Antisymmetric $\mathcal{PT}$-photonic structures with balanced positive and negative index materials

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In this Letter we study a new class of synthetic materials in which the refractive index satisfies a special symmetry, $n(-x) = -n^*(x)$, which we term antisymmetric parity-time ($\mathcal{APT}$) systems. Unlike $\mathcal{PT}$-symmetric systems which require balanced gain and loss, i.e. $n(-x) = n^*(x)$, $\mathcal{APT}$ systems consist of balanced positive and negative index materials (NIMs). Despite the seemingly $\mathcal{PT}$-symmetric optical potential $V(x) \equiv n(x)^2 \omega^2 / c^2$, such systems are not invariant under combined $\mathcal{PT}$ operation due to the discontinuity of the spatial derivative of the wavefunction. We show that $\mathcal{APT}$ systems display intriguing properties such as spontaneous phase transition of the scattering matrix, bidirectional invisibility, and a continuous lasing spectrum.

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Parity-time ($\mathcal{PT}$) symmetric systems have attracted considerable interests since the original work by Bender and colleagues [1–3], both in the fields of quantum mechanics and photonics. In $\mathcal{PT}$-symmetric quantum mechanics the Schrödinger equation does not have time-reversal symmetry but is invariant under combined parity-time operation. The corresponding Hamiltonian is non-Hermitian, yet it displays regions of real energy spectrum and was speculated as an alternative to canonical quantum mechanics. The idea was introduced into photonics by Christodoulides and coworkers [4, 5] utilizing balanced gain and loss ($n(-x) = n^*(x)$) to satisfy the $\mathcal{PT}$-symmetry. Many intriguing optical phenomena have since been predicted and observed, such as double refraction [5], power oscillations [6, 7], coexistence of coherent perfect absorption [8, 9] and lasing states [10, 11], spontaneous symmetry breaking of the scattering matrix [11], and unidirectional transmission resonances [12, 13].

Another field that has attracted tremendous research interests is negative index materials (NIM) [14–17]. In such materials the refractive index is negative over some frequency range, achieved by using engineered resonances of nanostructures in metamaterials. NIMs have been proposed to realize subwavelength imaging (i.e. “superlens” [14]) and dramatically change light-matter interaction via altered local density of optical states [18]. NIMs are strongly absorptive, but recent advances with imbedded active media have demonstrated reduced intrinsic loss and even net gain [19].

In this Letter we study a class of new systems bridging NIMs and $\mathcal{PT}$-symmetric photonics. Their refractive index is antisymmetric under combined $\mathcal{PT}$ operation, $n(-x) = -n^*(x)$, i.e. with balanced positive index materials (PIMs) and NIMs; the imaginary part of $n(x)$ is symmetric, which can be positive (loss), negative (gain), zero, or any complicated spatial function. Outside the system we assume that the refractive index is uniform, symmetric, and positive. We term such synthetic systems antisymmetric parity-time ($\mathcal{APT}$) systems, which display intriguing features such as bidirectional invisibility, spontaneous phase transition of the $S$-matrix, and a continuous lasing spectrum.

We base our discussion on the scalar wave equation for the electric field

$$\nabla^2 + n(x)^2 (\omega^2 / c^2) E(x; \omega) = 0,$$  \hspace{1cm} (1)

which describes steady-state solutions in one-dimensional (1D) and two-dimensional (2D) systems. Henceforth we set $c = 1$ and assume that the corresponding $\mu_{\text{NIM}} = -|\mu_{\text{PIM}}|$ if $n_{\text{NIM}} = -n_{\text{PIM}}^*$. By first glance one may misjudge that all the intriguing phenomena found in conventional $\mathcal{PT}$-symmetric systems would survive since the optical potential $V(x) \equiv n(x)^2 \omega^2$ is still invariant under $\mathcal{PT}$ operation. We note that this is not true since the boundary condition at PIM and NIM interfaces is now different. Take a 1D $\mathcal{APT}$ heterostructure for example (see Fig. 1), the electric field itself is still continuous at PIM and NIM interfaces, but its spatial derivation now satisfies [20, 21]

