DIAGONALIZABILITY OF NON HOMOGENEOUS QUANTUM MARKOV STATES AND ASSOCIATED VON NEUMANN ALGEBRAS

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Abstract. We clarify the meaning of diagonalizability of quantum Markov states. Then, we prove that each non homogeneous quantum Markov state is diagonalizable. Namely, for each Markov state \( \varphi \) on the spin algebra \( \mathcal{M} := \bigotimes_{j \in \mathbb{Z}} M_d(C) \), there exists a suitable maximal Abelian subalgebra \( \mathcal{D} \subset \mathcal{M} \), a Umegaki conditional expectation \( E : \mathcal{M} \mapsto \mathcal{D} \), and a Markov measure \( \mu \) on \( \text{spec}(\mathcal{D}) \) such that \( \varphi = \varphi_\mu \circ E \), the Markov state \( \varphi_\mu \), being the state on \( \mathcal{D} \) arising from the measure \( \mu \). An analogous result is true for non homogeneous quantum processes based on the forward or the backward chain. Besides, we determine the type of the von Neumann factors generated by GNS representation associated with translation invariant or periodic quantum Markov states.

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1. Introduction

It is known that, in quantum statistical mechanics, concrete systems are identified with states on corresponding algebras. In many cases, the algebra can be chosen to be a quasi–local algebra of observables. The states on these algebras satisfying Kubo–Martin–Schwinger boundary condition, as is known, describe equilibrium states of the quantum system under consideration. On the other hand, for classical systems with the finite radius of interaction, limiting Gibbs measures are know to be Markov random fields, see e.g. [15, 22, 29]. In connection with this, there is a problem to construct analogues of non commutative Markov chains, which arise from quantum statistical mechanics and quantum field theory in a natural way. This problem was firstly explored in [1] by introducing quantum Markov chains on the algebra of quasi–local...
observables. In the last decades, the investigation of quantum Markov processes had a considerable growth, in view of natural applications to quantum statistical mechanics, quantum field theory and quantum information theory. The reader is referred to [3]–[8], [12], [18] and the references cited therein, for recent development of the theory of quantum stochastic processes and their applications.

The investigation of a particular class of quantum Markov chains, called quantum Markov states, was pursued in [6, 7], where connections with properties of the modular operator of the states under consideration were established. This provides natural applications to temperature states arising from suitable quantum spin models, that is natural connections with the KMS boundary condition.\footnote{Most of the states arising from Markov processes considered in [18] describe ground states (i.e. states at zero temperature) of certain models of quantum spin chains.}

In [3], the most general one dimensional quantum Markov state has been considered. Among the other results concerning the structure of such states, the connection with classes of local Hamiltonians satisfying certain commutation relations and quantum Markov states has been obtained. The situation arising from quantum Markov states on the chain, describes one dimensional models of statistical mechanics with mutually commuting nearest neighbour interactions. Namely, one dimensional quantum Markov states are very near to be (diagonal liftings of) “Ising type” models, apart from noncommuting boundary terms, see Section 6 of [3].

In the present paper, we clarify the meaning of diagonalizability of one dimensional non homogeneous quantum Markov states. Namely, in Section 3 we prove that each non homogeneous quantum Markov state is diagonalizable, that is, for each Markov state $\varphi$ on the spin algebra $\mathcal{M} := \bigotimes_{j \in \mathbb{Z}} M_{d_j} (\mathbb{C})$ there exists a suitable maximal Abelian subalgebra $\mathcal{D} \subset \mathcal{M}$ (called diagonal in the sequel), a Umegaki conditional expectation $\mathcal{E} : \mathcal{M} \mapsto \mathcal{D}$ and a Markov measure $\mu$ on $\text{spec} (\mathcal{D})$ such that $\varphi = \varphi_{\mu} \circ \mathcal{E}$, the Markov state $\varphi_{\mu}$ being the state on $\mathcal{D}$ arising from the measure $\mu$. This allows us also to clarify a question raised in Section 6 of [3], relative to the rôle played by the non commuting boundary terms naturally arising from quantum Markov states, see Section 4 below.

The first diagonalizability result for quantum Markov states is contained in [31] for quantum Markov states generated by a Markov operator. In [20], the diagonalizability of more general one dimensional translation invariant quantum Markov states on the forward chain was
proved, without any statement about the Markovianity of the underlying classical measure. The proof in [20] of diagonalizability heavily depend on the commuting square condition (3.11) for the increasing sequence of Umegaki conditional expectations. The proof of (3.11), omitted in [20], easily follows by the direct ispection of the structure of local expected subalgebras and potentials, the last investigated in detail in [3].

It is known that factors of type III naturally arise in quantum field theory, statistical physics, representations of groups, see e.g. [24] and the references cited therein. Basically, the systematic investigation of the type of factors generated by the GNS representations of states naturally appearing in quantum field theory and in quantum statistical mechanics, was an interesting problem since the pioneering work of Araki and Wyss [11]. In [10, 28], a family of representations of uniformly hyperfinite algebras was constructed. They can be treated as free quantum lattice systems. In this case, most of the factors corresponding to these representations are of type III. However, the product states can be viewed as Gibbs states of Hamiltonian systems in which interactions between particles of the system are absent. So, it is natural to consider quantum lattice systems with nontrivial interactions, which lead us to treat firstly Markov states, as it was mentioned above. Simple examples of such systems are the Ising and Potts models. The quantum version of the last ones are diagonal liftings (i.e. they are constructed in a trivial way from the corresponding classical models, see e.g. Section 5) of classical processes. They have been studied in several papers, see e.g. [19, 25, 26, 32] and the reference cited therein.

Full analysis relative to the type of von Neumann algebras arising from general Markov states, or even states associated to quantum Markov processes on multidimensional lattices, is still an open problem.

In Section 5, we can partially solve this problem for physically relevant one dimensional quantum Markov states, that is for some examples of translation invariant or periodic states. Namely, Section 5 of this paper is devoted to determine the type of von Neumann factors arising from the GNS representations associated to quantum Markov states (for the classification of the type III factors, see [14]). This is done by using the simultaneous diagonalizability of the nearest neighbour terms of the interaction associated to quantum Markov states. This classification result in the corrected form established in Theorem 5.3) seems to be not known even for the Ising model, or for states arising from classical Markov chains, the last treated in some detail in Section 5.
Contrary to the situations present in literature, the states considered here appears as nondiagonal liftings of classical Markov processes which are diagonalizable in a nontrivial way by the result proven in Section 3, see Section 4 for a discussion about this point.\textsuperscript{2} However, it should be also noted that in [35], some properties of general diagonal state were studied in relation to representations of “large” groups of unitaries on Hilbert spaces, but concrete constructions of states were not considered there.

2. PRELIMINARIES

We start by recalling some well–known facts about inclusions of finite dimensional $C^*$–algebras.

