ON THE VISCOSITY SOLUTIONS TO SOME NONLINEAR ELLIPTIC EQUATIONS

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Abstract. We consider viscosity solutions of a class of nonlinear degenerate elliptic equations on bounded domains. We prove comparison principles and a priori supremum bounds for the solutions. We also address the eigenvalue problem and, in many instances, show the existence of a first eigenvalue and a first positive eigenfunction.

1. Introduction and statements of main results

In this work, we study issues related to the eigenvalue problem for some nonlinear degenerate elliptic operators. This may be considered as a follow-up of the work in [5], where we used the setting of viscosity solutions to show the existence of the first eigenvalue and a positive first eigenfunction of the infinity-Laplacian. The current work continues the effort of studying similar questions for a more general class of nonlinear elliptic and, possibly degenerate, operators including the $p$-Laplacian. See [3, 7, 9, 10, 11].

To state our results more precisely, we introduce notations that will be used throughout this work. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain, $\overline{\Omega}$ its closure and $\partial \Omega$ its boundary. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be continuous, $\mathbb{S}^n$ denote the set of $n \times n$ symmetric matrices and $H(p, X)$ be continuous, for $(p, X) \in \mathbb{R}^n \times \mathbb{S}^n$. We study properties of viscosity solutions of problems of the type

\begin{equation}
H(Du, D^2u) + f(x, u) = 0, \quad \text{in } \Omega, \text{ and } u = h \text{ on } \partial \Omega,
\end{equation}

where $h \in C(\partial \Omega)$. By a solution we mean a function $u \in C(\overline{\Omega})$ that solves (1.1) in the viscosity sense.

We require that the operator $H$ satisfy monotonicity in $X$, homogeneity in $p$ and $X$, a kind of coercitivity and, in some instances, will be taken to be invariant under reflections and rotations, see below. Our main effort in this work is to study questions related to the eigenvalue problem in (1.1) although some of our work applies to a more general class of functions $f$.

We now discuss conditions that $H$ will satisfy in this work and state the main results of this work. Let $o$ denote the origin in $\mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ will be occasionally written as $(x_1, x_2, \cdots, x_n)$. By $I$ we denote the $n \times n$ identity matrix, $O$ will denote the $n \times n$ matrix with all entries equalling zero. Also, $e$ will always stand for a unit vector in $\mathbb{R}^n$.

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Throughout this work we take $H \in C(\mathbb{R}^n \times S^n, \mathbb{R})$ and we require that $H(p, O) = 0$, $\forall p \in \mathbb{R}^n$. We now describe the conditions that $H$ satisfies.

**Condition A (Monotonicity):** We assume that $H(p, X)$ is continuous at $p = 0$ for any $X \in S^n$. In addition, for any $X, Y \in S^n$ with $X \leq Y$,

$$H(p, X) \leq H(p, Y), \quad \forall p \in \mathbb{R}^n.$$ (1.2)

It is clear that if $X \geq 0$ then $H(p, X) \geq 0$, for any $p$.

**Condition B (Homogeneity):** We assume here are constants $k_1 \geq 0$ and $k_2 > 0$, an odd integer, such that for any $\theta > 0$ and any $(p, X) \in \mathbb{R}^n \times S(n)$,

$$H(\pm p, X) = H(p, X) \quad \text{and} \quad H(\theta p, X) = |\theta|^{k_1} H(p, X), \quad \forall \theta \in \mathbb{R},$$ (1.3)

$$H(p, \pm X) = \pm H(p, X) \quad \text{and} \quad H(p, \theta X) = \theta^{k_2} H(p, X), \quad \forall \theta > 0.$$

We define $k = k_1 + k_2$ and $\gamma = k_1 + 2k_2$. For this work, we take $k_2 = 1$, although many of our results continue to hold if $k_2 > 1$. Set

$$k = k_1 + 1 \quad \text{and} \quad \gamma = k_1 + 2.$$ (1.4)

**Condition C (Coercivity):** We require that $H$ be coercive in the following sense. Let $e$ denote a unit vector in $\mathbb{R}^n$. Recall that $(e \otimes e)_{ij} = e_i e_j$. For every $-\infty < s < \infty$, we set

$$m_1(s) = \inf_{|e|=1} H(e, I - s e \otimes e) \quad \text{and} \quad m_2(s) = \sup_{|e|=1} H(e, I - s e \otimes e).$$

Let $k_1$ and $k = k_1 + 1$ be as in (1.3) and (1.4). Set $\hat{s} = k_1/k$. We impose that

$$0 < m_1(\hat{s}) \leq m_2(\hat{s}) < \infty.$$ (1.5)

More generally, we require that there are $-\infty < s_1 \leq 1 \leq s_0 < \infty$ (see (1.2)) such that

$$0 < m_1(s) \leq m_2(s) < \infty \quad \forall \ s \leq s_1, \quad \text{and} \quad m_2(s) < -\ell, \quad \forall \ s \geq s_0,$$ (1.6)

where $0 < \ell < \infty$. With (1.5) in view, we set

$$m_1 = m_1(\hat{s}), \quad m_2 = m_2(\hat{s}), \quad \alpha = \frac{k_1 + 2}{k} = \frac{\gamma}{k}, \quad \text{and} \quad \sigma = \frac{1}{\alpha m_1^{1/k}}.$$ (1.7)

In Part II of work, we will distinguish between the following two cases.

$$\exists 1 < \tilde{s} < 2 \text{ such that } m_2(\tilde{s}) < 0, \text{ or } \exists \bar{s} \geq 2 \text{ such that } m_2(s) < 0, \ \forall s > \bar{s}.$$ (1.8)

Note that in (i), if $s < 1$ then $I - s e \otimes e$ is a positive definite matrix. In (ii), $\tilde{s}$ is minimal in the sense that $m_1(s) \geq 0$ for $s < \tilde{s}$. As an example, condition (i) holds for the $p$-Laplacian when $p > n$, and (ii) holds when $2 \leq p \leq n$. We discuss (1.8) further in Section 2.

**Condition D (Symmetry):** We assume that $H$ is invariant under rotations and reflections. As a result if $v(x) = v(r)$, where $r = |x - z|$ for some $z \in \mathbb{R}^n$, then

$$H(Dv, D^2v) = G(r, v'(r), v''(r)).$$ (1.9)
It is not difficult to check that some well-known operators such as the \( p \)-Laplacian \( (p \geq 2) \) and the infinity-Laplacian satisfy the four conditions A, B, C and D, see \[1\].

We now address the eigenvalue problem. We take \( a \in C(\Omega) \cap L^\infty(\Omega), \inf_{\Omega} a > 0 \). Consider the problem of finding \((\lambda, u)\) where \( \lambda \in \mathbb{R} \) and \( u \in C(\Omega) \) solve
\[
H(Du, D^2u) + \lambda a(x)|u|^{k-1}u = 0, \quad \text{in } \Omega, \quad \text{and } u = h \text{ on } \partial \Omega,
\]
where \( h \in C(\partial \Omega) \) and \( h > 0 \). We say \( \lambda \) is an eigenvalue of the operator \( H \) and \( u \neq 0 \) an eigenfunction corresponding to \( \lambda \), if \((\lambda, u)\) solves (1.10) with \( h = 0 \). Our main effort is to characterize first eigenvalue and the first eigenfunction, see \[3, 5, 9\]. To this end, we study (1.10) and show the existence of positive solutions when \( h > 0 \) and when \( \lambda \) is less than a certain value \( \lambda_1 > 0 \) which turns out to be the first eigenvalue of \( H \).

It turns out that when (1.8)(i) holds, conditions A, B and C suffice. However, (1.8)(ii) appears to be less tractable and additional conditions are required. This is because at this time it is not clear to us as to how to prove a Harnack’s inequality for non-negative super-solutions.

We now state the main results of this work. The set \( \Omega \subset \mathbb{R}^n, n \geq 2, \) will always stand for a bounded domain in this work. By \( \text{usc}(\Omega) \) we denote the class of all upper semi-continuous functions on \( \Omega \), and \( \text{lsc}(\Omega) \) will denote the class of all lower semi-continuous functions on \( \Omega \). The first result is a quotient type comparison principle for positive solutions. Let \( g, h : \Omega \times \mathbb{R} \to \mathbb{R} \) and \( a : \Omega \to \mathbb{R}, a > 0, \) be continuous. Suppose that \( m > 0 \) is such that
\[
(1.11) \quad h(x, t) \geq a(x)|t|^{m-1}t > g(x, t), \quad \forall (x, t) \in \Omega \times \mathbb{R}.
\]

**Theorem 1.1.** Let \( H \) satisfy conditions A and B, and \( g \) and \( h \) be as in (1.11). Suppose that \( u \in \text{usc}(\Omega), \) and \( v \in \text{lsc}(\Omega), \) \( v > 0, \) solve
\[
H(Du, D^2u) + g(x, u) \geq 0, \quad \text{and } H(Dv, D^2v) + h(x, v) \leq 0, \quad \text{in } \Omega.
\]
Recall \( k \) from (1.4). Then either \( u \leq 0 \) in \( \Omega \), or the following conclusions hold.

(a) Suppose that \( k = m \). (i) If \( U \subset \Omega \) is a compactly contained sub-domain of \( \Omega \) such that \( u > 0 \) somewhere in \( U \) then
\[
\sup_{U} \frac{u}{v} = \sup_{\partial U} \frac{u}{v} > 0.
\]
(ii) Assume that \( u > 0 \) somewhere in \( \Omega \), and \( U_j \subset U_{j+1} \subset \Omega, \) \( j = 1, 2, \cdots, \) are compactly contained sub domains of \( \Omega \), with \( \cup_j U_j = \Omega \). If \( \lim_{j \to \infty} \sup_{U_j} u/v < \infty, \) then
\[
0 < \sup_{\Omega} \frac{u}{v} = \lim_{j \to \infty} \left( \sup_{U_j} \frac{u}{v} \right).
\]

(b) Take \( k \neq m \). We assume further that either (i) \( k > m \) and \( (u/v)(z) > 1, \) for some \( z \) in \( \Omega \), or (ii) \( k < m \) and \( \sup_{\Omega} u/v < 1 \). Then the conclusions in (i) and (ii) of part (a) hold.
For the rest of the results, we assume that \( H \) satisfies conditions \( A, B \) and \( C \).

We now state a result on a priori supremum bounds and is useful for the eigenvalue problem for \( H \) on \( \Omega \). Let \( f \in C(\Omega \times \mathbb{R}, \mathbb{R}) \) and

\[
(1.12) \quad \sup_{\Omega \times [t_1, t_2]} |f(x, t)| < \infty, \quad \forall t_1, t_2 \text{ such that } -\infty < t_1 \leq t_2 < \infty.
\]

Assume that there are constants \(-\infty < s_1 \leq 0 \leq s_2 < \infty\) such that

\[
(1.13) \quad \limsup_{t \to \infty} \left( \sup_{\Omega} f(x, t) \right) \leq s_2 \quad \text{and} \quad \liminf_{t \to -\infty} \left( \inf_{\Omega} f(x, t) \right) \geq s_1.
\]

**Theorem 1.2.** Let \( \lambda \in \mathbb{R}, f : \Omega \times \mathbb{R} \to \mathbb{R} \) be as in (1.12) and (1.13), and \( h \in C(\partial \Omega) \). Suppose that \( u \in C(\Omega) \) solves

\[
H(Du, D^2 u) + \lambda f(x, u) = 0, \quad \text{in } \Omega, \quad \text{and } u = h \text{ on } \partial \Omega.
\]

(a) If \( |\lambda| \) is small enough then \( u \) is a priori bounded and \( \sup_{\Omega} |u| \leq K \), where \( K \) depends on \( \lambda, s_1, s_2, k, h \) and \( \Omega \).

(b) If \( s_1 = s_2 = 0 \) then, for any \( \lambda \), \( u \) is a priori bounded and \( \sup_{\Omega} |u| \leq K \), where \( K \) depends on \( \lambda, k, h \) and \( \Omega \).

We now provide existence results for the eigenvalue problem in (1.10). The conditions (1.8) (i) and (ii) play a crucial role in these statements and the following hold throughout this work.

\( \Omega \) is any bounded domain if (1.8) (i) holds, and \( \Omega \) satisfies a uniform outer ball condition if (1.8) (ii) holds.

**Theorem 1.3.** Suppose that \( \lambda > 0 \), and \( a(x) \in C(\Omega) \) with \( \inf_{\Omega} a > 0 \). For \( h \in C(\partial \Omega) \) with \( \inf_{\partial \Omega} h > 0 \), consider the boundary value problem

\[
(1.14) \quad \text{condition if (1.8) (ii) holds.}
\]

\[
(1.15) \quad H(Du, D^2 u) + \lambda a(x)|u|^{k-1}u = 0, \quad \text{in } \Omega \text{ and } u = h \text{ on } \partial \Omega.
\]

Set \( R = \text{diam}(\Omega) \) and \( \nu = \sup_{\Omega} a(x) \). Recall (1.14).

(a) If (1.8) (i) holds and \( 0 < \lambda < |m_2(s)|(2 - \bar{s})^k(\nu R^\gamma)^{-1} \) then (1.15) has a unique positive solution.

(b) Suppose that (1.8) (ii) holds and \( \Omega \) satisfies a uniform outer ball condition with optimal radius \( 2\rho > 0 \). Fix \( \beta > \bar{s} - 2 \) and \( s = \beta + 2 \). If

\[
0 < \lambda < \frac{|m_2(s)| \beta^k}{\nu R^\gamma} \left( \frac{\rho}{R} \right)^{k\beta}
\]

then (1.15) has a unique positive solution. Moreover, \( u > \inf_{\partial \Omega} h \) in \( \Omega \).

Let \( \delta > 0 \) and \( \lambda > 0 \). Consider the problem of finding a positive solution \( u_\lambda \in C(\overline{\Omega}) \) of

\[
(1.16) \quad H(Du_\lambda, D^2 u_\lambda) + \lambda a(x)u_\lambda^k = 0, \quad \text{in } \Omega, \quad \text{and } u_\lambda = \delta \text{ on } \partial \Omega.
\]
If we call
\begin{equation}
\lambda_\Omega = \sup \{ \lambda : \text{(1.16) has a positive solution } u_\lambda \} ,
\end{equation}
then \( \lambda_\Omega > 0 \) by Theorem 1.3. Our next result shows that the bound in (1.17) is independent of the boundary data.

**Theorem 1.4.** Suppose that (1.14) holds. Let \( \delta > 0, a(x) \in C(\Omega), \inf_\Omega a > 0, \) and \( h \in C(\partial \Omega), \) with \( \inf_{\partial \Omega} h > 0. \) Suppose that, for some \( \lambda > 0, \) the problem \( H(Du, D^2u) + \lambda a(x)|u|^{k-1}u = 0, \) in \( \Omega, \) and \( u = \delta \) on \( \partial \Omega, \) has a positive solution. Then the problem
\begin{equation}
H(Dv, D^2v) + \lambda a(x)|v|^{k-1}v = 0, \quad \text{in } \Omega \text{ and } v = h \text{ on } \partial \Omega,
\end{equation}
also has a positive solution.

The boundedness of \( \lambda_\Omega \) is shown in

**Theorem 1.5.** Suppose that \( H \) satisfies conditions A, B and C. Let \( \delta > 0 \) and \( a(x) \in C(\Omega) \cap L^\infty(\Omega), \inf_\Omega a > 0. \) Recall (1.14).

(a) Suppose that (1.8) (i) holds then \( \lambda_\Omega < \infty. \)  (b) Suppose that (1.8) (ii) holds and \( H \) also satisfies D then \( \lambda_\Omega < \infty. \)

Finally, we show

**Theorem 1.6.** Suppose that \( H \) satisfies conditions A, B and C. Let \( a \in C(\Omega, \mathbb{R}), \inf_\Omega a > 0. \) Consider the problem
\begin{equation}
H(Du, D^2u) + \lambda_\Omega a(x)|u|^{k-1}u = 0, \quad \text{in } \Omega \text{ and } u = 0 \text{ on } \partial \Omega,
\end{equation}
where \( u \in C(\overline{\Omega}). \) Recall (1.14).

