A more abstract bounded exploration postulate

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Abstract

In article “Sequential abstract state machines capture sequential algorithms”, one of us axiomatized sequential algorithms by means of three postulates: sequential time, abstract state, and bounded exploration postulates. Here we give a more abstract version of the bounded exploration postulate which is closer in spirit to the abstract state postulate. In the presence of the sequential time and abstract state postulates, our postulate is equivalent to the original bounded exploration postulate.

1 Introduction

This paper is essentially an oversized footnote to [1] where sequential algorithms are axiomatized by means of three postulates: sequential time, abstract state, and bounded exploration postulates. To make our exposition more self-contained, we restate the postulates.

Definition 1.1 (Sequential Algorithms). A sequential algorithm \( A \) is defined by means of the sequential time, abstract state, and bounded exploration postulates below.

Postulate 1 (Sequential Time). \( A \) is associated with a nonempty set \( \mathcal{S}(A) \) (of states), a nonempty subset \( \mathcal{I}(A) \) (of initial states), and a map \( \tau_A : \mathcal{S}(A) \to \mathcal{S}(A) \) (the one-step transformation).

We write simply \( \mathcal{S}, \mathcal{I}, \) and \( \tau \) when \( A \) is clear from the context. The original version of the postulate in [1] did not require that \( \mathcal{S} \) and \( \mathcal{I} \) be nonempty. This reasonable modification is due to [3].

Postulate 2 (Abstract State).

- The states are first-order structures of the same vocabulary \( \Upsilon \) (or \( \Upsilon(A) \), the vocabulary of \( A \)), and \( \mathcal{S}, \mathcal{I} \) are closed under isomorphisms.

- The one-step transformation \( \tau \) does not change the base set of any state, and any isomorphism from a state \( X \) to a state \( Y \) is also an isomorphism from \( \tau(X) \) to \( \tau(Y) \).

Symbols in \( \Upsilon \) are function symbols; relation symbols are viewed as function symbols whose interpretations taking Boolean values. Each function symbol \( f \) has some number \( j \) of argument places; \( j \) is the \textit{arity} of \( f \). The arity may be zero; in the logic literature nullary functions symbols are often called constants.
Terms are built as usual from nullary symbols but means of symbols of positive arity.

The vocabulary $\Upsilon$ contains logical symbols $\text{true}$, $\text{false}$, undef, the equality sign, and the standard propositional connectives; the other symbols in $\Upsilon$ are nonlogical. In any (first-order) $\Upsilon$ structure, the values of true, false, and undef are distinct logical elements; the other elements are nonlogical.

Let $X$ be a state, $f$ range over $\Upsilon$, $j$ be the arity of $f$, and $x_0, x_1, \ldots, x_j$ range over the base set $|X|$ of $X$. A triple $(f, \bar{a}, b)$ is a nontrivial update if $x_0 \neq f(x_1, \ldots, x_j)$ in $\tau(X)$. The update set $\Delta(X)$ of the algorithm $A$ at state $X$ is the set of nontrivial updates of $X$.

States $X$ and $Y$ coincide over a set $T$ of $\Upsilon$ terms, symbolically $X \equiv^T Y$, if every $t \in T$ has the same value in $X$ and $Y$.

**Postulate 3** (Bounded Exploration). There exists a finite set $T$ of closed $\Upsilon$ terms (called a bounded exploration witness) such that

$$X \equiv^T Y \implies \Delta(X) = \Delta(Y) \text{ for all states } X, Y. \quad \triangleright$$

If $T \subseteq T'$ are sets of terms and $T$ is a bounded exploration witness, then $T'$ is also a bounded exploration witness. For example, $T'$ could comprise $T$ and all subterms of terms in $T$, so that $T'$ is closed under subterms.

This completes the definition of sequential algorithms.

The bounded exploration postulate arguably contradicts the spirit of the abstract state postulate according to which a state is just a presentation of its isomorphism type so that only the isomorphism type of the state is important. In the bounded exploration postulate above, it is essential that the states $X$ and $Y$ are concrete.

In technical report [2] we gave a more abstract form of the bounded exploration postulate that is in the spirit of the abstract state postulate and that is equivalent to the original bounded exploration postulate in the presence of the sequential time and abstract state postulates. We republish the relevant part (Part 1) of the technical report here (with slight modifications) to make it easier to access.

## 2 The new bounded exploration postulate

If $f$ is a function symbol in $\Upsilon$, $X$ is a state of $A$, $t$ is an $\Upsilon$ term, and $T$ is a set of $\Upsilon$ terms, then $f_X$ is the interpretation of $f$ in $X$, $\nu_X(t)$ is the value of $t$ in $X$, and $\nu_X(T) = \{\nu_X(t) : t \in T\}$.

