Baryons in O(4) and Vibron Model

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The structure of the reported excitation spectra of the light unflavored baryons is described in terms of multi-spin valued Lorentz group representations of the so called Rarita-Schwinger (RS) type $\left(\frac{3}{2}, \frac{3}{2}\right) \otimes \left[\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\right]$ with $K = 1, 3, \text{and } 5$. We first motivate legitimacy of such pattern as fundamental fields as they emerge in the decomposition of triple fermion constructs into Lorentz representations. We then study the baryon realization of RS fields as composite systems by means of the quark version of the $U(4)$ symmetric diatomic rovibron model. In using the $U(4) \supset O(4) \supset O(3) \supset O(2)$ reduction chain, we are able to reproduce quantum numbers and mass splittings of the above resonance assemblies. We present the essentials of the four dimensional angular momentum algebra and construct electromagnetic tensor operators. The predictive power of the model is illustrated by ratios of reduced probabilities concerning electric de-excitations of various resonances to the nucleon.

I. O(4) DEGENERACY MOTIF IN BARYON SPECTRA: AN INTRODUCTION

One of the basic quality tests for any model of composite baryons is the level of accuracy reached in describing the nucleon and $\Delta$ excitation spectra. In that respect, the knowledge on the degeneracy group of baryon spectra appears as a key tool in constructing the underlying Hamiltonian of the strong-interaction dynamics as a function of the Casimir operators of the symmetry group. To uncover the latter, one can analyze isospin by isospin how the quantum numbers of the resonances belonging to a particular cluster fit into O(1,3) Lorentz group representations of the so called Rarita-Schwinger (RS) type $\left(\frac{3}{2}, \frac{3}{2}\right)$

$$\Psi_{\mu_1 \mu_2 \ldots \mu_K} := \left(\frac{K}{2}, \frac{K}{2}\right) \otimes \left[\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\right]. \quad (1)$$

To be specific, one finds the three RS clusters with $K = 1, 3, \text{and } 5$ in both the nucleon ($N$) and $\Delta$ spectra. As long as the Lorentz group is locally isomorphic to O(4), multiplets with the quantum numbers of the RS representations also appear in typical O(4) problems such as the levels of an electron with spin in the hydrogen atom. There, the principal quantum number of the Coulomb problem is associated with $K + 1$ while the role of the boost generators is taken by the components of the Runge-Lenz vector. The Rarita-Schwinger fields are the so-called “diagonal case” (i.e. $a = b = \frac{K}{2}$) of the more general representations $\left(a, b\right) \otimes \left[\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\right]$.

A. Rarita-Schwinger Fields as Multi-Spin-Parity States

The RS fields are described in terms of totally symmetric traceless rank-$K$ Lorentz tensors with Dirac spinor components that satisfy the Dirac equation for each Lorentz index, $\mu_i$, associated with a four-vector $(\frac{1}{2}, \frac{1}{2})$ space

$$(i\partial_\lambda \gamma^\lambda - M) \Psi_{\mu_1 \mu_2 \ldots \mu_K} = 0. \quad (2)$$

The fields of the type in Eq. (1) were considered six decades earlier by Rarita and Schwinger [3], the most popular being the $K = 1$-field that has been frequently applied to the description of spin-3/2 particles. Around mid sixties, Weinberg [4] continued the tradition of the original Rarita-Schwinger work [3] and considered $\Psi_{\mu_1 \mu_2 \ldots \mu_K}$ as fields suited for the description of pure spin-$J = K + \frac{1}{2}$ states of fixed parity. The conjecture that $\Psi_{\mu_1 \mu_2 \ldots \mu_K}$ can be reduced to a single-spin state was based upon the belief that its lower-spin components are redundant, unphysical
states which can be removed by means of the two auxiliary conditions \( \partial^{\mu_1} \Psi_{\mu_1 \ldots \mu_K} = 0 \), and \( \gamma^{\mu_1} \Psi_{\mu_1 \ldots \mu_K} = 0 \). That these conditions do not serve the above purpose, was demonstrated in Ref. [8]. There, the first auxiliary condition was shown to solely test consistency with the mass-shell relation \( E^2 - \vec{p}^2 = m^2 \), while the second one amounted to the acausal energy-momentum dispersion relation \( E = -m \pm \sqrt{\vec{p}^2} \). It is that type of acausality that must be at the heart of the Velo-Zwanziger problem [6]. The RS fields in \( O(4) \) are in fact compilations of fermions of different spins and parities. To illustrate this statement, and for the sake of concreteness, we here consider the coupling of, say, a positive parity Dirac fermion to the \( \left( \frac{K}{2}, \frac{K}{2} \right) \) hyper-boson the latter being composed of \( O(3) \) states of either natural \((\eta = +)\), or, unnatural \((\eta = -)\) parities. These \((\text{mass degenerate})\) \( O(3) \) states carry all integer internal angular momenta, \( l \), with \( l = 0, \ldots, K \) and transform (for the odd \( K \)'s of interest) with respect to the space inversion operation \( \mathcal{P} \) according to
\[
\mathcal{P}|K; \eta; lm\rangle = \eta e^{i\pi l}|K; \eta; l - m\rangle, \quad l^P = 0^0, 1^-\eta, \ldots, K^-\eta, \quad m = -l, \ldots, l.
\]  
In coupling now the Dirac spinor to \( \left( \frac{K}{2}, \frac{K}{2} \right) \) from above, the following spin \((J)\) and parity \((P)\) quantum numbers are created
\[
J^P = \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots, \left( K + \frac{1}{2} \right) \right)^{-\eta}.
\]  
In the following, we will use for the spin-sequence in Eq. (2) the short-hand notation \( \sigma_{2I, \eta} \), with \( \sigma = K + 1 \), or, equivalently
\[
\sigma_{2I, \eta} = \left( \frac{\sigma - 1}{2}, \frac{\sigma - 1}{2}, \frac{1}{2}, \ldots, 0, \frac{1}{2} \right) \otimes \left[ \frac{1}{2}, 0 \right] \otimes \left( 0, \frac{1}{2} \right) \chi^I.
\]  
Here, \( \chi^I \) stands for the isospin spinor attributed to the states under consideration.

A glance at the baryon spectra teaches us that actually Nature strongly favors the excitations of multi-spin-valued resonance clusters over that of pure higher-spin states. This circumstance suggests a new data supported interpretation of the RS fields as complete resonance packages.

### B. Clustering Principle for Baryon Resonances

In terms of the notations introduced above, all reported light-quark baryons with masses below 2500 MeV (up to the \( \Delta (1600) \) resonance that is most probably an independent quark-gluon hybrid state [8]), have been shown in Ref. [2] to be completely accommodated by the RS clusters \( 2_{2I, +}, 4_{2I, -} \), and \( 6_{2I, -} \), having states of highest spin-3/2\(^2\), 7/2\(^+\), and 11/2\(^+\), respectively (see Fig. 1). In each one of the nucleon, \( \Delta \), and \( \Lambda \) hyperon spectra, the natural parity cluster \( 2_{2I, +} \) is always lowest mass. We consider it to reside in a Fock space, \( \mathcal{F}_+ \), built on top of a scalar vacuum.

