Abstract

We study (strictly) join irreducible varieties in the lattice of subvarieties of residuated lattices. We explore the connections with well-connected algebras and suitable generalizations, focusing in particular on representable varieties. Moreover we find weakened notions of Halldén completeness that characterize join irreducibility. We characterize strictly join irreducible varieties of basic hoops, and use the generalized rotation construction to find strictly join irreducible varieties in subvarieties of $\mathsf{MTL}$-algebras. We also obtain some general results about linear varieties of residuated lattices, with a particular focus on representable varieties, and a characterization for linear varieties of basic hoops.

1 Introduction

Substructural logics constitute a large class of logical systems algebraizable in the sense of Blok-Pigozzi, where the semantical characterization of provability of the Lindenbaum-Tarski algebraization extends to a characterization of logical deducibility via the algebraic equational consequence (see [25] for a detailed investigation). Substructural logics encompass classical logic, intuitionistic logic, fuzzy logics, relevance logics and many other systems, all seen as logical extensions of the Full Lambek calculus $\mathcal{FL}$. As a consequence of algebraizability, all extensions of $\mathcal{FL}$ are also algebraizable, and the lattice of axiomatic extensions is dually isomorphic to the subvariety lattice of the algebraic semantics, given by the variety of
FL-algebras. In this work we are interested in the positive fragment of FL (the system obtained by removing the constant 0, and consequently negation, from the language), FL+, whose corresponding algebraic semantics is given by the variety of residuated lattices RL.

Our investigation will be carried on in the algebraic framework, and goes in the direction of gaining a better understanding of the lattice of subvarieties of residuated lattices (thus, equivalently, the lattice of axiomatic extensions of the corresponding logics). In particular we study properties, and in some relevant cases we find characterizations, of those varieties that in the lattice of subvarieties are join irreducible or strictly join irreducible. Kihara and Ono showed that in the presence of integrality and commutativity join irreducibility of a variety is characterized by both a logical property, Halládén completeness, and by an algebraic property of the generating algebras, well-connectedness ([32], anticipated in [25]).

We show how both these notions can be generalized for non-integral, non-commutative subvarieties of RL, characterizing join irreducibility in a large class of residuated lattices, that include for instance all normal varieties, representable varieties, and ℓ-groups. Moreover, we answer two questions left open in [32] in our more general setting. In particular, we show that in our more general framework being join irreducible is equivalent to being generated by a subdirectly irreducible algebra, and that in general not all subdirectly irreducible algebras generate strictly join irreducible varieties. A key role is played by results implicit in [23] concerning the axiomatization of the join of varieties of residuated lattices.

We then focus on representable varieties, where well-connectedness is shown to correspond to being totally ordered. In the further presence of divisibility, we use the ordinal sum construction to find sufficient conditions for strict join irreducibility. Then we focus on the variety of basic hoops, where a characterization of strictly join irreducible varieties is given. The representation shows to be analogous to the one found in [11] for BL-algebras (of which basic hoops are 0-free subreducts) and is constructed based off the representation of totally ordered hoops with finite index given in [4].

Given the results in [7], using the generalized rotation construction in [10], we are able to lift the results found about the lattice of subvarieties of basic hoops to particular intervals of the lattice of subvarieties of MTL-algebras (representable, commutative, integral FL-algebras). In the end we focus on particular join irreducible varieties, namely linear varieties, finding some general characterizations in the presence of representability and a characterization for linear varieties of basic hoops, that again we lift to particular subvarieties of MTL.

2 Preliminaries

A residuated lattice is an algebra $A = \langle A, \vee, \wedge, \cdot, /, \backslash, 1 \rangle$ where

1. $\langle A, \vee, \wedge \rangle$ is a lattice;
2. $\langle A, \cdot, 1 \rangle$ is a monoid;
3. $/$ and $\backslash$ are the right and left divisions w.r.t. $\cdot$, i.e. $x \cdot y \leq z$ iff $y \leq x \backslash z$ iff $x \leq z/y$. 

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Residuated lattices form a variety $\mathbf{RL}$ and an axiomatization, together with the many equations holding in these very rich structures, can be found in [15].

A residuated lattice $A$ is integral if it satisfies the equation $x \leq 1$; it is commutative if $\cdot$ is commutative; it is divisible if the ordering of $A$ is the inverse divisibility ordering i.e. for $a, b \in A$

$$a \leq b \quad \text{if and only if} \quad \text{there are } c, d \in A \text{ with } a = cb = bd.$$ The first two properties are clearly equational, while the third is equivalent to the equations ([31])

$$(x/y)y \approx x \land y \approx y(y\backslash x).$$

Finally a residuated lattice is cancellative if the underlying monoid is cancellative in the usual sense; it turns out that this property is equational as well [13]. A residuated lattice is cancellative if and only if it satisfies

$$xy/y \approx x \quad x\backslash xy \approx y.$$ So the classes of residuated lattices that satisfy any combination of integrality, commutativity, divisibility or cancellativity are subvarieties of $\mathbf{RL}$. We shall call the variety of integral residuated lattices $\mathbf{IRL}$, commutative residuated lattices $\mathbf{CRL}$, and their intersection $\mathbf{CIRL}$.

Residuated lattices with an extra constant 0 in the language are called FL-algebras, since they are the equivalent algebraic semantics of the Full Lambek calculus $\mathcal{FL}$ (for precise definition of this calculus, or more details on algebraizability, see [25]). Residuated lattices are then the equivalent algebraic semantics of the positive (i.e., 0-free) fragment of $\mathcal{FL}$, $\mathcal{FL}^+$.

In particular, for every variety of residuated lattices $V$ over a language $\mathcal{L}$, its corresponding logic is $\mathcal{L}_V = \{ \varphi \in Fm_{\mathcal{L}} : V \models \varphi \geq 1 \}$, where $Fm_{\mathcal{L}}$ is the set of formulas over $\mathcal{L}$. Conversely, given any extension $\mathcal{L}$ of $\mathcal{FL}^+$ axiomatized by the set of formulas $\Phi$, its equivalent algebraic semantics is the subvariety $V_{\mathcal{L}}$ of $\mathbf{FL}$ axiomatized by the set of equations $\{ \varphi \geq 1 : \varphi \in \Phi \}$: Moreover, the following is shown in [27].

**Theorem 2.1.** The maps $L : V \mapsto L_V$ and $V : \mathcal{L} \mapsto V_{\mathcal{L}}$ are mutually inverse, dual lattice isomorphisms between the lattice of subvarieties of $\mathbf{RL}$ and the lattice of logical extensions of $\mathcal{FL}^+$.

It is useful to observe that a conjunction of a finite number of equations is equivalent in $\mathbf{RL}$ to a single inequality of the kind $p \geq 1$, or equivalently to an equation of the kind $p \land 1 \approx 1$ (Lemma 3.1 in [24]).

With respect to their structure theory, residuated lattices are congruence permutable and congruence point-regular at 1, and also congruence distributive (since they have lattice terms). It follows that FL-algebras and residuated lattices are ideal determined [29] and have a good theory of ideals in the sense of [8]. The role of the ideals is played by the congruence filters; we call a filter $F$ of $A$ a subset of $A$ that is a lattice filter containing 1 and is closed under multiplication. Both filters and congruence filters form algebraic lattices; if $A^+ = \{ a : a \geq 1 \}$, then for any $\theta \in \text{Con}(A)$,

$$A^+/\theta = \bigcup \{ a/\theta : a \geq 1 \}$$
is a congruence filter. Moreover the two mappings

\[ \theta \mapsto A^+ / \theta \quad F \mapsto \theta_F = \{(a, b) : a/b, b/a \in F\} \]

are mutually inverse order preserving maps from \( \text{Con}(A) \) to the congruence filter lattice of \( A \), that are therefore isomorphic. Notice that \( A^+ / \theta \) defines the same set as \( \lceil 1 \rceil_\theta \), used for example in \([23]\).

A residuated lattice is normal if every filter is a congruence filter and a variety of residuated lattices is normal if each of its members is normal. To determine which varieties are normal a description of the congruence filter generated by a subset is in order. Following \([31]\) we define the principal filter \( \Gamma^k(A) \) of iterated conjugates in \( A \). Let \( a \in A \) be a residuated lattice; then a filter \( F \) of \( A \) is a congruence filter and (1) holds. Assume then (2), and let \( a \in A \). This lemma allows a handy description of the congruence filter generated by a subset \( X \) of a residuated lattice \( A \); an iterated conjugate in \( A \) is a unary term \( \gamma_{a_1}(\gamma_{a_2}(\ldots \gamma_{a_n}(x))) \) where \( a_1, \ldots, a_n \in A \) and \( \gamma_{a_i} \in \{l_b(a), r_b(a)\} \) for \( i = 1 \ldots n, n \in \mathbb{N} \). We will denote with \( \Gamma^n(A) \) the set of iterated conjugates in \( A \) of length \( n \) (i.e. a composition of \( n \) left and right conjugates).

**Corollary 2.3.** Let \( X \subseteq A \); then the congruence filter generated by \( X \) in \( A \) is the set

\[ F(X) = \{b \in A : \gamma_1(a_1) \ldots \gamma_n(a_n) \leq b, n \in \mathbb{N}, a_1, \ldots, a_n \in X \cup \{1\}, \gamma_i \in \Gamma^k(A) \text{ for some } k \in \mathbb{N}, i = 1 \ldots n\} \]

Section 4 of \([26]\) contains an implicit characterization of normal residuated semilattices. Let’s make it explicit for residuated lattices.

**Lemma 2.4.** For a residuated lattice \( A \) the following are equivalent:

1. \( A \) is normal;
2. every principal filter of \( A \) is a congruence filter;
3. for all \( a \in A \), given any \( b \in A \) there exist \( n, m \in \mathbb{N} \) (possibly depending on \( b \)) such that \((a \land 1)^n b \leq ba \) and \( b(a \land 1)^m \leq ab \).

**Proof.** (1) clearly implies (2). Assume then (2), and let \( a \in A \); then the principal filter \( F \) generated by \( a \) is a congruence filter. If \( a \geq 1 \) (3) can be easily checked. From Lemma \([22]\) it follows that for any \( b \in A \) \( l_b(a), r_b(a) \in F \). This implies that there are \( n, m \in \mathbb{N} \) such that \( a^n \leq l_b(a) \) and \( a^m \leq r_b(a) \), from which (3) follows by residuation and order preservation. Finally assume (3) and let \( F \) be a filter of \( A \); if \( a \in F \) and \( b \in A \) then there is an \( n \in \mathbb{N} \) with \((a \land 1)^n b \leq ba \). It follows that \((a \land 1)^m \leq ba/b \) and so \( r_b(a) \in F \) and, by similar argument, \( l_b(a) \in F \) as well. By Lemma \([22]\) \( F \) is a congruence filter and (1) holds.

Of course there might be no bound on \( n \) in the above lemma; but if there is one (for instance, if \( A \) is finite), then \( V(A) \) is a normal variety of.
residuated lattices. On the other hand if an entire variety is normal then we can say more. Let $\text{RL}^n$ be the variety of residuated lattices satisfying:

$$(x \land 1)^n y \leq yx$$

$$y(x \land 1)^n \leq xy.$$ 

**Lemma 2.5.** A variety $V$ of residuated lattices is normal if and only if $V \subseteq \text{RL}^n$ for some $n$.

**Proof.** One direction is obvious. Assume then that $V$ is normal and let $F_V(x,y)$ be the free 2-generated algebra in $V$. Then $F_V(x,y)$ is normal and by Lemma 2.4 there are $m,k$ with $(x \land 1)^m y \leq yx$ and $y(x \land 1)^k \leq xy$. Take $n = \max\{m,k\}$; then in $F_V(x,y)$

$$(x \land 1)^n y \leq yx$$

$$y(x \land 1)^n \leq xy.$$ But since $F_V(x,y)$ is free, these inequalities hold in the entire $V$, i.e. $V \subseteq \text{FL}^n$.

Thus in particular, every variety of commutative residuated lattices is normal.

3 Join irreducible varieties and iterated conjugates

Given any variety $V$ we can consider its lattice of subvarieties $\Lambda(V)$; we will say that a subvariety $W \subseteq V$ is join irreducible (strongly join irreducible) if $W$ is a join irreducible (strongly join irreducible) member of $\Lambda(V)$. We will see how in relevant cases (strong) join irreducibility can be characterized by both a property of the corresponding logic and an algebraic description of the generating algebras.

In particular, we start by discussing the concept of well-connected algebra, that was introduced by L. Maksimova [33] to characterize the disjunction property in intermediate logics, i.e. those logics $\mathcal{L}$ for which $V_{\mathcal{L}}$ is a variety of Heyting algebras. A substructural logic $\mathcal{L}$ has the disjunction property if whenever $\varphi \lor \psi$ is a theorem of $\mathcal{L}$, in symbols $\mathcal{L} \vdash \varphi \lor \psi$, then either $\mathcal{L} \vdash \varphi$ or $\mathcal{L} \vdash \psi$. Likewise a Heyting algebra $A$ is well-connected if 1 is join irreducible, i.e. $a \lor b = 1$ implies either $a = 1$ or $b = 1$. A weaker property is Haldén completeness; a logic $\mathcal{L}$ is Haldén complete if it has the disjunction property w.r.t. to any pair of formulas that have no variables in common. Classical logic is Haldén complete but does not have the disjunction property, thus differentiating the two concepts. As shown in [32] these concepts are connected in commutative integral residuated lattices.

**Theorem 3.1.** (Theorem 2.5 in [32]) For a variety $V$ of commutative and integral residuated lattices the following are equivalent:

1. $\mathcal{L}_V$ is Haldén complete;
2. $V$ is join irreducible;
3. $V = V(A)$ for some well-connected algebra $A$. 

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How can we extend the definition of well-connected to the nonintegral case? The solution proposed in [32] (and later followed in [17]) is to define a residuated lattice $A$ to be well-connected if 1 is join prime in $A$, i.e. $a \lor b \geq 1$ implies $a \geq 1$ or $b \geq 1$.

We observe straight away that neither integrality nor commutativity are needed to prove that (3) implies (2).

**Lemma 3.2.** Let $V$ be a variety of residuated lattices; if $V = V(A)$ for some well-connected algebra $A \in V$ then $V$ is join irreducible.

*Proof*. Let $V = V(A)$ for some well-connected algebra $A \in V$, and suppose by way of contradiction that $V$ is not join irreducible, i.e. $V = W \lor Z$ for some proper subvarieties $W$ and $Z$. Then $W$ and $Z$ must be incomparable, hence there is an equation $p(x_1, \ldots, x_n) \geq 1$ holding in $W$ but not in $Z$ and an equation $q(y_1, \ldots, y_m) \geq 1$ holding in $Z$ but not in $W$. Clearly neither equation can hold in $V$, so there are $a_1, \ldots, a_n, b_1, \ldots, b_m \in A$ with $p(a_1, \ldots, a_n) \geq 1$ and $q(b_1, \ldots, b_m) \geq 1$; since $A$ is well-connected we must have $p(a_1, \ldots, a_n) \lor q(b_1, \ldots, b_m) \geq 1$ hence $V = V(A) \not\vdash p(x_1, \ldots, x_n) \lor q(y_1, \ldots, y_m) \geq 1$. On the other hand clearly $W, Z \models p(x_1, \ldots, x_n) \lor q(y_1, \ldots, y_m) \geq 1$ and since $V = W \lor Z$ we must have $V \models p(x_1, \ldots, x_n) \lor q(y_1, \ldots, y_m) \geq 1$. This is a contradiction, derived from the assumption that $V$ was not join irreducible.

