Simple scheme generating an axisymmetrically anisotropic initial flow of incompressible turbulence using a normal random number vector

Hiroki Suzuki¹-⁴, Koudai Hasebe², Yutaka Hasegawa³, Tatsuo Ushijima³ and Shinsuke Mochizuki¹

¹Graduate School of Sciences and Technology for Innovation, Yamaguchi University, 2-16-1 Tokiwadai, Ube-shi, Yamaguchi 755-8611, Japan
²Department of Engineering Physics, Electronics and Mechanics, Nagoya Institute of Technology, Gokiso-cho, Showa-ku, Nagoya-shi, Aichi 466-8555, Japan
³Department of Electrical and Mechanical Engineering, Nagoya Institute of Technology, Gokiso-cho, Showa-ku, Nagoya-shi, Aichi 466-8555, Japan
⁴Email: h.suzuki@yamaguchi-u.ac.jp

Abstract. The present study introduces a simple scheme for generating an axisymmetrically anisotropic incompressible initial flow. In addition, the proposed scheme is used to examine the effects of kinetic energy conservation errors on the time evolution of the anisotropic initial flows. In the analysis of the present study, an inviscid flow is analyzed. In this inviscid flow, the kinetic energy characteristics are known analytically. The kinetic energy conservation error is obtained using the Crank-Nicolson (CN) method. The scheme of the present study generates an axisymmetrically anisotropic initial flow by obtaining the weights of normal random vector components using an anisotropic parameter. The analysis of the flow field is performed using a fourth-order differential scheme that explicitly conserves kinetic energy with a fourth-order Runge-Kutta scheme. Because of the kinetic energy conservation error, the kinetic energy obtained using the CN method increases with time. The magnitude of the initial anisotropy obtained by this scheme decreases with time. This result is only slightly affected by kinetic energy conservation errors. On the other hand, kinetic energy conservation error affects the magnitude of the pressure-strain correlation term.

1. Introduction
Incompressible turbulent flow is a fundamental phenomenon in fluids engineering applications [1]. By analyzing the governing equations of the incompressible turbulent flow numerically, phenomena in the incompressible turbulent flow can be examined and investigated. The governing equations for the incompressible turbulent flow represent conservation laws of mass and momentum of the fluid [2]. Bared on these conservation laws, a conservation law of kinetic energy can be derived. By enforcing these conservation laws in the discretized governing equations, sufficiently accurate numerical simulations are considered to become possible [3]. In addition, incompressible turbulent flows to be investigated in engineering applications are often anisotropic, even in the absence of mean flow shear [4-6]. In anisotropic turbulent flows, in contrast to isotropic turbulent flows, kinetic energy is statistically exchanged among three components of velocity fluctuation intensity due to the pressure-strain correlation terms in the intensity equations. In the absence of mean flow shear, the generated...
turbulence decays as time proceeds. This decay characteristic of decaying turbulence may depend on the magnitude of the initial flow field with and without anisotropy.

Taylor-Green vortex flow has often been used as an initial velocity field for numerical analysis of decaying turbulence (e.g., [7]). This flow is based on the analytical solution of the Taylor-Green vortex and is commonly used in numerical benchmarks. At large scale, this Taylor-Green vortex flow is anisotropic. As a spatial discretization scheme for analyzing an incompressible flow, a discretization scheme has been developed that satisfies not only the conservation law described by the governing equations but also the conservation law of kinetic energy [3]. Here, this discretization scheme has an accuracy order higher than the fourth-order accuracy and is suitable for use in large-eddy simulation. In order to validate this discretization scheme, a simple scheme for generating the initial velocity field is used. On the other hand, a turbulence-generating grid is used to generate decaying turbulence in wind tunnel experiments. The decay characteristics of the grid-generated turbulence may depend on the shape of the turbulence-generating grid, which produces the initial velocity field. The significant variation among previous studies implies that the decay characteristics of the grid-generated turbulence depend on the initial velocity field (e.g., [8]). In addition to conventional turbulent grids, turbulent grids with fractal shapes have also been used in previous studies (e.g., [4-6]).

A discretization scheme that explicitly satisfies the conservation law of turbulent kinetic energy for analyzing incompressible turbulence has been validated in an inviscid flow. In this validation, an initial velocity field is obtained by rotating the field of a random number vector [3]. Note that this initial velocity field satisfies the continuity equation of an incompressible flow. In the validation of previous studies, this scheme generates an isotropic initial flow. On the other hand, actual turbulent flows without mean shear are often anisotropic (e.g., [4-6]). Thus, a scheme for generating an anisotropic initial flow appears to be useful. As an anisotropic initial flow, the Taylor-Green vortex has often been used in previous studies. The anisotropy of this flow is decided, and the value of the anisotropic factor is not set arbitrarily. In order to investigate the effect of anisotropy on the turbulent flow field, the value of the anisotropy factor should be set arbitrarily.

