Maximal randomness from partially entangled states

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Entangled quantum systems can be used to violate Bell inequalities. According to Bell’s theorem, whenever we see a Bell violation we can be sure that the measurement outcomes are not the result of an underlying deterministic process, regardless of internal details of the devices used in the test. Entanglement can thus be exploited as a resource for the generation of randomness that can be certified device independently. A central question then is how much randomness we can extract from a given entangled state using a well-chosen Bell test. In this work we show that up to two bits of randomness – the maximum theoretically possible – can be extracted from any partially entangled pure state of two qubits from the joint outcome of projective two-outcome measurements. We also show that two bits of randomness can be extracted locally using a four-outcome non-projective measurement. Both results are based on a Bell test, which we introduce, designed to self-test any partially entangled pure two-qubit state and measurements spanning all three dimensions of the Bloch sphere.

Although it was not the original motivation [1], Bell’s theorem [2] allows for a very strong test of quantum randomness. By preparing an entangled quantum system and exhibiting a Bell inequality violation with it, we can immediately know that the measurement outcomes were not the result of an underlying deterministic process. Notably, the identification of randomness that this gives is independent of any internal physical details of the devices used in the test. This observation is the basis of a class of quantum cryptography protocols, called device independent, that incorporate a Bell test as a self-test of the correct functioning of the implementation. The class includes device-independent versions of quantum key distribution and random number generation [3–6].

This perspective prompts an obvious question: How much randomness can we extract from a given entangled state? Previous work (see table 1) has shown that the two do not seem strongly related; we cannot necessarily get more randomness from a maximally entangled state than a weakly entangled one of the same dimension. This point was made with a proposed Bell test in [7] with which one could extract a uniformly random bit from any partially entangled pure state of two qubits from one of the measurements. An extension of the Bell test, also described in [7], showed that potentially up to two uniformly random bits could be extracted from a pair of projective measurements. The test, however, only strictly demonstrates this for the maximally entangled state $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, while it is shown that the amount of randomness generated by the measurements tends to 2 random bits for a very weakly entangled state $|\psi^\theta\rangle = \cos(\theta/2) |00\rangle + \sin(\theta/2) |11\rangle$ in the limit $\theta \to 0$ where it becomes separable. Therefore, the question of how much randomness one can extract from a generic entangled two-qubit pure state remains open.

The main result of this work is to solve this question and prove that the maximum of two bits of randomness can be certified device independently from any entangled two-qubit pure state. To do so, we introduce a Bell-type test that could be performed by two parties, traditionally called Alice and Bob, sharing any partially entangled pure qubit state and show that it can be used to nearly perform tomographic reconstruction of an arbitrary measurement performed on one of the subsystems. We exploit this to show that, alternatively, two uniformly random bits can be obtained by performing a suitable four-outcome measurement, defined by a Positive-Operator Valued Measure (POVM), on one side, generalising a result previously obtained in [8] for the maximally entangled state.

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|                  | $|\phi^+\rangle$ | $|\psi^\theta\rangle$ |
|------------------|------------------|-----------------------|
| **Local**        |                  |                       |
| PROJ             | 1 bit [5]        | 1 bit [7]             |
| POVM             | 2 bits [8]       | 2 bits                |
| **Global**       |                  |                       |
| PROJ             | 2 bits [7]       | 2 bits                |
| POVM             | 2.8997 bits [8]  | –                     |

Table 1: Amount of randomness known to be extractable from one (local) or jointly from two (global) projective (PROJ) or non-projective (POVM) measurements from the maximally ($|\phi^+\rangle$) and any partially ($|\psi^\theta\rangle$) entangled two-qubit state. The results in bold are known to be optimal. This work proves optimal bounds in two new scenarios (italic).
in its Schmidt decomposition as

$$|\psi_0\rangle = \cos(\frac{\theta}{2})|0\rangle + \sin(\frac{\theta}{2})|1\rangle$$

(1)

for an angle $\theta$ that, without loss of generality, we can and hereafter will take to be in the range $0 < \theta \leq \frac{\pi}{2}$. The same state is equivalently represented by its density operator $\rho_0 = |\psi_0\rangle \langle \psi_0|$, which we can express as

$$\rho_0 = \frac{1}{4} \left[ I \otimes I + \cos(\theta) (1 \otimes Z + Z \otimes 1) + \sin(\theta) (X \otimes X - Y \otimes Y) + Z \otimes Z \right]$$

(2)

in terms of the identity and Pauli operators $1$, $X$, $Y$, and $Z$ acting on each subsystem. We can see that Alice and Bob will have to perform measurements in the $X$-$Y$ plane, for example $A = X$ and $B = Y$, in order to extract two uniformly random bits from this state, since this is the only way to have $\langle A \rangle = \langle A \otimes B \rangle = \langle B \rangle = 0$. We would, however, intuitively expect the maximum violation of a Bell inequality on $\psi_0$ to be attained with measurements having a component in the $Z$ direction, since the correlation terms involving $Z$ in (2) are larger in magnitude than the analogous terms involving $X$ and $Y$. As such, we anticipate that we will need a Bell experiment engineered to exploit the entire Bloch sphere.

