Properties and construction of extreme bipartite states having positive partial transpose

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We consider a bipartite quantum system $H_A \otimes H_B$ with $M = \text{Dim } H_A$ and $N = \text{Dim } H_B$. We study the set $\mathcal{E}$ of extreme points of the compact convex set of all states having positive partial transpose (PPT) and its subsets $\mathcal{E}_r = \{\rho \in \mathcal{E} : \text{rank } \rho = r\}$. Our main results pertain to the subsets $\mathcal{E}_{r_{M,N}}$ consisting of states whose reduced density operators have ranks $M$ and $N$, respectively. The set $\mathcal{E}_1$ is just the set of pure product states. It is known that $\mathcal{E}_{r_{M,N}} = \emptyset$ for $1 < r \leq \min(M, N)$ and for $r = MN$. We prove that also $\mathcal{E}_{MN-1} = \emptyset$. Leinaas, Myrheim and Sollid have conjectured that $\mathcal{E}_{M+N-2} \neq \emptyset$ for all $M, N > 2$ and that $\mathcal{E}_{r_{M,N}} = \emptyset$ for $1 < r < M + N - 2$. We prove the first part of their conjecture. The second part is known to hold when $\min(M, N) = 3$ and we prove that it holds also when $\min(M, N) = 4$. This is a consequence of our result that $\mathcal{E}_{N+1} = \emptyset$ if $M, N > 3$.

We introduce the notion of “good” states, show that all pure states are good and give a simple description of the good separable states. For a good state $\rho \in \mathcal{E}_{M+N-2}$, we prove that the range of $\rho$ contains no product vectors and that the partial transpose of $\rho$ has rank $M + N - 2$ as well. In the special case $M = 3$, we construct good $3 \times N$ extreme states of rank $N + 1$ for all $N \geq 4$.

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Contents

I. Introduction 2

II. Preliminaries 5
    A. Mathematics 5
    B. Quantum information 7

III. Good and bad states 10

IV. $M \times N$ PPT states of rank $M + N - 2$ 13
    A. Product vectors in the kernel 13
    B. Good states 15

V. $M \times N$ PPT states of rank $N + 1$ 17

VI. Examples of $M \times N$ PPT states of rank $M + N - 2$ 22
    A. Good case: finitely many product vectors in the kernel 22
    B. Bad case: infinitely many product vectors in the kernel 24

VII. Some open problems 29

Acknowledgments 30

References 30

VIII. Appendix 31

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I. INTRODUCTION

Let us consider a finite-dimensional bipartite quantum systems represented by the complex Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with Dim $\mathcal{H}_A = M$ and Dim $\mathcal{H}_B = N$. A state of this system is a positive semidefinite linear operator $\rho : \mathcal{H} \to \mathcal{H}$ with $\text{Tr} \rho = 1$. A pure state is a state $\rho = |\psi\rangle \langle \psi|$ where $|\psi\rangle \in \mathcal{H}$ is a unit vector. A product state is a state $\rho = \rho_1 \otimes \rho_2$ where $\rho_1$ is a state on $\mathcal{H}_A$ and $\rho_2$ a state on $\mathcal{H}_B$. If moreover $\rho_1$ and $\rho_2$ are pure states, then we say that $\rho = \rho_1 \otimes \rho_2$ is a pure product state. For any nonzero vector $|x, y\rangle := |x\rangle \otimes |y\rangle$ we say that it is a product vector. By definition, a separable state, say $\sigma$, is a convex linear combination of pure product states $\rho_i$:

$$\sigma = \sum_{i=1}^{k} p_i \rho_i, \quad p_i \geq 0, \quad \sum_{i=1}^{k} p_i = 1. \quad (1)$$

A state is entangled if it is not separable. It is a highly nontrivial task to determine whether a given bipartite state is separable \cite{16}.

We can write any linear operator $\rho$ on $\mathcal{H}$ as

$$\rho = \sum_{i,j=0}^{M-1} |i\rangle \langle j| \otimes \rho_{ij}, \quad (2)$$

where $\{|i\rangle : 0 \leq i < M\}$ is an orthonormal (o.n.) basis of $\mathcal{H}_A$ and $\rho_{ij} = \langle i| \rho |j\rangle$ are linear operators on $\mathcal{H}_B$. Then the partial transpose $\rho^T$ of $\rho$ is defined by

$$\rho^T = \sum_{i,j=0}^{M-1} |j\rangle \langle i| \otimes \rho_{ij}. \quad (3)$$

The reduced operators $\rho_A$ and $\rho_B$ of $\rho$ are defined by

$$\rho_A = \text{Tr}_B(\rho) = \sum_{i,j=0}^{M-1} \text{Tr}(\rho_{ij}) |i\rangle \langle j|, \quad \rho_B = \text{Tr}_A(\rho) = \sum_{i=0}^{M-1} \rho_{ii}. \quad (4)$$

where $\text{Tr}_A$ and $\text{Tr}_B$ are partial traces. We refer to rank $\rho_A$ as the A-local rank and to rank $\rho_B$ as the B-local rank of $\rho$. We shall use the following very convenient but non-standard terminology.

**Definition 1** A bipartite state $\rho$ is a $k \times l$ state if $\text{rank} \rho_A = k$ and $\text{rank} \rho_B = l$.

If $\rho$ is a separable state, then necessarily $\rho^T \succeq 0$ (i.e., $\rho^T$ is positive semidefinite). This necessary condition for separability is due to Peres \cite{37}. If $\text{Dim} \mathcal{H} \leq 6$ then this separability condition is also sufficient (but not otherwise) \cite{0, 21, 22, 44}. We say that a state $\rho$ is PPT if it satisfies the Peres condition $\rho^T \succeq 0$. A state $\rho$ is NPT if $\rho^T$ is not positive semidefinite.

We are interested in the problem of describing the set, $\mathcal{E}$, of extreme points of the compact convex set consisting of all PPT states. We shall refer to any $\rho \in \mathcal{E}$ as an extreme state. Since every PPT state is a convex linear combination of extreme states, it is important to understand the structure of $\mathcal{E}$. The rank function provides the partition

$$\mathcal{E} = \bigcup_{r=1}^{MN} \mathcal{E}_r, \quad \mathcal{E}_r := \{ \rho \in \mathcal{E} : \text{rank} \rho = r \}. \quad (5)$$

The first part, $\mathcal{E}_1$, is the set of pure product states. Let us briefly explain this observation. Let $\rho \in \mathcal{E}_1$. Since any state of rank one is pure, $\rho$ is a pure PPT state. It follows from the Schmidt decomposition that any pure PPT state is necessarily a product state. Thus $\rho$ is a pure product state. Conversely, let $\rho$ be a pure product state. As $\rho$ has rank one, it is extremal among all states, and in particular it is extremal among all PPT states. Thus $\rho \in \mathcal{E}_1$.

It follows easily that $\mathcal{E}_1$ is also the set of extreme points of the compact convex set consisting of all separable states. Consequently, for $r > 1$, the set $\mathcal{E}_r$ contains only entangled states. Since all PPT states of rank less than four are separable \cite{26, 27}, we have $\mathcal{E}_2 = \mathcal{E}_3 = \emptyset$ (see also Proposition \cite{11} below).

We can further partition the subsets $\mathcal{E}_r$ by using the local ranks

$$\mathcal{E}_r = \bigcup_{(k,l)} \mathcal{E}^{k,l}_r, \quad \mathcal{E}^{k,l}_r := \{ \rho \in \mathcal{E}_r : \text{rank} \rho_A = k, \text{rank} \rho_B = l \}. \quad (6)$$
For \( r = 1 \) we have \( \mathcal{E}_1 = \mathcal{E}_1^{1,1} \) and for \( r = 4 \) we have \( \mathcal{E}_4 = \mathcal{E}_4^{3,3} \).

Assume that \( r > 4 \). Then the problem of deciding which sets \( \mathcal{E}_r^{k,l} \) are nonempty is apparently hard. If \( kl \leq 6 \) then any \( k \times l \) PPT state is separable, and so \( \mathcal{E}_r^{k,l} = \emptyset \). It is easy to see that the condition \( \min(k, l) > 1 \) is necessary for \( \mathcal{E}_r^{k,l} \) to be nonempty. The condition \( \max(k, l) < r \) is also necessary. This follows from the well known facts that any \( k \times l \) state of rank \( r \) is NPT if \( r < \max(k, l) \) and is separable if \( r = \max(k, l) \). See Theorem \([9]\) and Proposition \([10]\) which are proved in \([27]\) and \([30]\), respectively.

As we shall see below, \( \mathcal{E}_5^{4,4} = \emptyset \), and so for \( r = 5 \) we have

\[
\mathcal{E}_5 = \mathcal{E}_5^{2,4} \cup \mathcal{E}_5^{3,2} \cup \mathcal{E}_5^{3,3} \cup \mathcal{E}_5^{4,4} \cup \mathcal{E}_5^{4,3}.
\]

It was shown in \([1]\) that the sets \( \mathcal{E}_5^{2,4} \) and \( \mathcal{E}_5^{3,2} \) are nonempty. The sets \( \mathcal{E}_5^{3,3} \) and \( \mathcal{E}_5^{4,3} \) are also nonempty, see Example \([18]\). Since the example of the \( 3 \times 3 \) PPTES of rank five constructed in \([10]\) is extreme, it follows that \( \mathcal{E}_5^{3,3} \neq \emptyset \).

The following conjecture was proposed recently by Leinaas, Myrheim and Sollid \([23]\) Sec. III, part D).

**Conjecture 2** \((M, N > 2)\)

(i) \( \mathcal{E}_{M+N-2}^{M,N} \neq \emptyset \);

(ii) \( \mathcal{E}_{r}^{M,N} = \emptyset \) for \( 1 < r < M + N - 2 \);

(iii) if \( \rho \in \mathcal{E}_{M+N-2}^{M,N} \) then \( \operatorname{rank}(\rho^T) = M + N - 2 \);

Note that part (ii) of this conjecture is true if \( \min(M, N) = 3 \). It is also true if \( \min(M, N) = 4 \). This is a consequence of the general fact, proved in Theorem \([13]\) which says that \( \mathcal{E}_{N+1}^{M,N} = \emptyset \) if \( M, N > 3 \). (In particular, we have \( \mathcal{E}_5^{4,4} = \emptyset \).) Consequently, if \( k, l > 3 \) then the condition \( \max(k, l) < r - 1 \) is necessary for \( \mathcal{E}_r^{k,l} \) to be nonempty. The question whether (ii) is valid when \( \min(M, N) > 4 \) remains open.

In Theorem \([57]\) we prove part (i) of the above conjecture. The proof is based on explicit construction of the required extreme states. An important tool used in this proof is the extremality criterion first discovered in \([32]\), and independently in \([1]\) (see Proposition \([17]\) for an enhanced version). It has been hard in the past to verify that a given PPT state is extreme, see e.g. \([12, 29]\). By using the extremality criterion, this is now a routine task. Proposition \([13]\) gives a simple necessary condition for extremality. The well known fact that \( \mathcal{E}_{3,3}^{M,N} = \emptyset \) is an immediate consequence of this proposition. By using the same proposition, we prove that also \( \mathcal{E}_{3,2}^{M,N} \neq \emptyset \), see Corollary \([33]\).

Extreme states have applications to some important problems of entanglement theory. First, it is known that extreme states of rank \( > 1 \) are also edge states \([1]\). We recall that a PPT state \( \rho \) is an edge state if there is no product vector \( (a, b) \in \mathcal{R}(\rho) \) such that \( |a^*, b \rangle \in \mathcal{R}(\rho^T) \). Note that any edge state is necessarily entangled. Second, entanglement distillation is a core task in quantum information theory \([3]\). Although not all entangled states can be distilled \([23]\), we will show that extreme states can play the role of activators in entanglement distillation in Sec. VII.

Third, characterizing extreme states is useful for solving the separability problem in some special cases, see Proposition \([13]\) (ii).

After examining many examples of bipartite states, we came to the conclusion that they should be divided into two broad categories. For lack of a better name, we refer to them as “good” and “bad” states. The characterization of these states, in particular good states, is the main problem of this paper. More generally, we shall first define these notions for vector subspaces of \( \mathcal{H} \). For this we shall make use of complex projective spaces and some basic facts from algebraic geometry. We shall recall these notions and facts in the next section. For more information the reader may consult \([19]\).

We denote by \( \mathcal{P}_{AB} \) the complex projective space of \((\text{complex})\) dimension \( MN - 1 \) associated to \( \mathcal{H} \), and denote by \( \mathcal{P}_A \) and \( \mathcal{P}_B \) the projective spaces associated to \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively. For any vector subspace \( K \subseteq \mathcal{H} \), we denote by \( K \) the projective subspace of \( \mathcal{P}_B \) associated to \( K \). By a projective variety we mean any Zariski closed subset, say \( X \), of \( \mathcal{P}_{AB} \). If \( X \) is not the union of two proper Zariski closed subsets, then we say that \( X \) is irreducible. Any projective variety \( X \) is a finite irredundant union of irreducible projective varieties \( X_i, i = 1, \ldots, s \). The \( X_i \) are unique up to indexing, and we refer to them as the irreducible components of \( X \).

The points of \( \mathcal{P}_{AB} \) which correspond to product vectors form a projective variety \( \Sigma = \Sigma_{M-1,N-1} \) known as the Segre variety. Thus a point of \( \Sigma \) is a 1-dimensional subspace spanned by a product vector. The variety \( \Sigma \) is isomorphic to the direct product \( \mathcal{P}_A \times \mathcal{P}_B \). Its complex dimension is \( M + N - 2 \), and its codimension in the ambient projective space \( \mathcal{P}_{AB} \) is \((M - 1)(N - 1)\). Let us briefly recall the concept of degree for a projective variety, say \( X \), of dimension \( k \) embedded in the projective space \( \mathcal{P}^n \). The degree of \( X \) can be defined as the number of intersection points of \( X \) with a general projective subspace \( L \) of complementary dimension, \( n - k \). For instance, for the Segre variety \( \Sigma \subset \mathcal{P}_{AB} \), we have to take \( L \) of dimension \((MN - 1) - (M + N - 2) = (M - 1)(N - 1) \). We also recall (see \([19]\) Example 18.15) that the degree of \( \Sigma \) is the binomial coefficient

\[
\delta = \delta(M, N) := \left(\begin{array}{c}
M + N - 2 \\
M - 1
\end{array}\right).
\]
Let $K \subseteq \mathcal{H}$ be a vector subspace of dimension $k+1$ and let $X = \mathcal{K} \cap \Sigma$. We say that $K$ is good in $\mathcal{H}$ if either $X = \emptyset$ or $\dim X = k - (M-1)(N-1) \geq 0$ and $K$ and $\Sigma$ intersect generically transversely. Otherwise, we say that $K$ is bad in $\mathcal{H}$.

Let us clarify this definition. If $k < (M-1)(N-1)$ then there are two possibilities: $X = \emptyset$ and so $K$ is good or $X \neq \emptyset$ and so $K$ is bad. Assume now that $k \geq (M-1)(N-1)$. Then necessarily $X \neq \emptyset$ (see Proposition 9 below), and let $X_i$, $i = 1, \ldots, s$, be the irreducible components of $X$. Moreover, we have $\dim X_i \geq k - (M-1)(N-1)$ for each index $i$. As $\dim X = \max_i \dim X_i$, the assertion $\dim X = k - (M-1)(N-1)$ is equivalent to the assertion that $\dim X_i = k - (M-1)(N-1)$ for each index $i$. Finally, the transversality condition means that for each $i$ there exists a point $x_i \in X_i$ such that $x_i \notin X_j$ for $j \neq i$ and the sum of the tangent spaces of $\mathcal{K}$ and $\Sigma$ at $x_i$ is equal to the whole tangent space of $\mathcal{P}_{AB}$ at the same point. To state an affine equivalent of this condition, for any product vector $|a,b\rangle$, we set $S_{a,b} = |a\rangle \otimes \mathcal{H}_B + \mathcal{H}_A \otimes |b\rangle$. If the point $x_i$ is represented by the product vector $|a_i,b_i\rangle$, then the transversality condition at $x_i$ is equivalent to the equality $K + S_{a_i,b_i} = \mathcal{H}$.

To any state $\rho$, we attach a (possibly empty) projective variety $X_\rho := \hat{K} \cap \Sigma$ where $K = \ker \rho$. We say that $\rho$ is good or bad in $\mathcal{H}$ if $K$ is good or bad in $\mathcal{H}$, respectively. In Theorem 30 we give a simple description of good separable states. In Theorem 32 we prove that if $\rho$ is a PPT state of rank $r$ and $\ker \rho$ contains no 2-dimensional subspace $V \otimes W$, then either $r = M + N - 2$ and $|X_\rho| = \delta$ or $r > M + N - 2$ and $|X_\rho| < \delta$. (By $|X|$ we denote the cardinality of a set $X$.)

The good states have many good properties. First note that if $\rho$ is a good state and $\sigma$ another state having the same kernel (or, equivalently, range) as $\rho$, then $\sigma$ is also good. Second, if $\rho$ is a good state then the same is true for its transform $\rho' = V_\rho V^\dagger$ by any invertible local operator (ILO) $V = A \otimes B$. Since $\ker \rho = V^\dagger \ker \rho'$, this follows from the fact that $V^\dagger = A^\dagger \otimes B^\dagger$ acts on $\mathcal{P}_{AB}$ as a projective transformation and maps $\Sigma$ onto itself. By using these two facts, we can construct new good states and study their properties. For instance, we prove in Theorem 24 that good entangled non-distillable states must have full local ranks. One of the most important conjectures in entanglement theory asserts that some of the Werner states [13], which have full rank and thus are good, are not distillable [21]. On the other hand, we show in Proposition 22 that all pure states (the basic ingredients for quantum-information tasks) are good. Third, in Theorem 31 we provide a family of good separable states, which turn out to be the generalized classical states which occur in the study of quantum discord [4]. Hence, these examples show the operational meaning of good states in quantum information.

Part (ii) of Theorem 34 states that if $\rho$ is a good $M \times N$ PPT state of rank $M + N - 2$, then the same holds true for $\rho^\dagger$. Consequently, part (iii) of Conjecture 2 is valid in the good case, while it remains open in the bad case. We say that a subspace of $\mathcal{H}$ is a completely entangled subspace (CES) if it contains no product vectors. It follows from part (iii) of Theorem 34 that if $\rho \in \mathcal{E}_{M+N-2}$ is good and $M, N > 2$, then $\mathcal{R}(\rho)$ and $\mathcal{R}(\rho^\dagger)$ are CES.

In the borderline case, $r = M + N - 2$, we shall propose another conjecture. Let us first introduce the following definition.

**Definition 4** A PPT state $\sigma$ is strongly extreme if there are no PPT states $\rho \neq \sigma$ such that $\mathcal{R}(\rho) = \mathcal{R}(\sigma)$.

Obviously any pure product state is strongly extreme. It follows from Proposition 9 that any $3 \times 3$ PPTES of rank four is also strongly extreme. The strongly extreme states are extreme, see Lemma 20. In the same lemma it is shown that the range of a strongly extreme state is a CES. There exist examples of extreme states which are not strongly extreme, e.g., $3 \times 3$ extreme states $\sigma$ of rank five or six, see [22] and its references. Indeed, since rank $\sigma \geq 5$, $\mathcal{R}(\sigma)$ is not a CES (see Proposition 6).

We can now state our conjecture which generalizes Proposition 13.

**Conjecture 5** Every state $\rho \in \mathcal{E}_{M+N-2}^{M,N}$, $M, N > 2$, is strongly extreme.

Theorem 34 also shows that Conjecture 5 is valid in the good case, but it remains open in the bad case.

The content of our paper is as follows.

Sec. 11 has two subsections. In the first one we describe the tools that enable us to represent bipartite density matrices and perform the basic local operations on them. We also introduce the necessary background and give references about complex projective varieties embedded in an ambient complex projective space, $\mathcal{P}_{AB}$ in our context. We also define the good and bad subspaces and states. In the second subsection we summarize some important facts from quantum information theory that we will need. We introduce the concept of reducible and irreducible bipartite states, present the extremality criterion and give a short proof.

Sec. 111 deals mostly with the properties of good states. We first single out a special class of good states which we call universally good. These are good states which remain good after embedding the original $M \otimes N$ quantum system into an arbitrary $M' \otimes N'$ quantum system with $M' \geq M$ and $N' \geq N$. Then we show that all pure states are good, and consequently they are also universally good (see Proposition 22). The universally good PPT states
are fully characterized in Theorem 51, in particular they are separable. The Proposition 25 relates the number, \( m \), of product vectors in a subspace \( H \subseteq \mathcal{H} \) to the dimension of \( H \). In particular, it is shown that if \( m = \delta \) then this dimension must be \((M - 1)(N - 1) + 1\). In Theorem 28 we show how one can find all irreducible components of the variety \( X_\rho \) for arbitrary separable state \( \rho \). It turns out each of these components is the Segre variety of a subspace \( V \otimes W \subseteq \ker \rho \). Finally, we obtain a very simple characterization of the good separable states in Theorem 30.

Sec. IV is mainly about the borderline case: the \( M \times N \) PPT states of rank \( M + N - 2 \). We need two results from algebraic geometry, which are proved in the Appendix. There are two subsections. In the first one we prove a general result which applies to all \( M \times N \) PPT states \( \rho \), namely Theorem 52. First, it shows that if \( X_\rho \) is an infinite set then \( \ker \rho \) contains a 2-dimensional subspace \( V \otimes W \). (For a stronger version of this result see Theorem 54.) Second, if \( m := |X_\rho| < \infty \) then either \( m = \delta \) and rank \( \rho = M + N - 2 \) or \( m < \delta \) and rank \( \rho > M + N - 2 \). By using Theorem 52 we prove in Corollary 33 that \( \varepsilon_{MN}^{M-N-1} = 0 \). In the second subsection we characterize good \( M \times N \) PPT states of rank \( M + N - 2 \), see Theorem 54. Proposition 37 shows that if \( \rho \) is an \( M \times N \) state, \( |X_\rho| < \infty \) and rank \( \rho^F = M + N - 2 \), then \( \rho \) must be a good PPT state of the same rank.

In Sec. V we investigate the \( M \times N \) PPT states \( \rho \) of rank \( N + 1 \). In Proposition 39 we characterize such states \( \rho \) when the range of \( \rho \) contains at least one product vector. In Theorem 42 we analyze further the case when \( \rho \) is entangled. The main result of this section is that \( \rho \) cannot be extreme when \( M, N > 3 \), see Theorem 43. Based on this result we construct a link between the good and extreme states in Proposition 44. Then in Theorem 45 we extend assertions (i)-ii) of Theorem 11 to \( M \otimes N \) systems. We also give a sufficient condition for extremality of \( 3 \times N \) states of rank \( N + 1 \), see Theorem 47.

