Ground States of the Yukawa Models with Cutoffs

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Abstract. Ground states of the so called Yukawa model is considered. The Yukawa model describes a Dirac field interacting with a Klein-Gordon field. By introducing both ultraviolet cutoffs and spatial cutoffs, the total Hamiltonian is defined as a self-adjoint operator on a boson-fermion Fock space. It is shown that the total Hamiltonian has a positive spectral gap for all values of coupling constants. In particular the existence of ground states is proven.

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1 Introduction

In this paper we investigate the existence of ground states of the Yukawa model which describes a Dirac field interacting with a Klein-Gordon field. Both Dirac field and Klein-Gordon field are massive, and ultraviolet cutoffs are imposed on both of them. The total Hamiltonian of the Yukawa model is the sum of the free Hamiltonian and the interaction Hamiltonian:

\[ H = H_{\text{Dirac}} \otimes I + I \otimes H_{\text{KG}} + \kappa H' \]  (1)
on \mathcal{F} = \mathcal{F}_{\text{Dirac}} \otimes \mathcal{F}_{\text{KG}}, where \kappa > 0 is a coupling constant. The free Hamiltonians \( H_{\text{Dirac}} \) and \( H_{\text{KG}} \) are given by formally

\[ H_{\text{Dirac}} = \sum_{s=\pm 1/2} \int_{\mathbb{R}^3} \sqrt{M^2 + p^2} \left( b_s^* (p) b_s (p) + d_s^* (p) d_s (p) \right) dp, \quad M > 0, \]
\[ H_{\text{KG}} = \int_{\mathbb{R}^3} \sqrt{m^2 + k^2} a^* (k) a(k) dk, \quad m > 0. \]

In the subsequent section, we give the rigorous definition of \( H_{\text{Dirac}} \) and \( H_{\text{KG}} \). The interaction Hamiltonian \( H' \) is defined by

\[ H' = \int_{\mathbb{R}^3} \chi_I (x) \psi_{\chi_{\text{Dir}}} (x) \psi_{\chi_{\text{Dir}}} (x) \otimes \phi_{\chi_{\text{KG}}} (x) dx, \]
where \( \psi_{\chi_{\text{Dir}}} (x) \) is a Dirac field with an ultraviolet cutoff \( \chi_{\text{Dir}} \), and \( \phi_{\chi_{\text{KG}}} (x) \) a Klein-Gordon field with an ultraviolet cutoff \( \chi_{\text{KG}} \). We furthermore introduce a spatial cutoff \( \chi_I (x) \) in \( H' \) to define \( H \) as a self-adjoint operator. Since the interaction \( H' \) is relatively bounded with respect to \( H_{\text{Dirac}} \otimes I + I \otimes H_{\text{KG}} \) by virtue of cutoffs, \( H \) is self-adjoint and bounded from below by the Kato-Rellich theorem. We say that a self-adjoint operator \( X \) bounded from below has a ground state, if the bottom of its spectrum is an eigenvalue,
and the difference between the bottom of the spectrum and that of the essential spectrum is called spectral gap. In this paper we show that $H$ has a positive spectral gap for all values of coupling constants. In particular the existence of ground states follows from this.

In the last decade, a system of quantum particles governed by a Schrödinger operator interacting with a massless bose field are successfully investigated. In particular the existence of ground states of some massless models in non-relativistic QED is proven in \cite{8, 10} for all values of coupling constants. It is also shown in \cite{1, 5, 4, 17} that ground states of a massless model in QED exist but for sufficiently small values of coupling constants. Since quantized radiation fields in QED and nonrelativistic QED are massless, the spectral gap of the free Hamiltonians is zero. Then all the results mentioned above are not trivial. For other topics on the system of fields interacting fields, refer to \cite{11, 5}. On the analysis of a field equation of the Yukawa model, called the Dirac-Klein-Gordon equation, see \cite{6, 12, 19}.

Now let us consider the existence of ground states of the Yukawa model $H$. Since $H_{\text{Dirac}}$ and $H_{\text{KG}}$ are massive, the spectral gap of $H_{\text{Dirac}} \otimes 1 + 1 \otimes H_{\text{KG}}$ is positive. Then the regular perturbation theory \cite{14} says that $H$ also has ground states for sufficiently small values of coupling constants. It is not obvious, however, whether $H$ also has ground states for all values of coupling constants. Moreover unfortunately we cannot directly apply methods developed in \cite{8, 10} to show the existence of ground states of $H$. Outline of our strategy is as follows. To prove the existence of ground states of $H$, we use a momentum lattice approximation \cite{9, 2}. Then $H$ can be approximated with some lattice parameters $V$ and $L$ as

$$H_{L,V} = H_{\text{Dirac},V} \otimes 1 + 1 \otimes H_{\text{KG}} + \kappa H_{L,V}^t.$$ 

It is shown that $H_{\text{Dirac},V}$ has a compact resolvent. Then from a standard argument as in \cite{8, 2}, it follows that $H_{L,V}$ has a positive spectral gap which is uniform with respect to $V$ and $L$ by positive masses $m$ and $M$. Since $H_{L,V}$ converges to $H$ in the uniform resolvent sense as $V \to \infty$ and $L \to \infty$, we can see that $H$ also has a positive spectral gap. In this paper integrable condition $\int_{\mathbb{R}^3} |x| |q(x)| dx < \infty$ is supposed. This assumption corresponds to the spatial localization discussed in \cite{3, 10}.

