Complementary means with respect to a nonsymmetric invariant mean

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Abstract. It is known that if a bivariate mean $K$ is symmetric, continuous and strictly increasing in each variable, then for every mean $M$ there is a unique mean $N$ such that $K$ is invariant with respect to the mean-type mapping $(M, N)$, which means that $K \circ (M, N) = K$ and $N$ is called a $K$-complementary mean for $M$ (Matkowski in Aequ Math 57(1):87–107, 1999). This paper extends this result for a large class of nonsymmetric means. As an application, the limits of the sequences of iterates of the related mean-type mappings are determined, which allows us to find all continuous solutions of some functional equations.

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1. Introduction

Let $K$, $M$ and $N$ be bivariate means in an interval $I$. The mean $K$ is called invariant with respect to the mean-type mapping $(M, N) : I^2 \to I^2$, briefly $(M, N)$-invariant, if $K \circ (M, N) = K$ (cf. [3]). This $K$ is sometimes called the Gauss composition of the means $M$ and $N$ (cf. [2]), and $N$ is referred to as a complementary mean to $M$ with respect to $K$ (briefly, $K$-complementary mean to $M$) ([3], see also [6]).

It is known that, under general conditions, the invariance relation guarantees the convergence of the sequence $((M, N)^n : n \in \mathbb{N})$ of iterates of the mean-type mapping $(M, N)$ to the mean-type $(K, K)$. This fact has important applications in effectively solving some functional equations.

In [3] it was shown that if the mean $K$ is symmetric, continuous and strictly increasing in each variable, then for every mean $M$ there is a unique $K$-complementary mean.
The main results of this paper, Theorems 1 and 2, omitting the symmetry condition, give essential generalizations of this result. In particular, Theorem 1, where the symmetry of $K$ is replaced by the implication “$x < y \Rightarrow K(x,y) \geq K(y,x)$ for all $x, y \in I$”, says that for every bivariate mean $M$ in $I$ there is a unique mean $M_{[K]}$ such that $K$ is $(M, M_{[K]})$-invariant. Moreover, $M_{[K]}$ is strict or continuous if so is $M$, and both $M$ and $M_{[K]}$ are symmetric if so is $K$. Theorem 2 is a dual counterpart of Theorem 1. Here the symmetry of $K$ is replaced by the implication “$x < y \Rightarrow K(x,y) \leq K(y,x)$ for all $x, y \in I$”. For every mean $M$ it provides a unique mean $M_{[K]}$ such that $K$ is $(M, M_{[K]})$-invariant, and has similar additional properties as $M_{[K]}$.

A simple example shows that the assumed implications in Theorems 1 and 2 are essential. Moreover it is shown (Remark 3 and Example 2) that in these results the assumption that $K$ is strictly increasing cannot be replaced by the strictness of $K$.

Applying Theorems 1 and 2 we give conditions under which the sequences of mean-type mappings $(M, M_{[K]})$ and $(M_{[K]}, M)$ converge to $(K, K)$ (Theorem 3 and Theorem 4). This permits, in particular, to determine effectively all continuous functions $F : I^2 \to \mathbb{R}$ satisfying each of the functional equations

\[
F \left( M(x, y), M_{[K]}(x, y) \right) = F(x, y), \quad x, y \in I,
\]

\[
F \left( M_{[K]}(x, y), M(x, y) \right) = F(x, y), \quad x, y \in I.
\]

2. Complementary means

In the sequel $I \subset \mathbb{R}$ stands for an interval.

A function $M : I \times I \to I$ is called a mean in $I$, if

\[
\min (x, y) \leq M(x, y) \leq \max (x, y), \quad x, y \in I.
\]

A mean $M$ is called strict, if for $x \neq y$, both these inequalities are sharp; and symmetric, if $M(x, y) = M(y, x)$ for all $x, y \in I$.

Every mean is reflexive, that is $M(x, x) = x$ for all $x \in I$.

