On the logical structure of Bell theorems

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Abstract. Bell theorems show how to experimentally falsify local realism. Conclusive falsification is highly desirable as it would provide support for the most profoundly counterintuitive feature of quantum theory—nonlocality. Despite the preponderance of evidence for quantum mechanics, practical limits on detector efficiency and the difficulty of coordinating space-like separated measurements have provided loopholes for a classical worldview; these loopholes have never been simultaneously closed. A number of new experiments have recently been proposed to close both loopholes at once. We show these novel designs fail in the most basic way, by not ruling out local hidden variable models, and we provide an explicit classical model to demonstrate this. These experiments share a common flaw, which reveals a basic misunderstanding of how nonlocality proofs work. Given the time and resources now being devoted to such experiments, theoretical clarity is essential. Our explanation is presented in terms of simple logic and should serve to correct misconceptions and avoid future mistakes.

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1. Introduction

Some specific predictions of quantum mechanics are inconsistent with local realism [1]. Either these predictions are false or else our world is not locally realistic. These predictions can be tested, as quantum mechanics is a physical theory; however they are hard to verify indisputably. A new kind of nonlocality proof has emerged in the recent literature, dubbed ‘EPR Bell inequalities’ [2] for their reliance on Einstein, Podolsky and Rosen’s criterion for the existence of elements of reality [3]. These proofs have received widespread attention for their ability to close the outstanding experimental ‘loopholes’ and dispel classical paranoia once and for all. Examples are the two-photon experiments proposed by Cabello [4, 5] and by Greenberger, Horne and Zeilinger [6, 7]. Considerable investment is also being made to experimentally realize these proofs [8]–[10].

All these proposals are flawed. They do not rule out the most general type of local theory, exposing an important misconception concerning the structure of nonlocality proofs. The shortcut they take necessarily introduces an additional assumption into the proof procedure, and however plausible this assumption may be it allows local realism to evade contradiction. Though our argument is based on simple reasoning we are not just splitting logical hairs. This flaw allows local models to pass these ‘nonlocality tests’ with flying colours, as we show by explicit construction. If progress is to be made towards understanding our fundamentally nonclassical world, it is of paramount importance we understand precisely the experimental evidence in favour of nonlocality. This is especially important in light of the considerable resources now being devoted to realizing loophole-free experiments. There is much value therefore in a detailed examination of the structure of nonlocality proofs, and in exposing a tempting shortcut as a logical, theoretical and experimental dead end.

We direct our attention to Cabello’s design for a loophole-free Bell experiment [4, 5]. Though our analysis is general we focus on this one example for clarity, because it is perhaps the most convincing of its class, and has been clearly presented on a number of occasions. In section 2, we recall the salient features of Cabello’s experimental proposal. In section 3, we study this purportedly ‘loophole-free’ two-party Bell experiment and provide a classical model that perfectly reproduces all of the observed correlations, showing the proof is not valid. We find the flaw lies in an unwarranted assumption about the nature of ‘local elements of reality’. Interestingly, although such assumptions are intuitively reasonable, they are fatal to nonlocality proofs.
2. Four-qubit nonlocality

‘Bell theorems without inequalities’, also called ‘nonlocality without inequalities experiments’, are promising candidates for a loophole-free local realism falsification. They identify sets of measurements upon an entangled system that produce a set of possible outcomes qualitatively different from any set of possible outcomes from any locally realistic model of the experiment. Were such an experiment performed many times with perfect apparatus, the list of recorded outcomes would quickly convince us whether our experiment was behaving in a locally realistic fashion or not. Standard Bell-inequality experiments, in contrast, have no sharp distinction between the sets of outcomes; rather, it is the frequency of certain outcomes that is inexplicable by local hidden variables.

Cabello presents two essentially identical nonlocality without inequalities experiments in a four-qubit setting [4, 5]. It is well known that entangled four-qubit systems can provide violations of local realism, and this system is no exception. Here we concisely recall the ingredients.

