Continuity equations for all Lipkin’s zilches from symmetries of the standard electromagnetic action and Noether’s theorem

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In 1964, Lipkin discovered a set of conserved quantities for free electromagnetism, known as the zilches. Although for over four decades a physical interpretation for the zilches was not clear, in 2010 Tang and Cohen realized that one of the zilches, termed as optical chirality, provides a measure of the handedness of light, motivating novel investigations into the interactions of light with chiral matter. In 2018, Smith and Strange provided a physical interpretation for the rest of the zilches for a certain class of electromagnetic (EM) fields. Except for the case of optical chirality, the important question of how to derive the conservation of all zilches from symmetries of the standard free EM action still remains open. In this Letter, we provide the answer to this long-standing question by identifying the corresponding zilch symmetry transformations of the EM four-potential \( A_\mu \) and using Noether’s theorem. We also uncover the relation of the zilch symmetry transformations to a “hidden” invariance algebra of free Maxwell’s equations in potential form. In the presence of electric charges and currents, we present a new derivation of the continuity equation for optical chirality from symmetries of the theory. We also derive new zilch continuity equations that can be used in experimental investigations into the role of all zilches in light-matter interactions.

I. INTRODUCTION

Noether’s seminal theorem \( \text{[1]} \) is the cornerstone in understanding the deep connection between symmetries of physical theories and conservation laws. Starting from continuous symmetries of the action functional of a theory, Noether’s theorem can be used to derive conservation laws for the associated Euler-Lagrange equations. In relativistic field theories, such as electromagnetism in Minkowski spacetime, the knowledge of a symmetry leads to a Noether (four-)current, \( V^\mu \), which is conserved \( (\partial_\mu V^\mu = 0) \). This conservation holds for fields satisfying the Euler-Lagrange equations - i.e. for on-shell field configurations - and the corresponding Noether charge, \( Q = \int d^3x V^0 \), is time-independent.

An example of little-known time-independent quantities in free electromagnetism is given by the ten zilches that were discovered by Lipkin in 1964 \( \text{[2]} \). One of the zilches, now known as optical chirality, started drawing renewed theoretical and experimental interest in 2010, when Tang and Cohen realized that this particular zilch provides a measure of the chirality (or handedness) of light \( \text{[3]} \). The optical chirality density for the free electromagnetic (EM) field is \( \text{[2,3]} \)

\[
C = \frac{1}{2} \left( -E \cdot \frac{\partial B}{\partial t} + B \cdot \frac{\partial E}{\partial t} \right),
\]

(1)

where \( E \) and \( B \) are the electric and magnetic fields, respectively \( \text{[4]} \). (Throughout this Letter, we adopt the system of units in which the speed of light and the permittivity of free space are \( c = \varepsilon_0 = 1 \).) The flux of optical chirality is given by the three-vector

\[
S = \frac{1}{2} E \times \frac{\partial E}{\partial t} + \frac{1}{2} B \times \frac{\partial B}{\partial t},
\]

(2)

while the differential conservation law for optical chirality \( \text{[2]} \)

\[
\frac{\partial}{\partial t} C + \nabla \cdot S = 0
\]

(3)

is satisfied if \( E \) and \( B \) obey the free Maxwell equations

\[
\nabla \times B = \frac{\partial E}{\partial t}, \quad \nabla \times E = -\frac{\partial B}{\partial t}, \quad \nabla \cdot E = 0, \quad \nabla \cdot B = 0.
\]

(4)

Optical chirality is given by the integral of \( C \) over the space, \( \int d^3x C \), and is a constant of motion for free electromagnetism \( \text{[2]} \).

In Ref. \( \text{[3]} \), Tang and Cohen demonstrated that, in the presence of an EM field, the dissymmetry in the excitation rate of two small chiral molecules that are related to each other by mirror reflection is determined by the optical chirality. These findings have motivated novel investigations into chiral light-matter interactions \( \text{[3,5–16]} \). Understanding these interactions is very important in various disciplines. For example, it is known that deriving products of a given handedness in chemical reactions can be crucial - because molecules of a given handedness must be used in order to design drugs without negative side-effects \( \text{[17]} \) - and chiral light has been suggested to serve as a useful tool in order to achieve this \( \text{[18,20]} \). Applications of chiral light to the detection and characterization of chiral biomolecules have been also discussed \( \text{[3]} \). Also, chiral light-matter interactions are useful in optical information processing \( \text{[21]} \).

