COMPLETE DESCRIPTIONS OF INTERMEDIATE OPERATOR ALGEBRAS BY INTERMEDIATE EXTENSIONS OF DYNAMICAL SYSTEMS

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Abstract. Practically and intrinsically, inclusions of operator algebras are of fundamental interest. The subject of this paper is intermediate operator algebras of inclusions. There are two previously known theorems which naturally and completely describe all intermediate operator algebras: the Galois Correspondence Theorem and the Tensor Splitting Theorem. Here we establish the third, new complete description theorem which gives a canonical bijective correspondence between intermediate operator algebras and intermediate extensions of dynamical systems. One can also regard this theorem as a crossed product splitting theorem, analogous to the Tensor Splitting Theorem. We then give concrete applications, particularly to maximal amenability problem and a new realization result of intermediate operator algebra lattice.

1. Introduction

In operator algebra theory, it is a fundamental problem to consider inclusions of operator algebras. They naturally arise from many subjects, including knot theory [28], operator theory [4], and algebraic quantum field theory [31]. Certainly inclusions also have intrinsic interest; since Jones’ index theorem [27], subfactor theory becomes one of the central subject in operator algebra theory. the Connes embedding problem [9] (see [44] for a survey) is one of the most important open problem in operator algebra theory— the highlight is Kirchberg’s theorem [34] which reveals unexpected connections of this problem with other important open problems. Also, not just being of intrinsic interest, but also nice inclusions often play crucial roles in the classification/structure theory of operator algebras. Here we list only a few of celebrated results in which inclusions/embeddings play crucial roles: Connes’ classification of injective factors [9] (cf. [44]), Popa’s deformation/rigidity theory [53], [52], the Akemann–Ostrand property for hyperbolic groups [19] (see [45], [48] for significant applications), characterizations of C*-simplicity of discrete groups [29], [7] (cf. [17], [18], [46]). These facts support the importance of the study of inclusions of operator algebras, particularly to give detailed analysis of inclusions and to construct inclusions with interesting properties.

The subject of the paper is the complete description problem of inclusions of operator algebras. Here we briefly recall some known major results on this subject. Galois correspondence results state that for compact groups $G$ and for discrete groups $\Gamma$, under certain assumptions, there is a natural bijective correspondence between intermediate von Neumann algebras of $M^G \subset M$, $M \subset M \rtimes \Gamma$ and closed subgroups of $G$, $\Gamma$ respectively. This reduces the complete description problem of intermediate operator algebras—usually very hard and almost impossible—to that of closed subgroups—algebraic, tractable problem. We refer the reader to [25] for backgrounds and history of this subject. The fundamental work of Izumi–Longo–Popa

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finally established the Galois correspondence for all minimal actions of $G$ and all outer actions of $\Gamma$ on arbitrary factors \cite{25}, Theorems 3.15 and 3.13. The $C^*$-algebra analogues of Galois correspondences are also studied by many authors. Izumi has shown the Galois correspondence for outer actions of finite groups on $\sigma$-unital simple $C^*$-algebras, both as a compact group and as a discrete group \cite{23}, Corollary 6.6.

Another description result of intermediate operator algebras arises from tensor products. George-Kadison’s tensor splitting theorem \cite{12} states that for any factor $M$ and for any von Neumann algebra $N$, any intermediate von Neumann algebra of the inclusion $M \subset M \otimes N$ splits into $M \otimes N_0$ for some von Neumann subalgebra $N_0 \subset N$. An analogous result for simple $C^*$-algebras is independently proved by Zacharias \cite{65} and Zsido \cite{69}. This reduces the problem on intermediate operator algebras to that on operator subalgebras. We later combine these results with our theorems to obtain interesting inclusions. See Examples 4.12 to 4.15.

In this paper, we establish a new complete description result from a different direction. Instead of changing groups, we change the coefficients in the crossed products. More precisely, we establish the bijective correspondence between intermediate operator algebras and intermediate extensions of dynamical systems for certain crossed product inclusions. This reduces the complete description problem of intermediate operator algebras to that of intermediate extensions, which is geometric/ergodic theoretical, relatively tractable, with various interesting examples. This establishes a new interaction between operator algebras and dynamical systems.

**Main Theorem.** We obtain the following bijective correspondences.

**$C^*$-case (Theorem 2.3):** Let $\Gamma$ be a discrete group satisfying the AP \cite{15}. Let $\alpha : \Gamma \curvearrowright X$ be a free action on a locally compact space $X$. Let $\beta : \Gamma \curvearrowright Y$ be an extension of $\alpha$. Let $\pi : Y \to X$ be a proper factor map. Then the map

$$C_0(Z) \mapsto C_0(Z) \rtimes_\Gamma$$

gives a bijective correspondence between the set of intermediate extensions $C_0(X) \subset C_0(Z) \subset C_0(Y)$ of $\pi$ and that of intermediate $C^*$-algebras of $C_0(X) \rtimes_\Gamma \subset C_0(Y) \rtimes_\Gamma$.

**$W^*$-case (Theorem 3.6):** Let $\Gamma$ be a countable group. Let $\alpha : \Gamma \curvearrowright (X, \mu)$ be an essentially free non-singular action on a standard probability space. Let $\beta : \Gamma \curvearrowright (Y, \nu)$ be an extension of $\alpha$. Let $\pi : Y \to X$ be a factor map. Then the map

$$L^\infty(Z) \mapsto L^\infty(Z) \rtimes_\Gamma$$

gives a bijective correspondence between the set of intermediate extensions $L^\infty(X) \subset L^\infty(Z) \subset L^\infty(Y)$ of $\pi$ and that of intermediate von Neumann algebras of $L^\infty(X) \rtimes_\Gamma \subset L^\infty(Y) \rtimes_\Gamma$.

To the author’s knowledge, this is the first result in this direction. (However we point out that Hamachi–Kosaki \cite{16} studied some of these inclusions in the context of subfactor theory.) The statement is almost complete; we cannot remove the freeness condition (even cannot be weakened to topological freeness (Proposition 2.6)), and the $C^*$-algebra case fails for non-exact groups (Proposition 2.7). Typical examples of free actions arise as the left translation actions associated to group embeddings into locally compact groups. Also, we have shown in \cite{59} that (countable, non-torsion, exact) groups admit many minimal free actions on various compact spaces (\cite{59}, Theorem B.1). Essentially free actions naturally appear as translation actions on homogeneous spaces and as Bernoulli shift actions; see \cite{68}.
As a special application, we give new examples of maximal amenable subfactors. We shortly recall a background of the maximal amenability problem. The first observation on maximal amenability is due to Fuglede–Kadison [11], where they observed that all II$_1$-factors contain a maximal AFD II$_1$ subfactor. In the seminal paper [51], Popa invented a powerful strategy (now called Popa’s asymptotic orthogonality property) which in particular confirms the maximal amenability of the inclusion $L\mathbb{Z} \subset L\mathbb{F}_2$; $Z = \langle a \rangle \subset \langle a, b \rangle = \mathbb{F}_2$. Remarkably, this settles Kadison’s problem asking if every amenable von Neumann subalgebra of a type II$_1$ factor admits an intermediate AFD II$_1$ factor. After his discovery, following basically the same line, many researchers study and construct various examples of maximal amenable von Neumann subalgebras. We refer the reader to [20] and references therein for some of them. Recently, based on the study of central states and C$^*$-algebra theory, Boutonnet–Carderi [6] gave a new method to confirm maximal amenability of certain group von Neumann algebra inclusions $L\Lambda \subset L\Gamma$. Our Main Theorem gives yet another method to provide maximal amenable subalgebras. Combining the Main Theorem and Margulis’ factor theorem [40], we obtain new examples of maximal amenable subfactors of full factors of ergodic theoretical origin. Notably, the lattices of intermediate von Neumann algebras of these inclusions are completely computable (cf. Proposition 7.76 of [38], Example 3.10). For simplicity, here we restrict the statement to simple Lie groups. See Corollary 3.8 for the full statement.

**Corollary A.** Let $G$ be a connected simple Lie group with trivial center of real rank at least 2. Let $\Gamma$ be a lattice in $G$. Let $P$ be a minimal parabolic subgroup of $G$. Let $Q$ be a proper intermediate closed group of $P \subset G$. Then the map

$$R \mapsto L^\infty(G/R)\rtimes\Gamma \ (\text{where the commutant is taken in } B(L^2(G/P)))$$

gives a lattice isomorphism from the lattice of intermediate closed groups $R \subset G$ onto that of intermediate von Neumann algebras of $L^\infty(G/P)\rtimes\Gamma \subset L^\infty(G/Q)\rtimes\Gamma$. In particular $L^\infty(G/P)\rtimes\Gamma$ is a maximal amenable subalgebra of the non-amenable factor $L^\infty(G/Q)\rtimes\Gamma$.

We also discuss a non-commutative generalization of the Main Theorem (Theorems 4.5 and 4.6). This produces further interesting examples of inclusions (Examples 4.10 to 4.15). As a byproduct, we obtain the following realization theorem.

**Corollary B (Example 4.15).** Let $G$ be a locally compact second countable group. Then one can realize the closed subgroup lattice of $G$ as the lattice of intermediate von Neumann algebras of an irreducible subfactor $N \subset M$.
Corollary C (Theorem 5.1). Every Kirchberg algebra admits an endomorphism without proper intermediate C*-algebras or expectation.

To the author’s knowledge, Corollary C is the first result in this direction.

Organization of the paper. In Section 2, we prove the Main Theorem for C*-algebras. We also demonstrate how freeness and the AP are essential by constructing counterexamples. Section 3 is devoted for von Neumann algebras. Corollary A then follows from Margulis’ factor theorem. In Section 4, we give a non-commutative generalization of the Main Theorem (Theorems 4.5 and 4.6). We then provide some applications, including Corollary B. In Section 5, we prove Corollary C. To construct the desired endomorphism, we first construct an appropriate actions of the infinite rank free group on a Kirchberg algebra.

Lattices associated to inclusions. For an inclusion $A \subset B$ of C*-algebras, the set of intermediate C*-algebras of $A \subset B$ has the lattice operation

$$C \vee D := C^*(C, D),$$

$$C \wedge D := C \cap D.$$ 

Similarly, for a closed subgroup of a topological group and for an inclusion of von Neumann algebras, there is a natural lattice structure on the set of intermediate objects.

2. INTERMEDIATE C*-ALGEBRAS AND TOPOLOGICAL DYNAMICAL SYSTEMS

In this section, we prove the Main Theorem for C*-algebras (Theorem 2.3).

For the basic facts and notations on C*-algebras, we refer the reader to [8]. Here we only fix some notations. Let $Γ$ be a discrete group. For a $Γ$-C*-algebra $A$, for $s \in Γ$, denote by $u_s$ the implementing unitary element for $s$ in the multiplier algebra $M(A, Γ)$. Denote by $E: A \rtimes_s Γ \to A$ the canonical conditional expectation $E(xu_s) = xδ_s(s); x \in A, s \in Γ$. For $a \in A \rtimes_s Γ$ and $s \in Γ$, denote by $E_s(a)$ the $s$-th coefficient $E(au_s^*)$ of $a$. For a family $S_1, \ldots, S_n$ of subsets or elements in a C*-algebra $A$, denote by $C^*(S_1, \ldots, S_n)$ the C*-subalgebra of $A$ generated by $S_1, \ldots, S_n$. The symbol ‘⊗’ stands for the minimal tensor product of C*-algebras.

