A nonlinear model of the non-isothermal slip flow between two parallel plates

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Abstract. We study a mathematical model describing the steady-state non-isothermal flow of a viscous fluid between two parallel plates under the Navier slip boundary condition. The flow is driven by an applied pressure gradient. The dependence of the viscosity, thermal conductivity and slip coefficients on the temperature is taken into account. The model under consideration is a boundary value problem for a system of nonlinear coupled ordinary differential equations. We give the weak formulation of this problem and establish sufficient conditions for the existence and uniqueness of a weak solution. To construct weak solutions, we propose an algorithm based on the Galerkin procedure, methods of the topological degree theory, and compactness arguments. Moreover, explicit expressions for the velocity and the temperature on the plates are obtained.

1. Introduction

It is well known that flows and heat transfer models for viscous fluids with variable physical properties are important for analyzing many engineering problems. For example, the strong dependence of viscosity and/or slipping on temperature plays a crucial role in many technological processes.

In this report, we consider a nonlinear mathematical model for the steady-state non-isothermal unidirectional flow of a viscous fluid between two parallel plates $z = -h$ and $z = h$ under Navier’s slip boundary condition taking into account the dependence of the viscosity, thermal conductivity and slip coefficients on the temperature:

\[
\begin{align*}
-(\mu[\theta(z)]u'(z))' &= \xi, \quad z \in (-h, h), \\
-(k[\theta(z)]\theta'(z))' &= \omega(z), \quad z \in (-h, h), \\
\mu[\theta(h)]u'(h) &= -\chi[\theta(h)]u(h), \\
\mu[\theta(-h)]u'(-h) &= \chi[\theta(-h)]u(-h), \\
k[\theta(h)]\theta'(h) &= -\beta\theta(h), \\
k[\theta(-h)]\theta'(-h) &= \beta\theta(-h),
\end{align*}
\]

where $u$ is the velocity component in $x$-direction, $\theta$ denotes the deviation from the average temperature value, $-\xi$ is the pressure gradient ($\xi > 0$), i.e., $\partial p/\partial x = -\xi$, $\mu$ denotes the viscosity.
coefficient, \(k\) represents the thermal conductivity, \(\beta\) is the positive coefficient characterizing the heat transfer on solid walls of the channel, \(\chi\) is the slip coefficient, \(\omega\) stands for the heat source intensity, and the symbol \(\prime\) denotes the differentiation with respect to \(z\). Figure 1 shows the chosen coordinate system and geometry of the physical problem considered.

In system (1), the unknowns are \(u\) and \(\theta\), while all other functions and constants are assumed to be given.

The main aim of this work is to prove the solvability of boundary value problem (1) in the weak formulation and propose an algorithm for finding of weak solutions. Moreover, we establish sufficient conditions for uniqueness of a weak solution and derive explicit expressions for the velocity and the temperature on the plates \(z = \pm h\).

