ON THE CENTERS OF THE WEIGHT–HOMOGENEOUS POLYNOMIAL VECTOR FIELDS ON THE PLANE

JAUME LLIBRE
Universitat Autònoma de Barcelona, Departament de Matemàtiques
08193 Bellaterra, Barcelona, Catalonia, Spain
jllibre@mat.uab.cat

CLAUDIO PESSOA
Universidade Federal de Uberlândia, Faculdade de Matemática
Campus Santa Mónica - Bloco 1F, Sala 1F118
Av. João Naves de Avila, 2121, 38.408–100, Uberlândia, MG, Brasil
pessoa@famat.ufu.br

Abstract. We classify all the centers of a planar weight–homogeneous polynomial vector field of weight degree 1, 2, 3 and 4.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In the qualitative theory of real planar polynomial differential systems two of the main problems are the determination of limit cycles and the center–focus problem; i.e. to distinguish when a singular point is either a focus or a center. The notion of center goes back at least to Poincaré in [17]. He defined it for a vector field on the real plane. This paper deals with the classification of the centers for a class of polynomial differential systems. The classification of the centers of the polynomial differential systems started with the quadratic ones with the works of Dulac [7], Kapteyn [10], [11], Bautin [4], Zoladek [19], ... see Schlomiuk [18] for an update on the quadratic centers. There are many partial results for the centers of polynomial differential systems of degree larger than 2, but we are very far to obtain a complete classification of all centers for the polynomial differential systems of degree ≥ 3.

We deal with polynomial differential systems of the form

\begin{align}
\dot{x} &= P(x, y), \\
\dot{y} &= Q(x, y),
\end{align}

where \( P \) and \( Q \) are polynomials in the variables \( x \) and \( y \) with real coefficients.

We say that system (1) is weight–homogeneous if there exist \( s = (s_1, s_2) \in \mathbb{N}^2 \) and \( d \in \mathbb{N} \) such that for arbitrary \( \lambda \in \mathbb{R}^+ = \{ \lambda \in \mathbb{R} : \lambda > 0 \} \),

\[ P(\lambda^{s_1}x, \lambda^{s_2}y) = \lambda^{s_1-1+d}P(x, y), \quad Q(\lambda^{s_1}x, \lambda^{s_2}y) = \lambda^{s_2-1+d}Q(x, y). \]

We call \( s = (s_1, s_2) \) the weight exponent of system (1) and \( d \) the weight degree with respect to the weight exponent \( s \). In the particular case that \( s = (1, 1) \) systems (1) are exactly the homogeneous polynomial differential systems of degree \( d \).

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A singular point $p$ of system (1) is a **center** if there is a neighborhood of $p$ fulfilled of periodic orbits with the unique exception of $p$. The **period annulus** of a center is the region fulfilled by all the periodic orbits surrounding the center. We say that a center located at the origin is **global** if its period annulus is $\mathbb{R}^2 \setminus \{(0,0)\}$.

Let $U$ be an open subset of $\mathbb{R}^2$. Here a nonconstant analytic function $H: U \to \mathbb{R}$ is called a **first integral** of system (1) on $U$ if it is constant on all solutions curves $(x(t), y(t))$ of the vector field $X$ associated to system (1) on $U$; i.e. $H(x(t), y(t))$ = constant for all values of $t$ for which the solution $(x(t), y(t))$ is defined in $U$. Clearly $H$ is a first integral of the vector field $X$ on $U$ if and only if

$$XH = P \frac{\partial H}{\partial x} + Q \frac{\partial H}{\partial y} = 0$$

on $U$.

The main goal of this paper is to classify all centers of the weight–homogeneous planar polynomial differential systems (1) of weight degree 1, 2, 3 and 4 with $P$ and $Q$ coprime.

For doing that we characterize first the normal forms of all the weight–homogeneous planar polynomial differential systems of weight degree 1, 2, 3 and 4. Thus from the definition of weight–homogeneous polynomial differential systems (1) with weight degree 1, 2, 3 and 4, the exponents $u$ and $v$ of any monomial $x^u y^v$ of $P$ and $Q$ are such that they satisfy respectively the relations

$$s_1 u + s_2 v = s_1,$$

for weight degree 1;

$$s_1 u + s_2 v = s_1 + 1,$$

for weight degree 2;

$$s_1 u + s_2 v = s_1 + 2,$$

for weight degree 3; and

$$s_1 u + s_2 v = s_1 + 3,$$

for weight degree 4. Moreover taking into account that we only consider the cases with $P$ and $Q$ coprime, it is easy to check that the systems:

(i) with weight degree 1 are the following ones with their corresponding values of $s$:

(2) $s = (1, 1)$:

$$\dot{x} = a_{10}x + a_{01}y,$$

$$\dot{y} = b_{10}x + b_{01}y,$$

(3) $s = (1, p)$:

$$\dot{x} = a_{10}x,$$

$$\dot{y} = b_{p0}x^p + b_{01}y,$$ with $p \in \mathbb{N}, p > 1$,

(ii) with weight degree 2 are the following ones with their corresponding values of $s$:

