Duality, Polite Water-filling, and Optimization
for MIMO B-MAC Interference Networks and
iTree Networks

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Abstract

The general MIMO interference network is considered and is named the B-MAC Network, for it is
a combination of multiple interfering broadcast channels and multiaccess channels. Included as special
cases are interference channels, X channels, X networks, and most practical wireless networks. The
optimization of this important class of networks has been hindered by that beyond the single antenna
multiaccess channels, little is known about the optimal input structure, which is found here to be the
polite water-filling, satisfied by all Pareto optimal input. This network version of water-filling is polite
because it optimally balances between reducing interference to others and maximizing a link’s own rate,
offering a method to decompose a network into multiple equivalent single-user channels and thus, paving
the way for designing/improving low-complexity centralized/distributed/game-theoretic algorithms for
most network optimization problems. Deeply connected is the rate duality extended to the forward and
reverse links of the B-MAC networks. As a demonstration, weighted sum-rate maximization algorithms
with superior performance and low complexity are designed for B-MAC networks and are analyzed
for Interference Tree (iTree) Networks, a sub-class of the B-MAC networks that possesses promising
properties for further information theoretic study.

Index Terms

Water-filling, Duality, MIMO, Interference Channel, One-hop Network, Transmitter Optimization,
Network Information Theory

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I. INTRODUCTION

A. System Setup

The paper considers the optimization of general one-hop multiple-input multiple-output (MIMO) interference networks. Such a network can be represented by a bipartite graph with multi-antenna transmitter nodes on the left and multi-antenna receiver nodes on the right, with non-zero channels between them as edges. Each transmitter may send independent data to multiple receivers and each receiver may collect independent data from multiple transmitters. Consequently, the network is a combination of multiple interfering broadcast channels (BC) and multiaccess channels (MAC) and we name it the Broadcast-Multiaccess Channel (B-MAC), the B-MAC network, or the B-MAC Interference Network. It includes BC, MAC, interference channels [1]–[3], X channels [4, 5], X networks [6], and most practical communication networks, such as (cooperative) cellular networks, wireless LAN, and digital subscriber line, as special cases. Therefore, optimization of such networks has both theoretical and practical impact.

We assume Gaussian input and that each interference is either completely cancelled or treated as noise. A wide range of interference cancellation is allowed, from no cancellation to any cancellation specified by a binary coupling matrix of the data links, as long as each signal is decoded and possibly cancelled at no more than one receiver. For example, simple linear receivers, dirty paper coding (DPC) [7] at transmitters, and/or successive interference cancellation (SIC) at receivers may be employed. For the more sophisticated Han-Kobayashi scheme where a common message is decoded at more than one receiver, the results in this paper may be generalized as discussed in Remark 7 of Section III-B and in Section VI. The reasons of considering the above setting are as follows. 1) The capacity regions and optimal transmission strategies of B-MAC networks are unknown except for the special cases of BC and MAC, where DPC and SIC are optimal. Even for the well studied MIMO BC and MAC, the optimal input structure is unknown in general, except for the water-filling structure for sum-rate optimal points [8]–[10]. 2) Low complexity is desired in practical systems. The more sophisticated techniques such as Han-Kobayashi scheme [2], [11], [12] may not be implementable if the transmission schemes of the interference are not available and thus, cannot be decoded. 3) Treating weak interference as noise is sometimes optimal [13]–[18]. 4) Limiting to the above setting enables us to make progress and gain insight into the problem. The results and insight serves as a stepping
stone to more sophisticated design.

B. Single-user (Selfish) Water-filling

The generally non-convex optimization of this important class of networks has been hindered by that beyond the full solution in the 1998 seminal work [19] on single antenna multiaccess channels, little is known about the structure of the Pareto optimal input of the achievable rate region. Finding the boundary points of a MAC/BC capacity region, can be done using modified general purpose convex programming, e.g., [20]. But it is desirable to exploit the structure of the problem to design more efficient and lower complexity algorithms that will work well for more general networks where the problems are likely non-convex. For the special case of sum-rate maximization in MIMO MAC, a water-filling structure is discovered in [8]. The sum-rate can be combined into one logdet function as in the single-user case. Then, one user’s optimal input has the single-user water-filling structure when other users’ signals are considered as noise. For individual power constraints, it results in a simple and efficient iterative water-filling algorithm [8]. Using duality, the approach is modified for sum-rate maximization for BC to take care of sum power constraint instead of individual power constraints [9], [10]. The above approach works because the objective function happens to look like a single-user rate and thus, cannot be used for weighted sum-rate maximization, which depends on other Pareto optimal points. The approach in [9] is generalized for weighted sum-rate maximization with a single antenna at receivers [21]. Its generalized water-filling structure no longer has an exact water-filling interpretation and cannot be used for the cases with more than one receive antennas.

Directly applying single-user water-filling to networks is referred to as selfish water-filling here. It is well known to be far from optimal [22]–[24]. The reason is that selfish water-filling only considers the interference from others, without considering the interference to others, when maximizing a link’s own rate. Based on selfish water-filling, the game-theoretic, distributed, and iterative algorithms have been well studied for the digital subscriber line [22], [25]–[29], which has parallel SISO (single-input single-output) interference channels, for MIMO interference channels, e.g., [30]–[33], and for multiaccess channels, e.g., [24]. The uniqueness of the Nash equilibrium is established and the algorithms converge only under stringent conditions. Even in the case of convergence, the performance is rarely optimal.

The importance of controlling interference to others has been recognized in literature, e.g.,
But a systematic, general, and optimal method has not been found. In [34], [35], [37], SISO (parallel) interference channels is considered. The idea in [34] is to minimize each link’s individual interference to the single most vulnerable user subject to this link’s minimum rate requirement. In [37], it is proposed to levy a tax on each user’s power. The tax rate of a user is the negative of the derivative of other users’ weighted sum-rate over the user’s power. In [35], Section IV], after directly solving the Karush–Kuhn–Tucker (KKT) conditions, a term $t_k(n)$, which is the same as the previous tax rate, appears in the solution and affects the water-filling level. In [36], it is recognized that the traditional water-filling is selfish and in a MIMO interference channel, the direction causing less interference to others is the same as the direction receiving less interference from others in the reverse link.

Consequently, the following problems have been open.

- In single-user MIMO channels or parallel channels, no one uses general purpose convex programing for transmitter optimization because we know the optimal input structure is water-filling and thus, we only need to optimize the parameters of the water-filling. Why haven’t we been able to do the same for networks?
- What is the optimal input structure of all boundary points of the MIMO MAC/BC capacity region?
- Is it possible that there exists some kind of water-filling that is different from the single-user water-filling and is optimal for all boundary points of the MAC capacity region? If so, it can be used to design low-complexity high-performance algorithms for all optimization problems related to the boundary points. General purpose convex programing is no longer necessary.
- Even more ambitious questions are for the B-MAC networks. These general interference networks are much less understood and the related optimization problems are generally non-convex. What is the optimal input structure of all the boundary points of the achievable region? What is the optimal method to control interference to others while avoiding interference from others in a network? Can we decompose a network to multiple equivalent single-user channels so that the (distributed) optimization can be made easy? Does there exist a network version of water-filling that is optimal? Can we use the optimal input structure to design low complexity algorithms for all related optimization problems, including those for cellular networks, multi-cell cooperative networks, cognitive radios, and digital subscriber
lines, so that it is hard not to find near optimal solutions even in non-convex cases?

The main contribution of this paper is an optimal network version of water-filling for the above problems.

Related method to water-filling is zero forcing, which has good performance at high signal-to-noise power ratio (SNR). Spatial interference alignment [38] can be viewed as the network counter part of zero-forcing in broadcast channels and can be found by transmitter optimization algorithms. In [36], the duality based alternating optimization is employed to design distributed interference alignment algorithms. However, interference alignment is usually suboptimal at moderate or low SNR. Better algorithms solving the problem of how to choose the right number of data streams for each link is given in [39], [40]. In [4], interference alignment/zero-forcing is employed to maximize the multiplexing gain of the X channel, which is a combination of two-user MAC and two-user BC. However, the iterative schemes in [4] are specialized for the two user X channel and generalizations to general networks are not straightforward. The new water-filling in this paper automatically results in near optimal zero forcing transmission at high SNR and may be modified to conduct zero forcing directly for general interference networks.

C. SINR and Rate Duality

In this paper, we extend the MAC-BC rate duality to the forward and reverse links of the B-MAC networks. The duality has a deep connection to the new water-filling structure and can be used to design low complexity iterative algorithms. The extension is a simple consequence of the signal-to-interference-plus-noise power ratio (SINR) duality. The SINR duality states that if a set of SINRs is achievable in the forward links, then the same SINRs can be achieved in the reverse links when the set of transmit and receive beamforming vectors are fixed and the roles of transmit and receive beams are exchanged in the reverse links. Thus, optimizing the transmit vectors of the forward links is equivalent to optimizing the receive vectors in the reverse links, which can reduce complexity because receive vector optimization is simpler. The SINR duality between single-input multiple-output (SIMO) MAC and multiple-input single-output (MISO) BC was first found in [41] and further developed in [42], [43]. Alternating optimization based on it has been employed in [42], [44]–[49]. Significant insight and progress was achieved in [50], where the SINR duality is extended to any one-hop MIMO networks with linear beamformers by the Lagrange duality of linear programming.
The MAC-BC rate duality has been established in [51]–[54]. The forward link BC and reverse link MAC have the same capacity region and the dual input covariance matrices can be calculated by different transformations. In [52], the dual input covariance transformation is derived based on the flipped channel and achieves the same rates as the forward links with equal or less power. The transformation cannot be calculated sequentially in general B-MAC networks, unless there is no loops in the interference graphs as discussed later. The MAC-BC duality can also be derived from the SINR duality as shown in [53], where a different dual covariance transformation is calculated from the MMSE receive beams and power allocation that makes the SINRs of the forward and reverse links equal. Such a transformation achieves equal or higher rates than the forward links under the same sum power. Because the transformation does not need to be calculated sequentially, it can be easily generalized to B-MAC networks as followed in this paper. Furthermore, the precoding matrices in the SINR duality based transformation can be designed such that there is no interference among the streams of a link [55]. This orthogonal stream technique is directly applicable to B-MAC networks. The above MAC-BC duality assumes sum power constraint. It can be generalized to a linear constraint using minimax duality and SINR duality [56], [57]. Efficient interior point methods have been applied to solve optimizations with multiple linear constraints in [58], [59], where the advantage is that the optimal primary variables and Lagrange multipliers are searched jointly. It is expected that exploiting optimal input structure will produce better algorithms that also work for more general networks, as discussed later.

D. Contributions

The following is an overview of the contributions of this paper.

- **Duality:** As a simple consequence of the SINR duality, we show that the forward and reverse links of B-MAC networks have the same achievable rate region in Section III-A. The dual input covariance matrices are obtained from a transformation based on the MMSE filtering and the SINR duality. Different from the input covariance transformation in [52], which achieves the same rate as the forward links but with equal or less power and cannot be calculated sequentially for general B-MAC networks, the transformation in this paper achieves equal or larger rates with the same power and can be calculated easily for general B-MAC networks. We show that the two transformation coincide at the Pareto rate points.
Polite Water-filling: The long sought-after network version of water-filling for all the Pareto optimal input of the achievable rate region of the B-MAC networks is found in Section III-B. Different from the traditional selfish water-filling, the polite water-filling optimally balances between reducing interference to others and maximizing a link’s own rate in a beautifully symmetric form, while the traditional selfish water-filling only considers interference from others. Conceptually, it tells us that the optimal method to control the interference to others is through optimal pre-whitening of the channel of a link. It offers an elegant method to decompose a network into multiple equivalent single-user channels and thus, paves the way for designing/improving low-complexity centralized or distributed/game-theoretic algorithms for almost all network optimization problems, e.g., those in [4], [22]–[24], [30], [36], [46], [60]. In fact, when imposing the optimal structure at every iteration of an algorithm, intuition and empirical results show that it is hard not to obtain a good solution even in non-convex cases. Consequently, general purpose convex or non-convex programming is no longer necessary. It also provides hints on the optimization of Han-Kobayashi transmission scheme [2]. The polite water-filling has the following structure. Consider link $l$ with channel matrix $H_{l,l}$ and Pareto optimal input covariance matrix $\Sigma_l$. The equivalent input covariance matrix $Q_l \triangleq \hat{\Omega}_l^{-1/2}\Sigma_l\hat{\Omega}_l^{1/2}$ is found to be the water-filling of the pre- and post-whitened equivalent channel $\bar{H}_l = \Omega_l^{-1/2}H_{l,l}\hat{\Omega}_l^{-1/2}$, where $\Omega_l$ is the interference-plus-noise covariance of the forward link, used to avoid interference from others. The physical meaning of $\hat{\Omega}_l$, used to control interference to others, is two folded. In terms of the reverse link, it is the interference-plus-noise covariance of the reverse link, resulted from the optimal dual input. In terms of the forward link, it is the Lagrangian penalty for causing interference to other links. The deep connection to duality is that the optimal Lagrange multipliers work out beautifully to be the optimal dual reverse link powers. A surprising property is that the forward and reverse link equivalent input powers satisfy $\text{Tr}(Q_l) = \text{Tr}(\hat{Q}_l)$ for all individual links under total or other power constraints. This property will be useful for designing distributed algorithms. For the sum-rate optimal input, the polite water-filling does not reduce to the water-filling in [3] because the polite water-filling is related to each individual link.

Extension to a Linear Constraint: We show that all results in the paper, including duality, polite water-filling structure, and algorithms, can be generalized from the case of a sum power constraint and white noise to the case of a linear constraint on all links’ input.
covariance matrices and colored noise in Section III-C. The dual of the linear constraint in the forward link is the colored noise in the reverse link, the same as in multiaccess and broadcast channels [56], [57]. The linear constraint result can be used as basis to handle multiple linear constraints, which arise in individual power constraints, per-antenna power constraints, and interference reduction to primary users in cognitive radios [56], and therefore, is useful for many practical applications. More details on the extension of the algorithms will be discussed in Section VI on future work.

- **Weighted Sum-Rate Maximization:** The polite water-filling can be applied to most existing optimization problems of B-MAC networks. In Section IV, the weighted sum-rate maximization is used as an example to illustrate its superiority. The lowest complexity per iteration of the designed algorithms is linear with respect to the total number of data links. They have superior accuracy and convergence speed, which, unlike generic convex programming, does not depend on the total number of data links. This is a result of enforcing optimal input structure by polite water-filling, where the interference is summarized by $\Omega_l$ and $\hat{\Omega}_l$ no matter how many interfering links are there.

- **iTree Networks:** The optimization for B-MAC networks is not convex in general. To analyze the algorithms and to provide a monotonically converging algorithm, we introduce a narrower class of networks, named Interference Tree (iTree) Networks in Section IV-B which seems worth studying in its own right. After interference cancellation, the remaining interference among links can be represented by a directional graph. If there is no directional loop in the graph, the network is called an iTREE network, where one can find an indexing such that any link is not interfered by the links with smaller indices. It appears to be a logical extension of MAC and BC. An approach to making progress in network information theory is to study special cases such as deterministic channels [11], [61] and degree of freedom [5], [11], [62]. iTREE networks looks promising in this sense.

The rest of the paper is organized as follows. Section II defines the achievable rate region and summarizes the preliminaries. Section III presents the theoretical results on the duality and polite water-filling. As an example, polite water-filling is applied to weighted sum-rate maximization in Section IV where iTREE networks is introduced for optimality analysis. The performance of the algorithms is verified by simulation in Section V. The results of this paper provides a stepping
stone for solving more general problems, such as the optimization of Han-Kobayashi transmission scheme and cognitive radios, distributed optimization, and analysis of iTree Networks. They are discussed in Section VI along with the conclusions.

II. SYSTEM MODEL AND PRELIMINARIES

We first define the achievable rate region studied in the paper. It is followed by a summary of the SINR duality in [50], which will be useful for the derivation in the next Section. We use $\|\cdot\|$ for $L_2$ norm and $\|\cdot\|_1$ for $L_1$ norm. The complex gradient of a real function will be used extensively in the paper and is defined as follows. Define $f(Z) : \mathbb{C}^{M \times N} \rightarrow \mathbb{R}$. The extension of the results in [63] gives the gradient of $f(Z)$ over $Z$ as

$$\nabla_Z f \triangleq \left( \frac{df(Z)}{dZ} \right)^*,$$

where $\frac{df(Z)}{dZ} \in \mathbb{C}^{M \times N}$ is defined as

$$\frac{df(Z)}{dZ} = \begin{bmatrix} \frac{\partial f}{\partial z_{1,1}} & \cdots & \frac{\partial f}{\partial z_{1,N}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial z_{M,1}} & \cdots & \frac{\partial f}{\partial z_{M,N}} \end{bmatrix},$$

and $z_{i,j} = x_{i,j} + jy_{i,j}$, $\frac{\partial f}{\partial z_{i,j}} = \frac{1}{2} \frac{\partial f}{\partial x_{i,j}} - j \frac{\partial f}{\partial y_{i,j}}$, $\forall i, j$. If $Z$ is Hermitian, it can be proved that the above formula can be used without change by treating the entries in $Z$ as independent variables.

A. Definition of the Achievable Rate Region

We consider the most general one-hop MIMO interference network with multiple transmitters and receivers, which is a combination of interfering Broadcast and Multiaccess Channels. We call it a B-MAC network where each physical transmitter may have data for multiple receivers and each physical receivers may want data from multiple transmitters. There are $L$ transmission links. Let $T_l$ and $R_l$ denote the virtual transmitter and receiver of link $l$ equipped with $L_{T_l}$ and $L_{R_l}$ antennas respectively. The received signal at $R_l$ can be expressed as

$$y_l = \sum_{k=1}^{L} H_{l,k} x_k + w_l,$$

where $x_k \in \mathbb{C}^{L_{T_k} \times 1}$ is the transmit signal of link $k$ and is assumed to be circularly symmetric complex Gaussian; $H_{l,k} \in \mathbb{C}^{L_{R_l} \times L_{T_k}}$ is the channel state matrix between $T_k$ and $R_l$; and $w_l \in \mathbb{C}^{L_{R_l} \times 1}$.
$\mathbb{C}^{L_R \times 1}$ is a circularly symmetric complex Gaussian noise vector with zero mean and identity covariance matrix.

To handle a wide range of interference cancellation, we define a coupling matrix $\Phi \in \mathbb{R}^{L \times L}$ as a function of the interference cancellation scheme. It specifies whether interference is completely cancelled or treated as noise: if $x_k$, after interference cancellation, still causes interference to $x_l$, $\Phi_{l,k} = 1$ and otherwise, $\Phi_{l,k} = 0$. For example, when no interference cancellation technique is used, $\Phi$ is a matrix of 1’s with zero diagonal entries. If the transmitters (receivers) of several links are associated with the same physical transmitter (receiver), interference cancellation techniques such as dirty paper coding (successive decoding and cancellation) can be applied at this physical transmitter (receiver) to improve the performance.

**Remark 1:** The coupling matrices valid for the results of this paper are those for which there exists a transmission and receiving scheme such that each signal is decoded and possibly cancelled by no more than one receiver, because in the achievable rate region defined later, a rate is determined by one equivalent channel. In the Han-Kobayashi scheme, a common message is decoded by more than one receiver. Future extension to the Han-Kobayashi scheme is discussed in Remark 7.

We give some examples of valid coupling matrices. For a BC (MAC) employing DPC (SIC) where the $l$th link is the $l$th one to be encoded (decoded), the coupling matrix is given by $\Phi_{l,k} = 0, \forall k \leq l$ and $\Phi_{l,k} = 1, \forall k > l$. In Fig. 1 we give an example of a B-MAC network employing DPC and SIC. Both $T_2$ and $T_3$ ($R_1$ and $R_2$) are associated with the same physical transmitter (receiver). The following $\Phi^a, \Phi^b, \Phi^c, \Phi^d$ are valid coupling matrices for link 1, 2, 3 under the corresponding encoding and decoding orders: $a$. $x_3$ is encoded after $x_2$ and $x_2$ is decoded after $x_1$; $b$. $x_2$ is encoded after $x_3$ and $x_2$ is decoded after $x_1$; $c$. $x_3$ is encoded after $x_2$ and $x_1$ is decoded after $x_2$; $d$. There is no interference cancellation.

$$
\Phi^a = \begin{bmatrix}
0 & 1 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}, \quad \Phi^b = \begin{bmatrix}
0 & 1 & 1 \\
0 & 0 & 0 \\
1 & 1 & 0
\end{bmatrix}, \\
\Phi^c = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}, \quad \Phi^d = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}.
$$
Remark 2: When DPC and SIC are combined, it is possible that an interference may not be fully cancelled under a specific encoding and decoding order. For example, in Fig. 1 if $x_2$ is encoded after $x_3$ and $x_1$ is decoded after $x_2$, the interference from $x_2$ and $x_3$ to $x_1$ may not be fully cancelled [4], [64]. Such case cannot be described by the coupling matrix of 0’s and 1’s defined above. However, one may employ the results in this paper to obtain the inner and outer bounds of rates using pessimistic or optimistic coupling matrices. Another possible solution is to decompose the network into sub-networks, such as the methods proposed in [4], [64] where the two-user X channel is decomposed into two BCs or two MACs using zero-forcing and alternating optimization. Then each sub-network has a valid coupling matrix and the results in this paper can be applied. It will be an interesting future research to design more sophisticated coding techniques to fully cancel the interference from $x_2$ and $x_3$ to $x_1$.

Remark 3: The assumption of either full or no cancellation of each interference can be relaxed. All results in the paper still hold when the interference from link $k$ to link $l$ is partially cancelled, as long as the covariance matrix of the remaining interference has the form of $\bar{H}_{l,k} \Sigma_k \bar{H}_{l,k}^\dagger$, where $\bar{H}_{l,k} \in \mathbb{C}^{L_l \times L_k}$ can be any constant matrix. This is because $\bar{H}_{l,k}$ can be treated as the equivalent channel from link $k$ to link $l$.