To this end, we propose the following Bell test in which Alice and Bob perform $\pm 1$-valued measurements $A_x$, $x = 1, 2, 3$ and $B_y$, $y = 1, \ldots, 6$, in each round. They use the statistics to estimate the values of three Bell expressions. The first two,

$$I_\beta = \langle \beta A_1 + A_1(B_1 + B_2) + A_2(B_1 - B_2) \rangle,$$

$$J_\beta = \langle \beta A_1 + A_1(B_3 + B_4) + A_3(B_3 - B_4) \rangle,$$

(3)

(4)

are modified CHSH expressions of the kind introduced in [7] while the third,

$$S = \langle A_2(B_5 + B_6) + A_3(B_5 - B_6) \rangle,$$

(5)

is an ordinary CHSH [9, 10] expression. We choose

$$\beta = \frac{2 \cos(\theta)}{\sqrt{1 + \sin(\theta)^2}},$$

(6)

for the value of the parameter $\beta$ in the definitions of $I_\beta$ and $J_\beta$, depending on the angle $\theta$ that identifies the intended state $|\psi_0\rangle$. Alice and Bob should in particular check that these Bell expressions attain the values

$$I_\beta = 2\sqrt{2} \sqrt{1 + \beta^2/4},$$

$$J_\beta = 2\sqrt{2} \sqrt{1 + \beta^2/4},$$

$$S = 2\sqrt{2} \sin(\theta).$$

(7)

(8)

(9)

The Bell expectation values (7), (8), and (9) can be attained by measuring

$$A_1 = Z, \quad A_2 = X, \quad A_3 = \pm Y$$

(10)

on Alice’s side and performing suitable measurements on Bob’s side on the partially entangled state $|\psi_0\rangle$ [7].

Crucially for the intended application to randomness generation this is, as we will show, effectively the only way to attain these expectation values, even with a high-dimensional quantum system.

**Self-testing the state and Pauli basis.**— Suppose now that Alice and Bob perform the above Bell test with unknown measurements on an unknown state $\rho$. We will prove in the following that, if the expectation values $I_\beta = J_\beta = 2\sqrt{2} \sqrt{1 + \beta^2/4}$ and $S = 2\sqrt{2} \sin(\theta)$ are attained, there is a choice of local bases in which the state takes the form

$$\rho = \psi_0 \otimes \sigma_A \otimes \sigma_B,$$

(11)

where $\psi_0$ is the partially entangled state (2) and $\sigma_A \otimes \sigma_B$ is an unspecified ancillary state, and Alice’s measurements have the form

$$A_1 = Z \otimes 1,$$

$$A_2 = X \otimes 1,$$

$$A_3 = Y \otimes A_Y$$

(12)

(13)

(14)

where $A_Y$ is a $\pm 1$-valued Hermitian operator. The sign ambiguity in $A_3$ is unavoidable due to the symmetry of the scenario with respect to complex conjugation [11].

Note that, for simplicity, when we give an explicit expression for the local observables, we restrict our attention to the support of the local marginals of $\rho$. This is not restrictive since we are not concerned with, and in any case can infer nothing about, how Alice’s measurements act on any part of the Hilbert space that does not contain the state.

To begin with, we use the fact that the first constraint $I_\beta = 2\sqrt{2} \sqrt{1 + \beta^2/4}$ already implies (11), (12), and (13), with $\theta$ related to $\beta$ according to (6) above. This can be inferred from the derivation of the quantum bound on $I_\beta$ that was originally done in [7]. Ref. [7] is however not very explicit about this so we have included a detailed rederivation as an appendix. Note in particular that the relation between $A_1$ and $A_2$ can be expressed basis independently as $A_1^2 = A_2^2 = 1 \otimes 1$ and $\{ A_1, A_2 \} = 0$ where $\{ \cdot, \cdot \}$ is the anticommutator.

The term $J_\beta$ is the same Bell expression as $I_\beta$ except with different measurements and the second condition $J_\beta = 2\sqrt{2} \sqrt{1 + \beta^2/4}$ implies the same relation between $A_1$ and $A_3$ as the first did between $A_1$ and $A_2$. Having already identified $\rho$ and fixed $A_1$, we can derive the most general $A_3$ that anticommutes with $A_1$. Writing generally

$$A_3 = 1 \otimes A_1 + X \otimes A_X + Y \otimes A_Y + Z \otimes A_Z$$

(15)

and requiring $A_3^\dagger = A_3$, $\{ A_1, A_3 \} = 0$ for $A_1 = Z \otimes 1$, and $A_3^2 = 1 \otimes 1$, we find that $A_3$ must have the form

$$A_3 = X \otimes A_X + Y \otimes A_Y$$

(16)

for Hermitian operators $A_X$ and $A_Y$ satisfying

$$A_X^2 + A_Y^2 = I,$$

(17)

$$[A_X, A_Y] = 0,$$

(18)

where $[\cdot, \cdot]$ is the commutator.
Let us now prove that satisfying the third condition $S = 2\sqrt{2}\sin(\theta)$ forces us to set $A_X = 0$ in (16). As with $A_k$, we can decompose Bob’s measurement operators as

$$B_y = 1 \otimes B_{y1} + X \otimes B_{yX} + Y \otimes B_{yY} + Z \otimes B_{yZ}.$$  \hspace{1cm} (19)

