1. Introduction

The purpose of this paper is to introduce new families of Bethe ansatz integrable exclusion and zero-range processes on the one-dimensional lattice $\mathbb{Z}$. Our construction generalizes the $q$-Hahn Boson (zero-range) process introduced in [Pov13] and the $q$-Hahn TASEP further studied in [Cor14], by allowing jumps in both directions. Under mild assumptions on the microscopic dynamics, such random particle systems are expected to lie in the KPZ universality class. In particular, when started from step initial data, the positions of particles in the bulk of the rarefaction fan are expected to fluctuate according to Tracy-Widom type statistics, up to scaling constants depending on microscopic dynamics. Presently, universality predictions can be confirmed only for a small number of exactly solvable models. A greater variety of well-understood integrable models, with more and more degrees of freedom, is certainly useful towards the study of interacting particle systems under general assumptions. Another application of the study of integrable models is to better understand the cases which are not covered by the KPZ scaling theory. For instance, for an exclusion process starting from step initial data, the statistics of the location of the first particle does not yet fit into a universal framework.

The $q$-Hahn TASEP is a discrete-time exclusion process on $\mathbb{Z}$, depending on three parameters $q \in (0,1)$ and $0 \leq \nu < \mu < 1$. Each particle jumps independently, and chooses randomly its next location on the right, according to a discrete probability distribution with parameters $(q, \mu, \nu)$.
(see [9] for the expression of the weights \( \varphi_{q,\mu,\nu}(j|m) \)). This distribution is very similar to the weight function for the \( q \)-Hahn orthogonal polynomials, hence the name. The main reason why the solvability of the \( q \)-Hahn TASEP extends to the partially asymmetric case we consider is that many properties of the transition matrix are preserved by inversion of the parameters \( q, \mu, \nu \). By taking a limit when \( \mu \) goes to \( \nu \) and rescaling the time, the resulting partially asymmetric process is solvable via Bethe ansatz. One obtains closed formulas for the expectation of observables such as \( q^{x_{n}(t)} \), where \( x_{n}(t) \) is the position of the \( n \)th particle at time \( t \), using techniques developed by Borodin and Corwin in the context of Macdonald processes [BC14]. Further following those techniques, one arrives at Fredholm determinant formulas for the distribution of \( x_{n}(t) \), which can be analysed asymptotically.

**Main results.** In order to give an overview of our results, let us introduce our main model in an informal setting. A more precise definition of the process, that we call the (continuous time) \( q \)-Hahn asymmetric exclusion process (\( q \)-Hahn AEP) as well as a discussion about its existence is provided in Section 4.1.

For any \( q \in (0,1) \) and \( 0 \leq \nu < 1 \) and asymmetry parameters \( R, L \geq 0 \) with \( R + L = 1 \), the \( q \)-Hahn AEP is a continuous-time Markov process on configurations of particles

\[
+\infty = x_{0}(t) > x_{1}(t) > x_{2}(t) > \cdots > x_{n}(t) > \cdots \ ; \ x_{i} \in \mathbb{Z}.
\]

The \( n \)th particle, located at \( x_{n}(t) \) jumps on the right to the location \( x_{n}(t)+j \) at rate (i.e. according to independent exponentially distributed waiting times with rate) \( \phi_{q,\nu}^{R}(j|x_{n-1}(t)-x_{n}(t)-1) \) for all \( j \in \{ 1, \ldots, x_{n-1}(t)-x_{n}(t)-1 \} \), and jumps on the left to the location \( x_{n}(t)-j' \) at rate \( \phi_{q,\nu}^{L}(j'|x_{n}(t)-x_{n+1}(t)-1) \) for all \( j' \in \{ 1, \ldots, x_{n}(t)-x_{n+1}(t)-1 \} \). Figure 1 shows two possible jumps for \( x_{n}(t) \).

The rates \( \phi_{q,\nu}^{R}(j|m) \) and \( \phi_{q,\nu}^{L}(j|m) \), defined for all integers \( 1 \leq j \leq m \), are not arbitrary. To ensure the exact solvability of the process, they are constructed as limits of the \( q \)-Hahn distribution:

\[
\phi_{q,\nu}^{R}(j|m) := R \frac{\nu^{j-1}(\nu; q)_{m-j}}{[j]_{q} (\nu; q)_{m} (q; q)_{m-j}},
\]

\[
\phi_{q,\nu}^{L}(j|m) := L \frac{1}{[j]_{q} (\nu; q)_{m} (q; q)_{m-j}},
\]

The \( q \)-Pochhammer symbol \( (a; q)_{n} \) is defined in Section 2. Note that the superscript \( R \) (resp. \( L \)) on \( \phi_{q,\nu}^{R} \) (resp. \( \phi_{q,\nu}^{L} \)) is not an exponent. It only highlights the dependency on the asymmetry parameters \( R, L \). The reader is referred to Section 4.1 for a further discussion on the definition of the \( q \)-Hahn AEP and the expression for the rates above. Before stating our main formulas regarding this model, we briefly introduce two degenerations.

Setting \( \nu = 0 \), if \( L = 0 \), the rates of jumps to the right have the simple form

\[
\phi_{q,\nu}^{R}(j|x_{n-1}(t)-x_{n}(t)-1) = (1 - q^{x_{n-1}(t)-x_{n}(t)-1}) \mathbb{1}_{\{j=1\}},
\]

matching those of \( q \)-TASEP [BC14]. A further limit when the parameter \( q \) goes to zero leads to the well-studied totally asymmetric simple exclusion process (TASEP). However, when \( L > 0 \), jumps on the left are long-range. Hence our two-sided dynamics are different from those of the classical asymmetric simple exclusion process, but rather generalize the PushASEP [BF08].

![Figure 1. Two admissible jumps for the \( n \)th particle in the \( q \)-Hahn asymmetric exclusion process.](image-url)
Setting $\nu = q$, the rates of jumps no longer depend on the distance to the neighbouring particles. The $n$th particle jumps on the right to the location $x_n(t) + j$ at rate $R/[j]_q$ and jumps on the left to the location $x_n(t) - j'$ at rate $L/[j']_q$, where $[j]_q$ and $[j']_q$ are $q$-deformed integers (see Section 2). An example of some possible jumps is shown in Figure 2. One of our motivations for studying this model is that it has been known to be exactly solvable for a long time. Indeed, Sasamoto and Wadati [SW98] introduced a one-parametric family of zero-range processes diagonalizable via Bethe ansatz, called the multi-particle asymmetric diffusion model (MADM). Using a classical coupling between zero-range and exclusion processes that maps the gaps between consecutive particles to location $x_i - x_{i+1} - 1$ in the exclusion process with the population of the $i$th site in the zero-range process, the MADM corresponds to the $q$-Hahn AEP with $R = q/(1 + q)$ and $L = 1/(1 + q)$ (and $\nu = q$). It was later extended to arbitrary asymmetry parameters $R, L > 0$ [AKK99], and further studied in [Lee12]. We call this model the MADM exclusion process. Until now, no formulas amenable to asymptotic analysis have been written down for these systems.

Referring to the general $q, \nu$ setting, we also introduce in Section 4.1 a $q$-Hahn asymmetric zero-range process ($q$-Hahn AZRP) on $\mathbb{Z}$ with a finite number of particles. The dynamics are defined in order to correspond to the $q$-Hahn AEP via exclusion/zero-range transformation. Owing to a Markov duality between the $q$-Hahn AEP and the $q$-Hahn AZRP, and the Bethe ansatz solvability of the $q$-Hahn AZRP, we are able to prove the following moment formula for the locations of particles in the exclusion process.

**Proposition 1.1.** Fix $q \in (0, 1)$, $0 \leq \nu < 1$, and an integer $k$. Consider the continuous time $q$-Hahn AEP started from step initial data (i.e. $x_n(0) = -n$ for $n \geq 1$). Then for any $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$,

\[
\mathbb{E} \left[ \prod_{i=1}^{k} q^{x_{n_i}(t) + n_i} \right] = (-1)^k q^{k(k-1)/2} \prod_{i=1}^{k} z_A z_B \prod_{1 \leq A < B \leq k} \frac{z_A - z_B}{z_A - qz_B} \prod_{j=1}^{k} \left( \frac{1 - \nu z_j}{1 - z_j} \right)^{n_j} \exp \left( (q - 1)t \left( \frac{R z_j}{1 - \nu z_j} - \frac{L z_j}{1 - z_j} \right) \right) \int \frac{dz_j}{z_j(1 - \nu z_j)}. \tag{1}
\]

where the integration contours $\gamma_1, \ldots, \gamma_k$ are chosen so that they all contain 1, $\gamma_A$ contains $q\gamma_B$ for $B > A$ and all contours exclude 0 and $1/\nu$.

In Section 3 we also introduce discrete-time versions of the $q$-Hahn AEP and the $q$-Hahn AZRP as two-sided generalizations of the discrete time $q$-Hahn TASEP and the $q$-Hahn Boson from [Pov13, Cor14]. Although the duality between both processes extends to the two-sided case, our method to derive (1) does not exactly apply to the discrete-time $q$-Hahn AEP, and it is not presently clear if such an integral formula holds in that case.

Following the techniques of [BC14] we deduce from Proposition 1.1 the following theorem, which provides an exact formula for the $q$-Laplace transform of $q^{x_n(t)}$. 

![Figure 2. Rates of a few admissible jumps in the exclusion process corresponding to the multi-particle asymmetric diffusion model (MADM exclusion process).](image-url)
Theorem 1.2. Consider the $q$-Hahn AEP started from step initial data: $\forall n \in \mathbb{Z}_{>0}, x_n(0) = -n$. Then for all $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, we have the “Mellin-Barnes-type” Fredholm determinant formula
\[
\mathbb{E}\left[ \frac{1}{(\zeta q^{x_n(t)+n}; q)_{\infty}} \right] = \det (I + K_{\zeta})
\]
where $\det (I + K_{\zeta})$ is the Fredholm determinant of $K_{\zeta} : L^2(C_1) \to L^2(C_1)$ for $C_1$ a positively oriented circle containing 1 with small enough radius so as to not contain 0, $1/q$ and $1/\nu$. The operator $K_{\zeta}$ is defined in terms of its integral kernel
\[
K_{\zeta}(w, w') = \frac{1}{2\pi i} \int_{i\infty+1/2}^{i\infty-1/2} \left( -\zeta \right)^s g(w) \frac{1}{g(q^s w) q^s w - w'} ds
\]
with
\[
g(w) = \left( \frac{(\nu w; q)_{\infty}}{(w; q)_{\infty}} \right)^n \exp \left( (q-1)t \sum_{k=0}^{\infty} \frac{R \nu w q^k}{1 - \nu w q^k} - L \frac{w q^k}{1 - w q^k} \right) \frac{1}{(\nu w; q)_{\infty}}.
\]

When the asymmetry parameters $R$ and $L$ of the $q$-Hahn AEP are set to $R = 1$ and $L = 0$, particles can jump only to the right. An application of the law of large numbers and the classical central limit theorem shows that there exist constants $\pi$ and $\sigma$ such that $x_1(t)/t$ converges almost surely to $\pi$ and we have the convergence in distribution as $t$ goes to infinity
\[
\frac{x_1(t) - \pi t}{\sigma \sqrt{t}} \Rightarrow N(0, 1).
\]
Such a result is true in particular for the TASEP, but what happens if one allows jumps to the left? Theorem 2 in [TW09] shows that for ASEP, that is if one allows nearest-neighbour jumps to the left, the position of the first particle still fluctuates on a $\sqrt{t}$ scale, but the limiting law is not Gaussian.

An asymptotic analysis of the Fredholm determinant in Theorem 1.2 when $\nu = q$ shows that the situation is very different when one allows long-range jumps to the left.

Theorem 1.3. Consider the MADM exclusion process started from step initial condition. For asymmetry parameters $R$ and $L = 1 - R$ such that $R_{\text{min}}(q) < R < 1$, where $R_{\text{min}}(q)$ is an explicit bound depending on the parameter $q$ (see Theorem 6.3 and Remark 6.8 for a more precise statement), we have
\[
\lim_{t \to \infty} \mathbb{P}\left( \frac{x_1(t) - \pi t}{\sigma t^{1/3}} \geq x \right) = F_{\text{GUE}}(-x),
\]
where $\pi$ and $\sigma > 0$ are explicit constants depending on $R$ and $q$, and $F_{\text{GUE}}(x)$ is the distribution function of the GUE Tracy-Widom distribution (see Definition 6.7).

Theorem 1.3 is proved in Section 6 as Theorem 6.3. The asymptotic analysis of the Fredholm determinant also allows for a similar result for particles in the bulk of the rarefaction fan. The following theorem about fluctuation of particles positions in the rarefaction fan is also proved in Section 6 as Theorem 6.2.

Theorem 1.4. Consider the MADM exclusion process started from step initial condition, for asymmetry parameters $R$ and $L = 1 - R$ such that $R > L \geq 0$. Assume that $\theta \in (0, +\infty)$ parametrizes the position in the rarefaction fan (see Section 5 and Theorem 6.3 for a more precise statement). There exists explicit constants $\kappa(\theta), \pi(\theta)$ and $\sigma(\theta)$, such that under the additional hypothesis $q^\theta > 2q/(1 + q)$, then for $n = \lfloor \kappa(\theta) t \rfloor$, we have
\[
\lim_{t \to \infty} \mathbb{P}\left( \frac{x_n(t) - \pi(\theta)t}{\sigma(\theta)t^{1/3}} \geq x \right) = F_{\text{GUE}}(-x).
\]
Figure 3. Density profile $x \mapsto \rho(x)$ for a $q$-Hahn AEP with $q = \nu = 0.6$, and asymmetry parameters $R = 0.8$ and $L = 0.2$, starting from step initial data. The density $\rho(x)$ has to be understood as the local density of particles at time $t$ around site $xt$ for very large $t$.

The expressions of the model-dependent constants $\kappa(\theta), \pi(\theta)$ and $\sigma(\theta)$ as functions of $\theta$ confirm the predictions of KPZ scaling theory (see Section 5).

**Remark 1.5.** Lee recently posted a preprint on arXiv [Lee14] where a similar asymptotic result is proposed for an infinite volume MADM which is different from the one discussed in the present paper. Although Theorem 1.4 is not in contradiction with [Lee14], the present authors pointed out fundamental issues in the proof. In particular, the weak law of large numbers implied by the limit theorem [Lee14, Theorem 1.3] does not agree with the particle dynamics considered. At the time of posting of the present article, no revision remedying these issues have been made.

Theorem 1.4 implies as a corollary a weak law of large numbers: for $n = \lceil \kappa(\theta)t \rceil$,

$$\frac{x_n(t)}{t} \overset{p}{\longrightarrow} \pi(\theta).$$

This law of large numbers implies a macroscopic density profile of the rarefaction fan as in Figure 3. The density profile in the partially asymmetric case (that is when $R > L > 0$) is discontinuous. Such a discontinuity of the macroscopic density profile has previously been exhibited in certain particle systems (e.g. [GKR10] studies a facilitated exclusion process for which the density of particles stays above 1/2 when starting from step initial condition). However, to the authors knowledge, Theorem 1.3 provides the first limit theorem for the fluctuations of locations of particles at a downward (i.e. antishock) discontinuity of the density profile.

One can give a soft argument explaining why a discontinuity is present in the density profile. The rate at which the first particle jumps to the right is

$$\sum_{j=1}^{\infty} \phi_{q,\nu}^R(j + \infty) < \infty.$$

The rate at which the first particle jumps to the left is

$$\sum_{j=1}^{m} \phi_{q,\nu}^L(j|m) \xrightarrow{m \to +\infty} +\infty,$$

where $m = x_1(t) - x_2(t) - 1$. Thus, even if particles have a drift to the right because $R > L$, the first particle stays with high probability at a bounded distance from the second particle, and hence the density around the first particles is strictly positive.
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Outline of the paper. In Section 2 we provide definitions and establish useful identities for some $q$-deformed special functions that appear naturally in the next sections. In Section 3, we define a two-sided generalization of the discrete time $q$-Hahn TASEP from [Pov13, Cor14]. In Section 4, we introduce the $q$-Hahn AEP as a continuous time limit of the discrete-time exclusion process defined in Section 3 and establish the Fredholm determinant identity of Theorem 1.2. In Section 5, we study this process from the point of view of the conjectural KPZ scaling theory, and we state the predicted limit theorems. We sketch an asymptotic analysis of the Fredholm determinant, leading to the predicted Tracy-Widom limit theorem. In Section 6, we make a rigorous asymptotic analysis in the case $\nu = q$, which corresponds to the MADM, thus proving Theorems 1.3 and 1.4.

2. Preliminaries on the $q$-deformed gamma and digamma functions

We first recall classical notations from the theory of $q$-analogues. Fix hence forth that $q \in (0, 1)$. For $a \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$, define the $q$-Pochhammer symbol

$$ (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i) \quad \text{and} \quad (a; q)_{\infty} = \prod_{i=0}^{\infty} (1 - aq^i). $$

For an integer $n$, the $q$-integer $[n]_q$ is

$$ [n]_q = 1 + q + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q}. $$

The $q$-factorial is defined as

$$ [n]_q! = [n]_q[n - 1]_q \cdots [1]_q = \frac{(q; q)_n}{(1 - q)^n}. \quad (3) $$

The $q$-binomial coefficients are

$$ \binom{n}{k}_q = \frac{[n]_q!}{[n - k]_q! [k]_q!} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}. $$

For $|z| < 1$, the $q$-binomial theorem [AAR99, Theorem 10.2.1] implies that

$$ \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} z^k = \frac{(az; q)_{\infty}}{(z; q)_{\infty}}. \quad (4) $$

The $q$-gamma function is defined by

$$ \Gamma_q(z) = (1 - q)^{1-z} \frac{(q; q)_{\infty}}{(z; q)_{\infty}}, $$

and the $q$-digamma function is defined by

$$ \Psi_q(z) = \frac{\partial}{\partial z} \log \Gamma_q(z). $$

From the definition of the $q$-digamma function, we have a series representation for $\Psi_q$,

$$ \Psi_q(z) = \frac{d}{dz} \log \Gamma_q(z) = -\log(1 - q) + \log(q) \sum_{k=0}^{\infty} \frac{q^{k+z}}{1 - q^{k+z}}. \quad (5) $$
Let us also define a closely-related series that will appear in Section 5,
\[ G_q(z) := \sum_{i=1}^{\infty} \frac{z^i}{[i]_q}. \]

**Lemma 2.1.** For \( z \in \mathbb{C} \) with positive real part,
\[ G_q(z) = \frac{1 - q}{\log q} \left( \Psi_q(z) + \log(1 - q) \right). \]  
(6)

For \( z \in \mathbb{C} \) with real part greater than \(-1\),
\[ G_q^{-1}(z) = \frac{q^{-1} - 1}{\log q} \left( \Psi_q(z + 1) + \log(1 - q) \right). \]  
(7)

**Proof.** Assume \( z \in \mathbb{C} \) with positive real part. Using the series representation (5), we have that
\[ \frac{1 - q}{\log q} \left( \Psi_q(z) + \log(1 - q) \right) = (1 - q) \sum_{k=0}^{\infty} \frac{q^{k+z}}{1 - q^{k+z}}. \]
Since \( z \) has positive real part, we can write for all \( k \geq 0 \)
\[ \frac{q^{k+z}}{1 - q^{k+z}} = \sum_{i=1}^{\infty} q^{(k+z)i}, \]
so that the right-hand-side in (6) equals
\[ (1 - q) \sum_{k=0}^{\infty} \sum_{i=1}^{\infty} q^{(k+z)i}. \]
Exchange the summations yields
\[ \frac{1 - q}{\log q} \left( \Psi_q(z) + \log(1 - q) \right) = (1 - q) \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} q^{(k+z)i} = \sum_{i=1}^{\infty} \frac{(q^z)^i}{[i]_q}. \]
Equation (7) can be deduced from (6) replacing \( z \) by \( z + 1 \). \( \square \)

A consequence of Lemma 2.1 is the following formula for the \( k \)-fold derivatives of the \( q \)-digamma function:
\[ \Psi_q^{(k)}(z) = (\log q)^{k+1} \sum_{n=1}^{\infty} \frac{n^k q^n z^n}{1 - q^n}. \]  
(8)

3. A discrete time asymmetric \( q \)-Hahn process

Let us recall the definition of the \( q \)-Hahn-TASEP [Pov13, Cor14]. Fix \( q \in (0, 1) \) and \( 0 \leq \nu < \mu < 1 \). Then the \( N \)-particle \( q \)-Hahn TASEP is a discrete time Markov chain \( \vec{x}(t) = \{ x_n(t) \}_{n=0}^{N} \in \mathbb{X}^N \) where the state space \( \mathbb{X}^N \) is
\[ \mathbb{X}^N = \{ +\infty = x_0 > x_1 > \cdots > x_N ; \forall n \geq 1, x_n \in \mathbb{Z} \}. \]
At time \( t+1 \), each coordinate \( x_n(t) \) is updated independently and in parallel to \( x_n(t+1) = x_n(t) + j_n \) where \( 0 \leq j_n \leq x_{n-1}(t) - x_n(t) - 1 \) is drawn according to the \( q \)-Hahn probability distribution. The \( q \)-Hahn probability distribution on \( j \in \{0, 1, \ldots, m \} \) is defined by
\[ \varphi_{q,\mu,\nu}(j|m) = \mu^j \left( \frac{\nu/\mu; q}_{j}\left( \mu; q \right) \right) \left( \frac{m}{j} \right)_q. \]  
(9)
The exact solvability of the \( q \)-Hahn TASEP comes from:
A Markov duality with a $q$-Hahn totally asymmetric zero-range process ($q$-Hahn TAZRP), which is essentially the same process, but described by the evolution of gaps between consecutive particles.

