Bootstrapping the superconformal index with surface defects

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Abstract:

The analytic properties of the $\mathcal{N} = 2$ superconformal index are given a physical interpretation in terms of certain BPS surface defects, which arise as the IR limit of supersymmetric vortices. The residue of the index at a pole in flavor fugacity is interpreted as the index of a superconformal field theory without this flavor symmetry, but endowed with an additional surface defect. The residue can be efficiently extracted by acting on the index with a difference operator of Ruijsenaars-Schneider type. By imposing the associativity constraints of S-duality, we are then able to evaluate the index of all generalized quiver theories of type $A$, for generic values of the three superconformal fugacities, with or without surface defects.
1. Introduction

In recent years much has been learned about rigid supersymmetric quantum field theories by placing them on curved compact manifolds, notably (squashed) spheres.\footnote{More general susy-preserving backgrounds are beginning to be explored [1, 2].} In several cases
localization techniques have facilitated the exact calculation of the partition functions on these backgrounds, possibly in the presence of supersymmetric defects. Four important classes of examples are expectation values of circular Wilson loops in 4d $\mathcal{N} = 2$ theories on $S^4$ [3], partition functions of 3d $\mathcal{N} = 2$ theories on $S^3$ [4, 5, 6], partition functions of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories on $S^3 \times S^1$ [7, 8] and partition functions of $\mathcal{N} = 2$ theories on $S^2 \times S^1$ [9, 10, 11, 12, 13, 14] (the latter are usually referred to as “superconformal indices” if the theory is conformal).

These observables have been indispensable tools in the exploration of the new classes of 4d and 3d theories, dubbed respectively class $\mathcal{S}$ [15, 16] and class $\mathcal{R}$ [17]\(^2\), which arise by twisted compactifications of the 6d (2,0) theory on Riemann surfaces and on hyperbolic three-manifolds. This line of enquiry has led to the discovery of remarkable 4d/2d and 3d/3d relations. For example, according to the AGT conjecture [20], the $S^4$ partition function of a class $\mathcal{S}$ theory is computed by a Liouville/Toda correlator on the associated Riemann surface, whose complex structure moduli correspond the exactly marginal 4d gauge couplings. In a similar spirit, the $S^3 \times S^1$ partition function of a class $\mathcal{S}$ theory, being independent of the exactly marginal gauge theory couplings, is computed by a topological quantum field theory correlator on the Riemann surface [21].

In this paper we present a complete characterization of this 2d TQFT, and thereby provide an algorithm to evaluate the superconformal index of the general theory of class $\mathcal{S}$.\(^3\) The superconformal index is superficially a simpler observable than the $S^4$ partition function, since it can be evaluated by a free field calculation in any theory that admits a Lagrangian description. It is still however a very non-trivial function of the fugacities associated to the flavor symmetries and of three additional superconformal fugacities $(p, q, t)$. Its expression for a generic class $\mathcal{S}$ theory, which is non-Lagrangian, is a priori unknown. For a two-dimensional slice $(q, t)$ of the three-dimensional superconformal fugacity space an explicit description of the 2d TQFT was proposed in [22, 23] and shown to pass many checks. The key was to find a complete basis of functions of the flavor fugacity variables that diagonalizes the topological structure constants. The diagonalizing functions turn out to be closely related to Macdonald polynomials. Here we obtain the general answer, valid for arbitrary $(p, q, t)$, by a completely different strategy. The new method sheds light on the appearance of the Macdonald polynomials and yields naturally their expected “elliptic” generalization, which would have been difficult to obtain in the previous approach.

\(^2\)More precisely, the class $\mathcal{R}$ consists of $\mathcal{N} = 2$ SCFTs which admit an abelian Chern-Simons-Matter description, deformed by a certain class of superpotentials. Conjecturally, it includes the 3d theories defined by the compactification of the (2,0) theories on 3d manifolds which admit an ideal triangulation. See also [18, 19] for related 3d discussions.

\(^3\)More accurately we focus on class $\mathcal{S}$ theories of type A, that is, the $SU(N)$ generalized quivers. The generalization to theories of type-D and E is in progress.
The route that will lead us to the derivation of the general index is somewhat indirect, but it is guided by some simple physical ideas. A first source of inspiration is the AGT correspondence, where normalizable Liouville vertex operators are associated with flavor symmetries of the 4d gauge theory, while degenerate vertex operators correspond to inserting extra surface defects in the 4d theory. The degenerate operators are the key to the solution of Liouville theory by the conformal bootstrap [24]: considering their fusion with normalizable vertex operators one can derive functional equations that admit a unique solution. By analogy, we expect that adding surface defects to the $S^3 \times S^3$ partition function should correspond to inserting special “degenerate” punctures in the 2d TQFT correlator, and that their fusion with the ordinary flavor punctures will lead to “topological” bootstrap equations. This is indeed what we find. Another useful heuristic principle is that since divergences in a partition function must be related to flat bosonic directions, it should be possible to interpret the residue of the index at any of its poles in terms of the behavior of the 4d field theory “far away” in moduli space.

Guided by this intuition, we set out to evaluate the superconformal index of class $S$ theories endowed with BPS surface defects.\(^4\) This is a very interesting observable in its own right. Surface defects are among the least studied objects in four-dimensional quantum field theory and only recently they have started to receive proper attention (see e.g. [28, 29, 30]). Surface defects are in fact the only defects compatible with localization with minimal ($\mathcal{N} = 1$) supersymmetry in four dimensions. While in this paper we focus on $\mathcal{N} = 2$ theories, we hope that our results will serve as a stepping stone towards the study of surface defects in $\mathcal{N} = 1$ theories.

In theories that admit a Lagrangian description, such as the class $S$ theories of type $A_1$, it should be possible to evaluate the index and the $S^4$ partition function in the presence of surface defects by localization techniques. This is a very interesting direction for future work. Here we resort instead to a less direct construction that however also applies to the non-Lagrangian higher-rank theories. We start with the physical picture of a surface defect as the IR end point of a BPS vortex solution. The construction proceeds by embedding a given SCFT $\mathcal{T}_{IR}$ into a larger theory $\mathcal{T}_{UV}$ such that turning on a spatially constant Higgs branch vacuum expectation value (vev) one flows back to the original $\mathcal{T}_{IR}$. If one then modifies the UV boundary conditions of the RG flow by considering instead a position-dependent vev, the infrared endpoint is $\mathcal{T}_{IR}$ endowed with an additional BPS surface defect. The theory $\mathcal{T}_{UV}$ has an additional $U(1)_f$ flavor symmetry with respect to $\mathcal{T}_{IR}$. We argue that the residues of the index of $\mathcal{T}_{UV}$ at some special poles in the $U(1)_f$ flavor fugacity capture the index of $\mathcal{T}_{IR}$ in the presence of surface defects. Although $\mathcal{T}_{UV}$ does not contain any surface defects, the

\(^4\)See [25] for a previous attempt to incorporate surface defects in the superconformal index. The index of some $\mathcal{N} = 2$ theories in the presence of loop operators has been evaluated in [26], while [27] studied the index in presence of duality domain walls.
analytic structure of its index encodes the possibility to generate them by RG flows. By this route we are led to formulate a precise prescription to evaluate the index of $\mathcal{T}_{IR}$ with extra surface defects.

The prescription can be formulated entirely in terms of the index of $\mathcal{T}_{IR}$. Remarkably, adding surface defects to $\mathcal{T}_{IR}$ amounts to acting on its index with certain difference operators $\mathcal{S}_{(r,s)}$, closely related to the Hamiltonians of the elliptic Ruijsenaars-Schneider (RS) model [31, 32, 33]. The difference operators act as shifts of one of the $SU(N)$ flavor fugacities. The analogy with AGT is compelling: the action of the difference operator on one of the flavor punctures corresponds to the fusion of a degenerate Liouville primary with one of the normalizable primaries. Generalized S-duality predicts that one should get the same result independently of which flavor puncture the difference operator is acting on. This immediately leads to the conclusion that the functions that diagonalize the topological structure constants must be eigenfunctions of the difference operators (under the assumption that the spectrum is non-degenerate). By this route we are led to a complete determination of the 2d TQFT and thus of the index of a general class $\mathcal{S}$ theory of type $A$, with or without surface defects. For the two-dimensional slice $(q,t)$ of superconformal fugacity space the eigenfunctions turn out to be proportional to Macdonald polynomials and we re-derive the results of [23]. For arbitrary values $(p,q,t)$ of the superconformal fugacities the eigenfunctions are not known in closed analytic form. As a demonstration of principle that they exist, that they have a non-degenerate spectrum and that they are in fact calculable, we discuss a perturbative scheme to determine the eigenfunctions for small $q$ or $p$ – an approach that may have independent mathematical interest.

The structure of this paper is as follows. In section 2 we present an RG construction of certain BPS surface defects. In section 3 we give a physical interpretation of some special poles of the index in terms of such surface defects. We spell out a precise prescription for evaluating the index of a theory in the presence of these defects. In section 4 we apply our prescription to the case of $A_1$ quivers of class $\mathcal{S}$ and recast it in terms of the action of difference operators of RS type. In section 5 we study the properties of these difference operators and interpret them physically. In section 6 we combine the consistency conditions of generalized S-duality and the explicit form of the difference operators to “bootstrap” the index of a a general $A_1$ quiver. In section 7 we dimensionally reduce our results to $S^3$ and interpret them in the context of 3d gauge theories. In section 8 we present the extensions to the higher-rank theories. We close in section 9 with a discussion and a list of open problems. We collect in three appendices some additional material not needed on a first reading. In appendix A we describe the embedding of the 2d superalgebra that lives at the location of the defect into the 4d superalgebra, and recall some results on 2d partition functions. In appendix B we present a perturbative approach to the calculation of the elliptic RS wavefunctions. Finally
in appendix C we describe the generalization of our results to the case of flavor punctures with reduced symmetry.

2. An RG construction of supersymmetric surface defects

We begin by discussing a general construction of BPS surface defects, applicable to a large class of $\mathcal{N} = 2$ superconformal field theories. The construction proceeds by embedding a given SCFT $\mathcal{T}_{IR}$ into a larger theory $\mathcal{T}_{UV}$, such that turning on a spatially constant Higgs branch vacuum expectation value (vev) one flows back to the original $\mathcal{T}_{IR}$. If one then modifies the UV boundary conditions of the RG flow by considering instead a position-dependent vev, the infrared endpoint is $\mathcal{T}_{IR}$ endowed with an additional BPS surface defect.

In the rest of the paper we will explain how the pole structure of the superconformal index can be understood physically in terms of these surface defects. We will need to make some assumptions about the flavor charges of the Higgs branch vevs which initiate the RG flow. These assumptions, and many of our calculations, can be elegantly stated in terms of gauging a $U(1)_f$ flavor symmetry of $\mathcal{T}_{UV}$. There are reasons for which gauging $U(1)_f$ should be a bad idea: the gauged theory is not UV complete, and has an anomalous $U(1)_r$ R-symmetry. We believe that these problems are less serious than they appear, and we will sketch how one could overcome them. In any case, we also present an alternative analysis that does not rely on gauging $U(1)_f$ and leads to the same conclusions.

2.1 RG flow by a constant Higgs branch vev: the gauged perspective

As our main example, we take $\mathcal{T}_{IR}$ to be a generalized superconformal quiver of type $A_{N-1}$. We focus on a link of the quiver, corresponding to an $SU(N)$ gauge group. We cut the link and insert an extra node corresponding to a free hypermultiplet in the bifundamental representation of $SU(N) \times SU(N)$, see figure 1. The resulting superconformal field theory is what we call $\mathcal{T}_{UV}$. Relative to $\mathcal{T}_{IR}$, $\mathcal{T}_{UV}$ has an extra $U(1)_f$ flavor symmetry acting on the bifundamental hypermultiplet only. We now describe a supersymmetric RG flow that connects $\mathcal{T}_{UV}$ and $\mathcal{T}_{IR}$. The flow is initiated by gauging the $U(1)_f$ symmetry in the presence of a Fayet-Iliopolous (FI) parameter, which introduces a scale and forces the scalars in the hypermultiplet associated to the extra node to acquire a vev. We are interested in the symmetry-breaking pattern that preserves the diagonal $SU(N)$ of the original $SU(N) \times SU(N)$ non-abelian flavor symmetry. This is achieved by choosing the vevs to be proportional to the unit matrix,

$$Q_{a\hat{a}} = q\delta_{a\hat{a}} , \quad \tilde{Q}_{a\hat{a}} = \tilde{q}\delta_{a\hat{a}} .$$

(2.1)
The non-abelian D-term constraints are then automatically satisfied. The $U(1)_f$ moment maps are

$$\mu_3 = \text{Tr} (|Q|^2 - |\tilde{Q}|^2), \quad \mu_1 + i\mu_2 = \text{Tr} Q\tilde{Q}. \quad (2.2)$$

By an $SU(2)_R$ rotation (i.e. a choice of an $\mathcal{N} = 1$ subalgebra) we align the FI parameters $v_i$ along $\mu_3$, so that the $U(1)_f$ D-term constraints read

$$\mu_3 = N(|q|^2 - |\tilde{q}|^2) = v, \quad \mu_1 + i\mu_2 = Nq\tilde{q} = 0, \quad (2.3)$$

which have a unique solution up to $U(1)_f$ gauge rotations. Taking for definiteness $v > 0$, the solution is $|q|^2 = v/N, \tilde{q} = 0$.

The $U(1)_f \times SU(N)_1 \times SU(N)_2$ gauge symmetry is higgsed to a diagonal $SU(N)$, and the bifundamental hypermultiplets are eaten up in the process. In other terms, the extra node that we added to the generalized quiver is removed by the RG flow and in the IR we recover the original theory $T_{IR}$. A comment about R-symmetry breaking and its restoration is in order. The FI parameter breaks explicitly the $SU(2)_R$ symmetry of the UV theory to an $SO(2)_R$ subgroup.\textsuperscript{5} The vev further breaks this $SO(2)_R$ spontaneously. There is however a linear combination $SO(2)_{\bar{R}}$ of $SO(2)_R$ and $U(1)_{\bar{r}}$ which is preserved in the new vacuum.

In our conventions we assign

$$R_Q = \frac{1}{2}, \quad f_Q = -1, \quad (2.4)$$

so that the linear combination

$$\bar{R} \equiv R + \frac{f}{2} \quad (2.5)$$

leaves $Q$ invariant. In the IR, one must recover the full $SU(2)_{\bar{R}} \times U(1)_{\bar{r}}$ R-symmetry of the $\mathcal{N} = 2$ superconformal algebra. We identify the $SO(2)_{\bar{R}}$ symmetry, which is preserved all along the flow, with the Cartan subalgebra of the infrared $SU(2)_{\bar{R}}$.

\textsuperscript{5}However, the theory has still exact $\mathcal{N} = 2$ supersymmetry, as explained in [34, 35].
We can also apply the same construction to the external leg of the generalized quiver that defines $\mathcal{T}_{IR}$. The external leg is associated to an $SU(N)$ flavor symmetry. To obtain $\mathcal{T}_{UV}$, we break the leg and insert again a free bifundamental hypermultiplet. Again $\mathcal{T}_{UV}$ has an extra $U(1)_f$ symmetry. One can repeat exactly the same steps as above and define an RG flow between $\mathcal{T}_{UV}$ with higgsed $U(1)_f$ and $\mathcal{T}_{IR}$.

While we have focused on two very concrete examples, the basic idea is more general. Given an $\mathcal{N} = 2$ SCFT with a $U(1)_f$ flavor symmetry, we can gauge the $U(1)_f$ and turn on an FI parameter. The $U(1)_f$ symmetry is then higgsed, four real directions in the Higgs branch are eaten, and at much lower energies the theory flows to a new $\mathcal{N} = 2$ SCFT $\mathcal{T}_{IR}$. In geometric terms, the IR Higgs branch is the hyperkahler quotient of the UV Higgs branch by the tri-holomorphic $U(1)_f$ isometry which we gauged: we set the moment maps equal to the FI parameters, and remove the orbits of the $U(1)_f$ action. In order to define the RG flow properly, we need to pick a vacuum for the theory, i.e. a choice of vev for the Higgs branch fields. With no loss of generality, we can pick a complex structure, or $\mathcal{N} = 1$ subalgebra, and only turn on a real FI parameter, so that a Cartan subgroup $SO(2)_R$ of $SU(2)_R$ is not explicitly broken. One can then show that there is always a choice of vacuum such that $SO(2)_R$ is not spontaneously broken, i.e. a point on the Higgs branch where an $SO(2)_R$ rotation can be compensated by a $U(1)_f$ gauge rotation. Indeed, the Higgs branch of the UV SCFT is a hyperkahler cone, with a radial coordinate $\rho$ which is the moment map for $SO(2)_R$. We can look for the point of minimum $\rho$ among solutions of the moment map constraints. At that point, the $SO(2)_R$ isometry is aligned with the $U(1)_f$ isometry, and there is a linear combination $SO(2)_{\bar{R}}$ which is unbroken. In general one expects that $SO(2)_{\bar{R}}$ should become the Cartan subgroup of the IR $SU(2)_{\bar{R}}$ R-symmetry group.

\[ \text{Figure 2: Our second RG example.} \]

\[ T_{IR} \quad T_{UV} \]

\[ Q_{a,b}, \hat{Q}_{a,b} \]

\[ T_{IR} \quad T_{UV} \]

\[ Q_{a,b}, \hat{Q}_{a,b} \]

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\[ ^6 \text{There is a small caveat here concerning the difference between } SU(N) \text{ and } PSU(N) \text{ flavor symmetry. We will come back to it in sec 3.1} \]
2.2 RG flow by a variable Higgs branch vev: the gauged perspective

We now consider the same setup of figure 1, but with a more interesting choice of UV boundary conditions. We still embed $T_{IR}$ into the larger theory $T_{UV}$, gauge the extra $U(1)_f$ flavor symmetry and turn on an FI parameter, but rather than choosing the trivial background that brings us back to $T_{IR}$, we consider a position-dependent profile for the fields. A very interesting class of backgrounds are the non-abelian vortex solutions that preserve $(2,2)$ 2d supersymmetry. Vortices are solutions that depend on two of the four Cartesian coordinates, which we parametrize by a complex coordinate $z$.

Vortex solutions localized in the $z$ plane exist in general only if the FI parameter $v$ is turned on. Indeed, the tension of a BPS vortex solution is $2\pi rv$, where the integer $r$ counts the units of magnetic $U(1)_f$ flux on the $z$ plane. Away from the location of the vortices the scalar fields asymptote to constant vevs selected by the choice of vacuum at infinity. This implies that the endpoint of the RG flow is still $T_{IR}$ away from the location of the vortices. As we flow to the IR, the tension of the vortices goes to infinity, and they become BPS surface defects in $T_{IR}$.

The BPS equations for a vortex solution have a simple structure, and can be organized into a set of F-term constraints and some D-term constraints. The F-term constraints force the complex scalars in the hypermultiplets to be covariantly holomorphic, and to satisfy the usual condition $\mu_1 + i\mu_2 = 0$ for all gauge groups. The D-term equation is deformed to

$$\mu_3 = F_{z,\bar{z}} + v$$  \hspace{1cm} (2.6)

for all gauge groups, where the FI term $v$ lives in the $U(1)_f$ subgroup.

