Surveillance of a Faster Fixed-Course Target

ISAAC E. WEINTRAUB, Senior Member, IEEE
ALEXANDER VON MOLL, Senior Member, IEEE
ELOY GARCIA, Senior Member, IEEE
DAVID W. CASBEER, Senior Member, IEEE
MEIR PACHTER, Life Fellow, IEEE

The maximum surveillance of a target which is holding course is considered, wherein an observer vehicle aims to maximize the time that a faster target remains within a fixed-range of the observer. This entails two coupled phases: 1) approach phase and 2) observation phase. In the approach phase, the observer strives to make contact with the faster target, such that in the observation phase, the observer is able to maximize the time where the target remains within range. Using Pontryagin’s Minimum Principle, the optimal control laws for the observer are found in closed-form. Example scenarios highlight various aspects of the engagement.

I. INTRODUCTION

Surveillance-evasion problems are an important class of trajectory-planning problems wherein an evader chooses its trajectory to hinder the surveillance of an enemy observer. Throughout this article, the terms surveillance and observation are used interchangeably. In an early report by Koopman, tactics and scenarios surrounding search and screening were described in detail [1]. Koopman outlined naval and aerial strategies for searching out stationary and mobile targets using range-limited means such as visual detection, radar, and sonar. In his report, the location of targets were considered to be unknown by the observer, and the derived strategies leveraged probability to locate targets of interest through mobile search. Later, Dobbie and Taylor posed and investigated surveillance-evasion [2], [3], [4] making use of differential game theory [5]. Their work posed a turn-limited observer with greater speed than the target and defined the detection region, surveillance region, and escape region. More recent works have contributed to the surveillance-evasion differential game for a turn-limited observer [6], [7], [8], [9]. Where Dobbie and Taylor were concerned with the barriers in the game dictating guaranteed regions of observation or escape by a faster turn-limited observer, Lewin and Breakwell drew their attention to the game of degree [6]—maximizing contact time assuming that the target was already in contact with the observer at the onset. Lewin and Olsder continued the work by changing the contact region from a circle to that of a 2-D cone [7]. Lewin and Olsder also added more states to the original surveillance-evasion differential game by considering the isotropic rocket pursuit evasion game [10]. In the isotropic surveillance-evasion game, the target has bounded speed and can turn instantaneously and the observer has bounded acceleration and can direct it to any desired direction. Greenfield constrained the target and the observer to have the same speed and turn radius. He investigated surveillance-evasion for the game of two-cars and presented the solution to the game of kind and the game of degree [9]. Some 40 years after, Koopman presented strategies for search under uncertainty; Gilles and Vladimirsky posed search-evasion as a differential game under uncertainty [11]. A recent paper by the authors considers a surveillance-evasion scenario with a model similar to that used in this article, but the observer’s control strategy is assumed to be pure pursuit [12].

Different from prior work, this article models the target to be superior—faster than the observer. This presents the observer with limited opportunity to make contact with the target, and optimal strategies for maximizing sustained contact are also investigated. Prior work that considered a superior evader can be found in an early work by Breakwell [13] wherein a single pursuer strives to capture a faster evader using point-capture. Other works have considered multiple pursuers, in an effort to contain the superior evader, include [14], [15], [16], [17], and [18]. More specifically, [14] and [15] focused on the so-called game of approach, a zero-sum differential game in which the fast evader must pass between two pursuers. The aim of the pursuers is to minimize the approach distance, i.e., the minimum distance...
from either pursuer to the evader at any time along the trajectory. On the other hand, [16], [17], and [18] are concerned with formations of pursuers, which encircle the superior evader and determining whether there is a possibility for the evader to escape in any of the gaps between the pursuers. As pointed out in [19], point capture of a superior evader is not possible, even with multiple pursuers—and thus, these works endow a finite capture radius to the pursuers. This finite capture radius is akin to the observation disk modeled in this article (hence, the appearance of the Cartesian oval, both here and in [17]).

Prior work in multiphase pursuit-evasion scenarios have been considered, but not for surveillance-evasion [20], [21], [22], [23], [24], [25], [26], [27]. In [20], Breakwell and Hagedorn considered the point capture of two evaders in succession. In this multiphase optimal control problem a faster pursuer aims to capture two equal-speed evaders in succession. Phase-I corresponds to the pursuit, and capture of the first evader, followed by Phase-II wherein the second evader is captured. The cost/reward function is based upon the capture time of the second evader, which implicitly depends on the agents’ trajectories in Phase-I. A similar setup is found in [21], but rather than maximizing capture time, the evaders seek to minimize their terminal y-coordinate.

In capture-the-flag games, the agents’ strategies change between two phases. In Phase-I, an attacker strives to capture an enemy’s flag while an opposing defender tries to intercept the attacker before the attacker can reach the flag. In Phase-II (conditioned by the outcome of Phase-I), the attacker then attempts to reach a safe zone while the defender strives to reach the attacker before the attacker can reach the safe zone [22], [28], [29]. The control strategies for both sides vary based upon which phase of the game is currently active.

In the work of Nath, a pursuit-evasion scenario between two nonholonomic agents was considered [23]. The scenario itself can be considered to be a single phase, but the proposed evader strategy is comprised of two phases: 1) for large separation distance; and 2) for when the pursuer is nearby. In the reference [30], a multi-player differential game is described where the overall game is played in two phases or stages, the attack stage and the retreat stage.

Additionally, the works by Shinar and Turetsky have focused on pursuit-evasion for hybrid and switched systems in which the cost/reward is based on the zero-effort miss distance [24], [25]. In [24], the pursuer had two sets of dynamics, which were both known to the evader, and could switch once during the game. However, the evader did not know when the pursuer’s dynamics switch. This setup was extended in [25] wherein the pursuer dynamics may switch many times; both the full information (where evader knows when the switches occur) and asymmetric information cases were considered.

Recently, some turret defense scenarios comprising multiple phases have been analyzed [26], [27]. In both scenarios, the transition between phases occurs when the turret aligns its look angle with a mobile agent. The solution methodology involves obtaining the Value function for Phase-II and using it as the terminal cost/reward for Phase-I. For example, in [26], the mobile agent was an attacker, which must decide between engaging the turret or retreating to a prespecified safe zone. The attacker’s instantaneous cost (for either choice) was a piecewise discontinuous function, which depended on whether the turret was aligned or not. Meanwhile, in [27], two attackers cooperated against the turret. In Phase-I, one of the attackers drew the turret away from the second attacker in an effort to better the position of the latter in Phase-II (once the first attacker has been neutralized).

Other relevant works have considered pursuit to consist of multiple phases to maximize an objective concerning and entire pursuit-evasion scenario. In [31], a pursuing missile delayed a target assignment to maximize the expectation of capture of two possible evaders. In Phase-I, a pursuer moved toward a virtual target, then made a decision for which evader to pursue. In Phase-II, the pursuer engaged a specific targeted evader. Later, this target assignment was scaled to consider multiple pursuers and many evaders [32].

This article extends prior work concerning only the observation phase, solved in [33] and [34], to include the approach phase and answer how the approach is related to the observation phase. The relationship between the phases is that the value of Phase-II is treated as the terminal cost for Phase-I, giving rise to the optimal control for the whole scenario via the principle of dynamic programming. The observer’s circular range has realistic implications including, but not limited to the following: 1) visual contact range, 2) sensor range, and/or 3) communication range. Moreover, the target is assumed to have a fixed course, coinciding with two realistic possibilities: 1) the target is unaware of the observer (and thus makes no maneuver to avoid the latter), or 2) the target is far less maneuverable than the observer, and therefore, its trajectory can be approximated as a straight line. This assumption appears also in several examples within the seminal work on missile control by Shneydor [35], as well as in examples in works by Barton and Eliezer concerning a pursuer that implements pure pursuit [36].