This paper is organized as follows. In Section 2, we introduce Dirac fields and Klein-Gordon fields with ultraviolet cutoffs. Then we define the Yukawa Hamiltonian with spatial cutoffs on a boson-fermion Fock space, and state a main result. In Section 3, we give the proof of the main theorem.

### 2 Definitions and Main Results

#### 2.1 Dirac Fields and Klein-Gordon Fields

We first consider Dirac fields. The state space defined by $\mathcal{F}_{\text{Dirac}} = \bigoplus_{n=0}^{\infty} (\otimes_n a L^2(\mathbb{R}^3; \mathbb{C}^4))$, where $\otimes_n a L^2(\mathbb{R}^3; \mathbb{C}^4)$ denotes the $n$-fold anti-symmetric tensor product of $L^2(\mathbb{R}^3; \mathbb{C}^4)$ with $\otimes_0 a L^2(\mathbb{R}^3; \mathbb{C}^4) := C$. Let $\mathcal{D}$ be the subset of $L^2(\mathbb{R}^3; \mathbb{C}^4)$. We define the finite particle subspace $\mathcal{F}^\text{fin}_{\text{Dirac}}(\mathcal{D})$ on $\mathcal{D}$ by the set of $\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty}$ satisfying $\Psi^{(n)} \in \otimes_n a \mathcal{D}$ and $\Psi^{(n')} = 0$ for all $n' > N$ with some $N \geq 0$. Let $B(\xi), \xi = (\xi_1, \cdots, \xi_3) \in L^2(\mathbb{R}^3; \mathbb{C}^4)$, and $B^*(\eta), \eta = (\eta_1, \cdots, \eta_4) \in L^2(\mathbb{R}^3; \mathbb{C}^4)$, be the annihilation operator and the creation operator on $\mathcal{F}_{\text{Dirac}}$, respectively. For $f \in L^2(\mathbb{R}^3)$ let us set

$$b_{1/2}^*(f) = B^*(\xi, f, 0, 0, 0), \quad b_{-1/2}^*(f) = B^*(\xi, 0, f, 0, 0),$$

$$d_{1/2}^*(f) = B^*(\xi, 0, 0, f), \quad d_{-1/2}^*(f) = B^*(\xi, 0, 0, 0, f).$$
Then they satisfy canonical anti-commutation relations:

\[ \{ b_\alpha(f), b^*_\beta(g) \} = \{ d_\alpha(f), d^*_\beta(g) \} = \delta_\alpha\beta \delta(f,g) L^2(\mathbb{R}), \]

\[ \{ b_\alpha(f), b_\beta(g) \} = \{ d_\alpha(f), d_\beta(g) \} = \{ b_\alpha(f), d_\beta(g) \} = \{ b_\beta(f), d_\alpha(g) \} = 0. \]

It is known that \( b_\alpha(\xi) \) and \( d_\alpha(\xi) \) are bounded with

\[ \| b_\alpha(\xi) \| = \| d_\alpha(\xi) \| = \| \xi \|. \]  \hspace{1cm} (2)

The one particle energy of Dirac field with momentum \( p \in \mathbb{R}^3 \) is given by \( E(p) = \sqrt{p^2 + M^2} \), where \( M > 0 \) denotes the mass of an electron. Let

\[ f^{l}_{s}(p) = \frac{\chi_{\text{Dir}}(p) u^{l}_{s}(p)}{\sqrt{(2\pi)^3 E(p)}}, \quad g^{l}_{s}(p) = \frac{\chi_{\text{Dir}}(p) v^{l}_{s}(-p)}{\sqrt{(2\pi)^3 E(p)}}, \]

\[ s = \pm 1/2, \quad l = 1, \ldots, 4, \]

where \( \chi_{\text{Dir}} \) is an ultraviolet cutoff, and \( u^{l}_{s}(p) \) and \( v^{l}_{s}(p) \) denote spinors with the positive and negative energy part of \( \alpha^l p + \beta M \) with spin \( s = \pm 1/2 \), respectively. Here \( \alpha^l, j = 1, 2, 3, \) and \( \beta \) are the \( 4 \times 4 \) matrix satisfying the canonical anti-commutation relation \( \{ \alpha^l, \alpha^j \} = 2 \delta_{ij}, \{ \alpha^l, \beta \} = 0, \beta^2 = I \). The Dirac field \( \psi(x) = (\psi_1(x), \ldots, \psi_4(x)) \) is defined by

\[ \psi_l(x) = \sum_{s = \pm 1/2} (b^l_s f^{l}_{s,x} + d^l_s g^{l}_{s,x}), \quad l = 1, \ldots, 4, \]

where \( f^{l}_{s,x}(p) = f^{l}_{s}(p)e^{-ipx} \) and \( g^{l}_{s,x}(p) = g^{l}_{s}(p)e^{-ipx} \). We introduce the following assumption.