We next recall some terminologies of topological dynamical systems. For a locally compact (Hausdorff) space $X$, by the Gelfand duality, there is a bijective correspondence between group actions on $X$ and those on $C_0(X)$. We use the same symbol for the corresponding action. For instance, for $α: Γ \acts X, f \in C_0(X), s \in Γ$, we set $α_s(f) = f \circ α_s^{-1}$. For two actions $α: Γ \acts X$ and $β: Γ \acts Y$ of a discrete group $Γ$ on locally compact spaces $X, Y$, $β$ is said to be an extension of $α$ if there is an equivariant quotient map $π: Y \to X$. We refer to $π$ as a factor map from $β$ onto $α$. We also refer to $π$ as an extension. Note that the choice of a factor map is not necessary unique. Hence giving an extension $π$ has more information than just stating “$β$ is an extension of $α.” A factor map is said to be proper if the preimage of any compact set is compact. Let $π: Y \to X$ be an extension. An intermediate extension of $π$ is a triplet $(γ, p, q)$ where $γ: Γ \acts Z$ is an action on a locally compact space, $p: Y \to Z, q: Z \to X$ are factor maps with $q \circ p = π$. Note that when $π$ is proper, so are $p, q$. We identify two intermediate extensions $(γ, p, q)$ and $(γ', p', q')$ if there is a $Γ$-homeomorphism $h: Z \to Z'$ satisfying $h \circ p = p', q = q' \circ h$. For a proper extension $π$, after this identification, the Gelfand duality gives a bijective correspondence between the intermediate extensions of $π$ and the intermediate
Γ-C*-algebras of the associated inclusion $C_0(X) \subset C_0(Y)$. This induces a lattice structure on the set of equivalence classes of intermediate extensions of $\pi$.

We start by the following simple lemma.

**Lemma 2.1.** Let $A$ be a $C^*$-algebra. Let $\Phi: A \to A$ be a contractive completely positive map. Let $S$ be a subset of $A$. Define

$$\ell^2(S)_1 := \left\{ (s_i)_{i=1}^{\infty} \in S^\mathbb{N} : \text{the series } S := \sum_{i=1}^{\infty} s_i^* s_i \text{ converges in norm and } \|S\| \leq 1 \right\}.$$ 

For $a \in A$, set

$$\mathfrak{N}(a; S) := \text{norm-closure} \left\{ \sum_{i=1}^{\infty} s_i^* a s_i : (s_i)_{i=1}^{\infty} \in \ell^2(S)_1 \right\}.$$ 

Then the set

$$\mathfrak{N}(\Phi; S) := \{ a \in A : \Phi(a) \in \mathfrak{N}(a; S) \}$$

is norm closed in $A$.

**Proof.** Observe that for any $(s_i)_{i=1}^{\infty} \in \ell^2(S)_1$, the map $\Psi: A \to A$ given by $\Psi(a) := \sum_{i=1}^{\infty} s_i^* a s_i$ defines a contractive completely positive map. With this observation, standard norm estimations complete the proof. \hfill \Box

The following lemma is well-known. For the reader’s convenience, we include the proof.

**Lemma 2.2.** Let $\alpha: \Gamma \curvearrowright X$ be a free action on a locally compact space. Then for any finite subset $F$ of $\Gamma \setminus \{1\}$ and for any compact subset $K$ of $X$, there is a finite open cover $U_1, \ldots, U_n$ of $K$ satisfying $\alpha_s(U_i) \cap U_i = \emptyset$ for all $i = 1, \ldots, n$ and $s \in F$.

**Proof.** Since $\alpha$ is free, for any $x \in X$ and $s \in F$, we have $\alpha_s(x) \neq x$. Since $\alpha$ is continuous, for any $x \in X$, one can choose an open neighborhood $U_x$ of $x$ with $\alpha_s(U_x) \cap U_x = \emptyset$ for all $s \in F$. Obviously $U := \{U_x\}_{x \in K}$ forms an open cover of $K$. Now take a finite subcover of $U$ for $K$. \hfill \Box

Haagerup–Kraus [15] introduced the approximation property (AP) for discrete (more generally, for locally compact) groups. The AP is weaker than weak amenability, stronger than exactness, and it is preserved under many operations. Various many groups are known to have the AP, but not all exact groups have the AP [99] (e.g., $\text{SL}(3, \mathbb{Z})$). See [14] and Section 12 of [8] for details. Haagerup–Kraus [15] related the AP with the validity of the Fubini-type theorem for the reduced group operator algebras. In [57], we observed that the AP is also useful to determine the position of an element in the reduced crossed product (Proposition 3.4 in [57]). We need this result, which is why the AP requirement in the following theorem.

**Theorem 2.3** (The Main Theorem, $C^*$-case). Let $\Gamma$ be a discrete group satisfying the AP. Let $X, Y$ be locally compact spaces. Let $\alpha: \Gamma \curvearrowright X$, $\beta: \Gamma \curvearrowright Y$ be free actions. Let $\pi: Y \to X$ be a proper factor map. We regard $C_0(X)$ as a $\Gamma$-$C^*$-subalgebra of $C_0(Y)$ via $\pi$. Then the map

$$C_0(Z) \mapsto C_0(Z) \rtimes_r \Gamma$$

gives a lattice isomorphism between the lattice of intermediate extensions $Z$ of $\pi$ and that of intermediate $C^*$-algebras of $C_0(X) \rtimes_r \Gamma \subset C_0(Y) \rtimes_r \Gamma$. 

Proof. Let \( a \in C_c(Y) \rtimes_{\text{alg}} \Gamma := \text{span}\{fu_s : f \in C_c(Y), s \in \Gamma\} \). We first show that \( E(a) \in \mathcal{N}(a; C_0(X)) \). Define
\[
F := \{ s \in \Gamma \setminus \{e\} : E_s(a) \neq 0 \}.
\]
By the choice of \( a \), \( F \) is finite. By Lemma 2.2 one can choose a finite open cover \( U_1, \ldots, U_n \) of \( \pi(\text{supp}(E(a))) \subset X \) satisfying \( \alpha_s(U_i) \cap U_i = \emptyset \) for all \( i \) and \( s \in F \). Choose non-negative functions \( f_1, \ldots, f_n \) in \( C_c(X) \) satisfying \( \text{supp}(f_i) \subset U_i \) for each \( i \), \( \| \sum_{i=1}^n f_i^2 \| \leq 1 \), and
\[
\sum_{i=1}^n f_i^2(y) = 1 \quad \text{for all } y \in \text{supp}(E(a)).
\]
By the choice of \( U_i \), for any \( s \in F \) and any \( i \), we have
\[
f_i u_s f_i = f_i \alpha_s(f_i) u_s = 0.
\]
This yields
\[
\sum_{i=1}^n f_i a f_i = \sum_{i=1}^n f_i^2 E(a) = E(a)
\]
as desired.

By Lemma 2.1, we have \( \mathcal{N}(E; C_0(X)) = C_0(Y) \rtimes_{\text{t}} \Gamma \). Consequently \( E(a) \in C^*(C_0(X), a) \) for all \( a \in C_0(Y) \rtimes_{\text{t}} \Gamma \). In particular, for any intermediate \( C^\ast \)-algebra \( A \) of \( C_0(X) \rtimes_{\text{t}} \Gamma \subset C_0(Y) \rtimes_{\text{t}} \Gamma \), we have \( E(A) \subset A \). (We remark that we do not use the assumption on \( \Gamma \) for this conclusion.) Proposition 3.4 in [57] then yields \( A = E(A) \rtimes_{\text{t}} \Gamma \). \( \square \)

Remark 2.4. It would be worth remarking that the arguments in Theorem 2.3 does not use self-adjointness. Thus the map
\[
A \mapsto A \rtimes_{\text{t}} \Gamma := \text{span}\{au_s : a \in A, s \in \Gamma\}
\]
gives a lattice isomorphism between the lattice of intermediate norm closed \(\Gamma\)-algebras of \( C_0(X) \subset C_0(Y) \) and that of intermediate norm closed algebras of \( C_0(Y) \rtimes_{\text{t}} \Gamma \subset C_0(Y) \rtimes_{\text{t}} \Gamma \).

Remark 2.5. The assumption on \( \Gamma \) in Theorem 2.3 is not necessary when every intermediate extension \( Z \) of \( \pi \) admits a \(\Gamma\)-conditional expectation \( C_0(Y) \rightarrow C_0(Z) \). Indeed, by Exercise 4.1.4 of [8], any \(\Gamma\)-conditional expectation \( E : C_0(Y) \rightarrow C_0(Z) \) extends to a conditional expectation \( \tilde{E} : C_0(Y) \rtimes_{\text{t}} \Gamma \rightarrow C_0(Z) \rtimes_{\text{t}} \Gamma \) satisfying \( \tilde{E}(fu_s) = E(f)u_s \) for all \( f \in C_0(Y) \) and \( s \in \Gamma \). This implies
\[
C_0(Z) \rtimes_{\text{t}} \Gamma = \{ a \in C_0(Y) \rtimes_{\text{t}} \Gamma : E_s(a) \in C_0(Z) \text{ for all } s \in \Gamma\}.
\]

It is interesting to compare Theorem 2.3 with the ideal structure results of the reduced crossed product \( C^\ast \)-algebras [32], [3] (see also [29], [7]). Unlike these results, our argument requires the freeness rather than just the topological freeness for actions. (Recall that an action \( \alpha : \Gamma \curvearrowleft X \) is said to be topologically free if for any \( \Gamma \)-invariant closed subset \( Y \) of \( X \) and for any \( s \in \Gamma \setminus \{e\} \), the set of fixed points of \( \alpha_s \) in \( Y \) has empty interior in \( Y \).) In fact, as the following proposition shows, the freeness assumption is essential in Theorem 2.3.

Proposition 2.6. Let \( \Gamma \) be a discrete group. Let \( \alpha : \Gamma \curvearrowleft X \) be a non-free action on a locally compact space. Then there is a free action \( \beta : \Gamma \curvearrowleft Y \) and a proper extension \( \pi : Y \rightarrow X \) satisfying the following property. There is an intermediate \( C^\ast \)-algebra \( A \) of \( C_0(X) \rtimes_{\text{t}} \Gamma \subset C_0(Y) \rtimes_{\text{t}} \Gamma \) not of the form \( C_0(Z) \rtimes_{\text{t}} \Gamma \) for any intermediate extension \( Z \) of \( \pi \).
Proof. Take \( s \in \Gamma \setminus \{e\} \) and \( x \in X \) satisfying \( sx = x \). Let \( \Lambda \) be the subgroup of \( \Gamma \) generated by \( s \). Let \( p : \Gamma/\Lambda \to X \) be the map given by \( p(s\Lambda) := sx \). Define
\[
\iota : C_0(X) \to C_0(X) \oplus C_0(\Gamma/\Lambda)
\]
to be \( \iota(f) := (f, f \circ p) \). Let \( C_0(Y_0) \) denote the C\( ^* \)-subalgebra of \( C_0(X) \oplus C_0(\Gamma/\Lambda) \) generated by \( \iota(C_0(X)) \) and \( 0 \oplus c_0(\Gamma/\Lambda) \). We identify \( C_0(X) \), \( c_0(\Gamma/\Lambda) \) with these \( \Gamma \)-C\( ^* \)-subalgebras of \( C_0(Y_0) \) (not with \( C_0(X) \oplus 0 \) respectively). As \( C_0(X)C_0(Y_0) \) is norm dense in \( C_0(Y_0) \), the inclusion \( C_0(X) \subset C_0(Y_0) \) induces a proper extension \( q : Y_0 \to X \). Let \( \Gamma \curvearrowright \beta \Gamma \) be the left translation action on the Stone–Cech compactification. Let \( \beta : \Gamma \curvearrowright Y := \beta \Gamma \times Y_0 \) denote the diagonal action. Note that \( \beta \) is free (see the proof of Lemma 2.3 in [58] for instance). Denote by \( r : Y \to Y_0 \) the projection onto the second coordinate. Put \( \pi := q \circ r : Y \to X \).

