This paper is dedicated to Professor Vladimir Gutlyanskii on the occasion of his 75-th anniversary.

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ON QUASINEARLY SUBHARMONIC FUNCTIONS

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ABSTRACT. We recall the definition of quasinearly subharmonic functions, point out that this function class includes, among others, subharmonic functions, quasisubharmonic functions, nearly subharmonic functions and essentially almost subharmonic functions. It is shown that the sum of two quasinearly subharmonic functions may not be quasinearly subharmonic. Moreover, we characterize the harmonicity via quasinearly subharmonicity.

1. SUBHARMONIC FUNCTIONS AND NEARLY SUBHARMONIC FUNCTIONS.

Denote by $\mathbb{R}^N$ the $N$-dimensional Euclidean space. If $x \in \mathbb{R}^N$, then the open ball centered at $x$ with radius $r > 0$ will be denoted by $B^N(x, r)$ and we will write $\overline{B^N(x, r)}$ for the closure of this ball.

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Let $D$ be a domain in $\mathbb{R}^N$, $N \geq 2$. An upper semicontinuous function $u : D \to [−\infty, +\infty)$ is subharmonic if the inequality

$$u(x) \leq \frac{1}{\nu_N r^N} \int_{B^N(x,r)} u(y) \, dm_N(y)$$

holds for all $B^N(x,r) \subset D$, where $\nu_N$ is the volume of the unit ball in $\mathbb{R}^N$.

The function $u \equiv −\infty$ is considered subharmonic. A function $u$ defined on an open set $\Omega \subseteq \mathbb{R}^N$ is subharmonic if the restriction of $u$ to arbitrary connected component of $\Omega$ is subharmonic.

**Definition 1.** A function $u : D \to [−\infty, +\infty)$ is nearly subharmonic, if $u$ is Lebesgue measurable, $u^+ \in L^1_{\text{loc}}(D)$ and

$$u(x) \leq \frac{1}{\nu_N r^N} \int_{B^N(x,r)} u(y) \, dm_N(y)$$

holds for all $B^N(x,r) \subset D$.

Observe that our definition is slightly nonstandard because in the standard definition of nearly subharmonic functions one uses the stronger assumption $u \in L^1_{\text{loc}}(D)$, see e.g. [16], p. 14.

The following lemma is an analog of Proposition 2.2 (vii) from [36], p. 55, and Proposition 1.5.2 (vii) from [39], p. e2615.

**Lemma 1.** Let $D$ be a domain in $\mathbb{R}^N$, $N \geq 2$, and let $u : D \to [−\infty, +\infty)$ be nearly subharmonic in the sense of Definition 1. Then either $u \in L^1_{\text{loc}}(D)$ or the equality $u(x) = −\infty$ holds for every $x \in D$.

**Proof.** Suppose $u \notin L^1_{\text{loc}}(D)$. Then there is a compact set $K \subset D$ such that

$$\int_K u(y) \, dm_N(y) = −\infty.$$  

Since $K$ is compact and $D$ is open, we have

$$\text{dist}(K, \partial D) = \inf_{x \in K, y \in \partial D} |x - y| > 0.$$  

Let $\varepsilon$ be a positive real number satisfying the inequality

$$3\varepsilon < \text{dist}(K, \partial D).$$

We can find a finite set of balls $B^N(x_1, \varepsilon), \ldots, B^N(x_m, \varepsilon)$ such that $x_i \in K$ for every $i \in \{1, \ldots, m\}$ and

$$K \subseteq \bigcup_{i=1}^m B^N(x_i, \varepsilon) \subseteq D.$$  

These inclusions and (2) imply

$$\int_{B^N(x_{i_0}, \varepsilon)} u(y) \, dm_N(y) = −\infty$$
for some $i_0 \in \{1, \ldots, m\}$. It follows from (3) and $x_{i_0} \in K$ that
\[ B_N(x_{i_0}, \varepsilon) \subseteq B_N(x, 2\varepsilon) \subseteq D \]
holds for every $x \in B_N(x_{i_0}, \varepsilon)$. Using (4) we obtain
\[ \int_{B_N(x, 2\varepsilon)} u(y) \, dm_N(y) = -\infty \]
for every $x \in B_N(x_{i_0}, \varepsilon)$. Since $u$ is nearly subharmonic, it follows that
\[ -\infty \leq u(x) \leq 1 - \nu_N(2\varepsilon)^N \int_{B_N(x, 2\varepsilon)} u(y) \, dm_N(y) = -\infty, \]
i.e., $u(x) = -\infty$ for every $x \in B_N(x_{i_0}, \varepsilon)$. Write
\[ A = \{ x \in D : u(x) = -\infty \}. \]
Since $B_N(x_{i_0}, \varepsilon) \subseteq A$, the interior of $A$ is non-void, $\text{Int}(A) \neq \emptyset$. To complete
the proof, it is sufficient to show that $\text{Int}(A) = D$. If the last equality does
not hold, then there is a point $y^* \in D \cap \partial \text{Int}(A)$. Let $0 < \delta^* < \frac{1}{2} \text{dist}(y^*, \partial D)$. Then for every $y \in B_N(y^*, \delta^*)$ we have
\[ D \supseteq B_N(y, 2\delta^*) \]
and $B_N(y, 2\delta^*) \cap \text{Int}(A) \neq \emptyset$. Consequently $u(y) = -\infty$ holds for every $y \in B_N(y^*, \delta^*)$. Thus $y^* \in \text{Int}(A)$, contrary to $y^* \in \partial \text{Int}(A)$. □

The following proposition is well known under the additional condition $u \in L^1_{\text{loc}}(D)$.

