Directed rough fuzzy graph with application to trade networking

Uzma Ahmad · Iqra Nawaz

Abstract
In the realm of linking networks to the real world, connectivity (strength of connectedness) plays a crucial role. In this article, we introduce three types of vertices based on the indegree and outdegree of the vertices of directed rough fuzzy networks (DRFNs). We also define the concept of strength-reducing sets (SRSs) of DRF-vertices, DRF-edges, and important related results using the strongest path in directed rough fuzzy graphs (DRFGs). We generalize the idea of Menger’s theorem of vertices and edges in directed fuzzy graphs to directed rough fuzzy graphs which are appropriate for dealing with uncertainty in information systems. Furthermore, we use \( \alpha \) strong DRF-edges, \( \beta \) strong DRF-edges and \( \delta \) DRF-edges to identify developing countries most affected by trade deficits during COVID-19. Finally, our research results are compared with existing methods to demonstrate their applicability and productivity.

Keywords
Directed rough fuzzy graphs · Strength-reducing set of vertices · Strength-reducing set of edges · Mengers theorem · Network problem

Mathematics Subject Classification
03E72 · 68R10 · 68R05

1 Introduction

Nowadays, the complexity of representing situations in the real world is rapidly increasing, and we are not able to find perfect solutions in complex mathematical models. Usually, knowledge about a situation is hazy and unclear. When exploring challenges in social sciences, artificial intelligence, engineering, medicine, environmental sciences, and economics, we constantly encounter many types of uncertainty. To deal with uncertainty or vagueness, Zadeh
(1996, 1997) introduced the fuzzy set in 1965. In 1973, Kaufmann (1973) introduced the concept of fuzziness into classical graph theory by assigning membership of \([0, 1]\) to vertices and edges. To deal with ambiguities in real-life situations, we use fuzzy graphs. Rosenfeld (1975a, b) introduced some important concepts and applications of fuzzy graph theory, such as path, bridges, and connectedness. Fuzzy graphs were single-handedly introduced by Yeh and Bang (1975). In 1987, Bhattacharya (1987) and Bhattacharya and Suraweera (1991) established some concepts based on fuzzy connectivity, such as fuzzy bridges and fuzzy cutvertex. To explain several network models, Kóczy (1992) developed the concepts of fuzzy vertex graphs, fuzzy edge graphs, and fuzzy graphs. Rosenfeld (1975a, b, c) introduced cut nodes, geodesics, and strong edges in fuzzy graphs. Mathew and Sunitha (2009, 2010) introduced types of arcs and connectivity of nodes and arcs in fuzzy graphs. Tong and Zheng (1996) developed an approach to determine the connectivity matrix of a fuzzy graph.

Rough means approximate, imprecise, or inaccurate. The rough set introduced by Pawlak (1982) and Pawlak and Skowron (2007) describes uncertainty in terms of boundary region rather than membership values. The set under consideration is classical if the boundary region is empty; otherwise, it is imprecise or rough. In the theory of rough sets, equivalence classes are an important element for upper and lower approximations. Kryszkiewicz (1998) presented the rough set approach for inference in incomplete information systems. Rough set has proven to be very useful in decision-making problems. Many researchers have addressed decision-making based on rough sets or their hybrid models in their research articles (Akram et al. 2022; Ma et al. 2017; Riaz et al. 2021). Chen et al. (2008) and Chen and Zhong (2011) worked on granular structures and a hypergraph model for granular computing in graphs. Dubois and Prade (1990) analyzed both rough fuzzy sets and fuzzy rough sets and concluded that these two hybrid models are unique methods for dealing with vagueness. Fuzzy rough set approximates a fuzzy set under a fuzzy environment, whereas rough fuzzy set approximates a fuzzy set under a crisp environment. Recently, Akram and Arshad (2018) apply the concept of fuzzy rough sets to graphs. Malik and Akram (2018) and Zhan et al. (2019) studied intuitionistic fuzzy rough graph and its applications to decision-making.

In networks, directed fuzzy graphs (DFGs) are unable to cope with real networks with incomplete data information. Therefore, this drawback opens a way to introduce directed rough fuzzy graphs (DRFGs). The directed rough fuzzy model is a unique and innovative hybrid model for dealing with more complex problems of uncertainty. Zafar and Akram (2018) proposed the idea of DRFGs as a generalization of rough set and directed fuzzy graph. Recent work on DRFGs was discussed in Akram and Zafar (2019a). Akram and Zafar (2020) introduced the idea and related properties of DRFGs. A DRFG can be helpful in uncertain problems with incomplete knowledge to obtain finer and more accurate results. Recently, Akram and Zafar (2019b) proposed the concept of strength or intensity of the connection between each pair of vertices in DRFGs and addressed several essential concepts such as strongest DRF-path, strong DRF-path and DRF-bridge, \(\alpha\) strong DRF-edge, \(\beta\) strong DRF-edge, \(\delta\) DRF-edge, and so on.

Connectivity (power of connection) analysis deals with decision-making problems. In crisp and fuzzy graph theory, there is the concept of strength-reducing sets \((SRSs)\) for vertices and edges, but these notions are not defined in DRFGs. In classical graph theory, Menger’s theorem for undirected and directed networks has numerous applications in decision-making and social networks. However, in fuzzy graph theory (FGT), Menger’s theorem (Mathew and Sunitha 2013; Mathew and Mordeson 2017) for undirected and directed networks is more flexible and appropriate than in classical graph theory.
Recently, Karunambigai and Buvaneswari (2017) presented the idea of Menger’s theorem for intuitionistic fuzzy graphs Akram et al. (2021) discussed Menger’s theorem for m-polar fuzzy graphs and application to road network. In this research study, we present the concept of SRSs of vertices and edges in DRFGs which have not yet been explored in the context of DRFGs. We generalize the concept Menger’s theorem from FGs to DRFGs to obtain better decision-making results.

The motivation for our work is that the strength-reducing set of vertices (SRSVs) and the strength-reducing set of edges (SRSEs) of crisp and fuzzy graphs are documented in the literature, while these terms are not known for DRFGs. These terms allow an in-depth study of connectivity in DRFGs. For this reason, we propose these terms for DRFGs. We extend the idea of Menger’s theorem of vertices and edges in directed fuzzy graphs to directed rough fuzzy graphs which is suitable for uncertainty in information systems. The goal of this work is to solve problems based on the connectivity of DRFGs that are not easy to solve with fuzzy graphs. At the end of this paper, the trade network in developing countries is studied and the DRF-edge study is used to analyze the trade deficits in the trade network in developing countries of Asia during COVID-19. Thanks to our approach, many researchers will be able to study DRFGs in more detail.

The paper is organized as follows. In Segment 2, we present some basic definitions and results that help construct the features relevant to our research. In Segment 3, we classify three diverse categories of vertices in DRF networks according to the indegree and outdegree of the vertices, namely Source, Sinks, and Intermediate vertices. In Segment 4, we first define the DRF-vertex and DRF-edge strength reduction terms. Based on these terms, we define a \( w - z \) strength-reducing set of vertices (SRSVs) and a \( w - z \) strength-reducing set of edges (SRSEs) in a DRFG. Internally disjoint \([w, z]-\)DRF-paths are always edge-disjoint \([w, z]-\)DRF-paths, although the converse is not necessarily true. We also define the strong size of DRFG. We then present the vertex version of Menger’s theorem for DRFGs, together with a proof and an illustration. The edge version of Menger’s theorem for DRFGs is also discussed, as is the contrast between the different versions. Segment 5 explores how different types of DRF-edges can be used to find different regions with trade deficit regions in a developing country trade network due to COVID-19. The similarity between our research and the surviving methods is presented in Segment 6. Segment 7 concludes our research by summarizing our findings.

# 2 Directed rough fuzzy graphs

This segment presents few fundamental explanations which help us to construct the properties related directed rough fuzzy graph.

**Definition 2.1** (Dubois and Prade 1990) Let \( \mathcal{X} \) be a universe and \( \mathcal{R} \) an equivalence relation on \( \mathcal{X} \). The lower approximation and upper approximation are a rough fuzzy set of a fuzzy set (FS) \( V \in \mathcal{F}(\mathcal{X}) \), denoted by \( \overline{\mathcal{R}}V \) and \( \Gamma V \), respectively, defined as fuzzy sets in \( \mathcal{X} \), such that

\[
(\overline{\mathcal{R}}V)(x) = \bigwedge_{y \in \mathcal{X}} ((1 - \mathcal{R}(x, y)) \lor V(y)),
\]

\[
(\Gamma V)(x) = \bigvee_{y \in \mathcal{X}} (\mathcal{R}(x, y) \land V(y)),
\]

for every \( x \in \mathcal{X} \). The pair \( \mathcal{R}V = (\overline{\mathcal{R}}V, \Gamma V) \) is called a rough fuzzy set (RFs).
Definition 2.2 (Zafar and Akram 2018) A directed rough fuzzy graph (DRFG) on Boolean set \( \mathcal{X} \) is an four-ordered tuple \( G = (\mathcal{R}, \mathcal{R}V, \mathcal{S}, \mathcal{SE}) \), such that

(i) \( \mathcal{R} \) is an equivalence relation on \( \mathcal{X} \),
(ii) \( \mathcal{S} \) is an equivalence relation on \( E \subseteq \mathcal{X} \times \mathcal{X} \),
(iii) \( \mathcal{RS} = (\mathcal{R}V, \mathcal{R}V) \) is RFs on \( \mathcal{X} \),
(iv) \( \mathcal{SE} = (\mathcal{SE}, \mathcal{SE}) \) is an RFR on \( \mathcal{X} \),
(v) \( G = (\mathcal{G}, \mathcal{G}) = (\mathcal{R}V, \mathcal{S}E) \) is a DRFG; here, \( G = (\mathcal{R}V, \mathcal{S}E) \) is lower approximated directed fuzzy graph (DFG) and \( G = (\mathcal{R}V, \mathcal{S}E) \) is upper approximated directed fuzzy graph (DFG) of \( G = (\mathcal{G}, \mathcal{G}) \), such that

\[
\mathcal{SE}(wz) \leq \min\{(\mathcal{R}V)(w), (\mathcal{R}V)(z)\},
\]
\[
\overline{\mathcal{SE}}(wz) \leq \min\{(\mathcal{R}V)(w), (\mathcal{R}V)(z)\}, \quad \forall w, z \in \mathcal{X}.
\]

Example 2.3 Let \( \mathcal{X} = \{v_1, v_2, v_3, v_4\} \) be a crisp set and equivalence relation on \( \mathcal{X} \) be \( \mathcal{R} \), as shown in Table 1 defined by

| \( \mathcal{R} \) | \( v_1 \) | \( v_2 \) | \( v_3 \) | \( v_4 \) |
|---|---|---|---|---|
| \( v_1 \) | 1 | 0 | 1 | 0 |
| \( v_2 \) | 0 | 1 | 0 | 1 |
| \( v_3 \) | 1 | 0 | 1 | 0 |
| \( v_4 \) | 0 | 1 | 0 | 1 |

Table 1  Equivalence relation \( \mathcal{R} \) on \( \mathcal{X} \)

Let \( \mathcal{V} = \{(v_1, 0.8), (v_2, 0.6), (v_3, 0.7), (v_4, 0.5)\} \) be an FS on \( \mathcal{X} \) and \( \mathcal{R}V = (\mathcal{R}V, \mathcal{R}V) \) be RFs. Also, \( \mathcal{R}V \) and \( \overline{\mathcal{R}V} \) are represented by

\[
\mathcal{R}V = \{(v_1, 0.7), (v_2, 0.5), (v_3, 0.7), (v_4, 0.5)\},
\]
\[
\overline{\mathcal{R}V} = \{(v_1, 0.8), (v_2, 0.6), (v_3, 0.8), (v_4, 0.6)\}.
\]

Let \( E = \{v_1v_2, v_2v_3, v_2v_4, v_3v_4\} \subseteq \mathcal{X} \times \mathcal{X} \) and equivalence relation on \( E \) be \( \mathcal{S} \) as shown in Table 2 represented by

| \( \mathcal{S} \) | \( v_1v_2 \) | \( v_2v_3 \) | \( v_2v_4 \) | \( v_3v_4 \) |
|---|---|---|---|---|
| \( v_1v_2 \) | 1 | 0 | 0 | 1 |
| \( v_2v_3 \) | 0 | 1 | 0 | 0 |
| \( v_2v_4 \) | 0 | 0 | 1 | 0 |
| \( v_3v_4 \) | 1 | 0 | 0 | 1 |

