Radical layer length and syzygy-finite algebras *†

Junling Zheng ‡

Department of Mathematics, China Jiliang University, Hangzhou, 310018, P. R. China

Abstract

Let Λ be an artin algebra. We obtain that Λ is syzygy-finite when the radical layer length of Λ is at most two; as two consequences, we give a new upper bound for the dimension of the bounded derived category of the category mod Λ of finitely generated right Λ-modules in terms of the projective of certain class of simple right Λ-modules and also get the left big finitistic dimension conjecture holds.

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1 Introduction

Given a triangulated category $\mathcal{T}$, Rouquier introduced in [18] the dimension $\dim \mathcal{T}$ of $\mathcal{T}$ under the idea of Bondal and van den Bergh in [6]. This dimension and the infimum of the Orlov spectrum of $\mathcal{T}$ coincide, see [16, 4]. Roughly speaking, it is an invariant that measures how quickly the category can be built from one object. Many authors have studied the upper bound of $\dim \mathcal{T}$, see [4, 5, 7, 8, 12, 15, 18, 19, 21, 23, 22] and so on. There are a lot of triangulated categories having infinite dimension, for instance, Oppermann and Šťovíček proved in [15] that all proper thick subcategories of the bounded derived category of finitely generated modules over a Noetherian algebra containing perfect complexes have infinite dimension.

Let Λ be an artin algebra. Let mod Λ be the category of finitely generated right Λ-modules and let $D^b(\text{mod } \Lambda)$ be the bounded derived category of mod Λ. The upper bounds for the dimensions of the bounded derived category of mod Λ can be given in terms of the Loewy length $\text{LL}(\Lambda)$ and the global dimension $\text{gl.dim } \Lambda$ of Λ.

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‡Email: zhengjunling@cjlu.edu.cn
For a length-category \( C \), generalizing the Loewy length, Huard, Lanzilotta and Hernández introduced in \([9, 11]\) the (radical) layer length associated with a torsion pair, which is a new measure for objects of \( C \). Let \( \Lambda \) be an artin algebra and \( V \) a set of some simple modules in \( \text{mod} \Lambda \). Let \( t_V \) be the torsion radical of a torsion pair associated with \( V \) (see Section 3 for details). We use \( \ell_{t_V}(\Lambda) \) to denote the \( t_V \)-radical layer length of \( \Lambda \). For a module \( M \) in \( \text{mod} \Lambda \), we use \( \text{pd} M \) to denote the projective dimensions of \( M \); in particular, set \( \text{pd} M = -1 \) if \( M = 0 \). For a subclass \( B \) of \( \text{mod} \Lambda \), the projective dimension \( \text{pd} B \) of \( B \) is defined as

\[
\text{pd} B = \begin{cases} 
\sup \{ \text{pd} M \mid M \in B \}, & \text{if } B \neq \emptyset; \\
-1, & \text{if } B = \emptyset.
\end{cases}
\]

Note that \( V \) is a finite set. So, if each simple module in \( V \) has finite projective dimension, then \( \text{pd} V \) attains its (finite) maximum.

Now, let us list some results about the upper bound of the dimension of bounded derived categories.

**Theorem 1.1.** Let \( \Lambda \) be an artin algebra and \( V \) a set of some simple modules in \( \text{mod} \Lambda \). Then we have

1. \((19 \text{ Proposition 7.37})\) \( \dim D^b(\text{mod} \Lambda) \leq \text{LL}(\Lambda) - 1; \)
2. \((19 \text{ Proposition 7.4} \text{ and } 12 \text{ Proposition 2.6})\) \( \dim D^b(\text{mod} \Lambda) \leq \text{gl.dim} \Lambda; \)
3. \((22 \text{ Theorem 3.8})\) \( \dim D^b(\text{mod} \Lambda) \leq (\text{pd} V + 2)(\ell_{t_V}(\Lambda) + 1) - 2; \)
4. \((21)\) \( \dim D^b(\text{mod} \Lambda) \leq 2(\text{pd} V + \ell_{t_V}(\Lambda)) + 1. \)

For an integer \( m \geq 0 \), we denote by \( \Omega^m(X) \) the \( m \)-th syzygy of \( X \in \text{mod} \Lambda \) and we denote by

\[
\Omega^m(\text{mod} \Lambda) = \{ M \mid M \text{ is a direct summand of } \Omega^m(N) \text{ for some } N \in \text{mod} \Lambda \}.
\]

Following \( 20 \text{ P. 834} \), \( \Lambda \) is called \emph{\( m \)-syzygy-finite} if there are only finitely many non-isomorphic indecomposable modules in \( \Omega^m(\text{mod} \Lambda) \). If there is some nonnegative integer \( m \), such that \( \Lambda \) is \( m \)-syzygy-finite, then \( \Lambda \) is said to be syzygy-finite.

