Generic Orthotopes

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Abstract

This article studies a large, general class of orthogonal polytopes which we may call generic orthotopes. These objects emerged from a desire to represent a Coxeter complex by an orthogonal polytope that is particularly nice with respect to traditional topological, structural, or combinatorial considerations. Generic orthotopes have a pleasant “homogeneity” property, somewhat like a smoothly bounded compact subset of Euclidean space. Thus, as soon as we demand that every vertex of an orthogonal polytope be a floral arrangement, as defined here, many derivative structures such as faces and cross-sections are also described by floral arrangements. We also give formulas for the volume and Euler characteristic of a generic orthotope using a couple of statistics that are defined naturally for floral arrangements.

1 Introduction

Suppose $d$ is a non-negative integer. By an orthogonal polytope we mean a union of finitely many axis-aligned boxes in Euclidean space $\mathbb{R}^d$. This article lays a foundation for a theory of a particular set of orthogonal polytopes which represents an elementary generalization of the $d$-dimensional cube to $d$-dimensional orthogonal polytopes. We summarize the salient properties of these “generic orthotopes”:

- Every face of a generic orthotope is a generic orthotope.
- Every orthographic cross-section of a generic orthotope is a generic orthotope.
- The vertex figure of every vertex a generic orthotope is a simplex.
- The 1-dimensional skeleton of a generic orthotope is a bipartite $d$-regular graph.
There are elementary formulas which relate the volume and Euler characteristic of a generic orthotope.

The structural and combinatorial properties of a generic orthotope remain intact through small perturbations of their facets.

We may approximate any compact subset of Euclidean space to any degree of accuracy with a generic orthotope.

Notice that all but the last of these properties remain valid when “generic orthotope” is replaced by the word “cube”. We establish all of these properties in this article. Moreover, by what seems like good fortune, all of these properties follow in an elementary manner, given an understanding of the local structure of a generic orthotope.

The local structure of a generic orthotope has a convenient construction using read-once Boolean functions. Thus, one finds “floral arrangements” and “floral vertices” at the core of this theory, where a floral arrangement is determined by applying a read-once Boolean function to a set of half-spaces possessing distinct supporting hyperplanes. We may encode a read-once Boolean function by a series parallel diagram, and this leads to another bit of good fortune: We may use a topological invariant of these diagrams, namely the number of loops modulo 2, to obtain an expression for the Euler characteristic of a generic orthotope using only the values of this invariant at its vertices. Our statement of this formula appears below as Theorem 4.9. If one accepts the thesis that generic orthotopes are analogous to smoothly-bounded subsets of Euclidean space, then one cannot help but recognize the similarity of this formula to the Poincaré-Hopf theorem which expresses the Euler characteristic as the sum of indices of a vector field with a finite number of singularities.

The emergence of generic orthotopes is somewhat convoluted. The original motivation came from this author’s desire to represent Coxeter complexes by orthogonal polytopes which are somehow “nice”. In conceiving this problem, however, it was not clear what “nice” should mean with regards to orthogonal polytopes. This author regards convex polytopes which are simple (having exactly $d$ edges at every vertex) as particularly “nice”, but it was not a priori clear what higher structure one might borrow to study orthogonal polytopes. The present article precisely develops what “nice” should mean for an orthogonal polytope, and we pose the general problem for Coxeter complexes in the concluding section.

### 1.1 Examples in Low Dimensions

In order to illustrate the main ideas of this article, we consider the cases $d = 2$ and $d = 3$. In two dimensions, we draw a contrast between the polygons which appear in Figure 1.1; one of these is homeomorphic to a disc, while the other has “singular points” where the boundary is self-intersecting. Using the terminology developed here, the former of these is a generic orthotope and the latter is not.
Figure 1.1: (a) A generic orthogon. (b) Self-intersecting boundary.

One may relate the numbers of corners of the two types that one sees in a generic orthogon. In the polygon on the left in Figure 1.1, one notices that there are \( n_1 = 9 \) corners that “point outward” and \( n_3 = 5 \) corners that “point inward”. The authors of [14, 4] call these “salient” and “reentrant” points, respectively. The subscripts 1 and 3 here specify the number of quadrants occupied by the polygon at that type of vertex. An immediate corollary of the 2-dimensional version of our formula in Theorem 4.9 is that one always has \( n_1 - n_3 = 4 \) for every generic orthogon.

As one would expect, the situation when \( d = 3 \) is more complicated. In the terminology developed here, each of the vertices which appear in Figure 1.2 is a floral vertex. Superficially, the properties that make these vertices “nice” are (a) there are easily identified faces incident to the vertex and (b) the faces incident to each vertex coincide with the face lattice of a 2-dimensional simplex (i.e. a triangle). By contrast, if a 3-dimensional orthogonal polytope has a “degenerate vertex” as one appearing in Figure 1.3, then we do not regard it as a generic orthotope. One can quickly conceive of other kinds of degenerate points in 3 dimensions, and one imagines that the number of types of degeneracies that might arise when \( d \geq 4 \) grows quickly, perhaps exponentially. Up to congruence, the only types of floral vertices when \( d = 3 \) appear in Figure 1.2.

We may relate the numbers of congruence types of vertices which appear in such a polytope. Thus, suppose \( P \) is a 3-dimensional orthogonal polytope that such that every vertex appears as one of the four congruence types as depicted in Figure 1.2. For each \( i \in \{1, 3, 5, 7\} \), let \( n_i \) denote the number of vertices of these corresponding types, where \( i \) indicates the number of octants occupied by its tangent cone. Then Theorem 4.9 yields, for \( d = 3 \),

\[
n_1 - n_3 - n_5 + n_7 = 8\chi(P),
\]

where \( \chi(P) \) is the (combinatorial) Euler characteristic of \( P \).

We illustrate some of these ideas with an example. Define an orthogonal polytope by \( P = \bigcup_{v \in S} (v + [0, 1]^3) \), where \( S \subset \mathbb{R}^3 \) appears in Figure 1.4 and \( v + [0, 1]^3 \) denotes the translation of the unit cube \([0, 1]^3\) by adding \( v \). One should imagine \( P \) as an assembly of several stacks of unit cubes resting “skyscraper style” on a flat surface representing the \((x, y)\)-plane in \( \mathbb{R}^3 \). A view of \( P \) “from
Figure 1.2: Three-dimensional floral vertices.

Figure 1.3: Degenerate vertices in 3 dimensions.

\[ S = \left\{ (0,0,0), (1,0,0), (2,0,0), (3,0,0), (4,0,0), (0,1,0), (1,1,0), \\
(2,1,0), (3,1,0), (0,2,0), (2,2,0), (0,3,0), (1,3,0), (2,3,0), \\
(1,0,1), (2,0,1), (3,0,1), (4,0,1), (2,1,1), (3,1,1), (2,2,1), \\
(0,3,1), (1,3,1), (2,3,1), (1,0,2), (2,0,2), (3,0,2), (4,0,2) \right\}. \]

Figure 1.4: Generating corners of the example.
above” appears in Figure 1.5. The numbers which appear in the figure give the heights of these stacks. In order to see that \( P \) is a generic orthotope, notice that every vertex of \( P \) is congruent to one vertices appearing in Figure 1.2. Here, we have \( n_1 = 15 \), \( n_3 = 11 \), \( n_5 = 5 \), \( n_7 = 1 \), and thus
\[
n_1 - n_3 - n_5 + n_7 = 15 - 11 - 5 + 1 = 0,
\]
which we expect because \( P \) is homeomorphic to a solid 3-dimensional torus.

### 1.2 Flowers

Dating to the 1950’s, the Kneser-Poulsen conjecture asserts roughly that the union of a finite set of Euclidean balls cannot increase if the distances between their centers decrease or remain equal. In their work on this problem for general \( d \), Bezdek and Connelly [5] demonstrated equivalences between various statements which generalize the conjecture to flower weight functions. In their terminology, which can be traced to work by Csikós [12] and earlier to Gordon and Meyer [17], a “flower” is a subset of Euclidean space that is obtained by applying a certain type of Boolean function to a collection of balls. In their description, Bezdek and Connelly used the phrase “exactly once” to describe the variables used for the Boolean functions which they employed to define flowers. This led the present author to [11], where such functions are described as “read-once Boolean functions”, terminology which this author employs throughout. Thus, the underlying construction of a floral arrangement as defined here is essentially identical with the construction of flowers.

