Geometric quantization of $N = 2, D = 3$ superanyon

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Abstract

A classical model of $N = 2, D = 3$ fractional spin superparticle (superanyon) is presented, whose first-quantization procedure combines the Berezin quantization for the superspin degrees of freedom and the canonical quantization for the space-time ones. To provide the supersymmetry for the quantised theory, certain quantum corrections are required to the $N = 2$ supersymmetry generators as compared to the Berezin procedure. The renormalized generators are found and the first quantised theory of $N = 2$ superanyon is constructed.

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Anyons, particles with fractional spin and statistics in $(1+2)$-dimensional space-time, made their appearance twenty years ago $[1]$. In the middle 80's, the anyon concept was applied to the explanation of the fractional quantum Hall effect $[2]$ and to the high-$T_c$ superconductivity models $[3]$.

From the group-theoretical viewpoint, a possibility of fractional spin emerges from the ordinary classification of the Poincaré group irreps. Spin is not quantised in $D = 1 + 2$ because the little group of the massive irrep $SO(2)$ is an abelian group. Fractional spin describes an appropriate representation of the universal covering group $ISO^\uparrow(1, 2)$.

The investigations of anyons in the field-theory $[4, 5]$ are supplemented by the study of the mechanical models and the corresponding first quantised theories $[6, 7, 8]$. One of the related problems is to realize the one-particle wave equations of anyon and the corresponding action functionals $[9]$ in the form to be convenient for the quantum field theory. Another interesting problem is to construct a consistent interaction of anyons to external electromagnetic or gravitational fields $[10, 11]$.

The study of the spinning particle models $[12]$ has a long manifold history. For a certain period these models served as test examples of application of the modern quantization methods.

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In particular, the spinning particles are treated as elementary systems within the Kostant-Souriau-Kirillov (KSK) quantization method \cite{10, 11}. Roughly speaking, the later is based on the observation that physical phase space of any elementary system is isomorphic to a coadjoint orbit of the symmetry group. In the framework of the KSK construction the symplectic action of the symmetry group (classical mechanics) lifts to a unitary irreducible representation of the group in a space of functions on the classical symplectic manifold (quantum mechanics). More perfect results give the Berezin quantization method \cite{12, 13}, which is powerful for the Kähler homogeneous spaces. In this case the one-to-one correspondence can be established between the phase-space functions (covariant Berezin symbols) and the linear operators in a Hilbert space, being realized by holomorphic functions on the classical manifold. Moreover, the multiplication of the operators induces a noncommutative binary ∗-operation for the covariant symbols, that provides the correspondence principle for the observables \cite{12}.

The classical mechanics of D = 3 spinning particle and superparticle has some peculiar features, which have no parallel in higher dimensions. In particular, one can conceive that the anyon (super)particle lives in the phase space of special (super)geometry, being a direct product of the cotangent bundle of Minkowski space $T^*(R^{1,2})$ (phase space of the spinless particle) to the curved inner symplectic (super)space, which provides the 1+2 particle with non trivial (super)spin and corresponding degrees of freedom. Moreover, the inner (super)manifold is endowed with a structure of a Kähler homogeneous (super)space. The structure found for anyon particle and N = 1 superparticle in Ref. \cite{14} provides a convenient tool for first quantization of the (super)anyon. One can combine the canonical quantization in $T^*(R^{1,2})$ and the geometric quantization of the superspin degrees of freedom \cite{14}. First class constraints of the classical mechanics are converted into one-particle wave equations of (super)anyon according to the Dirac quantization prescription. The first quantization procedure \cite{14} gives results in agreement with the known description of the fractional (super)spin by the use of the unitary representations of the discrete series of $su(1, 1) \cong so(1, 2)$ algebra \cite{3} and of the unitary irreps of $osp(2|2)$ superalgebra and the deformed Heisenberg algebra \cite{13}.

