A semifilter approach to selection principles II: 
\(\tau^*\)-covers

Lyubomyr Zdomskyy

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Abstract

Developing the ideas of \[23\] we show that every Menger topological space has the property \(\bigcup \text{fin}(\mathcal{O}, \mathcal{T}^*)\) provided \((u < g)\), and every space with the property \(\bigcup \text{fin}(\mathcal{O}, \mathcal{T}^*)\) is Hurewicz provided \((\text{Depth}^+(\omega^{\omega}) \leq \mathfrak{b})\). Combining this with the results proven in cited literature, we settle all questions whether (it is consistent that) the properties \(P\) and \(Q\) [do not] coincide, where \(P\) and \(Q\) run over \(\bigcup \text{fin}(\mathcal{O}, \Gamma), \bigcup \text{fin}(\mathcal{O}, \mathcal{T}), \bigcup \text{fin}(\mathcal{O}, \mathcal{T}^*), \bigcup \text{fin}(\mathcal{O}, \Omega), \text{and } \bigcup \text{fin}(\mathcal{O}, \mathcal{O})\).

Introduction

Following \[15\] we say that a topological space \(X\) has the property \(\bigcup \text{fin}(\mathcal{A}, \mathcal{B})\), where \(\mathcal{A}\) and \(\mathcal{B}\) are collections of covers of \(X\), if for every sequence \((u_n)_{n \in \omega} \in \mathcal{A}\) there exists a sequence \((v_n)_{n \in \omega}\), where each \(v_n\) is a finite subset of \(u_n\), such that \(\{\bigcup v_n : n \in \omega\} \in \mathcal{B}\). Throughout this paper “cover” means “open cover” and \(\mathcal{A}\) is equal to the family \(\mathcal{O}\) of all open covers of \(X\). Concerning \(\mathcal{B}\), we shall also consider the collections \(\Gamma, \mathcal{T}, \mathcal{T}^*, \mathcal{T}^*\), and \(\Omega\) of all open \(\gamma\), \(\tau\), \(\tau^*\), \(\tau^*\), and \(\omega\)-covers of \(X\). For technical reasons we shall use the collection \(\Lambda\) of countable large covers. The most natural way to define these types of covers uses the Marczewski “dictionary” map introduced in \[13\]. Given an indexed family \(u = \{U_n : n \in \omega\}\) of subsets of a set \(X\) and element \(x \in X\), we define the Marczewski map \(\mu_u : X \to \mathcal{P}(\omega)\) letting \(\mu_u(x) = \{n \in \omega : x \in U_n\}\) (\(\mu_u(x)\) is nothing else but \(I_x(x, u)\) in notations of \[23\]). Recall, that \(A \subset^* B\) means that \(|A \setminus B| < \aleph_0\). A family \(A \subset \mathcal{P}(X)\) of subsets of a set \(X\) is a refinement of a family \(B \subset \mathcal{P}(X)\), if for every \(B \in \mathcal{B}\) there exists \(A \in \mathcal{A}\) such that \(A \subset B\). Depending on the properties of \(\mu_u(X)\) a family \(u = \{U_n : n \in \omega\}\) is defined to be

- a large cover of \(X\) \[15\], if for every \(x \in X\) the set \(\mu_u(x)\) is infinite;
- a \(\gamma\)-cover of \(X\) \[10\], if for every \(x \in X\) the set \(\mu_u(x)\) is cofinite in \(\omega\), i.e. \(\omega \setminus \mu_u(x)\) is finite;

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• a $\tau$-cover of $X$ [19], if it is a large cover and the family $\mu_u(X)$ is linearly preordered by the almost inclusion relation $\subset^*$ in sense that for all $x_1, x_2 \in X$ either $\mu_u(x_1) \subset^* \mu_u(x_2)$ or $\mu_u(x_2) \subset^* \mu_u(x_1)$:

• a $\tau^*$-cover of $X$ [19], if there exists a linearly preordered by $\subset^*$ refinement $\mathcal{J}$ of $\mu_u(X)$ consisting of infinite subsets of $\omega$;

• an $\omega$-cover [9], if the family $\mu_u(X)$ is centered, i.e. for every finite subset $K$ of $X$ the intersection $\bigcap_{x \in K} \mu_u(x)$ is infinite.