For the above B-MAC network, we define an achievable rate region. Unless explicitly specified, all achievable regions in this paper refer to the following one. Note that $\Phi_{l,l} = 0$ by definition. The interference-plus-noise of the $l$th link is $\sum_{k=1}^{L} \Phi_{l,k} H_{l,k} x_k + w_l$, whose covariance matrix is

$$\Omega_l = I + \sum_{k=1}^{L} \Phi_{l,k} H_{l,k} \Sigma_k H_{l,k}^\dagger,$$

where $\Sigma_k$ is the covariance matrix of $x_k$. We denote all the covariance matrices as

$$\Sigma_{1:L} = (\Sigma_1, \Sigma_2, \ldots, \Sigma_L).$$
Then the mutual information (rate) of link $l$ is given by a function of $\Sigma_{1:L}$ and $\Phi$

\[
I_l(\Sigma_{1:L}, \Phi) = \log \left| \mathbf{I} + \mathbf{H}_{l,l} \Sigma_{l} \mathbf{H}_{l,l}^\dagger \Omega_l^{-1} \right|.
\]  

(4)

**Definition 1:** The Achievable Rate Region with a fixed coupling matrix $\Phi$ and sum power constraint $P_T$ is defined as

\[
\mathcal{R}_\Phi(P_T) \triangleq \bigcup_{\Sigma_{1:L} : \sum_{i=1}^L \text{Tr} (\Sigma_i) \leq P_T} \{ \mathbf{r} \in \mathbb{R}_+^L : r_l \leq I_l(\Sigma_{1:L}, \Phi), 1 \leq l \leq L \}.
\]  

(5)

Assuming the sum power constraint is the necessary first step for more complicated cases. It has its own theoretical value and the result can be easily used for individual power constraints and the more general multiple linear constraints as discussed in Section III-C and VI.

**Definition 2:** If $\left[ I_l(\Sigma_{1:L}, \Phi) \right]_{l=1:L}$ is a Pareto rate point of $\mathcal{R}_\Phi(P_T)$, the input covariance matrices $\Sigma_{1:L}$ are said to be Pareto optimal.

A bigger achievable rate region can be defined by the convex closure of $\bigcup_{\Phi \in \Xi} \mathcal{R}_\Phi(P_T)$, where $\Xi$ is a set of valid coupling matrices. For example, if DPC and/or SIC are employed, $\Xi$ can be a set of valid coupling matrices corresponding to various encoding and/or decoding orders. In this paper, we focus on a fixed coupling matrix $\Phi$ and fixed channel matrices. The optimal coupling matrix $\Phi$, or equivalently, the optimal encoding and/or decoding order of the weighted sum-rate maximization problem is partially characterized in Section IV-A. It is straightforward to extend the results in this paper to random channel cases like fading channels. The reason to consider sum power constraints is given in Section III-C.

The above definition ensures the following.

**Theorem 1:** The boundary points of the region $\mathcal{R}_\Phi(P_T)$ are Pareto optimal. That is for each boundary point $\mathbf{r}$, there does not exist another $\mathbf{r}' \in \mathcal{R}_\Phi(P_T)$ satisfying $\mathbf{r}' \geq \mathbf{r}$ and $\exists k$, s.t. $r_k' > r_k$, i.e., all boundary rate points are strong Pareto optimums.

The proof is given in Appendix A.

In Section III we will characterize the Pareto optimal input covariance matrices based on a rate duality between the achievable rate regions of the forward and reverse links. The reverse links are obtained by reversing the directions of the forward links and replacing the channel matrices by their conjugate transposes. The coupling matrix for the reverse links is the transpose.
of that for the forward links. We use the notation \( \hat{\cdot} \) to denote the corresponding terms in the reverse links. For example, in the reverse links of the B-MAC network in Fig. 1, \( T_2/T_3 (R_1/R_2) \) becomes the receiver (transmitter), and \( \hat{x}_2 \) is decoded after \( \hat{x}_3 \) and \( \hat{x}_1 \) is encoded after \( \hat{x}_2 \), if in the forward links, \( x_3 \) is encoded after \( x_2 \) and \( x_2 \) is decoded after \( x_1 \). The interference-plus-noise covariance matrix of reverse link \( l \) is

\[
\hat{\Omega}_l = I + \sum_{k=1}^{L} \Phi_{k,l} H_{k,l}^\dagger \hat{\Sigma}_k H_{k,l}.
\]

(6)

And the rate of reverse link \( l \) is given by

\[
\hat{I}_l (\hat{\Sigma}_{1:L}, \Phi^T) = \log \left| I + H_{l,l}^\dagger \hat{\Sigma}_l H_{l,l} \hat{\Omega}_l^{-1} \right|.
\]

For a fixed \( \Phi^T \), the achievable rate region for the reverse links is defined as:

\[
\hat{\mathcal{R}}_{\Phi^T} (P_T) \triangleq \bigcup_{\hat{\Sigma}_{1:L}, \sum_{l=1}^{L} \text{Tr}(\hat{\Sigma}_l) \leq P_T} \{ \hat{r} \in \mathbb{R}_+^L : \hat{r}_l \leq \hat{I}_l (\hat{\Sigma}_{1:L}, \Phi^T), 1 \leq l \leq L \}.
\]

B. SINR Duality for MIMO B-MAC Networks

The above achievable rate region can be achieved by a spatial multiplexing scheme with successive interference cancellation (SIC) using minimum mean square error (MMSE) filtering and decoding.

**Definition 3:** The Decomposition of a MIMO Link to Multiple SISO Data Streams is defined as, for link \( l \) and \( M_l \geq \text{Rank}(\Sigma_l) \), finding a precoding matrix \( T_l = [\sqrt{p_{l,1}} t_{l,1}, \ldots, \sqrt{p_{l,M_l}} t_{l,M_l}] \) satisfying

\[
\Sigma_l = T_l T_l^\dagger = \sum_{m=1}^{M_l} p_{l,m} t_{l,m} t_{l,m}^\dagger,
\]

(8)

where \( t_{l,m} \in \mathbb{C}^{L \times 1} \) is a transmit vector with \( \|t_{l,m}\| = 1 \); and \( \{p_{l,m}\} \) are the transmit powers. We also define \( p = [p_{1,1}, \ldots, p_{1,M_1}, \ldots, p_{L,1}, \ldots, p_{L,M_L}]^T \) to be the transmit power vector of all links.
Note that the precoding matrix is not unique because \( T'_l = T_l V \) with unitary \( V \in \mathbb{C}^{M_l \times M_l} \) also gives the same covariance matrix in [5]. Without loss of generality, we assume the intra-signal decoding order is that the \( m \)th stream is the \( m \)th to be decoded and cancelled. The receive vector \( r_{l,m} \in \mathbb{C}^{L \times 1} \) for the \( m \)th stream of link \( l \) is obtained by the MMSE filtering as

\[
r_{l,m} = \alpha_{l,m} \left( \sum_{i=m+1}^{M_l} H_{l,i} p_{l,i} t_{l,i} t_{l,i}^H + \Omega_l \right)^{-1} H_{l,l} t_{l,m},
\]

(9)

where \( \alpha_{l,m} \) is chosen such that \( \| r_{l,m} \| = 1 \). This is referred to as MMSE-SIC (MMSE Successive Interference Cancellation) receiver in this paper.

For each stream, one can calculate its SINR and rate. For convenience, define the collections of transmit and receive vectors as

\[
T = \begin{bmatrix} t_{l,m} \end{bmatrix}_{m=1,...,M_l,l=1,...,L},
\]

(10)

\[
R = \begin{bmatrix} r_{l,m} \end{bmatrix}_{m=1,...,M_l,l=1,...,L}.
\]

(11)

The cross-talk matrix \( \Psi(T,R) \in \mathbb{R}_{+}^{\sum_l M_l \times \sum_i M_i} \) between different streams [46] is a function of \( T, R \), and, assuming unit transmit power, the element of the \( \left( \sum_{i=1}^{l-1} M_i + m \right) \)th row and \( \left( \sum_{i=1}^{k-1} M_i + n \right) \)th column of \( \Psi \) is the interference power from the \( k \)th link’s \( n \)th stream to the \( l \)th link’s \( m \)th stream and is given by

\[
\Psi_{k,n}^{l,m} = \begin{cases} 
0 & k = l \text{ and } m \geq n, \\
| r_{l,m}^H H_{l,l} t_{l,n} |^2 & k = l, \text{ and } m < n, \\
\Phi_{l,k} | r_{l,m}^H H_{l,k} t_{k,n} |^2 & \text{otherwise}.
\end{cases}
\]

(12)

Then the SINR and the rate for the \( m \)th stream of link \( l \) can be expressed as a function of \( T, R \) and \( p \),

\[
\gamma_{l,m}(T,R,p) = \frac{p_{l,m} | r_{l,m}^H H_{l,l} t_{l,m} |^2}{1 + \sum_{k=1}^{L} \sum_{n=1}^{M_k} p_{k,n} \Psi_{k,n}^{l,m}},
\]

(13)

\[
r_{l,m}(T,R,p) = \log \left( 1 + \gamma_{l,m}(T,R,p) \right).
\]

(14)

Such decomposition of data to streams with MMSE-SIC receiver is information lossless due to the following fact.
Fact 1: The mutual information in (4) is achieved by the MMSE-SIC receiver \[65\], i.e., it is equal to the sum-rate of all streams of link \( l \):

\[
r_s^l \equiv \sum_{m=1}^{M_l} r_{l,m}(T, R, p) = I_l(\Sigma_{1:L}, \Phi).
\]

In the reverse links, we can obtain SINRs using \( R \) as transmit vectors and \( T \) as receive vectors. The transmit powers is denoted as \( q = [q_{1,1}, \ldots, q_{1,M_1}, \ldots, q_{L,1}, \ldots, q_{L,M_L}]^T \). The decoding order of the streams within a link is the opposite to that of the forward link, i.e., the \( m \)th stream is the \( m \)th last to be decoded and cancelled. Then the SINR for the \( m \)th stream of reverse link \( l \) can be expressed as

\[
\hat{\gamma}_{l,m}(R, T, q) = \frac{q_{l,m} |t_{l,m}^\dagger H_{l,l}^\dagger r_{l,m}|^2}{1 + \sum_{k=1}^{L} \sum_{n=1}^{M_k} q_{k,n} \Psi_{l,m}^{k,n}}.
\]

For simplicity, we will use \( \{T, R, p\} \) (\( \{R, T, q\} \)) to denote the transmission and reception strategy described above in the forward (reverse) links.

The achievable SINR regions of the forward and reverse links are the same. Define the achievable SINR regions \( T_\Phi(P_T) \) and \( \hat{T}_\Phi^T(P_T) \) as the set of all SINRs that can be achieved under the sum power constraint \( P_T \) in the forward and reverse links respectively. For given set of SINR values \( \gamma^0 = [\gamma_{l,m}^0]_{m=1,\ldots,M_l, l=1,\ldots,L} \) define a diagonal matrix \( D(T, R, \gamma^0) \in \mathbb{R}_+^{\sum_i M_i \times \sum_i M_i} \) being a function of \( T, R \) and \( \gamma^0 \), where the \( (\sum_{i=1}^{i-1} M_i + m) \)th diagonal element is given by

\[
D_{\sum_{i=1}^{i-1} M_i + m, \sum_{i=1}^{i-1} M_i + m} = \gamma_{l,m}^0 / |r_{l,m}^\dagger H_{l,l}^\dagger t_{l,m}|^2.
\]

The SINR duality in \[50\] is restated in the following lemma.

Lemma 1: If a set of SINRs \( \gamma^0 \) is achieved by the transmission and reception strategy \( \{T, R, p\} \) with \( \|p\|_1 = P_T \) in the forward links, then \( \gamma^0 \) is also achievable in the reverse links with \( \{R, T, q\} \), where the transmit power vector \( q \) satisfies \( \|q\|_1 = P_T \) and is given by

\[
q = (D^{-1}(T, R, \gamma^0) - \Psi^T(T, R))^{-1} 1.
\]

And thus, one has \( T_\Phi(P_T) = \hat{T}_\Phi^T(P_T) \).
III. Theory

In this section, we first establish a rate duality, which is a simple consequence of the SINR duality in [50]. Then based on the rate duality, the polite water-filling structure and properties of the Pareto optimal input are characterized. Finally, we discuss the extension from a sum power constraint to a linear constraint and the extension from white noise to colored noise.

A. A Rate Duality for B-MAC Networks

The rate duality is a simple consequence of the SINR duality.

Theorem 2: The achievable rate regions of the forward and reverse links of a B-MAC network defined in (5) and (7) respectively are the same, i.e., $R_{\Phi}(P_T) = \tilde{R}_{\Phi,T}(P_T)$.

Proof: For any rate point $r$ in the region $R_{\Phi}(P_T)$ achieved by the input covariance matrices $\Sigma_{1:L}$, the covariance matrix transformation defined below can be used to obtain $\hat{\Sigma}_{1:L}$ for the reverse links such that a reverse link rate point $\hat{r} \geq r$ under the same sum power constraint $P_T$ can be achieved, according to Lemma 2. The same is true for the reverse links. Therefore, we have $R_{\Phi}(P_T) = \tilde{R}_{\Phi,T}(P_T)$. □

Definition 4: Let $\Sigma_l = \sum_{m=1}^{M_l} p_{l,m} t_{l,m} t_{l,m}^\dagger$, $l = 1, \ldots, L$ be a decomposition of $\Sigma_{1:L}$. Compute the MMSE-SIC receive vectors $R$ from (9) and the transmit powers $q$ in the reverse links from (18). The Covariance Transformation from $\Sigma_{1:L}$ to $\hat{\Sigma}_{1:L}$ is defined as

$$\hat{\Sigma}_l = \sum_{m=1}^{M_l} q_{l,m} r_{l,m} r_{l,m}^\dagger, \quad l = 1, \ldots, L.$$  \hspace{1cm} (19)

And $\hat{\Sigma}_{1:L}$ is called the dual input covariance matrices of $\Sigma_{1:L}$.

Lemma 2: For any input covariance matrices $\Sigma_{1:L}$ satisfying the sum power constraint $P_T$ and achieving a rate point $r \in R_{\Phi}(P_T)$, the corresponding dual input covariance matrices $\hat{\Sigma}_{1:L}$ achieves a rate point $\hat{r} \geq r$ in the reverse links under the same sum power constraint.

Proof: According to fact[1] $r$ is achieved by the transmission and reception strategy $\{T, R, p\}$ with $\|p\|_1 = \sum_{l=1}^{L} \text{Tr}(\Sigma_l)$. It follows from Lemma[1] that $\{R, T, q\}$ with $\|q\|_1 = \sum_{l=1}^{L} \text{Tr}(\Sigma_l)$ can also achieve the same rate point $r$ in the reverse links. Note that $T$ may not be the MMSE-SIC receive vectors, which is optimal for the reverse links. Therefore, the reverse link rates may be improved with a better receiver to obtain $\hat{r} \geq r$. □

The following corollary follows immediately from Lemma[2] and Theorem[2].
**Corollary 1:** For any input covariance matrices $\Sigma_{1:L}$ achieving a Pareto rate point, the corresponding dual input covariance matrices $\hat{\Sigma}_{1:L}$ achieves the same Pareto rate point in the reverse links.

The following makes connection of the covariance transformation (19) to the existing ones. First, it is essentially the same as the MAC-BC transformations in [56] and [55] which are also based on SINR duality. The only difference is that we do not restrict the choice of the precoding matrix $T_l$ in (8). In [56], the precoding matrix $T_l$ is chosen as the eigenvectors of $\Sigma_l$. And in [55], the precoding matrix $T_l$ is chosen to decorrelate each link $l$ such that the streams of link $l$ are orthogonal to each other. Second, we show that the MAC-BC transformation in [52] is equivalent to the covariance transformation in (19) at the Pareto rate point.

**Theorem 3:** For any input covariance matrices $\Sigma_{1:L}$ achieving a Pareto rate point, the dual input covariance matrices $\hat{\Sigma}_{1:L}$ produced by the covariance transformation (19) are the same as that produced by the MAC-BC transformation in [52], i.e.,

$$\hat{\Sigma}_l = \Omega_l^{-1/2} F_l G_l^\dagger \Omega_l^{1/2} \hat{\Sigma}_l \Omega_l^{-1/2} G_l F_l^\dagger, \quad l = 1, ..., L,$$

where $F_l \in \mathbb{C}^{L_{R_l} \times N_l}$, $G_l \in \mathbb{C}^{L_{T_l} \times N_l}$ are obtained by the thin singular value decomposition (SVD): $\Omega_l^{-1/2} H_{l,l} \Omega_l^{-1/2} = F_l \Delta_l G_l^\dagger$; and $N_l$ is the rank of $H_{l,l}$.

**Proof:** The proof relies on Theorem 5 in Section III-B which states that at the Pareto rate point, both $\Sigma_{1:L}$ and its dual input covariance matrices $\hat{\Sigma}_{1:L}$ satisfy the polite water-filling structure as in (29) and (30) respectively. Then we have

$$\Omega_l^{1/2} \hat{\Sigma}_l \Omega_l^{1/2} = F_l D_l F_l^\dagger = F_l G_l^\dagger G_l D_l G_l^\dagger G_l F_l^\dagger = F_l G_l^\dagger \Omega_l^{1/2} \hat{\Sigma}_l \Omega_l^{1/2} G_l F_l^\dagger,$$

where the first and last equations follow from (29) and (30) respectively.

**Remark 4:** However, at the inner point of the rate region, the above two transformations are different. The transformation in (19) keeps the sum transmit power unchanged while not decreasing the achievable rate, while the MAC-BC transformation in [52] keeps the achievable rate the same while not increasing the sum transmit power. Furthermore, the extension of MAC-BC transformation in [52] to B-MAC networks can only be sequentially calculated for iTree
networks defined in Section IV-B, a sub-class of B-MAC networks whose interference graph has no loops.

B. Characterization of the Pareto Optimal Input

In the following, we show that the Pareto optimal input covariance matrices have a polite water-filling structure. It generalizes the well known water-filling solution to networks. For single-user channel, the polite water-filling structure reduces to the conventional water-filling solution. First, we formally define the term polite water-filling structure.

Definition 5: Given input covariance matrices \( \Sigma_1: L \), obtain the dual input covariance matrices \( \hat{\Sigma}_1: L \) by the covariance transformation in (19). Let \( \Omega_l \)'s and \( \hat{\Omega}_l \)'s respectively be the corresponding interference-plus-noise covariance matrices in the forward and reverse links. For each link \( l \), pre- and post-whiten the channel \( H_{l,l} \) to produce an equivalent single-user channel \( \tilde{H}_l = \Omega_l^{-1/2} H_{l,l} \Omega_l^{-1/2} \). Define \( Q_l = \hat{\Omega}_l^{1/2} \tilde{\Sigma}_l^{1/2} \hat{\Omega}_l^{-1/2} \) as the equivalent input covariance matrix of the link \( l \). The input covariance matrix \( \Sigma_l \) is said to possess a polite water-filling structure if \( Q_l \) satisfies the structure of water-filling over \( \tilde{H}_l \), i.e.,

\[
Q_l = G_l D_l G_l^\dagger, \\
D_l = (\nu_l I - \Delta_l^{-2})^+, 
\]

where \( \nu_l \geq 0 \) is the water-filling level; the equivalent channel of link \( l \) is decomposed using the thin SVD as \( \tilde{H}_l = F_l \Delta_l G_l^\dagger \), where \( F_l \in \mathbb{C}^{L_l \times N_l}, \ G_l \in \mathbb{C}^{L_l \times N_l}, \ \Delta_l \in \mathbb{R}^{N_l \times N_l}_+; \) and \( N_l \) is the rank of \( H_{l,l} \). If all \( \Sigma_l \)'s possess the polite water-filling structure, then \( \Sigma_1:L \) is said to possess the polite water-filling structure.

For B-MAC, the polite water-filling structure of the Pareto optimal input can be proved by observing that the optimal input covariance matrix for each link is the solution of some single-user optimization problem. We use the notation \( \tilde{\Sigma}_l \) for the optimal variables corresponding to a Pareto rate point. Without loss of generality, assume that \( \tilde{\Sigma}_l = \sum_{m=1}^{M_l} \tilde{\phi}_{l,m} \tilde{t}_{l,m} \tilde{t}_{l,m}^\dagger, \ l = 1, ..., L, \) where \( \tilde{\phi}_{l,m} > 0, \ \forall l, m, \) achieves a Pareto rate point \( [\tilde{r}_l > 0] \) and \( \tilde{\Sigma}_l = \sum_{m=1}^{M_l} \tilde{q}_{l,m} \tilde{r}_{l,m} \tilde{r}_{l,m}^\dagger, \ l = 1, ..., L \) are the corresponding dual input covariance matrices. For the Pareto rate point where the rates of some links are zero, the proof also holds by simply deleting these links and applying the proof for the resulting smaller network. The corresponding interference-plus-noise covariance matrices are denoted as \( \tilde{\Omega}_l \) and \( \tilde{\Omega}_l \) respectively. Then it can be proved by contradiction that the
Pareto optimal input covariance matrix $\tilde{\Sigma}_l$ is the solution of the following single-user optimization problem for link $l$, where the transmission and reception schemes of other links are fixed as the Pareto optimal scheme, $\{\tilde{p}_{k,m}, \tilde{t}_{k,m}, \tilde{r}_{k,m}, k \neq l\}$.

$$\begin{align*}
\min_{\Sigma_l \succeq 0} & \quad \text{Tr} (\Sigma_l) \\
\text{s.t.} & \quad \log \left| I + H_{l,l} \Sigma_l H_{l,l}^H \tilde{\Omega}_l^{-1} \right| \geq \tilde{l}_l \\
& \quad \text{Tr} \left( \Sigma_l A_{k,m}^{(l)} \right) \leq \text{Tr} \left( \tilde{\Sigma}_l A_{k,m}^{(l)} \right) \quad m = 1, \ldots, M_k, k = 1, \ldots, L, k \neq l,
\end{align*}$$

where $A_{k,m}^{(l)} = \Phi_{k,l} H_{k,l}^H \tilde{r}_{k,m} \tilde{r}_{k,m}^H H_{k,l}$, and $\text{Tr} \left( \Sigma_l A_{k,m}^{(l)} \right)$ is the interference from link $l$ to the $m$th stream of link $k$ and is constrained not to exceed the optimal value $\text{Tr} \left( \tilde{\Sigma}_l A_{k,m}^{(l)} \right)$. The constraints force the rates to be the Pareto point while the power of link $l$ is minimized. The Lagrangian of problem (22) is

$$L_l (\lambda, \nu_l, \Theta, \Sigma_l) = \text{Tr} \left( \Sigma_l (A_l (\lambda) - \Theta) \right) - \sum_{k \neq l} \sum_{m=1}^{M_k} \lambda_{k,m} \text{Tr} \left( \tilde{\Sigma}_l A_{k,m}^{(l)} \right)$$

$$+ \nu_l \tilde{l}_l - \nu_l \log \left| I + H_{l,l} \Sigma_l H_{l,l}^H \tilde{\Omega}_l^{-1} \right|,$$

where the dual variables $\nu_l \in \mathbb{R}_+$ and $\lambda = [\lambda_{k,m}]_{m=1,\ldots,M_k,k \neq l} \in \mathbb{R}_+^{\sum_{k \neq l} M_k \times 1}$ are associated with the rate constraint and the interference constraints in (23) respectively; $A_l (\lambda) \triangleq I + \sum_{k \neq l} \sum_{m=1}^{M_k} \lambda_{k,m} A_{k,m}^{(l)}$ is a function of $\lambda$; $\Theta$ is the matrix dual variables associated with the positive semidefiniteness constraint on $\Sigma_l$. Note that problem (22) is convex, and thus, the duality gap is zero [66] and $\tilde{\Sigma}_l$ minimizes the Lagrangian (24) with optimal dual variables $\tilde{\nu}_l$ and $\tilde{\lambda} = [\tilde{\lambda}_{k,m}]_{m=1,\ldots,M_k,k \neq l}$, i.e., it is the solution of

$$\min_{\Sigma_l \succeq 0} \quad \text{Tr} \left( \Sigma_l A_l (\tilde{\lambda}) \right) - \tilde{\nu}_l \log \left| I + H_{l,l} \Sigma_l H_{l,l}^H \tilde{\Omega}_l^{-1} \right|.$$

Note that in the objective function of problem (25), the constant terms in the Lagrangian (24) have been deleted for simplicity of notations, and the term associated with the positive semidefiniteness constraint $\text{Tr} (\Sigma_l \Theta)$ is explicitly handled by adding the constraint $\Sigma_l \succeq 0$.

