A gauge invariant regulator for the ERG

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A gauge invariant regularisation for dealing with pure Yang-Mills theories within the exact renormalization group approach is proposed. It is based on the regularisation via covariant higher derivatives and includes auxiliary Pauli-Villars fields which amounts to a spontaneously broken $SU(N|N)$ super-gauge theory. We demonstrate perturbatively that the extended theory is ultraviolet finite in four dimensions and argue that it has a sensible limit when the regularization cutoff is removed.

1 Introduction

In an earlier work, one of us proposed a manifestly gauge invariant formulation of the exact renormalization group (ERG) approach for pure Yang-Mills theories. In the same article it was suggested to use the gauge invariant regularisation by higher covariant derivatives within this formulation. However, as it is known, this regularisation fails to regulate certain one-loop ultraviolet divergences. They are regularised by introducing additional regulating fields. It was realized that adding bosonic as well as fermionic fields to cancel ultraviolet divergences in such a way as to maintain the gauge invariance results in a spontaneously broken $SU(N|N)$ gauge theory.

In the present contribution we start with the $SU(N)$ pure gauge theory in $D$ space-time dimensions with the action

$$S_{YM} = \frac{1}{2} \int d^Dx \, tr \left( F_{\mu\nu} F^{\mu\nu} \right)$$

and extend it so as to include additional fields and covariant higher derivatives as regulators. The latter introduce also a scale $\Lambda$ which plays the rôle of the effective momentum cutoff. We also add a scalar Higgs field to give masses of order $\Lambda$ to some of the regulating fields, so that the massive ones behave precisely as Pauli-Villars fields. Our aim is to check the consistency of such a regularisation scheme. We show that the extended theory in four dimensions is indeed ultraviolet finite when the cutoff $\Lambda$ is kept finite. In the continuum limit, $\Lambda \to \infty$, the massive unphysical fields become infinitely heavy and decouple from the theory. In addition the unphysical gauge field sector decouples from the sector of physical fields in this limit. We argue that in this way the initial theory is recovered in the continuum limit.

The plan of the article is as follows. In sec. 2 we describe in more detail the main idea of the gauge invariant regularisation scheme. In sec. 3 the graded Lie algebra of $SU(N|M)$ is discussed and the field content and the action of the $SU(N|N)$ gauge

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theory are presented. In sec. 4 we consider some potential problems due to the presence of the Pauli-Villars fields using the example of $U(1|1)$ quantum mechanics and discuss the decoupling of the unphysical sector. Sec. 5 is devoted to the proof of finiteness of the regularised theory. Sec. 6 contains a summary of results.

2 Regularised extension of the $SU(N)$ Yang-Mills theory: general idea

As with the case of scalar and fermionic theories, our aim is to regularise a pure Yang-Mills theory in a way appropriate for the ERG approach, i.e. introducing a physical cutoff $\Lambda$ which sets the physical scale. Another requirement is that the regularisation should be manifestly gauge invariant.

We will adopt the point of view that a suitable regularisation for applications within the ERG approach must make the theory finite at least in the perturbative sense, i.e. at each order of perturbation theory. We believe that this is sufficient to make the theory finite within the non-perturbative treatment. Indeed, the perturbative sector is regularised by construction. As expected, if Feynman diagrams in the perturbative sector are regulated so as to be free of ultraviolet divergences, then all contributions in the expansion around, for example, an instanton, are also finite.

We implement the regularisation in two steps. The first step is to introduce the higher covariant derivative regularisation. For this we modify the canonical part of the effective action and the propagator as follows:

$$S = \frac{1}{2} \text{tr} \int d^Dx \ F_{\mu\nu} c^{-1} \left( -\frac{\nabla^2}{\Lambda^2} \right) \cdot F^{\mu\nu} + \ldots, \quad (2)$$

where $\nabla_{\mu} = \partial_{\mu} - igA_{\mu}$ and the dot means that the covariant derivative acts by commutation. Here $c \left( -\nabla^2/\Lambda^2 \right)$ is a (smooth) ultraviolet cutoff profile satisfying $c(0) = 1$ so that at low energies the propagator is unaltered, and $c(z) \to 0$ as $z \to \infty$ sufficiently fast so that all Feynman diagrams are expected to be ultraviolet regulated.

