On convergence of infinite matrix products

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Abstract
A necessary and sufficient condition for the convergence of an infinite right product of matrices of the form

\[ A := \begin{bmatrix} I & B \\ 0 & C \end{bmatrix}, \]

with (uniformly) contracting submatrices \( C \), is proven.

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1 Introduction
Consider the set of all matrices in \( \mathbb{C}^{d \times d} \) of the form

\[ A := \begin{bmatrix} I_s & B \\ 0 & C \end{bmatrix}, \tag{1} \]

where \( I_s \) denotes the identity matrix of order \( s < d \).

Matrices (1) are known (e.g., [1]) to form an LCP set whenever the submatrices \( B \) are uniformly bounded and the submatrices \( C \) are uniformly contracting, that is, satisfy the condition \( \| C \| \leq r \) for some fixed matrix (i.e., submultiplicative) norm \( \| \cdot \| \) on \( \mathbb{C}^{(d-s) \times (d-s)} \) and some constant \( r < 1 \). To recall, a set \( \Sigma \) has the LCP (RCP) property if all left (right) infinite products formed from matrices in \( \Sigma \) are convergent.

Matrices of the form (1), with uniformly bounded submatrices \( B \) and uniformly contracting submatrices \( C \), do not necessarily form an RCP set. (They do form such a set if and only if they satisfy a very stringent condition given in Corollary 2.3 below.) However, there exists a simple criterion that can be used to check whether a particular right infinite product formed from such matrices converges.

2 A convergence test

Theorem 2.1 Let \( (A_n)_{n \in \mathbb{N}} \) be a sequence of matrices of the form (1) and let

\[ \| C_n \| \leq r < 1 \quad \text{for all} \quad n \in \mathbb{N} \]

for some matrix norm \( \| \cdot \| \). The sequence \( (P_n := A_1A_2 \cdots A_n) \) converges if and only if so does the sequence \( (B_n(I - C_n)^{-1}) \). In this event,

\[ \lim_{n \to \infty} P_n = \begin{bmatrix} I & \lim_{n \to \infty} B_n(I - C_n)^{-1} \\ 0 & 0 \end{bmatrix}. \]

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Proof. To prove the necessity, partition \( P_n \) conformably with \( A_n \). Then

\[
P_n = \begin{bmatrix} I & X_n \\ 0 & C_1C_2 \cdots C_n \end{bmatrix}
\]

where \( X_n := \sum_{i=0}^{n} B_{n-i}(C_{n+1-i}C_{n+2-i} \cdots C_n) \).

If \( (P_n) \) converges, then \( \lim_{n \to \infty} (X_n - X_{n-1}) = 0 \). Also, \( \|(I - C_n)^{-1}\| \leq 1/(1 - r) \) for all \( n \in \mathbb{N} \). But \( X_n = B_n + X_{n-1}C_n \), so

\[
B_n(I - C_n)^{-1} - X_{n-1} = (X_n - X_{n-1})(I - C_n)^{-1} \xrightarrow{n \to \infty} 0.
\]

Hence

\[
\lim_{n \to \infty} B_n(I - C_n)^{-1} = \lim_{n \to \infty} X_n.
\]

Now prove the sufficiency. Without loss of generality one can assume that \( s = d - s \). Indeed, simply replace each \( A_n \) by

\[
\widetilde{A}_n := \begin{bmatrix} I_{\max\{s,d-s\}} & \widetilde{B}_n \\ 0 & \widetilde{C}_n \end{bmatrix}
\]

where

\[
\widetilde{B}_n := \begin{cases} B_n & \text{if } s \geq d - s \\ 0_{s \times (2s-d)} & \text{if } s < d - s
\end{cases}, \quad \widetilde{C}_n := \begin{cases} C_n & \text{if } s \geq d - s \\ 0_{(2s-d) \times (d-s)} & \text{if } s < d - s
\end{cases}
\]

Then the matrices \( \widetilde{A}_n \) satisfy all the assumptions of the theorem and the sequence \( (B_n(I - C_n)^{-1}) \) (the product \( P_n \)) converges iff so does the sequence \( (\widetilde{B}_n(I - \widetilde{C}_n)^{-1}) \) (the product \( \widetilde{P}_n \)).

