Thin II$_1$ factors with no Cartan subalgebras

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Abstract

It is a wide open problem to give an intrinsic criterion for a II$_1$ factor $M$ to admit a Cartan subalgebra $A$. When $A \subset M$ is a Cartan subalgebra, the $A$-bimodule $L^2(M)$ is “simple” in the sense that the left and right action of $A$ generate a maximal abelian subalgebra of $B(L^2(M))$. A II$_1$ factor $M$ that admits such a subalgebra $A$ is said to be $s$-thin. Very recently, Popa discovered an intrinsic local criterion for a II$_1$ factor $M$ to be $s$-thin and left open the question whether all $s$-thin II$_1$ factors admit a Cartan subalgebra. We answer this question negatively by constructing $s$-thin II$_1$ factors without Cartan subalgebras.

1 Introduction

One of the main decomposability properties of a II$_1$ factor $M$ is the existence of a Cartan subalgebra $A \subset M$, i.e. a maximal abelian subalgebra (MASA) whose normalizer $\mathcal{N}_M(A) = \{ u \in \mathcal{U}(M) \mid uAu^* = A \}$ generates $M$ as a von Neumann algebra. Indeed by [FM75], when $M$ admits a Cartan subalgebra, then $M$ can be realized as the von Neumann algebra $L_{\Omega}(\mathcal{R})$ associated with a countable equivalence relation $\mathcal{R}$, possibly twisted by a scalar 2-cocycle $\Omega$. If moreover this Cartan subalgebra is unique in the appropriate sense, this decomposition $M = L_{\Omega}(\mathcal{R})$ is canonical.

Although a lot of progress on the existence and uniqueness of Cartan subalgebras has been made (see e.g. [OP07, PV11]), there is so far no intrinsic local criterion to check whether a given II$_1$ factor admits a Cartan subalgebra. When $A \subset M$ is a Cartan subalgebra, then $A \subset M$ is in particular an $s$-MASA, meaning that the $A$-bimodule $\mathcal{A}L^2(M)\mathcal{A}$ is cyclic, i.e. there exists a vector $\xi \in L^2(M)$ such that $A\xi A$ spans a dense subspace of $L^2(M)$. Although it was already shown in [Pu59] that the hyperfinite II$_1$ factor $R$ admits an $s$-MASA $A \subset R$ that is singular (i.e. that satisfies $\mathcal{N}_R(A)^\prime = A$), all examples of $s$-MASAs so far were inside II$_1$ factors that also admit a Cartan subalgebra.

Very recently in [Po16], Popa discovered that the existence of an $s$-MASA in a II$_1$ factor $M$ is an intrinsic local property. He proved that a II$_1$ factor $M$ admits an $s$-MASA if and only if $M$ satisfies the $s$-thin approximation property: for every finite partition of the identity $p_1, \ldots, p_n$ in $M$, every finite subset $\mathcal{F} \subset M$ and every $\varepsilon > 0$, there exists a finer partition of the identity $q_1, \ldots, q_m$ and a single vector $\xi \in L^2(M)$ such that every element in $\mathcal{F}$ can be approximated up to $\varepsilon$ in $\| \cdot \|_2$ by linear combinations of the $q_i\xi q_j$.

Although an $s$-MASA can be singular and although it is even proved in [Po16, Corollary 4.2] that every $s$-thin II$_1$ factor admits uncountably many non conjugate singular $s$-MASAs, as said above, all known $s$-thin factors so far also admit a Cartan subalgebra and Popa poses as [Po16, Problem 5.1.2] to give examples of $s$-thin factors without Cartan subalgebras. We solve this problem here by constructing $s$-thin II$_1$ factors $M$ that are even strongly solid: whenever $B \subset M$ is a diffuse amenable von Neumann subalgebra, the normalizer $\mathcal{N}_M(B)^\prime$ stays amenable. Clearly, nonamenable strongly solid II$_1$ factors have no Cartan subalgebras.

We obtain this new class of strongly solid II$_1$ factors by applying Popa’s deformation/rigidity theory to Shlyakhtenko’s $A$-valued semicircular systems (see [Sh97] and Section 3 below). When

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A is abelian, this provides a rich source of examples of MASAs with special properties, like MASAs satisfying the s-thin approximation property of \([Po16]\).

Generalizing Voiculescu’s free Gaussian functor \([Vo83]\), the data of Shlyakhtenko’s construction consists of a tracial von Neumann algebra \((A, \tau)\) and a symmetric \(A\)-bimodule \(AH_A\), where the symmetry is given by an anti-unitary operator \(J : H \to H\) satisfying \(J^2 = 1\) and \(J(a \cdot \xi \cdot b) = b^* \cdot J\xi \cdot a^*\). The construction produces a tracial von Neumann algebra \(M\) containing \(A\) such that \(AL^2(M)_A\) can be identified with the full Fock space

\[
L^2(A) \oplus \bigoplus_{n \geq 1} (H \otimes_A \cdots \otimes_A H)^n.
\]

In the same way as the free Gaussian functor transforms direct sums of real Hilbert spaces into free products of von Neumann algebras, the construction of \([Sh97]\) transforms direct sums of \(A\)-bimodules into free products that are amalgamated over \(A\). Therefore, the deformation/rigidity results and methods for amalgamated free products introduced in \([IPP05, Io12]\), and in particular Popa’s s-malleable deformation obtained by “doubling and rotating” the \(A\)-bimodule, can be applied and yield the following result, proved in Corollaries 4.2 and 6.2 below (see Theorem 6.1 for the most general statement).

**Theorem A.** Let \((A, \tau)\) be a tracial von Neumann algebra and let \(M\) be the von Neumann algebra associated with a symmetric \(A\)-bimodule \(AH_A\). Assume that \(AH_A\) is weakly mixing (Definition 2.2) and that the left action of \(A\) on \(H\) is faithful. Then, \(M\) has no Cartan subalgebra. If moreover \(AH_A\) is mixing and \(A\) is amenable, then \(M\) is strongly solid.

In the particular case where \(A\) is diffuse abelian and the bimodule \(AH_A\) is weakly mixing, we get that \(A \subset M\) is a singular MASA. Very interesting examples arise as follows by taking \(A = L^\infty(K, \mu)\) where \(K\) is a second countable compact group with Haar probability measure \(\mu\). Whenever \(\nu\) is a probability measure on \(K\), we consider the \(A\)-bimodule \(H_\nu\) given by

\[
H_\nu = L^2(K \times K, \mu \times \nu) \quad \text{with} \quad (F \cdot \xi \cdot G)(x, y) = F(xy)\xi(x, y)G(x),
\]

for all \(F, G \in A\) and \(\xi \in H_\nu\). We assume that \(\nu\) is symmetric and use the symmetry

\[
J_\nu : H_\nu \to H_\nu : (J\xi)(x, y) = \overline{\xi(xy, y^{-1})} \quad \text{for all} \quad x, y \in K.
\]

We denote by \(M\) the tracial von Neumann algebra associated with the \(A\)-bimodule \((H_\nu, J_\nu)\).

The \(A\)-bimodule \(H_\nu\) is weakly mixing if and only if the measure \(\nu\) has no atoms, while \(H_\nu\) is mixing when the probability measure \(\nu\) is \(c_0\), meaning that the convolution operator \(\lambda(\nu)\) on \(L^2(K)\) is compact (see Definition 7.2 and Proposition 7.3). So for all \(c_0\) probability measures \(\nu\) on \(K\), we get that \(M\) is strongly solid.

On the other hand, when the measure \(\nu\) is concentrated on a subset of the form \(F \cup F^{-1}\), where \(F \subset K\) is free in the sense that every reduced word with letters from \(F \cup F^{-1}\) defines a nontrivial element of \(K\), then \(A \subset M\) is an s-MASA.

In Theorem 7.5, we construct a compact group \(K\), a free subset \(F \subset K\) generating \(K\) and a symmetric \(c_0\) probability measure \(\nu\) with support \(F \cup F^{-1}\). For this, we use results of \([AR92, GHSSV07]\) on the spectral gap and girth of a random Cayley graph of the finite group \(\text{PGL}(2,\mathbb{Z}/p\mathbb{Z})\). As a consequence, we obtain the first examples of s-thin \(\text{II}_1\) factors that have no Cartan subalgebra, solving \([Po16, \text{Problem 5.1.2}]\), which was the motivation for our work.

**Theorem B.** Taking a compact group \(K\) and a symmetric probability measure \(\nu\) on \(K\) as above, the associated \(\text{II}_1\) factor \(M\) is nonamenable, strongly solid and the canonical subalgebra \(A \subset M\) is an s-MASA.
As we explain in Remark 3.5, the so-called free Bogoljubov crossed products \( L(\mathbb{F}_\infty) \rtimes G \) associated with an (infinite dimensional) orthogonal representation of a countable group \( G \) can be written as the von Neumann algebra associated with a symmetric \( A \)-bimodule where \( A = L(G) \). Therefore, our Theorem A is a generalization of similar results proved in [Ho12b] for free Bogoljubov crossed products. Although free Bogoljubov crossed products \( M = L(\mathbb{F}_\infty) \rtimes G \) with \( G \) abelian provide examples of MASAs \( L(G) \subset M \) with interesting properties (see [HS09, Ho12a]), \( L(G) \subset M \) can never be an \( s \)-MASA (see Remark 7.4).

The point of view of \( A \)-valued semicircular systems is more flexible and even offers advantages in the study of free Bogoljubov crossed products \( M = L(\mathbb{F}_\infty) \rtimes G \). Indeed, in Corollary 6.4, we prove that these \( \Pi_1 \) factors \( M \) never have a Cartan subalgebra, while in [Ho12b], this could only be proved for special classes of orthogonal representations.

In Theorem 5.1, we prove several maximal amenability results for the inclusion \( A \subset M \) associated with a symmetric \( A \)-bimodule \((H, J)\), by combining the methods of [Po83, BH16]. Again, these results generalize [Ho12a, Ho12b] where the same was proved for free Bogoljubov crossed products.

We finally make some concluding remarks on the existence of \( c_0 \) probability measures supported on free subsets of a compact group. On an abelian compact group \( K \), a probability measure \( \nu \) is \( c_0 \) if and only if its Fourier transform \( \hat{\nu} \) tends to zero at infinity as a function from \( \hat{K} \) to \( \mathbb{C} \). Of course, no two elements of an abelian group are free, but the abelian variant of being free is the so-called independence property: a subset \( F \) of an abelian compact group \( K \) is called independent if any linear combination of distinct elements in \( F \) with coefficients in \( \mathbb{Z} \setminus \{0\} \) defines a non zero element in \( K \). It was proved in [Ru60] that there exist closed independent subsets of the circle group \( \mathbb{T} \) that carry a \( c_0 \) probability measure. It would be very interesting to get a better understanding of which, necessarily non abelian, compact groups admit \( c_0 \) probability measures supported on a free subset and we conjecture that these exist on the groups \( \text{SO}(n), n \geq 3 \).

2 Preliminaries

Let \((A, \tau)\) be a tracial von Neumann algebra.

**Definition 2.1.** A symmetric \( A \)-bimodule \((H, J)\) is an \( A \)-module \( A H A \) equipped with an anti-unitary operator \( J: H \to H \) such that \( J^2 = 1 \) and

\[
J(a \cdot \xi \cdot b) = b^* \cdot J\xi \cdot a^*, \quad \forall a, b \in A.
\]

A vector \( \xi \) in a right (resp. left) \( A \)-module \( H \) is said to be right (resp. left) bounded if there exists a \( \kappa > 0 \) such that \( \|\xi a\| \leq \kappa \|a\|_2 \) (resp. \( \|a\xi\| \leq \kappa \|a\|_2 \)) for all \( a \in A \). Whenever \( \xi \) is right bounded, we denote by \( \ell(\xi) \) the map \( L^2(A) \to H: a \mapsto \xi a \). Similarly, when \( \xi \) is left bounded, we denote by \( r(\xi) \) the map \( L^2(A) \to H: a \mapsto a\xi \).

Given right bounded vectors \( \xi, \eta \), the operator \( \ell(\xi)^* r(\eta) \) belongs to \( A \) and is denoted \( \langle \xi, \eta \rangle_A \). This defines an \( A \)-valued scalar product associated with the right \( A \)-module \( H \). Similarly, if \( \xi, \eta \in H \) are left bounded vectors, we define an \( A \)-valued scalar product associated with the left \( A \)-module \( H \) by \( A(\xi, \eta) = J r(\xi)^* r(\eta) J \in A \). Here, \( J \) denotes the canonical involution on \( L^2(A) \).

Popa’s non intertwinability condition (see [Po03, Section 2]) saying that \( B \not\prec_M A \) is equivalent with the existence of a sequence of unitaries \( b_n \in U(B) \) such that \( \lim_n \|E_A(x b_n y)\|_2 = 0 \) for all \( x, y \in M \) can be viewed as a weak mixing condition for the \( B \)-\( A \)-bimodule \( B L^2(M)_A \) (cf. the notions of relative (weak) mixing in [Po05, Definition 2.9]). This then naturally lead to the notion of a mixing, resp. weakly mixing bimodule in [PS12].
Definition 2.2 ([PS12]). Let \((A, \tau)\) and \((B, \tau)\) be tracial von Neumann algebras and \(B H_A\) a \(B\)-\(A\)-bimodule.

1. \(B H_A\) is called **left weakly mixing** if there exists a net of unitaries \(b_n \in \mathcal{U}(B)\) such that for all right bounded vectors \(\xi, \eta \in H\), we have
   \[
   \lim_n \|\langle b_n \xi, \eta \rangle_A\|_2 = 0.
   \]
2. \(B H_A\) is called **left mixing** if every net \(b_n \in \mathcal{U}(B)\) tending to 0 weakly satisfies
   \[
   \lim_n \|\langle b_n \xi, \eta \rangle_A\|_2 = 0
   \]
   for all right bounded vectors \(\xi, \eta \in H\).

We similarly define the notions of **right (weak) mixing**. When \(A H_A\) is a symmetric \(A\)-bimodule, left (weak) mixing is equivalent with right (weak) mixing and we simply refer to these properties as (weak) mixing.

In [Po03, Section 2], Popa proved that the intertwining relation \(B \prec_M A\) is equivalent with the existence of a nonzero \(B\)-\(A\)-subbimodule of \(L^2(M)\) having finite right \(A\)-dimension. In the same way, one gets the following characterization of weakly mixing bimodules. For details, see [PS12] and [Bo14, Theorem A.2.2].

Proposition 2.3 ([Po03, PS12, Bo14]). Let \((A, \tau)\) and \((B, \tau)\) be tracial von Neumann algebras and \(B H_A\) a \(B\)-\(A\)-bimodule. The following are equivalent:

1. \(B H_A\) is left weakly mixing;
2. \(\{0\}\) is the only \(B\)-\(A\)-subbimodule of \(B H_A\) of finite \(A\)-dimension;
3. \(B(H \otimes_A \mathcal{H})_B\) has no nonzero \(B\)-central vectors.

3 Shlyakhtenko’s \(A\)-valued semicircular systems

We first recall Voiculescu’s free Gaussian functor from the category of real Hilbert spaces to the category of tracial von Neumann algebras. Let \(H_\mathbb{R}\) be a real Hilbert space and let \(H\) be its complexification. The **full Fock space** of \(H\) is defined as

\[
\mathcal{F}(H) = \mathbb{C} \Omega \oplus \bigoplus_{n=1}^{\infty} H^\otimes_n.
\]

The unit vector \(\Omega\) is called the **vacuum vector**. Given a vector \(\xi \in H\), we define the **left creation operator** \(\ell(\xi) \in B(\mathcal{F}(H))\) by

\[
\ell(\xi)(\Omega) = \xi \quad \text{and} \quad \ell(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.
\]

Put

\[
\Gamma(H_\mathbb{R})^\tau := \{\ell(\xi) + \ell(\xi)^* \mid \xi \in H_\mathbb{R}\}^\tau.
\]

This von Neumann algebra is equipped with the faithful trace given by \(\tau(\cdot) = \langle \cdot \Omega, \Omega \rangle\). In [Vo83], it is proved that the operator \(\ell(\xi) + \ell(\xi)^*\) has a semicircular distribution with respect to the trace \(\tau\) and that \(\Gamma(H_\mathbb{R})^\tau \cong L(\mathcal{F}_{\dim H_\mathbb{R}})\). By the functoriality of the construction, any orthogonal transformation \(u\) of \(H_\mathbb{R}\) gives rise to an automorphism \(\alpha_u\) of \(\Gamma(H_\mathbb{R})^\tau\) satisfying
\( \alpha_a(\ell(\xi) + \ell(\xi)^*) = \ell(u\xi) + \ell(u^*\xi)^* \) for all \( \xi \in H_R \). So, every orthogonal representation \( \pi : G \to O(H_R) \) of a countable group \( G \) gives rise to the free Bogolyubov action \( \sigma_\pi : G \curvearrowright \Gamma(H_R)^\prime\prime \) given by \( \sigma_\pi(g) = \alpha_{\pi(g)} \) for all \( g \in G \).

In [Sh97], Shlyakhtenko introduced a generalization of Voiculescu’s free Gaussian functor, this time being a functor from the category of symmetric \( A \)-bimodules (where \( A \) is any von Neumann algebra) to the category of von Neumann algebras containing \( A \). We will here repeat this construction in the case where \( A \) is a tracial von Neumann algebra.

Let \((A, \tau)\) be a tracial von Neumann algebra and let \((H, J)\) be a symmetric \( A \)-bimodule. We denote by \( H(3.2) \) the \( n \)-fold Connes tensor product \( H \otimes_A H \otimes_A \cdots \otimes_A H \). The full Fock space of the \( A \)-bimodule \( _A H_A \) is defined by

\[
\mathcal{F}_A(H) = L^2(A) \oplus \bigoplus_{n=1}^{\infty} H(3.2) .
\]  

We denote by \( H \) the set of left and right \( A \)-bounded vectors in \( H \). Since \( A \) is a tracial von Neumann algebra, \( H \) is dense in \( H \). Given a right bounded vector \( \xi \in H \), we define the left creation operator \( \ell(\xi) \) analogous to the case where \( A = \mathbb{C} \) by

\[
\ell(\xi)(a) = \xi a, \quad a \in A ,
\]

\[
\ell(\xi)(\xi_1 \otimes_A \cdots \otimes_A \xi_n) = \xi \otimes_A \xi_1 \otimes_A \cdots \otimes_A \xi_n , \quad \xi, i \in H .
\]

Note that \( a\ell(\xi) = \ell(a\xi) \) and \( \ell(\xi)a = \ell(\xi)a \) for \( a \in A \) and that the adjoint map \( \ell(\xi)^* \) satisfies

\[
\ell(\xi)^*(a) = 0 \quad \text{for all} \quad a \in L^2(A) ,
\]

\[
\ell(\xi)^*(\xi_1 \otimes_A \cdots \otimes_A \xi_n) = \langle \xi, \xi_1 \rangle_A \xi_2 \otimes_A \cdots \otimes_A \xi_n \quad \text{for} \quad \xi, i \in H .
\]

**Definition 3.1.** Given a tracial von Neumann algebra \((A, \tau)\) and a symmetric \( A \)-bimodule \((H, J)\), we consider the full Fock space \( \mathcal{F}_A(H) \) given by (3.1) and define

\[
\Gamma(H, J, A, \tau)^\prime\prime := A \vee \{ \ell(\xi) + \ell(\xi)^* \mid \xi \in H, J\xi = \xi\}'' \subset B(\mathcal{F}_A(H)),
\]

where \( A \subset B(\mathcal{F}_A(H)) \) is given by the left action on \( \mathcal{F}_A(H) \). We also have

\[
\Gamma(H, J, A, \tau)^\prime\prime = A \vee \{ \ell(\xi) + \ell(\xi)^* \mid \xi \in H\}'' .
\]

We denote by \( \Omega \) the vacuum vector in \( \mathcal{F}_A(H) \) given by \( \Omega = 1_A \in L^2(A) \). We define \( \tau \) as the vector state on \( M = \Gamma(H, J, A, \tau)^\prime\prime \) given by the vacuum vector \( \Omega \). Whenever \( n \geq 1 \) and \( \xi_1, \ldots, \xi_n \in H \), we define the Wick product as in [HR10, Lemma 3.2] by

\[
W(\xi_1, \ldots, \xi_n) = \sum_{i=0}^{n} \ell(\xi_1) \cdots \ell(\xi_i)\ell(J\xi_{i+1})^* \cdots \ell(J\xi_n)^* .
\]

As in [HR10, Lemma 3.2], we get that \( W(\xi_1, \ldots, \xi_n) \in M \) and

\[
W(\xi_1, \ldots, \xi_n)\Omega = \xi_1 \otimes_A \cdots \otimes_A \xi_n .
\]

These elements, with \( n \geq 1 \), span a \( \| \cdot \|_2 \)-dense subspace of \( M \otimes A \). Together with \( A \), they span a \( \| \cdot \|_2 \)-dense \( * \)-subalgebra of \( M \).

**Proposition 3.2** ([Sh97]). The state \( \tau(\cdot) = \langle \cdot, \Omega \rangle \) defined above is a faithful trace on \( M \).
Proof. Define $\mathcal{J} : \mathcal{F}_A(H) \to \mathcal{F}_A(H)$ by $\mathcal{J}(a) = a^*$ for $a \in A$ and

$$\mathcal{J}(\xi_1 \otimes_A \cdots \otimes_A \xi_n) = J\xi_n \otimes_A \cdots \otimes_A J\xi_1$$

for $\xi_1, \ldots, \xi_n \in \mathcal{H}$. Then $\mathcal{J}$ is an anti-unitary map satisfying $\mathcal{J}^2 = 1$. One easily checks that $\mathcal{J}\ell(\xi)\mathcal{J} = r(J\xi)$ for all $\xi \in \mathcal{H}$ and that $\mathcal{J}a\mathcal{J}$ is just right multiplication by $a^*$ on $\mathcal{F}_A(H)$. This implies that $\mathcal{J}\mathcal{M}\mathcal{J}$ commutes with $M$. Indeed, for $\xi, \eta \in \mathcal{H}$ with $J\xi = \xi$ and $J\eta = \eta$, we have $(\xi, a\eta)_A = A(\xi, \eta)$ since

$$\langle Jr(\xi)^*r(\eta)Jx, y \rangle = \langle r(\xi)^*y^*, r(\eta)x^* \rangle = \langle y^*\xi a, x^*\eta \rangle = \langle J(x^*\eta), J(y^*\xi a) \rangle = \langle \eta x, a^*\xi y \rangle = \langle \ell(\xi)^*\ell(\eta)x, y \rangle ,$$

for all $x, y \in A$. It follows that

$$(\ell(\xi)^*r(\eta) + \ell(\xi)r(\eta)^*)(a) = (\xi, an)_A = A(\xi, a\eta) = (\xi, a\eta) = (r(\eta)^*\ell(\xi) + r(\eta)\ell(\xi)^*)(a), \quad \forall a \in A .$$

Since $\ell(\xi)$ and $r(\eta)^*$ clearly commute when restricted to $\mathcal{F}_A(H) \otimes L^2(A)$, it follows that $\ell(\xi) + \ell(\xi)^*$ commutes with $r(\eta)^* + r(\eta)$. We conclude that $M$ commutes with $\mathcal{J}\mathcal{M}\mathcal{J}$.

Next, we show that $\mathcal{J}(x\Omega) = x^*\Omega$ for all $x \in M$. This clearly holds for $x \in A$ so it suffices to prove it for $x$ of the form $x = W(\xi_1, \ldots, \xi_n)$ with $\xi_i \in \mathcal{H}$. We have

$$\mathcal{J}(W(\xi_1, \ldots, \xi_n)\Omega) = \mathcal{J}(\xi_1 \otimes_A \cdots \otimes_A \xi_n) = J\xi_n \otimes_A \cdots \otimes_A J\xi_1$$

$$= W(J\xi_n, \ldots, J\xi_1)\Omega = W(\xi_1, \ldots, \xi_n)^*\Omega .$$

We now get that

$$\tau(xy) = \langle xy\Omega, \Omega \rangle = \langle x, y^*\Omega, \Omega \rangle = \langle x, y^*J\Omega, \Omega \rangle = \langle Jy^*Jx\Omega, \Omega \rangle$$

$$= \langle x\Omega, JyJ\Omega \rangle = \langle x\Omega, y^*\Omega \rangle = \langle yx\Omega, \Omega \rangle = \tau(yx) ,$$

for all $x, y \in M$ and hence $\tau$ is a trace.

