Collision of High Frequency Plane Gravitational and Electromagnetic Waves

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Abstract

We study the head–on collision of linearly polarized, high frequency plane gravitational waves and their electromagnetic counterparts in the Einstein–Maxwell theory. The post–collision space–times are obtained by solving the vacuum Einstein and Einstein–Maxwell field equations in the geometrical optics approximation. The head–on collisions of all possible pairs of these systems of waves is described and the results are then generalized to non–linearly polarized waves which exhibit the maximum two degrees of freedom of polarization.

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1 Introduction

The primary purpose of this work is to determine the vacuum gravitational field produced by the head–on collision of high frequency plane gravitational waves. Although a large number of collision space–times are known (see for example [1]), this is a collision scenario which seems so far not to have received attention. High frequency gravitational waves are produced in the final stages of a binary neutron star merger, for example, when the orbital period of the stars is of the order of 100Hz. Such radiation will ultimately be detected as high frequency plane waves by a new generation of distant earth–bound detectors. A qualitative analytical description of the process of high frequency radiation, in–fall and merger, followed by low frequency ring–down to the formation of a black hole by an isolated gravitating system is given in [2]. With estimates of between three and one hundred such cataclysmic events per year detectable within a distance of hundreds of Mpc [3] [4] it is of some interest to explore the characteristics of the gravitational field produced by colliding high frequency gravitational waves.

2 High Frequency Plane Waves

To describe high frequency plane gravitational waves we begin with the space–time model of the gravitational field of linearly polarized plane gravitational waves propagating through a vacuum. This space–time has a line–element which can be put in Rosen [5] form

\[ ds^2 = -F^2 dx^2 - G^2 dy^2 + 2 du dv , \]  

(2.1)

where \( F, G \) are functions of \( u \) only. Einstein’s vacuum field equations are equivalent to

\[ \ddot{F} - h F = 0 , \quad \ddot{G} + h G = 0 . \]  

(2.2)

Here and throughout dots indicate differentiation with respect to \( u \) and \( h(u) \) is the freely specifiable profile of the waves. If in place of \( F(u), G(u) \) we use \( B(u), w(u) \) given by \( F = B e^w, G = B e^{-w} \) then (2.1) becomes

\[ ds^2 = -B^2 (e^{2w} dx^2 + e^{-2w} dy^2) + 2 du dv . \]  

(2.3)

Calculation of the Ricci tensor with the metric tensor given via this line–element results in

\[ R_{ij} = 2 \left( \frac{\ddot{B}}{B} + w^2 \right) u_i u_j , \]  

(2.4)
where the comma denotes partial differentiation with respect to the coordinates \( x' = (x, y, u, v) \). On account of (2.2) we have \( R_{ij} = 0 \) and thus
\[
\ddot{B} + \dot{w}^2 B = 0 . \tag{2.5}
\]

In Newman–Penrose notation the only non–vanishing component of the Riemann curvature tensor is
\[
\Psi_4 = -\ddot{w} - 2\dot{w} \frac{\dot{B}}{B} = -h(u) , \tag{2.6}
\]
with the last equality here coming from (2.2). This is a Petrov type N curvature tensor (the type associated with pure gravitational radiation) having the vector field \( \partial/\partial v \) as four–fold degenerate principal null direction. It is readily seen from (2.1) or (2.3) that the integral curves of \( \partial/\partial v \) are shear–free, expansion–free and twist–free null geodesics that generate the null hypersurfaces \( u = \text{constant} \). These null hypersurfaces are the histories of the wave fronts. They are null hyperplanes and the property of the field (2.6) that \( \Psi_4 = \text{constant} \) when \( u = \text{constant} \) distinguishes them as the histories of plane gravitational waves.

The specialization of these plane waves to high frequency plane waves has been carried out in the following elegant fashion by Burnett [6]: First replace (2.3) by a family of line–elements parametrized by a real parameter \( \lambda \geq 0 \). Thus we write
\[
d s^2 = -B^2_\lambda(u)(e^{2w_\lambda(u)}dx^2 + e^{-2w_\lambda(u)}dy^2) + 2 du dv , \tag{2.7}
\]
with
\[
w_\lambda(u) = \lambda \alpha_0(u) \sin \frac{u}{\lambda} , \tag{2.8}
\]
and
\[
\ddot{B}_\lambda + \dot{w}_\lambda^2 B_\lambda = 0 . \tag{2.9}
\]
Here \( \alpha_0(u) \) is an arbitrary (integrable) function of \( u \) for some range of \( u \) that includes \( u = 0 \). We shall be interested in these plane waves for small values of \( \lambda > 0 \), for which \( w_\lambda(u) = O(\lambda) \), and also in the limit \( \lambda \to 0 \). We think of \( \lambda \) as representing the wave–length of the waves. We assume that \( \lim_{\lambda \to 0} B_\lambda(u) = B_0(u) \) exists and is uniform on the range of \( u \). Writing (2.9) as an integral equation using an appropriate Green’s function, taking the limit \( \lambda \to 0 \) and using the Riemann–Lebesgue theorem, yields an integral equation for \( B_0(u) \) from which one can show that \( B_0(u) \) satisfies the differential equation
\[
\ddot{B}_0 + \frac{1}{2} \alpha_0^2 B_0 = 0 . \tag{2.10}
\]
The procedure described here leading to (2.10) is given in detail in [7]. For \( \lambda > 0 \) and small we can write the Isaacson–type expansion \([8]\)

