Semialgebraic Solution of Linear Equation with Continuous Semialgebraic Coefficients

Marcello Malagutti

13th September 2022

1 Introduction

C. Fefferman gave in [1], by means of analysis techniques, a necessary and sufficient condition for the existence of a continuous solution \((\phi_1, \cdots, \phi_s)\) of the system

\[
\phi = \sum_{i=1}^{s} \phi_i f_i
\]

(1.1)
given the continuous functions \(\phi\) and \(f_i\). More precisely, C. Fefferman, applying the theory of the Glaeser refinements for bundles, proved that system (1.1) has a continuous solution iff the affine Glaeser-stable bundle associated with system (1.1) has no empty fiber.

Moreover J. Kollár, in the same (joint) paper [1], starting from the above result and making use of algebraic geometry techniques as blowing up and at singular points, proved that fixed the polynomials \(f_1, \ldots, f_s\) and assuming system (1.1) solvable, then:

1) if \(\phi\) is semialgebraic then there is a solution \((\psi_1, \cdots, \psi_s)\) of \(\phi = \sum_{i} \psi_i f_i\) such that the \(\psi_i\) are also semialgebraic;

2) let \(U \subset \mathbb{R}^n \setminus Z\) (where \(Z := (f_1 = \cdots = f_r = 0)\)) be an open set such that \(\phi\) is \(C^m\) on \(U\) for some \(1 \leq m \leq \infty\) or \(m = \omega\). Then there is a solution
ψ = (ψ_1, ⋯, ψ_s) of φ = \sum_{i=1}^{s} ψ_i f_i such that the ψ_i are also C^m on U.

In this paper we generalise and prove, by using Fefferman’s techniques, a part of the above results shown by Kollár. More in details, we consider a compact metric space \( Q \subseteq \mathbb{R}^n \) and a system of linear equations

\[
A(x) \phi(x) = \gamma(x), \quad x = (x_1, \ldots, x_n) \in Q
\]

where

\[
Q \ni x \mapsto A(x) = (a_{ij}(x)) \in M_{r,s}(\mathbb{R})
\]

is continuous semialgebraic, with \( M_{r,s}(\mathbb{R}) \) denoting the set of real \( r \times s \) matrices and

\[
Q \ni x \mapsto \gamma(x) \in \mathbb{R}^r, \quad \gamma(x) = \begin{bmatrix} \gamma_1(x) \\ \vdots \\ \gamma_r(x) \end{bmatrix} \in \mathbb{R}^r
\]

being themselves continuous semialgebraic functions on \( Q \subseteq \mathbb{R}^n \).

Our aim is to find a necessary and sufficient condition for the existence of a solution \( Q \ni x \mapsto \phi(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_s(x) \end{bmatrix} \in \mathbb{R}^s \) of system (1.2), with the \( \phi_i : Q \rightarrow \mathbb{R} \) continuous and semialgebraic.

In particular, we find that a continuous and semialgebraic solution exists if and only if there exits a continuous solution and a semialgebraic one (they may possibly be different) under the hypothesis that

\[
B(x, r_{v_x}) \ni y \mapsto \gamma_{v_x}(y) = \Pi_{H^{(0)}_y} v_x
\]

is discontinuous at most at isolated points for each \( x \in Q \) where:

- \( H^{(0)}_y \) is the fiber at \( y \in Q \) of the singular affine bundle associated to system (1.2);
- \( B(x, r_{v_x}) \subset Q \) is an euclidean ball of small radius;
- \( v_x \) a vector of \( H^{(0)}_x \);
\[ - \Pi_{H^0_y} v_x \text{ the projection of } v_x \text{ on } H^0_y. \]

Moreover, we show how to calculate a continuous and semialgebraic solution of system (1.2).

2 The setting

Let us start by setting some notations and by showing some important preliminary results that will be used to pursue our goal. We shall endow every \( \mathbb{R}^s \) used here with euclidean norm.

**Notation 1:** Let \( V \subseteq \mathbb{R}^s \) be an affine space in \( \mathbb{R}^s \) and \( w \in \mathbb{R}^s \). We denote the projection of \( w \) on \( V \) (i.e. the point \( v \in V \) that makes the euclidean norm of \( v - w \) as small as possible) by \( \Pi_V w \).

