A LOCALIZATION IN MV-ALGEBRAS

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Abstract. In this document we consider a way of localizing an MV-algebra. Given any prime filter $F$ we find a local MV-algebra which has the same poset of prime filters as the poset of prime filters comparable to $F$.

1. Introduction

A local MV-algebra is one with a single maximal implication filter. Such algebras are of interest in the representation theory of MV-algebras (see [7] for example).

The set of prime implication filters of an MV-algebra forms a spectral root system, ordered by set-inclusion. The existence of a unique maximal filter is equivalent to the stem of this root system being nonempty. (The stem is the set $\text{Stem} = \{ P \mid P \text{ is a prime filter comparable to every other prime filter} \}$.)

Whenever the stem is non-empty it has a least element, the Conrad filter (defined below). This filter can be characterized in several ways, as we show in section 2 below. This work is heavily based on work of Conrad on lattice-ordered groups (see [5]), recasting his material in terms of implication filters in MV-algebras.

In the last section we consider how to invert this characterization to get a prime filter into the stem of an MV-algebra. This localization takes a prime implication filter $P$ and finds a quotient in which the maximal filter over $P$ is the unique maximal filter, and the prime filter structure of the quotient is isomorphic to the set of prime filters comparable to $P$.

In most of what follows the filters are taken to be implication filters rather than lattice filters. We recall that an implication filter is a lattice filter closed under powers.
Given an MV-algebra $\mathcal{L}$, there are several sets of filters that we are interested in:

$$\text{PSpec} = \{P \mid P \text{ is a prime implication filter of } \mathcal{L}\}$$

= the prime spectrum;

$$\text{PSpec}(F) = \{P \mid P \text{ is a prime implication filter of } \mathcal{L} \text{ comparable to } F\}$$;

$$\mu S = \{P \mid P \text{ is a minimal prime filter of } \mathcal{L}\}$$

= the minimal spectrum;

$$\mu S(F) = \{P \mid P \text{ is a minimal prime filter of } \mathcal{L} \text{ comparable to } F\}$$.

Our notation usually follows that of [3] with the exception that we use $\otimes$ instead of $\circ$.

2. Counits

Definition 2.1. $u \in \mathcal{L}$ is a counit iff $u < 1$ and there exists some $v < 1$ with $u \lor v = 1$.

Definition 2.2. The Conrad filter of an MV-algebra is the implication filter generated by the counits.

We usually denote it by $\mathcal{N}(\mathcal{L})$ or $N$.

If $N = \mathcal{N}(\mathcal{L})$ then $N$ is prime as $a \lor b = 1, a, b < 1$ implies $a$ and $b$ are counits and so in $N$.

All implication filters that contain $N$ form a chain. The following lemma provides an alternative characterization of the prime filters in this chain.

Lemma 2.3. Let $P$ be a prime implication filter. Then $P$ contains all counits iff for all $x \notin P$ and all $p \in P$ $p \geq x$.

Proof. Suppose that $x \notin P$ and $y \in P$ with $x \notin y$. We know that $(x \rightarrow y) \lor (y \rightarrow x) = 1$.

As $x \notin y$ we have $x \rightarrow y < 1$, and $y \notin x$ implies $y \rightarrow x < 1$ and so $y \rightarrow x$ is a co-unit.

If it is in $P$ then so is $x \wedge y = (y \rightarrow x) \otimes y$, contradicting $x \notin P$. Thus $P$ cannot contain all co-units.

Conversely if $a$ is a co-unit and $a \lor b = 1$ for some $b > 0$. One of $a$ or $b$ is in $P$ (as $P$ is prime). If $a \notin P$ then $a \leq b$ which is impossible, so $a \in P$. □

A slight variation of this proof lets us see that filters are incomparable because of counits.

Lemma 2.4. Let $P$ and $Q$ be incomparable implication filters. Then there is a counit in $Q \setminus P$. 

Proof. Suppose not, ie every counit in $Q$ is also in $P$. As $P$ and $Q$ are incomparable we can find $x \in Q \setminus P$ and $y \in P \setminus Q$. Thus $x \not\leq y$ and $y \not\leq x$ and so $x \to y < 1$ and $y \to x < 1$ and $(x \to y) \lor (y \to x) = 1$. So $y \to x$ is a counit in $Q$ and (by assumption) must be in $P$. As $y \in P$ we now have $x \land y = y \otimes (y \to x) \in P$ contradicting $x \not\in P$.

The next two results show that $N$ is actually the minimum prime filter comparable to all prime filters.

**Proposition 2.5.** Let $P$ be any prime implication filter that does not contain all counits. Then there is a prime implication filter incomparable to $P$.

*Proof.* As $P$ does not contain all counits we know that there is some $g \not\in P$ that is not below $P$, ie there is some $p \in P$ with $p \not\leq g$. Of course $g \not\leq p$.