(2) The solvability of this $q$-Hahn zero-range process via the Bethe ansatz. Indeed the $q$-Hahn TAZRP was introduced by Povolotsky in [Pov13] as the most general parallel update discrete time totally asymmetric ‘chipping’ model on a ring lattice with factorized invariant measures which is solvable via Bethe ansatz.

(3) The ability to express the $q$-Laplace transform as a Fredholm determinant using techniques introduced in the context of Macdonald processes [BC14].

In this section, we introduce a generalization of the $q$-Hahn TASEP allowing jumps in both directions such that the duality is preserved. Proposition 1.2 in [Cor14], shows that certain ‘q-moments’ of the $q$-Hahn probability distribution enjoy a symmetry relation, which is ultimately responsible for an intertwining (and hence Markov duality) of the Markov generators of the $q$-Hahn Boson model and the $q$-Hahn TASEP:

$$\sum_{j=0}^{m} \varphi_{q,\mu,\nu}(j|m)q^{jm} = \sum_{j=0}^{y} \varphi_{q,\mu,\nu}(j|y)q^{jm}. \quad (10)$$

The same identity replacing all variables by their inverse also holds:

$$\sum_{j=0}^{m} \varphi_{q^{-1},\mu^{-1},\nu^{-1}}(j|m)q^{-jm} = \sum_{j=0}^{y} \varphi_{q^{-1},\mu^{-1},\nu^{-1}}(j|y)q^{-jm}. \quad (11)$$

The weights $\varphi_{q,\mu,\nu}(j|m)$ and $\varphi_{q^{-1},\mu^{-1},\nu^{-1}}(j|m)$ define probability distributions on $j \in \{0, 1, \ldots, m\}$. Notice also that

$$\varphi_{q^{-1},\mu^{-1},\nu^{-1}}(j|m) = \left(\frac{\nu}{\mu}\right)^m \frac{1}{\mu^j} \varphi_{q,\mu,\nu}(j|m).$$

These observations motivate the introduction of a two-sided $q$-Hahn process where jumps to the left are distributed according to a $q$-Hahn distribution with parameters $q^{-1}, \mu^{-1}, \nu^{-1}$.

Notice that one can extend the $q$-Hahn weights by continuity when $\nu$ goes to zero. Thus,

$$\varphi_{q,\mu,0}(j|m) = \mu^j (\mu; q)_{m-j} \left[ \frac{m}{j} \right]_q \quad \text{and} \quad \varphi_{q^{-1},\mu^{-1},\infty}(j|m) = \mathbb{1}_{j=m}. \quad (12)$$

We now define two-sided generalizations of the $q$-Hahn Boson process and the $q$-Hahn TASEP.

**Definition 3.1.** The discrete-time $q$-Hahn asymmetric zero-range process (abbreviated discrete $q$-Hahn AZRP) is a discrete-time Markov chain $\vec{y}(t) = \{y_i(t)\}_{i=0}^{\infty} \in \mathbb{Y}^\infty$, where

$$\mathbb{Y}^\infty = \left\{ (y_0, y_1, \ldots) ; \forall i \in \mathbb{Z}_{\geq 0}, \, y_i \in \mathbb{Z}_{\geq 0} \text{ and } \sum_{i=0}^{\infty} y_i < \infty \right\}.$$

At time $t$, $y_i(t)$ particles are above site $i$ for all $i \geq 0$. At time $t+1$, $\vec{y}(t)$ is updated to another state $\vec{y}(t+1)$ according to the following dynamics. For each site $i$, independently and in parallel, $s_i \in \{0, \ldots, y_i(t)\}$ particles are transferred to site $i-1$ with probability $a \cdot \varphi_{q,\mu,\nu}(s_i|y_i(t))$ or $t_i \in \{0, \ldots, y_i(t)\}$ particles are transferred to site $i+1$ with probability $b \cdot \varphi_{q^{-1},\mu^{-1},\nu^{-1}}(t_i|y_i(t))$. The
variables $a$ and $b$ are asymmetry parameters such that $a + b = 1$, so that all of the above probabilities sum to 1. No particles are transferred out of site zero. Note that when $b = 0$, the $q$-Hahn AZRP reduces to the $q$-Hahn TAZRP from [Pov13]. Following the notations used in [Cor14], we denote by $P_{q,\mu,\nu}^{AZRP}$ the transition matrix of the discrete $q$-Hahn AZRP. It acts on functions $f : \mathbb{Y}^\infty \to \mathbb{R}$.

The discrete-time $q$-Hahn asymmetric exclusion process (abbreviated discrete $q$-Hahn AEP) is a discrete time Markov chain $\bar{x}(t) = \{x_n(t)\}_{n=0}^\infty \in \mathbb{X}^\infty$, where

$$
\mathbb{X}^\infty = \left\{ +\infty = x_0 > x_1 > \cdots > x_n > \cdots \mid \forall n \geq 1, x_n \in \mathbb{Z} \right\} \cup \left\{ \mathbb{Z}^N > 0, \forall n \geq N, x_n - x_{n+1} = 1 \right\}.
$$

In words, $\mathbb{X}^\infty$ is the space of particle configurations that have a right-most particle and a left-most empty site.

At time $t$, $x_n(t)$ records the location of particle $n$, for each $n \geq 0$. At time $t + 1$, $\bar{x}(t)$ is updated to another state $\bar{x}(t + 1)$ according to the following dynamics. Each particle $n$ can jump, independently and in parallel, to the position $x_n(t + 1) = x_n(t) + j_n \cdot \epsilon_n$, $j_n \in \{0, \ldots, x_n(t) - x_{n-1}(t) - 1\}$, with probability $a \cdot \phi_{q,\mu,\nu}(j_n | x_{n-1}(t) - x_n(t) - 1)$ or to the position $x_n(t + 1) = x_n(t) - k_n$ if $x_n(t) - x_{n+1}(t) - 1$, with probability $b \cdot \phi_{q^{-1},\mu^{-1},\nu^{-1}}(k_n | x_n(t) - x_{n+1}(t) - 1)$, where $a + b = 1$. We denote by $P_{q,\mu,\nu}^{AEP}$ the transition matrix of the discrete $q$-Hahn AEP. It acts on functions $f : \mathbb{X}^\infty \to \mathbb{R}$.

**Remark 3.2.** The discrete-time $q$-Hahn AZRP (resp. $q$-Hahn AEP) could be defined on a larger state space including configurations with an infinite number of particles (resp. an infinite number of positive gaps between consecutive particles). Such a more general definition would add some complexity in several of the later statements. In the following, we study the zero-range processes only with a finite number of particles and the exclusion process starting only from the step-initial condition $(\forall n > 0, x_n(0) = -n)$, thus we prefer to restrict our definition to the state-spaces $\mathbb{X}^\infty$ and $\mathbb{Y}^\infty$.

**Remark 3.3.** Because of the two-sided jumps, the study of asymmetric exclusion processes such as the $q$-Hahn AEP cannot be reduced to the study of finitely many particles, in contrast with totally asymmetric processes such as (q-Hahn) TASEP.

Proposition 3.3 states an intertwining of the transition matrices of the $q$-Hahn AEP and $q$-Hahn AZRP, which implies a Markov duality between these processes.

**Proposition 3.4.** Fix the asymmetry parameters $a, b \geq 0$ such that $a + b = 1$, $q \in (0, 1)$, and $0 < \nu < \mu < 1$. Define $H : \mathbb{X}^\infty \times \mathbb{Y}^\infty \to \mathbb{R}$ as

$$
H(\bar{x}, \bar{y}) := \prod_{i=0}^{\infty} q^{(x_i + i)y_i}, \quad (13)
$$

with the convention that the product is 0 when $y_0 > 0$. Then, $H$ intertwines the discrete $q$-Hahn AZRP and the discrete $q$-Hahn AEP in the sense that

$$
P_{q,\mu,\nu}^{AEP} H = H(P_{q,\mu,\nu}^{AZRP})^T,
$$

where $T$ denotes the matrix transpose.

**Remark 3.5.** There exists a simpler relation between the discrete $q$-Hahn AZRP and the discrete $q$-Hahn AEP by coupling the gaps between consecutive particles in the exclusion process and the number of particles on the sites of the zero-range process. This relation holds in general for exclusion and zero range processes. Our Proposition 3.4 is independent of this exclusion/zero-range transformation, and does not hold for any exclusion process.
Proof. This proof is adapted from that of Theorem 2.1 in [Cor14]. Since the jumps of particles occur in parallel, the generator $P_{q,\mu,\nu}^{AEP}$ acts on each coordinate independently, and we have

$$P_{q,\mu,\nu}^{AEP} H(\vec{x}, \vec{y}) = \prod_{i=1}^{\infty} \left( a \cdot \sum_{j_i=0}^{y_i} \varphi_{q,\mu,\nu}(j_i|x_{i-1} - x_i - 1)q^{j_i}y_i + b \cdot \sum_{k_i=0}^{x_i-x_{i+1}-1} \varphi_{q^{-1},\mu^{-1},\nu^{-1}}(k_i|x_i - x_{i+1} - 1)q^{-k_i}y_i \right) \prod_{i=0}^{\infty} q^{(x_{i+1})y_i}.$$

Applying Equations (10) and (11) to the terms inside the parenthesis, we find that

$$P_{q,\mu,\nu}^{AEP} H(\vec{x}, \vec{y}) = \prod_{i=1}^{\infty} \left( a \cdot \sum_{s_i=0}^{y_i} \varphi_{q,\mu,\nu}(s_i|y_i)q^{s_i(x_{i-1} - x_i - 1)} + b \cdot \sum_{t_i=0}^{y_i} \varphi_{q^{-1},\mu^{-1},\nu^{-1}}(t_i|y_i)q^{-t_i(x_i-x_{i+1}-1)} \right) \prod_{i=0}^{\infty} q^{(x_{i+1})y_i} = H(P_{q,\mu,\nu}^{AZRP})^T.$$

\[\square\]

Remark 3.6. One can see from the proof of Proposition 3.4 that our statement could be generalized:
- The duality still holds when the parameters $\mu$ and $\nu$ are not the same for the jumps to the left and the jumps to the right.
- The parameters $\mu$ and $\nu$ and the asymmetry parameters $a$ and $b$ could also depend on time and site/particle provided that the parameters corresponding to the $i^{th}$ particle in the exclusion process equal the parameters corresponding to the $i^{th}$ site in the zero-range process.

It is not presently clear if the solvability of the discrete-time $q$-Hahn AZRP (resp. discrete-time $q$-Hahn AEP) process extends to the general time and site-dependent (resp. particle-dependent) parameters beyond duality, see [Cor14, Section 2.4] for a related discussion in the $q$-Hahn TASEP case.

Definition 3.7. We say that $h : \mathbb{Z}_{\geq 0} \times \mathbb{Y}^\infty \to \mathbb{R}_+$ solves the true evolution equation with initial data $h_0 : \mathbb{Y}^\infty \to \mathbb{R}_+$ if it satisfies the conditions
(1) for all $\vec{y} \in \mathbb{Y}^\infty$ and $t \in \mathbb{Z}_{\geq 0}$, $h(t + 1; \vec{y}) = P_{q,\mu,\nu}^{AZRP} h(t; \vec{y})$;
(2) for all $\vec{y} \in \mathbb{Y}^\infty$, $h(0; \vec{y}) = h_0(\vec{y})$.

Corollary 3.8. For any fixed $\vec{x} \in \mathbb{X}^\infty$, denote $\mathbb{E}^{\vec{x}}[\cdot]$ as the expectation with respect to the initial data $\vec{x}(0) = \vec{x}$. Then,

$$h(t; \vec{y}) = \mathbb{E}^{\vec{x}}[H(\vec{x}(t), \vec{y})]$$

is the unique solution to the true evolution equation with initial data $h_0(\vec{y}) = H(\vec{x}, \vec{y})$.

Proof. Using Proposition 3.4

$$P_{q,\mu,\nu}^{AZRP} h(t; \vec{y}) = \sum_{\vec{x}'} \mathbb{E}^{\vec{x}}(\vec{x}(t) = \vec{x}') H(P_{q,\mu,\nu}^{AZRP})^T(\vec{x}', \vec{y})$$

$$= \sum_{\vec{x}'} \mathbb{E}^{\vec{x}}(\vec{x}(t) = \vec{x}') P_{q,\mu,\nu}^{AEP} H(\vec{x}', \vec{y})$$

$$= h(t + 1; \vec{y}).$$
To show the uniqueness of solutions to the true evolution equation, notice that \( h(t + 1; \vec{y}) \) depend only upon the values of \( h(t; \vec{y}') \) for a finite collection of configurations \( \vec{y}' \). By recurrence on \( t \), this implies the existence and uniqueness of solutions to the true evolution equation.

At this point, for the discrete time \( q \)-Hahn AZRP we come to an impediment which is not present in the totally asymmetric case. In [Cor14], it is shown that in order to solve the true evolution equation (written in terms of the \( k \) ordered particle locations, as in Proposition 4.3 for the continuous-time case) it suffices to solve a free evolution equation subject to \( k - 1 \) two-body boundary conditions (such as the one in the statement of Proposition 4.6 which corresponds to the continuous-time case). This sets the stage for the application of Bethe ansatz. Unfortunately, this sort of rewriting does not appear to be valid for the two-sided process. For further discussion, see Appendix A.

4. A CONTINUOUS-TIME ASYMMETRIC SYSTEM SOLVABLE VIA BETHE ANSATZ

In the previous section, the main obstacle to the solvability of the \( q \)-Hahn AZRP process by Bethe ansatz is the fact that one has to consider transitions from a state \( \vec{n} \) to \( \vec{n}' \) where some particles have jumped to the left and and some other particles to the right. This never happens for continuous time Markov dynamics. A general procedure to build continuous time dynamics out of the family of Markov chains that we have defined in Section 3 is to scale the parameters so that the jump probabilities \( \varphi(j|y) \) are of order \( \epsilon \) for all \( j \geq 1 \), and rescale time by setting \( \tau := t \epsilon^{-1} \). As \( \epsilon \to 0 \), the process converges to a continuous time Markov process.

4.1. General \( \nu \) case. Let us fix \( q, \nu \in (0, 1) \) and set \( \mu = \nu + (1-q)\epsilon \). Then for all \( j \geq 1 \), the jump probabilities become jump rates given by the limits,

\[
a \cdot \varphi_{q, \mu, \nu}(j|m)/\epsilon \to \epsilon \to 0_a \nu^{j-1} \left( \frac{1-q}{1-q^2} \right) \left( \frac{\nu}{\nu'; \nu} \right)_{m-j} \left( \frac{q}{q'; q} \right)_m,
\]

\[
b \cdot \varphi_{q^{-1}-\mu-1, \nu^{-1}}(j|m)/\epsilon \to \epsilon \to 0_b \nu^{-j} \left( \frac{1-q}{1-q^2} \right) \left( \frac{\nu}{\nu'; \nu} \right)_{m-j} \left( \frac{q}{q'; q} \right)_m.
\]

Let us fix some notation and write these limiting rates as \( \phi_{q, \nu}^R(j|m) \) and \( \phi_{q, \nu}^L(j|m) \):

\[
\phi_{q, \nu}^R(j|m) := R \left[ \frac{\nu^{j-1}}{\nu; \nu} \left( \frac{\nu}{\nu'; \nu} \right)_{m-j} \left( \frac{q}{q'; q} \right)_m \right],
\]

\[
\phi_{q, \nu}^L(j|m) := L \left[ \frac{1}{\nu; \nu} \left( \frac{\nu}{\nu'; \nu} \right)_{m-j} \left( \frac{q}{q'; q} \right)_m \right].
\]

The letters \( R \) and \( L \) stand for “right" and “left" as well as denote the values of the relative rates of jumps of particles in the process in those respective directions. Note that we deliberately removed the factor \( \nu^{-1} \) (present in the \( \epsilon \to 0 \) limit) from \( \phi_{q, \nu}^L(j|m) \) to be consistent with models previously introduced in the particle system literature (see Section 4.3). In this way, the rates are well-defined for \( \nu = 0 \) and all results of this section hold for \( \nu = 0 \) as well. It is useful for later calculations to notice that

\[
R^{-1} \phi_{q^{-1}-\nu^{-1}}^R(j|m) = \frac{\nu}{q} L^{-1} \phi_{q, \nu}^L(j|m).
\]

**Definition 4.1.** We define the (continuous time) \( q \)-Hahn asymmetric zero-range process (abbreviated \( q \)-Hahn AZRP) as a Markov process \( \vec{y}(t) \in \mathbb{Y}_k^\infty \) with infinitesimal generator \( B_{q, \nu} \) defined in (17). Before stating this generator, we must introduce some notation. For a vector \( \vec{y} = (y_0, y_1, \ldots) \), and any \( j \leq y_i \) we denote

\[
\vec{y}^0_{i,i-1} = (y_0, \ldots, y_{i-1} + j, y_i - j, y_{i+1}, \ldots),
\]

\[
\vec{y}^0_{i,i+1} = (y_0, \ldots, y_{i-1}, y_i - j, y_{i+1} + j, \ldots).
\]
The operator $B_{q,\nu}$ is defined by its action on functions $\mathbb{Y}^\infty \to \mathbb{R}$ by

$$
(B_{q,\nu}f)(\vec{y}) = \sum_{i=1}^{\infty} \left( \sum_{j=1}^{y_i} \phi^R_{q,\nu}(j|y_i) \left( f(\vec{y}_{i+1}^j) - f(\vec{y}_{i-1}^j) \right) + \sum_{j=1}^{y_i} \phi^L_{q,\nu}(j|y_i) \left( f(\vec{y}_{i+1}^{j+1}) - f(\vec{y}_{i-1}^{j+1}) \right) \right). \tag{17}
$$

Informally, if the site $i$ is occupied by $y$ particles, $j$ particles move together to site $i - 1$ with rate $\phi^R_{q,\nu}(j|y)$ whereas $j'$ particles move together to site $i + 1$ with rate $\phi^L_{q,\nu}(j'|y)$, for all $1 \leq j, j' \leq y$ (see Figure 5).

