SUBADDITIVITY OF KODAIRA DIMENSIONS FOR
FIBRATIONS OF THREE-FOLDS IN POSITIVE
CHARACTERISTICS

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Abstract. In this paper, we will study subadditivity of Kodaira dimensions in positive characteristics. We prove that for a separable fibration \( f : X \to Y \) from a smooth projective three-fold to a smooth projective surface or a curve, over an algebraically closed field \( k \) with \( \text{char} \ k > 5 \), if \( Y \) is of general type and \( S^0(X_{\bar{\eta}}, lK_{X_{\bar{\eta}}}) \neq 0 \) for some positive integer \( l \) where \( X_{\bar{\eta}} \) denotes the geometric generic fiber, then

\[ \kappa(X) \geq \kappa(Y) + \kappa(X_{\bar{\eta}}, K_{X_{\bar{\eta}}}) \]

under certain technical assumptions. We also get some general results under nefness and relative semi-ampleness conditions.

At the end of this paper we show a numerical criterion for a fibration to be birationally isotrivial, and give a new proof to \( C_{3,1} \) under the situation that general fibers are surfaces of general type, which has been proven by Ejiri (13).

Keywords: Kodaira dimension; positive characteristic; weak positivity; minimal model.

MSC: 14E05; 14E30.

1. Introduction

Let \( X \) be a projective variety over a field \( k \), \( D \) a \( \mathbb{Q} \)-Cartier divisor on \( X \). The \( D \)-dimension \( \kappa(X, D) \) is defined as

\[ \kappa(X, D) = \begin{cases} -\infty, & \text{if for every integer } m > 0, |mD| = \emptyset; \\ \max \{ \dim \Phi_{|mD|}(X) | m \in \mathbb{Z} \text{ and } m > 0 \}, & \text{otherwise.} \end{cases} \]

If \( X \) has a smooth projective birational model \( \tilde{X} \), the Kodaira dimension \( \kappa(X) \) of \( X \) is defined as \( \kappa(\tilde{X}, K_{\tilde{X}}) \) where \( K_{\tilde{X}} \) denotes the canonical divisor. Kodaira dimension is one of the most important birational invariant in the classification theory.

Let \( f : X \to Y \) be a morphism between two schemes. For \( y \in Y \), let \( X_y \) denote the fiber of \( f \) over \( y \); and for a divisor \( D \) (resp. a sheaf \( \mathcal{F} \)) on \( X \), let \( D_y \) (resp. \( \mathcal{F}_y \)) denote the restriction of \( D \) (resp. \( \mathcal{F} \)) on the fiber \( X_y \). Throughout this paper, since \( Y \) frequently appears as an integral scheme, we use the special notation \( \eta \) and \( \bar{\eta} \) for the generic and geometric generic point of \( Y \) respectively. We say \( f \) is a fibration if \( f \) is a projective morphism such that \( f_* \mathcal{O}_X = \mathcal{O}_Y \).

For a fibration between two projective varieties over the field of complex numbers \( \mathbb{C} \), Iitaka conjectured that subadditivity of Kodaira dimensions holds:

Conjecture 1.1 (Iitaka conjecture). Let \( f : X \to Y \) be a fibration between two smooth projective varieties over \( \mathbb{C} \), with \( \dim X = n \) and \( \dim Y = m \). Then

\[ C_{n,m} : \kappa(X) \geq \kappa(Y) + \kappa(X_{\bar{\eta}}). \]
This conjecture has been studied by Kawamata ([23], [24], [25]), Kollár ([27]), Viehweg ([39], [40], [40]), Birkar ([5]), Chen and Hacon ([9]), etc. We refer readers to [12] for a collection of results over C.

In positive characteristics, analogously it is conjectured that

**Conjecture 1.2** (Weak Subadditivity). Let \( f : X \to Y \) be a fibration between smooth projective varieties over an algebraically closed field \( k \) of positive characteristic, with \( \dim X = n \) and \( \dim Y = m \). Assume that the geometric generic fibre \( X_\bar{\eta} \) is integral and has a smooth projective birational model \( \tilde{X}_\bar{\eta} \). Then

\[
WC_{n,m} : \kappa(X) \geq \kappa(Y) + \kappa(\tilde{X}_\bar{\eta}).
\]

**Remark 1.3.** The condition that \( X_\bar{\eta} \) is integral is equivalent to that \( X_\bar{\eta} \) is reduced, and also is equivalent to that \( f \) is separable ([30], Sec. 3.2.2). If \( \dim Y = 1 \) then the fibration \( f \) is separable ([3], Lemma 7.2], thus \( X_\bar{\eta} \) is integral.

The reason why we assume the existence of smooth birational models is to guarantee that \( WC_{n,m} \) makes sense, because the geometric generic fibre \( X_\bar{\eta} \) is not necessarily smooth (which is true over C). In positive characteristics, smooth resolution of singularities has been proved in dimension \( \leq 3 \) ([10] and [11]).

Notice that if both \( X \) and \( Y \) are smooth, then the dualizing sheaf of \( X_\bar{\eta} \) is invertible, which corresponds to a Cartier divisor \( K_{X_\bar{\eta}} \). It is reasonable to ask whether the following is true.

**Conjecture 1.4.** Let \( f : X \to Y \) be a fibration between smooth projective varieties over an algebraically closed field \( k \) of positive characteristic, with \( \dim X = n \) and \( \dim Y = m \). Then

\[
C_{n,m} : \kappa(X) \geq \kappa(Y) + \kappa(X_\bar{\eta}, K_{X_\bar{\eta}}).
\]

It is known that \( C_{n,m} \) implies \( WC_{n,m} \) by [12], Corollary 2.5, and we call the inequality \( WC_{n,m} \) weak subadditivity. Up to a power of Frobenius base changes and a smooth resolution, to prove weak subadditivity \( WC_{n,m} \) is equivalent to prove \( C_{n,m} \) for another fibration with smooth geometric generic fiber ([7], proof of Corollary 1.3]). It is much easier to treat a fibration with smooth geometric generic fiber, because then one can take advantage of moduli theory and positivity results proved recently by Patakfalvi [32] and Ejiri [14]. Up to now, the following results have been proved:

1. \( WC_{n,n-1} \) and \( C_{2,1} \) by Chen and Zhang ([12]);
2. \( WC_{3,1} \) by Birkar, Chen and Zhang over \( \mathbb{F}_p, p > 5 \) ([7]);
3. \( WC_{3,1} \) under the situation that \( \tilde{X}_\bar{\eta} \) is of general type and \( \text{char} k > 5 \) by Ejiri ([14], a new proof is given in Appendix 7);
4. \( C_{n,m} \) under the situation that \( f \) is separable, \( \dim_{k(\bar{\eta})} S^0(\tilde{X}_\bar{\eta}, K_{\tilde{X}_\bar{\eta}}) > 0 \) and \( K_Y \) is big by Patakfalvi ([33]).

This paper aims to study subadditivity of Kodaira dimensions for fibrations with singular geometric generic fibers. Our main result is the following theorem, which generalizes the result (iv) above due to Patakfalvi, and can be applied to study fibrations of 3-folds.

**Theorem 1.5.** Let \( f : X \to Y \) be a separable fibration between two normal projective varieties over an algebraically closed field \( k \) with \( \text{char} k = p > 0 \). Assume either that \( Y \) is smooth or that \( f \) is flat. Let \( D \) be a Cartier divisor on \( X \).
If there exist an effective \( \mathbb{Q} \)-Weil divisor \( \Delta \) on \( X \) and a big \( \mathbb{Q} \)-Cartier divisor \( A \) on \( Y \) such that

1. \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier and \( p \nmid \text{ind}(K_X + \Delta) \); 
2. \( D - K_{X/Y} - \Delta - f^*A \) is nef and \( f \)-semi-ample; 
3. \( \dim \kappa(i) \mathbb{S}_0^0(X_{\bar{\eta}}, D_{\bar{\eta}}) > 0 \),

then

\[ \kappa(X, D) \geq \dim Y + \kappa(X_{\bar{\eta}}, K_{X_{\bar{\eta}}}). \]

In particular, if \( D \) is nef and \( f \)-big, and conditions (1) and (2') \( D - K_{X/Y} - \Delta - f^*A \) is nef hold, then \( D \) is big.

**Remark 1.6.** Setting \( \Delta = 0, D = K_X \) and \( A = K_Y \), applying the theorem above we get the result (iv) mentioned above ([33], Theorem 1.1).

**Remark 1.7.** For a separable fibration \( f : X \to Y \), there always exists a birational modification \( Y' \to Y \) such that the main component \( X' \) of \( X \times_Y Y' \) is flat over \( Y' \) ([1], Lemma 3.4]). Since Kodaira dimension is invariant under birational modification, we can reduce to a flat fibration. The advantage of flat fibrations lies in that the relative canonical sheaves behave well under base changes (cf. Proposition 2.1).

Combining recent results of minimal model theory in dimension 3 (cf. [18], [6]), we can prove

**Corollary 1.8.** Let \( f : X \to Y \) be a separable fibration from a smooth projective 3-fold to a smooth projective curve or a surface over an algebraically closed field \( k \) with \( \text{char } k = p > 5 \). Assume that

1. \( K_Y \) is big; 
2. \( S^0(X_{\bar{\eta}}, lK_{X_{\bar{\eta}}}) \neq 0 \) for some positive integer \( l \); and 
3. if \( \dim Y = 2 \), assume moreover that \( K_X \) is pseudo-effective, and that there exists a birational map \( \sigma : X \to \bar{X} \) to a minimal model of \( X \) such that, the restriction \( \sigma|X_{\bar{\eta}} \) is an isomorphism to its image.

Then

\[ \kappa(X) \geq \kappa(Y) + \kappa(X_{\bar{\eta}}, K_{X_{\bar{\eta}}}). \]

**Remark 1.9.** If \( Y \) is non-uniruled, then assumption (3) above is satisfied (Theorem 2.17). Varieties of maximal Albanese dimension are non-uniruled. The result above may be used to study abundance of a 3-fold with non-trivial Albanese map.

**Corollary 1.10.** Let \( f : X \to Y \) be a separable fibration from a smooth projective 3-fold to a smooth projective curve or a surface, over an algebraically closed field \( k \) with \( \text{char } k = p > 5 \). Assume that \( K_Y \) is big and \( Y \) is non-uniruled. Then

1. \( C_{3,n} \) is true if \( K_{X/Y} \) is \( f \)-big; and 
2. \( WC_{3,1} \) is true.

**Idea of the proof:** By a standard approach proposed by Viehweg in [12], granted the bigness of \( K_Y \), to prove subadditivity of Kodaira dimensions, we only need to prove the weak positivity of \( f_*\omega^d_{X/Y} \). Unfortunately, in positive characteristics, if fibers have bad singularities, the sheaf \( f_*\omega^d_{X/Y} \) is not necessarily weakly positive (see Raynaud’s example 1.14 below). To overcome this difficulty, stimulated by [34] and [33], we prove a positivity result (Theorem 1.11 below) without singularity conditions, but at the cost of assuming other conditions like nefness and relative semi-ampleness. These conditions are closely related to minimal model
theory. For a fibration of a 3-fold, by passing to a minimal model, we can prove that the sheaf $F^q_Y f_* (\omega^l_{X/Y} \otimes \omega_Y^{-1})$ contains a non-zero weakly positive sub-sheaf under certain situations (say, when $\omega^l_{X/Y}$ is $f$-big), which plays a similar role as the sheaf $f_* \omega^l_{X/Y}$ does in proving subadditivity of Kodaira dimensions.

The positivity result mentioned above is stated as follows.

**Theorem 1.11.** Let $f : X \to Y$ be a separable surjective projective morphism between two normal projective varieties over an algebraically closed field $k$ with char $k = p > 0$. Assume that $Y$ is Gorenstein. Let $\Delta$ be an effective $\mathbb{Q}$-Weil divisor on $X$ such that $K_{X/Y} + \Delta$ is $\mathbb{Q}$-Cartier and $p \nmid \ind(K_{X/Y} + \Delta)$. If $D$ is a Cartier divisor on $X$ such that $D-K_{X/Y} - \Delta$ is nef and $f$-semi-ample, then for sufficiently divisible $g$, the sheaf $F^q_Y f_* \mathcal{O}_X(D)$ contains a weakly positive sub-sheaf $S^q f_* \mathcal{O}_X(D)$ of rank $\dim_{k(g)} S^q_{\Delta}(X, D)$. Moreover if $Y$ is smooth, then $S^q f_* \mathcal{O}_X(D)$ is FWP.

**Remark 1.12.** (1) Please refer to Sec. 2.2 and 2.3 for the definitions of $S^q_{\Delta} f_* \mathcal{O}_X(D)$, $S^q_{\Delta}(X, D)$ and FWP. The property FWP is mildly stronger than weak positivity, and is easier to lead to subadditivity of Kodaira dimension in positive characteristics (cf. Theorem 1.4).

(2) In [12], Theorem D and Theorem E], the authors got similar results under the assumptions that $f$ is flat, relatively $G_1$ and $S_2$, $p \nmid \ind(K_{X/Y} + \Delta)$ and $D-K_{X/Y} - \Delta$ is nef and $f$-ample. And in [33, Sec. 6], Patakfalvi proved the weak positivity of $S^q f_* \omega^l_{X/Y}$ under some mild assumptions.