In Sec. VI we construct many examples of good and bad \( M \times N \) PPT states of rank \( M + N - 2 \). There are two subsections; the first contains good cases and the second bad cases. The most important are the infinite families given in Examples 51 and 56. The first of these families consists of strongly extreme \( 3 \times N \) states of rank \( N + 1, N > 3 \), and we prove that all of these states are good (see Theorem 52). The second family consists of bad \( M \times N \) PPT states of rank \( N + 2 \). In Theorem 57 we prove that all of these states are extreme. Thus, we confirm part (i) of Conjecture 2.

In Sec. VII we propose some open problems.

II. PRELIMINARIES

In this section we state our conventions and notation, and review known and derive some new results which will be used throughout the paper.

We shall write \( I_k \) for the identity \( k \times k \) matrix. We denote by \( \mathcal{R}(\rho) \) and \( \ker \rho \) the range and kernel of a linear map \( \rho \), respectively. Many of the results will begin with a clause specifying the assumptions on \( M \) and \( N \). The default will be that \( M, N > 1 \). From now on, unless stated otherwise, the states will not be normalized.

We say that a non-normalized state \( \rho \) is extreme if its normalization is an extreme point of the set of normalized PPT states. Equivalently, a non-normalized state \( \rho \) is extreme if it is PPT and cannot be written as the sum of two non-proportional PPT states.

The rest of this section is divided into two parts. The first part deals with mathematical topics and the second one with quantum information.

A. Mathematics

We shall denote by \( \{|i\}_A : i = 0, \ldots, M - 1 \} \) and \( \{|j\}_B : j = 0, \ldots, N - 1 \} \) o.n. bases of \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively. The subscripts A and B will be often omitted. Any state \( \rho \) of rank \( r \) can be represented as (see [7, Proposition 6])

\[
\rho = \sum_{i,j=0}^{M-1} |i\rangle \langle j| \otimes C_i^\dagger C_j, \tag{9}
\]

where the \( C_i \) are \( R \times N \) matrices and \( R \) is an arbitrary integer \( \geq r \). In particular, one can take \( R = r \). We shall often consider \( \rho \) as a block matrix \( \rho = C^\dagger C = |C_i^\dagger C_j| \), where \( C = |C_0 C_1 \cdots C_{M-1}| \) is an \( R \times MN \) matrix. Thus \( C_i^\dagger C_j \) is the matrix of the linear operator \( \langle i|_A \rho |j\rangle_A \) acting on \( \mathcal{H}_B \). For the reduced density matrices, we have the formulae

\[
\rho_B = \sum_{i=0}^{M-1} C_i^\dagger C_i; \quad \rho_A = [\text{Tr} C_i^\dagger C_j], \quad i,j = 0, \ldots, M - 1. \tag{10}
\]
It is easy to verify that the range of $\rho$ is the column space of the matrix $C^\dagger$ and that
\[
\ker \rho = \left\{ \sum_{i=0}^{M-1} |i\rangle \otimes |y_i\rangle : \sum_{i=0}^{M-1} C_i |y_i\rangle = 0 \right\}. \tag{11}
\]
In particular, if $C_i |j\rangle = 0$ for some $i$ and $j$ then $|i, j\rangle \in \ker \rho$.

For any bipartite state $\rho$ we have
\[
\begin{align*}
(\rho^T)_B &= \text{Tr}_A (\rho^T) = \text{Tr}_{AB} \rho = \rho_B, \tag{12} \\
(\rho^T)_A &= \text{Tr}_B (\rho^T) = (\text{Tr}_{BP})^T = (\rho_A)^T. \tag{13}
\end{align*}
\]
(The exponent $T$ denotes transposition.) Consequently,
\[
\begin{align*}
\text{rank} (\rho^T)_A &= \text{rank} \rho_A, \quad \text{rank} (\rho^T)_B = \text{rank} \rho_B. \tag{14}
\end{align*}
\]
If $\rho$ is an $M \times N$ PPT state, then $\rho^T$ is too. If $\rho$ is a PPTES so is $\rho^T$, but they may have different ranks. We refer to the ordered pair $(\text{rank} \rho, \text{rank} \rho^T)$ as the \textit{birank} of $\rho$.

For counting purposes, we do not distinguish two product vectors which are scalar multiples of each other. The maximum dimension of a CES is $(M-1)(N-1)$, see \cite{zhang1999, kosloff1989}. For an explicit and simple construction of CES with this dimension see \cite{zhang1999, kosloff1989}. For convenience, we shall represent the pure state $\sum_{i,j} \xi_{ij} |i\rangle \otimes |j\rangle$ also by the $M \times N$ matrix $[\xi_{ij}]$. Then pure product states are represented by matrices of rank one.

The \textit{partial conjugate} of a product vector $|a,b\rangle$ is the product vector $|a^*, b\rangle := |a^*\rangle \otimes |b\rangle$, where $|a^*\rangle$ is the conjugate of the vector $|a\rangle$ computed in the basis $\{|i\rangle_A : i = 0, \ldots, M-1\}$. Since $|a,b\rangle = |za, z^{-1}b\rangle$ for any nonzero $z \in \mathbb{C}$, and the partial conjugate of $|za, z^{-1}b\rangle$ is $|z^*a^*, z^{-1}b\rangle = (z^*/z)|a^*, b\rangle$, we see that the partial conjugation operation on product vectors is well-defined only up to a phase factor. Thus, the partial conjugation is an involutory automorphism of $\Sigma$ viewed as a real (but not complex) manifold.

In some of the proofs we shall use some basic facts about the intersection of two projective varieties embedded in a bigger projective space. Let us briefly describe these facts. The degree of a projective variety, say $X$, of dimension $k$ embedded in the projective space $\mathbb{P}^n$ can be defined as the number of intersection points of $X$ with a general projective subspace $L$ of complementary dimension, $n-k$. For instance, for the Segre variety $\Sigma \subset \mathbb{P}_{AB}$, we have to take $L$ of dimension $(MN-1) - (M+N-2) = (M-1)(N-1)$. Recall that the degree of $\Sigma$ is given by the formula \cite{segre1951}, while every projective subspace has degree 1.

The following proposition is a special case of some basic and well-known facts from algebraic geometry, see e.g. \cite[Theorem 7.2]{harris1995}. \par

\textbf{Proposition 6} \textit{For any projective subspace $L \subseteq \mathbb{P}_{AB}$ of dimension $k \geq (M-1)(N-1)$, the intersection $X = L \cap \Sigma$ is nonempty. Equivalently, any vector subspace of $\mathcal{H}$ of dimension $> (M-1)(N-1)$ must contain at least one product vector. More precisely, if $X_i$ ($i = 1, \ldots, s$) are the irreducible components of $X$, then}
\[
\text{Dim} \ X_i \geq k - (M-1)(N-1), \quad i = 1, \ldots, s. \tag{15}
\]
\textit{In particular, any vector subspace of $\mathcal{H}$ of dimension $> (M-1)(N-1) + 1$ contains infinitely many product vectors.}

If a state $\rho$ has kernel of dimension $(M-1)(N-1) + 1$ then $\text{rank} \rho = M + N - 2$. This partially motivates our interest in states $\rho$ of rank $M + N - 2$: their kernels must contain at least one product vector.

Let $\rho$ be an $M \times N$ state of rank $r \leq M + N - 2$ and let $r' = M + N - 1 - r$ and $K = \ker \rho$. Denote by $X_i$, $i = 1, \ldots, s$, the irreducible components of $X_\rho = K \cap \Sigma$ and let $d_i$ be the degree of $X_i$. If $r \leq M + N - 2$ then, by Proposition 6, $\text{Dim} \ X_i \geq r' - 1$ for each $i$.

In order to be able to apply the version of the Bézout’s theorem as stated in \cite[Bezout’s Theorem, pp. 80-81]{harris1995}, we need two conditions: (a) $\text{Dim} X_\rho = r' - 1$ and (b) $K$ and $\Sigma$ intersect generically transversely. When the conditions (a) and (b) hold, then the Bézout’s theorem asserts that the following degree formula is valid
\[
\delta = \sum_{i=1}^{s} d_i. \tag{16}
\]
At the end of Sec. 11 we shall verify that good separable states indeed satisfy this equation.
The more general version of the Bézout’s theorem [19, Theorem 18.4] can be applied assuming only condition (a). Then the degree formula has to be replaced by the more general one

\[ \delta = \sum_{i=1}^{s} \mu_i d_i, \]

where \( \mu_i \geq 1 \) is the intersection multiplicity of \( L \) and \( \Sigma \) along \( X_i \). The condition (b) implies that all \( \mu_i = 1 \) and so (17) reduces to (16). The converse is also valid, i.e., if (a) holds then (b) is equivalent to the assertion that all \( \mu_i = 1 \), see [33, p. 79] and [13, p. 137-138].

Note that \( X_\rho = \emptyset \) implies that \( r' \leq 0 \), i.e., \( r > M + N - 2 \). For the case \( r = M + N - 2 \), we have the following simple test.

**Proposition 7** If \( \rho \) is an \( M \times N \) (PPT or NPT) state of rank \( r = M + N - 2 \), then \( \rho \) is good if and only if \( |X_\rho| = \delta \).

**Proof.** Let \( K = \ker \rho \). In view of the hypotheses, \( \rho \) is good if and only if \( K \) and \( \Sigma \) intersect transversely. We can apply Eq. (17). Since the irreducible components \( X_i \) of \( X_\rho \) are just points, each \( d_i = 1 \) and \( s = |X_\rho| \). It follows that the equality \( |X_\rho| = \delta \) holds if and only if each \( \mu_i = 1 \), i.e., if and only if \( K \) and \( \Sigma \) intersect transversely.

As a basic example, we consider the \( 2 \times 2 \) separable state \( \rho = |00\rangle\langle 00| + |11\rangle\langle 11| \). We claim that \( \rho \) is good. We have to prove that \( K := \ker \rho = \text{span}\{|01\rangle, |10\rangle\} \) is a good subspace. The variety \( X_\rho \) consists of only two points, namely the points corresponding to product vectors \( |a_1,b_1\rangle = |01\rangle \) and \( |a_2,b_2\rangle = |10\rangle \). It is easy to verify that \( K + S_{a_1,b_1} = K + S_{a_2,b_2} = \mathcal{H} \). Hence, the transversality condition is satisfied, and so \( \rho \) is good.

Another basic example is the \( 1 \times 2 \) separable state \( \rho = |00\rangle\langle 00| + |01\rangle\langle 01| \) in the \( 2 \otimes 2 \) space. This is a bad state because \( \ker \rho = |1\rangle \otimes \mathcal{H}_B \), and so \( \dim X_\rho = 1 \).

Let us mention more examples of good states that are well known in quantum information. An o.n. set of product vectors \( \{\psi\} := \{|\psi_i\rangle : i = 1, \ldots, k\} \subset \mathcal{H} \) is an unextendible product basis (UPB) [2] if the subspace \( \{\psi\}^\perp \) is a CES. If \( \{\psi\} \) is a UPB, then the orthogonal projector onto \( \{\psi\}^\perp \) is a PPTES. It is known that any two-qutrit PPTES \( \rho \) of rank four can be constructed by using the fact that there are exactly six product vectors in \( \ker \rho \) [8]. Any five of these six product vectors can be converted to an UPB [2,8]. These \( \rho \) are good PPTES of the simplest kind. The UPB construction of PPTES works also in higher dimensions but is no longer universal. Indeed, we construct in Example [28] a good \( 3 \times 4 \) PPTES \( \sigma \) of rank five with \( |X_\sigma| = \delta(3,4) = 10 \). Thus \( \ker \sigma \) contains exactly ten product vectors. Moreover any seven of them are linearly independent. However, Lemma [29] shows that no seven of these ten product vectors can be simultaneously converted by an ILO to scalar multiples of vectors of an UPB.

In the case \( r > M + N - 2 \) there exist good as well as bad \( M \times N \) PPT states of rank \( r \). As examples of good states, we mention the \( 3 \times 3 \) edge PPTES of birank \( (7,6) \), (7,5) and \( (5,8) \) constructed in [13, Eqs. 5, 6], as well as the one of birank \( (6,8) \) constructed very recently [31, Eq. 1]. One can check that the kernels of these states are CES, and so they are good by the above definition. As an example of a bad state, we mention the \( 3 \times 3 \) edge state of rank five constructed in [10, Sec. II]. Its kernel has dimension four and contains exactly two product vectors. (It is known and easy to check that this state is extreme.)

To avoid possible confusion we give a formal definition of the term “general position”.

**Definition 8** We say that a family of product vectors \( \{|\psi_i\rangle = |\phi_i\rangle \otimes |\chi_i\rangle : i \in I\} \) is in general position (in \( \mathcal{H} \)) if for any \( J \subset I \) with \( |J| \leq M \) the vectors \( |\phi_j\rangle \), \( j \in J \), are linearly independent and for any \( K \subset I \) with \( |K| \leq N \) the vectors \( |\chi_k\rangle \), \( k \in K \), are linearly independent.

We warn the reader that it is possible for a family of product vectors contained in a subspace \( V \otimes W \subset \mathcal{H} \) to be in general position in \( V \otimes W \) but not in general position in \( \mathcal{H} \).

**B. Quantum information**

Let us now recall some basic results from quantum information, for proving the separability, distillability and PPT properties of some bipartite states.

We say that two \( n \)-partite states \( \rho \) and \( \sigma \) are equivalent under stochastic local operations and classical communications (or SLOCC-equivalent) if there exists an ILO \( \mathbb{A} = \bigotimes_{i=1}^n A_i \) such that \( \rho = A \sigma A^\dagger \) [14]. They are \( LU \)-equivalent if the \( A_i \) can be chosen to be unitary. In most cases of the present work, we will have \( n = 2 \). It is easy to see that any ILO transforms PPT, entangled, or separable state into the same kind of states. We shall often use ILOs to simplify the density matrices of states.

From [27, Theorem 1] we have
Theorem 9  The $M \times N$ states of rank less than $M$ or $N$ are distillable, and consequently they are NPT.

The next result follows from [20, Theorem 3], see also [7, Proposition 6 (ii)].

Proposition 10  If $\rho$ is an $M \times N$ PPT state of rank $N$, then $\rho$ is a sum of $N$ pure product states. Consequently, the rank of any PPTES is bigger than any of its local ranks, and any PPT state of rank at most three is separable.

(By Theorem 9 the hypothesis of this proposition implies that $M \leq N$.)

Let us recall from [7, Theorem 22] and [8, Theorems 17,22] the main facts about the $3 \times 3$ PPT states of rank four. Let $M = N = 3$ and let $U$ denote the set of UPBs in $H = \mathcal{H}_A \otimes \mathcal{H}_B$. For $\{\psi\} \in U$ we denote by $\Pi\{\psi\}$ the normalized state $(1/4)P$, where $P$ is the orthogonal projector onto $\{\psi\}^\perp$.

Theorem 11  ($M = N = 3$) For a $3 \times 3$ PPT state $\rho$ of rank four, the following assertions hold.

(i) $\rho$ is entangled if and only if $R(\rho)$ is a CES.

(ii) If $\rho$ is separable, then it is either the sum of four pure product states or the sum of a pure product state and a $2 \times 2$ separable state of rank three.

(iii) If $\rho$ is entangled, then

- (a) $\rho$ is extreme;
- (b) $\text{rank} \rho^* = 4$;
- (c) $\rho = A \otimes B \Pi\{\psi\} A^\dagger \otimes B^\dagger$ for some $A, B \in \text{GL}_3$ and some $\{\psi\} \in U$;
- (d) ker $\rho$ contains exactly 6 product vectors, and these vectors are in general position.

In Sec. 1V we shall generalize the results (i) and (ii) to arbitrary bipartite systems. On the other hand, the assertion (iii)(c) does not extend to $3 \times 4$ PPTES of rank five, see Example 50. Thus, there exist PPTES in higher dimensions which cannot be constructed via the UPB approach. So, the higher dimensional cases are essentially different from the two-qutrit case [8]. Finally the assertion (iii)(d) does not extend to $3 \times N$ PPTES of rank $N + 1$ when $N > 3$. Indeed, such state may contain infinitely many product vectors in the kernel, see Example 54.

Let $\sigma$ be an $M \times N$ PPT state of rank $N$. By Proposition 11 $\sigma$ is separable. Moreover, $\sigma$ is SLOCC-equivalent to a state $\rho$ given by Eq. 9 where all $C_i$ are diagonal matrices. This fact follows from [7, Proposition 6 (ii)], and will be used in several proofs in this paper.

We need the following simple fact.

Lemma 12  Let $\rho, \rho'$ be bipartite states. If $R(\rho') \subseteq R(\rho)$ then $R(\rho'_A) \subseteq R(\rho_A)$.

Proof.  For small $\varepsilon > 0$ we have $\rho - \varepsilon \rho' \succeq 0$. Hence, $\rho_A - \varepsilon \rho'_A \succeq 0$ and the assertion follows.

As an application, we have the following fact.

Proposition 13  If the normalized states $\rho$ and $\rho'$ are $3 \times 3$ PPTES of rank four with the same range, then $\rho = \rho'$.

By Lemma 12, we must have $R(\rho_A) = R(\rho'_A)$ and $R(\rho_B) = R(\rho'_B)$. Then the result follows from [8, Theorem 22].

We also need the concept of irreducibility for bipartite states introduced in [8, Definition 11]. We extend the definition of A and B-direct sums to arbitrary linear operators.

Definition 14  We say that a linear operator $\rho : \mathcal{H} \rightarrow \mathcal{H}$ is an A-direct sum of linear operators $\rho_1 : \mathcal{H} \rightarrow \mathcal{H}$ and $\rho_2 : \mathcal{H} \rightarrow \mathcal{H}$, and we write $\rho = \rho_1 \oplus_A \rho_2$, if $R(\rho_A) = R((\rho_1)_A) \oplus R((\rho_2)_A)$. (Note that we do not require the ranges of $(\rho_1)_A$ and $(\rho_2)_A$ to be orthogonal to each other.) A bipartite state $\rho$ is A-reducible if it is an A-direct sum of two states; otherwise $\rho$ is A-irreducible. One defines similarly the B-direct sum $\rho = \rho_1 \oplus_B \rho_2$.

We say that a state $\rho$ is reducible if it is either $A$ or $B$-reducible. We say that $\rho$ is irreducible if it is not reducible. We write $\rho = \rho_1 \oplus \rho_2$ if $\rho = \rho_1 \oplus_A \rho_2$ and $\rho = \rho_1 \oplus_B \rho_2$, and in that case we say that $\rho$ is a direct sum of $\rho_1$ and $\rho_2$.

The definitions of “reducible”, “irreducible” and “direct sum” of two states are designed for use in the bipartite setting and should not be confused with the usual definitions of these terms where $\mathcal{H}$ is not equipped with the tensor product structure. If $\rho_1$ and $\rho_2$ are states on the same Hilbert space, which represents a bipartite quantum system, then it is straightforward to check whether their sum is A-direct. However, if $\rho_1$ and $\rho_2$ act on two different Hilbert spaces representing two different bipartite quantum systems, one may wish to embed these two Hilbert spaces into a larger one, $\mathcal{H}$, which also represents a bipartite quantum system, such that the sum of $\rho_1$ and $\rho_2$ becomes an A-direct sum. This can be accomplished in many different ways, but there is no natural or canonical way to select such a construction. For that reason there is no operation of “forming” the A-direct sum of $\rho_1$ and $\rho_2$, and in each case such a construction has to be explained in more details. Of course, this warning applies also to B-direct sums.
Let \( A = B + C \) where \( B \) and \( C \) are Hermitian matrices and rank \( A = \text{rank} \, B + \text{rank} \, C \). Then it is easy to show that \( A \geq 0 \) implies that \( B \geq 0 \) and \( C \geq 0 \). Consequently, if \( \rho = \rho_1 \oplus_A \rho_2 \) or \( \rho = \rho_1 \oplus_B \rho_2 \) with \( \rho_1 \) and \( \rho_2 \) Hermitian and \( \rho \geq 0 \), then also \( \rho_1 \geq 0 \) and \( \rho_2 \geq 0 \).

Let us recall a related result [5, Corollary 16] to which we will refer in many proofs.

**Lemma 15** Let \( \rho = \sum_i \rho_i \) be an \( A \) or \( B \)-direct sum of the states \( \rho_i \). Then \( \rho \) is separable [PPT] if and only if each \( \rho_i \) is separable [PPT]. Consequently, \( \rho \) is a PPTES if and only if each \( \rho_i \) is PPT and at least one of them is entangled.

It follows from this lemma that any extreme state is irreducible. We insert here a new lemma.

**Lemma 16** Let \( \rho_1 \) and \( \rho_2 \) be linear operators on \( \mathcal{H} \).

(i) If \( \rho = \rho_1 \oplus_B \rho_2 \), then \( \rho^T = \rho_1^T \oplus_B \rho_2^T \).

(ii) If \( \rho_1 \) and \( \rho_2 \) are Hermitian and \( \rho = \rho_1 \oplus_A \rho_2 \), then \( \rho^T = \rho_1^T \oplus_A \rho_2^T \).

(iii) If a PPT state \( \rho \) is reducible, then so is \( \rho^T \).

**Proof.** (i) follows from the fact that \( (\sigma^T)_B = \sigma_B \) for any state \( \sigma \) on \( \mathcal{H} \), see Eq. (12). (ii) First observe that \( (\sigma^T)_A = (\sigma_A)^T \) for any state \( \sigma \) on \( \mathcal{H} \), see Eq. (13). Then the assertion follows from the fact that \( \mathcal{R}(\sigma^T) = \mathcal{R}(\sigma)^* \) for any Hermitian operator \( \sigma \) on \( \mathcal{H}_A \).

(iii) follows immediately from (i) and (ii).

Let \( \rho \) be any \( M \times N \) state and \( |a\rangle \in \mathcal{H}_A \) a nonzero vector. Then it is easy to verify that \( \langle a|\rho|a\rangle \neq 0 \). (Similarly, \( \langle b|\rho|b\rangle = 0 \) for any nonzero vector \( |b\rangle \in \mathcal{H}_B \).) The following two assertions are equivalent to each other:

(i) rank\( (\rho|a\rangle \langle a|) = 1 \);

(ii) \( |a\rangle \otimes \mathcal{H} \subseteq \ker \rho \) for some hyperplane \( \mathcal{H} \subset \mathcal{H}_B \).

Let us state the general extremality criterion which was discovered recently by Leinaas, Myrheim and Ovrum [32], and independently by Augusiak, Grabowski, Kus and Lewenstein [1]. We offer an enhanced version of this criterion and give a short proof. The new assertion (ii) in this criterion plays an essential role in the proof of Theorem [37].

**Proposition 17** (Extremality Criterion) For a PPT state \( \rho \), the following assertions are equivalent to each other.

