COMBINATORICS OF THE PERMUTATION TABLEAUX OF TYPE $B$

SYLVIE CORTEEL, MATTHIEU JOSUAT-VERGES AND JANG SOO KIM

ABSTRACT. Permutation tableaux are combinatorial objects related with permutations and various statistics on them. They appeared in connection with total positivity in Grassmannians, and stationary probabilities in a PASEP model. In particular they gave rise to an interesting $q$-analog of Eulerian numbers. The purpose of this article is to study some combinatorial properties of type $B$ permutation tableaux, defined by Lam and Williams, and links with signed permutation statistics.

We show that many of the tools used for permutation tableaux generalize in this case, including: the Matrix Ansatz (a method originally related with the PASEP), bijections with labeled paths and links with continued fractions, bijections with signed permutations. In particular we obtain a $q$-analog of the type $B$ Eulerian numbers, having a lot in common with the previously known $q$-Eulerian numbers: for example they have a nice symmetry property, they have the type $B$ Narayana numbers as constant terms.

The signed permutation statistics arising here are of several kinds. Firstly, there are several variants of descents and excedances, and more precisely of flag descents and flag excedances. Other statistics are the crossings and alignments, which generalize a previous definition on (unsigned) permutations. There are also some pattern-like statistics arising from variants of the bijection of Françon and Viennot.

CONTENTS

1. Introduction 1
2. Statistics of permutation tableaux of type $B$ and signed permutations 5
3. $q$-Eulerian numbers of type $B$ 7
4. Crossings and alignments 11
5. Solutions of the Matrix Ansatz 14
6. Interpretation of $B_n(y,t,q)$ via labeled Motzkin paths 15
7. Enumeration formula 20
8. Open problems 23
Acknowledgements. 24
References 24

1. INTRODUCTION

Permutation tableaux of type $B$ have been introduced in [19], and further studied in [8, 9]. We start this introduction by some definitions. A Ferrers diagram is a top and left justified arrangement of square cells with possibly empty rows and columns. The length of a Ferrers diagram is the sum of the number of rows and the number of columns. If a Ferrers diagram is of length $n$, we label the steps in the south-east border of the Ferrers diagram with $1, 2, \ldots, n$ from north-east to south-west. We will also call the step labeled with $i$ the $i$th step. The following is

All authors are partially supported by the grant ANR08-JCJC-0011. The second author was supported by the Austrian Science foundation FWF (START grant Y463) while he was postdoctoral researcher at the university of Vienna.

1
an example of a Ferrers diagram.

(1)

A permutation tableau is a 0,1-filling of a Ferrers diagram satisfying the following conditions: (1) each column has at least one 1 and (2) there is no 0 which has a 1 above it in the same column and a 1 to the left of it in the same row. The following is an example of a permutation tableau.

(2)

For a Ferrers diagram $F$ with $k$ columns, the \textit{shifted} Ferrers diagram of $F$, denoted $\overline{F}$, is the diagram obtained from $F$ by adding $k$ rows of size 1, 2, \ldots, $k$ above it in increasing order. The \textit{length} of $F$ is defined to be the length of $\overline{F}$. A diagonal cell is the rightmost cell of an added row.

For example, if $F$ is the Ferrers diagram in (1) then $\overline{F}$ is the diagram in (3) without the 0’s and 1’s.

A \textit{type B permutation tableau} of length $n$ is a 0,1-filling of a shifted Ferrers diagram of length $n$ satisfying the following conditions: (1) each column has at least one 1, (2) there is no 0 which has a 1 above it in the same column and a 1 to the left of it in the same row, and (3) if a 0 is in a diagonal cell, then it does not have a 1 to the left of it in the same row. The following is an example of a type B permutation tableau.

(3)

We respectively denote by $\mathcal{PT}(n)$ and $\mathcal{PT}_B(n)$ the sets of permutation tableaux and type B permutation tableaux of length $n$. For $T \in \mathcal{PT}(n)$ or $T \in \mathcal{PT}_B(n)$, a 1 is called \textit{superfluous} if it has a 1 above it in the same column. Let $\text{so}(T)$ denote the number of superfluous ones in $T$ in the same column. Let $\text{row}(T)$ denote the number of rows of $T$ except the added rows. For $T \in \mathcal{PT}_B(n)$, let $\text{diag}(T)$ be the number of ones on the diagonal of the added rows. Let

$$B_n(y, t, q) = \sum_{T \in \mathcal{PT}_B(n)} y^{2 \text{row}(T)+\text{diag}(T)} t^{\text{diag}(T)} q^{\text{so}(T)}.$$ 

For example:

$$B_0(y, t, q) = 1,$$

$$B_1(y, t, q) = y^2 + yt,$$

$$B_2(y, t, q) = y^4 + (2t + tq)y^3 + (t^2q + t^2 + 1)y^2 + ty.$$ 

Let also $B_{n,k}(t, q) = [y^k]B_n(y, t, q)$, which is nonzero when $n = k = 0$ or $1 \leq k \leq 2n$. Here $[y^k]f(y)$ means the coefficient of $y^k$ in $f(y)$. 
In the case $t = 0$, we have $B_{n,2k+1}(0,q) = 0$, and $B_{n,2k}(0,q)$ is the $q$-Eulerian number $E_{n,k}(q)$ defined by Williams [29] such that:

$$E_{n,k}(q) = \sum_{T \in \mathcal{PT}(n)} q^{\text{row}(T)}.$$

Indeed by condition (3) in the definition, a type $B$ permutation tableau with no 1 in a diagonal cell has only 0’s in the added row and is equivalent to a permutation tableau. Some properties are following:

- There is a symmetry such that $E_{n,k}(q) = E_{n,n+1-k}(q)$, and some special values are: $E_{n,k}(-1) = \binom{n}{k-1}$, $E_{n,k}(0) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$ (the Narayana number) [29].
- There is a continued fraction expansion for the generating function $\sum_{k=0}^{n} E_{n,k}(q)y^kz^n$.

Equivalently, we have

$$\text{Theorem 1.1.}$$

For a nonnegative integer $n$ we have

$$B_n(y, t, q) = \sum_{\pi \in B_n} y^{\text{fwex}(\pi)} (q) t^{\text{neg}(\pi)} q^{\text{cr}(\pi)}.$$ (4)

Equivalently, we have

$$B_{n,k}(t, q) = \sum_{\pi \in B_n} t^{\text{neg}(\pi)} q^{\text{cr}(\pi)}.$$ (5)

Here, $\text{fwex}(\pi)$ denotes the number of flag weak excedances, i.e. a variant of flag excedances, and two combinatorial interpretation in terms of permutations are known: a first one with descents and crossings, a second one with descents and the pattern 31-2 [7].

We will see throughout the article that all these properties can be generalized to $B_{n,k}(t, q)$ or $B_{n,k}^{*}(t, q)$ which is a variation of $B_{n,k}(t, q)$.

However the link with type $B$ Eulerian numbers is not completely immediate. Adin et al. [1] defined $\text{fdes}(\pi)$, the number of flag descents of $\pi \in B_n$. This refines $\text{des}_B(\pi)$ the Coxeter theoretic definition of the number of descents (see [4]) in the sense that $\lceil \text{fdes}(\pi)/2 \rceil = \text{des}_B(\pi)$. Here $B_n$ denotes the set of signed permutations of $[n]$, see Section 2 for the definition of the statistics. There is also a definition of flag excedances [11]. These flag statistics are the relevant ones in this context.

Using a zigzag bijection of [19], we will show in Section 2 the following theorem.

**Theorem 1.1.**

For a nonnegative integer $n$ we have

$$B_n(y, t, q) = \sum_{\pi \in B_n} y^{\text{fwex}(\pi)} (q) t^{\text{neg}(\pi)} q^{\text{cr}(\pi)}.$$ (4)

Equivalently, we have

$$B_{n,k}(t, q) = \sum_{\pi \in B_n} t^{\text{neg}(\pi)} q^{\text{cr}(\pi)}.$$ (5)

Here, $\text{fwex}(\pi)$ denotes the number of flag weak excedances, i.e. a variant of flag excedances, equidistributed with (fdes + 1). Theorem 1.1 is a type $B$ analog of a result of Steingrımsson and Williams [29]. In Section 2 we study how $\text{fwex}$ is related with the type $B$ descent $\text{des}_B$.

In Section 3 we define a $q$-Eulerian number of type $B$ by

$$E_{n,k}^B(q) = \sum_{\pi \in B_n} q^{\text{cr}(\pi)},$$ (6)

which is equivalent to $E_{n,k}^B(q) = B_{n,2k+1}(1,q) + B_{n,2k+1}(1,q)$. We then introduce a “pignose diagram” representing a signed permutation and prove the following theorem.

**Theorem 1.2.** For any $0 \leq k \leq n$, we have $E_{n,k}^B(-1) = \binom{n}{k}$, $E_{n,k}^B(0) = \binom{n}{k}^2$ (the Narayana numbers of type $B$), $E_{n,k}(1)$ is the type $B$ Eulerian number (see [4]), and

$$E_{n,k}^B(q) = E_{n-k}(q).$$ (7)

In fact we will prove a more general symmetry than (7). Observe that the identity $B_{n,k}(1,q) = B_{n,2n+1-k}(1,q)$ implies (7). However, for general $t$, we have $B_{n,k}(t,q) \neq B_{n,2n+1-k}(t,q)$. For instance, $B_{1,1}(t,q) = t$ and $B_{1,2}(t,q) = 1$. There is a way to fix this discrepancy. Let

$$B_{n,k}^*(t,q) = \sum_{\pi \in B_n} t^{\text{neg}(\pi)+\chi(\pi_1>0)} q^{\text{cr}(\pi)}.$$
where \( \chi(\pi_1 > 0) \) is 1 if \( \pi_1 > 0 \) and 0 otherwise. We will prove the following symmetry by a combinatorial argument:

**Theorem 1.3.** For \( 1 \leq k \leq 2n \), we have

\[
B_{n,k}(t, q) = B_{n,2n+1-k}(t, q).
\]

In particular, when \( t = 1 \), we have \( B_{n,k}(1, q) = B_{n,2n+1-k}(1, q) \).

Note that \( E_{n,k}(q) \) and \( E_{n,k}^B(q) \) are generating functions in which \( q \) records the statistic \( cr(\pi) \).

There is another statistic \( al(\pi) \) counting the number of alignments. Corteel [7] showed that the two statistics \( cr(\pi) \) and \( al(\pi) \) in \( S_n \) are closely related:

**Proposition 1.4.** [7] If \( \sigma \in S_n \) has \( k \) weak excedances, then

\[
\text{cr}(\sigma) + \text{al}(\sigma) = (k - 1)(n - k).
\]

In Section 3 we prove the following proposition which is a type B analog of the above result.

**Proposition 1.5.** For \( \pi \in B_n \) with \( \text{fwex}(\pi) = k \), we have

\[
2 \text{cr}(\pi) + \text{al}(\pi) = n^2 - 2n + k.
\]

Besides the zigzag bijection, another important result coming from permutation tableaux is the following:

**Theorem 1.6.** (Matrix Ansatz [8]) Let \( D \) and \( E \) be matrices, \( \langle W \rangle \) a row vector, and \( |V| \) a column vector, such that:

\[
DE = qED + D + E, \quad D|V| = |V|, \quad \langle W \rangle E = yt\langle W|D.
\]

Then we have:

\[
B_n(y, t, q) = \langle W| (y^2D + E)^n |V\rangle.
\]

This can be seen as an abstract rule to compute \( B_n(y, t, q) \), but it is also useful to have explicitly \( D, E, \langle W \rangle \) and \( |V| \) satisfying the relations (i.e. solutions of the Matrix Ansatz), see [8]. We will give two such solutions in Section 5. The following induction formula appears in relation with one of the solutions. We use the standard notation \( [n]_q = (1 - q^n)/(1 - q) \).

