Double-toric code

Komal Kumari, Garima Rajpoot, and Sudhir Ranjan Jain

1 Theoretical Nuclear Physics and Quantum Computing Section
Nuclear Physics Division, Bhabha Atomic Research Centre, Mumbai 400085, India
2 Homi Bhabha National Institute, Training School Complex, Anushakti Nagar, Mumbai 400094, India

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Abstract

We construct a double-toric surface code by exploiting the planar tessellation using a rhombus-shaped tile. With $n$ data qubits, we are able to encode at least $n/3$ logical qubits or quantum memories. By a suitable arrangement of the tiles, the code achieves larger distances, leading to significant error-correcting capability. We demonstrate the robustness of the logical qubits thus obtained in the presence of external noise. We believe that the optimality of the code presented here will pave the way for design of efficient scalable architectures.

1 Introduction

Computations and errors go hand in hand. It is well-known that quantum error correction has paved the way for fault-tolerant computation [1, 2]. The challenges posed by no-cloning theorem, existence of continuous errors, and projective measurements (collapse of state) have been overcome by the usage of entanglement, Hadamard basis, and syndrome decoding based on parity check matrices (PCM). This was realized by introducing the quantum stabilizer codes (QSC) [3], a profound generalization of classical coding [4, 5]. Essentially, $[[n, k]]$ QSC maps the state carrying information (logical qubit) in $2^k$-dimensional Hilbert space onto a codeword in $2^n$-dimensional Hilbert space. The redundancy in superpositions of the entangled states and measurement of the ancillae allows one to extract information from the system without knowing the state of individual qubits. Perhaps the most useful approach to achieve fault-tolerance is via developing a surface code [7, 8]. It is a topological code where the principle of building the code is to “patch” together repeated elements. It consists of a lattice of squares with alternating physical qubits acting as data qubits and ancillary qubits, the latter measuring $X$ and $Z$. Planarity helps in realizing an efficient engineering design of a chip accommodating qubits and ancillae. This modular approach leads to scalability while ensuring stabilizer commutativity. Surface code requires only nearest-neighbour interaction which facilitates implementation on hardware. While this code (a) is provably fault-tolerant under general noise models [9], (b) supports efficient decoding algorithms [10] and a fault-tolerant implementation of gates [11, 12], it requires a very large overhead in the number of physical qubits [8]. It was shown experimentally [13] that in surface code of 9 data qubits, the decoherence time of encoded logical qubits is much higher than that of physical qubits. For lesser overhead, using Hadamard rotation on alternate data qubits and syndrome measurement given by the product $XZZX$ on each qubit, an efficient fault-tolerant variant of the surface code [14] was presented. Further improvement in the overhead was achieved by the topological stabilizer code, $XYZ^2$ on a honeycomb structure [15]. Both of these codes have better performance for biased noise.
Figure 1: We show how the \((\pi/3, 2\pi/3)\)-rhombus shape, upon three successive reflections about its sides returns to a flipped configuration. It takes three more to restore the configuration, albeit at the expense of losing single-valuedness. Thus, the vertex marked with a black dot is like a branch point. Tessellation with this shape leads to visiting certain straight segments twice - these form the branch cuts (see Fig. 3).

Figure 2: Six copies of the rhombus are required to construct an invariant surface. By identifying the edges labelled by the same arrows, the shape shown folds into a surface topologically equivalent to a sphere with two handles (genus, \(g = 2\)).
The most striking consequence of this design is to look for “unit shapes” that can tessellate the plane upon successive reflections. Based on seminal work by Weyl [16] and Cartan [17], Coxeter [18] pointed out a one-to-one correspondence between Lie groups and reflection groups whose fundamental regions are simplexes in Euclidean space. These fundamental regions generate tori for “unit shapes” like square, equilateral triangle, right isosceles triangle, or a hemi-equilateral triangle [6]. To illustrate, if we consider a square tile, upon successive reflections about its sides, it can be easily seen that four copies, making a larger square with an edge-length twice the length of the original square, forms a unit of tessellation - the fundamental domain, identifying the pairs of parallel edges gives a torus, which is characterized by a topological invariant, the genus being equal to one. Beautiful multiply-connected shapes form by identifying edges of rational triangles and rhombi in a fundamental domain in a certain way [19, 20]. Thus, we may say that surface codes hitherto considered correspond to tori, and make what is well-known as “toric code” [21].

