A LMI condition for the Robustness of Linear Predictor Feedback with Respect to Time-Varying Input Delays

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Abstract

This paper discusses the robustness of the predictor feedback in the case of an unknown time varying delay. Specifically, we study the stability of the closed-loop system when the predictor feedback is designed based on the knowledge of the nominal value of the time-varying delay. By resorting to an adequate Lyapunov-Krasovskii functional, we derive a LMI-based sufficient condition ensuring the asymptotic stability of the closed-loop system for small enough variations of the time-varying delay around its nominal value. These results are extended to the feedback stabilization of a class of diagonal infinite-dimensional boundary control systems in the presence of a time-varying delay in the boundary control input.

Key words: Time-varying delay control; Predictor feedback; Robust stability; PDEs; Boundary control.

1 Introduction

Originally motivated by the work of Artstein [1], linear predictor feedback is an efficient tool for the feedback stabilization of Linear Time-Invariant (LTI) systems with constant input delay. In particular, predictor feedback can be used for controlling plants that are open-loop unstable and in the presence of large input delays. Many extensions have been reported (see, e.g., [13] and the references therein). These include the case of time-varying delays [15]; robustness with respect to disturbance signals [5]; truncated predictor [21]; predictor observers in the case of sensor delays [13]; predictors for nonlinear systems [3,14]; dependence of the delay on the state [2]; and the boundary control of Partial Differential Equations (PDEs) [16,19].

Most of the predictor feedback strategies reported in the literature assume a perfect knowledge in real-time of the input delay. However, such an assumption might be difficult to fulfill in practice. Consequently, there has been an increased interest in the last decade for the study of the robustness of the predictor feedback with respect to delay mismatches. An example of such a problem was investigated in [12] where the exponential stability of the closed-loop system was assessed for unknown constant delays with small enough deviations from the nominal value. The study of the impact of an unknown time-varying delay, but with known nominal value which is used to design the predictor feedback, on the system closed-loop stability was reported in [3]. In particular, it was shown that the exponential stability of the closed-loop system is guaranteed for sufficiently small variations of the delay in both amplitude and rate of variation. Such an approach was further investigated in [10] where a small gain condition on the only amplitude of variation of the delay around its nominal value was derived for ensuring the exponential stability of the closed-loop system. However, as underlined in [18], such a small gain condition might be conservative as it involves norms of matrices which generally grow quickly with their dimensions. In order to reduce such a conservatism, it was proposed in [18] to resort to a Lyapunov-Krasovskii functional approach in the case of constant uncertain delays. By doing so, a LMI-based sufficient condition, was derived, for ensuring the asymptotic stability of the closed-loop system with constant uncertain delays.

The first contribution of this paper deals with the study of the robustness of the predictor feedback that has been
designed based on the nominal value of an uncertain and time-varying input delay. By taking advantage of classical Lyapunov-Krasovskii functionals [9], we derive a LMI-based sufficient condition on the amplitude of variation of the input delay around its nominal value that ensures the asymptotic stability of the closed-loop system. Three examples are developed showing that, for these case studies, the proposed LMI condition provides less conservative results (for assessing the asymptotic stability) than the small gain condition reported in [10] (which ensures the exponential stability).

The second contribution of this paper deals with the extension of the above result to the feedback stabilization of a class of diagonal infinite-dimensional boundary control systems [7] in the presence of a time-varying delay in the boundary control input. The control strategy consists in 1) the use of a predictor feedback to stabilize a finite-dimensional subsystem capturing the unstable modes of the infinite-dimensional system; 2) ensuring that the control law designed on a finite-dimensional truncated models successfully stabilizes the full infinite-dimensional system. Such a control strategy, inspired by [20] in the case of a delay-free feedback control, was first reported in [19] for the exponential stabilization of a reaction-diffusion equation with a constant delay in the boundary control. This idea was extended to the exponential stabilization of a class of diagonal infinite-dimensional boundary control system with constant delay in the boundary control in [16]. In this paper, we assess the robustness of this control strategy in the case of an uncertain and time-varying input delay. Specifically, we show that for time-varying delays presenting 1) a sufficiently small amplitude of variation around its nominal value (with sufficient condition provided by the LMI condition discussed above); 2) a rate of variation that is bounded by an arbitrarily large constant; the infinite-dimensional closed-loop system is asymptotically stable.

The remainder of this paper is organized as follows. The robustness of the predictor feedback with respect to uncertain and time-varying delays is investigated in Section 2. The extension of this result to the feedback stabilization of a class of diagonal infinite-dimensional boundary control systems is presented in Section 3. The obtained results are applied in Section 4 for the feedback stabilization of an unstable reaction-diffusion equation. Finally, concluding remarks are provided in Section 5.

Notation. The sets of non-negative integers, positive integers, real, non-negative real, positive real, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_+^*$, and $\mathbb{C}$, respectively. The real and imaginary parts of a complex number $z$ are denoted by $\text{Re} z$ and $\text{Im} z$, respectively. The field $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. The set of $n \times m$ matrices over $\mathbb{K}$ is denoted by $\mathbb{K}^{n \times m}$ and is endowed with the Euclidean norm $||x|| = \sqrt{x^* x}$. The set of $n \times m$ matrices over $\mathbb{K}$ is denoted by $\mathbb{K}^{n \times m}$ and is endowed with the induced norm denoted by $\| \cdot \|$. For any matrix $P \in \mathbb{R}^{n \times n}$, $P > 0$ (resp. $P \geq 0$) means that $P$ is symmetric positive definite (resp. symmetric positive). The set of symmetric positive definite matrices of order $n$ is denoted by $\mathbb{S}_n^+$. For $M = (m_{i,j}) \in \mathbb{C}^{n \times m}$, we introduce

$$R(M) \triangleq \begin{bmatrix} \text{Re} M & -\text{Im} M \\ \text{Im} M & \text{Re} M \end{bmatrix} \in \mathbb{R}^{2n \times 2m}$$

where $\text{Re} M \triangleq (\text{Re} m_{i,j}) \in \mathbb{R}^{n \times m}$ and $\text{Im} M \triangleq (\text{Im} m_{i,j}) \in \mathbb{R}^{n \times m}$. For any $t_0 > 0$, we say that $\varphi \in C^0([0,t_0];\mathbb{R})$ is a transition signal over $[0,t_0]$ if $0 \leq \varphi \leq 1$, $\varphi(-\infty,0) = 0$, and $\varphi|_{[t_0,\infty)} = 1$. In Section 3, the notations and terminologies for infinite-dimensional systems are retrieved from [7].

2 Delay-robustness of predictor feedback for LTI systems

2.1 Problem setting and existing result

The first part of this paper deals with the feedback stabilization of the following LTI system with delay control:

$$\dot{x}(t) = Ax(t) + Bu(t-D(t)), \quad t \geq 0, \quad (1)$$

with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ such that the pair $(A,B)$ is stabilizable. Vectors $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ denote the state and the control input, respectively. The command input is subject to a time-varying delay $D \in C^0(\mathbb{R}_+;\mathbb{R}_+)$. We assume that there exist $D_0 > 0$ and $0 < \delta < D_0$ such that $|D(t) - D_0| \leq \delta$ for all $t \geq 0$. In this context, the following linear predictive feedback, which is based on the knowledge of the constant nominal value $D_0$, has been proposed in [3] for $t \geq 0$:

$$u(t) = K \left\{ e^{D_0 A} x(t) + \int_{t-D_0}^t e^{(t-s) A} B u(s) \, ds \right\}, \quad (2)$$

where $K \in \mathbb{R}^{n \times m}$ is a feedback gain such that $A_{cl} \triangleq A + BK$ is Hurwitz. The validity of such a control strategy was assessed in [10] via a small gain argument.