**Proposition 1.** Let $D$ be a domain in $\mathbb{R}^N$, $N \geq 2$ and let $u : D \to [-\infty, +\infty)$
be Lebesgue measurable. Then $u$ is nearly subharmonic in $D$ if and only if there
exists a subharmonic in $D$ function $u^*$ such that $u^*(x) \geq u(x)$ for all $x \in D$
and $u^*(x) = u(x)$ holds Lebesgue almost everywhere.

**Proof.** If $u(x) \equiv -\infty$ then, the proposition is evident. In the opposite case
by Lemma 1 we have $u \in L^1_{\text{loc}}(D)$, and it is a reformulation of Theorem 1
from [16], p. 14. □

**Remark 1.** In particular, if $u$ is nearly subharmonic, then we can take $u^*$ as
the lowest upper semicontinuous majorant of $u$: $u^*(x) = \limsup_{x' \to x} u(x')$.

Observe also that the *almost subharmonic functions*, by Szpirajn [17], are
included in Definition 1 in the following sense. Let $u : D \to [-\infty, +\infty)$ be
almost subharmonic, that is, $u \in L^1_{\text{loc}}(D)$ and inequality (1) is satisfied for
Lebesgue almost every $x \in D$ with all $B_N(x, r) \subseteq D$. Let
\[ D_1 := \{ x \in D : u(x) \leq \frac{1}{\nu_N r^N} \int_{B_N(x,r)} u(y) \, dm_N(y) \text{ for all } B_N(x,r) \subseteq D \}. \]
Define $\tilde{u} : D \to [-\infty, +\infty)$ as

$$\tilde{u}(x) := \begin{cases} u(x), & \text{when } x \in D_1, \\ -\infty, & \text{when } x \in D \setminus D_1. \end{cases}$$

By assumption $m_N(D \setminus D_1) = 0$, it is easy to see that $\tilde{u}$ is nearly subharmonic in $D$.

In the connection with almost subharmonic functions see also [3] and [26], p. 20, and [20], p. 238. Lieb and Loss even call the almost subharmonic functions briefly subharmonic functions.

2. QUASINEARLY SUBHARMONIC FUNCTIONS

Let $D$ be a domain in $\mathbb{R}^N$, $N \geq 2$. It is an important fact that if $u : D \to [0, +\infty)$ is subharmonic and $p > 0$, then there exists a constant $K = K(N, p) > 0$ such that the inequality

$$u(x)^p \leq \frac{K}{v_N r^N} \int_{B^N(x, r)} u(y)^p \, dm_N(y)$$

holds for all $B^N(x, r) \subset D$. In the case of $p = 1$ and $K = 1$, inequality (5) is just the familiar mean value inequality for (nonnegative) subharmonic functions. The case $p > 1$ follows from the the case $p = 1$ with the aid of Jensen’s inequality. The case $0 < p < 1$ has been given in Fefferman and Stein [12], Lemma 2, p. 172 and in [19], Theorem 1, p. 529, where, however, only absolute values of harmonic functions were considered. The proofs in [12] and in [19] are somewhat long. See also [13], Lemma 3.7, p. 116, and [2], (1.5), p. 210. In [27], Lemma, p. 69, it was pointed out that the proof in [12] includes the case of nonnegative subharmonic functions, too. See also [43], p. 271, [46], p. 114, [15], Lemma 1, p. 113, [44], Lemma 3, p. 305, [41], p. 794, [43], Lemma 1, p. 363, [44], Lemma 2.1, p. 7, [3], Theorem A, p. 413, and [1], p. 132. Observe that a possibility for an essentially different proof was pointed out already in [48], pp. 188-190. Later other different proofs were given in [23], p. 18 and Theorem 1, p. 19 (see also [24], Theorem A, p. 15), [28], pp. 233-234, [29], p. 188. The results in [23], [28] and [29] hold in fact for more general function classes than just for nonnegative subharmonic functions. Compare also [4], [7], p. 430, and [8], p. 485.

Inequality (5) has many applications. Among others, it has been applied to the weighted boundary behavior of subharmonic functions and to the nonintegrability of subharmonic or superharmonic functions.

It is therefore relevant to find a generalization of results related to inequality (5). We will do this in the following way.

Let $D$ be a domain in $\mathbb{R}^N$, $N \geq 2$. For every $u : D \to [-\infty, +\infty)$ and $M \geq 0$ we write $u_M := \max\{u, -M\} + M$. 

**Definition 2.** Let $K \in [1, +\infty)$. A Lebesgue measurable function $u : D \to [-\infty, +\infty)$ is $K$-quasinearly subharmonic, if $u^+ \in L^1_{\text{loc}}(D)$ and the inequality

$$u_M(x) \leq \frac{K}{v_N r^N} \int_{B^N(x,r)} u(y) \, dm_N(y)$$

holds for all $M \geq 0$ and $B^N(x,r) \subset D$. A function $u : D \to [-\infty, +\infty)$ is quasinearly subharmonic, if $u$ is $K$-quasinearly subharmonic for some $K$.