Similarly, we define the continuous time two-sided $q$-Hahn exclusion process as a Markov process $\vec{x}(t) \in \mathbb{X}^\infty$ by the action of its infinitesimal generator $T_{q,\nu}$. For a vector $\vec{x} = (x_0, x_1, \ldots)$ we denote for any $j \in \mathbb{Z}$ and $i \geq 1$

$$\vec{x}_i^j = (x_0, \ldots, x_{i-1}, x_i + j, x_{i+1}, \ldots).$$

The operator $T_{q,\nu}$ is defined by its action on functions $\mathbb{X}^\infty \to \mathbb{R}$ by

$$
(T_{q,\nu}f)(\vec{x}) = \sum_{i=1}^{\infty} \left( \sum_{j=1}^{x_{i-1} - x_i - 1} \phi^R_{q,\nu}(j|x_{i-1} - x_i - 1) \left( f(\vec{x}_i^{j+1}) - f(\vec{x}_i) \right) + \sum_{j=1}^{x_{i} - x_{i+1} - 1} \phi^L_{q,\nu}(j|x_i - x_{i+1} - 1) \left( f(\vec{x}_{i+1}^{j+1}) - f(\vec{x}_{i+1}) \right) \right). \tag{18}
$$

Before going further into the analysis of the $q$-Hahn AEP and AZRP, one should justify that they are well defined.

**Existence of the $q$-Hahn AZRP.** Observe that the (finite) number of particles is conserved by the dynamics. Let $k$ the number of particles in the initial condition. Then, each entry of the transition matrix of the process is bounded by

$$k \cdot \max_{m \in \{1, \ldots, k\}} \sum_{j \leq m} (\phi^R_{q,\nu}(j|m) + \phi^L_{q,\nu}(j|m)) < \infty.$$

Then, the existence of a Markov process with the generator (17) follows from the classical construction of Markov chains on a denumerable state space with bounded generator (see e.g. [EK09] Chap. 4 Section 2).

**Existence of $q$-Hahn AEP.** Although it should be possible to show that the generator (18) defines uniquely a Markov semi-group (Using e.g. [BO12] Proposition 4.3), we prefer to give a probabilistic construction of the $q$-Hahn AEP that corresponds to the generator. Fix some $T > 0$ and let us show that the processes is well-defined on the time interval $[0, T]$. Then, the construction will extend to any time $t \in \mathbb{R}_+$ by the Markov property. We prove that the construction on $[0, T]$ is actually that of a continuous-time Markov chain on a finite (random) state space. Consider a (possibly random) initial condition in $\mathbb{X}^\infty$. By the definition of the state space $\mathbb{X}^\infty$, there exists a (possibly random) integer $N$ such that for all $n > N$, $x_n(0) - x_{n+1}(0) = 1$. We claim that almost surely, there exists an integer $n > N$ such that the particle labelled by $n$ does not move on the time interval $[0, T]$.
Indeed, if this particle moves, then it has to move at least once to its right, since there is no room to its left. The rates at which a jump on the right occurs is bounded by

\[ M := \sup_{m \geq 1} \sum_{j=1}^{m} \phi_R^{q,\nu}(j|m) < \infty. \]

Since all particles are equipped with independent Poisson clocks, there exists almost surely a particle that does not jump to the right. Finally, the \( q \)-Hahn AEP can be constructed on \([0,T]\) as a Markov chain on a finite state-space.

We come now to the duality between the \( q \)-Hahn AEP and the \( q \)-Hahn AZRP.

**Proposition 4.2.** For any \((\vec{x}, \vec{y})\) in \(X^\infty \times Y^\infty\), we have that

\[ B_{q,\nu} H(\vec{x}, \vec{y}) = T_{q,\nu} H(\vec{x}, \vec{y}), \]

where \(B_{q,\nu}\) acts on the \(\vec{y}\) variable, \(T_{q,\nu}\) acts on the \(\vec{x}\) variable, and the function \(H\) is given in (13).

**Proof.** This duality is a consequence of the duality for the two-sided \( q \)-Hahn processes given in Proposition 3.4, but we choose to give a direct proof as well. Under the scalings above and when \( \epsilon \) goes to zero, identities (10) and (11) degenerate to

\[ \sum_{j=1}^{m} \phi_R^{q,\nu}(j|m) (q^jy - 1) = \sum_{j=1}^{y} \phi_R^{q,\nu}(j|y) (q^{jm} - 1), \]  

(19)

and

\[ \sum_{j=1}^{m} \phi_L^{q,\nu}(j|m) (q^{-j}y - 1) = \sum_{j=1}^{y} \phi_L^{q,\nu}(j|y) (q^{-jm} - 1). \]  

(20)

Let us explain how (19) is obtained. From the limit (14), we know that for \( j \geq 1 \),

\[ \varphi_{q,\mu,\nu}(j|m) = \epsilon R^{-1} \phi_R^{q,\nu}(j|m) + o(\epsilon). \]

Since \( \sum_{j=0}^{m} \varphi_{q,\mu,\nu}(j|m) = 1 \), we know that

\[ \varphi_{q,\mu,\nu}(0|m) = 1 - \sum_{j=1}^{m} \epsilon R^{-1} \phi_R^{q,\nu}(j|m) + o(\epsilon). \]

Finally, in terms of \( \epsilon \), identity (10) writes

\[ 1 - \sum_{j=1}^{m} \epsilon R^{-1} \phi_R^{q,\nu}(j|m) + \sum_{j=1}^{m} \epsilon R^{-1} \phi_R^{q,\nu}(j|m) q^jy + o(\epsilon) = \]

\[ 1 - \sum_{j=1}^{y} \epsilon R^{-1} \phi_R^{q,\nu}(j|y) + \sum_{j=1}^{y} \epsilon R^{-1} \phi_R^{q,\nu}(j|y) q^{jm} + o(\epsilon). \]

Subtracting 1 from both sides and keeping only terms of order \( \epsilon \), one gets identity (19). Identity (20) is obtained in a similar way.

Applying generators \( B_{q,\nu} \) and \( T_{q,\nu} \) to the function \( H(\vec{x}, \vec{y}) = \prod_{i=0}^{\infty} q^{(x_i+y)_i} \) and using Equations (19) and (20) to each term of the sum, one gets that \( B_{q,\nu} H = T_{q,\nu} H. \)

□
The $k$-particle $q$-Hahn AZRP process can be alternatively described in terms of ordered particle locations $\vec{n}(t) = \vec{n}(\vec{y}(t))$. The bijection between $\vec{n}$ coordinates and $\vec{y}$ coordinates is such that $n_i(t) = n$ if and only if $\sum_{j>n} y_j < i \leq \sum_{j\geq n} y_j$ and we impose that $\vec{n} \in \mathbb{W}^k$ where the Weyl chamber $\mathbb{W}^k$ is defined as
\[
\mathbb{W}^k = \{ n_1 \geq n_2 \geq \cdots \geq n_k ; \ n_i \in \mathbb{Z}_{\geq 0}, 1 \leq i \leq k \}.
\] (21)

For a subset $I \subset \{1, \ldots, k\}$ and $\vec{n} \in \mathbb{W}^k$, we introduce the vector $\vec{n}^+_I$ obtained from $\vec{n}$ by increasing by one all coordinates with index in $I$; and the vector $\vec{n}^-_I$ obtained from $\vec{n}$ by decreasing by one all coordinates with index in $I$. As an example,
\[
\vec{n}^+_I = (n_1, \ldots, n_{i-1}, n_i + 1, n_{i+1}, \ldots, n_k).
\]

With a slight abuse of notations, we will use the same symbol $B_{q,\nu}$ for the generator of the $q$-Hahn AZRP described in terms of variables in either $\mathbb{Y}_\infty$ or $\mathbb{W}^k$.

**Definition 4.3.** We say that $h : \mathbb{R}_+ \times \mathbb{W}^k$ solves the $k$-particle true evolution equation with initial data $h_0$ if it satisfies the conditions that:

1. For all $\vec{n} \in \mathbb{W}^k$ and $t \in \mathbb{R}_+$,
\[
\frac{d}{dt} h(t, \vec{n}) = B_{q,\nu} h(t, \vec{n}),
\]
2. For all $\vec{n} \in \mathbb{W}^k$, $h(t, \vec{n}) \xrightarrow{t \to 0} h_0(\vec{n})$,
3. For any $T > 0$, there exists constants $c, C > 0$ such that for all $\vec{n} \in \mathbb{W}^k$, $t \in [0, T]$,
\[
|h(t, \vec{n})| \leq C e^{c\|\vec{n}\|},
\]
and for all $\vec{n}, \vec{n}' \in \mathbb{W}^k$, $t \in [0, T]$,
\[
|h(t, \vec{n}) - h(t, \vec{n}')| \leq C |e^{c\|\vec{n}\|} - e^{c\|\vec{n}'\|}|,
\]
where we define the norm of a vector in $\mathbb{W}^k$ by $\|\vec{n}\| = \sum_{i=1}^k n_i$.

**Proposition 4.4.** Consider any initial data $h_0$ such that there exists constants $c, C > 0$ such that for all $\vec{n} \in \mathbb{W}^k$, $|h_0(\vec{n})| \leq C e^{c\|\vec{n}\|}$, and for all $\vec{n}, \vec{n}' \in \mathbb{W}^k$, $|h_0(\vec{n}) - h_0(\vec{n}')| \leq C |e^{c\|\vec{n}\|} - e^{c\|\vec{n}'\|}|$. Then the solution of the true evolution equation is unique.

**Proof.** We provide a probabilistic proof adapted from [BCST14, Appendix C]. Given $\vec{n}(t)$, a $q$-Hahn AZRP started from initial condition $\vec{n}(0) = \vec{n}$, we use a representation of any solution to the true evolution equation as a functional of the $q$-Hahn AZRP.

Let $h^1$ and $h^2$ two solutions of the true evolution equation with initial data $h_0$. Then $g := h^1 - h^2$ solves the true evolution equation with zero initial data. Let $T > 0$. Our aim is to prove that for any $\vec{n} \in \mathbb{W}^k$, $g(T, \vec{n}) = 0$. The idea is the following: By formally differentiating the function $t \mapsto E_{\vec{n}} [g(t, \vec{n}(T - t))]$ we find a zero derivative. Thus we expect that this function is constant, and hence its value for $t = T$, which is $g(T, \vec{n})$, equals the limit when $t$ goes to zero, which is expected to be 0. Of course, these formal manipulations need to be justified and we will see how condition (3) of the true evolution equation applies.

By condition (3) of the true evolution equation, there exist constants $c, C > 0$ such that for $t \in [0, T]$,
\[
|g(t, \vec{n})| \leq C e^{c\|\vec{n}\|}.
\] (22)

Let us first prove that on $[0, T]$, $\|\vec{n}(t)\| - \|\vec{n}\|$ can be bounded by a Poisson random variable $N_T$. Indeed, we have that for any $0 \leq t \leq T$,
\[
\mathbb{P}^\vec{n}(\|\vec{n}(t)\| - \|\vec{n}\| = N) \leq \mathbb{P}\left( \text{at least} \frac{N}{k} \text{ events on the right occurred on } [0, T] \right).
\]
The rate of an event on the right is crudely bounded by \(k\lambda\) where \(\lambda = \max_{j \leq m \leq k} \phi^k(j|m) < \infty\). Thus, \(|\bar{n}(t)| - |\bar{n}|\) can be bounded by a Poisson random variable \(N_T\) depending only on the horizon time \(T\).

Consider the function \([0, T] \to \mathbb{R}, t \mapsto \mathbb{E}|n(t, \bar{n}(T - t))|\). Given the exponential bound (22) and the inequality \(|\bar{n}(t)| \leq |\bar{n}| + N_T\), this function is well-defined. Moreover, one can apply dominated convergence to show that it is continuous. Thus, the limit when \(t\) goes to zero is zero (because of the initial condition for \(g\)).

Let us show that the function is constant. First, observe that for \(t \in [0, T]\),

\[
B_{q, \nu} g(t, \bar{n}) \leq \sum_{\bar{n} \to \bar{n}'} 2k\lambda|g(t, \bar{n}')| \leq (2k)^2 \lambda C e^{c(\|\bar{n}\| + k)}.
\]

Since \(\bar{n}(T - t)\) can be bounded by \(|\bar{n}| + N_T\),

\[
|B_{q, \nu} g(t, \bar{n}(T - t))| \leq (2k)^2 \lambda C e^{c(\|\bar{n}\| + k + N_T)}.
\] (23)

Consider the function \(\phi : [0, T]^2 \to \mathbb{R}\) defined by \(\phi(t, s) = \mathbb{E}[\bar{n}[g(t, \bar{n}(s))]\). Since the right-hand-side of (23) is integrable, one can take the partial derivative of \(\phi\) with respect to \(t\) inside the expectation, and we get

\[
\frac{\partial \phi}{\partial t}(t, s) = \mathbb{E}\bar{n}[B_{q, \nu} g(t, \bar{n}(s))]
\]

The equality comes from condition (1) of true evolution equation, using dominated convergence. By condition (3) of the true evolution equation, we also have that for \(t \in [0, T]\),

\[
|g(t, \bar{n}) - g(t, \bar{n}')| \leq C|e^{c\|\bar{n}\|} - e^{c\|\bar{n}'\|}|
\] (24)

Hence, for any fixed \(t \in [0, T]\), we have for \(0 < s < s' < T\)

\[
\frac{|\phi(t, s') - \phi(t, s)|}{s' - s} \leq C\mathbb{E}\bar{n}\left[\frac{|e^{c\|\bar{n}(s')\|} - e^{c\|\bar{n}(s)\|}|}{s - s'}\right].
\] (25)

Since one can bound \(|\|\bar{n}(s)\| - \|\bar{n}(s')\|||\) by a Poisson random variable with parameter proportional to \(s' - s\), the right-hand-side of (25) has a limit when \(s'\) goes to \(s\). This means that for any \(t \in [0, T]\), the function \(\bar{n} \mapsto g(t, \bar{n})\) is in the domain of the semi-group (of the \(q\)-Hahn AZRP). Thus, applying Kolmogorov backward equation and using the commutativity of the generator with the semi-group, we have that

\[
\frac{\partial \phi}{\partial s}(t, s) = \mathbb{E}\bar{n}[B_{q, \nu} g(t, \bar{n}(s))].
\]

Consequently the derivative of \(t \mapsto \mathbb{E}\bar{n}[g(t, \bar{n}(T - t))\) is zero. Hence the function is constant, and the value at \(t = T\), \(g(T, \bar{n})\) equals the limit when \(t \to 0\) which is zero. \(\square\)

**Corollary 4.5.** For any fixed \(\bar{x} \in \mathbb{X}^\infty\), the function \(u : \mathbb{R}_+ \times \mathbb{W}^k \to \mathbb{R}\) defined by

\[
u(t, \bar{n}) = \mathbb{E}\bar{x}[H(\bar{x}(t), \bar{n})]
\]

satisfies the true evolution equation with initial data \(h_0(\bar{n}) = H(\bar{x}, \bar{n})\). As a consequence, the \(q\)-Hahn AEP and the \(k\)-particle \(q\)-Hahn AZRP are dual with respect to the function \(H\), that is for any \(\bar{x} \in \mathbb{X}^\infty\) and \(\bar{n} \in \mathbb{W}^k\),

\[
\mathbb{E}\bar{x}[H(\bar{x}(t), \bar{n})] = \mathbb{E}\bar{n}[H(\bar{x}, \bar{n}(t))].
\]

**Proof.** By the Kolmogorov backward equation for the \(q\)-Hahn AZRP, it is clear that \((t, \bar{n}) \mapsto \mathbb{E}\bar{x}[H(\bar{x}, \bar{n}(t))]\) satisfies the true evolution equation with initial data \(\mathbb{E}[H(\bar{x}, \bar{n})]\) (the growth condition is clear). On the other hand, Kolmogorov backward equation for the \(q\)-Hahn AEP yields

\[
\frac{d}{dt} \mathbb{E}\bar{x}[H(\bar{x}(t), \bar{n})] = T_{q, \mu} \mathbb{E}\bar{x}[H(\bar{x}(t), \bar{n})] = \mathbb{E}\bar{x}[T_{q, \nu} H(\bar{x}(t), \bar{n})].
\]
Proposition 4.2 then implies

\[
\frac{d}{dt} u(t, \vec{n}) = \mathbb{E}^\vec{x} [B_{q,\nu} H(\vec{x}(t), \vec{n})] = B_{q,\nu} u(t, \vec{n}).
\]

Since \( u \) satisfies the growth condition and the initial condition, \( u \) solves the true evolution equation. Hence, by Proposition 4.4 we have that for all \( \vec{x} \in \mathbb{X}^\infty \) and \( \vec{n} \in \mathbb{W}^k \),

\[
\mathbb{E}^\vec{x} [H(\vec{x}(t), \vec{n})] = \mathbb{E}^{\vec{n}} [H(\vec{x}, \vec{n}(t))].
\]

In order to compute the observables \( \mathbb{E} \left[ \prod_{i=1}^k q^{x_{n_i}(t)+n_i} \right] \), it would be natural to solve the true evolution equation. However, it is not clear how to proceed directly, and Proposition 4.6 provides an important reduction by rewriting the \( k \)-particle true evolution equation as a \( k \)-particle free evolution equation with \( k - 1 \) two-body boundary conditions.

**Proposition 4.6.** Let \( \vec{x}(\cdot) \) denote the \( q \)-Hahn AEP. If \( u : \mathbb{R}^+ \times \mathbb{Z}^k \to \mathbb{C} \) solves:

1. \((k\text{-particle free evolution equation})\) for all \( \vec{n} \in \mathbb{Z}^k \) and \( t \in \mathbb{R}^+ \),

\[
\frac{d}{dt} u(t, \vec{n}) = \frac{1 - q}{1 - q\nu} \sum_{i=1}^k \left[ R \left( u(t; \vec{n}_i^-) - u(t; \vec{n}) \right) + L \left( u(t; \vec{n}_i^+) - u(t; \vec{n}) \right) \right];
\]

2. \((k - 1 \text{two-body boundary conditions})\) for all \( \vec{n} \in \mathbb{Z}^k \) and \( t \in \mathbb{R}^+ \) if \( n_i = n_{i+1} \) for some \( i \in \{1, \ldots, k - 1\} \) then

\[
\alpha u(t; \vec{n}_{i,i+1}^-) + \beta u(t; \vec{n}_{i,i+1}^+) + \gamma u(t; \vec{n}) - u(t; \vec{n}_i^-) = 0
\]

where the parameters \( \alpha, \beta, \gamma \) are defined in terms of \( q \) and \( \nu \) as

\[
\alpha = \frac{\nu(1 - q)}{1 - q\nu}, \quad \beta = \frac{q - \nu}{1 - q\nu}, \quad \gamma = \frac{1 - q}{1 - q\nu};
\]

3. \((\text{initial data})\) for all \( \vec{n} \in \mathbb{W}_k \), \( u(0; \vec{n}) = \mathbb{E} \left[ \prod_{i=1}^k q^{x_{n_i}(0)+n_i} \right] \);

4. \((\text{for any} \ T > 0, \ \text{there exists constants} \ c, C > 0 \ \text{such that for all} \ \vec{n} \in \mathbb{W}_k, \ t \in [0, T], \)

\[
|u(t; \vec{n})| \leq Ce^{c||\vec{n}||},
\]

and for all \( \vec{n}, \vec{n}' \in \mathbb{W}^k \), \( t \in [0, T] \),

\[
|h(t, \vec{n}) - h(t, \vec{n}')| \leq C(e^{c||\vec{n}||} - e^{c||\vec{n}'||};
\]

then for all \( \vec{n} \in \mathbb{W}_k \) and all \( t \in \mathbb{R}^+ \), \( u(t; \vec{n}) = \mathbb{E} \left[ \prod_{i=1}^k q^{x_{n_i}(t)+n_i} \right] \).

**Proof.** In the totally asymmetric case, that is when \( R = 1 \) and \( L = 0 \), this result can be seen as a degeneration of Proposition 1.7 in [Cor13].