We also introduce a new type of covers situated between $\tau$- and $\tau^*$-covers. A family $u = \{U_n : n \in \omega\}$ is

• a $\tau^\star$-cover of $X$, if there exists a linearly preordered by $\subset^*$ refinement $\mathcal{J} \subset \mu_u(X)$ of $\mu_u(X)$ consisting of infinite subsets of $\omega$.

Recall, that $\bigcup_{\text{fin}} (\mathcal{O}, \Gamma)$ and $\bigcup_{\text{fin}} (\mathcal{O}, \mathcal{O})$ are nothing else but the well-known Hurewicz and Menger covering properties introduced in [10] and [14] respectively at the beginning of 20-th century.

Since every $\gamma$-cover is a $\tau$-cover, every $\tau$-cover is a $\tau^*$-cover, every $\tau^*$-cover is an $\omega$-cover, the above properties are related as follows:

\[
\bigcup_{\text{fin}} (\mathcal{O}, \Gamma) \Rightarrow \bigcup_{\text{fin}} (\mathcal{O}, \mathcal{T}^\star) \Rightarrow \bigcup_{\text{fin}} (\mathcal{O}, T^\star) \Rightarrow \bigcup_{\text{fin}} (\mathcal{O}, \Omega)
\]

By a tower we understand a $\subset^*$-decreasing transfinite sequence of infinite subsets of $\omega$, i.e. a sequence $(T_\alpha)_{\alpha < \lambda}$ such that $T_\alpha \subset^* T_\beta$ for all $\alpha \geq \beta$. The cardinality $\lambda$ is called the length of this tower. The subsequent theorem, which is the main result of this paper, describes when some of the above properties coincide.

**Theorem 1.** (1) Under $u < g$ the selection principles $\bigcup_{\text{fin}} (\mathcal{O}, T^\star)$ and $\bigcup_{\text{fin}} (\mathcal{O}, \mathcal{O})$ coincide.

(2) Under Filter Dichotomy the selection principles $\bigcup_{\text{fin}} (\mathcal{O}, T^\star)$ and $\bigcup_{\text{fin}} (\mathcal{O}, \Omega)$ coincide.

(3) Selection principles $\bigcup_{\text{fin}} (\mathcal{O}, \Gamma)$ and $\bigcup_{\text{fin}} (\mathcal{O}, T^\star)$ coincide iff each semifilter generated by a tower is meager.

The following statement describes some partial cases of Theorem 1:

**Corollary 1.** (1) Selection principles $\bigcup_{\text{fin}} (\mathcal{O}, \Gamma)$ and $\bigcup_{\text{fin}} (\mathcal{O}, T^\star)$ coincide if the inequality $\text{Depth}^+ ([\omega]^{\aleph_0}) \leq b$ holds.
(2) Under \((b < d)\) (resp. \((t = d)\)) there exists a set of reals with the property 
\(\bigcup_{\text{fin}} (O, T^*)\) which fails to satisfy \(\bigcup_{\text{fin}} (O, \Gamma)\) (resp. \(\bigcup_{\text{fin}} (O, T)\)).

Theorem 1 gives a partial answer of Problem 5.2 from [3]. Namely, it implies the subsequent

**Corollary 2.** It is consistent that the property \(\bigcup_{\text{fin}} (O, T)\) is closed under unions of families of subspaces of the Baire space of size \(< b\).

**Proof.** Follows immediately from Theorem 1(3) and the fact that the property 
\(\bigcup_{\text{fin}} (O, \Gamma)\) is preserved by unions of less than \(b\) subspaces of the Baire space, see [11].

We refer the reader to [22] for definitions of all small cardinals and related notions we use. All notions concerning semifilters may be found in [1] and will be defined in the next section. The condition \((u < g)\) is known to be consistent: \(u = b = s < g = d\) in Miller’s model and the inequality \((u < g)\) implies \(u = b < g = d\), see [1] and [22]. Moreover, \((u < g)\) is equivalent to an assertion that all upward-closed neither meager nor comeager families of infinite subsets of \(\omega\) are “similar”, see [12, 4.9.22], [1] 7.6.4, 12.2.4, or Theorem 3. This assertion together with the Talagrand’s characterization of meager and comeager upward-closed families is a so-called trichotomy for upward-closed families or Semifilter Trichotomy in terms of [1]. The Filter Dichotomy follows from the Semifilter Trichotomy and is formally stronger than the principle NCF introduced by A. Blass, see [4, §9] and references there in.