The other implications in the general case however do not hold; an analysis of the Kihara-Ono construction reveals at once that there are two critical points. If $V$ is a variety of commutative residuated lattices then:

- every subdirectly irreducible algebra in $V$ is well-connected ([32], Lemma 2.2);
- if $W, Z$ are subvarieties of $V$ axiomatized (relative to $V$) by $p \geq 1$ and $q \geq 1$ (and we make sure that $p$ and $q$ have no variables in common), then $W \lor Z$ is axiomatized relative to $V$ by $p \lor q \geq 1$ ([32], Lemma 2.1).

Both statements are false if we remove commutativity; for the first it is easy to find a finite and integral residuated lattice that is simple but not well-connected (for instance the example below Lemma 3.60 in [25], while the second fails for more general reasons discussed at length in [24]).

In what follows we will describe classes of varieties of residuated lattices for which an analogous of Theorem 3.1 can be proved. To do so we will adapt to our purpose part of the theory developed in [24] about satisfaction of formulas generated by iterated conjugates.

We define a set $B^n(x, y)$ of equations in two variables $x, y$ for all $n \in \mathbb{N}$ in the following way; let $\Gamma^n$ be the set of iterated conjugates of length $n$ (i.e. a composition of $n$ left and right conjugates) over the appropriate language, with $\Gamma^1 = \{1\}$ (for a more general definition, here not needed, see [24], page 229). For all $n \in \mathbb{N}$

$$B^n(x, y) = \{ \gamma_1(x) \lor \gamma_2(y) \approx 1 : \gamma_1, \gamma_2 \in \Gamma^n \}.$$

Let $A$ be a residuated lattice and $a, b \in A$; we say that $A$ satisfies $B^n(a, b)$, in symbols $A \models B^n(a, b)$ if $A, a, b \models B^n(x, y)$, i.e. $\gamma_1(a) \lor \gamma_2(b) =$
1 for all $\gamma_1, \gamma_2 \in \Gamma^n(A)$. We say that $A$ satisfies $(G_{n,k})$ if for all $a, b \in A$, if $A \models B^n(a, b)$, then $A \models B^n(a, b)$.

**Lemma 3.3.** Let $A$ be a residuated lattice.

1. for all $n \in \mathbb{N}$, for all $a, b \in A$, if $A \models B^n(a, b)$ then $A \models B^{n-1}(a, b)$;
2. for all $n \in \mathbb{N}$ if $A$ satisfies $(G_{n,n+1})$ then it satisfies $(G_{n,k})$ for all $k \geq n$.

**Proof.** The proof of (1) is straightforward, once we have observed that $l_1(a) = a \land 1$ and all conjugates are smaller than 1.

For (2), suppose that $A$ satisfies $(G_{n,n+1})$, we will show that $A$ satisfies $(G_{n+1,n+2})$. As the relation $\to$ is clearly transitive this implies that $(G_{n,n+2})$ and hence, by induction, $(G_{n,k})$ for $k \geq n$.

Let then $a, b \in A$ such that $A \models B^{n+1}(a, b)$. Given any $\hat{\gamma}_1, \hat{\gamma}_2 \in \Gamma^{n+2}$, there are $\delta_1, \delta_2, \delta_1', \delta_2' \in \Gamma^1$, $\gamma_1, \gamma_2 \in \Gamma^n$ such that

$$\hat{\gamma}_1(x) = \delta_1 \gamma_1 \delta_2(x) \quad \hat{\gamma}_2(y) = \delta_1' \gamma_2 \delta_2'(y).$$

Since $A \models B^{n+1}(a, b)$, then $\gamma_1 \delta_2(a) \land \gamma_2 \delta_2'(b) = 1$ for all $\gamma_1, \gamma_2 \in \Gamma^n(A)$. If $u = \delta_2(a)$ and $v = \delta_2'(b)$, then $A$ satisfies $B^n(u, v)$; but since $A$ satisfies $(G_{n,n+1})$, then it satisfies $B^{n+1}(u, v)$ as well. This implies

$$\hat{\gamma}_1(a) \land \hat{\gamma}_2(b) = \delta_1 \gamma_1(u) \delta_1' \gamma_2(v) = 1,$$

which in turn yields that $A \models B^{n+2}(a, b)$, and then $A$ satisfies $(G_{n+1,n+2})$.

Thus, by transitivity and induction, it satisfies $(G_{n,k})$ for $k \geq n$. \hfill $\square$

Let’s look closely at the condition $(G_{0,1})$; first we observe the following.

**Lemma 3.4.** Let $V$ be a variety of residuated lattices; $V$ satisfies $(G_{0,1})$ (i.e. $A$ satisfies $(G_{0,1})$ for all $A \in V$) if and only if it satisfies the quasi equation

$$x \lor y \approx 1 \implies l_w(x) \lor r_z(y) \approx 1. \quad (G)$$

**Proof.** One direction is easy to check. Indeed, suppose $V$ satisfies $(G_{0,1})$, and let $A \in V, a, b \in A$ such that $a \lor b = 1$. Then $a = a \land 1, b = b \land 1$, and $(G)$ directly follows from $(G_{0,1})$, that corresponds to the set of quasi equations:

$$(x \land 1) \lor (y \land 1) \approx 1 \implies \gamma_1(x) \lor \gamma_2(y) \approx 1, \text{ for } \gamma_1, \gamma_2 \in \Gamma^1.$$  

For the other, suppose $(G)$ holds, and let $A \in V$ and $a, b \in A$; then $A \models B^0(a, b)$ if and only if $(a \land 1) \lor (b \land 1) = 1$. Then by $(G)$ for all $u, v \in A$

$$1 = l_u(a \land 1) \lor r_u(b \land 1) \leq l_u(a) \lor r_u(b) \leq 1.$$  

Thus $l_u(a) \lor r_u(b) = 1$, and in particular for all $u \in A$, $l_u(a) \lor r_1(b) = l_u(a) \lor (b \land 1) = 1$ and so, applying again the same reasoning since $l_u(a) \leq 1$, for all $v \in A$

$$1 = l_u(a) \lor l_u(b \land 1) \leq l_u(a) \lor l_u(b) \leq 1.$$  

The proof that $r_u(a) \lor r_u(b) = 1$ is similar, so we conclude that $A \models B^1(a, b)$; therefore $A$ satisfies $(G_0)$ and so does $V$. \hfill $\square$
(G) is trivially satisfied by any commutative variety of residuated lattices, but moreover:

**Lemma 3.5.** Any normal variety V of residuated lattices satisfies (G).

**Proof.** Let $A \in V$, and consider $x, y \in A$ with $x \lor y = 1$, thus clearly $x, y \leq 1$. Since both conjugates are below 1, $l_w(x) \lor r_z(y) \leq 1$. Thus we need to show that $1 \leq l_w(x) \lor r_z(y)$.

By Lemma 2.5, $(y \land 1 \land z \leq zy$ and $w(x \land 1 \land xw$, thus $(x \land 1 \land x \land 1 \land y \leq r_z(y)$. From [24], Lemma 3.20, we get that if $x \lor y = 1$ then $x^w \lor y^w = (x \land 1)^w \lor (y \land 1)^w = 1$. Thus,

$$1 \leq (x \land 1)^w \lor (y \land 1)^w \leq l_w(x) \lor r_z(y)$$

and the proof is completed. □

There are also non normal varieties satisfying (G), for instance, representable varieties (see Section 6 below). We observe also that the variety of $\ell$-groups satisfies $(G_{1,2})$ (see [24], p. 235). The last lemma we need is implicit in [24].

**Lemma 3.6.** Let $V$ be a variety of residuated lattices and let $p(x_1, \ldots, x_n) \geq 1$, $q(y_1, \ldots, y_m) \geq 1$ be two inequalities not holding in $V$. If $W$ and $Z$ are the subvarieties axiomatized by $p \land 1 \approx 1$ and $q \land 1 \approx 1$ respectively, then $W \lor Z$ is axiomatized by the set $B(p, q) = \bigcup_{n \in \mathbb{N}} B^n(p, q)$. Moreover if $V$ satisfies $(G_{l+1})$ for some $l \in \mathbb{N}$ then $W \lor Z$ is axiomatized by the finite set $B^l(p, q)$.

**Proof.** The first part is a consequence of Corollary 4.3 in [24]. The second part follows from Theorem 4.4 (1) in [24], where it is shown that $W \lor Z$ is axiomatized by $B^l(p, q)$ plus the (finite set of) equations implying $(G_{l+1})$, which here can be omitted since $(G_{l+1})$ holds in $V$. □

## 4 Γ-connectedness and Γ-completeness

The **negative cone** of a residuated lattice $A$ is an integral residuated lattice $A^{-}$ whose universe is $A^{-} = \{a \in A : a \leq 1\}$ and the operations are defined as follows: the lattice operations and the product are the same as in $A$ but the residuations are defined as $a_1/b := a/b \land 1$ and $a \land 1 : b := a \land b \land 1$.

The connections between $A^{-}$ and $A$ are strict; for instance it can be shown that Con$(A)$ and Con$(A^{-})$ are isomorphic [25]. A residuated lattice $A$ is **weakly well-connected** if $A^{-}$ is well-connected; we have the following obvious lemma.

**Lemma 4.1.** For a residuated lattice $A$ the following are equivalent:

1. $A$ is weakly well-connected;
2. 1 is join irreducible in $A$;
3. for $a, b \in A$ if $(a \land 1) \lor (b \land 1) = 1$, then either $a \geq 1$ or $b \geq 1$.
Clearly if $A$ is well connected, then it is weakly well-connected and the two concepts coincide in varieties that are 1-distributive, i.e. satisfy the equation

$$(x \lor y) \land 1 \leq (x \land 1) \lor (y \land 1).$$

We want to generalize these concepts; we say that a residuated lattice $A$ is $\Gamma^n$-connected if for all $a, b \in A$, if $\gamma_1(a) \lor \gamma_2(b) = 1$ for all $\gamma_1, \gamma_2 \in \Gamma_n(A)$, then either $a \geq 1$ or $b \geq 1$. To prove the next result we need a lemma.

**Lemma 4.2.** Let $A$ be a residuated lattice and let $a_1, \ldots, a_m, b_1, \ldots, b_l \in A$; if $a_i \lor b_j = 1$ for all $i \leq l$ and $j \leq m$, then for all $r$ and $d_1, \ldots, d_r \subseteq \{a_1, \ldots, a_l\}$ and $e_1, \ldots, e_k \subseteq \{b_1, \ldots, b_m\}$ we have

$$e_1 \ldots e_k \lor d_1 \ldots d_r = 1.$$

The proof is a simple finite induction using the equations holding in residuated lattices (see Lemma 3.2 in [24]).

**Lemma 4.3.** Let $V$ be a variety of residuated lattices that satisfies $(G_{n, n+1})$. Then every subdirectly irreducible algebra in $V$ is $\Gamma^n$-connected.

**Proof.** Let $A$ be subdirectly irreducible and let $a, b \in A$ with $\gamma_1(a) \lor \gamma_2(b) = 1$ for all $\gamma_1, \gamma_2 \in \Gamma^n(A)$, we will show that then either $a \geq 1$ or $b \geq 1$. Pick any $\gamma_1, \gamma_2 \in \Gamma^n(A)$, then if $d \in Fil_A(\gamma_1(a)) \cap Fil_A(\gamma_2(b))$ by using the description of the filter generated by an element in Corollary 2.3 we get that

$$\delta_1(\gamma_1(a)) \ldots \delta_l(\gamma_1(a)) \lor \delta_1(\gamma_2(b)) \ldots \delta_m(\gamma_2(b)) \leq d$$

for some $\delta_i, \delta'_j \in \Gamma^k$, for $i = 1 \ldots l, j = 1 \ldots m$ and $k \in \mathbb{N}$. Indeed the iterated conjugates can all be considered of the same (maximum) length $k$.

Let $\varepsilon_i = \delta_i \gamma_1$ and $\varepsilon'_j = \delta'_j \gamma_2$, then $\varepsilon_i, \varepsilon_j \in \Gamma^{n+k}$ for $i = 1 \ldots l, j = 1 \ldots m$. As $A$ satisfies $(G_{n, n+1})$ by Lemma 3.3 it satisfies $(G_{n, n+k})$ and then

$$\varepsilon_i(a) \lor \varepsilon'_j(b) = 1 \quad \text{for all } i, j;$$

now we apply Lemma 4.2 and we get

$$\varepsilon_1(a) \ldots \varepsilon_l(a) \lor \varepsilon'_1(b) \ldots \varepsilon'_m(b) = 1$$

so $d \geq 1$. Since $d$ was generic we must have $Fil_A(\gamma_1(a)) \cap Fil_A(\gamma_2(b)) = A^+$; but since $A$ is subdirectly irreducible the isomorphism between the congruence lattice and the filter lattice of $A$ forces either $Fil_A(\gamma_1(a)) = A^+$ or $Fil_A(\gamma_2(b)) = A^+$. This works for any choice of $\gamma_1, \gamma_2 \in \Gamma^n$, thus in particular it works for $\gamma_1 = \gamma_2 = l_1 \circ \ldots \circ l_1 = l_1$, thus we get that either $a \land 1 \geq 1$ or $b \land 1 \geq 1$, that is, either $a \geq 1$ or $b \geq 1$ as desired. Thus $A$ is $\Gamma^n$-connected.

Finally let’s complete the connection with logic. Let $\mathcal{L}$ be a substructural logic over $\mathcal{F}L^+$; given any two axiomatic extensions $\mathcal{L}_1$ and $\mathcal{L}_2$ axiomatized by formulas $\phi$ and $\psi$ respectively, for any $n$ Lemma 3.6 implicitly gives a set of formulas $B^n_\phi(\phi, \psi)$ such that $B_\mathcal{L}(\phi, \psi) = \bigcup_{n \in \mathbb{N}} B^n_\phi(\phi, \psi)$ axiomatizes the intersection $\mathcal{L}_1 \cap \mathcal{L}_2$, corresponding to the join of the varieties $V_{\mathcal{L}_1} \lor V_{\mathcal{L}_2}$. We say that $\mathcal{L}$ is $\Gamma^n$-complete if for all formulas $\phi$ and $\psi$ which have no variables in common, if $\mathcal{L} \vdash B^n_\phi(\phi, \psi)$ then either $\mathcal{L} \vdash \phi$ or $\mathcal{L} \vdash \psi$. 


Theorem 4.4. Let \( V \) be a variety of residuated lattices satisfying \((G_{n,n+1})\) for some \( n \in \mathbb{N} \); then the following are equivalent.

