Cancellation of Infrared divergences to all orders in LFQED

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Abstract

Coherent state approach has been proposed as an alternate way to deal with the true infrared divergences in light front field theory. We show that infrared divergences in fermion mass renormalization are eliminated to all orders in light front time ordered perturbation theory if one uses coherent state basis instead of the usual Fock basis to calculate the Hamiltonian matrix elements.

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INTRODUCTION

It is well known that the infrared (IR) divergences in quantum electrodynamics (QED) are eliminated to all orders due to a cancellation between the real and virtual contributions by virtue of the famous Bloch and Nordseick Theorem [1]. The theorem is based on the idea that in an actual experiment involving charged particles, one cannot specify the final state completely as due to the finite size of the detector, the charged particle can be accompanied by any number of soft photons. Therefore, in cross section calculations, one must sum over all possible final states taking into account emission of soft real photons which might have escaped detection. The Bloch-Nordseick mechanism takes into account all states with any number of soft photons below experimental resolution thus leading to cancellation of IR divergences.

In QED, a general approach to treat IR divergences was given by Yennie et al. [2]. In this approach, IR divergences are factored out and then treated to all order of covariant perturbation theory to give a residual perturbative expansion which is IR finite. These IR factors are then expressed in exponential form leading to cancellation of IR divergences between the real and virtual photon contributions. These factors depends only on the external momenta of charged particles and are independent of the momentum of the intermediate interaction terms.

Following the work of Yennie et al., Chung [3] showed that IR divergences indeed cancel to all orders in perturbation theory at the level of amplitude itself provided the initial and final states are chosen properly. The condition for this cancellation constrains the initial and final states, which are actually charged particles with a superposition of an infinitely large number of soft photons, to belong to a new space instead of the usual Fock space.

Kulish and Faddeev (KF) [4] developed the method of asymptotic dynamics and obtained a set of asymptotic states starting with a modified, relativistically and gauge invariant definition of S–matrix and showed the cancellation of IR divergences at amplitude level using this basis.

KF were the first to show that in QED, the asymptotic Hamiltonian does not coincide with the free Hamiltonian. They constructed the asymptotic Hamiltonian $V_{as}$ for QED thus modifying the asymptotic condition to introduce a new space of asymptotic states given by

$$|n; \pm \rangle = \Omega^A_\pm |n\rangle$$
where $\Omega_A^{\pm}$ defined by

$$\Omega_A^{\pm} = T \exp \left[ -i \int^{0}_{\mp \infty} V_{as}(t) dt \right]$$

is the asymptotic evolution operator and $|n\rangle$ is the Fock state. The transition matrix elements formed by using these states are IR finite. KF approach was applied by Greco et al. [5] to study the IR behavior of non abelian gauge theories using coherent states of definite color and factorized in fixed angle regime. The matrix elements using these coherent states are shown to be IR finite, first to the lowest order and then to all orders under the condition that the soft meson formula for real gluons is valid to all orders.

In this work, we demonstrate the cancellation of IR divergences in fermion mass renormalization in Light Front Quantum Electrodynamics (LFQED) to all orders by using a coherent state basis (CSB) for calculating the Hamiltonian matrix elements.

A coherent state formalism for Light Front Field Theory (LFFT) was developed and applied to show the cancellation of true IR divergences in one loop vertex correction in LFQED in Ref. [6]. Subsequently, the formalism was applied to show the same in LFQCD [7], and also to show the cancellation of IR divergences in fermion self energy correction at two loop order in LFQED [8, 9]. Possibility of practical application of the method was discussed in Ref. [10] where the coherent state method was applied to obtain an IR divergence free light-cone Schrödinger equation for positronium. In the following, we review the formalism developed in Ref. [6] and generalize the results of Ref. [8] to all orders.

