ON ACTION FUNCTIONALS FOR INTERACTING BRANE SYSTEMS

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We present an action functional and derive equations of motion for a coupled system of a bosonic Dp–brane and an open string ending on the Dp-brane. With this example we address the key issues of the recently proposed method for the construction of manifestly supersymmetric action functionals for interacting superbrane systems. We clarify, in particular, how the arbitrariness in sources localized on the intersection is related to the standard description of the flat D-branes as rigid planes where the string for endpoints ‘live’.

Introduction

Recently an approach to obtain a supersymmetric action functional for a coupled system of interacting superbranes has been proposed. The aim of the study was to construct a basis for a future search for new solitonic solutions of worldvolume equations of superbranes and of supergravity equations with a complicated system of brane sources.

The system of open fundamental superstring ending on a super–Dp-brane was considered in as a generic case of interacting brane system in the frame of the ‘brane democracy’ concept. On the other hand, the manifestly supersymmetric description of such system can be useful in a search for a quasiclassical effective action of the system of coincident branes which are conjectured to carry non-Abelian gauge fields (non-Abelian Dirac–Born–Infeld action).
In this contribution we would like to discuss the key issues of our approach by studying a pure bosonic limit of the interacting brane system. We elaborate the case of open bosonic string with endpoints living on the dynamical bosonic Dp–brane and discuss the relation of the approach with the standard description where the Dp-branes are treated as rigid p-dimensional hyperplanes.

1 Standard description of open string

To clarify the problem and to establish the notations let us begin with the standard consideration of the open bosonic string in flat D–dimensional space–time \( \mathcal{M}^D \). The action can be written as

\[
S_{\text{str}} = \frac{1}{2} \int_{\mathcal{M}^2} d^2 \xi \sqrt{-g} g^{mn}(\xi) \partial_m \hat{X}^m(\xi) \partial_n \hat{X}^n(\xi) \eta_{mn}. \tag{1}
\]

Here \( \{ (\xi^m) \} = \{ (\tau, \sigma) \} \) \( (m = 0, 1) \) are local coordinates on the string world-sheet \( \mathcal{M}^2 \), \( \hat{X}^m(\xi) \) \( (m = 0, \ldots, (D - 1)) \) are coordinate (embedding) functions which define (locally) an embedding of the worldsheet \( \mathcal{M}^2 \) into the D–dimensional space–time (target) \( \mathcal{M}^D \)

\[
X^m = \hat{X}^m(\xi) : \mathcal{M}^2 \to \mathcal{M}^D. \tag{2}
\]

The flat metric of \( \mathcal{M}^D \) is chosen to be 'mostly minus' \( \eta_{mn} = \text{diag}(+, - , \ldots, -) \).

When the worldsheet has a boundary \( \partial \mathcal{M}^2 \), the variation of the action (1) acquires the form

\[
\delta S_{\text{str}} = \frac{1}{2} \int_{\mathcal{M}^2} d^2 \xi \sqrt{-g} \delta g^{mn}(\xi) \left( \partial_m \hat{X}^m \partial_n \hat{X}^n - \frac{1}{2} g_{mn} g^{pq} \partial_p \hat{X}^m \partial_q \hat{X}^n \right)
- \int_{\partial \mathcal{M}^2} d^2 \xi \delta \hat{X}^m \partial_m \left( \sqrt{-g} g^{mn} \partial_n \hat{X}^n \right) - \int_{\partial \mathcal{M}^2} d^2 \xi \epsilon_{mn} \sqrt{-g} g^{nk} \partial_k \hat{X}^m \delta \hat{X}^n. \tag{3}
\]

The variation with respect to the auxiliary intrinsic metric \( g_{mn}(\xi) \) results in the equation

\[
\partial_m \hat{X}^m(\xi) \partial_n \hat{X}^n(\xi) = \frac{1}{2} g_{mn}(\xi) g^{pq}(\xi) \partial_p \hat{X}^m(\xi) \partial_q \hat{X}^n(\xi). \tag{4}
\]

One easily finds that the trace part of Eq. (4) is satisfied identically. This is the Noether identity reflecting the gauge Weyl symmetry of the action (1)

\[
g_{mn}(\xi) \to g'_{mn}(\xi) = e^{A} g_{mn}(\xi), \quad X^m(\xi) \to X^m(\xi). \tag{5}
\]
With this symmetry one can fix the gauge
\[ g^{mn}(\xi) \partial_m \hat{X}^m(\xi) \partial_m \hat{X}_n(\xi) = 2. \]  \hfill (6)
Then Eq. (4) becomes
\[ g_{mn}(\xi) = \partial_m \hat{X}^m(\xi) \partial_m \hat{X}_n(\xi), \]  \hfill (7)
and implies that the metric is induced by the embedding.

