Conservation laws and the equivalence group

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Abstract. The paper generalizes differential conservation laws for the two-dimensional hydrodynamic Euler equations and their equivalent conservation laws for families of streamlines previously derived by the author. Three-dimensional analogues of these results, i.e., differential conservation laws for the three-dimensional hydrodynamic Euler equations, are presented. All conservation laws are divergent identities of the form $\text{div}\, F = 0$ and are found in two equivalent forms: in “mathematical-physical” form, where the vector field $F$ is expressed in terms of quantities included in the Euler equations and their partial derivatives, and in geometric form. The second form represents conservation laws for families of streamlines where the vector field $F$ under the divergence sign is expressed in terms of the classical geometric characteristics of curves — their Frenet unit vectors $\tau$, $\nu$, and $\beta$ (the unit tangent, principal normal, and binormal vectors), the first curvature $k$, and the second curvature $\kappa$. Special attention is paid to the geometric interpretation of the obtained conservation laws and their constituent expressions. Issues related to the equivalence group of the eikonal equation and other equations are discussed.

All results were obtained using the general vector and geometric formulas (differential conservation laws and other formulas), obtained by the author for families of arbitrary smooth curves, families of arbitrary smooth surfaces, and arbitrary smooth vector fields.

1. Introduction

This paper is a continuation of the previous work of the author [1–9]. The line of research developed in these papers can be defined as the study of differential equations (DEs) of mathematical physics (the theory of propagation of waves of different nature in inhomogeneous media) based on geometric and group analysis. The solution of the problem of deriving a conservation law for a given mathematical object occupies an important place in this study. Conservation laws and their applications are of great significance in mathematical physics [10–12], continuum mechanics [12, 13], and computational mathematics [14, 15] (the list of references is, of course, incomplete).

We use a mathematical definition of differential conservation laws for DEs which is close to the definition given in [16]. Consider a DE (system) of the form $L[u, a] = 0$, where $u = u(x, t)$ is the solution of the DE (corresponding to the physical field or process considered), $a = a(x, t)$ are parameters, $x = (x_1, x_2, \ldots, x_n)$, $t$ are independent variables, $L$ is the given differential operator. The conservation law for such a DE is an identity of the form $\text{div}\, F + \partial R/\partial t = 0$, where the vector field $F$ and the scalar function $R$ are expressed in terms of the quantities $x$, $u$, and $a$ included in the DE and their partial derivatives. For example, the known conservation law for the motion of an incompressible fluid has the form $\text{div}\, v = 0$, where $v$ is the velocity of its
particles. Generalizing this definition, by the law of conservation for some mathematical object we will mean the above-mentioned identity where the vector field $\mathbf{F}$ and the scalar function $R$ are expressed in terms of characteristics of this object. In this paper, as in [3–8], as such an object we consider a particular DE, a family of curves, a family of surfaces, or a vector field.

The purpose of this work is to derive three-dimensional analogues of the results obtained in [4, 6] for the two-dimensional case, i.e., to derive differential conservation laws for the three-dimensional hydrodynamic Euler equations and their associated conservation laws for families of streamlines. The results are described in Section 3. Their derivation is based, in particular, on the results of [4, 8], which are briefly described in Section 2 in order to provide a comparison of the corresponding formulas to the two-dimensional and three-dimensional cases. In addition, in this paper, we give one more (compared to [6]) geometric interpretation for the conservation law for families of plane curves which is associated with the Gaussian curvature of some surfaces and with the equivalence group.

The vector lines $L_\tau$ of the physical vector fields corresponding to the solutions of the DEs of mathematical physics form a family of curves $\{L_\tau\}$ and continuously fill a domain $D$. For example, for hydrodynamic Euler equations the role of the curves $L_\tau$ is played by streamlines. Therefore, in this paper, as in [3–8], we study the properties of families of curves $\{L_\tau\}$, rather than the properties of fixed curves. All the quantities $\tau, \nu, \beta, k,$ and $\kappa$ are the vector and scalar fields in the domain $D$. The symbols $(\mathbf{a} \cdot \mathbf{b})$ and $\mathbf{a} \times \mathbf{b}$ denote the scalar and vector products of the vectors $\mathbf{a}$ and $\mathbf{b}$, $\nabla$ is the Hamiltonian operator, $(\mathbf{v} \cdot \nabla)\mathbf{a}$ is the derivative of the vector $\mathbf{a}$ in the direction of the vector $\mathbf{v}$. Group terms are understood in the sense of [10].