Requiring $B_y^2 = 1 \otimes 1$ implies among other things that

$$B_{y1}^2 + B_{yX}^2 + B_{yY}^2 + B_{yZ}^2 = 1$$  \hspace{1cm} (20)

and, in particular,

$$B_{yX}^2 + B_{yY}^2 \leq 1$$  \hspace{1cm} (21)

for the $X$ and $Y$ components. Using the expression (2) for $\psi_0$ in the Pauli basis we find

$$\langle A_2 B_y \rangle = \sin(\theta) \langle 1 \otimes B_{yX} \rangle$$  \hspace{1cm} (22)

$$\langle A_1 B_y \rangle = \sin(\theta) \langle A_X \otimes B_{yX} \rangle$$  \hspace{1cm} (23)

$$y = 5, 6,$$ for the correlation terms $\langle AB \rangle = \text{Tr}[\psi_0 \otimes \sigma_B]$, appearing in (5). Since $A_X$, $A_Y$, and $A_Y$ commute, we can further express them together as

$$1 = \sum_k |k\rangle \langle k|,$$  \hspace{1cm} (24)

$$A_X = \sum_k x_k |k\rangle \langle k|, \quad A_Y = \sum_k y_k |k\rangle \langle k|,$$  \hspace{1cm} (25)

where $\{|k\rangle\}$ is a basis of $A'$ and $x_k^2 + y_k^2 = 1$ due to the condition $A_X^2 + A_Y^2 = 1$. We can then express $\langle A_X \otimes B_{yX} \rangle$, for example, as

$$\langle A_X \otimes B_{yX} \rangle = \sum_k x_k \langle B_{yX} \rangle_k,$$  \hspace{1cm} (26)

where the expectation values $\langle \cdot \rangle_k = \text{Tr}[\cdot \sigma_k]$ are evaluated on the states

$$\sigma_k = \text{Tr}_{A'}[(|k\rangle \langle k| \otimes 1) \sigma_{A'W}].$$  \hspace{1cm} (27)

defined on the $B'$ subsystem. Note that they are normalised such that $\sum_k \text{Tr}[\sigma_k] = 1$.

Using all this in the CHSH expectation value and applying the Cauchy-Schwarz inequality a few times yields

$$\frac{S}{\sin(\theta)} = \langle (1 + A_X) \otimes B_{yX} \rangle - \langle A_Y \otimes B_{yY} \rangle$$

$$+ \langle (1 - A_X) \otimes B_{yX} \rangle + \langle A_Y \otimes B_{yY} \rangle$$

$$= \sum_k \left( (1 + x_k) \langle B_{yX} \rangle_k - y_k \langle B_{yY} \rangle_k \right.$$  \hspace{1cm} (28)

$$+ (1 - x_k) \langle B_{yX} \rangle_k + y_k \langle B_{yY} \rangle_k \left. \right)$$

$$\leq \sum_k \left( \sqrt{2(1 + x_k)} \sqrt{\langle B_{yX} \rangle_k^2 + \langle B_{yY} \rangle_k^2} \right.$$

$$+ \sqrt{2(1 - x_k)} \sqrt{\langle B_{yX} \rangle_k^2 + \langle B_{yY} \rangle_k^2} \left. \right)$$

$$\leq \sum_k \left( \sqrt{2(1 + x_k)} \right. \text{Tr}[\sigma_k] + \sqrt{2(1 - x_k)} \text{Tr}[\sigma_k] \left. \right)$$

$$\leq \sum_k 2\sqrt{2} \text{Tr}[\sigma_k]$$

$$= 2\sqrt{2},$$  \hspace{1cm} (28)

where we used that $x_k^2 + y_k^2 = 1$ to get to the third expression and that $\langle B \rangle_k \leq \sqrt{\langle B^2 \rangle_k} \sqrt{\langle 1 \rangle_k}$ and

$$\langle B_{yX}^2 + B_{yY}^2 \rangle_k \leq \langle 1 \rangle_k \text{Tr}[\sigma_k]$$  \hspace{1cm} (29)

to get to the fourth.

In order to actually attain $S = 2\sqrt{2}\sin(\theta)$, all the bounds applied in (28) have to hold with equality. This requires in particular $\sqrt{2(1 + x_k)} = \sqrt{2(1 - x_k)}$, i.e., $x_k = 0$ and $y_k = \pm 1$. We thus conclude that $A_X = 0$ and $A_Y = Y \otimes A_Y$ with $A_Y^2 = 1$.