As mentioned in the introduction, some one-loop diagrams remain unregularised whatever the choice of the regulating function $c^{-1}$: they require some additional regularisation. Such regularisation can be provided by adding some auxiliary fields. Introducing these fields just formally - in the way it is done in many text books - breaks gauge invariance. To preserve it, we introduce them by extending the theory (2) in a particular way. Thus, the second step is to supplement the theory with regulating fields. We will see that they include

- an additional bosonic field $A_{\mu}^2$;
- fermionic (Pauli-Villars) fields $B_{\mu}, \bar{B}_{\mu}$;
- two bosonic (Pauli-Villars) scalar fields $C_1$.

The fields $B_{\mu}, C_1$ and $C_2$ have masses of order $\Lambda$.

It turns out that this scheme eventually results in the spontaneously broken $SU(N|N)$ gauge theory. In particular, the physical field, i.e. the gauge field $A_{\mu}$
$A_1$, of the initial $SU(N)$ Yang-Mills theory (2), together with the fields $A_\mu^2$ and $B_\mu$, $\bar{B}_\mu$ form a super-gauge multiplet of $SU(N|N)$, and $B_\mu$, $C^1$ and $C^2$ get their masses via the Higgs mechanism.

3 Graded Lie algebra of $SU(N|M)$ and $SU(N|N)$ gauge theory

Elements of the graded Lie algebra of $SU(N|M)$ and $SU(N|N)$ gauge theory

Elements of the graded Lie algebra of $SU(N|M)$ are given by Hermitian $(N+M) \times (N+M)$ matrices

$$\mathcal{H} = \begin{pmatrix} H_1 & \theta \\ \theta^\dagger & H_2 \end{pmatrix}.$$  (3)

Here $H_N$ ($H_M$) is an $N \times N$ ($M \times M$) Hermitian matrix with complex bosonic elements, $\theta$ is an $N \times M$ matrix composed of complex Grassmann numbers. $\mathcal{H}$ is required to be supertraceless

$$\text{str}(\mathcal{H}) := \text{tr}(H_1) - \text{tr}(H_2) = 0.$$  (4)

The bosonic sector of the $SU(N|M)$ algebra forms the $SU(N) \times SU(M) \times U(1)$ subalgebra.

Let us now specialise to $M = N$, the case we will be interested in. The $2N \times 2N$ identity matrix, $\mathbb{1}_{2N}$, satisfies $\text{str} \mathbb{1}_{2N} = 0$ and, therefore, is an element of $SU(N|N)$. An arbitrary $2N \times 2N$ supermatrix $\mathcal{X}$ can be written as

$$\mathcal{X} = \frac{1}{2N} \text{str}(\mathcal{X}) \sigma_3 + \frac{1}{2N} \text{tr}(\mathcal{X}) \mathbb{1}_{2N} + \mathcal{X}^A T_A,$$  (5)

where

$$\sigma_3 = \begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_N \end{pmatrix},$$  (6)

and the other $SU(N|N)$ generators, $T_A$, are made of complex numbers. They can be chosen such that $\text{str}(T_A) = 0 = \text{tr}(T_A)$. The index $A$ runs over $2(N^2-1)$ bosonic and $2N^2$ fermionic indices. For $\mathcal{H} \in SU(N|N)$

$$\mathcal{H} = \mathcal{H}_0 \mathbb{1}_{2N} + \mathcal{H}^A T_A.$$  (7)

We also define the Killing super-metric in the $T_A$ subspace

$$g_{AB} = \frac{1}{2} \text{str}(T_AT_B).$$  (8)

g_{AB} is symmetric when both indices $A$ and $B$ are bosonic, antisymmetric when both are fermionic and is zero when one is bosonic and another is fermionic.

Let us turn to the construction of the $SU(N|N)$ extension of the regularised $SU(N)$ gauge theory (2). A basic ingredient is the super-gauge field $A_\mu$ which takes values in the graded Lie algebra $SU(N|N)$ and therefore can be written as

$$A_\mu = A_\mu^0 \mathbb{1}_{2N} + \tilde{A}_\mu,$$

where

$$\tilde{A}_\mu = \begin{pmatrix} A_\mu^1 & B_\mu \\ \bar{B}_\mu & A_\mu^2 \end{pmatrix} = A_\mu^A T_A.$$  (9)
Here $A_1^1 \equiv A_\mu$ is the physical $SU(N)_1$ gauge field, \textit{i.e.} the gauge field in the initial theory $\mathbb{B}$, while the fields $A_2^0$ and $B_\mu$ are part of the regulating structure. It will turn out later that the field $A_\mu^0$ is not dynamical and being a $U(1)$ factor does not couple to the other fields. Therefore in the rest of the article we will consider the $A_\mu$ field only. We will also omit the tilde to ease notation.

