RECENT PROGRESS IN AdS/CFT

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Abstract

The study of AdS/CFT (or gauge/gravity) duality has been one of the most active and illuminating areas of research in string theory over the past decade. The scope of its relevance and the insights it is providing seem to be ever expanding. In this talk I briefly describe some of the attempts to explore how the duality works for maximally supersymmetric systems.

Contribution to proceedings of the symposium Shifmania
celebrating the 60th birthday of M. A. Shifman
1 Introduction

Misha Shifman is one of the most productive (and nicest) theoretical physicists that I know. It is an honor to have been asked to speak at this gathering celebrating his 60th birthday. At this stage of my life, that age seems much younger than it used to.

Recently, in my capacity as an editor of Reviews of Modern Physics I persuaded Misha to coauthor a review article entitled Supersymmetric Solitons and How They Help Us Understand Non-Abelian Gauge Theories. I am very pleased to report that this article has now been published [1]. It is already highly cited.

I have chosen to present an overview of recent progress in AdS/CFT. An alternative, and more inclusive, name for this subject is gauge/gravity duality. There has been progress on various fronts. In the time available to me, I will only be able to scratch the surface of this fascinating subject. There is a great deal of impressive work that I will not mention at all. Further discussion of some of these topics, such as the Konishi multiplet, can be found in Arkady Tseytlin’s contribution [2].

2 Review of Some Basic Facts

In Maldacena’s original paper [3], he proposed three maximally supersymmetric examples of AdS/CFT duality. A basic indication that the dualities (or equivalences) are plausible is that the symmetries match. In each case, there is a supergroup, which describes the isometries of the string theory or M-theory background geometry. The same supergroup appears as the superconformal symmetry group of the dual quantum field theory. Also, the string theory or M-theory solution has $N$ units of flux threading the sphere factor in the geometry. In fact, the background configuration corresponds to the near-horizon geometry of $N$ coincident branes, each of which contributes one unit of flux. The dual conformal field theory, which also depends on the integer $N$, is the low energy world-volume theory on the branes.

- **M2-brane Duality**: M-theory on $AdS_4 \times S^7$ is dual to a superconformal field theory (SCFT) in three dimensions. The supergroup is $OSp(8|4)$.

- **D3-brane Duality**: Type IIB superstring theory on $AdS_5 \times S^5$ is dual to a SCFT in four dimensions, specifically $\mathcal{N} = 4$ super Yang–Mills (SYM) theory. The supergroup is $PSU(2,2|4)$.

- **M5-brane Duality**: M theory on $AdS_7 \times S^4$ is dual to a SCFT in six dimensions. The supergroup is $OSp(6,2|4)$.
2.1 The type IIB / $\mathcal{N} = 4$ SYM example

This by far the most studied, and best understood, example. The $N$ units of flux ($\int_{S^5} F_5 \approx N$) in the superstring solution correspond to the gauge group $SU(N)$ in the $\mathcal{N} = 4$ super Yang–Mills theory [4]. The gauge theory has a well-known large-$N$ topological (’t Hooft) expansion [5]. The expansion is in powers of $1/N$ for large $N$ at fixed $\lambda$, where the ’t Hooft parameter is

$$\lambda = g_{\text{YM}}^2 N. \quad (1)$$

This expansion corresponds to the loop expansion of the string theory. One also identifies

$$R^2/\alpha' \approx \sqrt{\lambda} \quad \text{and} \quad g_s \approx \lambda/N, \quad (2)$$

where $R$ is the radius of the $S^5$ and the $AdS_5$. $g_s$ is the string coupling constant determined by the value of the dilaton field.

2.2 The type IIA / ABJM example

There has been significant progress in the last couple of years in understanding the M2-brane duality. The suggestion [6] that the three-dimensional SCFT should be Chern–Simons gauge theory was implemented for maximal supersymmetry ($\mathcal{N} = 8$) by Bagger and Lambert [7] and by Gustavsson [8]. However, their construction only works for the gauge group $SO(4)$, and it does not provide the desired dual to M-theory on $AdS_4 \times S^7$.

