Stochastic processes on non-Archimedean spaces with values in non-Archimedean fields.

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27 October 2001 *

Abstract

Stochastic processes on topological vector spaces over non-Archimedean fields and with transition measures having values in non-Archimedean fields are defined and investigated. For this the non-Archimedean analog of the Kolmogorov theorem is proved. The analogous of Markov and Poisson processes are studied. For Poisson processes the corresponding Poisson measures are considered and the non-Archimedean analog of the Lévy theorem is proved. Wide classes of stochastic processes are constructed.

1 Introduction.

Classical stochastic analysis for real and complex vector spaces and real or complex transition measures is well developed [6, 16, 17, 20, 34, 35, 37], but the stochastic analysis on topological vector spaces over non-Archimedean fields and with transition measures having values in non-Archimedean fields was not studied. There are many differences of classical and non-Archimedean functional analysis [4, 18, 19, 38, 39, 40, 41] and many theorems of classical functional analysis are not true in their classical form in the non-Archimedean case, for example, measure theory, operator theory, theory of function spaces. This paper is devoted to such new non-Archimedean variant of stochastic

*Mathematics subject classification (1991 Revision) 28C20 and 46S10.
analysis and continues papers [33], where real and complex valued transition measures of stochastic processes on non-Archimedean spaces were considered. Stochastic differential equations on real Banach spaces and manifolds are widely used for solutions of mathematical and physical problems and for construction and investigation of measures on them. On the other hand, non-Archimedean functional analysis develops fastly in recent years and also its applications in mathematical physics [1, 11, 43, 41, 45, 23, 22]. Wide classes of quasi-invariant measures including analogous to Gaussian type on non-Archimedean Banach spaces, loops and diffeomorphisms groups were investigated in [26, 27, 28, 29, 30, 31]. Quasi-invariant measures on topological groups and their configuration spaces can be used for the investigations of their unitary representations (see [28, 29, 30, 31] and references therein).

In view of this developments non-Archimedean analogs of stochastic equations and diffusion processes need to be investigated. Some steps in this direction were made in [3, 12], where non-Archimedean time was considered, but stochastic processes there were on spaces of complex valued functions and transition measures were real or complex valued. At the same time measures may be real, complex or with values in a non-Archimedean field.

In the classical stochastic analysis indefinite integrals are widely used, but in the non-Archimedean case the field of $p$-adic numbers $\mathbb{Q}_p$ has not linear order structure apart from $\mathbb{R}$. This work treats the case which was not considered by another authors. These investigations are not restricted by the rigid geometry class [15], since it is rather narrow. Wider classes of functions and manifolds are considered. This is possible with the use of Schikhof’s works on classes of functions $C^n$ in the sense of difference quotients, which he investigated few years later the published formalism of the rigid geometry.

Here are considered spaces of functions with values in Banach spaces over non-Archimedean local fields, in particular, with values in the field $\mathbb{Q}_p$ of $p$-adic numbers. For this non-Archimedean analogs of stochastic processes are considered on spaces of functions with values in the non-Archimedean field such that a parameter analogous to the time is either real, $p$-adic or more generally can take values in any group (see §§4.1, 4.2). Certainly this encompasses cases of the time parameter with values in adeles and ideles. Their existence is proved in Theorem 4.3.

This became possible due to results of §2, where the non-Archimedean variant of the Kolmogorov theorem was proved.
In §3 non-Archimedean analogs of Markov cylindrical distributions are defined and Propositions 3.3.1 and 3.3.2 about their boundedness and unboundedness are proved.

Poisson measures and processes play very important role in classical stochastic analysis [21, 5]. In Section 5 their non-Archimedean analogs are considered. All results of this paper are obtained for the first time.

2 \( p \)-Adic probability measures.

Let \( X \) be a set and \( \mathcal{R} \) be a covering ring of \( X \) such that elements of \( \mathcal{R} \) are subsets of \( X \). Consider a field \( K \) with a nontrivial non-Archimedean valuation such that \( K \supset \mathbb{Q}_p \). Suppose that \( K \) is complete as the ultrametric space.

2.1. Definition. Suppose that \( S \) is a subfamily of \( \mathcal{R} \) such that for each \( A \) and \( B \) in \( S \) there exists \( C \in S \) with \( C \subset A \cap B \), then \( S \) is called shrinking. For a function \( f : \mathcal{R} \to K \) or \( f : \mathcal{R} \to \mathbb{R} \) the notation \( \lim_{A \in S} f(A) = 0 \) means that for each \( \epsilon > 0 \) there exists \( B \in S \) such that \( |f(A)| \leq \epsilon \) for each \( A \in S \) with \( A \subset B \).

2.2. Definition. A mapping \( \mu : \mathcal{R} \to K \) is called a measure if it satisfies the following conditions:

(i) \( \mu(A \cup B) = \mu(A) + \mu(B) \) for each \( A \) and \( B \) in \( \mathcal{R} \) such that \( A \cap B = \emptyset \);

(ii) for each \( A \in \mathcal{R} \) its \( \mu \)-norm \( \| A \|_\mu := \sup \{|\mu(B)| : B \in \mathcal{R}, B \subset A\} < \infty \) is bounded;

(iii) if \( S \subset \mathcal{R} \) is shrinking and \( \cap S := \bigcap_{S \in S} = \emptyset \), then \( \lim_{A \in S} \mu(A) = 0 \).

2.3. Note. These conditions are called respectively additivity, boundedness, and continuity. Condition (iii) is equivalent to \( \lim_{A \in S} \| A \|_\mu = 0 \) for each shrinking subfamily \( S \) in \( \mathcal{R} \) with \( \cap S = \emptyset \).

2.4. Definition. A measure \( \mu : \mathcal{R} \to K \) is called a probability measure if \( \mu(X) = 1 \) and \( \| X \|_\mu := \| \mu \| = 1 \).

2.5. Remarks. For functions \( f : X \to K \) and \( \phi : X \to [0, \infty) \) put \( \| f \|_\phi := \sup_{x \in X} |f(x)|\phi(x) \). Consider the following function:

\[
(1) \quad N_\mu(x) := \inf_{U : x \in U \in \mathcal{R}} \| U \|_\mu
\]

for each \( x \in X \). Put \( \| f \|_\mu := \| f \|_{N_\mu} \). Then for each \( A \subset X \) the function \( \| A \|_\mu := \sup_{x \in A} N_\mu(x) \) is defined such that its restriction on \( \mathcal{R} \) coincides
with that of given by Equation 2.2.(ii) (see also Chapter 7 [11]). A \( R \)-step function \( f \) is a function \( f : X \to K \) such that it is a finite linear combination over \( K \) of characteristic functions \( Ch_U \) of \( U \in R \). A function \( f \) is called \( \mu \)-integrable if there exists a sequence \( \{f_n : n \in N\} \) of step functions such that \( \lim_{n \to \infty} \|f - f_n\|_{R,\mu} = 0 \). The Banach space of \( \mu \)-integrable functions is denoted by \( L(\mu) := L(X, R, \mu, K) \). There exists a ring \( R_\mu \) of subsets \( A \) in \( X \) for which \( Ch_A \in L(\mu) \). The ring \( R_\mu \) is the extension of the ring \( R \) such that \( R_\mu \supset R \).

For example, if \( K \) is locally compact, then the valuation group \( \Gamma_K := \{|x| : x \in K, x \neq 0\} \) is discrete in \((0, \infty) \subset R \). If \( \mu \) is a measure such that \( 0 < \|\mu\| < \infty \), then there exists \( a \in K \) such that \( |a| = \|\mu\|^{-1} \), since \( \|\mu\| \in \Gamma_K \) for discrete \( \Gamma_K \), hence \( a\mu \) is also the measure with \( \|\mu\| = 1 \). If \( \|\mu\| = 1 \), then \( \mu \) is the nonzero measure. For such \( \mu \) with \( \mu(X) := b_X \in K \) if \( b_X \neq 1 \) we can take new set \( Y \) and define on \( X_0 := Y \cup X \) a minimal ring \( R_0 \) generated by \( R \) and \( \{Y\} \), that is, \( R_0 \cap Y = \{\emptyset, \{Y\}\} \) and \( R_0 = R \cup \{Y\} \). Since \( \|\mu\| = 1 \), then \( |b_X| \leq 1 \). Put \( \mu(Y) := 1 - b_X \), then there exists the extension of \( \mu \) from \( R \) on \( R_0 \) such that \( \|\mu\| = 1 \) and \( \mu(X_0) = 1 \), since \( |1 - b_X| \leq \max(1,|b_X|) = 1 \). In particular, we can take a singleton \( Y = \{y\} \). Therefore, probability measures are rather naturally related with nonzero bounded measures. This also shows that from \( \|\mu\| = 1 \) in general does not follow \( \mu(X) = 1 \). Evidently, from \( \mu(X) = 1 \) in general does not follow \( \|\mu\| = 1 \), for example, \( X = \{0, 1\}, \mathcal{R} = \{\emptyset, \{0\}, \{1\}, X\}, \mu(\{0\}) = a, \mu(\{1\}) = 1 - a \), where \( |a| > 1 \), hence \( \|\mu\| = |a| > 1 \). Wide class of probability \( \text{Q}_p \)-valued measures on non-Archimedean Banach spaces was constructed in \( \text{§II.3.15} \) [12].

Consider a nonvoid topological space \( X \). A topological space is called zero-dimensional if it has a base of its topology consisting of clopen subsets. A topological space \( X \) is called a \( T_0 \)-space if for each two distinct points \( x \) and \( y \) in \( X \) there exists an open subset \( U \) in \( X \) such that either \( x \in U \) and \( y \notin X \setminus U \) or \( y \in U \) and \( x \notin X \setminus U \).

A covering ring \( R \) of a space \( X \) defines on it a base of zero-dimensional topology \( \tau_R \) such that each element of \( R \) is considered as a clopen subset in \( X \). If \( \pi : X \to Y \) is a mapping such that \( \pi^{-1}(\mathcal{R}_Y) \subset \mathcal{R}_X \), then a measure \( \mu \) on \((X, \mathcal{R}_X)\) induces a measure \( \nu := \pi(\mu) \) on \((Y, \mathcal{R}_Y)\) such that \( \nu(A) = \mu(\pi^{-1}(A)) \) for each \( A \in \mathcal{R}_Y \).

2.6. Proposition. Let \((X, \mathcal{R}, \mu)\) be a measure space. Then there exists a quotient mapping \( \pi : X \to Y \) on a Hausdorff zero-dimensional space \((Y, \tau_G)\)
and \( \pi(\mu) := \nu \) is a measure on \( Y \) such that \( \mathcal{G} = \pi(\mathcal{R}) \), where \( (Y, \mathcal{G}, \nu) \) is the measure space.

**Proof.** Suppose that \( (Y, \tau_\mathcal{G}) \) is a \( T_0 \)-space, where \( \mathcal{G} \) is a covering ring of \( Y \). For each two distinct points \( x \) and \( y \) in \( Y \) there exists a clopen subset \( U \) in \( Y \) such that either \( x \in U \) and \( y \in Y \setminus U \) or \( y \in U \) and \( x \in Y \setminus U \), since the base of topology \( \tau_\mathcal{G} \) in \( Y \) consists of clopen subsets. On the other hand, \( Y \setminus U \) is also clopen, since \( U \) is clopen. Therefore, \( Y \) is the Hausdorff space. Clearly this implies that \( Y \) is the Tychonoff space (see §6.2 [9], but it is necessary to note that we consider the definition of the zero-dimensional space more general without \( T_1 \)-condition in §2.5).

Now we construct a \( T_1 \)-space \( Y \), that is a quotient space of \( X \). For this consider the relation in \( X \):

\[ x \kappa y \text{ if and only if for each } S \in \mathcal{R} \text{ with } x \in S \text{ there is the inclusion } \{x, y\} \subset S. \]

Evidently, \( x \kappa x \), that is, \( \kappa \) is reflexive. The relation \( x \kappa y \) means, that \( y \in V_x := \bigcap_{x \in S \in \mathcal{R}} S \), where \( V_x \) is closed in \( X \), then from \( y \in S \) it follows, that \( x \in S \), since otherwise \( y \notin V_x \), because \( \mathcal{R} \) is a covering ring. Therefore, \( V_x = V_y \) and \( y \kappa x \), hence \( \kappa \) is symmetric. Let \( x \kappa y \) and \( y \kappa z \), then \( V_x = V_y = V_z \), consequently, \( x \kappa z \) and \( \kappa \) is transitive. Therefore, \( \kappa \) is the equivalence relation.

Let \( \pi : X \to Y := X/\kappa \) be the quotient mapping and \( Y \) be supplied with the zero-dimensional topology generated by the covering ring \( \mathcal{G} \) such that \( \pi^{-1}(\mathcal{G}) = \mathcal{R} \), since each \( A \in \mathcal{R} \) is clopen, then for each \( x \in A \in \mathcal{R} \) we have \( V_x \subset A \). Then \( \pi^{-1}([y]) = V_y \) for each \( y \in X \) and \( [y] := \pi(y) \). Hence each point \( [y] \in Y \) is closed, hence \( Y \) is the \( T_1 \)-space. The topology in \( Y \) is generated by the covering ring \( \mathcal{G} \), consequently, \( Y \) is the Hausdorff space (see above), since from the \( T_1 \) separation property it follows the \( T_0 \) separation property.

If \( \mathcal{S} \) is the shrinking family with zero intersection in \( Y \) such that \( \mathcal{S} \subset \mathcal{G} \), then \( \pi^{-1}(\mathcal{S}) \) is also the shrinking family with zero intersection in \( X \) such that \( \pi^{-1}(\mathcal{S}) \subset \mathcal{R} \), hence from \( \lim_{A \in \pi^{-1}(\mathcal{S})} \mu(A) = 0 \) it follows \( \lim_{A \in \mathcal{S}} \nu(A) = 0 \). Therefore, Condition (\( iii \)) from §2.2 is satisfied. Evidently, \( \|\nu\| = \|\mu\| \) and \( \nu \) is additive on \( \mathcal{G} \), hence \( \nu \) is the measure.

2.7.1. **Note.** In view of Proposition 2.6 we consider henceforth Hausdorff zero-dimensional measurable \((X, \mathcal{R})\) spaces if another is not specified.