It is worth pointing out that mathematical models of non-isothermal flows of a viscous fluid through a given domain have addressed by many authors in a wide variety of contexts. We mention here only a few works and refer an interested reader to the literature quoted in these papers. Bostandzhyan et al. [1] have investigated the plane Couette flow, the axial flow in an annulus between two infinite cylinders, and the flow between two rotating cylinders with temperature-dependent viscosity that satisfies Reynolds’s relation \(\mu = \mu_0 \exp(-b\theta)\), where \(\mu_0\) and \(b\) are some constants. In the linear formulation a problem of the convection of a viscous incompressible fluid in a bounded domain under specified regimes of inflow and outflow on parts of its boundary was considered by Krein and Chan Tkhu Kha [2]. Optimal control problems of non-isothermal flows are studied in [3–6]. The paper [7] deals with the hysteresis phenomenon in the non-isothermal channel flow of a non-Newtonian fluid. Farooq et al. [8] have examined the Poiseuille flow between two heated parallel inclined plates by using the perturbation technique to obtain the approximate analytical expressions for velocity and temperature distributions. Schwarz [9] obtained an exact solution of the Navier–Stokes–Boussinesq equations that describes the plane-parallel advective flow in a horizontal incompressible fluid layer with rigid boundaries, assuming that the viscosity and thermal diffusivity coefficients are constant. In the paper [10], the flow in the entry region of a straight channel in the presence of viscous heating is studied via direct numerical simulations and a temporal linear stability analysis. Cheremnykh [11] derived a priori estimates of a solution to the problem of the unidirectional thermogravitational motion of a viscous fluid in a plane channel. The Couette–Poiseuille non-isothermal flow of couple stress fluid with magnetic field between two parallel plates was investigated in [12] by using the Runge–Kutta scheme with shooting to solve the corresponding non-linear system of equations. Domnich et al. [13] have proved the existence of weak solutions to a nonlinear system of equations describing steady creeping flows of a non-uniformly heated incompressible fluid through a bounded three-dimensional domain with locally Lipschitz boundary. Burmasheva
and Prosviryakov [14] have obtained exact solutions to the description of the unidirectional flow of a viscous fluid in an infinite horizontal layer of a given thickness, assuming that the flow is induced by the thermocapillary effect specified at the upper boundary of the layer and taking into account the condition of fluid slipping at the lower boundary. A novel mathematical model for flows of a heat-conducting fluid in a 3D pipeline network is proposed by Artemov et al. [15].

2. Preliminaries: notations, function spaces, and auxiliary results

Suppose $E$ and $F$ are Banach spaces. By $\mathcal{L}(E, F)$ we denote the space of all bounded linear mappings from $E$ to $F$. The space $\mathcal{L}(E, F)$ is equipped with the norm

$$
\|A\|_{\mathcal{L}(E, F)} \stackrel{\text{def}}{=} \sup_{\|w\|_E \neq 0} \frac{\|Aw\|_F}{\|w\|_E}.
$$

As usual, the strong (weak) convergence in a Banach space is denoted by $\to$ ($\rightharpoonup$). The mean of a function $v: [-h, h] \to \mathbb{R}$ is denoted by $\overline{v}_h$, i.e.,

$$
\overline{v}_h \stackrel{\text{def}}{=} \frac{1}{2h} \int_{-h}^{h} v(z) \, dz.
$$

By $H^1[-h, h]$ we denote the Sobolev space $W^{1,2}[-h, h]$ with the scalar product defined according to the relation

$$(v, u)_{H^1[-h, h]} \stackrel{\text{def}}{=} \int_{-h}^{h} v'(z)u'(z) \, dz + v(h)u(h) + v(-h)u(-h).$$

It is well known that any function from the space $H^1[-h, h]$ is absolutely continuous. The next lemma establishes the compactness of embedding for $H^1[-h, h]$ (see, e.g., [16, Chap. III, § 2.8]).

**Lemma 1.** The space $H^1[-h, h]$ is compactly embedded in the H"older space $C^{0,1/2}[-h, h]$.

Let us introduce the following subspaces of $H^1[-h, h]$:

$$
\begin{align*}
H^1_0[-h, h] & \stackrel{\text{def}}{=} \{ v \in H^1[-h, h] : v(-h) = v(h) = 0 \}, \\
K[-h, h] & \stackrel{\text{def}}{=} \{ v \in H^1[-h, h] : v = \text{const} \}, \\
H^1_{\text{even}}[-h, h] & \stackrel{\text{def}}{=} \{ v \in H^1[-h, h] : v(-z) = v(z) \ \forall z \in [-h, h] \}, \\
H^1_{0,\text{even}}[-h, h] & \stackrel{\text{def}}{=} \{ v \in H^1_0[-h, h] : v(-z) = v(z) \ \forall z \in [-h, h] \}.
\end{align*}
$$

The scalar products in these subspaces are induced from the space $H^1[-h, h]$. We have the decomposition

$$
H^1_{\text{even}}[-h, h] = H^1_{0,\text{even}}[-h, h] \oplus K[-h, h],
$$

where the symbol $\oplus$ denotes the orthogonal sum.