(3) Patakfalvi [32] and Ejiri [14] proved that, under assumptions that $(X, \Delta)$ is sharply $F$-pure, $K_X + \Delta$ is ample and $p \nmid \ind(K_{X/Y} + \Delta)$, the sheaf $f_* \mathcal{O}_X(m(K_{X/Y} + \Delta))$ is weakly positive for sufficiently divisible $m$.

(4) The main idea of the proof is to consider the trace maps of relative Frobenius iterations, similarly as in [33] and [34], and the proof is simplified by use of the criterion of weak positivity suggested by Ejiri ([14, Sec. 4]).

Applying the theorem above to log minimal models, immediately we get

**Corollary 1.13.** Let $f : X \to Y$ be a separable surjective projective morphism between two normal projective varieties over an algebraically closed field $k$ with char $k = p > 0$. Let $\Delta$ be an effective $\mathbb{Q}$-Weil divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $p \nmid \ind(K_X + \Delta)$. Assume that $K_X + \Delta$ is nef and $f$-semi-ample and $Y$ is Gorenstein. Then for a positive integer $l$ such that $l(K_X + \Delta)$ is Cartier and sufficiently divisible $g$, the sheaf $F^q_Y (\mathcal{O}_X(l(K_{X/Y} + \Delta))) \otimes \omega_Y^{-1})$ contains a weakly positive sub-sheaf of rank $\dim_{k(g)} S^q_{\Delta}(X, l(K_{X/Y} + \Delta))$.

Let’s recall Raynaud’s example, which gives a minimal surface $S$ of general type over a curve $C$, with $F^q C f_* \omega^l_{S/C}$ being negative while $f_* \omega^l_{S/C} \otimes \omega_C^{-1}$ being nef for $l \geq 2$.

**Example 1.14** ([35, 13 Theorem 3.6]). Let $C$ be a Tango curve with $g(C) \geq 2$ over an algebraically closed field $k$ with char $k = p \geq 3$. Then there exists a line bundle $L$ on $C$ such that $K_C \sim pL$. We have a non-trivial extension

$$0 \to \mathcal{O}_C \to E \to L \to 0$$

such that $\Sym^p E \otimes L^{-p}$ has a non-zero section.
Let $X = \mathbb{P}_C(\mathcal{E}^*) = \text{Proj}_{\mathcal{O}_C} \oplus \text{Sym}^1 \mathcal{E}$, $g : X \to C$ the natural projection, $E$ the natural section such that $E \sim \mathcal{O}_X(1)$ and $C'$ a smooth curve on $X$ such that $C' \sim pE - pf^*L$. Then $E$ and $C'$ are disjoint to each other. Let

$$M \sim \frac{p+1}{2}E - pf^*L'$$

where $L'$ is a line bundle on $C$ such that $2L' \sim L$. Denote by $\pi : S \to X$ the smooth double cover induced by the relation $2M \sim E + C'$, and by $f : S \to C$ the natural fibration. Then we have that

$$\pi_* \omega^l_{S/C} \cong \mathcal{O}_X((lK_{X/C} + M)) \oplus \mathcal{O}_X((lK_{X/C} + (l-1)M)),$$

thus $K_{X/C} \sim -2E + g^* \text{det } \mathcal{E} \sim -2E + g^*L$,

$$f_* \omega^l_{S/C} \cong \mathcal{O}_X((l(K_{X/C} + M)) \oplus \mathcal{O}_X((lK_{X/C} + (l-1)M))$$

$$\cong \mathcal{O}_X\left(\frac{(p-3)}{2}E + g^*((2-p)(l')\right)$$

$$\oplus \mathcal{O}_X\left(\frac{lp-p-3l-1}{2}E + g^*((2-p)(l+p))\right)$$

$$\cong (\text{Sym}\frac{lp-3}{2} \mathcal{E} \otimes (2-p)l') \oplus (\text{Sym}\frac{lp-p-3l-1}{2} \mathcal{E} \otimes ((2-p)(l+p))\right).$$

We can see that for any positive integers $e$ and $l$, the sheaf $F^e f_* \omega^l_{S/C}$ is negative, while the sheaf

$$f_* \omega^l_{S/C} \otimes \omega^{-1}_{C} \cong \text{Sym}\frac{lp-3}{2} \mathcal{E} \otimes (lp+2l-2p)l' \oplus \text{Sym}\frac{lp-p-3l-1}{2} \mathcal{E} \otimes (lp+2l-p)l'$$

is nef for $l \geq 2$.

**Conventions:**

For a morphism between schemes, we often use the same notation for the restriction map to a sub-scheme.

For a notherian scheme $X$, we denote by $\text{CDiv}(X)$ the additive group of Cartier divisors. An element $D$ in $\text{CDiv}(X) \otimes \mathbb{Q}$ is called a $\mathbb{Q}$-Cartier divisor, the index of $D$ is the smallest positive integer $n$ such that $nD$ is Cartier, which is denoted by $\text{ind}(D)$.

Let $X$ be a noetherian $G_1$ and $S_2$ scheme over a field $k$ of finite type and of pure dimension. An almost Cartier divisor (AC divisor for short) on $X$ is a reflexive coherent $\mathcal{O}_X$-submodule of the sheaf of total quotient ring $K(X)$ such that invertible in codimension one. Denote by $\text{WSh}(X)$ the set of AC divisors, which is an additive group. For $D \in \text{WSh}(X)$, we denote by $\mathcal{O}_X(D)$ the coherent sheaf defining $D$; if $\mathcal{O}_X \subset \mathcal{O}_X(D)$, we say $D$ is effective. For two AC divisors $D_1$ and $D_2$ on $X$ such that $E = D_2 - D_1$ is an effective AC divisor, then $\mathcal{O}_X \subset \mathcal{O}_X(E)$ induces a natural inclusion $\mathcal{O}_X(D_1) \subset \mathcal{O}_X(D_2)$. An element of $\text{WSh}(X) \otimes \mathbb{Q}$ is called a $\mathbb{Q}$-AC divisor. Naively we can define effectiveness in $\text{WSh}(X) \otimes \mathbb{Q}$. For a flat morphism $g : W \to X$, the pull-back $g^* : \text{WSh}(X) \otimes \mathbb{Q} \to \text{WSh}(W) \otimes \mathbb{Q}$ makes sense. For more details, please refer to for example [41].

Let $X$ be a normal variety. Denote by $\text{WDiv}(X)$ the additive group of Weil divisors. A Weil divisor $D$ on $X$ defines a reflexive, invertible in codimension one, coherent sub-sheaf $\mathcal{O}_X(D)$ of the constant sheaf $K(X)$, via

$$\mathcal{O}_X(D)_x := \{ f \in K(X) | (f) + D |_U \geq 0 \text{ for some open set } U \text{ containing } x \}.$$
So we can regard a Weil divisor as an AC divisor. For $D \in \text{WDiv}(X) \otimes \mathbb{R}$, $[D]$ denotes the integral part of $D$.

We use $\sim$ (resp. $\sim_\mathbb{Q}$) for linear (resp. $\mathbb{Q}$-linear) equivalence between AC (resp. $\mathbb{Q}$-AC) divisors. We use $\simeq$ for quasi-isomorphism between objects in a derived category, and use $\cong$ for the isomorphism between sheaves or schemes.

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## 2. Preliminaries

### 2.1. Canonical sheaf and relative canonical sheaf

Let $f : X \to Y$ be a projective morphism between noetherian schemes of pure dimension. Let $r = \dim X - \dim Y$. By Grothendieck duality theory (cf. [19]), there exists a functor $f^! : D^+(Y) \to D^+(X)$ such that for $F \in D^- (X)$ and $G \in D^+ (Y)$,

$$Rf_* R\text{Hom}_X (F,f^! G) \simeq R\text{Hom}_Y (Rf_! F,G).$$

The relative dualizing sheaf is defined as $\omega^0_{X/Y} = H^0 (f^! \mathcal{O}_Y [-r])$.

Canonical divisor and relative canonical divisor are defined as follows.

1. If $Y$ is the spectrum of a field and $X$ is $G_1$ and $S_2$, then there exists a Gorenstein open set $X_0 \subset X$ such that $\text{codim}_{X} (X \setminus X_0) \geq 2$. So $f^! \mathcal{O}_Y |_{X_0} \cong \omega^0_{X/Y}[r]|_{X_0}$, and $\omega^0_{X/Y}|_{X_0}$ is an invertible sheaf on $X_0$. The canonical sheaf of $X$ is defined as $\omega_X = i_* \omega^0_{X/Y}|_{X_0}$ where $i : X_0 \hookrightarrow X$ denotes the open immersion. The canonical divisor $K_X$ is defined as an AC divisor such that $\mathcal{O}_X (K_X) \cong \omega_X$.

2. If either $Y$ is Gorenstein and $X$ is $G_1$ and $S_2$, or $f$ is flat and relatively $G_1$ and $S_2$, similarly there exists an open immersion $i : X_0 \hookrightarrow X$ such that $\text{codim}_{X} (X \setminus X_0) \geq 2$, $f^! \mathcal{O}_Y |_{X_0} \cong \omega^0_{X/Y}[r]|_{X_0}$, and $\omega^0_{X/Y}|_{X_0}$ is an invertible sheaf. The relative canonical sheaf $\omega_{X/Y}$ and relative canonical divisor $K_{X/Y}$ can be defined similarly as in (1). If moreover $X$ and $Y$ are projective schemes over a field and $Y$ is Gorenstein, then $K_{X/Y} \sim K_X - f^* K_Y$.

3. If $f : X \to Y$ is a finite morphism, then $\omega_{X/Y}$ is defined via $f_* \omega_{X/Y} \cong \mathcal{H}om_{\mathcal{O}_Y} (f_* \mathcal{O}_X, \mathcal{O}_Y) \cong f_* f^! \mathcal{O}_Y$. And if moreover both $X$ and $Y$ are $G_1$ and $S_2$ projective schemes over a field, then we can define canonical divisor $K_{X/Y}$ as in (1), which satisfies that $K_{X/Y} \sim K_X - f^* K_Y$.

It is known that relative dualizing sheaf is compatible with a flat base change (cf. [19], Chap. III Sec. 8). For a non-flat base change, we have the following result which is similar to [12], Theorem 2.4.

**Proposition 2.1.** Let $f : X \to Y$ be a flat projective morphism between two normal varieties. Let $\Delta$ be an effective $\mathbb{Q}$-Weil divisor on $X$ such that $K_{X/Y} + \Delta$ is $\mathbb{Q}$-Cartier. Let $\pi : Y' \to Y$ be a smooth modification, $X' = X \times_Y Y'$ and $\sigma : X' \to X$ the normalization morphism, which fit into the following commutative diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{\sigma} & X \\
\downarrow{\pi'} & & \downarrow{\pi} \\
Y' & \xrightarrow{f'} & Y \\
\end{array}
$$

where $\sigma$ and $\pi$ are isomorphisms, $\pi'_* \omega_{X'/Y'} = \omega_{X/Y}$.
where \( \pi' \) and \( \tilde{f} \) denote the natural projections, and \( f' = \tilde{f} \circ \sigma \).

Then there exist an effective \( \sigma' \)-exceptional Cartier divisor \( E' \) and an effective divisor \( \Delta' \) on \( X' \) such that

\[
K_{X'/Y'} + \Delta' = \sigma^*(K_{X/Y} + \Delta) + E'.
\]

**Proof.** Denote by \( Y_0 \) the smooth locus of \( Y \), and let \( Y_0' = \pi^{-1}Y_0 \), \( X_0 = X \times_Y Y_0 \), \( X_0' = X' \times_Y Y_0' \) and \( X''_0 = X'' \times_Y Y_0' \). By arguing in codimension one, we assume \( X_0 \) is Gorenstein, hence \( f|_{X_0} \) is a flat Gorenstein morphism by [19, p.298 (Ex. 9.7)]. Then by remarks of [19, p.388], we have

\[
K_{\tilde{X}'_0/Y_0} = \pi'^*K_{X/Y}|_{\tilde{X}'_0}.
\]

Since \( X'_0 \to \tilde{X}'_0 \) is the normalization, by results of [36, Sec. 2], there exists an effective divisor \( \Delta'_0 \) on \( X'_0 \) such that

\[
\sigma'^*K_{X'/Y'}|_{\tilde{X}'_0} = \sigma^*K_{\tilde{X}'_0/Y'_0} = K_{X'_0/Y'_0} + C'.
\]

Since \( K_{X'/Y'} + \Delta \) is assumed to be \( \mathbb{Q} \)-Cartier, its pull-back makes sense. By argument above, there exists an effective divisor \( \Delta'_0 \) on \( X'_0 \) such that

\[
K_{X'_0/Y'_0} + \Delta'_0 = \sigma'^*(K_{X'/Y'} + \Delta)|_{\tilde{X}'_0}.
\]

Let \( D' \) be the closure of \( \Delta'_0 \) in \( X' \), which is a \( \mathbb{Q} \)-Weil divisor. Let \( B' = \sigma'^*(K_{X'/Y'} + \Delta) - (K_{X'/Y'} + D') \). If \( B' = 0 \), then we are done. Otherwise, since \( f' \) is equi-dimensional, the support of \( B' \) is mapped via \( f' \) to a codimension one cycle contained in \( Y' \setminus Y'_0 \). Since \( Y' \) is smooth, we can find an effective \( \pi' \)-exceptional Cartier divisor \( E \) on \( Y' \) such that \( D'' = f'^*E - B' \geq 0 \). Let \( E'' = f'^*E \) which is \( \sigma' \)-exceptional and \( \Delta = D' + D'' \). Then we are done.

\[\square\]

2.2. **Trace maps of Frobenius iterations.** Throughout this subsection, let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Let \( f : X \to Y \) be a morphism of schemes over \( k \). We will use the following notation:

1. \( F_X^e : X \to X \) for the \( e \)-th absolute Frobenius iteration, and sometimes, to avoid confusions, we use \( X^e \) for the source scheme in the morphism \( F_X^e : X \to X \);
2. \( X_Y^e \) for the fiber product \( X \times_Y Y^e \) of morphisms \( f : X \to Y \) and \( F_Y^e : Y \to Y \), \( f_e : X_Y^e \to Y \) and \( \pi_Y^e : X_Y^e \to X \) for the natural projections;
3. \( F_{X/Y}^e : X \to X_{Y^e} \) for the \( e \)-th relative Frobenius iteration over \( Y \).