We can see how to use read-once functions to define 3-dimensional floral vertices. Thus, denote \( H_1 = \{(x, y, z) : x \geq 0\} \), \( H_2 = \{(x, y, z) : y \geq 0\} \), and \( H_3 = \{(x, y, z) : z \geq 0\} \) as three half-spaces in \( \mathbb{R}^3 \). Using union \( \cup \) and \( \cap \) for union and intersection respectively, we may describe these four configurations by
\[
H_1 \cap H_2 \cap H_3, \ (H_1 \cup H_2) \cap H_3, \ (H_1 \cap H_2) \cup H_3, \text{ and } H_1 \cup H_2 \cup H_3.
\]
By contrast, neither of the degenerate vertices depicted in Figure 1.3 possesses such a representation. The graphs which appear in Figure 1.2 are the representations of these read-once functions by series-parallel diagrams, where \( \cap \) is
interpreted as series connection and \( \cup \) is interpreted as parallel connection. In the formula \( n_1 - n_3 - n_5 + n_7 = 8 \chi(P) \) for a 3-dimensional generic orthotope, the signs appearing as coefficients on \( n_i \) are determined simply by counting the number of loops in the corresponding diagrams modulo 2.

There is a substantial body of literature on read-once functions (with [11] as a good starting point), but most of this is not relevant for the present work. A reader of this work should acquaint themself with the basic notions of read-once functions, as these are fundamental for the definitions of floral arrangements and generic orthotopes. By a similar token, this article does not address the Kneser-Poulsen conjecture. The works cited above are relevant only insofar as these led this author to incorporate read-once functions into the theory of orthogonal polytopes.

### 1.3 Contexts

The audience for this work includes anyone who has interest in orthogonal polytopes generally. This author is impressed by the intrinsic beauty of generic orthotopes and believes they are worthy of study for their own sake. The contexts for orthogonal polytopes are certainly myriad, although we observe that there is a considerable gap in the theory of general \( d \)-dimensional orthogonal polytopes. Thus, whereas many workers have studied rectilinear polygons, polyominoes, polycubes, rectangular layouts, orthogonal graph drawings, orthogonal polyhedra/surfaces, \( xyz \) polyhedra, 3D staircase diagrams, and so on, comparatively few have focused on the general case when \( d \geq 4 \).

With that said, the literature on general \( d \)-dimensional orthogonal polytopes is not completely barren. Breen has a series of studies, starting with [9], which purport to seek analogues of Helly’s theorem on convex sets in Euclidean space for orthogonal polytopes, extending work of of Danzer and Grünbaum [13]. Starting approximately with [2] and [3], Barequet and several of his colleagues have worked on enumerating polycubes (also known as lattice animals) for general \( d \); see the aforementioned articles and [18] for more details, especially as these problems arise in statistical physics. Werman and Wright [25] study probabilistic aspects of random cubical complexes; their approach parallels and complements the present work as they also use the language of valuations for subsets of \( \mathbb{R}^d \).

Bournez, Maler, and Pnueli [8], concerned with devising “hybrid systems” in control theory and recognizing a gap in the general theory, develop algorithms for membership, face-detection and Boolean operations for representing these systems by orthogonal polytopes. Quoting from [8], “Beyond the original motivation coming from computer-aided control system design, we believe that orthogonal polyhedra and subsets of the integer grid are fundamental objects whose computational aspects deserve a thorough investigation.” This author believes that the present work advances this project significantly. Like the present work, the authors of [8] use ideas from Boolean algebra, although they do not use read-once functions to describe local structure of an orthogonal polytope.
Pérez-Aguila and his colleagues have devoted significant energies to studying general orthogonal polytopes from the perspective of computer science and computer engineering, [20, 21, 22, 23]. Their approach is largely founded on Pérez-Aguila’s $d$-dimensional generalization of the Extreme Vertices Model for 3-dimensional orthogonal polytopes (cf. [1]). This model appears closely related to the treatment of floral vertices shown here. Moreover, their perspective also shares similar significant structural and combinatorial considerations of orthogonal polytopes with this author. These works also employ Boolean algebra extensively, although again we notice a lack of emphasis on read-once Boolean functions in particular.

Orthogonal polytopes arise in toric geometry, where they are called “staircases”. A basic idea in this theory is that we can gain some algebraic insight by modeling a square-free monomial ideal in a polynomial ring by studying an associated orthogonal polytope lying in the primary orthant (where all coordinates are non-negative) of $\mathbb{R}^d$. The fascinating text [19] expounds on these ideas at great length. However, we stress that the orthogonal polytopes most often encountered in toric geometry appear to be “totally spherical” in the sense that every face is homeomorphic to a closed cell, whereas the objects studied here are considerably more general.

A generic orthotope shares some attributes with a smoothly bounded compact Euclidean set. For example, the tangent cone at every point on the boundary of any given generic orthotope is homeomorphic to a half-space. Similarly, the bipartiteness of the 1-dimensional skeleton of a generic orthotope is reminiscent of the orientability the boundary of a smoothly bounded compact set. Moreover, our function $\sigma$ seems to measure a discrete analogue of curvature at each point. In the same vein, we also mention this author’s recent work [24] on generic rectangulations. In its roughest description, a generic rectangulation is a subdivision of a rectangle of a rectangle into rectangles, where the descriptor “generic” means that no four constituent rectangles share a common corner. In [24], this author demonstrated that one may perform a “central involution” on a generic rectangulation about any one of its constituent rectangles, analogous to linear fractional transformations of the complex projective line $\mathbb{C}P^1$. Since generic orthotopes are defined discretely, this indicates a context in discrete differential geometry.

1.4 Organization

This article is organized as follows. First there is a brief section on background on orthogonal polytopes; this defines tangent cones and the face poset of an orthogonal polytope. Following this are two long sections which describe the foundations of generic orthotopes. These are divided according to local theory versus global theory. The section on local theory defines floral arrangements, studies the structure of a floral arrangement, and introduces two “local” valuations $\mu$ and $\tau$ which will be required in the section on global theory. The section on global theory defines generic orthotopes (in terms of floral arrangements), studies some of their properties, and gives several formulas relating the volume
and Euler characteristic of a generic orthotope. We conclude with a few open questions about generic orthotopes.

Throughout this article, denote $[d] = \{1, 2, 3, ..., d\}$.

2 General orthogonal polytopes

Denote the standard basis for $\mathbb{R}^d$ by $\{e_i : i \in [d]\}$, where $e_i$ is the unit vector pointing along the positive $x_i$-axis for each $i$. The cardinal directions are the $2d$ vectors $\{\pm e_i : i \in [d]\}$. The cardinal ray of $\delta$ is the cone generated by the cardinal direction $\delta$.

For each $i \in [d]$ and for each $\lambda \in \mathbb{R}$, let $\Pi_{i,\lambda}$ be the $(d-1)$-dimensional hyperplane defined by the equation $x_i = \lambda$, (i.e. the null-space of the functional $x_i - \lambda$). Define the $i$th canonical orthographic projection by

$$\pi_i : (x_1, x_2, ..., x_i, ..., x_d) \mapsto (x_1, x_2, ..., 0, ..., x_d).$$

We identify each hyperplane $\Pi_{i,\lambda}$ with $\mathbb{R}^{d-1}$ via the $i$th orthographic projection. If $I \subset [d]$ is any subset and $\lambda : I \to \mathbb{R}$ is a tuple, let

$$\Pi_{I,\lambda} = \bigcap_{i \in I} \Pi_{i,\lambda(i)}$$

be the generalized hyperplane determined by $(I, \lambda)$.

