Further results on multiple coverings of the farthest-off points

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Abstract

Multiple coverings of the farthest-off points \((R, \mu)\)-MCF codes and the corresponding \((\rho, \mu)\)-saturating sets in projective spaces \(PG(N, q)\) are considered. We propose and develop some methods which allow us to obtain new small \((1, \mu)\)-saturating sets and short \((2, \mu)\)-MCF codes with \(\mu\)-density either equal to 1 (optimal saturating sets and almost perfect MCF-codes) or close to 1 (roughly \(1 + 1/cq, c \geq 1\)). In particular, we provide new algebraic constructions and some bounds. Also, we classify minimal and optimal \((1, \mu)\)-saturating sets in \(PG(2, q)\), \(q\) small.
1 Introduction

A code $C$ is called \((R, \mu)-\text{multiple covering of the farthest-off points}\) (or \((R, \mu)\)-MCF for short) if for every word $x$ at distance $R$ from $C$ there are at least $\mu$ codewords in the Hamming sphere $S(x, R)$, where $R$ is the covering radius of $C$.

Multiple coverings can be viewed as a generalization of covering codes, see [5, 6]. Motivations for studying MCF codes come from the generalized football pool problem (see e.g. [20, 21, 25] and the references therein) and list decoding (see e.g. [31]).

In [6, 18, 19, 23, 24, 28, 29] results on MCF codes, mostly concerning the binary and the ternary cases, can be found. The development of this topic for arbitrary $q$ was presented in [2, 15, 27] and in the recent paper [3]. In particular, important parameters of \((R, \mu)\)-MCF codes such as the \(\mu\)-density and the \(\mu\)-length function have been introduced in [3]. In the same paper the notion of a \((\rho, \mu)\)-saturating set as the geometrical counterpart of \((\rho + 1, \mu)\)-MCF codes was proposed. Many useful results and constructions of MCF codes were obtained in [3] by geometrical methods. For an introduction to projective spaces over finite fields see [22].

The \(\mu\)-density of an \((R, \mu)\)-MCF code $C$ is the average value of \(\frac{1}{\mu} \#(S(x, R) \cap C)\), where $x$ is a word at distance $R$ from $C$. The \(\mu\)-density is greater than or equal to 1. If the minimum distance $d$ of $C$ is at least $2R - 1$, then the best $\mu$-density among linear $q$-ary codes with same codimension $r$ and covering radius $R$ is achieved by the shortest ones. An important class of MCF codes are almost perfect and perfect MCF codes which correspond to optimal saturating sets. For these codes each word at distance $R$ from the code belongs to exactly $\mu$ spheres centered in codewords; they have the best possible $\mu$-density, i.e. equal to 1. The $\mu$-length function $\ell_\mu(R, r, q)$ is defined as the smallest length $n$ of a linear \((R, \mu)\)-MCF code with parameters $[n, n-r, d]_q$, $d \geq 3$.

In this paper, we continue and develop the geometrical approach of [3] for constructing MCF codes with small $\mu$-density. We present a number of \((1, \mu)\)-saturating sets (and the corresponding \((2, \mu)\)-MCF codes) with good parameters.

In the space $PG(N, q)$, $q > 2$ even, we obtain \((1, \mu)\)-saturating sets with $\mu = \frac{q-2}{2}$ such that the $\mu$-density of the corresponding \((2, \mu)\)-MCF code tends to 1 when $N$ is fixed and $q$ tends to infinity, see Section 4.

New results concerning \((1, \mu)\)-saturating sets in planes $PG(2, q)$ are presented in Section 6. We give some upper and lower bounds on the size of \((1, \mu)\)-saturating sets, see Subsection 6.1. Also, we present many
examples of optimal saturating sets using classical geometrical objects such as partitions of $PG(2, q)$ in Singer point-orbits and sets of convenient lines, see Sections 5 and 7 and Subsection 6.3. Unfortunately, it is not always possible to construct almost perfect codes; in some cases we construct examples of $(1, \mu)$-saturating sets with $\mu$-density roughly of the same order of magnitude of $1 + 1/cq, c > 1$. In general, we give families of $\mu$-saturating sets of size less than $\mu \ell(2, 3, q)$, where $\ell(2, 3, q)$ is the minimum known size of a 1-saturating set in $PG(2, q)$, see e.g. [1, 11, 13] and the references therein.

Another achievement of this paper is the classification of minimal and optimal $(1, \mu)$-saturating sets in $PG(2, q)$ for small $q$, see Section 7.

The paper is organized as follows. In Section 2 we recall some definitions and results from [3] concerning MCF codes and $(\rho, \mu)$-saturating sets; in Section 3 we focus on $(1, \mu)$-saturating sets. In Section 4 we deal with $(1, \mu)$-saturating sets in $PG(N, q)$, $q$ even, having small size. In Section 5 perfect and almost perfect $(2, \mu)$-MCF codes are constructed from classical geometrical objects. In Section 6 we present some constructions and bounds on $(1, \mu)$-saturating sets in $PG(2, q)$. Finally, in Section 7, computational results on the classification of minimal and optimal $(1, \mu)$-saturating sets in $PG(2, q)$ are presented.

## 2 Multiple coverings and $(\rho, \mu)$-saturating sets

In the following we use the same notation as in [3]. An $(n, M, d)_q R$ code $C$ is a code of length $n$, cardinality $M$, minimum distance $d$, and covering radius $R$, over the finite field $F_q$ with $q$ elements. If $C$ is linear of dimension $k$ over $F_q$, then $C$ is also said to be an $[n, k, d, q] R$ code. When either $d$ or $R$ are not relevant or unknown they can be omitted in the above notation. Let $F_q^n$ be the linear space of dimension $n$ over $F_q$, equipped with the Hamming distance. The Hamming sphere of radius $j$ centered at $x \in F_q^n$ is denoted by $S(x, j)$. The size $V_q(n, j)$ of such a sphere is

$$V_q(n, j) = \sum_{i=0}^{j} \binom{n}{i} (q - 1)^i.$$  

Let $S(x, R)$ be the surface of the sphere $S(x, R)$. For an $(n, M)_q R$ code $C$, $A_w(C)$ denotes the number of codewords in $C$ of weight $w$, and
\( f_\theta(e, C) \) denotes the number of codewords at distance \( \theta \) from a vector \( e \) in \( \mathbb{F}_q^n \); equivalently, \( f_\theta(e, C) = \#(S(e, \theta) \cap C) \). Let

\[
\delta(C, R) = \frac{\sum_{x \in \mathbb{F}_q^n} \#\{c \in C \mid d(c, x) \leq R\}}{q^n} = \frac{M \cdot V_q(n, R)}{q^n}
\]

be the density of an \((n, M)qR\) covering code \( C \). In general, \( \delta(C, R) \geq 1 \), and equality holds if and only if \( C \) is a perfect code.

**Definition 2.1** ([3, 6, 19, 20]).

1. An \((n, M)qR\) code \( C \) is said to be an \((R, \mu)\) multiple covering of the farthest-off points \(((R, \mu)\)-MCF code for short) if for all \( x \in \mathbb{F}_q^n \) such that \( d(x, C) = R \) the number of codewords \( c \) such that \( d(x, c) = R \) is at least \( \mu \).

2. An \((n, M, d(C))qR\) code \( C \) is said to be an \((R, \mu)\) almost perfect multiple covering of the farthest-off points \(((R, \mu)\)-APMCF code for short) if for all \( x \in \mathbb{F}_q^n \) such that \( d(x, C) = R \) the number of codewords \( c \) such that \( d(x, c) = R \) is exactly \( \mu \). If, in addition, \( d(C) \geq 2R \) holds, then the code is called \((R, \mu)\) perfect multiple covering of the farthest-off points \(((R, \mu)\)-PMCF code for short).

In the literature, MCF codes are also called multiple coverings of deep holes, see e.g. [6, Chapter 14].

As already pointed out in [3, Sect. 2], there exists a connection between \(((R, \mu)\)-MCF and \(((R, \mu)\)-PMCF with weighted \( m \)-coverings; see e.g. [6, Sect. 13.1].

**Definition 2.2** ([3]). Let \( C \) be a \(((R, \mu)\)-MCF code. Let \( \{x_1, \ldots, x_{N_R(C)}\} \) be the set of vectors in \( \mathbb{F}_q^n \) with distance \( R \) from \( C \). The \( \mu \)-density of \( C \) is

\[
\gamma_\mu(C, R) = \frac{\sum_{i=1}^{N_R(C)} f_R(x_i, C)}{\mu N_R(C)}.
\]

(2.1)

It is easily seen that \( \gamma_\mu(C, R) \geq 1 \), and that \( C \) is an \((R, \mu)\) APMCF code precisely when equality holds. In general, this parameter is a measure of the quality of an \(((R, \mu)\)-MCF code.