The aim of this paper is to prove the following

**Theorem 1.2.** (see Theorem 3.4 and Corollary 3.7) Let \( A \) be an artin algebra. Let \( V \subseteq S^{<\infty} \). If \( \ell_{t_V}(A_A) \leq 2 \), then \( A \) is \((\text{pd} V + 2)\)-syzygy-finite and \( \dim D^b(\text{mod} A) \leq \text{pd} V + 3 \) and the left big finitistic dimension conjecture holds.

We also give examples to explain our results. In this case, we may be able to get a better upper bound on the dimension of the bounded derived category of \( \text{mod} \Lambda \).

## 2 Preliminaries

### 2.1 The dimension of a triangulated category

We recall some notions from \([18, 19, 14]\). Let \( T \) be a triangulated category and \( \mathcal{I} \subseteq \text{Ob} T \). Let \( \langle \mathcal{I} \rangle \) be the full subcategory consisting of \( T \) of all direct summands of finite direct sums of shifts
of objects in $\mathcal{I}$. Given two subclasses $\mathcal{I}_1, \mathcal{I}_2 \subseteq \text{Ob}\mathcal{T}$, we denote $\mathcal{I}_1 \ast \mathcal{I}_2$ by the full subcategory of all extensions between them, that is,

$$\mathcal{I}_1 \ast \mathcal{I}_2 = \{ X \mid X_1 \to X \to X_2 \to X_1[1] \text{ with } X_1 \in \mathcal{I}_1 \text{ and } X_2 \in \mathcal{I}_2 \}.$$ 

Write $\mathcal{I}_1 \diamond \mathcal{I}_2 := \langle \mathcal{I}_1 \ast \mathcal{I}_2 \rangle$. Then $(\mathcal{I}_1 \diamond \mathcal{I}_2) \diamond \mathcal{I}_3 = \mathcal{I}_1 \diamond (\mathcal{I}_2 \diamond \mathcal{I}_3)$ for any subclasses $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$ of $\mathcal{T}$ by the octahedral axiom. Write

$$\langle \mathcal{I} \rangle_0 := 0, \langle \mathcal{I} \rangle_1 := \langle \mathcal{I} \rangle \text{ and } \langle \mathcal{I} \rangle_{n+1} := \langle \mathcal{I} \rangle_n \diamond \langle \mathcal{I} \rangle_1 \text{ for any } n \geq 1.$$ 

**Definition 2.1.** ([18 Definition 3.2]) The dimension $\dim \mathcal{T}$ of a triangulated category $\mathcal{T}$ is the minimal $d$ such that there exists an object $M \in \mathcal{T}$ with $\mathcal{T} = \langle M \rangle_{d+1}$. If no such $M$ exists for any $d$, then we set $\dim \mathcal{T} = \infty$.

### 2.2 Radical layer lengths and torsion pairs

We recall some notions from [9]. Let $\mathcal{C}$ be a length-category, that is, $\mathcal{C}$ is an abelian, skeletally small category and every object of $\mathcal{C}$ has a finite composition series. We use $\text{End}_Z(\mathcal{C})$ to denote the category of all additive functors from $\mathcal{C}$ to $\mathcal{C}$, and use $\text{rad}$ to denote the Jacobson radical lying in $\text{End}_Z(\mathcal{C})$. For any $\alpha \in \text{End}_Z(\mathcal{C})$, set the $\alpha$-radical functor $F_\alpha := \text{rad} \circ \alpha$.

**Definition 2.2.** ([9 Definition 3.1]) For any $\alpha, \beta \in \text{End}_Z(\mathcal{C})$, we define the $(\alpha, \beta)$-layer length $\ell_{\alpha \beta} : \mathcal{C} \to \mathbb{N} \cup \{\infty\}$ via $\ell_{\alpha \beta}(M) = \inf\{ i \geq 0 \mid \alpha \circ \beta^i(M) = 0 \}$; and the $\alpha$-radical layer length $\ell_\alpha := \ell_{\alpha \alpha}$.

For more information about radical layer length, we can see [11] [9] [22] [23].