An axis-aligned box is a cartesian product of closed intervals $\prod_{i=1}^d [a_i, b_i]$ such that $a_i \leq b_i$ for all $i$ and each interval $[a_i, b_i]$ is embedded in the $i$th summand of the direct sum $\mathbb{R}^d = \bigoplus_{i=1}^d \mathbb{R}$. Call such a box pure $d$-dimensional if $a_i < b_i$ for all $i$. An orthogonal polytope is a subset of $\mathbb{R}^d$ that has an expression as the union of a finite set of axis-aligned boxes. Call an orthogonal polytope pure $d$-dimensional if it has an expression as a union of pure $d$-dimensional axis-aligned boxes and singular if it admits no such expression.

Axis-aligned boxes are the fundamental examples of orthogonal polytopes. (In fact, the term “orthotope” has been used for these objects, cf. [10].) The standard unit cube is the cartesian product $I^d$, where $I = [0, 1]$ is the unit interval. An orthogonal polytope $P$ is integral if it is a union of translates

$$P = \bigcup_{v \in S} (v + f_v) \text{ (Minkowski sum)},$$

where $S$ is a finite subset of the lattice $\mathbb{Z}^d \subset \mathbb{R}^d$ and $f_v$ is a face of the standard unit cube $I^d$ for all $v \in S$. Call an orthogonal polytope $P$ rational if there is a positive integer $n$ such that $nP$ is integral.

2.1 Tangent cones and faces

We aim here to define the face poset of an orthogonal polytope.

Given $s = (s_1, s_2, ..., s_d) \in \{\pm 1\}^d$, the orthant represented by $s$ is $\Omega_s = \{(x_1, x_2, ..., x_d) : s_i x_i \geq 0 \text{ for all } i\}$. A local orthotopal arrangement in $\mathbb{R}^d$ is a
union of orthants. Local orthotopal arrangements appear in bijective correspondence with Boolean functions. Define the sign of \( s \) as 
\[
(-1)^s = \prod_{i=1}^{d} s_i.
\]

Suppose \( P \subset \mathbb{R} \) is a pure \( d \)-dimensional orthotope and \( v \in P \). The tangent cone at \( v \) is the local orthotopal arrangement \( \alpha \) such that there exists \( \delta > 0 \) such that
\[
P \cap [-\delta, \delta]^d = v + (\alpha \cap [-\delta, \delta]^d).
\]

If \( v \in P \), define the genericity region of \( v \) as the set of all points \( w \) that can be joined by a path \( \gamma \) for which the tangent cone at every point along \( \gamma \) is congruent to the tangent cone at \( v \). Evidently every genericity region is path-connected and the genericity regions partition \( P \). The degree of a genericity region is the smallest dimension among the hyperplanes which contain it. Thus, if \( v \) lies interior to \( P \) if \( v \) has genericity degree \( d \), and \( v \) is a singular point of \( P \) if its degree of genericity is zero. Define a \( k \)-dimensional face of \( P \) as the closure of a \( k \)-dimensional genericity region. The face poset of \( P \) is the set of all faces of \( P \), partially ordered by inclusion.

Suppose \( \alpha \subset \mathbb{R}^d \) is a local orthotopal arrangement. Define \( \mu_d(\alpha) \) as the number of orthants occupied by \( \alpha \) and let \( \tau_d(\alpha) \) denote the sum of the signs of the orthants occupied by \( \alpha \). Apparently these functions satisfy inclusion-exclusion identity
\[
f(\alpha) + f(\beta) = f(\alpha \cap \beta) + f(\alpha \cup \beta),
\]
for \( f \in \{\mu_d, \tau_d\} \) and local orthotopal arrangements \( \alpha, \beta \).

3 Generic Orthotopes: Local Theory

The purpose here is to introduce and study floral arrangements, which will be needed in order to define generic orthotopes. First we explain how to use series-parallel diagrams to define floral arrangements and floral vertices. Then we describe some structure of the face lattice of a floral arrangement. Finally we introduce some “local” valuations and relate them to the functions \( \mu_d \) and \( \tau_d \) defined above.

3.1 Series-parallel diagrams

Define a series-parallel diagram (SPD for short) inductively as either

1. A single edge \( \leftrightarrow \) joining two terminals (the vertices),

2. a series connection of series-parallel diagrams, or

3. a parallel connection of series-parallel diagrams.

We admit the single vertex \( \bullet \) as an “honorary” SPD, even though it does not possess two distinct terminals.
3.1.1 Bouquet sign

If $\Delta$ is an SPD, let $E(\Delta)$ denote the edges of $\Delta$, and let $e(\Delta) = |E(\Delta)|$ and $v(\Delta)$ denote the numbers of edges and vertices of $\Delta$ respectively. Use the symbols $\land$ and $\lor$ to denote series and parallel connection, respectively. Then we have

$$v(\rightarrow) = 2, \quad e(\rightarrow) = 1,$$

$$e(\Delta_1 \lor \Delta_2) = e(\Delta_1 \land \Delta_2) = e(\Delta_1) + e(\Delta_2),$$

and

$$v(\Delta_1 \land \Delta_2) = v(\Delta_1) + v(\Delta_2) - 1,$$

for any SPD’s $\Delta_1$ and $\Delta_2$.

Define the bouquet rank of an SPD $\Delta$ as

$$\rho(\Delta) := e(\Delta) - v(\Delta) + 1.$$

The terminology comes from the fact that an SPD $\Delta$ is homotopy equivalent to a bouquet of $\rho(\Delta)$ circles. Evidently we have

$$\rho(\Delta_1 \land \Delta_2) = \rho(\Delta_1) + \rho(\Delta_2) \quad \text{and} \quad \rho(\Delta_1 \lor \Delta_2) = \rho(\Delta_1) + \rho(\Delta_2) + 1$$

for any $\Delta_1$, $\Delta_2$. Define the bouquet sign of $\Delta$ by

$$\sigma(\Delta) := (-1)^{\rho(\Delta)}.$$

Evidently we have

$$\sigma(\Delta_1 \land \Delta_2) = \sigma(\Delta_1)\sigma(\Delta_2) \quad \text{and} \quad \sigma(\Delta_1 \lor \Delta_2) = -\sigma(\Delta_1)\sigma(\Delta_2).$$

for all $\Delta_1$, $\Delta_2$.

3.1.2 Duality

Suppose $\Delta$ is an SPD. The dual of $\Delta$ is the SPD $\overline{\Delta}$ obtained by interchanging the roles of series and parallel connection in its parse tree. If $\Delta$ has $d$ edges, then the dual $\overline{\Delta}$ also has $d$ edges, and the bouquet rank $\rho$ satisfies $\rho(\Delta) + \rho(\overline{\Delta}) = d - 1$ for all $\Delta$. Accordingly, the bouquet sign $\sigma$ satisfies $\sigma(\Delta)\sigma(\overline{\Delta}) = (-1)^{d-1}$ for every $\Delta$ with $d$ edges.

A signed SPD is a pair $(\Delta, s)$, where $\Delta$ is an SPD and $s : E(\Delta) \to \{\pm 1\}$. In drawing signed SPD’s by hand, it is convenient to indicate negative edges with overline bars and positive edges without such marks or by using distinguishing colors. Define the dual of $(\Delta, s)$ as the signed SPD $(\overline{\Delta}, -s)$. Notice one obtains the dual $(\overline{\Delta}, -s)$ by applying DeMorgan’s laws when we interpret $(\Delta, s)$ as a Boolean function. Figure 3.1 displays an example of a signed series-parallel diagram and its dual.
3.2 Floral arrangements and floral vertices

Suppose \((\Delta, s)\) is a signed SPD with edges \(E \subset [d]\), and let \(\{H_i : i \in E\}\) be the positive closed half-spaces in \(\mathbb{R}^d\) such that \(H_i\) is supported by the hyperplane \(\Pi_{i,0}\) for each \(i\). The **floral arrangement** determined by \((\Delta, s)\) is the local orthotopal arrangement obtained by interpreting series connection as intersection, parallel connection as union, and each \(i \in E(\Delta)\) is evaluated as \(s_i H_i\). We use the term **floral vertex** in the case when \(E = [d]\). Figure 3.2 illustrates this for \(d = 2\). It is apparent that every floral arrangement \(\alpha = \alpha(\Delta, s)\) has an expression \(\alpha \sim \alpha' = \mathbb{R}^d - |E| \times \alpha'\), where \(\alpha' \subset \mathbb{R}^{|E|}\) is a floral vertex on \(|E|\) half-spaces.