The following theorem states that the physical meaning of the optimal dual variables $[\tilde{\lambda}_{k,m}]_{k \neq l}$ is exactly the optimal power allocation in the reverse links.
Theorem 4: The optimal dual variables of problem (22) are given by

$$\tilde{\lambda}_{k,m} = \tilde{q}_{k,m}, \quad m = 1, \ldots, M, \quad \forall k \neq l \quad (26)$$

$$\tilde{\nu}_l = \frac{\tilde{q}_{l,m} \tilde{p}_{l,m} (1 + \tilde{\gamma}_{l,m}) |\tilde{r}_{l,m}^\dagger H_{l,l} \tilde{r}_{l,m}|^2}{\tilde{\gamma}_{l,m}^2}, \quad \forall m \quad (27)$$

where $\tilde{\gamma}_{l,m}$ is the SINR of the $m^{th}$ stream of forward link $l$ achieved by $\{\tilde{p}_{k,m}, \tilde{t}_{k,m}, \tilde{r}_{k,m}\}$. Therefore,

$$A_l \left( \tilde{\lambda} \right) = A_l \left( [\tilde{q}_{k,m}]_{k \neq l} \right) = \tilde{\Omega}_l.$$

The proof is given in Appendix B.

The polite water-filling structure is shown by a single-user-channel view using the above results. Let $\tilde{H}_l = \tilde{\Omega}_l^{-1/2} H_{l,l} \tilde{\Omega}_l^{-1/2}$ and $\tilde{Q}_l = \tilde{\Omega}_l^{1/2} \tilde{\Sigma}_l \tilde{\Omega}_l^{1/2}$ respectively be the equivalent channel and the equivalent input covariance matrix for link $l$. Since $\tilde{\Omega}_l$ is non-singular, problem (25) is equivalent to a single-user optimization problem

$$\min_{\tilde{Q}_l \succeq 0} \text{Tr} (\tilde{Q}_l) - \tilde{\nu}_l \log \left| I + \tilde{H}_l \tilde{Q}_l \tilde{H}_l^\dagger \right|, \quad (28)$$

of which $\tilde{\Omega}_l^{1/2} \tilde{\Sigma}_l \tilde{\Omega}_l^{1/2}$ is an optimal solution. Since the optimal solution to problem (28) is unique and is given by the water-filling over $\tilde{H}_l$ with $\tilde{\nu}_l$ as the water-filling level [67], the following theorem is proved.

Theorem 5: For each $l$, perform the thin SVD as $\tilde{H}_l = F_l \Delta_l G_l^\dagger$, where $F_l \in \mathbb{C}^{L_l \times N_l}$, $G_l \in \mathbb{C}^{L_l \times N_l}$, $\Delta_l \in \mathbb{R}^{N_l \times N_l}_{++}$, and $N_l$ is the rank of $H_{l,l}$. At a Pareto rate point, the input covariance matrix $\tilde{\Sigma}_l$ must have a polite water-filling structure, i.e., the equivalent input covariance matrix $\tilde{Q}_l \triangleq \tilde{\Omega}_l^{1/2} \tilde{\Sigma}_l \tilde{\Omega}_l^{1/2}$ satisfies

$$\tilde{Q}_l = G_l D_l G_l^\dagger, \quad (29)$$

$$D_l = (\tilde{\nu}_l I - \Delta_l^{-2})^+. \quad (30)$$

Similarly, the corresponding $\tilde{\Sigma}_l$ produces $\tilde{Q}_l \triangleq \tilde{\Omega}_l^{1/2} \tilde{\Sigma}_l \tilde{\Omega}_l^{1/2}$, which satisfies

$$\tilde{Q}_l = F_l D_l F_l^\dagger. \quad (30)$$

Remark 5: The insight given by the proof of Theorem 5 is that restricting interference to other links can be achieved by pre-whitening the channel with reverse link interference-plus-noise covariance matrix. And thus, the mutual interfering B-MAC can be converted to virtually independent equivalent channels $\tilde{H}_l$, $l = 1, \ldots, L$. 

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Remark 6: The restriction of interference to others is achieved in two steps. First, in $\tilde{H}_l = \tilde{\Omega}_l^{-1/2} H_{l,l} \tilde{\Omega}_l^{-1/2}$, the multiplication of $\tilde{\Omega}_l^{-1/2}$ reduces the channel gain in the interfering directions so that in $\tilde{Q}_l$, less power will be filled in these directions. Second, in $\tilde{\Sigma}_l = \tilde{\Omega}_l^{-1/2} \tilde{Q}_l \tilde{\Omega}_l^{-1/2}$, the power to the interfering directions is further reduced by the multiplication of $\tilde{\Omega}_l^{-1/2}$.

Remark 7: The Lagrangian interpretation of $\tilde{\Omega}_l$ makes it possible to extend the duality and polite water-filling to Han-Kobayashi transmission scheme. Cancelling the interference requires the interference power to be greater than a threshold rather than less than it. Therefore, some Lagrange multipliers are negative in $A_l(\lambda)$. If we still interpret the Lagrange multiplier as reverse link power, we must introduce the concept of negative power. With the new concept of negative power and the modified coupling matrix, one may establish the duality and polite water-filling for Han-Kobayashi scheme. The matrix $\tilde{\Omega}_l$ likely remains positive definite. Otherwise, the solution to problem (25) has infinite power, which suggests there is no feasible power allocation to satisfy all the constraints.

Theorem 5 says that at the Pareto rate point, it is necessary that $\Sigma_{1:L}$ and $\hat{\Sigma}_{1:L}$ have the polite water-filling structure. The following theorem states that $\Sigma_l$ having the polite water-filling structure suffices for $\hat{\Sigma}_l$ to have the polite water-filling structure even at a non-Pareto rate point. This enables the optimization of the network part by part. A lemma is needed for the proof and reveals more insight to the duality. Although the covariance transformation preserves total power such that $\sum_{l=1}^L \text{Tr} (\Sigma_l) = \sum_{l=1}^L \text{Tr} (\hat{\Sigma}_l)$, in general, $\text{Tr} (\Sigma_l) = \text{Tr} (\hat{\Sigma}_l)$ is not true. Surprisingly, $\text{Tr} (Q_l) = \text{Tr} (\hat{Q}_l)$, $\forall l$ is true.

Lemma 3: Let $\hat{\Sigma}_{1:L}$ be the covariance matrices obtained from $\Sigma_{1:L}$ by the transformation in Definition 4. Define the equivalent covariance matrices $Q_l \triangleq \Omega_l^{1/2} \Sigma_l \Omega_l^{1/2}$ and $\hat{Q}_l \triangleq \Omega_l^{1/2} \hat{\Sigma}_l \Omega_l^{1/2}$. The power of the forward and reverse link equivalent covariance matrices of each link is equal, i.e., $\text{Tr} (Q_l) = \text{Tr} (\hat{Q}_l)$, $\forall l$.

The proof is given in Appendix C.

Theorem 6: If one input covariance matrix $\Sigma_l$ has the polite water-filling structure while other $\Sigma_k, \hat{\Sigma}_k, k \neq l$ are fixed, so does its dual $\hat{\Sigma}_l$, i.e., $\hat{Q}_l \triangleq \Omega_l^{1/2} \hat{\Sigma}_l \Omega_l^{1/2}$ is given by water-filling over the reverse equivalent channel $\hat{H}_l^\dagger \triangleq \Omega_l^{-1/2} \hat{H}_{l,l}' \Omega_l^{-1/2}$ as in (30).

Proof: Because water-filling uniquely achieves the single-user MIMO channel capacity [67], $Q_l$ achieves the capacity of $\tilde{H}_l$. Since the capacities of $\tilde{H}_l$ and $\tilde{H}_l^\dagger$ are the same under the same power constraint [67], $\hat{Q}_l$ achieves the capacity of $\tilde{H}_l^\dagger$ with the same power by Lemma 3.
Lemma 2. Therefore, $\hat{Q}_l$ is a water-filling over $H_l^\dagger$.

Decompose a MIMO problem to SISO problems using beams can often reduce the complexity. But for different decompositions of the input covariance matrices $\Sigma_{1:L}$, $\{r_{l,m}\}$ and $\{q_{l,m}\}$ are different in general, and thus the covariance transformation may also be different. However, as shown by the theorem below, the covariance transformation of (19) is unique and has an explicit matrix expression if the input covariance matrices have the polite water-filling structure.

**Theorem 7:** For any input covariance matrices $\Sigma_{1:L}$ satisfying the polite water-filling structure, the covariance transformation (19) is unique, i.e., for all decompositions of $\Sigma_{1:L}$, it will be transformed to the same dual input $\hat{\Sigma}_{1:L}$, and vice versa. Furthermore, the dual input covariance matrices $\hat{\Sigma}_{1:L}$ satisfies the following matrix equation

$$\hat{\Omega}_l^{-1}H_{l,l}^\dagger\hat{\Sigma}_lH_{l,l} = \Sigma_lH_{l,l}^\dagger\Omega_l^{-1}H_{l,l}, \quad l = 1, \ldots, L,$$

and can be explicitly expressed as

$$\hat{\Sigma}_l = \nu_l \left( \Omega_l^{-1} - \left( H_{l,l}\Sigma_lH_{l,l}^\dagger + \Omega_l \right)^{-1} \right), \quad l = 1, \ldots, L$$

where $\nu_l$ is the water-filling level in the polite water-filling structure of $\Sigma_l$ in (21).

The proof is given in Appendix D. The matrix equation in (31) is natural. At the Pareto rate point, the covariance transformation will give the same set of rates for the forward and reverse links. Hence we have

$$\log \left| I + H_{l,l}^\dagger\hat{\Sigma}_lH_{l,l}\hat{\Omega}_l^{-1} \right| = \log \left| I + H_{l,l}\Sigma_lH_{l,l}^\dagger\Omega_l^{-1} \right|$$

$$\Rightarrow \log \left| I + \hat{\Omega}_l^{-1}H_{l,l}^\dagger\hat{\Sigma}_lH_{l,l} \right| = \log \left| I + \Sigma_lH_{l,l}^\dagger\Omega_l^{-1}H_{l,l} \right|. \quad (33)$$

Theorem 7 shows that not only the determinant but also the corresponding matrices are equal, i.e., $\hat{\Omega}_l^{-1}H_{l,l}^\dagger\hat{\Sigma}_lH_{l,l} = \Sigma_lH_{l,l}^\dagger\Omega_l^{-1}H_{l,l}$.

**C. Extension to a Linear Constraint and Colored Noise**

So far we have assumed sum power constraint $\sum_{l=1}^L \text{Tr} \left( \Sigma_l \right) \leq P_T$ and white noise for simplicity. But individual power constraints is more common in a network, which can be easily handled by the following observation. All results in this paper can be directly applied in a much larger class of problems with a linear constraint $\sum_{l=1}^L \text{Tr} \left( \Sigma_l \hat{\mathbf{W}}_l \right) \leq P_T$ and/or colored noise with covariance $E \left[ \mathbf{w}_l\mathbf{w}_l^\dagger \right] = \mathbf{W}_l$, where $\hat{\mathbf{W}}_l$ and $\mathbf{W}_l$ are assumed to be non-singular.

$^1$A singular constraint or noise covariance matrix may result in infinite power or infinite capacity.
and Hermitian. The single linear constraint appears in Lagrangian functions for multiple linear constraints, which arise in cases of individual power constraints, per-antenna power constraints, interference reduction in cognitive radios, etc. [56], [57]. Combined with a simple Lagrange multiplier update, the algorithms in this paper can be generalized to solve the cases of multiple linear constraints.

For a single linear constraint and colored noise, we denote
\[
[H_{l,k}], \sum_{l=1}^{L} \text{Tr} \left( \Sigma_l \hat{W}_l \right) \leq P_T, [W_l], (34)
\]
as a network where the channel matrices is given by \([H_{l,k}];\) the input covariance matrices must satisfy the linear constraint \(\sum_{l=1}^{L} \text{Tr} \left( \Sigma_l \hat{W}_l \right) \leq P_T;\) and the covariance matrix of the noise at the receiver of link \(l\) is given by \(W_l\). If some links have the same physical receiver and thus the same noise, the noise covariance matrices of these links are implicitly assumed to be the same. Similarly, if some links have the same physical transmitter, the linear constraint matrices \(\hat{W}_l\)'s associated with these links are also assumed to be the same. The extension is facilitated by the following lemma which can be proved by variable substitutions.

**Lemma 4:** The achievable rate region of the network (34) is the same as the achievable rate region of the network with sum power constraint and white noise
\[
\left( \left[ W_l^{-1/2} H_{l,k} W_k^{-1/2} \right], \sum_{l=1}^{L} \text{Tr} \left( \Sigma_l \hat{W}_l \right) \leq P_T, \hat{W}_l \right), (35)
\]
If \(\Sigma_{1:L}^{'}\) achieves certain rates and satisfies the sum power constraint in network (35), \(\Sigma_{1:L}\) obtained by \(\Sigma_l = \hat{W}_l^{-1/2} \Sigma_l \hat{W}_l^{-1/2}, \forall l\) achieves the same rates and satisfies the linear constraint in network (34) and vice versa.

The above implies that the dual of colored noise in the forward link is a linear constraint in the reverse link and the dual of the linear constraint in the forward link is colored noise in the reverse link as stated in the following theorem.

**Theorem 8:** The dual of the network (34) is the network
\[
\left( \left[ H_{k,l}^\dagger \right], \sum_{l=1}^{L} \text{Tr} \left( \hat{\Sigma}_l \hat{W}_l \right) \leq P_T, [\hat{W}_l] \right), (36)
\]
in the sense that 1) both of them have the same achievable rate region; 2) If \(\Sigma_{1:L}\) achieves
certain rates and satisfies the linear constraint in network (34), its covariance transformation
\[ \hat{\Sigma}_{1:L} \]
achieves better rates, satisfies the dual linear constraint in network (36), and
\[ \sum_{l=1}^{L} \text{Tr}\left( \hat{\Sigma}_l \hat{W}_l \right) = \sum_{l=1}^{L} \text{Tr}\left( \Sigma_l \hat{W}_l \right) \leq P_T. \]

Proof: Apply Lemma 4 to networks (34) and (36) to produce a network and its dual with
sum power constraint and white noise. Then the result follows from Lemma 2.

Remark 8: For the special case of a BC with a single linear constraint, the duality result here
reduces to that in [56].

IV. ALGORITHMS

In this section, we use the weighted sum-rate maximization of B-MAC networks as an example
to illustrate the benefit of polite water-filling. We first present the simpler case of a sub-class
of B-MAC networks, the interference tree (iTree) networks, with a concave objective function.
Then the algorithm is modified to find sub-optimal but good solutions for the general B-MAC
networks. Readers who are interested in algorithms implementation only may directly go to
Section IV-D and read Table II for Algorithm PT and Table III for Algorithm PP. Algorithm P
and P1 are more of theoretic value.

A. The Optimization Problem

We consider the following weighted sum-rate maximization problem (WSRMP) with a fixed
coupling matrix \( \Phi \),

\[ \text{WSRMP: } g(\Phi) = \max_{\Sigma_{1:L}} f(\Sigma_1, \ldots, \Sigma_L, \Phi) \]

s.t.

\[ f(\Sigma_1, \ldots, \Sigma_L, \Phi) \]

\[ \triangleq \sum_{l=1}^{L} w_l I_l(\Sigma_{1:L}, \Phi), \]

\[ \Sigma_l \succeq 0, \forall l, \]

\[ \sum_{l=1}^{L} \text{Tr}(\Sigma_l) \leq P_T, \]

\[ \sum_{l=1}^{L} \text{Tr}(\Sigma_l) \leq P_T. \]

\[ \hat{\Sigma}_{1:L} \]

The covariance transformation for this case is also calculated from the MMSE receive beams and power allocation that makes
the SINRs of the forward and reverse links equal, just as in (19). The only difference is that the identity noise covariance in
\( \Omega_l \) is replaced by \( \hat{W}_l \) and the all-one vector \( 1 \) in (18) is replaced by the vector
\[ \left[ \hat{t}_{l,m}^\dagger \hat{W}_l \hat{t}_{l,m} \right]_{m=1, \ldots, M_l, l=1, \ldots, L}. \]
where \( w_l \geq 0 \) is the weight for link \( l \).

For the special case of DPC and SIC, we give a partial characterization of the optimal \( \Phi \), or equivalently, the optimal encoding/decoding order. It is in general a difficult problem because the encoding and decoding orders at the BC transmitters and the MAC receivers need to be solved jointly. However, for each pseudo BC transmitter/pseudo MAC receiver defined below in the B-MAC network, the optimal encoding/decoding order maximizing \( \Phi(\pi) \) is easily determined by the weights and is consistent with the optimal order of an individual BC or MAC, as proved in Theorem 9 below.

**Definition 6:** In a B-MAC network, a physical transmitter with a set of associated links, whose indices forms a set \( \mathcal{L}_B \), is said to be a pseudo BC transmitter if either all links in \( \mathcal{L}_B \) completely interfere with a link \( k \) or all links in \( \mathcal{L}_B \) do not interfere with a link \( k \), \( \forall k \in \mathcal{C}_B \), i.e., the columns with indices in \( \mathcal{L}_B \) of the coupling matrix \( \Phi \), excluding rows in \( \mathcal{L}_B \), are the same. A physical receiver with a set of associated links, whose indices forms a set \( \mathcal{L}_M \), is said to be a pseudo MAC receiver if either all links in \( \mathcal{L}_M \) are completely interfered by a link \( k \) or all links in \( \mathcal{L}_M \) are not interfered by a link \( k \), \( \forall k \in \mathcal{C}_M \), i.e., the rows with indices in \( \mathcal{L}_M \) of the coupling matrix \( \Phi \), excluding columns in \( \mathcal{L}_M \), are the same.

For example, if \( \forall l \in \mathcal{L}_B \), link \( l \) is the last one to be decoded at its receiver, then the corresponding physical transmitter is a pseudo BC transmitter. If \( \forall l \in \mathcal{L}_M \), link \( l \) is the first one to be encoded at its transmitter, then the corresponding physical receiver is a pseudo MAC receiver.

**Theorem 9:** In a B-MAC network employing dirty paper coding and successive decoding and cancellation, if there exists a pseudo BC transmitter (pseudo MAC receiver), its optimal encoding (decoding) order \( \pi \) of the following problem

\[
\max_{\pi} \ g(\Phi(\pi)) \tag{38}
\]

is that the signal of the link with the \( n^{th} \) largest (smallest) weight is the \( n^{th} \) one to be encoded (decoded).

**Proof:** It is proved by isolating the links associated with a pseudo BC transmitter or a pseudo MAC receiver from the network to form an individual BC or MAC. Let \( \mathcal{L}_B \) be a set of links associated with a pseudo BC transmitter. In the optimal solution of (38), \( \{ \Sigma_l : l \in \mathcal{L}_B \} \) and \( \pi \) are also the optimal input and encoding order that maximizes the weighted sum-rate of
a BC with fixed interference from links in \( L_B^C \) and under multiple linear constraints that the interference to links in \( L_B^C \) must not exceed the optimal values. The known result on BC with multiple linear constraints in [68] implies that the optimal encoding order is as stated in the theorem. For a set of links belonging to a pseudo MAC receiver, using similar method and generalizing the result in Section III-C to multiple linear constraints gives the desired decoding order.

All the rest of the paper is for a fixed coupling matrix \( \Phi \) and the argument \( \Phi \) in \( f (\Sigma_1, ..., \Sigma_L, \Phi) \) is omitted. In the simulation, DPC and SIC are employed and the encoding/decoding order is chosen to satisfy Theorem 9. We consider centralized algorithms with global channel knowledge. Distributed algorithms with partial knowledge can be developed based on them. More is discussed in Section VI.

B. iTree Networks

iTree networks appears to be a natural extension of MAC and BC. We define it below.

Definition 7: A B-MAC network with a fixed coupling matrix is called an Interference Tree (iTree) Network if after interference cancellation, the links can be indexed such that any link is not interfered by the links with smaller indices.

Definition 8: In an Interference Graph, each node represents a link. A directional edge from node \( i \) to node \( j \) means that link \( i \) causes interference to link \( j \).