Thus, assume that \( s = d - s \). Note that if the sequence \( (B_n(I - C_n)^{-1}) \) converges, then the sequence \( (B_n) \) is bounded, since \( ||I - C_n|| \leq 1 + r \) for all \( n \). Now, let

\[
D_n := X_n - B_n(I - C_n)^{-1}, \quad Y_n := B_{n+1}(I - C_{n+1})^{-1} - B_n(I - C_n)^{-1}
\]

for all \( n \in \mathbb{N} \). Then

\[
D_{n+1} = (D_n - Y_n)C_{n+1}, \tag{2}
\]

hence

\[
\|D_{n+1}\| \leq (\|D_n\| + \|Y_n\|)||C_{n+1}|| \leq (\|D_n\| + \|Y_n\|)r.
\]

Repeated use of this inequality gives

\[
\|D_n\| \leq \sum_{i=1}^{n-1} \|Y_{n-i}\| r^i.
\]

This implies, in particular, that

\[
S := \lim_{n \to \infty} \sup \|D_n\| < \infty.
\]

Since \( \lim_{n \to \infty} Y_n = 0 \), the identity (2) and the upper bound on \( \|C_n\| \) imply that \( S \leq rS \), therefore \( S = 0 \), that is,

\[
\lim_{n \to \infty} D_n = 0.
\]

The obtained criterion of convergence can be used to make two more observations in the same spirit.
Corollary 2.2 Let \((A_n)_{n \in \mathbb{N}}\) be a sequence of matrices of the form (1) such that the sequence \((C_n)\) converges to a matrix \(C\) with spectral radius smaller than 1. Then the sequence \((P_n := A_1 A_2 \cdots A_n)\) converges if and only if so does the sequence \((B_n)\). In this event,

\[
\lim_{n \to \infty} P_n = \begin{bmatrix} I & \lim_{n \to \infty} B_n (I - C)^{-1} \\ 0 & 0 \end{bmatrix}.
\]

Proof. If \(\sigma(C) < 1\), then there exists a matrix norm \(\| \cdot \|\) on \(\mathbb{C}^{(d-s) \times (d-s)}\) such that \(\|C\| < 1\) (e.g., [p.297, Lemma 5.6.10] [2]). So, \(\|C_n\| \leq r\) for all \(n \geq N\) for some \(r < 1\) and some \(N \in \mathbb{N}\), so the assumption of the theorem is then satisfied. The product \(P_n\) converges whenever so does the product \(A_N A_{N+1} \cdots\), so \((P_n)\) has a limit whenever \((B_n)\) has one. By the same reason, the sequence \((I - C_n)^{-1}\) is bounded, so the necessity argument from the proof of the Theorem shows that the convergence of \((B_n)\) is also necessary. \(\square\)

Corollary 2.3 A set \(\Sigma\) consisting of matrices of the form (1) with uniformly contracting submatrices \(C\), is an RCP set if and only if

\[
B_1 (I - C_1)^{-1} = B_2 (I - C_2)^{-1} \quad \text{for all } A_1, A_2 \in \Sigma,
\]

where

\[
A_i = \begin{bmatrix} I & B_i \\ 0 & C_i \end{bmatrix}, \quad i = 1, 2.
\]

Proof. Given \(A_1, A_2 \in \Sigma\), apply Theorem 2.1 to the product \(A_1 A_2 A_1 A_2 \cdots\) to see that the condition (11) is necessary and sufficient for the convergence of such a product. But if it is satisfied for all pairs of matrices from \(\Sigma\), then it is sufficient for the convergence of any right product of matrices from \(\Sigma\). \(\square\)

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References

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