DESCRIPTIVE CELLULAR HOMOLOGY

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Dedicated to J.H.C. Whitehead and Som Naimpally

ABSTRACT. This article introduces descriptive cellular homology on cell complexes, which is an extension of J.H.C. Whitehead’s CW topology. A main result is that a descriptive cellular complex is a topology on fibres in a fibre bundle. An application of two forms of cellular homology is given in terms of the persistence of shapes in CW spaces.

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1. INTRODUCTION

Homology separates cycles (connected paths) from boundaries of holes in shapes in topological spaces. Descriptive cellular homology provides a means of characterizing and comparing cycles and boundaries in terms of descriptions of cell complexes. A cell complex $K$ on a space $X$ is a finite collection of subsets of $X$ called cells such as 0-cells (vertices) and 1-cells (open arcs). A descriptive cellular homology is an extension of Whitehead Closure-finite Weak (CW) topology [5], which is a form of descriptive topology introduced in [3]. A straightforward application of these two forms of homology is the study persistence of shapes in CW spaces.

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2. Preliminaries

A Hausdorff space is a nonempty set $K$ in which every pair of distinct elements in $K$ is contained in disjoint open set. A decomposition of $K$ is a partition of $K$ into subsets called cells. Let $clA, bdyA$ denote the closure and boundary of a cell $A$, respectively. The interior of a cell $A$ (denoted by $intA$) is defined by $intA = clA - bdyA$. A complex with $n$ cells in $K$ is denoted by $σ^n$. The closure of $σ^n$ (denoted by $clσ^n$) is the image of an $n$-simplex $σ^n$ in a continuous homomorphic map $f : σ^n → clσ^n$. A Hausdorff space $K$ with a cell decomposition is called a CW complex [5] (briefly, complex) in which the following conditions hold.

1. For each cell $σ ∈ K$, $clA$ intersects a finite number of other cells (Closure finiteness).
2. A cell $A ∈ K$ is closed, provided $A ∩ clσ^n ≠ ∅$ is also closed (Weak topology).

The $n$-skeleton $K^n$ is the union of all $σ^j ∈ K$, such that $j ≤ n$. A fibre bundle $(E, B, π, F)$ is a structure, where $E$ is the total space, $B$ is the base space, $π : E → B$ is a continuous surjection on $E$ onto $B$ and $F ⊂ E$ is called the fibre. A region based probe function $ϕ : 2^K → R^n$ attaches description to sets. A descriptive cell complex $K_ϕ$ is a fibre bundle, $(K_ϕ, K, π, φ(U))$ on a complex $σ^n ∈ K$, and $U ⊂ K$.

Example 1. Sample Descriptive Cell Complex.

In Fig. 1, $K$ is a CW complex. A region-based probe function $ϕ : 2^K → R$ is used to attach description to each of the cells in $K$ (denoted by $K_ϕ$). Probes define a signature for a cell complex [4]. In $K_ϕ$, every cell has a description defined by an appropriate function, as opposed to $K$, where the description is assumed to be constant. The map $π : K_ϕ → K$, is a continuous surjective projection map. Thus, the descriptive CW complex is a fibre bundle $(K_ϕ, K, π, φ(U))$, where $U ⊂ K$. The region-based probe is the section of the bundle. Moreover, we illustrate a family of maps $\{να, s.t. α ∈ img ϕ(K)\}$, each of which maps from $K_ϕ$ to a CW complex $K_α$. A cycle of $σ^1$ is illustrated in blue color. The idea is that $p$-cycle is a chain or a formal sum of $σ^p$ which starts and ends at the same point. Each region of constant description in $K_ϕ$ is a descriptive hole and the family of maps $\{να\}_α$ maps $K_ϕ$ in such a way that a particular descriptive hole is projected to a hole (a void) in an abstract CW complex $K_α$.

A descriptive cellular complex is represented as $F → E → B$. The fibre bundle must satisfy the local trivialization condition, i.e., in a small neighborhood $U ⊂ B$, such that $π^{-1}(U)$ is homeomorphic to $U × F$. This means that the diagram in Fig. 2, commutes, i.e., each directed path with same endpoints lead to same result. Also from this diagram, the function $ϕ$ is the section, i.e. a continuous map from the base space to the total space, of the descriptive cellular complex $(X_ϕ, X, πi, φ(U))$.