Depth\(^+\)(\([\omega]^{\aleph_0}\)) denotes the smallest cardinality \(\kappa\) such that there is no tower of length \(\kappa\). Thus \(t < \text{Depth}^+([\omega]^{\aleph_0})\). A model with \(b \geq \text{Depth}^+([\omega]^{\aleph_0})\) was constructed in [6]. Some other applications of \(\text{Depth}^+([\omega]^{\aleph_0})\) in Selection Principles may be found in [16].

Theorem 1 with results proven in [11, 19, 21, and 23], enable us to settle almost all questions whether (it is consistent that) the properties \(P\) and \(Q\) [do not] coincide, where \(P\) and \(Q\) run over \(\bigcup_{\text{fin}} (O, O)\), \(\bigcup_{\text{fin}} (O, \Omega)\), \(\bigcup_{\text{fin}} (O, T^*)\), \(\bigcup_{\text{fin}} (O, T)\), and \(\bigcup_{\text{fin}} (O, \Gamma)\). (In fact, we settle all of the questions omitting \(\bigcup_{\text{fin}} (O, T^*)\).) Some sufficient conditions for \(P = Q\) and \(P \neq Q\) are summarized in Table 1. Each entry \(((i), (j)), i \neq j\) contains:

- A condition which implies \(((i) = (j))\) (resp. \(((i) \neq (j))\) provided \(i < j\) (resp. \(i > j\)) or “?” if no such a condition is known;
- \(\text{ZFC}\), if \((i) \neq (j)\) in \(\text{ZFC}\) and \(i > j\);
- \(\neg\), if \((i) \neq (j)\) in \(\text{ZFC}\) and \(i < j\);

and a reference to where this is proven. For example, “\(\llbracket x \rrbracket + \llbracket y \rrbracket, \llbracket z \rrbracket\)” means that the sufficiency of the corresponding condition was proven in \([z]\), and it can be simply derived by combining results of \(\llbracket x \rrbracket\) and \(\llbracket y \rrbracket\). Throughout the table, \(\lambda\) stands for \(\text{Depth}^+([\omega]^{\aleph_0})\).
Semiﬁlters

Our main tool is the notion of a semiﬁlter. Following [1] a family \( \mathcal{F} \) of nonempty subsets of \( \omega \) is called a semiﬁlter, if for every \( F \in \mathcal{F} \) and \( A \supseteq F \) the set \( A \) belongs to \( \mathcal{F} \). For example, each family \( \mathcal{A} \) of inﬁnite subsets of \( \omega \) generates the minimal semiﬁlter \( \uparrow \mathcal{A} = \{ B \subseteq \omega : \exists A \in \mathcal{A}(A \supseteq B) \} \) containing \( \mathcal{A} \). The family \( \mathcal{SF} \) of all semiﬁlters contains the smallest element \( \mathfrak{r} \) consisting of all cofinite subsets of \( \omega \), and the largest one, \( [\omega]^{\aleph_0} \), i.e. the family of all inﬁnite subsets of \( \omega \). Throughout this paper by a ﬁlter we understand a semiﬁlter which is closed under ﬁnite intersections of its elements.

Since every semiﬁlter \( \mathcal{F} \) on \( \omega \) is a subset of the powerset \( \mathcal{P}(\omega) \), which can be identified with the Cantor space \( \{0, 1\}^\omega \), we can speak about topological properties of semiﬁlters. Recall, that a subset of a topological space is meager, if it is a union of countably many nowhere dense subsets. The complements of meager subsets are called comeager. We shall often use the subsequent characterization of meagerness of semiﬁlters due to Talagrand, see [18] and [1, 5.3.1].

**Theorem 2.** A semiﬁlter \( \mathcal{F} \) on \( \omega \) is meager if and only if there exists an increasing number sequence \( (k_n)_{n \in \omega} \) such that every \( F \in \mathcal{F} \) meets all but ﬁnitely many half-intervals \([k_n, k_{n+1})\).

A crucial role in the proof of Theorem 2 belongs to the following fundamental result of C. Laflamme [12]. Following [1], semiﬁlter \( \mathcal{F} \) on \( \omega \) is said to be bi-Baire, if it is neither meager nor comeager. Note, that there is no comeager ﬁlter on \( \omega \), see [1] 5.3.2.