The action of the super-gauge field is taken to be

$$S_{YM} = \frac{1}{2} F_{\mu\nu} \{ e^{-1} \} F^{\mu\nu}, \quad (10)$$

where the gauge field strength is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$. For a given kernel $W$ and two super-matrices $u(x)$ and $v(x)$ the action $u\{W\}v$ is defined as

$$u\{W\}v := \text{str} \int d^D x \ u(x)W \left( -\nabla^2 \right) \cdot v(x). \quad (11)$$

The regulating function $c^{-1}$ is chosen to be a polynomial in $(-\nabla^2/\Lambda^2)$ of rank $r$. Next we introduce a super-scalar field

$$\mathcal{C} = \begin{pmatrix} C^1 & D \\ \bar{D} & C^2 \end{pmatrix}, \quad (12)$$

with the action

$$S_{\mathcal{C}} = \nabla_\mu \cdot \mathcal{C} \{ e^{-1} \} \nabla^\mu \cdot \mathcal{C} + \frac{\Lambda}{4} \text{str} \int d^D x \ (C^2 - \Lambda^2)^2, \quad (13)$$

where $\tilde{e}^{-1}$ is another regulating function. It is assumed to be a polynomial in $(-\nabla^2/\Lambda^2)$ of rank $\tilde{r}$. The field $\tilde{c}^{12}$ is not assumed supertraceless. It acquires a vacuum expectation value which may be taken to be $\langle \tilde{c}^{12} \rangle = \Lambda \sigma_3$. Shifting $\mathcal{C} \rightarrow \Lambda \sigma_3 + \mathcal{C}$, the action in the scalar sector becomes

$$S_{\mathcal{C}} = -g^2 \Lambda^2 \{ A_\mu, \sigma_3 \} \{ e^{-1} \} \{ A_\mu, \sigma_3 \} - 2ig\Lambda \{ A_\mu, \sigma_3 \} \{ e^{-1} \} \nabla^\mu \cdot \mathcal{C} + \nabla_\mu \cdot \mathcal{C} \{ e^{-1} \} \nabla^\mu \cdot \mathcal{C} + \frac{\Lambda}{4} \text{str} \int d^D x \ (\Lambda \{ \sigma_3, \mathcal{C} \} + C^2)^2. \quad (14)$$

One can check that the fields $B_\mu$, $C^1$ and $C^2$ acquire masses of order $\Lambda$, thus behaving precisely as Pauli-Villars fields.

It is convenient to impose ’t Hooft’s gauge fixing condition

$$\partial_\mu A^\mu + i\frac{\Lambda}{\xi} \tilde{e}^{-1} \mathcal{C}[\sigma_3, \mathcal{C}] = 0, \quad (15)$$

where $\tilde{e}^{-1}(-\partial^2/\Lambda^2)$ is a polynomial of rank $\tilde{r}$. Note that here the regulating function is not covariantised. The corresponding gauge fixing term in the action is

$$S_{GF} = \xi \partial_\mu A^\mu \cdot \tilde{e}^{-1} \cdot \partial_\nu A^\nu + 2ig\Lambda(\partial_\nu A^\nu) \cdot \tilde{e}^{-1} \cdot [\sigma_3, \mathcal{C}]$$

$$- g^2 \frac{\Lambda^2}{\xi} \mathcal{C}[\sigma_3, \mathcal{C}] \cdot \tilde{e}^{-2} \tilde{c} \cdot [\sigma_3, \mathcal{C}], \quad (16)$$

where $f \cdot W \cdot g \doteq \text{str} \int d^D x d^D y f(x)W_{xy} g(y)$ and $W_{xy}$ is the inverse Fourier transform of the kernel $W \left( \frac{q^2}{\Lambda^2} \right)$. The Faddeev-Popov ghost super-fields form the
SU(N|N) supermatrix
\[ \eta = \begin{pmatrix} \eta^1 & \phi \\ \psi & \eta^2 \end{pmatrix}. \] (17)

The action of the ghost sector is given by
\[
S_{\text{ghost}} = -\bar{\eta} \cdot \hat{c}^{-1} \hat{c} \cdot \partial_\mu \nabla^\mu \cdot \eta
- \int d^Dx \text{str} \left\{ \frac{\Lambda}{\xi} [\sigma_3, \bar{\eta}] \left( \Lambda[\sigma_3, \eta] + [C, \eta] \right) \right\}. \quad (18)
\]

In order to keep the high momentum behaviour of the $A$ propagator unchanged
by the introduction of the $C$ field and gauge fixing, we require the ranks of our
polynomial cutoff functions to be bounded as
\[ \hat{r} \geq r \geq \tilde{r}. \] (19)