**IR DIVERGENCES AND COHERENT STATE FORMALISM**

The coherent state method is based on the observation that for theories with long range interactions or theories having bound states as asymptotic states, the total Hamiltonian does not reduce to the free field Hamiltonian in the limit $|t| \to \infty$. In LFFT, the asymptotic Hamiltonian $H_{as}$, is evaluated by taking the limit $|x^+| \to \infty$ of the interaction Hamiltonian. Each term in the interaction Hamiltonian $H_{int}$ has a light-cone time dependence of the form $\exp[-i(p_1^- + p_2^- + \cdots + p_n^-)x^+]$ and therefore, if $(p_1^- + p_2^- + \cdots + p_n^-)$ vanishes at some vertex, then the corresponding term in $H_{int}$ does not vanish in large $x^+$ limit. Thus, the total Hamiltonian can be written as

$$H = H_{as} + H'_{I}$$  \hspace{1cm} (1)
where
\[ H_{as}(x^+) = H_0 + V_{as}(x^+) \] (2)
The associated \( x^+ \) evolution operator \( U_{as}(x^+) \) in the Schrödinger representation, which satisfies the equation
\[ i\frac{dU_{as}(x^+)}{dx^+} = H_{as}(x^+)U_{as}(x^+) \] (3)
can then be used to generate an asymptotic space
\[ \mathcal{H}_{as} = \exp[-\Omega^A(x^+)]\mathcal{H}_F \] (4)
from the usual Fock space \( \mathcal{H}_F \), in the limit \( x^+ \to -\infty \), where \( \Omega^A(x^+) \) is the asymptotic evolution operator defined by
\[ U_{as}(x^+) = \exp[-iH_0x^+]\exp[\Omega^A(x^+)] \] (5)
The asymptotic evolution operator is then used to define the coherent states
\[ |n : coh\rangle = \exp[-\Omega^A]|n\rangle \] (6)
The method of asymptotic dynamics, proposed originally by Kulish and Faddeev [4] in the context of equal time QED consists of identifying the terms in the interaction Hamiltonian which do not vanish at infinitely large times and then using them to construct the asymptotic Möller operator and hence the coherent states. In LFFT’s, the true IR divergences corresponding to \( k^+ \), \( k_\perp \to 0 \) are not expected to appear when one uses this CSB to calculate the transition matrix elements. In the following sections, we will prove this statement to all orders for fermion mass renormalization in LFQED.

Interaction Hamiltonian of LFQED in light front gauge is given by [11]
\[ H_I(x^+) = V_1(x^+) + V_2(x^+) + V_3(x^+) \]
where
\[ V_1(x^+) = e \sum_{i=1}^{4} \int d\nu_i^{(1)}[e^{-i\bar{\nu}_i^{(1)}x^+}\bar{h}_i^{(1)}(\nu_i^{(1)}) + e^{i\nu_i^{(1)}x^+}\bar{h}_i^{(1)*}(\nu_i^{(1)})] \] (7)
\( \bar{h}_i^{(1)}(\nu_i^{(1)}) \) and \( \nu_i^{(1)} \) are three point QED interaction vertex and the light front energy transferred at the vertex \( \bar{h}_i^{(1)} \) respectively. \( V_2 \) and \( V_3 \) are the non-local 4–point instantaneous vertices. Here, we will focus on construction of asymptotic Hamiltonian using 3–point vertex.
The same procedure can be used to construct the asymptotic Hamiltonian terms corresponding to the 4–point interaction vertices also. The details of the calculation of the four point asymptotic interaction can be found in Ref. [8]. One can notice, from the time dependence of $V_1(x^+)$ that it does not become zero at large (light-cone) times whenever $\nu_i^{(1)} = 0$. For example, the three point interaction Hamiltonian has a term with light-cone time dependence of the form $\exp[-i(p^- - k^- - (p - k)^-)]$ and therefore, if $(p^- - k^- - (p - k)^-) \to 0$ then this term does not vanish at infinite times. Thus, the asymptotic Hamiltonian has a contribution from this term which can be written as

