SHARP NORM ESTIMATE OF SCHWARZIAN DERIVATIVE FOR A CLASS OF CONVEX FUNCTIONS

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ABSTRACT. We establish a sharp norm estimate of the Schwarzian derivative for a function in the classes of convex functions introduced by Ma and Minda [Proceedings of the Conference on Complex Analysis, International Press Inc., 1992, 157-169]. As applications, we give sharp norm estimates for strongly convex functions of order $0 < \alpha < 1$, and for uniformly convex functions.

1. BACKGROUND AND MAIN RESULT

Let $\mathcal{A}$ be the class of analytic functions $f$ on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ satisfying the normalization conditions $f(0) = 0$ and $f'(0) = 1$ and let $\mathcal{S}$ be the class of univalent functions in $\mathcal{A}$. The Schwarzian derivative

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2} \left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'}\right)^2$$

and its norm (the hyperbolic sup-norm)

$$\|S_f\| = \sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |S_f(z)|$$

play an important role in the theory of Teichmüller spaces. Key results concerning the Schwarzian derivative are summarized in the following theorem.

**Theorem 1.1** (Nehari, Kühnau, Ahlfors-Weill). Let $f \in \mathcal{A}$. If $f$ is univalent, then $\|S_f\| \leq 6$. Conversely, if $\|S_f\| \leq 2$, then $f$ is univalent. Moreover, let $0 \leq k < 1$. If $f$ extends to a $k$-quasiconformal mapping of the Riemann sphere $\hat{\mathbb{C}}$ then $\|S_f\| \leq 6k$. Conversely, if $\|S_f\| \leq 2k$, then $f$ extends to a $k$-quasiconformal mapping of $\hat{\mathbb{C}}$.

Here, a mapping $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called $k$-quasiconformal if $f$ is a sense-preserving homeomorphism of $\hat{\mathbb{C}}$ and has locally integrable partial derivatives on $\mathbb{C} \setminus \{f^{-1}(\infty)\}$ with $|f| \leq k |f|$ a.e. The best reference to the above theorem is Lehto’s book [1].

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The universal Teichmüller space $T$ can be identified with the set of Schwarzian derivatives of univalent analytic functions on $D$ with quasiconformal extensions to $\hat{C}$. It is known that $T$ is a bounded domain in the Banach space of analytic functions on $D$ with finite hyperbolic sup norm (see [L2]).

In connection with Teichmüller spaces, it is an interesting problem to consider estimation of the norm of the Schwarzian derivatives of typical subclasses of univalent functions. A function $f \in \mathcal{A}$ is called starlike (resp. convex) if $f$ is univalent and the image $f(D)$ is starlike with respect to the origin (resp. convex). The classes of starlike and convex functions are denoted by $S^*$ and $K$, respectively. It is well known that $f \in \mathcal{A}$ is starlike (resp. convex) if and only if $\text{Re} \left[ \frac{zf'(z)}{f(z)} \right] > 0$ (resp. $\text{Re} \left[ 1 + \frac{zf''(z)}{f'(z)} \right] > 0$). These notions have been refined and generalized in many ways (see [G1]). In the present note, we are mainly concerned with strongly starlike and convex functions. A function $f \in \mathcal{A}$ is called strongly starlike (resp. strongly convex) of order $\alpha$ ($0 < \alpha < 1$) if $|\text{arg} \left[ \frac{zf'(z)}{f(z)} \right]| < \pi \alpha / 2$ (resp. $|\text{arg} \left[ 1 + \frac{zf''(z)}{f'(z)} \right]| < \pi \alpha / 2$) in $|z| < 1$. The classes of strongly starlike and convex functions of order $\alpha$ will be denoted by $S^*_\alpha$ and $K_\alpha$, respectively. See [Su2] for geometric characterizations of functions in $S^*_\alpha$.

Define $\gamma(\beta)$ for $0 < \beta < 1$ by

$$\gamma(\beta) = \frac{\pi}{2} \arctan \left[ \tan \left( \frac{\pi \beta}{2} \right) + \frac{\beta}{(1 + \beta) \sin(\pi / 2) (1 - \beta) \cos(\pi / 2)} \right].$$

Note that $\gamma(\beta)$ increases from 0 to 1 when $\beta$ moves from 0 to 1. Mocanu [Mo] found the following relation.

