The Darboux coordinates for a new family of Hamiltonian operators and linearization of associated evolution equations

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Abstract

A de Sole, V G Kac and M Wakimoto have recently introduced a new family of compatible Hamiltonian operators of the form

\[ H(N,0) = D^2 \circ ((1/u) \circ D)^{2n} \circ D, \]

where \( N = 2n+3, n = 0, 1, 2, \ldots, u \) is the dependent variable and \( D \) is the total derivative with respect to the independent variable. We present a differential substitution that reduces any linear combination of these operators to an operator with constant coefficients and linearizes any evolution equation which is bi-Hamiltonian with respect to a pair of any nontrivial linear combinations of the operators \( H(N,0) \). We also give the Darboux coordinates for \( H(N,0) \) for any odd \( N \geq 3 \).

Mathematics Subject Classification: 37K05, 37K10

1. Introduction

The Hamiltonian evolution equations are well known to play an important role in modern mathematical physics. Indeed, a Hamiltonian operator maps the variational derivatives of the conserved quantities into symmetries; this plays an important role in the theory of integrable systems which often turn out to be bi-Hamiltonian, see e.g. [5, 6, 8, 13] and references therein. It is thus no wonder that the study and, in particular, the classification of Hamiltonian operators is a subject of ongoing interest, see for instance [1, 3, 6, 7, 11, 12] and the works cited there.

Recently, de Sole, Kac and Wakimoto have made a major advance in this area. Namely, in [4] they gave a conjectural classification of Poisson vertex algebras in one differential variable, i.e. of scalar Hamiltonian operators. Inter alia, they have come up with a new infinite family of compatible Hamiltonian operators \( H^{(N,0)} = D^2 \circ ((1/u) \circ D)^{2n} \circ D, \) where
\( N = 2n + 3, \ n = 0, 1, 2, \ldots \), and \( D \) denotes the total derivative with respect to the space variable.

Although it is well known \([15, 16]\) that there is no proper counterpart of the Darboux theorem on canonical forms of finite-dimensional Poisson structures for Hamiltonian operators associated with evolutionary PDEs, it is often possible to find new variables in which a Hamiltonian operator takes a simpler form. One such form is the Gardner operator \( D \); in analogy with the finite-dimensional case the associated new variables are often called the Darboux coordinates, see e.g. \([3, 15]\). Bringing a Hamiltonian operator into the Gardner form enables one, e.g., to render the associated Hamiltonian systems into the canonical Hamiltonian form and construct Lagrangian representations (modulo potentialization) for these systems \([16]\).

In this paper we present the transformations that bring the operators \( H^{(N,0)} \) into the Gardner form, see corollary 1. Moreover, in theorem 1 we give a differential substitution which simultaneously turns the operators \( H^{(N,0)} \) for all odd \( N \geq 3 \) into operators with constant coefficients \( \tilde{H}^{(N,0)} = -D_{2n+1} \).

These results could be employed, e.g., for the study of (co)homology of the Poisson complexes associated with the operators \( H^{(N,0)} \). The cohomologies in question play an important role, e.g., in finding all Hamiltonian operators compatible with a given Hamiltonian operator and the associated multi-Hamiltonian systems, see for example \([10, 14, 17]\) and references therein. Another possible application of the results in question is, e.g., the construction of new integrable systems in spirit of \([9]\) in the new variables from theorem 1 or corollary 1 with a subsequent pullback to the original variables. Last but not least, the differential substitution from theorem 1 linearizes any evolution equation which is bi-Hamiltonian with respect to a pair of any nontrivial linear combinations of the operators \( H^{(N,0)} \), thus exhibiting a broad class of somewhat unusual (in that they are \( C \)-integrable rather than \( S \)-integrable) integrable bi-Hamiltonian systems; see corollary 2 for details.