In this letter, we suggest N = 2 superextended anyon model, generalizing the respective N = 1 one \cite{14}. We construct the embedding of the maximal coadjoint orbit of N = 2 Poincaré supergroup into the phase space $T^*(R^{1,2}) \times L^{1|2}$, where the inner supermanifold $L^{1|2} \cong OSp(2|2)/U(1) \times U(1)$, to be associated to the particle superspin, is a Kähler $OSp(2|2)$-homogeneous superspace, a typical orbit of the coadjoint representation of $OSp(2|2)$. The supergeometry underlying $L^{1|2}$ and the geometric quantization is studied in \cite{16, 17}. The first quantization procedure has some new features for N = 2 superparticle if compared to the N = 1 case \cite{14}. Surprisingly, certain quantum corrections are required to the N = 2 supercharge’s operators, being originally constructed from the respective classical values by an ordinary correspondence rule. It should be noted that there is no general prescription to make such corrections in the framework of Berezin quantization. Nevertheless, these corrections, being crucial for a compatibility of conventional quantum theory, can be exactly computed in this case.
Turn to explicit constructions. As is known \cite{1, 2, 3, 4}, the dynamics of the D = 3 spinning particle of mass $m$ and spin $s$ can be realized in the six dimensional phase space with symplectic two-form

$$\Omega_s = -dx^a \wedge dp_a + s\Omega_m \quad \Omega_m = \frac{1}{2} \epsilon^{abc} p_a dp_b \wedge dp_c$$

and is generated by the only constraint

$$p^2 + m^2 = 0.$$  \hfill (2)

The closed form $\Omega_m$ is called the Dirac monopole two-form. This formulation of the classical dynamics of anyon (known as canonical) is well suited for the introducing of external fields \cite{4}, but it is inconvenient for first quantization, because in quantum theory the covariant realization of the nonlinear Poisson structure (1) remains unsolved problem. To avoid this problem, one may equivalently reformulate the model in an extended phase space \cite{5, 6}.

The following reformulation is suitable for constructing the superextension of the model. Let us observe that $\Omega_m$ (1) is a Kähler two-form in a Lobachevsky plane $L$ realized as a mass hyperboloid \cite{4}. Really, consider the stereographic mapping of the mass hyperboloid onto an open unit disc in complex plane

$$p_a \equiv mn_a$$

$$n_a \equiv -\left(\frac{1 + z\bar{z}}{1 - z\bar{z}}, \frac{z + \bar{z}}{1 - z\bar{z}}, \frac{z - \bar{z}}{1 - z\bar{z}}\right) \quad |z| < 1 \quad n^2 \equiv -1.$$  \hfill (3)

Thus we are arriving at the Poincaré realization of $L$. Using the complex coordinate $z$, we have $\Omega_m = 1/2\epsilon^{abc} n_a dn_b \wedge dn_c = -2i(1 - z\bar{z})^{-2} dz \wedge d\bar{z}$. One can reformulate now the canonical model (1) in terms of the eight dimensional phase space $\mathcal{M}^8 = T^*(R^{1,2}) \times L$ with the symplectic two-form

$$\Omega_s = -dx^a \wedge dp_a - 2is \frac{dz \wedge d\bar{z}}{(1 - z\bar{z})^2}$$

and three constraints \cite{4}, two of which are of the second class and one of the first class. These constraints are equivalent to the following two first class constraints

$$p^2 + m^2 = 0 \quad (p, n) + m = 0,$$  \hfill (5)

that should mean the identical conservation of the particle mass and spin in $\mathcal{M}^8$. This constrained Hamiltonian theory could be derived directly from the first-order covariant Lagrangian

$$L = m(\dot{x}, n) + is \frac{\bar{z}\dot{z} - z\dot{\bar{z}}}{1 - z\bar{z}}.$$  \hfill (6)

The geometry of the symplectic manifold $\mathcal{M}^8$ is well adapted for the first quantization. We can canonically quantize the Poisson bracket in $T^*(R^{1,2})$ and to apply the Berezin quantization

\footnote{We use Latin letters $a, b, c, \ldots$ to denote vector indices and Greek letters $\alpha, \beta, \gamma \ldots$ for spinor ones; the space–time metric is $\eta_{ab} = \text{diag}(-, +, +)$, the antisymmetric Levi-Civita tensor $\epsilon^{abc}$ is normalized by the condition $\epsilon^{012} = -1$; the spinor indices are raised and lowered with the use of the spinor metric by the rule $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$, $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$, $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} = -\epsilon_{\alpha\beta}$, $\epsilon^{01} = -1$.}
method \cite{12} in the Lobachevsky plane \cite{12, 13}. The constraints \cite{5} are converted into the anyon wave equations at the quantum level. The quantization of the model \cite{1} is described in Ref. \cite{14}.

