ON THE RESTRICTED INVERTIBILITY PROBLEM WITH AN ADDITIONAL ORTHOGONALITY CONSTRAINT FOR RANDOM MATRICES

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Abstract. The Restricted Invertibility problem is the problem of selecting the largest subset of columns of a given matrix $X$, while keeping the smallest singular value of the extracted submatrix above a certain threshold. In this paper, we address this problem in the simpler case where $X$ is a random matrix but with the additional constraint that the selected columns be almost orthogonal to a given vector $v$. Our main result is a lower bound on the number of columns we can extract from a normalized i.i.d. Gaussian matrix for the worst $v$.

Keywords: Restricted Invertibility, Column selection, Random matrices.

1. Introduction

Let $X \in \mathbb{R}^{n \times p}$. The goal of this short note is to study the following quantity, denoted by $\gamma_{s,\rho_-}(X)$, defined for any $s \leq n$ and $\rho_- \in (0, 1)$ as

$$\gamma_{s,\rho_-}(X) = \sup_{v \in B(0,1)} \inf_{I \subset S_{s,\rho_-}} \|X_I^t v\|_{\infty},$$

where $S_{s,\rho_-}(X)$ is the family of all $S$ of $\{1, \ldots, p\}$ with cardinal $|S| = s$, such that $\sigma_{\min}(X_S) \geq \rho_-$. The meaning of the index $\gamma_{s,\rho_-}$ is the following: for any $v \in \mathbb{R}^n$, we look for the "almost orthogonal" family inside the set of columns of $X$ with cardinal $s$, which is the most orthogonal to $v$.

1.1. The constrained restricted invertibility problem. Once we have an idea of the behavior of $\gamma_{s,\rho_-}(X)$ as a function of $s$, we can derive a lower bound on the number of columns sufficiently orthogonal to a given vector which can be extracted from a given matrix and which form a well conditioned submatrix. This problem is a constrained counterpart to the well known Restricted Invertibility problem which has a long history starting with the seminal work of Bourgain and Tzafriri [1]. In particular, Bourgain and Tzafriri [1] obtained the following result for square matrices:

**Theorem 1.1 ([1]).** Given a $p \times p$ matrix $X$ whose columns have unit $\ell_2$-norm, there exists $I \subset \{1, \ldots, p\}$ with $|I| \geq d \frac{p}{\|X\|^2}$ such that $C \leq \lambda_{\min}(X_I^t X_I)$, where $d$ and $C$ are absolute constants.

See also [10] for a simpler proof. Vershynin [14] generalized Bourgain and Tzafriri’s result to the case of rectangular matrices and the estimate of $|T|$ was improved. Recently, Spielman and Srivastava proposed in [9] a deterministic construction of $T$. Using the same techniques, Youssef [15] was able to improve Vershynin’s result.

Applications of such results are well known in the domain of harmonic analysis [1]. The study of the condition number is also a subject of extensive study in statistics and signal processing [11].

In the present paper, we focus on the case where the matrix $X$ is random (which makes the problem a lot easier a priori) but we address restricted invertibility with the additional almost orthogonality constraint that $\gamma_{s,\rho_-}(X)$ should be small. Such types of results will find applications in applied harmonic analysis as well. Some applications to sparse recovery and the LASSO procedure in computational statistics are discussed in [4].
1.2. Main results.

**Definition 1.2.** The index $\gamma_{s,\rho_-}(X)$ associated with the matrix $X$ in $\mathbb{R}^{n \times p}$ is defined by

$$\gamma_{s,\rho_-}(X) = \sup_{v \in B(0,1)} \inf_{I \subseteq S_{s,\rho_-}} \|X_I^Tv\|_\infty. \tag{1.2}$$

An important remark is that the function $X \mapsto \gamma_{s,\rho_-}(X)$ is nonincreasing in the sense that if we set $X'' = [X, X']$, where $X'$ is a matrix in $\mathbb{R}^{n \times p'}$, then $\gamma_{s,\rho_-}(X) \geq \gamma_{s,\rho_-}(X')$.