Let $N \subset M$ be an inclusion of finite dimensional $C^*$–algebras. Consider the finite sets $\{p_i\}, \{q_j\}$ of all the minimal central projections of $M, N$ respectively. We symbolically write

$$\sum_j q_j N \subset \sum_i p_i M.$$  

Let us set $M_i := M_{p_i}, N_j := N_{q_j}, M_{ij} := M_{p_i q_j}, N_{ij} := N_{p_i q_j}$. Then, we have inclusions $N_{ij} \subset M_{ij}$ of finite dimensional factors. Hence,

$$M_{ij} \sim N_{ij} \otimes \tilde{N}_{ij} \quad (2.1)$$

for other finite dimensional factors $\tilde{N}_{ij}$.

Consider the canonical traces $\text{Tr}_M, \text{Tr}_N$, that is the traces which assign unit values on minimal projections. Notice that $\text{Tr}_M = \text{Tr}_M \circ E$ where $E$ is the conditional expectation of $M$ onto $\sum_{i,j} q_j (p_i M) q_j$ given by

$$E(x) = \sum_{i,j} q_j p_i x q_j.$$  

Taking into account the identification (2.1) and the last considerations, one can write symbolically

$$\text{Tr}_M = \bigoplus_{i,j} \left( \text{Tr}_{N_{ij}} \otimes \text{Tr}_{\tilde{N}_{ij}} \right).$$

\textsuperscript{2}Other nontrivial quantum liftings of classical Markov chains are constructed and studied in [5]. Apart from the standard applications to statistical mechanics, possible applications to quantum information theory are expected for the last processes.

\textsuperscript{3}The square root of the dimension of $\tilde{N}_{ij}$ is precisely the multiplicity of which the piece $q_j N \subset N$ appears into the piece $p_i M \subset M$. 
Further, the completely positive, \((\Tr_M, \Tr_N)\)-preserving linear map \(E_N^M\) of \(M\) onto \(N\) is given by
\[
E_N^M = \bigoplus_{i,j} \left( \id_{N_{ij}} \otimes \Tr_{\bar{N}_{ij}} \right).
\] (2.2)

Let \(\varphi\) be a positive functional on \(M\), together with its restriction \(\varphi\big|_N\) to \(N\). Consider the corresponding Radon–Nikodym derivatives \(T^\varphi_M, T^\varphi_N\) w.r.t. the canonical traces \(\Tr_M, \Tr_N\) respectively. We get
\[
T^\varphi_N = E_N^M(T^\varphi_M).
\] (2.3)

The starting point of our analysis is the \(C^*\)–infinite tensor product
\[
\mathcal{M} := \bigotimes_{j \in \mathbb{Z}} M_j^{C^*}
\]
where for \(j \in \mathbb{Z}\),
\[
M_j = M_{d_j}(\mathbb{C}).
\] (2.4)

With an abuse of notations, we denote with the same symbols elements of local algebras, and their canonical embeddings into bigger (local) algebras if this cause no confusion. For \(k \leq l\), we denote by \(M_{[k,l]}\) the local algebra relative to the segment \([k, l] \subset \mathbb{Z}\). Let \(\mathcal{S}(\mathcal{M})\) be the set of all states on \(\mathcal{M}\). The restriction of a state \(\varphi \in \mathcal{S}(\mathcal{M})\) to \(M_{[k,l]}\) will be denoted by \(\varphi_{[k,l]}\).

Suppose we have an increasing sequence \(\{N_{[k,l]}\}_{k \leq l}\) of local algebras such that
\[
N_{[k,k]} \subset M_{[k,k]} \equiv M_k, \quad N_{[k,k+1]} \subset M_{[k,k+1]},
\]
\[
M_{[k,l]} \subset N_{[k-1,l+1]} \subset M_{[k-1,l+1]}, \quad k \leq l.
\]

Consider an increasing sequence of \(C^*\)–algebras \(\{D_{[k,l]}\}_{k \leq l}\) where \(D_{[k,l]}\) is maximal Abelian in \(N_{[k,l]}\).

A diagonal algebra \(\mathcal{D} \subset \mathcal{M}\) is the Abelian \(C^*\)–subalgebra of \(\mathcal{M}\) obtained as
\[
\mathcal{D} := \left( \lim_{[k,l] \uparrow \mathbb{Z}} D_{[k,l]} \right)^{C^*}
\]
for \(D_{[k,l]}, N_{[k,l]}\) as above.

We deal only with \textit{locally faithful} states (i.e. states on \(\mathcal{M}\) with faithful restrictions to local subalgebras), even if most of the forthcoming analysis applies to non faithful states as well. For \(\varphi \in \mathcal{S}(\mathcal{M})\), locally faithful, the \textit{generalized conditional expectation}, or \(\varphi\)–expectation, \(\epsilon^\varphi_{k,l} : M_{[k,l+1]} \mapsto M_{[k,l]}\) is the completely positive \(\varphi\)–preserving linear
map associated to the inclusion $M_{[k,l]} \subset M_{[k,l+1]}$ defined in [2]. We refer the reader to that paper for the precise definition and further details on the Accardi–Cecchini generalized conditional expectations.

3. diagonalizability of Markov states

Let $\varphi \in \mathcal{S}(\mathcal{M})$ be a locally faithful state.

**Definition 3.1.** The state $\varphi \in \mathcal{S}(\mathcal{M})$ is said to be a Markov state if, for $k, l \in \mathbb{Z}, k < l$, we have

$$\epsilon_{k,l}^\varphi [M_{[k,l-1]}] = \text{id} M_{[k,l-1]}.$$

Quantum Markov states was firstly studied in [1, 6]. They are relevant in quantum statistical mechanics. The structure of quantum Markov states was intensively studied in [3, 7], where most of their properties were understood.

We briefly report useful results relative to the structure of Markov states. We refer the reader to [3] for details and proofs.