In the classical case the principal role in stochastic analysis plays the Kolmogorov theorem, that gives the possibility to construct a stochastic process on the basis of a system of finite dimensional (real-valued) probability distributions (see §III.4 [25]). The following three theorems resolve this problem.
for \( \mathbf{K} \)-valued measures in cases of a product of measure spaces, a consistent family of measure spaces and in cases of bounded cylindrical distributions. Finally Theorem 2.15.2 (the non-Archimedean analog of the Kolmogorov theorem) as the particular case of Theorem 2.14 is formulated.

Consider now a family of probability measure spaces \( \{(X_j, \mathcal{R}_j, \mu_j) : j \in \Lambda\} \), where \( \Lambda \) is a set. Suppose that each covering ring \( \mathcal{R}_j \) is complete relative to the measure \( \mu_j \), that is, \( \mathcal{R}_j = \mathcal{R}_{\mu_j} \), where \( \mathcal{R}_{\mu_j} \) denotes the completion of \( \mathcal{R}_j \) relative to \( \mu_j \). Let \( X := \prod_{j \in \Lambda} X_j \) be the product of topological spaces supplied with the product (Tychonoff) topology \( \tau_X \), where each \( X_j \) is considered in its \( \tau_{\mathcal{R}_j} \)-topology. There is the natural continuous projection \( \pi_j : X \to X_j \) for each \( j \in \Lambda \). Let \( \mathcal{R} \) be the ring of the form \( \bigcup_{j_1, \ldots, j_n \in \Lambda} \bigcap_{l=1}^n \pi_{j_l}^{-1}(\mathcal{R}_{j_l}) \).

**2.7.2. Definition.** A triple \((X, \mathcal{R}, \mu)\) is called a cylindrical distribution if it satisfies the following condition:

\[
\mu|_{\bigcap_{l=1}^n \pi_{j_l}^{-1}(\mathcal{R}_{j_l})} = \prod_{l=1}^n \tilde{\mu}_{j_l} \text{ for each } j_1, \ldots, j_n \in \Lambda \text{ and } n \in \mathbb{N}, \text{ where } \tilde{\mu}_j(\pi_j^{-1}(A)) := \mu_j(A) \text{ for each } A \in \mathcal{R}_j; \tilde{\mu}_j \text{ is the measure on } (X, \pi_j^{-1}(\mathcal{R}_j)).
\]

**2.8. Theorem.** A cylindrical distribution \( \mu \) on \((X, \mathcal{R})\) has an extension up to a probability measure \( \mu \) on \((X, \mathcal{R}_\mu)\), where \( \mu \) and \( X \) are the same as in §2.7.

**Proof.** For each \( j \in \Lambda \) we have the ring \( \mathcal{R}_j \). Let \( A \) and \( B \) be in \( \bigcap_{l=1}^n \pi_{j_l}^{-1}(\mathcal{R}_{j_l}) \), where \( j_1, \ldots, j_n \in \Lambda \) and \( n \in \mathbb{N} \). Then \( A = \bigcap_{l=1}^n \pi_{j_l}^{-1}(A_l) \), where \( A_l \in \mathcal{R}_{j_l} \) for each \( l = 1, \ldots, n \), analogously for \( B \) with \( B_l \) instead of \( A_l \). Such subsets \( A \) form the base of the topology \( \tau_X \) such that \( \tau_X \supset \mathcal{R} \). Therefore, \( A \cap B \) and \( A \setminus B \) and hence \( A \cup B \) are in \( \mathcal{R} \), since \( \mathcal{R}_{j_1} \times \ldots \times \mathcal{R}_{j_n} \) is the ring. Therefore, \( \mathcal{R} \) is the ring. The space \( X \) in the topology \( \tau_X \) is zero-dimensional, since the base \( \mathcal{R} \) of \( \tau_X \) consists of clopen subsets in \( X \). It is necessary to verify that the triple \((X, \mathcal{R}, \mu)\) satisfies Conditions 2.2.(i – iii). In general \( \tau_X \) and \( \mathcal{R} \) may not coincide, but as it is shown below the usage of the inclusion \( \tau_X \supset \mathcal{R} \) is sufficient for the proof. On the other hand, \((X_{j_1} \times \ldots \times X_{j_n}, \mathcal{R}_{j_1} \times \ldots \times \mathcal{R}_{j_n}, \mu_{j_1} \times \ldots \times \mu_{j_n})\) is the measure space for each \( j_1, \ldots, j_n \in \Lambda \) and \( n \in \mathbb{N} \), consequently, \( \mu \) on \( \mathcal{R} \) is additive. For each \( A \) of the outlined above form we have

(i) \( \|A\|_\mu = \prod_{l=1}^n \|A_l\|_{\mu_{j_l}} \leq 1 \). Such elements \( A \) in \( \mathcal{R} \) form the base of Tychonoff topology in \( X \), consequently, \( \|X\|_\mu \leq 1 \). For each \( j \in \Lambda \) we have \( \mu_j(X_j) = 1 \), hence \( \mu(X) = 1 \) and \( \|X\|_\mu = 1 \). Therefore, \( \mu \) satisfies Conditions 2.2.(i, ii). For each \( A \in \mathcal{R} \) the norm \( \|A\|_\mu \) is defined.

Consider now the function \( N_\mu(x) \) on \((X, \mathcal{R})\), that is defined by the For-
mula 2.5.(1). For each $\epsilon > 0$ and $x \in X$ there exists $A \in \mathcal{R}$ such that

$(ii) \|A\|_\mu - \epsilon < N_\mu(x) \leq \|A\|_\mu$. Each function $N_{\mu_j}(x_j)$ is upper semicontinuous on $(X_j, \mathcal{R}_j)$ by Theorem 7.6 [11]. In view of Lemma 7.2 [11] and Formula $(i)$ for each $x \in X$ and each $\epsilon > 0$ there exists its neighborhood $A \in \mathcal{R}$ such that

$(iii) \prod_{j=1}^n N_{\mu_j}(y_{j_k}) < N_\mu(x) + \epsilon$ for each $y_j := \pi_j(y)$ for each $j \in \Lambda$. Hence for each $x \in X$ and each $\epsilon > 0$ there exists its basic neighborhood $A$ such that

$(iv) N_\mu(y) < N_\mu(x) + \epsilon$ for each $y \in A$, that is, $N_\mu(x)$ is upper semicontinuous on $(X, \mathcal{R})$, since $0 \leq N_{\mu_j}(x_j) \leq 1$ for each $x_j \in X_j$ and $j \in \Lambda$. From Formulas $(i, ii, iii)$ and 2.2.(ii) we have

$(v) \|A\|_\mu = \sup_{x \in X} N_\mu(x)$ for each $A \in \mathcal{R}$, since $\|A\|_\mu = \sup_{x \in X} \prod_{j=1}^n N_{\mu_j}(x_i)$.

Let $V$ be a compact subset of $X$. Then for each $\epsilon > 0$ it has a covering by clopen subsets $E_x \in \mathcal{R}$ with $x \in V$ and $x \in E_x$ such that Inequalities $(ii - iv)$ are satisfied for $E_x$ instead of $A$. In view of compactness of $V$ the covering $\{E_x : x \in V\}$ has a finite subcovering $\{E_1, ..., E_m\}$. Then $\bigcup_{l=1}^m E_l \in \mathcal{R}$. Therefore,

$(vi) \sup_{x \in V} N_\mu(x) \leq \max_{l=1, ..., m} \|E_l\|_\mu \leq \sup_{x \in V} N_\mu(x) + 2\epsilon$, since $\sup_{x \in V} N_\mu(x) \leq \|\bigcup_{l=1}^m E_l\|_\mu$ due to Inequality $(ii)$,

$\|\bigcup_{l=1}^m E_l\|_\mu = \max_{l=1, ..., m} \|E_l\|_\mu = \sup_{x \in E_l} N_\mu(x)$ due to Formula $(v)$ and Condition 2.2.(ii), since $E_l \in \mathcal{R}$ for each $l = 1, ..., m$, but for each $E_l$ there exists $x_l \in E_l \cap V$ such that $N_\mu(y) < N_\mu(x_l) + \epsilon$ for each $y \in E_l$ due to Inequality $(iv)$ and the choice of $\{E_1, ..., E_m\}$ as the finite subcovering of the covering $\{E_x : x \in E_x \cap V, E_x \in \mathcal{R}\}$ (see above). Since $\epsilon > 0$ is arbitrary and $\|\bigcup_{l=1}^m E_l\|_\mu = \max_{l=1, ..., m} \|E_l\|_\mu$, then

$(vii) \sup_{x \in V} N_\mu(x) = \inf_{V \supseteq \bigcup_{l=1}^m E_l} \|A\|_\mu$, since for each $A \in \mathcal{R}$ such that $V \subset A$ there exists $\{E_l : l = 1, ..., m\}$ with $m \in \mathbb{N}$, where each $E_l$ is as above, such that $V \subset \bigcup_{l=1}^m E_l \subset A$. Though the compact subset $V$ is not necessarily in $\mathcal{R}$ we take Equation $(vii)$ as the definition of $\|V\|_\mu := \inf_{V \supseteq \bigcup_{l=1}^m E_l} \|A\|_\mu$.

Now we verify, that $\mu$ on $(X, \mathcal{R})$ satisfies Condition 2.2.(iii). Let $S$ be a shrinking subfamily in $\mathcal{R}$ with $\cap S = \emptyset$. In view of Theorem 7.12 [11] for each $\epsilon > 0$ the set $X_{\delta, \epsilon} := \{x : x \in X_j, N_{\mu_j}(x) \geq \epsilon\}$ is $\mathcal{R}_{\mu_j}$-compact. For each $\delta > 0$ choose a sequence $\{\epsilon_j : \epsilon_j > 0, j \in \Lambda\}$ with $\sup_{j \in \Lambda} \epsilon_j < \delta$. In view of the Tychonoff theorem (see §3.2.4 in [11]) $\prod_{j \in \Lambda} X_{\delta, \epsilon_j} =: X_{(\epsilon_j, \delta)}$ is the compact subset in $X$. Since for each $A$ and $B$ in $S$ there exists $C \in S$ such that $C \subset A \cap B$ and $\mathcal{R}$ is the ring, then consider finite intersections of finite
families in $S$, hence there exists the minimal family $S_0$ generated by $S$ such that $S_0 \subset R$, $S \subset S_0$ and $S_0$ is centered, that is, $A \cap B \in S_0$ for each $A$ and $B$ in $S_0$. Evidently, $\lim_{A \in S} \|A\|_\mu = 0$ is equivalent to $\lim_{A \in S_0} \|A\|_\mu = 0$, since $\|B\|_\mu \leq \|A\|_\mu$ for each $B \subset A$ (see also \S\S2.3 and 2.5). Denote $S_0$ by $S$ also.

Each element $S \in S$ is clopen in the centered family $S$ and $X_{\{\epsilon_j;\}}$ has the empty intersection with $\cap S$. In view of compactness of $X_{\{\epsilon_j;\}}$ there exists a finite subfamily $S_1, \ldots, S_n$ in $S$ such that $X_{\{\epsilon_j;\}} \cap (\cap_{j=1}^n S_i) = \emptyset$, hence $\lim_{A \in S \cap X_{\{\epsilon_j;\}}} \|A\|_\mu = 0$, so $0 = 0$. As above for each $\delta > 0$ choose a finite covering $E_1, \ldots, E_m \in R$ of $V := X_{\{\epsilon_j;\}}$ such that

$$(viii) \quad \|V\|_\mu \leq \max_{i=1,\ldots,m} \|E_i\|_\mu < \|V\|_\mu + \delta$$ (see Equation (vi)), hence there exists $A \in S \cap (\cup_{i=1}^m E_i)$ such that $\|A\|_\mu \leq \delta$, where $(\cup_{i=1}^m E_i) \in R$.

For each $x \in X \setminus X_{\{\epsilon_j;\}}$ there exists a basis neighbourhood $U = \cap_{i=1}^n \pi^{-1}(U_l)$ such that $U \cap X_{\{\epsilon_j;\}} = \emptyset$, since $X$ is Hausdorff and $X_{\{\epsilon_j;\}}$ is compact, where $U_l \in R$ and $j_l \in \Lambda$ for each $l = 1, \ldots, n$. Therefore,

$(ix) \quad N_{\mu}(x) \leq \delta$ for each $x \in X \setminus X_{\{\epsilon_j;\}}$, since $\|X_j\|_{\mu_j} = 1$ and $\sup_{x \in X \setminus X_{\{\epsilon_j;\}}} \|X_{\{\epsilon_j;\}}(x) \leq \epsilon_j < \delta$ for each $j \in \Lambda$. In view of Equations ($v, vi$) we have $\|A\|_\mu \leq \delta$ for each $A \subset X \setminus X_{\{\epsilon_j;\}}$ such that $A \in R$. Then applying Equation (viii) to $V = X_{\{\epsilon_j;\}}$ we get $\|(\cap_{i=1}^n S_i) \cap (\cup_{k=1}^m E_k)\|_\mu \leq \delta$ and $\|(\cup_{k=1}^m E_k) \setminus (\cup_{k=1}^m E_k)\|_\mu \leq \delta$ due to Inequality (ix), where $X \setminus (\cup_{k=1}^m E_k) \in R$, hence $\|(\cap_{i=1}^n S_i) \setminus (\cup_{k=1}^m E_k)\|_\mu \leq \delta$, since $\|A\|_\mu \leq \max(\|A\cap B\|_\mu, \|A\cap (X \setminus B)\|_\mu)$, where $(\cap_{i=1}^n S_i) \in S$. On the other hand, $\delta > 0$ is arbitrary, consequently, $\lim_{A \in S} \|A\|_\mu = 0$. This means that $(X, R, \mu)$ is the measure space. In view of Theorem 7.4 [1, [1]] the measure $\mu$ has an extension $\mu$ on the completion $R_\mu$ of $R$ relative to $\mu$, moreover, $\sup_{R \subset \mu \subset A} |\mu(B)| = \sup_{x \in A} N_{\mu}(x)$ for each $A \in R_\mu$.