2. Two-dimensional case. Basic formulas and conservation laws

The presentation in this section largely follows [6].

2.1. Conservation law for the two-dimensional case

Consider a family $\{L_\tau\}$ of curves $L_\tau$ which continuously fill some domain $D$ in a plane with rectangular coordinates $x$ and $y$ and unit vectors $\mathbf{i}$ and $\mathbf{j}$ along the $x$ and $y$ axes. Previously [5, 6], it has been found that for any smooth vector field of unit vectors $\tau = (\tau \times \tau - \tau \div \tau)$ and for any family $\{L_\tau\}$ of smooth plane curves $L_\tau$ with Frenet unit vectors $\tau(x, y)$ and $\nu(x, y)$ (tangent and normal vectors) or for any two mutually orthogonal families of smooth curves $\{L_\tau\}$ and $\{L_\nu\}$, the following divergent identity (in $D$) or the conservation law holds:

$$\text{div} \mathbf{S}(\tau) = 0 \iff \text{div} \mathbf{S}^* = 0, \quad (1)$$

where $\mathbf{S}(\tau) \overset{\text{def}}{=} \text{rot} \tau \times \tau - \tau \div \tau$, $\mathbf{S}^* \overset{\text{def}}{=} \mathbf{K}_\tau + \mathbf{K}_\nu = (\tau \cdot \nabla)\tau + (\nu \cdot \nabla)\nu = k\nu + k_\nu \eta = \text{rot} \tau \times \tau + \text{rot} \nu \times \nu$, and $\mathbf{S}(\tau) = \mathbf{S}^* = -\text{rot} \{\alpha(x, y)\mathbf{k}\}$. Here $\mathbf{K}_\tau = (\tau \cdot \nabla)\tau = \text{rot} \tau \times \tau = k\nu$ and $\mathbf{K}_\nu = (\nu \cdot \nabla)\nu = \text{rot} \nu \times \nu = k_\nu \nu = -k_\nu \tau$ are, respectively, the curvature vectors of the curves $L_\tau$ with curvature $k$ and the curves $L_\nu$ orthogonal to them with unit tangent vector $\nu$, unit normal vector $\eta = -\tau$, and curvature $k_\nu$, $\alpha(x, y)$ is the angle between the vector $\tau$ and the $Ox$ axis, $k = (\text{grad} \alpha \cdot \tau) = k(x, y)$; $\mathbf{k}$ is the unit vector along the $z$ axis, which plays the role of the binormal for the curves $L_\tau$.

2.2. Two geometric meanings of the conservation law (1)

The first geometric meaning. It was established in [6]. Identity (1) implies that for the family of smooth curves $\{L_\tau\}$, there exists a vector field $\mathbf{S}^* = \mathbf{K}_\tau + \mathbf{K}_\nu$ which is the sum of the curvature vectors of two plane curves $L_\tau$ and $L_\nu$ from the mutually orthogonal families of curves $\{L_\tau\}$ and $\{L_\nu\}$ and is a solenoidal vector field in $D$.

This property can be interpreted as the existence of the conservation law of differential form (1) for the vector field $\mathbf{S}^*$ in the differential geometry of plane curves. (Since, for any
smooth vector field $\mathbf{a}$, the identity $\text{div} \mathbf{a} = 0$ is a differential conservation law with the integral form for the flux (in the two-dimensional case) $\int_S (\mathbf{a} \cdot \mathbf{n}) \, dS = 0$, where $S$ is an arbitrary piecewise smooth closed curve in the plane $x, y$, $dS$ is an element of the length $S$, and $\mathbf{n}$ is the unit normal to $S$.

Identity (1) has a purely geometric meaning. However, it can be translated “into physical language” in the case where the family $\{L_a\}$ is a family of vector lines of some smooth vector field $\mathbf{v} = \mathbf{v}(x, y) = |\mathbf{v}| \mathbf{\tau}$ with modulus $|\mathbf{v}|$ and direction $\mathbf{\tau}$ ($|\mathbf{\tau}| = 1$) with the property $|\mathbf{v}| \neq 0$. It has been found [6] that identity (1) in terms of this field $\mathbf{v}$ is equivalent to the identity

$$\text{div} \mathbf{T}(\mathbf{v}) = 0, \quad \mathbf{T}(\mathbf{v}) \stackrel{\text{def}}{=} \frac{\text{rot} \mathbf{v} \times \mathbf{v} - \mathbf{v} \text{div} \mathbf{v}}{|\mathbf{v}|^2} + \text{grad} \ln |\mathbf{v}|,$$

(2)

which is the differential conservation law for the field $\mathbf{v}(x, y)$.