$$V_{1as}(x^+) = e \sum_{i=1,4} \int d\nu_i^{(1)} \Theta_\Delta(k)[e^{-i\nu_i^{(1)} x^+} \tilde{h}_i^{(1)}(\nu_i^{(1)})$$

$$+ e^{i\nu_i^{(1)} x^+} \tilde{h}_i^\dagger(\nu_i^{(1)})]$$

(8)

where $\Theta_\Delta(k)$ is the region of momentum space in which the energy difference $(p^- - k^- - (p - k)^-)$ becomes vanishingly small. It can be shown that this condition is satisfied in the region defined by

$$k_\perp^2 < \frac{k^+ \Delta}{p^+}, \quad k^+ < \frac{p^+ \Delta}{m^2}.$$  

We call this region the asymptotic region. $\Theta_\Delta(k)$ in Eq.(8) is given by

$$\Theta_\Delta(k) = \theta\left(\frac{k^+ \Delta}{p^+} - k_\perp^2\right) \theta\left(\frac{p^+ \Delta}{m^2} - k^+\right)$$

Substituting $k^+ \to 0$, $k_\perp \to 0$ in all slowly varying functions of $k$ and performing the $x^+$ integration, one obtains the asymptotic Möller operator which gives the asymptotic states as

$$\Omega_{\pm}^A |n: p_i\rangle = \exp\left[-\int dp^+ d^2 \mathbf{p}_\perp \sum_{\lambda=1,2} [d^3 k][f(k, \lambda : p)a^\dagger(k, \lambda) - f^*(k, \lambda : p)a(k, \lambda)]

+ e^2 \int dp^+ d^2 \mathbf{p}_\perp \sum_{\lambda_1, \lambda_2=1,2} [d^3 k_1][d^3 k_2][g_1(k_1, k_2, \lambda_1, \lambda_2 : p)a^\dagger(k_2, \lambda_2)a(k_1, \lambda_1) - g_2(k_1, k_2, \lambda_1, \lambda_2 : p)a(k_2, \lambda_2)a^\dagger(k_1, \lambda_1)]\rho(p) \right] |n: p_i\rangle$$

(9)

where

$$f(k, \lambda : p) = \frac{p_\mu \epsilon^\mu(k)}{p \cdot k} \theta\left(\frac{k^+ \Delta}{p^+} - k_\perp^2\right) \theta\left(\frac{p^+ \Delta}{m^2} - k^+\right),$$

(10)

The second term here arises from the 4–point instantaneous interaction. Expressions for $g_1$ and $g_2$ can be found in Ref. [8]. Following the same procedure as in Ref. [9], we have used
these asymptotic states to calculate the transition matrix elements and to demonstrate the absence of IR divergences in fermion self energy correction up to two loop level \[8, 9\]. In this work, we present an all order proof of cancellation of IR divergences in fermion self energy correction in CSB using the method of induction.

**GRAPHICAL METHOD OF PROVING IR FINITENESS**

In light-front time ordered perturbation theory, the transition matrix is given by the perturbative expansion

\[
T = V + V \frac{1}{p^2 - H_0} V + \cdots
\]

The electron mass shift is obtained by calculating \( T_{pp} \) which is the matrix element of the above series between the initial and the final electron states \( |p, s\rangle \) and it is given by,

\[
\delta m^2 = p^+ \sum_s T_{pp}
\]

where

\[
T_{pp} = \langle p, s | T | p, s\rangle
\]

\( T_{pp} \) can be written in powers of \( e^2 \) as

\[
T_{pp} = T^{(1)} + T^{(2)} + \cdots
\]

In general, \( T^{(n)} \) gives the \( O(e^{2n}) \) contribution to lepton self energy correction. Here, the initial (or final) lepton momentum is

\[
p = \left[ p^+, \frac{\mathbf{p}_\perp^2 + m^2}{2p^+}, \mathbf{p}_\perp \right],
\]