We consider a four-qubit state prepared upon two photons entangled in both their polarization \((H, V)\) and their path \((u, d)\) degrees of freedom

\[
|\psi\rangle = \frac{1}{2} (|Hu\rangle_A |Hu\rangle_B + |Hd\rangle_A |Hd\rangle_B + |Vu\rangle_A |Vu\rangle_B - |Vd\rangle_A |Vd\rangle_B).
\]  

(1)

Rewriting this explicitly as a four-qubit state, we have

\[
|\psi\rangle = \frac{1}{2} (|0\rangle_1 |0\rangle_2 |0\rangle_3 |0\rangle_4 + |0\rangle_1 |1\rangle_2 |0\rangle_3 |1\rangle_4 + |1\rangle_1 |0\rangle_2 |1\rangle_3 |0\rangle_4 - |1\rangle_1 |1\rangle_2 |1\rangle_3 |1\rangle_4).
\]  

(2)

Qubits 1 and 2 correspond to the polarization and path of Alice’s photon respectively, and likewise for qubits 3 and 4 for Bob. Now consider the following three measurements \(X_j, Y_j\) and \(Z_j\), performed individually on qubits \(j (j = 1 \ldots 4)\): \(X_j = |0\rangle_j \langle 1| + |1\rangle_j \langle 0|\), \(Y_j = i(|1\rangle_j \langle 0| - |0\rangle_j \langle 1|)\) and \(Z_j = |0\rangle_j \langle 0| + |1\rangle_j \langle 1|\). Each of these measurements has two possible outcomes which we label +1 and −1. Let the outcome of measurement \(X_j\) be written \(x_j \in \{+1, -1\}\), and similarly for \(Y_j\) and \(Z_j\). Quantum mechanics tells us that when appropriate measurements are made on state \(|\psi\rangle\), the following fourteen equalities will always be found to hold

\[
z_1 = z_3,
\]  

(3)

\[
z_2 = z_4,
\]  

(4)

\[
x_1 = x_3 z_4,
\]  

(5)

\[
x_2 = z_3 x_4,
\]  

(6)

\[
x_1 z_2 = x_3.
\]  

(7)

\[
z_1 x_2 = x_4.
\]  

(8)
There is no way to allot the values $-1$ and $+1$ to the twelve outcomes $\{x_j, y_j, z_j\}$ and satisfy all these equations simultaneously. A subset of just four equations, for instance (5), (9), (13) and (15) already leads to a contradiction. Therefore, any physical theory that demands these values be preassigned before the measurement choices $\{X_j, Y_j, Z_j\}$ are made is not consistent with quantum mechanics.

This inconsistency can indeed be exploited to obtain an all-versus-nothing nonlocality proof. We must be careful, however, that the measurements $\{X_j, Y_j, Z_j\}$ are performed in such a way that local realism requires the values $\{x_j, y_j, z_j\}$ be preassigned. This is easy to guarantee if the four qubits are space-like separated, but a complication arises when the four qubit state $|\psi\rangle$ is instantiated upon Cabello’s two-photon system. Qubits 1 and 2, the polarization and the path of Alice’s photon, clearly cannot be measured at space-like separation. The same clearly applies to Bob’s photon, so rather than making four independent qubit measurements chosen from three alternatives, we are really making two independent photon measurements chosen from nine alternatives: $X_1X_2$, $X_1Y_2$, $X_1Z_2$, $Y_1X_2$, $Y_1Y_2$, $Y_1Z_2$, $Z_1X_2$, $Z_1Y_2$ and $Z_1Z_2$. Cabello permits Alice and Bob to refrain from measuring one of their qubits, which leads to 9 + 6 = 15 possible local measurements, but this complication does not affect the analysis. These two measurements each have four possible outcomes: $\{(-1, -1), (-1, +1), (+1, -1), (+1, +1)\}$

There is no logical reason to assume that just because $x_1 = 1$ when $X_1X_2$ is measured, $x_1$ would have equalled 1 if we had measured $X_1Y_2$. Perhaps the different apparatus required to measure different path observables affects the photon’s observed polarization? If we want to rule out this possibility, we must design our experiment very carefully. Quantum mechanics may tell us these measurements are independent, but nothing prevents local hidden variables from disobeying this rule!
3. The logic of nonlocality proofs

The ‘nonlocality proof’ of section 2 works as follows. (Cabello’s two papers provide two different descriptions of essentially the same proof; for ease of reference we discuss only that formulated in [4], but our objection and counterexample apply equally to the equivalent formulation in [5].) Alice randomly chooses to perform one of the following two measurements

1a. \( X_1 \) and \( X_2 \)?
1b. \( X_3 \) and \( Y_4 \)?

2a. \( Y_1 \) and \( X_2 \)?
2b. \( X_3 \) and \( Z_4 \)?
3a. \( Y_3 \) and \( Y_4 \)?
3b. \( Y_3 \) and \( Z_4 \)?