(4) $s = (1, 1)$:

$$\dot{x} = a_{20}x^2 + a_{11}xy + a_{02}y^2,$$

$$\dot{y} = b_{20}x^2 + b_{11}xy + b_{02}y^2,$$

(5) $s = (1, 2)$:

$$\dot{x} = a_{20}x^2 + a_{01}y,$$

$$\dot{y} = b_{20}x^2 + b_{11}xy,$$

(6) $s = (2, 3)$:

$$\dot{x} = a_{01}y,$$

$$\dot{y} = b_{20}x^2,$$
(iii) with weight degree 3 are the systems with weight degree 2 and additionally the following ones with their corresponding values of s (see [5]):

\begin{align}
(7) & \quad s = (1, 1) : \quad \dot{x} = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\
& \quad \dot{y} = b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3; \\
(8) & \quad s = (1, 2) : \quad \dot{x} = a_{30}x^3 + a_{11}xy, \\
& \quad \dot{y} = b_{40}x^4 + b_{21}x^2y + b_{02}y^2, \\
(9) & \quad s = (1, 3) : \quad \dot{x} = a_{30}x^3 + a_{01}y, \\
& \quad \dot{y} = b_{50}x^5 + b_{21}x^2y, \\
(10) & \quad s = (1, 1) : \quad \dot{x} = a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4, \\
& \quad \dot{y} = b_{40}x^4 + b_{31}x^3y + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4, \\
(11) & \quad s = (1, 2) : \quad \dot{x} = a_{40}x^4 + a_{21}x^2y + a_{02}y^2, \\
& \quad \dot{y} = b_{50}x^5 + b_{31}x^3y + b_{12}xy^2, \\
(12) & \quad s = (1, 3) : \quad \dot{x} = a_{40}x^4 + a_{11}xy, \\
& \quad \dot{y} = b_{60}x^6 + b_{31}x^3y + b_{02}y^2, \\
(13) & \quad s = (1, 4) : \quad \dot{x} = a_{40}x^4 + a_{01}y, \\
& \quad \dot{y} = b_{70}x^7 + b_{31}x^3y, \\
(14) & \quad s = (2, 3) : \quad \dot{x} = a_{11}xy, \\
& \quad \dot{y} = b_{30}x^3 + b_{02}y^2, \\
(15) & \quad s = (2, 5) : \quad \dot{x} = a_{01}y, \\
& \quad \dot{y} = b_{40}x^4,
\end{align}

(iv) with weight degree 4 are the systems [5], [11], with weight degree 2 and additionally the following ones with their corresponding values of s:

\begin{align}
(16) & \quad \dot{x} = p_1x^3 + p_2x^2y + p_3xy^2 + \mu y^3, \\
& \quad \dot{y} = \mu x^3 + p_1x^2y + p_2xy^2 + p_3y^3, \quad \text{with } \mu \neq 0; \\
(17) & \quad \dot{x} = p_1x^3 + (p_2 + 3\alpha)x^2y + (p_3 - 6\alpha)xy^2 - 6\alpha y^3, \\
& \quad \dot{y} = p_1x^2y + (p_2 - 3\alpha)xy^2 + (p_3 + 6\alpha)y^3, \quad \text{with } \alpha = \pm 1; \\
(18) & \quad \dot{x} = p_1x^3 + p_2x^2y + (p_3 + 2)xy^2 - 4y^3, \\
& \quad \dot{y} = p_1x^2y + p_2xy^2 + (p_3 - 2)y^3.
\end{align}

For studying the centers of systems (i), (ii), (iii) and (iv), we first simplify the weight–homogeneous planar polynomial differential systems of weight degree 3 with weight exponent (1, 1) having 8 parameters to some normal forms with at most 4 independent parameters. After using these normal forms we characterize which of these systems have a center. The proposition that gives us this normal forms is the following (see [8] for the proof).

**Proposition 1.** For any cubic homogeneous system there exists some linear transformation and a rescaling of the independent variable which transforms the system into one and only one of the following canonical forms:

\begin{align}
(16) & \quad \dot{x} = p_1x^3 + p_2x^2y + p_3xy^2 + \mu y^3, \\
& \quad \dot{y} = \mu x^3 + p_1x^2y + p_2xy^2 + p_3y^3, \quad \text{with } \mu \neq 0; \\
(17) & \quad \dot{x} = p_1x^3 + (p_2 + 3\alpha)x^2y + (p_3 - 6\alpha)xy^2 - 6\alpha y^3, \\
& \quad \dot{y} = p_1x^2y + (p_2 - 3\alpha)xy^2 + (p_3 + 6\alpha)y^3, \quad \text{with } \alpha = \pm 1; \\
(18) & \quad \dot{x} = p_1x^3 + p_2x^2y + (p_3 + 2)xy^2 - 4y^3, \\
& \quad \dot{y} = p_1x^2y + p_2xy^2 + (p_3 - 2)y^3.
\end{align}
Theorem 2. \( \text{alent normal form (16)–(23) that have atmost four parameter s.} \)