The goal of this paper is the construction of \(((R, \mu)\)-MCF codes with small \( \mu \)-density.

We recall the following proposition from [3] concerning the \( \mu \)-density of an \(((R, \mu)\)-MCF code.
Proposition 2.3. Let $C$ be a linear $[n,k,d(C)]_q R$ code with $d(C) \geq 2R - 1$. If $C$ is $(R,\mu)$-MCF, then

$$\gamma_\mu(C, R) = \left( \binom{n}{R} \cdot (q-1)^R - \binom{2R-1}{R-1} \cdot A_{2R-1}(C) \right) / \mu \cdot (q^{n-k} - V_q(n, R - 1)) .$$

In the rest of the paper we will assume that

$$d(C) \geq 2R - 1. \quad (2.2)$$

Let $t = \left\lfloor \frac{d - 1}{2} \right\rfloor$ be the number of errors that can be corrected by a code with minimum distance $d$. Note that under Condition (2.2), $R = t + 1$; equivalently, $C$ is a quasi-perfect code in the classical sense.

The following definition of a $(\rho, \mu)$-saturating set in $PG(N,q)$ is given as in [3].

Definition 2.4. Let $S = \{P_1, \ldots, P_n\}$ be a subset of points of $PG(N,q)$. Then $S$ is said to be $(\rho, \mu)$-saturating if:

(M1) $S$ generates $PG(N,q)$;

(M2) there exists a point $Q$ in $PG(N,q)$ which does not belong to any subspace of dimension $\rho - 1$ generated by the points of $S$;

(M3) every point $Q$ in $PG(N,q)$ not belonging to any subspace of dimension $\rho - 1$ generated by the points of $S$, is such that the number of subspaces of dimension $\rho$ generated by the points of $S$ and containing $Q$, counted with multiplicity, is at least $\mu$. The multiplicity $m_T$ of a subspace $T$ is computed as the number of distinct sets of $\rho + 1$ independent points contained in $T \cap S$.

Note that if any $\rho + 1$ points of $S$ are linearly independent (that is, the minimum distance of the corresponding code is at least $\rho + 2$), then

$$m_T = \left( \frac{\#(T \cap S)}{\rho + 1} \right).$$

A $(\rho,\mu)$-saturating $n$-set in $PG(N,q)$ is called minimal if it does not contain a $(\rho,\mu)$-saturating $(n-1)$-set in $PG(N,q)$.

Let $S$ be a $(\rho,\mu)$-saturating $n$-set in $PG(n-k-1,q)$. The set $S$ is called optimal $(\rho,\mu)$-saturating set ($(\rho,\mu)$-OS set for short) if every point $Q$ in $PG(n-k-1,q)$ not belonging to any subspace of dimension $\rho - 1$ generated by the points of $S$, is such that the number of subspaces
of dimension \( \rho \) generated by the points of \( S \) and containing \( Q \), counted with multiplicity, is exactly \( \mu \).

An \([n, k]_q R\) code \( C \) with \( R = \rho + 1 \) corresponds to a \((\rho, \mu)\)-saturating \( n \)-set \( S \) in \( PG(n - k - 1, q) \) if \( C \) admits a parity-check matrix whose columns are homogeneous coordinates of the points in \( S \).

By [3, Proposition 3.6], a linear \([n, k]_q R\) code \( C \) corresponding to a \((\rho, \mu)\)-saturating \( n \)-set \( S \) in \( PG(n - k - 1, q) \) is a \((\rho + 1, \mu)\)-MCF code. Also, if \( S \) is a \((\rho, \mu)\)-OS set, the corresponding linear \([n, k]_q R\) code \( C \) is a \((\rho + 1, \mu)\) APMCF code with \( \gamma_\mu(C, \rho + 1) = 1 \). If in addition \( d(C) = 2R \), then it is a \((\rho + 1, \mu)\) PMCF code.

### 3 \((1, \mu)\)-saturating sets

For \( \rho = 1 \) Conditions (M1)-(M3) read as follows:

(M1) \( S \) generates \( PG(N, q) \);

(M2) \( S \) is not the whole \( PG(N, q) \);

(M3) every point \( Q \) in \( PG(N, q) \) not belonging to \( S \) is such that the number of secants of \( S \) through \( Q \) is at least \( \mu \), counted with multiplicity. The multiplicity \( m_\ell \) of a secant \( \ell \) is computed as

\[
m_\ell = \left( \frac{\#(\ell \cap S)}{2} \right).
\]

As already observed in [3], from Conditions (M1)-(M3) the following holds.

**Proposition 3.1.** \((i)\) Let \( C \) be the linear \([n, n - N - 1]_q 2\) code corresponding to a \((1, \mu)\)-saturating \( n \)-set \( S \). Then \( \mu \gamma_\mu(C, 2) \) is equal to the average number of secants of \( S \), counted with multiplicity, through a fixed point \( Q \in PG(N, q) \setminus S \).

\((ii)\) Let \( S \) be a \((1, \mu)\)-saturating set in \( PG(N, q) \). Then \( S \) is a \((1, \mu)\)-OS set precisely when each point \( Q \in PG(N, q) \setminus S \) belongs to exactly \( \mu \) secants of \( S \), counted with multiplicity.

Note that as \( R = 2 \), the condition \( d(C) > 2R - 1 \) reads as \( d(C) > 3 \).

Let \( B_3(S) \) denote the number of triples of collinear points in \( S \). The following is a characterization of \((1, \mu)\)-OS sets in \( PG(N, q) \); see [3].
Proposition 3.2. Let $S$ be a $(1, \mu)$-saturating set in $PG(N, q)$. Let $C_S$ be the $[n, n - N - 1]_q 2$ code corresponding to $S$.

(i) For the $\mu$-density of $C_S$ it holds that
\[
\gamma_{\mu}(C_S, 2) = \frac{n-1}{2}(q - 1) - \frac{3}{n}B_3(S) - \frac{\mu}{\mu \cdot \left(\frac{\#PG(N, q)}{n} - 1\right)}.
\]

(ii) The set $S$ is a $(1, \mu)$-OS set if and only if
\[
\frac{n - 1}{2}(q - 1) - \frac{3}{n}B_3(S) = \mu \cdot \left(\frac{\#PG(N, q)}{n} - 1\right).
\]

It is clear that if $q, N, \mu$ are fixed, then the best $\mu$-density is achieved for small $n$ and therefore, the following parameter seems to be relevant in this context.

Definition 3.3. The $\mu$-length function $\ell_{\mu}(2, r, q)$ is the smallest length $n$ of a linear $(2, \mu)$-MCF code with parameters $[n, n - r, d]_q 2$, $d \geq 3$, or equivalently the smallest cardinality of a $(1, \mu)$-saturating set in $PG(r - 1, q)$. For $\mu = 1$, $\ell_1(2, r, q)$ is the usual length function $\ell(2, r, q)$ [5, 6, 11] for 1-fold coverings.

Remark 3.4. [3] A number $\mu$ of disjoint copies of a 1-saturating set in $PG(N, q)$ give rise to a $(1, \mu)$-saturating set in $PG(N, q)$. Therefore,
\[
\ell_{\mu}(2, r, q) \leq \mu \ell(2, r, q).
\]

Denote by $\gamma_{\mu}(2, r, q)$ the minimum $\mu$-density of a linear $(2, \mu)$-MCF code of codimension $r$ over $\mathbb{F}_q$. Let $\delta(2, r, q)$ be the minimum density of a linear code with covering radius 2 and codimension $r$ over $\mathbb{F}_q$, then
\[
\gamma_{\mu}(2, r, q) \leq \frac{\frac{1}{2}(\mu \ell(2, r, q) - 1)(q - 1)}{\mu \cdot \left(\frac{\#PG(r - 1, q)}{\mu \ell(2, r, q)} - 1\right)} - 1 \sim \mu \delta(2, r, q).
\]

The same inequalities clearly hold for the best known lengths and densities, denoted, respectively, by $\overline{\ell}_{\mu}(2, r, q)$, $\overline{\ell}(2, r, q)$, $\overline{\mu}(2, r, q)$, and $\overline{\delta}(2, r, q)$:
\[
\overline{\ell}_{\mu}(2, r, q) \leq \mu \overline{\ell}(2, r, q).
\]
\[
\overline{\mu}(2, r, q) \leq \mu \overline{\delta}(2, r, q).
\]
From Equations (3.2)–(3.5), results for parameters \( \ell_{\mu}(2, r, q) \), \( \gamma_{\mu}(2, r, q) \), and \( \overline{\gamma}_{\mu}(2, r, q) \), can be immediately obtained from the vast body of literature on 1-saturating sets in finite projective spaces; see e.g. [1, 4–7, 10, 11, 13–17, 26, 30].

