pi : X^H \to X \) need not be surjective. For example, if \( X \) is of finite type over a field then \( \pi \) is birational if \( X \) is pure dimensional. Otherwise \( \pi \) maps birationally onto the union of those irreducible components of \( X \) that do not intersect any larger dimensional irreducible component. The same holds for excellent
schemes. There are, however, 2-dimensional, noetherian, integral schemes \( X \) where the sole coherent, torsion free, \( S_2 \) sheaf is the zero sheaf; see (45.2). For these \( X^H = \emptyset \).

In general, the most useful dualizing object on a scheme is Grothendieck’s dualizing complex \([Sta15, Tag 0A7B]\). However, the existence of a dualizing complex is a difficult question in general and for our purposes it is an overkill. If \( X \) has a dualizing complex \( \omega_X \) then \( \tilde{\omega}_X := H^{-\dim X}(\omega_X^*) \) is a \( T\!F\!S_2 \)-dualizing sheaf, but it turns out that if one aims to get duality only for torsion free, \( S_2 \) sheaves, then the required “dualizing sheaf” exists in greater generality. The following is a special case, a necessary and sufficient condition is given in (43).

**Theorem 2.** Let \( X \) be a noetherian, \( S_2 \) scheme such that the normalization of its underlying reduced scheme \( \pi: \bar{X} \to \text{red } X \) is finite. Then there is a coherent, torsion free, \( S_2 \) sheaf \( \tilde{\omega}_X \) such that the torsion free, \( S_2 \)-hull (13) of any coherent sheaf is given by

\[
F^H := \mathcal{H}om_X(\mathcal{H}om_X(F,\tilde{\omega}_X),\tilde{\omega}_X).
\]

Such an \( \tilde{\omega}_X \) is called a \( T\!F\!S_2 \)-dualizing sheaf on \( X \).

In particular, if \( F \) itself is torsion free and \( S_2 \) then

\[
F = \mathcal{H}om_X(\mathcal{H}om_X(F,\tilde{\omega}_X),\tilde{\omega}_X).
\]

The finiteness of \( \pi: \bar{X} \to \text{red } X \) is called condition N-1; this is the minimal assumption needed for Theorems 3 and 4 to make sense. If \( X \) is excellent, or universally Japanese, or Nagata then this condition holds; see \([Sta15, Tag 0BI1]\) and \([Sta15, Tag 033R]\) for the definitions and their basic properties.

In the literature, duality is usually stated either in the derived category of coherent sheaves as in \([Sta15, Tag 0A7C]\), or for maximal CM modules over CM rings as in \([Bh93, 3.3.10]\) and \([Eis95, Sec.2.19]\).

After establishing these results, we revisit the “\( n_Q = 2d_Q \)” theorem of \([Ser59, Sec.IV.11]\) (who credits earlier works of Severi, Kodaira, Samuel and Gorenstein) and \([Rei94, Sec.3]\) (who credits Serre). Although our statements are more general than the usual forms, the gist of the proof is classical.

**Theorem 3.** Let \( X \) be a noetherian, reduced, \( S_2 \) scheme. Assume that the normalization \( \pi: \bar{X} \to X \) is finite with conductors \( D \subset X \) and \( \bar{D} \subset X \) \( (46) \). Then

\begin{enumerate}
  \item \( \pi_*[\bar{D}] \geq [D] \) and
  \item equality holds iff the semilocal ring \( O_{D,X} \) is Gorenstein.
\end{enumerate}

We also give a characterization of seminormal schemes.

**Theorem 4.** Let \( X \) be a noetherian, reduced, \( S_2 \) scheme whose normalization \( \pi: \bar{X} \to X \) is finite with conductors \( D \subset X \) and \( \bar{D} \subset X \). Let \( \tilde{\omega}_X \) be a \( T\!F\!S_2 \)-dualizing sheaf on \( X \) and \( \hat{\omega}_X := \pi^!\omega_X = \mathcal{H}om_X(\pi_*O_{\bar{X}},\tilde{\omega}_X) \) the corresponding \( T\!F\!S_2 \)-dualizing sheaf on \( \bar{X} \) \( (48) \). The following are equivalent.

\begin{enumerate}
  \item \( X \) is seminormal.
  \item \( \bar{D} \) is reduced.
  \item \( \tilde{\omega}_X \subset \pi_*\hat{\omega}_X(\text{red } \bar{D}) \).
\end{enumerate}

In both of these results, the key step is to understand the dualizing sheaf of a 1-dimensional scheme. As a byproduct, we get the following in \( (63) \).
Proposition 5. Let $X$ be a noetherian, 1-dimensional, $S_1$ scheme. Then $X$ has a dualizing sheaf iff the following hold.

1. The local ring $\mathcal{O}_{x,X}$ has a dualizing module for every point $x \in X$.
2. There is an open and dense subset $U \subset X$ such that $\text{red } U$ is Gorenstein.

The standard definition of a dualizing module $\omega_R$ over a CM local ring $(R, m)$ (see, for instance, [BH93, p.107]) requires the vanishing of $\text{Ext}^i_R(R/m, \omega_R)$ for all $i \neq \dim R$. In Section 6 we discuss how to get by if we know vanishing only for $i < \dim R$, or even without any vanishing. See (66) and (67) for the complete statements.

6 (CM-dualizing sheaf). Let $X$ be a CM scheme. If $M$ is a torsion free CM sheaf then so is $\text{Hom}_X(M, \omega_X)$, see for example [BH93, 3.3.10]. The Appendix by Hailong Dao shows that if $\dim X \geq 3$ then $\omega_X$ is essentially the only coherent sheaf with this property.

This is also related to a question posed by Hochster in a lecture in 1972 whether the set $\{ L \in \text{Cl}(X) : L \text{ is CM} \}$ finite?

For cones this is proved in [Kar09, Thm.6.11]; we outline a more geometric argument in [73]. The case of 3-dimensional isolated hypersurface singularities is settled in [DK16, Cor.4.8]; various special examples were treated earlier by [Knö87] and [EP03].

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Assumptions. Unless otherwise specified, from now on all schemes and rings are assumed to be noetherian.

1. Torsion free, $S_2$ sheaves

We recall some well-known definitions and results first, see [Kol13, 2.58–63] or [Sta15, Tag 033P] for details.

Definition 7. A coherent sheaf $F$ satisfies Serre’s property $S_1$ if the following equivalent conditions hold.

1. $\text{depth}_x F_x \geq \min \{1, \text{codim}(x \in \text{Supp } F)\}$ for every $x \in X$.
2. $F$ has no embedded associated primes.
3. If $\text{codim}(x \in \text{Supp } F) \geq 1$ then $H^0_x(F_x) = H^0_x(F'_x) = 0$.

Definition 8. A coherent sheaf $F$ satisfies Serre’s property $S_2$ if the following equivalent conditions hold.

1. $\text{depth}_x F_x \geq \min \{2, \text{codim}(x \in \text{Supp } F)\}$ for every $x \in X$.
2. $F$ is $S_1$ and $F|_D$ is also $S_1$ whenever $D \subset U \subset X$ is a Cartier divisor in an open subset $U$ and $D$ does not contain any associated prime of $F$.
3. If $\text{codim}(x \in \text{Supp } F) \geq 1$ then $H^0_x(F_x) = 0$ and if $\text{codim}(x \in \text{Supp } F) \geq 2$ then $H^1_x(F_x) = 0$.
4. An exact sequence $0 \to F \to F' \to Q \to 0$ splits whenever $\text{Supp } Q \subset \text{Supp } F$ has codimension $\geq 2$.
5. $F$ is $S_1$ and for every open $U \subset X$ and every closed $Z \subset U$ of codimension $\geq 2$, the restriction map $H^0(U, F|_U) \to H^0(U \setminus Z, F|_{U \setminus Z})$ is an isomorphism.
The following is easiest to prove using (8.3).

**Lemma 9.** Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be an exact sequence of coherent sheaves.

1. If $F_1, F_3$ are $S_2$, so is $F_2$.
2. Assume that $F_2$ is $S_2$ and if $Z \subseteq \text{Supp} F_1$ has codimension $\geq 2$ in $\text{Supp} F_1$ the $Z$ is not an associated point of $F_3$. Then $F_1$ is $S_2$. \hfill \Box

We need the following form of Grothendieck’s dévissage, see for instance [Sta15 Tag 01YC] or [Kol17b, Sec.10.3].

**Lemma 10.** Let $F$ be a coherent sheaf with associated subschemes $X_j \subset X$. Then $F$ has an increasing filtration $0 = F_0 \subset F_1 \subset \cdots$ such that each $F_{i+1}/F_i$ is a coherent, torsion free, rank 1 sheaf over some $X_j$. Moreover, we can choose $\text{Supp}(F_1/F_0)$ arbitrarily and if $F$ is $S_2$ then the $F_{i+1}/F_i$ are also $S_2$. \hfill \Box

**Definition 11** (Torsion free sheaves). A coherent sheaf $T$ on $X$ is called a torsion sheaf if $T$ is nowhere dense in $X$. Every coherent sheaf $F$ has a largest torsion subsheaf, denoted by tors($F$). $F$ is called torsion free if tors($F$) $= 0$, equivalently, if every associated point of $F$ is a generic point of $X$.

*Warning on terminology.* Many authors define torsion and torsion free only over integral schemes, see for example [Sta15 Tag 0549] and [Sta15 Tag 0AVR].

**Definition 12** ($T^hS_2$ sheaves). We will be especially interested in coherent sheaves that are torsion free and $S_2$, abbreviated as $T^hS_2$. Note that if $X$ is normal, these are exactly the reflexive sheaves on $X$. We claim that, on non-normal schemes, $T^hS_2$ sheaves are the correct analogs of reflexive sheaves.

**Definition 13** (Torsion free $S_2$-hull). Let $F$ be a coherent sheaf on $X$. Its torsion free $S_2$-hull or $T^hS_2$-hull, $F^h$ is a coherent sheaf $F^h$ with map $q : F \to F^h$ such that

1. $F^h$ is $T^hS_2$,
2. $\ker q = \text{tors}(F)$,
3. $\text{Supp} F^h = \text{Supp}(F/\text{tors}(F))$ and
4. $\text{codim}_X \text{Supp}(F^h/F) \geq 2$.

It is clear that a torsion free $S_2$-hull is unique and its existence is a local question on $X$. Furthermore, $F^h = (F/\text{tors}(F))^H$.

*Warning on terminology.* If $X$ is pure dimensional and $F$ is torsion free, then this agrees with every notion of hull that I know of. If $F$ is torsion, then by our definition $F^h = 0$.

Another notion of $S_2$-hull is defined in [Kol08, Kol17a] where $T^hS_2$ in (1) is replaced by $S_2$ and $\text{tors}(F)$ in (2–3) is replaced by $\text{Emb}(F)$.

I could not think of a better notation for the torsion free $S_2$-hull then $F^h$, though this is the same as used in [Kol17a, Kol17b]. $F^h$ looks like the Henselization, $F^{**}$ or $F^{[*]}$ like the reflexive hull and $F^{hS_2}$ is way too cumbersome.

**Example 14.** The following examples show that in some cases there are fewer coherent, $S_2$ sheaves than one might expect.

1. Let $X \subset \mathbb{A}^3$ be the union of a plane $P$ and a line $L$ meeting at a point $p$. Let $G$ be a nonzero, coherent, $S_2$ sheaf on $X$. Then $\text{Supp} G$ is the union of $P$ and of finitely many points in $L \setminus \{p\}$.

   Indeed, let $G_L \subset G$ be the largest subsheaf supported on $L$. If $G$ is $S_2$ then $H^1_p(G) = 0$ hence $H^0_p(G_L) = 0$. But $p$ has codimension 1 in $L$, so $H^1_p(G_L) = H^0(L \setminus \{p\}, G_L|_{L \setminus \{p\}})/H^0(L, G_L)$.
is infinite, unless \( p \not\in \text{Supp} \mathcal{G}_L \). In particular, there is no coherent \( S_2 \) or \( TFS_2 \) sheaf on \( X \) whose support equals \( X \).

(14.2) Let \( \pi : \bar{X} \to X \) denote the normalization of the previous example. Then \( \mathcal{O}_{\bar{X}} \) is \( S_2 \) and \( TFS_2 \) but \( \pi_* \mathcal{O}_{\bar{X}} \) is neither.

