Color structure of quantum $SU(N)$ Yang-Mills theory

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Color confinement is the most puzzling phenomenon in the theory of strong interaction, quantum chromodynamics, based on a quantum $SU(3)$ Yang-Mills theory. The origin of color confinement supposed to be intimately related to non-perturbative features of the non-Abelian gauge theory, and touches very foundations of the theory. We revise basic concepts underlying QCD concentrating mainly on concepts of gluons and quarks and color structure of quantum states. Our main idea is that a Weyl symmetry is the only color symmetry which determines all color attributes of quantum states and physical observables. We construct an ansatz for classical Weyl symmetric dynamical solutions in $SU(3)$ Yang-Mills theory which describe one particle color singlet quantum states for gluons and quarks. Abelian Weyl symmetric solutions provide microscopic structure of a color invariant vacuum and vacuum gluon condensates. This resolves a problem of existence of a gauge invariant and stable vacuum in QCD. Generalization of our consideration to $SU(N)$ ($N = 4, 5$) Yang-Mills theory implies that the color confinement phase is possible only in $SU(3)$ Yang-Mills theory.

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A gauge principle is a fundamental guiding principle which provides construction of strict mathematical theories of fundamental interactions [1]. A quantum chromodynamics (QCD) is a well established theory of strong interaction which represents a quantum theory of classical Yang-Mills gauge theory with a color group $SU(3)$ [2], the strict formulation of which is unknown so far [3]. The gauge symmetry plays an important role in determination of dynamical laws in the theory. Formally, the gauge symmetry looks redundant, since it leads to unphysical pure gauge degrees of freedom which are usually removed from the theory by applying a gauge fixing procedure in such a way that final physical quantities do not depend on a chosen gauge. So that, the local gauge symmetry governs the time evolution of the physical system, but can not serve as a symmetry describing color attributes of fundamental particles and quantum states.

On the other hand, a global color symmetry $SU(3)$ can not be a real symmetry of particles as well, since (i) color particles are not observed due to color confinement phenomenon, (ii) there is no color charge conservation law like the electric charge conservation, and (iii) an escape from such paradoxical situation like that “a color symmetry exists but it is spontaneously broken” fails because it is in contradiction with the gauge invariant structure of the QCD vacuum which provides the color confinement [4, 5].

In the present paper we demonstrate that a Weyl group of the structure group $SU(N)$ represents a finite color symmetry which is the only color symmetry available after gauge fixing and which describes color properties of quantum states and physical observables in a consistent manner with the color confinement phenomenon.

In Section I we consider a general structure of quantum Yang-Mills theory and main requirements to classical and quantum fields necessary for construction of one particle color singlet quantum states. In Section II we consider a minimal ansatz for stationary axially symmetric solutions of magnetic type in $SU(2)$ Yang-Mills theory. We show that solutions admit a global color symmetry $SO(2)$ which remains after the gauge fixing and leads to a degenerated vacuum structure. This implies that $SU(2)$ Yang-Mills theory does not admit the color confinement phase. An $SU(3)$ Yang-Mills theory with fermions, i.e., quantum chromodynamics with one flavor quark, is considered in Section III. We verify uniqueness of the Weyl symmetric ansatz and show that there are two possible classes of Weyl symmetric solutions, but only one class is consistent with the Weyl symmetric structure for quarks. Our approach is non-perturbative in a sense that the Weyl symmetric structure of solutions is consistent with the whole non-Abelian structure of equations of motion, and our ansatz describes exact solutions to equations of motion. Applying a non-trivial Abelian projection defined by the Weyl symmetric ansatz we obtain unexpected results: all Weyl symmetric solutions for gluons and quarks belong to irreducible Weyl representations leading to color singlet one particle states contrary to commonly accepted view that gluons and quarks represent color particles as members of color octet and triplet respectively. In Section IV we generalize our results to a case of $SU(4)$ Yang-Mills theory and construct the most general Weyl symmetric ansatz. The obtained ansatz defines a two parametric family of solutions, so that vacuum solutions are degenerated. In general, this leads to spontaneous color symmetry breaking. Generalization to $SU(5)$ Yang-Mills theory and conclusions are contained in the last section.
I. STRUCTURE OF QUANTUM YANG-MILLS THEORY

We consider a standard Lagrangian of $SU(N)$ ($N = 2, 3, 4, 5$) Yang-Mills theory in the presence of one complex multiplet of Dirac fermions in the fundamental representation of $SU(N)$

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_{f},$$

$$\mathcal{L}_{YM} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu},$$

$$\mathcal{L}_{f} = \bar{\Psi} \left[ i \gamma^\mu \left( \partial_\mu - \frac{i g}{2} A_\mu^a \lambda^a \right) - m \right] \Psi. \quad (1)$$

where $(\mu, \nu = 0, 1, 2, 3; a = 1, 2, ..., N^2 - 1)$. Corresponding equations of motion describe a coupled system of a gluon field $A_\mu^a$ and Dirac spinor fields $\Psi_\alpha (\alpha = 1, 2 ... N)$

$$(D_\mu \vec{F}_{\mu\nu})^a = -\frac{g}{2} j_\nu^a \equiv -\frac{g}{2} \bar{\Psi} \gamma_\nu \lambda^a \Psi, \quad (2)$$

$$[i \gamma^\mu \left( \partial_\mu - \frac{i g}{2} A_\mu^a \lambda^a \right) - m] \Psi = 0. \quad (3)$$

A general quantization formalism based on functional integral provides a principal structure of quantum $SU(N)$ Yang-Mills theory. Following this approach a gauge connection (gauge potential) $A_\mu^a$ is split into two parts

$$A_\mu^a = A_\mu^{cl} + A_\mu^{quant}, \quad (4)$$

where the quantum field $A_\mu^{quant}$ describes quantum fluctuations and stands for arbitrary field configurations over which the functional integral is performed. The integration over quantum fields represents summation over all gauge equivalence classes of the gauge fields which are extracted by a gauge fixing procedure in a gauge invariant manner. So that for description of quantum fields one can use an arbitrary complete basis in the space of vector and spinor fields for gluons and quarks respectively.

In particular, in perturbative quantum chromodynamics (QCD) a basis of free plane waves is usually applied which leads to concepts of eight color gluons and three color quarks. Certainly, such defined gluons and quarks provide proper description of quantum gluons and quarks in internal lines and loops of Feynman diagrams and represent virtual quantum objects which are not observable directly. Observable particles are represented by a complete set of classical dynamical solutions which produce a Hilbert space of quantum states. A spectrum and properties of one particle quantum states depend on a chosen gauge fixing procedure and selected solution basis. Due to lack of a linear superposition principle in the non-Abelian theory a choice of different solution bases lead to non-equivalent particle contents or even unphysical particles. In a particular, the basis of plane wave solutions leads to unphysical concepts for gluon and quark particles. Therefore, first of all one should define necessary requirements to classical solutions which provide consistent definitions of corresponding particles or quantum states.