The aim of the present paper is to construct \((1, \mu)\)-saturating sets in \( PG(N, q) \) giving rise to \((2, \mu)\)-MCF codes with cardinality and density smaller to those in Inequalities (3.2)–(3.5).

4 Small \((1, \mu)\)-saturating sets in \( PG(N, q) \), \( q \) even

In \( PG(N, q) \), \( q \) even, small 1-saturating sets have been constructed; see [10, 14]. Roughly speaking, if \( N \) is odd, then in \( PG(N, q) \), \( q \) even, there are 1-saturating sets whose size is of the same order of magnitude as \( 2^{q^{(N-1)/2}} \). If \( N \) is even, then there exist 1-saturating sets of size about \( t_2(q)q^{(N-2)/2} \), where \( t_2(q) \) denotes the size of the smallest saturating set in \( PG(2, q) \). When \( q \) is a square, \( t_2(q) \leq 3\sqrt{q} - 1 \) holds [7]. Therefore, for \( q \) a square,

\[
\overline{\gamma}(2, r, q) \sim c(r)q^{(r-2)/2}, \quad \text{with} \quad c(r) = \begin{cases} 2 & \text{for even } r \\ 3 & \text{for odd } r \end{cases},
\]

and the following results on density about the order of magnitude of \( \overline{\gamma}_{\mu}(2, r, q) \) can be easily obtained, see (3.5):

\[
\overline{\gamma}_{\mu}(2, r, q) \sim \frac{1}{2} \mu c(r)^2. \quad (4.1)
\]

In this section we significantly improve (4.1) for the case \( \mu = \frac{q-2}{2} \), see (4.2) and (4.3).

For \( i = 0, \ldots, N \), let \( \pi_i \) be the subset of \( PG(N, q) \) defined as follows:

\[
\pi_i := \{(x_0, \ldots, x_N) \mid x_0 = x_1 = \ldots = x_{i-1} = 0, x_i \neq 0\},
\]

where \( x_0, \ldots, x_N \) are the homogeneous coordinates of a point in \( PG(N, q) \). Clearly, \( PG(N, q) \) is the disjoint union of \( \pi_0 \cup \pi_1 \cup \ldots \cup \pi_N \), also, each \( \pi_i \) with \( i < N \) can be viewed as an affine space \( AG(N-i, q) \), whereas \( \pi_N \) consists of a single point, namely \((0, \ldots, 0, 1)\).
Lemma 4.1. In an affine space $AG(N,q)$ with $q$ even, $q > 2$, there exists a subset $K$ of size less than or equal to

$$\left\lfloor \frac{1 + \sqrt{4q - 7}}{2} \right\rfloor q^{(N-1)/2}$$

such that every point of $AG(N,q) \setminus K$ belongs to at least $(q - 2)/2$ distinct secants of $K$.

Proof. Assume that $N$ is even. Then there exists a translation cap $K$ in $AG(N,q)$ of size $q^{N/2}$ (see (2.6) in [14]). By Proposition 2.5 in [14], every point of $AG(N,q) \setminus K$ belongs to $(q - 2)/2$ distinct secants of $K$. Then the assertion follows from

$$\sqrt{q} \leq \left\lfloor \frac{1 + \sqrt{4q - 7}}{2} \right\rfloor.$$

To deal with the case of odd dimension $N$, we set $s = \left\lfloor \frac{1 + \sqrt{4q - 7}}{2} \right\rfloor$ and we fix $s$ distinct elements $a_1, \ldots, a_s$ in $\mathbb{F}_q$. Note that

$$\binom{s}{2} \geq \frac{q - 2}{2}.$$

Let $K'$ be a translation cap in $AG(N - 1, q)$ of size $q^{(N-1)/2}$. Let

$$K = \{(P, a_i) \mid P \in K', i = 1, \ldots, s\} \subset AG(N, q).$$

By the doubling construction (see e.g. Remark 2.14 in [14]), we have that for each pair of integers $i, j$ with $1 \leq i < j \leq s$, the subset of $K$

$$K_{i,j} = \{(P, a_i), (P, a_j) \mid P \in K'\}
$$

is a complete cap of size $q^{(N-1)/2}$ in $AG(N, q)$.

Therefore, every point of $AG(N,q) \setminus K$ is covered by at least $\binom{s}{2} \geq (q - 2)/2$ secants of $K$. \hfill \square

A slight improvement of Lemma 4.1 can be obtained when $q$ is a square and there exists a translation cap of size $q^{3/2}$ in $AG(3, q)$.

Lemma 4.2. Assume that $q$ is a square and there exists a translation cap of size $q^{3/2}$ in $AG(3, q)$. Then in an affine space $AG(N,q)$ with $q$ even, $N > 2$, there exists a subset $K$ of size less than or equal to $q^{N/2}$ such that every point of $AG(N,q) \setminus K$ belongs to at least $(q - 2)/2$ distinct secants of $K$. 
Proof. The assertion for $N$ even follows from the proof of Lemma 4.1. Here it is possible to construct a translation cap in $AG(N, q)$ of size $q^{N/2}$ also for odd $N > 1$ by using Proposition 2.8 in [14].

Remark 4.3. The hypothesis of Lemma 4.2 are satisfied for instance for $q = 16$ (see [16, Lemma 3.1]).

Consider the partition

$$PG(N, q) = \pi_0 \cup \pi_2 \cup \ldots \cup \pi_{N-1} \cup \{(0, \ldots, 0, 1)\}.$$ 

As each $\pi_i$ is an $AG(N - i, q)$, by Lemma 4.1 we are able to find sets $K_i$ contained in $\pi_i$, of size at most

$$\left\lceil \frac{1 + \sqrt{4q - 7}}{2} \right\rceil q^{(N-i-1)/2},$$

and such that each point of $\pi_i \setminus K_i$ belongs at least $(q - 2)/2$ distinct secants of $K_i$.

Then the following result is obtained by considering the union of the $K_i$ ’s, together with the point $\{(0, \ldots, 0, 1)\}$.

Theorem 4.4. In a projective space $PG(N, q)$ with $q$ even, $q > 2$, there exists a subset $S$ of size equal to

$$1 + \left\lceil \frac{1 + \sqrt{4q - 7}}{2} \right\rceil (q^{N-1}/2 + q^{N-2}/2 + \ldots + q^{1/2} + 1)$$

such that every point of $PG(N, q) \setminus S$ belongs to at least $(q - 2)/2$ distinct secants of $S$.

If in addition $q$ is a square and there exists a translation cap of size $q^{3/2}$ in $AG(3, q)$, then $S$ can be chosen in such a way that

$$|S| = q^{N} + q^{N-1} + q^{N-2} + \ldots + q + q^{1/2} + 1 = \frac{q^{(N+1)/2} - 1}{q^{1/2} - 1}.$$ 

As for the density, we have that if $C(N, q)$ is the code corresponding to the set $S$ of Theorem 4.4, then by Proposition 3.2(i),

$$\gamma_{\frac{q-1}{2}}(C(N, q), 2) \leq \frac{|S|-1}{q-2} \left( \frac{|PG(N, q)|}{|S|} - 1 \right) = \frac{q-1}{q-2} \cdot \frac{|S|(|S|-1)}{|PG(N, q)| - |S|},$$

whence

$$\gamma_{\frac{q-2}{2}}(C(N, q), 2) < \frac{q-1}{q-2} \cdot \frac{q^{N} + 10q^{N-\frac{3}{2}} + 25q^{N-1}}{q^{N} + q^{N-1} + \ldots + q + 1 - q^{N/2} - 5q^{N-1}}.$$ (4.2)
Interestingly, if we fix codimension $N$ and let $q$ vary, we get that

$$\lim_{q \to \infty} \gamma_{2-2}(C(N, q), 2) = 1.$$ 

The situation is even more interesting if the hypothesis of Lemma 4.2 are satisfied. In this case we obtain the following very nice formula for $\gamma_{2-2}(C(N, q), 2)$, which is independent of $N$:

$$\gamma_{2-2}(C(N, q), 2) = \frac{q - 1}{q - 2} \cdot \frac{q^{(N+1)/2} - 1}{q^{(N+1)/2} + 1 - 1} = \frac{q - 1}{q - 2} \cdot \frac{\sqrt{q} + 1}{\sqrt{q} - 1} = \frac{(\sqrt{q} + 1)^2}{q - 2}. \quad (4.3)$$

For instance for $q = 16$ we have $|S| = \frac{4^{N+1} - 1}{3}$, and we get that $\gamma_{2-2}(C(N, q), 2)$ is equal to 25/14 independently of $N$.

