Maximal Efficiency of Collisional Penrose Process with Spinning Particle II
Collision with a Particle in ISCO

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We analyze the collisional Penrose process between a particle on the ISCO orbit around an extreme Kerr black hole and a particle impinging from infinity. We consider both cases with non-spinning and spinning particles. We evaluate the maximal efficiency, \( \eta_{\text{max}} = \frac{\text{(extracted energy)}}{\text{(input energy)}} \), for the elastic collision of two massive particles and for the photoemission process, in which the ISCO particle will escape to infinity after the collision with a massless impinging particle. For non-spinning particles, the maximum efficiency is \( \eta_{\text{max}} \approx 2.562 \) for the elastic collision and \( \eta_{\text{max}} \approx 7 \) for the photoemission process. While for spinning particles we obtain the maximal efficiency \( \eta_{\text{max}} \approx 8.442 \) for the elastic collision and \( \eta_{\text{max}} \approx 12.54 \) for the photoemission process.

I. INTRODUCTION

The energy extraction from a rotating black hole is one of the most fundamental issues in black hole physics. It may also be important for relativistic astrophysics. The Penrose process, in which we can extract the rotational energy by use of an ergo region of a black hole, plays a key role in the extraction. In particular, the collisional Penrose process may be more important because it may give more efficient energy extraction. Although it was pointed out that the center of mass energy diverges in the collision at the horizon of an extreme Kerr black hole, it does not mean that we can extract the energy as large as one likes because the infinite energy near the horizon will be red-shifted away. Hence, it is important to calculate how much energy we can really extract to infinity. Then the maximal efficiency \( \eta = \frac{\text{(extracted energy)}}{\text{(input energy)}} \) has been calculated in the collision between particles impinging from infinity by many authors.

In general relativity, the equation of motion for a spinning test particle is totally different from a test particle, and the effect of spin is non-trivial. Hence the effect of spin was also discussed just for the collision between particles impinging from infinity. In our previous paper (paper 1), we considered the collision between spinning test particles impinging from infinity and discussed the energy efficiency for various processes: the elastic collision, Compton scattering, and inverse Compton scattering. In the cases of the elastic collision and Compton scattering, we found that those extraction efficiencies give twice as larger as the cases of the collision of non-spinning particles.

However, we may have to tune the initial conditions for such collisions to be possible. In order to make the collisional process more realistic, we may discuss a collision between a particle on the innermost stable circular orbit (ISCO) and a particle impinging from infinity. Although it was shown that the center of mass energy diverges when the collision with a particle on the ISCO in the extreme Kerr black hole, the efficiency has not been so far calculated in the collision between a particle on the ISCO and a spinning particle. We then consider the collisional Penrose process between a spinning particle on its ISCO and a particle impinging from infinity and calculate the energy efficiency of the ejected particle. This is the second paper of a series of papers which provides the maximal efficiency of the collisional Penrose process with spinning particles.

In Sec. II, we briefly summarize the properties of a spinning particle, especially of the ISCO particle, in the extreme Kerr spacetime, and provide the timelike condition of the orbit. In Sec. III, we study the collision in the extreme Kerr geometry between a particle on the ISCO and a spinning particle plunging from the radial infinity and analyze the maximal efficiency. We also discuss the collision of one spinning massive particle and one massless particle (the photoemission process). Section IV is devoted to concluding remarks. Throughout this paper, we use the geometrical units of \( c = G = 1 \) and follow for the notations.

II. BASIC EQUATIONS

A. Equations of motion of a spinning particle

First, we briefly summarize the equation of motion of a spinning particle moving on the equatorial plane in the Kerr spacetime. We consider only the particle motion on the equatorial plane in an extreme Kerr black hole.
Introducing the tetrad basis
\[ e^{(a)}_{\mu} = \begin{pmatrix} \frac{r-M}{r} & 0 & 0 & -\frac{M(r-M)}{r^2} \\ 0 & \frac{r}{r-M} & 0 & 0 \\ 0 & 0 & \frac{r}{r-M} & \frac{r^2+M^2}{r^2} \\ -\frac{M}{r-M} & 0 & 0 & \frac{r^2+M^2}{r-M} \end{pmatrix}, \]

we discuss the equation of motion by use of the tetrad components, which are described by the latin indecies with a bracket.

We define a specific spin vector \( s^{(a)} \) by
\[ s^{(a)} = \frac{1}{2\mu} \epsilon^{(a)}_{(b)(c)(d)} u^{(b)} S^{(c)(d)}, \]
where \( \mu \) is the mass of a spinning particle, \( u^{(a)} \) is the specific momentum defined by \( u^{(a)} := p^{(a)}/\mu \) (the 4-momentum \( p^{(a)} \) divided by the mass of a spinning particle \( \mu \)), and \( S^{(a)(b)} \) is the spin tensor. \( \epsilon^{(a)(b)(c)(d)} \) is the totally antisymmetric tensor with \( \epsilon^{(0)(1)(2)(3)} = 1 \).