The rest of this article is organized as follows. We extend prior art by inclusion of an approach phase to the known results for the optimal observation phase, thereby extending the solution of the maximum observation problem to cover the entire state space. This article also provides a closed-form approach for solving a multiphase optimal control problem.

In order to solve the two-phase optimal control problem for maximizing surveillance of a faster fixed-course target, optimal control theory is leveraging, specifically Pontryagin’s Minimum Principle, [37]. In Section II, the optimal control problem is defined. In Section III, the optimal strategy for Phase-II is solved; then, in Section IV the optimal strategy for the observer in Phase-I is solved. The unified optimal strategy for both phases is presented in Section V. Three scenarios are presented in Section VI, highlighting the optimal strategy for the observer for various initial conditions. Finally, Section VII concludes this article and identifies the future extensions.
II. PROBLEM FORMULATION

Consider the optimal observation of a faster, nonmaneuvering target by a slower observer. The observer seeks to maximize the observation time of the target by controlling its heading. The speed of the observer (O) and the target (T) are \( v_O \) and \( v_T \), respectively. Also, define the speed ratio parameter: \( \alpha \equiv v_O / v_T \). Because the observer is slower than the target, \( 0 < \alpha < 1 \). Further, and without loss of generality, consider a Cartesian coordinate frame whose y-axis is aligned with the velocity vector of the nonmaneuvering constant-speed target. The state of the observer–target scenario is

\[
\mathbf{x}(t) = [x_O(t), y_O(t), y_T(t)]^T \in \mathbb{R}^3
\]

where \( t = [t_2, t_f] \) when it is within a range, \( R > 0 \), of the observer. The observation set, \( \mathcal{O} \), is modeled as a disk of radius, \( R \)

\[
\mathcal{O} = \{ \mathbf{x}(t) | x_O(t) + (y_O(t) - y_T(t))^2 - R^2 \leq 0, \ t \in [t_2, t_f] \}. \tag{2}
\]

The positions of \( O \) and \( T \) in the reference frame are specified by the Cartesian coordinates \( (x_O(t), y_O(t)) \) and \( (0, y_T(t)) \), respectively. The observer’s control variable is its instantaneous heading angle \( u(t) = \psi_O(t) \). The dynamics for this scenario are

\[
\dot{x}_O(t) = \alpha \cos \psi_O(t), \quad \dot{y}_O(t) = \alpha \sin \psi_O(t), \quad \dot{y}_T(t) = \dot{v}_T. \tag{3}
\]

Since the target is faster than the observer, escape of the target from the observer is unavoidable. Escape occurs when \( T \) is no longer within the range, \( R, O \); this determines the final time, \( t_f \). Thus, the terminal manifold is

\[
\mathcal{C} = \{ \mathbf{x}(t) | R^2 - x_O(t)^2 - (y_O(t) - y_T(t))^2 < 0, \ t \geq t_2 \}. \tag{4}
\]

The objective of the observer is to maximize the time for which the target remains inside its observation disk

\[
\psi_O^* \equiv \arg  \min_{\psi_O(t)} J = \int_{t_2}^{t_f} -1 \ dt = t_2 - t_f. \tag{5}
\]

The cost functional in (5) states that the observer aims to choose headings that maximize the observation time in the second phase. The observation time in Phase-II inherently depends on the observer’s trajectory in Phase-I. In order to model the observer’s objectives for Phase-I and Phase-II, specify the approach time as \( t_{ap} \equiv t_2 - t_1 \) and the observation time \( t_{obs} \equiv t_f - t_2 \). These time intervals pertain to Phase-I and Phase-II, respectively. The two phases occur in succession, and therefore: \( t_f = t_{ap} + t_{obs} \), where \( t_1 \), being the instant of initiation of the engagement, is set to \( t_1 = 0 \). In the best tradition of dynamic programming/backward induction, the analysis starts with Phase-II or “end game.”

For the sake of compactness, the explicit time dependence of the states, costates, and control will henceforth be suppressed.

III. PHASE-II: OBSERVATION PHASE

First, consider the observation phase wherein the observer aims to maximize the time that the target remains inside the observation disk. Leveraging dynamic programming, the approach taken in this article is to work backward from final time to initial time. The Hamiltonian for Phase-II is

\[
\mathcal{H}_B = p_{x_O} \alpha \cos \psi_{O,2} + p_{y_O} \alpha \sin \psi_{O,2} + p_{y_T} \tag{6}
\]

and the costates are

\[
\mathbf{p} = [p_{x_O} \ p_{y_O} \ p_{y_T}]^T. \tag{7}
\]

The Pontryagin Minimum Principle (PMP) yields necessary conditions for optimality

\[
\dot{\mathbf{x}}(t) = \frac{\partial \mathcal{H}_B}{\partial \mathbf{p}}(\mathbf{x}(t), \mathbf{p}(t), \psi_{O,2}(t), t), \tag{8}
\]

\[
\dot{\mathbf{p}}(t) = -\frac{\partial \mathcal{H}_B}{\partial \mathbf{x}}(\mathbf{x}(t), \mathbf{p}(t), \psi_{O,2}(t), t), \tag{9}
\]

\[
0 = \frac{\partial \mathcal{H}_B}{\partial \psi_{O,2}}(\mathbf{x}(t), \mathbf{p}(t), \psi_{O,2}(t), t). \tag{10}
\]
and $H_0(t_f) = 0$. The superscript * represents optimality. Evaluating the stationarity condition specified in (10)

$$0 = p_{x_0} \cos \psi^*_O, t_2 - p_{y_0} \sin \psi^*_O, t_2.$$  (11)

Evaluating the necessary conditions in (9), the costates are found to be constant, as expected

$$\dot{p}_{x_0}(t) = 0, \quad \dot{p}_{y_0}(t) = 0, \quad \dot{p}_\psi(t) = 0.$$  (12)

Solving for the optimal observer’s heading $\psi^*_O, t_2$ using (11), the following is obtained:

$$\sin \psi^*_O, t_2 = \frac{p_{x_0}}{\sqrt{p_{x_0}^2 + p_{y_0}^2}}, \quad \cos \psi^*_O, t_2 = \frac{p_{y_0}}{\sqrt{p_{x_0}^2 + p_{y_0}^2}}.$$  (13)

From (13) and (12), the costates are constant, therefore, the optimal heading of the observer is constant.

Next, consider the transversality conditions, which are used to formulate the relationship between the states and costates at final time, $t_f$

$$\frac{\partial}{\partial t_f} (x^*(t_f), t_f) - p(t_f) = \sigma \frac{\partial}{\partial t_f} (x^*(t_f), t_f)$$  (14)

where $h(\cdot)$ is the terminal cost of the object functional; $\sigma$ is a slack variable; $m$ is the terminal manifold, (4)

$$m(x(t_f), t_f) = x^2_O(t_f) + (y^2_O(t_f) - y^2_T(t_f))^2 - R^2 = 0.$$  (15)

Evaluation of the transversality conditions in (14)

$$-p^*(t_f) = \sigma \left[ \frac{\partial m}{\partial x_f} \frac{\partial m}{\partial y_f} \frac{\partial m}{\partial y_T} \right]_{t=t_f}.$$  (16)

Therefore

$$p^*_{x_0}(t_f) = -\sigma \frac{\partial m}{\partial x_f} \big|_{t=t_f} = -2\sigma x_0(t_f),$$  

$$p^*_{y_0}(t_f) = -\sigma \frac{\partial m}{\partial y_f} \big|_{t=t_f} = 2\sigma (y_T(t_f) - y_0(t_f)),$$  

$$p^*_\psi(t_f) = -\sigma \frac{\partial m}{\partial y_T} \big|_{t=t_f} = 2\sigma (y_0(t_f) - y_T(t_f)).$$  (17)

Substitution of $p^*_{x_0}$ and $p^*_{y_0}$ from (17) into the optimal heading obtained in (13), the optimal heading for the second phase is

$$\sin \psi^*_O, t_2 = \frac{\pm(y_T(t_f) - y_0(t_f))}{R}.$$  (18)

When the sign of (18) is +, the observer is heading toward the target and when the sign is –, the observer is headed away from the target, which is not a case of interest. The observer needs to have a positive component of velocity along the target’s direction for it to be a viable optimal path.