Note that the sequence of real-valued functions $\{\psi_j\}_{j=1}^{\infty}$, defined by

$$
\psi_j(z) \stackrel{\text{def}}{=} \frac{2\sqrt{h}}{\pi(2j-1)} \sin \left( \frac{(2j-1)\pi}{2h} (z + h) \right), \quad z \in [-h, h],
$$

is an orthonormal basis of the space $H^1_{0,\text{even}}[-h, h]$.
Let \( \psi_0 \equiv 1/\sqrt{2} \). Taking into account (2), it can easily be checked that the sequence \( \{\psi_j\}_{j=0}^{\infty} \) is an orthonormal basis of the space \( H^1_{\text{even}}[-h,h] \).

Using direct calculations, one can derive the following estimate.

**Lemma 2.** Let \( I : H^1_{\text{even}}[-h,h] \to C[-h,h] \) be the embedding operator. Then

\[
\|I\|_{L(H^1_{\text{even}}[-h,h],C[-h,h])} \leq \max\{1, 2\sqrt{r}\}. \tag{3}
\]

The proof of the solvability of (1) relies on the following statement.

**Lemma 3.** Assume that \( r > 0 \) and \( B_r \overset{\text{def}}{=} \{x \in \mathbb{R}^n : |x| < r\} \). Let \( \{F_t : \mathbb{R}^n \to \mathbb{R}^n\}_{t \in [0,1]} \) be a family of mappings such that

- the mapping \( F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n \), \( F(x,t) \overset{\text{def}}{=} F_t(x) \) is continuous;
- \( F_t(x) \neq 0 \) for any \( x \in \partial B_r \) and \( t \in [0,1] \);
- the mapping \( F_0 : \mathbb{R}^n \to \mathbb{R}^n \) is linear.

Then the equation \( F_1(x) = 0 \) has at least one solution in the ball \( B_r \).

This lemma can be proved by methods of the theory of topological degree [17].

3. The weak formulation of problem (1) and the main results

Assume that the following conditions hold.

- (i) the function \( \omega : [-h, h] \to \mathbb{R} \) is even and belongs to the Lebesgue space \( L^1[-h, h] \);
- (ii) the functions \( \chi : \mathbb{R} \to \mathbb{R} \), \( \mu : \mathbb{R} \to \mathbb{R} \), and \( k : \mathbb{R} \to \mathbb{R} \) are continuous;
- (iii) there exists a constant \( k_0 \) such that \( 0 < k_0 \leq k(s) \) for all \( s \in \mathbb{R} \);
- (iv) for any number \( r > 0 \) there exist constants \( \chi_r \) and \( \mu_r \) such that \( 0 < \chi_r \leq \chi(s) \) and \( 0 < \mu_r \leq \mu(s) \) for all \( s \in [-r,r] \).

**Definition.** We shall say that a pair \( (u, \theta) \in H^1_{\text{even}}[-h,h] \times H^1_{\text{even}}[-h,h] \) is a weak solution of boundary value problem (1) if the equalities

\[
\int_{-h}^{h} \mu \chi(\theta(z))u'(z)\psi(z)\,dz + 2\chi \theta \mu \psi(h) = \xi \int_{-h}^{h} \psi(z)\,dz, \tag{4}
\]

\[
\int_{-h}^{h} k \chi(\theta(z))\theta'(z)\psi'(z)\,dz + 2\beta \theta \mu \psi(h) = \int_{-h}^{h} \omega(z)\psi(z)\,dz \tag{5}
\]

hold for any \( \psi \in H^1_{\text{even}}[-h,h] \).