(i) \( \rho \) is not extreme.

(ii) There is a PPT state \( \sigma \), not a scalar multiple of \( \rho \), such that \( \mathcal{R}(\sigma) = \mathcal{R}(\rho) \) and \( \mathcal{R}(\sigma^T) = \mathcal{R}(\rho^T) \).

(iii) There is a Hermitian matrix \( H \), not a scalar multiple of \( \rho \), such that \( \mathcal{R}(H) \subseteq \mathcal{R}(\rho) \) and \( \mathcal{R}(H^T) \subseteq \mathcal{R}(\rho^T) \).

**Proof.** (i) \( \Rightarrow \) (ii). We have \( \rho = \rho_1 + \rho_2 \) where \( \rho_1 \) and \( \rho_2 \) are non-parallel PPT states. We also have \( \rho^T = \rho_1^T + \rho_2^T \). Then the state \( \sigma := \rho + \rho_1 \) is not a scalar multiple of \( \rho \) and satisfies \( \mathcal{R}(\sigma) = \mathcal{R}(\rho) \) and \( \mathcal{R}(\sigma^T) = \mathcal{R}(\rho^T) \).

(ii) \( \Rightarrow \) (iii) is trivial.

(iii) \( \Rightarrow \) (i). It follows from (iii) that there exists \( \varepsilon > 0 \) such that \( \rho + \varepsilon H \geq 0 \) and \( \rho^T + \varepsilon H^T \geq 0 \) for \( t \in [-\varepsilon, \varepsilon] \). Then \( \rho_1 = \rho - \varepsilon H \) and \( \rho_2 = \rho + \varepsilon H \) are non-parallel PPT states and \( \rho_1 + \rho_2 = 2\rho \). Hence (i) holds.

The equivalence of (i) and (ii) is a trivial consequence of the description of the faces of the convex cone of non-normalized PPT states given in [15].

The following necessary condition for extremality was first discovered by Leinaas, Myrheim and Ovrum [32]. Our concise proof below is essentially the same as their proof.

**Proposition 18** Let \( \rho \) be a PPT state of birank \( (r, s) \). If \( r^2 + s^2 > M^2N^2 + 1 \) then \( \rho \) is not extreme.

**Proof.** Let \( Q \) be the real vector spaces of all Hermitian matrices of size \( M \times N \). Denote by \( Y \subseteq [Z] \) the subspace of \( Q \) consisting of all Hermitian matrices whose range is contained in \( \mathcal{R}(\rho) \) \( [\mathcal{R}(\rho^T)] \). Note that \( \dim Y = r^2 \), \( \dim Z = s^2 \) and \( \dim Q = M^2N^2 \). The subspace \( Z^T := \{ H^T : H \in Y \} \subseteq Q \) also has dimension \( s^2 \). We need to estimate the dimension of the subspace \( V := \{ H \in Z : H^T \in Z \} \). Since \( V = Y \cap Z^T \), we have

\[
\dim V = \dim (Y \cap Z^T) \\
\geq \dim Y + \dim Z^T - \dim Q \\
= r^2 + s^2 - M^2N^2 > 1. \tag{18}
\]

Hence, the assertion (iii) of Proposition [17] holds, and so \( \rho \) is not extreme.

For instance, when \( M = 2 \) and \( N = 4 \) we see immediately that there are \( 2 \times 4 \) extreme states of birank \((6,6)\).

For a PPT state \( \rho \), if \( |a, b\rangle \in \ker \rho \) then \( |a^*, b\rangle \in \ker \rho^T \), see [31, Lemma 5]. Thus the partial conjugation automorphism \( \Sigma \to \Sigma \) maps \( X_\rho \) onto \( X_{\rho^T} \), and so part (i) of the following lemma holds.
Lemma 19 Let \( \rho \) be a PPT state of rank \( r \). Then

(i) \( |X_{\rho}| = |X_\rho| \) and \( \text{Dim } X_{\rho} = \text{Dim } X_\rho \);
(ii) if \( \rho \) is good and \( r \leq M + N - 2 \), then \( \text{rank } \rho^T \geq r \).

Proof. We prove (ii). It follows from the hypothesis that \( \text{Dim } X_\rho = M + N - 2 - r \). Assume that \( \text{rank } \rho^T < r \). Then
\[
\text{Dim } X_\rho \geq M + N - 2 - \text{rank } \rho^T > M + N - 2 - r = \text{Dim } X_\rho,
\]
which contradicts (i).

Finally, let us prove two basic facts about strongly extreme states.

Lemma 20 Let \( \sigma \) be a strongly extreme state.

(i) If \( \rho \) is a PPT state and \( \mathcal{R}(\rho) \subseteq \mathcal{R}(\sigma) \), then \( \rho \propto \sigma \). In particular, a strongly extreme state is extreme.
(ii) If \( \text{rank } \sigma > 1 \) then \( \mathcal{R}(\sigma) \) is a CES.

Proof. (i) Since \( \rho + \sigma \) is PPT, we must have \( \rho + \sigma \propto \sigma \). Hence \( \rho \propto \sigma \).

(ii) Assume that \( \mathcal{R}(\sigma) \) contains a product vector \( |a,b\rangle \). Then \( \rho = |a,b\rangle \langle a,b| \) is a PPT state and \( \mathcal{R}(\rho) \subseteq \mathcal{R}(\sigma) \). By (i) we have \( \rho \propto \sigma \), which contradicts the hypothesis that \( \text{rank } \sigma > 1 \). Hence \( \mathcal{R}(\sigma) \) must be a CES.

III. GOOD AND BAD STATES

We have divided the bipartite states into good and bad ones. Good states are of more interest since they share many good properties. The main result of this section is the characterization of good separable states, see Theorem 30.

We give explicit expression for any good separable state by using the product vectors contained in the range. Some preliminary results, like Proposition 22 and Lemma 23, treat general vector subspaces of \( \mathcal{H} \) and will be useful later. The proof of Proposition 25 is based on two facts from algebraic geometry for which we could not find a reference. Their proofs are given in the appendix. We also show that all pure states are good.

Definition 21 We say that a state \( \rho \) acting on \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \) is universally good if it is good and remains good whenever our \( M \otimes N \) system \( \mathcal{H} \) is embedded in a larger \( M' \otimes N' \) system.

To take care of the trivial cases \( M = 1 \) or \( N = 1 \), we observe that in these cases \( \mathcal{P}_{AB} = \Sigma \) and so every subspace of \( \mathcal{H} \) is good, and every state on \( \mathcal{H} \) is good.

Let us now show that all pure states are universally good.

Proposition 22 (\( M, N \geq 1 \)) Every pure state is good and, consequently, it is universally good.

Proof. Let \( \rho = |\Psi\rangle \langle \Psi| \) be any pure state. We may assume that \( M, N > 1 \). Since \( \text{rank } \rho = 1, K := \ker \rho \) is a hyperplane of \( \mathcal{H} \). After applying an ILO, we may assume that \( |\Psi| = \sum_{i=0}^{r-1} |ii| \), where \( r \leq \min(M,N) \) is the Schmidt rank of \( \Psi \). For \( |x| = \sum \xi_i |i\rangle_A |y| = \sum \eta_j |j\rangle_B \), we have \( |x,y| \in K \) if and only if \( \langle \Psi|x,y| = 0, i.e., \sum_{i=0}^{r-1} \xi_i \eta_b = 0 \). Hence, \( X_\rho \) is an irreducible hypersurface in \( \Sigma \) except when \( r = 1 \) in which case it has two irreducible components: \( \xi_0 = 0 \) and \( \eta_b = 0 \). Hence \( |00\rangle \notin K \), we see that \( |0\rangle \otimes \mathcal{H}_B \subset K \). Consequently, \( K \) and \( \Sigma \) intersect transversely at the point represented by \( |01\rangle \) and our claim is proved if \( r > 1 \). If \( r = 1 \) the above argument shows that the transversality condition is satisfied by the component \( \eta_0 = 0 \). The other component can be dealt with in the same manner.

Thus the difference \( M - \text{rank } \rho_A \) (and \( N - \text{rank } \rho_B \)) may be arbitrarily large for good states \( \rho \). On the other hand, we will now show that good PPTES must have full local ranks (i.e., \( \text{rank } \rho_A = M \) and \( \text{rank } \rho_B = N \)). To do this we need a preliminary fact.

Lemma 23 (\( M, N \geq 1 \)) Let \( \rho \) be a good state of rank \( r > \text{rank } \rho_A \). Then \( \text{rank } \rho_A = M \) and, consequently, \( \rho \) is not universally good: it becomes bad when our \( M \otimes N \) system is embedded in any larger system \( M' \otimes N' \) with \( M' > M \).

Proof. The first assertion is trivial when \( M = 1 \). Let \( M > 1 \). Assume that \( \text{rank } \rho_A < M \). Since \( H := \mathcal{R}(\rho_A) \subset \ker \rho \) is nonzero, we have \( X_\rho \neq \emptyset \). As \( \rho \) is good, it follows that \( r := \text{rank } \rho \leq M + N - 2 \) and \( \text{Dim } X_\rho = M + N - 2 - r \).

Since \( X_\rho \supseteq H \cap \Sigma \) and \( \text{Dim } H \cap \Sigma = M + N - 2 - \text{rank } \rho_A \), we have \( \text{rank } \rho_A \geq r \), which contradicts the hypothesis. Thus \( \text{rank } \rho_A = M \) and the first assertion is proved.

The second assertion follows from the first.

Thus, a product state \( \rho = \rho_A \otimes \rho_B \) of rank bigger than one is not universally good.

We now exhibit a link between goodness and distillability properties of entangled bipartite states.
Theorem 24 If a good entangled state $\rho$ is not distillable (e.g., if it is PPT), then it must have full local ranks (i.e., $\rho$ must be an $M \times N$ state).

Proof. Since $\rho$ is not distillable, we have rank $\rho \geq \text{rank } \rho_A$ by Theorem 9. If $\rho$ is PPT, Proposition 10 implies that rank $\rho \neq \text{rank } \rho_A$. By [7, Theorem 10], this is also true when $\rho$ is NPT. Thus rank $\rho > \text{rank } \rho_A$ and Lemma 24 implies that $\rho_A$ has full rank. Similarly, $\rho_B$ has full rank. □

Proposition 25 Let $H$ be a vector subspace of $\mathcal{H}$ of dimension $d$ containing exactly $m$, $0 \leq m < \infty$, product vectors.

(i) Then $m \leq \delta$ and $d \leq (M - 1)(N - 1) + 1$.

(ii) If $m = \delta$ then $d = (M - 1)(N - 1) + 1$, $H$ is spanned by product vectors and no proper subspace $V \otimes W$ of $\mathcal{H}$ contains $H$.

(iii) If $d \leq (M - 1)(N - 1)$ then $m < \delta$.

Proof. It follows from Proposition 6 that $d \leq (M - 1)(N - 1) + 1$. By Proposition 58, there exists a vector subspace $H' \supseteq H$ of dimension $(M - 1)(N - 1) + 1$ such that $H'$ contains only finitely many, say $m'$, product vectors. By the Bézout’s theorem, we have $m' \leq \delta$. Since $m \leq m'$, we also have $m \leq \delta$. Thus (i) is proved.

If $m = \delta$ then also $m' = \delta$ and Theorem 59 implies that $H' = H$, i.e., $d = (M - 1)(N - 1) + 1$, and that $H$ is spanned by product vectors. Thus (ii) is proved. The assertion (iii) follows from (i) and (ii). □

This proposition will be used in the proofs of Theorems 22 and 24 which are our main results regarding Conjectures 2 and 4.

Let us make a comment about the case $m = \delta$. In that case $H$ has a basis consisting of product vectors, say $|a_i, b_i\rangle$, $i = 1, \ldots, d$. If $V |W\rangle$ is the subspace of $\mathcal{H}_A [\mathcal{H}_B]$ spanned by the $|a_i\rangle \{ |b_i\rangle\}$, then each $|a_i, b_i\rangle \in V \otimes W$. It follows that $V = \mathcal{H}_A$ and $W = \mathcal{H}_B$. Thus, we have the following corollary.

Corollary 26 Let $\rho$ be a good $M \times N$ state of rank $M + N - 2$. Then ker $\rho$ is spanned by product vectors and $|X_\rho\rangle = \delta$.

If $\{ |a_i, b_i\rangle\}$ is any basis of ker $\rho$ consisting of product vectors, then the $|a_i\rangle$ span $\mathcal{H}_A$ and the $|b_i\rangle$ span $\mathcal{H}_B$.

The hypothesis that $\rho$ is good is essential as the following example shows.

Example 27 ($M = N = 3$) Consider the $3 \times 3$ separable state $\rho = \sum_{i=0}^{2} |i\rangle\langle i| + |a, b\rangle\langle a, b|$ of rank four, where $|a\rangle = |1\rangle_A + |2\rangle_A$ and $|b\rangle = |1\rangle_B + |2\rangle_B$. One can check that a product vector $|x, y\rangle$ belongs to ker $\rho$ if and only if the vectors $|x\rangle = \sum_i \xi_i |i\rangle$ and $|y\rangle = \sum_i \eta_i |i\rangle$ satisfy the equations

$$
\xi_0 \eta_0 = \xi_1 \eta_1 = \xi_2 \eta_2 = \xi_3 \eta_3 = 0.
$$

Hence such $|x, y\rangle$ belongs to one of the subspaces $|0\rangle \otimes |0\rangle^\perp$ or $|0\rangle^\perp \otimes |0\rangle$. Since these subspaces are contained in ker $\rho$, $\rho$ is bad. As ker $\rho$ has dimension five, it is not spanned by product vectors. We mention that $\rho^T = \rho$ in this example. □

The projective variety $X_\rho$ of the separable state $\rho$ in this example has only two irreducible components, the Segre varieties of the subspaces $|0\rangle \otimes |0\rangle^\perp$ and $|0\rangle^\perp \otimes |0\rangle$.

We can extend this observation to any separable state $\rho$ of rank $r$. We can write $\rho$ as a sum of pure product states $\rho = \sum_{i=1}^{m} |a_i, b_i\rangle\langle a_i, b_i|$, $m \geq r$. (21)

For any subsets $P, Q \subseteq I := \{1, \ldots, m\}$ we set

$$
V_P = \{ |a_j\rangle j \in P \}^\perp \subseteq \mathcal{H}_A, \quad W_Q = \{ |b_k\rangle k \in Q \}^\perp \subseteq \mathcal{H}_B.
$$

For simplicity, let us denote by $\Sigma_{P, Q}$ the Segre variety of the tensor product $V_P \otimes W_Q$. (If $V_P = 0$ or $W_Q = 0$ then $\Sigma_{P, Q} = 0$.) It is obvious that if $P \subseteq P' \subseteq I$ and $Q \subseteq Q' \subseteq I$, then $\Sigma_{P', Q'} \subseteq \Sigma_{P, Q}$.

Theorem 28 Let $\rho$ be a separable state given by Eq. (27). Then any irreducible component of $X_\rho$ is one of the Segre varieties $\Sigma_{P, Q}$, where $(P, Q)$ runs through all partitions of the index set $I = \{1, \ldots, m\}$.

Proof. Our first claim is that if $I = P \cup Q$, then $V_P \otimes W_Q \subseteq \ker \rho$ and so $\Sigma_{P, Q} \subseteq X_\rho$. For any $i \in I$ we have $i \in P$ or $i \in Q$, say $i \in P$. By definition of $V_P$, $|a_i\rangle$ is orthogonal to $V_P$, and so $|a_i, b_i\rangle$ is orthogonal to $V_P \otimes W_Q$. As the $|a_i, b_i\rangle$ span $\mathcal{R}(\rho)$, our first claim follows.

Our second claim is that for any product vector $|a, b\rangle \in \ker \rho$ there exists a partition $(P, Q)$ of $I$ such that $|a, b\rangle \in V_P \otimes W_Q$. To prove this claim, let $P \{ Q\}$ be the set of indexes $j \{ k\}$ such that $\langle a | a_j \rangle = 0$ $\{ | b | b_k \rangle = 0\}$. Since
\begin{equation}
\langle a_i, b_i \rangle = 0 \quad \text{for each } i \in I, \quad \text{we have } P \cup Q = I. \quad \text{By replacing } Q \text{ with } Q \setminus P, \text{ we obtain a partition of } I \text{ and our second claim follows.}
\end{equation}

Hence, the variety $X_J$ is the union of the Segre subvarieties $\Sigma_{P,Q}$ where $(P,Q)$ runs through all partitions of $I$. The assertion of the theorem follows because there are only finitely many partitions $(P,Q)$ of $I$ and each Segre variety $\Sigma_{P,Q}$ is irreducible.

\[\square\]

We need the following lemma, where we use the concept of “general position” (see Definition \[\text{Definition}\])

\begin{lemma}
Let $V \subseteq H$ be a subspace spanned by the product vectors $|a_i, b_i\rangle$, $i = 1, 2, \ldots, L$, in general position. If $L \leq M + N - 2$ then the $|a_i, b_i\rangle$ are linearly independent and any product vector in $V$ is a scalar multiple of some $|a_i, b_i\rangle$.
\end{lemma}

\begin{proof}
We may assume that $M \leq N$. The proof is by induction on $L$. Both assertions are true if $L = 1$. Now let $L > 1$. By the induction hypothesis, the vectors $|a_i, b_i\rangle$, $1 \leq i < L$, are linearly independent and $|a_L, b_L\rangle$ is not their linear combination. Thus the vectors $|a_i, b_i\rangle$, $1 \leq i \leq L$, are linearly independent. It remains to prove the second assertion.

Suppose there exists a product vector $|a, b\rangle \in V$ which is not a scalar multiple of any $|a_i, b_i\rangle$. We have $|a, b\rangle = \sum_i \xi_i |a_i, b_i\rangle$, $\xi_i \in \mathbb{C}$. The induction hypothesis implies that all $\xi_i \neq 0$. Assume that $L = N$. Since the $|a_i, b_i\rangle$ are in general position, the vectors $|b_1\rangle, \ldots, |b_N\rangle$ are linearly independent. As $|a, b\rangle$ is a product vector, it follows that each of the vectors $|a_1\rangle, \ldots, |a_N\rangle$ must be a scalar multiple of $|a\rangle$. Thus we have a contradiction, and we conclude that $L > N$.

Since $\{b_1, \ldots, b_N\}$ is a basis of $H_B$, we have
\begin{equation}
|b_i\rangle = \sum_{j=1}^N \eta_{ij} |b_j\rangle, \quad \eta_{ij} \in \mathbb{C}, \quad N < i \leq L;
\end{equation}
\begin{equation}
|a, b\rangle = \sum_{j=1}^N \left( \xi_j |a_j\rangle + \sum_{i=N+1}^L \xi_i \eta_{ij} |a_i\rangle \right) \otimes |b_j\rangle.
\end{equation}

As $\xi_1, \ldots, \xi_N$ are nonzero, Eq. (24) implies that the vectors $|a_1\rangle, \ldots, |a_N\rangle$ belong to the subspace spanned by the $|a_i\rangle$ with $N < i \leq L$ and $|a\rangle$. Since the dimension of this subspace is at most $L - N + 1 \leq M - 1$ and $M \leq N$, we conclude that $|a_1\rangle, \ldots, |a_M\rangle$ are linearly dependent. This contradicts our hypothesis, and proves that the second assertion is also valid.

\[\square\]

We can now characterize the good separable states.

\begin{theorem}
(M, N \geq 1) Let $\rho$ be a separable state of rank $r$.
\begin{enumerate}
\item[(i)] If $r \leq M + N - 2$ then $\rho$ is good if and only if $\rho = \sum_{i=1}^r |a_i, b_i\rangle \langle a_i, b_i|$, where the product vectors $|a_i, b_i\rangle$, $i = 1, \ldots, r$, are in general position.
\item[(ii)] If $r > M + N - 2$ then $\rho$ is good if and only if $\rho = \sum_{j=1}^m |a_j, b_j\rangle \langle a_j, b_j|$ and, for any partition $I = P \cup Q$ of the index set $I = \{1, \ldots, m\}$, either the $|a_j\rangle$, $j \in P$, span $H_A$ or the $|b_k\rangle$, $k \in Q$, span $H_B$.
\item[(iii)] If $\rho$ is good then so is $\rho^F$.
\end{enumerate}
\end{theorem}

\begin{proof}
(i) \text{Necessity.} Let $\rho$ be given by Eq. (21) where the $|a_i, b_i\rangle$ are pairwise non-parallel. We may assume that the $|a_i, b_i\rangle$, $i = 1, \ldots, r$ span $R(\rho)$. Assume that these $r$ product vectors are not in general position, say $|a_1\rangle, \ldots, |a_N\rangle$ are linearly dependent. Set $P = \{1, \ldots, n\}$ and $Q = \{n+1, \ldots, r\}$. Define the subspaces $V_P$ and $W_Q$ as in Eq. (22). We have $\text{Dim } V_P \geq M - n + 1$, $\text{Dim } W_Q \geq n + N - r$ and $V_P \otimes W_Q \subseteq \ker \rho$. Hence $\text{Dim } X_\rho \geq \text{Dim } \Sigma_{P,Q} \geq M + N - r - 2$, which contradicts the hypothesis that $\rho$ is good. Thus, the product vectors $|a_i, b_i\rangle$, $i \leq r$, must be in general position.

Now Lemma 29 implies that $m = r$.

\textit{Sufficiency.} We may assume that $M, N > 1$. By Theorem 28 every irreducible component of $X_\rho$ is the Segre variety $\Sigma_{P,Q}$ for some partition $(P,Q)$ of $\{1, \ldots, r\}$. We may assume that $|P| < M$ and $|Q| < N$ since otherwise $\Sigma_{P,Q} = \emptyset$. Note that then the $\Sigma_{P,Q}$ have dimension $M + N - 2 - r$, and so $\text{Dim } X_\rho = M + N - 2 - r$. It remains to verify the transversality condition. We choose $|a\rangle \in V_P$ and $|b\rangle \in W_Q$ such that $\langle a_k |a\rangle \neq 0$ for $k \in Q$ and $\langle b_j |b\rangle \neq 0$ for $j \in P$. We have to show that $\ker \rho + S_{a,b} = H$. For this it suffices to show that $\ker \rho \cap S_{a,b} \subseteq V_P \otimes |b\rangle + |a\rangle \otimes W_Q$. Let $|\psi\rangle = |a, y\rangle + |x, b\rangle \in \ker \rho$. Then $\langle \psi |\psi\rangle = 0$, which gives the equations $\langle b_j \rangle \langle a, |x\rangle + \langle a, |a\rangle \langle b_j |b\rangle = 0$ for $i = 1, \ldots, r$. Since $\langle b_j |b\rangle = 0$ for $i \in Q$ and $\langle a, |a\rangle = 0$ for $i \in P$, we get the equations $\langle a, |x\rangle = 0$ for $j \in P$ and $\langle b, |b\rangle = 0$ for $k \in Q$. Thus $|\psi\rangle \in V_P \otimes |b\rangle + |a\rangle \otimes W_Q$. Hence, the transversality condition is satisfied and so $\rho$ is good.

