Uniqueness and delayed blow-up of solutions for fractional stochastic differential equations with multiplicative noise

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Abstract

The solution of some deterministic equation without noise may not be unique or existential. We study a nonlinear fractional partial differential equation which is driven by multiplicative noise of the form

\[ D^\beta_t u = \left[ -(-\Delta)^s u + \zeta(u) \right] dt + A \sum_{m \in \mathbb{Z}}^{d-1} \sum_{j=1}^{d} \theta_m \sigma_{m,j}(x) \circ dW^m_{t,j}, \quad s \geq 1, \quad \frac{1}{2} < \beta < 1, \]

where \( A > 0 \) is a constant depending on the noise intensity, \( \circ \) represent the Stratonovich-type stochastic differential. We prove that under some extra hypotheses about \( \zeta \), the multiplicative noise can delay the blow-up of the deterministic solution, and the above equation admits a pathwise unique solution with infinite life time with large probability. The existence and uniqueness of the solutions of the above stochastic equation are proved by using Galerkin approximations and priori estimates. We also verify the validation of hypotheses in the time fractional Keller-Segel and time fractional Fisher-KPP equations in 3D case.

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1. Introductions

In this paper, we consider the blow-up problem for nonlinear partial differential equations

\[ D_t^\beta u = \left[ -(-\Delta)^s u + \zeta(u) \right] dt, \quad s \geq 1, \quad \frac{1}{2} < \beta < 1, \]  

(1.1)

with initial data \( u(x, 0) = u_0(x) \in L^2(T^d) \). Defined on the tours \( T^d = \mathbb{R}^d / \mathbb{Z}^d \), we discuss the effect on the life span of the solution when equation (1.1) is perturbed by stochastic noise with the following form

\[
\begin{cases}
D_t^\beta u = \left[ -(-\Delta)^s u + \zeta(u) \right] dt + A \sum_{m \in \mathbb{Z}^d} \sum_{j=1}^{d-1} \theta_m \sigma_{m,j} (x) \circ dW_t^{m,j}, \quad s \geq 1, \quad \frac{1}{2} < \beta < 1, \\
u(x, 0) = u_0(x), \quad u_0 \in L^2(T^d),
\end{cases}
\]

(1.2)

where \( D_t^\beta \) is the left sided Caputo fractional derivative of order \( \beta \in \left( \frac{1}{2}, 1 \right) \), \( d \geq 2 \), and \( s \geq 1 \) is fixed, \( A = \sqrt{\frac{d}{(d-1)(\| \theta \|_{L^2}^2)} \cdot b} \), note that \( v \) represents the noise intensity. \( \zeta \) is a nonlinear term that satisfies specific assumptions. The meaning of the parameters in the equation is detailed in Section 3.

The phenomenon that the solutions of ordinary or partial differential equations diverge in finite time is known as the blow up of solutions [1], which exists in various fields, such as chemotaxis in biology [2], curvature flow in geometry [3], and fluid mechanics [4], etc. Recently, there are numerous research results for the solution of the blow-up phenomena, the conditions for the blow-up, the blow-up moment, the blow-up rate and the set of blow-up solutions have been studied widely. The sufficient conditions for the blow-up solutions of semi-discrete partial differential equations are given in [5, 6]. Later on, they proposed an upper bound for the blow-up time of solutions of fully discrete partial differential equations [7]. The relationship between the blow-up set of
semi-discrete equations and the continuous equations was discussed in [8], moreover, they pointed out that the blow-up set of semi-discrete equations converges to the blast set of continuous equations when the spatial grid parameters are sufficiently small. A classical blow-up equation is

\[ u_t = \Delta u + u^p, \]

which is defined in a bounded domain with zero Dirichlet condition. For any \( p \in (1, (n + 2)/n) \), all non-negative solutions of classical blow-up in finite time. The type of equation in which all solutions explode in finite time is also known as the Fujita problem [9]. The blow-up about the solution of the equation is divided into four cases, namely, no blow-up, blow-up occurred in a limited time, blow-up when time tends to infinity and the instantaneous blow-up that occurs at \( t = 0 \) [10].

Compared with integer-order models, scholars have stated that fractional-order models can preserve the genetic and memory properties of functions in practical problems [11, 12], and the physical meaning of parameters in fractional models also more explicit. Consequently, motivated by practical applications in various fields such as electrical engineering, physics, control theory and even finance (see [13], [14], [15], [16]) fractional models have been significantly developed in the past decades. There are also extensive results on the blow-up of fractional deterministic systems.

Recently, Zhang and Sun in [17] studied the blow-up of equation

\[
\begin{aligned}
D_t^\beta u(x, t) &= \Delta u(x, t) + |u(x, t)|^{p-1} u(x, t), \quad x \in \mathbb{R}^n, \ t > a > 0 \\
u(x, 0) &= u_0(x) \geq 0, \quad \lim_{|x| \to \infty} u_0(x) = 0.
\end{aligned}
\]

(1.3)

where \( 0 < \beta < 1, p > 1 \). Similar to the Fujita exponent, for any exponential \( p \in (1, (n + 2)/n) \), the solution of Eq. (1.3) blow-up at finity. In addition, for \( p \geq 1 + \frac{2}{n} \), sufficiently small \( \|u_0\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \), there exists a global solution.

Zhang and Li [18] discussed a time fractional subdiffusion equation with
nonlinear memory in a bounded domain
\[
\begin{cases}
D_t^\beta u = \Delta u + I_t^{1-\gamma} \left( |u|^{p-1} u \right), & x \in \Omega \subset \mathbb{R}^n, \ t > 0 \\
u(x,0) = u_0(x) \geq 0, & x \in \Omega \subset \mathbb{R}^n \\
u(x,t) = 0, & x \in \partial \Omega, \ t > 0,
\end{cases}
\] (1.4)
where \( I_t^{1-\gamma} u = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{u(s)}{(t-s)^\gamma} ds, 0 < \beta < 1 \) and \( 0 \leq \gamma < 1 \). They proved in what relations the parameters \( p, \gamma, \beta, u_0 \) satisfy, all solutions of the Eq. (1.4) blow-up at \( t < \infty \), and in what conditions, the equation has global solution.

Li and Li in [19] considered the blow-up and global existence of the solution to a semilinear time-space Caputo–Hadamard fractional diffusion equation with fractional Laplacian
\[
\begin{cases}
CHD_{a,t}^{\beta} u(x,t) = -(-\Delta)^s u(x,t) |u(x,t)|^{p-1} u(x,t), & x \in \mathbb{R}^d, \ t > a > 0, \\
u(x,a) = u_a(x), & x \in \mathbb{R}^d,
\end{cases}
\] (1.5)
where \( \beta \in (0,1), s \in (0,1), p > 1, d \in \mathbb{N} \). By using contraction mapping principle, they studied the local existence and uniqueness of the mild solution, finally, showed the mild solution is equivalent to a weak solution.

In [20], Cao et al. analyzed the properties of the solution of the equation with the form
\[
\begin{cases}
D_t^\beta u = d\Delta u + a(x) u + u^p, & x \in \Omega \subset \mathbb{R}^n, \ t > 0 \\
u(x,0) = u_0(x) \geq 0, & x \in \Omega \subset \mathbb{R}^n \\
u(x,t) = 0, & x \in \partial \Omega, \ t > 0.
\end{cases}
\] (1.6)
They proved the sufficient conditions for the blow-up phenomenon of the solution under large initial conditions and studied the global existence of the solution under small initial conditions.

Take the nonlinear term \( \zeta = -u (1-u) \) in Eq. (1.1), the Eqs. (1.4) is called the time fractional Fisher-KPP reaction-diffusion equation, which has been studied by Alsaedi and al. in [21]
\[
\begin{cases}
D_t^\beta u = -(-\Delta)^s u - u (1-u), & x \in \Omega, \ t > 0, \\
u(x,t) = 0, x \in \mathbb{R}^N \setminus \Omega, \ t > 0, \\
u(x,0) = u_0(x), & x \in \Omega,
\end{cases}
\] (1.7)
with \( s \in (0,1] \). They gave the initial conditions that make the solution of system (1.7) with finite-time blow-up, and analyzed the asymptotic behavior of solutions, which are bounded.

Scholars believe that the explosion of the solution may be affected by some deterministic perturbations or stochastic perturbations, and examples of deterministic perturbations affecting the blow-up phenomenon can be found in [22]. In practical problems, complex systems often have a large amount of noise perturbations, so stochastic models have attracted the interest of many scholars. The properties of these systems, such as well-posedness, invariant measure, stability and invariant manifold, have been discussed (see ref [23, 24, 25, 26, 27]). In contrast to deterministic fractional differential equations, which have a wealth of results, there are relatively few theories for fractional stochastic differential equations in the Caputo sense, and most of the existing studies focus on investigating the existence and uniqueness of the solutions (see for instance [28], [29], [30], [31] and references therein).

Doan and al. in [32] studied the equation of the form

\[
\begin{cases}
D^{\beta} u_t = f(t, u_t) + g(t, u_t) \frac{dW_t}{dt}, & t \geq 0, \\
u_0 = u(0), & u_0 \in L^2(\Omega, H),
\end{cases}
\]  

(1.8)

where \( \beta \in (\frac{1}{2}, 1) \) and \( (W_t)_{t \in [0, \infty)} \) represents a standard scalar Brownian motion. They proved the solutions of the Eq. (1.8) are uniqueness and global existence if the function \( f, g \) satisfy a standard Lipschitz condition. Moreover, they indicate the asymptotic distance between time variable \( t \) and two different solutions that satisfying the above system.

In [33], Zhang and al. considered a nonlinear variable-order fractional stochastic differential equations

\[
\begin{cases}
\frac{du}{dt} = \left( -\lambda(t) D^{\beta(t)}_t (t) u + f(t, u) \right) dt + \sigma(t, u) dW, & t \in [0, T], \\
u(0) = u_0,
\end{cases}
\]  

(1.9)

where \( D^{\beta(t)}_t \) denotes the variable-order fractional derivative in the Riemann-
Liouville sense, and it holds
\[ D_t^\beta(t) g := \left[ \frac{1}{\Gamma(1-\beta(t))} \frac{d}{d\xi} \int_0^\xi g(s)(\xi-s)^{-\beta(t)} ds \right] \bigg|_{\xi=t} \]
which implies that the equation (1.9) can be transformed into the form in Caputo sense. The above equation is driven by a multiplicative white noise, they showed the well-posedness and the regularity of its solutions.

For a class of nonlinear time fractional stochastic partial differential equations
\[ D_t^\beta u_t(x) = -v(-\Delta)^{s/2} u_t(x) + I_t^{1-\beta} \left[ \lambda \sigma(u) \dot{W}(t, x) \right], \]  
(1.10)
where \( v > 0, \beta \in (0, 1), s \in (0, 2], d < \min\{2, \beta^{-1}\}, I_t^{1-\beta} \) is the Riesz fractional integral operator, \( \dot{W}(t, x) \) is a space-time white noise, and \( \sigma : \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous. In [34], Mijena and Erkan studied the Eq.(1.10) in a \( d+1 \) dimensions for \( \lambda = 1 \), and claimed that the absolute moments of the solutions of the equations grow exponentially and the distance to the origin of the farthest peak of these moments grows linearly with time. Later on, as a further extension of the results in [34], Foondun and al. in [35] studied the asymptotic behavior of the solution when the initial data \( u_0 \) satisfied some reasonable assumptions, which is related to the time and parameters \( \lambda \).

It is known that, noise has a significant effect on the properties of the solution of the deterministic equation. For example, the suitably non-degenerate additive noise can improve the pathwise non-uniqueness for some deterministic equation [36]. The strong uniqueness of a class of SPDE with a non-Lipschitz drift and an additive noise was studied in [37], which is also called the regularization by noise. Multiplicative noise makes the solution of the stochastic transport linear equation converge to the deterministic equation when appropriate assumptions are satisfied [38]. And noise can improve the well-posedness for stochastic scalar conservation laws [39]. Motivated by the above results, in this study we consider the nonlinear time fractional stochastic partial differential system (1.2), prove that the suitably multiplicative noise can prolong the lifespan of the deterministic solution under some assumptions about \( \zeta \). Then
we use Galerkin approximations and priori estimates to prove the existence and uniqueness of the solution of Eq. (1.2), and state that the equation (1.2) has a pathwise unique solution with infinite life time with large probability.

The paper is organized as follows. In Section 2, we will first state some definitions relative to (1.2). In Section 3, we will introduce our model, hypotheses and main results. In Section 4, we prove that the assumptions in Section 3 hold for two examples. In Section 5, we will present the proofs for our main results based on some lemmas.

2. Preliminaries and notations

In this section, let us recall some necessary definitions of fractional order operators, present some symbols and auxiliary lemmas used later.

