$N = 2$ Superfields and the M-Fivebrane

N.D. Lambert
and
P.C. West

Department of Mathematics
King’s College, London
England
WC2R 2LS

ABSTRACT

In this paper we provide a manifestly $N = 2$ supersymmetric formulation of the M-fivebrane in the presence of threebrane solitons. The superspace form of four-dimensional effective equations for the threebranes are readily obtained and yield the complete Seiberg-Witten equations of motion for $N = 2$ super-Yang-Mills. A particularly simple derivation is given by introducing an $N = 2$ superfield generalisation of the Seiberg-Witten differential.

* lambert, pwest@mth.kcl.ac.uk
1. Introduction

One of the most interesting recent developments is the connection between classical brane dynamics and quantum field theories. This work was initiated by Witten [1] who studied a configuration of NS-fivebranes and D-fourbranes in type IIA string theory, for which the low energy effective action is a four-dimensional $N = 2$ Yang-Mills theory. From the M theory point of view this configuration is a single M-fivebrane with a complicated set of self-intersections and it was argued in [1] that it is in fact wrapped on a Riemann surface $\Sigma$. This identification provides a systematic explanation of the origin of the auxiliary curve in the Seiberg-Witten solution of $N = 2$ Yang-Mills [2] for various gauge groups and matter content and also illuminates many qualitative features of quantum Yang-Mills theories. A similar role for the Riemann surface also appeared in [3] when considering Calabi-Yau compactifications of type II strings.

In [4] the worldvolume soliton solution for the intersection of two M-fivebranes (or self-intersection of a single M-fivebrane) was found. The Riemann surface then appears naturally as a consequence of the Bogomoln’yi condition. It was further shown in [5,6] that the classical effective action for this soliton could be calculated from the M-fivebrane’s equations of motion and the lowest order terms are precisely the full Seiberg-Witten effective action for quantum $N = 2$ Yang-Mills theory. Thus the classical M-fivebrane dynamics contains the entire quantum effects of low energy $N = 2$ $SU(2)$ Yang-Mills theory. This correspondence not only predicts the correct known perturbative contributions to the $\beta$-function [7] but also an infinite number of instanton corrections, of which only the first two have been found by explicit calculation [8].

In this paper we will restrict our attention to the case of two threebranes whose centre of mass is fixed. However our work can be readily extended to the more general case. In the corresponding type IIA picture one sees two NS-fivebranes with two D-fourbranes suspended between them so that the low energy effective action is $N = 2$ $SU(2)$ Yang-Mills [1]. We provide a complete $N = 2$ superspace
proof of the equivalence between the low energy motion of threebranes in the M-fivebrane and the low energy Seiberg-Witten effective action for quantum $N = 2$ $SU(2)$ super-Yang-Mills, extending the results of [5,6] to include the fermionic zero modes. However our primary motivation for this work is to see what insights and simplifications can be gained into the M-fivebrane/Seiberg-Witten correspondence from $N = 2$ superspace, which provides a geometrical unification of all of the zero modes. Indeed many of the subtleties which appear in the purely bosonic formulations can be clearly observed and understood in $N = 2$ superspace.

2. $N = 2$ and $N = 1$ Superfields

Let us first recall how the $N = 2$ chiral effective action decomposes in terms of $N = 1$ superfields. In this paper $N = 2$ superfields are denoted by calligraphic or bold faced letters, $N = 1$ superfields by upper case letters and $N = 0$ fields by lower case letters. $N = 2$ Yang-Mills theory is described by a chiral superfield $\mathcal{A}$ [9]

$$D^j_B A = 0 , \quad (2.1)$$

which also satisfies the constraint

$$D^{ij} A = \bar{D}^{ij} \bar{A} , \quad (2.2)$$

where $D^{ij} = D^{AI} D^j_A$ and $\bar{D}^{ij} = \bar{D}^{AI} \bar{D}^j_A$. This constraint then ensures that the vector component obeys the Bianchi identity and that the auxiliary field is real. The chiral effective action may then be written as