The quantity $\gamma_{s,\rho_-}(X)$ is very small for $p$ sufficiently large, at least for random matrices such as normalized standard Gaussian matrices as shown in the following theorem.

**Theorem 1.3.** Assume that $X$ is random matrix in $\mathbb{R}^{n \times p}$ with i.i.d. columns with uniform distribution on the unit sphere of $\mathbb{R}^n$. Let $\rho_-$ and $\varepsilon \in (0,1)$, $C_\kappa \in (0, +\infty)$ and assume that $p \geq \lfloor e^{\sqrt{2\pi}} \rfloor$. Set

$$K_\varepsilon = \sqrt{\frac{2\pi}{6}} \left( (1 + C_\kappa) \log \left( 1 + \frac{2}{\varepsilon} \right) + C_\kappa + \log \left( \frac{C_\kappa}{4} \right) \right).$$

Assume that $n, \kappa$ and $s$ satisfy

$$n \geq 6, \tag{1.3}$$

$$\kappa = \max \left\{ 4e^{-2(\ln 2 - 1)}, \frac{4e^3}{(1 - p_+)^2} \frac{(1 + K_\varepsilon)(1 + C_\kappa)}{\varepsilon(1 - \varepsilon)} \right\}^2 \log^2(p) \log(C_\kappa n), \tag{1.4}$$

$$\frac{\max\{ks, 2 \times 36 \times 3 \times 3, \exp((1 - \rho_-)/2)\}}{C_\kappa} \leq n \leq \min \left\{ \left( \frac{p}{\log(p)} \right)^2, \exp \left( \frac{1 - \rho_-}{\sqrt{2}} \right) \frac{C_\kappa}{p} \right\}. \tag{1.5}$$

Then, we have

$$\gamma_{s,\rho_-}(X) \leq 80 \frac{\log(p)}{p} \tag{1.6}$$

with probability at least $1 - 5 \frac{n}{p \log(p)^{n-1}} - 9 p^{-n}$.

**Corollary 1.4.** We can take $s$ as large as

$$\left\lfloor \frac{n}{C_s \log^2(p) \log(C_\kappa n)} \right\rfloor$$

with

$$C_s = \frac{e^2(1 - \rho_-)^2(1 - \varepsilon)^8}{4e^3} \frac{C_\kappa}{(1 + K_\varepsilon)^2(1 + C_\kappa)^2}. \tag{1.7}$$

**Proof.** Notice that the constraints (1.4) and (1.5) together imply the following constraint on $s$:

$$s \leq \frac{n}{C_s \log^2(p) \log(C_\kappa n)} \tag{1.8}.$$

The result follows immediately. \hfill \square

2. Proof of Proposition 1.3

2.1. Constructing an outer approximation for $I$ in the definition of $\gamma_{s,\rho_-}$. Take $v \in \mathbb{R}^n$. We construct an outer approximation $\tilde{I}$ of $I$ into which we be able to extract the set $I$. We procede recursively as follows: until $|\tilde{I}| = \min\{ks, p/2\}$, for some positive real number $\kappa$ to be specified later, do

- Choose $j_1 = \arg\min_{j=1,\ldots,p} |\langle X_j, v \rangle|$ and set $\tilde{I} = \{j_1\}$
- Choose $j_2 = \arg\min_{j=1,\ldots,p, j \notin \tilde{I}} |\langle X_j, v \rangle|$ and set $\tilde{I} = \tilde{I} \cup \{j_2\}$
- $\ldots$
- Choose $j_k = \arg\min_{j=1,\ldots,p, j \notin \tilde{I}} |\langle X_j, v \rangle|$ and set $\tilde{I} = \tilde{I} \cup \{j_k\}$. 