Here we introduce a surface code corresponding to a surface with genus two - topologically equivalent to a sphere with two handles. The basic tile shape is a \((\pi/3, 2\pi/3)\)-rhombus, the intricacy of this simple shape is illustrated in Fig. 1. The fundamental domain is constructed by stitching six copies which, via an appropriate identification of edges (Fig. 2), creates what we term as a “double torus” [19, 22, 6], hence the name of the code. Filling of the plane is shown in Fig. 3 where the bold segments correspond to regions visited twice upon successive reflections. However, as shown below, a stabilizer code can be built nevertheless.

![Figure 3: A plane can be filled by successively reflecting a \(\pi/3\)-rhombus about its sides. The arrangement of the rhombi is shown here with bold segments representing the regions visited twice. Upon identification of corresponding sides, one obtains the surface of a sphere with two handles (see Fig. 2). To compare, with square as a basic unit, one obtains the surface of a torus. See further details in Ref. [6]. Also see Fig. 1](image)

## 2 A single unit of the code

Let us begin by a simple illustration of the structure of the code where a single unit consists of 10 physical qubits, comprising of \(n = 6\) data \((D)\) qubits \((m = 4\) ancilla, \((A))\), represented by circles (squares), see Fig. 4. The bold (dotted) lines represent \(X\) (\(Z\)) - measurement. For instance, ancilla qubits \(A1\) and \(A4\) \((A2\) and \(A3\)) perform \(X\) \((Z)\) measurement on data qubits. States of data qubits
are acted upon by the operators belonging to the Pauli group $G_6$ [23]. We look for operators which would preserve logical qubit, $|\psi\rangle$ i.e., satisfy $P_i|\psi\rangle = (+1)|\psi\rangle$. For this example, such a set of stabilizers is $P = \{X_1X_2X_3X_4, X_3X_4X_5X_6, Z_1Z_3Z_5, Z_2Z_4Z_6\}$. A logical $|0\rangle_L$ qubit of this code is [3]:

$$|0\rangle_L = \frac{1}{\mathcal{N}} \prod_{P_i \in \{P\}} (I^{\otimes n} + P_i) |0^{\otimes n}\rangle$$

$$= \frac{1}{\mathcal{N}} (I^{\otimes 6} + X_1X_2X_3X_4)(I^{\otimes 6} + X_3X_4X_5X_6)$$

$$= \frac{1}{\mathcal{N}} (I^{\otimes 6} + Z_1Z_3Z_5)(I^{\otimes 6} + Z_2Z_4Z_6)|0^{\otimes 6}\rangle$$

$$= \frac{1}{\mathcal{N}} (|000000\rangle + |001111\rangle + |111100\rangle + |110011\rangle),$$

where $\mathcal{N}$ is an overall normalization constant. As can be easily verified, all stabilizers commute with each other, i.e., $[P_i, P_j] = 0, \forall \ i, j$. To transform $|0\rangle_L$ to $|1\rangle_L$, we would have to construct a logical-X operator, and similarly, a logical-Z operator. Such pairs of logical operators $\bar{X}$’s and $\bar{Z}$’s must i) commute with all the stabilizers in $P$; and ii) anti-commute pairwise, i.e. $\{\bar{X}_i, \bar{Z}_j\} = 0 \forall i, [\bar{X}_i, \bar{Z}_j] = 0 \forall i \neq j$. This, then, is a $[6, 2, 2]$ stabilizer code.