2.1. Preliminaries

Definition 2.1. [40] The left sided Riemann-Liouville fractional integral operator \( I_0^\beta \) of order \( \beta > 0 \), of a function \( u \in C_\kappa, \kappa \geq -1 \) is defined as

\[
I_0^\beta u(t) = \begin{cases} 
\frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} u(\tau) d\tau, & \beta > 0, t > 0, \\
1 & \beta = 0.
\end{cases}
\]

where \( \Gamma(\beta) = \int_0^\infty e^{-t}t^{\beta-1}dt \) is a Gamma function.

Definition 2.2. [41] The left sided Riemann-Liouville fractional derivative operator \( D_0^\beta \) of order \( 0 < \beta < 1 \), of a function \( u \in C_\kappa \), is defined as

\[
D_0^\beta u(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-\tau)^{-\beta} u(\tau) d\tau.
\]

Definition 2.3. [40] The Caputo time-fractional derivative operator \( D_0^\beta \) of order \( \beta > 0 \), is defined as

\[
D_0^\beta u(x, t) = \frac{\partial^n u(x, t)}{\partial t^n} = \begin{cases} 
\frac{1}{\Gamma(n-\beta)} \int_a^t (t-\tau)^{n-\beta-1} \frac{\partial^n u(x, \tau)}{\partial \tau^n} d\tau, & n-1 < \beta < n, t > 0, \\
\frac{\partial^n u(x, t)}{\partial t^n}, & \beta = n \in \mathbb{N}.
\end{cases}
\]
Definition 2.4. The one- and two-parameter Mittag-Leffler function is defined as

\[
\begin{align*}
E_{\beta} (z^\beta) &= \sum_{k=0}^{\infty} \frac{z^\beta}{\Gamma(\beta k + 1)}, \quad \beta > 0 \\
E_{\beta,\gamma} (z^\beta) &= \sum_{k=0}^{\infty} \frac{z^\beta}{\Gamma(\beta k + \gamma)}, \quad \beta, \gamma > 0
\end{align*}
\]

Definition 2.5. For given \( f \in L^2 (\mathbb{T}^d; \mathbb{R}^d) \), \( f \) is divergence free in sense if

\[
(f, \nabla g) = 0 \quad \forall g \in C^\infty (\mathbb{T}^d).
\]

Definition 2.6. The orthogonal projection \( \Pi \) is given by

\[
\Pi : f = \sum_{k \in \mathbb{Z}^d} f_k e_k \mapsto \Pi f = \sum_{k \in \mathbb{Z}^d} P_k f_k e_k, \quad (2.1)
\]

where \( P_k \in \mathbb{R}^d \times \mathbb{R}^d \) is the \( d \)-dimensional projection on \( k \perp \), \( P_k = I - \frac{k}{|k|} \otimes \frac{k}{|k|} (k \neq 0) \). Moreover, \( \Pi_N \) is given by

\[
f = \sum_{k \in \mathbb{Z}^d} f_k e_k \mapsto \Pi_N f = \sum_{k \in \mathbb{Z}^d} f_k e_k, \quad (2.2)
\]

where \( \Pi_N : C^\infty (\mathbb{T}^d)' \to C^\infty (\mathbb{T}^d) \).

Definition 2.7. We say that a random time \( \tau \) is a blow-up time (or explosion time) of the solution \( u(t,x) \) to (1.2), if the following two conditions are fulfilled:

(i) For any \( t < \tau \), \( \sup_{x \in \Omega} |u(t,x)| < \infty \) a.s.;

(ii) If \( \tau < \infty \), then \( \lim_{t \to \tau} \sup_{x \in \Omega} |u(t,x)| = \infty \).

Lemma 2.1. Suppose that \( q > 1 \), \( p \in [q, +\infty) \) and \( \frac{1}{p} + \frac{q}{2} = \frac{1}{q} \). Suppose that \( \Lambda^\sigma f \in L^q \), then \( f \in L^p \) and there is a constant \( C \geq 0 \) such that

\[
\|f\|_{L^p} \leq C \|\Lambda^\sigma f\|_{L^q}.
\]

Lemma 2.2. (Fractional comparison principle) Let \( u(0) = v(0) \), \( u(t) \) and \( v(t) \) satisfies

\[
D^\beta_t u(t) \geq D^\beta_t v(t)
\]
Lemma 2.3. [45] For \( u(t) \geq 0 \),

\[
D_t^\beta u(t) + c_1 u(t) \leq c_2(t)
\]

(2.3)

for almost all \( t \in [0, T] \), where \( c_1 > 0 \) and the function \( c_2(t) \) is non-negative and integrable for \( t \in [0, T] \). Then

\[
u(t) \leq u(0) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} c_2(s) \, ds.
\]

(2.4)

Lemma 2.4. [46] For any function \( v(t) \) absolutely continuous on \([0, T]\), one has the inequality

\[
v(t) D_t^\alpha v(t) \geq \frac{1}{2} D_t^\alpha v^2(t), \quad 0 < \alpha < 1.
\]

(2.5)

Lemma 2.5. [47] Let \( s \in (0, 1) \) and \( p \in [1, +\infty) \) be such that \( sp < n \). Let \( \Omega \subseteq \mathbb{R}^n \) be an extension domain for \( W^{s,p} \). Then there exists a positive constant \( C = C(n, p, s, \Omega) \) such that, for any \( f \in W^{s,p}(\Omega) \), we have

\[
\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{s,p}(\Omega)}.
\]

(2.6)

for any \( q \in [p, p^*) \), where \( p^* = p^*(N, s) = \frac{Np}{N-sp} \) is the so-called fractional critical exponent; i.e., the space \( W^{s,p}(\Omega) \) is continuously embedded in \( L^q(\Omega) \) for any \( q \in [p, p^*) \). If, in addition, \( \Omega \) is bounded, then the space \( W^{s,p}(\Omega) \) is continuously embedded in \( L^q(\Omega) \) for any \( q \in [1, p^*) \).

Lemma 2.6. [48] Let \( a(t) \) be a continuous on \([0, T]\) \( (0 < T \leq \infty) \), \( l(t) \) is non-negative and locally integrable on \([0, T]\), and suppose \( u(t) \) be a continuous non-negative function on \([0, T]\) with

\[
u(t) \leq a(t) + \int_0^t l(s)u(s)ds, t \in [0, T).
\]

(2.7)

Then

\[
u(t) \leq a(t) + \int_0^t l(s)a(s) \exp\left(\int_s^t l(\tau) \, d\tau \right) ds, t \in [0, T).
\]

(2.8)

If \( a(t) \) is a nonnegative non-decreasing on \([0, T)\), the inequality \ref{2.8} is reduced to

\[
u(t) \leq a(t) \exp\left(\int_0^t l(s) \, ds \right)
\]

(2.9)

If \( a(t) \equiv 0 \), then we can get \( u(t) \equiv 0 \) on \([0, T)\).
Property 2.1. [44] If $0 < \alpha < 1, u \in AC^1[0, T]$ or $u \in C^1[0, T]$, then the equality

$$I^\alpha_t \left( D^\alpha_t u \right) (t) = u(t) - u(0),$$

and

$$D^\alpha_t \left( I^\alpha_t u \right) (t) = u(t),$$

hold almost everywhere on $[0, T]$. In addition

$$D_t^{1-\beta} \int_0^t D^\beta_t u(\tau) \, d\tau = \left( I^\beta_t \frac{d}{dt} I^\beta_t D_t^{1-\beta} \frac{d}{dt} u \right)(t) = u(t) - u(0). \quad (2.10)$$

2.2. Notations

Note that $L^p(T^d, \mathbb{R}^d)$ represents the set of $p$-th integrable functions, which is defined on $T^d$ and real-valued and has norm

$$\|f\|_{L^p} = \left( \int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad \|f\|_{L^\infty} = \text{ess sup}_{x \in \Omega} |f(x)|.$$

We abbreviate it as $L^p(T^d)$. The Sobolev spaces $H^s(T^d)$ $(s \in \mathbb{R})$ is given by

$$H^s(T^d) = \left\{ f = \sum_k f_k e_k |f_{-k} = \bar{f}_k, \sum_k \left( 1 + |k|^2 \right)^s |f_k|^2 < \infty \right\},$$

where $f \in L^2(T^d; \mathbb{R})$, $\{e_k\}_{k \in \mathbb{Z}^d}$ given by $e_k = e^{ik \cdot x}$ is a complete orthonormal system [38]. For any $f$ on $\Omega$, $\alpha \in \mathbb{R}$, set $\Lambda = (-\Delta)^{\frac{\alpha}{2}}$, it holds $\|f\|_{H^\alpha} = \|\Lambda^s f\|_{L^2}$ (see [44]). The norm of $f \in L^p\left(0, T; H^s(T^d)\right)$ is abbreviated as $\|f\|_{L^p H^s}$, other similar representations have the same meaning. For any $f, g \in L^2(T^d)$, the inner product is denoted as $\langle f, g \rangle$. For the prevention of symbol abuse, we also have the symbol $\langle f, g \rangle$ for the dual product of $f \in H^s(T^d)$ and $g \in H^{-s}(T^d)$. The notation $[\cdot, \cdot]$ means the quadratic covariation process.

In the following representations, the symbol $\lesssim$ indicates that the number and equation on the left is less than or equal to $K$ times the number and equation on the right, where $K$ is a constant greater than 0, and its value will change under different circumstances. The dependence of the constants on parameters will be clearly written only when necessary.
3. Models and main result

Firstly, we discuss the time fractional deterministic equation

\[ D^\beta_t u = \left[ (-\Delta)^s u + \zeta(u) \right] dt, \tag{3.1} \]

where \( \Delta \) is the periodic Laplacian operator, \( s \geq 1 \) is fixed, \( \frac{1}{2} < \beta < 1 \), \( \zeta(u) \) is a nonlinear function with respect to \( u \), and the above equation is defined on the torus \( \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \). In order to make the conclusions more general, we assume that the nonlinear function \( \zeta \) satisfies some assumptions that are motivated by [50] and [51]. We will test the validity of this hypothesis with some classical equations in Section 4.

**Hypothesis 3.1.** The unknown nonlinear term \( \zeta \) satisfies the assumptions (i)-(iv).

(i) The unknown nonlinear function \( \zeta \) is a continuous mapping from Sobolev space \( H^{s-\gamma_1} \) to Sobolev space \( H^{-\gamma} \), and the Sobolev norm of the function \( \zeta \) satisfies the inequality

\[ \| \zeta(u) \|_{H^{-\gamma}} \leq C \left( 1 + \| u \|_{H^{s}}^{\frac{1}{2}} \right) \left( 1 + \| u \|_{H^{s}} \right), \]

with parameters \( a_1 \geq 0 \) and \( 0 < \gamma_1 < s \).

(ii) The inner product satisfies the inequality

\[ |\langle \zeta(u), u \rangle| \leq C \left( 1 + \| u \|_{L^2}^{2a_2} \right) \left( 1 + \| u \|_{H^{s}}^{2} \right), \]

with parameters \( a_2 \geq 0 \), \( 0 < \gamma_2 < 2 \). More generally, scaling \( L^2 \)-norm of \( u(t) \) further by the interpolation inequality gives

\[ |\langle \zeta(u), u \rangle| \leq C \left( 1 + \| u \|_{H^{s+\frac{\delta}{s}}}^{\frac{a_2+1}{2}} \right) \left( 1 + \| u \|_{H^{-\frac{\delta}{s}}}^{\frac{a_2+1}{2}} \right) \left( 1 + \| u \|_{H^{-\frac{\delta}{s}}}^{\frac{a_2+1}{2}} \right), \]

where \( \delta \) can be taken to be small enough.

(iii) Extending the first inequalities in condition (ii), it holds

\[ |\langle u - v, \zeta(u) - \zeta(v) \rangle| \leq C \| u - v \|_{L^2}^{a_3} \| u - v \|_{H^{s}}^{\frac{\gamma_3}{2}} \left( 1 + \| u \|_{H^{s}} + \| v \|_{H^{s}} \right), \]

\[ |\langle u - v, \zeta(u) - \zeta(v) \rangle| \leq C \| u - v \|_{L^2}^{a_3} \| u - v \|_{H^{s}}^{\frac{\gamma_3}{2}} \left( 1 + \| u \|_{H^{s}} + \| v \|_{H^{s}} \right), \]

\[ |\langle u - v, \zeta(u) - \zeta(v) \rangle| \leq C \| u - v \|_{L^2}^{a_3} \| u - v \|_{H^{s}}^{\frac{\gamma_3}{2}} \left( 1 + \| u \|_{H^{s}} + \| v \|_{H^{s}} \right). \]
where the parameters satisfy the system

\[
\begin{align*}
\gamma_3 + \eta &\leq 2, & \eta &\geq 0, \\
\gamma_3 + \eta &\leq 2, & \eta &\geq 0.
\end{align*}
\]

(iv) For a sufficiently large positive constant \(b\), the deterministic equation

\[
D_t^\beta u = -(-\Delta)^s u + b\Delta u + \zeta(u), \quad u(x,0) = u_0
\]

has a unique global solution \(u\) with trajectories in \(L^2(0,T;H^s(\mathbb{T}^d)) \cap C([0,T];L^2(\mathbb{T}^d))\), and the \(L^2\)-norm of the solution \(u\) is less than infinity when the initial data \(u(x,0) = u_0\) is bounded, closed and convex.