First we show that conditions (1) and (2) imply that \( u \) satisfies condition (1) of the true evolution equation in Definition 4.3. Condition (2) in Proposition 4.6 says that for all \( \vec{n} \) such that \( n_i = n_{i+1} \),

\[
\frac{\nu(1 - q)}{1 - q\nu} u(t; \vec{n}_{i,i+1}^-) + \frac{q - \nu}{1 - q\nu} u(t; \vec{n}_{i,i+1}^+) + \frac{1 - q}{1 - q\nu} u(t; \vec{n}) - u(t; \vec{n}_i^-) = 0.
\]

This is equivalent to saying that for all \( \vec{n} \) such that \( n_i = n_{i+1} \),

\[
\frac{\nu^{-1}(1 - q^{-1})}{1 - q^{-1}\nu^{-1}} u(t; \vec{n}_{i,i+1}^-) + \frac{q^{-1} - \nu^{-1}}{1 - q^{-1}\nu^{-1}} u(t; \vec{n}_{i,i+1}^+) + \frac{1 - q^{-1}}{1 - q^{-1}\nu^{-1}} u(t; \vec{n}) - u(t; \vec{n}_{i,i+1}^+) = 0.
\]
Indeed, if we set $\vec{m} := \vec{n}_{i,i+1}$ in (26), we have that $\vec{n}_{i+1} = \vec{m}_{i,i+1}^\dagger, \vec{n} = \vec{m}_{i,i+1}$ and $\vec{n}_i = \vec{m}_{i,i+1}^\dagger$. Dividing the numerator and the denominator of each coefficient in (26) by $X$ we have

\[
\frac{\nu(1-q)}{1-q} u(t; \vec{n}_{i,i+1}) = \frac{1-q^{-1}}{1-q^{-1}\nu-1} u(t; \vec{m}),
\]

\[
\frac{q-\nu}{1-q} u(t; \vec{n}_{i,i+1}) = \frac{q^{-1}-\nu^{-1}}{1-q^{-1}\nu-1} u(t; \vec{m}_i^\dagger),
\]

\[
\frac{1-q}{1-q} u(t; \vec{n}) = \frac{\nu^{-1}\nu(1-q^{-1})}{1-q^{-1}\nu-1} u(t; \vec{m}_{i,i+1}^\dagger).
\]

Finally we get exactly (27) with $\vec{n}$ replaced by $\vec{m}$.

The next lemma explains the effect of the boundary condition.

**Lemma 4.7.** Suppose that a function $f : \mathbb{Z}^m \to \mathbb{R}$ satisfies the boundary conditions that for all $\vec{n}$ such that $n_i = n_{i+1}$ for some $i \in \{1, \ldots, k-1\}$,

\[
\alpha f(\vec{n}_{i,i+1}) + \beta f(\vec{n}_{i+1}) + \gamma f(\vec{n}) - f(\vec{n}_{i}) = 0.
\]

Then for $\vec{n} = (n, \ldots, n)$, the function $f$ also satisfies

\[
\sum_{i=1}^{m} R \frac{1-q}{1-\nu} (f(\vec{n}_i) - f(\vec{n})) = \sum_{j=1}^{m} \phi^{R}_{q,\nu}(j|m)f(n, \ldots, n, n-1, \ldots, n-1),
\]

(28)

and

\[
\sum_{i=1}^{m} L \frac{1-q}{1-\nu} (f(\vec{n}_i^\dagger) - f(\vec{n})) = \sum_{j=1}^{m} \phi^{L}_{q,\nu}(j|m)f(n+1, \ldots, n+1, n, \ldots, n).
\]

(29)

**Proof.** Equation (28) is exactly the conclusion of Lemma 2.4 in Cor14 with $\mu = \nu + (1-q)\epsilon$ and keeping only the terms of order $\epsilon$. For completeness, we will give a direct proof as well. Theorem 1 in Pov13 states that an associative algebra generated by $A, B$ obeying the quadratic homogeneous relation

\[
BA = \alpha AA + \beta AB + \gamma BB,
\]

(30)

enjoys the following non-commutative analogue of Newton binomial expansion

\[
\left( \frac{\mu-\nu}{1-\nu} A + \frac{1-\mu}{1-\nu} B \right)^m = \sum_{j=0}^{m} \phi^{R}_{q,\nu}(j|m) A^j B^{m-j}.
\]

Let $\mu = \nu + (1-q)\epsilon$ and consider only the terms of order $\epsilon$ as $\epsilon \to 0$ in the above expression. By identification of $O(\epsilon)$ terms, we have

\[
\sum_{i=1}^{m} \frac{1-q}{1-\nu} B^{i-1} A B^{m-i} = \sum_{j=1}^{m} R^{-1} \phi^{R}_{q,\nu}(j|m) A^j B^{m-j}.
\]

(31)

Interpreting each monomial of the form $X_1 X_2 \ldots X_m$ with $X_i \in \{A, B\}$ as $f(n_1, \ldots, n_m)$ where $n_i = n$ if $X_i = B$ and $n_i = n-1$ if $X_i = A$, the boundary condition in the statement of the Lemma corresponds algebraically to the quadratic relation (30). Thus we find that for $\vec{n} = (n, \ldots, n)$, $f$ satisfies

\[
\sum_{i=1}^{m} R \frac{1-q}{1-\nu} (f(\vec{n}_i) - f(\vec{n})) = \sum_{j=1}^{m} \phi^{R}_{q,\nu}(j|m)f(n, \ldots, n, n-1, \ldots, n-1).
\]
Since \((31)\) is true as an identity in an algebra over the field of rational fractions in \(q\) and \(\nu\), we can certainly replace \(q\) and \(\nu\) by their inverses. Keeping in mind \((16)\), we find that
\[
\sum_{i=1}^{m} \frac{1 - q^{-1}}{1 - \nu^{-1}} B_{i-1} A B_{m-i} = \frac{\nu}{q} \sum_{j=1}^{m} L^{-1} \phi_{q,\nu}^j(j|m) A^j B^{m-j}.
\]
(32)

Interpreting the monomials as \(f(n_1, \ldots, n_m)\) with \(n_i = n\) or \(n + 1\), we get that
\[
\sum_{i=1}^{m} L \frac{1 - q}{1 - \nu} \left( f(\vec{n}_i) - f(\vec{n}) \right) = \sum_{j=1}^{m} \phi_{q,\nu}^j(j|m) f(n + 1, \ldots, n + 1, n, \ldots, n).
\]

The application of Lemma 28 for each cluster of equal elements in \(\vec{n}\) shows that under conditions (1) and (2), \(u(t; \vec{n})\) satisfies condition (1) of Definition 4.3
\[
\frac{d}{dt} h(t, \vec{n}) = B_{q,\nu} h(t, \vec{n}).
\]

The growth condition (3) of the true evolution equation is exactly the same as condition (4) of the Proposition with the same constants \(c, C\), and the initial data are the same. Hence, if \(u\) satisfies the conditions of the Proposition, it solves the true evolution equation with initial data \(h_0(\vec{y}) = H(\vec{x}, \vec{y})\), and by Proposition 4.4 \(u(t; \vec{n}) = \mathbb{E} \left[ \prod_{i=1}^{k} q^{x_n(t)+n} \right]\).

**Remark 4.8.** In the case \(\nu = q\), the system of ODEs with two-body boundary conditions in Proposition 4.6 was already known, see (10) and (12) in SW98.

Proposition 4.9 provides an exact contour integral formula for the observables \(\mathbb{E} \left[ \prod_{i=1}^{k} q^{x_n(t)+n} \right]\). We simply check that the formula is a solution to the true evolution equation, using Proposition 4.6. The form of this formula originates in the theory of Macdonald processes [BC14] and similar formulas have been obtained as solutions of true evolution equations in [BCSt14] and subsequent papers.

**Proposition 4.9.** Fix \(q \in (0, 1), 0 \leq \nu < 1\), and integer \(k\). Consider the continuous time \(q\)-Hahn exclusion process started from step initial data (i.e. \(x_n(0) = -n\) for \(n \geq 1\)). Then for any \(\vec{n} \in \mathbb{W}^k\),
\[
\mathbb{E} \left[ \prod_{i=1}^{k} q^{x_n(t)+n} \right] = \frac{(-1)^k q^{-k(k-1)/2}}{(2\pi i)^k} \oint_{\gamma_1} \cdots \oint_{\gamma_k} \prod_{1 \leq A < B \leq k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \left( 1 - \frac{1}{z_j} \right)^{n_j} \exp \left( (q-1)t \left( \frac{Rz_j}{1-\nu z_j} - \frac{Lz_j}{1-z_j} \right) \right) \frac{dz_j}{(1-\nu z_j)}. \tag{33}
\]

where the integration contours \(\gamma_1, \ldots, \gamma_k\) are chosen so that they all contain \(1, \gamma_A\) contains \(q\gamma_B\) for \(B > A\) and all contours exclude \(0\) and \(1/\nu\).

**Proof.** We prove that the right-hand-side of (33) verifies the conditions of Proposition 4.6. Note that (33) is very similar with the result of Theorem 1.9 in [Cor14] for \(q\)-Hahn TASEP, the only difference being that the factor \(((1 - \mu z_j)/(1 - \nu z_j))^t\) is replaced by
\[
\exp \left( (q-1)t \left( \frac{Rz_j}{1-\nu z_j} - \frac{Lz_j}{1-z_j} \right) \right).
\]

Let us explain briefly why conditions (2) and (3) are verified: As it is explained in the proof of Theorem 1.9 in [Cor14], the application of the boundary condition to the integrand brings out an additional factor
\[
\frac{(1 - \nu)^2}{(1-q\nu)(1-\nu z_i)(1-\nu z_{i+1})} (z_i - q z_{i+1}).
\]
The factor \((z_i - qz_{i+1})\) cancels out the pole separating the contours for the variables \(z_i\) and \(z_{i+1}\). We may then take the same contour and use antisymmetry to prove that the integral is zero. To check the initial data, one may observe by residue calculus that the integral is zero when \(n_k \leq 0\) since there is no pole at 1 for the \(z_k\) integral; and one verifies that the integral equals 1 in the alternative case by sending the contours to infinity (this is the same calculation as in [Cor14]).

Let us check the free evolution equation. The generator of the free evolution equation can be written as a sum \(\sum_{i=1}^k L_i\) where \(L_i\) acts by

\[
L_i f = \frac{1 - q}{1 - \nu} \left[ R \left( f(n_i^-) - f(n_i) \right) + L \left( f(n_i^-) - f(n_i) \right) \right].
\]

Applying \(L_i\) to the R.H.S of (33) brings inside the integration a factor

\[
\frac{1 - q}{1 - \nu} \left( R \left( \frac{1 - z_i}{1 - \nu z_i} - 1 \right) + L \left( \frac{1 - \nu z_i}{1 - z_i} - 1 \right) \right)
\]

which is readily shown to equal the argument of the exponential.

Finally, let us check the growth condition. Let us denote by \(\tilde{u}(t, \vec{n})\) the right-hand-side of (33). One can choose the contours \(\gamma_1, \ldots, \gamma_k\) such that for all \(1 \leq A < B \leq k\) and \(1 \leq j \leq k\), \(|z_A - qz_B|\), \(|1 - z_j|\), \(|1 - \nu z_j|\) and \(|z_j|\) are uniformly bounded away from zero. Since the contours are finite, one can find constants \(c_1, c_2\) and \(c_3\), such that for any \(t\) smaller that some arbitrary but fixed constant \(T\),

\[
|\tilde{u}(t, \vec{n})| \leq c_1 \prod_{j=1}^k \left( e_n^j \exp((1 - q)t c_3) \right),
\]

and

\[
|\tilde{u}(t, \vec{n}) - \tilde{u}(t, \vec{n}')| \leq c_1 \exp(k(1 - q)t c_3) e_2^{\|\vec{n}\|} - c_2^{\|\vec{n}'\|},
\]

where \(c_1, c_2\) and \(c_3\) depend only on the parameters \(q, \nu\), the choice of contours and the horizon time \(T\).

Proposition 4.9 provides a formula for all integer moments of the random variable \(q^{x_n(t)+n}\) when the continuous time two-sided \(q\)-Hahn exclusion process is started from step initial condition. Since \(q \in (0, 1)\) and \(x_n(t) + n \geq 0\), this completely characterizes the law of \(x_n(t)\). In order to extract information out of these expressions, we give a Fredholm determinant formula for the \(q\)-Laplace transform of \(q^{x_n(t)+n}\), following an approach designed initially for the study of Macdonald processes [BC14]. The reader is referred to [BC14 Section 3.22] for some background on Fredholm determinants. In the totally asymmetric case (\(L = 0\), Theorem 4.10 can also be seen as a degeneration when \(\epsilon\) goes to zero of Theorem 1.10 in [Cor14].

**Theorem 4.10.** Fix \(q \in (0, 1)\) and \(0 \leq \nu < 1\). Consider step initial data. Then for all \(\zeta \in \mathbb{C} \setminus \mathbb{R}_+\), we have the “Mellin-Barnes-type” Fredholm determinant formula

\[
\mathbb{E} \left[ \frac{1}{(\zeta q^{x_n(t)+n}; q)_{\infty}} \right] = \det (I + K_\zeta)
\]

where \(\det (I + K_\zeta)\) is the Fredholm determinant of \(K_\zeta : L^2(C_1) \rightarrow L^2(C_1)\) for \(C_1\) a positively oriented circle containing 1 with small enough radius so as to not contain 0, \(1/q\) and \(1/\nu\). The operator \(K_\zeta\) is defined in terms of its integral kernel

\[
K_\zeta(w, w') = \frac{1}{2\pi i} \int_{i\infty+1/2}^{i\infty+1/2} \frac{\pi}{\sin(-\pi s)} (-\zeta)^s \frac{g(w)}{g(q^s w)} q^s w - w' ds
\]

with

\[
g(w) = \left( \frac{q w; q}{w; q} \right)^n \exp \left( (q - 1)t \sum_{k=0}^\infty \frac{R}{\nu} \frac{\nu w q^k}{1 - \nu w q^k} - L \frac{w q^k}{1 - w q^k} \right) \frac{1}{(q w; q)_{\infty}}.
\]
The following “Cauchy-type” formula also holds:

\[
\mathbb{E} \left[ \frac{1}{(\zeta q^{x_n(t)} + n; q)_\infty} \right] = \frac{\det (I + \zeta \tilde{K})}{(\zeta; q)_\infty},
\]

where \( \det (I + \zeta \tilde{K}) \) is the Fredholm determinant of \( \zeta \) times the operator \( \tilde{K} : L^2(C_{0,1}) \to L^2(C_{0,1}) \) for \( C_{0,1} \) a positively oriented circle containing 0 and 1 but not \( 1/\nu \), and the operator \( \tilde{K} \) is defined by its integral kernel

\[
\tilde{K}(w, w') = \frac{g(w)/g(qw)}{qw' - w}.
\]

**Proof.** We will sketch the main deductions which occur in the proof of the Mellin-Barnes type formula \( (34) \). Similar derivations (with all details given) of such Fredholm determinants from moment formulas can be found in [BCS14, Theorem 3.18], [BCS14, Theorem 1.1] or more recently [Cor14, Theorem 1.10] and the proofs always follow the same general scheme (cf. [BCS14, Section 3.1]). Propositions 3.2 to 3.6 in [BCS14] show that for \( |\zeta| \) small enough and \( C_1 \) a positively oriented circle containing 1 with small enough radius,

\[
\sum_{k=0}^{\infty} \mathbb{E} \left[ q^{k(x_n(t) + n)} \right] \frac{\zeta^k}{[k]_q!} = \det (I + K_\zeta),
\]

with \([k]_q!\) as in \( (3) \). The only technical condition to verify is that

\[
\sup \{|g(w)/g(qw^s)| : w \in C_1, \ k \in \mathbb{Z}_{>0}, \ s \in D_{R,d,k}\} < \infty.
\]

Here, \( D_{R,d,k} \) is the contour depicted in [BCS14, Figure 3]. Note that here \( R \) is not the asymmetry parameter of the process but the radius of the circular part of the contour \( D_{R,d,k} \). If one chooses \( R \) large enough, \( d \) small enough, and the radius of \( C_1 \) small enough, then \( q^s w \) stay in a neighbourhood of the segment \([0, \sqrt{d}]\). The function \( g \) has singularities at \( q^{-n} \) and \( \nu^{-1} q^{-n} \) for all \( n \in \mathbb{Z}_{>0} \). Hence for \( w \in C_1 \) a small but fixed circle around 1, one can choose \( R \) and \( d \) such that \( q^s w \) stay in a compact region of the complex plane away from all singularities, and thus the ratio \(|g(w)/g(qw^s)|\) remains bounded.

By an application of the \( q \)-binomial theorem \( (4) \), for \(|\zeta| < 1\) we also have that

\[
\sum_{k=0}^{\infty} \mathbb{E} \left[ q^{k(x_n(t) + n)} \right] \frac{\zeta^k}{[k]_q!} = \mathbb{E} \left[ \frac{1}{(\zeta q^{x_n(t)} + n; q)_\infty} \right],
\]

proving that \( (34) \) holds for \(|\zeta|\) sufficiently small. Both sides of \( (34) \) can be seen to be analytic over \( \mathbb{C} \setminus \mathbb{R}_+ \). The left-hand side equals

\[
\sum_{k=0}^{\infty} \mathbb{P}(x_n(t) + n = k) \frac{1}{(1 - \zeta q^k)(1 - \zeta q^{k+1}) \ldots}.
\]

For any \( \zeta \in \mathbb{C} \setminus \mathbb{R}_+ \), the infinite products are uniformly convergent and bounded away from zero on a neighbourhood of \( \zeta \), which implies that the series is analytic. The right-hand side of \( (34) \) is

\[
\det (I + K_\zeta) = 1 = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{C_1} dw_1 \ldots \int_{C_1} dw_n \det (K_\zeta(w_i, w_j))_{i,j=1}^n.
\]

Due to exponential decay in \(|s|\) in the integrand of \( K_\zeta \), \( \det (K_\zeta(w_i, w_j))_{i,j=1}^n \) is analytic in \( \zeta \) for all \( w_1, \ldots, w_n \in C_1 \). Analyticity of the Fredholm expansion proceeds from absolute and uniform convergence of the series on a neighbourhood of \( \zeta \notin \mathbb{R}_+ \). This can be shown using that \(|g(w)/g(qw^s)| < const\) for \( w \in C_1 \) and \( s \in 1/2 + i\mathbb{R} \) and Hadamard’s bound to control the determinant.
4.2. Two-sided generalizations of q-TASEP. The limits of the $q$-Hahn weights when $\nu$ goes to zero and and when $\epsilon = (\mu - \nu)/(1 - q)$ goes to zero do not commute, thus several choices are possible in order to build two-sided version of the $q$-TASEP and the $q$-Boson process (see, e.g. [BCS14]).

4.2.1. Case $\nu = 0$ for $q$-Hahn AEP. The limit when we make first $\epsilon$ tend to zero corresponds to taking $\nu = 0$ in the rates $\phi_{q,i}^R$ and $\phi_{q,i}^L$. We have

$$
\phi_{q,0}^R(j|m) = R(1 - q^m) \mathbb{1}_{\{j=1\}} \quad \text{and} \quad \phi_{q,0}^L(j|m) = \frac{L(q;q)_m}{[j]_q (q;q)_{m-j}}. \tag{37}
$$

Let us describe the associated exclusion process. Independently for each $n \geq 1$, the particle at location $x_n(t)$ jumps to $x_n(t) + 1$ at rate $R(1 - q^{\text{gap}})$, the gap being here $x_{n-1}(t) - x_n(t) - 1$, and jumps to the location $x_n - j$ at rate $L/[j]_q (q;q)_{\text{gap}}/(q;q)_{\text{gap}-j}$, for all $j \in \{1, \ldots, n-n+1\}$, the gap being here $x_n(t) - x_{n+1}(t) - 1$. All the result in Section 4.1 apply for the case $\nu = 0$, and one could study this system in more details by analyzing the Fredholm determinant formula of Theorem 4.10.

A motivation for studying this process is the observation that as $q$ goes to 1,

$$
\phi^R(j|m) \approx R(1 - q)m \mathbb{1}_{\{j=1\}} \quad \text{and} \quad \phi^L(j|m) \approx L(1 - q)m \mathbb{1}_{\{j=1\}}. \tag{38}
$$

Thus, the rates on the left and on the right have the same expression at the first order in $1 - q$, and the limit of this process when $q \rightarrow 1$ may be interesting. Several scaling limits are possible.