\[
B_\lambda(u) = B_0(u) + \lambda B_1(u, \lambda) + \lambda^2 B_2(u, \lambda) + \ldots ,
\]

(2.11)

with \( B_n(u, \lambda) = O(\lambda^0) \), \( \dot{B}_n = O(\lambda^{-1}) \), \( \ddot{B}_n = O(\lambda^{-2}) \) for \( n = 1, 2, 3, \ldots \). For future use we shall require that \( B_\lambda(0) = 1 \) for all \( \lambda \geq 0 \). It thus follows from (2.8) and (2.9) that \( B_1 = 0 \) and so we have, for small \( \lambda \),

\[
B_\lambda(u) = B_0(u) + O(\lambda^2) .
\]

(2.12)

Substituting (2.8) and (2.12) into the line–element (2.7) we see that for small \( \lambda \) the line–element splits into a ‘background’ and a small perturbation as

\[
ds^2 = ds_0^2 - 2 \lambda B_0^2(u) \alpha_0(u) \sin \frac{u}{\lambda} (dx^2 - dy^2) + O(\lambda^2) ,
\]

(2.13)

where \( ds_0^2 \) is the line–element of the ‘background’ space–time given by

\[
ds_0^2 = -B_0^2(u) (dx^2 + dy^2) + 2 \, du \, dv .
\]

(2.14)

The space–time with line–element (2.7) is a vacuum space–time to all orders in \( \lambda \). The background space–time with line–element (2.14) is not a vacuum space–time. The Ricci tensor calculated with the metric tensor given via (2.14) has components

\[
R_{ij}^{(0)} = -\alpha_0^2(u) \, u_i \, u_j .
\]

(2.15)

The Riemann tensor of the space–time with line–element (2.13) has one non–vanishing Newman–Penrose component (given by (2.6) with \( w, B \) replaced by \( w_\lambda, B_\lambda \)) which for small \( \lambda \) reads

\[
\Psi_4 = \frac{1}{\lambda} \alpha_0(u) \sin \frac{u}{\lambda} + O(\lambda^0) .
\]

(2.16)

The key elements here which one associates with the geometrical optics approximation are the strong type N gravitational field of the waves given by (2.16) for small \( \lambda \) (this field then determines the ‘rays’ associated with the waves, which are the integral curves of the degenerate principal null direction \( \partial \partial / \partial v \)), and the classical form of the line–element (2.13) as a small perturbation of a background whose Ricci tensor is proportional to the ‘square’ of the (covariant) propagation direction in space–time of the histories of the wave fronts. In section 3 we will derive the approximate vacuum space–time model of the gravitational field which arises following the head–on collision of two families of these high frequency plane waves.
As a final preliminary we note that the Einstein–Maxwell vacuum field equations with linearly polarized plane electromagnetic waves as source are satisfied by a metric tensor given by the conformally flat line–element

\[ ds^2 = -B^2(u) \left( dx^2 + dy^2 \right) + 2 \, du \, dv \, , \]  

and a type N Maxwell field with one non–vanishing Newman–Penrose component \( \phi_2(u) \) which is real–valued and is given by

\[ \ddot{B} + \phi_2^2 B = 0 \, . \]  

(2.18)

If the electromagnetic waves are not linearly polarized and thus have the maximum two degrees of freedom of polarization then \( \phi_2(u) \) is a complex–valued function of the real variable \( u \) and (2.18) is replaced by

\[ \ddot{B} + |\phi_2|^2 B = 0 \, . \]  

(2.19)

Thus the high frequency case here, analogous to the high frequency gravitational waves described above, is obtained by replacing \( B(u) \) by \( B_\lambda(u) \) for \( \lambda \geq 0 \) and taking

\[ \phi_2(u) = a_0(u) \cos \frac{u}{\lambda} = O(\lambda^0) \, , \]  

(2.20)

with \( a_0 \) an arbitrary real–valued function of \( u \) in the linearly polarized case and a complex–valued function of \( u \) in the general case. Also for small \( \lambda \) the line–element (2.17) takes the form

\[ ds^2 = ds_0^2 + O(\lambda^2) \, , \quad ds_0^2 = -B_0^2(u) \left( dx^2 + dy^2 \right) + 2 \, du \, dv \, , \]  

(2.21)

with \( B_0(u) \) satisfying

\[ \ddot{B}_0 + \frac{1}{2} a_0^2 B_0 = 0 \, . \]  

(2.22)

It is interesting to consider the head–on collision problem for these waves. This is discussed in section 4 where we also consider the case in which these waves share their wave fronts with the high frequency gravitational waves.