Introduce the following $N = 2$ superextension of the Lagrangian \cite{3}:

$$L = m(\dot{x}, n) - im(n_{\alpha\beta}\theta^{\alpha}\dot{\theta}^{\beta} + n_{\alpha\beta}\chi^{\alpha}\dot{\chi}^{\beta}) + mb(\theta_{\alpha}\chi^{\alpha} - \chi_{\alpha}\dot{\theta}^{\alpha}) - mb\theta^{\alpha}n_{\alpha\gamma}\dot{n}_{\gamma}\chi^{\beta} + is\frac{z\dot{z} - \dot{z}z}{1 - z\bar{z}} , \quad (7)$$

where $n_{\alpha\beta} = (n^\alpha\sigma_a)_{\alpha\beta}$, the explicit form of $\sigma$-matrices

$$(\sigma_0)_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\sigma_1)_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (\sigma_2)_{\alpha\beta} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

is compatible with the SU(1, 1) spinor formalism; $\theta^{\alpha}, \chi^{\alpha}$ are $D = 3$ real odd Grassmann spinors. As is shown below, the real parameter $b$ is related to the superparticle’s central charge. Besides the Poincaré group, the model \cite{3} is invariant under the supertranslations of the form

$$\delta_{\xi}x^{\alpha} = i(\sigma^\alpha)_{\alpha\beta}\epsilon^{\alpha}\theta^{\beta} + ib\epsilon^{abc}n_b(\sigma_c)_{\alpha\beta}\epsilon^{\alpha}\chi^{\beta} - bn^{\alpha}\epsilon^{\alpha}\chi_{\alpha} \quad \delta_{\eta}x^{\alpha} = i(\sigma^\alpha)_{\alpha\beta}\eta^{\alpha}\chi^{\beta} - ib\eta^{abc}n_b(\sigma_c)_{\alpha\beta}\eta^{\alpha}\theta^{\beta} + bn^{\alpha}\eta^{\alpha}\theta_{\alpha} \quad (8)$$

The corresponding supercharges $Q^I_{\alpha}, I = 1, 2$ generate the Poisson brackets

$$\{Q^I_{\alpha}, Q^J_{\beta}\} = -2i\delta^{IJ}p_{\alpha\beta} - 2\epsilon^{IJ}\epsilon_{\alpha\beta}Z \quad Z = -b(p, n) \approx mb , \quad (9)$$

where $Z$ is a central charge and $\approx$ means a weak equality (modulo constraints). Because of the Bogomol’nyi-Prassad-Sommerfield bound $m \geq |Z|$ (see, for instance, \cite{19}) one may take here $|b| \leq 1$. Moreover, one can easy verify that in the BPS limit, when $|b| = 1$, half of the odd degrees of freedom drops out of the Lagrangian \cite{5} and the model reduces to the one of $N = 1$ fractional spin superparticle considered in Ref. \cite{14}. This case will not discussed again here and we assume $|b| < 1$ below.

In the terms of symplectic geometry, the Hamiltonian dynamics of the superparticle model \cite{3} is realized in $(8|4)$-dimensional supermanifold $M^{8|4}$ of a special structure, $M^{8|4} \cong T^*(\mathbb{R}^{1,2}) \times L^{1|2}$, where $L^{1|2}$ is an inner supermanifold of real dimension $(2|4)$. This means that the symplectic two-form on $M^{8|4}$ reads

$$\Omega_{\mathrm{SUSY}}^{\mathrm{SUSY}} = -dx^{\alpha} \wedge dp_{\alpha} + s\Omega_{L^{1|2}} , \quad (10)$$

where $\Omega_{L^{1|2}}$ does not depend on the space-time coordinates and momenta. $N = 2$ Poincaré supersymmetry leaves invariant the constraint surface \cite{5} in $M^{8|4}$. The Hamiltonian representation of the Poincaré superalgebra, being described below, shows that the constraints \cite{5} provide the identical conservation laws for the superparticle mass and superspin. These conserved values coincide respectively with constants $m$ and $s$ entering the original Lagrangian \cite{4}.