$$\frac{1}{\mu_{\text{PIM}}} \left. \frac{\partial E(x; \omega)}{\partial x} \right|_{x\in\text{PIM}} = \frac{1}{\mu_{\text{NIM}}} \left. \frac{\partial E(x; \omega)}{\partial x} \right|_{x\in\text{NIM}},$$  \hspace{1cm} (2)

which changes abruptly due to the sign difference of $\mu_{\text{PIM}}$ and $\mu_{\text{NIM}}$. Below we first analyze wave propagation and lasing in such 1D $\mathcal{APT}$ heterostructures, followed by a discussion of pseudo-$\mathcal{APT}$ symmetry for wave propagation in 2D with the paraxial approximation.

In Ref. [11] the phase transition of the scattering matrix ($S$-matrix) in a $\mathcal{PT}$-symmetric system is predicted based on the invariance of the system under combined

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FIG. 1. (Color online) Schematic of a 1D $\mathcal{APT}$ photonic heterostructure, consisting of $2N$ layers that satisfy $n(x) = -n(-x)^*$. 
PT operations; We would not expect a similar phase transition in \APT systems, since flipping the sign of the real part of the refractive index is not related to any symmetry operation of the physical state. However, there do exist some special properties of the transmission coefficient \( t \) and the left and right reflection coefficients \( r_L, r_R \) in a 1D \APT heterostructure:

\[
  r_L = r_R^*, \quad \text{Im}[t] = 0. \tag{3}
\]

To understand these properties, we start by noting one observation: By changing the refractive index of each layer in an arbitrary photonic heterostructure to its negative complex conjugate and flipping the sign the magnetic permeability, i.e. \( n_i \to -n_i^*, \mu_i \to -\mu_i \), the transfer matrix \( M \) \[\text{[22]}\], defined by

\[
\begin{pmatrix}
  A \\
  B
\end{pmatrix} = M
\begin{pmatrix}
  C \\
  D
\end{pmatrix},
\]

becomes its complex conjugate at the same real frequency:

\[
M(\omega) \to M^*(\omega), \quad \text{Im}[\omega] = 0. \tag{5}
\]

The field amplitudes \( A, B, C, \) and \( D \) are defined in Fig. 1 or more specifically,

\[
E(x; \omega) = \begin{cases}
  A e^{-i n_0 \omega (x+L/2)} + B e^{i n_0 \omega (x+L/2)}, & x < -L/2, \\
  C e^{-i n_0 \omega (x-L/2)} + D e^{i n_0 \omega (x-L/2)}, & x > L/2,
\end{cases}
\]

where \( n_0 \) is the refractive index outside the heterostructure and we have assumed \( \mu_0 = 1 \). The proof of Eq. 6 is straightforward from the analytical expression of \( M \):

\[
M(\omega) = D_0^{-1} \prod_{i=m}^N M_i D_0,
\]

obtained from the continuity of \( E(x; \omega) \) and Eq. 2. The matrices \( D_0 \) and \( m_i \) are given by

\[
D_0 = \begin{pmatrix}
  1 & 1 \\
  n_0 & -n_0
\end{pmatrix},
\]

\[
m_i(\omega) = \begin{pmatrix}
  \cos(n_i \omega \Delta_i) & i n_i \sin(n_i \omega \Delta_i) \\
  i n_i \sin(n_i \omega \Delta_i) & \cos(n_i \omega \Delta_i)
\end{pmatrix},
\]

where \( \Delta_i \) is the width of the \( i \)th layer. Under the transformation \( n_i \to -n_i^*, \mu_i \to -\mu_i \) \( (i = 1, \ldots, N) \), one finds \( m_i(\omega) \to m_i^*(\omega) \) at a real frequency and so does \( M(\omega) \). Since \( M(\omega) \) determines the wave propagation, all related quantities, such as \( r_L, r_R, t \) or equivalently the S-matrix, become their complex conjugate under this transformation.