2.2. An upper bound on $\|X_j^t v\|_{\infty}$. If we denote by $Z_j$ the quantity $|\langle X_j, v \rangle|$ and by $Z_{(r)}$ the $r^{th}$ order statistic, we get that

$$\|X_j^t v\|_{\infty} = Z_{(\kappa s)}.$$  

Since the $X_j$'s are assumed to be i.i.d. with uniform distribution on the unit sphere of $\mathbb{R}^n$, we obtain that the distribution of $Z_{(r)}$ is the distribution of the $r^{th}$ order statistics of the sequence $|X_j^t v|$, $j = 1, \ldots, p$. By (5) p.147 [6], $|X_j^t v|$ has density $g$ and CDF $G$ given by

$$g(z) = \frac{1}{\sqrt{\pi \Gamma \left( \frac{n-1}{2} \right)}} \left( 1 - z^2 \right)^{\frac{n-3}{2}} \quad \text{and} \quad G(z) = 2 \int_0^z g(\zeta) \, d\zeta.$$ 

Thus,

$$F_{Z_{(r)}}(z) = \mathbb{P}(B \geq r)$$

where $B$ is a binomial variable $B(p, G(z))$. Our next goal is to find the smallest value $z_0$ of $z$ which satisfies

$$F_{Z_{(\kappa s)}}(z_0) \geq 1 - p^{-n}. \quad (2.7)$$

We have the following standard concentration bound for $B$ (e.g. [5]):

$$\mathbb{P}(B \leq (1 - \varepsilon)\mathbb{E}[B]) \leq \exp \left( -\frac{1}{2} \varepsilon^2 \mathbb{E}[B] \right)$$

which gives

$$\mathbb{P}(B \geq (1 - \varepsilon)pG(z)) \geq 1 - \exp \left( -\frac{1}{2} \varepsilon^2 pG(z) \right)$$

We thus have to look for a root (or at least an upper bound to a root) of the equation

$$G(z) = \frac{1}{\sqrt{\pi \Gamma \left( \frac{n-1}{2} \right)}} \left( 1 - z^2 \right)^{\frac{n-3}{2}} \quad (2.8)$$

for $\varepsilon \leq 1/\sqrt{2}$. By a straightforward application of Stirling’s formula (see e.g. (1.4) in [8]), we obtain

$$\frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n-1}{2} \right)} \geq \frac{e^{2\ln(2)}}{2} \frac{(n-3)^{3/2}}{(n-2)^{1/2}}.$$ 

Thus, any choice of $z_0$ satisfying

$$z_0 \geq \frac{2 \sqrt{\pi}}{e^{2\ln(2)}} \frac{(n-2)^{1/2}}{(n-3)^{3/2}} \frac{1}{\frac{1}{2} \varepsilon^2 \log(p)} \quad (2.9)$$

is an upper bound to the quantile for $(1 - \varepsilon)pG(z_0)$-order statistics at level $p^{-n}$. We now want to enforce the constraint that

$$(1 - \varepsilon)pG(z_0) \leq \kappa s.$$ 

By again a straightforward application of Stirling’s formula, we obtain

$$G(z) \leq \frac{1}{\sqrt{\pi}} \frac{e^2}{2} \frac{(n-3)^{3/2}}{(n-2)^{1/2}} z \quad \text{for} \ n \geq 4.$$ 

Thus, we need to impose that

$$z_0 \leq \frac{2 \sqrt{\pi}}{e^2} \frac{(n-2)^{1/2}}{(n-3)^{3/2}} \frac{\kappa s}{(1 - \varepsilon)p} \quad (2.9)$$
Notice that the constraints (2.8) and (2.9) are compatible if
\[ \kappa \geq \frac{4}{e^{2\ln(2)-1}} \frac{1 - \varepsilon}{\varepsilon^2} \frac{n}{s} \log(p). \]
Take \( \varepsilon = 1 - \frac{1}{n/s \log(p)} \) and obtain
\[ \mathbb{P} \left( \|X_t^I v\|_{\infty} \geq \frac{8 \sqrt{\pi}}{e^{2\ln(2)}} \frac{(n-2)^{1/2}}{(n-3)^{3/2}} \frac{n}{p} \log(p) \right) \leq p^{-n} \]
for
\[ \kappa = \frac{4}{e^{2\ln(2)-1}} \]
for any \( p \) such that \( n/s \log(p) \geq \sqrt{2} \), which is clearly the case as soon as \( p \geq e^{4/\sqrt{2}} \) for \( s \leq n \) as assumed in the proposition.
If \( n \geq 6 \), we can simplify (2.10) with
\[ \mathbb{P} \left( \|X_t^I v\|_{\infty} \geq 80 \frac{\log(p)}{p} \right) \leq p^{-n} \]
2.3. Extracting a well conditioned submatrix of \( X_f \).