It is easy to check that $\Omega \in \mathcal{F}_A(H)$ is a cyclic vector for both $M$ and $\mathcal{J}\mathcal{M}\mathcal{J}$. Hence $\Omega$ is also separating for $M$ and it follows that $\tau$ is faithful. 

By construction, we have that $L^2(M) \cong \mathcal{F}_A(H)$ as $A$-bimodules.

In [Sh97], Shlyakhtenko used the terminology $A$-valued semicircular system for the family $\{\ell(\xi) + \ell(\xi)^* | \xi \in \mathcal{H}, J\xi = \xi\}$, as an analogue to the free Gaussian functor case, where the operator $\ell(\xi) + \ell(\xi)^*$ has a semicircular distribution with respect to $\tau$.

**Example 3.3.** 1. When $H = L^2(A)$ is the trivial $A$-bimodule with $J(a) = a^*$, we simply get

$$\Gamma(H, J, A, \tau)^\prime\prime = A \boxtimes L^\infty[0, 1] .$$

Indeed, $A$ commutes with $\ell(1) + \ell(1)^*$ and they together generate $\Gamma(H, J, A, \tau)^\prime\prime$. In particular, we see that $\Gamma(H, J, A, \tau)^\prime\prime$ is not always a factor.

2. When $H = L^2(A) \otimes L^2(A)$ is the coarse $A$-bimodule with $J(a \otimes b) = b^* \otimes a^*$, we get

$$\Gamma(H, J, A, \tau)^\prime\prime = (A, \tau) \ast L^\infty[0, 1] .$$

This example shows that the construction of $\Gamma(H, J, A, \tau)^\prime\prime$ may depend on the trace on $A$. Indeed, if $A = \mathbb{C}^2$ we can consider the trace $\tau_\delta$ for any $\delta \in (0, 1)$ given by $\tau_\delta(a, b) = \delta a + (1 - \delta) b$, $a, b \in \mathbb{C}$. By [Dy92, Lemma 1.6] we have that $L(\mathbb{Z}) \ast (A, \tau_\delta) = L(F_{1+2\delta(1-\delta)})$, the
interpolated free group factor. It is wide open whether the interpolated free group factors are all isomorphic. So at least, there is no obvious isomorphism between $\Gamma(H, J, A, \tau_{\delta_1})''$ and $\Gamma(H, J, A, \tau_{\delta_2})''$ for $\delta_1 \neq \delta_2$. In Example 3.6, we shall actually see that even the factoriality of $\Gamma(H, J, A, \tau)'''$ may depend on the choice of the trace $\tau$. For a general factoriality criterion for $\Gamma(H, J, A, \tau)'''$, see Theorem 6.1.

Note that the construction of $\Gamma(H, J, A, \tau)'''$ is functorial in the following sense. If $U \in \mathcal{U}(H)$ is a unitary operator that is $A$-bimodular and commutes with $J$, then $U$ defines a trace-preserving automorphism of $M = \Gamma(H, J, A, \tau)'''$ in the following way. Since $U$ is $A$-bimodular, we can define a unitary $U^n$ on $H^\otimes A$ by $U^n(\xi_1 \otimes_A \cdots \otimes_A \xi_n) = (U\xi_1 \otimes_A \cdots \otimes_A U\xi_n)$. The direct sum of these unitaries (and the identity on $L^2(A)$) then gives an $A$-bimodular unitary operator on $\mathcal{F}_A(H)$, which we will still denote by $U$. Note that $U\ell(\xi)U^* = \ell(U\xi)$ for all $\xi \in H$. Since $U$ commutes with $J$, it follows that $UMU^* = M$ so that $\text{Ad}U$ defines an automorphism of $M$.

Recall that for Voiculescu’s free Gaussian functor, we have that the direct sum of Hilbert spaces translates into the free product of von Neumann algebras, in the sense that $\Gamma(H_1 \oplus H_2) = \Gamma(H_1) \ast \Gamma(H_2)$. In the setting of $A$-bimodules in general, we instead get the amalgamated free product over $A$ as stated in the following proposition.

**Proposition 3.4 ([Sh97, Proposition 2.17]).** Let $(H_1, J_1)$ and $(H_2, J_2)$ be symmetric $A$-bimodules. Put $H = H_1 \oplus H_2$ and $J = J_1 \oplus J_2$. Then

$$\Gamma(H, J, A, \tau)''' \cong \Gamma(H_1, J_1, A, \tau)''' \ast_A \Gamma(H_2, J_2, A, \tau)''',$$

with respect to the unique trace-preserving conditional expectation onto $A$.

**Remark 3.5.** As we recalled in the beginning of this section, to every orthogonal representation $\pi : G \to O(K_\mathbb{R})$ of a countable group $G$ on a real Hilbert space $K_\mathbb{R}$ is associated the free Bogoljubov action $\sigma_\pi : G \curvearrowright \Gamma(K_\mathbb{R})'\ast$. Write $A = L(G)$ and $A$ with its canonical tracial state $\tau$. Denote by $K$ the complexification of $K_\mathbb{R}$ and equip the symmetric $A$-bimodule $A^H_A$ given by

$$H = \ell^2(G) \otimes K$$

with $u_g \cdot (\delta_h \otimes \xi) \cdot u_k = \delta_{ghk} \otimes \pi(g)\xi$

and $J(\delta_h \otimes \xi) = \delta_{h^{-1}} \otimes \pi(h^{-1})\xi$

(3.3)

where $(\delta_g)_{g \in G}$ denotes the canonical orthonormal basis of $\ell^2(G)$. It is now straightforward to check that there is a canonical trace preserving isomorphism

$$\Gamma(H, J, A, \tau)''' \cong \Gamma(K_\mathbb{R})'\ast \sigma_\pi G$$

that maps $A$ onto $L(G)$ identically.

**Example 3.6.** This final example illustrates that even the factoriality of $\Gamma(H, J, A, \tau)'''$ may depend on the choice of $\tau$. Take $A = \mathbb{C}^2$, $\alpha \in \text{Aut}(A)$ the flip automorphism and $H = \mathbb{C}^2$ with $A$-bimodule structure given by $a \cdot \xi \cdot b = \alpha(a)\xi b$. Define $J : H \to H : J(a) = \alpha(a)^\ast$. The $n$-fold tensor power $H^\otimes A$ can be identified with $\mathbb{C}^2$ with the bimodule structure given by

$$a \cdot \xi \cdot b = \begin{cases} a\xi b & \text{if } n \text{ is even}, \\ \alpha(a)\xi b & \text{if } n \text{ is odd}. \end{cases}$$

We denote by $\{e_n, f_n\}$ the canonical orthonormal basis of $H^\otimes A$ under this identification. For every $0 < \delta < 1$, denote by $\tau_\delta$ the trace on $A$ given by $\tau_\delta(a, b) = \delta a + (1 - \delta)b$. With respect to the canonical trace $\tau = \tau_{1/2}$, the left and right creation operators associated with the identity $1 \in A = H$ then become

$$\ell(e_n) = e_{n+1}, \quad \ell(f_n) = f_{n+1}, \quad r(e_n) = f_{n+1}, \quad r(f_n) = e_{n+1}.$$
for all \( n \geq 0 \).

By symmetry, it suffices to consider the case \( 0 < \delta \leq 1/2 \). With respect to the trace \( \tau_\delta \), the left and right creation operators \( \ell_\delta \) and \( r_\delta \) can be realized on the same Hilbert space and are given by

\[
\ell_\delta = \ell \lambda(D^{-1/2}) \quad \text{and} \quad r_\delta = r \rho(D^{-1/2}) ,
\]

where \( D = (2\delta, (1-\delta)) \) is the Radon-Nikodym derivative between \( \tau_\delta \) and \( \tau_{1/2} \) and where we denote by \( \lambda(\cdot) \) and \( \rho(\cdot) \) the left, resp. right, action of \( A \). Then,

\[
M_\delta := \Gamma(H, J, A, \tau_\delta)'' = \lambda(A) \vee \{ \ell_\delta + \ell_\delta^* \}'' = \lambda(A) \vee \{ S_\delta \}'' ,
\]

where \( S_\delta = \lambda(\Delta^{-1/4}) + \ell^* \lambda(\Delta^{1/4}) \) and \( \Delta = (\delta/(1-\delta), (1-\delta)/\delta) \). We still denote by \( \tau_\delta \) the canonical trace on \( M_\delta \).

Note that \( S_\delta = S_\delta^* \). Denoting by \( e = (1,0) \) and \( f = (0,1) \) the minimal projections in \( A \), we have that \( S_\delta e = f S_\delta \). When \( \delta = 1/2 \), the operator \( S_\delta \) is nonsingular and diffuse. When \( 0 < \delta < 1/2 \), the kernel of \( S_\delta \) has dimension 1 and \( S_\delta \) is diffuse on its orthogonal complement. We denote by \( z_\delta \) the projection onto the kernel of \( S_\delta \). Then \( z_\delta \) is a minimal and central projection in \( M_\delta \) with \( \tau_\delta(z_\delta) = 1 - 2\delta \). We conclude that there is a trace preserving \(*\)-isomorphism

\[
(M_\delta, \tau_\delta) \cong \big( M_\delta(\mathbb{C}) \big) \otimes B \bigoplus \big( \mathbb{C} , \delta(\Tr_{\tau_\delta}) \big) \bigoplus \mathbb{C} , \quad 1 - 2\delta \tag{3.4}
\]

where \((B_0, \tau_0)\) is a diffuse abelian von Neumann algebra with normal faithful tracial state \( \tau_0 \) and where we emphasized the choice of trace at the right hand side. Under the isomorphism (3.4), we have that

\[
e \mapsto (e_{11} \otimes 1) + 0 \quad f \mapsto (e_{22} \otimes 1) + 1 \quad S_\delta \mapsto ((e_{12} + e_{21}) \otimes b) + 0 \quad z_\delta \mapsto 0 + 1
\]

where \( b \in B \) is a positive nonsingular element generating \( B \).

Next, taking \( H \oplus H \) and \( J \oplus J \), it follows from Proposition 3.4 that

\[
M_\delta := \Gamma(H \oplus H, J \oplus J, A, \tau_\delta)'' = M_\delta *_A M_\delta ,
\]

where we used at the right hand side the amalgamated free product w.r.t. the unique \( \tau_\delta \)-preserving conditional expectations. We denote with superscripts \((1)\) and \((2)\) the elements of \( M_\delta \) viewed in the first, resp. second copy of \( M_\delta \) in the amalgamated free product. Note that \( f^{(1)} = f^{(2)} \) and that, denoting this projection as \( f \), we get that \( f M_\delta^{(1)} f \) and \( f M_\delta^{(2)} f \) are free inside \( f M_\delta f \). It then follows from [Vo86] that the projection \( z := z_\delta^{(1)} \wedge z_\delta^{(2)} \) is nonzero if and only if \( \delta < 1/3 \). Using the diffuse subalgebras \( B^{(1)} \) and \( B^{(2)} \), we get that \( \mathcal{Z}(M_\delta) = \mathbb{C} z + \mathbb{C}(1-z) \).

We conclude that \( \Gamma(H \oplus H, J \oplus J, A, \tau_\delta)'' \) is a factor if and only if \( 1/3 \leq \delta \leq 2/3 \).

## 4 Normalizers and (relative) strong solidity

The main result of this section is the following dichotomy theorem for \( A \)-valued semicircular systems. In the special case of free Bogoljubov crossed products (see Remark 3.5), this result was proven in [Ho12b, Theorem B]. As explained in the introduction, the \( A \)-valued semicircular systems fit perfectly into Popa’s deformation/rigidity theory. The proof of Theorem 4.1 therefore follows closely [IPP05, HS09, HR10, Io12, Ho12b], using in the same way Popa’s \( \ast \)-malleable deformation given by “doubling and rotating” the initial \( A \)-bimodule \( A H_A \) (see below).

We freely use Popa’s intertwining-by-bimodules (see [Po03, Section 2]) and the notion of relative amenability (see [OP07, Section 2.2]).
Theorem 4.1. Let \((A, \tau)\) be a tracial von Neumann algebra and \((H, J)\) a symmetric \(A\)-bimodule. Put \(M = \Gamma(H, J, A, \tau)''\). Let \(q \in M\) be a projection and \(B \subset qMq\) a von Neumann subalgebra. If \(B\) is amenable relative to \(A\), then at least one of the following statements holds: \(B \prec_M A\) or \(N_M(B)''\) stays amenable relative to \(A\).

As a consequence of Theorem 4.1, we get the following strong solidity theorem.

Corollary 4.2. Let \((A, \tau)\) be a tracial von Neumann algebra and \((H, J)\) a symmetric \(A\)-bimodule. Denote \(M = \Gamma(H, J, A, \tau)''\). Assume that \(\alpha H_A\) is mixing.

If \(B \subset M\) is a diffuse von Neumann subalgebra that is amenable relative to \(A\), then \(N_M(B)''\) stays amenable relative to \(A\).

So if \(A\) is amenable and \(\alpha H_A\) is mixing, we get that \(M\) is totally solid.

Proof. Denote \(P := N_M(B)''\). Since \(B \vee (B' \cap M) \subset P\), we have \(P' \cap M = Z(P)\). Denote by \(z \in Z(P)\) the smallest projection such that \(Pz \neq_M A\). Then, \(P(1 - z)\) fully embeds into \(A\) inside \(M\) and, in particular, \(P(1 - z)\) is amenable relative to \(A\). It remains to prove that also \(Pz\) is amenable relative to \(A\).

Since the bimodule \(\alpha H_A\) is mixing, the inclusion \(A \subset M\) is mixing in the sense of [Po03, Proof of Theorem 3.1] and [Io12, Definition 9.2]. Since \(N_M(Z)'' = Pz\), since \(Bz\) is diffuse and since \(Pz \neq_M A\), it follows from [Io12, Lemma 9.4] that \(Bz \neq_M A\). It then follows from Theorem 4.1 that \(Pz\) is amenable relative to \(A\).

To prove Theorem 4.1, we fix a tracial von Neumann algebra \((A, \tau)\) and a symmetric \(A\)-bimodule \((H, J)\). Put \(M = \Gamma(H, J, A, \tau)''\) as in Definition 3.1. Recall that \(L\(A\) = \mathcal{F}_A(H) = \bigoplus_{n=1}^{\infty} H^\otimes n\).

We construct as follows an \(s\)-malleable deformation of \(M\) in the sense of [Po03]. Put \(\mathcal{M} = \Gamma(H \oplus H, A, J \oplus J)''\).

By Proposition 3.4, we have \(\mathcal{M} = M \ast_A M\). We denote by \(\pi_1\) and \(\pi_2\) the two canonical embeddings of \(M\) into \(\mathcal{M}\). When no embedding is explicitly mentioned, we will always consider \(M \subset \mathcal{M}\) via the embedding \(\pi_1\).

Let \(U_t \in \mathcal{U}(H \oplus H), t \in \mathbb{R}\), be the rotation with angle \(t\), i.e.,

\[U_t(\xi, \eta) = (\cos(t)\xi - \sin(t)\eta, \sin(t)\xi + \cos(t)\eta) \quad \text{for } \xi, \eta \in H.\]

Since the construction of \(\Gamma(H, J, A, \tau)''\) is functorial, this gives rise to an automorphism \(\theta_t := \text{Ad} U_t \in \text{Aut}(\mathcal{M})\). Note that \(\theta_{t/2} \circ \pi_1 = \pi_2\).

Define \(\beta \in \mathcal{U}(H)\) by \(\beta(\xi, \eta) = (\xi, -\eta)\) for \(\xi, \eta \in H\). Again by functoriality, we have that \(\beta\) defines an automorphism of \(\mathcal{M}\). Now, \(\beta\) satisfies \(\beta(x) = x\) for all \(x \in \pi_1(M)\), \(\beta^2 = \text{id}\) and \(\beta \circ \theta_t = \theta_{-t} \circ \beta\) for all \(t\). Hence \((\mathcal{M}, (\theta_t)_{t \in \mathbb{R}})\) is an \(s\)-malleable deformation of \(M\).

The following two lemmas are the key ingredients in the proof of Theorem 4.1.

Lemma 4.3. Let \(q \in M\) be a projection and \(P \subset qMq\) a von Neumann subalgebra. If \(\theta_t(P) \prec_{\mathcal{M}} \pi_i(M)\) for some \(i \in \{1, 2\}\) and some \(t \in (0, \frac{\pi}{2})\), then \(P \prec_M A\).

Lemma 4.4. Let \(q \in M\) be a projection and \(P \subset qMq\) a von Neumann subalgebra. If \(\theta_t(P)\) is amenable relative to \(A\) inside \(\mathcal{M}\) for all \(t \in (0, \frac{\pi}{2})\), then \(P\) is amenable relative to \(A\) inside \(M\).

Before proving Lemma 4.3 and Lemma 4.4, we first show how Theorem 4.1 follows from these two lemmas and we deduce a relative strong solidity theorem for \(A\)-valued semicircular systems.
Proof of Theorem 4.1. Put $P = \mathcal{N}_{qMq}(B)'$. We apply [Va13, Theorem A] to the subalgebra $\theta_t(B) \subset M *_A M$ for a fixed $t \in (0, \frac{\pi}{2})$. Note that $\theta_t(B)$ is normalized by $\theta_t(P)$. So, we get that one of the following holds:

1. $\theta_t(B) \preceq_M A$.
2. $\theta_t(P) \preceq_M \pi_i(M)$ for some $i \in \{1, 2\}$.
3. $\theta_t(P)$ is amenable relative to $A$ inside $M$.

If 1 or 2 holds, it follows by Lemma 4.3 that $B \preceq_M A$. So, if we assume that $B \not\preceq_M A$, we get that $\theta_t(P)$ is amenable relative to $A$ inside $M$ for all $t \in (0, \frac{\pi}{2})$. It then follows from Lemma 4.4 that $P = \mathcal{N}_{qMq}(B)'$ is amenable relative to $A$ inside $M$. \qed

Proof of Lemma 4.3

We now turn to the proof of Lemma 4.3. We first give a sketch of the proof. For each $k \in \mathbb{N}$, we let $p_k \in B(L^2(M))$ denote the projection onto $H^\otimes_k A$. Given a von Neumann subalgebra $P \subset qMq$, we first show that if $\theta_t(P) \preceq_M \pi_i(M)$ for some $i \in \{1, 2\}$ and some $t \in (0, \frac{\pi}{2})$, then $P$ has “bounded tensor length”, in the sense that there exists $k \in \mathbb{N}$ and $\delta > 0$ such that $\|\sum_{i=0}^k p_i(a)\|_2 \geq \delta$ for all $a \in \mathcal{U}(P)$ (see Lemma 4.6). Next, we reason exactly as in the proof of [Po03, Theorem 4.1]. Since $\theta_t$ converges uniformly to id on the unit ball of $p_k(M)$ for any fixed $i \in \mathbb{N}$, we get a $t \in (0, \frac{\pi}{2})$ and a nonzero partial isometry $v \in \mathcal{M}$ such that $\theta_t(a)v = v a$ for all $a \in \mathcal{U}(P)$. Using the automorphism $\beta$, we can even obtain $t = \pi/2$, i.e., $\pi(a)v = v \pi(a)$ for all $a \in \mathcal{U}(P)$. Using results of [IPP05] on amalgamated free product von Neumann algebras, this implies that $P \preceq_M A$.

For simplicity, we put $M_i = \pi_i(M) \subset \mathcal{M}$ for $i \in \{1, 2\}$. Note that

$$L^2(M_1) = L^2(A) \oplus \bigoplus_{k=1}^{\infty} (H \oplus 0)^\otimes_k A, \quad L^2(M_2) = L^2(A) \oplus \bigoplus_{k=1}^{\infty} (0 \oplus H)^\otimes_k A,$$

as subspaces of $L^2(M) = \mathcal{F}_A(H \oplus H)$. Denote by $e_i, \in B(L^2(M))$ the projection onto $L^2(M_i)$.

Lemma 4.5. If $\mu_n \in L^2(M_1)$ is a bounded sequence such that $\lim_{n \to \infty} \|p_k(\mu_n)\| = 0$ for all $k \geq 0$, then for all $i = 1, 2$, $0 < t < \frac{\pi}{2}$, integers $a, b, c, d \geq 0$ and vectors $\xi, \eta, \gamma, \rho_i \in \mathcal{H} \oplus \mathcal{H}$, we have

$$\lim_{n \to \infty} \|e_i(\ell(\xi_1) \cdots \ell(\xi_a)\ell(\eta_b) \cdots \ell(\eta_1)\ell(\gamma_c) \cdots r(\gamma_1)r(\rho_1) \cdots r(\rho_d)U_i(\mu_n))\| = 0.$$

Proof. Fix $t \in (0, \frac{\pi}{2})$ and define $\delta_1 = \cos t$ and $\delta_2 = \sin t$. Define the operator $Z_i \in B(L^2(M))$ for $i = 1, 2$ by

$$Z_i = \bigoplus_{e \geq b+d} \delta_i^{e-b-d} (U_t \otimes_A 1^{(e-b-d)} \otimes_A U_t^*).$$

Denote $p_{\geq b} = \sum_{i=b}^{\infty} p_i$ and $p_{<b} = \sum_{i=0}^{b-1} p_i$. When $\kappa \geq b+d$, we have $\|Z_p(\mu_n)\| = \delta_i^{\kappa-b-d}$. Since $\lim_{n} \|p_{<\kappa}(\mu_n)\| = 0$ for every $\kappa$, we get that $\lim_n \|Z_i(\mu_n)\| = 0$. So, it suffices to prove that

$$e_i(\ell(\xi_1) \cdots \ell(\xi_a)\ell(\eta_b) \cdots \ell(\eta_1)\ell(\gamma_c) \cdots r(\gamma_1)r(\rho_1) \cdots r(\rho_d)U_i(\mu_n)) = \ell(q_1 \xi_1) \cdots \ell(q_1 \xi_a)\ell(\eta_b) \cdots \ell(\eta_1)\ell(q_1 \gamma_c) \cdots r(q_1 \gamma_1)r(\rho_1) \cdots r(\rho_d)Z_i(\mu)$$

for all $\mu \in L^2(M_1)$, where $q_1, \text{resp.} \ q_2$, denotes the orthogonal projection of $H \oplus H$ onto $H \oplus 0$, resp. $0 \oplus H$. It is sufficient to check this formula for $\mu = \mu_1 \otimes_A \cdots \otimes_A \mu_e$ with $\mu_i \in \mathcal{H} \oplus 0$ and $e \geq b+d$, where it follows by a direct computation. \qed

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Lemma 4.6. If \(a_n \in M\) is a bounded sequence with \(\lim_n \|p_k(a_n)\|_2 = 0\) for all \(k \geq 0\), then
\[
\lim_{n \to \infty} \|E_M(x\theta_t(a_n))y\|_2 = 0
\]
for all \(i \in \{1, 2\}\), \(0 < t < \frac{\pi}{2}\) and \(x, y \in \mathcal{M}\).

Proof. It suffices to take \(x = W(\xi_1, \ldots, \xi_k)\) and \(y = W(\eta_1, \ldots, \eta_m)\) with \(\xi_i, \eta_i \in \mathcal{H} \oplus \mathcal{H}\) (as defined in Section 3), since these elements span a \(\| \cdot \|_2\)-dense subspace of \(\mathcal{M} \ominus A\). Then,
\[
E_M(x\theta_t(a_n)y) = e_M(xJy^*JU_t(a_n)\Omega)
\]
\[
= \sum_{s=0}^k \sum_{r=0}^m e_M(\ell(\xi_1) \cdots \ell(\xi_k) \ell(J\xi_{k+1})^* \cdots \ell(J\xi_k)^* r(\eta_m) \cdots r(\eta_{r+1}) r(J\eta_r)^* \cdots r(J\eta_1)^* U_t(a_n)\Omega),
\]
and the result now follows from Lemma 4.5.