We will discuss the trace maps of (relative) Frobenius iterations in different settings. Please refer to [32], [33], [34] and [14] for more details and related results.

2.2.1. **Trace maps of absolute Frobenius iterations.**

**Notation 2.2.** Let \( X \) be a \( G_1 \) and \( S_2 \) projective scheme over \( k \) of finite type and of pure dimension. Denote by \( X_0 \) a Gorenstein open subset of \( X \) such that \( \text{codim}_X(X \setminus X_0) > 1 \). Let \( \Delta \) be an effective \( \mathbb{Q} \)-AC divisor such that \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier and \( p \nmid \text{ind}(K_X + \Delta) \). Then there exists a positive integer \( g \) such that \( (1-p^g)(K_X + \Delta) \) is Cartier for every positive integer \( e \), in particular \( (p^g - 1)\Delta|_{X_0} \) is an effective Cartier divisor. Let \( D \) be a Cartier divisor on \( X \).

Since \( X \) is \( G_1 \) and \( S_2 \), the composite map of the natural inclusion

\[
F_{X_0}^e \mathcal{O}_X((1 - p^g)(K_X + \Delta))|_{X_0} \to F_{X_0}^e \mathcal{O}_{X_0}((1 - p^g)K_{X_0})
\]
and the trace map \( Tr_{F_X^g} : F_{X^g}^g \mathcal{O}_X,((1-p^g)K_X) \rightarrow \mathcal{O}_X \) extends to a map on \( X \):

\[
Tr_{X,\Delta}^{g^\prime} : F_{X^g}^g \mathcal{O}_X((1-p^g)(K_X + \Delta)) \rightarrow \mathcal{O}_X.
\]

Twisting the trace map \( Tr_{X,\Delta}^{g^\prime} \) above by \( \mathcal{O}_X(D) \) induces a map

\[
Tr_{X,\Delta}(D) : F_{X^g}^g \mathcal{O}_X((1-p^g)(K_X + \Delta) \otimes \mathcal{O}_X(D)) 
\cong F_{X^g}^g \mathcal{O}_X((1-p^g)(K_X + \Delta) + p^g D) \rightarrow \mathcal{O}_X(D),
\]

then taking global sections gives

\[
H^0(Tr_{X,\Delta}(D)) = H^0(X, F_{X^g}^g \mathcal{O}_X((1-p^g)(K_X + \Delta) + p^g D)) \rightarrow H^0(X, D).
\]

Let

\[
S^g_{\Delta}(X, D) = \text{Im} H^0(Tr_{X,\Delta}(D)) \text{ and } S^g_{\Delta}(X, D) = \cap_{c \geq 0} S^g_{\Delta}(X, D).
\]

If \( \Delta = 0 \), we usually use the notation \( S^0(X, D) \) instead of \( S^g_{\Delta}(X, D) \).

For \( e' > e \), the map \( Tr_{X,\Delta}^{e'}(D) \) factors as

\[
Tr_{X,\Delta}(D) : F_{X^g}^g \mathcal{O}_X(1-p^g)(K_X + \Delta) + p^g D) \rightarrow F_{X^g}^g \mathcal{O}_X((1-p^g)(K_X + \Delta) + p^g D)
\]

So there is a natural inclusion \( S^e_{\Delta}(X, D) \subset S^g_{\Delta}(X, D) \), thus for sufficiently large \( e \), \( S^e_{\Delta}(X, D) = S^g_{\Delta}(X, D) \).

**Proposition 2.3.** Let the notation be as in Notation 2.2. Then

1. There exists an ideal \( \sigma(X, \Delta) \), namely, the non-F-pure ideal of \( (X, \Delta) \), such that for sufficiently divisible \( e \),

\[
\text{Im} Tr_{X,\Delta}^{g^\prime} = \sigma(X, \Delta) = Tr_{X,\Delta}^{g^\prime} F_{X^g}^g(\sigma(X, \Delta) \cdot \mathcal{O}_X((1-p^g)(K_X + \Delta))).
\]

2. If \( D \) is ample, then for sufficiently large \( l \)

\[
S^0_{\Delta}(X, lD) = H^0(X, \sigma(X, \Delta) \cdot \mathcal{O}_X(lD)).
\]

3. If \( X \) is integral and \( D \) is big, then for sufficiently large \( l \), \( S^0_{\Delta}(X, lD) \neq 0 \).

4. Assume moreover that \( X \) is integral. And let \( X' \) be another integral \( G_1 \) and \( S_2 \) projective scheme over \( k \), granted a birational morphism \( \sigma : X' \rightarrow X \). If there exist an effective Cartier divisor \( E \) and an effective \( \mathbb{Q} - AC \) divisor \( \Delta' \) on \( X' \) such that

\[
K_{X'} + \Delta' = \sigma^*(K_X + \Delta) + E,
\]

then

\[
\dim S^0_{\Delta}(X, D) \leq \dim S^0_{\Delta'}(X', \sigma^* D + E).
\]

In particular, if \( X' \) is the normalization of \( X \), then there exists an effective \( \mathbb{Q} - AC \) divisor \( \Delta' \) such that

\[
K_{X'} + \Delta' = \sigma^*(K_X + \Delta),
\]

thus

\[
\dim S^0_{\Delta}(X, D) \leq \dim S^0_{\Delta'}(X', \sigma^* D).
\]
Proof: For (1), please refer to [14, Lemma 13.1].

For (2), fix a sufficiently divisible \( g \) such that for every positive integer \( e \) the trace map below is surjective

\[
F_{X,*}^g(\sigma(X, \Delta) \cdot \mathcal{O}_X((1 - p^e)(K_X + \Delta))) \to \sigma(X, \Delta),
\]
and denote by \( B^g \) the kernel. By Fujita vanishing, we can find an integer \( l_0 \) such that for every integer \( l > l_0 \),

\[
H^1(X, B^g \otimes \mathcal{O}_X(lD)) = 0,
\]
thus for \( e = 1 \) the trace map

\[
H^0(Tr_{X,\Delta}^g(lD)) : H^0(X, F_{X,*}^g(\sigma(X, \Delta) \cdot \mathcal{O}_X((1 - p^e)(K_X + \Delta))) \otimes \mathcal{O}_X(lD)) \to H^0(X, \sigma(X, \Delta) \otimes \mathcal{O}_X(lD))
\]
is surjective. Then applying the arguments of [32, Sec. 2.E], by induction we can prove the surjection of \( H^0(Tr_{X,\Delta}^g(lD)) \) for every \( e \).

For (3), assume that \( X \) is integral and \( D \) is big. Then we can find an ample \( \mathbb{Q} \)-Cartier divisor \( H \) such that \( D - H \) is effective. Take a sufficiently divisible positive integer \( l \) such that, both \( E = l(D - H) \) and \( lH \) are Cartier. By the natural map

\[
H^0(X, F_{X,*}^g \mathcal{O}_X((1 - p^e)(K_X + \Delta)) \otimes \mathcal{O}_X(lH)) \otimes H^0(X, \mathcal{O}_X(E)) \to H^0(X, F_{X,*}^g \mathcal{O}_X((1 - p^e)(K_X + \Delta)) \otimes \mathcal{O}_X(lH + E))
\]

for a non-zero \( s_E \in H^0(X, \mathcal{O}_X(E)) \), we get an injective map below

\[
S_{\Delta}^0(X, lH) \otimes s_E \to S_{\Delta}^0(X, lH + E) = S_{\Delta}^0(X, lD).
\]

Then we can conclude assertion (3) by (2).

We are left to prove (4). Let \( K = K(X) \) and \( K_{\pi^+} = K(X^g) \). Denote by \( \zeta = \text{spec}K \) the generic point of \( X \) and \( X' \). Regard \( \text{Hom}_K(K_{\pi^+}, K) \) and \( K \) as constant quasi-coherent sheaves of both \( X \) and \( X' \). We have the following natural commutative diagram

\[
\begin{array}{ccc}
F_{X,*}^g \mathcal{O}_X((1 - p^e)(K_X + \Delta)) & \xrightarrow{Tr_{X,\Delta}^g} & F_{X,*}^g \omega_X^{1-p^e} \cong \text{Hom}_{\mathcal{O}_X}(F_{X,*}^g \mathcal{O}_X, \mathcal{O}_X) \\
\omega_X^{1-p^e} \cong \text{Hom}_{K}(K_{\pi^+}, K) & \xrightarrow{ev(1)} & K \\
F_{X',*}^g \mathcal{O}_{X'}((1 - p^e)(K_{X'} + \Delta')) & \xrightarrow{Tr_{X',\Delta'}^g} & F_{X',*}^g \omega_{X'}^{1-p^e} \cong \text{Hom}_{\mathcal{O}_{X'}}(F_{X',*}^g \mathcal{O}_{X'}, \mathcal{O}_{X'}) \end{array}
\]

where \( ev(1) \) denotes the evaluation map at \( 1 \in K_{\pi^+} \).

Since \( (1 - p^e)(K_X + \Delta) + p^e(\sigma^*D + E) - \sigma^*((1 - p^e)(K_X + \Delta) + p^eD) = E \) is an effective Cartier divisor, with the two sheaves below naturally seen as sub-sheaves of \( \text{Hom}_K(K_{\pi^+}, K) \cong K_{\pi^+} \) on \( X' \), we see that

\[
F_{X',*}^g \sigma^* \mathcal{O}_X((1 - p^e)(K_X + \Delta) + p^eD) \subseteq F_{X',*}^g \mathcal{O}_{X'}((1 - p^e)(K_{X'} + \Delta') + p^e(\sigma^*D + E)).
\]
By the natural inclusions $O_{X'}(\sigma^*D) \subseteq O_{X'}(\sigma^*D + E) \subseteq K$, we have the following commutative diagram

$$
\begin{array}{c}
\sigma^*O_X((1 - p^g)(K_X + \Delta) + p^gD) \\
\downarrow \\
O_{X'}((1 - p^g)(K_{X'} + \Delta') + p^g(\sigma^*D + E)) \\
\rightarrow \\
\sigma^*Tr_{X}(\sigma^*D) \\
\end{array}
\begin{array}{c}
\rightarrow \\
Tr_{X'}(\sigma^*D + E) \\
\rightarrow \\
O_{X'}(\sigma^*D + E)
\end{array}
$$

Taking global sections of the traces maps above, we conclude the injection

$$
\sigma^*S^0_\Delta(X, D) \hookrightarrow S^0_\Delta(X', \sigma^*D + E),
$$

thus

$$
\dim S^0_\Delta(X, D) \leq \dim S^0_\Delta(X', \sigma^*D + E).
$$

In particular, if $\sigma : X' \to X$ is the normalization map, then by [[36], Sec. 2] there exists an effective AC divisor $C$ on $X'$ such that

$$
\sigma^*\omega_X = \omega_{X'}(C).
$$

The remaining assertion is an easy consequence. \qed

2.2.2. Trace maps of relative Frobenius iterations I.

Notation 2.4. Let $f : X \to Y$ be a surjective projective morphism between two schemes over $k$ of finite type and of pure dimension. Assume that $X$ is $G_1$ and $S_2$ and that $Y$ is integral and regular. Let $\Delta$, $g$ and $D$ be assumed as in Notation 2.2.

By assumption $F^{eg}_Y$ is a flat morphism, so $X_{Y^{eg}}$ also satisfies $G_1$ and $S_2$, and $K_{X_{Y^{eg}}} = \pi^{eg}_Y K_{X/Y}$. By easy calculation we have that

$$
K_{X_{Y^{eg}}/X} = (1 - p^g)K_{X_{Y^{eg}}/Y}^{eg}$$

and $F^{eg}_{X/Y}\pi^{eg}_Y D = p^gD$.