**Notation 2:** If \( x \in Q \), consider \( Q \ni x \mapsto -\rightarrow A(x) \in M_{r,s}(\mathbb{R}) \), we denote \( \Pi_1(x) w = \Pi_{\text{Ker}A(x)^\perp} w, \quad \Pi_2(x) w = \Pi_{\text{Ker}A(x)^\perp} w. \)

**Notation 3:** We denote the \( i \)-th column of a matrix \( A \) by \( A_i \).

**Lemma 2.1.** Consider

- \( Q \subseteq \mathbb{R}^n \) a compact space,
- \( Q \ni x \mapsto A(x) \in M_{r,s}(\mathbb{R}) \) a matrix valued semialgebraic function,
- \( \phi : Q \to \mathbb{R}^s \) a map,

and let \( Q \ni x \mapsto p(x) = \Pi_1(x) \phi(x) \) be the projection of \( \phi \) on \( \text{Ker}A(x)^\perp \). If \( Q \ni x \mapsto \phi(x) \) is a semialgebraic function on \( Q \) then \( Q \ni x \mapsto p(x) \) is a semialgebraic function on \( Q \).

**Proof.** First of all notice that

\[ \text{Ker}A(x)^\perp = \text{Span}\{A(x)^T_i\}_{i \in \{1, \ldots, r\}} = \text{Span}\{\text{rows of } A(x)\}. \]

Then consider for a given \( I \subseteq \{1, \ldots, r\} \)

\[ K(I) = \{x \in Q : (A(x)^T_i)_{i \in I} \text{ is a basis of } \text{Im}A(x)^T\}. \]
The idea we want to pursue is to project the solution $Q \ni x \mapsto \phi(x)$ on $\ker A(x)^\perp$ and apply the Tarski-Seidenberg theorem, thus concluding that the projection is semialgebraic. Actually, to apply the Tarski-Seidenberg theorem we need that the dimension of the projection space be independent of $x \in Q$, so we will be localizing the problem on the $K(I)$. As a matter of fact on $K(I)$ the dimension of $\ker A(x)^\perp = \text{Span}\{A(x)^T_i\}_{i \in \{1, \ldots, r\}} = \text{Im}A(x)^T$ is constant by definition of $K(I)$. From the semialgebraicity of $Q \ni x \mapsto A(x)$ it is then trivial to deduce that $K(I)$ is a semialgebraic (possibly empty) subset on $Q$ since:

- the linear independence of $(A(x)^T_i)_{i \in I}$ can be translated in terms of the minors of $A(x)$;
- the condition that the vectors of $(A(x)^T_i)_{i \in I}$ are generators of $\text{Im}A(x)^T$ can be expressed by the following first-order formula

$$\forall v \in \text{Im}A(x)^T, \exists (\lambda^T_i)_{i \in I} \text{ real numbers s.t. } v = \sum_{i} \lambda_i A(x)^T_i.$$ 

As $\text{Im}A(x)^T$ is semialgebraic by definition we get that that condition is semialgebraic.

This is the reason why it suffices to prove the lemma for $K(I)$ since if a function is semialgebraic on a finite collection of semialgebraic sets then it is semialgebraic on their union:

- if $K(I) = \emptyset$ there is nothing to prove;
- if $K(I) \neq \emptyset$ let

$$\Gamma(I) = \{(x, \phi(x)) : x \in K(I)\}.$$

- If $s \geq r$ we define

$$V = \left\{(x,y) : y = A(x)^T \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{pmatrix}, x \in K(I), \lambda_i \in \mathbb{R} \right\} \subseteq \mathbb{R}^n \times \mathbb{R}^s.$$
If \( s < r \) we define

\[
V = \left\{ (x, y) : \quad y = \tilde{A}(x)^T \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{|I|} \end{pmatrix}, \quad x \in K(I), \quad \lambda_i \in \mathbb{R} \right\} \subseteq \mathbb{R}^n \times \mathbb{R}^s
\]

where \( \tilde{A}(x) \in M_{|I|, s}(\mathbb{R}) \) and the columns of \( \tilde{A}(x)^T \) form a basis of \( \text{Im}A(x)^T \) given by \( (A(x)^T)^i \) for \( i \in I \). In this case we will still write \( A(x) \) in place of \( \tilde{A}(x) \) and we will still write \( r \) for \( |I| \).