**Definition 2.1.** States $X, Y$ are $T$-similar if

$$\nu_X(s) = \nu_X(t) \iff \nu_Y(s) = \nu_Y(t) \text{ for all } s, t \in T. \quad \triangleright$$

If states $X, Y$ are $T$-similar then

$$\sigma(\nu_X(t)) = \nu_Y(t)$$  

(1)

is a bijection, the similarity function, from $\nu_X(T)$ to $\nu_Y(T)$.

\triangleright
Lemma 2.2. Suppose that $T$ is closed under subterms, $X$ and $Y$ are $T$-similar, and let term $t = f(t_1, \ldots, t_j) \in T$ and $x_i = \mathcal{V}_X(t_i)$ for $i = 1, \ldots, j$. Then

$$\sigma(f_X(x_1, \ldots, x_j)) = f_Y(\sigma(x_1), \ldots, \sigma(x_j)).$$  \hspace{1cm} (2)

Proof.

$$\sigma(f_X(x_1, \ldots, x_j)) = \sigma(f_X(\mathcal{V}_X(t_1), \ldots, \mathcal{V}_X(t_j))) = \sigma(\mathcal{V}_X(t)) = \mathcal{V}_Y(t) = f_Y(\mathcal{V}_Y(t_1), \ldots, \mathcal{V}_Y(t_j)) = f_Y(\sigma(x_1), \ldots, \sigma(x_j)). \quad \square$$

Remark 2.3. One may think that, under the hypotheses of Lemma 2, $\sigma$ is a partial isomorphism from $X$ to $Y$, so that (2) holds whenever $x_1, \ldots, x_j$ and $f_X(x_1, \ldots, x_j)$ are in the domain of $\sigma$. But this is not necessarily true. For example, let $a, b$ be nonlogical nullary symbols and $f$ a unary functional symbol in $\mathcal{Y}$. Set $T = \{a, b\}$ and consider states $X$ and $Y$ with nonlogical elements 1, 2, 3 where $f_X(1) = f_Y(1) = 2$, $f_X(2) = f_Y(2) = 3$, $f_X(3) = f_Y(3) = 1$, and

$$a_X = a_Y = 1, \quad b_X = 2, \quad b_Y = 3.$$  

The states $X$ and $Y$ are $T$-similar; in both cases the values of $a, b$ are distinct. But $\sigma$ is not a partial isomorphism because $\sigma(f_X(1)) = \sigma(2) = \sigma(b_X) = b_Y = 3$ while $f_Y(\sigma(1)) = f_Y(\sigma(a_X)) = f_Y(a_Y) = 2$. \hspace{1cm} \square

Definition 2.4. An element $x$ of state $X$ is $T$-accessible if $x = \mathcal{V}_X(t)$ for some $t \in T$. An update $u = (f, (x_1, \ldots, x_j), x_0)$ of $X$ is $T$-accessible if all elements $x_i$ are $T$-accessible. A set of updates of $X$ is $T$-accessible if every update in the set is $T$-accessible.

Further, if states $X, Y$ are $T$-similar, $\sigma : \mathcal{V}_X(T) \rightarrow \mathcal{V}_Y(T)$ is the similarity function, and if $u = (f, (x_1, \ldots, x_j), x_0)$ is a $T$-accessible update of $X$, define

$$\sigma(u) = (f, (\sigma(x_1), \ldots, \sigma(x_j)), \sigma(x_0)).$$  \hspace{1cm} \square

Postulate 4 (New Bounded Exploration Postulate). There exists a finite set $T$ of $\mathcal{Y}$ terms that is closed under subterms and such that

(i) $\Delta(X)$ is $T$-accessible for every state $X$, and

(ii) if states $X, Y$ are $T$-similar, $\sigma : \mathcal{V}_X(T) \rightarrow \mathcal{V}_Y(T)$ is the similarity function, and $u$ is an accessible update of $X$, then

$$u \in \Delta(X) \iff \sigma(u) \in \Delta(Y).$$

In other words, if terms $t_0$ and $f(t_1, \ldots, t_j)$ belong to $T$, $x_i = \mathcal{V}_X(t_i)$, and $y_i = \mathcal{V}_Y(t_i)$, then

$$(f, (x_1, \ldots, x_j), x_0) \in \Delta(X) \iff (f, (y_1, \ldots, y_j), y_0) \in \Delta(Y). \quad \square$$

The original bounded exploration postulate did not require the accessibility of updates. The accessibility was derived \[1\].

If $T \subseteq T'$ are sets of terms closed under subterms, and if $T$ is a bounded exploration witness, then $T'$ is also a bounded exploration witness.
Example 2.5. We illustrate the necessity of requirement (i). The vocabulary \( \Upsilon \) of our system \( A \) comprises a single nonlogical function symbol \( f \) which is nullary. Every state \( X \) of \( A \) has exactly two nonlogical elements, and the element \( f_X \) is nonlogical; all states are initial. Every transition of \( A \) changes the value of \( f_X \); if \( a, b \) are the nonlogical elements of \( X \) and \( f_X = a \), then \( f_{\tau(X)} = b \). Clearly, \( A \) satisfies the abstract state postulates.