We show that \( \pi \) is the desired extension. To see this, observe that \( c_0(\Gamma/\Lambda) \rtimes \Gamma \) is not simple. (Indeed \( \delta \Delta_c \in c_0(\Gamma/\Lambda) \) denotes the Dirac function at \( \Lambda \in \Gamma/\Lambda \).) Choose a proper ideal \( I \) of \( c_0(\Gamma/\Lambda) \). Note that \( I \cap c_0(\Gamma/\Lambda) = 0 \), \( E(I) = c_0(\Gamma/\Lambda) \) as \( c_0(\Gamma/\Lambda) \) has no proper \( \Gamma \)-ideal. Now define
\[
A := C_0(X) \rtimes \Gamma + I \subset C_0(Y) \rtimes \Gamma.
\]
As \( C_0(X) \rtimes \Gamma \) is contained in the idealizer of \( I \), \( A \) is a C\( ^* \)-algebra. Also,
\[
A \cap C_0(Y) = C_0(X) \subset C_0(X) + c_0(\Gamma/\Lambda) = E(A).
\]
Therefore \( A \) is not of the form \( C_0(Z) \rtimes \Gamma \) for any intermediate extension \( Z \) of \( \pi \). \( \qed \)

We also remark that Theorem 2.3 fails for non-exact groups ([14], [43]).

**Proposition 2.7.** Let \( \Gamma \) be a countable non-exact group. Then there are compact spaces \( X, Y, Z \), free actions \( \alpha : \Gamma \curvearrowright X, \beta : \Gamma \curvearrowright Y, \) and a factor map \( \pi : Y \to X \) satisfying the following property. There is an intermediate C\( ^* \)-algebra of the inclusion \( C(X) \rtimes \Gamma \subset C(Y) \rtimes \Gamma \) not of the form \( C(Z) \rtimes \Gamma \) for any intermediate extension \( Z \) of \( \pi \).

**Proof.** Take a unital separable \( \Gamma \)-C\( ^* \)-subalgebra \( C(X) \) of \( \ell^\infty(\Gamma) = C(\beta \Gamma) \) whose associated action \( \Gamma \curvearrowright X \) is free (see Lemma 2.3 of [58]). By replacing \( C(X) \) by \( C(X) + c_0(\Gamma) \) if necessary, we may assume that \( c_0(\Gamma) \subset C(X) \). Denote by \( \pi : \beta \Gamma \to X \) the factor map associated to the inclusion \( C(X) \subset C(\beta \Gamma) \). Define
\[
G^*(\Gamma) := \{ a \in C(\beta \Gamma) \rtimes \Gamma : E_s(a) \in c_0(\Gamma) \text{ for all } s \in \Gamma \}.
\]
Note that this ideal corresponds to the ideal of all Ghost operators ([54], Definition 1.2) in the uniform Roe algebra \( C^*_u(\Gamma) \) under the canonical isomorphism \( C^*_u(\Gamma) \cong C(\beta \Gamma) \rtimes \Gamma \) (see [8], Remark 5.5.4). Since \( \Gamma \) is non-exact, by Theorem 5.5.7 of [8], it does not have property A (with respect to a proper left invariant metric). Therefore, by Remark 4.3 of [54], \( G^*(\Gamma) \) is non-separable. Since \( C(X) \rtimes \Gamma \) is separable, one can choose an element
\[
a \in G^*(\Gamma) \setminus C(X) \rtimes \Gamma.
\]
We define the intermediate C\( ^* \)-algebra \( A \) of \( C(X) \rtimes \Gamma \) to be \( A := C^*(a, C(X), \Gamma) \).

Then, by the choice of \( a \), we have \( E(A) = C(X) \), \( A \neq C(X) \rtimes \Gamma \). Therefore \( A \) is not of the form \( C(Z) \rtimes \Gamma \) for any intermediate extension \( Z \) of \( \pi \). \( \qed \)
3. Intermediate von Neumann Algebras and Measurable Dynamical Systems

In this section, we prove the Main Theorem for von Neumann algebras. The proof follows basically the same line as that of Theorem 2.3. However, since the \( \sigma \)-strong topology is more delicate than the norm topology, we need slightly more arguments.

As usual, we only consider von Neumann algebras with separable predual (though most results have an appropriate extension to the general case). All inclusions of von Neumann algebras are assumed to be unital. The basic facts and notations on von Neumann algebras can be found in [61]. For a family \( S_1, \ldots, S_n \) of subsets or elements in a von Neumann algebra \( M \), denote by \( W^*(S_1, \ldots, S_n) \) the von Neumann subalgebra of \( M \) generated by \( S_1, \ldots, S_n \). Let \( \Gamma \) be a discrete group. Similar to the \( C^* \)-algebra case, for a \( \Gamma \)-von Neumann algebra \( M \), denote by \( E, E_s ; s \in \Gamma \) the canonical conditional expectation and the \( s \)-th coefficient map on \( M \) respectively.

We refer the reader to [13] and [68] for basic facts and definitions of ergodic theory. In measurable spaces, without stating,

- considered subsets are assumed to be measurable,
- we ignore null sets (for instance, the symbol ‘\( A \subset B \)’ means that \( A \setminus B \) is a null set),
- all actions on a standard Borel space are assumed to be non-singular,
- all factor maps \( \pi : (Y, \nu) \to (X, \mu) \) are assumed to preserve the measure class (that is, \( \pi_*(\nu) = \pi_*(\mu) \) share null sets).

In the measurable setting, we define the notion of intermediate extensions and their equivalence analogous to the topological setting (see the beginning of Section 2). In particular, for an extension \( \pi : (Y, \nu) \to (X, \mu) \), there is a bijective correspondence between the set of intermediate \( \Gamma \)-von Neumann algebras of \( L^\infty(X) \subset L^\infty(Y) \) and the set of equivalence classes of intermediate extensions of \( \pi \). With this correspondence, we introduce the lattice structure on the set of equivalence classes of intermediate extensions of \( \pi \).

We first recall the following standard lemma.

Lemma 3.1. Let \( \alpha : \Gamma \curvearrowright (X, \mu) \) be an essentially free action of a countable group \( \Gamma \) on a standard probability space \( (X, \mu) \). Then, for any finite subset \( F \) of \( \Gamma \setminus \{ e \} \), one can choose a partition \( X = X_1 \sqcup \cdots \sqcup X_{3^{|F|}} \) of \( X \) satisfying \( X_i \cap \alpha_s(X_1) = \emptyset \) for all \( s \in F \) and \( i = 1, \ldots, 3^{|F|} \).

Proof. Obviously it suffices to show the claim for singletons \( F \). Let \( s \in \Gamma \setminus \{ e \} \) be given. Set \( D_s := \{ A \subset X : A \cap \alpha_s(A) = \emptyset \} \).

Observe that by essential freeness, any non-null set \( A \subset X \) contains a non-null set in \( D_s \). We show that \( D_s \) contains a maximal element with respect to the inclusion order. To see this, take any chain \( \mathcal{C} \) in \( D_s \). Then there is an increasing sequence \( (A_n)_n \) in \( \mathcal{C} \) with \( \lim_{n \to \infty} \mu(A_n) = \sup_{A \in \mathcal{C}} \mu(A) \). Now \( A = \bigcup_{n=1}^\infty A_n \) gives an upper bound of \( \mathcal{C} \) in \( D_s \). By Zorn’s lemma, one can find a maximal element \( A \) in \( D_s \). We claim that \( \alpha_s(A) \cup A \cup \alpha_s^{-1}(A) = X \). Indeed, if it fails, one can choose a non-null subset \( B \) of \( X \setminus \alpha_s(A) \cup A \cup \alpha_s^{-1}(A) \) contained in \( D_s \). Then \( B \cup A \) gives an element of \( D_s \) larger than \( A \), a contradiction. Thus \( \alpha_s(A) \cup A \cup \alpha_s^{-1}(A) = X \). Now \( X_1 := A, X_2 := \alpha_s(A) \setminus X_1, X_3 := \alpha_s^{-1}(A) \setminus (X_1 \cup X_2) \) gives the desired partition. \( \square \)

For a faithful normal state \( \varphi \) on a von Neumann algebra \( M \), define the norm \( \| \cdot \|_\varphi \) on \( M \) by

\[
\| a \|_\varphi := \varphi(a^*a)^{\frac{1}{2}} \text{ for } a \in M.
\]
Obviously \( \| \cdot \|_\varphi \) is continuous with respect to the \( \sigma \)-strong topology. Moreover \( \| \cdot \|_\varphi \) induces the \( \sigma \)-strong topology on any (operator norm) bounded subset of \( M \). This norm is convenient (and will be used) to estimate the \( \sigma \)-strong convergence of bounded nets in \( M \). The set
\[
M_\varphi := \{ a \in M : \varphi(ab) = \varphi(ba) \text{ for all } b \in M \}
\]
is called the centralizer of \( \varphi \). Obviously \( M_\varphi \) is a finite von Neumann subalgebra of \( M \).

Let \((a_i)_{i \in I}\) be a bounded family in \( M \) with \( a_i^*a_j = a_ja_i^* = 0 \) for all \( i \neq j \). Then the series of \((a_i)_{i \in I}\) \( \sigma \)-strongly converges in \( M \). We denote it by \( \sum_{i \in I} a_i \).

**Lemma 3.2.** Let \( M \) be a von Neumann algebra with a faithful normal state \( \varphi \). Let \( \Phi : M \to M \) be a normal unital completely positive map. Let \( P \) be a set of projections in \( M_\varphi \). Set
\[
P^{N,\perp} := \{ (p_i)_{i=1}^\infty \in P^N : p_i \perp p_j \text{ for } i \neq j \}.
\]
For \( a \in M \), define
\[
\mathcal{G}(a; P) := \sigma\text{-strong-closure}\left\{ \sum_{i=1}^\infty p_iap_i : (p_i)_{i=1}^\infty \in P^{N,\perp} \right\}.
\]
Then the set
\[
\mathcal{G}(\Phi; P) := \{ a \in M : \Phi(a) \in \mathcal{G}(a; P) \}
\]
is \( \sigma \)-strongly closed.