In the nonsuperextended model on $M^{8}$, a Kähler geometry of the inner manifold $L$ makes possible to apply the Berezin quantization method for the construction of the first quantised theory of anyon. Let us study a supergeometry underlying $L^{1|2}$ and clarify its relationship to $N = 2$ Poincaré superalgebra.
The supermanifold $\mathcal{L}^{1|2}$ is considered in Refs. [16, 17] as a typical coadjoint orbit of the OSp(2|2) supergroup, $\mathcal{L}^{1|2} \cong \text{OSp}(2|2)/U(1) \times U(1)$. Moreover, $\mathcal{L}^{1|2}$ is shown to be a Kähler homogeneous OSp(2|2)-superspace of complex dimension (1|2), a $N = 2$ superextension of the Lobachevsky plane. $\mathcal{L}^{1|2}$ is called as $N = 2$ superunit disc. In holomorphic coordinates $z, \theta, \chi$ in $\mathcal{L}^{1|2}$, the Kähler superpotential reads [17]

$$\Phi = -2 \ln(1 - z\bar{z}) - (1 + b) \frac{\theta\bar{\theta}}{1 - z\bar{z}} - (1 - b) \frac{\chi\bar{\chi}}{1 - z\bar{z}} + \frac{1 - b^2}{2} \frac{\theta\bar{\theta}\chi\bar{\chi}}{(1 - z\bar{z})^2},$$  \hspace{1cm} (11)

and the symplectic two-superform is defined with respect to the Kähler superpotential in the standard way\footnote{We use the left derivatives only.}

$$s\Omega_{1|2} = \frac{is}{2} \delta\Phi \hspace{1cm} \delta = dz \frac{\partial}{\partial z} + d\theta \frac{\partial}{\partial \theta} + d\chi \frac{\partial}{\partial \chi}. \hspace{1cm} (12)$$

The complex odd variables $\theta, \chi$ are in one-to-one correspondence with the Majorana spinors $\theta^\alpha, \chi^\alpha$ used before in Eq. (5)

$$\theta = \sqrt{\frac{m}{s}} (z_\alpha \theta^\alpha - iz_\alpha \chi^\alpha) \left[ 1 + m - \frac{b}{4s} (\theta^\alpha \theta_\alpha + \chi^\alpha \chi_\alpha) \right]$$

$$\chi = \sqrt{\frac{m}{s}} (z_\alpha \chi^\alpha - iz_\alpha \theta^\alpha) \left[ 1 + m - \frac{b}{4s} (\theta^\alpha \theta_\alpha + \chi^\alpha \chi_\alpha) \right] \hspace{0.5cm} z_\alpha \equiv (z, -1). \hspace{1cm} (13)$$

Our main interest here is in the supergroup of the superholomorphic canonical transformations on $\mathcal{L}^{1|2}$. We have found that this supergroup, denoted SU(1, 1|2), is wider than OSp(2|2) is. We consider here the corresponding superalgebra su(1, 1|2). The even part $\text{su}(1,1|2)_0 = \text{span}\{J, P_1, P_4, Z; I = 1, 2, 3\}$ of su(1, 1|2) is a direct sum of the Lorentz algebra su(1, 1), the isotopic algebra u(2) and the one-dimensional centre of the superalgebra, whereas the odd part $\text{su}(1,1|2)_1 = \text{span}\{E^\alpha, F^\alpha, G^\alpha, H^\alpha\}$ is an eight dimensional module of the even part. The Hamiltonian supergenerators read

$$J_a = -s \pi_{a} \left( 1 - \frac{1 + b}{2} \frac{\theta\bar{\theta}}{1 - z\bar{z}} - \frac{1 - b}{2} \frac{\chi\bar{\chi}}{1 - z\bar{z}} + \frac{1 - b^2}{2} \frac{\theta\bar{\theta}\chi\bar{\chi}}{(1 - z\bar{z})^2} \right) \hspace{1cm} Z = s$$