(14.3) Set \( K := k(x_i : i \in \mathbb{N}) \). In \( \mathbb{A}^2_K \) with coordinates \( y_1, y_2 \) let \( p_1 \) be the point \((1,1)\) and \( p_2 \) the generic point of \((y_2 = 0)\). Then \( k(p_1) \cong K = k(x_i : i \in \mathbb{N}) \) and \( k(p_2) \cong \mathbb{K}(y_1) = k(y_1, x_i : i \in \mathbb{N}) \). Choose an isomorphism \( \phi : k(p_1) \cong k(p_2) \) and set

\[
R := \{ f \in K[y_1, y_2] : \phi(f(p_1)) = f(p_2) \}.
\]

Set \( X := \text{Spec}_k R \). Then \( \mathbb{A}^2_K \) is the normalization of \( X \), the normalization map \( \mathbb{A}^2_K \to X \) is finite but there is no coherent \( S_2 \) or \( TFS_2 \) sheaf on \( X \) whose support equals \( X \). Note that \( \phi \) is not \( K \)-linear, so \( X \) is not a \( K \)-scheme, not even of finite type over any field.

15 (Construction of \( TFS_2 \)-hulls). There are several well-known constructions of hulls.

(15.1) Let \( F \) be a coherent sheaf on \( X \). Assume that there is a dense, open subset \( j : U \subset X \) such that \( F|_U \) is \( TFS_2 \) and \( \text{codim}_X(X \setminus U) \geq 2 \). Then \( F \) has a \( TFS_2 \)-hull iff \( j_*(F|_U) \) is coherent, and then it is the \( TFS_2 \)-hull of \( F \). This follows from (14.4).

(15.2) Let \( G \) be a coherent \( TFS_2 \) sheaf on \( X \) and \( F \subset G \) a coherent subsheaf. Then \( F \) has a \( TFS_2 \)-hull and, using (15.2), it can be constructed as follows.

Let \( T \subset F/G \) denote the largest subsheaf whose support has codimension \( \geq 2 \) in \( X \). Then \( F^H \) is the preimage of \( T \) in \( G \).

(15.3) Let \( F \) be a coherent sheaf on \( X \). Assume that there is a dense, open subset \( U \subset X \) such that \( F|_U \) is \( TFS_2 \). Then \( F^H \) can be constructed as follows.

As we noted in (14.3), we may assume that \( X \) is affine. Let \( g \) be an equation of \( X \setminus U \). Let \( W \subset X \setminus U \) denote the closure of the union of those associated points of \( F/gF \) that have codimension \( \geq 2 \) in \( X \). Let \( j : X \setminus W \to X \) be the open embedding. Then \( F|_{X \setminus W} \) is \( TFS_2 \) and \( F \) has a \( TFS_2 \)-hull iff \( F^H := j_*(F|_{X \setminus W}) \) is coherent, and then it is the \( TFS_2 \)-hull of \( F \).

(15.4) Let \( 0 \to F_1 \to F_2 \to F_3 \) be an exact sequence of coherent sheaves. If \( F_1, F_3 \) have a \( TFS_2 \)-hull then so does \( F_2 \).

To see this note that, after killing the torsion parts, by (13.4) there is a dense, open subset \( j : U \subset X \) such that \( F_1|_U, F_3|_U \) are \( TFS_2 \) and \( \text{codim}_X(X \setminus U) \geq 2 \). Using (1) we get an exact sequence \( 0 \to F^H_1 \to F^H_2 \to F^H_3 \).

Before stating the main equivalence theorem, we need a definition.

**Definition 16.** A scheme \( X \) is formally \( S_2 \) at a point \( x \in X \) iff the completion \( \hat{X}_x \) does not have associated primes of dimension \( \leq 1 \). (This is probably not the best terminology but it is at least short. If \( \hat{X}_x \) is \( S_2 \) then it is also formally \( S_2 \) (see the argument in (14.1)) but the converse does not hold.)

**Theorem 17.** For a noetherian scheme \( X \) the following are equivalent.

1. Every coherent sheaf has a \( TFS_2 \)-hull.
2. \( \mathcal{O}_X \) has a \( TFS_2 \)-hull.
3. \( \mathcal{O}_{\text{red}, X} \) has a \( TFS_2 \)-hull.
4. There is a finite, surjective morphism \( p : Y \to X \) such that \( Y \) is \( S_2 \) and if \( \text{codim}_Y y = 0 \ (\text{resp.} = 1) \) then \( \text{codim}_X p(y) = 0 \ (\text{resp.} = 1) \).
(5) There is a $TfS_2$ sheaf whose support equals $X$.

(6) The $TfS_2$ locus of any coherent sheaf $F$ contains an open dense set and red $X$ is formally $S_2$ at every $x \in X$ of codimension $\geq 2$.

(7) The $TfS_2$ locus of any $S_1$ coherent sheaf $F$ is an open, dense set whose complement $Z(F)$ has codimension $\geq 2$ and red $X$ is formally $S_2$ at every $x \in Z(F)$.

(8) The $S_2$ locus of $X$ contains an open dense set and red $X$ is formally $S_2$ at every $x \in X$ of codimension $\geq 2$.

(9) The $S_2$ locus of red $X$ is an open, dense set whose complement $Z(\text{red } X)$ has codimension $\geq 2$ and red $X$ is formally $S_2$ at every $x \in Z(\text{red } X)$.

Proof. It is clear that (1) implies (2) and (3). For (4) one can then take either $Y := \text{Spec}_X \mathcal{O}_X^J$ or $Y := \text{Spec}_X \mathcal{O}_{\text{red } X}^H$. If (4) holds then $p_* \mathcal{O}_Y$ is $TfS_2$ by 13 and we get (5).

If $F$ has an $TfS_2$-hull $q : F \to F^H$ then $Z(F) = \text{Supp}(F^H/q(F))$, thus (1) and (20) imply (6–9).

The claims (6–9) are local, so we may assume that $X$ is affine. Let $F$ be a coherent, $S_1$ sheaf and $U \subset X$ an open, dense subset such that $F|_U$ is $TfS_2$. Pick $x \in X \setminus U$ and let $g$ be a local equation of $X \setminus U$. Then $F$ is not $S_2$ at $x$ if $x$ is an associated prime of $F/gF$. There are only finitely many such primes, let $W \subset X$ be the union of their closures. Then the $S_2$-locus of $F$ is $X \setminus W$, hence (6) implies (7). Similarly (8) implies (9) and (6) $\Rightarrow$ (8) and (7) $\Rightarrow$ (9) are clear.

If (9) holds then let $U$ denote the $S_2$ locus and $j : U \hookrightarrow X$ the natural embedding. Then $\mathcal{O}_U^J = j_*(\mathcal{O}_U)$ is coherent by (20), hence (9) implies (2).

It remains to prove that (5) implies (1). Let $X_i$ be the irreducible components of $X$. By 10 every red $X_i$ supports a rank 1 $TfS_2$ sheaf $L_i$. We claim that every rank 1 sheaf $M_i$ on $X_i$ has a $TfS_2$-hull. To see this, cover $X_i$ with open affine subsets $U_{ik}$. For every $k$ we can realize $M_i|_{U_{ik}}$ as a subsheaf of $L_i|_{U_{ik}}$. Thus $M_i|_{U_{ik}}$ has a $TfS_2$-hull by (13.2) and so does $M_i$. Finally using (10) and (15.4) we obtain that every torsion free coherent sheaf has a $TfS_2$-hull, and so does every coherent sheaf by 13.

**Lemma 18.** Let $p : X \to Y$ be a finite morphism and $F$ a coherent, $TfS_2$ sheaf on $X$. The following are equivalent.

1. The push-forward $p_* F$ is also a $TfS_2$ sheaf on $Y$.
2. For every $x \in \text{Supp } F$, if $\text{codim}_X x = 0$ (resp. $\text{codim}_X x = 1$) then $\text{codim}_Y p(x) = 0$ (resp. $\text{codim}_Y p(x) = 1$).

**Corollary 19.** Let $p : X \to Y$ be a finite morphism that maps generic points to generic points. If $Y$ satisfies the conditions of (17), then $p$ satisfies (18.2).

Proof. The codimension 0 case holds by the birationality assumption. Assume that $\text{codim}_X x = 1$. Then $X$ is a 1-dimensional irreducible component of the local scheme $Y_{p(x)}$ and the same holds after completion. Thus (17.4) implies that $\text{dim } Y_{p(x)} = 1$ hence $\text{codim}_Y p(x) = 1$.

We have used the following result on the coherence of push-forwards, which is a sharpening of [Gro60] IV.5.11.1.

**Proposition 20.** [Kol17a Thm.2] Let $X$ be a scheme, $Z \subset X$ a closed subset of codimension $\geq 2$ and $j : X \setminus Z \hookrightarrow X$ the natural injection. Assume that there is a $TfS_2$ sheaf on $X \setminus Z$ whose support is $X \setminus Z$. The following are equivalent.
(1) $j_* F$ is coherent for every $\mathcal{T}_2$ sheaf $F$ on $X \setminus Z$.
(2) There is a $\mathcal{T}_2$ sheaf on $X$ whose support is $X$.
(3) $X$ is formally $S_2$ at every $x \in Z$. □

**Corollary 21.** Let $X$ be a scheme satisfying the conditions \[\text{(17)\]} . Let $j : U \hookrightarrow X$ be an open subscheme and $G$ a $\mathcal{T}_2$ sheaf on $U$.

1. $G$ can be extended to a $\mathcal{T}_2$ sheaf on $X$.
2. If $\text{codim}_X (X \setminus U) \geq 2$ then $j_* G$ is the unique $\mathcal{T}_2$ extension.

Proof. Let $G_X$ be any coherent extension, then $(G_X)^H$ is a $\mathcal{T}_2$ extension. (2) was already noted in \[\text{(15)\]} . □

**Definition 22** ($S_2$-hull of a scheme). Let $X$ be a scheme. There is a unique largest subscheme $X_1 \subset X$ such that $\text{Supp}(X_1)$ is the support of a $\mathcal{T}_2$ sheaf on $X$.

Then $X^H := \text{Spec}_X \mathcal{O}_{X_1}^H$ is called the $\mathcal{T}_2$-hull of $X$. By construction, $X^H$ is $S_2$ and the natural map $\pi : X^H \to X_1$ is finite and birational.

**Warning.** The construction of $X^H$ is local on $X_1$ but not on $X$. If $U \subset X$ is an open subset that is disjoint from $X_1$ then there may well be a nontrivial $\mathcal{T}_2$ sheaf on $U$ that does not extend to a $\mathcal{T}_2$ sheaf on $X$. □

**23** (Proof of Theorem \[\text{1}\]). Let $X$ be a noetherian scheme and $X_1 \subset X$ as in \[\text{(22)\]} . Thus every $\mathcal{T}_2$ sheaf on $X$ is the push forward of a $\mathcal{T}_2$ sheaf on $X_1$. After replacing $X$ by $X_1$ we may as well assume that $X$ satisfies the the equivalent conditions \[\text{(17)\]} 1–9). The theorem them amounts to saying that every $\mathcal{T}_2$ sheaf on $X$ has a natural structure as an $\mathcal{O}_X^H$-module. This is a local question, so we may assume that $X$ is affine. Let $s \in H^0(X, \mathcal{O}_X^H)$. By \[\text{(14)\]} and \[\text{(21)\]} there is open subset $U \subset X$ such that $X \setminus U$ has codimension $\geq 2$ and $H^0(U, \mathcal{O}_U) = H^0(X, \mathcal{O}_X^H)$. Let $F$ be a $\mathcal{T}_2$ sheaf on $X$. Given $s \in H^0(X, \mathcal{O}_X^H)$ and $\sigma \in H^0(X, F)$, the product $(s|_U) \cdot (\sigma|_U)$ is a section of $H^0(U, F|_U)$. By \[\text{(3)\]} it uniquely extends to a section $s(\sigma) \in H^0(X, F)$. This defines the $\mathcal{O}_X^H$-module structure on $F$. □

**2. Duality for torsion free, $S_2$ sheaves**

**Definition 24.** Let $X$ be a scheme. A $\mathcal{T}_2$-dualizing sheaf on $X$ is a $\mathcal{T}_2$ sheaf $\hat{\omega}_X$ such that for every $\mathcal{T}_2$ sheaf $F$ the natural map

$$ j_F : F \to \text{Hom}_X (\text{Hom}_X (F, \hat{\omega}_X), \hat{\omega}_X) \quad \text{\[24\]} \]$$

is an isomorphism. If $X = \text{Spec} R$ is affine and $\hat{\omega}_X$ is the sheaf corresponding to an $R$-module $\hat{\omega}_R$, then the latter is called a $\mathcal{T}_2$-dualizing module of $R$.

Another way to define duality as an anti-equivalence $F \mapsto D(F)$ of the category of $\mathcal{T}_2$-sheaves such that $D(D(F)) = F$; see \[\text{Eis95\]} Sec.21.1 for this approach. Then $\text{Hom}_X (F_1, F_2) = \text{Hom}_X (D(F_2), D(F_1))$. In particular

$$ \text{Hom}_X (F, D(\mathcal{O}_X)) = \text{Hom}_X (\mathcal{O}_X, D(F)) = D(F). $$

Thus $D(\mathcal{O}_X)$ is a $\mathcal{T}_2$-dualizing sheaf on $X$. Conversely, if $\hat{\omega}_X$ is $\mathcal{T}_2$-dualizing then $D(F) := \text{Hom}_X (F, \hat{\omega}_X)$ is a duality on the category of $\mathcal{T}_2$-sheaves.