The collisions of the linearly polarized gravitational and electromagnetic plane waves described in sections 3 and 4 can be generalized to allow the maximum two degrees of freedom of polarization in each system of waves. This generalization is given in section 5.

### 3 Collision of Gravitational Waves

The line–element of the space–time containing the histories of the wave fronts of the incoming high frequency plane gravitational waves, as well as modeling
their gravitational fields and the gravitational field after the waves collide, has the Rosen–Szekeres form

\[ ds^2 = -e^{-U}(e^V dx^2 + e^{-V} dy^2) + 2e^{-M} du dv , \]  

(3.1)

with \( U, V, M \) functions of \( u, v \) in general. To set up the head–on collision of linearly polarized high frequency plane gravitational waves we take the line–element (3.1) in the region of space–time \( u \geq 0, v \leq 0 \) to coincide with (2.7) above. Thus in this region

\[ U = -2 \log B_\lambda(u) , \quad V = 2 w_\lambda(u) = 2 \lambda \alpha_0(u) \sin \frac{u}{\lambda} , \quad M = 0 . \]  

(3.2)

For the region \( u \leq 0, v \geq 0 \) we have similar waves traveling in the opposite direction by taking

\[ U = -2 \log D_\lambda(v) , \quad V = 2 k_\lambda(v) = 2 \lambda \beta_0(v) \sin \frac{v}{\lambda} , \quad M = 0 . \]  

(3.3)

In the region \( u \leq 0, v \leq 0 \) we have flat space–time with \( U = V = M = 0 \). We thus have these families of plane waves propagating into a region free of any gravitational field and then engaging in a head–on collision. To find the line–element of the space–time in the region \( u > 0, v > 0 \) after the collision we solve Einstein’s vacuum field equations for \( U(u, v), V(u, v), M(u, v) \) subject to the boundary conditions: when \( v = 0 \) and \( u \geq 0 \) the functions \( U, V, M \) are given by (3.2) and when \( u = 0 \) and \( v \geq 0 \) the functions \( U, V, M \) are given by (3.3). We will be content to satisfy the vacuum field equations approximately for small \( \lambda \) in the form

\[ R_{ij} = O(\lambda) . \]  

(3.4)

Thus we look for \( U, V, M \) in \( u > 0, v > 0 \) satisfying the boundary conditions above and the equations (see [1], p.39)

\[ 2 U_{uu} - U_u^2 - V_u^2 + 2 U_u M_u = O(\lambda) , \]  

(3.5)

\[ 2 U_{vv} - U_v^2 - V_v^2 + 2 U_v M_v = O(\lambda) , \]  

(3.6)

\[ 2 U_{uv} - U_u V_v - U_v V_u = O(\lambda) , \]  

(3.7)

\[ 2 M_{uv} + U_u U_v - V_u V_v = O(\lambda) , \]  

(3.8)

and

\[ U_{uv} - U_u U_v = O(\lambda) . \]  

(3.9)

Here the subscripts denote partial derivatives. These are the exact vacuum field equations if the \( O(\lambda) \)–terms are replaced by zeros. In fact (3.9) with zero on the right hand side is well known to yield \( U \) in the form

\[ U = -\log(f(u) + g(v)) , \]  

(3.10)
for some functions $f(u)$, $g(v)$. To ensure a smooth transition to Minkowskian space–time in $u < 0$, $v < 0$ on the boundaries $u = 0$, $v \leq 0$ and $v = 0$, $u \leq 0$ of this region, we must have the functions $B_\lambda(u)$, $D_\lambda(v)$ in (3.2) and (3.3) satisfy

\[ B_\lambda(0) = D_\lambda(0) = 1, \]

for $\lambda \geq 0$. Indeed to ensure that no impulsive light–like signals exist on the boundaries $u = 0$, $-\infty < v < +\infty$ and $v = 0$, $-\infty < u < +\infty$ we take

\[ \dot{B}_\lambda(0) = \dot{D}_\lambda(0) = \alpha_0(0) = \beta_0(0) = 0. \]

We also note that $B_\lambda(u)$ satisfies (2.9) and (2.12) with $B_0(u)$ satisfying (2.10), while $D_\lambda(v)$ satisfies corresponding equations

\[ \ddot{D}_\lambda + \dot{k}_\lambda^2 D_\lambda = 0, \quad \ddot{D}_0 + \frac{1}{2} \beta_0^2 D_0 = 0, \]

and

\[ D_\lambda(v) = D_0(v) + O(\lambda^2), \]

with a dot on a function of one variable denoting differentiation with respect to that variable.