2.9. Note. Theorem 2.8 has an evident generalization for bounded measures $\mu_j$ if two products $\Pi_{j' \in A_0} \mu_{j'}(X_{\{j;\}}) \in K$ and $\Pi_{j \in \Lambda} \|X_{\{j;\}}\|_{\mu_j} < \infty$ converge, where $A_0 := \{j' : j \in \Lambda, \mu_{j'}(X_{\{j;\}}) \neq 0\}$, when $\Lambda \setminus A_0$ is finite. Since $\mu$ is defined on $R$ and bounded on it, then $\mu$ has an extension to the bounded measure $\mu$ on $R_\mu$ such that $\mu(X) = \Pi_{j \in \Lambda} \mu_j(X_{\{j;\}})$ and $\|X\|_\mu = \Pi_{j \in \Lambda} \|X_{\{j;\}}\|_{\mu_j}$, where $R_\mu$ is the completion of $R$ relative to $\mu$.

The conditions imposed in \S\S2.7 on the family of measures are natural. In view of \S\S2.8 if $\Lambda = \mathbb{N}$ and $\Pi_{j=1}^\infty \|X_{\{j;\}}\|_{\mu_j} = 0$, then for each $x \in X$ and each $\epsilon > 0$ there exists $A \in R$ such that $\Pi_{j=1}^\infty \|\pi_j(A)\|_{\mu_j} < \epsilon$, hence $\|X\|_\mu = 0$. On the other hand, if $\Pi_{j=1}^\infty \|X_{\{j;\}}\|_{\mu_j} = \infty$, then for each $r > 0$ there exists $X_{\{j;\}}$ and $A \supset X_{\{j;\}}$ with $A \in R$ such that $\Pi_{j=1}^\infty \|\pi_j(A)\|_{\mu_j} > r$, hence $\|X\|_\mu = \infty$. 

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If \( \mu \) is a measure on \((X, \mathcal{R})\), then \( \mu(A) \in K \) for each \( A \in \mathcal{R} \). In the case of the product measure \( \mu \) this leads to the restriction, that \( \prod_{j \in \Lambda_0} \mu_j(X_j) \) is convergent, when \( \Lambda \setminus \Lambda_0 \) is finite. For infinite \( \Lambda \setminus \Lambda_0 \) we have \( \mu(A) = 0 \) for each \( A \in \mathcal{R} \). The condition \( ||X_j||_{\mu_j} = 1 \) for each \( j \) does not guarantee this convergence. For example, if \( K = Q_p \) with the prime \( p > 1 \) and the set \( \{ j : \mu_j(X_j) \in \{2, ..., p - 1\} \} \) is infinite, then \( \prod_{j \in \Lambda_0} \mu_j(X_j) \) diverges, since the multiplicative group \((\mathbb{Z}/p\mathbb{Z}) \setminus \{0\}\) of the quotient ring \((\mathbb{Z}/p\mathbb{Z})\) is cyclic.

2.10. **Note.** A set \( \Lambda \) is called directed if there exists a relation \( \leq \) on it satisfying the following conditions:

1. If \( j \leq k \) and \( k \leq m \), then \( j \leq m \);
2. For every \( j \in \Lambda \), \( j \leq j \);
3. For each \( j \) and \( k \) in \( \Lambda \) there exists \( m \in \Lambda \) such that \( j \leq m \) and \( k \leq m \). A subset \( \Upsilon \) of \( \Lambda \) directed by \( \leq \) is called cofinal in \( \Lambda \) if for each \( j \in \Lambda \) there exists \( m \in \Upsilon \) such that \( j \leq m \). Suppose that \( \Lambda \) is a directed set and \( \{(X_j, \mathcal{R}_j, \mu_j) : j \in \Lambda\} \) is a family of probability measure spaces, where \( \mathcal{R}_j \) is the covering ring (not necessarily separating). Supply each \( X_j \) with the topology \( \tau_j \) such that its base is the ring \( \mathcal{R}_j \) as in §2.5. Let this family be consistent in the following sense:

1. there exists a mapping \( \pi_j^k : X_k \to X_j \) for each \( k \geq j \) in \( \Lambda \) such that \( (\pi_j^k)^{-1}(\mathcal{R}_j) \subset \mathcal{R}_k \), \( \pi_j^k(x) = x \) for each \( x \in X_j \) and each \( j \in \Lambda \), \( \pi_j^m \circ \pi_j^l = \pi_l^m \) for each \( m \geq k \geq l \) in \( \Lambda \);
2. \( \pi_l^k(\mu_k) = (\mu_l) \) for each \( k \geq l \) in \( \Lambda \). Such family of measure spaces is called consistent.

2.11. **Theorem.** Let \( \{(X_j, \mathcal{R}_j, \mu_j) : j \in \Lambda\} \) be a consistent family as in §2.10. Then there exists a probability measure space \((X, \mathcal{R}_\mu, \mu)\) and a mapping \( \pi_j : X \to X_j \) for each \( j \in \Lambda \) such that \( (\pi_j)^{-1}(\mathcal{R}_j) \subset \mathcal{R} \) and \( \pi_j(\mu) = \mu_j \) for each \( j \in \Lambda \).

**Proof.** We have \( (\pi_j^k)^{-1}(\mathcal{R}_j) \subset \mathcal{R}_k \) for each \( k \geq j \) in \( \Lambda \), then \( (\pi_j^k)^{-1}(\tau_j) \subset \tau_k \) for each \( k \geq j \) in \( \Lambda \), since each open subset in \( (X_j, \tau_j) \) is the union of some subfamily \( \mathcal{G} \) in \( \mathcal{R}_j \) and \( (\pi_j^k)^{-1}(\bigcup \mathcal{G}) = \bigcup_{A \in \mathcal{G}} (\pi_j^k)^{-1}(A) \). Therefore, each \( \pi_j^k \) is continuous and there exists the inverse system \( \mathcal{S} := \{X_k, \pi_j^k, \Lambda\} \) of the spaces \( X_k \). Its limit \( \lim S =: X \) is the topological space with the topology \( \tau_X \) and continuous mappings \( \pi_j : X \to X_j \) such that \( \pi_j^k \circ \pi_k = \pi_j \) for each \( k \geq j \) in \( \Lambda \) (see §2.5 in [3]). Each element \( x \in X \) is the thread \( x = \{x_j : x_j \in X_j \}\) for each \( j \in \Lambda \). Then \( \pi_j^{-1}(\mathcal{R}_j) =: \mathcal{G}_j \) is the ring of subsets in \( X \) such that \( \mathcal{G}_j \subset \tau_X \) for each \( j \in \Lambda \). The base of topology
Consider on $\mathcal{R}$ a function $\mu$ with values in $\mathbf{K}$ such that $\mu(\pi_j^{-1}(A)) := \mu_j(A)$ for each $A \in \mathcal{R}_j$ and each $j \in \Lambda$. If $A$ and $B$ are disjoint elements in $\mathcal{R}$, then there exists $j \in \Lambda$ such that $A$ and $B$ are in $\mathcal{G}_j$, hence

(i) $A = \pi_j^{-1}(C)$ and $B = \pi_j^{-1}(D)$ for some $C$ and $D$ in $\mathcal{R}_j$, consequently, $\mu(A \cup B) = \mu_j(C \cup D) = \mu_j(C) + \mu_j(D) = \mu_j(A) + \mu_j(B)$, that is, $\mu$ is additive. Moreover, $\|A\|_{\mu} = \|C\|_{\mu_j}$ for each $A = \pi_j^{-1}(C)$ with $C \in \mathcal{R}_j$, hence $\|X\|_{\mu} = 1$. Since $\mu(X) = \mu_j(X_j)$ and $\mu_j(X_j) = 1$ for each $j \in \Lambda$, then $\mu(X) = 1$. Therefore, $\mu$ satisfies Conditions 2.2.(i, ii, iii). It remains to verify Condition 2.2.(iii). By Formula 2.5.(1) we have the function $N_\mu(x)$ on $(X, \mathcal{R})$ such that for each $x \in X$ and $\epsilon > 0$ there exists $A \in \mathcal{R}$ such that

(ii) $\|A\|_{\mu} - \epsilon < N_\mu(x) \leq \|A\|_{\mu}$. In view of (i) and upper semicontinuity of $N_{\mu_j}(x_j)$ on $(X_j, \mathcal{R}_j)$ for each $x \in X$ and $\epsilon > 0$ there exists $j \in \Lambda$ and its neighborhood $A = \pi_j^{-1}(C) \in \mathcal{R}$ such that

(iii) $N_{\mu_j}(y_j) < N_\mu(x) + \epsilon$ for each $y \in A$, where $y_j := \pi_j(y)$. Hence for each $x \in X$ and each $\epsilon > 0$ there exists its basic neighborhood $A$ such that

(iv) $N_{\mu_j}(y_j) < N_\mu(x) + \epsilon$ for each $y \in A$, that is, $N_\mu(x)$ is upper semicontinuous on $(X, \mathcal{R})$, since $0 \leq N_{\mu_j}(x_j) \leq 1$ for each $x_j \in X_j$ and $j \in \Lambda$. From Formulas (i, ii, iii) and 2.2.(ii) we have

(v) $\|A\|_{\mu} = \sup_{x \in X} N_\mu(x)$ for each $A \in \mathcal{R}$, since $\|A\|_{\mu} = \sup_{x \in C} N_{\mu_j}(x)$.

For a compact subset $V$ in $X$ and each $\epsilon > 0$ there exists a finite covering $\{E_1, \ldots, E_m\} \subset \mathcal{R}$ of $V$ such that inequalities (ii - iv) are satisfied for each $E_i$ instead of $A$. Therefore,

(vi) $\sup_{x \in V} N_\mu(x) \leq \max_{i=1, \ldots, m} \|E_i\|_{\mu} \leq \sup_{x \in V} N_\mu(x) + 2\epsilon$ and

(vii) $\sup_{x \in V} N_\mu(x) = \inf_{\mathcal{R} \supseteq V} \|A\|_{\mu}$. Though the compact subset $V$ is not necessarily in $\mathcal{R}$ we take Equation (vii) as the definition of $\|V\|_{\mu} := \inf_{\mathcal{R} \supseteq V} \|A\|_{\mu}$.

Choose a sequence $\epsilon_j = \delta > 0$ for each $j \in \Lambda$, where $\delta > 0$ is independent from $j$. For each $\epsilon_j > 0$ a subset $X_{j, \epsilon_j} := \{x_j : x_j \in X_j, N_{\mu_j}(x_j) \geq \epsilon_j > 0\}$ is compact. If $x_k \in X_{k, \epsilon_k}$, then $N_{\mu_j}(\pi_j^k(x_k)) \geq \epsilon_k$ for each $j < k$, since $(\pi_j^k)^{-1}(\mathcal{R}_j) \subset \mathcal{R}_k$ and $\|B\|_{\mu_k} \leq \|A\|_{\mu_k}$ for each $B$ and $A$ in $\mathcal{R}_k$ with $B \subset A$. 

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Hence $\pi^k_j(X_{k,l}) \subseteq X_{j,l}$ for each $j \leq k$ in $\Lambda$. Since $\pi^m_l \circ \pi^k_j = \pi^m_k$ for each $m \geq k \geq l$ in $\Lambda$, then $\{X_{k,\delta}, \pi^k_j, \Lambda\}$ is the inverse system. The image $\pi^k_j(X_{k,\delta})$ of each compact set $X_{k,\delta}$ is compact for each $k \geq j$ (see Theorem 3.1.10 [9]), since each $(X_k, \tau_k)$ is the Hausdorff space in our consideration. Since the limit of an inverse mapping system of compact spaces is compact (see Theorem 3.2.13 [9]), then the limit $\pi_X = \lim_{\Lambda} X_{\{\epsilon_j, \theta\}}$, since each $(X_j, \mu)$ is the Hausdorff space in our consideration. Since each $(X_k, \tau_k)$ is the Hausdorff space in our consideration. The limit $\pi_X = \lim_{\Lambda} X_{\{\epsilon_j, \theta\}}$ is the Hausdorff space in our consideration. 

For a shrinking family $S$ in $\mathcal{R}$ consider all finite intersections of finite families in $S$, this gives a centered family $\mathcal{R}_0$ in $\mathcal{R}$ and denote it also by $S$. Applying $(i - vi)$ to $V = X_{\{\epsilon_j, \theta\}}$ and using basic neighborhoods $U = \pi^{-1}_k(U_k)$, where $U_k \in \mathcal{R}_k$, we get analogously to §2.8 that for each shrinking family $S$ in $\mathcal{R}$ with $\cap S = \emptyset$ there exists $\lim_{n \in S} \|A\|_{\mu} = 0$, since due to (vi) we have

(viii) $N_{\mu}(x) \leq \delta$ for each $x \in X \setminus X_{\{\epsilon_j, \theta\}}$, since $N_{\mu_j}(x_j) < \delta$ for each $x_j \in X \setminus X_{j,\delta}$ and each $j \in \Lambda$. Using the completion of $\mathcal{R}$ relative to $\mu$ we get the probability measure space $(X, \mathcal{R}_{\mu}, \mu)$.

2.12. Remark. Each family $\{G_j : j \in \Lambda\}$ such that for each $j$ and $k$ in $\Lambda$ there exists $m$ such that $G_m \supseteq G_k \cup G_j$ defines on the set $\Lambda$ the structure of the directed set: $k \geq j$ if and only if $G_k \supseteq G_j$. If there exists a (continuous) retraction $r$ of $\prod_{j \in \Lambda} X_j =: Y$ on $\lim\{X_j, \pi^k_j, \Lambda\} =: X$, that is, $r(Y) = X$ and $r(x) = x$ for each $x \in X$, then a measure $\nu$ on $(Y, \mathcal{R}_Y)$ induces a measure $\mu = r(\nu)$ on $(X, \mathcal{R}_X)$, since $r^{-1}(\mathcal{R}_X) \subset \mathcal{R}_Y$, such that in this particular case Theorem 2.11 follows from Theorem 2.8. On the other hand, Theorem 2.8 can be deduced from Theorem 2.11, since a product of topological spaces is the particular case of a limit of an inverse system, but the direct proof of §2.8 is simpler.