$$S = \text{Im} \int d^4 x d^4 \theta F(\mathcal{A}) . \quad (2.3)$$

The solution to the above constraints in the Abelian case is of the form [10]

$$\mathcal{A} = \bar{D}^4 D^{ij} \mathcal{V}_{ij} , \quad (2.4)$$

where $\mathcal{V}_{ij}$ is an unconstrained superfield of mass dimension $-2$. Varying the action
(2.3) with respect to $\mathcal{V}_{ij}$ yields

$$\text{Im} \int d^4x d^4\theta \bar{D}^j D^i \delta \mathcal{V}_{ij} \frac{dF(A)}{dA} = 16 \text{Im} \int d^4x d^8\theta \delta \mathcal{V}_{ij} D^{ij} \left( \frac{dF(A)}{dA} \right), \quad (2.5)$$

and hence we find the equation of motion to be

$$D^{ij} \frac{dF}{dA} = \bar{D}^{ij} \frac{d\bar{F}}{d\bar{A}}. \quad (2.6)$$

We observe that if we write $A_D = dF/dA$ and $A$ as a doublet then the constraint (2.2) and the equations of motion (2.6) can be written as

$$D^{ij} \begin{pmatrix} A_D \\ A \end{pmatrix} = \bar{D}^{ij} \begin{pmatrix} \bar{A}_D \\ \bar{A} \end{pmatrix}. \quad (2.7)$$

Clearly this system has a set of equations invariant under

$$\begin{pmatrix} A_D \\ A \end{pmatrix} \rightarrow \Omega \begin{pmatrix} A_D \\ A \end{pmatrix}, \quad (2.8)$$

where $\Omega \in SL(2, \mathbb{Z})$. Thus an $SL(2, \mathbb{Z})$ symmetry is naturally realised in $N = 2$ superspace [11].

For a free theory $F = iA^2$ and we find the equation of motion and constraint imply $D^{ij}A = \bar{D}^{ij}A = 0$. At $\theta^A = 0$ this equation sets the auxiliary field to zero and ensures the correct equation of motion for the Yang-Mills theory.

We now solve the $N = 2$ superspace constraints in terms of $N = 1$ superfields. Let us label the two superspace Grassmann odd coordinates which occur in the $N = 2$ superspace as $\theta^{A1} = \theta^A$ and $\theta^{A2} = \eta^A$ and similarly for their conjugates. We also introduce the derivatives $D_A = D_A^1$ and $\nabla_A = D_A^2$ for which $D^2 = D^A D_A$, $\bar{D}^2 = \bar{D}^A \bar{D}_A$ and similarly for $\nabla^2$ and $\bar{\nabla}^2$. We associate the coordinates $\theta^A$ and $\bar{\theta}^A$ with those of the $N = 1$ superspace, which we will keep manifest. The $N = 2$ superfield $A$ can be decomposed into the $N = 1$ superfields $A = A |_{\eta=0}$,
\( W_A = \nabla_A A \mid_{\eta=0} \) and \( G = -\frac{1}{2} \nabla^2 A \mid_{\eta=0} \). These \( N=1 \) superfields are also \( N=1 \) chiral since \( A \) satisfies (2.1);

\[
\bar{D}_B A = 0 , \quad \bar{D}_B W_C = 0 .
\]

(2.9)

It remains to solve the constraint (2.2). Taking \( i = 1, j = 2 \) we find the constraint \( D^B W_B = \bar{D}^B \bar{W}_B \). Taking \( i = j = 2 \) we find that

\[
\nabla^2 A = \bar{D}^2 \bar{A} ,
\]

(2.10)

which implies \( G = -\frac{1}{2} \bar{D}^2 \bar{A} \) and so \( G \) is not an independent superfield.

