PARAMETRIC MANIFOLDS I: Extrinsic Approach

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ABSTRACT

A parametric manifold can be viewed as the manifold of orbits of a (regular) foliation of a manifold by means of a family of curves. If the foliation is hypersurface orthogonal, the parametric manifold is equivalent to the 1-parameter family of hypersurfaces orthogonal to the curves, each of which inherits a metric and connection from the original manifold via orthogonal projections; this is the well-known Gauss-Codazzi formalism. We generalize this formalism to the case where the foliation is not hypersurface orthogonal. Crucial to this generalization is the notion of deficiency, which measures the failure of the orthogonal tangent spaces to be surface-forming, and which behaves very much like torsion. Some applications to initial value problems in general relativity will be briefly discussed.

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1. Introduction

Associated with a foliation of spacetime by spacelike hypersurfaces is the dual foliation by timelike curves orthogonal to the hypersurfaces, i.e. the trajectories of observers whose instantaneous rest spaces consist precisely of the given hypersurfaces. But how does one describe physics as seen by observers whose trajectories are not hypersurface orthogonal? It is the goal of this paper to describe one possible framework for answering such questions.

The decomposition of various fields on a manifold into data on a hypersurface is not merely of interest for spacetimes. Given a (non-degenerate) metric of any signature on a manifold \( M \), the Gauss-Codazzi equations show how to project the geometry of \( M \) orthogonally onto a hypersurface \( \Sigma \). This paper generalizes the Gauss-Codazzi equations which describe the geometry orthogonal to a given family of curves to include the case when these curves fail to be hypersurface-orthogonal.

The term parametric manifold has been recently coined by Perjés [1] in this setting. He traces some of the geometric ideas back to Zel’manov [2]; similar ideas can also be found in some work of Einstein and Bergmann [3] on Kaluza-Klein theories. However, none of these authors describe the parametric theory in modern mathematical language, as tensors are given in terms of their components in a coordinate basis and their abstract properties are not clear. In particular, defining torsion in this setting, and especially distinguishing it from the new concept of deficiency, is hard to do without a basis-free approach. This paper presents one way of unifying these earlier ideas into a rigorous mathematical framework.

We start by reviewing some basic properties of connections in Section 2, followed by a description of the usual Gauss-Codazzi formalism in Section 3. We have deliberately presented some fairly standard material in considerable detail so that the comparison with the generalized Gauss-Codazzi framework, described in Section 4, will be clear. In Section 5, we express our results in a coordinate basis so that it can be more easily compared to earlier work. In Section 6 we then show that our framework does not, in fact, completely reproduce the earlier results cited above, and we further show how this can be remedied. Finally, in Section 7, we discuss our results.

2. Background

Let us begin by reviewing the standard notion of a connection on a manifold \( M \), together with some relevant properties of connections. For the following definitions, let \( M \) be a smooth manifold with (Lorentzian or Riemannian) metric \( g \) denoted by \( \langle \cdot , \cdot \rangle \). Also, let \( \mathcal{X}(M) \) denote the set of all smooth vector fields on \( M \) and \( \mathcal{F}(M) \) the ring of all smooth real-valued functions defined on \( M \).

**Definition 1** An (affine) connection \( \nabla \) on \( M \) is a mapping \( \nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M) \), usually denoted by \( \nabla(X,Y) = \nabla_X Y \), which satisfies the following axioms:

i. **Linearity over \( \mathcal{F}(M) \):** \( \nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z \)

ii. **Linearity:** \( \nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z \)

iii. **Product rule:** \( \nabla_X(fY) = f\nabla_X Y + X(f)Y \) for all \( X,Y,Z \in \mathcal{X}(M) \) and \( f,g \in \mathcal{F}(M) \).

The existence of a connection on \( M \) provides a way of differentiating vector fields along curves, which can be extended in the usual way to be a derivation on all tensor fields. Although traditionally one defines the concept of metric compatibility in terms of parallel vector fields along curves in \( M \), it can be restated (cf. [4]) as
Definition 2 An affine connection $\nabla$ is compatible with the metric of $\mathcal{M}$ provided
\[ X\left(\langle Y, Z \rangle \right) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \] (1)
for $X, Y, Z \in \chi(\mathcal{M})$.

Definition 3 A connection $\nabla$ is said to be torsion-free when
\[ \nabla_X Y - \nabla_Y X = [X, Y] \]
for all $X, Y \in \chi(\mathcal{M})$.

The action of $[X, Y]$ on functions $f \in \mathcal{F}(\mathcal{M})$ is defined by the action of the commutator
\[ [X, Y] f = XY f - YX f. \] (2)

Although it is not \textit{a priori} clear that with this definition $[X, Y]$ is a vector field, it can be shown (cf. [5]) that there exists a unique vector field, also written $[X, Y]$, satisfying (2).

A fundamental result in the theory of connections is

Theorem 4 There exists a unique connection on $\mathcal{M}$ which is compatible with the metric $g$ and torsion-free.

Definition 5 This unique connection is called the Levi-Civita connection.

The Levi-Civita connection can be given explicitly as (\textit{e.g.} [6])
\[ \langle Z, \nabla_Y X \rangle = \frac{1}{2} \left( X \left( \langle Y, Z \rangle \right) + Y \left( \langle Z, X \rangle \right) - Z \left( \langle X, Y \rangle \right) \right. \]
\[ \left. + \langle [Z, X], Y \rangle - \langle [Y, Z], X \rangle - \langle [X, Y], Z \rangle \right). \]

The notions of curvature and torsion play an interesting role in the development of a parametric theory. A clear understanding of the relationships between them will be useful when defining parametric curvature. Using the definitions in [7], rewritten in terms of an affine connection, we have

Definition 6 The torsion $T$ and curvature $R$ of $\nabla$ are given by
\[ T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \] (3)
and
\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \] (4)
for $X, Y, Z \in \chi(\mathcal{M})$. 
The case where $T(X, Y) \equiv 0$ agrees with the earlier notion of torsion-free.

Consider the components of $T$ and $R$ in some patch with coordinates $\{x^\alpha\}$, so that the coordinate vector fields $\{\partial_\alpha\}$ form a (local) basis of $\chi(M)$. Defining the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ by $\nabla_{\partial_\beta} \partial_\gamma = \Gamma^\alpha_{\beta\gamma} \partial_\alpha$ we have

$$T(\partial_\beta, \partial_\gamma) = \nabla_{\partial_\beta} \partial_\gamma - \nabla_{\partial_\gamma} \partial_\beta - 0 = (\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta}) \partial_\alpha = T^\alpha_{\beta\gamma} \partial_\alpha.$$ 

A torsion-free connection is thus sometimes referred to as a *symmetric connection*. While it is trivially true that mixed partial derivatives commute, the torsion tensor may be thought of as measuring the failure of mixed covariant derivatives to commute. As we see from above

$$\left(\nabla_{\partial_\beta} \nabla_{\partial_\gamma} - \nabla_{\partial_\gamma} \nabla_{\partial_\beta}\right)(f) = T^\alpha_{\beta\gamma} \partial_\alpha f$$

where we have used the fact that $\nabla_{\partial_\alpha} f = \partial_\alpha f = \frac{\partial f}{\partial x^\alpha}$