The plane-filling using the rhombus as a unit is known to “re-visit” certain segments forming periodically arranged branch cuts [19, 6] (this is not “tessellation” where there is unique filling of the plane). However, this periodic arrangement helps us to identify the edges. Analogous to the surface code [8], we need to specify the boundaries, the analogues of $X$- and $Z$-edges, which are unions of control-$X(Z)$ edges here, respectively the bold and dashed lines (Fig. 4). We define a path by connecting a data vertex of a rhombus to another data vertex of a corresponding copy with respect to the Fundamental Domain of the rhombus. Each path consists of product of operators which commute with the stabilizers, thus entailing the $Z(X)$ logical operators.

For instance, the paths containing $X$-ancillae start from the edge $D1 - A1$ or $A1 - D2$ and terminate at the edge $D5 - A4$ or $A4 - D6$. The paths for finding $\bar{Z}$ are $D1 - A1 - D3 - A4 - D5$, $D1 - A1 - D3 - A4 - D6$, $D1 - A1 - D4 - A4 - D5$, $D1 - A1 - D4 - A4 - D6$, ... where $A$’s are $X$-ancillae. These are the eight paths for the operator $\bar{Z}$. In terms of $Z$, these paths give us eight arrangements of which two are stabilizers ($Z_1Z_3Z_5$ and $Z_2Z_4Z_6$) and the remaining six qualify as logical $\bar{Z}$ operators ($Z_1Z_3Z_5$, $Z_1Z_4Z_6$, $Z_2Z_3Z_5$, $Z_2Z_4Z_6$, $Z_3Z_4Z_6$, and $Z_3Z_4Z_5$). They commute with the stabilizers. In a similar manner, there is another set of paths for finding $\bar{X}$. These are $D1 - A2 - D3$, $D1 - A2 - D5$, $D2 - A3 - D6$, $D2 - A3 - D5$, $D3 - A2 - D4$, ... where $A$’s are $Z$-ancillae. In terms of $X$, these paths give us six possible arrangements ($X_1X_3$, $X_1X_5$, $X_3X_5$, $X_2X_4$, $X_2X_6$, and $X_4X_6$) which commute with the stabilizers.

Figure 4: The data qubits are represented by $D$ and ancillary qubits by $A$. The solid (dashed) lines represent $X (Z)$-ancilla qubits. The data and ancillary qubits are placed alternately in spirit of [8].
We find that there are two pairs of logical operators satisfying above conditions for logical operators: \{\hat{X}_1 = X_1 X_3, \hat{Z}_1 = Z_1 Z_4 Z_5\} and \{\hat{X}_2 = X_1 X_6, \hat{Z}_2 = Z_2 Z_4 Z_5\}. We may find two different pairs for which logical states remain the same, thereby not producing any new codewords.

Let us consider that errors \(E_a\) and \(E_b\) occurred. According to the Knill-Laflamme theorem [24], the code should be able to distinguish error \(E_a\) acting on a basis codeword \(|\psi_i\rangle\) from error \(E_b\) acting on a different codeword \(|\psi_j\rangle\), where \(|\psi_i\rangle (i = 1, 2, \ldots, k)\), spans the codeword space. With \(E = E_a^\dagger E_b\), Knill-Laflamme conditions,

\[\langle \psi_i | E | \psi_j \rangle = C_{ab} \delta_{ij}, \forall |\psi\rangle,\] (2)

allow us to ascertain the code distance, where \(C_{ab}\) is a Hermitian matrix. The Knill-Laflamme condition is a necessary condition for the code to correct errors \(\{E_a\}\). It is also a sufficient condition. Two errors may act on a code if \(C_{ab}\) does not have a maximum rank. A code for which \(C_{ab}\) is (not) singular is (non)degenerate code [3]. The weight of the shortest \(E\) in the group containing all stabilizers and codewords for which (2) does not hold gives the distance of the code. For the code in Fig. 4, this weight is found to be two, and hence the distance of the code is two. Therefore, this is a \([6, 2, 2]\) code. It can provide two encryptions and single-qubit error detection but no correction.