We can now evaluate the action (2.3) in terms of \( N=1 \) superfields [12]

\[
S = \text{Im} \int d^4 x d^2 \theta d^2 \eta F ,
\]

\[
= \text{Im} \int d^4 x d^2 \theta \left\{ -\frac{\nabla^2 A}{4} \frac{dF}{dA} - \frac{1}{4} \nabla^B A \nabla_B A \cdot \frac{d^2 F}{dA^2} \right\} \mid_{\eta=0} ,
\]

(2.11)

\[
= \text{Im} \int d^4 x d^2 \theta \left\{ \frac{\bar{D}^2 \bar{A}}{4} \frac{dF}{dA} - \frac{1}{4} \frac{d^2 F}{dA^2} W^B W_B \right\} ,
\]

\[
= \text{Im} \left\{ \int d^4 x d^4 \theta \frac{dF}{dA} \frac{d^2 F}{dA^2} \frac{d^2 F}{d^2 W^B W_B} \right\} .
\]

This is easily recognised as the standard form for the action of an \( N=2 \) Abelian Yang-Mills multiplet written in \( N=1 \) superfields.

At this point we remark on an important notational oddity. Consider the co-efficient of \( D_A W_B \mid_{\theta=0} \). Following standard conventions the generators of the four-dimensional Lorentz group \( \sigma^{\mu \nu} \) satisfy \( \sigma^{\mu \nu} = -\frac{i}{2} \epsilon^{\mu \nu \rho \lambda} \sigma_{\rho \lambda} \). Therefore \( D_A W_B \mid_{\theta=0} = \sigma^{\mu \nu}_A (F_{\mu \nu} - i \star F_{\mu \nu}) \), where \( F_{\mu \nu} \) is the curl of a four-dimensional gauge field. Now the M-fivebrane field content contains a self-dual three form \( H \) and one can check that \( F_{\mu \nu} - i \star F_{\mu \nu} \) must come from the \( H_{\mu \nu \bar{z}} \) component of \( H \) (this was also shown in [6]). Thus one sees that the lowest component of \( A \) must be \( \bar{a}(\bar{z}) \). In this paper then unbarred superfields depend upon \( \bar{z} \) and barred superfields depend on \( z \). This
problem has its origins in the two uses for the bar symbol; as complex conjugation on the Riemann surface, or as Hermitian conjugation in $N = 2$ superspace. We are choosing the later definition to have precedence.

### 3. The M-Fivebrane and Seiberg-Witten

We refer the reader to [5,6] for a detailed discussion of the threebrane and its zero modes. The Riemann surface $\Sigma$ for two threebranes is given by

$$t^2 - 2(z^2 - u)t + \Lambda^4 = 0,$$

where $t = e^{-s}$, $z$ is a coordinate of the surface and $u$ is a modulus related to the relative separation of the two threebranes. From a supersymmetric point of view it is natural to take the complex scalar $\bar{s}(\bar{z})$ to be the lowest component of a chiral $N = 2$ superfield $S$ with independent $N = 1$ components

$$S |_{\theta=0} = S, \quad \nabla_A S |_{\eta=0} = H_A \bar{z}, \qquad (3.1)$$

with $S |_{\theta=0} = \bar{s}$ and $D_A H_B \bar{z} |_{\theta=0} = \sigma^{\mu\nu}_{AB} H_{\mu\nu} \bar{z}$. Similarly we promote the moduli $\bar{u}$ of the Riemann surface to an $N = 2$ superfield $U$ with the independent $N = 1$ components

$$U |_{\theta=0} = U, \quad \nabla_A U |_{\eta=0} = T_A, \qquad (3.2)$$

and $U |_{\theta=0} = \bar{u}$. We can then define the $N = 2$ superfields

$$A = \oint_A S d\bar{z}, \quad A_D = \oint_B S d\bar{z}, \qquad (3.3)$$

where $A$ and $B$ are a basis of one cycles of $\Sigma$. Given these definitions one can see that the $\eta^4$ components of $A$ and $A_D$ are

$$W_A = \nabla_A \oint_A S d\bar{z} = \oint_A \Lambda(U) T_A = \frac{dA}{dU} T_A, \quad (3.4)$$

$$W^D_A = \nabla_A \oint_B S d\bar{z} = \oint_B \Lambda(U) T_A = \frac{dA_D}{dU} T_A.$$