**Theorem 1 ([10])** Let $D_0 > 0$ be given and let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $K \in \mathbb{R}^{n \times m}$ be such that $A_{cl} \triangleq A + BK$ is Hurwitz. Let $\delta > 0$ be such that

$$M_\| e^{D_0 A} BK \| \left\{ e^{\| A_{cl} \| \delta} - e^{-\mu \delta} \right\} < \mu, \quad (3)$$

where $M, \mu > 0$ are constants satisfying $\|e^{A_{cl} t}\| \leq Me^{-\mu t}$ for all $t \geq 0$. Then, there exists $N, \sigma > 0$ such that for all $x_0 \in \mathbb{R}, u_0 \in C^0([D_0 - \delta,0];\mathbb{R}^m)$ with $u_0(0) = K \left\{ e^{D_0 A} x_0 + \int_{t-D_0}^t e^{-(t-s) A} B u_0(s) \, ds \right\}$, and $d \in C^0(\mathbb{R}_+;\mathbb{R}_+)$ with $|D_0 - D| \leq \delta$, the solution of (1-2) associated with the initial conditions $x(0) = x_0$ and
u(t) = u_0(t) for \(-D_0 - \delta \leq t \leq 0\) satisfies for all \(t \geq 0\) the following estimate:
\[
\begin{align*}
\|x(t)\| + & \max_{t-D_0-\delta \leq s \leq t} \|u(s)\| \\
\leq N e^{-\sigma t} \left\{ \|x_0\| + \max_{-D_0-\delta \leq s \leq 0} \|u_0(s)\| \right\}.
\end{align*}
\]

As the left hand-side of (3) is equal to zero when \(\delta = 0\), a continuity argument shows that there always exists a \(\delta > 0\) such that (3) holds true. Therefore, Theorem 1 ensures the existence of a sufficiently small amplitude of perturbation \(\delta > 0\) of the delay \(D(t)\) around its nominal value \(D_0\) such that the nominal linear predictor feedback (2) ensures the exponential stability of the closed-loop system. However, due to the nature of the small gain condition (3) that involves the norm of matrices (which generally grow quickly as a function of the matrices dimensions \(n, m\)), the admissible values of \(\delta\) might be conservative (see [3]). In particular, from the fact that \(M \geq 1\) and \(0 < \mu \leq \kappa_0(A_{d1}) \triangleq -\max\{\Re \lambda: \lambda \in \text{spe}(A_{d1})\}\), any \(\delta > 0\) such that the small gain condition (3) holds true satisfies the following upper-estimate:
\[
\delta < \frac{1}{\|A_{d1}\|} \left( 1 + \frac{\kappa_0(A_{d1})}{\|e^{D_0\delta}ABK\|} \right). \quad (4)
\]

In this context, the purpose of the first part of this paper is to derive a LMI-based sufficient condition ensuring the asymptotic stability of the closed-loop system with predictor feedback.

### 2.2 Preliminary results

For \(h > 0\), we denote by \(W\) the space of absolutely continuous functions \(\psi: [-h, 0] \to \mathbb{R}^n\) with square-integrable derivative endowed with the norm \(\|\psi\|_W \triangleq \sqrt{\|\psi(0)\|^2 + \int_{-h}^0 \|\psi'(\theta)\|^2 \, d\theta}\) (see [11, Chap. 4, Sec. 1.3]).

**Lemma 1.** Let \(M, N \in \mathbb{R}^{n \times n}\), \(D_0 > 0\), and \(\delta \in (0, D_0)\) be given. Assume that there exist \(P_1, Q \in \mathbb{S}^n_+\) and \(P_2, P_3 \in \mathbb{R}^{n \times n}\) such that the following LMI holds true:
\[
\begin{bmatrix}
M^T P_2 + P_2^T M & P_1 - P_2^T P_3 + M^T P_2 & \delta P_2^T N \\
- P_1 - P_2 - P_3^T M & -P_3 - P_3^T + 2\delta Q & \delta P_3^T N \\
\delta N^T P_2 & \delta N^T P_3 & -\delta Q
\end{bmatrix} < 0.
\]

Then, for any \(D \in C^0(\mathbb{R}_+; \mathbb{R}_+)\) such that \(|D - D_0| \leq \delta\), the system\(\hat{x}(t) = Mx(t) + N \{x(t-D(t)) - x(t-D_0)\}\), \(t \geq 0\);
\(x(t) = x_0(t), \quad t \in [-D_0-\delta, 0]\)

with initial condition \(x_0 \in W\) (for \(h = D_0 + \delta\)) is asymptotically stable in the sense that 1) for all \(\epsilon > 0\), there exists \(\eta > 0\) such that \(\|x_0\|_W \leq \eta\) implies \(|x(t)| \leq \epsilon\) for all \(t \geq 0\); 2) for all \(x_0 \in W\), \(x(t) \to 0\) as \(t \to +\infty\).

**Proof.** For all \(t \geq 0\), one has
\[
\dot{x}(t) = Mx(t) + N \{x(t-D(t)) - x(t-D_0)\}
\]
\[
= Mx(t) + N \int_{t-D_0}^{t-D(t)} \dot{x}(\tau) \, d\tau.
\]

Inspired by classical Lyapunov-Krasovskii functional depending on time derivative for systems with fast varying delays, see [9, Sec. 3.2], we introduce
\[
V(t) = x(t)^T P_1 x(t) + \int_{t-D_0}^{t-D_0+\delta} \dot{x}(s)^T Q \dot{x}(s) \, ds \, d\theta,
\]
where \(P_1, Q \in \mathbb{S}^n_+\). We have for all \(t \geq 0\) that
\[
\dot{V}(t) = 2x(t)^T P_1 \dot{x}(t) + 2\delta \dot{x}(t)^T Q \dot{x}(t)
\]
\[
- \int_{t-D_0}^{t-D_0+\delta} \dot{x}(t + \theta)^T Q \dot{x}(t + \theta) \, d\theta.
\]

The remaining of the proof is now an adaptation of [8, Proof of Th. 1]. Introducing
\[
P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix},
\]

where \(P_2, P_3 \in \mathbb{R}^{n \times n}\) are “slack variables” [9], we have
\[
x(t)^T P_1 \dot{x}(t)
\]
\[
\begin{align*}
(x(t))^T P_T & \begin{bmatrix} \dot{x}(t) \\ \dot{x}(t) \\ -\dot{x}(t) + Mx(t) + N \int_{t-D_0}^{t-D(t)} \dot{x}(\tau) \, d\tau \end{bmatrix} \\
(x(t))^T P_T & \begin{bmatrix} 0 & I \\ M & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\
+ & \int_{t-D_0}^{t-D(t)} (x(t))^T P_T \begin{bmatrix} 0 \\ N \end{bmatrix} \dot{x}(\tau) \, d\tau.
\end{align*}
\]