**Theorem 1.2** (Mocanu). $K_{\gamma(\beta)} \subset S^*_\beta$ for $0 < \beta < 1$.

In other words, $K_{\alpha} \subset S^*_{\gamma^{-1}(\alpha)}$ for $0 < \alpha < 1$, where $\gamma^{-1}$ denotes the inverse function of $\gamma : [0, 1] \to [0, 1]$. For sharp or improved relations of this kind, see a paper [KS2] of the authors.

We summarize important properties of strongly starlike functions as follows.

**Theorem 1.3.** A strongly starlike function $f$ of order $\alpha \in (0, 1)$ extends to a $\sin(\pi \alpha / 2)$-quasiconformal mapping of $\hat{C}$ and therefore $\|S_f\| \leq 6 \sin(\pi \alpha / 2)$.

The first part is due to Fait, Krzyż and Zygmunt [FKZ] and the second one is obtained from the first in combination with Theorem 1.1 (as was pointed out by Chiang [Ch]).

By Theorems 1.2 and 1.3 we see that a function $f \in K_{\alpha}$ extends to a $\sin(\pi \gamma^{-1}(\alpha) / 2)$-quasiconformal mapping of $\hat{C}$ and satisfies $\|S_f\| \leq$
6 \sin(\pi \gamma^{-1}(\alpha)/2). \) On the other hand, we have the following norm estimate for convex functions.

**Theorem 1.4.** A convex function \( f \) satisfies \( \|S_f\| \leq 2 \). The bound is sharp.

This result was repeatedly proved in the literature (see [Rob], [N2], [L1]), and was refined by Suita [NS] in the following form: the sharp inequality \( \|S_f\| \leq 8\alpha(1 - \alpha) \) holds for a function \( f \in \mathcal{A} \) with \( \text{Re} \left[ 1 + z f''(z)/f'(z) \right] > \alpha \) and \( 1/2 \leq \alpha < 1 \).

Obviously, the estimate \( \|S_f\| \leq 6 \sin(\pi \gamma^{-1}(\alpha)/2) \) for \( f \in \mathcal{K}_\alpha \) is not better than Theorem 1.4 when \( \alpha \) is close to 1. We will give a sharp norm estimate for \( f \in \mathcal{K}_\alpha \).

**Main Theorem 1.5.** Let \( f \) be a strongly convex function of order \( \alpha \) for \( 0 < \alpha < 1 \). Then the sharp inequality \( \|S_f\| \leq 2\alpha \) holds.

Define a function \( f_\alpha \in \mathcal{K}_\alpha \) by the relation

\[
1 + \frac{zf''_\alpha(z)}{f'_\alpha(z)} = \left( \frac{1 + z^2}{1 - z^2} \right)^\alpha.
\]

Then a simple computation gives

\[
f_\alpha(z) = z + \alpha z^3/3 + \alpha^2 z^5/5 + \alpha(1 + 8\alpha^2)z^7/63 + \cdots
\]

and thus \( S_{f_\alpha}(0) = 2\alpha \). Therefore, we see that this satisfies \( \|S_{f_\alpha}\| = 2\alpha \).

Combining Theorem 1.5 with the Ahlfors-Weill theorem (Theorem 1.1), we obtain the following result.

**Corollary 1.6.** A function \( f \in \mathcal{K}_\alpha \) extends to an \( \alpha \)-quasiconformal mapping of \( \hat{\mathbb{C}} \) for \( 0 < \alpha < 1 \).

By using Mathematica Ver. 7, we found that \( \sin(\pi \gamma^{-1}(\alpha)/2) < \alpha \) when \( 0 < \alpha < 0.3354 \) (see Figure 1). Therefore, the corollary gives a better bound only when \( \alpha > 0.3355 \), though it has obvious merit of simplicity.

For some reason, the second author was even led in [Su1] to the expectation that each function in \( \mathcal{K}_\alpha^* \) might extend to an \( \alpha \)-quasiconformal mapping of \( \hat{\mathbb{C}} \). This was recently disproved by Yuliang Shen [S] for every \( 0 < \alpha < 1 \).