Equations (3) and (4) illustrate how the \( 2_{2I, +} \) clusters (with \( I = 1/2, 3/2, 0 \)) always unite the first spin-1/2\(^+\), 1/2\(^-\), and 3/2\(^-\) resonances. For the non-strange baryons, \( 2_{2I, +} \) is followed by the unnatural parity clusters \( 4_{2I, -} \), and \( 6_{2I, -} \), which we view to reside in a different Fock space, \( \mathcal{F}_- \), built on top of a pseudoscalar vacuum that is orthogonal (for an ideal \( O(4) \) symmetry) to the previous scalar vacuum. To be specific, one finds all the seven \( \Delta \)-baryon resonances \( S_{31}, P_{31}, P_{33}, D_{33}, D_{35}, F_{35} \), and \( F_{37} \) from \( 4_{3, -} \) to be squeezed within the narrow mass region from 1900 MeV to 1950 MeV, while the \( I = 1/2 \) resonances parrellising them, of which only the \( F_{17} \) state is still “missing” from the data, are located around 1700\(^{+20}_{-50} \) MeV (see left Fig. 1).

Therefore, the \( F_{17} \) resonance is the only non-strange state with a mass below 2000 MeV which is “missing” for the completeness of the present RS classification scheme. In further paralleling baryons from the third nucleon and \( \Delta \) clusters with \( K + 1 = 6 \), one finds in addition the four states \( H_{111}, P_{31}, P_{33} \), and \( D_{33} \) with masses above 2000 MeV to be “missing” for the completeness of the new classification scheme. The \( H_{111} \) state is needed to parallel the well established \( H_{311} \) baryon, while the \( \Delta \)-states \( P_{31}, P_{33} \), and \( D_{33} \) are required as partners to the (less established) \( P_{11}(2100), P_{13}(1900), \) and \( D_{13}(2080) \) nucleon resonances. For \( \Lambda \) hyperons, incomplete data prevent a conclusive analysis. Even so, Fig. 2 (left) indicates that the RS motif may already show up in the reported spectrum. The \( \approx \)? degereneracy group of baryon spectra as already suggested in Refs. [4], is, therefore, confirmed to be
\[
SU(2)_I \otimes O(1, 3) \simeq SU(2)_I \otimes O(4),
\]  
i.e., Isospin\(\otimes\)Space-Time symmetry. To summarize, we here state the principle that light unflavored baryon excitations are patterned after Lorentz-multiplets. For example, the Rarita–Schwinger spinors \( \Psi_{\mu_1 \ldots \mu_K} \) with \( K = 1, 3, \) and \( 5 \) accommodate all the \( \pi N \) resonances according to:
\[ \mathcal{F}_+ : 2f_{I,+} : \Psi_{\mu_1} : P_{2I,1}; S_{2I,1}, D_{2I,3}, \text{ for } I = 0, \frac{1}{2}, \frac{3}{2}, \text{ and} \]
\[ \mathcal{F}_- : 4f_{I,-} : \Psi_{\mu_1\mu_2\mu_3} : S_{2I,1}; P_{2I,1}P_{2I,3}; D_{2I,3}; D_{2I,5}; F_{2I,5}, F_{2I,7}, \]
\[ \mathcal{F}_- : 6f_{I,-} : \Psi_{\mu_1\mu_2...\mu_5} : S_{2I,1}; P_{2I,1}P_{2I,3}; D_{2I,3}; D_{2I,5}; F_{2I,5}, F_{2I,7}; \]
\[ G_{2I,7}, G_{2I,9}; H_{2I,9}, H_{2I,11}, \text{ for } I = \frac{1}{2}, \frac{3}{2}, \]

with the five “missing” states: \( F_{17}, H_{1,11}, P_{31}, P_{33}, D_{33} \).

Occasionally, the above structures will be referred to as LAMPF clusters to emphasize their close relationship to LAMPF physics.

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Fig. 1 Rarita-Schwinger clustering of light unflavored baryon resonances. The full bricks stand for three-to four-star resonances, the empty bricks are one- to two-star states, while the triangles represent states that are “missing” for the completeness of the three RS clusters. Note that “missing” \( F_{17} \) and \( H_{1,11} \) nucleon excitations (left figure) appear as four-star resonances in the \( \Delta \) spectrum (right figure). The “missing” \( \Delta \) excitations \( P_{31}, P_{33}, \) and \( D_{33} \) from \( 6_{1,-} \) are one-to two star resonances in the nucleon counterpart \( 6_{1,-} \). The \( \Delta(1600) \) resonance (shadowed oval) drops out of our RS cluster systematics and we view it as an independent hybrid state.

The scalar vacuum in the first Fock space reflects the Nambu-Goldstone mode of chiral symmetry near the ground state. As argued in Ref. [8], its change to a pseudoscalar between the 1st and 2nd clusters, may be related to a change of the mode of chiral symmetry realization in baryonic spectra.

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Fig. 2 Clustering traces in the \( \Lambda \) hyperon spectrum (left). \( O(4) \) rotational bands of nucleon (N) and (\( \Delta \)) excitations (right). Notations as in Fig. 1.
Within our scheme, the inter-cluster spacing of 200 to 300 MeV is larger by a factor of 3 to 6 as compared to the mass spread within the clusters. For example, the $2_{1,+}$, $2_{3,+}$, $4_{1,-}$, and $4_{3,-}$ clusters carry the maximal internal mass splitting of 50 to 70 MeV.

Finally, the reported mass averages of the resonances from the RS multiplets with $K = 1, 3$, and 5 are well described by means of the following simple empirical relation:

$$M_{\sigma;I} = M_I - m_1 \frac{1}{\sigma^2} + m_2 \frac{\sigma^2 - 1}{4}, \quad I = \frac{1}{2}, \frac{3}{2},$$

where, again, $\sigma = K + 1$. The parameters take for the nucleon ($I = \frac{1}{2}$) the values $m_1 = 600$ MeV, $m_2 = 70$ MeV, and $M_{\sigma} = M_N + m_1$, respectively. The $\Delta$ spectrum ($I = \frac{3}{2}$) is best fitted by the smaller $m_2$ value of $m_2 = 40$ MeV and $M_{\Delta} = M_a + m_1$ (right Fig. 2).

It is the goal of this paper to develop a constituent model for baryons that explains the observed clustering in the spectra of the light unflavored baryons. The paper is organized as follows. In Section II we motivate legitimacy of fundamental fields of specified mass and unspecified spin as they emerge in the decomposition of a triple-Dirac-fermion system into Lorentz group representations. In Section III we present the quark version of the diatomic rovibron model and study its excitation modes. There we also establish correspondence between excited rovibron states and the baryonic RS clusters. We further make all the observed and some of the “missing” resonances distinguishable in organizing them into different rovibron modes. We construct the relevant quark Hamiltonian and recover Eq. (8).

We finally outline the construction of electric transition operators and calculate selected electric transitions of cluster inhabitants to the nucleon. The paper is finished by a brief summary and outlook.

II. MULTI-SPIN STATES AS LORENTZ COVARIANT REPRESENTATIONS

The relativistic description of three-Dirac-spinor systems has been studied in detail in Ref. [10]. Starting with the well known Lorentz invariance of the ordinary Dirac equation

$$(\gamma^\mu p_\mu - m) u(\vec{p}) = 0,$$  

the authors show that the direct product of three Dirac spinors gives rise to a 64-dimensional linear equation of the type

$$(\Gamma^\mu p_\mu - m) U(\vec{p}) = 0; \quad \text{with} \quad \Gamma^\mu = \sum_{r=1}^{3} \gamma^\mu_r$$

Here, $\gamma^\mu_r$ is the four dimensional unit matrix, while the index $r$ indicates position of the Dirac matrix $\gamma^\mu$ in $\gamma^\mu_r$. Under Lorentz transformations ($a_\mu^\nu$) of the $\gamma$ matrices, the matrices $\Gamma^\mu$ from Eq. (10) change according to $\Gamma^\mu' = U \Gamma^\nu U^{-1}$ with $U = U_1 \otimes U_2 \otimes U_3$, and $U_\nu$ defined as the matrix that covers the Lorentz transformation $\gamma^\mu' = a_\mu^\nu \gamma^\nu = U_\nu \gamma^\nu U^{-1}$ of $\gamma_r$. Equation (11) is therefore Lorentz invariant. Moreover, it was demonstrated that Eq. (11) has $U(4)$ as an additional dynamical symmetry.