1. \( \mathcal{L}_V \) is \( \Gamma^n \)-complete;
2. \( V \) is join irreducible;
3. \( V = \mathcal{V}(A) \) for some \( \Gamma^n \)-connected algebra \( A \).

Proof. We show first that (1) implies (2). Assume that (2) fails; i.e. \( V = W \lor Z \) and \( V \neq W, Z \). Let \( p \geq 1 \) be an equation holding in \( W \) but not in \( Z \) and \( q \geq 1 \) an equation holding in \( Z \) but not in \( W \), respectively; if \( W', Z' \) are the varieties axiomatized relative to \( V \) by \( p \geq 1, q \geq 1 \) respectively, we have
\[
V = W \lor Z \subseteq W' \lor Z' \subseteq V;
\]
so we may assume that \( W, Z \) are axiomatized relative to \( V \) by \( p \geq 1 \) and \( q \geq 1 \). Let \( \varphi := p \geq 1 \) and \( \psi := q \geq 1 \); then \( \mathcal{L}_W \cap \mathcal{L}_Z = \mathcal{L}_V \) is axiomatized by \( B_{\mathcal{L}_V}(\varphi, \psi) \). Hence \( \mathcal{L}_V \vdash B_{\mathcal{L}_V}(\varphi, \psi) \) but neither \( \mathcal{L}_V \vdash \varphi \) nor \( \mathcal{L}_V \vdash \psi \). Hence \( \mathcal{L}_V \) does not have the \( \Gamma^n \)-completeness property.

We now show that (2) implies (1). Assume that (1) fails; hence there exist formulas \( \phi \) and \( \psi \) with no common variables such that \( \mathcal{L}_V \vdash B_{\mathcal{L}_V}(\phi, \psi) \) but neither \( \mathcal{L}_V \vdash \phi \) nor \( \mathcal{L}_V \vdash \psi \). Then we can consider the two axiomatic extensions of \( \mathcal{L}_V, \mathcal{L}_1 \) and \( \mathcal{L}_2 \), axiomatized by \( \phi \) and \( \psi \) respectively. This corresponds to taking two subvarieties of \( V \), \( W \) and \( Z \), defined by identities \( p \land 1 \approx 1 \) and \( q \land 1 \approx 1 \) respectively. Since \( V \) satisfies \((G_{n,n+1})\), by Lemma 3.6 \( W \lor Z \) is finitely axiomatizable relative to \( V \) by \( B^n(p, q) \) and the corresponding formula axiomatizing \( \mathcal{L}_1 \cap \mathcal{L}_2 \) is \( B^n_{\mathcal{L}_V}(\phi, \psi) \). But we are assuming that \( \mathcal{L}_V \vdash B_{\mathcal{L}_V}(\phi, \psi) \), which implies that \( W \lor Z = V \); thus \( V \) is not join irreducible, a contradiction. Hence (2) implies (1), and (1) and (2) are equivalent.

We now show the equivalence of (2) and (3). Assume (2) and let \( A \) be any algebra such that \( \mathcal{V} = \mathcal{V}(A) \); let \( \{\theta_i : i \in I\} \) be the set of all congruences of \( A \) such that \( \mathcal{A}/\theta_i \) is subdirectly irreducible. Next for any equation \( \varepsilon \) in the language of \( V \) we define \( L_\varepsilon = \{i \in I : \mathcal{A}/\theta_i \not\models \varepsilon\} \); we claim that \( \Delta = \{L_\varepsilon : V \not\models \varepsilon\} \) is a collection of nonempty subsets with the finite intersection property.

First note that if \( V \not\models \varepsilon \) then \( A \not\models \varepsilon \) and hence (by Birkhoff’s Theorem) \( A/\theta_i \not\models \varepsilon \) for some \( i \); therefore \( L_\varepsilon \neq \emptyset \). Next we prove that for any \( \varepsilon, \delta \) such that \( V \not\models \varepsilon \) there is a \( \gamma \) such that \( V \not\models \gamma \) and \( L_\varepsilon \subseteq L_\gamma \cap L_\delta \).

Let \( W, Z \) be the subvarieties of \( V \) axiomatized by \( \varepsilon \) and \( \delta \) respectively; since \( V \not\models \varepsilon, \delta \) they are both proper subvarieties of \( V \). Let \( \varepsilon \) be \( p(x_1, \ldots, x_n) \geq 1 \) and \( \delta \) be \( q(y_1, \ldots, y_m) \geq 1 \); since \( V \) satisfies \((G_{n,n+1})\), by Lemma 3.6 \( W \lor Z \) is axiomatized by the finite set of equations \( B^n(p, q) \), which contains in particular the equation \( (p \land 1) \lor (q \land 1) \approx 1 \).

Let \( \gamma \) be the inequality equivalent to the conjunction of the equations in \( B^n(p, q) \); if \( i \notin L_\varepsilon \) then \( A/\theta_i \not\models p \geq 1 \), and thus also \( A/\theta_i \not\models \gamma_1(p) \geq 1 \) for any \( \gamma_1 \in \Gamma^n(A/\theta_i) \). Hence \( A_i/\theta_i \not\models \gamma_1(p) \lor \gamma_2(q) \approx 1 \) for all \( \gamma_1, \gamma_2 \in \Gamma^n \), and those are exactly the equations in \( B^n(p, q) \), thus \( i \notin L_\gamma \). Since the same can be said for \( \delta \) it follows by counterpositive that \( L_\gamma \subseteq L_\varepsilon \cap L_\delta \).

If \( V \not\models \gamma \), then \( V = W \lor Z \), contrary to the hypothesis that \( V \) is join irreducible. Hence \( V \not\models \gamma \) and so \( \Delta \) has the finite intersection property.
So we can take an ultrafilter $U$ on $I$ containing $\Delta$; let $B = \prod_{i \in I} A_i / \theta_i / U$. Since $A_i / \theta_i$ is subdirectly irreducible, then by Lemma 4.3 it is $\Gamma^n$-connected. But being $\Gamma^n$-connected is a first order property, so $B$ is $\Gamma^n$-connected as well and clearly $B \in \mathcal{V}(A)$. Moreover if an equation $\varepsilon$ fails in $A$, then $L_\varepsilon \in \Delta$, so $L_\varepsilon \in U$ and so $B \not\vDash \varepsilon$. This proves that $\mathcal{V}(B) = \mathcal{V}(A) = \mathcal{V}$ and so (3) holds.

Finally the proof that (3) implies (2) is similar to the one of Lemma 3.2. Suppose that $\mathcal{V} = \mathcal{V}(A)$ for some $\Gamma^n$-connected algebra $A$, and that, by way of contradiction, $\mathcal{V}$ is not join irreducible, i.e. $\mathcal{V} = \mathcal{W} \lor \mathcal{Z}$ for some proper subvarieties $\mathcal{W}$ and $\mathcal{Z}$. Then there is an equation $p(x_1, \ldots, x_n) \geq 1$ holding in $\mathcal{W}$ but not in $\mathcal{Z}$ and an equation $q(y_1, \ldots, y_m) \geq 1$ holding in $\mathcal{Z}$ but not in $\mathcal{W}$. Clearly neither equation can hold in $\mathcal{V}$, so there are $a_1, \ldots, a_n, b_1, \ldots, b_m \in A$ with $p(a_1, \ldots, a_n) \not\geq 1$ and $q(b_1, \ldots, b_m) \not\geq 1$; since $A$ is $\Gamma^n$-connected, there must be $\hat{\gamma}_1, \hat{\gamma}_2$, instances in $A$ of iterated conjugates $\gamma_1, \gamma_2 \in \Gamma^n$, such that

$$\hat{\gamma}_1(p(a_1, \ldots, a_n)) \lor \hat{\gamma}_2(q(b_1, \ldots, b_m)) \not\approx 1,$$

hence $\mathcal{V} = \mathcal{V}(A) \not\vDash \gamma_1(p(x_1, \ldots, x_n)) \lor \gamma_2(q(y_1, \ldots, y_m)) \approx 1$. On the other hand, since conjugates of elements in the positive cone of an algebra are also in the positive cone, $\gamma_1(p(x_1, \ldots, x_n)) \approx 1$ holds in $\mathcal{W}$ and $\gamma_2(q(y_1, \ldots, y_m)) \approx 1$ holds in $\mathcal{Z}$. Thus $\mathcal{W}, \mathcal{Z} \vDash \gamma_1(p(x_1, \ldots, x_n)) \lor \gamma_2(q(y_1, \ldots, y_m)) \approx 1$ and since $\mathcal{V} = \mathcal{W} \lor \mathcal{Z}$ we must have that $\mathcal{V} \vDash \gamma_1(p(x_1, \ldots, x_n)) \lor \gamma_2(q(y_1, \ldots, y_m)) \approx 1$. This is a contradiction, derived from the assumption that $\mathcal{V}$ was not join irreducible. Thus, also (2) and (3) are equivalent and the proof is completed.

Observe that $\Gamma^0$-connectedness is weak well-connectedness and $\Gamma^0$-completeness is the so called weak Halldén completeness for a substructural logic $\mathcal{L}$: if $\mathcal{L} \vdash (\phi \land 1) \lor (\psi \land 1)$ then either $\mathcal{L} \vdash \phi$ or $\mathcal{L} \vdash \psi$. So our result extends both Theorem 2.5 and 2.8 in [32], giving an (admittedly less meaningful) logical characterizations of the property of being join irreducible for a variety satisfying $(G_{n,n+1})$ for some $n \in \mathbb{N}$.

5 Join irreducibility and subdirect irreducibility

It is a straightforward consequence of Birkhoff’s Theorem that if $\mathcal{V}$ is any variety of algebras that is strictly join irreducible, then $\mathcal{V} = \mathcal{V}(A)$ for some subdirectly irreducible algebra $A$. In [32] the authors asked if the converse might hold for varieties of residuated lattices. We will see in Section 7 below that not all the varieties generated by a subdirectly irreducible basic hoop are strictly join irreducible so in general the converse does not hold even in very friendly varieties.

Next we point out a corollary of Lemma 4.3 and Theorem 4.4.

**Corollary 5.1.** Let $\mathcal{V}$ be a variety of residuated lattices satisfying $(G_{n,n+1})$ for some $n \in \mathbb{N}$. If there is a subdirectly irreducible algebra $A$ with $\mathcal{V} = \mathcal{V}(A)$, then $\mathcal{V}$ is join irreducible.
This is the analogous of Lemma 2.6(2) in [32] and again the authors asked if it was possible to invert it; we will show that our (more general) version is indeed invertible, thus answering their question as well. First observe that:

**Lemma 5.2.** Let $V$ be a variety of residuated lattices. If $A \in V$ is $\Gamma^n$-connected, then there exists a countable $\Gamma^n$-connected algebra $B \in V$ with $V(A) = V(B)$.

**Proof.** Suppose that $A$ is $\Gamma^n$-connected; it is clear that the class of $\Gamma^n$-connected in $V(A)$ is axiomatized by a set of first-order sentences, namely the axioms defining $V(A)$ and the universal sentence

$$\forall x \forall y [\&B^n(x, y) \Rightarrow ((x \geq 1) \text{ or } (y \geq 1))].$$

Where by $\&B^n(x, y)$ we mean the conjunction of the equations in $B^n(x, y) = \{\gamma_1(x) \lor \gamma_2(y) \approx 1 : \gamma_1, \gamma_2 \in \Gamma^n\}$. The downward version of the Löwenheim-Skolem Theorem implies that if $A$ is $\Gamma^n$-connected, then there is also a countable $\Gamma^n$-connected algebra $B$ that is elementarily equivalent to $A$. In particular $A$ and $B$ must satisfy the same equations, so $V(A) = V(B)$. \[\square\]

In [9] the authors proved an interesting result about FLew-algebras: if a variety is generated by a well-connected algebra, then it is also generated by a subdirectly irreducible algebra, which implies that Lemma 2.6(2) in [32] can indeed be inverted. By looking at their proofs it is clear that they do not rely on zero-boundedness and commutativity. Generalizing it for non-integral structures takes a little more work and we need to clarify the context. If $A$ is a residuated lattice we can define a set of equations $D_A$, called the **diagram** of $A$ in the following way: for every $a \in A$ we pick a variable $v_a$ and

$$D_A = \{v_a \ast v_b \leftrightarrow v_{a+b} \geq 1 : a, b \in A, \ast \in \{\lor, \land, \to, /, \}\}$$

where $a \leftrightarrow b = (a \setminus b) \land (b \setminus a)$. It is clear that the diagram encodes the operation table of $A$; moreover if $A$ is countable then all the terms involved are elements of the free algebra over $\omega$ generators of any variety to which $A$ belongs.

So if $A \in V$, we can use the semantical consequence relation $\models$ on $V$ and the expression $D_A \models p \geq 1$ makes sense for any term $p$ in the language of residuated lattices. Likewise $B \in V$ is a model of $D_A \models p \geq 1$ if any homomorphism $h : F_V(\omega) \rightarrow B$, say $h(v_a) = c_a$, such that $c_a \ast c_b = c_{a+b}$ for all $a, b \in A$, is also such that $h(p) \geq 1$. Now it is straightforward to check that the class of models of $D_A \models p \geq 1$ is closed under subalgebras and direct products; it follows that if there is a countermodel in $V$ of $D_A \models p \geq 1$, then there is a subdirectly irreducible countermodel. This circle of ideas is the core of the following lemma.

**Lemma 5.3.** Let $V$ be any variety of residuated lattices and let $A$ be a countable $\Gamma^n$-connected algebra in $V$; then there exists a subdirectly irreducible algebra $B \in V$ such that $A \in S(B)$.
Proof. Let $A$ be a countable $\Gamma^n$-connected algebra in $V$ and let $V = \{ v_a : a \in A \}$ be the set of variables defining the diagram $D_A$.

Now we define a set of equations $\Psi$ not valid in $A$ because of its $\Gamma^n$-connectedness. In particular, in $\Psi$ we write the meets of all possible joins of iterated conjugates of length $n$ of $m$ given variables, for each $m \in \mathbb{N}$. Notice that given $m \in \mathbb{N}$, such joins are in a finite number. As argued in [24] (p. 232), one can enumerate the indices of the iterated conjugates of a certain length and they will form an initial segment of the natural numbers. Let us call $\Gamma^m_n$ the finite set of all possible $m$-uples of iterated conjugates of length $n$, and $J_m$ the enumeration of the $m$-uples of indices of the conjugates in $\Gamma^m_n$. That is, each $j \in J_m$ is of the kind $j = (j_1, \ldots, j_m)$ and identifies an $m$-uple $(\gamma_{j_1}, \ldots, \gamma_{j_m}) \in \Gamma^m_n$. Then

$$\Psi = \{ \bigwedge_{j \in J_m} (\gamma_{j_1} (v_{a_1}) \lor \ldots \lor \gamma_{j_m} (v_{a_m})) \approx 1 : m \in \mathbb{N}, (\gamma_{j_1}, \ldots, \gamma_{j_m}) \in \Gamma^m_n, a_i \in A, a_i \not\approx 1, \text{ for } i = 1 \ldots m \}.$$ 

Notice that $\Psi$ is directed in the following sense:

If $\bigwedge_{j \in J} p_j \approx 1 \in \Psi$ and $\bigwedge_{k \in K} q_k \approx 1 \in \Psi$, then $\bigwedge_{j \in J} \bigwedge_{k \in K} p_j \lor q_k \approx 1 \in \Psi$.