For curvature,

$$R(\partial_\alpha, \partial_\beta) \partial_\gamma = (\nabla_{\partial_\alpha} \nabla_{\partial_\beta} - \nabla_{\partial_\beta} \nabla_{\partial_\alpha}) \partial_\gamma - 0 = R^\mu_{\gamma\alpha\beta} \partial_\mu. \quad (5)$$

As is often done, $R^\nu_{\delta\beta\alpha}$ may be expressed in terms of the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$,

$$R^\nu_{\delta\beta\alpha} = \frac{\partial \Gamma^\mu_{\delta\alpha}}{\partial x^\beta} - \frac{\partial \Gamma^\mu_{\delta\beta}}{\partial x^\alpha} + \Gamma^\nu_{\mu\beta} \Gamma^\mu_{\delta\alpha} - \Gamma^\nu_{\mu\alpha} \Gamma^\mu_{\delta\beta}.$$ 

It is worth noting that in the definition (4) of $R$, as well as in the formula (5) for the components $R^\mu_{\gamma\alpha\beta}$, there is no explicit mention of the torsion. The case is different when using the abstract index notation (see [8]), which closely parallels component notation in a coordinate basis.

In the abstract index notation, the vector field $\nabla_X Y$ is represented by $X^a \nabla_a Y^b$. In a coordinate basis (with coordinates $\{x^\alpha\}$), $X^a$ is the vector field $X^\alpha \partial_\alpha$. Furthermore, in this notation $\nabla_a Z^b = \partial_a Z^b + \Gamma^b_{ca} Z^c$ would represent the tensor with components

$$\frac{\partial Z^\beta}{\partial x^\alpha} + \Gamma^\beta_{\gamma\alpha} Z^\gamma.$$ 

In the absence of torsion, one often defines the action of the Riemann curvature tensor by

$$R^n_{cba} Z_n = (\nabla_a \nabla_b - \nabla_b \nabla_a) Z_c.$$ 

However, in terms of the Christoffel symbols $\Gamma^b_{ca}$

$$\nabla_a \nabla_b Z_c = \partial_a (\partial_b Z_c - \Gamma^m_{cb} Z_m) - \Gamma^m_{ba} (\partial_m Z_c - \Gamma^m_{cm} Z_n) - \Gamma^m_{ca} (\partial_b Z_m - \Gamma^m_{mb} Z_n).$$
yielding

\[(\nabla_a \nabla_b - \nabla_b \nabla_a)Z_c = (\Gamma^m_{ab} - \Gamma^m_{ba})\nabla_m Z_c + (\partial_b \Gamma^m_{ca} - \partial_a \Gamma^m_{cb})Z_m + (\Gamma^m_{ca} \Gamma^m_{mb} - \Gamma^m_{cb} \Gamma^m_{ma})Z_n \]

\[= T^m_{ab} \nabla_m Z_c + R^n_{cba} Z_n. \quad (6)\]

Rewriting equation (6) yields the correct abstract index expression for the curvature tensor in the presence of torsion:

\[R^n_{cba} Z_n = (\nabla_a \nabla_b - \nabla_b \nabla_a - T^m_{ab} \nabla_m)Z_c. \quad (7)\]

Thus, there is quite a difference between the treatment of torsion in the two notational schemes. While the first definition of curvature (equation (4)) proved to be valid with or without torsion, if one adopts the abstract index notation to describe a theory involving torsion, one must also re-define the curvature tensor to take this into account. While the abstract index notation is usually used to describe torsion-free theories (e.g., general relativity), we will see that the presence of “deficiency” in a parametric theory of spacetime has analogous consequences.

We conclude this discussion of torsion by stating the symmetries of the curvature tensor when torsion is present (cf. [9]).

**Theorem 7** \(R\) and \(T\) have the following symmetries:

i. \(T(X, Y) = -T(Y, X)\)

ii. \(R(X, Y) Z = -R(Y, X) Z\)

iii. \(\langle R(X, Y) Z, W \rangle = \langle R(X, Y) W, Z \rangle\) if \(\nabla\) is compatible with \(<, >\).

iv. (the first Bianchi identity)

\[R(X, Y) Z + R(Y, Z) X + R(Z, X) Y = \nabla_X T(Y, Z) + \nabla_Y T(Z, X) + \nabla_Z T(X, Y) + T(X, [Y, Z]) + T(Y, [Z, X]) + T(Z, [X, Y]) \quad (8)\]

**Proof:** Symmetries i and ii are immediate. To show iv just write out the cyclic sum, use the definition of \(T\), and keep in mind the Jacobi identity for bracket. Explicitly we have,

\[R(X, Y) Z + R(Y, Z) X + R(Z, X) Y = \nabla_X (\nabla_Y Z - \nabla_Z Y) + \nabla_Y (\nabla_Z X - \nabla_X Z) + \nabla_Z (\nabla_X Y - \nabla_Y X) - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X\]

\[= \nabla_X (T(Y, Z) + [Y, Z]) + \nabla_Y (T(Z, X) + [X, Z]) + \nabla_Z (T(X, Y) + [X, Y]) - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[X,Z]} Z - \nabla_{[Y,Z]} X\]

\[= \nabla_X T(Y, Z) + \nabla_Y T(Z, X) + \nabla_Z T(X, Y) + T(X, [Y, Z]) + T(Y, [Z, X]) + T(Z, [X, Y]) + [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]].\]
where the last three terms add to zero. To prove \( iii \) we need to assume that \( \nabla \) is compatible with the metric \( \langle , \rangle \), thus writing
\[
\langle \nabla_X \nabla_Y Z, W \rangle = X \langle \nabla_Y Z, W \rangle - \langle \nabla_Y Z, \nabla_X W \rangle \\
= X \langle \nabla_Y Z, W \rangle - Y \langle Z, \nabla_X W \rangle + \langle Z, \nabla_Y \nabla_X W \rangle
\]
and
\[
\langle \nabla_{[X,Y]} Z, W \rangle = [X, Y] \langle Z, W \rangle - \langle Z, \nabla_{[X,Y]} W \rangle
\]
we have
\[
\langle R(X, Y) Z, W \rangle = \langle \nabla_Y \nabla_X W, Z \rangle - \langle \nabla_X \nabla_Y W, Z \rangle + \langle \nabla_{[X,Y]} W, Z \rangle \\
+ X \langle \nabla_Y Z, W \rangle - Y \langle Z, \nabla_X W \rangle - Y \langle \nabla_X Z, W \rangle \\
+ X \langle Z, \nabla_Y W \rangle - [X,Y] \langle Z, W \rangle \\
= - \langle R(X, Y) W, Z \rangle + X Y \langle Z, W \rangle - X \langle Z, \nabla_Y W \rangle \\
- Y \langle Z, \nabla_X W \rangle - Y X \langle Z, W \rangle + Y \langle Z, \nabla_X W \rangle \\
+ X \langle Z, \nabla_Y W \rangle - [X,Y] \langle Z, W \rangle \\
= - \langle R(X, Y) W, Z \rangle.
\]

3. The Standard Gauss-Codazzi Formalism

The Gauss-Codazzi equations relate the geometry of a manifold (with metric) to the geometry of an embedded submanifold. Specifically, the higher-dimensional manifold induces a metric on the embedded surface, and thus gives rise to a unique derivative operator (on the surface) and finally a curvature tensor. The Gauss-Codazzi equations relate these induced quantities to the higher-dimensional quantities.