Here $\Lambda = \frac{dS}{dA} d\bar{z}$ is an $N = 1$ superfield whose lowest component is the anti-
holomorphic one form \( \bar{\lambda} \) of the Riemann surface. Furthermore it follows from

\[
H_{A\bar{z}} = \nabla_A S|_{\eta=0} = \frac{dS}{dU} \nabla_A U|_{\eta=0} = \frac{dS}{dU} T_A ,
\]  

(3.5)

and (3.4) that

\[
H_{\mu\nu\bar{z}} = \left( \frac{dS}{dA} \right) (F_{\mu\nu} - i \star F_{\mu\nu}) = \left( \frac{dA}{dU} \right)^{-1} (F_{\mu\nu} - i \star F_{\mu\nu}) \Lambda_{\bar{z}} ,
\]  

(3.6)

which agrees with the ansatz used in [6] for the vector zero modes.

Now we wish to obtain a manifestly \( N = 2 \) formulation of the equations of motion for the threebranes of the M-fivebrane [6]. To this end we postulate the following equation

\[
\mathcal{E} \equiv D^{ij} S - R^2 \Lambda^4 \partial_{\bar{z}} \left( \frac{D^A S D_A j \partial_{\bar{z}} S}{1 + R^2 \Lambda^4 |\partial_{\bar{z}} S|^2} - \frac{D^A S D_A \bar{j} \partial_{\bar{z}} S}{1 + R^2 \Lambda^4 |\partial_{\bar{z}} S|^2} \right) = 0 .
\]  

(3.7)

First we take the \( i = j = 1 \) component of (3.7). This gives at \( \eta = 0 \)

\[
D^2 S = R^2 \Lambda^4 \partial_{\bar{z}} \left( \frac{D^A S D_A \partial_{\bar{z}} \bar{S}}{1 + R^2 \Lambda^4 |\partial_{\bar{z}} S|^2} - \frac{T^A \bar{T}_A \partial_{\bar{z}} S}{1 + R^2 \Lambda^4 |\partial_{\bar{z}} S|^2} \right) = 0 .
\]  

(3.8)

Next we act on (3.8) with \( \bar{D}^2 \) and set the fermions to zero to obtain

\[
\bar{D}^2 D^2 S + 2 R^2 \Lambda^4 \partial_{\bar{z}} \left( \frac{\bar{D}^B D^A S D_B D_A S \partial_{\bar{z}} \bar{S}}{1 + R^2 \Lambda^4 |\partial_{\bar{z}} S|^2} - \frac{\bar{T}^A \bar{T}_A \partial_{\bar{z}} S}{1 + R^2 \Lambda^4 |\partial_{\bar{z}} S|^2} \right) = 0 .
\]  

(3.9)

This equation can then be evaluated to give

\[
\partial_{\mu} \partial^\mu \bar{s} - R^2 \Lambda^4 \partial_{\bar{z}} \left( \frac{\partial_{\rho} \bar{s} \partial^\rho \bar{s} \partial_{\bar{z}} \bar{s}}{1 + R^2 \Lambda^4 |\partial_{\bar{z}} S|^2} + H_{\mu\nu\bar{z}} H^{\mu\nu} \frac{\partial_{\bar{z}} \bar{s}}{1 + R^2 \Lambda^4 |\partial_{\bar{z}} S|^2} \right) = 0 ,
\]  

(3.10)

which is precisely the equation for the scalar zero modes obtained in [6], provided that we rescale \( H_{\mu\nu\bar{z}} \rightarrow 4H_{\mu\nu\bar{z}} \).