Goodman [G2] introduced the class \( \mathcal{UCV} \) of uniformly convex functions. Here, a function \( f \in \mathcal{A} \) is called *uniformly convex* if every (positively oriented) circular arc of the form \( \{z \in \mathbb{D} : |z - \zeta| = r\}, \zeta \in \mathbb{D}, 0 < r < |\zeta| + 1 \), is mapped by \( f \) univalently onto a convex arc. In particular, \( \mathcal{UCV} \subset \mathcal{K} \). See also [KW], [KS1] and [K] for \( k \)-uniform convexity (\( 0 \leq k < +\infty \)), a more refined notion of convexity, and related results. We have the following sharp norm estimate for \( \mathcal{UCV} \).
Main Theorem 1.7. Let \( f \) be a uniformly convex function. Then the sharp inequality \(|S_f| \leq \frac{8}{\pi^2}\) holds. In particular, \( f \) extends to a \( 4/\pi^2 \)-quasiconformal mapping of \( \hat{\mathbb{C}} \).

In \([K]\) the first author observed that a uniformly convex function extends to a \( k_1 \)-quasiconformal mapping of \( \hat{\mathbb{C}} \), where \( k_1 = \sin(\pi \gamma^{-1}(1/2)/2) \approx 0.52311 \). Therefore, the above bound \( 4/\pi^2 \approx 0.40528 \) is slightly better. (Note also that a numerical computation of \( K(1) = (1 + k_1)/(1 - k_1) \approx 3.19387 \) was not correct in \([K]\).)

In Section 2, we provide a principle leading to a sharp norm estimate of the Schwarzian derivative for a subclass of \( \mathcal{K} \) given in a specific way. By making use of it, we prove Theorem 1.5 in Section 3 and Theorem 1.7 in Section 4.

2. General norm estimate for convex functions

Ma and Minda \([MM1]\) introduced a unifying way of treatment of various subclasses of \( \mathcal{K} \). Let \( \varphi \) be an analytic function on \( \mathbb{D} \) with \( \varphi(0) = 1 \). The class \( \mathcal{K}(\varphi) \) is defined to be the set of functions \( f \in \mathcal{A} \) with \( 1 + zf''(z)/f'(z) \prec \varphi(z) \). Here, an analytic function \( g \) on \( \mathbb{D} \) is said to be subordinate to another \( h \) and denoted by \( g \prec h \) or \( g(z) \prec h(z) \) if \( g = h \circ \omega \) for an analytic function \( \omega \) on \( \mathbb{D} \) with \( \omega(0) = 0 \) and \( |\omega| < 1 \). When \( h \) is univalent, it is useful to note the following fact: \( g \prec h \) if and only if \( g(0) = h(0) \) and \( g(\mathbb{D}) \subset h(\mathbb{D}) \).

Let

\[
P_\alpha(z) = \left( \frac{1 + z}{1 - z} \right)^\alpha
\]
for a constant $\alpha > 0$. Then $P_\alpha$ maps $\mathbb{D}$ univalently onto the sector $|\arg w| < \pi \alpha/2$ for $0 < \alpha \leq 1$. Therefore, $\mathcal{K}(P_\alpha) = \mathcal{K}_\alpha$ for $0 < \alpha < 1$ and $\mathcal{K}(P_1) = \mathcal{K}$.

Ma and Minda [MM2] and Rønning [Ron] found the following characterization of the class $\mathcal{UCV}$. A function $f \in \mathcal{A}$ is uniformly convex if and only if $\Re \left[ 1 + z f''(z)/f'(z) \right] > \left| z f''(z)/f'(z) \right|$, $z \in \mathbb{D}$. Noting the fact that the function

\[(2.1) \quad P(z) = 1 + \frac{2}{\pi^2} \left( \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2\]

maps $\mathbb{D}$ univalently onto the domain $\{w : \Re w > |w - 1|\}$, we have $\mathcal{UCV} = \mathcal{K}(P)$ (see [Ron, p. 191]).

