Bloch band dynamics of a Josephson junction in an inductive environment

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We have measured the current-voltage characteristics of a Josephson junction with tunable Josephson energy $E_J$ embedded in an inductive environment provided by a chain of SQUIDs. Such an environment induces localization of the charge on the junction, which results in an enhancement of the zero-bias resistance of the circuit. We understand this result quantitatively in terms of the Josephson effect in circuits containing large Josephson energy $E_J > E_C$. Therefore, they could be used in principle in quantum metrology to realize a current standard, analogous to the way Josephson oscillations are used to realize a voltage standard.

A possible strategy to obtain long charge relaxation times consists of integrating a highly resistive element with Josephson junctions with a well-defined charge state. This is due to the macroscopic coherence of the superconducting wave function. In Josephson junction (JJ) circuits, the phase difference $\dot{\phi}$ between the two superconductors forming the junction and $\dot{N}$ the number of Cooper pairs that tunnel through the junction are quantum conjugate variables satisfying $[\dot{\phi}, \dot{N}] = i$. The relative strength of the fluctuations is proportional to $\sqrt{E_C/E_J}$ and the Josephson energy $E_J$, $\Delta \phi/\Delta N \sim \sqrt{E_C/E_J}$. Therefore, Josephson junctions ($E_J > E_C$) enable the control of a well-defined dissipationless phase state. This feature allowed the realization of the metrological Volt standard using networks of JJ. Decreasing the junction size into the submicron region decreases the ratio $E_J/E_C$ and, as a consequence the quantum fluctuations of the phase, $\Delta \phi$, start to play a dominant role. Large quantum phase fluctuations can produce $\Delta \varphi \approx 2\pi$ windings, a phenomenon known as quantum phase-slips. Since increasing phase fluctuations implies decreasing fluctuations of its conjugate variable, the charge, the resulting well-defined charge state is expected to yield insulating behaviour of the junction. However, such a state is not easily observed for a single Josephson junction, which is typically measured in a superconducting low-impedance environment. The associated charge relaxation time $\tau_q$ is too short to preserve the well-defined charge state on the measurement time scales.

The relevance of Josephson junction circuits with a well-defined charge state has been pointed out in pioneering articles by Averin, Likharev and Zorin. Indeed, Josephson junctions with a well-defined charge should sustain Bloch oscillations, consisting of voltage oscillations on the junction due to a periodic motion of the charge in the lowest Bloch band. These Bloch oscillations are dual to the standard Josephson oscillations. Therefore, they could be used in principle in quantum metrology to realize a current standard, analogous to the way Josephson oscillations are used to realize a voltage standard.

I. INTRODUCTION

Superconductors provide the unique possibility to create dissipationless macroscopic electrical quantum circuits that are characterized by the dynamics of a well-defined single degree of freedom, the superconducting phase. This is due to the macroscopic coherence of the superconducting wave function. In Josephson junction (JJ) circuits, the phase difference $\dot{\phi}$ between the two superconductors forming the junction and $\dot{N}$ the number of Cooper pairs that tunnel through the junction are quantum conjugate variables satisfying $[\dot{\phi}, \dot{N}] = i$. The relative strength of the fluctuations is proportional to the square root of the ratio of the charging energy $E_C$ and the Josephson energy $E_J$, $\Delta \phi/\Delta N \sim \sqrt{E_C/E_J}$. Therefore, Josephson effect in circuits containing large Josephson junctions ($E_J \gg E_C$) enables the control of a well-defined dissipationless phase state. This feature allowed the realization of the metrological Volt standard using networks of JJ. Decreasing the junction size into the submicron region decreases the ratio $E_J/E_C$ and, as a consequence the quantum fluctuations of the phase, $\Delta \phi$, start to play a dominant role. Large quantum phase fluctuations can produce $\Delta \varphi \approx 2\pi$ windings, a phenomenon known as quantum phase-slips. Since increasing phase fluctuations implies decreasing fluctuations of its conjugate variable, the charge, the resulting well-defined charge state is expected to yield insulating behaviour of the junction. However, such a state is not easily observed for a single Josephson junction, which is typically measured in a superconducting low-impedance environment. The associated charge relaxation time $\tau_q$ is too short to preserve the well-defined charge state on the measurement time scales.

