The classical-statistical limit of quantum mechanics.

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Abstract

The classical-statistical limit of quantum mechanics is studied. It is proved that the limit $\hbar \rightarrow 0$ is the good limit for the operators algebra but it si not so for the state compact set. In the last case decoherence must be invoked to obtain the classical-statistical limit.

1 Introduction

In this talk, using the methods of papers [1], [2], and [3], we will consider the classical-statistical limit of quantum mechanics for a system with continuous evolution spectrum (We have discussed how classical-statistical limit of quantum mechanics becomes the proper classical limit, i.e. the localization phenomenon, in paper [4]).

This study seems necessary because the existent einselection-approach theory is far from being perfect since, at least, the following features must be better explained:

i.- Einselection is based in the decomposition of the system in both a relevant part, the proper system, and an irrelevant one: the environment. This decomposition it not always possible, e.g. in the case of the universe, in fact:
"...the Universe as a whole is still a single entity with no 'outside' environment, and therefore, any resolution involving its division in unacceptable" (Zurek [5], p. 181).

ii.- Moreover there is not a clear criterion to define the "cut" between the proper system and its environment. In fact:

"In particular, one issue which has been often taken for granted is looming big, as a foundation of the whole decoherence problem. It is the question of what are the 'systems' which play a such crucial role in the discussions of the emergence classicality. This issue was raised earlier, but the progress to date has been slow at best" (Zurek [6], p. 122).

iii.- The definition of the basis where the system becomes classical, i.e. the pointer basis, is based in a predictability sieve which would produce the set of most stable possible states. But in practice this definition seems very difficult to implement. In fact, the basis vectors o are just good candidates for reasonable stable states [7].

None of these problems seem solved by the einselection approach today [7]. Nevertheless when in particular models the three mentioned problems can be bypassed the einselection approach works extraordinary well.

On the other hand in the "self-induced" approach that we are about to explain all these problems are absent: it can be used in close systems as the universe[8], the definition of a convenient subalgebra takes the role of the coarse-graining induced by the environment [9], and the pointer basis for the classical limit is perfectly defined (we will not consider this problem here since it is fully explained in [4]). In this way the new approach is justified.

2 The formalism for the observables.

Let us begin defining the quantum and classical operator algebra and their limits. We will consider a system with a complete set of commuting observables (CSCO) \{\hat{H}, \hat{P}_1, ..., \hat{P}_{N-1}\}, where \(\hat{H}\) has a continuous spectrum \(0 \leq \omega < \infty\) and also the \(\hat{P}_i\) have continuous spectra. Since the energy continuous spectrum is essential for our treatment we present the simplest model with just the continuous spectrum (we have added discrete spectra in ref. [4] and [10]). To simplify the notation we will just call \{\hat{H}, \hat{P}\} the set \{\hat{H}, \hat{P}_1, ..., \hat{P}_{N-1}\}. We will consider the orthonormal eigen basis \{|\omega, p\rangle\} of
\{\hat{H},\hat{P}\}$ and write the hamiltonian and $\hat{P}$ as

$$\hat{H} = \int_{p} \int_{0}^{\infty} \omega|\omega, p}\langle \omega, p|d\omega dp^{N-1} \quad \hat{P} = \int_{p} \int_{0}^{\infty} p|\omega, p}\langle \omega, p|d\omega dp^{N-1}$$  \hspace{1cm} (1)

Furthermore we will consider the algebra $\hat{A}$ of the operator (as we have said we are using the "final pointer basis" of paper [4]):

$$\hat{O} = \int_{p} \int_{0}^{\infty} O(\omega, p)|\omega, p}\langle \omega, p|d\omega dp^{N-1} + \int_{p} \int_{p'} \int_{0}^{\infty} \int_{0}^{\infty} O(\omega, \omega', p, p')|\omega, p}\langle \omega', p'|d\omega d\omega' dp^{N-1} dp'^{N-1}$$  \hspace{1cm} (2)

where the first term in the r.h.s. will be called $\hat{O}_{S}$ or the singular component and the second term will be called $\hat{O}_{R}$ or the regular component. Functions $O(\omega, p)$ and $O(\omega, \omega', p, p')$ are regular (see [1] for details), $[\hat{H}, \hat{O}_{S}] = 0$, $\hat{O}_{S} \in \hat{L}_{S}$, the singular space, $\hat{O}_{R} \in \hat{L}_{R}$, the regular space, and $\hat{A} = \hat{L}_{S} \oplus \hat{L}_{R}$. The observables are the self adjoint operators of $\hat{A}$. As $O(\omega, \omega', p, p')$ is a regular function $\hat{L}_{R}$ can be promoted to a space of operators on a Hilbert space $\mathcal{H}$ and the usual theory of the Wigner function [11] can be implemented in $\hat{L}_{R}$.