Algebraic structures are defined over CW approximations of topological spaces. A $p$-chain is a formal sum of $p$-cells of $σ^p$ in $X$, $c_p = \sum a_iσ^p_i$. For computational simplicity, $a_i ∈ \{0, 1\}$. Under componentwise addition, the set of $p$-chains would form an abelian group $(C_p^+, +)$. These are to be distinguished from the simplicial and singular chain complexes $(C_p^+, +)$. Boundary map is defined as, $∂^p : σ^p_α \mapsto \sum_β d_{αβ}σ^p_{αβ}$. Here $σ^p_α ∈ \{σ^p_β\}_α$ is the set of all $σ^p ∈ K$, and $σ^p_β ∈ \{σ^p_β\}_β$, is the set of all $σ^p ∈ K$. $d_{αβ}$
is the degree of the map $f : S^{p-1}_\alpha \to K^{p-1} \to S^{p-1}_\beta$.

Observe that $S^{p-1}_\alpha \to K^{n-1}$ is the attachment map of $\sigma^n_\alpha$ and $K^{p-1} \to S^{p-1}_\beta$ is the quotient map that collapses $\{K^{p-1} - \sigma^n_\alpha\}$ to a point. The map $f : S^{p-1}_\alpha \to S^{p-1}_\beta$ induces a homomorphism between infinite cyclic groups as $f_* : H_n(S^{p-1}_\alpha) \to H_n(S^{p-1}_\beta)$ and hence should be of the form $f_*(\alpha) = d\alpha$. The constant $d$ is called the degree of the map $f$ [2, § 2.2]. The boundary map is a homomorphism between chain groups, $\partial^p_\alpha : C_p \to C_{p-1}$. The group of $p$-cycles is defined as the kernel of boundary homomorphism, $Z^p_\alpha = \ker \partial^p_\alpha$. The group of $p$-boundaries are the defined as, $B^p_\alpha = \text{img} \partial^p_\alpha$. The $p^{th}$ homology group is the quotient group, $H^p_\alpha = Z^p_\alpha / B^p_\alpha$. The rank of $H^p_\alpha$ is the $p^{th}$ Betti number, $\beta^p = \text{rank} H^p_\alpha$.

Next, we establish the connection between singular homology and cellular homology. The chain complexes in cellular homology, $(C_p^n, +)$ are the relative homology groups $H_n(K^n, K^{n-1}) \equiv H_n(K^n / K^{n-1})$. The underlying chain groups are $(C_p^n(K^n)) / (C_p^n(K^{n-1}), +)$. The elements in $H_n(K^n, K^{n-1})$ are represented by relative chains, $c_r \in C_n(K^n)$ such that $\partial c_r \in C_n(K^{n-1})$. The relative boundaries are defined as $c_r = \partial c_r + \gamma$ such that $\partial c_r \in C_n(K^n)$, $c_r \in C_{n+1}(K^{n+1})$ and $\gamma \in C_{n-1}K^{n-1}$. The relative boundaries are trivial elements in $H_n(K^n, K^{n-1})$. Thus, $H_n(K^n, K^{n-1})$ has $\sigma^n \in K$ as its basis. For the details, see A. Hatcher [2].

3. Main Results

This section introduces the main results for a descriptive cellular homology. Let $\nu_\alpha : K_\Phi \to \hat{K}_\alpha$ be a mapping on a descriptive cellular complex($K_\Phi$) into abstract cellular complex($\hat{K}_\alpha$).

Here, $\hat{K}_\alpha = \{ K \setminus \sigma^2 \text{ s.t. } |\phi(\sigma^2) - \alpha| \leq \delta \}$, and $\delta$ is an arbitrary constant.

From this definition, $\hat{K}_\alpha \subset K_\Phi$, where $\alpha \in \text{img} \phi$, and $X_\phi = \bigcup \hat{K}_\alpha$. We define homology groups, $H^n_\alpha$, $H_{n-1}^\alpha$ on each of these complexes $\hat{K}_\alpha$. These groups require cellular chain groups, $H_n(\hat{K}_\alpha, \hat{K}_{\alpha-1})$ and boundary homomorphisms, $\partial^\alpha$.