The usual way to deal with the coordinate variation of the action (1) (presented in the second line of Eq. (3)) is to assume that the bulk and the boundary inputs should vanish separately. Then one arrives at the well known result that

- proper equations of motion have the form
  \[ \partial_m \left( \sqrt{-g} g^{mn}(\xi) \partial_n \hat{X}^m(\xi) \right) = 0; \]  \hfill (8)
- the boundary conditions should be chosen in a way which provides
  \[ d\xi^m \varepsilon_{mn} \sqrt{-g} g^{nk} \partial_k \hat{X}^m |_{\partial M^2; \xi^m = \hat{\xi}^m(\tau)} = 0 \]  \hfill (9)
  where the functions \( \hat{\xi}^m(\tau) \) define parametrically the 'embedding' of (a connected piece of) the world sheet boundary \( \partial M^2 \) into the worldsheet \( \xi^m = \hat{\xi}^m(\tau) : \partial M^2 \rightarrow M^2. \)  \hfill (10)

To satisfy Eq. (9) in a simple way one can assume that for some part \( \hat{X}^i(\xi) \) of the embedding functions
\[ \hat{X}^m = (\hat{X}^a, \hat{X}^i), \quad a = 0, \ldots, p, \quad i = (p + 1), \ldots, (D - 1) \]  \hfill (11)
the Neumann boundary conditions are imposed
\[ d\xi^m \varepsilon_{mn} \sqrt{-g} g^{nk} \partial_k \hat{X}^a |_{\xi^m = \hat{\xi}^m(\tau)} = 0, \quad \Leftrightarrow \quad \partial_\perp \hat{X}^a |_{\partial M^2} = 0, \]  \hfill (12)
while the remaining equations from the set (9) are satisfied due to the Dirichlet boundary conditions. The latter can be treated as ones imposed on the variation \( \delta \hat{X}^i(\hat{\xi}^m(\tau)) = 0 \) or, equivalently, as the boundary conditions imposed on the coordinate functions \( \hat{X}^i(\xi) \) directly
\[ \hat{X}^i(\hat{\xi}^m(\tau)) = \hat{X}^i(= \text{const}) \quad \Leftrightarrow \quad \delta \hat{X}^i(\hat{\xi}^m(\tau)) = 0 \]  \hfill (13)
Note that for the Dirichlet boundary problem the equations (8) appear formally only on the proper open subset of the worldsheet, i.e. only outside the boundary, because the variation on the boundary vanishes just due to Eq. (13). In the Neumann problem we can regard Eq. (8) as valid on the whole worldsheet. But in this case we shall assume that the boundary conditions (12) are imposed after the variation of the action has been performed (as we did not restrict the variations by the condition (12) when derived the equations of motion).

The boundary conditions (12), (13) certainly break Lorentz invariance for $p \neq (D - 1)$. Such a breaking reflects the existence of some $d = (p + 1)$-dimensional defects where the string endpoints move. These are worldvolumes of Dp-branes considered as flat hyperplanes.

### 2 Extended variational problem

However, we can proceed in a different manner which we will call extended variational problem or extended variational approach.

Let us introduce the current density distribution with support on the boundary of the worldsheet ($d^{2}\xi \wedge d^{2}\xi = d^{2}\xi_{\mu\nu}$)

$$j_{1} = d^{2}\xi_{m} \varepsilon_{mn} \int_{\partial M^{2}} d^{2}\xi_{n}^{}(\tau) \delta^{2}(\xi - \tilde{\xi}(\tau)) = \varepsilon_{mn} d^{2}\xi_{m} j^{n}, \quad (14)$$

Then one easily find that

$$\int_{M^{2}} j_{1} \wedge \hat{A}_{1} = \int_{\partial M^{2}} \hat{A}_{1} \quad (15)$$

holds for any one-form $\hat{A}_{1} = d^{2}\xi_{m} A_{m}(\xi)$ defined on the worldsheet, $\hat{A}_{1} = d^{2}\xi_{m}(\tau) A_{m}(\hat{\xi}(\tau))$.