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A possible strategy to obtain long charge relaxation times consists of integrating a highly resistive element with Josephson junctions with a well-defined charge state. This is due to the macroscopic coherence of the superconducting wave function. In Josephson junction (JJ) circuits, the phase difference $\dot{\phi}$ between the two superconductors forming the junction and $\dot{N}$ the number of Cooper pairs that tunnel through the junction are quantum conjugate variables satisfying $[\dot{\phi}, \dot{N}] = i$. The relative strength of the fluctuations is proportional to the square root of the ratio of the charging energy $E_C$ and the Josephson energy $E_J$, $\Delta \phi/\Delta N \sim \sqrt{E_C/E_J}$. Therefore, Josephson effect in circuits containing large Josephson junctions ($E_J \gg E_C$) enables the control of a well-defined dissipationless phase state. This feature allowed the realization of the metrological Volt standard using networks of JJ. Decreasing the junction size into the submicron region decreases the ratio $E_J/E_C$ and, as a consequence the quantum fluctuations of the phase, $\Delta \phi$, start to play a dominant role. Large quantum phase fluctuations can produce $\Delta \varphi \approx 2\pi$ windings, a phenomenon known as quantum phase-slips. Since increasing phase fluctuations implies decreasing fluctuations of its conjugate variable, the charge, the resulting well-defined charge state is expected to yield insulating behaviour of the junction. However, such a state is not easily observed for a single Josephson junction, which is typically measured in a superconducting low-impedance environment. The associated charge relaxation time $\tau_q$ is too short to preserve the well-defined charge state on the measurement time scales.

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Another possible strategy to obtain long charge relaxation times is the additional integration of a large inductive environment. We demonstrate that they result in a renormalization of the Josephson coupling energy
\[ J \sim E_C \]
and with gaps
\[ E_{\text{gap}} \sim E_J. \]
If \( E_J/E_C \) is large, the bands are sinusoidal with a width \( \Delta_0 \approx (E_J^2/E_C)^{1/4} e^{-\sqrt{8E_J/E_C}} \)
and with gaps \( E_{\text{gap}} \sim \hbar \omega_p \), where \( \omega_p = \sqrt{8E_JE_C}/\hbar \) is the junction’s plasma frequency. As long as LZ transitions between the bands can be ignored, the behavior of the junction is completely governed by the properties of the lowest band. Outside the parameter range \( E_J \gg E_C \), the bands are not sinusoidal and the dependence of the lowest band \( U_0 \) on the quasi-charge \( q \) is given by
\[ U_0(q) = \sum_{k=1}^{\infty} v_k \cos(k\pi q/e). \]

The non-sinusoidal nature of the bands is reflected by the summation over higher harmonics with index \( k \), which corresponds to the quantum phase slip processes in the junction where the phase winds by an amount \( 2\pi k \). The energies \( v_k \) are the amplitudes for these multiple windings to occur. By embedding the Josephson junction in an inductive environment, it is possible to induce dynamics of the quasi-charge \( q \) in the lowest band. The characteristic kinetic energy is then given by
\[ E_L = (\Phi_0/(2\pi))^2/L, \]
where \( \Phi_0 \) is the superconducting flux quantum. The full Hamiltonian for the corresponding circuit, shown in Fig. 1, for a voltage-biased configuration, is given by
\[ H = -\left( \frac{\hbar^2}{2L} \right) \frac{\partial^2}{\partial q^2} + U_0(q) - V_{\text{bias}}q. \]

The inductance plays the role of the mass of a fictitious particle with coordinate \( q \), moving within the potential energy \( U_0(q) \). We denote \( \Delta_0 \) the barrier height separating the minima of the potential \( U_0(q) \). In the tight-binding limit, \( E_L \ll \Delta_0 \), at vanishing voltage bias \( V_{\text{bias}} = 0 \), we can use the harmonic approximation for the potential \( U_0(q) \approx q^2/(2C_q) \) where the effective capacitance is defined as
\[ C_q^{-1} = (\partial^2 U_0/\partial q^2)^{1/2}. \]
In this case, the ground-state wave function is a Gaussian whose width equals to \( \Delta^2 = (e^2/\pi)R_Q/\sqrt{L/C_q} \) with \( R_Q = \hbar/(4e^2) \approx 6.45 \) kΩ. The ground-state energy equals \( \hbar \omega_q/2 \), where
\[ \omega_q = 1/\sqrt{LC_q} \]
is the associated plasma frequency.

In our experiment we can change the ratio \( E_J/E_C \) of the Josephson junction in situ for a given inductance \( L \), thus
changing the barrier height $\Delta_0$, and explore in particular the regime $\Delta_0 > h\omega_p/2$ where a localized charged-state is expected to appear, see Fig. 1b. We have measured the zero-bias resistance as a function of the ratio $E_J/E_C$ and analyze the result in terms of a possible charge localization.