Similarly as in 2.2.1 we get the trace map

$$
T^{eg}_{X/Y, \Delta}(D) : F^{eg}_{X/Y}O_X((1 - p^g)(K_X + \Delta) + p^gD) \to O_{X_{Y^{eg}}} (\pi^{eg}_Y D).
$$

Applying $f_{eg*}$ to the above map, we get

$$
f_{eg*}T^{eg}_{X/Y, \Delta}(D) : f_{eg*}O_X((1 - p^g)(K_X + \Delta) + p^gD) \to f_{eg*}O_{X_{Y^{eg}}} (\pi^{eg}_Y D) \cong F^{eg}_{Y}\pi_{Y^{eg}} D = f_{eg*}O_X(D).
$$

where $S^0_{\Delta}f_{*}O_X(D)$, introduced by [34, Def. 6.4] with slightly different notation, denotes the image of $f_{*}T^{eg}_{X/Y, \Delta}(D)$. If $\Delta = 0$, we use the notation $S^{eg}f_{*}O_X(D)$.
Let the notation be as in Notation 2.4. Then for every positive integer $e$, according to the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi_Y^{(e'-e)e}} & X \\
| & | & |
X_Y^{(e'-e)e} & \xrightarrow{F_Y^{(e'-e)e}} & X \\
Y & \xrightarrow{f} & Y
\end{array}
\]

the trace map $f_*\text{Tr}_{X/Y}^{(e'-e)e}D$ factors as

\[
f_*\mathcal{O}_X((1-p^{(e'-e)e})(K_{X/Y} + \Delta) + p^{(e'-e)e}D)
\]

\[
\xrightarrow{f_*\text{Tr}_{X/Y}^{(e'-e)e}((1-p^{(e'-e)e})(K_{X/Y} + \Delta) + p^{(e'-e)e}D)}
\]

\[
f_Y^{(e'-e)e}f_*\mathcal{O}_X((1-p^{(e'-e)e})(K_{X/Y} + \Delta) + p^{(e'-e)e}D)
\]

Then we conclude a natural inclusion

\[
S_\Delta^{(e'-e)e}f_*\mathcal{O}_X(D) \hookrightarrow F_Y^{(e'-e)e}S_\Delta^{(e'-e)e}f_*\mathcal{O}_X(D).
\]

**Proposition 2.5.** Let the notation be as in Notation 2.4. Then for every positive integer $e$,

\[
\dim_{k(\bar{\eta})} S_\Delta^{e\ast} (X_{\bar{\eta}}, D_{\bar{\eta}}) = \operatorname{rank} S_\Delta^{e\ast} f_*\mathcal{O}_X(D).
\]

Consequently for sufficiently large $e$, $\operatorname{rank} S_\Delta^{e\ast} f_*\mathcal{O}_X(D)$ is stable, which equals to

\[
\dim_{k(\bar{\eta})} S_\Delta^{e\ast} (X_{\bar{\eta}}, D_{\bar{\eta}}).
\]

**Proof.** Consider the following commutative diagram

\[
\begin{array}{ccc}
X_{\bar{\eta}} & \xrightarrow{F_{X_{\bar{\eta}}/\bar{\eta}}} & X_{\tilde{\eta}}^{e\ast} \\
| & | & |
\bar{\eta} & \xrightarrow{\pi_{\bar{\eta}}^{e\ast}} & \tilde{\eta}^{e\ast}
\end{array}
\]

Let $D_{\tilde{\eta}} = \pi_{\bar{\eta}}^{e\ast} D$. Then the trace map

\[
\text{Tr}_{\pi_{\bar{\eta}}^{e\ast}} : H^0(X_{\tilde{\eta}}^{e\ast}, D_{\tilde{\eta}}) \rightarrow H^0(X_{\bar{\eta}}, D_{\bar{\eta}})
\]

is an isomorphism since $k(\bar{\eta})$ is algebraically closed. By $F_{X_{\bar{\eta}}}^{e\ast} = \pi_{\tilde{\eta}}^{e\ast} \circ F_{X_{\bar{\eta}}/\tilde{\eta}}^{e\ast}$, the trace map

\[
H^0(\text{Tr}_{X_{\bar{\eta}}/\Delta_{\bar{\eta}}}^{e\ast}D) : H^0(X_{\bar{\eta}}, F_{X_{\bar{\eta}}/\bar{\eta}}^{e\ast}\mathcal{O}_{X_{\bar{\eta}}}((1-p^{e\ast})(K_{X_{\bar{\eta}} + \Delta_{\bar{\eta}}}) + p^{e\ast}D_{\bar{\eta}})) \rightarrow H^0(X_{\bar{\eta}}, D_{\bar{\eta}})
\]
factors as
\[ H^0(X_{\tilde{\eta}}, F_{\tilde{\eta}} e g, \mathcal{O}_{X_{\tilde{\eta}}}(1 - p^{eg}) (K_{X_{\tilde{\eta}}} + \Delta_{\tilde{\eta}} + p^{eg} D_{\tilde{\eta}})) \]
\[ \xrightarrow{H^0(\text{Tr}^{eg}_{X_{\tilde{\eta}}/\tilde{\eta}, \Delta_{\tilde{\eta}}}(D_{\tilde{\eta}}))} H^0(X_{\tilde{\eta}}, D_{\tilde{\eta}}) \xrightarrow{\text{Tr}^{eg}_{X_{\tilde{\eta}}/\tilde{\eta}}(D_{\tilde{\eta}})} H^0(X_{\tilde{\eta}}, D_{\tilde{\eta}}). \]

Since the morphism \( i : \tilde{\eta} \rightarrow Y \) is flat, the trace map \( \text{Tr}^{eg}_{X_{\tilde{\eta}}/\tilde{\eta}, \Delta_{\tilde{\eta}}}(D_{\tilde{\eta}}) \) coincides with the pull-back map via \( i^* \) of
\[ \text{Tr}^{eg}_{X/Y, \Delta}(D) : f_* \mathcal{O}_X((1 - p^{eg})(K_{X/Y} + \Delta) + p^{eg}D) \]
\[ \xrightarrow{\text{Tr}^{eg}_{X/Y, \Delta}(D)} S^{eg}_\Delta f_* \mathcal{O}_X(D) \hookrightarrow F_Y^{eg} f_* \mathcal{O}_X(D). \]

Then we conclude the proof by
\[ \text{rank} S^{eg}_\Delta f_* \mathcal{O}_X(D) = \dim_{k(\tilde{\eta})} \text{Im} \text{Tr}^{eg}_{X_{\tilde{\eta}}/\tilde{\eta}, \Delta_{\tilde{\eta}}}(D_{\tilde{\eta}}) = \dim_{k(\tilde{\eta})} S^{eg}_\Delta(X_{\tilde{\eta}}, D_{\tilde{\eta}}). \]

\[ \square \]

2.2.3. Trace maps of relative Frobenius iterations II: in the normal setting.

**Notation 2.6.** Let \( f : X \rightarrow Y \) be a Weil projective morphism between two schemes over \( k \) of finite type and of pure dimension. Assume that \( X \) is normal and \( Y \) is integral and regular. Let \( D \) be a Weil divisor on \( X \) and \( \Delta \) an effective \( \mathbb{Q} \)-Weil divisor on \( X \). Assume that \( K_X + \Delta = \mathbb{Q} \)-Cartier. It is known that \( X_{\mathbb{Q}} := S_2 \), \( \pi_{\mathbb{Q}}^* D \) is an AC divisor on \( X_{\mathbb{Q}} \), and the sheaf \( \mathcal{O}_{X_{\mathbb{Q}}}(\pi_{\mathbb{Q}}^* D) \) is isomorphic to \( \pi_{\mathbb{Q}}^* \mathcal{O}_X(D) \).

Replacing \((p^{eg} - 1)\Delta \) by \((p^{eg} - 1)\Delta\), analogously to Sec. 2.2.2, we get the trace maps
\[ \text{Tr}^{eg}_{X/Y, \Delta}(D) : F^{eg}_{X/Y, \mathcal{O}_X((1 - p^{eg})(K_{X/Y} - ([p^{eg} - 1] \Delta) + p^{eg}D) \rightarrow \mathcal{O}_{X_{\mathbb{Q}}}(\pi_{\mathbb{Q}}^* D) \]
and
\[ f_* \text{Tr}^{eg}_{X/Y, \Delta}(D) : f_* \mathcal{O}_X((1 - p^{eg})(K_{X/Y} - ([p^{eg} - 1] \Delta) + p^{eg}D) \]
\[ \rightarrow S^{eg}_\Delta f_* \mathcal{O}_X(D) \hookrightarrow f_* \mathcal{O}_{X_{\mathbb{Q}}}(\pi_{\mathbb{Q}}^* D) \equiv F^{eg}_Y f_* \mathcal{O}_X(D). \]
where \( S^{eg}_\Delta f_* \mathcal{O}_X(D) \) denotes the image of \( f_* \text{Tr}^{eg}_{X/Y, \Delta}(D) \). Note that for \( \epsilon' > \epsilon \) the divisor below is effective
\[ ([p^{\epsilon'} - 1] \Delta) - p^{(\epsilon' - \epsilon)g}([p^{\epsilon} - 1] \Delta). \]

We deduce natural inclusions
\[ S^{\epsilon'}_\Delta f_* \mathcal{O}_X(D) \hookrightarrow F^{(\epsilon' - \epsilon)g}_Y S^{eg}_\Delta f_* \mathcal{O}_X(D) \hookrightarrow F^{\epsilon' g}_Y f_* \mathcal{O}_X(D) \]
by the factorization
\[ f_* \text{Tr}^{\epsilon' g}_{X/Y, \Delta}(D) : f_* \mathcal{O}_X((1 - p^{\epsilon' g})(K_{X/Y} - ([p^{\epsilon'} - 1] \Delta) + p^{\epsilon' g}D) \]
\[ \hookrightarrow f_* \mathcal{O}_X((1 - p^{\epsilon' g})(K_{X/Y} - p^{(\epsilon' - \epsilon)g}([p^{\epsilon' g} - 1] \Delta) + p^{\epsilon' g}D) \]
\[ \xrightarrow{f_* \text{Tr}^{(\epsilon' - \epsilon)g}_{X/Y, \Delta}} f_* \mathcal{O}_X((1 - p^{eg})(K_{X/Y} - ([p^{eg} - 1] \Delta) + p^{eg}D) \]
\[ \xrightarrow{F^{(\epsilon' - \epsilon)g}_Y f_* \mathcal{O}_X(D)} F^{(\epsilon' - \epsilon)g}_Y S^{eg}_\Delta f_* \mathcal{O}_X(D) \hookrightarrow F^{\epsilon' g}_Y f_* \mathcal{O}_X(D). \]
Proposition 2.7. Let the notation be as in Notation 2.6. Assume moreover that $D$ is Cartier and $p 
otm | \text{ind}(K_X + \Delta)_\eta$. Let $g$ be a positive integer such that $(1 - p^g)(K_X + \Delta)_\eta$ is Cartier. Then for every positive integer $e$,

$$\dim_{k(\eta)} S^e_{\Delta}(X_{\eta}, D_{\eta}) = \text{rank} S^0_{\Delta} f_* O_X(D).$$

Consequently for sufficiently large $e$, $\text{rank} S^0_{\Delta} f_* O_X(D)$ is stable, which equals to $\dim_{k(\eta)} S^0_{\Delta}(X_{\eta}, D_{\eta})$.

Proof. Shrinking $Y$ and $X$, we can assume that $(1 - p^g)(K_X + \Delta)$ is Cartier. Then we are done by applying Proposition 2.6. $\square$

2.3. Weak positivity.

Definition 2.8. A torsion free coherent sheaf $\mathcal{F}$ on a normal quasi-projective variety $Y$ is said to be generically globally generated if for general closed point $y \in Y$ the homomorphism $H^0(Y, \mathcal{F}) \rightarrow \mathcal{F} \otimes k(y)$ is surjective; and is said to be weakly positive, if for every ample line bundle $H$ on $Y$ and positive integer $n$, there exists sufficiently large integer $n$ such that, $S^n(H \otimes S^m(\mathcal{F})^{**})$ is generically globally generated, where for a coherent sheaf $\mathcal{G}$, $\mathcal{G}^{**} := \text{Hom}(\text{Hom}(\mathcal{G}, O_Y), O_Y)$ denotes the double dual.

Let’s recall Ejiri’s criterion for the weak positivity introduced in [14], Sec. 4.

Definition 2.9. Let $Y$ be a quasi-projective variety, $\mathcal{F}$ a torsion free coherent sheaf and $H$ an ample $\mathbb{Q}$-Cartier divisor on $Y$. Let

$$t(Y, \mathcal{F}, H) = \text{sup}\{a \in \mathbb{Q}| \text{the sheaf } (F_Y^a \mathcal{F}) \otimes O_Y([-p^a H])$$

is generically globally generated for some $e > 0\}.$

We say $\mathcal{F}$ is FWP (Frobenius weakly positive) if for an ample $\mathbb{Q}$-Cartier divisor $H$, $t(Y, \mathcal{F}, H) \geq 0$, equivalently, there exist a sequence of positive integers $\{n_e| e = 1, 2, 3, \ldots\}$ such that $n_e H$ is Cartier, the sheaf $(F_Y^{n_e} \mathcal{F}) \otimes O_Y(n_e H)$ is generically globally generated and $\frac{t(Y, \mathcal{F}, H)}{p^e} \rightarrow 0$ as $e \rightarrow +\infty$.

This property is independent of the choice of the ample divisor $H$.