**Theorem 3.** The following conditions are equivalent:

1. \((u < g)\);

2. for any bi-Baire semiﬁlters \( \mathcal{F} \) and \( \mathcal{U} \) there exists an increasing number sequence \( (k_n)_{n \in \omega} \) such that the sets \( \{n \in \omega : F \cap [k_n, k_{n+1}) \neq \emptyset \} : F \in \mathcal{F} \) and \( \{n \in \omega : U \cap [k_n, k_{n+1}) \neq \emptyset \} : U \in \mathcal{U} \) coincide.

Thus the inequality \((u < g)\) implies the Filter Dichotomy [4] 9.16], which is the abbreviation of the assertion of Theorem 2 for bi-Baire ﬁlters.
For arbitrary bi-Baire filters $F$ and $U$ there exists an increasing number sequence $(k_n)_{n \in \omega}$ such that the sets \(\{n \in \omega : F \cap [k_n, k_{n+1}) \neq \emptyset\}\) and \(\{n \in \omega : U \cap [k_n, k_{n+1}) \neq \emptyset\}\) coincide.

The main idea of the semifilter approach to selection principles is to assign to a topological space $X$ the family \(\{\uparrow \mu_u(X) : u \in \Lambda(X)\}\). As it was shown in \[23\], the property \(\bigcup \text{fin}(\mathcal{O}, \mathcal{T})\) of a space $X$ may be characterized in terms of topological properties of elements of the above family.

**Theorem 4.** (\[23, Th. 3\]) Let $X$ be a Lindelöf topological space. Then $X$ has the property \(\bigcup \text{fin}(\mathcal{O}, \mathcal{T})\) if and only if for every $u \in \Lambda(X)$ so does the semifilter $\uparrow \mu_u(X)$.

And finally, we define some properties of semifilters closely related to \(\bigcup \text{fin}(\mathcal{O}, \mathcal{T}^*\ast)\) and \(\bigcup \text{fin}(\mathcal{O}, \mathcal{T}^\ast)\). We say that a family $B \subset F$ is a base of a semifilter $F$, if $F = \uparrow B$.

**Definition 1.** A filter $F$ on $\omega$ is defined to be a simple $P$-filter, if there exists a linearly preordered with respect to $\subset^*$ base of $F$.

The subsequent observation explains the importance of simple $P$-filters in studying of properties \(\bigcup \text{fin}(\mathcal{O}, \mathcal{T}^*\ast)\) and \(\bigcup \text{fin}(\mathcal{O}, \mathcal{T}^\ast)\).

**Observation 1.** A family $u = \{U_n : n \in \omega\}$ of subsets of $X$ is a $\tau^*$- (resp. $\tau^\ast$-) cover of $X$ if and only if $\mu_u(X)$ can be enlarged to (resp. generates) a simple $P$-filter.

We shall also use the subsequent characterization of simple $P$-filters.

**Theorem 5.** (\[1, 3.2.3\]) A filter $F$ is a simple $P$-filter if and only if $F$ has a base $B = (B_\alpha)_{\alpha < \chi(F)}$ such that $B_\alpha \subset^* B_\beta$ for all $\beta \leq \alpha < \chi(F)$.

Next, we shall search for conditions when there are nonmeager simple $P$-filter, or conditions which imply that all of them are meager.

**Proposition 1.** If $\text{Depth}^+([\omega]^{\aleph_0}) \leq b$, then each simple $P$-filter is meager.

**Proof.** Easily follows from Theorem 5, the definition of the cardinal $\text{Depth}^+([\omega]^{\aleph_0})$, and the fact that each semifilter with character $< b$ is meager, see \[23, 8.3.1\] or \[17\].

**Proposition 2.** There exists a nonmeager simple $P$-filter provided $b < d$ or $t = b$.

**Proof.** Follows immediately from \[1, 8.3.2, 11.2.3\].

The following simple characterization of the property $\bigcup \text{fin}(\mathcal{O}, \Gamma)$ is of crucial importance for the proof of Theorem 4.3. Let $u$ be a cover of a set $X$. A subset $B$ of $X$ is $u$-bounded, if $B \subset \cup v$ for some finite $v \subset u$. 

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Proposition 3. A topological space $X$ has the property $\bigcup_{\infty}(O, \Gamma)$ if and only if for every sequence $(u_n)_{n \in \omega}$ of open covers of $X$ there exists a sequence $(v_n)_{n \in \omega}$ such that each $v_n$ is a finite subset of $u_n$ and a semifilter $\uparrow \mu_{\{u_n, n \in \omega\}}(X)$ is meager.