Using the boundary conditions on $U$ along with (3.10) and (3.11) we see that in the region $u > 0$, $v > 0$ we have

\[ U = -\log\{B_\lambda^2(u) + D_\lambda^2(v) - 1\}. \]

This holds for $\lambda \geq 0$. On account of (2.12) and (3.14) we can write this as

\[ U = -\log\{B_0^2(u) + D_0^2(v) - 1\} + O(\lambda^2), \]

which is sufficiently accurate for our purposes, although the exact expression (3.15) is also very useful to have. With $U$ given by (3.16) we can solve (3.7) for $V(u, v)$ to arrive at

\[ V = \frac{2\lambda}{\sqrt{B_0(u)^2 + D_0(v)^2} - 1} \{B_0(u) \alpha_0(u) \sin \frac{u}{\lambda} + D_0(v) \beta_0(v) \sin \frac{v}{\lambda}\}. \]

Since $B_0(0) = D_0(0) = 1$ we see that this $V$ matches $V$ in (3.2) and (3.3) on $v = 0$ and $u = 0$ respectively. Now we turn our attention to equation (3.8) to determine $M(u, v)$. Using (3.9) and (3.17) we can write this equation as

\[ 2 M_{uv} + U_{uv} = \frac{4B_0 D_0 \alpha_0 \beta_0}{(B_0^2 + D_0^2 - 1)} \cos \frac{u}{\lambda} \cos \frac{v}{\lambda} + O(\lambda), \]

for
from which we obtain
\[ 2M + U = \frac{4\lambda^2 B_0 D_0 \alpha_0 \beta_0}{(B_0^2 + D_0^2 - 1)} \sin \frac{u}{\lambda} \sin \frac{v}{\lambda} + F_\lambda(u) + G_\lambda(v) + O(\lambda^3) \tag{3.19} \]

where \( F_\lambda, G_\lambda \) are functions of integration. By (3.2) and (3.3) we must have \( M \) vanishing when \( u = 0 \) and when \( v = 0 \). Using the exact expression for \( U \) in (3.15) we find that
\[ F_\lambda(u) + G_\lambda(v) = -2 \log(B_\lambda(u)D_\lambda(v)) + O(\lambda^3) \tag{3.20} \]

Thus \( M \) is given by
\[ M = -\log \frac{B_\lambda D_\lambda}{\sqrt{B_\lambda^2 + D_\lambda^2 - 1}} + \frac{2\lambda^2 B_0 D_0 \alpha_0 \beta_0}{(B_0^2 + D_0^2 - 1)} \sin \frac{u}{\lambda} \sin \frac{v}{\lambda} + O(\lambda^3) \tag{3.21} \]

Now using (2.12) and (3.15) we can write this as
\[ e^{-M} = \frac{B_0(u)D_0(v)}{\sqrt{B_0^2(u) + D_0^2(v) - 1}} + O(\lambda^2) \tag{3.22} \]

We note that (3.22) is sufficiently accurate for substitution into the line–element (3.1) for \( u > 0, v > 0 \) but (3.21) is needed to verify that the field equation (3.8) is satisfied. Substitution of (3.16), (3.17) and (3.22) into (3.1)
gives the line–element of the space–time in the region \( u > 0, v > 0 \) after the collision of the waves as
\[ ds^2 = ds_0^2 - (B_0^2(u) + D_0^2(v) - 1) V (dx^2 - dy^2) + O(\lambda^2) \tag{3.23} \]

with \( V = O(\lambda) \) given by (3.17). This is a small perturbation of a background space–time with line–element
\[ ds_0^2 = -\{B_0^2(u) + D_0^2(v) - 1\} (dx^2 + dy^2) + \frac{2B_0(u)D_0(v)}{\sqrt{B_0^2(u) + D_0^2(v) - 1}} du dv \tag{3.24} \]

With \( U, V, M \) given by (3.15), (3.16), (3.17) and (3.22) we must now substitute these functions into the two remaining field equations (3.5) and (3.6).

We find that since \( B_\lambda(u) \) satisfies (2.9) and (2.12) we have (3.5) now automatically satisfied and (3.6) is automatically satisfied because \( D_\lambda(v) \) satisfies (3.13) and (3.14).

The Newman–Penrose components \( \Psi_A \) (\( A = 0, 1, 2, 3, 4 \)) of the gravitational field of the waves (the Riemann curvature tensor) in the post–collision
Thus for small $\lambda$ the field is dominated by two systems of waves, one described by $\Psi_0$ corresponding to waves with propagation direction $\partial/\partial u$ in the space–time and the other described by $\Psi_4$ corresponding to waves with propagation direction $\partial/\partial v$ in the space–time. It looks very like a straightforward superposition of waves traveling in opposite directions were it not for the presence of the factor $(B_0^2(u) + D_0^2(v) - 1)^{-1/2}$. Through this factor each wave system interferes with the other and a curvature singularity appears when

$$B_0^2(u) + D_0^2(v) = 1 .$$

(3.28)

This is where the two systems of ‘rays’ (the integral curves of the vector fields $\partial/\partial u$ and $\partial/\partial v$) converge. The appearance of a curvature singularity following the collision is to be expected [1].