(ii) When $r \leq M + N - 2$, it follows from (i) that $\rho^F = \sum_{i=1}^r |a^*_i, b_i\rangle \langle a^*_i, b_i|$, where the $|a^*_i, b_i\rangle$ are in general position. By Lemma 29 we have rank $\rho^F = r$, and (i) shows that $\rho^F$ is good.
If \( r > M + N - 2 \) then \( X_\rho = \emptyset \) and, by Lemma 19, also \( X_\rho^r = \emptyset \). Hence, \( \rho^r \) is good. \( \square \)

Although, for separable \( \rho \), either both \( \rho \) and \( \rho^r \) are good or both bad, they may have different ranks in case (ii). A well-known example is the separable two-qubit Werner state \( \rho = I \otimes I + \sum_{i,j=0}^{1} |ij\rangle \langle ji| \). It is good since \( X_\rho = \emptyset \), but its birank is \((3,4)\).

As a simple corollary, we show that good separable states in (i) indeed satisfy the degree formula (10). There are \( \binom{r}{k} \) partitions \((P,Q)\) of \( \{1, \ldots, r\} \) such that \(|P| = k \). For such partitions \((P,Q)\), the degree of \( \Sigma_{P,Q} \) is \( \binom{M+N-2-r}{M-1-k} \). Hence, the sum of the degrees of all irreducible components of \( X_\rho \) is the left hand side of the identity

\[
\sum_{k=r-N+1}^{M-1} \binom{r}{k} \cdot \binom{M+N-2-r}{M-1-k} = \binom{M+N-2}{M-1}.
\] (25)

It is easy to verify this identity, and so Eq. (16) is satisfied.

We can now characterize the universally good PPT states.

**Theorem 31** A PPT state \( \rho \) is universally good if and only if \( \rho = \sum |a_i, b_i\rangle\langle a_i, b_i| \) where the \( |a_i\rangle \) and the \( |b_i\rangle \) are linearly independent.

**Proof.** Let \( r_A = \text{rank} \rho_A \), \( r_B = \text{rank} \rho_B \) and \( r = \text{rank} \rho \).

*Necessity.* Suppose \( \rho \) is universally good. By Lemma 23 we have \( r \leq \min(r_A, r_B) \). Since \( \rho \) is PPT, Theorem 9 shows that \( r \geq \max(r_A, r_B) \). Hence, we have \( r_A = r_B = r \) and the assertion follows from Proposition 10.

*Sufficiency.* When \( r = 1 \), the claim follows from Proposition 22. When \( r > 1 \), Theorem 24(ii) applies. \( \square \)

So far, pure entangled states are the only known NPT states, with rank \( \rho \leq \min(\text{rank} \rho_A, \text{rank} \rho_B) \), which are universally good. Constructing more examples of such states is an interesting problem.

**IV. \( M \times N \) PPT STATES OF RANK \( M + N - 2 \)**

This section is split into two subsections. In the first subsection we prove the basic property of \( M \times N \) PPT states \( \rho \) of rank \( M + N - 2 \), namely that if \( X_\rho \) is a finite set then \(|X_\rho| = \delta \). See Theorem 32 below for a stronger version of this result. In the second subsection we prove that part (iii) of Conjecture 2 and Conjecture 3 are valid in the good case.

**A. Product vectors in the kernel**

Motivated by Conjecture 2, we shall prove a general theorem about arbitrary \( M \times N \) PPT states. The proof is an extension of the proof of [8, Theorem 20]. We recall that the Segre variety \( \Sigma = \Sigma_{M-1,N-1} \) and the number \( \delta \) were defined in Section 1, see formula (8). Note that if the kernel of a state \( \rho \) contains a 2-dimensional subspace \( V \otimes W \), then the variety \( X_\rho \) contains a projective line and so \( \text{Dim} X_\rho \geq 1 \).

**Theorem 32** If \( \rho \) is an \( M \times N \) PPT state of rank \( r \) such that \( \ker \rho \) contains no 2-dimensional subspace \( V \otimes W \), then either \( r = M + N - 2 \) and \(|X_\rho| = \delta \) or \( r > M + N - 2 \) and \(|X_\rho| < \delta \).

**Proof.** If \( K = \ker \rho \) is a CES, then \( r > M + N - 2 \) and the assertion of the theorem holds. Thus we may assume that \( K \) contains a product vector. We choose an arbitrary product vector in \( K \). By changing the o.n. bases of \( H_A \) and \( H_B \), we may assume that the chosen product vector is \( |00\rangle \). By using Eq. (9), we may assume that \( \rho = C^\dagger C \), where \( C = [C_0 \ C_1 \ \cdots \ C_{M-1}] \) and the \( C_i \) are \( r \times N \) matrices. Since \(|00\rangle \in K \), the first column of \( C_0 \) is 0. The hypothesis (with \( \text{Dim} V = 1 \) and \( \text{Dim} W = 2 \)) implies that \( \text{rank} |a\rangle\rho|a\rangle \geq N - 1 \) for all nonzero vectors \(|a\rangle \in H_A \). As \( \langle 0|_A \rho |0\rangle_A = C_0^\dagger C_0 \), the block \( C_0 \) must have rank \( N - 1 \), and so we may assume that

\[
C_0 = \begin{bmatrix}
0 & I_{N-1} \\
0 & 0
\end{bmatrix}; \quad C_i = \begin{bmatrix}
u_i \ * \\
v_i \ *
\end{bmatrix}, \quad 0 < i < M,
\] (26)

where \( u_i \in \mathbb{C}^{N-1} \) and \( v_i \in \mathbb{C}^{r-N+1} \) are column vectors.

Observe that the first entry of the matrix \( \rho \) is 0. Since \( \rho^F \geq 0 \), the first row of \( \rho^F \) must be 0. We deduce that \( u_i = 0 \) for \( i > 0 \). The hypothesis (this time with \( \text{Dim} V = 2 \) and \( \text{Dim} W = 1 \)) implies that the first columns of the \( C_i \), \( 0 < i < M \), must be linearly independent. In particular, we must have \( r - (N - 1) \geq M - 1 \), i.e., \( r \geq M + N - 2 \).
Let \( \{ e_i : 1 \leq i < M \} \) be the standard basis of \( \mathbb{C}^{r-N+1} \). By using an ILO on system \( \mathbf{A} \), we may assume that \( \psi_i = e_i \) for \( 0 < i < M \). Thus we have

\[
C_i = \begin{bmatrix} 0 & * \\ e_i & * \end{bmatrix}, \quad 0 < i < M. \tag{27}
\]

The range of \( \psi \) is the subspace of dimension \( r \) spanned by the vectors \( |\psi_i\rangle \), \( i = 1, \ldots, r \), given by the columns of \( \mathbf{C}^\dagger \). Each of these columns can be split into \( N \) pieces of height \( M \) and the pieces arranged in natural order to form an \( M \times N \) matrix. By using this notation, we have

\[
|\psi_j\rangle = \begin{bmatrix} 0 & f_j^T \\ 0 & B_j \end{bmatrix}, \quad j = 1, \ldots, N - 1;
\]

\[
|\psi_{N+i-1}\rangle = \begin{bmatrix} 0 & \epsilon_i & B_{N+i-1} \end{bmatrix}, \quad i = 1, \ldots, M - 1;
\]

\[
|\psi_{N+i-1}\rangle = \begin{bmatrix} 0 & 0 & B_{N+i-1} \end{bmatrix}, \quad i = M, \ldots, r - N + 1,
\]

where \( \{ f_j \} \) is the standard basis of \( \mathbb{C}^{N-1} \) and the \( B_k = [b_{ij}^k] \) are \( (r - N + 1) \times (N - 1) \) matrices.

Let \( \mathcal{K} \) be the projective space associated to \( \mathcal{K} \). We introduce the homogeneous coordinates \( \xi_{ij} \) for the projective space \( \mathcal{P}_{AB} \) associated to \( \mathcal{H} \): If \( |\psi\rangle = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \alpha_{ij} |ij\rangle \) then the homogeneous coordinates of the corresponding point \( |\psi\rangle \in \mathcal{P}_{AB} \) are \( \xi_{ij} = \alpha_{ij} \).

We claim that \( \text{Dim} X_\rho = 0 \), i.e., \( X_\rho \) is a finite set. To prove this claim, we shall use the affine chart defined by \( \xi_{00} \neq 0 \) which contains the chosen point \( P = |00\rangle \). We introduce the affine coordinates \( x_{ij} \), \( (i, j) \neq (0, 0) \), in this affine chart by setting \( x_{ij} = \xi_{ij}/\xi_{00} \). Thus \( P \) is the origin, i.e., all of its affine coordinates \( x_{ij} = 0 \). Since \( \ker \rho = \mathcal{R}(\rho)^\perp \), the subspace \( \mathcal{K} \) is the zero set of the ideal \( J_1 \) generated by the \( r \) linear polynomials on the left hand side of the equa:

\[
x_{0k} + \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} (b_{ij}^k)^* x_{ij} = 0, \quad k = 1, \ldots, N - 1;
\]

\[
x_{k0} + \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} (b_{ij}^N)^* x_{ij} = 0, \quad k = 1, \ldots, M - 1;
\]

\[
\sum_{i=1}^{M-1} \sum_{j=1}^{N-1} (b_{ij}^k)^* x_{ij} = 0, \quad k = M, \ldots, r - N + 1.
\]

The piece of \( \Sigma \) contained in our affine chart consists of all \( M \times N \) matrices

\[
\begin{bmatrix}
1 & x_{01} & x_{02} & \cdots \\
x_{10} & x_{11} & x_{12} & \\
x_{20} & x_{21} & x_{22} & \\
\vdots & \end{bmatrix}
\]

of rank one. It is the zero set of the ideal \( J_2 \) generated by the \( (M - 1)(N - 1) \) quadratic polynomials \( x_{ij} - x_{0i} x_{0j}, \quad 1 \leq i < M, \quad 1 \leq j < N \). By substituting \( x_{ij} = x_{0i} x_{0j} \) \( (i, j > 0) \) into Eqs. \( (31-32) \), we obtain a system of \( M + N - 2 \) equations in \( M + N - 2 \) variables to which we can apply Theorem (1.16) of Mumford. By that theorem, the singleton set \( \{ P \} \) is an irreducible component of the affine variety defined by the \( M + N - 2 \) equations mentioned above. This remains true if we enlarge this set of equations with those in \( (33) \) because all of them vanish at the origin. We conclude that \( \{ P \} \) is also an irreducible component of \( X_\rho \). Since the point \( P \) was chosen arbitrarily in \( X_\rho \), our claim is proved.

If \( r > M + N - 2 \) then the fact that \( X_\rho \) is a finite set and Proposition \( (25) \) (iii) imply that \( \ker \rho \) contains at most \( \delta - 1 \) product vectors. It remains to consider the case \( r = M + N - 2 \). Note that now the set of equations \( (33) \) is empty.

Next we claim that the intersection multiplicity of \( \mathcal{K} \) and \( \Sigma \) at the point \( P = |00\rangle \) is 1. The computation of this multiplicity is carried out in the local ring, say \( R \), at the point \( P \). This local ring consists of all rational functions \( f/g \) such that \( g \) does not vanish at the origin, i.e., \( f \) and \( g \) are polynomials (with complex coefficients) in the affine coordinates \( x_{ij} \) and \( g \) has nonzero constant term. By expanding these rational functions in the Taylor series at the
origin, one can view $R$ as a subring of the power series ring $\mathbb{C}[[x_{ij}]]$ in the $MN-1$ affine coordinates $x_{ij}$. We denote by $m$ the maximal ideal of $R$ generated by the $x_{ij}$.

The quotient space $m/m^2$ is a vector space of dimension $MN-1$ with the images of the $x_{ij}$ as its basis. It is now easy to see that the images of the generators of $J_1$ and $J_2$ also span the space $m/m^2$. Hence, by Nakayama’s Lemma (see [11, p. 225]) we have $J_1 + J_2 = m$. Consequently, $R/(J_1 + J_2) \cong \mathbb{C}$ and so our claim is proved.

Recall that we chose in the beginning an arbitrary product vector in $\ker \rho$ and by changing the coordinates we were able to assume that this product vector is $|00\rangle$. Since the intersection multiplicity is invariant under coordinate changes, this means that we have shown that the intersection multiplicity is 1 at each point of $X_{\rho}$. By the Bézout’s theorem the sum of the multiplicities at all intersection points is $\delta$, and since all of the multiplicities are equal to 1 we conclude that $|X_{\rho}| = \delta$. This concludes the proof.

By Theorem 32 an $M \times N$ PPT state $\rho$ of rank $M+N-2$ is good if and only if $|X_{\rho}| < \infty$. The analogous assertion for NPT states is false. For counter-examples see the proof of [8, Theorem 10].

By Proposition 18 we have $E_{MN} = \emptyset$.

**Corollary 33** We have $E_{MN} = \emptyset$.

**Proof.** Assume that there exists $\rho \in E_{MN}$ and let $s = \text{rank} \rho^F$. Obviously, we must have $MN > 6$. Since $\rho^F$ is extreme and $E_2 = E_3 = \emptyset$, we must have $s > 3$. By Proposition 18 we have $s^2 \leq M^2 N^2 + 1 - (MN - 1)^2 = 2MN$. If $M = N = 3$, the only possibility is $s = 4$. Then Theorem 11 gives a contradiction. Thus, the case $M = N = 3$ is ruled out. In the remaining cases we have $2MN \leq (M + N - 2)^2$ and so $s \leq M + N - 2$. It follows from Theorem 32 that $|X_{\rho^F}| \geq \delta$. By Lemma 19 we have $|X_{\rho}| \geq \delta$. Since $\delta > 1$ and $\dim \ker \rho = 1$, we have a contradiction.

Thus if $E_{MN} = \emptyset$ then $r \leq M + N - 2$. This upper bound is sharp in the sense that $E_{MN}$ may be nonempty. Indeed there is a family of $3 \times 3$ edge states $\rho$ of birank $(5, 7)$ depending on a real parameter $\theta$ [31]. By using the extremality criterion, one can show that the state $\rho$ is extreme when $\cos \theta = 9/14$.

Assuming that part (ii) of Conjecture 2 is valid, Proposition 18 gives a stronger (at least for large $M, N$) upper bound $r^2 \leq M^2 N^2 + 1 - (M + N - 2)^2$. Without this conjecture, this stronger bound is valid for $M \leq 4$.

### B. Good states

Next we prove that part (iii) of Conjecture 2 and Conjecture 3 hold for good states.

**Theorem 34** $(M, N > 2)$ Let $\rho$ be a good $M \times N$ PPT state of rank $M + N - 2$.

(i) Then $\rho$ is irreducible.

(ii) $\rho^F$ is also a good $M \times N$ PPT state of rank $M + N - 2$.

(iii) If $\rho$ is extreme, then $\rho$ and $\rho^F$ are strongly extreme.

(iv) If $\rho_1$ is an $M_1 \times N_1$ state (acting on $\mathcal{H}$) of rank $r_1 > N_1$ and $\mathcal{R}(\rho_1) \subseteq \mathcal{R}(\rho)$, then $N_1 = N$.

(v) If $\rho$ is entangled and $E_r = \emptyset$ for $1 < r < M + N - 2$, then $\rho$ is extreme.

**Proof.** (i) Suppose $\rho$ is reducible, say $\rho = \rho_1 \otimes_B \rho_2$. Let $V_i = \mathcal{R}(\rho_i)_{\mathcal{A}}$ and $W_i = \mathcal{R}(\rho_i)_{\mathcal{B}}$, $i = 1, 2$. By Lemma 15 $\rho_i$ is an $M_i \times N_i$ PPT state of rank $r_i$ where $M_i = \dim V_i$ and $N_i = \dim W_i$. By using [1, Proposition 15], we may assume that $W_1 \perp W_2$. Let the state $\rho_i$ be the restriction of $\rho_1$ to the subspace $V_i \otimes W_i$. Since $V_i \otimes W_i$ is orthogonal to $V_2 \otimes W_2$, we have $\ker \rho_i \subseteq \ker \rho$, and so $\ker \rho_i$ contains only finitely many product vectors. Then Proposition 14 implies that $M_i + N_i - 3 \leq r_i$. Since $N_i = N$ and $r_1 + r_2 = M + N - 2$, it follows that $M_1 + M_2 \leq M + 2 < 2M$. Thus we may assume that $M_1 \leq M - 1$. Since $V_1^\perp \otimes W_1 \subseteq \ker \rho$, we must have $\dim V_1^\perp \otimes W_1 = 1$, i.e., $M_1 = M - 1$ and $N_1 = 1$. Since $r_1 \leq M_1 N_1 = M - 1$, Theorem 31 implies that $r_1 = M - 1$ and so $r_2 = N - 1 = N_2$. By Proposition 10 $\rho_2$ is a sum of $N - 1$ pure product states $\{|a_i, b_i\rangle\}$, $i = 1, \ldots, N - 1$. Since $N_1 = 1$, there exists a nonzero vector $|b\rangle \in W_1^\perp$ which is orthogonal to all $|b_i\rangle$ with $i < N - 1$. Then $|a_{N-1}\rangle \otimes |b\rangle \subseteq \ker \rho$ which contradicts the hypothesis of the theorem. Hence, the assertion (i) is proved.

(ii) By Eq. (26), $\rho^F$ is an $M \times N$ state. By Theorem 32 $\ker \rho$ contains exactly $\delta$ product vectors. By Lemma 19 $\ker(\rho^F)$ also contains exactly $\delta$ product vectors. By Proposition 26 $\rho^F$ has rank $M + N - 2$.

(iii) In view of (ii), it suffices to prove this assertion for $\rho$ only. Let $\sigma$ be a PPT state such that $\mathcal{R}(\sigma) = \mathcal{R}(\rho)$. By (ii), all three states $\rho^F$, $\sigma^F$ and $\rho^F + \sigma^F$ have rank $M + N - 2$, and so they must have the same range. By Proposition 17 $\sigma \propto \rho$. Thus $\rho$ is strongly extreme.

(iv) The assertion is true if $\rho_1 \propto \rho$. Hence, we shall assume that this is not the case. Without any loss of generality, we may assume that $\rho = \rho_1 + \rho_2$ where $\rho_2$ is an $M_2 \times N_2$ state of rank $r_2$. 


Assume that \( N_1 < N \). By using Eq. (10), we can write the matrix of \( \rho = C^\dagger C \), where \( C = [C_0 \ C_1 \cdots C_{M-1}] \) and the matrices \( C_i \), of size \((r_1 + r_2) \times N\), are block-triangular

\[
C_i = \begin{bmatrix} C_{i0} & 0 \\
C_{i1} & C_{i3} \end{bmatrix}
\]

(35)

with \( C_{i0} \) of size \( r_1 \times N_1 \). The top \( r_1 \) [bottom \( r_2 \)] rows of \( C \) represent \( \rho_1 \) [\( \rho_2 \)]. Consequently, the matrix \([C_{00} \ C_{10} \cdots C_{M-1,0}]\) has rank \( r_1 \). Let \( R \) be the rank of the matrix \([C_{03} \ C_{13} \cdots C_{M-1,3}]\). Since \( C \) has rank \( M + N - 2 \), it is clear that \( M + N - 2 \geq r_1 + R \). We can choose a unitary matrix \( U \) such that the top \( r_2 - R \) rows of \( U[C_{03} \ C_{13} \cdots C_{M-1,3}] \) are 0. Thus we may assume that

\[
C_i = \begin{bmatrix} C_{i0} & 0 \\
C_{i1} & C_{i3} \end{bmatrix}
\]

(36)

where \( C_{i3} \) is of size \( R \times (N - N_1) \).

Consider the PPT state \( \sigma \) of rank \( R \) defined by \( \sigma = (C')^\dagger C' \), where \( C' = [C'_{03} \ C'_{13} \cdots C'_{M-1,3}] \). Since \( \sigma \) is PPT and \( \ker \rho \) contains only finitely many product vectors, we have \( R \geq \max \sigma_A \geq M - 1 \). Moreover, if \( \max \sigma_A = M - 1 \) then we must have \( N_1 = N - 1 \). In that case we have

\[
N_1 = N - 1 \geq M + N - 2 - R \geq r_1 > N_1.
\]

(37)

which is a contradiction. We conclude that \( \max \sigma_A = M \).

Since \( \sigma_B \) is a principal submatrix of \( \rho_B \), we have \( \max \sigma_B = N - N_1 \) and so

\[
\max \sigma_A + \max \sigma_B - 2 = M + (N - N_1) - 2 \geq r_1 + R - N_1 > R.
\]

(38)

Hence by Theorem 33, \( (\mathcal{R}(\sigma_A) \otimes \mathcal{R}(\sigma_B)) \cap \ker \sigma \) contains infinitely many product vectors. As this subspace is contained in \( \ker \rho \), we have again a contradiction. Thus we have proved that \( N_1 = N \).

(v) Suppose \( \rho \) is not extreme. Then \( \rho = \sum_{i=1}^k \rho_i \) with pairwise non-parallel \( \rho_i \in \mathcal{E} \) and, say, \( \rho_1 \) entangled. By Proposition 10, \( r_1 := \max \rho_i \) is bigger than any of the two local ranks of \( \rho_1 \). Hence, by (iv), \( \rho_1 \) must be an \( M \times N \) state. The hypothesis of (v) implies that \( r_1 = M + N - 2 \) and (iv) shows that \( \rho_1 \) is strongly extreme. As \( \mathcal{R}(\rho_3) \subseteq \mathcal{R}(\rho) = \mathcal{R}(\rho_1) \), it follows that \( \rho_2 \propto \rho_1 \) which is a contradiction.

The assertion (ii) of Theorem 34 may fail if \( \rho \) is bad. In the following example \( \rho \) is a reducible PPTES. For another example with \( \rho \) reducible and separable see Example 10.

**Example 35** \((M = N = 4)\) Consider the \( 4 \times 4 \) reducible state \( \rho = |00\rangle\langle 00| \oplus \sigma \) of rank six, where \( \sigma \) is a \( 3 \times 3 \) edge state of rank five. Such \( \sigma \) may have the birank \((5,l)\) where \( 5 \leq l \leq 8 \) [18, 28]. Thus the birank of \( \rho \) is \((6,l+1)\), and so \( \max \rho < \max \rho^\dagger \) if \( l \neq 5 \). As \( |0,i\rangle \in \ker \rho \) for \( i = 1,2,3 \), \( \rho \) is bad.