Let us now factorize \( A(x)^T \) by QR decomposition: \( A(x)^T = Q(x)R(x) \). As a matter of fact if \( m \geq n \) every \( m \times n \) matrix can be written as the product of a squared orthogonal \( m \times m \) matrix \( Q \) and a rectangular \( m \times n \) matrix \( R \) with the blockwise structure

\[
R = \begin{pmatrix} R_1 \\ 0 \end{pmatrix}
\]

where \( R_1 \) is an \( n \times n \) upper triangular matrix and 0 is the \( (m - n) \times n \) zero matrix.

The set \( \Gamma(I) \) is semialgebraic since it is the graph of a semialgebraic function, hence

\[
\Gamma'(I) = \left\{ \begin{pmatrix} I_n & 0 \\ 0 & Q(x)^T \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} : \quad (x, y) \in \Gamma(I) \right\}
\]

is semialgebraic too. In fact, \( Q(x) \) has semialgebraic entries since the ones of \( A(x) \) are semialgebraic and the QR decomposition can be computed by multiplying iteratively \( A(x) \) by appropriate Householder matrices. These matrices are constructed in the following way.

Letting \( \| \cdot \| \) be the euclidean norm, \( z = (A(x)^T)^1, v = z + \| z \| e_1, \alpha = \frac{\| v \|^2}{2} \) and \( U_1 = I_s - \frac{v e^T}{\alpha} \) where \( I_s \) is the \( s \times s \) identity matrix we have that \( U_1 z = -\| z \| e_1 \).

Now, repeating the procedure on the minor of \( A(x)^T \) obtained by eliminating the first row and column (the new \( U_2 \) matrix has the form \( \begin{pmatrix} I_1 & 0 \\ 0 & U' \end{pmatrix} \) where \( U' \) is calculated as \( U_1 \) mutatis mutandis) we reach the goal. The \( Q \) matrix of the QR decomposition is then the transpose of the product of all the \( U_i \).
constructed in the previous way. The result is semialgebraic because in the
construction we used only sums, products and square root extractions of
semialgebraic functions (as the entries of \( A(x) \) are semialgebraic) that are
semialgebraic by definition of semialgebraic function.

We next put
\[
V' = \left\{ \begin{pmatrix} I_n & 0 \\ 0 & Q(x)^T \end{pmatrix} v : v \in V \right\}
\]
and see that if \( z \in V' \) then, by the definition of \( V' \), we can write
\[
z = (z_1, \ldots, z_{n+r}, 0, \ldots, 0)^T.
\]
It is important to observe that \( V' \) is obviously a vector space of \( \mathbb{R}^{n+s} \).

Now, recalling the Tarski-Seidenberg theorem\(^1\), we notice that the pro-
jection \( \widehat{\Gamma}'(I) \) of \( \Gamma'(I) \) onto \( V' \) is semialgebraic. It follows that
\[
\widehat{\Gamma}'(I) = \left\{ \begin{pmatrix} I_n & 0 \\ 0 & Q(x) \end{pmatrix} v, \ v \in \widehat{\Gamma}'(I) \right\}
\]
is semialgebraic too. By construction, \( \widehat{\Gamma}'(I) \) is the graph of \( \Pi_1(x) \phi(x), x \in K(I) \) and so the proof of Lemma 2.1 is complete.

\[\text{Theorem 2.2.} \text{ Consider a compact metric space } Q \subseteq \mathbb{R}^n \text{ and the system } \begin{align*}
A(x) \phi(x) &= \gamma(x), \ x \in Q,
\end{align*}\]
for semialgebraic continuous \( A \) and \( \gamma \).

The system has a semialgebraic solution \( \phi_0 : Q \to \mathbb{R}^s \) iff, given a solution
\( \phi_1 : Q \to \mathbb{R}^s \) of the system,
\[
p(x) = \Pi_1(x) \phi_1(x) \text{ is semialgebraic.}
\]

---

\(^1\text{Tarski-Seidenberg Theorem} \) Let \( A \) a semialgebraic subset of \( \mathbb{R}^{n+1} \) and \( \pi : \mathbb{R}^{n+1} \to \mathbb{R}^n \), the projection on the first \( n \) coordinates. Then \( \pi(A) \) is a semialgebraic subset of \( \mathbb{R}^n \).