**Proof.** We first observe that for any \((p_i)_{i=1}^\infty \in P^{N,\perp} \), the map \( \Psi : M \to M \) defined by \( \Psi(a) := \sum_{i=1}^\infty p_iap_i \) satisfies \( \| \Psi(a) \|_\varphi \leq \| a \|_\varphi \). Indeed, as \( p_i \in M_\varphi \) for all \( i \),
\[
\varphi(\Psi(a)^*\Psi(a)) = \varphi(\sum_{i,j} p_i a^* p_j p_i a p_j) = \varphi(\sum_i p_i a^* p_i a p_i) \leq \varphi(\sum_i a^* a) = \varphi(a^*a).
\]

Let \( a \) be taken from the \( \sigma \)-strong closure of \( \mathcal{G}(\Phi; P) \). Then for any \( \varepsilon > 0 \), one can choose \( b \in M \) and \((p_i)_{i=1}^\infty \in P^{N,\perp} \) satisfying
\[
\| b - a \|_\varphi < \varepsilon,
\]
\[
\| \Phi(b) - \Phi(a) \|_\varphi < \varepsilon,
\]
\[
\| \Phi(b) - \sum_{i=1}^\infty p_i bp_i \|_\varphi < \varepsilon.
\]
By the inequality proved in the previous paragraph, we have
\[
\| \sum_{i=1}^\infty p_iap_i - \sum_{i=1}^\infty p_ibp_i \|_\varphi \leq \| a - b \|_\varphi < \varepsilon.
\]
Combining these inequalities, we obtain
\[
\| \Phi(a) - \sum_{i=1}^\infty p_iap_i \|_\varphi < 3\varepsilon.
\]
As the set \( \{ \sum_{i=1}^\infty p_iap_i : (p_i)_{i=1}^\infty \in P^{N,\perp} \} \) is bounded, this proves \( a \in \mathcal{G}(\Phi; P) \) as desired. \( \square \)

We recall the following description of the commutant of the crossed product von Neumann algebra. For a discrete group \( \Gamma \), denote by \( \rho \) the right regular representation of \( \Gamma \) on \( \ell^2(\Gamma) \).
Proposition 3.3 ([61], Chapter V, Proposition 7.14). Let $M$ be a $\Gamma$-von Neumann algebra represented on a Hilbert space $H$. Let $v$ be a unitary representation of $\Gamma$ on $H$ implementing the $\Gamma$-action on $M$. Let $M \rtimes \Gamma \to \mathcal{B}(H \otimes \ell^2(\Gamma))$ be the left regular representation. Then the commutant of $M \rtimes \Gamma$ in $\mathcal{B}(H \otimes \ell^2(\Gamma))$ is equal to $M' \rtimes \Gamma$. Here $\Gamma$ acts on $M'$ through $v$, and $M' \rtimes \Gamma$ is represented on $H \otimes \ell^2(\Gamma)$ by the covariant representation $(\iota, \tilde{\rho})$ where $\iota: M' \to \mathcal{B}(H) \otimes \operatorname{Cid}_{\ell^2(\Gamma)}$ is the amplified representation and $\tilde{\rho} := v \otimes \rho$.

The following result immediately follows from Proposition 3.3 and the bicommutant theorem.

Corollary 3.4. Let $N \subset M$ be an inclusion of $\Gamma$-von Neumann algebras. Then

$$N \bar{\rtimes} \Gamma = \{ a \in M \bar{\rtimes} \Gamma : E_s(a) \in N \text{ for all } s \in \Gamma \}.$$

Remark 3.5. The $C^*$-algebra analogue of Corollary 3.4 holds true when the acting group has the AP (Proposition 3.4 of [57]). However it fails for non-exact groups, even when the coefficients are commutative as shown in Section 5 of [54]. In the intermediate cases ([39]), nothing is known.

Theorem 3.6 (The Main Theorem, $W^*$-case). Let $\Gamma$ be a countable group. Let $\alpha: \Gamma \curvearrowright (X, \mu)$, $\beta: \Gamma \curvearrowright (Y, \nu)$ be essentially free actions on a standard probability space. Let $\pi: Y \to X$ be a factor map. We regard $L^\infty(X)$ as a $\Gamma$-von Neumann subalgebra of $L^\infty(Y)$ via $\pi$. Then the map

$$L^\infty(Z) \mapsto L^\infty(Z) \bar{\rtimes} \Gamma$$

gives a lattice isomorphism between the lattice of intermediate extensions $Z$ of $\pi$ and that of intermediate von Neumann algebras of $L^\infty(X) \bar{\rtimes} \Gamma \subset L^\infty(Y) \bar{\rtimes} \Gamma$.

Proof. Let $a \in L^\infty(Y) \bar{\rtimes}_{\text{alg}} \Gamma$ be given. Set $F := \{ s \in \Gamma \setminus \{ e \} : E_s(a) \neq 0 \}$. By the choice of $a$, $F$ is finite. Choose a partition $(X_i)_{i=1}^n$ of $X$ as in Lemma 3.1 for $F$. Set $p_i := \chi_{X_i}$ for each $i$. Define $\varphi := \nu \circ E$. A direct computation shows that $L^\infty(Y) \subset (L^\infty(Y) \bar{\rtimes} \Gamma)_\varphi$. Let $P(X)$ denote the set of projections in $L^\infty(X)$. Then $(p_i)_{i=1}^n \in P(X)^{\bar{\rtimes} \Gamma}$ and $\sum_{i=1}^n p_i = 1$. By the choice of $(X_i)_{i=1}^n$, for any $s \in F$, we have

$$\sum_{i=1}^n p_i u_s p_i = \sum_{i=1}^n p_i \alpha_s(p_i) u_s = 0.$$

As a result, we obtain

$$\sum_{i=1}^n p_i a p_i = E(a).$$

Hence

$$L^\infty(Y) \bar{\rtimes}_{\text{alg}} \Gamma \subset \mathcal{G}(E; P(X)).$$

Lemma 3.2 then implies

$$\mathcal{G}(E; P(X)) = L^\infty(Y) \bar{\rtimes} \Gamma.$$

In particular, for any intermediate von Neumann algebra $Q$ of $L^\infty(X) \bar{\rtimes} \Gamma \subset L^\infty(Y) \bar{\rtimes} \Gamma$, we have $E(Q) \subset Q$. Corollary 3.4 then yields $Q = E(Q) \bar{\rtimes} \Gamma$. □

Theorem 3.6 and the following significant theorem of Margulis [40] lead to new natural and concrete examples of maximal amenable subalgebras. It is remarkable that the lattices of their intermediate von Neumann algebras are completely describable. Some of them are extremal in the sense that they admit no proper intermediate $W^*$-algebras.
For a locally compact second countable group $G$ and its closed subgroup $H$, the homogeneous space $G/H$ admits a $G$-quasi-invariant probability measure, which is unique up to measure class. We denote (one of) this measure by $m_{G/H}$, and regard $G/H$ as the standard probability space $(G/H, m_{G/H})$. We refer the reader to the book [38] for basic facts on Lie groups. In particular, for parabolic subgroups, see VII.7 of [38].

**Theorem 3.7** (Margulis’ factor theorem [41], cf. Theorem 8.1.3 in [68]). Let $G$ be a connected semisimple Lie group of real rank at least 2 with no compact factors and trivial center. Let $\Gamma$ be an irreducible lattice of $G$. Let $P$ be a minimal parabolic subgroup of $G$. Then the map $Q \mapsto L^\infty(G/Q)$ gives a lattice anti-isomorphism from the lattice of intermediate closed subgroups $Q$ of $P \subset G$ onto that of $\Gamma$-invariant von Neumann subalgebras of $L^\infty(G/P)$.

**Corollary 3.8.** Let $G, \Gamma, P$ be as in Theorem 3.7. Let $Q$ be a proper intermediate closed group of $P \subset G$ with $\bigcap_{g \in G} gQ g^{-1} \cap \Gamma = \{1\}$ (this is automatic when $G$ is simple). Then the map $R \mapsto L^\infty(G/R)^{\Gamma} \rtimes \Gamma$ (where the commutant is taken in $\mathcal{B}(L^2(G/P))$) gives a lattice isomorphism between the lattice of intermediate closed groups $R$ of $P \subset Q$ and that of intermediate von Neumann algebras of $L^\infty(G/P)^{\Gamma} \rtimes \Gamma$. In particular $L^\infty(G/P)^{\Gamma} \rtimes \Gamma$ is a maximal amenable subalgebra of the non-amenable factor $L^\infty(G/Q)^{\Gamma} \rtimes \Gamma$.

**Proof.** The assumption on $Q$ yields the essential freeness of $\Gamma \acts G/Q$; see Remark 13 of [47] for instance. Theorems 3.6, 3.7, and the bicommutant theorem confirm that the stated map indeed defines a lattice isomorphism. We also recall that minimal parabolic subgroups of $G$ are maximal amenable in $G$. In particular all intermediate closed groups of the inclusion $P \subset Q$ but $P$ are non-amenable. By Theorem 2.1 of [67], Theorem 3.6, and the maximal amenability of $P \subset G$, we conclude the maximal amenability of $L^\infty(G/P)^{\Gamma} \rtimes \Gamma \subset L^\infty(G/Q)^{\Gamma} \rtimes \Gamma$. Factoriality of these crossed products follows from Moore’s ergodicity theorem ([68], Corollary 2.2.6). □

**Remark 3.9** (Explicit presentation of $L^\infty(G/Q)^{\Gamma}$). Let $P \subset Q \subset G$ be closed inclusions of locally compact second countable groups. Here we present $L^\infty(G/Q)^{\Gamma} \subset \mathcal{B}(L^2(G/P))$ as the algebra of “block-diagonal operators along the decomposition $G/P = \bigsqcup_{x \in G/Q} x/P$. To see this, fix a $Q$-quasi-invariant probability measure $\nu_x$ on $Q/P$. Take a measurable cross section $s: G/Q \to G$ ([68], Corollary A.8). Define $\nu_x := s(x)_*(\nu_x)$ for $x \in G/Q$. Note that each $\nu_x$ is concentrated on $x/P$. The integral

$$\int_{G/Q} \nu_x \, dm_{G/Q}$$

then defines a $G$-quasi-invariant measure on $G/P$. We employ this integral for $m_{G/P}$. Along this integral decomposition, we obtain the direct integral decomposition

$$L^2(G/P) = \int_{G/Q} L^2(x/P, \nu_x) \, dm_{G/Q}.$$
(See Chapter IV.8 of [61] for direct integrals.) This leads to the direct integral decomposition

$$L^{\infty}(G/Q)' = \int_{G/Q}^\oplus \{B(L^2(x/P, \nu_x)), L^2(x/P, \nu_x)\} \, dm_{G/Q}.$$ 

The left translations define a left $G$-action on the right hand side, and it describes the left $G$-action on $L^{\infty}(G/Q)'$ via the above identification.

**Example 3.10.** Here we demonstrate a few explicit consequences of Corollary 3.8.

1. Put $G = \text{SL}(3, \mathbb{R})$, $\Gamma = \text{SL}(3, \mathbb{Z})$,
   
   $P := \{[a_{i,j}]_{i,j} \in G : a_{i,j} = 0 \text{ for all } i < j\}$,
   
   $Q = \{[a_{i,j}]_{i,j} \in G : a_{1,j} = 0 \text{ for } j = 2, 3\}$. We then have natural identifications
   
   $G/P = \text{Fl}_3(\mathbb{R})$ (the full flag manifold), $G/Q = \mathbb{P}^2 \mathbb{R}$,
   
   given by
   
   $AP \mapsto (A(\mathbb{R} \oplus 0_2), A(\mathbb{R}^2 \oplus 0_1), \mathbb{R}^3)$, $AQ \mapsto A(\mathbb{R} \oplus 0_2)$ for $A \in G$. The factor map $G/P \to G/Q$ corresponds to the map
   
   $[V_1, V_2, V_3] \mapsto [V_1]$.
   