In the mathematical study of the zilches, there is an important open question. That is: what are the symmetry transformations of the electric scalar potential \( \phi \)
and magnetic EM vector potential \((A)\) that leave invariant the standard EM action
\[
S[\phi, A] = \frac{1}{2} \int d^4x \left( E \cdot E - B \cdot B \right)
\]
with
\[
E = -\frac{\partial A}{\partial t} - \nabla \phi, \quad B = \nabla \times A
\]  
and that give rise to all zilch conservation laws with the use of Noether’s theorem? The full answer to this question has been unknown since the discovery of the zilches in 1964. Although the derivation and generalization of zilch conservation laws have been studied in various earlier works \([22–31]\), the only zilch conservation law that has hitherto been derived from symmetries of the action \([5]\) is the one concerning the conservation of optical chirality \([32]\). In particular, Philbin showed that optical chirality is the Noether charge corresponding to the following symmetry transformations \([32]\):
\[
\delta \phi = 0, \quad \delta A = \nabla \times \frac{\partial A}{\partial t}.
\]  

In this Letter, we provide the full answer to the aforementioned question for the first time, by identifying the underlying symmetries of the action \([5]\) that give rise to all zilch conservation laws. We call these symmetries the zilch symmetries of the action and the corresponding transformations of the EM four-potential \(A_\mu = (-\phi, A)\) the zilch symmetry transformations \([see \ Eq. (19)]\). We show that the zilch symmetry transformations of the four-potential belong to the enveloping algebra of a “hidden” invariance algebra of free Maxwell’s equations. This “hidden” algebra closes on the 30-dimensional real Lie algebra \(so(6, \mathbb{C})_\mathbb{R}\) (i.e. the ‘realification’ of the complex Lie algebra \(so(6, \mathbb{C})\)) up to gauge transformations of the four-potential. Our results for the EM four-potential provide an important insight into the connection between symmetries and the conservation of all zilches since the EM four-potential is the fundamental field variable in electromagnetism, not the electric and magnetic fields \([33]\).

We also study the zilch continuity equations in the presence of electric charges and currents. More specifically, we present a new way to derive the known continuity equation for optical chirality \([34]\)
\[
\frac{\partial}{\partial t} C + \nabla \cdot S = \frac{1}{2} \left( j \frac{\partial B}{\partial t} - \frac{\partial j}{\partial t} \cdot B \right)
\]  
(\(j\) is the material electric current density), from symmetries of the standard interacting EM action \([Eq. (38)]\) in which the EM four-potential couples to a non-dynamical material four-current. In Ref. \([34]\) the continuity equation \([7]\) was obtained from the complementary fields formalism, while a similar continuity equation had been first obtained in Ref. \([3]\). Apart from Eq. \((7)\), in this Letter, we also obtain new continuity equations for the rest of the zilches in the presence of electric charges and currents from symmetries of the interacting EM action \([35]\). Before concluding this Letter, we will pose the interesting open question of whether the invariance of the interacting EM action with a non-dynamical material four-current can be extended to the case where the material four-current is dynamical. Then, we will conclude by discussing why our results are interesting and useful in view of future investigations into the role of all zilches in light-matter interactions.

Note that, recently, Smith and Strange shed light on the mystery of the physical meaning of all zilches for certain topologically non-trivial vacuum EM fields \([35]\), which is something that is not widely known. Thus, we now know that all zilches - not only optical chirality - are conserved quantities with physical significance and, hence, it is of fundamental importance to uncover the symmetries of the basic dynamical variable, \(A_\mu\), underlying their conservation. These symmetries are presented in this Letter.

In what follows, after presenting our conventions and introducing the zilch tensor \([Eq. (12)]\) \([2, 36]\), we summarize the main results of this Letter, namely the derivation of all zilch conservation laws for free electromagnetism, as well as the derivation of zilch continuity equations in the presence of electric charges and currents, from symmetries of the standard action functional.