The 64 states from above are distributed over different irreducible representations (irreps) of $U(4)$ and the permutation group group $S_3$ as well. To be specific, one finds two $20$plets in turn associated with the Young schemes $[3000]$, and $[2100]$. They are completed by the quartet $[1110]$. The three-Dirac spinor state (denoted by $s^3$) can be characterized by the set of quantum numbers

$$|s^3[f]X, \{f\} R\rangle.$$

Here, $X$ stands for a set of quantum numbers characterizing the $U(4)$ basis vectors of the $[f]$ irrep, while $R$ denotes the Yamanouchi symbol labeling the basis vectors of the $S_3$ representation $\{f\}$ [13]. The Yamanouchi symbols for the $[3000]$, $[2100]$, and $[1110]$ are 1; 2, 1, and 1, respectively. The complete number $(N_{s^3})$ of 64 states of the three-Dirac-fermion $(s^3)$ system is then encoded by the relation

$$N_{s^3} = \sum_{[f]} \dim [f] \dim \{f\}$$

(12)
where \( \text{dim}[f] \) and \( \text{dim}\{f\} \) are in turn the dimensionalities of the \( U(4) \) irrep \( [f] \), and the \( S_3 \) irrep \( \{f\} \), respectively. In considering now the reduction chain \( U(4) \supset O(5) \), allows for a more detailed specification of the spin content of the \( U(4) \) multiplets from above (see Ref. [10] for details).

The quantum numbers of the irreducible representations (irreps) of \( O(5) \) are labeled by the two numbers \((\lambda_1, \lambda_2)\), which can be either integer, or half-integer. The states participating a given \( O(5) \) irrep can be further specified by the quantum numbers of the irreps of the \( O(5) \) subgroups appearing in the reduction chain \( O(5) \supset O(4) \supset O(3) \supset O(2) \). To specify the \( O(4) \) irreps in the context of the \( O(5) \) reduction down to \( O(2) \) it is more convenient to use instead of the pair \((a, b)\) from above, rather the pair \((m_1, m_2)\) with the mapping
\[
m_1 = a + b, \quad m_2 = a - b.
\]
Finally, the \( O(3) \) irreps in the \( O(3) \supset O(2) \) reduction scheme are labeled by the well known spin \((J)\) and magnetic quantum number \((M)\). The complete set of quantum numbers specifying a member of a \( O(5) \) multiplet \(|(\lambda_1, \lambda_2); (m_1, m_2); JM\rangle\) satisfy the inequalities
\[
\lambda_1 \geq m_1 \geq \lambda_2 \geq |m_2|, \quad m_1 \geq J \geq |m_2|, \quad J \geq M \geq -J.
\]
The \( U(4) \) irrep \([2100]\) is of particular interest for the present work. In the \( U(4) \supset O(5) \) reduction chain it splits into \( O(5) \) irreps according to
\[
[2100] \rightarrow \left( \begin{array}{c} 3 \ 1 \ 2 \ 2 \end{array} \right) \oplus \left( \begin{array}{c} 1 \ 1 \ 2 \ 2 \end{array} \right) . \tag{15}
\]

The first irrep on the rhs of the last equation is 16-dimensional, while the second is four dimensional and associated with a Dirac spinor. As we shall see below, the \( O(5) \) 16plet \( \left( \begin{array}{c} 3 \ 1 \ 2 \ 2 \end{array} \right) \) is nothing but the RS field with \( K = 1 \). Indeed, from Eq. (15) follows that
\[
\frac{3}{2} \geq m_1, \quad m_1 \geq \frac{1}{2}, \quad \text{and} \quad \frac{1}{2} \geq |m_2| . \tag{16}
\]
The inequalities in the latter equation are satisfied for \( m_1 = 3/2, 1/2 \), and for \( m_2 = 1/2, -1/2 \). In accordance with the 2nd equation in \([14]\), \( J \) can take the three values \( J = 3/2, 1/2, \) and \( J = 1/2 \). Thus, the \( \left( \begin{array}{c} 3 \ 1 \ 2 \ 2 \end{array} \right) \) irrep of \( O(5) \) describes a spin-3/2 and two spin-1/2 states and coincides with the lowest 16-dimensional Rarita-Schwinger field.

The above consideration gives an idea of how Lorentz representations of the RS type can emerge as fundamental free particles of definite mass and indefinite spin within the context of a relativistic space-time treatment. Though such point-like particles have not been detected so far, the \( \Lambda \) and \( \Delta \) spectra strongly indicate existence of composite RS fields. In the following, we shall focus onto that very realization of multi-spin Lorentz representations and explore their internal structure by means of constituent models. For a more profound textbook presentation on the various aspects of higher-dimensional relativistic supermultiplets, the interested reader is referred to Ref. [12].

III. THE QUARK VERSION OF THE DIATOMIC ROVIBRON MODEL AND THE RS CLUSTERING IN BARYON SPECTRA

Baryons in the quark model are considered as constituted of three quarks in a color singlet state. It appears naturally, therefore, to undertake an attempt of describing the baryonic system by means of algebraic models developed for the purposes of triatomic molecules, a path already pursued by Refs. [14]. There, the three body system was described in terms of two vectorial \((\vec{p}^+)\) and one scalar \((s^+)\) boson degrees of freedom that transform as the fundamental \( U(7) \) septet. In the dynamical symmetry limit
\[
U(7) \rightarrow U(3) \times U(4) \tag{17}
\]
the degrees of freedom associated with the one vectorial boson factorize from those associated with the scalar boson and the remaining vectorial boson. Because of that the physical states constructed within the \( U(7) \) IBM model are often labeled by means of \( U(3) \times U(4) \) quantum numbers. Below we will focus on that very sub-model of the IBM and show that it perfectly accommodates the RS clusters from above and thereby the LAMPF data on the non-strange baryon resonances.

The dynamical limit \( U(7) \rightarrow U(3) \times U(4) \) corresponds to the quark–diquark approximation of the three quark system, when two of the quarks reveal a stronger pair correlation to a diquark (Dq) \([14]\), while the third quark (q)
acts as a spectator. The diquark approximation turned out to be rather convenient in particular in describing various properties of the ground state baryons \[15,16\]. Within the context of the quark–diquark (q-Dq) model, the ideas of the rovibron model, known from the spectroscopy of diatomic molecules \[9\], can be applied to the description of the rotational-vibrational (rovibron) excitations of the q-Dq system.