$$P_1 = \frac{s \sqrt{1 - b^2} \theta\bar{\theta}}{2} \left( 1 - \frac{1 - b}{2} \frac{\chi\bar{\chi}}{1 - z\bar{z}} \right)$$

$$P_2 = \frac{is \sqrt{1 - b^2} \theta\bar{\theta}}{2} \left( 1 - \frac{1 - b}{2} \frac{\chi\bar{\chi}}{1 - z\bar{z}} \right)$$

$$P_3 = \frac{s \sqrt{1 - b^2} \theta\bar{\theta}}{2} \left( 1 - \frac{1 - b}{2} \frac{\chi\bar{\chi}}{1 - z\bar{z}} \right)$$

$$P_4 = \frac{s \sqrt{1 - b^2} \theta\bar{\theta}}{2} \left( 1 - \frac{1 - b}{2} \frac{\chi\bar{\chi}}{1 - z\bar{z}} \right) \hspace{1cm} F^\alpha = i \alpha^\beta E^\beta$$

$$E^\alpha = s \sqrt{1 + b} \left( \frac{z_\alpha \bar{\theta} - \bar{z}_\alpha \theta}{1 - z\bar{z}} \right) \left( 1 - \frac{1 - b}{2} \frac{\chi\bar{\chi}}{1 - z\bar{z}} \right)$$

$$G^\alpha = s \sqrt{1 - b} \left( \frac{z_\alpha \bar{\theta} - \bar{z}_\alpha \theta}{1 - z\bar{z}} \right) \left( 1 - \frac{1 - b}{2} \frac{\chi\bar{\chi}}{1 - z\bar{z}} \right) \hspace{1cm} H^\alpha = i \alpha^\beta G^\beta,$$

where $z_\alpha \equiv (1, z), \bar{z}_\alpha \equiv (\bar{z}, 1)$, and form $\text{su}(1,1|2)$ superalgebra with respect to the graded
Poisson bracket on \( N = 2 \) superunit disc \( \{ , \} \) \( \mathcal{L}_{1/2} \equiv \{ , \} \):

\[
\{ J_a, J_b \} = \epsilon_{abc}J^c \quad \{ J_a, E^\alpha \} = \frac{i}{2} (\sigma_\alpha)_{\beta} E^\beta \quad \{ J_a, F^\alpha \} = \frac{i}{2} (\sigma_\alpha)_{\beta} F^\beta \\
\{ P_1, P_2 \} = -\epsilon_{1JK}P^K \quad \{ J_a, G^\alpha \} = \frac{i}{2} (\sigma_\alpha)_{\beta} G^\beta \\
\{ E^\alpha, P_2 \} = 1/2 H^\alpha \quad \{ E^\alpha, P_3 \} = -1/2 G^\alpha \\
\{ F^\alpha, P_2 \} = -1/2 H^\alpha \quad \{ F^\alpha, P_3 \} = 1/2 E^\alpha \\
\{ G^\alpha, P_2 \} = 1/2 F^\alpha \quad \{ G^\alpha, P_3 \} = 1/2 H^\alpha \\
\{ H^\alpha, P_2 \} = -1/2 E^\alpha \quad \{ H^\alpha, P_3 \} = -1/2 G^\alpha \\
\{ E^\alpha, F^\beta \} = \epsilon^{\alpha\beta}(Z - P_3) \quad \{ E^\alpha, G^\beta \} = -\epsilon^{\alpha\beta}P_2 \quad \{ E^\alpha, H^\beta \} = \epsilon^{\alpha\beta}P_1 \\
\{ G^\alpha, H^\beta \} = \epsilon^{\alpha\beta}(Z + P_3) \quad \{ F^\alpha, H^\beta \} = -\epsilon^{\alpha\beta}P_2 \quad \{ F^\alpha, G^\beta \} = \epsilon^{\alpha\beta}P_1 \\
\{ E^\alpha, E^\beta \} = \{ F^\alpha, F^\beta \} = \{ G^\alpha, G^\beta \} = \{ H^\alpha, H^\beta \} = i(\sigma_\alpha)_{\beta} J^\alpha \\
\{ J_a, P_1 \} = 0 \quad \{ P_1, P_4 \} = 0 \quad \{ J_a, P_4 \} = 0 \\
\{ Z, \text{anything} \} = 0.
\]