In QCD the classical field $A_\mu^f$ plays another important role, due to the presence of a nontrivial vacuum $\bar{\Psi}$. Such a vacuum generates non-zero vacuum gluon and quark condensates, and we need some classical solutions which describe microscopic structure of the vacuum condensates. This implies a well known long-standing problem of construction of a vacuum fields which are stable against quantum fluctuations $\bar{\Psi}$. Moreover, it has been observed by Polyakov, \cite{4}, that color confinement phenomenon is provided by a color invariant vacuum. ‘t Hooft conjectured that such vacuum structure is described by Abelian fields according to a so-called Abelian projection of QCD, which was confirmed later by numerical study \cite{5, 9}. Such a resolution to the vacuum problem is not merely satisfactory, because obviously any non-trivial vacuum solution is not invariant under color transformation. A possible way to resolving this problem is given by H. Weyl emphasizing in his book \cite{10} the importance of symmetric configurations invariant under a subgroup of the initial symmetry group. Following this idea, the problem of construction of an invariant vacuum solution reduces to finding irreducible representations of a finite color subgroup of $SU(3)$ which admit a one-dimensional space of field variables. Then such one-dimensional irreducible representations will define color singlet vacuum and quantum states.

We impose the following main requirements to classical solutions describing one particle quantum states for gluons and quarks:

(a) the consistency with quantum mechanical principles implies that structure elements, elementary particles, must be vibrating at microscopic space-time level \cite{11, 12}. This leads to a condition that classical solutions describing fundamental particles and non-trivial vacuum must be time dependent. By this way one can obtain microscopic description of vacuum quark and gluon condensates;

(b) solutions must be exact solutions to non-linear equations of motion. Solutions obtained in some approximation (like the plane waves in perturbation theory) may not exist as strict solutions in the full theory and cause inconsistency with non-perturbative features;

(c) solutions must possess classical and quantum stability. Some unstable classical solutions are admissible as physical ones since they may correspond to metastable states. However, the quantum stability against quantum fluctuations is obligatory since it guarantees stability of the vacuum in the theory;

(d) solutions must admit localization of particles, in particular the energy, total angular momentum must be conserved in a finite space regions, since all known hadrons are localized objects;

Strictly speaking, in quantum theory one should consider classical solutions to equations of motion obtained from the functional of quantum effective action $\Gamma_{eff}$ which contains quantum corrections. In the present paper we constrain our consideration within the quasiclassical consideration neglecting quantum corrections. An important
role of quantum corrections in providing localized solutions is demonstrated in [15].

(e) any classical solution breaks the initial color symmetry. To classify color attributes quantized solutions one needs solutions which still possess a symmetry subgroup of $SU(N)$. Our main assumption is that a Weyl symmetry provided by the Weyl group of $SU(N)$ determines color properties of quantum states and color confinement structure.

II. COLOR STRUCTURE OF $SU(2)$

YANG-MILLS THEORY

A generalized axially symmetric Dashen-Hasslacher-Neveu (DHN) ansatz for stationary solutions of magnetic type is defined by the following non-vanishing components of the gauge potential [13]

$$
A_i^2 = K_0(r, \theta, t), \quad A_i^3 = K_1(r, \theta, t), \quad A_i^2 = K_2(r, \theta, t),
$$

$$
A_i^3 = K_3(r, \theta, t), \quad A_i^4 = K_4(r, \theta, t), \quad (5)
$$

where non-vanishing component fields of the gauge potential are presented by one Abelian magnetic potential, $K_1$, and by four off-diagonal fields $K_{0,1,2,3}$ in standard spherical coordinates $(r, \theta, \varphi)$. One can verify, the ansatz admits a non-trivial residual local $U(1)$ symmetry with a gauge parameter $\lambda(r, \theta, t)$

$$
K_1' = K_0 + \partial_\lambda, \quad K_1' = K_1 + \partial_\lambda, \quad K_2' = K_2 + \partial_\theta, \quad K_3' = K_3 \cos \lambda + K_4 \sin \lambda,
$$

$$
K_4' = K_4 \cos \lambda - K_3 \sin \lambda. \quad (6)
$$

Due to the presence of $U(1)$ local symmetry one has pure gauge degrees of freedom which can be eliminated from the space of classical solutions by fixing a gauge. It is suitable to add additional Lorenz type gauge fixing terms $\mathcal{L}_{gf}$ to the original Yang-Mills Lagrangian $\mathcal{L}_Y$

$$
\mathcal{L}_{gf} = -\frac{1}{2}(\partial_\lambda K_0 - \partial_\lambda K_1 - \frac{1}{r^2} \partial_\theta K_2)^2. \quad (7)
$$

With this the ansatz [5] is consistent with all equations of motion corresponding to the original Yang-Mills Lagrangian with gauge fixing terms and leads to five independent equations for the fields $K_{0,1,2,3}$. One has still a global $SO(2)$ symmetry with a constant parameter $\lambda$ which can be fixed by imposing a constraint on azimuthal components of the gauge potential

$$
K_3 = \tilde{\lambda} K_4. \quad (8)
$$

where $\tilde{\lambda}$ is an arbitrary real constant. After imposing constraint [5] the five equations for fields $K_{0,1,2,3,4}$ reduce to four independent second order hyperbolic differential equations for fields $K_{0,1,2,3}$

$$
r^2 \partial_\lambda^2 K_1 - r^2 \partial_\lambda^2 K_1 - \partial_\theta^2 K_1 + 2r(\partial_\lambda K_0 - \partial_\lambda K_1)
$$

$$
+ \cot \theta (\partial_\lambda K_2 - \partial_\theta K_1) + \frac{9}{2} \csc^2 \theta K_4^2 K_1 = 0, \quad (9)
$$

$$
r^2 \partial_\lambda^2 K_2 - r^2 \partial_\lambda^2 K_2 - \partial_\theta^2 K_2 + r^2 \cot \theta (\partial_\lambda K_0
$$

$$
- \partial_\lambda K_1) - \cot \theta \tilde{\lambda} K_2 + \frac{9}{2} \csc^2 \theta K_4^2 K_2 = 0, \quad (10)
$$

$$
r^2 \partial_\lambda^2 K_4 - r^2 \partial_\lambda^2 K_2 - \partial_\theta^2 K_4 + \cot \theta \tilde{\lambda} K_4
$$

$$
+ 3r^2(\partial_\lambda^2 - \partial_\theta^2) K_4 + 3K_4^2 K_4 = 0, \quad (11)
$$

$$
r^2 \partial_\lambda^2 K_0 - r^2 \partial_\lambda^2 K_0 - \partial_\theta^2 K_0 + 2r(\partial_\lambda K_1 - \partial_\lambda K_0)
$$

$$
+ \cot \theta (\partial_\lambda K_2 - \partial_\theta K_0) + \frac{9}{2} \csc^2 \theta K_4^2 K_0 = 0, \quad (12)
$$

and one quadratic constraint

$$
2r^2(\partial_\lambda K_0 - \partial_\lambda K_4 - K_2(\cot \theta K_4 - 2\partial_\theta K_4)
$$

$$
+ K_4(-\partial_\theta K_2 + r^2(\partial_\lambda K_0 - \partial_\lambda K_1)) = 0. \quad (13)
$$