## 5 Perfect and almost perfect $(2, \mu)$-MCF codes from classical geometrical objects in $PG(N, q)$

In this section we construct optimal $(1, \mu)$-saturating $n$-sets ($(1, \mu)$-OS $n$-sets) in $PG(N, q)$. Recall that an $[n, n - (N + 1), d(C)]$ code $C$ corresponding to a $(1, \mu)$-OS set is an almost perfect $(2, \mu)$-MCF code (APMCF code) if $d(C) = 3$ or perfect $(2, \mu)$-MCF code (PMCF code) if $d(C) = 4$, see Sections 2 and 3. The $\mu$-density of any APMCF or PMCF code $C$ is $\gamma_{\mu}(C, 2) = 1$.

In Proposition 5.7 we obtain MCF codes $C$ with $\mu$-density $\gamma_{\mu}(C, 2) = 1 + \frac{1}{q}$.

**Proposition 5.1.** Let $q = 2^v$ be even. Let $s = 2^k$, $1 \leq k \leq v$. Finally, let $n = (s-1)q + s$ and let $K$ be a maximal $(n, s)$-arc in $PG(2, q)$. Then $K$ is a $(1, \mu)$-OS $n$-set with parameters

$$n = (s-1)q + s, \quad \mu = \frac{1}{2}(s-1)n.$$ 

An $[n, n - 3, d(C_K)]$ code $C_K$ corresponding to $K$ is a $(2, \mu)$-PMCF code if $s = 2$ and a $(2, \mu)$-APMCF code if $s \geq 4$.

**Proof.** Every line of $PG(2, q)$ meets a maximal $(n, s)$-arc either in zero or in $s$ points whence it follows that every point of $PG(2, q)$ outside the arc lies on $\frac{n}{s}$ $s$-secant. Therefore, $R = 2$ and $\mu = \frac{s(n)}{2} = \frac{1}{2}(s-1)n$. 


For $s = 2$ the minimum distance $d(C_K)$ is 4 as in this case the arc $K$ is a hyperoval.

**Proposition 5.2.** An elliptic quadric $Q$ in $PG(3, q)$ is a $(1, \mu)$-OS $n$-set with

$$n = q^2 + 1, \quad \mu = \frac{1}{2}(q^2 - q).$$

A $[n, n - 4, 4]_q$ code $C_Q$ corresponding to $Q$ is a $(2, \mu)$-PMCF code.

*Proof.* Every point outside of the elliptic quadric $Q$ lies on $\frac{1}{2}(q^2 - q)$ bisecants.

**Proposition 5.3.** Let $q$ be square. A Hermitian curve $H$ in $PG(2, q)$ is a $(1, \mu)$-OS $n$-set with parameters

$$n = q\sqrt{q} + 1, \quad \mu = \frac{1}{2}(q^2 - q).$$

An $[n, n - 3, 3]_q$ code $C_H$ corresponding to $H$ is a $(2, \mu)$-APMCF code.

*Proof.* Every point outside of the Hermitian curve lies on $q - \sqrt{q}$ lines that are $(\sqrt{q} + 1)$-secants and on $\sqrt{q} + 1$ tangent lines to $H$. Therefore

$$\mu = (q - \sqrt{q})(\sqrt{q} + 1) = \frac{1}{2}(q^2 - q).$$

**Proposition 5.4.** Let $q$ be square. A Baer subplane $B$ in $PG(2, q)$ is a $(1, \mu)$-OS $n$-set with parameters

$$n = q + \sqrt{q} + 1, \quad \mu = \frac{1}{2}(q + \sqrt{q}).$$

An $[n, n - 3, 3]_q$ code $C_B$ corresponding to $B$ is a $(2, \mu)$-APMCF code.

*Proof.* Here $\mu = (\sqrt{q} + 1) = \frac{1}{2}(q + \sqrt{q})$ as every point outside the Baer subplane lies exactly on one $(\sqrt{q} + 1)$-secant of the subplane and on $q$ tangents to $B$.

**Proposition 5.5.** Let $S \subset PG(N, q)$ be a set such that through each point $P \in S$ the number of $i$-secants of $S$ is a fixed integer $x_i$. Then $PG(N, q) \setminus S$ is a $(1, \mu)$-OS $n$-set with

$$n = \frac{q^{N+1}}{q - 1} - \left|S\right|, \quad \mu = \sum_{i=1}^{\frac{q^{N+1}}{q - 1}} x_i \left(\frac{q + 1 - i}{2}\right).$$
Proof. It is enough to observe that each $i$-secant of $S$ becomes a $(q + 1 - i)$-secant of $PG(N, q) \setminus S$.

**Corollary 5.6.**

(i) Let $q = 2^v$ be even and $s = 2^k$, $1 \leq k \leq v - 1$. Consider $n = (s - 1)q + s$ and let $K$ be a maximal $(n, s)$-arc in $PG(2, q)$. The set $S = PG(2, q) \setminus K$ is a $(1, \mu)$-OS $n$-set with

$$n = q^2 + q + 1 - (s - 1)q - s, \quad \mu = (q + 1)\left(\frac{q + 1 - s}{2}\right).$$

An $[n, n - 3, 3]_2$ code $C_S$ corresponding to $S$ is a $(2, \mu)$-APMCF code.

(ii) Let $q$ be square and let $H$ be a Hermitian curve in $PG(2, q)$. The set $S = PG(2, q) \setminus H$ is a $(1, \mu)$-OS $n$-set with parameters

$$n = q^2 - q\sqrt{q} + q, \quad \mu = \left(\frac{q}{2}\right) + q\left(\frac{q - \sqrt{q}}{2}\right).$$

An $[n, n - 3, 3]_2$ code $C_S$ corresponding to $S$ is a $(2, \mu)$-APMCF code.

(iii) Let $q$ be square and consider a a Baer subplane $B$ in $PG(2, q)$. The set $S = PG(2, q) \setminus B$ is a $(1, \mu)$-OS $n$-set with parameters

$$n = q^2 - \sqrt{q}, \quad \mu = (\sqrt{q} + 1)\left(\frac{q - \sqrt{q}}{2}\right) + (q - \sqrt{q})\left(\frac{q}{2}\right).$$

An $[n, n - 3, 3]_2$ code $C_S$ corresponding to $S$ is a $(2, \mu)$-APMCF code.

(iv) Let $S \subset PG(N, q)$ be a set such that $PG(N, q) \setminus S$ is a $k$-cap, $N \geq 2$, $k \geq 2$. Then $S$ is a $(1, \mu)$-OS $n$-set with parameters

$$n = \frac{q^{N+1} - 1}{q - 1} - k, \quad \mu = (k - 1)\left(\frac{q - 1}{2}\right) + \left(\frac{q^N - 1}{q - 1} - k + 1\right)\left(\frac{q}{2}\right).$$

An $[n, n - N - 1, 3]_2$ code $C_S$ corresponding to $S$ is a $(2, \mu)$-APMCF code.

Proof. The claims follow directly from Proposition 5.5 and properties of the geometrical objects as pointed out in the previous propositions. □
Proposition 5.7. Let $q$ be odd. An oval $O$ in $PG(2, q)$ is a $(1, \mu)$-saturating $n$-set with parameters

$$n = q + 1, \quad \mu = \frac{1}{2}(q - 1).$$

An $[n, n - 3, 4]_q 2$ code $C_O$ corresponding to $O$ is a $(2, \mu)$-MCF code with $\mu$-density $\gamma_\mu(C_O, 2) = 1 + \frac{1}{q}$.

Proof. Each internal point of the oval lies on $\frac{1}{2}(q + 1)$ bisecants, whereas every external point lies on $\frac{1}{2}(q - 1)$ bisecants. Therefore, $\mu = \frac{1}{2}(q - 1)$.