\textbf{Corollary} If \( A \) is a semialgebraic subset of \( \mathbb{R}^{n+k} \), its image by the projection on the space of the first \( n \) coordinates is a semialgebraic subset of \( \mathbb{R}^n \).
Proof. First of all we show that given a solution \( \phi_1 : Q \rightarrow \mathbb{R}^s \) of the system, if \( Q \ni x \mapsto p(x) = \Pi_1(x) \phi_1(x) \) is semialgebraic then there exists a semialgebraic solution of the system \( \text{(1.2)} \). Notice that

\[
\phi_1(x) = \Pi_2(x) \phi_1(x) + \Pi_1(x) \phi_1(x), \quad \forall x \in Q.
\]

From this we get that

\[
\gamma(x) = A(x) \phi_1(x) = A(x) \Pi_1(x) \phi_1(x), \quad \forall x \in Q
\]

So \( Q \ni x \mapsto p(x) \) is a semialgebraic solution of the system.

Conversely we show that, given a solution \( \phi_1 : Q \rightarrow \mathbb{R}^s \) of the system, if the system has a semialgebraic solution \( \phi_0 : Q \rightarrow \mathbb{R}^s \) then

\[
Q \ni x \mapsto p(x) = \Pi_1(x) \phi_1(x) \text{ is semialgebraic.}
\]

In fact

\[
A(x)(\phi_0(x) - \phi_1(x)) = \gamma(x) - \gamma(x) = 0, \quad \forall x \in Q.
\]

Decomposing \( \phi_0(x) \) and \( \phi_1(x) \) into their components onto \( \text{Ker} A(x) \) and \( \text{Ker} A(x)^\perp \) we get, respectively,

\[
\phi_0(x) = \Pi_2(x) \phi_0(x) + \Pi_1(x) \phi_0(x), \quad \forall x \in Q,
\]

\[
\phi_1(x) = \Pi_2(x) \phi_1(x) + \Pi_1(x) \phi_1(x), \quad \forall x \in Q.
\]

Hence

\[
A(x)(\Pi_1(x) \phi_0(x) - \Pi_1(x) \phi_1(x)) = 0, \quad \forall x \in Q,
\]

so that

\[
\Pi_1(x) \phi_0(x) - \Pi_1(x) \phi_1(x) \in \text{Ker} A(x) \bigcap \text{Ker} A(x)^\perp = \{0\}.
\]

Therefore

\[
\Pi_1(x) \phi_1(x) = \Pi_1(x) \phi_0(x),
\]

and since \( Q \ni x \mapsto \Pi_1(x) \phi_0(x) \) is semialgebraic by Lemma \([2.1]\) so is
Let us notice that Theorem 2.2 shows also that, given system (1.2), \( Q \ni x \mapsto \Pi_1(x) \phi_0(x) \) is uniquely defined on \( Q \) since it is independent of the solution \( \phi_0 \).

At this point let us consider a singular affine bundle (or bundle for short) (see [1]), meaning a family \( \mathcal{H} = (H_x)_{x \in Q} \) of affine subspaces \( H_x \subseteq \mathbb{R}^s \), parametrized by the points \( x \in Q \). The affine subspaces

\[
H_x = \{ \lambda \in \mathbb{R}^s : A(x) \lambda = \gamma(x) \}, \quad x \in Q
\]

are the fibers of the bundle \( \mathcal{H} \). (Here, we allow the empty set \( \emptyset \) and the whole space \( \mathbb{R}^s \) as affine subspaces of \( \mathbb{R}^s \).)

Now we define \( \mathcal{H}^{(k)} \) to be the \( k \)-th Glaeser refinement of \( \mathcal{H} \) and \( \mathcal{H}^{Gl} \) to be the Glaeser-stable refinement of \( \mathcal{H} \) (their fibers will respectively be denoted by \( H^{(k)}_x \) and \( H^{Gl}_x \), \( \forall x \in Q \)). Notice that the projection on the fibers of \( \mathcal{H} \) is not linear as the fibers are affine spaces and not vector spaces.