A total Lagrangian $\mathcal{L}_Y + \mathcal{L}_{gf}$ takes a final form

$$
\mathcal{L}_{tot} = \mathcal{L}_0(K) + \mathcal{L}_{gf},
$$

$$
\mathcal{L}_0(K) = \frac{1}{2r^2} \left[ r^2(\partial_\lambda K_1 - \partial_\lambda K_0)^2 - (\partial_\theta K_1)^2 + (\partial_\theta K_0)^2 \right]
$$

$$
+ \frac{1}{2r^2} \left[ \partial_\lambda K_2(\partial_\lambda K_2 - 2\partial_\theta K_0) - \partial_\theta K_2(\partial_\theta K_2 - 2\partial_\theta K_0) \right]
$$

$$
+ \frac{3}{4r^4 \sin^2 \theta} \left[ r^2((\partial_\lambda K_4)^2 - (\partial_\lambda K_4)^2) - (\partial_\theta K_4)^2 \right]
$$

$$
- \frac{3}{4r^4 \sin^2 \theta} \left[ K_4^2(K_4^2 + r^2(K_4^2 - K_0^2)) \right], \quad (14)
$$

The Lagrangian can be treated as a non-linear realization of $U(1)$ gauge theory for the vector field $K_{0,1,2,3}$ in the Lorenz type gauge, and it reproduces the equations [12].

It has been proved that stationary solutions satisfying equations [12] are stable under vacuum gluon fluctuations [13]. The solutions are the only known classical solutions which describe a stable vacuum in a pure Yang-Mills theory in quasiclassical approximation. Each solution to equations [12] implies one parametric family of solutions obtained by $SO(2)$ global rotation [22].

Since the global symmetry represents a subgroup of the initial color group $SU(2)$ we conclude that $SU(2)$ Yang-Mills theory possesses a degenerated vacuum which leads to spontaneous color symmetry breaking and absence of color confinement phase.

III. WEYL SYMMETRIC ANSATZ IN $SU(3)$

YANG-MILLS THEORY WITH DIRAC FERMIONS

1. A pure $SU(3)$ Yang-Mills theory

A minimal Weyl symmetric ansatz for stable axially symmetric stationary solutions in $SU(3)$ Yang-Mills theory has been constructed in [14]. In this section we derive
the most general minimal Weyl symmetric ansatz and show that there are two types of Weyl symmetric ansatz in a pure $SU(3)$ Yang-Mills theory. However, as we will see later, the consistency with the Weyl symmetric structure for fermions implies uniqueness of the ansatz. We define a general Weyl symmetric ansatz by introducing the following non-vanishing field components of the gauge potential $A^a_{\mu}(r, \theta, \varphi, t)$ ($a = 1, 2, \ldots, 8$) in three color subspaces corresponding to $I, U, V$-types subgroups $SU(2)$ ($i = 0, 1, 2$ denotes the space-time coordinates ($t, r, \theta$) respectively)

\[ I : A^2_r = a_1 K_i, \quad A^1_{\varphi} = q_1 K_4, \]
\[ U : A^3_r = -q_i, \quad A^4_{\varphi} = Q_4, \]
\[ V : A^7_t = S_i, \quad A^6_{\varphi} = S_4, \]
\[ A^p_{\alpha} = A^a_{\alpha}, \quad A^3 = K_3, \quad A^8 = K_8, \quad (15) \]

where $a_1, q_1$ are number parameters, index $p = 1, 2, 3$ denotes $I, U, V$ components of vectors in the Cartan plane \{T^3, T^8\}, and $r^a_\alpha$ ($a = 3, 8$) are components of symmetric $I, U, V$-root vectors $r^1 = (1, 0)$, $r^2 = (-1/2, \sqrt{3}/2)$, $r^3 = (-1/2, -\sqrt{3}/2)$. The ansatz is explicit invariant under Weyl transformations which are presented by a symmetric permutation group $S_3$ acting on fields located in $I, U, V$ sectors. One has three Weyl multipoles (enumerated by index $i = (0, 1, 2)$) of off-diagonal fields, \{K_i, Q_i, S_i\}, each of them forms a reducible representation of the symmetric group $S_3$. Linear combinations of two Abelian fields $A^{3,8}$ corresponding to Cartan subalgebra generators form a triplet of $I, U, V$-vectors, \{A^I, A^U, A^V\}, which satisfy a constraint

\[ A^I_{\varphi} + A^U_{\varphi} + A^V_{\varphi} = 0. \quad (16) \]

The equation is explicit invariant under permutations of $I, U, V$-fields and defines a two-dimensional standard irreducible representation of $S_3$. The remaining fields \{K_4, Q_2, S_4\} forms a three-dimensional reducible representation of $S_3$. A pair of Cartan generators $T = (T^3, T^8)$, acts on generators from the coset $su(3)/u_3(1) \times u_8(1)$ in adjoint representation in the Cartan basis as follows

\[ [T, T^p_\alpha] = r^p_\alpha T^p_\alpha. \quad (17) \]

The eigenvalues $r^p_\alpha$ composed from the root vectors define color charges $C^p$ of off-diagonal $I, U, V$ fields (we set $g = 1$)

\[ g^I = (1, 0), \]
\[ g^U = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \]
\[ g^V = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right). \quad (18) \]

$I, U, V$ components of the Abelian fields have the same charges. Total color charges of all Weyl multipoles vanish identically. One can check consistence of the ansatz with all equations of motion corresponding to the $SU(3)$ Yang-Mills Lagrangian with Lorenz type gauge fixing terms $L_{gf}$ added in each $I, U, V$ sector.