**Theorem 1.7.** We have \( B_0 = 1 \), and

\[
B_{n+1}(y, t, q) = (y + t)D_q[(1 + yt)B_n(y, t, q)],
\]

where \( D_q \) is the \( q \)-derivative with respect to \( t \), i.e. the linear operator that sends \( t^n \) to \( [n]_q t^{n-1} \).

As another consequence of the solutions of the Matrix Ansatz, we will see \( B_n(y, t, q) \) as a generating function of labeled Motzkin paths or suffixes of labeled Motzkin paths. In particular, we obtain in this way a continued fraction:

**Theorem 1.8.** Let \( \gamma_h = y^2[h+1]_q + [h]_q + tyh^h([h]_q + [h+1]_q) \) and \( \lambda_h = y[h]_q(y + tq^{-1})(1 + ytq^h) \).

Then we have

\[
\sum_{n \geq 0} B_n(y, t, q)z^n = \frac{1}{1 - \gamma_0 z - \frac{\lambda_1 z^2}{1 - \gamma_1 z - \frac{\lambda_2 z^2}{1 - \gamma_2 z - \cdots}}}
\]

We use the notation \( \frac{a_1}{b_1 - \frac{a_2}{b_2 - \cdots}} = (a_1/(b_1 - (a_2/(b_2 - \cdots)))) \) Note that this kind of continued fraction (called J-fraction) are related with moments of orthogonal polynomials [27].

We will show in Section 5 that the two kinds of paths are in bijection with signed permutations, using variants of classical bijections of Françon and Viennot [13], Foata and Zeilberger [12]. We obtain two other interpretations of \( B_n(y, t, q) \) where \( y \) follows a descent statistic and \( q \) a pattern statistic. Once again, this is a type B analog of results on permutation tableaux and permutations [10] [20]. The results are the following (see Section 6 for the definitions):

**Theorem 1.9.** For \( n \geq 1 \), we have:

\[
B_n(y, t, q) = \sum_{\pi \in B_n} y^\text{hasc}(\pi) t^{\text{neg}(\pi)} q^{\text{pat}(\pi)} = \sum_{\pi \in B_n} y^\text{ides}(\pi) + 1 t^{\text{neg}(\pi)} q^{2\lambda_1 + 2^+(\pi)}.
\]
where each integer may be negated. For example, $\pi$ of \[6\].

For a nonnegative integer $n$, we have

$$B_n(y, 1, q) = \frac{1}{(1-q)^n} \sum_{k=0}^{n} \left( \sum_{i=0}^{2n-2k} y^i \binom{n}{k+\lfloor i/2 \rfloor} \binom{n}{\lfloor i/2 \rfloor} \right) \left( \sum_{j=0}^{2k} y^j q^{(j+1)(2k-j)/2} \right).$$

This will be given in Section 8 and the proof uses techniques developed in \[16\]. We conclude in Section 8 by a list of open problems.

2. Statistics of permutation tableaux of type $B$ and signed permutations

2.1. The zigzag bijection. Steingrimsson and Williams \[26\] defined a bijection $\Phi : \mathcal{PT}(n) \to S_n$ using zigzag paths. For given $T \in \mathcal{PT}(n)$, the corresponding permutation $\Phi(T) = \pi = \pi_1 \cdots \pi_n \in S_n$ is obtained as follows. Here $S_n$ denotes the set of permutations of $[n] = \{1, 2, \ldots, n\}$. For each $i \in [n]$, we travel from the $i$th step of the Ferrers diagram. If the $i$th step is horizontal (resp. vertical), then travel to the topmost (resp. leftmost) 1 in the column (resp. row) containing the $i$th step, and then travel to the east (resp. south) changing the direction to south or east whenever we meet a 1. Then $\pi_i$ is defined to be the label of the step at which we finally arrive. If there is no 1 in the row containing the $i$th step, then $\pi_i = i$. For example, if $T$ is the permutation tableau in \[2\], then $\Phi(T) = 7, 1, 6, 5, 3, 4, 2, 8$.

The bijection $\Phi$ is naturally extended to the bijection $\Phi_B : \mathcal{PT}_B(n) \to B_n$, where we denote by $B_n$ the set of signed permutations on $[n]$. A signed permutation on $[n]$ is a permutation of $[n]$ where each integer may be negated. For example, $\pi = 4, 2, -5, 6, -1, 3$ is a signed permutation of $[6]$. For $T \in \mathcal{PT}_B(n)$, we construct $\pi \in B_n$ in a similar way using zigzag paths except the following. If the column containing the $i$th step has a 1 in the diagonal, then we travel from the $i$th step to the 1 in the diagonal, travel to the leftmost 1 in the row containing the 1 and then travel to the south changing the direction to south or east whenever we meet a 1. If we arrive at the $j$th step then $\pi_j = -j$. For example, if $T$ is the type $B$ permutation tableau in Figure 1, we have $\Phi_B(T) = -3, 6, 2, 5, -4, 1, 7$.

Let $\pi = \pi_1 \cdots \pi_n \in B_n$. An integer $i \in [n]$ is called a weak excedance if $\pi_i \geq i$, and an excedance if $\pi_i > i$. An integer $i \in [n-1]$ is called a descent of $\pi$ if $\pi_i > \pi_{i+1}$. There are various statistics

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{zigzag_bijection.png}
\caption{An illustration of the zigzag bijection $\Phi_B$.}
\end{figure}
We denote by \( cr(\pi) \) such that Proposition 2.1.

Remark 1. There is another statistic \( cr(\pi) \) that was first defined in [S].

Definition 1. [S] A crossing of a signed permutation \( \pi = \pi_1 \cdots \pi_n \) is a pair \((i, j)\) with \(i, j > 0\) such that

- \( i < j \leq \pi_i < \pi_j \) or
- \( -i < j \leq -\pi_i < \pi_j \) or
- \( i > j > \pi_i > \pi_j \).

We denote by \( cr(\pi) \) the number of crossings of \( \pi \).

Steingrímsson and Williams [20] showed that if \( \pi = \Phi(T) \), then \( so(T) = cr(\pi) \) and \( wex(\pi) = row(T) \). Corteel et al. [S] found the following type \( B \) analog of this result.

Proposition 2.1. [S] The zigzag map \( \Phi_B : PT_B(n) \to B_n \) is a bijection. Moreover, if \( \pi = \Phi_B(T) \), then \( wex(\pi) = row(T) \), \( neg(\pi) = diag(T) \), and \( cr(\pi) = so(T) \).

Remark 1. In [S] Theorem 4] Corteel et al. only show \( neg(\pi) = diag(T) \) and \( cr(\pi) = so(T) \). However, one can easily see from the definition of \( \Phi_B \) that we also have \( wex(\pi) = row(T) \).

2.2. Equidistribution of some statistics in \( B_n \). We now find a relation between \( f\text{wex}, f\text{exc}, \) and \( f\text{des} \). For permutations, it is well known [25] 1.4.3 Proposition] that

\[
\begin{align*}
\#\{\pi \in S_n : wex(\pi) = k\} &= \#\{\pi \in S_n : des(\pi) = k - 1\} = \#\{\pi \in S_n : exc(\pi) = k - 1\}.
\end{align*}
\]

We will use the following result of Foata and Han.

Lemma 2.2. [11] Section 9] There is a bijection \( \psi : B_n \to B_n \) such that \( f\text{exc}(\pi) = f\text{des}(\psi(\pi)) \).

For \( \pi = \pi_1 \cdots \pi_n \in B_n \), we define \( -\pi \in B_n \) by \((-\pi)_i = -(\pi)_i \). We also define \( \pi^{\text{tr}} \in B_n \) to be the signed permutation such that \( \pi^{\text{tr}}_i = \epsilon \cdot j \) if and only \( \pi_j = \epsilon \cdot i \) for \( \epsilon \in \{1,-1\} \) and \( i,j \in [n] \). In other words, if \( M(\pi) \) is the signed permutation matrix of \( \pi \), then \( M(-\pi) = -M(\pi) \) and \( M(\pi^{\text{tr}}) = M(\pi)^{\text{tr}} \). Here, the signed permutation matrix \( M(\pi) \) is the \( n \times n \) matrix whose \((i, j)\)-entry is 1 if \( \pi_i = j \), \(-1 \) if \( \pi_i = -j \), and 0 otherwise. The following lemma is easy to prove.

Lemma 2.3. For \( \pi \in B_n \), we have

\[
\begin{align*}
f\text{des}(\pi) + f\text{des}(-\pi) &= 2n - 1, \quad f\text{wex}(\pi) + f\text{exc}(\pi^{\text{tr}}) = 2n.
\end{align*}
\]

We now are ready to prove a type \( B \) analog of [9].

Proposition 2.4. We have

\[
\begin{align*}
\#\{\pi \in B_n : f\text{wex}(\pi) = k\} &= \#\{\pi \in B_n : f\text{des}(\pi) = k - 1\} = \#\{\pi \in B_n : f\text{exc}(\pi) = k - 1\},
\end{align*}
\]

and

\[
\begin{align*}
\#\{\pi \in B_n : des_B(\pi) = k\} &= \#\{\pi \in B_n : [f\text{wex}(\pi)/2] = k\}.
\end{align*}
\]

Proof. By Lemmas 2.2 and 2.3 we have

\[
\begin{align*}
\#\{\pi \in B_n : f\text{wex}(\pi) = k\} &= \#\{\pi \in B_n : f\text{exc}(\pi) = 2n - k\} = \#\{\pi \in B_n : f\text{des}(\pi) = 2n - k\} \\
&= \#\{\pi \in B_n : f\text{des}(\pi) = 2n - 1 - (2n - k)\} \\
&= \#\{\pi \in B_n : f\text{des}(\pi) = k - 1\} = \#\{\pi \in B_n : f\text{exc}(\pi) = k - 1\}.
\end{align*}
\]
Equation (11) follows from Equation (10) and the fact that \( \text{des}_B(\pi) = \lceil (\text{des}(\pi) + 1)/2 \rceil \). \( \square \)

3. \( q \)-Eulerian numbers of type \( B \)

Let \( k \) and \( n \) be integers with \( 1 \leq k \leq n \). The (type \( A \)) Eulerian number \( E_{n,k} \) is the number of \( \pi \in S_n \) with \( \text{des}(\pi) = k - 1 \). Equivalently, \( E_{n,k} \) is the number of \( \pi \in S_n \) with \( \text{wex}(\pi) = k \). The \( q \)-Eulerian number \( E_{n,k}(q) \) originally defined in [29] is as follows:

\[
E_{n,k}(q) = B_{n,2k}(0,q) = \sum_{\pi \in S_n, \text{wex}(\pi) = k} q^{\text{cr}(\pi)}.
\]

Now let \( k \) and \( n \) be integers with \( 0 \leq k \leq n \). The type \( B \) Eulerian number \( E^B_{n,k} \) is the number of \( \pi \in B_n \) with \( \text{des}_B(\pi) = k \). By Equation (11), \( E^B_{n,k} \) is also the number of \( \pi \in B_n \) with \( \lfloor \text{fwex}(\pi)/2 \rfloor = k \).