Since \( P \) and \( Q \) are coprime it follows that the origin is the center at the origin. So the function \( g(\theta) \) is either positive, or negative for all \( \theta \in \mathbb{S}^1 \); otherwise if \( g(\theta^*) = 0 \) for some \( \theta^* \in \mathbb{S}^1 \), then the straight line \( \theta = \theta^* \) is invariant by the flow of system (1), in contradiction with the fact that the origin is a center. So the homogeneous polynomial \( xQ(x,y) - yP(x,y) \) of degree \( m + 1 \) has no real factors of degree 1, and consequently \( m \) must be odd. In the rest of this section we only consider homogeneous systems (1) having a center at the origin. So the function \( g(\theta) \) is either positive, or negative for all \( \theta \in \mathbb{S}^1 \). Hence system (1) is equivalent to

\begin{align*}
\dot{x} &= p_1 x^3 + (p_2 - 3\alpha)x^2 y + p_3 xy^2 - 6y^3, \quad \text{with } \alpha = \pm 1; \\
\dot{y} &= p_1 x^2 y + (p_2 + 3\alpha)xy^2 + p_3 y^3, \\
(19) \\
\dot{x} &= p_1 x^3 + p_2 x^2 y + p_3 xy^2 - \alpha y^3, \quad \text{with } \alpha = \pm 1; \\
\dot{y} &= p_1 x^2 y + p_2 xy^2 + p_3 y^3, \\
(20) \\
\dot{x} &= p_1 x^3 + (p_2 - 3\alpha\mu)x^2 y + p_3 xy^2 - \alpha y^3, \quad \text{with } \alpha = \pm 1, \mu > -\frac{1}{2} \\
\dot{y} &= \alpha x^3 + p_1 x^2 y + (p_2 + 3\alpha\mu)xy^2 + p_3 y^3, \quad \text{and } \mu \neq \frac{1}{2}; \\
(21) \\
\dot{x} &= p_1 x^3 + (p_2 - \alpha)x^2 y + p_3 xy^2 - \alpha y^3, \quad \text{with } \alpha = \pm 1; \\
\dot{y} &= \alpha x^3 + p_1 x^2 y + (p_2 + \alpha)xy^2 + p_3 y^3, \\
(22) \\
\dot{x} &= x(p_1 x^2 + p_2 xy + p_3 y^2), \\
\dot{y} &= y(p_1 x^2 y + p_2 xy + p_3 y^2). \\
(23)
\end{align*}

In what follows instead of working with system (1) we shall work with the equivalent normal form (16)–(23) that have atmost four parameters.

The main results of this paper is the following one.

**Theorem 2.** The following statements hold.

(a) Systems (2) has a center at the origin if and only if \( b_{01} = -a_{10} \text{ and } a_{10}b_{01} - a_{01}b_{10} > 0 \). System (3) has no centers.

(b) Systems (4) and (5) have no centers. System (6) has a center at the origin if and only if \( a_{11}b_{10} = 0 \text{ and } (2a_{20} + b_{11})^2 + 8(b_{10}a_{10} - b_{11}a_{20}) < 0 \).

(c) Systems (8), (16), (17), (18), (19), (20) and (23) have no centers. System (9) has a center at the origin if and only if \( a_{01}b_{50} = 0 \text{, } (3a_{30} + b_{21})^2 + 12(b_{50}a_{10} - b_{21}a_{30}) < 0 \text{ and } 3a_{30} + b_{21} = 0 \). Systems (21) and (22) have a center if and only if \( p_3 = -p_1 \).

(d) Systems (10), (11), (12), (13) and (15) have no centers. System (18) has a center at the origin if and only if \( a_{01}b_{70} = 0 \text{ and } (4a_{40} + b_{31})^2 + 16(b_{70}a_{10} - b_{31}a_{40}) < 0 \).

The centers of the planar weight–homogeneous polynomial vector fields of weight degree 1 are characterized by statement (a), of weight degree 2 by statement (b), of weight degree 3 by statement (c) and of weight degree 4 by statement (d).

2. Homogeneous centers

In this section we assume that \( P \) and \( Q \) are coprime homogeneous polynomials of degree \( m \). We want to characterize the homogeneous systems (1) which have a center at the origin. Since \( P \) and \( Q \) are coprime it follows that the origin is the unique (real and finite) singular point of system (1).