The first consequence of these equations is that the vev of gauge-invariant chiral operators, \textit{i.e.} chiral gauge-invariant operators in $T_{UV}$ with zero $U(1)_f$ gauge charge, is holomorphic on the $z$ plane. As it goes to a constant vev to infinity, the vev is constant everywhere. The moduli space of BPS solutions can be identified with the space of solutions to the F-term constraints modulo complexified gauge transformations, dropping the D-term equations. One can gauge away the anti-holomorphic component of the gauge connection, $A_{\bar{z}} \equiv 0$, so that the scalar vevs can be taken to be holomorphic, and actually polynomial in $z$. One is left with a quotient by polynomial complex gauge transformations.

The moduli space of BPS solutions with a given vortex number $r$ can be rather intricate. We wish to focus on backgrounds where the only scalars that receive a vev at infinity belong to hypermultiplets of the extra node in figure 1. The baryon operator $B = \det Q$ only carries $U(1)_f$ gauge charge. The only invertible holomorphic $U(1)_f$ transformations are constant

\footnote{At this stage we consider gauge theories in Minkowski space.}
rescalings, so we can write

\[ B(z) = P_r(z) = \prod_{i=1}^{r} (z - z_i), \quad (2.7) \]

to an overall rescaling. The degree \( r \) is the vortex number, and the \( z_i \) are gauge-invariant parameters identified with the position of the \( r \) vortices.

Although we cannot generally describe the full moduli space of vortex solutions for a generic \( \mathcal{T}_{UV} \), we can describe a universal subspace, consisting of solutions where only \( Q \) receives a vev. Then we are effectively considering solutions in \( \mathcal{N} = 2 \) SQCD with \( N_f = N_c \) flavors. Supersymmetric vortices in this model have been extensively studied. We refer to [36, 37, 38] for reviews and only recall some of the basic facts here.

In addition to the \( 2r \) (real) position moduli, there are \( 2(N-1)r \) internal orientational moduli, encoded in the holomorphic matrix \( Q(z) \) modulo gauge transformations that leave \( \det Q(z) = P_r(z) \) fixed. For example, for \( r = 1 \) there is a \( \mathbb{C} \times \mathbb{CP}^{N-1} \) worth of vortex solutions: the \( \mathbb{C} \) factor describes the position, while the \( \mathbb{CP}^{N-1} \) factor parametrizes the breaking of the diagonal \( SU(N) \) gauge group of \( \mathcal{T}_{IR} \) down to \( U(1) \times SU(N-1) \). For general \( r \), the vortex moduli space can be described by a \( U(r) \) gauged linear sigma model coupled to \( N \) fundamentals and one adjoint chiral multiplet (which contains the center of mass degrees of freedom) [39].

As we flow to the IR, we can keep all the \( r \) vortices at the origin, and set \( P_r(z) = z^r \). The background preserves a diagonal combination \( \bar{j} \) of the angular momentum \( j \) in the \( z \) plane and of the flavor symmetry \( f \),

\[ \bar{j} \equiv j + \frac{r}{N} f, \quad (2.8) \]

which should be identified with the angular momentum of the IR theory. There is also (for any choice of \( P_r(z) \), in fact) a preserved combination of the R and flavor symmetries,

\[ \bar{R} \equiv R + \frac{f}{2}, \quad (2.9) \]

which we identify with the IR R-symmetry. As the vortices have a tension which is proportional to the FI parameter, they become infinitely heavy surface defects in the IR SCFT \( \mathcal{T}_{IR} \).

We have just described the construction of an infinite family of surface defects \( \mathcal{S}_{(r,0)} \) in \( \mathcal{T}_{IR} \), labeled by the number of vortices \( r \). We can repeat the construction by considering \( s \) vortices in the plane \( w \), orthogonal to the \( z \) plane in \( \mathbb{R}^4 \); we consider the background with \( B(w) = w^s \), and obtain surface defects in \( \mathcal{T}_{IR} \) which we label as \( \mathcal{S}_{(0,s)} \). Finally we can consider surface defects on both planes at the same time, labelled as \( \mathcal{S}_{(r,s)} \).

We expect that the flat directions corresponding to the transverse positions of the vortices (the coefficients \( \{ z_i \} \) of the polynomial \( P_r(z) \)) will flow to \( r \) 2d free chiral multiplets \( \phi_i \) in the...
IR. It should be possible to strip off these free chiral multiplets, and define reduced surface defects $\mathcal{S}_{(r,0)}$ in $\mathcal{T}_{IR}$.

While the difference in the holomorphic gauge couplings of the two $SU(N)$ gauge factors coupled to the extra node of figure 1 does not affect the bulk IR SCFT, it is a marginal deformation parameter of the $\mathcal{S}_{(r,0)}$ defects, which behaves like a Kahler parameter for the $2d$ $(2,2)$ theory. Thus the $\mathcal{S}_{(r,0)}$ defects should support an exactly marginal twisted chiral operator.\(^8\)

### 2.3 RG flow by Higgs branch vevs: the ungauged perspective

In order to define our RG flow, we do not really need to gauge $U(1)_f$. We may simply turn on the appropriate Higgs branch vev in $\mathcal{T}_{UV}$, and look at physics below the scale set by the vev. Let us first consider the constant-vev background. Gauging the $U(1)_f$ ate up three scalar fluctuations transverse to the moment map level set, and one along the $U(1)_f$ orbit. Thus we expect to see in the IR the same $\mathcal{T}_{IR}$ we encountered in the previous section, together with a hypermultiplet which captures these four extra directions. This hypermultiplet transforms with unusual quantum numbers under the $SO(2)_{\tilde{R}}$ symmetry $\tilde{R} = R + f/2$: it has two components of charge 0, and components of charge $\pm 1$. The $SO(2)_{\tilde{R}}$ is promoted in the IR to an $SU(2)_{\tilde{R}}$ which acts as the standard R-symmetry of $\mathcal{T}_{IR}$. Under $SU(2)_{\tilde{R}}$ the hypermultiplet scalars transform as a singlet plus a triplet.\(^9\)

In the absence of the $U(1)_f$ gauge field, there are no dynamical vortex solutions. But we can still consider the RG flow triggered by turning on a position-dependent background with $B(z) = P_r(z) = z^r$ in $\mathcal{T}_{UV}$. Away from $z = 0$, the theory still flows to $\mathcal{T}_{IR}$ times the free hypermultiplet. The bulk theory is however modified at the origin by surface defects, which we are led to identify with the surface defects defined in the previous subsection. More precisely, we expect them to correspond to $\mathcal{S}_{(r,0)}$ (as opposed to $\mathcal{S}_{(r,0)}$), since the coefficients of $P_r$ are not fluctuating degrees of freedom in this setup. It is less obvious to describe the fate of the free hypermultiplet near the origin. The size of the gauge orbits now scales to zero as we approach the origin, as the Higgs branch vev goes to zero there. This suggests that the $\mathcal{S}_{(r,0)}$ defects are coupled to a surface defect for the free hypermultiplet theory. Consistency of this picture with the “gauged” picture of the previous subsection suggests that the degrees

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\(^8\)We could also construct more intricate variants of the $\mathcal{S}_{(r,0)}$ surface defects, starting from a general linear quiver with $k$ $SU(N)$ nodes, and applying our construction to each of the $U(1)$ flavor symmetry groups of the bifundamental hypermultiplets. We expect to obtain surface defects with $k-1$ exactly marginal twisted chiral operators. We leave the exploration of this construction for future work.

\(^9\)This is not in contradiction with the standard R-charge assignment for a hypermultiplet, since this hypermultiplet, being free and decoupled from $\mathcal{T}_{IR}$, has an accidental $Sp(1)$ flavor symmetry. The standard R-symmetry for the hypermultiplet is recovered as a diagonal combination of $SU(2)_{\tilde{R}}$ and $Sp(1)$. 
of freedom of the free hyper that survive at the origin should match with the center-of-mass motion degrees of freedom of the vortices in the previous description.

3. The prescription

In this section, we give a physical interpretation of a class of poles of the superconformal index in terms of the surface defects defined in the previous section. We will be led to a precise prescription to evaluate the index in the presence of surface defects.

Let us first review the definition of the superconformal index and fix our basic conventions. The index of an $\mathcal{N} = 2$ SCFT can be thought of as a trace over states of the theory in the radial quantization, i.e. a partition function on $S^3 \times S^1$ \[ I = \text{Tr}(-1)^F \left( \frac{t}{pq} \right)^r p^{j_{12}} q^{j_{34}} t^R \prod_i a_i^{f_i}. \] (3.1)

We denoted as $j_{12}$ as $j_{34}$ the rotation generators in two orthogonal planes: $j_{12} = j_2 + j_1$ and $j_{34} = j_2 - j_1$ with $j_{1,2}$ being the Cartans of the Lorentz $SU(2)_1 \times SU(2)_2$ isometry of $S^3$. $r$ is the $U(1)_r$ generator, and $R$ the $SU(2)_R$ generator of R-symmetries. The $a_i$ are fugacities for the flavor symmetry generators $f_i$. We will always assume that $|p| < 1$, $|q| < 1$, $|t| < 1$, $|a_i| = 1$, $|pq| < 1$. (3.2)

The particular index (3.1) counts states annihilated by supercharge $\tilde{Q}_1 \hat{-}$ (and its Hermitian conjugate): this charge has $SU(2)_R$ charge $\frac{1}{2}$, $r$-charge $-\frac{1}{2}$, and $SU(2)_1 \times SU(2)_2$ charges $(0, -\frac{1}{2})$. Other choices of the supercharge will give an equivalent index for $\mathcal{N} = 2$ theories [7]. Thus the states which contribute to the index (3.1) satisfy [23]

\[ 2 \left\{ \tilde{Q}_1 \hat{-}, \left( \tilde{Q}_1 \hat{-} \right)^\dagger \right\} = E - 2j_2 - 2R + r = 0. \] (3.3)

Let us also mention here the single letter indices (partition functions) of the basic ingredients of $\mathcal{N} = 2$ field theories, the hypermultiplet and the vector multiplet,

\[ I_{H}^{t}(p, q, t, a) = \frac{\sqrt{t} - \frac{pq}{\sqrt{t}}}{(1-p)(1-q)} (a + a^{-1}), \] (3.4)

\[ I_{V}^{t}(p, q, t) = -\frac{p}{1-p} - \frac{q}{1-q} + \frac{pq}{(1-p)(1-q)}. \]

\[ ^{10} \text{In this paper we follow the notations of [23].} \]
Here \( a \) is the fugacity for the \( U(1) \) charge of the half-hypers. To obtain the multi-particle index one computes the plethystic exponent the single-letter partition function. For example, the full, multi-particle, index of the vector multiplet is

\[
\mathcal{I}_V = \text{PE} \left[ \mathcal{I}_V^{s.t.}(p, q, t) \right] = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \mathcal{I}_V^{s.t.}(p^n, q^n, t^n) \right).
\]  

Given an index \( \mathcal{I}(a, \ldots) \) of a theory with \( SU(N) \) flavor symmetry labeled by fugacities \( a \) the index of a theory with this flavor symmetry gauged is given by

\[
\oint \left[ \prod_{i=1}^{N-1} \frac{da_i}{2\pi ia_i} \right] \Delta(a) \mathcal{I}_V(a) \mathcal{I}(a, \ldots),
\]

where

\[
\Delta(a) = \frac{1}{N!} \prod_{i \neq j} (1 - a_i/a_j),
\]

is the familiar \( SU(N) \) Haar measure, and

\[
\mathcal{I}_V(a) \equiv \text{PE} \left[ \mathcal{I}_V^{s.t.}(p, q, t) \left( \sum_{i,j=1}^{N} a_i/a_j - 1 \right) \right],
\]

is the index of an \( SU(N) \) vector multiplet.

### 3.1 Some important poles

The superconformal index has a rich analytical structure, in particular it exhibits many poles in the flavor fugacities. A divergence in the index is associated to the integration over a bosonic zero mode in the \( S^3 \times S^1 \) partition function. The introduction of a maximal set of superconformal and flavor fugacities is in fact motivated by the desire to regulate these divergences.

We wish to give a physical interpretation of the simple poles of the index and characterize their residues. Not to obscure the main point with heavy notations, consider a schematic definition of the index,

\[
\mathcal{I}(a, b) = \text{Tr}(-1)^F a^f b^g,
\]

where \( f \) and \( g \) are some charges. Let us assume that \( \mathcal{I} \) has a pole in fugacity \( a \),

\[
\mathcal{I} = \frac{\tilde{\mathcal{I}}(a, b)}{1 - a^{f^{\circ}} b^{g^{\circ}}}.
\]
The pole is naturally associated to a bosonic operator, $O$, with charges $f = f_O$ and $g = g_O$. The fact that we have the divergence means that arbitrary high powers of the operator $O$ contribute to the index. Although for generic values of the fugacities fluctuations are massive and the $S^3 \times S^1$ partition function is finite, for $a^{f_O} b^{g_O} = 1$ a flat direction opens up and the partition function diverges. The residue at the pole is given by $\tilde{I}(b^{-g_O/f_O}, b)$, so it can be written
\[
\tilde{I} = \text{Tr}(-1)^F b^{g'} .
\] (3.11)
where $g'$ denotes the shifted charge
\[
g' = g - \frac{g_O}{f_O} f .
\] (3.12)
The shifted charge is preserved in the background where $O$ has a non-zero vev. To characterize the residue, we can use the understanding of the divergence as arising from a zero-mode. We then expect the divergence of the partition function to be controlled by the behavior of theory “at infinity” in the moduli space parametrized by the vev of $O$, that is, by the properties of the IR theory reached at the endpoint of the RG flow triggered by giving $O$ a vev. We should interpret $\tilde{I}$ as the index of the IR fixed of this RG flow.

By computing the residue of fugacity $a$ we have lost the $U(1)$ symmetry corresponding to it in $\tilde{I}$ and thus the IR theory has smaller global symmetry, i.e. giving vev to $O$ explicitly breaks some of the symmetries of the UV theory. Note that the number of states contributing to $\tilde{I}$ is smaller (or equal) to the number of states contributing to $I$ since after shifting the charges some states might cancel out but new states do not start contributing. This is similar in spirit to Romelsberger’s prescription for computing indices of IR fixed points in the $\mathcal{N} = 1$ setups [8, 40, 41, 1].

Let us specialize this general framework to the flavor fugacities poles that arise from Higgs-branch zero modes. Consider a general SCFT with a Higgs branch of vacua $\mathcal{M}$, parameterized by a ring of BPS operators $\{O^\alpha\}$, which sit in spin $R^\alpha$ representations of $SU(2)_R$, have charges $f_1^\alpha$ under the flavor symmetries and carry no angular momenta or $r$-charge. The index receives a contribution from the highest weight component in the $SU(2)_R$ multiplet, complex chiral fields which we still denote as $O^\alpha$. The operator ring matches the ring of holomorphic functions on $\mathcal{M}$. For every generator $O^\alpha$, all the powers of operator $O^\alpha$ contribute to the index, and summing the geometric series this implies that the index has a pole\(^{12}\) at $1$ in the simplest case, this operator generates a ring and all the its powers contribute. More generally, there can be relations, which are taken into account by the numerator $\tilde{I}(a, b)$.

\(^{11}\)If we specialize to $p = q = 0$, so that the chiral operators on the Higgs branch are the only operators which contribute to the index, the index takes in general the form of a polynomial in $t$ over the product of factors $(1 - t^{\alpha_\alpha} \prod_{i} a_i^{f_i})$. In some cases, this specialization of the superconformal fugacities relates the index to the Hibert series (as defined e.g. in [42]) of the Higgs branch [23]. This relation was explored recently in [43, 44, 45].
\( t^{R_a} \prod_i a^{f_{B_i}} = 1. \)

We now focus on the constant Higgs branch vevs that we have considered in section 2, where we have a clear understanding of how the R-charges of the UV theory match the R-charge of the effective IR description. We can then hope to relate the residue of UV index at the pole with the index of the IR theory. Let us consider the setup of figure 2, and study the divergence of the UV index associated to the baryon operator \( B \). The position of the pole can be written as \( a^{f_B} = 1 \), where \( a \) is the fugacity for \( U(1)_f \). The \( SU(2) \) Cartan generator is \( R = R - R_B f_B \), so if we sit at the pole \( a = t^{-R_B / f_B} \) we are simply replacing \( t^R \) with \( t^\bar{R} \), i.e. evaluating the index with the fugacities which are appropriate for the IR theory.

Using \( R_B = N/2 \) and \( f_B = N \), the equation \( t^{N/2} a^N = 1 \) implies that the index has \( N \) poles at the locations

\[
a = \exp \left[ \frac{2\pi i}{N} \ell \right] t^{1/2}, \quad \ell = 0, \ldots, N - 1. \tag{3.13}
\]

The residues of all these poles quantify the same physics: the IR fixed point of a theory with a vev for baryonic operator. This baryonic operator breaks the \( U(1) \) flavor symmetry parametrized by \( a \) to \( Z_N \) symmetry, and looking at different poles of the index corresponds to the index of the IR theory with one of the \( N \) group elements of \( Z_N \) inserted in the trace formula. Thus, it makes sense to either sum over the \( N \) poles or consider the contribution of one of the poles multiplied by \( N \). In our setup of figure 2 the former corresponds to gauging the \( Z_N \) center of the \( SU(N) \) flavor symmetry and thus leaving only \( PSU(N) \) as the flavor symmetry in the IR: this scenario is the one naturally occuring in the gauged perspective of section 2.1. The latter point of view however corresponds to keeping the full \( SU(N) \) flavor symmetry which is more natural when considering flows initiated by a vev: this will be the point of view we will adopt in the rest of the paper.

In the IR we also expect to have the free hypermultiplet with scalars transforming in the \( 1 + 3 \) irreps of \( SU(2) \). The index of such a hypermultiplet captures exactly the divergent part of the UV index at \( t^{N/2} a^{-N} = 1 \), as it contains exactly the massless fluctuation which causes the divergence. If we start from the index of a standard free hypermultiplet, we can get the index of the \( 1 + 3 \) hypermultiplet by setting \( a = t^{1/2} \). Up to the divergent zero mode, we get precisely \( I_{V^{-1}} \), the inverse of the index of a \( U(1) \) vector multiplet. This motivates our final prescription:

\[
I[T_{IR}] = N I_{V} \text{Res}_{a = t^{1/2}} \frac{1}{a} I[T_{UV}]. \tag{3.14}
\]

The origin of the factor \( N \) is explained in the paragraph above. In summary, extracting the residue at \( a = t^{1/2} \) in the index of \( T_{UV} \) simply amounts to removing the extra node in the quiver (see figure 2) and recovering the original theory \( T_{IR} \). This is in perfect agreement with the intuition coming from the RG with a constant vev for the baryon operator.
The index also receives contributions from holomorphic derivatives of the baryon operator in the 12 and 34 planes, \( \partial_{12} \partial_{34} B \), so by the same logic as above we expect poles at \( t^R p^q q^a \). We wish to relate the residues at these poles to the theories obtained in the IR by turning on a space-dependent vev \( B(z, w) = z^w w^s \). As we have explained in section 2, the IR theory is \( \mathcal{T}_{IR} \) in the presence of an additional surface defect \( \mathcal{S}_{(r, s)} \), with some extra decoupled matter coming from the free hypermultiplet. In the \( S^3 \times S^1 \) geometry, the surface defects fill the “temporal” \( S^1 \) and (for \( r, s \neq 0 \)) the two maximal circles inside the \( S^3 \) fixed by the \( j_{12} \) and \( j_{34} \) rotations, respectively (or only one of them for \( r = 0 \) or \( s = 0 \)).