The strategy used to develop a proof of cancellation of IR divergences to all orders in LFQED is based on the method of induction. To begin with, we have shown the cancellation of IR divergences up to \( O(e^4) \) using CSB \[8\]. Now, we assume that the IR divergences have been cancelled up to \( O(e^{2n}) \) and represent the \( O(e^{2n}) \) IR finite amplitude by a blob i.e. a blob represents the sum of the Fock and coherent state contributions to the self energy correction which, on being added up together, give IR finite amplitude. The blob is of \( O(e^{2n}) \) and contains a maximum of \( n \) photon lines or \( 2n \) 3-point interaction vertices. In case of diagrams containing 4-point instantaneous interaction vertices or having soft photon
FIG. 1: Basic diagram representing the sum of all diagrams of $O(e^{2n})$ in Fock state and CSB
insertions, these numbers will be less than $n$ and $2n$ respectively. Then the final task would
be to express the $O(e^{2(n+1)})$ contributions in terms of this blob and show the cancellation of
IR divergences in $O(e^{2(n+1)})$ in CSB. The general expression for transition matrix element
in $O(e^{2n})$ is a sum of terms of the form:

$$T_j^{(n)} = -\frac{e^{2n}}{2p^+(2\pi)^3} \int \prod_i \frac{d^3k_i}{2k_i^+2p_{2i-1}^+} \prod_r \frac{\pi(p,s)f_1(p_1 + m)f_2(p_2 + m) \cdots \cdots \cdots (p_\ell + m)f_\ell u(p,s)}{\prod_{r}(p^- - p_r - \sum_i k_i)} \quad (16)$$

where $r$, $i$ and $\ell$ depend on the kind of diagram $T_j^{(n)}$ represents. This enables us to express
the $O(e^{2n})$ contribution as

$$T^{(n)} = \sum_j T_j^{(n)} = \sum_j \frac{\pi(p,s)M_n^{(j)} u(p,s)}{\mathcal{D}^{(j)}} \quad (17)$$

where $j$ is summed over all possible diagram in $O(e^{2n})$. $T^{(n)}$ will be assumed to be IR
divergence free. Here

$$\mathcal{D}^{(j)} = \prod_r \mathcal{D}_r^{(j)} \quad (18)$$

and $\mathcal{D}^{(j)}$ corresponds to the product of all the energy denominators, $\mathcal{D}_r^{(j)}$, corresponding to
$r$ different intermediate states in $j^{th}$ diagram. The graphical representation of the blob is
shown in Fig. 1. $T^{(n)}$ will be assumed to be IR divergence free.

AN EXAMPLE: CANCELLATION OF INFRARED DIVERGENCES UP TO
FOURTH ORDER

Before presenting the all order proof, we will first revisit the proof of cancellation of IR
divergences in $\delta m^2$ up to $O(e^4)$ to illustrate our strategy. In Ref. [8, 9] we showed that the
true infrared divergences in $\delta m^2$ get cancelled up to $O(e^4)$ if one uses CSB instead of Fock
basis to calculate the transition matrix elements [8]. Now we sketch the proof for the same based on graphical method which will be generalized to all orders in the next section. For the sake of simplicity, we will consider the contribution due to the 3–point interaction terms only.

![Diagram](image1)

**FIG. 2**: IR finite $O(e^2)$ blob which represents the sum of Fock state and coherent state contributions.

To begin with, consider the $O(e^2)$ corrections which are represented by the two diagrams on the r.h.s. of Fig. 2. It has been shown in Ref. [8, 9] that the sum of these two is finite. In our notation, it is equal to

$$T^{(2)} = \sum_j \frac{\pi(p, s) \mathcal{M}_j^{(2)} u(p, s)}{\mathcal{D}_j}$$  \hspace{1cm} (19)$$

where the sum runs over 1 and 2 corresponding to the two diagrams. The sum is represented by the IR finite blob on the l.h.s. in Fig. 2.