Formula (2) was found in [2] using group analysis for a potential plane field $\mathbf{v} = \text{grad} \, u(x, y)$, and in [4] for an arbitrary smooth plane field $\mathbf{v}(x, y)$. The equivalence of the identities (1) and (2) follows from the following assertion in [4–7]. For any plane vector field $\mathbf{v}(x, y)$ with components $v_j(x, y) \in C^1(D)$ ($j = 1, 2$), modulus $|\mathbf{v}| \neq 0$ in $D$, and direction $\mathbf{\tau} = \mathbf{\tau}(\alpha) = \mathbf{v}/|\mathbf{v}|$, the following identity holds:

$$\mathbf{T}(\mathbf{v}) \equiv \mathbf{S}(\mathbf{\tau}) \iff \mathbf{Q}(\mathbf{v}) = \mathbf{P}(|\mathbf{v}|) - \mathbf{S}(\mathbf{\tau}),$$

(3)

where $\mathbf{T}(\mathbf{v}) = \mathbf{P}(|\mathbf{v}|) - \mathbf{Q}(\mathbf{v})$, $\mathbf{Q}(\mathbf{v}) \stackrel{\text{def}}{=} (\mathbf{v} \text{div} \mathbf{v} - \text{rot} \mathbf{v} \times \mathbf{v})/|\mathbf{v}|^2$ and $\mathbf{P}(|\mathbf{v}|) \stackrel{\text{def}}{=} \text{grad} \ln |\mathbf{v}|$. If $v_j(x, y) \in C^2(D)$ ($j = 1, 2$), the following identities hold: $\text{div} \mathbf{S}(\mathbf{\tau}) = 0$, $\text{rot} \mathbf{S}(\mathbf{\tau}) = (\Delta \mathbf{\tau}) \mathbf{k}$, and $\text{div} \mathbf{Q} = \Delta \ln |\mathbf{v}|$, $\text{rot} \mathbf{Q} = - (\Delta \mathbf{\tau}) \mathbf{k} \Rightarrow \Delta \ln \{ |\mathbf{v}| e^{\pm i\alpha} \} = \text{div} \mathbf{Q} \equiv i(\text{rot} \mathbf{Q} \cdot \mathbf{k})$ (i is the imaginary unit) (see [5, 6]). Identity (3) is also valid for the field $\mathbf{v}(x, y, z)$ [4].

The second geometric meaning of the conservation law $\text{div} \mathbf{S}(\mathbf{\tau}) = 0 \iff \text{div} \mathbf{S}^* = 0 \iff \text{div} \mathbf{T} = 0$ and its group content. In [2] (see also [1, 3]), the following formula was obtained:

$$J^{11} = K(x, y) = \frac{1}{2} \frac{\Delta \ln J^7}{n^2} - A_2 \left( J^4 \right)^2 J^7 \iff \Delta \ln \sqrt{u_x^2 + u_y^2},$$

(4)

where $J^{11} = K(x, y) = - \Delta \ln n^2/(2n^2)$ is the Gaussian curvature of a surface in three-dimensional Euclidean space with a linear element (Riemannian metric) $dl^2 = n^2(x, y) (dx^2 + dy^2)$, $J^7 = \Delta u = (u_x^2 + u_y^2)/n^2$ and $J^4 = \Delta u/n^2$ are the first and the second differential Beltrami parameters of the function $u(x, y)$ for this surface, $A_2 = n^{-2}(u_x \partial/\partial x + u_y \partial/\partial y)$ is one of the three operators of invariant differentiation of the Lie group $G$; $J^4$, $J^7$, and $J^{11}$ are three of its 15 differential invariants.

The group $G$ is described and investigated in [1] and is the infinite Lie group of transformations of the space of five variables $x$, $y$, $t$, $u^3$, and $u^2$ for which the infinitesimal operator $X$ of any of its one-parameter subgroup has the form $X = \Phi(x, y) \partial/\partial x + \Psi(x, y) \partial/\partial y - 2 \Phi_x(x, y) u_x^2 \partial/\partial u^2$, where $\Phi$ and $\Psi$ are arbitrary conjugate harmonic functions. The basis of the differential invariants of the group $G$ is formed by the invariants $J^1 = t$ and $J^2 = u^1 = u$.

