Bending Instability of Stellar Disks: The Stabilizing Effect of a Compact Bulge

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Abstract

The saturation conditions for bending modes in inhomogeneous thin stellar disks that follow from an analysis of the dispersion relation are compared with those derived from $N$-body simulations. In the central regions of inhomogeneous disks, the reserve of disk strength against the growth of bending instability is smaller than that for a homogeneous layer. The spheroidal component (a dark halo, a bulge) is shown to have a stabilizing effect. The latter turns out to depend not only on the total mass of the spherical component, but also on the degree of mass concentration toward the center. We conclude that the presence of a compact (not necessarily massive) bulge in spiral galaxies may prove to be enough to suppress the bending perturbations that increase the disk thickness. This conclusion is corroborated by our $N$-body simulations in which we simulated the evolution of almost equilibrium, but unstable finite-thickness disks in the presence of spheroidal components. The final disk thickness at the same total mass of the spherical component (dark halo + bulge) has been found to be much smaller than that in the simulations where a concentrated bulge is present.

Key words: galaxies, stellar disks, bending instabilities.
A peculiarity of the disks in spiral galaxies is that these are rather thin objects. Their thickness is several times smaller than the radial scale length. How far stars can go from the principal galactic plane due to their vertical random velocity component determines the disk thickness at fixed star surface density. The larger the velocity dispersion, the thicker the disk. The random velocities of young stars are known to be low, but the stellar ensemble can subsequently heat up through various relaxation processes; i.e., the random velocity dispersion can increase. Thus, the stellar disk thickness depends on how effective the relaxation processes are in galaxies, and it is ultimately determined by the factors that suppress or trigger the various heating mechanisms. Three basic stellar disk heating mechanisms are commonly discussed in the literature: the scattering of stars by giant molecular clouds (Spitzer and Schwarzschild 1951, 1953), the scattering by transient spiral density waves (among the first results of numerical simulations are those obtained by Sellwood and Carlberg (1984)), and the heating of the ensemble of stars that constitute the disks of spiral galaxies as they interact with external sources, for example, with low-mass satellites (see, e.g., Walker et al. 1999; Velasquez and White 1999).

A second remarkable peculiarity of the stellar disks is that their structure is unusually “fragile”. This peculiarity was revealed by a linear analysis of the collisionless Boltzmann equation and has been repeatedly illustrated by N-body simulations. Numerous studies have shown that the initially regular structure of the stellar disks can change radically due to the growth of various instabilities, which give rise to large-scale structures both in the disk plane (bars, spiral arms, rings) and in the vertical direction (warps). The analytically and numerically obtained saturation conditions for unstable modes impose the most severe restrictions on the global structural and dynamical parameters of the stellar disks.

A local analysis suggests that the stars at a given distance $R$ from the disk center must have a radial velocity dispersion $\sigma_R(R)$ larger than some minimum critical value $\sigma^\text{cr}_R(R)$ (Toomre 1964) for the disk to be gravitationally stable in the region under consideration (at least against the growth of axisymmetric perturbations). In addition, for the disk to be stable against the growth of bending perturbations, the ratio of the vertical and radial velocity dispersions $\sigma_z/\sigma_R$ must also be larger than some value approximately equal to 0.2 – 0.37 (Toomre 1966; Kulsrud et al. 1971; Polyachenko and Shukhman 1977; Araki 1985). The latter quantity determines the minimum thickness of the galactic disk, and its comparison with the observed value allows the contribution of the bending instability to the vertical disk heating to be estimated and compared with the contribution of other relaxation mechanisms.

On the other hand, if we exclude the heating mechanisms mentioned above from our analysis and take, as is commonly done, the condition for marginal disk stability against the growth of bending modes by fixing $\sigma_z/\sigma_R$ at a level of the linear approximation, 0.29 – 0.37, then the velocity dispersion $\sigma_z$ at the same star surface density will decrease with decreasing $\sigma_R$ (see, e.g., Zasov et al. 1991). In this case, the disk will have a smaller thickness.

The presence of a spheroidal component, for example, a dark halo is known to produce a stabilizing effect and to decrease the minimum value of $\sigma^\text{cr}_R$ required for gravitational stability. Consequently, the disks embedded in a massive halo, on average, must have low
values of $\sigma_z$ and be, on average, thinner. Based on similar reasoning, Zasov et al. (1991, 2002) showed that the relative disk thickness $z_0/h$ ($z_0$ is the half-thickness of the disk, and $h$ is the radial exponential scale length) is proportional to $M_d/M_t$, where $M_d$ and $M_t$ are, respectively, the mass of the disk and the total mass of the galaxy within a fixed radius. These authors also concluded that a small disk thickness suggests the existence of a massive dark halo in the galaxy. Moreover, based on $N$-body simulations, Zasov et al. (1991) and Mikhailova et al. (2001) constructed a dependence that allows the relative mass of the dark halo $M_h/M_d$ to be estimated from $z_0/h$. However, as we show below, the relationship between the relative disk thickness and the mass of the spheroidal component is more complex.

In this paper, we numerically analyze the saturation conditions for bending instability in inhomogeneous three-dimensional stellar disks at nonlinear stages in the presence of a spheroidal component of different nature (a stellar bulge and a dark halo) and the constraints imposed on the final disk thickness.

2 PECULIARITIES OF THE GROWTH OF BENDING INSTABILITY IN INHOMOGENEOUS THIN STELLAR DISKS

2.1 Global Modes

To understand how the growth of bending instability in inhomogeneous disks differs from that in homogeneous disks, let us first turn to the result of Toomre (1966). Toomre was the first to derive the dispersion relation for long-wavelength bending perturbations in an infinitely thin gravitating layer with a nonzero velocity dispersion

$$\omega^2 = 2\pi G \Sigma |k| - \sigma_x^2 k^2. \quad (1)$$

where $\Sigma$ is the star surface density of the layer, and $\sigma_x^2$ is the velocity dispersion along a particular coordinate axis in the plane of the layer. It follows from Eq. (1) that the perturbations with a wavelength $\lambda = 2\pi/k > \lambda_1 = \sigma_x^2/G \Sigma$ are stable, since $\omega^2 > 0$ in this range.