By Proposition 3.1(i),

$$\mu \gamma_\mu(C_O, 2) = \frac{\frac{1}{2}q(q - 1) \cdot \frac{1}{2}(q + 1) + \frac{1}{2}q(q + 1) \cdot \frac{1}{2}(q - 1)}{\frac{1}{2}q(q - 1) + \frac{1}{2}q(q + 1)} = \frac{(q + 1)(q - 1)}{2q}.$$

\[ \square \]

6 Constructions of small $(1, \mu)$-saturating sets in $PG(2, q)$

In this section, we summarize some results concerning $(1, \mu)$-saturating sets in projective planes $PG(2, q)$.

6.1 Bounds

In this subsection we present some upper and lower bounds on the size of minimal $(1, \mu)$-saturating sets in $PG(2, q)$.

Proposition 6.1. In $PG(2, q)$, for a minimal $(1, \mu)$-saturated set $S$ the following holds.

(i)

$$\mu \leq (q + 1) \left( \frac{q}{2} \right).$$

(ii)

$$|S| \leq \begin{cases} 
q + \mu + 1 & \text{if } \mu \leq q + 2 \\
\min\{q + \mu, q^2 + q\} & \text{if } \mu \geq q + 3
\end{cases}.$$
Proof. (i) Any \((q^2 + q)\)-set in \(PG(2, q)\) is a \((q + 1)\(\frac{q}{2}\))-saturating set.

(ii) Let \(S\) be \((q + \mu + 1)\)-set in \(PG(2, q)\), \(q > 2, \mu \leq q + 2\) and consider a point \(P \in PG(2, q) \setminus S\). On the \(q + 1\) lines through \(P\) there are at least \(\mu\) pairs of points of \(S\) and therefore \(S\) is a \((1, \mu)\)-saturating set, possibly not minimal.

If \(\mu \geq q + 3\) and \(|S| = q + \mu\), then at least one triple of points of \(S\) lies on the same line through \(P\). So, in total there are at least \(\binom{3}{2} + (\mu - 3) = \mu\) pairs of points of \(S\). The bound \(|S| \leq q^2 + q\) holds due to Condition (M2).

We recall the following proposition from [3] concerning bounds on the smallest possible size of \((1, \mu)\)-saturating sets.

**Proposition 6.2.** For the length function \(\ell_\mu(2, 3, q)\), the following relations hold.

(i) Trivial bound:

\[
\ell_\mu(2, 3, q) \geq \sqrt{2\mu q}. \tag{6.1}
\]

(ii) Probabilistic bound:

\[
\ell_\mu(2, 3, q) < 66\sqrt{\mu q \ln q}, \quad \text{if } \mu < 121q \log q. \tag{6.2}
\]

(iii) Baer bound for \(q\) a square:

\[
\ell_\mu(2, 3, q) \leq \mu(3\sqrt{q} - 1).
\]

In some cases we can do better than the trivial lower bound mentioned above.

**Proposition 6.3.** Let \(A\) be a \((1, \mu)\)-saturating set in \(PG(2, q)\) of size \(k\). Suppose that \(\ell\) and \(\ell'\) are an \(r\)-secant and an \(s\)-secant of \(A\) respectively, with \(s \geq r\). Then

\[
k \geq \min \left\{ \frac{r + \frac{3}{2} + \sqrt{(s - r)(s + r - 2) + 2\mu(q - r + 1) + \frac{5}{4}}}{r + \frac{3}{2} + \sqrt{(s - r)(s + r - 1) + 2\mu(q - r) + \frac{1}{4}}} \right\}.
\]
Proof. There are $k - r - s + 1$ or $k - r - s$ points of $A$ not contained in $\ell \cup \ell'$, depending on the point $\ell \cap \ell'$ belonging or not to $A$. Since $A$ is a $(1, \mu)$-saturating set, then each point of $\ell$ not belonging to $A$ is covered at least $\mu$ times. Therefore, in the first case we obtain

$$(k - r - s + 1)(s - 1) + \binom{k - r - s + 1}{2} + \binom{r}{2} \geq \mu(q + 1 - r)$$

which implies

$$k \geq r + \frac{1}{2} + \sqrt{(s - r)(s + r - 2) + 2\mu(q - r + 1) + \frac{5}{4}},$$

whereas in the second case

$$(k - r - s)s + \binom{k - r - s}{2} + \binom{r}{2} \geq \mu(q - r)$$

implies

$$k \geq r + \frac{1}{2} + \sqrt{(s - r)(s + r - 1) + 2\mu(q - r) + \frac{1}{4}}.$$ 

Note that in the second case we do not consider how many times the intersection point of $\ell$ and $\ell'$ is covered. \qed

6.2 Constructions

In this subsection we explicitly construct examples of $(1, 2)$-saturating sets in $PG(2, q)$ of sizes $q + 2$ and $q + 3$.

Theorem 6.4. There exists a minimal $(1, 2)$-saturating set of size $q + 3$ in $PG(2, q)$, with $q = p^h$, $p$ prime. Its stabilizer in $PGL(3, q)$ has size $hq(q - 1)$. The corresponding $[q + 3, q]_q$ code $C$ is a $(2, 2)$-MCF with $\mu$-density $\gamma_2(C, 2) \approx 1 + \frac{1}{2q}.$

Proof. Let $\ell$ be a line and consider two points $P_1, P_2 \notin \ell$. It is straightforward to check that $A = \ell \cup \{P_1, P_2\}$ is a minimal $(1, 2)$-saturating set. Consider now two of such sets $A_1 = \ell_1 \cup \{P_1, Q_1\}$ and $A_2 = \ell_2 \cup \{P_2, Q_2\}$, with $P_i, Q_i \notin \ell_i$ and let $X_i, Y_i \in \ell_i$. The two sets $A_1$ and $A_2$ are projective equivalent since $\varphi : PG(2, q) \to PG(2, q)$ such that $\varphi(P_1) = P_2$, $\varphi(Q_1) = Q_2$, $\varphi(X_1) = X_2$ and $\varphi(Y_1) = Y_2$ is a collineation sending $A_1$ in $A_2$.  


The line ℓ can be chosen in \( q^2 + q + 1 \) different ways. \( P_1 \) and \( P_2 \) can be taken in an arbitrary way in \( PG(2, q) \setminus ℓ \). Therefore \( A \) can be chosen in \( q^2(q^2 - 1)(q^2 + q + 1) \). Hence, its stabilizer in \( PGL(3, q) \) has size

\[
\frac{|PΓL(3, q)|}{hq^2(q^2 - 1)(q^2 + q + 1)} = \frac{hq^3(q^2 - 1)(q^3 - 1)}{q^2(q^2 - 1)(q^2 + q + 1)} = hq(q - 1).
\]

The \( μ \)-density of the code \( C \) can be calculated by (3.1) where clearly \( n = q + 3, B_3(S) = 1 + \binom{q+1}{3}, \; \#PG(N, q) = q^2 + q + 1 \).

**Theorem 6.5.** Let \( ℓ \) be a line and \( P, Q, R, S, T \) points such that \( P, R, S \) and \( Q, R, T \) are collinear, and \( P, Q \in ℓ \). Then \( A = (ℓ \setminus \{P, Q\}) \cup \{R, S, T\} \subset PG(2, q) \) is a minimal \((1, 2)\)-saturating \((q + 2)\)-set for all \( q \geq 4 \).

**Proof.** All the points on the line \( PR \) are covered once by \( RS \) and once by the lines joining \( T \) and the points of \( ℓ \setminus \{P\} \). The points of the line \( PT \) are covered twice by the lines through \( R \) and \( S \) and the the points of \( ℓ \setminus \{P\} \). A similar argument holds for the points on the lines \( QR \) and \( QS \). The points on the lines through \( P \) distinct from \( ℓ, PR, \) and \( PT \) are covered at least two times by the lines through \( T, R, S \) and the points of \( ℓ \setminus \{P\} \). A similar argument holds for the points on the lines through \( Q \) distinct from \( ℓ, QR, \) and \( QS \).
This example is minimal. In fact it is not possible to delete $T$ (resp. $S$) since the points on the line $SR$ (resp. $RT$) would not be covered twice. $A \setminus \{R\}$ does not cover $R$. Let $X \in \ell \setminus \{P, Q\}$, then $A \setminus \{X\}$ does not cover twice the point $TX \cap m$.