Now, from the fact that, for any \(a, b \in \mathbb{R}^n\), \(2a^T b \leq \|a\|^2 + \|b\|^2\), we obtain that
\[
2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T P_T \begin{bmatrix} 0 \\ N \end{bmatrix} \dot{x}(\tau)
\]
\[\begin{align*}
&= 2 \left( Q^{-1/2} \begin{bmatrix} 0 \\
N \end{bmatrix} P \begin{bmatrix} x(t) \\
\dot{x}(t) \end{bmatrix} \right)^T \left( Q^{1/2} \dot{x}(\tau) \right) \\
&\leq \begin{bmatrix} x(t) \\
\dot{x}(t) \end{bmatrix}^T P \begin{bmatrix} 0 & I \\
N & M - I \end{bmatrix} \begin{bmatrix} x(t) \\
\dot{x}(t) \end{bmatrix} + 2 \delta \dot{x}(t)^T Q \dot{x}(t) \\
&\quad + |D(t) - D_0| \begin{bmatrix} x(t) \\
\dot{x}(t) \end{bmatrix}^T P \begin{bmatrix} 0 \\
N \end{bmatrix} Q^{-1} \begin{bmatrix} 0 \\
N \end{bmatrix} P \begin{bmatrix} x(t) \\
\dot{x}(t) \end{bmatrix} \\
&\quad + \int_{t-D(t)}^{t-D(t)+\delta} \dot{x}(\tau)^T Q \dot{x}(\tau) \, d\tau - \int_{t-D(t)-\delta}^{t-D(t)} \dot{x}(\tau)^T Q \dot{x}(\tau) \, d\tau \\
&\leq \begin{bmatrix} x(t) \\
\dot{x}(t) \end{bmatrix} \left( \Psi(\delta) + \delta P^T \begin{bmatrix} 0 \\
N \end{bmatrix} Q^{-1} \begin{bmatrix} 0 \\
N \end{bmatrix} P \right) \begin{bmatrix} x(t) \\
\dot{x}(t) \end{bmatrix},
\end{align*}\]

where it has been used the fact that the sum of the two integral terms is always non positive, and with

\[\Psi(\delta) \triangleq P^T \begin{bmatrix} 0 & I \\
M - I \end{bmatrix} + \begin{bmatrix} 0 & I \\
M - I \end{bmatrix} P + 2\delta \begin{bmatrix} 0 & 0 \\
0 & 0 \end{bmatrix}.\]

Assuming that the LMI condition (5) holds true, the Schur complement yields the existence of \(\alpha(\delta) > 0\) such that \(\dot{V}(t) \leq -\alpha(\delta) \|x(t)\|^2\). The conclusion follows from [11, Chap. 4, Th. 1.6]. \(\square\)

**Remark 1** The LMI (5) is independent of \(D_0\). Consequently, the stability result provided by Lemma 1 depends only on the amplitude of variation \(\delta\) of the delay \(D(t)\) around \(D_0\), but not on its nominal value \(D_0\).

**Lemma 2** Let \(D_0 > 0\) and \(M, N \in \mathbb{R}^{n \times n}\) with \(M\) Hurwitz be given. Introducing\(^1\)

\[\delta^* \triangleq \min \left( \frac{D_0}{2\sqrt{2}} \left\| N^T \begin{bmatrix} P_2 - (M^{-1})^T P_2 \end{bmatrix} \right\| \right) > 0\]

where \(P_2 \in \mathbb{S}^n_{++}\) is the unique solution of the Lyapunov equation

\[M^T P_2 + P_2 M = -I,\]

(10)

the LMI (5) is feasible for all \(\delta \in (0, \delta^*)\). Thus the conclusions of Lemma 1 hold true for any \(\delta \in (0, \delta^*)\).

\(^1\) With the convention \(\delta^* = D_0\) in the case \(N = 0\).

**Proof.** As \(M\) is Hurwitz, let \(P_2 \in \mathbb{S}^n_{++}\) be the unique solution of the Lyapunov equation (10). We introduce \(P_1 = 2P_2 \in \mathbb{S}^n_{++}\), \(P_3 = -(M^{-1})^T P_2\), and \(Q = \alpha I \in \mathbb{S}^n_{++}\) with \(\alpha > 0\). The LMI condition (5) becomes:

\[\begin{bmatrix}
-I & 0 & \delta P_2 N \\
0 & -(M^{-1})^T M^{-1} + 2\alpha \delta I & -\delta P_2 M^{-1} N \\
\delta N^T P_2 & -\delta N^T (M^{-1})^T P_2 & -\alpha \delta I
\end{bmatrix} < 0.
\]

(11)

As \(\alpha, \delta > 0\), the Schur complement shows that the LMI above is equivalent to

\[\begin{bmatrix}
-I & 0 \\
0 & -(M^{-1})^T M^{-1} + 2\alpha \delta I
\end{bmatrix}
\]

\[+ \frac{\delta}{\alpha} \begin{bmatrix} P_2 N \\
-P_2 M^{-1} N \end{bmatrix} \begin{bmatrix} P_2 N \\
-P_2 M^{-1} N \end{bmatrix}^T < 0.
\]

Noting that

\[\begin{bmatrix}
-I & 0 \\
0 & -(M^{-1})^T M^{-1}
\end{bmatrix} \preceq -\min \{1, \lambda_m ((M^{-1})^T M^{-1})\} I
\]

with \(\lambda_m ((M^{-1})^T M^{-1}) > 0\), and

\[\begin{bmatrix}
P_2 N \\
-P_2 M^{-1} N
\end{bmatrix} \begin{bmatrix}
P_2 N \\
-P_2 M^{-1} N
\end{bmatrix}^T \preceq \beta I,
\]

with \(\beta \triangleq \left\| N^T \begin{bmatrix} P_2 - (M^{-1})^T P_2 \end{bmatrix} \right\|^2 \geq 0\), we obtain the following sufficient condition guaranteeing that the LMI (11) holds true:

\[\delta < \frac{\min \{1, \lambda_m ((M^{-1})^T M^{-1})\}}{2\alpha + \beta/\alpha},\]

where \(\alpha > 0\) can be freely selected. In the case \(N = 0\), we obtain that \(\beta = 0\) and thus, by letting \(\alpha \to 0^+\), \(\delta^* = D_0\). In the case \(N \neq 0\), we have \(\beta > 0\). Indeed, by contradiction, \(\beta = 0\) implies \(N^T P_2 = P_2 N = 0\). Multiplying (10) from the left side by \(N^T\) and from the right side by \(N\), we obtain that \(N^T N = 0\) yielding \(N = 0\). To conclude the proof, it is sufficient to note that, for any given \(a, b > 0\), the function \(f(\alpha) = a\alpha + b/\alpha\) is such that \(f(\alpha) \geq f(\sqrt{ab}) = 2\sqrt{ab}\) for all \(\alpha > 0\). \(\square\)

**2.3 Robustness of linear predictor feedback with respect to time-varying delay in the control input**

We can now introduce the main result of this section.
Theorem 2. Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $K \in \mathbb{R}^{m \times n}$ be such that $A_{cl} \triangleq A + BK$ is Hurwitz. Let $\varphi$ be a transition signal over $[0, t_0]$ with $t_0 > 0$ and let $D_0 > 0$ be a given nominal delay. Then, there exists $\delta \in (0, D_0)$ such that for any $D \in C^0(\mathbb{R}_+; \mathbb{R}_+)$ with $|D - D_0| \leq \delta$, the closed-loop system given for any $t \geq 0$ by

$$
\dot{x}(t) = Ax(t) + Bu(t - D(t)),
$$

with initial condition $x_0 \in \mathbb{R}^n$ is asymptotically stable in the sense that 1) for all $t > 0$, there exists $\eta > 0$ such that $\|x(t)\| \leq \eta$ implies $\|x(t)\| + \|u(t)\| \leq \epsilon$ for all $t \geq 0$; 2) for all $x_0 \in \mathbb{R}^n$, $\|x(t)\| + \|u(t)\| \xrightarrow{t \to \infty} 0$.

In particular, the above conclusion holds true for any $\delta \in (0, D_0)$ such that there exist $P_1, Q \in \mathbb{S}^+_n$ and $P_2, P_3 \in \mathbb{R}^{n \times n}$ for which the LMI (5) holds true with $M = A_{cl}$ and $N = e^{D_0}BK$.