Proof. Only the “if” part needs a proof. Let $(u_n)_{n \in \omega}$ be a sequence of open covers of $X$. Without loss of generality, $u_{n+1}$ is a refinement of $u_n$ for all $n \in \omega$. Let $w = \{B_n : n \in \omega\}$ be such that each $B_n$ is $u_n$-bounded and $\uparrow \mu_w(X)$ is meager. Then there is an increasing number sequence $(k_n)_{n \in \omega}$ such that each element of $\uparrow \mu_w(X)$ meets all but finitely many half-intervals $[k_n, k_{n+1})$. Since $u_{n+1}$ is a refinement of $u_n$ for all $n \in \omega$, the union $C_n = \bigcup_{k \in [k_n, k_{n+1})} B_k$ is $u_n$-bounded. We claim that $\{C_n : n \in \omega\}$ is a $\gamma$-cover of $X$. Indeed, given any $x \in X$ find $n_0 \in \omega$ such that $\mu_w(x) \cap [k_n, k_{n+1}) \neq \emptyset$ for all $n \geq n_0$. The above means that for every $n \geq n_0$ we can find $k_x(n) \in [k_n, k_{n+1})$ with the property $x \in B_{k_x(n)}$, and hence $x \in B_{k_x(n)} \subset \bigcup_{k \in [k_n, k_{n+1})} B_k = C_n$ for all $n \geq n_0$.

In the proofs of Theorem 1 we shall use some properties of the eventual dominance relation $\leq^*$ on $\mathbb{N}$ defined as follows: $x \leq^* y$ whenever the set $\{n \in \omega : x_n > y_n\}$ is finite. A subset $A$ of $\mathbb{N}$ is said to be

- bounded, if there exists $x \in \mathbb{N}$ such that $a \leq^* x$ for every $a \in A$;
- dominating, if for every $x \in \mathbb{N}$ there exists $a \in A$ such that $x \leq^* a$;
- a scale, if there exists an ordinal $\alpha$ and a bijection $\varphi : \alpha \to A$ such that $\varphi(\beta) \leq^* \varphi(\eta)$ for all $\beta < \eta$. In case $\alpha = b$ the set $A$ is said to be a $b$-scale.

Proof of Theorem 1. Let $X$ be a topological space and $(u_n)_{n \in \omega}$ be a sequence of open covers of $X$ such that $u_{n+1}$ is an refinement of $u_n$ for all $n \in \omega$.

1. As it was mentioned in Introduction, $(u < \varpi)$ implies $(b < \varpi)$, and therefore there exists a nonmeager simple $P$-filter $F$ by Proposition 2. By the definition of the property $\bigcup_{\infty}(O, \Omega)$ there exists and a large cover $w_0 = \{B_n : n \in \omega\}$ of $X$ such that each $B_n$ is $u_n$-bounded, see [15]. Applying Theorem 3 we conclude that the semifilter $U = \uparrow \mu_{w_0}(X)$ has the property $\bigcup_{\infty}(O, \Omega)$, and consequently it is not comeager by [23] Prop. 2. Two cases are possible.

a) $U$ is bi-Baire. Then Theorem 8 supplies us with an increasing sequence $(k_n)_{n \in \omega}$ such that $G := \phi(U) = \phi(F)$, where $\phi : \omega \to \omega$ is such that $\phi^{-1}(n) = [k_n, k_{n+1})$ for all $n \in \omega$, and $\phi(A) = \{\phi(A) : A \in A\}$ for any family $A$ of subsets of $\omega$. Note that $G$ is a simple $P$-filter being an image of $F$ under $\phi$.

Let $C_n = \bigcup_{k \in [k_n, k_{n+1})} B_k$. By our choice of $(u_n)_{n \in \omega}$, each $C_n$ is $u_n$-bounded. We claim that $w_2 = \{C_n : n \in \omega\}$ is a $\tau^*$-cover of $X$. Indeed, since $G = \phi(U)$, $U$ is generated by $\mu_{w_1}(X)$, and $\mu_{w_2}(x) = \phi(\mu_{w_1}(x))$ for all $x \in X$, we conclude that $G$ is generated by $\mu_{w_2}(X)$. Now it suffices to apply Observation 1.

b) $\uparrow \mu_{w_1}(X)$ is meager. Then in the same way as in the proof of Proposition 3 we can construct a $\gamma$-cover $\{C_n : n \in \omega\}$ of $X$ such that each $C_n$ is $u_n$-bounded.
2. In this case it suffices to find an \( \omega \)-cover \( w_1 = \{ B_n : n \in \omega \} \) of \( X \) such that each \( B_n \) is \( u_n \)-bounded and apply to the filter \( \uparrow \mu_{w_1}(X) \) the same arguments as in the proof of the first item.