Relation (1) was derived for an infinitely thin disk and, when applied to finite-thickness disks, is valid only for perturbations with a wavelength longer than the vertical scale height of the system, i.e., $\lambda >> z_0$, where $z_0$ is the half-thickness of the layer. It can be shown from general considerations that the longest-wavelength perturbations in finite-thickness disks with a wavelength $\lambda < \lambda_2 \approx z_0 \frac{\sigma_x}{\sigma_z}$ are stable, since they are smeared by thermal motions in the plane of the layer (see, e.g., Polyachenko and Shukhman 1997). Having derived the corresponding dispersion relation, Polyachenko and Shukhman (1977) were the first to find the exact location of the stability boundary in the short-wavelength range for a homogeneous flat finite-thickness layer. Araki (1985) (see also Merritt and Sellwood 1994) obtained a similar result for a homogeneous layer with a vertical density profile close to the observed one in real galaxies:\footnote{This profile corresponds to the model of an isothermal layer (Spitzer 1942) and describes well the vertical density variations observed in galaxies (van der Kruit and Searle 1981).} $\rho(R, z) = \rho(R, 0) \text{sech}^2(z/z_0)$. As regards the
intermediate-wavelength ($\lambda_2 < \lambda < \lambda_J$) perturbations, they are unstable, as follows from the results of the studies mentioned above.

As the disk thickness $z_0$ increases, the wavelength $\lambda_2$ increases and tends to $\lambda_J$. When $\lambda_2 = \lambda_J$, the disk is stabilized against bending perturbations of any wavelengths. The following analytical estimate in the linear approximation obtained both from qualitative considerations (Toomre 1966; Kulsrud et al. 1971) and from an accurate analysis of the dispersion relation for a finite-thickness layer (Polyachenko and Shukhman 1977; Araki 1985) is valid:

$$(\sigma_z/\sigma_x)_{cr} \approx 0.29 - 0.37.$$  

The instability is completely suppressed if $\sigma_z/\sigma_x > (\sigma_z/\sigma_x)_{cr}$ and grows if $\sigma_z/\sigma_x < (\sigma_z/\sigma_x)_{cr}$.

As regards the inhomogeneous models, the first thing that radically distinguishes the growth of bending instability in inhomogeneous disks from that in a homogeneous layer is the existence of global unstable bending modes with a wavelength longer than the disk scale length. This conclusion follows from an analysis of the equation that describes the evolution of long-wavelength bending perturbations in an infinitely thin disk with a radially decreasing density (Polyachenko and Shukhman 1977; Merritt and Sellwood 1994). For special disk models, it was shown that the region of stable long-wavelength perturbations narrows significantly in this case (see, e.g., Fig. 2 from Merritt and Sellwood (1994)). This is attributed to the fact that the restoring force from the perturbation that grows in an inhomogeneous disk (Merritt and Sellwood 1994) or in a radially bounded disk (Polyachenko and Shukhman 1977) is always weaker than the corresponding restoring force in a homogeneous infinite layer. This fact was demonstrated more clearly by Sellwood (1996), who noted that the dispersion relation could be used to analyze the bending instability in inhomogeneous disks (at least qualitatively) if another term related to the restoring force from the unperturbed disk is added to it:

$$\omega^2 = \nu_d^2 + 2\pi G\Sigma |k| - \sigma_x^2 k^2,$$  

where $\nu_d = \sqrt{\partial^2 \Phi_d(R, z)/\partial z^2}$ is the vertical oscillation frequency of the stars, and $\Phi_d(R, z)$ is the potential of the disk.

For disks with a nonflat rotation curve, the additional term $\nu_d^2$ can play a destabilizing role. As was noted by Sellwood (1996), for an infinitely thin inhomogeneous disk,

$$\nu_d^2 = \left. \left( \frac{\partial^2 \Phi_d(R, z)}{\partial z^2} \right) \right|_{z=0^+} = -\left. \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \Phi_d}{\partial R} \right) \right|_{z=0} = -\frac{1}{R} \frac{dv_{c,d}^2}{dR},$$  

($v_{c,d}$ is the circular rotational velocity of the disk), and $\nu_d^2 < 0$ for $dv_{c,d}^2/dR > 0$.

Thus, an additional expulsive force from the unperturbed disk emerges in the central regions where the rotation curve rises. The destabilizing effect of the disk in the central regions gives rise to another region at small wave numbers (long $\lambda$) where $\omega^2 < 0$. This region is responsible for the growth of largescale bending instability and the emergence of global modes. In this case, one might expect a larger disk thickness and a larger value $\sigma_z/\sigma_x$ ($\sigma_z/\sigma_R$) than those in homogeneous models to be required to suppress the instability.

Applying all of this reasoning to finite-thickness disks is not quite obvious. On the one hand, it may be assumed that the restoring force from the unperturbed disk $F_z =$
−\partial \Phi_d/\partial z \text{ for } R \to 0 \text{ behaves as } F_z \simeq -GM_d z/|z|^3 \text{ (here, } M_d \text{ is the total mass of the disk) starting from some } z, \text{ i.e., decreases (in magnitude) with increasing } z. \text{ Consequently, } \nu_d^2 = \partial^2 \Phi_d/\partial z^2 = -\partial F_z/\partial z \text{ becomes negative (Sellwood 1996).}

On the other hand, we can exactly calculate the } z \text{ dependence of } \nu_d^2 \text{ for a given } R \text{ from the general formula}

\[ \nu_d^2(r) = -\partial F_z/\partial z = -\frac{\partial}{\partial z} \int \frac{G \rho_d(r')(z' - z)}{|r' - r|^3} \, d^3r', \]

(5)

for the density profile that is commonly used to describe the disks of spiral galaxies:

\[ \rho_d(R, z) = \frac{M_d}{4\pi h^2 z_0} \cdot \exp\left(\frac{-R}{h}\right) \cdot \text{sech}^2\left(\frac{z}{z_0}\right) \]