System (1) in polar coordinates \((\rho, \theta)\) defined by \( x = \rho \cos \theta, \ y = \rho \sin \theta \), becomes

\begin{equation}
\dot{\rho} = f(\theta)\rho^m, \ \ \dot{\theta} = g(\theta)\rho^{m-1}.
\end{equation}

If the origin of system (1) is a center, then from the expression of \( \dot{\theta} \) it follows that the function \( g(\theta) \) is either positive, or negative for all \( \theta \in \mathbb{S}^1 \); otherwise if \( g(\theta^*) = 0 \) for some \( \theta^* \in \mathbb{S}^1 \), then the straight line \( \theta = \theta^* \) is invariant by the flow of system (1), in contradiction with the fact that the origin is a center. So the homogeneous polynomial \( xQ(x,y) - yP(x,y) \) of degree \( m + 1 \) has no real factors of degree 1, and consequently \( m \) must be odd. In the rest of this section we only consider homogeneous systems (1) having a center at the origin. So the function \( g(\theta) \) is either positive, or negative for all \( \theta \in \mathbb{S}^1 \). Hence system (1) is equivalent to
the equation
\[ \frac{d \rho}{d \theta} = \frac{f(\theta)}{g(\theta)} \rho. \]

Separating the variables \( \rho \) and \( \theta \) of this equation, and integrating between 0 and \( \theta \) we get that any solution of the previous differential equation is
\[ \rho(\theta) = \begin{cases} 
\rho(0) \exp \left( \int_0^\theta \frac{f(r)}{g(r)} dr \right) & \text{if } f(\theta) \neq 0, \\
\rho(0) & \text{if } f(\theta) \equiv 0.
\end{cases} \]

From (25) it follows immediately the next proposition.

**Proposition 3.** Suppose that \( P \) and \( Q \) are coprime homogeneous polynomials of degree \( m \). Then the origin of system (1) is a global center if and only if the polynomial \( xQ(x, y) - yP(x, y) \) has no a real factors of degree 1 (in particular \( m \) is odd) and
\[ \int_0^{2\pi} \frac{f(r)}{g(r)} dr = 0. \]

This proposition is well known, see for instance [6] or [12].

### 3. Monodromic singularities

In this section we assume that \( P \) and \( Q \) are coprime polynomials and let \( m = \max\{\deg P, \deg Q\} \). Then we say that system (1) has degree \( m \).

Consider the polynomial vector field \( X = (P, Q) \) associated to system (1). The study of the local phase portrait at the singular points of the vector field \( X \) is a problem almost completely solved. In most of the cases one can know which is the behavior of the solutions in a neighborhood of a singular point. The only case that remains open is the monodromic one. In this case the orbits turn around the singular point. The difficulty is in distinguishing when the orbits spiral toward or backward the singular point (i.e. when the origin is a focus) and when the origin is a center. This problem is know as the center–focus problem, see [15] for a survey on this problem.

Suppose that the origin is a singular point of \( X \). Let \( \gamma(t) \) be an orbit of \( X \) defined in a neighborhood of the origin that tends to it when \( t \) tends to \( +\infty \) and such that \( \lim_{t \to +\infty} \gamma(t)/\|\gamma(t)\| \in S^1 \), where \( S^1 \) is the unit circle, with \( \| \cdot \| \) the Euclidean norm. In this case \( \gamma(t) \) is said a characteristic orbit. We may also consider orbits tending to the origin as \( t \) tends to \( -\infty \) but we can always change the sign of the parameter \( t \) in system (1) and assume that \( t \) tends to \( +\infty \). We have that the origin is a monodromic singular point of \( X \) if there is no characteristic orbit associated to it.

We write \( P(x, y) = \sum_{i=n}^m P_i(x, y) \), \( Q(x, y) = \sum_{i=0}^m Q_i(x, y) \), where \( P_i \) and \( Q_i \) are the respective homogeneous part of degree \( i \) of \( P \) and \( Q \), with \( 1 \leq n \leq m \). It is possible that \( P_n(x, y) \) or \( Q_n(x, y) \) is null, but both of them cannot be null. A characteristic direction for the origin of \( X \) is a root \( \omega \in S^1 \) of the homogeneous polynomial \( xQ_n(x, y) - yP_n(x, y) \), which can be written as \( \omega = (\cos \theta, \sin \theta) \) with \( \theta \in [0, 2\pi) \).

It is obvious that, unless \( xQ_n(x, y) - yP_n(x, y) \equiv 0 \), the number of characteristic directions for the origin of \( X \) is less than or equal to \( n+1 \). The following well-known result relates characteristic orbits with characteristic directions of singular points.
Proposition 4. Let \( \gamma(t) \) be a characteristic orbit for the origin of \( X \) associated to system (11) and \( \omega = \lim_{t \to +\infty} \frac{\gamma(t)}{t} \). Then \( \omega \) is a characteristic direction for \( X \).

This proposition is proved in (3). The reciprocal is not true, see (15) or (8).

