Notes of Boundedness on Cauchy Integrals on Lipschitz Curves

\( (p = 2) \)

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August 10, 2018

Abstract

We provide the details of the first proof in [1], which proved that Cauchy transform of \( L^2 \) functions on Lipschitz curves is bounded. We then prove that every \( L^2 \) function on Lipschitz curves is the sum of non-tangential boundary limit of functions in \( H^2(\Omega_{\pm}) \), the Hardy spaces on domains over and under the Lipschitz curve. We also obtain a more accurate boundary of Cauchy transform under the condition that the Lipschitz curve is the real axis.

Keywords: Cauchy Integral, Lipschitz curve, Hardy space, non-tangential boundary limit,

2010 Mathematics Subject Classification: Primary: 30H10, Secondary: 30E20, 30E25

1 Introduction

Paper [1] offered two elementary proofs of the boundedness of Cauchy integral (or transform) on Lipschitz curve \( \Gamma \), with integral index \( p = 2 \). The first one in which we are interested is succinct, thus without many details. In this paper, we give the full version of that proof. Since the Cauchy integral is actually analytic on two domains over and under \( \Gamma \) which we denote as \( \Omega_{\pm} \), it is in \( H^2(\Omega_{\pm}) \), the Hardy spaces on \( \Omega_{\pm} \), hence has non-tangential boundary limits from above and below \( \Gamma \) [2]. Then we could reach the result that every function in \( L^2(\Gamma) \) is the sum of two functions in \( H^2(\Omega_{+}) \) and \( H^2(\Omega_{-}) \), respectively. That result is usually written as \( L^2(\Gamma) = H^2(\Omega_{+}) + H^2(\Omega_{-}) \). We also apply the same method to the special case of \( \Gamma \) be \( \mathbb{R} \), and obtain a more accurate boundary of the Cauchy transform.

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2 Definitions

Let $\Gamma = \{\zeta(u) = u + ia(u) : u \in \mathbb{R}\}$ be a Lipschitz curve in the complex plain $\mathbb{C}$, where $\|a'\|_\infty = M < \infty$, $\Omega_\pm = \{\zeta(u) = u + i\tau : u \in \mathbb{R}, \tau > 0\}$ be the two domains lying above and below $\Gamma$, $\Phi_\pm$ be the two conformal representations from $\mathbb{C}_\pm = \{x + iy : x \in \mathbb{R}, y > 0\}$ onto $\Omega_\pm$, which both extend to the boundary, such that $\Phi_\pm(\mathbb{R}) = \Gamma$ and $\Phi_\pm(\infty) = \infty$, $\Psi_\pm : \Omega_\pm \rightarrow \mathbb{C}_\pm$ be the inverse mappings of $\Phi_\pm$. More detail of $\Phi_\pm$ and $\Psi_\pm$ are in [4] and [2].

For $w \in \mathbb{C}$, define $d(w)$ as the distance from $w$ to the curve $\Gamma$, that is

$$d(w) = \inf\{|w - \zeta| : \zeta \in \Gamma\},$$

which implies that

$$d(w) \leq |w - \zeta(0)| \leq |w| + |a(0)|.$$

For $r > 0$, denote $\{\{z| < r : z \in \mathbb{C}\}$ as $D(0,r)$, and $D(0,1)$, the unit disk of $\mathbb{C}$, as $\mathbb{D}$. For domain $D \subset \mathbb{C}$ and measure $dm$ on it, let $L^2(D, dm)$ be the function space of all complex valued, $dm$ measurable functions on $D$, equipped with norm

$$\|f\|_{L^2(D, dm)} = \left( \int_D |f|^2\, dm \right)^{\frac{1}{2}}, \text{ for } f \in L^2(D, dm).$$

Thus we could consider function spaces like $L^2(\mathbb{R}, dx)$, $L^2(\mathbb{C}_\pm, d\lambda)$, where $d\lambda$ is the area measure on $\mathbb{C}$, and a few more which will appear later in this paper.

Let $F(w)$ be a function analytic on $\Omega_+$, if

$$\sup_{\Gamma_r} \left( \int_{\Gamma_r} |F(w)|^p\, dw \right)^{\frac{1}{p}} = \|F\|_{H^p(\Omega_+)} < \infty, \text{ for } 0 < p < \infty,$$

where $\Gamma_r = \{\zeta + i\tau : \zeta \in \Gamma\}$, then we say that $F(w) \in H^p(\Omega_+)$. Fix $u_0 \in \mathbb{R}$ such that $\zeta'(u_0) = |\zeta'(u_0)|e^{i\phi_0}$ exists, and choose $\phi \in (0, \frac{\pi}{2})$, we denote $\zeta_0 = \zeta(u_0)$ and let

$$\Omega_\phi(\zeta_0) = \{\zeta_0 + re^{i\theta} : r > 0, \theta - \phi_0 \in (\phi, \pi - \phi)\},$$

then we say that a function $F(w)$ on $\Omega^+$ has non-tangential boundary limit $l$ at $\zeta_0$ if for $w \in \Omega_\phi(\zeta_0) \cap \Omega^+$,

$$\lim_{w \rightarrow \zeta_0} F(w) = l, \text{ for any } \phi \in \left(0, \frac{\pi}{2}\right).$$

It is not difficult to verify that for fixed $\phi \in \left(0, \frac{\pi}{2}\right)$, there exists constant $\delta > 0$, such that, if $|z| < \delta$ and $\zeta_0 + z \in \Omega_\phi(\zeta_0)$, then $\zeta_0 + z \in \Omega^+$ and $\zeta_0 - z \in \Omega^-$. Let $\zeta, \zeta_0 \in \Gamma$, we define

$$K_z(\zeta, \zeta_0) = \frac{1}{2\pi i} \left( \frac{1}{\zeta - (\zeta_0 + z)} - \frac{1}{\zeta - (\zeta_0 - z)} \right),$$
for $z \in \mathbb{C}$ and $z \neq \pm(\zeta - \zeta_0)$, then $K_z(\zeta, \zeta_0)$ is well-defined and we could write

$$K_z(\zeta, \zeta_0) = \frac{1}{\pi i} \cdot \frac{z}{(\zeta - \zeta_0)^2 - z^2}. \quad (1)$$

We could also verify that, if $\zeta_0 + z \in \Omega^+$ and $\zeta_0 - z \in \Omega^-$, then

$$\int_\Gamma K_z(\zeta, \zeta_0) d\zeta = 1.$$

3 Lemmas

Let $f$ be a univalent holomorphic function on $\mathbb{D}$. If $f(0) = 0$ and $f'(0) = 1$, we say that $f$ is in $S$. In other words, we define

$$S = \{ f(z) = z + a_2 z^2 + \cdots : f \text{ is holomorphic and univalent on } \mathbb{D} \}.$$

Lemma 3.1. If $f \in S$ and is continuous to the boundary $\partial \mathbb{D}$, then

$$\frac{1}{4} \leq \inf\{|f(z)| : |z| = 1\} \leq 1.$$

Proof. The first part of the above inequality comes from the Koebe $\frac{1}{4}$ theorem [3], and we only need to prove the second part. Let $\inf\{|f(z)| : |z| = 1\} = b$, then $b \geq \frac{1}{4}$ and $\overline{D(0, b)} \subset f(\overline{D})$, or $f^{-1}(\overline{D(0, b)}) \subset \overline{D}$. Define $g(z) = f^{-1}(bz)$ on $\mathbb{D}$, then $g(\mathbb{D}) \subset \mathbb{D}$, and $g(0) = f^{-1}(0) = 0$. By Schwarz lemma, $|g'(0)| \leq 1$. Since $f'(0) = 1$, $g'(z) = (f^{-1})'(bz) \cdot b$, and $(f^{-1})'(0)$ = $(f'(0))^{-1} = 1$, we have $|g'(0)| = |b| \leq 1$, and the lemma is proved. \qed

Some results below contain “±” as subscript and the two cases usually could be proved by using the same method. Then we will prove only one case and write the other one as a corollary.