   A direct computation shows that the inclusion $P \subset Q$ has no proper intermediate groups. Hence, by Corollary 3.8 the amenable subfactor
   
   $L^{\infty}(\text{Fl}_3(\mathbb{R})) \rtimes \Gamma \subset L^{\infty}(\mathbb{P}^2 \mathbb{R})' \rtimes \Gamma$
   
   has no proper intermediate von Neumann algebras. It is worth mentioning that Ozawa [47] recently showed that $L^{\infty}(\mathbb{P}^2 \mathbb{R}) \rtimes \Gamma$, hence $L^{\infty}(\mathbb{P}^2 \mathbb{R})' \rtimes \Gamma$, is a full factor.

2. For $n \geq 3$, the following quadruplet $(G, \Gamma, P, R)$ satisfies the assumptions of Corollary 3.8.
   
   $G = \text{PSL}(n, \mathbb{R})$, $\Gamma = \text{PSL}(n, \mathbb{Z})$,
   
   $P = \{[a_{i,j}]_{i,j} \in G : a_{i,j} = 0 \text{ for all } i < j\}$,
   
   $R = \{[a_{i,j}]_{i,j} \in G : a_{n,j} = 0 \text{ for all } j < n\}$. We completely describe the lattice of intermediate von Neumann algebras of
   
   $N := L^{\infty}(G/P) \rtimes \Gamma \subset L^{\infty}(G/R)' \rtimes \Gamma := M$.
   
   For $S \subset \{1, \ldots, n-1\}$, define
   
   $G_S := \{A \in G : A(\mathbb{R}^d \oplus 0_{n-d}) = \mathbb{R}^d \oplus 0_{n-d} \text{ for all } d \in \{1, \ldots, n-1\} \setminus S\}$.
   
   Denote by $\mathcal{B}(n-2)$ the lattice of subsets of $\{1, \ldots, n-2\}$. Then the map $S \mapsto G_S$ gives a lattice isomorphism from $\mathcal{B}(n-2)$ onto the lattice of intermediate subgroups of $P \subset R$ (see [38], VII.7 Examples (1)). Consequently the map
   
   $S \mapsto L^{\infty}(G/G_S)' \rtimes \Gamma$
   
   gives a lattice isomorphism from $\mathcal{B}(n-2)$ onto $\mathcal{L}$. By Theorem 11 of [17], all intermediate von Neumann subalgebras of $N \subset M$ but $N$ are full. Note that $G/G_S$ is identified with the partial flag manifold of signature $t_1, \ldots, t_r$ where $t_1 < \cdots < t_r$ denote the elements of $\{1, \ldots, n\} \setminus S$. 

(3) Let $G, P, R, G_S$ be as in (2). Set $\mathcal{O} := \mathbb{Z}[\sqrt{2}]$. Let $\sigma$ be the ring automorphism on $\mathcal{O}$ given by $\sigma(\sqrt{2}) = -\sqrt{2}$. Define

$$\Lambda := \{(A, \sigma(A)) : A \in \text{PSL}(n, \mathcal{O})\}.$$ 

Then the quadruplet $(G^2, \Lambda, P^2, G \times R)$ satisfies the assumptions of Corollary 2.8 (see Example 2.2.5 of [68]). Denote by $M$ the lattice of intermediate von Neumann algebras of

$$L^\infty(G^2/P^2) \rtimes \Lambda \subset L^\infty(G^2/G \times R)' \rtimes \Lambda$$

Put $G_{S,T} := G_S \times G_T$ for $(S,T) \in \mathfrak{P}(n-1) \times \mathfrak{P}(n-2)$. Then the map

$$(S,T) \mapsto L^\infty(G^2/G_{S,T})' \rtimes \Lambda$$

gives a lattice isomorphism from $\mathfrak{P}(n-1) \times \mathfrak{P}(n-2)$ onto $M$.

We end this section by giving a remark on non-free actions. Similar to the C$^*$-algebra case (Proposition 2.6), Theorem 3.6 fails for non-free actions.

**Proposition 3.11.** Let $\Gamma$ be a countable group. Let $\alpha : \Gamma \acts (X, \mu)$ be a non-essentially free action on a standard probability space. Then there is an essentially free action $\beta : \Gamma \acts (Y, \nu)$ on a standard probability space and a factor map $\pi : Y \to X$ with the following property. There is an intermediate von Neumann algebra $M$ of $L^\infty(X) \rtimes \Gamma \subset L^\infty(Y) \rtimes \Gamma$ not of the form $L^\infty(Z) \rtimes \Gamma$ for any intermediate extension $Z$ of $\pi$.

**Proof.** The proof is basically the same as that of Proposition 2.6. However, since single points no longer make sense in the measurable setting, we need a slight modification.

Take $s \in \Gamma \setminus \{e\}$ which trivially acts on a non-null set $U \subset X$. Let $\Lambda$ be the subgroup of $\Gamma$ generated by $s$. Define $p : \Gamma/\Lambda \times U \to \Gamma U$ by $p(s\Lambda, u) := su$. We equip $\Gamma/\Lambda \times U$ with the $\Gamma$-action $s(t\Lambda, u) := (st\Lambda, u); s, t \in \Gamma, u \in U$. Then $p$ is a factor map. Define a unital normal embedding

$$\iota : L^\infty(X) \to L^\infty(X) \oplus L^\infty(\Gamma/\Lambda \times U)$$

to be $\iota(f) := (f, f \circ p)$. Let $L^\infty(Y_0)$ denote the von Neumann algebra generated by $\iota(L^\infty(X))$ and $0 \oplus L^\infty(\Gamma/\Lambda \times U)$. We regard $L^\infty(X)$ as a (unital) von Neumann subalgebra of $L^\infty(Y_0)$ via $\iota$. We also identify $L^\infty(\Gamma/\Lambda \times U)$ with $0 \oplus L^\infty(\Gamma/\Lambda \times U)$. Now consider the diagonal action $\beta : \Gamma \acts Y := \Gamma \times Y_0$. This is a free action on the standard probability space $Y$. Let $q : Y \to Y_0$ be the projection onto the second coordinate. Define $\pi := p \circ q : Y \to X$.

We show that $\pi$ is the desired extension. To show this, observe that, since $L^\infty(\Gamma/\Lambda) \rtimes \Gamma \cong B(\ell^2(\Gamma/\Lambda)) \otimes L(\Lambda)$ is not a factor, one can choose a weakly closed ideal $I$ of $L^\infty(\Gamma/\Lambda \times U) \rtimes \Gamma$ satisfying $I \cap L^\infty(\Gamma/\Lambda \times U) = 0$, $E(I) = L^\infty(\Gamma/\Lambda \times U)$. Then, similar to the proof of Proposition 2.6

$$M := L^\infty(X) \rtimes \Gamma + I$$

gives the desired intermediate von Neumann algebra. $\square$

4. Non-commutative dynamical systems

In this section we extend the Main Theorem to non-commutative dynamical systems. We then give some applications, including the lattice realization theorem Corollary 13.
To formulate a non-commutative variant of the Main Theorem, we first need to introduce a few definitions. For a C*-algebra $A$, define
\[
\ell^2(A) := \left\{ a = (a_i)_{i=1}^\infty \in A^\mathbb{N} : \text{the series} \sum_{i=1}^\infty a_i^*a_i \text{ converges in norm} \right\}.
\]
We equip $\ell^2(A)$ with the norm $\|a\|_2 := \| \sum_{i=1}^\infty a_i^*a_i \|^{1/2}$. When $A$ is a $\Gamma$-C*-algebra, we equip $\ell^2(A)$ with the pointwise $\Gamma$-action. For $a = (a_i)_{i=1}^\infty, b = (b_i)_{i=1}^\infty \in \ell^2(A)$, $c \in A$, we define
\[
a^*b := \sum_{i=1}^\infty a_i^*b_i \in A,
\]
\[
ac := (a_ic)_{i=1}^\infty, ca := (ca_i)_{i=1}^\infty \in \ell^2(A).
\]
It is not hard to check
\[
\|a^*b\| \leq \|a\|_2\|b\|_2, \quad \max\{\|ac\|_2, \|ca\|_2\} \leq \|a\|_2\|c\|.
\]

**Definition 4.1.** Let $A \subset B$ be an inclusion of $\Gamma$-C*-algebras. We say that the inclusion is **centrally $\Gamma$-free** if for any $b, c \in B$, $\epsilon > 0$, $s \in \Gamma \setminus \{e\}$, there is an element $a \in \ell^2(A)_1(\subset \ell^2(B))$ satisfying $\|a^*ba - b\| < \epsilon$, $\|a^*cs(a)\| < \epsilon$.

For a $\Gamma$-C*-algebra $A$, we say that $A$ is **centrally $\Gamma$-free**, or say that the underlying $\Gamma$-action is **centrally free**, if the identity inclusion $A \subset A$ is centrally $\Gamma$-free. Note that for commutative C*-algebras, central freeness is equivalent to the freeness of the action on the Gelfand spectrum.

**Remark 4.2.** Unlike the measurable case (Lemma 3.1), we have no universal bound of the cardinals of open covers in Lemma 2.2. (The situation remains the same in the non-commutative case. See Remark 4.4) This is the reason why we adopt $\ell^2(A)$ in the formulation of central freeness. Its importance can be read from Examples 4.10 and 4.11.

For von Neumann algebras, it is convenient to use the notion of ultraproduct. Throughout this section, we fix a free ultrafilter $\omega$ on $\mathbb{N}$. Here we recall a few definitions. For complete treatments, we refer the reader to [2]. Let $M$ be a von Neumann algebra. Let $\ell^\infty(M)$ denote the C*-algebra of bounded sequences in $M$. Set
\[
\mathcal{I}_\omega(M) := \left\{ (a_i)_{i=1}^\infty \in \ell^\infty(M) : \ast\text{-strong-} \lim_{i \to \omega} a_i = 0 \right\},
\]
\[
\mathcal{M}_\omega(M) := \left\{ a \in \ell^\infty(M) : a\mathcal{I}_\omega(M), \mathcal{I}_\omega(M)a \subset \mathcal{I}_\omega(M) \right\},
\]
\[
\mathcal{M}_\omega(M) := \left\{ (a_i)_{i=1}^\infty \in \ell^\infty(M) : \lim_{i \to \omega} \| \varphi(a_i \cdot) - \varphi(\cdot a_i) \| = 0 \text{ for all } \varphi \in M_* \right\}.
\]
The quotient C*-algebras $M^\omega := \mathcal{M}_\omega(M)/\mathcal{I}_\omega(M)$, $M_\omega := \mathcal{M}_\omega(M)/\mathcal{I}_\omega(M)$, are called Ocneanu’s ultraproduct of $M$ and Connes’ asymptotic centralizer of $M$ respectively. It is known that $M^\omega$ is a von Neumann algebra (possibly with non-separable predual). Moreover $M_\omega$ sits in $M^\omega$ as a von Neumann subalgebra. For any faithful normal state $\varphi$ on $M$, the map $(a_i)_{i=1}^\infty \in \ell^\infty(M) \mapsto \lim_{i \to \omega} \varphi(a_i)$ induces a faithful normal state $\varphi^\omega$ on $M^\omega$ with $M_\omega \subset (M^\omega)_{\varphi^\omega}$. We identify $M$ with a von Neumann subalgebra of $M^\omega$ via the map $a \mapsto (a)_{i=1}^\infty + \mathcal{I}_\omega(M)$. For a $\Gamma$-von Neumann algebra $M$, we equip $M^\omega$ with the $\Gamma$-action induced from the pointwise action.