With the use of $j_{1}$ the coordinate variation of the string action can be written in the form

$$\delta S_{str} = - \int_{M^{2}} \left( d^{2}\xi \partial_{m} \left( \sqrt{-g}g^{mn} \partial_{n} \dot{X}^{m} \right) - j_{1} \wedge d^{2}\xi_{m} \varepsilon_{mn} \sqrt{-g}g^{nk} \partial_{k} \dot{X}^{m} \right) \delta \dot{X}^{m}. \quad (16)$$

The equations of motion are (7) and

$$d^{2}\xi \partial_{m} \left( \sqrt{-g}g^{mn}(\xi) \partial_{n} \dot{X}^{m}(\xi) \right) = j_{1} \wedge d^{2}\xi_{m} \varepsilon_{mn} \sqrt{-g}g^{nk} \partial_{k} \dot{X}^{m} \quad (17)$$

$$\Leftrightarrow \partial_{m} \left( \sqrt{-g}g^{mn} \partial_{n} \dot{X}^{m} \right) = - \int_{\partial M^{2}} d^{2}\xi_{m} \varepsilon_{mn} \sqrt{-g}g^{nk} \partial_{k} \dot{X}^{m} \delta^{2}(\xi - \tilde{\xi}(\tau)).$$
Thus no boundary conditions appear, but the equations of motion acquire a source localized at the boundary of the worldsheet. The result (17) of such an 'extended variational problem' looks different from the standard one (8), (12), (13).

To clarify the situation let us, for the moment, fix the conformal gauge and accept the parametrization where the string endpoints correspond to the values $\sigma = 0$ and $\sigma = \pi$. Then the standard approach produces the free (Laplace) equations with boundary conditions

$$
(\partial^2_\tau - \partial^2_\sigma) \hat{X}^a(\tau, \sigma) = 0, \quad \partial_\sigma \hat{X}^a|_{M^2} = 0,
$$

(18)

$$
(\partial^2_\tau - \partial^2_\sigma) \hat{X}^i(\tau, \sigma) = 0, \quad \hat{X}^i(\tau, 0) = \hat{X}^i_0, \quad \hat{X}^i(\tau, \pi) = \hat{X}^i_\pi,
$$

(19)

while the extended variational problem results in the equations with sources for both types of the coordinate functions

$$
(\partial^2_\tau - \partial^2_\sigma) \hat{X}^a(\tau, \sigma) = \partial_\sigma \hat{X}^a (\delta(\sigma) - \delta(\sigma - \pi))
$$

(20)

$$
(\partial^2_\tau - \partial^2_\sigma) \hat{X}^i(\tau, \sigma) = \partial_\sigma \hat{X}^i (\delta(\sigma) - \delta(\sigma - \pi))
$$

(21)

It is evident that the problem (18) can be obtained as a particular case of (20) because, after imposing an additional condition $\partial_\sigma \hat{X}^a|_{M^2} = 0$, the r.h.s. of Eq. (20) vanishes. As in such a way we arrive at the free equations on the whole worldsheet, this corresponds just to the result of Neumann variation problem with the supposition that the boundary conditions are imposed after the variation has been performed (i.e. imposed 'by hand').

This does not hold for (19), because $\hat{X}^i|_{M^2} = \text{const}$ does not imply $\partial_\sigma \hat{X}^i|_{M^2} = 0$. Thus the value of Laplace operator acting on the coordinate function is not indefinite on the boundary, as it is in the classical Dirichlet problem, but is determined by the value of the derivative $\partial_\sigma \hat{X}^i|_{M^2}$.

The fact that the straightforward application of the extended variational approach does not produce the Dirichlet boundary problem even as a particular case does not look not so surprising if one remembers that it assumes that the variations of the coordinate functions are unrestricted everywhere, while the Dirichlet boundary conditions (13) come just as restrictions on the variations.