The correct construction was eventually obtained by Aharony, Bergman, Jafferis, and Maldacena (ABJM) [9]. One key step in their work was to consider a more general problem: M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$, with $N$ units of flux. This gives $3/4$ maximal supersymmetry for $k > 2$. Thus, the dual gauge theory is an $\mathcal{N} = 6$ superconformal Chern–Simons theory in three dimensions. The appropriate gauge group turns out to be $U(N)_k \times U(N)_{-k}$, where the subscripts are the levels of the Chern–Simons terms. The ABJM theory also contains bifundamental scalar and spinor fields. This theory has a topological expansion, just like the usual ones in four dimensions, with an ’t Hooft parameter

$$\lambda = N/k. \quad (3)$$

The only unusual feature is that the ’t Hooft parameter is rational. The extension of the supersymmetry from $\mathcal{N} = 6$ to $\mathcal{N} = 8$ for $k = 1, 2$ is a nontrivial property of the quantum theory.
The orbifold $S^7/\mathbb{Z}_k$ can be described as a circle bundle over a $CP^3$ base. The circle has radius $R/k$, where $R$ is the $S^7$ radius. When $k^5 \gg N$, there is a weakly coupled type IIA superstring interpretation with string coupling constant

$$g_s \approx (N/k^5)^{1/4}.$$  \hspace{1cm} (4)

One then obtains the correspondences

$$R^2/\alpha' \approx \sqrt{\lambda} \quad \text{and} \quad g_s \approx \lambda^{5/4}/N,$$  \hspace{1cm} (5)

which is very similar to the previous duality, but not precisely the same.

### 2.3 AdS energies and conformal dimensions

The metric of $AdS_{p+2}$ in global coordinates is

$$ds^2[AdS_{p+2}] = d\rho^2 - \cosh^2 \rho dt^2 + \sinh^2 \rho ds^2[S^p].$$  \hspace{1cm} (6)

Here, $ds^2[S^p]$ denote the metric of a unit $p$-dimensional sphere. Actually, AdS/CFT duality requires taking the covering space of AdS, which means that the time coordinate $t$ runs from $-\infty$ to $+\infty$.

Witten [10] and Gubser, Klebanov, Polyakov [11] gave a prescription for relating $n$-point correlation functions in the gauge theory to corresponding quantities in the string theory. In the case of two-point functions, the duality relates the energy $E_A$ of a string state $|A\rangle$ (defined with respect to the global time coordinate $t$)

$$H_{\text{string}}|A\rangle = E_A|A\rangle,$$  \hspace{1cm} (7)

to the conformal dimensions $\Delta_A$ of the corresponding gauge-invariant local operator $O_A$ defined by

$$\langle O_A(x)O_B(y) \rangle \approx \frac{\delta_{AB}}{|x-y|^{2\Delta_A}}.$$  \hspace{1cm} (8)

Specifically, the duality requires that

$$\Delta_A(\lambda, 1/N) = E_A(R^2/\alpha', g_s).$$  \hspace{1cm} (9)

The ’t Hooft expansion of the dimension of $O_A$ is

$$\Delta_A(\lambda, 1/N) = \Delta_A^{(0)} + \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \sum_{l=1}^{\infty} \lambda^l \Delta_{l,g}. \hspace{1cm} (10)$$

$\Delta_A^{(0)}$ is the classical (engineering) dimension, and the rest is called the anomalous dimension.
Almost all studies have focused on the planar approximation, \( g = 0 \), which is dual to free string theory. This restriction may make the problem fully tractable, but it is certainly not easy. After all, it would be an extraordinary achievement to solve an interacting four-dimensional quantum field theory even in the planar approximation.

### 2.4 Approaches to testing the dualities

Given that it is not possible to completely solve any of these theories, the question arises how best to test and explore the workings of AdS/CFT duality. The most obvious things—matching symmetries and the dimensions of chiral primary operators—have been done long ago. One wants to dig deeper. One approach is to match, as much as possible, energies and dimensions of fields/operators that are not protected by supersymmetry. It should be noted, however, that a complete test of the duality would also require matching three-point correlators, since a conformal field theory is completely characterized by its two-point and three-point functions. There has been much less progress on this front.

One approach that has been quite successful is the following. First, identify tractable examples of classical solutions of the string world-sheet theory. Next, examine the spectrum of small excitations about these solutions and compute their energies \( E_A \). Finally, identify the corresponding class of operators in the dual gauge theory and compute their dimensions \( \Delta_A \) in the planar approximation. Then compare to \( E_A \). One subtlety in this analysis is that this comparison requires an extrapolation from large \( \lambda \), where the classical world sheet theory is valid, to small \( \lambda \), where the gauge theory can be studied perturbatively. Thus, one needs to identify examples in which this is possible. As we will see, in practice this has conjectural aspects.