\[ \text{(A.1) (Ultrasound cutoff for Dirac fields)} \quad \chi_{\text{Dir}} \text{ satisfies that} \]

\[ \int_{\mathbb{R}^3} \frac{|\chi_{\text{Dir}}(p) u^{l}_{s}(p)|^2}{E_M(p)} dp < \infty, \quad \int_{\mathbb{R}^3} \frac{|\chi_{\text{Dir}}(p) v^{l}_{s}(-p)|^2}{E_M(p)} dp < \infty. \]

We secondly define Klein-Gordon fields. The state space is defined by \( \mathcal{F}_{\text{KG}} = \bigoplus_{n=0}^{\infty} (\otimes^n L^2(\mathbb{R}^3)), \) where \( \otimes^n L^2(\mathbb{R}^3) \) denotes the \( n \)-fold symmetric tensor product of \( L^2(\mathbb{R}^3) \) with \( \otimes^n L^2(\mathbb{R}^3) := C \). In a similar way to the case of Dirac fields, we define the finite particle subspace \( \mathcal{F}_{\text{KG}}^{\text{fin}}(\mathcal{M}) \) on \( \mathcal{M} \subset L^2(\mathbb{R}^3) \) but anti-symmetric tensor products is replaced by symmetric tensor products. Let \( a(\xi), \xi \in L^2(\mathbb{R}^3), \) be the annihilation operator and the creation operator on \( \mathcal{F}_{\text{KG}}, \) respectively. Then they satisfy canonical commutation relations on \( \mathcal{F}_{\text{KG}}^{\text{fin}}(L^2(\mathbb{R}^3)) \):

\[ [a(\xi), a^*(\eta)] = (\xi, \eta), \quad [a(\xi), a(\eta)] = [a^*(\xi), a^*(\eta)] = 0. \]

Let \( S \) be a self-adjoint operator on \( L^2(\mathbb{R}^3) \). The second quantization of \( S \) is defined by

\[ d\Gamma(S)_{|\mathcal{F}_{\text{KG}}} = \bigoplus_{n=0}^{\infty} \left( \sum_{j=1}^{n} (I \otimes \cdots I \otimes S \otimes I \cdots \otimes I) \right)_{|\mathcal{F}_{\text{KG}}}. \]
Similarly, we can define the second quantization $d\Gamma(A)_{|_{\mathcal{F}_{\text{Dirac}}}}$ of the Dirac field for a operator $A$ on $L^2(\mathbb{R}^3;\mathbb{C}^4)$. For $\eta \in \mathcal{D}(S^{-1/2})$, $a(\eta)$ and $a^*(\eta)$ are relatively bounded with respect to $d\Gamma(S)_{|_{\mathcal{F}_{\text{KG}}}}$ with
\[ ||a(\eta)\Psi|| \leq ||S^{-1/2}\eta|| ||d\Gamma(S)_{|_{\mathcal{F}_{\text{KG}}}}\Psi||, \quad \Psi \in \mathcal{D}(d\Gamma(S)_{|_{\mathcal{F}_{\text{KG}}}}), \tag{3} \]
\[ ||a^*(\eta)\Psi|| \leq ||S^{-1/2}\eta|| ||d\Gamma(S)_{|_{\mathcal{F}_{\text{KG}}}}\Psi|| + ||\eta||||\Psi||, \quad \Psi \in \mathcal{D}(d\Gamma(S)_{|_{\mathcal{F}_{\text{KG}}}}). \tag{4} \]

The one particle energy of Klein-Gordon field with momentum $k \in \mathbb{R}^3$ is given by $\omega(k) = \sqrt{k^2 + m^2}$, $m > 0$. Let us define the field operator $\phi(x)$ by
\[ \phi(x) = \frac{1}{\sqrt{2}} \left( a(h_x) + a^*(h_x) \right), \]
where $h_x(k) = h(k)e^{ikx}$ with $h(k) = \frac{\chi_{\text{KG}}(k)}{\sqrt{(2\pi)^3\omega(k)}}$, and $\chi_{\text{KG}}$ is an ultraviolet cutoff function. We assume the following condition:

(A.3) (Ultraviolet cutoffs for Klein-Gordon fields) $\chi_{\text{KG}}$ satisfies that
\[ \int_{\mathbb{R}^3} \frac{|\chi_{\text{KG}}(k)|^2}{\omega(k)} < \infty, \quad \int_{\mathbb{R}^3} \frac{|\chi_{\text{KG}}(k)|^2}{\omega(k)^2} < \infty. \]

### 2.2 Total Hamiltonian and Main Theorem

The state space of the interaction system between Dirac fields and Klein Gordon fields is given by
\[ \mathcal{F} = \mathcal{F}_{\text{Dirac}} \otimes \mathcal{F}_{\text{KG}}, \]
and the free Hamiltonian by
\[ H_0 = H_{\text{Dirac}} \otimes I + I \otimes H_{\text{KG}}, \]
where $H_{\text{Dirac}} = d\Gamma(E)_{|_{\mathcal{F}_{\text{Dirac}}}}$ and $H_{\text{KG}} = d\Gamma(\omega)_{|_{\mathcal{F}_{\text{KG}}}}$. To define the interaction, we introduce a spatial cutoff satisfying the following condition:

(A.3) (Spatial cutoffs) $\chi_1$ satisfies that $\int_{\mathbb{R}^3} |\chi_1(x)|dx < \infty$.