In addition to the above defined class of quasinearly subharmonic functions, we will consider their proper subclass.

**Definition 3.** A Lebesgue measurable function $u : D \to [-\infty, +\infty)$ is $K$-quasinearly subharmonic n.s. (in the narrow sense), if $u^+ \in L^1_{\text{loc}}(D)$ and if there is a constant $K = K(N,u,D) \geq 1$ such that the inequality

$$u(x) \leq \frac{K}{v_N r^N} \int_{B^N(x,r)} u(y) \, dm_N(y)$$

holds for all $B^N(x,r) \subset D$. A function $u : D \to [-\infty, +\infty)$ is quasinearly subharmonic n.s., if $u$ is $K$-quasinearly subharmonic n.s. in $D$ for some $K$.

For a function $u$ is defined on an open set $\Omega \subseteq \mathbb{R}^n$, the quasinearly subharmonicity (quasinearly subharmonicity n.s.) of $u$ means that the restriction of $u$ to arbitrary connected component of $\Omega$ is quasinearly subharmonic (quasinearly subharmonic n.s.).

Observe that if $u : D \to [0, +\infty)$ is subharmonic and $p > 0$, then $u^p$ is quasinearly subharmonic n.s. and thus also quasinearly subharmonic, see statement (1) and statement (4) of Proposition 2 below and also [11].

More generally, the class of quasinearly subharmonic functions includes, among others the subharmonic and nearly subharmonic functions and also the quasisubharmonic functions (for the definition of this see [36] and [16]), also functions satisfying certain natural growth conditions, especially certain eigenfunctions, polyharmonic functions, subsolutions of certain general elliptic equations.

Let $D$ be a domain in $\mathbb{R}^N$, $N \geq 2$. Recall that a continuous function $u : D \to [0, +\infty)$ is said to be a Harnack function if there are $\lambda \in (0, 1)$ and $C_\lambda \in [1, +\infty)$ such that the following Harnack inequality

$$\max_{z \in B^N(x,\lambda r)} u(z) \leq C_\lambda \min_{z \in B^N(x,\lambda r)} u(z)$$

holds whenever $B^N(x,r) \subseteq D$. See [49], p. 259. Every Harnack function is quasinearly subharmonic. This implies the quasinearly subharmonicity of nonnegative harmonic functions as well as nonnegative solutions of some elliptic equations. In particular, the partial differential equations associated with quasiregular mappings belong to this family of elliptic equations, see Vuorinen [49] and the above references.
Observe that already Domar [7] has pointed out the relevance of the class of (nonnegative) quasinearly subharmonic functions. For, at least partly, more general function class, see [8].

We list below four simple examples of quasinearly subharmonic functions.

**Example 1.** Let \( D \) be a domain in \( \mathbb{R}^N \), \( N \geq 2 \). Any Lebesgue measurable function \( u : D \to [m, M] \), where \( 0 < m \leq M < +\infty \), is quasinearly subharmonic n.s. and, because of Proposition 2 (see below), also quasinearly subharmonic. If \( u \) is moreover continuous, then \( u \) is a Harnack function.

**Example 2.** The function \( u : \mathbb{R}^2 \to \mathbb{R} \)

\[
  u(x, y) := \begin{cases} 
-1, & \text{when } y < 0 \\
1, & \text{when } y \geq 0 
\end{cases}
\]

is 2-quasinearly subharmonic, but not quasinearly subharmonic n.s..

**Example 3.** Let \( D = (0, 2) \times (0, 1) \subset \mathbb{R}^2 \) and let \( c < 0 \) be arbitrary. Let \( E \subset D \) be a Borel set of zero Lebesgue measure. Let \( u : D \to [-\infty, +\infty) \)

\[
  u(x, y) := \begin{cases} 
  c, & \text{when } (x, y) \in E, \\
  1, & \text{when } (x, y) \in D \setminus E \text{ and } 0 < x < 1, \\
  2, & \text{when } (x, y) \in D \setminus E \text{ and } 1 \leq x < 2. 
\end{cases}
\]

The function \( u \) attains both negative and positive values, is 2-quasinearly subharmonic n.s., but not nearly subharmonic.

**Example 4.** Let \( D \) be a domain in \( \mathbb{R}^N \), \( N \geq 2 \), and let \( u : D \to [-\infty, +\infty) \) be a quasinearly subharmonic function (resp. quasinearly subharmonic n.s.). Let \( E \subset D \) be a Borel set of zero Lebesgue measure. Let \( v : D \to [-\infty, +\infty) \)

\[
  v(x) := \begin{cases} 
  -\infty, & \text{when } x \in E, \\
  u(x), & \text{when } x \in D \setminus E. 
\end{cases}
\]

The function \( v \) is quasinearly subharmonic (resp. quasinearly subharmonic n.s.).

The term quasinearly subharmonic function was first introduced in [30]. Quasinearly subharmonic functions (sometimes with a different terminology), or, essentially, perhaps just functions satisfying a certain generalized mean value inequality, have previously been considered, or used, in addition to the above listed references at least in [22, 30, 31, 32, 34, 35, 5, 17, 37, 38, 9, 10, 18] and [21].