We call $\bigwedge_{j \in J} \bigwedge_{k \in K} p_j \lor q_k \approx 1$ the upper bound of $\bigwedge_{j \in J} p_j \approx 1$ and $\bigwedge_{k \in K} q_k \approx 1$, since both

$$\bigwedge_{j \in J} p_j \leq \bigwedge_{j \in J} \bigwedge_{k \in K} p_j \lor q_k \quad \text{and} \quad \bigwedge_{k \in K} q_k \leq \bigwedge_{j \in J} \bigwedge_{k \in K} p_j \lor q_k.$$

Moreover, $D_A \not\models p \approx 1$ for each $p \approx 1 \in \Psi$, since $A$ naturally gives a countermodel. Indeed, with the assignment $g : F_V(\omega) \rightarrow A$ such that $g(v_a) = a$ for all $a \in A$, $A$ satisfies all inequalities in $D_A$ by construction, but it does not satisfy any of the equations $p \approx 1 \in \Psi$: if $p = \bigwedge_{j \in J} (\gamma_{j_1} (v_{a_1}) \lor \ldots \lor \gamma_{j_m} (v_{a_m})) \approx 1$ is satisfied with respect to the assignment $g$, then all of the meetands $(\gamma_{j_1} (a_1) \lor \ldots \lor \gamma_{j_m} (a_m)) = 1$, for all $(\gamma_{j_1}, \ldots, \gamma_{j_m}) \in \Gamma^m_n$, and thus at least one of the elements $a_i \geq 1$, since $A$ is $\Gamma^n$-connected. But by the definitions of $\Psi$ and $g$, $a_i \not\approx 1$ for all $i = 1 \ldots m$, a contradiction.

Let $v$ be a variable different from all the $v_a$, and let

$$D' = D_A \cup \{ p \land v \geq 1 : p \approx 1 \in \Psi \}.$$ 

We claim that $D' \not\models v \geq 1$. Suppose that $D' \models v \geq 1$, then by compactness there is a finite subset $D'' \subseteq D'$ such that $D'' \models v \geq 1$; let $\forall r \approx 1$ be the upper bound of the finite set of equalities $\{ p \approx 1 \in \Psi : p \land v \geq 1 \in D'' \}$ (such upper bound exists as $\Psi$ is directed). Since $D_A \not\models \psi$ for each $\psi \in \Psi$, in particular $D_A \not\models r \approx 1$; so there is an algebra $C \in V$ and a homomorphism $k : F_V(\omega) \rightarrow C$ such that all identities in $D_A$ are satisfied in $C$ but $k(r) \neq 1$, or equivalently $k(r) \not\approx 1$. Now if we extend $k$ by setting $k(v) = k(r)$ then $C$ becomes a countermodel of $D' \models v \geq 1$ as well, a

---

1. This part of the proof is essentially the same as in Lemma 3.4 of [IS], which we rewrite in this more general setting.
contradiction. Thus \( D' \not\models v \geq 1 \). It follows that there is a subdirectly irreducible algebra \( B \in V \) that is a countermodel of \( D' \models v \geq 1 \), with respect to an homomorphism \( h : F\nu(\omega) \rightarrow B \).

Observe that, since \( D_A \subseteq D' \), \( B \) also satisfies all inequalities in \( D_A \) with respect to \( h \), but \( h(p) \neq 1 \) for all \( p \approx 1 \in \Psi \). Indeed if it were \( h(p) = 1 \) it would follow that \( h(v) \geq h(p \backslash v) \geq 1 \), a contradiction.

Let’s define \( f : A \rightarrow B \) as \( f(a) = h(v_a) \), and show that it is an ambedding, which would settle the proof. Since \( B \) satisfies \( D_A \) with respect to \( h \), \( f \) is a homomorphism. To show that \( f \) is injective suppose that \( f(a) = f(b) \) given \( a, b \in A \); then by residuation \( 1 \leq f(a \backslash b), 1 \leq f(b \backslash a) \). Note that if \( c \geq 1 \), then \( v_c \wedge 1 \approx 1 \in \Psi \), thus \( f(c) = h(v_c) \geq 1 \); in fact if \( h(v_c) \geq 1 \) we would have \( h(v_c \wedge 1) = h(v_c) \wedge 1 = 1 \), a contradiction, as \( B \) satisfies no equality in \( \Psi \) with respect to \( h \). This implies that \( 1 \leq a \backslash b \) and \( 1 \leq b \backslash a \), thus \( a = b \) and the proof is completed.

**Theorem 5.4.** Let \( V \) be a variety of residuated lattices that satisfies \((G_{n,n+1})\) for some \( n \in \mathbb{N} \); if \( V \) is join irreducible, then there is a subdirectly irreducible algebra \( B \in V \) such that \( V(B) = V \).

**Proof.** Since \( V \) is join irreducible and satisfies \((G_{n,n+1})\), by Theorem 4.3 there is a \( \Gamma'' \)-connected algebra \( A \) with \( V(A) = V \). By Lemma 5.2 we may assume without loss of generality that \( A \) is countable; hence by Lemma 5.3 there is a subdirectly irreducible \( B \in V \) with \( A \in S(B) \). Clearly \( V = V(A) = V(B) \) and the thesis holds.

Thus combining Theorem 5.4 and Corollary 5.1 we obtain the following.

**Corollary 5.5.** Let \( V \) be a variety of residuated lattices that satisfies \((G_{n,n+1})\) for some \( n \in \mathbb{N} \). Then \( V \) is join irreducible if and only if there is a subdirectly irreducible algebra \( A \in V \) such that \( V(A) = V \).

Of course the same results we presented in this section hold for varieties of FL-algebras. In particular then, the previous corollary applies to all varieties satisfying \((G)\), thus for example: normal varieties of RL and FL, and thus CIRL, CIRL, FL_{aw}, FL_{aw} (answering then to Kihara and Ono question); representable subvarieties of RL and FL. Moreover, the theorem also applies to subvarieties of \( \ell \)-groups, since they satisfy \((G_{1,2})\).

### 6 Representable varieties

A residuated lattice is **representable** if it is a subdirect product of totally ordered residuated lattices; a variety is representable if each of its members is representable, which is equivalent to say that each subdirectly irreducible member of the variety is totally ordered. Being representable for a variety is equationally definable; indeed for any \( V \in \text{RL} \), \( V \) is representable if and only if \( V \models (x \wedge y) \vee x \wedge (x \wedge y \backslash y) \vee y \) (see 31, 15).

A residuated lattice is **prelinear** if it satisfies

\[(x/y) \vee (y/x) \geq 1.\]
A variety of residuated lattices is prelinear if each of its members is prelinear. Since any totally ordered residuated lattice is clearly prelinear, any representable variety is prelinear. It is well-known that any representable variety satisfies (G) (see for instance Theorem 6.7 in [15]) so we may apply Theorem 4.4 to conclude that each join irreducible subvariety is generated by a single weakly well-connected algebra. Moreover, since clearly the underlying lattices of any representable residuated lattice is distributive (as a subdirect product of chains) the concepts of well-connected and weakly well-connected coincide. Moreover the following holds.

Lemma 6.1. Let $V$ be a prelinear variety of residuated lattices and let $A \in V$. Then the following are equivalent:

1. $A$ is well-connected;
2. $A$ is totally ordered.

Proof. The implication (2) implies (1) clearly follows from the definition of well-connectedness. Suppose now that $A$ is prelinear and well-connected.

If the variety we are dealing with is also integral, then we can use ordinal sums, of which we recall the definition for the reader’s convenience. Let $(I, \leq)$ be a totally ordered set. For all $i \in I$ let $A_i$ be a totally ordered integral residuated ($\wedge$-semi)lattice, such that for $i \neq j$, $A_i \cap A_j = \{1\}$. Then $\bigoplus_{i \in I} A_i$, the ordinal sum of the family $(A_i)_{i \in I}$, is the structure whose base set is $\bigcup_{i \in I} A_i$ and the operations are defined as follows:

$$
x \rightarrow y = \begin{cases} 
x \rightarrow^{A_i} y & \text{if } x, y \in A_i, 
1 & \text{if } x \in A_i \setminus \{1\} \text{ and } y \in A_j \text{ with } i < j.
\end{cases}
$$

$$
x \cdot y = \begin{cases} 
x \cdot^{A_i} y & \text{if } x, y \in A_i, 
1 & \text{if } x \in A_i \setminus \{1\} \text{ and } y \in A_j \text{ with } i > j.
\end{cases}
$$

It is easily seen that join and meet are defined by the ordering obtained by stacking the various algebras one over the other. Moreover we call sum-irreducible an algebra that is not isomorphic to an ordinal sum of nontrivial components. We recall that:

Theorem 6.2. ([2], Theorem 3.2) Any integral residuated semilattice is the ordinal sum of sum-irreducible residuated semilattices.

In particular if $V$ is integral and representable, then every strictly join irreducible subvariety of $V$ is generated by a single totally ordered algebra $A$, which is the ordinal sum of sum-irreducible totally ordered algebras in $V$. Therefore knowing the sum-irreducible algebras in a variety of integral and representable residuated lattices is paramount. Though this is usually a difficult task there are cases in which it can be achieved.
A pseudohoop is a an integral and divisible residuated lattice; the variety of representable pseudohoops will be denoted by $\text{RPsh}$. A Wajsberg pseudohoop is a representable pseudohoop satisfying
\[(y/x)\backslash y \approx (x/y)\backslash x;\]
\[y/(x\backslash y) \approx x/(y\backslash x).\]

It turns out that totally ordered Wajsberg pseudohoops are precisely the totally ordered sum irreducible pseudohoops. This implies that:

**Theorem 6.3.** \[\text{[20]}\] Let $A$ be a totally ordered pseudohoop; then $A = \bigoplus_{i \in I} A_i$, where each $A_i$ is a (totally ordered) Wajsberg pseudohoop (and the decomposition is unique up to isomorphism of the components).

If $A$ is a totally ordered pseudohoop and $A = \bigoplus_{i \in I} A_i$ is its decomposition into Wajsberg pseudohoops, the index of $A$ is $n$ if $|I| = n$ and infinite if $I$ is infinite. Let’s consider the equation
\[\bigwedge_{i=0}^{n-1} x_i / (x_i / (x_{i+1} \backslash x_i)) \leq \bigvee_{i=0}^{n} x_i.\]

**Lemma 6.4.** (Lemma 6.1 in \[\text{[2]}\]) For any totally ordered pseudohoop $A$, the index of $A$ is less or equal to $n \in \mathbb{N}$ if and only if $A \models \lambda_n$.

By Theorem 4.4 and Lemma 6.1 if $A$ is a totally ordered pseudohoop, $\mathcal{V}(A)$ is join irreducible in $\Lambda(\text{RPsh})$.

**Theorem 6.5.** Let $A = \bigoplus_{i \in I} A_i$ be a totally ordered pseudohoop; if the index of $A$ is infinite, then $\mathcal{V}(A)$ is not strictly join irreducible.

**Proof.** Let $C$ be the set of all chains in $\mathcal{V}(A)$ having finite index; by Lemma 6.1, $\mathcal{V}(C) \neq \mathcal{V}(A)$ for all $C \in C$. Suppose that an equation $p(x_1, \ldots, x_n) \approx 1$ fails in $A$; then there are $a_1, \ldots, a_n \in A$ such that $p(a_1, \ldots, a_n) < 1$. If $C$ is the subalgebra of $A$ generated by $a_1, \ldots, a_n$, then $C \not\models p(x_1, \ldots, x_n) \approx 1$ and $C$ cannot have index greater than $n$. Hence $C \in C$ and $\bigvee_{C \in C} \mathcal{V}(C) = \mathcal{V}(A)$; hence $\mathcal{V}(A)$ is not strictly join irreducible.

So if $A$ is a totally ordered pseudohoop and $\mathcal{V}(A)$ is strictly join irreducible in $\Lambda(\text{RPsh})$, then $A$ must have finite index. Chains of finite index have been studied in \[\text{[5]}\] in a different context (BL-algebras and basic hoops); we will show that the results therein extend to totally ordered pseudohoops in a straightforward way.

Let $P$ be a poset; a join dense completion of $P$ is a complete lattice $L$ with an order embedding $\alpha : P \rightarrow L$ such that $\alpha(P)$ is join dense in $L$, i.e. for every $x \in L$, $x = \bigvee \{\alpha(p) : p \in P, \alpha(p) \leq x\}$. For any subvariety $\mathcal{V}$ of $\text{RPsh}$ we define
\[\mathcal{F}_\mathcal{V} = \{C \in \mathcal{V} : C \text{ is a chain of finite index}\}\]
and
\[\Phi_\mathcal{V} = \{\mathcal{V}(A) : A \in \mathcal{F}_\mathcal{V}\};\]

$\Phi_\mathcal{V}$ is clearly a poset under inclusion and moreover.
Theorem 6.6. For any subvariety $V$ of $\mathbf{RPsH}$, $\Lambda(V)$ is the join dense completion of $\Phi_V$.

Proof. Let $A$ be a finitely generated totally ordered pseudohoop in $V$; then, $A$ has index $\leq n$ and so $A \in \mathcal{F}_V$. Since every subvariety $W$ of $V$ is generated by its totally ordered members and every algebra is in the variety generated by its finitely generated subalgebras, every subvariety $W$ of $V$ is the supremum of all varieties $V(A)$, where $A \in \mathcal{F}_W$. We have thus shown that for every $W \in \Lambda(V)$

$$W = \bigvee \{ V(A) : A \in \mathcal{F}_W \}.$$ 

This implies that $\Lambda(V)$ is the join dense completion of $\Phi_V$.

Corollary 6.7. Let $V$ be any variety of representable pseudohoops; if $\Lambda(V)$ is finite then $V$ satisfies $\lambda_n$ for some $n \in \mathbb{N}$, and for all $A \in \mathcal{F}_V$, $V(A)$ is strictly join irreducible in $\Lambda(V)$.

Proof. If $\Lambda(V)$ is finite then it must satisfy $\lambda_n$ for some $n \in \mathbb{N}$; otherwise in $V$ there would be chains of arbitrarily large index and they would generate infinitely many distinct varieties. Moreover if $A \in \mathcal{F}_V$, then by Theorem 4.4 and Lemma 6.1, $V(A)$ is join irreducible. Thus since $\Lambda(V)$ is finite, $V(A)$ is strictly join irreducible in $\Lambda(V)$.

If $A, B \in \mathcal{F}_{\mathbf{RPsH}}$ we define $A \preceq B$ iff $HSP_u(A) \subseteq HSP_u(B)$; then $\preceq$ is a preorder with associated equivalence relation $\equiv$ and by Jónsson Lemma $V(A) = V(B)$ if and only if $A \equiv B$. It follows that to determine which varieties of representable pseudohoops generated by a chain of finite index are strictly join irreducible we may restrict to a representative of each $\equiv$-class. To do so we must have some ways of classifying the pseudo Wajsberg components in the decomposition.