**Remark.** Any classical solution to (1) is a weak solution. On the other hand, if \( (u, \theta) \) is a weak solution and the functions \( u, \theta, \mu, k \) are sufficiently smooth, then \( (u, \theta) \) is a classical solution to (1).

The following theorems give the main results of this work.

**Theorem 1.** Under conditions (i)–(iv), we have

(a) problem (1) has at least one weak solution;
(b) if \((u, \theta)\) is a weak solution to (1), then
\[
u(h) = u(-h) = \frac{\xi h}{\sqrt{\omega h \beta^\gamma}}, \tag{6}\]
\[\theta(h) = \theta(-h) = \omega h \beta^{-1}; \tag{7}\]

(c) if \((u, \theta)\) is a weak solution to (1), then
\[
\int_{-h}^{h} \mu[\theta(z)]u'(z)dz + 2\chi[\theta(h)]u^{2}(h) = 2h\xi \omega, \tag{8}\]
\[
\int_{-h}^{h} k[\theta(z)]\theta'(z)^{2}dz + 2\beta \theta^{2}(h) = \int_{-h}^{h} \omega(z) \theta(z)dz. \tag{9}\]

**Theorem 2.** If conditions (i)–(iv) hold and the function \(k\) satisfies the Lipschitz condition
\[
|k(s_{1}) - k(s_{2})| \leq M|s_{1} - s_{2}|, \quad \forall s_{1}, s_{2} \in \mathbb{R},
\]
with a constant \(M\) such that
\[
0 \leq M < \frac{\min\{k^{2}, \beta^{2}\}}{\|\omega\|_{L^{1}[-h,h]} \max\{1, 4h\}}, \tag{10}\]
then problem (1) admits a unique weak solution.

4. Sketch of the proof of Theorems 1 and 2

Using the approach proposed in [18, §3], we will perform the proof of the existence result (a) in three steps.

**Step 1. The Galerkin approximation of problem (1).** Let us fix an arbitrary integer \(m\). We consider an auxiliary problem with a parameter \(t \in [0, 1]^{3}\):

Find a vector \(\mathbf{q}_{m} = (a_{m0}, a_{m1}, \ldots, a_{mm}, b_{m0}, b_{m1}, \ldots, b_{mm})^{\top} \in \mathbb{R}^{2m+2}\) such that
\[
\int_{-h}^{h} \mu[\theta_{m}(z)]u'_{m}(z)\psi'_{j}(z)dz + 2\chi[\theta_{m}(h)]u_{m}(h)\psi_{j}(h) = \xi \int_{-h}^{h} \psi_{j}(z)dz, \quad j = 0, 1, \ldots, m, \tag{11}\]
\[
\int_{-h}^{h} k[\theta_{m}(z)]\theta'_{m}(z)\psi'_{j}(z)dz + 2\beta \theta^{2}(h) \psi_{j}(h) = \int_{-h}^{h} \omega(z) \psi_{j}(z)dz, \quad j = 0, 1, \ldots, m, \tag{12}\]
where
\[
u_{m}(z) \stackrel{\text{def}}{=} \sum_{j=0}^{m} a_{mj} \psi_{j}(z), \quad z \in [-h, h],
\]
\[
\theta_{m}(z) \stackrel{\text{def}}{=} \sum_{j=0}^{m} b_{mj} \psi_{j}(z), \quad z \in [-h, h].
\]

**Step 2. A priori estimates for the Galerkin solutions \(\mathbf{q}_{m} \).** Suppose that the vector \(\mathbf{q}_{m}\) satisfies equations (11) and (12) with some \(t \in [0, 1]\). Since \(\{\psi_{j}\}_{j=0}^{\infty}\) is an orthonormal basis of \(H^{1}_{even}[-h, h]\), we see that
\[
\|\mathbf{q}_{m}\|_{\mathbb{R}^{2m+2}}^{2} = \sum_{j=0}^{m} a_{mj}^{2} + \sum_{j=0}^{m} b_{mj}^{2} = \|u_{m}\|_{H^{1}_{even}[-h, h]}^{2} + \|\theta_{m}\|_{H^{1}_{even}[-h, h]}^{2}. \tag{13}\]
Let us multiply (12) by $b_{mj}$ and add the results for $j = 0, 1, \ldots, m$:

$$
\int_{-h}^{h} k[t\theta_m(z)]|\theta_m'(z)|^2 \, dz + 2\beta|\theta_m(h)|^2 = \int_{-h}^{h} \omega(z)\theta_m(z) \, dz.
$$