**Lemma 2.3.** Consider a compact metric space \( (Q, d_Q), Q \subseteq \mathbb{R}^n \) and an \( r \times s \) system of linear equations

\[
A(x) \phi(x) = \gamma(x), \quad x \in Q
\]

where for each \( x \in \mathbb{R}^r \) the entries of

\[
A(x) = (a_{ij}(x)) \in M_{r,s}(\mathbb{R}) \quad \text{and} \quad \gamma(x) = (\gamma_i(x)) \in \mathbb{R}^r
\]

are themselves semialgebraic functions on \( \mathbb{R}^n \).

If there is a semialgebraic solution \( \phi : Q \to \mathbb{R}^s \) of the system then \( \forall x \in Q, \forall v_x \in H^{Gl}_x \):

\[
Q \ni y \mapsto \gamma_{v_x}(y) := \Pi_{H^{(0)}_y} v_x \text{ is semialgebraic.}
\]
Proof. Consider $x \in Q$ and $v_x \in H^G_x$. We define

$$\gamma_{v_x}(y) := \Pi_{H^G_y}v_x, \quad \forall y \in Q.$$ 

By the definition of $H^G_y$ it is true that

$$H^G_y = \text{Ker}A(y) + \Pi_1(y)\phi(y), \quad \forall y \in Q,$$

and from this expression that

$$\Pi_{H^G_y}v_x = \Pi_1(y)\phi(y) + \Pi_2(y)v_x, \quad \forall y \in Q,$$

Now, $Q \ni y \mapsto \Pi_2(y)v_x = v_x - \Pi_1(y)v_x$ is semialgebraic, as $Q \ni y \mapsto v_x$ is semialgebraic (since it is constant) and so $Q \ni y \mapsto \Pi_1(y)v_x$ is semialgebraic by Lemma 2.1. In conclusion, $Q \ni y \mapsto \Pi_{H^G_y}v_x$ is semialgebraic, for it is the sum of semialgebraic functions (recall that $Q \ni y \mapsto \Pi_1(y)\phi(y)$ is also semialgebraic by Lemma 2.1 since $\phi$ is a semialgebraic solution of system (1.2)).

After this let us introduce a new notion.

**Definition 2.4.** Consider a compact metric space $(Q,d), Q \subseteq \mathbb{R}^n$. Given $\mathcal{H} = (H_x)_{x \in Q}$ whose Glaeser-stable subbundle is denoted by $(H^G_x)_{x \in Q}$, a **semialgebraic Glaeser-stable bundle** associated with the system (1.2) is a family $\tilde{\mathcal{H}}^G = (\tilde{H}^G_x)_{x \in Q}$ of affine subspaces $\tilde{H}^G_x \subseteq \mathbb{R}^s$, parametrized by the points $x \in Q$, where the fibers $\tilde{H}^G_x$ are given by:

$$\tilde{H}^G_x = \{ v \in H^G_x : \exists r_v \in \mathbb{R}^+ \text{ s.t. } \}$$

$$B(x,r_v) \ni y \mapsto \gamma_v(y) = \Pi_{H^G_y}v \text{ is semialgebraic}.\}

It is important to notice that $\tilde{H}^G$ is indeed a bundle, for $\tilde{H}^G_x$ is an affine space, for all $x \in Q$. As a matter of fact:

- if $\tilde{H}^G_x = \emptyset$ the space is affine as we assume the empty space to be an affine space;
− if $\bar{H}_x^G \neq \emptyset$, given $v_0, v_1, v_2 \in \bar{H}_x^G$ and $\lambda \in \mathbb{R}$, we have that

$$(v_1 - v_0) + \lambda(v_2 - v_0) + v_0 \in \bar{H}_x^G.$$  \hspace{1cm} (2.1)

Property (2.1) holds because $H_x^G$ is an affine space and since $v_0, v_1, v_2 \in \bar{H}_x^G \subseteq H_x^G$ we have

$$(v_1 - v_0) + \lambda(v_2 - v_0) + v_0 \in H_x^G.$$  

Moreover

$$\forall i \in \{0, 1, 2\}, \exists r_{v_i} \in \mathbb{R}^+ \text{ s.t. } B(x, r_{v_i}) \ni y \mapsto \gamma_i(y) = \Pi_{H_y^G} v_i \text{ is semialgebraic.}$$