We are almost ready to compute the index of the theory \( \mathcal{T}_{IR} \) in the presence of surface defects. What remains to be done is to figure out which contribution from the free hypermultiplet should we strip off. A simple choice is to just strip off the bulk contribution. We will later identify that with the index for the theory in the presence of the \( \mathcal{S}_{(r, s)} \) surface defect,

\[
\mathcal{I}[\mathcal{T}_{IR}, \mathcal{S}_{(r, s)}] = N \mathcal{I}_V \text{Res}_{a=t^{1/2} p^{r/N} q^{s/N}} \frac{1}{a} \mathcal{I}[\mathcal{T}_{UV}] .
\]  

(3.15)

Again, when we set \( a = t^{1/2} p^{r/N} q^{s/N} \) we are shifting the UV rotation generators to become the rotation generators appropriate to the IR SCFT, as explained in section 2.

### 3.2 A toy model: the free hypermultiplet

To understand which terms should be stripped off from the index to get rid of the decoupled degrees of freedom, it is useful to look at a toy-model where the UV theory coincides with a single hypermultiplet. The index of a free hypermultiplet is

\[
\mathcal{I}_H = \prod_{n, m \geq 0} \frac{1 - t^{-1/2} p^{n+1} q^{m+1} a^{\pm 1}}{1 - t^{1/2} p^{n} q^{m} a^{\pm 1}} \equiv \Gamma \left( t^{1/2} a^{\pm 1}; q, p \right) .
\]  

(3.16)

Here and throughout the paper we use a shorthand notation where the \( \pm \) exponent means that we take the product over both choices of sign. This index is thus given by a product of two elliptic Gamma functions.¹³ The denominator of (3.16) comes from the scalar fields in the hypermultiplet and their derivatives. If we specialize to \( a = t^{1/2} p^{n} q^{m} \) in the free hypermultiplet index, remove the zero mode and multiply it by \( \mathcal{I}_V \), we get some very suggestive results. For example, at \( a = t^{1/2} p \)

\[
R_{1,0} = \prod_{m \geq 0} \frac{(1 - t^{-1} q^{m+1})(1 - t q^{m})}{(1 - p^{-1} q^{m})(1 - pq^{m+1})} = \frac{\theta(t; q)}{\theta(p^{-1}; q)} .
\]  

(3.17)

¹³The relevance of these functions to the index computations was observed in [41]. See also [46] for a nice review of the properties of elliptic Gamma functions.
Here the theta-function is defined as

$$\theta(z; q) \equiv \prod_{m=0}^{\infty} (1 - zq^m)(1 - z^{-1}q^{m+1}), \quad \theta(qz; q) = \theta(z^{-1}; q). \quad (3.18)$$

We will also use the Pochhammer symbol,

$$\displaystyle \prod_{\ell=0}^{\infty} (1 - a \ b^\ell). \quad (3.19)$$

The residue $R_{1,0}$ takes the form of a 2d index for a free 2d chiral multiplet, living on the circle fixed by the $j_{12}$ rotations. We present the details of the precise relation between (3.17) and the 2d index in appendix A. The bosonic scalar field in the chiral multiplet has charge 1 under rotations in the 12 plane: it is exactly the 2d chiral field which we planned to strip off from $\hat{\mathfrak{S}}_{(1,0)}$ to get $\hat{\mathfrak{S}}_{(1,0)}$.

In order to read off the residue at $a = t^{1/2} p^r$, we can simplify some manipulations by looking at the single-letter partition function. We specialize to $a = t^{1/2} p^r$, subtract 1 to remove the pole

$$-\frac{p}{1-p} - \frac{q}{1-q} + \frac{pq/t - t}{(1-p)(1-q)} + \frac{tp^r - p^{r+1}q + p^{-r} - t^{-1}p^{1-r}q}{(1-p)(1-q)} - 1 \quad (3.20)$$

or

$$\frac{1-p^r}{1-p} \frac{pq - t}{1-q} + \frac{1-p^{-r}p^{-1} - qt^{-1}}{1-p^{-1}} \quad (3.21)$$

which gives the residue

$$R_{r,0} = \prod_{u=0}^{r-1} \prod_{m=0}^{\infty} \frac{(1 - t^{-1}p^{-u}q^{m+1})(1 - tp^u q^m)}{(1 - p^{-u-1}q^m)(1 - p^u + 1 q^{m+1})} = \prod_{u=0}^{r-1} \frac{\theta(tp^u; q)}{\theta(p^{-1-u}; q)}. \quad (3.22)$$

This has the form of a 2d index for $r$ free chiral multiplets, living on the circle fixed by the $j_{12}$ rotations (see appendix A for the definition of the 2d index). The 2d chiral fields we see in $R_{r,0}$ carry charges $1, \cdots, r$ under the 12 rotations, and so look exactly like the 2d chirals which we planned to strip off from $\hat{\mathfrak{S}}_{(r,0)}$ to get $\hat{\mathfrak{S}}_{(r,0)}$.

Finally, in order to read off the residue at $a = t^{1/2} p^r q^s$ we can use again the single-letter partition function

$$-\frac{p}{1-p} - \frac{q}{1-q} + \frac{pq/t - t}{(1-p)(1-q)} + \frac{tp^r q^s - p^{r+1}q^{s+1} + p^{-r}q^{-s} - t^{-1}p^{1-r}q^{1-s}}{(1-p)(1-q)} - 1 \quad (3.23)$$

We can write that as a sum of three terms. The first term corresponds to $R_{r,0}$. The second to $R_{0,s}$, which is the same as $R_{s,0}$ with $p$ and $q$ exchanged,

$$R_{0,s} = \prod_{u=0}^{s-1} \prod_{m=0}^{\infty} \frac{(1 - t^{-1}q^{-u}p^{m+1})(1 - tq^u p^m)}{(1 - q^{-u-1}p^m)(1 - q^{u+1}p^{m+1})} = \prod_{u=0}^{s-1} \frac{\theta(tq^u; p)}{\theta(q^{-1-u}; p)}. \quad (3.24)$$
The third piece is
\[
(t - pq) \frac{1 - p^r 1 - q^s}{1 - p 1 - q} - (t^{-1} - p^{-1} q^{-1}) \frac{1 - p^{-r} 1 - q^{-s}}{1 - p^{-1} 1 - q^{-1}}
\] (3.25)

This contributes to the residue through $rs$ ratios of the form
\[
\frac{(1 - p^{n+1} q^{m+1})(1 - t^{-1} p^{-n} q^{-m})}{(1 - p^{-n+1} q^{-m+1})(1 - t p^n q^m)} = \frac{pq}{t}
\] (3.26)

Hence the total residue is
\[
R_{r,s} = \left( \frac{pq}{t} \right)^r s R_{r,0} R_{0,s}
\] (3.27)

Up to the prefactor, this has the form of the product of two 2d indices, one for a 2d theory on the circle fixed by $j_{12}$, the other living on the circle fixed by $j_{34}$. The prefactor corresponds to a re-definition of the $U(1)_r$ charge by a constant shift. Such shifts of an abelian symmetry generator are common in the presence of defects, unless the generator sits in a non-abelian factor of the subgroup of the superconformal group preserved by the defect. Here we have two distinct surface defects, in a quarter-BPS configuration.

Inspired by this calculation, we propose our final recipe to completely strip off the free hypermultiplet contributions:
\[
\mathcal{I}[\mathcal{T}_{IR}, \hat{S}_{(r,s)}] = N R_{r,s}^{-1} \mathcal{I}_V \text{Res}_{a = \frac{t^{1/2} p^{r/N} q^{s/N}}{a}} \frac{1}{a} \mathcal{I}[\mathcal{T}_{UV}].
\] (3.28)

### 3.3 Gauging $U(1)_f$ and the index

Our basic prescription (3.28) has a suggestive interpretation in terms of the index of the theory where the $U(1)_f$ symmetry is gauged,
\[
\int \frac{da}{2\pi i a} a^\xi \mathcal{I}_V \mathcal{I}[\mathcal{T}_{UV}].
\] (3.29)

Here the contour integral is on the unit circle $|a| = 1$ and we added a FI parameter $\xi$ (in units where the $S^3$ has unit radius), which can be turned on in the index.\(^1\)

Notice that (3.29) only makes sense if $\xi$ is quantized. If $\xi > 0$, it is natural to try and evaluate the contour integral by picking the residues at $|a| < 1$. In particular, we will pick the residues at $a = \frac{t^{1/2} p^{r/N} q^{s/N}}{a}$. In general there could be other poles. The dominant contribution at large $\xi$ will be the residue at $a = t^{1/2}$, which is precisely our prescription for

\(^1\)We thank N. Seiberg for comments on this point.
the index of $T_{IR}$, while the subleading contributions (or a subset of them) correspond to the index of $T_{IR}$ with the insertions of $G_{(r,s)}$. The UV index can then be written as a sum over the indices of the IR theories associated to the different supersymmetric backgrounds.

One may worry that the $U(1)_f$ gauged theory is not well defined since it has both a Landau pole and an anomalous $U(1)_r$ symmetry. The presence of a Landau pole may not be a serious problem, as the index is independent of the gauge coupling, and we can suppress the Landau pole arbitrarily by making the gauge coupling smaller and smaller. The anomaly in $U(1)_r$ would appear to be a more serious obstruction. In principle, one can reabsorb the Tr$F \wedge F$ anomaly for the $U(1)_r$ symmetry by redefining the $U(1)_r$ charge as

$$r \to r + b_0 \int_{S^3} A \wedge F,$$

where $b_0$ is some proportionality constant. The Chern Simons action is usually not gauge invariant, but for an Abelian gauge theory ($F = dA$) $U(1)_f$, on a three manifold with no two-cycles, such as $S^3$, the gauge variation

$$\int_{S^3} dA \wedge F$$

is actually zero.

The redefinition of the $r$ charge will place an overall power of $pq/t$ in front of the contribution to the index of interesting topological sectors for $U(1)_f$. A pair of linked vortices may exactly fit the bill. Each vortex contains a unit of flux in the core, i.e. the integral of $A$ around the vortex should be one. If we have linked vortices, at the core of each we have flux $F$ from that vortex, and potential $A$ from the other vortex, which conspire to give a unit contribution to $\int_{S^3} A \wedge F$. This may neatly explain the prefactor $(\frac{pq}{t})^{r,s}$ we encountered in $R_{r,s}$.

4. Residues and Ruijsenaars-Schneider models: $A_1$ theories

We now turn to a detailed analysis of the analytic properties of the index for the $A_1$ theories of class $S$, the $A_1$ generalized quivers of [15]. The generalization to higher rank is conceptually straightforward and will be presented in section 8. The $A_1$ case is technically the simplest as there is only one type of flavor puncture, carrying an $SU(2)$ flavor symmetry. The basic building block is the free half-hypermultiplet in the trifundamental representation of $SU(2)^3$, associated to the three-punctured sphere. The general $A_1$ theory of class $S$ is associated to a pair-of-pants decomposition of a genus $g$ surface with $p$ punctures.
We consider the setup of figure 2. The SCFT $\mathcal{T}_{UV}$ is taken to be an $A_1$ class $S$ theory with $p = s + 1$ punctures, $s \geq 1$. We are interested in the degeneration limit where a three-punctured sphere is connected by a long cylinder to the rest of the surface, an $s$-punctured Riemann surface which we denote by $C$, see figure 3. The field theory interpretation of this geometry is familiar: the SCFT associated to $C$ is coupled to a trifundamental half-hyper, by weakly gauging a diagonal $SU(2)$ group. In the notation of figure 2, the theory associated to $C$ is $\mathcal{T}_{IR}$.

We then focus on the functional dependence of the superconformal index of $\mathcal{T}_{UV}$ on either one of the flavor fugacities of the weakly-coupled hyper. As outlined in sections 3 and 4, we expect to find poles at certain values of the flavor fugacities, and to be able to interpret the residue at each of these poles as the index of $\mathcal{T}_{IR}$ in the presence of additional surface defects. The analysis will be somewhat technical but the main idea is simple. The basic point is that since the index of $\mathcal{T}_{UV}$ is a contour integral in the fugacity of the gauged $SU(2)$ associated to the long cylinder, a pole in the flavor fugacity arises when the integration contour is “pinched” by two singularities. A little calculation reveals that the residue takes the form of a difference operator acting on the index of $\mathcal{T}_{IR}$. Our main claim is that the action of this difference operator introduces a surface defect in $\mathcal{T}_{IR}$, equation (4.18). Remarkably, the difference operator turns out to be the Hamiltonian of the relativistic Calogero-Moser model, also known as the elliptic Ruijsenaars-Schneider (RS) model.\footnote{The RS integrable models appear in several contexts in the study of $\mathcal{N} = 2$ gauge theories, see e.g. [47, 48].}
In the following sections we will employ these difference operators to extract several physical statements about the index of theories of class $S$. In particular we will gather evidence for the interpretation of the difference operators as operators introducing surface defects into the index computation. The difference operator can act on any one of the $s$ flavor fugacities of $T_{IR}$, and by generalized S-duality we expect to obtain the same result regardless of which fugacity we decide to act on. This constraint is so powerful that it allows to completely fix the index for any $A_1$ generalized quiver, with or without surface defects. Of course, in the $A_1$ case all theories have an explicit Lagrangian description, and the index of any quiver admits an explicit representation as a matrix integral. What this new logic buys us is a representation of the index where the structure constants of the associated 2d TQFT have been diagonalized. More importantly, this algorithm will generalize easily to the higher-rank quivers, which generically do not admit a direct Lagrangian description.

For general values of the three superconformal fugacities ($p, q, t$), the answer is given in terms of eigenfunctions and eigenvalues of the elliptic RS model, whose analytic form is not fully known. We construct however approximate eigenfunctions in a certain limit, which serves as a proof of concept of their existence and uniqueness (see appendix B). For $p \to 0$ (or equivalently $q \to 0$), the elliptic RS model degenerates to the much better studied trigonometric RS model, whose eigenfunctions are the Macdonald’s polynomials. By this route we recover from “first principles” the results of [23]. Finally we consider the dimensional reduction of the 4d superconformal index to the 3d partition function.

Let us introduce the basic ingredients. The index associated to the three-punctured sphere, the index of the free trifundamental half-hypermultiplet, is given by

$$I_{hyp}(a, b, c) = \prod_{m,n \geq 0} \frac{1 - p^{n+1} q^{m+1} t^{-\frac{1}{2}} a^{\pm 1} b^{\pm 1} c^{\pm 1}}{1 - p^n q^m t^{\frac{1}{2}} a^{\pm 1} b^{\pm 1} c^{\pm 1}} = \Gamma\left(t^{\frac{1}{2}} a^{\pm 1} b^{\pm 1} c^{\pm 1}; q, p\right).$$

We couple this free SCFT to a general $A_1$ quiver SCFT, which we call $T_{IR}$, associated to a Riemann surface $C$, by gauging fugacity $c$. The result is the SCFT that we call $T_{UV}$, see figures 2 and 3. The index $\mathcal{I}[T_{UV}] \equiv \mathcal{I}$ for short) reads

$$\mathcal{I}(a, \ldots) = \oint \frac{dc}{2\pi ic} \Delta(c) \mathcal{I}_V(c) \mathcal{I}_{hyp}(a, b, c) \mathcal{I}_C(c^{-1}, \ldots),$$

We refer to [49, 50]. The same quantum mechanical integrable models are also related to gauge theories in lower dimensions (see e.g. [51, 52, 53, 54, 55, 56]). It would be interesting to understand the connection between these lower-dimensional gauge theories and the 2d TQFT that computes the index, which for $q = t$ is known to reduce to 2d $q$-deformed Yang-Mills in the zero area limit [22].
where $\mathcal{I}_C \equiv \mathcal{I}[\mathcal{T}_{IR}]$. From the physical considerations of sections 2 and 3, we expect $\mathcal{I}(a, \ldots)$ to have simple poles at $a = t^{1/2}r^{1/2}q^{1/2}$, for $r$ and $s$ non-negative integers.\(^{16}\)

Let us start with the simplest case, the pole at

$$a = t^{\frac{1}{2}}. \quad (4.3)$$

To verify that $\mathcal{I}$ has indeed a pole at this value of the fugacity, we proceed as follows. The integrand has (among others) simple poles in $c$ at

$$c = t^{\frac{1}{2}}b^\pm_1a^{-1}, \quad c = t^{-\frac{1}{2}}b^\pm_1a. \quad (4.4)$$

Setting $a = t^{\frac{1}{2}}$ these two poles collide and pinch the $c$ integration contour producing a simple pole in $a$. To compute the residue at $a = t^{\frac{1}{2}}$ of $\mathcal{I}$ we can pick up the residues at the poles \((4.4)\) in $c$ inside the integration contour. The other poles in $c$ do not contribute to the residue in $a$. Explicit evaluation gives

$$\text{Res}_{a=t^{\frac{1}{2}}} \frac{1}{a} [\mathcal{I}] = \sum_{\pm} \text{Res}_{a=t^{\frac{1}{2}}; c=t^{\frac{1}{2}}b^\pm_1a^{-1}} \frac{1}{a} \left[ (\Delta(c)) \mathcal{I}_V(c) \mathcal{I}_{hyp}(a, b, c) \mathcal{I}_C(c^{-1}, \ldots) \right]$$

$$= \frac{1}{2} \frac{1}{2} \mathcal{I}_C(b, \ldots) \Delta(b) \mathcal{I}_V(b) \prod_{\pm} (1 - b^{1/2}) \quad PE \left[ \frac{p + q - 2pq + t - \frac{pq}{I}}{(1 - p)(1 - q)} (b^2 + b^{-2} + 2) \right] \quad (4.5)$$

$$= \frac{1}{2} \mathcal{I}_V^{-1} \mathcal{I}_C(b, \ldots).$$

The origin of the different numeric factors is as follows. The factor of 2 comes from summing over the two poles in \((4.4)\). One factor of $\frac{1}{2}$ comes from the Haar measure of $SU(2)$ and the second factor of $\frac{1}{2}$ comes from evaluating the hypermultiplet index at the pole.

There is also a pole at $a = -t^{\frac{1}{2}}$, whose residue is $\frac{1}{2} \mathcal{I}_V^{-1} \mathcal{I}_C(-b, \ldots)$. As we discussed in section 3 we need only consider the pole \((4.3)\) since with either sign the poles describe the same physics.\(^{17}\)

All in all, we can summarize our calculation by the simple relation\(^{18}\)

$$\mathcal{I}[\mathcal{T}_{IR}] = 2 \mathcal{I}_V \text{Res}_{a=t^{\frac{1}{2}}} \frac{1}{a} [\mathcal{I}[\mathcal{T}_{UV}]], \quad (4.6)$$

\(^{16}\)We do not claim these are the only poles in general. The poles we discuss here are the ones for which we found a simple interpretation in terms of the behavior at infinity of the Higgs branch. Other poles may have different interpretations: see e.g. [43] for some special examples.

\(^{17}\)Let us reiterate in this concrete context the logic of section 3.1: in the gauged perspective, as the contour integral corresponding to the gauging of $U(1)_f$, is around the unit circle both poles at $\pm t^{\frac{1}{2}}$ contribute. The index of the IR theory is invariant under negating any of the flavor fugacities. In other words the flavor symmetry has additional discrete $Z_2$ component which is gauged when we compute the residues. Thus, after gauging the $U(1)_f$ flavor symmetry (which for $A_1$ quivers is enhanced to $SU(2)$), we only have $PSU(2)$ flavor symmetry, as opposed to $SU(2)$, associated with one of the punctures. In the un-gauged perspective it is more natural to consider one of the poles, say at $a = t^{\frac{1}{2}}$, taken twice and thus keeping the center of $SU(2)$ not gauged.