Lemma 2.10. [13], Proposition 4.7] Let $Y$ be a normal quasi-projective variety, $Y_0 \subset Y$ an open set such that $\text{codim}_Y(Y \setminus Y_0) \geq 2$, $\mathcal{F}$ a torsion free coherent sheaf on $Y$. If $\mathcal{F}|_{Y_0}$ is FWP, then $\mathcal{F}$ is weakly positive.

2.4. Surjection of restriction maps. Recall a Keeler’s result.

Lemma 2.11 (26, Theorem 1.5). (Relative Fujita Vanishing) Let $f : X \rightarrow Y$ be a projective morphism over a noetherian scheme, $H$ an $f$-ample line bundle and $\mathcal{F}$ a coherent sheaf on $X$. Then there exists a positive integer $N$ such that, for every $n > N$ and every nef line bundle $L$

$$R^i f_*(\mathcal{F} \otimes H^n \otimes L) = 0,$$ if $i > 0$.

Lemma 2.12. Let $f : X \rightarrow Y$ be a surjective projective morphism between two projective varieties. Let $H$ be an ample line bundle and $\mathcal{F}$ a coherent sheaf on $X$. Then there exist a positive integer $N$ and a non-empty Zariski open set $Y_0 \subset Y$ such that, for every $n > N$, every nef line bundle $L$ on $X$ and every closed point $y \in Y_0$ the restriction map

$$r^{L,n}_{y} : H^0(X, \mathcal{F} \otimes H^n \otimes L) \rightarrow H^0(X_y, \mathcal{F} \otimes H^n \otimes L \otimes O_{X_y})$$
is surjective.

In particular for two nef Cartier divisors $A_1, A_2$ on $X$, if $A_1 + A_2$ is ample, then there exists an integer $M$ such that, for integers $m, n > M$ and closed point $y \in Y_0$ the restriction map below is surjective

$$r_{m,n}^y : H^0(X, \mathcal{F} \otimes \mathcal{O}_X(mA_1 + nA_2)) \to H^0(X_y, \mathcal{F} \otimes \mathcal{O}_X(mA_1 + nA_2) \otimes \mathcal{O}_{X_y}).$$

**Proof.** Consider the following commutative diagram

$$
\begin{array}{ccc}
X_\Delta & \xrightarrow{j} & X \times Y \\
\downarrow i & & \downarrow p_1 \\
\Delta & \xrightarrow{i} & Y \times Y
\end{array}
$$

where $i : \Delta \hookrightarrow Y \times Y$ denotes the diagonal embedding of $Y$, $X_\Delta = (X \times Y) \times Y \times Y \Delta$, $p_i, q_i, i = 1, 2$ denote the projection from $X \times Y, Y \times Y$ to the $i$th factors respectively.

Denote by $K$ the kernel of the restriction homomorphism $p_1^* \mathcal{F} \to p_1^* \mathcal{F} \otimes \mathcal{O}_{X_\Delta}$. Applying relative Fujita vanishing above, since $p_1^* \mathcal{H}$ is $p_2$-ample, we can find a positive integer $N$ such that, for every $n > N$, $i > 0$ and every nef line bundle $L$ on $X$

$$R^ip_{2,*}(K \otimes p_1^*(H^n \otimes L)) = 0 \text{ and } R^if_*((\mathcal{F} \otimes H^n \otimes L) = 0 \ (\clubsuit)).$$

Let $n > N$. Tensoring the exact sequence

$$0 \to K \to p_1^* \mathcal{F} \to p_1^* \mathcal{F} \otimes \mathcal{O}_{X_\Delta} \to 0$$

by the line bundle $p_1^*(H^n \otimes L)$ yields the exact sequence

$$0 \to K \otimes p_1^*(H^n \otimes L) \to p_1^*(\mathcal{F} \otimes H^n \otimes L) \to p_1^*(\mathcal{F} \otimes H^n \otimes L) \otimes \mathcal{O}_{X_\Delta} \to 0.$$

Applying the derived functor $Rp_{2,*}$ to the exact sequence above, by vanishing $\clubsuit$ we get a surjection

$$\alpha^{n,L} : p_{2,*}p_1^*(\mathcal{F} \otimes H^n \otimes L) \to p_{2,*}(p_1^*(\mathcal{F} \otimes H^n \otimes L) \otimes \mathcal{O}_{X_\Delta}).$$

Identifying $X_\Delta$ with $X$ via the isomorphism $p_1 \circ j$, and $p_2|_{X_\Delta} : X_\Delta \to Y$ with $f$, we can identify $p_{2,*}(p_1^*(\mathcal{F} \otimes H^n \otimes L) \otimes \mathcal{O}_{X_\Delta})$ with $f_*(\mathcal{F} \otimes H^n \otimes L)$. Then by

$$p_{2,*}p_1^*(\mathcal{F} \otimes H^n \otimes L) \cong H^0(X, \mathcal{F} \otimes H^n \otimes L) \otimes \mathcal{O}_Y \cong H^0(Y, f_*(\mathcal{F} \otimes H^n \otimes L)) \otimes \mathcal{O}_Y,$$

we can identify the map $\alpha^{n,L}$ with the natural map

$$\beta^{n,L} : H^0(Y, f_*((\mathcal{F} \otimes H^n \otimes L)) \otimes \mathcal{O}_Y \to f_*(\mathcal{F} \otimes H^n \otimes L).$$

There exists a non-empty open set $Y_0 \subset Y$ such that $\mathcal{F}$ is flat over $Y_0$. Since $R^1f_*(\mathcal{F} \otimes H^n \otimes L) = 0$ by $\spadesuit$, applying [20, Theorem 12.11] we have that for every closed point $y \in Y_0$,

$$f_*((\mathcal{F} \otimes H^n \otimes L) \otimes k(y) \cong H^0(X_y, \mathcal{F} \otimes H^n \otimes L \otimes \mathcal{O}_{X_y}).$$

Since $\beta^{n,L}$ is a surjection, we conclude that the restriction map

$$r_{y}^{n,L} : H^0(X, \mathcal{F} \otimes H^n \otimes L) \to H^0(X_y, \mathcal{F} \otimes H^n \otimes L \otimes \mathcal{O}_{X_y}).$$

is a surjection.

The remaining assertion is an easy consequence of the first one. $\square$
2.5. Minimal models of 3-folds. We collect some results on minimal models of 3-folds below, which will be used in this paper.

First recall a result of Kawamata adapted to char \( p > 0 \), and please refer to [8, Lemma 5.6] for a proof.

Lemma 2.13. Let \( f : X \to Z \) be a fibration between normal quasi-projective varieties over an algebraically closed field \( k \) with char \( k = p > 0 \). Let \( L \) be a nef \( \mathbb{Q} \)-Cartier divisor on \( X \) such that \( L|_F \sim \mathbb{Q} 0 \) where \( F \) is the generic fibre of \( f \). Assume \( \dim Z \leq 3 \). Then there exist a diagram

\[
\begin{array}{ccc}
X' & \phi \rightarrow & X \\
\downarrow f' & & \downarrow f \\
Z' & \psi \rightarrow & Z
\end{array}
\]

with \( \phi, \psi \) projective birational, and an \( \mathbb{R} \)-Cartier divisor \( D \) on \( Z' \) such that \( \phi^* L \sim \mathbb{Q} f'^* D \).

Theorem 2.14. Let \( (X, B) \) be a projective \( \mathbb{Q} \)-factorial klt pair of dimension 3 and \( f : X \to Y \) a fibration over an algebraically closed field \( k \) with \( \text{char} k = p > 5 \).

(1) If \( K_X + B \) is pseudo-effective over \( Y \), then \( (X, B) \) has a log minimal model over \( Y \).

(2) If \( K_X + B \) is not pseudo-effective over \( Y \), then \( (X, B) \) has a Mori fibre space over \( Y \).

(3) Assume that \( K_X + B \) is nef over \( Y \).

(3.1) If \( K_X + B \) or \( B \) is big over \( Y \), then \( K_X + B \) is semi-ample over \( Y \).

(3.2) If \( Y \) is a smooth curve and \( \kappa(X_\eta, (K_X + B)_\eta) \geq 0 \), then \( (K_X + B)_\eta \) is semi-ample on \( X_\eta \).

(3.3) If \( Y \) is a smooth curve and \( \kappa(X_\eta, (K_X + B)_\eta) = 0 \) or 2, then \( K_X + B \) is semi-ample over \( Y \).

(3.4) If \( Y \) is a smooth curve with \( g(Y) \geq 1 \) and \( \kappa(X_\eta, (K_X + B)_\eta) \geq 0 \), then \( K_X + B \) is nef.

(4) If \( Y \) is a non-uniruled surface and \( K_X + B \) is pseudo-effective over \( Y \), then \( K_X + B \) is pseudo-effective, and there exists a map to a minimal model \( \sigma : X \dashrightarrow \tilde{X} \) such that, the restriction \( \sigma|_{X_\eta} \) is an isomorphism to its image.

Proof. For (1) please refer to [15] and [6].

For (2), refer to [8].

For (3.1), please refer to [4], [14] and [8].

For (3.2) and (3.3) please refer to [7], Theorem 1.5 and 1.6 and the remark below 1.6. And (3.2) also can be obtained from [10].

Assertion (3.4) follows from the cone theorem [8, Theorem 1.1]. Indeed, otherwise we can find an extremal ray \( R \) generated by a rational curve \( \Gamma \), so \( \Gamma \) is contained in a fiber of \( f \) since \( g(Y) > 0 \), this contradicts that \( K_X + B \) is \( f \)-nef.

For (4), \( K_X + B \) is obviously pseudo-effective because otherwise, \( X \) will be ruled by horizontal rational curves (w.r.t. \( f \)) by (2), which contradicts that \( Y \) is non-uniruled. Notice that the locus contracted by an extremal contraction is uniruled (see the proof of [8, Lemma 3.2]). Then by running MMP, we get a needed map \( \sigma : X \dashrightarrow \tilde{X} \).
2.6. Covering Theorem. The result below is [21, Theorem 10.5] when \( X \) and \( Y \) are both smooth, and the proof also applies when they are normal.

**Theorem 2.15 (Covering Theorem).** Let \( f : X \rightarrow Y \) be a proper surjective morphism between normal projective varieties. If \( D \) is a Cartier divisor on \( Y \) and \( E \) an effective \( f \)-exceptional Cartier divisor on \( X \). Then

\[
\kappa(X, f^* D + E) = \kappa(Y, D).
\]

2.7. Easy subadditivity of Kodaira dimensions. The following result is known to experts, please refer to [7, Lemma 2.22] or [33, Lemma 4.2] for a proof.

**Lemma 2.16.** Let \( f : X \rightarrow Y \) be a fibration between normal projective varieties. Let \( D \) be an effective \( \mathbb{Q} \)-Cartier divisor on \( X \) and \( H \) a big \( \mathbb{Q} \)-Cartier divisor on \( Y \). Then

\[
\kappa(D + f^* H) \geq \kappa(X, D) + \dim Y.
\]

3. Proof of Theorem 1.11

Let \( Y_0 \) be a smooth open subset of \( Y \) such that \( \text{codim}_Y(Y \setminus Y_0) \geq 2 \).

By Proposition 2.7 we can assume \( g \) is divisible enough that for every positive integer \( e \), the sheaf \( S_{\Delta}^g f_* \mathcal{O}_X(D) \) has the stable rank \( \dim_k S_{\Delta}^g (X_\eta, D_\eta) \). Then for every integer \( e > 0 \), the composite homomorphism below is generically surjective

\[
\alpha^g : f_* \mathcal{O}_X((1 - p^g)(K_{X/Y}) - (p^g - 1)\Delta + p_{\alpha} D)|_{Y_0} \rightarrow (S_{\Delta}^g f_* \mathcal{O}_X(D))|_{Y_0} \rightarrow (F_Y^{(e-1)g} S_{\Delta}^g f_* \mathcal{O}_X(D))|_{Y_0},
\]

because the two sheaves \( S_{\Delta}^g f_* \mathcal{O}_X(D)|_{Y_0} \) and \( (F_Y^{(e-1)g} S_{\Delta}^g f_* \mathcal{O}_X(D))|_{Y_0} \) have the same rank.

Let \( H \) be an ample Cartier divisor on \( Y \). Tensoring the map \( \alpha^g \) by \( \mathcal{O}_Y(eH) \), we get a generically surjective homomorphism

\[
\beta^g : f_* \mathcal{O}_X((1 - p^g)(K_{X/Y}) - (p^g - 1)\Delta + p_{\alpha} D + e f^* H)|_{Y_0} \rightarrow F_Y^{(e-1)g} S_{\Delta}^g f_* \mathcal{O}_X(D) \otimes \mathcal{O}_Y(eH)|_{Y_0}.
\]

**Claim 3.1.** There is an integer \( e_0 \) such that, for every integer \( e > e_0 \) the sheaf \( f_* \mathcal{O}_X((1 - p^g)(K_{X/Y}) - (p^g - 1)\Delta + p_{\alpha} D + e f^* H) \) is generically globally generated.