We refer to the left half of an \APT system \( U \) and the right half \( V \). Eq. 6 implies that the transfer matrices of \( U \) and \( V \) are related by

\[
M_V(\omega) = \sigma M_U^{-1}(\omega)^* \sigma, \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

If we define \( M_U(\omega) \equiv (m_{11} \ m_{12}, m_{21} \ m_{22}) \) and the transfer matrix of the whole \APT system \( M_{\APT}(\omega) \equiv (m_{11}^M \ m_{12}^M, m_{21}^M \ m_{22}^M) \) we then find through \( M_{\APT}(\omega) = M_U(\omega) M_V(\omega) \) that

\[
m_{11}' = |m_{11}|^2 - |m_{12}|^2, \quad m_{22}' = |m_{22}|^2 - |m_{21}|^2, \quad m_{12}' = m_{12} m_{22}^* - m_{11} m_{21}^* = -(m_{21}').
\]

The S-matrix defined by

\[
\begin{pmatrix}
  A \\
  D
\end{pmatrix} = S_{\APT} \begin{pmatrix}
  B \\
  C
\end{pmatrix} \equiv \begin{pmatrix}
  r_L & t \\
  t & r_R
\end{pmatrix} \begin{pmatrix}
  B \\
  C
\end{pmatrix},
\]

(12)

can be easily expressed in terms of \( M_{\APT} \):

\[
S_{\APT}(\omega) = \frac{m_{12}^M}{m_{22}^M} \begin{pmatrix}
  m_{21}^M & 1 \\
  1 & -m_{21}^M
\end{pmatrix},
\]

from which we immediately find \[\text{[3]}\] using \[\text{[11]}\].

The phase transition of the S-matrix can be inferred from the relations \[\text{[5]}\], which suggest the parametrization of the S-matrix by three independent real quantities: \( t, a \equiv \text{Re}[r_L], \) and \( b \equiv \text{Im}[r_L] \), i.e.

\[
S = \begin{pmatrix}
  a + i b & t \\
  t & a - i b
\end{pmatrix},
\]

(14)

which is pseudo-Hermitian \[\text{[23]}\], i.e. \( S^* = \eta S \eta^{-1} \) with \( \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). The eigenvalues of the S-matrix are given by

\[
s_\pm = a \pm \sqrt{t^2 - b^2},
\]

(15)

which have two phases and the phase transitions occur at \( t = \pm b \). The scattering eigenstates \( \psi_\pm(\omega) = \begin{pmatrix} B_\pm \\ C_\pm \end{pmatrix} \) display a transition simultaneously:

\[
p_\pm \equiv \frac{B_\pm}{C_\pm} = \frac{1}{t} [ib \pm \sqrt{t^2 - b^2}].
\]

(16)

In one phase \( (|t| > |b|) \) the intensities of the two incident beams in a scattering eigenstate are the same, i.e. \( |p_\pm| = 1 \) (see Fig. 2). Thus we refer to this phase as the symmetric scattering phase and the other as the symmetry-broken phase \( (|t| < |b|) \). In the symmetric phase \( s_\pm \) are real, meaning that the symmetric inputs are either amplified or damped equally with no phase added during the scattering process. In the symmetric-broken phase the two scattering eigenstates have the same scattering strength \( |s_\pm| = |s_\mp| \) but with \( |p_\pm| = |p_\mp|^{-1} \).

Tuning to the phase transition points requires adjusting either the gain/loss parameter of the system, the frequency of incident beam, or scaling the system size. Given that it is challenging to maintain \APT (or \PT) symmetry in the first two approaches due to material dispersion, the third approach is probably most practical, i.e. by fabricating multiple scaled heterostructures and fixing the frequency of incident beams at a value that achieves the \APT symmetry.
The phase transition discussed above is a general property of all 1D \APT systems, independent of whether the system has net gain or loss. By flipping the sign of \( \Im [n(x)] \), i.e. changing local gain into loss and vice versa, the system merely undergoes a time reversal and the phase transitions happen at exactly the same locations.

