Relativistic Viscous Hydrodynamics with Angular Momentum

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Hydrodynamics is a general theoretical framework for describing the long-time large-distance behaviors of various macroscopic physical systems, with its equations based on conservation laws such as energy-momentum conservation and charge conservation. Recently there has been significant interest in understanding the implications of angular momentum conservation for a corresponding hydrodynamic theory. In this work, we examine the key conceptual issues for such a theory in the relativistic regime where the orbital and spin components get entangled. We derive the equations for relativistic viscous hydrodynamics with angular momentum through Navier-Stokes type of gradient expansion analysis and find five new transport coefficients for angular momentum diffusion modes.

INTRODUCTION

Hydrodynamics is a general theoretical framework for describing the long-time large-distance behaviors of macroscopic physical systems. It has many important applications in various branches of physics, from cosmic expansion and galaxy/star evolutions at the very large scales to relativistic nuclear collisions at the very small scales [1–8]. The core of hydrodynamics is about physical quantities protected by exact conservation laws, such as energy, momentum and conserved charge. Past hydrodynamic studies almost entirely focus on the energy-momentum conservation and charge conservation. Only very recently, there has been a rapidly increasing interest in understanding the implication of angular momentum conservation and developing a corresponding relativistic hydrodynamic theory [9–29]. Such interest is strongly fueled by experimental observations as well as phenomenological studies of spin polarization phenomena in rotating matter, with examples ranging from condensed matter flow systems to subatomic fluids in relativistic nuclear matter, with examples ranging from condensed matter to relativistic nuclear collisions [30–38]. See recent reviews in e.g. [39–48].

To develop a hydrodynamic theory with angular momentum appears both conceptually and technically challenging, despite intensive recent efforts. This is especially so in the relativistic regime where the separation between spin and orbital components becomes subtle. There are also confusions surrounding the property of angular momentum carried by a local fluid cell. This

GENERALITY

Hydrodynamic equations are nothing but macroscopic conservation laws. For example, the usual relativistic hydrodynamics consists of equations for energy-momentum as well as charge conservation:

\begin{align}
\partial_\mu T^{\mu\nu} &= 0, \\
\partial_\mu N^\mu &= 0,
\end{align}

where \( T^{\mu\nu} \) is energy-momentum tensor and \( N^\mu \) is the conserved charge current. Here we consider only a single conserved charge, and generalization to multiple currents \( N^\mu_i (i = 1, \ldots, k) \) is trivial.

Now we want to further include the angular momentum conservation:

\[ \partial_\mu J^{\mu\alpha\beta} = 0, \]

where the tensor \( J^{\mu\alpha\beta} \) is the local current associated with angular momentum transport. It should contain all contributions such as spin and orbital angular momentum.

A conceptual discussion is now in order. To construct a hydrodynamic description with angular momentum, one starts by assuming a separation between the macroscopic scale \( L \) (e.g. the system size) and the microscopic scale \( \lambda \), which is determined by underlying dynamical interactions relevant for the thermal relaxation and equilibration among both spin and orbital angular momentum of the fluid constituents. This allows introducing an intermediate hydrodynamic scale \( l \) for defining local fluid cells, with \( \lambda \ll l \ll L \), a coarse-graining process as illustrated in Fig. 1. Each fluid cell should be very close to local thermal equilibrium and can be represented by locally-defined hydrodynamic fields/variables such as temperature \( T(x^\mu) \) (or equivalently energy density \( \epsilon(x^\mu) \)), chemical potential \( \mu(x^\mu) \) (or charge density \( n(x^\mu) \)), pressure \( p(x^\mu) \), entropy density \( s(x^\mu) \), fluid velocity \( \mathbf{v}(x^\mu) \), etc.

Along the same line of consideration, let us examine the angular momentum carried by a local fluid cell. This

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However is tricker and there are two contributions:

$$J^{\alpha\beta} = \left( \varepsilon^{\alpha}T^{\mu\beta} - x^{\beta}T^{\mu\alpha} \right) + \Sigma^{\mu\alpha\beta}.$$ (4)

where both $J^{\mu\alpha\beta}$ and $\Sigma^{\mu\alpha\beta}$ are anti-symmetric in $\alpha \leftrightarrow \beta$. The first part, in the parentheses, comes from the angular momentum associated with the orbital motion of the fluid cell as a “whole object”. Note the $T^{\mu\nu}$ here is the canonical energy-momentum tensor which could in principle have both symmetric and antisymmetric components: $T^{\mu\nu} \equiv T^{\mu\nu}_{(s)} + T^{\mu\nu}_{(a)}$. The second part, i.e. $\Sigma^{\mu\alpha\beta}$, counts all the internal angular momentum from within the fluid cell. Microscopically it includes both orbital and spin contributions from all microscopic constituents in that cell, as illustrated in Fig. 1.