Proof. Let $\delta \in (0, D_0)$ be such that $D_0 > 0$ holds true, and we have $x \in C^1(\mathbb{R}_+; \mathbb{R}^n)$ and $u \in C^0(\mathbb{R}_+; \mathbb{R}^n)$. We introduce $z \in C^1(\mathbb{R}_+; \mathbb{R}^n)$ defined for all $t \geq 0$ by (see [1]):

$$
z(t) = e^{D_0}x(t) + \int_{t-\delta}^t e^{(t-s)}B_{\varphi}u(s)ds.
$$

As $u = \varphi Kz$, straightforward computations show that, for all $t \geq 0$,

$$
\dot{z}(t) = (A + \varphi(t)BK)z(t) + e^{D_0}BK\{[\varphi z(t - D(t))] - [\varphi z(t - D_0)]\}.
$$

In particular, we have for all $t \geq t_1 \triangleq t_0 + D_0 - \delta$ that

$$
\dot{z}(t) = A_{cl}z(t) + e^{D_0}BK\{z(t - D(t)) - z(t - D_0)\}
$$

with $A_{cl} \triangleq A + BK$ Hurwitz and the continuously differentiable initial condition $z|_{t_0,t_1}$. From Lemma 1 we obtain that $z(t) \xrightarrow{t \to \infty} 0$. Using $u = \varphi Kz$ and (12), we obtain that $\|x(t)\| + \|u(t)\| \leq \gamma_0 \sup_{s \leq \xi [\max(t_{t0},0), t]} \|z(s)\|$, for all $t \geq 0$ with $\gamma_0 = e^{D_0}\|A\| \|1 + D_0BK\| + \|K\| > 0$. This yields $\|x(t)\| + \|u(t)\| \xrightarrow{t \to \infty} 0$.

It remains to show the stability of the closed-loop system. To do so, we introduce $V(t) = \|z(t)\|^2/2$. The use of the Young’s inequality shows that there exist constants $\gamma_1, \gamma_2 > 0$, independent of $x_0$, such that for all $t \geq 0$,

$$
\dot{V}(t) \leq \gamma_1 V(t) + \gamma_2 [\varphi(t - D(t))]^2 V(t - D(t)) + \gamma_2 [\varphi(t - D(t))]^2 V(t - D(t)).
$$

We show by induction that, for any $n \in \mathbb{N}^*$, there exists a constant $C_n > 0$, independent of the initial condition $x_0$, such that $V(t) \leq C_n^2 \|x_0\|^2/2$ for all $t \in [0, n(D_0 - \delta)]$. In the case $n = 1$, we have for all $t \in [0, D_0 - \delta]$, $\varphi(t - D(t)) = \varphi(t - D_0) = 0$. Thus $\dot{V}(t) \leq \gamma_1 V(t)$ and we obtain that the property holds true with $C_1 = e^\gamma(D_0/\gamma)^2/2$. Assume that $V(t) \leq C_n^2 \|x_0\|^2/2$ for all $t \in [0, n(D_0 - \delta)]$. Then, for all $t \in [0, (n + 1)(D_0 - \delta)]$, we have $t - D(t) \leq n(D_0 - \delta)$ and $t - D_0 \leq (n(D_0 - \delta))$, yielding $\dot{V}(t) \leq \gamma_1 V(t) + \gamma_2 C_n^2 \|x_0\|^2$. A straightforward integration shows the existence of the claimed $C_{n+1} > 0$.

Let $n_0 \geq 1$ be such that $n_0(D_0 - \delta) \geq t_1$. This yields $\sup_{t \in [0, t_1]} \|z(t)\| \leq C_n \|x_0\|$. From (13), we deduce the existence of a constant $C_0 > 0$, independent of $x_0$, such that $\sup_{t \in [0, t_1]} \|\dot{z}(t)\| \leq C_0 \|x_0\|$. From the definition of $\|\cdot\|$, we deduce the existence of a constant $C_1 > 0$, independent of $x_0$, such that $\|z(t_1 + \cdot)\| \leq C_1 \|x_0\|$.

Let $\epsilon > 0$ be arbitrarily given. By applying Lemma 1 to (14), there exists $\eta' > 0$ such that $\|z(t_1 + \cdot)\| \leq \eta'$ implies $\|\dot{z}(t)\| \leq \epsilon/\gamma_0$ for all $t \geq t_1$. Introducing $\gamma = \min(\eta'/C_1, \epsilon/(C_n \gamma_0)) > 0$, we obtain that $\|x_0\| \leq \eta$ implies $\|\dot{z}(t)\| \leq \epsilon/\gamma_0$ and thus $\|x(t)\| + \|u(t)\| \leq \epsilon$ for all $t \geq 0$. □

Corollary 1. If we assume that $A, B, K$, and $x_0$ are complex-valued, i.e., $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times m}$, $K \in \mathbb{C}^{m \times n}$, and $x_0 \in \mathbb{C}^n$, the conclusions of Theorem 2 hold true with $M = \mathcal{R}(A_{cl})$ and $N = \mathcal{R}(e^{D_0}BK)$. In this case, the unknowns of the LMI (5) are such that $P_1, Q \in \mathbb{S}^+_{2n}$ and $P_2, P_3 \in \mathbb{R}^{2n \times 2n}$.

Proof. For $z(t) \in \mathbb{C}^n$, we infer that (14) is equivalent to

$$
\dot{Z}(t) = \mathcal{R}(A_{cl})Z(t) + \mathcal{R}(e^{D_0}BK)\{Z(t)(t - D(t)) - Z(t)(t - D_0)\}
$$

with $Z(t) = \left[\text{Re} z(t)^T \text{Im} z(t)^T\right]^T \in \mathbb{R}^{2n}$. Furthermore, as $A_{cl}$ is assumed Hurwitz, so is $\mathcal{R}(A_{cl})$. Then, the conclusion follows from the proof of Theorem 2. □

2.4 Applications

We compare the application of the results of Theorem 1 taken from [10] and Theorem 2 based on two examples extracted from [3].

Example 1. With the matrices

$$
A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} -1 & -3 \end{bmatrix},
$$
the closed-loop poles are located in $-1 \pm j$. For $D_0 = 1$, we obtain from Theorem 1 the exponential stability for \( \delta = 0.0212 \) (with upper-bound of 0.0440 given by (4)) while the application of Theorem 2 yields the asymptotic stability for \( \delta = 0.1031 \).

**Example 2** With the matrices

\[
A = \begin{bmatrix} -2/3 & -1 & 5/3 \\ 0 & -1 & 0 \\ 1/3 & -1 & 2/3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -2 & 1 \end{bmatrix},
\]

\[
K = \begin{bmatrix} 0.3572 & -0.4853 & 1.1281 \\ 0.3925 & -0.5660 & 0.4535 \end{bmatrix},
\]

the closed-loop poles are located in $-1 \pm j$ and $-2$. For $D_0 = 1$, we obtain from Theorem 1 the exponential stability for \( \delta = 0.0147 \) (with upper-bound of 0.0391 given by (4)) while the application of Theorem 2 yields the asymptotic stability for \( \delta = 0.1033 \).

### 3 Extension to the feedback stabilization of a class of diagonal infinite-dimensional systems

In this section, we extend the results of Theorem 2 to the feedback stabilization of a class of diagonal (infinite-dimensional) boundary control systems exhibiting a finite number of unstable modes by means of a boundary control input that is subject to a time-varying delay. In the upcoming developments, \((\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H})\) denotes a separable \(\mathbb{K}\)-Hilbert space.