Table 2  Equivalence relation \( \mathcal{S} \) on \( \mathcal{E} \)

Let \( \mathcal{E} = \{(v_1v_2, 0.4), (v_2v_3, 0.5), (v_2v_4, 0.2), (v_3v_4, 0.3)\} \) be a fuzzy set on \( E \) and \( \mathcal{SE} = (\mathcal{SE}, \mathcal{SE}) \) be rough fuzzy relation, where \( \mathcal{SE} \) and \( \overline{\mathcal{SE}} \) are defined by

\[
\mathcal{SE} = \{(v_1v_2, 0.3), (v_2v_3, 0.5), (v_2v_4, 0.2), (v_3v_4, 0.3)\},
\]
\[
\overline{\mathcal{SE}} = \{(v_1v_2, 0.4), (v_2v_3, 0.5), (v_2v_4, 0.2), (v_3v_4, 0.4)\}.
\]

Thus, \( G = (\mathcal{R}V, \mathcal{SE}) \) and \( \mathcal{G} = (\mathcal{R}V, \overline{\mathcal{SE}}) \) are DFGs shown in Fig. 1. Hence, \( G = (\mathcal{G}, \mathcal{G}) \) is a DRFG.
Definition 2.4 (Akram and Zafar 2019b) A path $P : w_0 \to w_1 \to \cdots \to w_n$ of length $n$ is directed rough fuzzy path (DRF-path) in $G = (\overrightarrow{G}, \overleftarrow{G})$ if $P$ is directed fuzzy path (DF-path) of length $n$ from $w_0$ to $w_n$ in $\overrightarrow{G}$ as well as in $\overleftarrow{G}$, where $G = (\overrightarrow{G}, \overleftarrow{G})$ is DRFG on non-empty set $\mathcal{X}$. A edge $w_0w_1 \in E$ of DRF-path $P$ in DRFG is weakest edge of $P$ in DRFG if $w_0w_1 \in E$ has lowest membership value between each possible edges of $P$ in $G$ as well as in $\overleftarrow{G}$. The strength or intensity of an DRF-path $P$ is explained as the total of the membership values of the weakest edge of $P$ in $G$ as well as in $\overleftarrow{G}$ and mathematically written as $S(P) = \bigwedge (\overrightarrow{S\mathcal{E}}(w_{i-1}w_i)) + \bigwedge (\overleftarrow{S\mathcal{E}}(w_{i-1}w_i))$, with $1 \leq i \leq n$. The strength or intensity of connection from vertices $w_0$ to $w_1$ in $G = (\overrightarrow{G}, \overleftarrow{G})$ the DRFG is represented by $\text{CONN}_G(w_0, w_1)$ and outlined as the strength or intensity of connection from $w_0$ to $w_1$ in $G$ as well as in $\overleftarrow{G}$ and represented by $\text{CONN}_G(w_0, w_1)$ in $G$ and $\text{CONN}_\overleftarrow{G}(w_0, w_1)$ in $\overleftarrow{G}$, where $\text{CONN}_G(w_0, w_1)$ is total to the maximum of strengths or intensities of all the paths from $w_0$ to $w_1$. Emphasize that, in DFGs, $\text{CONN}_G(w_0, w_1) \neq \text{CONN}_\overleftarrow{G}(w_0, w_1)$. An DRF-path $P$ in $G = (\overrightarrow{G}, \overleftarrow{G})$ is said to be strongest DRF-path in $G = (\overrightarrow{G}, \overleftarrow{G})$ if $S(P) = \text{CONN}_G(w_0, w_1)$. Let $G = (\overrightarrow{G}, \overleftarrow{G})$ be DRFG on non-empty set $\mathcal{X}$; therefore, $G = (\overrightarrow{G}, \overleftarrow{G})$ is consider to be connected DRFG if there is an edge between every pair of vertices in $G$ as well as in $\overleftarrow{G}$.

In this research, we suppose that DRFG is a connected DRFG.

Example 2.5 Consider $G = (\overrightarrow{G}, \overleftarrow{G})$ a DRFG given in Example 2.3 on $\mathcal{X} = \{v_1, v_2, v_3, v_4\}$. Therefore, $\overrightarrow{G} = (\overrightarrow{\mathcal{X}}, \overrightarrow{\mathcal{S}\mathcal{E}})$ and $\overleftarrow{G} = (\overleftarrow{\mathcal{X}}, \overleftarrow{\mathcal{S}\mathcal{E}})$ in Fig. 1 depicts DFGs. Subsequently, $v_1 \to v_2 \to v_4$ is a DRF-path $P$ of length 3 from $v_1$ to $v_4$ in both $G$ and $\overleftarrow{G}$. Therefore, $v_1 \to v_2 \to v_4$ is a DRF-path of length 3 from $v_1$ to $v_4$ in DRFG. Also, it is visible that $v_1 \to v_2 \to v_4$ have diverse strengths, 0.3 and 0.4 in $G$ and $\overleftarrow{G}$, respectively. Therefore, the strength of DRF-path $P : v_1 \to v_2 \to v_4$ in $G = (\overrightarrow{G}, \overleftarrow{G})$ the DRFG is $S(P) = 0.3 + 0.4 = 0.7$. By direct computations, we have strength of connection between each pair of vertices in $G$ as well as in $\overleftarrow{G}$ as follows:

$$
\begin{align*}
\text{CONN}_G(v_1, v_2) &= 0.3, \\
\text{CONN}_G(v_1, v_2) &= 0.4, \\
\text{CONN}_G(v_2, v_1) &= 0.0, \\
\text{CONN}_G(v_2, v_1) &= 0.0, \\
\text{CONN}_G(v_1, v_3) &= 0.3, \\
\text{CONN}_G(v_1, v_3) &= 0.4, \\
\text{CONN}_G(v_1, v_3) &= 0.4, \\
\text{CONN}_G(v_3, v_1) &= 0.0, \\
\text{CONN}_G(v_3, v_1) &= 0.0, \\
\text{CONN}_G(v_1, v_4) &= 0.3, \\
\text{CONN}_G(v_1, v_4) &= 0.3, \\
\text{CONN}_G(v_1, v_4) &= 0.4, \\
\text{CONN}_G(v_1, v_4) &= 0.0, \\
\text{CONN}_G(v_2, v_3) &= 0.3, \\
\text{CONN}_G(v_2, v_3) &= 0.4, \\
\text{CONN}_G(v_2, v_3) &= 0.4, \\
\text{CONN}_G(v_3, v_2) &= 0.0, \\
\text{CONN}_G(v_3, v_2) &= 0.0, \\
\text{CONN}_G(v_3, v_2) &= 0.5, \\
\text{CONN}_G(v_3, v_2) &= 0.5, \\
\text{CONN}_G(v_4, v_2) &= 0.0, \\
\text{CONN}_G(v_4, v_2) &= 0.0, \\
\text{CONN}_G(v_4, v_3) &= 0.0, \\
\text{CONN}_G(v_4, v_3) &= 0.0, \\
\text{CONN}_G(v_4, v_3) &= 0.0, \\
\text{CONN}_G(v_4, v_3) &= 0.3, \\
\text{CONN}_G(v_4, v_3) &= 0.3, \\
\text{CONN}_G(v_4, v_3) &= 0.4.
\end{align*}
$$

There are two directed paths (di-paths) from $v_1$ to $v_3$ in $G$ as well as in $\overleftarrow{G}$. Therefore, di-paths from $v_1$ to $v_3$ in $G = (\overrightarrow{G}, \overleftarrow{G})$ the DRFG are as follows:

(i) $v_1 \to v_2 \to v_3$ with strength 0.4.
(ii) \( v_1 \rightarrow v_2 \rightarrow v_4 \rightarrow v_3 \) with strength 0.7. Thus, \( v_1 \rightarrow v_2 \rightarrow v_4 \rightarrow v_3 \) is a strongest path in \( G = (\overline{G}, \overline{G}) \) the DRFG from \( v_1 \) to \( v_3 \).

**Definition 2.6** (Akram and Zafar 2019b) A DRFG \( \tilde{G} = (\overline{G}, \overline{G}) \) is said to be a partial directed rough fuzzy subgraph (partial DRF-subgraph) of \( G = (G, \overline{G}) \) the DRFG, if \( \tilde{G} \) is partial directed fuzzy subgraph (partial DF-subgraph) of \( G \) and \( \overline{G} \) is partial directed fuzzy subgraph (partial DF-subgraph) of \( \overline{G} \). In other words, for all \( w_0 \in \tilde{V} \) and \((w_0, w_1) \in \tilde{E} \), the following inequalities hold:

(i) \( (\overline{R}\tilde{V})(w_0) \leq (\overline{R}V)(w_0) \),  
(ii) \( (\overline{S}\tilde{E})(w_0, w_1) \leq (\overline{S}E)(w_0, w_1) \),  
(iii) \( (\overline{R}\tilde{V})(w_0) \leq (\overline{R}V)(w_0) \),  
(iv) \( (\overline{S}\tilde{E})(w_0, w_1) \leq (\overline{S}E)(w_0, w_1) \).

Obviously, \( \tilde{G} = (\overline{R}\tilde{V}, \overline{S}\tilde{E}) \), \( \overline{G} = (\overline{R}V, \overline{S}E) \), \( \tilde{G} = (\overline{R}\tilde{V}, \overline{S}\tilde{E}) \) and \( \overline{G} = (\overline{R}V, \overline{S}E) \) are DFGs, and \( \tilde{V} \subseteq V \), \( \tilde{E} \subseteq E \).

**Definition 2.7** (Akram and Zafar 2019b) A partial DRF-subgraph \( \tilde{G} = (\overline{G}, \overline{G}) \) is considered to be a directed rough fuzzy subgraph (DRF-subgraph) of partial DRFG \( G = (G, \overline{G}) \) if \( \tilde{G} \) is directed fuzzy subgraph (DF-subgraph) of \( G \) and \( \overline{G} \) is directed fuzzy subgraph (DF-subgraph) of \( \overline{G} \). In other words, for all \( w_0 \in \tilde{V} \) and \((w_0, w_1) \in \tilde{E} \), the following terms hold:

(i) \( (\overline{R}\tilde{V})(w_0) = (\overline{R}V)(w_0) \),  
(ii) \( (\overline{S}\tilde{E})(w_0, w_1) = (\overline{S}E)(w_0, w_1) \),  
(iii) \( (\overline{R}\tilde{V})(w_0) = (\overline{R}V)(w_0) \),  
(iv) \( (\overline{S}\tilde{E})(w_0, w_1) = (\overline{S}E)(w_0, w_1) \).

Obviously, \( \tilde{G} = (\overline{R}\tilde{V}, \overline{S}\tilde{E}) \) is partial DF-subgraph of \( G = (R\overline{V}, S\overline{E}) \) and \( \overline{G} = (\overline{R}V, \overline{S}E) \) is partial DF-subgraph of \( \overline{G} = (\overline{R}V, \overline{S}E) \).

**Definition 2.8** (Akram and Zafar 2019b) A DRF-subgraph \( \tilde{G} = (\overline{G}, \overline{G}) \) is considered to be a spanning directed rough fuzzy subgraph (spanning DRF-subgraph) of partial DRFG \( G = (G, \overline{G}) \) the DRFG if \( \tilde{G} \) and \( \overline{G} \) are spanning directed fuzzy subgraphs (spanning DF-subgraphs) of \( G \) and \( \overline{G} \), respectively. In other words, for all \( w_0 \in \mathcal{X} \), the following terms hold:

(i) \( (\overline{R}\tilde{V})(w_0) = (\overline{R}V)(w_0) \),  
(ii) \( (\overline{R}\overline{V})(w_0) = (\overline{R}V)(w_0) \).

Obviously, \( \tilde{G} = (\overline{R}\tilde{V}, \overline{S}\tilde{E}) \) is DF-subgraph of \( G = (R\overline{V}, S\overline{E}) \) and \( \overline{G} = (\overline{R}V, \overline{S}E) \) is DF-subgraph of \( \overline{G} = (\overline{R}V, \overline{S}E) \).

**Definition 2.9** (Akram and Zafar 2019b) Let \( G = (G, \overline{G}) \) be a DRFG. An edge \( w_0w_1 \in E \) is defined to be strong directed rough fuzzy edge (strong DRF-edge) of \( G \) if it is strong directed fuzzy edge (strong DF-edge) in \( \overline{G} \) as well as in \( \overline{G} \), that is

\[
\overline{S}E(w_0, w_1) \geq \text{CONN}_{\overline{G}-w_0w_1}(w_0, w_1)
\]

and

\[
\overline{S}E(w_0, w_1) \geq \text{CONN}_{\overline{G}-w_0w_1}(w_0, w_1),
\]

respectively.