2 bits of global randomness.— A slight extension to the Bell experiment we have introduced allows Alice and Bob to extract two bits and certify that they are random and uncorrelated. In addition to checking that (7), (8), and (9) are met, Bob can perform a seventh measurement, $B_7$, and check that its correlation with $A_3$ is

$$\langle A_3 B_7 \rangle = -\sin(\theta).$$  \hspace{1cm} (30)

As before, we can generally express $B_7$ as

$$B_7 = 1 \otimes B_{71} + X \otimes B_{7X} + Y \otimes B_{7Y} + Z \otimes B_{7Z}.$$  \hspace{1cm} (31)

Direct computation of $|\langle A_3 B_7 \rangle|$ with $A_3 = Y \otimes A_Y$ on the state $\rho = \psi_0 \otimes A_{W'}$ gives

$$|\langle A_3 B_7 \rangle| = \sin(\theta) |\langle A_Y \otimes B_{7Y} \rangle|$$

$$\leq \sin(\theta) \sqrt{\langle A_Y^2 \rangle} \sqrt{\langle B_{7Y}^2 \rangle}$$

$$= \sin(\theta) \sqrt{\langle B_{7Y}^2 \rangle}.$$  \hspace{1cm} (32)

The constraint $|\langle A_3 B_7 \rangle| = \sin(\theta)$ thus implies $\langle B_{7Y}^2 \rangle = 1$, which allows us to deduce $B_{7Y}^2 = 1$ (recall that we restrict ourselves to the support of Bob’s marginal $\rho_{BW'}$). Eq. (20) then implies

$$B_7 = Y \otimes B_{7Y}.$$  \hspace{1cm} (33)

With this information we can prove that the results of measuring $A_2$ and $B_7$ are maximally random. The probabilities of the four possible outcomes are

$$P(ab|27) = \frac{1}{4} \langle (1 + a A_2) \otimes (1 + b B_7) \rangle,$$  \hspace{1cm} (34)

$$a, b \in \{\pm 1\}.$$ For $A_2 = X \otimes 1$ and $B_7 = Y \otimes B_{7Y}$, direct calculation gives

$$P(ab|27) = \frac{1}{4}.$$  \hspace{1cm} (35)

Importantly, the fact that we can derive $P(ab|27) = 1/4$ from $I_3 = J_3 = 2\sqrt{2}\sqrt{1 + \beta^2/4}$, $S = 2\sqrt{2}\sin(\theta)$, and $\langle A_3 B_7 \rangle = -\sin(\theta)$ shows that these conditions together are extremal, i.e., they cannot be attained by averaging quantum strategies that give different values of these quantities. This rules out the possibility of a more detailed underlying explanation of the correlations that might allow better predictions to be made about $A_2$ and $B_7$.

2 bits of local randomness.— An alternative way to extract up to two random bits is for Bob to perform a POVM with four outcomes. We should first see how this works in the ideal case that Alice and Bob share the
partially entangled state $|\psi_0\rangle$. In this case Bob has access to the marginal state

$$\psi_{\text{B}} = \frac{1}{2}(1 + \cos(\theta)Z)$$

and can extract the equivalent of two random bits with a suitable POVM $\{R^b_{\text{B}}\}_{b \in \{1, \ldots, 4\}}$ satisfying

$$\text{Tr}[\psi_{\text{B}} R^b_{\text{B}}] = \frac{1}{4}.$$  

In order to rule out a better underlying explanation of the result, we will also need a POVM that is extremal, i.e., it must not be possible to express it as a convex sum of POVMs other than itself. Fortunately, it is not difficult to find POVMs that satisfy these requirements. Any rank-one POVM

$$R^b_{\text{B}} = \alpha_b \phi_b,$$  

$\alpha_b > 0$, is extremal provided that the projectors $\phi_b$ are linearly independent [12]. An example of such a POVM is given by

$$\alpha_1 = \frac{1}{2 + 2 \cos(\theta)}, \quad \phi_1 = \frac{1}{2}(1 + Z)$$

for $b = 1$ and, for $b \in \{2, 3, 4\},$

$$\alpha_b = \frac{3 + 4 \cos(\theta)}{6 + 6 \cos(\theta)}$$

and

$$\phi_b = \frac{1}{2}[1 + \cos(\lambda)Z + \sin(\lambda)(\cos(\mu_0)X + \sin(\mu_0)Y)]$$

with $\cos(\lambda) = [-3 + 4 \cos(\theta)]^{-1}$ and, for example, angles $\mu_2, \mu_3, \mu_4 = 0, \pm 2\pi/3$.