The parameter \(C\) in each of the above inequalities are not the same and are constants independent of the parameters, in order to avoid the abuse of symbols, uniformly denoted by the symbol \(C\).

Remark 3.1. By combining hypothesis (ii), Lemma 2.4 and Young inequality, we can conclude that exist some real constants \(Q > 0\), \(a' = \frac{2a_2}{2 - \gamma_2} > 0\), any solutions of (3.2) hold

\[
D_t^\beta \left\| u \right\|_{L^2}^2 = \int_{\mathbb{T}^d} D_t^\beta u^2 \, dx \leq \int_{\mathbb{T}^d} 2uD_t^\beta u \, dx
\]

\[
= -2 \left\| (-\Delta)^{s/2} u \right\|_{L^2}^2 - 2b \left\| \nabla u \right\|_{L^2}^2 + 2 \left\langle \zeta(u), u \right\rangle, \\
\leq -2 \left\| (-\Delta)^{s/2} u \right\|_{L^2}^2 - 2b \left\| \nabla u \right\|_{L^2}^2 + 2 (1 + \left\| u \right\|_{H^s}^2) (1 + \left\| u \right\|_{L^2}^{2^*})
\]

Combining the inequality (3.3) (see [51]),

\[
\left\| u \right\|_{H^s}^2 \leq 2^{s-1} \left( \left\| u \right\|_{L^2}^2 + \left\| (-\Delta)^{s/2} u \right\|_{L^2}^2 \right)
\]

and Poincaré inequality that we take the constants as \(\frac{1}{Z}\), namely

\[
\left\| u - \int_{\mathbb{T}^d} u(x,t) \, dx \right\|_{L^2}^2 \leq \frac{1}{Z} \left\| \nabla u \right\|_{L^2}^2,
\]

the above inequality can be simplified as

\[
D_t^\beta \left\| u \right\|_{L^2}^2 \leq -2 \left\| (-\Delta)^{s/2} u \right\|_{L^2}^2 - 2b \left\| \nabla u \right\|_{L^2}^2 + 2^{2-s} \left\| u \right\|_{H^s}^2 + Q \left( 1 + \left\| u \right\|_{L^2}^{2^*} \right) - 2 Z \left( 2Zb - 1 \right) \left\| u \right\|_{L^2}^2 \left( 1 + \left\| u \right\|_{L^2}^{2^*} \right).
\]
Solutions of fractional deterministic differential equations may experience blow-up, we add the Stratonovich-type stochastic term to the deterministic equation (3.6), i.e., shaped as

\[ D^\beta_t u = \left[ -(-\Delta)^s u + \zeta(u) \right] dt + A \sum_{m \in \mathbb{Z}_d^d} \sum_{j=1}^{d-1} \theta_m \sigma_{m,j}(x) \circ dW_{t}^{m,j}, \]

where \( A \) is taken as \( \sqrt{\frac{d}{(d-1)\|\theta\|_{\ell^2}}} \cdot b \) in order to simplify the following proof. We shall discuss whether the presence of stochastic noise can affect the explosion phenomenon of deterministic equation solutions. Next, we introduce the meaning of the other parameters in the equation (3.6) and the conditions that the parameters satisfy, which are similar to those in [51, 52].

Denoting positive constants by \( \mathbb{Z}_d^d_+ \), correspondingly, we denote negative constants by \( \mathbb{Z}_d^d_- \). For \( m \in \mathbb{Z}_d^d_+ \), let \( \theta = \{ \cdots, \theta_{-m}, \cdots, \theta_m, \cdots \} \) be a square addable sequence that satisfies \( \|\theta\|_{\ell^2} < \infty \) and \( \theta_m = \theta_{-m} \). The vector fields \( \sigma_{m,j} \) are defined as \( \sigma_{m,j} = q_{m,j}e^{2\pi i m \cdot x} \). To ensure that \( \sigma_{m,j} \) are divergence free, we take

\[
\begin{cases}
q_{m,j} \cdot m = 0, & m \in \mathbb{Z}_d^d_+,
q_{m,j} = q_{-m,j}, & m \in \mathbb{Z}_d^d_-,
\end{cases}
\]

where \( j = 1, 2, \cdots, d-1 \). Summarizing the above, for any \( x \in \mathbb{R}^d/\mathbb{Z}^d \), we have

\[
\sum_{m \in \mathbb{Z}_d^d} \sum_{j=1}^{d-1} \theta^2_m (\sigma_{m,j}(x) \otimes \sigma_{-m,j}(x)) = \sum_{m \in \mathbb{Z}_d^d} \theta^2_m \sum_{j=1}^{d-1} (q_{m,j}e^{2\pi i m \cdot x} \otimes q_{-m,j}e^{-2\pi i m \cdot x})
\]

\[
= \sum_{m \in \mathbb{Z}_d^d} \theta^2_m (q_{m,1} \otimes q_{-m,1} + \cdots + q_{m,d-1} \otimes q_{-m,d-1}) = \frac{d-1}{d} \|\theta\|_{\ell^2}^2 I_d.
\]

(3.7)

Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space with filtration \( \{\mathcal{F}_t : t \geq 0\} \), \( W_{t}^{m,j} \) is the complex Brownian motion defined on this probability space [52], namely

\[
W_{t}^{m,j} = \begin{cases}
B_{t}^{m,j} + iB_{t}^{-m,j}, & m \in \mathbb{Z}_d^d_+;\\
B_{t}^{-m,j} - iB_{t}^{m,j}, & m \in \mathbb{Z}_d^d_-.
\end{cases}
\]

\[1\] The choice of vector fields and complex Brownian motion is guided by Section 2.2 in [38], and based on the method used in Section 2.3 for converting the Stratonovich form to the Itô form, the equation is further simplified by choosing parameters \( A \) of a particular form.

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Hence it clearly holds $W_{t}^{m,j} = W_{t}^{-m,j}$. Furthermore, we assume that $W_{m,j}^{m_{1},j_{1}}, W_{m,j}^{m_{2},j_{2}}$ are independent whenever $j_{1} \neq j_{2}$ or $m_{1} \neq -m_{2}$, namely, we have

$$[W_{m,j}^{m_{1},j_{1}}, W_{m,j}^{m_{2},j_{2}}]_{t} = 2t \delta_{j_{1}-j_{2}} \delta_{m_{1}+m_{2}}.$$

Based on the relationship between the Stratonovich and Itô integrals [38], we can convert the equation (3.6) to an equivalent Itô form, we have

$$D_{t}^{\beta} u = \left[ -(-\Delta)^{\beta} u + \zeta(u) \right] dt + A \sum_{m \in \mathbb{Z}^{d}_{0}} \sum_{j=1}^{d-1} \theta_{m} \sigma_{m,j}(x) \nabla u \cdot dW_{t}^{m,j}$$

$$= \left[ -(-\Delta)^{\beta} u + \zeta(u) \right] dt + A \sum_{m \in \mathbb{Z}^{d}_{0}} \sum_{j=1}^{d-1} \theta_{m} \sigma_{m,j}(x) \nabla u \cdot dW_{t}^{m,j} + \frac{1}{2} \sum_{m \in \mathbb{Z}^{d}_{0}} \sum_{j=1}^{d-1} \theta_{m} d \left[ \sigma_{m,j}(x) \nabla u, W_{t}^{m,j} \right]$$

$$= \left[ -(-\Delta)^{\beta} u + \zeta(u) + b \Delta u \right] dt + A \sum_{m \in \mathbb{Z}^{d}_{0}} \sum_{j=1}^{d-1} \theta_{m} \sigma_{m,j}(x) \nabla u \cdot dW_{t}^{m,j}$$

The simplification of the last step holds due to the condition [37] and the value of the constant $A$.

In the following, we concentrate on the equation

$$D_{t}^{\beta} u = \left[ -(-\Delta)^{\beta} u + \zeta(u) + b \Delta u \right] dt + A \sum_{m \in \mathbb{Z}^{d}_{0}} \sum_{j=1}^{d-1} \theta_{m} \sigma_{m,j}(x) \nabla u \cdot dW_{t}^{m,j} \tag{3.8}$$

whose initial value is $u_{0}$. To prevent confusion, denote the solution of equation [3.8] by $u_{t}(u_{0}, \theta)$ and the solution of equation [3.2] by $u_{t}(u_{0})$. Here are the two main conclusions of this paper.

**Theorem 3.1.** Assume the initial data $u(x, 0) = u_{0}$ is bounded, closed and convex, there exists a square summable real sequence $\{\theta_{m}\}_{m}$, such that the equation

$$D_{t}^{\beta} u = \left[ -(-\Delta)^{\beta} u + \zeta(u) \right] dt + A \sum_{m \in \mathbb{Z}^{d}_{0}} \sum_{j=1}^{d-1} \theta_{m} \sigma_{m,j}(x) \nabla u \cdot dW_{t}^{m,j}$$

have a unique solution, where $\zeta$ is an unknown nonlinear function satisfies Hypothesis [7]. Moreover, for any $0 < T < \infty$, we can find positive parameters $S$ and $b$ big enough, the solution satisfies

$$\mathbb{P} \left( \|u_{t}(u_{0}, \theta)\|_{C([0,T];H^{-\gamma})} < S \right) \geq 1 - \varepsilon, \quad \forall t \in [0,T], \quad \forall \varepsilon \in (0,1). \tag{3.9}$$
Combining Theorem 3.1, we discuss the key question of this study, namely whether the blow-up time of the solution can be delayed in the presence of stochastic noise perturbations. We make further assumptions to show that the lifespan of the solution can be extended under these assumptions. Suppose the special sequence \( \{\theta_m\} \) taken in the above theorem is \( \{\theta^S\}_S \).

**Theorem 3.2.** \( \zeta \) satisfies Hypothesis 3.1, \( b > 0 \) big enough and choose \( u_0 \) is bounded, convex and closed. Assume there exist constants \( \lambda \) and \( K \) greater than \( 0 \) and can be sufficiently large, such that the \( L^2 \)-norm of the solution of the time fractional deterministic equation

\[
D^\beta_t u = -(-\Delta)^s u + b\Delta u + \zeta(u)
\]

holds the inequality \( \|u_t(u_0)\|_{L^2} \leq K\|u_0\|_{L^2} e^{-\lambda t} \). And the solution of the time fractional stochastic differential equation

\[
D^\beta_t u = [-(\Delta)^s u + \zeta(u)] dt + A \sum_{m \in \mathbb{Z}^d} \sum_{j=1}^{d-1} \theta^S \sigma_{m,j}(x) \circ dW^m_t (3.10)
\]

exists a pathwise unique global solution for sufficiently small initial data \( u_0 \). Then the lifespan of the pathwise unique solution of the Eqs. (3.10) can be taken to infinity with a high probability.

### 4. Examples

In this section, we verify that Hypothesis 3.1 holds for fractional Keller-Segel and fractional Fisher-KPP equations when \( d = 3 \).

#### 4.1. Fractional Keller-Segel Equation

Keller and Segel proposed a partial differential equation for modeling the chemotaxis behavior in cellular systems, which is known as the Keller-Segel model [53]. The simplified form is shown as

\[
\begin{align*}
\partial_t \rho &= \Delta \rho - \chi \nabla \cdot (\rho \nabla c) \quad \text{in } \Omega, \\
\partial_t c &= \gamma \Delta c + \beta \rho - \mu c, \\
\rho (0, \cdot) &= \rho_0 \geq 0, \quad c (0, \cdot) = c_0 \geq 0, \\
\partial_x \rho (t, \cdot) &= \partial_x c (t, \cdot) = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(4.1)
where \( \chi, \gamma, \mu, \beta \) are positive constants. This system is also one of the most important partial differential systems for learning about chemotactic aggregation. For dimension \( d = 1 \), the solutions of the system (5.10) are regular, while \( d \) takes other values, the solutions with finite-time blow-up if initial data big enough, see [54, 55, 56] for more details. Moreover, [56] gives the set of initial values that make the solutions of the equation (4.1) blow up.

The fractional Keller-Segel model is an extension of the Keller-Segel system for the case that the motion of the cell cannot be described by random walk [54]. For example, [57] introduced the mild solution of the equation

\[
\begin{align*}
D_\beta^t u - v\Delta u + (u \cdot \nabla) u + \nabla p &= f, \quad \text{in } \mathbb{R}^n, \quad t > 0, \\
 u(x,0) &= u_0(x) \geq 0, \quad \text{in } \mathbb{R}^n, \quad t > 0 \\
 \nabla \cdot u &= 0, \quad \text{in } \mathbb{R}^n,
\end{align*}
\]

where \( \beta \in (0,1) \). They also proved existence, uniqueness, decay, and regularity properties of mild solutions. Next, we discuss a class of fractional Keller-Segel equations and verify that Hypothesis 3.1 holds for such equations.