3. Let us assume that each simple \( P \)-filter is meager and \( X \) has the property \( \bigcup_{n \in \omega} (O, T^*) \). Then there exists a \( \tau^* \)-cover \( w = \{ B_n : n \in \omega \} \) of \( X \) such that each \( B_n \) is \( u_n \)-bounded. By Observation \[1\] this implies that the semifilter \( U := \uparrow \mu_{w}(X) \) can be enlarged to a simple \( P \)-filter \( F \), which is meager by our assumption, and hence so is \( U \). Applying Proposition \[3\] we conclude that \( X \) has the property \( \bigcup_{n \in \omega} (O, \Gamma) \).

Next, suppose that there exists a nonmeager simple \( P \)-filter \( F \). The rest of the proof falls naturally into two parts.

a) \( (b = 0) \). In this case the assertion follows from [21, 8.10], which supplies us with a subspace \( Y \) of the Baire space with the following properties:

(i) \( Y \) does not have the property \( \bigcup_{n \in \omega} (O, T) \);
(ii) for any sequence \( (w_n)_{n \in \omega} \) of open covers of \( Y \) there exists a family \( w = \{ B_n : n \in \omega \} \) such that each \( B_n \) is \( w_n \)-bounded and \( \uparrow \mu_{w}(X) \subset F \).

b) \( (b < 0) \). In this case the assertion follows from the subsequent two statements.

(i) There exists a subspace of the Baire space of size \( b \) which does not have the property \( \bigcup_{n \in \omega} (O, \Gamma) \).
(ii) \( (b < 0) \) implies that every subspace \( Y \) of the Baire space satisfies \( \bigcup_{n \in \omega} (O, T^*) \) provided \( |Y| \leq b \).

The first of them may be found in [13]. To prove the second one, find a (probably not bijective) enumeration \( \{ y_\alpha : \alpha < b \} \) of \( Y \). Recall from [19] that a subset \( Z \subset \omega^\omega \) has a weak excluded middle property if there exists \( x \in \omega^\omega \) such that the family \( \{ [z \leq x] : z \in Z \} \) can be enlarged to a simple \( P \)-filter, where for a relation \( R \) on \( \omega \) \( [z R x] = \{ n \in \omega : z(n) R x(n) \} \).

Let \( f : Y \to \omega^\omega \) be continuous. By transfinite induction over \( b \) construct a \( b \)-scale \( B = \{ b_\alpha : \alpha < b \} \) such that \( f(y_\alpha), b_\beta \leq^* b_\alpha \) for all \( \beta \leq \alpha < b \). Since \( b < 0 \), \( B \) is not dominating, which means that there exists \( c \in \omega^\omega \) such that \( c \leq^* b_\alpha \) for no \( \alpha < b \), and hence \( [b_\alpha < c] \) is infinite for all \( \alpha \). Observe that for arbitrary \( \beta \leq \alpha < b \) the equation \( b_\beta \leq^* b_\alpha \) implies \( [b_\alpha < c] \subseteq [b_\beta < c] \), and therefore \( \mathcal{T} = ([b_\alpha < c])_{\alpha < b} \) is a tower. Moreover, \( [b_\alpha < c] \subseteq^* [f(y_\alpha) \leq c] \), consequently the family \( \{ [f(y_\alpha) \leq c] : \alpha < b \} = \{ [f(y) \leq c] : y \in Y \} \) is a subset of the simple \( P \)-filter generated by \( \mathcal{T} \), and hence \( f(Y) \) has a weak excluded middle property. Applying [19, Th. 7.8] asserting that a subset \( Z \) of the Baire space satisfies \( \bigcup_{n \in \omega} (O, T^*) \) provided for every continuous \( \phi : Z \to \omega^\omega \) the image \( \phi(Z) \) has the weak excluded middle property, we conclude that \( Y \) has the property \( \bigcup_{n \in \omega} (O, T^*) \). \( \square \)
Proof of Corollary 1

1. Follows immediately from Proposition 1 and Theorem 1(3).

2. Under \((b < d)\) the assertion follows from Proposition 2 and Theorem 1(3). Under \((t = d)\) it suffices to use \((t = b)\)--part of Proposition 2 to find a non-meager simple \(P\)-filter and then apply the same arguments as in the proof of the \((b = d)\)--part of Theorem 1(3).