There is only one exception which occurs when the local gain/loss is zero \([22]\), i.e. \( \Im [n(x)] = 0 \). In this case an \APT heterostructure is always in the symmetric phase. More strikingly, the relations \([3]\) now take a special form:

\[
 t = 1, \quad r_L = r_R = 0, \tag{17}
\]

i.e. an \APT system becomes invisible, and it is independent of the complexity and size of the heterostructure and at what frequency the \APT symmetry occurs. This phenomenon is robust upon a slight breakdown of the \APT-symmetry or in the presence of a small \( \Im [n(x)] \neq 0 \) (see Fig. 3(a)). In addition, the invisibility is independent of whether the \APT structure is standalone or integrated in a photonic environment, as long as the nearest elements have the same refractive index (see Fig. 3(b)). If the \APT symmetry can be maintained over a finite frequency range, a pulse transmitted within this frequency window will be exactly the same as the initial pulse, with no pulse distortion or shrinking/expansion. This phenomenon is independent of the propagation direction, in contrast to the one-way invisibility found in \PT-symmetric heterostructures \([12]\).

The relations \([17]\) can be treated as a generalization of the phenomenon of vanished reflection that happens at the interface of two impedance matched PIM and NIM materials (see Ref. \([14]\) for example). There is at least one such interface in a \APT system, i.e. at \( x = 0 \), but multiple reflections occur at other interfaces between two NIMs, two PIMs, and a NIM and a PIM of different \( |n| \). One way to prove \([17]\) is from the transfer matrix \( M_{\APT}(\omega) \) directly:

\[
 M_{\APT}(\omega) = D_0^{-1} \left[ \Pi_{i=-N}^{N} m_i \right] D_0, \quad i \neq 0. \tag{18}
\]

Note that

\[
 m_{-i}(\omega)m_i(\omega) = 1 \tag{19}
\]

when \( n_i \) is real. Therefore,

\[
 M_{\APT}(\omega) = D_0^{-1} \left[ \Pi_{i=-N}^{N} m_i \right] D_0 = \ldots = 1, \tag{20}
\]

which implies \([17]\).

Net gain has been demonstrated in NIMs by embedding an active medium \([13]\), which opens the possibility of achieving lasing in metamaterials. Lasing modes are given by the poles of the S-matrix on the real frequency axis, which in general lead to discrete set of solutions containing the lasing frequency \( \omega_L^{(m)} \) and the corresponding threshold \( \tau^{(m)} = -\Im [n(x)] > 0 \) assuming a spatially uniform gain. Each lasing mode has its distinct intensity profile and roughly speaking the mode order \( m \) indicates the number of peaks inside the cavity. To determine \{\( \omega_L, \tau \)\} for each mode, one can, for example, solve the two equations given by the real and imaginary part of \( m_{22}' \) in 1D heterostructures (see Eq. \([13]\)). However,
the resonances of the tiny section of length $\delta$ achieve single-mode lasing. These modes originate from $\Delta n = \delta n/m$, which act as an external cavity for frequency selection. In this example, the variation of $\delta$ (or $L$) is enhanced by four times in the wavelength of the fundamental mode ($m = 1$) since $\Delta \lambda = 4\Delta n/m$, which can be easily measured. As a comparison, the sensitivity to detect $\delta$ is reduced by a factor of $\delta/L = 10^{-3}$ in a uniform PIM cavity of length $L$, which also has a much denser spectrum to analyze. We note that the laser linewidths are comparable in the two systems, since the thresholds of the corresponding modes are about the same as discussed.