**Definition 2.** For \( 0 \leq k \leq n \), we define the type \( B \) \( q \)-Eulerian number \( E^B_{n,k}(q) \) as follows:

\[
E^B_{n,k}(q) = \sum_{\pi \in B_n, \lfloor \text{fwex}(\pi)/2 \rfloor = k} q^{\text{cr}(\pi)}.
\]

By Theorem 1.1 we can write

\[
E^B_{n,k}(q) = B_{n,2k}(1,q) + B_{n,2k+1}(1,q).
\]

By Proposition 2.1 we have a different expression for \( E^B_{n,k}(q) \) using permutation tableaux of type \( B \):

\[
E^B_{n,k}(q) = \sum_{T \in \mathcal{PT}_{n,n}(n)} q^{\text{cr}(T)}.
\]

The \( q \)-Eulerian number \( E_{n,k}(q) \) becomes the binomial coefficient \( \binom{n}{k} \) when \( q = -1 \), the Narayana number \( \frac{1}{k} \binom{n}{k} \binom{n}{k-1} \) when \( q = 0 \), and the Eulerian number \( E_{n,k} \) when \( q = 1 \). Thus we have \( E_{n,k}(q) = E_{n,n+1-k}(q) \) in the special cases \( q \in \{-1,0,1\} \). More generally:

**Proposition 3.1** (Williams [29]). For any \( 1 \leq k \leq n \), we have

\[
E_{n,k}(q) = E_{n,n+1-k}(q).
\]

Our \( q \)-analog of the type \( B \) Eulerian numbers also has such properties.

**Theorem 3.2** (Theorem 1.2 in the introduction). For any \( 0 \leq k \leq n \), we have \( E^B_{n,k}(-1) = \binom{n}{k} \), \( E^B_{n,k}(0) = \binom{n}{k}^2 \) (the Narayana numbers of type \( B \)), \( E^B_{n,k}(1) \) is the type \( B \) Eulerian number, and

\[
E^B_{n,k}(q) = E^B_{n,n-k}(q).
\]

The values at \( q = -1 \) and \( q = 0 \) will be obtained respectively in Section 5 and Section 7. Recall that

\[
B^*_{n,k}(t,q) = \sum_{\pi \in \mathcal{B}_n, \text{fwex}(\pi) = k} t^{\text{neg}(\pi)+\chi(\pi_1>0)} q^{\text{cr}(\pi)}.
\]

Since \( B^*_{n,k}(1,q) = B_{n,k}(1,q) \), the symmetric property (13) is a direct consequence of (12) together with the following result:

**Theorem 3.3** (Theorem 1.3 in the introduction). For \( 1 \leq k \leq 2n \), we have

\[
B^*_{n,k}(t,q) = B^*_{n,2n+1-k}(t,q).
\]

In order to prove Theorem 3.3 we introduce a diagram representing a signed permutation.
3.1. Pignose diagrams. Given a set \( U \) of \( 2n \) distinct integers, an ordered matching on \( U \) is a set of ordered pairs \((i,j)\) of integers such that each integer in \( U \) appears exactly once. For an ordered matching \( M \) on \( U \) containing \( 2n \) integers \( a_1 < a_2 < \cdots < a_{2n} \), we define the standardization \( \text{st}(M) \) of \( M \) to be the ordered matching on \([2n]\) obtained from \( M \) by replacing \( a_i \) with \( i \) for each \( i \in [2n] \). For example, if \( M = \{(2,6),(5,3),(9,4),(7,8)\} \), then \( \text{st}(M) \) is the ordered matching \( \{(1,5),(4,2),(8,3),(6,7)\} \) on \([8]\).

We represent an ordered matching \( M \) on \( U \) as follows. Arrange the integers in \( U \) on a horizontal line in increasing order. For each pair \((i,j)\in M\), connect \( i \) and \( j \) with an upper arc if \( i < j \), and with a lower arc if \( i > j \). For example, the following represents \( \{(1,5),(4,2),(8,3),(6,7)\} \).

![Pignose diagram example](image)

We also define a crossing of an ordered matching \( M \) to be two intersecting arcs and denote by \( \text{cr}(M) \) the number of crossings of \( M \).

For \( \pi = \pi_1 \cdots \pi_n \in B_n \), the pignose diagram of \( \pi \) is defined as follows. First we arrange \( 2n \) vertices in a horizontal line where the \((2i-1)\)th vertex and the \(2i\)th vertex are enclosed by an ellipse labeled with \( i \) which we call the \( i \)th pignose. The left vertex and the right vertex in a pignose are called the first vertex and the second vertex respectively. For each \( i \in [n] \), we connect the first vertex of the \( i \)th pignose and the second vertex of the \( \pi_i \)th pignose with an arc in the following way. If \( \pi_i > 0 \), then draw an arc above the horizontal line if \( \pi_i \geq i \) and below the horizontal line if \( \pi_i < i \). If \( \pi_i < 0 \), then we draw an arc starting from the first vertex of the \( i \)th pignose below the horizontal line to the second vertex of the \( \pi_i \)th pignose above the horizontal line like a spiral oriented clockwise. We draw these spiral arcs so that these are not crossing each other below the horizontal line. For example, the following is the pignose diagram of \( \pi = (4,-6,1,-5,-3,7,2) \).

![Pignose diagram example](image)

For \( \pi \in B_n \), one can easily check that
- \( \text{cr}(\pi) \) is the number of unordered pairs of two arcs crossing each other in the pignose diagram of \( \pi \),
- \( \text{fwex}(\pi) \) is twice the number of upper arcs plus the number of spiral arcs, or equivalently, the number of vertices with a half arc above the horizontal line,
- \( \text{neg}(\pi) \) is the number of spiral arcs.

Since \( S_n \) is contained in \( B_n \), the pignose diagram for \( \pi \in S_n \) is also defined. Note that the pignose diagram of \( \pi \in S_n \) can be considered as an ordered matching on \([2n]\) by removing the ellipses enclosing two vertices and labeling the \( 2n \) vertices with \( 1,2,\ldots,2n \) from left to right. We call an ordered matching that can be obtained in this way a pignose matching. Not all ordered matchings are pignose matchings. One can easily prove the following proposition.

**Proposition 3.4.** An ordered matching \( M \) on \([2n]\) is a pignose matching if and only if in the representation of \( M \) each odd integer has a half arc of type \( \swarrow \) or \( \searrow \), and each even integer has a half arc of type \( \nearrow \) or \( \nwarrow \).

The following lemma was first observed by Médicis and Viennot [22 Lemme 3.1 Cas b.4]. For reader’s convenience we include a proof as well.

**Lemma 3.5.** For \( \pi \in S_n \) and an integer \( k \in [n] \), the number of integers \( i \in [n] \) with \( i \leq k \leq \pi_i \) is equal to the number of integers \( i \in [n] \) with \( \pi_i < k < i \) plus 1. Pictorially, this means that in the
pignose diagram of $\pi$ if we draw a vertical line dividing the $k$th pignose in the middle, then the number of upper arcs intersecting with this line is equal to the number of lower arcs intersecting with this line plus 1 as shown below.

\[
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\]
\[
x + 1
\]
\[
x
\]

Proof. Consider the pignose diagram of $\pi$ with a vertical line dividing the pignose of $k$ in the middle. Let $x$ (resp. $y$) be the number of lower (resp. upper) arcs intersecting with the vertical line. Let $a, b, c, d$ be, respectively, the number of half arcs to the left of the vertical line of the following types:

\[
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\]

For each pignose, the first vertex has a half edge of the first or the second type and the second vertex has a half edge of the third or the fourth type. Thus $a + b = c + d + 1$. Since $x = a - c$ and $y = d - b$, we are done. \(\square\)

3.2. Proof of Theorem 3.3. Let

\[
B_n^+ = \{ \pi \in B_n : \pi_1 > 0 \}, \quad B_n^- = \{ \pi \in B_n : \pi_1 < 0 \},
\]

\[
B_{n,k}^+(t,q) = \sum_{\pi \in B_n^+, \text{fwex}(\pi) = k} t^{\text{neg}(\pi)} q^{\text{cr}(\pi)}, \quad B_{n,k}^-(t,q) = \sum_{\pi \in B_n^-, \text{fwex}(\pi) = k} t^{\text{neg}(\pi)} q^{\text{cr}(\pi)}.
\]

Then

\[
B_{n,k}^+(t,q) = tB_{n,k}^+(t,q) + B_{n,k}^-(t,q).
\]

In this subsection we will prove the following proposition.

Proposition 3.6. There is a bijection $\phi : B_n^+ \to B_n^-$ such that

\[
\text{cr}(\phi(\pi)) = \text{cr}(\pi), \quad \text{neg}(\phi(\pi)) = \text{neg}(\pi) + 1, \quad \text{fwex}(\phi(\pi)) = 2n + 1 - \text{fwex}(\pi).
\]

Thus,

\[
tB_{n,k}^+(t,q) = B_{n,2n+1-k}^-(t,q), \quad B_{n,k}^-(t,q) = tB_{n,2n+1-k}^+(t,q).
\]

Note that Theorem 3.3 follows from (14) and (16). In order to prove Proposition 3.6 we need some definitions and lemmas.

Given an ordered matching $M$ on $[2n]$, we define $\rho(M)$ to be the ordered matching $st(M')$ where $M'$ is the ordered matching on $\{2,3,\ldots,2n+1\}$ obtained from $M$ by replacing 1 with $2n+1$. For example, if $M = \{(1,5), (4,2), (8,3), (6,7)\}$, then $M' = \{(9,5), (4,2), (8,3), (6,7)\}$ and $\rho(M) = st(M') = \{(8,4), (3,1), (7,2), (5,6)\}$. Pictorially, $\rho(M)$ is obtained from $M$ by moving the first vertex to the end and reflecting the arc adjacent to this vertex as follows.

\[
\begin{array}{c}
1 \xrightarrow{k} 2 \xrightarrow{\rho} 3 \xrightarrow{\rho} \cdots \xrightarrow{\rho} 8
\end{array}
\]

We denote $\rho^{(k)} = \rho \circ \cdots \circ \rho$.

Lemma 3.7. Let $M$ be a pignose matching on $[2n]$. Then $\rho^{(k)}(M)$ has the same number of crossings as $M$ for all positive integers $k$. 

Proof. Note that $\rho^{(2)}(M)$ is a pignose matching. Thus it suffices to prove for $k = 1$ and $k = 2$.

Considering $M$ as a pignose diagram of a permutation in $S_n$, assume that 1 is connected to $i$. By Lemma 3.5, if we draw a vertical line between the two vertices in the $i$th pignose, the number, say $x$, of upper arcs above $i$ except $(1,i)$ is equal to the number of lower arcs below $i$. Therefore, when we go from $M$ to $\rho(M)$, we lose $x$ crossings and obtain new $x$ crossings as shown below.

This proves the assertion for $k = 1$.

To prove for $k = 2$ we define the following. Given an ordered matching $N$, let $N^r$ be the ordered matching obtained by reflecting $N$ along the horizontal line. It is easy to see that $\cr(N) = \cr(N^r)$ and $\rho(N) = \rho(N^r)$. Moreover if $N$ is a pignose matching then so is $\rho(N)$. Thus we have

$$\cr(\rho^{(2)}(M)) = \cr(\rho(M)) = \cr(M).$$

Since $\rho(M)$ is a pignose matching, using the assertion for $k = 1$, we obtain that (17) is equal to

Thus $\cr(\rho^{(2)}(M)) = \cr(M)$ and we are done. □

For an ordered matching $M$ on $[2n]$ we define $M^r$ to be the ordered matching obtained from $M$ by replacing $i$ with $2n + 1 - i$ for each $i \in [2n]$. Pictorially $M^r$ is obtained from $M$ by taking a 180° rotation.

**Lemma 3.8.** Let $M$ be a pignose matching on $[2n]$. Then $N = (\rho^{(2k+1)}(M))^r$ is also a pignose matching and $\cr(M) = \cr(N)$.

**Proof.** Note that $\rho^{(2)}(M)$ is a pignose matching. Thus it suffices to prove that $(\rho(M))^r$ is a pignose matching, which easily follows from Proposition 3.4. □

For $\pi \in B_n$, let $\pi^- = (-\pi_1)\pi_2 \cdots \pi_n$.