We first formulate the sharp norm estimate for the class $\mathcal{K}(\varphi)$. To this end, we consider the quantity

\[F(s,t) = \frac{(1 - t^2)^2}{2t^2} A(s) + (1 - t^2) \left( 1 - \frac{s^2}{t^2} \right) B(s),\]

where

\[A(s) = \sup_{|z|=s} |2z\varphi'(z) + 1 - \varphi(z)^2|, \quad \text{and} \quad B(s) = \sup_{|z|=s} |\varphi'(z)|.\]

Define

\[N(\varphi) = \sup_{0<s<t<1} F(s,t).\]

Then we have the following.

**Main Theorem 2.1.** Let $\varphi$ be an analytic function on the unit disk with $\varphi(0) = 1$. Then, the sharp inequality $\|S_f\| \leq N(\varphi)$ holds for $f \in \mathcal{K}(\varphi)$.

Note here that $\varphi$ is not required to satisfy $\Re \varphi > 0$ in the theorem, though there is no guarantee for finiteness of $N(\varphi)$ in this general case.

**Proof.** Denote by $\mathcal{W}$ the set of analytic functions $\omega$ on $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega| < 1$.

Let $f \in \mathcal{K}(\varphi)$. Since $1 + zf''(z)/f'(z) \prec \varphi(z)$ by definition, we have $f''(z)/f'(z) = (\varphi(\omega(z)) - 1)/z$ for an $\omega \in \mathcal{W}$. Set $w = \omega(z)$ for a fixed $z \in \mathbb{D}$. Then the Schwarzian derivative $S_f$ of $f$ can be expressed by

\[S_f(z) = \frac{\varphi'(w)\omega'(z)}{z} - \frac{\varphi(w)^2 - 1}{2z^2}.\]

We now recall Dieudonné’s lemma (cf. [D]): for a fixed pair of points $z, w \in \mathbb{D}$ with $|w| \leq |z|$, one has

\[\{\omega'(z) : \omega \in \mathcal{W}, \omega(z) = w\} = \left\{ v \in \mathbb{C} : \left| v - \frac{w}{z} \right| \leq \frac{|z|^2 - |w|^2}{|z|(1 - |z|^2)} \right\}.\]
This means that \( \zeta = \omega'(z) - w/z \) varies over the closed disk
\[
|\zeta| \leq (t^2 - |w|^2)/t(1 - t^2)
\]
for fixed \(|z| = t < 1\). Then we can write
\[
S_f(z) = \frac{\varphi'(w)}{z} \left( \zeta + \frac{w}{z} \right) - \frac{\varphi(w)^2 - 1}{2z^2} + \frac{\varphi'(w)}{z}.
\]
Therefore, for \(|z| = t < 1\), we have the sharp inequality
\[
|S_f(z)| \leq \left| \frac{2w\varphi'(w) + 1 - \varphi(w)^2}{2t^2} + \frac{t^2 - |w|^2}{t^2(1 - t^2)} |\varphi'(w)| \right|
\]
for \(f \in \mathcal{K}(\varphi)\). This fact can be expressed by
\[
\sup_{f \in \mathcal{K}(\varphi)} |S_f(z)| = \sup_{s \leq t} \left[ \frac{A(s)}{2t^2} + \frac{t^2 - s^2}{t^2(1 - t^2)} B(s) \right].
\]
Hence, we have
\[
\sup_{f \in \mathcal{K}(\varphi)} (1 - t^2)|S_f(z)| = \sup_{s < t} F(s, t)
\]
for a fixed point \(z \in \mathbb{D}\) with \(|z| = t\) and the required relation. \(\square\)

As we will see below, we often have the relations
\[
A(s) = 2s\varphi'(s) + 1 - \varphi(s)^2 \quad \text{and} \quad B(s) = \varphi'(s)
\]
for \(0 \leq s < 1\). Then, by a simple computation, we have the expression
\[
F(s, t) = \frac{(1 - t^2)^2}{2t^4} (1 - \varphi(s)^2) + \frac{(1 - t^2)(1 - s)(s + t^2)}{t^2} \varphi'(s).
\]
Observe that \(F(s, t)\) is described in terms of \(s\) and \(t^2\) in this case.