For a spinning particle to move just on the equatorial plane, the direction of spin should be perpendicular to the equatorial plane. Hence we find only one component of \( s^{(a)} \) is non-trivial, i.e.,
\[ s^{(2)} = -s. \]

When \( s > 0 \), the particle spin is parallel to the direction of the black hole rotation, while it is anti-parallel if \( s < 0 \). From the supplementary condition; \( S^{(a)(b)} p^{(b)} = 0 \), which fixes the center of mass of a spinning particle, the spin tensor is described as
\[ S^{(0)(1)} = -sp^{(3)}, \quad S^{(0)(3)} = sp^{(1)}, \quad S^{(1)(3)} = sp^{(0)}. \]

For a Killing vector \( \xi^{(a)} \), we find a conserved quantity of a spinning particle
\[ Q_\xi = p^{(a)} \xi^{(a)} + \frac{1}{2} S^{(a)(b)} \left( \epsilon^{(b)(c)} \xi^{(c)} + \epsilon^{(a)(b)} \xi^{(b)} \right), \]
where Ricci rotation coefficient is defined by \( \epsilon^{(a)(b)(c)} := \epsilon^{(c)(b)(a)} \). Since there are two Killing vectors in the present spacetime, we have two conserved quantities; the energy \( E \) and the total angular momentum \( J \) along the world line of a spinning particle as follows;
\[ E = \frac{r-M}{r} p^{(0)} + \frac{M(r+s)}{r^2} p^{(3)}, \quad J = \frac{r-M}{r} (M+s) p^{(0)} + \frac{r(r^2+M^2) + M s (r+M)}{r^2} p^{(3)}. \]

In what follows, we shall use the normalized variables as
\[ \tilde{E} = \frac{E}{\mu}, \quad \tilde{J} = \frac{J}{\mu M}, \quad \tilde{s} = \frac{s}{M}, \quad \tilde{t} = \frac{t}{M}, \quad \tilde{r} = \frac{r}{M}, \quad \tilde{\tau} = \frac{\tau}{M}. \]

where \( \tau \) is the proper time. We will drop the tilde just for brevity.

From Eqs. (2.1) and (2.2), we find
\[ u^{(0)} = \frac{[r^3 + (1+s)r + s]}{r^2(r-1) (1 - \frac{r^2}{\Sigma^2})}, \]
\[ u^{(3)} = \frac{J - (1+s) E}{r (1 - \frac{r^2}{\Sigma^2})}. \]

Since we have the normalization condition \( u^{(a)} u^{(a)} = -1 \), the radial component of the specific momentum is given by
\[ u^{(1)} = \sigma \sqrt{(u^{(0)})^2 - (u^{(3)})^2 - 1}, \]
where \( \sigma = \pm 1 \) correspond to the outgoing and ingoing motions, respectively.

We should note that the 4-velocity \( u^{(a)} = dz^{(a)}/d\tau \), in which \( z(\tau) \) is an orbit of a particle, is not always parallel to the specific 4-momentum \( u^{(a)} \). When we normalize the affine parameter \( \tau \) as
\[ u^{(a)} u^{(a)} = -1, \]
the difference between \( v^{(a)} \) and \( u^{(a)} \) is given by
\[ u^{(a)} - v^{(a)} = \frac{S^{(a)(b)} R^{(b)(c)(d)(e)} u^{(c)} S^{(d)(e)}}{2\mu^2 + \frac{1}{4} R^{(b)(c)(d)(e)} S^{(b)(c)} S^{(d)(e)}}. \]

For the present setting, this relation between the 4-velocity and the specific 4-momentum is reduced to
\[ u^{(0)} = \Lambda_s^{-1} u^{(0)}, \quad u^{(1)} = \Lambda_s^{-1} u^{(1)}, \quad u^{(3)} = \frac{\Lambda_s^{-1} u^{(3)}}{1 - s^2/\Sigma^2}. \]

Hence, we finally obtain the equations of motion of the spinning particle as
\[ \Sigma_s \Lambda_s \frac{dt}{d\tau} = \left( 1 + \frac{3 s^2}{r \Sigma_s} \right) [J - (1+s) E] + \frac{r^2 + 1}{(r-1)^2} P_s, \]
\[ \Sigma_s \Lambda_s \frac{d\phi}{d\tau} = \pm \sqrt{R_s}, \]
\[ \Sigma_s \Lambda_s \frac{d\phi}{d\tau} = \frac{\Sigma_s}{r \Sigma_s} [J - (1+s) E] + \frac{1}{(r-1)^2} P_s. \]

where
\[ P_s = \left( r^2 + 1 + \frac{s}{r} (r+1) \right) E - \left( 1 + \frac{s}{r} \right) J, \]
\[ R_s = P_s^2 - (r-1)^2 \left[ \frac{\Sigma_s^2}{r^2} + -(1+s) E + J^2 \right]. \]
By using $\Sigma_s$ and $R_s$, the radial component of the specific momentum is written by
\[
u^{(1)} = \frac{r\sqrt{R_s}}{(r-1)\Sigma_s}. \tag{2.7}
\]

**B. The innermost stable circular orbit**

Since the orbit of a particle on the ISCO is circular, $\nu^{(1)}$ vanishes and then $\nu^{(1)}$ does so as well. We then find a constraint on the conserved quantities $E$ and $J$ from Eqs. (2.3) and (2.4) as follows: We first write down Eq. (2.7) as
\[
(u^{(1)})^2 = A(E - U_+(E - U_-)) \tag{2.8}
\]
where
\[
A(r, s) = \frac{r^2B}{(r - 1)^2\Sigma_s^2},
\]
\[
B(r, s) = \left[ r^2 + 1 + s\left(1 + \frac{1}{r}\right)^2 - (r - 1)^2(1 + s)^2, \right.
\]
and
\[
U_+(r, J, s) = XJ \pm \sqrt{(X^2 - Y)J^2 - Z}, \tag{2.9}
\]
with
\[
X(r) = \frac{1}{B}\left[ \left( r^2 + 1 + s\left(1 + \frac{1}{r}\right)^2 \right) \times (1 + \frac{s}{r}) - (r - 1)^2(1 + s)^2, \right],
\]
\[
Y(r) = \frac{1}{B}\left[ \left(1 + \frac{s}{r}\right)^2 - (r - 1)^2 \right],
\]
\[
Z(r) = -\frac{1}{B}\frac{(r - 1)^2\Sigma_s^2}{r^2}.
\]