**Lemma 2.3.** ([22 Lemma 2.6]) Let $\alpha, \beta \in \text{End}_Z(\mathcal{C})$. For any $M \in \mathcal{C}$, if $\ell_{\alpha \beta}(M) = n$, then $\ell_{\alpha \beta}(M) = \ell_{\alpha \beta}(\beta^j(M)) + j$ for any $0 \leq j \leq n$; in particular, if $\ell_{\alpha \beta}(M) = n$, then $\ell_{\alpha \beta}(\mathcal{F}_\alpha^n(M)) = 0$.

Recall that a torsion pair (or torsion theory) for $\mathcal{C}$ is a pair of classes $(\mathcal{T}, \mathcal{F})$ of objects in $\mathcal{C}$ satisfying the following conditions.

1. $\text{Hom}_\mathcal{C}(M, N) = 0$ for any $M \in \mathcal{T}$ and $N \in \mathcal{F}$;
2. an object $X \in \mathcal{C}$ is in $\mathcal{T}$ if $\text{Hom}_\mathcal{C}(X, -)|_{\mathcal{F}} = 0$;
3. an object $Y \in \mathcal{C}$ is in $\mathcal{F}$ if $\text{Hom}_\mathcal{C}(-, Y)|_{\mathcal{T}} = 0$.

For a subfunctor $\alpha$ of $1_\mathcal{C}$, we write $q_\alpha := 1_\mathcal{C}/\alpha$. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair for $\mathcal{C}$. Recall that the torsion radical $t$ is a functor in $\text{End}_Z(\mathcal{C})$ such that

$$0 \to t(M) \to M \to q_t(M) \to 0$$

is a short exact sequence and $q_t(M)(= M/t(M)) \in \mathcal{F}$. 

2.3 Some facts

In this section, $\Lambda$ is an artin algebra. Then $\text{mod } \Lambda$ is a length-category. For a module $M$ in $\text{mod } \Lambda$, we use $\text{rad } M$ and $\text{top } M$ to denote the radical, socle and top of $M$ respectively. For a subclass $\mathcal{W}$ of $\text{mod } \Lambda$, we use $\text{add } \mathcal{W}$ to denote the subcategory of $\text{mod } \Lambda$ consisting of direct summands of finite direct sums of modules in $\mathcal{W}$, and if $\mathcal{W} = \{M\}$ for some $M \in \text{mod } \Lambda$, we write $\text{add } M := \text{add } \mathcal{W}$.

Let $\mathcal{S}^{\infty}$ be the set of the simple modules with infinite projective dimension and $\mathcal{S}^{<\infty}$ be the set of the simple module with finite projective dimension. Let $\mathcal{S}$ be the set of the simple modules in $\text{mod } \Lambda$, and let $\mathcal{V}$ be a subset of $\mathcal{S}$ and $\mathcal{V}'$ the set of all the others simple modules in $\text{mod } \Lambda$, that is, $\mathcal{V}' = \mathcal{S} \setminus \mathcal{V}$. We write $\mathfrak{F}(\mathcal{V}) := \{M \in \text{mod } \Lambda \mid \text{there exists a finite chain} \}$ of submodules of $M$ such that each quotient $M_i/M_{i-1}$ is isomorphic to some module in $\mathcal{V}\}$. By \cite{9} Lemma 5.7 and Proposition 5.9, we have that $(\mathcal{T}_\mathcal{V}, \mathfrak{F}(\mathcal{V}))$ is a torsion pair, where

$$\mathcal{T}_\mathcal{V} = \{M \in \text{mod } \Lambda \mid \text{top } M \in \text{add } \mathcal{V}'\}.$$ 

We use $t_{\mathcal{V}}$ to denote the torsion radical of the torsion pair $(\mathcal{T}_\mathcal{V}, \mathfrak{F}(\mathcal{V}))$. Then $t_{\mathcal{V}}(M) \in \mathcal{T}_\mathcal{V}$ and $q_{t_{\mathcal{V}}}(M) \in \mathfrak{F}(\mathcal{V})$ for any $M \in \text{mod } \Lambda$.

2.4 Short exact sequences and radical layer length

Lemma 2.4. For any module $X \in \text{mod } \Lambda$. We have

1. $t_{\mathcal{V}}(\Lambda \Lambda)$ is a two side ideal and $t_{\mathcal{V}}(X) = X t_{\mathcal{V}}(\Lambda \Lambda)$.
2. $\text{rad } X = X \text{rad}(\Lambda \Lambda)$.
3. $\text{rad } t_{\mathcal{V}}(\Lambda \Lambda) = t_{\mathcal{V}}(\Lambda \Lambda) \text{rad}(\Lambda \Lambda)$.
4. $t_{\mathcal{V}} \text{rad } t_{\mathcal{V}}(\Lambda \Lambda) = t_{\mathcal{V}}(\Lambda \Lambda) \text{rad}(\Lambda \Lambda)t_{\mathcal{V}}(\Lambda \Lambda)$.
5. $t_{\mathcal{V}} F^i_{t_{\mathcal{V}}}(\Lambda \Lambda)$ is an ideal of $\Lambda$ for each $i \geq 0$.