**Lemma 3.9.** The map $\pi \mapsto \pi^-$ is a bijection from $B_n^+$ to $B_n^-$. Moreover, we have $\cr(\pi) = \cr(\pi^-)$, $\neg(\pi) = \neg(\pi^-) - 1$, and $\fwex(\pi) = \fwex(\pi^-) + 1$.

**Proof.** This is an immediate consequence of the following observation: if $\pi \in B_n^+$, the pignose diagram of $\pi^-$ is obtained from the pignose diagram of $\pi$ by changing the upper arc adjacent to 1 to a spiral arc as follows.

Now we are ready to define the map $\phi : B_n^+ \to B_n^-$ in Proposition 3.6. Suppose $\pi \in B_n^+$ and $\neg(\pi) = m$. We make the pignose diagram of $\pi$ to be a pignose matching on $[2m+2n]$ by dividing each spiral arc into one upper arc and one lower arc so that the left endpoint of the upper arc is
to the left of the left endpoint of the lower arc as shown below.

\[(18)\]

Let \( M \) be the pignose matching obtained in this way, and let \( N = (\rho'(2m+1)(M))^r \). By Lemma 3.8, \( N \) is also a pignose matching on \( [2m+2n] \) and \( \text{cr}(M) = \text{cr}(N) \). It is straightforward to check that \( N \) satisfies the following properties.

1. For each \( i \in [2m] \), the \( i \)th vertex is connected to the \( j \)th vertex for some \( j > 2m \).
2. The first \( m \) lower arcs do not cross each other.
3. The \( (2m+1) \)st vertex has an upper half arcs.
4. The number of upper half arcs adjacent to the last \( 2n \) vertices is \( 2n + 2 - k \).

By the first and the second properties, we can make \( N \) to be the pignose diagram of a signed permutation, say \( \sigma \in B_n \), by identifying the \((2i-1)\)th vertex and the \((2i)\)th vertex for each \( i \in [m] \). Then \( \text{neg}(\pi) = \text{neg}(\sigma) \). By the third property, we have \( \sigma \in B_n^+ \), and by the fourth property, we have \( \text{fwex}(\sigma) = 2n + 2 - k \). We define \( \phi(\pi) \) to be \( \sigma^- \). Clearly \( \phi \) is a bijection from \( B_n^+ \) to \( B_n^- \). By Lemma 3.9, \( \phi \) satisfies (15). This finishes the proof of Proposition 3.6.

Example 1. Let \( \pi = 3, -4, -2, 1 \in B_4^+ \). The pignose diagram of \( \pi \) and \( M \) are shown in (18). Then we can compute \( \rho'(5)(M) \), \( (\rho'(5)(M))^r \), \( \sigma \), and \( \sigma^- \) as follows.

\[
\rho'(5)(M) = \\
\( (\rho'(5)(M))^r = \) \\
\sigma = \\
\sigma^- =
\]

4. CROSSINGS AND ALIGNMENTS

For a permutation \( \sigma \in S_n \), an alignment is an unordered pair of two arcs in the pignose diagram of \( \sigma \) which look like one of the following figures:

\[(19)\]

Let \( \text{al}(\sigma) \) denote the number of alignments of \( \sigma \). The following proposition was first proved by the first author [7] using rather technical calculation. Here we provide another proof which is more combinatorial.
Proposition 4.1. If \( \sigma \in S_n \) has \( k \) weak excedances, then
\[
\text{cr}(\sigma) + \text{al}(\sigma) = (k - 1)(n - k).
\]

Proof. Since \( \text{wex}(\pi) = k \), we have \( k \) upper arcs and \( n - k \) lower arcs in the pignose diagram of \( \pi \).

Let \( A \) be the set of pairs \( (U, L) \) of an upper arc \( U \) and a lower arc \( L \). Then there are \( k(n - k) \) elements in \( A \). We define \( A_1 \), \( A_2 \) and \( A_3 \) to be the subsets of \( A \) consisting of the pairs of arcs whose relative locations look like the following:

\[
\begin{align*}
\text{or } \quad & \text{for } A_1, \\
\text{or } \quad & \text{for } A_2, \\
\text{or } \quad & \text{for } A_3,
\end{align*}
\]

Observe that \( A = A_1 \cup A_2 \cup A_3 \). Fix an upper arc \( U \) whose right endpoint is the second vertex of the \( i \)th pignose. Then by Lemma 3.5, the number of elements \( (U, L) \in A_2 \) is equal to the number of pairs \( (U, U') \) of upper arcs whose relative locations are the following:

\[
\begin{align*}
\text{or } \quad & \text{for } (20), \\
\text{or } \quad & \text{for } (21).
\end{align*}
\]

Now fix a lower arc \( L \) whose right endpoint is the first vertex of the \( j \)th pignose. Again by Lemma 3.5, the number of elements \( (U, L) \in A_3 \) is one more than the number of pairs \( (L, L') \) of lower arcs whose relative locations are the following:

\[
\begin{align*}
\text{or } \quad & \text{for } (20), \\
\text{or } \quad & \text{for } (21).
\end{align*}
\]

Observe that a crossing or an alignment is either an element in \( A_1 \), or a pair of arcs as shown in (20) or (21). Since we have \( n - k \) lower arcs, we obtain that the total number of crossings and alignments is equal to
\[
|A_1| + |A_2| + |A_3| - (n - k) = k(n - k) - (n - k) = (k - 1)(n - k).
\]

Now we define another representation of a signed permutation. Note that a signed permutation \( \pi = \pi_1 \cdots \pi_n \in B_n \) can be considered as a bijection on \( [\pm n] = \{1, 2, \ldots, n, -1, -2, \ldots, -n\} \) with \( \pi(i) = \pi_i \) and \( \pi(-i) = -\pi_i \) for \( i \in [n] \).

We define the full pignose diagram of \( \pi \in B_n \) as follows. First we arrange \( 2n \) pignoses in a horizontal line which are labeled with \( n, -(n - 1), \ldots, 1, 2, \ldots, n \). The first vertex and the second vertex of a pignose labeled \( i \) are, respectively, defined to be the left vertex and the right vertex of the pignose if \( i > 0 \), and to be the right vertex and the left vertex of the pignose if \( i < 0 \). For each \( i \in [\pm n] \), we connect the first vertex of the pignose labeled with \( i \) and the second vertex of the pignose labeled with \( \pi(i) \) with an arc in the following way. If \( \pi_i > 0 \), draw an arc above the horizontal line if \( \pi(i) \geq i \), and below the horizontal line if \( \pi(i) < i \). For example, the following is the full pignose diagram of \( \pi = [4, -6, 1, -5, -3, 7, 2] \).
The following lemma can be shown by the same argument in the proof of Lemma 3.3.

**Lemma 4.2.** Let \( \pi \in B_n \) and \( k \in [n] \). In the full pignose diagram of \( \pi \) if we draw a vertical line dividing the pignose labeled \(-k\) (resp. \( k \)) in the middle, then the number of upper arcs intersecting with this line is equal to the number of lower arcs intersecting with this line minus 1 (resp. plus 1) as shown below.

In a full pignose diagram, a *positive half arc* (resp. *negative half arc*) is a half arc attached to a vertex of a pignose labeled with a positive (resp. negative) integer. Note that the number of two arcs intersecting with each other in the full pignose diagram of \( \pi \in B_n \) is equal to \( 2 \text{cr}(\pi) \).

An *alignment* of \( \pi \in B_n \) is an unordered pair of arcs in the full pignose diagram of \( \pi \) whose relative locations are as shown in (19). We denote by \( \text{al}(\pi) \) the number of alignments of \( \pi \).

**Proposition 4.3.** For \( \pi \in B_n \) with \( \text{fwex}(\pi) = k \), we have

\[
2 \text{cr}(\pi) + \text{al}(\pi) = n^2 - 2n + k.
\]

**Proof.** Since the number of positive upper half arcs is equal to \( \text{fwex}(\pi) = k \), the number of positive lower half arcs is equal to \( 2n - k \).

Let \( A \) be the set of pairs \((U, L)\) of an upper arc \( U \) and a lower arc \( L \). Since we have \( n \) upper arcs and \( n \) lower arcs in total, there are \( n^2 \) elements in \( A \). Using Lemma 4.2 and the same argument as in the proof of Proposition 4.1, one can easily see that

\[
2 \text{cr}(\pi) + \text{al}(\pi) = n^2 - a - b,
\]

where \( a \) is the number of negative upper half arcs of the form

\[
\begin{array}{c}
\vdots \\
\end{array}
\]

and \( b \) is the number of positive lower half arcs of the form

\[
\begin{array}{c}
\vdots \\
\end{array}
\]

By the symmetry \( a \) is equal to the number of positive lower half arcs of the following form.

\[
\begin{array}{c}
\vdots \\
\end{array}
\]

Thus \( a + b \) is the number of positive half arcs, which is \( 2n - k \), and we obtain the desired formula. \( \square \)
5. Solutions of the Matrix Ansatz

Although it might be easy to check that some matrices satisfy the Matrix Ansatz (3), it is not obvious how to find explicitly such matrices. We provide two solutions in the form of semi-infinite tridiagonal matrices. They can be obtained using the following observation: if \( X \) and \( Y \) are such that \( XY - qYX = I \) (where \( I \) is the identity), then \( D = X(I + Y) \) and \( E = YX(I + Y) \) satisfy

\[
DE - qED = X(I + Y)YX(I + Y) - qYX(I + Y)X(I + Y) = (XY + XYY - qYXY)X(I + Y) = (I + Y)X(I + Y) = D + E.
\]

Then we can look for \( \langle W \rangle \) (respectively, \( \vert V \rangle \)) as a left (respectively, right) eigenvector of \( ytD - E \) (respectively, \( D \)).

5.1. Solution 1. Let \( X = (X_{i,j})_{i,j \geq 0} \) and \( Y = (Y_{i,j})_{i,j \geq 0} \) where

\[
X_{i,i+1} = [i + 1]_q \quad \text{and} \quad X_{i,j} = 0 \quad \text{otherwise}, \quad Y_{i+1,i} = 1, \quad Y_{i,i} = tyq^i \quad \text{and} \quad Y_{i,j} = 0 \quad \text{otherwise},
\]

and

\[
\langle W \rangle = (1, 0, 0, \ldots), \quad \vert V \rangle = (1, 0, 0, \ldots)^t.
\]

We can check that \( XY - qYX = I \), and that \( D = X(I + Y) \) and \( E = YX(I + Y) \) together with \( \langle W \rangle \) and \( \vert V \rangle \) provide a solution of (3). The coefficients are:

\[
(22) \quad D_{i,i} = [i + 1]_q, \quad D_{i,i+1} = (1 + ytq^{i+1})[i + 1]_q, \quad D_{i,j} = 0 \quad \text{otherwise},
\]

and

\[
(23) \quad E_{i,i} = (1 + ytq)[i]_q + ytq[i + 1]_q, \quad E_{i,i+1} = ytq(1 + ytq^{i+1})[i + 1]_q, \quad E_{i+1,i} = [i + 1]_q, \quad E_{i,j} = 0 \quad \text{otherwise}.
\]

Then, \( y^2D + E \) can be seen as a transfer matrix for “walks” in the nonnegative integers. Since the matrix is tridiagonal and because of the particular choice of \( \langle W \rangle \) and \( \vert V \rangle \), this shows that \( B_n(y, t, q) \) count some weighted Motzkin paths of length \( n \) (see the next section for more on the combinatorics of these paths). By standard methods, this gives a continued fraction for the generating function:

**Theorem 5.1** (Theorem 1.8 in the introduction). Let \( \gamma_h = y^2[h + 1]_q + [h]_q + ytq^h([h]_q + [h + 1]_q) \) and \( \lambda_h = y[h]_q^2(ty + tq^{-h})(1 + ytq^h) \), then we have:

\[
\sum_{n \geq 0} B_n(y, t, q)z^n = \frac{1}{1 - \gamma_0z} \frac{\lambda_1z^2}{1 - \gamma_1z} \frac{\lambda_2z^2}{1 - \gamma_2z} \cdots.
\]