We can regard $U_{(\pm)}(r, J, s)$ as the effective potential of a particle on the equatorial plane. Since $U_-(r)$ usually does not have an extremum and it is much less than the unity, we shall consider only $U_{(\pm)}(r)$. For a circular orbit with the radius $r = r_0$, the following conditions must be satisfied:
\[
U_+(r_0) = E, \text{ and } \frac{dU_+(r_0)}{dr} = 0. \tag{2.10}
\]
In addition, for the stability of the orbit, we have to impose
\[
\frac{d^2U_+}{dr^2}(r_0) > 0.
\]
Since the ISCO is the inner boundary of stable circular orbits, it must satisfy
\[
U_+(r_{\text{ISCO}}) = 0, \tag{2.10}
\]
\[
\frac{dU_+(r_{\text{ISCO}})}{dr} = 0, \tag{2.11}
\]
\[
\frac{d^2U_+}{dr^2}(r_{\text{ISCO}}) = 0. \tag{2.12}
\]

where $r_{\text{ISCO}}$ is the ISCO radius. From the above three conditions (2.10), (2.11) and (2.12), we find the energy $E_{\text{ISCO}}$ and angular momentum $J_{\text{ISCO}}$ of the particle on the ISCO, and the ISCO radius $r_{\text{ISCO}}$ in terms of spin $s$. In the extreme Kerr spacetime, we find $r_{\text{ISCO}} = 1$ (the horizon radius), and then
\[
E_{\text{ISCO}} = \frac{(1 - s^2)}{\sqrt{3(1 + 2s)}}, \tag{2.13}
\]
\[
J_{\text{ISCO}} = 2E_{\text{ISCO}}. \tag{2.14}
\]

From Eq. (2.6), we find
\[
v_{\text{ISCO}}^{(0)} = \frac{u_{\text{ISCO}}^{(0)}}{\delta}(1 - \frac{s^2}{r^3}),
\]
\[
v_{\text{ISCO}}^{(3)} = \frac{u_{\text{ISCO}}^{(3)}}{\delta}(1 + \frac{2s^2}{r^3}),
\]
where
\[
\delta = 1 - \frac{s^2}{r^3}\left[1 + 3(v_{\text{ISCO}}^{(3)})^2\right].
\]

**C. Constraints on the Orbits**

Here, we shall stress two important points as follows:

1. In order for a particle to reach the horizon $r_H := 1$, the radial function $R_s$ must be non-negative for $r \geq r_H$.

2. As seen in the previous section, the 4-velocity $v^{(a)}$ is not always parallel to the specific 4-momentum. Hence, we have to impose the timelike condition $v^{(a)}v_{(a)} < 0$ even though $v^{(a)}u_{(a)} = -1$ is imposed.

Before discussing the collisional Penrose process, we must treat these points properly. Since we have already discussed these points for the particles plunging from infinity in [9], we shall discuss only the case for the ISCO particle. For the first condition, $R_s(r_{\text{ISCO}}) = 0$ always holds by its definition and then it is always satisfied.

As for the timelike condition $v^{(a)}v_{(a)} < 0$, we find
\[
E_{\text{ISCO}}^2 < \frac{(1 - s)^2(1 + s)^4}{3s^2(1 + 2s)}. \tag{2.15}
\]

Since the conserved energy $E_{\text{ISCO}}$ is a function of the spin $s$, this inequality is reduced into
\[
(1 - s^2)^2(4 - 2s^3 - 3s^2 - 4s - 1) < 0.
\]

From this inequality, we obtain the constraint of the spin $s$ for the ISCO particle, i.e.,
\[
s_{\text{ISCO}}^{\min} \leq s \leq s_{\text{ISCO}}^{\max}, \tag{2.16}
\]
where $s_{\text{ISCO}}^{\min} \approx -0.302776$ and $s_{\text{ISCO}}^{\max} = 1$. The energy of the ISCO particle is bounded as
\[
0 \leq E_{\text{ISCO}} \leq E_{\text{ISCO}}^{\max} \approx 0.8340996.
\]
Note that for a particle plunging from infinity, we have the constraint such that
\[ s_{\text{min}} \leq s \leq s_{\text{max}}, \]
where \( s_{\text{min}} \approx -0.2709 \) and \( s_{\text{max}} \approx 0.4409 \) are the solutions of \( s^6 + 2s^5 - 4s^4 - 4s^3 - 7s^2 + 2s + 1 = 0 \) with the condition \(-1 \leq s_{\text{min}} < s_{\text{max}} \leq 1\).

### III. MAXIMAL EFFICIENCY OF COLLISION OF PARTICLES

Now, we discuss the collision of two particles 1 and 2, whose 4-momenta are \( p_1^{(a)} \) and \( p_2^{(a)} \). We then assume that the particle 1 is on the ISCO, while the particle 2 is impinging from infinity. In Appendix [A], we show that the center of mass energy can take arbitrary value in the collision between those two particles just as usual BSW effect. We will then analyze the maximal efficiency of the energy extraction, i.e., how much energy we can extract from an extreme black hole. For this purpose, we shall follow the same procedure as the paper I. The difference is that the particle 1 is moving on the ISCO orbit with the radius \( r_{\text{ISCO}} \). In the extreme Kerr spacetime, the ISCO radius is \( r_{\text{ISCO}} = 1 \). Hence, the collision must take place very close to the horizon (\( r_H = 1 \)). We then assume that the collisional point is given by \( r_c = 1/(1 - \epsilon) \) (\( 0 < \epsilon \ll 1 \)).