2.13. Note. Theorem 2.11 has an evident generalization to the following case: $\|X_j\|_{\mu_j} < \infty$ for each $j$ and there exist two limits $\lim_{j \in \Lambda_0} \mu_j(X_j) \in K$ and $\lim_{j \in \Lambda} \|X_j\|_{\mu_j} < \infty$, where $\Lambda_0 := \{j : j \in \Lambda, \mu_j(X_j) \neq 0\}$ and $\Lambda \setminus \Lambda_0$ is bounded in $\Lambda$. We have $\|X_j\|_{\mu_j} \leq \|X_k\|_{\mu_k}$ for each $j \leq k$ in $\Lambda$, since $\pi^k_j(\mu_k) = \mu_j$ and $\pi^k_\Lambda(\mathcal{R}_k) \subset \mathcal{R}_j$. Since $\Lambda$ is directed, then $\lim_{j \in \Lambda} \|X_j\|_{\mu_j} = \sup_{j \in \Lambda} \|X_j\|_{\mu_j}$. Since a cylindrical distribution $\mu$ is defined on $\mathcal{R}$ and bounded on it, then $\mu$ has an extension to the bounded measure $\mu$ on $\mathcal{R}$ such that $\mu(X) = \lim_{j \in \Lambda} \mu_j(X_j)$ and $\|X\|_\mu = \lim_{j \in \Lambda} \|X_j\|_{\mu_j}$, where $\mathcal{R}_\mu$ is the completion of $\mathcal{R}$ relative to $\mu$.

Let now $X$ be a set with a covering ring $\mathcal{R}$ such that $X \in \mathcal{R}$. Let also $\{(X, G_j, \mu_j) : j \in \Lambda\}$ be a family of measure spaces such that $\Lambda$ is directed.
and $G_j \subset G_k$ for each $j \leq k \in \Lambda$, $R = \bigcup_{j \in \Lambda} G_j$. Suppose $\mu : R \to K$ is such that $\mu|_{G_j} = \mu_j$ and $\mu_k|_{G_j} = \mu_j$ for each $j \leq k$ in $\Lambda$. Then the triple $(X,R,\mu)$ is called the cylindrical distribution. For each $A \in R$ there exists $j \in \Lambda$ such that $A \in G_j$, hence $\|A\|_\mu = \|A\|_{\mu_k}$ for each $k \geq j$ in $\Lambda$, consequently, $\|A\|_\mu := \lim_{k \in \Lambda} \|A\|_{\mu_k}$ is correctly defined. Suppose $\mu$ is bounded, that is, $\|X\|_\mu < \infty$. (A particular simpler case is given below in §2.15).

2.14. Theorem. Let $(X,R,\mu)$ be a bounded cylindrical distribution as in §2.13. Then $\mu$ has an extension to a bounded measure $\mu$ on the completion $R_\mu$ of $R$ relative to $\mu$.

Proof. Let $\tau_X$ be a topology on $X$ generated by the base $R$. In view of Proposition 2.6 each covering ring $G_j$ of $X$ produces an equivalence relation $\kappa_j$ and a quotient mapping $\pi_j : X \to X_j$ such that $\pi_j(G_j) =: R_j$ is a separating covering ring of $X_j$, where $X_j$ is zero-dimensional and Hausdorff. Moreover, $R_j$ is the base of topology $\tau_j$ on $X_j$. Since $G_k \supset G_j$ for each $k \geq j$, then on $(X_k, (\pi_k^j)^{-1}(R_j))$ there exists an equivalence relation $\kappa_k^j$ and a quotient (continuous) mapping $\pi_k^j : X_k \to X_j$ such that $\pi_k^m \circ \pi_k^j = \pi_j^m$ for each $j \leq k \leq m$ in $\Lambda$. Hence there exists an inverse mapping system $\{X_k, \pi_k^j, \Lambda\}$.

Therefore, the set $X$ in the topology $\tau_X$ generated by its base $R$ consisting of clopen subsets is homeomorphic with $\bigcup \{X_k, \pi_k^j, \Lambda\}$. Each $\pi_j(\mu) = \mu_j$ is a bounded measure on $(X_j, R_j)$ such that $\pi_k^j(\mu_k) = \mu_j$ and $(\pi_k^j)^{-1}(R_j) \subset R_k$ for each $k \geq j \in \Lambda$. Therefore, $\{(X_j, R_j, \mu_j) : j \in \Lambda\}$ is the consistent family of measure spaces. From the definition of $\mu$ it follows that $\mu$ is additive, hence $\|X\|_\mu$ is correctly defined. From $\|X\|_\mu < \infty$ it follows $\|X_j\|_{\mu_j} < \infty$ for each $j \in \Lambda$ and there exists $\lim_{j \in \Lambda} \|X_j\|_{\mu_j} = \|X\|_\mu$. From $X \in R$ it follows, that $\mu(X) = \mu_j(X_j)$ for each $j \in \Lambda$. Then this Theorem follows from Theorem 2.11.

2.15.1. Note. Let $X := \prod_{t \in T} X_t$ be a product of sets $X_t$ and on $X$ a covering ring $R$ be given such that for each $n \in \mathbb{N}$ and pairwise distinct points $t_1, \ldots, t_n$ in a set $T$ there exists a $K$-valued measure $P_{t_1,\ldots,t_n}$ on a covering ring $\mathcal{R}_{t_1,\ldots,t_n}$ of $X_{t_1} \times \ldots \times X_{t_n}$ such that $\pi_{t_1,\ldots,t_n}^{t_1,\ldots,t_n+1}(\mathcal{R}_{t_1,\ldots,t_n+1}) = \mathcal{R}_{t_1,\ldots,t_n}$ for each $t_{n+1} \in T$ and $P_{t_1,\ldots,t_n+1}(A_1 \times \ldots \times A_n \times X_{t_{n+1}}) = P_{t_1,\ldots,t_n}(A_1 \times \ldots \times A_n)$ for each $A_1 \times \ldots \times A_n \in \mathcal{R}_{t_1,\ldots,t_n}$, where $\pi_{t_1,\ldots,t_n}^{t_1,\ldots,t_n+1} : X_{t_1} \times \ldots \times X_{t_{n+1}} \rightarrow X_{t_1} \times \ldots \times X_{t_n}$ is the natural projection, $A_l \subset X_{t_l}$ for each $l = 1, \ldots, n$. Suppose that the cylindrical distribution is bounded, that is, 

$\sup_{t_1,\ldots,t_n \in T, n \in \mathbb{N}} \|P_{t_1,\ldots,t_n}\| < \infty$ and there exists
\[
\lim_{t_1, \ldots, t_n \in T_0; n \in \mathbb{N}} P_{t_1, \ldots, t_n}(X_{t_1} \times \ldots \times X_{t_n}) \in \mathcal{K}, \text{ where } T_0 := \{ t \in T : P_t(X_t) \neq 0 \}, \quad T \setminus T_0 \text{ is finite.}
\]

2.15.2. **Theorem** (the non-Archimedean analog of the Kolmogorov theorem). The cylindrical distribution \( P_{t_1, \ldots, t_n} \) from §2.15.1 has an extension to a bounded measure \( P \) on the completion \( \mathcal{R}_P \) of \( \mathcal{R} := \bigcup_{t_1, \ldots, t_n \in T, n \in \mathbb{N}} \mathcal{G}_{t_1, \ldots, t_n} \) relative to \( P \), where \( \mathcal{G}_{t_1, \ldots, t_n} := (\pi_{t_1, \ldots, t_n})^{-1}(\mathcal{R}_{t_1, \ldots, t_n}) \) and \( \pi_{t_1, \ldots, t_n} : X \to X_{t_1} \times \ldots \times X_{t_n} \) is the natural projection.

3 Markov distributions for a non-Archimedean Banach space.

3.1. **Remark.** Let \( H = c_0(\alpha, \mathbf{K}) \) be a Banach space over a non-Archimedean field \( \mathbf{K} \) with an ordinal \( \alpha \) (that is useful due to Kuratowski-Zorn lemma, see §1.3) and the standard orthonormal base \( \{e_j : j \in \alpha\} \), \( e_j = (0, \ldots, 0, 1, 0, \ldots) \) with 1 on the \( j \)-th place, that is, \( c_0(\alpha, \mathbf{K}) = \{x : x = (x_i : i \in \alpha, x_i \in \mathbf{K}), \sup_{i \in \alpha} |x_i| =: \|x\| < \infty, \text{ for each } b > 0 \text{ a set } \{i : |x_i| > b\} \text{ is finite }\}\). Suppose that \( \mathbf{K} \) is complete as the ultrametric space. For example, \( \mathbf{K} \) is such that \( \mathbf{Q}_p \subset \mathbf{K} \subset \mathbf{C}_p \) or \( \mathbf{F}_p(\theta) \subset \mathbf{K} \), where \( p \) is a prime number, \( \mathbf{Q}_p \) is the field of \( p \)-adic numbers, \( \mathbf{C}_p \) is the field of complex \( p \)-adic numbers, \( \mathbf{F}_p(\theta) \) is the field of formal power series by an indeterminate \( \theta \) over the finite field \( \mathbf{F}_p \) consisting of \( p \) elements. Let \( \mathcal{U}_p \) be a cylindrical ring generated by projections \( \pi_F : H \to F \) on finite dimensional over \( \mathbf{K} \) subspaces \( F \) in \( H \) and rings \( \text{Bco}(F) \) of clopen subsets. This ring \( \mathcal{U}_p \) is the base of the weak topology \( \tau_{H,w} \) in \( H \). Each vector \( x \in H \) is considered as continuous linear functional on \( H \) by the formula \( x(y) = \sum_j x^j y^j \) for each \( y \in H \), so there is the natural embedding \( H \hookrightarrow H^* = l^\infty(\alpha, \mathbf{K}) \), where \( x = \sum_j x^j e_j, x^j \in \mathbf{K}, l^\infty(\alpha, \mathbf{K}) := \{x : x = (x_i : i \in \alpha, x_i \in \mathbf{K}), \sup_{i \in \alpha} |x_i| =: \|x\| < \infty\} \). This justifies the following generalization.

3.2. **Notes and definitions.** Let \( T \) be a subset in \( \Lambda \) and containing a point \( t_0 \) and \( X_t = X \) be a locally \( \mathbf{K} \)-convex space for each \( t \in T \), where \( \Lambda \) is an additive group, for example, \( \Lambda \) is contained in \( \mathbf{R} \) or \( \mathbf{C} \) or a non-Archimedean field. Put \( (\hat{X}_T, \hat{U}) := \prod_{t \in T}(X_t, U_t) \) be a product of measurable spaces, where \( U_t \) are rings of clopen subsets of \( X_t \), \( \hat{U} \) is the ring of cylindrical subsets of \( \hat{X}_T \) generated by projections \( \hat{\pi}_q : \hat{X}_t \to X^q, X^q := \prod_{t \in q} X_t, q \subset T \) is a finite subset of \( T \) (see §1.1.3 [3]). Let \( \mathbf{K}_s \) be a subfield of \( \mathbf{C}_s \) such that \( \mathbf{K}_s \)
is complete as the ultrametric space, where $s$ is a prime number. A function $P(t_1, x_1, t_2, A)$ with values in $K_s$ for each $t_1 \neq t_2 \in T$, $x_1 \in X_{t_1}$, $A \in \mathcal{U}_{t_2}$ is called a transition measure if it satisfies the following conditions:

(i) the set function $\nu_{x_1,t_1,t_2}(A) := P(t_1, x_1, t_2, A)$ is a measure on $(X_{t_2}, \mathcal{U}_{t_2})$;
(ii) the function $\alpha_{t_1,t_2,A}(x_1) := P(t_1, x_1, t_2, A)$ of the variable $x_1$ is $\mathcal{U}_{t_1}$-measurable, that is, $\alpha_{t_1,t_2,A}(Bco(K_s)) \subseteq \mathcal{U}_{t_1}$;

(iii) $P(t_1, x_1, t_2, A) = \int_{X_x} P(t_1, x_1, z, dy)P(z, y, t_2, A)$ for each $t_1 \neq t_2 \in T$, that is, $P(z, y, t_2, A)$ as the function by $y$ is in $L((X_z, \mathcal{U}_z), \nu_{x_1,t_1,z}, K_s)$. A transition measure $P(t_1, x_1, t_2, A)$ is called normalised if

(iv) $P(t_1, x_1, t_2, X_{t_2}) = 1$ for each $t_1 \neq t_2 \in T$.

For each set $q = (t_0, t_1, \ldots, t_{n+1})$ of pairwise distinct points in $T$ there is defined a measure in $X^g := \prod_{t \in g} X_t$ by the formula

$$(v) \mu_{x_0}^g(E) = \int_E \prod_{k=1}^{n+1} P(t_{k-1}, x_{k-1}, t_k, dx_k), \quad E \in \mathcal{U}_g := \prod_{t \in g} \mathcal{U}_t,$$

where $g = q \setminus \{t_0\}$, variables $x_1, \ldots, x_{n+1}$ are such that $(x_1, \ldots, x_{n+1}) \in E$, $x_0 \in X_{t_0}$ is fixed.

Let $E = E_1 \times X_{t_1} \times E_2$, where $E_1 \in \prod_{t=t_1}^{t_0-1} \mathcal{U}_t$, $E_2 \in \prod_{t=t_0+1}^{t_0+n+1} \mathcal{U}_t$, if the transition measure $P(t, x_1, t_2, dx_2)$ is normalised, then

$$(vi) \mu_{x_0}^g(E) = \int_{E_1 \times X_{t_1}} \prod_{k=1}^{n-1} P(t_{k-1}, x_{k-1}, t_k, dx_k) \times \int_{X_{t_1}} P(t_{n-1}, x_{n-1}, t_n, dx_n) \prod_{k=n+1}^{n+1} P(t_{n-1}, x_{n-1}, t_k, dx_k) = \mu_{x_0}^g(E_1 \times E_2),$$

where $r = q \setminus \{t_n\}$. From Equation (vi) it follows, that

$$(vii) [\mu_{x_0}^g]_{x_0}^g = \mu_{x_0}^v$$

for each $v < q$ (that is, $v \subset q$), where $\pi_q^g : X^g \to X^v$ is the natural projection, $g = q \setminus \{t_0\}$, $w = v \setminus \{t_0\}$. Therefore, due to Conditions (iv, v, vii) : $\{\mu_{x_0}^q, \pi_q^g, \gamma_T\}$ is the consistent family of measures, which induce the cylindrical distribution $\tilde{\mu}_{x_0}$ on $(\tilde{X}_T, \tilde{\mathcal{U}})$ such that $\tilde{\mu}_{x_0}(\pi_q^{-1}(E)) = \mu_{x_0}^q(E)$ for each $E \in \mathcal{U}_q$, where $\gamma_T$ is the family of all finite subsets $q$ in $T$ such that $t_0 \in q \subset T$, $v \leq q \in \gamma_T$, $\pi_q : \tilde{X}_T \to X^q$ is the natural projection, $g = q \setminus \{t_0\}$.