2. Preliminaries

In what follows we deal with Hamiltonian operators and associated Hamiltonian evolution equations involving a single spatial variable \( x \) and a single dependent variable \( u \). An evolution equation of this kind has the form

\[
\frac{du}{dt} = K[u] = K(x, u, u_x, u_{xx}, \ldots),
\]

where the square brackets indicate that \( K \) is a differential function in the sense of \([13]\), meaning that it depends on \( x, u \), and finitely many derivatives of \( u \) with respect to the space variable \( x \).

Recall (see e.g. \([5, 6, 13]\) for details) that an evolution equation is said to be Hamiltonian with respect to the Hamiltonian operator \( D \) if it can be written in the form

\[
\frac{du}{dt} = D\delta_u T[u],
\]

where \( T = \int T[u] \, dx \) is the Hamiltonian functional, and \( \delta_u \) denotes the variational derivative with respect to \( u \).

**Lemma 1** ([11]). Let \( L_1 \) be a Hamiltonian operator in the variables \( x, u \). Under the transformation

\[
x = \varphi(y, v, v_y, \ldots, v_m), \quad u = \psi(y, v, v_y, \ldots, v_n),
\]

where \( v_j = D_y^i(v) \), and \( D_y \) is the total derivative with respect to \( y \), the operator \( L_1 \) goes into the Hamiltonian operator \( L_2 \) defined by the formula

\[
\mathcal{T}_1 = (D_y(\varphi))^{-1}K^* \circ L_2 \circ K,
\]

where \( K = T[u] \).
where $\mathcal{L}_1$ is obtained from $L_1$ upon using (1) and setting $D_x = (D_y(\psi))^{-1}D_y$.

$$K = \sum_{i=0}^{\max(m,n)} (-1)^i D^i \circ \left( \frac{\partial \psi}{\partial v_i} D_y(\psi) - \frac{\partial \psi}{\partial v_i} D_x(\psi) \right),$$

and $K^*$ is the formal adjoint of $K$.

**Remark 1.** Note that in general the operator $L_2$ may contain nonlocal terms unless (1) is a contact transformation, cf e.g. [1, 4, 11].

### 3. The main result

**Theorem 1.** The transformation $x = v, u = 1/v_y$ turns the $N$th order Hamiltonian operator $H^{(N,0)} = D_x^2 \circ ((1/u) \circ D_y)^{2n} \circ D_x$, where $N = 2n + 3, n = 0, 1, 2, \ldots$, into the Hamiltonian operator with constant coefficients $\tilde{H}^{(N,0)} = -D_y^{2n+1}$.

The proof is obtained by a straightforward application of lemma 1. Even though the transformation from theorem 1 is not contact, in the particular case under study the transformed operators $\tilde{H}^{(N,0)}$ happen to be free of nonlocal terms (cf remark 1).

Using lemma 1 we can further amplify the result of theorem 1 by providing the Darboux coordinates (cf section 1) for the operators $H^{(N,0)}$.

**Corollary 1.** The transformation $x = (-1)^{n+1} w_n, u = (-1)^{n+1} w_{n+1}$, where $w_k = D^k_1(w)$, and $z$ is the new independent variable, maps the Hamiltonian operator $H^{(N,0)} = D_x^2 \circ ((1/u) \circ D_y)^{2n} \circ D_z$ to the first-order Gardner operator $D_z$ for any odd $N \geq 3$.

**Remark 2.** The inverse of the transformation $x = v, u = 1/v_y$ is nothing but the extended hodograph transformation in the sense of [2]. Let us stress that, unlike the original transformation, the said inverse is not a differential substitution, i.e. it cannot be written in the form $y = \xi(x, u, u_x, \ldots, u_p), v = \chi(x, u, u_x, \ldots, u_p)$.

In [4] the following conjecture is stated:

**Conjecture 1 (De Sole, Kac and Wakimoto).** For any translation-invariant Hamiltonian operator $H$ of order $N \geq 7$ there exists a contact transformation that brings $H$ to either a quasiconstant coefficient skew-adjoint differential operator, or to a linear combination of the operators $H^{(j,0)}$ with $3 \leq j \leq N, j$ odd.