From this equality we derive that

$$
\min \{k_0, \beta\} \|\theta_m\|_{H^1_{\text{even}}[-h,h]} \leq \min \{k_0, \beta\} \left( \int_{-h}^{h} |\theta_m'(z)|^2 \, dz + 2|\theta_m(h)|^2 \right)
$$

$$
\leq \int_{-h}^{h} k[t\theta_m(z)]|\theta_m'(z)|^2 \, dz + 2\beta|\theta_m(h)|^2
= \int_{-h}^{h} \omega(z)\theta_m(z) \, dz
\leq \int_{-h}^{h} |\omega(z)| \, dz \max_{s \in [-h,h]} |\theta_m(s)| = \|\omega\|_{L^1[-h,h]} \|\theta_m\|_{C[-h,h]}
\leq \|\omega\|_{L^1[-h,h]} \|I\|_{L^2(H^1_{\text{even}}[-h,h],C[-h,h])} \|\theta_m\|_{H^1_{\text{even}}[-h,h]},
$$

and hence

$$
\|\theta_m\|_{H^1_{\text{even}}[-h,h]} \leq \frac{\|\omega\|_{L^1[-h,h]} \|I\|_{L^2(H^1_{\text{even}}[-h,h],C[-h,h])}}{\min \{k_0, \beta\}}. \quad (14)
$$

Moreover, we have

$$
\|\theta_m\|_{C[-h,h]} \leq \frac{\|\omega\|_{L^1[-h,h]} \|I\|_{L^2(H^1_{\text{even}}[-h,h],C[-h,h])}}{\min \{k_0, \beta\}}. \quad (15)
$$

Next, let us multiply (11) by $a_{mj}$ and add the results for $j = 0, 1, \ldots, m$; this gives

$$
\int_{-h}^{h} \mu[t\theta_m(z)]|u_m'(z)|^2 \, dz + 2\chi[t\theta_m(h)]u_m^2(h) = \xi \int_{-h}^{h} u_m(z) \, dz.
$$

By using the last equality, it is easily shown that

$$
\min \left\{ \inf_{s \in [-r,r]} \mu_0(s), \inf_{s \in [-r,r]} \chi_0(s) \right\} \|u_m\|_{H^1_{\text{even}}[-h,h]}^2 \leq \int_{-h}^{h} \mu[t\theta_m(z)]|u_m'(z)|^2 \, dz + 2\chi[t\theta_m(h)]u_m^2(h)
= \xi \int_{-h}^{h} u_m(z) \, dz
\leq \xi \int_{-h}^{h} \max_{s \in [-h,h]} |u_m(s)| \, dz
= 2h\xi \max_{s \in [-h,h]} |u_m(s)|
= 2h\xi \|u_m\|_{C[-h,h]}
\leq 2h\xi \|I\|_{L^2(H^1_{\text{even}}[-h,h],C[-h,h])} \|u_m\|_{H^1_{\text{even}}[-h,h]},
$$

where

$$
\tau = \frac{\|\omega\|_{L^1[-h,h]} \|I\|_{L^2(H^1_{\text{even}}[-h,h],C[-h,h])}}{\min \{k_0, \beta\}}.
$$
Taking into account condition (iv), we derive that
\[ \|u_m\|_{H^1_{\text{even}}[-h,h]} \leq \frac{2h\xi \|I\|_{\mathcal{L}(H^1_{\text{even}}[-h,h],C[-h,h])}}{\min \{\mu_r, \chi_r\}}. \] (16)