#### 3.1 Problem setting

Let $D_0 > 0$ and \( \delta \in (0, D_0) \) be given. We consider the abstract boundary control system [7]:

\[
\begin{cases}
\frac{dX}{dt}(t) = AX(t), & t \geq 0 \\
BX(t) = \ddot{u}(t) \triangleq u(t - D(t)), & t \geq 0 \\
X(0) = X_0
\end{cases}
\]

with

- \( A : D(A) \subset \mathcal{H} \to \mathcal{H} \) a linear (unbounded) operator;
- \( B : D(B) \subset \mathcal{H} \to \mathbb{K}^m \) with \( D(A) \subset D(B) \) a linear boundary operator;
- \( u : [-D_0 - \delta, +\infty) \to \mathbb{K}^m \) with \( u|_{[-D_0 - \delta, 0)} = 0 \) the boundary control;
- \( D : \mathbb{R}_+ \to (D_0 - \delta, D_0 + \delta) \) a time-varying delay.

It is assumed that \((A, B)\) is a boundary control system:

1. the disturbance free operator \( A_0 \), defined over the domain \( D(A_0) \triangleq D(A) \cap \ker(B) \) by \( A_0 \triangleq A|_{D(A_0)} \), is the generator of a \( C_0 \)-semigroup \( S \) on \( \mathcal{H} \);
2. there exists a bounded operator \( B \in \mathcal{L}(\mathbb{K}^m, \mathcal{H}) \) called a lifting operator, such that \( R(B) \subset D(A) \), \( AB \in \mathcal{L}(\mathbb{K}^m, \mathcal{H}) \), and \( BB = I_{\mathbb{K}^m} \);

where \( \ker(B) \) stands for the kernel of \( B \) and \( R(B) \) denotes the range of \( B \).

In the following developments, we assume that the boundary control system exhibits a diagonal structure:

**Assumption 1** The disturbance free operator \( A_0 \) is a Riesz spectral operator [7], i.e., is a linear and closed operator with simple eigenvalues \( \lambda_n \) and corresponding eigenvectors \( \phi_n \in D(A_0) \), \( n \in \mathbb{N}^* \), that satisfy:

1. \( \{ \phi_n, n \in \mathbb{N}^* \} \) is a Riesz basis [6]:
   - (a) \( \text{span}_n \phi_n = \mathcal{H} \);
   - (b) there exist constants \( m_R, M_R \in \mathbb{R}_+ \) such that for all \( N \in \mathbb{N}^* \) and all \( \alpha_1, \ldots, \alpha_N \in \mathbb{K} \),
     \[
     m_R \sum_{n=1}^N |\alpha_n|^2 \leq \left\| \sum_{n=1}^N \alpha_n \phi_n \right\|_\mathcal{H}^2 \leq M_R \sum_{n=1}^N |\alpha_n|^2.
     \]

2. The closure of \( \{ \lambda_n, n \in \mathbb{N}^* \} \) is totally disconnected, i.e., for any distinct \( a, b \in \{ \lambda_n, n \in \mathbb{N}^* \} \), \( [a, b] \not\subset \{ \lambda_n, n \in \mathbb{N}^* \} \).

We also assume that the system presents a finite number of unstable modes and that the set composed of the real part of the stable modes does not accumulate at 0:

**Assumption 2** There exist \( N_0 \in \mathbb{N}^* \) and \( \alpha \in \mathbb{R}_+ \) such that \( \text{Re} \lambda_n \leq -\alpha \) for all \( n \geq N_0 + 1 \).

As \( \{ \phi_n, n \in \mathbb{N}^* \} \) is a Riesz basis, we can introduce its biorthogonal sequence \( \{ \psi_n, n \in \mathbb{N}^* \} \), i.e., \( \langle \phi_k, \psi_l \rangle_\mathcal{H} = \delta_{k,l} \). Then, we have for all \( x \in \mathcal{H} \) the following series expansion:

\[
x = \sum_{n \geq 1} \langle x, \psi_n \rangle_\mathcal{H} \phi_n.
\]

As \( A_0 \) is a Riesz-spectral operator, then \( \psi_n \) is an eigenvector of the adjoint operator \( A_0^* \) associated with the eigenvalue \( \overline{\lambda}_n \).

#### 3.2 Spectral decomposition and finite dimensional truncated model

Under the assumption that \( \ddot{u} \in C^2([0, +\infty); \mathbb{K}^m) \) and \( X_0 \in D(A) \) such that \( BX_0 = \ddot{u}(0) = u(0 - D(0)) = 0 \) (i.e., \( X_0 \in D(A_0) \)), there exists a unique classical solution \( X \in C^0(\mathbb{R}_+; D(A)) \cap C^1(\mathbb{R}_+; \mathcal{H}) \) of (15); see, e.g., [7, Th. 3.3.3]. Then,

\[
X(t) = \sum_{n \in \mathbb{N}^*} \langle X(t), \psi_n \rangle_\mathcal{H} \phi_n = \sum_{n \in \mathbb{N}^*} c_n(t) \phi_n,
\]
where $c_n(t) = (X(t), \psi_n)_{\mathcal{H}}$. We infer that $c_n \in C^1(\mathbb{R}_+; \mathbb{K})$ and, from (15), we have for all $t \geq 0$, the following spectral decomposition [17]:

\[
\hat{c}_n(t) = \langle \frac{dX}{dt}(t), \psi_n \rangle_{\mathcal{H}} = \langle AX(t), \psi_n \rangle_{\mathcal{H}} = \langle A\{X(t) - B\tilde{u}(t)\}, \psi_n \rangle_{\mathcal{H}} + \langle AB\tilde{u}(t), \psi_n \rangle_{\mathcal{H}} = \langle A \{X(t) - B\tilde{u}(t)\}, \psi_n \rangle_{\mathcal{H}} + \langle AB\tilde{u}(t), \psi_n \rangle_{\mathcal{H}} = \langle X(t) - B\tilde{u}(t), \psi_n \rangle_{\mathcal{H}} + \langle AB\tilde{u}(t), \psi_n \rangle_{\mathcal{H}}
\]

where it has been used that $\mathcal{B}\{X(t) - B\tilde{u}(t)\} = \tilde{u}(t) - \tilde{u}(t) = 0$, showing that $X(t) - B\tilde{u}(t) \in D(A) \cap \ker(\mathcal{B}) = D(A_0)$.

Let $E = (e_1, e_2, \ldots, e_m)$ be the canonical basis of $\mathbb{K}^m$. Introducing $b_{n,k} = -\lambda_n \langle Be_k, \psi_n \rangle_{\mathcal{H}} + \langle ABe_k, \psi_n \rangle_{\mathcal{H}}$, we obtain from (17) that the following linear ODE holds for all $t \geq 0$

\[
\dot{Y}(t) = A_0 Y(t) + B_{N_0} u(t - D(t)), \quad \text{(18)}
\]

where $A_0 = \text{diag}(\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^{n \times n}$, $B_{N_0} = \{b_{n,k}\}_{1 \leq n \leq N_0, 1 \leq k \leq m} \in \mathbb{K}^{n \times m}$, and

\[
Y(t) = \begin{bmatrix}
    c_1(t) \\
    \vdots \\
    c_{N_0}(t)
\end{bmatrix} = \begin{bmatrix}
    \langle X(t), \psi_1 \rangle_{\mathcal{H}} \\
    \vdots \\
    \langle X(t), \psi_{N_0} \rangle_{\mathcal{H}}
\end{bmatrix} \in \mathbb{K}^{N_0}. \quad \text{(19)}
\]

Under the following assumption, we obtain the existence of a feedback gain $\mathbb{K}^{m \times N_0}$ such that $A_{cl} = A_0 + B_{N_0} K$ is Hurwitz.