Lemma 3.2. If $z = x + iy \in \mathbb{C}_+$, where $x \in \mathbb{R}$, $y > 0$, then

$$|y\Phi'(z)| \leq 2d(\Phi(z)) \leq 4|y\Phi'(z)|,$$

and

$$|y\Phi''(z)| \leq 3|\Phi'(z)|.$$

Consequently,

$$|y^2\Phi''(z)| \leq 6d(\Phi(z)).$$

Proof. Fix $z_0 = x_0 + iy_0 \in \mathbb{C}_+$, and define

$$z = T(\xi) = \frac{-y_0 \xi - z_0}{\xi - 1}, \quad \text{for } \xi \in \mathbb{D},$$

for $y > 0$, then

$$|y\Phi'(z)| \leq 2d(\Phi(z)) \leq 4|y\Phi'(z)|,$$
then $T$ is a fractional linear mapping from $\mathbb{D}$ onto $\mathbb{C}_+$, $T(0) = z_0$, and

$$T'(\xi) = \frac{z_0 - \overline{z}_0}{(\xi - 1)^2}, \quad \xi = T^{-1}(z) = \frac{z - z_0}{z - \overline{z}_0}.$$ 

Denote $\Phi_+$ as $\Phi$ and let

$$f(\xi) = \frac{\Phi(T(\xi)) - \Phi(z_0)}{(z_0 - \overline{z}_0)\Phi'(z_0)}$$

for $\xi \in \mathbb{D}$, then $f$ is univalent, $f(0) = 0$, and

$$f'(\xi) = \frac{\Phi'(T(\xi)) \cdot T'(\xi)}{(z_0 - \overline{z}_0)\Phi'(z_0)} = \frac{\Phi'(T(\xi))}{(\xi - 1)^2\Phi'(z_0)},$$

thus $f'(0) = 1$ and $f \in \mathcal{S}$. By Lemma 3.1,

$$\frac{1}{4} \leq \inf\{|f(\xi)| : |\xi| = 1\} \leq 1,$$

which is, by (2) and $z_0 - \overline{z}_0 = 2iy_0$,

$$\frac{1}{2}|y_0\Phi'(z_0)| \leq \inf\{|\Phi(z) - \Phi(z_0)| : z \in \mathbb{R}\} \leq 2|y_0\Phi'(z_0)|,$$

and it follows that

$$|y_0\Phi'(z_0)| \leq 2d(\Phi(z_0)) \leq 4|y_0\Phi'(z_0)|.$$

We also have, by Bieberbach theorem,

$$\left|\frac{f''(0)}{2!}\right| \leq 2, \quad \text{or} \quad |f''(0)| \leq 4.$$

Since

$$f''(\xi) = \frac{1}{(\xi - 1)^2\Phi'(z_0)}(\Phi''(T(\xi)) \cdot T'(\xi)(\xi - 1) - 2\Phi'(T(\xi))),$$

$T(0) = z_0$ and $T'(0) = z_0 - \overline{z}_0$, then

$$f''(0) = \frac{1}{\Phi'(z_0)}(\Phi''(z_0)(z_0 - \overline{z}_0) + 2\Phi'(z_0))$$

$$= 2iy_0 \frac{\Phi''(z_0)}{\Phi'(z_0)} + 2,$$

and, by (3),

$$\left|2iy_0 \frac{\Phi''(z_0)}{\Phi'(z_0)} + 2\right| \leq 4, \quad \text{or} \quad |y_0\Phi''(z_0)| \leq 3|\Phi'(z_0)|.$$

The last inequality of the lemma is an easy consequence of the former two. \hfill \Box

**Corollary 3.3.** If $z = x + iy \in \mathbb{C}_-$, where $x \in \mathbb{R}$, $y < 0$, then

$$|y\Phi'_-(z)| \leq 2d(\Phi'_-(z)) \leq 4|y\Phi'_-(z)|,$$

and

$$|y\Phi''_-(z)| \leq 3|\Phi'_-(z)|.$$

Consequently,

$$|y^2\Phi''_-(z)| \leq 6d(\Phi'(z)).$$
Let \( \{\gamma_n = \gamma'_n \cup [-a_n, a_n]\} \) be two series of rectifiable simple Jordan curves, which will eventually surround any compact subset of \( \mathbb{C}_\pm \). Here, \( \gamma'_n, \gamma_n \subset \mathbb{C}_\pm, \ a_n > 0 \) and \( a_n \to \infty \) as \( n \to \infty \), \([-a_n, a_n]\) is a straight segment on the real axis. Denote the area measure on \( \mathbb{C} \) as \( d\lambda \), and measure \( |y| d\lambda(z) \) as \( d\mu(z) \) for simplicity. Here, \( y = \text{Im } z \).

**Lemma 3.4.** If \( H_1, H_2 \) are two holomorphic functions on \( \mathbb{C}_+ \), which are both continuous to the boundary, \( z = x + iy \in \mathbb{C}_+ \), and

1. \( \int_{\gamma_n^+} H_1 \overline{H}_2 \, dz \to 0 \) as \( n \to \infty \);
2. \( H_1 \overline{H}_2 y \to 0 \) as \( |z| \to \infty \) and either \( x \) or \( y \) is fixed.

Then

\[
\int_R H_1 \overline{H}_2 \, dx = 4 \int_{\mathbb{C}_+} H'_1 \overline{H}'_2 \, d\mu(z) = \int_{\mathbb{C}_+} \Delta(H_1 \overline{H}_2) \, d\mu(z),
\]

where \( \Delta = 4 \frac{\partial^2}{\partial x \partial y} \) is the Laplace operator.

**Proof.** We will write \( \gamma_n^+ = \gamma'_n \cup [-a_n, a_n] \), and denote the domain which \( \gamma_n \) surrounds as \( D_n \), for fixed \( n \). We have, by Green’s theorem [3],

\[
\int_{\gamma_n} H_1 \overline{H}_2 \, dz = 2i \int_{D_n} \frac{\partial}{\partial y} (H_1 \overline{H}_2) \, d\lambda(z).
\]

Let \( n \to \infty \), then by condition (1) and the definition of \( \gamma_n^+ \), the above equation becomes

\[
\int_R H_1 \overline{H}_2 \, dx = 2i \int_{\mathbb{C}_+} \frac{\partial}{\partial y} (H_1 \overline{H}_2) \, d\lambda(z).
\]

Since \( H_1 \overline{H}_2 y \to 0 \) as \( y \to \infty \) and \( x \) fixed, \( \frac{\partial f}{\partial y} = i f' \) if \( f \) is holomorphic, then

\[
\int_{\mathbb{C}_+} \frac{\partial}{\partial y} (H_1 \overline{H}_2) \, d\lambda(z) = \int_{-\infty}^{+\infty} \int_0^{+\infty} H_1 \overline{H}_2 \, dy \, dx
\]

\[
= \int_{-\infty}^{+\infty} \left( H_1 \overline{H}_2 y \bigg|_{y=+\infty}^{y=0} - \int_0^{+\infty} y \frac{\partial}{\partial y} (H_1 \overline{H}_2) \, dy \right) \, dx
\]

\[
= - \int_{\mathbb{C}_+} y (H'_1 \overline{H}_2 + H_1 \overline{H}'_2) \, d\lambda(z)
\]

\[
= -i \int_{\mathbb{C}_+} y (H'_1 \overline{H}_2 - H_1 \overline{H}'_2) \, d\lambda(z),
\]

and since \( H_1 \overline{H}_2 y \to 0 \) as \( |x| \to \infty \) and \( y \) fixed, \( \frac{\partial f}{\partial x} = f' \) if \( f \) is holomorphic,

\[
\int_{\mathbb{C}_+} y H_1 \overline{H}_2 \, d\lambda(z) = \int_{-\infty}^{+\infty} \int_0^{+\infty} y H_1 \frac{\partial}{\partial x} \overline{H}_2 \, dx \, dy
\]

\[
= \int_{-\infty}^{+\infty} \left( y H_1 \overline{H}_2 \bigg|_{x=+\infty}^{x=-\infty} - \int_{-\infty}^{+\infty} \overline{H}_2 \frac{\partial}{\partial x} (y H_1) \, dx \right) \, dy
\]

\[
= - \int_{\mathbb{C}_+} y H'_1 \overline{H}_2 \, d\lambda(z),
\]
thus
\[ \iint_{C} \frac{\partial}{\partial z}(H_1 \overline{H_2}) \, d\lambda(z) = -2i \iint_{C} yH_1' \overline{H_2} \, d\lambda(z). \]
Together with (5), we get that
\[ \int_{\mathbb{R}} H_1 \overline{H_2} \, dx = 4 \iint_{C} H_1' \overline{H_2} \, d\lambda(z) = 4 \iint_{C} H_1' \overline{H_2} \, d\mu(z). \]
Since \( \Delta \) is the Laplace operator,
\[ \Delta(H_1 \overline{H_2}) = 4 \frac{\partial^2}{\partial z \partial \overline{z}}(H_1 \overline{H_2}) = 4 H_1' \overline{H_2}, \]
and the second equation of (4) is obvious.

Corollary 3.5. If \( H_1, \, H_2 \) are two holomorphic functions on \( C_- \), which are both continuous to the boundary, \( z = x + iy \in C_- \), and

1. \( \int_{\gamma_{\lambda,n}^-} H_1 \overline{H_2} \, dz \to 0 \) as \( n \to \infty \);
2. \( H_1 \overline{H_2} \to 0 \) as \( |z| \to \infty \) and either \( x \) or \( y \) is fixed.