The purposes of the present study are to propose a simple scheme to generate an anisotropic initial flow and to investigate the effects of the error of the conservation law of turbulent kinetic energy using the present scheme for generating an initial flow. In the present study, a method for generating an initial field is developed based on the previous scheme, which uses the rotation of a random vector. Specifically, an anisotropic initial flow is generated using the weights of the components of the random vector taken by the rotation. Here, the anisotropy of the initial flow field is set as axisymmetric in the present study. Using the present scheme for generating the initial flow, the effects of the conservation error of the turbulent kinetic energy on the flow field are investigated. Here, the Crank–Nicolson method is used to generate an error in the conservation law of turbulent kinetic energy. An inviscid flow in a periodic box is used to precisely examine the effects of error in the turbulent energy conservation law. In addition, the fourth-order discretization scheme is used in the present study as a spatial discretization scheme. In the results and discussion section of the present study, the anisotropic factor and the pressure-strain correlation terms are used.

2. Methods

2.1. Scheme generating an axisymmetrically anisotropic initial flow

The present study starts by focusing on the normal random vector $\Phi$. Here, the components of this normal random number vector are given as follows: $(\Phi_1, \Phi_2, \Phi_3)$. A coordinate system $(x, y, z)$ is also used. By rotating this normal random number vector, an initial velocity vector field $\mathbf{u}$ is calculated as follows:

$$
\mathbf{u} = \nabla \times \Phi.
$$

(1)
Figure 1. Values of weight components on the normal random number vector, where (a), (b), and (c) show the results for $a = 0, 0.1$, and $-0.1$, respectively. Here, weight components $(C_1, C_2, C_3)$ are the components for $(x, y, z)$.

Here, components of the initial velocity vector field $u$ are $(u, v, w)$. This initial velocity vector field satisfies the continuity equation: $\nabla \cdot u = 0$. The standard deviation of each component of this normal random vector is the same among the three components. Therefore, the initial velocity vector field can be set to be isotropic. The present study attempts to generate an axisymmetrically anisotropic initial velocity vector field. The intensities of the velocity fluctuation components of an axisymmetrically anisotropic flow field are given as follows:

$$\langle u^2 \rangle = (1 + 2a^2) \frac{2K}{3}, \langle v^2 \rangle = (1 - a^2) \frac{2K}{3}, \text{ and } \langle w^2 \rangle = (1 - a^2) \frac{2K}{3},$$

(2)

where $K$ is the kinetic energy defined as $K = \frac{1}{2} (\langle u^2 \rangle + \langle v^2 \rangle + \langle w^2 \rangle)$, and $\langle \rangle$ is the ensemble average. Here, $a$ is a parameter that determines the magnitude of the anisotropy of the initial flow field. From these forms, the anisotropy factors of an axisymmetrically anisotropic flow field $v_{rms}/u_{rms}$ and $w_{rms}/u_{rms}$ are given as follows:

$$v_{rms}/u_{rms} = w_{rms}/u_{rms} = (1 - a) / (1 + 2a),$$

(3)

where $u_{rms} = \langle u^2 \rangle^{1/2}$, $v_{rms} = \langle v^2 \rangle^{1/2}$, and $w_{rms} = \langle w^2 \rangle^{1/2}$. Here, the condition of $a = 0$ makes the velocity fluctuation intensities coincide with each other and gives an isotropic flow field. For an anisotropic flow field, the value of the anisotropy parameter is set to $a \neq 0$.

The velocity fluctuation intensities in the anisotropic flow are given by the weight components of the normal random number vector $(c_1, c_2, c_3)$. The weight components for giving the velocity fluctuation intensities shown in Equation (2) are derived as follows:

$$(c_1, c_2, c_3) = ((1 - 8a + 2a^2)^{1/2}/3^{1/2}, (1 + 2a)/3^{1/2}, (1 + 2a)/3^{1/2}).$$

(4)

Here, the axis of symmetry of the anisotropic initial flow field is taken to coincide with the $x$ direction. Using the condition $1 - 8a + 2a^2 \geq 0$, the range to be satisfied by the parameter value for determining the magnitude of anisotropy is given as follows:

$$- (14^{1/2} - 4)/2 \leq a \leq (14^{1/2} + 4)/2.$$

(5)
Figure 2. Temporal evolutions of kinetic energy normalized by the initial kinetic energy, where (a) and (b) show the results obtained by the RK4 and CN methods, respectively.

Here, in the case of $a < 0$, the initial velocity field is a pancake-type anisotropic flow field. On the other hand, when $a > 0$, the initial velocity field is a cigar-type anisotropic flow field. By setting the values of the weight components using Equation (4), an axisymmetric anisotropic flow based on Equation (2) is generated.