We are now ready to finish the proof of Lemma 4.3.

Proof of Lemma 4.3. Assume that \(\theta_i(P) \prec M_i\) for some \(i \in \{1, 2\}\) and \(t \in (0, \frac{\pi}{2})\). By Lemma 4.6, we get a \(\delta > 0\) and \(\kappa > 0\) such that \(\| \sum_{i=0}^\kappa p_i(a)\|_2 \geq 2\delta\) for all \(a \in \mathcal{U}(P)\). Note that \(\langle U_t(p_i(a)), p_j(a) \rangle = 0\) if \(i \neq j\) and that \(\langle U_t(p_i(a)), p_i(a) \rangle = \cos(t)\|p_i(a)\|_2^2\). Choose \(t_0 \in (0, \frac{\pi}{2})\) such that \(\cos(t_0)^2 \geq 1/2\) for all \(i = 0, \ldots, \kappa\). Note that we may choose \(t_0\) of the form \(t_0 = \pi/2^n\). For all \(a \in \mathcal{U}(P)\), we then have
\[
\tau(\theta_{t_0}(a^*a)) = \langle U_{t_0}(a^*a), a \rangle = \sum_{i=0}^\infty \langle U_{t_0}(p_i(a)), p_j(a) \rangle = \sum_{i=0}^\infty \cos(t_0)^i \|p_i(a)\|_2^2 \\
\geq \sum_{i=0}^\kappa \cos(t_0)^i \|p_i(a)\|_2^2 \geq \frac{1}{2}2\delta = \delta.
\]

Let \(v\) be the unique element of minimal \(\| \cdot \|_2\)-norm in the \(\| \cdot \|_2\)-closed convex hull of \(\{\theta_{t_0}(a^*a) \mid a \in \mathcal{U}(P)\}\). Then \(v \in \mathcal{M}\) and \(\theta_{t_0}(a)v = va\) for all \(a \in \mathcal{U}(P)\). Moreover, \(v \neq 0\) since \(\tau(v) > \delta\).

Put \(w_1 = \theta_{t_0}(v\beta(v))\). Then \(w_1\) satisfies \(w_1a = \theta_{2t_0}(a)w_1\) for all \(a \in \mathcal{U}(P)\). However, we do not know yet that \(w_1\) is nonzero. Assuming that \(P \nabla M A\), we have from Proposition 3.4 and [IPP05, Theorem 1.2.1] that \(P' \cap qMq \subset qMq\), hence \(v^*v \in qMq\). Thus \(w_1^*w_1 = \theta_{t_0}(\beta(v^*)v^*v(\beta(v^*))) = \theta_{t_0}(\beta(vv^*)) \neq 0\).

By iterating this process, we obtain \(w = w_{n-1} \neq 0\) such that \(wa = \theta_{\pi/2}(a)w\), i.e., \(w_{n+1}(a) = \pi_2(a)w\) for all \(a \in P\). This means that \(P \nabla M A\). As in [Ho07, Claim 5.3]), this is incompatible with our assumption \(P \nabla M A\). So it follows that \(P \nabla M A\) and the lemma is proved.

Proof of Lemma 4.4

Proof. Let \(P \subset qMq\) and assume that \(\theta_t(P)\) is amenable relative to \(A\) in \(M\) for all \(t \in (0, \frac{\pi}{2})\). As in the proof of [Io12, Theorem 5.1] (and [Val13, Theorem 3.4]), we let \(I\) be the set of all quadruples \((X, Y, \delta, t)\) where \(X \subset M\) and \(Y \subset \mathcal{U}(P)\) are finite subsets, \(\delta \in (0, 1)\) and \(t \in (0, \frac{\pi}{2})\). Then \(I\) is a directed set when equipped with the ordering \((X, Y, \delta, t) \leq (X', Y', \delta', t')\) if and only if \(X \subset X', Y \subset Y', \delta' \leq \delta\) and \(t' \leq t\).
By [OP07, Theorem 2.1], we can for each $i = (X, Y, \delta, t) \in I$ choose a vector $\xi_i \in \theta_t(q)L^2(M) \otimes_A L^2(M)\theta_t(q)$ such that $\|\xi_i\|_2 \leq 1$ and

$$\|x\xi_i, \xi_i - x\theta_t(q)\| \leq \delta \quad \text{for every } x \in X \text{ or } x = (\theta_t(y) - y)^*(\theta_t(y) - y) \text{ with } y \in Y,$$

$$\|\theta_t(y)\xi_i - \xi_i\theta_t(y)\|_2 \leq \delta \quad \text{for every } y \in Y.$$

We now prove that $qMqL^2(qMq)_p$ is weakly contained in $qMq(qL^2(M) \otimes_A L^2(M)q)_p$. For this, it suffices to show that

$$\lim_i\langle x\xi_i, \xi_i \rangle = \tau(x) \quad \text{for every } x \in qMq,$$

$$\lim_i \|y\xi_i - \xi_i y\|_2 = 0 \quad \text{for every } y \in P.$$  \hspace{1cm} (4.1)

Let $y \in \mathcal{U}(P)$ and $\varepsilon > 0$ be given. Choose $t > 0$ small enough so that $\|\theta_t(y) - y\|_2^2 \leq \varepsilon/6$. We have

$$\|y\xi_i - \xi_i y\|_2 \leq \|(y - \theta_t(y))\xi_i\|_2 + \|\theta_t(y)\xi_i - \xi_i\theta_t(y)\|_2 + \|\xi_i(\theta_t(y) - y)\|_2$$

for any $i \in I$. Moreover,

$$\|(y - \theta_t(y))\xi_i\|_2^2 = \|((\theta_t(y) - y)^*(\theta_t(y) - y))\xi_i\| \leq \|((\theta_t(y) - y)^*\theta_t(q))\|_2^2 + \varepsilon/6 \leq \varepsilon/3,$$

for $i \geq (\{0\}, \{y\}, \varepsilon/6, t)$ in $I$. Similarly, we get that $\|\xi_i(\theta_t(y) - y)\|_2 \leq \varepsilon/3$. Thus, we conclude that $\|y\xi_i - \xi_i y\|_2 \leq \varepsilon$ for $i \geq (\{0\}, \{y\}, \varepsilon/6, t)$ and so the second assertion of (4.1) holds true. The first assertion is proved similarly, using that $\|\theta_t(q) - q\|_2 \to 0$ as $t \to 0$.

By Proposition 3.4, we have $M = M_1 \ast_A M_2$. Under our identification $M = M_1$, we then get that $\mathcal{M}_1 L^2(M)_A \cong \mathcal{M}_1 L^2(M \otimes_A K)_A$, where $\mathcal{K}_A$ is the $A$-bimodule defined as the direct sum of $L^2(A)$ and all alternating tensor products $L^2(M \otimes A) \otimes_A L^2(M_1 \otimes A) \otimes_A \cdots$ starting with $L^2(M_2 \otimes A)$. We conclude that $qMqL^2(qMq)_p$ is weakly contained in $qMq(qL^2(M) \otimes_A (K \otimes_A L^2(M)q))_p$. It then follows from [PV11, Proposition 2.4] that $P$ is amenable relative to $A$ inside $M$. This finishes the proof of Lemma 4.4.

\section{Maximal amenability}

Fix a tracial von Neumann algebra $(A, \tau)$ and a symmetric Hilbert $A$-bimodule $AHa$ with symmetry $J : H \to H$. Denote by $M = \Gamma(H, J, A, \tau)'$ the associated von Neumann algebra with faithful normal tracial state $\tau$. We prove the following maximal amenability property by combining Popa’s asymptotic orthogonality [Po83] with the method of [BH16]. In the special case of free Bogoljubov crossed products (see Remark 3.5), part 3 of Theorem 5.1 was proved in [Ho12b, Theorem D].

\textbf{Theorem 5.1.} Assume that $AHa$ is weakly mixing. Then the following properties hold.

1. $\mathcal{Z}(M) = \{a \in \mathcal{Z}(A) \mid a\xi = \xi a \text{ for all } \xi \in H\}$.

2. If $B \subset M$ is a von Neumann subalgebra that is amenable relative to $A$ inside $M$ and if the bimodule $B \cap AHa$ is left weakly mixing, then $B \subset A$.

3. A von Neumann subalgebra of $M$ that properly contains $A$ is not amenable relative to $A$ inside $M$. If the $A$-bimodule $AHa$ is faithful\footnote{A $P$-$Q$-bimodule $PHQ$ is called faithful if the $*$-homomorphisms $P \to B(H)$ and $Q^{op} \to B(H)$ are faithful.}, then $M$ has no amenable direct summand. If $A$ is amenable, then $A \subset M$ is a maximal amenable subalgebra.
Proof. As above, identify

\[ L^2(M) = L^2(A) \oplus \bigoplus_{n \geq 1} (H \otimes_A \cdots \otimes_A H) \]

and denote by \( \mathcal{H} \subset H \) the subspace of vectors that are both left and right bounded.

1. Since \( AH_A \) is weakly mixing, it follows from Proposition 2.3 that the \( n \)-fold tensor products \( H \otimes_A \cdots \otimes_A H \) (with \( n \geq 1 \)) have no \( A \)-central vectors. Therefore, \( A' \cap M = \mathcal{Z}(A) \). Looking at the commutator of \( a \in \mathcal{Z}(A) \) and \( \ell(\xi) + \ell(J\xi)^* \), the conclusion follows.

2. Since \( B \) is amenable relative to \( A \) inside \( M \), we can fix a \( B \)-central state \( \omega \in \langle M, e_A \rangle^* \) such that \( \omega |_M = \tau \).

Claim I. For every \( \xi \in \mathcal{H} \) and every \( \varepsilon > 0 \), there exists a projection \( p \in A \) such that \( \tau(1-p) < \varepsilon \) and such that

\[ \omega(\ell(\xi)p\ell(\xi)^*) < \varepsilon . \]

To prove this claim, fix \( \xi \in \mathcal{H} \) and \( \varepsilon > 0 \). Define \( a = \sqrt{\langle \xi, \xi \rangle} \) and denote by \( q \in A \) the support projection of \( a \). Take a projection \( q_1 \in qAq \) that commutes with \( a \), such that \( \tau(q - q_1) < \varepsilon/2 \) and such that \( aq_1 \) is invertible in \( q_1Aq_1 \). Denote by \( b \in q_1Aq_1 \) this inverse and define \( \eta = \xi b \). By construction, \( \ell(\eta)^*\ell(\eta) = q_1 \) and \( \xi_1 = \eta a \).

Pick a positive integer \( N \) such that \( 2^{-N} < \varepsilon/(2\|a\|^2) \). Put \( \kappa = 2^N \). Then pick \( \delta > 0 \) such that \( \delta < \varepsilon/(\kappa^2\|a\|^2) \). We start by constructing unitary operators \( v_1, \ldots, v_\kappa \in U(A \cap B) \) and a projection \( q_2 \in q_1Aq_1 \) such that \( \tau(q_1 - q_2) < \varepsilon/2 \) and such that the vectors \( \eta_i = v_i\eta \) satisfy

\[ \|q_2\langle \eta_i, \eta_j \rangle q_2\| < \delta \quad \text{whenever } i \neq j \quad (5.1) \]

(\text{and where we indeed use the operator norm at the left hand side of (5.1)}).

We put \( c_0 = q_1 \) and \( v_1 = 1 \). Denoting by \( (a_i) \) the net of unitaries in \( B \cap A \) witnessing the left weak mixing of \( B \cap A H_A \), we get that \( \lim_i \| \langle \eta, a_i\eta \rangle \|_2 = 0 \). So we find a net of projections \( r_i \in q_1Aq_1 \) such that \( \tau(q_1 - r_i) \to 0 \) and

\[ \|r_i \langle \eta, a_i\eta \rangle r_i\| < \delta \quad \text{for every } i. \]

Take \( i \) large enough such that \( \tau(q_1 - r_i) < \varepsilon/4 \) and define \( e_1 := r_i \) and \( v_2 := a_i \). We have now constructed \( v_1, v_2 \). Inductively, we double the length of the sequence, until we arrive at \( v_1, \ldots, v_{2^n} \).

After \( k \) steps, we have constructed the projections \( e_1 \geq \cdots \geq e_k \) and unitaries \( v_1, \ldots, v_{2^k} \) in \( U(B \cap A) \) such that \( \tau(e_{j-1} - e_j) < 2^{-j-1} \varepsilon \) and such that the vectors \( \eta_i = v_i\eta \) satisfy

\[ \|e_k \langle \eta_i, \eta_j \rangle e_k\| < \delta \quad \text{whenever } i \neq j . \]

As in the first step, we can pick a unitary \( a \in U(B \cap A) \) and a projection \( e_{k+1} \in e_k A e_k \) such that \( \tau(e_k - e_{k+1}) < 2^{-k-2} \varepsilon \) and such that

\[ \|e_{k+1} \langle \eta_i, a\eta_j \rangle e_{k+1}\| < \delta \]

for all \( i, j \in \{1, \ldots, 2^k\} \). It now suffices to put \( v_{2^k+1} = av_i \) for all \( i = 1, \ldots, 2^k \). We have doubled our sequence. We continue for \( N \) steps and put \( q_2 = e_N \). So, (5.1) is proved.

Put \( \mu_i = \eta_i q_2 = v_i \eta q_2 \). Define the projections \( P_i = \ell(\mu_i)\ell(\mu_i)^* \) and note that \( P_i = v_i P_i v_i^* \). By construction, \( \|P_i P_j\| < \delta \) whenever \( i \neq j \). Writing \( P = \sum_{i=1}^\kappa P_i \) it follows that \( \|P^2 - P\| < \kappa^2 \delta \).

Since \( P \) is a positive operator, we conclude that \( \|P\| < 1 + \kappa^2 \delta \). Since \( \omega \) is \( B \)-central and \( v_i \in B \) for all \( i \), we get that

\[ \kappa \omega(P_1) = \sum_{i=1}^\kappa \omega(P_1) = \omega(P) \leq \|P\| \leq 1 + \kappa^2 \delta . \]
Therefore, \( \omega(P_1) < \kappa^{-1} + \kappa \delta < \|a\|^{-2} \varepsilon \).

Since \( q_1 \) and \( a \) commute, the right support of \( (q_1 - q_2)a \) is a projection of the form \( q_1 - p_0 \) where \( p_0 \in q_1 A q_1 \) is a projection with \( \tau(q_1 - p_0) \leq \tau(q_1 - q_2) < \varepsilon / 2 \). By construction, \( q_1 a p_0 = q_2 a p_0 \). Since \( p_0 \leq q_1 \) and \( \eta = \eta q_1 \), it follows that

\[
\xi p_0 = \xi q_1 p_0 = \eta a p_0 = \eta q_1 a p_0 = \eta q_2 a p_0.
\]

Define the projection \( p \in A \) given by \( p = (1 - q) + p_0 \). Since \( \xi (1 - q) = 0 \), we still have \( \xi p = \eta q_2 a p_0 \). Because \( 1 - p = (q - q_1) + (q_1 - p_0) \), we get that \( \tau(1 - p) < \varepsilon \). Finally,

\[
\omega(\ell(\xi p) \ell(\xi p)^*) = \omega(\ell(\eta q_2) a p_0 a^* \ell(\eta q_2)^*) < \|a\|^2 \omega(\ell(\eta q_2) \ell(\eta q_2)^*) = \|a\|^2 \omega(P_1) < \varepsilon.
\]

So, we have proven Claim I.

**Claim II.** For every \( \xi \in \mathcal{H} \) and every \( \varepsilon > 0 \), there exists a projection \( p \in A \) such that \( \tau(1 - p) < \varepsilon \) and such that \( \omega(\ell(\xi p) \ell(\xi p)^*) = 0 \).

For every integer \( k \geq 1 \), Claim I gives a projection \( p_k \in A \) with \( \tau(1 - p_k) < 2^{-k \varepsilon} \) and \( \omega(\ell(\xi p_k) \ell(\xi p_k)^*) < 1 / k \). Defining \( p = \bigwedge_k p_k \), we get that \( \tau(1 - p) < \varepsilon \) and, for every \( k \geq 1 \),

\[
\omega(\ell(\xi p) \ell(\xi p)^*) = \omega(\ell(\xi p_k) \ell(\xi p_k)^*) \leq \omega(\ell(\xi p_k) \ell(\xi p_k)^*) = \omega(\ell(\xi p_k) \ell(\xi p_k)^*) < 1 / k.
\]

So, \( \omega(\ell(\xi p) \ell(\xi p)^*) = 0 \) and Claim II is proved.

We can now conclude the proof of 2. Denote by \( E_A : M \rightarrow A \) and \( E_B : M \rightarrow B \) the unique trace preserving conditional expectations. It is sufficient to prove that \( E_B \circ E_A = E_B \). So we have to prove that \( E_B(x) = 0 \) for all \( x \in M \ominus A \). Using the Wick products defined in (3.2), it suffices to prove that \( E_B(W(\xi_1, \ldots, \xi_k)) = 0 \) for all \( k \geq 1 \) and all \( \xi_1, \ldots, \xi_k \in \mathcal{H} \).

Since \( \omega \) is \( B \)-central and \( \omega|_M = \tau \), there is a unique conditional expectation \( \Phi : (M, e_A) \rightarrow B \) such that \( \Phi|_M = E_B \) and \( \omega = \tau \circ \Phi \).

We first consider \( k \geq 2 \) and \( \xi_1, \ldots, \xi_k \in \mathcal{H} \). By Claim II, we can take sequences of projections \( p_n, q_n \in A \) such that \( p_n \rightarrow 1 \) and \( q_n \rightarrow 1 \) strongly and

\[
\Phi(\ell(\xi_1 p_n) \ell(\xi_1 p_n)^*) = 0 = \Phi(\ell((J \xi_1) q_n) \ell((J \xi_1) q_n)^*)
\]

for all \( n \). Then also \( \Phi(\ell(\xi_1 p_n) T) = 0 = \Phi(T \ell((J \xi_1) q_n)^*) \) for all \( n \) and all \( T \in (M, e_A) \). We conclude that

\[
E_B(W(\xi_1 p_n, \xi_2, \ldots, \xi_{k-1}, q_n \xi_k)) = \Phi(W(\xi_1 p_n, \xi_2, \ldots, \xi_{k-1}, q_n \xi_k)) = 0
\]

for all \( n \). Since \( E_B \) is normal, it follows that \( E_B(W(\xi_1, \ldots, \xi_k)) = 0 \).

We next consider the case \( k = 1 \). So it remains to prove that \( E_B(\ell(\xi) + \ell(J \xi)^*) = 0 \) for all \( \xi \in \mathcal{H} \). For this, it is sufficient to prove that \( \Phi(\ell(\xi)) = 0 \) for all \( \xi \in \mathcal{H} \). By Claim II and reasoning as above, we find a sequence of projections \( p_n \in A \) such that \( p_n \rightarrow 1 \) strongly and \( \Phi(\ell(\xi p_n) T) = 0 \) for all \( n \) and all \( T \in (M, e_A) \). In particular, we can take \( T = 1 \) and get that \( \Phi(\ell(\xi) p_n) = 0 \) for all \( n \). Write \( e_n = 1 - p_n \). Then,

\[
\Phi(\ell(\xi)^* \Phi(\ell(\xi)) = \Phi(\ell(\xi) e_n)^* \Phi(\ell(\xi) e_n) \leq \|\ell(\xi)\|^2 \Phi(e_n) = \|\ell(\xi)\|^2 E_B(e_n)
\]

Since \( E_B(e_n) \rightarrow 0 \) strongly, we conclude that \( \Phi(\ell(\xi)) = 0 \). This concludes the proof of 2.

3. It follows from 2 that a von Neumann subalgebra of \( M \) properly containing \( A \) is not amenable relative to \( A \) and thus, not amenable itself. Whenever \( H \neq \{0\} \), we have \( A \neq M \) and we conclude that \( M \) is not amenable. By 1, any direct summand of \( M \) is given as the von Neumann algebra associated with the symmetric weakly mixing \( A_z \)-bimodule \( H_z \) where \( z \in Z(A) \) is a nonzero central projection satisfying \( \xi z = z \xi \) for all \( \xi \in H \). If \( A H_A \) is faithful, we have \( H z \neq \{0\} \) and it follows that this direct summand is not amenable. The final statement is an immediate consequence of 2. \( \square \)
6 Absence of Cartan subalgebras

In this section, we give a complete description of the structure of the von Neumann algebra $M = \Gamma(H, J, A, \tau)^\prime\prime$ associated with an arbitrary symmetric $A$-bimodule $(H, J)$. We describe the trivial direct summands of $M$ and then prove that the remaining direct summand never has a Cartan subalgebra and describe its center (see Theorem 6.1). In all interesting cases, there are no trivial direct summands and this allows us to prove absence of Cartan subalgebras whenever $H$ is a weakly mixing $A$-bimodule (Corollary 6.2), when $A$ is a II$_1$ factor and $H$ is not the trivial bimodule nor the bimodule given by a period 2 automorphism of $A$ (Corollary 6.3), and finally for arbitrary free Bogoljubov crossed products (Corollary 6.4). This last result improves [Ho12b, Corollary C].

To prove our general structure theorem, we need the following terminology. Fix a tracial von Neumann algebra $(A, \tau)$. We say that an $A$-bimodule $H$ is given by a partial automorphism if one of the following two equivalent conditions holds.

- The commutant of the right $A$ action on $H$ equals the left $A$ action, and vice versa.
- There exists a projection $e \in B(\ell^2(\mathbb{N})) \overline{\otimes} A$, a central projection $z \in Z(A)$ and a surjective *-isomorphism $\alpha : Az \to e(B(\ell^2(\mathbb{N})) \overline{\otimes} A)$ such that $eH A \cong e(\ell^2(\mathbb{N}) \otimes L^2(A))$ with the bimodule structure given by $a \cdot \xi \cdot b = \alpha(a)\xi b$.

Fix a symmetric $A$-bimodule $(H, J)$ and denote $M = \Gamma(H, J, A, \tau)^\prime\prime$. Then, $M$ has two trivial direct summands. First denote by $z_0 \in Z(A)$ the largest projection such that $z_0H = \{0\}$. Then, $z_0 \in Z(M)$ and $Mz_0 = Az_0$. Next, there is a largest projection $z_1 \in Z(A)(1 - z_0)$ such that $z_1H = Hz_1$ and such that the $A$-bimodule $Hz_1$ is given by a partial automorphism of $A$ (see Lemma 6.6 for details). Again $z_1 \in Z(M)$ and $Mz_1$ can be computed by the methods of Example 3.6. In a way, $Mz_1$ is not very interesting, since it is always a direct sum of a corner of $A$ and a corner of $A \overline{\otimes} L^\infty([0, 1])$ or of an index 2 extension of this.

Writing $z_2 = 1 - (z_0 + z_1)$, we thus get that

$$M = Az_0 \oplus \Gamma(Hz_1, J, A_{z_1}, \tau)^\prime\prime \oplus \Gamma(Hz_2, J, A_{z_2}, \tau)^\prime\prime$$

and only the third direct summand is “interesting and nontrivial”. By Lemma 6.6, the symmetric $A_{z_2}$-bimodule $H_{z_2}$ is completely nontrivial in the following sense: the left action of $A_{z_2}$ on $H$ is faithful and there are no nonzero projections $e, f \in Z(A)_{z_2}$ such that $eH = Hf$ and such that $eH$ is given by a partial automorphism of $A_{z_2}$. So it suffices to describe the structure of the von Neumann algebra associated with an arbitrary completely nontrivial symmetric $A$-bimodule.