Lemma 4.1. Hypothesis 3.1 holds for the time fractional Keller-Segel system

\[
\begin{align*}
D_\beta^t \rho &= \Delta \rho - \nabla \cdot (\rho \nabla c), \\
-\Delta c &= \rho - \int_\Omega \rho(x) \, dx, \\
\nabla \cdot \rho &= 0, \tag{4.2}
\end{align*}
\]

where \( \beta \in (0,1) \), \( \Omega = \mathbb{T}^3 \).

Proof. It is obvious that we can simplify the above system (4.2) to

\[
D_\beta^s \rho = \Delta \rho - \nabla \cdot (\rho \nabla c) = \Delta \rho - \nabla \cdot [\rho \nabla \cdot (\rho^{-1}) (\rho - \rho_\Omega)] \\
= \Delta \rho - \nabla \cdot [\rho \nabla^{-1} (\rho - \rho_\Omega)],
\]

which is equivalent to taking \( s = 1 \), \( \zeta(\rho) = -\nabla \cdot [\rho \nabla^{-1} (\rho - \rho_\Omega)] \) in equation (3.1). We first prove that the mapping \( \zeta \) satisfies the conditions (i)-(iii). From the Sobolev embedding inequality, we obtain the inclusion relation between different Sobolev spaces as follows.

\[
H^1(\mathbb{T}^3) \hookrightarrow H^{3/4}(\mathbb{T}^3) \hookrightarrow L^4(\mathbb{T}^3), \quad H^{7/4}(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3), \quad H^{1/2}(\mathbb{T}^3) \hookrightarrow L^3(\mathbb{T}^3). \tag{4.3}
\]
By considering the properties of the norm, we deduce that

\[
\| \zeta (\rho_1) - \zeta (\rho_2) \|_{H^{-1}} = \| - \nabla \cdot [\rho_1 \nabla^{-1} (\rho_1 - \rho_{2\Omega})] + \nabla \cdot [\rho_2 \nabla^{-1} (\rho_2 - \rho_{2\Omega})] \|_{H^{-1}} \\
= \| \nabla \cdot (\rho_1 \nabla^{-1} \rho_1) - \nabla \cdot (\rho_1 \nabla^{-1} \rho_{2\Omega}) - \nabla \cdot (\rho_2 \nabla^{-1} \rho_2) + \nabla \cdot (\rho_2 \nabla^{-1} \rho_{2\Omega}) \|_{H^{-1}} \\
\leq \| \nabla \cdot (\rho_1 \nabla^{-1} \rho_1) - \nabla \cdot (\rho_2 \nabla^{-1} \rho_2) \|_{H^{-1}} + \| \nabla \cdot (\rho_1 \nabla^{-1} \rho_{2\Omega}) - \nabla \cdot (\rho_2 \nabla^{-1} \rho_{2\Omega}) \|_{H^{-1}} \\
\leq \| (\rho_1 \nabla^{-1} \rho_1) - (\rho_2 \nabla^{-1} \rho_2) \|_{L^2} + \| \rho_1 \nabla^{-1} \rho_{2\Omega} - \rho_2 \nabla^{-1} \rho_{2\Omega} \|_{L^2}.
\]

Next we scale the norm with the triangle inequality and results 1,3, the following formula holds \[1,3]\]

\[
\| (\rho_1 \nabla^{-1} \rho_1) - (\rho_2 \nabla^{-1} \rho_2) \|_{L^2} = \| \rho_1 \nabla^{-1} \rho_1 - \rho_2 \nabla^{-1} \rho_1 + \rho_2 \nabla^{-1} \rho_1 - \rho_2 \nabla^{-1} \rho_2 \|_{L^2} \\
\leq \| \rho_1 - \rho_2 \|_{L^2} \| \nabla^{-1} \rho_1 \|_{L^4} + \| \rho_2 \|_{L^4} \| \nabla^{-1} (\rho_1 - \rho_2) \|_{L^4} \\
\leq \| \rho_1 - \rho_2 \|_{L^2} \| \rho_1 \|_{H^{3/4}} + \| \rho_2 \|_{H^{3/4}} \| (\rho_1 - \rho_2) \|_{L^2}.
\]

Note that \( \rho_{2\Omega} (t) = \rho_{2\Omega} (0) = \int_{\Omega} \rho_1 (t, x) \, dx \), \( \rho_{2\Omega} (t) = \rho_{2\Omega} (0) = \int_{\Omega} \rho_2 (t, x) \, dx \), and both are constants. From the above statement the following inequality can be obtained,

\[
\| \zeta (\rho_1) - \zeta (\rho_2) \|_{H^{-1}} \leq \| \rho_1 - \rho_2 \|_{L^2} (1 + \| \rho_1 \|_{H^{3/4}} + \| \rho_2 \|_{H^{3/4}}).
\]

To obtain the condition (i), we let \( \rho_2 = 0 \) in the above equation, then

\[
\| \zeta (\rho_1) \|_{H^{-1}} \leq \| \rho_1 \|_{L^2} (1 + \| \rho_1 \|_{H^{3/4}}) \leq (1 + \| \rho_1 \|_{L^2}) (1 + \| \rho_1 \|_{H^{3/4}}),
\]

which means that (i) holds with \( a_1 = 1, \gamma_1 = 1/4 \).

Using the formula \( \langle f, \nabla^{-1} f \cdot \nabla f \rangle = -\frac{1}{2} \langle f^2, \nabla \cdot (\nabla^{-1} f) \rangle \) and the nature of inner product, it holds

\[
|\langle \zeta (\rho), \rho \rangle| = |\langle - \nabla \cdot [\rho \nabla^{-1} (\rho - \rho_{\Omega})] , \rho \rangle| \\
= |\langle - \nabla \rho \cdot \nabla^{-1} (\rho - \rho_{\Omega}) , \rho \rangle + \langle - \rho (\rho - \rho_{\Omega}) , \rho \rangle| \\
\leq \| \rho \|_{L^3}^3.
\]

We can obtain from the embedding theorem that the norm satisfies the formula

\[
\| \rho \|_{H^{3/2}}^3 \leq \| \rho \|_{L^2}^{3/2} \| \rho \|_{H^{3/2}}^{3/2} \leq \| \rho \|_{L^2}^{3/2} \| \rho \|_{H^{3/2}}^{3/2}, \quad (4.4)
\]

\[2\]Decomposition and scaling of the norm by means of a method similar to that of Lemma 2.1 in [51].
therefore it clearly holds
\[ |\langle \zeta (\rho) , \rho \rangle | \lesssim \left( 1 + \| \rho \|_{L^2}^{3/2} \right) \left( 1 + \| \rho \|_{H^1}^{3/2} \right), \]
which means that (ii) holds with \( a_2 = \gamma_2 = 3/2 \).

It is clear that
\[ |\langle \zeta (\rho_1) - \zeta (\rho_2) , \rho_1 - \rho_2 \rangle | \lesssim \| \rho_1 - \rho_2 \|_{L^2} \| \rho_1 - \rho_2 \|_{H^1} \left( 1 + \| \rho_1 \|_{H^1} + \| \rho_2 \|_{H^1} \right), \]
then we can use the result of condition (i) to get the inequality easily
\[ |\langle \zeta (\rho_1) - \zeta (\rho_2) , \rho_1 - \rho_2 \rangle | \lesssim \| \rho_1 - \rho_2 \|_{L^2} \| \rho_1 - \rho_2 \|_{H^1} \left( 1 + \| \rho_1 \|_{H^1} + \| \rho_2 \|_{H^1} \right), \]
which means that (iii) is satisfied with \( a_3 = \gamma_3 = \eta = 1 \).

Substituting \( \rho (t) - \rho_{T3} (t) = \rho (t) - m \) into the equation, we can obtain the same equation of the form 3.2, then from the Remark 3.1 it follows that
\[ D^\beta \| \rho - m \|_{L^2}^2 \leq -2 (4Zb - 1) \| \rho - m \|_{L^2}^2 + Q \left[ 1 + \left( \| \rho - m \|_{L^2}^2 \right)^{\beta/2} \right]. \]
We set \( D^\beta f (t) = -2 (4Zb - 1) f (t) + Q \left[ 1 + x(t)^{\beta/2} \right] \), by using Property 2.1 then the following identity holds
\[ f (t) = f (0) + \int_0^t (t-s)^{\beta-1} \left[ Q \left( 1 + f (s)^{\beta/2} \right) - 2 (4Zb - 1) x (s) \right] ds. \]
Assume that \( f (0) = \| \rho (0) - \rho_{T3} (0) \|_{L^2}^2 \) is finite, \( \rho_{T3} (0) \in \mathbb{R} \), we can take \( b \) sufficiently large such that \( f (t) < \infty \). As an application of Lemma 2.2 (iv) holds.

4.2. Fractional Fisher-KPP Equation

The reaction-diffusion equation controls the temporal evolution of the concentration or population density of species spreading, and its applications include the spatial and temporal spread of epidemics, the spatial spread of invasive species, etc [58, 59]. A well-known example of reaction-diffusion equation is the Fisher-KPP model, which is named after Fisher [60], Kolmogorov, and Petrovsky and Piskunov [61]. The standard reaction-representation of this equation [3] is
\[ \frac{\partial u (x,t)}{\partial t} = D \frac{\partial^2 u (x,t)}{\partial x^2} + ru (x,t) (1 - u (x,t)), \quad D > 0, r > 0. \]
As the research progressed, scholars found that the classical Fisher-KPP model has great limitations in modeling practical problems, so they combined the fractional order method with the reaction diffusion equation, and then focus more attention on the fractional order diffusion problem. Furthermore, they claim that the later method is more suitable for modeling sub-diffusion problems [62]. Regarding the blow up and global existence of the solution of the system, a large number of results have been presented. Ahmad at al. [63] have studied a reaction diffusion
\[
D_t^\beta u = \Delta u + u^2 - u
\] (4.5)
with a Caputo fractional derivative in time and various boundary conditions. With some conditions on the initial data, they demonstrated the solutions of Eq. (4.6) with finite-time blow-up. And for \( u_0 \in [0,1] \), there exists a global and bounded solution. Xu at al. [64] showed the blow-up phenomenon and the conditions for its appearance. By fixing the other parameters in the model, they found that the lower the order, the earlier the blow-up comes. Next, we also consider the above equation in 3D case.

Lemma 4.2. **Hypothesis 3.1 holds for the time fractional deterministic system**

\[
\begin{cases}
D_t^\beta u = \Delta u + u^2 - u, \\
\int_{\mathbb{R}^3} u(x,0) dx \in (-\infty, 1), \\
\|u(x,0) - \int_{\mathbb{R}^3} u(x,0) dx\|_{L^2} \leq \sqrt{\delta_0},
\end{cases}
\] (4.6)

where \( 0 < \beta < 1, \ 0 \leq \delta_0 < \infty \).

**Proof.** The system (4.6) is equivalent to taking \( s = 1, \ \zeta(\rho) = u^2 - u \) in equation (3.1). To begin with, we list several important inequalities used in the following. The interpolation inequality [65]

\[
\|u\|_{L^2} \leq \|u\|_{H^{\beta/\delta_0}}^{\beta/\delta_0} \|u\|_{H^{-\beta/\delta_0}}^{\delta_0 - \beta/\delta_0},
\] (4.7)

the Sobolev inequality (see [66])

\[
\|u\|_{L^p(R^n)} \leq \|u\|_{L^{p\beta} \cap L^{p(1-\gamma_1)}(R^n)} \|u\|_{L^{p(1-\gamma_1)}(R^n)},
\] (4.8)

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with \( \varsigma_1 = \left( \frac{1}{p_{k-1}} - \frac{1}{p_k} \right) / \left( \frac{1}{p_{k-1}} - \frac{2-n}{np_k} \right) \sim O(1) \), \( 1 - \varsigma_1 \sim O(1) \), \( p_k \to \infty \).