**Proof.** It suffices to check \( \gamma_h = y^2D_{h,h} + E_{h,h} \) and \( \lambda_h = (y^2D_{h,h+1} + E_{h,h+1})E_{h+1,h} \). \( \square \)

In particular, we have

\[
\sum_{n \geq 0} B_n(y, 1, -1)z^n = \frac{1}{1 - (y + y^2)z - \frac{1-y(1+y)(1-y)z^2}{1-(1+y)z}} = \frac{1 - z + yz}{1 - z - y^2z} = 1 + \frac{(y + y^2)z}{1 - (1 + y)z}.
\]

It is then easy to obtain

\[
B_n(y, 1, -1) = \sum_{k=1}^{2n} y^k \binom{n - 1}{\lfloor k/2 \rfloor - 1}.
\]

Using (12), we thus obtain as mentioned in Theorem 1.2 that the value at \( q = -1 \) of the \( q \)-Eulerian numbers of type \( B \) are binomial coefficients:

\[
(24) \quad E_{n,k}^B(-1) = \lfloor y^{2k} \rfloor B_n(y, 1, -1) + \lfloor y^{2k+1} \rfloor B_n(y, 1, -1) = \binom{n - 1}{k - 1} \binom{n - 1}{k} = \binom{n}{k}.
\]
5.2. Solution 2. Let \( X = (X_{i,j})_{i,j \geq 0} \) and \( Y = (Y_{i,j})_{i,j \geq 0} \) where
\[
X_{i,i+1} = [i + 1]_q \text{ and } X_{i,j} = 0 \text{ otherwise,} \quad Y_{i+1,i} = 1 \text{ and } Y_{i,j} = 0 \text{ otherwise,}
\]
and
\[
(W) = (1, yt, (yt)^2, \ldots), \quad |V| = (1, 0, 0, \ldots)^{tr}.
\]
We can check that \( XY - qYX = I \), and that \( D = X(I + Y) \) and \( E = YX(I + Y) \) together with \( (W) \text{ and } |V| \) provide a solution of (8). The coefficients are:
\[
D_{i,i} = D_{i,i+1} = [i + 1]_q, \quad D_{i,j} = 0 \text{ otherwise},
\]
\[
E_{i,i-1} = E_{i,i} = [i]_q, \quad E_{i,j} = 0 \text{ otherwise}.
\]

**Theorem 5.2** (Theorem 1.7 in the introduction). We have \( B_0 = 1 \), and the recurrence relation:
\[
(25) \quad B_{n+1}(y, t, q) = (y + t)D_q[(1 + yt)B_n(y, t, q)],
\]
where \( D_q \) is the \( q \)-derivative with respect to \( t \), which sends \( t^n \) to \([n]_q t^n - 1\).

**Proof.** The choice of \( (W) \) leads us to consider a basis for polynomials in \( t \) which is \( 1, yt, (yt)^2, \ldots \). Then \( v \mapsto (W)|v| \) is just the realization of the identification between column vectors and polynomials in \( t \). In this basis, \( Y \) is the matrix of multiplication by \( yt \), and \( X \) is the matrix of \( y^{-1}D_q \). We have:
\[
B_n(y, t, q) = (W)((y^2D + E)^n|V| = (W)((y^2I + Y)(I + Y)^n)|V|
\]
and the result follows. \( \square \)

In this case, seeing \( y^2D + E \) as a transfer matrix shows that \( B_n(y, t, q) \) counts some Motzkin suffixes.

**Definition 3.** A Motzkin suffix of length \( n \) and starting height \( k \) is a path in \( \mathbb{N}^2 \) from \( (0, k) \) to \( (n, 0) \) with steps \((1, 1), (1, 0) \) and \((1, -1)\), respectively denoted \( \nearrow, \rightarrow \) and \( \searrow \). The Motzkin paths are the particular cases where the starting height is 0. We denote by \( \text{sh}(p) \) the starting height of the path \( p \).

Then, expanding \( (W)((y^2D + E)^n|V) \) shows that \( B_n(y, t, q) \) is the generating function of Motzkin suffixes \( p \) with a weight \((yt)^{\text{sh}(p)}\), and weighs \( y^2D_{h,h+1} + E_{h,h} \) for a step \( \nearrow \) from height \( h \) to \( h + 1 \), \( y^2D_{h,h} + E_{h,h} \) for a step \( \rightarrow \) at height \( h \), and \( E_{h+1,h} \) for a step \( \searrow \) from height \( h + 1 \) to \( h \) (see the next section for more on the combinatorics of these paths).}

6. Interpretation of \( B_n(y, t, q) \) via labeled Motzkin paths

We show here that some known bijections from [7] can be adapted to the case of signed permutations. In this reference, the first author obtains refinements of two bijections originally given by Françon and Viennot [13], Foata and Zeilberger [12]. In the case of signed permutations, we will see that each of these two bijections has two variants, corresponding to the two kinds of paths obtained in the previous section.

The two variants of the Françon-Viennot bijection give new interpretations \( B_n(y, t, q) \) in terms ascent-like and pattern-like statistics in signed permutations. On the other side, the two variants of the Foata-Zeilberger bijection permit to recover the known interpretation as in [4] (so we omit details in this case).

6.1. Labeled Motzkin paths. Let \( \mathcal{M}_n \) be the set of weighted Motzkin paths of length \( n \), where each step is either:

- \( \nearrow \) from height \( h \) to \( h + 1 \) with weight \( q^i, 0 \leq i \leq h \) (type 1),
- \( \rightarrow \) from height \( h \) to \( h + 1 \) with weight \( y^i t q^{h+1+i}, 0 \leq i \leq h \) (type 2),
- \( \rightarrow \) at height \( h \) with weight \( y^2q^i, 0 \leq i \leq h \) (type 3),
- \( \rightarrow \) at height \( h \) with weight \( y^i t q^{h+i}, 0 \leq i \leq h \) (type 4),
- \( \rightarrow \) at height \( h \) with weight \( q^i, 0 \leq i \leq h - 1 \) (type 5),
- \( \rightarrow \) at height \( h \) with weight \( y^i t q^{h+i}, 0 \leq i \leq h - 1 \) (type 6),
- \( \searrow \) from height \( h + 1 \) to \( h \) with weight \( y^2q^i, 0 \leq i \leq h \) (type 7),
• \( \downarrow \) from height \( h + 1 \) to \( h \) with weight \( y^i t q^{h+i} \), \( 0 \leq i \leq h \) (type 8).

This set has generating function \( B_\eta(y,t,q) \) since the weights correspond to coefficients of \( y^2D + E \), where we use the first solution of the Matrix Ansatz from the previous section. More precisely, the correspondence is: \( E_{h,h-1} \rightarrow \) types 1 and 2, \( D_{h,h} \rightarrow \) type 3, \( E_{h,h} \rightarrow \) types 4, 5 and 6, \( D_{h,h+1} \rightarrow \) type 7, \( E_{h,h+1}\rightarrow \) type 8. Note that the weight on a step does not always determine its type (compare type 4 and 6), so that we need to think in terms of labeled paths where each step has a label between 1 and 8 to indicate its type.

6.1.1. The Françon-Viennot bijection, first variant. There is a bijection between \( S_n \) and weighted Motzkin paths which follows the number of descents and the number of 31-2 patterns in a permutation \([\mathbb{D}]\). The path is obtained by “scanning” the graph of a permutation from bottom to top, \( i.e. \) the \( i \)-th step is obtained by examining \( \sigma^{-1}(i) \). In the case of a signed permutation \( \pi \), we use the representation as in Figure \([\mathbb{D}]\) we place \( a \) or \( a^{-} \) in the \( i \)-th square in the \( i \)-th column and \( |\pi_i| \)th row depending on the sign of \( \pi(i) \).

![Figure 2](image_url)

**Figure 2.** The first variant the Françon-Viennot bijection, with the signed permutation \( \pi = 3, -5, -2, 4, 1 \).

Definition 4. For any \( \sigma \in S_n \) and \( 1 \leq i \leq n \), let \( 31-2(\sigma,i) = \# \{ j : 1 \leq j < i - 1 \text{ and } \pi_j > \pi_i > \pi_{j+1} \} \), and let \( 231(\sigma,i) = \# \{ j : i < j < n \text{ and } \pi_j > \pi_i > \pi_{j+1} \} \). Let also \( 31-2(\sigma) = \sum_{i=1}^{n} 31-2(\sigma,i) \).

Let \( \pi \in B_n \). We denote by \( |\pi| \) the permutation obtained by removing the negative signs of \( \pi \), that is, \( |\pi_i| = |\pi_i| \) for \( i = 1, 2, \ldots, n \). We take the convention that \( \pi_0 = 0 \) and \( \pi_{n+1} = n + 1 \). The bijection is defined in the following way. The path corresponding to \( \pi \) is of length \( n \) such that, if \( j = |\pi_i| \) and denoting \( s \) the \( j \)-th step, we have:

- If \( |\pi_{i-1}| > |\pi_i| < |\pi_{i+1}| \) and \( \pi_i > 0 \), then \( s \) is of type 1 with weight \( q^{31-2(|\pi|,i)} \).
- If \( |\pi_{i-1}| > |\pi_i| < |\pi_{i+1}| \) and \( \pi_i < 0 \), then \( s \) is of type 2 with weight \( q^{231(|\pi|,i)} \).
- If \( |\pi_{i-1}| < |\pi_i| < |\pi_{i+1}| \) and \( \pi_i > 0 \), then \( s \) is of type 3 with weight \( y^{2}q^{31-2(|\pi|,i)} \).
- If \( |\pi_{i-1}| < |\pi_i| < |\pi_{i+1}| \) and \( \pi_i < 0 \), then \( s \) is of type 4 with weight \( q^{431(|\pi|,i)} \).
- If \( |\pi_{i-1}| > |\pi_i| > |\pi_{i+1}| \) and \( \pi_i > 0 \), then \( s \) is of type 5 with weight \( q^{31-2(|\pi|,i)} \).
- If \( |\pi_{i-1}| > |\pi_i| > |\pi_{i+1}| \) and \( \pi_i < 0 \), then \( s \) is of type 6 with weight \( q^{31-2(|\pi|,i)} \).
- If \( |\pi_{i-1}| < |\pi_i| > |\pi_{i+1}| \) and \( \pi_i > 0 \), then \( s \) is of type 7 with weight \( y^2q^{31-2(|\pi|,i)} \).
- If \( |\pi_{i-1}| < |\pi_i| > |\pi_{i+1}| \) and \( \pi_i < 0 \), then \( s \) is of type 8 with weight \( y^2q^{31-2(|\pi|,i)} \).

In the case where \( \pi \) has no negative entry, this defines a bijection with the paths having steps of types 1, 3, 5, 7 only. This is a result from \([\mathbb{D}]\), and what we present here is a variant, so we refer to this work for more details. In the case of signed permutations, since each entry can be negated, it is natural that each step of type (1, 3, 5, or 7) has a respective variant (type 2, 4, 6, or 8). So the fact that the map is a bijection can be deduced from the case of unsigned permutations. It is also clear that \( t \) follows the number of negative entries of \( \pi \) since there is a factor \( t \) only in steps of type 2, 4, 6, and 8.