3 Planar code structure: Stacking and encryption

To construct the planar code using the above unit, we have to make copies by successive reflections about edges of a unit. The positions of each ancilla and each data qubit are given by specifying the coordinates on the plane. Denoting a point on this planar structure by the coordinates \((p, q)\), \(X\)-ancillae are placed at \((\pm 3p, \pm q \sqrt{3})\), \(Z\)-ancillae are correspondingly placed at coordinates \((\pm 3(p + 1/2), \pm \sqrt{3}(q + 1/2))\), where \(p = 0, 1, 2, \ldots\) and \(q = 0, 1, 2, \ldots\).

For the positions of data qubits, we have to consider two pairs of coordinates: \((\pm p; p \mod 3 \neq 0, \pm q \sqrt{3})\) and \((\pm (2p - 1)/2; (2p - 1) \mod 3 \neq 0, \pm \sqrt{3}(q - 1/2))\) with \(p, q = 1, 2, \ldots\).

To begin with, let us place a copy of the unit, Fig. 4, horizontally, thereby increasing the number of data qubits to \(n = 12\) and ancilla qubits to \(m = 7\) (Fig. 5). The set of stabilizers is \(P = \{X_1 X_2 X_3 X_4, X_3 X_4 X_5 X_6, X_7 X_8 X_9 X_{10}, X_9 X_{10} X_{11} X_{12}, Z_1 Z_3 Z_5, Z_2 Z_4 Z_6 Z_7 Z_9 Z_{11}, Z_8 Z_{10} Z_{12}\}\). The directed paths for finding \(\bar{Z}\) except for the trivial stabilizer path are \(-Z_1 Z_3 Z_6, Z_1 Z_4 Z_5, Z_1 Z_4 Z_6, Z_2 Z_3 Z_5, Z_2 Z_4 Z_6, Z_2 Z_5 Z_7, Z_2 Z_6 Z_9 Z_{11}, Z_3 Z_{10} Z_{12}\) and \(Z_8 Z_{10} Z_{12}\) and \(Z_8 Z_{10} Z_{12}\). The paths for \(\bar{X}\) are \(-X_1 X_3, X_1 X_5, X_3 X_5, X_2 X_4, X_2 X_6, X_4 X_6, X_7 X_9, X_7 X_{11}, X_9 X_{11}, X_8 X_{10}, X_8 X_{12}\) and \(X_1 X_{12}\). From these possible paths, we construct a complete set of logical operators which commute with the stabilizers and anti-commute pairwise: (i) \(\{\hat{X}_1 = X_2 X_6, \hat{Z}_1 = Z_1 Z_4 Z_6\}\), (ii) \(\{\hat{X}_2 = X_4 X_6, \hat{Z}_2 = Z_1 Z_4 Z_5\}\), (iii) \(\{\hat{X}_3 = X_7 X_{11}, \hat{Z}_3 = Z_8 Z_9 Z_{11}\}\), (iv) \(\{\hat{X}_4 = X_9 X_{11}, \hat{Z}_4 = Z_8 Z_9 Z_{12}\}\), (v) \(\{\hat{X}_5 = X_2 X_7, \hat{Z}_5 = Z_2 Z_4 Z_6\}\). The corresponding logical states constitute the logical codespace, denoted \(C\).

With the number of codewords \(k = 5\), this is a \([12, 5, 2]\) code, where \(k = 5\) is actually \(n - m = 12 - 7\). In the space \(C\), the minimum weight of error \(E\), and hence the distance turns out to be two, in accordance with 2. The total number of encodings, \(k\), is equal to the difference between the number of data qubits, \(n\) and ancilla qubits, \(m\). In \([6, 2, 2]\) code, we have \(n - m = 6 - 4 = 2 = k\) and in \([12, 5, 2]\) code, we have \(n - m = 12 - 7 = 5 = k\).