This decomposition marks a key conceptual difference between our approach and that in recent literature. The $\Sigma^{\mu\alpha\beta}$ should not be considered as simply a spin current. In fact, attempt to separate the spin content out of this total internal angular momentum may be dubious. To better illustrate this point and avoid confusion, let us consider an explicit example of a 3-particle cluster in classical mechanics as shown in Fig. 2. The total angular momentum with respect to the origin of the global frame is:

$$\vec{J} = \sum_{i=1,2,3} (\vec{x}_i \times \vec{p}_i + \vec{s}_i)$$

where $\vec{x}_i$ and $\vec{p}_i$ are the individual particle’s position and momentum in the global frame, respectively. The $\vec{s}_i$ is the individual particle’s intrinsic angular momentum, which could arise from e.g. microscopic spin or rotational motion within a composite particle (like the rotational modes of molecules). Now consider the center-of-mass (CM) of this cluster, with CM position $\vec{x}$ and total cluster momentum $\vec{p}$ (in the global frame). We further introduce the position and momentum of the individual particle with respect to the CM as $\vec{x}'_i = \vec{x}_i - \vec{x}$ and $\vec{p}'_i = m_i \vec{x}'_i$ with $m_i$ being the particle mass. Then the angular momentum can be rewritten as

$$\vec{J} = \vec{x}' \times \vec{p}' + \sum_{i=1,2,3} \left( \vec{x}'_i \times \vec{p}'_i + \vec{s}_i \right).$$

One can thus define an internal angular momentum for the cluster within the CM frame as $\vec{\Sigma} = \sum_{i=1,2,3} \left( \vec{x}'_i \times \vec{p}'_i + \vec{s}_i \right)$, such that one has the decomposition $\vec{J} = \vec{x} \times \vec{p} + \vec{\Sigma}$. It shall be obvious that the $\vec{\Sigma}$ includes contributions not only from the intrinsic angular momentum of the underlying particles but also from the rotational motion of these particles relative to the cluster CM. It is straightforward to see that this decomposition can be generalized to a fluid dynamical context for a fluid cell containing many particles, corresponding to that in Eq. (4). In our view, it may not be feasible to further separate out the intrinsic part and the internal motion part in a meaningful way for a coarse-grained fluid cell. It is worth emphasizing that a general description of fluid systems with angular momentum shall encompass not only quantum field systems with spins but also classical fluids which certainly could also possess angular momentum from both fluid orbital motion and intrinsic angular momentum.

The conserved quantity related to $\Sigma^{\mu\alpha\beta}$ is introduced as the antisymmetric tensor field $\sigma^{\alpha\beta}(x^\mu)$ which should represent the local angular momentum density, akin to the local charge density. It should also have a corresponding angular momentum chemical potential $\omega_{\alpha\beta}(x^\mu)$. We
emphasize the $\sigma^{\alpha\beta}$ and $\omega_{\alpha\beta}(x^\mu)$ should not be viewed as just spin tensor or spin chemical potential.

The hydrodynamic variables are also subject to thermodynamic constraints, that shall be generalized to include angular momentum. For example, the first law of thermodynamics requires:

$$\epsilon = -p + Ts + \mu n + \omega_{\alpha\beta}\sigma^{\alpha\beta}, \quad (5)$$

$$dp = sdT + nd\mu + \sigma^{\alpha\beta}d\omega_{\alpha\beta}, \quad (6)$$

$$de = Tsds + d\mu n + \omega_{\alpha\beta}d\sigma^{\alpha\beta}. \quad (7)$$

Furthermore, the second law of thermodynamics requires that the entropy can not decrease, i.e.

$$\partial_{\mu} S^\mu \geq 0. \quad (8)$$

where $S^\mu$ is the entropy current. Additionally, one can use the thermal equation of state to express e.g. pressure and entropy density in terms of $T, \mu, \omega_{\alpha\beta}$.