And the Hölder’s inequality (see [67])

\[
\int_a^b |f(x)g(x)| \, dx \leq \left( \int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} \left( \int_a^b |g(x)|^q \, dx \right)^{\frac{1}{q}},
\]

where \( p, q > 0 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

The proof of the (i)-(iii) of Hypothesis 3.1 are similar to that of [51]. For condition (i), using the fact that \( H^{1/2}(\mathbb{T}^3) \hookrightarrow L^3(\mathbb{T}^3) \), we have

\[
\| \zeta(u_1) - \zeta(u_2) \|_{H^{-1}} = \| (u_1^2 - u_1) - (u_2^2 - u_2) \|_{H^{-1}} \\
\lesssim \| (u_1^2 - u_1) - (u_1 - u_2) \|_{H^{-1/2}} \\
\lesssim \| u_1 + u_2 \|_{H^{1/2}} \| u_1 - u_2 \|_{H^{1/2}} + \| u_1 - u_2 \|_{H^{1/2}} \\
\lesssim \| u_1 - u_2 \|_{H^{1/2}} (1 + \| u_1 \|_{H^{1/2}} + \| u_2 \|_{H^{1/2}}).
\]

Then we take \( u_2 = 0 \), namely

\[
\| \zeta(u_1) \|_{H^{-1}} \lesssim \| u_1 \|_{H^{1/2}} (1 + \| u_1 \|_{H^{1/2}}),
\]

which indicates that the mapping \( \zeta : H^{1/2} \to H^{-1} \) is continuous, so (i) is satisfied with \( a_1 = 1, \gamma_1 = 1/2 \).

With respect to hypothesis (ii), we can easily conclude

\[
|\langle \zeta(u), u \rangle| = |\langle u^2 - u, u \rangle| = \int_{\mathbb{T}^3} (u^2 - u) \, u \, dx \\
\leq \| u \|_{L^3}^3 \lesssim \| u \|_{H^{1/2}}^{3/2} \| u \|_{L^2}^{3/2}
\]

by using the inequality (4.3), which implies (ii) is satisfied with \( a_2 = 3/2, \gamma_2 = 3/2 \).

Then we test assumption (iii), since

\[
|\langle \zeta(u_1) - \zeta(u_2), u_1 - u_2 \rangle| = |\langle (u_1^2 - u_1) - (u_2^2 - u_2), u_1 - u_2 \rangle| \\
= |\langle (u_1^2 - u_2^2) - (u_1 - u_2), u_1 - u_2 \rangle| \\
\lesssim \| u_1 + u_2 \|_{L^3} \| u_1 - u_2 \|_{L^3}^2 \\
\lesssim \| u_1 + u_2 \|_{H^{1/2}} \| u_1 - u_2 \|_{H^{1/2}}^2 \\
\lesssim (\| u_1 \|_{H^{1/2}} + \| u_2 \|_{H^{1/2}}) \| u_1 - u_2 \|_{H^{1/2}} \| u_1 - u_2 \|_{L^2},
\]

which implies (iii) is satisfied with \( a_3 = 1, \gamma_3 = 1, \eta = 1 \).
Finally, we verify that (iv) holds. Integrating $D_t^2 u = \Delta u + b \Delta u + u^2 - u$ over $T^3$ yields
\[
D_t^3 u_{T^3} = (1 + b) \Delta u_{T^3} + \|u\|_{L^2}^2 - u_{T^3} = \|u\|_{L^2}^2 - u_{T^3}^2 + u_{T^3}^2 - u_{T^3} = \|u - u_{T^3}\|_{L^2}^2 + u_{T^3}^2 - u_{T^3}.
\]
Subtracting the two equations, we say
\[
D_t^3 (u - u_{T^3}) = (1 + b) \Delta (u - u_{T^3}) + u^2 - u - \left(\|u\|_{L^2}^2 - u_{T^3}\right)
= (1 + b) \Delta (u - u_{T^3}) + \left(u^2 - \|u\|_{L^2}^2\right) - (u - u_{T^3}),
\]

since $\|u - u_{T^3}\|_{L^2}^2 \geq 0$ and $\|u\|_{L^2}^2 \geq 0$ hold constantly, as an application of Lemma 2.4, we can easily deduce
\[
D_t^3 \|u - u_{T^3}\|_{L^2}^2 \leq 2 \int_{T^3} (u - u_{T^3}) D_t^3 (u - u_{T^3}) \, dx
\]
\[
= -2 (1 + b) \|\nabla (u - u_{T^3})\|_{L^2}^2 + 2 \|u - u_{T^3}\|_{L^2}^2 + 2 \int_{T^3} \left(u^2 - \|u\|_{L^2}^2\right) (u - u_{T^3}) \, dx
\]
\[
\leq -2 (1 + b) \|\nabla (u - u_{T^3})\|_{L^2}^2 + 2 \int_{T^3} u^2 (u - u_{T^3}) \, dx.
\]
Next, we work on the term $\int_{T^3} u^2 (u - u_{T^3}) \, dx$, note that
\[
u^2 (u - u_{T^3}) = (u - u_{T^3})^3 + 2u_{T^3} u^2 - 3u_{T^3}^2 (u - u_{T^3}) + u_{T^3}^3
= (u - u_{T^3})^3 + 2u_{T^3} (u - u_{T^3})^2 + u_{T^3}^2 (u - u_{T^3})
\]

since $\int_{T^3} (u - u_{T^3}) \, dx = 0$, then, combining (4.7) and (4.8) yields
\[
\int_{T^3} u^2 (u - u_{T^3}) \, dx = \int_{T^3} (u - u_{T^3})^3 \, dx + 2u_{T^3} \int_{T^3} (u - u_{T^3})^2 \, dx
\]
\[
\leq \|u - u_{T^3}\|_{L^3}^3 + 2u_{T^3} \|u - u_{T^3}\|_{L^2}^2
\]
\[
\leq \|u - u_{T^3}\|_{H^{1/2}}^3 + 2u_{T^3} \|u - u_{T^3}\|_{L^2}^2
\]
\[
\leq \|u - u_{T^3}\|_{L^2}^{3/2} \|u - u_{T^3}\|_{H^{1/2}}^{3/2} + 2u_{T^3} \|u - u_{T^3}\|_{L^2}^2
\]
\[
\leq P \|u - u_{T^3}\|_{L^2}^2 + \|\nabla (u - u_{T^3})\|_{L^2}^2 + 2u_{T^3} \|u - u_{T^3}\|_{L^2}^2,
\]
where $P$ is an unimportant constant. Then, the above inequality can be trans-
formed into
\[
D_t^\beta \|u - u_{T^3}\|_{L^2}^2 \leq -2(1 + b) \|\nabla (u - u_{T^3})\|_{L^2}^2 + 2P \|u - u_{T^3}\|_{L^2}^6 \\
+ 2\|\nabla (u - u_{T^3})\|_{L^2}^2 + 4u_{T^3} \|u - u_{T^3}\|_{L^2}^2
\]
\[
= -2b \|\nabla (u - u_{T^3})\|_{L^2}^2 + 2P \|u - u_{T^3}\|_{L^2}^6 + 4u_{T^3} \|u - u_{T^3}\|_{L^2}^2
\]
\[
\leq -2b \|\nabla u\|_{L^2}^2 + 2 \|u - u_{T^3}\|_{L^2}^6 + 4u_{T^3} \|u - u_{T^3}\|_{L^2}^2
\]
\[
\leq (-2bZ + 4u_{T^3}) \|u - u_{T^3}\|_{L^2}^2 + 2P \|u - u_{T^3}\|_{L^2}^6,
\]
the last step is obtained from (3.4). In summary, we can obtain the system of equations
\[
\begin{aligned}
D_t^\beta u_{T^3} &= \|u - u_{T^3}\|_{L^2}^2 + u_{T^3}^2 - u_{T^3}, \\
D_t^\beta \|u - u_{T^3}\|_{L^2}^2 &\leq (-2bZ + 4u_{T^3}) \|u - u_{T^3}\|_{L^2}^2 + 2P \|u - u_{T^3}\|_{L^2}^6.
\end{aligned}
\]

Since the proof methods of the two cases \(0 \leq \int_{T^3} u_0 (x) \, dx < 1\) and \(\int_{T^3} u_0 (x) \, dx < 0\) are not the same, we shall discuss them separately.

For \(0 \leq \int_{T^3} u_0 (x) \, dx \leq k_0 < 1\), we set
\[
M_1 = \{ t \geq 0 ; 0 \leq u_{T^3} (t) < k_0 + \xi \} \cap \{ t \geq 0 ; \|u - u_{T^3}\|_{L^2} < \delta_0 + \xi \}, \quad \forall \xi \in (0, 1),
\]
for any \(t \in M_1\), it holds
\[
D_t^\beta \|u - u_{T^3}\|_{L^2}^2 \leq \left[ -2bZ + 4 (k_0 + \xi) + 2P (\delta_0 + \xi)^2 \right] \|u - u_{T^3}\|_{L^2}^2.
\]

Take constant \(b\) large enough such that \(2bZ - 4 (k_0 + \xi) - 2P (\delta_0 + \xi) > 0\), as an application of Lemma 2.3, we have
\[
\|u (t, x) - u_{T^3} (t)\|_{L^2}^2 \leq \|u_0 - u_{T^3} (0)\|_{L^2}^2 \leq \delta_0,
\]
moreover,
\[
D_t^\beta u_{T^3} = \|u - u_{T^3}\|_{L^2}^2 + u_{T^3}^2 - u_{T^3} \Rightarrow D_t^\beta u_{T^3} + u_{T^3} = \|u - u_{T^3}\|_{L^2}^2 + u_{T^3}^2 \leq \delta_0 + 1 + e^{-\lambda t},
\]
where \(\lambda\) is a non-negative constant. Applying Lemma 2.3 and Property 2.1 we get
\[
u_{T^3} (t) \leq u_{T^3} (0) + \frac{1}{T (\beta)} \int_0^t (t - s)^{\beta - 1} \left( \delta_0 + 1 + e^{-\lambda s} \right) ds
\]
\[
\leq u_{T^3} (0) + \frac{K}{T (\beta)} \int_0^t (t - s)^{\beta - 1} e^{-\lambda s} ds
\]
\[
\leq k_0 + \xi < 1,
\]
since the growth rate of exponential function is faster than that of power function, therefore, it is possible to take $K$ and $\lambda$ that satisfy inequalities $e^{-\lambda s} + 1 + \delta_0 \leq Ke^{-\lambda s}$ and $\frac{K}{1(\beta)} \int_0^t (t-s)^{\beta-1} e^{-\lambda s} ds \leq \xi$, which implies the conclusion holds.

Then we discuss the case that $u_T^3 < 0$. Set

$$M_2 = \{ t \geq 0 | u_{T^3} (t) < 0 \} \cap \left\{ t \geq 0 | \| u - u_{T^3} \|_{L^2}^2 < \delta_0 + \xi \right\}, \forall \xi \in (0, 1),$$

for any $t \in M_2$, it holds

$$D_t^\beta \| u - u_{T^3} \|_{L^2}^2 \leq \left[ -2bZ + 2P(\delta_0 + \xi)^2 \right] \| u - u_{T^3} \|_{L^2}^2.$$

As above, when $b$ is large enough such that the inequality $2bZ - 2P(\delta_0 + \xi)^2 > 0$ holds, we can obtain

$$\| u (t, x) - u_{T^3} (t) \|_{L^2}^2 \leq \| u_0 - u_{T^3} (0) \|_{L^2}^2 \leq \delta_0.$$

Combining the above two cases, it is clear that

$$\begin{cases} u_{T^3} (t) \leq 1, \\ \| u (t, x) - u_{T^3} (t) \|_{L^2}^2 \leq \| u_0 - u_{T^3} (0) \|_{L^2}^2 \leq \delta_0, \end{cases}$$

according to the triangle inequality, we have

$$\sup_{0 \leq t \leq T} \| u (t) \|_{L^2} \leq \sup_{0 \leq t \leq T} \| u (t, x) - u_{T^3} (t) \|_{L^2} + \sup_{0 \leq t \leq T} \| u_{T^3} (t) \|_{L^2} < \infty.$$ 

Therefore, (iv) holds. 

\[ \square \]

**Remark 4.1.** The proof of Lemma 4.2 is based on the verification in [51], the main feature of this paper is applying the triangle inequality, Cordoba-Cordoba inequality and Lemma 2.3 to scale the $L^2$-norm of the solution for the equation (4.6) to obtain the conclusion.

In the above discussion, we restricted $\int_{T^3} u_0 (x) dx$ to the range less than 1. The following proposition will explain why the value $\int_{T^3} u_0 (x) dx \geq 1$ cannot be taken.

\[23\]
Proposition 4.1. The time fractional equation
\[
\begin{aligned}
D_t^\beta u(t) &= (1 + b) \Delta u(t) + u^2(t) - u(t), \\
\omega_{T^3}(0) &= \int_{T^3} u(x,0) \, dx > 1,
\end{aligned}
\] (4.9)
experience an explosion in a finite time.