The randomness certification we wish to show is based on the fact that we can reconstruct a POVM performed by Bob, such as $\{R^b_{\text{B}}\}_b$, from its correlations with Pauli measurements on Alice’s side on the state $|\psi_0\rangle$. Writing our ideal POVM $\{R^b_{\text{B}}\}_b$ as

$$R^b_{\text{B}} = r^b_{\mu} \sigma_\mu$$

in the identity and Pauli basis $\{\sigma_\mu\} = \{1, X, Y, Z\}$, where we use implicit summation over the repeated Greek index $\mu$, we get

$$r_{\mu b} = \langle \sigma_\mu \otimes R^b_{\text{B}} \rangle_{\psi_0} = \eta_{\mu \nu} r^\nu_{b}$$

for coefficients $\eta_{\mu \nu} = \langle \sigma_\mu \otimes \sigma_\nu \rangle_{\psi_0}$ that can be read off (2). For $\theta \neq 0$, the coefficients $\eta_{\mu \nu}$ make up the components of an invertible matrix (e.g., its determinant is $-\sin(\theta)^4$). The conditions (43) thus uniquely identify the POVM elements $R^b_{\text{B}}$.

Returning to the device-independent case, Bob, as part of the Bell test, can perform a four-outcome measurement $B_7 = \{R_b\}$ and check with Alice that the local and two-body statistics are compatible with the ideal qubit POVM $\{R^b_{\text{B}}\}_b$, i.e., that

$$\langle A_\mu \otimes R_b \rangle_{\psi_0} = \langle \sigma_\mu \otimes R^b_{\text{B}} \rangle_{\psi_0} = r_{\mu b}$$

where $A_\mu = (1 \otimes 1, X \otimes 1, Y \otimes 1, Z \otimes 1)$ denotes the identity and Alice’s measurements. It will be useful in the following to express these all together as

$$A_\mu = \sigma_\mu \otimes \hat{A}_+ + \sigma_\mu^* \otimes \hat{A}_-,$$  

where $\hat{A}_+ \otimes \hat{A}_-$ are the positive and negative parts of $A_\mu$, such that $1_{A \hat{\mu}} = \hat{A}_+ + \hat{A}_-$ and $A_\mu = \hat{A}_+ - \hat{A}_-$, and $\sigma_\mu^* = \pm \sigma_\mu$ is the complex conjugate (in the standard basis) of $\sigma_\mu$.

The condition (44) gives sufficient information about the measurement $\{R_b\}$ to show that it yields an outcome that is intrinsically random, as we can show by adapting a proof in [8]. To model the problem, we can suppose that Alice and Bob share a purification $|\Psi\rangle = |\psi_0 \otimes |\chi\rangle_{A'B'E}$ of the state identified by the Bell test with an adversary, Eve, who attempts to guess Bob’s outcome. The probability that Eve is successful is

$$P(B_7 = E) = \sum_b P_{BE}(bb|y = 7) = \sum_b \text{Tr}[\Psi_{BB'E}(R_b \otimes \Pi_y)]$$

where $\{\Pi_y\}$ is a four-outcome POVM performed by Eve. Inserting $1_{A'} = \hat{A}_+ + \hat{A}_-$ we can rewrite this as

$$P(B_7 = E) = \sum_{ab} \text{Tr}[\Psi_{A'B'B'E}(A_a \otimes R_b \otimes \Pi_b)] = \sum_{ab} p_{ab} \text{Tr}[\psi_{\text{B}} R_{b|ab}],$$

where $a \in \{\pm\}$, where in the second line we introduced probabilities $p_{ab}$ and POVM elements $R_{b|ab}$ on the B system defined by

$$p_{ab} = \text{Tr}[\langle A_a \otimes 1_{BB'} \otimes \Pi_y \rangle(B \otimes \chi_{A'B'E})],$$

$$p_{ab} R_{b|ab} = \text{Tr}_{A'B'E}[\langle A_a \otimes 1_{BB'} \otimes \Pi_y \rangle(1_{B} \otimes \chi_{A'B'E})].$$

For $p_{ab} \neq 0$, the $R_{b|ab}$ defined this way form a POVM. Expanding $R_b$ as

$$R_b = \sigma_\mu \otimes R^b_{\text{B}},$$

we can identify the $R_{b|ab}$ by

$$p_{ab} R_{b|ab} = \sigma_\mu \langle A_\mu \otimes R^b_{\text{B}} \otimes \Pi_y \rangle_{A'B'E}.$$  

At this point we consider what we learn from the constraint $\langle A_\mu \otimes R_b \rangle = r_{\mu b}$. Multiplying both sides by $\sigma_\nu \otimes \sigma_\nu$ where $(\sigma_\nu)$ is the matrix inverse of $(\eta_{\mu \nu})$ and then substituting in (50) we get

$$R^b_{\text{B}} = \sigma_\nu^* \langle A_\mu \otimes R_{\nu b} \rangle = \sigma_\nu^* \langle \sigma_\mu \otimes \sigma_\nu \rangle_{\psi_0} \langle \hat{A}_+ \otimes R^b_{\nu} \rangle_{A'B'} + \sigma_\nu^* \langle \sigma_\mu \otimes \sigma_\nu \rangle_{\psi_0} \langle \hat{A}_- \otimes R^b_{\nu} \rangle_{A'B'}$$