4.2.2. Another two-sided $q$-TASEP preserving the duality. When sending first $\nu$ to zero, we have already noted in Section 3 that

$$
\varphi_{q,0}(j|m) = \mu^j(\mu; q)_m \begin{bmatrix} m \\ j \end{bmatrix}_q \quad \text{and} \quad \varphi_{q^{-1},\infty}(j|m) = \mathbb{1}_{j=m}.
$$

Hence, one can set $\mu = (1 - q)\epsilon$, $b = L\epsilon$, $a = 1 - L\epsilon$, $\tau = t\epsilon^{-1}$ and send $\epsilon$ to zero. This limit suggests the definition of continuous time Markov processes described as follows. In this two-sided $q$-TASEP, the particle at location $x_n(t)$ jumps to $x_n(t) + 1$ at rate $1 - q^{\text{gap}}$, the gap being here $x_{n-1}(t) - x_n(t) - 1$, and jumps to the location $x_{n+1}(t)$ at rate $L$. The Boson system is defined in order to preserve the duality property: if $y_i(t)$ particles occupy site $i$ then one particle jumps to site $i - 1$ at rate $1 - q^{\text{gap}}(t)$ and $y_i$ particles jump to site $i + 1$ at rate $L$. Indeed, the exclusion process is dual to the Boson process with respect to the function $H$, but one would need additional boundary conditions to write the generator of the Boson system as a $k$-particle free evolution generator subject to two-body boundary conditions (see Appendix A).

Remark 4.11. A third way to define a two-sided version of the $q$-TASEP consists in noticing that in the usual $q$-TASEP, jumps to the right have rate $(1 - q)[\text{gap}]_q$, thus one could give rate proportional to $(q^{-1} - 1)[\text{gap}]_q^{-1}$ to the jumps to the left. In this case as well, the duality between the exclusion and the zero-range processes is still preserved, but one needs additional boundary conditions to write the evolution of the zero-range system as a free evolution equation plus two-body boundary conditions. More precisely, one has to impose $(\Delta^-_i - q\Delta^-_{i+1})|_{n_i = n_{i+1}} u = 0$ for the jumps to the right, and $(\Delta^+_i - q\Delta^+_{i+1})|_{n_i = n_{i+1}} u = 0$ for the jumps to the left. These two conditions together impose that $q = 1$ and $u$ is symmetric. In the zero-range formulation, it corresponds to the non-interacting case where all particles perform independent random walks.

\footnote{That is, if one is searching for boundary conditions that do not involve the asymmetry parameters. More sophisticated boundary conditions involving asymmetry parameters might work.}
Remark 4.12. It is not always necessary for Bethe ansatz solvability that the true evolution equation can be factored as a free evolution equation subject to two-body boundary conditions. A case in which this does not happen is studied in [BCG14].

4.3. Degenerations to known systems.

4.3.1. Totally asymmetric case. When $R = 1$ and $L = 0$, we say that we are in the totally asymmetric case. This degeneration of the $q$-Hahn AZRP was studied by Takeyama in [Tak14]. Indeed, the particle system defined in [Tak14] is a zero-range process defined on $\mathbb{Z}$ controlled by two parameters $s$ and $q$. Particles move from site $i$ to $i - 1$ independently for each $i \in \mathbb{Z}$, and the rate at which $j$ particles move to the left from a site occupied by $m$ particles is given by

$$s^{j-1} \frac{j-1}{j} \prod_{i=0}^{j-1} \frac{[m-i]_q}{1 + s[m-1-i]_q}.$$

Setting $s = (1 - q)\nu$, one notices that

$$s^{j-1} \frac{j-1}{j} \prod_{i=0}^{j-1} \frac{[m-i]_q}{1 + s[m-1-i]_q} = \phi^R_{q,\nu}(j|m).$$

Remark 4.13. The totally asymmetric $q$-Hahn AEP, is also the natural continuous time limit of the (discrete-time) $q$-Hahn TASEP, and it was already noticed in [Pov13] that letting $\mu \to \nu$ and rescaling time was the right way of defining such a continuous time limit.

4.3.2. Multiparticle asymmetric diffusion model. When $\nu = q$, the jump rates of the $q$-Hahn AZRP and AEP no longer depend on the gap between consecutive particles (or the number of particles on each site in the zero-range formulation). The rates are now given by $R/[j]_q$ and $L/[j]_q$. The zero-range model with $N$ particles is exactly the “multi-particle asymmetric diffusion model” introduced by Sasamoto and Wadati in [SW98] and further studied by Lee [Lee12] (see also [AKK99, AKK98]). For the corresponding exclusion process, we prove (by an asymptotic analysis of the Fredholm determinant in (34)) in Section 6 that the rescaled positions of particles converge to a Tracy-Widom law.

4.3.3. Push-ASEP. As we have already noticed in Section 4.2.1, it can be interesting to examine alternative description of exclusion processes by applying particle-hole inversion. Let us consider the $q$-Hahn AEP when $\nu = 0$ (see Section 4.2.1), and let further $q = 0$. The process obtained after particle-hole inversion is known. Indeed, when $\nu = q = 0$, $\phi^R(j|m) = 1_{j=1}$ and $\phi^L(j|m) = 1$ for all $m \geq 1$. This corresponds to the Push-ASEP introduced in [BF08], wherein convergence to the Airy process is proved.

5. Predictions from the KPZ scaling theory

In this section, we explain how asymptotics of our Fredholm determinant formula (Theorem 4.10) confirms the universality predictions from the physics literature KPZ scaling theory [KMHH92, Spo12]. Although the original paper [KMHH92] on the KPZ scaling theory deals only with so-called single step models and directed random polymers, the predictions can be straightforwardly adapted to any exclusion process. In particular, we compute the non-universal constants arising in one-point limit theorems for the two-sided $q$-Hahn process. In Section 6 we provide a rigorous confirmation in a particular case.
Following [Spo12], we present the predictions of KPZ scaling theory in the context of exclusion processes. Assume that the translation invariant stationary measures for an exclusion process on \( \mathbb{Z} \) with local dynamics are precisely labelled by the density of particles \( \rho \), where

\[
\rho = \lim_{n \to \infty} \frac{1}{2n+1} \# \{ \text{particles between } -n \text{ and } n \}.
\]

We define the average steady-state current \( j(\rho) \) as the expected number of particles going from site 0 to 1 per unit time, for a system distributed according to the stationary measure indexed by \( \rho \). We also define the integrated covariance \( A(\rho) \) as

\[
A(\rho) = \sum_{j \in \mathbb{Z}} \text{Cov}(\eta_0, \eta_j),
\]

where \( \eta_0, \eta_j \in \{0, 1\} \) are the occupation variables of the exclusion system at sites 0 and \( j \), and the covariance is taken under the \( \rho \)-indexed stationary measure. One expects that the rescaled particle density \( \varrho(x, \tau) \), given heuristically by

\[
\varrho(x, \tau) = \lim_{\tau \to \infty} P(\text{There is a particle at } \lfloor xt \rfloor \text{ at time } t\tau)
\]

satisfies the conservation equation

\[
\frac{\partial}{\partial \tau} \varrho(x, \tau) + \frac{\partial}{\partial x} j(\varrho(x, \tau)) = 0,
\]

with initial condition which is \( \varrho(x, 0) = \mathbbm{1}_{x<0} \) for the step initial condition. The solution of this PDE yields a law of large numbers for the position of particles. For \( \kappa \geq 0 \), if \( n \) and \( t \) go to infinity with \( n = \lfloor \kappa t \rfloor \), then one has

\[
\frac{x_n(t)}{t} \xrightarrow{a.s.} \pi(\kappa).
\]

It turns out that instead of expressing \( \pi \) as a function of \( \kappa \), it is more convenient to parametrize \( \pi \) and \( \kappa \) by the local density \( \rho \). The existence of such a parametrization is given by the solution of the PDE (39): \( \pi(\rho) \) is implicitly determined by \( \rho = \varrho(\pi(\rho), 1) \) and \( \kappa(\rho) \) is determined by \( \pi(\kappa(\rho)) = \pi(\rho) \).

Let \( \lambda(\rho) = -j''(\rho) \). For \( \rho \) such that \( \lambda(\rho) \neq 0 \), the KPZ class conjecture states that

\[
\lim_{t \to \infty} P \left( \frac{x_n(t) - t\pi(\rho)}{\varrho(t)^{1/3}} \geq x \right) \xrightarrow{t \to \infty} F_{\text{GUE}}(-x),
\]

where \( \pi(\rho) = \frac{\partial j(\rho)}{\partial \rho} \),

\[
\sigma(\rho) := \left( \frac{\lambda(\rho)(A(\rho))^2}{2\rho^3} \right)^{1/3}
\]

and \( n \) goes to infinity with \( n = \lfloor \kappa(\rho)t \rfloor \). The precise definition of \( F_{\text{GUE}} \) is given in Definition 6.1.

**Remark 5.1.** The magnitude of fluctuations in [Spo12, Equation (2.14)] slightly differs from our expression \( \lambda(\rho) (A(\rho))^2/(2\rho^3) \). This is because [Spo12] considers fluctuations of the height function. The fluctuation of the height function is twice the fluctuations of the integrated current. And the fluctuations of the current are, on average, \( \rho \) times the fluctuations of a tagged particle. Then, the quantities \( j(\rho) \) and \( A(\rho) \) defined in [Spo12] differs from ours by a factor 2 and 4 respectively. Moreover, since we consider step-initial condition with particles on the left, it is more convenient to drop the minus sign. That is why the scale \( -\frac{1}{2} \lambda(\rho)^2 \) becomes \( \left( \lambda(\rho)(A(\rho))^2/(2\rho^3) \right)^{1/3} \).
5.1. **Hydrodynamic limit.** In the case of the continuous time \( q \)-Hahn AEP, there exist translation invariant and stationary measures \( \mu_\alpha \) indexed by a parameter \( \alpha \in (0, 1) \) such that the gaps between particles \( (x_n - x_{n+1} - 1) \) are independent and identically distributed according to

\[
\mu_\alpha (\text{gap} = m) = \alpha^n \frac{(\nu; q)_m (\alpha; q)^\infty}{(\nu; q)_m (\alpha

These measures are obviously translation invariant. Let us explain why they are stationary: It is known [Pov13] [Cor14] that this measures are stationary for the totally asymmetric \( q \)-Hahn TASEP. Thus they are also stationary for the totally asymmetric continuous time case. Since the family of measures \( \mu_\alpha \) is stable by inversion of the parameters \( q \) and \( \nu \), they are also stationary in the two-sided case.

From now on, we consider step initial data. Fix \( q \in (0, 1), \nu \in [0, 1) \) and assume \( L = 1 - R \), without loss of generality. Under the local stationarity hypothesis, the density \( \rho \) is given by

\[
\rho = \frac{1}{1 + \mathbb{E}[\text{gap}]}.
\]

Assuming that the gap is distributed according to \( (42) \), we find

\[
\mathbb{E}[\text{gap}] = \sum_{m=0}^{\infty} m \alpha^n \frac{(\nu; q)_m (\alpha; q)^\infty}{(\nu; q)_m (\alpha \nu; q)^\infty} = \alpha \frac{d}{d\alpha} \log \left( \frac{\alpha \nu; q)^\infty}{\alpha; q)^\infty} \right) = \frac{1}{\log(q)} (\Psi_q(\theta) - \Psi_q(\theta + V));
\]

where \( \theta = \log_q(\alpha) \) and \( V = \log_q(\nu) \). Hence, around a position where gaps between particles are distributed according to \( (42) \), the density \( \rho \) is related to the parameter \( \theta \) through

\[
\rho(\theta) = \frac{\log(q)}{\log(q) + \Psi_q(\theta) - \Psi_q(\theta + V)}.
\]

Let us compute the average steady-state current \( j(\rho) \). We have

\[
j(\rho) = \rho \cdot \mathbb{E}[\text{drift}] = \rho \cdot \sum_{m=0}^{\infty} \alpha^n \frac{(\nu; q)_m (\alpha; q)^\infty}{(\nu; q)_m (\alpha \nu; q)^\infty} \left( \sum_{j=1}^{m} j \phi^R_q(j|m) - \sum_{j'=1}^{m} j' \phi^L_q(j'|m) \right),
\]

\[
= \rho \alpha \frac{d}{d\alpha} \left( \frac{R}{\nu} G_q(\alpha \nu) - L G_q(\alpha) \right),
\]

\[
= \rho \frac{1 - q}{\log(q)^2} \left( \frac{R}{\nu} \Psi'_q(\theta + V) - L \Psi'_q(\theta) \right);
\]

where we have used the \( q \)-binomial theorem \( (4) \) to sum over \( m \) in the second equality and we have used Lemma \( 2.1 \) for the third equality. The functions \( \pi, \kappa \) and \( \sigma \) that arise in limit theorems \( (40) \) and \( (41) \) are a priori functions of the density \( \rho \), but given the formula \( (44) \), one can express all quantities as functions of the \( \theta \) variable. Also, all quantities should depend on the parameters \( q, \nu \) and \( R \) (we have assumed that \( L = 1 - R \)). In the following, the quantities \( \pi, \kappa \) and \( \sigma \) are denoted \( \pi_{\theta, \nu, \theta} ; \kappa_{\theta, \nu, \theta} \) and \( \sigma_{\theta, \nu, \theta} \).

5.1.1. **Computation of \( \pi_{\theta, \nu, \theta}(\theta) \).** Let \( \varrho : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \) a solution of the conservation PDE \( (39) \), with initial data corresponding to step initial condition. If a law of large numbers holds as in \( (40) \), then we should have for all \( t > 0 \)

\[
\varrho(\pi_{\theta, \nu, \theta}(\theta)t, t) = \rho(\theta).
\]
Taking derivatives with respect to $\theta$ in (45) yields
\[ \pi'_{q,\nu,R}(\theta) \frac{\partial q}{\partial x}(\pi_{q,\nu,R}(\theta)t, t) = \frac{\partial \rho(\theta)}{\partial \theta}, \]
where $\pi'_{q,\nu,R}(\theta) := \frac{d\pi_{q,\nu,R}(\theta)}{d\theta}$. Taking derivative with respect to $t$ in (45) yields
\[ \pi_{q,\nu,R}(\theta) \frac{\partial q}{\partial x}(\pi_{q,\nu,R}(\theta)t, t) + \frac{\partial q}{\partial t}(\pi_{q,\nu,R}(\theta)t, t) = 0. \]
This implies
\[ \pi'_{q,\nu,R}(\theta) \frac{\partial q}{\partial t}(\pi_{q,\nu,R}(\theta)t, t) = -\pi_{q,\nu,R}(\theta) \frac{\partial q(\theta)}{\partial \theta}. \] (46)
On the other hand, we also expect
\[ j(\rho(\pi_{q,\nu,R}(\theta)t, t)) = j(\rho(\theta)). \] (47)
Taking derivative with respect to $\theta$ in (47) yields
\[ \pi'_{q,\nu,R}(\theta) \frac{\partial j(\rho)}{\partial x}(\pi_{q,\nu,R}(\theta)t, t) = \frac{\partial j(\rho(\theta))}{\partial \theta}, \] (48)
where $j$ is considered as a function of the variable $\rho$ in the left-hand-side. Finally (46) and (48), together with the fact that $\rho$ solves the PDE (39), imply that
\[ \pi_{q,\nu,R}(\theta) = \frac{\partial j(\rho(\theta))}{\partial \theta} \left/ \frac{\partial \rho(\theta)}{\partial \theta} \right., \]
which yields the formula
\[ \pi_{q,\nu,R}(\theta) = \frac{1 - q}{\log(q)^2} \left[ R/\nu \left( \Psi_q'(\theta + V) + \Psi_q''(\theta + V) \frac{\log q + \Psi_q(\theta) - \Psi_q(\theta + V)}{\Psi_q'(\theta + V)} \right) \right. \]
\[ \left. - L \left( \Psi_q'(\theta) + \Psi_q''(\theta) \frac{\log q + \Psi_q(\theta) - \Psi_q(\theta + V)}{\Psi_q'(\theta + V)} \right) \right] \]. (49)

5.1.2. Computation of $\kappa_{q,\nu,R}(\theta)$. We are searching for a function $\kappa_{q,\nu,R}(\theta)$ such that asymptotically when $t$ goes to infinity, the particle indexed by $[\kappa_{q,\nu,R}(\theta)t]$ is asymptotically at the position $\pi_{q,\nu,R}(\theta)t$, when the system starts from step initial condition. For this purpose we can integrate the density between the position $\pi(\theta)$ and a position where we know $\kappa$. We claim that since we start from step initial condition, at the left end of the rarefaction fan the density is continuous and equals 1, and thus $\alpha = 0$ (i.e. $\theta = +\infty$). We see later that a discontinuity on the right end of the rarefaction fan can occur. Since we consider step initial data, for any fixed $t$, $x_N(t) = -N$ for $N$ large enough. Thus one has $\kappa(\theta = +\infty) = -\pi(\theta = +\infty)$. Integrating the density,
\[ \kappa(\theta) - (-\pi(\infty)) = \int_{\theta}^{\infty} \rho(\theta') \frac{d\pi(\theta')}{d\theta'} d\theta', \]
where we have kept fixed the variables $q$, $\nu$ and $R$. Integrating by parts, this gives
\[ \kappa(\theta) = -\pi(\infty) + [\rho \pi - j(\rho)]^\infty_{\theta} = -\rho(\theta) \pi(\theta) + j(\rho(\theta)) \]
This yields the formula
\[ \kappa_{q,\nu,R}(\theta) = \frac{1 - q}{\log(q)^2} \frac{R}{\nu} \frac{\Psi_q''(\theta + V) - L \Psi_q''(\theta)}{\Psi_q'(\theta) - \Psi_q'(\theta + V)}. \] (50)
In order to make sense physically, the quantity $\kappa_{q,\nu,R}(\theta)$ must be positive, at least for $\theta$ belonging to some interval $(\bar{\theta}, +\infty)$. Since $\kappa_{q,\nu,R}(\theta)$ tends to $R - L$ when $\theta$ tends to infinity (equivalently $\alpha \to 0$), this requires that $R > L$ and suggests that the particles lie on a support of size $O$(time) with high probability only if $R > L$.

Now assume that $R > L > 0$. Then $\kappa_{q,\nu,R}(\theta)$ tends to $-\infty$ when $\theta$ tends to 0. The local behaviour of particles around the first particles is described by the stationary measure $\mu_{\alpha_0}$, where $\alpha_0 = q^{\theta_0}$ is
such that $\kappa_{q,\nu,R}(\theta_0) = 0$. If $R > L > 0$, then $0 < \theta_0 < \infty$, which means that the density of particles $\rho$ is strictly positive around the first particle and that the density profile exhibit a discontinuity at the first particle, see Figure 3. (Note that the curved section in Figure 3 is the parametric curve $(\pi_{q,\nu}(\theta), \rho(\theta))$ for $\theta \in (\theta_0, +\infty)$ where $\theta_0$ is such that $\kappa_{q,\nu}(\theta_0) = 0$. This density profile is proved as a consequence of Theorem 6.2 in the case $q = \nu$.) Figure 8 provides an additional confirmation using simulation data.

The macroscopic position of the first particle is then given by

$$\pi(\theta_0) = \frac{1 - q}{(\log q)^2} \left( \frac{R}{\nu} \Psi_q'(\theta_0 + V) - L \Psi_q'(\theta_0) \right),$$

where $\theta_0 = \log_2(\alpha_0)$. Not surprisingly, this is also the drift of a particle in an environment given by $\mu_{\alpha_0}$.

### 5.2. Magnitude of fluctuations

One first needs to compute $\lambda = -j''(\rho)$. We have expressions for $j(\rho(\theta))$ and $\rho(\theta)$ but we take the second derivative of the function $j$ with respect to the variable $\rho$. We have that

$$j''(\rho(\theta)) = \frac{1 - q}{(\log q)^3} \left( \Psi_q'(\theta + V) - \Psi_q'(\theta) \right)^2 \times
\left( \frac{R}{\nu} \Psi_q''(\theta + V) - L \Psi_q''(\theta) - \left( \frac{R}{\nu} \Psi_q''(\theta + V) - L \Psi_q''(\theta) \right) \frac{\Psi_q''(\theta) - \Psi_q''(\theta + V)}{\Psi_q'(\theta) - \Psi_q'(\theta + V)} \right).$$

Note that by Lemma 5.2 $j''(\rho) \neq 0$ so that the main assumption of the KPZ class conjecture is satisfied.