By this definition, if the 0 sheaf is the only $\mathcal{T}_2$ sheaf on $X$ then it is also a $\mathcal{T}_2$-dualizing sheaf. This causes uninteresting exceptions in several statements. Thus we focus on $S_2$ schemes form now on. The following observation shows that this is not a restriction on the generality.
25. Let $X$ be a scheme and $\pi : X^H \to X$ its $S_2$-hull. If $X^H$ has a $TfS_2$-dualizing sheaf $\tilde{\omega}_X^H$ then, as an immediate consequence of (1), we get that

$$\tilde{\omega}_X := \pi_* \tilde{\omega}_X^H$$

is a $TfS_2$-dualizing sheaf on $X$. Thus in studying $TfS_2$-duality, we may as well restrict ourselves to $S_2$ schemes. If $X$ is $S_2$ then applying (24.1) to $F = \mathcal{O}_X$ gives that

$$\mathcal{H}om_X(\tilde{\omega}_X, \tilde{\omega}_X) \cong \mathcal{O}_X.$$  

A map of finite modules over a local ring is an isomorphism iff its completion is an isomorphism. Thus, once we have a candidate for a $TfS_2$-dualizing module, we can check it formally.

Lemma 26. Let $(R, m)$ be a local ring and $M$ a finite $R$-module. If $\tilde{M}$ is a $TfS_2$-dualizing module over $\tilde{R}$ then $M$ is a $TfS_2$-dualizing module over $R$. \hfill $\square$

The converse of (26) holds if $\dim R = 1$ by (30) and (35). There are, however, 2-dimensional normal rings $R$ whose completions are not Gorenstein at their generic points [FR70]. For these $R$ this $TfS_2$-dualizing module over $R$ by (28.1), but $\tilde{R}$ is not a $TfS_2$-dualizing module over $\tilde{R}$ by (35). This is in marked contrast with the dualizing complex, which is preserved by completion [Sta15, Tag 0DWD].

Lemma 27. Let $X$ be an $S_2$ scheme. A coherent, $TfS_2$ sheaf $\tilde{\Omega}_X$ is $TfS_2$-dualizing over $X$ iff its pull-back to the localization $\tilde{X}_x$ is $TfS_2$-dualizing for all points $x \in X$ of codimension $\leq 1$.

Proof. By (35) a map between coherent $TfS_2$ sheaves is an isomorphism iff it is an isomorphism at all points of codimension $\leq 1$. The converse is established in (30). \hfill $\square$

Example 28. The basic examples are the following.

(28.1) If $X$ is normal then $\mathcal{O}_X$ is a $TfS_2$-dualizing sheaf on $X$.

(28.2) If $X$ is CM then a dualizing sheaf [33] is also a $TfS_2$-dualizing sheaf; this follows from [BH93, 3.3.10] and (26). More generally, if an arbitrary scheme $X$ has a dualizing complex $\omega_X$ then $\tilde{\omega}_X := \mathcal{H}^{-\dim X}(\omega_X)$ is a $TfS_2$-dualizing sheaf. This follows from [Sta15, Tag 0A7C]. While these are the main examples, we try to work out the theory of $TfS_2$-dualizing sheaves without using the general theory of dualizing complexes.

(28.3) Note that usually there are many non-isomorphic $TfS_2$-dualizing sheaves. If $\tilde{\omega}_X$ is a $TfS_2$-dualizing sheaf and $L$ is invertible then $L \otimes \tilde{\omega}_X$ is also a $TfS_2$-dualizing sheaf since

$$\mathcal{H}om_X(\mathcal{H}om_X(F, L \otimes \tilde{\omega}_X), L \otimes \tilde{\omega}_X)$$

$$\mathcal{H}om_X(\mathcal{H}om_X(F, \tilde{\omega}_X) \otimes L, \tilde{\omega}_X) \otimes L$$

$$\mathcal{H}om_X(\mathcal{H}om_X(F, \omega_X), \tilde{\omega}_X) \otimes L^{-1} \otimes L$$

$$\mathcal{H}om_X(\mathcal{H}om_X(F, \tilde{\omega}_X), \tilde{\omega}_X).$$

We will give a precise characterization in (39).

(28.4) We see in (40) that on a regular scheme, $TfS_2$-dualizing $= $ invertible.

(28.5) [FR70] constructs a 1-dimensional integral scheme $X$ over $\mathbb{C}$ that has no dualizing sheaf. By (36) this implies that $X$ has no $TfS_2$-dualizing sheaf either.

(28.6) As far as I can tell, $TfS_2$-dualizing modules are not closely related to the semidualizing modules considered in [Fox72, SW07].
Lemma 29. Let $X$ be an $S_2$-scheme or, more generally, a scheme satisfying the conditions (17). Let $\omega_X$ be a $TfS_2$-dualizing sheaf and $j : U \to X$ an open embedding. Then $j^*\omega_X$ is a $TfS_2$-dualizing sheaf on $U$.

Proof. Let $F_U$ be a $TfS_2$ sheaf on $U$. By (21) we can extend $F_U$ to a $TfS_2$ sheaf $F_X$ on $X$. Thus
\[ \text{Hom}_U(\text{Hom}_U(F_U, j^*\omega_X), j^*\omega_X) \cong j^*\text{Hom}_X(\text{Hom}_X(F_X, \omega_X), \omega_X) \cong j^*F_X \cong F_U. \]

By passing to the direct limit, we get the following consequence.

Corollary 30. Let $X$ be an $S_2$-scheme or, more generally, a scheme satisfying the conditions (17), with a $TfS_2$-dualizing sheaf $\omega_X$. Let $j : W \to X$ be a direct limit of open embeddings. Then $j^*\omega_X$ is a $TfS_2$-dualizing sheaf on $W$.

For ease of reference, we recall the following from [Har77, Exrc.II I.6.10].

Lemma 31. Let $p : X \to Y$ be a finite morphism and $G$ a coherent sheaf on $Y$. Then
\begin{enumerate}
    
    \item The formula $p^!G := \text{Hom}_Y(p_*\mathcal{O}_X, G)$ defines a coherent sheaf on $X$.
    
    \item There is a trace map $\text{tr}_{X/Y} : p_*(p^!G) \to G$ obtained by sending a section $\sigma \in H^0(X, p^!G) = \text{Hom}_Y(p_*\mathcal{O}_X, G)$ to $\text{tr}_{X/Y}(\sigma) := \sigma(1) \in H^0(Y, G)$.
    
    \item For any coherent sheaf $F$ on $X$ there is a natural isomorphism $p_*\text{Hom}_X(F, p^!G) \cong \text{Hom}_Y(p_*F, G)$.
    
    \item For any coherent sheaf $F$ on $X$ there are natural maps $\phi_i : p_*\text{Ext}^i_X(F, p^!G) \to \text{Ext}^i_Y(p_*F, G)$.
    
    \item If $p$ is also flat then the $\phi_i$ are isomorphisms.
\end{enumerate}

Proposition 32. Let $p : X \to Y$ be a finite morphism satisfying (18.2). Let $\omega_Y$ be a $TfS_2$-dualizing sheaf on $Y$. Then $\omega_X := p^!\omega_Y$ is a $TfS_2$-dualizing sheaf on $X$.

Proof. Applying (31.3) twice we get that
\[ p_*\text{Hom}_X(\text{Hom}_X(F, \omega_X), \omega_X) \cong \text{Hom}_Y(p_*\text{Hom}_X(F, \omega_X), \omega_Y) \cong \text{Hom}_Y(\text{Hom}_Y(p_*F, \omega_Y), \omega_Y) \cong F. \]

Corollary 33. Let $X$ be a quasi-projective scheme. Then $X$ has a $TfS_2$-dualizing sheaf.

Proof. By (25.1) we may assume that $X$ is $S_2$ and then its connected components are pure dimensional by (34). If $X$ is projective and pure dimensional, Noether’s normalization theorem gives a finite morphism $p : X \to \mathbb{P}^{\dim X}$ that maps generic points to generic points. Thus $X$ has a $TfS_2$-dualizing sheaf by (25.2) and (25.3). The quasi-projective case is now implied by (30).

The following is a slightly stronger formulation of [Har62].

Lemma 34. Let $X$ be a connected, $S_2$ scheme with a dimension function. Then $X$ is pure dimensional and connected in codimension 1.

Low dimensions.

Over Artin schemes every coherent sheaf is $S_2$ and $TfS_2$-duality is the same as Matlis duality; see, for instance, [BH93, Sec.3.2] or [Eis95, Sec.21.1].

Lemma 35. Let $(A, m)$ be an Artin, local ring, $k := A/m$ and $\Omega$ a finite $A$-module. The following are equivalent.
\begin{enumerate}
    
    \item $\Omega$ is a $TfS_2$-dualizing module.
\end{enumerate}
\( \Omega \cong E(k) \), the injective hull of \( k \).

3. \( \text{Hom}_A(k, \Omega) \cong k \) and \( \text{Ext}^1_A(k, \Omega) = 0 \).

4. \( \text{Hom}_A(k, \Omega) \cong k \) and \( \text{Ext}^i_A(k, \Omega) = 0 \) for \( i > 0 \).

5. \( \text{Hom}_A(k, \Omega) \cong k \) and \( \text{length}(\Omega) = \text{length}(A) \).

6. \( \text{Hom}_A(k, \Omega) \cong k \) and \( \text{length}(\Omega) \geq \text{length}(A) \). \( \square \)

**Theorem 36.** Let \((R, m)\) be a 1-dimensional, \( S_1 \) local ring, \( k := R/m \) and \( \Omega \) a finite \( R \)-module. The following are equivalent.

1. \( \Omega \) is a \( Tfs_2 \)-dualizing module.

2. \( \text{Ext}_R^i(k, \Omega) = \delta_{i,1} \cdot k \) for every \( i \).

Note that (36.2) is one of the usual definitions of a dualizing module, see (65).

**Proof.** Assume (1) and write \( A^* := \text{Hom}_R(A, \Omega) \). From any exact sequence of torsion free modules

\[ 0 \to A \to B \to C \to 0 \]

we get an \( A^+ \subset A^* \) such that

\[ 0 \to C^* \to B^* \to A^+ \to 0 \]

is exact.

Applying duality again gives an exact sequence

\[ 0 \to (A^+)^* \to B \to C \to 0 \]

Thus \( (A^+)^* = A \) hence \( A^+ = A^* \) which proves the following.

**Claim 36.3.** Duality sends exact sequences of torsion free modules to exact sequences of their duals. \( \square \)

Thus if we have an exact sequence of torsion free modules

\[ 0 \to \Omega \to B \to C \to 0 \]

then its dual is

\[ 0 \to C^* \to B^* \to R \to 0, \]

hence they both split. That is, \( \text{Ext}_R^1(C, \Omega) = 0 \) for any torsion free module \( C \).

Chasing through a projective resolution of a module \( M \) we conclude the following.

**Claim 36.4.** Let \( M \) be a finitely generated \( R \)-module. Then \( \text{Ext}_R^i(M, \Omega) = 0 \) for \( i \geq 2 \). If \( M \) is torsion free then \( \text{Ext}_R^i(M, \Omega) = 0 \) for \( i \geq 1 \). \( \square \)

**Claim 36.5.** Let \( Q_1 \subset Q_2 \) be torsion free \( R \)-modules such that \( \text{length}(Q_2/Q_1) \) is finite. Then \( \text{length}(Q_2/Q_1) = \text{length}(Q_2^*/Q_1^*) \).

Proof. \( M \to M^* \) gives a one-to-one correspondence between intermediate modules \( Q_1 \subset M \subset Q_2 \) and \( Q_2^* \subset M^* \subset Q_1^* \). \( \square \)

Let \( T \) be a torsion \( R \)-module and write it as

\[ 0 \to K \to R^n \to T \to 0. \]

Duality gives

\[ 0 \to \Omega^n \to K^* \to \text{Ext}_R^1(T, \omega) \to 0. \]

Combining with (36.5) we get that

\[ \text{length}(\text{Ext}_R^1(T, \Omega)) = \text{length}(T). \]

In particular, \( \text{Ext}_R^1(R/m, \Omega) \cong R/m \). This completes the proof of (2).
The implication (2) \(\Rightarrow\) (1) was already noted in (28.2). I did not find any shortcuts to the proofs given in the references and there is no point repeating what is there.

**Proposition 37.** Let \(X\) be a 1-dimensional, \(S_1\) scheme and \(\tilde{\omega}_X\) a \(TfS_2\)-dualizing sheaf. Then every \(TfS_2\)-dualizing sheaf is of the form \(L \otimes \tilde{\omega}_X\) for some line bundle \(L\).