Recall that a differential function is called quasiconstant [4] if it depends only on $x$.

If we further allow for the transformation of the form $x = v, u = 1/v_y$ from theorem 1 and use the latter, we can state a somewhat stronger conjecture:

**Conjecture 2.** Any translation-invariant Hamiltonian operator $H$ of order $N \geq 7$ can be transformed into a quasiconstant coefficient skew-adjoint differential operator using either a contact transformation or a composition thereof with the transformation $x = v, u = 1/v_y$.

The transformation $x = v, u = 1/v_y$ can be written as a composition of the potentiation $x = z, u = w_z$ and of the hodograph transformation $z = v, w = y$. The latter is a contact (in fact, even a point) transformation, so, with the obvious change of notation, we can recast the above conjecture as follows:

**Conjecture 3.** Any translation-invariant Hamiltonian operator $H$ of order $N \geq 7$ can be transformed into a quasiconstant coefficient skew-adjoint differential operator using either a contact transformation or a composition thereof with the potentiation $x = y, u = v_y$. 
Thus, assuming that the above conjecture holds true, extending the class of allowed equivalence transformation using potentiation enables us to have just one type of normal forms for translation-invariant Hamiltonian operators, namely the skew-adjoint differential operators with coefficients that depend only on $x$.

Upon further allowing for the transformations of the form $x = y, v = \sum_{i=0}^{k} b_i(x) v_i$ and using the following

**Lemma 2.** Let $H = \sum_{i=0}^{N} a_i(x) D_i$ be quasiconstant skew-adjoint differential operator. There exists a transformation of the form $x = y, v = \sum_{i=0}^{k} b_i(x) v_i$ that turns $H$ to the Gardner operator $D_y$.

we arrive at a yet stronger conjecture:

**Conjecture 4.** Any translation-invariant Hamiltonian operator of order $N \geq 7$ can be transformed into the Gardner operator $D$ by either a transformation from conjecture 3 or a composition thereof with a transformation of the form $x = y, v = \sum_{i=0}^{k} a_i(x) v_i$.

### 4. Applications to evolution equations

Recall that the transformation $x = v, u = 1/v_y$ can be written as the composition of the potentiation $x = z, u = w_z$ and of the hodograph transformation $z = v, w = y$. The first of these is nothing but introduction of the potential $w$ for $u$. It is readily seen that a bi-Hamiltonian evolution equation

\[ u_t = H_1 \delta_u J_1 = H_2 \delta_u J_2, \]

where $H_i = \sum_{j=1}^{k_i} c_{ij} H^{(N_{ij}, 0)}$, $i = 1, 2, k_i$ are arbitrary natural numbers, and $c_{ij}$ are arbitrary constants, is nothing but the pullback of the bi-Hamiltonian equation

\[ u_t = \tilde{H}_1 \delta_u \tilde{J}_1 = \tilde{H}_2 \delta_u \tilde{J}_2, \]

where $\tilde{H}_i = \sum_{j=1}^{k_i} c_{ij} \tilde{H}^{(N_{ij}, 0)}$, $i = 1, 2, \tilde{H}^{(N, 0)} = -D_z \circ \left( \frac{1}{w_z} D_z \right)^{2n}$, and $\tilde{J}_i$ are obtained from $J_i$ using the substitution $x = z, u = w_z$. It is natural to refer to (4) as to the potential form of (3).

**Proposition 1.** The transformation $z = v, w = y$, where $y$ is the new independent variable, linearizes the potential form (4) of the bi-Hamiltonian evolution equation (3).

Before proving this let us point out the following important consequence of this result.

**Corollary 2.** The differential substitution $x = v, u = 1/v_y$ relates any equation of the form (3) to a linear evolution equation with constant coefficients.

Informally, this just means that the inverse of the transformation $x = v, u = 1/v_y$ linearizes (3), but this statement should be treated with some care, as this transformation is not uniquely invertible, and the inverse is not a differential substitution, cf remark 2 and [18].