Let us check what is the statistic followed by \( y \). There is a factor \( y \) on each step of type 2, 4, 6, 8 so that this statistic is the sum of \( \text{neg}(\pi) \) and other terms coming from the factors \( y^2 \) in steps of type 3 and 7. Note that the \( j \)-th step is of type 3 or 7 if and only if \( |\pi_{i-1}| < |\pi_i| \). This leads us to define an ascent statistic \( \text{hasc}(\pi) \) as

\[
\text{hasc}(\pi) = 2 \times \# \{ i \colon 0 \leq i \leq n-1 \text{ and } |\pi_i| < |\pi_{i+1}| \} + \text{neg}(\pi).
\]
It remains only to check what is the statistics followed by \( q \). Since there is always a weight \( q^{31-2(|\pi|, i)} \) on the \( i \)th step, this statistic is the sum of 31-2(|\( \pi \)|) and other terms. It remains to take into account the weights \( q^h \) or \( q^{h+1} \) that appear on each step of type 2,4,6,8. From the properties of the bijection in the unsigned case, we have that the “minimal” height \( h \) of the \( i \)th step is 31-2(|\( \pi \)|, \( i \)) + 2-31(|\( \pi \)|, \( i \)), plus 1 in the case where \( |\pi_i-1| > |\pi_i| > |\pi_{i+1}| \). So, apart a factor \( q \) on each step of type 2 and 6, we obtain the statistic

\[
31-2(|\pi|) + \sum_{1 \leq i < n} \left( 31-2(|\pi|, i) + 2-31(|\pi|, i) \right).
\]

Note that to take into account the factor \( q \) on each step of type 2 and 6, we can count \( i \) such that \( 1 \leq i < n \) and \( |\pi_{i-1}| > -\pi_i > 0 \). The statistic we eventually obtain can be rearranged as follows:

\[
\text{pat}(\pi) = \# \{ (i, j) : 1 \leq i < j \leq n, \text{ and } |\pi_i| > |\pi_j| > |\pi_{i+1}| \} + \\
\# \{ (i, j) : 1 \leq i, j \leq n, \text{ and } |\pi_i| > -\pi_j \geq |\pi_{i+1}| \}.
\]

This is a “pattern” statistic that is somewhat similar to the 31-2 statistic of the unsigned case (and indeed identical if \( \pi \) has only positive entries). Eventually, our first variant of the Françon-Viennot bijection gives the following.

**Proposition 6.1.** (First equality of Theorem [13] in the introduction.)

\[
B_n(y, t, q) = \sum_{\pi \in B_n} y^{\text{hasc}(\pi)} t^{\text{neg}(\pi)} q^{\text{pat}(\pi)}.
\]

Note that in the case \( t = 0 \), we recover the known result:

\[
\sum_{k=1}^{n} y^k E_{n,k}(q) = \sum_{\sigma \in S_n} y^{\text{asc}(\sigma)+1} q^{31-2(\sigma)}
\]

where \( \text{asc}(\sigma) = n - 1 - \text{des}(\sigma) = \# \{ i : 1 \leq i \leq n - 1 \text{ and } \sigma_i < \sigma_{i+1} \} \).

6.1.2. *The Foata-Zeilberger bijection, first variant.* In case of unsigned permutations, this bijection follows the number of weak excedances and crossings, see the bijection \( \Psi_{FZ} \) in [7]. To extend it, we use a representation of a signed permutation as an arrow diagram (this is equivalent to the pignose diagrams used earlier: if \( \sigma \in S_n \), we put \( n \) dots in a row, and draw an arrow from \( i \) to \( \sigma(i) \) which is above the axis if \( i \leq \sigma(i) \) and below otherwise, see [7]). The idea is to draw the arrow diagram of \( |\pi| \) and label an arrow from \( i \) to \( |\pi(i)| \) by \( + \) or \( - \) depending on the sign of \( \pi(i) \), see Figure 3.

![Figure 3](image)

**Figure 3.** The first variant of the Foata-Zeilberger bijection, with \( \pi = 3, -4, -2, 5, 1 \).

The crossings can be read in this representation as follows. The proof can be done by distinguishing all the possible cases for the signs of \( i \) and \( j \) for each type of crossing \((i,j)\). We omit details.

**Proposition 6.2.** Each crossing \((i,j)\) in the signed permutation \( \pi \) corresponds to one of the six configurations in Figure 4 where \( \pm \) means that the label of the arrow can be either \( + \) or \( - \), and the dots indicate that there might be an equality of the endpoint of an arrow and the startpoint of the other arrow.
For example, let us consider the signed permutation in Figure 3. The dots 1,2,3 (respectively, 1,2,5 and 1,2,3,5) correspond to a crossing as in the first (respectively, second and fourth) configuration of Figure 4. The dots 2,4 are also a crossing as in (the limit case of) the first configuration, as if the step \( \rightarrow \) have weight 1 and the facing step \( \rightarrow \) has weight \((a,b)\). But we still consider the steps \( \to \) of types 3, 4, 5, 6 as before.

Once we have this new description of the crossings, it is possible to encode the arrow diagram as in Figure 3 by elements in \( \mathcal{M}_n \). Actually it will be more convenient to see these paths in a slightly different way: each step \( \rightarrow \) (with weight \( a \)) faces a step \( \rightarrow \) (weight \( b \)), but now we think as if the step \( \rightarrow \) have weight 1 and the facing step \( \rightarrow \) has weight \((a,b)\). But we still consider the steps \( \to \) of types 3, 4, 5, 6 as before.

Now, the path is obtained from the signed permutation by “scanning” the arrow diagram from right to left, so that after scanning \( i \) nodes in the diagram we have built a Motzkin suffix of length \( i \). Suppose that there are \( h \) unconnected strands after scanning these \( i \) nodes, and accordingly the Motzkin suffix starts at height \( h \). When scanning the following node, we have several possibilities:

- If the \( i \)th node is \( \rightarrow \) then we add a step \( \rightarrow \) (with weight 1) to the Motzkin suffix,
- If the \( i \)th node is \( \dowarrow \) or \( \Uparrow \), then we add a step \( \to \) (type 3) with weight \( y^2 q^i \) where \( i \) counts the number of crossings as in the 5th configuration that appear when adding this \( i \)th node to the ones at its right.
- If the \( i \)th node is \( \dowarrow \) then we add a step \( \to \) (type 4) with weight \( y t \) to the Motzkin suffix, and if it is \( \Uparrow \) then we add a step \( \to \) (type 4) with weight \( y t q^{h+i} \), where \( i \) counts the number of crossings as in 3rd configuration that appear when adding this \( i \)th node to the ones to its right. Note that \( h \) is the number of crossings as in 2nd configuration that appear.
- If the \( i \)th node is \( \Uparrow \), then we add a step \( \rightarrow \) (type 5) with weight \( q^i \) where \( i \) counts the number of crossings as in the 6th configuration that appear.
- If the \( i \)th node is \( \dowarrow \), then we add a step \( \to \) (type 6) with weight \( y t q^{h+i} \) where \( i \) counts the number of crossings as in 4th configuration that appear. Note that \( h \) is the number of crossings as in 1st configuration that appear.
- If the \( i \)th node is \( \rightarrow \), then we add a step \( \rightarrow \) with a weight \((a,b)\) where \( a \) and \( b \) are as follows (this is similar to the previous cases but here we need to encode information about both the ingoing and outgoing arrow). The possibilities for \( a \) are \( q^0, \ldots, q^{h-1} \), or \( y t q^h, \ldots, y t q^{2h-1} \), such that there is a factor 1 (resp. \( y t \)) if the label of the ingoing arrow is + (resp. −), and \( q \) counts the crossings of the 6th (resp. 1st and 4th) configuration. The possibilities for \( b \) are \( y^2 q^0, \ldots, y^2 q^{h-1} \), or \( y t q^{h-1}, \ldots, y t q^{2h-2} \), such that there is a factor \( y^2 \) (resp. \( y t \)) if the label of the outgoing arrow is + (resp. −), and \( q \) counts the crossings of the 5th (resp. 2nd and 3rd) configuration.

See Figure 3 for an example. Let \( i < j \), if there is an arrow from \( i \) to \( j \) with a label + then the \( i \)th step gets a weight \( y^2 \), and if there is an arrow from \( i \) to \( j \) or from \( j \) to \( i \) with a label −, then the \( i \)th step gets a weight \( y t \). It follows that the parameters \( y \) and \( t \) correspond to \( \text{fwex}(\pi) \) and \( \text{neg}(\pi) \) in signed permutations. By the design of the bijection, the parameter \( q \) corresponds to \( \text{cr}(\pi) \). Hence we recover [4].
6.2. Suffixes of labeled Motzkin paths. We can see a signed permutation as a permutation on \([±n]\) or on \([2n]\) with a symmetry property. Then, applying the bijections of the unsigned case gives some weighted Motzkin paths with a vertical symmetry. It is natural to keep only the second half a vertically-symmetric path, and obtain suffixes of Motzkin paths.

**Definition 5.** Let \(\mathcal{N}_n\) be the set of weighted Motzkin suffixes with weights:
- either \(y^2 q^i\) with \(0 \leq i \leq h\), or \(q^i\) with \(0 \leq i \leq h - 1\), for a step \(\to\) at height \(h\),
- \(y^2 q^i\) with \(0 \leq i \leq h\), for a step \(\nearrow\) at height \(h\) to \(h + 1\),
- \(q^i\) with \(0 \leq i \leq h\), for a step \(\searrow\) at height \(h + 1\) to \(h\).

For any \(p \in \mathcal{N}_n\), let \(sh(p)\) be its initial height, and let \(w(p)\) be its total weight, i.e. the product of the weight of each step.

Then \(B_n(y, t, q)\) is the generating function \(\sum_{p \in \mathcal{N}_n} (yt)^{sh(p)} w(p)\), because the paths are the ones arising from the second solution of the Matrix Ansatz in the previous section.

6.2.1. The Françon-Viennot bijection, second variant. This second variant is a bijection between \(\mathcal{N}_n\) and \(B_n\), and it gives a combinatorial model of \(B_n(y, t, q)\) involving the flag descents, and different from the previous ones. The second bijection is done using the diagram of a signed permutation as in Figure 5.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
-5 & -4 & -3 & -2 & -1 & 1 \\
\hline
5 & 4 & 3 & 2 & 1 & 2 & 3 & 4 & 5 \\
\hline
\end{array}
\]

**Figure 5.** The second variant of the Françon-Viennot bijection, with \(\pi = 3, -5, -2, 4, 1\)

**Definition 6.** For a signed permutation \(\pi\), let \(f_{\text{neg}}(\pi)\) be the number of positive integers followed by a negative integer in the sequence \(\pi(-n), \ldots, \pi(-1), \pi(1), \ldots, \pi(n)\). Note that \(f_{\text{neg}}(\pi) = 0\) if and only if \(\pi\) is actually an unsigned permutation. Let

\[
31-2^+ (\pi) = \sum_{1 \leq i \leq 2n, \tilde{\pi}(i) > n} 31-2(\tilde{\pi}, i)
\]

where \(\tilde{\pi} \in S_{2n}\) is the permutation corresponding to \(\pi\) via the order-preserving identification \([±n] \to \{1, \ldots, 2n\}\).

The properties of the Françon-Viennot bijection show the following.

**Proposition 6.3** (Second equality of Theorem 1.9 in the introduction). For \(n \geq 1\),

\[
(29) \qquad B_n(y, t, q) = \sum_{\pi \in B_n} y^{\text{des}(\pi) + 1} t^{f_{\text{neg}}(\pi)} q^{31-2^+(\pi)}.
\]

Note that once again, we recover (28) in the case \(t = 0\).
6.2.2. The Foata-Zeilberger bijection, second variant. Let \( \pi \in B_n \), we use here the arrow diagram as in Figure 6, where the dots are labeled by \(-n, \ldots, -1, 1, \ldots, n\) and there is an arrow from \(i\) to \(\pi(i)\) above the axis if \(i \leq \pi(i)\) and below otherwise (this is equivalent to the full pignoze diagram used in Section 4 where each pignoze collapses to a single dot). The path \(p \in \mathcal{N}_n\) is as follows. If at the \(i\)th node in the diagram (where \(1 \leq i \leq n\)), there is an arrow arriving from the left and an arrow going to the left, then the \(i\)th step is \(\searrow\). If there is an arrow going to the right and arriving from the right, then it is a step \(\nearrow\). In all other cases it is a step \(\to\). Then, for each arrow going from \(i\) to \(j\) with \(0 < i < j\), we give a weight \(y^2\) to the \(i\)th step in the path; for each crossing \(i < j \leq \pi(i) < \pi(j)\) with \(j > 0\) we give a weight \(q\) to the \(j\)th step; and for each crossing \(i > j > \pi(i) > \pi(j)\) with \(j > 0\) we give a weight \(q\) to the \(j\)th step. See Figure 6 for example.