A variant of the preceding procedure is to compare equations that determine \( E_A \) and \( \Delta_A \) rather than the solutions. Approaches based on integrability and algebraic curves try to obtain equations of “Bethe type” on both sides and to match them. This is a very active area of research, but I will not be able to review it here. One important issue is that it is much easier to study the world-sheet theory when the range of \( \sigma \) is infinite (rather than a circle). In other words, the string itself is infinite, rather than a loop. In the gauge theory analysis this corresponds to the thermodynamic limit of the Bethe equations arising from a spin-chain analysis. There has been progress recently in extending the integrability techniques to the compact case [12]. However, the story is quite technical, and I don’t think it is completely settled.
3 Classical String Solutions

For the reasons outlined above, we want to identify classical string solutions in the $AdS_5 \times S^5$ background that can be used to test the duality. The discussion that follows largely follows an excellent review article by Plefka [13]. Other useful reviews include [14, 15].

The bosonic part of the string world-sheet action has six cyclic coordinates:

$$(t, \varphi_1, \varphi_2; \phi_1, \phi_2, \phi_3),$$ \hfill (11)

where the first three coordinates pertain to $AdS_5$ and the second three to $S^5$. Specifically, we parametrize $S^5$ as follows:

$$ds^2(S^5) = d\gamma^2 + \cos^2 \gamma d\phi_3^2 + \sin^2 \gamma ds^2(S^3),$$ \hfill (12)

where

$$ds^2(S^3) = d\psi^2 + \cos^2 \psi d\phi_1^2 + \sin^2 \psi d\phi_2^2.$$ \hfill (13)

Associated to these cyclic coordinates one has conserved charges

$$(E, S_1, S_2; J_1, J_2, J_3).$$ \hfill (14)

$E$ is the energy and the other five charges are angular momenta.

One much-studied class of string solutions involves a line up the center of $AdS_5$, described by $\rho = 0$ and $t = \kappa \tau$, where $\kappa$ is a constant and $\tau$ is the world-sheet time coordinate. These configurations have $S_1 = S_2 = 0$.

3.1 Point-particle solutions

The simplest solution is a point particle (collapsed string) encircling the sphere. In addition to $\rho = 0$ and $t = \kappa \tau$, this is described by

$$\gamma = \pi/2, \quad \phi_1 = \kappa \tau, \quad \psi = 0.$$ \hfill (15)

This has $J_2 = J_3 = 0$.

The quantum excitations of this solution have energies that can be expanded in powers of $1/J$ for large $J = J_1$, where

$$\kappa = J/\sqrt{\lambda}$$ \hfill (16)

is held fixed. This is equivalent to the Berenstein, Maldacena Nastase (BMN) analysis of strings in a plane-wave background [16]. One obtains

$$E - J \approx E_2(\kappa) + \frac{1}{J} E_4(\kappa) + \ldots$$ \hfill (17)
The exact BMN result is
\[ E_2 = \sum_{n=-\infty}^{\infty} \sqrt{n^2 + \kappa^2} N_n, \tag{18} \]
where \( N_n = \sum_{i=1}^{8} \alpha_n^i \alpha_n^i + \text{fermions} \) is expressed in terms of ordinary oscillators
\[ [\alpha_n^i, \alpha_n^{ij}] = \delta^{ij} \delta_{mn}. \tag{19} \]
The level matching condition is \( \sum n N_n = 0. \)

The BMN paper proposed a scaling rule, known as BMN scaling, which predicts agreement with the anomalous dimensions of operators in the dual gauge theory, even though one calculation is valid for large \( \lambda \) and the other for small \( \lambda \). In other words, their scaling hypothesis, if valid, would justify the extrapolation from small \( \lambda \) to large \( \lambda \). In fact, it turns out that \( E_2 \) agrees perfectly, but agreement for \( E_4 \) breaks down at three loops [17]. This is not a problem for AdS/CFT duality, only for the BMN scaling conjecture.\(^1\)

### 3.2 Spinning string solutions

A class of interesting generalizations of the preceding solution describes circular or folded strings that are extended on the \( S^3 \subset S^5 \). These have \( t = \kappa \tau, \rho = 0, \) and \( \gamma = \pi/2, \) as before. But now one takes
\[ \phi_1 = \omega_1 \tau, \quad \phi_2 = \omega_2 \tau, \quad \psi = \psi(\sigma). \tag{20} \]
For these choices, the string equation of motion gives
\[ \psi'' + \omega_{21}^2 \sin \psi \cos \psi = 0, \tag{21} \]
where \( \omega_{21}^2 = \omega_2^2 - \omega_1^2. \) This is the well-known pendulum equation.