3. Proof of Theorem 1.5

We begin with the following properties of \(P_\alpha\).

Lemma 3.1. The functions \(P_\alpha(z)\) and \(Q_\alpha(z) = 2zP_\alpha'(z) + 1 - P_\alpha(z)^2\) have non-negative Taylor coefficients about \(z = 0\) for \(0 < \alpha \leq 1\).

Proof. Since
\[
\log \frac{1 + z}{1 - z} = 2 \sum_{n=1}^{\infty} \frac{z^{2n-1}}{2n - 1},
\]
the function \(P_\alpha(z) = \exp(\alpha \log \frac{1 + z}{1 - z})\) has positive Taylor coefficients. Note also, by this expression, that \(P_\alpha\) satisfies the differential equation
\[
\frac{P_\alpha'(z)}{P_\alpha(z)} = \frac{2\alpha}{1 - z^2},
\]
and therefore,

\[ P''_\alpha(z) = \frac{2(\alpha + z)}{1 - z^2} P'_\alpha(z), \]

We next observe the expansion

\[ P_\alpha(z) = 1 + \sum_{n=1}^{\infty} a_n z^n. \]

Since \( a_1 = 2\alpha \) and \( P_\alpha \) maps \( \mathbb{D} \) univalently onto a convex domain for \( 0 < \alpha \leq 1 \), by a theorem of Löwner (cf. [D]), we have

\[ 0 \leq a_n \leq 2\alpha \quad (n = 1, 2, 3, \ldots). \]

By using (3.1) and (3.2), we now have the expression

\[ Q'_\alpha(z) = 2zP''_\alpha(z) + 2P'_\alpha(z)(1 - P_\alpha(z)) = 2P'_\alpha(z) \left( 1 - P_\alpha(z) + \frac{2z(\alpha + z)}{1 - z^2} \right). \]

Since \( P'_\alpha(z) \) has positive Taylor coefficients and

\[ 1 - P_\alpha(z) + \frac{2z(\alpha + z)}{1 - z^2} = \sum_{n=1}^{\infty} (2\alpha - a_{2n-1}) z^{2n-1} + \sum_{n=1}^{\infty} (2 - a_{2n}) z^{2n}, \]

the required assertion for \( Q_\alpha \) is deduced from (3.3).

We are now ready to prove our main theorem.

**Proof of Theorem 1.5** In view of Theorem 2.1, we need to show that \( N(P_\alpha) = 2\alpha \).

By Lemma 3.1, we can apply (2.2) for \( \varphi = P_\alpha \):

\[
F(s, t) = \frac{(1 - t^2)^2}{2t^2} (1 - P_\alpha(s))^2 + \frac{(1 - t^2)(1 - s)(s + t^2)}{t^2} P'_\alpha(s) \\
= \frac{(1 - t^2)^2}{2t^2} (1 - P_\alpha(s))^2 + 2\alpha \frac{(1 - t^2)(s + t^2)}{t^2(1 + s)} P_\alpha(s).
\]

Here, we have used (3.1).

Since \( F(0, t) = 2\alpha(1 - t^2) \to 2\alpha \) as \( t \to 0 \), it is enough to show the inequality \( F(s, t) \leq 2\alpha \) when \( 0 < s < t < 1 \). Letting \( x = 1 - t^2 \), we see now that

\[ F(s, t) \leq 2\alpha \]

\[ \Leftrightarrow \quad (1 - P_\alpha(s))^2 x^2 + \frac{4\alpha x(1 + s - x)}{1 + s} P_\alpha(s) \leq 4\alpha(1 - x) \]

\[ \Leftrightarrow \quad \left( P_\alpha(s)^2 + \frac{4\alpha}{1 + s} P_\alpha(s) - 1 \right) x^2 - 4\alpha(1 + P_\alpha(s)) x + 4\alpha \geq 0. \]
The left-hand side in the last inequality can be regarded as a quadratic polynomial in $x$ of the form

$$Kx^2 - 4Mx + 4L = K \left( x - \frac{2M}{K} \right)^2 + \frac{4}{K}(KL - M^2)$$

with $K > 0$. We now compute $KL - M^2$ as follows:

$$h(s) := \alpha \left( P_\alpha(s)^2 + \frac{4\alpha}{1+s}P_\alpha(s) - 1 \right) - \alpha^2(1 + P_\alpha(s))^2
= \alpha \left( (1 - \alpha)P_\alpha(s)^2 + \frac{2\alpha(1-s)}{1+s}P_\alpha(s) - (1 + \alpha) \right).$$