One can reduce the general ansatz to a minimal Weyl symmetric ansatz by extracting irreducible representations. Due to Lorentz covariance the reduction of the general ansatz to a minimal one can be done by imposing the following constraints

\[ Q_i = a_2 K_i, \quad S_i = a_3 K_i, \]
\[ Q_4 = q_2 K_4, \quad S_4 = q_3 K_4, \]
\[ K_3 = c_3 K_4, \quad K_3 = c_8 K_8. \quad (19) \]

Consistence of the full ansatz \[15\] \[19\] with all equations of motion is provided by the following values of number parameters $a_i, q_i, c_3, c_8$

\[ a_i = b_i = 1, \]
\[ q_2 = r_3^2 + r_8^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right), \]
\[ q_3 = r_3^2 + r_8^2 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right), \]
\[ c_8 = 1, \]
\[ c_3 = \frac{\sqrt{3}}{2}, \quad c_3^{II} = \sqrt{3}, \quad (24) \]

where the numbers $q_i$ are composed from the roots. All parameter values are defined uniquely except the parameter $c_3$ which admits two different values, $c_3^{I, II}$. In a pure Yang-Mills theory both values lead to the same quantum theory due to normalization freedom of the fields $K_i$ ($i = 0, 1, 2, 4$). With this, the $SU(3)$ Yang-Mills Lagrangian reduces to a simple form in terms of four independent fields $K_i$

\[ L_{red} = \frac{3}{2g^2} \left[ r^2 (\partial_\alpha K_1 - \partial_\alpha K_0)^2 - (\partial_\alpha K_2)^2 + (\partial_\alpha K_0)^2 \right] + \frac{3}{2r^2} \left[ (\partial_i K_2)(\partial_i K_2 - 2\partial_0 K_0) - \partial_i K_2(\partial_i K_2 - 2\partial_0 K_1) \right] + \frac{9}{4r^4 \sin^2 \theta} \left[ r^2 ((\partial_\alpha K_1)^2 - (\partial_\alpha K_0)^2) - (\partial_\alpha K_4)^2 \right] - \frac{1}{4r^4 \sin^2 \theta} \left[ K_2^2 (K_2^2 + r^2 (K_1^2 - K_0^2)) \right]. \quad (25) \]

The reduced Lagrangian does not contain cubic interaction terms which disappear due to mutual cancellation, one has only a quartic interaction with a Weyl invariant coupling constant $g_4$

\[ g_4^2 = \frac{3}{2} (q_1^2 + q_2^2 + q_3^2 - q_1 q_2 - q_2 q_3 - q_3 q_4) = \frac{27}{4}. \quad (26) \]

The Lagrangian $L_{red}$ is equivalent to $SU(2)$ Lagrangian $L_{0}$, \[20\] up to rescaling $K_i \rightarrow 1/\sqrt{3} K_i$. So the minimal Weyl symmetric $SU(3)$ Yang-Mills theory has the same equations and solutions as ones in $SU(2)$ gauge theory. The difference is that $SU(2)$ solutions admit the global
SO(2) color symmetry, whereas SU(3) solutions are non-degenerate and form invariant space under all Weyl color transformations.

Let us consider color properties of solutions defined by the ansatz \( [15] [19] \). The color structure of solutions is determined by Weyl representations which are described by representations of the symmetric permutation group \( S_3 \). The same values of parameters \( a_i, b_i, [20] \) imply that \( I, U, V \) components of the off-diagonal fields \( \{ K_i, Q_i, S_i \} \) are identical to each other

\[
K_i = Q_i = S_i. \tag{27}
\]

Due to this one has three one dimensional irreducible Weyl representations \( \{ (K_1, K_1, K_1) = (1, 1, 1)K_1 \} \) corresponding to three irreducible singlet representations \( \{ \Gamma_1 \} \) of the symmetric group \( S_3 \). Therefore we have three Weyl symmetric color singlet fields \( K_i \). Parameters \( q_1, q_2 \) in \( [21] [22] \) are made from the root vectors \( r^2, r^3 \) implying an equation

\[
K_4 + Q_4 + S_4 = 0. \tag{28}
\]

This equation defines a two dimensional plane in the space spanned by \( I, U, V \) components \( \{ K_4, Q_4, S_4 \} \). The equation is explicit invariant under permutations and defines a standard two-dimensional irreducible representation \( \Gamma_2(K_4) \) of \( S_3 \) which in general contains two independent fields. In our case this irreducible representation is realized in terms of only one independent field \( K_4 \), i.e., it is a singlet. In a similar manner the \( I, U, V \) combinations of the Abelian fields \( A^{3,8}_i \) form an equivalent standard two-dimensional irreducible representation \( \Gamma_2(K_3) \). Consistence of the ansatz with equations of motion implies that there is only one independent Abelian field in the Cartan plane, \( A^{3}_{\nu} = A^{8}_{\nu} \equiv K_3 \), \( [19] [20] \). Finally, two representations \( \Gamma_2(K_4) \) and \( \Gamma_2(K_3) \) are equivalent due to the constraint \( [19] [23] \). Let us show that field \( K_4 \) (and \( K_3 \) equivalently) represents a Weyl invariant field. We consider eigenvalues of a Lie algebra valued Abelian vector field, \( A_\mu = (A^3_\mu T^3, A^8_\mu T^8) \), acting in adjoint representation in the Cartan basis. In a case of equalled Abelian fields, \( A^{3}_\nu = A^{8}_\nu \), one can find

\[
[A^p_\mu, T^p_\nu] = K_3 r^p T^p_\nu. \tag{29}
\]

The eigenvalues \( K_3 r^p \) match the symmetric roots \( r^p \) with a normalization function \( K_3 \), so that the \( I, U, V \)-components of the Abelian field possess the same Weyl symmetry as the roots, and the field \( K_3 \) represents a Weyl invariant normalization function which produces singlet quantum states for Abelian gluon (more details on singlet structure of representations of the permutation group are presented in the Appendix).

We conclude, the Weyl symmetry implies that one has three color singlet fields \( K_i \) for off diagonal fields, corresponding to singlet irreducible representations \( \{ \Gamma_1 \} \), and one color singlet Abelian field \( K_3 \) (or \( K_4 \) equivalently) entering two equivalent standard irreducible representations \( \Gamma_2(K_4) \) and \( \Gamma_2(K_3) \).

It is remarkable, the Weyl symmetric ansatz \( [16] [14] \) leads to reduction of the initial eight dimensional SU(3) octet \( A^a_\mu \) to four irreducible singlet Weyl representations for four fields \( K_1, K_4 \) belonging to invariant non-intersecting subspaces under color Weyl transformations (permutations of \( S_3 \)).