The method for extracting \( X_f \) from \( X_f^I \) uses random column selection. For this purpose, we will need to control the coherence and the norm of \( X_f^I \).

Step 1: The coherence of \( X_f^I \). Let us define the spherical cap
\[ C(v, h) = \{ w \in \mathbb{R}^n \mid \langle v, w \rangle \geq h \}. \]
The area of \( C(v, h) \) is given by
\[ \text{Area}(C(v, h)) = \text{Area}(S(0, 1)) \int_0^{2h^2} t^{n-1}(1 - t)^{\frac{n}{2}} dt. \]
Thus, the probability that a random vector \( w \) with Haar measure on the unit sphere \( S(0, 1) \) falls into the spherical cap \( C(v, h) \) is given by
\[ \mathbb{P}(w \in C(v, h)) = \frac{C(v, h)}{S(0, 1)} = \frac{\int_0^{2h^2} t^{n-1}(1 - t)^{\frac{n}{2}} dt}{\int_0^1 t^{n-1}(1 - t)^{\frac{n}{2}} dt}. \]
The last term is the CDF of the Beta distribution. Using the fact that
\[ \mathbb{P}(X_j \in C(X_j', h)) = \mathbb{P}(X_j' \in C(X_j, h)) \]
the union bound, and the independence of the \( X_j \)'s, the probability that \( X_j \in C(X_j', h) \) for some \( (j, j') \) in \( \{1, \ldots, p\}^2 \) can be bounded as follows
\[ \mathbb{P} \left( \bigcup_{j \neq j' = 1}^p \{ X_j \in C(X_{j'}, h) \} \right) = \mathbb{P} \left( \bigcup_{j < j' = 1}^p \{ X_j \in C(X_{j'}, h) \} \right) \leq \sum_{j < j' = 1}^p \mathbb{P} \left( \{ X_j \in C(X_{j'}, h) \} \right) = \sum_{j < j' = 1}^p \mathbb{E} \left[ \mathbb{P} \left( \{ X_j \in C(X_{j'}, h) \} \mid X_{j'} \right) \right] = \frac{p(p-1)}{2} \int_0^{2h^2} t^{n-1}(1 - t)^{\frac{n}{2}} dt. \]
Our next task is to choose \( h \) so that
\[ \frac{p(p-1)}{2} \int_0^{2h^2} t^{n-1}(1 - t)^{\frac{n}{2}} dt \leq p^{-n}. \]
Let us make the following crude approximation

\[ \frac{p(p-1)}{2} \int_0^{2h-h^2} t^{\frac{1}{2}} (1-t)^{\frac{1}{2}} dt \leq \frac{p^2}{2} (2h)^{\frac{1}{2}} (2h-h). \]

Thus, taking

\[ h \geq \frac{1}{2} \exp \left( -2 \left( \log(p) + \frac{\log(p) - \log(2)}{n+1} \right) \right) \]

will work. Moreover, since \( p \geq 2 \), we deduce that

\[ \mu(X_j) \leq \frac{1}{2} p^{-2} \]

with probability at least \( 1 - p^{-n} \).

**Step 2: The norm of** \( X_{\tilde{I}} \).

The norm of any submatrix \( X_S \) with \( n \) rows and \( \kappa_s \) columns of \( X \) has the following variational representation

\[ \|X_S\| = \max_{\substack{v \in \mathbb{R}^n, \|v\|=1 \atop w \in \mathbb{R}^{\kappa_s}, \|w\|=1}} v^t X_S w. \]

We will use an easy \( \varepsilon \)-net argument to control this norm. For any \( v \in \mathbb{R}^n, v^t X_j, j \in S \) is a sub-Gaussian random variable satisfying

\[ \mathbb{P} \left( |v^t X_j| \geq u \right) \leq 2 \exp \left( -c n u^2 \right), \]

for some constant \( c \). Therefore, using the fact that \( \|w\|=1 \), we have that

\[ \mathbb{P} \left( \left| \sum_{j \in S} v^t X_j w \right| \geq u \right) \leq 2 \exp \left( -c n u^2 \right). \]

Let us recall two useful results of Rudelson and Vershynin. The first one gives a bound on the covering number of spheres.