Because of (18), it is observed that $\bar{M}\bar{Q}$ and $\bar{O}\bar{S}$ are collinear. Therefore, one can make use of $\Delta LMS$ for analysis in Phase-II. More details concerning Phase-II are in an earlier work [33], and for the 3-D case, refer to [34].

**Lemma 1** The optimal heading of the observer that maximizes observation time for Phase-II is

$$\psi^*_O, t_2 = \cos^{-1} \left( \frac{(\alpha^2 - 1) \sin \lambda_{TO, 2}}{\alpha^2 + 2\alpha \cos \lambda_{TO, 2} + 1} \right),$$

where $\alpha \in (0, 1)$ is the speed ratio between the observer and the target and $\lambda_{TO, 2} \in [-\pi, \pi]$ is the relative bearing from the target to the observer.

**Proof** Using the law of cosines for the triangle $\Delta MSL$ from Fig. 1

$$\bar{M}\bar{S}^2 = \bar{L}\bar{S}^2 + R^2 - 2\bar{R}\bar{L}\bar{S} \cos \lambda_{TO, 2}.$$  (19)

From the speed ratio

$$\bar{M}\bar{S} = \alpha \bar{L}\bar{S} + R.$$  (20)

Substitution of (20) into (19), and solving for $\bar{L}\bar{S}$, the following is obtained:

$$\bar{L}\bar{S} = \frac{2R(\alpha \cos \lambda_{TO, 2})}{1-\alpha^2}. $$  (21)

Since the cosine of an angle is the adjacent distance over the hypotenuse, the following is obtained:

$$\cos(\pi - \psi^*_O, t_2) = \frac{R \sin \lambda_{TO, 2}}{\alpha \bar{L}\bar{S} + R}.$$  (22)

Inserting (21) into (22)

$$-\cos \psi^*_O, t_2 = \frac{R \sin \lambda_{TO, 2}}{2R(\alpha \cos \lambda_{TO, 2}) + 1-\alpha^2}.$$  (23)

Through algebraic manipulation of (23), one finally obtains the optimal observer heading for the second phase

$$\psi^*_O, t_2 = \cos^{-1} \left( \frac{(\alpha^2 - 1) \sin \lambda_{TO, 2}}{\alpha^2 + 2\alpha \cos \lambda_{TO, 2} + 1} \right).$$  (24)

**Lemma 2** Once the target is within the range of the observer (under optimal play), the target remains inside the observation disk until the state, $x(t)$, reaches the terminal manifold, $\mathcal{G}$ at time, $t_f$ – observation is invariant.

**Proof** Using the law of cosines for $\Delta MSL$, the following relationship is obtained:

$$R^2 = \bar{L}\bar{S}^2 + (\alpha \bar{L}\bar{S} + R)^2 - 2\bar{R}\bar{L}\bar{S} (\alpha \bar{L}\bar{S} + R) \cos \omega.$$  (25)

Solving (25) for $\cos \omega$, the following is obtained:

$$\cos \omega = \frac{\bar{L}\bar{S} \bar{S} \bar{L} \bar{S}}{2(\alpha \bar{L}\bar{S} + R)}.$$  (26)

Now, consider a future time $t \in (t_2, t_f)$. Using the law of cosines for $\Delta FHS$

$$\rho^2 = \bar{F}\bar{S}^2 + \bar{H}\bar{S}^2 - 2\bar{F}\bar{H}\bar{S} \cos \omega.$$  (27)

Recognizing that $\bar{M}\bar{Q} = \alpha \bar{F}\bar{S}$ and $\bar{H}\bar{S} = \bar{H}\bar{Q} + R$

$$\rho^2 = \bar{F}\bar{S}^2 + (\alpha \bar{F}\bar{S} + R)^2 - 2\bar{F}\bar{H}\bar{S} (\alpha \bar{F}\bar{S} + R) \cos \omega.$$  (28)

Substituting (26) into (28), the following is obtained:

$$\rho^2 = R^2 + \frac{\bar{F}\bar{R}}{\alpha \bar{L}\bar{S} + R} \left( \bar{F}\bar{S} - \bar{L}\bar{S} \right)(1 - \alpha^2).$$  (29)

Therefore, the values of $t \in (t_2, t_f)$, $\rho < R$.

The function that describes the target distance while being observed is given in (21). The target is moving with unity speed, and therefore, the observation time is

$$t_{obs} = t_f - t_2 = \frac{2R(\alpha \cos \lambda_{TO, 2})}{1-\alpha^2}. $$  (30)

**Lemma 3** The observation time monotonically increases over the interval $\lambda_{TO, 2} \in [-\pi, 0]$ and monotonically decreases over the interval $\lambda_{TO, 2} \in [0, \pi]$ with a maximum at $\lambda_{TO, 2} = 0$.

**Proof** The observation time is as given in (30). Taking the partial derivative of the observation time, $t_{obs}$, with respect to
\( \lambda_{TO,2} \) and setting equal to zero provides candidate extremals for \( \lambda_{TO,2} \in [-\pi, \pi] \)

\[
\frac{\partial \psi}{\partial \lambda_{TO,2}} = 0 \Rightarrow -\frac{2R}{1-\alpha} \sin \lambda_{TO,2} = 0. \tag{31}
\]

Therefore, the observation time is a maximum when \( \lambda_{TO,2} = 2\pi n, n \in \mathbb{Z} \). The only candidate in the domain of \( \lambda_{TO,2} \in [-\pi, \pi] \) is

\[
\lambda_{TO,2} = 0. \tag{32}
\]

The speed ratio \( \alpha \in (0, 1) \) and \( R > 0 \). The observation time is a maximum when \( \lambda_{TO,2} = 0 \) and monotonically decreases from 0 over the domain \( \lambda_{TO,2} \in [-\pi, \pi] \). Further, investigating the sign of the partial in (31)

\[
-\frac{2R}{1-\alpha} \sin \lambda_{TO,2} > 0, \lambda_{TO,2} \in [-\pi, 0), \tag{33}
\]

\[
-\frac{2R}{1-\alpha} \sin \lambda_{TO,2} < 0, \lambda_{TO,2} \in (0, \pi]. \tag{34}
\]

From (33), observation time monotonically increases over the interval \( \lambda_{TO,2} \in [-\pi, 0) \). From (34), observation time monotonically decreases over the interval \( \lambda_{TO,2} \in (0, \pi] \). Lastly, from (32), the maximum observation time occurs when \( \lambda_{TO,2} = 0 \).

**Lemma 4** Observation time is zero for Phase-II when \( \lambda_{TO,2} \in [\cos^{-1}(-\alpha), \pi] \cup [-\pi, -\cos^{-1}(-\alpha)] \).