(i) Suppose that (1.8) (i) holds then (1.19) has a positive eigenfunction \( u. \)  (ii) Suppose that \( H \) also satisfies D and (1.8) (ii) holds then (1.19) has a positive radial eigenfunction when \( \Omega \) is a ball.

At this time, it is not clear to us as to how to extend part (ii) to general domains. Also, our work does not address whether \( \lambda_\Omega \) is simple or isolated.

We describe the lay out of the work. In Section 2, we include some definitions, additional notations and useful calculations. Section 3 presents comparison principles when \( H \) satisfies condition A are of some what general nature. The remaining work is divided into two parts. Part I has Sections 4 and 5. Sections 6-9 are in Part II. Section 4 lists additional comparison principles under the conditions A and B and the proof of Theorem 1.1. We include a change of variables formula, important for Theorem 1.5. Sections 5-9 require that \( H \) satisfies A, B and C. Section 5 contains proofs of Theorems 5.4 and 1.2. Section 6 contains a discussion of questions related to the problem and shows that solutions \( u_\lambda \) of (1.16) are Lipschitz continuous in \( \lambda. \) Proofs of
Theorems 1.3 and 1.4 are in Section 7. Theorem 1.5 is proven in Section 8. We present a proof of Theorem 1.5 in Section 9.

Finally, we comment that our work is more along the lines of [5] and contain analogues of the results in [9].

2. Additional notations, definitions and calculations

We introduce additional notations, provide definitions for viscosity solutions of (1.1) and include some useful calculations. We use $o$ to denote the origin. By $B_s(p)$, $s > 0$, we will mean the ball of radius $s$ centered at $p$. In this work, all differential equations and inequalities will be understood in the sense of viscosity, see below and [8]. We assume throughout that $H \in C(\mathbb{R}^n \times S^n, \mathbb{R})$ satisfies condition $A$, see (1.2).

We define the notion of a viscosity solution $u$ to the following in $\Omega$,

$$H(Du, D^2u) + f(x, u) = 0 \text{ in } \Omega \text{ and } u = h \text{ on } \partial \Omega,$$

where $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ and $h \in C(\partial \Omega)$.

A function $u \in usc(\Omega)$ is said to be a viscosity sub-solution of the equation in (2.1), in $\Omega$, or solves $H(Du, D^2u) + f(x, u) \geq 0$, in $\Omega$, if the following holds. For any $\psi \in C^2(\Omega)$ such that $u - \psi$ has a maximum at a point $y \in \Omega$, we have

$$H(D\psi(y), D^2\psi(y)) + f(y, u(y)) \geq 0.$$ 

Similarly, $u \in lsc(\Omega)$ is said to be a viscosity super-solution of the equation (2.1) or solves $H(Du, D^2u) + f(x, u) \leq 0$, in $\Omega$, if, for any $\psi \in C^2(\Omega)$ such that $u - \psi$ has a minimum at $y \in \Omega$, we have

$$H(D\psi(y), D^2\psi(y)) + f(y, u(y)) \leq 0.$$ 

A function $u \in C(\Omega)$ is a viscosity solution if it is both a sub-solution and a super-solution.

We define $u \in usc(\overline{\Omega})$ to be a viscosity sub-solution to the problem (2.1) if $u$ is a sub-solution in $\Omega$ and $u \leq h$ on $\partial \Omega$. Similarly, $u \in lsc(\overline{\Omega})$ is a super-solution of (2.1) if $u$ is a super-solution in $\Omega$ and $u \geq h$ on $\partial \Omega$. We define $u \in C(\overline{\Omega})$ to be a solution to (2.1), if it is both a sub-solution and a super-solution of (2.1).

In this work, we will utilize radial sub-solutions and super-solutions. We discuss (1.8) in this context. Let $v(x) = v(r)$ where $r = |x - z|$, for some $z \in \mathbb{R}^n$. Set $e = (e_1, e_2, \cdots, n)$ where $e_i = (x - z)_i/r$, $i = 1, 2, \cdots, n$. Then for $x \neq z$,

$$H(Dv, D^2v) = H \left( v'(r)e, \frac{v'(r)}{r} (I - e \otimes e) + v''(r)e \otimes e \right),$$

where $I$ is the $n \times n$ identity matrix. We now impose conditions $A$, $B$ and $C$ on $H$. Recall (1.3)-(1.8),

$$k = k_1 + 1, \quad \gamma = k_1 + 2 \quad \text{and} \quad \alpha = \gamma/k.$$
We take \( v(r) = c \pm dr^\beta \), where \( d > 0 \) and \( \beta > 0 \). Using (2.2), we obtain, in \( r > 0 \),
\[
H(Dv, D^2v) = H \left[ \pm d^\beta r^{\beta-1}e, \pm d^\beta r^{\beta-2}\{I + (\beta - 2)e \otimes e\} \right] \\
= \pm d^\beta \beta^k r^{(\beta-1)k_1+\beta-2} H(e, I + (\beta - 2)e \otimes e) \\
(2.3) \\
= \pm (d^\beta)^k r^{k\beta-\gamma} H(e, I + (\beta - 2)e \otimes e).
\]
If (1.8)(i) holds and \( v \) is such that
\[
\text{Remark 2.1. Our results on existence use Perron's method, see [8]. Consider the problem of} \\
\text{the existence of a solution} \\
\text{for each} \quad c, \quad \text{we take} \\
\text{such that} \\
\text{where} \quad H = \frac{(d^\beta)^k}{r^{k\beta+\gamma}} H(e, I + (\beta - 2)e \otimes e).
\]
If (1.8)(ii) holds and \( v = c \pm dr^\beta \) with \( \beta = 2 - \bar{s} > 0 \), then \( 0 < \beta < 1 \) and (2.3) leads to
\[
H(Dv^+, D^2v^+) \leq -\frac{(\beta)^k}{r^{\gamma-k\beta}} |m_2(s)| < 0, \quad \text{and} \quad H(Dv^-, D^2v^-) \geq \frac{(\beta)^k}{r^{\gamma-k\beta}} |m_2(s)| > 0.
\]
If (1.8)(ii) holds and \( v = c \pm dr^\beta \) with \( \beta > \bar{s} - 2 > 0 \), then (2.4) leads to
\[
H(Dv^+, D^2v^+) \geq \frac{(\beta)^k}{r^{k\beta+\gamma}} |m_2(s)| > 0, \quad \text{and} \quad H(Dv^-, D^2v^-) \leq -\frac{(\beta)^k}{r^{k\beta+\gamma}} |m_2(s)| < 0,
\]
where \( s = \beta + 2 \). As a second application, set \( \beta = \alpha = \gamma/k \) in (2.3) (see (1.7)) and take \( v = c \pm dr^\alpha \) to obtain
\[
H(Dv, D^2v) = (d\alpha)^k H(e, I - \frac{k_1}{k} e \otimes e) \geq (d\alpha)^k m_1 = \left(\frac{d}{\sigma}\right)^k > 0.
\]
Remark 2.1. Our results on existence use Perron’s method, see [8]. Consider the problem of showing the existence of a solution \( u \in C(\Omega) \) of
\[
H(Du, D^2u) + f(x, u) = 0, \quad \text{in} \quad \Omega \quad \text{and} \quad u = h \quad \text{on} \quad \partial \Omega,
\]
where \( h \in C(\partial \Omega) \). Assume that the above admits a comparison principle. Let \( \varepsilon > 0 \), be a given small number. For each \( y \in \partial \Omega \), we construct (i) a sub-solution \( v \) such that \( v(y) = h(y) - \varepsilon \) and \( v \leq h \) on \( \partial \Omega \), and (ii) a super-solution \( w \) such that \( w(y) = h(y) + \varepsilon \) and \( w \geq h \) on \( \partial \Omega \). We obtain the existence of a solution \( u \).

3. Comparison principles under Condition A

We assume \( H \) satisfies condition A (see (1.2)) i.e., \( H(p, X) \) is continuous on \( \mathbb{R}^n \times S^n \), \( H(p, O) = 0 \) and, for any \( X, Y \in S^n \) with \( X \leq Y \), we have \( H(p, X) \leq H(p, Y) \), \( \forall p \in \mathbb{R}^n \). We prove some comparison principles for \( H \). The section begins with a version of a comparison principle that is used often in this work, also see [5].

Theorem 3.1. (Comparison Principle) Let \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) and \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) be continuous. Suppose that \( u \in \text{usc}(\Omega) \) and \( v \in \text{lsc}(\Omega) \) satisfy in the viscosity sense,
\[
H(Du, D^2u) + f(x, u(x)) \geq 0 \quad \text{and} \quad H(Dv, D^2v) + g(x, v(x)) \leq 0, \quad \text{in} \ \Omega.
\]
If \( \sup_{\Omega}(u - v) > \sup_{\partial\Omega}(u - v) \) then there is a point \( z \in \Omega \) such that
\[
(u - v)(z) = \sup_{\Omega}(u - v) \quad \text{and} \quad g(z, v(z)) \leq f(z, u(z)).
\]

Equivalently, if \( \inf_{\Omega}(v - u) < \inf_{\partial\Omega}(v - u) \) then there is a point \( z \in \Omega \) such that \( (v - u)(z) = \inf_{\Omega}(v - u) \) and \( g(z, u(z)) \leq f(z, v(z)) \).

Proof. We employ the ideas in [8] (also see [5]) and use the concept of sub-jets and sup-jets. We prove part (a), the proof of part (b) follows in an analogous manner. Let \( M = \sup_{\Omega}(u - v) \); define, for \( \varepsilon > 0 \),
\[
(3.1) \quad w_{\varepsilon}(x, y) := u(x) - v(y) - \frac{1}{2\varepsilon}|x - y|^2, \quad \forall (x, y) \in \Omega \times \Omega.
\]
Set \( M_{\varepsilon} := \sup_{\Omega \times \Omega} w_{\varepsilon}(x, y) \), and let \((x_{\varepsilon}, y_{\varepsilon}) \in \overline{\Omega} \times \overline{\Omega}\) be such that \( M_{\varepsilon} \) is attained at \((x_{\varepsilon}, y_{\varepsilon})\). The following are well-known, see [8].
\[
\lim_{\varepsilon \to 0} M_{\varepsilon} = \lim_{\varepsilon \to 0} \left( u(x_{\varepsilon}) - v(y_{\varepsilon}) - \frac{|x_{\varepsilon} - y_{\varepsilon}|^2}{2\varepsilon} \right) = M, \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{|x_{\varepsilon} - y_{\varepsilon}|^2}{2\varepsilon} = 0.
\]
Let \( z \in \overline{\Omega} \) be such that \( x_{\varepsilon} \) and \( y_{\varepsilon} \to z \), as \( \varepsilon \to 0 \). Clearly, \( M = (u - v)(z) \). Since \( M > \sup_{\partial\Omega}(u - v) \), there is an open set \( O \) such that \( z, x_{\varepsilon} \) and \( y_{\varepsilon} \in O \subset \subset \Omega \).

Since \((x_{\varepsilon}, y_{\varepsilon})\) is a point of maximum of \( w_{\varepsilon}(x, y) \), there exist \( X_{\varepsilon} \) and \( Y_{\varepsilon} \) such that \((x_{\varepsilon} - y_{\varepsilon})/\varepsilon, X_{\varepsilon}) \in \overline{J}^{2,+}u(x_{\varepsilon})\) and \((x_{\varepsilon} - y_{\varepsilon})/\varepsilon, Y_{\varepsilon}) \in \overline{J}^{2,-}v(y_{\varepsilon})\).

Moreover, we have, see [8],
\[
(3.2) \quad -\frac{3}{\varepsilon} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_{\varepsilon} & 0 \\ 0 & -Y_{\varepsilon} \end{pmatrix} \leq \frac{3}{\varepsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.
\]

The above clearly implies \( X_{\varepsilon} \leq Y_{\varepsilon} \), and using the definitions of \( \overline{J}^{2,+} \) and \( \overline{J}^{2,-} \), we see that (recall definitions of sub-jets and sup-jets)
\[
(3.3) \quad -f(x_{\varepsilon}, u(x_{\varepsilon})) \leq H((x_{\varepsilon} - y_{\varepsilon})/\varepsilon, X_{\varepsilon}) \leq H((x_{\varepsilon} - y_{\varepsilon})/\varepsilon, Y_{\varepsilon}) \leq -g(y_{\varepsilon}, v(y_{\varepsilon})).
\]

Now let \( \varepsilon \to 0 \) to conclude that \( g(z, v(z)) \leq f(z, u(z)) \). \qed

We now record consequences of Theorem 3.1.

Lemma 3.2. (Maximum Principle) Let \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) be continuous.

(a) Suppose that \( f(x, t) < 0, \forall (x, t) \in \Omega \times \mathbb{R}, \) and \( u \in \text{lsc}(\overline{\Omega}) \). If \( u \) solves \( H(Du, D^2u) + f(x, u) \geq 0 \), in \( \Omega \), then \( \sup_{\Omega} u = \sup_{\partial\Omega} u \).

(b) Analogously, let \( f(x, t) > 0, \forall (x, t) \in \Omega \times \mathbb{R}, \) and \( v \in \text{lsc}(\overline{\Omega}) \). If \( v \) solves \( H(Dv, D^2v) + f(x, v) \leq 0 \), in \( \Omega \), then \( \inf_{\Omega} u = \inf_{\partial\Omega} u \).

Proof. We prove part (a), part (b) follows in analogous manner. Let \( f < 0 \) and set \( m = \sup_{\partial\Omega} u \). Suppose that \( \sup_{\Omega} u > m \).

Set \( v(x) = m, \forall x \in \overline{\Omega} \) and \( g(x, t) = 0 \). Then \( H(Dv, D^2v) + g(x, v) = 0 \) and \( H(Du, D^2u) + f(x, u) \geq 0 \), in \( \Omega \). By our hypothesis, \( \sup_{\Omega}(u - v) > \sup_{\partial\Omega}(u - v) = 0 \). By Theorem 3.1 there
is a point \( z \in \Omega \) such that \((u - v)(z) = \sup_{\Omega}(u - v) > 0 \) and \( f(z, u(z)) \geq g(z, v(z)) = 0 \). This contradicts that \( f < 0 \). Hence, the conclusion holds. \( \square \)

The next result is a version of the strong maximum principle that holds under some restrictions on \( f \) and the boundary data. A more general version may be found in [4].

**Lemma 3.3. (Strong Maximum Principle)** Let \( f \in C(\Omega \times \mathbb{R}, \mathbb{R}) \). Assume that there is a \( c \in \mathbb{R} \) such that \( \inf_{\Omega} |f(x, t)| = 0 \) if and only if \( t = c \).