At the collisional point \( r_c \), we impose the following conservations:
\[
\begin{align*}
E_1 + E_2 &= E_3 + E_4, \\
J_1 + J_2 &= J_3 + J_4, \\
s_1 + s_2 &= s_3 + s_4, \\
p_1^{(1)} + p_2^{(1)} &= p_3^{(1)} + p_4^{(1)}. 
\end{align*}
\]

After the collision, we assume that the particles 3 with the 4-momentum \( p_3^{(a)} \) is going away to infinity, while the particle 4 with the 4-momentum \( p_4^{(a)} \) falls into the black hole. In order for the particle 2 to reach the horizon from infinity, the particle 2 must be subcritical \((J_2 < 2E_2)\) and ingoing \((\sigma_2 = -1)\). So we assume that \( \sigma_4 = \sigma_2 = -1 \).

We then expand the radial component of the 4-momentum \( p_1^{(1)} \) in terms of \( \epsilon \) as
\[ p_1^{(1)} \approx \sigma \frac{|2E - J|}{\epsilon(1 - s)} + \cdots. \]

The conservation of the radial components of the momenta \( (p_1^{(1)} + p_2^{(1)}) = p_3^{(1)} + p_4^{(1)} \) yields
\[
\frac{|2E_2 - J_2|}{1 - s_2} = \sigma_3 \frac{|2E_3 - J_3|}{1 - s_3} - \frac{|2E_4 - J_4|}{1 - s_4} + O(\epsilon),
\]
in which we use \( J_1 = 2E_1 \). Here, we also assume \( s_3 = s_4 \) for simplicity. This means that the particle 1 becomes the particle 3 without change of the spin angular momentum after the collision. Then, the above conservation is written as
\[
\left[ \sigma_3 \frac{\text{sign}[2E_3 - J_3]}{1 - s_3} + \frac{1}{1 - s_4} \right] (2E_3 - J_3) = O(\epsilon).
\]

The above setting gives
\[ J_3 = 2E_3(1 + \alpha_3 \epsilon + \beta_3 \epsilon^2 + \cdots), \]
where \( \alpha_3 \) and \( \beta_3 \) are parameters of \( O(\epsilon^0) \). Since the particle 2 is subcritical \((J_2 < 2E_2)\), the angular momentum \( J_2 \) is written as
\[ J_2 = 2E_2(1 + \zeta), \]
where \( \zeta < 0 \) with \( \zeta = O(\epsilon^0) \). From the conservation laws, we find
\[ E_4 = E_1 + E_2 - E_3, \quad J_4 = J_1 + J_2 - J_3, \]
giving
\[ J_4 = 2E_4 \left( 1 + \frac{E_2}{E_4} \zeta + \cdots \right). \]

In what follows, we discuss two cases: [A] collision of two massive particles (MMM), and [B] collision of massless and massive particles (MPM), where we use the symbols of MMM and MPM following [5]. \( \mathbf{P} \) and \( \mathbf{M} \) respectively mean a massless particle (a photon) and a massive particle. The first and the second letters describe the particle on the ISCO and the particle from infinity before the collision, respectively. The third letter shows an escaped particle after the collision. For the case of MMM, we assume all masses of the particles are the same, i.e., \( \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu \). For the case of MPM, the particles 2 and 4 are massless and non-spinning, because a photon has no stable circular orbit, while the particles 1 and 3 are massive with the same mass, i.e., \( \mu_1 = \mu_3 = \mu \). From the conservation of the spin, we obtain \( s_1 = s_3 \). Since the massive particle is ejected by the incoming photon, we shall call this process MPM photoemission. Note that in the first paper, we call the process the inverse Compton scattering because a massive particle with large energy is going out after the collision.

Next, we evaluate \( E_2 \) and \( E_3 \) for the cases [A] and [B] separately.
A. Maximal Efficiency in Case [A] MMM (Collision of two massive particles)

For the massive particle, the radial component of the specific 4-momentum is given as

\[ u^{(1)} = \frac{\sigma r \sqrt{R_s}}{\Sigma_3 \sqrt{\Delta}} = \frac{\sigma \sqrt{r^2 [(r^3 + (1 + s)r + s)] E - (r + s)J^2} - (r - 1)^2 [r^4 2^2 + r^4 (J - (1 + s)) E^2]}{(r - 1)(r^3 - s^2)}. \]  

(3.5)

By expanding \( u^{(1)} \) with the conditions \( \xi_2 \) and \( \xi_3 \) in terms of \( \epsilon \), we find Eqs. (3.15)-(3.17) in the paper I. Note that for the particle 1 on the ISCO, \( u_1^{(1)} = 0 \).

Since \( u_1^{(1)} + u_2^{(1)} = u_3^{(1)} + u_4^{(1)} \), we find the leading order of \( \epsilon^{-1} \) is trivial. From the next leading order, i.e., \( O(\epsilon^0) \), we find

\[ \sigma_3 \frac{f(s_1, E_3, \alpha_3)}{1 - s_1^2} = \frac{[E_1 (2 + s_2) - E_3 g_1(s_2, \alpha_3)]}{1 - s_2^2}, \]

where

\[ f(s, E, \alpha) := \sqrt{E^2 [3 - 2 \alpha (1 + s)] [1 + 2 s - 2 \alpha (1 + s)] - (1 - s^2)^2}, \]

\[ g_1(s, \alpha) := 2 + s - 2 \alpha (1 + s), \]

\[ g_2(s, \alpha) := \alpha (2 + s - 2 \alpha). \]

This equation is reduced to

\[ \mathcal{A} E_3^2 - 2 \mathcal{B} E_3 + \mathcal{C} = 0, \]

where

\[ \mathcal{A} = -[3 - 2 \alpha_3 (1 + s_1)] [1 + 2 s_1 - 2 \alpha_3 (1 + s_1)] + \frac{(1 - s_3^2)^2}{(1 - s_1^2)^2} g_1(s_2, \alpha_3) \]