**Proof** From (30), the observation time is zero when \( t_{obs} \leq 0 \); namely

\[
\frac{2R(\alpha+ \cos \lambda_{TO,2})}{1-\alpha^{2}} \leq 0 \Rightarrow \alpha + \cos \lambda_{TO,2} \leq 0 \tag{35}
\]

rearranging (35), the conditions for which observation time is zero occurs when

\[
\lambda_{TO,2} \geq \cos^{-1}(-\alpha) \tag{36}
\]

and by symmetry

\[
\lambda_{TO,2} \leq -\cos^{-1}(-\alpha). \tag{37}
\]

Since the domain of \( \lambda_{TO,2} \in [-\pi, \pi] \), the regions for which observation time is zero occurs when \( \lambda_{TO,2} \in [\cos^{-1}(-\alpha), \pi] \cup [-\pi, -\cos^{-1}(-\alpha)] \).

**Lemma 5** The maximum possible observation time is

\[
t_{obs} = \frac{2R}{1-\alpha}. \tag{38}
\]

**Proof** From (30), the observation time is

\[
t_{obs} = \frac{2R(\alpha+ \cos \lambda_{TO,2})}{1-\alpha^{2}}. \tag{39}
\]

From Lemma 3, the maximum observation time occurs when \( \lambda_{TO,2} = 0 \). Substitution of \( \lambda_{TO,2} = 0 \) into (30) results in

\[
t_{obs} = \frac{2R(\alpha+1)}{1-\alpha} = \frac{2R(\alpha+1)}{(1-\alpha)(1+\alpha)} = \frac{2R}{1-\alpha}. \tag{39}
\]

\[\text{IV. PHASE-I: APPROACH PHASE}\]

The objective in Phase-I is to place the observer in a favorable position, to maximize the time of observation in Phase-II. From Lemma 3, more favorable positions are located lower on the observation disk for Phase-II. Much like Phase-II, because the observer is holonomic; the costates of Phase-I are constant; and, therefore, the optimal strategy for Phase-I is a straight-line trajectory. This stems from the Hamiltonian for Phase-I

\[
H_{1} = p_{\phi_{1}} \cos \psi_{1,2} + p_{\phi_{2}} \sin \psi_{1,2} + p_{\gamma}. \tag{40}
\]

Just as shown in (11), (12), and (13), the optimal heading of the observer for Phase-I is a straight line. Therefore, the selection of the optimal heading \( \psi_{1,2} \) that maximizes the overall observation time in Phase-II is of interest—namely by (38)

\[
\psi_{1,2} = \arg \max \psi \quad \psi_{1,2} \quad \text{arg max} \quad \frac{\cos \lambda_{TO,2}}{1-\alpha}. \tag{41}
\]

As shown in Lemma 5, the maximum observation in the second phase occurs when \( \lambda_{TO,2} = 0 \). Furthermore, by Lemma 3, the observation time monotonically increases over the interval \( \lambda_{TO,2} \in [-\pi, 0) \) and decreases over the interval \( \lambda_{TO,2} \in [0, \pi] \). In the problem definition, \( \alpha \) and \( R \) are constant and do not depend upon the heading of the observer. This allows (41) to be separated and rewritten in a simpler form as follows:

\[
\psi_{1,2} = \arg \max \psi \quad \psi_{1,2} \quad \text{arg max} \quad \frac{2R + \cos \lambda_{TO,2}}{1-\alpha}. \tag{42}
\]

Over the interval \( \lambda_{TO,2} \in [-\pi, \pi] \), the maximum of \( \cos \lambda_{TO,2} \) occurs when \( \lambda_{TO,2} = 0 \). And, due to the symmetry of the \( \cos(\cdot) \) function over the domain \( [-\pi, \pi] \), the absolute value of the argument should be minimized to maximize the \( \cos(\cdot) \) of that argument. Because the domain of \( \lambda_{TO,2} \in [-\pi, \pi] \), this fact allows the maximization problem in (42) to be equivalently written as a minimization problem

\[
\psi_{1,2} = \arg \min \psi \quad \psi_{1,2} \quad \text{arg min} \quad \frac{\cos \lambda_{TO,2}}{1-\alpha}. \tag{43}
\]

This minimization means that the optimal strategy for Phase-I is for the observer to take a heading for which the target contacts the observation disk as low as possible, thus, maximizing the total observation time in Phase-II.

**A. Decision Line**

The state space is partitioned into regions \( \mathcal{R}_{1}, \mathcal{R}_{2}, \) and \( \mathcal{R}_{3} \). The Decision Line, defined below, is used for partitioning the state space for obtaining the optimal control of the observer; its definition makes use of the well-known Apollonius circle, whose definition and derivation can be found in [38, Appendix 2]. Repeated here, the definition of the Apollonius Circle (as used in this work) is as follows.

**Definition 1** The Apollonius circle provides the locus of points describing the intersection of two agents taking
A straight-line constant velocity path in a Euclidean plane. Called foci, the initial condition of the two agents described by the target (T) and a point on the observation disk (I) and the relative speed of the observer to the target (α) are used to define the circle. The location of the Apollonius circle’s origin (C) lies on the ray from T pointing toward I and is a distance

$$\overline{TC} = \frac{2}{1 - \alpha^2} \overline{TI}$$

from I, opposite T, as pictured in Figs. 1 and 2; and whose radius is

$$\overline{CL} = \frac{\alpha}{1 - \alpha^2} \overline{TI}.$$

**Definition 2** The Decision Line (DL) is the locus of points, DL ⊂ \{Z\}; where the Apollonius circle, whose foci are Z and T with associated speed ratio parameter, α, is tangent to the target’s path.

Points in the (î, î̂) frame that are above the DL will have an Apollonius circle (w.r.t. the target, T, and associated speed ratio parameter, α) that crosses the î̂-axis, while points below the DL will have Apollonius circles, which do not intersect the î̂-axis. The DL will be useful in determining which points on the observation disk (if any) can make contact with the target at the end of Phase-I (beginning of Phase-II).

**Definition 3** The point W is the point on the observation disk centered at O, whose radius is R, that is located at \((x_0, y_0) - R\).

The point W is located at the bottom of the observation disk and is important for determining the optimal strategy for the observer in the Phase-I, as will be seen later.

**Lemma 6** The angle of the DL with respect to the î̂-axis is \(\theta_{DL} = \cos^{-1} \alpha\).

**Proof** Because the target is moving vertically along the î̂-axis of the Cartesian fixed frame, the center of the tangent circle to the target’s path is horizontal to the point of tangency. The radius of the Apollonius circle which is tangent to the target’s path has radius, \(R = \frac{ad}{1 - \alpha^2}\), where \(\alpha\) is the speed ratio parameter and \(d\) is the separation distance between the foci T and Z. Furthermore, the distance between Z and the center of the Apollonius circle is \(\overline{ZC} = \frac{2}{1 - \alpha^2} \alpha d\). These distances are labeled in Fig. 3 to assist the reader in their visualization of the Apollonius circle, which is tangent to the target’s path. From the geometry of Apollonius Circle, Apol(T, Z) and \(\Delta CAT\); as shown in Fig. 3, the angle of the decision line is found from solving the following for \(\theta_{DL}\):

$$\cos \theta_{DL} = \frac{\frac{da}{1 - \alpha^2}}{d + \frac{da}{1 - \alpha^2}} = \alpha. \tag{44}$$

Therefore, the angle of the decision line with respect to the target’s location is

$$\theta_{DL} = \cos^{-1} \alpha. \tag{45}$$

Furthermore, the DL is a line because \(\theta_{DL}\) is independent of the distance between T, Z, and d. Thus, any point whose angle with respect to T is \(\theta_{DL}\) is in DL. Therefore, DL, must be a straight line.