Let \( \Sigma \) be a (nondegenerate) hypersurface in \( \mathcal{M} \), \textit{i.e.} an embedded submanifold of codimension 1 such that the metric \( k \) induced on \( \Sigma \) by the metric \( g \) on \( \mathcal{M} \) is nondegenerate. The induced metric is of course the pullback of \( g \) along the embedding, but it can also be expressed as a projection operator as follows.

Let \( n \) be the unit normal vector to \( \Sigma \). \(^2\) Then the induced metric is given by
\[
k = g \pm n^b \otimes n^b
\]
where \( n^b \) is the 1-form dual (with respect to \( g \)) to \( n \) and where the sign depends on whether \( n \) is timelike (\( + \)) or spacelike (\( - \)).

For any point \( p \in \Sigma \), the tangent space \( T_p \mathcal{M} \) may be written as a direct sum
\[
T_p \mathcal{M} = T_p \Sigma \oplus (T_p \Sigma)^\perp \\
= (T_p \mathcal{M})^\perp \oplus (T_p \mathcal{M})^\top
\]

\(^2\) In the Lorentzian case, where \( (\mathcal{M}, g) \) is a \textit{spacetime}, \( \Sigma \) is typically a spacelike hypersurface, \textit{i.e.} a Riemannian manifold in its own right. In this case, it is customary to choose \( n \) to be the \textit{future-pointing} timelike unit vector field orthogonal to \( \Sigma \).
where \((T_p \Sigma)^\perp\) is the orthogonal complement of \(T_p \Sigma\) in \(T_p \mathcal{M}\) (with respect to the spacetime metric \(g\)). For any \(v \in T_p \mathcal{M}\), let \(v^\top\) and \(v^\perp\) be the obvious projections so that
\[
v = v^\perp + v^\top
\]
where we have used \(\perp\) to denote the projection to the tangent space of \(\Sigma\) (to agree with the notation of the next section).

Given vector fields \(X\) and \(Y\) on \(\Sigma\), one may define a connection on \(\Sigma\) by
\[
D_X Y = (\nabla_X Y)^\perp.
\]   \(\text{(9)}\)
Equation (9) not only defines an affine connection on \(\Sigma\), but, as is shown e.g. in [4], \(D\) is the unique Levi-Civita connection associated with the induced metric \(k\). ³ One may define the curvature of \(D\) in the usual manner:
\[
R_D(X, Y) Z = D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z
\]   \(\text{(10)}\)
where \(X, Y, Z\) are tangent to \(\Sigma\). Since \(\Sigma\) is a hypersurface, \([X, Y]\) denotes a vector field tangent to \(\Sigma\) and, hence, \(D_{[X, Y]} Z\) is well-defined. Using \(\langle , \rangle\) to denote the spacetime metric, one can show, e.g. [4], that the curvature \(R\) of \(\mathcal{M}\) and the curvature \(R_D\) of the surface \(\Sigma\) are related by Gauss’ equation
\[
\langle R(X, Y) Z, W \rangle = \langle R_D(X, Y) Z, W \rangle
\]
\[
- \langle B(Y, W), B(X, Z) \rangle + \langle B(X, W), B(Y, Z) \rangle
\]   \(\text{(11)}\)
where all the vectors \(X, Y, Z, W\) are assumed to be tangent to \(\Sigma\) and \(B(X, Y)\) is the tensor defined by
\[
B(X, Y) = \nabla_X Y - D_X Y
\]
\[
= (\nabla_X Y)^\top.
\]
Notice that \(B(X, Y)\) is orthogonal to \(\Sigma\).

**Theorem 8** Taking \(\nabla\), \(D\), and \(B\) as defined above, if \(\nabla\) is torsion-free then

i. \(D\) is torsion-free and

ii. \(B\) is symmetric.

**Proof:**
\[
T_D(X, Y) := D_X Y - D_Y X - [X, Y]
\]
\[
= (\nabla_X Y)^\perp - (\nabla_Y X)^\perp - [X, Y]
\]
\[
= (T(X, Y))^\perp
\]
\[
= 0
\]
since \([X, Y]^\perp = [X, Y]\) by Frobenius’ theorem. The symmetry of \(B\) follows from the torsion-free properties of both connections. We have
\[
B(X, Y) - B(Y, X) = \nabla_X Y - \nabla_Y X - (D_X Y - D_Y X)
\]
\[
= \left(T(X, Y) + [X, Y]\right) - \left(T_D(X, Y) + [X, Y]\right)
\]
\[
= 0.
\]

³ It is shown below that \(D\) is torsion-free; metric compatibility follows as a special case of Proposition 10 in the next section.
$B$ is closely related to the extrinsic curvature $K$ of $\Sigma$, which is defined by

$$K(X, Y) = \langle -\nabla_X Y, n \rangle.$$ 

The relationship between $K$ and $B$ is given by

$$K(X, Y) = \langle -\nabla_X Y, n \rangle = \langle -B(X, Y) - D_X Y, n \rangle = \langle -B(X, Y), n \rangle - \langle D_X Y, n \rangle = \langle -B(X, Y), n \rangle$$

so that the symmetry of $K$ follows directly from the symmetry of $B$ when $\nabla$ is torsion-free.

$B$ can be thought of as measuring the difference between the geometries of $M$ and $\Sigma$. In fact, $B$ is identically zero if (and only if) every geodesic of $\Sigma$ is also a geodesic of $M$.

It is worth mentioning that the tensor $B$ fails to be symmetric if $\nabla$ possesses torsion. If we let $T$ and $T_D$ represent the torsion tensors associated with the respective connections $\nabla$ and $D$, then the above calculation shows that

$$B(X, Y) - B(Y, X) = T(X, Y) - T_D(X, Y).$$

Therefore, the failure of $B$ to be symmetric is to be expected in the most general setting.

4. A Generalized Gauss-Codazzi Formalism

The above formalism lends itself nicely to the slicing viewpoint, in which a manifold is foliated with (usually spacelike) hypersurfaces. Both the slicing and Gauss-Codazzi formalisms focus on decomposing the geometry into a piece tangent to $\Sigma$ and a piece orthogonal to $\Sigma$. One can view these decompositions as a place to begin an initial value formulation; the Gauss-Codazzi relations impose certain constraints on the initial data.

One may instead consider the threading viewpoint [10], which is dual to slicing in that the manifold is now (regularly) foliated with a (non-null) vector field. If this vector field is hypersurface orthogonal, then the orthogonal hypersurfaces can be used as in the slicing scenario. But what happens if the vector field is not hypersurface orthogonal?

Given a non-null vector field $A$ (not necessarily unit), at each point $p$ in $M$ one still has the decomposition

$$T_p M = (T_p M)^\perp \oplus (T_p M)^\top.$$ 

For $v \in T_p M$, write

$$v = v^\perp + v^\top$$

with $v^\perp$ orthogonal to $A$ and $v^\top$ parallel to $A$. As before, the spacetime metric induces a metric $h$ on $(T_p M)^\perp$ defined by

$$h = g - \frac{A^\flat \otimes A^\flat}{\langle A^\flat, A^\flat \rangle}.$$ (12)

\footnote{A more complete discussion of the relationship between slicing and threading appears in [11].}
where $A^\sharp$ is the 1-form which is dual (with respect to the metric $g$) to the vector field $A$.