Then
\[ \int_{\mathbb{R}} H_1 \overline{H_2} \, dx = 4 \iint_{C_-} H_1' \overline{H_2} \, d\mu(z) = \iint_{C_-} \Delta(H_1 \overline{H_2}) \, d\mu(z), \tag{6} \]
where \( \Delta = 4 \frac{\partial^2}{\partial z \partial \overline{z}} \) is the Laplace operator.

Define \( T(D) \) as the holomorphic function space on domain \( D \), which satisfies that if \( F \in T(D) \), then there exist two constants \( A, \, r > 0 \), depending on \( F \) only, such that
\[ |F(z)| \leq \frac{A}{|z|}, \quad \text{and} \quad |F'(z)| \leq \frac{A}{|z|^2}, \quad \text{for} \quad |z| > r \quad \text{and} \quad z \in D. \]

Corollary 3.6. If \( F \in T(\Omega_+) \) is continuous to the boundary \( \Gamma \), then
\[ \int_{\mathbb{R}} |F(\Phi^+)|^2 \Phi'_+ \, dx = \iint_{C_+} \Delta(|F(\Phi^+)|^2 \Phi'_+) \, d\mu(z). \]

Proof. Suppose \( A, \, r \) are the two constants related to \( F \) in the definition of \( T(\Omega_+) \). Let \( R > r, \)
\[ l = R\sqrt{1 + M^2} + |a(0)|, \]
and
\[ E(R) = \{u + iv: \, |u| < R, \, |v| < l\} \cap \Omega_+, \]
then \( E(R) \neq \emptyset \), and \( \partial E(R) \) consists of a curve segment and three straight segments. Denote \( \partial E(R) \) as \( BCC'B' \), where
\[ B = \{u + ia(u): \, |u| \leq R\}, \quad CC' = \{R + iv: \, v \in [a(R), l]\}, \]
\[ C'B' = \{u + il: \, |u| \leq R\}, \quad B'B = \{-R + iv: \, v \in [a(-R), l]\}. \]
We then consider $\gamma_R = \partial \Psi(E(R)) \subset C_+$, and let $H_1 = F(\Phi)\Phi'$, $H_2 = F(\Phi)$. In order to invoke Lemma 3.4, it is sufficient to verify the two conditions in that lemma. Let $R \to \infty$, we have, by the definition of $F \in T_+$,

$$
\left| \int_{\Psi(C')} H_1 \overline{H_2} \, dz \right| \leq \int_{\Psi(C')} |F(\Phi)|^2 |\Phi'| \, dz = \int_{C'} |F(w)|^2 |w| \, dw
\leq \int_{C'} \frac{A^2}{|w|^2} \, dw \leq \int_{-\infty}^{+\infty} \frac{A^2}{R^2 + v^2} \, dv
= \frac{A^2 \pi}{R} \to 0,
$$

and

$$
\left| \int_{\Psi(C'B')} H_1 \overline{H_2} \, dz \right| \leq \int_{\Psi(C'B')} |F(\Phi)|^2 |\Phi'| \, dz \leq \int_{C'B'} \frac{A^2}{|w|^2} \, dw
\leq \int_{-\infty}^{+\infty} \frac{A^2}{u^2 + l^2} = \frac{A^2 \pi}{l} \to 0,
$$

since $l \to \infty$ as $R \to \infty$. We also have $\int_{\Psi(B')} H_1 \overline{H_2} \, dz \to 0$ by applying the same method.

For the second condition in Lemma 3.4, since $H_2' = F'(\Phi)\Phi'$ and $\Phi(\infty) = \infty$, we have, by Lemma 3.2,

$$
|H_1 \overline{H_2} y| = |F(\Phi)F'(\Phi)(\Phi')^2 y|
\leq \frac{A^2 y |\Phi'|^2}{|\Phi|^3} \leq \frac{A^2}{|\Phi|^3 y} \cdot |2d(\Phi)|^2
\leq \frac{4A^2}{|\Phi|^3 y} \cdot (|\Phi| + |\alpha(0)|)^2 \to 0,
$$

if $z = x + iy \to \infty$ with either $x \in \mathbb{R}$ or $y > 0$ fixed.

Thus Lemma 3.4 implies that

$$
\int_R H_1 \overline{H_2} \, dx = \int_{C_+} \Delta(H_1 \overline{H_2}) \, d\mu(z),
$$

which is the desired equation after replacing $H_1$ with $F(\Phi)\Phi'$ and $H_2$ with $F(\Phi)$.

\begin{proof}
\end{proof}

\begin{corollary}
If $F \in T(\Omega)$ is continuous to the boundary $\Gamma$, then

$$
\int_R |F'(\Phi)|^2 \Phi'' \, dx = \int_{C_-} \Delta(|F(\Phi)|^2 \Phi) \, d\mu(z).
$$

The following lemma has been proved in [2].

\begin{lemma}
For the conformal representation $\Phi_+: \mathbb{C}_+ \to \Omega_+$, we have

$$
\text{Re} \Phi_+(z) > 0, \quad \text{and} \quad |\Phi_+(z)| \leq M \text{Re} \Phi_+(z),
$$
or

$$
|\arg \Phi_+(z)| \leq \arctan M < \frac{\pi}{2},
$$

for all $z \in \mathbb{C}_+$.
\end{lemma}
Corollary 3.9. For the conformal representation $\Phi_- : \mathbb{C}_- \to \Omega_-$, we have

$$\Re \Phi'_-(z) > 0, \quad \text{and} \quad |\Im \Phi'_-(z)| \leq M \Re \Phi'_-(z),$$
or

$$|\arg \Phi'_-(z)| \leq \arctan M < \frac{\pi}{2},$$

for all $z \in \mathbb{C}_-$. Since $\Phi'_\pm \neq 0$ on $\Omega_\pm$, we could let $\Phi'_\pm = e^{V_\pm}$ and $D_\pm = e^{iV_\pm}$, then

$$\Phi''_\pm = V'_\pm e^{V_\pm} = V'_\pm \Phi'_\pm,$$

and

$$D'_\pm = iV'_\pm e^{iV_\pm} = iV'_\pm D_\pm = iD_\pm \Phi''_\pm (\Phi'_\pm)^{-1}.$$  

Denote $\theta_0 = \arctan M$, by Lemma 3.8,

$$|\Im V_\pm| = |\arg \Phi'_\pm(z)| \leq \theta_0,$$

and

$$|D_\pm| = e^{-\Im V_\pm} \in [e^{-\theta_0}, e^{\theta_0}],$$

then

$$|\Phi''_\pm| = \left|\frac{D'_\pm}{iD_\pm} \cdot \Phi'_\pm\right| \leq e^{\theta_0} |D'_\pm \Phi'_\pm|.$$  

Corollary 3.10. If $F \in \mathcal{T}(\Omega_+)$ is continuous to the boundary $\Gamma$, $D_+$ is defined as above, and let $H = F(\Phi_+)(\Phi'_+)^{\frac{1}{2}}$, then

$$\|H\|_{L^2(\mathbb{R},dx)} = 2\|H'\|_{L^2(\mathbb{C}_+,d\mu)},$$

and

$$\|HD_+\|_{L^2(\mathbb{R},dx)} = 2\|(HD_+)'\|_{L^2(\mathbb{C}_+,d\mu)}.$$  

Consequently,

$$\|HD'_+\|_{L^2(\mathbb{R},dx)} \leq e^{\theta_0} \|H\|_{L^2(\mathbb{R},dx)}.$$  

Proof. Suppose $F \in \mathcal{T}(\Omega_+)$ and denote $\Phi_+$ as $\Phi$, we consider the same $\partial \Psi(E(R))$ as in Corollary 3.6, and let $H_1 = H_2 = H = F(\Phi)(\Phi')^{\frac{1}{2}}$, then the first condition in Lemma 3.4 is verified in the same way as in that corollary. For the second condition, since

$$H'_2 = H' = F'(\Phi)(\Phi')^{\frac{3}{2}} + \frac{1}{2} F(\Phi)(\Phi')^{-\frac{1}{2}} \Phi''_2,$$
we have, by Lemma 3.2 and \( d(\Phi) \leq |\Phi| + |a(0)|, \)
\[
|H_1H_2 y| = y |F(\Phi)(\Phi')^{\frac{1}{2}}| \cdot \left| F'(\Phi)(\Phi')^{\frac{1}{2}} + \frac{1}{2} F(\Phi)(\Phi')^{-\frac{1}{2}} \Phi' \right|
\leq y \frac{A|\Phi'|^2}{|\Phi|} \left( \frac{A|\Phi'|^2}{|\Phi|^2} + \frac{A|\Phi''|}{2|\Phi| \cdot |\Phi'|^2} \right)
= \frac{yA^2}{|\Phi|^3} \left( |\Phi'|^2 + \frac{1}{2} |\Phi \Phi''| \right)
= \frac{yA^2}{|\Phi|^3} \left( \left| y \Phi' \right|^2 + \frac{1}{2} |\Phi \cdot y^2 \Phi''| \right)
\leq \frac{yA^2}{|\Phi|^3} \left( |2d(\Phi)|^2 + 3|\Phi|d(\Phi) \right)
= \frac{A^2}{y|\Phi|^3} \cdot d(\Phi) \left( 4d(\Phi) + 3|\Phi| \right)
\leq \frac{A^2}{y|\Phi|^3} \left( |\Phi| + |a(0)| \right) \left( 7|\Phi| + 3|a(0)| \right) \rightarrow 0,
\]
as \( z = x + iy \rightarrow \infty \) with either \( x \in \mathbb{R} \) or \( y > 0 \) fixed. Then, by Lemma 3.4,
\[
\int_{\mathbb{R}} |H|^2 \, dx = 4 \int_{\mathbb{C}^+} |H'|^2 \, d\mu(z),
\]
which is
\[
\|H\|_{L^2(\mathbb{R}, dx)} = 2\|H\|_{L^2(\mathbb{C}^+, d\mu)}.
\]