**Conventions.** — Greek tensor indices run from 0 to 3 and Latin tensor indices from 1 to 3. We follow the Einstein summation convention, while indices are raised and lowered with the mostly plus Minkowski metric \(g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)\). A spacetime point in standard Minkowski coordinates is \(x^\mu = (x^0, x^1, x^2, x^3) \equiv (t, x')\). The totally antisymmetric tensors in 4 and 3 dimensions are \(\epsilon^{\mu\nu\rho\sigma}\) and \(\epsilon^{ijk}\), respectively \((\epsilon^{0123} = -\epsilon^{123} = -1)\).

From this point, let us follow the relativistic notation for electromagnetism with \(A_\mu = (-\phi, A)\) being the EM four-potential. The action \([3]\) is expressed as
\[
S[A_\mu] = -\frac{1}{4} \int d^4x \, F^{\mu\nu} F_{\mu\nu},
\]  
where the antisymmetric EM tensor is defined as \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) (with \(F_{0i} = -E_i\) and \(F_{ik} = e_{ikm} B^m\)). The free Maxwell’s equations \(\partial^\nu F_{\nu\mu} = 0\) are expressed in potential form as
\[
\Box A_\mu - \partial_\nu \partial^\nu A_\mu = 0,
\]  
where \(\Box = \partial^\nu \partial_\nu\). Because of the definition of \(F_{\mu\nu}\) in terms of the four-potential, the equation
\[
\partial_\nu F_{\mu\nu} + \partial_\nu F_{\nu\mu} + \partial_\mu F_{\nu\nu} = 0
\]  
is identically satisfied. Equation \((9)\), as well as the action \([3]\), are invariant under infinitesimal gauge-transformations
\[
\delta_{\text{gauge}} A_\mu = \partial_\mu a,
\]  
where \(a\) is an arbitrary scalar function.
II. THE ZILCH TENSOR AND THE ZILCHES

It is known that the zilch conservation laws can be conveniently described in terms of the zilch tensor \( Z_{\mu \nu} \) \[2, 36\]. This tensor is expressed in terms of \( F_{\mu \nu} \) and its dual, \( *F_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} F^{\rho \sigma} \), as follows \[36\]:

\[
Z_{\mu \nu} = - * F_{\mu \rho} \partial_\rho F_{\nu \lambda} + F_{\mu \lambda} \partial_\rho * F_{\lambda \nu}.
\] (12)

Using the following identity \[36\]:

\[
\partial_\rho (* F_{\rho \lambda} F^{\mu \lambda}) = - \frac{1}{4} \delta^\mu_\rho \partial_\rho (* F^{\lambda \kappa} F_{\kappa \lambda}),
\] (13)

the zilch tensor \[12\] can be equivalently expressed as

\[
Z_{\mu \nu} = - * F_{\mu \lambda} \partial_\rho F_{\nu \lambda} - * F_{\nu \lambda} \partial_\rho F_{\mu \lambda} - \frac{1}{2} \delta^\mu_\rho * F^{\lambda \kappa} \partial_\rho F_{\lambda \kappa}.
\] (14)

This expression makes manifest that the properties \( Z_{\mu \nu} = Z_{\nu \mu} \) and \( Z_{\mu \nu} = 0 \) are identically satisfied. Moreover, by using free Maxwell’s equations, it is straightforward to show that the zilch tensor is divergence-free with respect to all of its indices and also satisfies \( Z^{\rho}_{\mu \nu} = 0 \) \[36\]. All components of the zilch tensor can be readily expressed in terms of the electric and magnetic fields and their first derivatives \[36\]. In particular, the optical chirality density \( C \) [Eq. (11)] is related to the zilch tensor as \( Z^{000} = 2C \).