The cylindrical distributions given by Equations (i – v, vii) are called Markov distributions (with time $t \in T$).
3.3. Proposition. If a normalized transition measure $P$ satisfies the condition
\[(i) C := \sup_q \left( \sum_{k=1}^{n} \ln(\sup_x \|\nu_{x,t_{k-1},t_k}\|) \right) < \infty,\]
where $q = (t_0, t_1, \ldots, t_n)$ with pairwise distinct points $t_0, \ldots, t_n \in T$ and $n \in \mathbb{N}$, then the Markov cylindrical distribution $\tilde{\mu}_{x_0}$ is bounded and it has an extension to a bounded measure $\tilde{\mu}_{x_0}$ on the completion $\tilde{\mathbb{U}}_{\tilde{\mu}_{x_0}}$ of $\tilde{\mathbb{U}}$ relative to $\tilde{\mu}_{x_0}$.

3.3.2. Proposition. If
\[(ii) C_x := \sup_q \left( \sum_{k=1}^{n} \ln(\|\nu_{x,t_{k-1},t_k}\|) \right) = \infty\]
for each $x$, where $q = (t_0, t_1, \ldots, t_n)$ with pairwise distinct points $t_0, \ldots, t_n \in T$ and $n \in \mathbb{N}$, then the Markov cylindrical distribution $\tilde{\mu}_{x_0}$ has the unbounded variation on each nonvoid set $E \in \mathbb{U}$.

Proof. (1). If $E \in \mathbb{U}$, then $E \in \mathbb{U}^g$ for some set $q = (t_0, t_1, \ldots, t_n)$ with pairwise distinct points $t_0, \ldots, t_n \in T$ and $n \in \mathbb{N}$ and $g = q \setminus \{t_0\}$, consequently, $|\mu_{x_0}^g(E)| \leq \prod_{k=1}^{n} \sup_x \|\nu_{x,t_{k-1},t_k}\| \leq \exp(C) < \infty$, since $t_k \in T$ for each $k = 0, 1, \ldots, n$, hence $\sup_{q,E} |\mu_{x_0}^g(E)| = \|\tilde{\mu}_{x_0}\| \leq \exp(C)$. In view of Theorem 2.15.2 we get an extension of $\tilde{\mu}_{x_0}$ to a bounded measure on $\tilde{\mathbb{U}}_{\tilde{\mu}_{x_0}}$.

(2). For each $(t_1, t_2, x)$ with $x$ in $\pi_{t_0,t_2}(E)$ there exists a set $\delta(t_1, t_2, x) \in \mathbb{U}_x \cap \pi_{t_0,t_2}(E)$ such that $|\delta(t_1, t_2, x)|_{\nu_{x_1,t_1,t_2}} > 1 + \epsilon(t_1, t_2, x)$, where $\epsilon(t_1, t_2, x) > 0$. In view of Condition (ii) for each $R > 0$ and $x$ we choose $q$ such that $\sum_{k=1}^{n} \epsilon(t_k, t_{k+1}, x_1, x) > R$. For chosen $u \neq u_1 \in T$ and $x \in \pi_{t_0,u}(E) \subset X_u$ we represent the set $\delta(u, u_1, x)$ as a finite union of disjoint subsets $\gamma_{j_1} \subset U_{u_1}$ such that for each $\gamma_{j_1}$ and $u_2 \neq u_1$ there is a set $\delta_{j_1} \subset U_{u_2} \cap \pi_{t_0,u_2}(E)$ satisfying $\|\delta_{j_1}\|_{\nu_{x_1,u_1,u_2}} \geq 1 + \epsilon(u_1, u_2, x)$ for each $x \in \gamma_{j_1}$. Then by induction $\delta_{j_1,\ldots,j_n} = \bigcup_{j_{n+1}=1}^{\gamma_{j_1,\ldots,j_n+1}} \gamma_{j_1,\ldots,j_n+1}$ so that for $u_{n+2} \neq u_{n+1} \in T$ there is a set $\delta_{j_1,\ldots,j_n+1} \subset U_{u_{n+1}} \cap \pi_{t_0,u_{n+1}}(E)$ for which $\|\delta_{j_1,\ldots,j_n+1}\|_{\nu_{x_{n+1},u_{n+1},u_{n+2}}} \geq 1 + \epsilon(u_{n+2} - u_{n+1}, x)$ for each $x \in \gamma_{j_1,\ldots,j_n}$, $\delta_{j_1,\ldots,j_n} \subset \gamma_{j_1,\ldots,j_n}$, $x(u_{n+1}) \subset \gamma_{j_1,\ldots,j_n}$, $\Gamma_{j_1,\ldots,j_n} := (\bigcup_{j_1,\ldots,j_n} \Gamma_{u_0,x_0,j_1,\ldots,j_n}) \subset \tilde{\mathbb{U}}$, since $m_1 \in N_{j_1}, \ldots, m_n \in N$. Then $\|\Gamma_{u,x_0}\|_{\tilde{\mu}_{x_0}} \geq \sup_{j_1,\ldots,j_n} \|\Pi_{k=1}^{j_k} \nu_{x_{k-1},x_{k-1},x_k}(dx_k)\|_{j_1,\ldots,j_n} \geq \prod_{k=1}^{n} [1 + \epsilon(u_{k-1} - u_k, x_{k-1}, x_k)] > R$, consequently, $\|E\|_{\tilde{\mu}_{x_0}} = \infty$, since $\|E\|_{\tilde{\mu}_{x_0}} \geq \sup_{u_0,x_0} \|\Gamma_{u,x_0}\|_{\tilde{\mu}_{x_0}}$ and $R > 0$ is arbitrary.
3.4. Let $X_t = X$ for each $t \in T$, $\tilde{X}_{t_0,x_0} := \{x \in \tilde{X}_T : x(t_0) = x_0\}$. We define a projection operator $\bar{\pi}_q : x \mapsto x_q$, where $x_q$ is defined on $q = (t_0, \ldots, t_{n+1})$ such that $x_q(t) = x(t)$ for each $t \in q$, that is, $x_q = x|_q$. For every $F : \tilde{X}_T \to C_s$ there corresponds $(S_q F)(x) := F(x_q) = F_q(y_0, \ldots, y_n)$, where $y_j = x(t_j), F_q : X^q \to C_s$. We put $F := \{F|F : X_T \to C_s, S_q F$ are $U^q$-measurable$\}$. If $F \in F$, $\tau = t_0 \in q$, then there exists an integral

$$(i) \ J_q(F) = \int_{X^q} (S_q F)(x_0, \ldots, x_n) \prod_{k=1}^{n+1} P(t_{k-1}, x_{k-1}, t_k, dx_k).$$

**Definition.** A function $F$ is called integrable with respect to the Markov cylindrical distribution $\mu_{x_0}$ if the limit

$$(ii) \ \lim_q J_q(F) =: J(F)$$

along the generalized net by finite subsets $q$ of $T$ exists. This limit is called a functional integral with respect to the Markov cylindrical distribution:

$$(iii) \ J(F) = \int_{\tilde{X}_{t_0,x_0}} F(x) \mu_{x_0}(dx).$$

3.5. **Remark.** Consider a $K_s$-valued measure $P(t, A)$ on $(X, \mathcal{U})$ for each $t \in T$ such that $A - x \in \mathcal{U}$ for each $A \in \mathcal{U}$ and $x \in X$, where $A \in \mathcal{U}$, $X$ is a locally $K$-convex space, $\mathcal{U}$ is a covering ring of $X$. Suppose $P$ be a spatially homogeneous transition measure (see also §3.2), that is,

$$(i) \ P(t_1, x_1, t_2, A) = P(t_2 - t_1, A - x_1)$$

for each $A \in \mathcal{U}$, $t_1 \neq t_2 \in T$ and $t_2 - t_1 \in T$ and every $x_1 \in X$, where $P(t, A)$ satisfies the following condition:

$$(ii) \ P(t_1 + t_2, A) = \int_X P(t_1, dy)P(t_2, A - y)$$

for each $t_1$ and $t_2$ and $t_1 + t_2 \in T$. Such a transition measure $P(t_1, x_1, t_2, A)$ is called homogeneous. In particular for $T = \mathbb{Z}_p$ we have

$$(iii) \ P(t + 1, A) = \int_X P(t, dy)P(1, A - y).$$
If \( P(t, A) \) is a continuous function by \( t \in T \) for each fixed \( A \in U \), then Equation \((iii)\) defines \( P(t, A) \) for each \( t \in T \), when \( P(1, A) \) is given, since \( Z \) is dense in \( Z_p \). §2.7 and 2.15 and 4.3 provide examples of Markov distributions. Examples of Markov distributions are also Poisson and Gaussian distributions given below and in a forthcoming paper.

3.6. Notes and definition. Let \( X \) be a locally \( K \)-convex space and \( P \) satisfies Conditions 3.2\((i - iii)\). For \( x \) and \( z \in Q^n_p \), we denote by \((z, x)\) the following sum: \( \sum_{j=1}^{n} x_j z_j \), where \( x = (x_j : j = 1, ..., n) \), \( x_j \in Q_p \). Each number \( y \in Q_p \) has a decomposition \( y = \sum_l a_l p^l \), where \( a_l \in (0, 1, ..., p - 1) \), \( \min(l : a_l \neq 0) =: ord_p(y) > -\infty \) for \( y \neq 0 \) and \( ord(0) := \infty \). We define a symbol \( \{y\}_p := \sum_{l < 0} a_l p^l \) for \( |y|_p > 1 \) and \( \{y\}_p = 0 \) for \( |y|_p \leq 1 \). We consider a character of \( X \), \( \chi_\gamma : X \to C_s \) given by the following formula:

\[
(i) \quad \chi_\gamma(x) = e^{\epsilon^{-1} \{e, \gamma(x)\}}_p
\]

for each \( \{e, \gamma(x)\}_p \neq 0 \), \( \chi_\gamma(x) := 1 \) for \( \{e, \gamma(x)\}_p = 0 \), where \( \epsilon = 1^z \) is a root of unity, \( z = p^{ord\{e, \gamma(x)\}_p} \), \( \gamma \in X^* \), \( X^* \) denotes the topologically conjugated space of continuous \( K \)-linear functionals on \( X \), the field \( K \) as the \( Q_p \)-linear space is \( n \)-dimensional, that is, \( dim_{Q_p} K = n \), \( K \) as the Banach space over \( Q_p \) is isomorphic with \( Q^n_p \), \( e = (1, ..., 1) \in Q^n_p \), where \( s \neq p \) are prime numbers (see [45] and [32]). Then

\[
(ii) \quad \phi(t_1, x_1, t_2, y) := \int_X \chi_y(x) P(t_1, x_1, t_2, dx)
\]

is the characteristic functional of the transition measure \( P(t_1, x_1, t_2, dx) \) for each \( t_1 \neq t_2 \in T \) and each \( x_1 \in X \). In the particular case of \( P \) satisfying Conditions 3.5\((i, ii)\) with \( t_0 = 0 \) its characteristic functional is such that

\[
(iii) \quad \phi(t_1, x_1, t_2, y) = \psi(t_2 - t_1, y) \chi_y(x_1), \text{ where }
\]

\[
(iv) \quad \psi(t, y) := \int_X \chi_y(x) P(t, dx) \quad \text{and }
\]

\[
(v) \quad \psi(t_1 + t_2, y) = \psi(t_1, y) \psi(t_2, y)
\]

for each \( t_1 \neq t_2 \in T \) and \( t_2 - t_1 \in T \) and \( t_1 + t_2 \in T \) respectively and \( y \in X^* \), \( x_1 \in X \).
4 Non-Archimedean stochastic processes.

4.1. Remark and definition. A measurable space \((\Omega, F)\) with a probability \(K_s\)-valued measure \(\lambda\) on a covering ring \(F\) of a set \(\Omega\) is called a probability space and is denoted by \((\Omega, F, \lambda)\). Points \(\omega \in \Omega\) are called elementary events and values \(\lambda(S)\) probabilities of events \(S \in F\). A measurable map \(\xi : (\Omega, F) \to (X, B)\) is called a random variable with values in \(X\), where \(B\) is a covering ring such that \(B \subset B\co(X)\), \(B\co(X)\) is the ring of all clopen subsets of a locally \(K\)-convex space \(X\), \(\xi^{-1}(B) \subset F\), where \(K\) is a non-Archimedean field complete as an ultrametric space.

The random variable \(\xi\) induces a normalized measure \(\nu_\xi(A) := \lambda(\xi^{-1}(A))\) in \(X\) and a new probability space \((X, B, \nu_\xi)\).