Next, denote \( D_+ \) as \( D \) and let \( H_3 = H_4 = HD \). Since \( |D| \leq e^{\theta_0} \), the first condition in Lemma 3.4 could be easily verified. We now turn to the second condition. Since \( D' = iD\Phi''(\Phi')^{-1} \), and
\[
H_4' = H'D + HD'
= F'(\Phi)(\Phi')^{\frac{1}{2}}D + \frac{1}{2} F(\Phi)(\Phi')^{-\frac{1}{2}} \Phi''D + F(\Phi)(\Phi')^{\frac{1}{2}} \cdot iD\Phi''(\Phi')^{-1}
= F'(\Phi)(\Phi')^{\frac{1}{2}}D + \left( \frac{1}{2} + i \right) F(\Phi)(\Phi')^{-\frac{1}{2}} \Phi''D,
\]
we have, by \( H_3 = F(\Phi)(\Phi')^{\frac{1}{2}}D, \)
\[
|H_3H_4 y| \leq e^{2\theta_0} y |F(\Phi)(\Phi')^{\frac{1}{2}}| \cdot \left( |F'(\Phi)(\Phi')^{\frac{1}{2}}| + \frac{3}{2} |F(\Phi)(\Phi')^{-\frac{1}{2}} \Phi'| \right)
\leq e^{2\theta_0} y \cdot \frac{A|\Phi'|^2}{|\Phi|} \cdot \left( \frac{A|\Phi'|^2}{|\Phi|^2} + \frac{3A|\Phi''|}{2|\Phi| \cdot |\Phi'|^2} \right)
\leq \frac{e^{2\theta_0} A^2}{y|\Phi|^3} \left( (2d(\Phi))^2 + 9d(\Phi)|\Phi| \right)
\leq \frac{e^{2\theta_0} A^2}{y|\Phi|^3} \left( 13|\Phi| + 4a(0) \right) (|\Phi| + |a(0)|) \rightarrow 0,
\]

\[9\]
as $z = x + iy \to \infty$ with either $x \in \mathbb{R}$ or $y > 0$ fixed. Then,
\[
\int_{\mathbb{R}} |HD|^2 \, dx = 4 \int_{\mathbb{C}_+} |(HD)'|^2 \, d\mu(z),
\]
or equivalently,
\[
\|HD\|_{L^2(\mathbb{R}, dx)} = 2\|(HD)'\|_{L^2(\mathbb{C}_+, d\mu)},
\]
which finishes the equation part of the corollary.

Since $|D| \leq e^{\theta_0}$, it follows that,
\[
\|HD\|_{L^2(\mathbb{R}, dx)} \leq e^{\theta_0}\|H\|_{L^2(\mathbb{R}, dx)},
\]
and
\[
\|(HD)'\|_{L^2(\mathbb{C}_+, d\mu)} = \|HD' + H'D\|_{L^2(\mathbb{C}_+, d\mu)}
\geq \|HD'\|_{L^2(\mathbb{C}_+, d\mu)} - \|H'D\|_{L^2(\mathbb{C}_+, d\mu)}
\geq \|HD'\|_{L^2(\mathbb{C}_+, d\mu)} - e^{\theta_0}\|H'\|_{L^2(\mathbb{C}_+, d\mu)}
= \|HD'\|_{L^2(\mathbb{C}_+, d\mu)} - \frac{1}{2}e^{\theta_0}\|H\|_{L^2(\mathbb{R}, dx)},
\]
then
\[
e^{\theta_0}\|H\|_{L^2(\mathbb{R}, dx)} \geq 2\|HD'\|_{L^2(\mathbb{C}_+, d\mu)} - e^{\theta_0}\|H\|_{L^2(\mathbb{R}, dx)},
\]
or
\[
e^{\theta_0}\|H\|_{L^2(\mathbb{R}, dx)} \geq \|HD'\|_{L^2(\mathbb{C}_+, d\mu)},
\]
which proves the corollary. \(\square\)

**Corollary 3.11.** If $F \in \mathcal{T}(\Omega_-)$ is continuous to the boundary $\Gamma$, $D_-$ is defined as above, and let $H = F(\Phi_-(\Phi'_-))^\frac{1}{2}$, then
\[
\|H\|_{L^2(\mathbb{R}, dx)} = 2\|H'\|_{L^2(\mathbb{C}_-, d\mu)},
\]
and
\[
\|HD_\perp\|_{L^2(\mathbb{R}, dx)} = 2\|(HD)_\perp'\|_{L^2(\mathbb{C}_-, d\mu)}.
\]
Consequently,
\[
\|HD'_\perp\|_{L^2(\mathbb{R}, dx)} \leq e^{\theta_0}\|H\|_{L^2(\mathbb{R}, dx)}.
\]

## 4 Proof of the Main Theorems

Denote the measure $d(w) \, d\lambda(w)$ on $\mathbb{C}$ as $d\nu(w)$ and consider the function spaces $L^2(\Omega_{\pm}, d\nu)$, thus, if $F \in L^2(\Omega_{\pm}, d\nu)$, then
\[
\|F\|_{L^2(\Omega_{\pm}, d\nu)} = \int_{\Omega_{\pm}} |F|^2 \, d\nu(w) = \int_{\Omega_{\pm}} |F|^2 \, d\lambda(w) < \infty.
\]
Theorem 4.1. If $F \in \mathcal{T}(\Omega_+)$ and $\tau > 0$ is fixed, then
\[
\|F(\cdot + i\tau)\|_{L^2(\Gamma,|d\zeta|)} \leq C\sqrt{1 + M^2}\|F'\|_{L^2(\Omega_+,d\nu)}.
\]
If, furthermore, $F$ is continuous to the boundary, then we also have
\[
\|F\|_{L^2(\Gamma,|d\zeta|)} \leq C\sqrt{1 + M^2}\|F'\|_{L^2(\Omega_+,d\nu)}.
\]
Here, $C = 7e^{2\theta_0}$ and $M = \|a'\|_\infty$.