Let $\chi^\bot \subset \chi(\mathcal{M})$ denote the set of all vector fields (everywhere) orthogonal to $A$. For $X, Y \in \chi^\bot$, one may define the operator

$$D_XY = (\nabla_X Y)^\bot.$$  

Proposition 9 $D$ satisfies the properties of an affine connection. Specifically:

1. $D_{fX+gY}Z = fD_XZ + gD_Y Z$
2. $D_X(Y + Z) = D_X Y + D_X Z$
3. $D_X(fY) = fD_X Y + X(f) Y$

for all vector fields $X, Y, Z \in \chi^\bot$.

Proof: This is just a consequence of the linearity of projections. First,

$$D_{fX+gY}Z = (\nabla_{fX+gY} Z)^\bot = (f\nabla_X Z + g\nabla_Y Z)^\bot = fD_X Z + gD_Y Z.$$

Second,

$$D_X(Y + Z) = (\nabla_X Y + \nabla_X Z)^\bot = D_X Y + D_X Z.$$

Finally,

$$D_X(fY) = (\nabla_X fY)^\bot = (f\nabla_X Y + X(f) Y)^\bot = fD_X Y + X(f) Y.$$

Therefore, $D$ is an affine connection. ♠

In the case where $(\mathcal{T}\mathcal{M})^\bot$ corresponded to the tangent space of some hypersurface, it was stated that $D$ was the Levi-Civita connection of the surface (with respect to the induced metric). Although (in the present scenario) $D$ is not, in general, the Levi-Civita connection on any submanifold, we may still investigate the familiar properties associated with the Levi-Civita connection. Using $\langle \,, \rangle$ to represent the metric $h$, we have

Proposition 10 If $\nabla$ is compatible with $g$, then $D$ is compatible with the metric $h$. That is,

$$X\left(\langle Y, Z \rangle\right) = \langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle$$

for $X, Y, Z \in \chi^\bot$.

Proof: For $X, Y \in \chi^\bot$, we have $\langle X, Y \rangle = \langle X, Y \rangle$. Since $D_X Y = \nabla_X Y - (\nabla_X Y)^\top$ and $\langle (\nabla_X Y)^\top, Z \rangle = 0$, we have $\langle D_X Y, Z \rangle = \langle \nabla_X Y, Z \rangle$. The fact that $D$ is compatible with $h$ is now a consequence of the fact that $\nabla$ is compatible with $g$. ♣
In the last section we showed that $D$ being torsion-free was an immediate consequence of $\nabla$ being torsion-free. In the present situation, progress is hindered by the fact that while $D_X Y - D_Y X$ represents a vector field orthogonal to $A$, $[X, Y]$ may not. In fact, $[X, Y] \in (TM)^\perp$ for all $X$ and $Y$ in $(TM)^\perp$ if and only if $(T, M)^\perp$ is surface-forming (Frobenius’ Theorem). Thus, it is quite fruitless to compare $D_X Y - D_Y X$ with $[X, Y]$. One may, however, decompose $[X, Y]$ as $[X, Y] = [X, Y]^\top + [X, Y]^\perp$.

We may now measure the fact that $(TM)^\perp$ is not surface-forming by the existence of $[X, Y]^\top$ and use $[X, Y]^\perp$ to measure the torsion of $D$.

**Definition 11** The (generalized) torsion, $^\perp T_D$, associated with the connection $D$ is defined by

$$ ^\perp T_D(X, Y) = D_X Y - D_Y X - [X, Y]^\perp. $$

**Lemma 12** The generalized torsion is precisely the projection of the torsion associated with $\nabla$.

*Proof:* We have,

$$ ^\perp T_D(X, Y) = D_X Y - D_Y X - [X, Y]^\perp = (\nabla_X Y - \nabla_Y X - [X, Y])^\perp = T(X, Y)^\perp. $$

♠

**Theorem 13** If $\nabla$ is torsion-free, then $^\perp T_D(X, Y) \equiv 0$ for all $X, Y \in \chi^\perp$.

*Proof:*

$$ ^\perp T_D(X, Y) = (T(X, Y))^\perp = 0. $$

♠

Therefore $D$ still inherits its (generalized) torsion only from $\nabla$.

We will show below that, in a coordinate basis, the connection symbols, $^\perp T^{i}_{jk}$, associated with $D$ obey the symmetry $^\perp T^{i}_{jk} = ^\perp T^{i}_{kj}$ if and only if $D$ is torsion-free ($^\perp T_D = 0$). Thus, the above definition of $^\perp T_D$ is quite reasonable.

**Definition 14** The deficiency, $\mathcal{D}$, of the connection $D$ is defined by

$$ \mathcal{D}(X, Y) = [X, Y]^\top. $$

**Theorem 15** The following statements are equivalent:

i. $(TM)^\perp$ is surface-forming.

ii. The generalized torsion $^\perp T_D$ associated with $D$ is the (standard) torsion $T_D$ as defined by (3).

iii. $\mathcal{D}(X, Y) \equiv 0$ for all $X, Y \in \chi^\perp$. 
Proof: This theorem is basically the vector field version of Frobenius’ theorem rewritten to emphasize the new definitions. By definition, $D(X, Y)\equiv 0$ if and only if $[X, Y]^{\top}\equiv 0$. Thus $D(X, Y)\equiv 0$ if and only if $[X, Y]\in \chi^\perp$, yielding $iii \Leftrightarrow i$ via Frobenius’ theorem. To show $iii \Rightarrow ii$, we again have $[X, Y]^{\top}\equiv 0$ so $[X, Y]^\perp\equiv [X, Y]$, making the two notions of torsion coincide. Since $\perp T_D(X, Y) - T_D(X, Y) = [X, Y]^\perp$, we also easily have $ii \Rightarrow iii$. ♠

For $X, Y \in \chi^\perp$, define as before

$$B(X, Y) = \nabla_X Y - D_X Y.$$ $B(X, Y)$ is again a vector field orthogonal to the vector fields $X$ and $Y$ and is in fact tangent to $A$. Even when $\nabla$ is torsion-free, $B$ may still fail to be symmetric.

$$B(X, Y) - B(Y, X) = T(X, Y) + [X, Y]^{\perp} - \perp T_D(X, Y) - [X, Y]^\perp$$

$$= D(X, Y) + T(X, Y) - \perp T_D(X, Y).$$

(13)

**Theorem 16** If $\nabla$ is torsion-free, then $B(X, Y) = B(Y, X)$ if and only if $D(X, Y) = 0$.

Proof: $T(X, Y) = 0$ implies that $\perp T_D(X, Y) = 0$ and, hence, equation (13) reduces to

$$B(X, Y) - B(Y, X) = D(X, Y).$$ ♠

We have that the deficiency of the connection $D$ measures the failure of $(T_M)^{\perp}$ to be surface-forming and, equivalently, the failure of the extrinsic curvature $B$ to be symmetric in a torsion-free setting.