(i) Suppose that \( f \leq 0 \) and \( u \in \text{usc}(\overline{\Omega}) \) solves \( H(Du, D^2u) + f(x, u) \geq 0 \), in \( \Omega \). If \( \sup_{\partial \Omega} u > c \) or \( \sup_{\Omega} u < c \) then

\[
u(x) < \sup_{\partial \Omega} u, \forall x \in \Omega.
\]

(ii) Analogously, suppose that \( f \geq 0 \) and \( u \in \text{lsc}(\overline{\Omega}) \) solves \( H(Du, D^2u) + f(x, u) \leq 0 \), in \( \Omega \). If \( \inf_{\partial \Omega} u < c \) or \( \inf_{\Omega} u > c \) then

\[
u(x) > \inf_{\partial \Omega} u, \forall x \in \Omega.
\]

**Proof.** We prove (i), the proof of (ii) is similar. Suppose that the claim is false i.e., there is a point \( z \in \Omega \) such that \( u(z) = \sup_{\Omega} u \geq \sup_{\partial \Omega} u \). By our hypothesis, \( u(z) \neq c \). Let \( \varepsilon > 0 \) be small, define \( \psi_{\varepsilon}(x) = u(z) + \varepsilon|x - z|^2 \) in \( \Omega \). Then \( \psi_{\varepsilon} \in C^2(\Omega) \), \((u - \psi_{\varepsilon})(z) = 0 \) and \((u - \psi_{\varepsilon})(x) \leq -\varepsilon|x - z|^2 < 0 \), \( \forall x \in \overline{\Omega}, x \neq z \). Thus, for any \( \varepsilon > 0 \), \( z \) is the only point of maximum of \( u - \psi_{\varepsilon} \) in \( \Omega \). Using the definition of a viscosity sub-solution, we have

\[
H(D\psi_{\varepsilon}(z), D^2\psi_{\varepsilon}(z)) + f(z, u(z)) = H(0, -2\varepsilon I) + f(z, u(z)) \geq 0.
\]

Letting \( \varepsilon \to 0 \), we get \( 0 = H(0, O) \geq -f(z, u(z)) > 0 \). Thus, the claim holds. \( \square \)

Finally,

**Lemma 3.4.** Let \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) and \( h : \Omega \times \mathbb{R} \to \mathbb{R} \) be continuous. Suppose that (i) \( g(x, t) < h(x, t), \forall (x, t) \in \Omega \times \mathbb{R} \) and at least one of \( g(\cdot, t) \) and \( h(\cdot, t) \) is non-increasing in \( t \), or (ii) \( g = h \) and \( g \) is strictly decreasing in \( t \). Let \( u \in \text{usc}(\overline{\Omega}) \) and \( v \in \text{lsc}(\overline{\Omega}) \) satisfy \( H(Du, D^2u) + g(x, u) \geq 0 \) and \( H(Dv, D^2v) + h(x, v) \leq 0 \), in \( \Omega \). If \( u \leq v \), on \( \partial \Omega \), then \( u \leq v \) in \( \Omega \). Moreover,

\[
\sup_{\Omega}(u - v)^+ = \sup_{\partial \Omega} (u - v)^+.
\]

**Proof.** Suppose that \( g \) is non-increasing in \( t \) and \( \sup_{\Omega}(u - v) > \sup_{\partial \Omega}(u - v) \). By Theorem 3.1 there is a point \( z \in \Omega \) such that \((u - v)(z) = \sup_{\Omega}(u - v) > 0 \) and \( g(z, u(z)) \geq h(z, v(z)) \). Since \( u(z) > v(z) \), we get \( g(z, v(z)) \geq g(z, u(z)) \geq h(z, v(z)) \), a contradiction. Thus, the claim holds.

Next, set \( \mu = \sup_{\partial \Omega} (u - v) \) and assume \( \mu > 0 \). Define \( u_\mu = u - \mu \) and observe that

\[
H(Du_\mu, D^2u_\mu) = H(Du, D^2u) \geq -g(x, u) \geq -g(x, u_\mu), \quad \text{and} \quad \sup_{\partial \Omega} (u_\mu - v) = 0.
\]
If \( \sup_{\Omega}(u_\mu - v) > 0 \) then, by Theorem 3.1, there is a point \( z \in \Omega \) such that \( (u_\mu - v)(p) > 0 \) and \( g(z, u_\mu(z)) \geq h(z, v(z)) \). Since, \( v(z) < u_\mu(z) \), \( g(z, v(z)) \geq g(z, u_\mu(z)) \geq h(z, v(z)) \), a contradiction. Thus the claim holds.

\[ \blacksquare \]

**PART I**

4. **Proof of Theorem 1.1 and a change of variables result under conditions A and B**

In this section, \( H \) will satisfy conditions A (monotonicity) and B (homogeneity), see (1.2) and (1.3). We show some additional comparison principles that follow from Theorem 3.1, including Theorem 1.1. A version was derived in [5]. We also discuss a change of variables result. Recall the definition of \( k = k_1 + 1 \) in (1.4).

**Proof of Theorem 1.1:** The proof is similar to the one in [5]. Let \( U \subset \Omega \) be a compactly contained sub-domain of \( \Omega \). We assume that \( u > 0 \) somewhere in \( U \) and show that \( \sup_U (u/v) = \sup_{\partial U} u/v \).

Suppose that \( p \in U \) is such that \( u(p)/v(p) = \sup_U (u/v) > \sup_{\partial U} u/v \). Set \( \tau = u(p)/v(p) \). Since \( v > 0 \), we have \( u(p) > 0 \) and \( \tau > 0 \). Thus,
\[(4.1) \quad w(x) = u(x) - \tau v(x) \leq 0, \quad x \in U.\]

Using (1.2), (1.3), (1.4) and (1.11), we have that
\[(4.2) \quad H(D\tau v, D^2\tau v) \leq -\tau^k h(x, v(x)) \leq -\tau^{k-m} a(x)(\tau v(x))^m \quad \forall x \in \Omega.\]

From (4.1), \( w(x) < 0 \), for any \( x \in \partial U \), and \( \sup_U w = w(p) = 0 \). Thus, \( \sup_U w \geq \sup_{\partial U} w \), and applying Theorem 3.1 and (1.11) to \( u \) and \( \tau v \) (see (4.2)) there is a \( z \in U \) such that
\[(4.3) \quad w(z) = \sup_U w = 0 \quad \text{and} \quad g(z, u(z)) \geq \tau^{k-m} a(z)(\tau v(z))^m > 0.\]

From (4.3) we get \( u(z) = \tau v(z) > 0 \) and
\[(4.4) \quad g(z, u(z)) \geq \tau^{k-m} a(z)(\tau v(z))^m = \tau^{k-m} a(z) u(z)^m > \tau^{k-m} g(z, u(z)) > 0.\]

The above leads to a contradiction for \( k = m \) and part (a)(i) holds. We now prove part (a)(ii). Set \( \mu_j = \sup_{\partial U_j}(u/v) \). By part(i), \( \mu_j \)'s are increasing and \( \mu = \sup_j \mu_j < \infty \). If \( \sup_{\Omega}(u/v) > \mu \) then one can find a set \( U_j \) such that \( \sup_{U_j}(u/v) > \mu \). This violates part (a)(i) since \( \sup_{\partial U_j}(u/v) \leq \mu \).

From (4.4), \( \tau^{k-m} < 1 \). This also leads to a contradiction for part (b). Thus, the theorem holds.

\[ \blacksquare \]

**Remark 4.1.** Let \( u, v, g \) and \( h \) be as in the statement of Theorem 1.1. Since \( v > 0 \), the quotient comparison principle implies that if \( u \) is positive somewhere in \( \Omega \) then \( u \) is positive somewhere on \( \partial \Omega \). As a result, if \( u = 0 \), on \( \partial \Omega \), then \( u \leq 0 \) in \( \Omega \).
The next lemma extends Theorem 1.1 when the condition (1.11) is relaxed to include the case \( g \leq h \).

**Lemma 4.2.** Let \( a \in C(\Omega) \), \( a > 0 \), \( u \in \text{usc}(\Omega) \), and \( v \in \text{lsc}(\Omega) \cap L^\infty(\Omega) \), \( \inf_{\Omega} v > 0 \). Assume that \( g(x,t) \leq a(x)|t|^{k-1}t \leq h(x,t) \), \( \forall (x,t) \in \Omega \times \mathbb{R} \), where \( k \) is as in (1.4). If \( u, v \) satisfy

\[
H(Du, D^2u) + g(x,u) \geq 0, \quad \text{and} \quad H(Dv, D^2v) + h(x,v) \leq 0, \quad \text{in} \ \Omega,
\]

and \( u > 0 \), somewhere in \( \Omega \), then \( \sup_{\Omega} (u/v) = \sup_{\partial \Omega} (u/v) \).

**Proof.** Since \( v > 0 \), we observe that \( h(x,v(x)) \geq a(x)v(x)^k > 0 \), \( \forall x \in \Omega \). Also, since \( \inf_{\Omega} v > 0 \), by re-defining \( h(x,t) = \inf_{s \geq \inf_{\Omega} v} h(x,s) \), for \( t < \inf_{\Omega} v \), we may apply Lemma 3.2 to conclude that \( v \geq \inf_{\partial \Omega} v \), in \( \Omega \).

Set \( \mu = \inf_{\partial \Omega} v \), \( \ell = \sup_{\Omega} v \), and \( v_s = v - s\mu \), for \( 0 < s < 1 \). Let \( \varepsilon > 0 \) be small, to be determined later. Recalling that \( h(x,t) \geq a(x)t^k \), \( \forall t \geq 0 \), we calculate

\[
H(Dv_s, D^2v_s) + (1 + \varepsilon)a(x)v_s^k \leq h(x,v) \left( \frac{(1 + \varepsilon)v_s^k}{v^k} - 1 \right) \leq \left( 1 + \varepsilon \right) \left( \frac{\ell - s\mu}{\ell} \right)^k h(x,v).
\]

Here we have used that \( (\theta - s\mu)/\theta \) is an increasing function in \( \theta \). We choose \( \varepsilon \) small enough so that

\[
H(Dv_s, D^2v_s) + (1 + \varepsilon)a(x)v_s^k \leq 0, \quad \text{in} \ \Omega.
\]

Proceeding as in the proof of Theorem 1.1, we take \( \tau = \sup_{\Omega} u/v \) and assume that \( \tau > \sup_{\partial \Omega} u/v \). Working with \( u, v_s \) and \( w = u - \tau v_s \), we obtain from there is a \( z \in \Omega \) such that \( w(z) = \sup_{\Omega} w = 0 \) and \( g(z,u(z)) \geq (1 + \varepsilon)a(z)(\tau v_s(z))^k > 0 \). Since \( \tau v_s(z) = u(z) \),

\[
g(z,u(z)) \geq (1 + \varepsilon)a(z)(\tau v_s(z))^k = (1 + \varepsilon)a(z)u(z)^k \geq (1 + \varepsilon)g(z,u(z)) > 0.
\]

This is a contradiction and \( \sup_{\Omega} (u/v_s) = \sup_{\partial \Omega} (u/v_s) \). Letting \( s \to 0 \) proves the claim. \( \square \)

We now present a result regarding a change of variables which will prove useful in Part II.

**Lemma 4.3.** Let \( f \in C(\Omega \times \mathbb{R}, \mathbb{R}) \) and \( \zeta : \mathbb{R} \to \mathbb{R} \), be a \( C^2 \) function with \( \zeta' \geq 0 \). Take \( u : \Omega \to \mathbb{R} \) and set \( v = \zeta(u) \). Call \( \eta = \zeta^{-1} \).

(a) If \( \zeta \) is convex and \( u \in \text{usc}(\Omega) \) satisfies \( H(Du, D^2u) + f(x,u) \geq 0 \), in \( \Omega \), then

\[
H(Dv, D^2v) + [\eta'(v(x))]^{-k} f(x,\eta(v(x))) \geq 0, \quad \text{in} \ \Omega.
\]

(b) If \( \zeta \) is concave and \( u \in \text{lsc}(\Omega) \) and \( H(Du, D^2u) + f(x,u) \leq 0 \), in \( \Omega \), then

\[
H(Dv, D^2v) + [\eta'(v(x))]^{-k} f(x,\eta(v(x))) \leq 0, \quad \text{in} \ \Omega.
\]
Proof. We prove part (a). Clearly, \( v \in \text{usc}(\Omega) \). Since \( \zeta \) is convex, we have

\[
\zeta(t) - \zeta(s) \geq \zeta'(s)(t-s).
\]

Suppose that \( \psi \in C^2(\Omega) \) and \( p \in \Omega \) are such that \( v(x) - \psi(x) \) has a maximum at \( p \), i.e., \( (v - \psi)(x) \leq (v - \psi)(p) \). Thus, we obtain \( \zeta'(u(p))(u(x) - u(p)) \leq \zeta(u(x)) - \zeta(u(p)) = \psi(x) - \psi(p) \), see (4.6). Rearranging,

\[
u(x) - \frac{\psi(x)}{\zeta'(u(p))} \leq u(p) - \frac{\psi(p)}{\zeta'(u(p))}.
\]

Since \( u \) is a sub-solution, we obtain

\[
H \left( \frac{D\psi(p)}{\zeta'(u(p))}, \frac{D^2\psi(p)}{\zeta'(u(p))} \right) + f(p, u(p)) \geq 0.
\]

Using \( \zeta'(\eta(t))\eta'(t) = 1 \) together with (1.2), (1.3) and (1.4) we rewrite the above as

\[
H \left( D\psi(p), D^2\psi(p) \right) + [\eta'(v(p))]^{-k} f(p, \eta(v(p))) \geq 0.
\]

To prove part (b), we observe that the inequalities in (4.6) and (4.7) are reversed. One may now argue similarly to show part (b). \( \square \)

Remark 4.4. Let \( g \) and \( h \), in \( C(\Omega, \mathbb{R}) \), be such that \( g(x, t) \leq a(x)t^k \leq h(x, t), \forall (x, t) \in \Omega \times \mathbb{R}^+ \).

Suppose that \( u : \Omega \to \mathbb{R}^+ \) and \( \zeta(t) = t^\beta, t \geq 0 \). Lemma 4.3 implies the following.

(i) Suppose that \( u \in \text{usc}(\Omega) \) solves \( H(Du, D^2u) + g(x, u) \geq 0 \), in \( \Omega \). If \( \beta > 1 \) then

\[
H(Dv, D^2v) + \beta^k a(x)v^k \geq 0,
\]

(ii) Suppose that \( u \in \text{lsc}(\Omega) \) solves \( H(Du, D^2u) + h(x, u) \leq 0 \), in \( \Omega \). If \( \beta < 1 \) then

\[
H(Dv, D^2v) + \beta^k a(x)v^k \leq 0. \quad \square
\]

Next, Theorem 1.1, Lemma 4.2 and Remark 4.4 imply the following comparison principle.

Lemma 4.5. Let \( g, h \) be in \( C(\Omega, \mathbb{R}) \) and \( g(x, t) \leq a(x)|t|^{k-1}t \leq h(x, t), \forall (x, t) \in \Omega \times \mathbb{R} \), where \( a \in C(\Omega) \) and \( a > 0 \). In addition, assume that (i) \( 0 < \beta < 1 \) is such that \( g(x, t) \leq \beta^k a(x)t^k, \forall t \geq 0 \), and (ii) \( \sup_{x \in \Omega} |h(x, t)| = 0 \) if and only if \( t = 0 \). Let \( u \in \text{usc}(\Omega) \) and \( v \in \text{lsc}(\Omega), v > 0 \), solve

\[
H(Du, D^2u) + g(x, u) \geq 0, \quad \text{and} \quad H(Dv, D^2v) + h(x, v) \leq 0, \quad \text{in} \ \Omega.
\]

Assume that \( u > 0 \) somewhere in \( \Omega \). If \( u \leq v \) on \( \partial \Omega \) then \( u \leq v \) in \( \Omega \). Also,

\[
\sup_{\Omega} \frac{u}{v^\beta} \leq \sup_{\partial \Omega} \frac{u}{v^\beta}.
\]

If \( 0 < \sup_{\partial \Omega} u \leq \inf_{\Omega} v, \text{ on } \partial \Omega, \text{ then } u < v, \text{ in } \Omega, \text{ and } u(z) \leq (\inf_{\partial \Omega} v)^{1-\beta} v(z)^\beta, \ \forall z \in \Omega. \)
Proof. By Theorem 1.1 and Lemma 1.2 if \( u \leq v \) on \( \partial \Omega \) then \( u \leq v \) in \( \Omega \). By Remark 1.4 \( w = v^\beta \) solves \( H(Dw, D^2w) + \beta^k a(x)w^k \leq 0 \). From Lemma 1.2 it is seen that \((1.8)\) holds. By Lemma 3.3 \( v > \inf_{\partial \Omega} v \). Next, using \( u \leq v \) in \( \Omega \), we get for any \( z \in \Omega \),
\[
\frac{u(z)}{v(z)^\beta} \leq \frac{\sup_{\partial \Omega} u}{\inf_{\partial \Omega} v^\beta} \leq \frac{\inf_{\partial \Omega} v}{\inf_{\partial \Omega} v^\beta} \leq \inf_{\partial \Omega} v^{1-\beta} < v(z)^{1-\beta}.
\]
Thus, \( u(z) < v(z) \). The above inequality also implies \( u(x) \leq \inf_{\partial \Omega} v^{1-\beta} v^\beta(x) \), \( \forall x \in \Omega \). \( \square \)

5. A priori bounds: Proof of Theorem 1.2

In this section, we derive some useful a priori bounds. We consider a fairly general class of functions \( f(x,t) \) for our results. We assume that \( H \) satisfies conditions A, B and C, see \((1.2)-(1.8)\). However, we make no use of \((1.8)\) in this section.