### A. Rovibron Model for the Quark–Diquark System

In the rovibron model (RVM) the relative q–Dq motion (see Fig. 3) is described by means of four types of boson creation operators \( s^+, p^+_1, p^+_0, \) and \( p^+_{-1} \) (compare \[9\]). The operators \( s^+ \) and \( p^+_m \) in turn transform as rank-0, and rank-1 spherical tensors, i.e. the magnetic quantum number \( m \) takes in turn the values \( m = 1, 0, \) and \( -1 \). In order to construct boson-annihilation operators that also transform as spherical tensors, one introduces the four operators \( \tilde{s} = s, \) and \( \tilde{p}_m = (-1)^m p_m \). Constructing rank-\( k \) tensor product of any rank-\( k_1 \) and rank-\( k_2 \) tensors, say, \( A_{k_1}^{m_1} \) and \( A_{k_2}^{m_2} \), is standard and given by

\[
[A_{k_1}^{m_1} \otimes A_{k_2}^{m_2}]^k_m = \sum_{m_1, m_2} (k_{1} m_1 k_{2} m_2 | k m) A_{m_1}^{k_1} A_{m_2}^{k_2} .
\]

Here, \( (k_{1} m_1 k_{2} m_2 | k m) \) are the well known \( O(3) \) Clebsch-Gordan coefficients.

![Fig. 3 Schematic presentation of a q-Dq two-body system.](image)

Now, the lowest states of the two-body system are identified with \( N \) boson states and are characterized by the ket-vectors \( |n_s n_p l m \rangle \) (or, a linear combination of them) within a properly defined Fock space. The constant \( N = n_s + n_p \) stands for the total number of \( s \)- and \( p \)-bosons and plays the role of a parameter of the theory. In molecular physics, the parameter \( N \) is usually associated with the number of molecular bound states. The group symmetry of the rovibron model is well known to be \( U(4) \). The fifteen generators of the associated \( su(4) \) algebra are determined as the following set of bilinears

\[
A_{00} = s^+ \tilde{s}, \quad A_{0m} = s^+ \tilde{p}_m, \\
A_{m0} = p^+_m \tilde{s}, \quad A_{mm'} = p^+_m \tilde{p}_{m'} .
\]

The \( u(4) \) algebra is then recovered by the following commutation relations

\[
[A_{a \beta}, A_{a \delta}] = \delta_{\beta \gamma} A_{a \delta} - \delta_{a \delta} A_{\gamma \beta} .
\]

The operators associated with physical observables can then be expressed as combinations of the \( u(4) \) generators. To be specific, the three-dimensional angular momentum takes the form

\[
L_m = \sqrt{2} [p^+ \otimes \tilde{p}]_{m}^{1} .
\]

Further operators are \( (D_m) \)– and \( (D'_m) \) defined as

\[
D_m = [p^+ \otimes \tilde{s} + s^+ \otimes \tilde{p}]_{m}^{1} ,
\]

\[
D'_m = i[p^+ \otimes \tilde{s} - s^+ \otimes \tilde{p}]_{m}^{1} ,
\]
The Casimir operator $C_N$ associated with the rovibron model as qRVM. It is of common knowledge that the totally symmetric irreps of the momenta in Eq. (32) to the spin-1/2 of the three quarks in the nucleon, the following sequence of states is obtained:

$$\eta$$

The parity carried by these levels is $\eta(-1)^l$ where $\eta$ is the parity of the relevant vacuum. In coupling now the angular momenta in Eq. (32) to the spin-1/2 of the three quarks in the nucleon, the following sequence of states is obtained:

Finally, a quadrupole operator $Q_m$ can be constructed as

$$Q_m = [p^+ \otimes p^2_m], \quad \text{with} \quad m = -2, \ldots, +2. \quad (24)$$

The $u(4)$ algebra has the two algebras $su(3)$, and $so(4)$, as respective sub-algebras. The $su(3)$ algebra is constituted by the three generators $L_m$, and the five components of the quadrupole operator $Q_m$. Its $so(4)$ subalgebra is constituted by the three components of the angular momentum operator $L_m$, on the one side, and the three components of the operator $D'_m$, on the other side. Thus there are two exactly soluble RVM limits that correspond to the two different chains of reducing $U(4)$ down to $O(3)$. These are:

$$U(4) \supset U(3) \supset O(3) \quad \text{and} \quad U(4) \supset O(4) \supset O(3), \quad (25)$$

respectively. The Hamiltonian of the RVM in these exactly soluble limits is then constructed as a properly chosen function of the Casimir operators of the algebras of either the first, or the second chain. For example, in case one approaches $O(3)$ via $U(3)$, the Hamiltonian of a dynamical $SU(3)$ symmetry can be cast into the form:

$$H_{SU(3)} = H_0 + \alpha C_2(SU(3)) + \beta C_2(SO(3)). \quad (26)$$

Here, $H_0$ is a constant, $C_2(SU(3))$, and $C_2(SO(3))$ are in turn the quadratic (in terms of the generators) Casimirs of the groups $SU(3)$, and $SO(3)$, respectively, while $\alpha$ and $\beta$ are constants, to be determined from data fits.

A similar expression (in obvious notations) can be written for the RVM Hamiltonian in the $U(4) \supset O(4) \supset O(3)$ exactly soluble limit:

$$H_{SO(4)} = H_0 + \tilde{\alpha} C_2(SO(4)) + \tilde{\beta} C_2(SO(3)). \quad (27)$$

The Casimir operator $C_2(SO(4))$ is defined accordingly as

$$C_2(SO(4)) = \frac{1}{4} \left( \vec{L}^2 + \vec{D}^{'2} \right) \quad (28)$$

and has an eigenvalue of $K \left( \frac{K}{2} + 1 \right)$. In molecular physics, only linear combinations of the Casimir operators are used, as a rule. However, as known from the hydrogen atom [17], the Hamiltonian is determined by the inverse power of $C_2(SO(4))$ according to

$$H_{\text{Coul}} = f (-4C_2(SO(4)) - 1)^{-1} \quad (29)$$

where $f$ is a parameter with the dimensionality of mass. This Hamiltonian predicts the energy of the states as $E_K = -f/(K + 1)^2$ and does not follow the simple linear pattern (see also Eq. (27)).

In order to demonstrate how the RVM applies to baryon spectroscopy, let us consider the case of $q$-D$q$ states associated with $N=5$ and for the case of a $SO(4)$ dynamical symmetry. From now on we shall refer to the quark rovibron model as qRVM. It is of common knowledge that the totally symmetric irreps of the $u(4)$ algebra with the Young scheme $[N]$ contain the $SO(4)$ irreps $(\frac{K}{2}, \frac{K}{2})$ with

$$K = N, N - 2, \ldots, 1 \quad \text{or} \quad 0. \quad (30)$$

Each one of these $SO(4)$ irreps contains $SO(3)$ multiplets with three dimensional angular momenta

$$l = K, K - 1, K - 2, \ldots, 1, 0. \quad (31)$$

In applying the branching rules in Eqs. (30), (31) to the case $N=5$, one encounters the series of levels

$$K = 1: \quad l = 0, 1;$$
$$K = 3: \quad l = 0, 1, 2, 3;$$
$$K = 5: \quad l = 0, 1, 2, 3, 4, 5. \quad (32)$$

The parity carried by these levels is $\eta(-1)^l$ where $\eta$ is the parity of the relevant vacuum. In coupling now the angular momenta in Eq. (32) to the spin-1/2 of the three quarks in the nucleon, the following sequence of states is obtained:
Thus rovibron states of half-integer spin will transform according to \((4, \frac{1}{2}) \otimes \left(\frac{1}{2}, 0\right) \oplus (0, \frac{1}{2})\) representations of \(SO(4)\). The isospin structure is accounted for pragmatically through attaching to the RS clusters an isospin spinor \(\chi'\) with \(I\) taking the values \(I = \frac{1}{2}\) and \(I = \frac{3}{2}\) for the nucleon, and the \(\Delta\) states, respectively. As illustrated by Fig. 1, the above quantum numbers cover both the nucleon and the \(\Delta\) excitations.