Now let us define the linear functional $\mathcal{F} \times (\mathcal{F}_{\text{Dirac}}^{\text{fin}}(\mathcal{D}(E)) \otimes \mathcal{F}_{\text{KG}}^{\text{fin}}(\mathcal{D}(\omega))) \to \mathbb{C}$, where $\otimes$ denotes the algebraic tensor product, by
\[ \ell_1(\Phi, \Psi) = \int_{\mathbb{R}^3} \chi_1(x) \left( \Phi, \overline{\psi(x)} \psi(x) \otimes \phi(x) \Psi \right) dx, \tag{5} \]
where $\overline{\psi(x)} = \psi^*(x) \gamma^0$ with $\gamma^0 = \beta$. By (2) we have
\[ ||\psi(x)|| \leq M_{\text{Dir}}^l, \tag{6} \]
where $M_{\text{Dir}}^l = \sum_{\gamma = \pm 1/2} (||f_\gamma^l|| + ||g_{\gamma}^l||)$. We also see that by (3) and (4),
\[ ||\phi(x)\Psi|| \leq \sqrt{2}M_{\text{KG}}^l ||H_{\text{KG}}^{1/2}\Psi|| + \frac{1}{\sqrt{2}}M_{\text{KG}}^0 ||\Psi||, \tag{7} \]
where $M_{\text{KG}}^j = \| \frac{h}{\sqrt{\omega_0^j}} \|$, $j \in \{0\} \cup \mathbb{N}$. By (6) and (7), we have

\[ |\ell_1(\Phi, \Psi)| \leq \left( L_4 \| (I \otimes H_{\text{KG}}^{1/2}) \Psi \| + R_1 \| \Psi \| \right) \| \Phi \|, \tag{8} \]

where $L_4 = \sqrt{2} \| \chi_1 \|_{L^1} \sum_{j,p} | \gamma_{j,p}^0 | M_{\text{Dir}}^1 M_{\text{Dir}}^0 M_{\text{KG}}^0$, and $R_1 = \frac{1}{\sqrt{2}} \| \chi_1 \|_{L^1} \sum_{j,p} | \gamma_{j,p}^0 | M_{\text{Dir}}^1 M_{\text{Dir}}^0 M_{\text{KG}}^0$. By the Riesz representation theorem, we can define the symmetric operator $H': \mathcal{F} \to \mathcal{F}$ such that

\[ (\Phi, H' \Psi) = \ell_1(\Phi, \Psi), \tag{9} \]

and

\[ \|H' \Psi\| \leq L_4 \| (I \otimes H_{\text{KG}}^{1/2}) \Psi \| + R_1 \| \Psi \|. \tag{10} \]

We see that $H'$ is formally denoted by

\[ H' = \int_{\mathbb{R}^3} \chi_1(x) \overline{\psi(x)} \psi(x) \otimes \phi(x) dx. \]

The total Hamiltonian of the Yukawa model is then defined by

\[ H = H_0 + \kappa H', \quad \kappa \in \mathbb{R}. \tag{11} \]

Let us consider the self-adjointness of $H$. For $\varepsilon > 0$, there exists $C_\varepsilon \geq 0$ such that for all $\Psi \in \mathcal{D}(H_{\text{KG}})$,

\[ \|H_{\text{KG}}^{1/2} \Psi\| \leq \varepsilon \|H_{\text{KG}} \Psi\| + c_\varepsilon \| \Psi \|. \tag{12} \]

Then by (12) and (10), we see that for $\Psi \in \mathcal{D}(H_0)$,

\[ \|H' \Psi\| \leq \varepsilon L_4 \|H_0 \Psi\| + (c_\varepsilon L_4 + R_1) \| \Psi \|. \tag{13} \]

Let us take sufficiently small $\varepsilon > 0$ such as $\varepsilon L_4 < 1$ in (13). Then by the Kato-Rellich theorem, $H$ is self-adjoint on $\mathcal{D}(H_0)$ and essentially self-adjoint on any core of $H_0$. In particular, $H$ is essentially self-adjoint on

\[ \mathcal{D}_0 = \mathcal{F}_{\text{Dirac}}^\text{fin}(\mathcal{D}(E)) \hat{\otimes} \mathcal{F}_{\text{KG}}^\text{fin}(\mathcal{D}(\omega)). \tag{14} \]

The Kato-Rellich theorem also shows that $H$ is bounded from below i.e. $\inf \sigma(H) > -\infty$.