The background space–time corresponding to the post–collision space–time region $u > 0, v > 0$ has line–element (3.24). This is not a vacuum space–time. Its Ricci tensor has components

$$R^{(0)}_{ij} = -\frac{\alpha_0^2(u) B_0^2(u)}{(B_0^2(u) + D_0^2(v) - 1)} u_i u_j - \frac{\beta_0^2(v) D_0^2(v)}{(B_0^2(u) + D_0^2(v) - 1)} v_i v_j .$$

(3.29)

This would be a simple superposition of the background Ricci tensors associated with the incoming high frequency gravitational waves were it not for the factor $(B_0^2(u) + D_0^2(v) - 1)^{-1}$ in each term. The Weyl tensor of this background has only one non–vanishing Newman–Penrose component

$$\Psi_2 = -\frac{B_0(u) D_0(v) \dot{B}_0(u) \dot{D}_0(v)}{(B_0^2(u) + D_0^2(v) - 1)^2} .$$

(3.30)

This is a type D Weyl tensor with $\partial/\partial u$, $\partial/\partial v$ as the two degenerate principal null directions. We see that (3.28) is a curvature singularity in this space–time also.
4 Collision of Electromagnetic Waves

We now consider the head–on collision of linearly polarized high frequency electromagnetic waves. In this case the line–element \( u \geq 0, v \leq 0 \) is taken to be the line–element \( (2.17) \) with \( B(u) = B_\lambda(u) \), where \( \dot{B}_\lambda + \phi_2^2 B_\lambda = 0 \) with \( \phi_2(u) \) given by \( (2.20) \). Thus in this region
\[
U = -2 \log B_\lambda(u), \quad V = M = 0, \quad \phi_2 = a_0(u) \cos \frac{u}{\lambda}. \tag{4.1}
\]
For \( u \leq 0, v \geq 0 \) we have
\[
U = -2 \log D_\lambda(v), \quad V = M = 0, \tag{4.2}
\]
and in this case the only non–vanishing Newman–Penrose component of the electromagnetic field is
\[
\phi_0(v) = b_0(v) \cos \frac{v}{\lambda}. \tag{4.3}
\]
As before we take the region \( u \leq 0, v \leq 0 \) to be flat space–time with \( U = V = M = 0 \) there. The functions \( B_\lambda(u), D_\lambda(v) \) for \( \lambda \geq 0 \) have the properties \( (2.12) \) and \( (3.14) \) with \( B_0(u) \) satisfying \( (2.22) \) and \( D_0(v) \) satisfying
\[
\ddot{D}_0 + \frac{1}{2}b_0^2 D_0 = 0. \tag{4.4}
\]
Also \( (3.11) \) and \( (3.12) \) continue to hold with \( \alpha_0, \beta_0 \) replaced by \( a_0, b_0 \) respectively.

We look for approximate solutions of the Einstein–Maxwell vacuum field equations in the post–collision region \( u > 0, v > 0 \). This involves determining \( U,V,M,\phi_0, \phi_2 \) for \( u > 0, v > 0 \) satisfying the boundary conditions: when \( u \geq 0, v = 0 \) these functions are given by \( (4.1) \) together with \( \phi_0 = 0 \), and when \( u = 0, v \geq 0 \) they are given by \( (4.2) \) and \( (4.3) \) along with \( \phi_2 = 0 \). The field equations to be satisfied are Einstein’s equations with an electromagnetic field as source:
\[
2 U_{uu} - U_u^2 - V_u^2 + 2 U_u M_u - 4 \phi_2^2 = O(\lambda), \tag{4.5}
\]
\[
2 U_{vv} - U_v^2 - V_v^2 + 2 U_v M_v - 4 \phi_0^2 = O(\lambda), \tag{4.6}
\]
\[
2 V_{uv} - U_u V_v = 0, \quad 2 U_u V_v = O(\lambda), \tag{4.7}
\]
\[
2 M_{uv} + U_u U_v - V_u V_v = O(\lambda), \tag{4.8}
\]
and
\[
U_{uv} = 0. \tag{4.9}
\]
together with Maxwell’s equations

\[ \frac{\partial \phi_0}{\partial u} - \frac{1}{2} U_u \phi_0 + \frac{1}{2} V_v \phi_2 = O(\lambda) \, , \quad (4.10) \]

\[ \frac{\partial \phi_2}{\partial v} - \frac{1}{2} U_v \phi_2 + \frac{1}{2} V_u \phi_0 = O(\lambda) \, . \quad (4.11) \]