Being an affine connection, $D$ must have an associated “curvature” tensor. However, the existence of the $[X, Y]^{\top}$ component prevents one from proceeding as before — “$D_{[X,Y]}$” doesn’t make sense! It appears as if this problem may be overcome simply by using the quantity $[X, Y]^\perp$ to represent the commutator of two vector fields orthogonal to the original vector field $A$. Armed with such a notion of “bracket”, the next step would be to define a curvature operator.

**Definition 17** Define the operator $S$ by

$$S(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}^\perp Z.$$ (14)

Unfortunately, such a definition immediately leads to problems.

**Proposition 18** $S(X, Y)Z$ is not function linear (unless $D = 0$). That is

$$S(X, fY)(gZ) \neq fg S(X, Y)Z.$$
Proof:

\[ S(X, fY)(gZ) = (D_X(fD_Y) - fD_YD_X - D_{f[X,Y]} + D_{X(f)}Y)(gZ) \]
\[ = fS(X,Y)(gZ) + (X(f)D_Y - X(f)D_Y)(gZ) \]
\[ = fS(X,Y)(gZ) \]
\[ = f\left(D_X(Y(g)Z + gD_Y Z) - D_Y(X(g)Z + gD_X Z) \right. \]
\[ - [X,Y] (g)Z - gD_{[X,Y]} Z \right) \]
\[ = f \left( [X,Y](g)Z - [X,Y] (g)Z + gS(S,Y)Z \right) \]
\[ = f gS(X,Y)Z + \mathcal{D}(X,Y)(gZ) \]

Therefore, in order to define a function linear curvature operator (tensor!), we must keep track of the \([X,Y]^\perp\) component (we cannot just project it away and forget about it). That is, the \(D_{[X,Y]} Z\) term in equation (14) is not complete. We do not want to project the vector field \([X,Y]\) too soon! We will, therefore, consider replacing the last term of (14) by the term \((\nabla_{[X,Y]} Z)^\perp\). This term is equivalent to the \(D_{[X,Y]} Z\) term in equation (10). However, since \([X,Y]\) is not necessarily orthogonal to \(A\) we cannot write \((\nabla_{[X,Y]} Z)^\perp\) in terms of the connection \(D\).

**Definition 19** The (generalized) curvature operator associated with \(D\) is defined by

\[ ^\perp R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - (\nabla_{[X,Y]} Z)^\perp. \]

**Proposition 20** \(^\perp R\) is function linear. That is, \(^\perp R\) is tensorial.

Proof:

\[ ^\perp R(X, fY)(gZ) = \]
\[ \left( D_X(fD_Y) - fD_YD_X - (\nabla_{f[X,Y]} + \nabla_{X(f)}Y)^\perp \right)(gZ) \]
\[ = f^\perp R(X,Y)(gZ) + (X(f)D_Y - (\nabla_{X(f)}Y)^\perp)(gZ) \]
\[ = f^\perp R(X,Y)(gZ) + \left( X(f)D_Y - X(f)D_Y \right)(gZ) \]
\[ = f^\perp R(X,Y)(gZ) \]
\[ = f \left( D_X(Y(g)Z + gD_Y Z) - D_Y(X(g)Z + gD_X Z) \right. \]
\[ - ([X,Y](g)Z + g\nabla_{[X,Y]} Z)^\perp \right) \]
\[ = f \left( g^\perp R(X,Y)Z + \right. \]
\[ \left. [X,Y](g)Z - ([X,Y](g)Z)^\perp \right) \]
\[ = fg^\perp R(X,Y)Z \]

where the linearity of the projection map was used throughout. ♠
Theorem 21 If $\nabla$ is metric compatible, then $^\perp R$ satisfies Gauss' Equation. That is,

$$\langle ^\perp R(X, Y)Z, W \rangle = \langle R(X, Y)Z, W \rangle + \langle B(Y, W), B(X, Z) \rangle - \langle B(X, W), B(Y, Z) \rangle$$

(15)

where $X, Y, Z$ and $W$ are orthogonal to $A$.

Proof: First, a few computational observations. Since $B(X, Y) = \nabla_X Y - D_X Y$ is orthogonal to $A$,

$$\langle \nabla_X Y, Z \rangle = \langle D_X Y, Z \rangle$$

(16)

for vector fields $X, Y, Z$ orthogonal to $A$. While we have shown that $D$ is compatible with the metric $\langle \langle , \rangle \rangle$, it is also true that, since the full metric $\langle \langle , \rangle \rangle$ agrees with the induced metric $\langle \langle , \rangle \rangle$ on $\chi^\perp$, one may write

$$X \left( \langle Y, Z \rangle \right) = \langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle.$$

$D$ is thus “compatible” with the metric $\langle \langle , \rangle \rangle$ when restricted to the subspace $\chi^\perp$. Using the definition of $R$ and $B$, we expand the right hand side of equation (15)

$$RHS = \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W \rangle
- \langle \nabla_Y W - D_Y W, \nabla_X Z - D_X Z \rangle
+ \langle \nabla_X W - D_X W, \nabla_Y Z - D_Y Z \rangle
= X \langle \nabla_Y Z, W \rangle - \langle \nabla_Y Z, \nabla_X W \rangle - Y \langle \nabla_X Z, W \rangle
+ \langle \nabla_X Z, \nabla_Y W \rangle - \langle \nabla_{[X,Y]} Z, W \rangle - \langle \nabla_Y W, \nabla_X Z \rangle
+ \langle \nabla_Y W, D_X Z \rangle + \langle D_Y W, \nabla_X Z \rangle - \langle D_Y W, D_X Z \rangle
+ \langle \nabla_X W, \nabla_Y Z \rangle - \langle \nabla_X W, D_Y Z \rangle - \langle D_X W, \nabla_Y Z \rangle
+ \langle D_X W, D_Y Z \rangle
= X \langle \nabla_Y Z, W \rangle - Y \langle \nabla_X Z, W \rangle - \langle \nabla_{[X,Y]} Z, W \rangle
+ \langle D_Y W, D_X Z \rangle - \langle D_X W, D_Y Z \rangle
= X \langle D_Y Z, W \rangle - \langle D_Y Z, D_X W \rangle
- Y \langle D_X Z, W \rangle + \langle D_Y W, D_X Z \rangle - \langle \nabla_{[X,Y]} Z \rangle^\perp, W \rangle
= \langle D_X D_Y Z - D_Y D_X Z - \nabla_{[X,Y]} Z \rangle^\perp, W \rangle
= \langle ^\perp R(X, Y)Z, W \rangle$$

where the second step involved the symmetry of the metric as well as equation (16). ☞
The above derivation of Gauss' equation only used the properties of metric compatibility (for both pairs of connections and metrics). In particular, the symmetry (torsion) of either connection was not a concern. Thus, we have further shown that Gauss' equation is valid in the presence of torsion.

Given \( \langle , \rangle \), \( R \), and \( B \), one may use Gauss' equation to define a curvature operator \( R_D \). In this context, we may view \( \perp R \) as the unique curvature tensor associated with \( D \) which satisfies Gauss' equation.