**Proof of the claim.** Since \( D - K_{X/Y} - \Delta \) is \( f \)-semi-ample, we have two morphisms \( h : X \rightarrow Z \) and \( g : Z \rightarrow Y \), such that \( D - K_{X/Y} - \Delta \sim q h^* A \) where \( A \) is an ample \( \mathbb{Q} \)-Cartier divisor on \( Z \), which is also nef by assumption. Take an integer \( d > 0 \) such that \( A = dA' \sim d(D - K_{X/Y} - \Delta) \) is Cartier. Write that \( p^g - 1 = q_e d + r_e \) where \( q_e \) and \( r_e \) are integers such that \( 0 \leq r_e < d \). Then by \( f_* = g_* \circ h_* \), we have

\[
f_* \mathcal{O}_X((1 - p^g)K_{X/Y} - (p^g - 1)\Delta + p_{\alpha} D + e f^* H) \\
\cong f_* \mathcal{O}_X(q_e d(D - K_{X/Y} - \Delta) + e f^* H + (r_e + 1)D - r_e K_{X/Y} - [r_e \Delta]) \\
\cong g_* h_* \mathcal{O}_X(h^* q_e A + e h^* g^* H + (r_e + 1)D - r_e K_{X/Y} - [r_e \Delta]) \\
\cong g_* (O_Z(q_e A + e g^* H) \otimes h_* \mathcal{O}_X((r_e + 1)D - r_e K_{X/Y} - [r_e \Delta]))
\]

where the last "\( \cong \)" is due to projection formula. Note that the set

\[
\{h_* \mathcal{O}_X((r_e + 1)D - r_e K_{X/Y} - [r_e \Delta])|_{e = 0, 1, 2, \cdots}\}
\]

contains finitely many coherent sheaves. Since \( A + g^* H \) is ample, and both \( A \) and \( g^* H \) are nef, by Lemma 2.12 there exist a positive integer \( e_0 \) and a non-empty
Zariski open subset $Y'_0 \subset Y$ such that for every $e > e_0$ and $y \in Y'_0$, the restriction map

$$H^0(Y, f_*O_X((1-p^g)(K_{X/Y}) - [(p^g - 1)\Delta] + p^g D + ef^*H))$$

$$\cong H^0(Z, O_Z(q_e A + eg^*H) \otimes h_*O_X((r_e + 1)D - r_e K_{X/Y} - [r_e\Delta]))$$

$$\cong H^0(Z, O_Z(q_e A + eg^*H) \otimes h_*O_X((r_e + 1)D - r_e K_{X/Y} - [r_e\Delta]) \otimes O_{Z_y})$$

$$\cong g_*O_Z(q_e A + eg^*H) \otimes h_*O_X((r_e + 1)D - r_e K_{X/Y} - [r_e\Delta]) \otimes k(y)$$

$$\cong f_*O_X((1-p^g)(K_{X/Y}) - [(p^g - 1)\Delta] + p^g D + ef^*H) \otimes k(y)$$

is surjective.

The claim above implies that the image of $\beta^g$ is generically globally generated, hence so is the sheaf $F_Y^{(e-1)p}S_A^g f_*O_X(D) \otimes O_Y(eH)|_{Y_0}$. Therefore, the sheaf $S_A^g f_*O_X(D)$ is FWP, which implies that $S_A^g f_*O_X(D)$ is weakly positive by Lemma 2.10.

If $Y$ is smooth, letting $Y_0 = Y$, the argument above shows that $S_A^g f_*O_X(D)$ is FWP. So we complete the proof of Theorem 1.1.

4. Subadditivity of Kodaira dimensions

In this section, we will prove Theorem 1.5 Let’s begin with a theorem which is stimulated by [23], Lemma 4.4.

**Theorem 4.1.** Let $f : X \to Y$ be a separable fibration between normal projective varieties, and let $D$ be a Cartier divisor on $X$. Assume that for some positive integer $e$, the sheaf $F_Y^{e} f_*O_X(D) \otimes O_Y(eH)|_{Y_0}$. Then for any big $\mathbb{Q}$-Cartier divisor $H$ on $Y$, we have

$$\kappa(D + H) \geq \kappa(X_n, D|_{X_n}) + \dim Y.$$ 

**Proof.** Let $A$ be an ample $\mathbb{Q}$-Cartier divisor on $Y$ such that $H \geq 2A$.

By assumption $F_Y^e f_*O_X(D)$ contains a non-zero FWP sub-sheaf $F$. We can find positive integers $g, n_g << p^g$ such that, the sheaf $F_Y^{e+g}F \otimes O_Y(n_gA)$ is generically globally generated.

Consider the following commutative diagram

$$\begin{array}{ccc}
X' & \xrightarrow{\sigma'} & X_{e+g} & \xrightarrow{\sigma} & X \\
\downarrow{f'} & & \downarrow{f_{e+g}} & & \downarrow{f} \\
Y_{e+g} & \xrightarrow{f'} & Y & \xrightarrow{f} & Y
\end{array}$$

where $X'$ denotes the normalization of $X_{e+g}$ and $\sigma, \sigma', f'$ denote the natural maps.

By the commutative diagram above, there are natural inclusions

$$F_Y^{e+g} f_*O_X(D) \hookrightarrow f_{e+g} \pi_Y^{e+g} f_*O_X(D) \hookrightarrow f'_*O_X((\sigma'* \pi_Y^{e+g} D) = f'_*O_X((\sigma*D).$$

Therefore, the sheaf $f'_*O_X((\sigma*D) \otimes O_Y(n_gA)$ contains a generically globally generated sub-sheaf $F_Y^{g+e}F \otimes O_Y(n_gA)$. We can find an effective divisor $D'$ on $X'$ such that

$$D' \sim \sigma^*D + n_gf'^*A.$$
By $F_Y^{c+g} H = p^{c+g} H > 2n_g A$, we conclude that
\[
\kappa(X, D + f^* H) = \kappa(X', \sigma^* D + f^* F_Y^{c+g} H) \geq \kappa(X', \sigma^* D + 2n_g f^* A) \\
= \kappa(X', D' + n_g f^* A) \geq \kappa(X'_{\eta+g}, D'|_{X'_{\eta+g}}) + \dim Y \cdots \text{by Lemma 2.16}
\]
\[
\geq \kappa(X, \eta, D|_{X_{\eta}}) + \dim Y \cdots \text{since } D'_{|_{X_{\eta+g}}} = \sigma^* (D|_{X_{\eta}}).
\]
We complete the proof. 

\[\square\]

4.1. Proof of Theorem 1.5. We break the proof into several steps.

**Step 1:** We first consider the case that $Y$ is not smooth, hence $f : X \to Y$ is flat by assumption. By [22], there exists a smooth modification $\mu_1 : Y_1 \to Y$.

Let $\sigma : X_1 \to X \times_Y Y_1$ be the normalization map, which fits into the following commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\sigma} & X \\
\downarrow{f_1} & & \downarrow{f} \\
Y_1 & \xrightarrow{\mu_1} & Y.
\end{array}
\]

Applying Proposition 2.1, there exist an effective $\mathbb{Q}$-Weil divisor $\Delta_1$ and an effective $\sigma$-exceptional Cartier divisor $E_1$ on $X_1$ such that
\[
K_{X_1/Y_1} + \Delta_1 = \sigma_1^*(K_{X/Y} + \Delta) + E_1.
\]

Then $p \nmid \ind(K_{X_1/Y_1} + \Delta_1)_{\eta_1}$ where $\eta_1$ denotes the generic point of $X_1$. Let
\[
A_1 = \mu_1^* A \text{ and } D_1 = \sigma_1^* D + E_1.
\]

Consider the birational morphism $(X_1)_{\tilde{\eta}} \to X_{\tilde{\eta}}$. By Proposition 2.3 (4) we have
\[
\dim_{k(\tilde{\eta})} S^0_{(\Delta_1)_{\tilde{\eta}}} ((X_1)_{\tilde{\eta}}, (D_1)_{\tilde{\eta}}) \geq \dim_{k(\tilde{\eta})} S^0_{\Delta_{\eta}} (X_{\tilde{\eta}}, D_{\tilde{\eta}}) > 0.
\]

Note that
\[
D_1 - (K_{X_1/Y_1} + \Delta_1) - f_1^* A_1 = \sigma_1^* D + E_1 - (\sigma_1^* (K_{X/Y} + \Delta) + E_1) - f_1^* A_1 \\
= \sigma_1^* (D - K_{X/Y} - \Delta - f^* A)
\]
is nef and $f_1$-semi-ample.

If $Y$ is smooth, then let $Y_1 = Y$, $\mu_1 = \id_Y$ and other notation be introduced as above.

**Step 2:** For sufficiently large integer $g_0$, the integral part $A_2 = [p^{g_0} A_1]$ is big. Let $B = p^{g_0} A_1 - A_2$. On $Y_2 = Y_1^{g_0}$, we can write that
\[
F_{Y_1}^{g_0} A_1 = A_2 + B.
\]
Let $X'_2$ be the normalization of $X_2 = (X_1)_{Y_1} = X_1 \times_{Y_1} Y_2$. We have the following commutative diagram

$$
\begin{array}{ccc}
X'_2 & \xrightarrow{\sigma_2} & X_2 = (X_1)_{Y_1} \\
| & \downarrow{f'_2} & | \\
Y'_2 = Y_1 & \xrightarrow{f_2} & Y_1
\end{array}
$$

where $\sigma_2, \sigma'_2, \pi_{Y_1}, f_2$ and $f'_2$ denote the natural morphisms. Since $F_{Y_1}^{g_0}$ is flat, we have $K_{X'/Y_2} = \pi_{Y_1}^{g_0*} K_{X_1/Y_1}$. Let

$$
\Delta_2 = f'_2 B + \pi_{Y_1}^{g_0*} \Delta_1 \text{ and } D_2' = \sigma_2^* \pi_{Y_1}^{g_0*} D_1 = \sigma_2^* D_1.
$$

Then applying Proposition 2.3 (4) again, there exists an effective $\mathbb{Q}$-Weil divisor $\Delta'_2$ on $X'_2$ such that

$$
K_{X'_2/Y_2} + \Delta'_2 = \sigma_2^*(K_{X_2/Y_2} + \Delta_2) = \sigma_2^*(\pi_{Y_1}^{g_0*} K_{X_1/Y_1} + f'_2 B + \pi_{Y_1}^{g_0*} \Delta_1) = \sigma_2^*(K_{X_1/Y_1} + \Delta_1) + f'_2 B,
$$

thus

$$p \nmid \text{ind}(K_{X'_2/Y_2} + \Delta'_2)_{\eta_2} \text{ and } S_{\Delta'_2}^0((X'_2)_{\eta_2}, (D'_2)_{\eta_2}) \neq 0.
$$

We also have that

$$\text{(D}_2' - f'_2^* A_2) - K_{X'_2/Y_2} - \Delta'_2 = \sigma_2^*(D_1 - (K_{X_1/Y_1} + \Delta_1) - f_1^* A_1)$$

is nef and $f'_2$-semi-ample.

**Step 3:** Applying Theorem 1.11 on the pair $(X'_2, \Delta'_2)$, by Step 2 we have that for sufficiently divisible $e$, the sheaf $F_{Y_2}^{g_0*} f_{2*} \mathcal{O}_{X'_2}(D'_2 - f'_2^* A_2)$ contains a non-zero sub-sheaf. Then we conclude that

$$
\kappa(X, D) = \kappa(X_1, D_1) = \sigma_2^* D + E_1 = \kappa(X'_2, D'_2 = \sigma_2^* D_1) \cdots \text{by Covering Theorem 2.15}
$$

$$= \kappa(X'_2, (D'_2 - f'_2^* A_2) + f'_2^* A_2)
$$

$$\geq \dim Y + \kappa((X'_2)_{\eta_2}, (D'_2)_{\eta_2}) \cdots \text{by Theorem 1.11}
$$

$$\geq \dim Y + \kappa(X_{\bar{\eta}}, D_{\bar{\eta}}) \cdots \text{since } (D'_2)_{\eta_2} \cong \sigma_2^* \sigma_1^* D_{\eta}
$$

$$= \dim Y + \kappa(X_{\bar{\eta}}, D_{\bar{\eta}}) \cdots \text{since the natural map } \bar{\eta} \rightarrow \eta \text{ is flat.}
$$

**Step 4:** We are left to consider the case that $D$ is nef and $f$-big and prove the remaining assertion.

Take an ample divisor $A_1$ on $Y$. Then $D + f^* A_1$ is big. We can write that

$$D + f^* A_1 \sim_{\mathbb{Q}} H_1 + B_1$$

where $H_1$ is ample and $B_1$ is an effective $\mathbb{Q}$-Cartier divisor with $p \nmid \text{ind}(B_1)$. Take a rational number $\delta > 0$ small enough such that

(i) $A' = A - \delta A_1$ is big on $Y$; and

(ii) $p \nmid \text{ind}(\delta B_1).

Let $\Delta' = \Delta + \delta B_1$. Then for sufficiently large $m$, since $D$ is nef and $f$-big we have

(a) $mD - (K_{X_1/Y} + \Delta') - f^* A' = (m - 1 - \delta)D + \delta(D + f^* A_1 - B_1) + (D - (K_{X_1/Y} + \Delta) - f^* A)$ is ample by condition (2'); and

(b) $S_{\Delta'}^0(X_{\bar{\eta}}, mD_{\bar{\eta}}) \neq 0$ by Proposition 2.3 (3).
Finally applying the result in previous steps on the pair \((X, \Delta')\), we show that \(mD\) is big and complete the proof.