3. Basic properties of quasinearly subharmonic functions

Recall that a function \( \varphi : [0, +\infty) \to [0, +\infty) \) satisfies a \( \Delta_2 \)-condition, if there is a constant \( C = C(\varphi) \geq 1 \) such that \( \varphi(2t) \leq C \varphi(t) \) for all \( t \in [0, +\infty) \).
Definition 4. A function $\psi : [0, +\infty) \to [0, +\infty)$ is permissible, if there exist an increasing (strictly or not), convex function $\psi_1 : [0, +\infty) \to [0, +\infty)$ and a strictly increasing surjection $\psi_2 : [0, +\infty) \to [0, +\infty)$ such that $\psi = \psi_2 \circ \psi_1$ and the following conditions hold:

(a) $\psi_1$ satisfies the $\Delta_2$-condition,
(b) $\psi^{-1}_2$ satisfies the $\Delta_2$-condition,
(c) the function $t \mapsto \frac{\psi_2(t)}{t}$ is quasi-decreasing, i.e. there is a constant $C = C(\psi_2) > 0$ such that

$$\frac{\psi_2(s)}{s} \geq C \frac{\psi_2(t)}{t}$$

whenever $0 < s \leq t$.

Permissible functions are necessarily continuous.

Examples of permissible functions are: $\psi_1(t) = t^p$, $p > 0$, and $\psi_2(t) = ct^{p\alpha}[\log(\delta + t^{p\gamma})]^{\beta}$, $c > 0$, $0 < \alpha < 1$, $\delta, \gamma \in \mathbb{R}$ such that $0 < \alpha + \beta \gamma < 1$, and $p \geq 1$. And also functions of the form $\psi_3 = \phi \circ \varphi$, where $\phi : [0, +\infty) \to [0, +\infty)$ is a concave surjection whose inverse $\phi^{-1}$ satisfies the $\Delta_2$-condition and $\varphi : [0, +\infty) \to [0, +\infty)$ is an increasing, convex function satisfying the $\Delta_2$-condition.

It is interesting to note the following fact, see [25], Lemma 1 and Remark 1, p. 93:

Let $\psi : [0, +\infty) \to [0, +\infty)$ be a permissible function. Then

(1) there are a number $p > 0$ and a convex function $M : [0, +\infty) \to [0, +\infty)$ satisfying the $\Delta_2$-condition such that $\psi(t) \asymp g(t^p)$, that is, there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 \leq \frac{\psi(t)}{g(t^p)} \leq C_2$$

for all $t > 0$;

(2) there are a number $p > 0$ and a convex function $\vartheta : [0, +\infty) \to [0, +\infty)$ satisfying the $\Delta_2$-condition such that $\psi(t) \asymp \vartheta(t^p)$.

Next we list certain basic properties of quasinearly subharmonic functions, see [36], Proposition 2.1 and Proposition 2.2 and [39], Proposition 1.5.1.

Proposition 2. Let $D$ be a domain in $\mathbb{R}^N$, $N \geq 2$.

(1) If $u : D \to [0, +\infty)$ is Lebesgue measurable and $u^+ \in L^1_{\text{loc}}(D)$, then $u$ is $K$-quasinearly subharmonic if and only if $u$ is $K$-quasinearly subharmonic n.s., that is, if

$$u(x) \leq \frac{K}{r_N^{1-N}} \int_{B^N(x,r)} u(y) \, dm_N(y)$$

holds for all $B^N(x,r) \subset D$. 
(2) If \( u : D \to [-\infty, +\infty) \) is \( K \)-quasinearly subharmonic n.s., then \( u \) is \( K \)-quasinearly subharmonic in \( D \), but not necessarily conversely.

(3) A function \( u : D \to [-\infty, +\infty) \) is \( 1 \)-quasinearly subharmonic if and only if it is nearly subharmonic, that is, \( 1 \)-quasinearly subharmonic n.s.

(4) If \( u : D \to [0, +\infty) \) is quasinearly subharmonic and \( \psi : [0, +\infty) \to [0, +\infty) \) is permissible, then \( \psi \circ u \) is quasinearly subharmonic in \( D \). Especially, if \( h : D \to \mathbb{R} \) is harmonic and \( p > 0 \), then \( |h|^p \) is quasinearly subharmonic.

(5) The Harnack functions are quasinearly subharmonic.

Proof. We leave statements (1), (2) and (5) to the reader. For the proof of statement (4), see [33], Lemma 2.1, p. 32. To prove statement (3) suppose that \( u \) is nearly subharmonic in \( D \). Then clearly \( u_M \) is nearly subharmonic for all \( M \geq 0 \), and thus for every \( B_N(x,r) \subset D \), one has
\[
u_N \left( \int_{B_N(x,r)} u_M(y) \, dm_N(y) \right).
\]
Hence \( u \) is \( 1 \)-quasinearly subharmonic.

On the other hand, if \( u \) is \( 1 \)-quasinearly subharmonic in \( D \), then one sees at once, with the aid of the Lebesgue Monotone Convergence Theorem, that \( u \) is nearly subharmonic in \( D \).

Proposition 3. Let \( D \) be a domain in \( \mathbb{R}^N, N \geq 2 \).

(1) If \( u : D \to [-\infty, +\infty) \) is \( K_1 \)-quasinearly subharmonic and \( K_2 \geq K_1 \), then \( u \) is \( K_2 \)-quasinearly subharmonic.