In what follows, we need the following version of the maximum principle, also see Lemmas 3.2 and 3.3 in this context.

Lemma 5.1. (Maximum principle) (i) If \( u \in \text{usc}(\overline{\Omega}) \) solves \( H(Du, D^2u) \geq 0 \), in \( \Omega \), then \( \sup_{\Omega} u = \sup_{\partial \Omega} u \). (ii) If \( u \in \text{lsc}(\overline{\Omega}) \) solves \( H(Du, D^2u) \leq 0 \), in \( \Omega \), then \( \inf_{\Omega} u = \inf_{\partial \Omega} u \).

Proof. Let \( q \in \mathbb{R}^n \setminus \overline{\Omega} \) and \( 0 < \rho < R < \infty \) be such that \( \Omega \subset B_R(q) \setminus B_\rho(q) \). We prove (i) by contradiction. Let \( \varepsilon > 0 \) and \( p \in \Omega \) be such that \( u(p) \geq \sup_{\partial \Omega} u + \varepsilon \). Define
\[
w(x) = \sup_{\partial \Omega} u + \frac{\varepsilon}{2} \left( \frac{R^2 - |x|}{R^2 - \rho^2} \right), \quad \forall x \in B_R(q) \setminus B_\rho(q).
\]
Thus, \( \sup_{\partial \Omega} u \leq w(x) \leq \sup_{\partial \Omega} u + \varepsilon/2 \), in \( B_R(q) \setminus B_\rho(q) \). Clearly, \( u(p) - w(p) > 0 \) and \( u - w \leq 0 \), on \( \partial \Omega \). Let \( z \in \Omega \) be a point of maximum of \( u - w \) on \( \overline{\Omega} \). By \((1.2)\), \((1.3)\), \((1.4)\) and \((1.7)\),
\[
H(Dw(z), D^2w(z)) = - \left( \frac{\varepsilon}{R^2 - \rho^2} \right)^k |x - q|^k H(e, I) \leq -m_1 \rho^k \left( \frac{\varepsilon}{R^2 - \rho^2} \right)^k < 0,
\]
where \( e \) is a unit vector in \( \mathbb{R}^n \) and \( I \) is the \( n \times n \) identity matrix. This is a contradiction and the claim holds. Proof of Part (ii) is similar. \( \square \)

We consider the problem
\[
(5.1) \quad H(Du, D^2u) + f(x,u) = 0, \quad \in \Omega \text{ and } u = h \text{ on } \partial \Omega,
\]
where \( h \in C(\partial \Omega) \) and \( f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R}) \). For a function \( g \), define \( g^+ = \max\{g,0\} \) and \( g^- = \min\{g,0\} \). We now present a priori supremum bounds when \( f(x,u) = f(x) \), also see \([6]\).

Lemma 5.2. Let \( f \in C(\Omega) \cap L^\infty(\Omega) \), \( h \in C(\partial \Omega) \), \( \alpha \) and \( \sigma \) be as in \((1.7)\). Suppose that \( B_{R_s}(z_0) \), for some \( z_0 \in \mathbb{R}^n \), is the out-ball of \( \Omega \). Consider the problem
\[
(5.2) \quad H(Du, D^2u) + f(x) = 0, \quad \forall x \in \Omega, \quad u = h, \text{ on } \partial \Omega,
\]
If \( u \in \text{usc}(\Omega) \) is a sub-solution of (5.2) then
\[
\sup_{\Omega} u \leq \sup_{\partial \Omega} h + \sigma(\sup_{\Omega} f^+)^{1/k} R_0^\alpha.
\]
Similarly, if \( u \in \text{lsc}(\Omega) \) is a super-solution of (5.2) then
\[
\inf_{\Omega} u \geq \inf_{\partial \Omega} h - \sigma|\inf_{\Omega} f^-|^{1/k} R_0^\alpha.
\]

**Proof.** We prove part (i). Let \( u \) be a sub-solution of (5.2). Fix \( \varepsilon > 0 \) and consider the function
\[
w_\varepsilon(x) = \sup_{\partial \Omega} h + \sigma(\sup_{\Omega} f^+)^{1/k} (R_0^\alpha - |x - z_0|^\alpha), \forall x \in \Omega.
\]

Applying (1.3), (1.4), (1.5) and (2.7), we have
\[
H(Dw_\varepsilon, D^2 w_\varepsilon) = - \left( \frac{\sup_{\Omega} f^+ + \varepsilon}{m_1} \right) H\left(e, I - \frac{k_1}{k} e \otimes e\right) \leq - \sup_{\Omega} f^+ - \varepsilon < -f, \quad \text{in } \Omega.
\]

Also, \( w_\varepsilon \geq h \), on \( \partial \Omega \). Thus, Lemma 3.4 implies \( u(x) \leq w_\varepsilon(x) \), in \( \Omega \). Since \( \varepsilon \) is arbitrary the claim follows. The estimate in part (ii) follows by taking
\[
\hat{w}_\varepsilon(x) = \inf_{\Omega} h - \sigma(\inf_{\Omega} f^-)^{1/k} (R_0^\alpha - |x - z_0|^\alpha), \forall x \in \Omega.
\]
\[\square\]

Let \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) be continuous and satisfy
\[
\sup_{\Omega \times [t_1, t_2]} |f(x, t)| < \infty, \quad \forall t_1, t_2 \text{ such that } -\infty < t_1 \leq t_2 < \infty.
\]

We apply Lemma 5.2 to prove Theorem 1.2. A related result is proven in [6].

**Proof of Theorem 1.2.** Set \( M = \sup_{\Omega} |u| \), \( L = \sup_{\partial \Omega} |h| \) and \( R_0 \) the radius of the out-ball of \( \Omega \). Let \( \varepsilon > 0 \), small, be fixed.

We prove part (a). By (1.13) there exists \( t_1 > 0 \) such that
\[
(s_1 - \varepsilon)|t|^k \leq \inf_{\Omega} f(x, t) \leq \sup_{\Omega} f(x, t) \leq (s_2 + \varepsilon)|t|^k, \quad \forall |t| > t_1.
\]

By (5.3), there is a \( 0 < s_3 < \infty \) such that \( \sup_{[-t_1, t_1] \times \Omega} |f(x, t)| \leq s_3 \). Define \( s_4 = \max(|s_1| + \varepsilon, s_2 + \varepsilon) \), then \( \sup_{\Omega} |f(x, t)| \leq s_4 |t|^k + s_3, \quad -\infty < t < \infty \).

Thus, we have \( |f(x, u)| \leq s_4 M^k + s_3 \) and
\[
-|\lambda|(s_4 M^k + s_3) \leq H(Du, D^2 u) \leq |\lambda|(s_4 M^k + s_3), \quad \text{in } \Omega.
\]

Next, using \( -L \leq u \leq L \) on \( \partial \Omega \) and applying the estimates of Lemma 5.2 to (5.3) we have
\[
0 \leq |u| \leq M \leq L + \sigma|\lambda|^{1/k}(s_4 M^k + s_3)^{1/k} R_0^\alpha \leq L + |\lambda|^{1/k} s_5 M + s_6,
\]
where \( s_5 > 0 \) and \( s_6 > 0 \) are independent of \( M \). Hence, \( M \leq (L + s_6)(1 - |\lambda|^{1/k} s_5)^{-1} \). It is clear that if \( |\lambda| \) is small enough \( u \) is a priori bounded.

To show (b), take \( \varepsilon = (2\sigma|\lambda|^{1/k} R_0^\alpha)^{-1} \), and we get from (5.4), \( |f(x, t)| \leq \varepsilon |t|^k, \quad |t| \geq t_1 > 0 \), where \( t_1 > L \). Suppose that \( M > t_1 \), then \( H(Du, D^2 u) \leq \varepsilon M^k \) in the set \( \{u > t_1\} \). Using Lemma 6.2 and the definition of \( \varepsilon \), we obtain that \( \sup_{\Omega} u \leq 2t_1 \). A similar argument can be used to obtain a lower bound for \( \inf_{\Omega} u \). \( \square \)
Part II

6. Estimates for the eigenvalue problem

Throughout Part II, we assume that $H$ satisfies conditions $A$, $B$ and $C$, see (1.2)-(1.7). From hereon, $\lambda \in \mathbb{R}$ stands for a parameter, $a \in C(\Omega, \mathbb{R})$ and $\delta \geq 0$. We assume throughout that there are $0 < \mu \leq \nu < \infty$ and

\begin{equation}
0 < \mu \leq a(x) \leq \nu < \infty, \quad \forall x \in \Omega.
\end{equation}

We study the problem

\begin{equation}
H(Du, D^2u) + \lambda a(x)|u|^{k-1}u = 0, \quad \text{in } \Omega \text{ and } u = h \text{ on } \partial \Omega,
\end{equation}

where $u \in C(\overline{\Omega})$, $h \in C(\partial \Omega)$ and $\inf_{\partial \Omega} h > 0$.

First, we make an observation relevant to the eigenvalue problem for $H$. More precisely, it shows that if there is a positive super-solution of (6.1), for some $\lambda > 0$, then any solution of (6.2) is necessarily positive. It follows that if a solution of (6.2) changes sign, for some $\lambda > 0$, then there are no positive super-solutions of (6.2). Also, see Lemma 6.2 (iii) below.

**Lemma 6.1.** Let $\lambda > 0$, $a(x)$ be as in (6.1) and $h \in C(\partial \Omega)$, $\inf_{\partial \Omega} h > 0$. Suppose that $u \in C(\overline{\Omega})$, $u > 0$, solves $H(Du, D^2u) + \lambda a(x)u^k \leq 0$, in $\Omega$, and $u \geq h$ on $\partial \Omega$. If $v \in C(\overline{\Omega})$ solves

\begin{equation}
H(Dv, D^2v) + \lambda a(x)|v|^{k-1}v = 0, \quad \text{in } \Omega \text{ and } v = h \text{ on } \partial \Omega,
\end{equation}

then $v > \inf_{\partial \Omega} h$ in $\Omega$ and is unique.

**Proof.** Suppose that $v$ changes sign in $\Omega$, set $\Omega^- = \{v < 0\}$. Take $w = -v$, then $w > 0$, in $\Omega^-$, $H(Dw, D^2w) + \lambda a(x)w^k = 0$, in $\Omega^-$, and $w = 0$, on $\partial \Omega^-$. Use Lemma 4.2 in every component of $\Omega^-$ to conclude that $\sup_{\Omega^-} \frac{w}{u} \leq \sup_{\partial \Omega^-} \frac{(w/u)}{\sup_{\partial \Omega} h} = 0$. Thus, $\Omega^- = \emptyset$ and $v \geq 0$ in $\Omega$. The minimum principle in Lemma 3.3 yields that $v > \inf_{\partial \Omega} h$. Uniqueness follows from Lemma 4.2. \hfill \Box

Next, recalling (6.1) and applying the estimates of Lemma 5.2 to a solution $u$ of (6.2), we get

\[
\inf_{\partial \Omega} h - \sigma |\nu\lambda|^{1/k}R_0^\alpha \inf_{\partial \Omega} u^- \leq u(x) \leq \sup_{\partial \Omega} h + \sigma |\nu\lambda|^{1/k}R_0^\alpha \sup_{\partial \Omega} u^+.
\]

where $\alpha$ and $\sigma$ are as in (1.7). Setting $\Lambda = \nu^{-1}(\sigma R_0^\alpha)^{-k}$, we obtain

\begin{equation}
\inf_{\partial \Omega} h - \left(\frac{\lambda}{\Lambda}\right)^{1/k} \inf_{\partial \Omega} u^- \leq u(x) \leq \sup_{\partial \Omega} h + \left(\frac{\lambda}{\Lambda}\right)^{1/k} \sup_{\partial \Omega} u^+.
\end{equation}

Our next result discusses the influence of $\lambda$ on the solutions of (6.2).

**Lemma 6.2.** Let $a \in C(\Omega)$ be as in (6.1) and $\Lambda$ be as in (6.3). Suppose that $u \in C(\overline{\Omega})$ solves

\begin{equation}
H(Du, D^2u) + \lambda a(x)|u|^{k-1}u = 0, \quad \text{in } \Omega, \text{ and } u = h \text{ on } \partial \Omega,
\end{equation}

where $h \in C(\partial \Omega)$. Set $\kappa_1 = \inf_{\partial \Omega} h$ and $\kappa_2 = \sup_{\partial \Omega} h$. Then the following hold.
(i) If \( \lambda \leq 0 \) then \( \min(0, \kappa_1) \leq u \leq \max(0, \kappa_2) \) in \( \Omega \). If \( \lambda = 0 \) then \( \kappa_1 \leq u \leq \kappa_2 \) in \( \Omega \).

(ii) If \( h = 0 \) and \( u \) is a non-zero solution then \( \lambda > 0 \).

(iii) If \( 0 < \lambda < \Lambda \) then

\[
\frac{\kappa_1}{1 - (\lambda/\Lambda)^{1/k}} \leq \inf_{\Omega} u^- \leq u(x) \leq \sup_{\Omega} u^+ \leq \frac{\kappa_2}{1 - (\lambda/\Lambda)^{1/k}}.
\]

In particular, if \( h \geq 0 \) then \( \kappa_1 \leq u \leq \theta \kappa_2 \), where \( \theta = (1 - (\lambda/\Lambda)^{1/k})^{-1} \).

Proof. We prove part (i). Let \( \lambda \leq 0 \) and \( \Omega^- = \{ x \in \Omega : u(x) < \min(0, \kappa_1) \} \) be non-empty. Then \( H(Du, D^2 u) \leq 0 \), in \( \Omega^- \), and this contradicts Lemma 5.1(ii). Next, if \( \Omega^+ = \{ x \in \Omega : u(x) > \max(0, \kappa_2) \} \) is non-empty then \( H(Du, D^2 u) \geq 0 \) in \( \Omega^+ \). This contradicts Lemma 5.1(i). Part (ii) now follows as a contrapositive of (i). To show (iii), we use (6.3) and conclude that

\[
\inf_{\partial \Omega} h \leq \inf_{\Omega} u^- \leq u(x) \leq \sup_{\Omega} u^+ \leq \sup_{\partial \Omega} h.
\]

If \( \inf_{\partial \Omega} h \geq 0 \) then \( \inf_{\Omega} u^- = 0 \) and we obtain the final estimate in the lemma.

In the next result, given a positive solution of (6.2) we construct a super-solution for slightly larger value of \( \lambda \), and a sub-solution for a smaller value of \( \lambda \).

**Theorem 6.3.** Let \( a \in C(\Omega) \) be as in (6.1), \( h \in C(\partial \Omega) \), \( \inf_{\partial \Omega} h > 0 \), and \( \lambda > 0 \). Suppose that \( u \in L^\infty(\Omega) \) and \( u > 0 \). Set \( \vartheta = \inf_{\partial \Omega} h \), and for \( s > 0 \), define \( v_s = (u - s \vartheta)/(1 - s) \) and \( w_s = (u + s \vartheta)/(1 + s) \).

(i) Fix \( 0 < s < 1 \). If \( u \in lsc(\Omega) \) solves \( H(Du, D^2 u) + \lambda a(x) u^k \leq 0 \), in \( \Omega \), and \( u \geq h \) on \( \partial \Omega \), then, for every

\[
0 < \varepsilon \leq s \lambda k \left( \frac{\vartheta \sup_{\Omega} u}{1 - s(\vartheta \sup_{\Omega} u)} \right),
\]

the function \( v_s \) solves \( H(Dv_s, D^2 v_s) + (\lambda + \varepsilon) a(x) v_s^k \leq 0 \), in \( \Omega \), and \( v_s \geq h \) on \( \partial \Omega \).

(ii) Suppose that \( u \in usc(\Omega) \) solves \( H(Du, D^2 u) + \lambda a(x) u^k \geq 0 \), in \( \Omega \), and \( u \leq h \) on \( \partial \Omega \). For every \( 0 < \varepsilon < \lambda \) there is an \( s > 0 \) such that \( w_s \) solves

\[
H(Dw_s, D^2 w_s) + (\lambda - \varepsilon) a(x) w_s^k \geq 0, \text{ in } \Omega, \text{ and } w_s \leq h \text{ on } \partial \Omega.
\]

Proof. Set \( m = \sup_{\Omega} u \).