Proof. (1) See \cite{9} Proposition 5.9(c)].
(2) See \cite{3} Proposition 3.5].
(3) Let $X = t_{\mathcal{V}}(\Lambda \Lambda)$ and By(2).
(4) By (1)(3).
(5) By (1)(2)(3)(4). \qed

Lemma 2.5. For any module $X \in \text{mod } \Lambda$, we have $t_{\mathcal{V}} F^i_{t_{\mathcal{V}}}(X) = X(t_{\mathcal{V}} F^i_{t_{\mathcal{V}}}(\Lambda \Lambda))$ for each $i \geq 0$.

Proof. If $i = 0$, by Lemma 2.4(1).
Suppose that if $i = n$, we have $t_{\mathcal{V}} F^n_{t_{\mathcal{V}}}(X) = X(t_{\mathcal{V}} F^n_{t_{\mathcal{V}}}(\Lambda \Lambda))$.
Theorem 2.10. Let

\[ \text{rad} \text{ preserve monomorphism and epimorphism} \] (see [9, Lemma 3.6(a)]) and epimorphism (see [2, Chapter V, Lemma 1.1]).

Proof. Note that rad preserve monomorphism (see [9, Lemma 3.6(a)]) and epimorphism (see [2, Chapter V, Lemma 1.1]).

Lemma 2.6. ([22, Lemma 3.3]) The functor \( t_V \) preserve monomorphism and epimorphism.

Lemma 2.7. The functor rad preserve monomorphism and epimorphism.

Proof. Note that rad preserve monomorphism (see [9, Lemma 3.6(a)]) and epimorphism (see [2, Chapter V, Lemma 1.1]).

Lemma 2.8. For each \( i \geq 0 \), \( F_{iV} = \text{rad} \circ t_V \) and \( t_V F_{iV} = \text{rad} \circ t_V \) preserve monomorphism and epimorphism.

Proof. By Lemma 2.6 and Lemma 2.7.

By Definition 2.2, we have the following observation.

Lemma 2.9. For any module \( X \in \text{mod} \Lambda \), we have \( t_V F_{iV} L^{\ell_iV}(X) = 0 \).

Now, we give the main theorem in this paper.

Theorem 2.10. Let \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) be an exact sequence in \( \text{mod} \Lambda \). Then

\[
\max\{\ell_iV(M), \ell_iV(N)\} \leq \ell^{\ell_iV}(M) \leq \ell^{\ell_iV}(L) + \ell^{\ell_iV}(N).
\]

In particular, if \( \ell^{\ell_iV}(L) = 0 \), then \( \ell^{\ell_iV}(N) = \ell^{\ell_iV}(M) \); if \( \ell^{\ell_iV}(N) = 0 \), then \( \ell^{\ell_iV}(L) = \ell^{\ell_iV}(M) \).

Proof. By Lemma 2.8, we know that \( F_{iV} = \text{rad} \circ t_V \) preserve monomorphism and epimorphism. Thus by [9, Lemma 3.4(b)(c)], we can obtain that \( \ell^{\ell_iV}(L) \leq \ell^{\ell_iV}(M) \) and \( \ell^{\ell_iV}(N) \leq \ell^{\ell_iV}(M) \), that is,

\[
\max\{\ell^{\ell_iV}(L), \ell^{\ell_iV}(N)\} \leq \ell^{\ell_iV}(M).
\]

Next, we will prove the second \( '\leq' \).