Remark 1. If a function $M : I \times I \to \mathbb{R}$ is reflexive and (strictly) increasing in each of the variables, then it is a (strict) mean in $I$.

Let $K, M, N : I^2 \to I$ be means. The mean $K$ is called \textit{invariant with respect to the mean-type mapping} $(M, N) : I^2 \to I^2$, briefly $(M, N)$-invariant, \textit{if} $K \circ (M, N) = K$ (cf. [3]). In the case when $K$ is unique, it is sometimes called the Gauss composition of the means $M$ and $N$ (cf. [2]). If $K$ is a unique $(M, N)$-invariant mean we say that $N$ is a \textit{complementary mean to} $M$ (briefly, $N$ is $K$-complementary to $M$).

Recall that if $M, N$ are continuous and strict means, then there exists a unique $(M, N)$-invariant mean (cf. [1], also [4], [2], [5]).
**Theorem 1.** Let a continuous mean $K : I^2 \to I$ be increasing in the first variable and strictly increasing in the second one. Suppose that $K$ satisfies the following condition:

$$x < y \implies K(x, y) \geq K(y, x), \quad x, y \in I. \tag{1}$$

Then

(i) for every mean $M : I^2 \to I$ there is a unique mean $M_{[K]} : I^2 \to I$ such that $K$ is $(M, M_{[K]})$-invariant, i.e.

$$K(M(x, y), M_{[K]}(x, y)) = K(x, y), \quad x, y \in I; \tag{2}$$

(ii) if for a symmetric mean $M$, the mean $M_{[K]}$ is symmetric, then $K$ is symmetric;

(iii) if $M$ is strict then so is $M_{[K]}$;

(iv) if $M$ is continuous then so is $M_{[K]}$.

**Proof.** Take an arbitrary mean $M : I^2 \to I$ and $x, y \in I$. We have

$$\min(x, y) \leq M(x, y) \leq \max(x, y).$$

By setting $z = M_{[K]}(x, y)$, the equation (2) becomes

$$K(M(x, y), z) = K(x, y).$$

If $x = y$ then $M(x, y) = K(x, y) = x$ so this equation simplifies to

$$K(x, z) = x.$$

Since, by the reflexivity of every mean, $K(x, x) = x$, the strict monotonicity of $K$ in the second variable implies that $M_{[K]}(x, y) := z = x$ is the only solution of this equation.

Assume that $x < y$.

In this case we have

$$x \leq M(x, y) \leq y.$$

Define the function $\varphi : [x, y] \to \mathbb{R}$ by

$$\varphi(t) := K(M(x, y), t) - K(x, y), \quad t \in [x, y].$$

Of course, $\varphi$ is continuous.

For $t = x$, making use first of the inequality $x \leq M(x, y)$ and implication (1), and then from the inequality $M(x, y) \leq y$ and the monotonicity of $K$ in the second variable, we have

$$\varphi(x) = K(M(x, y), x) - K(x, y) \leq K(x, M(x, y)) - K(x, y) < 0.$$

For $t = y$, the inequality $x \leq M(x, y)$ and the monotonicity of $K$ in the first variable give

$$\varphi(y) = K(M(x, y), y) - K(x, y) \geq K(x, y) - K(x, y) = 0.$$
The Darboux property of \( \varphi \) implies that there is \( z \in [x, y] \) such that \( \varphi(z) = 0 \), i.e. such that

\[
K(M(x, y), z) = K(x, y).
\]

By the strict monotonicity of \( K \) in the second variable, such a \( z \) is unique. Setting here \( M_{[K]}(x, y) := z \), we get

\[
K(M(x, y), M_{[K]}(x, y)) = K(x, y).
\]

Since in the case \( y < x \) we can argue similarly, the proof of (i) is complete.

(ii) This is a trivial consequence of (2).