Proof. We first assume $F \in \mathcal{T}(\Omega_+)$ is continuous to the boundary and denote $\Phi_+$ as $\Phi$. Since $\Gamma = \{\zeta(u) = u + ia(u) : u \in \mathbb{R}\}$ is a Lipschitz curve and, by Lemma 3.8, $|\text{Im} \Phi'(z)| \leq M\text{Re} \Phi(z)$, we have
\[
\|F\|^2_{L^2(\Gamma,|d\zeta|)} = \int_{\Gamma} |F|^2|d\zeta| = \int_{\mathbb{R}} |F(\Phi)|^2|\Phi'| \, dx \\
\leq \sqrt{1 + M^2} \int_{\mathbb{R}} |F(\Phi)|^2 \text{Re} \Phi' \, dx \\
\leq \sqrt{1 + M^2} \left| \int_{\mathbb{R}} |F(\Phi)|^2 \Phi' \, dx \right|.
\]
Denote $\int_{\mathbb{R}} |F(\Phi)|^2|\Phi'| \, dx$ as $I_1$, $\sqrt{1 + M^2}$ as $M_1$, $F(\Phi)$ as $H$, then, by Corollary 3.6,
\[
I_1 \leq M_1 \left| \int_{\mathbb{C}_+} \Delta(|H|^2\Phi') \, d\mu(z) \right| \\
= 4M_1 \left| \int_{\mathbb{C}_+} (|H'|^2\Phi' + H\overline{H}\Phi'') \, d\mu(z) \right| \\
= 4M_1 \left( \int_{\mathbb{C}_+} |H'|^2\Phi' \, d\mu(z) + \int_{\mathbb{C}_+} |H\overline{H}\Phi''| \, d\mu(z) \right).
\]
We denote the first integral right above as $I_2$. It follows that, by Lemma 3.2,
\[
I_2 = \|H'(\Phi')^{1/2}\|^2_{L^2(\mathbb{C}_+,d\mu)} \\
= \int_{\mathbb{C}_+} |F'(\Phi)|^2 |\Phi'|^3 y \, d\lambda(z) \\
\leq 2 \int_{\mathbb{C}_+} |F'(\Phi)|^2 |\Phi'|^2 d(\Phi) \, d\lambda(z) \\
= 2 \int_{\Omega_+} |F'(w)|^2 d(w) \, d\lambda(w) \\
= 2 \|F'||^2_{L^2(\Omega_+,d\nu)}.
\]
Since $\Phi' = e^V$ and $\Phi'' = V'\Phi'$, Hölder’s inequality implies that
\[
\int_{\mathbb{C}_+} |H\overline{H}\Phi''| \, d\mu(z) = \int_{\mathbb{C}_+} |HV'(\Phi')^{1/2} \cdot \overline{H'}(\Phi')^{1/2}| \, d\mu(z) \\
\leq \|HV'(\Phi')^{1/2}\|_{L^2(\mathbb{C}_+,d\mu)} \cdot \|\overline{H'}(\Phi')^{1/2}\|_{L^2(\mathbb{C}_+,d\mu)} \\
= \|HV'(\Phi')^{1/2}\|_{L^2(\mathbb{C}_+,d\mu)} \cdot I_2^{1/2}.
\]
Notice that \( D = e^{i\mathbf{V}} \) and \(|V'| = |D'D^{-1}| \leq e^{\theta_0} |D'|\), then
\[
\|HV'(\Phi')^{\frac{1}{2}}\|_{L^2(C_+,d\mu)} \leq e^{\theta_0} \|H(\Phi')^{\frac{1}{2}}D'\|_{L^2(C_+,d\mu)}
\[
\leq e^{2\theta_0} \|H(\Phi')^{\frac{1}{2}}\|_{L^2(\mathbb{R},dx)}
\[
= e^{2\theta_0} I_1^{\frac{1}{2}},
\]
by Corollary 3.10, since \( H(\Phi')^{\frac{1}{2}} = F(\Phi')(\Phi')^{\frac{1}{2}} \).

Thus, inequality (7) becomes
\[
I_1 \leq 4M_1 (I_2 + I_2^{\frac{1}{2}} \cdot e^{2\theta_0} I_1^{\frac{1}{2}}),
\]
and we rewrite it as
\[
I_1 - 8M_1 e^{2\theta_0} I_1^{\frac{1}{2}} I_2^{\frac{1}{2}} \leq 8M_1 I_2 - I_1,
\]
or
\[
(I_1^{\frac{1}{2}} - 4M_1 e^{2\theta_0} I_2^{\frac{1}{2}})^2 \leq (8M_1 + 16M_1^2 e^{4\theta_0}) I_2 - I_1,
\]
and then
\[
I_1 \leq 8M_1 (1 + 2M_1 e^{4\theta_0}) I_2 \leq 24e^{4\theta_0} M_1^2 I_2,
\]
as \( M_1 e^{4\theta_0} = e^{4\theta_0} \sqrt{1 + M^2} \geq 1 \). Together with (8), we have
\[
\|F\|^2_{L^2(\Gamma,|d\zeta|)} = I_1 \leq 48e^{4\theta_0} M_1^2 \|F'\|^2_{L^2(\Omega_+,d\nu)},
\]
and
\[
\|F\|^2_{L^2(\Gamma,|d\zeta|)} \leq 7e^{2\theta_0} M_1 \|F'\|^2_{L^2(\Omega_+,d\nu)} = C \sqrt{1 + M^2} \|F'\|^2_{L^2(\Omega_+,d\nu)},
\]
where \( C = 7e^{2\theta_0} \).

For the general case of \( F \in \mathcal{T}(\Omega_+) \), fix \( \tau > 0 \) and define \( G(w) = F(w+i\tau) \) for \( w \in \Omega_+ \), then \( G \in \mathcal{T}(\Omega_+) \) and is continuous to the boundary. By what we have proved,
\[
\|F(-i\tau)\|^2_{L^2(\Gamma,|d\zeta|)} = |G|_{L^2(\Gamma,|d\zeta|)} \leq C \sqrt{1 + M^2} \|G'\|_{L^2(\Omega_+,d\nu)}.
\]
Since \( G'(w) = F'(w+i\tau) \) and \( d(w) \leq d(w+i\tau) \) for \( w \in \Omega_+ \),
\[
\|G'\|^2_{L^2(\Omega_+,d\nu)} = \int_{\Omega_+} |F'(w+i\tau)|^2 d(w) d\lambda(w)
\[
\leq \int_{\Omega_+} |F'(w+i\tau)|^2 d(w+i\tau) d\lambda(w)
\[
= \int_{\Omega_+ + i\tau} |F'(w)|^2 d(w) d\lambda(w)
\[
\leq \int_{\Omega_+} |F'(w)|^2 d(w) d\lambda(w)
\[
= \|F'\|^2_{L^2(\Omega_+,d\nu)},
\]

where \( \Omega_+ + i\tau = \{ w + i\tau : w \in \Omega_+ \} \subset \Omega_+ \) for \( \tau > 0 \). Thus
\[
\| F(\cdot + i\tau) \|_{L^2(\Gamma, |d\zeta|)} \leq C\sqrt{1 + M^2} \| F' \|_{L^2(\Omega_+, d\nu)},
\]
and the theorem is proved.

**Corollary 4.2.** If \( F \in T(\Omega_-) \) and \( \tau > 0 \) is fixed, then
\[
\| F(\cdot - i\tau) \|_{L^2(\Gamma, |d\zeta|)} \leq C\sqrt{1 + M^2} \| F' \|_{L^2(\Omega_-, d\nu)}.
\]
If, furthermore, \( F \) is continuous to the boundary, then we also have
\[
\| F \|_{L^2(\Gamma, |d\zeta|)} \leq C\sqrt{1 + M^2} \| F' \|_{L^2(\Omega_-, d\nu)}.
\]
Here, \( C = 7e^{2\theta_0} \) and \( M = \| a' \|_\infty \).

**Theorem 4.3.** Let \( f \in L^2(\Omega_+, d\nu) \) be compactly supported, and define, for \( w_2 \in \Omega_- \),
\[
Tf(w_2) = \iint_{\Omega_+} \frac{f(w_1)d(w_1)}{(w_1 - w_2)^2} \, d\lambda(w_1) = \iint_{\Omega_+} \frac{f(w_1) \, d\nu(w_1)}{(w_1 - w_2)^2}.
\]
Then, \( \| Tf \|_{L^2(\Gamma, |d\zeta|)} \leq C\sqrt{1 + M^2} \| f \|_{L^2(\Omega_+, d\nu)} \), where \( C = 56\pi e^{2\theta_0} \).

**Proof.** Suppose \( E = \text{supp} f \subset D(0, R) \), where \( R > 0 \). Since \( E \subset \Omega_+ \) is compact, \( Tf \) is holomorphic on a neighborhood of \( \Omega_- \), thus continuous to the boundary \( \Gamma \). If \( |w_2| > 2R \), then \( |w_2 - w_1| > \frac{1}{2}|w_2| \) for \( |w_1| < R \). Since \( d(w_1) \leq |w_2 - w_1| \) for \( w_1 \in \Omega_+ \) and \( w_2 \in \Omega_- \), we have, by Hölder’s inequality,
\[
|Tf(w_2)| \leq \frac{2}{|w_2|} \iint_E \frac{f(w_1)}{d(w_1)} \, d\lambda(w_1)
\leq \frac{2}{|w_2|} \left( \iint_E |f(w_1)|^2 \, d\lambda(w_1) \right)^{\frac{1}{2}} \left( \iint_E \frac{d\lambda(w_1)}{d(w_1)} \right)^{\frac{1}{2}}
= \frac{2}{|w_2|} \| f \|_{L^2(\Omega_+, d\nu)} \left( \iint_E \frac{d\lambda(w_1)}{d(w_1)} \right)^{\frac{1}{2}}
= \frac{2A}{|w_2|},
\]
where \( A = \| f \|_{L^2(\Omega_+, d\nu)} (\iint_E \frac{d\lambda(w_1)}{d(w_1)})^{\frac{1}{2}} \), and
\[
|(Tf)'(w_2)| = \left| \frac{2}{w_2} \iint_{\Omega_+} \frac{f(w_1)d(w_1)}{(w_1 - w_2)^3} \, d\lambda(w_1) \right|
\leq \frac{8}{|w_2|^2} \iint_E |f(w_1)| \, d\lambda(w_1)
= \frac{8A}{|w_2|^2}.
\]
Thus $Tf \in \mathcal{T}(\Omega_-)$ and by Corollary 4.2,
\[
\|Tf\|_{L^2(\Gamma,|d\zeta|)} \leq C \sqrt{1 + M^2}\|Tf\|\|\|_{L^2(\Omega_-,\text{div})},
\]  
(9)
where $C = 7e^{2b_0}$.