We denote by $\dim_{-A}(K)$ the right $A$-dimension of a right Hilbert $A$-module $K$. Recall that the value of $\dim_{-A}(K)$ depends on the choice of the trace $\tau$. We similarly define $\dim_{A-}(K)$ for a left Hilbert $A$-module $K$. As in (6.10), for every $A$-bimodule $H$, there is a unique element $\Delta_H^\ell$ in the extended positive part of $Z(A)$ characterized by $\tau(\Delta_H^\ell e) = \dim_{-A}(eH)$ for every projection $e \in Z(A)$.

**Theorem 6.1.** Let $(A, \tau)$ be a tracial von Neumann algebra and $(H, J)$ a completely nontrivial symmetric $A$-bimodule. Write $M = \Gamma(H, J, A, \tau)^\prime\prime$. There is a canonical central projection $q \in Z(M)$ (which, most of the time, is zero) such that the following holds.

(a) No direct summand of $M(1 - q)$ is amenable relative to $A(1 - q)$.

(b) No direct summand of $M(1 - q)$ admits a Cartan subalgebra.

(c) $Mq = Aq$ and the support of $E_A(1 - q)$ equals 1.
(d) Defining $C := \{a \in Z(A) \mid a\xi = \xi a \text{ for all } \xi \in H\}$, we get that $Z(M) = Z(A)q + C(1-q)$.

Moreover, we have that $E_A(q) = Z(\Delta^t_H)$, where $Z : (0, +\infty) \to \mathbb{R}$ is the positive function given by $Z(t) = 1 - t$ when $t \in (0, 1)$ and $Z(t) = 0$ when $t \geq 1$.

**Corollary 6.2.** Let $(A, \tau)$ be a tracial von Neumann algebra and $(H, J)$ a symmetric $A$-bimodule. Put $M = \Gamma(H, J, A, \tau)^\prime\prime$. If $A\mathcal{H}_A$ is weakly mixing and faithful, then no direct summand of $M$ has a Cartan subalgebra and $Z(M) = \{a \in Z(A) \mid a\xi = \xi a \text{ for all } \xi \in H\}$.

**Proof.** Let $z \in Z(A)$ be a nonzero central projection. Since $zH \neq \{0\}$ and $zH$ is still left weakly mixing as an $A$-bimodule, we have that $\dim_{-A}(zH) = +\infty$ and that $zH$ is not given by a partial automorphism of $A$. So the conclusions follow from Theorem 6.1.

When $A$ is a II$_1$ factor, the results of Theorem 6.1 can be formulated more easily as follows.

**Corollary 6.3.** Let $A$ be a II$_1$ factor with its unique tracial state $\tau$ and let $(H, J)$ be a symmetric $A$-bimodule. Denote $M = \Gamma(A, \tau, H, J)^\prime\prime$. Unless $H$ is zero or $H$ is the trivial $A$-bimodule or $H$ is the symmetric $A$-bimodule associated with a period 2 outer automorphism of $A$, the following holds: $M$ is a factor, $M$ is not amenable relative to $A$ and $M$ has no Cartan subalgebra.

**Proof.** Since $A$ is a II$_1$ factor, the only symmetric $A$-bimodules given by a partial automorphism of $A$ are the trivial $A$-bimodule and the $A$-bimodule given by $a \in \text{Aut}(A)$ with $\alpha \circ \alpha$ being inner. When a symmetric $A$-bimodule $H$ is not given by a partial automorphism of $A$, we have that $\dim_{-A}(H) > 1$. So, the conclusion follows from Theorem 6.1.

We finally deduce that free Bogoljubov crossed products never have a Cartan subalgebra. In [Ho12b, Corollary C], this was proven under extra assumptions on the underlying orthogonal representation.

**Corollary 6.4.** Let $G$ be an arbitrary countable group and $\pi : G \to O(K_K)$ an orthogonal representation of $G$ with $\dim(K_K) \geq 2$. Denote by $\sigma_* : G \curvearrowright \Gamma(K_K)'' \cong L(F_{\dim(K_K)})$ the associated free Bogoljubov action with crossed product $M := \Gamma(K_K)'' \rtimes \sigma_* G$ (see Remark 3.5). Then no direct summand of $M$ has a Cartan subalgebra. Also, $M$ is a factor if and only if $\pi(g) \neq 1$ for every $g \in G \setminus \{e\}$ that has a finite conjugacy class.

**Proof.** Write $A = L(G)$ with its canonical tracial state $\tau$. By Remark 3.5, we can view $M = \Gamma(H, J, A, \tau)^\prime\prime$ where the symmetric $A$-bimodule $(H, J)$ is given by (3.3). Denote by $K$ the complexification of $K_K$. Observe that $H \cong \ell^2(G) \otimes K$ with bimodule structure $a \cdot \xi \cdot b = \alpha(a)\xi b$, where $\alpha : L(G) \to L(G) \otimes B(K)$ is given by $\alpha(u_g) = u_g \otimes \pi(g)$ for all $g \in G$. Since $(\tau \otimes \text{id})\alpha(a) = \tau(a)1$ for all $a \in L(G)$, it follows that $\Delta^t_H = \dim(K_K)1$.

The left and right actions of $A$ on $H$ are faithful. Since $H \otimes_A \overline{H}$ can be identified with the bimodule associated with the representation $\pi \otimes \pi$, the center valued dimension of $H \otimes_A \overline{H}$ as a left $A$-module equals $\dim(K_K)^21$. It follows from Lemma 6.5 below that $H$ is completely nontrivial. So, all conclusions follow from Theorem 6.1.

We now prove Theorem 6.1, using several lemmas that we prove at the end of this section.

**Proof of Theorem 6.1.** Let $K \subset H$ be the maximal left weakly mixing $A$-subbimodule of $H$, i.e. the orthogonal complement of the span of all $A$-submodules of $H$ having finite right $A$-dimension. Denote by $z_0 \in Z(A)$ the support of the left $A$ action on $K$. In the first part of the proof, assuming $z_0 \neq 0$, we show that

$$(1) \quad Z(M)z_0 \subset Z(A)z_0,$$
(2) every $M$-central state $\omega$ on $(M, e_A)$ that is normal on $M$ satisfies $\omega(z_0) = 0$.

Note that $K \subset z_0 H$. Denote by $K \subset K$ the dense subspace of vectors that are both left and right bounded. Define the von Neumann subalgebra $N \subset z_0 M z_0$ given by

$$N := (A z_0 \cup \{W(\xi, J(\mu)) \mid \xi, \mu \in K\})'' ,$$

(6.1)

where we used the notation of (3.2). Then, the linear span of $A z_0$ and elements of the form $W(\xi_1, J(\mu_1), \ldots, \xi_k, J(\mu_k))$, $k \geq 1$, $\xi_i, \mu_i \in K$, is a dense *-subalgebra of $N$.

Whenever $K_1, \ldots, K_n \subset H$ are $A$-submodules, we denote by concatenation $K_1 \cdots K_n$ the $A$-submodule of $L^2(M)$ given by

$$K_1 \cdots K_n := K_1 \otimes_A \cdots \otimes_A K_n \subset H \otimes_A \cdots \otimes_A H \subset L^2(M) .$$

In the same way, we write powers of $A$-submodules and when $K_i \subset H^{k_i}$ are $A$-submodules, then $K_1 \cdots K_n \subset H^{k_1 + \cdots + k_n}$ is a well defined $A$-submodule.

Using this notation, note that $L^2(N)$ is the direct sum of $L^2(A z_0)$ and the spaces $L_n := (K J(K))^n$, $n \geq 1$. Since $K$ is a left weakly mixing $A$-bimodule, it follows that $N \cap (A z_0)' = Z(A) z_0$.

We claim that

(3) $N \not\subset A z_0$, meaning that the $N$-$A$-bimodule $L^2(N)$ is left weakly mixing.

Since $N \cap (A z_0)' = Z(A) z_0$, to prove this claim, it suffices to show that $\dim_{-A}(L^2(N)e) = +\infty$ for every nonzero projection $e \in Z(A) z_0$. Since the left action of $A z_0$ on $K$ is faithful and $K$ is left weakly mixing, we get that $\dim_{-A}(K J(K)e) = +\infty$. So certainly $\dim_{-A}(L^2(N)e) = +\infty$ and the claim follows.

Proof of (1). Define the $A$-submodule $R \subset L^2(M)$ given as

$$R := (H \oplus (K + J(K))) \oplus \bigoplus_{n=0}^\infty (H \otimes K) H^n (H \otimes J(K)) .$$

Since $K$ is left weakly mixing and $J(K)$ is right weakly mixing, all $A$-central vectors in $L^2(M)$ belong to $L^2(A) + R$. Next note that left, resp. right multiplication by elements of $N$ induces an $N$-bimodular unitary operator

$$L^2(N) \otimes_A R \otimes_A L^2(N) \to \overline{N R N} \subset L^2(z_0 M z_0) .$$

Since the $N$-$A$-bimodule $L^2(N)$ is left weakly mixing, it follows that $\overline{N R N}$ has no nonzero $N$-central vectors. Every element $x \in Z(M) z_0$ defines a vector in $L^2(z_0 M z_0)$ that is both $A$-central and $N$-central. By $A$-centrality, we conclude that $x \in A z_0 + z_0 R z_0$. In particular, $x \in L^2(N) + \overline{N R N}$. Since $x$ is $N$-central and $\overline{N R N}$ has no nonzero $N$-central vectors, we get that $x \in L^2(N)$ and thus, $x \in Z(A) z_0$.

Proof of (2). Denote $L_{\text{even}} := L^2(N)$ and define $L_{\text{odd}}$ as the direct sum of the $A$-bimodules $(K J(K))^n K$, $n \geq 0$. Note that both $L_{\text{even}}$ and $L_{\text{odd}}$ are $A$-$N$-bimodules. The same argument as in the proof of Theorem 5.1, using the left weak mixing of $K$, shows that the von Neumann algebras $B(L_{\text{even}}) \cap (A^{op})'$ and $B(L_{\text{odd}}) \cap (A^{op})'$ admit no $N$-central states that are normal on $N$. Note that we have the following decomposition of $L^2(z_0 M)$ as an $N$-$A$-bimodule:

$$L^2(\zeta_0 M) = \left( L_{\text{even}} \otimes_A \left( L^2(A) \oplus \bigoplus_{n \geq 0} (H \otimes K) H^n \right) \right) \oplus \left( L_{\text{odd}} \otimes_A \left( L^2(A) \oplus \bigoplus_{n \geq 0} (H \otimes J(K)) H^n \right) \right) .$$
This decomposition induces *-homomorphisms from $B(L_{\text{even}}) \cap (A^{\text{op}})'$ and $B(L_{\text{odd}}) \cap (A^{\text{op}})'$ to $B(z_0 L^2(M)) \cap (A^{\text{op}})' = z_0(M, e_A)z_0$. So, $z_0(M, e_A)z_0$ admits no $N$-central state that is normal on $N$. A fortiori, (2) holds.

Next we define the projection $z_1 \in Z(A)(1 - z_0)$ given by

$$z_1 = 1_{(1, +\infty)}(\Delta^e_{(1-z_0)H}). \quad (6.2)$$

We also write $z = z_0 + z_1$ and $z_2 = 1 - z$.

Denote by $e' \in Z(A)z_1$ the maximal projection with the following properties: the right support $f \in Z(A)$ of $e'H$ satisfies $e'H = zHf$ and the $A$-bimodule $e'H$ is given by a partial automorphism of $A$. Define $e = z_1 - e'$.

By the definition of $z_0$, we get that the $A$-bimodule $(1 - z_0)H$ is a sum of $A$-bimodules that are finitely generated as a right Hilbert $A$-module. It then follows from the definition of $z_1$ that we can choose a projection $e_1 \in Z(A)z_1$ that lies arbitrarily close to $z_1$ and for which there exists an $A$-subbimodule $L_1 \subset z_1 H$ with the following properties:

- the left support of $L_1$ equals $e_1$,
- $L_1$ is finitely generated as a right Hilbert $A$-module,
- $\Delta^e_{L_1}$ is bounded and satisfies $\Delta^e_{L_1} \geq \delta_1 e_1$ for some real number $\delta_1 > 1$.

Denote by $e_2$ the left support of $e_1(H \ominus L_1)$. Making $e_1$ slightly smaller, but still arbitrarily close to $z_1$, we may assume that $e_2$ is the left support of an $A$-subbimodule $L_2 \subset e_1(H \ominus L_1)$ with the following properties: $L_2$ is finitely generated as a right Hilbert $A$-module and $\Delta^e_{L_2}$ is bounded. By construction, $e_2 \leq e_1$. Since $e_2 L_1$ and $L_2$ are orthogonal and have the same left support $e_2$, it follows that for nonzero projections $s \in Z(A)e_2$, the $A$-bimodule $sH$ is not given by a partial automorphism of $A$. This means that $e_2 \leq e$ and thus, $e_2 \leq e e_1$. Define $L = L_1 + L_2$. Using notation $(6.12)$, it follows from Lemma 6.5 that the left support of $e_2 L J(L)e_2 \cap (t_{e_2}LA)^\perp$ equals $e_2$. A fortiori, the left support of $e_2 L H z \cap (t_{e_2}LA)^\perp$ equals $e_2$.

We put $e_3 = ee_1 - e_2$. Since $e_2$ is the left support of $e_1(H \ominus L_1)$, we get that $e_3 H = e_3 L_1 = e_3 L$. Since $e_3 \leq e$, applying Lemma 6.5 to the $A$-bimodule $zH$, we conclude that the left support of $e_3 L H z \cap (t_{e_3}HA)^\perp$ equals $e_3$. Summarizing, $L$ has the following properties:

- the left support of $L$ equals $e_1$,
- $L$ is finitely generated as a right Hilbert $A$-module,
- $\Delta^e_L$ is bounded and satisfies $\Delta^e_L \geq \delta e_1$ for some real number $\delta > 1$,
- the left support of $L H z \cap (t_L A)^\perp$ equals $e_1$.

Denote by $s \in Z(A)$ the left support of $L H (z_0 + e_1) \cap (t_LA)^\perp$. Since $e_1$ could be chosen arbitrarily close to $z_1$, it follows that $s$ lies arbitrarily close to $e$.

We next prove that

(4) $\ Z(M)s \subset Z(A)s,$

(5) every $M$-central state $\omega$ on $\langle M, e_A \rangle$ that is normal on $M$ satisfies $\omega(s) = 0$.

Write $\Delta := \Delta^e_L$, choose a Pimsner-Popa basis $(\xi_i)_{i=1}^n$ for the right Hilbert $A$-module $L$ and put

$$t := t_L = \sum_{i=1}^n \xi_i \otimes_A J(\xi_i).$$
Since $\Delta$ is bounded, the vectors $\xi_i \in H$ are both left and right bounded.

Denoting by $P_t$ the orthogonal projection onto a Hilbert subspace $T$, the main properties of $t$, used throughout the proof, are:

$$
(t, t)_A = A(t, t) = \Delta , \quad \ell(t)^* t = J(P_L(t)) \quad \text{and} \quad r(t)^* t = P_L(J(t)) ,
$$

for all left and right bounded vectors $\xi \in \mathcal{H}$.

Since the vectors $\xi_i$ are both left and right bounded, we can define the self-adjoint element $S_1 \in e_1 Me_1$ given by

$$
S_1 := \sum_{i=1}^n W(\xi_i, J(\xi_i)) .
$$

By Lemma 6.8, the von Neumann algebra $D := \{S_1\}'$ is a subalgebra of $e_1 Me_1 \cap (Ae_1)'$ that is diffuse relative to $Ae_1$. We fix a unitary $u \in \mathcal{U}(D)$ satisfying $E_{Ae_1}(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$.

Defining

$$
S_k := \sum_{i_1, \ldots, i_k=1}^n W(\xi_{i_1}, J(\xi_{i_1}), \ldots, \xi_{i_k}, J(\xi_{i_k})) ,
$$

and denoting by $\Omega \in L^2(M)$ the vacuum vector, we get that

$$
t_k := S_k \Omega = t \bigotimes_A \cdots \bigotimes_A t . \quad \text{(6.3)}
$$

With the convention that $S_0 = e_1$, the elements $S_k, k \geq 0$ span a dense $*$-subalgebra of $D$ and are orthogonal in $L^2(D)$.

Proof of (4). We start by proving that an element $x \in \mathcal{Z}(M)e_1$ must be of a special form.

Define the von Neumann subalgebra $E \subset e_1 Me_1$ given by $E := Ae_1 \vee D$. Define $T_0 \subset H^2$ as the closure of $tA$. Note that $\ell(t)\ell(t)^* \Delta^{-1}$ is the orthogonal projection of $H^2$ onto $T_0$. Then define $T_2 := H^2 \ominus T_0$ and $T_3 := H^3 \ominus (T_0H + HT_0)$. Observe that $L^2(e_1 Me_1 \ominus E)$ is spanned by the $D$-subbimodules

$$
DHH \quad DT_2D , \quad DT_3D \quad DT_2H^nT_2D \quad \text{with} \quad n \geq 0 . \quad \text{(6.4)}
$$

Each of the $D$-bimodules in (6.4) is contained in a multiple of the coarse $D$-bimodule $L^2(D) \otimes L^2(D)$. This is only nontrivial for the first one $DHH$. Fix a left and right bounded vector $\mu \in H$ with $\|\mu\| \leq 1$. Using the notation $t_k$ introduced in (6.3), one checks that

$$
S_k W(\mu) \Omega = t_k \otimes_A \mu + t_{k-1} \otimes_A P_L(\mu) \quad \text{and} \quad W(\mu) S_k \Omega = \mu \otimes_A t_k + P_{J(\mu)}(\mu) \otimes_A t_{k-1} .
$$

When $\mu, \eta \in H$ are left and right bounded vectors, we have $\langle t_k \otimes_A \mu, \eta \otimes_A t_l \rangle = 0$ if $k \neq l$, while

$$
\langle t_k \otimes_A \mu, \eta \otimes_A t_k \rangle = \langle \ell(\eta)^* (t_k \otimes_A \mu), t_k \rangle
= \langle J(P_L(\eta)) \otimes_A t_{k-1} \otimes_A \mu, t_k \rangle
= \langle J(P_L(\eta)) \otimes_A t_{k-1}, r(\mu)^* t_k \rangle = \langle J(P_L(\eta)) \otimes_A t_{k-1}, t_{k-1} \otimes P_L(J(\mu)) \rangle .
$$

We can continue inductively and find complex numbers $\alpha_k, \beta_k, \gamma_k$ with modulus at most 1, depending on the vector $\mu$ that we keep fixed, such that

$$
\langle S_k W(\mu) S_l, W(\mu) \rangle = \begin{cases} 
\alpha_k & \text{if } k = l \text{ and } k \geq 0, \\
\beta_k & \text{if } k = l + 1 \text{ and } l \geq 0, \\
\gamma_k & \text{if } k = l - 1 \text{ and } l \geq 1.
\end{cases}
$$
We next claim that
\[ \xi := \sum_{k=0}^{\infty} \left( \alpha_k (\Delta^{-k} S_k \otimes \Delta^{-k} S_k) + \beta_k (\Delta^{-k-1} S_{k+1} \otimes \Delta^{-k} S_k) + \gamma_k (\Delta^{-k} S_k \otimes \Delta^{-k-1} S_{k+1}) \right) \]
is a well defined element in \( L^2(E) \otimes L^2(E) \). This follows because \( E_A(S_k^2) = \langle t_k, t_k \rangle_A = \Delta^k \) and thus
\[ \| \Delta^{-k} S_k \|_2^2 = \tau(\Delta^{-2k} S_k^2) = \tau(\Delta^{-k}) \leq \delta^{-k} , \]
where \( \delta > 1 \). By construction,
\[ \langle S_k W(\mu) S_l, W(\mu) \rangle = \tau(e_1)^{-2} (\tau \otimes \tau)((S_k \otimes S_l) \xi) . \]
So, the \( D \)-bimodule \( D_A D \) is contained in the coarse \( D \)-bimodule \( L^2(E) \otimes L^2(E) \).

We have thus proved that all \( D \)-bimodules in (6.4) are contained in a multiple of the coarse \( D \)-bimodule. Since \( D \) is diffuse, it follows that \( e_1 M e_1 \cap D' \subset E \). In particular, \( E(M)e_1 \subset E \).

We are now ready to prove (4). Fix \( x \in Z(M) \). We have to prove that \( xs \in A \). Because of (1) and the previous paragraphs, we can uniquely decompose \( x(z_0 + e_1) \) as the \( \| \cdot \|_2 \)-convergent sum
\[ x(z_0 + e_1) = a_0 + \sum_{k=1}^{\infty} S_k a_k \]  
with \( a_0 \in A(z_0 + e_1) \) and \( a_k \in A e_1 \) for all \( k \geq 1 \). Note that \( a_0 = E_A(x)(z_0 + e_1) \) and \( a_k = \Delta^{-k} E_A(S_k x) \) for all \( k \geq 1 \).

Let now \( \eta \in LH(z_0 + e_1) \cap (tA)^{\perp} \) be an arbitrary left and right bounded vector. Note that
\[ \eta = \sum_{i=1}^{n} \xi_i \otimes_A J(\eta_i) \]  
where the vectors \( \eta_i \in (z_0 + e_1)H \) are both left and right bounded. Define
\[ W(\eta) := \sum_{i=1}^{n} W(\xi_i, J(\eta_i)) \]
and note that \( W(\eta) \in sM(z_0 + e_1) \subset e_1 M(z_0 + e_1) \).

Using that \( W(\eta) \) commutes with \( x \) and using the decomposition of \( x(z_0 + e_1) \) in (6.5), we find that
\[ W(\eta)x\Omega = W(\eta)(z_0 + e_1)x\Omega = W(\eta)a_0 \Omega + \sum_{k=1}^{\infty} W(\eta)S_k a_k \Omega \]
\[ = \eta(a_0 + a_1) + \sum_{k=1}^{\infty} \eta \otimes_A t_k(a_k + a_{k+1}) , \]
\[ xW(\eta)\Omega = xe_1 W(\eta)\Omega = a_0 e_1 W(\eta)\Omega + \sum_{k=1}^{\infty} a_k S_k W(\eta)\Omega \]
\[ = (a_0 + a_1)\eta + \sum_{k=1}^{\infty} (a_k + a_{k+1}) t_k \otimes_A \eta . \]
In this last expression for \( xW(\eta)\Omega \), all terms except \((a_0 + a_1)\eta \) are orthogonal to \( W(\eta)x\Omega \). We conclude that \((a_k + a_{k+1}) t_k \otimes_A \eta = 0 \) for all \( k \geq 1 \) and for all choices of \( \eta \). Since the left support
of \( LH(z_0 + e_1) \cap (tA)^{-1} \) equals \( s \), it follows that \( (a_k + a_{k+1})s = 0 \) for all \( k \geq 1 \). This means that \( a_k s = (-1)^{k-1}a_1 s \) for all \( k \geq 1 \).

Since,
\[
+\infty > \|x\|_2^2 \geq \sum_{k=1}^{\infty} \|S_k a_k s\|_2^2 = \sum_{k=1}^{\infty} \tau(s a_k^* a_k s) = \sum_{k=1}^{\infty} \delta_k \|a_1 s\|_2^2 ,
\]
it follows that \( a_1 s = 0 \). So, \( a_k s = 0 \) for all \( k \geq 1 \). From (6.5), it follows that \( xs \in A \), so that (4) is proved.