![Figure 6](https://via.placeholder.com/150)

**Figure 6.** The second variant of the Foata-Zeilberger bijection in the case of \(\pi = -5, 4, 2, -3, 1\).

The number of arrows that join a positive integer to a negative one is \(\text{neg}(\pi)\). From the construction, this is also the initial height of the path we have built. So giving a weight \((yt)^{\text{sh}(p)}\) to the path \(p\) ensures that \(y\) and \(t\) respectively follow \(\text{fwex}(\pi)\) and \(\text{neg}(\pi)\). Also from the definition of the bijection, \(q\) counts the number of crossings. Hence we recover (31).

7. Enumeration formula

In this section, we prove a formula for \(B_n(y, 0, q)\). The formula itself is somewhat similar to the following one for \(B_n(y, 1, q)\) proved in [15] but the proof is different:

\[
B_n(y, 0, q) = \frac{1}{(1-q)^n} \sum_{k=0}^{n} \left( \sum_{j=0}^{n-k} y^j \left( \binom{n}{j+k} - \binom{n}{j-1} \binom{n}{j+k+1} \right) \right) \left( \sum_{i=0}^{k} (-1)^i q^i (k+1-i) \right).
\]

**Theorem 7.1** (Theorem 1.10 in the introduction).

\[
B_n(y, 1, q) = \frac{1}{(1-q)^n} \sum_{j=0}^{n} (-1)^j \left( \sum_{i=0}^{2n-2j} y^i \left( \binom{n}{j+\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor} \right) \right) \left( \sum_{\ell=0}^{2j} y^\ell q^{\ell(2j-\ell+1)} \right).
\]

Or, equivalently:

\[
B_n(y, 1, q) = \frac{1}{(1-q)^n} \sum_{k=0}^{n} \sum_{i=0}^{n-k} y^{2i} \left( \binom{n}{i+k} - \binom{n}{i-1} \binom{n}{i+k+1} \right) \left( \sum_{j=0}^{k-j} q^{2j} \right) \left( \sum_{\ell=0}^{2j} y^\ell q^{\ell(2j-\ell+1)} \right).
\]

We can obtain (31) from (32) by simplifying a summation as follows. Let \(P_j = \sum_{\ell=0}^{2j} y^\ell q^{\ell(2j-\ell+1)}\), then the right-hand side of (32) is

\[
\frac{1}{(1-q)^n} \sum_{j=0}^{n} (-1)^j P_j \sum_{k=j}^{n-2j} y^{k-j+2i} \left( \binom{n}{i+k} - \binom{n}{i-1} \binom{n}{i+k+1} \right) \left( \sum_{m=0}^{[m/2]} y^m \sum_{i=0}^{[m/2]} \left( \binom{n}{i} \binom{n}{m+j-i} - \binom{n}{i-1} \binom{n}{m+j-i+1} \right) \right).
\]

Here we have introduced the new index \(m = 2i + k - j\), and the condition \(k - j \geq 0\) gives the condition \(m \geq 2i\), i.e. \(i \leq \left\lfloor m/2 \right\rfloor\). But the \(i\)-sum in the latter formula is actually telescopic and only the term \(\binom{n}{i} \binom{n}{m+j-i} \) with \(i = \left\lfloor m/2 \right\rfloor\) remains. Using the fact that \(m - \left\lfloor m/2 \right\rfloor = \left\lfloor m/2 \right\rfloor\), we obtain (31).
So, (31) is a simpler formula, but (32) is the one which is conveniently proved, using results from [16]. The theorem follows from the two lemmas below (and a third lemma is needed to prove the second lemma). The first lemma was essentially present in [17].

**Lemma 7.2.** Suppose that two sequences \((b_n)_{n \geq 0}\) and \((c_n)_{n \geq 0}\) are such that:

\[
\sum_{n \geq 0} b_n z^n = \frac{1}{1 - \gamma_0 z} - \frac{\lambda_1 z^2}{1 - \gamma_1 z - \gamma_2 z - \ldots}
\]

and

\[
\sum_{n \geq 0} c_n z^n = \frac{1}{(1 + z)(1 + y^2 z) - \gamma_0 z} - \frac{\lambda_1 z^2}{(1 + z)(1 + y^2 z) - \gamma_1 z} - \frac{\lambda_2 z^2}{(1 + z)(1 + y^2 z) - \gamma_2 z - \ldots}
\]

Then we have:

\[
b_n = \sum_{k=0}^{n} \left( \sum_{j=0}^{n-k} y^{2j} \binom{n}{j+k} - \binom{n}{j-1} \binom{n}{j+k+1} \right) c_k.
\]

**Proof.** Let \(f(z)\) and \(g(z)\) respectively denote the generating functions of \((b_n)_{n \geq 0}\) and \((c_n)_{n \geq 0}\) as in Equations (33) and (34). Divide by \((1 + z)(1 + y^2 z)\) the numerator and denominator of each fraction in (34), this gives an equivalence of continued fractions so that:

\[
z g(z) = \frac{z}{(1 + z)(1 + y^2 z)} f \left( \frac{z}{(1 + z)(1 + y^2 z)} \right).
\]

It follows that

\[
z f(z) = C(z) g(C(z))
\]

where \(C(z)\) is the compositional inverse of \(\frac{z}{(1 + z)(1 + y^2 z)}\). It remains to show that

\[
C(z)^{k+1} = \sum_{n \geq 0} \left( \sum_{j=0}^{n-k} y^{2j} \binom{n}{j+k} - \binom{n}{j-1} \binom{n}{j+k+1} \right) z^{n+1}.
\]

Indeed, the lemma follows from taking the coefficient of \(z^{n+1}\) in both sides of (36) and using (34) to simplify the right-hand side. Showing (36) can be conveniently done using Lagrange inversion. For example, with [15] p.148, Theorem A, we obtain

\[
[z^{n+1}]C(z)^{k+1} = \frac{k+1}{n+1} [z^{n-k}][(1 + z)(1 + y^2 z)]^{n+1} = \frac{k+1}{n+1} \sum_{j=0}^{n-k} y^{2j} \binom{n+1}{j} \binom{n+1}{n-k-j},
\]

and it is straightforward to check that

\[
\binom{k+1}{n+1} \binom{n+1}{n-k-j} = \binom{n}{j} \binom{n}{j+k} - \binom{n}{j-1} \binom{n}{j+k+1}.
\]

This completes the proof.

We can apply this lemma with \(b_n = (1 - q)^n B_n(y, 1, q)\). The continued fraction expansion of \(\sum b_n z^n\) is immediately obtained from the one of \(\sum B_n(y, t, q) z^n\) obtained in Theorem 1.8. More precisely, it is exactly Equation (33) with the values

\[
\gamma_n = y^2 (1 - q^{h+1}) + (1 - q^h) + yq^h (2 - q^{h+1} - q^h),
\]

\[
\lambda_n = y (1 - q^h) (y + q^{h-1}) (1 + yq^h).
\]

The theorem is now a consequence of another lemma which gives the value of \(c_k\).

**Lemma 7.3.** With \(\gamma_n\) and \(\lambda_n\) as in (37), we have:

\[
\frac{1}{(1 + z)(1 + y^2 z) - \gamma_0 z} - \frac{\lambda_1 z^2}{(1 + z)(1 + y^2 z) - \gamma_1 z} - \frac{\lambda_2 z^2}{(1 + z)(1 + y^2 z) - \gamma_2 z - \ldots}
\]

\[
= \sum_{k \geq 0} z^k \sum_{j=0}^{k} (-1)^j \sum_{i=0}^{j} y^{i} q^{(2i-j+1)/2}.
\]
Proof. After multiplying by $1 - yz$, this is equivalent to
\begin{equation}
\frac{1 - yz}{(1 + z)(1 + y^2z) - \gamma_0 z - (1 + z)(1 + y^2z) - \gamma_1 z - (1 + z)(1 + y^2z) - \gamma_2 z - \cdots} = \sum_{j \geq 0} (-z)^j \sum_{i=0}^{2j} y^i q^i(2j-i+1)/2.
\end{equation}

A continued fraction expansion of the right-hand side is obtained in the work of the second and third authors [16]. More precisely, the substitution $(z, y, q) \mapsto (-yz, yq^2, q^2)$ in [16, Theorem 7.1] gives
\begin{equation}
\sum_{j \geq 0} (-z)^j \sum_{i=0}^{2j} y^i q^i(2j-i+1)/2 = \frac{1 - dz}{1 - yz + 1 - yz + 1 - yz + 1 - yz + \cdots},
\end{equation}
where $d_{2h+1} = (1 + yq^{h+1})(y + q^h)$ and $d_{2h} = (1 - q^h)^2$. It is immediate to check the relations
\begin{equation}
\lambda_h = d_{2h-1}d_{2h}, \quad \gamma_h = (1 + y)^2 - d_{2h} - d_{2h+1}
\end{equation}
(with the convention $d_0 = 0$). It remains to identify the left-hand side of (38) with the right-hand side of (39). The end of the proof follows from the lemma below.

Lemma 7.4. Let $(\gamma_h)_{h \geq 0}$, $(\lambda_h)_{h \geq 1}$ and $(d_h)_{h \geq 0}$ be three sequences satisfying (40) with $d_0 = 0$. Then we have:
\begin{equation}
\frac{1 - yz}{(1 + z)(1 + y^2z) - \gamma_0 z - (1 + z)(1 + y^2z) - \gamma_1 z - (1 + z)(1 + y^2z) - \gamma_2 z - \cdots} = \frac{1}{1 - yz + 1 - yz + 1 - yz + 1 - yz + \cdots}.
\end{equation}

Proof. In the left-hand side, divide the numerator and denominator of each fraction by $1 - yz$. Using that
$$\frac{(1 + z)(1 + y^2z)}{1 - yz} = 1 - yz + \frac{z(1 + y)^2}{1 - yz},$$
the left-hand side of (41) is
$$\frac{1 - yz + ((1 + y)^2 - \gamma_0)\frac{z}{1 - yz}}{1 - yz + ((1 + y)^2 - \gamma_1)\frac{z}{1 - yz}} - \cdots,$$
hence it is equal to
\begin{equation}
\frac{1}{1 - yz + d_1 \frac{z}{1 - yz}} - \frac{d_1 d_2 \left(\frac{z}{1 - yz}\right)^2}{1 - yz + (d_2 + d_3)\frac{z}{1 - yz}} - \frac{d_3 d_4 \left(\frac{z}{1 - yz}\right)^2}{1 - yz + (d_4 + d_5)\frac{z}{1 - yz}} - \cdots.
\end{equation}

This can be shown to be equal to the right-hand side of (41), using the combinatorics of weighted Schröder paths. A Schröder path (of length $2n$) is a path from $(0, 0)$ to $(2n, 0)$ in $\mathbb{N}^2$ with steps $(1, 1)$, $(1, -1)$ and $(2, 0)$, respectively denoted by $\nearrow$, $\searrow$, $\rightarrow$. We set that each step $\rightarrow$ has a weight $y$, each step $\nearrow$ from height $h - 1$ to $h$ has weight $-d_h$. Then by standard methods, the weighted generating function of all Schröder paths is the continued fraction in the right-hand side of (41). By counting differently, we can obtain (42). The idea is to split each Schröder path into some subpaths, by putting a splitting point each time the path arrives at even height. This way, we can see a Schröder path as an ordered sequence of:
- A subsequence $\nearrow \rightarrow \ldots \rightarrow \searrow$ starting at height $2h$ whose generating function is $-d_{2h+1} \frac{z}{1 - yz}$.
- A subsequence $\searrow \rightarrow \ldots \rightarrow \nearrow$ starting at height $2h$ whose generating function is $-d_{2h+1} \frac{z}{1 - yz}$.
- A subsequence $\nearrow \rightarrow \ldots \rightarrow \searrow$ starting at height $2h$ whose generating function is $d_{2h+1} d_{2h+2} \frac{z}{1 - yz}$.
- A subsequence $\searrow \rightarrow \ldots \rightarrow \nearrow$ starting at height $2h$ whose generating function is $\frac{z}{1 - yz}$.
- A step $\rightarrow$ at height $2h$ whose generating function is $yz$. 