So far we have considered only transformation properties of field configurations satisfying the Weyl symmetric ansatz \( [14] [16] \) under Weyl transformations (or permutations of the symmetric group \( S_3 \)). We did not take into account dynamical properties of solutions which determine the number of dynamical degrees of freedom. These issues are considered in \( [14] [16] \) where it has been demonstrated that among four fields \( K_1 \) there are only two dynamical propagating modes, \( K_2 \) and \( K_4 \). This can be easily understood from the fact that a final reduced Lagrangian is identical to SU(2) Lagrangian \( L_{\text{tot}} \), \( [25] \), which represents a non-linear U(1) gauge theory. Due to the presence of a local U(1) symmetry one dynamical degree of freedom represents a pure gauge field, and another degree of freedom is removed on-shell, leaving only two dynamical degrees of freedom, \( K_2, K_4 \). Moreover, one would expect that magnetic type non-Abelian solution should contain only one dynamical degree of freedom, as in the Maxwell theory the vector spherical harmonics of magnetic and electric types correspond to two possible photon polarizations, one polarization mode for magnetic and one mode for electric type harmonic. Indeed, the magnetic type non-Abelian Weyl symmetric solution admits only one independent dynamic field which can be chosen as Abelian field \( K_3 \). A source of such a phenomenon is very non-trivial and related to another important requirement for physical solutions - solutions must admit localization of corresponding quantum states in a finite space regions. Indeed, it has been proved \( [15] \), that solutions for a given set of quantum numbers admit energy conservation law inside a finite space region if amplitudes of fields \( K_{2,4} \) are correlated, as a consequence, one has finally only one independent field \( K_3 \). In a similar manner one can consider electric type solutions due to electric-magnetic duality \( [14] \).

2. Weyl symmetric structure of SU(3) Yang-Mills theory with fermions

Let us consider Weyl symmetric structure of a coupled system of gluon and quarks presented by one SU(3) complex triplet of Dirac spinor fields (quarks) in fundamental representation \( [2] [3] \).

A commonly accepted simple Abelian projection with two independent Abelian fields \( A^{3,8}_i \) corresponding to the Cartan generators immediately results in three independent Dirac equations for three independent color quarks

\[
[i\gamma^\mu\partial_\mu - \frac{i}{2} \sum_p A^p_\mu w^p_\mu - m] \Psi_i = 0, \tag{30}
\]

where color charges of quarks are defined by com-
ponents $w^p_\alpha (\alpha = 3, 8)$ of weight vectors $w^p = \{(1, 1/\sqrt{3}), (-1, 1/\sqrt{3}), (0, -2/\sqrt{3})\}$. It is important to observe, that a Weyl symmetric ansatz for the whole set of components of the gauge potential leads to the Weyl symmetric ansatz \[15, 19\] which contains only one Abelian field. To preserve a Weyl symmetry of the total Lagrangian including fermions, we must apply a Weyl symmetric ansatz which is consistent with a full non-linear structure of the Yang-Mills Lagrangian. Substituting the Weyl symmetric ansatz \[15, 19\] into the equation for quark,\[13\], one obtains a drastically different equation

$$\left[ i \gamma^\mu \partial_\mu - m + \frac{g}{2} \gamma^\mu A_\mu G + \frac{g}{2} \gamma^\mu K_\mu Q \right] \Psi = 0, \quad (31)$$

where $A_\mu = \delta_{\mu 3} K_3$, $K_\mu = \delta_{\mu i} K_i (i = 0, 1, 2)$. A color charge matrix $Q$ defines interaction of quarks with off-diagonal gluon fields $K_i$

$$Q = \begin{pmatrix} 0 & -i a_1 & i a_2 \\ -i a_1 & 0 & -i a_3 \\ -i a_2 & i a_3 & 0 \end{pmatrix}, \quad (32)$$

and a color charge matrix $G$ determines interaction with the Abelian gluon field $K_3$

$$G = \begin{pmatrix} \tilde{w}^1 & c_3^{-1}(r_3^3 + r_8^3) & c_3^{-1}(r_3^2 + r_8^2) \\ c_3^{-1}(r_3^3 + r_8^3) & \tilde{w}^2 & c_3^{-1}(r_3^1 + r_8^1) \\ c_3^{-1}(r_3^2 + r_8^2) & c_3^{-1}(r_3^1 + r_8^1) & \tilde{w}^3 \end{pmatrix}. \quad (33)$$

Substituting the value $c_3 = c_3^f$ and explicit expressions for the root vectors $r^f_\pm$ the charge matrix $G$ obtains a simple form which is explicit invariant under permutations of matrix elements in each row and column

$$G = \begin{pmatrix} \tilde{w}^1 & \tilde{w}^3 & \tilde{w}^2 \\ \tilde{w}^3 & \tilde{w}^2 & \tilde{w}^1 \\ \tilde{w}^2 & \tilde{w}^1 & \tilde{w}^3 \end{pmatrix}. \quad (34)$$

This provides a Weyl symmetric structure of the equation for quarks, and implies irreducible Weyl representations which describe singlet quarks. A total sum of elements in each row and column vanishes identically due to properties of weights. This defines a one dimensional irreducible representation of the permutation group $S_3$ with a basis color eigenvector $u^0 = (1, 1, 1)$

$$\Psi_0 = u^0 \psi_0, \quad (35)$$

where $\psi_0$ is a Dirac spinor field with space-time coordinate dependence. The eigenvector $\Psi_0$ corresponds to a zero eigenvalue, i.e., zero color charge. One can verify that the charge matrix $Q$ has the same zero mode $\Psi_0$, so the neutral quark does not interact with Abelian and off-diagonal gluons. The neutral quark satisfies a free Dirac equation and determines a color singlet vacuum structure. In a particular it describes free vacuum quark condensate. One can verify that equation for gluon,\[2\], decouples from equations for quarks,\[4\], so the coupled system of non-linear equations for gluon and quarks admits solutions for non-interacting free gluon and quark condensates. The color charge matrix $G$ has other two non-zero eigenvalues

$$\lambda_{\pm} = \pm \tilde{g} = \pm \sqrt{6}, \quad (36)$$

where

$$\tilde{g}^2 = (\tilde{w}^1)^2 + (\tilde{w}^2)^2 + (\tilde{w}^3)^2 - \tilde{w}^1 \tilde{w}^2 - \tilde{w}^2 \tilde{w}^3 - \tilde{w}^3 \tilde{w}^1 \quad (37)$$

is a Weyl invariant color charge. One can find corresponding two color eigenvectors

$$u^\pm = \begin{pmatrix} \tilde{w}^1 \tilde{w}^3 + \tilde{w}^2 (\pm \tilde{g} - \tilde{w}^2) \\ \tilde{w}^3 \tilde{w}^2 + \tilde{w}^1 (\pm \tilde{g} - \tilde{w}^1) \end{pmatrix}, \quad (38)$$

which form an orthogonal basis for a two dimensional standard irreducible representation $\Gamma_2$. Both eigenvectors satisfy an equation for the same two-dimensional plane $P^2$

$$u_1^+ + u_2^+ + u_3^+ = 0. \quad (39)$$

Note that each color eigenvector $u^\pm$ does not form an irreducible one dimensional representation of $S_3$ (see Appendix for more details).