Instead, if we stack a unit vertically on the single unit, Fig. 4, the number of physical qubits \(n = 10\), while the number of ancilla qubits \(m = 7\). (Fig. 6). The stabilizers are simply written, viz., \(P = \{X_1 X_2 X_3 X_4, X_3 X_4 X_5 X_6 X_7 X_8, X_7 X_8 X_9 X_{10}, Z_1 Z_3 Z_5, Z_2 Z_4 Z_6, Z_5 Z_7 Z_9, Z_6 Z_8 Z_{10}\}\). Following the arguments presented above for identifying paths, we obtain \(X\) and \(Z\); the complete set of logical operators commuting with the stabilizers and anti-commuting pairwise is thus (i) \(\{\hat{X}_1 = X_2 X_6 X_8, \hat{Z}_1 = Z_1 Z_4 Z_6\}\),...
Figure 5: Single unit in Fig. 4 is stacked horizontally so that we have two copies connected at the Z-ancilla, \( A_3 \). Making another copy along the horizontal direction increases the number of encodings to \( k = 5 \) while keeping the distance, \( d = 2 \). Clearly, this does not aid much to error correction, but it certainly increases the number of encodings substantially.

\[ \bar{Z}_1 = Z_1 Z_4 Z_9 Z_9 \}
\[ \bar{X}_2 = X_2 X_6 X_{10}, \bar{Z}_2 = Z_5 Z_7 Z_{10} \}
\[ \bar{X}_3 = X_4 X_6 X_8, \bar{Z}_3 = Z_2 Z_3 Z_6 \}

The distance of this code is three and this is a \([10, 3, 3]\) code. Thus, it allows single-qubit error correction. This is an outcome of the increased path length between one \( X \) ancilla boundary to another, thereby increasing the weight of possible \( X \) itself. Therefore, to increase the number of encodings as well as distance, we need to increase the number of units across the plane, vertically as well as horizontally. For this, we have to fill the plane and use more units as shown in Fig. 7.

There are \( n = 20 \) data qubits and \( m = 12 \) ancillae, with \( k = 8 \) encodings and a code distance, \( d = 3 \). The set of stabilizers is \( P = \{ X_1 X_2 X_3 X_4, X_3 X_4 X_5 X_6 X_7 X_8, X_7 X_8 X_9 X_{10}, X_{11} X_{12} X_{13} X_{14}, X_{13} X_{14} X_{15} X_{16} X_{17} X_{18}, X_{17} X_{18} X_{19} X_{20}, Z_1 Z_3 Z_5, Z_2 Z_4 Z_6 Z_7 Z_8 Z_{11} Z_{12} Z_{15}, Z_5 Z_7 Z_9, Z_6 Z_8 Z_{10} Z_{15} Z_{17} Z_{19}, Z_{12} Z_{14} Z_{16}, Z_{16} Z_{18} Z_{20} \} \). The set of logical operators is (i) \( \{ \bar{X}_1 = X_1 X_3 X_9, \bar{Z}_1 = Z_1 Z_3 Z_7 Z_{15} \} \),
(ii) \( \{ \bar{X}_2 = X_3 X_5 X_7, \bar{Z}_2 = Z_1 Z_3 Z_8 Z_9 \} \),
(iii) \( \{ \bar{X}_3 = X_4 X_6 X_{17}, \bar{Z}_3 = Z_1 Z_4 Z_8 Z_9 \} \),
(iv) \( \{ \bar{X}_4 = \}

Figure 6: The same unit in Fig. 4 has been copied vertically. This increases the number of encryptions by only one, giving a total of three logical operators, but the distance has now become three, providing us the possibility of single-qubit error correction.
two vertical units shown in Fig. 6, and thereby fill the whole plane, increasing the encryption. For instance, by placing four copies vertically, we could realize a distance of four, allowing at least a two-qubit error correction. Placing six copies vertically would give a distance of five, and so on.