Theorem 6.8. \cite{19,13} Let $A$ be any totally ordered Wajsberg pseudohoop. If $A$ is cancellative, then there is a totally ordered group $G$ such that $A$ is isomorphic with $G^-$, the negative cone of $G$. If $A$ is bounded, then there is a totally ordered group with strong unit $u$, such that $A \cong \Gamma(G,u)$.

$\Gamma(G,u)$ is often called the Mundici’s functor, see \cite{19} for a definition. Since by Lemma 4.5 in \cite{1} each Wajsberg pseudohoops is either bounded or cancellative, the theorem gives a complete representation of totally ordered Wajsberg pseudohoops. However this is less useful than it seems, at least in general. First totally ordered groups are common (every free group is totally orderable); second, if $G$ is a totally ordered group we have no idea in general of what $\mathbf{ISP}_u(G^-)$ is (except for one case, see next section). We believe that a careful analysis of the techniques employed in \cite{21} and \cite{22} could lead to a better understanding of the problem; however we will not explore this path here, leaving it open for future investigations. From now on we will deal only with commutative structures, first because it is possible to prove meaningful results about them and second because they are more suited for the applications we have in mind.
7 Strictly join irreducible varieties of basic hoops

From now on we will stay within the variety CIRL of commutative and integral residuated lattices; in this case $a \setminus b = b/a$ and we will use the symbol $a \to b$ for both. For any commutative residuated lattice the concepts of prelinear and representable coincide (see for instance [4]); a basic hoop is a commutative and representable pseudohoop and a Wajsberg hoop is a commutative Wajsberg pseudohoop.

By Theorem 6.3 any totally ordered basic hoop is the ordinal sum of Wajsberg hoops (and this is how the result was originally formulated in [4]); hence by the same argument as before (Theorem 6.5), a strictly join irreducible variety $V$ of basic hoops must be $V(A)$ for some chain of finite index $A$. In the commutative case however we can choose representatives of the $\equiv$-classes in a very nice and uniform way.

We recall that given any totally ordered hoop of finite index $\mathcal{A} = \bigoplus_{i=1}^{n} A_i$ we have [4]

$$\text{HSP}_u(\mathcal{A}) = \bigcup_{i=1}^{n-1} \left( \bigoplus_{j=1}^{i-1} \text{ISP}_u(A_j) \right) \oplus \text{HSP}_u(A_n)$$

Therefore if $\mathcal{A} = \bigoplus_{i=1}^{n} A_i$ and $\mathcal{B} = \bigoplus_{j=1}^{m} B_j$ belong to the class of totally ordered basic hoops of finite index $\mathcal{F}_{BH}$, then $\mathcal{A} \equiv \mathcal{B}$ if and only if $n = m$, $\text{ISP}_u(A_i) = \text{ISP}_u(B_i)$ for $i = 1, \ldots, n-1$ and $\text{HSP}_u(A_n) = \text{HSP}_u(B_n)$.

First we need to straighten out some technical details; a BL-algebra is a zero-bounded basic hoop, i.e. a hoop with a nullary operation, denoted by 0, satisfying $0 \leq x$; a Wajsberg algebra is a zero-bounded Wajsberg hoop. It is clear that any zero-free subreduct of a BL-algebra is a basic hoop, and that any bounded (i.e. with a smallest element) basic hoop is a reduct of a BL-algebra; moreover the variety BH is the class of zero-free subreducts of BL-algebras [3]. The operator ISP$_u$ on Wajsberg hoops has been extensively studied first in [28] and then in [4]; really the results in [28] are about MV-algebras but we can use them anyway and this is why. MV-algebras are termwise equivalent to Wajsberg algebras which in turn are polynomially equivalent to bounded Wajsberg hoops. The following lemma is easy to show, and is often implicitly used in the study of Wajsberg hoops and Wajsberg algebras.

**Lemma 7.1.** Let $O$ be a class operator that is a composition of $I, H, S, P, P_u$. Let $A, B$ be Wajsberg chains and $A_0, B_0$ be their Wajsberg hoop reducts. Then $O(A) \subseteq O(B)$ if and only if $O(A_0) \subseteq O(B_0)$.

This allows us to consider totally ordered bounded Wajsberg hoops as if they were Wajsberg algebras. In other words to check that ISP$_u(A) = ISP_u(B)$ when they are both bounded totally ordered Wajsberg hoops is enough to check the equality by considering them as Wajsberg algebras.

Since a totally order Wajsberg hoop is either bounded or cancellative [14] we can use Gispert’s results in [28] for Wajsberg algebras and integrate them with the cancellative case.

In [5] an algorithm is given to choose a representative for each $\equiv$-class of finite ordinal sums of BL-algebras; while we reassure that the same
The radical $A$ of a bounded Wajsberg hoop $A$, in symbols $Rad(A)$, is the intersection of the maximal filters of $A$; it is easy to see that $Rad(A)$ is a cancellative basic subhoop of $A$. We say that a bounded Wajsberg hoop $A$ has rank $n$, if $A/Rad(A) \cong L_n$, otherwise it has infinite rank. For any bounded Wajsberg hoop $A$, $d_A$, called the divisibility index, is the maximum $k$ such that $L_k$ is embeddable in $A$ if any, otherwise $d_A = \infty$.

**Lemma 7.2.**

1. $L_n$ has rank $n$ and divisibility index $n$.

2. For any $k \geq 0$, $L_{n,k}$ has rank $n$ and $d_{L_{n,k}} = gcd(n,k)$; in particular $d_{L_n} = n$.

3. $S_n$ has infinite rank and $L_k \in S(S_n)$ if and only if $k \mid n$; hence $d_{S_n} = n$.

4. If $A$ is a nontrivial totally ordered cancellative hoop then $ISP_u(A) = ISP_u(C_\omega)$.

5. If $A$ is a non-simple bounded Wajsberg chain of finite rank $n$, then $ISP_u(A) = ISP_u(L_n^\infty)$ if and only if $d_A = n$.

6. If $A$ is a non-simple bounded Wajsberg chain of finite rank $k$, then $d_A$ divides $k$, and $ISP_u(A) = ISP_u(L_{k,d_A})$.

7. If $A$ is a Wajsberg chain of infinite rank then $ISP_u(A) = ISP_u(S_n)$ if and only if $d_A = n$. 

The universe is the cartesian product, the group operations are defined componentwise and the ordering is the lexicographic ordering (w.r.t. the natural ordering of $Z$); then $Z \times I Z$ is a totally ordered abelian group and we can apply $\Gamma$ to it. We now define some useful Wajsberg chains:

- the finite Wajsberg chain with $n + 1$ elements $L_n = \Gamma(Z,n)$;
- the infinite Wajsberg chain $L_n^\infty = \Gamma(Z \times I Z, (n,0))$;
- the infinite Wajsberg chain $L_{n,k} = \Gamma(Z \times I Z, (n,k))$;
- the infinite bounded Wajsberg chain $[0,1]_L = \Gamma(R,1)$, i.e. the real interval with operations induced by the Wajsberg $t$-norm, i.e $xy = \max(x+y-1,0)$, $x \rightarrow y = \min(1 + x - y,1)$;
- the infinite bounded Wajsberg chain $Q = \Gamma(Q,1) = Q \cap [0,1]_L$;
- if $U$ is a set of positive integers we denote by $Q(U)$ the subalgebra of $Q$ generated by $\{L_k : k \in U\}$, where $L_k$ is seen as an algebra with universe $\{\frac{0}{n}, \frac{1}{n}, \ldots, \frac{n}{n}\}$;
- the unbounded Wajsberg chain $C_\omega$ that has as universe the free group on one generator, where the product is the group product and $a^l \rightarrow a^m = a^{\max(m-l,0)}$;
- finally we fix once and for all an irrational number $\alpha \in [0,1]$ and we let $X$ be the totally ordered dense subgroup of $\mathbb{R}$ generated by $\alpha$ and $1$; then $S_n = \Gamma(X,n)$. 

Let $G$ be a lattice ordered abelian group; by [22], if $u$ is a strong unit of $G$ we can construct a bounded Wajsberg hoop $\Gamma(G,u) = \langle \{0, u\}, \rightarrow, 0, u \rangle$ where $ab = \max\{a + b - u, 0\}$ and $a \rightarrow b = \max\{u - a + b, u\}$. Let now $Z \times I Z$ denote the lexicographic product of two copies of $Z$. In other words, 

the universe is the cartesian product, the group operations are defined componentwise and the ordering is the lexicographic ordering (w.r.t. the natural ordering of $Z$); then $Z \times I Z$ is a totally ordered abelian group and we can apply $\Gamma$ to it. We now define some useful Wajsberg chains:

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- if $U$ is a set of positive integers we denote by $Q(U)$ the subalgebra of $Q$ generated by $\{L_k : k \in U\}$, where $L_k$ is seen as an algebra with universe $\{\frac{0}{n}, \frac{1}{n}, \ldots, \frac{n}{n}\}$;
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The **radical** of a bounded Wajsberg hoop $A$, in symbols $Rad(A)$, is the intersection of the maximal filters of $A$; it is easy to see that $Rad(A)$ is a cancellative basic subhoop of $A$. We say that a bounded Wajsberg hoop $A$ has rank $n$, if $A/Rad(A) \cong L_n$, otherwise it has infinite rank. For any bounded Wajsberg hoop $A$, $d_A$, called the divisibility index, is the maximum $k$ such that $L_k$ is embeddable in $A$ if any, otherwise $d_A = \infty$.

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3. $S_n$ has infinite rank and $L_k \in S(S_n)$ if and only if $k \mid n$; hence $d_{S_n} = n$.

4. If $A$ is a nontrivial totally ordered cancellative hoop then $ISP_u(A) = ISP_u(C_\omega)$.

5. If $A$ is a non-simple bounded Wajsberg chain of finite rank $n$, then $ISP_u(A) = ISP_u(L_n^\infty)$ if and only if $d_A = n$.

6. If $A$ is a non-simple bounded Wajsberg chain of finite rank $k$, then $d_A$ divides $k$, and $ISP_u(A) = ISP_u(L_{k,d_A})$.

7. If $A$ is a Wajsberg chain of infinite rank then $ISP_u(A) = ISP_u(S_n)$ if and only if $d_A = n$. 

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8. If $A$ is a bounded Wajsberg chain of infinite rank and $d_A = \infty$, then $\text{ISP}_u(A) = \text{ISP}_u(Q(U))$ where $U = \{k : L_k \in S(A)\}$.

9. If $A$ is an infinite subdirectly irreducible Wajsberg algebra of finite rank $k$, then $\text{HSP}_u(A) = \text{HSP}_u(L_{k,k})$.

10. If $A$ is a subdirectly irreducible Wajsberg algebra of infinite rank, then $\text{HSP}_u(A) = \text{HSP}_u([0,1]_L)$.

Proof. Claims (1) and (3) are obvious, while (2), (4) and (5) are in [4]. The proofs of (6), (7) and (8) are in [28].

For (9) note that both $A$ and $L_{k,k}$ are subdirectly irreducible and generate the same variety, namely the variety of Wajsberg algebras $A$ such that $d_A$ divides $k$ (see Lemmas 6.1 and 6.3 in [4]). For (10) it is well-known that any two algebras of infinite rank generate the entire variety of Wajsberg algebras (see for instance Theorem 2.5 in [6]). Since any variety of Wajsberg hoops is congruence distributive, (9) and (10) follow from Jónsson’s Lemma.

**Theorem 7.3.** Let $A = \bigoplus_{i=1}^n A_i$ be any totally ordered basic hoop of finite index; then there is an unique (up to $\equiv$) $B \in \mathcal{F}_{BH}$ such that $A \equiv B$. More precisely if $B = \bigoplus_{i=1}^n B_i$, then

\[
B_i = \begin{cases} 
C_\omega, & \text{if } A_i \text{ is cancellative;} \\
L_n, & \text{if } A_i \cong L_n; \\
L_{k,d}, & \text{if } A_i \text{ is infinite, has rank } k \text{ and } d = d_{A_i}; \\
Q(U), & \text{if } A_i \text{ has infinite rank and } U = \{k : L_k \in S(A)\} \text{ is infinite}; \\
S_k, & \text{if } A_i \text{ has infinite rank and } d_{A_i} = k.
\end{cases}
\]

for $i < n$ and

\[
B_n = \begin{cases} 
L_n, & \text{if } A_n \cong L_n; \\
L_{k,k}, & \text{if } A_n \text{ is infinite and has rank } k; \\
C_\omega, & \text{if } A_n \text{ is cancellative}; \\
[0,1]_L, & \text{if } A_n \text{ has infinite rank}.
\end{cases}
\]

Proof. That $\text{HSP}_u(A) = \text{HSP}_u(B)$ follows from Lemma 7.2 and the description of $\text{HSP}_u(A)$. To prove uniqueness let $C = \bigoplus_{i=1}^m C_i \in \mathcal{F}_{BH}$; if $m \neq n$, then $\text{ISP}_u(B) \neq \text{ISP}_u(C)$. If they have the same index $n$ and they are different, then they must differ on some Wajsberg component. The first possibility is that there exists an $i < n$ such that

- $B_i$ is cancellative and $C_i$ is not (or viceversa);
- for some $k$, $B_i$ has at most $k$ elements and $C_i$ has more (or viceversa);
- $L_k$ is embeddable in $B_i$ and it is not embeddable in $C_i$ (or viceversa);
- $\{k : L_k \text{ is embeddable in } B_i\}$ is infinite and $\{k : L_k \text{ is embeddable in } C_i\}$ is not (or viceversa);
- $B_i$ and $C_i$ have different divisibility index.
But it is easy to see that any of this conditions implies $\text{ISP}_\nu(B_\nu) \neq \text{ISP}_\nu(C_\nu)$, so $\text{HSP}_\nu(B_\nu) \neq \text{HSP}_\nu(C)$. If this is not the case, then $B_\nu$ and $C_\nu$ must differ; a similar argument shows that in this case $\text{HSP}_\nu(B_\nu) \neq \text{HSP}_\nu(C_\nu)$. Hence $B$ is unique up to $\equiv$.

Now we can characterize the subvarieties of $\text{BH}$ that are strictly join irreducible (the reader can compare this with the analogous result for BL-algebras proved in [11]).

**Theorem 7.4.** A subvariety $V$ of $\text{BH}$ is strictly join irreducible if and only if it is equal to $V(A)$ for some $A \in \mathcal{F}_V$ such that its Wajsberg components are either cancellative or are bounded and have finite rank.

**Proof.** Let’s prove necessity first. We can work modulo $\equiv$ i.e. we may suppose that $A = \bigoplus_{i=0}^{n-1} A_i$ has the “canonical form” suggested by Theorem 7.3. So suppose that $A_n = [0, 1]_L$ and for any $k \in \mathbb{N}$ let $C_k = \bigoplus_{i=0}^{n-1} A_i \oplus L_k$; then clearly $C_k \neq A$ for all $k$. However $\text{HSP}_\nu(\{C_k : k \in \mathbb{N}\}) = \text{HSP}_\nu(A)$ since it is well known that $\text{HSP}_\nu(\{L_k : k \in \mathbb{N}\}) = \text{HSP}_\nu([0, 1])$. Hence $\bigvee_{k \in \mathbb{N}} V(C_k) = V(A)$ and $V(A)$ is not strictly join irreducible.