Thus $g \to p < 1$ and $p \to g < 1$ and $(g \to p) \lor (p \to g) = 1$.

As $(p \to g) \otimes (p \lor g) = g$ we must have $p \to g \not\in P$.

Let $Q$ be maximal avoiding $g \to p$. Then $Q$ is prime and as $(g \to p) \lor (p \to g) = 1 \in Q$ we have $p \to g \in Q \setminus P$. By construction $g \to p \in P \setminus Q$ and so these two ideals are incomparable. □

**Proposition 2.6.** If $P$ is a prime implication filter then either $N \subseteq P$ or $P \subseteq N$.

*Proof.* If $P$ is not a subset of $N$ then we can find $p \in P \setminus N$. $p \not\in N$ implies $p$ is below $N$ and so $N \subseteq [p, 1] \subseteq P$.

Thus $N$ is the minimum prime implication filter comparable to all others. The existence of such a filter implies that $N$ is a proper filter, as if we have a minimal prime implication filter $F$ comparable to all others then it must contain all counits – by proposition 2.5 and so $N$ exists and so $F$ equals $N$.

Since any desired root system is the root system of an MV-algebra ([4]), we see that it is possible to have non-trivial $N$.

**Proposition 2.7.** $N$ is a minimal prime implication filter iff $N = \{1\}$.

*Proof.* The right to left implication is immediate.

If $N$ is minimal then it is the unique minimal implication filter and so must equal $\{1\}$ – as we know the intersection of all minimal implication filters is $\{1\}$.

Or just notice that $\mathcal{L}$ embeds into $\prod_{m \in S} \mathcal{L}/m = \mathcal{L}/N$ is linearly ordered, and so $\mathcal{L}$ is linearly ordered which implies $N = \{1\}$.

We also note that if $N$ is proper then there is a unique maximal implication filter – the one that contains $N$. We also have the converse.

**Proposition 2.8.** If there is exactly one maximal proper implication filter then it contains all counits.
Proof. Let $M$ be the maximum implication filter. Let $a, b < 1$ with $a \vee b = 1$. Let $F_b = \{ x \mid x \vee b = 1 \}$. Then $0 \notin F_b, a \in F_b$ and it is easy to see that $F_b$ is a lattice filter. Also, $x \in F_b$ implies $x^a \vee b \geq (x \vee b)^a = (x \vee b)^a = 1$ and so $F_b$ is an implication filter. Hence $a \in F_b \subseteq M$. □

Thus, if there is a maximum implication filter $M$ then $N \subseteq M$ and $N$ is proper.

3. Localization

Let $P$ be a prime implication filter. We seek a quotient of $L$ in which $P$ contains the Conrad filter. The construction we give below also preserves the structure of $\text{PSpec}(P)$.

Definition 3.1. Let $P$ be a prime implication filter. Then

$$\ell(P) = \left[ \left\{ x \rightarrow p \mid p \in P \text{ and } x \notin P \right\} \right].$$

Because of lemma 2.4 we need to quotient out by at least $\ell(P)$ in order to make $P$ contain all counits in a quotient.

It is clear that $\ell(P) \subseteq P$ as $x \rightarrow p \geq p$ for any $p \in P$. In general this inclusion is strict, with the only exception being minimal prime filters.

Lemma 3.2. $P$ is minimal prime iff $\ell(P) = P$.

Proof. If $P$ is minimal prime and $p \in P$ then there is some $t \notin P$ with $t \lor p = 1$. Therefore $t \rightarrow p = 1 \rightarrow p = p \in \ell(P)$.

If $\ell(P) = P$ and $p \in P$ then $p \geq x \rightarrow p'$ for some $x \notin P$ and $p' \in P$. Now $p' \rightarrow x \notin P$ else we would have $p' \otimes (p' \rightarrow x) = p' \land x \in P$ and so $x \in P$. Also $p \lor (p' \rightarrow x) \geq (x \rightarrow p') \lor (p' \rightarrow x) = 1$. Thus $P$ must be minimal. □

The next few lemmas show the relationship of $\ell(P)$ to the minimal filters below $P$.

Lemma 3.3. If $m \subseteq P$ is minimal prime then $\ell(P) \subseteq m$.

Proof. Let $x \notin P$ and $p \in P$. Then $p \otimes (p \rightarrow x) = p \land x$ implies $p \rightarrow x \notin P$ and so is not in $m$. But $(x \rightarrow p) \lor (p \rightarrow x) = 1 \in m$ and $m$ is prime, so $x \rightarrow p \in m$. □

Lemma 3.4. Let $p \in P \setminus \ell(P)$. Then there is some minimal prime filter $m \subseteq P$ with $p \notin m$.