Proof of (5). Fix an \( M \)-central state \( \omega \) on \( (M, e_A) \) that is normal on \( M \). We have to prove that \( \omega(s) = 0 \). Recall that we defined \( T_0 \subset H^2 \) as the closure of \( tA \). Consider the following orthogonal decomposition of \( e_1 L^2(M) \) as an \( A \)-bimodule:
\[
e_1 L^2(M) = V_0 \oplus V_1 \oplus V_2 \quad \text{where} \quad V_0 := \bigoplus_{n=0}^{\infty} T_0 H^n ,
\]
\[
V_1 := L^2(A e_1) \oplus \bigoplus_{n=0}^{\infty} (e_1 H \ominus L) H^n , \quad V_2 := L \oplus \bigoplus_{n=0}^{\infty} (L H \ominus T_0) H^n .
\]

Denote by \( Q_i \in e_1 (M, e_A) e_1 \) the projections onto \( V_i \), for \( i = 0, 1, 2 \). So, \( e_1 = Q_0 + Q_1 + Q_2 \). Also note that the projections \( Q_i \) commute with \( A \). We prove below that \( \omega(sQ_0) = \omega(Q_1) = \omega(Q_2) = 0 \). Once these statements are proved, (5) follows.

To prove that \( \omega(Q_1) = 0 \), note that for all \( \mu \in V_1 \) and all \( k \geq 1 \), we have that \( S_k \mu = t_k \otimes_A \mu \) and thus, \( S_k \mu \) is orthogonal to \( V_1 \). So, for all \( \mu, \mu' \in V_1 \) and \( d \in D \), we get that
\[
\langle d \mu, \mu' \rangle = \tau(e_1)^{-1} \tau(d) \langle \mu, \mu' \rangle .
\]

Above we introduced the unitary element \( u \in U(D) \) satisfying \( \tau(u^k) = 0 \) for all \( k \in \mathbb{Z} \setminus \{0\} \).

It follows that the subspaces \( u^k V_1 \) are all orthogonal. So, the projections \( u^k Q_1 u^{-k} \) are all orthogonal. By \( M \)-centrality, \( \omega \) takes the same value on each of these projections. So, \( \omega(Q_1) = 0 \).

To prove that \( \omega(Q_2) = 0 \), we argue similarly. For all \( \mu \in V_2 \) and all \( k \geq 2 \), we have that \( S_k \mu = t_k \otimes_A \mu + t_{k-1} \otimes_A \mu \) and thus, \( S_k \mu \) is orthogonal to \( V_2 \). On the other hand, \( S_1 \mu = t \otimes_A \mu + \mu \) and here, only \( t \otimes_A \mu \) is orthogonal to \( V_2 \). It follows that for all \( \mu, \mu' \in V_2 \) and \( d \in D \),
\[
\langle d \mu, \mu' \rangle = \gamma(d) \langle \mu, \mu' \rangle ,
\]
where \( \gamma : D \to C \) is the normal state given by \( \gamma(e_1) = \gamma(S_1) = 1 \) and \( \gamma(S_k) = 0 \) for all \( k \geq 2 \).

Note that \( \gamma \) can be defined as well as the vector state on \( D \) implemented by any choice of unit vector in \( V_2 \). Since \( D \) is diffuse, we can choose a unitary \( v \in U(D) \) such that \( \gamma(v^k) = 0 \) for all \( k \in \mathbb{Z} \setminus \{0\} \). It follows that the subspaces \( v^k V_2 \) are all orthogonal. As in the previous paragraph, we get that \( \omega(Q_2) = 0 \).

It remains to prove that \( \omega(s Q_0) = 0 \). Fix \( \eta \in LH(z_0 + e_1) \ominus T_0 \) as in (6.6) and define
\[
\eta' = \sum_{i=1}^{n} \eta_i \otimes_A J(\xi_i) .
\]

Note that \( \eta' \in (z_0 + e_1) H J(L) \ominus T_0 \). From (2), we already know that \( \omega(z_0) = 0 \). Since \( e_1 \eta' \in V_1 + V_2 \), we also know that \( \omega(\ell(e_1 \eta') \ell(e_1 \eta')^*) \). Both together imply that \( \omega(\ell(\eta') \ell(\eta')^*) \). For all \( n \geq 0 \) and \( \mu \in H^n \), we have that
\[
W(\eta)(\eta' \otimes_A t \otimes_A \mu) = \eta \otimes_A \eta' \otimes_A t \otimes_A \mu + \sum_{i=1}^{n} \ell(\xi_i) \ell(\eta_i)^* (\eta' \otimes_A t \otimes_A \mu) + \langle \eta', \eta' \rangle_A (t \otimes_A \mu) .
\]
Since
\[ \ell(t)^* \sum_{i=1}^{n} \ell(\xi_i)^* \ell(\eta_i) \eta' = \sum_{i=1}^{n} \ell(J(\xi_i))^* \ell(\eta_i)^* \eta' = \ell(\eta')^* \eta' = \langle \eta', \eta' \rangle_A \]
and since the projection \( Q_0 \) is given by \( Q_0 = \Delta^{-1} \ell(t) \ell(t)^* \), we get that
\[ Q_0 W(\eta)(\eta' \otimes_A t \otimes_A \mu) = \langle \eta', \eta' \rangle_A \Delta^{-1} (t \otimes_A t \otimes_A \mu) + \langle \eta', \eta' \rangle_A (t \otimes_A \mu) \]
for all \( n \geq 0 \) and all \( \mu \in H^n \). This means that
\[ Q_0 W(\eta) \ell(\eta' \otimes_A t) = \langle \eta', \eta' \rangle_A \left( \Delta^{-1} (t \otimes_A t + \ell(t)) = \ell(t) \langle \eta', \eta' \rangle_A (1 + \Delta^{-1} \ell(t)) \right). \]
Because
\[ \| \Delta^{-1} \ell(t) \|^2 = \| \Delta^{-2} \ell(t)^* \ell(t) \| = \| \Delta^{-1} \| \leq \delta^{-1} < 1, \]
the operator \( R := 1 + \Delta^{-1} \ell(t) \) is invertible. Also note that there exists a \( \kappa > 0 \) such that
\[ \ell(\eta' \otimes_A t) \ell(\eta' \otimes_A t)^* \leq \kappa \ell(\eta')^* \ell(\eta'). \]
So, we find \( \varepsilon > 0 \) and \( \kappa > 0 \) such that
\[ \varepsilon \ell(t) \left( \langle \eta', \eta' \rangle_A \right)^2 \ell(t)^* \leq \ell(t) \langle \eta', \eta' \rangle_A RR^* \langle \eta', \eta' \rangle_A \ell(t)^* \]
\[ = Q_0 W(\eta) \ell(\eta' \otimes_A t) \ell(\eta' \otimes_A t)^* W(\eta)^* Q_0 \]
\[ \leq \kappa Q_0 W(\eta) \ell(\eta')^* W(\eta)^* Q_0. \quad (6.7) \]
We already proved that \( \omega(\ell(\eta')^* \ell(\eta')) = 0 \). Since \( \omega \) is \( M \)-central, also
\[ \omega(W(\eta) \ell(\eta')^* W(\eta)^*) = 0. \]
Because \( e_1 = Q_0 + Q_1 + Q_2 \) and \( \omega(Q_1) = \omega(Q_2) = 0 \), the Cauchy-Schwarz inequality implies that \( \omega(Y) = \omega(Q_0 Y) = \omega(Y Q_0) \) for all \( Y \in e_1 \langle M, e_A \rangle e_1 \). Therefore,
\[ \omega(Q_0 W(\eta) \ell(\eta')^* W(\eta)^* Q_0) = \omega(W(\eta) \ell(\eta')^* W(\eta)^*) = 0. \]
It then follows from (6.7) that
\[ \omega((\langle \eta', \eta' \rangle_A)^2 \Delta Q_0) = 0 \]
for all bounded vectors \( \eta' \in (z_0 + e_1)H J(L) \otimes T_0 \). By the Cauchy-Schwarz inequality and the normality of \( \omega \) restricted to \( M \), we get that \( \omega(a_i Q_0) \to \omega(a Q_0) \) whenever \( a_i \in A \) is a bounded sequence such that \( \| a_i - a \|_2 \to 0 \). Since the right support of the \( A \)-bimodule \( (z_0 + e_1)H J(L) \otimes T_0 \) equals \( s \), it follows that \( \omega(s Q_0) = 0 \). Since we already proved that \( \omega(Q_1) = \omega(Q_2) = 0 \), it follows that (5) holds.

Since \( s \) lies arbitrarily close to \( e \), it follows from (1)-(2) and (4)-(5) that
\[ (6) \quad \mathcal{Z}(M)(z_0 + e) \subset \mathcal{Z}(A)(z_0 + e), \]
\[ (7) \quad \text{every } M \text{-central state } \omega \text{ on } \langle M, e_A \rangle \text{ that is normal on } M \text{ satisfies } \omega(z_0 + e) = 0. \]
Recall that \( z = z_0 + z_1 \) and \( z_2 = 1 - (z_0 + z_1) \). Note that \( \Delta^\ell_{z_2 H} \leq z_2 \). We claim that \( z_2 H z_2 = \{0\} \). Denote by \( e_0 \in \mathcal{Z}(A)_{z_2} \) the left support of \( z_2 H z_2 \). Note that by symmetry, \( e_0 \) also is the right support of \( z_2 H z_2 \). By Lemma 6.7, we get that \( \Delta^\ell_{e_0 H e_0} = e_0 \) and that \( e_0 H e_0 \) is given by a partial automorphism of \( A \). Since
\[ \Delta^\ell_{e_0 H} = \Delta^\ell_{e_0 H e_0} + \Delta^\ell_{e_0 H (1-e_0)} = e_0 + \Delta^\ell_{e_0 H (1-e_0)} \]
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and since $\Delta_{e_0 A}^{e_0 A} \leq e_0$, we get that $e_0 A (1 - e_0) = \{0\}$. We conclude that $e_0 A = H e_0 = e_0 A e_0$ and that this $A$-bimodule is given by a partial automorphism of $A$. Since $H$ is assumed to be completely nontrivial, we get that $e_0 = 0$ and the claim is proved.

Recall that $e \in Z(A)_{z_1}$ was defined as $e = z_1 - e'$ where $e' \in Z(A)_{z_1}$ has the following properties: denoting by $f \in Z(A)$ the right support of $e'H$, we have that $e'H = z H f$ and that the $A$-bimodule $e'H$ is given by a partial automorphism of $A$. We claim that $f \leq z$. To prove this claim, denote $f_1 := f z_2$. If $f_1 \neq 0$, we find a nonzero projection $e'' \in Z(A) e'$ such that $e'' H = z H f_1$ and such that this $A$-bimodule is given by a partial automorphism of $A$. Above, we have proved that $z_2 H z_2 = \{0\}$. A fortiori, $z_2 H f_1 = \{0\}$, meaning that $H f_1 = z H f_1$. But then, $e'' H = H f_1$, contradicting the complete nontriviality of $H$. So, we have proved that $f \leq z$.

We next claim that $f \leq z_0 + e$. To prove this claim, assume that $f' := f e'$ is nonzero. Then, $f' H = f e' H = f z H f \subset H z$ because $f \leq z$. Applying the symmetry $J$, it follows that $H f' = z H f'$ and thus $e'' H = H f'$ for some nonzero projection $e'' \in Z(A) e'$, again contradicting the complete nontriviality of $H$. So, we have proved that $f \leq z_0 + e$.

Since $e'H$ is given by a partial automorphism of $A$, we can take projections $e'' \in Z(A) e'$ arbitrarily close to $e'$ such that $e'' H$ is finitely generated as a right Hilbert $A$-module and $\Delta_{e'' H}^{e'' H}$ is bounded. Denote by $f' \in Z(A) f$ the right support of $e'' H$ and denote by $\alpha : Z(A) e'' \to Z(A) f'$ the corresponding surjective $*$-isomorphism satisfying $a \xi = \xi \alpha (a)$ for all $a \in Z(A) e''$. Let $(\gamma_i)_{i=1}^n$ be a Pimsner-Popa basis of the right $A$-module $e'' H$ and define

$$R_i = \ell (\gamma_i) + \ell (J(\gamma_i))^* \quad \text{and} \quad R = \sum_{i=1}^n R_i R_i^* = \Delta_{e'' H}^{e'' H} + \sum_{i=1}^n W(\gamma_i, J(\gamma_i)).$$

Note that $R_i \in e'' M f'$ and $R \in e'' M e''$. Since $\Delta_{e'' H}^{e'' H} = e'' \Delta_{H}^{e''} e''$, it follows from Lemma 6.8 that the support projection of $R$ equals $e''$.

Let $x \in Z(M)$ and using (6), take $a \in Z(A) (z_0 + e)$ such that $(z_0 + e) x = a$. Since $f' \leq z_0 + e$, we have $f' x = a f'$ and thus

$$x R = \sum_{i=1}^n R_i x R_i^* = \sum_{i=1}^n R_i a f' R_i^* = \alpha^{-1} (a f') R.$$

Since the support projection of $R$ equals $e''$, we have proved that $Z(M) e'' \subset Z(A) e''$. Since $e''$ lies arbitrarily close to $e'$, together with (6), it follows that

$$(8) \quad Z(M) z \subset Z(A) z.$$

A similar reasoning using (7) then implies that

$$(9) \quad \text{every } M\text{-central state } \omega \text{ on } (M, e_A) \text{ that is normal on } M \text{ satisfies } \omega(z) = 0.$$

To prove the first two statements of the theorem, it remains to see what happens under the projection $z_2$.

Denote $\Delta_2 := \Delta_{z_2 H}^{z_2 H}$. By the definition of $z_2$, we have that $\Delta_2 \leq z_2$. Let $(\mu_i)_{i \in I}$ be a (possibly infinite) Pimsner-Popa basis for the right $A$-module $z_2 H$. Since $\Delta_2$ is bounded, we may choose the vectors $\mu_i$ to be left and right bounded. For the same reason,

$$s := \sum_{i \in I} \mu_i \otimes_A J(\mu_i)$$
is a well defined bounded $A$-central vector in $z_2H H z_2$ and the infinite sums

$$G_n = \sum_{i_1, \ldots, i_n} W(\mu_{i_1}, J(\mu_{i_1}), \ldots, \mu_{i_n}, J(\mu_{i_n}))$$

are well defined bounded operators in $z_2 M z_2 \cap (A z_2)'$ satisfying

$$G_n \Omega = s_n := s \otimes_A \cdots \otimes_A s \cdot \text{n times} \ .$$

By convention, we put $G_0 = z_2$. From the definition of $G_n$, we obtain the recurrence relation

$$G_1 G_n = G_{n+1} + G_n + \Delta_2 G_{n-1} \quad (6.8)$$

for all $n \geq 1$, and thus, $G_{n+1} = (G_1 - 1)G_n - \Delta_2 G_{n-1}$ for all $n \geq 1$.

Denote by $q \in z_2 M z_2$ the projection onto the kernel of $G_1 + \Delta_2$. Although the sum defining $G_1$ is infinite, the computations in the proof of Lemma 6.8 remain valid and it follows that the kernel of $(G_1 + \Delta_2) 1_{\{1\}}(\Delta_2)$ is reduced to zero. So, $q \leq 1_{\{0,1\}}(\Delta_2)$.

With the convention that $s_0 = z_2 \Omega$, we claim that

$$q \Omega = \sum_{k=0}^{\infty} (-1)^k(z_2 - \Delta_2)s_k = \sum_{k=0}^{\infty} (-1)^k s_k(z_2 - \Delta_2) \ . \quad (6.9)$$

Because

$$\sum_{k=0}^{\infty} ||(z_2 - \Delta_2)s_k||_2^2 = \sum_{k=0}^{\infty} \tau(\langle s_k, s_k \rangle_A (z_2 - \Delta_2)^2) = \sum_{k=0}^{\infty} \tau(\Delta_2^k(z_2 - \Delta_2)^2) = \tau(z_2 - \Delta_2) < \infty ,$$

the right hand side of (6.9) is a well defined element $p \in L^2(z_2 M z_2)$ satisfying, with $\| \cdot \|_2$-convergence,

$$p = \sum_{k=0}^{\infty} (-1)^k(z_2 - \Delta_2) G_k \ .$$

Note that $p = p^*$. Using the recurrence relation (6.8), it follows that $(G_1 + \Delta_2)p = 0$ and thus $p = qp$. Taking the adjoint, also $p = pq$.

On the other hand, because $(G_1 + \Delta_2)q = 0$, we have $G_1q = -\Delta_2q$. Using the recurrence relation (6.8), it follows that $G_kq = (-1)^k \Delta_2^kq$ for all $k \geq 0$. It then follows that

$$pq = \sum_{k=0}^{\infty} (z_2 - \Delta_2) \Delta_2^k q = 1_{\{0,1\}}(\Delta_2) q = q \ .$$

We already proved that $pq = p$, so that $p = q$ and (6.9) is proved.

From (6.9), we get for all $\xi \in H$ that

$$(\ell(\xi) + \ell(J(\xi))^*) q \Omega = (\ell(\xi z_2) + \ell(J(\xi z_2))^*) q \Omega = 0 \ .$$

So, for all $x \in M$, we have that $x q = E_A(x) q$. Taking the adjoint, also $qx = q E_A(x)$ for all $x \in M$. Since $q$ commutes with $A$, it follows that $q \in Z(M)$ and $Mq = Aq$. From (6.9), we also get that $E_A(q) = z_2 - \Delta_2$ and thus $E_A(q) = Z(\Delta_2^q)$ where $Z : (0, +\infty) \to \mathbb{R}$ is defined as in the formulation of the theorem. So, $E_A(1 - q) = z + \Delta_2$ and this operator has support equal to 1. Statement (c) of the theorem is now proven.

We next prove that
(10) \( \mathcal{Z}(M)(z_2 - q) \subset \mathcal{Z}(A)(z_2 - q) \).

Take \( x \in \mathcal{Z}(M) \) and write

\[
xz_2\Omega = \sum_{n=0}^{\infty} \zeta_n \quad \text{with} \quad \zeta_n \in z_2H^n.
\]

Using (8), take \( a \in \mathcal{Z}(A)z \) such that \( xz = a \). Also write \( a_0 = E_A(xz_2) \) and note that \( \zeta_0 = a_0\Omega \). Since \( z_2H_2z_2 = 0 \), we have \( z_2H = z_2Hz \) and we get, for every \( \xi \in \mathcal{H} \), that

\[
\sum_{n=0}^{\infty} (\ell(\xi)^* + \ell(J(\xi))) \zeta_n = (\ell(\xi)^* + \ell(J(\xi))) xz_2\Omega
\]

\[
= x(\ell(\xi)^* + \ell(J(\xi))) z_2\Omega = xJ(z_2\xi) = xzJ(z_2\xi) = aJ(z_2\xi).
\]

Comparing the components in \( H^n \) for all \( n \geq 0 \), we find that

\[
\ell(\xi)^{n} \zeta_1 = 0, \quad \ell(\xi)^{n} \zeta_2 = aJ(\xi) - J(\xi)a_0, \quad \ell(\xi)^{n} \zeta_{n+1} = -J(\xi) \otimes A \zeta_{n-1}
\]

for all \( \xi \in z_2\mathcal{H} \) and all \( n \geq 2 \). Since \( \zeta_n \in z_2H^n \) for all \( n \), it first follows that \( \zeta_1 = 0 \) and then inductively, that \( \zeta_n = 0 \) for all odd \( n \).

Next, we get that \( \zeta_2 = sa - sa_0 \), where

\[
sa := \sum_{i \in I} \mu_i \otimes A aJ(\mu_i)
\]

is a well defined \( A \)-central vector in \( z_2H^2z_2 \).

Before continuing the proof, we give another expression for \( sa \). For all \( \mu, \mu' \in z_2\mathcal{H} = z_2Hz \), we have that \( W(J(\mu), \mu') \in zMz \). Since \( xz = a \) and \( x \in \mathcal{Z}(M) \), it follows that \( a \) commutes with \( W(J(\mu), \mu') \). This means that

\[
aJ(\mu) \otimes A \mu' = J(\mu) \otimes A \mu' a \quad \text{for all} \quad \mu, \mu' \in z_2\mathcal{H}.
\]

It follows that \( aJ(\mu) \otimes A sa = J(\mu) \otimes A sa \) for all \( \mu \in z_2\mathcal{H} \). Defining the normal completely positive map \( \varphi : Az \to A_{z_2} \) given by

\[
\varphi(b) = \sum_{i \in I} (J(\mu_i), bJ(\mu_i))_A \quad \text{for all} \quad b \in Az,
\]

we get that \( \varphi(a) s = \Delta_2 s_a \). Since \( \varphi(z) = \Delta_2 \), there is a unique normal completely positive map \( \psi : Az \to A_{z_2} \) such that \( \psi(b)\Delta_2 = \varphi(b) \) for all \( b \in Az \). We conclude that \( sa = \psi(a) s = s\psi(a) \).

Writing \( a_1 = \psi(a) - a_0 \), we get that \( \zeta_2 = sa_1 \). We then conclude that \( \zeta_{2n} = (-1)^{n+1} s_a a_1 \) for all \( n \geq 1 \). Define the spectral projection \( r = 1_{(1)}(\Delta_2) \). Since

\[
\langle \zeta_{2n}, \zeta_{2n} \rangle_A = a_1^* \langle s_n, s_n \rangle_A a_1 = a_1^* \Delta_2^n a_1,
\]

we get that \( \|\zeta_{2n}r\| = \|a_1r\| \) for all \( n \). Since \( \sum_n \|\zeta_{2n}r\|^2 < \infty \), we conclude that \( a_1r = 0 \) and thus \( xr \in A \).

Using (6.9), it follows that \( x(z_2 - \Delta_2) = qa_1 + a_2 \) for some element \( a_2 \in A \). Since \( xr \in A \), it follows that \( x(z_2 - q) \in A(z_2 - q) \). Since the support of \( E_A(z_2 - q) \) equals \( z_2 \), it follows that (10) holds.

Using (8) and (10), to conclude the proof of statement (d), it suffices to prove that for any \( a \in \mathcal{Z}(A) \), we have \( a(1 - q) \in \mathcal{Z}(M) \) if and only if \( a \in C \), where \( C \) is defined in the formulation.
of the theorem. This follows immediately by expressing the commutation with \( \ell(\xi) + \ell(J(\xi))^* \) for all \( \xi \in \mathcal{H} \) and using that \( (\ell(\xi) + \ell(J(\xi))^*)^* q = 0 \), as shown above.

Let \( \omega \) be an \( M \)-central state on \((M, e_A)\) that is normal on \( M \). To conclude the proof of statement (a), we have to show that \( \omega(1 - q) = 0 \). By (9), we already know that \( \omega(z) = 0 \). With \( \mu_i \in \mathcal{H} = \mathcal{H}^* \) as above, define \( y_i := \ell(\mu_i) + \ell(J(\mu_i))^* \). Note that \( y_i \in \mathcal{H}^* \) and that \( G_1 + \Delta_2 = \sum_i y_i y_i^* \). By \( M \)-centrality and normality of \( \omega \) on \( M \), and because \( y_i^* y_i \in \mathcal{H} \), we get that \( \omega(G_1 + \Delta_2) = 0 \). So, \( \omega(z_2 - q) = 0 \). Since we already know that \( \omega(z) = 0 \), we conclude that \( \omega(1 - q) = 0 \).

It remains to prove statement (b). Assume that \( s \in \mathcal{Z}(M)(1 - q) \) is a nonzero projection and that \( B \subset M \) is a Cartan subalgebra. Since \( \mathcal{N}_{M_{\mathcal{H}}}(B)'' = M \), a combination of statement (a) and Theorem 4.1 implies that \( B \prec_M A(1 - q) \). The \( A \)-subbimodule \( z_2H = z_2H \mathcal{H} \) of \( L^2(M) \) has finite right \( A \)-dimension equal to \( \tau(\Delta_2) \) and realizes a full intertwining of \( A(z_2 - q) \) into \( A_2 \). It then follows that \( B \prec_M A_2 \).

By [Po03, Theorem 2.1], we can take projections \( q_1 \in B, p \in A_2 \), a faithful normal unital \(*\)-homomorphism \( \theta : Bq_1 \to pAp \) and a nonzero partial isometry \( v \in q_1Mp \) such that \( bv = v\theta(b) \) for all \( b \in Bq_1 \). Since \( B \subset M \) is maximal abelian, we may assume that \( vv^* = q_1 \). By [Io11, Lemma 1.5], we may assume that \( B_0 := \theta(Bq_1) \) is a maximal abelian subalgebra of \( pAp \). Write \( q_2 = v^* v \) and note that \( q_2 \in B_0' \cap pMp \). We may assume that the support projection of \( E_A(q_2) \) equals \( p \).