This equation has a first integral
\[ \psi' = \omega_{21} \sqrt{q - \sin^2 \psi}, \quad q = (\kappa^2 - \omega_1^2)/\omega_{21}^2. \tag{22} \]
The solution for \( q < 1 \), which involves the elliptic integrals \( E(q) \) and \( K(q) \), describes a folded string. It corresponds to a pendulum that oscillates back and forth. The solution for \( q > 1 \), which involves the elliptic integrals \( E(q^{-1}) \) and \( K(q^{-1}) \), describes a circular string. It corresponds to a pendulum that goes round and round. In the classical limit, the energy has the form
\[ E = \sqrt{\lambda} F(J_1/\sqrt{\lambda}, J_2/\sqrt{\lambda}). \tag{23} \]

\(^1\)Perhaps it would be more fair to say that the BMN scaling conjecture was made for the plane-wave limit only, which corresponds to \( E_2 \); what fails is an attempt to generalize the scaling conjecture beyond that.
3.3 Dual gauge theory analysis

This string theory result can be extrapolated to small $\lambda$ and compared to the dual gauge theory. The operators that carry $J_1, J_2$ charges have the form

$$O_{\alpha}^{J_1,J_2} = \text{Tr} \left( Z^{J_1} W^{J_2} \right) + \ldots \quad (24)$$

where $Z$ and $W$ are complex scalar fields in the adjoint of $SU(N)$. The additional terms denoted by dots involve different orderings of the $Z$s and $W$s. Such a trace can be viewed as a ring configuration of an $S = 1/2$ quantum spin chain, where $W$ corresponds to spin up and $Z$ corresponds to spin down.

The conformal dimensions of operators $O_{\alpha}^{J_1,J_2}(x)$ with these charges are eigenvalues of the dilatation operator

$$\mathcal{D}O_{\alpha}^{J_1,J_2}(x) = \sum_{\beta} D_{\alpha\beta} O_{\beta}^{J_1,J_2}(x). \quad (25)$$

In the planar one-loop approximation the equations are precisely those of a ferromagnetic Heisenberg spin chain, which is a well-known integrable system, whose Hamiltonian is proportional to

$$\mathcal{H} = \sum_{i=1}^{J} \left( \frac{1}{4} - \vec{\sigma}_i \cdot \vec{\sigma}_{i+1} \right). \quad (26)$$

This can be solved using Bethe ansatz techniques, thereby obtaining conformal dimensions that can be compared (successully) with energies of the corresponding string solutions. Higher-order terms, which correspond to more complicated spin-chain Hamiltonians, have also been studied.

3.4 Strings spinning in AdS

Another interesting class of classical string solutions are ones in which the string position is extended in the AdS space and a point moving on the sphere (rather than the other way around). The first example of this type is the straight folded string rotating in $AdS_3 \subset AdS_5$ [18]. One finds that for large $S$

$$E = 2\Gamma(\lambda) \log S + O(S^0), \quad (27)$$

where

$$\Gamma(\lambda) = \frac{\sqrt{\lambda}}{2\pi} + O(\lambda^0) \quad \text{for} \quad \lambda \gg 1. \quad (28)$$

The dual gauge theory operators are

$$\text{Tr}(D_+^{s_1} Z D_+^{s_2} Z) \quad s_1 + s_2 = S. \quad (29)$$
Their anomalous dimensions take the same form as the energy with
\[ \Gamma(\lambda) = \frac{\lambda}{4\pi^2} + O(\lambda^2) \quad \text{for} \quad \lambda \ll 1. \] (30)

In order to compare these, one needs a procedure to extrapolate between small and large \( \lambda \). In fact, an exact formula for the \textit{cusp anomalous dimension} \( \Gamma(\lambda) \) has been deduced using the assumption of exact integrability [19]. It passes all tests and is likely to be correct.

The generalization of this duality to twist \( J \) operators, which have the form
\[ \text{Tr}(D_{s_1} Z D_{s_2} Z \ldots D_{s_J} Z) \quad \text{where} \quad \sum s_i = S, \] (31)
has been explored by Dorey and Losi [20]. They computed the corresponding conformal dimensions using an \( SL(2) \) spin chain model. For large \( S \) the correspond classical string solutions are \textit{spiky strings} with \( J \) cusps. The duality predictions are verified to the extent that they have been explored.

4 Conclusion

There has been a lot of progress in testing AdS/CFT in various special cases for maximally supersymmetric theories. Much of this progress has exploited the integrability of the string world-sheet theory on the one hand and the integrability of various spin-chain models that arise in studies of the dual gauge theory in the planar approximation on the other hand.

I have described classical string solutions and the dual gauge theory operators that are relevant to the D3-brane duality. There has also been very interesting work exploring analogous constructions for the M2-brane duality following the discovery of the ABJM theory. Much less is known about the M5-brane theory, though there has been significant progress when two of the dimensions wrap a Riemann surface [21, 22].

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