By Lemma 2.6(5), we know that \( t_V F_{iV}(\Lambda) \) is an ideal of \( \Lambda \) for each \( i \geq 0 \). By assumption, we have \( M/L \cong N \). Moreover, we get

\[
(t_V F_{iV}^{\ell_{iV}}(N) + L)/L = (M(t_V F_{iV}^{\ell_{iV}}(N)(\Lambda)) + L)/L
\]

(by Lemma 2.5)

\[
= (M/L)(t_V F_{iV}^{\ell_{iV}}(N)(\Lambda))
\]

\[
\cong N(t_V F_{iV}^{\ell_{iV}}(N)(\Lambda))
\]

(by \( M/L \cong N \))

\[
= t_V F_{iV}^{\ell_{iV}}(N)
\]

(by Lemma 2.5)

\[
= 0
\]

(by Lemma 2.9).
Corollary 2.12. Let $t_Y F^\ell t_Y(N)(M) + L = L$. Moreover, $t_Y F^\ell t_Y(N)(M) \subseteq L$. And by Lemma 2.8 and Lemma 2.10, we have

$$t_Y F^\ell t_Y(L)(t_Y F^\ell t_Y(N)(M)) \subseteq t_Y F^\ell t_Y(L)(L) = 0,$$

where we use the fact that $t_Y$ is idempotent, that is, $t_Y^2 = t_Y$. That is,

$$t_Y F^\ell t_Y(L + \ell t_Y(N))(M) = t_Y F^\ell t_Y(L)(t_Y F^\ell t_Y(N)(M)) \subseteq t_Y F^\ell t_Y(L)(L) = 0.$$

Thus, $\ell t_Y(M) \leq \ell t_Y(L) + \ell t_Y(N)$ by Definition 2.2. \hfill $\Box$

Remark 2.11. Note that the functions Loewy length $\text{LL}$ and infinite layer length $\ell t_{\infty}$ are particular radical layer length, more details see [9, 11]. Corollary 2.12(1) is a classical result. The first “$\leq$” in Corollary 2.12(2) is first established in [11, Proposition 4.5(a)(b)].

Corollary 2.12. Let $0 \to L \to M \to N \to 0$ be an exact sequence in $\text{mod } \Lambda$. Then

1. $\max\{\text{LL}(L), \text{LL}(N)\} \leq \text{LL}(M) \leq \text{LL}(L) + \text{LL}(N)$.
2. $\max\{\ell t_{\infty}(L), \ell t_{\infty}(N)\} \leq \ell t_{\infty}(M) \leq \ell t_{\infty}(L) + \ell t_{\infty}(N)$.
3. If $\ell t_{\infty}(L) = 0$, then $\ell t_{\infty}(N) = \ell t_{\infty}(M)$; if $\ell t_{\infty}(N) = 0$, then $\ell t_{\infty}(L) = \ell t_{\infty}(M)$.

Proof. (1) are particular cases of Theorem 2.10. (3) if $\ell t_{\infty}(L) = 0$, by (2), we have

$$\ell t_{\infty}(N) = \max\{\ell t_{\infty}(L), \ell t_{\infty}(N)\} \leq \ell t_{\infty}(M) \leq \ell t_{\infty}(L) + \ell t_{\infty}(N) = \ell t_{\infty}(N),$$

that is, $\ell t_{\infty}(N) = \ell t_{\infty}(N)$. Similarly, if $\ell t_{\infty}(N) = 0$, then $\ell t_{\infty}(L) = \ell t_{\infty}(M)$. \hfill $\Box$

3 Main results

Lemma 3.1. ([10, Lemma 3.6]) Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\text{mod } \Lambda$. Then we have the following:

1. If $\text{pd } Z$ is finite, then, for any $m$ with $\text{pd } Z \leq m$, there are projective $\Lambda$-modules $P_m$ and $P'_m$ such that $\Omega^m(X) \oplus P_m \cong \Omega^m(Y) \oplus P'_m$
2. If $\text{pd } X$ is finite, then, for any $m$ with $\text{pd } X \leq m$, there are projective $\Lambda$-modules $P_m$ and $P'_m$ such that $\Omega^{m+1}(Y) \oplus P_m \cong \Omega^{m+1}(Z) \oplus P'_m$

The following lemma is a special case of [9, Lemma 6.3].

Lemma 3.2. ([9, Lemma 6.3]) Let $\mathcal{V} \subseteq S^{<\infty}$ and $M \in \text{mod } \Lambda$. If $t_Y(M) \neq 0$, then $\ell t_{\infty}(\Omega t_Y(M)) \leq \ell t_{\infty}(A_{\Lambda}) - 1$.

Lemma 3.3. Let $M, N \in \text{mod } \Lambda$. If $M \in \text{add}(N)$, then for any $n \geq 0$, we have $\Omega^n(M) \in \text{add}(\Omega^n(N))$.