The next key step is to find the constitutive relations that specify conserved quantities, i.e. $T^\mu{}^\nu$, $N^\mu$ and $\Sigma^{\mu\nu\beta}$ in terms of hydrodynamic variables. This can be done through a systematic gradient expansion. One starts by assuming perfect local thermal equilibrium to write down these entirely in terms of local variables and obtain the ideal hydrodynamics. One next introduces viscous terms into constitutive relations, involving only single-derivative terms of hydrodynamic variables to obtain Navier-Stokes type of viscous hydrodynamics. See e.g. [1, 2] for textbook examples of such derivation in absence of angular momentum. In the rest of the paper, we will derive a relativistic viscous hydrodynamics with angular momentum.

For later convenience, we introduce a number of notations: (a) metric tensor $g^{\mu\nu} = (+, -, -, -)$, covariant four-velocity $u^\mu = (\gamma, \gamma v)$ (with $\gamma = 1/\sqrt{1 + v^2}$) and its full derivative $\theta = \partial_{\mu} u^\mu$; (b) decomposition of covariant gradient, $\partial_{\mu} = u_{\nu} D + \nabla_{\mu}$ where $D = u_{\nu} \partial^\nu$, and $\nabla_{\mu} = \Delta_{\mu\nu} \partial^\nu$, with the transverse projector $\Delta_{\mu\nu} = (g_{\mu\nu} - u_{\mu} u_{\nu})$; (c) $\beta = \frac{1}{T}$, $\alpha = \frac{\mu}{T}$ and $\beta^\mu = \frac{u^\mu}{T}$.

**IDEAL HYDRODYNAMICS WITH ANGULAR MOMENTUM**

In the ideal hydrodynamic limit, all conserved quantities can be uniquely determined from local variables. For example, it is well known that the energy-momentum tensor and charge current are given by:

$$T^{\mu\nu}(0) = \epsilon u^\mu u^\nu - p\Delta^{\mu\nu}, \quad N^\mu(0) = n u^\mu. \quad (9)$$

Substituting these into Eqs. (1)(2) leads to the usual relativistic ideal hydrodynamic equations.

For angular momentum, one can similarly write down

$$\Sigma^{\mu\nu\beta}(0) = \sigma^{\alpha\beta}u^\mu. \quad (10)$$

Combining the above with Eqs. (3)(4), one thus obtains equation for the angular momentum conservation:

$$\partial_{\mu} J^{\mu\nu\beta}(0) = \sigma^{\alpha\beta} \theta + D\sigma^{\alpha\beta} = 0. \quad (11)$$

Note that in terms of gradient expansion counting, both $\sigma^{\alpha\beta}$ and $\omega_{\alpha\beta}$ are considered zeroth order terms consistently throughout the present work. This differs from a number of treatments in the literature. For example, the $\omega_{\alpha\beta}$ is treated as first order term in gradient expansion in [16, 17]. It shall be emphasized that conceptually the above thermodynamic relations are meant to relate locally defined quantities for fluid cells in local thermal equilibrium which are generally considered as zeroth order terms in the hydrodynamic context.

At this order, the entropy current is $S^\mu(0) = su^\mu$. By using the thermodynamic relations as well as the ideal hydrodynamic equations for $T^{\mu\nu}(0)$, $N^\mu(0)$ and $J^{\mu\nu\beta}(0)$, it is straightforward to verify that $\partial_{\mu} S^\mu(0) = \partial_{\mu} (su^\mu) = 0$. That is, entropy stays unchanged in ideal hydrodynamic limit, as it should be.

**VISCIOUS HYDRODYNAMICS WITH ANGULAR MOMENTUM**

We next consider dissipative corrections to the linear order of gradient terms by adding viscous currents $\tilde{T}^{\mu\nu}, \tilde{N}^\mu, \tilde{\Sigma}^{\mu\nu\beta}$ and $\tilde{S}^\mu$ to the ideal currents, i.e.:

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - p\Delta^{\mu\nu} + \tilde{T}^{\mu\nu}, \quad (12)$$

$$N^\mu = n u^\mu + \tilde{N}^\mu, \quad (13)$$

$$\Sigma^{\mu\nu\beta} = u^\mu \sigma^{\alpha\beta} + \tilde{\Sigma}^{\mu\nu\beta}, \quad (14)$$

$$S^\mu = su^\mu + \tilde{S}^\mu. \quad (15)$$

These viscous currents are subject to matching conditions for mapping the energy density, charge density and angular momentum density between a non-equilibrium state and the corresponding equilibrium state:

$$\tilde{T}^{\mu\nu} u_{\mu} u_{\nu} = \tilde{N}^\mu u_{\mu} = \tilde{\Sigma}^{\mu\nu\beta} u_{\mu} = \tilde{S}^\mu u_{\mu} = 0. \quad (16)$$