If \(\alpha\) is a floral arrangement defined by the signed SPD \((\Delta, s)\), let \(\overline{\alpha}\) denote the complementary arrangement defined by the dual \((\overline{\Delta}, -s)\).

At this point, it is important to note a particular usage of the symbols \(\wedge, \vee, \cap,\) and \(\cup\). If \(\alpha, \beta \subset \mathbb{R}^d\) are floral arrangements, then \(\alpha \cap \beta, \alpha \cup \beta \subset \mathbb{R}^d\) are interpreted using ordinary intersection and union. However, we cannot expect \(\alpha \cap \beta\) or \(\alpha \cup \beta\) to be a floral arrangement in general. For example, if \(\alpha\) is represented by \((1 \vee 2) \wedge 3\) and \(\beta\) is represented by \((1 \vee 3) \wedge 2\), then \(\alpha \cup \beta\) is the degenerate arrangement with 6 edges as depicted in Figure 1.3. On the other hand, if \(\alpha\) and \(\beta\) are floral vertices, then we use \(\alpha \wedge \beta\) or \(\alpha \vee \beta\) to denote the floral vertex obtained by joining the SPD’s for \(\alpha\) and \(\beta\) in series or parallel, respectively. In particular, if \(\alpha\) and \(\beta\) are floral vertices, then \(\alpha \wedge \beta\) is a cartesian product of \(\alpha\) and \(\beta\). This is evident, for example, in Figure 3.2.

We consider congruence types of floral arrangements. The group of symmetries of the cube \([-1,1]^d\) acts on acts on floral arrangements \(\alpha \subset \mathbb{R}^d\) in an apparent way. Denote this group by \(BC_d\), and recall that \(BC_d\) is isomorphic to the wreath product \(S_2 \wr S_d = S_d \ltimes S_2^d\), where \(S_d\) acts on \(S_2^d\) by permuting the coordinates and \(S_2^d\) acts by altering the signs of the coordinates. Every element \(g \in BC_d\) may thus be regarded as an ordered pair \(g = (f, s)\), where \(s \in \{\pm 1\}^d\) is a tuple of signs and \(f\) is a permutation of \([d]\). Suppose \(g = (f, s) \in BC_d\) and \((\Delta, s')\) is a signed SPD with edge set \(I \subset [d]\). Then the action of \((f, s)\) on \((\Delta, s')\) is the signed SPD with edge set \(f(I)\) and sign function \(ss'\) obtained by
coordinate-wise multiplication. We may represent each orbit in this action by an SPD where we do not distinguish the edges. We refer to the dimension of a floral vertex $\alpha$ as the number $\dim \alpha = \ell(\Delta)$ of edges used in the diagram $\Delta$ which defines $\alpha$.

Let $A_d$ denote the number of congruence classes of floral vertices on $d$ edges. This coincides with the number of unmarked SPD’s and appears as sequence A000084 in the Online Encyclopedia of Integer Sequences. Thus, the number of congruence classes of floral arrangements in $\mathbb{R}^d$ is $\sum_{k=0}^{d} A_k$. Several values of these sequences appear in the table in Figure 3.3.

### 3.3 Facets of a floral vertex

We demonstrate here that the faces of a floral vertex coincide with the faces of a simplex and that every face is also described by a floral vertex. Throughout this section, we assume that $\alpha \subset \mathbb{R}^d$ is a floral vertex determined by a signed SPD.
\((\Delta, s)\) with edges \(E = [d]\). Without loss of generality, we assume that every component of \(s\) is +1, so that all of the corresponding half-spaces are positive. For each \(i \in E\), denote \(\delta_i(\alpha) = (\partial \alpha) \cap \Pi_{i,0}\).

**Proposition 3.1.** Suppose a floral vertex has an expression \(\alpha = \alpha_1 \wedge \alpha_2\), where \(\alpha_1\) is a floral vertex on \(\{H_1, H_2, \ldots, H_k\}\). For each \(i \in \{1, 2, 3, \ldots, k\}\), we have \(\delta_i(\alpha) = \delta_i(\alpha_1) \wedge \alpha_2\).

**Proof.** This follows from the fact that the floral vertex \(\alpha_1 \wedge \alpha_2\) is the cartesian product of the floral vertices \(\alpha_1\) and \(\alpha_2\). \(\square\)

We may now state:

**Proposition 3.2.** Let \(\alpha \subset \mathbb{R}^d\) be a floral vertex, and suppose \(k\) is an integer with \(0 < k < d\). Then (i) every degree-\(k\) genericity region of \(\alpha\) is a \(k\)-dimensional face of \(\alpha\), and (ii) every generalized coordinate hyperplane of dimension \(k\) contains precisely one face of \(\alpha\).

**Proof.** This follows by induction. The statement is apparently valid when \(d = 1\). Suppose \(d \geq 2\) is fixed and assume the statement is true for all values less than \(d\), and let \(\alpha\) be a floral vertex on \(d\) half-spaces. Suppose first that \(\alpha\) is a conjunction, say \(\alpha = \alpha_1 \wedge \alpha_2\). The induction hypothesis then holds for \(\alpha_1\) and \(\alpha_2\). However, since \(\alpha\) is then a cartesian product, so the conclusion holds. On the other hand, if \(\alpha\) is a disjunction \(\alpha_1 \lor \alpha_2\), then one may apply this argument to the complementary arrangement \(\overline{\alpha}\), which is necessarily a conjunction. \(\square\)

As an immediate corollary of the preceding results on the structure of a local floral arrangement, we see:

**Proposition 3.3.** Let \(\alpha \subset \mathbb{R}^d\) be a floral vertex on the half-spaces \(H_1, H_2, \ldots, H_d\). Then each facet \(\delta_i(\alpha)\) is a floral vertex on the subspaces \(H_j \cap \Pi_{i,0}\), with \(j \neq i\).

To elucidate this, we combine the preceding results about the structure of a floral arrangement to describe an algorithm for computing \(\delta_i(\alpha)\). The input of the algorithm includes a positive integer \(d\), a floral vertex \(\alpha\) described by an SPD \(\Delta\), and an integer \(i \in [d]\). In the first iteration, one checks whether or not \(\Delta\) is a conjunction or a disjunction. If \(\Delta\) is a conjunction, then we let \(\alpha_0 = \alpha\). If \(\Delta\) is a disjunction, then let \(\alpha_0 = \overline{\alpha}\). Next, assuming that we know \(\alpha_k\) from a previous iteration, we define two new floral arrangements \(\beta_k\) and \(\gamma_k\) by the requirement that \(\alpha_k = \beta_k \lor \gamma_k\), where \(\beta_k\) and \(\gamma_k\) are floral arrangements such that \(\beta_k\) is a disjunction which contains \(i\) or \(\overline{i}\). Next, we define \(\alpha_{k+1} = \overline{\beta_k}\). We iterate this until \(\alpha_k\) is a floral arrangement on a single half-space. The floral arrangement representing the facet \(\delta_i(\alpha)\) is then the conjunction \(\gamma_1 \land \gamma_2 \land \cdots \land \gamma_k\), where \(\gamma_k\) is the last “difference” obtained from these iterations.