So far we have discussed the $\mathcal{APT}$ symmetry with balanced PIMs and NIMs. One may attempt to realize a “pseudo-$\mathcal{APT}$” symmetry using only PIMs (or NIMs), satisfying $n(x) = -n(-x)^*$ but with $\mu(x) = \mu(-x)$. One example in which such symmetry can be realized is wave propagation in 2D paraxial geometry [4, 5], with transverse index variation $n(x) = n_0 + \delta n(x)$ satisfying $|\delta n(x)| \ll n_0$ and $\delta n(x) = -\delta n(-x)^*$. The Helmholtz equation [1] in this case becomes

$$
\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + \delta n(x)k_0 \phi = 0, \tag{21}
$$

where $\phi(x, z)$ is the slowly varying part of the electric field $E(x, z) = \phi(x, z)e^{ik_0z}$, and $|\delta n(x)|$ is the coordinate in the transverse direction. The transverse optical potential now is proportional to $\delta n(x)$ instead of index squared in the original Heltmoltz equation [1], and the intriguing phenomena discussed above disappear. The only exception happens when the system becomes equivalent to a conventional $\mathcal{PT}$-symmetric structure. The latter occurs, for example, if $\delta n(x) = A\sin \tilde{x} + iB \cos \tilde{x}$ ($A, B \in \mathbb{R}$); shifting $\tilde{x}$ by $\pi/2$ transforms $\delta n(x)$ to $A\cos \tilde{x} - iB \sin \tilde{x}$, satisfying $n(x) = n(-x)^*$.

In summary, we propose a new class of synthetic materials which are antisymmetric under a combined parity-time operation, i.e. $n(x) = -n(-x)^*$. $\mathcal{APT}$ systems demonstrate interesting features such as bidirectional invisibility, spontaneous phase transition of the S-matrix, and a continuous lasing spectrum. Properties of $\mathcal{APT}$ systems in higher dimensions are under investigation and will be reported elsewhere.

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Appendix A: Threshold correspondence in a simple $\mathcal{APT}$ heterostructure and a PIM cavity

The threshold $\tau(\omega_L)$ of the continuous lasing spectrum in a simple 2-layer cavity discussed in the main text is given by the solution of the real equation

$$
|\cos \alpha|^2 + \frac{n_r^2 - \tau^2}{n_r^2 + \tau^2} |\sin \alpha|^2 \neq -\operatorname{Im} \left[ \left( n_r - i\tau + \frac{1}{n_r - i\tau} \right) \sin \alpha (\cos \alpha)^* \right], \tag{A1}
$$

in which $n_r$ is the real part of the refractive index in the PIM, $\alpha \equiv (n_r - i\tau)\omega_L/L/2$, and $L$ is the cavity length. In comparison, the threshold and the discrete lasing frequency in an uniform PIM cavity of the same length

![FIG. 4. (Color online) (a) Threshold value $\tau(\omega_L)$ for the continuous lasing spectrum in a 2-layer $\mathcal{APT}$ heterostructure. Each layer is 500nm thick and the refractive indices are $\pm 2 - i\tau(\omega_L)$ at threshold. Squares indicate the discrete lasing solutions $\omega L = 1.735, 3.245, 4.786$ in a uniform PIM cavity of the same length. Inset: Intensity profiles at $\omega L = 1.5, 3, 4.5$ marked by the red dots in the main figure. Shadowed areas indicate the cavity. (b) Number of lasing modes in the region $kL_0 < 30$ versus the width of one layer in (a).](image-url)
are simultaneously determined by the following complex equation

\[
cos(2\alpha) = i \left[ \left( n_r - i\tau + \frac{1}{n_r - i\tau} \right) \sin \alpha \cos \alpha \right]. \quad (A2)
\]

It implies that

\[
\tan \alpha = -i(n_r - i\tau), \quad -\frac{i}{n_r - i\tau}, \quad (A3)
\]
or

\[
\text{Re} \left[ \frac{\tan \alpha}{(\tan \alpha)^*} \right] = \frac{n_r^2 - \tau^2}{n_r^2 + \tau^2}. \quad (A4)
\]

By taking the real part of both sides of (A2) after multiplying them by \((\cos \alpha)^* / \cos \alpha\), we find

\[
|\cos \alpha|^2 + \text{Re} \left[ \frac{\tan \alpha}{(\tan \alpha)^*} \right] |\sin \alpha|^2
= -\text{Im} \left[ \left( n_r - i\tau + \frac{1}{n_r - i\tau} \right) \sin \alpha (\cos \alpha)^* \right], \quad (A5)
\]
from which we recover (A1) using (A4).

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