Proof. Integrating the first term of the Eq. (4.9) on torus $T^3$, we have
\[
D_t^\beta \omega_{T^3} = (1 + b) \Delta \omega_{T^3} + \int_{T^3} u^2 \, dx - \omega_{T^3},
\]
recall the Jensen inequality (see [68]), it holds
\[
\int_{T^3} u^2(t, x) \, dx \geq \left( \int_{T^3} u(t, x) \, dx \right)^2 = \omega_{T^3}^2(t).
\]
Note that $\Delta \omega_{T^3} = \int_{T^3} \Delta u(x, t) \, dx = 0$, combining the above conclusions, we know from the Theorem 3.2 in [69] that for initial values satisfying the condition $\omega(0) > 1$, the solution of the equation
\[
D_t^\beta \omega(t) = \omega^2(t) - \omega(t)
\] (4.10)
satisfies the identity
\[
\lim_{t \to T_1} \omega(t) = +\infty,
\]
where $T_1$ is a constant less than infinity, which implies that the blow-up of the solution of Eq. (4.10) occurs in a limited time. This also means that the blow-up of the solution of Eq. (4.10) occurs in a limited time.

We set $\omega(0) = \int_{T^3} u(x, 0) \, dx$, as an application of Lemma 2.2, it holds
\[
\lim_{t \to T_1} \| u(t) \|_{L^2} \geq \lim_{t \to T_1} \omega_{T^3}(t) \geq \lim_{t \to T_1} \omega(t) = +\infty,
\]
namely, the conclusion holds.

\[\square\]

5. Proof of main conclusions

The emphasis in this study is on the impact of the presence of the stochastic term on the blow-up time of the deterministic equation. In this section, we prove that Theorem 3.1 and Theorem 3.2 holds.
Due to the existence of nonlinear parts, fractional stochastic differential equations may only have local solutions for general initial value conditions [51]. In order to discuss the global solution of the equation, we introduce a smooth non-increasing cut-off function $L_S$, which has the form

$$L_S (||u||_{H^{-\gamma}}) = \begin{cases} 
1, & 0 \leq ||u||_{H^{-\gamma}} \leq S, \\
0, & ||u||_{H^{-\gamma}} > S + 1,
\end{cases}$$

(5.1)

Note that $L_S (||u||_{H^{-\gamma}})$ is equivalent to $L_S (u)$ when take $\gamma$ small enough. Next, we work on fractional stochastic differential equations with cut-off function $L_S$, the equation can be rewritten as

$$\begin{aligned}
D_t^\beta u &= \{-(\Delta)^\beta u + b\Delta u + L_S (||u||_{H^{-\gamma}}) \zeta (u)\} dt + A \sum_{m,j=1}^{d-1} \theta_m \sigma_{m,j} \nabla u (t) dW^{m,j}_t, \\
u (x, 0) &= u (0).
\end{aligned}$$

(5.2)

where $s \geq 1, \frac{1}{2} < \beta < 1, A = \sqrt{\frac{d}{(d-1)||\theta||_2}}, b, b > 0$, then set $\Lambda = (-\Delta)^{1/2}$.

**Definition 5.1.** A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is considered, $\{W^{m,j} : m \in \mathbb{Z}^d_0, j = 1, \cdots, d - 1\}$ is a collection of independent complex Brownian motions, which is defined on $\Omega$.

The $(\mathcal{F}_t)$-progressively measurable process $u$ is said as a solution to the equation (5.2), with trajectories of class $L^\infty (0, T; L^2 (\mathbb{T}^d)) \cap L^2 (0, T; H^s (\mathbb{T}^d)), \mathbb{P} - a.s$,

$$\langle u (t), \varphi \rangle = \langle u_0, \varphi \rangle - \frac{1}{\Gamma (\beta)} \int_0^t (t - \tau)^{\beta - 1} [\langle (\Lambda^s u (\tau), \Lambda^s \varphi \rangle + b \langle \nabla u (\tau), \nabla \varphi \rangle] d\tau$$

$$+ \frac{1}{\Gamma (\beta)} \int_0^t (t - \tau)^{\beta - 1} L_S (||u||_{H^{-\gamma}}) \langle \zeta (u (\tau)), \varphi \rangle d\tau - \frac{1}{\Gamma (\beta)} \sum_{m,j=1}^{d-1} \theta_m \int_0^t (t - \tau)^{\beta - 1} \langle u (\tau), \sigma_{m,j} \cdot \nabla \varphi \rangle dW^{m,j}_\tau$$

for all $t \in [0, T]$, where $u_0 \in L^2 (\mathbb{T}^d)$ and the function $\varphi \in H^s (\mathbb{T}^d)$ is divergence free.

**Remark 5.1.** To facilitate the following discussion, we briefly note

$$G (t) = \frac{1}{\Gamma (\beta)} \int_0^t (t - \tau)^{\beta - 1} [\Lambda^{2s} u (\tau) + b\Delta u (\tau) + L_S (||u (\tau)||_{H^{-\gamma}}) \zeta (u (\tau))] d\tau,$$

$$M (t) = \frac{A}{\Gamma (\beta)} \int_0^t \sum_{m,j=1}^{d-1} \theta_m \sigma_{m,j} (t - \tau)^{\beta - 1} \cdot \nabla u (\tau) dW^{m,j}_\tau.$$
The problem is first solved on a finite dimensional space, namely employing the Galerkin approximations. Set $H_M = \text{span}\{e^{2\pi mi \cdot x} : |m| \leq M\}$ is a finite dimensional space, which is also the subspace of $L^2(\mathbb{T}^d)$. Define the orthogonal projection $\Pi_M : H \to H_M$, it can be be expressed as

$$\Pi_M h = \sum_{m \in \mathbb{Z}^d} h_m e^{2\pi mi \cdot x}, \quad \forall h \in H.$$ 

From the above definition, it is clear that for any $h \in H_M$, $\|h\|_{L^2} = \|h\|_{H_x}$. Then, we discuss the fractional stochastic differential equations on $H_M$.

**Lemma 5.1.** For any $\theta \in \ell^2, u_0 \in L^2(\mathbb{T}^d)$, the time fractional stochastic differential equation

$$D^\beta_t u_M (t) = \left[ -\Lambda^x u_M (t) + b \Delta u_M (t) + L_S (\|u_M (t)\|_{H^{-\gamma}}) \Pi_M \xi (u_M (t)) \right] dt + A \sum_{m} \theta_m \Pi_M (\sigma_{m,j} \cdot \nabla u_M (t)) dW^{m,j}_t,$$

$$u_M (0) = \Pi_M u_0,$$

(5.3)

admits a unique solution. Moreover, it holds the inequality

$$\sup_{t \in [0, T]} \|u_M (t)\|_{L^2}^2 \leq K \left( \|u_0\|_{L^2}^2, T, \gamma \right)$$

(5.4)

with probability no less than $1 - \varepsilon$, where $K \left( \|u_0\|_{L^2}^2, T, \gamma \right)$ is a constant related to the parameters $\|u_0\|_{L^2}^2, T, \gamma$.

**Proof.** In the following expression, we shall denote the solution of Eq.(5.3) by $u_M$. By using Lemma 2.3 we have

$$D^\beta_t \|u_M\|_{L^2}^2 \leq 2 \int_{T_0} u_M (t) D^\beta_t u_M (t) dx = -2 \|\Lambda^x u_M (t)\|_{L^2}^2 - 2b \|\nabla u_M (t)\|_{L^2}^2 + 2L_S (\|u_M (t)\|_{H^{-\gamma}}) \langle \xi (u_M (t)), u_M (t) \rangle$$

$$- 2A \sum_{m} \theta_m \langle u_M (t), \Pi_M (\sigma_{m,j} \cdot \nabla u_M (t)) \rangle dW^{m,j}_t.$$

Since the projection $\Pi_M$ is orthogonal, and $\sigma_{m,j}$ is divergence free by the definition in Section 3, we state

$$\langle u_M (t), \Pi_M (\sigma_{m,j} \cdot \nabla u_M (t)) \rangle = \langle u_M (t), \sigma_{m,j} \cdot \nabla u_M (t) \rangle = 0,$$

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considering the condition $b > 0$, the above inequality can be simplified as

$$D_t^\beta \|u_M\|^2_{L^2} \leq -2 \|\Lambda^s u_M\|^2_{L^2} + 2L_S (u_M (t)) \|\zeta (u_M (t)), u_M (t)\) \quad (5.5)$$

Next, we mainly work on the second term of (5.5). Applying the inequality in the Hypothesis \ref{Hypothesis 3.1}(ii), we state

$$2L_S (\|u_M (t)\|_{H^{-\gamma}}) \langle \zeta (u_M (t)), u_M (t)\rangle \lesssim 2L_S (\|u_M (t)\|_{H^{-\gamma}}) \left(1 + \|u_M (t)\|_{H^{\gamma+\sigma_2/\sigma'}}^2 \right) \times \left(1 + \|u_M (t)\|_{H^{-\gamma}}^2 \right) \lesssim 2 \left(1 + S^\sigma_{2/\sigma'} \right) \left(1 + \|u_M (t)\|_{H^{\gamma+\sigma_2/\sigma'}}^2 \right),$$

combining Young’s inequality and \ref{3.3}, the above equation can be further simplified as

$$2L_S (\|u_M (t)\|_{H^{-\gamma}}) \langle \zeta (u_M (t)), u_M (t)\rangle \leq 2 \left[1 + S^\sigma_{2/\sigma'} + C \left(1 + S^\sigma_{2/\sigma'} \right) \frac{2}{3-\gamma-\sigma_2/\sigma'} \right]
+ \|u_M (t)\|^2_{L^2} + \|\Lambda^s u_M (t)\|^2_{L^2}.$$ 

Now we abbreviate $2 \left[1 + S^\sigma_{2/\sigma'} + C \left(1 + S^\sigma_{2/\sigma'} \right) \frac{2}{3-\gamma-\sigma_2/\sigma'} \right]$ as the constant $\tilde{C}$, it is easy to obtain

$$D_t^\beta \|u_M\|^2_{L^2} + \|\Lambda^s u_M (t)\|^2_{L^2} \leq \|u_M (t)\|^2_{L^2} + \tilde{C}.$$ 

Since $\|\Lambda^s u_M (\tau)\|^2_{L^2} \geq 0$, we say $D_t^\beta \|u_M\|^2_{L^2} \leq \|u_M (t)\|^2_{L^2} + \tilde{C}$. Recall Property \ref{2.4} yields

$$\|u_M (t)\|^2_{L^2} \leq \frac{1}{\Gamma (\beta)} \int_0^t (t - \tau)^{\beta - 1} \|u_M (\tau)\|^2_{L^2} d\tau + C' t^\beta \|u (0)\|^2_{L^2}$$

where $\beta \in (1/2, 1)$ and $C' > 0$. We apply the Theorem 2.4 in \cite{48}, the inequality \ref{5.4} holds.

We prove the pathwise uniqueness of the solution using a priori estimation. Suppose that $u_1$ and $u_2$ are two solutions of Eq. \ref{Eq. 5.4}, which correspond to the same Brownian motion and initial data $u_0 \in L^2 (\mathbb{T}^d)$. Set $\xi = u_1 - u_2$, the following identity holds,

$$\langle \xi (t), \varphi \rangle = -\frac{1}{\Gamma (\beta)} \int_0^t (t - \tau)^{\beta - 1} \langle \Lambda^s \xi (\tau), \Lambda^s \varphi \rangle d\tau + \frac{b}{\Gamma (\beta)} \int_0^t (t - \tau)^{\beta - 1} \langle \nabla \xi u (\tau), \nabla \varphi \rangle d\tau + \frac{1}{\Gamma (\beta)} \int_0^t (t - \tau)^{\beta - 1} \langle L_S (\|u_1\|_{H^{-\gamma}}) \zeta (u_1 (\tau)) - L_S (\|u_2\|_{H^{-\gamma}}) \zeta (u_2 (t)), \varphi \rangle d\tau
- \frac{A}{\Gamma (\beta)} \sum_{m=1}^{d-1} \sum_{j=1}^{\theta_m} \int_0^t (t - \tau)^{\beta - 1} \langle \xi (\tau), \sigma_{m,j} \cdot \nabla \varphi \rangle dW^{m,j}_t,$$

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for any $\varphi \in H^s(\mathbb{T}^2)$. Similar to (5.5), it holds

$$D^2_t \| \xi (t) \|^2_{L^2} \leq -2 \| \Lambda^s \xi (t) \|^2_{L^2} + 2 (L_S (\| u_1 \|_{H^{-s}}) \zeta (u_1 (t)) - L_S (\| u_2 \|_{H^{-s}}) \zeta (u_2 (t)), \xi (t)).$$

(5.6)

Note that the second term in the Eq. (5.6) can be split into

$$L_S (\| u_1 \|_{H^{-s}}) \zeta (u_1 (t)) - L_S (\| u_2 \|_{H^{-s}}) \zeta (u_2 (t)) = L_S (\| u_1 \|_{H^{-s}}) \zeta (u_1 (t)) - L_S (\| u_2 \|_{H^{-s}}) \zeta (u_1 (t))

+ L_S (\| u_2 \|_{H^{-s}}) \zeta (u_1 (t)) - L_S (\| u_2 \|_{H^{-s}}) \zeta (u_2 (t))

= [L_S (\| u_1 \|_{H^{-s}}) - L_S (\| u_2 \|_{H^{-s}})] \zeta (u_1 (t)) + L_S (\| u_2 \|_{H^{-s}}) [\zeta (u_1 (t)) - \zeta (u_2 (t))],$$

we shall discuss them separately.