Figure 3.4 illustrates an example of this algorithm. The red numerals indicate negative half-spaces, and the red arrows indicate complementation. Here, the input arrangement is

\[
\alpha = (((((1 \lor 2) \land 3) \lor 4) \land 5) \lor 6) \land (7 \lor 8),
\]

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Figure 3.4: Illustration of the facet algorithm, for $\delta_4(\alpha)$.

represented by the SPD in the figure, and the output arrangement is

\[ \delta_4(\alpha) = (\bar{1} \lor \bar{2}) \land 5 \land \bar{6} \land (7 \lor 8). \]

The reader is urged to use this algorithm to compute $\delta_i(\alpha)$ for $i \neq 4$.

### 3.4 Edges and cross-sections

In this section we explain some structure concerning edges and cross-sections of a floral vertex. First we show how to determine the edges emanating from a floral vertex. Then we show that each cross-section perpendicular to a given edge is described by a floral arrangement.

#### 3.4.1 Edges of a floral vertex

Let $\alpha$ be a floral vertex. It is a consequence of the analysis above that for each $i \in [d]$, exactly one of the cones generated by either $e_i$ or $-e_i$ is an edge of $\alpha$. We demonstrate here how to determine the directions of these edges.

A precise statement of this result is facilitated with a method of deleting an edge from an SPD, as follows. Suppose $\Delta$ is an SPD with edges $E = [d]$. Choose and fix an edge $i \in E$. Call $i$ disjunctive if there is another subdiagram that is connected in parallel with $i$. Call $i$ conjunctive if it is not disjunctive. By De Morgan’s laws, an edge $i$ of $\Delta$ is is conjunctive in $\Delta$ if and only if $i$ is disjunctive in the dual $\Delta$. 

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If \( i \) is an edge of \( \Delta \), define the deletion of \( i \) as the SPD \( \Delta \setminus i \) as follows:

- If \( i \) is conjunctive, then \( \Delta \setminus i \) is the SPD obtained by deleting \( i \) from \( \Delta \) and identifying the terminals of \( i \).
- If \( i \) is disjunctive, then \( \Delta \setminus i \) is the series parallel diagram obtained by deleting \( i \) from \( \Delta \).

**Proposition 3.4.** Suppose \( \alpha \) is a floral vertex determined by the signed series-parallel diagram \((\Delta, s)\), with \( s = (s_1, s_2, \ldots, s_d) \) and \( i \) is an edge of \( \Delta \). Then the cone generated by \( s_i e_i \) is an edge of \( \alpha \) if and only if \( \sigma(\Delta) = \sigma(\Delta \setminus i) \).

**Proof.** We use an observation about the facet-computing algorithm described above. Each iteration in the algorithm involves complementation of a subdiagram of \( \Delta \) or \( \bar{\Delta} \) which contains \( i \) or \( \bar{i} \). Thus, in the last iteration, one is left with the series-parallel graph consisting of a single edge that is marked with either \( i \) or \( \bar{i} \). Thus, the sign is determined by the parity of the number of iterations required. Note that a floral vertex and its complementary arrangement have the same boundary complex, so, in particular, they have the same edges. Note also that cartesian products preserve edges in the following sense: If \( e_i \) generates an edge of a floral vertex, say \( \alpha \), then \( e_i \) also generates an edge in a conjunction \( \alpha \land \beta \).

From this, we see that another way to write this result is that, under the given hypotheses, the product \( \sigma(\Delta)\sigma(\Delta \setminus i)s_i e_i \) generates an edge for each \( i \).

### 3.4.2 Cross-sections

Suppose \( \alpha \subset \mathbb{R}^d \) is a floral arrangement. For our purposes, a cross-section of \( \alpha \) is an intersection of the form \( \alpha \cap \Pi_{I,\lambda} \), where \( \Pi_{I,\lambda} \) is a generalized hyperplane as defined earlier. We show that every cross-section parallel to a generalized coordinate hyperplane of a floral arrangement is described by another floral arrangement.

Assume \( \alpha \subset \mathbb{R}^d \) is a floral vertex determined by the signed SPD \((\Delta, s)\). Without loss of generality, assume that every component of \( s \) is +1. Suppose \( i \in E(\Delta) \) is given. The structure of a given cross-section \( \alpha \cap \Pi_{I,\lambda} \) depends on the sign of \( \lambda \) and the orientation of the edge of \( \alpha \) corresponding to \( i \). To distinguish the cases, define an edge cross-section of \( \alpha \) as a cross-section that passes through the relative interior of the edge of \( \alpha \) corresponding to \( i \), and define a residual cross-section of \( \alpha \) as a cross section \( \alpha \cap (\Pi_{I,\lambda}) \) such that \( \alpha \cap (\Pi_{I,-\lambda}) \) is an edge cross-section. Following immediately from the definitions, we notice:

**Proposition 3.5.** Suppose \( \alpha \) is a floral vertex represented by the series-parallel diagram \( \Delta \) and \( i \) is an edge of \( \Delta \). The edge cross-section of \( \alpha \) corresponding to \( i \) is congruent to the floral vertex represented by the series parallel diagram \( \Delta \setminus i \).

Next we describe the residual cross-section of \( \alpha \) for the edge \( i \). Given an SPD \( \Delta \), define \( \Delta_i \) as follows:
If $i$ is conjunctive, then $\Delta_i$ is the maximal subdiagram of $\Delta$ connected in series with $i$.

If $i$ is disjunctive, then $\Delta_i$ is the maximal subdiagram of $\Delta$ connected in parallel with $i$.

Notice if $i$ is disjunctive, then $\Delta_i$ is the parallel connection of all of the subdiagrams that share terminals with $i$. If $i$ is conjunctive, then one may employ the same idea using duality. Thus, if $i$ is conjunctive in $\Delta$, then $i$ is disjunctive in the dual $\overline{\Delta}$, and one may define the dual of $\Delta_i$ as the maximal subdiagram of $\overline{\Delta}$ connected in parallel with $i$, then dualize to obtain $\Delta_i$. We define the deletion $\Delta \setminus \Delta_i$ analogous to our definition of $\Delta \setminus i$ above. Now we may state:

**Proposition 3.6.** Suppose $\alpha$ is a floral vertex determined by the series-parallel diagram $\Delta$ and suppose $i$ is an edge of $\Delta$. Then the residual cross-section of $\alpha$ corresponding to $i$ is congruent to the floral arrangement determined by the series parallel diagram $\Delta \setminus \Delta_i$.

**Proof.** By our result above on the orientation of the edge of $\alpha$ corresponding to $i$, one may obtain the diagram of a residual cross-section by conjoining the diagram $\Delta$ with either $i$ or $i$ depending respectively on whether $i$ is conjunctive or disjunctive. If $i$ is conjunctive, then conjoining $i$ with $\Delta$ kills all subdiagrams that are connected in series with $i$. This is a consequence of the set-theoretic fact that $(\alpha \cap \beta) \cap \overline{\alpha}$ is an empty set for any $\alpha, \beta$. If $i$ is disjunctive, then conjoining $i$ with $\Delta$ kills all subdiagrams that are connected in parallel with $i$. This is a consequence of the set-theoretic fact that $(\alpha \cup \beta) \cap \alpha = \alpha \cap \beta$ for any $\alpha, \beta$. The result then follows by induction.

Figures 3.5 and 3.6 illustrate examples of cross-sections of the four types. In both figures, one is given a floral vertex $\alpha \subset \mathbb{R}^8$. Figure 3.5 displays the resulting cross-sections for the conjunctive edge numbered 6, and Figure 3.6 displays the resulting cross-sections for the disjunctive edge numbered 1. In both figures, the edge cross-sections appear on the left while the residual cross-sections appear on the right. The reader is urged to compute edge cross-sections and residual cross-sections for the other edges.

Finally we describe the cross-sections of the form $\alpha \cap \Pi_{i \neq 0}$. Since we defined a local arrangement as being a union of closed orthants, each such cross-section is congruent to exactly one of either an edge cross-section or a residual cross-section of $\alpha$. It can’t be both because we assumed that $\alpha$ is a floral vertex. Thus, in either case, we can describe every cross-section by a floral arrangement.