To simultaneously increase the number of encryptions and the distance of the code, we must stack the unit (Fig. 4) horizontally as well as vertically. An equal number of vertical and horizontal stacking arranges the unit structures (Fig. 4) in equal number of rows and columns. If the number of these rows and columns is \( p \) (1, 2, 3, …), then this planar code structure has \( p^2 \) units. \( p = 1 \) corresponds to the unit structure Fig. 4 of this planar code and \( p = 2 \) corresponds to the planar code structure Fig. 7. To construct this planar code with \( p^2 \) units, the required number of data qubits, \( n = 2p(2p + 1) \) and the required number of ancilla qubits, \( m = 2p(p + 1) \). The number of logical encryptions, \( k = 2p^2 \). The code distance of this planar code structure is \( d = \left\lfloor \frac{p+2}{2} \right\rfloor + 1 \), where \( \lfloor \cdot \rfloor \) is the floor function. So the general form of the code is \( \left\{ 2p(2p + 1), 2p^2, \left\lfloor \frac{p+2}{2} \right\rfloor + 1 \right\} \). The encoding rate of this code is \( k/n = p/(2p + 1) \). For a single unit \( (p = 1) \), encoding rate is \( 1/3 \) and for a large structure, \( p \to \infty \), the encoding rate has the maximum value \( k/n = 1/2 \).

We can also stack the unit vertically and horizontally unequally. Let us consider a single unit as the structure in Fig. 6, say \( R \), where \( n = 10, m = 7 \) and \( d = 3 \). If we stack \( v \) layers vertically, we will have to add \( 8v \) data qubits and \( 6v \) ancillae, where \( v = 0, 1, \ldots \) If we add \( h \) \( R \)'s horizontally, we will have to add \( 10h \) data qubits and \( 5h \) ancillae, where \( h = 0, 1, \ldots \) This will create an L-shaped structure. So, for this kind of structure, we have, \( n = 10 + 8v + 10h \) and \( m = 7 + 6v + 5h \), so that \( k = 3 + 2v + 5h \) and \( d = v + 3 \).

Now, if we have to make a \( (v + 1) \times (h + 1) \) type of matrix of \( R \)'s, we will have to add another \( 8vh \) data qubits and \( 4vh \) ancillae. So, we will have \( n = 10 + 8v + 10h + 8vh \), \( m = 7 + 6v + 5h + 4vh \), \( k = 3 + 2v + 5h + 4vh \) and \( d = v + 3 \).

To incorporate a \( v' \times h' \) type matrix anywhere in between the L-shaped structure, where \( v \) and \( h \) include \( v' \) and \( h' \), these numbers will change as \( n = 10 + 8v + 10h + 8v'h' \) and \( m = 7 + 6v + 5h + 4v'h' \). The distance is still \( d = v + 3 \).
For a \((v + 1) \times (h + 1)\) type of matrix, the encoding rate \(k/n\) is thus \((3 + 2v + 5h + 4vh)/(10 + 8v + 10h + 8vh)\) which is \(\sim 3/10\) for \(R\) and \(\sim 1/2\) for very large values of \(v, h\).

We thus find that the encoding rate, \(k/n\), of this planar code structure is always \(\geq 0.3\) and for a large structure, it tends to 0.5.