In particular, \( \text{osp}(2|2) \) subsuperalgebra is generated by \( J_a, B, \sqrt{ms} V^\alpha, \sqrt{ms} W^\alpha \), where \( V^\alpha, W^\alpha \) are defined below in Eqs. (17) and \( B = P_3 - bZ \). These \( \text{osp}(2|2) \) supergenerators were evaluated in Ref. [17] in the framework of the theory of the supercoherent states. We conceive here that \( N = 2 \) superunit disc is not only the typical coadjoint orbit of the \( \text{OSp}(2|2) \) supergroup, \( \mathcal{L}_{1/2} \cong \text{OSp}(2|2)/(U(1) \times U(1)) \), but it can be treated as an atypical orbit of the supergroup \( \text{SU}(1,1|2) \) as well, \( \mathcal{L}_{1/2} \cong \text{SU}(1,1|2)/(U(2|2) \times U(1)) \).

The supersymplectic action of the \( N = 2 \) Poincaré superalgebra in \( \mathcal{M}^{8|4} \) is generated by the following combinations of the space-time variables and the “inner” \( \text{su}(1,1|2) \)-generators:

\[
J_a = \epsilon_{abc} J^c + J_a \\
P_a = p_a \\
Q^1_\alpha = (i\alpha_\beta W^\beta + m\tilde{W}_\alpha)[1 + q^c(bP_3 - \sqrt{1 - b^2} P_2 - P_4)] \\
Q^2_\alpha = (i\alpha_\beta V^\beta + m\tilde{V}_\alpha)[1 + q^c(bP_3 + \sqrt{1 - b^2} P_2 - P_4)]
\]

where

\[
W^\alpha = \frac{1}{2\sqrt{ms}}(\sqrt{1 + b} E^\alpha + \sqrt{1 - b} H^\alpha) \\
V^\alpha = \frac{1}{2\sqrt{ms}}(\sqrt{1 + b} F^\alpha + \sqrt{1 - b} G^\alpha) \\
\tilde{W}^\alpha = \frac{1}{2\sqrt{ms}}(\sqrt{1 + b} F^\alpha - \sqrt{1 - b} G^\alpha) \\
\tilde{V}^\alpha = \frac{1}{2\sqrt{ms}}(\sqrt{1 + b} H^\alpha - \sqrt{1 - b} E^\alpha)
\]

and

\[
q^c = \frac{1}{4s}
\]

is a parameter, which should be renormalized later to provide the supersymmetry in the quantum theory.

Relations (16) assume that the problem of operator realization of \( N = 2 \) Poincaré supersymmetry can be solved in the quantum theory if appropriate realization is constructed for
su(1,1|2) superalgebra. The later problem may admit an elegant solution in the framework of the geometric quantization on the Kähler homogeneous superspaces. For N = 2 superunit disc the geometric quantization is constructed in Ref. [17], where the classical symbols of osp(2|2) are lifted to the operators acting in the space $O_{s,b}$ of the antiholomorphic functions of the form

$$F(z, \bar{\theta}, \bar{\chi}) = F_0(z) + \sqrt{s(1 + b)} \bar{\theta} F_1(z) + \sqrt{s(1 - b)} \chi F_2(z) + \sqrt{s(2s + 1)(1 - b^2)/2} \bar{\theta} \chi F_3(z).$$