Remark 9: The iTree network is related to but different from the network with tree topology in terms of nonzero channel gain. A network with tree topology implies iTree network only if the interference cancellation order is chosen properly. For example, a MAC which has tree topology is not an iTree network if the successive decoding is not employed at the receiver. Similarly, a BC is not an iTree network if dirty paper coding is not employed. On the other hand, even if there are loops in a network, it may be an iTree network if the interference cancellation order is right. We give such an example in Fig. 2 where there are three physical transmitters and three physical receivers with four transmission links 1, 2, 3, and 4, and dirty paper coding and successive decoding and cancellation are employed. Let \( x_l \) be the transmit signal of link \( l \). For encoding/decoding order A, at the physical receiver corresponding to \( R_1 \) and \( R_2 \), the signal \( x_2 \) is decoded after \( x_1 \). At the physical transmitter corresponding to \( T_2 \) and \( T_3 \), the signal \( x_3 \) is encoded after \( x_2 \). With this encoding and decoding order, each link \( l \in \{2, 3, 4\} \) is not interfered
by the first \( l - 1 \) links. Therefore, the network in Fig. 2 is still an iTree network even though it has a loop of nonzero channel gains. However, for encoding/decoding order B, SIC is not employed at \( R_1/R_2 \), and \( x_2 \) is encoded after \( x_3 \) at \( T_2/T_3 \). The network in Fig. 2 is no longer an iTree network because the interference graph has directional loops, making the iTree indexing impossible.

iTee networks can be equivalently defined using their interference graphs.

**Lemma 5:** A B-MAC network with a fixed coupling matrix is an iTree network if and only if after interference cancellation, its interference graph does not have any directional loop.

**Proof:** If the interference graph has no loops, the following algorithm can find indices satisfying the definition of the iTree network: 1) \( l = 1 \); 2) index a node whose edges are all incoming by \( l \); 3) delete the node and all the edges connected to it; 4) let \( l = l + 1 \) and repeat 2) until all nodes are indexed. On the other hand, the interference graph of an iTree network only has edges from a node to the nodes with lower indices, making them impossible to form a directional loop.

\[ \Phi_{l,k} = 0, \forall k \leq l \text{ and } \Phi_{l,k} = 1, \forall k > l. \]

**Lemma 6:** If in an iTree network, the \( l \)th link is not interfered by the links with lower indices, in the reverse links, the \( l \)th link is not interfered by the links with higher indices.

The following is obvious.
Lemma 7: The interference graph of the reverse links can be obtained from the interference graph of the forward links by reversing the directions of all the edges.

Two examples of iTree networks with concave weighted sum-rate functions are as follows. An obvious example is the MAC with the decoding order equal to the ascending order of the weights \( \{w_l\} \). The second example is the two-user Z channel, where the cross link channel gain satisfies \( H_{2,1} = 0 \), providing that some further conditions are satisfied as stated in the following theorem.

**Theorem 10:** The weighted sum-rate of a two-user Z channel is concave function of the input covariance matrices if the following are satisfied.

1) The channel matrices \( H_{1,2} \) and \( H_{2,2} \) are invertible and satisfy \( H_{2,2}^\dagger H_{2,2} \succeq H_{1,2}^\dagger H_{1,2} \).

2) The weights satisfy \( w_1 \leq w_2 \).

The proof is given in Appendix E. The important question of a complete characterization of the subset of iTree networks with concave weighted sum-rate function, i.e., finding the necessary and sufficient conditions on \( H_{k,l} \) and \( \Phi \) for the subset, is out of the scope of this paper and is left for future work.

C. Algorithms for iTree Networks with Concave Objective Functions

The algorithm in this section illustrates the usage of polite water-filling and is a nontrivial generalization of the algorithm in [9]. First we show that WSRMP (37) of an iTree network with concave objective function \( f(\cdot) \) can be equivalently solved by the following convex optimization problem.

\[
\max_{\Sigma(1:L)} f\text{mod} (\Sigma (1), \Sigma (2), \ldots, \Sigma (L)) \quad (39)
\]

\[
\text{s.t.} \forall k, l, \Sigma_l (k) \succeq 0, \text{ and } \forall k \sum_{l=1}^{L} \text{Tr}(\Sigma_l (k)) \leq P_T,
\]

where \( \Sigma (k) \triangleq (\Sigma_1 (k), \ldots, \Sigma_L (k)) \) for \( k = 1, \ldots, L \) with \( \Sigma_l (k) \in \mathbb{C}^{L_l \times L_l} \). The objective function is defined as

\[
f\text{mod} (\Sigma (1), \Sigma (2), \ldots, \Sigma (L)) \triangleq \frac{1}{L} \sum_{i=1}^{L} f (\Sigma_1 ([1 - i]_L), \ldots, \Sigma_l ([l - i]_L), \ldots, \Sigma_L ([L - i]_L)),
\]
where \([n]_L = (n \mod L) + 1\).

The physical meaning of the optimization problem (39) is that it is a weighted sum-rate maximization problem of \(L\) networks, all of which are identical to the network in the original problem in (37). The purpose of expanding the single-network optimization problem in (37) to the multiple-network optimization problem in (39) is to decouple the power constraints so that the input covariance matrices of a network belong to different power constraints and the input covariance matrices of a power constraint belong to different networks. In problem (39), the update of the input covariance matrices of a power constraint is easier because there is no interference among the networks, and other input covariance matrices not belonging to this power constraint can be treated as constants, resulting in monotonically convergent iterative algorithm. A similar method is also used in [9], [21]. We use the subscript \(i\) to denote the terms corresponding to the \(i^{th}\) network, e.g., the input covariance matrices for the \(i^{th}\) network is denoted as \(\Sigma_{i,1:L} = (\Sigma_{i,1}, \ldots, \Sigma_{i,L})\). The one-to-one mapping between \((\Sigma_{1,1:L}, \ldots, \Sigma_{L,1:L})\) and \((\Sigma(1), \ldots, \Sigma(L))\) is given by

\[
\Sigma_{i,l} = \Sigma_{l \left( \lfloor \frac{l-i}{L} \rfloor \right)}, \quad \Sigma_{l}(k) = \Sigma_{\left\lfloor \frac{l-k}{L} \right\rfloor,i}, \quad \forall i, k, l \in \{1, \ldots, L\}.
\]

Then we have

\[
f_{\text{mod}}(\Sigma(1), \Sigma(2), \ldots, \Sigma(L)) = \frac{1}{L} \sum_{i=1}^{L} \sum_{l=1}^{L} w_{l,i} I_{i,l}(\Sigma_{i,1:L}, \Phi).
\]

Note that the power constraint \(\sum_{i=1}^{L} \text{Tr}(\Sigma_i(k)) \leq P_T, \forall k\) couples the networks together. The following example illustrates the mapping in (40) for \(L = 4\):

| 1st link | 2nd link | 3rd link | 4th link |
|----------|----------|----------|----------|
| 1st network | \(\Sigma_1(1)\) | \(\Sigma_2(2)\) | \(\Sigma_3(3)\) | \(\Sigma_4(4)\) |
| 2nd network | \(\Sigma_1(4)\) | \(\Sigma_2(1)\) | \(\Sigma_3(2)\) | \(\Sigma_4(3)\) |
| 3rd network | \(\Sigma_1(3)\) | \(\Sigma_2(4)\) | \(\Sigma_3(1)\) | \(\Sigma_4(2)\) |
| 4th network | \(\Sigma_1(2)\) | \(\Sigma_2(3)\) | \(\Sigma_3(4)\) | \(\Sigma_4(1)\) |

where the element at the \(l^{th}\) column and \(i^{th}\) row is equal to \(\Sigma_{i,l}\), the input covariance matrix for the \(l^{th}\) link of the \(i^{th}\) network.

The following lemma holds for \(f_{\text{mod}}(\cdot)\).
Lemma 8: The function $f_{\text{mod}}(\cdot)$ satisfies the following properties:

1) Let $\Sigma_l \succeq 0$, $\forall l$, satisfy $\sum_{l=1}^L \trace(\Sigma_l) \leq P_T$. The mapping $\Sigma(k) = \Sigma_{1:L}$, $\forall k$, results in

$$\sum_{l=1}^L \trace(\Sigma_l(k)) \leq P_T, \forall k, \text{ and } f_{\text{mod}}(\Sigma(1 : L)) \text{ satisfying } f_{\text{mod}}(\Sigma_{1:L}, \Sigma_{1:L}, ..., \Sigma_{1:L}) = f(\Sigma_{1:L}).$$

2) Let $\Sigma_l(k) \succeq 0$, $\forall l, k$, satisfy $\sum_{l=1}^L \trace(\Sigma_l(k)) \leq P_T, \forall k$. The mapping $\Sigma_{1:L} = (1/L) \sum_{k=1}^L \Sigma(k)$ results in $\sum_{l=1}^L \trace(\Sigma_l) \leq P_T$ and $f(\Sigma_{1:L})$ satisfying

$$f_{\text{mod}}(\Sigma(1), \Sigma(2), ..., \Sigma(L)) \leq f(\Sigma_{1:L}).$$

Proof: The first property is obvious and the second property holds because

$$f_{\text{mod}}(\Sigma(1), \Sigma(2), ..., \Sigma(L)) \leq f\left(\frac{1}{L} \sum_{k=1}^L \Sigma_1(k), ..., \frac{1}{L} \sum_{k=1}^L \Sigma_L(k)\right)$$

where (41) follows from the concavity of $f(\cdot)$.

The lemma says that the problems (37) and (39) are equivalent and every optimal solution $\tilde{\Sigma}(1), \tilde{\Sigma}(2), ..., \tilde{\Sigma}(L)$ of problem (39) maps directly to the optimal solution $\tilde{\Sigma}_{1:L} = (1/L) \sum_{k=1}^L \tilde{\Sigma}(k)$ of the WSRMP (37).

We first obtain insight of the problem by finding the necessary and sufficient conditions satisfied by the optimum. With the insight, we design an algorithm which monotonically increases the objective function until convergence to the optimum.

Theorem 11: Necessity: If $\tilde{\Sigma}(1), ..., \tilde{\Sigma}(L)$ is an optimum of problem (39), then $\forall k, \tilde{\Sigma}(k)$ must satisfy the following optimality conditions:

1) For any $1 \leq l \leq L$, $\tilde{\Sigma}_l(k)$ possesses the polite water-filling structure as in Definition 5 i.e., $\hat{\Omega}_{[l-k]L,L}^{1/2} \tilde{\Sigma}_l(k) \hat{\Omega}_{[l-k]L,L}^{1/2}$ is given by water-filling over the equivalent channel $\hat{\Omega}_{[l-k]L,L}^{-1/2} H_l \hat{\Omega}_{[l-k]L,L}^{-1/2}$.

2) The water-filling level for $\tilde{\Sigma}_l(k)$ is given by $\tilde{\nu}_l = w_l / \tilde{\mu}$, where $\tilde{\nu}_l$ does not depend on $k$ and $\tilde{\mu} > 0$ is chosen such that $\sum_{l=1}^L \trace(\tilde{\Sigma}_l(k)) = P_T$.

 Sufficiency: If certain $\tilde{\Sigma}(1), ..., \tilde{\Sigma}(L)$ satisfies the above optimality conditions, then it must satisfy the KKT conditions of problem (39), and thus is the global optimum of problem (39).

We go over the proof in detail as it helps design the algorithm. It can be proved by contradiction that the optimum $\tilde{\Sigma}(1), ..., \tilde{\Sigma}(L)$ must satisfy $\sum_{l=1}^L \trace(\tilde{\Sigma}_l(k)) = P_T, \forall k$. Otherwise, if
\[ \sum_{l=1}^{L} \text{Tr} \left( \hat{\Sigma}_{i} (k) \right) < P_T \] for some \( k \), we can strictly improve the rate of the first link in the \((1 - k)_L\)th network by increasing the power of \( \hat{\Sigma}_{1} (k) \) until \( \sum_{l=1}^{L} \text{Tr} \left( \hat{\Sigma}_{l} (k) \right) = P_T \). All other rates are not affected because the first link causes no interference to other links. Then the objective function can be strictly increased\(^3\), which contradicts that \( \hat{\Sigma}_{1}, \ldots, \hat{\Sigma}_{L} \) is an optimum.

The remaining necessity part is proved by showing that if for some \( k \), \( \Sigma_{i} (k) \) does not satisfy the polite water-filling structure in Theorem 11, the objective function can be strictly increased by enforcing this structure on \( \Sigma_{i} (k) \). Without loss of generality, we only need to prove this for \( k = 1 \) due to the circular structure of \( f_{\text{mod}}(\cdot) \), i.e.,

\[ f_{\text{mod}}(\Sigma_{1}, \Sigma_{2}, \ldots, \Sigma_{L}) = f_{\text{mod}}(\Sigma_{1}, \Sigma_{L}, \Sigma_{1}, \Sigma_{2}, \ldots, \Sigma_{k-1}) \] .

We define some notations and give two useful lemmas. For the \( i \)th network, fixing the input covariance matrices \( \Sigma_{i,j} \), \( j = i + 1, \ldots, L \) for the last \( L - i \) links, the first \( i \) links form a sub-network

\[ \left( [H_{i,l}]_{k,l=1,\ldots,i}, \sum_{l=1}^{i} \text{Tr} (\Sigma_{i,l}) = P_{T,i} [W_{i,l}]_{l=1,\ldots,i} \right) \] .

where \( W_{i,l} = I + \sum_{j=i+1}^{L} \Phi_{i,j} H_{i,j} \Sigma_{i,j} H_{i,j}^{\dagger} \) is the covariance matrix of the equivalent colored noise of link \( l \); \( P_{T,i} = \sum_{j=1}^{i} \text{Tr} (\Sigma_{i,j}) \). It is obvious that this sub-network is still an iTree network.

By Theorem 8, the corresponding dual sub-network is

\[ \left( [H_{i,l}^{\dagger}]_{k,l=1,\ldots,i}, \sum_{l=1}^{i} \text{Tr} \left( \Sigma_{i,l} W_{i,l} \right) = P_{T,i} [I]_{l=1,\ldots,i} \right) \] .

Denote \( T_{i,l} (\Sigma_{i,1:i}, \Phi) \) and \( T_{i,l} (\Sigma_{i,1:i}, \Phi^{T}) \) the forward and reverse link rates of the \( l \)th link of the \( i \)th sub-network \( (42) \) achieved by \( \Sigma_{i,1:i} \) and \( \hat{\Sigma}_{i,1:i} \) respectively. In contrast, \( T_{i,l} (\Sigma_{i,1:L}, \Phi) \) and \( T_{i,l} (\hat{\Sigma}_{i,1:L}, \Phi^{T}) \) denote the forward and reverse link rates of the \( l \)th link of the \( i \)th network.

**Lemma 9:** Let \( \hat{\Sigma}_{i,1:L} = \left( \hat{\Sigma}_{i,1}, \hat{\Sigma}_{i,2}, \ldots, \hat{\Sigma}_{i,L} \right) \) be the covariance transformation of \( \Sigma_{i,1:L} = (\Sigma_{i,1}, \Sigma_{i,2}, \ldots, \Sigma_{i,L}) \), applied to the \( i \)th network. Then \( \hat{\Sigma}_{i,1:i} = \left( \hat{\Sigma}_{i,1}, \hat{\Sigma}_{i,2}, \ldots, \hat{\Sigma}_{i,i} \right) \) is also the covariance transformation of \( \Sigma_{i,1:i} = (\Sigma_{i,1}, \Sigma_{i,2}, \ldots, \Sigma_{i,i}) \), applied to the \( i \)th sub-network \( (42) \).

**Proof:** The interference from link \( i + 1, \ldots, L \) is counted in the colored noise in the \( i \)th sub-network \( (42) \). Therefore, the MMSE-SIC receive vectors \( \{r_{i,l,m}, l = 1, \ldots, i\} \) are the same in

\(^3\)If \( H_{1,1} = 0 \), we can still increase the objective function by improving the rate of the second link in the \((2 - k)_L\)th network, and so on.
both $i^{th}$ network and its sub-network. Because the $i^{th}$ network is an iTree network, in its dual network, there is no interference from link $i+1, ..., L$ to link $1, ..., i$ by Lemma 6. Therefore, for link $1, ..., i$, there is no difference between the $i^{th}$ dual sub-network (43) and the $i^{th}$ dual network. In both dual networks, the transmit powers of the first $i$ links given by (18) in the covariance transformation achieve the same SINRs as in the $i^{th}$ network. But for fixed $\{r_{i,l,m}, l = 1, ..., i\}$ and $\{t_{i,l,m}, l = 1, ..., i\}$, the transmit powers producing the same SINRs are unique [50, Lemma 1]. Therefore the transmit powers of the first $i$ links must be the same for the two dual networks, which implies that Lemma 9 must hold.

Lemma 10: The necessary and sufficient conditions for $\tilde{\Sigma}_{1:L}$ to be the optimal solution of problem (37) for an iTree network with concave objective function are listed below.

1) It possesses the polite water-filling structure as in Definition 5.

2) The water-filling level $\tilde{\nu}_l$ for $\tilde{\Sigma}_l$ is given by $\tilde{\nu}_l = w_l/\tilde{\mu}$, where $\tilde{\mu} > 0$ is chosen such that $\sum_{l=1}^L \text{Tr}(\tilde{\Sigma}_l) = P_T$.

The proof is given in Appendix F.

Then we give an algorithm to improve the objective function if $\Sigma(1)$ does not satisfy the two necessary conditions in Theorem 11. It contains three steps. Note that $\Sigma(1)$ contains the input covariance matrices of the $i^{th}$ link of the $i^{th}$ network for $i = 1, ..., L$, i.e., $\Sigma(1) = (\Sigma_{1,1}, \Sigma_{2,2}, ..., \Sigma_{L,L})$.

Step 1: For $i = 1, ..., L$, calculate $\tilde{\Sigma}_{i,1,i-1}$ by the covariance transformation applied to the $i^{th}$ sub-network. Due to the special interference structure of the iTree network, the calculation of the reverse transmit powers of the covariance transformation can be simplified to be calculated one by one as follows. When calculating $q_{i,l,m}$, the transmit powers $\{q_{i,k,m}: m = 1, ..., M_{i,k}, k = 1, ..., l-1\}$ and $\{q_{i,n}: n = 1, ..., m-1\}$ have been calculated. Therefore, we can calculate $\tilde{\Sigma}_{i,k} = \sum_{m=1}^{M_{i,k}} q_{i,k,m} r_{i,k,m} r_{i,k,m}^H$, $k = 1, ..., l-1$ and obtain the interference-plus-noise covariance matrix of the reverse link $l$ as $\tilde{\Omega}_{i,l} = I + \sum_{k=1}^{l-1} \Phi_{i,k} \Sigma_{i,k} \Phi_{i,k}^H$. Then $q_{i,l,m}$ is given by

$$q_{i,l,m} = \gamma_{i,l,m} \left( t_{i,l,m}^H \tilde{\Omega}_{i,l} t_{i,l,m} + \sum_{n=1}^{m-1} q_{i,l,n} \left| t_{i,l,m}^H H_{i,l}^H r_{i,l,n} \right|^2 \right) \left( t_{i,l,m}^H H_{i,l}^H r_{i,l,m} \right)^2 \gamma_{i,l,m}$$

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and correspondingly
\[ \hat{\Sigma}_{i,l} = \sum_{m=1}^{\bar{M}_{i,l}} q_{i,l,m} r_{i,l,m} r_{i,l,m}^\dagger. \] (45)

By Theorem 8 we have \( \hat{I}_{i,l} \left( \hat{\Sigma}_{i,1;i}, \Phi^T \right) \geq I_{i,l} \left( \Sigma_{i,1;i}, \Phi \right) = I_{i,l} \left( \Sigma_{i,1:L}, \Phi \right), \ l = 1, ..., i \) and \( \sum_{i=1}^{L} \text{Tr} \left( \hat{\Sigma}_{i,l} W_{i,i} \right) = \sum_{i=1}^{L} \text{Tr} \left( \Sigma_{i,l} \right) = P_T^i. \)

**Step 2:** Improve \( \sum_{i=1}^{L} w_l \hat{I}_{i,l} \left( \hat{\Sigma}_{i,1;i}, \Phi^T \right) \) by enforcing the polite water-filling structure on \( \hat{\Sigma}_{i,j}, \forall i. \) By Lemma 6 in the \( i^{th} \) sub-network, the reverse link \( i \) causes no interference to the first \( i-1 \) reverse links. Fixing \( \hat{\Sigma}_{i,l}, l = 1, ..., i-1, \forall i, \) we can improve \( \sum_{i=1}^{L} w_l \hat{I}_{i,l} \left( \hat{\Sigma}_{i,1;i}, \Phi^T \right) \) without reducing the rates of reverse link 1, ..., \( i-1 \) in the \( i^{th} \) sub-network for all \( i \) by solving the following weighted sum-rate maximization problem for \( L \) parallel channels with colored noise:

\[
\begin{align*}
\max_{\Sigma_{i,l} \succeq 0, i=1,...,L} & \sum_{i=1}^{L} w_i \log \left| I + H_{i,i}^\dagger \hat{\Sigma}_{i,i} H_{i,i} \hat{\Omega}_{i,i}^{-1} \right| \\
\text{s.t.} & \sum_{i=1}^{L} \text{Tr} \left( \hat{\Sigma}_{i,i} W_{i,i} \right) \leq \sum_{i=1}^{L} \left( P_T^i - \sum_{i=1}^{L} \text{Tr} \left( \hat{\Sigma}_{i,l} W_{i,l} \right) \right),
\end{align*}
\] (46)

where \( \hat{\Omega}_{i,i} = I + \sum_{i=1}^{L} \hat{\Phi}_{i,i} H_{i,i}^\dagger \hat{\Sigma}_{i,i} H_{i,i} \) and \( W_{i,i} = I + \sum_{j=i+1}^{L} \hat{\Phi}_{i,j} H_{i,j} \Sigma_{i,j} H_{i,j}^\dagger \). This is a classic problem with unique water-filling solution. The solution with sum power constraint is obtained in [9], which can be extended to the case of linear constraint here. Alternatively, the solution can be obtained from Theorem 5 and Lemma 4 for a network of parallel channels. Perform the thin SVD \( \hat{\Omega}_{i,i}^{-1/2} H_{i,i} \hat{\Omega}_{i,i}^{-1/2} = F_i \Delta_i G_i^\dagger \). Let \( N_i = \text{Rank} \left( H_{i,i} \right) \) and \( \delta_{i,j} \) be the \( j^{th} \) diagonal element of \( \Delta^2_i \). Obtain \( D_i \) as

\[
\begin{align*}
D_i &= \text{diag} \left( d_{i,1}, ..., d_{i,N_i} \right), \\
d_{i,j} &= \left( \frac{w_i}{\mu} - \frac{1}{\delta_{i,j}} \right)^+, \ j = 1, ..., N_i,
\end{align*}
\] (47)

where \( \mu \) is chosen such that \( \sum_{i=1}^{L} \sum_{j=1}^{N_i} d_{i,j} = \sum_{i=1}^{L} \left( P_T^i - \sum_{i=1}^{L} \text{Tr} \left( \hat{\Sigma}_{i,l} W_{i,l} \right) \right) \). The calculation of \( \mu \) is easy: 1) Let \( (i, j) \) index the \( j^{th} \) eigen-channel of the \( i^{th} \) network’s \( i^{th} \) link. Initialize the set of indices of channels with nonnegative power as \( \Gamma = \{(1,1), ..., (1,N_1), ..., (L,1), ..., (L,N_L)\} \) to include all eigen-channels; 2) Calculate \( \mu \) for the channels in \( \Gamma \) without \((\cdot)^+\) operation; 3) For all \( (n,m) \in \Gamma, \) if \( d_{n,m} < 0 \), fix it with zero power \( d_{n,m} = 0 \), delete \( (n,m) \) from \( \Gamma \). Repeat step 2) until all \( d_{n,m} \geq 0 \). Then the optimal solution of problem (46) is given.
by
\[ \hat{\Sigma}_{i,i} = (\Sigma_{i,i} - w_i \hat{T}_{i,i} \Phi \hat{T}_{i,i}^T) \]
where \( w_i = \frac{1}{\Omega_{i,i}} \) and \( \Omega_{i,i} \) is given by
\[ \Omega_{i,i} = \frac{(\Sigma_{i,i} - w_i \hat{T}_{i,i} \Phi \hat{T}_{i,i}^T)}{w_i} \]
for all the \( i = 1, \ldots, L \). Combining the above facts and the second property in Lemma 8, the objective function can be strictly increased by updating \( \Sigma(k) \)’s as
\[ \Sigma(k) = \frac{1}{L} \sum_{l=1}^{L} \Sigma'(l), \quad 1 \leq k \leq L, \]
where \( \Sigma'(1), \ldots, \Sigma'(L) \) is obtained from \( \Sigma'(1:L), \ldots, \Sigma'(L:L) \) according to (40); and the updated \( \Sigma(k) \)’s satisfy \( \sum_{l=1}^{L} \text{Tr}(\Sigma(l)) = P_T, \forall k \).