The ten zilches are given by the following ten time-independent quantities \[2, 36\]:

\[
\mathcal{Z}^{\mu \nu} = \mathcal{Z}^{\nu \mu} = \int d^3x Z^{\mu \nu 0},
\] (15)

with \( \partial / \partial t = 0 \). Only nine zilches in Eq. (15) are independent since \( Z^{\mu \nu 0} = 0 \). The \( \mu 0 \)-component \( (Z^{\mu 0}) \) of the zilch tensor is the spatial density of the zilch \( \mathcal{Z}^{\mu \nu} \), and the \( \mu \nu \)-components \( (Z^{\mu \nu}) \) are the components of the three-vector describing the corresponding flux \[2\]. The time-independence of the ten zilches follows from the ten differential conservation laws described by \( \partial_\rho Z^{\mu \nu \rho} = 0 \). The conservation law \[36\] for optical chirality corresponds to \( \frac{1}{2} (\partial_0 Z^{000} + \partial_\nu Z^{00\nu}) = 0 \), while optical chirality is \( \frac{1}{2} Z^{000} = \int d^3x C \).

For later convenience, note that the integral in Eq. (15) has the symmetry property

\[
\int d^3x Z^{\mu \nu 0} = \int d^3x Z^{\nu \mu 0} = \int d^3x Z^{\mu \nu 0} = \int d^3x Z^{\mu 0 \nu} = \int d^3x Z^{0 \nu \mu}.
\] (16)

because the difference \( Z^{\mu 0 \nu} - Z^{0 \nu \mu} \) can always be expressed as a spatial divergence \[36\]:

\[
Z^{\mu 0 \nu} - Z^{0 \nu \mu} = \partial_\lambda \Lambda_{\mu \nu \lambda},
\]

where the explicit expression for the tensor \( \Lambda \) is not needed for the present discussion. It immediately follows that the difference \( Z^{\mu 0 \nu} - Z^{0 \nu \mu} \) can also be written as a spatial divergence. Hence, the \( \mu \nu \)-zilch, \( \mathcal{Z}^{\mu \nu} \), can be actually interpreted as the time-independent quantity that corresponds to any of the three differential conservation laws: \( \partial_\nu Z_{\mu \nu} = 0 \) (which is the one used by Lipkin \[2\]), \( \partial_\nu Z^{\mu \nu} = 0 \) and \( \partial_\nu Z^{0 \nu \mu} = 0 \). These differential conservation laws are not independent of each other. For example, the conservation law \( \partial_\nu Z^{\mu \nu} = 0 \) can be re-written as \( \partial_\nu Z^{\mu 0 \nu} = 0 \) by using the relations

\[
\partial_\nu Z^{\mu 0 \nu} = \partial_\nu \left( Z^{\mu 0 \nu} + \partial_\nu \Lambda_{\mu \nu \lambda} \right)
\] (17)

and

\[
\partial_\nu Z^{\mu \nu} = \partial_\nu \left( Z^{\mu \nu} - \partial_\nu \Lambda_{\mu \nu \lambda} \right)
\] (18)

for the corresponding spatial densities and fluxes, respectively.

III. NOETHER SYMMETRIES FOR ALL ZILCHES

The zilch symmetries of the standard free action \[48\] underlying the conservation of the zilch correspond to the following infinitesimal transformations:

\[
\Delta A_\nu = n^\rho \tilde{n}^\mu \epsilon_{\mu \nu \sigma \lambda} \partial^\sigma \partial_\rho A_\lambda,
\] (19)

where \( \tilde{n}^\mu \) and \( n^\nu \) are two arbitrary constant four-vectors. We call the transformations in Eq. (19) the zilch symmetry transformations of the four-potential. Note that the symmetry transformation given by Eq. (19) corresponds to a special case of the zilch symmetry transformation \[19\] with \( \tilde{n}^\mu = n^\mu = \delta^\mu_0 \).