7
Now we take the $i = 1, j = 2$ component of (3.7). At $\eta = 0$ we find

$$D^A T_A + R^2 \Lambda^4 \partial \bar{z} \left( \frac{D^A S T_A \partial \bar{z} \bar{S}}{1 + R^2 \Lambda^4 |\partial \bar{z} \bar{S}|^2} + \frac{T^A \bar{D}^A \bar{S} \partial \bar{z} \bar{S}}{1 + R^2 \Lambda^4 |\partial \bar{z} \bar{S}|^2} \right) = 0 . \quad (3.11)$$

Next we act with $D_C \bar{D}_B$ on (3.11) and set the fermions to zero to obtain

$$D_C \bar{D}_B D^A T_A - R^2 \Lambda^4 \partial \bar{z} \left( \frac{\partial \bar{z} \bar{D}_B D^A S D_C T_A}{1 + R^2 \Lambda^4 |\partial \bar{z} \bar{S}|^2} - \frac{\partial \bar{z} S D_C \bar{D}^A \bar{S} \bar{D}_B T_A}{1 + R^2 \Lambda^4 |\partial \bar{z} \bar{S}|^2} \right) = 0 . \quad (3.12)$$

Evaluating (3.12) we then arrive at the equation for the vector zero modes

$$\partial^\nu H_{\mu \nu \bar{z}} - R^2 \Lambda^4 \partial \bar{z} \left( \frac{\partial \bar{z} \bar{s} \partial^\nu \bar{s} H_{\mu \nu \bar{z}}}{1 + R^2 \Lambda^4 |\partial \bar{z} \bar{s}|^2} - \frac{\partial \bar{z} \bar{s} \partial^\nu \bar{s} H_{\mu \nu \bar{z}}}{1 + R^2 \Lambda^4 |\partial \bar{z} \bar{s}|^2} \right) = 0 , \quad (3.13)$$

which is again precisely the same equation that was found in [6]. Since the scalar and vector component equations agree with the correct equations of motion, it follows that (3.7) describes the all zero modes of a threebrane soliton in the M-fivebrane worldvolume.

Now that we have shown that (3.7) reproduces the M-fivebrane equations of motion we can determine the equations of motion for the massless modes of the threebrane solitons by reducing $\mathcal{E} = 0$ over the Riemann surface. Thus we consider

$$0 = \int_\Sigma \mathcal{E} d\bar{z} \wedge \bar{\Lambda} , \quad (3.14)$$

where, according to our conventions $\bar{\Lambda} = \frac{d\bar{S}}{dt} dz$ is an $N = 2$ superfield whose lowest component is the holomorphic one form $\lambda$. Expanding out the terms in (3.7) one finds

$$0 = D^{\bar{i} \bar{j}} \bar{U} \bar{I} + D^{\bar{A} \bar{i}} \bar{U} \bar{D}_\bar{A} \bar{j} \bar{U} \frac{d\bar{I}}{d\bar{U}} - D^{\bar{A} \bar{i}} \bar{U} \bar{D}_\bar{A} \bar{j} \bar{U} \bar{J} + \bar{D}^{\bar{A} \bar{j}} \bar{U} \bar{D}_\bar{A} \bar{j} \bar{U} \bar{K} , \quad (3.15)$$

where the $I, J$ and $K$ integrals have appeared before in [6] where their values have
also been deduced;

\[
I \equiv \int_{\Sigma} \Lambda \wedge \bar{\Lambda} = \frac{dA \ d\bar{A}}{dU \ d\bar{U}} (\tau - \bar{\tau}) ,
\]

\[
J \equiv R^2 \Lambda^4 \int_{\Sigma} \partial_{\bar{z}} \left( \frac{\Lambda^2 \partial_{\bar{z}} S}{1 + R^2 \Lambda^4 \partial_{\bar{z}} S} \right) d\bar{z} \wedge \bar{\Lambda} = 0 ,
\]

\[
K \equiv R^2 \Lambda^4 \int_{\Sigma} \partial_{\bar{z}} \left( \frac{\Lambda^2 \partial_{\bar{z}} S}{1 + R^2 \Lambda^4 \partial_{\bar{z}} S} \right) d\bar{z} \wedge \bar{\Lambda} = -\frac{d\bar{\tau}}{d\bar{U}} \left( \frac{d\bar{A}}{d\bar{U}} \right)^2 ,
\]

where \( \tau(U) = \frac{dA_D}{dA} \). The first integral is easily evaluated using the Riemann bilinear identity, however in [6] the two non-holomorphic integrals required a rather indirect method to evaluate them. In the appendix to this paper we provide a direct proof of the above expressions for \( J \) and \( K \).