Let \( T \) be an arbitrary set of \( \Upsilon \) terms. Then \( T \subseteq \{ \text{true}, \text{false}, \text{undef}, f \} \) and \( T \) is closed under subterms. If \( X \) is a state with nonlogical elements \( a, b \) and \( f_X = a \), then the unique update \((f, b)\) of \( X \) is \((f, b)\) which is not \( T \)-accessible. Accordingly, \( T \) fails requirement (i). But, since there are no accessible updates, \( T \) satisfies requirement (ii).

\( A \) does not satisfy the original bounded exploration postulate either. Indeed, let \( X \) be a state with nonlogical elements \( a, b \) where \( f_X = a \), and let \( Y \) be obtained from \( X \) by replacing \( b \) with a fresh element \( c \). Then \( X \) and \( Y \) coincide over every set \( T \) of terms but \( \Delta(X) = \{(f, b)\} \neq \{(f, c)\} = \Delta(Y) \). \( \triangleright \)

3 Equivalence of two bounded exploration postulates

We abbreviate “bounded exploration” to BE.

Theorem 3.1. Suppose that \( A \) satisfies the sequential state and abstract state postulates. Then \( A \) satisfies the new BE postulate if and only if it satisfies the original one.

Proof.

Only if. We assume that \( A \) satisfies the new BE postulate with some BE witness \( T \), and we prove that it satisfies the original one with the same BE witness \( T \). Suppose that the states \( X \) and \( Y \) of \( A \) coincide over \( T \). Then \( X, Y \) are \( T \)-similar and the similarity function \( \sigma \) is the identity function from \( \mathcal{V}_X(T) \) onto \( \mathcal{V}_Y(T) \). By the new BE postulate, \( \Delta(X) = \Delta(Y) \).

If. We assume that \( A \) satisfies the original BE postulate with a BE witness \( T \). Without loss of generality, \( T \) is closed under subterms. We prove that \( A \) satisfies the original BE postulate with the same BE witness \( T \).

Statement (i) is proven in \( \Pi \) Lemma 6.2. To prove statement (ii), suppose that \( X \) and \( Y \) are \( T \)-similar states of \( A \), \( t_0 = f(t_1, \ldots, t_j) \in T \), \( x_i = \mathcal{V}_X(t_i) \), and \( y_i = \mathcal{V}_Y(t_i) \). By symmetry, it suffices to prove that \((f, (y_1, \ldots, y_j), y_0) \in \Delta(Y)\) if \((f, (x_1, \ldots, x_j), x_0) \in \Delta(X)\). Suppose that \((f, (x_1, \ldots, x_j), x_0) \in \Delta(X)\).

Case 1: \( \mathcal{V}_X(T) \cap \mathcal{V}_Y(T) = \emptyset \). Create a new state \( X' \) from \( X \) by replacing \( \mathcal{V}_X(t) \) with \( \mathcal{V}_Y(t) \) for every \( t \in T \). States \( X' \) and \( Y \) coincide over \( T \). By the old BE postulate, \( \Delta(X') = \Delta(Y) \).

There is an isomorphism \( \xi : X \rightarrow X' \) that coincides with the similarity function on \( \mathcal{V}_X(T) \) and is identity otherwise. \( \xi \) naturally lifts to locations, updates and sets of updates, and we have

\[(f, (y_1, \ldots, y_j), y_0) = \xi((f, (x_1, \ldots, x_j), x_0)) \in \xi(\Delta(X)) = \Delta(X') = \Delta(Y) \]

Case 2: \( \mathcal{V}_X(T) \cap \mathcal{V}_Y(T) \neq \emptyset \). Let \( \eta \) be an isomorphism from \( X \) to a state \( X' \) of \( A \) such that \( \mathcal{V}_{X'}(T) \cap \mathcal{V}_Y(T) = \emptyset \). Lifting \( \eta \) as above, we have \( \mathcal{V}_{X'}(t_i) = \eta x_i \)
\[ (f, (\eta x_1, \ldots, \eta x_j), \eta x_0) = \eta((f, (x_1, \ldots, x_j), x_0)) \in \eta(\Delta(X)) = \Delta(X'). \]

Obviously \(X'\) and \(Y\) are \(T\)-similar. We have Case 1 with \(X'\) playing the role of \(X\) and \(\eta x_i\) playing the role of \(x_i\). Thus \((f, (y_1, \ldots, y_j), y_0) \in \Delta(Y)\).

\section*{References}

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