A vertex \( w_1 \) is strong directed rough fuzzy neighbor (strong DRF-neighbor) of vertex \( w_0 \) in \( G = (G, \overline{G}) \) if \( w_0w_1 \) is strong DRF-edge in \( G = (G, \overline{G}) \), in other words that is strong DF edge in \( G \) as well as in \( \overline{G} \).  

\( \square \) Springer
Definition 2.10 (Akram and Zafar 2019b) A DRF-edge \( w_0w_1 \in E \) is characterized to be an \( \alpha \)-strong DRF-edge in \( G = (\mathcal{G}, \mathcal{G}) \) the DRFG if that is \( \alpha \)-strong DF-edge in \( \mathcal{G} \) as well as in \( \mathcal{G} \), that is

\[
\mathcal{SSE}(w_0, w_1) > \text{CONN}_{\mathcal{G}-w_0w_1}(w_0, w_1)
\]

and

\[
\overline{\mathcal{SSE}}(w_0, w_1) > \text{CONN}_{\mathcal{G}-w_0w_1}(w_0, w_1), \text{ respectively.}
\]

Definition 2.11 (Akram and Zafar 2019b) A DRF-edge \( w_0w_1 \in E \) is characterized to be an \( \beta \)-strong DRF-edge in \( G = (\mathcal{G}, \mathcal{G}) \) the DRFG, if that is \( \beta \)-strong DF-edge in \( \mathcal{G} \) as well as in \( \mathcal{G} \), that is

\[
\mathcal{SSE}(w_0, w_1) = \text{CONN}_{\mathcal{G}-w_0w_1}(w_0, w_1)
\]

and

\[
\overline{\mathcal{SSE}}(w_0, w_1) = \text{CONN}_{\mathcal{G}-w_0w_1}(w_0, w_1), \text{ respectively.}
\]

Definition 2.12 (Akram and Zafar 2019b) A DRF-edge \( w_0w_1 \in E \) is characterized to be an \( \delta \)-strong DRF-edge in \( G = (\mathcal{G}, \mathcal{G}) \) the DRFG, if that is \( \delta \)-DF-edge in \( \mathcal{G} \) as well as in \( \mathcal{G} \), that is

\[
\mathcal{SSE}(w_0, w_1) < \text{CONN}_{\mathcal{G}-w_0w_1}(w_0, w_1)
\]

and

\[
\overline{\mathcal{SSE}}(w_0, w_1) < \text{CONN}_{\mathcal{G}-w_0w_1}(w_0, w_1), \text{ respectively.}
\]

Definition 2.13 (Akram and Zafar 2019b) A edge \( x_0x_1 \in E \) is directed rough fuzzy bridge (DRF-bridge) of \( G = (\mathcal{G}, \mathcal{G}) \) the DRFG if its deletion decreases the strength of connection between several pair of vertices in \( G = (\mathcal{G}, \mathcal{G}) \) the DRFG, that is

\[
\text{CONN}_{\mathcal{G}-x_0x_1}(w_0, w_1) < \text{CONN}_{\mathcal{G}}(w_0, w_1) \quad \text{and} \quad \text{CONN}_{\overline{\mathcal{G}}-x_0x_1}(w_0, w_1) < \text{CONN}_{\overline{\mathcal{G}}}(w_0, w_1).
\]

Definition 2.14 (Akram and Zafar 2019b) A vertex \( x_0 \in \mathcal{X} \) is directed rough fuzzy cutvertex (DRF-cutvertex) of \( G = (\mathcal{G}, \mathcal{G}) \) the DRFG whose deletion from \( G = (\mathcal{G}, \mathcal{G}) \) decreases the strength of connection between several pair of vertices apart from \( x_0 \) in \( G = (\mathcal{G}, \mathcal{G}) \) the DRFG, that is

\[
\text{CONN}_{\mathcal{G}-x_0}(w_0, w_1) < \text{CONN}_{\mathcal{G}}(w_0, w_1) \quad \text{and} \quad \text{CONN}_{\overline{\mathcal{G}}-x_0}(w_0, w_1) < \text{CONN}_{\overline{\mathcal{G}}}(w_0, w_1).
\]

Example 2.15 Consider \( G = (\mathcal{G}, \mathcal{G}) \) a DRFG on \( \mathcal{X} = \{v_1, v_2, v_3, v_4\} \). Therefore, \( \mathcal{G} = (\overline{\mathcal{R}}, \mathcal{SSE}) \) and \( \overline{G} = (\overline{\mathcal{R}}, \overline{\mathcal{SSE}}) \) in Fig. 2 depicts DFGs. The strength of connection between
vertices of DRFG $G = (\overline{G}, \overline{\varepsilon})$ for edges is as follows:

\[
\begin{align*}
\text{CONN}_{G-(v_1, v_2)}(v_1, v_2) &= 0.1, \\
\text{CONN}_{\overline{G}-(v_1, v_2)}(v_1, v_2) &= 0.3,
\end{align*}
\]

It is easy to verify that the edges $v_1v_4$, $v_2v_3$, $v_2v_1$, $v_4v_1$, $v_3v_2$ are \(\alpha\) strong DF-edges in $G$ and $v_1v_4$, $v_2v_1$, $v_4v_1$, $v_3v_2$ are \(\alpha\) strong DF-edges in $\overline{G}$, respectively. Since $v_1v_4$, $v_2v_1$, $v_4v_1$, $v_3v_2$ are \(\alpha\) strong DRF-edges in both $G$ and $\overline{G}$. Therefore, the edges $v_1v_4$, $v_2v_1$, $v_4v_1$, $v_3v_2$ are \(\alpha\) strong DRF-edges in $G = (\overline{G}, \overline{\varepsilon})$ the DRFG. In similar manner, the edges $v_1v_2$, $v_4v_3$ are \(\beta\) strong DF-edges in $G = (\overline{G}, \overline{\varepsilon})$ the DRFG and $v_2v_4$ is $\delta$ DRF-edge in $G = (\overline{G}, \overline{\varepsilon})$ the DRFG. All \(\alpha\) strong DRF-edges and \(\beta\) strong DRF-edges are called strong DRF-edges of $G = (\overline{G}, \overline{\varepsilon})$ the DRFG. Therefore, $v_1v_4$, $v_2v_3$, $v_2v_1$, $v_4v_1$, $v_3v_2$, $v_1v_2$, $v_4v_3$ are strong DRF-edges of $G = (\overline{G}, \overline{\varepsilon})$ the DRFG. By routine calculations, it is clear that the edge $v_3v_1$ and vertex $v_2$ are DRF-bridge and DRF-cutvertex of graph $G = (\overline{G}, \overline{\varepsilon})$, respectively. In other words, the edge $v_4v_1$ and vertex $v_2$ are DF-bridge and DF-cutvertex in both $G$ and $\overline{G}$ DFGs, respectively. Similarly, other DRF-bridge and DRF-cutvertex can find easily.

3 Types of directed rough fuzzy vertices

In classical graph theory networking, the strength of a vertex is assumed to be infinite, meaning that any flow arriving at it will be diverted through various output edges. However, with a vertex capitated network, this is not possible. Furthermore, we require a directed rough fuzzy
model, which is different from an undirected rough fuzzy model, when the potential function of whether vertices or edges turns directed rough fuzzy (DRF). In directed network, if \( u_0u_1 \) is an edge, we say \( u_0 \) is tail and \( u_1 \) is head of the edge. In study of graph networks, there are three different categories of vertices in directed rough fuzzy network (DRF-network), namely, sources, sinks, and intermediate vertices. The following is a list of their definitions:

**Definition 3.1** Consider \( G = (G, \overline{G}) \) the DRFG on classical set \( \mathcal{X} \). For a vertex \( u \in \mathcal{X} \), we specify the rough fuzzy (RF) indegree for vertex \( u \) as the total of membership values of each edges towards \( u \) from other vertices in \( G \) and \( \overline{G} \), individually, denoted by \( d^{-u}_G(u) \) and delineated by follows:

\[
d^{-u}_G(u) = d^{u}_G(u) + d^{-u}_G(u),
\]

where

\[
d^{-u}_G(u) = \sum_{u, z \in (\mathcal{R}V)''} zu \quad \text{and} \quad d^{u}_G(u) = \sum_{u, z \in (\mathcal{R}V)''} zu.
\]

Similarly, we specify the rough fuzzy (RF) outdegree for vertex \( u \) is the total of membership values of each edges outward from \( u \) to other vertices in \( G \) and \( \overline{G} \), individually, denoted by \( d^{+u}_G(u) \) and delineated by follows:

\[
d^{+u}_G(u) = d^{+u}_G(u) + d^{-u}_G(u),
\]

where

\[
d^{+u}_G(u) = \sum_{u, z \in (\mathcal{R}V)''} uz \quad \text{and} \quad d^{+u}_G(u) = \sum_{u, z \in (\mathcal{R}V)''} uz.
\]

**Note:** With each \( G = (G, \overline{G}) \) the DRFG, we associate a directed crisp graph of \( G \) signified by \( G'' = (G'', \overline{G}'') \), delineated by \( G'' = \overline{G''} \) or \( G'' = \overline{G''} \) as

\[
(\mathcal{R}V)' = \{u \in \mathcal{X}|(\mathcal{R}V)(u) > 0\},
\]

\[
(\overline{\mathcal{R}V)' = \{u \in \mathcal{X}|(\overline{\mathcal{R}V})(u) > 0\},
\]

\[
(\mathcal{S}\mathcal{E})'' = \{uz \in E|(\mathcal{S}\mathcal{E})(uz) > 0\},
\]

\[
(\overline{\mathcal{S}\mathcal{E})'' = \{uz \in E|(\overline{\mathcal{S}\mathcal{E}})(uz) > 0\}.
\]

We call \( G'' = (G'', \overline{G''}) \), the underlying directed crisp graph of DRFG having no weights or fuzzy membership value.

**Definition 3.2** Consider \( G = (G, \overline{G}) \) the DRFG on crisp set \( \mathcal{X} \). For any vertex \( w \in \mathcal{X} \) in \( G = (G, \overline{G}) \) the DRFG is a source DRF-vertex if \( d^{-w}_G(w) = 0 \).

**Definition 3.3** Consider \( G = (G, \overline{G}) \) the DRFG on crisp set \( \mathcal{X} \). For any vertex \( w \in \mathcal{X} \) in \( G = (G, \overline{G}) \), the DRFG is a sink DRF-vertex if \( d^{+w}_G(w) = 0 \).

**Definition 3.4** Consider \( G = (G, \overline{G}) \) the DRFG on crisp set \( \mathcal{X} \). For any vertex \( w \in \mathcal{X} \) in \( G = (G, \overline{G}) \), the DRFG is a intermediate DRF-vertex if \( d^{-w}_G(w) > 0 \) as well as \( d^{+w}_G(w) > 0 \).

**Example 3.5** Suppose \( G = (G, \overline{G}) \) a DRFG on \( \mathcal{X} = \{v_1, v_2, v_3, v_4\} \). Therefore, \( G = (\mathcal{R}V, \mathcal{S}\mathcal{E}) \) and \( \overline{G} = (\overline{\mathcal{R}V}, \overline{\mathcal{S}\mathcal{E}}) \) in Fig. 3 depicts DFGs. By direct computations, the indegree \( d^{-v_1}_G(v_1) \) and outdegree \( d^{+v_1}_G(v_1) \) for vertex \( v_1 \) in \( G = (G, \overline{G}) \) are as follows:

\[
d^{-v_1}_G(v_1) = d^{v_1}_G(v_1) + d^{-v_1}_G(v_1) = 0.2 + 0.4 = 0.6,
\]
as well as in $G$.

**Note:** If $G = (G, \overline{G})$, the DRFG is di-connected graph or, in other words, if there is a di-path joining any of two vertices in DRFG, then there exist no sources and sinks in $G = (G, \overline{G})$ the DRFG. In this case, all vertices in $G = (G, \overline{G})$ are intermediate vertices.