(6)

where } h \text{ is the exponential scale length of the disk, } z_0 \text{ is the vertical scale height, and } M_d \text{ is the total mass of the disk. The derived dependence}^2 \text{ for various } R \text{ is shown in Fig. 1b. Figure 1b shows the } z \text{ dependence of the vertical force } F_z, \text{ which necessarily has an extremum at some } z. \text{ In the central regions, } \nu_d^2 \text{ is negative at } z > 2z_0 \text{ (on the periphery, the passage to the negative region occurs at even larger } z/z_0). \text{ This implies that all of the above reasoning for finitethickness disks is directly applicable only to largeamplitude perturbations. It is also clear that the bellshaped (or axisymmetric) mode with the azimuthal number } m = 0 \text{ must be most unstable in the central regions. If the amplitude of the axisymmetric bend increases significantly during the growth of bending instability, then it can raise the central stars above the plane of the disk and bring them into a “dangerous” zone where } \nu_d^2 < 0; \text{ subsequently, the self-gravity of the disk itself will contribute to the growth of a bend to the point of saturation.}

2.2 The Central Regions of Hot Infinitely Thin Disks

The second peculiarity of the growth of bending instability in inhomogeneous infinitely thin disks is as follows. The degree of disk instability against bending perturbations depends on the degree of disk “heating” in the plane. Exact and approximate analyses of the dispersion relation for an inhomogeneous infinitely thin disk (Polyachenko and Shukhman 1977; Merritt and Sellwood 1994) show that the modes with increasingly large azimuthal numbers in the central regions become unstable for all wavelengths as the fraction of the kinetic energy contained in the random motions in the disk plane increases. The conclusion about the existence of a Toomre parameter } Q_T \text{ (Toomre 1964), which characterizes the degree of heating of the stellar disk in the plane, at which the disk cannot be stabilized against bending perturbations of any wavelengths can also be drawn from Eq. (3), i.e., the equation written for long-wavelength perturbations in a homogeneous layer, but “rectified” by the term } \nu_d^2 \text{ to include the inhomogeneity effects.}

\[ \text{We see from Eq. (6) that triple integrals over infinite intervals must generally be calculated to determine } \nu_d^2. \text{ This can be easily done by using adaptive algorithms for calculating the integrals. In our integration, we used the gsl library (information about the gsl (GNU Scientific Library) project can be found at } \text{http://sources/redhat/com/gsl). For models with known analytical density-potential pairs, for example, the Miyamoto-Nagai disk (see Binney and Tremaine (1987), p. 44), a comparison of the analytical } z \text{ dependence of } \nu_d^2 \text{ at a given } R \text{ with that calculated numerically by triple integration yielded a close match.} \]
Indeed, $\omega^2 < 0$ for any $k$ if

$$\nu_d^2 + \frac{(\pi G \Sigma)^2}{\sigma_x^2} < 0.$$  

Given (4),

$$\nu_d^2 = 2\Omega_d^2 - \kappa_d^2,$$

where $\Omega_d = \frac{v_{c,d}}{R}$ is the angular velocity, and $\kappa_d^2 = 2\left(\frac{v_{c,d}}{R^2}\left(1 + \frac{R}{v_{c,d}} \frac{dv_{c,d}}{dR}\right)\right)$ is the square of the epicyclic frequency. Then,

$$\omega^2 < 0 \quad 2\Omega_d^2 - \kappa_d^2 + \frac{(\pi G \Sigma)^2}{\sigma_x^2} < 0.$$  

For the central regions of a rigidly rotating disk ($\kappa_d^2 = 4\Omega_d^2$), we obtain from (7)

$$\omega^2 < 0 \quad \sigma_x > \sqrt{2} \frac{\pi G \Sigma}{\kappa_d} \approx \sqrt{2} \sigma_R^c,$$

or

$$\omega^2 < 0 \quad Q_T > \sqrt{2} \text{ for any wavelength},$$

where $Q_T = \sigma_x/\sigma_R^c$ is the Toomre parameter (Toomre 1964).

Condition (8) is not an exact criterion; it is only an estimation relation. However, it shows that the central regions of hot stellar disks ($Q_T >> 1$) with a large reserve of strength against the growth of instabilities in the plane of the disk (bars, spiral arms) cannot be stabilized against the growth of bending perturbations of any wavelengths. The theory constructed for a homogeneous infinitely thin layer does not yield this regime. It should be borne in mind, however, that the contribution of the destabilizing term ($\nu_d^2$) must decrease with increasing disk thickness; therefore, the instability will be saturated at a large, but finite disk thickness. However, the following might be expected: other things being equal, the hotter the initial model in the plane, i.e., the larger the Toomre parameter $Q_T$, the higher the saturation level.

### 3 BENDING INSTABILITY: NUMERICAL SIMULATIONS OF THREE-DIMENSIONAL DISKS

The nonlinear growth stages of bending instability in inhomogeneous finite-thickness stellar disks have been extensively investigated by numerically solving the gravitational $N$-body problem for various stellar disk models. Raha et al. (1991) first observed the bending instability of bars in their numerical simulations; Sellwood and Merritt (1996) and Merritt and Sellwood (1994) studied the nonlinear regime of bending instability in nonrotating disks with the radial density profiles that corresponded to the Kuzmin–Toomre model (see, e.g., Binney and Tremaine (1987), p. 43); Griv et al. (1998) numerically analyzed the development of a bend in a layer of newly formed stars; Tseng (2000) simulated the evolution of the vertical structure of a homogeneous, initially thin disk of finite radius;

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3Actually, the Toomre parameter should have been defined as $\sigma_R/\sigma_R^c$, but the difference does not matter in our case.
Sotnikova and Rodionov (2003) considered a rotating disk with an exponential density profile along the $R$ axis and assumed the presence of a dark halo in the system. The latter authors analyzed the question of how the evolution of initially equilibrium thin disks depends on the governing parameters of the bending instability, which include the initial disk half-thickness $z_0$, the Toomre parameter $Q_T$, and the relative mass of the dark halo $M_h/M_d$ within a fixed radius. Below, we list the most important conclusions that follow from the $N$-body simulations described in the literature. These conclusions in many respects agree with those given in the section entitled “Peculiarities of the growth of bending instability...” of this paper for inhomogeneous infinitely thin disks.

(1) In inhomogeneous models, all of the experimentally observed modes are global, i.e., the scale length of the unstable perturbations is larger than the disk scale. The linear theory constructed for homogeneous models yields no such result.