\(^{18}\)Recall the symbols $\mathcal{I}[\mathcal{T}_{IR}] \equiv \mathcal{I}_C$ are used interchangeably.
which precisely confirms (3.28) for \( r = s = 0 \). Taking the residue of the UV index at the “extra” \( U(1)_f \) flavor fugacity \( a = t^\frac{1}{2} \) we have “completely closed” the \( U(1)_f \) puncture and got back the IR index.

Let us now consider the poles with non-trivial \( p \) and \( q \) dependence,

\[
a = \pm t^\frac{1}{2} \frac{p^{r/2} q^{s/2}}{r^2}, \tag{4.7}
\]

with non-negative integers \( r \) and \( s \). We need only focus on the positive sign. We give the details of the calculation for \( s > 0, r = 0 \), as the generalization to non-zero \( r \) is straightforward we will simply quote the result at the end of this subsection. We start by considering the poles of the integrand of (4.2) at

\[
\begin{align*}
  c &= t^\frac{1}{2} q^{m_1} \frac{1}{ab}, \\
  c^{-1} &= t^\frac{1}{2} q^{m_2} \frac{b}{a}, \\
  c &= t^\frac{1}{2} q^{m_1} \frac{b}{a}, \\
  c^{-1} &= t^\frac{1}{2} q^{m_2} \frac{1}{ab}.
\end{align*} \tag{4.8}
\]

If the non-negative integers \( m_i \) are chosen such that

\[
s = m_1 + m_2, \tag{4.9}
\]

then the pairs of poles in each line of (4.8) collide when \( a = t^\frac{1}{2} q^\frac{1}{2}s \). Further, since all expressions are symmetric under \( c \to c^{-1} \) the residues at the poles corresponding to the two lines in (4.8) are equal under exchange of \( m_1 \) with \( m_2 \). It follows that to compute the residue at (4.7), we need to keep track of the terms in the \( c \) contour integral classified by a partition of \( s \) into two non-negative parts. E.g. for \( s = 1 \) we get two different choices, \( m_1 = 1, m_2 = 0 \) and \( m_1 = 0, m_2 = 1 \). Let us then compute the residues. The contribution of the hyper is given by

\[
A_{(m_i)} = 2 \frac{1}{2} \mathcal{I}_C(q^{\frac{1}{2} - m_i} b_i) \prod_{i,j=1}^{2} \prod_{m,n \geq 0} \frac{1 - p^{n+1} q^{m-m_j+1} t^{-1} b_j/b_i}{1 - p^n q^{m+m_j} t b_i/b_j} \frac{1 - p^{n+1} q^{m+m_j+1} b_i/b_j}{1 - p^n q^{m-m_j} b_j/b_i}, \tag{4.10}
\]

where we have defined \( b_1 \equiv b, b_2 \equiv b^{-1} \) and the prime over the second product indicates that we are omitting the diverging term. The factor of 2 in front is coming from the two lines in (4.8), while the factor of half arises since the pole in \( a \) appears as \( 1/(1 - t q^s a^{-2}) \).

We have to multiply each of these factors by the index of the vector multiplet and the Haar measure evaluated at the pole and sum over all the partitions of \( s \), which we will denote by \( \pi(s) \). The index of the vector multiplet together with the Haar measure evaluated at the
Thus the residue at a pole is given by

$$
\mathcal{B}_{(m_i)} = \frac{I^*_V}{2^{\frac{n}{2}}} \prod_{i,j=1}^{\frac{n}{2}} \prod_{m,n \geq 0} \frac{1 - p^n q^{m+m_j-m_i} b_i/b_j}{1 - p^{n+1} q^{m+m_j-m_i+1} t^{-1} b_i/b_j} \frac{1 - p^n q^{m+m_i-m_j} b_j/b_i}{1 - p^{n+1} q^{m+m_i-m_j+1} b_j/b_i}.
$$

(4.11)

Combining the two factors (4.10) and (4.11), and further multiplying by twice the index of free vector multiplet, we finally have

$$
2 \mathcal{I}^*_V \text{Res}_{a = t^{\frac{1}{2}} q^{^\frac{r}{2}}} \frac{1}{a} [\mathcal{I}] = \sum_{\{m_i\} \in \pi(r)} \mathcal{I}^*_V \mathcal{A}_{(m_i)} \mathcal{B}_{(m_i)} = \sum_{\{m_i\} \in \pi(r)} f^{(r)}_{\{m_i\}}(b) \mathcal{I}^*_C(q^{\frac{1}{2}r-m_i} b_i).
$$

(4.12)

We see that the computation of the residue at the pole (4.7) amounts to applying a difference operator $\mathcal{S}_{(0,s)}$ to the index,

$$
\mathcal{S}_{(0,s)} \mathcal{I}_C = 2 \mathcal{I}^*_V \text{Res}_{a = t^{\frac{1}{2}} q^{^\frac{r}{2}}} \frac{1}{a} [\mathcal{I}].
$$

(4.13)

The generalizations of this calculation to non-zero $r$ in (4.7) is immediate. The residue is given by a sum over terms coming from partitions of both $s$ and $r$,

$$
s = \sum_{i=1}^{2} m_i, \quad r = \sum_{i=1}^{2} m^*_i.
$$

(4.14)

We will denote the operator which computes the residue at (4.7) by $\mathcal{S}_{(r,s)}$.

Let us now compute explicitly the basic operator $\mathcal{S}_{(0,1)}$. First the factors $f^{(s=1)}_{\{m_i\}}$ are given by

$$
f^{(s=1)}_{\{m_i\}} = \prod_{m \geq 0} \frac{(1 - p^m t)(1 - p^{m+1} t^{-1})}{(1 - p^{m+1} q)(1 - p^{m+1} q^{-1})} \frac{1 - p^m t q^{-1} b_2/b_1}{1 - p^m t q^{-1} b_1/b_2} \frac{1 - p^m t q b_1/b_2}{1 - p^m t b_1/b_2}.
$$

(4.15)

Thus the residue at $a = t^{\frac{1}{2}} q^{^\frac{1}{2}}$ is given by the action of the following operator

$$
\mathcal{S}_{(0,1)} \mathcal{I}_C = \frac{\theta(t^2 p)}{\theta(q^{-1} p)} \sum_{i=1}^{2} \prod_{j \neq i} \frac{\theta(b_i/b_j; p)}{\theta(b_j/b_i; p)} \mathcal{I}_C(b_i \to q^{-\frac{1}{2}} b_i, b_j \neq i \to q^{\frac{1}{2}} b_j),
$$

(4.16)

$$
= \frac{\theta(t^2 p)}{\theta(q^{-1} p)} \left[ \frac{\theta(b^2 q^{-1} p)}{\theta(b^2 p)} \mathcal{I}_C(b q^{1/2}, \ldots) + \frac{\theta(b^2 p)}{\theta(b^{-2} p)} \mathcal{I}_C(b q^{-1/2}, \ldots) \right].
$$
The operator $\mathcal{S}_{(0,1)}$ is, up to a conjugation by a simple function (see appendix B), the basic difference operator of the $A_1$ RS model! We recognize the prefactor $\frac{\theta(tp)}{\theta(q^{-1};p)}$ as the function $R_{0,1}$ introduced in (3.24). This is the 2d index of one free chiral field in two dimensions (see appendix A) and we interpret it as the center of mass degree of freedom of the surface defect. As we have discussed in section 3, it is natural to strip off this factor from the definition of the defect. We can then define the difference operator associated to the “bare” defect, $\bar{\mathcal{S}}_{(0,1)} \equiv R_{0,1}^{-1} \mathcal{S}_{(0,1)}$. However, we will mostly write equations for the “full” operators $\mathcal{S}_{(r,s)}$, since they are somewhat easier to manipulate algebraically.

The operator corresponding to residue at $a = \frac{1}{t^s} p^\frac{1}{2}$, $\mathcal{S}_{(1,0)}$, is simply obtained by exchanging $q$ and $p$ in (4.16). It is not difficult to write down the operators for general $s$,

$$
\mathcal{S}_{(0,s)} \mathcal{I}_C = \sum_{m=0}^s \prod_{n=0}^{m-1} \theta(tq^n; p) \prod_{n=0}^{s-m-1} \theta(q^{-n}; p) \times \left[ \prod_{n=0}^{m-1} \frac{\theta(tq^{-2m+n}b^2; p)}{\theta(q^{-m}nmb^2; p)} \right] \mathcal{I}_C(b \to q^{\frac{s}{2}-m}b). \tag{4.17}
$$

Finally we quote the general result for all $r$ and $s$,

$$\mathcal{I}[\mathcal{T}_{IR}, \mathcal{S}_{(r,s)}] = 2 \mathcal{I}_V \operatorname{Res}_{a = \frac{1}{t^s} p^{r/2} q^{s/2}} \frac{1}{a} [\mathcal{I}[\mathcal{T}_{UV}]] = \mathcal{S}_{(r,s)} \mathcal{I}[\mathcal{T}_{IR}] = \sum_{\sum_{i=1}^r n_i = r} \sum_{\sum_{i=1}^s m_i = s} \mathcal{I}_C(p^r q^{s} q^{\frac{r}{2}-m_i} b_i) \times \left[ \prod_{i,j=1}^{2} \prod_{m=0}^{n_i-1} \frac{\theta(p^{n_j} q^{m_j} b_i / b_j; q)}{\theta(p^{-n_j} q^{m_j} b_i / b_j; q)} \right] \left[ \prod_{n=0}^{n_i-1} \frac{\theta(q^{n_j-m_i} p^{n_j-n_i} b_i / b_j; q)}{\theta(q^{m_j-n_i} p^{n_j-n_i} b_i / b_j; q)} \right], \tag{4.18}
$$

where the prime on the product means as always that we omit the divergent term. Let us stress here that although in the analysis of this section we have used the jargon of theories of class $\mathcal{S}$ the final result (4.18) applies to any theory of the setup of figure 2.

5. Properties of the operators $\mathcal{S}_{(r,s)}$

We now proceed to investigate the algebraic properties of the difference operators $\mathcal{S}_{(r,s)}$ obtained in the previous section. These operators have many beautiful mathematical properties. Some of these properties are to be anticipated on purely physical grounds following the fact that $\mathcal{S}_{(r,s)}$ compute residues for theories enjoying generalized S-duality.
We will find further evidence for our main claim, that $\mathcal{G}_{(r,s)} \mathcal{I}[T_{IR}]$ is the index of a $T_{IR}$ in the presence of two surface defects along the two orthogonal circles of $S^3$ fixed by $j_{12}$ and $j_{34}$. We will see that the two defects are naturally associated to the spin $r/2$ and spin $s/2$ representations of $SU(2)$.

**Factorization**

We claim

$$\mathcal{G}_{(r,s)} = \left( \frac{pq}{t} \right)^r s \mathcal{G}_{(r,0)} \mathcal{G}_{(0,s)} = \left( \frac{pq}{t} \right)^s r \mathcal{G}_{(0,s)} \mathcal{G}_{(r,0)}.$$  \hspace{1cm} (5.1)

To show this we write

$$\mathcal{G}_{(0,s)} \mathcal{G}_{(r,0)} \mathcal{I} = \sum_{n_i=0}^{n_i=1} \sum_{m_i=0}^{m_i=1} \mathcal{I}(q^{2n_i} p^{2m_i} t^{2}) \prod_{i,j=1}^{2} \left[ \prod_{n=0}^{n=1} \frac{\theta(p^{n+j-n_i} t b_i / b_j ; q)}{\theta(p^{n+j-n_i} b_i / b_j ; q)} \right] \left[ \prod_{m=0}^{m=1} \frac{\theta(p^{n_j-n_i} q^{m+j-n_i} t b_i / b_j ; p)}{\theta(p^{n_j-n_i} q^{m} b_i / b_j ; p)} \right],$$

and then using

$$\theta(x; q) = (-1)^n x^n q^{\frac{1}{2} n(n-1)} \theta(q^n x; q),$$

we establish (5.1). This factorization property has a natural physical interpretation. We have claimed that the operators $\mathcal{G}_{(r,s)}$ introduce two surface defects, labeled by integers $r$ and $s$, along the two orthogonal planes, and thus can be introduced (almost) without interfering with each other. The two defects feel each other's presence only through the proportionality factor of $\left( \frac{pq}{t} \right)^{r+s}$, which as explained in section 3.3 amounts to a redefinition of the IR r-charge.

**Commutativity of the operators**

The two operators $\mathcal{G}_{(1,0)}$ and $\mathcal{G}_{(0,1)}$ commute,

$$[\mathcal{G}_{(1,0)}, \mathcal{G}_{(0,1)}] = 0.$$  \hspace{1cm} (5.4)

Mathematically this follows from simple theta-function identities. For example considering the term proportional to $\mathcal{I}(q^2 p^2 a, \ldots)$ in the commutator we get

$$\frac{\theta(q^{2} a^{-2}; q)}{\theta(a^2; q)} \frac{\theta(q^{2} p a^{-2}; p)}{\theta(p a^2; p)} - \frac{\theta(q^{2} a^{-2}; p)}{\theta(a^2; p)} \frac{\theta(q^{2} a^{-2}; q)}{\theta(p a^2; q)} = 0,$$

etc. In more generality one can show that all the operators $\mathcal{G}_{(r,s)}$ commute with each other. Physically this result is expected since these operators compute residues in theories which enjoy S-duality and we can employ this duality to change the order in which the residues are taken, see figure 4.
Figure 4: By generalized S-duality, the order in which one extracts the residues in fugacities $a$ and $b$ is expected to be immaterial, indeed the three different decompositions of the surface shown in the figure are topologically equivalent. This implies $[\mathcal{S}_{(r,s)} \cdot \mathcal{S}_{(r',s')}]=0$.

**Self-adjointness**

The operator $\mathcal{S}_{(1,0)}$ is self-adjoint under the natural propagator measure $\Delta \cdot \mathcal{I}_V$,

$$\oint \frac{da}{a} \Delta(a) \mathcal{I}_V(a) f(a) \left[ \mathcal{S}_{(0,1)} \cdot g(a) \right] = \oint \frac{da}{a} \Delta(a) \mathcal{I}_V(a) \left[ \mathcal{S}_{(0,1)} \cdot f(a) \right] g(a). \quad (5.6)$$

This can be shown simply by a change of the integration variable $a \rightarrow a q^{\pm \frac{1}{2}}$, if one assumes that the test functions $f(a)$ and $g(a)$ do not have poles in the strip $|q|^\frac{1}{2} \leq |a| \leq |q^{-\frac{1}{2}}|$. Under this assumption we write

$$\oint \frac{da}{a} \mathcal{I}_V(a) \Delta(a) f(a) \left[ \frac{\theta(t a^{-2}; p)}{\theta(a^2; p)} g(a)q^{1/2} + \frac{\theta(t a^2; p)}{\theta(a^{-2}; p)} g(a)q^{-1/2} \right] = \quad (5.7)$$

$$\oint \frac{db}{b} \mathcal{I}_V(bq^{-\frac{1}{2}}) \Delta(bq^{-\frac{1}{2}}) f(bq^{-\frac{1}{2}}) \left[ \frac{\theta(t b^{-2}; p)}{\theta(b^2 q^{-1}; p)} g(b) \right] + \oint \frac{db}{b} \mathcal{I}_V(bq^{\frac{1}{2}}) \Delta(bq^{\frac{1}{2}}) f(bq^{\frac{1}{2}}) \left[ \frac{\theta(t b^2; p)}{\theta(b^{-2} q^{-1}; p)} g(b) \right].$$

The integrand of the first integral satisfies

$$\mathcal{I}_V(bq^{-\frac{1}{2}}) \Delta(bq^{-\frac{1}{2}}) \left[ \frac{\theta(t b^{-2}; p)}{\theta(b^2 q^{-1}; p)} \right] = \frac{1}{2} (1-qb^2)(1-b^2 q^{-1}) PE \left( \frac{qp-t}{1-p} \right) b^{-2} \times$$

$$PE \left( \frac{q^{-1} - pt^{-1}}{1-p} \right) b^2 \left[ \frac{1}{2} (1-q)(1-b^2 q^{-1}) \frac{t}{1-p} \left( \frac{p}{1-p} - \frac{q}{1-q} \right) + \frac{p q - t}{(1-p)(1-q)} (q^{-1} b^2 + b^{-2} - 1) \right] = \mathcal{I}_V(b) \Delta(b) \frac{\theta(t b^2; p)}{\theta(b^{-2}; p)}, \quad (5.8)$$

and analogously for the second term thus establishing the self-adjointness of $\mathcal{S}_{(0,1)}$. This property is to be expected from how we introduced the difference operator, namely as a way
Figure 5: Three different but equivalent ways to introduce a surface defect in a tube. First by coupling the two edges of the tube to a free hypermultiplet and computing a $U(1)_f$ residue of the index. Then by acting with the relevant difference operator on either of the edges and then gauging. The equivalence of these different procedures implies that the index is self-adjoint under the natural measure.

to evaluate residues in the $SU(2)_f$ flavor fugacity of $\mathcal{T}_{UV}$. Consider the setup of the first row in figure 5, where two theories $\mathcal{T}$ and $\mathcal{T}'$ are connected to a bifundamental hypermultiplet by gauging two $SU(2)$ groups. The index can be written as double-contour integral for the two $SU(2)$ gauge fugacities. To extract the residue in the $SU(2)_f$ fugacity, we can follow the procedure of the previous subsection and pinch one of the two contours for fixed values of the second integration variable. The result is the second or third column in figure 5, according to the order in which we perform the countour integrals. Since the setup is symmetric under exchange of $\mathcal{T}$ and $\mathcal{T}'$, it follows that $S_{(r,s)}$ must be self-adjoint under the propagator measure.

Acting with difference operators on the trinion

In the $2d$ TQFT interpretation of the index of theories of class $\mathcal{S}$, the action of the difference operator $S_{(r,s)}$ on a flavor fugacity corresponds to the fusion of the “degenerate puncture” associated to $(r,s)$ surface defect with a flavor puncture. The choice of flavor puncture on which the operator acts is then expected to be immaterial. This follows directly from our construction of the operators and from the generalized S-duality enjoyed by theories of class $\mathcal{S}$. The difference operators compute a residue of the index at a pole of one of the flavor fugacities. To obtain the operators we have singled out an additional flavor fugacity by decoupling a free hypermultiplet as in figure 3. Invariance of the index under S-duality means that our choice
of the additional flavor fugacity must be immaterial. The crucial claim is that the difference operators must give the same result when acting on any of the three legs of the elementary “trinions”. By decomposing the Riemann surface into pairs of pants, it follows from this claim that the difference operators can be freely moved around the surface, indeed we have already shown that they are are self-adjoint with respect to the propagator measure.