Since

$$h'(s) = \frac{4\alpha^2(1-\alpha)}{(1+s)^2} P_{\alpha+1}(s) - 1 \cdot P_\alpha(s) > 0$$

for $s > 0$, the function $h(s)$ is monotone increasing in $0 < s < 1$. Thus $h(s) > h(0) = 0$ for $0 < s < 1$. Therefore, we have seen that $KL - M^2 \geq 0$, which implies $F(s,t) \leq 2\alpha$ as expected. 

Obviously, the above proof covers the case when $\alpha = 1$. Thus, we have obtained yet another proof of Theorem 1.4.

4. PROOF OF THEOREM 1.7

We will need the following estimate.

**Lemma 4.1.** For every non-negative integer $n$, the following inequality holds:

$$\sum_{0 \leq k,l,m \atop k+l+m=n} \frac{1}{(2k+1)(2l+1)(2m+1)} \leq 1.$$

**Proof.** We denote by $A_n$ the summation in question. Also, let

$$B_n = \sum_{0 \leq k,l \atop k+l=n} \frac{1}{(2k+1)(2l+1)}.$$

By the partial fraction decomposition, we observe

$$B_n = \frac{1}{2n+2} \sum_{k+l=n} \left( \frac{1}{2k+1} + \frac{1}{2l+1} \right)$$

$$= \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{2k+1}.$$

Since $1/(2k+1) \leq 1/3$ for $k \geq 1$, we have

$$(4.1) \quad B_n \leq \frac{1}{n+1} \left( 1 + \frac{n}{3} \right) \leq \frac{2}{3} \quad \text{for } n \geq 1.$$
Similarly, by the partial fraction decomposition, we have
\[
\frac{2n+3}{(2k+1)(2l+1)(2m+1)} = \frac{1}{(2k+1)(2l+1)} + \frac{1}{(2l+1)(2m+1)} + \frac{1}{(2m+1)(2k+1)}
\]
for \(k + l + m = n\). We now apply it and take into account the symmetry in \(k, l, m\) and \(n\) to obtain finally
\[
A_n = \frac{3}{2n+3} \sum_{j=0}^{n} B_j \leq \frac{3}{2n+3} \left(1 + \frac{2}{3^n}\right) = 1.
\]

\[\square\]

Note. In the previous manuscript, we had only a lengthy proof for Lemma 4.1. The second author asked for an elegant proof of it in *Sugaku Seminar*, a mathematical monthly magazine published in Japan. Several readers gave nice proofs as above. See an article (in Japanese) of the second author in *ibid*. vol. 50, no. 3 (2011) for details. The authors would like to express their thanks to the readers of the magazine.

We next show a result similar to Lemma 3.1.

**Lemma 4.2.** The functions \(P\) given in (2.1) and \(Q(z) = 2zP'(z) + 1 - P(z)^2\) have non-negative Taylor coefficients about \(z = 0\).