The rules according to which these subsequences can be put together is conveniently encoded in a continued fraction so that we obtain exactly (42) . More precisely, let $F_h$ be the generating function of Schröder paths from height $2h$ to $2h$ and staying at height $\geq 2h-1$ . We have:

$$F_h = \frac{1}{1 - yz + (d_{2h} + d_{2h+1})\frac{y}{1-yz} - d_{2h+1}d_{2h+2}\left(\frac{y}{1-yz}\right)^2F_{h+1}} .$$

Indeed, we can see a Schröder paths from height $2h$ to $2h$ and staying at height $\geq 2h-1$ as an ordered sequence of:

- steps $\rightarrow$ at height $2h$ whose generating function is $yz$,
- subsequences $\overset{\rightarrow}{\ldots} \rightarrow \overset{\rightarrow}{\ldots}$ starting at height $2h$ whose generating function is $-d_{2h+1}\frac{y}{1-yz}$,
- subsequences $\overset{\rightarrow}{\ldots} \rightarrow \overset{\rightarrow}{\ldots}$ starting at height $2h$ whose generating function is $-d_{2h}\frac{y}{1-yz}$,
- subsequences $\overset{\rightarrow}{\ldots} \rightarrow \overset{\rightarrow}{\ldots} \overset{\rightarrow}{\ldots} \rightarrow \overset{\rightarrow}{\ldots}$ where $P$ is a path from height $2h+2$ to $2h+2$ staying above height $2h+1$ whose generating function is $d_{2h+1}d_{2h+2}\left(\frac{y}{1-yz}\right)^2F_{h+1}$.

Hence we obtain (43) . Using (43) for successive values of $h$ , we obtain that $F_0$ is the continued fraction in (42) . This completes the proof.

Let us examine the case $q = 0$ in the formula (41) . We obtain:

$$B_n(y,1,0) = \sum_{k=0}^{n} (-1)^k y^i \binom{n}{k+i/2} \binom{n}{i/2} = \sum_{i=0}^{2n} y^i \binom{n}{i/2} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k+i/2} .$$

The alternating sum of binomial coefficients is itself a binomial coefficient, and we obtain:

$$B_n(y,1,0) = \sum_{i=1}^{2n} y^i \binom{n}{i/2} \left( \frac{n+1}{i/2} \right) .$$

We thus obtain, as mentioned in Theorem 1.2 that the value at $q = 0$ of the $q$-Eulerian numbers of type $B$ are the Narayana numbers of type $B$ :

$$E_{n,k}^B(0) = [y^{2k}]B_n(y,1,0) + [y^{2k+1}]B_n(y,1,0) = \binom{n}{k} \binom{n-1}{k-1} + \binom{n}{k} \binom{n-1}{k} = \binom{n}{k}^2 .$$

Remark 2 . One can prove the identity $E_{n,k}^B(0) = \binom{n}{k}^2$ bijectively as follows . Considering $\pi \in B_n$ as a permutation on $\pm n = \{ \pm 1, \pm 2, \ldots , \pm n \}$ , define $f(\pi)$ to be the partition of $\pm n$ obtained by making cycles of $\pi$ into blocks . It is not difficult to show that the map $f$ is a bijection from the set of $\pi \in B_n$ with $\text{cr}(\pi) = 0$ to the set of type $B$ noncrossing partitions of $\pm n$ such that if $\lfloor \text{fwex}(\pi)/2 \rfloor = k$ then $f(\pi)$ has $2k$ nonzero blocks . It is well known that $\binom{n}{k}^2$ is the number of type $B$ noncrossing partitions of $\pm n$ with $2k$ nonzero blocks , see [23] .

8. Open problems

We conclude this paper by a list of open problems.

**Problem 1 .** Since the introduction of permutation tableaux in [29, 26] , several variants have been defined [2, 3, 28] . A nice feature of these variants is that the permutation statistics arise naturally, from a recursive construction of the tableaux via an insertion algorithm [3] . The type $B$ version of these tableaux can be defined with the condition of being conjugate-symmetric. A natural question is to check whether the insertion algorithm can be used to recover some of our results.

**Problem 2 .** One key feature of our new $q$-Eulerian polynomials of type $B$ is their symmetry, i.e. we have $B_{n,k}^B(t,q) = B_{n,2n+1-k}^B(t,q)$ . We prove this symmetry using the pignose diagram of a signed permutation. It would be interesting to show this symmetry using the permutation tableaux of type $B$.

**Problem 3 .** We have defined alignments in Section 4 and showed that for a signed permutation $\pi$ with $\text{fwex}(\pi) = k$, we have $2\text{cr}(\pi) + \text{al}(\pi) = n^2 - 2n + k$ . A similar identity exists for
the type $A$, see Proposition [4] and can be shown on permutations or directly on permutation tableaux. It would be elegant to show our identity directly on the permutation tableaux of type $B$.

**Problem 4.** A notion that is closely related to alignments and in some sense dual to the crossing, is the one of *nesting* [7]. When we introduce a parameter $p$ counting the number of crossings in permutations, there are continued fractions containing $p,q$-integers rather than the $q$-integers, see [7, 24]. A definition of nestings in signed permutations have been given by Hamdi [14]. It would be interesting to check if our results can be generalized to take into account these nestings.

**Problem 5.** In the last section, we have obtained a formula for $B_n(y,1,q)$. We can ask if there is a more general formula for $B_n(y,t,q)$, but it seems that the present methods do not generalize in this case.

**Problem 6.** Recently Kim and Stanton [18] gave a combinatorial proof of the formula (30) for $B_n(y,0,q)$, which is a generating function for type $A$ permutation tableaux. It is worth asking whether this combinatorial approach can be generalized for $B_n(y,1,q)$ and possibly $B_n(y,t,q)$.

**Acknowledgements.**

We thank Philippe Nadeau and Lauren Williams for constructive discussions during the elaboration of this work.

**References**

[1] R.M. Adin, F. Brenti and Y. Roichman. Descent numbers and major indices for the hyperoctahedral group. *Adv. in Appl. Math.*, 27(2-3):210–224, 2001; Special issue in honor of Dominique Foata’s 65th birthday (Philadelphia, PA, 2000).

[2] J.-C. Aval, A. Boussicault and S. Dasse-Hartaut. Dyck tableaux. *Preprint*, 2011.

[3] J.-C. Aval, A. Boussicault and P. Nadeau. Tree-like tableaux. *Proceedings of FPSAC2011*, 2011.

[4] A. Björner and F. Brenti. Combinatorics of Coxeter groups. Springer, 2005.

[5] L. Carlitz. A combinatorial property of $q$-Eulerian numbers. *Amer. Math. Monthly*, 82:51–54, 1975.

[6] L. Comtet. Advanced combinatorics. Reidel Publishing Company, 1974.

[7] S. Corteel, Crossings and alignments of permutations. *Adv. in Appl. Math.*, 38(2):149–163, 2007.

[8] S. Corteel, M. Josuat-Verges and L.K. Williams. Matrix Ansatz, orthogonal polynomials and permutations. *Adv. in Appl. Math.*, 46:209–225, 2011.

[9] S. Corteel and J.S. Kim. Combinatorics on permutation tableaux of type A and type B. *European J. Combin.*, 32(4):563–579, 2011.

[10] S. Corteel and P. Nadeau. Bijectios for permutation tableaux. *European J. Combin.*, 30:295–300, 2009.

[11] D. Foata and G. Han. Signed words and permutations, V; a sextuple distribution. *Ramanujan J.*, 19(1):29–52, 2009.

[12] D. Foata and D. Zeilberger. Denert’s permutation statistic is indeed Euler-Mahonian. *Stud. Appl. Math.*, 83(1):31–59, 1990.

[13] J. Françon and X.G. Viennot. Permutations selon leurs pics, creux, doubles montées et double descentes, nombres d’Euler et nombres de Genocchi. *Discrete Math.*, 28(1):21–35, 1979.

[14] A. Hamdi. Symmetric Distribution of Crossings and Nestings in Permutations of Type B. *Electr. J. Comb.*, 18(1): Article P200, 2011.

[15] M. Josuat-Vergès. Rooks placements in Young diagrams and permutation enumeration. *Adv. in Appl. Math.* 47:1–22, 2011.

[16] M. Josuat-Vergès and J.S. Kim. Touchard-Riordan formulas, T-fractions, and Jacobi’s triple product identity. *Proceedings of FPSAC2011*, 2011.

[17] M. Josuat-Vergès and M. Rubey. Crossings, Motzkin paths, and moments. *Discrete Math.*, 311:2064–2078, 2011.

[18] J.S. Kim and D. Stanton. On enumeration formulas for weighted Motzkin paths. *in preparation*.

[19] T. Lam and L.K. Williams. Total positivity for cominuscule Grassmannians. *New York J. of Math.*, 14:53–99, 2008.

[20] P. Leroux and X.G. Viennot. Combinatorial resolution of systems of differential equations, I. Ordinary differential equations. In “Combinatoire énumérative”, eds. G. Labelle and P. Leroux, Lecture Notes in Maths. 1234, Springer-Verlag, Berlin, 1986, pp. 210–245.

[21] Lothaire. Combinatorics on words. Addison–Wesley, 1983.

[22] A. de Médicis and X.G. Viennot. Moments des $q$-polynômes de Laguerre et la bijection de Foata-Zeilberger. *Adv. in Appl. Math.*, 15(3):262–304, 1994.
[23] V. Reiner. Non-crossing partitions for classical reflection groups. *Discrete Math.*, 177:195–222, 1997.

[24] H. Shin and J. Zeng. The symmetric and unimodal expansion of Eulerian polynomials via continued fractions. *European J. Combin.*, 33(2):111–127, 2012.

[25] R.P. Stanley. *Enumerative Combinatorics. Vol. 1. second edition (version of 15 July 2011).* to be published in Cambridge University Press, 2011.

[26] E. Steingrímsson and L.K. Williams. Permutation tableaux and permutation patterns. *J. Combin. Theory Ser. A*, 114:211–234, 2007.

[27] X.G. Viennot. Une théorie combinatoire des polynômes orthogonaux. Lecture notes, UQÀM, Montréal, 1984.

[28] X.G. Viennot. Alternative tableaux and partially asymmetric exclusion process. Talk in Isaac Newton institute, April 2008.

[29] L.K. Williams. Enumeration of totally positive Grassmann cells. *Adv. Math.*, 190:319–342, 2005.

LIAFA, CNRS and Université Paris-Diderot, Case 7014, 75205 Paris Cedex 13, FRANCE  
E-mail address: corteel@liafa.jussieu.fr

Institut Gaspard Monge, CNRS and Université de Marne-la-Vallée, FRANCE  
E-mail address: matthieu.josuat-verges@univ-mlv.fr

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455, USA  
E-mail address: kimjs@math.umn.edu