A convenient technique is to decompose any Lorentz index into longitudinal and transverse (with respect to $u^\mu$) components, e.g. $V^{\mu} = A^{\mu} + B^{\mu}$ with $A^{\mu} = u^{\nu}V_{\nu}$ and $B^{\mu} = \Delta_{\mu\nu}V_{\nu}$. This helps simplify the implementation of the matching conditions in Eq.(13), as we shall see later.

The analysis with only $\tilde{T}^{\mu\nu}$ and $\tilde{N}$ is well known. Here we focus on the new ingredient i.e. the angular momentum, whose conservation Eq.(3) now gives

$$\partial_{\mu} J^{\mu\alpha\beta} = 2T_{(a)}^{\alpha\beta} + \theta \sigma^{\alpha\beta} + D\sigma^{\alpha\beta} + \partial_{\mu} \Sigma^{\mu\alpha\beta} = 0, \quad (17)$$

where $T_{(a)}^{\alpha\beta}$ is the anti-symmetric part of $\tilde{T}^{\alpha\beta}$ tensor.

We now analyze the entropy current up to linear-order gradient terms, given by:

$$S^\mu = p\beta^\mu + \beta_{(a)} T^{\mu\nu} - \alpha N^\nu - \beta \omega_{\alpha\beta} \Sigma^{\mu\alpha\beta} \quad (18)$$

What is important is the divergence of entropy current $\partial_{\mu} S^\mu(0)$ which now contains gradient terms up to quadratic order. Let us first re-examine the ideal part, $\partial_{\mu} S^\mu(0)$, which is in linear order of gradient expansion and could
potentially receive contributions from the newly introduced viscous terms. By using thermal relations, hydrodynamic equations and consistently keeping up to linear-order gradient terms, one obtains:

$$\partial_\mu S^\mu_{(0)} = 2\beta\omega_{\alpha\beta}\widetilde{T}^{\alpha\beta}.$$  \hspace{1cm} (19)

To make sure that the lowest-order entropy evolution corresponds to reversible ideal hydrodynamics, the above expression should vanish thus requiring \(T^{\alpha\beta}_{(a)} = 0\). So the viscous tensor \(\widetilde{T}^{\mu\nu}\) up to linear order of gradient expansion must be symmetric, \(\widetilde{T}^{\mu\nu} = \widetilde{T}^{\nu\mu}\). (This though does not rule out possible anti-symmetric component in higher order.) As a consequence, the angular momentum conservation Eqs. (3)(4) become simply \(\partial_\mu\Sigma^{\alpha\mu\beta} = 0\). See also pertinent discussions on this point by Fukushima and Pu in [17].

The next step is to calculate the entropy current divergence up to the quadratic order of gradient terms:

$$\partial_\mu S^\mu = \partial_\mu \left( p^\mu + \beta T^{\mu\nu} - \alpha N^\mu - \beta\omega^{\alpha\beta}\Sigma^{\mu\alpha\beta} \right)$$

$$= \widetilde{T}^{\mu\nu}\partial_\nu\beta_\mu - \widetilde{N}^{\mu}\partial_\mu\beta_\nu - \Sigma^{\mu\alpha\beta}\partial_\mu (\beta\omega_{\alpha\beta}), \hspace{1cm} (20)$$

where we have used the previously mentioned thermal dynamic constraints as well as hydrodynamic equations to arrive at the last line.

The key step toward uniquely determining the constitutive relations for the various viscous terms (at Navier-Stokes level) can be done by analyzing how the above Eq.(20) can ensure the condition in Eq.(8). To do this, one needs a careful discussion of the frame choice. In viscous hydrodynamics, the definition of local rest frame (LRF) and the corresponding fluid velocity \(u^\mu\) become subtle. This is because the physically observable conserved currents associated with a given fluid cell are typically not aligned, e.g. the energy current \(T^{\mu\nu}u_\nu\) and the charge current \(N^\mu\) are not parallel. This leaves room for different frame choices such as the well-known Eckart frame and Landau frame [1, 6, 49–52]. In the following we shall derive the viscous hydrodynamics with angular momentum in both frames.