**Definition 4** \(\mathcal{R}_1\) is the region of the state space where no observation is possible: \(\mathcal{R}_1 \triangleq \{x \mid t_{obs} = 0\}\).

**Definition 5** \(\mathcal{R}_3\) is the region of the state space where the optimal observation time is the maximum possible observation time: \(\mathcal{R}_3 \triangleq \{x \mid t_{obs} = \frac{2R}{1 - \alpha}\}\).

**Definition 6** \(\mathcal{R}_2\) is the region of the state space where the optimal observation time is bounded between zero and the maximum observation time: \(\mathcal{R}_2 \triangleq \{x \mid 0 < t_{obs} < \frac{2R}{1 - \alpha}\}\).

**Lemma 7** If the point at the bottom of the observation disk, W, is on or above the decision line, then the state \(x \in \mathcal{R}_3\).

**Proof** By Lemma 6, any point in the Cartesian space above or on the decision line may reach the target. If a point on the bottom of the observation disk lies on or above the decision line, then the bottom of the observation disk may reach the target, corresponding to \(t_{obs,2} = 0\), and the observation time is, therefore, \(t_{obs} = \frac{2R}{1 - \alpha}\).

**Lemma 8** If the observation disk lies beneath the decision line and intersects with the decision line at 1 or fewer points then the state \(x \in \mathcal{R}_1\).

**Proof** By Lemma 6, any point in the Cartesian space above or on the decision line may reach the target. If the entire
observation disk lies beneath the decision line, then the observer is unable to reach the target for any amount of time. If the decision line intersects the observation at one point, then it does so tangentially. Because the decision line has an angle $\theta_{DL} = \cos^{-1} \alpha$, this means that the tangent point occurs at the angle defined by the limit of the observation time from Lemma 4.

**LEMMA 9** If the decision line intersects the observer’s disk at two points, then the state $x \in B_2 \cup B_1$. Consequently, the optimal observation time is nonzero.

**PROOF** If a line intersects a circle at two real points, it cannot be tangential to the circle, and by definition must intersect the circle [39, pp. 459]. This requires the line to create a chord. By Lemma 6, any point in the Cartesian space above or on the decision line may reach the target. Since two such points exist where the DL crosses the observation disk, via a chord, it is possible for the observer to reach the target and $t_{obs} \neq 0$. By Definition 4, all states where $t_{obs} = 0$ belong to $B_1$; therefore, $x \notin B_1$, and therefore, by Definition 6 and Definition 10 $x \in B_2 \cup B_3$.

**LEMMA 10** If the decision line intersects the observer’s disk at two points and the x-coordinate of either of the said points is less than the x-coordinate of $O$, then the state $x \in B_2$.

**PROOF** By Lemma 9, If the decision line intersects the observer’s disk at two points, then the state $x \in B_2 \cup B_3$. Consequently, the optimal observation time is nonzero. Consider each intersection point: $I_1$ and $I_2$ whose coordinates are $(x_{I_1}, y_{I_1})$ and $(x_{I_2}, y_{I_2})$, respectively. Recall that the observer location is $O$ and its Cartesian coordinate is $(x_0, y_0)$. Consider the following by contradiction: If both $x_{I_1} \geq x_0$ and $x_{I_2} \geq x_0$, then the DL creates a chord whose points are to the right of $O$. Therefore, the point at the bottom of the observation disk is on or above the DL. Therefore, from Lemma 7, $x \in B_3$.

If either $x_{I_1} < x_0$ or $x_{I_2} < x_0$, then the state does not belong to $B_3$ (because there are two points) or $B_3$, for this would require both $x_{I_1} \geq x_0$ and $x_{I_2} \geq x_0$. Therefore, by Definition 6, $x \in B_2$.

**B. Optimal Observer Strategy - Phase 1**

The optimal heading for $O$ in the first phase is dependent upon the speed ratio parameter, $\alpha$, the observation range, $R$, and the initial state, $x(t_0) = [x_0(t_0), y_0(t_0), y_r(t_0)]^T$. The regions $B_1$, $B_2$, and $B_3$, and the decision line are shown in Fig. 4. Three targets for an observer with observation range, $R$, and speed ratio parameter, $\alpha$, are shown in order to highlight the various regions where the optimal observation time is maximum, nonzero, and zero.

1) **Case 1:** First, consider $T_6$. At initial time, $x_0 \in B_1$. By Lemma 8, the observer cannot reach the target and the $t_{obs} = 0$. Consequently, the optimal control, $\psi_{DL,1}$, is undefined.

If the target is located in the yellow or green regions, relative to the observer’s location $O$, then the observer can observe the target for some nonzero amount of time.

2) **Case 2:** Next, consider $T_1$. At initial time, $x_1 \in B_2$. By Definition 6, the observer reaches the target at some angle that is not directly in front of the target.

![Fig. 4. Outcome of the optimal observation scenario is dictated by the relative position of the target to the observer, the speed ratio parameter, $\alpha$, and the observation range, $R$. At initial time, if the target is located in the gray region, then the observer is unable to reach the target. If the target is located in the yellow or green regions, relative to the observer’s location $O$, then the observer can observe the target for some nonzero amount of time.](image-url)

**LEMMA 11** If $x \in B_2$, the optimal observation time is

$$t_{obs} = \frac{2R}{1-\alpha^2 \left(\frac{x_0 + y_0m_{DL}}{R}\right)}$$

where

$$m_{DL} = \frac{\sqrt{1 - \alpha^2}}{\alpha} \quad \sigma = x_0^2 + y_0^2 - R^2.$$  

**PROOF** If $x \in B_2$, then the DL must intersect the observation disk at two points (from Lemma 9) and one or both of these intersections has an x-coordinate less than that of the observer, $O$ (from Lemma 10). Let the point $I$ be defined as the intersection of the DL with the observation disk with the smaller x-coordinate (c.f. Fig. 1). The coordinates of $I$ can be found by computing the intersection of the DL with the observation disk. The equations describing the DL and observation disk can be written, respectively, as

$$y = \frac{\sqrt{1 - \alpha^2}}{\alpha} x$$

$$y = \frac{\sqrt{1 - \alpha^2}}{\alpha} x$$

Substituting (48) into (49) yields a quadratic equation in $x$ corresponding to the two intersections of the DL and
and the expression for \( L = y - y_0 = \sin \alpha - \sin \beta = x - x_0 \) (with associated speed ratio \( \frac{\dot{x}}{\dot{y}} = \frac{x_0 - y_0}{y_0 - y} \)).

Finally, the associated observation time in the subsequent phase is found by substituting \( x_t \) and \( y_t \) into the above equation and then into (30), yielding (46).

**Remark 1** By Lemma 11, the bottom-most intersection point between the DL and the observation disk is the optimal interception point. While points on the observation disk that reside above the DL represent suboptimal intersection points as by (43) and by definition theses intersection points are not lower than the optimal one on the observation disk.

**Remark 2** By (46) it can be seen that as the chosen interception point is selected higher on the observation disk (i.e., as \( y_t \) is increased), the observation time decays (linearly with \( y_t \)).

**Lemma 12** If \( x \in B_2 \), the optimal observer heading is

\[
\psi_{o,1}^* = \cos^{-1} \alpha + \frac{\pi}{2}. \tag{50}
\]

**Proof** This proof makes extensive use of the geometry depicted in Fig. 1. Recall that the point \( I \) is defined as the leftmost intersection of the DL with the observation disk. Because the point \( I \) is fixed w.r.t. the observer, it moves at speed \( \alpha \) relative to the target. Also, from Definition 2, the Apollonius circle whose foci are \( I \) and \( T \) must be tangent to target’s path. Let \( L \) be this tangent point. Then, by the definition of an Apollonius circle it must be that \( \Delta TL = IL \).