A word of caution is necessary at this point. If torsion is present in either (or both) of the connections, the tensor \( B(X,Y) \) will no longer be symmetric. This affects the symmetries of the tensors \( \perp R \) and \( R \). In particular, as we shall see, \( \perp R \) may not enjoy the familiar cyclic symmetry

\[
\perp R(X,Y)Z + \perp R(Y,Z)X + \perp R(Z,X)Y = 0
\]
even if \( R \) does! However, the other symmetries are immediate. More precisely,

**Theorem 22** Let \( \nabla \) be a torsion-free Riemannian connection associated with the metric \( \langle , \rangle \), with curvature tensor \( R \). Using \( D, \langle \langle , \rangle \rangle, B, \) and \( D \) as defined above, if \( \hat{R} \) is an induced curvature operator associated with the connection \( D \) and \( \hat{R} \) and \( R \) satisfy Gauss' equation, then \( \hat{R} \) has the following symmetries:

i. \( \langle \hat{R}(X,Y)Z, W \rangle = - \langle \hat{R}(Y,X)Z, W \rangle \)

ii. \( \langle \hat{R}(X,Y)Z, W \rangle = - \langle \hat{R}(X,Y)W, Z \rangle \)

iii. (first Bianchi identity)

\[
\langle \hat{R}(X,Y)Z + \hat{R}(Y,Z)X + \hat{R}(Z,X)Y, W \rangle \\
= \langle B(X,W), D(Y,Z) \rangle + \langle B(Y,W), D(Z,X) \rangle \\
+ \langle B(Z,W), D(X,Y) \rangle \\
= - \langle \nabla_X D(Y,Z), W \rangle - \langle \nabla_Y D(Z,X), W \rangle \\
- \langle \nabla_Z D(X,Y), W \rangle
\] .

**Proof:** The symmetries in i and ii can be read off directly from equation (11), keeping in mind that \( \hat{R} \) satisfies all of the symmetries of the usual Riemann curvature tensor (in the absence of torsion). To prove (iii), just cyclicly permute \( X, Y, \) and \( Z \) in the terms on the right hand side of equation (11) and add, obtaining

\[
\langle \hat{R}(X,Y)Z + \hat{R}(Y,Z)X + \hat{R}(Z,X)Y, W \rangle \\
= 0 - \langle B(Y,W), B(X,Z) \rangle - \langle B(Z,W), B(Y,X) \rangle \\
- \langle B(X,W), B(Z,Y) \rangle + \langle B(X,W), B(Y,Z) \rangle \\
+ \langle B(Y,W), B(Z,X) \rangle + \langle B(Z,W), B(X,Y) \rangle \\
= \langle B(X,W), D(Y,Z) \rangle + \langle B(Y,W), D(Z,X) \rangle \\
+ \langle B(Z,W), D(X,Y) \rangle .
\]

which is the first line in iii. However, this cyclic sum involving \( B \) and \( D \) may be rewritten in terms of \( \nabla \) and \( D \). Thus written, claim iii resembles the standard
cyclic symmetry of $R$ (see equation (8)). Keep in mind, however, that neither $\nabla$ nor $D$ possess torsion, although deficiency is present. We have

$$\langle B(X, W), D(Y, Z) \rangle = \langle \nabla_X W - D_X W, D(Y, Z) \rangle$$

$$= \langle \nabla_X W, D(Y, Z) \rangle$$

$$= X \langle W, D(Y, Z) \rangle - \langle W, \nabla_X D(Y, Z) \rangle$$

$$= -\langle \nabla_X D(Y, Z), W \rangle$$

since $D(Y, Z)$ is orthogonal to $W$. Thus the second equation in $iii$ is true. ♠

Note that $W$ is arbitrary in $i$ and $iii$, so that these can be rewritten in the same form as Theorem 7 (except of course for the intermediate result in (17)).

One further comment on the similarities between equations (8) and (17) is worth making. In equation (8) there are three extra terms of the form $T(X, [Y, Z])$ (and cyclic permutations). One might expect analogous terms in equation (17) involving $D(X, [Y, Z]^\perp)$ and cyclic permutations. However, since $D(X, Y)$ represents a vector field orthogonal to $A$,

$$\langle D(X, [Y, Z]^\perp), W \rangle = 0.$$

We see that the new concept of deficiency does indeed appear in the first Bianchi identity in just the way torsion would.

5. Coordinate Expressions

Let us now work in a coordinate patch and investigate the components of the above operators. For simplicity, we will consider the coordinate system inherited from a threading decomposition of spacetime. Let $A$ be timelike (and $\Sigma$ spacelike) with norm $1/M$, i.e. $\langle A, A \rangle = -1/M^2$. We now introduce coordinates $x^\alpha = (x^0, x^i) = (t, x^i)$ such that the given vector field $A$ can be written $A^\alpha = M^\alpha_0 (\partial t)^\alpha$. $M$ is the threading lapse; note that $A^0 = 1$ and $A^i = 0$. The coordinates $x^i$ are constant along integral curves of $\partial_t$ and can thus be thought of as coordinates on the (local) surfaces $\{t \equiv \text{constant}\}$. We assume throughout that a (Lorentzian) metric $g$ is given, that $\nabla$ is its associated Levi-Civita connection, and that all other tensors are as defined in the previous section.

Letting $m$ be the metric dual of the unit vector $A/\langle A, A \rangle$, the threading shift 1-form is given by

$$M_i dx^i := dt + \frac{1}{M} m$$

Thus,

$$A_0 = -1 \quad \text{and} \quad A_i = M_i.$$

In these coordinates, the spacetime metric $g$ takes the form

$$(g_{\alpha\beta}) = \begin{pmatrix} -M^2 & M^2 M_j \\ M^2 M_i & h_{ij} - M^2 M_i M_j \end{pmatrix}$$

5 A more complete discussion of threading and its relationship to parametric manifolds appears in [11].
The functions \( h_{ij} = g_{ij} + M^2 M_i M_j \) correspond to the components of the \textit{threading metric}, the metric on \( \chi^\perp \) induced by \( g \) (equation (12)). These functions can also be thought of as the nonzero components of the tensor

\[
h_{\alpha\beta} = g_{\alpha\beta} + M^2 A_\alpha A_\beta
\]

which is associated with the projection operator

\[
P^\beta_\alpha = h^\beta_\alpha = \delta^\beta_\alpha + M^2 A_\alpha A^\beta
\]

where \( \delta^\beta_\alpha \) is the Kronecker delta symbol. Being a projection operator guarantees that \( P^\beta_\alpha X^\alpha = X^\beta \) for \( X \in \chi^\perp \). It is easy to show that a spacetime vector field \( X = X^\alpha \partial_\alpha = X^0 \partial_t + X^i \partial_i \) is orthogonal to \( A \) if and only if \( X^0 = M_i X^i \).

To simplify notation we will introduce a “starry” derivative notation in all coordinate directions. Define

\[
\partial^*_\alpha = \partial_\alpha + A_\alpha \partial_t.
\]

Notice that since \( A_0 = -1 \) and \( A_i = M_i \), we have

\[
\partial^*_0 = 0
\]

and

\[
\partial^*_i = \partial_i + M_i \partial_t.
\]

and we will often write \( \partial^*_i f \) as \( f^*_i \).