4 Potential problems in the unphysical sector

The quadratic part of the action (10) is equal to
\[
S_{YM} = \int d^Dx \left[ \frac{1}{2} \text{str} (F_{\mu\nu}^1)^2 - \text{str} (F_{\mu\nu}^2)^2
- 2\text{tr} \left( \partial_\mu B_\nu - \partial_\nu B_\mu \right) (\partial_\mu B_\nu - \partial_\nu B_\mu) + \ldots \right].
\]

The appearance of the term with negative sign could potentially be a problem due
to an instability in the theory. Below we consider the example of $U(1|1)$ quantum
mechanics which shows that it is rather a sign of the loss of unitarity.

Let us define the Hermitian super-position as
\[ \mathcal{X} = \begin{pmatrix} x_1 & \psi \\ \bar{\psi} & x_2 \end{pmatrix}, \] (20)

and consider the model with a simple harmonic potential. The Lagrangian is given by
\[ L = \frac{1}{2} \text{str} \dot{\mathcal{X}}^2 - \frac{1}{2} \text{str} \mathcal{X}^2. \] (21)

The conjugate momentum variables are equal to
\[
p_{x_1} = \dot{x}_1, \quad p_{x_2} = -\dot{x}_2, \quad [x_j, p_{x_j}] = i, \\
p_\psi = \dot{\psi}, \quad p_\bar{\psi} = -\dot{\psi}. \] (22)

Next we define the $a_j, a_j^\dagger$ operators ($j = 1, 2$)
\[ a_j = \frac{1}{\sqrt{2}} (x_j + ip_{x_j}), \quad a_j^\dagger = \frac{1}{\sqrt{2}} (x_j - ip_{x_j}), \] (23)

with the commutation relations $[a_i, a_j^\dagger] = \delta_{ij}$. In terms of these operators the Hamiltonian reads
\[ H = \frac{1}{2} (a_1^\dagger a_1 + a_1 a_1^\dagger) - \frac{1}{2} (a_2^\dagger a_2 + a_2 a_2^\dagger) + \text{(fermionic part)}. \] (24)
Now we introduce the vacuum states $|0\rangle_1$, $|0\rangle_2$ satisfying the relations
\[ a_1 |0\rangle_1 = 0, \quad a_+^\dagger |0\rangle_2 = 0, \] (25)
and build up sets of $n$-particle states according to the formulae
\[ |n\rangle_1 = \frac{1}{\sqrt{n!}} (a_+^\dagger)^n |0\rangle_1, \]
\[ |n\rangle_2 = \frac{1}{\sqrt{n!}} (a_2)^n |0\rangle_2. \] (26)

Note that $a_2$ plays the rôle of the creation operator of the particle of the second type. With such definitions the Hamiltonian of the system is bounded from below. In particular,
\[ H |n\rangle_2 = +n |n\rangle_2. \] (27)

Furthermore, it can be shown that these definitions ensure that the vacuum preserves the $U(1|1)$ symmetry. However the $n$-particle states in the ‘$2$’-sector with odd $n$ possess negative norms:
\[ 2 |n\rangle_2 = \frac{1}{n^2} < 0 (a_+^\dagger)^n (a_2)^n |0\rangle_2 = (a_2)^n < 0 |0\rangle_2. \] (28)

This can be referred to as violation of unitarity (negative probability).

We would like to mention that the appearance of negative norm states as a consequence of a wrong sign in part of the action is not that unusual. The Gupta-Bleuler quantization procedure\cite{4} relies on a modification of the Lagrangian which results in a wrong sign appearing in the $A_0$ part of the action. Nonetheless, in the present case there is no analogue of the Gupta-Bleuler condition.

At finite $\Lambda$, transitions between the $A_1^\mu$- and the $A_2^\mu$-sector via (massive) Pauli-Villars fields are not forbidden - they are just suppressed, thus leading to violation of unitarity. In the continuum limit ($\Lambda \to \infty$), however, the amplitudes for such transitions vanish and the physical sector decouples from the unphysical one.

To see an example let us consider $D = 4$. The lowest order $A_1^\mu A_2^\mu$ amplitude comes from the term
\[ \text{str} (A A) \text{str} (A A) \times \text{(IR and UV finite coefficient)} \] (29)
in the effective action. Gauge invariance and dimensional considerations imply that this term is in fact
\[ \sim \int d^4 x \frac{1}{\Lambda^4} \text{str} (FF) \text{str} (FF). \] (30)

Therefore, it vanishes as $\Lambda \to \infty$.  