Next consider an inclusion $N \subset M$ of von Neumann algebras. When it admits a faithful normal conditional expectation, we have $N^\omega \subset M^\omega$. In general, this is not true. We thus define
\[
N^\omega \cap M_\omega := \left[ \ell^\infty(N) \cap \mathcal{M}_\omega(M) \right]/\mathcal{I}_\omega(N).
\]
It is not hard to show that $N^\omega \cap M_\omega$ is a von Neumann subalgebra of $M_\omega$. Indeed, this follows from a slight modification of the proof of the fact that $M_\omega$ is a von Neumann algebra.

**Definition 4.3.** Let $N \subset M$ be an inclusion of $\Gamma$-von Neumann algebras. We say that the inclusion is centrally $\Gamma$-free if for any $s \in \Gamma \setminus \{e\}$, there is a sequence $(p_i)_{i=1}^\infty$ of projections in $N^\omega \cap M_\omega$ satisfying $\sum_{i=1}^\infty p_i = 1$ and $p_j s(p_i) = 0$ for all $j \in \mathbb{N}$.

**Remark 4.4.** Here we list a few remarks related to Definition 4.3.

1. By Theorem 1.6 and Lemma 2.6 in Chapter XVII of [62], Lemma 3.1, and standard reindexation arguments, the central $\Gamma$-freeness of $N \subset M$ is equivalent to the proper outerness of $\Gamma \preceq N^\omega \cap M_\omega$. In particular, the central freeness of a group action on a von Neumann algebra $M$ ([12], Section 5.2) is equivalent to the central $\Gamma$-freeness of the identity inclusion $M \subset M$.

2. By the same reason, one can replace $\infty$ in Definition 4.3 by 3.

3. When $M$ is finite, $N^\omega \cap M_\omega = N^\omega \cap M'$. Hence our notion of central freeness is compatible with central triviality for automorphisms of type $\Pi_1$ subfactors [30].

We now state non-commutative analogues of the Main Theorem.

**Theorem 4.5.** Let $\Gamma$ be a discrete group with the AP. Let $A \subset B$ be an inclusion of $\Gamma$-$C^*$-algebras. Assume that the inclusion is centrally $\Gamma$-free. Then the map

$$D \mapsto D \rtimes_\Gamma$$

gives a lattice isomorphism between the lattice of intermediate $\Gamma$-$C^*$-algebras $D$ of $A \subset B$ and that of intermediate $C^*$-algebras of $A \rtimes_\Gamma \Gamma \subset B \rtimes_\Gamma \Gamma$.

**Theorem 4.6.** Let $\Gamma$ be a countable discrete group. Let $N \subset M$ be an inclusion of $\Gamma$-von Neumann algebras. Assume that the inclusion is centrally $\Gamma$-free. Then the map

$$P \mapsto P \rtimes_\Gamma$$

gives a lattice isomorphism between the lattice of intermediate $\Gamma$-von Neumann algebras $P$ of $N \subset M$ and that of intermediate von Neumann algebras of $N \rtimes_\Gamma \Gamma \subset M \rtimes_\Gamma \Gamma$.

Theorems 4.5 and 4.6 are proved in a similar way to the Main Theorem.

**Proof of Theorem 4.5.** Similar to the proof of Theorem 2.3, it suffices to show $B \rtimes_{\text{alg}} \Gamma \subset M(\mathcal{E};A)$. Let $a \in B \rtimes_{\text{alg}} \Gamma$ be given. Set $F := \{s \in \Gamma \setminus \{e\} : E_s(a) \neq 0\}$. Put $k := |F|$. Enumerate $F$ as $s_1, \ldots, s_k$. We will show $a \in M(\mathcal{E};A)$ by induction on $k$. The case $k = 0$ is trivial. Suppose we have shown the claim for $k-1$. Let $\epsilon > 0$ be given. Put $a_k := E_{s_k}(a)$. Then by applying the induction hypothesis to $a - a_k u_{s_k}$, one can choose $b \in \ell^2(A)_1$ satisfying

$$\|b^* (a - a_k u_{s_k}) b - E(a)\| < \epsilon.$$ 

Since $A \subset B$ is centrally $\Gamma$-free, one can choose $c \in \ell^2(A)_1$ satisfying

$$\|c^* (a - a_k u_{s_k}) c - E(a)\| < \epsilon, \quad \|c^* (b^* a_k s_k(b)) s_k(c)\| < \epsilon.$$ 

Combining these three inequalities, we obtain

$$\|c^* (b^* a b) c - E(a)\| \leq \|c^* (b^* (a - a_k u_{s_k}) b - E(a)) c\| + \|c^* E(a) c - E(a)\| + \|c^* (b^* a_k s_k(b)) s_k(c)\| < 3\epsilon.$$
Since $\epsilon > 0$ is arbitrary, we conclude $a \in \mathcal{N}(E; A)$.

**Lemma 4.7.** Let $N \subset M$ be an inclusion of $\Gamma$-von Neumann algebras. Assume that $N \subset M$ is centrally $\Gamma$-free. Then for any finite subset $F \subset \Gamma \setminus \{e\}$, there is a sequence $(p_i)_{i=1}^\infty$ of projections in $N^\omega \cap M_\omega$ satisfying $\sum_{i=1}^n p_i = 1$ and $p_j s(p_j) = 0$ for all $s \in F$ and $j \in \mathbb{N}$.

**Proof.** This follows from standard reindexation arguments.

**Lemma 4.8.** Let $N \subset M$ be an inclusion of $\Gamma$-von Neumann algebras. Let $P_{N^\omega \cap M_\omega} \subset (M \rtimes \Gamma)^\omega$ denote the set of projections in $N^\omega \cap M_\omega$. Then for any $a \in M \rtimes \Gamma$, we have

$$\mathcal{S}(a; P_{N^\omega \cap M_\omega}) \subset (M \rtimes \Gamma)^\omega \subset W^*(N; a).$$

**Proof.** Fix a faithful normal state $\varphi$ on $M$. Put $\psi := \varphi \circ E$. Let $b \in \mathcal{S}(a; P_{N^\omega \cap M_\omega}) \cap (M \rtimes \Gamma)^\omega$ be given. Then, by assumption, for any $\epsilon > 0$, there are pairwise orthogonal projections $p_1, \ldots, p_k \in P_{N^\omega \cap M_\omega}$ satisfying $\|\sum_{i=1}^k p_i a p_i - b\|_{\psi} < \epsilon$. Then, by a standard functional calculus argument, one can choose pairwise orthogonal projections $q_1, \ldots, q_k$ in $N$ with $\|\sum_{i=1}^k q_i a q_i - b\|_{\psi} < \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude $b \in W^*(N; a)$.

**Proof of Theorem 4.6.** We continue the notations of Lemma 4.8. Note that $P_{N^\omega \cap M_\omega} \subset (M^\omega)^{\omega^\omega}$. Hence, by Lemmas 3.2 and 4.8, it suffices to show $E(a) \in \mathcal{S}(a; P_{N^\omega \cap M_\omega})$ for any $a \in M \rtimes_{\text{alg}} \Gamma$. (We remark that Lemma 3.2 is valid without separability of the predual.)

Let $a \in M \rtimes_{\text{alg}} \Gamma$ be given. Set $F := \{s \in \Gamma \setminus \{e\} : E_s(a) \neq 0\}$. Then by Lemma 4.7 there is a sequence $(p_i)_{i=1}^\infty$ of projections in $N^\omega \cap M_\omega$ satisfying $\sum_{i=1}^\infty p_i = 1$, $p_j s(p_j) = 0$ for all $j \in \mathbb{N}$ and $s \in F$. These relations yield $\sum_{i=1}^\infty p_i a p_i = E(a)$ as desired.

In the rest of this section, we give examples of centrally $\Gamma$-free inclusions. We first record the following permanence property of central freeness.

**Remark 4.9** (Stability under tensor products and quotients). Given a centrally $\Gamma$-free inclusion $A \subset B$ of $\Gamma$-$C^*$-algebras. Then, for any non-degenerate inclusion $C \subset D$ of $\Gamma$-$C^*$-algebras and for any $\Gamma$-ideal $I$ of the maximal tensor product $B \otimes_{\text{max}} D$, the inclusion

$$C^*(A : C, I) / I \subset (B \otimes_{\text{max}} D) / I$$

is again centrally $\Gamma$-free.

**Example 4.10** (Non-commutative Bernoulli shifts for $C^*$-algebras). Let $A \subset B$ be a unital inclusion of $C^*$-algebras and let $\Gamma$ be an infinite group. Assume that $A$ is simple (and non-commutative). We will show that the inclusion of the non-commutative Bernoulli shifts $\otimes_\Gamma A \subset \otimes_\Gamma B$ is centrally $\Gamma$-free. Let $\sigma, \varphi : \Gamma \curvearrowright \otimes_\Gamma A$ denote the left and right shift action. Let $s \in \Gamma \setminus \{e\}$ be given. We first observe that there is a nonzero positive element $a \in \otimes_\Gamma A$ with $a a_s(a) = 0$. To see this, for $t \in \Gamma$, let $\iota_t : A \rightarrow \otimes_\Gamma A$ denote the canonical embedding into the $t$-th tensor product factor. Choose two nonzero positive elements $a_1, a_2 \in A$ with $a_1 a_2 = 0$. Then $a := \iota_{e}(a_1) \iota_{e}(a_2)$ possesses the desired property. Now, since $\otimes_\Gamma A$ is unital and simple, one can choose a sequence $x_1, \ldots, x_n \in A$ satisfying $\sum_{i=1}^n x_i a^2 x_i = 1$. Set $x_i := 0$ for $i > n$ and define $c := (a x_i)_{i=1}^\infty \in \ell^2(A)$. Then it is clear that $c^* c = 1, c^* \sigma_s(c) = 0$. Now for any given $b_1, b_2$ in $\otimes_\Gamma B$ and $\epsilon > 0$, choose $t \in \Gamma$ satisfying, with $d := g_t(c), \|d b_i^* - b_i^* d\| < \epsilon$ for $i = 1, 2$. As $g_t$ commutes with $\sigma_s$, we have $d^* \sigma_s(d) = g_t(c^* \sigma_s(c)) = 0$. Hence

$$\|d b_1^* - b_1^* d\| \leq \|d b_1^* - b_1^* d\|_2 < \epsilon.$$
\[ \|d^*b_2\sigma_s(d)\| \leq \|db_2^* - b_2^*d\|_2 + \|b_2\|\|d^*\sigma_s(d)\| < \varepsilon. \]

This proves the central \( \Gamma \)-freeness of \( \bigotimes_i A \subset \bigotimes_i B \).

**Example 4.11** (Infinite tensor product actions). Let \( I \) be an infinite set. For each \( i \in I \), let \( B_i \) be a unital \( \Gamma \)-C*-algebra, \( A_i \) be a unital \( \Gamma \)-C*-subalgebra of \( B_i \), and assume that infinitely many \( A_i \) admit a unital centrally \( \Gamma \)-free C*-subalgebra \( C_i \). Put \( A := \bigotimes_{i \in I} A_i, B := \bigotimes_{i \in I} B_i \). We equip \( A, B \), with the diagonal \( \Gamma \)-action. Then the inclusion \( A \subset B \) is centrally \( \Gamma \)-free.