(i) Fix \( 0 < s < 1 \) and set \( \eta = u - s \vartheta \). By Lemma 5.1 \( u \geq \vartheta \), and \( \eta \geq (1 - s) \vartheta \), in \( \Omega \), and \( \eta \geq (1 - s) h \) on \( \partial \Omega \). Observe that \( (t - s \vartheta)/t \), \( t \geq \vartheta \), is increasing. Calculating,

\[
H(D\eta, D^2 \eta) + (\lambda + \varepsilon) a(x) \eta^k \leq a(x) \left\{ (\lambda + \varepsilon) \eta^k - \lambda u^k \right\} = a(x) u^k \left\{ (\lambda + \varepsilon) \left( \frac{u - s \vartheta}{u} \right)^k - \lambda \right\}
\]

\[
\leq a(x) u^k \left\{ (\lambda + \varepsilon) \left( \frac{m - s \vartheta}{m} \right)^k - \lambda \right\} \leq 0,
\]
if we choose $0 < \varepsilon \leq \lambda \left\{ m^k (m - s \theta)^{-k} - 1 \right\}$. Using the lower bound $k(t - 1) \leq t^k - 1$, $t \geq 1$, we take
\[
0 < \varepsilon \leq s \lambda k \left( \frac{(\theta/m)}{1 - s(\theta/m)} \right).
\]
The homogeneity of $H$ shows that the function $v_\varepsilon(x) = \eta/(1 - s) = (u - s \theta)/(1 - s)$ \(\forall x \in \Omega\), solves $H(Dv_\varepsilon, D^2v_\varepsilon) + (\lambda + \varepsilon)a(x)u^k \leq 0$, in $\Omega$, and $v_\varepsilon \geq h$, on $\partial \Omega$.

(ii) Let $0 < \varepsilon < \lambda$ be fixed and $s > 0$, to be determined. Set $\varphi = u + s \theta$, in $\overline{\Omega}$ and calculate to obtain
\[
H(D\varphi, D^2 \varphi) + (\lambda - \varepsilon)a(x) \varphi^k \geq a(x)u^k \left( (\lambda - \varepsilon) \frac{\varphi^k}{u^k} - \lambda \right) = a(x)u^k \left( (\lambda - \varepsilon) \left( \frac{u + s \theta}{u} \right)^k - \lambda \right) \geq 0,
\]
if we choose $s$ such that $0 < \varepsilon \leq \lambda \left\{ (m + s \theta)^k m^{-k} - 1 \right\}$. Now observe that $w_\varepsilon \geq \theta$, in $\Omega$, and $w_\varepsilon \leq h$ on $\partial \Omega$. \(\square\)

We now introduce a quantity that will be useful for the eigenvalue problem. Let $\delta > 0$ and $\lambda > 0$. Consider the problem of finding a positive solution $u_\lambda \in C(\Omega)$ of
\[
H(Du_\lambda, D^2u_\lambda) + \lambda a(x)|u_\lambda|^{k-1}u_\lambda = 0, \quad in \ \Omega, \quad and \ \partial \Omega.
\]
Define
\[
\lambda_\Omega = \sup\{\lambda : (6.4) \ has \ a \ positive \ solution \ u_\lambda\}.
\]
We will show in Sections 7 and 8 that $0 < \lambda_\Omega < \infty$. For the next result we assume this fact.

**Theorem 6.4.** Let $0 < \lambda < \lambda_\Omega$, where $\lambda_\Omega$ is as in (6.2), $a \in C(\Omega)$ be as in (6.1) and $u_\lambda > 0$ be a solution of (6.4). The following hold.

(i) A solution $u_\lambda$ is unique and $u_\lambda > \delta$. (ii) For every $x \in \Omega$, the function $u_\lambda(x)$ increases as $\lambda$ increases. (iii) Suppose that $0 < \lambda_\Omega < \infty$. If $m_\lambda = \sup_{\Omega} u_\lambda$ then
\[
m_\lambda \geq \delta \left( 1 + \frac{k \lambda}{\lambda_\Omega - \lambda} \right), \quad 0 < \lambda < \lambda_\Omega.
\]

Thus, $m_\lambda \to \infty$ as $\lambda \to \lambda_\Omega$.

(iv) The set of $\lambda$’s for which (6.4) has a positive solution is the interval $[0, \lambda_\Omega)$.

**Proof.** Parts (i) and (ii) follow from Lemma 6.1 and Theorem 1.1. We use Theorem 6.3(i) to prove part (iii). To see this, let $u_\lambda$ be the solution of (6.4) for some $\lambda < \lambda_\Omega$. Let $0 < s < 1$ and $\varepsilon$ be as in part (i) of Theorem 6.3. Then $v_\varepsilon$ is a super-solution of (6.4) with $\lambda + \varepsilon, v_\varepsilon = \delta$ on $\partial \Omega$, and the function $v = \delta$ is a sub-solution in $\Omega$. By Lemma 4.2 and Remark 2.1 there is a positive solution of $H(Dw, D^2w) + (\lambda + \varepsilon)a(x)w^k = 0$, in $\Omega$, and $w = \delta$ on $\partial \Omega$. From Theorem 6.3(i)
\[
0 < \varepsilon \leq s \lambda k \left( \frac{(\delta/m_\lambda)}{1 - s(\delta/m_\lambda)} \right) \leq \lambda k \left( \frac{(\delta/m_\lambda)}{1 - (\delta/m_\lambda)} \right), \quad \forall \ 0 < s < 1.
\]
Clearly, $\lambda + \varepsilon \leq \lambda_\Omega$. Thus,
\[
\lambda_\Omega - \lambda \geq \lambda k \left( \frac{\delta/m_\lambda}{1 - (\delta/m_\lambda)} \right).
\]
Rearranging, we obtain the estimate in part (iii). To show part (iv), let $\lambda < \lambda_\Omega$ and \(6.4\) have a solution $u_\lambda$. If $0 < \lambda < \lambda_\Omega$ then $H(Du_\lambda, D^2u) + \lambda a(x)u_\lambda(x)^k \leq 0$, with $u_\lambda = \delta$ on $\partial \Omega$. Thus, $u_\lambda$ is super-solution and $v = \delta$ is a sub-solution. By Lemma \(4.2\) and Remark \(2.1\) the problem $H(Du, D^2w) + \lambda a(x)w^k = 0$, in $\Omega$, with $w = \delta$, has a positive solution.

Finally, as a consequence of Theorem \(6.3\), we prove that $u_\lambda$, a solution of \(6.4\), is an increasing Lipschitz continuous function of $\lambda$. We comment that the lower bound proves a strong comparison principle for \(6.6\). Assume that $\lambda_\Omega > 0$, which is to be proven in Section 7.

**Theorem 6.5.** Let $\lambda > 0$, $\delta > 0$ and $u_\lambda \in C(\Omega)$, $u_\lambda > 0$, solve
\[
H(Du_\lambda, D^2u_\lambda) + \lambda a(x)u_\lambda^k = 0, \quad \text{in } \Omega \text{ and } u_\lambda = \delta \text{ on } \partial \Omega.
\]
Set $v_\lambda(x) = u_\lambda(x)$, $\forall x \in \Omega$ and $M_\lambda = \sup_\Omega u_\lambda$. Then for each $x \in \Omega$, $v_\lambda(x)$ is a non-decreasing Lipschitz continuous function of $\lambda$ and for a.e. $\lambda$,
\[
\max \left( \frac{v_\lambda \log v_\lambda(x)}{k\lambda}, 0 \right) \leq \frac{dv_\lambda}{d\lambda} \leq \left( \frac{M_\lambda}{k\delta} \right) \frac{v_\lambda(x) - \delta}{\lambda}, \quad 0 < \lambda < \lambda_\Omega.
\]

**Proof.** By Theorem \(6.4\)(iv), $\lambda$ is an interior point. By Theorem \(1.1\) for each $x \in \Omega$, $v_\lambda(x) = u_\lambda(x)$ is increasing in $\lambda$. Fix $x$ and a $\lambda_0$ such that \(6.6\) has a solution. Set $M_0 = M_{\lambda_0}$.

We make repeated use of Theorem \(6.3\) (i). Recall that if $u_\lambda$ solves \(6.6\) then for $0 < s < 1$ and $0 < \varepsilon \leq sk\lambda(\delta/M_\lambda)$ then there is a solution $u_{\lambda + \varepsilon} > 0$ of
\[
H(Du_{\lambda + \varepsilon}, D^2u_{\lambda + \varepsilon}) + (\lambda + \varepsilon)a(x)u_{\lambda + \varepsilon}^k = 0, \quad \text{in } \Omega, \text{ and } u_{\lambda + \varepsilon} = \delta.
\]
Also, $w(x) = (u_\lambda - s\delta)(1 - s)^{-1}$ is a super-solution of \(6.7\), and by Lemma \(4.2\)
\[
u_{\lambda + \varepsilon} \leq w, \quad \text{in } \Omega.
\]

**Upper Bound** By \(6.7\) and \(6.8\), for $0 < s < 1$ and $0 < \varepsilon \leq sk\lambda_0(\delta/M_0)$, we have
\[
v_\lambda(x_0 + \varepsilon) = u_{\lambda_0 + \varepsilon}(x_0) \leq \frac{u_{\lambda_0}(x_0) - s\delta}{1 - s} = \frac{v_\lambda(x_0) - s\delta}{1 - s}.
\]
Taking $\varepsilon = ks\lambda_0(\delta/M_0)$, $s$ small, we get
\[
0 \leq \frac{v_\lambda(x_0 + \varepsilon) - v_\lambda(x_0)}{\varepsilon} \leq \frac{s(v_\lambda(x_0) - \delta)}{(1 - s)^\varepsilon} = \left( \frac{M_0}{k\delta} \right) \frac{v_\lambda(x_0) - \delta}{\lambda_0(1 - s)}
\]
Thus, the right hand derivative $D_+^x v_\lambda(x_0)$ is $\leq (M_0/k\delta)(v_\lambda(x_0) - \delta)/\lambda_0$, by letting $s \to 0$.

We now compute the left hand derivative $D_-^x M_{\lambda_0}$ as follows. Fix $0 < s < 1$, small, and choose $\lambda < \lambda_0$ such that (see \(6.7\) and \(6.8\))
\[
\lambda = \lambda_0 \left( 1 - ks \left( \frac{\delta}{M_0} \right) \right) \quad \text{and} \quad \varepsilon = ks \left( \frac{\delta}{M_0} \right)
\]
Thus, $\lambda_0 = \lambda + \varepsilon$. 

Using (6.3), (6.9), Theorem 6.4 and Lemma 4.2 yield \( v_x(\lambda + \varepsilon) = v_x(\lambda_0) \leq (v_x(\lambda) - s\delta)/(1 - s) \), and
\[
0 \leq \frac{v_x(\lambda + \varepsilon) - v_x(\lambda)}{\varepsilon} = \frac{v_x(\lambda_0) - v_x(\lambda)}{\varepsilon} \leq \frac{s(v_x(\lambda) - s\delta)}{\varepsilon(1 - s)} \leq \left( \frac{M_0}{k\delta} \right) \frac{v_x(\lambda_0) - \delta}{\lambda_0(1 - s)}.
\]
Letting \( s \to 0 \), we get \( D_\lambda^{-} v_x(\lambda_0) \leq (M_0/k\delta)(v_x(\lambda_0) - \delta)/\lambda_0 \). Clearly, \( v_x(\lambda) \) is Lipschitz continuous, for a fixed \( x \) and \( \delta > 0 \), and the upper bound in the theorem holds.

**Lower Bound.** We prove the lower bound using some simple considerations. Let \( 0 < \lambda_1 < \lambda_2 < \lambda_\Omega \). Using (6.6) and Remark 4.4
\[
v_x(\lambda_1)/\delta \leq (v_x(\lambda_2)/\delta)^\tau, \quad \text{where} \quad \tau = (\lambda_1/\lambda_2)^{1/k} < 1.
\]
We obtain \( \log v_x(\lambda_1) \leq \tau \log v_x(\lambda_2) \). Subtracting \( \tau \log v_x(\lambda) \) from both sides, rearranging and noting that \( v_x(\lambda) \) is Lipschitz continuous, we see that
\[
\log v_x(\lambda_2) - \tau \log v_x(\lambda_1) - \log v_x(\lambda_1) \geq \left( \frac{\lambda_2^{1/k} - \lambda_1^{1/k}}{\lambda_1^{1/k}(\lambda_2 - \lambda_1)} \right) \log v_x(\lambda_1).
\]
The conclusion follows by letting \( \lambda_2 \to \lambda_1 \).

---

7. **Existence: Proofs of Theorems 1.3 and 1.4**

We now present the proof of Theorem 1.3 and show the existence of a solution of (6.2), for small \( \lambda > 0 \). This is done by constructing suitable sub-solutions and super-solutions. The two cases in (1.8) will be addressed separately. Note that this shows that \( \lambda_\Omega > 0 \).

**Proof of Theorem 1.3.** Set
\[
m = \inf_{\partial \Omega} h, \quad \bar{m} = \sup_{\partial \Omega} h \quad \text{and} \quad R = \text{diam}(\Omega).
\]
Using (2.5) and (2.6), we construct suitable sub-solutions and super-solutions to achieve our goal. Let \( y \in \partial \Omega \) and \( \varepsilon > 0 \), small, be such that \( m - 2\varepsilon > 0 \). Let \( r = |x - y| \). By continuity, there is a \( \eta > 0 \) such that
\[
(7.1) \quad h(y) - \varepsilon \leq h(x) \leq h(y) + \varepsilon, \quad \forall x \in \overline{B}_\eta(y) \cap \partial \Omega.
\]

**Case (a):** Suppose that (1.8)(i) holds. Fix \( \beta = 2 - \bar{s} \). Let \( v^\pm = c \pm dr^\beta \), by (2.5),
\[
(7.2) \quad H(Dv^+, D^2v^+) \leq \frac{(d\beta)^k}{r^{\gamma-k\beta}} m_2(\bar{s}) < 0, \quad \text{and} \quad H(Dv^-, D^2v^-) \geq \frac{(d\beta)^k}{r^{\gamma-k\beta}} |m_2(\bar{s})| > 0.
\]
Note that \( 0 < \beta < 1, k = k_1 + 1, \gamma = k_1 + 2 \) and \( \gamma - k\beta > 0 \), see (1.7).

We construct a sub-solution \( v \). We assume that \( h(y) > m \), otherwise we take \( v(x) = m \) in \( \Omega \).

Note that \( h(y) - \varepsilon - (m/2) > 0 \). Set \( r = |x - y| \) and
\[
(7.3) \quad v(x) = \begin{cases} 
  h(y) - \varepsilon - (h(y) - \varepsilon - m/2) (r/\eta)^\beta, & \text{in} \ \overline{B}_\eta(y) \cap \Omega, \\
  m/2, & \text{in} \ \overline{\Omega} \setminus \overline{B}_\eta(y).
\end{cases}
\]
Then \( v(y) = h(y) - \varepsilon \) and \( v = m/2 \), on \( \partial B_\eta(y) \). Applying (7.2), we obtain that \( H(Dv, D^2v) \geq 0 \), thus \( v \) is a sub-solution of (1.15) in \( B_\eta(y) \cap \Omega \). Also, \( v \) is a sub-solution in \( \Omega \setminus B_\eta(y) \) and by (7.1) \( v \leq h \) on \( \partial \Omega \). To show that \( v \) is a sub-solution in \( \Omega \), let \( p \in \partial B_\eta(y) \cap \Omega \) and \( \psi \in C^2 \) be such that \((v - \psi)(x) \leq (v - \psi)(p)\). Since \( v(p) = m/2 \) and \( v(x) \geq v(p) \), we get \( 0 \leq (D\psi(p), x - p) + o(|x - p|) \) as \( x \to p \). It follows that \( D\psi(p) = 0 \) and a second order expansion shows that \( D^2\psi(p) \geq 0 \).