(2) If \( u_1 : D \to [-\infty, +\infty) \) and \( u_2 : D \to [-\infty, +\infty) \) are \( K \)-quasinearly subharmonic n.s., then \( \lambda_1 u_1 + \lambda_2 u_2 \) is \( K \)-quasinearly subharmonic n.s. for all \( \lambda_1, \lambda_2 \geq 0 \).

(3) If \( u : D \to [-\infty, +\infty) \) is quasinearly subharmonic, then \( u \) is locally bounded above.

(4) If \( u_j : D \to [-\infty, +\infty), j = 1, 2, \ldots, \) are \( K \)-quasinearly subharmonic (resp. \( K \)-quasinearly subharmonic n.s.), and \( u_j \searrow u \) as \( j \to +\infty \), then \( u \) is \( K \)-quasinearly subharmonic (resp. \( K \)-quasinearly subharmonic n.s.).

(5) If \( u : D \to [-\infty, +\infty) \) is \( K_1 \)-quasinearly subharmonic and \( v : D \to [-\infty, +\infty) \) is \( K_2 \)-quasinearly subharmonic, then \( \max\{u, v\} \) is \( K \)-quasinearly subharmonic in \( D \) with \( K = \max\{K_1, K_2\} \). Especially, \( u^+ := \max\{u, 0\} \) is \( K_1 \)-quasinearly subharmonic.

(6) Let \( \mathcal{F} \) be a family of \( K \)-quasinearly subharmonic (resp. \( K \)-quasinearly subharmonic n.s.) functions in \( D \) and let \( w := \sup_{u \in \mathcal{F}} u \). If \( w \) is Lebesgue measurable and \( w^+ \in L^1_{\text{loc}}(D) \), then \( w \) is \( K \)-quasinearly subharmonic (resp. \( K \)-quasinearly subharmonic n.s.).
If \( u : D \to [-\infty, +\infty) \) is quasinearly subharmonic n.s., then either \( u \equiv -\infty \) or \( u \) is finite almost everywhere, and \( u \in L^1_{\text{loc}}(D) \).

We leave the simple statements (1)–(6) to the reader and note only that a proof of (7) is completely similar to the proof of Lemma 1.

Remark 2. If \( u : D \to [-\infty, +\infty) \) is strictly negative, finite and constant, then \( u \) is nearly subharmonic but, for every \( K > 1 \), \( u \) is not \( K \)-nearly subharmonic n.s. Thus, the analog of statement (1) does not hold for functions which are quasinearly subharmonic in narrow sense.

Remark 3. Related to statement (2) above, it is easy to see that, if \( u : D \to [-\infty, +\infty) \) is \( K \)-quasinearly subharmonic, then \( \lambda u + C \) is \( K \)-quasinearly subharmonic for all \( \lambda \geq 0 \) and \( C \geq 0 \).

The following example shows that the sum of two quasinearly subharmonic functions can be not quasinearly subharmonic.

Example 5. The function \( u : \mathbb{R}^2 \to \mathbb{R} \),
\[
u(x, y) := \begin{cases} 3, & \text{when } x = 0, \\ 1, & \text{when } x \neq 0, \end{cases}
\]
is 3-quasinearly subharmonic. The constant function \( v : \mathbb{R}^2 \to \mathbb{R} \),
\[
v(x, y) \equiv -2,
\]
is harmonic. Then we have
\[
r(x, y) := \begin{cases} 1, & \text{when } x = 0 \\ -1, & \text{when } x \neq 0 \end{cases}
\]
and
\[
(u + v)_M = \max\{u + v, -M\} + M = (u + v + M)^+
\]
for every \( M \geq 0 \). In particular for \( M = 1 \) we obtain
\[
r(x, y) := \begin{cases} 2, & \text{when } x = 0 \\ 0, & \text{when } x \neq 0 \end{cases}
\]
Since \( (u + v)_1(0, 0) > 1 \) and the double integral \( \int_B (u + v)_1(x, y)dA \) is zero for every ball \( B \subset \mathbb{R}^2 \), the function \( (u + v)_1 \) is not quasinearly subharmonic. Hence \( (u + v) \) is also not quasinearly subharmonic.

Remark 4. It is easy to see that the analog of statement (7) from Proposition does not hold for quasinearly subharmonic functions. A counterexample is the function \( u : \mathbb{R}^2 \to [-\infty, +\infty) \),
\[
u(x, y) := \begin{cases} -\infty, & \text{when } y \leq 0, \\ 1, & \text{when } y > 0, \end{cases}
\]
which is 2-quasinearly subharmonic, but surely not quasinearly subharmonic n.s.
4. CHARACTERIZATION OF HARMONIC FUNCTIONS VIA QUASINEARLY SUBHARMONIC FUNCTIONS

A subharmonic function \( u : \Omega \rightarrow [−\infty, \infty) \) defined on an open \( \Omega \subseteq \mathbb{R}^N \) is harmonic if and only if the function \( −u \) is also subharmonic, [14], p. 54. In this section we show that this remains true if one uses quasinearly subharmonic in the narrow sense functions instead of subharmonic functions.

**Proposition 4.** Let \( D \) be a domain in \( \mathbb{R}^N, N \geq 2 \). Then the following statements are equivalent for every \( u : D \rightarrow [−\infty, \infty) \).