At the classical statistical level $\hat{L}_{R}$ will have a classical analogue $\mathcal{L}_{R}$, a space of integrable $L_{1}$ - functions over the phase space $\mathcal{M} = \mathcal{M}_{2N} \equiv \mathbb{R}^{2N}$. These functions will be called $O_{R}(\phi)$ where $\phi$ symbolizes the coordinates over $\mathcal{M}$, $\phi^{a} = (q^{1}, ..., q^{N}, p^{1}, ..., p^{N})$ with $a = 1, 2, ... 2N$. As it is known we can map $\hat{L}_{R}$ on $\mathcal{L}_{R}$ via the usual Wigner transformation or symbol as $symb : \hat{L}_{R} \rightarrow \mathcal{L}_{R}$

$$symb\hat{f}_{R} = f_{R}(\phi) = \int_{-\infty}^{\infty} \langle q - y|\hat{f}|q + y\rangle e^{2q.y/h} d^{N}y.$$  \hspace{1cm} (3)

On $\mathcal{L}_{R}$ we can define the "star product" (i.e. the classical operator related with the multiplication on $\hat{L}_{R}$): $symb(\hat{f}\hat{g}) = symb\hat{f} \ast symb\hat{g} = (f \ast g)(\phi)$. It can be proved ([11], eq. (2.59)) that $(f \ast g)(\phi) = f(\phi) \exp \left(\frac{i}{\hbar} \frac{\partial}{\partial a} \omega^{ab} \frac{\partial}{\partial b}\right) g(\phi) = g(\phi) \exp \left(\frac{i}{\hbar} \frac{\partial}{\partial a} \omega^{ab} \frac{\partial}{\partial b}\right) f(\phi)$. We also define the Moyal bracket as the symbol corresponding to the commutator in $\hat{L}_{R}$: $\{f, g\}_{mb} = \frac{1}{i\hbar}(f \ast g - g \ast f) =$
symb \left( \frac{1}{\hbar} [f, g] \right). In the limit \( \hbar \to 0 \) the star product becomes the ordinary product and the Moyal bracket the Poisson bracket

\[(f \ast g)(\phi) = f(\phi)g(\phi) + 0(\hbar), \quad \{f, g\}_{mb} = \{f, g\}_{pb} + 0(\hbar^2). \tag{4}\]

Let us now consider the singular space \( \hat{L}_S \), the space of the operators that commute with \( \hat{H} \) and \( \hat{P} \). We will see that the mapping symb given by (3) is also well defined for the observables in \( \hat{L}_S \). In fact, from eq. (2) we know that \( \hat{O}_S = \int_p \int_0^\infty O(\omega, p)|\omega, p\rangle \langle \omega, p| d\omega dp^{N-1} \) so using well known procedures we can conclude that \( \hat{O}_S = O(\hat{H}, \hat{P}) \). But, if the \( \hat{f}, \hat{g} \) commute, as the members of the CSCO do, we have \( \text{symb} \langle \hat{f}, \hat{g} \rangle = (f \ast g)(\phi) = f(\phi)g(\phi) + 0(\hbar^2) \) then (using the same procedure as before) symb\( \hat{O}_S = O_S(\phi) = O(H(\phi), P(\phi)) + 0(\hbar^2) \). We have succeeded in computing all the symbs of the observables of \( \hat{L}_S \) up to \( 0(\hbar^2) \), so when \( \hbar \to 0 \) they are just \( O(H(\phi), P(\phi)) \), and we have defined the mapping symb : \( \hat{L}_S \rightarrow L_S \), symb\( \hat{O}_S = O_S(\phi) = O(H(\phi), P(\phi)) + 0(\hbar^2) \).