The group and geometric meaning of identity (4) is that it expresses the differential invariant $J^{11}$, defined by only one function $u^2 = n^2(x, y)$ and coinciding with Gaussian curvature $K$, in terms of the other invariants $J^4$ and $J^7$ of the group $G$, defined by the two functions $u^4 = u(x, y, t)$ and $u^2 = n^2(x, y)$ — the Beltrami parameters $\Delta u$ and $\Delta u^2$. The group $G$ is an extension of the group of conformal transformations of the plane $(x, y)$ to the space of the variables $x$, $y$, $t$, $u^1 = u$, and $u^2 = n^2$, and, at the same time, it is the equivalence group of the eikonal equation $(u_x^2 + u_y^2)/n^2(x, y) = 1$, the wave equation $(u_{xx} + u_{yy})/n^2(x, y) = u_{tt}$, and other DEs of mathematical physics [1].
It has been found [2] that formula (4) can be represented as the equivalent vector divergent identity \( \text{div} \mathbf{T} = 0 \), where \( \mathbf{T} = T(\text{grad } u) = \text{grad } \ln |\text{grad } u| - \frac{\Delta u}{|\text{grad } u|^2} \text{grad } u \), for any plane potential vector field \( \mathbf{v} = \text{grad } u(x, y) \). From this, in view of identity (3), we obtain the conservation law \( \text{div} \mathbf{S}(\tau) = 0 \Leftrightarrow \text{div } \mathbf{S}^* = 0 \) for the field of directions \( \tau(x, y) \) of the vector field \( \mathbf{v} = \text{grad } u(x, y) \), i.e., for the family of vector lines \( L_{\tau} \) of this field. Then in [4], identity (3) was established for any three-dimensional vector field \( \mathbf{v}(x, y, z) = |\mathbf{v}|\tau \) with direction \( \tau \) \( (|\tau| = 1) \), and identity (2) was established for any smooth plane field \( \mathbf{v}(x, y) = |\mathbf{v}|\tau(x, y) \).

The properties of the group \( G \) were used in [9].

2.3. Conservation laws for the hydrodynamic Euler equations

The hydrodynamic Euler equations have the form [18] \( \partial \mathbf{v}/\partial t + \text{grad } \mathbf{v}^2/2 - \mathbf{v} \times \text{rot } \mathbf{v} = \mathbf{F} - \text{grad } p/\rho \), where (in the two-dimensional case) \( \mathbf{v} = \mathbf{v}(x, y, t) \) is the velocity, \( \mathbf{v} = |\mathbf{v}| \), \( \rho = \rho(x, y, t) \) is the density of the fluid, \( p = p(x, y, t) \) is the pressure, \( \mathbf{F} = \mathbf{F}(x, y, t) \) is the body force vector per unit mass. In [4], the following theorem was proved.

**Theorem 1.** For any plane motion of an ideal fluid \( (\mathbf{v} = \mathbf{v}(x, y, t) = v_i \mathbf{i} + v_j \mathbf{j} = v \tau, \|\tau\| \neq 0 \text{ in } D, v_j(x, y) \in C^2(D), \ j = 1, 2) \) the Euler equations can be represented as

\[
\mathbf{G} = -\mathbf{S} - \mathbf{P} - \mathbf{Q} = -\text{rot } \{\alpha(x, y, t)k\} \Rightarrow \text{div } \mathbf{G} = 0, \text{rot } \mathbf{G} = -\Delta \alpha k k \Rightarrow \Delta \ln |\mathbf{v}| = \text{div } \mathbf{Q},
\]

where \( \mathbf{G} = \frac{1}{v^2} \left\{ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \times \mathbf{v} + \frac{1}{\rho}\text{grad } p - \mathbf{F} \right\} \). The fields \( \mathbf{S}, \mathbf{Q}, \text{ and } \mathbf{P} \) are defined in (1) and (3), and \( \alpha = \alpha(x, y, t) \) is the angle between the vector \( \mathbf{v} \) or the streamline and the Ox axis. In particular, for steady plane motion of an incompressible fluid in the absence of body forces (i.e., for \( v_i = 0, \text{div } \mathbf{v} = 0, \text{ and } \mathbf{F} = 0 \)), the following identities hold:

\[
\text{grad } |\mathbf{v}| = \text{rot } (\alpha k) \Rightarrow \text{div } \text{grad } p/\rho = 0 \text{ and rot } \text{grad } p/\rho = -\Delta \alpha k.
\]

The streamlines are vector lines \( L_{\tau} \) of the velocity vector field \( \mathbf{v}(x, y, t) \) (for \( t = \text{const} \)); therefore, Theorem 1 and formulas (1) lead to