Proof. Look in $L/\ell(P)$. Then $[p] \neq 1$ and is in $P/\ell(P)$. We also know that the Conrad filter of $L/\ell(P)$ is contained in $P/\ell(P)$ – since $x \notin P$ and $p \in P$ implies $x \rightarrow p \in \ell(P)$ and so $x \leq p \text{ mod } \ell(P)$. All minimal filters must be subsets of the Conrad filter and so take $M$ to be a minimal prime filter.
of $\mathcal{L}/\ell(P)$ that avoids $[p] < [1]$. Then $M \subseteq P/\ell(P)$ and so the preimage $M'$
that avoids $p$.

Any minimal filter of $\mathcal{L}$ contained in $M'$ works. □

**Theorem 3.5.**

$$\ell(P) = \bigcap \{m \mid m \in \mu S \text{ and } m \subseteq P\}.$$ 

**Proof.** By lemma 3.3 we know that LHS $\subseteq$ RHS.

From lemma 3.4 we know that $p \not\in \text{LHS}$ implies $p \not\in \text{RHS}$, i.e. RHS $\subseteq$ LHS. □

We can now define the localization of an MV-algebra at a prime implication filter.

**Definition 3.6.** Let $P$ be a prime implication filter of an MV-algebra $\mathcal{L}$. Then the localization of $\mathcal{L}$ at $P$ is the MV-algebra $\mathcal{L}/\ell(P)$.

If $Q \subseteq P$ are two prime implication filters then we have $\{m \mid m \in \mu S \text{ and } m \subseteq Q\} \subseteq \{m \mid m \in \mu S \text{ and } m \subseteq P\}$ and so $\ell(P) \subseteq \ell(Q)$ (from the theorem). Hence there is a natural MV-morphism $\mathcal{L}/\ell(P) \to \mathcal{L}/\ell(Q)$.

And finally a universal property of this localization.

We recall that if $f : \mathcal{L} \to \mathcal{M}$ is an MV-morphism then the shell of $f$ is

$$\text{sh}(f) = f^{-1}[1] = \{x \mid f(x) = 1\}$$

is an implication filter in $\mathcal{L}$.

**Theorem 3.7.** Let $P$ be any filter and $f : \mathcal{L} \to \mathcal{M}$ such that $\text{sh}(f) \subseteq P$ and $\mathcal{N}(\mathcal{M}) \subseteq f[P] \uparrow$.

Then $\ell(P) \subseteq \text{sh}(f)$.

**Proof.** Let $x \not\in P$ and $p \in P$. If $f(x) \not\in f[P]$ then $f(x) \leq f(p)$ and so $f(x \to p) = 1$, i.e. $x \to p \in \text{sh}(f)$.

If $f(x) \in f[P]$ then for some $p \in P$ we have $x \to p$ and $p \to x$ both in the shell of $f$ and hence in $P$. But then $x \land p = p \otimes (p \to x) \in P$ – contradiction. □

From the theorem we see that if $f$ takes $P$ to a filter containing all counits then $f$ factorizes through $\mathcal{L}/\ell(P)$, and so, in some sense, $\mathcal{L}/\ell(P)$ is the largest quotient in which $P$ contains all counits (or dominates its complement).

The assumption that $\text{sh}(f) \subseteq P$ is essential, else the theorem yields only that the smaller set $\ell(P \lor \text{sh}(f)) \subseteq \text{sh}(f)$. Indeed if $P, Q$ are incomparable prime filters then $\mathcal{N}(\mathcal{L}/Q) = \{1\} \subseteq P/Q$ but if $q \in Q \setminus P$ and $p \in P \setminus Q$ then $q \to p \in \ell(P) \setminus Q$ – else $p \land q = q \otimes (q \to p) \in Q$, contradicting $p \not\in Q$.

**Lemma 3.8.** Let $F$ be a prime filter. Then $\ell(P) \subseteq F$ iff $F$ is comparable to $P$.
Proof. If $P \subseteq F$ then $\ell(P) \subseteq P \subseteq F$. If $F \subseteq P$ then $\ell(P) \subseteq \ell(F) \subseteq F$.

Conversely, if $\ell(P) \subseteq F$ then $F/\ell(P)$ is prime in $\mathcal{L}/\ell(P)$ and so comparable to $P/\ell(P)$. Hence $F = \eta^{-1}[F/\ell(P)]$ is comparable to $\eta^{-1}[P/\ell(P)] = P$. \hfill \Box

**Theorem 3.9.** $\text{PSpec}(P)$ is order-isomorphic to $\text{PSpec}(\mathcal{L}/\ell(P))$.

**Proof.** We know that $\text{PSpec}(\mathcal{L}/\ell(P))$ is order-isomorphic to 
$\{F \mid F$ is a prime filter with $\ell(P) \subseteq F\}$ and from the lemma the latter set is $\text{PSpec}(P)$. \hfill \Box

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