3 Lagrange multiplier approach for Dirichlet problem.

If one would like to reproduce the Dirichlet problem (13) from the extended variational approach, one can try to incorporate the boundary conditions (13) with Lagrange multiplier one-form $P^i_\tau = d\tau P^i(\tau)$ into the action written in the
conformal gauge $g_{mn} = \partial_m X^m \partial_n X^m = \eta_{mn} \equiv \text{diag}(+1, -1)$ (cf. [1])

$$S_{str} = \frac{1}{2} \int_{\mathcal{M}^2} d^2 \xi \left( \partial_m \hat{X}^a \partial^m \hat{X}_a - \partial_m \hat{X}^i \partial^m \hat{X}^i \right) + \int_{\partial \mathcal{M}^2} d\tau P_i \left( \hat{X}^i(\xi(\tau)) - \hat{X}^i \right)$$

(22)

Then the Dirichlet boundary conditions are produced by the variation with respect to Lagrange multiplier $P_i(\tau)$ or, more precisely, two Lagrange multipliers $P^0_i(\tau)$ and $P^\pi_i(\tau)$ (each living on the worldline of the corresponding endpoint of the string), while the dynamical equations for the coordinate functions $\hat{X}^i$ read

$$\left( \partial_\tau^2 - \partial_\sigma^2 \right) \hat{X}^i(\tau, \sigma) = \left( \partial_\sigma \hat{X}^i - P_i^0 \right) \delta(\sigma) - \left( \partial_\sigma \hat{X}^i - P_i^\pi \right) \delta(\sigma - \pi).$$

(23)

What one can see from the very beginning is the arbitrariness in the Lagrangian multipliers $P^0_i(\tau)$ which cannot be fixed by any gauge symmetry of the action. So one can conclude that the Lagrange multipliers carry some degrees of freedom and some doubts may arise concerning the applicability of the Lagrangian multiplier method to the problem.

However if one notes that
i) Eq. (23) becomes free when considered outside the boundary (i.e. on the open proper subset of the string worldsheet),
ii) the boundary conditions (13) are produced as equations of motion for Lagrange multipliers,
iii) the action of the Laplace operator on the coordinate functions $X^i$ is indefinite on the boundary just due to the arbitrariness of the Lagrange multiplier, one easily concludes that the Dirichlet boundary problem is indeed reproduced.

When one can find a solution for it (as it can be done in the case of string), then, substituting it into the equations, one can fix the value of the Lagrange multiplier $P^i(\tau)$.

This way to reproduce the Dirichlet boundary problem could be regarded as an artificial one. However it just provides the possibility to describe the system of open (super)string and dynamical (super-)Dp–brane and, more generally, of open (super–)brane(s) ending on closed 'host' (super–)brane(s) on the level of (quasi)classical action functional [1].

4 Dynamical string with the ends on dynamical D-brane

The action for free Dp-brane [3, 8] has the form

$$S_{Dp} = \int_{\mathcal{M}^{p+1}} d^{p+1} \zeta \sqrt{|G|}, \quad |G| = (-)^p \text{det}(G_{\tilde{m} \tilde{n}}),$$

(24)
where
\[ G_{\tilde{m}\tilde{n}} \equiv \partial_{\tilde{m}} \tilde{X}^m(\zeta) \partial_{\tilde{n}} \tilde{X}^m(\zeta) - \mathcal{F}_{\tilde{m}\tilde{n}} \] (25) is the non-symmetric \( \text{open string induced metric} \) \( ^9 \), the coordinate functions \( \tilde{X}^m(\zeta) \) define (locally) an embedding of the \( d = (p + 1) \)-dimensional world-volume \( \mathcal{M}^{p+1} = \{(\zeta)\} (\tilde{m} = 0, \ldots, p) \) of the Dp-brane into the D(=10)-dimensional space–time \( \mathcal{M}^D \)
\[ X^m = \tilde{X}^m(\zeta) : \mathcal{M}^{p+1} \Rightarrow \mathcal{M}^D, \] (26)
\[ \mathcal{F} \equiv dA = \frac{1}{2} d\zeta^\tilde{m} d\zeta^\tilde{n} \mathcal{F}_{\tilde{m}\tilde{n}} \] (27) is the field strength of the worldvolume gauge field of the Dp–brane \( A = d\zeta^\tilde{m} A_{\tilde{m}}(\zeta) \).