(iii) Assume, on the contrary, that \( M_{[K]} \) is not strict, so there are \( x_0, y_0 \in I \), \( x_0 < y_0 \), such that \( M_{[K]}(x_0, y_0) = x_0 \) or \( M_{[K]}(x_0, y_0) = y_0 \).

In the first case, from (2), we would have

\[
K(M(x_0, y_0), x_0) = K(M(x_0, y_0), M_{[K]}(x_0, y_0)) = K(x_0, y_0),
\]

and, from (1),

\[
K(x_0, y_0) \geq K(y_0, x_0),
\]

whence \( K(M(x_0, y_0), x_0) \geq K(y_0, x_0) \). But this is a contradiction as \( K \) is strictly increasing in the first variable and \( x_0 < M(x_0, y_0) < y_0 \).

Since in the case when \( M_{[K]}(x_0, y_0) = y_0 \) we can argue similarly, this completes the proof of (iii).

(iv) We omit the simple argument of this part. \( \square \)

**Definition 1.** Under the conditions of Theorem 1, the mean \( M_{[K]} \) is called \( K \)-complementary (or \( K \)-right complementary) to the mean \( M \).

Under the conditions of this theorem, part (i) can be strengthened as follows:

**Remark 2.** For every mean \( M : I^2 \to I \) there is a unique function \( N : I^2 \to I \) such that

\[
K(M(x, y), N(x, y)) = K(x, y), \quad x, y \in I,
\]

holds and, moreover, \( N \) is a mean.

To show that assumption (1) is essential, consider the following

**Example 1.** Let \( I = \mathbb{R} \) and \( K : \mathbb{R}^2 \to \mathbb{R} \) be a weighted arithmetic mean, i.e. \( K(x, y) = ax + (1 - a)y \) for some fixed \( a \in (0, 1) \), and let \( M : \mathbb{R}^2 \to \mathbb{R} \) be a mean. A simple calculation shows that a function \( N : \mathbb{R}^2 \to \mathbb{R} \) satisfies (2) if and only if

\[
N(x, y) = \frac{a(x - M(x, y))}{1 - a} + y, \quad x, y \in \mathbb{R}.
\]

(3)
Assume that for every weighted arithmetic mean

\[ M(x, y) = bx + (1 - b)y, \quad b \in [0, 1], \]

the function \( N \) is a mean in \( \mathbb{R} \), that is that

\[ N(x, y) = \frac{a(1-b)}{1-a}x + \left(1 - \frac{a(1-b)}{1-a}\right)y, \quad x, y \in \mathbb{R}, \]

is a mean in \( \mathbb{R} \). Clearly, \( N \) is a mean for every \( b \in [0, 1] \) if, and only if,

\[ \frac{a(1-b)}{1-a} \leq 1, \quad b \in [0, 1]. \]

Since the function \([0, 1] \ni b \mapsto \frac{a(1-b)}{1-a}\) is decreasing, this inequality holds true iff \( \frac{a(1-b)}{1-a} \leq 1 \), that is iff

\[ a \leq \frac{1}{2}. \]

We can also argue as follows. The function \( N \) given by (3) is a mean for every mean \( M \) if and only if

\[ \min (x, y) \leq \frac{a(x - M(x, y))}{1-a} + y \leq \max (x, y), \quad x, y \in \mathbb{R}, \]

or, equivalently, if and only if, for every mean \( M \), and for all \( x, y \in \mathbb{R} \),

\[ ax + (1-a)y - (1-a)\max (x, y) \leq aM(x, y) \leq ax + (1-a)y - (1-a)\min (x, y). \]

Taking here first \( M = \min \) and then \( M = \max \) we obtain, respectively, the inequalities

\[ ax + (1-a)y \leq a\min (x, y) + (1-a)\max (x, y), \quad x, y \in \mathbb{R}, \]

\[ (1-a)\min (x, y) + a\max (x, y) \leq ax + (1-a)y, \quad x, y \in \mathbb{R}. \]

It is easy to see that each of these inequalities holds, if and only if \( a \leq \frac{1}{2} \).