Finally, combining (13), (14), and (16), we obtain
\[ \|q_m\|^2_{\mathbb{R}^{2m+2}} \leq \frac{4h^2\xi^2 \|I\|_{\mathcal{L}(H^1_{\text{even}}[-h,h],C[-h,h])}^2}{\min \{\mu_r, \chi_r\}^2} + \frac{\|\omega\|_{L^1[-h,h]}^2 \|I\|_{\mathcal{L}(H^1_{\text{even}}[-h,h],C[-h,h])}^2}{\min \{k_0, \beta^2\}}. \]

Note that the right-hand side of the last inequality does not depend on \( m \) and \( t \). Therefore, applying Lemma 3, we deduce that problem (11)–(12) is solvable for each \( m \in \{1,2,\ldots\} \) and \( t \in [0,1] \).

**Step 3. Passing to the limit \( m \to \infty \).** By \( \tilde{q}_m = (\tilde{a}_{m0}, \tilde{a}_{m1}, \ldots, \tilde{a}_{mm}, \tilde{b}_{m0}, \tilde{b}_{m1}, \ldots, \tilde{b}_{mm})^\top \) denote a solution of problem (11)–(12) with \( t \). Let
\[ \tilde{u}_m(z) \overset{\text{def}}{=} \sum_{j=0}^{m} \tilde{a}_{mj} \psi_j(z), \quad z \in [-h,h], \]
\[ \tilde{\theta}_m(z) \overset{\text{def}}{=} \sum_{j=0}^{m} \tilde{b}_{mj} \psi_j(z), \quad z \in [-h,h]. \]

Then we have
\[ \int_{-h}^{h} \mu[\tilde{\theta}_m(z)]\tilde{u}_m'(z)\psi'_j(z) \, dz + 2\chi[\tilde{\theta}_m(h)]\tilde{u}_m(h)\psi_j(h) = \xi \int_{-h}^{h} \psi_j(z) \, dz, \quad j = 0,1,\ldots,m, \] (17)
\[ \int_{-h}^{h} k[\tilde{\theta}_m(z)]\tilde{\theta}_m'(z)\psi'_j(z) \, dz + 2\tilde{\theta}_m(h)\psi_j(h) = \int_{-h}^{h} \omega(z)\psi_j(z) \, dz, \quad j = 0,1,\ldots,m. \] (18)

Consider the sequence \( \{(\tilde{u}_m, \tilde{\theta}_m)\}_{m=1}^\infty \). From estimates (14) and (16) it follows that this sequence is bounded in the space \( H^1_{\text{even}}[-h,h] \times H^1_{\text{even}}[-h,h] \). Hence there exist functions \( u_* \) and \( \theta_* \) belonging to \( H^1_{\text{even}}[-h,h] \) and a subsequence \( \{m_i\}_{i=1}^\infty \) such that \( \tilde{u}_{m_i} \rightharpoonup u_* \) and \( \tilde{\theta}_{m_i} \rightharpoonup \theta_* \) weakly in \( H^1_{\text{even}}[-h,h] \) as \( i \to \infty \). Without loss of generality it can be assumed that
\[ \tilde{u}_m \rightharpoonup u_* \text{ weakly in } H^1_{\text{even}}[-h,h], \] (19)
\[ \tilde{\theta}_m \rightharpoonup \theta_* \text{ weakly in } H^1_{\text{even}}[-h,h], \] (20)
as \( m \to \infty \).

Moreover, since the embedding operator \( I : H^1_{\text{even}}[-h,h] \to C[-h,h] \) is compact (see Lemma 1), we have
\[ \tilde{u}_m \to u_* \text{ strongly in } C[-h,h], \] (21)
\[ \tilde{\theta}_m \to \theta_* \text{ strongly in } C[-h,h], \] (22)
as \( m \to \infty \).