1. The function \( u \) is harmonic.
2. There is \( K \geq 1 \) such that the functions \( u \) and \( −u \) are \( K \)-quasinearly subharmonic n.s.

**Proof.** The implication (1) \( \Rightarrow \) (2) is trivial. Suppose statement (2) holds. Since \( u \) and \( −u \) are \( K \)-quasinearly subharmonic n.s., we have

\[
\frac{K}{\nu_N r^N} \int_{B^N(x,r)} u(y) \, dm_N(y) \leq u(x) \leq \frac{K}{\nu_N r^N} \int_{B^N(x,r)} u(y) \, dm_N(y)
\]

and, consequently,

\[
u_N r^N \int_{B^N(x,r)} u(y) \, dm_N(y) \leq u(x) = \frac{K}{\nu_N r^N} \int_{B^N(x,r)} u(y) \, dm_N(y)
\]

holds whenever \( B^N(x,r) \subset D \). Using statement (7) from Proposition 3 we see that \( u \in L^1_{\text{loc}}(D) \). It follows from (6) for all \( x, z \in D \) and sufficiently small \( r > 0 \) that

\[
|u(x) − u(z)| \leq \frac{K}{\nu_N r^N} \int_{B^N(x,r) \triangle B^N(z,r)} |u(y)| \, dm_N(y),
\]

where \( B^N(x,r) \triangle B^N(z,r) \) is the symmetric difference of the balls \( B^N(x,r) \) and \( B^N(z,r) \). Since

\[
\lim_{x \rightarrow z} m_N(B^N(x,r) \triangle B^N(z,r)) = 0,
\]

the absolute continuity of the Lebesgue integral and the condition \( u \in L^1_{\text{loc}}(D) \) imply that \( f \) is continuous on \( D \). Let \( x \in D \) and \( u(x) \neq 0 \). Equality (6) and continuity of \( u \) at the point \( x \) imply that \( K = 1 \). Every continuous function \( u \) satisfying (6) with \( K = 1 \) for all \( B^N(x,r) \subset D \) is harmonic. Statement (1) follows. \( \square \)

**Corollary 1.** Let \( D \) be a domain in \( \mathbb{R}^N, N \geq 2 \). Then a function \( u : D \rightarrow [−\infty, \infty) \) is harmonic if and only if the functions \( u \) and \( −u \) are nearly subharmonic.

**Lemma 2.** Let \( D \) be a domain in \( \mathbb{R}^N, N \geq 2 \), let \( u : D \rightarrow [−\infty, \infty) \) be \( K_1 \)-quasinearly subharmonic n.s. and let \( −u \) be \( K_2 \)-quasinearly subharmonic n.s. If there is a point \( y_0 \in D \) such that \( u(y_0) > 0 \), then the inequality \( K_1 \geq K_2 \) holds.
Proof. Let $y_0 \in D$ and $u(y_0) > 0$. Then for sufficiently small $r > 0$ we have the double inequality

$$K_2 \frac{1}{\nu N r^N} \int_{B_N(y_0,r)} u(y) \, dm_N(y) \leq u(y_0) \leq K_1 \frac{1}{\nu N r^N} \int_{B_N(y_0,r)} u(y) \, dm_N(y),$$

thus

$$K_2 \frac{1}{\nu N r^N} \int_{B_N(y_0,r)} u(y) \, dm_N(y) \leq K_1 \frac{1}{\nu N r^N} \int_{B_N(y_0,r)} u(y) \, dm_N(y).$$

Inequality (8), $u^+ \in \mathcal{L}_{1,\text{loc}}^1(D)$ and $u(y_0) > 0$ imply that

$$0 < \int_{B_N(y_0,r)} u(y) \, dm_N(y) < +\infty.$$  

Now $K_2 \leq K_1$ follows from (9). $\square$

Using this lemma and Proposition 2 we obtain the following

**Proposition 5.** Let $D$ be a domain in $\mathbb{R}^N$, $N \geq 2$. Let $u : D \to [-\infty, \infty)$ be a function such that there are $x_1, x_2 \in D$ satisfying the double inequality

$$u(x_1) > 0 > u(x_2).$$

Then the function $u$ is harmonic if and only if the functions $u$ and $-u$ are quasinearly subharmonic n.s.

**Proof.** It suffices to show that $u$ is harmonic if $u$ and $-u$ are quasinearly subharmonic n.s. Let $u$ be $K_1$-quasinearly subharmonic n.s. and $-u$ be $K_2$-quasinearly subharmonic n.s. Then double inequality (10) and Lemma 2 imply the equality $K_1 = K_2$. Now the harmonicity of $u$ follows from Proposition 4. $\square$

The following example shows that there is $u : D \to (0, \infty)$ such that $u$ and $(-u)$ are quasinearly subharmonic n.s. but $u$ is not harmonic.

**Example 6.** Let $D = \mathbb{R}^n$ and

$$u(x) := \begin{cases} 2, & \text{when } x = 0, \\ 1, & \text{when } x \neq 0. \end{cases}$$

Then $u$ is 2-quasinearly subharmonic n.s. and $(-u)$ is 1-quasinearly subharmonic n.s., but $u$ is discontinuous at zero.