III. SYSTEM

The experiments were performed on the circuit shown in Fig. 2. In order to realize a Josephson junction with tunable $\Delta_0$, we designed it in the form of a SQUID such that the ratio $E_J/E_C \propto |\cos \pi \Phi/\Phi_0|$, where $\Phi$ is the magnetic flux threading the SQUID loop. For future use we define the flux frustration parameter $f = \Phi/\Phi_0$. The junction capacitance is estimated from the design to be $C = 6.9 \text{ fF}$ such that $E_C/h = 2.8 \text{ GHz}$. The series inductance consists of a linear chain of $N$ SQUID junctions, each characterized by a Josephson inductance $L_{SO} = 10 \text{ nH}$ estimated from resistance measurements at $T = 1.5 \text{ K}$. This chain provides therefore a total inductance of $L = NL_{SO}$. This inductance is also flux-dependent. The SQUID loops forming the chain are 1.6 times smaller than the SQUID loop forming the tunable junction. The inductance of the chain $L$ changes only by 10% in the small flux frustration range $f = [0.46, 0.5]$ where the tunable junction is probed, so that in first order it can be considered as constant. We fabricated chains containing a different number $N$ of junctions $N = 28, 38, 48, 68, 88$ and 108\text{a}. The capacitance of the SQUIDs in the chain is equal to the one forming the tunable junction. The capacitance to ground of the islands between the SQUIDs is estimated to be $C_0 \simeq C_{SO}/75$. The sample is shunted by a home-made $NbTi/AlO_3/NbTi$ parallel capacitor $C_{sh} = 200 \text{ pF}$. This shunt capacitor, together with a resistance $R = 3 \text{ k}\Omega$, provides a low-pass filter with a cut-off frequency of about 60 MHz, thus protecting the junction and its inductive environment from spurious noise above this frequency. The measurements were carried out in a two-point configuration using high-frequency filters in form of thermocoax cables. In the small region at $f = [0.46, 0.5]$, where we will analyse our data, the circuit is effectively voltage biased. The sample is kept at the base temperature of about 50 mK of our dilution refrigerator.$^{25}$

IV. EXPERIMENTAL RESULTS AND QUALITATIVE DISCUSSION

We start our discussion by focusing on the results obtained for the single junction in series with the 48 junction chain as an example to illustrate the results. Fig. 3a shows the zero-bias resistance of this circuit as a function of the flux frustration parameter $f$. First of all we note that the resistance is a few k\Omega for zero frustration. We attribute this finite resistance value to the occurrence of residual incoherent quantum phase-slips in the junctions forming the chain. We can estimate the rates$^{26}$ for a phase winding by $\pm 2\pi$ for a given junction as $\Gamma_{qps}^{\pm} = A^{\pm} e^{-B^{\pm}}$, where

$$A^{\pm} = 12\sqrt{6\pi} \frac{\omega_p}{2\pi} \sqrt{\frac{\Delta U^{\pm}}{h\omega_p}}, B^{\pm} = \frac{36}{5} \frac{\Delta U^{\pm}}{h\omega_p}. \quad (6)$$

Here, $\Delta U^{\pm} \sim 2E_J - h\omega_p/2 \pm \pi h/2e$ is the effective barrier for tunnelling of the phase of the junction by $\mp 2\pi$. Since the temperature is lower than the plasma frequency $\omega_p/2\pi = 25.4 \text{ GHz}$, thermal activation can be ignored. Quantum phase-slips give rise to a voltage $V = h(\Gamma_{qps}^{+} - \Gamma_{qps}^{-})/2e$. Linearizing the rates $\Gamma_{qps}^{\pm}$ with respect to the bias current $I$ we find $V = R_{qps}I$ with the following estimate for the zero-bias resistance:

$$R_{qps} \simeq R_Q \frac{36\pi}{5} \frac{\Gamma_{qps}}{\omega_p}. \quad (7)$$

Using the parameters of the experiment, we find a residual resistance per junction of $\approx 65 \text{ }\Omega$. For a chain with 48 junctions this indeed yields a total residual resistance of 3.1 k\Omega. This zero-bias resistance at small values of the frustration changes linearly with the number $N$ of the junctions (see inset of Fig 3b). From the experimental $N$ dependence we extract a resistance of 59 $\Omega$ per junction.

Upon increasing the frustration $f$, we observe a significant increase of the zero-bias resistance up to 4 M\Omega reached for the maximal frustration $f = 0.5$. Fig. 3b shows the current-voltage characteristics, taken at the two flux frustrations corresponding to the onset of the resistance increase ($f = 0.42$, left arrow in Fig. 3b) and at its maximum ($f = 0.5$, right arrow in Fig. 3b). We understand this resistance increase by three orders of magnitude as a result of the enhanced charge localization. As the flux frustration $f$ is increased, the ratio $E_J/E_C$ of the junction decreases, thereby increasing the width $\Delta_0$ of the lowest Bloch band. As soon as the bandwidth $\Delta_0$ exceeds the kinetic energy $E_k$ due to the inductance, the charge becomes more and more localized. More precisely, in this limit the dual plasma frequency becomes
An estimation of the ratio $\Delta_0/E_L = 0.6$ for our quantum phase-slip junction results in a situation where only one level, with energy $\hbar \omega_q/2$, is located in the potential $U_0(q)$, see Fig. 4.