**Theorem 6.6.** Let $\ell$ be a line and $P, Q, R, S, T$ points such that $P, R, S, T$ are collinear, and $P, Q \in \ell$. Then $A = (\ell \setminus \{P, Q\}) \cup \{R, S, T\}$ is a minimal $(1, 2)$-saturating $(q + 2)$-set for all $q \geq 4$.

*Proof.* Let $s$ be a line through $Q$ different from $\ell$ and let $m$ be the line containing $R, S, T$.

Let $X = m \cap s$. If $X \in \{R, S, T\}$, then every point of $s \setminus \{Q\}$ is covered twice by the lines through the points $\{R, S, T\} \setminus \{X\}$ and the the $q - 1$ points of $\ell$. If $X \notin \{R, S, T\}$, then every point of $s \setminus \{Q, X\}$ is covered three times by the lines through $R, S, T$ and the $q - 1$ points of $\ell$.

It is not possible to delete $R, S, T$ since in this case the points on $m$ are covered only once. Also, it is not possible to delete a point $X \in \ell$, since in this case in the line $XT$ only $(q - 2)$ points are covered twice. Hence $A$ is a minimal $(1, 2)$-saturating set of size $(q + 2)$. □
6.3 (1, \mu)-saturating sets and partitions of $PG(2, q)$ in Singer point-orbits

In [3,8,9,12], partitions of $PG(2, q)$ by Singer subgroups are considered. Methods of [8,12], allow us to represent an incidence matrix of the plane $PG(2, q)$ as a BDC matrix defined below. We present some new results; see [12, Sec. 7.3] for comparison. We recall the following definition.

**Definition 6.7.** [8] Let $v = td$. A $v \times v$ matrix $A$ is said to be block double-circulant matrix (or BDC matrix) if

$$A = \begin{bmatrix}
C_{0,0} & C_{0,1} & \cdots & C_{0,t-1} \\
C_{1,0} & C_{1,1} & \cdots & C_{1,t-1} \\
\vdots & \vdots & \ddots & \vdots \\
C_{t-1,0} & C_{t-1,1} & \cdots & C_{t-1,t-1}
\end{bmatrix}, \quad (6.3)$$

$$W(A) = \begin{bmatrix}
w_0 & w_1 & w_2 & w_3 & \cdots & w_{t-2} & w_{t-1} \\
w_{t-1} & w_0 & w_1 & w_2 & \cdots & w_{t-3} & w_{t-2} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
w_1 & w_2 & w_3 & w_4 & \cdots & w_{t-1} & w_0
\end{bmatrix}, \quad (6.4)$$

where $C_{i,j}$ is a circulant $d \times d$ $0,1$-matrix for all $i, j$; submatrices $C_{i,j}$ and $C_{t,m}$ with $j-i \equiv m-l$ (mod $t$) have equal weights; $W(A)$ is a circulant $t \times t$ matrix whose entry in a position $i, j$ is the weight of $C_{i,j}$. $W(A)$ is called a weight matrix of $A$. The vector $W(A) = (w_0, w_1, \ldots, w_{t-1})$ is called a weight vector of $A$.

Let $q^2 + q + 1 = dt$. Then, by using the cyclic Singer group of $PG(2, q)$, the incidence matrix of the plane $PG(2, q)$ can be constructed as a BDC matrix $A$ of the form (6.3). In this plane, we number the points $P_1, \ldots, P_{q^2+q+1}$ and the lines $\ell_1, \ldots, \ell_{q^2+q+1}$ so that $P_i$ corresponds to the $i$-th column of $A$ and $\ell_i$ corresponds to the $i$-th row of $A$. Denote by $P_v = \{P_{dv+1}, \ldots, P_{dv+d}\}$, $0 \leq v \leq t - 1$, the point set corresponding to the $(v+1)$-th block column of $A$. Let $L_u = \{\ell_{du+1}, \ldots, \ell_{du+d}\}$, $0 \leq u \leq t - 1$, be the line set corresponding to the $(u+1)$-th block row of $A$. Here and further addition and subtraction of indices are expressed modulo $t$.

We give a development of [3, Lemma 7.7].

**Lemma 6.8.** Let $1 \leq m \leq t - 1$. An md-set

$$P^{(m)} = P_0 \cup P_1 \cup \ldots \cup P_{m-1}$$

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corresponding to the first \( md \) columns of \( A \) in (6.3) is a \((1, \mu)\)-saturating set \( S \) in \( PG(2, q) \) with

\[
\mu = \min_v N_v^{(m)}, \quad 1 \leq m \leq v \leq t - 1, \tag{6.5}
\]

\[
N_v^{(m)} = \sum_{u=0}^{t-1} w_{t-u+v} \left( \frac{w_u^{(m)}}{2} \right) \geq 0, \quad w_u^{(m)} = \sum_{j=0}^{m-1} w_{t-u+j}.
\]

Moreover, an \([md, md - 3, 3]q\) code \( C_S \) corresponding to \( S \) is a \((2, \mu)\)-MCF code with minimum distance \( d = 3 \) and \( \mu \)-density

\[
\gamma_{\mu}(C_S, 2) = \frac{1}{\mu} \cdot \frac{\sum_{v=m}^{t-1} N_v^{(m)}}{t - m}. \tag{6.6}
\]

**Proof.** Every line of \( L_u \) is a \( w_u^{(m)} \)-secant of \( P^{(m)} \). Let \( v \geq m \). Every point of \( P_v \) is covered by \( w_{t-u+v} \) specimens of \( w_u^{(m)} \)-secants of \( P^{(m)} \) with multiplicity \( \left( \frac{w_u^{(m)}}{2} \right) \) for \( 0 \leq u \leq t - 1 \). So, every point of \( P_v \), \( m \leq v \leq t - 1 \), is covered by \( N_v^{(m)} \) secants of \( P^{(m)} \). This implies (6.5) and, together with Proposition 3.1, gives rise to (6.6). \( \square \)

By [3, Th. 7.8], the Singer partition of \( PG(2, q) \) gives a \((1, \mu)\)-OS set and the corresponding \((1, \mu)\)-APMCF code in the following cases:

- \( m = t - 1 \) for an arbitrary weight vector;
- \( 1 \leq m \leq t - 1 \) and the weight vector has the form \( \overline{W}(A) = (w_0, w, \ldots, w) \);
- \( m = 1 \) and the weight vector contains exactly two distinct weights.

**Example 6.9.** All the \((1, \mu)\)-saturating \( md \)-sets below are optimal by [3, Th. 7.8]. The corresponding \([md, md - 3, 3]q\) codes \( C \) are \((1, \mu)\)-APMCF with \( \gamma_{\mu}(C, 2) = 1 \). The multiplicity \( \mu \) has been calculated by (6.5) or by [3, Th. 7.8].

(i) Let \( q \) be square. Then \((q^2 + q + 1) = (q + \sqrt{q} + 1)(q - \sqrt{q} + 1)\). Let \( d = q + \sqrt{q} + 1 \), \( t = q - \sqrt{q} + 1 \). There is a partition of \( PG(2, q) \) such that all the subsets \( P_v \) are disjoint Baer subplanes. We have \( \overline{W}(A) = (\sqrt{q} + 1, 1, \ldots, 1) \) whence

\[
\mu = m \left( \frac{\sqrt{q} + m}{2} \right) + (q + 1 - m) \left( \frac{m}{2} \right), \quad 1 \leq m \leq t - 1.
\]

The case \( m = 1 \) coincides with code of Proposition 5.4.
(ii) Let \( q = p^{4v+2}, p \equiv 2 \mod 3 \). Then by [12, Prop. 4],

\[
t = 3, \quad d = \frac{q^2 + q + 1}{3}, \quad w_0 = \frac{q + 2\sqrt{q} + 1}{3}, \quad w_1 = w_2 = w = \frac{q - \sqrt{q} + 1}{3}.
\]

For \( m = 1 \) we have

\[
\mu = w\left(\frac{w_0}{2}\right) + \left(\frac{w}{2}\right)(w_0 + w) = \frac{1}{18}(q^3 - q\sqrt{q} - 2).
\]

(iii) Let \( q = p^{4v}, p \equiv 2 \mod 3 \). Then by [12, Prop. 4],

\[
t = 3, \quad d = \frac{q^2 + q + 1}{3}, \quad w_0 = \frac{q - 2\sqrt{q} + 1}{3}, \quad w_1 = w_2 = w = \frac{q + \sqrt{q} + 1}{3}.
\]