Considering $\tau = \min\{r_{v_0}, r_{v_1}, r_{v_2}\}$ we get that on $B(x, \tau)$

$$\gamma(y) := \Pi_{H_y^G}(v_1 - v_0) + \lambda(v_2 - v_0) + v_0).$$

We now consider the orthogonal decomposition of $\gamma(y)$ on to $\ker A(y)$ and $\ker A(y)^\perp$

$$\gamma(y) = \Pi_1(y)\Pi_{H_y^G}(v_1 + \lambda(v_2 - v_0)) + \Pi_2(y)\Pi_{H_y^G}(v_1 + \lambda(v_2 - v_0)).$$

Recalling then that the projection of a solution of system (1.2) on $\ker A(y)^\perp$ is unique and considering $B(x, \tau) \ni y \mapsto p(y) = \Pi_1\gamma_{v_i}(y), \ i = 0, 1, 2$, gives

$$\gamma(y) = p(y) + \Pi_2(y)(v_1 + \lambda(v_2 - v_0))$$

Note that $p$ is semialgebraic on $B(x, \tau)$ by Lemma 2.1 and by the uniqueness of $p$ (that is $p(y) = \Pi_1(y)\Pi_{H_y^G} v_0$ and $\Pi_{H_y^G} v_0$ is semialgebraic by hypothesis). Moreover

$$B(x, \tau) \ni y \mapsto \Pi_2(y)(v_1 + \lambda(v_2 - v_0)) = (v_1 + \lambda(v_2 - v_0)) - \Pi_1(y)(v_1 + \lambda(v_2 - v_0))$$

is semialgebraic by Lemma 2.1 as $B(x, \tau) \ni y \mapsto v_1 + \lambda(v_2 - v_0)$ is semialgebraic since it is constant.
Remark 2.5. By Lemma 2.3 it follows that if a semialgebraic solution of system (1.2) exists then $\tilde{H}_x^{Gl} = H_x^{Gl}$. Moreover if for an $x \in Q$ there is a $v_x \in H_x^{Gl}$ such that $B(x, r_{v_x}) \ni y \mapsto \gamma_{v_x}(y) = \Pi_{H_y^{(0)}}v_x$ is semialgebraic then $\tilde{H}_x^{Gl} = H_x^{Gl}$ since if $v' \in H_x^{Gl}$ then

$$B(x, r_{v_x}) \ni y \mapsto \Pi_{H_y^{(0)}}v' = \Pi_1(y) \Pi_{H_y^{(0)}}v' + \Pi_2(y) \Pi_{H_y^{(0)}}v'$$

$$= \Pi_1(y) \gamma_{v_x}(y) + \Pi_2(y) v'$$

$$= \Pi_1(y) \gamma_{v_x}(y) + v' - \Pi_1(y) v'$$

where the equality in the second line is due to the uniqueness of the projection of a solution and to the fact that $H_y^{(0)}$ is parallel to $\text{Ker} A(y)$. Therefore $B(x, r_{v_x}) \ni y \mapsto \Pi_{H_y^{(0)}}v' = \Pi_1(y) \Pi_{H_y^{(0)}}v' + \Pi_2(y) \Pi_{H_y^{(0)}}v'$ is semialgebraic as it is a sum of semialgebraic functions by Lemma 2.1 and thus $v_x \in \tilde{H}_x^{Gl}$.

3 Existence of a continuous semialgebraic solution

Theorem 3.1. Consider a compact metric space $Q \subseteq \mathbb{R}^n$ and a system of linear equations

$$A(x) \phi(x) = \gamma(x), \quad x \in Q$$

(3.1)

where the entries of

$$A(x) = (a_{ij}(x_1, \ldots, x_n)) \in M_{r,s}(\mathbb{R}) \quad \text{and} \quad \gamma(x) = (\gamma_i(x)) \in \mathbb{R}^r$$

are themselves semialgebraic functions on $\mathbb{R}^n$. Assume that for every given $x \in Q$ such that $\tilde{H}_x^{Gl} \neq \emptyset$ there exists $v_x \in \tilde{H}_x^{Gl}$ such that $B(x, r_{v_x}) \ni y \mapsto \gamma_{v_x}(y) = \Pi_{H_y^{(0)}}v_x$ is discontinuous, at most, at isolated points (that therefore must be finitely many). Then system (3.1) has a continuous semialgebraic solution $\phi : Q \rightarrow \mathbb{R}^s$ iff $\tilde{H}_x^{Gl}$ has no empty fiber.

Proof.