### 4 Strength-reducing sets

In classical network notion, a $w - z$ separating set of vertices ($SSV$s) $\mathcal{X}'$ is a set or collection of vertices in $\overline{G} = (\mathcal{X}', \mathcal{E})$ whose elimination disconnects $\overline{G}$ and, $w$ and $z$ members to diverse components of $\overline{G}\backslash \mathcal{X}'$. Likewise, $w - z$ separating set of edges ($SSEs$) $\mathcal{E}'$ is outlined. In graphs and networks, the study of strength reduction is much important; consequently, we define $w - z$ strength-reducing sets of vertices ($SRSVs$) and $w - z$ strength-reducing set of edges ($SRSEs$) in DRF-graph $G = (G, \overline{G})$ as follows:

**Definition 4.1** Let $G = (G, \overline{G})$ be DRFG on crisp set $\mathcal{X}$. For each two vertices $w, z \in \mathcal{X}$ in such manner the edge $wz$ is not strong edge in $G = (G, \overline{G})$ or $wz$ is not strong edge in both FGs, $G$ and $\overline{G}$. A set $\mathcal{X}' \subseteq \mathcal{X}$ of vertices is said to be a $w - z \ SRSVs$ of $G = (G, \overline{G})$ if $\mathcal{X}'$ is $SRSVs$ in $G$ as well as in $\overline{G}$, mathematically represented as follows:

$$\text{CONN}_{G-\mathcal{X}'}(w, z) < \text{CONN}_{G}(w, z)$$

$$\text{CONN}_{\overline{G}-\mathcal{X}'}(w, z) < \text{CONN}_{\overline{G}}(w, z),$$

where $G - \mathcal{X}'$ and $\overline{G} - \mathcal{X}'$ are DF-subgraphs of $G$ as well as $\overline{G}$, respectively, received by deleting every vertices in $\mathcal{X}'$.

**Definition 4.2** Let $G = (G, \overline{G})$ be DRFG on crisp set $\mathcal{X}$. A collection $E' \subseteq E$ of edges is said to be a $w - z \ SRSEs$ of $G$ if $E'$ is $SRSEs$ in $G$ as well as in $\overline{G}$, mathematically
represented as follows:

\[
\text{CONN}_{G - E'}(w, z) < \text{CONN}_{\overline{G}}(w, z)
\]

\[
\text{CONN}_{\overline{G} - E'}(w, z) < \text{CONN}_{G}(w, z),
\]

where \( G - E' \) and \( \overline{G} - E' \) are DF-subgraphs of \( G \) as well as \( \overline{G} \), respectively, received by deleting every edges in \( E' \).

**Definition 4.3** A \( w - z \) \( SRS\bar{V}s \) by \( n \) elements is characteristic to be a minimum \( w - z \) \( SRS\bar{V}s \) in \( G \) if \( w - z \) \( SRS\bar{V}s \) is minimum \( w - z \) \( SRS\bar{V}s \) in both \( G \) and \( \overline{G} \). That is, there does not exist any \( w - z \) \( SRS\bar{V}s \) having less than \( n \) elements in \( G \). A minimum \( w - z \) \( SRS\bar{V}s \) is denoted by \( \text{X'}_{G = (\overline{G}, \overline{G})}(w, z) \).

**Definition 4.4** A \( w - z \) \( SRS\bar{E}s \) by \( n \) elements is characteristic to be a minimum \( w - z \) \( SRS\bar{E}s \) in \( G \) if, in both \( G \) and \( \overline{G} \), the \( w - z \) \( SRS\bar{E}s \) is minimum \( w - z \) \( SRS\bar{E}s \). That is, there does not exist any \( w - z \) \( SRS\bar{E}s \) having less than \( n \) elements in \( G \). A minimum \( w - z \) \( SRS\bar{E}s \) is denoted by \( E'_{G = (\overline{G}, \overline{G})}(w, z) \).

**Example 4.5** Consider \( G = (\overline{G}, \overline{G}) \) a DRFG on \( X = \{v_1, v_2, v_3, v_4\} \). Therefore, \( G = (\overline{\overline{G}}, \overline{\overline{G}}) \) and \( \overline{G} = (\overline{G}, \overline{G}) \) in Fig. 4 depicts DFGs. Calculations made directly, the strength of connectedness between vertices of \( G = (\overline{G}, \overline{G}) \) the DRFG for strong edges is as follows:

\[
\text{CONN}_{G - (v_1, v_2)}(v_1, v_2) = 0.4, \quad \text{CONN}_{\overline{G} - (v_1, v_2)}(v_1, v_2) = 0.5,
\]

\[
\text{CONN}_{G - (v_2, v_1)}(v_2, v_1) = 0.0, \quad \text{CONN}_{\overline{G} - (v_2, v_1)}(v_2, v_1) = 0.0,
\]

\[
\text{CONN}_{G - (v_1, v_3)}(v_1, v_3) = 0.4, \quad \text{CONN}_{\overline{G} - (v_1, v_3)}(v_1, v_3) = 0.5,
\]

\[
\text{CONN}_{G - (v_3, v_1)}(v_3, v_1) = 0.4, \quad \text{CONN}_{\overline{G} - (v_3, v_1)}(v_3, v_1) = 0.5,
\]

\[
\text{CONN}_{G - (v_1, v_4)}(v_1, v_4) = 0.0, \quad \text{CONN}_{\overline{G} - (v_1, v_4)}(v_1, v_4) = 0.0,
\]

\[
\text{CONN}_{G - (v_4, v_1)}(v_4, v_1) = 0.4, \quad \text{CONN}_{\overline{G} - (v_4, v_1)}(v_4, v_1) = 0.5,
\]

\[
\text{CONN}_{G - (v_2, v_3)}(v_2, v_3) = 0.5, \quad \text{CONN}_{\overline{G} - (v_2, v_3)}(v_2, v_3) = 0.6,
\]

\[
\text{CONN}_{G - (v_3, v_2)}(v_3, v_2) = 0.0, \quad \text{CONN}_{\overline{G} - (v_3, v_2)}(v_3, v_2) = 0.0,
\]

\[
\text{CONN}_{G - (v_4, v_2)}(v_4, v_2) = 0.4, \quad \text{CONN}_{\overline{G} - (v_4, v_2)}(v_4, v_2) = 0.5,
\]

\[
\text{CONN}_{G - (v_4, v_3)}(v_4, v_3) = 0.4, \quad \text{CONN}_{\overline{G} - (v_4, v_3)}(v_4, v_3) = 0.5,
\]

\[
\text{CONN}_{G - (v_4, v_3)}(v_4, v_3) = 0.0, \quad \text{CONN}_{\overline{G} - (v_4, v_3)}(v_4, v_3) = 0.5.
\]

By direct calculations, the edges \( v_1v_2, v_1v_3, v_1v_4, v_2v_1, v_3v_2, v_4v_3 \) are \( \alpha \)-edges in both \( G \) and \( \overline{G} \), respectively. Therefore, the edges \( v_1v_2, v_1v_3, v_1v_4, v_2v_1, v_3v_2, v_4v_3 \) are \( \alpha \)-edges in \( G = (\overline{G}, \overline{G}) \) the DRFG. In similar manner, the edge \( v_4v_1 \) is \( \beta \)-edge in \( G = (\overline{G}, \overline{G}) \) the DRFG. Thus, all edges are strong DRF-edges in \( G = (\overline{G}, \overline{G}) \) except \( v_2v_3 \) edge. Therefore, the strength of connectedness between vertices \( v_2 \) to \( v_3 \) for \( G \) and \( \overline{G} \), respectively, is as follows:

\[
\text{CONN}_{G}(v_2, v_3) = 0.5,
\]

\[
\text{CONN}_{\overline{G}}(v_2, v_3) = 0.7.
\]

Let \( X' = \{v_1\} \) be a subset of vertex set \( X = \{v_1, v_2, v_3, v_4\} \). Therefore, the connection strength between vertices \( v_2 \) to \( v_3 \) for \( G - X' \) and \( \overline{G} - X' \), respectively, is as follows:

\[
\text{CONN}_{G - X'}(v_2, v_3) = 0.4.
\]
CONNG\alpha’(v_2, v_3) = 0.5.

It is clear from the following that \( \alpha’ = \{v_1\} \) is a \( v_2 - v_3 \) \( SRSV \) in both \( G \) and \( \overline{G} \):

\[
\text{CONNG}\alpha’(v_2, v_3) = 0.4 < \text{CONNG}(v_2, v_3) = 0.5,
\]
\[
\text{CONNG}\alpha’(v_2, v_3) = 0.5 < \text{CONNG}(v_2, v_3) = 0.7.
\]

Therefore, \( \alpha’ = \{v_1\} \) is a \( v_2 - v_3 \) \( SRSV \) in \( G = (G, \overline{G}) \). Note that \( v_1v_2 \) is strong DRF-edge in \( G = (G, \overline{G}) \); therefore, there are no \( v_1 - v_2 \) \( SRSV \) in \( G = (G, \overline{G}) \).

Let \( \epsilon’ = \{v_1v_3\} \) be a subset of edge set \( \epsilon \). Therefore, the connection strength between vertices \( v_2 \) to \( v_3 \) for \( G - \epsilon’ \) and \( \overline{G} - \epsilon’ \), respectively, is as follows:

\[
\text{CONNG}\epsilon’(v_2, v_3) = 0.4,
\]
\[
\text{CONNG}\epsilon’(v_2, v_3) = 0.5.
\]

It is clear from the following that \( \epsilon’ = \{v_1v_3\} \) is a \( v_2 - v_3 \) \( SRS\epsilon \) in both \( G \) and \( \overline{G} \):

\[
\text{CONNG}\epsilon’(v_2, v_3) = 0.4 < \text{CONNG}(v_2, v_3) = 0.5,
\]
\[
\text{CONNG}\epsilon’(v_2, v_3) = 0.5 < \text{CONNG}(v_2, v_3) = 0.7.
\]

Therefore, \( \epsilon’ = \{v_1v_3\} \) is a \( v_2 - v_3 \) \( SRS\epsilon \) in \( G = (G, \overline{G}) \). Also, \( \alpha’ = \{v_1\} \) and \( \epsilon’ = \{v_1v_3\} \) are minimum \( v_2 - v_3 \) \( SRSV \) and minimum \( v_2 - v_3 \) \( SRS\epsilon \), respectively, in \( G = (G, \overline{G}) \).

**Remark** In \( G = (G, \overline{G}) \) the DRFG, the \( w - z \) \( SRSV \)s are possibly distinct. Also, in \( G = (G, \overline{G}) \), the minimum \( w - z \) \( SRSV \)s are possibly distinct but comprise similar number (No.) of vertices. Similarly, in \( G = (G, \overline{G}) \), the \( w - z \) \( SRS\epsilon \)s are possibly distinct. Additionally, while the minimum \( w - z \) \( SRS\epsilon \)s in \( G = (G, \overline{G}) \) are distinct, they all have the equal number of edges.

**Theorem 4.6** (Rule to find out \( w - z \) \( SRSV \)s in \( G = (G, \overline{G}) \) the DRFG) Let \( G = (G, \overline{G}) \) be a connected DRFG and \( wz \) be a DRF-edge of \( G = (G, \overline{G}) \), such that it is not a strong (\( \alpha \)-edge or \( \beta \)-edge) DRF-edge. If \( \alpha’ \subseteq \alpha \) is an \( w - z \) \( SRSV \)s in \( G = (G, \overline{G}) \), then from vertex \( w \) to \( z \), every strongest DRF-path comprises at least one vertex from collection \( \alpha’ \) and vice verse.
**Proof** Let $G = (G, \overline{G})$ be a connected DRFG and $wz$ be an DRF-edge of $G = (G, \overline{G})$ in such a manner it is not a strong ($\alpha$-edge or $\beta$-edge) DRF-edge. Furthermore, assume this $\lambda'$ is a $w-z$ $\sigma\tau\sigma\tau \nu \nu s$ and $P$ is a strongest $[w, z]$ DRF-path in $G = (G, \overline{G})$. Assume on contrary that $P$ contains no vertex of $\lambda'$. That is, removing $\lambda'$ does not harm $P$; therefore, the partial DRF-subgraph $G = (G, \overline{G}) - \lambda'$ still contains $[w, z]$ DRF-$P$. Thus, $\text{CONN}_{G=(G,\overline{G})}^{-\lambda'}(w, z) = \text{CONN}_{G=(G,\overline{G})}((w, z)$, which contradicts the reality that $\lambda'$ is a $w-z$ $\sigma\tau\sigma\tau \nu \nu s$. Therefore, $P$ should comprise at least one vertex from $\lambda'$. It is clear that this outcome is not correct when $wz$ is strong DRF-edge. Every DRF-edge which is strong edge in DRFG is DRF-strongest $w-z$ path comprising no vertex from $\lambda'$. Straightaway, inversely assume that every strongest DRF-path from $w$ to $z$ comprises at least one vertex of $\lambda'$, where $\lambda' \subseteq \lambda'$ and $w, z$ not in $\lambda'$. Therefore, the withdrawal of $\lambda'$ demolishes possible strongest $w-z$ paths in $G = (G, \overline{G})$. Consequently, $\text{CONN}_{G=(G,\overline{G})}^{-\lambda'}(w, z) < \text{CONN}_{G}((w, z)$. Thus, it follows that $\lambda'$ is a $w-z$ $\sigma\tau\sigma\tau \nu \nu s$ in $G = (G, \overline{G})$. □

**Example 4.7** Consider $G = (G, \overline{G})$ the DRFG given in Example 4.5 on $\lambda' = \{v_1, v_2, v_3, v_4\}$. Therefore, $G = (\overline{R} \nu, S \epsilon)$ and $\overline{G} = (\overline{R} \nu, S \epsilon)$ in Fig. 4 depicts DFGs. Since $v_2 v_3$ is not a strong edge, there are two DRF-paths from $v_2$ to $v_3$ in $G = (G, \overline{G})$ which are as follows:

(i). $v_2 \rightarrow v_1 \rightarrow v_3$ with strength 1.2,

(ii). $v_2 \rightarrow v_1 \rightarrow v_4 \rightarrow v_3$ with strength 0.9. So, from $v_2$ to $v_3$, $v_2 \rightarrow v_1 \rightarrow v_3$ is a strongest path in $(G, \overline{G})$. Note that there is only one strongest DRF-path from $v_2$ to $v_3$, and this strongest DRF-path from $v_2$ to $v_3$ contains vertex from $v_2 - v_3$ $\sigma\tau\sigma\tau \nu \nu s \lambda' = \{v_1\}$.