5. Application to three-folds

5.1. Proof of Corollary \([1, 8]\) We can assume \(\kappa(X_\eta, K_{X_\eta}) \geq 0\). By assumption, \(K_X\) is pseudo-effective, hence \(X\) has a minimal model \(\tilde{X}\) by Theorem \([2, 14]\). Denote by \(\sigma : X \to \tilde{X}\) the natural map, which is assumed to be a morphism by blowing up \(X\) if necessary.

Case \(\dim Y = 1\).

In this case, by Theorem \([2, 14]\) the map \(f\) factors through a fibration \(\tilde{f} : \tilde{X} \to Y\). Since \(X_\eta\) is regular, it has a regular minimal model by \([30], \text{Sec. 9.3}\). We can assume that \(\tilde{X}_\eta\) is regular, and if necessary, replacing \(X\) by some resolution of \(\tilde{X}\), assume that \(\tilde{X}_\eta = X_\eta\). So the \(\sigma\)-exceptional divisors are contained in finitely many fibers of \(f\). With the proof postponed, we claim that

Claim 5.1. If necessary, blowing up \(X\) along centers contained in finitely many fibers of \(f\), we can assume there exists an effective \(\mathbb{Q}\)-Cartier divisor \(B\) contained in finitely many fibers of \(f\) such that, for every rational number \(\delta \in (0, 1]\), the divisor \(\sigma^*K_{\tilde{X}} - \delta B\) is \(f\)-semi-ample.

Let

\[E_1 = (l - 1)K_X - \sigma^*(l - 1)K_{\tilde{X}}\]

which is an effective \(\sigma\)-exceptional \(\mathbb{Q}\)-Cartier divisor, thus is contained in finitely many fibers of \(f\). Let

\[\Delta = E_1 + \delta B\]

and \(D^l = lK_X = \sigma^*(l - 1)K_{\tilde{X}} + K_X + E_1\).

Take a closed point \(P\) on \(Y\). Then

1. since \(K_{\tilde{X}}\) is nef, taking rational numbers \(0 < t << 1, 0 < \delta << t\), we see that for \(l \geq 2\), the divisor

\[D^l - (K_{X/Y} + \Delta) - f^*(K_Y - P)\]

is \(f\)-semi-ample by the 1st \(\sim_\mathbb{Q}\) and nef by the 2nd \(\sim_\mathbb{Q}\);

2. \(S^R_{\eta} = (X_\eta, (D^l)_{X_\eta}) = S^0(\eta, lK_{X_\eta}) \neq 0\) since \(\Delta_\eta = 0\) and \((D^l)_{X_\eta} = lK_{X_\eta}\).

Since \(g(Y) > 1\), \(K_Y - P\) is ample, applying Theorem \([1, 5]\) on the pair \((X, \Delta)\), we conclude that

\[\kappa(X) = \kappa(X, lK_X) \geq \kappa(X_\eta, K_{X_\eta}) + \dim Y.\]

Proof of the claim above. By Theorem \([2, 14]\) \(K_{X_\eta}\) is semi-ample, and \(K_{X/Y}\) is \(f\)-semi-ample if \(\kappa(X_\eta, K_{X_\eta}) = 0\) or 2. So in case \(\kappa(X_\eta, K_{X_\eta}) = 0\) or 2, we can set \(B = 0\).

In the following we assume \(\kappa(X_\eta, K_{X_\eta}) = 1\). Consider the relative Iitaka fibration \(h : X \to Z\) of \(X\) over \(Y\), which is assumed to be a morphism by blowing up \(X\) along centers contained in finitely many fibers of \(f\). So \(f\) factors as

\[f = g \circ h : X \to Z \to Y.\]

Notice that \(\sigma^*K_{\tilde{X}}|_{X_\xi} \sim_\mathbb{Q} 0\) where \(X_\xi\) denotes the generic fiber of \(h : X \to Z\). Applying Lemma \([2, 13]\) by blowing up \(X\) and \(Z\) along centers contained in finitely many
many fibers of $f$ and $g$ respectively, we can assume $Z$ is smooth, and there exists a $g$-big divisor $H$ on $Z$ such that $\sigma^*K_{\tilde{X}} = h^*H$. Then $H$ is nef and $H|_{Z_{\eta}}$ is ample. The relative exceptional locus $E(H)$ of $H$ is contained in finitely many fibers of $g$, but does not contain a whole fiber. Write that $E(H) = E_1 \cup E_2 \cup \cdots \cup E_n$ where the $E_i$’s are reduced integral curves on $Z$. The matrix $(E_i \cdot E_j)_{ij}$ is negative definite by [30, Chap. 9 Theorem 1.23]. Then we can find sufficiently small positive rational numbers $a_i$’s such that $H - \sum a_i E_i$ is $g$-ample. We see that for every rational number $0 < \delta \leq 1$, the divisor $H - \delta \sum a_i E_i$ is $g$-ample. Set $B = h^* \sum a_i E_i$. Then we are done. □

**Case dim $Y = 2$.**

In this case, $X_{\eta}$ is a regular curve over the field $k(\eta)$ and $\sigma^*K_{\tilde{X}}|_{X_{\eta}} = K_{X_{\eta}}$ by assumption (3). We argue case by case according to $\kappa(X_{\eta}, K_{X_{\eta}})$ is 0 or 1.

**Subcase** $\kappa(X_{\eta}, K_{X_{\eta}}) = 0.$ Then $\sigma^*K_{\tilde{X}}|_{X_{\eta}} = K_{X_{\eta}}$ is trivial. Since $K_{\tilde{X}}$ is nef, by Lemma 2.13 we get a fibration $f_1 : X_1 \to Y_1$ between two smooth projective varieties and two birational morphisms $\sigma_1 : X_1 \to X$ and $\eta_1 : Y_1 \to Y$, which fit into the following commutative diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\sigma_1} & X \\
| & f_1 \downarrow & \sigma \downarrow \\
Y_1 & \xrightarrow{\eta_1} & Y \\
\end{array}
$$

such that, there is a divisor $A_1$ on $Y_1$ such that $\sigma_1^*\sigma^*K_{\tilde{X}} = f_1^*A_1$. Obviously, $(X_1)_{\eta} = X_{\eta}$.

There is an effective $\sigma \circ \sigma_1$-exceptional divisor $E_1 \sim (l-1)K_{X_1} - \sigma_1^*\sigma^*(l-1)K_{\tilde{X}}$. Let

$$
\Delta_1 = E_1 \text{ and } D_1 = lK_{X_1} = \sigma_1^*\sigma^*(l-1)K_{\tilde{X}} + X_{\eta} + E_1.
$$

Then

$$
D_1 - (K_{X_1/Y_1} + \Delta_1) - f_1^*K_{Y_1} = \sigma_1^*\sigma^*(l-1)K_{\tilde{X}}
$$

is nef and $f_1$-semi-ample; and by assumption (2)

$$
S^0_{(\Delta_1)_{\bar{\eta}}}((X_1)_{\bar{\eta}}, (D_1)_{\bar{\eta}}) = S^0_{(X_{\eta}, lK_{X_{\eta}})} \neq 0.
$$

Finally applying Theorem 1.5 on the pair $(X_1, \Delta_1)$, we prove that

$$
\kappa(X) = \kappa(X_1, lK_{X_1}) \geq \kappa(X_{\eta}, K_{X_{\eta}}) + \dim Y.
$$

**Subcase** $\kappa(X_{\eta}, K_{X_{\eta}}) = 1$: Then by assumption (3), the divisor $\sigma^*K_{\tilde{X}}$ is nef and $f$-big. We will imitate Step 4 of proof of Theorem 1.5.

Take an ample divisor $A_1$ on $Y$. Then $\sigma^*K_{\tilde{X}} + f^*A_1$ is big. We can write that $\sigma^*K_{\tilde{X}} + f^*A_1 \sim_{\mathbb{Q}} H_1 + B_1$ where $H_1$ is ample and $B_1$ is an effective $\mathbb{Q}$-Cartier divisor $p \cdot \text{ind}(B_1)$. Take a rational number $\delta > 0$ small enough such that

(i) $A = K_Y - \delta A_1$ is big on $Y$; and

(ii) $p \cdot \text{ind}(\delta B_1)$.

Let $\Delta = \delta B_1$. Then for sufficiently large $m$ such that $mK_{\tilde{X}}$ is Cartier, we have

(a) $(m\sigma^*K_{\tilde{X}} + K_X) - (K_{X'/Y} + \Delta) - f^*A = (m - \delta)\sigma^*K_{\tilde{X}} + \delta(\sigma^*K_{\tilde{X}} + f^*A_1 - B_1)$ is ample; and

(b) $S^0_{(X_{\eta}, (m\sigma^*K_{\tilde{X}} + K_X)_{\bar{\eta}})} = S^0_{(X_{\eta}, (m + 1)K_{X_{\eta}})} \neq 0$ by Proposition 2.3 (3).
Applying Theorem 1.15 on the pair $(X, \Delta)$, we conclude that
\[ \kappa(X) = \kappa(X, m\sigma^*K_X + K_X) \geq \dim Y + 1. \]

5.2. Proof of Corollary 1.10. Denote by $\sigma : X \to \tilde{X}$ a map to the minimal model of $X$ which can be assumed to be a morphism. We get a fibration $\tilde{f} : \tilde{X} \to Y$.

Combining Corollary 1.8 Proposition 2.3 (3) and Theorem 2.14 (3.4, 4), we can prove assertion (1).

We are left to prove assertion (2). Assume $g(Y) > 1$ and $\kappa(\tilde{X}_{\bar{\eta}}) \geq 0$ where $\tilde{X}_{\bar{\eta}}$ is a smooth projective model of $X_{\bar{\eta}}$. Up to a power of Frobenius base changes and a smooth resolution ([17], proof of Corollary 1.3]), we reduce to considering the case that geometric generic fiber $X_{\bar{\eta}}$ is smooth, i.e., $X_{\eta}$ is smooth over $\eta$. We only need to consider the cases $\kappa(X_{\bar{\eta}}) = 0$ or 1.

If $\kappa(X_{\bar{\eta}}) = 0$, then $K_{\tilde{X}/Y}$ is $f$-semi-ample by Theorem 2.14. Notice that general fibers of $\tilde{f}$ have canonical singularities, which are strongly $F$-regular because $\text{char} k > 5$. Applying [32, Theorem 3.16], we have that $K_{\tilde{X}/Y}$ is nef, so there exists a nef $\mathbb{Q}$-Cartier divisor $M$ on $Y$ such that
\[ K_{\tilde{X}/Y} \sim_\mathbb{Q} \tilde{f}^*M. \]

It is easy to conclude that
\[ \kappa(X) = \kappa(X, K_{\tilde{X}}) = \kappa(X, K_{\tilde{X}/Y} + \tilde{f}^*K_Y) = \kappa(Y, K_Y + M) = 1 = \dim Y. \]

If $\kappa(X_{\bar{\eta}}) = 1$, then $X_{\bar{\eta}}$ is a smooth surface over $k(\bar{\eta})$, whose geometric generic Iitaka fiber is a smooth elliptic curve. We break the proof into three steps.

Step 1: The generic fiber $X_{\bar{\eta}}$ has been assumed to be smooth over $k(\eta)$, so it has a smooth minimal model with the canonical divisor being semi-ample. So in the following, we can assume $X_{\bar{\eta}}$ is smooth over $k(\eta)$, and $X_{\eta} \cong X_{\bar{\eta}}$.

Similarly as in the proof of Claim 5.1 considering the relative Iitaka fibration, if necessary blowing up $X$ along centers contained in finitely many fibers of $f$, we can assume $f = g \circ h : X \to Z \to Y$ where $Z$ is a smooth projective surface and $h : X \to Z$ is an elliptic fibration, such that $\sigma^*K_X \sim_\mathbb{Q} h^*H$ for a nef and $g$-big $\mathbb{Q}$-Cartier divisor $H$ on $Z$. By Nakayama’s proof (cf. [12, Lemma 7.3]), flattening $\sigma$ by means of Hilbert scheme, then resolving singularities, we get the following commutative diagram
\[
\begin{array}{ccccccc}
X' & \xrightarrow{\mu} & X & \xrightarrow{\sigma} & \tilde{X} \\
\downarrow{\nu} & & \downarrow{\kappa} & & \downarrow{\tilde{f}} \\
Z' & \xrightarrow{\nu'} & Z & \xrightarrow{g} & Y
\end{array}
\]

where $\nu : Z' \to Z$ and $\mu : X' \to X$ are two birational morphisms, $X'$ and $Z'$ are assumed to be smooth projective varieties, $h' : X' \to Z'$ is a fibration such that every prime $h'$-exceptional divisor is also exceptional with respect to $\mu$, and $\sigma'$, $g'$ denote the composite morphisms $\sigma \circ \mu, g \circ \nu$ respectively.