The assertion (iii) of Theorem 34 does not hold when \( \max \rho > M + N - 2 \). For example the kernel of the \( 3 \times 3 \) edge state of rank five constructed in [10] Sec. II has dimension four but it contains only two product vectors. On the other hand, its range contains a product vector so it is not strongly extreme. (It is known and easy to check that this state is extreme.)

We shall now strengthen the assertion of Theorem 34 in the case \( r < M + N - 2 \).

**Theorem 36** Let \( \rho \) be an \( M \times N \) PPT state of rank \( r < M + N - 2 \) and let \( r' = M + N - 1 - r \). Then

(i) \( \ker \rho \) contains subspaces \( |a\rangle \otimes W \) and \( V \otimes |b\rangle \) of dimension \( r' \).

(ii) The subspaces in (i) can be chosen so that \( |a\rangle \in V \) and \( |b\rangle \in W \).

**Proof.** (i) By symmetry, it suffices to prove only the assertion that \( \ker \rho \) contains a subspace \( V \otimes |b\rangle \) of dimension \( r' \). By Theorem 3 we have \( r \geq \max(M,N) \) and so \( \min(M,N) \geq 3 \). We can write \( \rho = C^\dagger C \), where \( C = [C_0 \cdots C_{M-1}] \) and the \( C_i \) are \( r \times N \) matrices. Since \( \ker \rho \) contains a product vector, we may assume that \( |0,N-1\rangle \in \ker \rho \). Hence, we may assume that the \( C_i \) have the form

\[
C_0 = \begin{bmatrix} I_R & 0 \\
0 & 0 \end{bmatrix}; \quad C_i = \begin{bmatrix} C_{i0} & C_{i1} \\
C_{i2} & C_{i3} \end{bmatrix}, \quad i > 0,
\]

(39)

where the blocks \( C_{i0} \) are \( R \times R \). Since \( \rho^\dagger \geq 0 \) we must have \( C_{i1} = 0, i > 0 \). Let \( m \) be the dimension of the matrix space spanned by the blocks \( C_{i3} \) and note that \( m \geq 1 \). We can now assume that the blocks \( C_{i3}, i = 1, \ldots, m \), are linearly independent and \( C_{i3} = 0 \) for \( i > m \).
We use the induction on $M+N$, and for fixed $M$ and $N$. This is vacuously true when $r = \max(M,N)$. If $r \geq M+N-1$ then the assertion follows from the observation that $\{1,\ldots,m\}^\perp \otimes \{N-1\} \subseteq \ker \sigma$ and $M-n \geq r'$. Assume that $r < M+N-1$ and let us apply the induction hypothesis to the PPT state $\sigma := (C')^\perp C'$ where $C' = [C_{13}\cdots C_{33}]$. This state acts on a $(M-1) \otimes (N-R)$ subsystem of our $M \otimes N$ system, and we have rank $\sigma \leq r-R$ and rank $\sigma_{A} = m$. Since $\sigma_{B} > 0$, and $\sigma_{A}$ is a principal submatrix of $\sigma_{B}$, it follows that $\sigma_{B} > 0$. In particular, rank $\sigma_{B} = N-R$. Thus $\sigma$ is an $m \times (N-R)$ PPT state of rank at most $r$. Since rank $\sigma \leq r-R \leq m+(N-R)-2$, we infer that there exists a subspace $V' \otimes |b\rangle \subseteq \ker \sigma$ of dimension at least $m+N-1-r$. (If rank $\sigma = m+(N-R) - 2$ we know this is true without using the induction hypothesis.)

By invoking (i) we can assume that $N-R \leq r'$. Then, because $|0\rangle_A \in V''$, we can choose $V$ so that $|0\rangle_A \in V$. Similarly, we can choose the first column of $r'$-dimensional subspace $W$. Then contained in the span of the basis vectors $|R_B,\ldots,|N-1\rangle_B$ such that $|N-1\rangle_B \in W$. It remains to observe that $|0\rangle \otimes W and $V \otimes |N-1\rangle$ are contained in ker $\sigma$.

The analog of assertion (i) for bad $M \times N$ PPT states of rank $r = M+N-2$ is not valid. A counter-example is the state $\rho$ in Example 53 with $a = b = c = d = e = f = g = 0$. On one hand we have $|0\rangle_A \otimes W \subseteq \ker \rho$ where $W$ is the span of $|2\rangle_B$ and $|3\rangle_B$. On the other hand, by using Eq. (11), it is not hard to show that ker $\rho$ contains no two-dimensional subspace $V \otimes |y\rangle_B$.

We conclude this section with another property of states whose kernel contains only finitely many product vectors.

**Proposition 37** Let $\rho$ be an $M \times N$ state such that $|X_{\rho}| < \infty$. If rank $\rho^\perp = M+N-2$ then $\rho$ is a good PPT state of rank $M+N-2$.

**Proof.** If $(x,y) \in \ker \rho^\perp$, then $(x^\ast,y^\ast)\rho(x,y) = (x,y)^\ast \rho^\perp(x,y) = 0$. As $\rho \geq 0$, we have $(x^\ast,y) \in \ker \rho$. We infer that $|X_{\rho}| < \infty$. Let $|\psi\rangle$ be an eigenvector of $\rho^\perp$ with eigenvalue $\lambda \neq 0$ and let $H = C|\psi\rangle + \ker \rho^\perp$. Since Dim $H \geq (M-1)(N-1)+1$, $H$ contains infinitely many product vectors. Hence, there exists $|\phi\rangle \in \ker \rho^\perp$ such that $|\psi\rangle + |\phi\rangle = |a,b\rangle$ is a product vector. Since $\rho^\perp|\phi\rangle = 0$, we have $\langle \phi|\rho^\perp|\psi\rangle = \langle a,b|\rho^\perp|a,b\rangle = \langle a^\ast,b\rangle \rho^\perp(a^\ast,b) \geq 0$. It follows that $\lambda > 0$. Hence $\rho^\perp \geq 0$, i.e., $\rho$ is a PPT state. By Theorem 33 $\rho^\perp$ is good. Thus we can apply Theorem 37 to $\rho^\perp$ to complete the proof. \(\square\)

**V. $M \times N$ PPT STATES OF RANK $N+1$**

In this section we focus on part (ii) of Conjecture 2. The Proposition 39 and Theorems 42 and 45 describe the structure of the $M \times N$ PPTES $\rho$ of rank $N+1$. The main result of this section is Theorem 46 in which we show that, for $M,N > 3$ any $M \times N$ PPTES $\rho$ of rank $N+1$ is reducible. Hence, such states cannot be extreme. From this result we deduce that $\mathcal{E}_{r}^{M,N} = \emptyset$ if min$(M,N) = 3,4$ and $1 < r < M+N-2$. Theorem 17 will be used in the next section for the construction of extreme states.

The following proposition is an analog of [2, Proposition 18]. Recall that the direct sum of two states was introduced in Definition 17.

**Proposition 38** Let $\rho$ be an $M \times N$ PPT state and let $|a\rangle \in H_A$ be such that rank$(|a\rangle \rho|a\rangle) = 1$. Then $\rho = \rho_1 \oplus \rho_2$ where $\rho_1$ is a pure product state. If $\rho$ is entangled and rank $\rho = N+1$, then $\rho = \rho_1 \oplus \rho_2$.

**Proof.** By using Eq. (9) we have $\rho = C^\perp C$, where $C = [C_0 C_1 \cdots C_{M-1}]$ and the $C_i$ are matrices of size $R \times N$, $R = \text{rank } \rho$. By choosing suitable bases we can assume that $|a\rangle = |0\rangle_A$ and ker$(|a\rangle \rho|a\rangle) = |0\rangle_B^\perp$. Consequently, only the first column of $C_0$ is nonzero. By replacing $C$ with $UC$ where $U$ is unitary, we may also assume that the first column of $C_0$ is 0 except for its first entry which is nonzero. By rescaling $C_0$, we may assume that this entry is 1. Since $\rho^\perp \geq 0$ and only the first entry of $C_0^\perp|0\rangle_C$ is nonzero, we infer that the first row of $C_i$, $i > 0$, is 0 except possibly its first entry. By subtracting suitable multiples of $C_0$ from the $C_i$, $i > 0$, we may assume that the first rows of these $C_i$ are 0. It is now easy to check that $\rho = |00\rangle|00\rangle \oplus \rho_2$ where $\rho_2 = C^\perp C'$ and $C'$ is the submatrix of $[C_1 C_2 \cdots C_{M-1}]$ obtained by deleting the first row. The first assertion is proved.

Now assume that $\rho$ is entangled and that rank $\rho = N+1$. Then rank $\rho_2 = N$ and $\rho_2$ must be entangled by Lemma 15. Clearly, we have rank$(\rho_2)_B \leq N$. On the other hand, since $\rho_B = |00\rangle+|\rho_2\rangle_B$ we have rank$(\rho_2)_B \geq \text{rank } \rho_B - 1 = N - 1$. Hence, Proposition 15 implies that rank$(\rho_2)_B = N-1$. It follows from the first assertion that rank$(\rho_2)_A = M-1$, and so the second assertion is proved. \(\square\)

We start by assuming that the range of a PPT state $\rho$ contains a product vector in which case it is relatively easy to describe the structure of $\rho$. 

Proposition 39 \((M, N > 2)\) Let \(\rho\) be an \(M \times N\) PPT state of rank \(N + 1\) such that \(\mathcal{R}(\rho)\) contains at least one product vector. If \(\rho\) is \(B\)-irreducible, then \(\rho\) is a sum of \(N + 1\) pure product states. Otherwise, \(\rho = \rho_1 \oplus_B \rho_2\) where \(\rho_1\) is a pure product state.

**Proof.** In order to prove the second assertion, let us assume that \(\rho = \rho' \oplus_B \rho''\). Since rank \(\rho' + \text{rank} \rho'' = N + 1\) and rank \(\rho'_B + \text{rank} \rho''_B = N\), we may assume that rank \(\rho' = \text{rank} \rho'_B\). Hence, we can apply Proposition 10 to \(\rho'\). As the sum in this proposition is necessarily \(B\)-direct, the second assertion is proved.

From now on we assume that \(\rho\) is \(B\)-irreducible. By Proposition 38, we have rank \(|b| \rho |b| \geq 2\) for all nonzero \(|b| \in \mathcal{H}_B\).

Using Eq. 19, we have \(\rho = C^\dagger C\) where \(C = [C_0 \ C_1 \ \cdots \ C_{M-1}]\) and the \(C_i\) are \((N + 1) \times N\) matrices.

Assume that there is an \(|a| \in \mathcal{H}_A\) such that rank \(|a| \rho |a| = 1\). By Proposition 38, \(\rho\) is an \(A\)-direct sum of a pure product state and an \((M - 1) \times P\) state \(\sigma\) of rank \(N\). Since \(\rho\) is \(B\)-irreducible, we must have \(P = N\). Hence, the first assertion holds in this case by Proposition 10. Thus we may assume that rank \(|a| \rho |a| \geq 2\) for all nonzero vectors \(|a| \in \mathcal{H}_A\). In particular, rank \(C_i\) \(\geq 2\) for each \(i\).

By the hypothesis, we may assume that the first row of \(C\) corresponds to the product vector in \(\mathcal{R}(\rho)\). By performing an ILO on system \(A\), we may also assume that the first row of each \(C_i\), \(i > 0\), is \(0\). The state \(\sigma := [C_1 \ \cdots \ C_{M-1}] [C_1 \ \cdots \ C_{M-1}]^\dagger\) is PPT and \(\sigma_B = \sum_{i > 0} C_i^\dagger C_i\). If \(\sigma_B |b| = 0\) for some \(|b| \neq 0\), then \(C_i |b| = 0\) for \(i > 0\) and so \(|0\rangle + \otimes |b| \rangle \subseteq \ker \rho\). This contradicts our assumption on the rank of \(|b| \rho |b|\). We conclude that rank \(\sigma_B = N\).

Since \(\sigma\) is PPT and rank \(\sigma \leq N\), Theorem 13 implies that rank \(\sigma = N\) and \(M \leq N\).

By dropping the first row of \(C_i\), we obtain the \(N \times N\) matrix \(C'_i\), \(i = 0, 1, \ldots, M - 1\). By applying Proposition 10 and Proposition 6 to the state \(\sigma\), we may assume that the matrices \(C'_i\), \(i > 0\), are diagonal. Since rank \(\sigma = N\), we may also assume that \(C'_1 = I_N\).

By simultaneously permuting the diagonal entries (if necessary) we may assume that

\[
C'_i = \lambda_{i1} I_{l_1} + \cdots + \lambda_{ik} I_{l_k}, \quad i > 0; \quad l_1 + \cdots + l_k = N,
\]

and that whenever \(r \neq s\) there exists an \(i > 1\) such that \(\lambda_{ir} \neq \lambda_{is}\). (Note that all \(\lambda_{ir} = 1\).) Since the \(C_i\) are linearly independent, each set \(\{\lambda_{ir} : r = 1, \ldots, k\}, i > 1\), must have at least two elements. In particular, we have \(k \geq 2\). The local transformations that we used to transform the \(C_i\), \(i > 0\), to this special form, can be performed on the entire matrices \(C_i\), \(i > 0\). In order to transform simultaneously the state \(\rho\), we have to perform the same local B-transformations on \(C_0\) as well as to multiply it by the same unitary matrices on the left hand side. The first rows of the \(C_i\), \(i > 0\), are not affected by any of these transformations and will remain 0.

We partition the matrix \(C'_0 = [A_{ii}]_{i,j=1}^k\) with \(A_{ii}\) square of order \(l_i\). We claim that \(A_{rs} = 0\) for \(r \neq s\). To prove this claim, recall that there exists an index \(i > 1\) such that \(\lambda_{ir} \neq \lambda_{is}\). We may assume temporarily that \(\lambda_{is} = 0\). (Just replace \(C_i\) with \(C_i - \lambda_{is} C_1\).) Then the \(sth\) diagonal block of order \(l_s\) in \(C_i^\dagger C_i\) is 0. Since

\[
\left[ \begin{array}{c} C'_0 C_0 \ C'_0 C_1 \\ C'_0 C_0 \ C'_0 C_1 \end{array} \right] = \left[ \begin{array}{c} C'_0 C_0 \ C'_i C_0 \\ C'_0 C_0 \ C'_i C_1 \end{array} \right] \geq 0,
\]

we deduce that the \(sth\) block-row of \(C'_0 C_1\) must vanish. In particular, \(\lambda_{ir} A_{ir}^\dagger = 0\). As \(\lambda_{ir} \neq \lambda_{is}\), our claim is proved.

Hence, we have \(C'_0 = B_1 + \cdots + B_k\) with \(B_1 = A_{ii}\) square of order \(l_i\). Let \(U_i\) be a unitary matrix such that \(U_i B_i U_i^\dagger\) is upper triangular and let \(U = U_1 + \cdots + U_k\). Note that the transformation \(C_i \rightarrow \{[I] + \otimes C_i\} U\) leaves the matrices \(C_i\), \(i > 0\), unchanged. Thus we may assume that all \(B_i\) are upper triangular. The first row of \(C_0\) consists of the vectors \(w_1, \ldots, w_k\) of lengths \(l_1, \ldots, l_k\), respectively. Let \(\mu_i\) and \(\nu_i\) be the first entries of \(w_i\) and \(B_i\), respectively.

If some \(\mu_i = 0\), say \(\mu_1 = 0\), then by subtracting from \(C_i\), \(i \neq 1\), a suitable scalar multiple of \(C_1\), we may assume that the first columns of these \(C_i\) are 0. This contradicts our assumption on the rank of \(|b| \rho |b|\). Hence, all \(\mu_i \neq 0\).

We claim that, for any \(s \in \{1, \ldots, k\}\), the matrix \(B_s\) is diagonal. To prove this claim, let us choose an \(r \in \{1, \ldots, k\}\) such that \(r \neq s\). Let us also fix an index \(i > 1\) such that \(\lambda_{ir} \neq \lambda_{is}\). (Recall that such \(i\) exists.) Since \([C_0 C_1 C_i] [C_0 C_1 C_i]^\dagger\) is a PPT state, so is \([C_0 C_i] [C_0 C_i]^\dagger\) where \(C_0 = C_0 - \nu_r C_1\) and \(C_i = C_i - \lambda_{ir} C_1\). Since \(\nu_r\) is the only nonzero entry in the \((l_1 + \cdots + l_{r-1} + 1)\)th column of \(C_0\), and the corresponding column of \(C_i\) is 0, we may assume that \(\nu_r\) is the only nonzero entry in the first row of \(C_0\). It follows that the state

\[
[C_s - \nu_r I_{l_s} (\lambda_{is} - \lambda_{ir}) I_{l_i}] [C_s - \nu_r I_{l_s} (\lambda_{is} - \lambda_{ir}) I_{l_i}]
\]

is PPT. Since \(\lambda_{ir} \neq \lambda_{is}\), the state \([C_s] [C_s]^\dagger\) is a \(2 \times l_s\) state of rank \(l_s\). Hence it is separable and, by Proposition 6, \(B_s\) is a normal matrix. Since it is also upper triangular, it must be diagonal. Hence, our claim is proved.

It follows that \(\rho\) is a sum of \(N + 1\) pure product states, which completes the proof of the first assertion. \(\square\)

Example 40 \((M = N = 3)\) As Proposition 39 suggests, a separable \(M \times N\) state of rank \(N + 1\) may fail to be the sum of \(N + 1\) pure product states. Indeed, the \(3 \times 3\) separable state \(\rho = 2 \sum_{i=0}^2 |ii\rangle \langle ii| + (|01\rangle + |10\rangle)(|01\rangle + |10\rangle)\) has rank four. As \(\rho^2\) has rank five, \(\rho\) is not a sum of four pure product states. \(\square\)
Example 41 \((M = 3, N = 4)\) As Proposition\(^{[29]}\) suggests, an \(M \times N\) PPTES of rank \(N + 1\) may be \(A\)-irreducible. As an example we can take the \(3 \times 4\) state \(\rho = |00\rangle\langle00| \oplus_B \sigma\) of rank five, where \(\sigma\) is a \(3 \times 3\) PPTES of rank four. Suppose \(\rho = \rho_1 \oplus_A \rho_2\). Then we have rank\((\rho)_A \leq 2\) and rank \(\rho_i \leq 4\). Thus both \(\rho_1\) and \(\rho_2\) are separable, and so is \(\rho\). We have a contradiction. \(\square\)

For any \(2 \times N\) state \(\rho\) of rank \(N + 1\), \(\mathcal{R}(\rho)\) contains infinitely many product vectors, see Eq. \(^{[5]}\). The first example \(\rho\) of \(2 \times 4\) PPTES was constructed in \(^{[44]}\) and \(^{[21]}\), Eq. \((32)\). This state has rank five and so \(\mathcal{R}(\rho)\) contains infinitely many product vectors. Moreover, we claim that \(\rho\) is irreducible. To prove this claim, assume that \(\rho\) is reducible. Then necessarily \(\rho = \rho_1 \oplus_B \rho_2\), and by Lemma\(^{[14]}\) \(\rho_1\) and \(\rho_2\) are PPT. Since their B-local ranks are at most three, they are separable. This is a contradiction and the claim is proved. Thus Proposition\(^{[39]}\) does not extend to the case \(M = 2\).

We can now characterize the reducible \(M \times N\) PPTES of rank \(N + 1\).

Theorem 42 \((M, N > 2)\) For an \(M \times N\) PPTES \(\rho\) of rank \(N + 1\), the following are equivalent to each other

(i) \(\rho\) is reducible;
(ii) \(\mathcal{R}(\rho)\) contains at least one product vector;
(iii) \(\rho = \rho_1 \oplus_B \rho_2\), where \(\rho_1\) is a pure product state.

Proof. \((i) \rightarrow (ii)\). Assume \(\rho = \rho' + \rho''\) is an A or B-direct sum. By Theorem\(^{[19]}\) we have rank \(\rho' \geq \text{rank} \rho_B\) and rank \(\rho'' \geq \text{rank} \rho_B\). Since rank \(\rho' + \text{rank} \rho'' = \text{rank} \rho = N + 1\), we have \(N + 1 \geq \text{rank} \rho_B + \text{rank} \rho_B = N\). Therefore, say, rank \(\rho' = \text{rank} \rho_B\). Hence \(\rho''\) is separable by Proposition\(^{[10]}\) and \(\rho''\) holds.

(ii) \rightarrow (iii) follows from Proposition\(^{[39]}\) because \(\rho\) is entangled.

(iii) \rightarrow (i) trivial. \(\square\)

Using these results, we now prove the main result of this section.

Theorem 43 \((M, N > 3)\) If \(\rho\) is an \(M \times N\) PPTES of rank \(N + 1\) then \(\rho = \rho_1 \oplus_B \rho_2\), where \(\rho_1\) is a pure product state. Consequently, \(\mathcal{S}_{N+1} = \emptyset\).

Proof. Let \(\rho\) be an \(M \times N\) PPTES of rank \(N + 1\). Suppose that the assertion is false. Then, by Theorem\(^{[12]}\) \(\rho\) is irreducible and \(\mathcal{R}(\rho)\) is a CES. For any \(|a\rangle \in \mathcal{H}_A\) let \(r_a\) be the rank of the linear operator \(\langle a|\rho|a\rangle\). Since \(\text{Dim ker} \rho = MN - N - 1 > (M - 1)(N - 1) + 1\), \(\ker \rho\) contains infinitely many product vectors. If \(|a, b\rangle \in \ker \rho\) is a product vector then \(\langle a|\rho|a\rangle\) kills the vector \(|b\rangle\), and so \(r_a < N\). Let \(R\) be the maximum of \(r_a\) taken over all \(|a\rangle \in \mathcal{H}_A\) such that \(r_a < N\). Thus \(R < N\). Without any loss of generality we may assume that \(\langle 0|A\rho|0\rangle\) has rank \(R\).

We can write \(\rho\) as in Eq. \(^{[9]}\). Thus \(\rho = C^\dagger C\) where \(C = [C_0 \cdots C_{M-1}]\) and the blocks \(C_i\) are \((N + 1) \times N\) matrices. By Proposition\(^{[38]}\) we have \(r_a > 1\) for all nonzero vectors \(|a\rangle \in \mathcal{H}_A\). In particular, rank \(C_i \geq 2\) for each \(i\). Consequently, we may assume that

\[
C_0 = \begin{bmatrix}
I_R & 0 \\
0 & 0
\end{bmatrix}; \quad C_i = \begin{bmatrix}
C_{i0} & C_{i1} \\
C_{i2} & C_{i3}
\end{bmatrix}, \quad i > 0,
\]

where the \(C_{i0}\) are \(R \times R\) matrices. Since \(\rho^\dagger \geq 0\), all \(C_{i1} = 0\).