The only relevant equations ever tested by this experiment are thus (5), (9), (13) and (15). Quantum mechanics predicts these equations will always be satisfied. For this to be a valid nonlocality proof, there must be no way for local hidden variables to achieve the same thing. Yet the following classical model does exactly that, and also manages to perfectly mimic the quantum measurement statistics!

Let \( \lambda_1, \lambda_2 \) and \( \mu \) be three independent random bits taking the values +1 or −1 with equal probability. These will be the local hidden variables of our classical model. Instead of two entangled photons, Alice and Bob share a two-part system each part of which carries a copy of \( \lambda_1, \lambda_2 \); Bob also has a copy of \( \mu \).

Alice’s part of the system behaves as follows—regardless of whether she performs measurement 1a or 2a, it will simply output ‘\( \lambda_1 \) and \( \lambda_2 \)’

1a. \( \rightarrow \lambda_1 \) and \( \lambda_2 \).
2a. \( \rightarrow \lambda_1 \) and \( \lambda_2 \).

Bob’s system produces the following measurement outcomes

1b. \( \rightarrow \mu \) and \( \mu \lambda_1 \lambda_2 \).
2b. \( \rightarrow \mu \) and \( \mu \lambda_1 \).
3b. \( \rightarrow \mu \) and \( \mu \lambda_1 \lambda_2 \).
4b. \( \rightarrow \mu \) and \( -\mu \lambda_1 \).

It is easy to see that in perfect agreement with quantum mechanics, the result of each individual ‘qubit’ measurement is completely random, yet the global correlations of equations (5), (9), (13) and (15) always hold. This local model is, in the context of this experiment, utterly indistinguishable from quantum mechanics itself, and this needs just two shared random bits and one private random bit to achieve. Since the experiment admits a simple locally realistic explanation, it cannot falsify local realism!

It is argued in [4, 5] that the nonlocality proof succeeds regardless, because local models such as this are forbidden. It is claimed Bob must always give the same answer to questions such
as ‘What is \(z_4\)’, regardless of the context in which that question is asked: ‘Since \(z_4\) represents a local element of reality, Bob’s answer to \(Z_4\) must be independent on whether \(Z_4\) is asked together with \(Y_3\) or \(X_3\)’ (emphasis added). This is exactly the misconception at the heart of this and other recent proposals for ‘improved’ nonlocality proofs. We must not make any assumptions about what constitutes a local element of reality! Any alleged proof that spends any time whatsoever establishing ‘what the local elements of reality must be’ is likely to be wrong, or, at the very least, not as general as it should be.

Nonlocality proofs share a simple logical structure: they are proofs by contradiction. Two assumptions are made—the assumption of locality and the assumption of realism—and a conclusion drawn concerning the possible outcomes of measurements upon causally unconnected systems. This conclusion is false if the predictions of quantum mechanics for entangled states are true. When these predictions are experimentally verified, the conclusion is experimentally refuted, and therefore at least one of our two premises must have been false.

The new model for nonlocality proofs has a different two-step structure. In the first step, some predictions of quantum mechanics for the behaviour of a specific state \(|\psi\rangle\) under a specific set of possible measurements \(\{X_j, Y_j, Z_j\}\) are assumed to be true. In the proposed model [4, 5], it is assumed that pairs of measurements upon different qubits encoded on the same photon are outcome independent; the outcome of the measurement on qubit 1 is shown to be independent of the choice of measurement on qubit 2, and vice versa. From this premise a preliminary conclusion is drawn concerning the nature of viable local hidden-variable models. In the second step, locality, realism, and the conclusion of the first step are assumed, and a deduction is made concerning the possible measurement outcomes. It is then shown that this conclusion is false if some other predictions of quantum mechanics for the state \(|\psi\rangle\) are true (to be specific, equations (5), (9), (13) and (15)).

The problem with this two-stage approach should be apparent. When we conduct the experiment presented in section 2 using two photons, our logical conclusion will be shown to be inconsistent with observable evidence. We can deduce that at least one of the premises of our overall argument must have been false. However, the proposed new type of nonlocality proof has a total of three premises, not two! In addition to locality and realism, it is assumed from the outset that individual qubit measurement outcomes to represent ‘local elements of reality’ (as defined by Einstein, Podolsky and Rosen) and must be assigned definite values. The proposed nonlocality proof never tests to see if this assumption is true for the system and measurements in question. Thus the ensuing experiment will not rule out local realism. The third assumption can act as a ‘logical shield’, protecting locality and realism from contradiction. (It must also be noted that the term coined by Cabello, ‘Einstein, Podolsky, Rosen local elements of reality’ or EPRLERs, is misleading. Einstein, Podolsky and Rosen never put forth a definition of a LER but only offered a criterion to recognize one [3]; they explicitly allowed for the possibility of other models.)