This contradiction proves that the optimal \( \hat{\Sigma}_l(k) \), \( \forall k, l \) must satisfy the polite water-filling structure and the water-filling level for \( \hat{\Sigma}_l(k) \) is given by \( w_l/\hat{\mu}_k \). The condition that \( \hat{\mu}_k = \bar{\mu}_k \), \( \forall k \) can also be proved by contradiction. If \( \hat{\mu}_k \)'s are different, the water-filling levels \( w_l/\hat{\mu}_k \)'s of the input covariance matrices \( \hat{\Sigma}_{1,1:L} \) for the first network are not proportional to the weights. Then by Lemma 10, \( \hat{\Sigma}_{1,1:L} \) is not the optimal solution of problem (37) with sum power constraint \( \sum_{l=1}^{L} \text{Tr}(\hat{\Sigma}_{1,l}) \) for the first network, which implies that there exists \( \Sigma'(1:L) \) achieving a higher weighted sum-rate for the first network with the same sum power. Similarly, for the \( l \)th network, \( \forall i \), there exists \( \Sigma'(1:L) \) achieving a higher weighted sum-rate with the same sum power. Then
After updating $\Sigma^{(k)}$’s from $\Sigma_i^{(1:L)}$’s using (49), the objective function of problem (39) is strictly increased. This contradiction completes the proof for the necessity part.

Now, we prove the sufficiency part. For convenience, we map $\tilde{\Sigma}_i^{(1)}, \ldots, \tilde{\Sigma}_i^{(L)}$ to $(\tilde{\Sigma}_i^{(1)}, \ldots, \tilde{\Sigma}_i^{(L)})$ using (40) and show that $(\tilde{\Sigma}_i^{(1:L)}, \ldots, \tilde{\Sigma}_i^{(L)})$ satisfies the KKT conditions of problem (39). The Lagrangian of problem (39) is

$$L(\mu_1^{1:L}, \Theta_1^{1:L}, \Sigma_1^{1:L}) = \sum_{i=1}^{L} \sum_{l=1}^{L} w_l \log |I + H_{l,l} \Sigma_i^{l,l} H_{l,l}^\dagger \Omega_{i,l}^{-1}|$$

$$+ \sum_{k=1}^{L} \mu_k \left( P_T - \sum_{l=1}^{L} \text{Tr} (\Sigma_i^{[l-k],l}) \right)$$

$$+ \sum_{i=1}^{L} \sum_{l=1}^{L} \text{Tr}(\Sigma_i^{l,l} \Theta_i^{l,l}).$$

The KKT conditions are

$$\nabla_{\Sigma_i^{l,l}} L = 0; \quad \sum_{l=1}^{L} \text{Tr}(\Sigma_i^{[l-k],l}) = P_T;$$

$$\text{Tr}(\Sigma_i^{l,l} \Theta_i^{l,l}) = 0; \quad \mu_k \geq 0, \Sigma_i^{l,l}, \Theta_i^{l,l} \succeq 0; \quad (50)$$

for all $i, k, l = 1, \ldots, L$. The condition $\nabla_{\Sigma_i^{l,l}} L = 0$ can be expressed as

$$\sum_{k \neq l} \frac{w_k}{\mu_{[l-i]}_{L}} \Phi_{k,l}^{i,i} H_{k,k}^\dagger$$

$$\times \left( \Omega_{i,k}^{-1} - \left( \Omega_{i,k} + H_{k,k} \Sigma_i^{k,k} H_{k,k}^\dagger \right)^{-1} \right) H_{k,k} + I$$

$$= \frac{w_l}{\mu_{[l-i]}_{L}} H_{l,l}^\dagger \left( \Omega_{i,l} + H_{l,l} \Sigma_i^{l,l} H_{l,l}^\dagger \right)^{-1} H_{l,l} + \frac{1}{\mu_{[l-i]}_{L}} \Theta_i^{l,l}. \quad (51)$$

Since $\tilde{\Sigma}_i$ satisfies the polite water-filling structure with water-filling levels given by $(w_l/\bar{\mu})$’s, by Theorem 7, the dual input covariance matrices $\tilde{\Sigma}_i^{(1:L)}$ can be expressed as

$$\tilde{\Sigma}_i^{k,l} = \frac{w_k}{\bar{\mu}} \left( \tilde{\Omega}_{i,k}^{-1} - \left( \tilde{\Omega}_{i,k} + H_{k,k} \tilde{\Sigma}_i^{k,k} H_{k,k}^\dagger \right)^{-1} \right), \forall k. \quad (52)$$
Then we have
\[
\sum_{k \neq l} \frac{w_k}{\mu} \Phi_{k,l} H_{k,l}^\dagger \times \left( \tilde{\Omega}_{i,k}^{-1} - \left( \tilde{\Omega}_{i,k} + H_{k,k} \tilde{\Sigma}_{i,k} H_{k,k}^\dagger \right)^{-1} \right) H_{k,l} + I
\]
\[
= \sum_{k \neq l} \Phi_{k,l} H_{k,l}^\dagger \tilde{\Sigma}_{i,k} H_{k,l} + I = \tilde{\Omega}_{i,l}.
\]

Choose the dual variables \( \mu_{1:L} \) as \( \mu_k = \tilde{\mu}, \forall k \) and substitute \( \tilde{\Sigma}_{i,1:L} \) into condition (51) to obtain
\[
\tilde{\Omega}_{i,l} = \frac{w_l}{\tilde{\mu}} H_{l,l}^\dagger \left( \tilde{\Omega}_{i,l} + H_{l,l} \tilde{\Sigma}_{i,l} H_{l,l}^\dagger \right)^{-1} H_{l,l} + \frac{1}{\mu} \Theta_{i,l}.
\] (53)

Because Equation (53) is also the KKT condition of the single-user polite water-filling problem over the channel \( \tilde{\Omega}_{i,l}^{-1/2} H_{l,l} \tilde{\Omega}_{i,l}^{-1/2} \) and \( \tilde{\Sigma}_{i,l} \) has polite water-filling structure over this channel with water-filling level \( w_l/\tilde{\mu} \) by the optimality condition, \( \left( \tilde{\Sigma}_{1,1:L}, \ldots, \tilde{\Sigma}_{L,1:L} \right) \) satisfies condition (53). It can be verified that \( \left( \tilde{\Sigma}_{1,1:L}, \ldots, \tilde{\Sigma}_{L,1:L} \right) \) also satisfies all other KKT conditions in (50). Since problem (39) is convex, KKT conditions are sufficient for global optimality. This completes the proof for Theorem 11.

Algorithm P: The proof of Theorem 11 actually gives an algorithm to find the optimal solution of problem (39) by monotonically increasing the objective function. We refer to it as Algorithm P and summarize it in Table I, where P stands for Polite. Note that although in each iteration, we only enforce the polite water-filling structure in Theorem 11 on \( \Sigma (1) \), all \( \Sigma (k) \)'s will eventually satisfy the polite water-filling structure due to (49).

The convergence and optimality of Algorithm P is proved in the theorem below.

Theorem 12: For any iTREE network with concave weighted sum-rate function \( f(\cdot) \), Algorithm P converges to the optimal solution of problem (39) and problem (37).

Proof: In each iteration, Algorithm P monotonically increases the objective function in problem (39). Since the objective is upper bounded, Algorithm P must converge to a fixed point. At the fixed point, \( \Sigma (1) \) must satisfy the optimality conditions in Theorem 11. Otherwise, the algorithm can strictly increase the objective function, which contradicts with the assumption of fixed point. Then at the fixed point, all \( \Sigma (k) \)'s satisfy the optimality conditions in Theorem 11 because at the end of each iteration, we make all \( \Sigma (k) \)'s the same in (49). By Theorem 11, the fixed point is the optimal solution of problem (39). And by Lemma 8, it must also be the optimal solution of problem (37).
Table I

Algorithm P (for iTree Networks with Concave $f(\cdot)$)

Choose a feasible initial point $\Sigma(k) = (\Sigma_1, ..., \Sigma_L), k = 1, ..., L$ such that $\Sigma_l \succeq 0, \forall l$ and $\sum_{l=1}^L \text{Tr}(\Sigma_l) \leq P_T$.

While not converge do
1. Calculate $\hat{\Sigma}_{L,1:L-1}$ by the covariance transformation applied to the $L$th sub-network. Because all $\Sigma(k)$’s are the same, the $\hat{\Sigma}_{i,1:i-1}$’s for other sub-networks can be obtained by $\hat{\Sigma}_{i,1:i-1} = (\hat{\Sigma}_{i,1}, ..., \hat{\Sigma}_{i,i-1}), i = 1, ..., L - 1$
2. Solve for the optimal $\hat{\Sigma}'_{i,i}$’s in the optimization problem (46) by polite water-filling.
3. For $\forall i$, calculate $\Sigma'_{i,1:i}$ by the covariance transformation of $\hat{\Sigma}'_{i,i}$ applied to the $i$th sub-network.
   Obtain $(\Sigma'(1), ..., \Sigma'(L))$ from $(\Sigma'_{1,1:L}, ..., \Sigma'_{L,1:L})$ according to (40), where $\Sigma'_{i,1:L} = (\Sigma'_{i,1}, ..., \Sigma'_{i,i}, \Sigma_{i,i+1}, ..., \Sigma_{i,L}), \forall i$.
4. Update $\Sigma(k)$’s using (49).
End

Algorithm P1: Another algorithm named P1 can be designed with improved convergence speed using a similar method as in [21] by modifying the updating step in (49) at the cost of a simple linear search. The modified update rule is

$$\Sigma(k) = \frac{\tilde{\beta} P_T}{P_1} \Sigma'(1) + \left(1 - \tilde{\beta}\right) \frac{P_T}{\sum_{l=2}^L P_l} \sum_{l=2}^L \Sigma'(l), \forall k,$$

where $P_k = \sum_{l=1}^L \text{Tr}(\Sigma_l'(k)), \forall k$; and $\tilde{\beta}$ is the optimal solution of the following optimization problem

$$\max_{\beta \in [P_1/(LP_T), 1]} \left[ \frac{\beta P_T}{P_1} \Sigma'(1) + \left(1 - \beta\right) \frac{P_T}{\sum_{l=2}^L P_l} \sum_{l=2}^L \Sigma'(l) \right].$$

It is clear that the update (54) yields at least the same improvement of the objective function as the update (49) because if we let $\tilde{\beta} = P_1/(LP_T)$, (54) reduces to (49). Algorithm P1 is provably convergent and is observed to have much faster convergence speed in the simulations.
D. Algorithms for General B-MAC Networks

Algorithm PT: We obtain Algorithm PT, standing for Polite-Transformation, for general B-MAC networks by a modification of Algorithm P1 so that each iteration is simply a polite water-filling in the forward links and a covariance transformation in the reverse links. It is observed that $\tilde{\beta}$ in (54) is always close to 1 in simulation, which suggests a more aggressive iterative algorithm by modifying Algorithm P1 as follows: Change the linear constraint of problem (46) in step 2 to sum power constraint: $\sum_{l=1}^{L} \text{Tr} \left( \hat{\Sigma}_{i,l} \right) = P_T$; In step 1, simply let $\hat{\Sigma}_{i,l} = \hat{\Sigma}_{l,l}'', l = 1, \ldots, i - 1, i = 1, \ldots, L$, where $\hat{\Sigma}_{l,l}'', l = 1, \ldots, L$ is the solution to the modified problem of (46) under sum power constraint in the step 2 of the previous iteration; In step 2, solve the optimal $\hat{\Sigma}_{i,i}'$’s for the modified problem of (46), where in (47), $\mu$ is chosen to satisfy the sum power constraint $\sum_{l=1}^{L} \text{Tr} \left( \hat{\Sigma}_{i,i}' \right) = P_T$; In step 3, treating $\left( \hat{\Sigma}_{1,1}', \hat{\Sigma}_{2,2}', \ldots, \hat{\Sigma}_{L,L}' \right)$ as the input covariance matrices of a single network, obtain $\Sigma'(1)$ from the covariance transformation of $\left( \hat{\Sigma}_{1,1}', \hat{\Sigma}_{2,2}', \ldots, \hat{\Sigma}_{L,L}' \right)$, and let $\tilde{\beta} = 1$ in the update (54); Because of the symmetry, we can change the roles of the forward and reverse links so that the water-filling is for the forward links. Algorithm PT can be reorganized in a simpler form as in Table II in terms of one network without expanding to $L$ networks.

Note that Algorithm PT does not require any special interference structure or concavity of the weighted sum-rate function. Therefore, it works for general MIMO B-MAC networks. After convergence, the solution of Algorithm PT satisfies the KKT conditions of WSRMP, and thus is a stationary point.

Theorem 13: Apply Algorithm PT to solve problem (37), whose Lagrange function is

$$L(\mu, \Theta_{1:L}, \Sigma_{1:L}) = \sum_{l=1}^{L} w_l \log \left| I + H_{l,l} \Sigma_l H_{l,l}^H \Omega_l^{-1} \right|$$

$$+ \mu \left( P_T - \sum_{l=1}^{L} \text{Tr} \left( \Sigma_l \right) \right) + \sum_{l=1}^{L} \text{Tr} \left( \Sigma_l \Theta_l \right),$$


**Table II**

**Algorithm PT (for B-MAC Networks)**

| Initialize $\Sigma_{1:L}$ and $\hat{\Sigma}_{1:L}$ with zeros or other values such that $\Sigma_l, \hat{\Sigma}_l \succeq 0, \forall l$. |
|---|
| **While** not converge **do** |
| 1. Polite water-filling in the forward links |
| a. For $\forall l$, obtain $\Omega_l$ and $\hat{\Omega}_l$ from $\Sigma_{1:L}$ and $\hat{\Sigma}_{1:L}$ using (2) and (6) respectively. Perform thin SVD $\Omega_l^{-1/2}H_{l,l}\hat{\Omega}_l^{-1/2} = F_l\Delta_l G_l^\dagger$. |
| b. Let $\delta_{l,i}$ be the $i$th diagonal element of $\Delta_l^2$ and let $\rho_{l,i}$ be the norm square of the $i$th column of $\hat{\Omega}_l^{-1/2}G_l$. Obtain $D_l$ as $D_l = \text{diag}(d_{l,1}, ..., d_{l,N_l})$, where $\mu$ is chosen such that $\sum_{l=1}^{L} \sum_{i=1}^{N_l} \rho_{l,i}d_{l,i} = P_T$. |
| c. Update $\Sigma_l$'s as $\Sigma_l = \hat{\Omega}_l^{-1/2}G_lD_lG_l^\dagger\hat{\Omega}_l^{-1/2}$, $\forall l$. |
| 2. Covariance transformation in the reverse links |
| Obtain $\hat{\Sigma}_{1:L}$ from the covariance transformation of $\Sigma_{1:L}$, using (19). |
| **End** |

and the KKT conditions are

\[
\nabla \Sigma_l L = w_l H_{l,l}^\dagger \left( \Omega_l + H_{l,l}\Sigma_l H_{l,l}^\dagger \right)^{-1} H_{l,l} + \Theta_l - \mu I - \sum_{k \neq l} w_k \Phi_{k,l} H_{k,l}^\dagger \\
\times \left( \Omega_k^{-1} - \left( \Omega_k + H_{k,k}\Sigma_k H_{k,k}^\dagger \right)^{-1} \right) H_{k,l} = 0; \\
\sum_{l=1}^{L} \text{Tr} (\Sigma_l) = P_T; \\
\text{Tr} (\Sigma_l \Theta_l) = 0; \\
\mu \geq 0; \quad \Sigma_l, \Theta_l \succeq 0; (55)
\]

for all $l = 1, ..., L$. If Algorithm PT converges, the solution of the algorithm satisfies the above KKT conditions, and thus is a stationary point.

**Proof:** Let $\bar{\Sigma}_{1:L}$ and $\bar{\hat{\Sigma}}_{1:L}$ be the solution of Algorithm PT after convergence. It is clear that $\bar{\Sigma}_{1:L}$ satisfies the polite water-filling structure and the water-filling levels $\bar{v}_l$'s are proportional.
to the weights, i.e., \( \bar{\nu}_l = w_l/\bar{\mu} \). Furthermore, we have \( \bar{\Sigma}_l = \bar{\nu}_l \left( \bar{\Omega}_l^{-1} - \left( H_{l,l} \bar{\Sigma}_l H_{l,l}^\dagger + \bar{\Omega}_l \right)^{-1} \right) \) by Theorem 7. The rest of the proof is similar to the proof of the sufficiency part in Theorem 11. Choose the dual variable as \( \mu = \bar{\mu} \) and substitute \( \bar{\Sigma}_1 : L \) into condition \( \nabla_{\Sigma} L = 0 \) to obtain

\[
\bar{\Omega}_l = \bar{\nu}_l H_{l,l}^\dagger \left( \bar{\Omega}_l + H_{l,l} \bar{\Sigma}_l H_{l,l}^\dagger \right)^{-1} H_{l,l} + \frac{1}{\bar{\mu}} \Theta_l,
\]

which holds because of the polite water-filling structure of \( \bar{\Sigma}_1 : L \). It can be verified that \( \bar{\Sigma}_1 : L \) satisfies all other KKT conditions in (55) as well.

If the weighted sum-rate function in WSRMP is concave, the problem is convex and any stationary point is also a global maximum. In other cases when the weighted sum-rate function is not concave, we cannot guarantee that Algorithm PT converges to the global optimum since it may get stuck at some other stationary point. Nonetheless, Algorithm PT gives very good solution.

We prove for two simple examples that Algorithm PT converges monotonically to a stationary point. More discussion on the convergence is given in the Remark 10.

**Example 1:** Consider weighted sum-rate maximization for a 2-user SISO MAC with \( w_2 > w_1 \), where the user 1 is decoded first. The optimization problem is

\[
\max_{p_1, p_2} f \triangleq w_1 \log \left( \frac{1 + p_1 g_1 + p_2 g_2}{1 + p_2 g_2} \right) + w_2 \log \left( 1 + p_2 g_2 \right),
\]

s.t. \( p_1 + p_2 \leq P_T \),

where \( p_i \) is the transmit power and \( g_i \) is the channel gain. Algorithm PT updates the transmit powers as

\[
p_1' = \left( \nu w_1 - \frac{1 + p_2 g_2}{g_1} \right)^+,
p_2' = \left( \nu w_2 - \frac{1 + p_2 g_2}{1 + p_1 g_1 + p_2 g_2} \right)^+,
\]

and \( \nu \) is chosen such that \( p_1' + p_2' = P_T \). Suppose \( \frac{\partial f}{\partial p_1} > \frac{\partial f}{\partial p_2} \). Then \( \frac{1 + p_1 g_1 + p_2 g_2}{w_1 g_1} > \frac{1 + p_1 g_1 + p_2 g_2}{w_2 g_2} \). If \( \nu = \frac{1 + p_1 g_1 + p_2 g_2}{w_1 g_1} \), then \( p_1' = p_1 \) and \( p_2' < p_2 \). Therefore we should increase the water-filling level. Similarly if \( \nu = \frac{1 + p_1 g_1 + p_2 g_2}{w_2 g_2} \), then \( p_1' > p_1 \) and \( p_2' = p_2 \). Therefore the water-filling level must be decreased. Then we have \( \frac{1 + p_1 g_1 + p_2 g_2}{w_1 g_1} < \nu < \frac{1 + p_1 g_1 + p_2 g_2}{w_2 g_2} \). Because of \( \nu > \frac{1 + p_1 g_1 + p_2 g_2}{w_1 g_1} \), \( p_1 \) is increased and thus the transmit powers are updated according to the gradient direction. But if
we increase $p_1$ too much, the direction of the gradient may change, then the objective may still be decreased. However it follows from $\nu < \frac{1+p_1 g_1 + p_2 g_2}{w_2 g_2}$ that $\frac{\partial f}{\partial p_1} > \frac{\partial f}{\partial p_2}$ after the update. Hence the objective is strictly increased after each update until $\frac{\partial f}{\partial p_1} = \frac{\partial f}{\partial p_2}$, which is the stationary point of problem (57).

Similarly, one can prove the result for sum-rate maximization of a 2-user SISO Z channel.

**Algorithm PP:** We also designed Algorithm PP, standing for Polite-Polite, that uses polite water-filling for both the forward and reverse links. It is modified from Algorithm PT, where each iteration is a polite water-filling in the forward link and a covariance transformation in the reverse link. The algorithm is shown in Table III. It can be proved that Theorem 13 also holds for Algorithm PP following similar proof as that for Algorithm PT.