### 3.5 Local valuations

In this section we introduce and develop functions $\mu$ and $\tau$ and relate these to the functions $\mu_d$ and $\tau_d$ introduced above. These functions will facilitate the study of global properties of generic orthotopes below.
3.5.1 Volume

First we define $\mu$ on SPDs as follows. Suppose $\Delta$ is a SPD. We define $\mu(\Delta)$ inductively as follows.

1. If $\Delta$ is the SPD with exactly one edge, then $\mu(\Delta) = 1$.
2. $\mu(\Delta_1 \land \Delta_2) = \mu(\Delta_1)\mu(\Delta_2)$ for all $\Delta_1, \Delta_2$.
3. $\mu(\overline{\Delta}) + \mu(\Delta) = 2^d$, where $\Delta$ has $d$ edges.

This function $\mu$ is evidently related to the functions $\mu_d$:

**Proposition 3.7.** Suppose $\alpha \subset \mathbb{R}^d$ is a floral arrangement that is given by a SPD $\Delta$ having $k$ edges. Then $\mu_d(\alpha) = 2^{d-k}\mu(\Delta)$.

If $(\Delta, s)$ is a signed SPD on $k$ edges and $\alpha \subset \mathbb{R}^d$ is a floral arrangement determined by $(\Delta, s)$, then we abuse notation slightly by defining $\mu(\alpha) := \mu(\Delta)$.

3.5.2 Signed volume

Suppose $(\Delta, s)$ is a signed SPD. We define $\tau(\Delta, s)$ inductively as follows.

1. If $\Delta$ is the diagram with a single edge, then $\tau(\Delta, 1) = 1$. 

Figure 3.5: Cross-sections for a conjunctive edge.

Figure 3.6: Cross-sections for a disjunctive edge.
Figure 3.7: Floral vertices in four dimensions

2. \( \tau(\Delta_1 \wedge \Delta_2, (s_1, s_2)) = \tau(\Delta_1, s_1)\tau(\Delta_2, s_2) \) for all \((\Delta_1, s_1), (\Delta_2, s_2)\).

3. \( \tau(\Delta, -s) + \tau(\Delta, s) = 0 \) for all \((\Delta, s)\).

As above, we can relate \( \tau \) and \( \tau_d \):

**Proposition 3.8.** Suppose \( \alpha \subset \mathbb{R}^d \) is a floral arrangement determined by a signed SPD \((\Delta, s)\) having \( k \) edges. on \( k \) half-spaces. Then

\[
\tau_d(\alpha) = \begin{cases} 
\tau(\Delta, s) & \text{if } k = d, \\
0 & \text{if } k < d. 
\end{cases}
\]

If \((\Delta, s)\) is a signed SPD on \( k \) edges and \( \alpha \subset \mathbb{R}^d \) is the floral arrangement determined \((\Delta, s)\), then we again abuse notation slightly by defining \( \tau(\alpha) := \tau(\Delta, s) \).

**Proposition 3.9.** If \( \alpha \) is a floral vertex determined by the signed series-parallel diagram \((\Delta, s)\), then \( \tau(\alpha) = (-1)^{d}\sigma(\Delta) \).

If \((\Delta, s)\) is a signed SPD with \( d \) edges and \( \alpha \) is the floral vertex determined by \((\Delta, s)\), then we once again abuse notation slightly by defining \( \sigma(\alpha) := \sigma(\Delta) \).

**Proposition 3.10.** The following hold: (i) If \( \alpha \) is the 1-dimensional half-space (i.e. a cardinal ray), then \( \sigma(\alpha) = 1 \). (ii) \( \sigma(\alpha \wedge \beta) = \sigma(\alpha)\sigma(\beta) \) for all floral vertices \( \alpha, \beta \). (iii) \( \sigma(\alpha) + (-1)^d\sigma(\alpha) = 0 \) for every floral vertex \( \alpha \).

Figure 3.7 displays all congruence types of floral vertices in 4 dimensions, together with the values of \( \sigma \) and \( \mu \) for each.

4 **Generic Orthotopes: Global Theory**

In this section we define:
Definition 4.1. A generic orthotope of dimension $d$ is an orthogonal polytope for which every singular point is a floral vertex on $d$ half-spaces.

From our analysis of the facets of a floral vertex, we see:

Proposition 4.2. If $P$ is a generic orthotope, then every face of $P$ of dimension $k$ is a generic orthotope of dimension $k$.

From our analysis of cross-sections of a floral arrangement, we see:

Proposition 4.3. Suppose $P \subset \mathbb{R}^d$ is a generic orthotope and $\Pi \subset \mathbb{R}^d$ is a generalized hyperplane. Then $P \cap \Pi$ is a generic orthotope.

We can also say:

Proposition 4.4. The 1-dimensional skeleton of a generic orthotope is a bipartite graph of degree $d$.

Proof. Let $P$ be a generic orthotope and suppose $\alpha_1$ and $\alpha_2$ are the floral arrangements of adjacent vertices of $P$. We will show that $\tau(\alpha_1) + \tau(\alpha_2) = 0$. Assume that the floral vertices $\alpha_1$, $\alpha_2$ are determined by the signed SPDs $(\Delta_1, s_1)$, $(\Delta_2, s_2)$ respectively. Assume that the edge joining the vertices is parallel to $e_i$. Let $(\Delta, s)$ be the SPD that represents the edge cross-section. Then, from our analysis of edge cross-sections, we have $\Delta = \Delta_1 \setminus i = \Delta_2 \setminus i$. Let $s_{1,i}, s_{2,i} \in \{\pm 1\}$ be the $i$th component of $s_1, s_2$, respectively. The key observation is that the cardinal direction of the edge starting at one of the vertices is the negative of the cardinal direction of the edge starting at the other vertex. Recall from our discussion of edge orientations that the cardinal ray generated by $\sigma(\Delta_1)\sigma(\Delta_1 \setminus i)s_{1,i}e_i$ is an edge of $\alpha_1$, while the cardinal ray generated generated by $\sigma(\Delta_2)\sigma(\Delta_2 \setminus i)s_{2,i}e_i$ is an edge of $\alpha_2$. Thus, whether $s_{1,i}$ and $s_{2,i}$ have equal or opposite sign, we have

$$\tau(\Delta_1, s_1) = (-1)^{s_1}\sigma(\Delta_1) = -(-1)^{s_2}\sigma(\Delta_2) = -\tau(\Delta_2, s_2).$$

\[\square\]

4.1 Approximation

For a pair $P, Q \subset \mathbb{R}^d$ of compact subsets, define the Hausdorff distance function by

$$\text{Hdist}(P, Q) = \max \left\{ \sup_{p \in P} d(p, Q), \sup_{q \in Q} d(q, P) \right\},$$

where

$$d(x, P) = \inf_{p \in P} \|x - p\|_\infty$$

denotes the distance from a point $x$ to $P$ induced by the $L^\infty$ norm.

Theorem 4.5. Given any compact subset $E \subset \mathbb{R}^d$ and any $\epsilon > 0$, there is a generic orthotope $P$ such that $\text{Hdist}(E, P) < \epsilon$. 

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Lemma 4.6. Suppose $P, Q \subset \mathbb{R}^d$ are generic orthotopes that have no supporting hyperplanes in common. Then $P \cap Q$ and $P \cup Q$ are generic orthotopes.