$$= \sigma_\nu^* \langle \hat{A}_+ \otimes R^b_{\nu} \rangle_{A'B'} + \sigma_\nu^* \langle \hat{A}_- \otimes R^b_{\nu} \rangle_{A'B'}$$

$$= \sum_p p_{b|e} R_{b|+e} + \sum_e p_{b|-e} R_{b|-e},$$

where we used that $\sigma_\nu^* = \pm \sigma_\nu$ in the same way as $\sigma_\mu$ and, in the last line, $R^*_{b|-e}$ is the complex conjugate of
Comparing the last and first lines and using that \( \{ R_b^\text{id}\} \) is supposed to be extremal we can conclude
\[
R_b|_{+ c} = R_b^\text{id} \quad \text{and} \quad R_b|_{- c} = R_b^\text{id*} \quad (53)
\]
for all values of \( c \). Using this in (47), we finally find
\[
P(B_T = E) = \sum_b p_{+ b} \text{Tr}[\psi_B^R R_b^\text{id}] + \sum_b p_{- b} \text{Tr}[\psi_B^R R_b^{\text{id*}}] = 1/4 \quad (54)
\]
for the local guessing probability.

**Conclusion.** — We have at this point proved what we set out to show. Up to two bits of global or local randomness can be extracted from any partially entangled pure two-qubit state from the corresponding variant of our Bell test. The analytic approach we used allowed us to show that the probabilities (34) and (54) are exactly 1/4 in either case if the ideal correlations are attained. For deviations from the ideal conditions, for instance due to noise, the randomness can still be bounded numerically using the NPA hierarchy [13–15].

Of possible independent interest, our Bell test allows its participants to infer that they must share a given partially entangled qubit state and are performing measurements spanning the entire Bloch sphere on it. Previous work has already shown that we can often infer substantial information about a quantum system from a Bell test [16–20]. Of particular relevance, tests had been designed to identify partially entangled qubit states [21, 22] or the Pauli measurements [8, 23, 24] but, before now, not both together in the same test.

Our work completely solves the problem of randomness certification from any entangled pure two-qubit state using projective measurements. This question however remains open for POVMs. While four random bits are potentially attainable [8], no construction has achieved this bound, nor it has been proven to be unattainable for some partially entangled states.

**Note added.** — While completing this work, we learned that a similar approach was developed independently in [25].

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where

\[ I_\beta = \beta(A) + \langle AB \rangle + \langle AB' \rangle + \langle A'B \rangle - \langle A'B' \rangle \] \hspace{1cm} (55)

allows us to infer substantial information about the underlying quantum state and measurements. More precisely, if the quantum bound

\[ I_\beta = 2\sqrt{2}\sqrt{1 + \beta^2/4} \] \hspace{1cm} (56)

is attained then, in a suitable choice of basis, the underlying state must be of the form

\[ \rho = \psi_\beta \otimes \sigma_{\text{junk}}, \] \hspace{1cm} (57)

where \( \psi_\beta = \langle \psi_\beta | \psi_\beta \rangle \) is the density operator associated to the state \( \langle \psi_\beta | = \cos(\theta_\beta/2) |00\rangle + \sin(\theta_\beta/2) |11\rangle \), and the measurements are

\[ A = Z \otimes 1 \oplus A_\perp, \] \hspace{1cm} (58)
\[ A' = X \otimes 1 \oplus A_\perp \] \hspace{1cm} (59)

and

\[ B = (\cos(\mu_\beta/2)Z + \sin(\mu_\beta/2)X) \otimes 1 \oplus B_\perp, \] \hspace{1cm} (60)
\[ B' = (\cos(\mu_\beta/2)Z - \sin(\mu_\beta/2)X) \otimes 1 \oplus B_\perp, \] \hspace{1cm} (61)

where the terms with the ‘\( \perp \)’ subscript act on the orthogonal complements of the supports of Alice’s and Bob’s marginals \( \rho_A = \text{Tr}_B[\rho] \) and \( \rho_B = \text{Tr}_A[\rho] \). The angles \( \mu_\beta \) and \( \theta_\beta \) are related to \( \beta \) by

\[ \sin(\theta_\beta) = \sqrt{1 - \beta^2/4}, \quad \cos(\theta_\beta) = \sqrt{2\beta^2/4}, \] \hspace{1cm} (62)
\[ \sin(\mu_\beta/2) = \sqrt{1 - \beta^2/4}, \quad \cos(\mu_\beta/2) = \sqrt{2\beta^2/4}. \] \hspace{1cm} (63)

Inversely, \( \beta \) and \( \mu_\beta \) are related to \( \theta_\beta \) by

\[ \beta = \frac{2\cos(\theta_\beta)}{\sqrt{1 + \sin(\theta_\beta)^2}}, \quad \tan(\mu_\beta/2) = \sin(\theta_\beta). \] \hspace{1cm} (64)

This result was essentially proved in the course of deriving the Tsirelson bound (56) for the more general family of \( I_\beta^2 \) expressions done in [7], particularly the steps around Eqs. (14)–(16). (The result is also closely related to the self-test based on \( I_3 \) in [22], although the formulation is slightly different.) Since [7] does not present this as a main result we review it here in more detail.