Step 2: By [12, 3.2], we have $\kappa(X', K_{X'/Z'}) \geq 0$, so $K_{X'/Z'} \sim_\mathbb{Q} B_1'$ where $B_1'$ is an effective divisor on $X'$. Let $H' = \nu'^*H$. There exists an effective $\sigma'$-exceptional divisor $E_1'$ such that $K_{X'/Z'} \sim_\mathbb{Q} \sigma'^*K_X + E_1'$. Then
\[ h'^*H' \sim_\mathbb{Q} \sigma'^*K_X \sim_\mathbb{Q} K_{X'/Z'} - E_1' \sim_\mathbb{Q} h'^*K_{Z'} + B_1' - E_1' = h'^*K_{Z'} + B_2' - E_2' (\bigcirc) \]
where $B'_2$ and $E'_2$ are assumed to be effective divisors without common components. Observe that both $B'_2$ and $E'_2$ are vertical with respect to $h'$. Then since $B'_2 - E'_2 = h'^*(H' - K_{Z'})$ is $h'$-nef, we conclude that $h'(B'_2)$ and $h'(E'_2)$ have no common divisorial component on $Z'$. So we can write that

$$B'_2 - E'_2 = h'^*(\Delta'_1 - \Delta'_2)$$

where $\Delta'_1$ and $\Delta'_2$ are effective $\mathbb{Q}$-Cartier divisors supported on $h'(B'_2)$ and $h'(E'_2)$ respectively. In particular, we can write that

$$h'^*(\Delta'_2) = E'_2 + N$$

where $N$ is an $h'$-exceptional divisor (not necessarily effective), thus the divisor $h'^*(\Delta'_2)$ is an effective $\sigma'$-exceptional divisor. Therefore, applying Covering Theorem 2.15 we conclude that for any positive integers $a$ and $b$,

$$\kappa(X) = \kappa(X', K_{X'}) = \kappa(X', a\sigma' K_{X'} + bh'^*(\Delta'_2)) = \kappa(Z', aH' + b\Delta'_2).$$

And by relation (5), we get that

$$h'^*(H' + \Delta'_2) \sim_\mathbb{Q} h'^*(K_{Z'} + \Delta'_1).$$

**Step 3:** Let $a, b$ be positive integers such that $aH'$ and $b\Delta'_2$ are Cartier divisors. For sufficiently large integer $v$, we have $p \nmid \text{ind}(\Delta'_1 + \frac{1}{p^{v-1}} \Delta'_1)$.

Take a closed point $P \in Y$. Since $H'$ is nef on $Z'$ and $H'|_{Z'}$ is ample, applying the proof of Claim 5.1, we can find an effective $\mathbb{Q}$-Cartier divisor $B'$ on $Z'$ such that

1. $B'$ is contained in the union of finitely many fibers of $g'$;
2. the divisor $H' - B' + \frac{1}{2}g'^*P$ is ample.

So for sufficiently large $v$, the divisor $H' + \frac{1}{2}g'^*P - B' - \frac{1}{p^{v-1}}\Delta'_1$ is ample too.

Let

$$D'_{a,b} = aH' + b\Delta'_2$$

and

$$\Delta' = (1 + \frac{1}{p^v - 1})\Delta'_1 + (b - 1)\Delta'_2 + 2B'.$$

Then for $a > 2$, the divisor

$$D'_{a,b} - (K_{Z'/Y} + \Delta') - g'^*(K_Y - P)$$

$$\sim_\mathbb{Q} \left( (a - 2)H' + \frac{1}{2}g'^*P - B' \right) + (H' + \frac{1}{2}g'^*P - B' - \frac{1}{p^{v-1}}\Delta'_1)$$

is nef and $g'$-ample.

Since $H'|_{Z'_{\eta}}$ is ample and $\Delta'_2|_{Z'_{\eta}} = 0$, applying Proposition 2.23 we have that for sufficiently divisible $a$

$$S^0_{\Delta'_1}(Z'_{\eta}, (D'_{a,b})_{\eta}) = S^0_{\Delta'_1}(Z'_{\eta}, aH'_\eta) \neq 0.$$

Then by the fact that $K_Y - P$ is ample on $Y$, applying Theorem 1.20 on the pair $(X', \Delta')$ we can prove that

$$\kappa(X) = \kappa(Z', aH' + b\Delta'_2) = \kappa(Z', D'_{a,b}) = 2.$$

Finally we complete the proof.
6. Further Questions

**Question 1.** Is subadditivity of Kodaira dimensions true for inseparable fibrations?

Our approach does not apply to inseparable fibrations mainly because, for an inseparable fibration \( f : X \to Y \) there exists a flat base change \( Y' \to Y \) such that the fiber product \( X \times_Y Y' \) is not reduced, but some techniques, say, Covering Theorem 2.13 and Lemma 2.16, do not hold for non-reduced varieties.

**Question 2.** Let \( f : X \to Y \) be a fibration from a klt projective 3-fold to a smooth projective surface. Assume \( \kappa(Y) \geq 0 \) and \( K_{X/Y} \) is \( f \)-nef. Is \( K_X \) pseudo-effective? And if moreover \( Y \) is minimal, then is \( X \) minimal too?

If the answer to this question is positive, then condition (3) in Corollary 1.8 can be removed, so \( C_{3,m}, m = 1, 2 \) holds true for a separable fibration under the situation that \( Y \) is of general type and \( K_X \) is \( f \)-big.

**Question 3.** For a fibration \( h : X \to Z \) between two klt projective varieties such that \( K_X \) is relatively \( \mathbb{Q} \)-trivial over \( Z \), is there a “canonical bundle formula”: \( K_X \sim_{\mathbb{Q}} h^*(K_Z + \Delta) \) where \( \Delta \) is an effective divisor on \( Z \)?

Our strategy to prove \( C_{n,m} \) heavily relies on non-vanishing of \( S^0(X_\eta,lK_{X_\eta}) \). However, this often fails, for example when \( X_\eta \) is a supersingular elliptic curve. We have known that \( S^0(X_\eta,lK_{X_\eta}) \) does not vanish if \( K_{X_\eta} \) is big and \( l >> 0 \). An idea to overcome this difficulty is to consider the relative Iitaka fibration \( h : X \to Z \), which has been used to prove \( WC_{3,1} \) in the previous section. If we have a “canonical bundle formula” as above, we can reduce to studying \( \kappa(Z,K_Z + \Delta) \) with \( K_Z + \Delta \) being relatively big over \( Y \). Over complex numbers this is true, more precisely \( \Delta = B + M \) is the sum of discriminant part and moduli part (cf. [15], Theorem 4.5] and [2, Theorem 0.2]). In positive characteristics, we have “canonical bundle formula” under the situation that the geometric generic fiber \( X_\xi \) of \( h \) is globally \( F \)-split (cf. [14], Theorem 3.18] or [13, Theorem B]), or that \( X_\xi \) is a smooth elliptic curve.

7. Appendix: A Numerical Criterion for a Fibration to be Isotrivial

This appendix, though having little relation with the main part of this paper from the point of view of the method, is written to suggest an idea to treat a fibration \( f : X \to Y \) to a curve with \( g(Y) = 1 \) and \( \deg f_*\omega^1_{X/Y} = 0 \) (or \( \deg S^g f_*\omega^1_{X/Y} = 0 \)). In particular, we give a new proof to Ejiri’s result \( C_{3,1} \) in [14] for a fibration \( f : X \to Y \) over the field \( k \) with \( \char k > 5 \), under the situation that general fibers are smooth surfaces of general type. According to Ejiri’s proof, the most difficult case happens when \( g(Y) = 1 \) and \( \deg f_*\omega^1_{X/Y} = 0 \). To treat this case, Ejiri [14] uses deep results of vector bundles on elliptic curves and a very clever trick, and our strategy is to show that \( f \) is birationally isotrivial.

Recall that over complex numbers \( \mathbb{C} \), if the geometric generic fiber has good minimal model and \( \deg f_*\omega^1_{X/Y} = 0 \) for sufficiently divisible positive integer \( l \), then \( f \) is birationally isotrivial, i.e., general fibers are birationally equivalent to each other (cf. [25], Theorem 1.1]). In positive characteristics, analogously we have the following proposition, which is related to [32, Corollary 4.1] and may be known to some experts.
Proposition 7.1. Let \( f : X \to Y \) be a fibration from a normal projective variety to a smooth projective curve over an algebraically closed field \( k \) with \( \text{char} \ k > 5 \). Assume that

1. \( K_{X/Y} \) is \( \mathbb{Q} \)-Cartier and \( f \)-ample;
2. a general fiber has strongly \( F \)-regular singularities;
3. for a general fiber \( X_y \) the automorphism group \( \text{Aut}(X_y) \) is a finite scheme;
4. \( \deg f_*\omega^{\text{ld}}_{X/Y} = 0 \) for sufficiently divisible positive integer \( l \).

Then \( f \) is isotrivial, i.e., general fibers are isomorphic to each other. Moreover, there exist a smooth projective curve \( Y' \) and a flat base change \( \pi : Y' \to Y \) such that \( X \times_Y Y' \) is birationally equivalent to \( F \times Y' \), where \( F \) denotes a general fiber of \( f \).

Proof. We will follow Kollár’s idea in [28, Sec. 3] and use the notation therein.

By the given conditions (1) and (2), we can find a sufficiently divisible \( d \) such that for every positive integer \( l \),

(a) the sheaf \( f_*\omega^{\text{ld}}_{X/Y} \) is nef (applying [32, Corollary 3.18]);
(b) the natural map \( \eta' : \text{Sym}^l(f_*\omega^{\text{ld}}_{X/Y}) \to f_*\omega^{\text{ld}}_{X/Y} \) is generically surjective.

Since the image of \( \eta' \) is nef and \( \deg f_*\omega^{\text{ld}}_{X/Y} = 0 \), \( \eta' \) must be surjective. Let \( r_l = \text{rank} f_*\omega^{\text{ld}}_{X/Y} \) and \( w_l = \text{rank} \text{Sym}^l(f_*\omega^{\text{ld}}_{X/Y}) \). Let \( G = \text{GL}_{r_l}, \) the structure group of \( f_*\omega^{\text{ld}}_{X/Y} \) and \( \text{Sym}^l(f_*\omega^{\text{ld}}_{X/Y}) \).

Then for sufficiently large \( l \), the classifying map
\[
\eta_{Gr} : \{ \text{closed point of } Y \} \to \text{Gr}(w_l, r_l)/G
\]
satisfies that
- every fiber of \( \eta_{Gr} \) is finite if \( f \) is not isotrivial since \( Y \) is a curve; and
- for general \( y \in Y \), only finitely many elements of \( G/k^* \cong PGL_{r_l} \) leave \( \eta_y \) invariant since \( \text{Aut}(X_y) \) is assumed to be finite.

Applying the proof of [28, Lemma 3.13], we can conclude that \( f \) is isotrivial.

For the remaining assertion, we consider the natural morphism
\[
F \times \text{Mor}(F, X) \to X
\]
where \( \text{Mor}(F, X) \) denotes the scheme parameterizing the morphisms from \( F \) to \( X \), every component of which is a scheme of finite type over \( k \). Since \( f \) is isotrivial, we can find a reduced component \( Y' \) of \( \text{Mor}(F, X) \) such that
- the natural map \( F \times Y' \to X \) is dominant, and
- for general closed point \( y' \in Y' \), \( y' \) corresponds to an isomorphism \( F \to X_y \) for some closed point \( y \in Y \), which defines a dominant map \( Y' \to Y \).

Since \( \text{Aut}(F) \) is finite, \( \dim Y' = 1 \). Replacing \( Y' \) by its smooth projective model, we complete the proof.

Corollary 7.2. Let \( f : X \to Y \) be a fibration from a smooth projective 3-fold to a smooth projective curve over an algebraically closed field \( k \) with \( \text{char} \ k > 5 \). Assume that general fibers are smooth and of general type.

If \( \deg f_*\omega_{X/Y} = 0 \) for sufficiently divisible positive integer \( l \), then \( f \) is birationally isotrivial, i.e., general fibers are birationally equivalent to each other.

Proof. Consider the relative canonical model \( \bar{f} : \bar{X} \to Y \). Then \( K_{\bar{X}/Y} \) is \( \bar{f} \)-ample, and general fibers of \( \bar{f} \) have canonical singularities, which are strongly \( F \)-regular since \( \text{char} \ k > 5 \). It is known that for a general fiber \( X_y \) the automorphism group
Aut($X_y$) is finite (cf. [4]). Then the assertion follows from applying Proposition 7.1.

**Corollary 7.3.** Let the notation be as in Corollary 7.2. Then

$$\kappa(X) \geq \kappa(Y) + \kappa(X_{\eta}).$$

**Proof.** Here we can assume that $g(Y) \geq 1$. By [32], Theorem 3.18, $f_*\omega_{X/Y}^l$ is a nef vector bundle on $Y$ for sufficiently divisible positive integer $l$. If $\deg f_*\omega_{X/Y}^l > 0$ for some $l$, then we are done by [7], Proposition 5.1.

So we can assume $\deg f_*\omega_{X/Y}^l = 0$ for any sufficiently divisible positive integer $l$. By Corollary 7.2, there exists a smooth projective curve $Y'$ and a flat base change $\pi: Y' \to Y$ such that $X \times Y'$ is birationally equivalent to $F \times Y'$ where $F$ is a general fiber of $Y$. Let $X'$ be a smooth resolution of $X \times Y'$. Then by Theorem 2.1 and Covering Theorem 2.15, we have that

$$\kappa(X, K_{X/Y}) \geq \kappa(X', K_{X'/Y'}) = \kappa(F) = \kappa(X_{\eta}) = 2,$$

and thus

$$\kappa(X, K_X) \geq \kappa(Y) + \kappa(X_{\eta}).$$

□

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