In polar coordinates \( x = \rho \cos \theta, y = \rho \sin \theta \), and doing a change of time \( t \mapsto \tau \) such that \( dt/d\tau = \rho^{n-1} \), system (11) becomes

\[
\begin{align*}
\dot{\rho} &= \rho (R_n(\theta) + \rho R_{n+1}(\theta) + \cdots + \rho^{m-n} R_m(\rho)), \\
\dot{\theta} &= F_n(\theta) + \rho F_{n+1}(\theta) + \cdots + \rho^{m-n} F_m(\rho),
\end{align*}
\]

where

\[
R_j(\theta) = P_j(\cos \theta, \sin \theta) \cos \theta + Q_j(\cos \theta, \sin \theta) \sin \theta,
\]

\[
F_j(\theta) = Q_j(\cos \theta, \sin \theta) \cos \theta - P_j(\cos \theta, \sin \theta) \sin \theta,
\]

are homogeneous trigonometric polynomials of degree \( j + 1 \), for \( n \leq j \leq m \).

Let \( \gamma(t) = (\rho(t), \theta(t)) \) be an orbit for system (11) written in polar coordinates. We have that \( \gamma(t) \) tends to the origin if \( \rho(t) \) is not identically zero and \( \lim_{t \to +\infty} \rho(t) = 0 \). Moreover \( \gamma(t) \) tends to the origin spirally if it tends to the origin and \( \lim_{t \to +\infty} \theta(t) = \pm \infty \). Therefore \( \gamma(t) \) is a characteristic orbit for the origin of \( X \) if it tends to the origin and \( \lim_{t \to +\infty} \theta(t) = \theta^* < \infty \). In this case we say that \( \theta^* \) is the tangent at the origin of \( \gamma(t) \).

The polynomial \( xQ_n(x, y) - yP_n(x, y) \), considered over \( \mathbb{S}^1 \), in polar coordinates is \( F_n(\theta) \) given in (26). Hence if \( \gamma(t) = (\rho(t), \theta(t)) \) is a characteristic orbit for the origin of \( X \) written in polar coordinates with tangent \( \theta^* \) at the origin, then \( F_n(\theta^*) = 0 \).

Now we include here a well-known property of monodromic singular points, see (8) for an easy proof.

Proposition 5. If the origin is monodromic, then \( F_n(\theta) \neq 0 \) and \( \dot{\theta}(\rho, \theta) \) is of definite sign for any \((\rho, \theta)\) verifying \( 0 < \rho < \epsilon \), for a certain \( \epsilon > 0 \) sufficiently small.

When the eigenvalues of the matrix \( DX(p) \) at the singular point \( p \) are complex and not real, we know that the origin is monodromic. Such class of singular points are called of linear type. If their real part is different from zero then the singular point is a focus, while if their real part is zero the singular point may be a center or a focus. This last case is the classical Lyapunov-Poincaré center problem. The study of this problem for a concrete family of differential equations passes through the calculation of the so-called Lyapunov constants. The problem of how many constants are needed to distinguish between a center of a focus is in general open.

When the matrix \( DX(p) \) has its two eigenvalues equal to zero but the matrix is not identically null, it is said that \( p \) is a nilpotent singular point. The monodromy problem in this case was solved in (2) and the center problem has been studied in (10) and (9). Nevertheless, in general, given a polynomial system with a nilpotent monodromic singular point it is not an easy task to know if it is a focus or a center.

Suppose that \( X \) has a singular point \( p \) such that the matrix \( DX(p) \) is a nilpotent matrix. If the singularity \( p \) is a center then \( p \) is called a nilpotent center. In this case using suitable coordinates system (11) can be written as

\[
\begin{align*}
\dot{x} &= y + P_2(x, y), \\
\dot{y} &= Q_2(x, y),
\end{align*}
\]

where \( P_2, Q_2 \) are polynomials of degree at least 2.

We state a theorem, proved in (2) (see also (3) and (11)), which solves the monodromic problem for nilpotent singular points.
Theorem 6. Consider system (27) and assume that the origin is an isolated singularity. Define the functions
\[
\begin{align*}
    f(x) &= Q_2(x, F(x)) = ax^\alpha + O(x^{\alpha+1}), \\
    \phi(x) &= \text{div}(y + P_2(x, y), Q_2(x, y)) \big|_{y=F(x)} = bx^\beta + O(x^{\beta+1}),
\end{align*}
\]
where \(a \neq 0, \alpha \geq 2, b \neq 0\) and \(\beta \geq 1\), or \(\phi(x) \equiv 0\). Here the function \(y = F(x)\) is the solution of \(y + P_2(x, y) = 0\) passing through \((0,0)\). Then the origin of \(\Phi\) is monodromic if and only if \(a\) is an odd number \((\alpha = 2n-1)\), and one of the following three conditions holds: 
\[\beta > n - 1; \beta = n - 1\] and \(b^2 + 4an < 0; \text{and } \phi \equiv 0.\]

Once we know how to distinguish between monodromic and not monodromic nilpotent singular points, we wish to solve the center–focus problem for the monodromic ones. Our approach to this problem passes through the computation of the Taylor expansion of the return map near the singular point. In the next section we define the generalized Lyapunov constants introduced to solve the stability problem. Instead of working with system (27), we shall use the special normal form given in the next statement, see [3] or [1] for the proof.