Note that in the present simple version of the rovibron model, the spin of the quark–diquark system is \(S = \frac{1}{2}\), and the total spin \(J\) takes the values \(J = L \pm \frac{1}{2}\) in accordance with Eqs. (32) and (33). The strong relevance of same picture for both the nucleon and the \(\Delta(1232)\) spectra (where the diquark is in a vector-isovector state) hints onto the dominance of a scalar diquark for both the excited nucleon– and \(\Delta(1232)\) states. This situation is reminiscent of the \(\gamma 1\) configuration of the \(70(1^+)\)plet of the canonical \(SU(6)_{SF} \otimes O(3)_L\) symmetry where the mixed symmetric/antisymmetric wave function in spin-space is compensated by a mixed symmetric/antisymmetric wave function in coordinate space, while the isotriplet \(I = 3/2\) part is totally symmetric.

We here will leave aside the discussion of the generic problem of the various incarnations of the IBM model regarding the symmetry properties of the resonance wave functions to a later date and rather concentrate in the next subsection onto the “missing” resonance problem.

### B. Observed and “Missing” Resonance Clusters within the Rovibron Model

The comparison of the states in Eq. (33) with the reported ones in Eq. (7) shows that the predicted sets are in agreement with the characteristics of the non-strange baryon excitations with masses below \(\sim 2500\) MeV, provided, the parity \(\eta\) of the vacuum changes from scalar (\(\eta = 1\)) for the \(K = 1\), to pseudoscalar (\(\eta = -1\)) for the \(K = 3, 5\) clusters. A pseudoscalar “vacuum” can be modeled in terms of an excited composite diquark carrying an internal angular momentum \(L = 1\) and maximal spin \(S = 1\). In one of the possibilities the total spin of such a system can be \(|L - S| = 0\). To explain the properties of the ground state, one has to consider separately even \(N\) values, such as, say, \(N' = 4\). In that case another branch of excitations, with \(K = 4, 2,\) and \(0\) will emerge. The \(K = 0\) value characterizes the ground state, \(K = 2\) corresponds to \((1, 1) \otimes \left(\frac{1}{2}, 0\right) \oplus (0, \frac{1}{2})\), while \(K = 4\) corresponds to \((2, 2) \otimes \left(\frac{1}{2}, 0\right) \oplus (0, \frac{1}{2})\). These are the multiplets that we will associate with the “missing” resonances predicted by the rovibron model. In this manner, reported and “missing” resonances fall apart and populate distinct \(U(4)\)- and \(SO(4)\) representations. In making observed and “missing” resonances distinguishable, reasons for their absence or, presence in the spectra are easier to be searched for. As to the parity of the resonances with even \(K\)’s, there is some ambiguity. As a guidance one may consider the decomposition of the three-quark \((q^3)\) Hilbert space into Lorentz group representations as performed in Ref. [3]. There, two states of the type \((1, 1) \otimes \left(\frac{1}{2}, 0\right) \oplus (0, \frac{1}{2})\) were found. The first one arose out of the decomposition of the \(q^3\)-Hilbert space spanned by the \(1s - 1p - 2s\) single-particle states. It was close to \((\frac{1}{2}, \frac{1}{2}) \otimes \left(\frac{1}{2}, 0\right) \oplus (0, \frac{1}{2})\) and carried opposite parity to the latter. It accommodated, therefore, unnatural parity resonances. The second \(K = 2\) state was part of the \((1s - 3s - 2p - 1d)\)- single-particle configuration space and was closer to \((\frac{3}{2}, \frac{1}{2}) \otimes \left(\frac{1}{2}, 0\right) \oplus (0, \frac{1}{2})\). It also carried opposite parity to the latter and accommodated natural parity resonances. Finally, the \(K = 4\) cluster \((2, 2) \otimes \left(\frac{1}{2}, 0\right) \oplus (0, \frac{1}{2})\) emerged in the decomposition of the one-particle-one-hole states within the \((1s - 4s - 3p - 2d - 1f - 1g)\) configuration space and carried also natural parity, that is, opposite parity to \((\frac{5}{2}, \frac{5}{2}) \otimes \left(\frac{1}{2}, 0\right) \oplus (0, \frac{1}{2})\). In accordance with the above results, we here will treat the \(N = 4\) states to be all of natural parities and identify them with the nucleon \((K = 0)\), the natural parity \(K = 2\), and the natural parity \(K = 4\) RS clusters.

The unnatural parity \(K = 2\) cluster from Ref. [3] could be generated through an unnatural parity \(N = 2\) excitation mode. However, this mode would require manifest chiral symmetry up to \(\approx 1550\) MeV which contradicts at least present data. With this observation in mind, we here will restrict ourselves to the consideration of the natural parity \(N = 4\) clusters. In this manner the unnatural parity \(K = 2\) state from Ref. [3] will be dropped out from the current version of the rovibron model. From now on we will refer to the excited \(N = 4\) states as to “missing” rovibron clusters.

Now, the qRVM Hamiltonian that reproduces the mass formula from Eq. (8) is given by the following function of \(C_2(SO(4))\)

\[
H_{qRVM} = H_0 - f_1 \left(4C_2(SO(4)) + 1\right)^{-1} + f_2 \left(C_2(SO(4))\right).
\]

(34)
The states in Eq. (33) are degenerate and the dynamical symmetry is SO(4). The parameter set

$$H_0 = M_{N/\Delta} + f_1, \quad f_1 = m_1, \quad f_2 = m_2,$$

with \( I = 1, \frac{3}{2}, \frac{5}{2}, \) recovers the empirical mass formula in Eq. (8). Thus, the SO(4) dynamical symmetry limit of the qRVM picture of baryon structure motivates existence of quasi-degenerate clusters of resonances in the nucleon- and \( \Delta \) baryon spectra. In Table I we list the masses of the RS clusters concluded from Eqs. (34), and (35).

**TABLE I.** Predicted mass distribution of observed (obs), and missing (miss) rovibron clusters (in MeV) according to Eq. (34,35). The sign of \( \eta \) in Eq. (3) determines natural- (\( \eta = +1 \)), or, unnatural (\( \eta = -1 \)) parity states. All \( \Delta \) excitations have been calculated with \( m_2 = 40 \) MeV rather than with the nucleon value of \( m_2 = 70 \) MeV. The experimental mass averages of the resonances from a given RS cluster have been labeled by “exp”. The nucleon and \( \Delta \) ground state masses \( M_N \) and \( M_\Delta \) were taken to equal their experimental values.

| K | sign \( \eta \) | \( \eta \) Nobs | \( \eta \) Nexp | \( \eta \) \( \Delta \) obs | \( \eta \) \( \Delta \) exp | \( \eta \) Nmiss | \( \eta \) \( \Delta \) miss |
|---|---|---|---|---|---|---|---|
| 0 | + | 939 | 939 | 1232 | 1232 | | |
| 1 | + | 1441 | 1498 | 1712 | 1690 | | |
| 2 | + | | | 1612 | 1846 | | |
| 3 | - | 1764 | 1689 | 1944 | 1922 | | |
| 4 | + | | | 1935 | 2048 | | |
| 5 | - | 2135 | 2102 | 2165 | 2276 | | |
The data on the $\Lambda$, $\Sigma$, and $\Omega^-$ hyperon spectra are still far from being as complete as those of the nucleon and the $\Delta$ baryons and do not allow, at least at the present stage, a conclusive statement on relevance or irrelevance of the rovibron picture. The presence of the heavier strange quark can significantly influence the excitation modes of the $q^3$-system. In case, the presence of the $s$ quark in the hyperon structure is essential, the $U(4) \supset U(3) \supset O(3)$ chain can be favored over $U(4) \supset O(4) \supset O(3)$ and a different clustering motif can appear here. For the time being, this issue will be dropped out of further consideration.