6
5 Counting of ultraviolet divergences

Using standard rules for calculating the superficial degree of divergence of a one-particle-irreducible (1PI) diagram in $D$ space-time dimensions, we get

\[
\mathcal{D}_f = DL - (2r + 2) I_A - (2\tilde{r} + 2) I_C - (2\tilde{r} - 2\tilde{r} + 2) I_\eta + \sum_{i=3}^{2r+4} (2\tilde{r} + 4 - i) V_A^i + \\
+ \sum_{j=2}^{2\tilde{r}+2} (2\tilde{r} + 2 - j) V_{A^iC} + \sum_{k=1}^{2\tilde{r}+2} (2\tilde{r} + 2 - k) V_{A^iC2} + (2\tilde{r} - 2\tilde{r} + 1) V_{\eta^2A},
\]

where $L$ is the number of loops and $I_f$ and $V_f$ correspond to the number of internal lines and vertices of $f$-type respectively. In eq. (31), inequalities (19) have already been assumed for the degree of divergence of the vector propagator to be counted properly.

As it stands, eq. (31) does not account properly for 1PI diagrams with external anti-ghost lines. In fact, the whole momentum dependence of the $V_{\eta^2A}$ vertex is counted as flowing into the loop, without taking into account the fact that such a dependence is actually only carried by $\eta$ lines and, thus, that one has to check whether such lines are external or not. This results in a systematic overestimate of $\mathcal{D}_f$. In order to remedy this and, thus, improve our upper bound, $\mathcal{D}_f$, we add $-(2\tilde{r} - 2\tilde{r} + 1) E_\eta A$, with $E_\eta A$ being the number of external anti-ghost lines which enter $V_{\eta^2A}$ vertices.

Therefore, the improved formula for the superficial degree of divergence is

\[
\mathcal{D}_f = DL - (2r + 2) I_A - (2\tilde{r} + 2) I_C - (2\tilde{r} - 2\tilde{r} + 2) I_\eta + \sum_{i=3}^{2r+4} (2\tilde{r} + 4 - i) V_A^i + \\
+ \sum_{j=2}^{2\tilde{r}+2} (2\tilde{r} + 2 - j) V_{A^iC} + \sum_{k=1}^{2\tilde{r}+2} (2\tilde{r} + 2 - k) V_{A^iC2} + (2\tilde{r} - 2\tilde{r} + 1) (V_{\eta^2A} - E_\eta A).
\]

The variables upon which $\mathcal{D}_f$ is dependent can be easily related to the number of external lines of each type, $E_f$, as

\[
L = 1 + I_A + I_C + I_\eta - \sum V_A - \sum V_{A^iC} - \sum V_{A^iC2} - V_{\eta^2A} - V_{\eta^2C} - V_{C3} - V_{C4},
\]

\[
E_A = -2I_A + \sum iV_A + \sum jV_{A^iC} + \sum kV_{A^iC2} + V_{\eta^2A},
\]

\[
E_C = -2I_C + \sum V_{A^iC} + 2 \sum V_{A^iC2} + 3V_{C3} + 4V_{C4} + V_{\eta^2C},
\]

\[
E_\eta = E_\eta A + E_\eta C + E_\eta A + E_\eta C = -2I_\eta + 2V_{\eta^2A} + 2V_{\eta^2C}.
\]

In the last of the above relations, to ensure consistency with previous notation we split external ghost lines according to the vertices they are attached to. Thus $E_\eta f$, $f = A, C$, is the number of external (anti-) ghost lines entering $V_{\eta^2f}$ vertices; they satisfy the expected constraint $E_\eta A + E_\eta C = E_\eta A + E_\eta C$. The first of eq. (33) is valid for connected diagrams only, as the first term in the r.h.s. - representing the number of connected components - has been set to 1.
By making use of the above formulae, it is possible to rewrite $\mathcal{D}_\Gamma$ in a more useful form, independent of internal lines,

$$\mathcal{D}_\Gamma = (D - 2r - 4)(L - 2) - E_A - (r - \tilde{r} + 1)E_C - 2(r + \hat{r} - \tilde{r} + 1)E_0^C - (2r + 3)E_0^A - (r - \tilde{r} + 1)\sum_j V_{AjC} + (r - 3\tilde{r} - 1)V_{C3} + 2(r - 2\tilde{r})V_{Cs} + (r + \hat{r} - 2\tilde{r} - 1)V_{\tilde{r}\hat{r}C} + 2(D - r - 2).$$

(34)

As stated by the convergence theorem, if the superficial degree of divergence of all the connected proper sub-diagrams of a given diagram $G$ is negative, then the Feynman integral corresponding to $G$ is absolutely convergent. Therefore, we have to find constraints (if they exist) over $r$, $\tilde{r}$ and $\hat{r}$, such that $\mathcal{D}_\Gamma$ is negative for all the possible diagrams at any loop order.