Taking into account (19)–(22), we can pass to the limit \( m \to \infty \) in (17) and (18); this gives
\[ \int_{-h}^{h} \mu[\theta_*(z)]u_*'(z)\psi'_j(z) \, dz + 2\chi[\theta_*(h)]u_*(h)\psi_j(h) = \xi \int_{-h}^{h} \psi_j(z) \, dz, \] (23)
\[ \int_{-h}^{h} k[\theta_*(z)]\theta'_*(z)\psi_j(z)\,dz + 2\beta\theta_*(h)\psi_j(h) = \int_{-h}^{h} \omega(z)\psi_j(z)\,dz, \quad (24) \]

for any \( j \in \{0, 1, 2, \ldots \} \).

Since the sequence \( \{\psi_j\}_{j=0}^{\infty} \) is an orthonormal basis of \( H^1_{\text{even}}[-h, h] \), equalities (23) and (24) remain valid if we replace \( \psi_j \) with an arbitrary function \( \psi \in H^1_{\text{even}}[-h, h] \). Thus, we have established that the pair \((u_*, \theta_*)\) is a weak solution to problem (1) and statement (a) is proved.

Next, substituting \( \psi \equiv 1 \) into (4) and (5), we derive (6) and (7). It is also easy to obtain energy equalities (8) and (9) by letting \( \psi = u \) in (5) and \( \psi = \theta \) in (6).

Now we must only show the uniqueness of weak solutions under the conditions of Theorem 2. Suppose that \((v_1, \eta_1)\) and \((v_2, \eta_2)\) are weak solutions of problem (1). Let us prove the equality \((v_1, \eta_1) = (v_2, \eta_2)\).

Note that if pairs \((u, \theta)\) and \((v, \theta)\) are weak solutions to (1), then \( u = v \). Therefore, it suffices to show that \( \eta_1 = \eta_2 \).

We obviously have

\[ \int_{-h}^{h} k[\eta_1(z)]\eta'_1(z)\psi(z)\,dz + 2\beta\eta_1(h)\psi(h) = \int_{-h}^{h} \omega(z)\psi(z)\,dz, \]

\[ \int_{-h}^{h} k[\eta_2(z)]\eta'_2(z)\psi(z)\,dz + 2\beta\eta_2(h)\psi(h) = \int_{-h}^{h} \omega(z)\psi(z)\,dz, \]

for any \( \psi \in H^1_{\text{even}}[-h, h] \). Subtracting the second equality from the first equality, we get

\[ \int_{-h}^{h} \left\{ k[\eta_1(z)]\eta'_1(z) - k[\eta_2(z)]\eta'_2(z) \right\} \psi(z)\,dz + 2\beta(\eta_1(h) - \eta_2(h))\psi(h) = 0. \]

Choosing \( \psi = \eta_1 - \eta_2 \) and using (3), from the last equality one can obtain

\[ \|\eta_1 - \eta_2\|_{H^1_{\text{even}}[-h, h]}^2 \leq \frac{M\|\omega\|_{L^1[-h, h]}\max\{1, 4h\}}{\min\{k_0^2, \beta^2\}} \|\eta_1 - \eta_2\|_{H^1_{\text{even}}[-h, h]}^2, \]

and hence

\[ \left(1 - \frac{M\|\omega\|_{L^1[-h, h]}\max\{1, 4h\}}{\min\{k_0^2, \beta^2\}}\right) \|\eta_1 - \eta_2\|_{H^1_{\text{even}}[-h, h]}^2 \leq 0. \quad (25) \]

On the other hand, it follows from (10) that

\[ 0 \leq \frac{M\|\omega\|_{L^1[-h, h]}\max\{1, 4h\}}{\min\{k_0^2, \beta^2\}} < 1. \quad (26) \]

Finally, combining inequalities (25) and (26), we easily obtain that \( \eta_1 = \eta_2 \). This means that the statement of Theorem 2 is valid.

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