Next suppose that for some $i < n$, $A_i$ is bounded and has infinite rank; modulo $\equiv$ either $A_i = Q(U)$, where $U = \{k : L_k \in S(A_i)\}$ is infinite, or else $A = S_m$ for some $m$. In the first case for $k \in U$ we let $C_k$ to be the chain in which we have replaced the $i$-th component of $A$ with $L_k$; clearly $C_k \neq A$ for all $k$. However, since $Q(U)$ is generated by $\{L_k : k \in U\}$ any equation that fails in $A$ must fail in $C_k$ for some $k$; this proves that $\bigvee_{k \in U} V(C_k) = V(A)$ and so $V(A)$ is not strictly join irreducible.

For the second case we let $C_k$ to be the chain in which we have replaced the $i$-th component of $A$ with $L_{m,k}$; again clearly $C_k \neq A$ for all $k$ and a very similar argument to the one above shows that $\bigvee_{k \in U} V(C_k) = V(A)$ (but see Proposition 8 in [11] for details). Hence also in this case $V(A)$ is not strictly join irreducible and the proof of necessity is finished.

Let $V = V(A)$ where $A$ satisfies the hypotheses of the theorem; since $A$ is well-connected by Lemma 6.1 then $V$ is join irreducible in $\Lambda(\text{BH})$ by Theorem 4.3. Therefore if we show that $\Lambda(V)$ is finite, then the thesis will hold. Now by Theorem 6.6 it is enough to show that $\mathcal{F}_V/\equiv$ is finite, i.e. that $\mathcal{F}_V/\equiv$ is a finite set. Now any “canonical” representative in $\mathcal{F}_V/\equiv$ must be a chain of index $\leq$ than the index of $A$ and whose components are either $C_\omega$ or whose rank cannot exceed the maximum rank of the non cancellative components of $A$. Clearly there are only finitely many choices, so $\mathcal{F}_V/\equiv$ is finite as wished.

**Corollary 7.5.** Let $V$ be any variety of basic hoops; then the following are equivalent:

1. $\Lambda(V)$ is finite;

2. $V$ satisfies $\lambda_n$ and for all $A \in \mathcal{F}_V$, $V(A)$ is strictly join irreducible in $\Lambda(V)$.
8 Some examples

The only subvariety of BH satisfying $\lambda_1$ is of course the variety WH of Wajsberg hoops; $\Lambda(WH)$ has been totally described in [6]. It turns out that a proper subvariety of WH is generated by finitely many totally ordered hoops among $C_\omega$ and $\{L_n, L_\infty^n : n \in \mathbb{N}\}$. This (by Jonsson Lemma) implies that for any proper subvariety $V$ of WH, $\Lambda(V)$ is finite and hence by Corollary [25] $V$ is strictly join irreducible if and only if it is join irreducible if and only if it is generated by a single totally ordered Wajsberg hoop that is either cancellative or has finite rank (and so it is either $L_n$ or $L_\infty^n$ for some $n$).

Let $C = V(C_\omega)$ be the variety of cancellative hoops and let for $n \in \mathbb{N}$, $W_n = V(L_n)$ and $W_\infty^n = V(L_\infty^n)$. In Figure 1 we see a picture of $\Lambda(W_\infty^6)$ where we have labeled the subdirectly irreducible elements.

Now we would like to examine a larger lattice of subvarieties and we need some definitions. A Gödel hoop is an idempotent basic hoop. Gödel hoops are termwise equivalent to relative Stone lattices [30] and the variety is denoted by $GH$; it is well-known that the only subdirectly irreducible in $GH$ is $2$ (the two element residuated lattice) and that $\Lambda(GH)$ is a chain of type $\omega + 1$. More precisely, if $G_n = \bigoplus_{i=1}^n 2$, then the proper subvarieties of $GH$ are $G_n = V(G_n)$.

A product hoop is a basic hoop satisfying

$$(y \rightarrow z) \lor ((y \rightarrow xy) \rightarrow x) \approx 1;$$
The variety $\text{PH}$ of product hoops has been examined in \([3]\); it turns out that $\Lambda(\text{PH})$ has five elements and the proper non trivial subvarieties are $\text{C}$, $\text{G}_1$ and $\text{C} \lor \text{G}_1$. Moreover $\text{PH} = \text{V}(2 \oplus \text{C}_\omega)$ so it is strictly join irreducible in $\Lambda(\text{BH})$. Let $L = \text{WH} \lor \text{PH} \lor \text{GH}$; we want to describe $\Lambda(L)$.

Since basic hoops are congruence distributive, $\Lambda(\text{BH})$ is distributive and since it is also dually algebraic it satisfies the infinite distributive law

$$x \lor \bigwedge_{i \in I} y_i \approx \bigwedge_{i \in I} (x \lor y_i).$$

Let $L$ be any lattice; an interval is the set $[a,b] = \{ c : a \leq c \leq b \}$. An interval $[a,b]$ is an upper transpose of and interval $[c,d]$ (and $[c,d]$ is a lower transpose of $[a,b]$) if $a \lor d = b$ and $a \land d = c$. Two intervals $[a,b], [c,d]$ are $n$-step projective if there exists intervals $[a,b] = [x_0,y_0],[x_1,y_1], \ldots, [x_n,y_n] = [c,d]$ such that $[x_i,y_i]$ is an upper or lower transpose of $[x_{i+1},y_{i+1}]$. Two intervals are projective if they are $n$-step projective for some $n$. It is a nontrivial fact that in a distributive lattice “projectivity = 2-step projectivity”. Moreover we say that $b$ is a cover of $a$ and we write $a \prec b$ if $[a,b] = \{a,b\}$.

Let us apply these facts to $L$. First observe that

$$\text{WH} \land \text{PH} = \text{C} \lor \text{G}_1 = \text{V}(2, \text{C}_\omega);$$

to simplify the notation we will call that variety $\text{CPH}$. Since $\text{WH} \land \text{PH} = \text{CPH}$ and $\text{CPH} \prec \text{PH}$ we have that $\text{WH} \prec \text{WH} \lor \text{PH}$ and $[\text{CPH}, \text{WH}] \approx [\text{PH}, \text{WH} \lor \text{PH}]$. The same conclusion holds if we substitute $\text{WH}$ with any variety of Wajsberg hoops $W$.

**Lemma 8.1.** Let $W$ any variety of Wajsberg hoops. If $V \in [\text{PH}, W \lor \text{PH}]$ then there exists a variety $W' \subseteq W$ such that $W' \lor \text{PH} = V$. If $V \in [\text{CPH}, W \lor \text{PH}]$, then either $\text{PH} \leq V$ or $V \leq \text{WH}$.

**Proof.** Let $W' = \text{WH} \land V$; then

$$W' \lor \text{PH} = (\text{WH} \land V) \lor \text{PH}
= (\text{WH} \lor \text{PH}) \land (V \lor \text{PH}) = (\text{WH} \lor \text{PH}) \land V = V.$$

For the second, let $W' = \text{WH} \land V$. By distributivity we see at once that $[W', W' \lor \text{PH}]$ and $[\text{CPH}, \text{PH}]$ are isomorphic. It follows that $W' \ll W' \lor \text{PH}$ and hence either $V = W'$ or $V = W' \lor \text{PH}$. The former implies $V \leq \text{WH}$ and from the latter

$$V = W' \lor \text{PH} = (\text{WH} \lor V) \lor \text{PH}
= (\text{WH} \lor V) \lor (V \lor \text{PH}) = \text{CPH} \lor (V \lor \text{PH}) = V \lor \text{PH}.$$

Thus $\text{PH} \leq V$.

The same argument can be applied as it is to the intervals $[W_2, \text{GH}]$, $[\text{CPH}, \text{CPH} \lor \text{GH}]$ and $[\text{PH}, \text{PH} \lor \text{GH}]$. Then to $[\text{CPH}, \text{GH} \lor \text{PH}]$ and $[\text{WH}, L]$, and finally to $[\text{CPH}, \text{WH} \lor \text{PH}]$ and $[\text{GH}, L]$. Repeating this process for any variety of Wajsberg hoops or Gödel hoops we obtain a picture of the lattice of subvarieties of $L$ (Figure 2).

Clearly Figure 2 is rather sketchy, since all the equal ovals are isomorphic intervals and they are all connected. However, using the same techniques as in \([3]\) it is not hard to prove the following:

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Figure 2: The lattice of subvarieties of $L$
Lemma 8.2. Let $V$ be a subvariety of $L$ not above WH or GH. Then there are finite $I, J \subseteq \mathbb{N}$ (possibly empty) and an $n \in \mathbb{N}$ (possibly 0) such that

$$V = V(\{G_n, 2 \oplus C_\omega, C_i, C_i^\omega : i \in I, j \in J\})$$

It follow that if $V \subseteq L$ and $V$ is not above WH or GH, then $\Lambda(V)$ is finite, so Corollary 7.5 applies and then all the subvarieties generated by a chain of finite index are strictly join irreducible. Note that GH and WH are join irreducible but not strictly join irreducible in $\Lambda(V)$, since WH is generated by any Wajsberg chain of infinite rank and GH is generated by the chain $[0, 1]_G$, where $xy = x \land y$ in the usual ordering.

9 Generalized rotations

In this section we are going to use the results contained in [7] to lift information about strictly join irreducible varieties of basic hoops to some subvarieties of MTL-algebras. In particular, starting from a variety of basic hoops (or more in general, of representable CIRLs), we can use their generalized rotation (introduced in [10]) to generate a variety of MTL-algebras, and we are able to describe the relations between the respective lattices of subvarieties. Let us be more precise.

Consider a class of representable commutative residuated integral lattices (RCIRLs for short) $K$, a natural number $n \geq 2$, and a term defined rotation, i.e.: a unary term $\delta$ in the language of residuated lattices that on every algebra in $A \in K$ defines a lattice homomorphism that is also a nucleus (a closure operator satisfying $\delta(x) \cdot \delta(y) \leq \delta(x \cdot y)$ for all $x, y \in A$). It is easy to see that $\delta(x) = x$ and $\delta(x) = 1$ are both term defined rotations on any class of CIRLs. Then for every algebra $A \in K$ we can consider its generalized $n$-rotation, whose definition we write here for the reader’s convenience (for more details see [16]).

The lattice structure is given by the disjoint union of $A$ and $\delta^\prime[A]$ with the dualized order:

$$a' < b \text{ and } a' \leq b' \text{ iff } b \leq a$$

to which we add a chain of $n - 2$ elements strictly between $A$ and $\delta[A]'$. Intuitively, the resulting algebra has as skeleton a Lukasiewicz chain of $n$ elements, $L_{n-1}$, where the top is 1 and the bottom is $0 = 1'$. Let us call the elements of the Lukasiewicz chain:

$$0 = l_0 < l_1 < \ldots < l_{n-2} < l_{n-1} = 1.$$

$A$ is a substructure, the products in $\delta[A]'$ are all defined to be the bottom element $0 = 1'$. Moreover

$$a \cdot b' = (a \rightarrow b)'', \quad a \rightarrow b' = (\delta(b \cdot a))'', \quad a' \rightarrow b' = b \rightarrow a$$

$$a'l_i = l_i = l_i a, \quad b'l_i = 0 = l_i b' \quad (\text{for } i \notin \{0, n - 1\}).$$

The obtained structure, that we shall denote with $A^{\delta_n}$, is an MTL-algebra. It is relevant to notice that the term defined rotation $\delta$ coincides exactly
with the double negation of $A^{\delta_n}$ when restricted to $A$. Moreover, $A$ is the radical of $A^{\delta_n}$ (and its only maximal filter).

We can now define subvarieties of MTL in the following way: let $K$ be any class of representable CIRLs, $n \geq 2$, $\delta$ a term defined rotation, and we define

$$K^{\delta_n} = \{ A^{\delta_m} : m - 1 | n - 1, A \in K \}.$$ 

From now on we will write $\delta$ for $\delta_2$ (this coincides with the construction in [12]). Whenever we consider a variety of CIRLs $V$, we will write $V^{\delta_n}$ for the generated variety $V(V^{\delta_n})$. Moreover, we will write $V_t$ for the class of totally ordered members in $V$.

When we use $\delta$ as the identity map, we find interesting examples where the algebras obtained by rotations are involutive. We shall call this particular instance of the construction a generalized disconnected $n$-rotation. For instance, with $V$ the variety of Gödel hoops, its generalized disconnected 3-rotation $V^{\delta_3}$ is the variety of nilpotent minimum algebras $NM$. Starting with a variety of basic hoops $V$, the only disconnected $n$-rotation that is a subvariety of BL-algebras is the variety generated by perfect MV-algebras, given by 2-rotations of cancellative hoops (Proposition 3.4 [7]). Thus with this particular construction we are able to move from basic hoops to MTL-algebras. Notice that if instead we use the term defined rotation $\delta(x) = 1$ on varieties of basic hoops, then we obtain varieties of BL-algebras. In particular, for $n = 2$, we get the variety of SBL of pseudocomplemented (or, equivalently, Stonean) BL-algebras.

Let us recall some results about the lattice of subvarieties of term-defined rotations obtained in [7].

**Lemma 9.1** ([7]). Let $V$ be a class of RCIRLs, $\delta$ a term-defined rotation and $n \geq 2$. Then the following are equivalent:

1. $A \in \text{HSP}_u(K)$
2. $A^{\delta_m} \in \text{HSP}_u(K^{\delta_n})$ for every $m : m - 1 | n - 1$.

In the case where $K$ consists of a single algebra, we can actually obtain some extra information.

**Lemma 9.2.** Let $\delta$ a term-defined rotation, $n \geq 2$ and $m - 1 | n - 1$. Then $D^{\delta_m} \in \text{HSP}_u(B^{\delta_n})$ if and only if $D^{\delta_n} \in \text{HSP}_u(B^{\delta_n})$.

**Proof.** The right-to-left direction is a consequence of Lemma 9.1. Now suppose $D^{\delta_m} \in \text{HSP}_u(B^{\delta_n})$ with $m - 1 | n - 1$; we want to show that also $D^{\delta_n} \in \text{HSP}_u(B^{\delta_n})$. Notice that $B^{\delta_n}$ is directly indecomposable (see [10]), which is a first order property in FL_{ew}-algebras, since it corresponds to the fact that the only complemented elements are 0 and 1, which can clearly be expressed with a first order sentence. Thus all ultrapowers of $B^{\delta_n}$ are also directly indecomposable. All directly indecomposable algebras in a variety generated by generalized rotations are generalized rotations themselves (Theorem 4.8, [10]). Moreover, having an MV-skeleton of $n$ elements can also be expressed as a first order property, using the term $\gamma_n$ defined in [10]. In generalized rotations the term $\gamma_n$ is the identity map only on the elements of the MV-skeleton, evaluates to 1 in the elements of the radical and evaluates to 0 in the elements of the coradical. Therefore if one wishes to describe with a first order sentence that a generalized rotation has
an MV-skeleton of \( n \) elements, one can say that there exist \( n \) elements \( x_1 \ldots x_n \), pairwise different, such that \( \gamma_n(x_i) = x_i \) for \( i = 1 \ldots k \), and that are the only ones with such property. So we can conclude that any ultrapower of \( B^\delta \) is a generalized rotation \( U^\delta \). Thus \( D^\delta \in \text{HS}(U^\delta) \) and of course \( D^\delta \in \text{HS}(U^\delta) \); this implies \( D^\delta \in \text{HSP}_u(B^\delta) \) and the proof is completed.