**Corollary 1.** The conservation laws \( \text{div } \mathbf{G} = 0 \) and \( \text{div } \{\text{grad } p/\rho^2 \} = 0 \) in Theorem 1 are equivalent to the following geometric property of the streamlines: the sum \( \mathbf{S}^* \) of the curvature vector \( \mathbf{K}_{\tau} = \text{rot } \tau \times \tau \) of the streamlines \( L_{\tau} \) and the curvature vector \( \mathbf{K}_{\nu} = \text{rot } \nu \times \nu \) of the curves \( L_{\nu} \) orthogonal to them is a solenoidal field for any fixed \( t \): \( \text{div } \mathbf{S}^* = 0, \mathbf{S}^* = -\text{rot } (\alpha k) \).

3. Three-dimensional case

3.1. Conservation laws of the form \( \text{div } \{\mathbf{S}(\tau) - \Phi\} = 0 \)

Suppose that \( D \) is a domain in the Euclidean space \( E^3 \) with Cartesian coordinates \( x, y, \) and \( z \); \( i, j \) and \( k \) are the unit vectors along the coordinate axes \( x, y, \) and \( z \), respectively; \( \tau = \tau(x, y, z) \) is the vector field of unit vectors defined in \( D, \|\tau\|^2 = 1 \). The geometry of vector fields (see [20]) considers the case of a holonomic field \( \tau \) for which there exists a family of surfaces \( S_\tau \) with normal \( \tau \) which are orthogonal to the field \( \tau \) and the general case where the field \( \tau \) can be non-holonomic. A necessary and sufficient condition for the the field \( \tau \) to be holonomic [20, Ch. 1, § 1] is the identity \( (\tau \cdot \text{rot } \tau) = 0 \) in \( D \). The geometry of vector fields introduces analogues of the classical characteristics of surfaces \( S_\tau \) for a non-holonomic field \( \tau \) [20]. For example, an analogue of the Gaussian curvature of a surface \( S_\tau \) is the total curvature of the second kind \( K \) [20]. In the case of a holonomic field \( \tau \), these analogues coincide with the corresponding classical characteristics of surfaces \( S_\tau \) with normal \( \tau \); for example, the above-mentioned quantity \( K \) coincides with the Gaussian curvature [20].

In the three-dimensional case, generally speaking, \( \mathbf{S}(\tau) \neq \mathbf{S}^* \), \( \text{div } \mathbf{S}(\tau) \neq 0 \), and \( \text{div } \mathbf{S}^* \neq 0 \). The relationship between \( \mathbf{S}(\tau) \) and \( \mathbf{S}^* \) is given by Theorem 2 below. The geometric reason that \( \text{div } \mathbf{S}(\tau) \neq 0 \) in \( D \) is generally due to the fact that, as found in [20], \( \text{div } \mathbf{S}(\tau) = -2K \); generally, \( K \neq 0 \) (in the holonomic case, \( K \) is the Gaussian curvature of surfaces orthogonal to the vector lines of the field \( \tau \) or to the curves \( L_{\tau} \)). In this case, as in the two-dimensional case,
we assume that $S(τ) = \text{rot } τ \times τ - τ \text{ div } τ = K_τ - τ \text{ div } τ$ and $S^* = K_ν + K_β$, where $K_τ = (τ \cdot \nabla)τ = \text{rot } τ \times τ = kν$, $K_ν = (ν \cdot \nabla)ν = ν \times ν$, $K_β = (β \cdot \nabla)β = β \times β$ are the curvature vectors of the vector lines $L_τ$, $L_ν$, and $L_β$ of the Frenet unit vector fields $τ$, $ν$, and $β$, respectively ($τ$ is the unit tangent vector, $ν$ is the principal normal vector, $β$ is the binormal vector) of the curves $L_τ$.