After taking the ergodic limit of the $\varphi$–expectations $\epsilon_{k,l}^\varphi$, and a decreasing martingale convergence theorem ([3], Section 5), it is possible to recover a sequence $\{\mathcal{E}^j\}_{j \in \mathbb{Z}}$ of transition expectations which are Umegaki conditional expectations

$$\mathcal{E}^j : M_j \otimes M_{j+1} \mapsto R_j \subset M_j$$

such that

$$\varphi_{[k,l]}(A_k \otimes \cdots \otimes A_l) = \varphi_{[k,k]}(\mathcal{E}^k(A_k \otimes \cdots \otimes \mathcal{E}^{l-1}(A_{l-1} \otimes A_l) \cdots))$$

for every $k, l \in \mathbb{Z}$ with $k < l$, and $A_k \otimes \cdots \otimes A_{l-1} \otimes A_l$ any linear generator of $M_{[k,l]}$. Let $\{P_{\omega_j}^j\}_{\omega_j \in \Omega_j}$ be the set of all minimal central projections of the range $R_j := \mathcal{R}(\mathcal{E}^j)$ of $\mathcal{E}^j$. Put

$$B_j := \sum_{\omega_j \in \Omega_j} P_{\omega_j}^j M_j P_{\omega_j}^j,$$

and

$$B_{[k,l]} := \bigoplus_{k \leq j \leq l} B_j.$$

Consider the conditional expectation $E^j : M_j \mapsto B_j$ given by

$$E^j(A) := \sum_{\omega_j \in \Omega_j} P_{\omega_j}^j a P_{\omega_j}^j.$$
Define
\[ E_{[k,l]} := \bigoplus_{k \leq j \leq l} E^j. \] (3.2)

By (3.1), it is easy to show that
\[ \varphi_{[k,l]} = \varphi_{[k,l]} \circ E_{[k,l]}. \]

After the identification \( M_j P^i \omega_j \cong P^i \omega_j M_j P^i \omega_j \) (i.e. the reduced algebra \( M_j P^i \omega_j \) acting on \( P^i \omega_j \mathbb{C}^d_j \)), we have
\[ M_j P^i \omega_j = N^i \omega_j \otimes \bar{N}^i \omega_j \]
for finite dimensional factors \( N^i \omega_j, \bar{N}^i \omega_j \). We can write after the last identifications,
\[ B_{[k,l]} := \bigoplus_{\omega_j, \ldots, \omega_l} (N^k \omega_k \otimes \bar{N}^k \omega_k) \otimes \cdots \otimes (N^l \omega_l \otimes \bar{N}^l \omega_l). \] (3.3)

Consider the potentials \( \{ h_{M[k,l]} \}_{k \leq l} \) obtained by the formula
\[ \varphi_{[k,l]} = \text{Tr}_{M_{[k,l]}} (e^{-h_{M[k,l]}}). \] (3.4)

Then \( h_{M[k,l]} \) has the nice decomposition
\[ h_{M[k,l]} = \bigoplus_{\omega_k, \ldots, \omega_l} h^k \omega_k \otimes h^k \omega_k, \omega_{k+1} \otimes \cdots \otimes h^{l-1} \omega_{l-1}, \omega_l \otimes \hat{h}^l \omega_l \] (3.5)
for selfadjoint elements \( h^j \omega_j, \hat{h}^j \omega_j, h^j \omega_j, \omega_{j+1} \) localized in \( N^j \omega_j, \bar{N}^j \omega_j, \hat{N}^j \omega_j \otimes N^j+1 \omega_j \) respectively.

Defining
\[ H_j := \sum_{\omega_j} P^i \omega_j (h^j \omega_j \otimes I) P^i \omega_j, \quad \hat{H}_j := \sum_{\omega_j} P^i \omega_j (I \otimes \hat{h}^j \omega_j) P^i \omega_j \]
\[ H_{j,j+1} := \sum_{\omega_j, \omega_{j+1}} (P^j \omega_j \otimes P^{j+1} \omega_{j+1}) (I \otimes h^j \omega_j, \omega_{j+1} \otimes I) (P^j \omega_j \otimes P^{j+1} \omega_{j+1}) \]
we find sequences of selfadjoint operators \( \{ H_j \}_{j \in \mathbb{Z}}, \{ \hat{H}_j \}_{j \in \mathbb{Z}} \) localized in \( M_{[i,j]} \equiv M_j \), and \( \{ H_{j,j+1} \}_{j \in \mathbb{Z}} \) localized in \( M_{[j,j+1]} \) respectively, satisfying the commutation relations
\[ [H_j, H_{j,j+1}] = [H_{j,j+1}, \hat{H}_j] = [H_j, \hat{H}_j] = [H_{j,j+1}, H_{j+1,j+2}] = 0, \] (3.6)
such that
\[ h_{M[k,l]} = H_k + \sum_{j=k}^{l-1} H_{j,j+1} + \hat{H}_l, \quad (3.7) \]
for each \( k \leq l \).

In Section 5 of [3] it is proven also the converse. Namely, if \( \varphi \in \mathcal{M} \) is locally faithful, with potentials having the form (3.7), for addenda localized as above, and satisfying the commutation relations (3.6), then it is a Markov state.

We are ready to prove the diagonalizability result for quantum Markov states.

**Theorem 3.2.** Let \( \varphi \in \mathcal{S}(\mathcal{M}) \) be a Markov state. Then there exists a diagonal algebra \( \mathcal{D} \subset \mathcal{M} \), a classical Markov process, with Markov measure \( \mu \) on \( \text{spec}(\mathcal{D}) \) w.r.t. the same order–localization of \( \mathbb{Z} \), and a Umegaki conditional expectation \( \mathcal{E} : \mathcal{M} \mapsto \mathcal{D} \) such that \( \varphi = \varphi_\mu \circ \mathcal{E} \), where \( \varphi_\mu \) is the state on \( \mathcal{D} \) corresponding to the measure \( \mu \).

**Proof.** Let \( R_j \) be the range of the (Umegaki) transition expectation \( \mathcal{E}_j \), with relative commutant \( R_j^c := R_j \mathcal{M} \). Define
\[ N_{[k,k]} := Z(R_k), \quad N_{[k,k+1]} := R_k^c \otimes R_{k+1}, \]
\[ N_{[k,l]} := R_k^c \otimes M_{[k+1,l-1]} \otimes R_l, \quad k < l + 1. \]

For each \( k \leq j < l \), and \( \omega_j \in \Omega_j \), choose a maximal Abelian subalgebra \( D_{\omega_j} \) of \( N_{\omega_j} \otimes N_{\omega_j+1} \) containing \( h_{\omega_j} \). Put
\[ D_{[k,k]} := N_{[k,k]} \equiv Z(R_k), \]
\[ D_{[k,l]} := \bigoplus_{\omega_k, \ldots, \omega_l} (D_{\omega_k} \otimes D_{\omega_{k+1}} \otimes \cdots \otimes D_{\omega_{l-1}} \otimes R_l), \quad k < l, \]
\[ \mathcal{D} := \left( \lim_{[k,l] \uparrow \mathbb{Z}} D_{[k,l]} \right)^{C^*}. \]

According to our definition, \( \mathcal{D} \) is a diagonal algebra of \( \mathcal{M} \) as the \( D_{[k,l]} \) are increasing and maximal Abelian in the \( N_{[k,l]} \). Consider the potentials \( h_{N_{[k,l]}} \) associated to the restrictions \( \varphi_{\mid N_{[k,l]} \rangle} \). We get by (2.3),
\[ e^{-h_{N_{[k,l]}}} = E_{N_{[k,l]}}^{M_{[k,l]}}(e^{-h_{M_{[k,l]}}}). \]