Proof. This is a special case of [Sta15, Tag 0A7F] or [BH93, 3.3.4], but here is a direct proof using the above computations.

Let \(\tilde{\Omega}_X\) be another \(TfS_2\)-dualizing sheaf, and set \(L := \mathcal{H}om_X(\tilde{\omega}_X, \tilde{\Omega}_X)\). If the claim holds for all localizations of \(X\) then \(L\) is a line bundle by (21.2), and then \(\tilde{\Omega}_X \cong L \otimes \tilde{\omega}_X\). Thus it is sufficient to prove the claim when \(X\) is local.

We keep \(*\) for \(\omega_X\)-duality. Write \(\tilde{\Omega}_X^*\) as

\[
0 \to K \to \mathcal{O}_X^* \to \tilde{\Omega}_X^* \to 0
\]

we get

\[
0 \to \tilde{\Omega}_X \to \tilde{\omega}_X^* \to K^* \to 0.
\]

Since \(\tilde{\Omega}_X\) is \(TfS_2\)-dualizing, these sequences split by (36.4). So \(\tilde{\Omega}_X = \mathcal{H}om_X(\tilde{\Omega}_X, \tilde{\omega}_X)\) is projective, hence isomorphic to \(\mathcal{O}_X\). Thus

\[
\tilde{\Omega}_X \cong (\tilde{\Omega}_X^*)^* \cong \mathcal{H}om_X(\mathcal{H}om_X(\tilde{\Omega}_X, \tilde{\omega}_X), \tilde{\omega}_X) \cong \mathcal{H}om_X(\mathcal{O}_X, \tilde{\omega}_X) \cong \tilde{\omega}_X. \quad \square
\]

We have not yet discussed the existence of \(TfS_2\)-dualizing sheaves and modules. By (23), in the 1-dimensional case this is equivalent to the existence of dualizing sheaves. We recall the main results about dualizing sheaves on 1-dimensional schemes in Section 3.

3. **Existence of \(TfS_2\)-dualizing sheaves**

We start with the uniqueness question for \(TfS_2\)-dualizing sheaves and then prove the main existence theorem. At the end we give a series of examples of noetherian rings without \(TfS_2\)-dualizing modules.

**Definition 38.** A coherent sheaf \(L\) on a scheme \(X\) is called *mostly invertible* if it is \(S_2\) and there is an open subset \(j : U \hookrightarrow X\) such that \(\text{codim}_X(X \setminus U) \geq 2\) and \(L|_U\) is invertible. Equivalently, if \(L\) is invertible at all points of codimension \(\leq 1\). Thus \(L\) is also \(TfS_2\).

If \(F\) is a \(TfS_2\) sheaf on \(X\) then we set \(L \otimes F := j_*(L|_U \otimes F|_U)\). If \(L_1, L_2\) are mostly invertible then so is \(L_1 \otimes L_2\) and \(L_1^{-1} := j_*(L|_U)^{-1}\).

**Theorem 39.** Let \(X\) be an \(S_2\) scheme with a \(TfS_2\)-dualizing sheaf \(\tilde{\omega}_X\).

1. If \(L\) is mostly invertible then \(L \otimes \tilde{\omega}_X\) is also a \(TfS_2\)-dualizing sheaf.
2. Every \(TfS_2\)-dualizing sheaf is obtained this way.

Proof. By assumption there is an open subset \(j : U \hookrightarrow X\) such that \(\text{codim}_X(X \setminus U) \geq 2\) and \(L|_U\) is a line bundle. Thus

\[
\mathcal{H}om_X(\mathcal{H}om_X(F, L \otimes \tilde{\omega}_X), L \otimes \tilde{\omega}_X)
= j_* \mathcal{H}om_U(\mathcal{H}om_U(F, L|_U \otimes \tilde{\omega}_U), L|_U \otimes \tilde{\omega}_U)
= j_* \mathcal{H}om_U(\mathcal{H}om_U(F, \tilde{\omega}_U), \tilde{\omega}_U)
= \mathcal{H}om_X(\mathcal{H}om_X(F, \tilde{\omega}_X), \tilde{\omega}_X),
\]

where we used (28.3) in the middle and (21.2) at the ends.
Let $\hat{\Omega}_X$ be another $\text{TfS}_2$-dualizing sheaf and set $L := \mathcal{H}om_X(\hat{\omega}_X, \hat{\Omega}_X)$. If $L$ is mostly invertible then $\Omega_X \cong L \otimes \hat{\omega}_X$. Thus we need to show that $L$ is invertible at all points of codimension $\leq 1$. This can be done after localization. The codimension 0 case follows from (35) and the codimension 1 case from (36).

**Corollary 40.** Let $X$ be a regular scheme. Then a coherent sheaf is $\text{TfS}_2$-dualizing iff it is invertible.

Next we show that $\text{TfS}_2$-dualizing sheaves exist for Nagata schemes, in a formulation that is slightly more general than Theorem 2.

**Theorem 41.** Let $X$ be a noetherian scheme whose normalization $\pi: \bar{X} \to X$ is finite. Then $X$ has a $\text{TfS}_2$-dualizing sheaf.

Note that we do not assume that $X$ is reduced, thus $\bar{X} = \text{red} \bar{X}$.

Proof. By (25) it is enough to prove this for the $S_2$-hull of $X$. Thus we may as well assume that $X$ is $S_2$.

By (42) there is a dense open subset $U \subset X$ with a $\text{TfS}_2$-dualizing sheaf $\hat{\omega}_U$. Let $p_i \in X \setminus U$ be a generic point that has codimension 1 in $X$. By assumption the normalization $\pi: X_{p_i} \to X_{p_i}$ is finite. If $X$ is reduced, then $X_{p_i}$ has a dualizing sheaf $\hat{\omega}_i$ by (52). In the non-reduced case we can use the more general (60).

For each generic point of $X_{p_i}$, the restrictions of $\hat{\omega}_U$ and of $\hat{\omega}_i$ are isomorphic. After fixing these isomorphisms, we can identify $\hat{\omega}_i$ with a subsheaf of $(\pi_\ast \hat{\omega}_U)_{p_i}$. We can now choose a coherent subsheaf $G \subset j_\ast \hat{\omega}_U$ such that $G|_U = \hat{\omega}_U$ and $G|_{p_i} = \hat{\omega}_i$ for every $i$. By (17) $G$ has a $\text{TfS}_2$-hull $G^H$ and $\hat{\Omega}_X := G^H$ is a $\text{TfS}_2$-dualizing sheaf on $X$ by (27).

**Lemma 42.** Let $X$ be a scheme and $\hat{\Omega}_X$ a coherent sheaf on $X$. Assume that $\text{red} X$ is normal and $\hat{\Omega}_X |_g$ is dualizing for every generic point $g \in X$.

Then there is a dense open subset $U \subset X$ such that $\hat{\Omega}_X |_U$ is $\text{TfS}_2$-dualizing.

**Warning.** A similar assertion does not hold for the dualizing sheaf, see (45.2).

Proof. Set $U_1 := X \setminus \text{Supp}(\mathcal{E}xt^1_X(\mathcal{O}_{\text{red} X}, \hat{\Omega}_X)) \setminus \text{Supp}(\text{tors}(\mathcal{O}_X))$ and then let $U \subset U_1$ be the open set where $\text{socle}(\hat{\Omega}_X)$ is locally free over $\text{red} X$.

Let $F$ be a coherent, $\text{TfS}_2$-sheaf on $U$. By assumption

$$j_F : F \to \mathcal{H}om_U(\mathcal{H}om_U(F, \hat{\Omega}_U), \hat{\Omega}_U)$$

is an isomorphism at the generic points. It remains to prove that it is also an isomorphism at codimension $\leq 1$ points. By (27), this can be checked after localizing at codimension $\leq 1$ points. At codimension 0 points we get a dualizing sheaf by assumption, at codimension 1 points we get a dualizing sheaf by (62).

The following existence result is a direct generalization of Proposition 5; it can be proved exactly as (111) and (183).

**Theorem 43.** Let $X$ be an $S_2$ scheme. Then $X$ has a $\text{TfS}_2$-dualizing sheaf iff the following hold.

1. $\mathcal{O}_{x, X}$ has a dualizing module for every codimension 1 point $x \in X$.
2. There is an open and dense subset $U \subset X$ such that $\text{red}(X_x)$ is Gorenstein for every codimension 1 point $x \in U$.

Note that, by (59), assumption (1) can be reformulated as...
Since \( r \) not free, but isomorphic to the module limit of the \( \{ R_i, m_i : i \in I \} \) and hence the so is and let \( K \). Thus Depending on the choice of the \( R \) in infinite set points, so it does not have a \( T^n_2 \)-dualizing sheaf. We use the following general construction, modeled on \([\text{Nag62, Appendix, Exmp.1}]\).

**Proposition 44.** Let \( k \) be a field, \( I \) an arbitrary set and \( \{ (R_i, m_i) : i \in I \} \) integral, essentially of finitely type \( k \)-algebras such that \( k \cong R_i/m_i \). Then there is a noetherian, integral \( k \)-algebra \( R = R(I, R_i, m_i) \) with the following properties.

1. The maximal ideals of \( R \) can be naturally indexed as \( \{ M_i : i \in I \} \).
2. \( R_{M_i} \cong R_i \otimes_k K'_i \) for some fields \( K'_i \supset k \).
3. Every nonzero ideal of \( F \) is contained in only finitely many maximal ideals.

Proof. For any finite subset \( J \subset I \) set \( R_J := \bigotimes_{j \in J} R_j \). If \( J_1 \subset J_2 \) then using the natural injections \( k \to R_j \), we get injections \( R_{J_1} \hookrightarrow R_{J_2} \). Let \( R_I \) denote the direct limit of the \( \{ R_J \} \). Usually \( R_I \) is not noetherian. Somewhat sloppily we identify \( R_J \) with its image in \( R_I \).

This also defines \( R_{I_1} \) for any subset \( I_1 \subset I \). We use the notation \( R'_{I,J} := R_{I \setminus J} \) and let \( K'_{i,J} \) be the quotient field of \( R'_{i,J} \). Note that

\[ R_I/(R_I m_i) \cong (R_i/m_i) \otimes_k R'_i, \]

hence the \( R_I m_i \) are prime ideals. We obtain \( R \) from \( R_I \) by inverting every element in \( R_I \setminus \cup_i R_I m_i \). Set \( M_i := R m_i \). By construction the \( M_i \) are the maximal ideals of \( R \).

Pick any \( r \in R_I \). Then \( r \in R_J \) for some finite subset \( J \subset I \). If \( J' \subset I \) is disjoint from \( J \) and \( s \in R_{J'} \setminus \{ 0 \} \) then \( r + s \) is not in any \( R_I m_i \), hence it is invertible in \( R \). Since \( r + s \equiv s \mod (r)R_I \) we see that

\[ 1 \equiv \frac{r}{s} \mod (r)R. \]

Thus \( R/(r) \cong (R_I/(r)) \otimes_k K'_J \). In particular, \( R/(r) \) is Noetherian for every \( r \) and so is \( R \). \( \square \)

**Example 45.** Depending on the choice of the \( R_i \) in \([11]\), we get many examples of noetherian domains with unexpected behavior.

\((45.1)\) A 1-dimensional integral domain without a dualizing module. Pick an infinite set \( I \) and for \( i \in I \) let \( R_i \) be the localization of \( k[t^i, t^i, t^3] \) at the origin. Note that \( R_i \) is not Gorenstein. The resulting \( R \) has a dense set of non-Gorenstein points, so it does not have a \( T^n_2 \)-dualizing module though all of its localizations at maximal ideals have one.

\((45.2)\) A 2-dimensional normal ring without a dualizing module. Pick an infinite set \( I \) and for \( i \in I \) let \( R_i \) be the localization of

\[ S := (x^a y^b : 3 \mid a + b) \subset k[x, y]. \]

Note that \( S \) is also the ring of invariants \( k[\langle x, y \rangle ]/\omega(1, 1) \). Its dualizing module is not free, but isomorphic to the module

\[ \omega_S \cong (x^a y^b : 3 \mid a + b - 1) \subset k[x, y]. \]

The resulting \( R \) has a dense set of non-Gorenstein points, so it does not have a dualizing module though all of its localizations at maximal ideals have one. By contrast, \( R \) has plenty of \( T^n_2 \)-dualizing modules, for example \( R \) itself.
A 2-dimensional integral domain without finite, torsion free, $S_2$ modules.

Pick an infinite set $I$ and for $i \in I$ let $R_i$ be the localization of

$$S := \langle x^a y^b : a + b \geq 2 \rangle \subset k[x, y].$$

Note that $S$ is not normal and not $S_2$. The resulting $R$ has a dense set of non-$S_2$ points, so it does not have a nonzero, finite, torsion free, $S_2$ module though all of its localizations at maximal ideals have one.