### 4 Robustness to dephasing noise

Any logical qubit should be robust against dephasing caused by an external noise. Recently, it has been shown [25] that certain observables formed by code space population and logical operators in the code space help determine the dynamical behavior of logical qubits. We consider a time-dependent external fluctuating magnetic field in \(z\)-direction, which acts on the qubits individually (or globally), leading to local (or global) dephasing. To estimate its effect, let us consider the logical \(|1\rangle_L:\)

\[
|1\rangle_L = X_1X_3|0\rangle_L = \frac{1}{\sqrt{\mathcal{N}}}(|101000\rangle + |100111\rangle + |010100\rangle + |011011\rangle).
\]

Initially, the logical state is \(|\psi\rangle_L = \cos \frac{\theta}{2} |0\rangle_L + e^{i\phi}\sin \frac{\theta}{2} |1\rangle_L\), where \(\theta \in [0, \pi]\) and \(\phi \in [0, 2\pi]\) are real parameters. The evolution of \(|\psi\rangle_L\) gives the logical Bloch sphere coordinates, \(X_L, Y_L\) and \(Z_L\).

Assuming the global dephasing process by a single fluctuating variable \(B(t)\) along the \(z\)-direction acting on all data qubits, the Hamiltonian representing the effect of noise may be written as \(H_G(t) = \frac{1}{2}B(t) \sum_{i=1}^{6} \sigma_z\). In case of local dephasing, the Hamiltonian reads as: \(H_L(t) = \frac{1}{2} \sum_{i=1}^{6} B_i(t) \sigma_z\). Dephasing noise changes the state \(|\psi\rangle_L\) to another state \(|\psi'\rangle\), thereby making the density matrix of the logical qubit as \(\rho' = \int |\psi'\rangle \langle \psi'| P(B)dB\).

In the case of global dephasing, the off-diagonal terms in the density matrix have coefficients, \(c_{\Delta n} = \exp \left[-\frac{1}{2} \left(\frac{\Delta n}{2}\right)^2 \gamma t\right]\), viz., \(1, e^{-2\gamma t}\) and \(e^{-8\gamma t}\) from Eq. (6) in 4. These correspond to the terms of form \(|b_i^m\rangle \langle b_i^m|, |b_i^m\rangle \langle b_i^{m+4}|\) and \(|b_i^m\rangle \langle b_i^{m\pm 8}|\), respectively. It may be noted that \(\Delta m = 0\) quantifies a decoherence free space.

In case of local dephasing, from Eq. (7) in 4 we see that for \(\gamma = 0\), the Bloch vector coordinates in the new state, \(|\psi'\rangle\) are the same as that in the old state, \(|\psi\rangle_L\) and \(\mathcal{P}\) has a pre-factor 1/4. The off-diagonal elements, \(c_{\Delta n} = \exp \left[-\frac{\Delta n}{2}\gamma t\right]\). Comparing with \(\exp [-\gamma t]\), we infer \(\Delta n = 2\), an analogue of “Hamming distance” of the code; it is also in agreement with the distance found using the Knill-Laflamme conditions, Eq. (2). As \(t \to \infty\), \(p_x = p_y = 0\), which shows that there is no leakage in the \(X - Y\) space. Moreover, even in the presence of noise, \(\langle R_Z\rangle\) remains unaffected. Thus the code is significantly robust against noise.

A lot of work has been done on stabilizer codes due to their significance in the development of fault-tolerant quantum computation [14, 15, 13]. We have presented a code which is inspired by the intricate topologies of the phase space surfaces of quantum billiards [6]. We have given complete description of the code - from organization of data and ancillary qubits to the construction of logical operators and entangled states. We have also shown that the code is scalable with a desirable high encoding rate. The design and optimality of the code illustrates, à la Kitaev, beautiful connections between quantum computation, information, geometry and topology.
Appendix: Effect of global and local dephasing noise

To study the effect of global and local dephasing noise on the logical qubit of Fig. 4, we write the randomly fluctuating variable $B(t)$, obeying the Gaussian distribution $P(B)$ [25]. Thus we have,

\[
\langle \exp \left( \pm i \int_0^t B(t')dt' \right) \rangle = \exp \left[ -\frac{1}{2} \left( \int_0^t B(t')dt' \right)^2 \right] = e^{-\gamma t/2}
\]

assuming the stationarity of the auto-correlation function of delta-correlated noise, with $\gamma = \langle [B(0)]^2 \rangle$.