**Lemma 1.** Let $τ = τ(x,y,z)$, $v = \cos α_1 i + \cos α_2 j + \cos α_3 k$ be the vector field of unit vectors ($|τ| = 1$) with the domain of definition $D$, where $α_1$, $α_2$, and $α_3$ are the direction angles between the vector $τ$ and the $x$, $y$, and $z$ axes, respectively, and $τ(x,y,z) \in C^2(D)$. Then the field $S(τ)$ can be represented in any of the forms $S(τ) = \sum_{j=1}^{3} \text{grad } \cos α_j \times (i_j × τ) = \sum_{j=1}^{3} \cos α_j \text{rot } (τ × i_j)$, $S(τ) = \Phi_1(τ) - \text{rot } Ψ(τ) = \Phi_2(τ)$, where $i_j = i$, $i_2 = j$, $i_3 = k$, $Φ_1(τ) = 2\{\cos α_3 \text{rot } (cos α_2 i) + \cos α_1 \text{rot } (cos α_3 j) + \cos α_2 \text{rot } (cos α_1 k)\}$, $Φ_2(τ) = -2\{\cos α_2 \text{rot } (cos α_3 i) + \cos α_3 \text{rot } (cos α_1 j) + \cos α_1 \text{rot } (cos α_2 k)\}$, and $Ψ(τ)$ is defined.

**Proof.** The theorem is proved by substituting the formula $τ = \cos α_1 i + \cos α_2 j + \cos α_3 k$ into the expression $S(τ) = \text{rot } τ \times τ - τ \text{ div } τ$ using the well-known formulas of vector analysis [18] $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$, $\text{rot } (φa) = φ \text{rot } a + \text{grad } φ \times a$.

Lemma 1 implies

**Theorem 2.** Under the conditions of Lemma 1 for $τ \in C^2(D)$, the following equivalent divergent identities (conservation law) for the field of unit vectors $τ = τ(x,y,z)$ hold in $D$: $\text{div } \{S(τ) - Φ_1(τ)\} = 0$, $i = 1,2,3$. If $τ = τ(x,y,z)$ is the field of unit tangent vectors to the curves $L_τ$ of the family $\{L_τ\}$ or the field of unit normals to the surfaces $S_τ$ of the family $\{S_τ\}$, it can be considered as the conservation law for the family of curves $\{L_τ\}$ or the family of surfaces $\{S_τ\}$. If $τ$ is the direction of the vector field $v = |v|τ = v_1 i + v_2 j + v_3 k$, then, under the conditions of Lemma 1, the following equivalent divergent identities for the vector field $v \in C^2(D)$ hold in $D$: $\text{div } \{T(v) - Φ_1(v)\} = 0, i = 1,2,3$, where $Φ_1(v)$ is obtained from $Φ_1(τ)$ by replacing $τ$ by $v/|v|$, and $Φ_2(v)$ is defined by $Φ_2(v) = -\frac{2}{|v|}\left\{v_2 \text{rot } \left(\frac{v_3}{|v|} i\right) + v_3 \text{rot } \left(\frac{v_1}{|v|} j\right) + v_1 \text{rot } \left(\frac{v_2}{|v|} k\right)\right\}$.

Here we have used the identity $S(τ) = T(v)$ for the the three-dimensional case obtained in [4]. In view of the above formula $K = -\frac{1}{2} \text{div } S(τ)$ [20], Theorem 2 implies

**Corollary 2** (the geometric meaning of the conservation law of Theorem 2). For a holonomic field $τ$, the Gaussian curvature $K$ of the surfaces $S_τ \in \{S_τ\}$ orthogonal to the field $τ$ admits the divergent representation $K = -\frac{1}{2} \text{div } Φ_1(τ)$ in $D$. If the surfaces $S_τ$ are orthogonal to the vector lines $L_τ$ of the vector field $v = |v|τ$, $|v| \neq 0$ in $D$ and $v \in C^2(D)$, then $K = -\frac{1}{2} \text{div } T(v) = -\frac{1}{2} \text{div } Φ_1(v)$.

3.2. The conservation law formula $\{T - Φ_1\} = 0$ for the hydrodynamic Euler equations

**Corollary 3.** Let $v = v(z,y,z) = vτ$ be the velocity in the hydrodynamic Euler equations which can be written as $G = -T(τ) = -S(τ)$, where $G = \{v_1 + v_2 \text{ div } v + \text{grad } p/ρ - F\}/v^2$, in the domain $D$; $v = |v| \neq 0$ in $D$, $v \in C^2(D)$, the pressure $p \in C^2(D)$, the density $ρ \in C^4(D)$, and the body force $F \in C^4(D)$. Then, in $D$, the conservation law $\text{div } \{G + Φ_1(v)\} = 0, i = 1,2$, holds, where the field $Φ_1(v)$ is defined in Theorem 2.

The geometric meaning of the conservation law $\text{div } \{T - Φ_1(τ)\} = 0, i = 1,2$ for the Euler equations is that the Gaussian curvature $K$ of the surfaces with the normal $τ = v/ν$ which are orthogonal to the streamlines admits the divergent representation $K = -\frac{1}{2} \text{div } T = -\frac{1}{2} \text{div } Φ_1(τ) = \frac{1}{2} \text{div } G$.