**Proposition 2.1.** ([13, Proposition 2.1]). For any positive integer \( d \), there exists an \( \varepsilon \)-net of the unit sphere of \( \mathbb{R}^d \) of cardinality

\[ 2d \left( 1 + \frac{2}{\varepsilon} \right)^{d-1}. \]

The second controls the approximation of the norm based on an \( \varepsilon \)-net.

**Proposition 2.2.** ([13, Proposition 2.2]). Let \( \mathcal{N} \) be an \( \varepsilon \)-net of the unit sphere of \( \mathbb{R}^d \) and let \( \mathcal{N}' \) be an \( \varepsilon' \)-net of the unit sphere of \( \mathbb{R}^{d'} \). Then for any linear operator \( A : \mathbb{R}^d \mapsto \mathbb{R}^{d'} \), we have

\[ \|A\| \leq \frac{1}{(1-\varepsilon)(1-\varepsilon')} \sup_{\substack{v \in \mathcal{N} \atop w \in \mathcal{N}'}} |v^t Aw|. \]

Let \( \mathcal{N} \) (resp. \( \mathcal{N}' \)) be an \( \varepsilon \)-net of the unit sphere of \( \mathbb{R}^{\kappa_s} \) (resp. of \( \mathbb{R}^{n} \)). On the other hand, we have that

\[ \mathbb{P} \left( \sup_{\substack{v \in \mathcal{N} \atop w \in \mathcal{N}'}} |v^t Aw| \geq u \right) \leq 2|\mathcal{N}| |\mathcal{N}'| \exp \left( -c n u^2 \right), \]

\[ \leq 8 n \kappa_s \left( 1 + \frac{2}{\varepsilon} \right)^{n+\kappa_s-2} \exp \left( -c n u^2 \right), \]