- At first we show that if $\tilde{H}_x^{Gl}$ has no empty fiber then the system (1.2) has
a continuous semialgebraic solution $\phi : Q \to \mathbb{R}^s$.

By hypothesis we have that $\forall x \in Q$, $\exists v_x \in H_x^G \neq \emptyset$, so that $\exists r_{v_x} \in \mathbb{R}^+$ s.t. $B(x, r_{v_x}) \ni y \mapsto \gamma_{v_x}(y) = \Pi_{H_y^{(0)}} v_x$ is semialgebraic and possibly discontinuous at isolated points.

We next claim that

$$\forall x \in Q, \forall v_x \in \tilde{H}_x^G \neq \emptyset, \exists r_{v_x} \text{ s.t.}$$

$$B(x, r_{v_x}) \ni y \mapsto \gamma_{v_x}(y) = \Pi_{H_y^{(0)}} v_x \text{ is continuous.}$$

To prove the claim, let us, by contradiction, assume that

$$\exists x \in Q, \exists v_x \in \tilde{H}_x^G : \forall n \in \mathbb{N}, \exists y_n \in B(x, r_{v_x}) \text{ s.t.}$$

$$B(x, r_{v_x}) \ni y \mapsto \gamma_{v_x}(y) \text{ is discontinuous at } y_n.$$

We have two cases:

1. $\forall n \in \mathbb{N}$ one has $y_n \neq x$. Then $\gamma_{v_x}(y)$ has infinitely many isolated discontinuity points which is impossible because $\gamma_{v_x}$ is semialgebraic;

2. $\exists n \in \mathbb{N}$ such that $y_n = x$. If $v_x \notin H_x^G$ then $v_x \notin \tilde{H}_x^G$ since $H_x^G \supseteq \tilde{H}_x^G$. This is clearly not possible as we took $v_x \in \tilde{H}_x^G$. Now, if we had $v_x \in H_x^G$ then on the one hand

$$\text{dist}(v_x; H_y^G) \xrightarrow{y \to x} 0,$$

and, on the other,

$$\left\| \Pi_{H_y^{(0)}} v_x - v_x \right\| = \text{dist}(v_x, H_y^{(0)}) \xrightarrow{H_y^{(0)} \subseteq H_y^G} d(v_x, H_y^{(0)}).$$

Therefore

$$\Pi_{H_y^{(0)}} v_x \xrightarrow{y \to x} v_x = \Pi_{H_x^{(0)}} v_x.$$  \hspace{1cm} \text{This is impossible since $\gamma_{v_x}(y)$ would be continuous at $x$, contrary to the assumption. Thus \[(3.2)\] holds.}$$

Now notice that the set of balls $\{B(x, r_{v_x})\}_{x \in Q}$, where $v_x$ is chosen in $\tilde{H}_x^G$,
is an open cover of the compact space $Q$. Then there is $N$ such that 
\{B(x_i, \tau_{v_{x_i}})\}_{i=1,\ldots,N}$ is an open cover of $Q$. Consider

$$
\tau_{(x,r)}(y) = \begin{cases} 
\sqrt{r^2 - \|y - x\|^2} & \text{for } y \in B(x, r), \\
0 & \text{for } y \notin B(x, r).
\end{cases}
$$

Notice that $\tau_{(x,r)}(y)$ is semialgebraic and continuous on $Q$, $\forall x \in Q$, $\forall r \in \mathbb{R}^+$ and that 
\[\sum_{i=1}^N \tau_{(x_i, \tau_{v_{x_i}})}(y) > 0 \text{ for each } y \in Q \text{ as } \tau_{(x,r)}(y) \geq 0 \text{ for every } y \in Q\] and $\tau_{(x,r)}(y) > 0$ for all $y \in B(x, r)$. Moreover, $\forall y \in Q$, $\exists B(x_i, \tau_{v_{x_i}})$ as above s.t. $y \in B(x_i, \tau_{v_{x_i}})$ since \{\{B(x_i, \tau_{v_{x_i}})\}_{i=1,\ldots,N}\} is an open covering of $Q$. Hence the function

$$
\phi(y) = \frac{1}{N} \sum_{i=1}^N \tau_{(x_i, \tau_{v_{x_i}})}(y) \prod_{j=1}^H \left( \left. H_y(0) \right|_{x_j} \right)
$$

is a semialgebraic and continuous solution of the system on $Q$.