The state \(\sigma = C^\dagger C\), where \(C' = [C_{13} \cdots C_{M-1,3}]\), is a PPT state of rank \(\leq N - R + 1\) which acts on a \((M - 1) \times (N - R)\) subsystem of our \(M \times N\) system. Since \(\rho_B > 0\) and \(\sigma_B\) is its principal submatrix, we have rank \(\sigma_B = N - R\). By using Theorem\(^{[9]}\) we deduce that the rank of \(\sigma\) must be either \(N - R\) or \(N - R + 1\). Assume that this rank is \(N - R\). Then, by Proposition\(^{[10]}\) \(\sigma\) is a sum of \(N - R\) pure product states. Consequently, we may assume that the blocks \(C_{i3}\) are diagonal matrices (with the zero last row). Moreover, we can assume that the first entry of \(C_{i3}\) is 1 for \(i = 1\) and 0 for \(i > 1\). Since \(\rho^\dagger \geq 0\), the first row of \(C_{i2}\), \(i > 1\), must be 0. Thus the nonzero entries of the \((R + 1)\)st row of \(C\) occur only inside the block \(C_1\). This means that \(\mathcal{R}(\rho)\) contains a product vector, which gives us a contradiction.

We conclude that \(\sigma\) must have rank \(N - R + 1\), and so \(m := \text{rank} \sigma_A\) is in the range \(1 < m < M\). Hence, we may assume that \(C_{i3} = 0\) for \(i > m\). Consequently, the matrices \(C_{i3}, 1 \leq i \leq m\) are linearly independent. Moreover, by using the definition of \(R\), we know that any nontrivial linear combination of these \(m\) matrices must have full rank, \(N - R\).

Assume now that \(m > 2\). We can consider the state \(\sigma\) as acting on the Hilbert space \(\mathcal{R}(\sigma_A) \otimes \mathcal{R}(\sigma_B)\) of dimension \(m(N - R)\). Then its kernel has the dimension \((m - 1)(N - R) - 1\) which is bigger than \((m - 1)(N - R - 1)\). Therefore this kernel contains a product vector. Equivalently (see Eq. \(^{[11]}\)), there exist scalars \(\xi_i, i = 1, \ldots, m\), not all 0, such that the matrix \(\sum_{i=1}^m \xi_i C_{i3}\) has rank less than \(m\). Thus we have a contradiction.

Consequently, we must have \(m = 2\). Since any nontrivial linear combination of \(C_{13}\) and \(C_{23}\) has rank \(N - R\), the matrix \([C_{13} C_{23}]\) must have rank \(N - R + 1\). For \(i > 2\) we have \(C_{i3} = 0\) and since \(\rho^\dagger \geq 0\), it follows that \(C_{i2}C_{13} = C_{i2}C_{23} = 0\). Hence, we have \(C_{i2} = 0\) for \(i > 2\).
The state $\tau := C''\pi C''^\dagger$, where $C'' = [I_R \ C_{30} \ \cdots \ C_{M-1,0}]$, is a PPT state of rank $R$. Note that rank $\tau_A \leq M - 2$ and rank $\tau_B = R$. By Proposition 10 we can assume that the matrices $C''_i, \ i > 2$, are diagonal. By simultaneously permuting their diagonal entries (if necessary) we may assume that

$$C''_i = \lambda_i I_{l_1} \oplus \cdots \oplus \lambda_k I_{l_k}, \ i > 2; \ l_1 + \cdots + l_k = R, \quad (44)$$

and that whenever $r \neq s$ there exists an $i$ such that $\lambda_r \neq \lambda_s$. Since the blocks $C_{ij} = 0$ when $i > 2$ and $j \neq 0$ and rank $\rho_A = M \geq 4$, we must have $k > 1$.

As in the proof of Proposition 33 we can show that the matrices $C_{10}$ and $C_{20}$ are direct sums

$$C_{10} = E_1 \oplus \cdots \oplus E_k, \ C_{20} = F_1 \oplus \cdots \oplus F_k, \quad (45)$$

where $E_i$ and $F_i$ are square blocks of size $l_i$, and we may assume the $E_i$ are lower triangular.

Let us write

$$C_{i2} = \begin{bmatrix} C_{i21} \\ C_{i22} \end{bmatrix}, \ i > 0; \ C_{i3} = \begin{bmatrix} C_{i31} \\ C_{i32} \end{bmatrix}, \ i = 1, 2; \quad (46)$$

where $C_{22}$ and $C_{32}$ are row-vectors. By multiplying $C$ on the left hand side by a unitary matrix $I_R \oplus U$, we may assume that $C_{32} = 0$. Since $C_{13}$ has rank $N - R$, the block $C_{131}$ is an invertible matrix. Consequently, we may assume that $C_{121} = 0$. We split the row-vector $C_{122}$ into $k$ pieces $w_1, \ldots, w_k$ of lengths $l_1, \ldots, l_k$, respectively. To summarize, the matrices $C_j, \ j > 0$, have the form:

$$C_1 = \begin{bmatrix} E_1 & & 0 \\ & \ddots & \\ 0 & & E_k \end{bmatrix}, \ C_2 = \begin{bmatrix} F_1 & & 0 \\ & \ddots & \\ C_{221} & C_{222} & C_{231} \end{bmatrix}, \ C_j = \begin{bmatrix} \lambda_1 I_{l_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_k I_{l_k} \end{bmatrix}, \ j > 2. \quad (47)$$

Since $R(\rho)$ is a CES, each $l_i > 1$ and at least one $w_i \neq 0$. As we can simultaneously permute the first $k$ diagonal blocks of the matrices $C_j$, we may assume that $w_1 \neq 0$. Let $w_1 = (a_1, \ldots, a_n, 0, \ldots, 0)$ where $a_n \neq 0$ and let us partition

$$E_1 = \begin{bmatrix} E_{10} \\ E_{12} \\ E_{13} \end{bmatrix},$$

where $E_{10}$ is of size $n \times n$.

If $n < l_1$ then the state $[I_{l_1-n} E_{13}]^\dagger [I_{l_1-n} E_{13}]$ is PPT and so the matrix $E_{13}$ must be normal. Since $E_{13}$ is also lower triangular, it must be a diagonal matrix. By using this fact and the observation that the state $[C_0 C_1]^\dagger [C_0 C_1]$ is PPT, one can easily show that $E_{12} = 0$. We conclude that except for $a_n$ and the last entry of $E_{10}$ all other entries of the $n$th column of $C_1$ are 0. This is trivially true also in the case $n = l_1$. By subtracting from $C_1$ a scalar multiple of $C_0$, we may assume that the last entry of $E_{10}$ is 0. Now $a_n$ is the only nonzero entry in the $n$th column of $C_1$.

We can choose an index $i > 2$ such that $\lambda_i \neq \lambda_k$. By replacing temporarily $C_i$ with $C_i - \lambda_i C_0$, the $n$th column of $C_i$ becomes 0. It follows easily that the state $[E_k, (\lambda_i - \lambda_k) I_{l_k}]^\dagger [E_k, (\lambda_i - \lambda_k) I_{l_k}]$ is a PPT state of rank $l_k$. Since its B-local rank is also $l_k$, the matrix $E_k$ must be normal. As it is also lower triangular, it must be a diagonal matrix. We can further assume that

$$E_k = \mu_1 I_{n_1} \oplus \cdots \oplus \mu_s I_{n_s}, \quad (48)$$

with each $G_j$ upper triangular of order $n_j$. Then the $R$th row of $C$ shows that $R(\rho)$ contains a product vector.

This contradicts Theorem 42 and so the proof is completed. \hfill $\square$

As a consequence, we obtain a link between the good and extreme states.

**Proposition 44** ($N \geq M = 3, 4$) Let $\rho$ be a good $M \times N$ PPT state of rank $M + N - 2$.

(i) If $\rho$ is entangled then it is strongly extreme.

(ii) If $R(\rho)$ contains a product vector, then $\rho$ is separable.

**Proof.** (i) follows from Theorem 43 and parts (iii) and (v) of Theorem 44.

(ii) If $M = 3$, it follows from Theorem 44(i) that a good $\rho$ is irreducible. Then $\rho$ is separable by Proposition 39.

Now let $M = 4$, and assume that $\rho$ is entangled. Since $\rho$ is good, part (iii) of Theorem 44 shows that $\rho$ is not extreme and so we have $\rho = \rho_1 + \rho_2$ with $\rho_1$ a PPTES. It follows from part (iv) of the same theorem that $\rho_1$ is a
Theorem 46 Let \( C \) such that both matrices as before, the blocks positive semidefinite for \( I \) one of them has rank \( \sigma \) rank has rank \( \rho \) is not extreme, \( \rho \) is also irreducible by Lemma 16. We may assume that \( \rho_2 := \rho - \rho_1 \geq 0 \). By Theorem 42 \( R(\rho) \) is a CES, and so both \( \rho_1 \) and \( \rho_2 \) must be entangled. Let \( R_i = \text{rank}(\rho_i) \) and \( r_i = \text{rank}(\rho_i)_B \), \( i = 1, 2 \). By the hypothesis we have \( R_1 > r_1 \).

Assume that \( r_1 < N \). By using Eq. 9, we may assume that \( \rho = C^\dagger C \), where \( C = [C_0 \ C_1 \cdots C_{M-1}] \) and

\[
C_i = \begin{bmatrix} C_{i0} & C_{i1} \\ 0 & C_{i2} \end{bmatrix} \tag{49}
\]

are matrices of size \((R_1 + R_2) \times N\), the blocks \( C_{i0} \) are of size \( n \times (N-r_1) \), \( n \leq R_2 \), and the matrix \([C_0 \ C_1 \cdots C_{M-1}]\) has rank \( n \). The bottom \( R_1 \) \( R_2 \) rows of \( C \) represent \( \rho_1 \) \( \rho_2 \). Since rank \( \rho = N+1 \), we can choose a unitary matrix \( U \) such that the bottom \( R_1 + R_2 - N - 1 \) rows of \( U[C_0 \ C_1 \cdots C_{M-1}] \) are 0. Then the last \( R_1 + R_2 - N - 1 \) rows of \((I_n \oplus U)C\) are 0, and by dropping them, we may assume that in Eq. (49) the \( C_i \) are of size \((N+1) \times N\) and, as before, the blocks \( C_{i0} \) have size \( n \times (N-r_1) \). Since

\[
C_i^\dagger C_j = \begin{bmatrix} C_{i0}^\dagger C_{j0} & C_{i0}^\dagger C_{j1} \\ C_{i1}^\dagger C_{j0} & C_{i1}^\dagger C_{j1} + C_{i2}^\dagger C_{j2} \end{bmatrix}, \tag{50}
\]

it follows immediately that the state \( \sigma := [C_{i0}^\dagger C_{j0}]_{i,j=0}^{M-1} \) is PPT and that \( \sigma_B = \sum_i C_{i0}^\dagger C_{i0} \) has rank \( N - r_1 \). By Theorem 42 we have \( N + 1 - R_1 \geq n = \text{rank} \sigma \geq \text{rank} \sigma_B = N - r_1 \). As \( R_1 > r_1 \), we must have \( R_1 = r_1 + 1 \) and rank \( \sigma = N - r_1 \). By Proposition 10 \( \sigma \) is separable and is the sum of \( N - r_1 \) pure product states. Consequently, as mentioned in Sect. 11, we may assume that the blocks \( C_{i0} \) are diagonal matrices. We may also assume that the first entry of \( C_0 \) is not 0. By subtracting a scalar multiple of \( C_0 \) from the \( C_i \), \( i > 0 \), we may assume that the first column of \( C_i \) is 0. Then Proposition 15 implies that \( \rho \) is reducible, which is a contradiction.

Thus we must have \( r_1 = N \), and so \( R_1 = N + 1 \). Since \( R(\rho_1) = R(\rho) \), Lemma 12 implies that rank \( \rho_A = \text{rank} \rho_A = M \). Thus (i) holds.

(ii) By Theorem 42 \( R(\rho) \) is a CES. Hence, \( \rho_1 \) is a PPTES. Theorem 9 and Proposition 10 imply that rank \( \rho_1 > \text{rank}(\rho_1)_B \). By part (i), \( \rho_1 \) is an \( M \times N \) state of rank \( N + 1 \). Finally, Theorem 42 implies that \( \rho_1 \) is irreducible. \( \square \)

Theorem 47 Any irreducible \( 3 \times N \) PPTES of birank \((N+1, N+1)\) is extreme.

Proof. Suppose there exists a counter-example, say \( \rho \). Since \( \rho \) is irreducible, \( \rho^\dagger \) is also irreducible by Lemma 16. Since \( \rho \) is not extreme, \( \rho = \rho_1 + \rho_2 \) where \( \rho_1 \) and \( \rho_2 \) are non-parallel PPT states. By Theorem 16 \( \rho_1 \) and \( \rho_2 \) are \( M \times N \) PPTES of rank \( N + 1 \). Consequently, \( \rho_1, \rho_2 \) and \( \rho \) have the same range, and the same is true for \( \rho_1^\dagger, \rho_2^\dagger \) and \( \rho^\dagger \). Consider the Hermitian matrix \( \sigma(t) = \rho_2 - t \rho_1 \) depending on the real parameter \( t \). Both \( \sigma(t) \) and \( \sigma(t)^\dagger \) are positive semidefinite for \( t \leq 0 \) and indefinite for \( t = t_0 := \text{Tr}(\rho_2)/\text{Tr}(\rho_1) \). Hence, there exists a unique \( t_1 \in (0, t_0) \) such that both matrices \( \sigma(t) \) and \( \sigma(t)^\dagger \) are positive semidefinite and have rank \( N + 1 \) for \( 0 \leq t < t_1 \), while at least one of them has rank \( \leq N \) for \( t = t_1 \). Since \( \rho = (1 + t_1)\rho_1 + \sigma(t_1) \) and \( \rho^\dagger = (1 + t_1)\rho_1^\dagger + \sigma(t_1)^\dagger \), Theorem 16 gives a contradiction. \( \square \)
VI. EXAMPLES OF $M \times N$ PPT STATES OF RANK $M+N-2$

If $\rho$ is an $M \times N$ PPT state of rank $M+N-2$, then according to Theorem 32 there are two possibilities: $\rho$ is good in which case $\ker \rho$ contains exactly $\delta$ product vectors or $\rho$ is bad in which case we know that $\ker \rho$ contains a 2-dimensional subspace $V \otimes \mathcal{W}$. Both cases occur even when $\rho$ is a PPTES, and we will construct a variety of examples. They are discussed in subsections A and B, respectively. It follows immediately from Theorem 34 that the states in the good case, namely Examples 48, 50, and 51, are strongly extreme.

A. Good case: finitely many product vectors in the kernel

Since we assume that $M,N > 2$, the smallest case is $M = N = 3$. Let $M = N = 3$ and let $\rho$ be a $3 \times 3$ PPTES of rank four. It is well-known that $\ker \rho$ contains exactly six product vectors. Hence, $\rho$ is a good state by Proposition 7.

Assuming that $M \leq N$, the next case is $M = 3$, $N = 4$. The state $\rho$ of Example 48 is extracted from the family GenTiles2 of UPB constructed in [12]. In this example, there are exactly ten product vectors in $\ker \rho$, which are in general position. Next in Example 50, we shall construct a $3 \times 4$ extreme PPTES of rank five, whose kernel contains exactly ten product vectors. However, these product vectors are in general position. This is the only known example of this kind. At the end of this subsection, in Example 51, we shall construct a $3 \times N$ extreme state of rank $N+1$ whose kernel contains exactly $N(N+1)/2$ product vectors.

Example 48 ($M = 3, N = 4$) Consider the 7-dimensional subspace $K$ of the space of complex $3 \times 4$ matrices (identified with $\mathcal{H}$):

$$
\begin{bmatrix}
\xi_1 + \xi_7 & \xi_4 + \xi_7 & -\xi_3 + \xi_7 & -\xi_4 + \xi_7 \\
-\xi_1 + \xi_7 & \xi_2 + \xi_7 & \xi_5 + \xi_7 & -\xi_5 + \xi_7 \\
\xi_6 + \xi_7 & -\xi_2 + \xi_7 & \xi_3 + \xi_7 & -\xi_6 + \xi_7
\end{bmatrix} = \sum_{i=1}^{7} \xi_i W_i. \quad (51)
$$

The $W_i$ form an orthogonal (non-normalized) basis of $K$ and each of them has rank one.

The $W_i$ form an orthogonal (non-normalized) basis of $K$ and each of them has rank one. Each of the matrices

$$
W_8 = 15(-W_1 + W_3 + W_5 + W_6) - 5W_4 + 3W_7, \quad (52)
$$

$$
W_9 = 15(W_1 - W_2 + W_4 + W_6) - 5W_5 + 3W_7, \quad (53)
$$

$$
W_{10} = 15(W_2 - W_3 + W_4 + W_5) - 5W_6 + 3W_7, \quad (54)
$$

also has rank one. The orthogonal projector, $\rho$, onto $K^\perp$ is a $3 \times 4$ PPT state of rank five. It is entangled because $K^\perp$ is a CES. It is not hard to verify that $\ker \rho = K$ contains only 10 matrices of rank one, namely the $W_i$, $i = 1, \ldots, 10$. Note that the 10 product vectors in $\ker \rho$ are not in general position. Indeed, if we write $W_i = |a_i\rangle \otimes |b_i\rangle$ for each $i$, then the $|a_i\rangle$ with $i = 1, 2, 3$ are linearly dependent (and the same is true for the $|b_j\rangle$ with $j = 2, 3, 4, 5$). \hfill \Box

We would like to construct examples of PPTES, $\rho$, of rank $M+N-2$ such that $\ker \rho$ contains exactly $\delta$ product vectors, and moreover these product vectors are in general position. An example will be given later (see Example 51). Unfortunately, the method of using UPB to produce such $\rho$ works only when $M = N = 3$. This follows from the following simple lemma.

Lemma 49 If a UPB consists of $(M-1)(N-1)+1$ product vectors in general position, then $M = N = 3$.

Proof. Let $\{ |a_i, b_i\rangle : i = 1, \ldots, (M-1)(N-1)+1 \}$ be a UPB. Since there are no UPB when $M = 2$ or $N = 2$, we have $M, N \geq 3$. Assume the $|a_i, b_i\rangle$ are in general position. Let $p$ be the number of indexes $i$ such that $|a_i\rangle \perp |a_i\rangle$ and $q$ be the number of indexes $j$ such that $|b_i\rangle \perp |b_j\rangle$. Our assumption implies that $p \leq M-1$ and $q \leq N-1$. Since $|a_i, b_i\rangle \perp |a_i, b_i\rangle$ for $i > 1$, we have

$$
(M-1)(N-1) \leq p + q \leq M + N - 2, \quad (55)
$$

Thus $(M-2)(N-2) \leq 1$ and so $M = N = 3$. \hfill \Box

Example 50 ($M = 3, N = 4$) We shall construct a good real $3 \times 4$ extreme PPTES $\rho$ of birank $(5,5)$ such that $\mathcal{R}(\rho)$ is a CES and the 10 product vectors belonging to $\ker \rho$ are in general position.

We write $\rho$ as in Eq. 13 where we set $M = 3$, $N = 4$, $R = 5$ and define the blocks $C_i$ by

$$
C_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 1 & 2 & 0 \\
1 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 \\
-3 & -1 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}. \quad (56)
$$
One can verify by direct computation that there are exactly 10 product vectors in \( \ker \rho \), and that they are in general position. Moreover, \( R(\rho) \) is a CES and \( \rho \) is entangled. Since \( C_i^0 C_1, C_0^1 C_2 \) and \( C_1^0 C_2 \) are real symmetric matrices, we have \( \rho^T = \rho \). Hence \( \rho \) is PPT and rank \( \rho^T = 5 \). By Theorem 42 \( \rho \) is irreducible, and by Theorem 47 it is extreme. □

Let \( \{a_i, b_i\}, \ i = 1, \ldots, 10 \), be the product vectors in this example. According to Lemma 49 there is no ILO \( A \otimes B \) such that some seven of the ten product vectors \( A \otimes B(a_i, b_i) \) (after normalization) form an UPB. As \( R(\rho) \) is a CES and the \( \{a_i, b_i\} \) span \( \ker \rho \), we can say that \( \ker \rho \) is spanned by a general UPB according to the following definition. A general UPB is a set of linearly independent product vectors \( \{\psi\} := \{|\psi_i\rangle : i = 1, \ldots, k\} \subset \mathcal{H} \) such that \( \{\psi\}^\perp \) is a CES 40.

**Example 51** (\( N > M = 3 \)) We shall construct a family of good real \( 3 \times N \) extreme states \( \rho \) of birank \( (N + 1, N + 1) \) depending on \( N - 3 \) parameters. By Theorem 44 these states are strongly extreme.

We write \( \rho \) as in Eq. (3) where we set \( M = 3, R = N + 1 \) and define the \( R \times N \) blocks \( C_0 = I_{N-1} \oplus 0 \) and

\[
C_1 = \begin{bmatrix}
(N - 3)(B^2 - I_{N-3}) & e & b & 0 \\
e^T & 0 & 0 & 0 \\
b^T & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 1 \\
\end{bmatrix},
C_2 = \begin{bmatrix}
B - I_{N-3} & 0 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 1 & -1 & 0 \\
e^T - b^T B & -1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix},
\]

(57)

where \( e \) is the all-one column vector, \( b \) column vector with components \( b_1, \ldots, b_{N-3} \), and \( B \) the diagonal matrix with diagonal entries \( b_1, \ldots, b_{N-3} \). We assume that the \( b_i \) are real, \( b_i^2 \neq 1 \), and that the \( b_i^2 \) are pairwise distinct.

We claim that \( R(\rho) \) is a CES. To prove this claim, it suffices to show that the matrix

\[
\begin{bmatrix}
\xi C_0 \\
\xi C_1 \\
\xi C_2 \\
\end{bmatrix}
\]

cannot have rank one for any row vector \( \xi \in \mathbb{C}^{N+1} \). We omit the straightforward and tedious verification.