In order to compute $A(\rho)$, we follow [Spo12] and define

$$Z(\alpha) = \frac{(\alpha \nu; q)_\infty}{(\alpha; q)_\infty},$$

the normalization constant in the definition of (42), and $G(\alpha) = \log(Z(\alpha))$. Then

$$A = \frac{\alpha G''(\alpha)}{(1 + \alpha G')^3},$$

where all derivatives are taken with respect to the variable $\alpha$. (The formula differs by a factor 4 with [Spo12] because we take occupation variables $\eta_i \in \{0, 1\}$ instead of $\{-1, 1\}$.) With $Z$ as in (51), we have

$$G'(\alpha) = \frac{1}{\alpha \log q} \left( \Psi_q'(\theta) - \Psi_q'(\theta + V) \right),$$

and

$$A(\theta) = \log q \frac{\Psi_q'(\theta) - \Psi_q'(\theta + V)}{\left( \log q + \Psi_q'(\theta) - \Psi_q'(\theta + V) \right)^3}.$$  

Finally, $\sigma_{q,\nu}(\theta) = \left( \frac{\lambda A^2}{2\rho^3} \right)^{1/3}$ with

$$\frac{\lambda A^2}{2\rho^3} = \frac{q - 1}{4(\log q)^4} \left( \frac{R}{\nu} \Psi_q''(\theta + V) - L \Psi_q''(\theta) - \left( \frac{R}{\nu} \Psi_q''(\theta + V) - L \Psi_q''(\theta) \right) \frac{\Psi_q''(\theta) - \Psi_q''(\theta + V)}{\Psi_q'(\theta) - \Psi_q'(\theta + V)} \right).$$

One should note that we have always $\sigma_{q,\nu}(\theta) > 0$ (see Lemma 5.2 for a proof of this claim).
5.3. Critical point Fredholm determinant asymptotics. We sketch an asymptotic analysis
of the Mellin-Barnes Fredholm determinant formula of Theorem 4.10 that confirms the KPZ class
conjecture for the continuous time-sided q-Hahn exclusion process. In particular, we recover
independently the functions \( \pi_{q,\nu}(\theta) \), \( \kappa_{q,\nu}(\theta) \) and \( \sigma_{q,\nu}(\theta) \) from [49], [50] and [53]. We do not provide
all necessary justifications to make this rigorous. However, in Section 6, we do so for the \( \nu = q \) case
under certain ranges of parameters.

The function \( x \mapsto 1/(−q^2; q)_\infty \) converges to 1 in \(+\infty\) and 0 in \(−\infty\) and the sequence of functions
\( \left( x \mapsto 1/(−q^{1/3}; q)_\infty \right)_{t>0} \) satisfies the hypotheses of Lemma 4.1.39 in [BC14]. On account of this,
if we set
\[
\zeta = -q^{-\kappa t - \pi t - i^{\frac{1}{3}} \sigma x},
\]
then it follows that for \( \sigma > 0 \),
\[
\lim_{t \to \infty} \mathbb{E} \left[ \frac{1}{(\zeta q^{\nu t} + n; q)_\infty} \right] = \lim_{t \to \infty} \mathbb{P} \left( \frac{x_n(t) - \pi t}{\sigma t^{1/3}} \geq x \right),
\]
with \( n = \lfloor \kappa t \rfloor \). For the moment, we let the constants \( \kappa, \pi \) and \( \sigma \) remain undetermined.

\[
\mathbb{E} \left[ \frac{1}{(\zeta q^{\nu t} + n; q)_\infty} \right]
\]
is given by det \((I + K_\zeta)\) as in (34). Assume for the moment that the contour
\( C_1 \) for the variable \( w \) is a very small circle around 1. Let us make the change of variables
\[
w = q^W, \quad w' = q^{W'}, \quad s + W = Z.
\]
Then the Fredholm determinant can be written with the new variables as det \((I + K_x)\) where \( K_x \) is
an operator acting on \( L^2(C_0) \) where \( C_0 \) is the image of \( C_1 \) under the mapping \( w \mapsto \log_q(w) \), defined
by its kernel
\[
K_x(W, W') = \frac{q^W \log q}{2 \pi i} \int_{D_W} \frac{\pi}{\sin(-\pi(Z-W))} \exp \left( t(f_0(Z) - f_0(W)) - t^{1/3} \sigma x \log(q)(Z-W) \right) \frac{1}{q^Z - q^{W'}} \frac{(\nu q^Z; q)_\infty}{(\nu q^W; q)_\infty} dZ,
\]
where the new contour \( D_W \) as the straight line \( W + 1/2 + i\mathbb{R} \), and the function \( f_0 \) is defined by
\[
f_0(Z) = \kappa \log \left( \frac{(q^Z; q)_\infty}{(\nu q^Z; q)_\infty} \right) + \frac{1 - q}{\log(q)} \left( \frac{R}{\nu} \Psi_q(Z + V) - L \left( \Psi_q(Z) \right) \right) - \log(q)(\kappa + \pi).
\]
Since \( C_1 \) was any small enough circle around 1, \( C_0 \) can be deformed to be a small circle around 0,
and we can also deform the contour for \( Z \) to be simply \( 1/2 + i\mathbb{R} \) without crossing any singularities.

The principle of Laplace’s method is to deform the integration contours so that they go across a
critical point of \( f_0 \), and then make a Taylor approximation around the critical point. Actually, we
know that the Airy kernel would occur in the limit if this critical point is a double critical point, so
we determine our unknown parameters \( \kappa, \pi, \sigma \) so as to have a double critical point. We have that
\[
f_0'(Z) = \kappa \left( \Psi'_q(Z + V) - \Psi_q(Z) \right) + \frac{1 - q}{\log(q)} \left( \frac{R}{\nu} \Psi'_q(Z + V) - L \left( \Psi'_q(Z) \right) \right) - \log(q)(\kappa + \pi),
\]
and
\[
f_0''(Z) = \kappa \left( \Psi''_q(Z + V) - \Psi'_q(Z) \right) + \frac{1 - q}{\log(q)} \left( \frac{R}{\nu} \Psi''_q(Z + V) - L \left( \Psi''_q(Z) \right) \right).
\]
We see that if \( \pi = \pi_{q,\nu}(\theta) \) and \( \kappa = \kappa_{q,\nu}(\theta) \) as in [49] and [50], then \( f_0'(\theta) = f_0''(\theta) = 0 \). Hence, up
to higher order terms in \((Z - \theta)\),
\[
f_0(Z) - f_0(W) \approx \frac{f_0''(\theta)}{6} ((Z - \theta)^3 - (W - \theta)^3).
\]
The next lemma, about the sign of \( f_0''' \), is proved in Section 6.3

**Lemma 5.2.** For any \( q \in (0, 1) \), \( \nu \in [0, 1) \), and any \( R, L \geq 0 \) such that \( R + L = 1 \), we have that for all \( \theta > 0 \), \( f_0'''(\theta) > 0 \).

Using Lemma 5.2, we know the behaviour of \( \text{Re}[f_0] \) in the neighbourhood of \( \theta \). To make Laplace’s method rigorous, we must control the real part of \( f_0 \) along the contours for \( Z \) and \( W \), and prove that only the integration in the neighbourhood of \( \theta \) has a contribution to the limit. We do not prove that here, and the rest of the asymptotic analysis presented in this section would require some additional effort to be completely rigorous.

Assume that one is able to deform the contours for \( Z \) and \( W \) passing through \( \theta \) so that
- The contour for \( Z \) departs \( \theta \) with angles \( \xi \) and \( -\xi \) where \( \xi \in (\pi/6, \pi/2) \), and \( \text{Re}[f_0] \) attains its maximum uniquely at \( \theta \),
- The contour for \( W \) departs \( \theta \) with angles \( \omega \) and \( -\omega \) where \( \omega \in (\pi/2, 5\pi/6) \), and \( \text{Re}[f_0] \) attains its minimum uniquely at \( \theta \).

Then, modulo some estimates that we do not state explicitly here, the Fredholm determinant can be approximated by the following. We make the change of variables \( Z - \theta = z t^{-1/3} \) and likewise for \( W \) and \( W' \). Taking into account the Jacobian of the \( W \) and \( W' \) change of variables, we get that the kernel has rescaled to

\[
\tilde{K}_x(w, w') = \frac{1}{2\ii \pi} \int_{-\infty}^{\infty} \frac{1}{w - z} \exp \left( f_0'''(\theta)/6(z^3 - w^3) - \sigma x \log(q)(z - w) \right) dz.
\]

Finally, if we set \( \sigma = \left( -\frac{f_0'''(\theta)}{2(\log q)^3} \right)^{1/3} \), and we make the change of variables replacing \( -z\sigma \log(q) \) by \( z \) and likewise for \( w \) and \( w' \), we get the kernel

\[
\tilde{K}_x(w, w') = \frac{1}{2\ii \pi} \int_{-\infty}^{\infty} \frac{1}{w - z} e^{3/3 - w^3/3 + x(z - w)} dz,
\]

acting on a contour coming from \( \infty e^{-2i\pi/3} \) to \( \infty e^{2i\pi/3} \) which does not intersect the contour for \( z \). Let us call \( G \) this contour. Using the “\( \text{det}(I - AB) = \text{det}(I - BA) \) trick” to reformulate Fredholm determinants, see e.g. Lemma 8.6 in [BCF14], one has that

\[
\det(I + \tilde{K}_x)_{L^2(G)} = \det(I - K_{AI})_{L^2(-x, +\infty)},
\]

where \( K_{AI} \) is the Airy kernel defined in 6.1. Since \( F_{\text{GUE}}(x) = \det(I - K_{AI})_{L^2(-x, +\infty)} \), we have that

\[
\lim_{t \to \infty} \mathbb{P} \left( \frac{x_n(t) - t\pi(\theta)}{\sigma(t)^{1/3}} \geq x \right) \xrightarrow{t \to \infty} F_{\text{GUE}}(-x)
\]

as claimed in (41).

The expression for \( \sigma_{q, \nu}(\theta) \) in (63) is the same as \( \sigma = \left( -\frac{f_0'''(\theta)}{2(\log q)^3} \right)^{1/3} \). Indeed, we have that

\[
f_0'''(Z) = \frac{1 - q}{\log q} \left( \frac{R}{\nu} \frac{\Psi_q'''(Z + V)}{\Psi_q'''(Z)} - L \frac{\Psi_q'''(\theta + V)}{\Psi_q'''(\theta)} \right) \frac{\Psi_q''(Z) - \Psi_q''(Z + V)}{\Psi_q''(\theta) - \Psi_q''(\theta + V)},
\]

so that \( \sigma_{q, \nu}(\theta)^3 = \left( -\frac{f_0'''(\theta)}{2(\log q)^3} \right)^{1/3} \).

6. Asymptotic Analysis

In this section, we make the arguments of Section 5.3 rigorous in the case \( \nu = q \), which, in light of Section 4.3.2, corresponds with the MADM. In order to simplify the notations we set \( \pi(\theta) = \pi_{q, q, R}(\theta) \), \( \kappa(\theta) = \kappa_{q, q, R}(\theta) \), and \( \sigma(\theta) = \sigma_{q, q, R}(\theta) \), without writing explicitly the dependency on the parameters \( q \) and \( R \).
Suppose additionally that 

\[ q \]

of the kernel ensures that we do not cross any residues when deforming the integration contour in the definition of the distribution function \( F_{\text{GUE}}(x) \) of the GUE Tracy-Widom distribution is defined by \( F_{\text{GUE}}(x) = \det(I - K_{\Lambda i})_{L^2(x, +\infty)} \) where \( K_{\Lambda i} \) is the Airy kernel,

\[
K_{\Lambda i}(u, v) = \frac{1}{(2\pi i)^2} \int e^{2ix/\beta} \int_{e^{-2ix/\beta}} \int_{e^{-i\beta x}} \frac{1}{e^{wz/\alpha} - w},
\]

where the contours for \( z \) comes from infinity with an angle \(-\pi/3\) and go to infinity with an angle \(\pi/3\) ; the contour for \( w \) comes from infinity with an angle \(-2\pi/3\) and go to infinity with an angle \(2\pi/3\), and both contours do not intersect.

**Theorem 6.2.** Fix \( q \in (0, 1) \), \( \nu = q \) and \( R > L \geq 0 \) with \( R + L = 1 \). Let \( \theta > 0 \) such that \( \kappa(\theta) \geq 0 \). Suppose additionally that \( q^\theta > 2q/(1 + q) \). Then, for \( n = \lfloor \kappa(\theta)t \rfloor \), we have

\[
\lim_{t \to \infty} \mathbb{P} \left( \frac{x_n(t) - \pi(\theta)t}{\sigma(\theta)t^{1/3}} \geq x \right) = F_{\text{GUE}}(-x).
\]

**Remark 6.3.** One can check on simulated numerical data that the sign of \( \sigma(\theta) \) must be positive. Indeed, on Figures 7 and 8, one can see that the simulated curve is above the limiting curve predicted from KPZ scaling theory. This is coherent with the positive sign of \( \sigma(\theta) \) and the fact that the Tracy-Widom distribution has negative mean.

**Theorem 6.4.** Fix \( q \in (0, 1) \), \( \nu = q \) and let

\[
R_{\min}(q) = \frac{q \Psi''_q \left( \log_q \left( \frac{2q}{1 + q} \right) \right)}{\Psi'_q \left( \log_q \left( \frac{2q}{1 + q} \right) \right) + q \Psi''_q \left( \log_q \left( \frac{2q}{1 + q} \right) \right)} \in \left( \frac{1}{2}, 1 \right).
\]

Then for \( R_{\min}(q) < R < 1 \) and \( L = 1 - R \), there exists a real number \( \theta_0 > 0 \) such that \( \kappa_{q, q, R}(\theta_0) = 0 \), and we have

\[
\lim_{t \to \infty} \mathbb{P} \left( \frac{x_1(t) - \pi(\theta_0)t}{\sigma(\theta_0)t^{1/3}} \geq x \right) = F_{\text{GUE}}(-x).
\]

**Remark 6.5.** We expect the same kind of result for the fluctuations of the position of the first particle in any \( q \)-Hahn AEP with positive asymmetry, when the parameter \( \nu \) is such that \( 0 < \nu < 1 \).

**Remark 6.6.** The condition \( q^\theta > 2q/(1 + q) \) in Theorem 6.2 is most probably purely technical. It ensures that we do not cross any residues when deforming the integration contour in the definition of the kernel \( K_{\zeta} \) in Theorem 4.10 (see Remark 6.8). The condition \( R_{\min}(q) < R \) in Theorem 6.4 is equivalent to \( q^\theta > 2q/(1 + q) \) in the particular setting of Theorem 6.4.

However, the condition \( R < 1 \) is really meaningful, since in the totally asymmetric case \( (R = 1) \), the first particle has Gaussian fluctuations.

### 6.1. Proof of Theorem 6.2

The proof uses Laplace’s method and follows the style of [BCF13] (similar proofs can be found in [Bar13] for \( q \)-TASEP with slow particles, in [BCF13] for the semi-discrete directed polymer, and in [Vet14] for the \( q \)-Hahn TASEP).

Fix \( q \in (0, 1) \), \( \nu = q \), \( R > L \geq 0 \) with \( R + L = 1 \) and \( \theta > 0 \) such that \( \kappa(\theta) \geq 0 \). In the particular case \( q = \nu \), Theorem 4.10 states that for all \( \zeta \in \mathbb{C} \setminus \mathbb{R}_+ \),

\[
\mathbb{E} \left[ \frac{1}{(\zeta q^x(t) + \nu; q)_\infty} \right] = \det(I + K_{\zeta})
\]

where \( \det(I + K_{\zeta}) \) is the Fredholm determinant of \( K_{\zeta} : L^2(C_1) \to L^2(C_1) \) for \( C_1 \) a positively oriented circle containing 1 with small enough radius so as to not contain 0, \( 1/q \). The operator \( K_{\zeta} \) is defined in terms of its integral kernel

\[
K_{\zeta}(w, w') = \frac{1}{2\pi i} \int_{-i\infty + 1/2}^{i\infty + 1/2} \frac{\pi}{\sin(-\pi s)}(-\zeta)^s \frac{g(w)}{g(q^s w)} \frac{1}{q^s w - w'} ds
\]
Figure 7. Comparison between simulated numerical data and predicted hydrodynamic limit. The black curve is \( (x_N(t)/t, N/t)_N \) for \( N \) ranging from 1 to \( t = 500 \) (which is fixed) in the totally asymmetric case \( (R = 1, L = 0) \), with \( \nu = q = 0.4 \). This is also the graph of the function \( x \mapsto N_{tx}(t)/t \), where by definition \( N_x(t) \) is the number of particles right to \( x \) at time \( t \). The gray curve is the parametric curve \( (\pi(\theta), \kappa(\theta))_{\theta \in (0, +\infty)} \) with \( \pi(\theta) \) and \( \kappa(\theta) \) as in (49) and (50).

Figure 8. The black curve is a simulation of \( (x_N(t)/t, N/t)_N \) for \( N \) ranging from 1 to \( (R - L)t \), with \( t = 1500 \) fixed, \( R = 0.9, L = .1 \) and \( \nu = q = 0.6 \). The gray curve is the parametric curve \( (\pi(\theta), \kappa(\theta))_{\theta \in (\theta_0, +\infty)} \) where \( \theta_0 \) is such that \( \kappa(\theta_0) \) as in Section 5.1.2. It goes from the point \( (L - R, R - L) \) to the point \( (\pi(\theta_0), 0) \). Since the slope of the curve \( x \mapsto N_{tx}(t)/t \) (or equivalently \( (x_N(t), N/t)_N \)) is the macroscopic density \( \rho(x, 1) \), this simulationally confirms the discontinuity of density at the point \( \pi(\theta_0) \) (see Figure 3).

with

\[
g(w) = \left( \frac{1}{1 - w} \right)^n \exp \left( \frac{(q - 1)t}{\log(q)} \left( \frac{R}{q} (\Psi_q(W + 1) + \log(1 - q)) - L (\Psi_q(W) + \log(1 - q)) \right) \right) \frac{1}{(qw; q)_\infty},
\]

where \( W = \log_q(w) \).

Remark 6.7. One notices that the argument of the exponential simplifies to \( t \frac{(1 - q)}{1 + q} \frac{w}{1 - w} \) when \( R/L = q \). This yields a simpler analysis, though we work with the general \( R, L \) case here.