Lemma 7. An analytic vector field having an isolated nilpotent monodromic singularity can be written as
\[
\begin{align*}
    \dot{x} &= y(-1 + \hat{P}(x, y)), \\
    \dot{y} &= f(x) + \phi(x)y + \hat{Q}(x, y)y^2,
\end{align*}
\]
where all the above functions are analytic at the origin and satisfy \(\hat{P}(0,0) = 0\), \(f(x) = x^{2n-1} + O(x^{2n})\) for some \(n \in \mathbb{N}, n \geq 2\), and \(\phi(x) \equiv 0\) or \(\phi(x) = bx^\beta + O(x^{\beta+1})\) for some \(\beta \in \mathbb{N}\) satisfying \(\beta \geq n - 1\). Moreover, if \(\beta = n - 1\) then \(b^2 - 4n < 0\).

4. Generalized Lyapunov constants

Using the notation of previous section we will introduce some generalized polar coordinates. As a starting point recall that given any natural number \(n \in \mathbb{N}\), the generalized trigonometric functions \(x(\theta) = \text{Cs}(\theta), y(\theta) = \text{Sn}(\theta)\), are defined as the unique solution of the Cauchy problem
\[
\begin{align*}
    \dot{x} &= \frac{dx}{d\theta} = -y, \\
    \dot{y} &= \frac{dy}{d\theta} = x^{2n-1},
\end{align*}
\]
with initial conditions \(x(0) = 1, y(0) = 0\).

Some properties of these functions are stated in the next proposition, see [14].

Proposition 8. Let \(n \in \mathbb{N}\) and let \(\text{Cs}(\theta)\) and \(\text{Sn}(\theta)\) be the generalized trigonometric functions determined by system (29). Then the following statement hold.
\[\begin{align*}
    (a) \quad &\text{Cs}^{2n}(\theta) + n\text{Sn}^2(\theta) = 1, \\
    (b) \quad &\text{Cs}(\theta) \text{ and } \text{Sn}(\theta) \text{ are } T\text{-periodic functions where} \\
    &T = 2\sqrt{\frac{\pi \Gamma\left(\frac{1}{2n}\right)}{\pi \Gamma\left(\frac{1}{2n} + 1\right)}}, \\
    (c) \quad &\text{Sn}(\theta + T/2) = -\text{Sn}(\theta), \text{Cs}(\theta + T/2) = -\text{Cs}(\theta), \text{Sn}(-\theta + T/2) = \text{Sn}(\theta) \text{ and} \\
    &\text{Cs}(-\theta + T/2) = -\text{Cs}(\theta).
\end{align*}\]
Here as is usual Γ denotes the gamma function, i.e. \( \Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt \).

We define for any given \( n \in \mathbb{N} \) the generalized polar coordinates \( r \) and \( \theta \) of the real plane \((x, y) \in \mathbb{R}^2\) as

\[
x = r \text{Cs}(\theta), \quad y = r^n \text{Sn}(\theta),
\]

where the function Sn and Cs are given through (29). Notice that \( x^{2n} + ny^{2n} = r^{2n} \).

Furthermore the following equalities hold

\[
\dot{r} = \frac{x^{2n-1} \dot{x} + y \dot{y}}{r^{2n-1}},
\]

\[
\dot{\theta} = \frac{x \dot{y} - ny \dot{x}}{r^{n+1}}.
\]

Introducing the coordinates (30), it follows from (31) that system (28) can be reduced to the following differential equation

\[
\frac{dr}{d\theta} = R(r, \theta) = \frac{G(r, \theta)}{1 + H(r, \theta)},
\]

where \( R \) is analytic at the origin. Thus the above equation can be written in the neighborhood of the origin as

\[
\frac{dr}{d\theta} = \sum_{i=1}^{\infty} R_i(\theta) r^i,
\]

where the functions \( R_i \) for \( i \geq 1 \) are \( T \)-periodic, being \( T \) the period of the generalized trigonometric functions. Now we can define the generalized Lyapunov constants see [14].

Consider the solution \( r(\theta, \rho) \) of (33) such that \( r(0, \rho) = \rho \). It can be written as

\[
r(\theta, \rho) = \sum_{i=1}^{\infty} u_i(\theta) \rho^i,
\]

where \( u_1(0) = 1 \) and \( u_k(0) = 0 \) for all \( k \geq 2 \). Hence the return map is given by the series

\[
P(\rho) = r(T, \rho) = \sum_{i=1}^{\infty} u_i(T) \rho^i.
\]

For a fixed system the only significant term is the first that makes the return map differ from the identity map, and this will determine the stability of the origin. On the other hand if we consider a family of systems depending on parameters, each of the \( u_i(T) \) depends on these parameters. Thus the stability of the origin is given by the first \( V_k = u_k(T) \neq 0 \) with \( u_1(T) = 1, u_2(T) = \cdots = u_{k-1}(T) = 0 \), which it is called \( k \)th generalized Lyapunov constant. Note that system (27) has a center if and only if \( V_1 = 1 \) and \( V_k = 0 \) for all \( k \geq 2 \).