(3.7)

\[ \mathcal{B} = g_1(s_2, \alpha_3) \frac{(1 - s_3^2)^2}{(1 - s_1^2)^2} (2 + s_2) E_1 \]

(3.8)

\[ \mathcal{C} = \frac{(1 - s_3^2)^2}{(1 - s_1^2)^2} (2 + s_2)^2 E_1^2 + (1 - s_1^2)^2, \]

(3.9)

with the condition such that \( E_3 \leq E_{3, \text{cr}} \) for \( \sigma_3 = 1 \), or \( E_3 \geq E_{3, \text{cr}} \) for \( \sigma_3 = -1 \), where

\[ E_{3, \text{cr}} := \frac{2 + s_2}{g_1(s_2, \alpha_3)} E_1. \]

Here we focus just into the case of \( \sigma_3 = -1 \). We should stress that for the outgoing particle 3 after collision (\( \sigma_3 = 1 \)), the energy \( E_3 \) has the upper bound \( E_{3, \text{cr}} \), which magnitude is the order of \( E_1 \). Hence we may not expect large efficiency. We will present the concrete analysis for the case of \( \sigma_3 = 1 \) in Appendix [3] in which we find the efficiency is not so high.

Since the particle 3 is assumed to be ingoing after the collision, the orbit must be supercritical, i.e., \( J_3 > 2 E_3 \), which means either \( \alpha_3 > 0 \) or \( \alpha_3 = 0 \) with \( \beta_3 > 0 \). Once we give \( \alpha_3 \), the value of \( E_3 \) is fixed in terms of \( s_1, s_2 \) and \( E_1 \) by

\[ E_3 = E_{3,+} := \frac{B + \sqrt{B^2 - 4 \mathcal{A} \mathcal{C}}}{A}, \]

(3.10)

where we have chosen the larger root because it gives the larger extracted energy as it turns out that \( \mathcal{A} \) is always positive.

The next leading order terms give

\[ \mathcal{P} E_2 = (1 - s_2)^3 (E_3 - E_1)^2, \]

(3.11)

where

\[ \mathcal{P} := 2 (E_3 - E_1) (1 - s_2)^3 \]

\[ + 4 \zeta \left\{ 2 (1 + s_2) E_3 [\alpha_3 (2 + s_2) - \beta_3 (1 - s_2)] \right\} \]

\[ - s_2 (2 + s_2) (E_3 - E_1) - \sigma_3 \frac{(1 - s_3^2)^2}{(1 - s_1^2)^2} \]

\[ \times \left( h(s_1) - 2 (1 + s_1)^2 (2 + s_2) g_2(s_1, \alpha_3) \right) \]

\[ + 2 \beta_3 (1 + s_1) (1 - s_3^2) g_1(s_1, \alpha_3) \} \right]. \]

(3.12)

Since this fixes the value of \( E_2 \), we obtain the efficiency by

\[ \eta = \frac{E_3}{E_1 + E_2}, \]

when \( \alpha_3, \beta_3 \) and \( \zeta \) are given.
1. Non-spinning particles

We first consider the collision of non-spinning particles: $s_1 = s_2 = 0$. In this case, the energy of the particle 1, $E_1$, is $1/\sqrt{3}$. Since the particle 2 is plunging from infinity, we have the constraint of $E_2 \geq 1$ for a massive particle. Hence, in order to obtain the maximal efficiency, we derive the maximal value of the energy of the particle 3 and then confirm that $E_2 = 1$ is possible.

The energy of the particle 3 is given by

$$E_3 = \frac{1}{\sqrt{3}} \left[ 4(1 - \alpha_3) + \sqrt{4(3 - 2\alpha_3)(1 - 2\alpha_3) - 3} \right].$$

Then we find that from Fig. 1, the maximal value of the energy of the particle 3 is given at $\alpha_3 = 0$.

![Image showing the energy of the particle 3, $E_3$, in terms of $\alpha_3$. The maximal value of this is given at $\alpha_3 = 0$.](image1)

FIG. 1. The energy of the particle 3, $E_3$, in terms of $\alpha_3$. The maximal value of this is given at $\alpha_3 = 0$.

Next, when $\alpha_3 = 0+$, the energy of the particle 2 is given by

$$E_2 = \frac{36\sqrt{3}}{36 + 7\zeta(4\beta_3 + 7)}, \quad (3.13)$$

which becomes unity when $\beta_3$ is taken to be

$$\beta_3 = \frac{1}{4} \left( \frac{36(\sqrt{3} - 1)}{7\zeta} - 7 \right).$$

As a result, we obtain the maximal efficiency for the collision of non-spinning particles, $\eta_{\text{max}} = 7(\sqrt{3} - 1)/2 \approx 2.562$.

2. Spinning particles

Next, we consider the collision of spinning particles. As we showed, giving two particle spins ($s_1$ and $s_2$), we find the energies of the particle 1, particle 2, and particle 3 in terms of the orbit parameters of the particles 2 and 3 ($\alpha_3$, $\beta_3$, and $\zeta$).

For a given value of $E_1$, the larger efficiency is obtained for larger $E_3$ as well as smaller $E_2$. Hence, in order to get the large efficiency, we first must find large extraction energy of the particle 3 ($E_3$) for given values of $E_1$ on the ISCO and $E_2$ of the ingoing particle. In our approach, the energy of the particle 1 ($E_1$) is a function of $s_1$ while $E_2$ is a function of $s_1$, $s_2$, $\alpha_3$, $\beta_3$ and $\zeta$. As the non-spinning particles, we have the constraint for the particle 2, $E_2 \geq 1$.