At higher voltages, the IV characteristic for $f = 0.5$ shows a current peak that we attribute to the existence of electromagnetic modes in the chain at higher frequencies. The effect of these modes on the behaviour found at low bias will be discussed in detail in Sec. V.

In Fig. 3, the temperature dependence of the zero-bias resistance is shown for $f = 0.5$. We see that with decreasing temperature $T$, the zero bias resistance increases, down to a temperature of about 80 mK. At lower temperatures, $R_0$ saturates. The effect of finite temperatures is to induce thermal fluctuations kicking the charge particle out of the minima. The gradual decrease of $R_0$ with increasing $T$ is therefore expected: charge localization is more pronounced at lower temperatures. The low-temperature saturation is most probably a result of the existence of a finite environment noise-temperature of about 80 mK. We estimated that quantum tunneling can still be ignored at this temperature. Similar findings were reported in reference 6.

V. CHARGE DIFFUSION

In order to account for the dependence of the measured zero-bias resistance on the flux frustration parameter $f$, we start by analyzing the simplest possible model describing the dynamics of the quasi-charge $q$, assuming its dynamics to be restricted to the lowest charge band. This amounts to ignoring Landau-Zener (LZ) transitions. We will come back to this assumption in section VII below. For now we consider the limit of low voltage bias and low temperature, $eV, k_B T < \Delta_0$. We include the effect of a small tilt due to finite bias, see Fig. 4. Classically, the charge will be localized in one of the minima of the band. This would give rise to a zero current state. However, driven by thermal or quantum fluctuations, the particle can hop between neighbouring minima. In the presence of a finite bias voltage, the rate $\Gamma^{-}$ for hopping from right to left differs from $\Gamma^{+}$ corresponding to hopping from left to right, giving rise to a finite current $I = 2e(\Gamma^{+} - \Gamma^{-})$. We calculate the rates $\Gamma^{\pm}$ using Kramers’s classical result for the escape of a particle from a potential well. We assume thermal activation to be dominant as the temperature is in the same orders of magnitude as the dual plasma frequency $\omega_q/2\pi = 4$ GHz so that the rates $\Gamma^{\pm}$ can be expressed as

$$\Gamma^{\pm} = \frac{\omega_q^2 e^{-E^{\pm}/k_B T}}{2\pi}. \quad (8)$$

Here we used the attempt frequency $\omega_q$, and $E^{\pm} = \Delta_0 \mp eV$ denote the barrier height for tunneling to the left and the right of the well in the presence of the tilt $V$ (see also Fig. 4). We recall that the parameter $\Delta_0$ depends on the
The current flowing through the system is

\[ R = \frac{R_0}{\Delta_0} e^{\Delta_0/\kappa_B T}. \]  \( \text{(9)} \)

In Fig. 5 the measured zero-bias resistance \( R_0 \) is shown as a function of the parameter \( E_J/E_C \) for the \( N = 48 \) junction chain, together with a fit based on Eq. (9). Similar fits have been performed for all the measured chains. These fits enable us to compare the behaviour of the fitted exponent \( \Delta_0/\kappa_B T \) as a function of the ratio \( E_J/E_C \) with the theoretically expected one \( \Delta_0 \), determined by solving the Mathieu equation corresponding to Hamiltonian (4) using the system’s parameters. From the fits we find that the actual values of the exponent and the prefactor differ greatly from the expected one. Specifically, we find that good fits can be obtained using a barrier height \( \Delta_0^{fit} = \alpha \Delta_0 \) that is \( \alpha \) times higher than the expected value \( \Delta_0 \). In Table I we list the multiplicative factors \( \alpha \) for the other samples together with the prefactor \( \omega_0^{fit} \) obtained by fitting and the corresponding prefactor expected from theory \( \omega_q \). Note that the factor \( \alpha \) is an almost monotonically increasing function of the chain length \( N \). We obtain values ranging from \( \alpha = 1.7 \) to \( \alpha = 6.3 \). As will be argued in the following, we can attribute this discrepancy between the experimentally found exponent and the theoretically expected one to two phenomena: (i) the renormalization of the Josephson coupling energy of the small junction due to electromagnetic modes propagating along the chain and (ii) the effect of interband transitions (LZ processes) that dominate the charge dynamics whenever the gap \( \sim E_J \) separating the lowest two charge bands becomes too small compared to the characteristic energy of the dynamics of the quasi-charge.