There is nothing logically invalid about using three assumptions, instead of just the two. We certainly do not reject the third premise because we are forbidden from making spurious and unsupported assumptions about the properties of reality. After all, the assumptions of locality and realism are (surprisingly!) poorly physically motivated, whereas Cabello’s additional assumption is experimentally verifiable. Ultimately, we can make any assumptions we want, but the conclusion we will end up drawing is that ‘one of our assumptions must be wrong’. If we want to rule out local realism, we had better not have any additional assumptions in the way that can act as sacrificial pawns. If we have assumed some quantum predictions without testing...
them, logic dictates that these predictions might be wrong, however reasonable they seem. In this case, the application of logic may appear physically counterintuitive: an implicit assumption that quantum mechanics describes what is really physically happening leads to a proof with a classical solution! Nevertheless the logic is indisputable: the classical model is extremely simple and perfectly reproduces the supposedly nonlocal quantum correlations; an experiment with a classical explanation cannot prove nonclassicality. This highlights the value of proper logical analysis. The existence of an additional necessary assumption can be used as a test for the possibility of a local hidden variable solution, saving one the effort of exhaustively constructing new local models of every specific case.

It is important to be very clear about our reasons for rejecting the additional assumption, so let us reiterate one last time. It is fatal to include an additional assumption in nonlocality proofs, even if that assumption is known to be true for quantum mechanics, because doing so can open the door to classical models for which that additional assumption is false. Cabello’s errant assumption is surely true, as it is a mathematical property of quantum mechanics. Nevertheless, when the validity of quantum mechanics itself is at issue, it is a mistake to assume it and not test it, as must be clear from the simple counterexample presented above—quantum assumptions have led to a classical solution.

How do we do things right? We must get rid of the additional assumption. We can redesign our experiment such that in parallel with everything else, it actually tests whether all the predicted behaviours of the quantum state $|\psi\rangle$ under measurements $\{X_j, Y_j, Z_j\}$ are observed, both equations (5), (9), (13), (15) and the independence of separate qubit measurements. This revision guarantees the only assumptions that might be false are locality and realism. Testing additional predictions means we will have to ask Alice and Bob to perform some additional measurements. It is exactly these measurements that Cabello adds to his original experiment in order to create a valid nonlocality proof in his recent response to criticism [11]. (Of course the validity of the extended experiment was never questioned, and does not imply the validity of the original smaller experiment. Half a valid proof is no proof at all.) However, the original proposal explicitly avoided these additional tests to reduce the supposed maximum classical success rate to $3/4$ and ease the burden on the photon detectors. As we have shown, this was unsuccessful. The valid extended experiment works because it tests all fourteen equations (3) to (16). A local hidden variable model can reproduce these correlations with probability at most $13/14$, significantly worse than other two-party proposals [12], and thus is not ‘stronger’ or ‘loophole-free’ in any meaningful sense.

There is a different way to make the original experiment valid. We can abandon the two-photon instantiation of $|\psi\rangle$ and consider four space-like separated qubits. The resulting experiment does not yield a better experimental proposal than the pseudotelepathy game of Greenberger, Horne and Zeilinger [13] and Mermin [14] if we are concerned with closing the detection loophole or with minimizing the number of participants.

4. Conclusion

We have shown that a conceptual error in the design of nonlocality proofs can be fatal to the ultimate goal of such a proof, which is to demonstrate that our world is not locally realistic. More precisely, we have elucidated why the description of a good nonlocality proof can (and should) be given without any discussion of quantum mechanics or the nature of local elements

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of reality. It is only the actual experimental setup, or the quantum winning strategy that needs to invoke quantum mechanics. We have shown that doing otherwise can fatally compromise the conclusions that can be drawn from nonlocality proofs. Because the two-participant nonlocality proofs of Cabello \([4, 5]\) need to invoke quantum predictions as assumptions, we conclude that these proposals do not achieve their purported goal of ruling out locally realistic descriptions of our world, in spite of the fact that they do rule out some small subclass of classical models. The same objection dooms all nonlocality proofs of this type.

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