4. Conductors

**Definition 46.** Let $X$ be a reduced scheme whose normalization $\pi : \bar{X} \to X$ is finite. Its conductor ideal sheaf is defined as

$$\text{cond}_{\bar{X}/X} := \mathcal{H}om_X(\pi_* \mathcal{O}_{\bar{X}}, \mathcal{O}_X).$$

It is the largest ideal sheaf on $X$ that is also an ideal sheaf on $\bar{X}$. We define the conductor subschemes as

$$D := \text{Spec}_X(\mathcal{O}_X/\text{cond}_{\bar{X}/X}) \quad \text{and} \quad \bar{D} := \text{Spec}_X(\mathcal{O}_X/\text{cond}_{\bar{X}/X}).$$

Since $1$ is a local section of $\text{cond}_{\bar{X}/X}$ over $U \subset X$ iff $\pi$ is an isomorphism over $U$, we see that

$$\text{Supp } D = \pi(\text{Supp } \bar{D}) = \text{Supp}(\pi_* \mathcal{O}_{\bar{X}}/\mathcal{O}_X).$$

If $X$ is $S_2$ then $D \subset X$ and $\bar{D} \subset \bar{X}$ are $S_1$ and of pure codimension 1. Indeed, let $q : \mathcal{O}_{\bar{X}} \to \mathcal{O}_{\bar{D}}$ be the quotient map and $T \subset \pi_* \mathcal{O}_D$ the largest subsheaf whose support has codimension $\geq 2$ in $X$. Assume that $T \neq 0$. Note that $q^{-1}(T) \subset \mathcal{O}_X$ is an ideal sheaf, so it is not contained in $\mathcal{O}_X$ by the maximality of the conductor. Then $\langle \mathcal{O}_X, q^{-1}(T) \rangle \supset \mathcal{O}_X$ is a nontrivial extension whose cosupport has codimension $\geq 2$, contradicting the $S_2$ condition for $X$.

Note also that $\pi_* \mathcal{O}_X \subset \mathcal{H}om_X(\text{cond}_{\bar{X}/X}, \mathcal{O}_X)$, thus the conductor is a coherent ideal sheaf iff $\pi$ is finite.

**Definition 47.** Let $X$ be a scheme and $F$ a coherent sheaf on $X$ such that $\text{Supp } F$ has codimension $\geq 1$. The divisorial support of $F$ is

$$[F] := \sum_x \text{length}_{k(x)}(F_x) \cdot [x],$$

where the summation is over all codimension 1 points $x \in X$ and $[x]$ denotes the Weil divisor defined by the closure of $x$. If $Z \subset X$ is a subscheme then we set $[Z] := [\mathcal{O}_Z]$.

**48 (Normalization and dualizing sheaf).** Let $X$ be a reduced scheme with finite normalization $\pi : \bar{X} \to X$ satisfying (35)2. If $X$ has a $TfS_2$-dualizing sheaf $\hat{\omega}_X$ then we always choose

$$\hat{\omega}_X := \pi^! \hat{\omega}_X = \mathcal{H}om_X(\pi_* \mathcal{O}_{\bar{X}}, \hat{\omega}_X)$$

as our $TfS_2$-dualizing sheaf on $\bar{X}$.

We can view sections of $\hat{\omega}_X$ as rational sections of $\hat{\omega}_X$ with poles along $\bar{D}$. Thus $\hat{\omega}_X \subset \pi_* \hat{\omega}_X(m\bar{D})$ for some $m > 0$.

We can also view $\hat{\omega}_X$ as a non-coherent subsheaf of $\hat{\omega}_X(m\bar{D})$, but we need to be careful since $\hat{\omega}_X$ is not even a sheaf of $\mathcal{O}_X$-modules.

There is, however, a smallest coherent subsheaf of $\hat{\omega}_X(m\bar{D})$ that contains $\hat{\omega}_X$, we denote it by $\mathcal{O}_X \cdot \hat{\omega}_X \subset \hat{\omega}_X(m\bar{D})$.

The following duality is quite useful.
Lemma 49. Let \( p : X \to Y \) be a finite morphism between \( S_2 \) schemes that maps generic points to generic points. Let \( \tilde{\omega}_Y \) be a \( T\!fS_2 \)-dualizing sheaf on \( Y \) and \( \tilde{\omega}_X := p^! \tilde{\omega}_Y \). Then

\[
\text{Hom}_Y(p_*\mathcal{O}_X, \mathcal{O}_Y) \cong \text{Hom}_Y(\tilde{\omega}_Y, p_*\tilde{\omega}_X).
\]

Proof. Note that \( \tilde{\omega}_X \) is a \( T\!fS_2 \)-dualizing sheaf by (52) and (19). The claim follows from the isomorphisms

\[
\text{Hom}_Y(\tilde{\omega}_Y, p_*\tilde{\omega}_X) = \text{Hom}_Y(\tilde{\omega}_Y, \text{Hom}_Y(p_*\mathcal{O}_X, \tilde{\omega}_Y)) = \text{Hom}_Y(\tilde{\omega}_Y \otimes p_*\mathcal{O}_X, \tilde{\omega}_Y) = \text{Hom}_Y(p_*\mathcal{O}_X, \text{Hom}_Y(\tilde{\omega}_Y, \tilde{\omega}_Y)) = \text{Hom}_Y(p_*\mathcal{O}_X, \mathcal{O}_Y),
\]

where at the end we used (24.2). \( \square \)

The next result is closely related to [Rei94, Thm.3.2].

Lemma 50. Let \( X \) be a reduced, \( S_2 \) scheme whose normalization \( \pi : \bar{X} \to X \) is finite with conductors \( D \subset X \) and \( \bar{D} \subset \bar{X} \). Then the following equivalent claims hold.

1. \( \bar{D} = \inf \{ E : \tilde{\omega}_X \subset \pi_* (\tilde{\omega}_X(E)) \} \).
2. \( \mathcal{O}_X \cdot \tilde{\omega}_X \subset \tilde{\omega}_X(\bar{D}) \) and the support of the quotient has codimension \( \geq 2 \).

Proof. Note that \( \mathcal{O}_X(\bar{D}) = \text{Hom}_X(\pi_*\mathcal{O}_{\bar{X}}, \mathcal{O}_X) \). Thus by (49) we get that \( \mathcal{O}_X(\bar{D}) \cdot \tilde{\omega}_X \subset \pi_*\tilde{\omega}_X \). Since \( \mathcal{O}_X(\bar{D}) = \pi_*\mathcal{O}_{\bar{X}}(-\bar{D}), \) this implies that \( \tilde{\omega}_X \subset \tilde{\omega}_X(\bar{D}) \).

Conversely, assume that \( \tilde{\omega}_X \subset \tilde{\omega}_X(E) \). Then \( \mathcal{O}_X(-E) \cdot \tilde{\omega}_X \subset \tilde{\omega}_X \), thus (49) shows that \( \mathcal{O}_X(-E) \subset \mathcal{O}_{\bar{X}} \). So \( \mathcal{O}_{\bar{X}}(-\bar{D}) \subset \mathcal{O}_X(-\bar{D}) \) and hence \( E \geq D \). \( \square \)

As a consequence we get the characterization of seminormal, \( S_2 \) schemes.

51 (Proof of Theorem 4). The equivalence of (4.2) and (4.3) is a special case of (51) and (4.1) \( \Leftrightarrow \) (4.2) follows from the equality

\[
X^{sn} = \text{Spec}_{\mathcal{O}_X}(\pi_*\mathcal{O}_{\bar{X}}(-\text{red } \bar{D})),
\]

which is a rewriting of the last formula on [Ko98, p.85]. \( \square \)

Next we focus on the 1-dimensional case. Then the dualizing sheaf usually exists.

Proposition 52. Let \( (0 \in C) \) be a local, 1-dimensional, reduced scheme whose normalization \( \pi : \bar{C} \to C \) is finite. Then \( C \) has a dualizing sheaf \( \omega_C \).

One can view the above claim as a very special converse to (32). (I do not know if its higher dimensional versions are true or not.) We give 2 proofs. The first, in (58) gives a concrete construction of the dualizing sheaf. More general results on 1-dimensional rings are treated in (60) and (59).

The following is taken from [Rei94, p.714].

Proposition 53. Let \( (0 \in C) \) be a local, 1-dimensional, reduced scheme whose normalization \( \pi : (0 \in \bar{C}) \to (0 \in C) \) is finite with conductors \( D \subset C \) and \( \bar{D} \subset \bar{C} \). Assume that the residue field \( k(0) \) is infinite and let \( \sigma \in \omega_C \) be a general section. Then

\[
\mathcal{O}_C(-\bar{D}) \cdot \sigma = \pi_* \omega_{\bar{C}} \subset \omega_C.
\]
Proposition 57. Let $\tilde{c}_i \in \tilde{C}$ be the preimages of $0 \in C$. By (50) for every $i$ there is a section $\sigma_i \in \omega_C$ such that $\sigma_i$ generates $\omega_C(\tilde{D})$ at $c_i$. If $k(0)$ is infinite the a general linear combination $\sigma := \sum_i \lambda_i \sigma_i$ generates $\omega_C(\tilde{D})$ everywhere.

We can now compute $\mathcal{O}_C(-D) \cdot \sigma$ on $\tilde{C}$ as

$$\mathcal{O}_C(-D) \cdot \sigma = \mathcal{O}_C(-D) \cdot \mathcal{O}_C \cdot \sigma = \mathcal{O}_C(-D) \cdot \omega_C(\tilde{D}) = \omega_C. \quad \square$$

Corollary 54. Let $(0 \in C)$ be a local, 1-dimensional, reduced scheme whose normalization $\pi : (0 \in \tilde{C}) \rightarrow (0 \in C)$ is finite with conductors $D \subset C$ and $\tilde{D} \subset \tilde{C}$. Then

1. $\text{length}(\omega_C/\pi_*\omega_C) \geq \text{length}(\mathcal{O}_D)$ and
2. equality holds iff $\omega_C$ is free.

Proof. If $k(0)$ is infinite then (53) gives an embedding $\mathcal{O}_D \cdot \sigma \hookrightarrow \omega_C/\pi_*\omega_C$, and equality holds iff $\sigma$ generates $\omega_C/\pi_*\omega_C$. By (55) $\pi_*\omega_C$ is contained in $\mathcal{O}_C \cdot \sigma$, thus equality holds iff $\sigma$ generates $\omega_C$.

If $k(0)$ is finite, then first we take $\lambda_i^1$ and localize at the generic point of the fiber over $0 \in C$. The residue field is now $k(0)(t)$, hence infinite, and the lengths are unchanged. \hfill \square

Lemma 55. Let $(0 \in C)$ be a local, 1-dimensional, reduced scheme whose normalization $\pi : (0 \in \tilde{C}) \rightarrow (0 \in C)$ is finite with conductors $D \subset C$ and $\tilde{D} \subset \tilde{C}$. Then $\text{length}(\omega_C/\pi_*\omega_C) = \text{length}(\mathcal{O}_D) - \text{length}(\mathcal{O}_D)$.

Proof. $\pi_*\omega_C$ is the dual of $\pi_*\mathcal{O}_D$ by (48.1), so (50) says that $\text{length}(\omega_C/\pi_*\omega_C) = \text{length}(\pi_*\mathcal{O}_C/\mathcal{O}_C)$ and $\pi_*\mathcal{O}_C/\mathcal{O}_C \cong \pi_*\mathcal{O}_D/\mathcal{O}_D$. \hfill \square

We can now prove the “$n_Q = 2\delta_Q$ theorem.”

56 (Proof of Theorem 3). The claim can be checked after localizing at various generic points of $D$. As we noted in (10), $D \subset X$ has pure codimension 1. Thus we may assume that $C := X$ is local and $\dim C = 1$. Since the normalization $\pi : \tilde{C} \rightarrow C$ is finite, $C$ has a dualizing sheaf $\omega_C$ by (52).

Combining (54) and (55) we get that

$$\text{length}(\pi_*\mathcal{O}_D) - \text{length}(\mathcal{O}_D) = \text{length}(\omega_C/\pi_*\omega_C) \geq \text{length}(\mathcal{O}_D),$$

and equality holds iff $C$ is Gorenstein. \hfill \square

Let us state another variant of [Rei94, 3.2.1].