The following is a consequence of Lemma 9.1.

**Proposition 9.3.** Let \( V, W \) be subvarieties of RCIRL, \( \delta \) a term-defined rotation and \( n \geq 2 \). Then \( V^\delta \subseteq W^\delta \) iff \( m - 1 \mid n - 1 \) and \( V \subseteq W \).

As a special case notice that, given a subvariety \( K \) of RCIRL and \( \delta \) a term defined rotation, the lattice of subvarieties of \( K \) and the lattice of subvarieties of \( K^\delta \) are isomorphic. In the case of varieties obtained as rotations of basic hoops, we can also say something more.

**Lemma 9.4.** For any subvariety \( V \) of BH, \( \Lambda(V^\delta) \) is the join dense completion of \( \Phi^\delta_V = \{ V(A^\delta_m) : A \in F_V, m - 1 \mid n - 1 \} \).

**Proof.** Let \( B \) be a totally ordered algebra in \( V^\delta \), then \( B \) is isomorphic to a generalized rotation \( A^\delta_m \) with \( m - 1 \mid n - 1 \). Moreover, if \( B \) is finitely generated, then \( A \in F_V \), since \( B \) is completely determined by its MV-skeleton \( L_{m-1} \) and the totally ordered basic hoop that is its radical \( A \), that cannot have infinitely many sum-irreducible components if it is finitely generated. Then the proof continues as in Lemma 6.6 and we can conclude that \( \Lambda(V^\delta) \) is the join dense completion of \( \Phi^\delta_V \).

**Lemma 9.5.** Let \( V, W_1 \ldots W_k \) be subvarieties of RCIRL, \( \delta \) a term-defined rotation and \( n \geq 2 \). Then

\[
V = \bigvee_{i \in I} W_i \text{ if and only if } V^\delta = \bigvee_{i \in I} W_i^\delta
\]

**Proof.** First, notice that given a RCIRL \( A, A \) is subdirectly irreducible if and only if \( A^\delta_m \) is subdirectly irreducible. Indeed, in the generalized rotation construction \( A \) is a congruence filter of \( A^\delta_m \), thus the former has a unique atomic filter if and only if the latter does.

Let us show the left-to-right direction first, thus we suppose that \( V = \bigvee_{i \in I} W_i \). Now, as a consequence of Lemma 9.1 \( V^\delta \) is the variety generated by algebras in the set \( \{ A^\delta_m : A \in V, A \text{ subdirectly irreducible} \} \). Notice that in particular all the algebras \( A^\delta_m \) for \( m - 1 \mid n - 1 \) are subalgebras of \( A^\delta_n \). Now we consider a subdirectly irreducible algebra \( A \in V \), and we call \( \mathfrak{s}W_i^\delta \) and \( \mathfrak{s}W_i \) the class of subdirectly irreducible algebras in
Proposition 9.6. Let \( W \subseteq \mathcal{W} \) and \( W_i \) respectively. We get:

\[
A^{\delta_m} \in V^{\delta_n} \iff A^{\delta_m} \in V^{\delta_n} \text{ for all } m - 1 \mid n - 1
\]

\[
\iff A \in V
\]

\[
\iff A \in \bigvee_{i \in I} W_i
\]

\[
\iff A \in \text{HSP}_a(\bigcup_{i \in I} \text{si}W_i)
\]

\[
\iff A^{\delta_m} \in \text{HSP}_a(\bigcup_{i \in I} \text{si}W_i^{\delta_n}) \text{ for all } m - 1 \mid n - 1
\]

\[
\iff A^{\delta_m} \in \bigvee_{i \in I} W_i^{\delta_n} \text{ for all } m - 1 \mid n - 1
\]

\[
\iff A^{\delta_m} \in \bigvee_{i \in I} W_i^{\delta_n}.
\]

The first and last equivalence follow from the fact that \( A^{\delta_m} \) is a subalgebra of \( A^{\delta_n} \) iff \( m - 1 \mid n - 1 \); the second and the third to last equivalences follow from Lemma 9.1. The third one follows by hypothesis. It remains to show the fourth equivalence (the proof of the second to last one being analogous). Since \( A \) is subdirectly irreducible and residuated lattices are congruence distributive, by Jónsson’s Lemma \( A \in V(K) \) iff \( A \in \text{HSP}_a(K) \) for any class of algebras \( K \). The equivalence then follows from the fact that \( \bigcup_{i \in I} \text{si}W_i \) is a set of generators for \( \bigvee_{i \in I} W_i \). Indeed, clearly \( V(\bigcup_{i \in I} \text{si}W_i) \subseteq \bigvee_{i \in I} W_i \). Moreover, for all \( i \in I \), \( W_i \subseteq V(\bigcup_{i=1}^k \text{si}W_i) \) thus \( V(\bigcup_{i \in I} \text{si}W_i) \).

We have showed that \( V^{\delta_m} = \bigvee_{i \in I} W_i^{\delta_n} \). The proof of the right-to-left direction uses the very same equivalences and is left to the reader.

Now we can prove the following.

**Proposition 9.6.** Let \( V \subseteq \mathcal{W} \) be subvarieties of \( \text{BH} \), \( \delta \) a term defined rotation and \( m, n \geq 2 \). Then \( V^{\delta_m} \) is strictly join irreducible in the lattice of subvarieties of \( W^{\delta_n} \) iff \( V \) is strictly join irreducible in the lattice of subvarieties of \( W \) and \( m - 1 \mid n - 1 \).

**Proof.** The left-to-right direction is relatively easy to see by contraposition. Indeed, first of all if \( m - 1 \) does not divide \( n - 1 \) then \( V^{\delta_m} \) is not a subvariety of \( W^{\delta_n} \) by Proposition 9.3. Moreover, if \( V \) is not strictly join irreducible, then \( V = \bigvee_{i \in I} W_i \) for some proper subvarieties \( W_i, i \in I \), but then \( V^{\delta_m} = \bigvee_{i \in I} W_i^{\delta_m} \) by Lemma 9.2. Next \( W_i^{\delta_m} \) is a subvariety of \( V^{\delta_m} \) by Proposition 9.3. It is also proper subvariety: let \( A \in V, A \notin W_i \). Then by Lemmas 9.1 and 9.2 \( A^{\delta_m} \in V^{\delta_m} \) but \( A^{\delta_m} \notin W_i^{\delta_m} \). Thus \( V^{\delta_m} \) is not strictly join irreducible.

Suppose now that \( V \) is strictly join irreducible and \( m - 1 \mid n - 1 \). Then \( V = V(A) \) for some chain \( A \) of finite order whose Wajsberg components are either cancellative or are bounded and have finite rank (Theorem 7.4). We can show that \( V^{\delta_m} = V(A^{\delta_m}) \). One inclusion is obvious, while for the other one we use the fact that subdirectly irreducible algebras in \( V^{\delta_m} \) are generalized rotations of subdirectly irreducible algebras in \( V \). So let’s take a subdirectly irreducible algebra \( C^{\delta_k} \in V^{\delta_m} \), with \( (k - 1) \mid (m - 1) \). Then \( C \) is a subdirectly irreducible algebra in \( V = V(A) \), which implies that
\(C \in \text{HSP}_u(A)\). Via Lemma 9.1 this implies that \(C^{j_k} \in \text{HSP}_u(A^{j_m})\), and thus every subdirectly irreducible in \(V^{j_m}\) belongs to \(V(A^{j_m})\), which completes the proof of the equality \(V^{j_m} = V(A^{j_m})\).

We have shown that \(V^{j_m}\) is generated by a totally ordered algebra, which by Corollary 6.1 is also well-connected; by applying Theorem 4.4 we get that \(V^{j_m}\) is join irreducible. We now show that \(\Lambda(V^{j_m})\) is finite, which will settle the proof. In particular, given Lemma 9.4, it suffices to show that \(\Phi_{V^{j_m}}\) is finite. In order to prove this, we will show that given \(B\) and \(C\) in \(\mathcal{F}_V\), for all \(k - 1 \mid m - 1\) we get

if \(\text{HSP}_u(B) = \text{HSP}_u(C)\) then \(\text{HSP}_u(B^{j_k}) = \text{HSP}_u(C^{j_k})\).

Since \(V\) is strictly join irreducible, \(\Phi_V\) is finite as shown in the proof of Theorem 7.3, which then implies that also \(\Phi_{V^{j_m}}\) is finite.

So let now \(B\) and \(C\) in \(\mathcal{F}_V\) such that \(\text{HSP}_u(B) = \text{HSP}_u(C)\). We show that for any \(k\) such that \(k - 1 \mid m - 1\), \(\text{HSP}_u(B^{j_k}) = \text{HSP}_u(C^{j_k})\).

Indeed consider an algebra in \(\text{HSP}_u(B^{j_k})\); then it is equal to \(D^{j_k}\) for some totally ordered basic hoop \(D\) and \(j - 1 \mid k - 1\). Thus by Lemma 9.2 \(D^{j_k} \in \text{HSP}_u(B^{j_k})\), and then \(D^{j_k} \subseteq \text{HSP}_u(B^{j_k})\) for all \(l - 1 \mid k - 1\) (since they are subalgebras of \(D^{j_k}\)); hence by Lemma 9.1 \(D \in \text{HSP}_u(B) = \text{HSP}_u(C)\) by hypothesis, and then (using again Lemma 9.1), \(D^{j_k} \subseteq \text{HSP}_u(C^{j_k})\), which implies that \(\text{HSP}_u(B^{j_k}) \subseteq \text{HSP}_u(C^{j_k})\). The equality follows from the fact that we can use the same reasoning starting with an algebra in \(\text{HSP}_u(C^{j_k})\). Therefore the proof is completed.

Moreover, we can also use the following result to obtain information on splitting algebras in varieties generated by generalized rotations. We recall that an algebra \(A \in V\) is splitting in the lattice of subvarieties of \(V\) if there is a subvariety \(W_A\) of \(V\) (the conjugate variety of \(A\)) such that for any variety \(U \subseteq V\) either \(U \subseteq W_A\) or \(A \notin U\).

**Proposition 9.7.** Let \(A\) be a totally ordered basic hoop of finite index such that its Wajsberg components are either cancellative or bounded with finite rank, and let \(V = V(A)\). Then for all suitable term defined rotations \(\delta, A^{j_m}\) is splitting in \(V^{j_m}\) for any \(m\) such that \(m - 1 \mid n - 1\).

**Proof.** It follows from Theorem 7.4 that \(V\) is strictly join irreducible in \(BH\), then the result follows by applying Lemma 3.3 in [7].

### 9.1 Some examples

We shall now give some specific applications of the theorems of this section. For details about the subvarieties mentioned here and their axiomatization, we refer the reader to [7]. Let us start considering the generalized rotations obtained with the term defined rotation \(\delta(x) = \ell(x) = 1\). Then \(BH^{\ell_n}\) is the subvariety of \(BL\) generated by all the so-called \(n\)-liftings of basic hoops: ordinal sums of the kind \(L_n \oplus B\), for a basic hoop \(B\). If \(t_\alpha(x) \approx 1\) is the equation axiomatizing \(V(L_n)\), then the variety \(BH^{\ell_n}\) is axiomatized by \(t_\alpha(\neg x) \approx 1\). Clearly, for \(n = 2\) we have the variety of Stonean \(BL\)-algebras. Then Proposition 9.6 applies to this variety, and we
can say that if \( V \) is a variety of basic hoops, then \( V^{\ell_m} \) is strictly join irreducible in the lattice of subvarieties of \( \text{BH}^{\ell_n} \) if and only if \( V \) is strictly join irreducible in the lattice of subvarieties of basic hoops and \( m - 1 \mid n - 1 \).

More in general, the following characterization is a direct consequence of Proposition 9.6 and Theorem 7.4.

**Corollary 9.8.** Let \( V \) be a subvariety of \( \text{BH} \), \( \delta \) a term-defined rotation and \( m, n \geq 2 \). \( V^{\delta_m} \) is strictly join irreducible in the lattice of subvarieties of \( \text{BH}^{\delta_n} \) if and only if \( m - 1 \mid n - 1 \) and \( V = V(A) \) for some \( A \in \mathcal{F}_V \) such that its Wajsberg components are either cancellative or are bounded and have finite rank.

For the rest of the section we are going to focus on term defined rotation the identity map, \( \delta = \text{id} \). Thus the resulting varieties will be involutive subvarieties of \( \text{MTL} \).

9.1.1 Nilpotent minimum varieties

Let us first consider involutive varieties generated by generalized disconnected rotations of Gödel hoops, that we called in [7] nilpotent minimum varieties, in similarity to the specific case of nilpotent minimum algebras \( \text{NM} = \text{GH}^{\delta_3} \).

Every proper subvariety \( G_k = V(G_k) \) of Gödel hoops is strictly join irreducible. Thus \( G_k^{\delta_m} \) is strictly join irreducible in the lattice of subvarieties of \( G_k^{\delta_n} \) whenever \( m - 1 \mid n - 1 \).

Moreover, all algebras \( G_k \) satisfy the hypothesis of Proposition 9.6, thus all rotations \( G_k^{\delta_m} \) are splitting in \( G_k^{\delta_n} \) whenever \( m - 1 \mid n - 1 \).

9.1.2 Nilpotent Lukasiewicz varieties

We call nilpotent Lukasiewicz varieties all subvarieties of \( \text{BH}^{\delta_n} \) generated by disconnected \( n \)-rotations of Wajsberg hoops. \( \text{WH}^{\delta_n} \) can be axiomatized by

\[
(\nabla_n(x) \land \nabla_n(y)) \rightarrow (((x \rightarrow y) \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x)) \approx 1. 
\]

Proper subvarieties of \( \text{WH} \) are all generated by finitely many chains [8] so their lattice of subvarieties is finite. Any proper variety of Wajsberg hoops \( V \) is axiomatized (modulo basic hoops) by a single equation in one variable of the form \( t_V(x) \approx 1 \), and \( V^{\delta_n} \) is axiomatized by \( \neg x \lor t_V(x) \approx 1 \) (modulo \( \text{BH}^{\delta_n} \)).

Now, the proper subvarieties of \( \text{WH} \) that are strictly join irreducible are the varieties generated by a single Wajsberg chain that is either cancellative or with finite rank, i.e., either \( L_n \) or \( L_n^\infty \) for some \( n \). Calling \( W_n \) and \( W_n^\infty \) the varieties generated by, respectively, \( L_n \) or \( L_n^\infty \), for all \( m \geq 2 \), \( W_n^{\delta_m} \) and \( (W_n^\infty)^{\delta_m} \) are strictly join irreducible in \( W_n^{\delta_n} \) whenever \( m - 1 \mid n - 1 \).