(2) The saturation level for the bending instability depends on $\sigma_R$ (or $Q_T$). The larger the reserve of disk strength against perturbations in the disk plane, the greater the difficulty to stabilize the disk against the growth of bending perturbations. A rapid and significant (occasionally catastrophic) increase in the disk thickness, particularly in the central regions, to values that are severalfold larger than those yielded by a linear analysis for homogeneous, moderately hot models (Toomre 1966; Kulsrud et al. 1971; Polyachenko and Shukhnam 1977; Araki 1985) was observed in all of the simulations with initially hot disks (Sellwood and Merritt 1994; Tseng 2000; Sotnikova and Rodionov 2003). This mechanism may be responsible for the formation of central bulges in spiral galaxies, at least it can feed the spherical component with new objects.

(3) The central regions of the disk are most unstable (Sellwood and Merritt 1994; Merritt and Sellwood 1994; Griv and Chiueh 1998; Griv et al. 2002; Sotnikova and Rodionov 2003). It is here that the bending modes are formed. Subsequently, their amplitude increases, sometimes significantly. This is particularly true for the perturbations with $m = 0$. The nonaxisymmetric modes (with the azimuthal numbers $m = 1$ and $m = 2$) drift to the disk periphery, temporarily creating the effect of a largescale warp of the entire galaxy, and are then damped$^4$. The instability in the central regions of the stellar disks is saturated at (Sotnikova and Rodionov 2003)

$$\frac{\sigma_z}{\sigma_R} \approx 0.75 - 0.8.$$  

(4) The presence of a massive dark halo, which was included in the models by Sotnikova and Rodionov (2003), has always been a stabilizing factor that suppresses the growth of bending modes. This effect appears to have been first described qualitatively by Zasov et al. (1991).

$^4$In contrast to the result of Griv et al. (2002), the largescale S-shaped or U-shaped warp of the galactic edge always disappeared on long time scales ($\gtrsim 5$ Gyr) in all of the simulations by Sotnikova and Rodionov (2003).
THE STABILIZING EFFECT OF THE SPHEROIDAL COMPONENT: A QUALITATIVE ANALYSIS

The remarkable agreement (at least on a qualitative level) between the conclusions that follow from the analysis of the dispersion relation for an infinitely thin disk and the results of numerical simulations with three-dimensional disks allows us to analyze the applicability of yet another conclusion that can be drawn from Eq. (3) to finite-thickness disks. Since the central regions of the disk are most unstable, it is important to separate out the factors that have a stabilizing effect precisely on these regions. An additional spherical component (a dark halo, a bulge) can be such a stabilizing factor. In this case, another term related to the restoring force exerted from the spherical component appears in Eq. (3), with

\[ \nu_{\text{sph}}^2 = \frac{\partial^2 \Phi_{\text{sph}}(r)}{\partial z^2} \bigg|_{z=0} > 0. \]  

(9)

This term was also introduced by Sellwood (1996) when analyzing the dispersion relation for the longwavelength bending perturbations of an infinitely thin disk, but its role was not studied.

Zasov et al. (1991, 2002) and Mikhailova et al. (2001) concluded that the minimum possible relative thickness of an equilibrium stellar disk, \( z_0/h \), decreases with increasing relative mass of the dark halo. Let us consider the relationship between the stellar disk thickness and the mass of the spheroidal component in terms of the stabilization conditions for the bending modes in inhomogeneous thin disks.

We take specific bulge and halo models (these models were subsequently used in our numerical calculations). A Plummer sphere is taken as the bulge model. Its potential is (see, e.g., Binney and Tremaine (1987), pp. 42–43)

\[ \Phi_b(r) = -\frac{GM_b}{(r^2 + a_b^2)^{1/2}}, \]  

(10)

where \( M_b \) is the total mass of the bulge, and \( a_b \) is the scale length of the matter distribution. We describe the potential of the dark halo in terms of the logarithmic potential (see, e.g., Binney and Tremaine (1987), p. 46)

\[ \Phi_h(r) = \frac{v_\infty^2}{2} \ln(r^2 + a_h^2) + \text{const}, \]  

(11)

where \( a_h \) is the scale length, and \( v_\infty \) is the velocity of a particle in a circular orbit of infinite radius. The parameter \( v_\infty \) is related to the mass of the halo with a sphere of given radius \( r \) by \( M_h(r) = \frac{v_\infty^2}{2} \frac{r^3}{G (r^2 + a_h^2)}. \)

For the additional stabilizing term in the dispersion relation, the models of the spherical components (the bulge and the halo) that we used yield

\[ \nu_b^2 = \frac{GM_b}{(R^2 + a_b^2)^{3/2}}; \quad \nu_h^2 = \frac{v_\infty^2}{R^2 + a_h^2}. \]
The stabilizing effect of the spherical component weakens at large \( R \), but the disk itself in the peripheral regions has a stabilizing effect: \( F_z \approx -GM_d z / R^3 \) at \( |z| << R \Rightarrow \nu_d^2 = -\partial F_z / \partial z > 0 \). On the other hand, the strength of the effect increases in the central (most unstable) regions (i.e., for \( R \to 0 \)); this strength depends not only on the total mass of the spherical component (\( M_b \) or \( v_\infty \)), but also on the degree of matter concentration toward the center, \( a_b \) and \( a_h \). It thus follows that the presence of a compact (not necessarily massive) bulge in galaxies may prove to be enough to suppress the bending perturbations. This implies that the disks of galaxies with compact bulges can be as thin as the disks embedded in a massive dark halo.

To test our conclusion, we carried out a series of numerical simulations.

## 5 THE STABILIZATION OF BENDING PERTURBATIONS BY A COMPACT BULGE: N-BODY SIMULATIONS

### 5.1 The Method

We used an algorithm based on the hierarchical tree construction method (Barnes and Hut 1986) to simulate the evolution of a self-gravitating stellar disk. In our calculations, we always included the quadrupole term in the Laplace expansion of the potential produced by groups of distant bodies. The parameter \( \theta \) (Hernquist 1993) that is responsible for the accuracy of calculating the gravitational force was chosen to be 0.7 in all our simulations. The NEMO software package (http://astro.udm.edu/nemo; Teuben 1995) was taken as the basis. We enhanced the capabilities of this package by including several original codes for specifying the equilibrium initial conditions in a flat stellar system\(^5\) and supplemented it with new codes that allow us to easily analyze the data obtained and to present them in convenient graphical and video formats.