In the next subsection, we shall outline the calculational scheme for branching ratios of reduced probabilities for electromagnetic transitions.

C. O(4) Angular Momentum Algebra and Multipole Operators

In the following, resonance states from a RS cluster will be denoted as

$$| N; 0^n; (a, b); l^z; S; J^z M_J \rangle$$  \hspace{1cm} (36)

Here, $\eta = \pm$ denotes the parity of the vacuum of the Fock space accommodating the RS cluster, $(a, b) = (\frac{K}{2}, \frac{K}{2})$, $l$ is the underlying three-dimensional angular momentum of parity $\eta(-1)^l$, $S$ is the quark spin, while $J^z$ and $M_J$ are in turn total spin and magnetic quantum numbers of the resonance under consideration. In fact, $K$ is nothing but the four-dimensional angular momentum.

Within the framework of the rovibron model one can describe three different types of transitions:

(i) Transitions without change of the quantum numbers $N$ and $K$, i.e. transitions between resonances from same cluster. In such a case, the transition operator is the $D^m_0$ generator of the $so(4)$ algebra and one can calculate the reduced probabilities $B(\alpha_1, J_1 \rightarrow \alpha_2, J_2; E1)$ for electric dipole transitions. Notice that the reduced transition probability of the multipolarity $\lambda$ as carried out by the operator $T^{\alpha, \lambda}$ between states of initial and final spins $J_1$ and $J_2$, respectively, is defined as \[19\]

$$B(\alpha_1, J_1 \rightarrow \alpha_2, J_2; T^{\alpha, \lambda}) = \frac{1}{2J_1 + 1} \left| \langle \alpha_2 J_2 | [T^{\alpha, \lambda} | \alpha_1 J_1] \right|^2 . $$  \hspace{1cm} (37)

Unfortunately, such transitions are difficult and perhaps even beyond any possibility of being observed.

(ii) Transitions between states of same number of bosons $N$ but of different four dimensional angular momenta, $\Delta K \neq 0$, i.e. transitions between resonances belonging to different RS clusters. Operators that can realize such transitions between different $O(4)$ multiplets are $U(4)$ generators (or, tensor products of them) lying outside of the $so(4)$ sub-algebra. The latter operators constitute the set

$$Q_m = [p^+ \otimes \bar{p}]^2_m, \hspace{0.5cm} E_m = \frac{1}{\sqrt{2}} D_m, \hspace{0.5cm} E0 = \frac{1}{2\sqrt{3}}(3n_s - n_p). $$  \hspace{1cm} (38)

It is not difficult to prove that the nine operators in Eq. (38) behave with respect to $SO(4)$ transformation as the components of the totally symmetric rank-2 tensor, $T^{(1,1)lm}$ where

$$T^{(1,1)2m} := Q_m, \hspace{0.5cm} T^{(1,1)1m} := E_m, \hspace{0.5cm} T^{(1,1)00} := E0. $$  \hspace{1cm} (39)

By the way, the tensor $T^{(1,1)lm}$ is the one of lowest rank that can realize transitions between $SO(4)$ multiplets having same number of bosons $N$ and differing by two units in $K$.

(iii) Transitions between $U(4)$ multiplets whose number of bosons differ by one unit ($\Delta N = 1$), the most interesting being resonance de-excitation modes into the nucleon

$$| N_1 = 5; 0^n; K_1; L_1; S_1 = \frac{1}{2}; J_1 M_1 \rangle \rightarrow | N_2 = 4; 0^+; K_2 = 0; L_2 = 0; S_2 = \frac{1}{2}; \frac{1}{2} \rangle. $$  \hspace{1cm} (40)

In the following we will be mainly interested in transitions of the third type. At the present stage, however, it is convenient to first outline the general scheme of the $SO(4)$ Racah algebra.

Tensor products $[T^{(a_1,b_2)} \otimes T^{(a_2,b_2)}]^{(a,b)lm}_{(a,b)lm}$ in $SO(4)$ are defined as (see Refs. [1],[18] for details)
In combining Eqs. (45) and (46) results into 

\[
\left[ T^{(a_2,b_2)} \otimes T^{(a_1,b_1)} \right]^{(a,b)lm} = \sum_{l_1, l, l_2, m_2} \left( a_1 b_1, l_1 m_1 \right) \left( a_2 b_2, l_2 m_2 \right) T^{(a_1,b_1)l_1 m_1} T^{(a_2,b_2)l_2 m_2}. \tag{41}
\]

The matrix elements of any tensor operator \( T^{(a,b)lm} \) between \( O(4) \) states are expressed as 

\[
\left\langle (a_1, b_1); l_1 m_1 \left| T^{(a,b)lm} \right| (a_2, b_2); l_2 m_2 \right\rangle = \left( a_2 b_2, l_2 m_2 \right) \left( a_1 b_1, l_1 m_1 \right) \left( T^{(a,b)lm} \right) \tag{42}
\]

The \( SO(4) \) Clebsch-Gordan coefficients entering the last equation are determined by 

\[
\left( (a_2 b_2) l_2 m_2 \right) \left( a_1 b_1 \right) \left( a_2 b_2 \right) \left( a_1 b_1 l_1 m_1 \right) = \\
\sqrt{(2l_1 + 1)(2l_2 + 1)(2l_1 + 1)(2a + 1)(2b + 1)} \quad (-1)^{l-m} \left( \begin{array}{ccc}
l_1 & l_2 & l \\
-m_1 & -m_2 & m
\end{array} \right) \left( \begin{array}{ccc}
a_1 & a_2 & a \\
b_1 & b_2 & b \\
l_1 & l_2 & l
\end{array} \right). \tag{43}
\]

The last equation shows that the ratios of the reduced probabilities of electromagnetic transitions between resonances with different \( K \) quantum numbers are determined as ratios of the squared \( SO(4) \) Clebsch-Gordan coefficients, as the triple barred transition matrix elements cancel out. As an example of that type of transitions let us consider the electromagnetic de-excitation of the natural parity resonances with spins \( 3/2^- \) and \( 1/2^- \) from the first cluster to the nucleon. Obviously, the relevant tensor operator in \( SO(4) \) space is \( T\left(\frac{3}{2},\frac{1}{2}\right) \). The latter should connect \( U(4) \) states with different numbers of bosons i.e. \( \Delta N = 1 \). Therefore, it can be taken in the form 

\[
T\left(\frac{3}{2},\frac{1}{2}\right)^{1m} = p^+_m, \quad T\left(\frac{3}{2},\frac{1}{2}\right)^{00} = s^+. \tag{44}
\]

Transitions of the above type can then be calculated by means of ordinary Racah algebra in considering \( \alpha_i := N(a_i, b_i) = N(K_i/2, K_i/2) \) (with \( i = 1, 2 \)) as an intrinsic quantum number according to:

\[
\left\langle \alpha_1, l_1; \frac{1}{2}, J^\pi M_j \left| T^{a,lm} \right| \alpha_2, 0; \frac{1}{2}, 1^+ \right\rangle = (-1)^{(J-J_n)} \left( \begin{array}{c}
J \\
-M_j
\end{array} \right) \left( \begin{array}{c}
l \\
m
\end{array} \right) \left( \alpha_1, l_1; \frac{1}{2}, J^\pi \right) \left( \alpha_2, 0; 1^+ \right). \tag{45}
\]