However, not all the diagrams are regularised this way. For example, the degree of divergence of the one-loop diagrams with just external $A$ lines is $D - E_A$, i.e. independent of the parameters $r$, $\tilde{r}$ and $\hat{r}$ and hence no conditions can be found for it to be negative. We will start with the analysis of diagrams with two or more loops in an arbitrary number of dimensions and, after, we will return to one-loop diagrams to show the finiteness of the theory in four dimensions only.

5.1 Multiloop graph analysis

In order for every possible 1PI diagram to have a negative $\mathcal{D}_\Gamma$, we can impose all coefficients in eq. (34) to be negative and, thus, get sufficient conditions. This amounts to the following relations

$$r > D - 2, \quad r < 2\tilde{r}, \quad \hat{r} < r + \tilde{r} + 1,$$

(35)

together with eq. (19). It is easy to see that there are integers $r$, $\tilde{r}$, $\hat{r}$ satisfying eqs (19), (35). To get proper bounds, $r$, $\tilde{r}$, $\hat{r}$ should be considered as real numbers, the restriction to integers being taken at the end. As a matter of fact it is consistent to take these parameters real having in mind more general cutoff functions (analytic around the origin, $p = 0$, and with asymptotic behaviour $e^{-1} \sim \frac{e^{-r}}{\Lambda^r}$ etc.).

The conditions (35) imply a lower bound on $\tilde{r}$, $\hat{r} > \frac{D}{2} - 1$, as well. Some of the relations (35) - those setting $D$-dependent lower bounds on $r$ and $\tilde{r}$ - may be expected to be also necessary, as the higher the space-time dimension, the more divergent the diagrams.

However, physics does not provide any reasonable arguments to explain upper bounds on $\tilde{r}$ and $r$, apart from $r \leq \hat{r}$ (cf eq. (19)). In fact they are not necessary, as we now show.

Let us denote by $\mathcal{S}$ the collection of triples $(r, \tilde{r}, \hat{r})$ such that $\mathcal{D}_\Gamma$ is negative for any given set of 1PI diagrams and eq. (13) holds.

Proposition: If $(r_0, \tilde{r}_0, \hat{r}_0) \in \mathcal{S}$, then the subset $\{(r, \tilde{r}, \hat{r}) \text{ s.t. } r \geq r_0, \tilde{r} = \tilde{r}_0, \hat{r} \geq \hat{r}_0, 0 < \tilde{r}_0 \leq r \leq \hat{r}_0\} \subset \mathcal{S}$. 

8
Proof:

The proof is essentially based on the one-particle-irreducibility of diagrams.

The whole dependence of eq. (34) on \( \hat{r} \) amounts to \( 2\hat{r} (E^C_\eta - V^C_\eta^2) \), which is always non-positive as it is not possible to have more external anti-ghost lines entering \( V^C_\eta^2 \) vertices than \( V^C_\eta^2 \) vertices themselves. Thus, increasing \( \hat{r} \) above \( \hat{r}_0 \) can only decrease an already negative \( D_\Gamma \).

As far as \( r \) is concerned, it enters eq. (34) as

\[
\begin{align*}
  r \left( -2L + 2 - E^C_\eta - 2E^C_\eta - 2E^A_\eta - \sum_j V_{\mathcal{A}j}^C + V_{\mathcal{C}j} + 2V_{\mathcal{C}j} + V_{\eta^2\mathcal{C}} \right) &= 2r \left( \sum_i V_{\mathcal{A}i} - I_\mathcal{A} \right),
\end{align*}
\]

(36)

where the last equality follows by using eq. (33) or directly from eq. (32).

This contribution is always non-positive as we know that in a 1PI diagram every \( V_{\mathcal{A}i} \) vertex must attach to at least two internal \( \mathcal{A} \) lines. Therefore increasing \( r \) above \( r_0 \) can only cause \( D_\Gamma \) to decrease further.