Based on the above argument, for an elastic scattering, we expect that the efficiency could take the maximum value as $E_3/(E_1 + 1)$ for some values of $s_1$, $s_2$, and $\alpha_3$ if $E_2 = 1$. After finding $s_1$, $s_2$, and $\alpha_3$, which give the maximal efficiency, it is sufficient to show that $E_2 = 1$ is possible for some choices of the remaining parameters ($\zeta$ and $\beta_3$).

The particle 3 energy ($E_3$) is determined by Eq. (3.10) for given value of $\alpha_3$. Since the orbit of the particle 3 is near critical, we have two constraints: $E_3 \geq E_{3,cr}$ for $\alpha_3 = -1$ and the timelike condition $s_3 = -1$.

Just as the paper I, in order to find the large value of $E_3$, from the timelike condition, we find that the spin magnitude $s_3(= s_1)$ must be small. For small value of $s_1$, we find $\alpha_3 \approx 0$ gives the largest efficiency of $E_3/(E_1 + 1)$, just as the paper I. Hence next setting $\alpha_3 = 0+$, we analyze the maximal efficiency. Here, $0+$ means that we assume $\alpha_3 > 0$ but take a limit of $\alpha_3 \rightarrow 0$ after taking the limit of $\epsilon \rightarrow 0$. This is justified because $E_2$ and $E_3$ change smoothly when we take the limit of $\alpha_3 \rightarrow 0$.

Assuming $\alpha_3 = 0+$, we look for the maximal value of $E_3/(E_1 + 1)$ for given $s_1$ and $s_2$. In Fig. 2 we show the contour map of $E_3/(E_1 + 1)$ in terms of $s_1$ and $s_2$. The red point, which is $(s_1, s_2) \approx (0.03196, s_{\text{min}})$, gives the maximal value of $E_3/(E_1 + 1)$.

![Image showing the contour map of $E_3/(E_1 + 1)$ in terms of $s_1$ and $s_2$. The timelike condition for the particle 3 orbit is satisfied in the light green region. As a result, the maximal value of $E_3/(E_1 + 1) \approx 8.442$ is obtained when $s_2 = s_{\text{min}} \approx -0.2709$ and $s_1 \approx 0.03196$ (the red point in the figure).](image2)

FIG. 2. The contour map of $E_3/(E_1 + 1)$ in terms of $s_1$ and $s_2$. The timelike condition for the particle 3 orbit is satisfied in the light green region. As a result, the maximal value of $E_3/(E_1 + 1) \approx 8.442$ is obtained when $s_2 = s_{\text{min}} \approx -0.2709$ and $s_1 \approx 0.03196$ (the red point in the figure).

Since $E_2 \geq 1$ when we plunge the particle 2 from infinity, we have to confirm that $E_2 = 1$ is possible for a certain values of remaining parameters ($\zeta$ and $\beta_3$).

The condition for $E_2 = 1$ in eq. (3.10) gives the relation between $\zeta$ and $\beta_3$, which is a linear equation of $\beta_3$. Hence we always find a real solution of $\beta_3$. While the timelike condition of the particle 2 gives the constraint
on $\zeta$, 

$$\zeta_{\text{min}} < \zeta < 0,$$

where

$$\zeta_{\text{min}} := -\frac{(1-s_2)}{2} \left[ 1 + \frac{(1-s_2)(1+s_2)^2}{\sqrt{3s_2^2(2+s_2^2)}} \right].$$

For the parameters giving the maximal value of $E_3$, we find the relation between $\zeta$ and $\beta_3$, which is shown in Fig. 3. From the timelike condition for the particle 2 orbit, we have the constraint of $\zeta_{\text{min}} < \zeta < 0$ where $\zeta_{\text{min}} \approx -1.271$.

Since there exists a possible range of parameters with $E_2 = 1$, we find the maximal efficiency is given by

$$\eta_{\text{max}} = \frac{E_3}{E_1 + 1} \approx 8.442.$$  \hspace{1cm} (3.14)

### B. Maximal Efficiency in Case [B] MPM (photoemission)

For collision of the massless particle (photon) and massive particle, we should assume the particle 1 (ISCO particle) is massive and the incoming particle is massless because there is no ISCO for massless particles. Hence we consider the photoemission process, i.e., a massive particle is emitted via a collision process by an incoming photon.

For the momenta of the massive particles 1 and 3, the radial components of 4-momenta do not change, while for the massless particles 2 and 4, we find

$$p_2^{(1)} = 2\epsilon^{-1}E_2\zeta - 2E_2(1+2\zeta) - \epsilon \frac{E_2(1-4\zeta^2)}{4\zeta} + O(\epsilon^2),$$

$$p_4^{(1)} = 2\epsilon^{-1}E_2\zeta - 2[E_4 + 2E_2\zeta + E_3\alpha_3] - \frac{E_2^2 - 8E_2E_3(2\alpha_3 - \beta_3)\zeta - 4E_2^2\zeta^2}{4E_2\zeta} + O(\epsilon^2).$$  \hspace{1cm} (3.15, 3.16)

where $E_4 = E_1 + E_2 - E_3$.

From the conservation of the radial components of the 4-momenta, we find

$$E_3 = \frac{B + \sqrt{B^2 - 4AC}}{2A} |_{s_2=0};$$

and

$$E_2 = \frac{(E_3 - E_1)^2}{P} |_{s_2=0},$$

where $A, B, C$ and $P$ are given by eqs. (3.7), (3.8), (3.9) and (3.12), which should be evaluated with $s_2 = 0$. Note that as a result, $E_2$ and $E_3$ coincide with those found at the collision of a spinning massive particle and a non-spinning massive particle.