It is useful to examine the way in which the zilch symmetry transformation \[19\] acts on the EM tensor for off-shell field configurations; that is

\[
\Delta F_{\mu \nu} = \partial_\rho A_\nu \partial^\rho A_\mu - \partial_\rho A_\mu \partial^\rho A_\nu = \tilde{n}^\rho n^\sigma \left( \partial_\rho A_\nu \partial^\sigma A_\mu - \epsilon_{\mu \nu \sigma \lambda} \partial_\rho \partial_\lambda F^{\rho \sigma} \right),
\] (20)

where we have made use of the following important off-shell identity \[48\]:

\[
\partial_\rho * F_{\mu \nu} + \partial_\nu * F_{\rho \mu} + \partial_\mu * F_{\nu \rho} = \epsilon_{\mu \nu \sigma \lambda} \partial^\sigma F^{\rho \lambda}.
\] (21)

We now proceed to demonstrate that the zilch symmetry transformation \[19\] is indeed a symmetry of the action \[48\] and then apply Noether’s theorem. We find that the variation

\[
\Delta S = - \frac{1}{2} \int d^4x F^{\mu \nu} \Delta F_{\mu \nu}
\] (22)

is given by a total divergence (without making use of the equations of motion), as

\[
\Delta S = \int d^4x \partial_\nu D^\nu
\] (23)
with \[39\]
\[
D^\nu = \frac{1}{2} \partial^\rho \hat{n}^\mu \left( 2 * F^{\lambda \nu} \partial_\rho F_{\mu \lambda} + Z_{\mu \nu} \rho + \delta_{\rho}^\nu F_{\mu \rho} \partial^\beta F_\beta^\sigma \right).
\]
(24)

Now, the usual procedure [40] can be followed in order to construct the conserved Noether current, \(V^\nu\), associated with the zilch symmetry transformation \([19]\), as
\[
V^\nu = \frac{\partial \mathcal{L}}{\partial (\partial_\rho A_\mu)} \Delta A_\mu - D^\nu,
\]
(25)
where \(\mathcal{L} = -\frac{1}{4} F^{\alpha \beta} F_{\alpha \beta}\) is the free EM Lagrangian density. Substituting the expression for \(D^\nu\) \([\text{Eq. (24)}]\) into \(\text{Eq. (25)}\) and making use of the identity \([13]\), we find
\[
V^\nu = \frac{1}{2} \hat{n}^\rho \partial^\nu \left( Z_{\mu \nu} \rho - \delta_{\rho}^\nu F_{\mu \rho} \partial^\beta F_\beta^\sigma \right).
\]
(26)

The definition of a conserved Noether current is not unique; we are free to add any term that vanishes on-shell and/or any term that is equal to the divergence of any rank-2 antisymmetric tensor to the expression for the Noether current \([41]\). Thus, we are allowed to express the Noether current in \(\text{Eq. (26)}\) as
\[
V^\nu_{\text{zilch}} = \frac{1}{2} \hat{n}^\rho \partial^\nu \left( Z_{\mu \nu} \rho \right)
\]
(27)
with \(\partial_\rho V^\nu_{\text{zilch}} = 0\). Since the constant four-vectors \(n^\rho\) and \(\hat{n}^\mu\) in \(\text{Eq. (27)}\) are arbitrary, we conclude that
\[
\partial_\rho Z^{\mu \nu \rho} = 0.
\]
(28)

In other words, the zilch tensor is the conserved Noether current corresponding to the zilch symmetries \([19]\) of the free action \([5]\), while the corresponding Noether charges are the zilches \([15]\).

### IV. A “HIDDEN” INVARIANCE ALGEBRA AND THE ZILCH TRANSFORMATIONS

Here we investigate the relation of the zilch symmetry transformations \([19]\) to a “hidden” invariance algebra of free Maxwell’s equations in potential form \([9]\). Let us first discuss this “hidden” algebra.

Let \(\xi^\mu\) be any of the fifteen conformal Killing vectors of Minkowski spacetime satisfying
\[
\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{\partial^2 \xi_\sigma}{2} \eta_{\mu \nu}.
\]
(29)
The conformal Killing vectors of Minkowski spacetime consist of \([42]\): the four generators of spacetime translations, the six generators of the Lorentz algebra so\((3,1)\), the generator of dilations and the four generators of special conformal transformations \([42]\). These fifteen form a basis for the algebra of infinitesimal conformal transformations of Minkowski spacetime which is isomorphic to so\((4,2)\).