We first consider the Eckart frame, defining local flow velocity \(u^\mu_E\) along the charge current:

$$u^\mu_E = \frac{N^\mu}{\sqrt{N_{\nu}N^\nu}}, \hspace{1cm} (21)$$

In this case the viscous diffusion flux \(\widetilde{N}^\mu\) is absent. For notational simplicity, we will just use symbol \(u^\mu\) in place of \(u^\mu_E\) in the derivations and results from Eckart frame. In this frame, we can carry out a general and systematic decomposition of the physical currents \(N^\mu, T^{\mu\nu}\) and \(\Sigma^{\mu\alpha\beta}\) into longitudinal and transverse components, obtaining the following form:

$$T^{\mu\nu} = \epsilon^{\mu\rho\nu\sigma}u^\rho u^\sigma - (p + \Pi)\Delta^{\mu\nu} + 2u^\mu q^\nu + \pi^{\mu\nu}, \hspace{1cm} (22)$$

$$N^\mu = nu^\mu, \hspace{1cm} (23)$$

$$\Sigma^{\mu\alpha\beta} = u^\mu \sigma^{\alpha\beta} + 2u^{[\alpha} \Delta^{\beta]_\sigma} \Phi + 2u^{[\alpha} \tau^{\beta]_\sigma} + \Theta^{\mu\alpha\beta}. \hspace{1cm} (24)$$

In the above, we recognize the familiar bulk viscous pressure \(\Pi\), shear viscous tensor \(\pi^{\mu\nu}\) and the diffusion flux \(q^\mu\). Furthermore, there emerge new dissipative quantities \(\Phi, \tau^{\mu_\alpha\beta}, \pi^{\mu_\alpha\beta}\) associated with angular momentum transport in Eq.(24). They are all considered linear-order terms in gradient expansion. They also satisfy the following properties: \(u^\mu q^\nu = u_\mu \pi^{\mu\nu} = u_\mu \pi^{\mu_\nu} = u_\mu \pi^{\mu_\nu}_{(s)} = u_\mu \pi^{\mu_\nu}_{(a)} = 0; \pi^{\mu\nu} = \pi^{\nu\mu}, \pi^{\mu_\nu}_{(s)} = \pi^{\nu_\mu}_{(s)} = \pi^{\mu_\nu}_{(s)}\).

It is also evident from the viscous transport in Eq.(24) that they are all considered linear-order terms in gradient expansion. They also satisfy the following properties: \(u^\mu q^\nu = u_\mu \pi^{\mu\nu} = u_\mu \pi^{\mu_\nu} = u_\mu \pi^{\mu_\nu}_{(s)} = u_\mu \pi^{\mu_\nu}_{(a)} = 0; \pi^{\mu\nu} = \pi^{\nu\mu}, \pi^{\mu_\nu}_{(s)} = \pi^{\nu_\mu}_{(s)} = \pi^{\mu_\nu}_{(s)}\).

Finally one substitutes the decomposition Eqs.(22)(23)(24) into the entropy divergence Eq.(20) and requires the positivity condition Eq.(8) to be satisfied for all possible fluid configurations. The only way for this to happen is to for all terms contributing to entropy evolution to be space-like quadratic terms. After a long calculation, one arrives at the following final results which uniquely fix various viscous terms in Eckart frame:

$$\Pi = -\zeta_0 \beta_\rho, \hspace{1cm} (25)$$

$$\pi^{\mu\nu} = 2\eta \nabla^\nu(u^\mu), \hspace{1cm} (26)$$

$$q^\mu = \lambda T \left( \frac{\nabla^\mu T}{T} - Du^\mu \right)$$

$$- \frac{\lambda h T^2}{\epsilon + \rho} \left[ \nabla^\mu \left( \frac{\mu}{T} \right) + \frac{\sigma^{\alpha\beta}}{n} \nabla^\mu \left( \omega_{\alpha\beta} \right) \right], \hspace{1cm} (27)$$

$$\Phi = -\chi_1 u^\alpha \nabla^\beta \left( \frac{\omega_{\alpha\beta}}{T} \right), \hspace{1cm} (28)$$