Taking into account (2.2), (3.5), we obtain
\[ h_{N_{[k,l]}} = K_k + \sum_{j=k}^{l-1} H_{j,j+1} + \hat{K}_l \quad (3.8). \]
for
\[
K_j := - \sum_{\omega_j} \ln \left( \text{Tr} \, N_{\omega_j}^j \, e^{-h_{\omega_j}^j} \right) P^j_{\omega_j},
\]
(3.9)
\[
\hat{K}_j := - \sum_{\omega_j} \ln \left( \text{Tr} \, \bar{N}_{\omega_j}^j \, e^{-\hat{h}_{\omega_j}^j} \right) P^j_{\omega_j}.
\]
(3.10)

Summarizing, by restricting ourselves to the sequence \(\{N_{[k,l]}\}_{k \leq l}\), we find a collection \(\{h_{N_{[k,l]}}\}_{k \leq l}\) of mutually commuting potentials, with \(h_{N_{[k,l]}} \in D_{[k,l]}\), arising from a nearest neighbour interaction, see (3.6), (3.8), (3.9). Namely, \(\{h_{N_{[k,l]}}\}_{k \leq l} \subset \mathfrak{D}\).

Let \(E_{k,l} : N_{[k,l]} \mapsto D_{[k,l]}\) be the canonical conditional expectation of \(N_{[k,l]}\) onto the maximal abelian subalgebra \(D_{[k,l]}\). We have
\[
\varphi|_{N_{[k,l]}} = \text{Tr} _{N_{[k,l]}} (e^{-h_{N_{[k,l]}}} \cdot ) = \text{Tr} _{N_{[k,l]}} (e^{-h_{N_{[k,l]}}} E_{k,l}(\cdot)) .
\]
(3.11)

Further,
\[
E_{k-1,l+1} [N_{[k,l]}] = E_{k,l}.
\]
(3.12)

Indeed by projectivity,
\[
E_{k,l} = E_{k,l} \circ E_{[k,l]}
\]
with \(E_{[k,l]}\) given in (3.2). The compatibility condition (3.11) immediately follows taking into account (3.3).

Let \(\varphi_\mu := \varphi|_{\mathfrak{D}}\), where \(\mu\) is the probability measure on \(\text{spec}(\mathfrak{D})\) associated to \(\varphi|_{\mathfrak{D}}\). By (3.11),
\[
\mathcal{E}_0 := \lim_{[k,l] \to \mathbb{Z}} E_{k,l}
\]
is well–defined on \(\bigcup_{k,l} N_{[k,l]}\) (which is a dense subalgebra of \(\mathfrak{M}\)), and extends by continuity to a Umegaki conditional expectation \(\mathcal{E}\) of \(\mathfrak{M}\) onto \(\mathfrak{D}\). Further, by (3.10), \(\varphi = \varphi \circ \mathcal{E}_0 \equiv \varphi_\mu \circ \mathcal{E}_0\) on localized elements of \(\mathfrak{M}\). By a standard continuity argument, we obtain \(\varphi = \varphi_\mu \circ \mathcal{E}\). The
The fact that $\mu$ is a Markov measure on $\text{spec}(D)$ w.r.t the order–localization of $Z$ is checked in the appendix. □

The diagonalizability result for translation invariant quantum Markov states is contained in [20] for homogeneous processes on the forward chain, without any mention about the Markovianity of the underlying classical processes. As in our situation, the proof of the diagonalizability in Theorem 4.1 of [20] heavily depends on the commuting square condition (3.11). In the most general situation considered here (hence, including the case considered in [20]), (3.11) easily follows by a direct inspection of the structure of local expected subalgebras and potentials investigated in detail in [3], and reported in the present paper for the convenience of the reader.

We end the present section by noticing that an analogous result can be proven for non homogeneous processes on one–side (forward or backward) ordered chains. By looking at support–projections of local restrictions of states (or equivalently by defining the Markov property directly in terms of Umegaki transition expectations, see [3], Definition 2.1), it is straightforward to prove the diagonalizability result for general (non necessarily locally faithful) Markov states on ordered chains.

4. FROM QUANTUM MARKOV STATES TO QUANTUM STATISTICAL MECHANICS

In standard models of statistical mechanics describing classical or quantum spin systems, one considers on a quasi–local algebra $\mathfrak{A}$, local Hamiltonians $\{h_\Lambda\}_{\Lambda \subset Z^d}$, $\Lambda$ bounded, satisfying suitable conditions. Then, one constructs the finite volume Gibbs states (to simplify matter we reduce ourselves to the case with inverse temperature $\beta = 1$)

$$\varphi_\Lambda := Z^{-1} \text{Tr}_{\mathfrak{A}_\Lambda} (e^{-h_\Lambda} \cdot),$$

(4.1)

$Z$ being the partition function, see e.g. [13, 30, 32]. The local Hamiltonians $h_\Lambda$ are usually based on a interaction term describing the mutual interaction of all spins in the volume $\Lambda$, and a boundary term arising from some fixed boundary conditions imposed to spins surrounding the bounded region $\Lambda$. After extending the $\varphi_\Lambda$ to all of $\mathfrak{A}$, each $*$–weak limit $\lim_{\Lambda_n \uparrow Z^d} \varphi_{\Lambda_n}$ of the net $\{\varphi_\Lambda\}_{\Lambda \subset Z^d}$ is an infinite volume Gibbs state, or a DLR state (KMS state in quantum setting) for the system under consideration see [16, 17, 21, 23].

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The proof of (3.11), missing in [20], would follow by general results contained in Section I.1 of [35].
In the classical case, it is stated for finite range interactions, that an infinite volume Gibbs state arises from a $\delta$-Markov process and vice-versa, $\delta$ being the range of the interaction, see e.g. [15, 22, 29]. For ordered unidimensional chains, a quantum analogue of that result is proven in [3], provided that the “leading” terms $\{H_{j,j+1}\}_{j \in \mathbb{Z}}$ commute with each other, see also [4] for connected results relative to the multidimensional case. In quantum setting, it can happen that $\{h_\Lambda\}_{\Lambda \subset \mathbb{Z}^d}$ does not generate a commutative algebra due to the boundary effects (see [3], Section 6).