Let $X$ be self-adjoint and bounded from below. Let us denote the infimum of the spectrum of $X$ by $E_0(X) = \inf \sigma(X)$. We say that $X$ has a ground state if $E_0(X)$ is an eigenvalue of $X$. Let

\[ \nu = \min \{ m, M \}. \tag{15} \]

Then it is known that the spectrum of $H_0$ is $\sigma(H_0) = \{0\} \cup [\nu, \infty)$. To prove the existence of the ground states of $H$, we introduce the additional condition on the spatial cutoff.

\[ (\text{A.4}) \text{ (Spatial localization)} \quad \chi_1 \text{ satisfies that } \int_{\mathbb{R}^3} |x| |\chi_1(x)| dx < \infty. \]

Now we are in the position to state the main theorem.

**Theorem 2.1**

Assume (A.1)-(A.4). Then $[E_0(H), E_0(H) + \nu] \cap \sigma(H)$ is purely discrete for all values of coupling constants. In particular $H$ has ground states for all values of coupling constants.
3 Proof of Main Theorem

Let us introduce some notations. Let $\Gamma_V$ be the set of lattice points

$$\Gamma_V = \{ q = (q_1, q_2, q_3) \mid q_j = \frac{2\pi}{V} n_j, \, n_j \in \mathbb{Z}, \, j = 1, 2, 3 \}.$$  

For each lattice point $q \in \Gamma_V$, set $C(q, V) = [q_1 - \frac{\phi}{2}, q_1 + \frac{\phi}{2}] \times [q_2 - \frac{\phi}{2}, q_2 + \frac{\phi}{2}] \times [q_3 - \frac{\phi}{2}, q_3 + \frac{\phi}{2}] \subset \mathbb{R}^3$ and $I_L = [-L, L] \times [-L, L] \subset \mathbb{R}^2$. For $\xi \in L^2(\mathbb{R}^3)$, we define the approximated functions $\xi_L$ and $\xi_{L,V}$ by

$$\xi_L(k) = \xi(k) \chi_L(k),$$

$$\xi_{L,V}(k) = \sum_{q \in \Gamma_V} \xi(q) \chi_{C(q, V)} \cap I_L(k),$$

where $\chi_j(k)$ denotes the characteristic function on $J \subset \mathbb{R}^3$. By considering the map $L^2(\mathbb{R}^3) \ni \xi = \sum_q \xi(q) \chi_{C(q, V)} \mapsto (\xi(q))_{q \in \Gamma_V} \in \ell^2(\Gamma_V)$, we can identify $\ell^2(\Gamma_V)$ as a closed subspace of $L^2(\mathbb{R}^3)$. Let us set

$$\mathcal{F}_V = \mathcal{F}_{\text{Dirac}, V} \otimes \mathcal{F}_{\text{KG}},$$

where $\mathcal{F}_{\text{Dirac}, V} = \bigoplus_{n=0}^\infty (\otimes \mathbb{R}^3 \ell^2(\Gamma_V; \mathbb{C}^4))$. Let us define $H_{0,V}$ on $\mathcal{F}$ by

$$H_{0,V} = H_{\text{Dirac},V} \otimes I + I \otimes H_{\text{KG}},$$

where $H_{\text{Dirac},V} = d \Gamma(E_V) : \Gamma_{\text{Dirac}}$ with $E_V(p) = \sum_{q \in \Gamma_V} E(q) \chi_{C(q, V)}(p)$. Approximated interaction Hamiltonians are also defined by

$$H'_{L,V} = \int \chi_i(x) \left( \frac{\psi_{L,V}(x) \psi_{L,V}(x) \otimes \phi(x)}{\mathcal{V}_V} \right) dx,$$

$$H'_{L} = \int \chi_i(x) \left( \frac{\psi_{L}(x) \psi_{L}(x) \otimes \phi(x)}{\mathcal{V}_V} \right) dx,$$

where $\psi_{L}(x) = (\psi^4_{L}(x))_{i=1}^4$ and $\psi_{L,V}(x) = (\psi^4_{L,V}(x))_{i=1}^4$. Approximated interaction Hamiltonians are also defined by

$$H_{L,V} = H_{0,V} + \kappa H'_{L,V},$$

$$H_{L} = H_0 + \kappa H'_{L}.$$  

In a similar way to the case of $H$, we can prove that $H_L$ and $H_{L,V}$ are essentially self-adjoint on $\mathcal{D}_0$ and $\mathcal{D}_{0,V} = \mathcal{F}_{\text{Dirac}} \otimes \mathcal{F}_{\text{KG}}$ of $\mathcal{D}(\omega)$, respectively.