It follows from this claim that \( \rho \) is entangled. Since \( C_i^0 C_1, C_0^1 C_2 \) and \( C_1^0 C_2 \) are real symmetric matrices, we have \( \rho^T = \rho \). Hence \( \rho \) is a PPTES of birank \( (N + 1, N + 1) \). This state is irreducible by Theorem 42 and extreme by Theorem 47. For \( N = 4, 5, 6 \) and several choices of the parameters \( b_i \), P. Sollid 41 has verified computationally that the \( \delta \) product vectors in \( \ker \rho \) are in general position. Whether this is true in general, remains an open question. □

**Theorem 52** All \( 3 \times N \) PPT states \( \rho \) constructed in Example 51 are good.

**Proof.** Since \( \rho \) has rank \( M + N - 2 (= N + 1) \), it suffices to prove that \( X_\rho \) is a finite set.

Assume that \( |X_\rho| = \infty \). By Theorem 42 there is a 2-dimensional subspace \( V \otimes W \subseteq \ker \rho \). Since \( \rho \) is irreducible, Proposition 33 implies that we must have \( \text{Dim} V = 1 \) and \( \text{Dim} W = 2 \). Let \( \xi_0 |0\rangle + \xi_1 |1\rangle + \xi_2 |2\rangle \in V \) be a nonzero vector. Then the matrix \( \Xi = \xi_0 C_0 + \xi_1 C_1 + \xi_2 C_2 \) has rank less than \( N - 1 \). Obviously, at most one \( \xi_i \) is 0. Suppose that \( \xi_1 = 0 \). Then the \( (N - 3) \times 3 \) submatrix in the upper right corner of \( \Xi \) is 0, the \( 4 \times 3 \) submatrix below it has rank 3, and the \( (N - 3) \times (N - 3) \) submatrix in the upper left corner of \( \Xi \) has rank \( \geq N - 4 \), and we have a contradiction. Hence, \( \xi_1 \neq 0 \) and we can assume that \( \xi_1 = 1 \). Thus we have

\[
\Xi = \begin{bmatrix}
\xi_0 I_{N-3} + (N - 3)(B^2 - I_{N-3}) + \xi_2 (B - I_{N-3}) & e & b & 0 \\
\xi_0 e^T & \xi_0 - \xi_2 & \xi_2 & 0 \\
\xi_2 e^T - b^T B & \xi_2 & 0 & -1 \\
\xi_2 & 1 + \xi_0 - \xi_2 & 0 & 0 \\
0 & -1 & 1 + \xi_2 & 0 \\
\end{bmatrix}.
\]

(59)

Suppose that \( \xi_2 = 0 \). Then the \( 2 \times (N - 2) \) submatrix in the lower left corner of \( \Xi \) is 0 and the \( 2 \times 2 \) submatrix in the lower right corner has rank 2. Since the \( b_i^2 \) are pairwise distinct, the \( (N - 3) \times (N - 2) \) submatrix in the upper left corner of \( \Xi \) has rank \( N - 3 \), and we have a contradiction. Hence, \( \xi_2 \neq 0 \).

Suppose that \( \xi_2 = -1 \). Then rank \( \Xi = 2 + r \), where \( r \) is the rank of the \( (N - 1) \times (N - 2) \) submatrix in the upper left corner of \( \Xi \). It is easy to show that \( r \geq N - 3 \), and so we have again a contradiction. Hence, \( \xi_2 \neq -1 \).

At most two of the first \( N - 3 \) diagonal entries \( d_i := (N - 3)(b_i^2 - 1) + \xi_0 + (b_i - 1)\xi_2, i = 1, \ldots, N - 3 \), of \( \Xi \) may be 0. Suppose that exactly one of the \( d_i \) is 0, say \( d_1 = 0 \). Then, by using the fact that \( \xi_2 \neq -1 \), we see that the \( (N - 1) \times (N - 1) \) submatrix of \( \Xi \) obtained by removing the rows \( N - 1 \) and \( N \) and column \( N - 1 \) is invertible, and we have a contradiction. Suppose that exactly two of the \( d_i \) are 0, say \( d_1 = d_2 = 0 \). Then the \( (N - 1) \times (N - 1) \) submatrix in the upper left corner of \( \Xi \) is invertible, and we have a contradiction. We conclude that all \( d_i \) are nonzero.


For \( i \in \{N - 2, N - 1, N \} \), let \( \Xi[i] \) denote the square submatrix of \( \Xi \) of order \( N - 2 \) obtained by deleting the last two columns and keeping only the \( i \)th row from the last four rows. Since \( \Xi[N - 2] \oplus [1 + \xi_2] \) is a \((N - 1) \times (N - 1)\) submatrix of \( \Xi \) and \( \xi_2 \neq -1 \), our assumption implies that \( \det \Xi[N - 2] = 0 \). Similarly, one can show that \( \det \Xi[N - 1] = \det \Xi[N] = 0 \). Hence we have

\[
\xi_0 - \xi_2 = \sum_i \frac{1}{d_i}, \quad \xi_2 = \sum_i b_i d_i, \quad 1 = \sum_i b_i^2 - 1. \quad (60)
\]

By multiplying these equations by \( \xi_0 - \xi_2 \), \( \xi_2 \), \( N - 3 \) respectively and adding them up, we obtain that \((\xi_0 - \xi_2)^2 + \xi_2^3 = 0\). Hence \( \xi_0 \) or \( \xi_2 \) is not real. Let us denote by \( \xi_0' \), \( \xi_2' \), \( d_i' \) the imaginary parts of \( \xi_0 \), \( \xi_2 \), \( d_i \), respectively. From the definition of the \( d_i \) we have \( d_i' = \xi_0' + (b_i - 1)\xi_2' \). From Eq. \((60)\) we have

\[
\xi_0' - \xi_2' = -\sum_i \frac{d_i'}{|d_i|^2}, \quad \xi_2' = -\sum_i \frac{b_i d_i'}{|d_i|^2}. \quad (61)
\]

By multiplying the first of these equations by \( \xi_0' - \xi_2' \) and the second by \( \xi_2' \) and adding them, we obtain that

\[
(\xi_0' - \xi_2')^2 + (\xi_2')^2 = -\sum_i \frac{(d_i')^2}{|d_i|^2}. \quad (62)
\]

As the left hand side is positive, we have a contradiction. Hence, our assumption is false and so \( \rho \) is good. \( \square \)

### B. Bad case: infinitely many product vectors in the kernel

The examples in this subsection will be all bad, i.e. the kernel of the state will contain infinitely many product vectors. The Examples 54, 55 and 56 cover all possible local ranks \((M, N)\), except \( M = N = 3 \) which is an exception (see [8, Theorem 22]). The first two examples are easily shown to be extreme. Moreover, we prove that all states in Example 56 are extreme (see Theorem 57) and thereby we confirm part (i) of Conjecture 2.

Since UPBs are used extensively in quantum information, we first consider the PPTES \( \rho \) associated to an arbitrary UPB of the family GenTiles2. Suppose that \( N \geq M \geq 3 \) and \( N > 3 \). Then by [12, Theorem 6] the following \( MN - 2M + 1 \) o.n. product vectors form a UPB:

\[
|S_j\rangle := \frac{1}{\sqrt{2}}((j) - |j + 1 \mod M\rangle \otimes |j\rangle), \quad 0 \leq j \leq M - 1; \quad (63)
\]

\[
|L_{jk}\rangle := |j\rangle \otimes \frac{1}{\sqrt{N - 2}} \left( \sum_{i=0}^{M-3} \omega^{ik} |i + j + 1 \mod M\rangle + \sum_{i=M-2}^{N-3} \omega^{ik} |i + 2\rangle \right), \quad (65)
\]

\[
\omega := e^{2 \pi i / N}, \quad 0 \leq j \leq M - 1, \quad 1 \leq k \leq N - 3; \quad (66)
\]

\[
|F\rangle := \frac{1}{\sqrt{NM}} \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} |i\rangle \otimes |j\rangle. \quad (67)
\]

**Lemma 53** Let \( \rho \) be the PPTES of rank \( 2M - 1 \) associated with this UPB, i.e.,

\[
\rho = I_{MN} - \sum_{j=0}^{M-1} |S_j\rangle\langle S_j| - \sum_{j=0}^{M-1} \sum_{k=1}^{N-3} |L_{jk}\rangle \langle L_{jk}| - |F\rangle\langle F|. \quad (68)
\]

Then \( \text{rank} \rho_A = M \) and, if \( N > M \), \( \text{rank} \rho_B = M + 1 \).

**Proof.** By a direct tedious computation, which we omit, we find that \( \rho_A = Z/2M \), where \( Z \) is the circulant matrix with the first row \([4M - 2, M - 2, -2, -2, \ldots, -2, M - 2]\). Hence \( \det Z = \prod_\zeta f(\zeta) \), where the product is taken over all \( M \)th roots of unity, \( \zeta \), and \( f(t) \) is the polynomial

\[
f(t) = 4M - 2 + (M - 2)t - 2t^2 - 2t^3 - \cdots - 2t^{M-2} + (M - 2)t^{M-1} = M(4 + t + t^{M-1}) - 2(1 + t + t^2 + \cdots + t^{M-1}). \quad (69)
\]
Since \( f(1) = 4M \) and \( f(\zeta) = M(4 + \zeta + \zeta^{-1}) \) when \( \zeta^M = 1 \) but \( \zeta \neq 1 \), all of these numbers are nonzero, and so rank \( \rho_\Lambda = M \).

Now assume that \( N > M \). By another straightforward tedious computation, we find that

\[
\rho_B = \frac{1}{N(N-2)} \begin{bmatrix} U & X \\ X^T & Y \end{bmatrix} = \frac{1}{N(N-2)} W, \tag{70}
\]

where \( U \) is a circulant matrix of order \( M \) with first row

\[
[(M + N - 5)N + 2, (M - 4)N + 2, (M - 5)N + 2, \ldots, (M - 5)N + 2, (M - 4)N + 2],
\]

and \( X \) and \( Y \) are matrices all of whose entries are equal to \( (M - 3)N + 2 \) and \( (M - 1)N + 2 \), respectively. Since the last \( N - M \) rows of \( W \) are all equal to \( \zeta \) to each other, it is clear that rank \( \rho_B \leq M + 1 \).

It remains to show that the matrix \( V \) of order \( M + 1 \), obtained by deleting the last \( N - M - 1 \) rows and columns of \( W \), is nonsingular. By subtracting \( \lambda := ((M - 3)N + 2)/((M - 1)N + 2) \) times the last column of \( W \) from all other columns, the problem reduces to proving that the matrix \( U' := (M - 1 + 2/N)(U - \lambda J) \) is nonsingular, where \( J \) is all-ones matrix. The matrix \( U' \) is also circulant with first row

\[
[N((M - 1)N - 2), (M - 5)N + 2, -4N, -4N, \ldots, -4N, (M - 5)N + 2].
\]

We can now prove that \( U' \) is nonsingular by using the same argument as for \( Z \).

The state \( \rho \) defined by Eq. (65) is \( \Gamma \)-invariant and, for \( N > 4 \), contains infinitely many product vectors in its kernel. Both assertions are immediate from the definition of the product vectors \( S_j, L_{jk} \), and \( F \). Apparently all these states \( \rho \) are extreme; we have verified it in several cases by using the Extremality Criterion (see Proposition 17). It follows easily from the above proof that

\[
\det \rho_A = 2^{M-1} \prod_{k=1}^{M-1} \left( 2 + \cos \frac{2\pi k}{M} \right). \tag{73}
\]

**Example 54** \((M = 3, N = 4)\) We shall construct a real 7-parameter family of \( 3 \times 4 \) extreme PPTES \( \rho \) of birank \((5, 5)\) such that \( \mathcal{R}(\rho) \) is a CES and ker \( \rho \) contains infinitely many product vectors.

The states \( \rho \) in this family are given by Eq. (44) with \( M = 3, N = 4 \) and \( R = 5 \). The blocks \( C_i \) are given by

\[
C_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -be/a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & a & 0 & 0 \\ a & f & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & bd & 0 & c \end{bmatrix}, \tag{74}
\]

where \( a, b, c, d, e, f, g \) are real parameters which are all nonzero except possibly \( f \) and \( g \).

Since \( |0, 2\rangle, |0, 3\rangle \in \ker \rho \), it is obvious that \( \rho \) is bad. If we identify \( \mathcal{H}_A \otimes \mathcal{H}_B \) with the space of \( 3 \times 4 \) complex matrices, then the five rows of \( C = [C_0 \ C_1 \ C_2] \) are represented by the matrices

\[
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -be/a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & a & 0 & 0 \\ a & f & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & bd & 0 & c \end{bmatrix}. \tag{75}
\]

It is obvious that these matrices are linearly independent, and so rank \( \rho = 5 \). It is easy to verify that the space spanned by these five matrices contains no matrix of rank one. Consequently, \( \mathcal{R}(\rho) \) is a CES and \( \rho \) is entangled. Since \( C_0^\dagger C_1, C_0^\dagger C_2 \) and \( C_1^\dagger C_2 \) are real symmetric matrices, we have \( \rho^T = \rho \). Hence \( \rho \) is PPT and rank \( \rho^T = 5 \). By Theorem 42 \( \rho \) is irreducible, and by Theorem 47 it is extreme.

We now specialize the values of the parameters in the above example and extend this particular case to obtain \( 3 \times N \) PPTES \( \rho^{(N)} \) for all \( N \geq 4 \). Each state \( \rho^{(N)} \) is extreme, \( \Gamma \)-invariant, has rank \( N + 1 \), its range is a CES, and its kernel contains infinitely many product vectors.
Example 55 \((N > M = 3)\) Let us denote by \(C_i^{(4)}, i = 0, 1, 2,\) the matrices \(74\) where we set \(a = b = c = d = e = 1\) and \(f = g = 0.\) For \(N > 4\) we define the \((N + 1) \times N\) matrices \(C_i^{(N)}, i = 0, 1, 2,\) as follows: \(C_0^{(N)} = I_{N-4} \oplus C_0^{(4)}\) and

\[
C_1^{(N)} = \begin{bmatrix}
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad C_2^{(N)} = \begin{bmatrix}
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} . \tag{76}
\]

Let \(\rho^{(N)} = \sum_{i,j} |i\rangle\langle j| \otimes C_i^{(N)} \otimes C_j^{(N)}, N \geq 4.\) It is not hard to verify that \(\rho^{(N)}\) is a \(3 \times N\) state. Indeed, we have

\[
\rho_A^{(N)} = \begin{bmatrix}
N - 2 & -1 & 0 \\
-1 & 2N - 4 & 2N - 7 \\
0 & 2N - 7 & 2N + 1
\end{bmatrix} > 0. \tag{77}
\]

Since \(C_i^{(N)} \otimes C_j^{(N)} = \mathbb{1} \oplus \text{diag}(2, 1, 1)\) with \(S \geq 0,\) and \(C_0^{(N)} \otimes C_0^{(N)} = I_{N-3} \oplus \text{diag}(0, 0, 0),\) we also have \(\rho_B^{(N)} > 0.\)

The matrices \(C_i^{(N)} \otimes C_j^{(N)}, i < j,\) are real and symmetric, and so \(\rho^{(N)}\) is \(\Gamma\)-invariant. In particular, \(\rho^{(N)}\) is a PPT state. Since the last two columns of \(C_0^{(N)}\) are 0, we have \(|0\rangle \otimes (\xi|N - 2\rangle + \eta|N - 1\rangle) \in \ker \rho^{(N)}\) for all \(\xi, \eta \in \mathbb{C}.\)

We claim that \(\rho^{(N)}\) has rank \(N + 1\) and that its range is a CES. Let \(C_i^{(N)} = [C_0^{(N)} \ C_1^{(N)} \ C_2^{(N)}].\) Each column of the \(3N \times (N + 1)\) matrix \(C_i^{(N)}\) represents a vector in \(\mathcal{H}.\) These vectors span the range of \(\tilde{\rho}^{(N)}\). We can represent these vectors by \(3 \times N\) matrices \(W_1, \ldots , W_{N+1}.\) It is easy to see that these matrices are linearly independent, and so \(\rho^{(N)}\) has rank \(N + 1.\) An arbitrary vector in \(\mathcal{R}(\rho^{(N)})\) is represented by a matrix \(\sum_i \xi_i W_i,\) where the \(\xi_i\) are complex scalars. It is now easy to verify that this matrix cannot have rank 1, i.e., \(\mathcal{R}(\rho^{(N)})\) is a CES.

As in the previous example, it follows that \(\rho^{(N)}\) is extreme. \(\Box\)

We now construct a new family of examples of PPTES which extends the one-parameter family \(\rho^{(N)}, N \geq 4,\) of Example 55. This new family depends on two discrete parameters \(M\) and \(N,\) and \(M - 3\) real parameters \(c_i, i = 3, 4, \ldots, M - 1.\) It consists of \(M \times N\) PPTES \(\rho^{(M, N)}\) of rank \(M + N - 2.\) Note that \(\rho^{(3, N)} = \rho^{(N)}\). We will prove that each state \(\rho^{(M, N)}\) is \(\Gamma\)-invariant, its range is a CES, and its kernel contains infinitely many product vectors. By using the Extremality Criterion (Proposition 17), we have verified that they are extreme for \(M + N \leq 27.\) We shall prove in Theorem 57 below that all of these states are extreme.

Example 56 \((M \geq 3, N \geq 4)\) We define the \((M + N - 2) \times N\) matrices

\[
C_i^{(M, N)} = \begin{bmatrix}
C_i^{(N)} \\
Q_i
\end{bmatrix}, \quad i = 0, 1, 2; \tag{78}
\]

\[
C_i^{(M, N)} = \begin{bmatrix}
P \\
Q_i
\end{bmatrix}, \quad i = 3, 4, \ldots, M - 1; \tag{79}
\]

where the \(C_i^{(N)}, i = 0, 1, 2,\) are given by Eqs. 76; the \((1, 2)\) entry of \(P\) is 1 and all other are 0; the first column of \(Q_0\) has all entries equal to 1 and all other columns are 0; \(Q_1 = Q_2 = 0\) and for \(i > 2\) each \(Q_i\) has exactly two nonzero entries, namely \((i - 2, 1)\)th entry is \(c_i\) and \((i - 2, 2)\)th entry is -1. The numbers \(c_i, i = 3, 4, \ldots, M - 1\) are required to be real, nonzero and distinct.

Let \(\rho^{(M, N)} = C \Gamma C\) where

\[
C := [C_0^{(M, N)} \ C_1^{(M, N)} \cdots C_{M-1}^{(M, N)}]. \tag{80}
\]

It is not hard to verify that \(\rho^{(M, N)}\) is an \(M \times N\) state of rank \(M + N - 2.\) The fact that rank \(\rho_B^{(M, N)} = N\) follows from Example 55 where we have shown that rank \(\rho_B^{(N)} = N.\) To prove that rank \(\rho_A^{(M, N)} = M,\) we first compute the
reduced density matrix

$$\rho_A^{(M,N)} = \begin{bmatrix} M + N - 5 & -1 & 0 & c_3 & c_4 & \cdots & c_{M-1} \\ -1 & 2N - 4 & 2N - 7 & 1 & 1 & 1 \\ 0 & 2N - 7 & 2N + 1 & 1 & 1 & 1 \\ c_3 & 1 & 1 & 2 + c_3^2 & 1 & 1 \\ c_4 & 1 & 1 & 1 & 2 + c_4^2 & 1 \\ \vdots \\ c_{M-1} & 1 & 1 & 1 & 1 & 2 + c_{M-1}^2 \end{bmatrix}.$$  \hspace{1cm} (81)

Since $$\begin{bmatrix} 1 & c_i & c_i^2 \end{bmatrix} \geq 0$$ for each $$i$$, it suffices to show that

$$\begin{bmatrix} \rho_A^{(N)} \\ ET \\ E \end{bmatrix} > 0,$$

where $$\rho_A^{(N)}$$ is as in Example [55] $$J_{M-3}$$ is all-ones matrix, and so is $$E$$ except that its first column is 0. By using [4, Proposition 8.2.3] and the fact that $$(I_{M-3} + J_{M-3})^{-1} = I_{M-3} - J_{M-3}/(M - 2)$$, one deduces that the inequality (82) is equivalent to

$$\begin{bmatrix} N - 2 & -1 & 0 \\ -1 & 2N - 5 + \frac{1}{M - 2} & 2N - 8 + \frac{1}{M - 2} \\ 0 & 2N - 8 + \frac{1}{M - 2} & 2N + \frac{1}{M - 2} \end{bmatrix} > 0.$$  \hspace{1cm} (83)

It suffices to verify the latter inequality for $$M = +\infty$$ only, which is straightforward. Finally, to prove that $$\text{rank} \rho_A^{(M,N)} = M + N - 2$$, we have to show that $$C$$ has full rank. It follows from Example [55] that the first $$N + 1$$ rows of $$C$$ are linearly independent. Then by inspecting the first columns of $$C_i^{(M,N)}$$, $$3 \leq i \leq M - 1$$, we see that $$C$$ indeed has full rank.

One can easily verify that the matrices $$C_i^{(M,N)}|C_j^{(M,N)}$$, $$i < j$$, are symmetric. Since they are also real, it follows that $$\rho^{(M,N)}$$ is $$\Gamma$$-invariant, and so $$\rho^{(M,N)}$$ is a PPT state.

Next we claim that the range of $$\rho_A^{(M,N)}$$ is a CES. Each column of $$C_i$$ represents a vector in $$\mathcal{H}$$. These $$M + N - 2$$ vectors span the range of $$\rho^{(M,N)}$$. We can represent these vectors by $$M \times N$$ matrices $$W_1, \ldots, W_{M+N-2}$$, respectively. An arbitrary vector in $$\mathcal{R}(\rho^{(M,N)})$$ is represented by a linear combination $$\Xi = \sum \xi_i W_i$$, where the $$\xi_i$$ are complex scalars. Now it suffices to verify that $$\Xi$$ cannot have rank 1. We briefly indicate the main steps. It follows from Example [55] that the claim holds if $$\xi_i = 0$$ for $$i > N + 1$$. Otherwise we must have $$\xi_i = 0$$ for $$4 \leq i \leq N - 2$$. Next one can deduce that $$\xi_{N-1} = \xi_N = \xi_{N+1} = 0$$, and then that also $$\xi_1 = \xi_2 = \xi_3 = 0$$. Thus only the variables $$\xi_{N+2}, \ldots, \xi_{M+N-2}$$ may be nonzero. However at least $$M - 4$$ of them, say $$\xi_{N+2}, \ldots, \xi_{M+N-3}$$ have to vanish because the $$c_i$$ are distinct. As rank $$W_i > 1$$ for each $$i$$, the last coefficient $$\xi_{M+N-2}$$ must also vanish.

Since the last two columns of $$C_0^{(M,N)}$$ are 0, we have $$|0 \rangle \otimes (\langle N | - 2 \rangle + \eta |N - 1\rangle) \in \ker \rho^{(M,N)}$$ for all $$\xi, \eta \in \mathbb{C}$$. Thus $$\rho^{(M,N)}$$ is bad.

**Theorem 57** For $$M, N > 2$$ we have $$\mathcal{E}^{M,N}_{M+N-2} \neq \emptyset$$, i.e., part (i) of Conjecture [2] is valid.

**Proof.** This is well known for $$M = N = 3$$. It suffices to prove that the states $$\rho = \rho^{(M,N)}$$ defined in Example [56] are extreme when $$M \leq N$$. For $$M = 3$$ this was shown in Example [55]. Hence, we may assume that $$M \geq 4$$.