Clearly, \( H(D\psi(p), D^2\psi(p)) + \lambda \alpha(p)v(p)^k \geq 0 \).

Next, we construct a super-solution \( w \). We assume that \( h(y) < \bar{m} \), otherwise take \( w(x) = \bar{m} \) in \( \Omega \). Let \( \lambda = \theta|m_2(\bar{s})|(2 - \bar{s})^k(R^*\nu)^{-1} \), for a fixed \( 0 < \theta < 1 \). Set \( r = |x - y| \) and

\[
w(x) = h(y) + \varepsilon + dr^\beta, \quad \text{in } \Omega,
\]

where \( \beta = 2 - \bar{s} \) and

\[
d \geq \max \left( \frac{2\bar{m} - h(y) - \varepsilon}{\eta^3}, \frac{2\bar{m}\theta^{1/k}}{(1 - \theta^{1/k})R^3} \right).
\]

It is easy to see that \( w(y) = h(y) + \varepsilon, \quad w \geq 2\bar{m}, \quad \Omega \setminus B_\eta(y) \), and by (7.1) \( w \geq h \) on \( \partial \Omega \). Set \( c = h(y) + \varepsilon \), and observing that \( \gamma - \bar{k}\beta > 0 \), we calculate, using the value of \( \lambda \),

\[
H(Dw, D^2w) + \lambda \alpha(x)w^k \leq \lambda \nu(c + dR^\beta)^k - \frac{\beta^k d^k}{R^{\gamma - \bar{k}\beta}}m_2(\bar{s})
\]

\[
= (c + dR^\beta)^k \left( \lambda \nu - \frac{\beta^k |m_2(\bar{s})|}{R^\gamma} \right)
\]

\[
\leq (c + dR^\beta)^k \left( \lambda \nu - \frac{|m_2(\bar{s})|\beta^k \theta}{R^\gamma} \right) = 0,
\]

where we have used \( t/(1 + t) \) is increasing in \( t \) and that \( dR^\beta \geq 2\bar{m}\theta^{1/k}(1 - \theta^{1/k})^{-1} \). Thus, \( w \) is a super-solution. Lemma 4.2 and Remark 2.1 imply existence of a solution \( u \) of (1.15).

**Case (b):** Let (1.8) (ii) hold. Since \( \Omega \) satisfies a uniform outer ball condition, there is an optimal radius \( 2\rho > 0 \) and a point \( z \in \mathbb{R}^n \) such that \( B_{2\rho}(z) \subset \mathbb{R}^n \setminus \Omega \) and \( y \in \partial B_{2\rho}(z) \cap \partial \Omega \). We choose \( \eta < \rho \). Observe from (2.6), if we fix \( \beta > \bar{s} - 2 \) and \( s = \beta + 2 \) then \( \beta > 0 \). Taking \( v^+ = c + dr^{-\beta} \),

\[
H(Dv^+, D^2v^+) \geq \frac{(d\beta)^k|m_2(s)|}{y^{k\beta + \gamma}} > 0, \quad \text{and} \quad H(Dv^-, D^2v^-) \leq -\frac{(d\beta)^k|m_2(s)|}{y^{k\beta + \gamma}} < 0.
\]

We construct a sub-solution as follows. We choose \( p \) on the segment \( yz \) such that \( |y - p| = \eta/4 \). Clearly, \( \Omega \cap B_{\eta/2}(p) \setminus B_{\eta/4}(p) \) is a non-empty open set. Set \( r = |x - p| \).

Assume that \( h(y) > m \) and \( m - 2\varepsilon > 0 \). We take \( v(x) = c + dr^{-\beta} \) in \( B_{\eta/2}(p) \), where

\[
c = h(y) - \varepsilon - \frac{4\beta d}{\eta^3}, \quad \text{and} \quad d = \eta^\beta \left( \frac{h(y) - \varepsilon - (m/2)}{4\beta^2 - 2\beta} \right) > 0.
\]

Clearly, \( v(y) = v(\eta/4) = h(y) - \varepsilon \) and \( v(x) = m/2 \) on \( \partial B_{\eta/2}(p) \). We now extend \( v = m/2 \) in \( \Omega \setminus B_{\eta/2}(p) \).

By (1.4), \( H(Dv, D^2v) + \lambda \alpha(x)v^k \geq 0 \), in \( B_{\eta/2}(p) \cap \Omega \). Since \( B_{\eta/2}(p) \subset B_\eta(y) \), (1.4) implies \( v \leq h \) on \( \partial \Omega \). The proof that \( v \) is a sub-solution in \( \Omega \) is similar to that in Case (a).
We construct super solutions as follows. Take $0 < \theta < 1$ and

\begin{equation}
\lambda = \frac{|m_2(s)| \beta^k}{\nu R^\gamma} \left( \frac{\rho}{R} \right)^{k\beta}.
\end{equation}

Recalling the outer ball condition, we select $q$ on the segment $\overline{y\xi}$ such that $|q - y| = \rho$. Set $r = |x - q|$ and notice that there is a $\bar{\rho} > \rho$ such that $B_{\bar{\rho}}(q) \cap \Omega \subset B_{\eta}(y) \cap \Omega$ (note that $2\rho$ is the optimal radius). Set $w(x) = c - dr^{-\beta}$, where

$$c = h(y) + \varepsilon + \frac{d}{\rho^\beta}, \quad \text{and} \quad d \geq \max \left\{ \frac{(\rho \bar{\rho})^\beta (2\bar{m} - h(y) - \varepsilon)}{\rho^\beta - \rho^\beta}, \frac{2\bar{m} \theta^{1/k} \rho^\beta}{1 - \theta^{1/k}} \right\}.$$ 

Clearly, $w(y) = w(\rho) = h(y) + \varepsilon$, $w(x) \geq 2\bar{m}$, in $\Omega \setminus B_{\bar{\rho}}(p)$, and (7.11) implies $w \geq h$ on $\partial \Omega$. Next, using (7.4) and (7.5), we calculate, in $\Omega$

$$H(Dw, D^2w) + \lambda a(x)w^k \leq \theta (2\bar{m} + d\rho^{-\beta})^{k} \frac{|m_2(s)| \beta^k}{R^{k\beta + \gamma}} \left( \frac{\rho}{R} \right)^{k\beta} - \frac{\beta^k d \beta^k |m_2(s)|}{R^{k\beta + \gamma}} \left( \theta - \left( \frac{d\rho^{-\beta}}{2\bar{m} + d\rho^{-\beta}} \right)^k \right) \leq 0.$$

The last inequality follows from the bound on $d\rho^{-\beta} \geq 2\bar{m} \theta^{1/k} (1 - \theta^{1/k})^{-1}$. Thus, $w$ is a super-solution. Existence now follows from Lemma 4.2 and Remark 2.1. \hfill \Box

**Remark 7.1.** It is clear from the proof of Theorem 1.3 that, unlike the super-solutions, the constructions of the sub-solutions in Cases (a) and (b) are independent of $\lambda$, except for $\lambda \geq 0$. Also, the upper bound for $\lambda$ in the two cases does not depend on the boundary data $h$. \hfill \Box

We now show a domain monotonicity property of $\lambda$. This plays an important role in proving Theorem 1.4.

**Lemma 7.2.** Let $\Omega' \subset \Omega$ be a sub-domain. Suppose that $\lambda > 0$, $\delta > 0$ and $a \in C(\Omega) \cap L^\infty(\Omega)$, $a > 0$. Assume that for some $0 < \lambda < \infty$ the problem

\begin{equation}
H(Du, D^2u) + \lambda a(x)|u|^{k-1}u = 0, \quad \text{in} \ \Omega', \ \text{and} \ u = \delta \ \text{on} \ \partial \Omega',
\end{equation}

has a positive solution $u \in C(\overline{\Omega'})$. Then the problem $H(Dv, D^2v) + \lambda a(x)|v|^{k-1}v = 0$, in $\Omega'$, and $v = \delta$ on $\partial \Omega'$, has a positive solution $v \in C(\overline{\Omega})$.

**Proof.** By Theorem 1.3 and (6.5), $\lambda_{\Omega'} > 0$. The lemma holds if $\lambda_{\Omega'} = \infty$. Assume that $\lambda_{\Omega'} < \infty$.

We argue by contradiction and suppose that $\lambda \geq \lambda_{\Omega'}$. By Theorem 6.4 (iv) and (6.5), for any $0 < t < \lambda_{\Omega'}$, there is a solution $v_t \in C(\overline{\Omega'})$, $v_t > 0$, of

$$H(Dv_t, D^2v_t) + ta(x)v_t^{k} = 0, \quad \text{in} \ \Omega', \ \text{and} \ v_t = \delta \ \text{on} \ \partial \Omega'.$$
By Theorem 6.4 (i), \( u \geq \delta \) on \( \partial \Omega' \). Applying the comparison principle in Theorem 1.1 in \( \Omega' \), \( v_t \leq u \) for every \( t \in (0, \lambda_{\Omega'}) \). Since \( u \) is bounded, this contradicts part (iii) of Theorem 6.4. The claim holds and \( \lambda < \lambda_{\Omega'} \).

We now prove Theorem 1.4 which shows that the definition of \( \lambda_{\Omega} \) in (6.5) is the same if \( \delta \) is replaced by any \( h \in C(\partial \Omega), \, h > 0 \). Also, see Remark 7.1.

**Proof of Theorem 1.4.** Our goal is to show that if, for some fixed \( \lambda > 0 \), there is a solution \( u \in C(\Omega) \), \( u > 0 \), of

\[
H(Du, D^2u) + \lambda a(x)u^k = 0, \quad \text{in } \Omega, \quad \text{and } u = \delta \text{ on } \partial \Omega,
\]

then the above problem can be solved for any \( h \in C(\partial \Omega), \, h > 0 \). As noted in Remark 7.1, the sub-solutions constructed in Theorem 1.3 place no restrictions on \( \lambda \) and can be utilized here. Thus, our effort here would be to construct super-solutions to (7.7) for a given boundary data \( h \).

Set

\[
\bar{m} = \sup_{\partial \Omega} h \quad \text{and} \quad R = \text{diam}(\Omega).
\]

By continuity, there is a \( \eta > 0 \) such that

\[
h(y) - \varepsilon \leq h(x) \leq h(y) + \varepsilon, \quad \forall x \in B_\eta(y) \cap \partial \Omega.
\]

Let \( \lambda > 0 \) and \( u \) solve (7.7) with \( \delta = 2\bar{m} \). Call \( M = \sup_{\Omega} u \). Set \( \Omega_\eta = \Omega \setminus \bar{B}_\eta(y) \). By our hypothesis and Lemma 7.2 there is a unique solution \( \phi > 0 \) of

\[
H(D\phi, D^2\phi) + \lambda a(x)\phi^k = 0, \quad \text{in } \Omega_\eta, \quad \text{and } \phi = 2\bar{m} \text{ on } \partial \Omega_\eta.
\]

By Lemma 5.1 \( u \geq 2\bar{m} \) on \( \partial \Omega_\eta \). Using Lemma 4.2, (7.9), Lemmas 5.3 and 5.1, \( 2\bar{m} < \phi \leq u \leq M \), for any \( \eta > 0 \), small.

**Case (a):** Suppose that (1.8) (i) holds and \( \Omega \) is a general domain. Call \( r = |x - y| \) and we take

\[
\bar{\phi}(x) = h(y) + \varepsilon + (2\bar{m} - h(y) - \varepsilon)\frac{r^\beta}{\eta^\gamma}, \quad \text{in } B_\eta(y) \cap \Omega,
\]

where \( \beta = 2 - \bar{s} \), see (1.8). Recalling (7.9) and (7.10), we define

\[
w(x) = \begin{cases} 
\phi(x), & \forall x \in \Omega, \\
\bar{\phi}(x), & \forall x \in B_\eta(y) \cap \Omega.
\end{cases}
\]

Note that \( w \in C(\Omega), \, w(y) = h(y) + \varepsilon, \, w = 2\bar{m}, \) on \( \partial B_\eta(y), \) and \( w \geq h \) on \( \partial \Omega \). We will choose \( \eta > 0 \), small, such that (7.11) holds and \( w \) is a super-solution in \( \Omega \). By (2.5) (also see (7.12)), we have, in \( B_\eta(y) \cap \Omega, \) for \( 0 < \eta \leq \eta_0, \) for some \( \eta_0 > 0 \), small enough,

\[
H(D\bar{\phi}, D^2\bar{\phi}) + \lambda a(x)\bar{\phi}^k \leq \lambda \nu \bar{\phi}^k - \left(\frac{2\bar{m} - h(y) - \varepsilon}{\eta^\gamma}\right)^k \beta^k \frac{|m_2(\bar{s})|}{\eta^\gamma} \leq 0.
\]
In the above we have used $0 < r < \eta$ and $\gamma - k\beta > 0$. To show that $w$ is a super-solution, we need to show that $w$ is a super-solution on $\partial B_\eta(y) \cap \Omega$. To do this, we choose a value of $\eta$ so that the radial derivative of $\bar{\phi}(\eta)$ is greater than the radial rate of increase of $\phi$ on $r = \eta$.

We estimate $\phi$ (in $\Omega_\eta$) from above, near $\partial B_\eta(y)$, as follows. Recalling from (7.10) that $2\bar{m} < \bar{\phi} \leq M$, choose $\theta > 1$ so that

$$(7.13) \quad \theta = \left(1 + \frac{2(1.5M - 2\bar{m})}{2\bar{m} - h(y) - \varepsilon}\right)^{1/\beta},$$

where $\beta = 2 - \bar{s}$, see (1.8)(i). In $\eta \leq r \leq \theta \eta$, set

$$\psi(x) = 2\bar{m} + d(r^\beta - \eta^\beta), \quad \text{where } d = \frac{1.5M - 2\bar{m}}{\eta^\beta(\theta^\beta - 1)}.$$

It is easily checked that $\psi = 2\bar{m}$, on $r = \eta$, $\psi = 1.5M$, on $r = \theta \eta$, and using (7.13),

$$(7.14) \quad d = \frac{1.5M - 2\bar{m}}{\eta^\beta(\theta^\beta - 1)} = \frac{2\bar{m} - h(y) - \varepsilon}{2\eta^\beta}.$$ 

Our goal is to choose $\eta < \eta_0$ so that $\psi$ is a super-solution in $\eta < r < \theta \eta$. This would then imply by Lemma 4.2 that $\phi \leq \psi$, in $\eta < r < \theta \eta$.

Employing (2.5) and (7.14),

$$H(D\psi, D^2\psi) + \lambda a(x)\psi^k \leq \lambda \nu M^k - \frac{(d\beta)^k|m_2(\bar{s})|}{r^{\gamma - k\beta}} \leq \lambda \nu M^k - \frac{(d\beta)^k|m_2(\bar{s})|}{\eta^{\gamma - k\beta}} = \lambda \nu M^k - \frac{(dy^\beta\beta)^k|m_2(\bar{s})|}{\eta^\gamma} = \lambda \nu M^k - \left(\frac{1.5M - 2\bar{m}}{\theta^\beta - 1}\right)^k \frac{\beta^k|m_2(\bar{s})|}{\eta^\gamma} \leq 0,$$

if $\eta \leq \eta_1$, small enough. We now choose $0 < \eta < \min(\eta_0, \eta_1)$, small enough so that (7.8) and (7.12) hold. This gives us the desired $\phi$, $\bar{\phi}$ and the upper bound $\psi$.

Next, we show that $w$ is a super-solution. Suppose that $w - \varphi, \varphi \in C^2$, has a minimum at some $p \in \partial B_\eta(y) \cap \Omega$. Then

$$(i) \quad \varphi(x) - \varphi(p) \leq w(x) - w(p) = \bar{\phi}(x) - \bar{\phi}(p) \leq 0, \quad \forall x \in B_\eta(y) \cap \Omega,$$

and

$$(ii) \quad \varphi(x) - \varphi(p) \leq w(x) - w(p) \leq \psi(x) - \psi(p), \quad \forall x \in \Omega_\eta.$$ 

Thus, using (7.10) and (7.14), the above observations yield respectively,

$$\frac{\partial \varphi}{\partial r}(p) \geq \bar{\phi}'(\eta-) = \beta \left(\frac{2\bar{m} - h(y) - \varepsilon}{\eta}\right) \quad \text{and} \quad \frac{\partial \varphi}{\partial r}(p) \leq \psi'(\eta+) = \beta \left(\frac{2\bar{m} - h(y) - \varepsilon}{2\eta}\right).$$

This is a contradiction and thus $w$ is a super-solution in $\Omega$. Lemma 4.2 and Remark 2.1 imply the existence of a solution of (1.18).

