UNIQUENESS OF AN ENTIRE FUNCTION SHARING TWO VALUES JOINTLY WITH ITS DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper, we continue to investigate the uniqueness problem when an entire function \( f \) and its linear differential polynomial \( L(f) \) share two distinct complex values CMW (counting multiplicities in the weak sense) jointly. Also, We investigate the same problem when \( f \) and its differential monomial \( M(f) \) share two distinct complex values CMW. Our results generalize the recent result of Lahiri (Comput. Methods Funct. Theory, https://doi.org/10.1007/s40315-020-00355-4).

1. Introduction, Definitions and Results

A function analytic in the open complex plane \( \mathbb{C} \) except possibly for poles is called meromorphic in \( \mathbb{C} \). If no poles occur, then the function is called entire. For a non-constant meromorphic function \( f \) defined in \( \mathbb{C} \) and for \( a \in \mathbb{C} \cup \{\infty\} \), we denote by \( E(a, f) \) the set of \( a \)-points of \( f \) counted multiplicities and \( \overline{E}(a, f) \) the set of all \( a \)-points ignoring multiplicities. If for two non-constant meromorphic functions \( f \) and \( g \), \( E(a, f) = E(a, g) \), we say that \( f \) and \( g \) share the value \( a \) CM (counting multiplicities). If \( \overline{E}(a, f) = \overline{E}(a, g) \), then we say that \( f \) and \( g \) are said to share the value \( a \) IM (ignoring multiplicities). Throughout the paper, the standard notations of Nevanlinna’s value distribution theory of meromorphic functions [5, 16] have been adopted. A meromorphic function \( a(z) \) is said to be small with respect to \( f \) provided that \( T(r, a) = S(r, f) \), that is \( T(r, a) = o(T(r, f)) \) as \( r \rightarrow \infty \), outside of a possible exceptional set of finite linear measure.

In 1976, it was shown by Rubel and Yang [14] that if an entire function \( f \) and its derivative \( f' \) share two values \( a, b \) CM, then \( f = f' \). After that Gundersen [4] improved the result by considering two IM shared Values. Yang [15] also extended the result of Rubel and Yang [14] by replacing \( f' \) with the \( k \)-th derivative \( f^{(k)} \). Since then the subject of sharing values between a meromorphic function and its derivatives has become one of the most prominent branches of the uniqueness theory. Mues and Steinmetz [13] showed that if a meromorphic function \( f \) shares three finite values IM with \( f' \), then \( f = f' \). Frank and Schwick [1] improved this result by replacing \( f' \) with \( f^{(k)} \), where \( k \) is a positive integer. After that many mathematicians spent their times towards the improvements of this result (see [2, 3, 8, 12]). In 2000, Li and Yang [9] improved the result of Yang [15] in the following.

**Theorem A.** [9] Let \( f \) be a non-constant entire function, \( k \) be a positive integer and \( a, b \) be distinct finite numbers. If \( f \) and \( f^{(k)} \) share \( a \) and \( b \) IM, then \( f = f^{(k)} \).
We now recall the notion of set sharing as follows: Let $S$ be a subset of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a \in S} E(a, f)$ and $\overline{E}_f(S) = \bigcup_{a \in S} \overline{E}(a, f)$. We say that two meromorphic functions $f$ and $g$ share the set $S$ CM or IM if $E_f(S) = E_g(S)$ or $\overline{E}_f(S) = \overline{E}_g(S)$, respectively.

Using the notion of set sharing instead of value sharing, Li and Yang [10] proved the following theorem.

**Theorem B.** [10] Let $f$ be a non-constant entire function and $a_1$, $a_2$ be two distinct finite complex numbers. If $f$ and $f^{(1)}$ share the set $\{a_1, a_2\}$ CM, then one and only one of the following holds:

(i) $f = f^{(1)}$
(ii) $f + f^{(1)} = a_1 + a_2$
(iii) $f = c_1 e^{cz} + c_2 e^{-cz}$ with $a_1 + a_2 = 0$, where $c$, $c_1$ and $c_2$ are non-zero constants satisfying $c^2 \neq 1$ and $4c^2 c_1 c_2 = a_1^2 (c^2 - 1)$.