Using the above proposition, we see that the second and the third inequalities in eq. (35) are not necessary, and we are thus left with the sufficient relations

\[
\begin{align*}
  r &> D - 2, \\
  \hat{r} &> \frac{D - 2}{2} - 1 \quad \text{and} \quad \hat{r} \geq r \geq \hat{r} > 0.
\end{align*}
\]

For the case that the inverse cutoff functions are polynomials these inequalities imply

\[
\begin{align*}
  r &\geq D - 1, \\
  \hat{r} &\geq \left\lfloor \frac{D - 2}{2} \right\rfloor + 1,
\end{align*}
\]

(37)

\([x]\) being the integer part of \( x \). The above conditions are also necessary, as they ensure finiteness in the two two-loop vacuum diagrams with only \( \mathcal{A}^3 \) and \( \mathcal{C}^4 \) vertices respectively.

5.2 One loop diagram analysis

To perform the analysis of one-loop diagrams we adopt the strategy of divide and conquer: we cut them up into tadpole-like pieces, defined as the sub-diagrams which contain just one internal propagator attached to one vertex. This can be done in two different ways, according to which propagator remains attached to the vertex being cut (see fig. 1 for an example).

![Figure 1. Tadpole-like pieces.](image-url)
We then compute the degree of divergence of every possible piece we can end up with, aiming to show that they all contribute negatively to the overall $D_{\Gamma}$. If this is the case - and it is indeed - we are just left with the analysis of the simplest possible one-loop graphs, as any other can be obtained by adding tadpole-like pieces, which causes $D_{\Gamma}$ to decrease further.

In other words, we can always bound from above the degree of divergence of a one-loop diagram by removing tadpole-like pieces one by one and joining together the rest of the diagram - hence increasing the overall $D_{\Gamma}$. Eventually we will be left with a very simple graph, usually a proper tadpole.

The first part of the analysis, that is calculating $D_{\Gamma}$ for such “components”, is straightforward: by inspection of eq. (31) it is easy to appreciate that all the possible sub-diagrams contribute negative terms within the bounds we have already set on $r$, $\tilde{r}$ and $\hat{r}$. Attaching an $A$ propagator to a $V_{\eta^2A}$ vertex is the only case which needs some comment. In this case $D_{\Gamma} = 2\hat{r} - 2r - 2\tilde{r} - 1$ can be positive if $\hat{r}$ is large enough. However, it is not possible to add just a single $V_{\eta^2A}$ vertex, since it would require the introduction of just a single external ghost line, which is forbidden. Adding two such vertices make such a contribution convergent as seen in the correction introduced in eq. (32).

The second part of the analysis, i.e. showing that all the simplest possible one-loop diagrams can be regulated by a suitable choice of $r$, $\tilde{r}$ and $\hat{r}$, is quite long but straightforward as well. We find two further constraints on the ranks of the cutoff functions

$$r - \tilde{r} > \frac{D}{2} - 1, \quad \hat{r} - \hat{r} > \frac{D}{4} - 1,$$

which come from the graphs sketched in fig. 2.

In order to deal with the diagrams that remain unregularised despite the introduction of (covariantised) cutoff functions, we now specialise to $D = 4$. Those graphs, with no external $C$ or $\eta$ lines and with up to 4 external $A$ legs, are finite due to the cancellation of the ultraviolet divergences between the contributions of the bosonic and fermionic propagators corresponding to internal lines. This cancellation will be referred to as the supertrace mechanism.

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*a*except those whose $D_{\Gamma}$ is independent of the ranks of the cutoff functions, e.g. graphs with no external $C$ or $\eta$ lines and with up to $D$ external $A$ legs.
To illustrate it, let us sketch the calculation of the one-loop diagram with two external $A$-lines and two internal $A$-lines. The terms of the perturbation theory expansion which generate this type of diagram, schematically omitting the Lorentz indices, involve the product of two vertices:

$$\text{str} ([A(x), A(x)] [A(y), A(y)]),$$

where $A$ stands for the gauge field or its derivative. The leading part of the propagator between the $A^A$ and $A^B$ fields in the momentum representation is proportional to $g^{AB}$. Using the completeness relation for the generators $T_A$ it is easy to show that by Wick pairing

$$\text{str}(XA(x))\text{str}(A(y)Y) = \left[ \frac{1}{2} \text{str}(X) - \frac{1}{4N} (\text{tr}X \text{str}Y + \text{str}X \text{tr}Y) \right] \times \Delta(x - y),$$

where $\Delta(x - y)$ is a space-time dependent factor coming from the propagator. Applying this formula to eq. (39) one can see that after the first pairing it reduces to