For \( m = 1 \) we have

\[
\mu = w\left(\frac{w_0}{2}\right) + \left(\frac{w}{2}\right)(w_0 + w) = \frac{1}{18}(q^3 + q\sqrt{q} - 1).
\]

(iv) Let \( q = p^{2c} \). Let \( t \) be a prime divisor of \( q^2 + q + 1 \). Then \( t \) divides either \( q + \sqrt{q} + 1 \) or \( q - \sqrt{q} + 1 \). Assume that \( p \) (mod \( t \)) is a generator of the multiplicative group of \( \mathbb{Z}_t \). By [12, Prop. 6], in this case \( w_0 = (q + 1 \pm (1 - t)\sqrt{q})/t, w_1 = \ldots = w_{t-1} = w = (q + 1 \pm \sqrt{q})/t \). For \( m = 1 \) we have

\[
\mu = w\left(\frac{w_0}{2}\right) + \left(\frac{w}{2}\right)(w_0 + w(t - 2)) = \frac{q^3 \pm (t - 2)q\sqrt{q} - t + 1}{2t^2}.
\]

Note that the hypothesis that \( p \) (mod \( t \)) is a generator of the multiplicative group of \( \mathbb{Z}_t \) holds, e.g. in the following cases: \( q = 3^4, t = 7; q = 2^8, t = 13; q = 5^4, t = 7; q = 2^{12}, t = 19; q = 3^8, t = 7; q = 2^{16}, t = 13; q = 17^4, t = 7; p \equiv 2 \) (mod \( t \), \( t = 3 \).

(v) Let \( q = 125 \). By [12, Tab. 1], there is the partition with \( t = 19, d = 829, \overline{W}(A) = (4, 9, 9, 9, 9, 4, 9, 9, 4, 9, 4, 9, 4, 9, 9, 4, 9, 4, 9) \).

For \( m = 1 \) we have \( \mu = 2706 \).

Partitions providing \((1, \mu)\)-OS sets are not always possible. But, as rule, the partitions provide “good” \((1, \mu)\)-saturating sets such that the corresponding \((1, \mu)\)-MCF codes have \(\mu\)-density \(\gamma_{\mu}(C, 2)\) of order of magnitude less than \(1 + \frac{1}{cq}, c \geq 1\). In the following we give examples of “good” \((1, \mu)\)-saturating sets.
Table 1: Values of $\mu$ and $\mu$-density for partitions with three distinct values of $w_i$

| $q$ | $w_0$ | $w_1$ | $w_2$ | $n = d$ | $\mu$ | $\gamma_{\mu}(C, 2)$ | < |
|-----|-------|-------|-------|--------|------|----------------|----|
| 7   | 1     | 4     | 3     | 19     | 18   | 1.0833         | 1 + 1/q |
| 13  | 4     | 7     | 3     | 61     | 117  | 1.0256         | 1 + 1/3q |
| 19  | 4     | 7     | 9     | 127    | 375  | 1.0200         | 1 + 1/2q |
| 31  | 7     | 12    | 13    | 331    | 1656 | 1.0045         | 1 + 1/7q |
| 37  | 13    | 9     | 16    | 469    | 2796 | 1.0075         | 1 + 1/3q |
| 43  | 19    | 13    | 12    | 631    | 4392 | 1.0024         | 1 + 1/9q |
| 49  | 13    | 21    | 16    | 817    | 6498 | 1.0046         | 1 + 1/4q |
| 61  | 16    | 25    | 21    | 1261   | 12570| 1.0036         | 1 + 1/4q |
| 67  | 28    | 19    | 21    | 1519   | 16653| 1.0019         | 1 + 1/7q |
| 73  | 19    | 28    | 27    | 1801   | 21627| 1.0008         | 1 + 1/16q |
| 79  | 31    | 21    | 28    | 2107   | 27363| 1.0019         | 1 + 1/6q |
| 97  | 28    | 31    | 39    | 3169   | 50601| 1.0013         | 1 + 1/7q |
| 103 | 28    | 37    | 39    | 3571   | 60708| 1.0008         | 1 + 1/11q |
| 127 | 36    | 49    | 43    | 5419   | 113673| 1.0012        | 1 + 1/6q |
| 139 | 39    | 49    | 52    | 6487   | 149175| 1.0007       | 1 + 1/11q |
| 157 | 61    | 48    | 49    | 8269   | 214848| 1.0002       | 1 + 1/35q |
| 163 | 63    | 49    | 52    | 8911   | 240387| 1.0005       | 1 + 1/12q |

Example 6.10. Using the approach of [12], we obtain by computer search partitions with $t = 3$ and with three distinct values of $w_i$, see also [12, Table 1]. We take $m = 1$ and $n = d$. The values of $q, w_i, n, \mu,$ and $\gamma_{\mu}(C, 2)$ are given in Table 1. The values of $\mu$ and $\gamma_{\mu}(C, 2)$ are obtained by (6.5) and (6.6) respectively. In the last column we write relation of the form $1 + \frac{1}{cq}$ such that $\gamma_{\mu}(C, 2) < 1 + \frac{1}{cq}$. One can see that $1 \leq c \leq 35$.

7 Classification of minimal and optimal $(1, \mu)$-saturating sets in $PG(2, q)$

We performed a computer based search for minimal $(1, 2)$-saturating sets. The results are collected in Table 2. In the 2-nd column, the values $\ell(2, 3, q)$ of the smallest cardinality of a 1-saturating set in $PG(2, q)$, taken from [1, 11], are given. The cases when $\ell(2, 3, q) = \overline{\ell}(2, 3, q)$ are marked by the dot “.”. In the 5-th column, we give some values of
Table 2: The number of nonequivalent minimal (1,2)-saturating $n$-sets in $PG(2,q)$ and the spectrum of sizes $n$.

| $q$ | $\ell(2, 3, q)$ | $2\sqrt{q}$ | $q + \mu + 1$ | Spectrum of $n$ |
|-----|-----------------|-------------|---------------|-----------------|
| 3   | $4^1, \cdot$    | 4           | 6             | $6^4, \cdot$    |
| 4   | $5^1, \cdot$    | 4           | 7             | $6^27^5, \cdot$ |
| 5   | $6^6, \cdot$    | 5           | 8             | $6^17^48^1$, *  |
| 7   | $6^3, \cdot$    | 6           | 10            | $8^139^564^10^424$, * |
| 8   | $6^1, \cdot$    | 6           | 11            | $8^29^154^10^3372^11^611$, * |
| 9   | $6^1, \cdot$    | 6           | 12            | $8^19^57^10^121^45^11^7674^12^3049$, * |
| 11  | $7^1, \cdot$    | 7           | 14            | $10^434^8[11 - 14]$, * |
| 13  | $8^2, \cdot$    | 8           | 16            | $10^211^50794[12 - 16]$, * |
| 16  | $9^4, \cdot$    | 8           | 19            | $11^52[12 - 19]$, * |
| 17  | $10^{3640}, \cdot$ | 9         | 20            | $[12 - 20]$, * |
| 19  | $10^46$, *      | 9           | 22            | $[13 - 22]$ |
| 23  | $10^1, \cdot$   | 10          | 26            | $[15 - 26]$ |
| 25  | 12              | 10          | 28            | $[17 - 28]$ |
| 27  | 12              | 11          | 30            | $[17 - 30]$ |
| 29  | 13              | 11          | 32            | $[19 - 32]$ |
| 31  | 14              | 12          | 34            | $[19, 21 - 34]$ |
| 32  | 13              | 12          | 35            | $[20 - 35]$ |
| 37  | 15              | 13          | 38            | $[23, 26 - 40]$ |
| 41  | 16              | 13          | 44            | $[25, 29 - 44]$ |
| 43  | 16              | 14          | 46            | $[25, 30 - 46]$ |
| 47  | 18              | 14          | 50            | $[27, 34 - 50]$ |
| 49  | 18              | 14          | 52            | $[29, 34 - 52]$ |

For $3 \leq q \leq 17$, we have found the complete spectrum of sizes $n$. This situation is marked by the dot “. ”. In the 2-nd and the 5-th columns, the superscript notes the numbers of nonequivalent sets of the corresponding size. For $3 \leq q \leq 9$, we obtain the complete classification of the spectrum of sizes $n$ of minimal (1,2)-saturating $n$-sets in $PG(2,q)$. This situation is marked by the asterisk *. In the 3-rd column the trivial lower bound (6.1) is given. Finally, the size $q + \mu + 1 = q + 3$ of the largest minimal (1,2)-saturating set in $PG(2,q)$, see Proposition 6.1(ii), is written in the 4-th column.
Table 3: Sizes of small (1,2)-saturating $n$-sets in $PG(2, q)$.

| $q$  | 53 59 61 67 71 73 79 81 83 89 97 101 103 109 113 125 127 131 139 |
|------|-----------------------------------------------------------|
| $n$  | 31 33 35 37 39 41 39 45 45 49 53 55 55 57 59 61 67 67 69 73 |

Using different constructions we obtained (1,2)-saturating $n$-sets in $PG(2, q)$ having several points on a conic, with sizes described in Table 3.