$$\tau^{\mu_\alpha\beta}_{(s)} = -\chi_2 u^\alpha \left[ (\Delta^{\beta_\rho\Delta_\gamma} + \Delta^{\alpha\rho} \Delta^{\beta_\gamma}) \nabla_\gamma \left( \frac{\omega_{\beta_\gamma}}{T} \right) \right]$$

$$- \frac{2}{3} \Delta^{\alpha\rho} g^{\beta_\gamma} \nabla_\gamma \left( \frac{\omega_{\beta_\gamma}}{T} \right), \hspace{1cm} (29)$$

$$\tau^{\mu_\alpha\beta}_{(a)} = -\chi_3 u^\alpha \left[ (\Delta^{\beta_\rho} \Delta^{\gamma_\nu} - \Delta^{\nu_\rho} \Delta^{\gamma_\nu}) \nabla_\gamma \left( \frac{\omega_{\beta_\gamma}}{T} \right) \right]$$

$$+ \chi_5 \Delta^{\alpha\delta} \Delta^{\beta_\rho} \Delta^{\gamma} \nabla_\gamma \left( \frac{\omega_{\beta_\gamma}}{T} \right). \hspace{1cm} (30)$$
momentum and temperature gradients as well as from bulk and shear gradient terms in the stress tensor.

If one instead uses the Landau frame or energy frame, the flow velocity is then specified by the energy flow:

$$u_L^\mu \equiv \frac{T_\mu^\nu u_L^\nu}{\sqrt{u_L^\gamma T_\gamma^\beta u_L^\beta}}.$$ (32)

In this frame, the conserved currents take the following form upon decomposition:

$$T^{\mu\nu} = \epsilon u_L^\mu u_L^\nu - (p + \Pi) \Delta_L^{\mu\nu} + \pi^{\mu\nu},$$ (33)

$$N^\mu = n u_L^\mu - n \frac{q^\mu}{\epsilon + p},$$ (34)

$$\Sigma^{\mu\alpha\beta} = u_L^\mu \sigma^{\alpha\beta} - \frac{q^\mu}{\epsilon + p} \sigma^{\alpha\beta} + 2 u_L^{[\alpha} \Delta_L^{\beta]} \Phi$$
$$+ 2 u_L^{[\alpha} \tau_L^{\beta]} + 2 u_L^{[\alpha} r_L^{\beta]} + \Theta^\alpha\beta.$$ (35)

Following the same step as in the Eckart frame, one obtains formally the same expressions for the set of viscous constitutive relations in Eqs.(25–30) albeit with all quantities in the Landau frame.

SUMMARY AND DISCUSSIONS

In summary, we’ve clarified conceptual issues and derived the equations for relativistic viscous hydrodynamics with angular momentum through a consistent gradient expansion up to the linear order. The key new results are constitutive equations for local angular momentum diffusion modes. The present work helps laying a firm foundation and paving ways for further theoretical developments as well as potential phenomenological applications.

We end this paper with discussions on a number of ideas for future study. For relativistic hydrodynamics, one needs to go beyond the Navier-Stokes order for causality consideration [53–60]. An obvious extension of the present result would be Israel-Stewart type of second order hydrodynamics, via e.g. $D\Sigma^{\mu\alpha\beta} = -\frac{(\Sigma^{\mu\alpha\beta} - \Sigma^{\mu a \beta})}{\tau_\Sigma}$. A full derivation of the second-order viscous hydrodynamics with angular momentum [20, 29], from either macroscopic or microscopic approach, would be highly desired. The values of five newly introduced transport coefficients for angular momentum transport are specific to various physical systems and can be computed from e.g. quantum transport theory with spin. Concerning frame choice, the presence of the angular momentum current brings the possibility of a new frame choice beyond the Eckart and Landau frames discussed in the present work, namely an angular momentum frame in which the flow velocity $u^\mu$ is defined along local angular momentum current. Another interesting problem would be the analysis of various new hydrodynamic modes associated with angular momentum component by linearizing the hydrodynamic equations obtained in this work. Last but not least, it is tempting to think about phenomenological applications e.g. for calculating polarization observables in heavy ion collisions. This can be done by giving a proper initial condition for angular momentum density, numerically solving its evolution and generalizing the Cooper-Frye procedure to include the angular momentum chemical potential $\omega_{\mu\beta}$ on the freeze-out hypersurface for computing the spin polarization of produced particles. We anticipate many new and interesting developments based on the present work to come soon and to be reported elsewhere.

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