The exact vacuum Einstein–Maxwell field equations are given by (4.5)–(4.11) with the \( O(\lambda) \)-terms replaced by zeros \([1]\). Just as in the gravitational case we conclude immediately that

\[ e^{-\nu} = B_\lambda^2(u) + D_\lambda^2(v) - 1 = B_0^2(u) + D_0^2(v) - 1 + O(\lambda^2) \, . \quad (4.12) \]

Working through the equations (4.5)–(4.8) and (4.10), (4.11) we find that in \( u > 0, \; v > 0, \)

\[ V = \frac{2 \lambda^2 B_0(u) D_0(v) a_0(u) b_0(v)}{B_0^2(u) + D_0^2(v) - 1} \sin \frac{u}{\lambda} \sin \frac{v}{\lambda} \, , \quad (4.13) \]

\[ \phi_0 = \frac{D_0(v) b_0(v)}{\sqrt{B_0^2(u) + D_0^2(v) - 1}} \cos \frac{v}{\lambda} \, , \quad (4.14) \]

\[ \phi_2 = \frac{B_0(u) a_0(u)}{\sqrt{B_0^2(u) + D_0^2(v) - 1}} \cos \frac{u}{\lambda} \, , \quad (4.15) \]

\[ e^{-M} = \frac{B_\lambda(u) D_\lambda(v)}{\sqrt{B_\lambda^2(u) + D_\lambda^2(v) - 1}} + O(\lambda^3) \, . \quad (4.16) \]

Clearly from (4.14) and (4.15) there are electromagnetic waves in this region traveling in opposite directions. They are not a superposition of the incoming electromagnetic waves however and they are singular on \( B_\lambda^2(u) + D_\lambda^2(v) = 1 \). The line–element of this post–collision region of the space–time is given by \( ds^2 = ds_0^2 + O(\lambda^2) \). This is in the form of a background having line–element \( ds_0^2 \) given again by (3.24) and a small second order perturbation. The Weyl tensor of this region of space–time has non–zero Newman–Penrose components

\[ \Psi_0 = \Psi_4 = \frac{B_0(u) D_0(v) a_0(u) b_0(v)}{B_0^2(u) + D_0^2(v) - 1} \sin \frac{u}{\lambda} \sin \frac{v}{\lambda} + O(\lambda) = O(\lambda^0) \, , \quad (4.17) \]

\[ \Psi_2 = -\frac{B_0(u) D_0(v) \dot{B}_0(u) \dot{D}_0(v)}{(B_0^2(u) + D_0^2(v) - 1)^2} + O(\lambda) = O(\lambda^0) \, . \quad (4.18) \]
Thus the collision of the electromagnetic waves has produced two systems of identical gravitational waves traveling in opposite directions together with a comparable magnitude ‘Coulomb’ term (4.18).

We can easily combine the results of this section with those of section 3. If the incoming high frequency, linearly polarized plane gravitational waves share their wave fronts with the high frequency, linearly polarized plane electromagnetic waves then the line–element in the region $u \geq 0$, $v \leq 0$ is given by (2.7) and (2.8) with $\phi_2$ given by (2.20). Now (2.9) is replaced by

$$\ddot{B}_\lambda + \left(\dot{w}_\lambda^2 + \dot{\phi}_2^2\right) B_\lambda = 0 . \quad (4.19)$$

As before we have $B_\lambda(u) = B_0(u) + O(\lambda^2)$ where now $B_0(u)$ satisfies

$$\ddot{B}_0 + \frac{1}{2}(\alpha_0^2 + a_0^2) B_0 = 0 . \quad (4.20)$$

In this case we have in the region $u \geq 0$, $v \leq 0$

$$U = -2 \log B_\lambda(u) \ , \ V = 2 \lambda \alpha_0(u) \sin \frac{u}{\lambda} , \ M = 0 , \ \phi_2 = a_0(u) \cos \frac{u}{\lambda} . \quad (4.21)$$

In the region $u \leq 0$, $v \geq 0$ we have

$$U = -2 \log D_\lambda(v) \ , \ V = 2 \lambda \beta_0(v) \sin \frac{v}{\lambda} , \ M = 0 , \ \phi_0 = b_0(v) \cos \frac{v}{\lambda} , \quad (4.22)$$

with $D_\lambda(v) = D_0(v) + O(\lambda^2)$ and $D_0(v)$ satisfying

$$\ddot{D}_0 + \frac{1}{2}(\beta_0^2 + b_0^2) D_0 = 0 . \quad (4.23)$$

Now the space–time in the post collision region has line–element (3.1) with $U$ given by (3.16) and

$$V = \frac{2 \lambda}{\sqrt{B_0^2(u) + D_0^2(v)} - 1} \left\{B_0(u) \alpha_0(u) \sin \frac{u}{\lambda} + D_0(v) \beta_0(v) \sin \frac{v}{\lambda}\right\} + \frac{2 \lambda^2 B_0(u) D_0(v) a_0(u) b_0(v)}{B_0^2(u) + D_0^2(v) - 1} \sin \frac{u}{\lambda} \sin \frac{v}{\lambda} , \quad (4.24)$$