With this notation the variation of the free Dp-brane action (24) can be written in the form
\[ \delta S_{Dp} = + \int_{\mathcal{M}^{p+1}} d^{p+1}\zeta \partial_{\tilde{m}} \left( \sqrt{|G|} G^{-1[\tilde{m}\tilde{n}]} \right) \delta \tilde{A}_{\tilde{n}}(\zeta) \] (28)
\[ - \int_{\mathcal{M}^{p+1}} d^{p+1}\zeta \partial_{\tilde{m}} \left( \sqrt{|G|} G^{-1(\tilde{m}\tilde{n})} \partial_{\tilde{n}} \tilde{X}^m \right) \delta \tilde{X}^m, \]
where
\[ G^{-1 mn} = G^{-1(mn)} + G^{-1[mn]} \]
is the matrix inverse to the open string metrics \( ^{23} \).

To describe the interacting system of the open string and the dynamical D-brane on the level of action functional one has to provide the counterpart of (13) with \textit{coordinate functions of D-brane} instead of constants \( \tilde{X}^i \). Namely we have to impose an identification
\[ \tilde{X}^m(\hat{\zeta}(\tau)) = \hat{X}^m(\tilde{\zeta}(\tau)), \] (29)
where \( \zeta^{\tilde{m}} = \hat{\zeta}^{\tilde{m}}(\tau) \) defines an embedding of the worldline(s) of string endpoints (i.e. the boundary of the worldsheet) into the worldvolume of the Dp-brane
\[ \zeta^{\tilde{m}} = \hat{\zeta}^{\tilde{m}}(\tau) : \partial \mathcal{M}^2 \Rightarrow \mathcal{M}^{p+1}. \] (30)

The problem is that, when \( ^{23} \) is taken into account the variations \( \delta \hat{X}^m_{\tilde{m}} \)
and \( \delta \hat{X}^m_{\tilde{m}}(\zeta) = \hat{\zeta}(\tau) \) cannot be treated as independent on the intersection \( \mathcal{M}^{1+p} \cup \mathcal{M}^2 = \partial \mathcal{M}^2 \). Indeed, varying \( ^{23} \) one finds
\[ \delta \hat{X}^m_{\zeta = \xi(\tau)} + \delta \hat{\zeta}^{\tilde{m}}(\tau) \partial_{\tilde{m}} \hat{X}^m(\hat{\zeta}(\tau)) = \delta \hat{X}^m_{\zeta = \xi(\tau)} + \delta \hat{\zeta}^{\tilde{m}}(\tau) \partial_{\tilde{m}} \hat{X}^m(\hat{\zeta}(\tau)). \] (31)
A method to implement (29) into the action principle was proposed in [1]. It consists in imposing the identification (29) with Lagrange multipliers on $P_m = d\tau \mathcal{P}_m(\tau)$ into the action (cf. with (22)). Thus the action for a coupled system of open string and dynamical Dp-brane can be written as

$$S = S_{str} + S_{Dp} + S_{int}. \quad (32)$$

Here $S_{str}$ is determined by [1] but with open worldsheet whose boundary is a set of two (or more!) worldlines which are embedded into the worldvolume of the Dp–brane [30]. $S_{Dp}$ is determined by Eq. (24). The last term in (32)

$$S_{int} = \int_{\partial M^2} \left( \hat{A} + P_{\hat{m}} \left( \hat{X}^\mu (\hat{\xi}) - \hat{X}^\mu (\hat{\zeta}) \right) \right), \quad (33)$$

describes, in particular, the interaction of the string endpoints with the worldvolume gauge field of the Dp–brane $A = d\zeta^\hat{m} A_{\hat{m}}(\zeta)$ whose pull-back on the intersection is $\hat{A} = d\zeta^\hat{m}(\tau) A_{\hat{m}}(\hat{\zeta}(\tau)) = d\tau \partial_{\tau} \zeta^\hat{m} A_{\hat{m}}(\hat{\zeta})$.