In the two-dimensional case, we have $v = v(x,y)$, $τ = τ(x,y)$, and $\text{div } Φ_1 = 0$, whence we
obtain the conservation law \( \text{div} \mathbf{G} = 0 \iff \text{div} \mathbf{T}(\tau) = 0 \iff \text{div} \mathbf{S}(\tau) = 0 \) found in [5] for the two-dimensional case.

4. Three-dimensional analogues of the conservation law \( \text{div} \mathbf{S}(\tau) = 0 \iff \text{div} \mathbf{S}^* = 0 \iff \text{div} \mathbf{T}(\nu) = 0 \) for the three-dimensional case and higher-order conservation laws of the form \( \text{div} \mathbf{F} = 0 \). The field \( \mathbf{R}^* \)

Suppose that \( \{L_\tau\} \) is a family of curves \( L_\tau \) which continuously fill the domain \( D \)

(A) one and only one curve \( L_\tau \in \{L_\tau\} \) passes through each point \( (x,y,z) \in D \);

(B) at each point \( (x,y,z) \) of any curve \( L_\tau \in \{L_\tau\} \), there exists the (right) Frenet basis \( (\tau,\nu,\beta) \) (\( \tau \) is the unit tangent vector, \( \nu \) is the principal normal vector, \( \beta \) is the binormal vector), so that in \( D \) three mutually orthogonal vector fields \( \tau, \nu, \) and \( \beta \) are defined so that \( \tau = \nu \times \beta, \ \nu = \beta \times \tau, \) and \( \beta = \tau \times \nu \):

(C) \( \tau(x,y,z) \in C^2(D) \). In [8] the following formulas were obtained.

**Theorem 3.** Under conditions (A)-(C), in \( D \) the following divergent identity (conservation law for the family of curves \( \{L_\tau\} \) with Frenet basis \( (\tau,\nu,\beta) \), the first curvature \( k \), and the second curvature \( \kappa \) and for the family of surfaces \( \{S_\tau\} \) if the curves \( L_\tau \) are taken to be the vector lines of the fields of the normals \( \tau \) to these surfaces \( S_\tau \) holds:

\[
\text{div} \{ \tau \text{div} \mathbf{S}^* - \kappa \text{rot} \tau - k \text{rot} \beta \} = 0
\]

\[
\text{div} \{ \{1/2\} \tau \text{div} \mathbf{S}(\tau) - k\nu(\nu \cdot \text{rot} \beta) - k\beta(\beta \cdot \text{rot} \beta + \kappa) \} = 0 \iff \text{div} \{ \tau(\kappa^2 - \kappa(\tau \cdot \text{rot} \tau) - (\tau \cdot [\text{rot} \nu \text{rot} \beta])) - k\nu(\nu \cdot \text{rot} \beta) - k\beta(\kappa + (\beta \cdot \text{rot} \beta)) \} = 0.
\]

Here the expression in brackets is equal to \( \text{rot} \mathbf{R}^* \) everywhere; the quantities \( \text{div} \mathbf{S}(\tau) \) and \( \text{div} \mathbf{S}^* \) can be expressed by the following formulas (three-dimensional analogues of the identities for the two-dimensional case): \( \text{div} \mathbf{S}(\tau) = 0 \text{ and } \text{div} \mathbf{S}^* = 0 \):

\[
\text{div} \mathbf{S}(\tau) = 2 \{ \kappa(\nu \cdot \text{rot} \beta) - (\tau \cdot [\text{rot} \nu \text{rot} \beta]) \}.
\]

\[
\text{div} \mathbf{S}(\tau) = 2(\tau \cdot \text{rot} \mathbf{R}^* + k(\tau \cdot \text{rot} \beta) + \kappa(\tau \cdot \text{rot} \beta)).
\]

The vector field \( \mathbf{R}^* \) is a measure of the difference of the fields \( \mathbf{S}^* \) and \( \mathbf{S}(\tau) \). In the two-dimensional case where \( \tau = \tau(x,y), \kappa = 0, \) and \( \beta = k = \text{const} \), we obtain \( \text{rot} \beta = 0, \mathbf{R}^* = 0 \Rightarrow \mathbf{S}(\tau) = \mathbf{S}^* = \mathbf{K}_\tau + \mathbf{K}_\nu, \text{div} \mathbf{S}(\tau) = 0 \iff \text{div} \mathbf{S}^* = 0.
\]

**Corollary 4.** In view of the formula \( \mathbf{K} = -\text{div} \mathbf{S}(\tau)/2 \) from [20, § 8], the Gaussian curvature \( \mathbf{K} \) of the surfaces \( S_\tau \) orthogonal to the family \( \{L_\tau\} \) of curves \( L_\tau \) (i.e., to the field \( \tau \)) is given by any of the formulas (6) for \( \text{div} \mathbf{S}(\tau) \) in which \( \text{div} \mathbf{S}(\tau) \) is replaced by \( -2\mathbf{K} \).