Note that with \( K(x, y) = ax + (1-a)y \), the inequality \( a \leq \frac{1}{2} \) is equivalent to implication (1).

If \( K \) satisfies the “opposite” condition to (1), interchanging the roles of variables in \( K \) and applying Theorem 1, we obtain the following

**Theorem 2.** Let a continuous mean \( K : I^2 \to I \) be strictly increasing in the first variable and increasing in the second one. Suppose that \( K \) satisfies the following condition:

\[ x < y \implies K(x, y) \leq K(y, x), \quad x, y \in I. \]

Then

(i) for every mean \( M : I^2 \to I \) there is a unique mean \( M^{[K]} : I^2 \to I \) such that \( K \) is \((M, M^{[K]})\)-invariant, i.e.

\[ K(M^{[K]}(x, y), M(x, y)) = K(x, y), \quad x, y \in I; \]
if for some symmetric mean $M$, the mean $M^{[K]}$ is symmetric, then $K$ is symmetric;
(iii) if $M$ is strict then so is $M^{[K]}$;
(iv) if $M$ is continuous then so is $M^{[K]}$.

**Definition 2.** Under the conditions of Theorem 2, the mean $M^{[K]}$ can be referred to as $K$-complementary (or $K$-left complementary) to the mean $M$.

From Theorems 1 and 2 we obtain the following improvement of Remark 1 in [3]:

**Corollary 1.** If $K : I^2 \to I$ is a continuous strictly increasing in each variable and symmetric mean, then for every mean $M : I^2 \to I$ there is a unique function $N : I^2 \to I$ such that $K \circ (M, N) = K$, moreover $N$ is a mean and

$$M^{[K]} = N = M^{[K]}.$$

We end this section with

**Remark 3.** In Theorems 1 and 2 (as well as in [3], Remark 1), the strict increasing monotonicity in each variable of the invariant mean $K : I^2 \to I$, cannot be weaken by the assumption that $K$ is a strict mean.

To show it consider the following

**Example 2.** It is known that the contra-harmonic mean $K : (0, \infty)^2 \to (0, \infty)$,

$$K(x, y) = \frac{x^2 + y^2}{x + y},$$

is not strictly increasing. The symmetry of $K$ implies that condition (1) of Theorem 1 is satisfied. Taking for $M$ the arithmetic mean $A(x, y) = \frac{x + y}{2}$ for $x, y > 0$, by a simple calculation, we get

$$K_{[A]}(x, y) = \frac{x^2 + y^2 + \sqrt{2} \sqrt{x^4 + y^4}}{2(x + y)}, \quad x, y > 0.$$

Of course, $K_{[A]} : (0, \infty)^2 \to (0, \infty)$ and $K_{[A]}$ is reflexive, i.e. $K_{[A]}(x, x) = x$; so $K_{[A]}$ is a bivariate pre-mean in $(0, \infty)$. But, as

$$K_{[A]}(1, 10) > 11,$$

$K_{[A]}$ is not a mean.

(It can be verified similarly, that $K_{[G]}$, $K_{[H]}$ where $G$ and $H$ stand, respectively, for the geometric and harmonic mean, are pre-means, but not means.)
### 3. Some applications

Applying Theorem 1 and the main result of [5] (see also [4]) we obtain the following

**Theorem 3.** Suppose that a mean \( K : I^2 \to I \) is continuous and strictly increasing in each variable.

(i) If \( K \) satisfies the condition
\[
x < y \implies K(x, y) \geq K(y, x), \quad x, y \in I,
\]
then for every continuous and strict mean \( M : I^2 \to I \), the sequence \((M, M[K])^n : n \in \mathbb{N}\) of iterates of the mean-type mapping \((M, M[K]) : I^2 \to I^2\) converges uniformly on compact sets to the mean-type map \((K, K)\).