### 5.2 The Numerical Model

When constructing the galaxy model, we distinguished two components in it: a self-gravitating stellar disk and a spherically symmetric component that was described in terms of the external static potential, which is a superposition of two potentials, \([\text{10}]\) for the bulge and \([\text{11}]\) for the dark halo. At large distances \( R \) from the center of the stellar system, in the region where the halo dominates, potential \([\text{11}]\) yields a flat rotation curve. The disk was represented by a system of \( N \) gravitating bodies with the density profile \([\text{6}]\).

The initial conditions in the \( N \)-body problem suggest specifying the mass, position in space, and three velocity components for each particle. The particle coordinates are naturally determined in accordance with the disk matter density profile \([\text{6}]\); the distant regions of the disk are disregarded. We took only those particles for which the cylindrical radius \( R < R_{\text{max}} \) and \( |z| < z_{\text{max}} \). The mass of all particles was assumed to be the

\(^5\)The mkexphot code for specifying equilibrium stellar disk models from the NEMO package has limitations on the parameters of the outer halo and does not enable the gravitational field of the bulge to be specified. The models constructed using our codes closely agree with those obtained using the mkexphot code for identical parameters.
same. The total mass of the particles was equal to the mass of the disk region under consideration (i.e., the disk region for which \( R < R_{\text{max}} \) and \( |z| < z_{\text{max}} \)). The particle velocities were specified using the equilibrium Jeans equations by the standard technique (see, e.g., Hernquist (1993), Section 2.2.3).

### 5.3 Specifying the Velocity Field in the Model Galaxy

To specify the initial particle velocities for a disk that is in equilibrium in the plane and in the vertical direction, we make the following assumptions:

1. The velocity distribution function is the Schwarzschild one; in other words, the particle velocity distribution function has only four nonzero moments: the mean azimuthal velocity \( \bar{v}_\phi \), the radial velocity dispersion \( \sigma_R \), the azimuthal velocity dispersion \( \sigma_\phi \), and the vertical velocity dispersion \( \sigma_z \).

2. All four moments depend only on the cylindrical radius \( R \) and do not depend on \( z \).

3. The epicyclic approximation is valid\(^7\).

4. \( \sigma_R^2 \) is proportional to the surface density of the stellar disk, i.e., \( \sigma_R \propto \exp \left( -R/2h \right) \) (this assumption agrees well with the observational data; see, e.g., van der Kruit and Searle 1981).

The following relations for the moments (in which the \( R \) dependence was omitted for simplicity) can then be derived from the Jeans equations (see, e.g., Binney and Tremaine 1987):

\[
\begin{align*}
\bar{v}_\phi^2 &= v_c^2 + \sigma_R^2 - \sigma_\phi^2 + \frac{R \Sigma d \sigma_R^2}{\Sigma d}, \\
\sigma_\phi &= \sigma_R \frac{\kappa}{2\Omega}, \\
\sigma_z^2 &= \pi \Sigma d z_0,
\end{align*}
\]

where \( v_c \) is the circular velocity of a particle placed in the total potential of the disk and the spherical component, the bulge and the halo, \( v_c^2 = v_{c,d}^2 + v_{c,b}^2 + v_{c,h}^2 \); \( v_{c,b}^2 = R \frac{\partial \Phi_b}{\partial R} \); \( v_{c,h}^2 = R \frac{\partial \Phi_h}{\partial R} \), \( \Omega = \frac{v_c}{R} \) is the angular velocity, and \( \kappa = \sqrt{2 \frac{v_c^2}{R^2} + \frac{1}{R} \frac{dv_c^2}{dR}} \) is the epicyclic frequency.

The circular velocities for the bulge (10) and the halo (11) have analytical expressions. The circular velocity for the disk (6) can be determined by numerical integration (see the section entitled “Global modes”) using the general formula

\[
v_{c,d}^2(r) = \int G \rho_d(r') \cdot \frac{(r' - r) \cdot R}{|r' - r|^3} \, d^3r',
\]

where \( R \) is the projection of the vector \( r \) onto the disk plane.

The Jeans equations that are used to derive relations (12) are known to provide no exact disk equilibrium (see, e.g., Binney and Tremaine 1987). Moreover, the last relation in (12), which follows from the vertical equilibrium condition for a disk with the density

\[^6\text{As many members of the astronomical society, we have the bad habit of calling the standard of the distribution function the dispersion.}

\[^7\text{In the central regions of the disk, this approximation breakdown.}\]
profile and a $z$-independent $\sigma_z$, was written without including the influence of the additional spheroidal components. The adjustment to equilibrium occurs on time scales of the order of several vertical oscillation times $1/\nu_d$. This time was always no longer than $100 - 120$ integration time steps for the equations of motion (the thinner the disk, the shorter this time) and much shorter than the instability growth time scale in the disk. Moreover, in the context of the problem of the growth and saturation of unstable modes, a small deviation of the disk from equilibrium at the initial time may be treated as an additional initial perturbation.

The fourth assumption (see above) about the radial velocity dispersion $\sigma_R^2(R)$ causes difficulties in calculating $\bar{v}_\phi$ in the central regions. In the first equation of system $\bar{v}_\phi$ is occasionally negative at small $R$ (since $\sigma_R^2(R)$ rapidly increases toward the center, the last term on the right-hand side can make a large negative contribution). For this reason, the dependence for $\sigma_R$ was reduced at the center (Hernquist 1993):

$$\sigma_R \propto \exp \left( -\sqrt{R^2 - 2a_s^2/2h} \right). \quad (14)$$

If the parameter $a_s$ is taken to be $h/4 - h/2$, then this proves to be enough to properly calculate $\bar{v}_\phi$. In the central regions of the disk, $\sigma_z$ adjusted to $\sigma_R$ in such a way that the ratio $\sigma_z/\sigma_R$ was constant at a given half-thickness $z_0 = \text{const}$ in the initial model throughout the disk.