**Theorem 4.8** (Rule to find out $w - z$ $\sigma\tau\sigma\tau \nu \nu s$ in $G = (G, \overline{G})$ the DRFG) Let $G = (G, \overline{G})$ be a connected DRFG. Also, for $G = (G, \overline{G})$ the DRFG, $w, z$ are any two vertices. If $E' \subseteq E$ is a $w-z$ $\sigma\tau\sigma\tau \nu \nu s$ in $G = (G, \overline{G})$ the DRFG, then from vertex $w$ to $z$, every strongest DRF-path comprises at least one edge from collection $E'$ and vice versa.

**Proof** Let $G = (G, \overline{G})$ be a connected DRFG and $wz$ be an DRF-edge of $G = (G, \overline{G})$ in such a manner it is not a strong ($\alpha$-edge or $\beta$-edge) DRF-edge. Furthermore, presume that $E'$ is $w-z$ $\sigma\tau\sigma\tau \nu \nu s$ and the strongest $[w, z]$ DRF-path in $G = (G, \overline{G})$ is $P$. Suppose on contrary that $P$ contains no edge of $\lambda'$. That is, removing $E'$ does not harm $P$; therefore, the partial DRF-subgraph $G = (G, \overline{G}) - E'$ still contains $[w, z]$ DRF-$P$. Thus, $\text{CONN}_{G=(G,\overline{G})}^{-E'}(w, z) = \text{CONN}_{G=(G,\overline{G})}((w, z)$, which contradicts the reality that $E'$ is a $w-z$ $\sigma\tau\sigma\tau \nu \nu s$. Therefore, $P$ should include at least one edge from collection $E'$. Straightaway, conversely assume that every strongest DRF-path should comprise at least one edge of $E'$, where $E' \subseteq E$. Therefore, the withdrawal of $E'$ demolishes possible strongest $w-z$ paths in $G = (G, \overline{G})$. Consequently, $\text{CONN}_{G=(G,\overline{G})}^{-E'}(w, z) < \text{CONN}_{G}((w, z)$. Thus, it follows that $E'$ is a $w-z$ $\sigma\tau\sigma\tau \nu \nu s$ in $G = (G, \overline{G})$ the DRFG. □

**Example 4.9** Consider $G = (G, \overline{G})$ the DRFG given in Example 4.5 on $\lambda' = \{v_1, v_2, v_3, v_4\}$. Consequently, $G = (\overline{R} \nu, S \epsilon)$ and $\overline{G} = (\overline{R} \nu, S \epsilon)$ in Fig. 4 depicts DFGs. There are two DRF-paths from $v_2$ to $v_3$ in $G = (G, \overline{G})$ which are as follows:

(i). $v_2 \rightarrow v_1 \rightarrow v_3$ with strength 1.2,

(ii). $v_2 \rightarrow v_1 \rightarrow v_4 \rightarrow v_3$ with strength 0.9. So, from $v_2$ to $v_3$, $v_2 \rightarrow v_1 \rightarrow v_3$ is a strongest path in $(G, \overline{G})$. Note that there is only one strongest DRF-path from $v_2$ to $v_3$, and this strongest DRF-path from $v_2$ to $v_3$ contains edge from $v_2 - v_3$ $\sigma\tau\sigma\tau \nu \nu s E' = \{v_1v_3\}$. 

\[\mathbb{Springer}\]
Definition 4.10 Let \( G = (G, \overrightarrow{G}) \) be an DRFG on non-empty crisp set \( \mathcal{X} \). Suppose that \( \mathcal{P}_1, \mathcal{P}_2 \) be two \([w-z]\) DRF-paths of \( G = (G, \overrightarrow{G}) \), where \( w, z \) are DRF-vertices belong to \( \mathcal{X} \), such that \( w \neq z \). In that case, \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are called edge-disjoint \([w,z]\) DRF-paths in \( G = (G, \overrightarrow{G}) \) provided that \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) edge-disjoint \([w,z]\) DF-paths in both \( G \) and \( \overrightarrow{G} \). Mathematically, define as \( E(\mathcal{P}_1) \cap E(\mathcal{P}_2) = \emptyset \), i.e., \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) have no common edges in \( G = (G, \overrightarrow{G}) \) the DRFG. Two edge-disjoint \([w,z]\) DRF-paths are called internally disjoint \([w,z]\) DRF-paths in \( G = (G, \overrightarrow{G}) \) if these two edge-disjoint \([w,z]\) DRF-paths are internally disjoint \([w,z]\) DF-paths in both \( G \) and \( \overrightarrow{G} \) or, equivalently, it is written as \( \mathcal{X}(\mathcal{P}_1) \cap \mathcal{X}(\mathcal{P}_2) = \{w, z\} \), i.e., \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) have no common vertices except \( w \) and \( z \) in \( G = (G, \overrightarrow{G}) \) the DRFG.

Remark In DRFG, internally disjoint \([w,z]\) DRF-paths are always edge-disjoint \([w,z]\) DRF-paths. The inverse, on the other hand, does not have to be true.

Example 4.11 Consider \( G = (G, \overrightarrow{G}) \) the DRFG given in Example 4.5 on \( \mathcal{X} = \{v_1, v_2, v_3, v_4\} \). Consequently, \( G = (\overrightarrow{R}v, \overrightarrow{S}E) \) and \( \overrightarrow{G} = (\overrightarrow{R}v, \overrightarrow{S}E) \) in Fig. 4 depicts DFGs. There are two DRF-paths from \( v_1 \) to \( v_3 \) in \( G = (G, \overrightarrow{G}) \) which are as follows:

\[ \mathcal{P}_1 = v_1 \rightarrow v_2 \rightarrow v_3 \]

\[ \mathcal{P}_2 = v_1 \rightarrow v_4 \rightarrow v_3. \]

Clearly, \( E(\mathcal{P}_1) \cap E(\mathcal{P}_2) = \emptyset \) and \( \mathcal{X}(\mathcal{P}_1) \cap \mathcal{X}(\mathcal{P}_2) = \emptyset \) for \([v_1-v_3]\)-DRF paths in \( G = (G, \overrightarrow{G}) \). Therefore, \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are edge-disjoint and internally disjoint \([v_1-v_3]\)-DRF-paths, respectively.

Definition 4.12 Let \( G = (G, \overrightarrow{G}) \) be an DRFG over a crisp set \( \mathcal{X} \) and \( E_\mathcal{X} \) be the collection of all strong DRF-edges of \( G = (G, \overrightarrow{G}) \) the DRFG. In that case, the number of strong DRF-edges of \( G = (G, \overrightarrow{G}) \) is called strong size of \( G = (G, \overrightarrow{G}) \) the DRFG and denoted by \( S_\mathcal{X}(G = (G, \overrightarrow{G})) \).

Example 4.13 Consider \( G = (G, \overrightarrow{G}) \) the DRFG given in Example 4.5 on \( \mathcal{X} = \{v_1, v_2, v_3, v_4\} \). Consequently, \( G = (\overrightarrow{R}v, \overrightarrow{S}E) \) and \( \overrightarrow{G} = (\overrightarrow{R}v, \overrightarrow{S}E) \) in Fig. 4 depicts DFGs. The set \( E_\mathcal{X} = \{v_1v_2, v_1v_3, v_1v_4, v_2v_1, v_3v_2, v_4v_3, v_4v_1\} \) is the collection of all strong DRF-edges of \( G = (G, \overrightarrow{G}) \). Therefore, \( S_\mathcal{X}(G = (G, \overrightarrow{G})) = 7 \) for given DRFG.

4.1 Menger’s theorem for DRFGs

In 1927, Karl. Menger presented very famous result of classical graph theory called Menger’s theorem. Now, we introduce the concept of Menger’s theorem for DRFGs.

Theorem 4.14 Let \( G = (G, \overrightarrow{G}) \) be a DRFG, and for every two DRF-vertices \( w, z \in \mathcal{X} \) as \( wz \in E \) is a weak DRF-edge, the highest number of internally disjoint strongest \([w,z]\) DRF-paths in \( G = (G, \overrightarrow{G}) \) is identical to the number of vertices in a minimal \( w-z SRSV \)s.

Proof Let \( G = (G, \overrightarrow{G}) \) be a DRFG and \( w, z \in \mathcal{X} \) be any two DRF-vertices as \( wz \in E \) is not strong or weak DRF-edge of \( G = (G, \overrightarrow{G}) \). (It is obvious \( wz \in E \) is also not a strong or weak DF-edge in both \( G \) and \( \overrightarrow{G} \).) We must prove that the number of DRF-vertices in a minimal \( w-z SRSV \)s in \( G = (G, \overrightarrow{G}) \) is identical to the highest number of internally disjoint strongest \([w,z]\) DRF-paths in \( G = (G, \overrightarrow{G}) \). We demonstrate this outcome via mathematical induction technique on the strong size of the \( G = (G, \overrightarrow{G}) \) the DRFG, i.e., \( S_\mathcal{X}(G = (G, \overrightarrow{G})) \).

Case 1.1. When \( S_\mathcal{X}(G = (G, \overrightarrow{G})) = 0 \), \( G = (G, \overrightarrow{G}) \) is such that the collection of entire strong DRF-edges \( E_\mathcal{X} = \emptyset \), and the outcome holds trivially for every pair of DRF-vertices \( w, z \in \mathcal{X} \).
Case 1.2. Suppose that the result holds for each DRFG with \( S_z(G = (G, G)) < m \) \( (m \geq 1) \).

Case 1.3. Let \( G = (G, G) \) be a DRFG of \( S_z(G = (G, G)) = m \) and \( w, z, \in \mathcal{X} \) be any two vertices as \( wz \) is not strong or weak DRF-edge of \( G = (G, G) \). When the vertices \( w \) and \( z \) belong to separate parts of \( G = (G, G) \), the outcome is evident. As a result, both DRF-vertices \( w \) and \( z \) are assumed to correspond to the same parts of \( G = (G, G) \). Then, \( wz \) is either not a \( G = (G, G) \) strong DRF-edge or it is a \( \delta \) DRF-edge. In both conditions, \( w - z \) SRSVs is present in \( G = (G, G) \). (If \( wz \) is a strong DRF-edge in \( G = (G, G) \) the DRFG, then removing some number of vertices will not diminish the strength or intensity of connectivity between vertices \( w \) and \( z \); hence, there are no \( w - z \) SRSVs in \( G = (G, G) \)).