- Conversely, we show that if system (1.2) has a continuous semialgebraic solution $\phi : Q \to \mathbb{R}^s$ then $\tilde{H}_x^{G_1}$ has no empty fiber.

By Remark 2.5, we have that $\tilde{H}_x^{G_1} = H_x^{G_1}$ but $H_x^{G_1}$ has no empty fiber because there is a continuous solution of the system (1.2) as shown in [1].

The proof of the theorem is complete. $\square$

Theorem 3.1 gives therefore an answer to the initial problem of determining a necessary and sufficient condition for the existence of a solution $Q \ni x \mapsto \phi(x) = \begin{bmatrix} \phi_1(x) \\
\vdots \\
\phi_s(x) \end{bmatrix} \in \mathbb{R}^s$ of the system (1.2), with the $\phi_i : Q \to \mathbb{R}$ continuous and semialgebraic.

**Remark 3.2.** Let $p$ be the projection of a solution of the system (1.2) on $\text{Ker}A(y)^\perp$, $y \in Q$. If $p$ is not semialgebraic the system has no semialgebraic solution by Theorem 2.2 and so it has no continuous and semialgebraic solution. Otherwise, if $p$ is semialgebraic then $\tilde{H}_x^{G_1} = H_x^{G_1}$ since if $v \in H_x^{G_1}$ we
may write

\[ Q \ni y \mapsto \Pi_{H_y^0} v = p(y) + \Pi_2(y) v = p(y) + v - \Pi_1(y) v \]

that is semialgebraic by Lemma 2.1. We have that \( \gamma_{v_x}(y) = p(y) + \Pi_1(y) v_x \), \( \forall y \in B(x, r_{v_x}) \) since \( \gamma_{v_x}(y) \) is the projection of \( v_x \) on \( H_y^0 \). This implies that for each \( x \in Q \) there is a \( v_x \) such that \( \gamma_{v_x} \) is semialgebraic iff \( p \) is semialgebraic and \( \gamma_{v_x} \) is discontinuous at most at isolated points for each \( x \in Q \) iff \( p \) and \( B(x, r_{v_x}) \ni y \mapsto \Pi_1(y) v_x \) are discontinuous at most at isolated points for each \( x \in Q \). Note that if there is a \( v_x \) such that \( \gamma_{v_x} \) is semialgebraic then \( \gamma_{v_x} \) is semialgebraic for all \( v \in H^G \) by Remark 2.5.

Hence, if we know a solution \( \phi \) of system (1.2) or, at least, its projection on \( \text{Ker}A(x)^\perp \), Theorem 3.1 can be written in the following (equivalent) form.

**Theorem 3.3.** Let \( p \) be the projection of a solution of the system on \( \text{Ker}A(x)^\perp \) and assume that for every given \( x \in Q \) there exists \( v_x \in \tilde{H}_x^G \) such that \( B(x, r_{v_x}) \ni y \mapsto \gamma_{v_x}(y) = \Pi_{H_y^0} v_x \) is discontinuous, at most, at isolated points (which is the same as saying that \( p \) and \( B(x, r_{v_x}) \ni y \mapsto \Pi_1(y) v_x \) are discontinuous at most at isolated points for every given \( x \in Q \)). We have that the following conditions are equivalent:

i) The system has a continuous and semialgebraic solution.

ii) \( H^G \) has no empty fiber.

iii) The Glaeser-stable refinement \( H \) associated with the system has no empty fibers and \( p \) is semialgebraic.

iv) The system has a continuous solution and a semialgebraic one (they may possibly be different).

**Proof.**

We showed that \( i) \Leftrightarrow ii) \) in Theorem 3.1 (using also Remark 3.2). The proof of \( ii) \Leftrightarrow iii) \) follows from Remark 3.2 and that of \( iii) \Leftrightarrow iv) \) from Fefferman’s result that a system like (1.2) has a continuous solution iff the Glaeser-stable refinement \( H \) associated with the system has no empty fibers (see [1]) and from Theorem 2.2. \( \square \)
References

[1] C. Fefferman, J. Kollár, *Continuous Solutions of Linear Equations*, From Fourier analysis and number theory to Radon transforms and geometry, Dev. Math., vol. 28, Springer, New York, 2013, pp. 233-282. MR 2986959