$$\phi_0 = \frac{D_0(v) b_0(v)}{\sqrt{B_0^2(u) + D_0^2(v)} - 1} \cos \frac{v}{\lambda} - \frac{\lambda D_0(v) B_0(u) \beta_0(v) a_0(u)}{B_0^2(u) + D_0^2(v) - 1} \cos \frac{u}{\lambda} \sin \frac{v}{\lambda} + O(\lambda^2) , \quad (4.25)$$
and $M$ is given by (4.16). This approximate solution of the Einstein–Maxwell equations gives the post collision region of the space–time in such a form that all possible combinations of pairs of colliding systems of waves can be extracted as special cases. For example, the collision of high frequency, linearly polarized plane electromagnetic waves with high frequency, linearly polarized plane gravitational waves corresponds to putting $\alpha_0 = b_0 = 0$; after collision these would superimpose were it not for the factor $(B_0^2(u) + D_0^2(v) - 1)^{-1/2}$ appearing in $V$ and $\phi_2$.

5 Arbitrary Polarization

All of the foregoing can be generalized to allow the gravitational and electromagnetic waves to have the maximum two degrees of freedom of polarization. We have already pointed out following (2.18) how to introduce this generalization in the case of electromagnetic waves. If the plane gravitational waves have general polarization then the line–element (2.3) must be generalized to the Rosen–Szekeres form

$$ds^2 = -B^2 \{ (e^w \cosh q \, dx + e^{-w} \sinh q \, dy)^2 + (e^w \sinh q \, dx + e^{-w} \cosh q \, dy)^2 \} + 2 \, du \, dv , \tag{5.1}$$

with $B, w, q$ functions of $u$ only. How this arises is outlined in some detail in [7]. The Ricci tensor calculated with this metric tensor has components

$$R_{ij} = 2 \left( \ddot{B} \frac{\dot{B}}{B} + \dot{w}^2 \cosh^2 2q + \dot{q}^2 \right) u_i \, u_j , \tag{5.2}$$

and the only non–vanishing Weyl tensor component in Newman–Penrose notation is

$$\Psi_4 = -\ddot{w} \cosh 2q - 4 \dot{w} \dot{q} \sinh 2q - 2 B^{-1} \dot{B} \dot{w} \cosh 2q + i (\ddot{q} + 2 B^{-1} \dot{B} \dot{q} - \dot{w}^2 \sinh 4q) . \tag{5.3}$$

Following the Burnett [6] approach described in section 2 we specialize to high frequency plane waves by making the replacements $B(u) \to B\lambda(u), \; w(u) \to \lambda w(\lambda u)$.
\(w_\lambda(u), \ q(u) \rightarrow q_\lambda(u)\) with these functions satisfying the Einstein vacuum field equation
\[
\ddot{B}_\lambda + (\dot{w}_\lambda^2 \cosh^2 2q_\lambda + \dot{q}_\lambda^2) B_\lambda = 0 ,
\tag{5.4}
\]
and
\[
w_\lambda(u) + i q_\lambda(u) = \lambda \alpha_0(u) \sin \frac{u}{\lambda} ,
\tag{5.5}
\]
where now \(\alpha_0(u)\) is a complex-valued function of the real variable \(u\). Following the procedure outlined after equation (2.9) [see reference [7]] we have \(B_\lambda(u) = B_0(u) + O(\lambda^2)\) and \(B_0(u)\) satisfies
\[
\ddot{B}_0 + \frac{1}{2} |\alpha_0|^2 B_0 = 0 .
\tag{5.6}
\]
The line-element (2.13) is now replaced by
\[
ds^2 = ds_0^2 - 2 \lambda B_0^2(u) \sin \frac{u}{\lambda} \Re\{\alpha_0(u) (dx - idy)^2\} + O(\lambda^2) ,
\tag{5.7}
\]
where \(\Re\) denotes the real part of the quantity following it in brackets and \(ds_0^2\) is given by (2.14). The background Ricci tensor (2.15) reads now
\[
R_{ij}^{(0)} = -|\alpha_0(u)|^2 u_i u_j ,
\tag{5.8}
\]
while the gravitational field (2.16) is generalized to
\[
\Psi_4 = \frac{1}{\lambda} \bar{\alpha}_0(u) \sin \frac{u}{\lambda} + O(\lambda^0) ,
\tag{5.9}
\]
with the bar denoting complex conjugation. The fact that \(\Psi_4\) here is complex shows that we are dealing with waves having two degrees of freedom of polarization.