**Assumption 3** $(A_{N_0}, B_{N_0})$ is stabilizable.

Then, we can employ the strategy presented in Section 2 to ensure the feedback stabilization of the finite-dimensional truncated dynamics (18). The objective is now to assess that such a strategy ensures the stabilization of the full infinite-dimensional system.

### 3.3 Dynamics of the closed-loop system

Let $D_0, t_0 > 0$ and $\delta \in (0, D_0)$ be given. Let $\varphi \in C^2(\mathbb{R}; \mathbb{R})$ be a transition signal over $[0, t_0]$ and $D \in C^2(\mathbb{R}_+; \mathbb{R})$ be a time-varying delay such that $|D - D_0| \leq \delta$. The dynamics of the closed-loop system is given by (see [16] for the nominal case $D(t) = D_0$):

\[
\begin{aligned}
\frac{dX}{dt}(t) &= AX(t), \\
\mathcal{B}X(t) &= \tilde{u}(t) = u(t - D(t)), \\
\theta(t) &= \varphi(t)K_{\mathcal{E}D_0 A_{N_0} Y(t)} \\
&\quad + \varphi(t)K \int_{t-D_0}^t e^{(t-s)A_{N_0}} B_{N_0} u(s) \, ds, \\
Y(t) &= \begin{bmatrix}
    \langle X(t), \psi_1 \rangle_{\mathcal{H}} \\
    \vdots \\
    \langle X(t), \psi_{N_0} \rangle_{\mathcal{H}}
\end{bmatrix}, \\
X(0) &= X_0
\end{aligned} \quad \text{(20)}
\]

for any $t \geq 0$. The feedback gain $K \in \mathbb{K}^{m \times N_0}$ is selected such that $A_{cl} = A_{N_0} + B_{N_0} K$ is Hurwitz.

**Lemma 3** Let $(A, B)$ be an abstract boundary control system such that Assumptions 1, 2, and 3 hold true. For any $X_0 \in D(A_0)$ and $D \in C^2(\mathbb{R}_+; \mathbb{R})$ such that $|D - D_0| \leq \delta < D_0$, the closed-loop system (20) admits a unique classical solution $X \in C^1([0, t_0]; D(A)) \cap C^1([0, t_0]; \mathcal{H})$. The associated control law $u$ is uniquely defined and is of class $C^1([-D_0 - \delta, +\infty); \mathbb{K}^m)$. It can be written under the form $u = \varphi KZ$ with, for all $t \geq 0$,

\[
Z(t) = e^{D_0 A_{N_0}} Y(t) + \int_{t-D_0}^t e^{(t-s)A_{N_0}} B_{N_0} u(s) \, ds, \quad \text{(21)}
\]

which is such that $Z \in C^2(\mathbb{R}_+; \mathbb{K}^{N_0})$ with for all $t \geq 0$,

\[
\dot{Z}(t) = (A_{N_0} + \varphi(t)B_{N_0} K) Z(t) + e^{D_0 A_{N_0}} B_{N_0} \{[\varphi Z](t - D(t)) - [\varphi Z](t - D_0)\}. \quad \text{(22)}
\]

The proof of Lemma 3 relies on the invertibility of the Artstein transformation [4] and on the fact that for any $t \in [n(D_0 - \delta), (n+1)(D_0 - \delta)]$ with $n \in \mathbb{N}$, the actual control input $\tilde{u}(t)$ is such that $\tilde{u}(t) = 0$ for $n = 0$ and depends only on the system state $X$ (via $Y$) over the range of time $[0, n(D_0 - \delta)]$ when $n \geq 1$. Therefore, the existence of a classical solution $X$ for the closed-loop system (20) can be shown by induction using classical results on boundary control systems with boundary input of class $C^2$ (see, e.g., [7, Th. 3.3.3]). Such a regularity of the control input follows first from the fact that the control law $u$ implicitly defined in (20) via the Artstein transformation is of class $C^0$ (see [4]) and then from (21-22). A detailed proof in the case $\delta = 0$, i.e., $D(t) = D_0$, can be found in [16, Lemma 4.1] and, based on the above remarks, can be extended in a straightforward manner to the configuration of Lemma 3.
3.4 Asymptotic stability of the closed-loop system

The exponential stability of the closed-loop system (20) in the nominal configuration \( D(t) = D_0 \) has been assessed in [16]. The contribution of this paper relies in the following result in the case of an unknown time-varying delay \( D(t) \) with known nominal value \( D_0 \).

**Theorem 3** Let \((A,B)\) be an abstract boundary control system such that Assumptions 1, 2, and 3 hold true. There exists \( \delta \in (0, D_0) \) such that for any \( D \in C^2(\mathbb{R}_+; \mathbb{R}) \) with \(|D - D_0| \leq \delta \) and \( \sup_{t \in \mathbb{R}_+} |\dot{D}(t)| < +\infty \), the closed-loop dynamics (20) with initial condition \( X_0 \in D(A_0) \) is asymptotically stable in the sense that 1) for all \( \epsilon > 0 \), there exists \( \eta > 0 \) such that \( \|X_0\|_H \leq \eta \) implies \( \|X(t)\|_H + \|u(t)\| \leq \epsilon \) for all \( t \geq 0 \); 2) for all \( X_0 \in D(A_0) \), \( \|X(t)\|_H + \|u(t)\| \xrightarrow{t \to \infty} 0 \). This conclusion holds true for any \( \delta \in (0, D_0) \) such that the LMI (5), with

- in the case \( \mathbb{K} = \mathbb{R} \), \( M = A_{N_0} + B_{N_0}K, N = e^{D_0A_{N_0}}B_{N_0}K, P_1, Q \in S^{+}_n \), and \( P_2, P_3 \in \mathbb{R}^{n \times n} \),
- in the case \( \mathbb{K} = \mathbb{C} \), \( M = \mathcal{R}(A_{N_0} + B_{N_0}K), N = \mathcal{R}(e^{D_0A_{N_0}}B_{N_0}K), P_1, Q \in S^{+}_2 \), and \( P_2, P_3 \in \mathbb{R}^{2n \times 2n} \),

is feasible.

**Proof.** Let \( \delta \in (0, D_0) \) be such that the LMI (5) is feasible (from Lemma 2, any \( \delta \in (0, \delta^*) \) satisfies such a condition). Let \( X_0 \in D(A_0) \) and \( D \in C^2(\mathbb{R}_+; \mathbb{R}) \) such that \(|D - D_0| \leq \delta \) and \( \sup_{t \in \mathbb{R}_+} |\dot{D}(t)| < +\infty \) be given.

From Lemma 3, we denote by \( X \in C^0(\mathbb{R}_+; D(A)) \cap C^1(\mathbb{R}_+; H) \) the unique classical solution of the closed-loop system (20) and \( u \in C^2([-D_0 - \delta, +\infty); \mathbb{K}^m) \) the associated control input. Thus \( \|X(t)\| + \|Z(t)\| + \|u(t)\| \xrightarrow{t \to +\infty} 0 \).