which gives

\[ \mathbb{P} \left( \sup_{\substack{v \in \mathcal{N} \atop w \in \mathcal{N}'}} |v^t Aw| \geq u \right) \leq 8 \frac{n \kappa_s \varepsilon^2}{(2+\varepsilon)^2} \exp \left( - \left( c n u^2 - (n + \kappa_s) \log \left( 1 + \frac{2}{\varepsilon} \right) \right) \right). \]
Using Proposition \[2.2\], we obtain that
\[
\mathbb{P} (\| X \| \geq u) \leq \mathbb{P} \left( \frac{1}{(1 - \varepsilon)^2} \sup_{v \in \mathbb{N}} |v^t A u| \geq u \right).
\]
Thus, we obtain
\[
\mathbb{P} (\| X \| \geq u) \leq 8 \frac{n \kappa s \varepsilon^2}{(2 + \varepsilon)^2} \exp \left( - \left( cn (1 - \varepsilon)^4 u^2 - (n + \kappa s) \log \left( 1 + \frac{2}{\varepsilon} \right) \right) \right).
\]
To conclude, let us note that
\[
\mathbb{P} (\| X \| \geq u) \leq \mathbb{P} \left( \max_{x \subset \{1, \ldots, p\}} \| X \| \geq u \right)
\]
\[
\leq \left( \frac{p}{\kappa s} \right) 8 \frac{n \kappa s \varepsilon^2}{(2 + \varepsilon)^2} \exp \left( - \left( cn (1 - \varepsilon)^4 u^2 - (n + \kappa s) \log \left( 1 + \frac{2}{\varepsilon} \right) \right) \right).
\]
and using the fact that
\[
\left( \frac{p}{\kappa s} \right) \leq \left( \frac{e p}{\kappa s} \right)^{\kappa s},
\]
once we finally obtain
\[
\mathbb{P} (\| X \| \geq u) \leq 8 \exp \left( - \left( cn (1 - \varepsilon)^4 u^2 - (n + \kappa s) \log \left( 1 + \frac{2}{\varepsilon} \right) - \kappa s \log \left( \frac{e p}{\kappa s} \right) - \log \left( \frac{n \kappa s \varepsilon^2}{(2 + \varepsilon)^2} \right) \right) \right).
\]
The right hand side term will be less than \( 8p^{-n} \) when
\[
n \log(p) \leq cn (1 - \varepsilon)^4 u^2 - (n + \kappa s) \log \left( 1 + \frac{2}{\varepsilon} \right) - \kappa s \log \left( \frac{e p}{\kappa s} \right) - \log \left( \frac{n \kappa s \varepsilon^2}{(2 + \varepsilon)^2} \right) .
\]
This happens if
\[
u^2 \geq \frac{1}{c(1 - \varepsilon)^4} \left( n \log(p) n + (1 + \frac{n}{\kappa s}) \log \left( 1 + \frac{2}{\varepsilon} \right) + \frac{\kappa s}{n} \log \left( \frac{e p}{\kappa s} \right) + \frac{n}{n} \log \left( \frac{n \kappa s \varepsilon^2}{(2 + \varepsilon)^2} \right) \right).
\]
Notice that
\[
(1 + \frac{\kappa s}{n}) \log \left( 1 + \frac{2}{\varepsilon} \right) + \frac{\kappa s}{n} \log \left( \frac{e p}{\kappa s} \right) + \frac{n}{n} \log \left( \frac{n \kappa s \varepsilon^2}{(2 + \varepsilon)^2} \right)
\]
\[
\leq (1 + C_n) \log \left( 1 + \frac{2}{\varepsilon} \right) + C_n + \frac{n}{n} \log \left( \frac{C_n \varepsilon^2}{4} \right),
\]
since \( n \geq 1 \). Now, since
\[
\frac{6}{\sqrt{2\pi}} \leq \log(p) \leq \frac{n + \kappa s}{n} \log(p),
\]
we finally obtain
\[
\mathbb{P} \left( \| X \| \geq \frac{1 + \frac{K_n}{c(1 - \varepsilon)^4}}{n} \log(p) \right) \leq \frac{8}{p^n}.
\]
\textbf{Step 3.} We will use the following lemma on the distance to identity of randomly selected submatrices.

\textbf{Lemma 2.3.} Let \( r \in (0, 1) \). Let \( n, \kappa \) and \( s \) satisfy conditions \[1.3\] and \[1.4\] assumed in Proposition \[1.3\]. Let \( \Sigma \subset \{1, \ldots, \kappa s\} \) be a random support with uniform distribution on index sets with cardinal \( s \). Then, with probability greater than or equal to \( 1 - 9 p^{-n} \) on \( X \), the following bound holds:
\[
\mathbb{P} (\| X_{\Sigma} X - \text{Id}_s \| \geq r | X) < 1.
\]
\textbf{Proof.} See Appendix.
Taking \( r = 1 - \rho_\cdot \), we conclude from Lemma 2.3 that, for any \( s \) satisfying (1.7), there exists a subset \( \tilde{I} \) of \( I \) with cardinal \( s \) such that

\[
\sigma_{\min}(X_{\tilde{I}}) \geq \rho_\cdot.
\]

### 2.3.1. The supremum over an \( \varepsilon \)-net.

Recalling Proposition 2.1, there exists an \( \varepsilon \)-net \( \mathcal{N} \) covering the unit sphere in \( \mathbb{R}^n \) with cardinal

\[
|\mathcal{N}| \leq 2n \left( 1 + \frac{2}{\varepsilon} \right)^{n-1}.
\]

Combining this with (2.10), we have that

\[
\mathbb{P} \left( \sup_{v \in \mathcal{N}} \inf_{I \subseteq S_{\varepsilon, \rho_\cdot}} \| X^t_I v \| \geq 80 \frac{\log(p)}{p} \right) \leq 2n \left( 1 + \frac{2}{\varepsilon} \right)^{n-1} p^{-n} + 9 p^{-n}.
\]