We proceed by first restricting to projective measurements on a bipartite pure qubit state before generalising to arbitrary dimension using the Jordan lemma and explicitly allowing for an underlying mixed state.

A. Qubit systems

The most general two-qubit pure state has the form

\[ |\psi\rangle = \cos\left(\frac{\theta}{2}\right)|00\rangle + \sin\left(\frac{\theta}{2}\right)|11\rangle, \] \hspace{1cm} (65)

for \( 0 \leq \theta \leq \pi/2 \), in its Schmidt decomposition, while the most general projective measurements worth considering are

\[ A = a \cdot \sigma, \quad B = b \cdot \sigma, \] \hspace{1cm} (66)
\[ A' = a' \cdot \sigma, \quad B' = b' \cdot \sigma \] \hspace{1cm} (67)

with \( \|a\| = \|a'\| = \|b\| = \|b'\| = 1 \), since we cannot exceed the classical bound if any of the measurements are \( \pm 1 \). We recall that the density operator associated with the state (65) can be written

\[ \psi = \frac{1}{4} \left[ I \otimes I + \cos(\theta)(Z \otimes I + I \otimes Z) \right. \]
\[ \left. + \sin(\theta)(X \otimes X - Y \otimes Y) + Z \otimes Z \right] \] \hspace{1cm} (68)

in terms of the Pauli operators \( X, Y, \) and \( Z \).

We write the expectation value of \( I_\beta \) as

\[ I_\beta = \beta \cos(\theta) a_x S + \] \hspace{1cm} (69)

where

\[ S = \langle A(B + B') + A'(B - B') \rangle \]
\[ = a \cdot T(b + b') + a' \cdot T(b - b') \] \hspace{1cm} (70)

and

\[ T = \begin{bmatrix} \sin(\theta) & 0 \\ 0 & -\sin(\theta) \end{bmatrix}, \]
\[ 0 & 1 \] \hspace{1cm} (71)

Substituting now

\[ b + b' = 2 \cos\left(\frac{\theta}{2}\right)b_+, \quad b - b' = 2 \sin\left(\frac{\theta}{2}\right)b_- \] \hspace{1cm} (72)

where \( b_\pm \) are normalised and orthogonal and we take \( \cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right) \geq 0 \),

\[ S = 2 \cos\left(\frac{\theta}{2}\right) a \cdot T b_+ + 2 \sin\left(\frac{\theta}{2}\right) a' \cdot T b_- \]
\[ \leq 2 \cos\left(\frac{\theta}{2}\right) \|T b_+\| + 2 \sin\left(\frac{\theta}{2}\right) \|T b_-\| \]
\[ \leq 2 \sqrt{\|T b_+\|^2 + \|T b_-\|^2} \]
\[ = 2 \sqrt{\text{Tr}[T^2(b_+ b_+^T + b_- b_-^T)']} \]
\[ \leq 2 \sqrt{1 + \sin(\theta)^2}. \] \hspace{1cm} (73)

Using this in (69),

\[ I_\beta \leq \beta \cos(\theta) a_x + 2 \sqrt{1 + \sin(\theta)^2} \]
\[ \leq \beta \cos(\theta) + 2 \sqrt{1 + \sin(\theta)^2} \]
\[ \leq 2 \sqrt{2} \sqrt{1 + \beta^2/4}. \] \hspace{1cm} (74)

In order to attain the quantum bound \( I_\beta = 2 \sqrt{2} \sqrt{1 + \beta^2/4} \), all of the inequalities used to get from (69) to (74) must hold with equality. Working backwards, we extract that

\[ 2 \cos(\theta) = \beta \sqrt{1 + \sin(\theta)^2}, \] \hspace{1cm} (75)
\[ a = 1 x, \] \hspace{1cm} (76)
\[ b_+ = 1 x, \] \hspace{1cm} (77)
\[ b = \cos(\varphi) 1_x - \sin(\varphi) 1_y, \] \hspace{1cm} (78)
\[ a' = \cos(\varphi) 1_x + \sin(\varphi) 1_y, \] \hspace{1cm} (79)
\[ \cos\left(\frac{\theta}{2}\right) \sin(\theta) = \sin\left(\frac{\theta}{2}\right). \] \hspace{1cm} (80)
Under the convention $\beta > 0$ and $0 \leq \theta, \phi, \theta', \phi' \leq \frac{\pi}{2}$ that we are working with, these imply the relations (62) and (63) for $\theta, \phi$ and $\mu, \ell$ given above. The remaining undetermined parameter $\phi$ can be set to 0 e.g. with the phase changes $|1\rangle_A \rightarrow e^{i\phi}|1\rangle_A$ and $|1\rangle_B \rightarrow e^{-i\phi}|1\rangle_B$, under which the Schmidt decomposition is invariant.