Since \( z = z_0 + z_1 \), at least one of the projections \( p_{z_0}, p_{z_1} \) is nonzero. Since we can cut down everything with the projections \( z_0 \) and \( z_1 \), we may assume that either \( p \leq z_0 \) or \( p \leq z_1 \).

**Proof in the case where \( p \leq z_0 \).** Recall that we denoted by \( K \subset H \) the largest \( A \)-submodule that is left weakly mixing and that \( z_0 \) is the left support of \( K \). First assume that the \( B_0 \)-bimodule \( pK \) is left weakly mixing. Define the orthogonal decomposition of the \( pAp \)-bimodule \( pL^2(M)p \) given by

\[
pL^2(M)p = U_1 \oplus U_2 \quad \text{with} \quad U_1 = \bigoplus_{n=0}^{\infty} pKH^n p \quad \text{and} \quad U_2 = L^2(pAp) \oplus \bigoplus_{n=0}^{\infty} p(H \ominus K)H^n p.
\]

We claim that \( v^* \mathcal{N}_{q_1Mq_1}(Bq_1)v \subset U_2 \). To prove this claim, take \( u \in \mathcal{N}_{q_1Mq_1}(Bq_1) \) and write \( uv^* bu = \alpha(b) \) for all \( b \in Bq_1 \). Put \( x = v^* uv \) and denote by \( y \) the orthogonal projection of \( x \) onto \( U_1 \). Since \( U_1 \) is a \( pAp \)-subbimodule of \( pL^2(M)p \), we get that \( y \) is a right \( pAp \)-bounded vector in \( U_1 \) and that \( \theta(b)y = y\theta(b) \) for all \( b \in Bq_1 \). Since the \( B_0 \)-bimodule \( pK \) is left weakly mixing, also \( U_1 \) is left weakly mixing as a \( B_0 \)-\( pAp \)-bimodule. So, we can take a sequence of unitaries \( b_n \in U(Bq_1) \) such that \( \lim_{n \to \infty} \| \theta(b_n)y, y \|_{pAp} \|_2 = 0 \). But,

\[
\theta(b_n)y, y \|_{pAp} = \| y\theta(\alpha(b_n)), y \|_{pAp} = \theta(\alpha(b_n)) \cdot \| y, y \|_{pAp}.
\]

Since \( \theta(\alpha(b_n)) \) is a unitary in \( B_0 \), we have \( \| \theta(\alpha(b_n)) \cdot \| y, y \|_{pAp} \|_2 = \| y, y \|_{pAp} \|_2 \) for all \( n \). We conclude that \( y = 0 \) and thus \( v^* uv \subset U_2 \). Since the linear span of \( \mathcal{N}_{q_1Mq_1}(Bq_1) \) is \( \| \cdot \|_2 \)-dense in \( q_1Mq_1 \), we get that \( q_2Mq_2 \subset U_2 \).

Again consider the von Neumann subalgebra \( N \subset z_0Mz_0 \) introduced in (6.1). Since

\[
P_{pL^2(N)p}(U_2) \subset L^2(pAp),
\]

we get that \( E_{pNq_2}(q_2Mq_2) \subset pAp \). Denote by \( N_0 \subset pAp \) the von Neumann algebra generated by the subspace \( E_{pNq_2}(q_2Mq_2) \). So, \( N_0 \subset pAp \). In particular, \( E_N(q_2) \subset A \), so that \( E_N(q_2) = E_A(q_2) \) and thus, \( E_N(q_2) \) has support \( p \). By [Io11, Lemma 1.6], the inclusion \( N_0 \subset pAp \) is essentially of finite index in the sense of Definition 6.9. A fortiori, \( pAp \subset pAp \) is essentially of finite index. This contradicts the left weak mixing of the \( N \)-\( A \)-bimodule \( L^2(N) \) that we obtained in (3).
Next assume that the $B_0$-$A$-bimodule $pK$ is not left weakly mixing and take a nonzero $B_0$-$A$-subbimodule $K_1 \subset pK$ that is finitely generated as a right Hilbert $A$-module. Denote by $z_0^* \in Z(B_0)$ the support projection of the left action of $B_0$ on $K_1$. Since $K_1 \neq \{0\}$, also $z_0^* \neq 0$. Since the support of $E_A(q_2)z_0^*$ equals $p$, we get that $E_A(q_2 z_0^*) = E_A(q_2)z_0^* \neq 0$. So, $q_2 z_0^* \neq 0$ and we can cut down everything by $z_0^*$ and assume that the left $B_0$ action on $K_1$ is faithful.

Put $P = N_{pAp}(B_0)''$. Whenever $u \in N_{q_1 M_{q_1}}(B_0)$ with $u b^* = \alpha(b)$ for all $b \in B_0$, we have $E_A(v^* w) \theta(b) = \theta(\alpha(b)) E_A(v^* w)$ for all $b \in B_0$. Since $B_0 \subset pAp$ is maximal abelian, it follows that $E_A(v^* w) \in P$. So $E_A(q_2 M_{q_2}) \subset P$. From [Io11, Lemma 1.6], we conclude that the inclusion $P \subset pAp$ is essentially of finite index in the sense of Definition 6.9. So, all conditions of Lemma 6.10 are satisfied and we can choose a diffuse abelian von Neumann subalgebra $D \subset B_0' \cap pMp$ that is in tensor product position w.r.t. $B_0$. Since $B_0 \subset q_1 M q_1$ is maximal abelian, also $B_0 q_2 \subset q_2 M_{q_2}$ is maximal abelian. So, $q_2(B_0' \cap pMp) q_2 = B_0 q_2$, contradicting Lemma 6.11 below.

**Proof in the case where** $p \leq z_1$. As proven above, we can find projections $e_1 \in Z(A) z_1$ that lie arbitrarily close to $z_1$ and for which there exists an $A$-subbimodule $L \subset z_1 H$ with the following properties: the left support of $L$ equals $e_1$, $L$ is finitely generated as a right Hilbert $A$-module, $\Delta^L_L$ is bounded and $\Delta^L_L \geq e_1$. Taking $e_1$ close enough to $z_1$ and cutting down with $e_1$, we may assume that $p \leq e_1$. By Lemma 6.8, we can choose a diffuse abelian von Neumann subalgebra $D \subset (Ae_1)' \cap e_1 M e_1$ that is in tensor product position w.r.t. $Ae_1$. Then $D_p \subset B_0' \cap pMp$ and $D_p$ is in tensor product position w.r.t. $B_0$. Since $D_p$ is diffuse abelian and $q_2 \in B_0' \cap pMp$ is a projection satisfying $q_2(B_0' \cap pMp) q_2 = B_0 q_2$, this again contradicts Lemma 6.11.

In the proof of Theorem 6.1, we needed several technical lemmas that we prove now.

Let $(A, \tau)$ be a tracial von Neumann algebra and denote by $\overline{Z}(A)$ the extended positive part of $Z(A)$, i.e. when we identify $Z(A) = L^\infty(X, \mu)$, then $\overline{Z}(A)$ consists of all measurable functions $f : X \to [0, +\infty]$ up to identification of functions that are equal almost everywhere.

Whenever $(B, \tau)$ and $(A, \tau)$ are tracial von Neumann algebras and $H$ is a $B$-$A$-bimodule, we denote by $\Delta_H^e \in \overline{Z}(B)$ the unique element in the extended positive part of $Z(B)$ characterized by

$$\tau(\Delta_H^e e) = \dim_{-A}(eH) \quad \text{for all projections} \ e \in Z(B). \quad (6.10)$$

Writing $H \approx p(L^2(N) \otimes L^2(A))$ with the bimodule action given by $b \cdot \xi \cdot a = \alpha(b) \xi a$ where $\alpha : B \to p(B(L^2(N)) \otimes A)p$ is a normal $*$-homomorphism, we get that $\tau(\Delta_H^e \cdot) = (\text{Tr} \otimes \tau) \alpha(\cdot)$ and this also allows to construct $\Delta_H^e$.

Recall that a **finitely generated** right Hilbert $A$-module $K$ admits a *Pimsner-Popa basis*, i.e. right bounded elements $\xi_1, \ldots, \xi_n$ such that

$$\xi = \sum_{i=1}^{n} \xi_i \langle \xi_i, \xi \rangle_A \quad (6.11)$$

for all right bounded elements $\xi \in K$. We denote by $t_K \in K \otimes_A \overline{K}$ the associated vector given by

$$t_K := \sum_{i=1}^{n} \xi_i \otimes_A \overline{\xi_i}. \quad (6.12)$$

When $K$ is an $A$-bimodule, then $t_K$ is an $A$-central vector and $\langle t_K, t_K \rangle_A = \Delta_K^e$. Recall from the beginning of this section the notion of an $A$-bimodule given by a partial automorphism of $A$. Given an $A$-bimodule $L$, denote by $\text{zdim}_{-A}(L)$, resp. $\text{zdim}_{A-}(L)$, the
center valued dimension of $L$ as a right, resp. left $A$-module. These are elements in the extended positive part of $Z(A)$. We have that $L$ is finitely generated as a right Hilbert $A$-module if and only if $\text{zdim}_{-A}(L)$ is bounded.

**Lemma 6.5.** Let $(A, \tau)$ be a tracial von Neumann algebra and $T$ an $A$-bimodule with left support $e$. Denote $\Sigma := \text{zdim}_{-A}(T \otimes_A \overline{T})$. Then, the support of $\Sigma$ equals $e$ and $\Sigma \geq e$. Defining $e_1 = 1_{(1)}(\Sigma)$, the following holds.

1. Denoting by $f_1 \in Z(A)$ the right support of $e_1 T$, we have that $e_1 T = T f_1$ and that the $A$-bimodule $e_1 T$ is given by a partial automorphism of $A$.

2. When $e_2 \in Z(A)e$ and $f_2 \in Z(A)$ are projections such that $e_2 T = T f_2$ and such that the $A$-bimodule $e_2 T$ is given by a partial automorphism of $A$, then $e_2 \leq e_1$.

3. If $e_0 \in Z(A)e$ is a projection such that $e_0 T$ is finitely generated as a right Hilbert $A$-module, then the left support of $e_0 T \otimes_A \overline{T} \cap (e_0 T A)^+$ equals $e_0(1 - e_1)$.

**Proof.** Choose a set $I$, a projection $p \in B(\ell^2(I)) \otimes A$ and a normal unital $*$-homomorphism $\alpha : A \to p(B(\ell^2(I)) \otimes A)p$ such that $T \cong p(B(\ell^2(I)) \otimes L^2(A))$ with the $A$-bimodule structure given by $a \cdot \xi \cdot b = \alpha(a) \xi b$. Note that $e$ equals the support of $\alpha$. Also note that $T \otimes_A \overline{T} \cong L^2(p(B(\ell^2(I)) \otimes A)p)$ with the $A$-bimodule structure given by $a \cdot \xi \cdot b = \alpha(a) \xi b$.

Define $e_0 = 1_{(0,1)}(\Sigma)$ and denote by $f_0 \in Z(A)$ the right support of $e_0 T$. Note that $(1 \otimes f_0)p$ is the central support of $\alpha(e_0)$ inside $p(B(\ell^2(I)) \otimes A)p$. By construction, $\text{zdim}_{A} - (e_0 T \otimes_A \overline{T}) \leq e_0$.

It follows that the commutant of the left action on $e_0 T \otimes_A \overline{T}$ is a finite von Neumann algebra. A fortiori, $p(B(\ell^2(I)) \otimes A)p(1 \otimes f_0)$ is a finite von Neumann algebra. We can thus choose a sequence of projections $q_n \in Z(A)f_0$ such that $q_n \to f_0$ and $p(1 \otimes q_n)$ has finite trace for all $n$.

Denote by $p_n \in Z(A)e_0$ the support of the homomorphism that maps $a \in A e_0$ to $\alpha(a)(1 \otimes q_n)$. It follows that $p_n \to e_0$.

Since the closure of $\alpha(A e_0)(1 \otimes q_n)$ inside $L^2(p(B(\ell^2(I)) \otimes A)p)$ has $\text{zdim}_{A} -$ equal to $p_n$, we conclude that $\Sigma p_n \geq p_n$ for all $n$ and thus $\Sigma e_0 \geq e_0$. From the definition of $e_0$, it then follows that $\Sigma e_0 = e_0$ and $e_0 = e_1$ (as defined in the formulation of the lemma), as well as $\Sigma \geq e$ and $f_0 = f_1$. Since $p_n \Sigma = p_0$ for all $n$, it also follows that $\alpha(A p_n)(1 \otimes q_n)$ is dense in $\alpha(p_n)L^2(B(\ell^2(I)) \otimes A)p$ for all $n$, because the orthogonal complement has dimension zero. This means that $\alpha(e_1) = (1 \otimes f_1)p$ and that $\alpha : A e_1 \to p(B(\ell^2(I)) \otimes A)p(1 \otimes f_1)$ is a surjective $*$-isomorphism. So, $e_1 T = T f_1$ and this $A$-bimodule is given by a partial automorphism of $A$.

So the first statement of the lemma is proved. Take $e_2 \in Z(A)e$ and $f_2 \in Z(A)$ as in the second statement of the lemma. It follows that $e_2 T \otimes_A \overline{T} = e_2 T \otimes_A e_2 T$ and that $\text{zdim}_{A} - (e_2 T \otimes_A \overline{T}) = e_2$. So, $e_2 \Sigma = e_2$, meaning that $e_2 \leq e_1$.

Finally take $e_0 \in Z(A)$ as in the last statement of the lemma. We have $(\text{Tr} \otimes \tau)\alpha(e_0) = \text{dim}_{-A}(e_0 T) < \infty$. Under the above isomorphism between $T \otimes_A \overline{T}$ and $L^2(p(B(\ell^2(I)) \otimes A)p)$, the vector $e_0 T$ corresponds to $\alpha(e_0)$. So we have to determine the left support $z$ of $\alpha(e_0)pL^2(B(\ell^2(I)) \otimes A)p \cap \alpha(A e_0)^{-1}$. A projection $e_3 \in Z(A)e_0$ is orthogonal to $z$ if and only if $\alpha(A e_3)$ is dense in $\alpha(e_3)pL^2(B(\ell^2(I)) \otimes A)p$. This holds if and only if there exists a projection $f_3 \in Z(A)$ such that $\alpha(f_3) = (1 \otimes f_3)p$ and $\alpha(A e_3) = p(B(\ell^2(I)) \otimes A)p(1 \otimes f_3)$. Since this is equivalent with $e_3 \leq e_1$, we have proved that $z = e_0(1 - e_1)$.

**Lemma 6.6.** Let $(A, \tau)$ be a tracial von Neumann algebra and $(H, J)$ a symmetric $A$-bimodule with left (and thus also, right) support $e \in Z(A)$. There is a unique projection $e_1 \in Z(A)$ such that $e_1 H = He_1$, the $A$-bimodule $e_1 H$ is given by a partial automorphism of $A$ and the $A(e_1 - e_1)$-bimodule $(1 - e_1)H$ is completely nontrivial.
Proof. By Lemma 6.5, we find projections $e_1, f_1 \in \mathcal{Z}(A)e$ such that $e_1 H = H f_1$, the $A$-bimodule $e_1 H$ is given by a partial automorphism of $A$ and writing $e_2 := e - e_1$, $f_2 = e - f_1$, the $A e_2 - A f_2$-bimodule $e_2 H = H f_2$ is completely nontrivial. Since $H \cong A$, we must have $e_1 = f_1$ and $e_2 = f_2$. The uniqueness of $e_1$ can be checked easily.

By symmetry, given an $A$-bimodule $H$, we can also define $\Delta_H^e \in \mathcal{Z}(A)$ characterized by the formula $\tau(\Delta_H^e) = \dim_{A^e}(H e)$ for every projection $e \in \mathcal{Z}(A)$.

Lemma 6.7. Let $(A, \tau)$ be a tracial von Neumann algebra and $T$ an $A$-bimodule with left support $e \in \mathcal{Z}(A)$ and right support $f \in \mathcal{Z}(A)$. If $\Delta_T^e \leq e$ and $\Delta_T^f \leq f$, then $\Delta_T^e = e$, $\Delta_T^f = f$ and $T$ is given by a partial automorphism of $A$.

Proof. Let $e_0 \in \mathcal{Z}(A)e$ be the maximal projection with the following properties: the right support $f_0 \in \mathcal{Z}(A)f$ of $e_0 T$ satisfies $e_0 T = T f_0$, the $A$-bimodule $e_0 T$ is given by a partial automorphism of $A$ and $\Delta_T^e = e_0$, $\Delta_T^f = f_0$. We have to prove that $e_0 = e$.

Assume that $e_0$ is strictly smaller than $e$. Since $e_0 T = T f_0$, also $f_0$ is strictly smaller than $f$. Denote $e_1 = e - e_0$ and $f_1 = f - f_0$. Note that $e_1 T = T f_1$. Since $\dim_{A^e}(T) = \tau(\Delta_T^e) \leq \tau(e) \leq 1$ and similarly $\dim_{A^f}(T) \leq 1$, it follows from [PSV15, Proposition 2.3] that there exists a nonzero $A$-subbimodule $K \subset e_1 T$ with the following properties: $K$ is finitely generated, both as a left Hilbert $A$-module and as a right Hilbert $A$-module, and denoting by $e_2 \in \mathcal{Z}(A)e_1$ and $f_2 \in \mathcal{Z}(A)f_1$ the left, resp. right, support of $K$, there is a surjective $*$-isomorphism $\alpha : \mathcal{Z}(A)f_2 \to \mathcal{Z}(A)e_2$ such that $\alpha a = \alpha(a) \xi$ for all $\xi \in K, a \in \mathcal{Z}(A)f_2$.

Denote by $D$ the Radon-Nikodym derivative between $\tau \circ \alpha$ and $\tau$, so that $\tau(b) = \tau(\alpha(b) D)$ for all $b \in \mathcal{Z}(A)f_2$. By a direct computation, we get that

$$\Delta_K^e = D \alpha(\dim_{A^e}(K)) \quad \text{and} \quad \alpha(\Delta_K^f) = D^{-1} \dim_{A^f}(K).$$

In particular, we get that

$$\Delta_K^e \alpha(\Delta_K^e) = \dim_{A^e}(K) \alpha(\dim_{A^e}(K)). \quad (6.13)$$

By Lemma 6.5 and the computation in the proof of [PSV15, Lemma 2.2], we have

$$\dim_{A^e}(K) \alpha(\dim_{A^e}(K)) = \dim_{A^e}(K) \otimes_A K \geq e_2. \quad (6.14)$$

Since $\Delta_K^e \leq e_2$ and $\Delta_K^f \leq f_2$, in combination with (6.13), it follows that $\Delta_K^e = e_2$ and $\Delta_K^f = f_2$. From (6.14), we then also get that $\dim_{A^e}(K) \otimes_A K \geq e_2$. By Lemma 6.5, $K$ is given by a partial automorphism of $A$.

Since $e_2 \geq \Delta_K^{e_2 T} = \Delta_K^{e_2 T} \otimes_A K = e_2 + \Delta_K^{e_2 T} \otimes_A K$, we conclude that $e_2 T \otimes K = \{0\}$. So, $e_2 T = K$ and $e_2 T$ is given by a partial automorphism of $A$. This then contradicts the maximality of $e_0$.

Lemma 6.8. Let $(A, \tau)$ be a tracial von Neumann algebra and $(H, J)$ a symmetric $A$-bimodule.

Write $M = \Gamma(H, J, A, \tau)^w$. Let $p \in A$ be a projection and $B \subset p A p$ a von Neumann subalgebra such that $B' \cap p A p = \mathcal{Z}(B)$. Let $K \subset p H$ be a $B$-$A$-subbimodule that is finitely generated as a right Hilbert $A$-module. Assume that $\Delta_K^e$ is bounded and satisfies $\Delta_K^e \geq p$, as $B$-$A$-bimodule.

Let $(\xi_k)_{k=1}^n$ be a Pimsner-Popa basis for $K$ as a right $A$-module. Then the vectors $\xi_k$ are also left $A$-bounded and using the notation of (3.2), we define $S \in p Mp$ given by

$$S := \sum_{k=1}^n W(\xi_k, J(\xi_k)). \quad (6.15)$$

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Then, $S \in B' \cap pMp$, $S$ is self-adjoint and $S$ is diffuse relative to $B$. More precisely, in the von Neumann algebra $D := \{S\}'$, there exists a unitary $u \in \mathcal{U}(D)$ satisfying $E_B(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$.

Proof. Given a Pimsner-Popa basis $(\xi_k)_{k=1}^n$ for the right Hilbert $A$-module $K$ is the same as defining a right $A$-linear unitary operator $\theta : e(C^n \otimes \ell^2(A)) \to K$ for some projection $e \in A^n := M_n(\mathbb{C}) \otimes A$, with $\xi_k = \theta(e\xi_k \otimes 1)$. Define the faithful normal $*$-homomorphism $\alpha : B \to eA^n e$ such that $\theta(\alpha(b)\xi) = b\theta(\xi)$ for all $b \in B$ and $\xi \in e(C^n \otimes \ell^2(A))$. View $\ell^n \otimes K$ as a $B$-$A^n$-submodule of $\ell^n \otimes pH$. Define the vector $\xi \in \ell^n \otimes K$ given by

$$\xi = \sum_{k=1}^{n} \overline{e_k} \otimes \xi_k .$$

Then, $b\xi = \xi \alpha(b)$ for all $b \in B$ and, in particular, $\xi \in (\ell^n \otimes K)e$.

Define the normal positive functional $\omega : pAp \to \mathbb{C} : \omega(a) = \langle a\xi, \xi \rangle$. Since $\omega$ is $B$-central and $B' \cap pAp = \mathcal{Z}(B)$, we find $\Delta \in L^1(\mathcal{Z}(B))^+$ such that $\omega(a) = \tau(a\Delta)$ for all $a \in pAp$. For but all projections $q \in B$, we have

$$\tau(q\Delta) = \omega(q) = \langle q\xi, \xi \rangle = \langle \xi \alpha(q), \xi \rangle = (\text{Tr} \otimes \tau)(\alpha(q)) = \dim_A(qK) .$$

This means that $\Delta = \Delta^L_K$. Since $\Delta^L_K$ is bounded, the vectors $\xi_k \in H$ are left $A$-bounded.

So, the vectors $\xi_k$ are both left and right $A$-bounded, so that the operator $S$ given by (6.15) is a well defined element of $pMp$. Since

$$S = \sum_{k=1}^{n} (\ell(\xi_k)\ell(J(\xi_k))) + \ell(\xi_k)(\ell(\xi_k))^* + \ell(J(\xi_k))^*\ell(\xi_k)^* ,$$

we get that $S = S^*$. From this formula, we also get that $S$ commutes with $B$. Put $S_1 := \Delta + S$. Since $\Delta \in \mathcal{Z}(B)$, it suffices to prove that $S_1$ is diffuse relative to $B$.