Now consider the triangle \( \Delta TL \); essentially, two sides and the angle \( \angle TLI \) are known. The associated Law of Cosines is

\[
\overline{TL}^2 = \overline{L}^2 + (x_t^2 + y_t^2) - 2\overline{TL} \sqrt{x_t^2 + y_t^2} \cos \left( \frac{\pi}{2} - \cos^{-1} \alpha \right). \tag{51}
\]

Substituting in \( \cos \left( \frac{\pi}{2} - \cos^{-1} \alpha \right) = \sqrt{1 - \alpha^2} \) along with \( \overline{TL} = \alpha \overline{LL} \) yields

\[
\overline{LL} = \sqrt{\frac{x_t^2 + y_t^2}{\sqrt{1 - \alpha^2}}} \tag{52}
\]

and thus, all three sides of the triangle \( \Delta TL \) are known. Now, the Law of Cosines may be used again to determine the angle \( \angle TLI \)

\[
\overline{TL}^2 = \overline{LL}^2 + (x_t^2 + y_t^2) - 2\overline{LL} \sqrt{x_t^2 + y_t^2} \cos (\angle TLI). \tag{53}
\]

Substituting in \( \overline{LL} = \alpha \overline{LL} \) and the expression for \( \overline{LL} \) yields \( \cos (\angle TLI) = 0 \), and therefore, \( \angle TLI = \frac{\pi}{2} \). Finally, the angle that \( \overline{I}L \) makes with the positive \( x \)-axis is given by (50).

**3) Case 3:** Finally, consider \( T_2 \) in Fig. 4. The point \( W \) (i.e., the lowest point of the observation disk) is on the decision line, which implies that \( x_2 \in B_3 \). From Definition 5, the optimal observation time must be the maximum possible observation time, \( \lambda_{obs} = \frac{2R}{\delta - \alpha} \).

When \( W \) is above the DL, then the Apollonius circle whose foci are \( W \) and \( T \) (with associated speed ratio \( \alpha \)) intersects the positive \( y \)-axis twice. By definition, the point \( W \) (moving with relative speed \( \alpha \)) can be moved onto any of the points along the positive \( y \)-axis that are inside this Apollonius circle before the target arrives at that point. All of these possibilities result in \( \lambda_{T,0,2} = 0 \), which corresponds to the maximum observation time. Therefore, all of these possibilities are equally optimal, and thus, the optimal heading for the observer is nonunique in this case.

**Lemma 13** If \( x \in B_3 \), any (constant) observer heading in the range

\[
\psi_{o,1}^* \in \left[ \frac{\pi}{2} - \xi + \sin^{-1} \left( \frac{\sin \delta}{\alpha} \right), \frac{\pi}{2} - \xi - \sin^{-1} \left( \frac{\sin \delta}{\alpha} \right) \right] \tag{54}
\]

where

\[
\xi = \sin^{-1} \left( \frac{x_0}{\sqrt{x_0^2 + y_0^2}} \right) = \sin^{-1} \left( \frac{x_0}{\sqrt{x_0^2 + (y_0 - \delta)^2}} \right) \tag{55}
\]

is optimal.

**Proof** Consider the general configuration for Case 3 given in Fig. 5. It is clear that all points along the line segment \( LL' \) lie inside the associated Apollonius circle, where \( L \) and \( L' \) are the two intersections of the Apollonius circle with the \( y \)-axis. Thus, it suffices to compute the headings associated with moving the point \( W \) to \( L' \) and to \( L \), respectively, as any heading between these will reach a point on \( LL' \) thereby achieving \( \lambda_{T,0,2} = 0 \), giving the maximum observation time.

4154 IEEE TRANSACTIONS ON AEROSPACE AND ELECTRONIC SYSTEMS VOL. 59, NO. 4 AUGUST 2023
First consider the triangle \( \triangle TLW \). From the Law of Sines and the definition of the Apollonius circle, it must be that
\[
\frac{\sin \gamma_L}{\sin \xi} = \frac{\sin \xi}{\sin \gamma_L'} = \frac{\sin \xi}{\sin \gamma_L}.
\]
giving \( \sin \gamma_L' = \frac{\sin \xi}{\sin \alpha} \). A similar relationship for the triangle \( \triangle TLW \) gives \( \sin \gamma_L = \frac{\sin \xi}{\sin \alpha} \) as well. Thus, it must be that \( \gamma_L' < \frac{\pi}{2} < \gamma_L \). The associated headings can be written as \( \frac{\pi}{2} - \xi + (\pi - \gamma_L) \) and \( \frac{\pi}{2} - \xi + (\pi - \gamma_L') \). Taking the arcsin of the \( \sin \gamma \) terms, accounting for the proper quadrant, and substituting into this expression yields (52).

**Lemma 14** If \( x \in B_1 \), then the time it takes for the observer to approach the target (time of Phase-I) is \( t_{\text{apr}} = \frac{y_0 - R}{1 - \sin \psi_{O,1}} \). Where the feasible domain of the observer headings, \( \psi_{O,1} \), are as defined in (52). Let \( B_1 \) be the target’s position and its trajectory is aligned with the \( y \)-axis.

Consider the \( \triangle TWL' \). The time that occurs until the observer reaches the target is \( t_{\text{apr}} = \frac{y_0 - R}{1 - \sin \psi_{O,1}} \). The following can be written about \( TL' \)
\[
t_{\text{apr}} = \frac{\sqrt{y_0^2 + (y_0 - R)^2}}{1 - \sin \psi_{O,1}}.
\]
(54)

By Lemma 13, the Apollonius circle whose foci are \( T \) and \( W \) and speed ratio is \( \alpha \) dictates the interception headings for the observer \( \psi_{O,1} \) in (52). Inspecting the Apollonius circle
\[
\alpha TL' = TL'.
\]
(55)

Substituting the speed ratio from (55) into (54)
\[
TL' = (y_0 - R) + \alpha TL' \sin(\pi - \psi_{O,1}).
\]
(56)

Solving (56) for \( TL' \) yields
\[
t_{\text{apr}} = \frac{y_0 - R}{1 - \sin \psi_{O,1}}.
\]
(57)

### V. COMPLETE SOLUTION

Using the results of the previous Lemmas, the full solution of multiphase optimal control problem is summarized in the following.

**Theorem 1** The optimal heading that maximizes the observation time in Phase-II, the corresponding approach time, and corresponding observation time are given by

\[
\psi_{O} = \begin{cases} 
\text{undef} & x \in \mathcal{B}_1 \ t \in [t_1, t_f] \\
\sin^{-1} \alpha + \pi/2 & x \in \mathcal{B}_2 \ t \in [t_1, t_f] \\
\text{Eq. (52)} & x \in \mathcal{B}_3 \ t \in [t_1, t_2] \\
\text{Eq. (24)} & x \in \mathcal{B}_2 \ t \in [t_2, t_f] \\
\pi/2 & x \in \mathcal{B}_3 \ t \in [t_2, t_f]
\end{cases}
\]
(58)

\[
t_{\text{apr}} = \begin{cases} 
\infty & x \in \mathcal{B}_1 \\
\sqrt{y_0^2 + (y_0 - R)^2} & x \in \mathcal{B}_2 \\
\text{otherwise} & x \in \mathcal{B}_3
\end{cases}
\]
(59)

\[
t_{\text{obs}} = \begin{cases} 
0 & x \in \mathcal{B}_1 \ t \in [t_1, t_f] \\
2R \alpha + 2\alpha R \sin \psi_{O} & x \in \mathcal{B}_2 \ t \in [t_2, t_f] \\
\text{otherwise} & x \in \mathcal{B}_3
\end{cases}
\]
(60)

where \( x_1 \) and \( y_1 \) are given in (47). \( t_{\text{obs}} \) is given in (39). \( \lambda_{TO,2} = \tan^{-1} \left( \frac{y_0 - y_T}{y_0 - y_1} \right) \) is given in (24), and \( \psi_{O,1}^* \) is given by (52) when \( x \in \mathcal{B}_3 \).