### TABLE I. Fitting parameters and errors for all measured chains. \( \alpha \) is the multiplicative factor in front of \( \Delta_0 \) used when fitting with Eq. (9).

| \( N \) | \( \alpha \) | \( \omega_0^{fit}/E_C \) | \( \omega_q/E_C \) |
|---|---|---|---|
| 28 | 1.7 ± 0.1 | 0.61 ± 0.02 | 1.17 |
| 38 | 4.1 ± 0.1 | 7.0 ± 0.4 | 1.01 |
| 48 | 4.6 ± 0.2 | 1.7 ± 0.5 | 0.99 |
| 68 | 5.4 ± 0.2 | 17 ± 4 | 0.84 |
| 88 | 6.3 ± 0.2 | 52 ± 12 | 0.74 |
| 108 | 5.5 ± 0.2 | 22 ± 7 | 0.67 |

VI. RENORMALIZATION OF THE BANDWIDTH

It is well-known that the Josephson coupling energy \( E_J \) of a Josephson junction connected to an external circuit is suppressed down to a value \( E_J^* \) by the quantum fluctuations induced by the corresponding environment.\(^{30-32} \) As a consequence, the bandwidth \( \Delta_0^* \) found for the lowest charge band will be larger than the bare width \( \Delta_0 \). Referring again to the measurement circuit as shown in Fig. 2 we first note that the junction is well-protected from high-frequency noise originating from the external leads by thermo coax and the RC filter down to 60 MHz. Since the resistance measurements were performed at flux frustration parameters \( f \) corresponding to a bare Josephson coupling energy in the GHz range, we suppose that the external noise does not account for any renormalization of \( E_J \). However, the chain providing the inductance is directly coupled to the tunable junction. The fact that the islands realize a finite capacitance \( C_0 \) to ground leads to the appearance of dispersive electromagnetic modes in the chain,\(^{20,33,34} \) with a dispersion
FIG. 6. Dispersion of the propagating modes on Josephson junction chains with a finite capacitance to ground \( C_0/C_SQ = 1/50 \).

relation given by

\[
\omega_k = \omega_p \sqrt{\frac{1 - \cos k}{1 - \cos k + C_0/2C_SQ}},
\]

where \( k = 2\pi n / N \) are the discrete wave-vectors of these modes. Here we assumed periodic boundary conditions as we do not know the exact boundary conditions in our experiment. The dispersion relation is shown in Fig. 6. It consists of a linear part, \( \omega_k = \omega_0 k \) for small \( k \)-vectors, with \( k < \sqrt{C_0/C_SQ} \). Here \( \omega_0 = 1/\sqrt{LC_0} \). The frequency \( \omega_l \) of the lowest mode is inversely proportional to the chain length \( N \). For the longest chains measured, this frequency is estimated to be about 40 % of the chain’s plasma frequency \( \omega_p \). For larger wave-vectors, the dispersion relation saturates at the plasma frequency \( \omega_p \). In units of temperature, the frequency range between \( \omega_l \) and \( \omega_p \) covered by the modes corresponds to a range between 300 mK and 1 K. Since the temperature at which the experiment is performed is lower than 300 mK, a zero-temperature treatment of the modes is adequate. The equivalent voltage range is between 30 \( \mu V \) and 100 \( \mu V \), which corresponds to a range where the current peak is found in the IV characteristic, see Fig. 3b, thereby providing indirect evidence for the existence of the modes. In this relatively high bias voltage range, Cooper pair transfer in the small junction is possible as the modes of the chain provide the necessary dissipation.

In the limit of small voltage biases, as it is argued in Appendix A, the modes induce zero point quantum phase fluctuations at the end of the chain that couple to the small junction. They add to the phase difference \( \phi \) across the junction, thereby renormalizing the Josephson coupling energy \( E_J \) of the tunable junction down to a value

\[
E_J^* = E_J \exp \left\{ -\frac{1}{N} \sum_k \frac{e^2}{\hbar} \sqrt{\frac{L}{2C(k)(1 - \cos k)}} \right\}. \tag{10}
\]

Using this result, we calculate the renormalized effective bandwidth \( \Delta_0^* \), corresponding to the spectrum of Hamiltonian (11). Fig. 7 shows the dependence of the ratio \( \Delta_0^* / \Delta_0 \) as a function of \( f \), for the \( N = 48 \) junction chain. We see that this ratio is larger than unity and a monotonically decreasing function of \( f \), reaching unity at \( f = 1/2 \), where the Josephson coupling energy vanishes and the bandwidth attains its largest possible value \( E_C \). The \( N \)-dependence of the ratio \( \Delta_0^* / \Delta_0 \) is plotted in Fig. 7b, for a fixed value of \( f = 0.46 \). Indeed, it is a monotonically increasing function of \( N \), however, even for \( N = 108 \) it does not exceed a value of about 1.7, and therefore can account only in part for the observed discrepancy discussed in section V above. In the next Section we will
show that LZ processes between the lowest charge band and the higher ones may account for the rest of the discrepancy.