Now, assume that \( X_{G\left( G, G \right)}'(w, z) \) is a least \( w - z \) SRSVs in \( G = (G, G) \) the DRFG with \( |X_{G\left( G, G \right)}'(w, z)| = t \geq 1 \). With Theorem 4.6, all strongest \( [w, z] \) DRF-paths should have at least one DRF-vertex from \( X_{G\left( G, G \right)}'(w, z) \). Every strongest \( [w, z] \) DRF-paths minimum have one DRF-vertex from \( X_{G\left( G, G \right)}'(w, z) \), according to Theorem 4.6. This means that every \( w - z \) SRSVs must have at least the number of DRF-vertices, such that the number of internally disjoint strongest \( [w, z] \) DRF-paths, indicating that there are maximum \( t \) internally disjoint strongest \( [w, z] \) DRF-paths. If \( t = 1 \), then \( |X_{G\left( G, G \right)}'(w, z)| = 1 \). Suppose that \( |X_{G\left( G, G \right)}'(w, z)| = \{w1\} \). Then, \( \text{CONN}_{G\left( G, G \right)} - \{w1\}(w, z) < \text{CONN}_{G\left( G, G \right)}(w, z) \), i.e., \( w1 \) is an DRF-cutvertex of \( G = (G, G) \). As a sense, each of the most powerful \( [w - z] \) DRF-paths must travel via vertex \( w1 \). As a result, there are just one internally disjoint strongest \( [w - z] \) DRF-path, and the outcome is correct. Assume that \( t \geq 2 \) is true. Then, there are three possibilities:

Case 2.1. \( G = (G, G) \) contain minimum \( w - z \) SRSVs with a DRF-vertex \( w1 \) that has \( \alpha \) strong DRF-neighbor of both DRF-vertices \( w \) and \( z \), i.e., \( w1 \) \( w1 \) and \( w1z \) both are \( \alpha \) DRF-edges.

Case 2.2. For all minimum \( w - z \) SRSVs \( X_{G\left( G, G \right)}'(w, z) \) in \( G = (G, G) \), either every vertex \( X_{G\left( G, G \right)}'(w, z) \) is an \( \alpha \) strong DRF-neighbor of \( w \), then \( wz \) is an edge which unique strongest \( [w, z] \) DRF-path but not of DRF-vertex \( z \) or every vertex in \( X_{G\left( G, G \right)}'(w, z) \) is an \( \alpha \) strong DRF-neighbor of \( z \), however, not of DRF-vertex \( w \).

Case 2.3. There are \( w - z \) SRSVs say \( V \) in \( G = (G, G) \) as no DRF-vertex from \( V \) is an \( \alpha \) strong DRF-neighbor of DRF-vertices \( w \) and \( z \), and \( V \) has minimum one DRF-vertex that is not \( \alpha \) strong DRF-neighbor of DRF-vertex \( w \) and minimum one DRF-vertex which is not \( \alpha \) strong DRF-neighbor of DRF-vertex \( z \).

Assert: There are \( t \) internally disjoint strongest \( [w, z] \) DRF-paths in \( G = (G, G) \) the DRFG in all three situations. Let \( X_{G\left( G, G \right)}'(w, z) \) be the minimum \( w - z \) SRSVs that satisfy the property specified in case 2.1. Then, \( X_{G\left( G, G \right)}'(w, z) \) \( \{w1\} \) is a minimum \( w - z \) SRSVs in partial DRF-subgraph \( G = (G, G) \) \( \{w1\} \) contains \( t - 1 \) DRF-vertices. Like \( w\), \( w1 \) and \( w1z \) equally are \( \alpha \) DRF-edge, they are clearly strong DRF-edges, and hence, \( S_z(G = (G, G) \) \{w1\}) < S_z(G = (G, G)). \) With induction, \( G = (G, G) \) \{w1\} own \( t - 1 \) internally disjoint strongest \( [w, z] \) DRF-paths. As well as \( w\), \( w1 \) \( w1z \) equally are \( \alpha \) DRF-edges, \( \mathcal{P} \) is a strongest \( [w, z] \) DRF-path in it.

Suppose \( X_{G\left( G, G \right)}'(w, z) \) be minimum \( w - z \) SRSVs by the feature mentioned in case 2.2. Assume that every DRF-vertex in \( X_{G\left( G, G \right)}'(w, z) \) is an \( \alpha \) strong DRF-neighbor of DRF-vertex \( w \). Suppose a strongest \( [w, z] \) DRF-path \( \mathcal{P} \) in \( G = (G, G) \). Let \( w1 \) be the starting DRF-vertex of \( \mathcal{P} \) in such a way that \( w1 \in X_{G\left( G, G \right)}'(w, z) \). Then, \( w\), \( w1 \) is a \( \alpha \) DRF-edge and
as a result of \( w_1 \) is not \( \alpha \) DRF-edge, there are at least one DRF-vertex \( w_2 \) apart from \( w \) and \( z \) like \( w_1 w_2 \) is \( \beta \) DRF-edge.

**Our assert:** We have to establish that every \( w - z \, SRSV \)s in \( G = (G, \overline{G})\{w_1 w_2\} \) contains precisely \( t \) DRF-vertices. We suppose on conversely that there are a minimum \( w - z \, SRSV \)s denoted by \( U = \{u_1, u_2, \ldots, u_{t-1}\} \) in \( G = (G, \overline{G})\{w_1 w_2\} \) with \( t - 1 \) DRF-vertices. Thus, \( U \cup \{w_1\} \) is minimum \( w - z \, SRSV \)s in \( G = (G, \overline{G}) \). As well as, notice that each DRF-vertex in the collection of \( U \cup \{w_1\} \), i.e., \( u_1, u_2, \ldots, U_{t-1} \) and \( w_1 \) is \( \alpha \) strong DRF-neighbor of DRF-vertex \( w \). Equally \( U \cup \{z\} \) is too a minimum \( w - z \, SRSV \)s in \( G = (G, \overline{G}) \). As a result, DRF-vertex \( w_2 \) is \( \alpha \) strong DRF-neighbor of DRF-vertex \( w \). The claim that \( w_1 w_2 \) is a \( \beta \) DRF-edge contradicts this. As a result, in \( G = (G, \overline{G})\{w_1\} \), \( t \) is the least number of DRF-vertices in a \( w - z \, SRSV \)s. As \( S_4(G = (G, \overline{G})\{w_1\}) < S_4(G = \overline{G}) \). As a result of induction hypothesis, \( G = (G, \overline{G})\{w_1 w_2\} \) contains \( t \) internally disjoint strongest \([w, z]\) DRF-paths and therefore in \( G = (G, \overline{G}) \).

Consider \( V = \{v_1, v_2, \ldots, v_t\} \) the minimum \( w - z \, SRSV \)s with \( t \) DRF-vertices that fulfill the condition stated in case 2.3. Consider every strongest DRF-paths from DRF-vertex \( w \) to DRF-vertex \( z \). Because \( V \) is a minimum \( w - z \, SRSV \)s, the DRF-vertices \( v_1, v_2, \ldots, v_t \) surely contain in one of these \([w, z]\) DRF-paths. Let \( G_w = (G, \overline{G})_w \) be the DRF-subgraph of \( G = (G, \overline{G}) \) that contains all \([w, v_t]\) DRF-subpaths of the strongest \([w - z]\) DRF-paths, where \( v_t \in V \) is the only DRF-vertex of the DRF-path that belongs to \( V \). Note that if \( CONN_{G = (G, \overline{G})}(w, z) = t \), then \( S_4(v_0, w_t) \geq t \) for all DRF-edges \((w_1, w_2)\) in these DRF-paths. Suppose that \( G'_w = (G', \overline{G'})_w \) is the DRF-subgraph constructed from \( G_w = (G, \overline{G})_w \) by considering a new DRF-vertex \( v' \) and joining \( v' \) to each DRF-vertex \( v_1, v_2, \ldots, v_t \). For all \( i = 1, 2, 3, \ldots, t \), let \( R_{\mathcal{V}}(v') = 1 \) and \( S_{\mathcal{E}}(v_i, v') = R_{\mathcal{V}}(v_i) \). \( G_z = (G, \overline{G})_z \) and \( G'_z = (G', \overline{G'})_z \) are also specified as DRF-subgraphs. \( V \) has a DRF-vertex that is not a \( \alpha \) strong DRF-neighbor of DRF-vertex \( w \) and another DRF-vertex that is not a \( \alpha \) strong DRF-neighbor of DRF-vertex \( z \). (It is worth noting that every recently introduced DRF-edge is a powerful DRF-edge.) This way, we have \( S_4(G'_w = (G', \overline{G'})_w) < S_4(G = (G, \overline{G})) \).

As a result, by induction hypothesis \( G'_w = (G', \overline{G'})_w \) contains \( t \) internally disjoint \([w, v']\) DRF-paths say \( P_{A_i}, i = 1, 2, \ldots, t \), in such a way that \( P_{A_i} \) have \( v_i \). As well as, we have \( S_4(G'_z = (G', \overline{G'})_z) < S_4(G = (G, \overline{G})) \). As a result, by induction that \( G'_z = (G', \overline{G'})_z \) have \( t \) internally disjoint \([v'_z, z]\) DRF-paths say \( P_{B_i}, i = 1, 2, \ldots, t \), such that \( P_{B_i} \) have \( v_i \). For \( i = 1, 2, \ldots, t \), let \( P'_{A_i} \) be the \([w, v_t]\) DRF-subpaths of \( P_{A_i} \) and \( P'_{B_i} \) be the \([v_i, z]\) DRF-subpaths of \( P_{B_i} \). Now, by combining \( P_{A_i} \) and \( P_{B_i}, i = 1, 2, \ldots, t, t \) internally disjoint strongest \([w, z]\) DRF-path can be created. As a result, the result is valid by induction.

The DRF-edge form of Menger’s theorem is then stated. The proof follows the same pattern as Theorem 4.14.

**Theorem 4.15** Let \( G = (G, \overline{G}) \) be an DRFG. For each two DRF-vertices, \( w, z \in X \). Then, highest number of edge disjoint strongest \([w, z]\) DRF-paths in \( G = (G, \overline{G}) \) is same to the number of edges in a minimal \( w - z \, SRSVs \).

**Example 4.16** (Specimen to Theorems 4.14 and 4.15) Consider \( G = (G, \overline{G}) \) the DRFG given in Example 4.5 on \( X = \{v_1, v_2, v_3, v_4\} \). Consequently, \( G = (R_{\mathcal{V}}, S_{\mathcal{E}}) \) and \( \overline{G} = (R_{\mathcal{V}}, S_{\mathcal{E}}) \) in Fig. 4 depicts DFGs. Since \( v_2 v_3 \) is not a strong DRF-edge. There are two DRF-paths from \( v_2 \) to \( v_3 \) in \( G = (G, \overline{G}) \) which are as follows:

(i). \( v_2 \rightarrow v_1 \rightarrow v_3 \) with strength 1.2,
(ii). \( v_2 \rightarrow v_1 \rightarrow v_4 \rightarrow v_3 \) with strength 0.9. Therefore, in \( G = (G, \overline{G}) \), there is just one strongest DRF-path from \( v_2 \) to \( v_3 \) in \( G = (G, \overline{G}) \) with strength 1.2. According to
Table 3  Equivalence relation $\mathcal{R}$ on $\mathcal{X}$

|       | India | Pakistan | Bangladesh | Iran | Sri Lanka |
|-------|-------|----------|------------|------|-----------|
| India | 1     | 0        | 1          | 0    | 0         |
| Pakistan | 0   | 1        | 0          | 1    | 0         |
| Bangladesh | 1 | 0        | 1          | 0    | 0         |
| Iran | 0     | 1        | 0          | 1    | 0         |
| Sri Lanka | 0 | 0        | 0          | 0    | 1         |

Menger’s theorem for vertices, any minimum $v_2 - v_3$ $SRSV$ must contain one vertex. It is simple to confirm. $\{v_1\}$ is minimum $v_2 - v_3$ $SRSV$ in $G = (\overline{G}, \overline{G})$.

Clearly, in any minimum $v_2 - v_3$ $SRS\xi$ in $G = (\overline{G}, \overline{G})$, there is only one edge, and that is the reason, there is only one edge-disjoint strongest DRF-path from $v_2$ to $v_3$ in $G = (\overline{G}, \overline{G})$. $\{v_1, v_3\}$ is minimum $v_2 - v_3$ $SRS\xi$ in $G = (\overline{G}, \overline{G})$.

5 Application to trade network

Coronavirus 2019 (COVID-19) is a pandemic disease caused by coronavirus 2 (severe acute respiratory syndrome) (SARS-CoV-2). The first reported case of COVID-19 was discovered in Wuhan, China, in December 2019. The COVID-19 pandemic is an endless global pandemic and expanding worldwide. COVID-19 symptoms are many, but some common symptoms contain breathing difficulties, fever, loss of smell, cough, fatigue, headache, and loss of taste. This ongoing global pandemic shows its impacts in every field of life like industry, culture and entertainment, education, society, politics, economy, science and technology, health issues, environment, transportation, and many more. The most serious impact of COVID-19 is on economy of whole world specially on developing countries of Asia. Since the COVID-19 worldwide pandemic has began to touch more and more countries in various regions of the world, its global socio-economic impact has begun to emerge, but the full amount of which will be unknown for some time. So far, it is known that many businesses have suffered losses as a result of forced business closures, consumer declines, business order cancellations, and other factors. Developing Asian countries have been particularly hard hit, both directly as a result of COVID-19 and more indirectly as a result of the detrimental impact on the global supply chain, which their economies are strongly reliant on.