In the present paper we have shown that, starting from a quantum Markov state on $\mathcal{M} \equiv \bigotimes_{j \in \mathbb{Z}} M_{d_j}$, we can recover a nontrivial filtration $\{N_{[k,l]}\}_{k \leq l}$ of $\mathcal{M}$ and an increasing sequence $\{D_{[k,l]}\}_{k \leq l}$ of Abelian algebras with the $D_{[k,l]}$ non trivial (i.e. not arising from the standard tensor structure of $\mathcal{M}$) maximal Abelian subalgebras of the $N_{[k,l]}$ such that $\varphi$ is the lifting of $\varphi_{[D]}$, the last one being a classical Markov state on $\mathcal{D} := \left(\lim_{[k,l] \uparrow \mathbb{Z}} D_{[k,l]}\right)^{C^*}$, constructed by the compatible sequence of U-megaki conditional expectations $E_{k,l} : N_{[k,l]} \mapsto D_{[k,l]}$ preserving the canonical trace $\text{Tr}_{N_{[k,l]}}$. This is possible as the (nearest neighbour) potentials $\{h_{N_{[k,l]}}\}_{k \leq l}$ generate a commutative subalgebra of $\mathcal{D}$.

As it is straightforwardly seen, the converse is also true. Namely, one can start with any fixed filtration $\{N_{[k,l]}\}_{k \leq l}$ as above, together with a nearest neighbour interaction

$$h_{k,l} = \sum_{j=k}^{l-1} H_{j,j+1} \quad (4.2)$$

with $\{H_{j,j+1}\}_{j \in \mathbb{Z}}$ mutually commuting. By adding boundary terms $K_k$ and $\tilde{K}_l$ to (4.2) such that all addenda commute with each other, one can construct for finite regions $\Lambda = [k,l]$, finite volume Gibbs states $\{\varphi_\Lambda\}_{\Lambda \subset \mathbb{Z}}$ as in (4.1), associated to the Hamiltonian

$$h_{[k,l]} = K_k + h_{k,l} + \tilde{K}_l$$

having the same form as in (3.8).

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6The restriction of $\text{Tr}_{N_{[k,l]}}$ to $D_{[k,l]}$ is the uniform measure which assigns the same weight 1 to the minimal projections of $D_{[k,l]}$.}
Each $\ast$–weak limit point of the sequence $\{\varphi_{\Lambda}\}_{\Lambda \subseteq \mathbb{Z}}$, gives rise to a Markov state on $\mathcal{M}$ which is the lifting of a classical Markov state on a “diagonal” algebra, due to the commutativity of the $h_{[k,l]}$.

Taking into account the above considerations, one can assert that each quantum Markov state on the ordered chain arises from some underlying (non trivial) classical Ising model.

The quantum character of such states manifests itself in the following way. In order to construct (or recover) such states, one should take into account various nontrivial local filtrations of $\mathcal{M}$, together with various (commuting) boundary terms.

If one chooses to investigate quantum Markov states by considering only the natural filtration $\{M_{[k,l]}\}_{k \leq l}$ of $\mathcal{M}$, one obtains a leading term as that in (4.2). But non commuting boundary terms could naturally arise in (3.7), see the examples in Section 6 of [3]. In the constructive approach, the appearance of such non commuting boundary terms cannot be disregarded in order to obtain general infinite volume Gibbs states for nearest neighbour interactions. Here, it should be noted that if the nearest neighbour model is translation invariant or periodic, then according to Theorem 1 of [9], we will have a unique quantum Markov state corresponding to the considered model. Namely, for translation invariant or periodic models, the construction of quantum Markov states does not depend on boundary terms.

5. Types of von Neumann algebras associated with quantum Markov states

In this section we investigate the type of von Neumann factors generated by the GNS representation associated with the quantum Markov states.

Let us consider the $C^*$–algebra $\mathcal{M}$ defined in Section 2. The shift automorphism of the algebra $\mathcal{M}$ will be denoted by $\theta$. A state $\varphi \in \mathcal{S}(\mathcal{M})$ is called $l$–periodic if $\varphi(\theta^j(x)) = \varphi(x)$ for all $x \in \mathcal{M}$. If $l = 1$, $\varphi \in \mathcal{S}(\mathcal{M})$ is translation invariant. Notice that, in order to have $l$–periodicity, it is necessary $d_{j+l} = d_j$, $j \in \mathbb{Z}$, for the $d_j$ in (2.4). Further, we have for localized Hamiltonians (3.5), and their leading terms (4.2),

$$h_{M_{[j+l,k+l]}} = h_{M_{[j,k]}} \quad h_{j+l,k+l} = h_{j,k}$$

Notice that, besides the limiting subsequence $[k_n, l_n] \uparrow \mathbb{Z}$, the thermodinamical limits might depend also on the chosen boundary terms.

For Markov states with multidimensional indices, where there is no canonical order (i.e. for the Markov fields considered in [4]), it is expected the appearance of non diagonalizable examples.
for all \( j, k \in \mathbb{Z} \). In order to avoid the trivial situation, we consider only non–tracial locally faithful translation invariant or \( l \)–periodic Markov states. This means that \( h_{0,l} \neq C I \), that is \( h_{0,l} \) is nontrivial.

We are going to connect the type of the von Neumann factor \( \pi_\varphi(\mathfrak{M})'' \) with properties of the spectrum \( \sigma(h_{0,l}) \) of the fundamental block \( h_{0,l} \) of the leading term (4.2) of the canonical Hamiltonian associated to \( \varphi \).

Due to commuting properties of the \( h_{M[-n,n]} \) (see (3.6)), the following strong limit
\[
\sigma_{\varphi}(A) = \lim_{n \to \infty} e^{ith_{M[-n,n]}A}e^{-ith_{M[-n,n]}}, \quad A \in \mathfrak{M}
\]
exists. Further, \( \varphi \) is a KMS (at inverse temperature 1) for \( \sigma_{\varphi} \). According to Theorem 1 of [9], it is the unique KMS state for \( \sigma_{\varphi} \), and \( \pi_\varphi(\mathfrak{M})'' \) is a factor. Notice that we have also
\[
\sigma_{\varphi}(A) = \lim_{n \to \infty} e^{ith_{-n,n}A}e^{-ith_{-n,n}}.
\]

The extension to all of \( \pi_\varphi(\mathfrak{M})'' \), denoted also by \( \sigma_{\varphi} \), is precisely the modular group associated to the normal extension of \( \varphi \) (denoted also by \( \varphi \)) to \( \pi_\varphi(\mathfrak{M})'' \).

Let \( sp(\tau) \) be the Arveson spectrum of the action \( \tau \) of a locally compact group on a \( C^* \)–algebra. Denote \( \sigma^n := \text{ad}(e^{ith_{-l,n}}) \), \( l \) being the period of the state under consideration.