It is important to stress, that each quark triplet $\Psi^\pm = u^\pm \psi^\pm (\vec{r}, \vec{t})$ contains a color vector $u^\pm$ which belongs to the irreducible two-dimensional representation $\Gamma_2$. However, the non-zero mode $\psi^\pm$ contains only one field variable $\psi^\pm$, so it is a singlet field. One can verify that two sets of vectors obtained by permutations of components of two basis vectors $u^\pm$ form two Weyl invariant sets which do not intersect each other, even though they form a complete basis in the two dimensional plane $P^2$. Therefore, two quark triplets $\Psi^\pm = u^\pm \psi^\pm (\vec{r}, \vec{t})$ represent two independent irreducible singlet representations with respect to a number of independent field variables, two single quarks, which interact to Abelian gluon with a Weyl invariant color charge $\pm \tilde{g}$. Interaction of non-zero quark modes with off-diagonal gluons is described by the charge matrix $Q$ which has a different non-zero Weyl invariant charge

$$\tilde{g}^2 = a_1^2 + a_2^2 + a_3^2 = 3. \quad (40)$$

The second type of Weyl symmetric solutions defined by the parameter value $c_3^f = \sqrt{3}$ does not provide Weyl symmetric form for the color charge matrix $G$. So that one has a unique Weyl symmetric structure with a unique non-degenerate vacuum. We conclude, the structure of color confinement and quantum states in $SU(3)$ Yang-Mills theory is determined completely by the Weyl symmetry which is the only available color symmetry resulting in color singlet representations for gluons and quarks. Definitions for color gluons and quarks defined as members of $SU(3)$ color multiplets represent artifacts of the
applied perturbation theory and simple Abelian projection which are not consistent with the whole non-Abelian structure of the total Lagrangian including gluons and quarks.

IV. WEYL SYMMETRIC STRUCTURE OF SU(4) YANG-MILLS THEORY

We use a standard Weyl basis \{\lambda^i\} (i = 1, 2, ..., 15) in the Lie algebra su(4). The Weyl group of SU(4) is presented by a symmetric permutation group \(S_4\). We consider a minimal Weyl symmetric ansatz for time dependent axially symmetric fields of magnetic type.

A Cartan subalgebra of group SU(3) contains three generators \(T^{3,8,15}\), so that for magnetic type fields one has corresponding three independent Abelian magnetic potentials \(A^\alpha_{\varphi}\). One has six root vectors \(r^\alpha_p\) (\(p = 1, 2, ..., 6\)), \((\alpha = 3, 8, 15)\), defined as eigenvalues of Cartan generators acting on complex generators of the coset space \(SU(4)/U_3(1) \times U_6(1) \times U_{15}(1)\) in adjoint representation

\[
\begin{align*}
\text{r}^1 &= (1, 0, 0), \\
\text{r}^2 &= (-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0), \\
\text{r}^3 &= (-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0), \\
\text{r}^4 &= (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{2}{\sqrt{6}}), \\
\text{r}^5 &= (-\frac{1}{2}, \frac{1}{2\sqrt{3}}, -\frac{2}{\sqrt{6}}), \\
\text{r}^6 &= (0, \frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{6}}). \\
\end{align*}
\]

A general Weyl symmetric ansatz contains three Abelian magnetic fields \(A^\alpha_{\varphi}\) in the Cartan space, additional six independent Abelian fields \(A^\alpha_{\varphi}\) in the coset subspace spanned by generators \(T^{1,4,6,9,11,13}\) and eighteen off-diagonal fields \(K_i, Q_i, S_i, P_i, T_i, R_i\) (\(i = 0, 1, 2\))

\[
\begin{align*}
A^1_t &= p_1 K_i, & A^1_s &= Q_i, & A^1_r &= S_i, \\
A^2_t &= Q_i, & A^2_s &= Q_i, & A^2_r &= R_i, \\
A^3_t &= q_1 K_1, & A^3_s &= Q_4, & A^3_r &= S_4, \\
A^4_t &= P_4, & A^4_s &= T_4, & A^4_r &= R_4, \\
A^5_t &= K_3, & A^5_s &= K_8, & A^5_r &= K_{15}, \\
\end{align*}
\]

where \(p_1, q_1\) are number parameters. One can verify that ansatz \((42)\) is consistent with Yang-Mills equations of motion, and symmetric under Weyl permutations in six dimensional spaces spanned by \(T^{1,4,6,9,11,13}\) and \(T^{2,5,7,10,12,14}\) and in the root space spanned by six linear combinations of Cartan generators \(r^\alpha_p T^\alpha\).

One can reduce the ansatz and construct a minimal Weyl symmetric ansatz with four independent fields \(K_i = K_0, K_r = K_1, K_0 = K_2, K_\varphi = K_4\) by imposing additional reduction constraints

\[
\begin{align*}
K_3 &= c_3 K_4, \\
K_8 &= c_8 K_4, \\
K_{15} &= c_{15} K_4, \\
Q_1 &= p_2 K_i, & S_1 &= p_3 K_i, \\
P_1 &= q_4 K_4, & S_4 &= q_4 K_4, \\
P_4 &= q_9 K_4, & T_4 &= q_{11} K_4, & R_4 &= q_{13} K_4, \\
\end{align*}
\]

where \(c_3, c_8, c_{15}, p_2, q_9\) are number parameters. In a case of SU(3) Yang-Mills theory the parameters \(q_1, q_4, q_8\) are defined by root vectors which provide a proper definition of an irreducible Weyl group representations. In a case of SU(4) Yang-Mills theory the parameters \(q_4\) cannot be set free by roots of SU(4) Lie algebra since the total sum of roots does not vanish contrary to the case of SU(3) roots, and, as a consequence, one meets an obstacle with constructing a non-degenerate five dimensional irreducible Weyl representation by setting a constraint

\[
K_4 + Q_4 + S_4 + P_4 + T_4 + R_4 = 0. \\
\]