In [1] it is proved the following result.

**Proposition 9.** Consider system (28) under the assumptions of Lemma 7. Then the origin is a monodromic singular point and its first generalized Lyapunov constants is

\[
V_1 = \exp \left( -\frac{2b\pi}{n\sqrt{4n - \beta^2}} \right),
\]

when \( \beta = n - 1 \) and \( n \) is odd; otherwise \( V_1 = 1 \).
5. Weight–homogeneous centers of Weight degree 1, 2, 3 and 4

In this section we shall prove the main results of this paper. We remark that for a weight–homogeneous polynomial vector field $X = (P, Q)$ with $P$ and $Q$ coprime the origin is the unique (real and finite) singularity of $X$. Moreover when it is a center it is a global center (see [12]).

Proof of Theorem 2(a). Consider system (2). This system is linear and we know that the origin is a center if and only if $b_{01} = -a_{10}$ and $a_{10}b_{01} - a_{01}b_{10} > 0$. Now system (3) does not have a center because the straight line $x = 0$ is invariant by its flow.

Proof of Theorem 2(b). Consider the vector field $X = (P, Q)$ associated to system (4). This vector field does not have centers by Proposition 3, because $m$ is even.

System (6) does not have a center, because the origin is always a cusp, by a result that can be find in [3] page 362. Note that this system has the first integral $H(x, y) = a_{01}y^2/2 - b_{20}x^3/3$.

Now we will study system (5). Note that $a_{01}b_{30} \neq 0$, otherwise the straight lines $x = 0$ or $y = 0$ should be invariant by the flow of this system, and so the origin cannot be a center. Therefore rescaling the independent variable system (5) becomes

$$
\dot{x} = y + P_2(x, y) = y + \frac{a_{20}}{a_{01}}x^2,
$$

$$
\dot{y} = Q_2(x, y) = \frac{b_{30}}{a_{01}}x^3 + \frac{b_{11}}{a_{01}}xy.
$$

We have that $F(x) = -a_{20}x^2/a_{01}$ is the solution of $y + P_2(x, y) = 0$. Hence

$$
f(x) = Q_2(x, F(x)) = \frac{b_{30}a_{01} - b_{11}a_{20}}{a_{01}^2}x^3;
$$

$$
\phi(x) = \text{div}(y + P_2(x, y), Q_2(x, y)) \big|_{y=F(x)} = \frac{2a_{20} + b_{11}}{a_{01}}x.
$$

Thus by Theorem 6 the origin of system (5) is monodromic if and only if

$$(2a_{20} + b_{11})^2 + 12(b_{30}a_{01} - b_{11}a_{20}) < 0.
$$

In this case the origin is a center of (5), because the system above is reversible, i.e. it is invariant by the change of variables $(x, y, t) \mapsto (-x, y, -t)$.

Proof of Theorem 2(c). We begin studying system (8). This system does not have centers, because the straight line $x = 0$ is invariant by its flow.

Consider system (9). Note that $a_{01}b_{50} \neq 0$, otherwise the straight lines $x = 0$ or $y = 0$ should be invariant by the flow of this system, and so the origin cannot be a center. Rescaling the independent variable system (9) becomes

$$
\dot{x} = y + \frac{a_{30}}{a_{01}}x^3,
$$

$$
\dot{y} = \frac{b_{21}}{a_{01}}x^2y + \frac{b_{50}}{a_{01}}x^5.
$$

In a similar way as in the study of system (5) in the proof of Theorem 2(b), we have that the origin of system (9) is monodromic if and only if

$$(3a_{30} + b_{21})^2 + 12(b_{50}a_{01} - b_{21}a_{30}) < 0.$$
Now by the change of variables
\[
\tilde{y} = \left( \frac{a_{30}b_{21} - a_{01}b_{50}}{a_{01}^2} \right)^{1/4} y + \frac{a_{30}}{a_{01}} \left( \frac{a_{30}b_{21} - a_{01}b_{50}}{a_{01}^2} \right)^{1/4} x^3,
\]
\[
\tilde{x} = -\left( \frac{a_{30}b_{21} - a_{01}b_{50}}{a_{01}^2} \right)^{1/4} x,
\]

system (35) can be written as
\[
(36) \begin{align*}
\dot{x} &= -y, \\
\dot{y} &= x^5 + \text{sign}(a_{01}) \frac{3a_{30} + b_{21}}{\sqrt{a_{30}b_{21} - b_{50}a_{01}}} x^2 y.
\end{align*}
\]

Therefore by Proposition 9 the first generalized Lyapunov constant \( V_1 \) of system (36) is
\[
V_1 = \exp \left( \frac{2 \left( \frac{\text{sign}(a_{01}) \frac{3a_{30} + b_{21}}{\sqrt{a_{30}b_{21} - b_{50}a_{01}}}}{\pi} \right)}{n \left( \sqrt{4m - \left( \frac{\text{sign}(a_{01}) \frac{3a_{30} + b_{21}}{\sqrt{a_{30}b_{21} - b_{50}a_{01}}}}{2} \right)^2} \right)} \right)
\]

Hence \( 3a_{30} + b_{21} = 0 \) is a necessary condition for the origin be a center of system (36). But this condition is also sufficient, because in this case system (36) has the first integral \( H(x, y) = y^2/2 + x^6/6 \). Note that the origin is a global minimum of the graphic of \( H \).