Proof. Since $M \in \text{add}(N)$, we can set $M \oplus L \cong N^s$ for some positive integer $n$ and $L \in \text{mod } \Lambda$. Thus, $\Omega^n(M) \oplus \Omega^n(L) \cong \Omega^n(M) \oplus \Omega^n(L) \cong \Omega^n(N^s) \cong (\Omega^n(N))^s$. That is, $\Omega^n(M) \in \text{add}(\Omega^n(N))$. \hfill $\Box$

Theorem 3.4. Let $\mathcal{V} \subseteq S^{<\infty}$. If $\ell t_{\infty}(A_{\Lambda}) \leq 2$, then $\text{mod } \Lambda$ is $\text{(pd } \mathcal{V} + 2)-\text{syzygy-finite}$. 
Proof. We set \( \delta = \text{pd} V \). If \( \ell^t_v(A_A) = 0 \). For any module \( M \in \text{mod} A \), we have \( \ell^t_v(M) \leq \ell^t_v(A_A) = 0 \), that is, \( \ell^t_v(M) = 0 \). And then \( M \in \mathcal{F}(V) \), moreover, \( \text{pd} M \leq \delta \).

Now consider the case \( 1 \leq \ell^t_v(A_A) = 2 \). We have the following two canonical two short exact sequences

\[
0 \longrightarrow t_V(M) \longrightarrow M \longrightarrow q_{t_V}(M) \longrightarrow 0, \tag{3.1}
\]

\[
0 \longrightarrow t_V\Omega_t V(M) \longrightarrow \Omega t_V(M) \longrightarrow q_{t_V}\Omega t_V(M) \longrightarrow 0, \tag{3.2}
\]

\[
0 \longrightarrow \text{rad}_t V\Omega t_V(M) \longrightarrow t_V\Omega t_V(M) \longrightarrow \text{top}_t V\Omega t_V(M) \longrightarrow 0. \tag{3.3}
\]

For any module \( M \in \text{mod} A \).

If \( \ell^t_v(\Omega t_V(M)) = 0 \), by Lemma 2.10 and sequence (3.2) we know that

\[
\ell^t_v(t_V\Omega t_V(M)) = \ell^t_v(\Omega t_V(M)) = 0;
\]

and by Lemma 2.10 and sequence (3.3) we know that

\[
0 \leq \ell^t_v(\text{rad}_t V\Omega t_V(M)) \leq \ell^t_v(\Omega t_V(M)) = 0.
\]

That is, \( \ell^t_v(\text{rad}_t V\Omega t_V(M)) = 0 \). And then \( \text{pd} \text{rad}_t V\Omega t_V(M) \leq \delta \).

If \( \ell^t_v(\Omega t_V(M)) = 1 \). By Lemma 2.10 and sequence (3.2) we have \( \ell^t_v(t_V\Omega t_V(M)) = \ell^t_v(\Omega t_V(M)) = 1 \). By Lemma 2.3 we have

\[
\ell^t_v(\text{rad}_t V\Omega t_V(M)) = \ell^t_v(t_V\Omega t_V(M)) - 1 = 1 - 1 = 0.
\]

Thus, \( \text{pd} \text{rad}_t V\Omega t_V(M) \leq \delta \).

By the short exact sequence (3.1) and Lemma 3.1(1), we have

\[
\Omega^{\delta+1}t_V(M) \oplus P_1 = \Omega^\delta(\Omega t_V(M)) \oplus P_1 \cong \Omega^{\delta+1}(M) \oplus P_2. \tag{3.4}
\]

By the short exact sequence (3.2) and Lemma 3.1(1), we have

\[
\Omega^{\delta+1}(t_V\Omega t_V(M)) \oplus P_3 \cong \Omega^{\delta+1}(\Omega t_V(M)) \oplus P_4 = \Omega^{\delta+2}t_V(M) \oplus P_4. \tag{3.5}
\]

By the short exact sequence (3.3) and Lemma 3.1(2), we have

\[
\Omega^{\delta+1}(t_V\Omega t_V(M)) \oplus P_3 \cong \Omega^{\delta+1}\text{top}_t V\Omega t_V(M) \oplus P_6. \tag{3.6}
\]