VII. CHARGE DIFFUSION IN THE PRESENCE OF LANDAU-ZENER PROCESSES

In this section we extend the charge diffusion model presented in Sec. III and include the effect of possible interband transitions ignored so far. The probability to pass from the lowest charge band to the next one is given by

$$P_Z = \exp \left\{ -\frac{\pi^2}{4} \frac{E_{\text{gap}}^2}{E_C \hbar \omega_x} \right\},$$

where $E_{\text{gap}} \approx E_J$ is the gap separating the two bands and $\omega_x$ the relevant frequency associated with the dynamics of the quasi-charge. At zero voltage bias, this frequency will be approximately given by the attempt frequency $\omega_q$ at which the quasi-charge tries to escape by thermal activation from the well formed by the minima of the lowest band. The probability to remain in the lowest band is given by $1 - P_Z$. We then expect the measured resistance $R_m$ to be given by the weighted sum

$$R_m = (1 - P_Z)R_0 + P_Z R_Z,$$

where $R_0$, the resistance associated with charge diffusion in the lowest band (Eq. 4) is now calculated with a multiplicative factor $\alpha = 1$ but taking into account the renormalized barrier height $\Delta_0$. $R_Z$ is a constant fit parameter, representing the resistance associated with charge dynamics in the higher bands. We assume $R_Z$ to be independent of the flux frustration parameter. When calculating $E_{\text{gap}}$ in Eq. 11, we solve the Mathieu equation associated with Hamiltonian (1) for the single junction, using the renormalized Josephson energy $E_J$, Eq. 10. In Fig. 8 we show a fit of the data for all our chains with Eq. 12. Table III shows the parameters used to obtain the best fits.

A few remarks are in order at this point. Note that $\omega_q^{\text{fit}}$ has the tendency to decrease with the chain length, a fact that is expected as the attempt frequency of the escaping particle decreases with increasing particle mass, the mass of the charge being given by the chain’s inductance. On the other hand, the frequency $\omega_q^{\text{fit}}$ (see Table III) is systematically smaller than the frequency $\omega_q$ (see Table I) associated with the curvature of the lowest Bloch band. This indicates that the charge motion is possibly overdamped. Such overdamped motion could result from a finite quality factor of the electromagnetic modes. Indeed, microwave transmission experiments done on a 200 Josephson junction chain with a similar ratio $E_J/E_C \approx 10$ as in the experiment here have shown an internal quality factor of about 100. We note that the fitting parameter $R_Z$ increases as a function of $N$ and takes a maximum value of $R_Z = 3500 \Omega$ for $N = 48$. Above $N = 48$ it is difficult to conclude a systematic behavior of $R_Z$. A more detailed understanding of the behavior of $R_Z$ requires a more detailed understanding of the processes responsible for charge relaxation in the higher bands, which is beyond the scope of the present paper.

In summary, as it can be seen in Fig. 8 the measured zero-bias resistance as a function of the flux frustration parameter shows three different behaviours. Close to
$f = 0.5$, where $E_J \ll E_C$, LZ processes dominate. The measured resistance has a peak, the form of which is entirely dominated by the flux-dependence of the LZ probability $P_Z$. Away from $f = 0.5$, the decrease of the resistance with $f$ becomes slower than the one predicted by $P_Z$ alone: the system enters the regime where charge diffusion within the lowest band dominates. Even farther away from $f = 0.5$, the charge diffusion model breaks down. Here, the bandwidth becomes smaller than the residual noise temperature of the system that we estimate to be about 80 mK. This happens at $f \approx 0.47$. The charge is no longer localized and our model based on simple escape rates underestimates the actual charge transfer rates. Hence the fit over-estimates the measured resistance.

VIII. CONCLUSIONS

Our transport measurements suggest the existence of a localized charge state on a Josephson junction due to an inductive environment. We could explain the measured zero-bias resistance with a model combining charge diffusion within the lowest Bloch band and LZ processes between bands. In order to reduce the charge dynamics to the lowest Bloch band, future experiments should study single Josephson junctions with a larger Josephson coupling $E_J$ hence avoiding Landau-Zener transitions. At the same time the inductance of the environment should be increased to ensure a localized charge state. Increasing the SQUID chain length to obtain a larger inductance reduces the frequencies of the electromagnetic modes and renormalisation effects will start to play a dominant role. Future experiments using the combination of a tunable junction with a controllable SQUID chain constitute an experimental test bed to explore the renormalization effects of the electromagnetic environment on a small junction with a controllable SQUID chain behave as inductances with inductance

\[ C_{nm} = (C_0 + 2C_{SQ})\delta_{m,n} - C_{SQ}\delta_{n+1,m} - C_{SQ}\delta_{n-1,m}. \]