The proportionality factor in (14) can be determined via the Toomre parameter $Q_T$ at some radius $R_{\text{ref}}$:

$$\sigma_R(R_{\text{ref}}) = Q_T \sigma_R^c(R_{\text{ref}}) = Q_T \frac{3.36 \cdot \Sigma_d(R_{\text{ref}})}{\kappa(R_{\text{ref}})}. \quad (15)$$

The sought proportionality factor can be obtained from (14) and (15). Specifying $Q_T$, which ensures disk stability in the plane, at $R_{\text{ref}} \approx 2.5h$ yields the condition $Q_T(R) \geq Q_T(R_{\text{ref}})$ (Hernquist 1993). The latter, in turn, ensures a stability level against perturbations in the disk plane no lower than that at $R_{\text{ref}}$. The initial half-thickness $z_0$ for the adopted $Q_T(R_{\text{ref}})$ was chosen in such a way that $\sigma_z/\sigma_R$ was less than $0.3 - 0.4$, which ensured initial instability against the growth of bending modes.

### 5.4 Parameters of the Problem

All of the results discussed below are presented in the following system of units: the gravitational constant is $G = 1$, the unit of length is $R_u = 1$ kpc, and the unit of time is $t_u = 1$ Myr. The unit of mass is then $M_u = R_u^3/Gt_u^2 = 22.2 \cdot 10^{10} M_\odot$, and the unit of velocity is $v_u = R_u/t_u = 978$ km s$^{-1}$.

The number of bodies in the simulations was $N = 300000$ (in several cases, 500000 and 600000). The force of interaction between two particles with coordinates $\mathbf{r}_i$ and $\mathbf{r}_j$ and masses $m_i$ and $m_j$ was modified, as is commonly done, as follows:

$$\mathbf{F}_{ij} = Gm_i m_j \frac{\mathbf{r}_j - \mathbf{r}_i}{(\mathbf{r}_j - \mathbf{r}_i)^2 + \epsilon^2)^{3/2}},$$

where $\epsilon$ is the softening length of the potential produced by an individual particle. When collisionless systems are simulated, this parameter is introduced for two reasons. First, the
divergence of the interaction force in close particle–particle encounters must be avoided when integrating the equations of motion. Second, when the phase density of a collisionless system is represented by a finite number of particles, the inevitable fluctuations in the particle distribution must be smoothed in such a way that the forces acting in the system being simulated are to the forces acting between the particles in a system with a smoother density profile. The softening length $\epsilon$ was chosen to be 0.02. This value is approximately a factor of 2 or 3 smaller than the mean separation between the particles (at $N = 300000$) within the region containing half of the disk mass. On the one hand, it matches the criterion for choosing $\epsilon$ based on minimization of the mean irregular force (Merritt 1996) and, on the other hand, allows the vertical structure of thin disks to be adequately resolved.

To integrate the equations of motion for particles in the self-consistent potential of the disk and the external field produced by the spheroidal component, we used a leapfrog scheme that ensured the second order of accuracy in time step. The time step was 0.5 (in several models, 0.25)\(^8\).

We constructed a total of about 60 models. The entire set of models can be arbitrarily divided into two classes: the models with and without bulges. The scale length of the density distribution in the bulge $a_b$ was assumed to be equal to 0.5 almost for all of the models with bulges. In several models without bulges, we chose a concentrated halo ($a_h = 2$). In all of the remaining cases, the halo was “looser” ($a_h = 10$). The total relative mass of the spheroidal components $\mu = M_{\text{sph}}(4h)/M_d(4h)$ was varied over the range 0.25 to 3.5

The disk in our models has the following parameters\(^9\): $h = 3.5$ and the disk mass (in dimensional units) $M_d(4h) = (4 - 8) \times 10^{10} M_\odot$. The initial thickness was varied over the range $z_0 = 0.1 - 0.5$. In order not to abruptly cut off the model disk at the radius corresponding to the optical radius ($\sim 4h$), we chose $R_{\text{max}} = 25$, with $z_{\text{max}} = 5$. The smoothing parameter of the initial radial profile of the velocity dispersion $\sigma_R$ is $a_s = 1$. The parameter $Q_T$ in the discussion of our simulations is given for the radius $R_{\text{ref}} = 8.5$.

### 5.5 Simulation Results

Previously (Sotnikova and Rodionov 2003), we showed that there are two distinct vertical stellar disk relaxation mechanisms related to bending instability: the bending instability of the entire disk and the bending instability of the bar forming in the disk. The former mechanism dominated in galaxies that are hot in the plane ($Q_T > 2.0$), and the bar formation was suppressed in this case; the latter mechanism dominated in galaxies with a moderate Toomre parameter $Q_T$ (such galaxies were unstable against the growth of a bar mode). In the simulations whose results are presented and analyzed below, we also considered two distinct cases: hot disks ($Q_T = 2.0$) in which only bending instability developed, and cooler models ($Q_T = 1.5$) — here, we observed the combined effect of the two types of instability.

**Hot disks.** For hot (in the plane) disks ($Q_T = 2.0$), we revealed distinct patterns of growth and saturation of bending perturbations that are consistent with the conclusions\(^8\)

\(^8\)The choice of the time step is limited above by $\epsilon$ — the particle must take at least one step on the smoothing length.

\(^9\)These parameters are close those of the disk in our Galaxy.
following from a qualitative analysis of the dispersion relation (3).

The stabilizing effect from the presence of a massive spherical component that was discussed in the section entitled “The stabilizing effect of the spheroidal component...” is clearly seen in Fig. 2. This figure shows the radial profile of \( \sigma_z/\sigma_R \) for model 12 with \( M_h = 0 \) and \( \mu = 3.0 \). Throughout the disk, \( \sigma_z/\sigma_R \) was set at \( \approx 0.35 \) and was determined not by the bending instability, but by the disk heating through the scattering of stars by inhomogeneities related to the different disk thickness in the spiral arms and in the interarm space (Sotnikova and Rodionov 2003).

We will demonstrate the stabilizing effect of a compact (not necessarily massive) bulge comparable to the effect of a massive dark halo with a broader density profile using the results obtained for the following group of models as an example: 50, 76, 75, 49, and 53. In all five models, the total mass of the spheroidal component is the same and accounts for half of the disk mass within four exponential disk scale lengths, \( \mu = 0.5 \), but it is differently distributed between the two spherical subcomponents. In model 50, all of the mass is contained in the halo \( \mu = \mu_h = M_h/(4h)/M_d(4h) \); in model 53, only a compact bulge is present \( \mu = \mu_b = M_b/(4h)/M_d(4h) \). The remaining models are intermediate between the two extreme models. The initial thickness for all of the models was chosen to be the same, \( z_0 = 0.1 \). The rotation curves for these models are shown in Fig. 3. The variety of the shown curves to some extent reflects the actual variety of rotation curves for spiral galaxies.