**Proof.** Let
\[
G(z) = \sum_{n=0}^{\infty} \frac{z^n}{2n+1} = \frac{1}{2\sqrt{z}} \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}}.
\]
Then
\[
P(z) = 1 + \frac{8}{\pi^2}zG(z)^2.
\]
Therefore, it is immediate to see that \(P(z)\) has positive Taylor coefficients about \(z = 0\). Furthermore, we can easily check the formula
\[
P'(z) = \frac{8G(z)}{\pi^2(1 - z)}.
\]
We also note that \(G^3\) can be described by
\[
G(z)^3 = \sum_{n=0}^{\infty} A_n z^n,
\]
where \(A_n\) is the number given in Lemma 4.1. With these facts in mind, we now compute
\[
Q(z) = 16 \pi^2 z G(z) \left( \frac{1}{1 - z} - G(z) - \frac{4z}{\pi^2} G(z)^3 \right)
\]
\[
= \frac{64}{\pi^4} z^2 G(z) \sum_{n=0}^{\infty} \left( \frac{\pi^2}{4} \cdot \frac{2n + 2}{2n + 3} - A_n \right) z^n.
\]
Since
\[
\frac{\pi^2}{4} \cdot \frac{2n + 2}{2n + 3} \geq \frac{\pi^2}{6} > 1 \geq A_n
\]
for \(n = 0, 1, 2, \ldots\) by Lemma 4.1, we see that the Taylor coefficients of \(Q\) about \(z = 0\) are non-negative. \(\square\)

We are now in a position to show the second main result.

Proof of Theorem 1.7. We take the same strategy as in the proof of Theorem 1.5. In view of Theorem 2.1, we only need to show that \(N(P) = 8/\pi^2\). By Lemma 4.2 and formulae (2.2), (4.2), (4.3), we now have
\[
F(s, t) = \frac{(1 - t^2)^2}{2t^2} (1 - P(s)^2) + \frac{(1 - t^2)(1 - s)(s + t^2)}{t^2} P'(s)
\]
\[
= \frac{8(1 - t^2)}{\pi^2 t^2} \left( (s + t^2)G(s) - s(1 - t^2)G(s)^2 - \frac{4}{\pi^2} s^2 (1 - t^2) G(s)^3 \right)
\]
\[
= \frac{8x}{\pi^2 (1 - x)} \left( (s + 1 - x) G(s) - s x G(s)^2 - \frac{4}{\pi^2} s^2 x G(s)^3 \right),
\]
where we put \(x = 1 - t^2\). Since \(F(0, t) = 8(1 - t^2)/\pi^2 \to 8/\pi^2\) as \(t \to 0\), it suffices to show that \(F(s, t) \leq 8/\pi^2\) for \(0 < s < t < 1\). This is equivalent to the inequality
\[
x \left( (s + 1 - x) G(s) - s x G(s)^2 - \frac{4}{\pi^2} s^2 x G(s)^3 \right) \leq 1 - x
\]
\[
\Leftrightarrow \left( G(s) + s G(s)^2 + \frac{4}{\pi^2} G(s)^4 \right) x^2 - (1 + (1 + s)G(s)) x + 1 \geq 0
\]
for \(0 < x < 1 - s^2\). The left-hand side in the last inequality is of the form \(Kx^2 - Mx + L\) with
\[
4KL - M^2 = \left( \frac{4s G(s)^2}{\pi} \right)^2 - \left( 1 - (1 - s) G(s) \right)^2.
\]
It is enough to show \(4KL - M^2 \geq 0\). Since \(G(s) < 1/(1 - s)\), we observe that \(4KL - M^2 \geq 0\) if and only if
\[
\frac{4s G(s)^2}{\pi} \geq 1 - (1 - s) G(s),
\]
which is equivalent to
\[
G(s) \geq \frac{\pi \sqrt{(1-s)^2 + 16s/\pi - 1} + s}{8s}.
\]
Since it is easily checked that \(\pi \left(\sqrt{(1-s)^2 + 16s/\pi - 1} + s\right)/8s < 1\) and \(G(s) > 1\) for \(0 < s < 1\), the inequality (4.4) certainly holds. Thus the proof is now complete. \(\square\)

Define a function \(f_0 \in \mathcal{UCV}\) by the relation
\[
1 + z\frac{f_0''(z)}{f_0'(z)} = P(z^2) = 1 + \frac{2}{\pi^2} \left(\log \frac{1+z}{1-z}\right)^2.
\]
Then we have
\[
f_0(z) = z + \frac{4}{3\pi^2} z^3 + \left(\frac{4}{15\pi^2} + \frac{8}{5\pi^4}\right) z^5 + \cdots
\]
and thus \(S_{f_0}(0) = \frac{8}{\pi^2}\) so that \(\|S_{f_0}\| = \frac{8}{\pi^2}\).

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