\textbf{Lemma 5.1.} \textit{In the above situation, we have}
\[
sp(\sigma_{\varphi}) \subset \bigcup_n \left( \sigma(h_{-ln,ln}) - \sigma(h_{-ln,ln}) \right).
\]

\textit{Proof.} By passing to the regrouped algebra, we can consider \( l = 1 \). Taking into account the commuting properties of the interaction, we have
\[
sp(\sigma_{\varphi}) = \bigcup_n \bigcup_{A \in M[-n,n]} sp(\sigma_{\varphi}(A)) = \bigcup_n \bigcup_{A \in M[-n,n]} sp(\sigma^{n+1}(A))
\]
\[
\subset \bigcup_n \bigcup_{A \in M[-n+1]} sp(\sigma^{n+1}(A)) = \bigcup_n sp(\sigma^{n+1}M[-n+1]).
\]

The proof follows by Proposition 14.13 of [34]. \hfill \Box

\textbf{Lemma 5.2.} \textit{Let} \( \{x_1, \ldots, x_n\} \subset \mathbb{R} \backslash \{0\} \) \textit{such that} \( x_i/x_j \in \mathbb{Q} \) \textit{for all} \( i, j \). \textit{Then}
\[
\{x_1, \ldots, x_n\} \subset \mathbb{Z} \ln \alpha \tag{5.1}
\]
\footnote{For the definition of the Arveson spectrum \( sp(\tau) \), as well as \( sp(\tau(A)) \), see e.g. [27].}
for some \( \alpha \in (0, 1) \).

Proof. From our assumptions, we have

\[
\frac{x_1}{x_i} = \frac{p_i}{q_i}, \quad i = 2, \ldots, n,
\]

where \( p_i \in \mathbb{N} \setminus \{0\}, q_i \in \mathbb{Z} \setminus \{0\} \). Define

\[
\alpha := e^{-\frac{\left| x_1 \right|}{\prod_{j=2}^{n} p_j}}.
\]

Then

\[
x_1 = -\text{sign}(x_1) \left( \prod_{j=2}^{n} p_j \right) \ln \alpha,
\]

\[
x_i = -q_i \left( \prod_{j=2, j \neq i}^{n} p_j \right) \ln \alpha, \quad i = 2, \ldots, n.
\]

Let \( h_{0, i} \) be the fundamental block of the leading term of the canonical Hamiltonian associated to the locally faithful Markov state \( \varphi \). Consider, for \( h, k, h', k' \in \sigma(h_{0, i}) \) with \( h \neq k, h' \neq k' \), the following fractions \( \frac{h-k}{h'-k'} \).

**Theorem 5.3.** Let \( \varphi \in S(\mathcal{M}) \) be a locally faithful Markov state. The following assertions hold true.

(i) If \( \left\{ \frac{h-k}{h'-k'} \right\} \subset \mathbb{Q} \), then \( \pi_{\varphi}(\mathcal{M})'' \) is a type III\( _\lambda \) factor for some \( \lambda \in (0, 1) \).

(ii) If \( \pi_{\varphi}(\mathcal{M})'' \) is a type III\( _1 \) factor, then \( \left\{ \frac{h-k}{h'-k'} \right\} \not\subset \mathbb{Q} \).

Proof. As before, we can consider only translation invariant Markov states. By applying Theorem 3.1 of [33], we get for the Connes invariant \( \Gamma \) (see [14]), \( \Gamma(\pi_{\varphi}(\mathcal{M})'') = \Gamma(\sigma^{\varphi}) = \text{sp}(\sigma^{\varphi}) \). Further, this means also that \( \pi_{\varphi}(\mathcal{M})'' \) is a type III\( _\lambda \) factor, \( \lambda \in (0, 1) \), as we are considering non–tracial states. Then, it is enough to prove the former, the latter being a direct consequence of the former.

Let \( \left\{ \frac{h-k}{h'-k'} \right\} \subset \mathbb{Q} \) be satisfied. By Lemma 5.2,

\[
\left\{ h - k \mid h, k \in \sigma(H_{0, 1}) \right\} \subset \mathbb{Z} \ln \alpha
\]

for some \( \alpha \in (0, 1) \).

\[
\text{The best } \alpha \text{ in (5.1) is the minimum of the } \alpha \in (0, 1) \text{ such that (5.1) is true.}
\]

\[\text{It can be obtained by changing the reference element, and compute all the corresponding } \alpha \text{ in (5.2) by taking relatively prime pairs } p_i, q_i. \quad \text{The minimum we are looking for, is precisely the smallest among all these } \alpha.\]
From the simultaneous diagonalizability of the $H_{i,i+1}$, we find that
\[ \sigma(h_{-n,n}) \subset \left\{ \sum_{i=-n}^{n-1} h_i \mid h_i \in \sigma(H_{0,1}) \right\}. \]

Then we have
\[ \sigma(h_{-n,n}) - \sigma(h_{-n,n}) \subset \left\{ \sum_{i=-n}^{n-1} (h_i - k_i) \mid h_i, k_i \in \sigma(H_{0,1}) \right\} \subset \mathbb{Z} \ln \alpha. \]

According to Lemma 5.1, we infer that $\text{sp}(\sigma) \subset \mathbb{Z} \ln \alpha$, that is $\text{sp}(\sigma)$ is discrete. Hence, there is a number $m \in \mathbb{N} \setminus \{0\}$ such that $\text{sp}(\sigma) = \mathbb{Z} \ln \lambda$, with $\lambda := \alpha^m$. Thus $\pi_\varphi(\mathcal{M})^n$ is a type III$_\lambda$ factor. \(\square\)

Here, it should be noted that one might argue that the spectrum $\sigma(h_{0,i})$ of the fundamental block of the Hamiltonian associated to the periodic Markov state $\varphi$, completely determines the type of $\pi_\varphi(\mathcal{M})^n$. Unfortunately, we are not able to prove the reverse statements in Theorem 5.3, contrarily to that is asserted in literature.\(^{11}\)

Even if one can construct by results in Section 4 of [3], a wide class of quantum Markov states to which the previous results apply, in order to explain some natural applications of Theorem 5.3 to pre–assigned models, we are going to consider some natural examples.

5.1. **Ising model.** In this situation,
\[ \mathcal{M} = \bigotimes_{\mathbb{Z}} M_2(\mathbb{C})^{C^*}. \]

The Ising model on $\mathbb{Z}$ is defined by the following formal Hamiltonian
\[ H = - \sum_{j \in \mathbb{Z}} J_{j,j+1} \sigma_z^j \sigma_z^{j+1}, \]
where $J_{j,j+1} \in \mathbb{R}$ are coupling constants and $\sigma_z^j$ is the Pauli matrix $\sigma_z$ on the $j$–th site. Further, we suppose that the coupling constants are defined by
\[ J_{j,j+1} = \begin{cases} J_1, & \text{if } j \in 2\mathbb{Z}, \\ J_2, & \text{if } j \in 2\mathbb{Z} + 1, \end{cases} \]
where $J_1, J_2 \in \mathbb{R}$. It is known (see [9]) that for the given Hamiltonian there exists a unique Gibbs state $\varphi$ on $\mathcal{M}$ which is 2–periodic. In this