Write $A_1 = pAp$ and $A_2 = eA^n e$. Equip $A_1$ and $A_2$ with the non normalized traces given by restricting $\tau$ to $A_1$ and $\text{Tr} \otimes \tau$ to $A_2$. View $\xi$ as a vector in the $A_1$-$A_2$-bimodule $(\ell^n \otimes pH)e$ and note that

$$\langle \xi, \xi \rangle_{A_2} = e , \quad A_1 \langle \xi, \xi \rangle = \Delta .$$

Denote $L := (\ell^n \otimes pH)e$. Recall that we view $L$ as an $A_1$-$A_2$-bimodule and that $\xi \in L$. Write $L' := e(\ell^n \otimes Hp)$, view $L'$ as an $A_2'$-$A_1'$-bimodule and note that the anti-unitary operator

$$J_1 : L \to L' : J_1(\sum_{k=1}^{n} \overline{e_k} \otimes \mu_k) = \sum_{k=1}^{n} e_k \otimes J(\mu_k)$$

satisfies $J_1(a\mu b) = b^*J_1(\mu)a^*$ for all $\mu \in L$, $a \in A_1$ and $b \in A_2$. Define $\xi' \in L'$ given by $\xi' = J_1(\xi)\Delta^{-1/2}$. Then $\xi'$ satisfies the following properties.

$$\langle \xi', \xi' \rangle_{A_1} = p , \quad A_2 \langle \xi', \xi' \rangle = \alpha(\Delta^{-1}) \text{ and } \alpha(b)\xi' = \xi'b \forall b \in B .$$

Define the Hilbert spaces

$$L_{\text{even}} = L^2(A_1) \oplus \bigoplus_{m=1}^{\infty} (L \otimes_{A_2} L')^m_{\otimes A_1} ,$$

$$L_{\text{odd}} = L' \otimes_{A_1} L_{\text{even}} = \bigoplus_{m=0}^{\infty} (L' \otimes_{A_1} (L \otimes_{A_2} L')^m_{\otimes A_1}) .$$
Note that $L_{\text{even}}$ is an $A_1$-bimodule, while $L_{\text{odd}}$ is an $A_2$-$A_1$-bimodule. Then,
\[
W := \ell(\xi')\Delta^{1/2} + \ell(\xi)^*
\]
is a well defined bounded operator from $L_{\text{even}}$ to $L_{\text{odd}}$ and $W^*W \in B(L_{\text{even}})$.

Using the natural isometry $L \otimes_{A_2} L' \hookrightarrow p(H \otimes A H)p$, we define the isometry $V : L_{\text{even}} \to pL^2(M)p$ given as the direct sum of the compositions of
\[
(L \otimes_{A_2} L')_{\otimes A_1} \hookrightarrow (p(H \otimes A H)p)_{\otimes A_1} \hookrightarrow p(H_{\otimes 2m})p.
\]

Then $V$ is $A_1$-bimodular and
\[
V W^*W = S_1 V.
\]

To compute the $*$-distribution of $B \cup \{S_1\}$ w.r.t. the trace $\tau$, it is thus sufficient to compute the $*$-distribution of $B \cup \{W^*W\}$ acting on $L_{\text{even}}$ and w.r.t. the vector functional implemented by $p \in L^2(A_1) \subset L_{\text{even}}$.

Define the closed subspaces $L_{\text{even}}^0 \subset L_{\text{even}}$ and $L_{\text{odd}}^0 \subset L_{\text{odd}}$ given as the closed linear span
\[
L_{\text{even}}^0 = \text{span}\{L^2(B), (\xi \otimes_{A_2} \xi')_{\otimes A_1} B \mid m \geq 1\},
\]
\[
L_{\text{odd}}^0 = \text{span}\{(\xi' \otimes_{A_1} (\xi \otimes_{A_2} \xi')_{\otimes A_1}) B \mid m \geq 0\}.
\]

Since $\xi \otimes_{A_2} \xi'$ is a $B$-central vector and since $\langle \xi, \xi \rangle_{A_2} = e$ and $\langle \xi', \xi' \rangle_{A_1} = p$, we find that $W(L_{\text{even}}^0) \subset L_{\text{odd}}^0$ and $W^*(L_{\text{odd}}^0) \subset L_{\text{even}}^0$. So to compute the $*$-distribution of $B \cup \{W^*W\}$, we may restrict $B$ and $W^*W$ to $L_{\text{even}}^0$.

Consider the full Fock space $\mathcal{F}(\mathbb{C}^2)$ of the 2-dimensional Hilbert space $\mathbb{C}^2$, with creation operators $\ell_1 = \ell(e_1)$ and $\ell_2 = \ell(e_2)$ given by the standard basis vectors $e_1, e_2 \in \mathbb{C}^2$. Denote by $\eta$ the vector state on $B(\mathcal{F}(\mathbb{C}^2))$ implemented by the vacuum vector $\Omega \in \mathcal{F}(\mathbb{C}^2)$. For every $\lambda \geq 1$, consider the operator $X(\lambda) \in B(\mathcal{F}(\mathbb{C}^2))$ given by $X(\lambda) = \sqrt{\lambda^2 + 1} 1$. We find that $X(\lambda)^*X(\lambda) = \lambda y^*y$ with $y = \ell_2 + \lambda^{-1/2} \ell_1$. It then follows from [Sh96, Lemma 4.3 and discussion after Definition 4.1] that the spectral measure of $X(\lambda)^*X(\lambda)$ has no atoms. Also for every $\lambda \geq 1$, $\eta$ is a faithful state on $\{X(\lambda)^*X(\lambda)\}''$.

Identify $Z(B) = L^\infty(Z, \mu)$ for some standard probability space $(Z, \mu)$. View $\Delta$ as a bounded function from $Z$ to $[1, +\infty]$ and define $Y \in B(\mathcal{F}(\mathbb{C}^2)) \otimes L^\infty(Z, \mu)$ given by $Y(z) = X(\Delta(z))$. We can view $Y$ as an element of $B(\mathcal{F}(\mathbb{C}^2)) \otimes B$ acting on the Hilbert space $\mathcal{F}(\mathbb{C}^2) \otimes L^2(B)$. Also, $\eta \otimes \tau$ is faithful on $(1 \otimes B \cup \{Y^*Y\})''$. Define the isometry
\[
U : L_{\text{even}}^0 \to \mathcal{F}(\mathbb{C}^2) \otimes L^2(B) : U((\xi \otimes_{A_2} \xi')_{\otimes A_1} b) = (e_1 \otimes e_2)_{\otimes m} \otimes b.
\]

By construction, $UW^*W = Y^*YU$ and $U$ is $B$-bimodular. It follows that the $*$-distribution of $B \cup \{S_1\}$ w.r.t. $\tau$ equals the $*$-distribution of $1 \otimes B \cup \{Y^*Y\}$ w.r.t. $\eta \otimes \tau$. So there is a unique normal $*$-isomorphism
\[
\Psi : (1 \otimes B \cup \{Y^*Y\})'' \to (B \cup \{S_1\})''
\]
satisfying $\Psi(1 \otimes b) = b$ for all $b \in B$ and $\Psi(Y^*Y) = S_1$. Also, $\tau \circ \Psi = \eta \otimes \tau$. Since for all $z \in Z$, the spectral measure of $Y(z)^*Y(z)$ has no atoms, there exists a unitary $v \in \{Y^*Y\}''$ such that $(\eta \otimes \tau)((1 \otimes b)v^k) = 0$ for all $b \in B$ and $k \in Z \setminus \{0\}$. Taking $u = \Psi(v)$, the lemma is proved. \hfill $\Box$

**Definition 6.9 ([Va07, Definition A.2]).** A von Neumann subalgebra $P$ of a tracial von Neumann algebra $(Q, \tau)$ is said to be of **essentially finite index** if there exist projections $q \in P' \cap Q$ arbitrarily close to 1 such that $Pq \subset qQq$ has finite Jones index.

To make the connection with [Io11, Lemma 1.6], note that $P \subset Q$ is essentially of finite index if and only if $qQq \prec_{qQq} Pq$ for every nonzero projection $q \in P' \cap Q$. 


Lemma 6.10. Let $(A, \tau)$ be a tracial von Neumann algebra and $(H, J)$ a symmetric $A$-bimodule. Write $M = \Gamma(H, J, A, \tau)''$.

Let $p \in A$ be a projection and $B \subset pAp$ a von Neumann subalgebra such that $B' \cap pAp = Z(B)$ and such that $N_{pAp}(B)''$ has essentially finite index in $pAp$. Let $K_1 \subset pH$ be a $B$-$A$-subbimodule satisfying the following three properties.

1. $K_1$ is a direct sum of $B$-$A$-subbimodules of finite right $A$-dimension.

2. The left action of $B$ on $K_1$ is faithful.

3. The $A$-bimodule $AK_1$ is left weakly mixing.

Then there exists a diffuse abelian von Neumann subalgebra $D \subset B' \cap pMp$ that is in tensor product position w.r.t. $B$. More precisely, there exists a unitary $u \in B' \cap pMp$ such that $E_B(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$.

Proof. We claim that for every $\varepsilon > 0$, there exists a projection $z \in Z(B)$ with $\tau(p - z) < \varepsilon$ and a $B$-$A$-subbimodule $L \subset zH$ such that $L$ is finitely generated as a right Hilbert $A$-module and such that $\Delta'_{L}$ is bounded and satisfies $\Delta'_{L} \geq z$. To prove this claim, denote $K := AK_1$ and let $(K_i)_{i \in I}$ be a maximal family of mutually orthogonal nonzero $B$-$A$-subbimodules of $pK$ that are finitely generated as a right $A$-module. Denote by $R$ the closed linear span of all $K_i$. Whenever $u \in N_{pAp}(B)$ and $i \in I$, also $uK_i$ is a $B$-$A$-subbimodule of $pK$ that is finitely generated as a right $A$-module. By the maximality of the family $(K_i)_{i \in I}$, we get that $uK_i \subset R$. So, $uR = R$ for all $u \in N_{pAp}(B)$. Writing $P := N_{pAp}(B)''$, we conclude that $R$ is a $P$-$A$-subbimodule of $pK$.

Since $P \subset pAp$ is essentially of finite index and since $AK_A$ is left weakly mixing, Lemma 6.12 says that for every projection $q \in P$, the right $A$-module $qR$ is either $\{0\}$ or of infinite right $A$-dimension. By the assumptions of the lemma and the maximality of the family $(K_i)_{i \in I}$, the left $B$-action on $R$ is faithful. So $qL \neq \{0\}$ and thus $\dim_-(qL) = \infty$ for every nonzero projection $q \in B$. This means that for every nonzero projection $q \in B$,

$$\sum_{i \in I} \tau(q\Delta'_{K_i}) = \sum_{i \in I} \dim_-(qK_i) = \dim_-(qR) = \infty.$$ 

So we can find a projection $z \in Z(B)$ and a finite subset $I_0 \subset I$ such that $\tau(p - z) < \varepsilon$ and such that the operator $\Delta := \sum_{i \in I_0} \Delta'_{K_i}$ is bounded and satisfies $\Delta \geq z$. Defining $L = \sum_{i \in I_0} zK_i$, the claim is proved.

Combining the claim with Lemma 6.8, we find for every $\varepsilon > 0$, a projection $z \in Z(B)$ with $\tau(p - z) < \varepsilon$ and a unitary $u \in (Bz)' \cap zMz$ such that $E_B(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. So, we find projections $z_n \in Z(B)$ and unitaries $u_n \in (Bz_n)' \cap z_nMz_n$ such that $E_B(u_n^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$ and such that $\bigvee_n z_n = p$. We can then choose projections $z'_n \in Z(B)$ with $z'_n \leq z_n$ and $\sum_n z'_n = p$. Defining $u = \sum_n u_n z'_n$, we have found a unitary in $B' \cap pMp$ satisfying $E_B(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. So, the lemma is proved.

Above we also needed the following two lemmas.

Lemma 6.11. Let $(N, \tau)$ be a tracial von Neumann algebra and $B \subset N$ an abelian von Neumann subalgebra. Assume that $D \subset B' \cap N$ is a diffuse abelian von Neumann subalgebra that is in tensor product position w.r.t. $B$. Then there is no nonzero projection $q \in B' \cap N$ satisfying $q(B' \cap N)q = Bq$. 

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Proof. Put $P = B' \cap N$ and assume that $q \in P$ is a nonzero projection such that $qPq = Bq$. Note that $B \subset \mathcal{Z}(P)$ because $B$ is abelian. Take a nonzero projection $z \in \mathcal{Z}(P)$ such that $z = \sum_{i=1}^{n} v_{i}v_{i}^{*}$ where $v_{1}, \ldots, v_{n}$ are partial isometries in $Pq$. Note that $zq \neq 0$ and write $p = zq$. Then,

$$Pp = Pzq = zPq = \text{span}\{v_{i}qPq \mid i = 1,\ldots,n\} = \text{span}\{v_{i}B \mid i = 1,\ldots,n\}. $$

So, $L^{2}(P)p$ is finitely generated as a right Hilbert $B$-module. Define $Q = B \cup D$ and denote by $e \in Q$ the support projection of $E_{Q}(p)$. Then $\xi \mapsto \xi p$ is an injective right $B$-linear map from $L^{2}(Q)e$ to $L^{2}(P)p$. So also $L^{2}(Q)e$ is finitely generated as a right Hilbert $B$-module. Since $Q \cong B \otimes D$ with $D$ diffuse and since $e$ is a nonzero projection in $Q \cong B \otimes D$, this is absurd. \hfill \Box

Lemma 6.12. Let $(A, \tau)$ be a tracial von Neumann algebra and $\mathcal{A} \mathcal{K}_{A}$ an $A$-bimodule that is left weakly mixing. Let $p \in A$ be a projection and $P \subset pA$ a von Neumann subalgebra that is essentially of finite index (see Definition 6.9). If $L \subset pK$ is a $P$-$A$-subbimodule and $q \in P$ is a projection such that $qL \neq \{0\}$, then the right $A$-dimension of $qL$ is infinite.

Proof. Assume for contradiction that $q \in P$ is a projection such that $qL$ is nonzero and such that $qL$ has finite right $A$-dimension. Since $P \subset pA$ is essentially of finite index, there exist projections $p_{1} \in P' \cap pA$ that lie arbitrarily close to $p$ such that $Ap_{1}$ is finitely generated as a right $PP_{1}$ module (purely algebraically using a Pimsner-Popa basis, see e.g. [Va07, A.2]). There also exist central projections $z \in \mathcal{Z}(P)$ that lie arbitrarily close to $p$ such that $Pzq$ is finitely generated as a right $qPq$-module. Take such $p_{1}$ and $z$ with $p_{1}zqL \neq \{0\}$. Then $Ap_{1}zq$ is finitely generated as a right $qPq$-module. Therefore, the closed linear span of $Ap_{1}zqL$ is a nonzero $A$-subbimodule of $K$ having finite right $A$-dimension. This contradicts the left weak mixing of $\mathcal{A}H_{A}$. \hfill \Box

7 Compact groups, free subsets, $c_{0}$ probability measures and the proof of Theorem B

For every second countable compact group $K$ with Haar probability measure $\mu$ and for every symmetric probability measure $\nu$ on $K$, we consider $A = L^{\infty}(K, \mu)$, the $A$-bimodule $H_{\nu} = L^{2}(K \times K, \mu \times \nu)$ given by (1.1) and the symmetry $J_{\nu}: H_{\nu} \to H_{\nu}$ given by (1.2). We put $M = \Gamma(H_{\nu}, J_{\nu}, A, \mu)^{\nu}$. In Proposition 7.3 below, we characterize when the bimodule $H_{\nu}$ is mixing (so that $M$ becomes strongly solid by Corollary 4.2) and when $A \subset M$ is an $s$-MASA. For the latter, the crucial property will be that the support $S$ of $\nu$ is of the form $S = F \cup F^{-1}$ where $F \subset K$ is a closed subset that is free in the following sense.

Definition 7.1. A subset $F$ of a group $G$ is called free if

$$g_{1}^{\epsilon_{1}} \cdots g_{n}^{\epsilon_{n}} \neq e$$

for all nontrivial reduced words, i.e. for all $n \geq 1$ and all $g_{1}, \ldots, g_{n} \in F$, $\epsilon_{1}, \ldots, \epsilon_{n} \in \{\pm 1\}$ satisfying $\epsilon_{i} = \epsilon_{i+1}$ whenever $1 \leq i \leq n-1$ and $g_{i} = g_{i+1}$.

On the other hand, the mixing property of $H_{\nu}$ will follow from the following $c_{0}$ condition on the measure $\nu$. 

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Whenever $K$ is a compact group, we denote by $\lambda : K \to U(L^2(K))$ the left regular representation. For every probability measure $\nu$ on $K$ and every unitary representation $\pi : K \to U(H)$, we denote

$$\pi(\nu) = \int_K \pi(x) \, d\nu(x) .$$

**Definition 7.2.** A probability measure $\nu$ on a compact group $K$ is said to be $c_0$ if the operator $\lambda(\nu) \in B(L^2(K))$ is compact.

Note that $\nu$ is $c_0$ if and only if $\lambda(\nu)$ belongs to the reduced group C*-algebra $C_r^*(K)$. Also, since the regular representation of $K$ decomposes as the direct sum of all irreducible representations of $K$, each appearing with multiplicity equal to its dimension, we get that a probability measure $\nu$ is $c_0$ if and only if

$$\lim_{\pi \in \text{Irr}(K), \pi \to \infty} \|\pi(\nu)\| = 0 ,$$

i.e. if and only if the map $\text{Irr}(K) \to \mathbb{R} : \pi \mapsto \|\pi(\nu)\|$ is $c_0$. In particular, when $K$ is an abelian compact group, a probability measure $\nu$ on $K$ is $c_0$ if and only if the Fourier transform of $\nu$ is a $c_0$ function on $\hat{K}$.

**Proposition 7.3.** Let $K$ be a second countable compact group $K$ with Haar probability measure $\mu$. Put $A = L^\infty(K, \mu)$. Let $\nu$ be a symmetric probability measure on $K$ without atoms. Define the $A$-bimodule $H_\nu$ with symmetry $J_\nu$ by (1.1) and (1.2). Denote by $M = \Gamma(H_\nu, J_\nu, A, \mu)^\pi$ the associated tracial von Neumann algebra. Let $S$ be the support of $\nu$, i.e. the smallest closed subset of $K$ with $\nu(S) = 1$.

1. The bimodule $H_\nu$ is weakly mixing, $A \subset M$ is a singular MASA, $M$ has no Cartan subalgebra and $A \subset M$ is a maximal amenable subalgebra.

2. The von Neumann algebra $M$ has no amenable direct summand. The center $Z(M)$ of $M$ equals $L^\infty(K/K_0)$ where $K_0 \subset K$ is the closure of the subgroup generated by $S$. So if $S$ topologically generates $K$, then $M$ is a nonamenable $II_1$ factor.

3. If $S$ is of the form $S = F \cup F^{-1}$ where $F \subset K$ is a closed subset that is free in the sense of Definition 7.1, then $A \subset M$ is an s-MASA.

4. If $\nu$ is $c_0$ in the sense of Definition 7.2, then the bimodule $H_\nu$ is mixing. So then, $M$ is strongly solid and whenever $B \subset M$ is an amenable von Neumann subalgebra for which $B \cap A$ is diffuse, we have $B \subset A$.

**Proof.** 1. Note that

$$H_\nu^{\otimes_n} \cong L^2(K \times K \times \cdots \times K, \mu \times \nu \times \cdots \times \nu) \tag{7.1}$$

with the $A$-bimodule structure given by

$$(F \cdot \xi \cdot G)(x, y_1, \ldots, y_n) = F(xy_1 \cdots y_n) \xi(x, y_1, \ldots, y_n) G(x) .$$

Define $D \subset K \times K$ given by $D = \{(y, y^{-1}) \mid y \in K\}$. Since $\nu$ has no atoms, we have $(\nu \times \nu)(D) = 0$. It then follows that $H_\nu \otimes_A H_\nu$ has no nonzero $A$-central vectors. By Proposition 2.3, the $A$-bimodule $H_\nu$ is weakly mixing. So also $L^2(M) \otimes L^2(A)$ is a weakly mixing $A$-bimodule, implying that $N_M(A) \subset A$. So, $A \subset M$ is a MASA and this MASA is singular. By Theorem 6.1, $M$ has no Cartan subalgebra. By Theorem 5.1, we get that $A \subset M$ is a maximal amenable subalgebra.

2. Since $H_\nu$ is weakly mixing, we get from Theorem 5.1 that $M$ has no amenable direct summand and that $Z(M)$ consists of all $a \in A$ satisfying $a \cdot \xi = \xi \cdot a$ for all $\xi \in H_\nu$. It is then clear that
$L^\infty(K/K_0) \subset Z(M)$. To prove the converse, fix $a \in A$ with $a \cdot \xi = \xi \cdot a$ for all $\xi \in H_\nu$. We find in particular that $a(xy) = a(x)$ for $\mu \times \nu$-a.e. $(x,y) \in K \times K$. Let $U_n$ be a decreasing sequence of basic neighborhoods of $e$ in $K$. Define the functions $b_n$ given by

$$b_n(y) = \mu(U_n)^{-1} \int_{U_n} a(xy) \, d\mu(x).$$

For every fixed $n$, the functions $b_n$ still satisfy $b_n(xy) = b(x)$ for $\mu \times \nu$-a.e. $(x,y) \in K \times K$. But the functions $b_n$ are continuous. It follows that $b_n(x) = b_n(y)$ for all $x \in K$ and all $y \in S$. So, $b_n \in C(K/K_0)$. Since $\lim_n ||b_n - a||_1 = 0$, we get that $a \in L^\infty(K/K_0)$.

3. Denote by $\nu_n := (\pi_n)s(\nu^n)$ and then $\eta = \frac{1}{2}\delta_0 + \sum_{n=1}^{\infty} 2^{-n-1}\nu_n$. Using (7.1), it follows that $A^LZ(M)_A$ is isomorphic with the $A$-bimodule

$$L^2(K \times K, \mu \times \eta) \quad \text{with} \quad (F \cdot \xi \cdot G)(x,y) = F(xy)\xi(x,y)G(x).$$

So, $A^LZ(M)_A$ is a cyclic bimodule and $A \subset M$ is an s-MASA.

4. Define $\xi_0 \in H_\nu$ by $\xi_0(x,y) = 1$ for all $x, y \in K$. Denote by $\varphi : A \to A$ the completely positive map given by $\varphi(a) = (\xi_0, a\xi_0)A$. To prove that $H_\nu$ is mixing, it is sufficient to prove that $\lim_n ||\varphi(a_n)||_2 = 0$ whenever $(a_n)$ is a bounded sequence in $A$ that converges weakly to 0. Denoting by $\rho : K \to L^2(K)$ the right regular representation, we get that $\varphi(a) = \rho(\nu)(a)$ for all $a \in A \subset L^2(K)$. Since $\rho(\nu)$ is a compact operator, we indeed get that $\lim_n ||\rho(\nu)(a_n)||_2 = 0$. So, $H_\nu$ is a mixing $A$-bimodule. By Corollary 4.2, $M$ is strongly solid. The remaining statement follows from Theorem 5.1.

**Remark 7.4.** In the special case where $K$ is abelian, we identify $L^\infty(K,\mu) = L(G)$, with $G := \bar{K}$ being a countable abelian group. Then the symmetric $L^\infty(K,\mu)$-bimodule $H_\nu$ given by (1.1) and (1.2) is isomorphic with the symmetric $L(G)$-bimodule associated, as in Remark 3.5, with the cyclic orthogonal representation of $G$ with spectral measure $\nu$. In particular, as in Remark 3.5, the von Neumann algebras $M = \Gamma(H_\nu, J_\nu, L^\infty(K), \mu)^\sigma$ can also be realized as a free Bogoljubov crossed product by the countable abelian group $G$. In this way, Proposition 7.3 generalizes the results of [HS09, Ho12a]. Note however that for a free Bogoljubov crossed product $M = \Gamma(K_\R)^\sigma \rtimes G$ with $G$ abelian, the subalgebra $L(G) \subset M$ is never an s-MASA. So our more general construction is essential to prove Theorem B.

For non abelian compact groups $K$, we can still view $K = \hat{G}$, but $G$ is no longer a countable group, rather a discrete Kac algebra. It is then still possible to identify the $\Pi_1$ factors $M$ in Proposition 7.3 with a crossed product $\Gamma(K_\R)^\sigma \rtimes G$, where the discrete Kac algebra action of $G$ on $\Gamma(K_\R)^\sigma$ is the free Bogoljubov action associated in [Va02] with an orthogonal corepresentation of the quantum group $G$.