In the $A_1$ case the index of the basic building block, the index of the trinion, is explicitly known, so we can establish this claim by direct computation. The $A_1$ trinion index is the index of a free tri-fundamental hypermultiplet,

$$\mathcal{I}(a,b,c) = \Gamma \left( t^2 a^\pm b^\pm c^\pm; p,q \right).$$

(5.9)

Dropping overall flavor-independent factors we obtain

$$\mathcal{S}_{(0,1)}(a)\mathcal{I}(a,b,c) \sim \frac{\theta(tq a^{-2}; p)}{\theta(a^2; p)} \Gamma(t^2 b^\pm c^\pm (aq^2)^\pm; p,q) + \frac{\theta(tq a^2; p)}{\theta(a^{-2}; p)} \Gamma(t^2 b^\pm c^\pm (aq^{-2})^\pm; p,q)$$

$$= \Gamma \left( \sqrt{\frac{t}{q}} a^\pm b^\pm c^\pm; p,q \right) \left[ \frac{\theta(tq a^{-2}; p)\theta(t^2 b^\pm c^\pm; p)}{\theta(a^2; p)} + \frac{\theta(tq a^2; p)\theta(t^2 a^{-2} b^\pm c^\pm; p)}{\theta(a^{-2}; p)} \right].$$

(5.10)

In the Macdonald limit $p = 0$ [23] it is easy to show that the above expression is symmetric under permutations of flavor fugacities,

$$\mathcal{S}_{(0,1)}(a)\mathcal{I}(a,b,c) \sim \Gamma \left( \sqrt{\frac{t}{q}} a^\pm b^\pm c^\pm, 0, q \right) \times$$

$$\left[ 1 + \left( \frac{t}{q} \right)^3 - (\chi_{adj}(a) + \chi_{adj}(b) + \chi_{adj}(c)) \left( \frac{t}{q} + \left( \frac{t}{q} \right)^2 \right) + 2\chi_f(a)\chi_f(b)\chi_f(c) \left( \frac{t}{q} \right)^{3/2} \right].$$

(5.11)

We can also easily check the $a \leftrightarrow b \leftrightarrow c$ symmetry Schur limit $q = t$ [23, 22],

$$\frac{\theta(a^{-2}; p)\theta(ab^\pm c^\pm; p)}{\theta(a^2; p)} = \frac{\theta(a^2; p)\theta(a^{-1} b^\pm c^\pm; p)}{\theta(a^{-2}; p)} = \sqrt{\theta(a^\pm b^\pm c^\pm; p)},$$

(5.12)

$$\Gamma \left( a^\pm b^\pm c^\pm; p,q \right) = \frac{1}{\theta(ab^\pm c^\pm; p)\theta(a^{-1} b^\pm c^\pm; p)} = \frac{1}{\sqrt{\theta(a^\pm b^\pm c^\pm; p)\theta(a^\pm b^\pm c^\pm; q)}},$$

we obtain

$$\mathcal{S}_{(0,1)}(a)\mathcal{I}(a,b,c) \sim \frac{2}{\sqrt{\theta(a^\pm b^\pm c^\pm; q)}}.$$  (5.13)

This expression is manifestly symmetric and also independent of $p$ as expected [23]. For arbitrary $(p,q,t)$ one has to verify that the combination of theta functions appearing in (5.11) is

---

[19] The same expression can be also obtained thus from (5.11) by setting $t = q$. 

---
symmetric under permutations of the three fugacities, an exercise we leave to our enthusiastic readers. We have checked this claim perturbatively in $p$.

Even if the theory has no flavor punctures, we can still define an operation that modifies its index to include surface defects. We consider any pair-of-pants decomposition of the Riemann surface, act with the difference operator on any of the open punctures and glue the pieces back together. The result is well-defined, since the properties just discussed guarantee that it does not depend on the specific decomposition or choice of open puncture.

In a 4d theory with surface defects, it is generally difficult to separate the 4d and 2d degrees of freedom counted by the index. In the simple case of the theory of a free hypermultiplet endowed with surface defect, we can however identify an interesting operator intrinsic to the defect. Let us consider the “Coulomb limit” $[23]^{20}$ (see appendix A): $t, p, q \to 0$ with $p^2/t$ and $q^2/t$ fixed. The index of the trinion acted upon by $S(1,0)$ (5.10) becomes

$$S_{(0,1)}(a, b, c) \sim 1 + \frac{pq}{t}.$$ (5.14)

The non-trivial term comes entirely from the difference operator itself, and we identify it with a Coulomb branch operator $X$ living on the defect, obeying the constraint $X^2 = 0$ (as natural for $SU(2)$ gauge group). Its 2d charges (see appendix A) are $L_0 = \frac{1}{2}$, $J_0 = -1$, and $f = 0$. It is a naturally interpreted as the exactly marginal twisted chiral field that couples to the relative gauge coupling, which we mentioned at the end of section 2.2.$^{21}$

**Recursion relations**

Finally, the difference operators satisfy interesting recursion relations. Consider the computation of $S_{(0,s)} S_{(0,1)} T$. Let $S_T(UV)$ be the index of the theory with one extra $U(1)_f$ puncture. To compute $S_{(0,s)} S_{(0,1)} T$ we extract the residue of $S_{(0,1)} S_T(UV)$ at $a = t^{1/2} q^{s/2}$. If we substitute the explicit expression for $S_{(0,1)}$ we get the neat relation,

$$\frac{\theta(q^{-1}; p)}{\theta(t; p)} S_{(0,s)} S_{(0,1)} = \frac{\theta(q^{-s-1}; p)}{\theta(t q^{-1}; p)} S_{(0,s+1)} + \frac{\theta(t^2 q^{s-1}; p)}{\theta(t q^{-1}; p)} S_{(0,s-1)}.$$ (5.15)

This recursion relation can be formally recast as a fusion product between operators corresponding to representation of $SU(2)$. We can write (5.15) as

$$[s] \times [1] = \mathcal{A}_{(s,1|s+1)} [s + 1] + \mathcal{A}_{(s,1|s-1)} [s - 1],$$ (5.16)

$^{20}$A related limit for $\mathcal{N} = 4$ SYM was discussed in [57].

$^{21}$For higher rank theories there is an analogous story. For instance the Coulomb index of a bi-fundamental free hypermultiplet with a basic surface defect is $\sum_{t=0}^{N-1} \binom{N}{t}$. Assuming that the Coulomb limit commutes with the residue computation this will be true for all indices.
where \([s]\) is an operator corresponding to an \(s + 1\)-dimensional irrep of \(SU(2)\). This is reminiscent of the fusion relation between degenerate operator in Liouville theory, which in the AGT correspondence are associated to surface defects \([58]\).

One can check (5.15) by using the explicit expression for \(\mathcal{S}_{(0,s)}\) (4.18). One then finds the same theta-function identity that we encountered in verifying the symmetry under permutation of flavor fugacities of the trinion when acted upon with a difference operator (5.10). This identity takes the following general form
\[
\theta(xb^{-1}a^{1}c^{1};p)\theta(x^{2}b^{2};p) + \theta(xba^{1}c^{1};p)\frac{\theta(x^{2}b^{-2};p)}{\theta(b^{2};p)} = \\
\theta(xa^{-1}b^{1}c^{1};p)\frac{\theta(x^{2}a^{2};p)}{\theta(a^{-2};p)} + \theta(xab^{1}c^{1};p)\frac{\theta(x^{2}a^{-2};p)}{\theta(a^{2};p)}. \tag{5.17}
\]

As suggested in section 3 it is natural to strip off a factor
\[
\frac{\theta(t;p)}{\theta(q^{-1};p)}\frac{\theta(tq;p)}{\theta(q^{-2};p)} \cdots \frac{\theta(tq^{s-1};p)}{\theta(q^{-s};p)} \tag{5.18}
\]

As a result, the recursion relation takes a slightly simpler form,
\[
\mathcal{S}_{(0,s)}\mathcal{S}_{(0,1)} = \mathcal{S}_{(0,s+1)} + \frac{\theta(t^{2}q^{s-1};p)}{\theta(1/tq^{s};p)}\frac{\theta(q^{-s};p)}{\theta(tq^{s-1};p)} \mathcal{S}_{(0,s-1)}. \tag{5.19}
\]

Remarkably, the ratios of \(\theta\)-functions can be interpreted as 2d indices \(\chi_{2d}\) of free chiral fields with a specific R-charge assignment (see appendix A)
\[
\frac{\theta(\frac{t^{2}}{q^{s}}x;p)}{\theta(x;p)} \equiv \chi_{2d}(x). \tag{5.20}
\]

In terms of these 2d indices we can write a final form of the recursion relation,
\[
\chi_{2d}(t)\mathcal{S}_{(0,s)}\mathcal{S}_{(0,1)} = \chi_{2d}(tq^{s})\mathcal{S}_{(0,s+1)} + \chi_{2d}(t^{-1}q^{-s})\mathcal{S}_{(0,s-1)}. \tag{5.21}
\]

6. Bootstrapping the index for \(A_1\) theories

We are now ready to implement the “bootstrap” of the 2d TQFT that computes the index of class \(S\) theories of type \(A_1\). In other terms, we will determine the index from consistency conditions alone. Our strategy is a simplified (topological) version of the “Teschner trick” \([24]\). A degenerate puncture associated to a surface defect can be fused with any of the
flavor punctures, and assuming S-duality the result must be independent of the chosen flavor puncture.\footnote{Of course, for the general $A_1$ theory, which has a Lagrangian description, the index is already explicitly known in terms of a matrix integral. Since everything is explicit, S-duality of the index for the $A_1$ quivers can be rigorously established \cite{21} using an identity for a certain integral of elliptic Gamma functions \cite{59}. However, the bootstrap approach generalizes easily to higher-rank theories, whose index is a priori unknown, and even for the $A_1$ theories it has the virtues of directly giving the index in the very useful “diagonal” form.}

To proceed, we make a plausible technical assumption: the difference operators $\mathcal{S}_{(r,s)}$ admit a complete set of eigenfunctions $\{\psi_{\lambda}(a)\}$, normalizable under the propagator measure, with non-degenerate eigenvalues $E_{(r,s)}^\lambda$. The label $\lambda$ runs over the irreducible $SU(2)$ representations. Since the difference operators are self-adjoint in the propagator measure, an implication of this assumption is that any two different eigenfunctions are orthogonal in the propagator measure, and can be normalized to be orthonormal. This assumption is certainly true in the degenerations limits $p = 0$ or $q = 0$, where the difference operators become the familiar Macdonald operators\footnote{More precisely, they are related to the Macdonald operators by a similarity transformation.} and the eigenfunctions are proportional to Macdonald polynomials. For arbitrary $(p,q,t)$, to the best of our knowledge the eigenfunctions are not known in closed form: they are expected to be an “elliptic” deformation of the Macdonald functions. In appendix B we describe an approximation scheme to determine the eigenfunctions in an expansion for small $q$ and $p$.

Expanding the index of the trinion,

$$I_{0,3} = \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma} \psi^\alpha(a) \psi^\beta(b) \psi^\gamma(c), \quad (6.1)$$

and acting with the difference operator one any of the flavor punctures, we have

$$\mathcal{S}_{(r,s)} I_{0,3} = \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma} E_{(r,s)}^\alpha \psi^\alpha(a) \psi^\beta(b) \psi^\gamma(c) = \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma} E_{(r,s)}^\gamma \psi^\alpha(a) \psi^\beta(b) \psi^\gamma(c) = \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma} E_{(r,s)}^\gamma \psi^\alpha(a) \psi^\beta(b) \psi^\gamma(c). \quad (6.2)$$

As by assumption the eigenvalues are non-degenerate, this implies that the index is diagonal in the basis $\{\psi^\alpha\}$,

$$I_{0,3} = \sum_{\alpha} C_{\alpha} \psi^\alpha(a) \psi^\alpha(b) \psi^\alpha(c). \quad (6.3)$$

It remains to fix the structure constants $C_{\alpha}$. Using orthonormality of the eigenfunctions under the propagator measure, the index of the theory associated to the four-punctured sphere is immediately given by

$$I_{0,4} = \sum_{\alpha} C_{\alpha}^2 \psi^\alpha(a) \psi^\alpha(b) \psi^\alpha(c) \psi^\alpha(d). \quad (6.4)$$
According to our prescription, extracting the residue at \( a = t^{1/2} \) closes a puncture, so

\[
\mathcal{I}_{0,3} = 2 \mathcal{I}_V \ \text{Res}_{a=t^{1/2}} \frac{1}{a} \mathcal{I}_{0,4}.
\]  

(6.5)

We conclude that

\[
C_\alpha = \frac{1}{2 \mathcal{I}_V} \left( \text{Res}_{a=t^{1/2}} \frac{1}{a} \psi^{\alpha}(a) \right).
\]

(6.6)

The index of general \( A_1 \) theory associated to a surface with \( k \) flavor punctures is given by

\[
\mathcal{I}_{g,k}(a_i) = \sum_\alpha (C_\alpha)^{2g-2+k} \prod_{i=1}^k \psi^{\alpha}(a_i).
\]

(6.7)

To introduce an \((r, s)\) surface defect, we are instructed to simply multiply by \( E^{\alpha}_{(r, s)} \) inside the summation sign.

We can also give expressions for the eigenvalues using the residue interpretation of the difference operators. For example

\[
E^{(0, s)}_\alpha = \frac{\text{Res}_{a=t^{1/2}} \frac{1}{a} \psi^{\alpha}(a)}{\text{Res}_{a=t^{1/2}} \frac{1}{a} \psi^{\alpha}(a)},
\]

(6.8)

In particular in the Macdonald limit, \( p = 0 \), we have

\[
\psi^{\alpha}(a) = \frac{1}{(ta^{1/2}; q)} P^{\alpha}(a; q, t), \quad E^{(0, s)}_\alpha = \left[ \prod_{i=0}^{s-1} \frac{1 - t^2 q^i}{1 - q^{i+1}} \right] \frac{P^{\alpha}(t^{1/2} q^{1/2} s; q, t)}{P^{\alpha}(t^{1/2}; q, t)}.
\]

(6.9)

Here \( P^{\alpha}(z; q, t) \) are Macdonald polynomials \((B.5)\) and we have defined as always

\[
(z; q) \equiv \prod_{\ell=0}^{\infty} (1 - z q^\ell).
\]

(6.10)

It is easy to verify that then the energies satisfy the recursion relation

\[
\frac{1 - q^{-1}}{1 - t} E^{(0, s)}_\alpha E^{(0, 1)}_\alpha = \frac{1 - q^{-s-1}}{1 - q^{s+1}} E^{(0, s+1)}_\alpha + \frac{1 - t^2 q^{-s-1}}{1 - q^{-s+1}} E^{(0, s-1)}_\alpha,
\]

(6.11)

which is a direct consequence of recursion relation satisfied by the operators \((5.15)\).

In the Macdonald limit the energies \( E^{\alpha}_{(r, 0)} \) can be related to modular S-matrix of the refined Chern-Simons theory \([60]\). Up to \( \alpha \) independent normalization factor these energies are given by

\[
\frac{P^{\alpha}(t^{1/2} q^{1/2} s; q, t)}{P^{\alpha}(t^{1/2}; q, t)} = S^{\alpha} S^{0}_{0} S^{s}_{0} S^{s}_{\alpha},
\]

(6.12)
where $S_\alpha^\lambda$ is the modular S-matrix of the refined Chern-Simons theory as defined in [60]. In the Schur limit ($q = t$) the refined Chern-Simons theory further reduces to the 2d q-deformed Yang-Mills (in the zero-area limit), as defined in [61]. The relation of 2d qYM to the index was discussed in [22, 23]. In the 2d YM language the flavor punctures correspond to fixing the holonomies of the gauge fields around the punctures. Thus, at least in the Macdonald limit, introducing a surface defect corresponds to adding a puncture with fixed holonomy for dual variables of the 2d theory, i.e. the canonical momenta dual to the gauge fields [62, 63, 61].

7. 3d reduction

Starting from the 4d index, which is the supersymmetric partition function on $S^3 \times S^1$, we can dimensionally reduce on the $S^1$ to obtain the partition function of a 3d theory on a squashed $S^3$. The reduction of the 4d index to 3d partition function is achieved by the following scaling of the fugacities,

$$q = e^{i\beta \omega_1}, \quad p = e^{i\beta \omega_2}, \quad t = e^{i\beta}, \quad a_\ell = e^{i\beta m_\ell},$$

and taking $\beta \to 0$ [64, 65, 66], where $a_\ell$ are 4d flavor fugacities and $m_\ell$ 3d real masses. The 4d index goes to the “ellipsoid partition function”, with squashing parameter

$$b = \frac{\omega_1}{\omega_2}. \quad (7.2)$$

The 3d theory which arise in the infrared from the circle reduction a 4d theory of class $S$ has a nice mirror description [67] in terms takes of a star-shaped quiver: a central node connected to a set of linear quivers, where each linear quiver is associated to a puncture of the Riemann surface. The wavefunction $\psi(\alpha)$ that we associate to a puncture in the TQFT description of 4d index reduces in the 3d limit to the $S^3$ partition function of the corresponding linear quiver tail [68].

The surface operators of the 4d theory are expected to become line defects of the 3d theory. Following the chain of dualities, one finds that the canonical surface defects for a 4d theory of class $S$, which arise from codimension four defects of the 6d (2,0) theory, go to Wilson loops for the central node of the quiver in the mirror 3d description. We would like to verify this fact.

For $A_1$ theories the mirror description involves a 3d gauge theory with a $U(1)$ gauge group for each puncture, and a single $SU(2)$ gauge group. The matter content includes a doublet of hypermultiplets for each puncture, with charge one under the corresponding

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24See also [69] for related work.
$U(1)$ gauge group, and $g$ adjoint hypermultiplets for the $SU(2)$ gauge group, where $g$ is the genus of the Riemann surface. Each “quiver tail” consists of a $U(1)$ gauge theory coupled to a doublet of hypermultiplets. This is the so-called $T[SU(2)]$ theory, which is intimately related to the S-duality operation in four-dimensional $\mathcal{N} = 4$ SYM [70]. In particular, its $S^3$ partition function is related by a generalization of AGT [71] to the integral kernel which implements the $S$ operation on conformal blocks on a one-punctured torus. This fact has a simple consequence: the $S^3$ partition function of $T[SU(2)]$ satisfies a functional equation, and intertwines the action of two operators, which are related to the action of 't Hooft and Wilson loops of $\mathcal{N} = 4$ SYM.

The Wilson loop operator multiplies the $T[SU(2)]$ partition function by a character of the $SU(2)$ flavor symmetry which rotates the doublet of hypermultiplets. In particular, the $S^3$ partition function of the full star-shaped quiver in the presence of a Wilson loop operator for the central $SU(2)$ node can be obtained by acting with the Wilson loop operator on any of the $T[SU(2)]$ linear quiver tails partition functions. This can be then rewritten as the action of a 't Hooft line operator on the partition function of that quiver tail. We will now show that 3d limit of the operator which inserts a surface defect in the 4d theory exactly reproduces the 't Hooft line operator. This will verify that the insertion of the surface defect goes to the insertion of a Wilson line defect for the middle node of the mirror star-shaper quiver.