In order to express double barred matrix element in terms of triple barred matrix elements, the following relations should be taken into account:

\[
\left( \alpha_1, l_1; \frac{1}{2}, J^\pi \right) \left( \alpha_2, 0; 1^+ \right) = \delta_{i1} \sqrt{2(2J + 1)} \left( \alpha_1, l \right) \left( T^{a,l} \right) \left( \alpha_2, 0 \right),
\]

\[
\left( N(a_1, b_1); l_1 \right) \left( \left| T^{(a,b)l} \right| N'(a_2, b_2); l_2 \right) = \sqrt{(2l_1 + 1)(2l_2 + 1)(2l_1 + 1)(2a_1 + 1)(2b_1 + 1)} \left( \begin{array}{c}
a_2 & b_2 & l_2 \\
a & b & l \\
a_1 & b_1 & l_1
\end{array} \right),
\]

\[
\left( \begin{array}{c}
0 & 0 & 0 \\
a & b & l \\
a_1 & b_1 & l_1
\end{array} \right) = \delta_{i1} \delta_{b_1} \delta_{a_1} \frac{1}{\sqrt{(2l_1 + 1)(2a_1 + 1)(2b_1 + 1)}}. \tag{46}
\]

In combining Eqs. (43) and (44) results into 

\[
\left| \left( N(a_1, b_1); l_1; \frac{1}{2}, J^\pi \right) \left( T^{(a,b)l} \right| N'(0, 0); 0^+, 1^+ \right) \right|^2 = (2J + 1) \left| \left( N(a, a) \right) \left( T^{(a,a)} \right) \left( N'(0, 0) \right) \right|^2. \tag{47}
\]
D. Electric De-excitations of Resonances to the Nucleon

Eqs. (45)-(47) can be applied to calculate the ratio of, say, the electric dipole de-excitations $D_{13}(1520) \rightarrow p + \gamma$, and $S_{11}(1535) \rightarrow p + \gamma$. In this case $l_d^2 = l^2 = 1^-$, $a_1 = a = \frac{1}{2}$, $b_1 = b = \frac{1}{2}$, and $J^\pi$ takes the two values $J^\pi = \frac{3}{2}^-$, and $\frac{1}{2}^-$, respectively.

Substitution of the relevant quantum numbers into Eqs. (45)-(47) followed by a calculation of the ratio of the squared values of the $J^\pi = \frac{3}{2}^-$, and $J^\pi = \frac{1}{2}^-$ matrix elements yields the theoretical ratio of the electric dipole widths of interest, $\Gamma_{D_{13}}$, and $\Gamma_{S_{11}}$ of the respective $D_{13}(1520)$ and $S_{11}(1535)$ states as

$$R_{\text{th}} = \left( \frac{\Gamma_{D_{13}}}{\Gamma_{S_{11}}} \right) = 1. \quad (48)$$

In order to compare it to data, one may approximate the dipole widths with the total $\gamma$ widths and obtain their experimental values from the full widths and the branching ratios listed in [2]. The full widths of the $D_{13}(1520)$ and $S_{11}(1535)$ resonances are reported as 120 MeV and 150 MeV, respectively. The $D_{13}(1520) \rightarrow p + \gamma$ branching ratio is reported as 0.46-0.56%, while the $S_{11}(1535)$ takes values within the broader range from 0.15% to 0.35%. The theoretical prediction corresponds to a $S_{11}(1535) \rightarrow p + \gamma$ ratio of 0.35% and lies thereby at the upper bound of the data range. This ratio is in fact $J$-independent. It shows that the purely algebraic description is insufficient to reproduce the electromagnetic properties of the resonances in great detail. In that regard, further development of the model is needed with the aim to account for the internal diquark structure.

Remarkably, the internal structure of the diquark does not show up in the spectra, and seems to be less important. The merit of the rovibron model is that there it can be treated as a correction rather than as a leading mechanism from the very beginning.

One can further compare gamma-widths of resonances carrying different internal $O(3)$ quantum numbers $l$. This effect is easiest to study on the example of the natural parity resonances from the “missing” rovibron clusters. To be specific, we will compare the reduced probabilities for the following two transitions:

$$|4; 0^+; (2, 2); 1^-; \frac{3}{2}; \frac{3}{2} \rangle \stackrel{T^{(2,2)1m}}{\longrightarrow} |4; 0^+; (0, 0); 0^+; \frac{1}{2}; \frac{1}{2} \rangle,$$

$$|4; 0^+; (2, 2); 3^-; \frac{5}{2}; \frac{5}{2} \rangle \stackrel{T^{(2,2)3m}}{\longrightarrow} |4; 0^+; (0, 0); 0^+; \frac{1}{2}; \frac{1}{2} \rangle \quad (49)$$

The relevant transition operator is

$$T^{(2,2)lm} = [T^{(1,1)} \otimes T^{(1,1)}]^{(2,2)lm}. \quad (50)$$

Here, $l$ can take the values $l = 0, 1, 2, 3$, and 4. The first of the transitions in Eq. (49) is governed by the electric dipole operator $T^{(2,2)1m}$, while the second is controlled by the electric octupole $T^{(2,2)3m}$. We are going to calculate the ratio $R_2$ of the quantities

$$R_2 = \frac{B \left( \alpha_1, \frac{3}{2}^- \rightarrow \alpha_2, \frac{1}{2}^+ ; T^{(2,2)1} \right)}{B \left( \alpha_1, \frac{3}{2}^- \rightarrow \alpha_2, \frac{1}{2}^+ ; T^{(2,2)3} \right)} \quad (51)$$

Here

$$B \left( \alpha_1, \frac{3}{2}^- \rightarrow \alpha_2, \frac{1}{2}^+ ; T^{(2,2)1} \right) = \frac{1}{4} \left( 4; 0^+; (1, 1); 1^-; \frac{3}{2}; \frac{3}{2} \right) \left| \left| T^{(2,2)1} \otimes I \right| \right|^2 \left| \left| 4; 0^+; (0, 0); 0^+; \frac{1}{2}; \frac{1}{2} \right| \right|^2,$$

$$B \left( \alpha_1, \frac{3}{2}^- \rightarrow \alpha_2, \frac{1}{2}^+ ; T^{(2,2)3} \right) = \frac{1}{6} \left( 4; 0^+; (1, 1); 3^-; \frac{5}{2}; \frac{5}{2} \right) \left| \left| T^{(2,2)3} \otimes I \right| \right|^2 \left| \left| 4; 0^+; (0, 0); 0^+; \frac{1}{2}; \frac{1}{2} \right| \right|^2. \quad (52)$$

Usage of Eq. (47) yields equal reduced probabilities for both the dipole and octupole de-excitations and thereby the unit value for $R_2$. Thus, within this early version of the rovibron model, a given RS cluster will have a common partial $(\gamma + p)$- decay width, that is insensitive to its $O(3)$ spin content.
A more interesting situation occurs in the case of LAMPF clusters, such like \( |5;0^-; (3\, 3/2, 3/2); 2^-; (1\, 1/2, 3^-\, m_{3/2}) \). There, one encounters a suppression of electromagnetic transitions to the nucleon. Indeed, in the rigorous case of an ideal O(4) symmetry, due to the unnatural parities of the nucleon resonances with masses above 1535 MeV (and the \( \Delta \) excitations with masses above 1700 MeV), transitions of the type

\[
|5;0^-; (3\, 3/2, 3/2); 2^-; (1\, 1/2, 3^-\, m_{3/2}) \to |4;0^+; (0,0);0^+; (1\, 1/2, 1^+\, m_{1/2})
\]