Case (b): Now assume that (1.8) (ii) holds and $\Omega$ satisfies a uniform outer ball condition. Fix $y \in \partial \Omega$ and $\varepsilon > 0$, small. Let $\rho > 0$ and $z \in \mathbb{R}^n$ be such that $B_\rho(z) \subset \mathbb{R}^n \setminus \Omega$ and $y \in \partial B_\rho(z) \cap \partial \Omega$. Fix $z$ and $\rho > 0$, and set $r = |x - z|$. We modify (7.8) as follows. We choose $\eta > 0$, small, such that

$$h(y) - \varepsilon \leq h(x) \leq h(y) + \varepsilon, \quad \forall x \text{ such that } \rho \leq |x - z| \leq \rho + \eta.$$
The basic ideas are similar to those in Case (a). Define $\Omega_\eta = \Omega \setminus B_{\rho+\eta}(x)$. For $\eta > 0$, to be determined later, we note that there is a solution $\phi > 0$ of

\[(7.15) \quad H(D\phi, D^2\phi) + \lambda a(x)\phi^k = 0, \quad \text{in } \Omega_\eta, \text{ and } \phi = 2\bar{m} \text{ on } \partial \Omega_\eta,\]

by our hypothesis and Lemma 7.2. As noted in (7.9), $2\bar{m} < \phi \leq M$ for any $\eta > 0$, small.

Call $A_\eta = \{x \in \Omega : \rho < |x - z| < \rho + \eta\}$. We fix $\beta > \bar{s} - 2$ and $s = \beta + 2$, and define

\[(7.16) \quad \bar{\phi}(x) = h(y) + \varepsilon + \left(\frac{2\bar{m} - h(y) - \varepsilon}{\rho^{\beta} - (\rho + \eta)^{-\beta}}\right) \left(\frac{1}{\rho^{\beta}} - \frac{1}{r^{\beta}}\right), \quad \rho \leq r \leq \rho + \eta.\]

It is clear that $\bar{\phi}(y) = h(y) + \varepsilon$ and $\bar{\phi} = 2\bar{m}$, on $r = \rho + \eta$. Using (7.15) and (7.16), we define

\[(7.17) \quad w(x) = \begin{cases} \phi(x), & \forall x \in \Omega_\eta, \\ \bar{\phi}(x), & \forall x \in \bar{A}_\eta. \end{cases}\]

As done in Case (a), our idea is to select $\eta > 0$ such that $w$ is a super-solution in $\Omega$. Clearly, $w(y) = h(y) + \varepsilon$ and $w \geq h$ on $\partial \Omega$. Using (2.6) (see (7.14)) in $\Omega_\eta$,

\[
H(D\bar{\phi}, D^2\bar{\phi}) + \lambda a(x)\bar{\phi}^k \leq \lambda \nu(2\bar{m})^k - \frac{\beta^k |m_2(s)|}{r^{k\beta+\gamma}} \left(\frac{2\bar{m} - h(y) - \varepsilon}{\rho^{\beta} - (\rho + \eta)^{-\beta}}\right)^k 
\leq \lambda \nu(2\bar{m})^k - \frac{\beta^k |m_2(s)|}{(\rho + \eta)^{k\beta+\gamma}} \left(\frac{2\bar{m} - h(y) - \varepsilon}{\rho^{\beta} - (\rho + \eta)^{-\beta}}\right)^k.
\]

Thus, $\bar{\phi}$ is a super-solution in $A_\eta$, if $\eta \leq \eta_0$, small enough.

Next we choose $0 < \eta_1 \leq \eta_0$ so that the quantity

\[(7.18) \quad K = \left(\frac{\rho^{\beta}}{(\rho + \eta)^{\beta} - \rho^{\beta}}\right) \left(\frac{2\bar{m} - h(y) - \varepsilon}{2(1.5M - 2\bar{m})}\right) > 1,\]

for $0 < \eta \leq \eta_1$. We calculate an upper bound for $\phi$ in $\Omega_\eta \cap B_{\rho+\theta\eta}(z)$, where $\theta > 1$ is to be determined. We take

\[(7.19) \quad \psi(x) = 2\bar{m} + \left(\frac{1.5M - 2\bar{m}}{(\rho + \eta)^{-\beta} - (\rho + \theta\eta)^{-\beta}}\right) \left(\frac{1}{(\rho + \eta)^{\beta}} - \frac{1}{r^{\beta}}\right), \quad \rho + \eta \leq r \leq \rho + \theta\eta.\]

Then $\psi = 2\bar{m}$, on $r = \rho + \eta$, and $\psi = 1.5M\rho + \theta\eta$. We calculate $\psi$ from the requirement that $\bar{\phi}(\rho + \eta) > \psi'(\rho + \eta)$, in particular, we impose that

\[
\frac{1.5M - 2\bar{m}}{(\rho + \eta)^{-\beta} - (\rho + \theta\eta)^{-\beta}} = \frac{1}{2} \left(\frac{2\bar{m} - h(y) - \varepsilon}{\rho^{\beta} - (\rho + \eta)^{-\beta}}\right),
\]

see (7.16) and (7.17). Rearranging and recalling (7.18),

\[
\frac{\rho^{\beta}}{(\rho + \theta\eta)^{\beta} - (\rho + \eta)^{\beta}} = \left(\frac{\rho^{\beta}}{(\rho + \eta)^{\beta} - \rho^{\beta}}\right) \left(\frac{2\bar{m} - h(y) - \varepsilon}{2(1.5M - 2\bar{m})}\right) = K.
\]

By (7.15), $(\rho + \theta\eta)^{\beta} = K(\rho + \eta)^{\beta}/(K - 1)$. Clearly, $\theta = \theta(\eta) > 1$ if $\eta \leq \eta_1$.

We now show that $\psi$ is super-solution in $\rho + \eta < r < \rho + \theta\eta$, if $\eta$ is small enough. We calculate using (2.6)

\[
H(D\psi, D^2\psi) + \lambda a(x)\psi^k \leq \lambda \nu M^k - \left(\frac{1.5M - 2\bar{m}}{(\rho + \eta)^{-\beta} - (\rho + \theta\eta)^{-\beta}}\right)^k \frac{\beta^k |m_2(s)|}{(\rho + \theta\eta)^{k\beta+\gamma}}.
\]
Thus, \( \psi \) is a super-solution if \( 0 < \eta \leq \eta_2 \), small. In particular, we choose \( 0 < \eta \leq \eta_2 \leq \eta_1 \). This now determines \( \theta \). Arguing as in Case (a), \( w \) is a super-solution in \( \Omega \). Lemma 4.2 and Remark 2.1 imply the existence of a solution. \( \square \)

8. Boundedness of \( \lambda_\Omega \): Proof of Theorem 1.5

In this section, we show that \( \lambda_\Omega \), defined in Theorem 6.4 is bounded. We have broken up the proof into two parts. The first part addresses the case when (1.8)(i) holds together with conditions \( A, B \) and \( C \). The second part addresses the case (1.8)(ii). In this case, however, conditions \( A, B, C \) and \( D \) will apply. The last condition guarantees that we have radial solutions on a ball. The proof presented here differs from the one in [5].

Let \( a(x) \in C(\Omega) \cap L^\infty(\Omega) \), \( \inf_{\Omega} a > 0 \), and \( \delta > 0 \); we consider the problem

\[
(8.1) \quad H(Du, D^2 u) + \lambda a(x) |u|^{k-1} u = 0, \quad \text{in } \Omega \text{ and } u = \delta \text{ on } \partial \Omega.
\]

We recall the definition of \( \lambda_\Omega \):

\[
(8.2) \quad \lambda_\Omega = \sup \{ \lambda : (8.1) \text{ has a positive solution} \}.
\]

By Theorem 6.4 if \( 0 < \lambda < \lambda_\Omega \), \( u \in C(\overline{\Omega}) \), \( u > \delta \) and is unique.

We make an observation. Let \( b \in C(\Omega) \) with \( a(x) \leq b(x) \), \( \forall x \in \Omega \). Suppose that, for some \( \lambda > 0 \), \( v \in C(\overline{\Omega}) \), \( v > 0 \), solves

\[
H(Dv, D^2 v) + \lambda b(x) |v|^{k-1} v = 0, \quad \text{in } \Omega \text{, and } v = \delta \text{ on } \partial \Omega.
\]

Then (8.1) has a solution \( u > 0 \) for \( \lambda \). To see this note that \( v \) is super-solution of (8.1), i.e.,

\[
H(Dv, D^2 v) + \lambda a(x) |v|^{k-1} v \leq 0, \quad \text{in } \Omega \text{, and } v = \delta \text{ on } \partial \Omega.
\]

Also, \( w = \delta \) is a sub-solution of (8.1). By Lemma 4.2 and Remark 2.1 there is solution \( w \leq u \leq v \) of (8.1). If we call \( \lambda_\Omega(a) \) as the bound in (8.2) for the weight function \( a(x) \) then

\[
(8.3) \quad \lambda_\Omega(b) \leq \lambda_\Omega(a), \text{ for } a(x) \leq b(x), \forall x \in \Omega.
\]

We now prove Theorem 1.5

**Proof of Theorem 1.5** Let \( B_R(y) \), where \( y \in \Omega \), denote the in-ball of \( \Omega \). Call \( B = B_R(y) \). If (8.1) has a solution \( u > 0 \) for some \( \lambda > 0 \) then, by Lemma 7.2, (8.1) has a positive solution in \( B \).

Set \( \mu = \inf_{\Omega} a(x) \) and consider the problem

\[
(8.4) \quad H(Dv, D^2 v) + \lambda \mu v^k = 0, \quad \text{in } B \text{, and } v = \delta \text{ on } \partial B.
\]

By (8.3), if (8.1) has a solution on \( B \) for some \( \lambda > 0 \) then (8.3) has a positive solution. Thus, if we show that \( \lambda_B < \infty \) then \( \lambda_\Omega < \infty \). We set \( \lambda = \lambda \mu \) and \( \lambda_R = \mu \lambda(\mu) \). We assume that \( \lambda_R = \infty \) and derive a contradiction.
Let $\lambda_1 > 0$ and set $\gamma = k_1 + 2$, see (1.7). Call $\lambda_\ell = \ell^\gamma \lambda_1$, $\ell = 1, 2, \ldots$. For every $\ell = 1, 2, \ldots$, there is an $u_\ell \in C(\overline{B})$, $u > 0$, that solves

(8.5) \[ H(Du_\ell, D^2u_\ell) = \lambda_\ell u_\ell^k = 0, \text{ in } B, \text{ and } u_\ell = \delta_\ell, \]

where $\delta_\ell > 0$ is so chosen that $u_\ell(y) = 1$ (using scaling if needed). This follows from Theorem 6.4(iv).

We claim that $\delta_\ell > 0$ for every $\ell$. For each $\ell$, (8.4) has a unique positive solution $v_\ell > 0$ with $v_\ell = 1$ on $\partial\Omega$. Thus by Lemma 4.2, $u_\ell/v_\ell \leq \delta_\ell$ and $v_\ell/u_\ell \leq 1/\delta_\ell$ implying $u_\ell = \delta_\ell v_\ell \neq 0$.

**Step 1:** We show that $\delta_\ell$ decreases to zero. Recall from (1.7) that $\alpha = \gamma/k$.

For each $\ell$, set $\tau_\ell = (\lambda_\ell/\lambda_{\ell+1})^{1/k}$, then $\tau_\ell = (\ell/(\ell + 1))^\alpha$, $\ell = 1, 2, \ldots$. Applying Remark 4.3(ii) and Lemma 4.5, $u_\ell(x)[u_{\ell+1}(x)]^{-\tau_\ell} \leq \delta_\ell(\delta_{\ell+1})^{-\tau_\ell}$. Taking $x = y$, we have $\delta_{\ell+1}^{\tau_\ell} \leq \delta_\ell$; iterating

(8.6) \[ 0 < \delta_{\ell+1} \leq (\delta_\ell)^{1/\tau_\ell} \leq (\delta_{\ell-1})^{1/(\tau_\ell \tau_{\ell-1})} \leq \cdots \leq (\delta_1)^{(\ell+1)^\alpha}. \]

Since $\delta_1 < 1$, the claim follows by letting $\ell \to \infty$.

**Step 2:** We employ scaling as follows. Define, for each $\ell = 1, 2, \ldots$, $R_\ell = \ell R$ and $v_\ell(z) = u_\ell(x)$, $\forall z \in B_{R_\ell}(y)$, where $z = y + \ell(x - y)$. Then $H(Du_\ell, D^2u_\ell) = \ell^\gamma H(Dv_\ell, D^2v_\ell)$, $\lambda_1 = \lambda_\ell/\ell^\gamma$ and (8.5) implies

(8.7) \[ H(Dv_\ell, D^2v_\ell) + \lambda_1 v_\ell^k = 0, \text{ in } B_{R_\ell}(y), \text{ and } v_\ell(R_\ell) = \delta_\ell. \]

Moreover, $v_\ell(y) = u_\ell(y) = 1$. Call $B_\ell = B_{R_\ell}(y)$.

We now address cases (i) and (ii) in (1.8) separately. Set $r = |z - y|$ and recall (2.5) and (2.6).

**Case (a):** Suppose that (1.8)(i) holds; fix $\beta = 2 - \bar{s}$. For $m$, large, take

$$ (8.8) \quad w_\ell(z) = w_m(r) = \frac{1}{2} \left( 1 - \left( 1 - \frac{r}{R_m} \right)^\beta \right), \quad 0 < r \leq R_m. $$

Note that $w_m(y) = 1/2$ and $w_m(R_m) = \delta_m/2$. Moreover, $w > 0$ and the calculation in (2.3) shows that $H(Dw_m, D^2w_m) + \lambda_1 \nu w_m^k \geq 0$, in $B_{R_m}(y) \setminus \{y\}$. Applying Lemma 4.2 to $w_m$ and $v_m$ in $B_m \setminus \{y\}$, we see that

(8.9) \[ w_\ell(z) \leq v_m(z)/2, \quad \forall z \in B_\ell(y). \]

**Step a(i):** Let $1 < \ell \leq m$. Applying Lemma 4.2 we see that

$$ (8.9) \quad \frac{v_\ell(z)}{v_m(z)} \leq \frac{\delta_\ell}{\inf_{r=R_\ell} v_m}, \quad \forall z \in B_\ell. $$

Taking $z = y$ in the above inequality and using (8.8), we obtain that for each $m \leq \ell$, $\inf_{r=R_\ell} v_m \leq \delta_\ell$ and

$$ (8.10) \quad \inf_{r=R_\ell} w_m \leq \delta_\ell/2. $$
Step a(ii): Take $m = 2\ell$. Using Step a(i) and Step 2,
\[
\frac{1}{2} \left( 1 - \frac{1 - \delta_{2\ell}}{2^\beta} \right) = w_{2\ell}(R_\ell) \leq \inf_{r = R_\ell} v_{2\ell} \leq \frac{\delta_{\ell}}{2}.
\]
Letting $\ell \to \infty$, we obtain a contradiction to (8.6).

Case (b): Assume that (1.8)(ii) holds. Fix $\beta = s - 2$, where $s > \bar{s}$. We impose conditions $A, B, C$ and $D$.

Step b(i): Since $H$ satisfies condition $D$, using the reflection and the rotation invariance of $H$ and the comparison principle in Lemma 4.2, $v_\ell$ is radial, see Step 2. By Lemma 5.1, $v_\ell(r) \geq v_\ell(\rho)$, for $0 \leq r \leq \rho \leq R_\ell$. Thus, $v_\ell(r)$ is non-increasing and $\sup v_\ell = v_\ell(y) = 1$.