Let us consider the 3d limit (7.1) for the operator $\tilde{S}_{(0,1)}$. The prefactors

$$\frac{\theta(\frac{1}{q}a^{-2}; p)}{\theta(a^2; p)} \rightarrow \frac{\sin \pi \frac{1-\omega_1-2m}{\omega_2}}{\sin 2\pi \frac{m}{\omega_2}},$$

while shift operators reduce to

$$f(q^{\frac{1}{m}}a) \rightarrow f(m + \frac{1}{2}\omega_1).$$

All in all,

$$\tilde{S}_{(0,1)} \rightarrow \frac{\sin \pi \frac{1-\omega_1-2m}{\omega_2}}{\sin 2\pi \frac{m}{\omega_2}} \Delta_{m\rightarrow m+\frac{1}{2}\omega_1} + \frac{\sin \pi \frac{1-\omega_1+2m}{\omega_2}}{\sin 2\pi \frac{m}{\omega_2}} \Delta_{m\rightarrow m-\frac{1}{2}\omega_1}.$$  

(In complete analogy, the limit of the operator $\tilde{S}_{(1,0)}$ is obtained by switching $\omega_1$ with $\omega_2$.) Under an appropriate identification of the parameters this coincides with the 't Hooft line defect, as desired. Line operators in the framework of AGT were studied in [72] for the $A_1$ case and in [73] for higher rank theories. One can compare the operator (7.5) to equation (5.25) in [72], which displays operator inserting a 't Hooft operator in the $\mathcal{N} = 2^*$ theory. The ratio of the two periods is the $b$ parameter of the underlying Liouville CFT, and the modular properties we will discuss below are related to the change $b \rightarrow b^{-1}$. Finally let us briefly discuss some mathematical properties of the 3d difference operators. The 3d reduction of $\tilde{S}_{(1,0)}$ is nothing but the basic Macdonald difference operator, with
effective parameters

\[ t = e^{2\pi i \frac{1}{\omega_2}}, \quad q = e^{2\pi i \frac{\omega_1}{\omega_2}}, \quad a = e^{2\pi i \frac{m}{\omega_2}}. \]  

(7.6)

Similarly, for the reduction of \( \tilde{S}_{(0,1)} \) we get the Macdonald operator with parameters

\[ t' = e^{2\pi i \frac{1}{\omega_1}}, \quad q' = e^{2\pi i \frac{\omega_2}{\omega_1}}, \quad a' = e^{2\pi i \frac{m}{\omega_1}}. \]  

(7.7)

Note that \( q \) and \( q' \) are related by a “modular” transformation,

\[ q = e^{2\pi \tau} \rightarrow q' = e^{-2\pi \tau}. \]  

(7.8)

Physically this follows from the fact that the (1,0) and (0,1) surface defects have support on the two maximal (linked) circles in \( S^3 \) fixed respectively by \( j_{12} \) and \( j_{34} \). Interchanging the two types of defects is the same as interchanging the circles. Let us also write down the 3d reduction of the elliptic measure,

\[ \Delta_I = \frac{1}{2} \frac{\Gamma(\frac{\theta}{T}; q, p)}{\Gamma(a_{\mp}; q, p)} \rightarrow \frac{1}{2} \prod_{i,j \geq 1} \frac{\Omega_{ij} + 1 \pm 2m}{\Omega_{i+1,j+1} - 1 \pm 2m} \frac{\Omega_{ij} \pm 2m}{\Omega_{i+1,j+1} \pm 2m}, \]  

(7.9)

where we have defined \( \Omega_{ij} = i\omega_1 + j\omega_2 \).

8. \( A_{N-1} \) theories

In this section we outline the generalization of our “topological bootstrap” procedure to the higher rank theories. We first discuss, but not review in detail, the six-dimensional construction of \( N = 2 \) four-dimensional gauge theories and surface defects in them, and the correspondence with two-dimensional Toda CFT correlation functions. We refer the reader to \([16, 74, 75]\) for a complete discussion of the relevant facts.

The (2,0) 6d SCFTs are labeled by a simply-laced Lie algebra \( g \). An important class of codimension-two (to be contrasted with codimension-four) superconformal defects is labeled by an embedding \( \rho \) of \( su(2) \) in \( g \), see e.g. \([74, 76]\). The properties of these defects can be motivated, for example, upon compactification on a circle shared by all the defects. The bulk theory becomes 5d SYM with gauge algebra \( g \). The codimension two defect induces a singularity of the 5d SYM fields which breaks the gauge symmetry to \( g(\rho) \) (possibly times some Abelian factors) at the defect. The codimension-four defect becomes a Wilson loop in the representation \( R \) of \( g \) or \( g(\rho) \). In the construction of four-dimensional \( N = 2 \) gauge theories, one considers the twisted compactification of the 6d SCFT on a Riemann surface \( C \),
with codimension two defects at points in $C$. Codimension four defects at points in $C$ produce surface defects in the 4d theory.

Nekrasov’s instanton partition function for the 4d theory is conjecturally the same as a conformal block for the $W_N$ current algebra [20, 77]. The vertex operators for the $W_N$ theory come in families associated to various patterns of degenerate vectors. The families are labeled by the embedding $\rho$ of $su(2)$ in $g$, and an appropriate choice of Toda momentum, a vector valued in the dual of the Lie algebra $g$. Non-degenerate vertex operators (corresponding to $\rho = 0$) are labeled by an imaginary Toda momentum. Semi-degenerate vertex operators are labeled by the sum of an imaginary momentum in $f(\rho)$, and a real momentum of the form $b\lambda + b^{-1}\lambda'$, where $\lambda, \lambda'$ are weights for $g(\rho)$. For a fully degenerate vertex operator, $f(\rho) = 0$ and $g(\rho) = g$.

The vertex operators map to a six-dimensional configuration with a codimension two defect labeled by $\rho$ and codimension four defects labeled by $\lambda$ and $\lambda'$, sitting on the 12 and 34 planes in flat space. In the Toda context one can change the type of a defect, by analytically continuing the continuous imaginary part of the momentum to some discrete choices of real momentum.

Much of our analysis concerning the analytic structure of the index can be repeated for the analytic structure of the Nekrasov’s partition function, i.e. for the properly normalized conformal blocks. The conformal blocks have poles at discrete values of the Toda momenta, which correspond to special values of the mass parameters of the four-dimensional theory. At these values, the divergence is caused by a flat directions in the Higgs branch. Thus the process of making a $W_N$ vertex operator more degenerate can be matched to a Higgsing process, in the presence of vortices which become surface defects in the IR.

Let us start from the basic example involving a bifundamental hypermultiplet. Our construction starts from a theory $T_{IR}$, which for simplicity we can take to have a six-dimensional description in terms of the $A_{N-1}$ theory on a Riemann surface $\mathcal{C}$ with at least one “full” puncture, which carries the $SU(N)$ flavor symmetry. Then $T_{UV}$ is a theory associated to $\mathcal{C}$ with one extra simple puncture inserted near the full puncture. The position of the simple puncture controls the gauge coupling of the auxiliary $SU(N)$ gauge group. The simple puncture has a $U(1)$ flavor symmetry, which we gauge in the Higgsing procedure.

In the context of conformal blocks, the Toda momentum of a simple puncture is an imaginary multiple of a single simple weight $w_1$. It can be analytically continued to the real momentum of a fully degenerate puncture, but only one with $\lambda = r w_1$ and $\lambda' = s w_1$. These surface defects are naturally associated to $r$-th and $s$-th symmetric powers of the fundamental

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25This assumption does not remove any generality from our analysis: no matter what $T_{IR}$ is, one can build an auxiliary six-dimensional theory in terms of a sphere with two full punctures, and $T_{IR}$ added by hand at one puncture, by gauging the diagonal $SU(N)$ flavor symmetry.
representation. In the context of the superconformal index, they are the defects introduced by the difference operators \( \mathcal{S}_{(r,s)} \).

Can we discuss surface defects labeled by other representations of \( SU(N) \)? In the context of 5d SYM, Wilson loops in representations \( R_1 \) and \( R_2 \) can be brought together to a Wilson loop in the representation \( R_1 \otimes R_2 \), which can then be decomposed to a sum of Wilson loops in irreducible representations

\[
R_1 \otimes R_2 = \sum_i N_{12}^i R_i. \tag{8.1}
\]

If we try to uplift such a formula to an “OPE” of codimension-four defects in 6d,

\[
S_{R_1} \otimes S_{R_2} = \sum_i \mathcal{T}_{12}^i \otimes S_{R_i}, \tag{8.2}
\]

we need to promote the coefficients \( N_{12}^i \) to 2d theories \( \mathcal{T}_{12}^i \). Here “OPE” is intended as a collision of the punctures in \( \mathcal{C} \), not in space-time: the configuration of two codimension-four defects represents a single, two-parameter surface defect in four-dimension, which simplifies to a sum of defects if the parameters are tuned appropriately.

At the level of the index, the location of the defects on \( \mathcal{C} \) is immaterial, and the relation will become

\[
\mathcal{S}_{(R_1,0)} \otimes \mathcal{S}_{(R_2,0)} = \sum_i \chi[\mathcal{T}_{12}^i] \otimes \mathcal{S}_{(R_i,0)}, \tag{8.3}
\]

where \( \chi[\mathcal{T}_{12}^i] \) is the 2d index of the “OPE coefficient theory”. We have discussed such relations in the \( A_1 \) case (see section 5).

Closing minimal punctures

In this section we discuss the generalization of the residue computation we performed for \( A_1 \) quivers to higher rank cases. The generalization is straightforward and we will only outline the essential steps. As before, we consider a theory of class \( \mathcal{S}, \mathcal{T}_{IR} = \mathcal{T}_C \) with at least one maximal puncture and couple to it a free bi-fundamental hypermultiplet. The index of the free bi-fundamental hypermultiplet is equal to

\[
\mathcal{I}_{hyp}(b, c; a) = \prod_{i,j=1}^{N} \prod_{m,n \geq 0} \frac{1 - p^{n+1} q^{m+1} t^{-\frac{1}{2}} (a b_i c_j)^{-1}}{1 - p^n q^m t^\frac{1}{2} a b_i c_j} \frac{1 - p^{n+1} q^{m+1} t^{-\frac{1}{2}} a b_i c_j}{1 - p^n q^m t^\frac{1}{2} (a b_i c_j)^{-1}}. \tag{8.4}
\]

Here \( b_i \) and \( c_i \) (\( \prod_{j=1}^{n} b_j = \prod_{j=1}^{n} c_j = 1 \)) are \( SU(N) \) fugacities and \( a \) is a \( U(1) \) fugacity. To couple this hypermultiplet to a general \( A_{N-1} \) quiver corresponding to Riemann surface \( \mathcal{C} \) we gauge a diagonal symmetry \( SU(N) \) by integrating over fugacity \( c \),

\[
\mathcal{I}_{UV} = \oint \prod_{i, j=1}^{N} \frac{dc_j}{2\pi i c_j} \Delta(c) \mathcal{I}_V(c) \mathcal{I}_{hyp}(b, c; a) \mathcal{I}_C(c^{-1}, \ldots). \tag{8.5}
\]
This index has poles at the following values of the $U(1)$ flavor fugacity

$$a = t^{1/2} p^{r/N} q^{s/N}, \quad (*)$$

with non-negative integers $r$ and $s$. Actually, there are poles also at $a \to \exp[\frac{2\pi i}{N} \ell] a$ as we discussed before (see section 3.1). A way to see this is again by studying the pinchings of $c$ contour integrals. Let us start by considering the poles of the integrand of (*). Let $c_i = t^{1/2} q^{m_i} p^{n_i} \frac{1}{a b^{\sigma(i)}}, \quad i = 1, \ldots, N$.

Here $\sigma(\cdot)$ is an element of $S(N)$. Notice that these combined poles have a straightforward physical interpretation, as contributions from a region in field space where complex bi-fundamental fields $Q_i^{\sigma(i)}$ receive a vev, which Higgses the $U(N)$ gauge symmetry completely. The $SU(N)$ gauge invariant operator is a determinant of the bifundamental fields. Thus the poles at $a^N t^{N/2} p^r q^s = 1$. The residue (multiplied by $N$ for future convenience) is $\mathcal{I}_V^{-1} \mathcal{I}_{IR}(b, \ldots)$. If we gauge the $U(1)$ flavor symmetry with fugacity $a$, the contribution to the index of the simple pole at $a = t^{1/2}$ is $N^{-1} \mathcal{I}_{IR}(b, \ldots)$. The other poles at $a = \exp[\frac{2\pi i}{N} \ell] t^{1/2}$ give the answer $N^{-1} \mathcal{I}_{IR}(\exp[-\frac{2\pi i}{N} \ell] b, \ldots)$, and they combine to give the $Z_N$ center gauging of $\mathcal{I}_{IR}$, as expected (see section 3.1).

Further specifying to $a = t^{1/2} p^{\sum_i m_i} q^{\sum_i n_i}$ we get that the simple poles above pinch the integration contour and become double poles on the unit circle, because we have only $N - 1$ independent $SU(N)$ flavor fugacities, if

$$s = \sum_{i=1}^N m_i, \quad r = \sum_{i=1}^N n_i. \quad (**)$$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{To compute the residue of the index at a pole of the $U(1)$ fugacity we consider the degeneration limit of the theory by decoupling a free-hypermultiplet.}
\end{figure}
This implies that the index has simple poles at (8.6). The residue computation is given by the action of the following operator on the index

\[ I[T_{IR}, \mathcal{G}_{(r,s)}] = \mathcal{G}_{(r,s)} I[T_{IR}] = \sum_{i=1}^{N} \sum_{m_{i}=r}^{N} \sum_{m_{i}=s}^{N} \mathcal{I}_{C}(q^{m_{i}+m_{j}-m_{i}t_{b_{i}}/b_{j}; p}) \times \prod_{i,j=1}^{N} \left[ \prod_{m=0}^{n_{i}-1} \theta(p^{n_{j}}q^{m_{j}-m_{i}t_{b_{i}}/b_{j}; p}) \right] \left[ \prod_{n=0}^{m_{i}-1} \theta(q^{m_{j}+m_{j}n_{j}-n_{i}t_{b_{i}}/b_{j}; q}) \right]. \] (8.9)

The basic properties of these operators are the same as for the $A_{1}$ case. These operators are all commuting and self-adjoint under the natural measure.

The index of a general theory of class $S$

We are finally ready to bootstrap the index of class $S$ theories of type $A_{N-1}$. The difference operators $\mathcal{G}_{(r,s)}$ have common eigenfunctions $\{\psi^{\alpha}(a)\}$, labelled by irreducible $SU(N)$ representations. The eigenfunctions can be chosen to be orthonormal under the propagator measure and we assume again that their eigenvalues are non-degenerate. By the usual logic of introducing a degenerate puncture and colliding it with flavor punctures, we deduce that the index of the trinion with three maximal punctures is diagonal in the $\{\psi^{\alpha}(a)\}$ basis,

\[ I_{trin}(b, c, d) = \sum_{\alpha} C_{\alpha} \psi^{\alpha}(b) \psi^{\alpha}(c) \psi^{\alpha}(d). \] (8.10)

The determination of the structure constants is somewhat more involved than in the $A_{1}$ case. Let us outline the procedure for the case of $A_{2}$.

The index of the free hypermultiplet of $A_{2}$ can be expanded as

\[ I_{hyp}(b, c; a) = \sum_{\alpha} \psi^{\alpha}(b) \psi^{\alpha}(c) \phi^{\alpha}_{(2,1)}(a). \] (8.11)

Here $b$ and $c$ are $SU(3)$ fugacities and $a$ is $U(1)$ fugacity. The label (2,1) denotes the auxiliary Young diagram corresponding to the $U(1)$ puncture. This equation defines the function $\phi^{\alpha}_{(2,1)}(a)$ since the left-hand side is explicitly known (8.4). (That $I_{hyp}(b, c; a)$ must take this diagonal form follows again by consistency of the theory with an extra degenerate puncture.) Consider now the theory associated to a sphere with two maximal and two minimal punctures. The index can be calculated in two different duality frames. In the degeneration
limit of the surface where a maximal puncture collides with a minimal puncture, the theory is obtained by gauging the diagonal $SU(3)$ of two bifundamental hypermultiplets,

$$A = \sum_{\alpha} \phi_{(2,1)}^\alpha (b/c) \phi_{(2,1)}^\alpha ((c b)^{-1}) \prod_{\ell=1}^2 \psi^\alpha (a_\ell) .$$

(8.12)

Here $b$ and $c$ denote two $U(1)$ fugacities. On the other hand, in the degeneration limit where two maximal punctures collide, the trinion with three maximal punctures is coupled to a free hypermultiplet by gauging a $SU(2)$ subgroup (this is the duality discovered in [78]). In this duality frame the index is given by

$$A = \sum_{\alpha} C_\alpha \prod_{\ell=1}^2 \psi^\alpha (a_\ell) \oint \frac{dz}{2 \pi i z} \Delta(z) I_V(z) \hat{I}_{hyp}(z, b^3) \psi^\alpha (z, z^{-1} c, c^{-2}) .$$

(8.13)

Here $\hat{I}_{hyp}(z, b^3)$ is a free hypermultiplet in the fundamental of $SU(2)$, where the $SU(2)$ fugacity is parametrized by $z$; the two half-hypermultiplets have charges $\pm 3$ under the $U(1)$ symmetry parametrized by $b$. The $U(1)$ charges of the different components in the two duality frames are linearly dependent [78], which results e.g. in the non-trivial power of $b$ in the free hypermultiplet $\hat{I}_{hyp}$. Comparing the indices in the two duality frames we obtain

$$C_\alpha = \frac{\phi_{(2,1)}^\alpha (b/c) \phi_{(2,1)}^\alpha ((c b)^{-1})}{\oint \frac{dz}{2 \pi i z} \Delta(z) I_V(z) \hat{I}_{hyp}(z, b^3) \psi^\alpha (z, z^{-1} c, c^{-2})} .$$

(8.14)

The right-hand side here is written as a function of $b$ and $c$ but is actually independent of these as an implication of S-duality; this can be explicitly checked [23].

Finally, by gluing elementary building blocks, we can write the index of a genus $g$ theory with $k$ maximal and $k'$ minimal punctures,

$$\sum_{\alpha} (C_\alpha)^{2g-2+k-k'} \prod_{\ell=1}^k \psi^\alpha (a_\ell) \prod_{m=1}^{k'} \phi_{(2,1)}^\alpha (b_m) .$$

(8.15)

This algorithm can be easily generalized to higher rank. The trinion with three maximal punctures is always diagonal in the $\{ \psi^\alpha (a) \}$ basis, see (8.10). The index of the free hypermultiplet has a diagonal expression analogous to (8.11), in terms of two $\psi^\alpha$ wavefunctions and one wavefunction $\phi_{(2,1,...,1)}^\alpha$ associated to the minimal puncture, which can in fact be fixed by the known expression of the index of the free hyper. Finally the structure constants $C_\alpha$ are determined by comparing two degeneration limits of the $N + 1$-punctured sphere with two maximal and $N - 1$ minimal punctures: in one duality frame we have a linear superconformal quiver with the two maximal punctures at the ends, and in the other one we have the $SU(N)$ trinion (the $T_N$ theory) coupled to the superconformal tail with a maximal puncture.
on one end and minimal puncture on the other end. The two duality frames are related by a
generalized Argyres-Seiberg duality \cite{15}. The former duality frame is completely Lagrangian
and the index can be computed explicitly. The index in the latter frame is written in terms
of the index of the trinion. Equating the indices in the two duality frames one determines the
trinion structure constants $C_\alpha$. These date are sufficient to fix the index of any theory con-
taining only maximal and minimal punctures. More general punctures can be incorporated
by a Higgsing procedure that involves more general superconformal tails. In appendix C we
give an example of this procedure. The final prescription is spelled out in \cite{43} and coincides
with the prescription of \cite{23}.