Proof. We must show that every vertex of $P \cap Q$ and every vertex of $P \cup Q$ is floral. Suppose $v$ is a vertex of $P \cap Q$. Then there are floral arrangements $\alpha$ and $\beta$ such that the tangent cone at $v$ in $P$ (respectively $Q$) is $\alpha$ (respectively $\beta$). However, $P$ and $Q$ have no supporting hyperplanes in common, so the floral arrangement at $v \in P \cap Q$ is $\alpha \wedge \beta$. However, since $P$ and $Q$ have no common supporting hyperplanes, $\alpha \wedge \beta$ is represented by a read-once Boolean function. The same argument holds for the case when $v$ is a vertex of the union $P \cup Q$. \[ \Box \]

Now we argue a proof of Theorem 4.5. Suppose a compact set $E \subset \mathbb{R}^d$ and $\epsilon > 0$ are given. First choose a rational orthogonal polytope $Q_0$ such that $Hdist(Q_0, E) < \frac{\epsilon}{2}$. That $Q_0$ exists is guaranteed by the compactness of $E$. Let $n$ be the least positive integer such that $Q_1 = nQ_0$ is an integral orthotope. Let $S \subset \mathbb{Z}^d$ be the finite set such that $Q_1$ is a union of translates $Q_1 = \bigcup_{v \in S} (v + f_v)$ where $f_v$ is a non-empty face of the standard unit cube $I^d$ for each $v \in S$. For each $v \in S$, choose an orthogonally aligned $d$-dimensional box $B_v$ such that (i) $v + f_v$ lies in the interior of $B_v$, (ii) $Hdist(B_v, v + f_v) < \frac{\epsilon}{2}$, and (iii) no two of the boxes $B_v$ share a supporting hyperplane. That these boxes $B_v$ exist is justified by the fact that $S$ is finite. Finally, let $P = \frac{1}{n} \bigcup_{v \in S} B_v$. Evidently we have $Hdist(P, Q_0) < \frac{\epsilon}{n}$. From the triangle inequality,

$$Hdist(P, Q) \leq Hdist(P, Q_0) + Hdist(Q_0, Q) < \frac{\epsilon}{2n} + \frac{\epsilon}{2} = \frac{1 + n}{2n} \epsilon \leq \epsilon.$$ 

Moreover, since the supporting hyperplanes of the boxes $B_v$ for $v \in S$ are distinct, $P$ is a generic orthotope.

The lemma above also helps to see why we regard generic orthotopes as “generic”: Suppose $P$ is a generic orthotope and $\Pi$ is a supporting hyperplane. Then we may “shift” $\Pi$ in a direction perpendicular to $\Pi$ while leaving all other supporting hyperplanes fixed. For example, one may accomplish such a transformation using a piece-wise linear function. One may use such a shift, provided it does not pass across another supporting hyperplane parallel to $\Pi$, to construct another generic orthotope $P'$ which has the same face poset as $P$.

The space of generic orthotopes is not open in the space of all orthogonal polytopes, as the following example demonstrates. Let $P = [0, 2] \times [0, 2] \times [0, 1]$ and for each $\epsilon > 0$ define $Q_\epsilon = P \cup ([0, 1] \times [0, 1] \times [1, 1 + \epsilon]) \cup ([1, 2] \times [1, 2] \times [1, 1 + \epsilon]).$
Then $P$ is a generic orthotope, but $Q_{\epsilon}$ is an orthogonal polytope which is never a generic orthotope. Moreover, we have $\lim_{\epsilon \to 0} \text{Hdist}(P, Q_{\epsilon}) = 0$.

### 4.2 Volume and Euler characteristic

This section shows several combinatorial formulas concerning generic orthotopes.

**Theorem 4.7.** Suppose $P$ is an integral generic orthotope. Then

$$\text{volume}(P) = 2^{-d} \sum_{v \in P \cap \mathbb{Z}^d} \mu_d(\alpha(v)),$$

where $\alpha(v)$ denotes the floral arrangement at $v$.

The formula is easy to understand: For any point $v \in P \cap \mathbb{Z}^d$, the fraction $\mu(v)/2^d$ is the volume of $(v + [-1/2, 1/2]^d) \cap P$. To compute the total volume, simply add all of these values.

If $P$ is an integral generic orthotope, let $n_\alpha$ denote the number of points in $P \cap \mathbb{Z}^d$ of floral type $\alpha$. Then we may write the formula above as

$$\text{volume}(P) = 2^{-d} \sum_{\alpha} \mu_d(\alpha)n_\alpha,$$

where we sum over all congruence types of floral arrangements.

We also have a determinantal expression for the volume of a generic orthotope:

**Theorem 4.8.** Suppose $P$ is a rational generic orthotope. For each vertex of $v$, denote the coordinates where $v = (v_1, v_2, ..., v_d)$. Then

$$\text{volume}(P) = \sum_v \tau(v) \prod_{i=1}^d v_i,$$

summing over all vertices $v \in P$ and $\tau(v)$ denotes the signed volume of the floral arrangement at $v$.

**Proof.** First we show that the formula holds for an integral generic orthotope $P$. Assuming this, we may subdivide $P$ as the union

$$P = \bigcup_{v \in S} (v + I^d),$$

where $S \subset \mathbb{Z}^d$ is finite. The formula holds for each unit cube $v + I^d$, so we have

$$1 = \text{volume}(v + I^d) = \prod_{i=1}^d ((v_i + 1) - v_i).$$
If we sum these over all of $S$, then we obtain an expression

$$\text{volume}(P) = \sum_{v \in P \cap \mathbb{Z}^d} c_v \prod_{i=1}^d v_i,$$

where $c_v$ denotes a coefficient that depends on $v$. The key observation is that the coefficient $c_v$ vanishes exactly when the floral arrangement at $v$ occupies an even number of orthants. Moreover, the floral arrangements of $P$ that are occupied by an odd number of orthants coincide with the (floral) vertices of $P$. One then verifies that the coefficient $c_v$ is indeed equal to $\tau(v)$ for every floral vertex. Since the formula (4.8) holds for every integral generic orthotope, it also holds for every rational generic orthotope.

We have a similar formula for the Euler characteristic of a generic orthotope. Suppose $P$ is a generic orthotope. Define

$$\sigma(P) = \sum_{v} \sigma(\alpha(v)) = \sum_{\alpha} \sigma(\alpha)n_{\alpha},$$

where the first sum is over all vertices $v \in P$ the second sum is over floral types $\alpha$.

**Theorem 4.9.** Suppose $P$ is a generic orthotope with Euler characteristic $\chi(P)$. Then

$$\chi(P) = 2^{-d} \sigma(P).$$

To establish this, we first prove that $\sigma$ is a valuation when restricted to generic orthotopes:

**Proposition 4.10.** If all four of $\{P, Q, P \cap Q, P \cup Q\}$ are generic orthotopes, then $\sigma(P) + \sigma(Q) = \sigma(P \cap Q) + \sigma(P \cup Q)$.

We facilitate this by use of a lemma.

**Lemma 4.11.** Suppose $(\alpha, \beta)$ is a pair of floral arrangements such that both of $\alpha \cap \beta$ and $\alpha \cup \beta$ are floral arrangements. Then $\alpha$ and $\beta$ have no opposite half-spaces.

**Proof.** In seeking a contradiction, assume that $\alpha$ and $\beta$ have opposite half-spaces. Let $f$ and $g$ be facets that have opposite half-planes, and let $\Pi$ be the hyperplane containing $f$ and $g$. Let $f^o$ and $g^o$ denote the relative interiors of $f$ and $g$ respectively with respect to $\Pi$. Suppose first that $f^o \cap g^o$ is empty. Then the disjunction $\alpha \lor \beta$ has two genericity regions with opposite outward normal vectors, contradicting the assumption that $\alpha \cap \beta$ and $\alpha \cup \beta$ are floral arrangements. On the other hand, suppose $f^o \cap g^o$ is non-empty. Then the join $\alpha \land \beta$ is not a pure $d$-dimensional orthotope, so again this contradicts the assumption that $(\alpha, \beta)$ is a floral pair. \hspace{1cm} \Box
Now we prove Proposition 4.10. This follows by relating $\sigma$ to the signed volume function $\tau_d$. Thus, suppose $\alpha, \beta, \alpha \cap \beta$, and $\alpha \cup \beta$ are floral arrangements. From the lemma, we may assume without loss of generality that $\alpha$ and $\beta$ are both represented by signed SPDs, where every edge is marked positive. Being a sum of signs of orthants, $\tau_d$ trivially satisfies the inclusion-exclusion rule. In this case, since all of the half-spaces are positive, this implies that $\sigma$ also satisfies the inclusion-exclusion rule. One may verify that $\sigma(B) = 2^d$ for every pure $d$-dimensional axis-aligned box $B \subset \mathbb{R}^d$. Thus, $\sigma$ is a valuation when restricted to generic orthotopes. Since $\sigma$ is constant on axis-aligned boxes, it yields a multiple of the Euler characteristic.