According to current research, the epidemic COVID-19 will push more than 80 million people in developing Asia into poverty by 2020. Given inequalities in sectors such as health, education, and employment, this figure may be much higher. Nearly 12 million people in Pakistan are expected to lose their jobs as a result of the shutdown. It is also expected that the percentage of people living in poverty will rise from 25% to roughly 55% if GDP growth remains between 0 and 1.5% due to the COVID-19 situation. The situation is similar in other developing Asian countries, such as India, Bangladesh, Sri Lanka, Iran, and Nepal. In this paper, we focus on the types of socio-economic status (SESS) in an economic network of developing countries in Asia using the connectivity between the countries that cross in the economic network. Consider the group of Asian developing countries $\mathcal{X} = \{\text{India, Pakistan, Bangladesh, Iran, Sri Lanka}\}$. We create an equivalence relation $\mathcal{R}$ as in Table 3, so that all developing countries whose trade was similarly affected by COVID are placed in the same equivalence class. Similarly, countries whose trade was affected by COVID in different ways are in a different equivalence class.
Let $V = \{(\text{India}, 0.7), (\text{Pakistan}, 0.6), (\text{Bangladesh}, 0.4), (\text{Iran}, 0.8), (\text{Sri Lanka}, 0.3)\}$ be an FS on $X$ which represents the net product to be traded to other countries and $\mathcal{R}X = (\mathcal{R}_X, \mathcal{R}_X')$ RFs. Here, $\mathcal{R}_X$ and $\mathcal{R}_X'$ are lower and upper approximations of $\mathcal{X}$, respectively, as follows:

$$\mathcal{R}V = \{(\text{India}, 0.4), (\text{Pakistan}, 0.6), (\text{Bangladesh}, 0.4), (\text{Iran}, 0.6), (\text{Sri Lanka}, 0.3)\},$$

$$\overline{\mathcal{R}}V = \{(\text{India}, 0.7), (\text{Pakistan}, 0.8), (\text{Bangladesh}, 0.7), (\text{Iran}, 0.8), (\text{Sri Lanka}, 0.3)\}.$$

Let $E = \{(\text{India}, \text{Pakistan}), (\text{India}, \text{Bangladesh}), (\text{India}, \text{Sri Lanka}), (\text{Pakistan}, \text{India}), (\text{Pakistan}, \text{Iran}), (\text{Bangladesh}, \text{India}), (\text{Bangladesh}, \text{Iran}), (\text{Iran}, \text{Pakistan}), (\text{Iran}, \text{Sri Lanka}), (\text{Sri Lanka}, \text{Pakistan}), (\text{Sri Lanka}, \text{Bangladesh}), (\text{Sri Lanka}, \text{Iran})\} \subseteq X \times X$ and on $E$, $S$ an equivalence relation as shown in Table 4 defined by

where $S$ represents the equivalence classes of trade relations between different developing countries during COVID-19. For example, the trade relations $(\text{India, Pakistan})$ and $(\text{India, Bangladesh})$ fit into identical corresponding class. This means that India’s linkage with Pakistan is identical to India’s linkage with Bangladesh. Let $E = \{(\text{India, Pakistan}, 0.35), (\text{India, Bangladesh}, 0.40), (\text{India, Sri Lanka}, 0.22), (\text{Pakistan, India}, 0.34), (\text{Pakistan, Iran}, 0.30), (\text{Bangladesh, India}, 0.25), (\text{Bangladesh, Iran}, 0.15), (\text{Iran, Pakistan}, 0.50), (\text{Iran, Sri Lanka}, 0.20), (\text{Sri Lanka, Pakistan}, 0.18), (\text{Sri Lanka, Bangladesh}, 0.24), (\text{Sri Lanka, Iran}, 0.24)\}$ be an FS on $E$, which represents membership of net trade between developing countries during COVID. Let $\mathcal{S}E = (\mathcal{SE}, \overline{\mathcal{SE}})$ be RFR, where $\mathcal{SE}$ and $\overline{\mathcal{SE}}$ are lower and upper approximations of $\mathcal{E}$, respectively, defined as follows:

$$\mathcal{SE} = \{(\text{India, Pakistan}, 0.30), (\text{India, Bangladesh}, 0.30), (\text{India, Sri Lanka}, 0.22), (\text{Pakistan, India}, 0.34), (\text{Pakistan, Iran}, 0.30), (\text{Bangladesh, India}, 0.25), (\text{Bangladesh, Iran}, 0.15), (\text{Iran, Pakistan}, 0.50), (\text{Iran, Sri Lanka}, 0.20), (\text{Sri Lanka, Pakistan}, 0.18), (\text{Sri Lanka, Bangladesh}, 0.24), (\text{Sri Lanka, Iran}, 0.24)\},$$

$$\overline{\mathcal{SE}} = \{(\text{India, Pakistan}, 0.40), (\text{India, Bangladesh}, 0.40), (\text{India, Sri Lanka}, 0.22), (\text{Pakistan, India}, 0.34), (\text{Pakistan, Iran}, 0.40), (\text{Bangladesh, India}, 0.25), (\text{Bangladesh, Iran}, 0.15), (\text{Iran, Pakistan}, 0.50), (\text{Iran, Sri Lanka}, 0.20), (\text{Sri Lanka, Pakistan}, 0.18), (\text{Sri Lanka, Bangladesh}, 0.24), (\text{Sri Lanka, Iran}, 0.24)\}.$$

As a result, the DFGs $G = (\mathcal{R}V, \mathcal{SE})$ and $\overline{G} = (\overline{\mathcal{R}}V, \overline{\mathcal{SE}})$ in Fig. 5 are DFGs. Therefore, $G = (G, \overline{G})$ is a DRFG, which is a trade network for developing countries. Since the vertices represent developing countries and the membership of each vertex in the lower and upper directed graphs is measured by net product trade with other countries. The approach for our application is now presented in the following algorithm. In this trading system, if there is trade between two vertices, they are adjacent to each other. Consider the two vertices India and Pakistan. Since there is export and import between India and Pakistan, there is an edge between the vertices of these two countries. In this trade network, we represent import with an inward edge and export with an outward edge. We can classify developing countries based on their trade deficits during COVID-19 to calculate the connectivity between distinct import and export flows. We refer to these regions with trade deficits as $\alpha$-region, $\beta$-region, and $\delta$-region. The region with no trade deficit is referred to as $\alpha$-region and the region with lower trade deficit is referred to as $\beta$-region. Regions with higher trade deficits, on the other hand, are referred to as $\delta$-regions.
| \( \mathcal{S} \) | (India, Pakistan) | (India, Bangladesh) | (India, Sri Lanka) | (Pakistan, India) | (Pakistan, Iran) | (Bangladesh, India, Iran) | (Iran, Sri Lanka) | (Iran, Pakistan) | (Sri Lanka, Bangladesh) | (Sri Lanka, Iran) |
|---------------------|-------------------|---------------------|-------------------|------------------|-----------------|------------------------|------------------|------------------|----------------------|------------------|
| (India, Pakistan)   | 1                 | 1                   | 0                 | 0                | 0               | 0                      | 0                | 0                | 0                    | 0                |
| (India, Bangladesh) | 1                 | 0                   | 0                 | 1                | 0               | 0                      | 0                | 0                | 0                    | 0                |
| (India, Sri Lanka)  | 0                 | 0                   | 1                 | 0                | 0               | 0                      | 0                | 0                | 0                    | 0                |
| (Pakistan, India)   | 0                 | 0                   | 0                 | 1                | 0               | 0                      | 0                | 0                | 0                    | 0                |
| (Pakistan, Iran)    | 0                 | 1                   | 0                 | 0                | 0               | 0                      | 0                | 0                | 0                    | 0                |
| (Bangladesh, 0 India) | 0               | 0                   | 0                 | 0                | 1               | 0                      | 0                | 0                | 0                    | 0                |
| (Bangladesh, 0 Iran) | 0                | 0                   | 0                 | 0                | 0               | 1                      | 0                | 0                | 0                    | 0                |
| (Iran, Pakistan)    | 0                 | 0                   | 0                 | 0                | 0               | 0                      | 1                | 0                | 0                    | 0                |
| (Iran, Sri Lanka)   | 0                 | 0                   | 0                 | 0                | 0               | 0                      | 1                | 0                | 0                    | 0                |
| (Sri Lanka, Pakistan) | 0               | 0                   | 0                 | 0                | 0               | 0                      | 0                | 1                | 0                    | 0                |
| (Sri Lanka, Bangladesh) | 0               | 0                   | 0                 | 0                | 0               | 0                      | 0                | 0                | 1                    | 1                |
| (Sri Lanka, Iran)   | 0                 | 0                   | 0                 | 0                | 0               | 0                      | 0                | 0                | 0                    | 1                |
Consider two countries: India and Pakistan. All feasible paths from India to Pakistan in DFG $\overline{\mathcal{G}}$ and DFG $\overline{\mathcal{G}}$, along with their strengths and the strength of the link between India and Pakistan, are listed below in Tables 5 and 6:

It is observed that the edge between India and Pakistan is a strong edge in DFG $\mathcal{G}$ and DFG $\overline{\mathcal{G}}$. Similarly, the edges between other countries can also be analyzed. It is difficult to deal unclear data information system and discuss the nature of each edge with fuzzy theory. A DRFG is an extended approach for a DFG and a rough set. In a fuzzy graph, the vertices and edges have a membership value from the interval $[0, 1]$, but rough fuzzy graphs...
Table 7  All paths from India to Pakistan in $G = (\overline{G}, G)$

| Path | Description |
|------|-------------|
| $P_1$ | India $\rightarrow$ Pakistan with strength 0.70 |
| $P_2$ | India $\rightarrow$ Sri Lanka $\rightarrow$ Pakistan with strength 0.36 |
| $P_3$ | India $\rightarrow$ Sri Lanka $\rightarrow$ Iran $\rightarrow$ Pakistan with strength 0.44 |
| $P_4$ | India $\rightarrow$ Sri Lanka $\rightarrow$ Bangladesh $\rightarrow$ Iran $\rightarrow$ Pakistan with strength 0.30 |
| $P_5$ | India $\rightarrow$ Sri Lanka $\rightarrow$ Bangladesh $\rightarrow$ Pakistan with strength 0.30 |
| $P_6$ | India $\rightarrow$ Bangladesh $\rightarrow$ Iran $\rightarrow$ Sri Lanka $\rightarrow$ Pakistan with strength 0.30 |

CONN $G = (\overline{G}, G)$ (India, Pakistan) = 0.70
CONN $G = (\overline{G}, G)$ (Pakistan, India) = 0.70

are important, because lower and upper approximations of vertices and edges have fuzzy membership values. Each $G$ and $\overline{G}$ of the membership value can be any real number in $[0, 1]$. A DRF diagram is an efficient approach to discuss incomplete knowledge relationships between countries and to control knowledge loss in a given system. All feasible paths from India to Pakistan in $G = (\overline{G}, G)$, along with their strengths and the strength of the link between India and Pakistan, are listed below in Table 7:

It can be observed that the DRF-edge between India and Pakistan is a strong DRF-edge. Similarly, the edges between other countries can also be analyzed. It helpful to discuss the nature of each edge in an DRFG. This proves the applicability and efficiency of our research work. From Fig. 5 and routine calculations, the connectivity between the vertices of $G$ and $\overline{G}$ is as follows:

CONN $G = (\overline{G}, G)$ (India, Pakistan) = 0.22,
CONN $G = (\overline{G}, G)$ (Pakistan, India) = 0.20,
CONN $G = (\overline{G}, G)$ (India, Bangladesh) = 0.22,
CONN $G = (\overline{G}, G)$ (Bangladesh, India) = 0.15,
CONN $G = (\overline{G}, G)$ (India, Sri Lanka) = 0.20,
CONN $G = (\overline{G}, G)$ (Pakistan, Iran) = 0.22,
CONN $G = (\overline{G}, G)$ (Iran, Pakistan) = 0.20,
CONN $G = (\overline{G}, G)$ (Bangladesh, Iran) = 0.25,
CONN $G = (\overline{G}, G)$ (Iran, Sri Lanka) = 0.22,
CONN $G = (\overline{G}, G)$ (Sri Lanka, Iran) = 0.24,
CONN $G = (\overline{G}, G)$ (Sri Lanka, Pakistan) = 0.24,
CONN $G = (\overline{G}, G)$ (Sri Lanka, Bangladesh) = 0.24,
By direct calculations, it is found that $\alpha$-regions are (India, Pakistan), (Pakistan, India), (India, Bangladesh), (India, Sri Lanka), (Pakistan, Iran), (Bangladesh, India), (Iran, Pakistan), $\beta$-regions are (Sri Lanka, Bangladesh), (Sri Lanka, Iran), and the $\delta$-regions are (Sri Lanka, Pakistan), (Bangladesh, Iran) in the DRFG network. This classification into different regions with trade deficits will be helpful to understand the situation of trade in developing countries due to COVID-19. We present our proposed method as an algorithm in Table 8. We also describe our method in a flowchart as shown in Fig. 6.