9. Discussion

In summary, the superconformal index, being a protected quantity, must be consistent with a
large class of deformations of the 4d field theory. Moving along an exactly marginal direction
the index does not change, so it must be the same when evaluated in different S-duality
frames. This has the non-trivial implication that the index of class $\mathcal{S}$ theories is computed
by the correlator of a 2d TQFT \cite{21}. In this paper we have considered a more interesting
class of deformations: we have studied how the index is affected by expectation values of
supersymmetric operators. We have argued that a pole of the index in flavor fugacity is
associated to a bosonic flat direction, parametrized by the vacuum expectation value of a
protected operator, and that the residue captures the IR physics reached at the endpoint of
the flow triggered by the vev. We have formulated a precise prescription to evaluate the index
of the IR theory.

We have focussed on a special class of poles, associated to RG flows that introduce
two-dimensional defects in the IR SCFT. In particular, starting from the index of a theory
with flavor symmetry $\mathcal{G} \times U(1)$ our residue calculus determines the index of the theory with
smaller flavor symmetry, $\mathcal{G}$, but endowed with BPS surface defects. In the language of the
2d TQFT, the surface defects are associated to special “degenerate” punctures. The fusion
of the degenerate punctures with the “ordinary” punctures associated to flavor symmetries
amounts to acting on the flavor fugacities with certain elliptic difference operators. Under a
minimal set of assumptions, namely consistency of the 2d TQFT structure and knowledge of
the index of free theories, we were then able to determine the index of a general theory of
class $\mathcal{S}$.

\footnote{The prescription of \cite{23} gave divergent results for a small subclass of theories of class $\mathcal{S}$, however following
the residue logic described in this paper it was explained in \cite{43} how to resolve those singularities.}
We can perhaps extract two general lessons for quantum field theory. First, in a given theory, even if ultimately we are most interested in the familiar correlators of local operators, it is fruitful to consider the larger set of observables that includes defects of various codimension. In our problem the introduction of surface defects was the key step. Second, to obtain results in a given theory, it is fruitful to enlarge the view to all the theories of the same class. Indeed our “bootstrap” is a conventional bootstrap for the 2d TQFT, but for the 4d SCFT it is really a bootstrap in theory space.

Some open problems

In closing, let us briefly mention several further research directions, questions, and speculations arising from our work.

- First and foremost, it would be very valuable to have a bona fide computation of the superconformal index for Lagrangian class $S$ theories endowed with surface defects. We have in mind a first-principles calculation directly from path integral definition, using the methods of supersymmetric localization.

- The difference operators that introduce surface defects are sums of terms shifting the values of flavor fugacities in different ways, with coefficients looking as indices of 2d free fields, i.e. combinations of theta-functions (see e.g. (8.9)).

  It is natural to wonder this sum has a more direct physical interpretation. What happens physically when the defect puncture collides with a flavor puncture? There is some evidence [75] that the surface defect may degenerate to the direct sum of simpler, decoupled defects. Schematically, we could write

  \[ \mathcal{S}_{(r,0)} \rightarrow \sum_i T_{r,i} \times \sigma_i \]  

  (9.1)

  where $\sigma_i$ are the simpler surface defects, and the “coefficients” $T_i$ are further decoupled 2d degrees of freedom. We can imagine that the simple defects $\sigma_i$ give the individual shift operators in the index fugacities, while the 2d index of the $T_{r,i}$ produces the coefficients in the expansion of the difference operator. It would be interesting to make sense of this conjectural structure.

  Similarly, we have seen that the difference operators introducing the defects satisfy recursion relations. The coefficients of these recursion relations are again combinations of theta-functions, and could perhaps be interpreted as indices of 2d theories.

- The index of class $S$ theories takes a diagonal form in the eigenfunctions $\psi^\alpha (a)$ of the $\mathcal{S}_{(r,s)}$ difference operators. The eigenfunctions do not depend on the specific four-
dimensional theory. It is tempting to interpret these wavefunctions as universal modules of some intricate algebra, built out of the flavour currents and their descendants.

• Finally, it will be important to generalize our results $\mathcal{N} = 1$ theories. The simplest route is to take the $\mathcal{N} = 1$ (“Sicilian”) limit of our construction, which corresponds to giving a mass to the chiral field in the $\mathcal{N} = 2$ vector multiplet [79]. At the level of the index this amounts to setting $p q = t^2$. It would be interesting to check if the index of surface defects remains sensible after this mass deformation to $\mathcal{N} = 1$.

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A. Embedding $2d$ in $4d$

Let us discuss the embedding of $2d$ $(2, 2)$ superalgebra in the $\mathcal{N} = 2$ $4d$ superalgebra. We choose to parametrize $S^3 \times S^1$ by $\sum_{i=1}^{4} x_i^2 = 1$ for $S^3$ and by an $x_0$ for the $S^1$. We start from the $\mathcal{N} = 2$ superconformal algebra $SU(2, 2|2)$, whose bosonic subgroup is $SU(2, 2) \times SU(2)_R \times U(1)_r$. A maximal subgroup of $SU(2, 2)$ is $SU(2)_{j_1} \times SU(2)_{j_2} \times U(1)_E$. The generators $j_{12} = j_1 + j_2$ is the rotation of the $(x_1, x_2)$ plane and $j_{34} = j_2 - j_1$ is the rotation on the $(x_3, x_4)$ plane.
Table 1: For each supercharge $Q$, we list its quantum numbers, the associated $\delta \equiv 2 \{Q, Q^\dagger\}$, and the other $\delta$s commuting with it. Here $I = 1, 2$ are $SU(2)_R$ indices and $\alpha = \pm, \dot{\alpha} = \pm$ Lorentz indices. $E$ is the conformal dimension, $(j_1, j_2)$ the Cartan generators of the $SU(2)_1 \otimes SU(2)_2$ isometry group, and $(R, r)$, the Cartan generators of the $SU(2)_R \otimes U(1)_r$ R-symmetry group.

The two dimensional defects wrap the $S^1$ and one of the equators: either $x_1 = x_2 = 0$ or $x_3 = x_4 = 0$. The surface defects preserve an $SU(1, 1|1) \times SU(1, 1|1) \times U(1)_J$ subalgebra, where the $U(1)_f$ commutes with all the preserved supercharges and can be considered as a flavor symmetry from the 2d perspective. The bosonic subalgebra is given by two copies, left and right, of $SU(1, 1) \times U(1)_J$ and by the flavor $U(1)_f$. The $SU(1, 1)$ is a chiral half of the global conformal group in two dimensions with Cartan generator $L_0$ ($\bar{L}_0$), while the $U(1)_J$ is generated by $J_0$ ($\bar{J}_0$). We denote the generator of the flavor $U(1)_f$ by $f$. The fermionic generators of the two $SU(1, 1|1)$ NSNS algebras are denoted by $G^\pm_r$, $\bar{G}^\pm_r$ with $r = \pm \frac{1}{2}$.

There are many equivalent ways to embed this into the 4d algebra but since we are computing the index with respect to $\bar{Q}_{1-}$ there are two choices which correspond to two possible choices of $f$,

$$ f = j_{12} + R, \quad f' = j_{34} + R. \quad (A.1) $$

The two choices are of course isomorphic and correspond to putting the defects on the two different equators. For concreteness we will focus on the former choice. The left and the right super-charges are then mapped as follows

$$ (G^+_{-\frac{1}{2}}, G^-_{-\frac{1}{2}}) = (Q_{1-}, \bar{Q}_{2+}), \quad (\bar{G}^+_{-\frac{1}{2}}, \bar{G}^-_{-\frac{1}{2}}) = (\bar{Q}_{1-}, Q_{2+}). \quad (A.2) $$

The $G^\pm_{\frac{1}{2}}$ and $\bar{G}^\pm_{\frac{1}{2}}$ are the superconformal counterparts. The map between charges

$$ J_0 = R + r + f, \quad \bar{J}_0 = R - r + f, \quad (A.3) $$

$$ L_0 + \bar{L}_0 = E, \quad L_0 - \bar{L}_0 = j_{34}, $$
generates the canonical \((2,2)\) superalgebra
\[
[L_0, G^\pm_r] = -rG^\pm_r, \quad [\bar{L}_0, \bar{G}^\pm_r] = -r\bar{G}^\pm_r, \\
[J_0, G^\pm_r] = \pm G^\pm_r, \quad [\bar{J}_0, \bar{G}^\pm_r] = \pm \bar{G}^\pm_r,
\]
\[
\{G^\pm_{1/2}, G^\mp_{-1/2}\} = L_0 \pm \frac{1}{2} J_0, \quad \{\bar{G}^\pm_{1/2}, \bar{G}^\mp_{-1/2}\} = \bar{L}_0 \pm \frac{1}{2} \bar{J}_0.
\]

The admixture of the flavor symmetry \(f\) inside \(J_0\) and \(\bar{J}_0\) is fixed by the last anticommutators.

Let us now consider our 4\(d\) index,
\[
\mathcal{I} = \text{Tr}(-1)^F p^{j_{12} - r} q^{j_{34} - r} t^{r + R}.
\]

We notice that we can identify the charges as,
\[
j_{34} - r = L_0 - \frac{1}{2} J_0 - (\bar{L}_0 - \frac{1}{2} \bar{J}_0), \quad j_{12} - r = 2f - J_0 \quad r + R = J_0 - f.
\]

Moreover only states with \(\bar{L}_0 = \frac{1}{2} \bar{J}_0\) contribute to the index due to (3.3). Thus, the index can be written as follows
\[
\mathcal{I} = \text{Tr}(-1)^F q^{L_0 + \frac{1}{2} J_0 - f} p^f \left( \frac{t}{pq} \right)^{J_0 - f}.
\]

This is an NSNS index of the \((2,2)\) theory. Choosing the symmetry \(f'\) in (A.1) to be the flavor symmetry the index becomes
\[
\mathcal{I} = \text{Tr}(-1)^F p^{L_0 + \frac{1}{2} J_0 - f'} q^{f'} \left( \frac{t}{pq} \right)^{J_0 - f'},
\]
i.e. we just switch between \(p\) and \(q\), and \(f\) with \(f'\).

We can give a 2\(d\) interpretation of the prefactor that we obtained in the residue computation,
\[
\frac{\theta(t p^u, q)}{\theta(p^{-1-u}, q)} = \frac{\theta(p q^{1-u} t, q)}{\theta(p^{-1-u}, q)} \equiv \chi_{2d}(p^{-1-u}),
\]
where \(u\) is non-negative integer. The numerator can be understood as the partition function generated by two fermions with the following charges
\[
\frac{q p^{-u}}{t} : \quad L_0 = -\frac{1}{2} u, \quad f = -1 - u, \quad J_0 = -2 - u, \\
t p^u : \quad L_0 = 1 + \frac{1}{2} u, \quad f = 1 + u, \quad J_0 = 2 + u,
\]
and their derivatives. The charges of the derivative which contribute to index are \(L_0 = 1, J_0 = 0, f = 0,\) and \(\bar{J}_0 = \bar{L}_0 = 0\). This derivative contributes a factor of \(q\) to (A.7) and
generates the infinite product in the theta-functions of (A.9). Similarly, the denominator is generated by bosons with the following charges

\begin{align*}
    p^{1-u} & : \quad L_0 = -\frac{1}{2} - \frac{1}{2} u, \quad f = -1 - u, \quad J_0 = -1 - u, \quad (A.12) \\
    p^{1+u} q & : \quad L_0 = \frac{3}{2} + \frac{1}{2} u, \quad f = 1 + u, \quad J_0 = 1 + u. \quad (A.13)
\end{align*}

These are unusual charge assignments. The standard charges of a free chiral field (free complex scalar and a complex fermions) are obtained in the formal limit \( u = -1 \): here the flavor charges are zero and the expression (A.9) diverges and hence the limit is formal. We take this as further indication that these partition functions describe decoupled fields with accidental symmetries, and that they should be stripped off from the definition of the surface defects.

If we redefine the charges as

\begin{align*}
    \hat{L}_0 &= L_0 - \frac{1}{2} f, \quad \hat{J}_0 = J_0 - f, \quad (A.14)
\end{align*}

then the new charges have natural values for the fields contributing to (A.9).

Note also that the combinations of theta-functions appearing in the basic difference operator (4.16) are \( \theta(t^{\frac{1}{2}b^{\pm 2}}; q)/\theta(b^{\pm 2}; q) \) which is again a free chiral field, with standard charges, also charged under flavor symmetry coupled to \( b \).

Finally, we consider different supersymmetric limits of the index. For concreteness we consider the index (A.7).

**“Higgs” index**

We take the limit \( q \to 0 \) while keeping the other fugacities finite. This is the 2d version of the 4d Macdonald index [23]. The index becomes

\[ I_H = \text{Tr}_H (-1)^F p^{2J_0 - J_0} t^{J_0 - f}. \quad (A.15) \]

Here \( \text{Tr}_H \) is summing over states satisfying

\[ L_0 = \frac{1}{2} J_0, \quad \bar{L}_0 = \frac{1}{2} \bar{J}_0. \quad (A.16) \]

Thus this index is half-BPS and gets contributions from states annihilated by two supercharges, one left and one right, of same R-symmetry charge. The index of the free 2d chiral field (A.9) here becomes

\[ \frac{\theta(t p^u, q)}{\theta(p^{-1-u}, q)} \to \frac{1 - t p^u}{1 - p^{-1-u}}. \quad (A.17) \]

For instance the index of the \( A_1 \) trinion with a surface defect in this limit is given in (5.11) and gets contributions from states charged under flavor symmetry and thus captures Higgs
branch states. Note that the 2d (2, 2) algebra does not constrain the value of charge \( f \) and thus there is no analogue here of the Hall-Littlewood index of [23].

“Coulomb” index

We take the limit \( \{ q, p, t \} \to 0 \) while keeping \( p/q \) and \( pq/t \) finite. This is the 2d version of the 4d Coulomb index [23]. The index becomes

\[
\mathcal{I}_C = \Tr_C (-1)^F \left( \frac{p}{q} \right)^f \left( \frac{t}{pq} \right)^{\bar{J}_0 - \bar{f}}.
\] (A.18)

Here \( \Tr_C \) is summing over states satisfying

\[
L_0 = -\frac{1}{2} J_0, \quad \bar{L}_0 = \frac{1}{2} \bar{J}_0.
\] (A.19)

Thus this index is half-BPS and gets contributions from states annihilated by two super-charges, one left and one right, of opposite R-symmetry charge. The index of the free 2d chiral field (A.9) here becomes

\[
\frac{\theta(tp^u, q)}{\theta(p^{-1-u}, q)} \quad \to \quad \left( \frac{pq}{t} \right)^{1+u} \frac{1 - \frac{pq}{t} \left( \frac{q}{p} \right)^{1+u}}{1 - \left( \frac{q}{p} \right)^{1+u}}.
\] (A.20)

The index of the \( A_1 \) trinion with basic surface defect in this limit (5.14) gets contributions from flavor singlets not counted by the Higgs index and thus naturally captures Coulomb branch physics.

“Schur” index

We take the limit \( p = t \) while keeping all the fugacities finite. This is the 2d version of the 4d Schur index [23]. The index becomes

\[
\mathcal{I}_S = \Tr (-1)^F q^{L_0 - \frac{1}{2} J_0} pf
\] (A.21)

Since \( f \) commutes with all the charges in the (2, 2) algebra this index is actually independent of \( q \). In other words this index is simply \( \Tr (-1)^F \) refined only by flavor symmetries from the 2d perspective. Given such an index care has to be taken in expanding it in fugacities since there is no a-priori good expansion parameter. It gets contributions only from states satisfying

\[
L_0 = \frac{1}{2} J_0, \quad \bar{L}_0 = \frac{1}{2} \bar{J}_0.
\] (A.22)
The index of the free 2d chiral field here (A.9) becomes
\[
\frac{\theta (tp^a, q)}{\theta (p^{-1-u}, q)} \rightarrow -p^{1+u}, \tag{A.23}
\]
and in particular all dependence on \(q\) factors out automatically as it should. Note that here the vacuum canceled out against a fermionic state with zero \(f\)-charge: this is an artifact of the issue mentioned above.

**B. Wavefunctions of the \(A_1\) elliptic RS model**

The difference operators that we defined through our residue prescription are closely related to well-known difference operators. Consider the similarity transformation
\[
\mathcal{S}_{(0,1)} = \frac{\theta (tp^2; p)}{\theta (q^2; p)} \hat{K}(b) H \hat{K}^{-1}(b), \quad \hat{K}(b) = \Gamma (tb^\pm 2; p, q), \tag{B.1}
\]
where \(H\) is given by
\[
H = \frac{\theta (tb^2; p)}{\theta (b^2; p)} \Delta_{b \rightarrow q^{1/2} b} + \frac{\theta (tb^{-2}; p)}{\theta (b^{-2}; p)} \Delta_{b \rightarrow q^{-1/2} b}. \tag{B.2}
\]
This operator is known as the elliptic Ruijsenaars-Schneider (RS) difference operator, which is a relativistic generalization of the Hamiltonian of the elliptic Calogero-Moser-Sutherland model \([31, 32, 33]\). In fact we have a second set of operators obtained by interchanging \(p\) with \(q\). All these operators are commuting as we have discussed in the bulk of the paper. The spectrum of the RS operator is well-known in certain degeneration limits. Taking \(p = 0\) the operator reduces to the Macdonald difference operator, whose eigenfunctions are the Macdonald polynomials. Conjugating with \(\hat{K}\) the propagator measure (the Haar measure times the index of the \(\mathcal{N} = 2\) vector multiplet), we find
\[
\Delta \mathcal{I}_V = \frac{1}{2} \Gamma (tb^{\pm 2}; p, q) \frac{1}{\Gamma (b^{\pm 2}; p, q)} \rightarrow \hat{\Delta} = \frac{1}{2} \Gamma (tb^{\pm 2}; p, q). \tag{B.3}
\]
The operator \(H\) is self-adjoint under this measure. Setting \(p = 0\) this becomes
\[
\hat{\Delta} = \frac{1}{2} (tb^{\pm 2}; q), \tag{B.4}
\]
\(\Delta \mathcal{I}_V\)

\footnote{A relation between Calogero-Moser-type models and elliptic Gamma functions, which are the building blocks of the index of the chiral superfield, was discussed in \([81]\).}
which is the measure under which Macdonald polynomials are orthogonal. The Macdonald polynomials are explicitly given by

$$P^\lambda(b|q, t) = (1 - q^\lambda t)^{\frac{1}{2}} \prod_{i=0}^{\lambda-1} \frac{\sqrt{1 - q^{i+1}(1 - q^2 t^2)}}{(1 - q^i t)} \prod_{j=0}^{\lambda-1} \left( \frac{1 - q^{2j+1}}{1 - t q^j} \right) \times$$

$$\sum_{i=0}^{\lambda} \prod_{j=0}^{\lambda-1} \frac{1 - t q^j}{1 - q^{2j+1}} \prod_{j=0}^{\lambda-1} \frac{1 - t q^j}{1 - q^{2j+1}} b^{2i-\lambda}.$$  \hspace{1cm} (B.5)

It is important that these eigenfunctions are non-degenerate.