**Lemma 3.1** Assume (A.1)-(A.3). Then $H_{L,V}$ is reduced to $\mathcal{F}_V$.

**(Proof)** Let us denote $p_V$ the orthogonal projections from $L^2(\mathbb{R}^3)$ to $\ell^2(\Gamma_V)$. Then $\Gamma(p_V) = \bigoplus_{n=0}^\infty (\otimes \mathbb{R}^3 p_V)$ is the projection from $\mathcal{F}_{\text{Dirac}}$ to $\mathcal{F}_{\text{Dirac},V}$. Let $\Psi \in \mathcal{D}_{0,V}$. Then it is easy to see that $(\Gamma(p_V) \otimes I)H_{0,V}\Psi = H_{0,V}(\Gamma(p_V) \otimes I)\Psi$. By using $p_V \chi_{C(q, V)} = \chi_{C(q, V)}$, we see that for all $\Phi \in \mathcal{F}$,

$$\int \chi_i(x)(\Phi, (\Gamma(p_V) \psi_{L,V}(x) \psi_{L,V}(x) \otimes \phi(x))\Psi) dx = \int \chi_i(x)(\Phi, (\psi_{L,V}(x) \psi_{L,V}(x) \otimes \phi(x))((\Gamma(p_V) \otimes I)\Psi) dx.$$

Hence $(\Gamma(p_V) \otimes I)H_{L,V} \Psi = H_{L,V}(\Gamma(p_V) \otimes I)\Psi$. Thus $(\Gamma(p_V) \otimes I)H_{L,V} \Psi = H_{L,V}(\Gamma(p_V) \otimes I)\Psi$ follows for all $\Psi \in \mathcal{D}_{0,V}$. Since $\mathcal{D}_{0,V}$ is a core of $H_{L,V}$, the lemma follows.
Proposition 3.2 Assume (A.1)-(A.4). Then $H_{L,V} \mid_{\mathcal{F}_V}$ has purely discrete spectrum in $[E_0(H_{L,V}), E_0(H_{L,V}) + \nu)$.

To prove Proposition 3.2 we also take the lattice approximation of Klein-Gordon fields. Let us set
\[
\mathcal{F}_{V,V'} = \mathcal{F}_{\text{Dirac},V} \otimes \mathcal{F}_{\text{KG},V'},
\]
where $\mathcal{F}_{\text{KG},V'} = \oplus_{n=0}^{\infty} (\otimes_{k} \ell^2(\Gamma_{V'}))$. Set
\[
H_{0,V,V'} = H_{\text{Dirac},V} \otimes I + I \otimes H_{\text{KG},V'},
\]
where $H_{\text{KG},V'} = d\Gamma(\omega_{V'}|_{\mathcal{F}_{\text{KG}}}$ with $\omega_{V'}(k) = \sum_{q \in \Gamma_{V'}} \omega(q) \chi_{C}(q,V')(k)$. Let
\[
H_{L,V',L'} = \int_{\mathbb{R}} \chi_{1}(x) \left( \overline{\psi_{L,V}(x)} \psi_{L,V}(x) \otimes \phi_{L'}(x) \right) dx,
\]
\[
H'_{L,V',L'} = \int_{\mathbb{R}} \chi_{1}(x) \left( \overline{\psi_{L,V}(x)} \psi_{L,V}(x) \otimes \phi_{L'}(x) \right) dx,
\]
where $\phi_{L'}(x) = \frac{1}{\sqrt{2}} \left\{ a((h_{L})_{L'}) + a^*(((h_{L})_{L'}) \right\}$ and $\phi_{L'}(x) = \frac{1}{\sqrt{2}} \left\{ a((h_{L})_{L'}) + a^*(((h_{L})_{L'}) \right\}$. Let
\[
H_{L,V',L'} = H_{0,V,V'} + \kappa H'_{L,V',L'}, \quad H_{L,L'} = H_{0,L'} + \kappa H'_{L,L'}.
\]
In a similar way to $H$, we can prove that $H_{L,V,L'}$ and $H_{L,V',L'}$ are essentially self-adjoint on $\mathcal{D}_{0,V}$ and $\mathcal{D}_{0,V'} = \mathcal{D}_{\text{fin}}(\mathcal{D}(E_{V})) \otimes \mathcal{D}_{\text{fin}}(\mathcal{D}(\omega_{V'}))$, respectively.

Lemma 3.3 Suppose (A.1)-(A.3). Then $H_{L,V',L'}$ is reduced to $\mathcal{F}_{V,V'}$, and $H_{L,V',L'}|_{\mathcal{F}_{V,V'}}$ has purely discrete spectrum in $[E_0(H_{L,V',L'}), E_0(H_{L,V',L'}) + \nu)$.

(Proof) In a similar way to the proof of Lemma 3.1 it is shown that $H_{L,V',L'}$ is reduced to $\mathcal{F}_{V,V'}$. Since $H_{0,V,V'|_{\mathcal{F}_{V,V'}}}$ has a compact resolvent, $H_{L,V',L'}|_{\mathcal{F}_{V,V'}}$ also has a compact resolvent by the general theorem [2 Theorem 3.8]. Hence, in particular, $H_{L,V',L'}|_{\mathcal{F}_{V,V'}}$ has purely discrete spectrum in $[E_0(H_{L,V',L'}), E_0(H_{L,V',L'}) + \nu)$.