6 Comparative analysis

The theory of rough sets uses the boundary region of a set to demonstrate ambiguity. Non-emptiness of the boundary region of a set indicates that it is a rough set, while emptiness indicates that it is a classical set. In rough sets, we have additional knowledge related to the elements. Elements with identical knowledge are stored in the same equivalence class, and we take lower and upper approximations of the set that control vagueness using these equivalence classes. In directed crisp theory, the strength of the connection for any pair of vertices is either 0 or 1, since there is or is not a directed edge between any two vertices. For this pretext, the study of edges in directed crisp graph theory provides no insight. The directed classical graph is an enhancement of DFG, and all edges and vertices in DFGs are ambiguous. The degree of membership of DFGs can be any real integer in the range $[0, 1]$, and the existence of the membership value is essential for the study of edges to determine the nature of each edge and the basic structure of the DFG. Rough set and fuzzy set are two alternative ways of dealing with ambiguity. They are not opposing theories, and to achieve profitable consequences, the two theories can be combined. The result of this analysis is a hybrid model of these theories showing the fuzziness of the rough set: the rough fuzzy set. The DRFGs is a combination of rough fuzzy sets and DFGs. In DRFGs, all edges and vertices in the lower and upper approximations are uncertain in nature. The membership value of the DRFG, like that of the DFG, can be any real number in the range $[0, 1]$. Studying the edges for lower and upper approximations of DFGs is significant for the study of edges to determine the nature of each edge and the basic structure of DRFG due to the availability of the membership value. At the same time, many real problems with fuzziness contain inaccurate, uncertain, or incomplete knowledge. FGs are not able to competently handle the uncertainty of fuzzy incomplete information of a problem. To deal with this kind of faults in real-world problems, we choose DRFGs that are able to deal with fuzziness in incomplete information.
Table 8  Algorithm

Algorithm: Identifying the affecting developing countries due to Covid-19 in trade network

Input
Step 1. Consider a directed rough fuzzy trade network of developing countries
Step 2. In the considered network, use an arrow to show the direction of import with an inside edge and export with an outward edge
Step 3. Insert the countries set $X$ and the equivalence relation on $X$ is $R$
Step 4. Insert the relations set $E$ and equivalence relation on $E$ is $S$
Step 5. On $X$, Insert $V$ the vertex FS and on $E$, insert $E$ the edge FS
Step 6. Calculate the membership of vertex $\overline{RV} = (RV_V, RV)$ in DRFGs by using the formula
\[
(\overline{RV})(x) = \bigwedge_{y \in X} ((1 - R(x, y)) \lor V(y)),
\]
\[
(\overline{RV})(x) = \bigvee_{y \in X} (R(x, y) \land V(y)),
\]
Step 7. Compute the membership of edges $\overline{SV} = (SV, \overline{SV})$ in DRFGs by using the formula
\[
(\overline{SE})(wz) = \bigwedge_{xy \in E} ((1 - S(wx, xy)) \lor E(xy)),
\]
\[
(\overline{SE})(wz) = \bigvee_{xy \in E} (S(wx, xy) \land E(xy)),
\]
Step 8. Step 6 and step 7 should be repeated for each pair of designated trade flows
Step 9. Calculate the connectivity strength or intensity between each pair of identified trade networks
Output
Step 10. (i): There is a $\alpha$-trade region if the intensity of connection inside two specified trade flow justified following:
\[
\overline{SE}(z_0, z_1) > CONN_{G_{w_0w_1}}(w_0, w_1),
\]
\[
\overline{SE}(z_0, z_1) > CONN_{G_{w_0w_1}}(w_0, w_1),
\]
(ii): The $\beta$-trade region exists if the intensity of connection inside two specified trade flow justified following:
\[
\overline{SE}(z_0, z_1) = CONN_{G_{w_0w_1}}(w_0, w_1),
\]
\[
\overline{SE}(z_0, z_1) = CONN_{G_{w_0w_1}}(w_0, w_1),
\]
(iii): The $\delta$-trade region exists if the intensity of connection inside two specified trade flow justified following:
\[
\overline{SE}(z_0, z_1) < CONN_{G_{w_0w_1}}(w_0, w_1),
\]
\[
\overline{SE}(z_0, z_1) < CONN_{G_{w_0w_1}}(w_0, w_1),
\]
Consider a DRF trade network of developing countries

Arrows indicate the import with inward edge export with outward edge

Input set $X$ of countries and equivalence relation $R$.

Input set $E$ of relations and equivalence relation $S$.

Input fuzzy vertex set $V$ on $X$.

Input fuzzy edge set $E$ on $E$.

Calculate the membership of vertex $RV = (R_V, R_V)$ by using the DRF-vertex formula

Calculate the membership of edge $SV = (S_V, S_V)$ by using the DRF-edge formula

Repeat this step for all vertices and edges

Calculate the intensity of connectedness between two vertices

Justified formula $\alpha$ DRF-edge

Justified formula $\beta$ DRF-edge

Justified formula $\delta$ DRF-edge

$\alpha$ - trade region

$\beta$ - trade region

$\delta$ - trade region

Fig. 6 Flowchart of trade network problem

7 Conclusion

A DRF is useful for simulating real-world challenges, including ambiguity and uncertainty. This model gives more compatibility, precisions, and flexibility to the system as compared to fuzzy model and rough model. There are still many directed fuzzy graph theoretic results which have not been yet studied in case of DRF-graph theory, e.g., vertex and edge connectivity parameters, strength-reducing sets, etc. In this work, we have introduced the idea of three different types of vertices based on the indegree and outdegree of vertices of directed rough fuzzy networks. We have proposed strength reducing sets ($SRS\delta$) of DRF-vertices and DRF-edges in DRFGs. We have also presented some important results on reducing the strength of DRFGs along with examples. The fuzzy version of Menger’s theorem is not suitable for dealing with problems where the information system is uncertain. That is why, we have generalized the idea of Menger’s theorem from vertices and edges in DFGs to DRFGs, which are suitable for dealing with uncertainty in information systems. We have presented an application based on strong and weak edges of DRFGs to determine the impact of COVID-19 on developing countries trade and to deal with vagueness and uncertainty in the economy. The knowledge of strong and weak edge highlights the significance of each edge, which results in controlling the trade deficits in the countries affected by COVID-19. Moreover, we have designed an algorithm (flowchart) to elaborate the procedure of our application. Finally, we made a detailed comparison between our research results and existing methods to demonstrate their applicability and productivity.

In the future, we plan to extend our research to the connectivity of vertices and edges of Soft Directed Rough Fuzzy Graphs (SDRGFs).
References

Akram M, Arshad M (2018) Fuzzy rough graph theory with applications. Int J Comput Intell Syst 12(1):90–107
Akram M, Zafar F (2019a) Rough fuzzy digraphs with application. J Appl Math Comput 59(1):91–127
Akram M, Zafar F (2019b) A new approach to compute measures of connectivity in rough fuzzy network models. J Intell Fuzzy Syst 36(1):449–465
Akram M, Zafar F (2020) Hybrid soft computing models applied to graph theory. Studies in fuzziness and soft computing, vol 380. Springer, Berlin, pp 1–434
Akram M, Siddique S, Ahmad U (2021) Menger’s theorem for m-polar fuzzy graphs and application of m-polar fuzzy edges to road network. J Intell Fuzzy Syst 41(1):1553–1574
Akram M, Siddique S, Alharbi MG (2022) Clustering algorithm with strength of connectedness for m-polar fuzzy network models. Math Biosci Eng 19(1):420–455
Bhattacharya P (1987) Some remarks on fuzzy graphs. Pattern Recognit Lett 6(5):297–302
Bhattacharya P, Suraweera F (1991) An algorithm to compute the supremum of max–min powers and a property of fuzzy graphs. Pattern Recognit Lett 12(7):413–420
Bhutani KR, Rosenfeld A (2003a) Fuzzy end nodes in fuzzy graphs. Inf Sci 152:323–326
Bhutani KR, Rosenfeld A (2003b) Geodesies in fuzzy graphs. Electron Notes Discrete Math 15:49–52
Bhutani KR, Rosenfeld A (2003c) Strong arcs in fuzzy graphs. Inf Sci 152:319–322
Chen G, Zhong N (2011) Granular structures in graphs. In: International conference on rough sets and knowledge–edge technology. Springer, pp 649–658
Chen G, Zhong N, Yao Y (2008) A hypergraph model of granular computing. In: IEEE International conference on granular computing, pp 130–135
Dubois D, Prade H (1990) Rough fuzzy sets and fuzzy rough sets. Int J Gen Syst 17(2–3):191–209
Karunambigai MG, Buvaneswari R (2017) Menger’s theorem for intuitionistic fuzzy graphs. Notes Intuit Fuzzy Sets 23(1):70–78
Kauffmann A (1973) Introduction à la théorie des sous-ensembles flous, Masson et Cie, 1
Kóczy L (1992) Fuzzy graphs in the evaluation and optimization of networks. Fuzzy Sets Syst 46(3):307–319
Kryszkiewicz M (1998) Rough set approach to incomplete information systems. Inf Sci 112(1–4):39–49
Ma X, Liu Q, Zhan J (2017) A survey of decision making methods based on certain hybrid soft set models. Artif Intell Rev 47(4):507–530
Malik HM, Akram M (2018) A new approach based on intuitionistic fuzzy rough graphs for decision-making. J Intell Fuzzy Syst 34(4):2325–2342
Mathew S, Mordeson JN (2017) Directed fuzzy networks as a model to illicit flows and max flow min cut theorem. New Math Natural Comput 13(03):219–229
Mathew S, Sunitha MS (2009) Types of arcs in a fuzzy graph. Inf Sci 179(11):1760–1768
Mathew S, Sunitha MS (2010) Node connectivity and arc connectivity of a fuzzy graph. Inf Sci 180(4):519–531
Mathew S, Sunitha MS (2013) Menger’s theorem for fuzzy graphs. Inf Sci 222:717–726
Mordeson JN, Nair PS (2000) Fuzzy graphs and fuzzy hypergraphs. Physica-verlay, Heidelberg
Pawlak Z (1982) Rough sets. Int J Comput Inf Sci 46(3):307–319
Riaz M, Karaaslan F, Nawaz I, Sohail M (2021) Soft multi-rough set topology with applications to multi-criteria decision-making problems. Soft Comput 25(1):799–815
Rosenfeld A (1975a) Fuzzy graphs. In: Zadeh LA, Fu KS, Shimura M (eds) Fuzzy sets and their applications. Physica, Heidelberg
Rosenfeld A (1975b) Fuzzy graphs. In: Zadeh LA, Fu K-S, Tanaka K, Shimura M (eds) Fuzzy sets and their applications to cognitive and decision processes. Academic Press, New York, pp 77–95
Tong Z, Zheng D (1996) An algorithm for finding the connectedness matrix of a fuzzy graph. Congr Numer 189–192
Yeh HT, Bang SY (1975) Fuzzy relations, fuzzy graphs, and their applications to clustering analysis. In: Zadeh LA, Fu K-S, Tanaka K, Shimura M (eds) Fuzzy sets and their applications to cognitive and decision processes. Academic Press, New York, pp 125–149
Zadeh LA (1996) Fuzzy sets. In: Kluwer Academic Publishers, Dordrecht, pp 394–432
Zadeh LA (1997) Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic. Fuzzy Sets Syst 90(2):111–127
Zafar F, Akram M (2018) A novel decision-making method based on rough fuzzy information. Int J Fuzzy Syst 20(3):1000–1014
Zhan J, Malik HM, Akram M (2019) Novel decision-making algorithms based on intuitionistic fuzzy rough environment. Int J Mach Learn Cybern 10(6):1459–1485
Publisher's Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.