Let us work out the action of the connection \( D \) in these coordinates. Given \( X \) and \( Y \) in \( \chi^\perp \), we defined

\[
D_X Y = (\nabla_X Y)^\perp
\]

\[
= P^\gamma_\alpha X^\beta \nabla_\beta Y^\gamma \partial_\alpha
\]

\[
= P^\gamma_\alpha P^\delta_\beta X^\beta \nabla_\delta Y^\gamma \partial_\alpha
\]

\[
= X^\beta P^\gamma_\alpha P^\delta_\beta \left( Y^\gamma_{\delta \beta} + \Gamma^\gamma_{\mu \lambda} Y^\mu \right) \partial_\alpha
\]

\[
= X^\beta \left( P^\gamma_\alpha (Y^\gamma_{\delta \beta} + M^2 A_\beta A^\delta Y^\gamma_{\delta \beta}) + P^\gamma_\alpha P^\delta_\beta \Gamma^\gamma_{\mu \lambda} Y^\mu \right) \partial_\alpha
\]

\[
= X^\beta \left( P^\gamma_\alpha Y^\gamma_{\beta \gamma} + P^\gamma_\alpha P^\delta_\beta P^\mu_\nu Y^\nu \Gamma^\gamma_{\mu \lambda} \right) \partial_\alpha
\]

\[
= X^\beta \left( Y^\gamma_{\beta \gamma} + M^2 A_\gamma A^\alpha Y^\gamma_{\beta \gamma} + \Gamma^\gamma_{\nu \beta} Y^\nu \right) \partial_\alpha
\]

\[
= X^\beta \left( Y^\gamma_{\beta \gamma} + \Gamma^\gamma_{\nu \beta} - M^2 A_\gamma A^\alpha Y^\gamma_{\beta \gamma} \right) \partial_\alpha
\]

where we have defined the symbol \( \Gamma^\gamma_{\nu \beta} \) by

\[
\Gamma^\gamma_{\nu \beta} = P^\gamma_\alpha P^\delta_\beta P^\mu_\nu \Gamma^\gamma_{\mu \lambda}.
\]

It can be show that the symbol \( \Gamma^\gamma_{\nu \beta} \) behaves like a projected tensor. That is,

\[
\Gamma^0_0 = \Gamma^0_0 = \Gamma^0_0 = \Gamma^0_0 = \Gamma^0_0 = \Gamma^0_0 = 0
\]
and
\[ \Gamma^0_{\alpha\beta} = A_i \Gamma^i_{\alpha\beta}. \]

Since \( \partial_s \) is a basis for \( \mathbf{\Gamma}^\perp \), we define the components of projected tensors by evaluating them on this basis. Using Lemma 12, it immediately follows that the components of the torsion \( \mathbf{T}_D \) of \( D \) are
\[ \mathbf{T}_D^k_{ij} \partial_{sk} = \mathbf{T}_D(\partial_s, \partial_s) \]
\[ = (\Gamma^k_{ij} - \Gamma^k_{ji}) \partial_{sk} \]
which also shows that a torsion-free projected connection is indeed symmetric as claimed.

Since \( D_X Y \) is orthogonal to \( A \), \( D_X Y \) is completely determined by its components \( (D_X Y)^i \). That is
\[ (D_X Y)^\alpha \partial_\alpha = M_i (D_X Y)^i \partial_t + (D_X Y)^i \partial_i \]
\[ = (D_X Y)^i \partial_s. \]
But we have shown above that
\[ (D_X Y)^i = X^j \left( Y^i s_j + \Gamma^i_{k j} Y^k \right) \]
where we have used the facts that \( Y^i s_0 \equiv 0 \) for all \( Y \) and \( A^i = 0 \). The above formula for \( (D_X Y)^i \) corresponds exactly to the parametric covariant derivative operator introduced by Perjés [1]. After a long but straightforward calculation, one may show that in the absence of torsion the terms \( \Gamma^i_{jk} \) may be written in a familiar form involving the parametric derivative operator and the components of the induced metric \( h_{ij} \), which again agrees with Perjés:
\[ \Gamma^i_{jk} = \frac{1}{2} h^{im} (h_{mj k} + h_{mk j} - h_{jk m}). \]

This provides covariant confirmation that Perjés’ parametric structure can be induced by a projective geometry of spacetime.

Continuing our coordinate description, let us calculate the components of the curvature tensor \( \mathbf{R} \) defined earlier. The components of \( \mathbf{R} \) are defined by
\[ \mathbf{R}(\partial_{s i}, \partial_{s j}) \partial_{s k} = \mathbf{R}^l_{kij} \partial_{s l} \]
\[ = \mathbf{R}^l_{kij} (\partial_t + M_i \partial_t). \]
Calculating the “spatial” components of \( D_{\partial_{s i}} D_{\partial_{s j}} \partial_{s k} \), we find:
\[ (D_{\partial_{s i}} D_{\partial_{s j}} \partial_{s k})^l = (D_{\partial_{s i}} (D_{\partial_{s j}} \partial_{s k}))^l \]
\[ = \partial_{s i} (D_{\partial_{s j}} \partial_{s k})^l + \Gamma^l_{nl} (D_{\partial_{s j}} \partial_{s k})^n \]
\[ = \partial_{s i} (\delta^l_{k j} + \Gamma^l_{mj} \delta^m_k) \]
\[ + \Gamma^l_{nl} (\delta^n_{k j} + \Gamma^n_{mj} \delta^m_k) \]
\[ = \Gamma^l_{j k s i} + \Gamma^l_{n i} \Gamma^n_{k j}. \]
Also,

\[ [\partial_{si}, \partial_{sj}] = (M_{ji} - M_{ij}) \partial_t \]

\[ = D_{ji} \partial_t \]

where we have introduced the notation \( D_{ji} = M_{ji} - M_{ij} \). Therefore,

\[
\nabla_{[\partial_{si}, \partial_{sj}]} \partial_{sk} = D_{ji} \left( \frac{\partial M_k}{\partial t} + \Gamma^0_{00} M_k + \Gamma^0_{k0} \right) \partial_t
\]

\[ + D_{ji} \left( \Gamma^l_{00} M_k + \Gamma^l_{k0} \right) \partial_t \]

thus yielding

\[
\left( \nabla_{[\partial_{si}, \partial_{sj}]} \partial_{sk} \right)^\perp = \left( (M_{ji} - M_{ij}) \left( \Gamma^l_{k0} + M_k \Gamma^l_{00} \right) \right) \partial_t.
\]

Writing everything out gives us

\[
\perp R^l_{kij} = \perp \Gamma^l_{kji} - \perp \Gamma^l_{kij} + \perp \Gamma^l_{ni} \Gamma^n_{kj} - \perp \Gamma^l_{nj} \Gamma^n_{ki}
\]

\[ + 2 \left( M_{ji} - M_{ij} \right) \left( \Gamma^l_{00} M_k + \Gamma^l_{k0} \right)
\]

\[ = \perp \Gamma^l_{kji} - \perp \Gamma^l_{kij} + \perp \Gamma^l_{ni} \Gamma^n_{kj} - \perp \Gamma^l_{nj} \Gamma^n_{ki}
\]

\[ + (M_{ji} - M_{ij}) h^l_{mn} \left( M^2 M_{mkn} - M^2 M_{knm} + \partial_t h_{km} \right) \quad (20)
\]

where the symbols \( \Gamma \) were replaced by the equivalent expressions involving the threading metric, lapse function, and shift 1-form. As we see, the components of \( \perp R \) are not quite as nice as in the case where the \( \partial_{si} \) span a hypersurface. The non-zero contribution of \( [\partial_{si}, \partial_{sj}] \) continues to complicate matters.