As we have explained in Section 5.3 if we set

\[
\zeta = -q^{-\kappa(\theta)t - \pi(\theta)t - t^{1/3} \sigma(\theta)x},
\]
then it follows that
\[
\lim_{t \to \infty} \mathbb{E} \left[ \frac{1}{(q^x(t)+n)^\infty} \right] = \lim_{t \to \infty} \mathbb{P} \left( \frac{x_n(t) - \pi(\theta)t}{\sigma(\theta)t^{1/3}} \geq x \right),
\]
with \( n = \lfloor \kappa(\theta) \rfloor \). Also from Section 5.3 making the change of variables:
\[
w = q^W, \quad w' = q^{W'}, \quad s + W = Z,
\]
The Fredholm determinant \( \det(I + K_{\xi}) \) equals \( \det(I + K_x) \) where \( K_x \) is an operator acting on \( L^2(C_0) \) where \( C_0 \) is a small circle around 0, defined by its kernel
\[
K_x(W, W') = \frac{q^W \log q}{2\pi i} \int_D \frac{\pi}{\sin(\pi(Z-W))} \times \exp \left( t(f_0(Z) - f_0(W)) - t^{1/3}\sigma(\theta) \log(q)x(Z-W) \right) \frac{1}{q^Z - q^{W'}} \frac{(q^{Z+1}; q)_{\infty}}{(q^{W+1}; q)_{\infty}} dZ,
\]
where the new contour \( D \) is the straight line \( 1/2 + i \mathbb{R} \), and the function \( f_0 \) is defined by
\[
f_0(Z) = \kappa(\theta) \log(1-q^{-Z}) + \frac{1-q}{\log(q)} \left( R \Psi_q(Z + V) - L \Psi_q(Z) \right) - Z \log(q) \left( \kappa(\theta) + \pi(\theta) \right).
\]
Using the expressions (50) and (49) for \( \kappa(\theta) \) and \( \pi(\theta) \) in terms of the \( q \)-digamma function, we have
\[
f_0(Z) = \frac{1-q}{\log(q)} \left( R \Psi_q(Z + 1) + \log(1-q) - Z \Psi_q'(\theta + 1) \right.
\]
\[
+ \frac{\Psi''_q(\theta + 1)}{\log q} \left( \frac{(1-\alpha)^2 \log(1-q^Z)}{\alpha} + Z(1-\alpha) \right) \right.
\]
\[
- L \left[ \Psi_q(Z) + \log(1-q) - Z \Psi_q'(\theta) + \frac{\Psi''_q(\theta)}{\log q} \left( \frac{(1-\alpha)^2 \log(1-q^Z)}{\alpha} + Z(1-\alpha) \right) \right],
\]
with \( \alpha = q^\theta \). For the derivatives, we have
\[
f_0'(Z) = \frac{1-q}{\log(q)} \left( \Psi_q'(Z + 1) - \Psi_q'(\theta + 1) \right)
\]
\[
+ \frac{\Psi''_q(\theta + 1)}{\log q} \left( (1-\alpha) - \frac{(1-\alpha)^2}{\alpha} q^Z \right) \right)
\]
\[
- \frac{1-q}{\log(q)} L \left[ \Psi_q'(Z) - \Psi_q'(\theta) + \frac{\Psi''_q(\theta)}{\log q} \left( (1-\alpha) - \frac{(1-\alpha)^2}{\alpha} q^Z \right) \right],
\]
\[
f_0''(Z) = \frac{1-q}{\log(q)} \left( \Psi''_q(Z + 1) - \frac{q^Z}{(1-q^Z)^2} \frac{(1-\alpha)^2}{\alpha} \Psi_q''(\theta + 1) \right)
\]
\[
- \frac{1-q}{\log(q)} L \left[ \Psi''_q(Z) - \frac{q^Z}{(1-q^Z)^2} \frac{(1-\alpha)^2}{\alpha} \Psi_q''(\theta) \right].
\]
Notice that the formulas become much simpler in the special case of Remark 6.7. Using the fact that \( \Psi_q'(Z) - \Psi_q'(Z + 1) = \log(q)^2 \frac{q^Z}{1-q^Z} \), one has
\[
f_0'(Z) = \frac{(1-q) \log(q)}{(1+q)(1-\alpha)^2} \left( q^Z \left( 1 - \alpha^2 - \frac{(1-\alpha)^2}{1-q^Z} \right) - \alpha^2 \right).
\]
One readily verifies that \( f_0'(\theta) = f_0''(\theta) = 0 \). Since the saddle-point is at \( \theta \), we need to deform the integration contours for the variables \( Z \) and \( W \) so that they pass through \( \theta \) and control the real part of \( f_0 \) along these contours.

Let \( C_\alpha \) be the positively oriented contour enclosing 0 defined by its parametrization
\[
W(u) := \log_q(1 - (1-\alpha)e^{iu})
\]
for $u \in (-\pi, \pi)$. Hence $q^{W(u)}$ ranges in a circle of radius $(1 - \alpha)$ centered at 1 (see Figure 9). In order to use $C_\alpha$ as the contour for $W$ in the definition of the Fredholm determinant $\det(I + K_x)$, one should not encounter any singularities of the kernel when deforming the contour. Hence $C_\alpha$ should not enclose $-1$ (this is the equivalent with the fact that the contour $C_1$ in Theorem 4.10 must not enclose $1/q$.) For the rest of this section, we impose the condition

$$2 - \alpha < 1/q,$$

so that our contour deformation is valid.

When deforming the contour for the variable $W$, one also have to deform the contour for the variable $Z$, since in the original definition of $K_\zeta$ in Equation (62), the only singularities of the integrand for the variable $s$ are for $s \in Z$. This means that the singularities at $W + 1, W + 2, \ldots$ for the variable $Z$ must be on the right of the contour for $Z$. Let us choose the contour $D_\alpha$ being the straight line parametrized by $Z(u) := \theta + iu$ for $u \in \mathbb{R}$. To ensure that $\text{Re}[W + 1] > \theta$, or equivalently that $qw < \alpha$ (see Figure 9), we impose the condition that

$$\alpha > \frac{2q}{1 + q}.$$  \hspace{1cm} (69)

Condition (69) implies in particular the previous condition $2 - \alpha < 1/q$.

**Remark 6.8.** Condition (69) is the same as condition (2.15) in [Vet14] (modulo a sign mistake in (the present version of) [Vet14, Equation (2.15)]). To get rid of this condition, one would need to add small circles around each pole in $W + 1, W + 2, \ldots$ in the definition of the contour $D$, as in [FV13]. The rest of the asymptotic analysis would remain almost unchanged provided one is able to prove that for any $W \in C_\alpha$ and $k \geq 1$ such that $|q^{W+k}| > \alpha$, $\text{Re}[f_0(W) - f_0(W + k)] > 0$. In our case, it appears that the analysis of $\text{Re}[f_0(W) - f_0(W + k)]$ is computationally difficult and we do not pursue that here.

One notices that $\text{Re}[f_0]$ is periodic with a period $i\frac{2\pi}{\log q}$. Moreover, $f_0(Z) = \overline{f_0(Z)}$ so that $\text{Re}[f_0]$ is determined by its restriction on the domain $\mathbb{R} + i[0, -\pi/\log q]$. The following results about the behaviour of $\text{Re}[f_0]$ along the contours are proved in Section 6.3.

**Lemma 6.9.** For any $R > L \geq 0$ with $R + L = 1$, we have $f_0'''(\theta) > 0$.

**Proof.** This is a particular case ($\nu = q$) of Lemma 5.2, which we prove in Section 6.2.

**Proposition 6.10.** Assume that (68) holds. For any $R > L \geq 0$ with $R + L = 1$, the contour $C_\alpha$ is steep-descent for the function $-\text{Re}[f_0]$ in the following sense: the function $u \mapsto \text{Re}[f_0(W(u))]$ is increasing for $u \in [0, \pi]$ and decreasing for $u \in [-\pi, 0]$.\hfill $\square$
Proposition 6.11. Assume that (68) holds. For any \( R > L \geq 0 \) with \( R + L = 1 \), the contour \( \mathcal{D}_\alpha \) is steep-descent for the function \( \text{Re}[f_0] \) in the following sense: the function \( t \mapsto \text{Re}[f_0(Z(t))] \) is decreasing for \( u \in [0, -\pi/\log q] \) and increasing for \( u \in [\pi/\log q, 0] \).

We are now able to prove that asymptotically, the contribution to the Fredholm determinant of the contours are negligible outside a neighbourhood of \( \theta \).

Proposition 6.12. For any fixed \( \delta > 0 \) and \( \epsilon > 0 \), there exists a real \( t_0 \) such that for all \( t > t_0 \)

\[
\left| \det(I + K_x)_{L^2(C_\alpha)} - \det(I + K_{x,\delta})_{L^2(C_{\alpha,\delta})} \right| < \epsilon
\]

where \( C_{\alpha,\delta} \) is the intersection of \( C_\alpha \) with the ball \( B(\theta, \delta) \) of radius \( \delta \) around \( \theta \), and

\[
K_{x,\delta}(W,W') = \frac{q^W \log q}{2\pi i} \int_{\mathcal{D}_\delta} \frac{\pi}{\sin(-\pi(Z - W))} \times \exp \left( t(f_0(Z) - f_0(W)) - t^{1/3} \sigma(\theta) \log(q)x(Z - W) \right) \frac{1}{q^{Z} - q^{W'}} \frac{(q^{Z+1};q)_\infty}{(q^{W+1};q)_\infty} dZ,
\]

where \( \mathcal{D}_\delta = \mathcal{D} \cap B(\theta, \delta) \).

Proof. We have the Fredholm determinant expansion

\[
\det(I + K_x)_{L^2(C_\alpha)} = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{-\pi}^{\pi} ds_1 \ldots \int_{-\pi}^{\pi} ds_k \det(K_x(W(s_i), W(s_j))) \frac{dW(s_i)}{ds_i},
\]

with \( W(s) \) as in (67). Let us denote by \( s_\delta \) the positive real number such that \( |W(s_\delta) - \theta| = \delta \).

We need to prove that one can replace all the integrations on \([-\pi, \pi]\) by integrations on \([-s_\delta, s_\delta]\), making a negligible error. By Propositions 6.10 and 6.11, we can find a constant \( c_\delta > 0 \) such that for \( |s| > s_\delta \) and for any \( Z \in \mathcal{D}_\alpha \),

\[
\text{Re}[f_0(Z) - f_0(W(s))] < -c_\delta.
\]

The integral in (63) is absolutely integrable due to the exponential decay of the sine in the denominator. Thus, one can find a constant \( C_\delta \) such that for \( |s| > s_\delta \), any \( W' \in C_\alpha \) and \( t \) large enough,

\[
|K(W(s), W')| < C_\delta \exp(-tc_\delta/2).
\]

By dominated convergence the error (that is the expansion (70) with integration on \([-\pi, \pi] \setminus [-s_\delta, s_\delta]\)) goes to zero for \( t \) going to infinity.

We also have to prove that one can localize the \( Z \) integrals as well. Recall that \( \text{Re}[f_0] \) is periodic on the contour \( \mathcal{D}_\alpha \). By the steep-descent property of Proposition 6.11 and the same kind of dominated convergence arguments, one can localize the integrations on

\[
\bigcup_{k \in \mathbb{Z}} I_k, \text{ where } I_k = \left[ \theta - i\delta + i2k\pi/\log q, \theta + i\delta + i2k\pi/\log q \right],
\]

making a negligible error. Since \( f_0(Z) - f_0(\theta) \approx f_0''(\theta)/6(Z - \theta)^3 \), by making the change of variables \( Z = \theta + i2\pi k/\log q + zt^{-1/3} \), we see that only the integral for \( Z \in [\theta - i\delta, \theta + i\delta] \) contributes to the limit. Indeed, for \( k \neq 0 \), and \( Z \in I_k \)

\[
\frac{dZ}{\sin(\pi(Z - W))} \approx t^{-1/3} \exp \left( -2\pi^2 k/\log(q) \right).
\]

Hence the sum of contributions of integrals over \( I_k \) for \( k \neq 0 \) is \( O(t^{-1/3}) \) and one can finally integrate over \( \mathcal{D}_{W,\delta} \) making an error going to 0 as \( t \to \infty \). It is not enough to show that the error made on the kernel goes to zero as \( t \) goes to infinity, but one can justify that the error on the Fredholm determinant goes to zero as well by a dominated convergence argument on the expansion (70). \( \square \)
By the Cauchy theorem, one can replace the contours $D_\delta$ and $C_{\alpha,\delta}$ by wedge-shaped contours $\hat{D}_\varphi,\delta := \{\theta + \delta e^{i\varphi \text{sgn}(y)} | y | y \in [-1,1]\}$ and $\hat{C}_{\psi,\delta} := \{\theta + \delta e^{i(\pi-\psi)\text{sgn}(y)} | y | y \in [-1,1]\}$, where the angles $\varphi, \psi \in (\pi/6, \pi/2)$ are chosen so that the endpoints of the contours do not change.

Let us make the change of variables 
\[ Z = \theta + \tilde{z}t^{-1/3}, \quad W = \theta + \tilde{w}t^{-1/3}, \quad W' = \theta + \tilde{w}'t^{-1/3}. \]

We define the corresponding rescaled contours 
\[ D^L_\varphi := \{Le^{i\varphi \text{sgn}(y)} | y | y \in [-1,1]\}, \]
\[ C^L_\psi := \{Le^{i(\pi-\psi)\text{sgn}(y)} | y | y \in [-1,1]\}. \]

**Proposition 6.13.** We have the convergence 
\[ \lim_{t \to \infty} \det(I + K_t)^{L_2(C_\alpha)} = \det(I + K_{t,\infty})^{L_2(C_\alpha^\infty)}, \]
where for $L \in \mathbb{R}_+ \cup \{\infty\}$,
\[ K_{t,L} = \frac{1}{2 \pi i} \int_{D_{\tilde{z}}} \frac{d\tilde{z}}{(\tilde{z} - \tilde{w})(\tilde{w} - \tilde{z})} \exp \left(\left(-1/3 - x\tilde{z}\sigma(\theta) \log q\right)\frac{3}{3} - x\tilde{z}\tilde{\sigma}(\theta) \log q\right) \exp \left(\left(-1/3 - x\tilde{w}\sigma(\theta) \log q\right)\frac{3}{3} - x\tilde{w}\tilde{\sigma}(\theta) \log q\right). \]

**Proof.** By the change of variables and the discussion about contours above,
\[ \det(I + K_{t,\delta})^{L_2(C_{\alpha,\delta})} = \det(I + K_{t,\infty})^{L_2(C_{\alpha,\infty})}, \]
where $K_{t,\delta}$ is the rescaled kernel
\[ K_{t,\delta}(\tilde{w}, \tilde{w}') = t^{-1/3} K_{t,\delta}(\theta + \tilde{w}t^{-1/3}, \theta + \tilde{w}'t^{-1/3}), \]
where we use the contours $D^L_{\tilde{z}}$ for the integration with respect to the variable $\tilde{z}$.

Let us estimate the error that we make by replacing $f_0$ by its Taylor approximation. We recall that with our definition of $\sigma(\theta)$ in (43),
\[ f_0''(\theta) = -2 (\sigma(\theta) \log(q))^3. \]

Using Taylor expansion, there exists $C_{f_0}$ such that
\[ |f_0(Z) - f_0(\theta) + (\sigma(\theta) \log(q)(Z - \theta))^2 / 3| < C_{f_0}|Z - \theta|^4, \]
for $Z$ in a fixed neighbourhood of $\theta$ (say e.g. $|Z - \theta| < \theta$). Hence for $Z = \theta + \tilde{z}t^{-1/3}, W = \theta + \tilde{w}t^{-1/3}, W' = \theta + \tilde{w}'t^{-1/3},$
\[ |t(f_0(Z) - f_0(W)) - ((-\sigma(\theta) \log(q)\tilde{z})^3/3 - (-\sigma(\theta) \log(q)\tilde{w})^3/3)| < t^{-1/3}C_{f_0}(|\tilde{z}|^4 + |\tilde{w}|^4) \leq \delta(|\tilde{z}|^3 + |\tilde{w}|^3). \quad (71) \]

To control the other factors in the integrand, let
\[ F(Z,W,W') := \frac{t^{-1/3}q^W \log(q)\pi t^{-1/3}}{q^Z - q^W W} \sin(\pi(Z - W)) \frac{(q^{Z+1};q)_\infty}{(q^{W+1};q)_\infty}. \]
we have that
\[ F(Z,W,W') \xrightarrow{t \to \infty} F_{\lim}(\tilde{z}, \tilde{w}, \tilde{w}') := \frac{1}{\tilde{z} - \tilde{w}'} \frac{1}{\tilde{z} - \tilde{w}}. \]

**Lemma 6.14.** For $\tilde{z} \in D^L_{\tilde{z}}$, and $\tilde{w}, \tilde{w}' \in C^L_{\psi}$, with $Z = \theta + \tilde{z}t^{-1/3}, W = \theta + \tilde{w}t^{-1/3}$ and $W' = \theta + \tilde{w}'t^{-1/3}$, we have that
\[ |F(Z,W,W') - F_{\lim}(\tilde{z}, \tilde{w}, \tilde{w}')| < Ct^{-1/3}P(|\tilde{z}|, |\tilde{w}|, |\tilde{w}'|)F_{\lim}(\tilde{z}, \tilde{w}, \tilde{w}'), \]
where and $P$ is a polynomial and $C$ is a constant independent of $t$ and $\delta$, as soon as $\delta$ belongs to some fixed neighbourhood of 0.
Proof. Since $|Z - \theta| < \delta, |W - \theta| < \delta$ and $|W' - \theta| < \delta$, there exist constants $C_1, C_2$ and $C_3$ such that

$$\left| \frac{q^W \log(q)(Z - W')}{q^Z - q^W} - 1 \right| \leq C_1(|Z - \theta| + |W' - \theta|),$$

$$\left| \frac{\pi(Z - W)}{\sin(\pi(Z - W))} - 1 \right| \leq C_2(|Z - \theta| + |W - \theta|),$$

$$\left| \frac{(q^{Z+1}; q)_\infty}{(q^{W+1}; q)_\infty} - 1 \right| \leq C_3(|Z - \theta| + |W - \theta|).$$

Hence there exists a constant $C$ and a polynomial $P$ of degree 3 such that

$$\left| \frac{F(Z, W, W')}{F^{\lim}(\tilde{z}, \tilde{w}, \tilde{w}')} - 1 \right| \leq C t^{-1/3} P(|\tilde{z}|, |\tilde{w}|, |\tilde{w}'|),$$

and the result follows. \qed

Now we estimate the difference between the kernels $K^t_{x, \delta}$ and $K^t_{x, \delta t^{1/3}}$. Let

$$f(Z, W, W') = t(f_0(Z) - f_0(W)) - t^{1/3} \sigma(\theta) \log(q) x(Z - W)$$

and

$$f^{\lim}(\tilde{z}, \tilde{w}, \tilde{w}') = ((-\tilde{z} \sigma(\theta) \log(q))^3/3 - x\tilde{z} \sigma(\theta) \log(q) - ((-\tilde{w} \sigma(\theta) \log(q))^3/3 - x\tilde{w} \sigma(\theta) \log(q)).$$

The difference between the kernels is estimated by

$$|K^t_{x, \delta}(\tilde{w}, \tilde{w}') - K^t_{x, \delta t^{1/3}}(\tilde{w}, \tilde{w}')| \leq \int_{D^t_{\delta^{1/3}}} d\tilde{z} \exp(f^{\lim})|F| \cdot |\exp(f^{\lim} - f)| - 1| + \int_{D^t_{\delta^{1/3}}} d\tilde{z} \exp(f^{\lim})|F - F^{\lim}|,$$

where we have omitted the arguments of the functions $f(Z, W, W')$, $f^{\lim}(\tilde{z}, \tilde{w}, \tilde{w}')$, $F(Z, W, W')$ and $F^{\lim}(\tilde{z}, \tilde{w}, \tilde{w}')$.

Using the inequality $|\exp(x) - 1| < |x| |\exp(|x|)|$ and (71), we have

$$|\exp(f^{\lim} - f)| - 1| < t^{-1/3} C_{f_0} (|\tilde{z}|^4 + |\tilde{w}|^4) \exp(\delta (|\tilde{z}|^3 + |\tilde{w}|^3)).$$

Hence, for $\delta$ small enough, the first integral in the right-hand-side of (72) have cubic exponential decay in $|\tilde{z}|$, and the limit when $t \to \infty$ is zero by dominated convergence. The second integral goes to zero as well by the same argument. We have shown pointwise convergence of the kernels.