It is observed in the simulations that both Algorithm PT and Algorithm PP have faster convergence speed and higher accuracy than the algorithms not exploiting the optimal input structure. In most cases, Algorithm PT and PP are observed to have similar convergence speed. But for networks with strong interference loops such as strong interference channel, Algorithm PP is observed to have a better convergence behavior than Algorithm PT. Another advantage of Algorithm PP is that the polite water-filling procedure to obtain $\hat{\Sigma}_1: L$ has lower complexity than the covariance transformation in Algorithm PT, as will be discussed in Section IV-E.

**Remark 10:** A nontrivial future work is to find the convergence conditions of Algorithm PT or PP for iTree networks and general B-MAC networks. It has been observed that Algorithm PT and PP always converge monotonically in iTree networks with concave or non-concave objective functions, just like the provably monotonically convergent algorithms designed for iTree networks in [40]. It is possible that the structure of the iTree networks guarantees it, which suggests that iTree networks has more useful properties to be found. For general B-MAC networks, one possible solution may be extending the approach used in [33] for iterative selfish water-filling to polite water-filling. It is proved in [33] that under certain stringent conditions, the selfish water-filling is a contraction mapping, and thus the uniqueness of the Nash equilibrium and the convergence of the iterative selfish water-filling is guaranteed by the fixed point theory. The physical interpretation of these conditions is that the interference among the links is sufficiently small. Because in the polite water-filling, the interference to other links is well taken care of by the pre-whitening of the channel using $\hat{\Omega}_t^{-1/2}$, we conjecture that the convergence conditions for Algorithm PT or PP will be much looser than iterative selfish water-filling. This conjecture is
**Algorithm PP (for B-MAC Networks)**

| Table III |
|-----------|
| Initialize $\hat{\Sigma}_l$ and $\Omega_l$'s such that $\sum_{l=1}^L \text{Tr} \left( \hat{\Sigma}_l \right) \leq P_T$. $\hat{\Sigma}_l \succeq 0, \forall l$ and $\Omega_l = I, \forall l$. |

**While** not converge **do**

1. Polite water-filling in the forward links
   
   a. For $\forall l$, obtain $\hat{\Omega}_l$ from $\hat{\Sigma}_1:L$ using (6).

   Perform thin SVD $\hat{\Omega}_l^{-1/2} H_l, \hat{\Omega}_l^{-1/2} = F_l \Delta_l G_l^\dagger$.

   b. Let $\delta_{l,i}$ be the $i^{th}$ diagonal element of $\Delta_l^2$ and let $\rho_{l,i}$ be the norm square of the $i^{th}$ column of $\hat{\Omega}_l^{-1/2} G_l$. Obtain $D_l$ as

   $$D_l = \text{diag} \left( d_{l,1}, ..., d_{l,N_l} \right)$$

   $$d_{l,i} = \left( \frac{w_l \mu - \frac{1}{\delta_{l,i}}}{\delta_{l,i}} \right)^+, i = 1, ..., N_l,$$

   where $\mu$ is chosen such that

   $$\sum_{l=1}^L \sum_{i=1}^{N_l} \rho_{l,i} d_{l,i} = P_T.$$

   c. Update $\hat{\Sigma}_l$'s as

   $$\hat{\Sigma}_l = \hat{\Omega}_l^{-1/2} F_l D_l F_l^\dagger \hat{\Omega}_l^{-1/2}, \forall l.$$  

2. Polite water-filling in the reverse links

   a. For $\forall l$, obtain $\hat{\Omega}_l$ from $\hat{\Sigma}_1:L$ using (2).

   Perform thin SVD $\hat{\Omega}_l^{-1/2} H_l, \hat{\Omega}_l^{-1/2} = F_l \Delta_l G_l^\dagger$.

   b. Let $\delta_{l,i}$ be the $i^{th}$ diagonal element of $\Delta_l^2$ and let $\hat{\rho}_{l,i}$ be the norm square of the $i^{th}$ column of $\hat{\Omega}_l^{-1/2} F_l$. Obtain $D_l$ as

   $$D_l = \text{diag} \left( d_{l,1}, ..., d_{l,N_l} \right)$$

   $$d_{l,i} = \left( \frac{w_l \mu - \frac{1}{\delta_{l,i}}}{\delta_{l,i}} \right)^+, i = 1, ..., N_l,$$

   where $\mu$ is chosen such that

   $$\sum_{l=1}^L \sum_{i=1}^{N_l} \hat{\rho}_{l,i} d_{l,i} = P_T.$$

   c. Update $\hat{\Sigma}_l$'s as

   $$\hat{\Sigma}_l = \Omega_l^{-1/2} F_l D_l F_l^\dagger \Omega_l^{-1/2}, \forall l.$$  

End

verified by the simulations in Section V where Algorithm PP is observed to converge for almost all general B-MAC networks simulated. Algorithm PT also converges for most general B-MAC networks simulated except for the interference channel with very strong interference. But even for this case, Algorithm PT still converges after a few trials, e.g., three trials, to select a good initial point. Another approach to convergence proof may be interpreting the polite water-filling as message passing in a factor graph and derive the conditions of convergence on the graph.

**Remark 11:** The following is the connection to another iterative water-filling algorithm pro-
posed in [35] Section IV] for parallel SISO interference networks. It is derived by directly solving the KKT conditions of the weighted-sum rate maximization problem. As expected, the result also has a term \( t_k(n) \) related to the interference to other nodes. In the SISO case, the matrix \( \hat{\Omega}_l \) equals to \( 1 + \frac{t_k(n)}{\lambda_k} \) in [35] only for stationary points. Algorithm PT or PP can be used there to reduce the inner and outer iterations to one iteration, reducing the complexity. On the other hand, we can design an algorithm by directly solving the KKT conditions for the MIMO case and by using the concept of polite water-filling. It results in an algorithm that has similar performance to Algorithm PT but does not take the advantage of the duality. In the algorithm, we replace the matrix \( \hat{\Omega}_l \) by the term

\[
I + \sum_{k \neq l} \frac{\mu_k}{\mu} \Phi_{k,l} \Sigma_{k,l}^\dagger \times \left( \Omega_k^{-1} - \left( \Omega_k + \Sigma_k \Sigma_{k,k} \right)^{-1} \right) \Sigma_{k,l}^\dagger \tag{58}
\]

in (55), avoiding the covariance transformation in Algorithm PT and reverse link polite water-filling in Algorithm PP. But as a result, the equivalent channel becomes a function of \( \mu \) and in order to perform the polite water-filling, SVD has to be repeatedly done while searching for \( \mu \) to satisfy the power constraint, increasing the overall complexity.

E. Complexity Analysis

We first give a brief analysis to show the complexity order per iteration of the proposed algorithms. The main computation complexity lies in the SVD and matrix inverse operations in the polite water-filling procedure and in the calculation of the MMSE-SIC receivers for the covariance transformation. Note that both SVD and matrix inverse are performed over the matrices whose dimensions are equal to the number of the transmit and receive antennas at each node and are not increased with the number of transmission links \( L \). This, however, is not true for generic optimization methods which do not take advantage of the polite water-filling structure of the optimal solution. Since the operations of SVD and matrix inverse have similar complexity, we use the order of the total number of SVD and matrix inverse operations to measure the complexity. Note that we need to calculate \( M_l \) MMSE-SIC receive vectors for the \( l^{th} \) link in the covariance transformation. However, as pointed out in [55], if the precoding matrix \( T_l \) is chosen to decorrelate the streams of link \( l \), we only need to perform one matrix inverse and one SVD to find such precoding matrices and the corresponding MMSE-SIC receive vectors.
• We give the complexity order of each iteration of Algorithm P. In step 1, we need to calculate $\hat{\Sigma}_{1:L-1}$ by the covariance transformation, which has a complexity order of $O(L)$. In step 2, the complexity order of the polite water-filling procedure to obtain $\hat{\Sigma}_{i,i}$'s is also $O(L)$. And in step 3, the covariance transformation to calculate $\Sigma_{i,1:L-1}$, $i = 1, ..., L$ has a complexity order of $O(L^2)$. Therefore, the total complexity order is $O(L^2)$.

• Algorithm P1 only adds a linear search compared to Algorithm P. Therefore, it has the same complexity order as Algorithm P.

• The complexity order of Algorithm PT depends on whether the network under optimization is iTree network or not. For iTree network, it is clear that the covariance transformation to obtain $\hat{\Sigma}_{1:L}$ in step 1 and the polite water-filling to obtain $\Sigma_{i}$'s in step 2 have a complexity order of $O(L)$ because the reverse link power in the covariance transformation can be calculated one by one. For other networks, to calculate the reverse transmit powers in the covariance transformation, we need to solve a $\sum_{l=1}^{L} M_l$-dimensional linear equation as in (18), whose complexity depends on the density and structure of the cross-talk matrix $\Psi(T,R)$. In the worst case, the complexity order is $O(L^3)$. In other cases such as with triangular or sparse $\Psi(T,R)$, the complexity is much lower. Fortunately, there are many fast algorithms to solve the linear equations even for relatively large dimensions, and in practice, $\Psi(T,R)$ is usually sparse for a large wireless network because of path loss or interference suppression techniques.

• The complexity order of Algorithm PP is $O(L)$ regardless of whether the network is an iTree network or not since the reverse input covariance matrices are also obtained by polite water-filling.

In practice, Algorithm PT is the choice for iTree networks because its performance is similar to Algorithm P1 but the complexity is much lower, while Algorithm PP, which has even lower complexity, is a better choice for general B-MAC networks because it also has fast convergence speed and the convergence behavior is less sensitive to the interference loops and interference strength. Therefore, Algorithm P and P1 are more of theoretic value. Although Algorithm P and P1 have higher complexity than the $O(L)$ algorithms in [9], [10], [20], [21], they are still much simpler than the standard interior point methods of complexity order of $O(L^3)$ [66] and can be used in more general cases compared to those for MIMO MAC/BC.
Second, we discuss the number of iterations needed or in other words, the convergence speed. The number of neighbors of a node in the network is expected to have little influence on the convergence speed of the designed algorithms because at each link, the forward link interference-plus-noise covariance matrix and its reverse link dual summarize all interference no matter how many neighbors of the link are there. The convergence speed is expected to depend on the girth of the network because each iteration propagates the influence of the transmission scheme of a transmitter to its closest neighbors. But in a wireless network, large girth with uniformly strong channel gains has small probability to happen. In addition, the optimal polite water-filling structure is forced on to the solution at each iteration. Therefore, the number of iterations needed is small. In simulations results, we observe that 1.5 iterations achieves much of the rate and 2.5 iterations achieves most of the rate of 50 user multiaccess channels and two or three user interference channels, where half an iteration refers to an update in the forward link or reverse link. Define the accuracy as the difference between the performance of a certain number of iterations and the maximum, if known. Accuracy versus iteration numbers gives a closer look at the asymptotic convergence behavior. As expected, the algorithms in this paper have superior accuracy because of the optimality of the polite water-filling structure.

In summary, Algorithm PT and PP are expected and have been verified by simulation to have superior performance, complexity per iteration, convergence speed, and accuracy than other algorithms that do not take the advantage of the optimal input structure.

V. SIMULATION RESULTS

In this section, simulations are used to verify the performance of the proposed algorithms. Block fading channel is assumed and the channel matrices are independently generated by

$$H_{l,k} = \sqrt{g_{l,k}} H_{l,k}^{(W)}, \forall k, l,$$

where $H_{l,k}^{(W)}$ has zero-mean i.i.d. Gaussian entries with unit variance and $g_{l,k}, \forall k, l$ is a constant. In most cases, we let $g_{l,k} = 0$ dB, $\forall k, l$ except for Fig. 10 and Fig. 11. For weighted sum-rate, the weights are randomly chosen between 0.8 and 1.2. In Fig. 3-5, Fig. 10 and Fig. 11, each simulation is averaged over 100 randomly generated channel realizations to show that the performance difference is not an accident. In all other figures, the simulation is performed over a single channel realization to show the details of the asymptotic accuracy or the effect of different initial points. In Fig. 9, Fig. 11 and Fig. 12 where we run Algorithm PT or PP with several initial points, the first initial point is chosen to be zero and other initial
Figure 3. Convergence speed comparison of the sum-rate for a 10-user MAC.

Figure 4. Convergence speed comparison of the sum-rate for a 50-user MAC.
Figure 5. Convergence speed comparison of the weighted sum-rate for a 10/50-user MAC.

Figure 6. Sum-rate accuracy of the algorithms for a 50-user MAC.
Figure 7. Weighted sum-rate accuracy of the algorithms for a 50-user MAC.

Figure 8. Convergence speed of the algorithms for a Z channel and an interference channel.

points are randomly generated. In all other cases, we use zero initial points for all algorithms.

To make fair comparison of the convergence speed by taking into account the complexity, we use effective iterations. Each iteration of the Algorithm PT or PP consists of an update in the forward links and an update in the reverse links. The rates are output at 0.5, 1.5, 2.5, etc., effective iterations. For Algorithm P1, the $i^{th}$ iteration is also the $i^{th}$ effective iteration because
each iteration consists of the update in the reverse and then the update in the forward links. For the other algorithms that we compared with in the literature, the $i^{th}$ iteration is counted as the $(i/2)^{th}$ effective iteration because they only have update in the forward links. This is a conservative approach because some of these other algorithms' complexity of each iteration is higher than that of one half iteration of Algorithm PT or PP. Nevertheless, Algorithms PT and
PP still show clear advantages.

We first demonstrate the superior convergence speed of Algorithms P1, PT and PP. Fig. 11 plot the (weighted) sum-rate versus effective iteration number for a MIMO MAC with 2 transmit antennas at each user and 8 receive antennas. The scenarios of sum-rate, weighted sum-rate, 10 users, and 50 users are simulated. The results are compared with those of the steepest ascent

Figure 11. Convergence behavior comparison with selfish water-filling for a 3-user interference channel.

Figure 12. Achieved rate regions of a two-user interference channel.
algorithm in [20] with Matlab code from [69], the ‘Original Algorithm’ and ‘Algorithm 2’ of JRVG in [9] with Matlab code from [70], and the dual decomposition algorithm in [10]. The ‘Original Algorithm’ in [9] is obtained by enforcing the water-filling structure for the sum-rate in [8] at each iteration. It is observed in [9] that the sum-rate of the ‘Original Algorithm’ grows fast initially and then may decrease and/or oscillate. Therefore, a hybrid algorithm called ‘Original + Algorithm 2’ is considered the best in [9], where the ‘Original Algorithm’ is performed for the first five iterations, and then ‘Algorithm 2’ in [9] is used for all the subsequent iterations. It is observed that Algorithms P1, PT and PP have similar initial convergence speed as that of ‘Original + Algorithm 2’ and outperform all other algorithms. As expected, the convergence speed of Algorithms P1, PT and PP appears to be independent of the number of users because the girth of the interference graph of the MAC is one, regardless of the number of users. In Fig. 6 and Fig. 7 we compare the asymptotic convergence speed, or accuracy, defined as the errors from the maximums of the sum-rate and the weighted sum-rate respectively. Algorithm P1, PT and PP have the highest accuracy than other algorithms except for the dual decomposition algorithm in [10]. This is not surprising because the dual decomposition algorithm is a water-filling directly on the objective function, sum-rate, while the polite water-filling is on each link. The errors accumulate when summing all the links’ rates together. In summary, for sum-rate, the best algorithm will be ‘Original Algorithm + Algorithm PT or PP’ for a compromise of initial convergence speed and accuracy. For weighted sum-rate, the best is Algorithm PT or PP.

In Fig. 8 we plot the sum-rate for a Z channel with concave sum-rate function and the weighted sum-rate for a 3-user interference channel. In both channels, each node has 4 antennas. For the Z channel, it can be observed that Algorithms P1 and PT have faster convergence speed than Algorithm P. For the 3-user interference channel where the problem may be non-convex, Algorithm PT and PP have the same performance and converge fast and monotonically to a stationary point.

In Fig. 9 we show the convergence speed of Algorithm PP for a nontrivial example of B-MAC networks, the two-user X channel, with 3 antennas at each transmitter and 3 antennas at each receiver. We plot the weighted sum-rate versus the effective iteration number with three different initial points for the same channel realization. For all initial points, Algorithm PP converges quickly to some stationary points. However, the convergence speed is observed to depend on the initial point. And due to the non-convexity of the problem, Algorithm PP with
different initial points may converge to different stationary points.

As expected, the polite water-filling is superior to the selfish water-filling. The selfish water-filling can be obtained by fixing the $\hat{\Omega}_l$'s in Algorithm PT or PP as identity matrices. Selfish water-filling is expected to converge in iTree networks but not in networks with strong interference loops as shown in Fig. 10 for the iTree network of Fig. 2 and in Fig. 11 for a 3-user interference channel. Each node is assumed to have four antennas. In the upper sub-plot of Fig. 10 we consider the moderate interference case, where we set $g_{l,k} = 0 \text{dB}, \forall k, l$. In the lower sub-plot of Fig. 10 we consider strong interference case, where we set $g_{l,3} = 10 \text{dB}, l = 1, 2$ for the interfering links, and $g_{l,k} = 0 \text{dB}$ for other $k, l$'s. It can be observed that the three algorithms converge in both cases. But Algorithm PT and PP achieve a much higher sum-rate than the selfish water-filling. In the upper sub-plot of Fig. 11 we set $g_{l,k} = 0 \text{dB}, \forall k, l$, and similar results as in Fig. 10 can be observed. In the lower sub-plot, we consider a strong interference channel, where we set $g_{l,k} = 10 \text{dB}, \forall k \neq l$, and $g_{l,k} = 0 \text{dB}, \forall k = l$. In this case, the selfish water-filling based algorithm no longer converges. Algorithm PT converges with a good initial point found by a typically three trials. But even with a non-converging initial point, Algorithm PT still achieves a higher sum-rate than the selfish water-filling. Algorithm PP converges for almost all channel realizations, which demonstrates its insensitivity to the interference loops and strength.

Algorithms PT and PP can be used to find the approximate boundary of the convex hull of the achievable rate region of MIMO B-MAC networks. In Fig. 12 we plot the approximate boundary achieved by Algorithm PP for a two-user interference channel with 3 antennas at each transmitter and 4 antennas at each receiver. The sum power constraint is $P_T = 10 \text{dB}$. The weights are $w_1 = \mu$ and $w_2 = 1 - \mu$, with $\mu$ varying from 0.01 to 0.99 with step 0.01. The boundary (dot line) obtained from a single initial point for Algorithm PP is close to the pseudo global optimum (solid line), which is the best solution of Algorithm PT with many randomly generated initial points. The result demonstrates that Algorithm PP can find good solutions even with single initial point. Not showing is that Algorithm PT with a single initial point also achieves near pseudo optimum boundary.

VI. CONCLUSIONS AND FUTURE WORK

This paper extends BC-MAC rate duality to the MIMO one-hop interference networks named B-MAC networks with Gaussian input and any valid coupling matrix. The main contribution is
the discovery that the Pareto optimal input has a polite water-filling structure, which optimally balances between reducing interference to others and maximizing a link’s own rate. It provides insight into understanding interference in networks and an elegant method to decompose a network to multiple equivalent single-user links. It can be employed to design/improve most network optimization algorithms. As an example, low complexity weighted sum-rate maximization algorithms are designed and demonstrated to have superior convergence speed and accuracy. A sub-class of the B-MAC networks, the interference tree (iTree) networks, is identified, for which the optimality of the algorithm is discussed. iTree networks appears to be a natural extension of broadcast and multiaccess networks and possesses many desirable properties.

The results in this paper is a stepping stone for solving many interesting problems. Some future work is listed below.

- **Extension to Han-Kobayashi Transmission Scheme:** Han-Kobayashi scheme cancels more interference and is especially beneficial when the interfering channel gain is large. But its optimization is still an open problem. As discussed in Remark 7 of Section III-B, the Lagrangian interpretation of the reverse link interference-plus-noise covariance matrix for the polite water-filling makes it possible to extend the duality and polite water-filling to Han-Kobayashi transmission scheme so as to help understand the optimal input structure and design low complexity optimization algorithms. The approach in this paper may also be useful in multi-hop networks, relay channels, and ad-hoc networks.

- **General Linear Constraints, Cognitive Radio, and Other Optimization Problems:** Polite water-filling is beneficial to most B-MAC network optimization problems. For example, the extension to a linear constraint in Section III-C means that the duality and polite water-filling can be powerful tools to optimize networks with general linear constraints, such as per antenna power constraints and interference constraints in cognitive radio. In these problems, the single linear constraint in this paper becomes a weighted sum of multiple linear constraints where the weights are Lagrange multipliers. Searching these Lagrange multipliers together with the primary variables is an advantage of the efficient interior point algorithms in [56], [57]. The techniques in this paper can lead to better iterative algorithms by taking advantage of the structure of the problems: Polite water-filling can solve the problem efficiently to obtain $Q_l$’s for fixed Lagrange multipliers, where the covariances $\hat{\Omega}_l$’s are functions of the Lagrange multipliers. Then, at each iteration and for fixed $Q_l$’s,
we may update the Lagrange multipliers by forcing $\Sigma_l = \hat{\Omega}_l^{-1/2} Q_l \hat{\Omega}_l^{-1/2}$ to satisfy the linear constraints so that both the optimal primary and dual variables can be found jointly. A better approach is to introduce virtual nodes for the constraints and take advantage of the polite water-filling and duality for the larger network where those Lagrange multipliers become some of the reverse link powers. Other optimization problem examples include ones with quality of service requirement, such as minimizing power under rate constraints and maximizing the minimum of weighted rates under a sum power constraint in our recent work [40].

- **Distributed Optimization and Finite Rate Message Passing:** This paper studies centralized optimization with global channel state information and a sum power constraint. In practice and for large networks, we have to design distributed/game-theoretic optimization algorithms with partial channel state information under individual or group power constraints [39], [40], [71]. The insight from polite water-filling structure turns out to be very useful as it turns the problem into single-user optimization problems under the influence of interference from and to others as summarized by the covariance matrices $\Omega_l$ and $\hat{\Omega}_l$. Partial or full knowledge of these covariance matrices can be obtained from reverse link transmission or pilot training in time division duplex (TDD) systems or from message passing among the nodes in frequency division duplex (FDD) systems [72]. The observation that very few iterations, usually two to three, suffices for the algorithms to achieve most of the gain makes the message passing approach meaningful. In practice, the message passing is further limited to finite rate. The single-user view of the polite water-filling structure makes it convenient to extend the results of the single-user finite rate feedback in [73] to B-MAC networks.