**Lemma 1.** Let $(K, K_\Phi, \pi)$ be a descriptive cellular complex, $\{\nu_\alpha : K_\Phi \to \hat{K}_\alpha \text{ s.t. } \alpha \in \text{img} \phi\}$ be a class of maps. Then, $H_n(\hat{K}_\alpha, \hat{K}_{\alpha-1})$ is a subgroup of $H_n(K^n, K^{n-1})$.

**Proof.** As $\hat{K}_\alpha \subset K$, all $\sigma^p \in \hat{K}_\alpha$ are included in $K$. Hence, $H_n(\hat{K}_\alpha, \hat{K}_{\alpha-1})$, with basis $\sigma^p \in \hat{K}_\alpha$, is a subset of $H_n(K^n, K^{n-1})$, a basis $\sigma^p \in K$. Equipped with a chain addition, we can identify the identity element. The identity element in $H_n(\hat{K}_\alpha, \hat{K}_{\alpha-1})$ can be constructed by setting the weight of each $\sigma^p \in \hat{K}_\alpha$ equal to 0. This identity can be seen as the result of restriction of the identity of $H_n(K^n, K^{n-1})$ to only those $\sigma \in K$ which are also $\hat{\sigma} \in \hat{K}_\alpha$. The inverse of each element $c = \sum_i a_i \hat{\sigma}^n_i \in H_n(\hat{K}_\alpha, \hat{K}_{\alpha-1})$ is $\hat{c} = \sum_i -a_i \hat{\sigma}^n_i$. The inverse of each element is by definition in $H_n(\hat{K}_\alpha, \hat{K}_{\alpha-1})$ as it is a formal sum of $\hat{\sigma}^n$. Assume two chains $c = \sum_j a_j \hat{\sigma}^n$ and $\hat{c} = \sum_j b_j \hat{\sigma}^n$, then $c + \hat{c} = \sum_j (a_j + b_j) \hat{\sigma}^n$. By definition $c + \hat{c} \in H_n(\hat{K}_\alpha, \hat{K}_{\alpha-1})$ as it is a formal sum of $\hat{\sigma}^n \in \hat{K}_\alpha$. Thus, $H_n(\hat{K}_\alpha, \hat{K}_{\alpha-1})$ is closed under chain addition. It can be seen from the definition of the cellular boundary map that coefficients of $\hat{\sigma}^n$ are integers. Thus associativity of chain addition in $H_n(\hat{K}_\alpha, \hat{K}_{\alpha-1})$ follows from the associativity of integers.
Lemma 2. Let us consider \( H_n(\hat{K}_n, \hat{K}_{n-1}) \) to be chain groups with boundary homomorphisms associated as follows:

\[
\cdots \to \ker \partial_n \xrightarrow{\partial_n} \ker \partial_{n+1} \xrightarrow{\partial_n} \ker \partial_{n+1} \xrightarrow{\partial_n} \ker \partial_{n+1} \to \ker \partial_{n+1} \to \ker \partial_{n+1} \to \cdots
\]

Then \( \partial_n \circ \partial_n = 0 \).

Proof. The composition \( \partial_n \circ \partial_n \) holds for \( H_n(K^n, K^{n-1}) \) as a result of Lemma 2.34 [2]. The proof via diagram chasing, is detailed in [2, § 2.2, p.139].

Using this proof as the basis and the Lemma 1 stating that \( H_n(\hat{K}_n, \hat{K}_{n-1}) \) is a subgroup of \( H_n(K^n, K^{n-1}) \), it can be concluded that \( \partial_n \circ \partial_n = 0 \) holds for \( H_n(\hat{K}_n, \hat{K}_{n-1}) \).

Theorem 1. \( \text{img} \partial_n \subset \text{ker} \partial_n. \)

Proof. From Lemma 2, we can conclude that \( \text{img} \partial_n \subset \text{ker} \partial_n. \) Thus, the sequence

\[
\cdots \to \ker \partial_n \xrightarrow{\partial_n} \ker \partial_n \xrightarrow{\partial_n} \ker \partial_n \to \ker \partial_n \to \ker \partial_n \to \cdots
\]

is an exact sequence.

Theorem 2. The condition \( |\phi(\sigma^2) - \alpha| \leq \delta \) is equivalent to \( \phi(\sigma^2) \in \tau_{std}^\phi \).