(53)
cannot proceed via electric \( E\lambda \)- nor via magnetic \( M\lambda \) multipoles (to be presented elsewhere). In the less rigid scenario of a violated \( O(4) \) symmetry, due to the unnatural parities of the nucleon resonances with masses above 1535 MeV (and the \( \Delta \) one encounters a suppression of electromagnetic transitions to the nucleon.

\[
|J^\pi = 3^-\, m_{3/2} = \sqrt{1 - \alpha^2} |5;0^-; (3\, 3/2, 3/2); 2^-; (1\, 1/2, 3^-\, m_{3/2}) + \alpha |4;0^+; (1,1);1^-; (1\, 1/2, 3^-\, m_{3/2})
\]

(54)
For similar reasons, also a mixing with \( K' = 4 \) states can take place. Within this mixing scheme, unnatural parity resonances can be excited electrically via their natural parity component. As long as the relevant transition operator for such transitions is \( T^{(3\, 3/2, 3/2)}_{lm} \), its matrix element between the nucleon and the resonance of interest will be proportional to the mixing parameter \( \alpha \). To be specific,

\[
\langle J^\pi = 3^-\, m_{3/2} |T^{(1,1)}_{lm}|4;0^+; (0,0);0^+; (1\, 1/2, 1^+\, m_{1/2})\rangle = \alpha \langle 4;0^+; (1,1);1^-; (1\, 1/2, 3^-\, m_{3/2}) |T^{(1,1)}_{lm}|4;0^+; (0,0);0^+; (1\, 1/2, 1^+\, m_{1/2})\rangle.
\]

(55)
It is obvious from the last equation, that electric excitations of the nucleon into the unnatural parity resonances will be suppressed by the factor of \( \alpha^2 \). At the present early stage of development of the quark rovibron model, the mixing parameter \( \alpha \) can not be calculated but has to be considered as free and determined from data. A theoretical prediction for \( \alpha \) would require more fundamental approach to the internal diquark dynamics. In case the \( O(4) \) symmetry is slightly violated, one may assume \( \alpha \) to be same for all cluster inhabitants and perform some calculations as to what extent such states can be linked via electromagnetic transitions to the nucleon.

IV. SUMMARY AND OUTLOOK

The results of the present study can be summarized as follows:

1. The present investigation communicated an idea of how Lorentz representations of the RS type can emerge as fundamental as well as composite free particles of definite mass and indefinite spin within the context of a relativistic space-time treatment of the three Dirac-fermion system. Though structureless RS particles have not been detected so far, the \( N \) and \( \Delta \) spectra strongly indicate existence of composite RS fields.

2. Excited light unflavored baryons preferably exist as multi-resonance clusters that are described in terms of RS multiplets such as the (predominantly) observed LAMPF clusters \( 2_{2I,+}, 4_{2I,-} \) and \( 6_{2I,-} \), and the “missing” clusters \( 3_{2I,+, and 5_{2I,+} \).

3. The above RS clusters accommodate all the resonances observed so far in the \( \pi N \) decay channel (up to the \( \Delta (1600) \) state). The LAMPF data constitute, therefore, an almost accomplished excitation mode in its own right, as only 5 resonances are “missing” for the completeness of this structure.

4. We modeled composite RS fields within the framework of the quark rovibron model and constructed a Hamiltonian that fits the masses of the LAMPF clusters.

5. In using that Hamiltonian we predicted, from a different but the \( SU(6)_{SF} \otimes O(3)_L \) perspective, the masses of two “missing” clusters of natural parity resonances, in support of the TJNAF “missing” resonance search program [22]. “Missing” resonances under debate in the literature, such like \( P_{13}(1880) [21] \) and \( P_{13}(1910) [22] \) could neatly fit into the \( (2,2) \otimes \left( \frac{1}{2},0 \right) \otimes \left( 0,\frac{1}{2} \right) \) RS cluster at 1935 MeV in Table I.
6. We constructed electric transition operators, outlined the essentials of the $O(4)$ Racah algebra, and calculated ratios of reduced probabilities of various resonance de-excitation to the nucleon. We found the internal structure of the diquark to be of minor importance for the gross features of the excitation modes. At the vertex level, however, a point-like diquark was shown to be insufficient to account for differences in the branching ratios of resonances from same cluster. It is that place where the present early version of the qRVM model of baryon structure needs further improvements. Treating the internal structure of the diquark as a correction rather than as a leading mechanism from the very beginning is a major merit of the quark rovibron model.

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[1] Particle Data Group, Eur. Phys. J. C15 (2000).
[2] M. Kirchbach, Mod. Phys. Lett. A12, 2373 (1997); Few Body Syst. Suppl. 11, 47 (1999).
[3] W. Rarita and J. Schwinger, Phys. Rev. 60, 61 (1941).
[4] S. Weinberg, Phys. Rev. 133, B318 (1964).
[5] D. V. Ahluwalia and M. Kirchbach, Mod. Phys. Lett. A16, 1377 (2001); M. Kirchbach and D. V. Ahluwalia, e-Print Archive: hep-ph/0108030.
[6] G. Velo and D. Zwanziger, Phys. Rev. 186, 1337 (1969).
[7] T. Barnes and F. E. Close, Phys. Lett. B123, 89 (1983); ibid. B128, 277 (1983).
[8] M. Kirchbach, Int. J. Mod. Phys. A15, 1435 (2000).
[9] F. Iachello, and R. D. Levin Algebraic Theory of Molecules (Oxford Univ. Press, N.Y.) 1992.
[10] M. Moshinsky, A. G. Nikitin, A. Sharma, and Yu. F. Smirnov, J. Phys. A:Math. Gen. 31, 6045 (1998).
[11] Bryan G. Wybourne, Classical Groups for Physicists (John Wiley&Sons, N.Y.) 1973, Chpt. 19.
[12] M. Moshinsky and Yu. F. Smirnov, The Harmonic Oscillator in Modern Physics (Harwood Academic Publishers, N. Y.) 1996.
[13] F. Iachello, Phys. Rev. Lett. 78, 13 (1989); R. Bijker, F. Iachello, and A. Leviatan, Phys. Rev. C54, 1935 (1996).
[14] Proc. Int. Conf. Diquarks 3, Torino, Oct. 28-30 (1996), eds. M. Anselmino and E. Predazzi, (World Scientific).
[15] C. Hellstern, R. Alkofer, M. Oettel, and H. Reinhardt, Nucl. Phys. A627, 679 (1997).
[16] K. Kusaka, G. Piller, A. W. Thomas, and A. G. Williams, Phys. Rev. D55, 5299 (1997).
[17] J. P. Elliott, and P. G. Dawber Symmetries in Physics (The MacMillan Press Ltd, London, 1979).
[18] G. F. Filippov, V. I. Ovcharenko, and Yu. F. Smirnov, Microscopic Theory of Collective Excitations of Atomic Nuclei (Naukova Dumka, Kiev) 1981 (in Russian).
[19] Kris L. G. Heyde, The Nuclear Shell Model (Springer Verlag, Berlin).
[20] V. Burkert, Nucl. Phys. A684, 16 (2001).
[21] S. Capstick, T. S. H. Lee, W. Roberts, and A. Svar caffe, Phys. Rev. C59, 3002 (1999).
[22] Yongseok Oh, A. I. Titov, and T. S. H. Lee, e-Print archive: nucl-th/0104046.