1. Non-spinning particles

We consider the collision between a non-spinning massive particle and a massless particle: $s_1 = 0$. As the non-spinning particles in the elastic collision, the energy of the particle 1, $E_1$, is given by $1/\sqrt{3}$ and the maximal value of the energy of the particle 3 is given at $\alpha_3 = 0$. On the other hand, the energy of the particle 2 is given by Eq. (3.13). Since the particles 2 plunges from infinity, we have the constraint for the energy of the particle 2, $E_2 \geq 0$ if $\zeta \beta_3 \rightarrow \infty$, $E_2 \rightarrow 0$ is possible and we obtain $\eta_{\text{max}} = 7$.

2. Spinning particle+massless particle

In this situation, we first discuss $E_2/E_1$ since it is determined only by $\alpha_3$ and $s_1$. After setting these parameters which give the maximum value of $E_3/E_1$, we confirm that $E_2 = 0$ is possible for some choice of the parameters $\zeta$ and $\beta_3$ since we have the constraint $E_2 \geq 0$ for a massless particle.

In Fig. 4 we show the contour map of $E_3/E_1$ in terms of $\alpha_3$ and $s_1$. The red point, which is $(\alpha_3, s_1) = (0,0.06360)$, gives the maximal value of $E_3/E_1$.

If $E_2 \rightarrow 0$ is possible, it gives the minimal value of $E_2$ and then the maximal efficiency is given by $\eta_{\text{max}} = E_3/E_1$. Hence, assuming $\alpha_3 = 0+$ and $s_1 = 0.06360$, we...
analyze whether \( E_2 \to 0 \) is possible or not. From Eq. (3.12), we find the asymptotic behavior of \( P \) as

\[
P \approx 8E_3\alpha \beta \left[ \frac{E_3(2 + s_1)}{(1 - s_1)f(s_1, E_3, 0)} - 1 \right],
\]

if \( \alpha \beta \to \infty \). It gives \( E_2 \to 0 \). \( \alpha \) is constrained as \( -\infty < \alpha < 0 \) because the particle 2 is non-spinning, while \( \beta \) is arbitrary as long as \( \alpha > 0 \). As a result, we obtain \( E_2 \to 0 \) in the limit of \( \alpha \beta \to \infty \). \( \beta \) must be negative. Hence, we find the maximum efficiency \( \eta_{\max} \approx 12.54 \) for the photoemission process.

![FIG. 4. The contour map of \( E_3/E_1 \) in terms of \( \alpha_3 \) and \( s_1 \). The timelike condition for the particle 3 is satisfied in the light-green shaded region. The maximum value of \( E_3/E_1 = 12.54 \) is obtained at the red point \((\alpha_3, s_1) = (0, 0.06360)\).](image)

### IV. CONCLUDING REMARKS

We have studied the collisional Penrose process for non-spinning and spinning particles around an extreme Kerr black hole. For the collision between a particle on the ISCO orbit and a particle impinging from infinity, we have evaluated the maximal efficiency of the energy extraction. We summarize our present result as well as the previous one in the paper I [9] in Table I.

In the non-spinning case, we find that the maximal efficiency is 2.562 for the elastic collision, and 7 for the photoemission process. While, for the cases of spinning particles, we obtain the maximal efficiency in the elastic scattering (MMMO) is \( \eta_{\max} \approx 8.442 \) and \( \eta \approx 12.54 \) for the photoemission process (MPM0). When we take into account the spin, we find that the efficiency becomes larger both in the elastic collision and for the photoemission process.

Note that for the collision between particles impinging from infinity, the maximal efficiency becomes the largest in the Compton scattering (PMP+) when the energy of a particle 1 \((E_1)\) takes \( E_1 \to \infty \). This result does not change even if the spin is taken into account. In the present case, however, the particle on the ISCO should be massive, which means that the PMP process is not possible.

In the current analysis, compared with the non-spinning case, both maximal efficiencies for the elastic scattering and for the photoemission process become twice larger than the non-spinning case. We can conclude that spin plays an important role also in the collision with the ISCO particle. Note that the efficiency does not change significantly in the case of the photoemission process. This is because the absorbed massless particle is non-spinning.

| Table I. The maximal efficiencies and energies for three collisions between a particle in its ISCO and a particle impinging from infinity. We include the previous result for the nonISCO orbit case. Following [9], we use the symbols of MMM0, PMP0, MPM0 for each process, where “0” means the collision with a particle on the ISCO orbit. We also include the cases of three collisions with particles impinging from infinity discussed in the paper I [9] as a reference. The maximal efficiencies and maximal energies are always enhanced when the spin effect is taken into account. |}
Our analysis is performed for an extreme Kerr black hole. Since the existence of an extreme black hole may not be likely [13], we will extend the present analysis into the case for a non-extreme black hole.

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Appendix A: BSW effect for the collision with a spinning particle on the ISCO

The center of mass energy \( E_{\text{cm}} \) is defined as

\[
E_{\text{cm}}^2 = -(p_{1}(a) + p_{2}(a))(p_{1}^{(a)} + p_{2}^{(a)}).
\]

We consider the collision between spinning particles which have the same mass \( \mu \). If the particle 1 is on the ISCO, \( p_1^{(1)} = 0 \) holds. Hence, we find

\[
\frac{E_{\text{cm}}^2}{2\mu^2} = 1 - \frac{r^4 A_s(s_1, E_1, J_1)A_s(s_2, E_2, J_2)}{(r^3 - s_1^2)(r^3 - s_2^2)} + \frac{r^2 B_s(s_1, E_1, J_1)B_s(s_2, E_2, J_2)}{(r - 1)^2(r^3 - s_1^2)(r^3 - s_2^2)}
\]

where

\[
A_s(s, E, J) = (1 + s)E - J,
B_s(s, E, J) = (r^3 + (1 + s)r + s)E - J(r + s).
\]

In addition, the relation between the angular momentum \( J_1 \) and the energy \( E_1 \) is given by \( J_1 = 2E_1 \). Assuming the collision takes place near the horizon, i.e., \( r = 1 + \epsilon \), the above equation becomes as follows:

\[
\frac{E_{\text{cm}}^2}{2\mu^2} = \frac{(2 + s_1)E_1(2E_2 - J_2)}{(1 - s_1^2)(1 - s_2^2)}/\epsilon + O(\epsilon^0).
\]

Thus, we find that the center of mass energy \( E_{\text{cm}} \) diverges at the horizon (\( \epsilon \to 0 \)).