The variation of the interaction term is

$$\delta S_{int} = \int_{\partial M^2} \delta P_{\hat{m}} \left[ \hat{X}^\mu (\hat{\xi}(\tau)) - \hat{X}^\mu (\hat{\zeta}(\tau)) \right] + \quad (34)$$

$$+ \int_{\partial M^2} P_{\hat{m}} \left[ \delta \hat{X}^\mu (\hat{\xi}(\tau)) - \delta \hat{X}^\mu (\hat{\zeta}(\tau)) \right] + \int_{\partial M^2} d\zeta^\hat{m}(\tau) \delta A_{\hat{m}} (\hat{\zeta}(\tau)) +$$

$$+ \int_{\partial M^2} \delta \hat{\zeta}^m(\tau) \left[ i_{\hat{m}} F - P_{\hat{m}} \partial_{\hat{m}} \hat{X}^\mu \right] +$$

$$+ \int_{\partial M^2} \delta \hat{\zeta}^m(\tau) \left[ P_{\hat{m}} \partial_{\hat{m}} \hat{X}^\mu - d\xi^\hat{p} \varepsilon_{\hat{p} \hat{m}_{p+1}} \sqrt{-g} g^{jk} \partial_k \hat{X}^\mu \partial_m \hat{X}^\nu \right].$$

The third and forth lines include the variation with respect to coordinate functions $\hat{\zeta}^\hat{m}(\tau)$ and $\hat{\zeta}^m(\tau)$ defining the embedding of the string boundary worldlines in the Dp–brane worldvolume [30] and in the worldsheet [10].

To vary the whole action it is convenient to introduce the following current density distribution form (see [1] and refs. therein)

$$j_p = d\zeta^\hat{m}_p \int_{\partial M^{p+1}} \delta \hat{\zeta}^\hat{m}(\tau) d\hat{p}^{p+1} \left( \zeta - \hat{\zeta}(\tau) \right) = (-)^p d\zeta^{\hat{m}_p} j_{\hat{m}_p}. \quad (35)$$

\[a\] In our conventions $d\zeta^{\hat{m}_1} \wedge \ldots \wedge d\zeta^{\hat{m}_{p+1}} \equiv \varepsilon^{\hat{m}_1 \ldots \hat{m}_{p+1}} (d\zeta)^{(p+1)} \equiv (-)^p \varepsilon^{\hat{m}_1 \ldots \hat{m}_{p+1}} d\zeta^{\hat{m}_1} \wedge \ldots \wedge d\zeta^{\hat{m}_{p+1}}, \ldots$
As \( j_p \wedge d\zeta^\hat{n} = d^{p+1}\zeta \ j^\hat{n} \), it is easy to see that
\[
\int_{\mathcal{M}^{p+3}} j_3 \wedge \hat{A}_1 = \int_{\partial \mathcal{M}^{p+2}} \hat{A}_1. \tag{36}
\]

With the use of (35) and the worldsheet current density distribution (14) one can write the variation of the action (32) in the form
\[
(\mathcal{G}_{\hat{m}\hat{n}} \equiv \partial_\hat{m} \hat{X}^\mu(\zeta) \partial_\hat{n} \hat{X}^\mu(\zeta) - \mathcal{F}_{\hat{m}\hat{n}} \tag{33}, \quad \mathcal{G}^{-1}_{\hat{m}\hat{n}} \mathcal{G}_{\hat{q}\hat{n}} = \delta^\hat{q}_{\hat{n}}) \]
\[
\delta S = \frac{1}{2} \int_{\mathcal{M}^2} d^2 \xi \sqrt{-g} \delta g^{\mu\nu}(\xi) \left( \partial_\mu \hat{X}^\nu \partial_\nu \hat{X}^\mu - \frac{1}{2} g^{\mu\nu} g^{kl} \partial_\mu \hat{X}^\nu \partial_\kappa \hat{X}^\kappa \right) - \int_{\mathcal{M}^2} \delta \hat{X}_m \left[ d^2 \xi \partial_\nu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \hat{X}_m \right) - j_1 \wedge \left( P^m_1 + d\epsilon^m \varepsilon_m \right) \sqrt{-g} \partial_\nu \hat{X}^\nu \right] - \int_{\mathcal{M}^{p+1}} d^{p+1}\zeta \left[ \partial_\hat{m} \left( \sqrt{|G|} G^{-1}[\hat{m}\hat{n}] \right) - j^\hat{n} \right] \delta A_\hat{n}(\zeta) - \int_{\mathcal{M}^{p+1}} \left[ d^{p+1}\zeta \partial_\hat{m} \left( \sqrt{|G|} G^{-1}[\hat{m}\hat{n}] \partial_\hat{n} \hat{X}^\mu \right) + j_p \wedge P^m_1 \right] \delta \hat{X}_m. \]