In 2020, Lahiri [6] introduced a new type of set sharing notion called CMW (counting multiplicities in the weak sense) as follows:

Let $f$ and $g$ be two non-constant meromorphic functions in $\mathbb{C}$ and $a \in \mathbb{C} \cup \{\infty\}$ and $B \subset \mathbb{C} \cup \{\infty\}$. We denote by $E_B(a; f, g)$ the set of those distinct $a$-points of $f$ which are the $b$-points of $g$ having the same multiplicity for some $b \in B$. For $A \subset \mathbb{C} \cup \{\infty\}$, we put $E_B(A; f, g) = \bigcup_{a \in A} E_B(a; f, g)$. Clearly $E_B(A; f, g) = E_B(A; g, f)$ for $A = B$. For $S \subset \mathbb{C} \cup \{\infty\}$ we define

$$Y = \{\overline{E}(S, f) \cup E(S, g)\} \setminus \overline{E}_S(S, f; g).$$

We say that $f$ and $g$ share the set $S$ with counting multiplicities in the weak sense (CMW) if $N_Y(r, a; f) = S(r, f)$ and $N_Y(r, a; g) = S(r, g)$ for every $a \in S$, where $N_Y(r, a; f)$ denotes the counting function, counted with multiplicities of those $a$-points of $f$ which lie in the set $Y$.

We note that $f$ and $g$ share the set $S$ with counting multiplicities if and only if $Y = \emptyset$.

Lahiri [6] greatly improved Theorem B by considering the higher order derivative $f^{(k)}$ and CMW in place of CM set sharing and proved the following theorem.

**Theorem C.** [6] Let $f$ be a non-constant entire function and $k$ be a positive integer such that

$$N\left(r, \frac{f^{(k)}}{f^{(1)}}\right) = S(r, f).$$

Suppose that $a_1$ and $a_2$ are two distinct finite complex numbers. If $f$ and $f^{(k)}$ share the set $\{a_1, a_2\}$ CMW, then only one of the following holds:

(i) $f = f^{(k)}$
(ii) $f + f^{(k)} = a_1 + a_2$
(iii) $f = c_1 e^{cz} + c_2 e^{-cz}$ with $a_1 + a_2 = 0$, where $c$, $c_1$ and $c_2$ are non-zero constants satisfying $c^{2k} \neq 1$ and $4c^{2k} c_1 c_2 = a_1^2 (c^{2k} - 1)$ and $k$ is an odd positive integer.
For further investigation of the above theorem, we now define a linear differential polynomial $L(f)$ and a differential monomial $M(f)$ of an entire function $f$ as follows:

$$L(f) = b_1 f^{(1)} + b_2 f^{(2)} + \cdots + b_k f^{(k)} = \sum_{j=1}^{k} b_j f^{(j)},$$

(1.2)

where $b_1, b_2, \ldots, b_k (\neq 0)$ are complex constants, and

$$M(f) = (f^{(1)})^{n_1} (f^{(2)})^{n_2} \cdots (f^{(k)})^{n_k},$$

(1.3)

where $k$ is a positive integer and $n_1, n_2, \ldots, n_k$ are non-negative integers, not all of them are zero. We call $k$ and $\lambda = \sum_{j=1}^{k} n_j$, respectively the order and the degree of the monomial $M(f)$.

From the above discussion it is natural to ask the following questions.

**Question 1.1.** What can be said about the uniqueness when an entire function $f$ share two values jointly CMW with its linear differential polynomial $L(f)$?

**Question 1.2.** What can be said about the uniqueness when an entire function $f$ share two values jointly CMW with its differential monomial $M(f)$?

In the present paper, we prove the following results which will answer the above questions positively. We use a methodology which is similar to [6] but with some modifications.