**Corollary 5.** As found in [8], the quantity \( (-\text{rot} \mathbf{R}^*) \) coincides with the so-called curvature vector of the vector field of unit vectors \( \tau(x,y,z) \), denoted by \( \mathbf{P} \) \([20, \S 5] \): \( \mathbf{P} = -\text{rot} \mathbf{R}^* \). In [20, § 10], the following invariant representation was obtained for the vector field \( \mathbf{P} \): \( \mathbf{P} = K(\tau - 2\text{div} \tau \mathbf{K}_\tau + (\mathbf{K}_\tau \cdot \nabla) \tau \), where \( \mathbf{K}_\tau = \text{rot} \tau \times \tau \) is the curvature vector of the vector lines of the field \( \tau \). Consequently, the conservation law of Theorem 3 has the following geometric meaning: the vector field \( \mathbf{P} \) of the curvature vectors of the field of unit vectors \( \tau \) (\( \tau \) is the unit tangent vector to the curves \( L_\tau \) or the unit normal to the surfaces \( S_\tau \)) is a solenoidal field and is equal to any expression in braces with the minus sign in the identities (5) of Theorem 3.

**Remark 1.** The Frenet unit vectors \( \nu \) and \( \beta \), the first curvature \( k \) of the curves \( L_\tau \), and the second curvature \( \kappa \) can be expressed in terms of \( \tau \) \([18, 19, 20, \S 15] \): \( \nu = (\text{rot} \tau \times \tau)/k, \beta = \tau \times \nu, k = |\text{rot} \tau \times \tau|, \) and \( \kappa = \{(\tau \cdot \text{rot} \tau) - (\nu \cdot \text{rot} \nu) - (\beta \cdot \text{rot} \beta)/2. \) Since, in view
of the formulas of Section 3.1 and Theorem 3, the quantities \( S(\tau) \), \( S^* \), \( R^* \), \( \text{div} S(\tau) \), \( \text{div} S^* \), and \( \text{rot} R^* \), \( K \) are expressed in terms of the unit vectors \( \tau \), \( \nu \), and \( \beta \), the first curvature \( k \), and the second curvature \( \kappa \) of the curves \( L_\tau \), all these quantities can ultimately be expressed only in terms of the field \( \tau \). Therefore, all formulas of Theorem 3 can be expressed only in terms of the field \( \tau \) (the field of unit tangent vectors to the curves \( L_\tau \) or the field of unit normals to the surfaces \( S_\tau \)).

Remark 2. To obtain a higher-order (than in Corollary 2) conservation law for the solutions of the Euler equations, everywhere in the formulas of Theorem 3 and Remark 1 we need to replace \( \tau \) by \( \nu/|\nu| \), replace the words “curves \( L_\tau \)” and “surfaces \( S_\tau \)” by the words “streamlines” and “surfaces orthogonal to them,” and set \( S(\tau) = T(\nu) = -G \), where the field \( G \) is defined in Corrolary 3. Formulas for the Gaussian curvature \( K \) of the surfaces with normal \( \tau \) which are orthogonal to the streamlines follow from the formulas of Corollary 3 and Remark 1 for \( \tau = \nu/|\nu| \).

The 10-parameter group \( G^{10} \) — a three-dimensional analogue of the equivalence group \( G \) (see Section 2.2, [1]) —has been studied; a three-dimensional analogue of formula (4) has been obtained. Presentation of these results requires a separate article.

5. Conclusions
The present and previous studies of the author [1–9] have shown the feasibility and effectiveness of integrating different mathematical branches such as the DEs of mathematical physics, vector analysis, group analysis of DEs based on the Lie equivalence groups, the geometry of a vector field, and differential geometry. Differential conservation laws for families of curves and surfaces were derived, and then these general vector and geometric formulas were used to derive differential conservation laws for the three-dimensional hydrodynamic Euler equations.

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