Then the equations of motion become evident. They are

- **The Born–Infeld equation** (with \( G_{mn} \) defined by (22))
  \[
  \partial_\mu \left( \sqrt{|G|} G^{-1}[\hat{m}\hat{n}] \right) = j^\hat{n} \equiv \int_{\partial \mathcal{M}^2} d\zeta^\hat{n}(\tau) \delta^{p+1} \left( \zeta - \hat{\zeta}(\tau) \right). \tag{38}
  \]
  It includes the definite source localized on the intersection (and thus outside the intersection the equation is free).

- **Equations for the Dp-brane coordinate functions** (with \( G_{mn} \) defined by Eq. (22))
  \[
  \partial_\mu \left( \sqrt{|G|} G^{-1}[\hat{m}\hat{n}] \partial_\nu \hat{X}_m \right) = - \int_{\partial \mathcal{M}^2} d\tau P^m_1(\tau) \delta^{p+1} \left( \zeta - \hat{\zeta}(\tau) \right) \tag{39}
  \]

---

\(^b\) To arrive at the expression for the r.h.s. one should perform some formal extension of the property (35), as the form \( P_1 = d\tau P(\tau) \) has not been defined as a pull–back of any 1-form on \( \mathcal{M}^2 \). Let us assume that such a form \( P_1 = d\zeta^\hat{m} P^\hat{m} \) is introduced. Then, by definition, \( P_1(\hat{\zeta}(\tau)) = d\zeta^\hat{m} P^\hat{m}(\hat{\zeta}(\tau)) = d\tau P^\hat{m}(\tau) \) and one finds \( \int_{\mathcal{M}^{p+1}} j_p \wedge P^m_1 = \int_{\partial \mathcal{M}^2} d\tau P^m_1(\tau) \delta^{p+1}(\zeta - \hat{\zeta}(\tau)) \).
The equation for the string coordinate functions (with $g_{mn}(\xi)$ defined by (7) or, equivalently, by (4) and the Weyl symmetry (5))

$$d^2 \xi \partial_m \left( \sqrt{-g} g^{mn}(\xi) \partial_n \hat{X}^m(\xi) \right) = j_1 \wedge \left( P^m_1 + d\xi^m \varepsilon_{mn} \sqrt{-g} g^{nk} \partial_k \hat{X}^m \right).$$

(40)

5 Reparametrization symmetry of the coupled system

The variations of the action (32) with respect to the worldvolume and worldsheet embedding coordinate functions $\delta \hat{\zeta}(\tau)$ (30) and $\tilde{\xi}^m(\tau)$ (10) is

$$\delta \tilde{\xi}, \hat{\zeta} S = \int_{\partial M^2} \delta \hat{\zeta}(\xi) \left[ i_{\hat{m}} F - P_{1m} \partial_m \hat{X}^m \right] + \int_{\partial M^2} \delta \tilde{\xi}^m(\tau) \left[ P_{1m} \partial_m \hat{X}^m - d\xi^p \sqrt{-g} g^{nk} \partial_k \hat{X}^m \partial_m \hat{X}^m \right].$$

(41)

The corresponding equations of motion

$$i_{\hat{m}} F|_{\partial M^2} = P_{1m} \partial_m \hat{X}^m(\hat{\zeta}(\tau))$$

(42)

$$d\xi^p(\tau) \sqrt{-g} g^{nk} \partial_k \hat{X}^m \partial_m \hat{X}^m = P_{1m} \partial_m \hat{X}^m$$

(43)

are dependent.