Proposition 57. Let $X$ be a reduced, $S_2$ scheme with finite normalization $\pi : \tilde{X} \rightarrow X$ and conductors $D \subset X$ and $\tilde{D} \subset \tilde{X}$. Let $\tilde{\omega}_X$ be a $TfS_2$-dualizing sheaf on $X$ and set $\tilde{\omega}_X := \pi^! \tilde{\omega}_X$. Then there is an exact sequence

$$0 \rightarrow \tilde{\omega}_X \rightarrow \pi_*\tilde{\omega}_X(\tilde{D}) \stackrel{T_D}{\rightarrow} \text{Ext}^1_X(\mathcal{O}_D, \tilde{\omega}_X) \rightarrow 0. \quad (57.1)$$

If $X$ is CM and $\tilde{\omega}_X = \omega_X$ is a dualizing sheaf then the sequence becomes

$$0 \rightarrow \omega_X \rightarrow \pi_*\omega_X(\tilde{D}) \stackrel{T_D}{\rightarrow} \omega_D \rightarrow 0. \quad (57.2)$$

Proof. Start with the exact sequence

$$0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0. \quad (57.3)$$

Take $\mathcal{H}om_X(\mathcal{O}_X(-D), \tilde{\omega}_X)$ to get

$$0 \rightarrow \tilde{\omega}_X \rightarrow \mathcal{H}om_X(\mathcal{O}_X(-D), \tilde{\omega}_X) \rightarrow \text{Ext}^1_X(\mathcal{O}_D, \tilde{\omega}_X) \rightarrow \text{Ext}^1_X(\mathcal{O}_X, \tilde{\omega}_X) = 0.
By (31.3),
\[
\text{Hom}_X(\mathcal{O}_X(-D), \hat{\omega}_X) = \text{Hom}_X(\pi_*\mathcal{O}_X(-D), \hat{\omega}_X)
\]
\[
= \text{Hom}_X(\mathcal{O}_X(-D), \hat{\omega}_X) = \hat{\omega}_X(\overline{D}),
\]
proving (1). If $X$ is CM then general duality shows that $\text{Ext}^1_X(\mathcal{O}_D, \omega_X) \cong \omega_D$; we will not use this part, see [BH93, 3.3.7] or [Sta15, Tag 0AX0] for proofs.

Note finally that the $\text{socle}$ of the trace $\text{map} tr :$ socle $(\overline{D})$ gives an isomorphism $\overline{\omega} \cong \text{Supp}(\overline{\omega})$ using that $\text{socle}(\overline{\omega}) \subseteq \text{Supp}(\overline{\omega})$.

We will write this down very explicitly in (58).

□

58 (Construction of $\omega_C$ from $\omega_{\overline{C}}$). Let $(0 \in C)$ be a local, 1-dimensional, reduced scheme whose normalization $\pi : C \to C$ is finite. Since $C$ is regular, $\omega_C$ exists, for example we can choose $\omega_{\overline{C}} = \mathcal{O}_{\overline{C}}$. We give a construction of $\omega_C$ starting with $\omega_{\overline{C}}$.

If $\omega_C$ exists then it sits in the exact sequence (57.2). The other 2 sheaves in (57.2) are $\pi_*\omega_D$ and $\omega_D$, whose existence is already known. Thus it is natural to set

\[
\Omega := \ker[\pi_*\omega_C(\overline{D}) \xrightarrow{\mathcal{R}} \pi_*\omega_D \xrightarrow{\text{tr}} \omega_D],
\]

and aim to show the following.

Claim 58.2. $\Omega$ is a dualizing sheaf over $C$.

Note that for curves over a field the Poincaré residue map $R_{\overline{C}/D}$ and the trace map $tr_{\overline{D}/D}$ are both canonical. Over a general scheme the dualizing sheaf is defined only up to tensoring with a line bundle, so, I believe that, depending on the choices we make, either $R_{\overline{C}/D}$ or $tr_{\overline{D}/D}$ in (58.1) is not canonical. This is not a problem since $\omega_C$ is not unique as a subsheaf of $\pi_*\omega_{\overline{C}}$; we can multiply it by any section of $\mathcal{O}_{\overline{C}}$.

Since $\pi$ is birational, $\Omega$ is dualizing at the generic points of $C$, thus, by (58), the following implies (58.2).

Claim 58.3. $\text{Ext}^1_C(k, \Omega) \cong k$.

Proof. To fix our notation, we have $\pi : (P \subset \overline{C}) \to (0 \in C)$ where $P = \{p_i \in I\} = \pi^{-1}(0)$ and with conductors $D \subset C$ and $\overline{D} \subset \overline{C}$. Set $k := k(0)$ and $k_i := k(p_i)$. The conductor can be written as $\overline{D} = \sum r_i[p_i]$ for some $r_i \geq 1$ and $P = \text{Supp }\overline{D}$. We can indentify $\omega_{\overline{D}} \cong \omega_{\overline{C}}(\overline{D})/\omega_{\overline{C}}$ and $\text{socle}(\omega_{\overline{D}}) \cong \omega_C(P)/\omega_{\overline{C}}$.

The residue map gives an isomorphism

\[
R : \text{socle}(\omega_{\overline{D}}) \cong \sum k_i,
\]

which is, however, not canonical.

Another way to represent $\omega_{\overline{D}}$ is using the isomorphism

\[
\omega_{\overline{D}} \cong \text{Hom}_D(\mathcal{O}_D, \omega_D).
\]

Using that $\text{socle}(\omega_{\overline{D}}) \cong k$, under this isomorphism the socle is represented as

\[
\text{socle}(\omega_{\overline{D}}) \cong \sum \text{Hom}_k(k_i, k).
\]

Using the form (58.5), the $\text{socle}$ of the trace map

\[
\text{tr} : \text{socle}(\omega_{\overline{D}}) \to \text{socle}(\omega_D) \cong k
\]
sends \( \{(\phi_i : k_i \to k) : i \in I\} \) to \( \sum_i \phi_i(1) \). Since the trace is non-degenerate, using the form \( (58) \) we get instead the representation
\[
\{(x_i \in k_i) : i \in I\} \mapsto \sum_i \text{tr}_{k_i/k}(c_i x_i) = 0 \quad \text{for some} \quad c_i \in k_i^*,
\]
where the \( c_i \) arise from the unknown isomorphism \( k_i \cong \text{Hom}_k(k_i, k) \). (Over a field, the correct choices lead to \( c_i = 1 \) for every \( i \), see \[\text{Ser59}\] Secs.IV.9–10.) We can summarize these discussions as follows.

Claim \( 58 \). \( \Omega \cap H^0(\bar{C}, \omega_{\bar{C}}(P)) \subset H^0(\bar{C}, \omega_{\bar{C}}(P)) \) is a \( k \)-hypersurface defined by the equation
\[
\sum_i \text{tr}_{k_i/k}(c_i \cdot R_{p_i}(\sigma)) = 0,
\]
that is, a section \( \sigma \) of \( H^0(\bar{C}, \omega_{\bar{C}}(P)) \) is in \( \Omega \) iff it satisfies \( (58) \).

Now back to the proof of \([58]3\). For any nonsplit extension \( 0 \to \Omega \to \Omega' \to k \to 0 \), we can view \( \Omega' \) uniquely as a subsheaf of \( \omega_C(\hat{D} + P) \). Pick a section \( \sigma \) of \( \Omega' \) mapping to \( 1 \in k \). At the points \( p_i \in \bar{C} \) we can write \( \sigma = v_i x_i^{-r_i} \) where \( x_i \) is a local parameter and \( v_i \) a local section of \( \omega_C \) at \( p_i \).

Since \( m_0 \sigma \in \Omega \) and \( \mathcal{O}_C(-D) \subset m_0 \), we see that \( \mathcal{O}_C(-D) \cdot \sigma \in \Omega \). For any \( a_i \in k_i \) there is a \( g \in \mathcal{O}_C(-D) \) such that \( g = u_i x_i^{r_i} \) where \( u_i \) is a local section of \( \mathcal{O}_C \) at \( p_i \) such that \( u_i(p_i) = a_i \in k_i \). Thus \( g \sigma \in \Omega \) and \( g \sigma = u_i v_i x_i^{-1} \) at \( p_i \) for every \( i \). By \( (58) \) we get the equation
\[
\sum_i \text{tr}_{k_i/k}(a_i c_i R_{p_i}(v_i x_i^{-1})) = 0 \quad \text{for every} \quad a_i \in k_i.
\]
Since the trace is non-degenerate, we conclude that \( R_{p_i}(v_i x_i^{-1}) = 0 \) for every \( i \). That is, \( v_i x_i^{-1} \) is a local section of \( \omega_{\bar{C}} \) at \( p_i \). Therefore \( \sigma = (v_i x_i^{-1}) x_i^{-r} \) is a local section of \( \omega_{\bar{C}}(\hat{D}) \) at each \( p_i \). Thus \( \Omega' \subset \omega_{\bar{C}}(\hat{D}) \) and we get a map \( \Omega' \to k \to \omega_D \).
The image of \( k \) is then the socle of \( \omega_D \), hence \( \Omega' \) is the preimage of \( \text{socle}(\omega_D) \cong k \). Thus \( \Omega' \) is unique and so \( \mathcal{E}xt_C^1(k, \Omega) \cong k \). This proves \( (58) \) and hence also \( (58) \).

5. Duality for 1-dimensional schemes

One can give a complete characterization of those 1-dimensional schemes that have a dualizing sheaf. This is probably known but I did not find a complete reference. The key results are the next local characterization and the global statement Proposition 5.

Theorem 59. \([\text{FFGR75}]\) \(5.3\) Let \( (R, m) \) be 1-dimensional local ring. Then \( R \) has a dualizing module iff the generic fibers of \( R \to \hat{R} \) are Gorenstein.

(Aside. By \([\text{Kaw00}]\) \(1.4\), an arbitrary local ring has a dualizing complex iff it is a quotient of a local, Gorenstein ring, but this is very hard to use in practice. See \([\text{BH93}]\) 3.3.6 for the simpler CM version.)

Note that if the normalization \( \hat{R} \) is finite over \( R \) then the completion of \( R/\sqrt{0} \) is reduced by \([\text{Kru30}]\) Satz 9 (see also \([\text{Kol07}]\) 1.101). Therefore the generic fibers of \( R \to \hat{R} \) are sums of fields, hence Gorenstein. Thus \( (59) \) is a much stronger existence result than \( (52) \), though the latter gives \( \omega_R \) in a more concrete form.

For completeness’ sake, we outline the standard proof of the following special case, which starts with complete local rings and descends from there.

Proposition 60. Let \( (R, m) \) be a 1-dimensional, \( S_1 \) local ring such that the completion of \( R/\sqrt{0} \) is reduced. Then \( R \) has a dualizing module.
Proof. Set $k = \frac{R}{m}$ and assume first that $\text{char } R = \text{char } k$. If $R$ is complete, then we can view $R$ as a $k$-algebra, cf. [Sta15 Tag 0323]. Let $y_1, \ldots, y_n \in m$ be a system of parameters, where $n = \dim R$. Then $R$ is finite over the power series ring $k[[y_1, \ldots, y_n]]$. Since $k[[y_1, \ldots, y_n]]$ is a dualizing module over itself by (28), $R$ has a $\text{TF}S_k$-dualizing module by (29).

In the non-complete case, let $Q(\hat{R})$ be the total ring of quotients of $R$, $Q(\hat{R})$ the total ring of quotients of $\hat{R}$. Let $\omega_{Q(R)}$ and $\omega_{Q(\hat{R})}$ be the corresponding dualizing modules. We check below that

$$Q(\hat{R}) \otimes_{Q(R)} \omega_{Q(R)} \cong \omega_{Q(\hat{R})}.$$  \hfill (60.1)

If this holds then let $\Omega_R$ be a finite $R$-module such that $Q(R) \otimes_R \Omega_R \cong \omega_{Q(R)}$.

By (61) we can realize $\omega_R$ as a submodule of $\Omega_R$ with finite quotient. So there is a submodule $\omega_R \subset \Omega_R$ such that $\omega_R = \omega_{\hat{R}}$. Thus $\omega_R$ is a dualizing module by (60).

In order to prove (60.1) we check the conditions of (35.5). We know that $\text{length}(\omega_{Q(R)}) = \text{length}(Q(R))$ and this is preserved when we tensor by $Q(\hat{R})$. We also know that

$$\text{Hom}_{Q(\hat{R})}(Q(\hat{R}) \otimes_{Q(R)} K(R), Q(\hat{R}) \otimes_{Q(R)} \omega_{Q(R)}) \cong Q(\hat{R}) \otimes_{Q(R)} K(R).$$

If the completion of $R/\sqrt{0}$ is reduced, then the generic fibers of $R \to \hat{R}$ are sums of fields, hence $K(\hat{R}) = Q(\hat{R}) \otimes_{Q(R)} K(R)$ and we are done. (A very similar argument proves (59). One needs to argue that the socle of $Q(\hat{R}) \otimes_{Q(R)} \omega_{Q(R)}$ is the socle of $Q(\hat{R}) \otimes_{Q(R)} K(R)$ and the latter is a sum of the residue fields if $Q(\hat{R}) \otimes_{Q(R)} K(R)$ is Gorenstein. See also [BH93 3.3.14] or [Sta15 Tag 0E4D].)