**Proof** The region in which the state lies is determined via Lemmas 7, 8, 10, and 9, and the associated optimal heading and observation times are given by Lemmas 11, 12, and 13.

Consider the optimal headings for Phase-II. From Lemma 8, observation time is zero therefore the associated optimal heading for the observer is undefined for Phase-II. By Lemma 1, the optimal heading is as described in (24), this applies when \( x \in \mathcal{B}_2 \cup \mathcal{B}_3 \). However, when \( x \in \mathcal{B}_3 \), the observer reaches the target so that \( \lambda_{TO,2} = 0 \) by Lemma 5 and therefore the optimal heading of the observer is \( \psi_{O,2}^* = \pi/2 \).

Next, consider the approach time for Phase-I. By Lemma 8, when \( x \in \mathcal{B}_1 \), the observation time is zero, and therefore, the approach phase never terminates; this is because the observer cannot reach the target. Next, by Lemma 12, when the distance taken by the target before being contacted by the observer is shown in (51). Dividing this by the speed of the target (unity speed) provides the approach time, \( t_{\text{apr}} \), when \( x \in \mathcal{B}_2 \). Lastly, by Lemma 14, the approach time is described by (57), when \( x \in \mathcal{B}_3 \).
Fig. 7. Example A: The observer and its observation disk lie below the decision line and \( x_0 \in \mathcal{B}_1 \), and therefore, observation is not possible.

A. Scenario A—No Observation

As described by (58), if \( x \in \mathcal{B}_1 \) then the optimal strategy is undefined and the observation time is zero. Although, this scenario is not as interesting as the others, it highlights when observation is not possible.

Consider Scenario A from Table I, the initial state of the system is \( x_0 = (8, 4, 0) \), the speed of the observer with respect to the target is \( \alpha = 0.60 \), and the observation range is \( R = 2.00 \).

The first step is to determine if the DL intersects the observation disk. By substituting (48) into (49) and solving for \( x \), the intersections are obtained from the resulting quadratic equation. The DL as shown in (48) for the initial conditions provided is: \( y = 1.333x \). The observation disk (centered at the location of the observer) as provided in (49) is \( (x - 8)^2 + (y - 4)^2 = 4 \). The observation disk and the DL are shown in Fig. 7. Solving the two equations and two unknowns results in the following imaginary coordinates: \((x, y) = (4.8000 \pm 2.0785i, 6.4000 \pm 2.7713i)\). In this example, the roots of the quadratic equation are complex, and therefore, the DL does not intersect the observation disk.

Evaluating the equation for the DL at the \( x \)-coordinate of the observer: \( y = 1.333\times(8) = 10.6667 \). This location is greater than the \( y \)-coordinate of the observer: 4.00. Therefore, the observation disk lies below the DL and \( x_0 \in \mathcal{B}_1 \). Therefore, by Lemma 8 and Theorem 1, observation is not possible no matter the strategy of the observer.

B. Scenario B—Limited Observation

As described by (58) in Theorem 1, if \( x \in \mathcal{B}_2 \) then there exists a unique optimal heading for the observer in Phase-I and another in Phase-II. These are used to provide the maximum possible observation of the faster nonmaneuvering target.

First, the intersections of the DL with the observation disk are found. In this example, the DL has equation \( y = 0.75x \) and the observation disk is \( (x - 5)^2 + (y - 2)^2 = 4 \). The DL and observation disk for this example are plotted in Fig. 8. By solving the quadratic equation from substituting the DL equation into the observation disk, the two intersections are: \((x, y) = (3.01737, 2.26303)\) and \((5.30263, 3.97697)\). By Lemma 10, since two intersections exist, \( x \in \mathcal{B}_2 \). Once the membership of \( x \) is obtained, the optimal strategies for Phase-I and Phase-II are provided by (58). Making the appropriate substitutions, the optimal strategy is

\[
\psi_{O_1}(t) = \begin{cases} 
131.4096^\circ & t \in [0, 6.2862] \\
104.4560^\circ & t \in [6.2862, 13.7138] 
\end{cases}
\]

The states at the critical times are as shown in Table II.

| Parameter | Value |
|-----------|-------|
| \( t_1 \) | 0.0000 |
| \( t_2 \) | 6.2862 |
| \( t_f \) | 13.714 |

The approach time is \( t_{apr} = 6.2862 \text{TU} \) and the observation time is \( t_{obs} = 7.4276 \text{TU} \).

C. Scenario C—Maximum Observation

As described by (58) in Theorem 1, if \( x \in \mathcal{B}_3 \) then there exists a range of optimal headings that the observer can take in Phase-I that ensure a maximum possible observation in Phase-II. Defined by \( \lambda_{TO_2} = 0 \) in (39), the resulting strategy for the observer in Phase-I is obtained using (58) and in Phase-II is \( \psi_{O_2} = 90^\circ \). In this example, the limiting cases for \( \psi_{O_1} \) are considered, highlighting the difference in

Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
outcome from implementing either limiting strategy. Both cases are plotted in Fig. 9 and presented in Table III.

In this example, the DL has equation \( y = 1.020x \) and the observation disk has equation \((x - 3)^2 + (y - 6)^2 = 4\). The intersection of the DL and observation disk have unreal solutions: \((x, y) = (4.4694 \pm 0.338326i, 4.5597 \pm 0.34516i)\). Because the point \(W\), whose location is \(W = (x_0, y_0 - R) = (3, 4)\) is above the DL, we know that no intersections exist and that \(x \in \mathbb{R}_3\). From Theorem 1, the limiting cases for optimal observer strategies are obtained

\[
\psi_{O,1} = [112.127^\circ, 174.133^\circ], \quad \psi_{O,2} = 90^\circ.
\]

As expected, \(t_{apr}\) varies, depending upon the heading taken for Phase-I. For the two limiting cases, \(t_{apr} = [4.3083, 11.378]\) TU. However, since \(x \in \mathbb{R}_3\) the maximum observation time is possible for the optimal range of observer headings; \(\tau_{obs} = 13.333\) TU.

VII. CONCLUSION

The optimal control laws for an observer to keep a nonmaneuvering constant speed target within an observation range for as long as possible have been obtained. The presented analysis and results show that the state space may be partitioned into three regions of space: 1) no observation; 2) limited observation, and 3) maximum observation. Depending upon the initial conditions and problem parameters (speed ratio, \(\alpha\), and observation range, \(R\)), this partitioning is obtained in closed form making it suitable for feedback strategies to be implemented. This is enabled by the construction of a decision line that is determined by the speed ratio parameter, \(\alpha\). In order to highlight the three separate regions, three scenarios are shown, demonstrating the solutions to this optimal control problem. Future extensions of this work include observation in 3-D, observation of a maneuvering target via a differential game formulation, and the inclusion of more observer agents.