For convenience, we set $$R = M + N - 2$$ and we switch the two blocks in Eqs. (78) and (79). Thus we define the $$R \times N$$ blocks

$$C_i = \begin{bmatrix} Q_i \\ C_i^{(N)} \end{bmatrix}, \quad i = 0, 1, 2;$$  \hspace{1cm} (84)

$$C_i = \begin{bmatrix} Q_i \\ P \end{bmatrix}, \quad i = 3, 4, \ldots, M - 1.$$  \hspace{1cm} (85)

The equality $$\rho = C^\dagger C$$ remains valid, with $$C := [C_0 \ C_1 \ \cdots \ C_{M-1}]$$. 
Let σ be a PPT state such that \( \mathcal{R}(\sigma) = \mathcal{R}(\rho) \) and \( \mathcal{R}(\sigma^T) = \mathcal{R}(\rho^T) \). Recall that \( \rho^T = \rho \). By Proposition \ref{prop:ptt_state} it suffices to show that σ must be a scalar multiple of ρ. Since ρ and σ have the same range, there exists an invertible matrix A such that

\[
\sigma = (AC)^tAC = C^tHC,
\]

where \( H = [h_{ij}] := A^tA > 0 \). Let us partition σ into \( M^2 \) blocks \( \sigma = [\sigma_{rs}]_{r,s=0}^{M-1} \) where \( \sigma_{rs} := C^t_iHC_s \). We use \( r[s] \) as the blockwise row [column] index. Then \( \sigma^T = [\sigma_{rs}]_{r,s=0}^{M-1} \) is the blockwise transpose of σ.

Let us partition the index set \( \{1, 2, \ldots, R\} \) into three subsets \( J_1, J_2, J_3 \) defined by inequalities \( i \leq M - 2, M - 2 < j < R - 2, k \geq R - 2 \), respectively.

The last two columns of \( C_0 \) are 0. Hence, the last two rows of the blocks of \( \sigma_{s0} = C^t_iHC_0 \) are 0. Since \( \sigma^T \geq 0 \), the last two rows of these blocks must be 0. By taking \( s = 1, 2 \) we see that \( h_{ij} = 0 \) for \( i \in J_2 \) and \( j \in J_3 \).

Assume that the index \( s > 2 \). Then all but the first two columns of \( C_0 \) are 0. Hence, all diagonal entries but the first two of \( \sigma_{ss} \) are 0. Consequently, the last \( N - 2 \) rows of the blocks of \( \sigma_{rs}, r = 1, 2 \), must be 0. Equating the last two rows of these blocks to 0, we see that \( h_{ij} = 0 \) for \( i \in J_1 \) and \( j \in J_3 \). If \( N > 4 \) then the equations provided by the middle \( N - 4 \) rows of the same blocks imply that \( h_{ij} = 0 \) for \( i \in J_1 \) and \( j \in J_2 \). The same conclusion is valid in the case \( N = 4 \) because in that case we have \( J_2 = \{M - 1\} \) and the first column of \( C_1 \) is 0, so the \( (N + 1) \) th row of \( \sigma^T \) must be 0.

Thus \( H = H_1 \oplus H_2 \oplus H_3 \), with square matrices \( H_1, H_2, H_3 \) of order \( M - 2, N - 3 \) and 3, respectively.

Let \( V_k, 1 \leq k \leq R \), be the \( M \times N \) matrix whose \( i \) th row is the \( k \) th row of \( C_{i-1}, i = 1, 2, \ldots, M \). These matrices represent vectors in \( \mathcal{H} \) and as such they form a basis of \( \mathcal{R}(\rho) \). An arbitrary vector in \( \mathcal{R}(\rho) \) is represented by the linear combination \( \Xi := \sum_k \xi_k V_k \), where \( \xi_k \) are complex scalars. Explicitly,

\[
\Xi = \begin{bmatrix}
\xi_1 + \cdots + \xi_{M-2} & \xi_{M-1} & \xi_M & \cdots & \xi_{R-6} \\
\xi_{M-1} & \xi_{M-2} + \xi_M & \xi_{M-1} + \xi_{M+1} & \xi_{R-7} + \xi_{R-5} \\
\xi_{M-1} & \xi_{M-2} + \xi_M & \xi_{M-1} + \xi_{M+1} & \xi_{R-7} + \xi_{R-5} \\
c_{3}\xi_1 & \xi_{M-2} - \xi_1 & 0 & 0 \\
\vdots & & & & \\
c_{M-1}\xi_{M-3} & \xi_{M-2} - \xi_{M-3} & 0 & 0 & \\
\xi_{R-5} & \xi_{R-4} & \xi_{R-3} & 0 & 0 \\
\xi_{R-6} + \xi_{R-4} & \xi_{R-5} & \xi_{R-3} & \xi_{R-2} & \xi_{R-1} \\
\xi_{R-6} + \xi_{R-4} + \xi_R & \xi_{R-5} + \xi_{R-3} + \xi_R & \xi_{R-4} + \xi_{R-2} + \xi_{R-1} & \xi_R & \xi_{R-1} + \xi_R \\
0 & 0 & 0 & 0 & 0 \\
\vdots & & & & \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}. \tag{87}
\]

The range of the matrix \( \sigma^T \) is spanned by its columns, which we represent by \( M \times N \) matrices. Let \( S_k \) be the \( M \times N \) matrix representing the \( k \) th column of \( \sigma^T \).

We have

\[
S_1 := \begin{bmatrix}
\sum H_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & h_{M-1,M-1} & h_{M,M-1} & \cdots & h_{R-4,M-1} & h_{R-3,M-1} & 0 & 0 \\
0 & h_{M-1,M-1} & h_{M,M-1} & \cdots & h_{R-4,M-1} & h_{R-3,M-1} & 0 & 0 \\
c_3H_1(1) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & & & & & & & & \\
c_{M-1}H_1(M-3) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \tag{88}
\]

where \( H_1(k) \) is the sum of the entries in the \( k \) th column of \( H_1 \) and \( \sum H_1 \) is the sum of all entries of \( H_1 \). Since \( \mathcal{R}(\sigma^T) = \mathcal{R}(\rho^T) \), this matrix must have the form \( \Xi \). From the equation \( S_1 = \Xi \) we obtain that \( \xi_i = 0 \) for \( i \geq M - 1, \xi_1 = \xi_2 = \cdots = \xi_{M-2} \) as well as that all off-diagonal entries of the first row of \( H_2 \) are 0, and that \( h_{M-1,M-1} = H_1(1) = H_1(2) = \cdots = H_1(M-2) \).
Next we have

\[ S_2 := \begin{bmatrix}
0 & h_{M-1,M-1} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & h_{M,M} & h_{M+1,M} & \cdots & h_{R-5,M} & h_{R-3,M} & 0 \\
0 & 0 & h_{M,M} & h_{M+1,M} & \cdots & h_{R-5,M} & h_{R-3,M} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}. \] (89)

From the equation \( S_2 = \Xi \) we obtain that \( \xi_i = 0 \) for \( i \neq M - 1 \), as well as that all off-diagonal entries of the second row of \( H_2 \) are 0, and that \( h_{M,M} = h_{M-1,M-1} \) (if \( N > 4 \)). Similarly, by using the fact that each of the matrices \( S_3, \ldots, S_{N-3} \) must have the form \( \Xi \), we conclude that \( H_2 = h_{M-1,M-1}I_{N-3} \).

From the equation \( S_{2N-3} = \Xi \) we obtain that \( h_{R-2,R} = h_{R-1,R} = 0 \) and \( h_{R,R} = h_{M-1,M-1} \). From \( S_{2N-1} = \Xi \) we obtain that \( h_{R-2,R-1} = 0 \) and \( h_{R-2,R-2} = h_{R,R} \), and from \( S_{2N} = \Xi \) that \( h_{R-1,R-1} = h_{R,R} \). Hence \( H_2 = h_{R,R}I_{N-3} \) and \( H_3 = h_{R,R}I_3 \).

From the equation \( S_{2N+1} = \Xi \) we first deduce that \( \xi_i = 0 \) for all \( i \neq M - 1, R \), and then that all off-diagonal entries of the last row of \( H_1 \) are 0. We also obtain the equations \( \xi_{M-1} = h_{M-2,M-2} \), \( \xi_R = h_{M-1,M-1} \), and \( \xi_{M-1} + \xi_R = 2h_{M-1,M-1} \) which imply that \( h_{M-2,M-2} = h_{M-1,M-1} \). From the matrix equation \( S_{2N+1} = \Xi \) we first deduce that \( \xi_i = 0 \) for \( i \geq M - 2 \), and then that all off-diagonal entries of the first row of \( H_1 \) are 0, and finally that \( h_{1,1} = h_{M-2,M-2} \). From the matrix equation \( S_{2N+1} = \Xi \) we first deduce that \( \xi_i = 0 \) for \( i \geq M - 2 \), and then that all off-diagonal entries of the second row of \( H_1 \) are 0, and finally that \( h_{2,2} = h_{M-2,M-2} \). Similarly, by using the equations \( S_{rN+1} = \Xi \), with \( r = 5, 6, \ldots, M - 1 \), we can deduce that \( H_1 = h_{1,1}I_{M-2} \). Since \( h_{1,1} = h_{M-1,M-1} \), we have \( H = h_{1,1}I_R \). Thus \( \sigma = h_{1,1} \rho \), which completes the proof.

\[ \square \]

VII. SOME OPEN PROBLEMS

The sum of two entangled extreme states is not necessarily an edge state. We shall construct an example. Let \( \rho_1 \) be any state belonging to the family [8, Eq. 108] of \( 3 \times 3 \) PPTES of rank four depending on four real parameters. We set \( \rho_2 = I_3 \otimes P \rho_1 I_3 \otimes I^1 \), where \( P \) is the cyclic permutation matrix with first row \([0, 0, 1]\). By Theorem [1], both \( \rho_1 \) and \( \rho_2 \) are entangled. One can easily verify that the PPT state \( \rho = \rho_1 + \rho_2 \) is a \( 3 \times 3 \) state of birank \((8, 8)\). It follows from [29, Theorem 2.3,(ii)] that \( \rho \) is not an edge state.

**Problem 1.** Can the sum of two entangled extreme states be separable?

Every separable state is a sum of pure product states, but such decomposition is not unique in general. (We assume that the summands are pairwise non-parallel.) We point out that the good \( M \times N \) separable states \( \sigma \) of rank \( r \leq M + N - 2 \) have this uniqueness property. Indeed, it follows from Theorem [24] (ii) that \( \sigma = \sum_{i=1}^{rN} |\psi_i\rangle \langle \psi_i| \), where \( |\psi_i\rangle \) are product vectors in general position. By Lemma [40] there are no other product vectors in \( R(\sigma) \). So the above decomposition of \( \sigma \) is unique. Every PPT state is a sum of extreme states.

**Problem 2.** Which PPTES have unique decomposition as a sum of extreme states?

Since most quantum-information tasks require pure states, the entanglement distillation (i.e., the task of producing pure entangled states) is a central problem in quantum information [8]. Mathematically, an entangled state \( \rho \) is \( n \)-distillable under LOCC if there exists a pure state \( |\psi\rangle \) of Schmidt rank two such that \( \langle \psi| (\rho^{\otimes n})^T |\psi\rangle < 0 \) [13]. A state \( \rho \) is distillable if it is \( n \)-distillable for some positive integer \( n \). Otherwise we say that \( \rho \) is non-distillable.

It follows easily from this definition that no PPTES is distillable [23]. It is also believed that entanglement distillation may fail for some NPT states [13]. Nevertheless, the PPTES of full rank can be used to activate the distillability of any NPT state [24, 42]. This means that, for any NPT state \( \rho_{A_1B_1} \), there exists a PPTES \( \rho_{A_2B_2} \) of full rank such that the bipartite state \( \rho_{A_1A_2:B_1B_2} := \rho_{A_1B_1} \otimes \rho_{A_2B_2} \) is 1-distillable. Hence, there exists a pure state \( |\psi\rangle \) of Schmidt rank two such that \( \langle \psi| (\rho_{A_1B_1} \otimes \rho_{A_2B_2})^T |\psi\rangle < 0 \). We can write \( \rho_{A_2B_2} = \sum_{i=1}^{k} \rho_i \), where \( \rho_i \) are extreme states. Necessarily \( k > 1 \) because extreme states cannot have full rank [1]. We deduce that for some \( i \) we have \( \langle \psi| (\rho_{A_1B_1} \otimes \rho_i)^T |\psi\rangle < 0 \). Therefore it suffices to use extreme states as the activators in entanglement distillation.

It is known that the distillable entanglement is upper bounded by the distillable key [25]. So extreme states can activate the distillable key of NPT states. Though any PPTES have zero distillable entanglement, there are PPTES with positive distillable key [25]. We may further ask

**Problem 3.** Can an entangled extreme state produce distillable key?

In connection with Theorem [50] we raise the following problem.

**Problem 4.** Construct good \( M \times N \) PPTES of rank \( M + N - 2 \) when \( N \geq M \geq 4 \).

We propose one more problem about extreme states.

**Problem 5.** If \( \rho \) is a strongly extreme state, is \( \rho^2 \) also strongly extreme?
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VIII. APPENDIX

Several proofs in the main part of the paper (Proposition 25 and Theorems 52 and 54) rely on two important facts related to the Bézout’s theorem. Our objective here is to state and prove these facts.

As both proofs make use of the linear projections in projective space, we shall first sketch their definition. Let $V \subseteq \mathbb{C}^{n+1}$ be a vector subspace of dimension $m + 1$ and $L \subseteq \mathbb{P}^n$ the projective subspace associated to $V$; its points are the one-dimensional subspaces of $V$. Let us choose $n - m$ linear forms $l_k: \mathbb{C}^{n+1} \to \mathbb{C}, \ k = 1, \ldots, n - m$ such that $V = \cap_k \ker l_k$. Then the map $\pi: \mathbb{P}^n \setminus L \to \mathbb{P}^{n-m-1}$ defined by $\pi(Cx) = [l_1(x) : \ldots : l_{n-m}(x)]$ is regular, and we refer to it as the projection with center $L$. It can be described geometrically as follows. We first fix a subspace $W \subseteq \mathbb{C}^{n+1}$ of dimension $n - m$ such that $V \cap W = 0$. Our projective space $\mathbb{P}^{n-m-1}$ will be the subspace of $\mathbb{P}^n$ associated to $W$. If $x \in \mathbb{C}^{n+1} \setminus V$ then the vector subspace spanned by $V$ and $x$, of dimension $m + 1$, meets $W$ in a one-dimensional subspace, say $Cy$, and we define $\pi(Cx) = Cy$.

The proof of the first proposition is due to David McKinnon 34.

Proposition 58 Let $X$ be an irreducible projective subvariety of $\mathbb{P}^n$, of dimension $k$, and let $L$ be a linear subspace of dimension $m$ (strictly less than $n - k$) such that $L \cap X$ is finite. Then there is a linear subspace $M$, containing $L$, whose intersection with $X$ is again finite, and such that the dimension of $M$ is exactly $n - k$.

Proof. Consider the linear projection $\pi: \mathbb{P}^n \setminus L \to \mathbb{P}^{n-m-1}$ with center $L$. The set $X^0 = X \setminus L$ is open in $X$ and so it is a quasi-projective variety. Since $\pi$ is a regular map, so is its restriction $f: X^0 \to \mathbb{P}^{n-m-1}$. The fibres of $f$ are the intersections of $X^0$ with linear subspaces of dimension $m + 1$ containing $L$. Since $m < n - k$, we deduce that $n - m - 1 \geq k$, so that $\mathbb{P}^{n-m-1}$ has dimension at least as large as the dimension of $X$.

Let $Y$ be the Zariski-closure of the set $f(X^0)$. If $Y$ is not equal to $\mathbb{P}^{n-m-1}$, then $f$ is not onto, and so there is some linear subspace of dimension $m + 1$, containing $L$, whose intersection with $X$ is contained in $L$, and therefore finite. If $Y$ is equal to $\mathbb{P}^{n-m-1}$, then there is some nonempty Zariski-open subset of $Y$ contained in $f(X^0)$ such that the dimension of the fibre over any of its points plus the dimension of $Y$ equals the dimension of $X$, see 35 Corollary (3.15)]. Consequently, all these fibres have dimension zero, which means that there exist linear subspaces of dimension $m + 1$ containing $L$ whose intersection with $X$ is finite. In either case, if $m$ is strictly less than $n - k$, we can construct a linear subspace of dimension $m + 1$ that contains $L$ and still intersects $X$ in a finite set of points. Continuing in this manner, we can construct the desired space $M$.

The question whether the theorem below is valid was posed on MathOverflow by the second author (under additional hypothesis that $X$ is smooth). The first proof was given by Mike Roth 38. Subsequently, together with Mike, we found another proof given below. We say that a projective subvariety $X$ of $\mathbb{P}^n$ is degenerate (in $\mathbb{P}^n$) if it is contained in a hyperplane of $\mathbb{P}^n$. 

Theorem 59 Let \( X \subseteq \mathbb{P}^n \) be an irreducible complex projective variety embedded in the \( n \)-dimensional projective space. Let \( k \) be the dimension of \( X \) and \( d \) its degree. Let \( L \subseteq \mathbb{P}^n \) be a linear subspace of dimension \( n - k \) and \( Z = L \cap X \). If \( X \) is not contained in any hyperplane of \( \mathbb{P}^n \) and \( Z \) is finite of cardinality \( d \), then \( Z \) spans \( L \).

Proof. Let \( M \) be the linear subspace spanned by \( Z \). Assume that \( M \neq L \), and let \( m (\leq n - k) \) be its dimension. We use induction on \( m \) to show that \( X \) is degenerate (which contradicts our hypothesis). The inductive steps will make use of a projection \( \pi : \mathbb{P}^n \setminus \{p\} \to \mathbb{P}^{n-1} \) from a suitably chosen point \( p \in M \setminus Z \). We define the maps \( f : X \to Y := \pi(X) \) and \( g : X \to \mathbb{P}^{n-1} \) to be the restrictions of \( \pi \). Since \( L \) and \( X \) intersect transversely at any \( z \in Z \), the differential of \( g \) at \( z \) will be injective. Hence, there will exist an open connected neighborhood \( W_z \subseteq X \) of \( z \) in analytic (i.e., ordinary) topology such that \( f(W_z) = g(W_z) \) is a complex submanifold of \( \mathbb{P}^{n-1} \) of dimension \( k \) and \( f \) induces an isomorphism of \( W_z \) and \( f(W_z) \) as complex manifolds.

First, suppose that \( m = 1 \), i.e., \( M \) is a projective line. Then we can choose for \( p \) any point in \( M \setminus Z \). Let \( z \in Z \) and observe that the fibre of \( f \) over the point \( y_0 = f(z) \) is \( Z \).

We claim that \( Y \) is a cone with vertex \( y_0 \). Let \( y \) be any other point of \( Y \) and \( \ell \) the line in \( \mathbb{P}^{n-1} \) joining \( y_0 \) and \( y \). Suppose that \( Y \cap \ell \) is a finite set, and let \( P \) be the 2-plane \( \{p\} \cup \pi^{-1}(\ell) \). Since each fibre of \( f \) is finite, \( P \cap X \) is a finite set. As \( P \supseteq Z \cup f^{-1}(y) \) and \( f^{-1}(y) \neq \emptyset \), we have \( |P \cap X| \geq d + 1 \). By Proposition 53 there is a \((n - k)\)-plane \( Q \) such that \( Q \supseteq P \) and \( Q \cap X \) is finite. This contradicts the Bezout’s theorem because \( |Q \cap X| > d \). We conclude that the set \( Y \cap \ell \) must be infinite, and so \( \ell \subseteq Y \) and our claim is proved.

We next claim that \( Y \) is locally irreducible near \( y_0 \) in the analytic topology. Suppose on the contrary that \( Y \) is locally reducible near \( y_0 \), and let \( h_1, \ldots, h_s \) be local analytic equations near \( y_0 \) cutting out an analytic component \( V \). The fact that \( Y \) is a cone with vertex \( y_0 \) then implies (by observing that the cone remains invariant under scaling) that the homogeneous pieces of each \( h_i \) vanish on \( V \), and hence cut out \( V \). Since the homogeneous pieces are homogeneous polynomials, this now implies that \( Y \) is reducible in the Zariski topology. Since \( Y \) is irreducible this is a contradiction and establishes the second claim.

From the preliminary remarks made above it follows that there exists an open connected neighborhood \( U \subseteq \mathbb{P}^{n-1} \) of \( y_0 \) in analytic topology such that \( U \cap Y \) is a union of \( d \) complex \( k \)-dimensional submanifolds (one for each point \( z \in Z \)) passing through \( y_0 \). Since the local analytic structure of \( Y \) near \( y_0 \) is a union of \( d \) submanifolds, the only way that \( Y \) can be irreducible near \( y_0 \) in the analytic topology is if all the submanifolds are the same, so that \( Y \) is smooth at \( y_0 \). This implies that \( Y \) is a linear space of dimension \( k \). Hence the Zariski closure of \( f^{-1}(Y) \) is a linear space of dimension \( k + 1 \). As this linear space contains \( X \) and \( k + 1 < n \) (since \( 1 = m < n - k \)), \( X \) is degenerate.

Next, suppose that \( m > 1 \). In this case we choose for \( p \) a point in \( M \setminus Z \) which is not on any line joining two points of \( Z \). Observe that, for \( z \in Z \), the fibre of \( f \) over \( f(z) = \{z\} \). Let \( W_z \) be chosen as in the beginning of the proof. Since the set \( X' = X \setminus W_z \) is compact in analytic topology, its image \( f(X') \) is also compact. Hence, the set \( U = Y \setminus f(X') \) is open in \( Y \) in analytic topology. Note that \( f(z) \in U \) and that for each \( y \in U \) there is a unique \( x \in W_z \) such that \( f(x) = y \) and \( f^{-1}(y) = \{x\} \). On the other hand there exists a nonempty Zariski open subset \( V \) of \( Y \) such that for all \( y \in V \) the fibre \( f^{-1}(y) \) consists of exactly \( d' \) points, where \( d' \) is the degree of \( f \). Since \( V \) is dense in \( Y \), not only in Zariski but also in analytic topology [33, Theorem (2.33)], we conclude that \( U \cap V \neq \emptyset \). Consequently, we have \( d' = 1 \) and the image \( Y \) must have degree \( d \) in \( \mathbb{P}^{n-1} \).

The image \( f(L \setminus \{p\}) \) is a linear subspace of \( \mathbb{P}^{n-1} \) of dimension \((n - 1) - k \). Its intersection with \( Y \) is the set \( f(Z) \) of cardinality \( d \). The span of \( f(Z) \) is the linear space \( f(M \setminus \{p\}) \) of dimension \( m - 1 \). By the induction hypothesis \( Y \) is degenerate in \( \mathbb{P}^{n-1} \), and so \( X \) is degenerate in \( \mathbb{P}^n \).

This completes the proof that \( X \) is degenerate, and we can conclude that our assumption is false, i.e., we must have \( M = L \). \( \square \)