Now, in order to assess the asymptotic stability of the full infinite-dimensional system, we introduce for all \( t \geq 0 \),

\[
V(t) = \frac{1}{2} \sum_{k \geq N_0 + 1} |\langle X(t) - \hat{B}(t), \psi_k \rangle|^2 \geq 0,
\]

which is such that \( V(t) \leq \|X(t) - \hat{B}(t)\|_H^2 / (2m_R) < +\infty \) and \( V \in C^1(\mathbb{R}_+; \mathbb{R}) \). The quantity \( V(t) \) is used to derive an upper bound of \( \|X(t)\|_H \) as follows. Noting that

\[
\frac{1}{2} \sum_{k=1}^{N_0} |\langle X(t) - \hat{B}(t), \psi_k \rangle|^2 \\
\leq \sum_{k=1}^{N_0} |\langle X(t), \psi_k \rangle|^2 + \sum_{k=1}^{N_0} |\langle B(t), \psi_k \rangle|^2
\]

we obtain that

\[
V(t) \geq \frac{1}{2} \sum_{k \geq 1} |\langle X(t) - \hat{B}(t), \psi_k \rangle|^2 - \frac{1}{2} \sum_{k=1}^{N_0} |\langle X(t) - \hat{B}(t), \psi_k \rangle|^2
\]

\[
\geq \frac{1}{2m_R} \|X(t) - \hat{B}(t)\|_H^2 - \|Y(t)\|_H^2 - \frac{1}{m_R} \|\hat{B}(t)\|_H^2.
\]

Using the triangular inequality, this yields for all \( t \geq 0 \),

\[
\|X(t)\|_H \leq \|\hat{B}(t)\|_H + 2M_R \left( V(t) + \|Y(t)\|_H^2 + \frac{1}{m_R} \|\hat{B}(t)\|_H^2 \right)
\]

As \( t - D(t) \geq t - D_0 - \delta \), we have that \( \hat{u}(t) = u(t - D(t)) \xrightarrow{t \to +\infty} 0 \). Recalling that \( B \) is a bounded operator, we deduce that \( \|X(t)\|_H \leq \sqrt{2M_R} \limsup_{t \to +\infty} \sqrt{V(t)} \).

We compute for all \( t \geq 0 \) the time derivative of \( V \) as follows:

\[
\dot{V}(t) = \sum_{k \geq N_0 + 1} \text{Re} \left\{ \left( \frac{dX}{dt}(t) - \hat{B}(t), \psi_k \right)_H \right\} \times \langle X(t) - \hat{B}(t), \psi_k \rangle_H
\]

where \( \hat{u}(t) = (1 - D(t))\hat{u}(t - D(t)) \). Using (17), Assumption 2, and the Young Inequality (Y.I), we obtain that

\[
\dot{V}(t) \overset{(17)}{=} \sum_{k \geq N_0 + 1} \text{Re}(\lambda_k) |\langle X(t) - \hat{B}(t), \psi_k \rangle_H|^2
\]

\[
+ \sum_{k \geq N_0 + 1} \text{Re} \left\{ \langle (AB\hat{u}(t), \psi_k)_H - \langle \hat{B}(t), \psi_k \rangle_H \rangle \times \langle X(t) - \hat{B}(t), \psi_k \rangle_H \right\}
\]

\[
\leq -2\alpha V(t)
\]

\[
+ \sum_{k \geq N_0 + 1} \left( |\langle AB\hat{u}(t), \psi_k \rangle_H| + |\langle \hat{B}(t), \psi_k \rangle_H| \right) \times |\langle X(t) - \hat{B}(t), \psi_k \rangle_H|
\]

\[
\leq -\alpha V(t) + \frac{1}{\alpha} \sum_{k \geq N_0 + 1} \left( |\langle AB\hat{u}(t), \psi_k \rangle_H|^2 + |\langle \hat{B}(t), \psi_k \rangle_H|^2 \right).
\]

\[
\overset{(Y.I)}{\leq} -\alpha V(t) + \frac{1}{\alpha} \sum_{k \geq N_0 + 1} \left( |\langle AB\hat{u}(t), \psi_k \rangle_H|^2 + |\langle \hat{B}(t), \psi_k \rangle_H|^2 \right).
\]
For all $t \geq D_0 + \delta + t_0$, as $t - D(t) \geq t_0$ and thus $\varphi(t - D(t)) = 1$, we have

$$u(t) = u(t - D(t)) = KZ(t - D(t)) = \sum_{i=1}^{m} \{K_iZ(t - D(t))\} e_i$$

where $K_i$ stands for $i$-th line of $K$. We deduce that

$$\sum_{k \geq N_0 + 1} |\langle ABu(t), \psi_k \rangle_H|^2 \leq m \sum_{i=1}^{m} \sum_{k \geq 1} |\langle ABc_i, \psi_k \rangle|^2 |K_iZ(t - D(t))|^2$$

$$\leq \frac{m}{m_R} \left( \sum_{i=1}^{m} \|ABc_i\|_{L^2}^2 \|K_i\|_2 \right) \|Z(t - D(t))\|^2.$$ (16)

Similarly, for all $t \geq t_1 = 2(D_0 + \delta) + t_0$,

$$\dot{u}(t - D(t)) = K \dot{Z}(t - D(t)) \equiv KA_0Z(t - D(t)) + \hat{K}_N_0K\{Z(t - 2D(t)) - Z(t - D(t) - D_0)\}$$

$$= \sum_{i=1}^{m} \{K_iA_0Z(t - D(t))\} e_i + \sum_{i=1}^{m} \{K_i\hat{B}_N_0K\{Z(t - 2D(t)) - Z(t - D(t) - D_0)\}\} e_i$$

with $\hat{B}_N_0 \triangleq e^{D_0A_0}B_N_0$. We deduce that

$$\sum_{k \geq N_0 + 1} |\langle B\dot{u}(t), \psi_k \rangle_H|^2 \leq \frac{2\beta m}{m_R} \left( \sum_{i=1}^{m} \|Bc_i\|_{L^2}^2 \|K_iA_0\|_2 \right) \|Z(t - D(t))\|^2$$

$$+ \frac{2\beta m}{m_R} \left( \sum_{i=1}^{m} \|Bc_i\|_{L^2}^2 \|K_i\hat{B}_N_0K\|_2 \right) \times \|Z(t - 2D(t)) - Z(t - D(t) - D_0)\|^2.$$ (22)

where $\beta \triangleq \left( 1 + \sup_{t \in \mathbb{R}_+} |\dot{D}(t)| \right)^2$. Thus, introducing the constants $C_1, C_2 \geq 0$ defined by:

$$C_1 = \frac{m}{\alpha m_R} \sum_{i=1}^{m} \left\{ \|ABc_i\|_{L^2}^2 \|K_i\|_2^2 + 2\beta \|Bc_i\|_{L^2}^2 \|K_iA_0\|_2 \right\},$$

$$C_2 = \frac{2\beta m}{m_R} \sum_{i=1}^{m} \|Bc_i\|_{L^2}^2 \|K_i\hat{B}_N_0K\|_2^2,$$

we obtain that, for all $t \geq t_1$, $\dot{V}(t) \leq -\alpha V(t) + \omega(t)$ with $\omega(t) \geq 0$ defined for $t \geq t_1$ by:

$$\omega(t) \triangleq C_1 \|Z(t - D(t))\|^2 + C_2 \|Z(t - 2D(t)) - Z(t - D(t) - D_0)\|^2 \to 0 \quad t \to +\infty$$

where the limits holds true because $D(t) \in (D_0 - \delta, D_0 + \delta)$ and $Z(t) \to 0$ as $t \to +\infty$. Now, as $V$ is of class $C^1$ over $\mathbb{R}_+$, we obtain after integration that, for all $t \geq t_1$,

$$0 \leq V(t) \leq e^{-\alpha(t-t_1)}V(t_1) + e^{-\alpha t} \int_{t_1}^{t} e^{\alpha \tau} \omega(\tau) d\tau. \quad (26)$$

As $\alpha > 0$ and because the limit (25) holds true, we deduce that $V(t) \to 0$ and thus $\|X(t)\|_H \to 0$.