Proof. For a space \( X \), we define a topology \( (\text{img} \phi(X), \tau_{std}^\phi) \). Any set \( U \in \tau_{std}^\phi \) is defined as: for all \( p \in X \), there exists an arbitrary positive real number \( \delta \) such that \( B^\phi_{\phi(p)} \subseteq U \). Here \( B^\phi_{\phi(p)} \) is a ball of radius \( r \) and centered on a point \( x \). Hence from this definition, we can see that \( |\phi(\sigma^2) - \alpha| \leq \delta \Rightarrow \phi(\sigma^2) \in \tau_{std}^\phi \). Moreover, the statement \( \phi(\sigma^2) \in \tau_{std}^\phi \Rightarrow |\phi(\sigma^2) - \alpha| \leq \delta \), also follows from the definition of \( \tau_{std}^\phi \). From this argument it follows that, \( |\phi(\sigma^2) - \alpha| \leq \delta \) is equivalent to \( \phi(\sigma^2) \in \tau_{std}^\phi \). Descriptive homology gives a local view of a space, since it is associated with subgroups \( H_n(\hat{K}_n, \hat{K}_{n-1}) \) of \( H_n(K^n, K^{n-1}) \). Next, we establish a relationship between \( H_p(K) \) and \( H^p_n(K) \). To do this, we assume a standard topology \( (\text{img} \phi(X), \tau_{std}^\phi) \), and define \( \hat{K}_n = \{ K \setminus \sigma^2 \text{ s.t. } \sigma^2 \in K \text{ and } |\phi(\sigma^2) - \alpha| \leq \delta \} \). The \( p^{th} \) homology group is denoted by \( H^p_v \), where \( V \in \tau_{std}^\phi \).

Theorem 3. Let \( (X, \pi, \phi(U)) \) be a descriptive cellular complex where \( U \subseteq X \). \( (\text{img} \phi(X), \tau_{std}^\phi) \) is the topology defined on the fibres. Then,

\[
H^p_v \equiv H_p(X) \text{ when } V = \bigcup \tau_{std}^\phi
\]
4.1: Changing temperature of regions in space
4.2: Changing area of regions in space

**Figure 4.** Persistence over time in CW complexes.

Proof. $H_p^\phi$ is defined as the cellular homology group associated with $K_\alpha = \{X \setminus \sigma^p \text{ s.t. } \sigma^p \in K \text{ and } |\phi(\sigma^p) - \alpha| \leq \delta\}$. If $V = \bigcup_{\sigma^p \in G} \text{img } \phi(K)$ Then, $K_\alpha = X$ as all the two simplices regardless of their description would be included. Thus, by definition it would equal the classical cellular homology group $H_\phi$ of space $K$.

**Remark 1.** Each fibre bundle has a local trivialization as illustrated in Fig. 2, and the base $K$ in the case of a cellular complex has intersecting subsets. This raises the question of how the fibre $\phi(U)$ transitions between such intersecting sets. This can be done by associating with $(K_\phi, K, \pi, \phi(U))$ a topological group $G$, which acts continuously on the fiber, $\phi(U)$ from the left. That is, for $e \in G$, the identity element, $ex = x$, where $x \in \phi(U)$. The notion of continuity requires the group $G$ to be a topological group. Let us formalize the action of group $G$ on the fibre.

**Theorem 4.** Let $(E, B, \pi, F)$ be the fibre bundle, where $E$ is the total space, $B$ is the base space, $\pi$ is a continuous surjection and $F$ is the fibre. Let $(U_i, \phi_i)$ and $(U_j, \phi_j)$ be two intersecting sets, $U_i \cap U_j \neq \emptyset$, in $B$ with their sections $\phi_i, \phi_j$. Then the following holds:

$$\phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times F \rightarrow (U_i \cap U_j) \times F$$

$$\phi_i \circ \phi_j^{-1}(x, f) \mapsto (x, t_{ij}(x)f) \text{ s.t. } t_{ij} : U_i \cap U_j \rightarrow G.$$

Proof. The fibre bundle satisfies local trivialization condition. Consequently, $\phi_i : U_i \rightarrow U_i \times F$ and $\phi_j : U_j \rightarrow U_j \times F$. Since we assume $U_j \cap U_i \neq \emptyset$, there are two different trivializations for the region $U_i \cap U_j$. The map $\phi_j^{-1} : U_j \times F \rightarrow U_j$. Since a region $U_i \cap U_j$ has two trivializations in the fibre bundle, we can shift between the two. Thus, we can define a composition map, $\phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times F \rightarrow (U_i \cap U_j) \times F$. Such a map can be defined as $\phi_i \circ \phi_j^{-1}(x, f) \mapsto (x, t_{ij}(x)f)$. In this case $t_{ij} \in G$ and $G$ is the structure group or the gauge group defined in [1].