If we wish to include high frequency plane electromagnetic waves of arbitrary polarization sharing their wave fronts with these gravitational waves then such waves have a Maxwell field with one non-vanishing Newman–Penrose component
\[
\phi_2(u) = a_0(u) \cos \frac{u}{\lambda} ,
\tag{5.10}
\]
where now \(a_0(u)\) is a complex-valued function of the real variable \(u\). The equations (5.7)–(5.9) [the latter is now the Weyl conformal curvature tensor] continue to hold but (5.4) is replaced by
\[
\ddot{B}_\lambda + (\dot{w}_\lambda^2 \cosh^2 2q_\lambda + \dot{q}_\lambda^2 + |\phi_2|^2) B_\lambda = 0 ,
\tag{5.11}
\]
and (5.6) is replaced by
\[
\ddot{B}_0 + \frac{1}{2} (|\alpha_0|^2 + |a_0|^2) B_0 = 0 ,
\tag{5.12}
\]
and so \((5.8)\) becomes in this case

\[ R_{ij}^{(0)} = -(|\alpha_0|^2 + |a_0|^2) u_{,i} u_{,j} . \]  

(5.13)

The line–element in the post collision part of the space–time in this case is the generalized Rosen–Szekeres form

\[ ds^2 = -e^{-U}\{e^{V} \cosh W \, dx^2 - 2 \sinh W \, dx \, dy + e^{-V} \cosh W \, dy^2\} 
+ 2 e^{-M} \, du \, dv , \]

(5.14)

where \(U, V, W, M\) are functions of \(u, v\). This coincides with (3.1) when \(W = 0\). As initial data on \(u \geq 0, \, v = 0\) we have

\[ U = -2 \log B_\lambda(u) , \, V - i W = 2 \lambda \alpha_0(u) \sin \frac{u}{\lambda} , \, M = 0 \, , \]

(5.15)

and

\[ \phi_2 = a_0(u) \cos \frac{u}{\lambda} , \, \phi_0 = 0 \, . \]

(5.16)

The data on \(u = 0, \, v \geq 0\) is

\[ U = -2 \log D_\lambda(v) , \, V - i W = 2 \lambda \beta_0(v) \sin \frac{v}{\lambda} , \, M = 0 \, , \]

(5.17)

and

\[ \phi_0 = b_0(v) \cos \frac{v}{\lambda} , \, \phi_2 = 0 \, . \]

(5.18)

Here \(\beta_0, \, b_0\) are complex–valued functions and \(D_\lambda(v) = D_0(v) + O(\lambda^2)\) with \(D_0(v)\) satisfying

\[ \dot{D}_0 + \frac{1}{2}(|\beta_0|^2 + |b_0|^2) D_0 = 0 \, . \]

(5.19)

We note that \(B_\lambda(0) = D_\lambda(0) = 1 \) and \(\dot{B}_\lambda(0) = \dot{D}_\lambda(0) = 0 \) for \(\lambda \geq 0\) while \(\alpha_0(0) = \beta_0(0) = a_0(0) = b_0(0) = 0\).

Solving this initial value problem for the functions \(U, V, W, M\) of \((u, v)\) in the line–element \((5.14)\), and for the Maxwell field described by \(\phi_0(u, v)\) and \(\phi_2(u, v)\), in the region \(u > 0, \, v > 0\), we find that \(U\) is once again given by \((4.12)\) while

\[ V - i W = \frac{2 \lambda}{\sqrt{B_\lambda^2(u) + D_\lambda^2(v) - 1}} \{D_0(v) \beta_0(v) \sin \frac{v}{\lambda} + B_0(u) \alpha_0(u) \sin \frac{u}{\lambda}\} 
+ \frac{2 \lambda^2 B_0(u) D_0(v) \bar{a}_0(u) b_0(v)}{B_\lambda^2(u) + D_\lambda^2(v) - 1} \sin \frac{u}{\lambda} \sin \frac{v}{\lambda} \, , \]

(5.20)
the vacuum Einstein–Maxwell field equations \([1]\). Since these functions satisfy the initial data given above and satisfy approximately \([103, 497]\)

\[ds^{2} = ds_{0}^{2} - (B_{0}^{2}(u) + D_{0}^{2}(v) - 1) \Re\{Z (dx - i dy)^{2}\} + O(\lambda^{2}) , \]

with \(ds_{0}^{2}\) given by \([3.24]\). The Weyl tensor is a generalization of \((3.25)–(3.27)\) with the dominant Newman–Penrose components given by

\[\Psi_{0} = \frac{\lambda^{-1} \beta_{0}(v) D_{0}(v)}{\sqrt{B_{0}^{2}(u) + D_{0}^{2}(v) - 1}} \sin \frac{u}{\lambda} + O(\lambda^{0}) , \]

\[\Psi_{4} = \frac{\lambda^{-1} \alpha_{0}(u) B_{0}(u)}{\sqrt{B_{0}^{2}(u) + D_{0}^{2}(v) - 1}} \sin \frac{u}{\lambda} + O(\lambda^{0}) . \]
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