And then we have the following isomorphisms

\[
\Omega^{\delta+2}(M) \oplus P_4 \oplus P_5 \cong \Omega(\Omega^{\delta+1}(M) \oplus P_2) \oplus P_4 \oplus P_5 \\
\cong \Omega(\Omega^{\delta+1}t_V(M) \oplus P_1) \oplus P_4 \oplus P_5 \ (\text{by (3.4)}) \\
\cong \Omega^{\delta+2}(t_V(M)) \oplus P_4 \oplus P_5 \\
\cong (\Omega^{\delta+1}t_V\Omega t_V(M) \oplus P_3) \oplus P_5 \ (\text{by (3.5)}) \\
\cong (\Omega^{\delta+1}t_V\Omega t_V(M) \oplus P_3) \oplus P_3 \\
\cong (\Omega^{\delta+1}\text{top}_t V\Omega t_V(M) \oplus P_6) \oplus P_3 \ (\text{by (3.6)}) \\
\in \text{add}(\Omega^{\delta+2}(A/\text{rad} A) \oplus A). \ (\text{by Lemma 3.3})
\]
By assumptions and Lemma 3.2, we always have $\ell^v(\Omega V(M)) \leq 1$. Thus, for any module $M \in \text{mod } A$, we have $\Omega^\delta + 2(M) \in \text{add}(\Omega^\delta + 2(A/\text{rad } A) \oplus A)$. That is, $\text{mod } A$ is $(\delta + 2)$-syzygy-finite.

**Corollary 3.5.** If $\ell^v(A_A) \leq 2$, then $A$ is syzygy-finite.

**Proof.** Let $V = S^{<\infty}$, we have $\ell^v(A_A) = \ell^v(A_A) \leq 2$ by [9, Example 5.8(1)]. And then by 3.4 we know that $A_A$ is syzygy-finite. □

The notion of the left big finitistic dimension conjecture can be seen in [17].

**Corollary 3.6.** Let $A$ be a finite dimensional algebra over a field $K$. Let $V \subseteq S^{<\infty}$. If $\ell^v(A_A) \leq 2$, then

$$1.\text{Fin.dim } A < \infty,$$

where $1.\text{Fin.dim } A = \sup\{\text{pd } M \mid M \text{ is a left } A\text{-module with } \text{pd } M < \infty\}$; that is, the left big finitistic dimension conjecture holds.

**Proof.** By [17] Definition 4.1, Definition 4.2, Corollary 7.3, Theorem 4.3 and Theorem 3.4. □

As a consequence we have the following upper bound on the dimension $\text{dim } D^b(\text{mod } A)$ of the bounded derived category of $\text{mod } A$ in the sense of Rouquier (see [19, 18, 12]). Here, we have an interesting corollary as follows

**Corollary 3.7.** Let $V \subseteq S^{<\infty}$. Suppose that $\ell^v(A_A) \leq 2$. Then $\text{dim } D^b(\text{mod } A) \leq \text{pd } V + 3$.

**Proof.** By Theorem 3.4 and [1] Corollary 3.6. □

**Corollary 3.8.** Let $A$ be an artin algebra. Let $V \subseteq S^{<\infty}$. If $\ell^v(A_A) \leq 2$, then

$$\Psi \text{dim(} \text{mod } A) < \infty,$$

where $\Psi \text{dim(} \text{mod } A)$ is defined in [13].

**Proof.** By Corollary 3.5 and [13] Theorem 3.2. □

**Example 3.9.** ([22]) Consider the bound quiver algebra $\Lambda = kQ/I$, where $k$ is an algebraically closed field and $Q$ is given by

```plaintext
\begin{align*}
&1 \quad \alpha_1 \\
\alpha_{m+1} &\quad 2 \quad \alpha_2 \\
\alpha_{m+2} &\quad 3 \quad \alpha_3 \\
m + 1 &\quad 4 \quad \alpha_4 \\
m + 2 &\quad \ldots \quad \alpha_m \\
&\quad m
\end{align*}
```
and $I$ is generated by $\{\alpha_1^2, \alpha_1 \alpha_{m+1}, \alpha_1 \alpha_{m+2}, \alpha_2 \alpha_3 \cdots \alpha_m\}$ with $m \geq 10$. Then the indecomposable projective $\Lambda$-modules are

$$
\begin{align*}
P(1) = & \begin{array}{c}
1 \\
2 \\
m-1
\end{array} & P(2) = & \begin{array}{c}
2 \\
3 \\
m
\end{array} & P(3) = & \begin{array}{c}
3 \\
3 \\
m
\end{array} & P(m+1) = m+1, & P(m+2) = m+2
\end{align*}
$$

and $P(i+1) = \text{rad} P(i)$ for any $2 \leq i \leq m-1$.

We have

$$
\text{pd} S(i) = \begin{cases}
\infty, & \text{if } i = 1; \\
1, & \text{if } 2 \leq i \leq m-1; \\
0, & \text{if } m \leq i \leq m+2.
\end{cases}
$$

So $S^\infty = \{S(1)\}$ and $S^{<\infty} = \{S(i) \mid 2 \leq i \leq m+2\}$.