REFERENCES

[1] B. O. Koopman, “Search and screening,” Washington, D.C.: Operations Evaluation Group, Office of the Chief of Naval Operations, Navy Dept., Washington, DC, USA, Tech. Rep. 214252, 1946.
[2] J. M. Dobbie, “Solution of some surveillance-evasion problems by the methods of differential games,” in Proc. 4th Int. Conf. Oper. Rese., 1966.
[3] J. M. Dobbie, “A survey of search theory,” Operations Res., vol. 16, no. 3, pp. 525–537, 1968.
[4] J. G. Taylor, “Application of differential games to problems of naval warfare: Surveillance-evasion: Part I,” Naval Postgraduate Sch., Monterey, CA, Tech. Rep. NPS-55TW70061A, 1970.
[5] R. Isaacs, Differential Games: A. Mathematical Theory With Applications to Optimization, Control and Warfare. New York NY, USA: Wiley, 1965.
[6] J. Lewin and J. Breakwell, “The surveillance-evasion game of degree,” J. Optim. Theory Appl., vol. 16, no. 3, pp. 339–353, 1975.
[7] J. Lewin and G. Obsder, “Conic surveillance evasion,” J. Optim. Theory Appl., vol. 27, no. 1, pp. 107–125, 1979.
[8] J. Lewin and G. J. Olsder, “The isotropic rocket–A surveillance evasion game,” Comput. Math. Appl., vol. 18, no. 1–3, pp. 15–34, 1989.
[9] I. Greenfeld, “A differential game of surveillance evasion of two identical cars,” J. Optim. Theory Appl., vol. 52, no. 1, pp. 53–79, 1987.
[10] P. Bernhard, “Linear pursuit-evasion games and the isotropic rocket,” Dept. of Aeronaut. and Astronaut., Stanford Univ., Stanford, CA, USA, Tech. Rep. AFAL-TR-71-30, 1970.
[11] M. Gilles and A. Vladimirska, “Evasive path planning under surveillance uncertainty,” Dyn. Games Appl., vol. 10, pp. 391–416, 2020.
[12] A. Von Moll, M. Pachter, and Z. Fuchs, “Pure pursuit with an effector,” Dyn. Games Appl., 2022.
[13] J. V. Breakwell, Pursuit of a Faster Evader. Amsterdam, The Netherlands: Springer, 1975, pp. 243–256.
[14] P. Hagedorn and J. V. Breakwell, “A differential game with two pursuers and one evader,” J. Optim. Theory Appl., vol. 18, no. 1, pp. 15–29, 1976.
[15] J. Szots, A. V. Savkin, and I. Harmati, “Revisiting a three-player pursuit-evasion game,” J. Optim. Theory Appl., vol. 190, no. 1, pp. 581–601, 2021.
[16] M. V. Ramana and M. Kothari, “Pursuit-evasion games of high speed,” J. Intell. Robot. Syst., vol. 85, no. 2, pp. 293–306, 2017.
[17] E. Garcia and S. D. Bopardikar, “Cooperative containment of a high-speed evader,” in Proc. IEEE Amer. Control Conf., 2021, pp. 4698–4703.
[18] S. Jin and Z. Qu, “Pursuit-evasion games with multi-pursuer vs. one fast evader,” in Proc. IEEE World Congr. Intell. Control, Automat., 2010, pp. 3184–3189.
[19] F. Chernous’ko, “A problem of evasion from many pursuers,” J. Appl. Math. Mechanics, vol. 40, no. 1, pp. 11–20, 1976.
[20] J. V. Breakwell and P. Hagedorn, “Point capture of two evaders in succession,” J. Optim. Theory Appl., vol. 27, no. 1, pp. 89–97, 1979.
[21] E. Garcia, A. Von Moll, D. Casbeer, and M. Pachter, “Strategies for defending a coastline against multiple attackers,” in Proc. IEEE Conf. Decis. Control, 2019, pp. 7319–7324.
[22] E. Garcia, D. W. Casbeer, and M. Pachter, “The capture-the-flag differential game,” in IEEE Conf. Decis. Control, 2018, pp. 4167–4172.

[23] S. Nath and D. Ghose, “A two-phase evasive strategy for a pursuit-evasion problem involving two non-holonomic agents with incomplete information,” Eur. J. Control, vol. 1, no. 3, pp. 1–6, 2022.

[24] J. Shinar, V. Y. Glizer, and V. Turetsky, “A pursuit-evasion game with hybrid pursuer dynamics,” Eur. J. Control, vol. 15, no. 6, pp. 665–684, 2009.

[25] V. Turetsky and T. Shima, “Pursuit-evasion guidance in a switched system,” SIAM J. Control Optim., vol. 56, no. 4, pp. 2613–2633, 2018.

[26] A. V. Moll and Z. Fuchs, “Turret lock-on in an engage or retreat game,” in Proc. IEEE Amer. Control Conf., 2021, pp. 3188–3195.

[27] A. Von Moll, D. Shishika, Z. Fuchs, and M. Dorothy, “The turret-runner-penetrator differential game with role selection,” IEEE Trans. Aerosp. Electron. Syst., vol. 58, no. 6, pp. 5687–5702, Dec. 2022.

[28] H. Huang, J. Ding, W. Zhang, and C. J. Tomlin, “Automation-assisted capture-the-flag: A differential game approach,” IEEE Trans. Control Syst. Technol., vol. 23, no. 3, pp. 1014–1028, May 2015.

[29] H. Huang, J. Ding, W. Zhang, and C. J. Tomlin, “A differential game approach to planning in adversarial scenarios: A case study on capture-the-flag,” in Proc. IEEE Int. Conf. Robot. Automat., 2011, pp. 1451–1456.

[30] E. Garcia, D. W. Casbeer, D. Tran, and M. Pachter, “A differential game approach for beyond visual range tactics,” in Proc. IEEE Amer. Control Conf., 2021, pp. 3210–3215.

[31] V. Turetsky, M. Weiss, and T. Shima, “Minimum effort pursuit guidance with delayed engagement decision,” J. Guid., Control, Dyn., vol. 42, no. 12, pp. 2664–2670, 2019.

[32] M. Weiss, V. Shalumov, and T. Shima, “Minimum effort pursuit guidance with multiple delayed engagement decisions,” J. Guid., Control, Dyn., vol. 45, no. 7, pp. 1310–1320, 2022.

[33] I. E. Weintraub, A. Von Moll, E. Garcia, D. Casbeer, Z. J. L. Demers, and M. Pachter, “Maximum observation of a faster non-maneuvering target by a slower observer,” in Proc. IEEE Amer. Control Conf., 2020, pp. 100–105.

[34] I. E. Weintraub, A. Von Moll, E. Garcia, and M. Pachter, “Maximum observation of a target by a slower observer in three dimensions,” J. Guid., Control, Dyn., vol. 44, no. 3, pp. 646–653, 2021.

[35] N. A. Shneydor, Missile Guidance and Pursuit: Kinematics, Dynamics and Control, Sawston, U.K.: Woodhead Publishing, 1998.

[36] J. C. Barton and C. J. Eliezer, “On pursuit curves,” J. Australian Math. Soc. Ser. B., Appl. Math., vol. 41, no. 3, pp. 358–371, 2000.

[37] D. Kirk, Optimal Control Theory: An Introduction, Englewood Cliffs, NJ, USA: Prentice-Hall, 1970.

[38] I. Weintraub, E. Garcia, and M. Pachter, “Optimal guidance strategy for the defense of a non-maneuverable target in 3-dimensions,” IET Control Theory Appl., vol. 14, pp. 1531–1538, Jul. 2020.

[39] R. Rhoad, G. Milauskas, and R. Whipple, Geometry for Enjoyment and Challenge. New ed. Evanston, IL, USA: McDougal, Littel & Company, 1997.