Step b(ii): Let $1 \leq \ell \leq m$, applying Lemma 4.2 in $B_\ell$ we obtain that
\[
1 = \frac{v_m(o)}{v_\ell(o)} \leq \frac{v_m(R_\ell)}{\delta_\ell} \quad \text{and} \quad 1 = \frac{v_\ell(o)}{v_m(o)} \leq \frac{\delta_\ell}{v_m(R_\ell)}.
\]
Clearly, $v_m(R_\ell) = \delta_\ell$. Lemma 4.2 implies that $v_m = v_\ell$, in $B_\ell$. Thus, $v_m$ extends $v_\ell$ to $B_m$.

Step b(iii): We claim that the decay estimate in Step 1 cannot hold, leading to a contradiction and thus proving that $\lambda_\Omega < \infty$. We proceed as follows. By Step b(i), there is a $0 < \rho < R$ such that for any $\ell$, $v_\ell > 1/2$, in $B_\rho(y)$. Consider the function
\[
\omega(z) = \frac{1}{2} \left( \delta_{2\ell} + \frac{r^{-\beta} - R_{2\ell}^{-\beta}}{r^{-\beta} - R_\ell^{-\beta}} \right), \quad \forall \rho \leq r \leq R_{2\ell}.
\]
Using (2.4), $H(D\omega, D^2\omega) + \lambda \nu \omega^k \geq 0$, in $\rho < r < R_{2\ell}$. Noting that $\omega(\rho) = 1/2$ and $\omega(R_{2\ell}) = \delta_{2\ell}/2$, and applying Lemma 4.2 in $B_{R_{2\ell}} \setminus B_\rho(y)$ shows that $\omega(z) \leq v_{2\ell}(z)$, in $\rho \leq r \leq R_{2\ell}$. Recall Steps 1, 2 and Step b(ii), and take $r = R_\ell$ to find
\[
\frac{C}{\delta_\ell} \leq \omega(R_\ell) \leq v_{2\ell}(R_\ell) = \delta_\ell \leq (\delta_1)^{\ell_\alpha},
\]
where $C > 0$, depends on $\beta, \rho$ and $R$. Letting $\ell \to \infty$, we obtain a contradiction. □

9. Existence of a positive first eigenfunction

This section has two sub-sections. In Sub-section 1, we show the existence of a positive eigenfunction on a general domain when (1.8) (i) holds, also see [5, 9]. Conditions $A, B$ and $C$ apply. In Sub-section 2, (1.8)(ii) applies and we impose conditions $A, B, C$ and $D$. We show the existence of a first radial eigenfunction when $\Omega$ is a ball.

Sub-section I: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We assume that (1.8) (i) holds. Fix $\beta = 2 - \bar{s}$ and recall (2.5). We take $a \in C(\Omega) \cap L^\infty(\Omega)$, $\inf_\Omega a > 0$. Also, see [1, 2].

Lemma 9.1. (Harnack’s inequality and Hölder Continuity) Let $w \in lsc(\Omega) \cap L^\infty(\Omega)$, $w \geq 0$, solve $H(Dw, D^2w) + \lambda a(x)|w|^{k-1}w \leq 0$, in $\Omega$. For any $y \in \Omega$ and $R > 0$ such that $B_{4R}(y) \subset \Omega$, for some $R > 0$, we have
\[
\sup_{B_R(y)} w \leq C \inf_{B_R(y)} w, \quad \text{and} \quad |w(x) - w(z)| \leq (3R)^{-\beta} (\sup_{B_R(y)} w)|x - z|^\beta, \quad \forall x, z \in B_R(y),
\]
where $C > 0$ is universal.

Proof. Assume that $w(y) > 0$, for some $y \in \Omega$, and $B_{4R}(y) \subset \Omega$. Set $r = |x - y|$ and take

$$\psi(x) = w(y) \left(1 - r^\beta (4R)^{-\beta}\right), \quad \text{in } B_{4R}(y).$$

Set $A = B_{4R}(y) \setminus \{y\}$. Then $\psi(y) = w(y)$, $\psi = 0$, on $r = 4R$, and using (2.5), $H(D\psi, D^2\psi) > 0$ in $A$. Clearly, $\inf(w - \psi) = 0$, on $\partial A$. Suppose that $\inf_{A}(w - \psi) < 0$ and $p \in A$ is a point of minimum. Then $H(\psi(p), D^2\psi(p)) + \lambda a(p)|w(p)|^{k-1}w(z) > 0$, which contradicts the definition of $w$. Thus, we get $w(x) \geq \psi(x) > 0$, in $B_{4R}(y)$.

Observe that for any $z \in B_R(y)$, $B_R(y) \subset B_{2R}(z)$. We obtain arguing as above,

$$w(x) \geq w(z) \left(1 - |x - z|^\beta (3R)^{-\beta}\right), \quad \text{for any } x, z \in B_R(y).$$

Noting that $|x - z| \leq 2R$, we get the claim. To show Hölder continuity, we rewrite (9.1) as

$$w(z) - w(x) \leq w(z)|x - z|^\beta (3R)^{-\beta} \leq (3R)^{-\beta} \left(\sup_{B_R(y)} w\right)|x - z|^\beta.$$

Taking $x \in B_R(y)$ and replacing $x$ by $z$, we get the claim. \qed

Proof of Theorem 1.6(i). Let $\delta > 0$ and $\lambda_m$, $m = 1, 2, \ldots$ be an increasing sequence such that $\lambda_m \to \lambda_\Omega$ as $m \to \infty$. For each $m$, there is a unique positive $u_m \in C(\Omega)$ such that

$$H(Du_m, D^2u_m) + \lambda a(x)u_m^k = 0, \quad \text{in } \Omega, \quad \text{and } u_m = \delta, \text{on } \partial \Omega.$$

Set $\theta_m = \sup_\Omega u_m$. By Theorem 6.4, $\theta_m \to \infty$ as $m \to \infty$. Call $v_m = u_m/\theta_m$. Then $\sup v_m = 1$ and on $\partial \Omega$, $v_m = \delta/\theta_m \to 0$. By Lemma 9.1, there is a sub-sequence (which we call) $v_m$ and a function $v \in C(\Omega)$ such that $v_m \to v$ locally uniformly. Clearly, $v \geq 0$ and $\sup_\Omega v = 1$.

(a) We show that $H(Dv, D^2v) + \lambda_\Omega a(x)v^k = 0$, in $\Omega$. Let $\phi \in C^2$ and $p \in \Omega$ be such that $v - \phi$ has a minimum at $p$. Set $B_\varepsilon = B_\varepsilon(p)$, for $\varepsilon > 0$, small. Let $\hat{\Omega}$ be a compact sub-domain of $\Omega$ containing $B_\varepsilon$. Set $d = \text{dist}(p, \partial \hat{\Omega})$ and $k = \max(3, 3d^{-4})$. Call $\psi = \phi(x) - k|x - p|^4$. By taking $m$ large, $\sup_{\hat{\Omega}}|v_m - v| \leq \varepsilon^4$ on $\hat{\Omega}$. Then

$$v_m(x) - \psi(x) \geq k|x - p|^4 + (v_m - v)(x) + (v - v_m)(p) + (v_m - \psi)(p).$$

Noting that $v_m - \psi > (v_m - \psi)(p)$, on $\Omega \setminus B_\varepsilon$, $v_m - \psi$ has a minimum at some $p_m \in B_\varepsilon$, and

$$H(D\psi(p_m), D^2\psi(p_m)) + \lambda_m a(p_m)v_m(p_m)^k \leq 0. \quad \text{Letting } \varepsilon \to 0, H(D\phi(p), D^2\phi(p)) + \lambda_\Omega a(p)v(p)^k \leq 0.$$

Rest of the proof is similar.

(b) We show that $v \in C(\Omega)$. Let $y \in \partial \Omega$ and set $r = |x - y|$. Recalling (1.3) (i) and (2.5), take $w(x) = \varepsilon + (1 - \varepsilon)(r/\rho)^\beta$, in $B_\rho(y)$, where $0 < \varepsilon < 1/4$ is small and $\rho$ is to be determined. Setting $\nu = \sup_\Omega a$ and recalling that $\gamma - k\beta > 0$, one can choose a $\rho > 0$, small and independent of $\varepsilon$ so that in $B_\rho(y) \cap \Omega$,

$$H(Dw, D^2w) + \lambda a(x)w^k \leq \lambda \nu - \frac{(\beta(1 - \varepsilon))^k|m_2(\tilde{s})|}{\rho^\gamma} \leq \lambda \nu - \left(\frac{3\beta}{4}\right)^k \frac{|m_2(\tilde{s})|}{\rho^\gamma} < 0.$$
Next, for large $m, w \geq v_m$ on $\partial(\Omega \cap B_\rho(y))$. By Lemma 4.2, $w \geq v_m$, in $B_\rho(y) \cap \Omega$. Letting $m \to \infty$, we obtain $w \geq v$, in $\Omega \cap B_\rho(y)$. Clearly, $0 \leq \liminf_{x \to y} v(x) \leq \limsup_{x \to y} v(x) \leq w(y) = \varepsilon$. Since $\varepsilon$ is arbitrary, $v(y) = 0$ and $v \in C(\overline{\Omega})$.

(c) Next, we show that $v > 0$ in $\Omega$. Let $p \in \Omega$ be such that $v(p) = 1$. Recall from (b) the bound $v \leq w = (r/\rho)^\beta$, in $B_\rho(y) \cap \Omega$. Clearly, if we take $r = \rho/2$, we have $v < 1$. Thus, $p$ is at least $\rho/2$ away from $\partial \Omega$. We now apply Harnack’s inequality in Lemma 9.1 to conclude $v > 0$ in $\Omega$. □

**Sub-section 2:** In this sub-section, $\Omega$ is the ball $B_R(o)$, where $R > 0$. We prove the existence of a positive and a radial first eigenfunction. Set $\lambda_R = \lambda_{B_R(o)}$, $r = |x|$, $\forall x \in \mathbb{R}^n$ and take $a(x) = 1$ to be the constant function. We assume that $H$ satisfies conditions $A$, $B$, $C$ and $D$. As observed in Step b(i) of Theorem 1.5 the positive solutions $u \in C(B_R(o))$ to

\begin{equation}
H(Du, D^2 u) + \lambda u^k = 0, \quad \text{in } B_R(o), \quad u = \delta \text{ on } \partial B_R(o), \quad 0 < \lambda < \lambda_R,
\end{equation}

are radial and non-increasing in $r$. Call $B_0 = B_R(o)$.

We record a scaling property. Let $0 < R_1 < R_2$ and $B_i = B_{R_i}(o), i = 1, 2$. By Lemma 4.2, $\lambda_{R_1} \geq \lambda_{R_2}$. Denote by $u_\lambda$ the solution of (9.2) in $B_1$, for $0 < \lambda < \lambda_{R_1}$. Define $v(y) = u_\lambda(x)$, $\forall y \in B_2$, where $y = (R_2/R_1)x$. Setting $s = R_2/R_1$ and $\gamma = k_1 + 2$, we see that $H(Du_\lambda, D^2 u_\lambda) = s^\gamma H(Dv, D^2 v)$ and

\begin{equation}
H(Dv, D^2 v) + s^{-\gamma} \lambda v^k = 0, \quad \text{in } B_2, \quad v = \delta, \quad \text{on } \partial B_2.
\end{equation}

By (8.2) and Theorem 6.4 (iv), $\lambda_{R_1} = s^\gamma \lambda_{R_2}$

\begin{equation}
\lambda_{R_1} \frac{R_1^\gamma}{R_2^\gamma} = \lambda_{R_2} R_2^\gamma.
\end{equation}

We now show existence of a first eigenfunction on a ball. We achieve the proof in 5 steps. Set $r = |x|$, $\forall x \in \mathbb{R}^n$.

**Step 1:** Fix $0 < \lambda < \lambda_R$ and let $u_0 = u_0(r) \in C(\overline{B})$, $u_0 > 0$, be the unique solution of

\begin{equation}
H(Du_0, D^2 u_0) + \lambda u_0^k = 0, \quad \text{in } B_0, \quad u_0(R) = \delta,
\end{equation}

where $\delta > 0$ such that $u_0(o) = 1$. Set $\hat{R} = (\lambda_R/\lambda)^{1/\gamma} R$. By (9.3), $\lambda_{\hat{R}} = \lambda$ and $\hat{R} > R$.

**Step 2:** Let $\{\lambda_\ell\}_{\ell=1}^\infty$ be a decreasing sequence such that $\lambda < \lambda_\ell < \lambda_R$ and $\lim_{\ell \to \infty} \lambda_\ell = \lambda$. Call $R_\ell$ such that $\lambda_\ell R_\ell^\gamma = \lambda R_\ell^\gamma$ and $B_\ell = B_{R_\ell}(o)$. By (9.3), $\lambda_\ell = \lambda_{B_\ell}$. By Step 1,

\begin{equation}
R < R_1 < \cdots < R_\ell < \cdots < \hat{R}, \quad \text{and } \lim_{\ell \to \infty} R_\ell = \hat{R}.
\end{equation}

Since $\lambda < \lambda_\ell, \forall \ell$, there is a unique $u_\ell \in C(\overline{B_\ell})$, $u_\ell > 0$, radial and non-increasing such that

\begin{equation}
H(Du_\ell, D^2 u_\ell) + \lambda u_\ell^k = 0, \quad \text{in } B_\ell, \quad u_\ell(R_\ell) = \delta_\ell.
\end{equation}
where $\delta_\ell > 0$ is so chosen that $u_\ell(o) = 1$, see (9.2). If $1 \leq m \leq \ell$ then $\lambda_m \geq \lambda_\ell$ and $R_m \leq R_\ell$. Using Lemma 4.2 in $B_m$, we see that

$$1 = \frac{u_m(o)}{u_\ell(o)} \leq \frac{u_m(R_m)}{u_\ell(R_m)} \quad \text{and} \quad 1 = \frac{u_\ell(o)}{u_m(o)} \leq \frac{u_\ell(R_m)}{u_m(R_m)}.$$  

Hence, $u_m(R_m) = u_\ell(R_m) = \delta_m$ and $u_m = u_\ell$ in $B_m$. Thus, $u_\ell$ extends $u_m$ to $B_\ell$, in particular, $u_\ell$ extends $u_0$ to $B_\ell$. Moreover, $\delta_\ell$ is decreasing.

**Step 3:** We claim that $\lim_{\ell \to \infty} \delta_\ell = 0$. Suppose not. Since $\delta_\ell$ is decreasing, for some $\eta > 0$, $\delta_\ell \geq \eta$, $\forall \ell$. Since $u_\ell \geq \delta_\ell \geq \eta$, by taking $s = 1/2$ in estimate in part (i) of Theorem 6.3 there is a solution $v$ to

$$H(Dv, D^2v) + \tilde{\lambda} v^k = 0, \quad \text{in } B_\ell, \quad \text{and } v(R_\ell) = \delta_\ell,$$  

where we may choose $\lambda(1 + k\eta/2) < \tilde{\lambda} < \lambda_\ell$. Since $\eta$ is independent of $\ell$, letting $\ell \to \infty$ and using Step 2, we obtain a contradiction.

**Step 4:** Recalling Step 2, for $x \in B_{\hat{R}}(o)$, define

$$u(x) = \lim_{\ell \to \infty} u_\ell(x).$$  

By (9.1), $H(Du, D^2u) + \lambda u^k = 0$, in $B_{\hat{R}}(o)$. Since $u$ is radial and decreasing, define $u(\hat{R}) = 0$. Also, $u \in C(B_{\hat{R}})$, since $u(R_\ell) = u_\ell(R_\ell) = \delta_\ell \to 0$, see Step 3.

**Step 5:** We scale $u$ as follows. Set $w(\rho) = u(r)$ where $\rho = rR/\hat{R}$. Thus, $w \in C(B)$, $w > 0$, solves $H(Dw, D^2w) + \lambda(\hat{R}/R)^\gamma w^k = 0$, in $B_R(o)$, and $w(R) = 0$. By Step 1, $\lambda_R = \lambda(\hat{R}/R)^\gamma$ and, thus, $w$ is a first eigenfunction on $B_R(o)$. \qed

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