$$\frac{1}{2} \text{str} ([A(x), A(x)][A(y), A(y)]) \Delta(x - y).$$

Here we have used the cyclicity property of the supertrace, $\text{str}(XY) = \text{str}(YX)$, which implies that $\text{str}([X, Y]) = 0$. For the next step we use the identity

$$\text{str}(XTA) = \frac{1}{2} \text{str}(X) - \frac{1}{4N} [\text{str}(X\sigma_3 Y) + \text{str}(X Y \sigma_3)],$$

valid for any super-matrices $X$ and $Y$, which follows from the already mentioned completeness relation. Using this identity we calculate the second $A$-$A$ pairing in eq. (41) and find that the $\sigma_3$ terms appearing in eq. (42) all cancel, as they must – to preserve the $SU(N|N)$ invariance, leaving only terms of the form $\text{str}A\text{str}A$ or $\text{str}A\text{str}1$, both of which vanish because $\text{str}A = \text{str}1 = 0$. This is a demonstration of the supertrace mechanism at work.

One can check by direct calculation that the supertrace mechanism ensures the finiteness of all the diagrams with two and three external $A$-lines. For the diagrams with 4 external $A$-lines the supertrace mechanism is not sufficient (at finite $N$). However, these are already finite. This follows because gauge invariant effective vertices containing less than four $A$s have already been shown to be finite but gauge invariant effective vertices with a minimum of four $A$s are already finite by power counting and the Ward identities for the $SU(N|N)$ gauge theory.

To summarise, in order to enforce finiteness on all diagrams in four dimensions we need to impose the following constraints (cf eq. 43)

$$r \geq 3, \quad \tilde{r} \geq 2, \quad r - \tilde{r} > 1, \quad \tilde{r} - \tilde{r} > 0, \quad \hat{r} \geq r \geq \tilde{r} > 0.$$  

6 Summary and conclusions

We have analyzed the $SU(N|N)$ gauge theory with the higher covariant derivative regulators $c(-\nabla^2/\Lambda^2)$ and Higgs field, viewed as a regularised version of the $SU(N)$

\footnote{Beware that the commutators do not vanish once these are taken into account!}
Yang-Mills theory. Its structure is determined by the requirement that it can be used within a manifestly gauge invariant formulation of the ERG.

The extension includes the physical Yang-Mills field \( A_\mu^1 \equiv A_\mu \) of the initial theory and the regulating fields: the bosonic gauge field \( A_\mu^2 \), the fermionic Pauli-Villars field \( B_\mu \) and the scalar Pauli-Villars fields \( C_i \). All the regulator fields except \( A_\mu^2 \) acquire masses proportional to the momentum cutoff \( \Lambda \) via the Higgs mechanism. The presence of the unphysical regulator fields lead to a source of unitarity violation in the theory with finite cutoff. However, when the regularisation is removed, i.e. in the limit \( \Lambda \to \infty \), the massive fields \( B_\mu \) and \( C_i \) become infinitely heavy and decouple. As a consequence the physical sector, which is the original \( SU(N) \) Yang-Mills theory, becomes decoupled from the unphysical sector of the field \( A_\mu^2 \).

In this way the unitarity of the theory is restored in the continuum limit, i.e. when \( \Lambda \to \infty \).

We showed that in four dimensions the one-loop one-particle-irreducible diagrams with two, three or four external \( A \)-lines are finite due to the peculiar structure of the \( SU(N|N) \) supergauge group. The rest of the one-loop one-particle-irreducible diagrams and all one-particle-irreducible diagrams with the number of loops \( L \geq 2 \) can be made finite by the proper choice of the regulating functions \( c^{-1} \), \( \tilde{c}^{-1} \) and \( \hat{c}^{-1} \). The necessary and sufficient conditions on the parameters of these functions are given in eqs (13). While we only demonstrate finiteness in four dimensions, we are confident it can be extended up to eight dimensions. At this point the Ward identities regarding the 1PI function with four external \( A \)-legs no longer guarantee the finiteness of such contribution.

We expect the use of the \( SU(N|N) \) regularised extension of the \( SU(N) \) pure Yang-Mills theory to open up new possibilities of non-perturbative and gauge-invariant treatment of Yang-Mills theories in the framework of the ERG approach.

Acknowledgements

We would like to thank J.I. Latorre and C. Wetterich for discussions and valuable comments. Yu.K. acknowledges financial support from the PPARC grant PPA/V/S/1998/00907 and from the Programme "Universities of Russia" (grant 990588). T.R.M. and S.A. acknowledge support from PPARC SPG PPA/G/S/1998/00527. J.F.T. thanks PPARC for support through a studentship.

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