In order to show that the Fredholm determinants also converge, we give a dominated convergence argument. The estimate (71) also shows that for $\delta$ small enough, one can bound the kernel $K^t_{x, \delta}$ by

$$|K^t_{x, \delta}(\tilde{w}, \tilde{w}')| < C \exp(\Re(\sigma(\theta) \log(q) \tilde{w}^3))/6)$$

for some constant $C$. Then, Hadamard’s bound yields

$$\det(K^t_{x, \delta}(\tilde{w}_i, \tilde{w}_j))^n_{i,j=1} \leq n^{n/2} C^n \prod_{i=1}^n \exp(\Re(\sigma(\theta) \log(q) \tilde{w}_i^3)/6).$$

It follows that the Fredholm determinant expansion

$$\det(I + K^t_{x, \delta})_{L^2(C^t_{\delta^{1/3}})} = \sum_{n=0}^\infty \frac{1}{n!} \int_{C^t_{\delta^{1/3}}} d\tilde{w}_1 \cdots \int_{C^t_{\delta^{1/3}}} d\tilde{w}_n \det(K^t_{x, \delta}(\tilde{w}_i, \tilde{w}_j))^n_{i,j=1},$$
is absolutely integrable and summable. Thus, by dominated convergence
\[
\lim_{t \to \infty} \det(I + K_x)_{L^2(C_\alpha)} = \lim_{t \to \infty} \det(I + K_{x,\Delta t^{1/3}})_{L^2(C_\alpha^{t^{1/3}})} = \det(I + K_x')_{L^2(C_\alpha^{t^{1/3}})}.
\]

Finally, using a reformulation of the Airy kernel as in Section 5.3 and a new change of variables \(\tilde{z} \leftarrow -z \sigma(\theta) \log q\), and likewise for \(\tilde{w}\) and \(\tilde{w}'\), one gets
\[
\det(I + K_x')_{L^2(C_\alpha^{t^{1/3}})} = \det(I - K_{Ai})_{L^2(-\infty, \infty)},
\]
which finishes the proof of Theorem 6.2.

6.2. **Proof of Theorem 6.4.** The condition \(R < 1\) ensures that there exists a solution \(\theta_0 > 0\) to the equation
\[
\kappa_{q,q,R}(\theta) = 0.
\]
The condition \(R > R_{\text{min}}(q)\) ensures that the solution \(\theta_0\) is such that \(q^{\theta_0} > \frac{2q}{1+q}\). Indeed, given the definition of \(\kappa_{q,\nu,R}(\theta)\) in (50), \(\theta_0\) satisfies
\[
\frac{\Psi''(\theta_0) + 1}{q \Psi''(\theta_0)} = 1 - \frac{R_{\text{min}}(q)}{R_{\text{min}}(q)}.
\]
If we set \(\theta_{\text{max}} = \log_q(2q/(1 + q))\), then
\[
\frac{\Psi''(\theta_{\text{max}}) + 1}{q \Psi''(\theta_{\text{max}})} = 1 - \frac{R_{\text{min}}(q)}{R_{\text{min}}(q)}.
\]
Since the function \(\theta \mapsto \Psi''(\theta + 1)/\Psi''(\theta)\) is increasing on \(\mathbb{R}_+\), the condition \(R > R_{\text{min}}(q)\) implies that \(\theta_0 < \theta_{\text{max}}\) and equivalently \(q^{\theta_0} > \frac{2q}{1+q}\).

If we set \(\zeta = -q^{-\pi(\theta_0) t - t^{1/3} \sigma(\theta_0) x}\), then
\[
\lim_{t \to \infty} \mathbb{E} \left[ \frac{1}{(\zeta q^{21(t+1)^{1/3}})^{\infty}} \right] = \lim_{t \to \infty} \mathbb{P} \left( \frac{x_1(t) - \pi(\theta_0) t}{\sigma(\theta_0) t^{1/3}} \leq x \right).
\]
The \(q\)-Laplace transform \(\mathbb{E} \left[ \frac{1}{(\zeta q^{21(t+1)^{1/3}})^{\infty}} \right]\) is the Fredholm determinant of a kernel written in terms of \(f_0\) exactly as in (63) with the only modification that the integrand should be multiplied by
\[
\frac{\nu q^W; q)^{\infty}}{(q^W; q)^{\infty}} / \frac{\nu q^Z; q)^{\infty}}{(q^Z; q)^{\infty}}.
\]
This additional factor does not perturb the rest of the asymptotic analysis, and disappears in the limit when we rescale the variables around \(\theta\). Since the condition \(q^{\theta_0} > 2q/(1 + q)\) is satisfied, Theorem 6.4 follows from the proof of Theorem 6.2.

6.3. **Proofs of Lemmas about properties of \(f_0\).**

**Proof of Lemma 5.2.** With \(R + L = 1\), the expression for \(f_0''(\theta)\) in Equation (60) is linear in \(R\). Hence we may prove the positivity only for the extremal values, i.e. \(R = 1\) and \(R = 0\).

We first prove that the function
\[
\theta \in \mathbb{R}_{>0} \mapsto \frac{\Psi''(\theta)}{\Psi_0''(\theta)}
\]
is strictly increasing. We show that the derivative is positive, that is for any \(\theta > 0\),
\[
\Psi''(\theta) \Psi''(\theta) > (\Psi_0''(\theta))^2.
\]
Proof of Proposition 6.10. It suffices to prove that for
\[ \frac{\sum_{n,m \geq 1} n^2 \alpha^n m^2 \alpha^m}{1 - q^n 1 - q^m} > \frac{\sum_{n,m \geq 1} n^3 \alpha^n m^3 \alpha^m}{1 - q^n 1 - q^m}, \]
for \( \alpha \in (0,1) \). Each side of (73) is a power series in \( \alpha \), and we claim that the inequality holds for each coefficient. Indeed, keeping only the coefficient of \( \alpha^k \), we have to prove that
\[ \sum_{n=1}^{k-1} \frac{n^2(k-n)^2}{(1-q^n)(1-q^{k-n})} \geq \frac{n^2(k-n)^2}{(1-q^n)(1-q^{k-n})}, \]
with strict inequality for at least one coefficient. Symmetrizing the left-hand-side, the inequality is equivalent to
\[ \sum_{n=1}^{k-1} \frac{n^2(k-n)^2}{(1-q^n)(1-q^{k-n})} \frac{n^2 + (k-n)^2}{2} \geq \frac{n^2(k-n)^2}{(1-q^n)(1-q^{k-n})} n(k-n), \]
which clearly holds, with strict inequality for \( k \geq 3 \).

Case \( R = 1 \). In that case, we have to prove that
\[ \Psi_q''(\theta + V) - \Psi_q''(\theta + V) \frac{\Psi_q''(\theta) - \Psi_q''(\theta + V)}{\Psi_q''(\theta) - \Psi_q''(\theta + V)} < 0. \]
Using Cauchy mean value theorem, the ratio can be rewritten as
\[ \frac{\Psi_q''(\theta) - \Psi_q''(\theta + V)}{\Psi_q''(\theta) - \Psi_q''(\theta + V)} = \frac{\Psi_q''(\tilde{\theta})}{\Psi_q''(\tilde{\theta})}, \]
for some \( \tilde{\theta} \in (\theta, \theta + V) \). Since \( \Psi_q''(x) < 0 \) for \( x \in (0, +\infty) \), the inequality reduces to
\[ \frac{\Psi_q''(\theta + V)}{\Psi_q''(\theta)} > \frac{\Psi_q''(\tilde{\theta})}{\Psi_q''(\tilde{\theta})}, \]
which is true by the first part of the proof.

Case \( R = 0 \). In that case, we have to prove that
\[ \Psi_q''(\theta) - \Psi_q''(\theta) \frac{\Psi_q''(\theta) - \Psi_q''(\theta + V)}{\Psi_q''(\theta) - \Psi_q''(\theta + V)} > 0. \]
Using the same argument, one is left with proving
\[ \frac{\Psi_q''(\theta)}{\Psi_q''(\theta)} < \frac{\Psi_q''(\tilde{\theta})}{\Psi_q''(\tilde{\theta})}, \]
which is already done as well.

The proof also applies to the \( \nu = 0 \) case, since the \( \nu \) in the denominator in Equation (60) can be cancelled by a factor \( \nu \) coming out from the \( q \)-digamma function.

Proof of Proposition 6.10. It suffices to prove that for \( u \in (0, \pi) \),
\[ \frac{d}{du} \text{Re}[f_0(W(u))] > 0. \]
We have
\[ \frac{d}{du} \text{Re}[f_0(W(u))] = \text{Re} \left[ \frac{dW(u)}{du} f_0'(W(u)) \right] = \text{Im} \left[ \frac{1}{\log q 1 - (1-\alpha)e^{iu}} f_0'(W(u)) \right]. \]
We use the linear dependence of \( f_0 \) on \( R \) as in the proof of Lemma 5.2.
Case $R = 1$. Using (65), one needs to prove that
\[
\text{Im} \left[ \frac{\Psi'_q(W(u) + 1)}{(\log q)^2} \frac{1 - q^W(u)}{q^W(u)} - \frac{\Psi'_q(A + 1)}{(\log q)^2} \frac{1 - q^W(u)}{q^W(u)} + \frac{\Psi''_q(A + 1)}{(\log q)^3} (1 - \alpha) \frac{1 - q^W(u)}{q^W(u)} \right] > 0.
\]

Using the series representation of the $q$-digamma function (5), the last inequality can be written as
\[
\text{Im} \left[ \sum_{k=1}^{\infty} \frac{(1 - \alpha)e^{iu}}{1 - (1 - \alpha)e^{iu}} \left( \frac{(1 - (1 - \alpha)e^{iu})q^k}{(1 - (1 - \alpha)e^{iu})q^k} - \frac{\alpha q^k}{(1 - \alpha q^k)^2} + \frac{\alpha q^k(1 + \alpha q^k)(1 - \alpha)}{(1 - \alpha q^k)^3} \right) \right] > 0.
\]

A computation – painful by hand, but easy for Mathematica – shows that the left-hand-side can be rewritten as
\[
\sum_{k=1}^{\infty} 4\sin(u)\sin^2(u/2)(1 - \alpha)^2\alpha q^k(1 - (2 - \alpha)q^k)h(\alpha, q^k, u) |1 - (1 - \alpha)e^{iu}|^2 |1 - (1 - (1 - \alpha)e^{iu})q^k|^2 |1 - \alpha q^k|^3,
\]
where
\[
h(\alpha, q, u) = 1 - \alpha q \left( 4 - \alpha (2 + 2q(1 - \alpha) + q^2(2 - q)(1 + (1 - \alpha)^2)) \right) + 2(1 - \alpha)\alpha^2 q^2(1 - q^2)\cos(u).
\]
For any $u \in (0, \pi)$, $\cos(u) \geq -1$, hence
\[
h(\alpha, q, u) \geq 1 - \alpha q(2 - \alpha) (2 - \alpha q^2(2 - \alpha)(2 - q))
\]
and for any $\alpha \in (0, 1), q \in (0, 1), 1 - \alpha q(2 - \alpha) (2 - \alpha q^2(2 - \alpha)(2 - q)) \geq 0$. Thus, if $(2 - \alpha)q < 1$, each term in (75) is positive.

Case $R = qL$. Since $R + L = 1$, this case corresponds to $R = q/(1 + q)$ and $L = 1/(1 + q)$. As we have noticed in Remark 6.7, we have the simpler expression (66) for $f_0'$ when $R = qL$. Hence it is enough to show that
\[
\text{Im} \left[ \frac{1 - q}{(1 + q)(1 - \alpha)^2} \left( 1 - \alpha^2 - \frac{(1 - \alpha)^2}{1 - q^W(u)} - \alpha^2 \frac{1 - q^W(u)}{q^W(u)} \right) \right] > 0
\]
or equivalently, that
\[
\frac{1 - q}{(1 + q)(1 - \alpha)^2} (1 - \alpha) \sin(u) \left( 1 - \frac{\alpha^2}{|q^W(u)|^2} \right) > 0
\]
which is true since $|q^W(u)| \leq \alpha$ by assumption.

To conclude, since $f_0$ is linear in $R$, the result is also proved for any value $R \in [q/(1 + q), 1]$. □

Proof of Proposition 6.11. It suffices to show that for $u \in (0, -\pi/\log(q))$,
\[
0 > \frac{d}{du} \text{Re}[f_0(Z(u))] = -\frac{1}{\log q} \text{Im}[f_0'(Z(u))],
\]
where $Z(u) = \theta + iu \ (u \in \mathbb{R})$. We use the linear dependence of $f_0$ on $R$ as in the proof of Lemma 5.2 and Proposition 6.10.

Case $R = 1$. Using (65), one has to show that
\[
\text{Im} \left[ \frac{\Psi'_q(Z(u) + 1)}{(\log q)^2} - \frac{\Psi'_q(A + 1)}{(\log q)^2} \frac{(1 - \alpha)^2}{\alpha} \frac{1 - q^Z(u)}{1 - q^Z(u)} \right] > 0.
\]

Using the series representation of the $q$-digamma function (5), the last inequality can be written
\[
\text{Im} \left[ \sum_{k=1}^{\infty} \frac{\alpha e^{iu}q^k}{(1 - \alpha e^{iu}q^k)^2} - \frac{\alpha q^k(1 + \alpha q^k)(1 - \alpha)^2 e^{iu}}{(1 - \alpha q^k)^3 (1 - \alpha e^{iu})} \right] > 0.
\]
The left-hand-side equals

\[
\sum_{k=1}^{\infty} \frac{\sin(u)\alpha(1 - \alpha q^k)(2 - \alpha - \alpha^2 q^k)(1 + (\alpha - 2)q^k)}{|1 - \alpha e^{iu}q^k|^4 (1 - \alpha q^k)^3|1 - \alpha e^{iu}|^2}.
\] (76)

If \((2 - \alpha)q < 1\), then for all \(k \geq 1\), \(1 + (\alpha - 2)q^k \geq 0\), and each term in (76) is positive.

**Case** \(R = qL\). Using (66), it is enough to show that

\[
\text{Im} \left[ \frac{q^Z(u)}{1 - q^Z(u)} \left( 1 - \alpha^2 - \frac{(1 - \alpha)^2}{1 - q^Z(u)} \right) - \alpha \right] > 0,
\]

which is true since the left-hand-side equals

\[
\frac{2\sin(u)\alpha^2(1 - \alpha^2)(1 - \cos(u))}{|1 - \alpha e^{iu}|^2}.
\]

To conclude, since \(f_0\) is linear in \(R\), the result is also proved for any value \(R \in [q/(1 + q), 1]\). \(\square\)

**APPENDIX A. FREE EVOLUTION WITH BOUNDARY CONDITIONS IN THE DISCRETE TIME CASE**

One cannot prove the analogue of Proposition 4.6 in the discrete time case using the same boundary conditions. The step which breaks down in the two-sided discrete time case is the ability to show that the transition matrix of the (discrete-time) zero-range process acts on functions \(W^k \rightarrow \mathbb{R}\) as the transition matrix of a free evolution of particles on \(\mathbb{Z}^k\) subject to a two-body interaction condition on the boundary of \(\mathbb{Z}^k\). We will demonstrate this through an example.

Consider the situation of discrete time \(q\)-Hahn AZRP with two particles. For compactness, let us denote \(\varphi_{jm} = a\varphi_{q,m,j}(j|m)\) and \(\varphi'_{jm} = b\varphi_{q^{-1},\nu^{-1},j}(j|m)\). The transition matrix \(\mathcal{B}\) of the zero-range process acts on functions \(f : \mathbb{W}^2 \rightarrow \mathbb{R}\) by

\[
\mathcal{B}f(n_1, n_2) = (\varphi_{11})^2 f(n_1 - 1, n_2 - 1) + \varphi_{11} (\varphi_{01} + \varphi'_{01}) f(n_1 - 1, n_2) + \varphi_{11} (\varphi_{01} + \varphi'_{01}) f(n_1, n_2 - 1)
\]

\[
+ (\varphi'_{11})^2 f(n_1 + 1, n_2 + 1) + \varphi'_{11} (\varphi_{01} + \varphi'_{01}) f(n_1 + 1, n_2) + \varphi'_{11} (\varphi_{01} + \varphi'_{01}) f(n_1, n_2 + 1)
\]

\[
+ \varphi_{11}\varphi'_{11} f(n_1 + 1, n - 2 - 1) + \varphi_{11}\varphi'_{11} f(n_1 + 1, n_2 + 1) + (\varphi_{01} + \varphi'_{01})^2 f(n_1, n_2),
\] (77)

when \(n_1 > n_2\) and \(\mathcal{B}\) acts by

\[
\mathcal{B}f(n, n) = \varphi_{12} f(n, n - 1) + \varphi_{22} f(n - 1, n - 1) + \varphi'_{12} f(n + 1, n)
\]

\[
+ \varphi_{22} f(n + 1, n + 1) + (\varphi_{02} + \varphi'_{02}) f(n, n),
\] (78)

when \(n_1 = n_2\).

When \(n_1 > n_2\), the action of the transition matrix of the free evolution in \(\mathbb{Z}^2\) – let’s call it \(\mathcal{F}\) – acts exactly as in (77). However when \(n_1 = n_2\), \(\mathcal{F}\) acts by

\[
\mathcal{F}f(n, n) = (\varphi_{11})^2 f(n - 1, n - 1) + \varphi_{11} (\varphi_{01} + \varphi'_{01}) f(n - 1, n) + \varphi_{11} (\varphi_{01} + \varphi'_{01}) f(n, n - 1)
\]

\[
+ (\varphi'_{11})^2 f(n + 1, n + 1) + \varphi'_{11} (\varphi_{01} + \varphi'_{01}) f(n + 1, n) + \varphi'_{11} (\varphi_{01} + \varphi'_{01}) f(n, n + 1)
\]

\[
\varphi_{11}\varphi'_{11} f(n + 1, n - 1) + \varphi_{11}\varphi'_{11} f(n - 1, n + 1) + (\varphi_{01} + \varphi'_{01})^2 f(n, n),
\] (79)

which involves states outside the Weyl chamber. Hence the boundary condition should be such that the right-hand side in (78) equals the right-hand-side in (79). When \(a\) or \(b\) equal zero, we know from the study of the \(q\)-Hahn TASEP that the suitable boundary condition is

\[
f(n - 1, n) = \alpha f(n - 1, n - 1) + \beta f(n, n - 1) + \gamma f(n, n),
\] (80)

where \(\alpha, \beta\) and \(\gamma\) have the same expressions as in the statement of Proposition 4.6.
If the boundary condition (80) holds, then one can rewrite the right-hand-side of (78) as
\[
\begin{align*}
\varphi_{11}\varphi_{01}(n-1,n) + \varphi_{11}\varphi_{01}(n,n-1) + (\varphi_{11})^2f(n-1,n-1) + (\varphi_{01})^2f(n,n) \\
+ \varphi_{11}'\varphi_{01}'(n+1,n) + \varphi_{11}'\varphi_{01}'(n,n+1) + (\varphi_{11})^2f(n+1,n+1) + (\varphi_{01})^2f(n,n).
\end{align*}
\] (81)

Hence, assuming that the boundary condition (80) holds, the difference of (78) and (79) is
\[
0 = 2\varphi_{01}\varphi_{01}'(n,n) + \varphi_{11}\varphi_{01}'(n-1,n) + \varphi_{11}\varphi_{01}'(n,n-1) \\
+ \varphi_{11}'\varphi_{01}'(n+1,n) + \varphi_{11}'\varphi_{01}'(n,n+1) \\
+ \varphi_{11}\varphi_{11}'(n+1,n,n-1) + \varphi_{11}\varphi_{11}'(n+1,n,n+1) \\
(82)
\]

which is significantly more complicated than the boundary condition (80). Here, we have searched for boundary conditions that do not depend on the asymmetry parameters, that is why we need the condition (80) when \(b = 0\). One might find a boundary condition depending on the asymmetry parameters that would be such that the difference between (78) and (79) is zero for all \(a, b\), but then one would need a more involved binomial identity generalizing Theorem 1 in [Pov13] to show that \(n\)-body interaction actually reduces to two-body interaction when dealing with system with more than two particles.

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G. Barraquand, Université Paris Diderot, PRES Sorbonne Paris Cité, Laboratoire de probabilités et modèles aléatoires, 5 Rue Thomas Mann, 75013, Paris, France

E-mail address: barraquand@math.univ-paris-diderot.fr

I. Corwin, Columbia University, Department of Mathematics, 2990 Broadway, New York, NY 10027, USA, and Clay Mathematics Institute, 10 Memorial Blvd. Suite 902, Providence, RI 02903, USA, and Institut Henri Poincaré, 11 Rue Pierre et Marie Curie, 75005 Paris, France

E-mail address: ivan.corwin@gmail.com