Following [25], we analyze the effect of noise on the $N$-qubit system by grouping the physical states by their magnetization, defined as the difference between the number of spins in the state $|0\rangle$, denoted by $n'$, and the remaining in state $|1\rangle$, $N - n'$. The magnetisation is, $m' = 2n' - N$. The logical state $|0\rangle_L$ is written as,

\[
|0\rangle_L = \sum_{m'} \sum_{l=1}^{N_m'} b_{l}^{m'} |b_l\rangle_m'.
\]

Dephasing noise changes the state $|\psi\rangle_L$ to another state $|\psi'\rangle_L$, where $|\psi'\rangle = \exp \left[ -i \int_0^t H_{L,G}(t')dt' \right] |\psi\rangle_L$. The density matrix corresponding to the logical qubit is $\rho' = \int |\psi'\rangle \langle \psi'| P(B)dB$. The Bloch coordinates $\mathcal{R} \equiv \{R_X, R_Y, R_Z\}$ in the new state are obtained by evaluating the expectation values of the logical operators in the evolved state, given by $\langle \mathcal{R} \rangle = Tr[\rho' \hat{\mathcal{L}}]$, where $\hat{\mathcal{L}} \equiv \{\hat{X}, \hat{Y}, \hat{Z}\}$ represents the logical Bloch vectors in the initial state, $|\psi\rangle$. Noise causes leakage from the code space, quantified by $P \equiv \{p_x, p_y, p_z\} = \langle \hat{\mathcal{L}} P_c \rangle$ where

\[
P_c = \frac{1}{2^n} \prod_{P_i \in \mathcal{P}} (I^{\otimes n} + P_i). \tag{5}
\]

For the single unit structure (Fig. 4), in the presence of global dephasing noise, the logical Bloch coordinates and the values quantifying leakage turn out to be

\[
\begin{align*}
\langle R_X \rangle &= \frac{1}{2} (1 + e^{-2\gamma t}) \sin \theta \cos \phi \\
\langle R_Y \rangle &= -\frac{1}{2} (1 + e^{-2\gamma t}) \sin \theta \sin \phi \\
\langle R_Z \rangle &= \cos \theta \\
p_x &= \frac{1}{32} (3 + 4e^{-2\gamma t} + e^{-8\gamma t}) \sin \theta \cos \phi \\
p_y &= \frac{1}{32} (3 + 4e^{-2\gamma t} + e^{-8\gamma t}) \sin \theta \sin \phi \\
p_z &= \frac{1}{64} (1 + 9 \cos \theta - 4e^{-2\gamma t}(1 - \cos \theta) + 3e^{-8\gamma t}(1 + \cos \theta)).
\end{align*} \tag{6}
\]

The terms in $P_c$ contribute to a factor of $2^4/2^6$ in $\langle \hat{\mathcal{L}} P_c \rangle$, thus giving a pre-factor, $1/4$ in $\mathcal{P}$. 

9
In the presence of local dephasing noise, we have

\[
\langle R_X \rangle = e^{-2\gamma t} \sin \theta \cos \phi \\
\langle R_Y \rangle = -e^{-2\gamma t} \sin \theta \sin \phi \\
\langle R_Z \rangle = \cos \theta \\
\]

\[
p_x = \frac{1}{8}(e^{-2\gamma t} + e^{-8\gamma t}) \sin \theta \cos \phi \\
p_y = -\frac{1}{8}(e^{-2\gamma t} + e^{-8\gamma t}) \sin \theta \sin \phi \\
p_z = \frac{1}{16}(1 + 3e^{-8\gamma t}) \cos \theta.
\]

(7)

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[23] $G_1 \equiv \{ \pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ \}$ is the set of matrices that forms a group under the operation of matrix multiplication on a single qubit.

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