For finishing the proof of statement (c) we must study system (7). We consider the normal forms of system (7) given by Proposition 1. Let \( X = (P, Q) \) be the vector field associated to the systems given by Proposition 1. System (16) does not have a center by Proposition 3, because due to the fact that
\[
xQ(x, y) - yP(x, y) = \mu(x - y)(x + y)(x^2 + y^2),
\]
it has the invariant straight lines \( x \pm y = 0 \) though the origin.

In the same away system (17) does not have a center, because
\[
xQ(x, y) - yP(x, y) = -6\alpha y^2(x^2 - 2xy - y^2).
\]

Now systems (18), (19), (20) and (23) do not have a center, because \( y = 0 \) is an invariant straight line for these systems.

For system (22) we have that
\[
xQ(x, y) - yP(x, y) = \alpha(x^2 + y^2)^2.
\]

On the other hand for this system we get that
\[
\int_\alpha^{2\pi} \frac{f(\theta)}{g(\theta)} d\theta = \frac{\pi(p_1 + p_3)}{\alpha},
\]

where \( f \) and \( g \) are given by (24). Therefore, by Proposition 3 system (22) has a center if and only if \( p_3 = -p_1 \).

Analogously for system (21) we have that
\[
xQ(x, y) - yP(x, y) = \alpha(x^4 + 6\mu x^2 y^2 + y^4).
\]

Hence \( xQ(x, y) - yP(x, y) = 0 \) if and only if
\[
x = \pm |y| \sqrt{-3\mu \pm \sqrt{9\mu^2 - 1}}.
\]
Thus, if $\mu > -1/3$ the polynomial $x^4 + 6\mu x^2 y^2 + y^4$ does not have a real factor of degree 1. On another hand for this system we get that
\[
\int_0^{2\pi} \frac{f(\theta)}{g(\theta)} \, d\theta = \frac{\sqrt{3\mu + \sqrt{9\mu^2 - 1}} - \sqrt{3\mu - \sqrt{9\mu^2 - 1}}}{\sqrt{9\mu^2 - 1}} \pi(p_1 + p_3),
\]
where $f$ and $g$ are given by (24). Therefore, by Proposition 3 system (21) has a center if and only if $p_3 = -p_1$. This finish the proof of Theorem 2 (c). □

Proof of Theorem 2 (d). Consider the vector field $X = (P, Q)$ associated to system (10). This vector field does not have centers by Proposition 3 because $m$ is even.

Systems (12) and (14) does not have centers, because the straight line $x = 0$ is invariant by its flow. System (15) does not have a center, because the origin is always a cusp, by a result that can be find in [3] page 362. Note that this system has the first integral $H(x, y) = b_{40}x^5/5 - a_{01}y^2/2$.

Consider system (13). Note that $a_{01}b_{70} \neq 0$, otherwise the straight lines $x = 0$ or $y = 0$ should be invariant by the flow of this system, and so the origin cannot be a center. Rescaling the independent variable system (13) becomes
\[
\begin{align*}
\dot{x} &= y + \frac{a_{40}}{a_{01}}x^4, \\
\dot{y} &= \frac{b_{11}}{a_{01}}x^3y + \frac{b_{70}}{a_{01}}x^7.
\end{align*}
\]
In a similar way as in the study of system (3) in the proof of Theorem 2 (b), we have that the origin of system (13) is monodromic if and only if
\[
(4a_{40} + b_{31})^2 + 16(b_{70}a_{01} - b_{31}a_{40}) < 0.
\]
Now if $a_{01}b_{70} \neq 0$ and the origin of system (13) is monodromic, then it is a center because system (37) is reversible, i.e. it is invariant by the change of variables $(x, y, t) \rightarrow (-x, y, -t)$.

For finishing the proof we must study system (11). Note that $a_{02}b_{50} \neq 0$, otherwise the straight lines $x = 0$ or $y = 0$ should be invariant by the flow of this system, and so the origin cannot be a center.

Now in polar coordinates system (11) is given by (20), where $n = 2$ and $F_2(\theta) = -a_{02} \sin^3(\theta)$. Hence, as $\dot{\theta}(\rho, \theta) = -a_{02} \sin^3(\theta) + \rho F_3(\theta) + \cdots$, it follows that $\dot{\theta}(\rho, \theta)$ changes of sign for $0 < \rho < \epsilon$ with $\epsilon$ small enough. Therefore, by Proposition 5 the origin cannot be a monodromic singular point of system (11). Hence system (11) has no centers. □

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