### 2.4. From the \( \varepsilon \)-net to the whole sphere.

For any \( v' \), one can find \( v \in \mathcal{N} \) with \( \| v' - v \|_2 \leq \varepsilon \). Thus, we have

\[
\| X^t_I v' \|_\infty \leq \| X^t_I v \|_\infty + \| X^t_I (v' - v) \|_\infty \\
\leq \| X^t_I v \|_\infty + \max_{j \in I} \| (X_j, (v' - v)) \| \\
\leq \| X^t_I v \|_\infty + \max_{j \in I} \| X_j \|_2 \| v' - v \|_2 \\
\leq \| X^t_I v \|_\infty + \varepsilon.
\]

Taking

\[
\varepsilon = 80 \frac{\log(p)}{p},
\]

we obtain from (2.16) and (2.15) that

\[
\mathbb{P} \left( \sup_{\| v \|_2 = 1} \inf_{I \subseteq S_{\varepsilon, \rho_\cdot}} \| X^t_I v \| \geq 80 \frac{\log(p)}{p} \right) \leq 20 n \left( 1 + \frac{2}{80 \log(p)} \right)^{n-1} p^{-n} + 9 p^{-n}
\]

and thus,

\[
\mathbb{P} \left( \sup_{\| v \|_2 = 1} \inf_{I \subseteq S_{\varepsilon, \rho_\cdot}} \| X^t_I v \| \geq 80 \frac{\log(p)}{p} \right) \leq 5 \frac{n}{p} \frac{\log(p)^{n-1}}{p} + 9 p^{-n},
\]

for \( p \geq \exp(6/\sqrt{2\pi}) \).

### Appendix A. Proof of Lemma 2.3

For any index set \( S \subset \{1, \ldots, \kappa s\} \) with cardinal \( s \), define \( R_S \) as the diagonal matrix with

\[
(R_S)_{i,i} = \begin{cases} 
1 & \text{if } i \in S, \\
0 & \text{otherwise.}
\end{cases}
\]

Notice that we have

\[
\| X_S^t X_S - I \| = \| R_S HR_S \|
\]

with \( H = X^t X - I \). In what follows, \( R_S \) simply denotes a diagonal matrix with i.i.d. diagonal components \( \delta_j, j = 1, \ldots, \kappa s \) with Bernoulli \( B(1, 1/\kappa) \) distribution. Let \( R' \) be an independent copy of \( R \). Assume that \( S \) is drawn uniformly at random among index sets of \( \{1, \ldots, \kappa s\} \) with cardinal \( s \). By an easy Poissonization argument, similar to [2] Claim (3.29) p.2173, we have that

(\text{A.17}) \quad \mathbb{P} \left( \| R_S HR_S \| \geq r \right) \leq 2 \mathbb{P} \left( \| RHR \| \geq r \right),

and by Proposition 4.1 in [3], we have that

(\text{A.18}) \quad \mathbb{P} \left( \| RHR \| \geq r \right) \leq 36 \mathbb{P} \left( \| RHR' \| \geq r/2 \right).
In order to bound the right hand side term, we will use Proposition 4.2. Set \( r' = r/2 \). Assuming that \( \kappa r'^2 \geq u^2 \geq \frac{1}{\kappa} \|X\|_4^4 \) and \( v^2 \geq \frac{1}{\kappa} \|X\|_2^2 \), the right hand side term can be bounded from above as follows:

(A.19) \[
\mathbb{P} \left( \| RHR' \| \geq r' \right) \leq 3 \kappa s \mathcal{V}(s, [r', u, v]),
\]

with

\[
\mathcal{V}(s, [r', u, v]) = \left( e \frac{1}{\kappa} \frac{u^2}{r'^2} \right)^{\frac{r^2}{v^2}} + \left( e \frac{1}{\kappa} \frac{\|M\|_4^4}{u^2} \right)^{\frac{u^2}{r'^2}} + \left( e \frac{1}{\kappa} \frac{1}{v^2} \right)^{\frac{v^2}{r'^2} \mu(M)^2}.
\]