We diagonalize the Hamiltonian (A1) with the help of the following mode expansions for $Q$ and $\phi$

\[ \phi_n = \frac{1}{\sqrt{N}} \sum_k \sqrt{\frac{2e^2}{C(k)\hbar \omega_k}} (a_k + a_k^\dagger) e^{ikn}, \quad (A2) \]

\[ Q_n = -\frac{ie}{\sqrt{N}} \sum_k \sqrt{\frac{C(k)\hbar \omega_k}{2e^2}} (a_k - a_k^\dagger) e^{ikn}. \quad (A3) \]

Here, $C(k) = C_0 + 2C_{SQ}(1 - \cos k)$ is the discrete Fourier transform of the capacitance matrix, $C(k) = (1/N) \sum_k e^{ik(n-m)}C_{nm}$. The dispersion relation is given by

\[ \omega_k = \omega_p \sqrt{\frac{2(1 - \cos k)}{C_0/C_{SQ} + 2(1 - \cos k)}}, \quad (A4) \]

with the plasma frequency $\omega_p = 1/\sqrt{L_{SQ}C_{SQ}}$. We use periodic boundary conditions, which implies that $k = 2\pi m/N$ with $m = \pm 1, \pm 2, \ldots, \pm N/2$. The diagonal form of $H_{ch}$ reads

\[ H_{ch} = \sum_k \hbar \omega_k (a_k^\dagger a_k + 1/2). \quad (A5) \]



FIG. 9. Josephson junction chain.

Appendix A: Harmonic modes of a Josephson junction chain

In this Appendix we briefly review the quantum theory of a harmonic Josephson junction chain. We consider a Josephson junction chain, consisting of $N$ junctions, each with a capacitance $C_{SQ}$ and a Josephson coupling energy $E_{J,ch}$. We denote the capacitance of the islands between the junctions to ground by $C_0$. In the harmonic limit, valid when $E_{J,ch} \gg E_C = e^2/2C$, the junctions forming the chain behave as inductances with inductance $L_{SQ} = (\hbar/2e)^2/E_{J,ch}$ each. Then the chain is described by the circuit model shown in Fig. 9. Its Hamiltonian can be written as

\[ H_{ch} = \frac{1}{2} \sum_{n,m} Q_n C_{nm}^{-1} Q_m + \frac{1}{2L_{SQ}} \left( \frac{\hbar}{2e} \right)^2 \sum_n (\phi_n - \phi_{n+1})^2, \quad (A1) \]

where $Q_n$ and $\phi_n$ denote the charge and the phase of the $n^{th}$ island, respectively. These variables satisfy the canonical commutation relation $[Q_n, \phi_m] = -2ie\delta_{n,m}$. The matrix $C_{nm}^{-1}$ is the inverse of the chain’s capacitance matrix

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The small Josephson junction is connected to one of the ends of the chain, say the one corresponding to \( n = 0 \). As a result, the phase difference \( \phi \) across the junction acquires a fluctuating part, \( \phi_0 \), and its Josephson coupling energy can be written as \(-E_J \cos(\phi + \phi_0)\). Upon averaging over the fluctuations \( \phi_0 \), we obtain the junction’s effective Josephson energy \( U(\phi) \) with a renormalized Josephson coupling energy \( E_J^* \),

\[
U(\phi) = -E_J \langle \cos(\phi + \phi_0) \rangle_{H_{ch}} = -E_J^* \cos \phi, \quad (A6)
\]

where \( E_J^* = E_J(\cos \phi_0)_{H_{ch}} \) and the average \( \langle \cdot \rangle_{H_{ch}} \) is taken with respect to the Hamiltonian \( H_{ch} \) at zero temperature. Using the fact that \( \langle \cos \phi_0 \rangle_{H_{ch}} = \exp\{-(\phi_0^2)_{H_{ch}}/2 \} \) and the mode expansion \( (A2) \) to calculate \( \langle \phi_0 \rangle_{H_{ch}} \) we finally obtain

\[
E_J^* = E_J \exp \left\{ \frac{1}{N} \sum_k e^2 \sqrt{\frac{L}{2C(k)(1 - \cos k)}} \right\}. \quad (A7)
\]