Define an operator $S: L^2(\Omega_+,d\lambda) \to L^2(\Omega_-,d\lambda)$ by
\[
SF(w_2) = d(w_2)^{\frac{1}{2}} \int_{\Omega_+} \frac{F(w_1)d(w_1)^{\frac{1}{2}}}{|w_1 - w_2|^3} d\lambda(w_1)
= \int_{\Omega_+} K(w_1,w_2)F(w_1) d\lambda(w_1),
\]
where $w_2 \in \Omega_-$ and $K(w_1,w_2) = d(w_1)^{\frac{1}{2}}d(w_2)^{\frac{1}{2}}|w_1 - w_2|^{-3}$. For $w_2 \in \Omega_-$ fixed, since $d(w_1) \leq |w_1 - w_2|$ and $\Omega_+ \subset \mathbb{C} \setminus D(w_2, d(w_2))$, then
\[
\int_{\Omega_+} K(w_1,w_2) d\lambda(w_1) \leq d(w_2)^{\frac{1}{2}}\int_{\mathbb{C}\setminus D(w_2, d(w_2))} |w_1 - w_2|^{-\frac{5}{2}} d\lambda(w_1)
= d(w_2)^{\frac{1}{2}}\int_0^{\infty} \int_{d(w_2)}^{2\pi} r^{-\frac{5}{2}} \cdot r \, dr \, d\theta
= 2\pi d(w_2)^{\frac{1}{2}} \int_{d(w_2)}^{\infty} r^{-\frac{3}{2}} dr
= 2\pi d(w_2)^{\frac{1}{2}} \cdot 2d(w_2)^{-\frac{1}{2}}
= 4\pi.
\]

The same computation yields that, for fixed $w_1 \in \Omega_+$,
\[
\int_{\Omega_+} K(w_1,w_2) d\lambda(w_2) \leq 4\pi.
\]

By Schur’s lemma [5], $S$ is a bounded operator from $L^2(\Omega_+,d\lambda)$ to $L^2(\Omega_-,d\lambda)$, and $\|S\| \leq 4\pi$.

If we let $F(w_1) = f(w_1)d(w_1)^{\frac{1}{2}}$, then
\[
\|SF\|_{L^2(\Omega_-,d\lambda)} \leq 4\pi\|F\|_{L^2(\Omega_-,d\lambda)} = 4\pi\|f\|_{L^2(\Omega_+,d\nu)},
\]
and
\[
\|(Tf)'(w_2)\| \leq 2\int_{\Omega_+} \frac{|f(w_1)|d(w_1)}{|w_1 - w_2|^3} d\lambda(w_1) = \frac{2(SF)(w_2)}{d(w_2)^{\frac{1}{2}}}.\]

It follows that,
\[
\|(Tf)'\|_{L^2(\Omega_-,d\nu)} \leq 2d^{-\frac{1}{2}}SF\|_{L^2(\Omega_-,d\nu)}
= 2\|SF\|_{L^2(\Omega_-,d\lambda)}
\leq 8\pi\|f\|_{L^2(\Omega_+,d\nu)}.
\]
thus, by (9),

\[
\| T f \|_{L^2(\Gamma, |d\zeta|)} \leq C \sqrt{1 + M^2} \cdot 8\pi \| f \|_{L^2(\Omega, d\nu)} \\
\leq C' \sqrt{1 + M^2} \| f \|_{L^2(\Omega, d\nu)},
\]

where \( C' = 56\pi e^{2\theta_0} \), and this proves the theorem.

\[ \square \]

**Corollary 4.4.** Let \( f \in L^2(\Omega, d\nu) \) be compactly supported, and define, for \( w_1 \in \overline{\Omega}_+ \),

\[
T_f(w_1) = \int_{\Omega_-} \frac{f(w_2) d(w_2)}{(w_2-w_1)^2} d\lambda = \int_{\Omega_+} \frac{f(w_2) d\nu(w_2)}{(w_2-w_1)^2}.
\]

Then, \( \| T f \|_{L^2(\Gamma, |d\zeta|)} \leq C \sqrt{1 + M^2} \| f \|_{L^2(\Omega, d\nu)} \), where \( C = 56\pi e^{2\theta_0} \).

For \( g(\zeta) \in L^2(\Gamma, |d\zeta|) \), we define the Cauchy integral, or Cauchy transform, of \( g \) on \( \Gamma \) as

\[
C g(w) = G(w) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(\zeta) d\zeta}{\zeta-w}, \quad \text{for } w \in \Omega_+,
\]

then \( G(w) \) is holomorphic on \( \Omega_+ \), and

\[
G'(w) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(\zeta) d\zeta}{(\zeta-w)^2}.
\]

The two estimates in Theorem 4.1 and Theorem 4.3 could now be combined to yield a proof of the following theorem, which shows the \( L^2 \) boundedness of Cauchy integrals on Lipschitz curves. The proof also implies that \( G \in T(\Omega_\pm) \).

**Theorem 4.5.** If \( g(\zeta) \in L^2(\Gamma, |d\zeta|) \) and \( G(w) \) the Cauchy integral of \( g \) on \( \Gamma \), then

\[
\sup_{\tau > 0} \| G(\cdot \pm i\tau) \|_{L^2(\Gamma, |d\zeta|)} \leq C(1 + M^2) \| g \|_{L^2(\Gamma, |d\zeta|)},
\]

where \( C = 196e^{4\theta_0} \).

**Proof.** We first assume \( g \) is compactly supported on \( \Gamma \), and suppose that \( E = \text{supp} g \subset D(0, R) \), where \( R > 0 \). If \( |w| > 2R \), then \( |w-\zeta| > \frac{1}{7} |w| \) for \( \zeta \in E \), and

\[
|G(w)| \leq \frac{1}{2\pi} \int_{E} \frac{|g(\zeta)|}{|\zeta - w|} |d\zeta|
\leq \frac{1}{\pi |w|} \int_{E} |g(\zeta)| |d\zeta|
\leq \frac{1}{\pi |w|} \left( \int_{E} |g(\zeta)|^2 |d\zeta| \right)^{\frac{1}{2}} \left( \int_{E} |d\zeta| \right)^{\frac{1}{2}}
\leq \frac{1}{\pi |w|} \| g \|_{L^2(\Gamma, |d\zeta|)} (2R \sqrt{1 + M^2})^{\frac{1}{2}}
= \frac{A}{|w|},
\]

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where \( A = \frac{1}{\pi} \| g \|_{L^2(\Gamma, |d\zeta|)} (2R\sqrt{1 + M^2})^{\frac{1}{2}} \). We also have

\[
|G'(w)| \leq \frac{1}{2\pi} \int_{\zeta} \frac{|g(\zeta)|}{|\zeta - w|^2} |d\zeta| \leq \frac{2}{\pi |w|^2} \int_{E} |g(\zeta)| |d\zeta| = \frac{2A}{|w|^2},
\]

then \( G \in \mathcal{T}(\Omega_+) \).

Next we will focus on the case of \( G \in \mathcal{T}(\Omega_+) \), and let

\[
B = \{ f \in L^2(\Omega_+, d\nu) : \| f \|_{L^2(\Omega_+, d\nu)} \leq 1, f \text{ is compactly supported in } \Omega_+ \},
\]

then

\[
\| G' \|_{L^2(\Omega_+, d\nu)} = \sup_{f \in B} \left| \int_{\Omega_+} G' d\nu \right|.
\]

Fix \( \tau > 0 \), by Theorem 4.1, Fubini’s theorem and Theorem 4.3, we obtain,

\[
\| G(\cdot + i\tau) \|_{L^2(\Gamma, |d\zeta|)} \\
\leq C_1 \left\| G' \right\|_{L^2(\Omega_+, d\nu)} \\
= \frac{C_1 \sqrt{1 + M^2}}{2\pi} \sup_{f \in B} \left| \int_{\Omega_+} \left( \int_{\Gamma} g(\zeta) \frac{d\zeta}{\zeta - w_1} \right) f(w_1) d\nu(w_1) \right| \\
= \frac{C_1 \sqrt{1 + M^2}}{2\pi} \sup_{f \in B} \left| \int_{\Gamma} g(\zeta) (T\mathcal{T})(\zeta) d\zeta \right| \\
\leq \frac{C_1 \sqrt{1 + M^2}}{2\pi} \sup_{f \in B} \left( \| g \|_{L^2(\Gamma, |d\zeta|)} \right) \left\| T\mathcal{T} \right\|_{L^2(\Gamma, |d\zeta|)} \| f \|_{L^2(\Omega_+, d\nu)} \\
\leq \frac{C_1 C_2 (1 + M^2)}{2\pi} \| g \|_{L^2(\Gamma, |d\zeta|)} \sup_{f \in B} \| f \|_{L^2(\Omega_+, d\nu)} \\
\leq C (1 + M^2) \| g \|_{L^2(\Gamma, |d\zeta|)},
\]

where \( C_1 = 7e^{2\theta_0}, C_2 = 56\pi e^{2\theta_0} \) and \( C = \frac{1}{\pi^2} C_1 C_2 = 196e^{4\theta_0} \).