Let us observe that if \( O(\omega, p) = \delta(\omega - \omega')\delta(p - p') \) we have from the last equation

\[
\text{symb} |\omega', p\rangle \langle \omega', p'| = \delta(H(\phi) - \omega')\delta(P(\phi) - p')
\]

where we have disregarded the \( 0(\hbar^2) \) as we will always do below. Therefore, from the eq. (3) we have defined a classical space \( \mathcal{A} = \mathcal{L}_R \oplus \mathcal{L}_S \) and a mapping symb : \( \hat{A} \rightarrow \mathcal{A} \), symb\( \hat{O} = O(\phi) \). Then we now have the limit \( \hbar \to 0 \) for \( \hat{A} \rightarrow \mathcal{A} \) (where also eq. (4) is valid). But, even if this limit is well defined and can be considered as the classical limit of the algebra of the operators, it is only the limit of the equations of the system, since these are a consequence of the algebra.

### 3 The formalism for the states.

Let us now consider the quantum and classical state sets and their limits. Let us introduce the symbols \( |\omega, p\rangle = |\omega, p\rangle \langle \omega, p| \) and \( |\omega, \omega', p, p'\rangle = |\omega, p\rangle \langle \omega', p'| \). \{\( |\omega, p, p'\rangle \)} is the basis of \( \hat{L}_S \) and \{\( |\omega, \omega', p, p'\rangle \)} is the basis of \( \hat{L}_R \), then eq. (2) reads

\[
\hat{O} = \int_p \int_0^\infty O(\omega, p)|\omega, p\rangle d\omega dp^{N-1}
\]
\[ \int_0^\infty \int_0^\infty O(\omega, \omega', p, p')|\omega, \omega', p, p'|d\omega dp = 1 + \int_0^\infty O(\omega, \omega', p, p')|\omega, \omega', p, p'|d\omega dp. \]

The state are functionals over the space \( \hat{A} = \hat{L}_S \oplus \hat{L}_R \). Therefore, let us consider the dual space \( \hat{A}' = \hat{L}'_S \oplus \hat{L}'_R \). We will call \( \{(\omega, p)\} \) the basis of \( \hat{L}'_S \) and \( \{(\omega, \omega', p, p')\} \) the basis of \( \hat{L}'_R \). Observe that \( (\omega, \omega', p, p') = \rho(p, \omega', \omega, p' \rangle \rho(p, \omega, \omega', p) \rangle \) \( \) but \( (\omega, p) \neq \rho(p, \omega) \rangle \langle \omega, p \) \. Moreover \( (\omega, \omega', p, p') = \rho(p, \omega', \omega, p) \rangle \langle \omega, \omega', p, p' \) \( \) and the rest of the (,|,) = 0. Then a generic functional of \( \hat{A}' \) reads

\[ \hat{\rho} = \int_0^\infty \int_\infty \rho(\omega, p)(\omega, p) d\omega dp + \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \rho(\omega, \omega', p, p')(\omega, \omega', p, p') d\omega dp d\omega' dp' \] (7)

As functions \( O(\omega, p), O(\omega, \omega', p, p') \) functions \( \rho(\omega, p), \rho(\omega, \omega', p, p') \) are regular and have all the mathematical properties to make the formalism successful [1]. Moreover the \( \hat{\rho} \) must be self-adjoint, and its diagonal \( \rho(\omega, p) \) must represent probabilities, thus

\[ \hat{\rho} = \hat{\rho}^+, \quad \int_0^\infty \rho(\omega, p) d\omega dp = 1, \quad \rho(\omega, p) \geq 0. \] (8)

The \( \hat{\rho} \) with these properties belongs to a convex set \( \hat{S} \), the set of the states. Also \( (\hat{\rho}|\hat{O}) = \int_0^\infty \rho(\omega, p)O(\omega, p) d\omega dp = \) \( + \int_0^\infty \int_0^\infty \rho(\omega, \omega', p, p')O(\omega', \omega, p') d\omega dp d\omega' dp' \) \( - N \). As \( \hat{L}_R \) is a space of operators on a Hilbert space \( \hat{H} \) so it is equal to its dual \( \hat{L}'_R \). Then it is known that the symbol for any \( \hat{\rho}_R \in \hat{L}'_R \) is defined as \( \rho_R(\phi) = (\pi h)^{-N} \text{symb} \hat{\rho}_R \) by the same equations (3). From this definition we have (cf. [11] eq. (2.13))

\[ (\hat{\rho}_R|\hat{O}_R) = (\text{symb} \hat{\rho}_R|\text{symb} \hat{O}_R) = \int d\phi^{2N} \rho_R(\phi) \hat{O}_R(\phi) \] (9)

and in \( \hat{L}_R \) and \( \hat{L}'_R \) all the equations are the usual ones (i.e. those of paper [11]).
Let us now consider the singular dual space $\mathcal{L}'_S$. In this space we will define the symbol $\tilde{\rho}_S$ as the function on $\mathcal{M}$ that satisfies a equation similar to (9) for any $\tilde{O}_S \in \mathcal{L}_S$, namely

$$ (\text{symb} \tilde{\rho}_S | \text{symb} \tilde{O}_S) \equiv (\tilde{\rho}_S | \tilde{O}_S) \quad (10) $$