For general \((p, q, t)\), although there is some discussion of wavefunctions in the literature [82, 83, 84] (see also [85, 86, 87]), to the best of our knowledge no explicit closed form is known. However, as a proof of concept that such eigenfunctions exist and are non-degenerate we can construct approximate wavefunction as follows. We solve for the eigenfunctions using the following ansatz

$$\psi_{\ell}(a) = \frac{a^{\ell+1} g(a) - a^{-\ell-1} g(a^{-1})}{a - a^{-1}}.$$  \hspace{1cm} (B.6)

Plugging this ansatz into the difference equation (B.2) we obtain,

$$q^{\frac{1}{2}} \left[ a^{\ell+1} g(q^\frac{1}{2} a) \theta(t a^2; p) + a^{-\ell} g(q^{-\frac{1}{2}} a^{-1}) \theta(t a^{-2}; p) \right] +$$

$$+ q^{-\frac{1}{2}} \left[ a^{-\ell} g(q^{-\frac{1}{2}} a^{-1}) \theta(t a^{-2}; p) + a^{\ell+1} g(q^\frac{1}{2} a^{-1}) \theta(t a^2; p) \right] = \mathcal{E}_q \frac{a^{\ell+1} g(a) - a^{-\ell-1} g(a^{-1})}{a - a^{-1}}.$$  \hspace{1cm} (B.7)

As a first approximation we neglect the term in the first line, assuming as always that \(|q| < 1\), and obtain the simple equation

$$\frac{g_0(q^{-\frac{1}{2}} a) \theta(t a^{-2}; p)}{(1 - qa^{-2}) \theta(a^{-2}; p)} = q^{\frac{1}{2}} \mathcal{E}_q^{(0)} \frac{g_0(a)}{1 - a^{-2}}.$$  \hspace{1cm} (B.8)

The subscript 0 on \(g_0(\cdot)\) signifies that we are making an approximation here. This equation is solved using the following property of the elliptic Gamma function

$$\Gamma(q z; p, q) = \theta(z; p) \Gamma(z; p, q).$$  \hspace{1cm} (B.9)

The solution is

$$g_0(a) = (1 - a^{-2}) \frac{\Gamma(a^{-2}; p, q)}{\Gamma(t a^{-2}; p, q)}, \quad \mathcal{E}_q^{(0)} = q^{\frac{1}{2} \ell}.$$  \hspace{1cm} (B.10)

This solution is a joint eigenfunction of the two hamiltonians \(\mathcal{E}_{(1,0)}\) and \(\mathcal{E}_{(0,1)}\), i.e. it is symmetric in \(p\) and \(q\). The eigenfunction are given thus by

$$\psi_{\ell}(a) \sim a^{\ell} \frac{\Gamma(a^{-2}; p, q)}{\Gamma(t a^{-2}; p, q)} + a^{-\ell} \frac{\Gamma(a^2; p, q)}{\Gamma(t a^2; p, q)}.$$  \hspace{1cm} (B.11)
Note that this solution is correct up to order $q^{\ell/2}$ $(p^{\ell/2})$, since we neglected a term of the form $q^{\ell/2}g_0(q^{1/2}a)$. The function $g_0(q^{1/2}a)$ actually is not regular in $q$: it has an expansion in negative powers of $q$ of arbitrary order with coefficients proportional to some power of $p$. This expansion has terms of the form $q^{-r}p^r$ along with terms with higher powers of $p$ and same power of $q$,

$$\left.\frac{\Gamma(a^{-2}; p, q)}{\Gamma(ta^{-2}; p, q)}\right|_{a \to q^{\ell/2}} = F_{\text{reg}}(a, q, p, t) \prod_{i,j \geq 0} \frac{1 - \frac{p}{q} q^i a^{-2}}{1 - \frac{p}{q} p^j q^2 a^{-2}}.$$  \hspace{0.5cm} (B.12)

Here $F_{\text{reg}}(\cdot)$ is regular in $q$-expansion. If we expand in $p$ to $r$th order then the term which we neglected on the left-hand-side of (B.7) contributes at order $\frac{1}{2} \ell - r$. On the other hand the second term on the left-hand-side of (B.7) contributes at order $-\frac{1}{2} \ell + r$. Thus, the highest order in $q$-expansion to which the eigen-function are consistent with the assumptions $r = \frac{1}{2} \ell$.

We can make several checks of this result. Note that $\mathcal{S}_{(1,0)}$ and $H$ can be mapped to each other by exchanging $t \to \frac{p^2}{t}$. Thus, the (exact) eigenfunctions have to satisfy

$$\Gamma(t a^{\pm2}; p, q) \psi(; p, q, t) = \psi(; p, q, \frac{p^2}{t}).$$  \hspace{0.5cm} (B.13)

This property is obviously satisfied in the above approximation. The approximate eigenfunctions are orthogonal under the natural measure (up to the order they are valid),

$$\oint da \frac{\Gamma(ta^{\pm2}; p, q)}{4\pi ia \Gamma(a^{\pm2}; p, q)} \psi_\ell(a) \psi_{\ell'}(a) = \delta_{\ell\ell'} n_\ell.$$  \hspace{0.5cm} (B.14)

The normalization is given by

$$\ell = 0 : \quad n_\ell = 1 + t,$$

$$\ell \neq 0 : \quad n_\ell = 1.$$  \hspace{0.5cm} (B.15)

In the special limit $p = 0$ the approximate eigenfunctions coincide with Macdonald polynomials up to order $\ell - 1$ in $q$. In the limit $p = q = 0$ they are identical to Hall-Littlewood polynomials (no approximation). Starting from these approximate eigenfunctions one should in principle be able to obtain corrections perturbatively in $q$ and $p$.

\section*{C. Non-maximal punctures}

Let us discuss the procedure of reducing the flavor symmetry of a puncture by our residue calculus. We consider a concrete example that captures the general algorithm. We will derive the prescription to partially-close a full puncture to obtain an $L$-shaped puncture with two rows (see figure 7).
Figure 7: Closing the full puncture down to $L$-shaped one by closing two minimal punctures. On the left we have a generic theory with a superconformal tail ending in an $L$-shaped puncture represented by a Riemann surface. On the right we have a Lagrangian representation of the superconformal tail: the squares denote hypermultiplets and circles gauge groups.

First we couple a theory with a maximal puncture to superconformal tail ending in an $L$-shaped puncture as depicted in figure 7. The figure on the left corresponds to a Riemann surface representation of the theory, and on the right we have a Lagrangian description for the tail. The flavor symmetry of the $L$-shaped puncture is $U(1) \times SU(N-2)$. The superconformal tail has two additional factors of $U(1)$ flavor symmetry: in the Riemann surface picture these correspond to the two minimal punctures, and in the Lagrangian representation the $U(1)$s are flavor symmetries of the hypermultiplets. To obtain the index with the $L$-shaped puncture only, we have to remove the two minimal punctures by computing the residue of the relevant $U(1)$ flavor fugacities at $t^{1/2}$, as we discussed in the bulk of the paper. We will implement this procedure in a convenient way to obtain an explicit expression for this index. The index of this theory is given by

$$
\oint \prod_{i=1}^{N-1} \frac{dc_i}{2\pi i c_i} \mathcal{I}_V(c) \Delta(c) \mathcal{I}_C(c^{-1},..) \times
$$

$$
\oint \prod_{j=1}^{N-2} \frac{db_j}{2\pi b_j} \mathcal{I}_V(b) \Delta(b) \mathcal{I}_{hyp}(b, d, z) \mathcal{I}_{hyp}(c, x, \{b, y\}).
$$

$\mathcal{I}_C$ denotes the index of the Riemann surface to which we glue the superconformal tail. Here the free hypermultiplet indices are given by

$$
\mathcal{I}_{hyp}(b, d, z) = \prod_{i=1}^{N-1} \prod_{j=1}^{N-2} \prod_{m,n \geq 0} \frac{1 - p^{n+1}q^{m+1}t^{1/2} (d_j b_i z)}{1 - p^n q^m t^{1/2} d_j b_i z} \frac{1 - p^{n+1}q^{m+1}t^{-1/2} d_j b_i z}{1 - p^n q^m t^{1/2} (d_j b_i z)^{-1}},
$$

$$
\mathcal{I}_{hyp}(c, x, \{b, y\}) = \prod_{i,j=1}^{N} \prod_{m,n \geq 0} \frac{1 - p^{n+1}q^{m+1}t^{-1/2} (x b_i c_j)}{1 - p^n q^m t^{1/2} x b_i c_j} \frac{1 - p^{n+1}q^{m+1}t^{1/2} x b_i c_j}{1 - p^n q^m t^{1/2} (x b_i c_j)^{-1}}.
$$
We have defined \( \{ \hat{b}_1, \ldots, \hat{b}_N \} = \{ y\hat{b}_1^{-1}, \ldots, y\hat{b}_{N-1}^{-1}, y^{1-N} \} \). Let us look for poles which can pinch the integration contours when a combination of the \( U(1) \) fugacities associated to the \( U(1) \) punctures is equal to \( t^{\frac{1}{2}} \). There are three \( U(1) \) fugacities in the tail denoted above by \( x, y \) and \( z \). Let us denote by \( \alpha \) and \( \beta \) the combinations of \( U(1) \) fugacities corresponding to the minimal punctures and by \( \gamma \) the \( U(1) \) fugacity inside the \( L \)-shaped puncture. Then we have the following identification of the \( U(1) \) fugacities in the two descriptions: the Riemann surface and the Lagrangian,

\[
\alpha = x, \quad \beta = \frac{1}{y} z^{\frac{N-2}{N}}, \quad \gamma = y^2 z^{\frac{2}{N}}.
\]

We will consider the poles at \( \alpha = \beta = t^{\frac{1}{2}} \) one after the other. Note that if the \( c \) fugacity is not gauged there are no such poles. The poles in the \( c \) integration occur at

\[
c_j = \frac{t^{\frac{1}{2}}}{xy} b_{\sigma(j)} \quad \text{or} \quad c_j = t^{\frac{1}{2}} y^{N-1} x^{-1}, \quad j = 1 \ldots N - 1; \quad c_N = \frac{t^{1-N}}{(xy)^{1-N}}.
\]

Thus if we choose

\[
x = t^{\frac{1}{2}},
\]

all the \( c \) contours are pinched and we obtain a pole. We evaluate then the integrand of the \( c \) contour integral at

\[
x = t^{\frac{1}{2}}; \quad c_j = y^{-1} b_{\sigma(j)}, \quad j = 1 \ldots N - 1; \quad c_N = y^{N-1}.
\]

We obtain

\[
N! \int \prod_{j=1}^{N-2} \frac{db_j}{2\pi b_j} \mathcal{I}_V(b^{-1}) \Delta(b^{-1}) \mathcal{I}_C(\hat{b},..) \times \mathcal{I}_V(b) \Delta(b) \mathcal{I}_{hyp}(b, d, z) \mathcal{I}'_{hyp}(\{b, y\}).
\]

The free hypermultiplet which coupled to \( c \) becomes

\[
\mathcal{I}_{hyp}(c, x, \{b, y\}) \to \mathcal{I}'_{hyp}(\{b, y\}) = \prod_{i,j=1}^{N} \prod_{m,n \geq 0} \frac{1 - p^{n+1} q^{m+1} t^{-1} \hat{b}_j/\hat{b}_i - 1 - p^{n+1} q^{m+1} \hat{b}_j/\hat{b}_i}{1 - p^n q^m \hat{b}_i/\hat{b}_j - 1 - p^n q^m \hat{b}_j/\hat{b}_i} = \frac{1}{N! \mathcal{I}_V(b^{-1}) \Delta(b^{-1}) \mathcal{I}_V}.
\]

In turn, the index becomes

\[
\int \prod_{j=1}^{N-2} \frac{db_j}{2\pi b_j} \mathcal{I}_C(\hat{b},..) \mathcal{I}_V(b) \Delta(b) \mathcal{I}_{hyp}(b, d, z).
\]
The poles in the $b$ integration contour now occur at

$$b_j = \frac{t^j}{z} d_{\sigma^{(j)}}, \quad j = 1 \ldots N - 2; \quad b_{N-1} = \frac{t^{2-N}}{z^2-N}.$$  \hspace{1cm} \text{(C.9)}

We have $\gamma = y^2 z^2/N$ and $\beta = \frac{1}{y} z^{N-2}$. Thus we set $\beta = t^{\frac{1}{2}}$

$$y z = t^{\frac{1}{2}} \gamma, \quad y^{N-1} = t^{-\frac{1}{2}} \gamma^{\frac{N-2}{2}}, \quad b_{N-1} y^{-1} = \frac{t^{2-N}}{y^{2-N}} = t^{\frac{1}{2}} \gamma^{\frac{N-2}{2}}.$$  \hspace{1cm} \text{(C.10)}

In particular

$$c_j = \frac{1}{\gamma} d_j^{-1}, \quad j = 1 \ldots N - 2, \quad c_{N-1} = t^{\frac{1}{2}} \gamma^{-\frac{N-2}{2}}, \quad c_N = t^{-\frac{1}{2}} \gamma^{-\frac{N-2}{2}}.$$  \hspace{1cm} \text{(C.11)}

If we only consider the poles in the decoupled piece (the sphere with the superconformal tail) there are no contour pinchings. However, since we know that these poles exist, say by decoupling each of the minimal punctures together with a maximal one as before, we deduce that $\mathcal{I}_C$ has to have a relevant pole. We note that

$$\mathcal{I}_V \Delta(b) \mathcal{I}_V(b) \mathcal{I}_{hyp}(b,d,z) \rightarrow \frac{1}{\Gamma(t^{\frac{1}{2}}(\gamma^{N/2} d_j)^{\pm 1};p,q)}.$$ \hspace{1cm} \text{(C.12)}

Thus we deduce that the index (times the two free $U(1)$ factors) is given by

$$\mathcal{I}_L = \mathcal{I}'_C((\frac{1}{\gamma} d_j^{-1}, t^{\frac{1}{2}} \gamma^{-\frac{N-2}{2}}, t^{-\frac{1}{2}} \gamma^{\frac{N-2}{2}}), \ldots) \frac{1}{\Gamma(t^{\frac{1}{2}}(\gamma^{N/2} d_j)^{\pm 1};p,q)}.$$ \hspace{1cm} \text{(C.13)}

To obtain the index of a theory with puncture corresponding to $L$-shaped Young diagram we have to compute a certain residue of the theory with full puncture and strip off a free hypermultiplet. This procedure can be generalized for arbitrary non-maximal punctures.

Let us compare the above result with the general prescription to compute the index of non maximal punctures in the Macdonald case suggested in [23]. Following the prescription of [23] the index of the theory with $L$-shaped puncture is given by

$$\mathcal{I}_L = \hat{K}_L(\gamma,d) \hat{K}(\gamma) \sum_{\lambda} \psi^\lambda(\frac{1}{\gamma} d^{-1}, t^{\frac{1}{2}} \gamma^{-\frac{N-2}{2}}, t^{-\frac{1}{2}} \gamma^{\frac{N-2}{2}}) \Psi^\lambda(\gamma).$$  \hspace{1cm} \text{(C.14)}

Where $\Psi^\lambda(\gamma)$ and $\hat{K}(\gamma)$ are combination of functions not depending on the flavor fugacities of the $L$ shaped puncture. The function $\psi^\lambda$ is a Macdonald polynomial and we also have\textsuperscript{28}

$$\hat{K}_L(\gamma,d) = PE \left[ \frac{t^{\frac{3}{2}}}{1-q} \sum_{i,j=1}^{N-2} d_i/d_j + \frac{t^{3/2}}{1-q} \sum_{i=1}^{N-2} (d_i \gamma^{\frac{1}{2}N} + d_i^{-1} \gamma^{-\frac{1}{2}N}) + \frac{t^2 + t}{1-q} \right].$$ \hspace{1cm} \text{(C.15)}

\textsuperscript{28}The plethystic exponential is defined as $PE[f(x)] = \exp(\sum_{i=1}^{\infty} \frac{1}{z} f(x^i))$ where we also have to specify what are the parameters $x$ with respect to which the plethystics is done: in index applications the parameters $x$ are all the fugacities appearing in index definition.
On the other hand the index of the same theory but with the $L$-shaped puncture traded with the maximal one is given by

$$ I = \hat{K}(a) \hat{K}(\cdot) \sum_{\lambda} \psi^\lambda(a_1, \ldots, a_N) \Psi^\lambda(\cdot). \quad (C.16) $$

The factor $\hat{K}$ for the maximal puncture has the following form

$$ \hat{K} = PE \left[ \frac{t}{1 - q} \sum_{i,j=1}^{N} a_i/a_j \right] = \prod_{i,j} \frac{1}{(t a_i/a_j; q)}. \quad (C.17) $$

We now want to obtain $I_L$ from a residue of $I$. $I$ has poles whenever

$$ a_i = t q^\ell a_j. \quad (C.18) $$

Since $\prod_{i=1}^{N} a_i = 1$ a class of poles is

$$ a_i = t^{1/2} q^{\ell/2} \frac{1}{\prod_{j \neq i}^{N-1} a_j^{1/2}}. \quad (C.19) $$

Let us consider the pole at $a_1$ with $\ell = 0$ and define $b_1 = \prod_{j \neq 1}^{N-1} a_j^{-1/2}$. Then the fugacity assignment in the orthogonal functions will be

$$ \psi^\lambda(a_1, \ldots, a_N) \rightarrow \psi^\lambda(t^{1/2}b_1, t^{-1/2}b_1, a_2 \ldots, a_{N-1}), \quad (C.20) $$

and of course by construction $b_1^2 \prod_{i=2}^{N-1} a_i = 1$. The above assignment corresponds to two row Young diagram with one box in the first row and $N - 1$ boxes in the second and defining a puncture with flavor symmetry $S(U(1) \times U(N-2))$. We see that it is the same as in (C.14) with the following identification of fugacities,

$$ a_i = d_i \gamma^{-1}, \quad b_1 = \gamma^{N-2}. \quad (C.21) $$

Let us now compute the residue of the index at this pole

$$ Res \left\{ \frac{1}{a_1} \hat{K}(a) \right\} = \frac{2}{(t; q)^2(t^2; q)(q; q)} \prod_{i,j=2}^{N-1} \frac{1}{(t a_i/a_j; q)} \prod_{i=2}^{N-1} \frac{1}{(t^{3/2}(a_i/b_1)^{\pm 1}; q)(t^{1/2}(b_1/a_i)^{\pm 1}; q)}. \quad (C.22) $$

On the other hand we also have from (C.15)

$$ \hat{K}_L(a) = \frac{1}{(t; q)(t^2; q)} \prod_{i,j=2}^{N-1} \frac{1}{(t a_i/a_j; q)} \prod_{i=2}^{N-1} \frac{1}{(t^{3/2}(a_i/b_1)^{\pm 1}; q)}. \quad (C.23) $$
The ratio of the two quantities above is simply given by

\[
\frac{\hat{K}_L(a)}{\text{Res}_{a_1\rightarrow t^{1/2}b_1} \{\frac{1}{a_1} \hat{K}(a)\}} = \frac{1}{2} \mathcal{I}_V \prod_{i=2}^{N-1} \left( t^{1/2} (b_1/a_i)^{\pm 1} \right) = \frac{\mathcal{I}_V}{2 \text{PE} \left[ \frac{t^{1/2}}{1-q} \frac{1}{b_1} (a_1 + b_1 a^{-1}) \right]}. \tag{C.24}
\]

Thus the prescription to obtain the index of a theory with \(S(U(1) \times U(N-2))\) puncture from the index of the theory with the maximal puncture is just to consider a certain pole of the latter discussed here and multiply the residue by the index of a free \(U(1)\) vector multiplet and divide by the index of an appropriate free hyper-multiplet. We get complete agreement with (C.13). The procedure of [23] to compute the index of theories with any non-maximal punctures can be phrased as a computation of residues as was discussed in [43].

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