The “hidden” invariance algebra of free Maxwell’s equations \([9]\) is generated by two types of infinitesimal symmetry transformations of the four-potential. The first type corresponds to the well-known infinitesimal conformal transformations (conveniently described by the Lie derivative)
\[
L_{\xi} A_\mu = \xi^\lambda \partial_\lambda A_\mu + A_\lambda \partial_\mu \xi^\lambda, \quad \xi \in \text{so}(4,2),
\]
(30)
which form a representation of so\((4,2)\) on the solution space of Maxwell’s equations \([9]\). The second type of transformations corresponds to the little-known (“hidden”) transformations \([44]\)
\[
T_{\xi} A_\mu = \xi^\rho \epsilon_{\rho \mu \sigma \lambda} \partial^\sigma A^\lambda, \quad \xi \in \text{so}(4,2).
\]
(31)

If \(A_\mu\) is a solution of Maxwell’s equations, then so are \(L_{\xi} A_\mu\) and \(T_{\xi} A_\mu\) for all \(\xi \in \text{so}(4,2)\) \([44]\). The effect of the “hidden” transformation \([31]\) on \(F_{\mu \nu}\) corresponds to the product of a duality transformation with an infinitesimal conformal transformation \([15]\) as
\[
T_{\xi} F_{\mu \nu} \equiv \partial_\rho T_{\xi} A_\rho - \partial_\nu T_{\xi} A_\mu = L_{\xi} \star F_{\mu \nu},
\]
(32)
where
\[
L_{\xi} \star F_{\mu \nu} = \xi^\rho \partial_\rho F_{\mu \nu} + \star F_{\mu \rho} \partial_\nu \xi^\rho + \star F_{\rho \sigma} \partial_\nu \xi^\rho.
\]
(33)
The structure of the “hidden” invariance algebra of Maxwell’s equations in potential form is determined by the Lie brackets:
\[
[L_{\xi'}, L_{\xi}] A_\mu = L_{[\xi', \xi]} A_\mu
\]
(34)
\[
[L_{\xi'}, T_{\xi}] A_\mu = T_{[\xi', \xi]} A_\mu
\]
(35)
and
\[
[T_{\xi'}, T_{\xi}] A_\mu = - L_{[\xi', \xi]} A_\mu + \partial_\mu \left( [\xi', \xi]^\rho A^\sigma - \eta^\rho_{\nu \sigma} \eta^\nu_{\mu \rho} \right).
\]
(36)
where, e.g., \([L_{\xi'}, L_{\xi}] = L_{\xi'} L_{\xi} - L_{\xi} L_{\xi'}\), while \(\xi\) and \(\xi'\) are any two basis elements of so\((4,2)\) with \([\xi', \xi]^\rho = L_{\xi'} \epsilon^\rho\) \([36,39]\). We observe the appearance of a gauge transformation of the form \([31]\) in the last term of \(\text{Eq. (36)}\) \([36]\). To the best of our knowledge, the explicit expressions for the commutators \([36,39]\) and \([36,39]\) appear here for the first time. The commutation relations in Eqs. \([33-36]\) coincide with the commutation relations of the 30-dimensional real Lie algebra so\((6,\mathbb{C})\) \([10]\) (up to the gauge transformation in \(\text{Eq. (39)}\)). The so\((6,\mathbb{C})\) invariance of Maxwell’s equations expressed in terms of the electric and magnetic fields has been demonstrated in Ref. \([46]\).

Now, let us denote the zilch symmetry transformation \([19]\) with associated Noether current corresponding to \(Z_{\alpha \beta}\) \((\alpha\) and \(\beta\) have fixed values) as \(\Delta_{(\beta, \alpha)} A_\mu\). The latter is readily expressed as [see \(\text{Eq. (19)}\)]
\[
\Delta_{(\beta, \alpha)} A_\mu = \partial_\beta \left( \epsilon_{\alpha \rho \sigma \lambda} \partial^\sigma A^\lambda \right) = L_{P_{(\beta)}} T_{P_{(\alpha)}} A_\mu.
\]
(37)
It is obvious from this expression that $\Delta_{(\beta,\alpha)}A_{\mu}$ is given by the product of a “hidden” transformation (31) with respect to the translation Killing vector $P_{(\alpha)} = \partial_\alpha$ and a Lie derivative (30) with respect to the translation Killing vector $P_{(\beta)} = \partial_\beta$. This makes clear that the zilch invariance algebra $[\Delta]_{(\alpha)}$ belongs to the enveloping algebra of our “hidden” invariance algebra [and so do all transformations of the form (19)].