**Remark 5.** The above functions $u$ and $(-u)$ are both 2-quasinearly subharmonic. Thus Proposition 4 becomes false if we replace the quasinearly subharmonicity n.s. by quasinearly subharmonicity.
References

[1] P. Ahern, J. Bruna, Maximal and area integral characterizations of Hardy-Sobolev spaces in the unit ball of $\mathbb{C}^n$, Revista Mat. Iberoamericana, 4, 123–153 (1988).
[2] P. Ahern, W. Rudin, Zero sets of functions in harmonic Hardy spaces, Math. Scand., 73, 209–214 (1993).
[3] B.J. Cole, T.J. Ransford, Subharmonicity without upper semicontinuity, J. Functional Anal., 147, 420–442 (1997).
[4] E. Di Benedetto, N.S. Trudinger, Harnack inequalities for quasi-minima of variational integrals, Ann. Inst. H. Poincar, Analyse Nonlineaire, 1, 295–308 (1984).
[5] O. Djordjević, M. Pavlović, Equivalent norms on Dirichlet spaces of polyharmonic functions on the ball in $\mathbb{R}^N$, Bol. Soc. Mat. Mexicana (3), 13, 307–319 (2007).
[6] O. Djordjević, M. Pavlović, $L^p$-integrability of the maximal function of a polyharmonic function, J. Math. Anal. Appl., 336, 411–417 (2007).
[7] Y. Domar, On the existence of a largest subharmonic minorant of a given function, Arkiv mat., 3 (39), 429–440 (1957).
[8] Y. Domar, Uniform boundedness in families related to subharmonic functions, J. London Math. Soc. (2), 38, 485–491 (1988).
[9] O. Dovgoshey, J. Riihentaus, Bi-Lipschitz mappings and quasinearly subharmonic functions, Int. J. Math. Math. Sci., 1–8 (2010).
[10] O. Dovgoshey, J. Riihentaus, A remark concerning generalized mean value inequalities for subharmonic functions, International Conference Analytic Methods of Mechanics and Complex Analysis, Dedicated to N.A. Kilchevskii and V.A. Zmorovich on the Occasion of their Birthday Centenary, Kiev, Ukraine, June 29 - July 5, 2009, Transactions of the Institute of Mathematics of the National Academy of Ukraine, 7 (2), 26–33 (2010).
[11] O. Dovgoshey, J. Riihentaus, Mean type inequalities for quasinearly subharmonic functions, Glasgow Math. J., 55, 349–368 (2013).
[12] C. Fefferman, E.M. Stein, $H^p$ spaces of several variables, Acta Math., 129, 137–192 (1972).
[13] J.B. Garnett, Bounded Analytic Functions, Springer, New York, 2007 (Revised First Edition).
[14] W.K. Hayman, P.B. Kennedy Subharmonic functions, Academic Press, London, 1, (1976).
[15] D.J. Hallenbeck, Radial growth of subharmonic functions, Pitman Research Notes, 262, 113–121 (1992).
[16] M. Herve, Analytic and Plurisubharmonic Functions in Finite and Infinite Dimensional Spaces, Lecture Notes in Mathematics 198, Springer-Verlag, Berlin, 1971.
[17] V. Kojić, Quasi-nearly subharmonic functions and conformal mappings, Filomat., 21 (2), 243–249 (2007).
[18] P. Koskela, V. Manojlović, Quasi-nearly subharmonic functions and quasiconformal mappings, Potential Anal., 37, 187–196 (2012).
[19] U. Kuran, Subharmonic behavior of $|h|^p$, ($p > 0$, $h$ harmonic), J. London Math. Soc. (2), 8, 529–538 (1974).
[20] E.H. Lieb, M. Loss, Analysis, Graduate Studies in Mathematics, 14, American Mathematical Society, Providence, Rhode Island, 2001.
[21] Olivera R. Mihić, *Some properties of quasinearly subharmonic functions and maximal theorem for Bergman type spaces*, ISRN Mathematical Analysis, 3 p. (2013).

[22] Y. Mizuta, *Potential Theory in Euclidean Spaces*, Gaguto International Series, Mathematical Sciences and Applications, 6, Gakkōtosho Co., Tokyo, 1996.

[23] M. Pavlović, *On subharmonic behavior and oscillation of functions on balls in \( \mathbb{R}^n \)*, Publ. Inst. Math. (Beograd), 55 (69), 18–22 (1994).

[24] M. Pavlović, *Subharmonic behavior of smooth functions*, Math. Vesnik, **48**, 15–21 (1996).

[25] M. Pavlović, J. Riihentaus, *Classes of quasi-nearly subharmonic functions*, Potential Anal., **29**, 89–104 (2008).

[26] T. Rado, *Subharmonic Functions*, Springer, Berlin, 1937.

[27] J. Riihentaus, *On a theorem of Avanissian–Arsove*, Expo. Math., **7**, 69–72 (1989).

[28] J. Riihentaus, *Subharmonic functions: non-tangential and tangential boundary behavior*, Function Spaces, Differential Operators and Nonlinear Analysis (FSDONA’99), Proceedings of the Syôte Conference 1999, V. Mustonen, J. Rákosník (eds.), Math. Inst., Czech Acad. Science, Praha, 2000, pp. 229–238. (ISBN 80-85823-42-X).

[29] J. Riihentaus, *A generalized mean value inequality for subharmonic functions*, Expo. Math., **19**, 187–190 (2001).