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1. M. Tinkham, *Introduction to superconductivity*, 2nd ed. (Dover publications, 2004).
2. R. L. Kautz, Reports on Progress in Physics **59**, 935 (1996).
3. K. Likharev and A. Zorin, *Journal of Low Temperature Physics* **59**, 347 (1985).
4. D. Averin, A. B. Zorin, and K. Likharev, JETP **61**, 407 (1985).
5. F. Nguyen, N. Boulant, G. Ithier, P. Bertet, H. Poitier, D. Vion, and D. Esteve, *Phys. Rev. Lett.* **99**, 187005 (2007).
6. C. H. Webster, J. C. Fenton, T. T. Hongisto, S. P. Giblin, A. B. Zorin, and P. A. Warburton, *Phys. Rev. B* **87**, 144510 (2013).
7. L. S. Kuzmin and D. B. Haviland, *Phys. Rev. Lett.* **67**, 2890 (1991).
8. M. Watanabe and D. B. Haviland, *Phys. Rev. Lett.* **86**, 5120 (2001).
9. S. Corlevi, W. Guichard, F. W. J. Hekking, and D. B. Haviland, *Phys. Rev. Lett.* **97**, 096802 (2006).
10. A. Ergül, J. Lidmar, J. Johansson, Y. A. Azizoglu, D. Schaeffer, and D. B. Haviland, *New Journal of Physics* **15**, 095014 (2013).
11. N. Vogt, R. Schafer, H. Rotzinger, W. Cui, A. Fiebig, A. Shnirman, and A. V. Ustino, arXiv:1407.3353 (2014).
12. K. Arutyunov, D. Golubev, and A. Zaikin, *Physics Reports* **464**, 1 (2008).
13. J. E. Mooij and Y. V. Nazarov, *Nat. Phys.* **2**, 169 (2006).
14. O. V. Astafiev, L. B. Ioffe, S. Kafanov, Y. A. Pashkin, K. Y. Arutyunov, D. Shahar, O. Cohen, and J. S. Tsai, *Nature* **484**, 355 (2012).
15. T. T. Hongisto and A. B. Zorin, *Phys. Rev. Lett.* **108**, 097001 (2012).
16. I. M. Pop, B. Douit, L. Ioffe, I. Protopopov, F. Lecocq, I. Matei, O. Buisson, and W. Guichard, *Phys. Rev. B* **85**, 094503 (2012).
17. J. S. Lehtinen, K. Zakharov, and K. Y. Arutyunov, *Phys. Rev. Lett.* **109**, 187001 (2012).
18. W. Guichard and F. W. J. Hekking, *Phys. Rev. B* **81**, 064508 (2010).
19. V. E. Manucharyan, J. Koch, L. I. Glazman, and M. H. Devoret, *Science* **326**, 113 (2009).
20. N. A. Mashuk, L. M. Pop, A. Kamal, Z. K. Minev, and M. H. Devoret, *Phys. Rev. Lett.* **109**, 137002 (2012).
21. M. T. Bell, I. A. Sadovsky, L. B. Ioffe, A. Y. Kitaev, and M. E. Gershenson, *Phys. Rev. Lett.* **109**, 137003 (2012).
22. I. M. Pop, K. Gerlings, G. Catelani, R. J. Schoelkopf, L. I. Glazman, and M. H. Devoret, *Nature* **508**, 369 (2014).
23. J. E. Mooij and C. J. P. M. Harmans, *New Journal of Physics* **7**, 219 (2005).
24. The measured samples comprise two sets of four chains (8, 18, 28, 38) and (48, 68, 88, 108). The two sets have slightly different parameters. As the chains with 8 and 18 squids do not show charge localization we did not include them in the discussion. The chains are actually measured in a configuration where one chain is connected in series with the parallel combination of the other three chains on the chip. We corrected the zero bias resistance for the presence of the other three chains.
25. F. Balestro, *Dynamique quantique d’un SQUID-DC*, Ph.D. thesis (2003).
26. A. Caldeira and A. Leggett, *Ann. Phys.* **149**, 374 (1983).
27. M. Hofheinz, F. Portier, Q. Baudouin, P. Joyez, D. Vion, P. Bertet, P. Roche, and D. Esteve, *Phys. Rev. Lett.* **106**, 217005 (2011).
28. G. L. Ingold and Y. V. Nazarov, *Single Charge Tunneling* edited by H. Grabert and M.H. Devoret, NATO Science Series 2009.
29. H. Kramers, *Phys. Rev. B* **7**, 284 (1940).
30. A. Schmid, *Phys. Rev. Lett.* **51**, 1506 (1983).
31. G. Schön and A. Zaikin, *Physics Reports* **198**, 237 (1990).
32. U. Weiss, *World Scientific Singapore*, 3rd edition (2008).
33. G. Rastelli, I. M. Pop, and F. W. J. Hekking, *Phys. Rev. B* **87**, 174513 (2013).
34. I. M. Pop, *Quantum phase-slips in Josephson junction chains*, Ph.D. thesis (2014).
35. T. Weißl, *Quantum phase and charge dynamics in Josephson junction chains*, Ph.D. thesis (2014).