Now consider the two diagrams on the r.h.s. of Fig. 3, which are actually Fig. 3(a) and 8(b) in Ref. [8] and have been shown to be equal to

$$T^{(2)}_{3b} = \frac{e^4}{(2\pi)^6} \int \frac{d^2k_{1\perp} d^2k_{2\perp}}{2p^+} \left[ \frac{dk_1^+ dk_2^+}{32k_1^+ k_2^+ p_1^+ p_2^+ p_3^+} \right]$$

\[ \times \frac{\pi(p, s)[\jmath_1^{\lambda_1}(k_1)(\phi_1 + m)\jmath_2^{\lambda_2}(k_2)(\phi_2 + m)\jmath_1^{\lambda_1}(k_1)] u(p, s)}{(p^+ - p_1^- - k_1^-)(p^- - p_2^- - k_2^-)(p^- - p_1^- - k_1^-)} \]  \hspace{1cm} (20)

and

$$T^{(2)}_{3c} = \frac{e^4}{(2\pi)^6} \int \frac{d^2k_{1\perp} d^2k_{2\perp}}{2p^+} \left[ \frac{dk_1^+ dk_2^+}{16k_1^+ k_2^+ p_1^+ p_2^+ p_3^+} \right]$$

\[ \times \frac{\pi(p, s)[\jmath_1^{\lambda_1}(k_1)(\phi_1 + m)\jmath_2^{\lambda_2}(k_2)(\phi_2 + m)\jmath_1^{\lambda_1}(k_1)] u(p, s)(p \cdot e^{\lambda_2}(k_2)) \Theta_\Delta(k_2)}{(p \cdot k_2)(p^- - p_1^- - k_1^-)(p^- - p_2^- - k_2^-)} \]  \hspace{1cm} (21)
respectively. Here, \( p_1 = p - k_1 \) and \( p_2 = p - k_1 - k_2 \). The sum of Eqs. (20) and (21) in our new notation is,

\[
T^{(2)}_{3a} = T^{(2)}_{3b} + T^{(2)}_{3c} = \frac{e^2}{(2\pi)^3} \int d^3k_1 \frac{\bar{u}(p, s)\not\epsilon(k_1)(\not\phi_1 + m)\mathcal{M}_{2}^{(j)}(\not\phi_1 + m)\epsilon(k_1)u(p, s)}{2k_1^+ (p \cdot k_1)^2 D^{(j)}}
\]

(22)

Note that the l.h.s. of Fig. 2 minus the external lines is \( \mathcal{M}_2^{(j)} \) for the jth diagram and hence the sum of the two \( O(e^4) \) diagrams on the r.h.s. of Fig. 3 is represented by Eq. (22). Since the blob is IR finite, the IR divergences can appear “only” from the vanishing energy denominators of the kind \( (p - k_1^- - (p - k_1)^-) \). All the other energy denominators inside the blob, which depend on \( k_1 \) and \( k_2 \) need not be taken into account. An additional contribution in \( O(e^4) \), when one uses the CSB, is shown in Fig. 4 and is given by

\[
T^{(2)}_{4a} = -\frac{e^2}{(2\pi)^3} \int d^3k_1 \frac{\bar{u}(p, s)\mathcal{M}_2^{(j)}(\not\phi_1 + m)\epsilon(k_1)u(p, s)(p \cdot k_1)}{2k_1^+ (p \cdot k_1)^2 D^{(j)}}
\]

(23)

In the limit, \( k_1^+ \to 0, k_{1\perp} \to 0 \), the numerator of Eqs. (22) and (23) become equal and the sum of their contributions is IR finite. At \( O(e^4) \), there are two other diagrams involving only the three point vertices which contribute to self energy correction. A similar argument can be constructed for the remaining two diagrams as well. The cancellation of IR divergences of these remaining diagrams is illustrated graphically in Fig. 5.