**Example 4.12** (Minimal ambient nuclear C*-algebras). We now construct new examples of minimal ambient nuclear C*-algebras (see [58] for the first examples). We emphasize that, in contrast to the von Neumann algebra case, the existence of a minimal ambient nuclear C*-algebra is not clear. Indeed, the class of nuclear C*-algebras does not form a monotone class [57]. Novelties of this approach are

(i) we only use the primeness of the action rather than the much stronger property called property \( \mathcal{R} \) (see [58], Proposition 3.3),

(ii) we do not use the Powers property of (subgroups of) the acting group.

(Here we recall that a topological dynamical system \( \alpha \) is said to be prime if it has no non-trivial proper factors.)

Let \( \Gamma \) be a discrete group with the AP. Take an amenable prime action \( \alpha : \Gamma \curvearrowright X \) on a compact space \( X \) (see Propositions 3.3 and 3.7 of [58] for existence results).

Take a simple nuclear centrally free \( \Gamma \)-C*-algebra \( A \) (cf. Examples 4.10, 4.11). Theorem 4.3.4 in [8] implies nuclearity of \( [A \otimes C(X)] \rtimes \Gamma \). By Theorem 4.5 and the tensor splitting theorem [65], [69], the inclusion \( A \rtimes \Gamma \subset [A \otimes C(X)] \rtimes \Gamma \) has no proper intermediate C*-algebras. Note that when \( \Gamma \) is non-amenable, typically \( A \rtimes \gamma \Gamma \) is non-nuclear (for instance when \( A \) has a \( \Gamma \)-invariant state). (However it is occasionally nuclear [60].)

**Example 4.13** (Non-commutative Bernoulli shifts for von Neumann algebras). Let \( Q \) be a von Neumann algebra with a faithful normal state \( \varphi \) satisfying \( Q_\varphi \neq C \). Here we only consider infinite groups \( \Gamma \). Let \( \sigma \) and \( \rho \) denote the left and right shift action of \( \Gamma \) on \( (M, \psi) := \bigotimes_\Gamma(Q, \varphi) \) respectively. Then \( \sigma \) is centrally free. To see this, choose an abelian von Neumann subalgebra \( C \neq A \subset Q_\varphi \). Then the action \( \Gamma \curvearrowright B := \bigotimes_\Gamma(A, \varphi|_A) \) comes from a Bernoulli shift on a non-trivial standard probability space, which is essentially free. Thus, by Lemma 4.11, for each \( s \in \Gamma \setminus \{e\} \), one can choose projections \( p_1, p_2, p_3 \) in \( B \) satisfying \( p_1 + p_2 + p_3 = 1, p_j \sigma_s(p_j) = 0 \) for each \( j \). Take a sequence \( (s_j)_{j=1}^\infty \) in \( \Gamma \) tending to infinity. The sequences \( (q_1(p_j))_{j=1}^\infty, j = 1, 2, 3 \), then define the projections \( q_1, q_2, q_3 \in M_\omega \) satisfying \( q_1 + q_2 + q_3 = 1 \) and \( q_1s,q_j = 0 \) for all \( j \).

**Example 4.14** (Twisted variant of Ge–Kadison’s splitting theorem). For any \( \Gamma \)-von Neumann algebra \( N \) and for any centrally free \( \Gamma \)-factor \( M \), by applying Theorem 4.6 and [12] to \( M \subset M \bar{\otimes} N \), we obtain the lattice isomorphism

\[ P \mapsto [M \bar{\otimes} P] \rtimes \Gamma, \]

between the lattice of \( \Gamma \)-von Neumann subalgebras \( P \subset N \) and that of intermediate von Neumann algebras of \( M \rtimes \Gamma \subset (M \bar{\otimes} N) \rtimes \Gamma \). This phenomenon can be seen as a twisted version of Ge–Kadison’s tensor splitting theorem [12]. Combining this with Corollary 4.1 (taking \( M \) to be amenable), we obtain further examples of maximal amenable subalgebras.

Now consider an amenable group \( \Gamma \). Let \( M \) be a centrally \( \Gamma \)-free AFD II_1 factor. Then, each measure preserving (not necessary free) prime action \( \alpha \) of \( \Gamma \) on a diffuse probability space
(X, μ) (e.g., the Chaçon system (13, Theorem 16.6)) associates an infinite index subfactor of the AFD II_1 factor with no proper intermediate von Neumann algebras. Indeed, by the primeness of α and the above observation, the inclusion $M \rtimes \Gamma \subset [M \otimes L^\infty(X)] \rtimes \Gamma$ has no proper intermediate von Neumann algebras.

**Example 4.15** (Realizations of the intermediate group lattices). Let G be a locally compact second countable group and H be a closed subgroup of G. We realize the lattice of intermediate closed groups of $H \subset G$ as the lattice of intermediate subfactors of an irreducible subfactor. (Recall that a subfactor $N \subset M$ is said to be irreducible if all intermediate von Neumann algebras are factors.) To see this, take a countable dense subgroup $\Gamma \subset G$. We regard $\Gamma$ as a discrete group. As $\Gamma$ is dense in $G$, any $\Gamma$-invariant von Neumann subalgebra of $L^\infty(G/H)$ is $G$-invariant. Hence it must be of the form $L^\infty(G/L)$ for some closed intermediate subgroup $L$ of $H \subset G$ (cf. [68], Appendix B). Now take a centrally free $\Gamma$-factor $N$ (see Example 4.13). As $[N \otimes L^\infty(G/L)] \rtimes \Gamma$ is a factor for any $L$ (Chapter V.7 of [61]), the inclusion $[N \otimes L^\infty(G/H)] \rtimes \Gamma \subset [N \otimes B(L^2(G/H))] \rtimes \Gamma$ possesses the desired properties. Note that thanks to Theorem 2 of [63], this result can be extended to general locally compact groups. However, if the group is not second countable, then the factors are no longer separable.

Here we record the following interesting observation on the Galois correspondence theorem, which is an immediate consequence of Example 4.15. Here we recall that an action of a locally compact group $G$ on a factor $M$ is said to be minimal if the fixed point subalgebra $M^G$ of $G$ is irreducible in $M$.

**Corollary 4.16.** Any locally compact group $G$ admits a minimal action on a factor $M$ with the following property. The map $H \mapsto M^H$ gives a lattice anti-isomorphism between the lattice of closed subgroups $H$ of $G$ and the lattice of intermediate von Neumann algebras of $M^G \subset M$.

**Proof.** We use the notations used in Example 4.15 with $H = \{e\}$. Set $M := [N \otimes B(L^2(G))] \rtimes \Gamma$. Consider the $G$-action

$$\text{id}_N \otimes \rho: G \curvearrowright N \otimes B(L^2(G)),$$

where $\rho: G \curvearrowright B(L^2(G))$ denotes the action induced from the right regular action. Then this action obviously commutes with the equipped $\Gamma$-action. Hence it extends to the $G$-action on $M$. By the argument in Example 4.15 (cf. [63]), this action satisfies the required properties. □

**Remark 4.17.** Related to Remark 4.9 and Examples 4.12 to 4.15, we remark that the equivariant version of the tensor splitting theorems [12], [65], [69] fail in general. More precisely, for a $\Gamma$-simple $C^*$-algebra $A$ (i.e., with no proper $\Gamma$-ideals) and for a unital $\Gamma$-$C^*$-algebra $B$, an intermediate $\Gamma$-$C^*$-algebra of $A \subset A \otimes B$ does not need to split into $A \otimes B_0$, $B_0 \subset B$, and similarly for the von Neumann algebra case. Indeed, let $\Gamma$ be a finite group of order at least 3. Consider the inclusion $\ell^\infty(\Gamma) \otimes C \subset \ell^\infty(\Gamma) \otimes \ell^\infty(\Gamma)$. Here $\Gamma$ is equipped with the left translation $\Gamma$-action. Then $\ell^\infty(\Gamma)$ is $\Gamma$-simple. On $\Gamma \times \Gamma$, we define the equivalence relation $\sim$ by declaring $(s, t) \sim (s', t')$ if $s = s'$ and either $t, t' \in \Gamma \setminus \{s\}$ or $t = t'$ holds. Define $Z := (\Gamma \times \Gamma) / \sim$. Then $\ell^\infty(Z)$ defines an intermediate $\Gamma$-$C^*$-algebra of the inclusion $\ell^\infty(\Gamma) \otimes C \subset \ell^\infty(\Gamma) \otimes \ell^\infty(\Gamma)$, while $\ell^\infty(Z)$ does not split into $\ell^\infty(\Gamma) \otimes \ell^\infty(W)$ for any factor $W$ of $\Gamma$. 
5. Constructions of exotic endomorphisms

In this final section we construct “exotic” endomorphisms on $C^*$-algebras. Thanks to Kirchberg’s tensor absorption theorem ([35], Theorem 3.15 of [36]), the next theorem implies Corollary [C]. The key idea is to realize the Cuntz algebra $\mathcal{O}_\infty$ as a corner of the crossed product of an appropriate free group action. This realization result itself would be of independent interest.

For the basic facts and notations on K-theory and KK-theory, we refer the reader to [5].

**Theorem 5.1.** Let $A$ be a simple $C^*$-algebra satisfying $A \otimes O_\infty \cong A$. Then there is an endomorphism $\sigma: A \to A$ with the following properties.

- The inclusion $\sigma(A) \subset A$ has no proper intermediate $C^*$-algebras.
- The inclusion $\sigma(A) \subset A$ does not admit a conditional expectation.

**Proof.** Let $\Gamma$ be a countable free group of infinite rank. Take an amenable minimal topologically free action $\alpha$ of $\Lambda = \mathbb{F}_2$ on the circle $\mathbb{T}$. (E.g., take a lattice $\Lambda$ in $\text{PSL}(2, \mathbb{R})$ isomorphic to $\mathbb{F}_2$ (see Example E.10 in [8]). Let $P$ denote the subgroup of upper triangular matrices in $\text{PSL}(2, \mathbb{R})$.

Then, thanks to Theorem 9.5.3 of [68], Theorem 5.4.1 of [8], and Remark 13 of [47], the left translation action $\alpha: \Lambda \curvearrowright \text{PSL}(2, \mathbb{R})/P \cong \mathbb{T}$ possesses the desired properties.) By the Pimsner–Voiculescu exact sequence ([50], Theorem 3.5), we obtain the exact sequence

$$0 \to \mathbb{Z} \to K_0(C(\mathbb{T}) \rtimes_r \Lambda) \to \mathbb{Z}^2.$$

From this exact sequence, one can find a homomorphism $K_0(C(\mathbb{T}) \rtimes_r \Lambda) \to \mathbb{Z}$ sending $[1]$ to $1$. It also follows from [50] that $C(\mathbb{T}) \rtimes_r \Lambda$ satisfies the universal coefficient theorem (cf. Corollary 7.2 of [55]). By [3] and [1] (Theorem 4.4.3 of [8]), $C(\mathbb{T}) \rtimes_r \Lambda$ is simple and nuclear. Now we embed $\Gamma$ into $\Lambda$. Then Theorem 4.1.1 of [49] gives a unital embedding $\iota: C(\mathbb{T}) \rtimes_r \Gamma \subset C(\mathbb{T}) \rtimes_r \Lambda \to O_\infty$.

Now define

$$\beta := \bigotimes_N \text{ad}(\iota|_\Gamma) : \Gamma \curvearrowright \bigotimes_N O_\infty.$$

Note that $\beta$ is centrally free by Theorem 1 of [41].