Let $V := \{S(i) \mid 3 \leq i \leq m-1\} \subseteq S^{<\infty}$. Then $\text{pd} V = 1$ and $\ell \ell^t V(\Lambda) = 2$ (see [22, Example 4.1])

(1) By Theorem 1.1(1), we have $\dim D^b(\text{mod } \Lambda) \leq \ell \ell(\Lambda) - 1 = m - 2$.

(2) By Theorem 1.1(3), we have $\dim D^b(\text{mod } \Lambda) \leq (\text{pd} V + 2)(\ell \ell^t V(\Lambda) + 1) - 2 = 7$.

(3) By Theorem 1.1(4), we have $\dim D^b(\text{mod } \Lambda) \leq 2(\text{pd} V + \ell \ell^t V(\Lambda)) + 1 = 7$.

(4) By Corollary 3.7, $\dim D^b(\text{mod } \Lambda) \leq \text{pd} V + 3 = 4$. That is, we can get a better upper bound.

**Example 3.10.** ([23, Example 3.21]) Consider the bound quiver algebra $\Lambda = kQ/I$, where $k$ is a field and $Q$ is given by

$$
\begin{align*}
\begin{array}{c}
2n + 1 \\
2n \\
n + 1
\end{array} & \begin{array}{c}
\alpha_2n+1 \\
\alpha_1 \\
\alpha_{n+1}
\end{array} & \begin{array}{c}
2 \\
\alpha_2 \\
\alpha_{n+2}
\end{array} & \begin{array}{c}
3 \\
\alpha_3 \\
\alpha_{n+3}
\end{array} & \cdots & \begin{array}{c}
\alpha_{n-1} \\
\alpha_{n+4} \\
\alpha_{2n-1}
\end{array} & \begin{array}{c}
n \\
n + 2 \\
n + 3
\end{array} & \cdots & \begin{array}{c}
n \\
n + 2 \alpha_{n+1} \\
\cdots
\end{array}
\end{align*}
$$

and $I$ is generated by $\{\alpha_i \alpha_{i+1} \mid n + 1 \leq i \leq 2n - 1\}$ with $n \geq 6$. Then the indecomposable
projective $\Lambda$-modules are

\[
\begin{array}{c c c c c}
1 & 2 & 2n & 2n + 1 & 2 \\
n + 1 & 3 & 3 & j & j \\
P(1) = 3 & P(2) = 4 & P(3) = 4 & P(j) = j + 1 & P(l) = l,
\end{array}
\]

where $n + 1 \leq j \leq 2n - 2$, $2n - 1 \leq l \leq 2n + 1$ and $P(i + 1) = \text{rad} P(i)$ for any $2 \leq i \leq n - 1$.

We have

\[
\text{pd} S(i) = \begin{cases} 
  n - 1, & \text{if } i = 1; \\
  1, & \text{if } 2 \leq i \leq n - 1; \\
  0, & \text{if } i = n, 2n, 2n + 1; \\
  2n - 1 - i, & \text{if } n + 1 \leq i \leq 2n - 1.
\end{cases}
\]

So $S^{\leq \infty} = \{\text{all simple modules in mod } \Lambda\}$. Let $\mathcal{V} := \{S(i) \mid 2 \leq i \leq n\}(\subseteq S^{\leq \infty})$. Then $\text{pd} \mathcal{V} = 1$ and $\ell^\mathcal{V}(\Lambda) = 2$ (see [23, Example 3.21])

1. By Theorem 1.1(1), we have $\dim D^b(\text{mod } \Lambda) \leq LL(\Lambda) - 1 = n - 1$.
2. By Theorem 1.1(2), we have $\dim D^b(\text{mod } \Lambda) \leq \text{gl.dim } \Lambda = n - 1$.
3. By Theorem 1.1(3), we have $\dim D^b(\text{mod } \Lambda) \leq (\text{pd } \mathcal{V} + 2)(\ell^\mathcal{V}(\Lambda) + 1) - 2 = 7$.
4. By Theorem 1.1(4), we have $\dim D^b(\text{mod } \Lambda) \leq 2(\text{pd } \mathcal{V} + \ell^\mathcal{V}(\Lambda)) + 1 = 7$.
5. By Corollary 3.7, $\dim D^b(\text{mod } \Lambda) \leq \text{pd } \mathcal{V} + 3 = 4$.

That is, we also can get a better upper bound than [22, Example 4.1].

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