V. ZILCH CONTINUITY EQUATIONS IN THE PRESENCE OF ELECTRIC CHARGES AND CURRENTS

In the presence of a non-dynamical material four-current, $J^\mu = (\rho, J)$, the standard interacting EM action is

$$S' = S + \int d^4x \, J^\nu A_\nu = \int d^4x \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\nu A_\nu \right).$$

(38)

Let us consider the variation of $S'$ under the following simultaneous transformations of $A_\nu$ and $J^\nu$:

$$\Delta A_\nu = n^\mu \bar{n}_\mu \epsilon_{\mu\nu\rho\sigma}; \partial_\rho A_\sigma,$$

(39)

$$\Delta J^\nu = n^\mu \bar{n}_\mu \epsilon_{\mu\nu\rho\sigma}; \partial_\rho J^\sigma,$$

(40)

where $n^\mu$ and $\bar{n}_\mu$ are two arbitrary constant four-vectors, while Eq. (39) coincides with the zilch symmetry transformation (19). The variation of the free action, $S$, is already known to be a total divergence [see Eq. (23)]. Also, after a straightforward calculation, we find that the variation of the interaction term is a total divergence, as

$$\Delta \left( \int d^4x J^\nu A_\nu \right) = \int d^4x \, \partial_\nu D_\nu,'$$

(41)

where

$$D_\nu,' = \bar{n}_\mu n^\mu (\delta_\nu^\lambda \epsilon_{\mu\nu\rho\sigma}; F_{\mu\lambda} - \partial_\rho F_\lambda J^\sigma \epsilon_{\mu\nu\rho\sigma}).$$

(42)

Thus, the variation of the interacting action is

$$\Delta S' = \int d^4x \, \partial_\nu (D_\nu' + D_\nu,),$$

(43)

where $D_\nu'$ is given by Eq. (24).

Now, by applying the standard Noether algorithm (40), we find the following continuity equations for the zilch tensor:

$$\partial_\lambda Z^{\mu\nu\lambda} = J_\lambda \partial^\nu F_{\mu\lambda} - * F_{\mu\lambda} \partial^\nu J_\lambda.$$

(44)

These continuity equations determine the rate of gain or loss of the quantity $\int d^4x Z^{\mu\nu\lambda}$ with spatial density given by $Z^{\mu\nu\lambda}$ and flux components given by $Z^{\mu\nu\lambda}$. The continuity equations (44) can be re-expressed in the form of continuity equations for the zilches (15), with spatial density given by $Z^{\mu\nu\lambda}$ and flux components given by $Z^{\mu\nu\lambda}$. as follows. First, we observe that, although in the presence of electric charges and currents the quantity $\int d^4x Z^{\mu\nu\lambda}$ and the $\mu\nu$-zilch, $\int d^4x Z^{\mu\nu\lambda}$ [Eq. (15)] are not equal to each other unless $\mu = \nu = 0$ (because the symmetries of (15) no longer holds), they are related to each other by (17)

$$Z^{\mu\nu\rho} - Z^{\mu\nu\rho} = \frac{1}{2} \left( \epsilon^{\mu\nu\rho} \partial_\nu S_\rho - \epsilon^{\lambda\nu\rho} \partial_\nu T_\rho - \epsilon^{\epsilon\nu\rho} \partial_\nu \partial_\rho T_\epsilon - \epsilon^{\nu\rho\lambda} \partial_\lambda T_\rho \right.$$

$$\left. - \epsilon^{\nu\rho\lambda} \partial_\lambda T_\rho + \epsilon^{\nu\rho\lambda} \partial_\lambda T_\rho \right).$$

(45)

where

$$T_\alpha = -F_{\alpha\lambda} F_{\beta\mu} - \frac{1}{4} \delta_\beta^\alpha F_{\kappa\lambda} F_{\kappa\lambda}$$