(ii) If \( K \) satisfies the condition
\[
x < y \implies K(x, y) \leq K(y, x), \quad x, y \in I,
\]
then for every continuous and strict mean \( M : I^2 \to I \), the sequence \((M[K], M)^n : n \in \mathbb{N}\) of iterates of the mean-type mapping \((M[K], M) : I^2 \to I^2\) converges uniformly on compact sets to the mean-type map \((K, K)\).

Using this result we prove the following

**Theorem 4.** Let a mean \( K : I^2 \to I \) be continuous and strictly increasing in each variable and \( M : I^2 \to I \) be an arbitrary strict and continuous mean.

(i) Suppose that \( K \) satisfies the condition
\[
x < y \implies K(x, y) \geq K(y, x), \quad x, y \in I.
\]
Then a function \( F : I^2 \to \mathbb{R} \) continuous at every point of the diagonal \( \Delta(I^2) := \{(x, x) : x \in I\} \) satisfies the functional equation
\[
F(M(x, y), M[K](x, y)) = F(x, y), \quad x, y \in I,
\]
if and only if there is a single variable continuous function \( \varphi : I \to \mathbb{R} \) such that
\[
F(x, y) = \varphi(K(x, y)), \quad x, y \in I.
\]

(ii) Suppose that \( K \) satisfies the condition
\[
x < y \implies K(x, y) \leq K(y, x), \quad x, y \in I.
\]
Then a function \( F : I^2 \to \mathbb{R} \) continuous at every point of the diagonal \( \Delta(I^2) \) satisfies the functional equation
\[
F\left(M[K](x, y), M(x, y)\right)
= F(x, y), \quad x, y \in I,
\]
if and only if there is a single variable continuous function \( \varphi : I \to \mathbb{R} \) such that

\[
F(x, y) = \varphi(K(x, y)), \quad x, y \in I.
\]

**Proof.** Assume first that \( F : I^2 \to \mathbb{R} \) is continuous on the diagonal \( \Delta(I^2) \) and satisfies equation (4), that is

\[
F \circ (M, M_{[K]}) = F.
\]

Hence, by induction,

\[
F = F \circ (M, M_{[K]})^n, \quad n \in \mathbb{N},
\]

where \( (M, M_{[K]})^n \) is the \( n \)th iterate of \( (M, M_{[K]}) \). By Theorem 3 the sequence of mean-type mappings \( (M, M_{[K]})^n \) converges to the mean-type mapping \( (K, K) : I^2 \to I^2 \); that is

\[
\lim_{n \to \infty} (M, M_{[K]})^n (x, y) = (K(x, y), K(x, y)), \quad (x, y) \in I^2.
\]

Since \( (K(x, y), K(x, y)) \) belongs to the diagonal \( \Delta(I^2) \) for every \( (x, y) \in I^2 \), by (5) and the continuity of \( F \) on \( \Delta(I^2) \) implies that for every \( (x, y) \in I^2 \),

\[
F(x, y) = \lim_{n \to \infty} F \left( (M, M_{[K]})^n (x, y) \right) = F \left( \lim_{n \to \infty} (M, M_{[K]})^n (x, y) \right)
\]

\[
= F(K(x, y), K(x, y)).
\]

Setting

\[
\varphi(t) := F(t, t), \quad t \in I,
\]

we conclude that \( F(x, y) = \varphi(K(x, y)) \) for all \( (x, y) \in I^2 \).

To prove the converse implication, take an arbitrary function \( \varphi : I \to \mathbb{R} \) and put \( F := \varphi \circ K \). Then, for all \( x, y \in I \), making use of the \( K \)-invariance with respect to \( M, M_{[K]} \), we have

\[
F(M(x, y), M_{[K]}(x, y)) = (\varphi \circ K)(M(x, y), M_{[K]}(x, y))
\]

\[
= \varphi(K(M(x, y), M_{[K]}(x, y))) = \varphi(K(x, y))
\]

\[
= F(x, y),
\]

which completes the proof of (i).

We omit the similar proof of (ii). \( \square \)

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