We traced the evolution of these models up to \( t = 5000 \). Figure 4 illustrates the variations in the dynamical parameters of the disk \( \sigma_R \) and \( \sigma_z \) the radial and vertical velocity dispersions calculated at \( R = 2h \). All of the models demonstrate an initial increase in \( \sigma_z \) and a decrease in \( \sigma_R \). Subsequently (after \( t \approx 1000 \)), the latter parameter reaches an approximately constant value, while \( \sigma_z \) for some of the models (this is primarily true for the model with a massive bulge) continues to slowly increase. The number of particles in our models and the softening length were chosen in such a way that the two-body relaxation time was much longer than the time scale on which we considered the evolution of our numerical models. The absence of heating related to numerical relaxation is confirmed by the behavior of \( \sigma_R \) and the preservation of the pattern of evolution of the system as the number of particles increases to \( N = 600000 \). The continuing small secular increase in the vertical velocity dispersion probably reflects the fact that some of our models did not reach a steady state\(^{11}\).

Figure 5 shows five frames that correspond to the late evolutionary stages of our model disks. As expected, the saturation level for the bending instability in model 50 was very high and did not match the standard linear criterion. The galaxy greatly thickened at the final evolutionary stages. However, when we transferred 50% of the mass from the halo to the bulge (model 49: \( \mu_h = \mu_b = 0.25 \)), the picture changed. The saturation level for the bending instability became much lower. At the final evolutionary stages, the disk was much thinner than that in model 50. In model 53, when we placed all of the mass of the spherical component in the bulge, the amplitude of the observed bend was very low, and the galaxy remained quite thin even at the late evolutionary stages.

\[^{10}\]In model 53, a bulge with a mass equal to half of the disk mass and a scale length of 500 pc is atypical of real galaxies. We consider this as a limiting case.

\[^{11}\]If the system has no third integral of motion, then its evolution to equilibrium must eventually lead to the relation \( \sigma_z = \sigma_R \).
The disk thickness can be quantitatively estimated as the root-mean-square (rms) value of the \( z \) coordinates of the disk particles, \( z_{\text{rms}} = \sqrt{< z^2 > - < z >^2} \). This estimate is commonly encountered in the literature. It can be shown that for the vertical density profile \( \rho \), the relationship between this parameter and \( z \) is given by \( z_{\text{rms}} = \frac{\pi}{2\sqrt{3}} z_0 \approx 0.91 z_0 \). In practice, however, this parameter proved to be a not very good characteristic of the thickness. First, the fluctuations in this parameter along \( R \) were found to be great even when using a large number of particles if only no averaging is performed in concentric rings of large width. Second, the thickness calculated in this way turns out to be systematically overestimated due to the existing of a significant tail of the particles that went far from the disk plane. For these reasons, we estimated the disk thickness at a given distance \( R \) through the median of the absolute value of \( z \) that was designated as \( z_{1/2} \). Twice the value of \( z_{1/2} \) is nothing but the disk thickness within which half of the particles is contained. For the density profile \( \rho \), \( 2z_{1/2} = z_0 \cdot \ln 3 \approx 1.1 z_0 \).

Figure 6 shows the differences between the radial disk thickness profiles for model 53 obtained by the two described methods (the averaging was performed in concentric rings; the ring width was \( \Delta R = 0.4 \)). We see that \( z_{1/2} \) behaves much more smoothly (we have in mind the overall monotonic dependence of the density and the fluctuation level) than does the rms value of the \( z \) coordinate commonly used to estimate the disk thickness in \( N \)-body simulations.

Figure 7 shows the radial disk thickness profile for models 50, 76, 75, 49, and 53 at the time \( t = 3000 \). Note that the thickness profile for model 50 is rather unusual in shape. This shape is most likely attributable to the existence of X-shaped stationary orbits in the central regions that arise at a certain disk thickness when there are conditions for the resonance between the stellar oscillation frequencies in the disk plane and in the vertical direction (see, e.g., Patsis et al. 2002). We see the following from Fig. 7 as well as from Fig. 6 which show the edge-on views of the model galaxies: the thinner the galactic disk, the larger the mass of the spheroidal component contained in a compact bulge. Since not all of our models reached a steady state, their thickness continues to slowly increase (Fig. 8), but the differences in thickness are always preserved. Similar results were obtained for all of the remaining such models with the same mass of the spherical component in the range \( M_{\text{sph}}(4h) = 0.25M_d(4h) \) to \( M_{\text{sph}}(4h) = 3.5M_d(4h) \). The stabilizing effect of a bulge was particularly pronounced in those cases where the bulk of the galactic mass was contained in the disk.

The final disk thickness at fixed initial \( Q_T \) was determined only by the relative mass of the spheroidal component and the contribution of the bulge to this mass and did not depend on how far from stability the initial state of the disk was chosen. The start from different initial disk thicknesses led to models without any systematic differences between them. Figure 9 illustrates this result, which is similar to that obtained in their numerical simulations by Sellwood and Merritt (1994).

Thus, our three-dimensional calculations are in good agreement with the conclusion following from our analysis of the dispersion relation for a thin disk that a bulge is an effective stabilizing factor during the growth of bending instability. Moreover, since the initial bend is formed in the most unstable central part of the galaxy (Sotnikova and Rodionov 2003), it is the central regions that must be stabilized. This does not require a massive dark halo; the presence of a compact spherical component like a bulge will suffice. This suggests that the final thickness of the model galaxy depends not only on the total
mass of the spherical component, but also on the mass distribution in it.

Barred galaxies. The bending instability of bars is an effective vertical disk heating mechanism for galaxies unstable against the formation of a bar (Raha et al. 1991; Sotnikova and Rodionov 2003). In contrast to the warp in the entire disk, the warp in the bar is formed not in the central regions, but in the entire bar simultaneously (this is seen particularly clearly in our color two-dimensional histograms of the warp accessible at http://www.astro.spbu.ru/staff/seger/articles/warps_2002/fig6_web.html and http://www.astro.spbu.ru/staff/seger/articles/warps_2002/fig7_web.html). Therefore, the conclusion that a compact bulge during the growth of bending instability in a bar will have the same effective stabilizing effect as that for hot stellar disks is not obvious in advance.