Figure 1 shows the values of the weight components on the normal random vector depending on the parameter values for determining the magnitude of the anisotropy. Here, the values of the parameter $a$ are set to $a = 0, 0.1, \text{and } -0.1$. As shown in the figure, in the isotropic initial velocity field given by the setting $a = 0$, the values of the weight components are the same among the three components. Under the condition $a = 0.1$, the weight components $C_2$ and $C_3$ are identical to each other and are larger than the value of $C_1$. On the other hand, under the condition $a = -0.1$, the weight components $C_2$ and $C_3$ are the same as each other and are smaller than the value of $C_1$. Using these values of weight components, an axisymmetric anisotropic flow field is obtained for the condition $a = 0.1$.

2.2. Numerical techniques
The governing equations are the continuity equation and the Navier-Stokes (NS) equation. These equations are discretized using the fourth-order spatial discretization scheme, which was proposed in a previous study and has been used in previous studies (e.g., [9]). Here, this discretization scheme explicitly satisfies the kinetic energy conservation law. In this case, the skew-symmetric type is used as a form for discretizing the convection terms. The present study investigates the effect of kinetic energy conservation error on the flow field. Therefore, the six-stage fourth-order Runge-Kutta (RK4) method [10] and the Crank-Nicolson (CN) method are used in this analysis. When RK4 is used, the kinetic energy conservation error is negligible (e.g., [2, 3]). In contrast to the use of RK4, the CN method causes significant kinetic energy conservation errors. The scheme proposed in the present study generates the initial velocity field. Here, a uniform random number is generated by the Mersenne twister scheme [11]. Normal random numbers can be obtained using Box-Muller transforms [12] for a series of uniform random numbers.
Figure 3. Temporal evolutions of velocity fluctuation intensities, \( \langle u^2 \rangle \) and \( \langle v^2 \rangle \). Here, (a) and (c) show the results obtained using the RK4 method, and (b) and (d) show the results obtained using the CN method.

The computational region is a periodic box. Therefore, periodic boundary conditions are set for all boundary conditions. The size of the periodic box is \( L^3 = 2\pi^3 \). This periodic box is filled with an inviscid fluid. This periodic box is discretized using equally spaced computational grid points. The number of computational grid points is \( N^3 = 16^3 \). This number of computational grids is set based on previous studies. As a computational condition for the anisotropic initial flow field, the value of the computational parameter for the anisotropy \( a \) was set to \( a = \pm 0.1 \). In this numerical analysis, 1,000 cases were carried out for each computational condition. Statistics were calculated using ensemble averages over these cases. The time increment \( \Delta t \) was set to \( \Delta t = 0.2 \) in the present study in order to set a significant conservation error of the turbulent kinetic energy due to the use of the CN method.
3. Results and Discussion

3.1. Temporal evolution of statistics

The present study starts by examining the conservation law of kinetic energy. Figure 2 shows temporal profiles of the kinetic energy obtained using RK4 and CN. Here, the kinetic energy is calculated as a spatial mean value. In the present study, kinetic energy is maintained analytically to be constant because an inviscid flow is used. As shown in the figure, the kinetic energy obtained by RK4 is constant as time increases. This result shows that the conservation law of turbulent energy is made to hold in this analysis using RK4. On the other hand, the kinetic energy obtained by the CN method increases with time. The temporal profile of this increasing kinetic energy is found among the cases of isotropic and anisotropic initial fields. As shown in the figure, a significant error of the conservation law of kinetic energy is generated using the CN method.

In Figure 3, the temporal evolutions of $\langle u^2 \rangle$ and $\langle v^2 \rangle$ are shown among cases of isotropic and anisotropic initial fields. Here, $\langle u^2 \rangle$ and $\langle v^2 \rangle$ are also calculated as spatial mean values. Since the initial velocity field is axisymmetrically anisotropic, the value of $\langle v^2 \rangle$ is equal to that of $\langle w^2 \rangle$. For the analysis using RK4, $\langle u^2 \rangle$ and $\langle v^2 \rangle$ are maintained to be constant in the isotropic initial field. In the case of an anisotropic initial field, $\langle u^2 \rangle$ and $\langle v^2 \rangle$ approach the value of the isotropic initial field as time increases. In the case of an anisotropic initial field, $\langle u^2 \rangle$ and $\langle v^2 \rangle$ are symmetric with respect to the value due to the isotropic initial field. In the analysis using the CN method, $\langle u^2 \rangle$ and $\langle v^2 \rangle$ increase with time in the case of an anisotropic initial field. As time increases, $\langle u^2 \rangle$ and $\langle v^2 \rangle$ in the anisotropic initial field approach a profile determined by the isotropic initial flow field. Due to the use of the CN method, errors of the conservation law for kinetic energy and velocity fluctuation intensities are allowed. As shown in the figure, errors of the conservation law errors significantly affect the temporal evolution of the velocity fluctuation intensities.