I do not know a similarly elementary proof in the mixed characteristic case. Here one writes $R$ as a quotient of a power series ring $S := \Lambda[[x_1, \ldots, x_r]]$ where $\Lambda$ is a complete DVR and then $\text{Ext}^2_S S_{-\dim R}(R, S)$ is a $\text{TF}S_k$-dualizing module; see [BH93 3.3.7] or [Eis95 Sec.21.6]. It is also a very special case of general duality theory as in [Sta15 Tag 0AX9]. The rest goes as before. \hfill \Box

Lemma 61. Let $R$ be a 1-dimensional local ring and $Q(R)$ its total ring of quotients. For finite $R$-modules $M, N$ the following are equivalent.

1. $Q(R) \otimes_R M \cong Q(R) \otimes_R N$.
2. There is a map $\phi: M \to N$ whose kernel and cokernel are torsion. \hfill \Box

For the global existence we need the following criterion for the dualizing module over a nonreduced ring.

Lemma 62. Let $R$ be a 1-dimensional, $S_1$, local ring and set $S := R/\sqrt{0}$. Let $\Omega_R$ be a finite $R$-module such that $\text{socle}(\Omega_R) \cong \omega_S$ and $\text{Ext}^1_R(S, \Omega_R) = 0$. Then $\Omega_R$ is a dualizing $R$-module.

Proof. By (34), $\Omega_R$ is dualizing at the generic points. For a maximal ideal $m \subset S$ with residue field $k$, duality gives the exact sequence

$$0 \to \text{Hom}_R(S, \Omega_R) \to \text{Hom}_R(m, \Omega_R) \to \text{Ext}^1_R(k, \Omega_R) \to 0.$$  \hfill (62.1)

The image of an $R$-homomorphism from an $S$-module to $\Omega_R$ is contained in the socle, which is $\omega_S$. Thus (62.1) can be rewritten as

$$0 \to \text{Hom}_S(S, \omega_S) \to \text{Hom}_S(m, \omega_S) \to \text{Ext}^1_R(k, \Omega_R) \to 0.$$  \hfill (62.2)
This shows that \( \text{Ext}^1_X(k, \Omega_R) \cong \text{Ext}^1_X(k, \omega_S) \cong k \), and so \( \Omega_R \) is a dualizing module by (60).

63 (Proof of Proposition 64). Let \( \omega_X \) be a dualizing sheaf. Then every localization has a dualizing module by (30) and \( \omega_{\text{red} X} := \text{Hom}_X(\mathcal{O}_{\text{red} X}, \omega_X) \) is a dualizing sheaf of red \( X \) by (62). It is a coherent, rank 1, torsion free sheaf, hence locally free over a dense, open subset, proving (62).

Conversely, let \( \Omega \) be a coherent, torsion free sheaf on \( X \) that is dualizing at all generic points of \( X \). This implies that the support of \( \mathcal{E}xt^1_X(\mathcal{O}_{\text{red} X}, \Omega) \) is nowhere dense. Let \( U \subset X \) be a dense open subset such that red \( U \) is Gorenstein, socle(\( \Omega \)) is invertible on red \( U \) and \( \mathcal{E}xt^1_U(\mathcal{O}_{\text{red} U}, \Omega|_U) = 0 \). Then \( \Omega|_U \) is dualizing by (62).

For each \( x \in X \setminus U \) let \( \omega_{X_x} \) be a dualizing sheaf on \( X_x \). Over the generic points \( g_x \in X_x \) we can fix isomorphisms of \( \Omega|_{g_x} \) and \( \omega_{X_x}|_{g_x} \) and glue the sheaves \( \Omega \) and \( \omega_{X_x} \) together. We get a dualizing sheaf on \( X \).

\[ \square \]

6. Dualizing module of CM rings

We start with 2 observations that allow us to reduce various questions about CM modules to 1-dimensional CM modules. Recall that a finite \( R \)-module \( N \) is Cohen-Macaulay or CM for short, if there is a system of parameters \( x_1, \ldots, x_n \) such that \( x_{i+1} \) is a non-zerodivisor on \( N/(x_1, \ldots, x_i)N \) for \( i = 0, \ldots, \dim N - 1 \).

64 (CM modules and dimension reduction). Let \( (R, m) \) be a local ring and \( M \) a finite \( R \)-module.

Assume that \( \dim M \geq 2 \) and \( x \in m \) is not contained in any of the positive dimensional associated primes of \( M \). Then \( M \) is CM iff \( M/xM \) is. Using this inductively we get the following.

Claim 64.1. Let \( (R, m) \) be a local ring, \( x_1, \ldots, x_n \) a system of parameters and \( M \) a finite \( R \)-module of dimension \( d \). Then for general \( x'_i \equiv x_i \mod m^2 \), the module \( M \) is CM iff \( M/(x'_1, \ldots, x'_{d-1})M \) is CM.

\[ \square \]

Next note that, by (10), \( M \) admits a filtration where each successive quotient \( G_j \) is a rank 1 torsion free module over \( R/P_j \) for some prime ideal \( P_j \). There is thus a non-zerodivisor \( g \in m \) such that each \( (G_j)_g \) is free over \( (R/P_j)_g \). Choose \( x \in m \) such that \( g \) is not contained in any of the minimal primes of \( M/xM \). Let \( P \) be a minimal associated prime of \( M \) and \( Q \) a minimal associated prime of \( M/xM \) that contains \( P \). Then

\[ \text{length}_Q(M/xM)_Q = \text{length}_Q(R_P/xR_P) \cdot \text{length}_P(M_P). \]

Using this inductively, we obtain the following.

Claim 64.2. Let \( (R, m) \) be a local ring, \( x_1, \ldots, x_n \) a system of parameters and \( M, N \) finite \( R \)-modules of dimension \( d \). Assume that \( \text{length}_P(M_P) \geq \text{length}_P(N_P) \) for every \( d \)-dimensional prime \( P \). Then, for general \( x'_i \equiv x_i \mod m^2 \),

\[ \text{length}_Q(M/(x'_1, \ldots, x'_{d-1})M) \geq \text{length}_Q(N/(x'_1, \ldots, x'_{d-1})N) \]

for every 1-dimensional prime \( Q \).

\[ \square \]

There are several standard definitions of a dualizing/canonical module.

Definition 65 (Dualizing or canonical module). Let \( (R, m) \) be a local, CM ring of dimension \( n \), \( k := R/m \) and \( M \) a finite \( R \)-module. Then \( M \) is a dualizing module or a canonical module iff any of the following equivalent conditions hold.
Theorem 66. Let \((R, m)\) be a local CM ring of dimension \(n\) and \(k := R/m\). Let \(\Omega\) be a finite \(R\)-module. The following are equivalent.

1. \(\Omega\) is dualizing.
2. \(\text{Ext}_R^i(k, \Omega) = \delta_{i0} \cdot k\) for every \(i\).
3. \(\text{Ext}_R^i(k, \Omega) = \delta_{i0} \cdot k\) for \(0 \leq i \leq n\) and \(\Omega_P\) is dualizing over \(R_P\) for every minimal prime \(P \subset R\).
4. \(\text{Ext}_R^i(k, \Omega) = \delta_{i0} \cdot k\) for \(0 \leq i \leq n\) and \(\text{length}_P \Omega_P = \text{length}_P R_P\) for every minimal prime \(P \subset R\).
5. \(\text{Ext}_R^i(k, \Omega) = \delta_{i0} \cdot k\) for \(0 \leq i \leq n\) and \(\text{length}_P \Omega_P \geq \text{length}_P R_P\) for every minimal prime \(P \subset R\).

Proof. The first 2 claims are equivalent by our definition (65.1) and it is clear that each assertion implies the next one. Thus it remains to prove that (5) \(\Rightarrow\) (2).

If \(n = 0\) then the first part of (5) says that \(\text{socle}(\Omega) \cong k\), so we can realize \(\Omega\) as a submodule of \(E(k)\), the injective hull of \(k\). Thus \(\text{length}\Omega \leq \text{length} E(k) = \text{length} R\), where the last equality holds by (83). The second part of (5) says that \(\text{length}\Omega \geq \text{length} R\). Thus \(\Omega = E(k)\) is dualizing.

If \(n = 1\) then let \(x \in m\) be a non-zerodivisor. A special case of (72) says that \(\text{socle}(\Omega/x\Omega) \cong k\). By (71) \(\text{length}_Q (\Omega/x\Omega) \geq \text{length}_Q (R/xR)\). Thus \(\Omega/x\Omega\) is dualizing over \(R/xR\) by the already settled \(n = 0\) case. In particular,

\[
\text{Ext}_R^i(k, \Omega/x\Omega) = \delta_{i0} \cdot k \quad \text{for every } i,
\]

hence \(\text{Ext}_R^i(k, \Omega) = \delta_{i1} \cdot k\) for every \(i\) by (72).

If \(n \geq 2\) then the first part of (5) says that \(\Omega\) is CM. By (64.2), we can choose a system of parameters \(x_1, \ldots, x_n\) such that

\[
\text{length}_Q (\Omega/(x_1, \ldots, x_{n-1})\Omega) \geq \text{length}_Q (R/(x_1, \ldots, x_{n-1})R)
\]

for every 1-dimensional prime \(Q\). Applying (72) gives that

\[
\text{Ext}_R^i(k, \Omega/(x_1, \ldots, x_{n-1})\Omega) = \delta_{i1} \cdot k \quad \text{for } i \leq 1.
\]

The \(n = 1\) case now gives that \(\Omega/(x_1, \ldots, x_{n-1})\Omega\) is dualizing over \(R/(x_1, \ldots, x_{n-1})R\) and applying (72) again shows that \(\text{Ext}_R^i(k, \Omega) = \delta_{i1} \cdot k\) for every \(i\). Thus \(\Omega\) is dualizing over \(R\).
The CM condition, that is, the vanishing of $\text{Ext}^i_R(k, \Omega) = \delta_{in} \cdot k$ for $0 \leq i < n$ can be checked in other ways.

**Lemma 67.** Let $(R, m)$ be a local ring of dimension $n$ and $x_1, \ldots, x_n$ a system of parameters. Let $M, N$ be finite $R$-modules of dimension $d$ such that $\text{length}_P M_P \geq \text{length}_P N_P$ for every $d$-dimensional prime $P \subset R$. Assume that $N$ is CM. Then

1. $\text{length}_0(M/(x_1, \ldots, x_n)M) \geq \text{length}_0(N/(x_1, \ldots, x_n))$ and
2. equality holds iff $M$ is also CM and $\text{length}_P M_P = \text{length}_P N_P$ for every $d$-dimensional prime $P$.

Note that $M, N$ need not be isomorphic in case (2).

**Proof.** There is nothing to prove if $d = 0$. If $d = 1$ then $M' := M/\text{tors}(M)$ is CM and, by (71),

$$
\text{length}_0(M'/x_1 M') = \sum_P \text{length}_P(M'_P) \cdot \text{length}_0((R/P)/x_1(R/P))
$$

and equality holds iff $\text{length}_P M_P = \text{length}_P N_P$ for every 1-dimensional prime $P$. Therefore

$$
\text{length}_0(M/x_1 M) = \text{length}_0(M'/x_1 M') + \text{length}_0(\text{tors}(M)/x_1 \text{tors}(M))
$$

and

$$
\text{length}_0(N/x_1 N) = \text{length}_0(N/x_1 N) + \text{length}_0(\text{tors}(M)/x_1 \text{tors}(M)).
$$

Since $\text{tors}(M)/x_1 \text{tors}(M) = 0$ iff $\text{tors}(M) = 0$, this settles the $d = 1$ case. As in the proof of (66), the $d \geq 2$ case reduces to the above using (64–1–2).

The following is an immediate combination of (65), (66) and (67).

**Corollary 68.** Let $(R, m)$ be a local, CM ring of dimension $n$ with residue field $k$ and $x_1, \ldots, x_n$ a system of parameters. Let $\Omega$ be a finite $R$-module. Then $\Omega$ is dualizing iff

1. $\text{length}_P \Omega_P \geq \text{length}_P R_P$ for every minimal prime $P \subset R$ and
2. $\text{socle}(\Omega/(x_1, \ldots, x_n)\Omega) \cong k$.

The above result and [Eis95 Sec.21.3] leads to the following.

**Question 69.** Let $(R, m)$ be a local, CM ring of dimension $n$ with dualizing module $\omega_R$ and $x_1, \ldots, x_n$ a system of parameters. Let $M$ be a finite $R$-module such that $M/(x_1, \ldots, x_n)M \cong E(k)$, the injective hull of $k := R/m$ over $R/(x_1, \ldots, x_n)$. Is then $M$ a quotient of $\omega_R$?

The next example shows that this is not the case.