Appendix B: The case (3) with \( \sigma = 1 \)

1. Case(A)MMM (Collision of two massive particles)

In this case, the condition \( E_3 \leq E_{3,\text{cr}} \) must be satisfied. As a result, \( E_{3,+} \), which is the larger root of Eq. (3.6), is excluded. The possible solution is

\[
E_3 = E_{3,-} := \frac{B - \sqrt{B^2 - AC}}{A}.
\]

\( E_{3,\text{cr}} \) increases monotonically with respect to \( \alpha_3 \). \( E_{3,\text{cr}} \) is positive for \( \alpha_3 < \alpha_{3,\infty} := \frac{2 + s_2}{2(s_1 + s_2)} \) and \( E_{3,\text{cr}} \to \infty \) as \( \alpha_3 \to \alpha_{3,\infty} \). Beyond \( \alpha_{3,\infty} \), \( E_{3,\text{cr}} \) becomes negative, and this case should be excluded. As \( \alpha_3 \) increases, \( E_3 \) also increases but faster than \( E_{3,\text{cr}} \) and reaches the upper bound \( E_{3,\text{cr}} \) at some value of \( \alpha_3 = \alpha_{3,\text{cr}} \). Hence, for given \( s_1 \) and \( s_2 \), it is sufficient to find \( \alpha_3 \) satisfying \( E_3 = E_{3,\text{cr}} \). From the condition \( E_3 = E_{3,\text{cr}} \) and Eq. (3.6), we find the quadratic equation of \( \alpha_3 \). Solving its equation, we obtain

\[
\alpha_3 = 0 \text{ or } \alpha_3(s_1, s_2),
\]

\[
\alpha_3(s_1, s_2) := \frac{(2 + s_2)(1 - s_2 - 3s_1s_2 + s_2^2(2 + s_2))(1 + 2s_1)(1 - 2s_2 - 2s_2^2 + s_2^2)}{(1 + 2s_1)(1 - 2s_2 - 2s_2^2 + s_2^2)}.
\]

Thus, we obtain the largest value of \( E_3 \) or \( E_{3,\text{cr}} \) by inserting the solutions. For \( \alpha_3 = 0 \), \( E_3 \) is \( E_1 \). Hence, we find the maximal value of \( E_3/(E_1 + 1) \):

\[
\frac{E_3}{E_1 + 1} = \frac{1}{1 + 1/E_1} \leq 0.4550 \quad (s_1 = s_{\text{ISCO}}^{\text{ISCO}}).
\]

On the other hand, for \( \alpha_3 = \alpha_3(s_1, s_2) \), we show the contour map of \( E_3/(E_1 + 1) \) in Fig. 5. As the maximum efficiency, \( E_3/(E_1 + 1) \approx 0.06587 \) is obtained at the red point \((s_1, s_2) \approx (0.08230, 0.449) \).

![Figure 4. The contour map of \( E_3/(E_1 + 1) \) for \( \alpha_3 = \alpha_3(s_1, s_2) \) in terms of \( s_1 \) and \( s_2 \). The red line \( E_3 \) is shown in Fig. 5. As the maximum point efficiency, \( E_3/(E_1 + 1) \approx 0.06587 \) is obtained at the red point \((s_1, s_2) \approx (0.08230, 0.449) \).](image)

Comparing these results, we find \( E_3/(E_1 + 1) \approx 0.4550 \) \((s_1 = s_\text{ISCO}^{\text{ISCO}}) \) as the maximum value for arbitrary \( s_2 \). Since we find that \( E_2 = 1 \) is possible from Fig. 6 \( \eta_{\text{max}} \approx 0.4550 \) is obtained as the maximum efficiency.

2. Case(B)MPM (photoemission)

In this case, the condition \( E_3 \leq E_{3,\text{cr}} \) must be satisfied. We show the contour map of \( E_3/E_1 \) in terms of \( \alpha_3 \) and \( s_1 \) in Fig. 6. As the maximum efficiency, \( E_3/E_1 \approx 1.000 \) is obtained at \( \alpha_3 \approx 0 \) for any \( s_1 \).
This result has physical meaning when $E_2 = 0$ is possible in terms of $\zeta$ and $\beta_3$. To see this, from Eq. 8.12 we find the asymptotic behavior of $\mathcal{P}$ becomes

$$
\mathcal{P} \approx -8E_3\zeta\beta_3 \left[ \frac{E_3(2 + s_1)}{(1 - s_1)f(s_1, E_3, 0)} + 1 \right],
$$

when we take a limit of $\zeta\beta_3 \to -\infty$. $\zeta$ is constrained as $-\infty < \zeta < 0$ because the particle 2 is non-spinning, while $\beta_3$ is also arbitrary. As a result, we obtain $E_2 \to 0$ in the limit of $\zeta\beta_3 \to -\infty$. Hence, we find the maximum efficiency $\eta_{\text{max}} \approx 1.000$ for the inverse Compton scattering. For $s_1 = 0$, the maximum efficiency also becomes $\eta_{\text{max}} \approx 1.000$ since it doesn’t depend on $s_1$.

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