Next choose a unital Kirchberg algebra $C$ satisfying the universal coefficient theorem with $K_0(C) \cong \mathbb{Z}[\Gamma]$ (as an additive group), $[1_C]_0 = 0$, $K_1(C) = 0$.

Take an action $\gamma: \Gamma \curvearrowright C$ whose induced action on $K_0(C)$ is conjugate to the left translation action on the group ring $\mathbb{Z}[\Gamma]$. (Such an action exists by Theorem 4.1.1 of [49].) Put

$$D := O_\infty \otimes C \otimes \bigotimes_N O_\infty.$$

We equip $D$ with the $\Gamma$-action

$$\eta := \text{id}_{O_\infty} \otimes \gamma \otimes \beta.$$

We claim that

$$K_0(D \rtimes_r \Gamma) \cong \mathbb{Z}, \quad K_1(D \rtimes_r \Gamma) = 0.$$

We fix a free basis $S$ of $\Gamma$. Then, thanks to the Pimsner–Voiculescu exact sequence [50], the claim follows from the following assertion. The additive map

$$\nu: \bigoplus_S \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma]$$
given by
\[(x_s)_{s \in S} \mapsto \sum_{s \in S} (x_s - sx_s)\]
satisfies
\[\ker(\nu) = 0, \ coker(\nu) \cong \mathbb{Z}.\]
(We remark that Theorem 3.5 of [50] is concentrated on the finite rank free groups. However the infinite rank case follows from the finite rank case as the exact sequences are compatible with the canonical inclusions \(\mathbb{F}_n \subset \mathbb{F}_{n+1}; \ n \in \mathbb{N}\).) We prove this assertion. For \(w \in \Gamma \setminus \{e\}\), denote by \(i(w) \in S \cup S^{-1}\) the initial alphabet of \(w\) with respect to \(S\). Also, for \(w \in \Gamma\), denote by \(|w|\) the length of the reduced word of \(w\) (with respect to \(S\)). For \(a \in \mathbb{Z}[\Gamma]\), define
\[\text{supp}(a) := \{s \in \Gamma : a(s) \neq 0\} .\]
To prove the first equality, assume we have \(x = (x_s)_{s \in S} \in \ker(\nu) \setminus \{0\}\). Choose \(s \in S\) and \(t \in \text{supp}(x_s)\) satisfying \(|t| \geq |u|\) for all \(u \in \bigcup_{w \in S} \text{supp}(x_w)\). As \(\sum_{u \in S} (x_u - ux_u) = 0\), the maximal property of \(t\) forces \(i(t) = s^{-1}\). (Otherwise \(|st| = |t| + 1\) and \(\nu(x)(st) = -x_s(t) \neq 0\). We then must have an element \(u \in S \setminus \{s\}\) with \(t \in \text{supp}(x_u) \cup \text{supp}(ux_u)\). Again by the maximal property of \(t\), the relation \(t \in \text{supp}(x_u)\) is impossible. Hence \(t \in \text{supp}(ux_u)\), or equivalently, \(u^{-1}t \in \text{supp}(x_u)\). This contradicts to the choice of \(t\) as \(|u^{-1}t| = |t| + 1\). Thus \(\ker(\nu) = 0\).
To prove \(coker(\nu) \cong \mathbb{Z}\), consider the homomorphism \(\tau_0 : \mathbb{Z}[\Gamma] \to \mathbb{Z}\) given by \(a \mapsto \sum_{s \in \Gamma} a(s)\). Obviously \(\tau_0\) is surjective and \(\text{im}(\nu) \subset \ker(\tau_0)\). As \(\delta_w - \delta_{s^{-1}} \in \ker(\tau_0)\) for all \(w \in \Gamma\) (by induction on \(|w|\)), we conclude \(\text{im}(\nu) = \ker(\tau_0)\). Thus \(\tau_0\) induces the desired isomorphism \(coker(\nu) \cong \mathbb{Z}\).

Let \(Q = [0, 1]^N\) denote the Hilbert cube. Note that \(Q\) is a contractible compact metrizable space. Since \(\text{Homeo}(Q)\) is a Polish (hence separable) group in the topology of uniform convergence (see [33], Chapter 1.9.B (3)), one can choose a homomorphism
\[h : \Gamma \to \text{Homeo}(Q)\]
with dense image. As the diagonal action
\[\text{Homeo}(Q) \curvearrowright \{(x_1, x_2) \in Q \times Q : x_1 \neq x_2\}\]
is transitive (see e.g., [37], Theorem 4.5), \(Q\) admits no non-trivial \(\Gamma\)-invariant closed equivalence relation. Thus \(h\) is minimal and prime. We equip \(D \otimes C(Q)\) with the \(\Gamma\)-action \(\eta \otimes h\). Since \(Q\) has no \(\text{Homeo}(Q)\)-invariant probability measure, the inclusion
\[D_1 := D \rtimes_{\eta} \Gamma \subset [D \otimes C(Q)] \rtimes_{\eta} \Gamma := D_2\]
does not admit a conditional expectation. Indeed, suppose we have a conditional expectation \(\Psi : D_2 \to D_1\). As the center of \(D\) only consists of scalars, the restriction \(E \circ \Psi|_{C(Q)}\) defines a \(\Gamma\)-invariant probability measure on \(Q\), a contradiction. Observe that \(D\), hence also \(D \otimes C(Q)\), can be presented as the norm closure of the increasing union of nuclear \(\Gamma\)-C*-subalgebras whose underlying \(\Gamma\)-action is amenable in the sense of Definition 4.3.1 in [8]. Indeed the following sequence has the desired properties.
\[\mathcal{O}_\infty \otimes C \otimes \left( \bigotimes_{k=1}^n \mathcal{O}_\infty \right) \otimes \iota(C(T)) \otimes \left( \bigotimes_{k=n+2}^\infty C_1 \mathcal{O}_\infty \right), \ n \in \mathbb{N} .\]
(Cf. Proposition B in [60].) Therefore by [1] (Theorem 4.3.4 of [8]), both \(D_i\) are nuclear. By Theorem 7.1 of [7], both \(D_i\) are simple. Since both \(D_i\) have \(\mathcal{O}_\infty\) as a tensor factor, they
are purely infinite. Thanks to the Pimsner–Voiculescu exact sequence \[50\], both \(D_i\) are KK-equivalent to \(O_\infty\) and the inclusion map \(D_1 \to D_2\) induces a KK-equivalence. Now choose a projection \(q \in D\) generating \(K_0(D_i)\) \([10]\). By the Kirchberg–Phillips classification theorem \([35]\), \([49]\), both \(qD_iq\) are isomorphic to \(O_\infty\).

We equip \(A \otimes D\), \(A \otimes D \otimes C(Q)\) with the \(\Gamma\)-action \(id_A \otimes \eta, id_A \otimes \eta \otimes h\) respectively. Consider the inclusion

\[D_{1,A} := (A \otimes D) \rtimes \Gamma \subset [A \otimes D \otimes C(Q)] \rtimes \Gamma := D_{2,A}.
\]

Since \(h\) is prime, Theorem 3.3 of \([65]\) implies the following statement. Any intermediate \(\Gamma\)-C*-algebra of \(A \otimes D \subset A \otimes D \otimes C(Q)\) is either equal to \(A \otimes D \otimes C(Q)\) or contained in \(F(A \otimes D, C(Q), \mathbb{C}) := \{a \in A \otimes D \otimes C(Q) : (\varphi \otimes id_{C(Q)})(a) \in \mathbb{C} \text{ for all } \varphi \in (A \otimes D)^*\}\).

(We remark that Theorem 3.3 of \([65]\) is only stated for unital C*-algebras. However we still can apply it as \(A \otimes D\) has approximate units of projections \([65]\).) Since the inclusion \(\mathbb{C} \subset C(Q)\) admits a conditional expectation, we have \(F(A \otimes D, C(Q), \mathbb{C}) = A \otimes D\). Hence, by Theorem \([45]\)
\(D_{1,A} \subset D_{2,A}\) has no proper intermediate C*-algebras. Observe that the inclusion \(D_{1,A} \subset D_{2,A}\) has no conditional expectations. Indeed, as \(A \subset D_{1,A}\) is a simple tensor factor of \(D_{2,A}\), a multiplicative domain argument shows that any conditional expectation \(\Phi: D_{1,A} \to D_{2,A}\) splits into \(id_A \otimes \Phi\) for some conditional expectation \(\Phi: D_2 \to D_1\). We have already seen that such a \(\Phi\) does not exist.

Now consider the projection

\[\tilde{q} := 1_{M(A)} \otimes q \in M(A \otimes D) (\subset M(D_{1,A}) \subset M(D_{2,A})).\]

Note that

\[\tilde{q}D_{1,A}\tilde{q} \cong A \otimes qD_iq \cong A \otimes O_\infty \cong A \text{ for } i = 1, 2.\]

We next show that the inclusion \(\tilde{q}D_{1,A}\tilde{q} \subset \tilde{q}D_{2,A}\tilde{q}\) has no proper intermediate C*-algebras. This follows from the next lemma. The proof is straightforward and we leave it to the reader.

**Lemma 5.2.** Let \(\mathfrak{A} \subset \mathfrak{B}\) be a non-degenerate inclusion of C*-algebras. Let \(r \in M(\mathfrak{A})\) be a projection. Then the map

\[\mathcal{C} \mapsto C^*(\mathfrak{A}, \mathcal{C})\]

gives an embedding of the lattice of intermediate C*-algebras \(\mathcal{C}\) of \(r\mathfrak{A}r \subset r\mathfrak{B}r\) into that of \(\mathfrak{A} \subset \mathfrak{B}\). Moreover, the map \(D \mapsto rDr\) gives a left inverse.

We remark that the above maps are bijective when \(r\) is full in \(A\) (i.e., when \(\overline{\text{span}}(ArA) = A\).)

**Proof of Theorem 5.1 (continuation).** By Lemma 5.2 the inclusion \(\tilde{q}D_{1,A}\tilde{q} \subset \tilde{q}D_{2,A}\tilde{q}\) has no proper intermediate C*-algebras. As \(D_{1,A} \subset D_{2,A}\) does not admit a conditional expectation, neither does \(\tilde{q}D_{1,A}\tilde{q} \subset \tilde{q}D_{2,A}\tilde{q}\). Indeed, since \(D\) is purely infinite, there is an isometry \(v\) in \(D\) satisfying \(vv^* \leq q\). Put \(\tilde{v} := 1_{M(A)} \otimes v\). Then \(\tilde{v}^*\tilde{q}D_{i,A}\tilde{q}\tilde{v} = D_{1,A}\) for \(i = 1, 2\). Hence any conditional expectation \(\Psi: \tilde{q}D_{2,A}\tilde{q} \to \tilde{q}D_{1,A}\tilde{q}\) induces a conditional expectation \(\tilde{\Psi}: D_{2,A} \to D_{1,A}\) by the formula \(\tilde{\Psi}(a) := \tilde{v}^*\Psi(\tilde{v}a\tilde{v}^*)\tilde{v}\). This is a contradiction.

Now for \(i = 1, 2\), choose an isomorphism \(\sigma_i: \tilde{q}D_{i,A}\tilde{q} \to A\). Denote by \(\iota: \tilde{q}D_{1,A}\tilde{q} \to \tilde{q}D_{2,A}\tilde{q}\) the inclusion map. The endomorphism \(\sigma := \sigma_2 \circ \iota \circ \sigma_1^{-1}\) then possesses the desired properties. \(\square\)
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