In Theorem 4, $t_{ij}$ is the transition function and $G$ is the structure group or the gauge group. Since $G$ is a group of transition functions $t_{ij} \in G$, $t_{ij}$ must satisfy certain conditions. That is, $t_{ii} = 1$ is the identity element and $t_{ij}t_{ji} = 1$ gives the inverse of each element. Moreover, there is a group operation such that $t_{ik} = t_{ij}t_{jk}$. If there is no other $g \in G$ except the identity, such that $gx = x$ for all $f \in F$, then $G$ is a group of homeomorphisms on $F$.

4. Application: Persistence in CW Spaces

We illustrate Theorem 4 and its implications, using an expansion of Fig. 1 shown in Fig. 4.
The probe function $\phi : 2^K \to \mathbb{R}^2$, maps each region to a feature vector, $[\text{temperature, area}]$. Consider, for example, two subsets $U_i, U_j \in 2^K$ such that $U_i \cap U_j \neq \emptyset$. $U_i, U_j$ are represented by two planar regions in Fig. 4, namely, $U_i$ in gray and $U_j$ in light gray are two regions in the base space $K$, which have been colored for the sake of distinction. The intersection in this case is the shared $\alpha^1$. We assume that the functions $\phi$ and $\pi$ change with respect to the value of a parameter, $\{\theta_i\}_i$.

The parameter $\{\theta_i\}_i$ can be thought of as time or spatial location. In this example, we consider $\{\theta_i\}_i$ to be time. We will consider the changes in each of the components of the feature vector; namely temperature (Fig. 4.1) and area (Fig. 4.2) separately. Using the local trivialization condition for $\theta_i$, the fibre for the region $U_i$, $\phi(U_i) = [\text{green, 0.25}]$ and the fibre for region $U_j$ is $\phi(U_j) = [\text{red, 0.75}]$. The temperature is represented by color, where temperature red > yellow > green. Thus, Theorem 4 states that the transition of the description from the region $U_i$ to $U_j$ across the common intersection is governed by transition functions that form a group $G$. Next, consider $U_i, U_j$ for different values of the parameters, namely, $\theta_i, \theta_k$.

Let us first look at Fig. 4.1, in which the areas remain unchanged while the temperature of $U_i$ decreases as it changes from red to yellow and then to green. In keeping with changing temperature, the transition functions that describe the changes across $U_i \cap U_j$ also change. This leads to a change in the group $G$ with the value of the temperature parameter. Thus, a change in $G$ is an indicator of transitions in the description of the topological space with respect to the value of the parameters. Again, for example, consider Fig. 4.2 in which the area of the $U_i$ reduces with time while the area of $U_j$ remains the same. This also results in a change in the transition function values and hence in the group $G$.

We can combine these observations in the study of persistence with a shape signature introduced in [4, §2.5], to develop shape signatures for topological spaces based on the description. Persistence in CW spaces focuses on the stability of topological signatures with respect to one or more parameters.

**References**

1. Y. Félix and J. Oprea, Rational homotopy of gauge groups, Proc. Amer. Math. Soc. 137 (2009), no. 4, 1519–1527, MR2465678.
2. A. Hatcher, Algebraic topology, Cambridge University Press, Cambridge, UK, 2002, xii+544 pp. ISBN: 0-521-79160-X, MR1867354.
3. S.A. Naimpally and J.F. Peters, Topology with applications. Topological spaces via near and far, World Scientific, Singapore, 2013, xv + 277 pp, Amer. Math. Soc. MR3075111.
4. J.F. Peters, Proximal planar shape signatures. Homology nerves and descriptive proximity, Advances in Math. Sci. J. 6 (2017), no. 2, 71–85, Cf. arXiv: 1711.07338v6, 2017.
5. J.H.C. Whitehead, Combinatorial homotopy. I., Bulletin of the American Mathematical Society 55 (1949), no. 3, 213–245, Part 1, MR0030759.

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