Let \(T\) be a set with a covering ring \(R\) and a measure \(\eta : R \to K_s\). Consider the following Banach space \(L^q(T, R, \eta, H)\) as the completion of the set of all \(R\)-step functions \(f : T \to H\) relative to the following norm:

(1) \[ \|f\|_{\eta,q} := \sup_{t \in T} \|f(t)\|_{H^\eta(t)}^{1/q} \] for \(1 \leq q < \infty\) and

(2) \[ \|f\|_{\eta,\infty} := \sup_{1 \leq q < \infty} \|f(t)\|_{\eta,q}, \] where \(H\) is a Banach space over \(K\) (see also §2.5). For \(0 < q < 1\) this is the metric space with the metric

(3) \[ \rho_q(f, g) := \sup_{t \in T} \|f(t) - g(t)\|_{H^\eta(t)}^{1/q}. \]

If \(H\) is a complete locally \(K\)-convex space, then \(H\) is a projective limit of Banach spaces \(H = \lim\{H, \pi_\beta^\alpha, \Upsilon\}\), where \(\Upsilon\) is a directed set, \(\pi_\beta^\alpha : H \to H_\beta\) is a \(K\)-linear continuous mapping for each \(\alpha \geq \beta\), \(\pi_\alpha : H \to H_\alpha\) is a \(K\)-linear continuous mapping such that \(\pi_\beta^\alpha \circ \pi_\alpha = \pi_\beta\) for each \(\alpha \geq \beta\) (see §6.205 [36]). Each norm \(\rho_\alpha\) on \(H_\alpha\) induces a prednorm \(\tilde{\rho}_\alpha\) on \(H\). If \(f : T \to H\), then \(\pi_\alpha \circ f = : f_\alpha : T \to H_\alpha\). In this case \(L^q(T, R, \eta, H)\) is defined as a completion of a family of all step functions \(f : T \to H\) relative to the family of prednorms

(1') \[ \|f\|_{\eta,q,\alpha} := \sup_{t \in T} \tilde{\rho}_\alpha(f(t))N_{\eta(t)}^{1/q}, \] \(\alpha \in \Upsilon\), for \(1 \leq q < \infty\) and

(2') \[ \|f\|_{\eta,\infty,\alpha} := \sup_{1 \leq q < \infty} \|f(t)\|_{\eta,q,\alpha}, \] \(\alpha \in \Upsilon\), or pseudometrics

(3') \[ \rho_{q,\alpha}(f, g) := \sup_{t \in T} \tilde{\rho}_\alpha(f(t) - g(t))N_{\eta(t)}^{1/q}, \] \(\alpha \in \Upsilon\), for \(0 < q < 1\). Therefore, \(L^q(T, R, \eta, H)\) is isomorphic with the projective limit

\[ \lim\{L^q(T, R, \eta, H_\alpha), \pi_\alpha^\alpha, \Upsilon\}. \] For \(q = 1\) we write simply \(L(T, R, \eta, H)\) and \(\|f\|_\eta\). This definition is correct, since \(\lim_{q \to \infty} a^{1/q} = 1\) for each \(\infty > a > 0\).

For example, \(T\) may be a subset of \(R\). Let \(R_d\) be the field \(R\) supplied with the discrete topology. Since the cardinality \(\text{card}(R) = c = 2^{8^n}\), then there are bijective mappings of \(R\) on \(Y_1 := \{0, \ldots, b\}^N\) and also on \(Y_2 := N^N\), where \(b\) is a positive integer number. Supply \(\{0, \ldots, b\}\) and \(N\) with the
discrete topologies and \( Y_1 \) and \( Y_2 \) with the product topologies. Then zero-dimensional spaces \( Y_1 \) and \( Y_2 \) supply \( \mathbb{R} \) with covering separating rings \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) contained in \( Bco(Y_1) \) and \( Bco(Y_2) \) respectively. Certainly this is not related with the standard (Euclidean) metric in \( \mathbb{R} \). Therefore, for the space \( L^q(T, \mathcal{R}, \eta, H) \) we can consider \( t \in T \) as the real time parameter. If \( T \subset F \) with a non-Archimedean field \( F \), then we can consider the non-Archimedean time parameter. If \( T \) is a zero-dimensional \( T_1 \)-space, then denote by \( C_0^0(T, H) \) the Banach space of continuous bounded functions \( f : T \to H \) supplied with the norm:

\[
(4) \quad \|f\|_{C_0^0} := \sup_{t \in T} \|f(t)\|_H < \infty.
\]

If \( T \) is compact, then \( C_0^0(T, H) \) is isomorphic with the space \( C_0^0(T, H) \) of continuous functions.

For a set \( T \) and a complete locally \( K \)-convex space \( H \) over \( K \) consider the product \( K \)-convex space \( H^T := \prod_{t \in T} H_t \) in the product topology, where \( H_t := H \) for each \( t \in T \).

Then take on either \( X := X(T, H) = L^q(T, \mathcal{R}, \eta, H) \) or \( X := X(T, H) = C_0^0(T, H) \) or on \( X = X(T, H) = H^T \) a covering ring \( \mathcal{B} \) such that \( \mathcal{B} \subset Bco(X) \).

Consider a random variable \( \xi : \omega \mapsto \xi(t, \omega) \) with values in \( (X, \mathcal{B}) \) and \( t \in T \).

Events \( S_1, \ldots, S_n \) are called independent in total if \( P(\Pi_{k=1}^{n} S_k) \). Subrings \( \mathcal{F}_k \subset \mathcal{F} \) are said to be independent if all collections of events \( S_k \in \mathcal{F}_k \) are independent in total, where \( k = 1, \ldots, n, n \in \mathcal{N} \). To each collection of random variables \( \xi_\gamma \) on \( (\Omega, \mathcal{F}) \) with \( \gamma \in \Upsilon \) is related the minimal ring \( \mathcal{F}_\Upsilon \subset \mathcal{F} \) with respect to which all \( \xi_\gamma \) are measurable, where \( \Upsilon \) is a set. Collections \( \{\xi_\gamma : \gamma \in \Upsilon_j\} \) are called independent if such are \( \mathcal{F}_{\Upsilon_j} \), where \( \Upsilon_j \subset \Upsilon \) for each \( j = 1, \ldots, n, n \in \mathcal{N} \).

Consider \( T \) such that \( \text{card}(T) > n \). For \( X = C_0^0(T, H) \) or \( X = H^T \) define \( X(T, H; (t_1, \ldots, t_n); (z_1, \ldots, z_n)) \) as a closed submanifold of \( f : T \to H, f \in X \) such that \( f(t_1) = z_1, \ldots, f(t_n) = z_n, \) where \( t_1, \ldots, t_n \) are pairwise distinct points in \( T \) and \( z_1, \ldots, z_n \) are points in \( H \). For pairwise distinct points \( t_1, \ldots, t_n \) in \( T \) with \( N_\eta(t_1) > 0, \ldots, N_\eta(t_n) > 0 \) define \( X(T, H; (t_1, \ldots, t_n); (z_1, \ldots, z_n)) \) as a closed submanifold which is the completion relative to the norm \( \|f\|_{\eta,q} \) of a family of \( \mathcal{R} \)-step functions \( f : T \to H \) such that \( f(t_1) = z_1, \ldots, f(t_n) = z_n. \) In these cases \( X(T, H; (t_1, \ldots, t_n); (0, \ldots, 0)) \) is the proper \( K \)-linear subspace of \( X(T, H) \) such that \( X(T, H) \) is isomorphic with \( X(T, H; (t_1, \ldots, t_n); (0, \ldots, 0)) \oplus H^n, \) since if \( f \in X, \) then \( f(t) - f(t_1) =: g(t) \in X(T, H; t_1; z_1) \) (in the third case we use that \( T \in \mathcal{R} \) and hence there exists the embedding \( H \hookrightarrow X \)). For \( n = 1 \) and \( t_0 \in T \) and \( z_1 = 0 \) we denote \( X_0 := X_0(T, H) := X(T, H; t_0; 0). \)
4.2. Definition. We define a (non-Archimedean) stochastic process \( w(t, \omega) \) with values in \( H \) as a random variable such that:

(i) the differences \( w(t_4, \omega) - w(t_3, \omega) \) and \( w(t_2, \omega) - w(t_1, \omega) \) are independent for each chosen \( \omega, (t_1, t_2) \) and \( (t_3, t_4) \) with \( t_1 \neq t_2, t_3 \neq t_4 \), either \( t_1 \) or \( t_2 \) is not in the two-element set \( \{ t_3, t_4 \} \), where \( \omega \in \Omega \);

(ii) the random variable \( \omega(t, \omega) - \omega(u, \omega) \) has a distribution \( \mu^{F,t,u} \), where \( \mu \) is a probability \( K_{\omega} \)-valued measure on \( (X(T, H), B) \) from §4.1, \( \mu^{g}(A) := \mu(g^{-1}(A)) \) for \( g : X \to H \) such that \( g^{-1}(\mathcal{R}_H) \subset B \) and each \( A \in \mathcal{R}_H \), a continuous linear operator \( F_{t,u} : X \to H \) is given by the formula \( F_{t,u}(w) := w(t, \omega) - w(u, \omega) \) for each \( w \in L^{q}(\Omega, F, \lambda; X_0) \), where \( 1 \leq q \leq \infty, X_0 \) is the closed subspace of \( X \) as in §4.1, \( \mathcal{R}_H \) is a covering ring of \( H \) such that \( F_{t,u}^{-1}(\mathcal{R}_H) \subset B \) for each \( t \neq u \) in \( T \);

(iii) we also put \( w(0, \omega) = 0 \), that is, we consider a \( K \)-linear subspace \( L^{q}(\Omega, F, \lambda; X_0) \) of \( L^{q}(\Omega, F, \lambda; X) \), where \( \Omega \neq \emptyset \).

It is seen that \( w(t, \omega) \) is a Markov process with transition measure \( P(u, x, t, A) = \mu^{F,t,u}(A \setminus x) \).

This definition is justified by the following Theorem.

4.3. Theorem. Let either \( X = C_{0}^{0}(T, H) \) or \( X = H^{T} \) or \( X = L^{q}(T, \mathcal{R}, \eta, H) \) with \( 1 \leq q \leq \infty \) be the same spaces as in §4.1, where the valuation group \( \Gamma_{K} \) is discrete in \((0, \infty)\). Then there exists a family \( \Psi \) of pairwise inequivalent (non-Archimedean) stochastic processes on \( X \) of the cardinality \( \text{card}(\Psi) \geq \text{card}(T) \text{card}(H) \) or \( \text{card}(\Psi) \geq \text{card}(\mathcal{R}) \text{card}(H) \) respectively.

Proof. Each complete locally \( K \)-convex space \( H \) is a projective limit of Banach spaces \( H_{\alpha} \). Therefore, due to §§2.25 and 4.2 it is sufficient to consider the case of the Banach space \( H \). Since \( H \) is over the field \( K \) with the valuation group \( \Gamma_{K} \) discrete in \((0, \infty)\), then \( H \) is isomorphic with the Banach space \( c_{0}(\alpha, K) \) (see Theorems 5.13 and 5.16 [14]), where \( \alpha \) is an ordinal.

Let \( \mathcal{R}_{K} \) be a covering separating ring of \( K \) such that elements of \( \mathcal{R}_{K} \) are clopen subsets in \( K \). Then there exists a lot of probability measures \( m \) on \( (K, \mathcal{R}_{K}) \) with values in \( K_{s} \), for example, atomic measure with atoms \( a_{j} \) such that for each \( U \in \mathcal{R}_{K} \) either \( a_{j} \subset U \) or \( a_{j} \subset K \setminus U \). For example, this can be done for singleton atoms. Let the family \( \Upsilon \) of \( a_{j} \) be countable and \( \lim_{j} m(a_{j}) = 0 \), when \( \Upsilon \) is infinite. Then \( m(S) := \sum_{a_{j} \subset S} a_{j} \) for each \( S \in \mathcal{R}_{K} \) and \( \|m\| = \sup_{j} |m(a_{j})| \). If \( K \) is infinite and contains a locally compact infinite subfield \( F \) with a nontrivial valuation, then \( K \) can be considered as a locally \( F \)-convex space. As the locally \( F \)-convex space \( K \) in its weak topology is isomorphic with \( F^{\Upsilon} \), since \( \Gamma_{F} \) is discrete in \((0, \infty)\) and there is the
non-Archimedean variant of the Hahn-Banach theorem, where \( \gamma \) is a set (see §8.203 in [36]). Having a measure on \( F \) we can construct a probability measure on \( K \) due to Theorem 2.8 and Note 2.9 and Remarks 2.5.

Therefore, consider also the particular case of the locally compact field \( K \). If \( K \) is infinite, then either \( K \supseteq \mathbb{Q}_p \) or \( K = F_p(\theta) \) with the corresponding prime number \( p \), since \( K \) is with the nontrivial valuation \( \mathbb{Q}_p \). If \( K \) is finite, then \( K = F_p \). Let \( s \) be a prime number such that \( s \neq p \), then \( K \) is \( s \)-free as the additive topological group (see the Monna-Springer theorem in §8.4 in [14]). Therefore, there exists the \( K_s \)-valued Haar measure \( w \) on \( K \), that is, the bounded measure on each clopen compact subset of \( K \) with \( w(B(K,0,1)) = 1 \) and \( w(y + A) = w(A) \) for each \( A \in Bco(K) \) and each \( y \in K \), where \( B(Y,y,r) := \{ z : z \in Y, d(y,z) \leq r \} \) is the ball in an ultrametric space \( Y \) with an ultrametric \( d \) and a point \( y \in Y \).

We have the following isomorphisms:
\[
L^\theta(T, \mathcal{R,} \eta, H) = L^\theta(T, \mathcal{R,} \eta, K) \otimes H \quad \text{and} \quad C_b^0(T, H) = C_b^0(T, K) \otimes H,
\]
moreover, \( L^\theta(T, \mathcal{R,} \eta, K) \) is isomorphic with \( c_0(\beta_L, K) \) and \( C_b^0(T, K) \) is isomorphic with \( c_0(\beta_C, K) \), where \( \beta_L \) and \( \beta_C \) are ordinals, since \( \Gamma_K \) is discrete (see Chapter 5 in [14]). The locally \( K \)-convex space \( HT \) is isomorphic with \( Y_1 \otimes H \), where \( Y_1 := K^T \). On the other hand, the Banach space \( c_0(\alpha, K) \) in its weak topology \( \tau_w \) is isomorphic with \( K^T \) for \( \text{card}(\alpha) = \text{card}(T) \), since the valuation group of \( K \) is discrete in \( (0, \infty) \) (see §8.203 in [36]). Therefore, the ring of clopen subsets in \( (c_0(\alpha, K), \tau_w) \) supplies \( Y_1 \) with the covering separating ring. If \( \mu_1 \) and \( \mu_2 \) are \( K_s \)-valued measures on Banach spaces \( Y_1 \) and \( Y_2 \) with covering rings \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) respectively, then \( \mu_1 \otimes \mu_2 \) is the \( K_s \)-valued measure on \( (Y_1 \otimes Y_2, \mathcal{R}_1 \times \mathcal{R}_2) \). In the Banach space \( c_0(\beta, K) \) there exists the canonical base \( (e_j : j \in \beta) \), where \( e_j := (0, \ldots, 0, 1, 0, \ldots) \) with 1 on the \( j \)-th place. With this standard base are associated projections \( \pi_{j_1, \ldots, j_n}(x) := \sum_{i=1}^{n} x^i e_{j_i} \) for each \( j_1, \ldots, j_n \in \beta \) and each \( n \in \mathbb{N} \) and for each vector \( x \in c_0(\beta, K) \) with coordinates \( x^j \in K \) in the standard base. Consider a covering separating ring \( \mathcal{R} \) of \( c_0(\beta, K) \) such that
\[
\mathcal{R} := \bigcup_{j_1, \ldots, j_n \in \beta; n \in \mathbb{N}} (\pi_{j_1, \ldots, j_n})^{-1}(Bco(span_{K}(e_{j_1}, \ldots, e_{j_n}))),
\]
where \( \text{span}_{K}(z_l : l \in \gamma) := \{ x : x \in c_0(\beta, K); x = \sum_{j \in \zeta} a^j z_j; a^j \in K; \text{card}(\zeta) < \aleph_0 \} \) for each \( \gamma \subseteq \beta \). On the completion \( \mathcal{R}_\mu \) there exists a probability \( K_s \)-valued measure \( \mu \) generated by a bounded cylindrical distribution as in §2.8, §2.9 or §2.15. For example, each \( \mu_j(dx) := f_j(x)w(dx) \) is a measure on \( K \), where \( f_j \in L(K, \mathcal{R}(K), w, K_s) \), \( w \) is either the Haar measure or any other probability measure on \( K \), \( \mu_j = \pi_j(\mu) \) for each \( j \in \beta \).
In particular, for \( \text{card}(\beta) \leq \aleph_0 \) and locally compact \( K \) non-Archimedean infinite field with nontrivial valuation there exists \( \mu \) such that \( \mathcal{R}_\mu \supset Bco(c_0(\beta, K)) \).