To prove the dependence of Eq. (43) one should contract Eq. (40) with $\partial_l \hat{X}^m(\xi)$ on the target space indices. Then, writing the left hand part as

$$\partial_m \left( \sqrt{-g} g^{mn}(\xi) \partial_n \hat{X}^m(\xi) \right) \partial_l \hat{X}^m(\xi) =$$

(44)

\[ = \partial_m \left( \sqrt{-g} g^{mn}(\xi) \partial_n \hat{X}^m(\xi) \right) \partial_l \hat{X}^m(\xi) - \left( \sqrt{-g} g^{mn}(\xi) \partial_n \hat{X}^m(\xi) \right) \partial_l \partial_m \hat{X}^m(\xi) \]

and using Eq. (4) (which remains the same for the coupled system), one obtains (in arbitrary space-time dimension D)

$$d^2 \xi \sqrt{-g} g^{D-2} D = \partial_l \left( \sqrt{-g} g^{mn}(\xi) \partial_m \hat{X}^m(\xi) \partial_n \hat{X}^m(\xi) \right) =$$

(45)

\[ - j_1 \wedge \left( P^m_1 + d\xi^m \varepsilon_{mn} \sqrt{-g} g^{nk} \partial_k \hat{X}^m \right). \]

However, using the Weyl symmetry one can fix the gauge (3), where the l.h.s. of Eq. (43) vanishes, and we arrive at (43).
To prove the dependence of Eq. (42) one should consider the longitudinal part of Eq. (39), i.e. the result of the contraction of Eq. (39) with \(\partial_m \hat{X}^m\) on the target space indices

\[
\partial_m \left( \sqrt{|G|} G^{-1(\hat{m} \hat{n})} \partial_n \hat{X}^m \right) \partial_l \hat{X}^m = -p^m \partial_l \hat{X}^m. \tag{46}
\]

Here we, for shortness, introduced the notation

\[
\int_{\partial M^2} d\tau P_{\hat{m}}(\tau) \delta^{\hat{p}+1} \left( \zeta - \hat{\zeta}(\tau) \right) = -p^m.
\]

Dealing with the l.h.s. of Eq. (46) in the same way as in (44) and using the identities

\[
\delta_{\hat{n}}^\hat{m} = G^{-1(\hat{m} \hat{k})} \partial_{\hat{k}} \hat{X}^\hat{m} \partial_h \hat{X}_{\hat{m}} + G^{-1[\hat{m} \hat{k}]} F_{k\hat{n}},
\]

\[
G^{-1[\hat{m} \hat{k}]} \partial_{\hat{k}} \hat{X}^\hat{m} \partial_h \hat{X}_{\hat{m}} = -G^{-1(\hat{m} \hat{k})} F_{\hat{k}\hat{n}},
\]

(which follow from \(G^{-1(\hat{m} \hat{k})} G_{\hat{n} \hat{k}} = \delta_{\hat{n}}^\hat{m}\) ) one arrives at

\[
\partial_m \left( \sqrt{|G|} G^{-1(\hat{m} \hat{n})} \partial_n \hat{X}^m \right) \partial_l \hat{X}^m = -\partial_m \left( \sqrt{|G|} G^{-1[\hat{m} \hat{k}]} \right) F_{k\hat{n}}. \tag{47}
\]

Substituting Eqs. (40) and (38) we obtain from Eq. (47)

\[
\int_{\partial M^2} \left( -i_{\hat{m}} F_2 + P_{\hat{m}} \partial_{\hat{m}} \hat{X}^\hat{m}(\zeta(\tau)) \right) \delta^{\hat{p}+1} \left( \zeta - \hat{\zeta}(\tau) \right) = 0. \tag{48}
\]

Eq. (48) is equivalent to (42).

The dependence of the equations (42), (43) should be regarded as Noether identities which reflect some gauge symmetries of the action.

Reversing the arguments, we can consider (42), (43) as independent equations, but conclude that the longitudinal parts of Eqs. (39), (40) are dependent as they are in the case of the free string and Dp-brane. From such point of view we can state that the dependence of Eq. (43) and Eq. (42) reflects the reparametrization symmetries of the string worldsheet and the Dp-brane worldvolume which, hence, survive when coupling is ‘switched on’.

6 Concluding remarks

Thus, applying the method of Ref. 1 we obtained the equations of motion (38) - (40) for the coupled system of an open bosonic string and a dynamical bosonic Dp–brane where the string endpoints live. In such a way we illustrated the key issues of the approach 1 and we are able to comment on its relation to the standard description of Dp–branes as rigid planes 3.

The investigation of Eqs. (38) - (40), (24) and their supersymmetric generalizations will be the subject of the forthcoming paper. Note that such a study simplifies essentially when the Lorentz–harmonic formulations of the (super–)string 4 and (super–)Dp–brane actions 4 are used instead of (1), (24).
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