Lemma 3.4 Assume (A.1)-(A.4). Then for all $z \in \mathbf{C} \setminus \mathbf{R}$, it follows that
\[
(1) \lim_{V' \to \infty} \|(H_{L,V',L'} - z)^{-1} - (H_{L,V,L'} - z)^{-1}\| = 0, \quad (2) \lim_{L' \to \infty} \|(H_{L,V,L'} - z)^{-1} - (H_{L,V,L'} - z)^{-1}\| = 0.
\]

(Proof) We see that
\[
(H_{L,V',L'} - z)^{-1} - (H_{L,V,L'} - z)^{-1} = (H_{L,V',L'} - z)^{-1} \left\{ I \otimes (H_{KG} - H_{KG,V'}) + \kappa (H'_{L,V',L'} - H'_{L,V',L'}) \right\} (H_{L,V,L'} - z)^{-1}.
\]
Let $C_{V,m} = \sqrt{3} \left( \frac{2^{4m}}{V} \right)^{3/2} \left( \frac{1}{2^{m}} \right) + 1$. It is shown in [2] Lemma 3.1 that
\[
\|(I \otimes (H_{KG} - H_{KG,V'}))(H_{L,V,L'} - z)^{-1}\| \leq \frac{2C_{V,m}}{1 - C_{V,m}} \|(I \otimes H_{KG})(H_{L,V,L'} - z)^{-1}\| \to 0.
\]
as $V' \to \infty$. By (5) and (4), we also see that

$$
\|(H_{L',V',L'}' - H_{L,V,L'}') (H_{L,V,L'} - z)^{-1}\| \leq \sum_{l,p} |\mathcal{P}_l |M_{l,2} M_{l,2}' \left\{ \beta_1 \int_{\mathbb{R}^3} |\chi_l(x)| \left\| \frac{(h_{L,v})_{L'}}{\sqrt{\partial}} - \frac{(h_{L,v})_{L',v'}}{\sqrt{\partial}} \right\| dx
+ \beta_2 \int_{\mathbb{R}^3} |\chi_l(x)| \left\| (h_{L,v})_{L'} - (h_{L,v})_{L',v'} \right\| \right\} dx,
$$

where $\beta_1 = \sqrt{2} \| \mathcal{I} \otimes H_{KG}^{1/2} (H_{L,V,L'} - z)^{-1} \|$ and $\beta_2 = \frac{1}{\sqrt{2}} \|(H_{L,V,L'} - z)^{-1} \|$. From Assumptions (A.2), (A.4) and the fact $|e^{ikx} - e^{ik'x}| \leq |k - k'| |x|$, it follows that $\lim_{V' \to \infty} \int_{\mathbb{R}^3} |\chi_l(x)| \left\| (h_{L,v})_{L'} - (h_{L,v})_{L',v'} \right\| dx = 0$ and

$$
\lim_{V' \to \infty} \int_{\mathbb{R}^3} |\chi_l(x)| \left\| \frac{(h_{L,v})_{L'}}{\sqrt{\partial}} - \frac{(h_{L,v})_{L',v'}}{\sqrt{\partial}} \right\| dx = 0.
$$

Hence we have $\|(H_{L',V',L'}' - H_{L,V,L'}') (H_{L,V,L'} - z)^{-1}\| \to 0$. Thus we obtain (1). In a similar way to (1), we can also prove (2). ■

\textit{(Proof of Proposition 3.2)}

The decomposition $L^2(\mathbb{R}^3) = \ell^2(\Gamma_{v}) \oplus \ell^2(\Gamma_{v'})$ yields that $\mathcal{T}_{KG} \simeq \mathcal{T}_{KG,v} \oplus (\oplus_{n=0}^{\infty} \ell^2(\Gamma_n))$. Then we have $\mathcal{T}_{v} \simeq \mathcal{T}_{v,v} \oplus \mathcal{T}_{v,v'}$, where $(\mathcal{T}_{v,v})' = \mathcal{T}_{v,v} \oplus (\oplus_{n=0}^{\infty} \ell^2(\Gamma_n))$. Then we have for $n \geq 1$,

$$
H_{L',V',L',V'} \simeq H_{L',V',V'} \otimes I_{|\mathcal{T}_{v,v'}|} + I_{|\mathcal{T}_{v,v'}|} \otimes d\Gamma(0) \otimes \ell^2(\Gamma_n) \geq E_0 (H_{L',V',V'}) + nm.
$$

Hence we have $H_{L',V',L',V'} \simeq E_0 (H_{L',V',V'}) \oplus \mathcal{T}_{v,v'}$. While $H_{L',V',L',V'}$ has purely discrete spectrum in $E_0 (H_{L',V',V'})$, $E_0 (H_{L',V',V'}) + V$ by Lemma 3.3. Then $H_{L',V',L',V'}$ also has purely discrete spectrum in $E_0 (H_{L',V',V'})$, $E_0 (H_{L',V',V'}) + V$. Since $H_{L',V'}$ converges to $H_{L',V'}$ as $V' \to \infty$ in the norm resolvent sense by Lemma 3.4, $H_{L',V'}$ has purely discrete spectrum in $E_0 (H_{L',V'})$, $E_0 (H_{L',V'}) + V$ by [15] Lemm 4.6. Since $H_{L',V'}$ converges to $H_{L'}$ in the norm resolvent sense as $L' \to \infty$ by Lemma 3.4, $H_{L'}$ has also purely discrete spectrum in $E_0 (H_{L'}), E_0 (H_{L'}) + V$.