For this consider on the Banach space \( c_0 := c_0(\omega_0, K) \) a linear operator \( J \in L_0(c_0) \), where \( L_0(H) \) denotes the Banach space of compact \( K \)-linear operators on the Banach space \( H \), such that \( Je_i = v_i e_i \) with \( v_i \neq 0 \) for each \( i \) and a measure \( \nu(dx) := f(x)w(dx) \), where \( f : K \to B(K, 0, r) \) with \( r \geq 1 \) is a function belonging to the space \( L(K, \mathcal{R}_w, w, K_\alpha) \) such that \( \lim_{|x| \to \infty} f(x) = 0 \) and \( \nu(K) = 1, \|S\|_\nu > 0 \) for each clopen subset \( S \) in \( K \), for example, when \( f(x) \neq 0 \) \( w \)-almost everywhere. In particular we can choose \( \nu \) with \( \|\nu\| = 1 \).

In view of Lemma 2.3 and Theorem 2.30 from II [32] and Theorem 2.8 there exists a product measure

\[
\begin{align*}
(i) & \quad \mu(dx) := \prod_{i=1}^\infty \nu_i(dx^i) \text{ on the ring } Bco(c_0) \text{ of clopen subsets of } c_0, \text{ where} \\
(ii) & \quad \nu_i(dx^i) := f(x^i/v_i)\nu(dx^i/v_i).
\end{align*}
\]

Consider, for example, the particular case of \( X = C^0(T, H) \) with compact \( T \). If \( t_0 \in T \) is an isolated point, then \( C^0(T, H) = C^0(T \setminus \{t_0\}, H) \oplus H \), so we consider the case of \( T \) dense in itself. Let \( Z \) be a compact subset without isolated points in a local field \( K \). Then the Banach space \( C^0(Z, K) \) has the Amice polynomial orthonormal base \( Q_m(x) \), where \( x \in Z, m \in \aleph_0 := \{0, 1, 2, \ldots\} \). Each \( f \in C^0 \) has a decomposition \( f(x) = \sum_m a_m(f)Q_m(x) \) such that \( \lim_{m \to \infty} a_m = 0 \), where \( a_m \in K \). These decompositions establish the isometric isomorphism \( \theta : C^0(T, K) \to c_0(\omega_0, K) \) such that \( \|f\|_{c^0} = \max_m |a_m(f)| = \|\theta(f)\|_{c_0}. \)

If \( u_i \) are roots of the polynomials \( Q_m \) as in \( \text{[1]} \), then \( Q_m(u_i) = 0 \) for each \( m > i \). The set \( \{u_i : i\} \) is dense in \( T \).

The locally \( K \)-convex space \( X = X(T, H) \) is isomorphic with the tensor product \( X(T, K) \otimes H \) (see §4.R [11] and [30]). If \( J_i \in L_0(Y_i) \) is nondegenerate for each \( i = 1, 2 \), that is, \( \text{ker}(J_i) = \{0\} \), then \( J := J_1 \otimes J_2 \in L_0(Y_1 \otimes Y_2) \) is nondegenerate (see also Theorem 4.33 [11]). If \( X(T, K) \) and \( H \) are of separable type over a non-Archimedean locally compact infinite field \( K \) with nontrivial valuation, then we can construct a measure \( \mu \) on \( X \) such that \( \mathcal{R}_\mu \supset Bco(X) \). The case \( H^T \) we reduce to \( (c_0(\alpha, K), \tau_\alpha) \otimes H \) as above.

Put \( Y_1 := X(T, K) \) and \( Y_2 := H \) and \( J := J_1 \otimes J_2 \in L_0(Y_1 \otimes Y_2) \), where \( J_1 e_m := \alpha_m e_m \) such that \( \alpha_m \neq 0 \) for each \( m \) and \( \lim_i \alpha_i = 0 \). Take \( J_2 \) also nondegenerate. Then \( J \) induces a product measure \( \mu \) on \( X(T, H) \) such that \( \mu = \mu_1 \otimes \mu_2 \), where \( \mu_i \) are measures on \( Y_i \) induced by \( J_i \) due to Formulas \( (i, ii) \).

Analogously considering the following subspace \( X_0(T, H) \) and operators \( J := J_1 \otimes J_2 \in L_0(X_0(T, K) \otimes H) \) we get the measures \( \mu \) on it also, where \( t_0 \in T \) is a marked point. On the other hand, the space \( X(T, H) \) is Lindelöf (see §3.8 [1]),
hence each subset $U$ open in $X(T, H)$ is a countable union of clopen subsets. Hence the characteristic function $Ch_U$ of $U$ belongs to $L(X, \mathcal{R}, \mu, K)$, since $||\mu|| = \sup_N N_\mu(x) < \infty$, consequently, $\mathcal{R}_\mu \ni U$. In general the condition $\mathcal{R}_\mu \supset Bco(X_0)$ is not imposed in §4.1 and in Definition 4.2, so $\mathcal{R}_\mu$ may be any covering ring of $X_0$.

For each finite number of pairwise distinct points $(t_0, t_1, \ldots, t_n)$ in $T$ and points $(0, z_1, \ldots, z_n)$ in $H$ there exists a closed subset $X(T, H; (t_0, t_1, \ldots, t_n); (0, z_1, \ldots, z_n))$ in $X(T, H)$ such that

$$X(T, H; (t_0, t_1, \ldots, t_n); (0, z_1, \ldots, z_n)) = (0, z_1, \ldots, z_n) + X(T, H; (t_0, t_1, \ldots, t_n); (0, 0, 0)).$$

Therefore, $X(T, H; (t_0, t_1, \ldots, t_n); (0, z_1, \ldots, z_n))$ is the $K$-linear subspace in $X(T, H)$.

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Therefore, $X(T, H; (t_0, t_1, \ldots, t_n); (0, z_1, \ldots, z_n))$ is the $K$-linear subspace in $X(T, H)$. Therefore, using cylindrical distributions we get examples of such measures $\mu$ for which stochastic processes exist. Hence to each such measure on $X_0(T, H)$ there corresponds the stochastic process.
Evidently, on the field $K$ there exists a family $\Psi_K$ of inequivalent $K_s$-valued measures of the cardinality $\text{card}(\Psi_K) \geq \text{card}(K)$, since the subfamily of atomic measures satisfies this inequality. In the particular case of $C_p \supset F \supset Q_p$ or $F = F_p(\theta)$ we use the Haar measure $w$ also for which $\text{card}(L_q(F, w, \text{Bco}(F), K_s)) = c := \text{card}(R)$ for each $1 \leq q \leq \infty$. In view of the non-Archimedean variant of the Kakutani theorem (see II.3.5 [32]) we get the inequalities for $\text{card}(\Psi)$, since $\text{card}(H) = \text{card}(\beta_H) \text{card}(K)$. In particular, for $T = B(K, 0, r)$ with $r > 0$ and a locally compact field $K$ either $K \supset Q_p$ or $K = F_p(\theta)$ considering all operators $J := J_1 \otimes J_2 \in L_0(Y_1 \otimes Y_2)$ and the corresponding measures as above we get $c^{\aleph_0} = c$ inequivalent measures for each chosen $f$.

Note. Evidently, this theorem is also true for $C^0(T, H)$, that follows from the proof. If take $\nu$ with $\text{supp}(\nu) = B(K, 0, 1)$, then repeating the proof it is possible to construct $\mu$ with $\text{supp}(\mu) \subset B(C^0(T, K), 0, 1) \times B(H, 0, 1)$. Certainly such measure $\mu$ can not be quasi-invariant relative to shifts from a dense $K$-linear subspace in $C^0(T, H)$, but (starting from the Haar measure $w$ on $F$) $\mu$ can be constructed quasi-invariant relative to a dense additive subgroup $G'$ of $B(C^0(T, K), 0, 1) \times B(H, 0, 1)$, moreover, there exists $\mu$ for which $G'$ is also $B(K, 0, 1)$-absolutely convex.

5 Poisson processes.

5.1. Definition. Let $T$ be an additive group such that $T \subset B(K_s, 0, r)$ and $0 \neq \rho \in K_s$ with $|\rho|r < s^{1/(1-s)}$, where $K_s$ is a field such that $Q_s \subset K_s \subset C_s$, $K_s$ is complete as the ultrametric space. Consider a stochastic process $\xi \in L^q(\Omega, F, \lambda, X_0(T, H))$ such that the transition measure has the form

$$P(t_1, x, t_2, A) := P(t_2 - t_1, x, A) := \text{Exp}(-\rho(t_2 - t_1))P(A - x)$$

(see §3.2 and §4.2) for each $x \in H$ and $A \in \mathcal{R}_H$ and $t_1$ and $t_2$ in $T$, where $\text{Exp}(x) := \sum_{n=0}^{\infty} x^n/n!$. Then such process is called the Poisson process.

5.2. Proposition. Let

$$P(A - x) = f_H P(-x + dy)P(A - y) \text{ for each } x \in H \text{ and } A \in \mathcal{R}_H$$

and $P(H) = 1$ and $\|P\| = 1$,

then there exists a measure $\mu$ on $X_0(T, H)$ for which the Poisson process exists.

Proof. The exponential function converges if $|x| < s^{1/(1-s)}$, since $|n!|_{s}^{-1} \leq s^{(n-1)/(s-1)}$ for each $0 < n \in \mathbf{Z}$ in accordance with Lemma 4.1.2 [24]. Hence
\[ |Exp(-\rho(t_2-t_1))-1| < 1 \text{ for each } t_1 \text{ and } t_2 \text{ in } T, \text{ consequently, } |Exp(-\rho(t_2-t_1))| = 1. \] 

Take 
\[ \mu_{t_1,\ldots,t_n} := P(t_2-t_1,0,*)\ldots P(t_n-t_{n-1},0,*), \]
for each pairwise distinct points \( t_1,\ldots,t_n \in T, \) where 
\[ \mu_{t_1,\ldots,t_n} = \pi_{t_1,\ldots,t_n}(\mu) \]
and 
\[ \pi_{t_1,\ldots,t_n} : X_0(T,H) \to H_{t_1} \times \ldots \times H_{t_n} \]
is the natural projection, \( H_t = H \) for each \( t \in T. \) There is a family \( \Lambda \) of all finite subsets of \( T \) directed by inclusion. In view of Theorem 2.14 the cylindrical distribution \( \mu \) generated by the family \( P(t_2-t_1,x,A) \) has an extension to a measure on \( X_0(T,H). \) All others conditions are satisfied in accordance with \( \S 4.2 \) and \( \S 5.1. \)

**5.3. Note.** Let \( K \) be a complete ultrametric space with an ultrametric \( d, \) that is, 
\[ d(x,y) \leq \max(d(x,z),d(y,z)) \text{ for each } x,y,z \in X. \]

Let 
\[ d(x,y) := \max_{1 \leq i \leq n} d(x_i,y_i) \]
be the ultrametric in \( K^n, \) where \( x = (x_i : i = 1,\ldots,n) \in K^n, x_i \in K. \) Put 
\[ K^n := \{ x \in K^n : x_i \neq x_j \text{ for each } i \neq j \}. \]
Supply \( \tilde{K}^n \) with an ultrametric 
\[ \delta^{(n)}_{\tilde{K}}(x,y) := d^n_{\tilde{K}}(x,y)/[\max(d^n_{\tilde{K}}(x,y),d^n_{\tilde{K}}(x,(\tilde{K}^n)^c),d^n_{\tilde{K}}(y,(\tilde{K}^n)^c))], \]
where \( A^c := K^n \setminus A \) for a subset \( A \subset K^n. \) Then \( (\tilde{K}^n,\delta^n_{\tilde{K}}) \) is the complete ultrametric space. Let also \( B^n_K \) denotes the collection of all \( n \)-point subsets of \( K. \) Then the ultrametric \( \delta^n_K \) is equivalent with the following ultrametric 
\[ d^{(n)}_K(\gamma,\gamma') := \inf_{\sigma \in \Sigma_n} d^n_K((x_1,\ldots,x_n),(x'_{\sigma(1)},\ldots,x'_{\sigma(n)})), \]
where \( \Sigma_n \) is the symmetric group of \( (1,\ldots,n), \sigma \in \Sigma_n, \sigma : (1,\ldots,n) \to (1,\ldots,n); \ gamma,\gamma' \in B^n_K. \) For each subset \( A \subset K \) a number mapping \( N_A : B^n_K \to \mathbb{N}_0 \) is defined by the following formula: \( N_A(\gamma) := card(\gamma \cap A), \) where \( \mathbb{N} := \{1,2,3,\ldots\}, \mathbb{N}_0 := \{0,1,2,3,\ldots\}. \) It remains to show, that \( \delta^n_K \) is the ultrametric for the ultrametric space \( (K,d). \) For this we mention, that 

(i) \( \delta^n_K(x,y) > 0, \) when \( x \neq y, \) and \( \delta^n_K(x,x) = 0. \)