Then we use Lagrange mean value theorem, hypothesis (i) and the condition that the $L^2$-norm of $u_i$ ($i = 1, 2$) is bounded, it is clear that

$$|L_S (\| u_1 \|_{H^{-s}}) - L_S (\| u_2 \|_{H^{-s}})| \| \zeta (u_1 (t)), \xi (t)) \| \leq \| L' \|_{\infty} \| u_1 \|_{H^{-s}} - \| u_2 \|_{H^{-s}} | \zeta (u_1 (t)) \|_{H^{-s}} \| \xi (t) \|_{H^s}

\leq \| \xi \|_{L^2} (1 + \| u_1 (t) \|_{H^s}) \| \xi (t) \|_{H^s}.$$

Combining the inequality (5.3), the above inequality can be deformed as

$$|L_S (\| u_1 \|_{H^{-s}}) - L_S (\| u_2 \|_{H^{-s}})| \| \zeta (u_1 (t)), \xi (t)) \| \leq K \left(1 + \| u_1 (t) \|^2_{H^s}\right) \| \xi (t) \|^2_{L^2} + \frac{1}{2} \| \Lambda^s \xi (t) \|^2_{L^2} .$$

(5.7)

For the another part, we apply the same inequalities as above and hypothesis (iii), thus having

$$L_S (u_2 (t)) | \zeta (u_1 (t)) - \zeta (u_2 (t)), \xi (t)) | \leq \| \xi (t) \|^2_{H^s} \| \zeta (u_1 (t)) \|^2_{H^s} + \| \zeta (u_2 (t)) \|^2_{H^s}

\leq \frac{1}{2} \| \Lambda^s \xi (t) \|^2_{L^2} + C \| \xi (t) \|^2_{L^2} \left(1 + \| u_1 (t) \|^2_{H^s} + \| u_2 (t) \|^2_{H^s}\right).$$

(5.8)

Based on the above discussion we know that $\| \Lambda^s \xi (t) \|^2_{L^2}$ is also bounded, then following the results of (5.7) and (5.8), yields

$$D^2_t \| \xi (t) \|^2_{L^2} \leq K \left(1 + \| u_1 (t) \|^2_{H^s} + \| u_2 (t) \|^2_{H^s}\right) \| \xi (t) \|^2_{L^2},$$

where $K$ stands for an insignificant constant. Because two different solutions of this equation have the same initial value, namely $\| \xi (0) \|_{L^2} = \| u_1 (0) - u_2 (0) \|_{L^2} = 0$. As an application of Gronwall inequality, see Lemma 2.10, it holds $\| \xi (t) \|_{L^2} \equiv 0$, which implies the uniqueness of the solution. □
Remark 5.2. We scale and deform fractional inequalities by comparison principle and Property 2.1, the similar method to that in \[51\] is used to prove that the lemma holds. From Lemma 5.1, we can derive the following theorem to hold.

To prove the main conclusions of this paper, we present several lemmas and corollaries.

Lemma 5.2. For any $\alpha, \delta, \kappa > 0, p < \infty$, let $K = C ([0, T]; \mathcal{L}^2) \cap \mathcal{C}^\kappa ([0, T]; H^{-\alpha}) \cap L^2 (0, T; H^\alpha)$ and $X = L^p (0, T; L^2) \cap \mathcal{C} ([0, T]; H^{-\delta}) \cap L^2 (0, T; H^{s-\delta})$. Then $K \subset \subset X$ (i.e. compact embedding). Moreover, for initial data $u_M (0) = \Pi_M u_0$, the solution of the time fractional stochastic equation with cut-off function

$$D_t^\beta u_M (t) = [ -\Lambda^{2s} u_M (t) + b \Delta u_M (t) + L_S (\|u_M (t)\|_{H^{-\gamma}}) \Pi_M \zeta (u_M (t))] dt$$

$$- A \sum_{m, j=1}^{d+1} \theta_m \Pi_M (\sigma_{m, j} \cdot \nabla u_M (t)) dW_{t}^m,$$

satisfy the inequality

$$\mathbb{E} (\|u_M\|_{L^2}^{2s} + \|u_M\|_{L^\infty}^{2s} + \|u_M\|_{C^\kappa H^{-\alpha}}) < \infty. \quad (5.9)$$

Proof. Take a sequence $\{h_n\}_n \subset K$ that is bounded. In $C^\kappa ([0, T]; H^{-\alpha})$, we can extract a subsequence $\{h_{n_k}\}_{n_k}$ from $\{h_n\}_n$ that converges to $h$, the conclusion follows from the Ascoli–Arzela theorem \[70\]. By using the inequality

$$\|h_{n_k} - h\|_{H^{-\delta}} \leq \|h_{n_k} - h\|_{L^2}^{1-\mu} \|h_{n_k} - h\|_{H^{-\alpha}}^{\mu}, \quad (5.10)$$

where $0 < \mu < 1$. In space $C ([0, T]; L^2)$, it holds

$$\|h_{n_k} (t) - h (t)\|_{L^2} \leq C, \quad (5.11)$$

and $C$ is a constant independent of the parameters. Meanwhile, we have

$$\|h_{n_k} (t) - h (t)\|_{H^{-\alpha}} \to 0. \quad (5.12)$$

\footnotetext{4}{The definition of the two topologies in this paper are similar to those of Lemma 3.3 in \[51\].}
From (5.10) - (5.12), we can easily deduce that the sequence \( \{ h_{nk} \}_{nk} \) converge to \( h \) in space \( C([0, T]; H^{-\delta}) \) for any \( \delta > 0 \). Recall the interpolation inequality, then

\[
\int_0^T \| h_{nk} (t) - h(t) \|^p_{L^2} \, dt \leq \int_0^T \| h_{nk} (t) - h(t) \|^p_{H^{-\beta}} \| h_{nk} (t) - h(t) \|^p_{H^{-\beta}} \, dt
\]

which implies convergence of \( h_{nk} (t) \) in the space \( L^p (0, T; L^2) \). Applying the inequality (5.11) again after taking the parameter \( p = 2 \), we state

\[
\int_0^T \| h_{nk} (t) - h(t) \|^2_{H^{-\delta}} \, dt \leq \int_0^T \| h_{nk} (t) - h(t) \|^{2(1-\delta)}_{H^{-\delta}} \| h_{nk} (t) - h(t) \|^{2\delta}_{L^2} \, dt.
\]

Similarly, since \( h_{nk} (t) \) converge to \( h(t) \) in \( L^2 (0, T; L^2) \) and the boundness of \( L^2 (0, T; H^s) \), the sequence converge in space \( L^2 (0, T; H^s) \).

Next, we verify the second conclusion. We divide the solution of the equation into two parts and discuss them separately. If \( u \in H^s \), then \( \Delta u \in H^{s-2} \subset H^{-s} \), according to the triangle inequality,

\[
\frac{1}{\Gamma^2(\delta)} \int_0^T \| (T - \tau)^{2\beta - 2} \left[ -(-\Delta)^s u (\tau) + b\Delta u (\tau) + L_S (\| u (\tau) \|_{H^{-\gamma}}) \right] \|_{H^{-\delta}}^2 \, d\tau
\]

by using the Hypothesis [3.1] and Hölder’s inequality it holds

\[
\frac{1}{\Gamma^2(\beta)} \int_0^T (T - \tau)^{2\beta - 2} \| u (\tau) \|^2_{H^s} \, d\tau \leq C_1 \| u \|^2_{L^2 H^s},
\]

Similarly, we can easily show that

\[
\frac{1}{\Gamma^2(\beta)} \int_0^T (T - \tau)^{2\beta - 2} \| \zeta (u (\tau)) \|^2_{H^{-\delta}} \, d\tau \leq C_2 \left( 1 + \| u \|^2_{L^2 H^s} \right) \left( 1 + \| u \|^{2a_1}_{L^\infty L^2} \right)
\]

where \( C_1, C_2 \) are both constants. Therefore, we deduce

\[
\| G (t) \|_{C^{1/2} H^{-\delta}} \leq \| G (t) \|_{W^{1,2} H^{-\delta}} \lesssim (1 + \| u \|^{a_1}_{L^\infty L^2}) (1 + \| u \|_{L^2 H^s}).
\]
For \( M(t) \), we have

\[
\left[ \langle M(t), e^{2\pi im \cdot x} \rangle \right](t) = \frac{A^2}{\Gamma(\beta)} \left[ \sum_{m} \frac{d-1}{j=1} \theta_m \int_{0}^{t} \left( (t-r)^{\beta-1} u_M(r), \sigma_{m,j} \right) \Pi_M \nabla e^{2\pi im \cdot x} \right] dW_{r,m,j}(t)
\]

\[
\leq \frac{2A^2}{\Gamma(\beta)} \sum_{m} \frac{d-1}{j=1} \theta_m \int_{0}^{t} \left( (t-r)^{2\beta-2} u_M(r), \sigma_{m,j} \right) \Pi_M \nabla e^{2\pi im \cdot x} \right] \frac{dr}{2}
\]

\[
\leq \frac{2A^2}{\Gamma(\beta)} \left[ \| \Pi_M \nabla e^{2\pi im \cdot x} \|_{L^\infty} \right] \int_{0}^{t} \left( (t-r)^{2\beta-2} \| u_M(r) \|_{L^2} \right) \frac{dr}{2}
\]

Since any order partial derivative of \( e^{2\pi im \cdot x} \) exists, \( \nabla e^{2\pi im \cdot x} = 2\pi mi e^{2\pi im \cdot x} \), \( \Delta e^{2\pi im \cdot x} = -4\pi^2 m^2 e^{2\pi im \cdot x} \), \( \sigma_{m,j} = q_{m,j} e^{2\pi im \cdot x} \). We shall use the Burkholder-Davis-Gundy inequality \([72]\) and Hölder’s inequality to estimate the martingale term, it holds

\[
E \left[ \| M(t) \|_{H^{-\alpha}}^2 \right] \leq \sum_{m} \left( 1 + |m|^2 \right)^{-\alpha} E \left[ \left( \langle M(t), e^{2\pi im \cdot x} \rangle \right)^2 \right] \lesssim \| \theta \|_{L^\infty} \left( t^{2\beta-1} \right)
\]

By using the Lemma \([5.1] \) it is obvious that the inequality \([5.13] \) holds. \( \square \)

**Remark 5.3.** This lemma is essential to prove the tightness of the distribution of solutions to the Eq. \([5.3] \). We use the similar method in \([5.1] \) to prove the embedding relation of two topologies. Based on this lemma we can get the following lemma to hold.

**Corollary 5.1.** Denote the law of \( u_M \) by \( \rho_M \), the family \( \{ \rho_M \}_M \) is tight on

\[ X = L^p \left( 0, T; L^2 \right) \cap C \left( [0, T]; H^{-\delta} \right) \cap L^2 \left( 0, T; H^{\delta} \right) \]

**Proof.** Recall the conclusion of Lemma \([5.2] \) that \( K \subset \subset X \), then

\[ \| \cdot \|_X \lesssim \| \cdot \|_K \]

it is obvious from the Prokhorov theorem (see \([73] \)) that the conclusion holds. \( \square \)

**Proposition 5.1.** Take the sequence \( \{ \theta^S \}_{S \geq 1} \subset \ell^2 \) that satisfy the limit \( \| \theta^S \|_{L^\infty}/\| \theta^S \|_{\ell^2} \to 0 \) as \( S \to \infty \). Assume the family \( \{ u^S_0 \}_{S \in \ell^2} \) is bounded, convex and closed.
that hold \( u^S_t \xrightarrow{W} u_0 \). In \( L^p \left( 0, T; L^2 \right) \cap C \left( [0, T]; H^{-s} \right) \cap L^2 \left( 0, T; H^{s/2} \right) \), the unique solution of fractional stochastic differential equation

\[
\begin{aligned}
D_t^\beta u &= \left[ -\Lambda^{2s} u (\tau) + b \Delta u (\tau) + L_S \left( \|u\|_{H^{-s}} \right) \zeta \left( u (\tau) \right) \right] dt + A \sum_{m,j=1}^{d-1} \theta^S_{m,j} \sigma_{m,j} \cdot \nabla u (t) dW_t^{m,j} \\
u_0 &= u_0^S
\end{aligned}
\]

converge to the unique solution of the time fractional deterministic equation

\[
D_t^\beta u = \left[ -\Lambda^{2s} u (\tau) + b \Delta u (\tau) + L_S \left( \|u\|_{H^{-s}} \right) \zeta \left( u (\tau) \right) \right] dt,
\]

in probability with initial data \( u_0 \).