**CANCELLATION OF INFRARED DIVERGENCES TO ALL ORDERS**

Now, we consider an \( O(e^{2n}) \) blob which we will assume to be free of IR divergences. We shall show that the cancellation of IR divergences in \( O(e^{2(n+1)}) \) contribution to fermion mass renormalization in LFQED follows from this assumption. To construct an \( O(e^{2(n+1)}) \) diagram in Fock basis, we can add a photon to \( n^{th} \) order blob in three different ways as
FIG. 5: Additional diagrams corresponding to $O(e^4)$ contributions in Fock basis and CSB.

![Diagrams](image)

FIG. 6: Addition of a photon line to $O(e^{2n})$ blob in Fock basis shown in Fig. [6]. The contributions coming from the diagram in Figs. 6(a), (b) and (c), in the limit $q^+ \to 0, q_\perp \to 0$ are given by

$$T^{(n+1)}_{6a} = \frac{e^2}{(2\pi)^3} \int \frac{d^3q}{2q^+} \frac{\bar{u}(p,s)\ell(q)(\slashed{P} + m)\mathcal{M}^{(j)}_n(\slashed{P} + m)\ell(q)u(p,s)}{(p \cdot q)^2 \mathcal{D}^{(j)}}$$  \hspace{1cm} (24)

$$T^{(n+1)}_{6b} = -\frac{e^2}{(2\pi)^3} \int \frac{d^3q}{2q^+} \frac{\bar{u}(p,s)\ell(q)(\slashed{P} + m)\ell(q)(\slashed{P'} + m)\mathcal{M}^{(j)}_n u(p,s)}{(p \cdot q)(p^- - p'^-) \mathcal{D}^{(j)}}$$ \hspace{1cm} (25)

$$T^{(n+1)}_{6c} = \frac{e^2}{(2\pi)^3} \int \frac{d^3q}{2q^+} \frac{\bar{u}(p,s)\mathcal{M}^{(j)}_n(\slashed{P} + m)\ell(q)u(p,s)}{(p \cdot q) \mathcal{D}^{(j)}}$$ \hspace{1cm} (26)

where $P = p - q$. The additional contributions in $(n + 1)^{th}$ order, when we use the CSB,
FIG. 7: Addition of a photon line to $O(e^{2n})$ blob in CSB are shown in Fig. 7 and are given by

\[
T_{7a}^{(n+1)} = - \frac{e^2}{(2\pi)^3} \int \frac{d^3 q}{2q^+} \frac{\bar{\pi}(p,s)\mathcal{M}^{(j)}_n(p+q)}{(p \cdot q)^2} \frac{\hat{\epsilon}(q)\Theta(q)}{D^{(j)}}
\]

(27)

\[
T_{7b}^{(n+1)} = - \frac{e^2}{(2\pi)^3} \int \frac{d^3 q}{2q^+} \frac{\bar{\pi}(p,s)\hat{\epsilon}(q)(q' + m)\mathcal{M}^{(j)}_n(p,s)}{(p \cdot q)(p^- - p'^-)D^{(j)}} \Theta(q)
\]

(28)

\[
T_{7c}^{(n+1)} = - \frac{e^2}{(2\pi)^3} \int \frac{d^3 q}{2q^+} \frac{\bar{\pi}(p,s)\mathcal{M}^{(j)}_n(p,s)}{(p \cdot q)} \frac{\hat{\epsilon}(q)\Theta(q)}{D^{(j)}}
\]

(29)

In the limit, $q^+ \to 0$, $q^\perp \to 0$, $\hat{P}\hat{\epsilon}(q) \to p \cdot \epsilon$. As a result, the sum of Eqs. (24)–(26) exactly cancels the sum of Eqs. (27)–(29).

It is to be noted that even if the blob contains 4–point instantaneous vertices, Figs. 6 and 7 are the only diagrams in next order in graphical notation and therefore, the argument presented in this section still holds.

**CONCLUSION**

We have demonstrated, using the coherent state formalism, that the true IR divergences in self energy correction cancel to all order in LFQED. The proof presented here can be extended to a general n-point amplitude. We plan to address this problem in future.
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