In the general case, we let \( g_n(\zeta) = \chi_{D(0, n)} g(\zeta) \) for \( \zeta \in \Gamma \), where \( n > 0 \) and \( \chi \) is the characteristic function of a set, then \( g_n \) is compactly supported on \( \Gamma \), and \( \| g_n - g \|_{L^2(\Gamma, |d\zeta|)} \to 0 \) as \( n \to \infty \). For \( \tau > 0 \) and \( \zeta_0 \in \Gamma \) both fixed, let \( w_0 = \zeta_0 + i\tau \in \Omega_+ \). Denote the Cauchy integral of \( g_n \) as \( G_n \), then we have

\[
|G_n(w_0) - G(w_0)| \leq \frac{1}{2\pi} \int_{\Gamma} \frac{|g_n(\zeta) - g(\zeta)|}{\zeta - w_0} |d\zeta| \\
\leq \frac{1}{2\pi} \| g_n - g \|_{L^2(\Gamma, |d\zeta|)} \left( \int_{\Gamma} \frac{|d\zeta|}{|\zeta - w_0|^2} \right)^{\frac{1}{2}} \\
\to 0 \text{ as } n \to \infty,
\]

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thus, by Fatou’s lemma,

\[
\|G(\cdot + i\tau)\|_{L^2(\Gamma, |d\zeta|)} \leq \liminf_{n \to \infty} \|G_n(\cdot + i\tau)\|_{L^2(\Gamma, |d\zeta|)} \\
\leq \liminf_{n \to \infty} C(1 + M^2)\|g_n\|_{L^2(\Gamma, |d\zeta|)} \\
= C(1 + M^2)\|g\|_{L^2(\Gamma, |d\zeta|)},
\]

and the theorem follows.

The following lemma is proved in [2].

**Lemma 4.6.** If \( F(\zeta) \in L^p(\Gamma, |d\zeta|) \), and \( u_0 \) is the Lebesgue point of \( F(u + ia(u)) \) such that \( \zeta'(u_0) = |\zeta'(u_0)|e^{i\phi_0} \) exists, where \( \phi_0 \in (-\theta_0, \theta_0) \), then for any \( \phi \in (0, \frac{\pi}{2}) \), we have

\[
\lim_{z \to \zeta_0 \in \Omega_+ \cap \Omega_+^\pm} \int_{\Gamma} K_\pm(\zeta, \zeta_0)F(\zeta)\,d\zeta = F(\zeta_0).
\]

Now we could prove that \( L^2(\Gamma, |d\zeta|) \) is the sum of \( H^2(\Omega_\pm) \) in the non-tangential boundary limit sense.

**Corollary 4.7.** Every function in \( L^2(\Gamma, |d\zeta|) \) is (a.e. on \( \Gamma \)) the sum of the non-tangential boundary limit of two functions in \( H^2(\Omega_+) \) and \( H^2(\Omega_-) \), respectively, or we could simply write

\[
L^2(\Gamma, |d\zeta|) = H^2(\Omega_+) + H^2(\Omega_-).
\]

**Proof.** For \( g \in L^2(\Gamma, |d\zeta|) \), let

\[
G_1(w_1) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(\zeta)\,d\zeta}{\zeta - w_1}, \quad \text{for } w_1 \in \Omega_+,
\]

and

\[
G_2(w_2) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(\zeta)\,d\zeta}{\zeta - w_2}, \quad \text{for } w_2 \in \Omega_-,
\]

then both \( G_1(w_1) \) and \( G_2(w_2) \) are analytic [2]. By Theorem 4.5, there exists constant \( C \), such that

\[
\sup_{\tau > 0} \|G_i(\cdot + i\tau)\|_{L^2(\Gamma, |d\zeta|)} \leq C(1 + M^2)\|g\|_{L^2(\Gamma, |d\zeta|)}, \quad \text{for } i = 1, 2.
\]

It means that \( G_1 \in H^2(\Omega_+) \) and \( G_2 \in H^2(\Omega_-) \), thus both of them have non-tangential boundary limit a.e. on \( \Gamma \). We still denote the limit functions as \( G_1 \) and \( G_2 \), respectively.

Now suppose \( u_0 \) is the Lebesgue point of \( g(\zeta(u)) \) and \( \zeta'(u_0) \) exists. Let \( \zeta_0 = \zeta(u_0) \), \( w_1 = \zeta_0 + z \) and \( w_2 = \zeta_0 - z \), where \( z \in \mathbb{C} \) and \( |z| \) is sufficiently small such that \( w_1 \in \Omega_+ \) and \( w_2 \in \Omega_- \), then

\[
G_1(w_1) - G_2(w_2) = G_1(\zeta_0 + z) - G_2(\zeta_0 - z) \\
= \frac{1}{2\pi i} \int_{\Gamma} \left( \frac{g(\zeta)}{\zeta - (\zeta_0 + z)} - \frac{g(\zeta)}{\zeta - (\zeta_0 - z)} \right)\,d\zeta \\
= \int_{\Gamma} K_\pm(\zeta, \zeta_0)g(\zeta)\,d\zeta.
\]
Lemma 4.6 implies that
\[ \lim_{z \to 0} |g(\zeta_0) - (G_1(w_1) - G_2(w_2))| = 0, \]
and \( g(\zeta_0) = G_1(\zeta_0) - G_2(\zeta_0) \) follows. Thus the corollary is proved.

5 The special case of “\( M = 0 \)”

In this section, we will obtain a more accurate upper bound of the norm of Cauchy transform under the assumption that \( \|a'\|_\infty = M = 0 \). Notice that in this case, we have
\[
d\mu(z) = |y|d\lambda(z) = d(z)d\lambda(z) = d\nu(z),
\]

\( \Gamma = \mathbb{R} \) and \( \Omega_\pm = \mathbb{C}_\pm \).

**Theorem 5.1.** If \( F \in T(\mathbb{C}_+) \) and \( \tau > 0 \) is fixed, then
\[
\|F(\cdot + i\tau)\|_{L^2(\mathbb{R},dx)} \leq 2\|F'\|_{L^2(\mathbb{C}_+,d\mu)}.
\]

If, furthermore, \( F \) is continuous to the boundary, then we also have
\[
\|F\|_{L^2(\mathbb{R},dx)} = 2\|F'\|_{L^2(\mathbb{C}_+,d\mu)}.
\]

**Proof.** The continuous case is just Corollary 3.6, since now \( \Phi_+(z) = z \) and \( \Delta(|F|^2) = 4|F'|^2 \). The general case is proved in the same way as in Theorem 4.1.

**Corollary 5.2.** If \( F \in T(\mathbb{C}_-) \) and \( \tau > 0 \) is fixed, then
\[
\|F(\cdot - i\tau)\|_{L^2(\mathbb{R},dx)} \leq 2\|F'\|_{L^2(\mathbb{C}_-,d\mu)}.
\]

If, furthermore, \( F \) is continuous to the boundary, then we also have
\[
\|F\|_{L^2(\mathbb{R},dx)} = 2\|F'\|_{L^2(\mathbb{C}_-,d\mu)}.
\]

The “\( M = 0 \)” version of Theorem 4.3 is the following theorem.