9.1.3 Nilpotent product varieties

We refer to the varieties \( C_n^{\delta_n} \) for \( n \geq 2 \) as nilpotent product varieties. Since cancellative hoops are axiomatized relative to Wajsberg hoop by
\[ (x \rightarrow x^2) \rightarrow x \approx 1 \] the variety \( C^{\delta_n} \) is axiomatized by
\[ \neg x^n \rightarrow ((x \rightarrow x^2) \rightarrow x) \approx 1. \]

The lattice of subvarieties can be easily described knowing that: \( C^{\delta_n} = V(C^{\delta_n}_\omega) \). For each \( C^{\delta_n} \), the varieties \( C^{\delta_m} \) where \( m - 1 \mid n - 1 \) are strictly join irreducible.

### 10 Linear varieties

A variety \( V \) of residuated lattices is **linear** if \( \Lambda(V) \) is totally ordered. Clearly each subvariety of a linear variety is join irreducible; via Theorem 4.4 each subvariety is then generated by a single \( \Gamma^n \)-connected algebra.

Since obviously if every subvariety of a variety is join irreducible, then the variety is linear, we have:

**Lemma 10.1.** A variety \( V \) is linear if and only if each subvariety of \( V \) is generated by a single \( \Gamma^n \)-connected algebra. Moreover if the order type of the chain is \( \omega + 1 \), then every subvariety is generated by a single subdirectly irreducible algebra.

If the variety is also representable, then each subvariety is generated by a single chain. The prototype of a variety satisfying this is the variety \(\text{GH} \) of Gödel hoops; \( \Lambda(\text{GH}) \) is the chain \( G_1 \subseteq G_2 \subseteq \cdots \subseteq G \) and each \( G_n \) is generated by \( \bigoplus_{i=1}^{n} 2 \) which is of course subdirectly irreducible. Therefore \( \text{GH} \) is generated by any infinite set of chain of finite index or else by a single chain of infinite index. This is no coincidence. First we need a simple lemma:

**Lemma 10.2.** Let \( A \) be totally ordered residuated lattice. Then \( A \) is finite if and only if \( A \models \bigvee_{i<n} x_i + 1/x_i \geq 1 \) for some \( n \).

**Proof.** Let \( A \) be finite, say \( |A| = n - 1 \). Therefore if \( a_1, \ldots, a_n \in A \) there must be some \( i < n \) with \( a_i \leq a_{i+1} \); this implies \( \bigvee_{i<n} a_{i+1}/a_i \geq 1 \) and so \( A \) satisfies the desired equation.

Conversely suppose that \( A \) is infinite; then for any \( n \) we can find \( a_1, \ldots, a_n \in A \) with \( a_n < \cdots < a_1 \), so that \( a_{i+1}/a_i < 1 \) for all \( i < n \). As \( A \) is a chain 1 is join prime, so \( \bigvee_{i<n} a_{i+1}/a_i < 1 \). Therefore \( A \) does not satisfy any of the desired equations.

**Lemma 10.3.** Let \( V \) be any linear variety of representable residuated lattices.

1. Given any infinite chain \( A \), \( V(A) \) contains the variety \( \mathcal{V}_c \) generated by all chains in \( V \).
2. If there is an infinite chain in \( V \), \( V \) has the finite model property (FMP for short) if and only if \( \mathcal{V}_c = V(A) \) for all infinite chains \( A \in V \).
3. For any fixed \( n \), all totally ordered members of \( V \) of cardinality \( n \) are isomorphic.
Proof. Let us prove (1) first. Let \( A \) be any infinite chain in \( V \) and let \( B \) be a finite chain in \( V \). Then since \( V \) is linear, either \( V(A) \subseteq V(B) \) or \( V(B) \subseteq V(A) \). But since \( B \) is finite, by Lemma 10.2 it satisfies an identity that \( A \) does not, thus \( V(A) \nsubseteq V(B) \) and then \( V(B) \subseteq V(A) \). Thus \( V(A) \) contains all finite chains in \( V \), which proves (1).

We now show (2). We have already shown that \( V_c \subseteq V(A) \) for all infinite chains \( A \in V \). We recall that a variety has the finite model property iff it is generated by its finite members. So the FMP holds for \( V \), then \( V_c = V \) and then \( V_c = V(A) = V \). Vice versa, suppose \( V_c = V(A) \) for all infinite chains \( A \in V \). If an equation fails in \( V \), it fails in some chain, possibly infinite, \( B \). But \( V(B) = V_c \), thus the equation must fail in a finite chain. This means that the FMP holds for \( V \).

Finally we prove (3). Let \( A, B \in V \) two chains having the same cardinality with \( A \ncong B \); as they are finite and totally ordered they are subdirectly irreducible and Jónsson’s Lemma implies that \( B \ncong V(A) \) and \( A \ncong V(B) \). Since \( V \) is linear this is a contradiction and (2) holds.

Corollary 10.4. Let \( V \) be a linear variety of representable residuated lattices with the finite model property (i.e. \( V \) is generated by its finite algebras).

1. Every infinite chain generates \( V \), and the only proper subvarieties of \( V \) are generated by a single finite chain (up to isomorphism).

2. If \( V \) is generated by a finite chain, then \( \Lambda(V) \) is finite.

3. If \( V \) is generated by an infinite chain, \( \Lambda(V) \) has order type \( \omega + 1 \).

Proof. Since \( V \) has the finite model property, by Lemma 10.3 if \( V \) has an infinite chain it generates the whole variety since it generates all finite chains. Therefore all the proper subvarieties must be generated by a finite chain. If \( V \) is generated by a finite chain \( A \) then by Jónsson’s Lemma any totally ordered member of \( V \) must be in \( HS(A) \); hence they are all finite and (1) holds.

If \( V \) is generated by a finite chain of cardinality \( n \), then in \( V \) there are only finite chains of cardinality at most \( n \) (by Lemma 10.2). Since finite chain of equal cardinality are isomorphic (by Lemma 10.3) \( \Lambda(V) \) must be finite.

Finally, if \( V \) is generated by an infinite chain, then the only proper subvarieties are generated by finite chains; by Lemma 10.3 we can pick a representative chain \( A_{n+1} \) with \( n \)-elements for any \( n \in \mathbb{N} \) and \( \Lambda(V) \) is

\[
V(A_0) < V(A_1) < \cdots < V(A_{n+1}) < \cdots < V.
\]

Therefore (3) holds.

Note that the proofs of Lemma 10.3 and Corollary 10.4 depend basically on the fact that we can distinguish equationally finite chains of different cardinalities and finite chains from infinite chains.

If \( V \) is a subvariety of \( RPS_H \) we can rephrase the previous results substituting the finiteness of the chain with the finiteness of the index. In other words:

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Corollary 10.5. Let $V$ be a linear variety contained in $\text{RPsH}$ with the finite model property.

1. $V$ is generated by each chain of infinite index $A$ (if any); hence the only proper subvarieties of $V$ are those generated by chains of finite index.

2. $\Lambda(V)$ is either finite or it is a chain of type $\omega + 1$.

Now we can use the classification of chains of finite index in $\text{BH}$ to describe all linear varieties of basic hoops. First we observe that the linear varieties of Wajsberg hoops are easy to classify simply by inspection using the description of $\Lambda(\text{WH})$ in [6]. We recall that we are denoting by $W_n$ the variety generated the Wajsberg hoop with $n + 1$-elements.

Theorem 10.6. The only linear varieties of Wajsberg hoops are the variety $C$ of cancellative hoops and $W_n$ where either $n = 1$ or $n$ is a prime power.

Observe that if $A = \bigoplus_{i \in I} A_i$ is a totally ordered hoop, then $V(A_i) \subseteq V(A)$ for all $i \in I$. Hence if $V(A)$ happens to be linear, then by Theorem 10.6 each $A_i$ must be either $C_\omega$ or $L_n$ where $n$ is a prime power.

Lemma 10.7. Let $A = \bigoplus_{i \in I} A_i$, with $|I| > 1$, be a totally ordered hoop such that $V(A)$ is linear. Then

1. either $A_i = 2$ for $i \in I$

2. or $A_i = C_\omega$ for $i \in I$.

Proof. It follows from Theorem 10.6 that the only possible components are either $C_\omega$, $2$ or $L_n$ for some prime power $n$. Note that since $V(2)$ and $V(C_\omega)$ are incomparable varieties of Wajsberg hoops, $2$ and $C_\omega$ cannot appear both in the decomposition of $A$. Since $2$ is a subalgebra of each $\text{Wa}_n$, also $\text{Wa}_n$ (for any $n$) and $C_\omega$ cannot appear both in the decomposition of $A$. Thus if there is $i$ such that $A_i = C_\omega$, then $A_i = C_\omega$ for all $i \in I$ and (2) holds.

Suppose now that $C_\omega$ is not in the decomposition of $A$, and suppose that for some $i$, $A_i = 2$. If there is a $j \neq i$ such that $A_j = L_n$ where $n$ is a prime power, $n > 1$, it follows immediately that $V(L_n)$ and $V(2 \oplus 2)$ are incomparable subvarieties of $V(2 \oplus L_n)$ so $V(A)$ is not linear. The same holds if there are two components of the kind $L_m$ and $L_n$, with $m, n$ (possibly equal) prime powers. Thus the only other possibility is that (1) holds.

Now we can characterize all the linear varieties of basic hoops. Let us call $\Omega(\mathbb{C})$ the variety generated by arbitrary ordinal sums of $C_\omega$.

Theorem 10.8. A variety $V$ of basic hoops is linear if and only if:

1. $V$ is a linear variety of Wajsberg hoops or

2. $V$ is a subvariety of $G$, or

3. $V$ is a subvariety of the variety $\Omega(\mathbb{C})$ of hoops satisfying $x \rightarrow x^2 \leq x$.  

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Proof. We have already seen that \( G \) is linear. For the variety \( \Omega(C) \) we simply note that any chain of finite index has only cancellative hoops as components, since the equation \( x \to x^2 \leq x \), in conjunction with Tanaka’s equation, implies cancellativity. Therefore, modulo \( \equiv \), any chain of index \( n \) is of the form \( \bigoplus_{i<n} C_\omega \); it is obvious now that \( \Omega(C) \) is linear and generated by the chain \( \bigoplus_{n\in\omega} C_\omega \).

Conversely assume that \( V \) is a linear variety of basic hoops that does not consist entirely of Wajsberg hoops. Then \( V \) is generated by a chain \( A \) of index \( > 1 \) (possibly infinite). Hence by Lemma 10.7 the components of \( A' \) are all equal to \( 2 \) or to \( C_\omega \) and the conclusion follows.

Moreover, similarly to the case of Wajsberg hoops the varieties of Wajsberg algebras that are linear can be found by simply inspecting the description of the lattice of subvarieties in [35]. Let us call \( Wa_n \) the variety generated by the \( n+1 \) elements MV-chain.

Lemma 10.9. Let \( V \) be a variety of Wajsberg algebras; then \( V \) is linear if and only if it is either \( Wa_n \) where either \( n = 1 \) or \( n \) is a prime power, or else it is generated by the Chang algebra (i.e. the zero-bounded version of \( L_2^\infty \)).

We can now apply our results to varieties obtained by generalized \( n \)-rotations. In particular, if we pick \( \delta \) to be any term-defined rotation on \( BH \) and \( n = 2 \), we can use the lattice isomorphism between \( \Lambda(BH) \) and \( \Lambda(BH^{\delta^2}) \) described after Proposition 9.3. In particular the following result applies if \( \delta = 1 \) or \( \delta = id \), that is to say, to \( SBL \) and the variety generated by disconnected rotations of basic hoops that we shall call \( bDL \).

Theorem 10.10. Let \( \delta \) be a term defined rotation, then a variety \( V \) contained in \( BH^{\delta^2} \) is linear if and only if \( V = W^{\delta^2} \) where \( W \) is a linear variety of basic hoops.

Thus in particular a subvariety of \( SBL \) is linear if and only if it is either

- a subvariety of the variety of Gödel algebras (bounded Gödel hoops), or
- the variety generated by \( 2 \oplus Wa_n \) where \( n \) is a prime power, or
- the variety generated by \( 2 \oplus A \), where \( A \) is a totally ordered member of \( \Omega(C) \).

What about \( BH^m \) for \( n > 2 \)? It turns out that the only linear varieties are those already described in Theorem 10.10.

Theorem 10.11. Let \( \delta \) be a term defined rotation, then a variety \( V \) contained in \( BH^m \) is linear if and only if either

1. \( V = Wa_n \) with \( n = 1 \) or \( n \) a prime power, or
2. \( V = W^{\delta^2} \) where \( W \) is a linear variety of basic hoops.

Proof. The right-to-left direction is easy to see, and is a consequence of Theorem 10.10 and Proposition 9.3. Suppose now that \( V \) contained in \( BH^m \) is linear, and it is not a linear variety of MV-algebras. Since it is linear it is also join irreducible, which means it is generated by a single subdirectly irreducible algebra (Theorem 9.3), say \( A^m \neq L_{m-1} \), with
\(m - 1 \mid n - 1\). If \(m > 2\), then both the MV-skeleton \(L_{m-1}\) and \(A^{\leq 2}\) are subalgebras of \(A^m\), and they generate incomparable subvarieties of \(V\). Thus necessarily \(m = 2\). Thus the result follows from Theorem 10.10.

Finally we show how we can get the classification of all linear varieties of BL-algebras (already obtained in [10]) using our techniques.

**Theorem 10.12.** A variety \(V\) of BL-algebras is linear if and only if:

1. \(V\) is a linear variety of Wajsberg algebras, or
2. \(V\) is a linear variety contained in \(SBL\).

**Proof.** Let \(A = \bigoplus_{i \in I} A_i\) be a BL-algebra such that \(V(A)\) is linear; if \(A\) is not a Wajsberg algebra, then \(|I| > 1\). Take \(k \in I, k \neq 0\); then \(V(A_0 \oplus A_k)\) must be a linear subvariety of \(V(A)\). If \(A_0 \neq 2\) then \(A_0 \not\in \text{HSP}_a(2)\), hence \(2 \oplus A_k\) and \(A_0\) generate incomparable subvarieties of \(V(A_0 \oplus A_k)\). But this is a contradiction so \(A_0 = 2\) and \(A \in SBL\).

11 Conclusions

We have investigated in a very general setting the properties of being, respectively, join irreducible and strictly join irreducible in subvarieties of residuated lattices. Moreover, we have applied the results found to representable varieties, and obtained a precise characterization of (strictly) join irreducible varieties and linear varieties in the case of basic hoops. Finally, we made use of the generalized rotation construction to lift some of these results to subvarieties of MTL-algebras.

As for future work, it is worth mentioning that a study of the techniques employed in [21] and [22] could lead to an analogue characterization for varieties of representable pseudohoops.

Finally, we believe that the results and techniques developed in the first sections of this work can pave the way for many other possible applications than the ones considered here.

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