\(^{11}\)For example, the proof of the connected results in [26] seems to be incomplete.
case, the operators $H_{j,j+1}$ have the following form

$$H_{j,j+1} = \begin{cases} 
\begin{pmatrix} J_1 & 0 & 0 & 0 \\
0 & -J_1 & 0 & 0 \\
0 & 0 & -J_1 & 0 \\
0 & 0 & 0 & J_1 
\end{pmatrix}, & \text{if } j \in 2\mathbb{Z}, \\
\end{cases}$$

$$\begin{cases} 
\begin{pmatrix} J_2 & 0 & 0 & 0 \\
0 & -J_2 & 0 & 0 \\
0 & 0 & -J_2 & 0 \\
0 & 0 & 0 & J_2 
\end{pmatrix}, & \text{if } j \in 2\mathbb{Z} + 1, \\
\end{cases}$$

The spectrum of $H_{j,j+1}$ is $\{J_1, -J_1\}$ if $j \in 2\mathbb{Z}$, $\{J_2, -J_2\}$ if $j \in 2\mathbb{Z} + 1$ respectively. Now if $J_1/J_2$ is rational, the rationality condition of Theorem 5.3 is satisfied, this infers that the von Neumann factor $\pi_\varphi(\mathcal{M})''$ is of type III$_\lambda$, for some $\lambda \in (0, 1)$.

5.2. Markov process. Consider a discrete Markov process with the state space $d := \{1, \ldots, d\}$ and the transition probabilities defined by means of a stochastic matrix $P = (p_{ij})_{i,j=1}^d$ with (not all equal) $p_{ij} > 0$ for all $i, j$. Consider the canonical inclusion

$$\mathcal{D} = \bigotimes_{\mathbb{Z}} \mathbb{C}_d^{c^*} \subset \mathcal{M} = \bigotimes_{\mathbb{Z}} \mathcal{M}_d(\mathbb{C})^{c^*}.$$

Here, $\mathcal{D} \sim C(\Omega)$, where $\Omega = \prod_{\mathbb{Z}} d$. Let $\mu_P$ be the translation invariant Markov measure on $\Omega$ determined by the transition matrix $P$. Define the diagonal lifting of the classical process associated to $P$ as

$$\varphi(A) := \int_\Omega \mathcal{E}(A)(\omega)\mu_P(d\omega),$$

where $\mathcal{E}$ is the canonical Umegaki conditional expectation of $\mathcal{M}$ onto the Abelian algebra $\mathcal{D}$. It is clear that such state is a translation invariant quantum Markov state.

It is not hard to check that the corresponding $H_{j,j+1}$ operator has the form

$$H_{j,j+1} = \begin{pmatrix} B^{(1)} & 0 & \cdots & 0 \\
0 & B^{(2)} & 0 & \cdots \\
& \cdots & \ddots & \cdots \\
0 & 0 & \cdots & B^{(d)} 
\end{pmatrix},$$
where $B^{(k)} = (b_{ij,k})_{i,j=1}^{d}$, $k = 1, \ldots, d$ are $d \times d$ diagonal matrices such that

$$b_{ij,k} = \begin{cases} -\ln p_{k,i}, & i = j, \ i = 1, \ldots, d \\ 0, & i \neq j \end{cases}$$

If there exist integers $m_{ij} \, i, j \in \{1, \ldots, d\}$, and some number $\alpha \in (0, 1)$ such that $p_{11} p_{i,j} = \alpha^{m_{ij}}$, then we easily see that the rationality condition of Theorem 5.3 is satisfied, this means that the von Neuman factor $\pi(\mathfrak{M})''$ is of type III$\lambda$, for some $\lambda \in (0, 1)$. This result extends a result of [19].

6. APPENDIX

For the convenience of the reader, we verify that the measure $\mu$ on $\mathfrak{D}$ associated to $\varphi|_{\mathfrak{D}}$ is a Markov measure on $\text{spec}(\mathfrak{D})$ w.r.t. the order–localization of $\mathfrak{Z}$.

For our purpose, it suffices to verify that for every $k \leq n \leq l$ in $\mathfrak{Z}$ and $A \in \text{spec}(D_{[k,n]})$ and $B \in \text{spec}(D_{[n,l]})$ we have for the conditional probability,

$$P(A \cap B|\bar{\omega}_n) = P(A|\bar{\omega}_n) P(B|\bar{\omega}_n),$$

where $\bar{\omega}_n$ is a fixed point in $\text{spec}(Z(R_n)) \equiv \Omega_n$. In order to make computations, we should see the past $D_{[k,n]}$, the present algebra $D_{[n,n]} \equiv Z(R_n)$, and the future algebra $D_{[n,l]}$ inside the ambient algebra $D_{[k,l]}$. In such a situation

$$\text{spec}(D_{[k,l]}) = \bigcup_{\omega_k, \ldots, \omega_l} S^{k}_{\omega_k, \omega_{k+1}} \times \cdots \times S^{l}_{\omega_{l-1}, \omega_l},$$

Taking into account Formulae (3.5) and (3.8), we compute for

$$f := \sum_{\omega_k, \ldots, \omega_l} \chi^{k}_{\omega_k, \omega_{k+1}} \times \cdots \times \chi^{l-1}_{\omega_{l-1}, \omega_l} f^{k}_{\omega_k, \omega_{k+1}} \otimes \cdots \otimes f^{l-1}_{\omega_{l-1}, \omega_l},$$

$$\varphi(f) = \sum_{\omega_k, \ldots, \omega_l} \left( \int_{S^{k}_{\omega_k, \omega_{k+1}}} T^{k}_{\omega_k, \omega_{k+1}} f^{k}_{\omega_k, \omega_{k+1}} \right) \times \cdots \times \left( \int_{S^{l-1}_{\omega_{l-1}, \omega_l}} T^{l-1}_{\omega_{l-1}, \omega_l} f^{l-1}_{\omega_{l-1}, \omega_l} \right) \quad (6.1)$$

where the densities $T$ are positive functions, and $\int$ assigns weight 1 to atoms.
We start by noticing that, inside $D_{[k,l]}$, we get for $P^n_{\bar{\omega}_n}$,

$$P^n_{\bar{\omega}_n} = \sum_{\omega_k, \ldots, \omega_{n-1}, \omega_{n+1}, \ldots, \omega_l} \mathcal{X}_{S_k}^{\omega_{k+1}} \otimes \cdots \otimes \mathcal{X}_{S_{n-1}}^{\bar{\omega}_n} \otimes \mathcal{X}_{S_n}^{\omega_{n+1}} \otimes \cdots \otimes \mathcal{X}_{S_{l-1}}^{\bar{\omega}_l}. \tag{6.2}$$