Multiplying by \((\frac{dA}{dU})^{-1}\) one can rewrite (3.15) in the simple form

\[
D^{ij} A_D - \bar{D}^{ij} \bar{A}_D - \bar{\tau} (D^{ij} A - \bar{D}^{ij} \bar{A}) = 0 .
\]

The real and imaginary parts of this equation are equivalent to

\[
D^{ij} A = D^{ij} \bar{A} , \quad D^{ij} A_D = D^{ij} \bar{A}_D ,
\]

respectively. Therefore if we introduce a function \( F(A) \) defined so that \( A_D = \frac{\partial F}{\partial A} \) then we see from (3.3) that these equations are precisely those of the Seiberg-Witten effective theory for \( N = 2 \) \( SU(2) \) Yang-Mills with \( N = 2 \) superspace action (2.3).

Finally let us consider the following \( N = 2 \) superspace generalisation of the Seiberg-Witten differential, \( \Lambda_{SW} = S d\bar{z} \). The lowest component of \( \Lambda_{SW} \) is the Seiberg-Witten differential \( \bar{\lambda}_{SW} \) and its \( \eta^A \) component is the form \( H_{A\bar{z}} d\bar{z} \). First
we note that for either the $A$ or $B$ cycle

$$
\oint (\mathcal{E}d\bar{z} - \mathcal{E}dz) = \oint (D^{ij}\Lambda_{SW} - \bar{D}^{ij}\bar{\Lambda}_{SW}) ,
$$

(3.19)

since the non-holomorphic terms in $\mathcal{E}$ collect into the form $d(f - \bar{f})$. We can discard the integral of $d(f - \bar{f})$ over the $A$ or $B$ cycles as the function $f(S, \bar{S})$ is non-singular in a neighbourhood of these cycles, which are therefore closed curves on $\Sigma$. A direct derivation of the Seiberg-Witten effective equations of motion comes from evaluating

$$
0 = \oint_A (\mathcal{E}d\bar{z} - \mathcal{E}dz) = \oint_A (D^{ij}\Lambda_{SW} - \bar{D}^{ij}\bar{\Lambda}_{SW}) ,
$$

$$
0 = \oint_B (\mathcal{E}d\bar{z} - \mathcal{E}dz) = \oint_B (D^{ij}\Lambda_{SW} - \bar{D}^{ij}\bar{\Lambda}_{SW}) ,
$$

(3.20)

which yields

$$
D^{ij} \left( \begin{array}{c} f_A \Lambda_{SW} \\ \bar{f}_B \bar{\Lambda}_{SW} \end{array} \right) = \bar{D}^{ij} \left( \begin{array}{c} f_A \bar{\Lambda}_{SW} \\ \bar{f}_B \Lambda_{SW} \end{array} \right) .
$$

(3.21)

Clearly an $SL(2, \mathbb{Z})$ transformation on the $(A, B)$ cycles generates the $SL(2, \mathbb{Z})$ transformation on $(A, A_D)$ discussed in the previous section. We note that the condition $D^{ij}\Lambda_{SW} = \bar{D}^{ij}\bar{\Lambda}_{SW}$ is simply the constraint (2.2) applied to $\Lambda_{SW}$. Thus the Seiberg-Witten equations of motion can be obtained by imposing the $N = 2$ superfield constraint (2.2) on the generalised Seiberg-Witten differential $\Lambda_{SW}$ and then integrating it over the cycles of the Riemann surface.