Figure 4. Temporal evolutions of an anisotropy factor $v_{rms}/u_{rms}$, where (a) and (b) show the results obtained using the RK4 and CN methods, respectively.
3.2. Anisotropy and pressure-strain correlation

Figure 4 shows the temporal evolutions of the anisotropic factor. Here, the anisotropic factor is calculated as Equation (3). In the obtained results using RK4, in the cases of an anisotropic initial field, the value of the anisotropic factor deviates from unity. The value of the anisotropy factor approaches unity with time in the cases of an anisotropic initial flow field. In addition, for \( t > 20 \), the magnitude of the deviation of the anisotropic factor from unity is almost the same for the positive and negative values of the anisotropy parameter. When the initial flow field is isotropic, the value of the anisotropy factor is maintained to be unity. These results, obtained using RK4, are also found in the results obtained using the CN method. In addition, the time evolutions of the anisotropic factor are consistent between the results obtained using the RK4 and CN methods. These results indicate that the conservation error of the kinetic energy resulting from the use of the CN method only slightly affects the temporal variation of the anisotropy factor.

The pressure-strain correlation terms exchange the velocity fluctuation intensities between the different components. The pressure-strain correlation terms are included in the governing equations of velocity fluctuation intensities. Figure 5 shows the temporal evolutions of the pressure-strain correlation term for \( \langle u^2 \rangle, \Pi_{11} \), for the flow field obtained using the RK4 and CN methods. If the initial flow field is isotropic, the pressure-strain correlation term is maintained to be zero. As shown in the figure, the magnitude of the pressure-strain correlation term is zero at the initial time and then becomes locally maximum or locally minimum around \( t = 4 \). Then, at a time larger than \( t = 4 \), the magnitude of the pressure-strain correlation term approaches zero. The temporal evolutions of the pressure-strain correlation term are symmetric with respect to the positive and negative values of \( a \).

The approximate shape of the temporal evolution of the pressure-strain correlation term appears to be the same between the results obtained using the RK4 and CN methods. However, in contrast to the temporal evolutions of the anisotropic factor, the temporal evolutions of the pressure-strain correlation term are affected by the kinetic energy conservation errors resulting from the use of the CN method. Specifically, the deviation of the pressure-strain correlation term obtained using the CN method from zero is larger than that obtained using the RK4 method. The factor causing this difference may be increased kinetic energy due to the use of the CN method. As shown in Figure 2, the use of the CN method increases the kinetic energy, even if the flow is inviscid due to the conservation error.

Figure 5. Temporal evolutions of a pressure-strain correlation term of \( \langle u^2 \rangle \), where (a) and (b) show the results obtained by the RK4 and CN methods, respectively.
4. Conclusions
The present study proposes a simple scheme for generating an axisymmetrically anisotropic initial flow field for simulating incompressible flows. Here, in the proposed scheme, the magnitude of the anisotropy can be set to an arbitrary value. From the normal random vector, an axisymmetrically anisotropic initial flow was generated. The generated initial flow satisfies the continuity equation. Using this scheme, the time evolution of the fluctuating velocity field in an inviscid flow is analyzed, and the effects of the kinetic energy conservation error on the time evolution of the initial anisotropic flow field were obtained. In order to accurately simulate the kinetic energy conservation error, a periodic box filled with the inviscid flow was simulated. The fourth-order discretization scheme was used as a spatial discretization scheme. In order to obtain statistics, 1,000 cases were set for each computational condition.

Using the scheme proposed in the present study, an axisymmetrically anisotropic initial velocity field based on pancake- and cigar-type fields was generated. In contrast to the case in which the fourth-order Runge-Kutta method is used, using the Crank-Nicolson method, due to the kinetic energy conservation error, the value of kinetic energy that should be constant increased with time. For the computational conditions for both time integration schemes, the value of the anisotropic factor of the initial velocity field decreases with time. This result is clearly shown by calculating the anisotropy factor. The anisotropy factor approaches unity with increasing time. Here, the time evolution of the anisotropy factor is only slightly affected by the kinetic energy conservation error. On the other hand, the kinetic energy conservation error affected the pressure-strain correlation term in the equation of velocity fluctuation intensity. When the kinetic energy increases with time due to kinetic energy conservation errors, the magnitude of the pressure-strain correlation term appears to be larger.

The present study proposed a simple scheme for generating an axisymmetrically anisotropic initial velocity field. In the present study, an initial velocity field was generated using a normal random vector. Using other vectors, various initial velocity fields with axisymmetric anisotropy may be generated as future work. The decay characteristics of decaying turbulence may be affected by differences in the initial velocity field (e.g., [8]). While the inviscid flow was used in the present study, the effect of the initial velocity field on the decaying turbulence in the viscous flow can be investigated using the scheme proposed in the present study.

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