To resolve this problem we treat the parameters \(p_1, q_1\) as free parameters providing a Weyl symmetric structure of the charge matrices \(G, Q\) in the Dirac equation for quarks in fundamental representation of SU(4) with applied Abelian projection defined by the ansatz \((42)\)

\[
\left[ i\gamma^\mu \partial_\mu - m + \frac{g}{2} \gamma^\mu A_\mu G + \frac{g}{2} \gamma^\mu K_\mu Q \right] \Psi = 0, \\
\]

where \(A_\mu = \delta_\mu 3 K_4, \ K_\mu = \delta_\mu i K_i\), the color charge matrix \(G\) reads

\[
G = \begin{pmatrix}
\tilde{w}^1 & q_1 & w^2 & q_4 & q_9 \\
q_1 & \tilde{w}^2 & q_6 & q_{11} & q_{13} \\
q_4 & q_6 & \tilde{w}^3 & q_{13} & \tilde{w}^4 \\
q_9 & q_{11} & q_{13} & \tilde{w}^4 & \end{pmatrix}, \\
\]

with diagonal elements \(\tilde{w}^\alpha\) representing linear combinations of weights of \(su(4)\) Lie algebra

\[
\begin{align*}
\tilde{w}^1 &= c_3 \cdot 1 + \frac{1}{\sqrt{3}} c_8 + \frac{1}{\sqrt{6}} c_{15}, \\
\tilde{w}^2 &= -c_3 \cdot 1 + \frac{1}{\sqrt{3}} c_8 + \frac{1}{\sqrt{6}} c_{15}, \\
\tilde{w}^3 &= -\frac{2}{\sqrt{3}} c_8 + \frac{1}{\sqrt{6}} c_{15}, \\
\tilde{w}^4 &= \frac{3}{\sqrt{6}} c_{15}. \\
\end{align*}
\]

A color charge matrix \(Q\) providing coupling constants of off-diagonal gluons with quarks has the following form

\[
Q = \begin{pmatrix}
0 & -ip_1 & -ip_2 & -ip_4 \\
ip_1 & 0 & -ip_3 & -ip_5 \\
ip_2 & ip_3 & 0 & -ip_6 \\
ip_4 & ip_5 & ip_6 & 0 \\
\end{pmatrix}, \\
\]

One can set \(q_1 = 1, p_1 = 1\) due to normalization freedom of the fields \(K_i, K_4\). Weyl symmetric structure of
the Dirac equation implies that a total sum of matrix elements in each row and column in matrices \(A_\gamma^p, A_\beta^p\) must vanish. Consistence equations for the matrix \(Q\) can be easily found:

\[
\begin{align*}
    p_3 &= -1 - p_2, \\
    p_4 &= -1 - p_2, \\
    p_5 &= 2 + p_2, \\
    p_6 &= -1,
\end{align*}
\]

Corresponding consistence equations for matrix \(G\) allow to find expressions for parameters \(c_3, c_8, c_{15}, c_{13}\) in terms of remaining parameters. With this one can solve analytically all constraints arising from consistence of the ansatz with all equations of motion. Final result is the following

\[
\begin{align*}
    c_3 &= \frac{p_2(2 + p_2)(-1 + p_2 - q_9) - 2(1 + q_9)}{2p_2(1 + p_2)}, \\
    c_8 &= \frac{2(1 + q_9) - p_2(2(1 + q_9) + p_2(5 + p_2 + 3q_9))}{2\sqrt{3}p_2(1 + p_2)}, \\
    c_{15} &= \frac{(2 + p_2)(2(1 + q_9) + p_2(3 + p_2 + 3q_9))}{\sqrt{6}p_2(1 + p_2)}, \\
    c_{13} &= \frac{1}{p_2}(2 + p_2 + 2q_9),
\end{align*}
\]

\[
q_4 = \frac{-2 + p_2(1 + p_2 + q_9)}{2(1 + p_2)}, \\
q_6 = \frac{1}{p_2}(2 + 2p_2 + 2q_9 + p_2q_9), \\
q_{11} = \frac{(2 + p_2)(1 + p_2 + q_9)}{2(1 + p_2)}, \\
q_{13} = \frac{1}{p_2}(2 + p_2 + 2q_9),
\]

The solution forms a two-parametric family defined by two arbitrary numbers \(p_2, q_9\). The parameters satisfy two relationships

\[
\begin{align*}
    q_1 + q_4 + q_6 + q_9 + q_{11} + q_{13} &= 0, \\
    p_1 + p_2 + p_3 + p_4 + p_5 + p_6 &= 0,
\end{align*}
\]

which provide existence of standard irreducible representations \(\Gamma_5\). One can find that color charge matrix \(G\) has two zero modes, one of which belongs to singlet irreducible representation containing a free zero quark mode. Other two color quark modes have opposite color charges. The matrix \(Q\) has also two zero modes for a free color singlet quarks, one of which coincides with the zero mode of the matrix \(G\). Other two non-zero quark modes have opposite color charges and can be decomposed in the basis of non-zero modes of the charge matrix \(G\). The obtained ansatz provides a Weyl symmetric structure of the gluon fields \(K_1, K_4\) and for quarks. However, due to presence of two free parameters \(p_2, q_9\) the ansatz defines solutions which are degenerated. In a general, this implies a spontaneous color symmetry breaking which excludes color confinement phase.

\section{Discussions}

A similar analysis can be performed for \(SU(5)\) Yang-Mills theory. We construct first a Weyl symmetric ansatz in a pure Yang-Mills theory. A general ansatz for magnetic type time-dependent solutions contains four Abelian potentials \(A_\gamma^p, A_\beta^p\) corresponding to the maximal Abelian subgroup of \(SU(5)\), ten Abelian potentials \(A_\gamma^p\) with an internal index \(p = (1, 4, 6, 9, 11, 13, 16, 18, 20, 22)\) which corresponds to coset generators \(T^{1,4,6,9,11,13,16,18,20,22}\) and it is used also for counting the root vectors, and ten off-diagonal vector fields \(A_\gamma^p(i = 0, 1, 2)\) corresponding to index values \(\vec{p} = (2, 5, 7, 10, 12, 14, 17, 19, 21, 23)\) corresponding to coset generators \(T^{2,5,7,10,12,14,17,19,21,23}\). A minimal ansatz is reduced by imposing reduction constraints

\[
\begin{align*}
    K_\alpha &= c_4 K_4, \\
    A_\gamma^p &= p^\gamma K_4, \\
    A_\gamma^p &= q^\gamma K_4,
\end{align*}
\]

where index \(\alpha = 3, 8, 15, 24\) denotes for generators of the Cartan subalgebra, and \(K_4\) is the only Abelian magnetic potential. Consistence of the minimal ansatz with all equations of motion leads to constraints on parameters \(c_4, p^\gamma, q^\gamma\). Additional constraints appear from the requirement that equation for a quark must possess Weyl symmetric structure.