The main result of this section says that in certain sufficiently non abelian compact groups $K$, one can find “large” free subsets $F \subset K$, where “large” means that $F$ carries a non atomic probability measure that is $c_0$. We conjecture that the compact Lie groups $SO(n)$, $n \geq 3$, admit free subsets carrying a $c_0$ probability measure. For our purposes, it is however sufficient to prove that these exist in more ad hoc groups.

For every prime number $p$, denote by $\Gamma_p$ the finite group $\Gamma_p = PGL_2(\Z/p\Z)$. The following is the main result of this section. Recall that the support of a probability measure $\nu$ on a compact space $K$ is defined as the smallest closed subset $S \subset K$ with $\nu(S) = 1$.
There exists a sequence of prime numbers $p_n$ tending to infinity, a closed free subset $F \subseteq K := \bigcap_{n=1}^{\infty} \Gamma_{p_n}$ topologically generating $K$ and a symmetric, non atomic, $\alpha_0$ probability measure $\nu$ on $K$ whose support equals $F \cup F^{-1}$.

We then immediately get:

**Proof of Theorem B.** Take $K$ and $\nu$ as in Theorem 7.5. Denote by $M$ the associated von Neumann algebra with abelian subalgebra $A \subseteq M$ as in Proposition 7.3. By Proposition 7.3, we get that $M$ is a nonamenable, strongly solid II$_1$ factor and that $A \subseteq M$ is an s-MAA. □

Before proving Theorem 7.5, we need some preparation.

The Alon-Roichman theorem [AR92] asserts that the Cayley graph given by a random and independent choice of $k \geq \epsilon \log |G|$ elements in a finite group $G$ has expected second eigenvalue at most $\epsilon$, with the normalization chosen so that the largest eigenvalue is 1. In [LR04, Theorem 2], a simple proof of that result was given. The same proofs yield the following result. For completeness, we provide the argument.

Whenever $G$ is a group, $\pi : G \to \mathcal{U}(H)$ is a unitary representation and $g_1, \ldots, g_k \in G$, we write

$$\pi(g_1, \ldots, g_k) := \frac{1}{k} \sum_{j=1}^{k} \pi(g_j). \quad (7.2)$$

**Lemma 7.6 ([LR04]).** Let $G_n$ be a sequence of finite groups and $k_n$ a sequence of positive integers such that $k_n/\log |G_n| \to \infty$. For every $\epsilon > 0$ and for a uniform and independent choice of $k_n$ elements $g_1, \ldots, g_{k_n} \in G_n$, we have that

$$\lim_{n \to \infty} P\{ \|\pi(g_1, \ldots, g_{k_n})\| \leq \epsilon \text{ for all } \pi \in \text{Irr}(G_n) \setminus \{\epsilon\} \} = 1.$$ 

**Proof.** Fix a finite group $G$ and a positive integer $k$. Let $g_1, \ldots, g_k$ be a uniform and independent choice of elements of $G$. Denote by $\lambda_0 : G \to \mathcal{U}(\ell^2(G) \otimes \mathbb{C}1)$ the regular representation restricted to $\ell^2(G) \otimes \mathbb{C}1$. Put $d = |G| - 1$. Both

$$T(g_1, \ldots, g_k) = \frac{1}{k} \sum_{j=1}^{k} \lambda_0(g_j) + \lambda_0(g_j)^*$$

and

$$S(g_1, \ldots, g_k) = \frac{1}{k} \sum_{j=1}^{k} i\lambda_0(g_j) - i\lambda_0(g_j)^*$$

are sums of $k$ independent self-adjoint $d \times d$ matrices of norm at most 1 and having expectation 0. We apply [AW01, Theorem 19] to the independent random variables

$$X_j = \frac{2 + \lambda_0(g_j) + \lambda_0(g_j)^*}{4},$$

satisfying $0 \leq X_j \leq 1$ and having expectation $1/2$. We conclude that for every $0 \leq \epsilon \leq 1/2$,

$$P\{ \|T(g_1, \ldots, g_k)\| \leq \epsilon \} = P\{ (1 - \epsilon)^{1/2} \leq \frac{1}{k} \sum_{j=1}^{k} X_j \leq (1 + \epsilon)^{1/2} \} \geq 1 - 2d \exp\left(-\frac{k \epsilon^2}{4 \log 2}\right).$$

The same estimate holds for $S(g_1, \ldots, g_k)$. Since $\lambda_0(g_1, \ldots, g_k) = T(g_1, \ldots, g_k) - iS(g_1, \ldots, g_k)$ and since $\lambda_0$ is the direct sum of all nontrivial irreducible representations of $G$ (all appearing with multiplicity equal to their dimension), we conclude that

$$P\{ \|\pi(g_1, \ldots, g_k)\| \leq \epsilon \text{ for all } \pi \in \text{Irr}(G) \setminus \{\epsilon\} \} \geq 1 - 4|G| \exp\left(-\frac{k \epsilon^2}{16 \log 2}\right).$$

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Taking $G = G_n$, $k = k_n$ and $n \to \infty$, our assumption that $k_n / \log |G_n| \to \infty$ implies that for every fixed $\varepsilon > 0$,

$$|G_n| \exp\left(-k_n \frac{\varepsilon^2}{16 \log 2}\right) \to 0$$

and thus the lemma follows. \hfill \Box

On the other hand in [GHSSV07], it is proven that random Cayley graphs of the groups $\text{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ have large girth. More precisely, we say that elements $g_1, \ldots, g_k$ in a group $G$ satisfy no relation of length $\leq \ell$ if every nontrivial reduced word of length at most $\ell$ with letters from $g_1^{\pm 1}, \ldots, g_k^{\pm 1}$ defines a nontrivial element in $G$.

The estimates in the proof of [GHSSV07, Lemma 10] give the following result. Again for completeness, we provide the argument.

**Lemma 7.7 ([GHSSV07]).** Let $p_n$ be a sequence of prime numbers tending to infinity and let $k_n$ be a sequence of positive integers such that $\log k_n / \log p_n \to 0$. Put $\Gamma_{p_n} = \text{PGL}_2(\mathbb{Z}/p_n\mathbb{Z})$. For every $\ell > 0$ and for a uniform and independent choice of $k_n$ elements $g_1, \ldots, g_k \in \Gamma_{p_n}$, we have that

$$\lim_{n \to \infty} \mathbb{P}\left( g_1, \ldots, g_k \text{ satisfy no relation of length } \leq \ell \right) = 1.$$ 

**Proof.** Let $G$ be a group. A law of length $\ell$ in $G$ is a nontrivial element $w$ in a free group $F_\ell$ such that $w$ has length $\ell$ and $w(g_1, \ldots, g_n) = e$ for all $g_1, \ldots, g_n \in G$. For example, if $G$ is abelian, the element $w = aba^{-1}b^{-1}$ of $F_2$ defines a law of length 4 in $G$. Since the labeling of the generators does not matter, any law of length $\ell$ can be defined by a nontrivial element of $F_\ell$ with $n \leq \ell$. In particular, there are only finitely many possible laws of a certain length $\ell$.

Since $F_\infty \hookrightarrow F_2 \hookrightarrow \text{PSL}_2(\mathbb{Z})$, the group PSL$_2(\mathbb{Z})$ satisfies no law. For every prime number $p$, write $\Gamma_p = \text{PGL}_2(\mathbb{Z}/p\mathbb{Z})$. Using the quotient maps $\text{PSL}_2(\mathbb{Z}) \to \text{PSL}_2(\mathbb{Z}/p\mathbb{Z})$, we get that a given nontrivial element $w \in F_n$ can be a law for at most finitely many $\Gamma_p$. So, for every $\ell > 0$, we get that $\Gamma_p$ satisfies no law of length $\leq \ell$ for all large enough primes $p$. (Note that [GHSSV07, Proposition 11] provides a much more precise result.)

Let $w = g_{i_1}^{\varepsilon_1} \cdots g_{i_k}^{\varepsilon_k}$ with $i_j \in \{1, \ldots, k\}$ and $\varepsilon_j \in \{\pm 1\}$ be a reduced word of length $\ell$ in $g_{i_1}^{\pm 1}, \ldots, g_{i_k}^{\pm 1}$. Let $p$ be a prime number and assume that $w$ is not a law of $\Gamma_p$. With the same argument as in the proof of [GHSSV07, Lemma 10], we now prove that for a uniform and independent choice of $g_1, \ldots, g_k \in \Gamma_p$, we have that

$$\mathbb{P}\left( w(g_1, \ldots, g_k) = e \text{ in } \Gamma_p \right) \leq \frac{\ell}{p} \left(1 + \frac{1}{p-1}\right)^{3k}. \quad (7.3)$$

Denote $F_p = \mathbb{Z}/p\mathbb{Z}$, not to be confused with the free group $F_p$. Write $G_p = \text{GL}_2(F_p) \subset F_p^{2\times 2}$. Define the map

$$W : (F_p^{2\times 2})^k \to F_p^{2\times 2} : W(a_1, \ldots, a_k) = b_1 \cdots b_\ell,$$

where $b_j = a_{i_j}$ when $\varepsilon_j = 1$ and $b_j$ equals the adjunct matrix of $a_{i_j}$ when $\varepsilon_j = -1$. Note that the four components $W_{st}$, $s, t \in \{1, 2\}$, of the map $W$ are polynomials of degree at most $\ell$ in the $4k$ variables $a \in (F_p^{2\times 2})^k$. Define the subset $\mathcal{W} \subset (F_p^{2\times 2})^k$ given by

$$\mathcal{W} = \{ a \in (F_p^{2\times 2})^k \mid W(a) \text{ is a multiple of the identity matrix} \}$$

$$= \{ a \in (F_p^{2\times 2})^k \mid W_{11}(a) - W_{22}(a) = W_{12}(a) = W_{21}(a) = 0 \}.$$ 

We also define $\mathcal{V} = \mathcal{W} \cap (G_p)^k$ and

$$\mathcal{U} = \{ g \in (\Gamma_p)^k \mid w(g_1, \ldots, g_k) = e \text{ in } \Gamma_p \}.$$
The quotient map $G_p \to \Gamma_p$ induces the $(p-1)^k$-fold covering $\pi : \mathcal{V} \to \mathcal{U}$.

The subset $\mathcal{W} \subset F_p^{4k}$ is the solution set of a system of three polynomial equations of degree at most $\ell$. If each of these polynomials is identically zero, we get that $\mathcal{W} = F_p^{4k}$ and thus $\mathcal{U} = (\Gamma_p)^k$. This means that $w$ is a law of $\Gamma_p$, which we supposed not to be the case. So at least one of the polynomials is not identically zero. The number of zeros of such a polynomial is bounded above by $\ell p^{4k-1}$ (and a better, even optimal, bound can be found in [Se89]). So, $|\mathcal{W}| \leq \ell p^{4k-1}$. Then also $|\mathcal{V}| \leq \ell p^{4k-1}$ and because $\pi$ is a $(p-1)^k$-fold covering, we find that

$$|\mathcal{U}| \leq \ell (p-1)^{-k} p^{4k-1}.$$

Since $|\Gamma_p| = (p-1)p(p+1)$, we conclude that

$$\mathbf{P}(w(g_1, \ldots, g_k) = e \text{ in } \Gamma_p) = \frac{|\mathcal{U}|}{|\Gamma_p|^k} \leq \frac{\ell}{p} (p-1)^{-2k} (p+1)^{-k} p^{3k} \leq \frac{\ell}{p} \left(1 + \frac{1}{p-1}\right)^{3k}.$$

So, (7.3) holds.

Now assume that $p_n$ is a sequence of prime numbers and $k_n$ are positive integers such that $p_n \to \infty$ and $\log k_n/\log p_n \to 0$. For all $n$ large enough, $3k_n \leq p_n - 1$ and for all $n$ large enough, as we explained in the beginning of the proof, $\Gamma_{p_n}$ has no law of length $\leq \ell$. Since $(1 + 1/x)^x < 3$ for all $x > 0$ and since there are less than $(2k)^{\ell + 1}$ reduced words of length $\leq \ell$ in $g_1^{\pm 1}, \ldots, g_k^{\pm 1}$, we find that for all $n$ large enough and a uniform, independent choice of $g_1, \ldots, g_k \in \Gamma_{p_n}$, we have

$$\mathbf{P}(g_1, \ldots, g_k \text{ satisfy a relation of length } \leq \ell \text{ in } \Gamma_{p_n}) \leq (2k_n)^{\ell + 1} \frac{3\ell}{p_n}.$$

By our assumption that $\log k_n/\log p_n \to 0$, the right hand side tends to 0 as $n \to \infty$ and the lemma is proved.

Combining Lemmas 7.6 and 7.7, we obtain the following.

**Lemma 7.8.** For all $\varepsilon > 0$ and all $k_0, p_0, \ell \in \mathbb{N}$, there exist a prime number $p \geq p_0$, an integer $k \geq k_0$ and elements $g_1, \ldots, g_k \in \Gamma_p = \text{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ generating the group $\Gamma_p$ such that

1. $\|\pi(g_1, \ldots, g_k)\| \leq \varepsilon$ for every nontrivial irreducible representation $\pi \in \text{Irr}(\Gamma_p)$,

2. $g_1, \ldots, g_k$ satisfy no relation of length $\leq \ell$.

**Proof.** Choose any sequence of prime numbers $p_n$ tending to infinity. Define $k_n = |(\log p_n)^2|$. Since $|\Gamma_{p_n}| = (p_n - 1)p_n(p_n + 1)$, we get that $k_n/\log |\Gamma_{p_n}| \to \infty$. Also, $\log k_n/\log p_n \to 0$. So Lemmas 7.6 and 7.7 apply and for a large enough choice of $n$, properties 1 and 2 in the lemma hold for $p = p_n$, $k = k_n$ and a large portion of the $k_n$-tuples $(g_1, \ldots, g_{k_n}) \in \Gamma_{p_n}$. The first property in the lemma is equivalent with

$$\left\| \left( \frac{1}{k} \sum_{j=1}^{k} \lambda(g_j) \right)_{\ell^2(\Gamma_p) \otimes \mathbb{C}1} \right\| \leq \varepsilon,$$

where $\lambda : \Gamma_p \to \ell^2(\Gamma_p)$ is the regular representation. If $\varepsilon < 1$, it then follows in particular that there are no non zero functions in $\ell^2(\Gamma_p) \otimes \mathbb{C}1$ that are invariant under all $\lambda(g_j)$, meaning that every element of $\Gamma_p$ can be written as a product of elements in $\{g_1, \ldots, g_k\}$. So, we get that $g_1, \ldots, g_k$ generate $\Gamma_p$.

Having proven Lemma 7.8, we are now ready to prove Theorem 7.5.
Proof of Theorem 7.5. As in (7.2), for every finite group $G$, subset $F \subset G$ and unitary representation $\pi : G \to \mathcal{U}(H)$, we write

$$\pi(F) := \frac{1}{|F|} \sum_{g \in F} \pi(g).$$

For every prime number $p$, we write $\Gamma_p = \operatorname{PGL}_2(\mathbb{Z}/p\mathbb{Z})$. We construct by induction on $n$ a sequence of prime numbers $p_n$ and a generating set

$$F_n \subset K_n := \prod_{j=1}^n \Gamma_{p_j}$$

such that, denoting by $\theta_{n-1} : K_n \to K_{n-1}$ to projection onto the first $n-1$ coordinates, the following properties hold.

1. $\theta_{n-1}(F_n) = F_{n-1}$ and the map $\theta_{n-1} : F_n \to F_{n-1}$ is an $r_n$-fold covering with $r_n \geq 2$.

2. If $\pi \in \operatorname{Irr}(K_n)$ and $\pi$ does not factor through $\theta_{n-1}$, then $\|\pi(F_n)\| \leq 1/n$.

3. The elements of $F_n$ satisfy no relation of length $\leq n$.

Assume that $p_1, \ldots, p_{n-1}$ and $F_1, \ldots, F_{n-1}$ have been constructed. We have to construct $p_n$ and $F_n$. Write $k_1 = |F_{n-1}|$ and put $k_0 = \max\{2n+1, k_1\}$. By Lemma 7.8, we can choose $k_2 > k_0$, a prime number $p_n$ and a subset $F \subset \Gamma_{p_n}$ with $|F| = k_2$ such that the elements of $F$ satisfy no relation of length $\leq 3n$ and such that $\|\pi(F)\| \leq 1/(4n)$ for every nontrivial irreducible representation $\pi$ of $\Gamma_{p_n}$.

Write $F_{n-1} = \{g_1, \ldots, g_{k_1}\}$ and $F = \{h_1, \ldots, h_{k_2}\}$. Note that we have chosen $k_2 > \max\{2n+1, k_1\}$. So we can define the subset $F_n \subset K_{n-1} \times \Gamma_{p_n} = K_n$ given by

$$F_n = \{(g_i, h_i h_j h_i^{-1}) \mid 1 \leq i \leq k_1, 1 \leq j \leq k_2, i \neq j\}.$$

Note that $\theta_{n-1}(F_n) = F_{n-1}$ and that the map $\theta_{n-1} : F_n \to F_{n-1}$ is a $(k_2-1)$-fold covering.

Every irreducible representation $\pi \in \operatorname{Irr}(K_n)$ that does not factor through $\theta_{n-1}$ is of the form $\pi = \pi_1 \otimes \pi_2$ with $\pi_1 \in \operatorname{Irr}(K_{n-1})$ and with $\pi_2$ being a nontrivial irreducible representation of $\Gamma_{p_n}$. Note that

$$\pi(F_n) = \frac{1}{k_1} \sum_{i=1}^{k_1} \pi_1(g_i) \otimes \pi_2(h_i) T_i \pi_2(h_i)^*,$$

where

$$T_i := \frac{1}{k_2-1} \sum_{1 \leq j \leq k_2, j \neq i} \pi_2(h_j).$$

For every fixed $i \in \{1, \ldots, k_1\}$, we have

$$T_i = \frac{k_2}{k_2-1} \pi_2(F) - \frac{1}{k_2-1} \pi_2(h_i).$$

Therefore,

$$\|T_i\| < 2 \|\pi_2(F)\| + \frac{1}{2n} \leq \frac{1}{n}. \quad (7.4)$$

It then also follows that $\|\pi(F_n)\| < 1/n$.

We next prove that $F_n$ is a generating set of $K_n$. Fix $i \in \{1, \ldots, k_1\}$. For all $s, t \in \{1, \ldots, k_2\}$ with $s \neq i$ and $t \neq i$, we have

$$(g_i, h_i h_s h_i^{-1}) (g_i, h_i h_t h_i^{-1})^{-1} = (e, h_i h_s h_t h_i^{-1} h_i^{-1}).$$
It thus suffices to prove that the set $H_i := \{ h_i h_t^{-1} \mid s, t \in \{1, \ldots, k_2\} \setminus \{i\} \}$ generates $\Gamma_{p_n}$ for each $i \in \{1, \ldots, k_1\}$.

Denote by $\lambda_0$ the regular representation of $\Gamma_{p_n}$ restricted to $\ell^2(\Gamma_{p_n}) \oplus \mathbb{C}1$. Define

$$R_i = \frac{1}{k_2 - 1} \sum_{1 \leq j \leq k_2, j \neq i} \lambda_0(h_j).$$

By (7.4), we get that $\|R_i\| < 1$. Then also $\|R_i R_i^*\| < 1$. So, there is no non-zero function in $\ell^2(\Gamma_{p_n}) \oplus \mathbb{C}1$ that is invariant under all $\lambda(h)$, $h \in H_i$. It follows that each $H_i$ is a generating set of $\Gamma_{p_n}$.

Denote by $\eta_n : K_n \to \Gamma_{p_n}$ the projection onto the last coordinate. If the elements of $F_n$ satisfy any relation of length $\leq n$, applying $\eta_n$ will give a non-trivial relation of length $\leq 3n$ between the elements of $F$. Since such relations do not exist, we have proved that the elements of $F_n$ satisfy no relation of length $\leq n$.

Define $K = \prod_{n=1}^{\infty} \Gamma_{p_n}$ and still denote by $\theta_n : K \to K_n$ the projection onto the first $n$ coordinates. Define

$$F = \{ k \in K \mid \theta_n(k) \in F_n \text{ for all } n \geq 1 \}.$$

Note that $F \subset K$ is closed and $\theta_n(F) = F_n$. Denoting by $\langle F \rangle$ the subgroup of $K$ generated by $F$, we get that $\theta_n(\langle F \rangle) = K_n$ for all $n$. So, $\langle F \rangle$ is dense in $K$, meaning that $F$ topologically generates $K$.

Since each map $\theta_{n-1} : F_n \to F_{n-1}$ is an $r_n$-fold covering, there is a unique probability measure $\nu_0$ on $K$ such that $(\theta_n)_*(\nu_0)$ is the normalized counting measure on $F_n$ for each $n$. Since $r_n \geq 2$ for all $n$, the measure $\nu_0$ is non-atomic. Note that the support of $\nu_0$ equals $F$. Define the symmetric probability measure $\nu$ on $K$ given by $\nu(U) = (\nu_0(U) + \nu_0(U^{-1}))/2$ for all Borel sets $U \subset K$. The support of $\nu$ equals $F \cup F^{-1}$. Since $\lambda(\nu) = (\lambda(\nu_0) + \lambda(\nu_0^*))/2$, to conclude the proof of the theorem, it suffices to prove that $F$ is free and that $\nu_0$ is a $c_0$ probability measure.

Let $g_1^{\varepsilon_1} \cdots g_m^{\varepsilon_m}$ be a reduced word of length $m$ with $g_1, \ldots, g_m \in F$. Take $n \geq m$ large enough such that $\theta_n(g_i) \neq \theta_n(g_{i+1})$ whenever $g_i \neq g_{i+1}$. We then get that $\theta_n(g_1)^{\varepsilon_1} \cdots \theta_n(g_m)^{\varepsilon_m}$ is a reduced word of length $m \leq n$ in the elements of $F_n$. It follows that

$$e \neq \theta_n(g_1)^{\varepsilon_1} \cdots \theta_n(g_m)^{\varepsilon_m} = \theta_n(g_1^{\varepsilon_1} \cdots g_m^{\varepsilon_m}).$$

So, $g_1^{\varepsilon_1} \cdots g_m^{\varepsilon_m} \neq e$ and we have proven that $F$ is free.

We finally prove that $\|\pi(\nu_0)\| < 1/m$ for every irreducible representation $\pi$ of $K$ that does not factor through $\theta_m : K \to K_m$. Since there are only finitely many irreducible representations that do factor through $\theta_m : K \to K_m$, this will conclude the proof of the theorem. Let $\pi$ be such an irreducible representation. There then exists a unique $n > m$ such that $\pi = \pi_0 \circ \theta_n$ and $\pi_0$ is an irreducible representation of $K_n$ that does not factor through $\theta_{n-1} : K_n \to K_{n-1}$. But then $\pi(\nu_0) = \pi_0(F_n)$ and thus

$$\|\pi(\nu_0)\| = \|\pi_0(F_n)\| \leq \frac{1}{n} < \frac{1}{m}.$$

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\square
\]

### References

[AW01] R. Ahlswede and A. Winter, Strong converse for identification via quantum channels. *IEEE Trans. Inform. Theory* 48 (2002), 569-579.
[Va02] S. Vaes, Strictly outer actions of groups and quantum groups. *J. Reine Angew. Math.* **578** (2005), 147-184.

[Va07] S. Vaes, Explicit computations of all finite index bimodules for a family of II$_1$ factors. *Ann. Sci. Éc. Norm. Supér.* **41** (2008), 743-788.

[Va13] S. Vaes, Normalizers inside amalgamated free product von Neumann algebras. *Publ. Res. Inst. Math. Sci.* **50** (2014), 695-721.

[Vo83] D. Voiculescu, Symmetries of some reduced free product C*-algebras. In *Operator algebras and their connections with topology and ergodic theory (Busteni, 1983)*, Lecture Notes in Math. **1132**, Springer, Berlin, 1985, pp. 556-588.

[Vo86] D. Voiculescu, Multiplication of certain noncommuting random variables. *J. Operator Theory* **18** (1987), 223-235.