[30] J. Riihentaus, *Subharmonic functions, mean value inequality, boundary behavior, nonintegrability and exceptional sets*, International Workshop on Potential Theory and Free Boundary Flows, Kiev, Ukraine, August 19-27, 2003, Transactions of the Institute of Mathematics of the National Academy of Sciences of Ukraine, 1 (3), 169–191 (2004).

[31] J. Riihentaus, *Weighted boundary behavior and nonintegrability of subharmonic functions*, International Conference on Education and Information Systems: Technologies and Applications (EISTA’04), Orlando, Florida, USA, July 21-25, 2004, Proceedings, M. Chang, Y-T. Hsia, F. Malpica, M. Suarez, A. Tremante, F. Welsch (eds.), vol. II, 2004, pp. 196–202.

[32] J. Riihentaus, *An integrability condition and weighted boundary behavior of subharmonic and \( M \)-subharmonic functions: a survey*, Int. J. Diff. Eq. Appl., **10**, 1–14 (2005).

[33] J. Riihentaus, *A weighted boundary limit result for subharmonic functions*, Adv. Algebra and Analysis, **1**, 27–38 (2006).

[34] J. Riihentaus, *Separately quasi-nearly subharmonic functions*, Complex Analysis and Potential Theory, Proceedings of the Conference Satellite to ICM 2006, Tahir Aliyev Azeroğlu, Promarz M. Tamrazov (eds.), Gebze Institute of Technology, Gebze, Turkey, September 8-14, 2006, World Scientific, Singapore, 156–165 (2007).

[35] J. Riihentaus, *On the subharmonicity of separately subharmonic functions*, Proceedings of the 11th WSEAS International Conference on Applied Mathematics (MATH’07), Dallas, Texas, USA, March 22-24, 2007, Kleanthis Psarris, Andrew D. Jones (eds.), WSEAS, 230–236 (2007).

[36] J. Riihentaus, *Subharmonic functions, generalizations and separately subharmonic functions*, The XIV-th Conference on Analytic Functions, July 22-28, 2007,
Chelm, Poland, Scientific Bulletin of Chelm, Section of Mathematics and Computer Science, 2, 49–76 (2007).

[37] J. Riihentaus, *Separately subharmonic functions and quasi-nearly subharmonic functions*, The 12th Worlds Multi-Conference on Systemics, Cybernetics and Informatics (WMSCI 2008), Orlando, Florida, USA, June 29th-July 2nd, 2008, Proceedings, N. Callaos, W. Lesso, C.D. Zinn, J. Baralt, J. Boukachour, C. White, T. Marwala, F. Nelwanondo (eds.), V, 53–56 (2008).

[38] J. Riihentaus, *On an inequality related to the radial growth of subharmonic functions*, CUBO, A Mathematical Journal, 11 (4), 127–136 (2009).

[39] J. Riihentaus, *Subharmonic functions, generalizations, weighted boundary behavior, and separately subharmonic functions: A survey*, Fifth World Congress of Nonlinear Analysts (WCNA 2008), Orlando, Florida, USA, July 2-9, 2008, Nonlinear Analysis, Series A: Theory, Methods & Applications, 71 (12), e2613–e26267 (2009).

[40] M. Stoll, *Weighted tangential boundary limits of subharmonic functions on domains in \( \mathbb{R}^n \) \((n \geq 2)\)*, Math. Scand., 83, 300–308 (1998).

[41] M. Stoll, *Harmonic majorants for eigenfunctions of the Laplacian with finite Dirichlet integrals*, J. Math. Anal. Appl., 274 (2), 788–811 (2002).

[42] M. Stoll, *On generalizations of the Littlewood-Paley inequalities to domains in \( \mathbb{R}^n \), \((n \geq 2)\)*, International Workshop on Potential Theory in Matsue, Shimane University, Matsue, Japan, August 23-28, 2004. A revised version of the paper *The Littlewood-Paley inequalities for Hardy-Orlicz spaces of harmonic functions on domains in \( \mathbb{R}^n \)* has been published in Advanced Studies in Pure Mathematics, 4, 363–376 (2006).

[43] M. Stoll, *On the Littlewood-Paley inequalities for subharmonic functions on domains in \( \mathbb{R}^n \)*, Recent Advances in Harmonic Analysis and Applications (eds. D. Bilyk et al.), Springer Proceedings in Mathematics & Statistics, 25 (2), 357–383 (2013).

[44] M. Stoll, *Littlewood-Paley theory for subharmonic functions on the unit ball in \( \mathbb{R}^N \)*, J. Math. Anal. Appl., 420 (1), 483–514 (2014).

[45] N. Suzuki, *Nonintegrability of harmonic functions in a domain*, Japan. J. Math., 16, 269–278 (1990).

[46] N. Suzuki, *Nonintegrability of superharmonic functions*, Proc. Amer. Math. Soc., 113 (1), 113–115 (1991).

[47] E. Szpilrajn, *Remarques sur les fonctions sousharmoniques*, Ann. Math., 34, 588–594 (1933).

[48] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Academic Press, London, 1986.

[49] M. Vuorinen, *On the Harnack constant and the boundary behavior of Harnack functions*, Ann. Acad. Sci. Fenn., Ser. A I, Math., 7, 259–277 (1982).