(46)

is the Maxwell stress-energy tensor (with $\partial_\alpha T_\beta = J_\alpha F_{\beta\mu}$). Then, by taking the divergence of Eq. (45) with respect to the index $\rho$ and using the continuity equation (41) we find

$$\partial_\nu Z^{\mu\nu\rho} = \eta^{\mu\nu} F_{\rho\lambda} \partial_\lambda J^\sigma - * F^{\mu\nu} (\partial_\nu J^\sigma - \partial_\sigma J^\nu)$$

$$- * F^{\mu\nu} (\partial_\nu J^\sigma - \partial_\sigma J^\nu).$$

(47)

These are the ten continuity equations determining the rate of gain (or loss) for the ten zilches (15) in the presence of electric charges and currents. For $\mu = \nu = 0$, both continuity equations (44) and (47) coincide with the known equation (17) for optical chirality. To the best of our knowledge, the other nine continuity equations in Eq. (47) are presented here for the first time. (The continuity equations [13] and [17] have been verified using the field equations.)

VI. AN INTERESTING OPEN QUESTION

Let us suppose that the EM field interacts with a dynamical matter field with corresponding four-current $J^\mu$. Now, the action of the full interacting theory is

$$\int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^\nu A_\nu \right) + S_{\text{matter}},$$

(48)

where $S_{\text{matter}}$ is the action corresponding to the free matter field. According to our earlier discussion, the simultaneous transformations (39) and (40) (with $J^\nu$ replaced by $J^\nu$) are symmetries of the first two terms in Eq. (48). Motivated by this observation, we may pose the question of whether one could identify symmetries of the full interacting theory (i.e. symmetries of all three terms in Eq. (48)). In other words, is it possible to identify a transformation of the matter field such that: this transformation is a symmetry of $S_{\text{matter}}$, while the four-current $J^\mu$ transforms as in Eq. (40)?
VII. DISCUSSION

Almost sixty years after Lipkin’s discovery [2], the symmetries of the standard EM action [5] underlying the conservation of all zilches were found [see Eq. (19)]. In the presence of a non-dynamical material four-current, we gave a new derivation of the continuity equation for optical chirality [7], and we also derived new zilch continuity equations [47], by identifying the underlying symmetries [Eqs. (39) and (40)] of the interacting action [38].

A particularly interesting uninvestigated question is the one concerning the role of all zilches in light-matter interactions - the case of optical chirality is the only exception since its role has been studied [3]. The importance of this question becomes manifest by considering the fact that a physical interpretation for all zilches has been recently provided [35]. In particular, in Ref. [35] it was found that the zilches of a certain class of topologically non-trivial EM fields in vacuum can be expressed in terms of energy, momentum, angular momentum and helicity of the fields. Also, it was demonstrated that the zilches of these fields encode information about the topology of the field lines. We hope that the results presented in this Letter will be useful in future attempts to study the role of all zilches in light-matter interactions.

More specifically, motivated by the interpretation and applications of the known continuity equation (7) for optical chirality [3, 48, 50], it is natural to interpret each of our new zilch continuity equations [Eq. (47)] as determining the rate of loss or gain of the corresponding “zilch quantity” of the EM field. Electrically charged matter acts as a source or sink for the zilch quantities, while the source (or sink) term in each continuity equation is expressed in terms of quantities that are both physical and observable (see the right-hand side of Eq. (47)). This makes the continuity equations a useful tool for experimental investigations concerning the role of all zilches in light-matter interactions (as in the case of optical chirality [3]).

The results summarized in the present Letter establish a clear connection between all zilch continuity equations and symmetries of the standard EM action. Having identified all zilches with Noether charges, we can interpret them as the generators of the corresponding symmetry transformations [19] of the four-potential in the standard (classical or quantum) EM theory [32, 40, 50]. In the case of optical chirality, the explicit knowledge of the underlying symmetry generator is known to offer physical insight, since it allows the identification of the optical chirality eigenstates with plane waves of circular polarization [32]. Similarly, the symmetry transformations [19] uncovered in this Letter can be used to identify the eigenstates of all zilches, which is something that we leave for future work.

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