6. Zel’manov curvature

In his work on parametric manifolds, which provided much of the motivation for this current work, Perjés [1] gives the following definition of the Zel’manov curvature \(^6\)

\[
[\nabla_k \nabla_j - \nabla_j \nabla_k + (\omega_{jki} - \omega_{kij}) \partial_t] X_i = Z^r_{ijk} X_r \quad (21)
\]

(compare (7)) where \( \omega_i \) can be identified with the \( M_i \) defined in the previous section. In components, this takes the form [1]

\[
Z^l_{kij} = \perp \Gamma^l_{kji} - \perp \Gamma^l_{kij} + \perp \Gamma^l_{ni} \Gamma^n_{jk} - \perp \Gamma^l_{nj} \Gamma^n_{ik}.
\]

The Zel’manov curvature thus does not contain the contribution from \( [\partial_{si}, \partial_{sj}] \). If one wants to relate the parametric tensor \( Z^i_{jkl} \) to a spacetime tensor, one must re-examine the story leading up to the definition of \( \perp R \).

It seemed most natural to define \( \perp R \) with the \((\nabla_{[X,Y]}^\perp) \) term, as this definition closely resembles the definition of the standard curvature tensor. However, consider the definition

\[
\perp \tilde{R}(X,Y)Z = D_X D_Y Z - D_Y D_X Z - (\mathcal{L}_{[X,Y]}^\perp Z) \perp
\]

\[ ^6 \text{ A similar expression appears in [3].} \]
where $\mathcal{L}$ denotes Lie differentiation.

The difference between the two curvature operators is

$$\perp R(X, Y)Z - \bar{\perp} R(X, Y)Z = - (\nabla_Z [X, Y])^\perp$$  \hspace{1cm} (23)

(assuming $\nabla$ is torsion-free). In light of our earlier comments, we know that $\perp \bar{\perp} R$ does not satisfy Gauss’ equation. However, there are the following similarities between $\perp R$ and $\perp \bar{\perp} R$.

1. $\perp R(X, Y)f = \bar{\perp} R(X, Y)f$ for $X, Y \in \chi^\perp$,

and in the case where $(T\mathcal{M})^\perp$ is surface-forming one has that $[\partial_{*i}, \partial_{*j}] = 0$ which implies

2. $\perp R(\partial_{*i}, \partial_{*j}) = \bar{\perp} R(\partial_{*i}, \partial_{*j})$

so the two tensors agree in this special case.

For the components of $\perp \bar{\perp} R$, we must calculate $(\mathcal{L}_{\partial_{*i} \partial_{*j}} \partial_{*k})^\perp$. If $\nabla$ is torsion-free, the definition of $\mathcal{L}$ yields

$$ (\mathcal{L}_{\partial_{*i} \partial_{*j}} \partial_{*k})^\perp = [[\partial_{*i}, \partial_{*j}], \partial_{*k}]$$

$$= 0$$  \hspace{1cm} (24)

Thus, the nonzero term $[\partial_{*i}, \partial_{*j}]$ does not contribute to the components of $\perp \bar{\perp} R$.

We now show that we have in fact defined the Zel’manov curvature.

**Theorem 23** $\perp \bar{\perp} R$ is the Zel’manov curvature.

**Proof:** Using equations (19) and (24), we have

$$\perp \bar{\perp} R_{ijkl}^i = \perp \Gamma_{kl*}^i - \perp \Gamma_{ki*}^l + \perp \Gamma_{ni}^l \perp \Gamma_{kj}^n - \perp \Gamma_{nj}^l \perp \Gamma_{ki}^n$$

$$= Z^l_{*ijk}.$$  \hspace{1cm} (25)

7. Discussion

The generalized Gauss-Codazzi approach seems to have been successful in defining a notion of projected connection $D$, with a corresponding notion of torsion. Moreover, $D$ was found to be torsion-free if $\nabla$ was torsion-free. Most importantly, the deficiency $D$ was explicitly defined in such a way as to make its relationship to the torsion tensor clear. While distinct from torsion, deficiency plays much the same role *e.g.* in the first Bianchi identity.

However, there is one very peculiar aspect of this formalism, namely the absence of an embedded hypersurface $\Sigma$ to which to project! We have chosen to interpret the Gauss-Codazzi formalism as providing a pointwise projection, so that one obtains tensor fields defined on all of $\mathcal{M}$, rather than on a preferred hypersurface $\Sigma$.

\footnote{While we have not yet checked explicitly, we expect the same will be true for the Codazzi equation and the second Bianchi identity.}
If the foliations are suitably regular, the orthogonal hypersurfaces in the standard setting will be diffeomorphic to each other. Thus, all projected tensors can be viewed as living on the same such hypersurface, but at different “times”. This viewpoint lends itself well to initial value problems. But this viewpoint also carries over to our generalized framework, the only change being that one must work on the manifold of orbits of the foliation, which is diffeomorphic (but not isometric) to any hypersurface of constant “time”.

A parametric manifold is, in this setting, the manifold of orbits, on which there are 1-parameter families of projected tensor fields. The geometric nature of its construction, using orthogonal projection, ensures that it is reparameterization invariant, i.e. that it is invariant under the special coordinate transformations which relabel “time”. This notion of 1-parameter families of tensor fields together with a suitable reparameterization invariance can be used to give an intrinsic description of a parametric manifold, without using projections; this will be published separately [12].

Finally, we note that we have two somewhat different candidates, $\perp R$ and $Z$, for the curvature of our projected connection. The difference between the two involves the deficiency, and hence vanishes in the hypersurface-orthogonal case. More importantly, the difference also involves the lapse function $M$, which relates our arbitrary parameter $t$ to arc length (proper “time”) along the given curves. For a given problem, which of these notions of curvature is “correct” may thus depend on whether a notion of distance orthogonal to the manifold of orbits is appropriate.

This brings us to the use of the Gauss-Codazzi formalism in initial value problems, especially in general relativity. The initial value formulation of Einstein’s equations imposes (consequences of) the Gauss-Codazzi equations as constraints (and then uses the Mainardi equations to determine the evolution). We thus conjecture that our framework can be used to generalize the initial value formulation for Einstein’s equations to appropriate data on the manifold of orbits. We further conjecture that it is precisely the (generalized) Gauss and Codazzi equations which will lead to the appropriate constraints. This would also provide some evidence in favor of $\perp R$, which does satisfy Gauss’ equation, rather than $Z$, which doesn’t. We are actively pursuing these ideas.
ACKNOWLEDGEMENTS

This work forms part of a dissertation submitted to Oregon State University (by SB) in partial fulfillment of the requirements for the Ph.D. degree in mathematics. This work was partially funded by NSF grant PHY-9208494.

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