It remains to show the stability of the closed-loop system. Note first that, from (19), we have the estimate $\|Y(0)\| \leq \|X_0\|/\sqrt{m_R}$. Thus, from the estimate (23), the bounded nature of $\dot{B}$, and the conclusion of Theorem 2 that applies to the truncated model (18), it is sufficient to show that for all $\epsilon > 0$, there exists $\eta > 0$ such that $\|X_0\|_H \leq \eta$ implies $V(t) \leq \epsilon$ for all $t \geq 0$. From the proof of Theorem 2 (but with a different value of $t_1 > 0$), there exist constants $\tilde{C}_1, \tilde{C}_2 > 0$, independent of $X_0$, such that $\sup_{t \in [0, t_1]} \|Z(t)\| \leq \tilde{C}_1 \|X_0\|_H$ and $\sup_{t \in [0, t_1]} \|\tilde{Z}(t)\| \leq \tilde{C}_2 \|X_0\|_H$. Then, from (24), the fact that $\sup_{t \in [0, t_1]} |\dot{D}(t)| < +\infty$ and $\sup_{t \in \mathbb{R}_+} |\dot{\varphi}(t)| < +\infty$, and $V(0) \leq \|X_0\|_H^2/(2m_R)$, we obtain the existence of a constant $\tilde{C}_3 > 0$, independent of $X_0$, such that $V(t) \leq \tilde{C}_3 \|X_0\|_H^2$ for all $t \in [0, t_1]$. We deduce from (26) that $V(t) \leq \tilde{C}_3 \|X_0\|_H^2 + e^{-\alpha t} \int_{t_1}^{t} e^{\alpha \tau} \omega(\tau) d\tau$ for all $t \geq 0$. Let $\epsilon > 0$ be arbitrarily given. From the proof of Theorem 2, there exists $\eta' > 0$ such that $\|X_0\|_H \leq \eta'$ implies $\omega(t) \leq \alpha \epsilon/2$ for all $t \geq t_1$. Introducing $\eta = \min\left(\eta', \epsilon/(2\tilde{C}_3)\right) > 0$, we obtain that $\|X_0\|_H \leq \eta$ implies $V(t) \leq \epsilon$ for all $t \geq 0$. \(\square\)

### 4 Illustrative example

We consider the following one-dimensional reaction-diffusion equation on $(0, L)$ with a delayed Dirichlet boundary control:

$$\begin{cases}
\begin{aligned}
y_{tt}(x,t) &= ay_{xx}(t,x) + cy(t,x), & (t,x) \in \mathbb{R}_+ \times (0, L) \\
y(t,0) &= u(t - D(t)), & t > 0
\end{aligned}
\end{cases}$$

$$y(t,L) = \dot{Y}(t)$$

and

$$y(0,x) = \dot{y}(0,x)$$

with $\dot{Y}(0) = \dot{y}(0)$. The control $u(t)$ is calculated using the solution $Y(t)$ of the adjoint equation that we shall establish later.
with \( a, c > 0, y(t, \xi) \in \mathbb{R}, \) and \( u(t) \in \mathbb{R}^2. \) Introducing the \( \mathbb{R}\)-Hilbert space \( \mathcal{H} = L^2(0, L) \) with \( \langle f, g \rangle_{\mathcal{H}} = \int_0^L f g \, dx, \) it is well-known that the above reaction-diffusion equation can be written under the form of the abstract boundary control system \((15)\) with \( X(t) = y(t, \cdot) \in \mathcal{H}, \) \( A\dot{f} = af'' + cf \) over the domain \( D(A) = H^2(0, L), \) and the boundary operator \( Bf = f(0), f(L) \) over the domain \( D(B) = H^1(0, L). \) An example of lifting operator \( B \) associated with \((A, B)\) is given for any \((u_1, u_2) \in \mathbb{R}^2 \) by \( \{B(u_1, u_2)\}(\xi) = u_1 + (u_2 - u_1)\xi/L \) with \( \xi \in (0, L). \) It is well-known that the disturbance free operator \( A_0 \) is a Riesz-spectral operator that generates a \( C_0\)-semigroup with \( \lambda_n = c - an^2\pi^2 L^2 \) and \( \phi_n(\xi) = \psi_n(\xi) = \sqrt{2/L} \sin(n\pi x/L), \) \( n \geq 1. \) Then, the boundary control system \((A, B)\) satisfies Assumptions 1 and 2. Furthermore, straightforward computations show that \( b_{n, 1} = an\pi \sqrt{2/L^3} \) and \( b_{n, 2} = (1) A_n^{1/2}n\pi \sqrt{2/L^3}. \) As the eigenvalues are simple and \( b_{n,k} \neq 0 \) for all \( n \geq 1 \) and \( k \in \{1, 2\}, \) we directly obtain via the Kalman condition that \((A_{N_0}, B_{N_0})\) is commandable, fulfilling Assumption 3. Thus, one can compute a feedback gain \( K \in \mathbb{R}^{m \times N_0} \) such that \( A_{N_0} + B_{N_0}K \) is Hurwitz and then apply the result of Theorem 3 for ensuring the asymptotic stability of the closed-loop system.

For numerical computations, we set \( a = c = 0.5, \) and \( L = 2\pi. \) In this configuration, we have two unstable modes \( \lambda_1 = 0.375 \) and \( \lambda_2 = 0 \) while the two first stable modes are such that \( \lambda_3 = -0.625 \) and \( \lambda_4 = -1.5. \) Setting \( N_0 = 3, \) the feedback gain \( K \in \mathbb{R}^{2 \times 3} \) is computed to place the poles of the closed-loop truncated model \( A_3 = A_{N_0} + B_{N_0}K \) at \(-0.75, -1, \) and \(-1.25. \) Figure 1 compares for \( D_0 \in (0, 5] \) the upper-bound \((4)\) on the admissible values of \( \delta \in (0, D_0) \) such that the small gain condition of Theorem 1 from \cite{27} holds true and the values of \( \delta \) computed based on Theorem 2 with MATLAB R2017b. For the studied example, the values of \( \delta \) provided by Theorem 2 are significantly less conservative.

For numerical simulations, we set the nominal value of the delay to \( D_0 = 1 \) s. In this case, we were able to assess the feasibility of the LMI \((5)\) for values of \( \delta \) up to \( \delta = 0.3177. \) We set the initial condition \( X_0(\xi) = -\xi(2L/3 - \xi)(L - \xi) \) and the time-varying delay \( D(t) = 1 + 0.3 \sin(3\pi t + \pi/4) \) which is of class \( C^2 \) and is such that \( |D(t) - D_0| \leq 0.3 \leq 0.3177 \) and \( |D(t)| \leq 0.3\pi < +\infty \) for all \( t \geq 0. \) The transition time \( t_0 \) is taken as \( t_0 = 0.5s \) while the transition signal \( \varphi|_{[0, t_0]} \) is selected as the restriction over \([0, t_0]\) of the unique quintic polynomial function \( f \) satisfying \( f(0) = f'(0) = f''(0) = f'''(0) = f''''(0) = 0 \) and \( f(t_0) = 1. \) The employed numerical scheme relies on the discretization of the reaction-diffusion equation using its first 10 modes. The time domain evolution of the closed-loop system is depicted in Figs. 2-3. As expected from Theorem 3, both the system state and the control input converge to zero.

5 Conclusion

This paper discussed first the use of predictor feedback for the stabilization of finite-dimensional LTI systems in the presence of an uncertain time varying-delay in the control input. By means of a Lyapunov-Krasovskii functional, it has been derived an LMI-based sufficient condition ensuring the asymptotic stability of the closed-loop system for small enough variations of the time-varying delay around its nominal value. Then, this result has been extended to the feedback stabilization of a class of diagonal infinite-dimensional boundary control systems.

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