B. Basis fixing

Assuming a pure system of two qubits, the preceding derivation shows that, if the quantum bound $I_\beta = 2\sqrt{2} \sqrt{1 + \beta^2/4}$ is attained, then there is a basis in which:

i) the state is

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|00\rangle + \sin\left(\frac{\theta}{2}\right)|11\rangle; \quad (81)$$

ii) Alice’s measurements are

$$A_1 = Z, \quad A_2 = X; \quad (82)$$

iii) Bob’s measurements are such that

$$B_1 + B_2 \propto Z, \quad B_1 - B_2 \propto X. \quad (83)$$

In order to generalise this is important to notice that imposing any two of these conditions implies the third. For example, fixing i) and ii) or iii) implies Eqs. (76)–(80) with $\phi = 0$, which completely determines the other party’s measurements. On the other hand, $A$ and $B + B'$ are diagonal in the same bases as the marginals $\psi_A = \text{Tr}_B[\psi]$ and $\psi_B = \text{Tr}_A[\psi]$. Fixing ii) and iii) thus implies

$$\psi_A = \psi_B \equiv \cos\left(\frac{\theta}{2}\right)|0\rangle\langle 0| + \sin\left(\frac{\theta}{2}\right)|1\rangle\langle 1|; \quad (84)$$

the most general pure state consistent with this is

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|00\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|11\rangle, \quad (85)$$

whose density operator can be expressed in terms of the Pauli operators as

$$\psi = \frac{1}{4} \left[ I \otimes 1 + \cos(\theta) (Z \otimes 1 + 1 \otimes Z) + Z \otimes Z + \sin(\theta) (\cos(\phi) X \otimes X + \sin(\phi) X \otimes Y + \sin(\phi) Y \otimes X - \cos(\phi) Y \otimes Y) \right]. \quad (86)$$

It’s then straightforward to see that we need $\phi = 0$ in order to maximise the expectation value of $A'(B - B') \propto X \otimes X$.

C. Arbitrary dimension and mixed states

According to the Jordan lemma, the measurement operators $A$, $A'$ and $B$, $B'$ can be block diagonalised in their respective Hilbert spaces into blocks no larger than $2 \times 2$. We express this as

$$A = \sum_j A_j \otimes |j\rangle\langle j|, \quad A' = \sum_j A'_j \otimes |j\rangle\langle j|, \quad (87)$$

$$B = \sum_k B_k \otimes |k\rangle\langle k|, \quad B' = \sum_k B'_k \otimes |k\rangle\langle k|, \quad (88)$$

where $A_j$, $A'_j$, $B_k$, and $B'_k$ are $2 \times 2$ dichotomic Hermitian operators and, with respect to this block diagonalisation, an arbitrary unknown state as

$$\rho = \sum_s p_s \Psi_s \quad (89)$$

with

$$|\Psi_s\rangle = \sum_{jk} \sqrt{q_{jks}} |\psi_{jks}\rangle |j\rangle |k\rangle. \quad (90)$$

We can then express the expectation value of $I_\beta$ as a convex sum

$$I_\beta = \sum_{jk} \rho_{jks} \langle \psi_{jks}| (A_j (B_k + B'_k) + A'_j (B_k - B'_k)) |\psi_{jks}\rangle \quad (91)$$

of contributions.

In order to attain the quantum bound $I_\beta = 2\sqrt{2} \sqrt{1 + \beta^2/4}$, for each contribution $(j, k, s)$ either we must have $I_{jks} = 2\sqrt{2} \sqrt{1 + \beta^2/4}$ or $p_j q_{jks} = 0$. For those contributions for which $p_j q_{jks} \neq 0$ and where $I_{jks}$ attains the quantum bound, we are free to choose the bases such that

$$A_j = Z, \quad A'_j = X, \quad (92)$$

and

$$B_k + B'_k = 2 \cos\left(\frac{\mu}{2}\right) Z, \quad (93)$$

$$B_k - B'_k = 2 \sin\left(\frac{\mu}{2}\right) X, \quad (94)$$

i.e., conditions ii) and iii) from the previous subsection, which fixes $|\psi_{jks}\rangle = |\psi_j\rangle$. For any remaining blocks, we necessarily have $\sum_k p_j q_{jks} = 0$ or $\sum_j p_j q_{jks} = 0$, i.e., the corresponding block $j$ or $k$ acts on the orthogonal complement to the support of Alice’s or Bob’s marginal of $\rho$. Removing these and collectively denoting them $A_j$, $A'_j$, $B_k$, and $B'_k$ gives the expressions (58)–(61) above for Alice’s and Bob’s measurements and

$$\rho = \psi_\beta \otimes \sum_s p_s |\text{junk}_s\rangle\langle \text{junk}_s| = \psi_\beta \otimes \sigma_\text{junk}, \quad (95)$$

with $|\text{junk}_s\rangle = \sum_k \sqrt{q_{jks}} |j\rangle |k\rangle$, for the state.