We must define the meaning of these symbols, precisely $|\text{symb} \tilde{O}_S\rangle$ and the inner product we are using in $(\text{symb} \tilde{\rho}_S | \text{symb} \tilde{O}_S)$. From eq. (5) we know that $\text{symb} |\omega', p'\rangle = \delta(H(\phi) - \omega')\delta^{N-1}(P(\phi) - p')$. It is clear that we cannot normalize this function with the variables and in the domain of integration of (9). In fact using the canonical variables $\alpha$, conjugated to the $P$, and $-t$, the canonical variable conjugated to $H$, for any function like $\text{symb} |\omega', p'\rangle = \delta(H(\phi) - \omega')\delta^{N-1}(P(\phi) - p')$, which is a constant for the $\alpha$ and the $t$, the integral will turn out to be infinity. So these functions $f(H, P)$ are not classical densities since they do not belong to $L_1$ and they must be normalized in a different way. But let us observe that $O_S(\phi) = O(H(\phi), P(\phi))$ also is a function of this class. Then we can normalize these functions if:

1) We only integrate over the momentum space $H, P$, precisely $|O_S(\phi)| = \int dH \int dP \int_0^{\infty} |O(\omega, p)|\delta(H-\omega')\delta^{N-1}(P-p')d\omega = \int dH \int dP \int_0^{\infty} |O(H, P)|$.

2) We choose the regular function $O(\omega, p)$ in the space $L_1$ of the momentum space $\omega, p$, precisely $\int d\omega \int dP |O(\omega, p)| < \infty$. So we will normalize the $f(H, P)$ in this way and we will perform all the integrations in the $\mathcal{L}_S$ space in the same way. Now we can calculate the function $\rho_{s\omega'}(\phi) = \text{symb}(\omega, p)$ which from eqs. (5) and (10) must satisfy

$$ \int dH \int dP \rho_{s\omega'}(\phi)\delta(H-\omega')\delta^{N-1}(P-p') = \delta(\omega-\omega')\delta(p-p') \quad (11) $$

where we have used the just defined way of integration. Then $\rho_{s\omega'}(\phi) = \text{symb}(\omega', p') = \delta(H(\phi) - \omega')\delta^{N-1}(P(\phi) - p')$. So finally we can say that $(\tilde{\rho}_S | \tilde{O}_S) \equiv (\text{symb} \tilde{\rho}_S | \text{symb} \tilde{O}_S) = \int dHdP\rho_S(\phi)O_S(\phi)$ the analogous of eq. (9) in space $\mathcal{L}_S$. Moreover

$$ \rho_S(\phi) = \int_0^{\infty} \rho(\omega, p)|\text{symb}(\omega, p)|d\omega dp $$

$$ = \int_0^{\infty} \rho(\omega, p)\delta(H(\phi) - \omega)\delta^{N-1}(P(\phi) - p) d\omega dp = \rho(H(\phi), P(\phi)) $$
and it is a constant of the motion. It can be normalized in space \( L_s \) as
\[
O(\omega, p) \text{ namely }
\int d\omega \int dp^{N-1} |\rho(\omega, p)| < \infty \quad \text{or simply} \quad \int d\omega \int dp^{N-1} \rho(\omega, p) = 1
\]

(12)

if we take into account eq. (8). Thus, according to the prescriptions above, we have defined the mapping of the quantum states space \( \hat{A}' \) on the “classical” state space \( A' \): symb : \( \hat{A}' \to A' \). In the limit \( h \to 0 \) equation (4) is always valid so we would have something like the classical limit for the states. But it is not so, because in general the obtained \( \rho(\phi) \) does not satisfy the condition \( \rho(\phi) \geq 0 \) (for every \( \phi \in \) ) even if \( h \to 0 \). So \( \rho(\phi) \) is not a density function and therefore mapping (3) is not a mapping of quantum mechanics on classical statistical mechanics. Application (3) does not lead to the classical world. So for \( h \to 0 \) the isomorphism (3) is a mapping of quantum mechanics on a certain quantum mechanics “alla classica”, namely formulated in phase space \( M \) (but not with \( \rho \) non-negative defined). Thus \( h \to 0 \) is not the classical limit. To obtain this limit we must introduce decoherence.