**Example 70.** Fix $m \geq 3$ and consider the monomial ring $R := k[t^i : i \geq m]$. Set $x = t^m$, then $R/xR = \langle 1, t^{m+1}, \ldots, t^{2m-1} \rangle$ and $(t^{m+1}, \ldots, t^{2m-1})^2 = 0$ in $R/xR$. We can write $\omega_R = \langle t^{-m}, \ldots, t^{-2}, 1, t, \ldots \rangle \cdot dt$. Then

$$
\omega_R/x\omega_R = \langle t^{-m}, \ldots, t^{-2}, t^{m-1} \rangle \cdot dt.
$$

Setting $\sigma_i = t^{-i} dt$, $\Sigma := \langle \sigma_m, \ldots, \sigma_2 \rangle$ and $s = t^{m-1} dt$, the module structure on $\omega_R/x\omega_R$ is given by $t^i \sigma_j = \delta_{i,j} \cdot s$.

**Claim** 70.2. $\dim \text{Ext}_R(\omega_R/x\omega_R, k) = m^2 - m - 1$. 

---

[71]: Footnote or reference text
[72]: Footnote or reference text
Proof. Consider an extension $0 \to k \to M \xrightarrow{s} \omega_R/x\omega_R \to 0$. If we fix a lifting $\bar{x} \in c^{-1}(s)$ then we get $k$-linear maps

$$\bar{x} := x \circ c^{-1} : \Sigma \to k \quad \text{and} \quad \tau_i := t^i \circ c^{-1} : \Sigma \to c^{-1}(s)/(\bar{s}),$$

the latter for $i = m+1, \ldots, 2m-1$. These maps can be chosen arbitrarily and they determine the $R$-module structure of $M$. \hfill $\square$

Note that if $\bar{x} \neq 0$ then $M/xM \cong \omega_R/x\omega_R$. Extensions as above that are quotients of $\omega_R$ correspond to maps $\text{Hom}_R(x\omega_R, k) \cong \text{Hom}_R(\Sigma, k) \cong k^{m-1}$. Comparing this with (70.2) gives a negative answer to (69).

Corollary 70.3. For $m \geq 3$ there are Artinian $R$-modules $M$ such that $M/xM \cong \omega_R/x\omega_R$ but $M$ is not a quotient of $\omega_R$. \hfill $\square$

The next computation shows what changes for torsion free $R$-modules. These are flat over $k[x]$; we determine the 1st infinitesimal extension.

Claim 70.4. Let $M$ be an extension of $\omega_R/x\omega_R$ by $\omega_R/x\omega_R$ such that $M/xM \cong \omega_R/x\omega_R$. Then $M \cong \omega_R/x^2\omega_R$.

Proof. By assumption we have an extension

$$0 \to \omega_R/x\omega_R \xrightarrow{i_x} M \to \omega_R/x\omega_R \to 0,$$

as a sequence of $R/x^2R$-modules. Apply duality to it over $R/x^2R$. Note that

$$\text{Hom}_{R/x^2R}(\omega_R/x\omega_R, \omega_R/x^2\omega_R) = \text{Hom}_{R/x^2R}(\omega_R/x\omega_R, x\omega_R/x^2\omega_R) \cong \text{Hom}_{R/x^2R}(\omega_R/x\omega_R, \omega_R/x\omega_R) = R/xR.$$ 

So we get $0 \to R/xR \to M^* \to R/xR \to 0$. The quotient map $R/x^2R \to R/xR$ lifts to $R/x^2R \to M^*$. Duality now gives $M \to \omega_R/x^2\omega_R$. Since $xM = i_x(\omega_R/x\omega_R)$ by assumption, the map $M \to \omega_R/x^2\omega_R$ is an isomorphism. \hfill $\square$

The following is a special case of Herbrand quotients (see [Ful84, A.1]).

Lemma 71. Let $(R, m)$ be a local, 1-dimensional, $S_1$ ring with minimal primes $P_i$. Let $F$ be a finite, torsion free $R$-module and $r \in m$ a non-zero divisor. Then

$$\text{length}_h(F/rF) = \sum_i \text{length}_{P_i}(F_{P_i}) \cdot \text{length}_h((R/P_i)/r(R/P_i)).$$

Proof. Both sides are additive on short exact sequences of finite, torsion free modules. Thus, by [L1], it is enough to prove the claim when $R$ is integral and $F$ has rank 1. Then we can realize $F$ as an ideal $F \subseteq R$ such that $R/F$ has finite length. Computing $\text{length}_h(R/rF)$ two ways we get that

$$\text{length}_h(R/rR) + \text{length}_h(rR/rF) = \text{length}_h(R/F) + \text{length}_h(F/rF).$$

Since multiplication by $r$ gives an isomorphism $R/F \cong rR/rF$, we are done. \hfill $\square$

We have repeatedly used the following result of [Ree53], see also [BH93, 3.1.16].

Lemma 72. Let $R$ be a ring, $N, M$ finite $R$-modules on $r \in R$. Assume that $r$ is a non-zero divisor on $R, M$ and $rN = 0$. Then there are canonical isomorphisms

$$\text{Ext}^i_{R}(N, M) \cong \text{Ext}^i_{R/rR}(N, M/rM)$$

for every $i \geq 0$. \hfill $\square$
73 (Cones over surfaces). Let us recall first the basic facts about cones as in [Kol13 Sec.3.1]. Let $S$ be a smooth, projective surface, $H$ an ample line bundle and

$$(0, X) := C(S, L) := \text{Spec } \sum_{m \geq 0} H^0(S, H^m)$$

the corresponding affine cone over $S$. Then $X$ is CM iff $H^1(S, H^m) = 0$ for every $m \in \mathbb{Z}$. If this holds then $\text{Cl}(X) \cong \text{Cl}(S)/[H] = \text{Pic}(S)/[H]$.

If $L$ is a line bundle on $S$ then let $C(L)$ denote the corresponding divisorial sheaf on $X$. It is the sheafification of the module $\sum_{m=-\infty}^{\infty} H^0(S, L \otimes H^m)$. Note that $C(L)$ is CM iff $H^1(S, L \otimes H^m) = 0$ for every $m \in \mathbb{Z}$.

Claim 73.1. Only finitely many of the $C(L)$ are CM.

Proof. Let $L$ be a line bundle on $S$ and $m := [(L \cdot H)/(H \cdot H)]$. Then $C(L) \cong C(L \otimes H^{-m})$ and the intersection number $((L \otimes H^{-m}) \cdot H)$ is between 0 and $(H \cdot H)$. Set

$$\text{Cl}^b(S) := \{(L) : 0 \leq (L \cdot H) \leq (H \cdot H)\} \subset \text{Cl}(S).$$

We have proved that $\text{Cl}^b(S) \to \text{Cl}(X)$ is surjective. It is thus enough to show that $h^1(S, L) = 0$ holds for only finitely many line bundles in $\text{Cl}^b(S)$.

By the Hodge index theorem, $L \mapsto (L \cdot L)$ is a negative definite quadratic function on $\text{Cl}^b(S)$. Thus, by Riemann-Roch, $L \mapsto \chi(S, L)$ is the sum of a negative definite quadratic function and of a linear function.

On the other hand, by the Matsusasaka inequality (see [Mat72] or [Kol96 VI.2.15.8])

$$h^0(S, L) \leq \frac{(L \cdot H)^2}{(H \cdot H)} + 2,$$

so both $h^0(S, L)$ and $h^2(S, L) = h^0(S, \omega_S \otimes L^{-1})$ are bounded on $\text{Cl}^b(S)$. Thus $L \mapsto h^1(S, L)$ is the sum of a positive definite quadratic function, a linear function and a bounded function on $\text{Cl}^b(S)$. Therefore it has only finitely many zeros. □

7. Appendix by Hailong Dao

Let $X$ be a CM scheme. We say that a coherent sheaf $\Omega$ is CM-dualizing if it is $TfS_2$-dualizing and the duality preserves CM sheaves. That is, if $M$ is torsion free and CM then so is $\text{Hom}_X(M, \Omega)$. Note that a dualizing sheaf $\omega_X$ is also CM-dualizing, see for example [BH93 3.3.10]. If $\dim X = 2$ then CM is the same as $S_2$, thus every $TfS_2$-dualizing sheaf is also CM-dualizing. The situation is, however, quite different if $\dim X \geq 3$, as shown by the following result that answers a question of Kollár that was posed in the first version of this paper.

Theorem 74. Let $(x, X)$ be a local, CM scheme of dimension $\geq 3$. Let $\Omega$ be a torsion free, coherent sheaf on $X$ such that, for every torsion free, CM sheaf $M$, its dual $\text{Hom}_X(M, \Omega)$ is also torsion free and CM. Then $\Omega$ is a direct sum of copies of $\omega_X$.

Proof. Note first that $\Omega$ is CM since $\Omega = \text{Hom}_X(O_X, \Omega)$. Therefore

$$\mathcal{E}xt^j_X(k, \Omega) = 0 \quad \text{for } j < \dim X.$$  \hspace{1cm} (74.1)

Let $k$ be the residue field at $x \in X$ and consider a free resolution of it

$$\cdots \to F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} k.$$
Let $K_i := \ker \phi_i$ be the $i$th syzygy module of $k$ and set $K_0 := k$. Note that the $K_i$ are locally free on $X \setminus \{x\}$, in particular $\mathcal{E}xt^1_X(K_i, M)$ is supported on $\{x\}$ for every $j \geq 1$.

From $0 \to K_{i+1} \to F_{i+1} \to K_i \to 0$ we get that $\mathcal{E}xt^{j-1}_X(k, K_i) \cong \mathcal{E}xt^j_X(k, K_{i+1})$ for every $j \leq \dim X - 1$; in particular $K_i$ is CM for $i \geq \dim X$.

Similarly, we get an exact sequence

$$0 \to \mathcal{H}om_X(K_i, \Omega) \to \mathcal{H}om_X(F_{i+1}, \Omega) \to \mathcal{H}om_X(K_{i+1}, \Omega) \to \mathcal{E}xt^1_X(K_i, \Omega) \to 0$$

and isomorphisms

$$\mathcal{E}xt^j_X(K_{i+1}, \Omega) \cong \mathcal{E}xt^{j+1}_X(K_i, \Omega) \quad \text{for } j \geq 1.$$

Breaking the 4-term sequence into 2 short exact sequence gives that if $\mathcal{H}om_X(K_{i+1}, \Omega)$ is CM then

$$\mathcal{E}xt^j_X(K_i, \Omega) \cong H^2_x(X, \mathcal{H}om_X(K_i, \Omega)).$$

If $i \geq \dim X$ and $\dim X \geq 3$ then this shows that $\mathcal{E}xt^1_X(K_i, \Omega) = 0$ for $i \geq \dim X$.

Using the isomorphisms (7.4.2) we get that

$$\mathcal{E}xt^j_X(k, \Omega) = 0 \quad \text{for } j > \dim X. \quad (7.4.3)$$

Combining this with (7.4.1) gives that $\Omega$ is a direct sum of copies of $\omega_X$ by [BH93, 3.3.28].


This immediately implies the following.

**Corollary 75.** Let $X$ be a CM scheme of pure dimension $\geq 3$ and $\tilde{\omega}_1, \tilde{\omega}_2$ CM-dualizing sheaves on $X$. Then $\tilde{\omega}_1 \cong \tilde{\omega}_2 \otimes L$ for some line bundle $L$ on $X$. \qed

**References**

[BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR 1251956 (95h:13020)

[DK16] Hailong Dao and Kazuhiro Kurano, *Boundary and shape of Cohen-Macaulay cone*, Math. Ann. 364 (2016) 713–736.

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. MR 1329060 (97a:13001)

[EP03] V. Ene and D. Popescu, *Rank one maximal Cohen-Macaulay modules over singularities of type $Y_1^3 + Y_2^3 + Y_3^3 + Y_4^3$*, NATO Sci. Ser. II Math. Phys. Chem. 115 Kluwer Acad. Publ., Dordrecht, (2003), 141–157.

[FFGR75] Robert Fossum, Hans-Bjørn Foxby, Phillip Griffith, and Idun Reiten, *Minimal injective resolutions with applications to dualizing modules and Gorenstein modules*, Inst. Hautes Études Sci. Publ. Math. (1975), no. 45, 193–215. MR 0396529

[Fox72] Hans-Bjørn Foxby, *Gorenstein modules and related modules*, Math. Scand. 31 (1972), 267–284 (1973). MR 0327752

[FR70] Daniel Ferrand and Michel Raynaud, *Fibres formelles d’un anneau local noethérien*, Ann. Sci. École Norm. Sup. (4) 3 (1970), 295–311. MR 0272779

[Ful84] William Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 2, Springer-Verlag, Berlin, 1984. MR MR732620 (85k:14004)

[Gro60] Alexander Grothendieck, *Eléments de géométrie algébrique. I–IV.*, Inst. Hautes Études Sci. Publ. Math. (1960), no. 4, 8,11,17,20,24,28,32.

[Har62] Robin Hartshorne, *Complete intersections and connectedness*, Amer. J. Math. 84 (1962), 497–508. MR 0142547 (26 #116)

[Har77] ——*, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 (57 #3116)