- **Extension to Fading Channels, OFDM Systems, and Adaptive Transmission:** Fixed channel matrices are considered in this paper. It is straightforward to extend the results to time varying channels like fading channels and OFDM system by considering parallel B-MAC networks. Then, the polite water-filling will be across users, space (antennas), time, and frequency. The optimization of encoding/decoding order in general is an open problem. Progress may be first made for the iTreed networks. In time varying channels, the insight of viewing reverse link power as Lagrange multipliers can be employed to design adaptive transmission algorithms where the Lagrange multipliers are adaptively adjusted. In fact, the iterative algorithms in this paper can be used with little modification by, e.g., updating to
the latest channel matrices whenever they are available.

- **iTree Networks**: A fruitful approach to the open problem of network information theory is to study various special cases, such as deterministic channels [11], [61] and degree of freedom [5], [11], [62], in order to gain insight into the problem. iTree networks appears to be a useful sub-class of networks to study. Because there is no interference loops, the optimization of a link will affect some other nodes, but which in turn will not change the interference to this link. Therefore, it is possible to say more about the optimality of the transmission and make more progress on network information theory for iTree networks. The tree structure also makes it easier to analyze the distributed optimization by iterative message passing [39], just like how the tree structure in LDPC decoding graph facilitates its analysis [74]. In fact, a factor graph interpretation of the polite water-filling may reveal some more insight. Because of mutual interference in wireless channels, there are likely many loops in the interference graph defined in Section IV-B. Interference cancellation techniques like the Han-Kobayashi scheme [2] can be viewed as a method to break the strong loops and make the graph more like a tree, making iTree networks relevant to practice and more meaningful to study.

**APPENDIX**

A. **Proof of Theorem 1**

Assume \( r \) is a boundary point and \( \Sigma'_{1:L} \) achieves a rate point \( r' \geq r \), and \( \exists k, \text{ s.t. } r'_{k} > r_{k} \). One can find an \( 0 < \alpha < 1 \) such that \( \mathcal{I}_{l} \left( (\Sigma'_{1}, ..., \alpha \Sigma'_{k}, ..., \Sigma'_{L}), \Phi \right) \geq r_{l}, \forall l \). Then, the extra power \( \text{Tr} \left( \Sigma'_{k} \right) - \text{Tr} \left( \alpha \Sigma'_{k} \right) \) can be used to improve all non-zero rate users’ rates over \( r \) using \( \beta \left( \Sigma'_{1}, ..., \alpha \Sigma'_{k}, ..., \Sigma'_{L} \right) \) as the input covariance matrices, where \( \beta > 1 \) is chosen such that \( \sum_{l \neq k} \text{Tr} \left( \beta \Sigma'_{l} \right) + \text{Tr} \left( \beta \alpha \Sigma'_{k} \right) = P_{T} \). By the definition that the region is a union of sets \( \{ x : 0 \leq x_{l} \leq \mathcal{I}_{l} \left( \Sigma'_{1:L}, \Phi \right), 1 \leq l \leq L \} \), \( r \) must not be a boundary point, which contradicts with the assumption. Therefore, the statement in Theorem 1 must be true.

B. **Proof of Theorem 4**

Without loss of generality, we only prove Theorem 4 for link 1. We assume \( M_{l} = M, l = 1, ..., L \) for simplicity of notations. The proof is obtained by solving the KKT conditions of
another sum power minimization problem as will be formulated later in (59), where the minimization is over \( \Sigma_1 \) and the transmit powers of other links \( p_{-1} = [p_{2,T}^T, \ldots, p_{L,T}^T]^T \in \mathbb{R}_{++}^{(L-1)M \times 1} \) with the transmit and receive vectors for other links fixed as \{\tilde{t}_{k,m}, \tilde{r}_{k,m}, k \neq 1\}. We first give Lemma 11 which states that there exist dual variables satisfying the KKT conditions of problem (59). Then we show that the dual variables in Lemma 11 are the optimal dual variables of problem (22) and they must be uniquely given by Theorem 4. Finally, we prove Lemma 11 by the enhanced Fritz John necessary conditions [75, Sec. 5.2].

First we define the notations used in the proof. Define four submatrices of the cross-talk matrix \( \tilde{\Psi} = \Psi (\tilde{T}, \tilde{R}) \) in (12) as

\[
\tilde{\Psi} = \begin{bmatrix}
\tilde{\Psi}_1 \in \mathbb{R}^{M \times M}_{+} & \tilde{\Psi}_{1,-1} \in \mathbb{R}^{M \times (L-1)M}_{+} \\
\tilde{\Psi}_{-1,1} \in \mathbb{R}^{(L-1)M \times M}_{+} & \tilde{\Psi}_{-1,-1} \in \mathbb{R}^{(L-1)M \times (L-1)M}_{+}
\end{bmatrix}
\]

where \( \tilde{T} = [\tilde{t}_{l,m}]_{m=1,\ldots,M, l=1,\ldots,L} \) and \( \tilde{R} = [\tilde{r}_{l,m}]_{m=1,\ldots,M, l=1,\ldots,L} \) are the Pareto optimal transmit and receive vectors. Let \( \tilde{D}_1 \in \mathbb{R}^{M \times M}_{+} \) and \( \tilde{D}_{-1} \in \mathbb{R}^{(L-1)M \times (L-1)M}_{+} \) respectively be the submatrices at the upper left and lower right corner of \( D (\tilde{T}, \tilde{R}, \gamma^0) \) in (17), where \( \gamma^0 \) is set as the SINR values achieved by \( \{\tilde{T}, \tilde{R}, \tilde{p}\} \), i.e., \( \gamma^0_{l,m} = \tilde{\gamma}_{l,m}, 1 \leq m \leq M, 1 \leq l \leq L \). Define

\[
B_{-1} = \tilde{D}_{-1}^{-1} - \tilde{\Psi}_{-1}.
\]

For \( k = 2, \ldots, L \), let \( b_{k,m}^{k,m} \in \mathbb{C}^{1 \times (L-1)M} \) be the \( ((k-2)M + m)^{th} \) row of \( B_{-1} \).

Then we formulate another optimization problem in (59) and show that the dual variables satisfying the KKT conditions of problem (59) is also the optimal dual variables of problem (22). Suppose for other links, the transmit and receive vectors are fixed as \( \tilde{T} \) and \( \tilde{R} \) respectively. In problem (22), the power allocation for other links is also fixed as \( \tilde{p}_{-1} = [\tilde{p}_{2,T}^T, \ldots, \tilde{p}_{L,T}^T]^T \). Now consider the following sum power minimization problem where the power allocation for other links is not fixed.

\[
\begin{aligned}
\min_{\Sigma_1 \succeq 0, p_{-1} \geq 0} & \quad \text{Tr} (\Sigma_1) + 1^T p_{-1} \\
\text{s.t.} & \quad \log \left| I + H_{1,1} \Sigma_1 H_{1,1}^\dagger \Omega_1^{-1} (p_{-1}) \right| \geq \tilde{\gamma}_1 \\
& \quad \text{Tr} \left( \Sigma_1 A_{k,m}^{(1)} \right) - b_{k,m} p_{-1} \leq -1, \quad k = 2, \ldots, L, \quad m = 1, \ldots, M,
\end{aligned}
\]
where
\[
\Omega_1(p_{-1}) = I + \sum_{k=2}^{L} \Phi_{1,k} H_{1,k} \sum_{m=1}^{M} p_{k,m} \tilde{t}_{k,m} \tilde{t}_{k,m}^* H_{1,k}^* ,
\]
is the interference-plus-noise covariance matrix of link 1. Note that the constraints in (60) imply the individual rate constraints: \(\sum_{m=1}^{M} \log (1 + \gamma_{k,m}) \geq \tilde{\gamma}_k = \sum_{m=1}^{M} \log (1 + \tilde{\gamma}_{k,m})\), \(k = 2, \ldots, L\), where
\[
\gamma_{k,m} = \frac{p_{k,m} \tilde{r}_{k,m}^* H_{k,k} \tilde{t}_{k,m}^2}{1 + \text{Tr} \left( \Sigma_1 A_{k,m}^{(1)} \right) + \sum_{l=2}^{L} \sum_{n=1}^{M} p_{l,n} \tilde{\Phi}_{k,m}^{l,n},}
\]
is the SINR for the \(m\)th stream of link \(k\) achieved by \(\Sigma_1\), \(p_{-1}\), and \(\{\tilde{t}_{k,m}, \tilde{r}_{k,m}, k \neq 1\}\). This is because from the definition of \(b_{k,m}^{-1}\), the constraint \(\text{Tr} \left( \Sigma_1 A_{k,m}^{(1)} - b_{k,m}^{-1} p_{-1} \right) \leq -1\) is equivalent to \(\gamma_{k,m} \geq \tilde{\gamma}_{k,m}\). It can be proved by contradiction that the Pareto optimal input \(\tilde{\Sigma}_1\) and \(\tilde{p}_{-1}\) is an optimal solution of (59). The Lagrangian of problem (59) is given by
\[
L(\lambda, \nu_1, \Theta, \Sigma_1, p_{-1}) = \text{Tr} \left( \Sigma_1 (A_{1,1}(\lambda) - \Theta) \right) + \nu_1 \tilde{\gamma}_1
\]
\[
- \nu_1 \log \left| I + H_{1,1} \Sigma_1 H_{1,1}^* \Omega_1^{-1}(p_{-1}) \right|
\]
\[
+ 1^T p_{-1} - \lambda^T B_{-1} p_{-1} + \lambda^T 1,
\]
where the dual variables \(\nu_1 \in \mathbb{R}_+\) and \(\lambda = [\lambda_{2,1}, \ldots, \lambda_{2,M}, \ldots, \lambda_{L,1}, \ldots, \lambda_{L,M}] \in \mathbb{R}^{(L-1)M \times 1}\), \(\Theta\) is the matrix dual variables associated with the positive semidefiniteness constraint on \(\Sigma_1\). Because this problem may not be convex, later, we will prove the following lemma which states that there exist dual variables satisfying the KKT conditions of problem (59).

**Lemma 11:** The KKT conditions are necessary for \(\tilde{\Sigma}_1, \tilde{p}_{-1}\) to be optimal for problem (59), i.e., there exist dual variables \(\tilde{\lambda} = [\tilde{\lambda}_{k,m}]_{k \neq 1} \in \mathbb{R}^{(L-1)M \times 1}\), \(\tilde{\nu}_1 \geq 0\) and \(\tilde{\Theta} \succeq 0\), \(\text{Tr} \left( \tilde{\Sigma}_1 \tilde{\Theta} \right) = 0\) such that
\[
\tilde{\nabla}_{\Sigma_1} L \triangleq \nabla_{\Sigma_1} L(\tilde{\lambda}, \tilde{\nu}_1, \tilde{\Theta}, \Sigma_1, p_{-1}) \bigg|_{\Sigma_1 = \Sigma_1} = 0,
\]
\[
\tilde{\nabla}_{p_{-1}} L \triangleq \nabla_{p_{-1}} L(\tilde{\lambda}, \tilde{\nu}_1, \tilde{\Theta}, \tilde{\Sigma}_1, p_{-1}) \bigg|_{p_{-1} = \tilde{p}_{-1}} = 0.
\]
Note that \(\Omega_1(\tilde{p}_{-1}) = \tilde{\Omega}_1\). Then (63) can be expressed as
\[
A_1(\tilde{\lambda}) - \tilde{\nu}_1 H_{1,1}^* \left( \tilde{\Omega}_1 + H_{1,1} \Sigma_1 H_{1,1}^* \right)^{-1} H_{1,1} - \tilde{\Theta} = 0.
\]
Because (65) is also the same as the equation that the derivative of Lagrangian of the problem (22) equals to zero, and \( \tilde{\Sigma}_1 \) is the optimal solution of problem (22), \( \tilde{\lambda}, \tilde{\nu}_1, \tilde{\Theta}, \tilde{\Sigma}_1 \) also satisfy the KKT conditions of problem (22). Since problem (22) is convex, \( \tilde{\lambda}, \tilde{\nu}_1, \tilde{\Theta} \) are the optimal dual variables of problem (22).

We show below that \( \tilde{\lambda} \) and \( \tilde{\nu}_1 \) in Lemma 11 is uniquely given by (26) and (27). Combining this fact and Lemma 11, we conclude that \( \tilde{\lambda} \) and \( \tilde{\nu}_1 \) given by (26) and (27) are the optimal dual variables of problem (22). This will complete the proof for Theorem 4. It is difficult to directly solve \( \tilde{\lambda} \) and \( \tilde{\nu}_1 \) from (63) and (64). To simplify the problem, without loss of generality, we restrict \( \Sigma_1 \) to be \( \Sigma_1 = \tilde{T}_1 \text{diag}(p_1) \tilde{T}_1^\dagger \) in the Lagrangian. For convenience, define

\[
\begin{align*}
\bar{r}_1^s(p) &= \sum_{m=1}^M \log \left( 1 + \gamma_{1,m}(\tilde{T}, \tilde{R}, p) \right),
\end{align*}
\]

where \( \gamma_{1,m}(\tilde{T}, \tilde{R}, p) \) is the SINR of the \( m \)th stream of link 1 achieved by \( \{\tilde{T}, \tilde{R}, p\} \) as in (13). Note that \( \log \left| I + H_{1,1} \Sigma_1 H_{1,1}^\dagger \Omega_1^{-1}(p) \right| = r_1^s(p) \), and \( \tilde{t}_1^\dagger_m \tilde{t}_1,m = 0 \) because \( \text{Tr}(\Sigma_1 \tilde{\Theta}) = 0 \) and \( \tilde{\Theta} \) is positive semidefinite. Then the Lagrangian (62) can be rewritten as a function of \( \tilde{\lambda}, \tilde{\nu}_1 \) and \( p = [p_1^T, p_1^T]^T \) as follow.

\[
\begin{align*}
\bar{L}(\tilde{\lambda}, \tilde{\nu}_1, p) &= \sum_{m=1}^M p_{1,m} \tilde{t}_1,m \Lambda_1^s \left( \tilde{\lambda} \right) \tilde{t}_1,m + 1^T p_{-1}^T \\
&\quad+ \tilde{\nu}_1 \left( \tilde{L}_1 - r_1^s(p) \right) - \lambda^T B_{-1} p_{-1} + \lambda^T \mathbf{1}.
\end{align*}
\]

Note that (63) and (64) imply

\[
\begin{align*}
\nabla_{p_1} \bar{L} \left( \tilde{\lambda}, \tilde{\nu}_1, [\bar{p}_1^T, \bar{p}_{-1}^T]^T \right) \bigg|_{p_1=\tilde{p}_1} &= 0, \\
\nabla_{p_{-1}} \bar{L} \left( \tilde{\lambda}, \tilde{\nu}_1, [\bar{p}_1^T, \bar{p}_{-1}^T]^T \right) \bigg|_{p_{-1}=\tilde{p}_{-1}} &= 0.
\end{align*}
\]

As will be shown below, the solution of the above equations is uniquely given by (26) and (27). Therefore, this unique solution must also be the solution of the more complex equations in (63) and (64).

For convenience, define \( \hat{q}_1 = [\hat{q}_{1,1}, \ldots, \hat{q}_{1,M}] \) and

\[
\hat{q}_{1,m} = \frac{\tilde{\nu}_1 \gamma_{1,m}}{\tilde{p}_{1,m} (1 + \tilde{\gamma}_{1,m}) G_{1,m}},
\]

where \( \gamma_{1,m} \) and \( \tilde{\gamma}_{1,m} \) are given by (66) and (67), and \( G_{1,m} \) is the MPP of the \( m \)th stream of link 1.
where \( \tilde{G}_{1,m} = \left| \tilde{r}_{1,m}^\dagger H_{1,1}\tilde{t}_{1,m} \right|^2 \). It can be derived from (66) and \( \tilde{\gamma}_{1,m} = \gamma_{1,m} \left( \tilde{T}, \tilde{R}, \tilde{p} \right) \) that

\[
\tilde{v}_1 \frac{\partial r_1^s (p)}{\partial p_{1,m}} \bigg|_{p=\tilde{p}} = \frac{\tilde{G}_{1,m} \tilde{q}_{1,m}}{\tilde{\gamma}_{1,m}} - \sum_{n=1}^{M} \tilde{\Psi}_{1,n}^1 \hat{q}_{1,n}, m = 1, \ldots, M, \\
\tilde{v}_1 \frac{\partial r_1^s (p)}{\partial p_{k,m}} \bigg|_{p=\tilde{p}} = - \sum_{n=1}^{M} \tilde{\Psi}_{k,n}^1 \hat{q}_{1,n}, m = 1, \ldots, M, k \neq 1.
\]

Note that \( \tilde{t}_{1,m}^\dagger A_1 \left( \tilde{\lambda} \right) \tilde{t}_{1,m} = \sum_{k=2}^{L} \sum_{n=1}^{M} \tilde{\lambda}_{k,n} \tilde{\Psi}_{1,n}^m + 1 \) and recall the definitions of the sub matrices of \( \tilde{\Psi} \) and \( D \left( \tilde{T}, \tilde{R}, \gamma^0 \right) \). Then equation (67) and (68) can be expressed as

\[
\tilde{D}_1^{-1} \hat{q}_1 - \tilde{\Psi}_{1,1}^T \tilde{\lambda} - \tilde{\Psi}_{1,-1}^T \hat{q}_1 = 1, \\
B_{1,1}^T \tilde{\lambda} - \tilde{\Psi}_{1,1}^T \hat{q}_1 = 1,
\]

respectively. Define \( \hat{q} = \left[ \hat{q}_1^T, \tilde{\lambda}^T \right]^T \). Then (69) and (70) together form the following linear equations

\[
\left( D^{-1} \left( \tilde{T}, \tilde{R}, \gamma^0 \right) - \Psi^T \left( \tilde{T}, \tilde{R} \right) \right) \hat{q} = 1.
\]

Since \( D^{-1} \left( \tilde{T}, \tilde{R}, \gamma^0 \right) - \Psi^T \left( \tilde{T}, \tilde{R} \right) \) is invertible \([50]\), \( \hat{q} \) is uniquely given by

\[
\hat{q} = \left( D^{-1} \left( \tilde{T}, \tilde{R}, \gamma^0 \right) - \Psi^T \left( \tilde{T}, \tilde{R} \right) \right)^{-1} 1 = \hat{q},
\]

which means \( \tilde{\lambda}, \tilde{v}_1 \) is uniquely given by (26) and (27).

The rest is to prove Lemma 11. First, we give some notations and definitions. Let \( \text{Vec} (M) \) be the column vector obtained by stacking the columns of the matrix \( M \) on top of each other. Define a convex set of \( \left( L^2_{T_1} + (L - 1) M \right) \)-dimensional complex vectors

\[
\mathcal{X} = \left\{ x = \left[ \text{Vec} \left( \Sigma_1 \right)^T, p_{-1}^T \right]^T : \Sigma_1 \succeq 0, p_{-1} \geq 0 \right\}.
\]

We also define the complex normal cone of a convex set as a natural generalization of the real one defined in \([75\), Sec. 5.2].

**Definition 9:** Denote \( \mathcal{X} \) a convex set of \( n \)-dimensional complex vectors. For \( x \in \mathcal{X} \),

\[
N_{\mathcal{X}} (x) = \left\{ z \in \mathbb{C}^n : \text{Re} \left[ z^\dagger (\tilde{x} - x) \right] \leq 0, \forall \tilde{x} \in \mathcal{X} \right\},
\]

is called the complex normal cone of \( \mathcal{X} \) at \( x \).