4 Time evolution and the decoherence

We only obtain the true classical limit for \( t \to \infty \) via the decoherence process. In fact, \( \rho(t) \) evolves in time as
\[
\rho(\phi, t) = \rho(H(\phi), P(\phi)) + \int_0^\infty \int_{0}^{\infty} \rho(\omega, \omega', p, p') e^{i(\omega - \omega')t/\hbar} \text{ symb}(\omega, \omega', p, p') \int_{0}^{\infty} \int_{0}^{\infty} \rho(\omega, \omega', p, p') e^{i(\omega - \omega')t/\hbar} d\omega d\omega' dp^{N-1} dp^{N-1}.
\]

This functional acts on spaces \( A \) giving
\[
(\rho(\phi, t)|O(\phi)) = \int dHdP^{N-1} \rho(H(\phi), P(\phi))O(H(\phi), P(\phi)) + \int_0^\infty \int_0^\infty \rho(\omega, \omega', p, p') e^{i(\omega - \omega')t/\hbar} \text{ symb}(\omega, \omega', p, p') \int_{0}^{\infty} \int_{0}^{\infty} \rho(\omega, \omega', p, p') e^{i(\omega - \omega')t/\hbar} O(\omega, \omega', p, p') d\omega d\omega' dp^{N-1} dp^{N-1}.
\]

Now if the regular functions \( \rho(\omega, \omega', p, p') \) and \( O(\omega', \omega', p, p') \) are endowed with adequate properties as those listed in [1] the Riemann-Lebesgue theorem can be used to obtain
\[
\lim_{t \to \infty} (\rho(\phi, t)|O(\phi)) = \int dHdP^{N-1} \rho(H(\phi), P(\phi)) O(H(\phi), P(\phi))
\]
\[
= (\rho_*(\phi)|O(\phi))
\]
for any \(O(\phi) \in \mathcal{A}\) and where we have defined the functional \(\rho_*(\phi) = \rho_S(\phi)\) and find the classical final limit. In fact, we have found the weak limit \(W\lim_{t \to \infty} (\rho(\phi, t)) = (\rho_*(\phi))\). As we can see only the singular part remains and therefore, in the quantum case, only the singular-diagonal part remains too, thus the time evolution has made the system decoheres. Of course, as this kind of decoherence is obtained when \(t \to \infty\) we can ask ourselves which the decoherence time is. The problem is solved in papers[1] and [10], where via an analytic continuation of the resolvent of the von Neumann-Liouville operator in the complex plane, it is shown that decoherence time is the inverse of the distance of the closer pole to the real axis. Systems with no poles (e.g. the free particle) have infinite decoherence time and therefore they do not decohere in practice.

Now we can find the property that was missing at the end of the last section, i.e. \(\rho(\phi) \geq 0\). In fact, condition (8) yields \(\rho_*(\phi) = \rho_S(\phi) = \rho(H(\phi), P(\phi)) \geq 0\) (cf. eq.(12)). So now we have the true statistical-classical limit when \(t \to \infty, \hbar \to 0\), in which case the mapping \(symb: \mathcal{A} \to \mathcal{A}'\) maps the quantum states with non-negative probability in the diagonal \((\rho(\omega, p) \geq 0)\) into final classical states with \(\rho_*(\phi) \geq 0\), which are non-negative density functions as they should be.

Moreover the \(\rho_*(\phi) = \rho_S(\phi)\) of eq. (12) has a clear physical meaning: It is the sum of densities strongly peaked in the classical trajectories defined by the constant of the motion \(H(\phi) = \omega, P(\phi) = p\) averaged by the classical density function \(\rho(\omega, p)\) which is properly normalized according eq. (12). This fact is essential in the localization process to obtain the final classical limit (see [4]).

In conclusion, we have demonstrated that “symb” and \(\hbar \to 0\) give the correct classical-statistical limit of quantum mechanics for the algebra of operators, but for the set of states it must be complemented with \(t \to \infty\), which produces the decohered final state with a non-negative density function.

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