\textbf{Lemma 3.5} Assume (A.1)-(A.4). For all $z \in \mathbb{C} \setminus \mathbb{R}$, it follows that

$$
(1) \lim_{V \to \infty} \|(H_{L,V} - z)^{-1} - (H_{L} - z)^{-1}\| = 0, \quad (2) \lim_{L \to \infty} \|(H_{L} - z)^{-1} - (H_{L} - z)^{-1}\| = 0.
$$

\textbf{(Proof)} The proof is quite parallel with that of Lemma 3.4. Let $C_{v,M} = \sqrt{\mathcal{T}} \left( \frac{V}{L} \right)^3 \left( \frac{1}{2M+1} \right)$. Then

$$
\|(H_{L,V} - z)^{-1} - (H_{L} - z)^{-1}\| \leq \frac{i}{|\text{Im} z|} \left\{ \frac{2C_{v,M}}{1 - C_{v,M}} \left\| (H_{\text{Dirac}} \otimes I) (H_{L} - z)^{-1} \right\|
+ C \sum_{l,p} |\mathcal{P}_l| \int_{\mathbb{R}^3} |\chi_l(x)| (\left\| (\psi_{L,V} - (\psi_{L,V}^0))^\ast \right\| (\psi_{L,V}^0 - (\psi_{L,V}^0)^\ast)| dx
+ \left\| (\psi_{L,V}^0 - (\psi_{L,V}^0)^\ast) \right\| (\psi_{L,V}^0 - (\psi_{L,V}^0)^\ast)| dx
\right\} \right\} dx,
$$

where $C = \sqrt{\mathcal{T}} M_{KG}^1 \left\| (I \otimes H_{KG}^{1/2}) (H_{L} - z)^{-1} \right\| + \frac{1}{\sqrt{2}} M_{0,KG} \left\| (H_{L} - z)^{-1} \right\|$. Here we used (7). By (2) and (A.4), there exists a constant $c' \geq 0$ such that $|\psi_{L,V}^0(x)| \leq c'$. Then $\int_{\mathbb{R}^3} |\chi_l(x)| \left\| (\psi_{L,V}^0 - (\psi_{L,V}^0)^\ast) \right\| dx \to 0$ as $V \to \infty$. Then $\|(H_{L,V} - H_{L'})^0 (H_{L} - z)^{-1}\| \to 0$ as $V \to \infty$ follows. Thus we obtain (1). Similarly we can prove (2). ■
(Proof of Theorem 2.1)
The proof is parallel with that of Proposition 3.2. From the decomposition $L^2(\mathbb{R}^3; \mathbb{C}^4) = \ell^2(\Gamma_V; \mathbb{C}^4) \oplus \ell^2(\Gamma_V; \mathbb{C}^4) \perp$, it follows that $\mathcal{F} \simeq \mathcal{F}_V \oplus (\mathcal{F}_V) \perp$, where $(\mathcal{F}_V) \perp = \bigoplus_{n=1}^{\infty} \mathcal{F}_V^{(n)}$ with $\mathcal{F}_V^{(n)} = \mathcal{F}_V \otimes (\bigotimes_{a=1}^{n} \ell^2(\Gamma_V; \mathbb{C}^4) \perp)$. Then we have for $n \geq 1$,

$$H_{L,V}|_{\mathcal{F}_V^{(n)}} \simeq H_{L,V}|_{\mathcal{F}_V} \otimes I|_{\bigotimes_{a=1}^{n} \ell^2(\Gamma_V; \mathbb{C}^4) \perp} + I|_{\mathcal{F}_V} \otimes d\Gamma(\omega)|_{\bigotimes_{a=1}^{n} \ell^2(\Gamma_V; \mathbb{C}^4) \perp} \geq E_0(\mathcal{H}_{L,V}) + nM,$$

and $H_{L,V}|_{\mathcal{F}_V} \geq E_0(\mathcal{H}_{L,V}) + \nu$. Then $H_{L,V}$ has purely discrete spectrum in $[E_0(\mathcal{H}_{L,V}), E_0(\mathcal{H}_{L,V}) + \nu]$, since $H_{L,V}|_{\mathcal{F}_V}$ has purely discrete spectrum in $[E_0(\mathcal{H}_{L,V}), E_0(\mathcal{H}_{L,V}) + \nu]$ by Proposition 3.2. Then Lemma 3.3 yields that $H$ has also purely discrete spectrum in $[E_0(H), E_0(H) + \nu]$.

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