**Proof of proposition 1.** The hodograph transformation $z = v, w = y$ sends (4) into

\[ v_t = \tilde{H}_1 \delta_v \tilde{J}_1 = \tilde{H}_2 \delta_v \tilde{J}_2, \]

where $\tilde{H}_i = \sum_{j=1}^{k_i} c_{ij} \tilde{H}^{(N_{ij}, 0)} = -\sum_{j=1}^{k_i} c_{ij} D_y^{N_{ij}-2}$ are linear differential operators with constant coefficients, and $\tilde{J}_i$ are obtained from $J_i$ using the transformation in question.
Lemma 3. Let $v_1 = K[v]$ be an $n$th order evolution equation which is bi-Hamiltonian with respect to a pair of Hamiltonian operators with constant coefficients. Then $v_i = K[v]$ is necessarily a linear equation with constant coefficients, i.e. we have $v_i = \sum_{i=0}^{\infty} c_i v_i$, where $c_i = \text{const}$.

Proof of the lemma. Denote the Hamiltonian operators in question by $D_i$, $i = 1, 2$. Since $v_1 = K[v]$ is bi-Hamiltonian with respect to these operators by assumption, their ratio $R = D_2 / D_1^{-1}$ is a (formal) recursion operator for this equation, that is (see e.g. [13] for details),

$$\text{pr} \ v_K(R) - [D_K, R] = 0,$$

where $D_K = \sum_{i=0}^{\infty} (\partial K / \partial v_i) D^i$ is the Fréchet derivative of $K$ and $\text{pr} \ v_K$ is the prolongation of the evolutionary vector field $v_K$ with the characteristic $K$. Since $D_i$, $i = 1, 2$, have constant coefficients by assumption, we have $\text{pr} \ v_K(D_i) = 0$, and therefore $\text{pr} \ v_K(R) = 0$, so $D_K$ commutes with $R$, whence it readily follows that $D_K$ has constant coefficients (recall that $K$ is independent of $t$ by assumption) and therefore $K$ indeed is a linear combination of $v_i$ with constant coefficients.

The desired result now readily follows from the above lemma.

Example 1. Consider a bi-Hamiltonian evolution equation

$$u_i = D^3_i(u^{-2}) = H^{(3)}_1 \delta_u T_1 = H^{(5)}_0 \delta_u T_2,$$

where $T_1 = -\int dx/u$ and $T_2 = \int x^2 u dx$.

The potential form (4) of (5) reads

$$v_i = D^2_i(w_i^{-2}) = \tilde{H}^{(3)}_5 \delta_w \tilde{T}_1 = \tilde{H}^{(5)}_0 \delta_w \tilde{T}_2.$$

Recall that $u = u_i$ and $x = z_i$; we have $\tilde{H}^{(3)}_i = -D_z$, $\tilde{H}^{(5)}_0 = -D_u \circ ((1/w_i)D_z)^2$, $\tilde{T}_1 = -\int dz_i/w_i$, and $\tilde{T}_2 = \int z_i^2 w_i dz$. Note that (6) has, up to a rescaling of $t$, the form (2.31) from [2].

In perfect agreement with proposition 1 (cf also proposition 2.2 in [2]) the hodograph transformation $z = v$, $w = y$ linearizes (6) into a (trivially) bi-Hamiltonian equation

$$v_i = -2v_{yyyy} = \tilde{H}^{(3)}_0 \delta_v \tilde{T}_1 = \tilde{H}^{(5)}_0 \delta_v \tilde{T}_2,$$

where $\tilde{H}^{(3)}_0 = -D_v$, $\tilde{H}^{(5)}_0 = -D^3_v$, $\tilde{T}_1 = -\int v_i^2 dy$, and $\tilde{T}_2 = \int v^2 dy$.

The transformation $x = v$, $u = 1/v_i$ relates (7) to (5), cf corollary 2, so (5) provides an explicit example of a $C$-integrable (rather than $S$-integrable) bi-Hamiltonian system, just as discussed in section 1.

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