The smallest cardinalities of (1,2)-saturating sets for each $q$ in Table 2 and sizes $n$ in Table 3 are smaller than $2^7(2, 3, q)$. So, for $\mu = 2$, $r = 3$, and $q$ from Tables 2 and 3, the goal formulated at the end of Section 3 is achieved.

In Table 4 a classification of minimal $m$-saturating sets for some values of $\mu \leq (q + 1)(\delta - 1)$, $q \leq 11$, is presented; the superscript over a size indicates the number of distinct $\mu$-saturating sets of that size (up to collineations). If the subscript is absent, then there exists at least a $(1, \mu)$-saturating set of that size.

**Remark 7.1.** By property (M3) of Definition 2.4, see also (M3) in Section 3, a $(1, \mu)$-saturating set is also a $(1, \mu - k)$-saturating set for $1 \leq k \leq \mu - 1$. Moreover, a minimal $(1, \mu)$-saturating set can also be a minimal $(1, \mu - k)$-saturating set for $k = 1, 2, \ldots, \delta, \delta \geq 1$. This happens when removing any point from this set we obtain a $(1, \mu - \delta - 1)$-saturating set. For example, let $q = 3$ and consider a line $\ell$. Then $S = PG(2, 3) \setminus \ell$ is a $(1,9)$-saturating set and removing any point from $S$ we obtain a $(1,4)$-saturating set. So, $S$ is a minimal $(1,9)$-, $(1,8)$-, $(1,7)$-, $(1,6)$-, and $(1,5)$-saturating set. This example and many other such situations are written in Table 4.

In Table 5 we give the classification of optimal $(1, \mu)$-saturating sets ($(1, \mu)$-OS) in $PG(2, q)$. For such sets $S$, every point of $PG(2, q) \setminus S$ is covered exactly $\mu$ times. An entry of the form $n^t_\mu$ means that there exist $t$ projectively distinct optimal $(1, \mu)$-saturating sets with size $n$. Entries provided by Propositions 5.1 – 5.4 and 7.3, Example 6.9, and Corollary 5.6 are written in bold font.

**Observation 7.2.** For $q = 3$, the $(1,3)$-OS of size 7 is 2 concurrent lines; the $(1,4)$-OS of size 7 is contained in 3 concurrent lines. For
Table 4: Classification of minimal \((1, \mu)\)-saturating sets in \(PG(2, q)\)

| \(\mu\) | \(3\)  | \(4\)  | \(5\)  | \(6\)  | \(7\)  | \(8\)  | \(9\)  | \(10\) |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \(q = 3\) | \(11\) | \(12\) | \(13\) | \(14\) | \(15\) | \(16\) | \(17\) | \(18\) |
| \(q = 4\) | \(12\) | \(-\)  | \(-\)  | \(-\)  | \(-\)  | \(-\)  | \(-\)  | \(-\)  |
| \(q = 5\) | \(14\) | \(15\) | \(16\) | \(17\) | \(18\) | \(19\) | \(20\) | \(21\) |
| \(q = 6\) | \(18\) | \(19\) | \(20\) | \(21\) | \(22\) | \(23\) | \(24\) | \(25\) |
| \(q = 7\) | \(21\) | \(22\) | \(23\) | \(24\) | \(25\) | \(26\) | \(27\) | \(28\) |
| \(q = 8\) | \(25\) | \(26\) | \(27\) | \(28\) | \(29\) | \(30\) | \(31\) | \(32\) |

| \(\mu\) | \(3\)  | \(4\)  | \(5\)  | \(6\)  | \(7\)  | \(8\)  | \(9\)  | \(10\) |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \(q = 3\) | \(43\) | \(44\) | \(45\) | \(46\) | \(47\) | \(48\) | \(49\) | \(50\) |
| \(q = 4\) | \(51\) | \(52\) | \(53\) | \(54\) | \(55\) | \(56\) | \(57\) | \(58\) |
| \(q = 5\) | \(51\) | \(52\) | \(53\) | \(54\) | \(55\) | \(56\) | \(57\) | \(58\) |

\(| 59 | 60 \)
Table 5: Classification of optimal $(1, \mu)$-saturating $n$-sets in $PG(2, q)$

| $q$ | $n^t_\mu$ |
|-----|------------|
| 3   | $7^1_3 \ 7^1_4 \ 9^1_6 \ 10^1_8 \ 9^1_9 \ 10^1_9 \ 11^1_{10} \ 12^1_{12}$ |
| 4   | $6^1_3 \ 7^1_4 \ 9^1_5 \ 9^1_6 \ 11^1_8 \ 12^1_9 \ 12^2_{10} \ 13^2_{12} \ 14^1_4 \ 15^1_5 \ 15^1_{15} \ 15^1_{16}$ |
|     | $15^1_{17} \ 16^1_{18} \ 17^1_{21} \ 16^1_{24} \ 17^1_{24} \ 18^1_{24} \ 18^1_{25} \ 19^1_{27} \ 20^1_{30}$ |
| 5   | $11^1_5 \ 14^1_{11} \ 15^1_{12} \ 16^1_{15} \ 16^1_{18} \ 19^1_{22} \ 19^2_{23} \ 21^2_{27} \ 21^2_{30} \ 22^3_{32} \ 23^3_{35}$ |
|     | $25^1_{40} \ 25^1_{41} \ 25^1_{42} \ 26^1_{44} \ 27^1_{48} \ 25^1_{50} \ 26^1_{50} \ 27^1_{51} \ 28^1_{52} \ 28^1_{53} \ 29^1_{56} \ 30^1_{60}$ |

$q = 4$ the $(1,3)$-OS of size 6 is the hyperoval, the $(1,4)$-OS of size 7 is a complete 3-arc, cf. with Proposition 5.1.

In all these 4 cases, the points external to the $(1, \mu)$-OS, say $S$, form a unique orbit of the stabilizer group of $S$.

In the following we give some constructions of optimal $(1, \mu)$-OS in $PG(2, q)$ providing many entries in Table 5.

**Theorem 7.3.** The following sets $S$ are $(1, \mu)$-OS in $PG(2, q)$.

(i) The set $S$ is the union of $L$ concurrent lines, $2 \leq L \leq q$. It holds that

$$|S| = 1 + Lq, \ \mu = \left(\begin{array}{c} L \\ 2 \end{array}\right)q.$$  

(ii) The set $S$ is the union of $q$ concurrent lines and $b$ other points on the $(q + 1)$-th one, $1 \leq b \leq q - 1$. It holds that

$$|S| = 1 + q^2 + b, \ \mu = \left(\begin{array}{c} b + 1 \\ 2 \end{array}\right) + \left(\begin{array}{c} q \\ 2 \end{array}\right)q.$$  

(iii) The set $S$ is a triangle. It holds that

$$|S| = 3q, \ \mu = 3 + (q - 2)\left(\begin{array}{c} 3 \\ 2 \end{array}\right) = 3(q - 1).$$  

(iv) The set $S = PG(2, q) \setminus T$ where $T$ is a vertex-less triangle. It holds that

$$|S| = q^2 - 2q + 4, \ \mu = 1 + \left(\begin{array}{c} q \\ 2 \end{array}\right) + (q - 1)\left(\begin{array}{c} q - 2 \\ 2 \end{array}\right).$$  

**Proof.** The sizes of $S$ are obvious. Let $P$ be a point of $PG(2, q) \setminus S$. Let $G$ be the intersection point of concurrent lines.
(i) Every line through $P$ distinct from $PG$ intersects $S$ in $L$ points.

(ii) Every line through $P$ distinct from $PG$ intersects $S$ in $q$ points. The line $PG$ provides is a $(b + 1)$-secant of $S$.

(iii) Three lines through $P$ and one of vertices of the triangle are bisecants of $S$, whereas every other line is a 3-secant.

(iv) This follows directly from Proposition 5.5.

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