To make contact with the previous discussion we note that because $dz \wedge \bar{\Lambda} = 0$ (3.14) can be rewritten as

$$
0 = \oint_{\Sigma} (\mathcal{E}d\bar{z} - \mathcal{E}dz) \wedge \bar{\Lambda}
$$

$$
= \oint_B (D^{ij}\Lambda_{SW} - \bar{D}^{ij}\bar{\Lambda}_{SW}) \oint_A \bar{\Lambda} - \oint_A (D^{ij}\Lambda_{SW} - \bar{D}^{ij}\bar{\Lambda}_{SW}) \oint_B \bar{\Lambda}
$$

$$
= (D^{ij}\mathcal{A}_D - \bar{D}^{ij}\mathcal{A}_D) \frac{d\bar{\Lambda}}{dU} - (D^{ij}\mathcal{A} - \bar{D}^{ij}\mathcal{A}) \frac{d\bar{\Lambda}_D}{dU} ,
$$

(3.22)
where we have applied the Riemann bilinear relation and again dropped the total derivative terms. In this way we arrive immediately at equation (3.17), whose real and imaginary parts are (3.21).

We would like to thank A. Pressley for discussions on Riemann surfaces.

Note Added

From the above superfield construction one sees a potential obstruction to obtaining the Seiberg-Witten effective action from a six-dimensional action. The chiral nature of $N = 2$ superspace requires that such an action could depend only upon $\Lambda_{SW} = Sdz$ and not $\bar{\Lambda}_{SW} = \bar{S}d\bar{z}$. However to obtain a covariant action in six dimensions requires both $dz$ and $d\bar{z}$ to appear.

APPENDIX

In this appendix we will explicitly derive the values of the $J$ and $K$ integrals which appear above. If we define

$$f = \frac{R^2 \Lambda^4 \lambda^2 \partial \bar{s}}{1 + R^2 \Lambda^4 |\partial \bar{s}|^2}, \quad (A.1)$$

then the integrals in question become (for simplicity we only consider the lowest component terms)

$$J = \int_{\Sigma} \partial \bar{z} \bar{f} d\bar{z} \wedge \lambda, \quad K = \int_{\Sigma} \partial \bar{z} f dz \wedge \lambda. \quad (A.2)$$

It is important to note that the function $f$ has singularities at the roots $e_i$ of $Q$. Thus to evaluate $J$ and $K$ we must first cut holes of radius $\epsilon$ around each of the $e_i$, $i = 1, 2, 3, 4$, forming a new Riemann Surface $\Sigma_\epsilon$ with a boundary. The actual values for $J$ and $K$ are then found by taking the limit $\epsilon \to 0$. Since $\lambda$ is a closed form the integrands in (A.2) can be expressed as $d(\bar{f}\lambda)$ and $d(f\lambda)$ respectively.
The surface integrals can then be reduced to an integral around the boundary $\partial \Sigma_\epsilon$. Thus we find

$$J = \sum_i \oint_{\gamma_i} dz \bar{f} \lambda_z, \quad K = \sum_i \oint_{\gamma_i} dz f \lambda_z,$$

(A.3)

where $\gamma_i$ is the $i$th component of $\partial \Sigma_\epsilon$. If we substitute in the expression for $f$ and expand in powers of $\epsilon$ we find

$$J = -\frac{1}{2} \sum_i \oint_{\gamma_i} dz \frac{1}{z Q} \frac{1}{Q} + O(\epsilon), \quad K = -\frac{1}{2} \sum_i \oint_{\gamma_i} dz \frac{1}{z Q^2} + O(\epsilon).$$

(A.4)