**Example.** Suppose $d = 4$. Then the formula in Theorem 4.9 says
\[
\begin{align*}
&n_{\ldots} - n_{\ldots} - n_{\ldots} + n_{\ldots} + n_{\ldots} - n_{\ldots} \\
&- n_{\ldots} - n_{\ldots} + n_{\ldots} + n_{\ldots} - n_{\ldots} \\
&= 2^4 \cdot \chi(P).
\end{align*}
\]
We invite the reader to attempt to assemble 4-dimensional generic orthotopes for experimentation.

5 Conclusion and open problems

Having established a theory of generic orthotopes, we pose several questions.

5.1 Generic polyconvex polytopes

Define a *polyconvex polytope* as a subset of $\mathbb{R}^d$ that can be formed as the union of finitely many convex polytopes. Define a *generic polyconvex polytope* as a polyconvex polytope such that the tangent cone at every vertex is described by applying a read-once Boolean function to a set of $d$ half-spaces with distinct supporting hyperplanes. Thus, in a generic polyconvex polytope, every vertex can be transformed via a linear transformation to a floral vertex. Clearly every face of a generic polyconvex polytope is a generic polyconvex polytope. Is a similar statement valid for cross-sections?

5.2 Discrete Morse theory

Suppose $P$ is a polyconvex polytope. As Bieri and Nef describe in [6] and [7], one may compute the volume and Euler characteristic of $P$ using “sweep plane” algorithms. Their algorithms compute the volume and Euler characteristic by adding local statistics as the level sets of a linear functional (essentially a discrete analogue of a Morse function) pass across the vertices of $P$. Can we refine these algorithms to handle generic orthotopes or generic polyconvex polytopes specifically? What effect does this have on the complexity of the problem of computing the volume and Euler characteristic?
5.3 Genericization

Let $P$ be an orthogonal polytope. The problem here is to study methods of approximating $P$ by a generic orthotope. How can we accomplish this in a general way? An obvious place to start is to analyze local orthotopal arrangements which are not floral arrangements. For example, it is not hard to imagine “perturbing” one or more of the supporting planes in the degenerate vertices appearing in Figure 5.3 to obtain a pair of nearby floral vertices. More generally, this author imagines “blow ups” along singular (non-floral) faces, obtained by systematically uniting $P$ with generic orthotopes which “cover” the singular faces. What are the most efficient algorithms for generizicing a given orthogonal polytope?

5.4 Simplicial orthotopal arrangements

Is every simplicial orthotopal arrangement floral? We have seen that the face lattice of every floral arrangement coincides with that of a simplex. We ask whether or not the converse is also valid. Thus, given an orthotopal arrangement $\alpha \subset \mathbb{R}^d$ such that the face poset of $\alpha$ coincides with a simplex, we wonder whether there is necessarily a read-once Boolean function which defines $\alpha$. An affirmative answer would significantly strengthen this author’s thesis that generic orthotopes represent an elementary generalization of the cube to general orthogonal polytopes.

5.5 Flag orthotopes

Suppose $P$ is a $d$-dimensional convex polytope with face lattice $\mathcal{L}(P)$ and $\mathcal{L}(P) \xrightarrow{\phi} \mathbb{R}$. For each complete flag $(\emptyset \subset f_0 \subset f_1 \subset f_2 \subset ... \subset f_{d-1} \subset P)$ in $\mathcal{L}(P)$ (where $\dim(f_i) = i$), this yields a point

$$(\phi(f_0), \phi(f_1), \phi(f_2), ..., \phi(f_{d-1})) \in \mathbb{R}^d,$$

and the assembly of these points may or may not coincide with the vertices of a generic orthotope. Which pairs $(P, \phi)$ does this yield the vertices of a generic orthotope that respects the face lattice $\mathcal{L}(P)$? Can we develop efficient algorithms for realizing $\mathcal{L}(P)$ by a generic orthotope?

Figure 5.1 shows an example of this idea when $d = 3$. The left part of this figure represents a drawing of a Schlegel diagram of a 3-dimensional convex polytope, say $P$. Notice that every vertex, edge, and facet is marked by a value $\phi(f_i)$. The right part of this figure illustrates an axonometric projection of the “flag orthotope” given by the data $(P, \phi)$. Notice that the polygonal regions on the right are marked with distances to the three coordinate planes and this yields a 3-dimensional generic orthotope which “displays” the entire face lattice of $P$. 
5.6 Shadows of 4D generic orthotopes

As this author noticed in [24], one may construct 3-dimensional flag orthotopes from generic rectangulations. This idea is easy to conceive due to the extremely limited number of 3-dimensional floral vertices. What about the next higher dimension? Thus, whereas a generic rectangulation represents a 2-dimensional projection of a flag orthotope of a non-separable planar map, what are the analogous configurations when considering 3-dimensional projections of a 4-dimensional generic orthotope? Due to the number of different types of 4-dimensional floral vertices, this project appears to be quite large.

5.7 Coxeter complexes

The original motivation of this work came from a desire to realize Coxeter complexes by orthogonal polytopes, and we may now state this problem precisely. Suppose \((G, S)\) is a Coxeter system of finite type, where \(S = \{\sigma_1, \sigma_2, ..., \sigma_d\}\) is the set of generating involutions of \(G\) and \(d\) is the rank of \((G, S)\). Define a generic orthotopal realization of \((G, S)\) as a generic orthotope \(P = P_{(G, S)} \subset \mathbb{R}^d\) such that there is a bijection \(\phi : G \rightarrow \mathcal{L}_0(P)\) (the vertices of \(P\)) where two vertices \(v_1 = \phi(g_1), v_2 = \phi(g_2)\) of \(P\) are connected by an edge parallel to the \(i\)th coordinate axis whenever \(g_1 = \sigma_i g_2\).

Figure 5.2 displays an example of this idea. The underlying Coxeter group \(G\) is the symmetric group on 4 letters and the generators are \(S = \{b, g, r\}\), subject to the relations

\[\{b^2, g^2, r^2, (bg)^3, (br)^2, (gr)^3\}\]
Figure 5.2: The Coxeter complex of type $A_3$.

The figure depicts an axonometric projection of a realization of $(G, S)$ as a generic orthotope. As in our discussion of flag orthotopes above, the polygonal regions are marked by distances to coordinate planes in $\mathbb{R}^3$. Notice in particular that there are 24 vertices, corresponding to the elements of $G$. One also notices that all of the edges sharing a common color are mutually parallel and that the 1-dimensional skeleton comprises a Cayley graph of $G$ with the generators $\{b, g, r\}$. Although this example was first conceived in the context of graph drawing (as in [15] and [16] for example), we notice that this representation is faithful to the entire Coxeter complex of the corresponding Coxeter system. Thus, for every $k \in \{0, 1, 2, 3\}$, the $k$-dimensional faces of this polytope correspond to cosets of the Coxeter subgroups generated by $k$ elements of $\{b, g, r\}$.

This author is interested in realizing every finite Coxeter complex by a generic orthotope. Ideally, one would like to see a uniform system for realizing each of the three infinite sequences $A_d$, $BC_d$, $D_d$ of spherical Coxeter systems. Aside from formulating it precisely, this author has made little progress on this problem. For example, it is possible to realize every finite rank-3 Coxeter system by a generic orthotope. (This is a good exercise for the interested reader.) However, the problem is daunting as soon as $d \geq 4$. For various reasons, this author suspects that no realization of the $D_4$ Coxeter complex as a generic orthotope exists. Since $D_4$ occurs as a subdiagram of $D_d$ for all $d \geq 5$ and of the exceptional series $E_d$, a negative result would imply that none of these particular Coxeter systems has a realization as a generic orthotope. How should we handle infinite Coxeter systems?
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