Then problem (59) can be rewritten as an optimization problem over the convex set \( \mathcal{X} \), where the cost function and constraint functions are real functions of a complex vector \( x \in \mathcal{X} \) and
an optimal solution is given by \( \tilde{x} = \left[ \text{Vec} \left( \tilde{\Sigma}_1 \right)^T, \tilde{p}_{-1}^T \right]^T \). The proof is based on the enhanced Fritz John necessary conditions in [75, Sec. 5.2], where they are expressed in real number. Because a real function of a complex vector \( x \) is equivalent to a real function of a real vector \( [\text{Re}(x)^T, \text{Im}(x)^T]^T \), for convenience, we will later rewrite the enhanced Fritz John necessary conditions in [75, Sec. 5.2] in complex form using the definition of the complex gradient of a real function in Section II. Define a function related to the Lagrangian function in (62)

\[
\tilde{J}(\eta, \lambda, \nu_1, x) \triangleq J(\eta, \lambda, \nu_1, \Sigma_1, p_{-1})
\]

\[
= \text{Tr} \left( \Sigma_1 \tilde{A} (\lambda, \eta) \right) + \eta^T p_{-1} - \lambda^T B_{-1} p_{-1} + \lambda^T 1
\]

\[
+ \nu_1 \left( \tilde{I}_1 - \log \left| I + H_{1,1} \Sigma_1 H_{1,1}^\dagger \Omega_{1,1}^{-1} (p_{-1}) \right| \right),
\]

where \( x = \left[ \text{Vec} \left( \Sigma_1 \right)^T, p_{-1}^T \right]^T \); and \( \tilde{A} (\lambda, \eta) = \sum_{k=2}^{L} \sum_{m=1}^{M} \lambda_k \eta_m A_{k,m}^{(1)} + \eta I \). Because the cost function and constraint functions in problem (59) are smooth over \( x \in \mathcal{X} \) and \( \mathcal{X} \) is a nonempty closed set, the optimal solution \( \tilde{x} \) must satisfy the enhanced Fritz John necessary conditions [75, Sec. 5.2]: there exist \( \eta, \lambda, \nu_1 \) such that

\[
-w \in N_{\mathcal{X}} (\tilde{x}),
\]

\[
\eta, \lambda, \nu_1 \geq 0 \text{ and not all } 0,
\]

where \( w \triangleq \nabla_x J(\eta, \lambda, \nu_1, \tilde{x}) \), and \( N_{\mathcal{X}} (\tilde{x}) \) is the complex normal cone of \( \mathcal{X} \) at \( \tilde{x} \). It can be shown that \( w^T (x - \tilde{x}), \forall x \in \mathcal{X} \) is real by the definitions of \( w \) and \( \mathcal{X} \). Then by Definition 9, the condition (72) can be expressed as

\[
w^T (x - \tilde{x}) \geq 0, \forall x \in \mathcal{X}.
\]

We first show that if \( \bar{\eta} > 0 \), the above enhanced Fritz John conditions are equivalent to the KKT conditions of problem (59). Then we show that \( \bar{\eta} = 0 \) is impossible. This concludes that \( \tilde{\Sigma}_1, \tilde{p}_{-1} \) must satisfy the KKT conditions in (63) and (64). For convenience, define

\[
\tilde{\nabla}_{\Sigma_1} J = \nabla_{\Sigma_1} J(\eta, \lambda, \nu_1, \tilde{\Sigma}_1, \tilde{p}_{-1}),
\]

\[
\tilde{\nabla}_{p_{-1}} J = \nabla_{p_{-1}} J(\eta, \lambda, \nu_1, \tilde{\Sigma}_1, \tilde{p}_{-1}).
\]

Then \( w = \nabla_x J |_{x=\tilde{x}} = \left[ \left( \text{Vec} \left( \tilde{\nabla}_{\Sigma_1} J \right) \right)^T, \left( \tilde{\nabla}_{p_{-1}} J \right)^T \right]^T \). For \( x = \left[ \text{Vec} \left( \Sigma_1 + Q \right)^T, \tilde{p}_{-1}^T \right]^T \in \mathcal{X} \), where \( Q \succeq 0 \) can be any positive semidefinite matrix, by (74), we have

\[
w^T (x - \tilde{x}) = \text{Tr} \left( Q \tilde{\nabla}_{\Sigma_1} J \right) \geq 0.
\]
In addition to that $\tilde{\nabla}_{\Sigma_i} J$ is Hermitian, it implies that $\tilde{\nabla}_{\Sigma_i} J$ is a positive semidefinite matrix. For $x = [0^T, \tilde{p}_{-1}^T]^T$, by (74), we have
\[
 w^\dagger (x - \tilde{x}) = -\text{Tr} \left( \Sigma_1 \tilde{\nabla}_{\Sigma_i} J \right) \geq 0.
\]
Since both $\tilde{\nabla}_{\Sigma_i} J$ and $\Sigma_i$ are positive semidefinite, we must have
\[
 \text{Tr} \left( \Sigma_1 \tilde{\nabla}_{\Sigma_i} J \right) = 0.
\]
For $x = \left[ \text{Vec} \left( \Sigma_1 \right)^T, p_{-1}^T \right]^T$, (74) becomes
\[
 \left( \tilde{\nabla}_{p_{-1}} J \right)^T (p_{-1} - \tilde{p}_{-1}) \geq 0.
\]
Since $\tilde{p}_{-1} > 0$ and $p_{-1}$ can be any nonnegative vector, we must have
\[
 \tilde{\nabla}_{p_{-1}} J = 0. \tag{77}
\]
If $\bar{\eta} > 0$, we can make $\bar{\lambda} = \lambda / \bar{\eta} \geq 0$, $\bar{\nu}_1 = \nu_1 / \bar{\eta} \geq 0$, $\bar{\Theta} = \tilde{\nabla}_{\Sigma_i} J / \bar{\eta} \geq 0$. Then, the KKT conditions in Lemma [11] are satisfied by observing
\[
 \tilde{\nabla}_{\Sigma_i} L = \tilde{\nabla}_{\Sigma_i} J / \bar{\eta} - \bar{\Theta} = 0,
\]
\[
 \tilde{\nabla}_{p_{-1}} L = \tilde{\nabla}_{p_{-1}} J / \bar{\eta} = 0,
\]
\[
 \text{Tr} \left( \Sigma_1 \bar{\Theta} \right) = \text{Tr} \left( \Sigma_1 \tilde{\nabla}_{\Sigma_i} J / \bar{\eta} \right) = 0.
\]
The rest is to prove that $\bar{\eta} = 0$ is impossible by contradiction. Suppose $\bar{\eta} = 0$. Define a function
\[
 J_0(\bar{\lambda}, \bar{\nu}_1, \Sigma_1, p_{-1}) = J(\bar{\eta}, \bar{\lambda}, \bar{\nu}_1, \Sigma_1, p_{-1}) - \text{Tr} \left( \Sigma_1 \tilde{\nabla}_{\Sigma_i} J \right).
\]
The it follows from (75) and (77) that
\[
 \nabla_{\Sigma_i} J_0(\bar{\lambda}, \bar{\nu}_1, \Sigma_1, \tilde{p}_{-1}) = 0, \tag{78}
\]
\[
 \nabla_{p_{-1}} J_0(\bar{\lambda}, \bar{\nu}_1, \Sigma_1, \tilde{p}_{-1}) = 0. \tag{79}
\]
Note that $\tilde{t}_{1,m}^\dagger \tilde{\nabla}_{\Sigma_i} J \tilde{t}_{1,m} = 0$ because $\text{Tr} \left( \Sigma_1 \tilde{\nabla}_{\Sigma_i} J \right) = 0$. If we restrict $\Sigma_1$ to be $\Sigma_1 = \tilde{T}_1 \text{diag} (p_1) \tilde{T}_1$, then log $\left| I + H_{1,1} \Sigma_1 H_{1,1}^\dagger \Omega_1^{-1} (p) \right| = r_1^* (p)$ and $J_0$ can be rewritten as a function of $p_1, p_{-1}$
\[
 J_0(\bar{\lambda}, \bar{\nu}_1, p_1, p_{-1}) = \sum_{m=1}^M p_{1,m} \tilde{t}_{1,m}^\dagger \tilde{A} (\bar{\lambda}, 0) \tilde{t}_{1,m} + \bar{\lambda}^T 1
\]
\[
 + \bar{\nu}_1 \left( \tilde{L}_1 - r_1^* (p) \right) - \bar{\lambda}^T 1^T B_{-1} p_{-1}.
\]
Then (78) and (79) imply
\[
\nabla p_1 J_0(\bar{\lambda}, \bar{\nu}_1, \tilde{p}_1, \tilde{p}_-1) = 0, \tag{80}
\]
\[
\nabla p_{-1} J_0(\bar{\lambda}, \bar{\nu}_1, \tilde{p}_1, \tilde{p}_{-1}) = 0. \tag{81}
\]

Let \(\bar{q}_1 = [\bar{q}_{1,1}, ..., \bar{q}_{1,M}]\), where \(\bar{q}_{1,m}, 1 \leq m \leq M\) is given by
\[
\bar{q}_{1,m} = \frac{\bar{\nu}_1 \gamma_{1,m}^2}{\tilde{p}_{1,m} (1 + \gamma_{1,m}) G_{1,m}}.
\]

Define \(\bar{q} = [\bar{q}_1^T, \bar{\lambda}^T]^T\). Then following similar steps as those solving (67) and (68), it can be shown that (80) and (81) form together the following linear equations
\[
\left( D^{-1} \left( \bar{T}, \bar{R}, \gamma^0 \right) - \Psi^T \left( \bar{T}, \bar{R} \right) \right) \bar{q} = 0.
\]

Because \(D^{-1} \left( \bar{T}, \bar{R}, \gamma^0 \right) - \Psi^T \left( \bar{T}, \bar{R} \right)\) is invertible [50], \(\bar{q} = 0\) must hold, which implies \(\bar{\lambda} = 0, \bar{\nu}_1 = 0\). This contradicts with the condition (73) that \(\bar{\eta}, \bar{\lambda}, \bar{\nu}_1\) can not be all zeros. Therefore, we must have \(\bar{\eta} > 0\).

C. Proof of Lemma 3

It is a consequence of the SINR duality applied to a single-user channel \(\bar{H}_l \triangleq \Omega_l^{-1/2} H_{l,t} \hat{\Omega}_l^{-1/2}\). Decompose \(Q_l\) and \(\hat{Q}_l\) to beams as \(Q_l = \sum_{m=1}^{M_l} d_{l,m} u_{l,m} u_{l,m}^\dagger, \) where \(d_{l,m} = p_{l,m} \left\| \hat{\Omega}_l^{1/2} t_{l,m} \right\|^2\) is the equivalent transmit power and \(u_{l,m} = \hat{\Omega}_l^{1/2} \sqrt{p_{l,m} t_{l,m}} / \sqrt{d_{l,m}}\) is the equivalent transmit vector; and \(\hat{Q}_l = \sum_{m=1}^{M_l} \hat{d}_{l,m} v_{l,m} v_{l,m}^\dagger, \) where \(\hat{d}_{l,m} = q_{l,m} \left\| \Omega_l^{1/2} r_{l,m} \right\|^2\) and \(v_{l,m} = \Omega_l^{1/2} \sqrt{q_{l,m} r_{l,m}} / \sqrt{\hat{d}_{l,m}}\). According to the covariance transformation, \(\{t_{l,m}\}, \{r_{l,m}\}, \{p_{l,m}\}\) and \(\{q_{l,m}\}\) achieves the same set of SINRs in the forward and reverse links respectively. Correspondingly, \(\{u_{l,m}\}, \{v_{l,m}\}, \{d_{l,m}\}\) and \(\{\hat{d}_{l,m}\}\) achieves the same set of SINRs in the single-user channel \(\bar{H}_l\) and \(\bar{H}_l^\dagger\) respectively. Apply Theorem 1 of [50] to this single-user network, we obtain \(\sum_{m=1}^{M_l} d_{l,m} = \sum_{m=1}^{M_l} \hat{d}_{l,m}\), i.e., \(\text{Tr} (Q_l) = \text{Tr} (\hat{Q}_l)\).

D. Proof of Theorem 7

The proof contains three parts. The uniqueness of the covariance transformation is shown by the idea that the covariance transformation for a network can be equivalently viewed as the covariance transformations for parallel single-user channels, where the capacity achieving
input covariance matrix for each single-user channel is unique. The other two parts, the matrix equations (31) and (32), are proved by utilizing the polite water-filling structure.

First we prove the uniqueness of the covariance transformation by showing that different decompositions of \( \Sigma_{1:L} \) produce the same \( \hat{\Sigma}_{1:L} \) through the covariance transformation. We will use the notations as in the proof of Lemma 3. Suppose decompositions of equations (31) and (32), are proved by utilizing the polite water-filling structure.

Let the capacity of the equivalent reverse channel \( \hat{H}_l \) be achieved through the covariance transformation. We will use the notations defined in Theorem 5. Noting that \( \hat{H}_l \) is the result of the equivalent input covariance matrices are \( \hat{Q}_l = \sum_{m=1}^{M_l} d_{l,m} u_{l,m} u_{l,m}^\dagger \) and \( \hat{Q}_l = \sum_{m=1}^{M_l} d_{l,m} v_{l,m} v_{l,m}^\dagger \). For a new decomposition \( \Sigma_l = \sum_{m=1}^{M_l} p_{l,m} t'_{l,m} t'_{l,m}^\dagger \), \( l = 1, \ldots, L \), we can change from the old decomposition to the new decomposition one link at a time. Without loss of generality, let \( l \) be the first link to be changed and prove \( \hat{\Sigma}_{1:L} \) remains the same. In this case, the transmission schemes of all other links are still \( \{ t_{k,m}, p_{k,m}, \forall k \neq l \} \), and the corresponding MMSE-SIC vectors are still \( \{ r_{k,m}, \forall k \neq l \} \) because the interference at link \( k(\neq l) \) does not depend on the decomposition of \( \Sigma_l \). Furthermore, we artificially fix the transmit powers of all other reverse links as \( \{ q_{k,m}, \forall k \neq l \} \). Then both \( \Omega_l \) and \( \hat{Q}_l \) are not changed. Find \( \hat{\Sigma}_l \) by calculating the MMSE-SIC receive vectors \( \{ r'_{l,m} \} \) and the unique \( q'_{l,m} \) such that the SINR of the \( m \)th stream of the \( l \)th forward and reverse link are the same, \( \forall m \). Since \( \hat{Q}_l \) is a water-filling solution over the forward equivalent channel \( \hat{H}_l \) and thus achieves the capacity of \( \hat{H}_l \) [67], the corresponding \( \hat{Q'}_l = \Omega_l^{1/2} \hat{\Sigma}_l \Omega_l^{1/2} \) achieves the capacity of the equivalent reverse channel \( \hat{H}_l \) with power \( \text{Tr} \left( \hat{Q'}_l \right) = \text{Tr} (\hat{Q}_l) \) by Lemma 5 applied to a single-user network of link \( l \). Therefore, \( \hat{Q'}_l = \hat{Q}_l \) because \( \hat{Q}_l \) also achieves the capacity of the \( \hat{H}_l \) with the same power \( \text{Tr} (\hat{Q}_l) = \text{Tr} (\hat{Q}_l) \) and the capacity achieving covariance matrix is unique [67]. It follows that \( \hat{\Sigma}_l = \hat{\Sigma}_l \), which implies the interference from reverse link \( l \) to other reverse links does not change. Then, \( \{ r'_{l,m}, q'_{l,m} \} \) and \( \{ r_{k,m}, q_{k,m}, \forall k \neq l \} \) achieves the same SINRs as the forward links and thus \( \{ \hat{\Sigma}_l = \hat{\Sigma}_l, \hat{\Sigma}_{k \neq l} \} \) is the result of the covariance transformation and is invariant with the decomposition of \( \Sigma_l \).

Second, we prove (31). In the rest of the proof, we will only use the terms corresponding to link \( l \). The subscript \( l \) will be omitted for simplicity. We will use the notations defined in Theorem 5. Noting that \( \hat{\Omega} = \hat{\Sigma}^{1/2} \hat{H}^\dagger \hat{\Omega}^{-1/2} \), \( \hat{\Sigma} = \hat{\Omega}^{-1/2} \hat{H} \hat{\Omega}^{-1/2} \), and \( \hat{\Sigma} = \hat{\Omega}^{-1/2} \hat{H} \hat{\Omega}^{-1/2} \),
\( F^\dagger F = I \) and \( G^\dagger G = I \), we have

\[
\Delta D \Delta = D \Delta^2
\]

\( \Rightarrow \quad \hat{\Omega}^{-\frac{1}{2}} G \Delta F^\dagger F D F^\dagger F \Delta G^\dagger = \hat{\Omega}^{-\frac{1}{2}} G D G^\dagger G \Delta^2 F^\dagger F \Delta G^\dagger \Rightarrow \quad \hat{\Omega}^{-\frac{1}{2}} \hat{\Omega}^{\ast \frac{1}{2}} H^{\ast \frac{1}{2}} \Omega^{-\frac{1}{2}} F D F^\dagger \Omega^{-\frac{1}{2}} H \Omega^{-\frac{1}{2}} = \hat{\Omega}^{-\frac{1}{2}} \hat{\Omega}^{\ast \frac{1}{2}} H \hat{\Omega}^{-\frac{1}{2}} \frac{1}{2} \Omega^{-\frac{1}{2}} \hat{\Omega}^{-\frac{1}{2}} = \hat{\Sigma}^{\ast \frac{1}{2}} \Omega^{-\frac{1}{2}} \hat{\Omega}^{-\frac{1}{2}} = \hat{\Sigma}
\]

where the last equation follows from (29) and (30) in Theorem 5.

Finally, we prove (32). Expand \( F \) and \( G \) to full unitary matrices \( \tilde{F} = [\bar{F} F] \in \mathbb{C}^{LR \times LR} \) and \( \tilde{G} = [\bar{G} G] \in \mathbb{C}^{LT \times LT} \). We also zero pad \( \Delta \) and \( D \) to \( \tilde{\Delta} \in \mathbb{C}^{LR \times LT} \) and \( \tilde{D} \in \mathbb{C}^{LT \times LT} \). Then we have

\[
\nu \left( \Omega^{-1} - (H \Sigma H^\dagger + \Omega)^{-1} \right)
\]

\( = \nu \Omega^{-1/2} \left( I - (\bar{H} Q \bar{H}^\dagger + I)^{-1} \right) \Omega^{-1/2} \)

\( = \nu \Omega^{-1/2} \left( I - \left( \tilde{F} \tilde{D} \tilde{D}^T \tilde{F}^\dagger + I \right)^{-1} \right) \tilde{F}^\dagger \Omega^{-1/2} \)

\( = \nu \Omega^{-1/2} \tilde{F} \left( I - \left( \tilde{\Delta} \tilde{D} \tilde{D}^T + I \right)^{-1} \right) \tilde{F}^\dagger \Omega^{-1/2} \)

\( = \Omega^{-1/2} \tilde{F} D F^\dagger \Omega^{-1/2} = \hat{\Sigma} \quad (82) \)

where the Equation (82) follows from \( Q = GDG^\dagger \) and the Equation (83) follows from \( D = (\nu I - \Delta^{-2})^+ \) and \( \hat{Q} = FDF^\dagger \).
E. Proof of Theorem 10

The weighted sum-rate function for a two-user Z channel can be expressed as

\[ f(\Sigma_1, \Sigma_2) = w_1 \log |I + H_{1,1} \Sigma_1 H_{1,1}^\dagger + H_{1,2} \Sigma_2 H_{1,2}^\dagger| \]

\[ + w_1 \left( \log |I + H_{2,2} \Sigma_2 H_{2,2}^\dagger| - \log |I + H_{1,2} \Sigma_2 H_{1,2}^\dagger| \right) \]

\[ + (w_2 - w_1) \log |I + H_{2,2} \Sigma_2 H_{2,2}^\dagger|. \]

Because the sum of concave functions is still concave, and the first and third terms in (84) are logdet functions and thus are concave, we only need to show that the second term

\[ g(\Sigma_2) = \log |I + H_{2,2} \Sigma_2 H_{2,2}^\dagger| - \log |I + H_{1,2} \Sigma_2 H_{1,2}^\dagger| \]

is a concave function.

Consider the convex combination of two different inputs, \( X_2 \succeq 0 \) and \( Z_2 \succeq 0 \),

\[ \Sigma_2 = tZ_2 + (1 - t)X_2 \]

\[ = X_2 + tY_2, \]

where \( Y_2 = Z_2 - X_2 \) is Hermitian and \( 0 \leq t \leq 1 \). Then \( g(\Sigma_2) \) is a concave function if and only if \( \frac{d^2}{dt^2} g(\Sigma_2) \leq 0 \) for any \( X_2 \succeq 0, Z_2 \succeq 0 \) and \( 0 \leq t \leq 1 \) [66].

It can be derived that [30]

\[ \frac{d^2}{dt^2} g(\Sigma_2) = - \text{Tr}[A Y_2 A Y_2] + \text{Tr}(B Y_2 B Y_2), \]

where \( A = \left( (H_{2,2}^\dagger H_{2,2})^{-1} + \Sigma_2 \right)^{-1} \) and \( B = \left( (H_{1,2}^\dagger H_{1,2})^{-1} + \Sigma_2 \right)^{-1} \) are positive definite matrices. If \( H_{2,2}^\dagger H_{2,2} \succeq H_{1,2}^\dagger H_{1,2} \), it can be proved that \( A \succeq B \). Then

\[ \text{Tr}(A Y_2 A Y_2) \]

\[ = \text{Tr} \left( A^{-1/2} Y_2 A Y_2 A^{-1/2} \right) \]

\[ \geq \text{Tr} \left( A^{-1/2} Y_2 B Y_2 A^{-1/2} \right) \]

\[ = \text{Tr} \left( B^{-1/2} Y_2 A Y_2 B^{-1/2} \right) \]

\[ \geq \text{Tr} \left( B^{-1/2} Y_2 B Y_2 B^{-1/2} \right) \]

\[ = \text{Tr}(B Y_2 B Y_2), \]

which implies that \( \frac{d^2}{dt^2} g(\Sigma_2) \leq 0 \) and \( g(\Sigma_2) \) is a concave function. This completes the proof.
F. Proof of Lemma 10

Without loss of generality, we consider an iTree network where the $l^{th}$ link is not interfered by the first $l-1$ links. Because problem (37) is convex, we only need to show that the conditions in Lemma 10 are necessary and sufficient to satisfy the KKT conditions of problem (37) given in (55). The sufficient part is proved in Theorem 13 for general B-MAC networks. We only need to prove the necessary part, i.e., if $\tilde{\Sigma}_1:L$ satisfy the KKT conditions with the optimal dual variable $\tilde{\mu}$, it must satisfy the conditions in Lemma 10. Due to the interference structure of iTree networks, we have $\Phi_{k,l} = 0$ for $k \geq l$, and thus the condition $\nabla \Sigma_l |_{\Sigma_1:L=\tilde{\Sigma}_1:L} = 0$ can be solved one by one from $l = 1$ to $l = L$. For $l = 1$, $\nabla \Sigma_1 |_{\Sigma_1:L=\tilde{\Sigma}_1:L} = 0$ can be expressed as

$$\tilde{\Omega}_1 = \frac{w_1}{\mu} H_{1,1}^\dagger \left( \tilde{\Omega}_1 + H_{1,1} \tilde{\Sigma}_1 H_{1,1}^\dagger \right)^{-1} H_{1,1} + \frac{1}{\mu} \Theta_1,$$

where $\tilde{\Omega}_1 = I$. Because Equation (85) is also the KKT condition of the single-user polite water-filling problem, the solution $\tilde{\Sigma}_1$ is unique and must satisfy the polite water-filling structure with water-filling level $w_1/\tilde{\mu}$ as in Lemma 10. By Theorem 7 we have

$$\tilde{\Sigma}_1 = \frac{w_1}{\tilde{\mu}} \left( \tilde{\Omega}_1^{-1} - \left( \tilde{\Omega}_1 + H_{1,1} \tilde{\Sigma}_1 H_{1,1}^\dagger \right)^{-1} \right).$$

Then for $l = 2$, $\nabla \Sigma_2 |_{\Sigma_1:L=\tilde{\Sigma}_1:L} = 0$ can be expressed as

$$\tilde{\Omega}_2 = \frac{w_2}{\tilde{\mu}} H_{2,2}^\dagger \left( \tilde{\Omega}_2 + H_{2,2} \tilde{\Sigma}_2 H_{2,2}^\dagger \right)^{-1} H_{2,2} + \frac{1}{\mu} \Theta_2,$$

where

$$\tilde{\Omega}_2 = I + H_{2,2}^\dagger \tilde{\Sigma}_2 H_{2,2}\left( \tilde{\Omega}_1^{-1} - \left( \tilde{\Omega}_1 + H_{1,1} \tilde{\Sigma}_1 H_{1,1}^\dagger \right)^{-1} \right) H_{1,2}.$$

Similarly, the solution $\tilde{\Sigma}_2$ must satisfy the polite water-filling structure with water-filling level $w_2/\tilde{\mu}$ as in Lemma 10. Following similar proof as above, it can be shown that for $l = 3, ..., L$, $\tilde{\Sigma}_l$ must satisfy the conditions in Lemma 10. This completes the proof.

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