In our simulations with $Q_T = 1.5$ and $\mu \lesssim 1.0$, the warp in the bar was formed early, at $t \approx 800$. The presence of a compact bulge eventually led to the formation of thinner disks, although the effect itself was fairly complex.

Within $R < 1.5h$, the most prominent features of the bars at late evolutionary stages were X-shaped structures. If the disk is viewed edge-on, then they manifest themselves as a bulge with an appreciable extent in the $z$ direction with boxy isophotes. In the region $R < 1.5h$ where the bar dominated, we failed to reveal any distinct patterns in the model disk thickness variations with increasing contribution of the bulge to the total mass of the spheroidal component. In general, the presence of a bulge “pushed forward” the bar formation time and caused the saturation level for the bending instability of a bar to lower. However, a further analysis is required to completely understand the processes during the interaction between a compact bulge and a bar.

As regards the peripheral regions of the disks ($R > 1.5h$), they were always appreciably thinner in the models with a compact bulge at late evolutionary stages. Figure 10 demonstrates this effect for two groups of models with different bulge contributions: models 52 and 51 with $\mu = 0.5$ and models 43 and 42 with $\mu = 0.875$. The differences in thickness show up most clearly in the models with a small relative mass of the spheroidal component. When $\mu$ increases, the disk becomes very thin, as might be expected, and its thickness ceases to depend on how the mass is distributed between the halo and the bulge.

6 CONCLUSIONS

A comparison of the conclusions that follow from a linear analysis with the results of numerical simulations for three-dimensional disks shows that, in contrast to homogeneous models, global bending modes with a wavelength longer than the disk scale length can arise in inhomogeneous disks. If the amplitude of the waves during the growth of instability increases significantly, then they heat it up significantly in the vertical direction as they pass through the entire disk. Hot disks are most unstable against the growth of bending perturbations. An additional spheroidal component, for example, a dark halo is a factor that stabilizes the bending perturbations.

Our additional qualitative analysis of the dispersion relation for inhomogeneous models led us to new conclusions regarding the stabilization conditions for the bending modes in stellar disks. These conclusions were confirmed in our numerical simulations.
(1) Since the central regions of the disk (particularly if the disk is hot) are most unstable, the conditions under which the growth of perturbations is suppressed are determined not only by the mass of the spherical component, but also by the density distribution in it. The suppressing effect is enhanced with increasing concentration toward the center.

(2) The presence of a compact and moderately massive bulge in a galaxy effectively prevents the growth of bending perturbations.

(3) It follows from an analysis of the entire set of our results that a more accurate approach to estimating the dark halo mass from the observed relative thickness of the stellar disk $z_0/h$ in spiral galaxies is required.

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Fig. 1: (a) Magnitude of the vertical gravitational force ($-F_z$) versus $z$ at various distances $R$ from the disk center; (b) the square of the vertical oscillation frequency $\nu^2 = -\partial F_z/\partial z$ versus $z$ for various $R$. We took $G = 1$ and the disk parameters $M_d = 1$ (the total disk mass), $h = 3.5$, and $z_0 = 0.1$. 
Fig. 2: Ratio $\sigma_z/\sigma_R$ versus $R$ for the time $t = 3000$ (model 12 with a massive halo: $\mu = 3.0, \mu_b = 0$).
Fig. 3: Initial rotation curves for models 50, 76, 75, 49, and 53. The ratio of the total mass of the spherical component to the mass of the disk within a radius of $4h$ is the same for all models, $\mu = 0.5$; $\mu_b = 0$ for model 50, $\mu_b = 0.0625$ for model 76, $\mu_b = 0.125$ for model 75, $\mu_b = 0.25$ for model 49, and $\mu_b = 0.5$ for model 53. The unit of velocity is $978$ km s$^{-1}$.
Fig. 4: Evolution of the velocity dispersions $\sigma_R$ and $\sigma_z$ at $R = 2h$ for the same models as those in Fig. 3.
Fig. 5: Edge-on view of the galaxy at the time $t = 3000$ for models 50, 76, 75, 49, and 53. The strength of the image blackening corresponds to the logarithm of the particle number per pixel. The horizontal and vertical scales are 60 and 10, respectively. The ratio of the total mass of the spherical component to the mass of the disk within a radius of $4h$ is the same for all models, $\mu = 0.5$. 
Fig. 6: Radial thickness profiles for the galaxy at the time $t = 3000$ for model 53. The thickness was determined by two methods: as the rms value of the $z$ coordinates of the disk particles, $z_{\text{rms}}$, and as twice the median of $|z|$, $2z_{1/2}$. 
Fig. 7: Radial thickness profiles for the galaxy ($2z_{1/2}$) at the time $t = 3000$ for models 50, 76, 75, 49, and 53.
Fig. 8: Evolution of the disk thickness ($2z_{1/2}$) at $R = 2h$ for models 50, 76, 75, 49, and 53 (see the caption to Fig. 3).
Fig. 9: Radial thickness profiles for the galaxy $(2z_{1/2})$ at the time $t = 3000$ for models that differ only by the initial thickness. For all of the models, $\mu = 0.6$ and $\mu_b = 0$. Models 11_1, 11_2, and 11_3 are different random realizations of a stellar system with $z_0 = 0.1$; models 25 and 25_1 are different random realizations of a system with $z_0 = 0.2$; model 26_1 is for $z_0 = 0.3$. 
Fig. 10: Radial thickness profile for the galaxy \((2z_{1/2})\) at the time \(t = 3000\) for models unstable against the growth of a bar mode \((Q_T = 1.5)\): (a) models with \(\mu = 0.5\) (model 52 with \(\mu_b = 0\), model 51 with \(\mu_b = 0.25\)), (b) models with \(\mu = 0.875\) (model 43 with \(\mu_b = 0.25\), model 42 with \(\mu_b = 0.25\)); the initial half-thickness for all of the models is \(z_0 = 0.1\).