**Proof** The symbol \( \{\rho^S\}_S \) is used to represent the law of \( \{u^S\}_S \). It follows from the Prokhorov theorem and above results that exists a subsequence \( \{\rho^S_i\}_{S_i} \subset \{\rho^S\}_S \), and holds \( \rho^S_i \xrightarrow{W} \rho \), while \( \rho \) is a probability measure. Then, Skorokhod’s representation theorem (see [74]) tells us, on another probability space \( \left( \hat{\Omega}, \hat{\mathcal{F}}, \hat{P} \right) \), it is possible to find the random variables \( \{\hat{u}^S_i\}_{S_i} \) and \( \hat{u} \), the former have the same distribution as \( \rho^S \), and the latter have the same distribution as \( \rho \), such that \( \hat{u}^S_i \xrightarrow{a.s.} \hat{u} \).

Next, we consider the new complex motions, which are defined on the probability space \( \left( \hat{\Omega}, \hat{\mathcal{F}}, \hat{P} \right) \), and write as \( \hat{W} = \left\{ \hat{W}^{m,j}_i : m \in \mathbb{Z}_0^d, j = 1, \ldots, d-1 \right\} \). It hold \( (u^S_i, W) \overset{d}{=} (\hat{u}^S_i, \hat{W}_S) \), in addition \( \hat{W}^{m,j}_S \xrightarrow{a.s.} \hat{W}^{m,j}_i \). Note that \( \hat{W}^{m,j}_i \) represents the subsequence of \( \left\{ \hat{W}_i^{m,j} : m \in \mathbb{Z}_0^d, j = 1, \ldots, d \right\} \), other conditions are the same as the family \( \{W^{m,j}_i\}_{m,j} \).

We shall prove that \( \hat{u} \) is the solution to deterministic equation (5.14). Obviously,

\[
\langle \hat{u}^S_i (t), \varphi \rangle = \left\langle u^S_i (t), \varphi \right\rangle - \frac{1}{\Gamma (\beta)} \int_0^t \left\langle (t-\tau)^{\beta-1} \Delta^{s} \hat{u}^S_i (\tau), \varphi \right\rangle + b \left\langle (t-\tau)^{\beta-1} \Delta^{s} \hat{u}^S_i (\tau), \varphi \right\rangle d\tau \\
+ \frac{1}{\Gamma (\beta)} \int_0^t L_S \left( \left\| \hat{u}^S_i \right\|_{H^{-s}} \right) \left\langle (t-\tau)^{\beta-1} \zeta \left( \hat{u}^S_i (\tau) \right), \varphi \right\rangle d\tau \\
- \frac{A}{\Gamma (\beta)} \sum_{m,j=1}^{d-1} \theta^S_{m,j} \int_0^t \left\langle (t-\tau)^{\beta-1} \hat{u}^S_i (\tau), \sigma_{m,j} \cdot \nabla \varphi \right\rangle dW^{m,j}_\tau
\]

for test function \( \varphi \in C^\infty \left( \mathbb{T}^d, \mathbb{R}^d \right) \), which is divergence free. For the last part of
the above inequality, we have

$$
\tilde{\mathbb{E}}(M^S_i (t))^2 = \mathbb{E} \left( \frac{A^2}{\Gamma(\beta)} \sum_{m=1}^{d-1} \sum_{j=1}^{d-1} \theta_{m,j}^j \int_0^t \left( (t-s)^{\beta-1} \tilde{u}^S_i(t), \sigma_{m,j} \cdot \nabla \varphi \right) dW^{m,j}_s \right)^2
$$

$$
= 2 \frac{A^2}{\Gamma(\beta)} \sum_{m=1}^{d-1} \sum_{j=1}^{d-1} \mathbb{E} \left( \int_0^t \left( (t-s)^{\beta-1} \tilde{u}^S_i(t), \sigma_{m,j} \cdot \nabla \varphi \right)^2 ds \right)
$$

$$
\leq \frac{A^2 \| \theta \|^2_{L^\infty}}{\Gamma(\beta)^2} \sum_{m=1}^{d-1} \sum_{j=1}^{d-1} \mathbb{E} \left( \int_0^t \left( (t-s)^{\beta-1} \tilde{u}^S_i(t), \sigma_{m,j} \cdot \nabla \varphi \right)^2 ds \right)
$$

where $\mathbb{E}$ represent the expectation on the new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Recall the definition of $\sigma_{m,j} \in L^2 (\mathbb{T}^d, \mathbb{R}^d)$, we show that

$$
\sum_{m=1}^{d-1} \sum_{j=1}^{d-1} \left| \left( (t-s)^{\beta-1} \tilde{u}^S_i(t), \sigma_{m,j} \cdot \nabla \varphi \right) \right|^2 = \sum_{m=1}^{d-1} \sum_{j=1}^{d-1} \left| \left( \nabla \varphi \cdot (t-s)^{\beta-1} \tilde{u}^S_i(t), \sigma_{m,j} \right) \right|^2
$$

$$
\leq \left\| \nabla \varphi \cdot (t-s)^{\beta-1} \tilde{u}^S_i(t) \right\|^2_{L^2} \leq \left\| \nabla \varphi \right\|^2_{L^\infty} \left\| (t-s)^{\beta-1} \tilde{u}^S_i(t) \right\|^2_{L^2}.
$$

According to the value of the parameter $A$ and the conditions satisfied by the special sequence $\{\theta_{i,j}\}_{i \geq 1}$, it is obvious that when $i$ tends to infinity,

$$
\frac{A^2 \| \theta \|^2_{L^\infty}}{\Gamma(\beta)^2} = \frac{d \cdot b \| \theta \|^2_{L^\infty}}{d - 1 \| \theta \|^2_{L^2}} \rightarrow 0.
$$

From the discussion above, the boundedness of $\left\| \tilde{u}^S_i \right\|_{L^2}$ is known, then combining the Hölder’s inequality, we deduce the expectation $\mathbb{E} (M^S_i (t))^2$ tends to 0 as $i$ tends to infinity. Recall the fact $\tilde{u}^S_i \overset{a.s.}{\rightarrow} \tilde{u}$ and $u^S_0 \overset{W}{\rightarrow} u_0$, we can conclude the following identity holds,

$$
\langle \tilde{u} (t), \varphi \rangle = \langle u_0, \varphi \rangle - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left[ \left\langle \Lambda^s \tilde{u} (t), \Lambda^s \varphi \right\rangle + b \left\langle \nabla \tilde{u} (t), \nabla \varphi \right\rangle \right] ds
$$

$$
+ \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} L_S (\left\| \tilde{u} \right\|_{L^\infty} \left\langle \zeta (\tilde{u} (t)), \varphi \right\rangle dt.
$$

Namely, $\tilde{u}$ is the solution to deterministic equation (5.13). Overall results of the above, the family $\{\rho^S_i\}_{i \geq 1}$ converge weakly to the Dirac measure $\delta_u$, which is also called the point mass at $u$, note that $u$ represents the unique solution of deterministic equation. From what we know from corollary 5.1, the $\{\rho^S\}_{i}$ is tight.

\footnote{Drawing from the method of Proposition 4.2 in [52] for estimating the martingale part, the conclusion of this paper can be seen as a special case when the parameter $\beta$ is not equal to 1.}
on $X$, from which we claim that $\{\rho^S\}_S \overset{W}{\rightarrow} \delta_u$. As the sequence of distribution functions converges weakly, the sequence of random variables $\{u^S\}_S$ converges in distribution to $u$, moreover, the sequence also converges in probability. \hfill $\Box$

**Remark 5.4.** Inspired by [51] and [52], we prove the convergence of the solution to the fractional stochastic equation (5.13), which can be seen as an extension of the conclusion about the fractional-order equation and of vital importance for the following proof of the main theorem in this paper.

From now on, the proof of Theorem 3.1 and Theorem 3.2 will be given.

**Proof of Theorem 3.1.** Recall the condition (iv) in the Hypothesis 3.1, for the bounded $u_0$, it is possible to find constant $S$ and $b$ that

$$
\|u_t(u_0)\|_{L^\infty(0,T;L^2)} \leq S - 1,
$$

By Proposition 5.1, it is clear that

$$
P \left( \|u^S_t(u_0, \theta) - u_t(u_0)\|_{C([0,T];H^{-\gamma})} \leq \varepsilon \right) \geq 1 - \varepsilon,
$$

for any $\varepsilon \in (0,1)$ and $T > 0$. Apply the properties of the norm, since

$$
\|u^S_t(u_0, \theta^S)\|_{C([0,T];H^{-\gamma})} - \|u_t(u_0)\|_{C([0,T];H^{-\gamma})} \leq \|u^S_t(u_0, \theta^S) - u_t(u_0)\|_{C([0,T];H^{-\gamma})},
$$

we can deduce that $\|u^S_t(u_0, \theta^S)\|_{C([0,T];H^{-\gamma})} < S$, which implies that

$$
P \left( \|u^S_t(u_0, \theta^S)\|_{C([0,T];H^{-\gamma})} < S \right) \geq P \left( \|u^S_t(u_0, \theta^S) - u_t(u_0)\|_{C([0,T];H^{-\gamma})} \leq \varepsilon \right) \geq 1 - \varepsilon.
$$

In other words,

$$
P \left( L_S \left( \|u^S_t(u_0, \theta^S)\|_{C([0,T];H^{-\gamma})} = 1 \right) \right) \geq 1 - \varepsilon,
$$

it also means that the Eq. (5.14) reduces to the deterministic equation without cut-off. The above result implies that the life span of solutions to (5.13) with initial data $u_0$ is greater than $T$. \hfill $\Box$

**Remark 5.5.** Compared with [51], the proof of Theorem 3.1 in this paper mainly employs the conclusion of Proposition 5.1, the triangle inequality and the inclusion relation of events to prove the boundedness of the norm of the solution to Eq. (5.14), and extends the result to the lifetime problem of the solution.

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Proof of Theorem 3.2. Since we have the inequality
\[ \|u_t(u_0)\|_{L^2} \leq K\|u_0\|_{L^2}e^{-\lambda t}, \]
where parameters \( K \) and \( \lambda \) can be taken big enough to hold
\[ \|u_t(u_0)\|_{L^2(T-1,T;L^2)} \leq \left[ \int_{T-1}^T (K\|u_0\|_{L^2}e^{-\lambda t})^2 dt \right]^{1/2} \leq \frac{K^2}{2\lambda} \|u_0\|_{L^2}^2e^{-2\lambda(T-1)} \leq \frac{\varepsilon}{2}. \]

From the Proposition 5.1, it is possible to find \( S \) big enough such that the equation reduces to (3.8), moreover, it holds
\[ P \left( \|u^S_t(u_0,\theta) - u_t(u_0)\|_{L^2(0,T;L^2)} \leq \varepsilon \right) \geq 1 - \varepsilon, \]

Let \( Z = \left\{ \|u^S_t(u_0,\theta) - u_t(u_0)\|_{L^2(0,T;L^2)} \leq \varepsilon \right\} \) be a set, then the probability of this event occurring is greater than \( 1 - \varepsilon \). It is easily to deduce from triangle inequality that
\[ \|u^S_t(u_0,\theta)\|_{L^2(T-1,T;L^2)} \leq \|u^S_t(u_0,\theta) - u_t(u_0)\|_{L^2(T-1,T;L^2)} + \|u_t(u_0)\|_{L^2(T-1,T;L^2)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \]

namely we can find \( t(\nu) \), which is taken in the interval \( [T-1,T] \) and \( \nu \in Z \), such that the \( L^2 \)-norm of \( u^S_t(\nu) (u_0, \theta) \) small enough. Let \( \|u^S_t(\nu) (u_0, \theta)\|_{L^2} \) be the initial data of Eq. (3.8), then the conclusion holds. \( \square \)

Future work

In this paper, we mainly focus on the fractional stochastic equation
\[
\begin{cases}
D_t^{\beta_1} u = \left(-(-\Delta)^{\beta_1} u + \zeta(u)\right) dt + A \sum_{m \in Z_0^d} \sum_{j=1}^{d-1} \theta_m \sigma_{m,j} \circ dW^m_t, \\
u|_{t=0} = u_0,
\end{cases}
\]
considering the uniqueness of its solution and the explosion phenomenon, as well as the influence of noise on its blow-up time. In the subsequent study, we will consider the blow-up problem in equation
\[
\begin{cases}
D_t^{\beta_1} u = \left(-(-\Delta)^{\beta_1} u + \zeta(u)\right) dt + A \int_0^t \sum_{m \in Z_0^d} \sum_{j=1}^{d-1} \theta_m \sigma_{m,j} \circ D_t^{\beta_2} W^m_t, \\
u|_{t=0} = u_0,
\end{cases}
\]
where \( \beta_1, \beta_2 \in (0,1) \).
Conflict of interest

The authors declare that they have no conflict of interest.

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