Now, if $A, B \in \text{spec}(D_{[k,l]})$ are localized in the past and in the future of $n$ respectively, we have inside $D_{[k,l]}$,

$$X_A P^n_{\bar{\omega}_n} = \sum_{a \in A, \rho(a) = \bar{\omega}_n} \mathcal{X}_{\{a_k^{\omega_k(a), \omega_{k+1}(a)}\}} \otimes \cdots \otimes \mathcal{X}_{\{a_{n-1}^{\omega_{n-1}(a), \bar{\omega}_n}\}} \otimes \mathcal{X}_{S_{n-1}}^{\omega_{n+1}} \otimes \cdots \otimes \mathcal{X}_{S_{l-1}}^{\bar{\omega}_l},$$

$$P^n_{\bar{\omega}_n} X_B = \sum_{b \in B, \lambda(b) = \bar{\omega}_n} \mathcal{X}_{S_k}^{\omega_{k+1}} \otimes \cdots \otimes \mathcal{X}_{S_{n-1}}^{\bar{\omega}_n} \otimes \mathcal{X}_{\{b_i^{\omega_i(b), \omega_i+1(b)}\}} \otimes \cdots \otimes \mathcal{X}_{\{b_{l-1}^{\omega_{l-1}(b), \omega_l(b)}\}} \tag{6.3}$$

$$X_A P^n_{\bar{\omega}_n} X_B = \sum_{a \in A, \rho(a) = \bar{\omega}_n} \mathcal{X}_{\{a_k^{\omega_k(a), \omega_{k+1}(a)}\}} \otimes \cdots \otimes \mathcal{X}_{\{a_{n-1}^{\omega_{n-1}(a), \bar{\omega}_n}\}} \otimes \mathcal{X}_{\{b_i^{\omega_i(b), \omega_i+1(b)}\}} \otimes \cdots \otimes \mathcal{X}_{\{b_{l-1}^{\omega_{l-1}(b), \omega_l(b)}\}}.$$

Here,

$$a = a_k^{\omega_k(a), \omega_{k+1}(a)} \times \cdots \times a_{n-1}^{\omega_{n-1}(a), \bar{\omega}_n},$$

$$b = b_i^{\omega_i(b), \omega_i+1(b)} \times \cdots \times b_{l-1}^{\omega_{l-1}(b), \omega_l(b)}.$$

are generic points of $A, B$ respectively, and with these notations, $\rho(a) := \omega_n(a), \lambda(b) := \omega_l(b)$. 
Taking into account (6.1), (6.2) and (6.3), we have the following computations.

\[
\varphi(P^n_{\omega_n}) = \sum_{\omega_{k_1}, \ldots, \omega_{n-1}, \omega_n, \ldots, \omega_l} \left( \int_{S_{k_1}^{\omega_{k_1}}} T_{\omega_{k_1}}^{k_1} \omega_{k_1+1} \right) \times \cdots \times \left( \int_{S_{n-1}^{\omega_{n-1}}} T_{\omega_{n-1}}^{n-1} \omega_n \right) \\
\times \left( \int_{S_{n}^{\omega_n}} T_{\omega_n}^{n} \omega_{n+1} \right) \times \cdots \times \left( \int_{S_{l-1}^{\omega_{l-1}}} T_{\omega_{l-1}}^{l-1} \omega_l \right) \\
= \sum_{a \in A, \rho(a) = \omega_n} T_{\omega_k(a), \omega_{k+1}(a)}^{k}(a_{\omega_k(a), \omega_{k+1}(a)}) \times \cdots \times T_{\omega_{n-1}(a), \omega_n}^{n-1}(a_{\omega_{n-1}(a), \omega_n}) \\
\times T_{\omega_n, \omega_{n+1}(a)}^{n}(b_{\omega_n, \omega_{n+1}(a)}) \times \cdots \times T_{\omega_{l-1}(b), \omega_l(b)}^{l-1}(b_{\omega_{l-1}(b), \omega_l(b)}) \\
= \sum_{b \in B, \lambda(b) = \omega_n} T_{\omega_k(a), \omega_{k+1}(a)}^{k}(a_{\omega_k(a), \omega_{k+1}(a)}) \times \cdots \times T_{\omega_{n-1}(a), \omega_n}^{n-1}(a_{\omega_{n-1}(a), \omega_n}) \\
\times T_{\omega_n, \omega_{n+1}(b)}^{n}(b_{\omega_n, \omega_{n+1}(b)}) \times \cdots \times T_{\omega_{l-1}(b), \omega_l(b)}^{l-1}(b_{\omega_{l-1}(b), \omega_l(b)}) \\
= \sum_{a \in A, \rho(a) = \omega_n} T_{\omega_k(a), \omega_{k+1}(a)}^{k}(a_{\omega_k(a), \omega_{k+1}(a)}) \times \cdots \times T_{\omega_{n-1}(a), \omega_n}^{n-1}(a_{\omega_{n-1}(a), \omega_n}) \\
\times \left( \int_{S_{k}^{\omega_n}} T_{\omega_n, \omega_{n+1}}^{n} \right) \times \cdots \times \left( \int_{S_{l-1}^{\omega_{l-1}}} T_{\omega_{l-1}}^{l-1} \right) \\
= \sum_{b \in B, \lambda(b) = \omega_n} T_{\omega_k(a), \omega_{k+1}(a)}^{k}(a_{\omega_k(a), \omega_{k+1}(a)}) \times \cdots \times T_{\omega_{n-1}(a), \omega_n}^{n-1}(a_{\omega_{n-1}(a), \omega_n}) \\
\times \left( \int_{S_{n}^{\omega_n}} T_{\omega_n, \omega_{n+1}}^{n} \right) \times \cdots \times \left( \int_{S_{l-1}^{\omega_{l-1}}} T_{\omega_{l-1}}^{l-1} \right).
\[ \varphi(P_n^X B) = \sum_{\omega_k, \ldots, \omega_{n-1}} (\int_{S_{\omega_k, \omega_{k+1}}} T_{\omega_k, \omega_{k+1}}^k) \times \cdots \times (\int_{S_{\omega_{n-1}, \omega_n}} T_{\omega_{n-1}, \omega_n}^{n-1}) 
\times T_{\omega_n, \omega_{n+1}}^n(b_n) \times \cdots \times T_{\omega_{l-1}, \omega_l}^{l-1}(b_l) \times (\int_{S_{\omega_{l-1}, \omega_l}} T_{\omega_{l-1}, \omega_l}^{l-1}) \times \cdots \times (\int_{S_{\omega_n, \omega_{n+1}}} T_{\omega_n, \omega_{n+1}}^n(b_n) \times \cdots \times T_{\omega_{l-1}, \omega_l}^{l-1}(b_l)) \right). \]

Collecting together the last computations, we get

\[ P(A \cap B \vert \bar{\omega}_n) = \frac{\varphi(X_A P_n^X B)}{\varphi(P_n^X)} \]

which is the assertion.

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