\(\square\)

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References

[1] Banakh T., Zdomsky L., Coherence of semifilters, http://www.franko.lviv.ua/faculty/mechmat/Departments/Topology/booksite.html

[2] Banakh T., Zdomsky L., Selection principles and infinite games on multicovered spaces and their applications, in preparation.

[3] Bartoszyński T., Shelah S., Tsaban B., Additivity properties of topological diagonalizations, J. Symbolic Logic 68 (2003), 1254–1260. (Full version: http://arxiv.org/abs/math.LO/0112262)

[4] Blass A., Combinatorial cardinal characteristics of the continuum, in Handbook of Set Theory (M. Foreman et. al., Eds.), to appear.

[5] Chaber J., Pol R., A remark on Fremlin-Miller theorem concerning the Menger property and Michael concentrated sets, preprint.

[6] Dordal P., A model in which the base-matrix tree cannot have cofinal branches, J. Symbolic Logic 52 (1987), 651–664.

[7] Dow A., Set theory in topology, in Recent Progress in General Topology (M. Husek et. al., Eds.), Elsevier Sci. Publ., Amsterdam, 1992, pp. 168–197.

[8] Miller A., Fremlin D., On some properties of Hurewicz, Menger, and Rothberger, Fund. Math. 129 (1988), 17–33.

[9] Gerlits J., Nagy Zs., Some properties of \(C(X)\), I, Topology Appl. 14 (2) (1982), 151–163.

[10] Hurewicz W., Über die Verallgemeinerung des Borelschen Theorems, Mathematische Zeitschrift 24 (1925) 401-421.

[11] Just W., Miller A., Scheepers M., Szeptycki S., The combinatorics of open covers II, Topology Appl. 73 (1996), 241–266.

[12] Laflamme C., Equivalence of families of functions on natural numbers, Trans. Amer. Math. Soc. 330 (1992), 307–319.

[13] Marczewski E. (Szpilrajn), The characteristic function of a sequence of sets and some of its applications, Fund. Math. 31 (1938), 207–233.

[14] Menger K., Einige Überdeckungssätze der Punktmengelehre, Sitzungsberichte. Abt. 2a, Mathemetic, Astronomie, Physic, Meteorologie und Mechanic (Wiener Akademie) 133 (1924) 421-444.

[15] Scheepers M., Combinatorics of open covers I: Ramsey Theory, Topology Appl. 69 (1996), 31–62.
[16] Shelah S., Tsaban B., *Critical cardinalities and additivity properties of combinatorial notions of smallness*, J. Appl. Anal. 9 (2003), 149–162.  
http://arxiv.org/abs/math.LO/0304019

[17] Solomon R., *Families of sets and functions*, Czechoslovak Math. J. 27 (1977), 556–559.

[18] Talagrand M., *Filtres: Mesurabilité, rapidité, propriété de Baire forte*, Studia Math. 74 (1982), 283–291.

[19] Tsaban B., *Selection principles and the minimal tower problem*, Note Math., to appear.  
http://arxiv.org/abs/math.LO/0105045

[20] Tsaban B., (eds.), *SPM Bulletin* 3 (2003)  
http://arxiv.org/abs/math.GN/0303057

[21] Tsaban B., Zdomsky L., *Scales, Fields, and a problem of Hurewicz*, submitted to J. Amer. Math. Soc..  
http://arxiv.org/abs/math.GN/0507043.

[22] Vaughan J., *Small uncountable cardinals and topology*, in Open problems in topology (J. van Mill, G.M. Reed, Eds.), Elsevier Sci. Publ., Amsterdam, 1990, pp. 195-218.

[23] Zdomsky L., *A semifilter approach to selection principles*, to appear in Comment. Math. Univ. Carolinae.  
http://arxiv.org/abs/math.GN/0412498

Department of Mechanics and Mathematics,  
Ivan Franko Lviv National University,  
Universytetska 1, Lviv, 79000, Ukraine.

*E-mail address: lzdonsky@rambler.ru*