(ii) \( \delta^n_K(x,y) = \delta^n_K(y,x), \) since this symmetry is true for \( d^n_K \) and for \([*] \) in the denominator in the formula defining \( \delta^n_K. \) To prove 

(iii) \( \delta^n_K(x,y) \leq \max(\delta^n_K(x,z),\delta^n_K(z,y)) \)
we consider the case \( \delta^n_K(x,z) \geq \delta^n_K(y,z), \) hence it is sufficient to show, that 
\( \delta^n_K(x,y) \leq \delta^n_K(x,z). \) Let

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(a) $d^n_K(x, z) \geq \max(d^n_K(z, (\tilde{K}^n)^c), d^n_K(x, (\tilde{K}^n)^c))$
then $\delta^n_K(x, z) = 1$, hence $\delta^n_K(x, y) \leq \delta^n_K(x, z)$, since $\delta^n_K(x, y) \leq 1$ for each $x, y \in \tilde{K}^n$. Let
(b) $d^n_K(x, (\tilde{K}^n)^c) > \max(d^n_K(x, z), d^n_K(z, (\tilde{K}^n)^c))$, then
$\delta^n_K(x, z) = d^n_K(x, z)/d^n_K(x, (\tilde{K}^n)^c) \leq 1$. Since $d^n_K(z, A) := \inf_{a \in A} d^n_K(z, a)$, then
$\delta^n_K(z, (\tilde{K}^n)^c) \leq \max(d^n_K(y, (\tilde{K}^n)^c), d^n_K(y, z))$.
If $d^n_K(x, z) < d^n_K(z, (\tilde{K}^n)^c)$ and $d^n_K(x, y) \leq d^n_K(x, z)$, then
$\delta^n_K(z, (\tilde{K}^n)^c) \leq \max(d^n_K(x, z), d^n_K(x, (\tilde{K}^n)^c), d^n_K(z, (\tilde{K}^n)^c))$.
Hence
$\delta^n_K(x, y) \max(d^n_K(x, z), d^n_K(x, (\tilde{K}^n)^c), d^n_K(z, (\tilde{K}^n)^c))$
$\leq d^n_K(x, z) \max(d^n_K(x, y), d^n_K(x, (\tilde{K}^n)^c), d^n_K(y, (\tilde{K}^n)^c))$.
With the help of (ii) the remaining cases may be lightly written.

5.4. Notes and definitions. As usually let
$B_K := \bigoplus_{n=0}^{\infty} B^n_K$,
where $B^n_K := \{\emptyset\}$ is a singleton, $B_K \ni x = (x_n : x_n \in B^n_K, n = 0, 1, 2, \ldots)$. If a complete ultrametric space $X$ is not compact, then there exists an increasing sequence of subsets $K_n \subset X$ such that $X = \bigcup_n K_n$ and $K_n$ are complete spaces in the induced uniformity from $X$. Moreover, $K_n$ can be chosen clopen in $X$. Then the following space
$\Gamma_X := \{\gamma : \gamma \subset X \text{ and } \text{card}(\gamma \cap K_n) < \infty \text{ for each } n\}$
is called the configuration space and it is isomorphic with the projective limit
$\pi - \lim \{B_{K_n}, \pi^n_{m}, N\}$, where $\pi^n_m(\gamma_m) = \gamma_n$ for each $m > n$ and $\gamma_n \in B_{K_n}$.
If $d_n$ denotes the ultrametric in $B_{K_n}$, then $d_{n+1}|B_{K_n} = d_n$, since $K_n \subset K_{n+1}$.
Then $\prod_{n=1}^{\infty} B_{K_n} := Y$ in the Tychonoff product topology is ultrametrizable, that induces the ultrametric in $\Gamma_X$, for example,
$\rho(x, y) := d_n(x_n, y_n)p^{-n}$ is the ultrametric in $\Gamma_X$,
where $n = n(x, y) := \min_{(x_j \neq y_j)} j, x = (x_j : j \in N, x_j \in B_{K_j}), 1 < p \in N$.

Let $K \in \{K_n : n \in N\}$, then $m_K$ denotes the restriction $m|_K$, where $m : R \rightarrow K$ is a measure on a covering ring $R_m$ of $X$, $K_n \in R_m$ for each $n \in N$. Suppose that $K_n \in R_m$ for each $n$ and $l$ in $N$, where $R_m$ is the completion of the covering ring $R_n$ of $X^n$ relative to the product measure $m^n = \otimes_{j=1}^{n} m_j$, $m_j = m$ for each $j$. Then $m^n_K := \otimes_{j=1}^{n} (m_K)_j$ is a measure on $K^n$ and hence on $\tilde{K}^n$, when $m$ is such that $\|m|_{(K^n \setminus \tilde{K}^n)}\| = 0$, for example, non-atomic $m$, where $(m_K)_j = m_K$ for each $j$. Let $m(K_l) \neq 0$ for each $l \in N$, $m(X) \neq 0$ and $\|m\| < s^{1/(1-s)}$. Therefore,
(i) $P_{K,m} := Exp(-m(K)) \sum_{n=0}^{\infty} m_K/n!$
is a measure on $\mathcal{R}(B_K)$, where

$$\mathcal{R}(B_K) = B_K \cap (\bigoplus_{n=0}^{\infty} \mathcal{R}_m^n),$$

$m_{K,0}$ is a probability measure on the singleton $B_K^0$, and $m_{K,n}$ are images of $m_K^n$ under the following mappings:

$$p_K^n : (x_1, \ldots, x_n) \in K^n \rightarrow \{x_1, \ldots, x_n\} \in B_K^n.$$

Such system of measures $P_{K,n}$ is consistent, that is,

$$\pi_l^n(P_{K_1,m}) = P_{K,n,m}$$

for each $n \leq l$.

This defines the unique measure $P_m$ on $\mathcal{R}(\Gamma_X)$, which is called the Poisson measure, where $\pi_n : Y \rightarrow B_{K_n}$ is the natural projection for each $n \in \mathbb{N}$ (see for comparison the case of real-valued Poisson measures in [44]). For each $n_1, \ldots, n_l \in \mathbb{N}_0$ and disjoint subsets $B_1, \ldots, B_l$ in $X$ belonging to $\mathcal{R}_m$ there is the following equality:

$$\pi_l^n(P_{K_1,m}) = P_{K,m}$$

for each $n \leq l$.

5.5. Corollary. Let suppositions of Proposition 5.2 be satisfied with $H = S_X$ for a complete $\mathbf{K}$-linear space $X$ and $P(A) = P_m(A)$ for each $A \in \mathcal{R}(S_X)$, then there exists a measure $\mu$ on $X_0(T, H)$ for which the Poisson process exists.

5.6. Definition. The stochastic process of Corollary 5.5 is called the Poisson process with values in $X$.

5.7. Note. If $\xi \in L^q(\Omega, \mathcal{F}, \lambda; X_0(T, H))$ is a stochastic process, then its mean value at the moment $t \in T$ is defined by the following formula:

$$(i) \quad M_t(\xi) := \int_{\Omega} \xi(t, \omega) \lambda(d\omega).$$

Let $H = \mathbf{K}$ be a field, where $\mathbb{Q}_p \subset \mathbf{K} \subset \mathbb{C}_p$, let also $\lambda$ be with values in $\mathbf{K}$. Suppose that
Let $\rho \in B(K, 0, c)$ and let $T$ be a subgroup of $B(K, 0, r)$, where $R \max(c, r) < p^{1/(1-p)}$, $K$ is the locally compact field. Let $\tilde{P}^1 : C^0(B(K, 0, c), K) \to C^1(B(K, 0, c), K)$ be an antiderivation operator (see also §§54, 80 [43]).

5.8. Theorem. (Non-Archimedean analog of the Lévy theorem.) Let $\psi$ be a continuously differentiable function, from $T$ into $K$ belonging to $\tilde{P}^1(C^0(B(K, 0, c), K))$ and $\psi(0) = 0$. Then there exists a stochastic process such that

$$M_t(\exp(-\rho \xi(t, \omega))) = \exp(-t\psi(\rho))$$

for each $t$ in $T$ and each $\rho \in B(K, 0, c)$.

Proof. For the construction of $\xi$ consider solution of the following equation

$$M_t[\exp(-\rho \xi(t, \omega))] = \exp(-t\psi(\rho)).$$

Then $e(t) = e(t-s) e(s)$ for each $t$ and $s$ in $T$ and each $\rho \in B(K, 0, c)$, where $e_\rho(t) := e(t) := M_t(\exp(-\rho \xi(t, \omega)))$.

Hence

$$\frac{\partial e_\rho(t)}{\partial \rho} = -t \psi'(\rho) \exp(-t\psi(\rho)),$$

$$\psi'(\rho) = t^{-1} \int_K l \, EXP(-\rho l) P(\{\omega : \xi(t, \omega) \in dl\})$$

for each $t \neq 0$, where $EXP$ is the locally analytic extension of $\exp$ on $K$ (with values in $\{x : x \in C_p, \ |x-1| < 1\}$, see [43]). In particular,

$$\lim_{t \to 0, t \neq 0} t^{-1} \int_K l \, EXP(-\rho l) P(\{\omega : \xi(t, \omega) \in dl\}).$$

By the conditions of this theorem we have

$$\psi(\rho) = \tilde{P}^1_\beta \psi'(\beta)|^0.$$  

Consider a measure $m$ on a separating covering ring $R(K)$ such that $R(K) \supset Bco(K) \cup \{0\}$ with values in $K$ such that

$$m(dl) := \lim_{t \to 0, t \neq 0} l P(\{\omega : \xi(t, \omega) \in dl\})/t.$$  

Therefore,

$$\psi(\rho) = \tilde{P}^1_\beta (\int_K EXP(-\beta l) m(dl))|^0.$$  

From $\psi(0) = 0$ we have $e_\rho(1) = 1$ for each $\rho$, consequently,

$$\psi(\rho) = \rho m_0 + \int_K [1 - EXP(-\rho l)] l^{-1} m(dl),$$

where $m_0 := m(\{0\})$, since

$$\lim_{t \to 0, t \neq 0} [1 - EXP(-\rho l)]/l = \rho$$

and

$$\lim_{\rho \to 0, \rho \neq 0} \int_{B(K, 0, k)} [1 - EXP(-\rho l)] l^{-1} m(dl) = 0.$$  

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for each $k > 0$. Define a measure $n(dl)$ such that $n(\{0\}) = 0$ and $n(dl) = l^{-1}m(dl)$ on $K \setminus \{0\}$, then
$$
\psi(\rho) = \rho m_0 + \int_K[1 - \text{EXP}(-\rho l)] n(dl).
$$
We search a solution of the problem in the form
$$
\psi(t, \omega) = tm_0 + \int_K \eta(t, dl, \omega),
$$
where $\eta(t, dl, \omega)$ is the measure on $R(K)$ for each fixed $t \in T$ and $\omega \in \Omega$ such that its moments satisfy the Poisson distribution with the Poisson measure $P_{tn}$, that is,
$$
M_t[\eta^k(t, dl, \omega)] = \sum_{s \leq k} a_{s,k}(tn)^s(dl)/s!
$$
for each $t \in T$, where $a_{0,j} = 0$, $a_{1,j} = 1$ and recurrently
$$
a_{k,j} = k^j - \sum_{s=1}^{k} \binom{k}{s} a_{k-s,j}
$$
for each $k \leq j$, in particular, $a_{j,j} = j!$, that is,
$$
a_{k,j} = \sum_{s_1+\ldots+s_k=j, s_1 \geq 1, \ldots, s_k \geq 1} [j!/(s_1!\ldots s_k!)].
$$
Using the fact that the set of step functions is dense in $L(K, R(K), n, C_p)$ we get
$$
M_t[\text{EXP}(-\rho \int_K l\eta(t, dl, \omega))] = \lim_{Z} M_t[\prod_j \text{EXP}(-\rho l_j \eta(t, \delta_j, \omega))]
$$
$$
= \lim_{Z} \prod_j M_t[\text{EXP}(-\rho l_j \eta(t, \delta_j, \omega))] = \lim_{Z} \text{EXP}(-\rho t \sum_j (1 - \text{EXP}(-\rho l_j)) n(\delta_j))
$$
$$
= \text{EXP}[-\rho t \int_K (1 - \text{EXP}(-\rho l)n(dl)],
$$
where $Z$ is an ordered family of partitions $U$ of $K$ into disjoint union of elements of $R(K)$, $U \leq V$ in $Z$ if and only if each element of the disjoint covering $U$ is a union of elements of $V$, $l_j \in \delta_j \in U \in Z$. The limit
$$
\lim_{U \in Z} f(U) =: \lim_{Z} f = a
$$
means that for each $\epsilon > 0$ there exists $U$ such that for each $V$ with $U \leq V$ we have
$$
|a - f(V)| < \epsilon,
$$
where $f(U)$ is one of the functions defined as above with $l_j \in \delta_j \in U$, that is,
$$
f(U) = M_t[g \circ h(\eta)],
$$
where $g \circ h(\eta)$ is the composition of the continuous function $g$ and of
\[ h(\eta) = \int_{K} \zeta(y)\eta(t,dy,\omega) \]

with the step function \( \zeta \). We get the equation

\[ M_t[\exp(-\rho\xi(t,\omega))] = \exp(-\rho t\tau_0)M_t[\exp(-\rho\int_{K} \ln(t,dl,\omega))]. \]

In view of Corollary 5.5 it defines the stochastic process with the probability space \((\Omega, F, \lambda)\), the existence of which follows from the second half of §4.3.

5.9. **Note.** From the preceding results it follows, that there are several specific features of non-Archimedean stochastic processes and in particular Poisson processes in comparison with the classical case. For this there are several reasons. The non-Archimedean infinite field \( K \) with nontrivial valuation has not any linear ordering compatible with its field structure. In the non-Archimedean case there is not any indefinite integral. Theory of analytic functions and elements has many specific features in the non-Archimedean case [11, 43]. Moreover, interpretations of probabilities also are different [23, 24].

We have started our collaboration with the investigation of one problem formulated few years ago by A. Khrennikov and A.C.M. van Rooij. It was in the study of non-Archimedean analogs of the Kolmogorov theorem for measures with values in non-Archimedean fields.

S. Ludkovsky is sincerely grateful to A. Khrennikov for his hospitality at International Center for Mathematical Modeling of Växjö University.
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