The number of independent constraints equals the number of free parameters, so we resolve all constraints numerically and have found a following solution for the parameters

\[
\begin{align*}
    p_1 &= -1, \\
    p_2 &= -0.138325, \\
    p_3 &= -0.535107, \\
    p_4 &= -0.277493, \\
    p_5 &= -0.247661, \\
    p_6 &= 0.182746, \\
    p_7 &= 0.535065, \\
    p_8 &= 0.577338, \\
    p_9 &= -0.366178, \\
    p_{10} &= 0.027693,
\end{align*}
\]

\[
\begin{align*}
    q_1 &= 1, \\
    q_2 &= 0.011256, \\
    q_3 &= 0.082786, \\
    q_4 &= 0.011342, \\
    q_5 &= 0.154373, \\
    q_6 &= 0.199056, \\
    q_7 &= 0.052219, \\
    q_8 &= -0.285323, \\
    q_9 &= -0.406182, \\
    q_{10} &= -0.715090,
\end{align*}
\]

\[
\begin{align*}
    c_3 &= -0.009271, \\
    c_8 &= -0.620184, \\
    c_{15} &= -0.583812, \\
    c_{24} &= -1.153294,
\end{align*}
\]

The color charge matrix \(G\) has three zero modes, and two non-zero modes with effective coupling constants \(\lambda = \pm 1.412\). The obtained parameter values provides a Weyl symmetric structure of the minimal ansatz for a pure Yang-Mills theory. In a case of presence of fermions
the number of consistence equations providing Weyl symmetric structure becomes more than number of free parameters. We have not found Weyl symmetric ansatz in a case of including fermions, however, we can not exclude completely its existence since the consistence equations have a high non-linear structure and there might be some special solutions for the parameters which are missed during numeric solving.

In conclusion, we have considered an ansatz for Weyl symmetric classical stationary solutions in $SU(N)$ ($N = 2, 3, 4, 5$) Yang-Mills theory which describe color singlet one particle states for gluons and quarks. Previous approaches to construction of a gauge invariant vacuum or gauge invariant quantum states, fail because they were based on a seemed natural assumption that color symmetry of quantum states, gluons and quarks, is provided by a global color group $SU(3)$. We have demonstrated that a Weyl symmetric ansatz in $SU(2)$ and $SU(4)$ Yang-Mills theory admits a global color symmetry which leads to degeneracy of solutions, as a consequence, the vacuum in these theories is degenerate. Selection of a single vacuum in general leads to spontaneous color symmetry breaking which excludes color confinement phase in the theory.

In a case of $SU(5)$ Yang-Mills theory we did not find an ansatz with a minimal set of four gluon fields $K_1, K_4$ which provides Weyl symmetry of the total Lagrangian including fermions. We suppose that Weyl symmetric color singlet solutions for gluons and quarks exist only in $SU(3)$ Yang-Mills theory, i.e., in QCD. Our results reveal a non-trivial color structure of solutions necessary for construction of color singlet one-particle quantum states for gluons and quarks. Abelian solutions provide color invariant structure of the vacuum, which determines the color confinement phase in QCD $\text{\cite{B170,B160}}$. This opens a novel way to description of hadron structure.

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Appendix

We use main definitions and results from the group theory related to representations of the symmetric group $S_N$.

A representation of the symmetric group $S_N$ is defined as a group homomorphism of the group $S_N$ to the group of automorphisms of a vector space $V$. In other words the symmetric group $S_N$ can be realized by a symmetric group $S_N$ acting by permutations on vector components of a vector $\vec{V} = (V_1, V_2, \ldots, V_N)$. A one-dimensional irreducible representation (the sign representation) $\Gamma_1$ is defined by a constraint imposed on the vector components

$$V_1 = V_2 = \ldots = V_N = V. \quad (53)$$

The vector $\vec{V} = (1, 1, \ldots, 1)V$ is invariant (symmetric) under permutations of its components and represents a singlet representation. A standard $N - 1$-dimensional irreducible representation $\Gamma_{N-1}$ is defined by equation

$$V_1 + V_2 + \ldots + V_N = 0. \quad (54)$$

The equation defines a $N - 1$ dimensional hyperplane in $R^N$ which represents an invariant subspace under permutations of vector components. A dimension of the standard representation $\Gamma_{N-1}$ is defined by dimension of the hyperplane and equals $N - 1$. The standard representation is irreducible.

Let us compare representations of the unitary group $SU(3)$ which is a color group for QCD, and representations of the corresponding Weyl group which is a finite
color subgroup of $SU(3)$. A quark multiplet in fundamental representation of $SU(3)$ is given by three complex Dirac spinor functions forming a fundamental color triplet

$$\Psi = (\psi_1, \psi_2, \psi_3),$$

where $\psi_i$ are Dirac spinor functions depending on space-time coordinates. The representation is irreducible and three-dimensional in color space. In addition one has three independent coordinate spinor functions which describe three independent particles, color quarks. In quantum Yang-Mills theory, after removing pure gauge fields by gauge fixing procedure, a color symmetry is given by a finite color Weyl group, which is equivalent to permutation group $S_3$, which has different representations. For instance, a complex quark triplet (55) corresponding to the fundamental representation of $SU(3)$ color group realizes a reducible representation of $S_3$. It contains one dimensional irreducible representation $\Gamma_1$ with a basis vector

$$\Psi_0 = (1, 1, 1)\psi^0 \equiv u^0 \psi^0$$

with a coordinate spinor function $\psi^0$. It is clear that representation describe one particle, i.e., it is a singlet representation containing a neutral quark with zero color charge, (35). Another irreducible representation is given by a standard two-dimensional representation defined by equation

$$\psi_1 + \psi_2 + \psi_3 = 0.$$ 

One has two independent spinor functions which describe two quarks, so the number of independent quarks is the same. Despite on a mathematical fact that representation $\Gamma_2$ is irreducible and can not be decomposed further into irreducible representations of smaller dimensions, we have shown in the paper that these two quarks correspond to two eigenvectors $u^\pm$, (38), of the color charge matrix $G$ with Weyl invariant color charges $\tilde{g} = \pm \sqrt{6}$, and the quarks are described by two complex triplets

$$\Psi^+ = u^+ \psi^+, \quad \Psi^- = u^- \psi^-,$$

where color and coordinate parts are factorized. The color vectors $u^{0\pm}$ form a complete orthogonal basis vectors in the color space. The color vectors $u^\pm$ are not singlets. Under permutations of their components each eigenvector generates a sextet consisting of six elements which lie in the two-dimensional plane and do not form a one-dimensional space. Elements of two sextets are pairwise orthogonal and form two Weyl invariant non-intersecting sets. Each quark triplet $\Psi^\pm$ contains one independent Dirac spinor $\psi^\pm$ which describes a single quark. Thus, each quark triplet $\Psi^\pm$ describes one color singlet quark $\psi^\pm$. Note that the presence of a sextet of color vectors $\{u^+\}$ obtained by permutations of vector components $u^+_i$ does not imply that quark solution is degenerated since one has only one single quark described by Weyl invariant spinor $\psi^+$. 