**Theorem 5.3.** Let \( f \in L^2(\mathbb{C}_+,d\mu) \) be compactly supported, and define, for \( z_2 \in \mathbb{C}_- \),
\[
Tf(z_2) = \iint_{\mathbb{C}_+} \frac{f(z_1)d(z_1)}{(z_1 - z_2)^2} \, d\lambda(z_1) = \iint_{\mathbb{C}_+} \frac{f(z_1) \, d\mu(z_1)}{(z_1 - z_2)^2}.
\]

Then, \( \|Tf\|_{L^2(\Gamma,d\zeta)} \leq 4\pi \|f\|_{L^2(\mathbb{C}_+,d\mu)} \).
Proof. We could still verify that $Tf \in T(\mathbb{C}_-) $ and is continuous to the boundary $\mathbb{R}$, then, by Corollary 5.2,

$$\|Tf\|_{L^2(\mathbb{R}dx)} = 2\|(Tf)'\|_{L^2(\mathbb{C}_-,d\nu)}.$$ 

Define an operator $S: L^2(\mathbb{C}_+,d\lambda) \to L^2(\mathbb{C}_-,d\lambda)$ by

$$SF(z_2) = d(z_2)^\frac{1}{2} \int_{\mathbb{C}_+} \frac{F(z_1) d(z_1)^\frac{1}{2}}{|z_1 - z_2|^3} d\lambda(z_1)$$

$$= \int_{\mathbb{C}_+} K(z_1, z_2) F(z_1) d\lambda(z_1),$$

where $z_1 \in \mathbb{C}_+, \ z_2 \in \mathbb{C}_-, \text{ and } K(z_1, z_2) = d(z_1)^\frac{1}{2} d(z_2)^\frac{1}{2} |z_1 - z_2|^{-3}$. Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, we have $d(z_1) = y_1$ and $d(z_2) = |y_2|$ by the definition of $d(z)$. Then fix $z_2$,

$$\int_{\mathbb{C}_+} K(z_1, z_2) d\lambda(z_1) = \int_{\mathbb{C}_+} \frac{|y_2|^\frac{1}{2} |y_1|^\frac{1}{2}}{\langle x_1 - x_2 \rangle^2 + \langle y_1 - y_2 \rangle^2} d\lambda(z_1)$$

$$= \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{dx_1}{\langle x_1 - x_2 \rangle^2 + \langle y_1 - y_2 \rangle^2} |y_2|^\frac{1}{2} |y_1|^\frac{1}{2} dy_1$$

$$= \int_0^{+\infty} \frac{|y_2|^\frac{1}{2} |y_1|^\frac{1}{2}}{\langle y_1 - y_2 \rangle^2} \int_{-\infty}^{+\infty} \frac{dt}{(t^2 + 1)^3}$$

$$= \frac{|y_2|^\frac{1}{2} |y_1|^\frac{1}{2}}{|y_2|^2} \int_0^{+\infty} \frac{t^{\frac{1}{2}} dt}{(t + 1)^2} \cdot 2 \int_0^{+\infty} \frac{t^{-\frac{1}{2}} dt}{(t + 1)^{\frac{3}{2}}}$$

$$= \int_0^{+\infty} \frac{t^{\frac{1}{2}} dt}{(t + 1)^2} \cdot \int_0^{+\infty} \frac{t^{-\frac{1}{2}} dt}{(t + 1)^{\frac{3}{2}}}.$$

Let $s = \frac{t}{1+t}$ for $t \in (0, +\infty)$, then $s \in (0, 1),

$$t = \frac{s}{1-s}, \ t+1 = \frac{1}{1-s}, \text{ and } dt = \frac{ds}{(1-s)^2},$$

By invoking the Euler’s Gamma function $\Gamma(\cdot)$ and Beta function $B(\cdot)$, we have

$$\int_0^{+\infty} \frac{t^{\frac{1}{2}} dt}{(t + 1)^2} = \int_0^1 \frac{s^{\frac{1}{2}} (1-s)^{-\frac{1}{2}}}{(1-s)^{-2}} \cdot (1-s)^{-2} ds$$

$$= \int_0^1 s^{\frac{1}{2}} (1-s)^{-\frac{1}{2}} ds$$

$$= B\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2\right)}$$

$$= \frac{1}{2} \Gamma\left(\frac{1}{2}\right)^2 = \frac{\pi}{2},$$

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and

\[
\int_{0}^{+\infty} \frac{t^{-\frac{3}{2}} \, dt}{(t + 1)\frac{3}{2}} = \int_{0}^{1} \frac{s^{-\frac{1}{2}} (1 - s)^{\frac{3}{2}}}{(1 - s)^{-\frac{3}{2}}} \cdot (1 - s)^{-2} \, ds \\
= \int_{0}^{1} s^{-\frac{1}{2}} \, ds \\
= 2,
\]

then

\[
\iint_{C_+} K(z_1, z_2) \, d\lambda(z_1) = \frac{\pi}{2} \cdot 2 = \pi.
\]

The same computation yields that, for fixed \( z_1 \in C_+ \),

\[
\iint_{C_-} K(z_1, z_2) \, d\lambda(z_2) = \pi.
\]

By Schur’s lemma, \( S \) is a bounded operator from \( L^2(C_+, d\lambda) \) to \( L^2(C_-, d\lambda) \), and \( \|S\| \leq \pi \).

Let \( F(z_1) = f(z_1) d(z_1)^{\frac{3}{2}} \), then

\[
\|SF\|_{L^2(C_-, d\lambda)} \leq \pi \|F\|_{L^2(C_+, d\lambda)} = \pi \|f\|_{L^2(C_+, d\mu)},
\]

and

\[
|(Tf)'(z_2)| \leq 2 \iint_{C_+} \frac{|f(z_1)| d(z_1)}{|z_1 - z_2|^3} \, d\lambda(z_1) = \frac{2(SF)(z_2)}{d(z_2)^{\frac{3}{2}}},
\]

which follows that,

\[
\|(Tf)'\|_{L^2(C_-, d\mu)} \leq 2 \|d^{-\frac{1}{2}} SF\|_{L^2(C_-, d\mu)} \\
= 2 \|SF\|_{L^2(C_-, d\lambda)} \\
\leq 2 \pi \|f\|_{L^2(C_+, d\mu)},
\]

Remember that \( \|Tf\|_{L^2(\mathbb{R}, dx)} = 2 \|(Tf)'\|_{L^2(C_-, d\mu)} \), then

\[
\|Tf\|_{L^2(\mathbb{R}, dx)} \leq 4 \pi \|f\|_{L^2(C_+, d\mu)},
\]

and the theorem is proved. \( \square \)

**Corollary 5.4.** Let \( f \in L^2(C_-, d\mu) \) be compactly supported, and define, for \( z_1 \in \overline{C_+} \),

\[
Tf(z_1) = \iint_{C_+} \frac{f(z_2) d(z_2)}{(z_2 - z_1)^2} \, d\lambda(z_2) = \iint_{C_+} \frac{f(z_2) d\mu(z_2)}{(z_2 - z_1)^2}.
\]

Then, \( \|Tf\|_{L^2(\Gamma, |d\zeta|)} \leq 4 \pi \|f\|_{L^2(C_+, d\mu)} \).

Now we could proof the boundedness of Cauchy integral on \( \mathbb{R} \), which is a special case of Theorem 4.5.
Theorem 5.5. If \( g(x) \in L^2(\mathbb{R}, dx) \) and \( G(z) \) the Cauchy integral of \( g \) on \( \mathbb{R} \), that is
\[
G(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(t)\,dt}{t - z}, \quad \text{for } z \in \mathbb{C}_\pm
\]
then
\[
\sup_{\tau > 0} \|G(\cdot \pm i\tau)\|_{L^2(\mathbb{R}, dx)} \leq 4\|g\|_{L^2(\mathbb{R}, dx)}.
\]

Proof. We will prove the theorem while supposing that \( g \) is compactly supported on \( \mathbb{R} \), and omit the proof of the general case, which could be treated by the same method as in Theorem 4.5. It has been proved in that theorem that \( G \in \mathcal{T}(\mathbb{C}_\pm) \) if \( g \) is non-zero on a compact interval of \( \mathbb{R} \). We suppose that \( G \in \mathcal{T}(\mathbb{C}_+) \), and let
\[
B = \{ f \in L^2(\mathbb{C}_+, d\mu) : \|f\|_{L^2(\mathbb{C}_+, d\mu)} \leq 1, f \text{ is compactly supported in } \mathbb{C}_+ \},
\]
then
\[
\|G'\|_{L^2(\mathbb{C}_+, d\mu)} = \sup_{f \in B} \left| \iint_{\mathbb{C}_+} G' f \, d\mu \right|.
\]
Fix \( \tau > 0 \), by Theorem 5.1, Fubini’s theorem and Theorem 5.3, we have,
\[
\|G(\cdot + i\tau)\|_{L^2(\mathbb{R}, dx)}
\]
\[
\begin{align*}
&= 2\|G'\|_{L^2(\mathbb{C}_+, d\mu)} \\
&= \frac{1}{\pi} \sup_{f \in B} \left| \iint_{\mathbb{C}_+} \left( \int_{\mathbb{R}} \frac{g(t)\,dt}{(t - z_1)^2} \right) f(z_1) d\lambda(z_1) \right| \\
&= \frac{1}{\pi} \sup_{f \in B} \left| \int_{\mathbb{R}} g(t)(Tf)(t) \, dt \right| \\
&\leq \frac{1}{\pi} \left( \|g\|_{L^2(\mathbb{R}, dx)} \|Tf\|_{L^2(\mathbb{R}, dx)} \right) \\
&\leq \frac{4\pi}{\pi} \|g\|_{L^2(\mathbb{R}, dx)} \sup_{f \in B} \|f\|_{L^2(\mathbb{C}_+, d\mu)} \\
&\leq 4\|g\|_{L^2(\mathbb{R}, dx)},
\end{align*}
\]
and this proves the theorem.

Funding
This work is supported by National Natural Science Foundation of China(Grant No. 11271045).

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