Using (2.11) and (2.13), we deduce that with probability at least \( 1 - 8p^{-n} - p^{-n} \), we have

\[
\mathcal{V}(s, [r', u, v]) = e \frac{1}{\kappa} \left( \frac{1 + K_{\varepsilon}}{c(1-\varepsilon)^4} \right)^{\frac{n + \kappa s}{n}} \left( \frac{1 + K_{\varepsilon}}{c(1-\varepsilon)^4} \right)^{\frac{n + \kappa s}{n}} \log(p) \frac{2\kappa s}{n} \frac{2\kappa s}{n}.
\]

Thus, we have that

\[
v^2 = r'^2 \frac{1}{\log(C_n n)}
\]

\[
u^2 = C_V \left( \frac{1 + K_{\varepsilon}}{c(1-\varepsilon)^4} \right)^2 \frac{n + \kappa s}{n} \log(p)
\]

\[
\kappa \geq e^3 \frac{C_V}{r'^2} \left( \frac{1 + K_{\varepsilon}}{c(1-\varepsilon)^4} \right)^2 \frac{n + \kappa s}{n} \log(p)^2
\]

for some \( C_V \) possibly depending on \( s \). Since \( \kappa s \leq C_n n \), this implies in particular that

(A.20) \[
\kappa \geq e^3 \frac{C_V}{r'^2} \left( \frac{1 + K_{\varepsilon}}{c(1-\varepsilon)^4} \right)^2 \frac{n + \kappa s}{n} \log(p)^2
\]

Thus, we obtain that

\[
\mathcal{V}(s, [r', u, v]) = \left( e \frac{1}{e^2} \frac{\log(C_n n)}{r'^2} \right)^{\frac{r'^2}{v^2}} + \left( e \frac{1}{e^2} \frac{\log(C_n n)}{e^2 \frac{C_V}{e^2 C_V}} \right)^{\frac{\log(C_n n)}{e^2 \frac{C_V}{e^2 C_V}}}.
\]

Using (A.17), (A.18) and (A.19), we obtain that

\[
\mathbb{P} \left( \| RHR_s \| \geq r' \right) \leq 2 \times 36 \times 3 \times \kappa s \left( \frac{1}{e^2} \frac{\log(C_n n)}{r'^2} \right)^{\frac{r'^2}{v^2}} + \left( e \frac{1}{e^2} \frac{\log(C_n n)}{r'^2} \frac{C_V}{e^2 \frac{C_V}{e^2 C_V}} \right)^{\frac{2r'^2 e^2}{\log(C_n n)}}.
\]

Take

(A.21) \[
C_V = \log(C_n n)
\]

and, since \( p > 1 \) and \( r \in (0, 1) \), we obtain

\[
\mathbb{P} \left( \| RHR_s \| \geq r' \right) \leq 2 \times 36 \times 3 \times \kappa s \left( \frac{1}{e^2} \frac{\log(C_n n)}{r'^2} \right)^{\frac{r'^2}{v^2}} + \left( e \frac{1}{e^2} \frac{\log(C_n n)}{e^2 \log(C_n n)} \right)^{\frac{2r'^2 e^2}{\log(C_n n)}}.
\]
Replace $r'$ by $r/2$. Since it is assumed that $n \geq \exp(r/2)/C_\kappa$ and $p \geq \sqrt{2\log(C_\kappa n)/r}$, it is sufficient to impose that
\[ C_\kappa^2 n^2 \geq (2 \times 36 \times 3 \times \kappa s \times 3)^{1/\log(e^2)}, \]
in order for the right hand side of (A.22) to be less than one. Since $\kappa s \leq C_\kappa n$, it is sufficient to impose that
\[ C_\kappa^2 n^2 \geq 2 \times 36 \times 3 \times C_\kappa n \times 3, \]
or equivalently,
\[ C_\kappa n \geq 2 \times 36 \times 3 \times 3. \]
This is implied by (1.3) in the assumptions. On the other hand, combining (A.20) and (A.21) implies that one can take
\[ \kappa = \frac{4e^3}{r^2} \left( \frac{(1 + K_\kappa)(1 + C_\kappa)}{c(1 - c)^4} \right)^2 \log^2(p) \log(C_\kappa n), \]
which is nothing but (1.4) in the assumptions.

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