We take $s > 0$ below; the case of $s < 0$ requires some inessential changes. The Hamiltonian generators (14) of $su(1,1|2)$ superalgebra may be lifted to the unitary irreducible representation in the space $O_{s,b}$. One gets

$$\mathbf{J}_a = -\bar{z} \bar{\xi}_a - (\bar{\partial} \bar{\xi}_a) \left( s + \frac{1}{2} \frac{\partial}{\partial \theta} + \frac{1}{2} \frac{\partial}{\partial \chi} \right)$$

where $\bar{\partial}$ is $\partial/\partial z$, $\bar{\xi}_a \equiv -1/2(2\bar{z}, 1 + \bar{z}, i(1 - \bar{z}))$. The (anti) commutation relations for these operators follow from Eqs. (15) by replacing $\{ , \} \rightarrow 1/i[ , ]_\mp$ (anticommutator for two odd operators and commutator in the rest cases). These operators are Hermitian with respect to an invariant inner product $\langle \cdot | \cdot \rangle_{L^2[1]}$ [13,17]. With respect to the su(1,1) subalgebra, the constructed representation is decomposed into the direct sum $D^+_\uparrow \oplus D^s_{\uparrow+1/2} \oplus D^s_{\uparrow+1/2} \oplus D^s_{\uparrow+1}$ of the unitary representations of discrete series.

As the geometry of the phase space and its symmetries are clear now, we are in position to study the quantization of the superanyon model. Consider the space of functions of the form $F(p, \bar{z}, \theta, \chi)$, where $F(p, \bar{z}, \theta, \chi) \in O_{s,b}$ for each fixed momentum $p$, and take, accounting for Eq. (16), the following ansatz for the generators of the N = 2 Poincaré superalgebra

$$\hat{J}_a = -i \epsilon_{abc} \hat{p}^b \frac{\partial}{\partial \hat{p}_c} + \mathbf{J}_a$$

$$\hat{P}_a = p_a$$

$$\hat{Z} = mb$$

$$\hat{Q}^a_b = (ip_{\alpha\beta} W^\beta + m \hat{W}_a)[1 + q(bP_3 - \sqrt{1 - b^2} P_2 - P_4)]$$

$$\hat{Q}^a_a = (ip_{\alpha\beta} V^\beta + m \hat{V}_a)[1 + q(bP_3 + \sqrt{1 - b^2} P_2 - P_4)].$$

The operators $W^a, \hat{W}^a, V^a, \hat{V}^a$ are expressed as linear combinations of $E^a, \tilde{E}^a, G^a, \tilde{H}^a$ according to relations (17). So, relations (21) represent the quantum operators resulting from the
classical Poincaré supergroup generators by a straightforward canonical quantization. However, examining the respective (anti)commutators, we find that the operators (21) do not generate a representation of \( N = 2 \) Poincaré superalgebra, if \( q = q^{cl} \) (18) as in the expressions (16). The problem is in anticommutators of the supercharges which have on shell the form

\[
[\hat{Q}^I_\alpha, \hat{Q}^J_\beta] = 2\delta^{IJ}p_{\alpha\beta} - 2i\text{img}\epsilon^{IJ}_\epsilon\epsilon_{\alpha\beta} + \mathcal{O}(s^{-2}),
\]  

(22)

(compare with Eq. (9)). Mention that the corrections \( \mathcal{O}(s^{-2}) \), which appear in r.h.s. of the anticommutators, should be expected in advance according to a correspondence principle. The latter follows naturally from Berezin quantization method for Kähler homogeneous manifolds [12], which implies that the classical symbol of quantum commutator of two bounded operators in a Hilbert space coincides with the Poisson bracket of respective covariant symbols only in first order in the “Planck constant”. In general, the corrections in higher orders vanish only for the generators of the Lie algebra of the symmetry group. It can be found that the correspondence principle holds for the \( N = 2 \) superunit disc \( \mathcal{L}^{1|2} \) too, where the parameter \( s^{-1} \) serves as a “Planck constant”. Thus, the quantum corrections in the anticommutators (22) originate from the nonlinearity of the Poincaré supercharge operators (21) in the generators (20) of the “inner” superalgebra \( \text{su}(1,1|2) \).

The conventional construction of the one-particle quantum mechanics for superanyon implies to have an exact realization of the Poincaré supersymmetry, without any disclosing corrections depending on the parameters of the model. To find the true realization, we can try, starting from Eqs. (21), (22), to introduce a renormalized terms in the observables (21) for the closure of the anticommutators (22). However, we don’t have any general reasons, which may ensure the consistency of the renormalization procedure; a structure of possible higher order corrections to (21) is unclear also. Surprisingly, exact corrections may be found in the simplest ansatz for the quantum observables. Namely, we find that the closure of the Poincaré superalgebra is achieved by the renormalization of the only parameter \( q \) entering the expressions (21) of the supercharges.