From here it is easy to see that the integrand of $J$ has no simple poles in $z$ and so $J = 0$. However $K$ does have simple poles. After taking the $\epsilon \to 0$ limit we find

$$K = -\pi i \sum_i \frac{1}{e_i} \prod_{j \neq i} \frac{1}{e_i - e_j},$$

(A.5)

in fact one can show that all the $\epsilon$ dependent terms vanish, where we have used the values $e_i = \pm \sqrt{u} \pm \Lambda^2$ of the four roots. Note that despite its complicated definition, $K$ depends holomorphically on $u$ and is independent of the constant $R^2 \Lambda^4$. Since the four roots $e_i$ are given by $\pm \sqrt{u} \pm \Lambda^2$ we obtain

$$K = \frac{\pi i}{u^2 - \Lambda^4}.$$  

(A.6)

This is a remarkably simple expression and up to a constant it is the inverse of the discriminant of $\Sigma$. Lastly we wish to show that $K$ is actually equal to

$$E = -\frac{d\tau}{du} \left( \frac{da}{du} \right)^2.$$  

(A.7)

To this end it can be shown that both $K$ and $E$ are modular functions of weight zero. Thus $K^{-1}E$ is also a modular function of weight zero. Clearly $K$ has simple poles at $u = \pm \Lambda^2$ and furthermore, as is well known [2], $\tau$ and $a$ also have singularities only at these points. From the known solution [2] one can explicitly check that $E$ has only simple poles at $u = \pm \Lambda^2$. Hence $K^{-1}E$ is non-singular and thus must be constant. Finally, by examining the perturbative regime where $u \to \infty$, $a = -2\pi i \sqrt{u}$ and $\tau = \tau_0 + \frac{i}{\pi} \ln(u/\Lambda^2)$, one can see that indeed $K = E$.  

12
REFERENCES

1. E. Witten, Nucl. Phys. B500 (1997) 3, hep-th/9703166

2. N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, hep-th/9407087

3. A. Klemm, W. Lerche, C. Vafa and N., Warner, Nucl. Phys. B477 (1996) 746, hep-th/9604034

4. P.S. Howe, N.D. Lambert and P.C. West, The Threebrane Soliton of the M-fivebrane, hep-th/9710033

5. P.S. Howe, N.D. Lambert and P.C. West, Classical M-Fivebrane Dynamics and Quantum N = 2 Yang-Mills, hep-th/9710032

6. N.D. Lambert and P.C. West, Gauge Fields and M-Fivebrane Dynamics, hep-th/9712040

7. P. Howe, K. Stelle and P. West, Phys. Lett. 124B (1983) 55; see also P. West, in Proceedings of the 1983 Shelter Island II Conference on Quantum Field Theory and Fundamental Problems of Physics, edited by R. Jackiw, N. Kuri, S. Weinberg and E. Witten (M.I.T. Press); P.S. Howe, K.S. Stelle and P.K. Townsend, Nucl. Phys. B236 (1984) 125, M. Grisaru and W. Siegel, Nucl. Phys B201 (1982)292.

8. N. Dorey, V. V. Khoze and M. P. Mattis, Phys. lett. B388 (1996) 324, hep-th/9607066; Phys. Rev. D54 (1996) 2921; Phys. Rev. D54 (1996) 7832, hep-th/9607202; K. Ito and N. Sasakura, Phys. Lett. B382 (1996) 95, hep-th/9602073; Nucl. Phys. B484 (1997) 141, hep-th/9608054; A. Yung, Nucl. Phys. B485 (1997) 38, hep-th/9605096; H. Aoyama, T. Harano, M. Sato and S. Wada, Phys. lett. B388 (1996) 331, hep-th/9607076; T. Harano and M. Sato, Nucl.Phys. B484 (1997) 167, hep-th/9608060

9. R. Grimm, M. Sohnius and J. Wess, Nucl. Phys. B133 (1978) 275

10. L. Mezincescu, JINR report P2-12572 (1979)

11. This follows a discussion between A. van Proeyen and P. West.
12. G. Sierra and P.K. Townsend, in Proceedings, *Supersymmetry and Supergravity 1983*, ed. B. Milewski, World Scientific, 1983; S.J. Gates Jr., Nucl. Phys. B238 (1984) 349