It is examined by a direct calculation that the generators (21) with renormalized value of \( q \)

\[
q^{quant} = 1 - \sqrt{1 - \frac{1}{2s + 1}} = q^{cl} + \mathcal{O}(s^{-2}) .
\]  

(23)

form the closed Poincaré superalgebra and, thus, they are treated as true quantum observables of \( N=2 \) superanyon.

The super Poincaré covariant equations for the wave function of the \( N = 2 \) superanyon have the form

\[
(p^2 + m^2)F^{phys}(p, \bar{z}, \bar{\theta}, \bar{\chi}) = 0 \\
[p, J] - mP_4 - msF^{phys}(p, \bar{z}, \bar{\theta}, \bar{\chi}) = 0 .
\]  

(24)

that appears when the constraint operators are imposed on the physical states \( F^{phys} \). It should be recognized that the \( N = 2 \) Poincaré supersymmetry is realized on shell only. The last remarkable

\footnote{The proof will be presented elsewhere.}
step is that the space of solutions of the wave equations is endowed with the structure of Hilbert space, where the operators (21) form a unitary representation. The respective inner product
\[ \langle F^{\text{phys}}, G^{\text{phys}} \rangle = \int \frac{d\vec{p}}{p^0} \langle F^{\text{phys}}|G^{\text{phys}} \rangle_{L^{1|2}} \quad p^0 = \sqrt{\vec{p}^2 + m^2} > 0 \] (25)
is constructed in terms of the Poincaré invariant measure on the mass shell and the \( su(1,1|2) \) invariant inner product in \( L^{1|2} \).

Thus, the first quantised theory of \( N = 2 \) superanyon has been constructed in general. The theory give the supersymmetric generalization of the well known description \([5, 6]\) of the fractional spin states using the unitary representations \( D^s \) of discrete series of \( SU(1,1) \). Each component of the expansion (19) of the wave function \( F^{\text{phys}} \) describes a particle of mass \( m \) and fractional spin \( s \), \( s + 1/2 \) (two states) or \( s + 1 \).

In this letter we have begun with a classical model (7), being \( N = 2 \) superextension of the canonical model of anyon, and arrive to the first quantised theory. The reverse way is in the following. The space of quantum states \( \{ |p, \bar{z}, \bar{\theta}, \bar{\chi} \rangle \} \) (which are solutions of Eq. (24)) is labeled by the points of the surface of constraints (5) of the classical phase superspace \( T^*(R^{1|2}) \times L^{1|2} \), similar to geometric quantization method. Moreover, we have
\[ \frac{\langle p, \bar{z}, \bar{\theta}, \bar{\chi}|J_a|p, \bar{z}, \bar{\theta}, \bar{\chi} \rangle}{\langle p, \bar{z}, \bar{\theta}, \bar{\chi}|p, \bar{z}, \bar{\theta}, \bar{\chi} \rangle} = J_a \quad \frac{\langle p, \bar{z}, \bar{\theta}, \bar{\chi}|P_a|p, \bar{z}, \bar{\theta}, \bar{\chi} \rangle}{\langle p, \bar{z}, \bar{\theta}, \bar{\chi}|p, \bar{z}, \bar{\theta}, \bar{\chi} \rangle} = P_a \]
\[ \frac{\langle p, \bar{z}, \bar{\theta}, \bar{\chi}|Q^I_{a'}|p, \bar{z}, \bar{\theta}, \bar{\chi} \rangle}{\langle p, \bar{z}, \bar{\theta}, \bar{\chi}|p, \bar{z}, \bar{\theta}, \bar{\chi} \rangle} = Q^I_{a'} + O(s^{-2}). \] (26)

The last correction \( O(s^{-2}) \) is related certainly to the renormalization of the parameter \( q \) in Eq. (21). The possibility of this renormalization is the most intriguing result of the geometric quantization of the superspin degrees of freedom of \( N = 2 \) superanyon.

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