2. **Main results**

**Theorem 2.1.** Let $f$ be a non-constant entire function and $L(f)$ be a linear differential polynomial defined as in (1.2) such that

$$\mathcal{N}(r, L(f)/f^{(1)}) = S(r, f).$$

(2.1)

Suppose that $a_1$ and $a_2$ are two distinct finite complex numbers. If $f$ and $L(f)$ share the set $\{a_1, a_2\}$ CMW, then only one of the following holds:

(i) $f = L(f)$

(ii) $f + L(f) = a_1 + a_2$

(iii) $f = c_1 e^{cz} + c_2 e^{-cz}$ with $a_1 + a_2 = 0$, where $c, c_1$ and $c_2$ are non-zero constants satisfying $(b_1 c + b_2 c^3 + \cdots + b_k c^k)^2 \neq 1$ and $4(b_1 c + b_2 c^3 + \cdots + b_k c^k)^2 c_1 c_2 = a_1^2 ((b_1 c + b_2 c^3 + \cdots + b_k c^k) - 1)$ and $k$ is an odd positive integer.

**Theorem 2.2.** Let $f$ be a non-constant entire function and $M(f)$ be a differential monomial defined as in (1.3) such that

$$\mathcal{N}(r, M(f)/(f^{(1)})^\lambda) = S(r, f).$$

(2.2)

Suppose that $a_1$ and $a_2$ are two distinct finite complex numbers. If $f^\lambda$ and $M(f)$ share the set $\{a_1, a_2\}$ CMW, then only one of the following holds:

(i) $f^\lambda = M(f)$

(ii) $f^\lambda + M(f) = a_1 + a_2$

(iii) $f^\lambda = c_1 e^{cz} + c_2 e^{-cz}$, $M(f) = \sqrt[k]{a_1 c_1 e^{2cz} - c_2}/e^{cz}$ with $a_1 + a_2 = 0$, where $A, c, c_1$ and $c_2$ are non-zero constants and $\lambda = \sum_{j=1}^{k} n_j$.

We give the following examples in the support of the main theorems.
Example 2.1. Let \( f = e^{\omega z} + a_1 + a_2 \), where \( \omega^k = -1 \), \( k \) is a positive integer and \( a_1, a_2 \) are any two finite distinct complex constants. If \( L(f) = M(f) = f^{(k)} \). Then all the conditions of Theorems 2.1 and 2.2 are satisfied. Here conclusion (iv) of Theorems 2.1 and 2.2 holds.

Example 2.2. Let \( f = e^{\lambda z} \), where \( \lambda^n = 1 \) and \( L(f) = M(f) = f^{(5)} \). Then all the conditions of the above two theorems are satisfied and conclusion (i) of the above two theorems holds.

Remark 2.1. By taking \( L(f) = f^{(k)} \) in Theorem 2.1, we get Theorem C, which is a particular case of our result.

3. Key lemmas

In this section, we present some necessary lemmas which will be required to prove the main results.

Lemma 3.1. Let \( f \) be a non-constant entire function and \( a_1, a_2 \) be two distinct finite complex numbers. If \( f \) and \( L(f) \) share the set \{\( a_1, a_2 \)\} CMW, then \( S(r, L(f)) = S(r, f) \).

Proof. Since \( f \) is entire, we have
\[
T(r, L(f)) = m(r, L(f)) \leq m\left( r, \frac{L(f)}{f} \right) + m(r, f) \leq T(r, f) + S(r, f).
\]
Again since \( f \) and \( L(f) \) share the set \{\( a_1, a_2 \)\} CMW, we get by second fundamental theorem
\[
T(r, f) \leq \overline{N}\left( r, \frac{1}{f - a_1} \right) + \overline{N}\left( r, \frac{1}{f - a_2} \right) + S(r, f) \leq 2T(r, L(f)) + S(r, f).
\]
From 3.1 and 3.2 we conclude that \( S(r, L(f)) = S(r, f) \). This proves the lemma.

Lemma 3.2. Let \( f \) be a non-constant entire function and \( a_1, a_2 \) be two distinct finite complex numbers. If \( f^\lambda \) and \( M(f) \), where \( \lambda = \sum_{j=1}^{k} n_j \) and \( M(f) \) is defined as in (3.3) share the set \{\( a_1, a_2 \)\} CMW, then \( S(r, M(f)) = S(r, f) \).

Proof. The proof of the lemma can be carried out in the line of the proof of Lemma 3.1. So, we omit the details.

Lemma 3.3. [11] [10] Let \( f \) be a non-constant meromorphic function and \( R(f) = P(f)/Q(f) \), where \( P(f) = \sum_{k=0}^{p} a_k f^k \) and \( Q(f) = \sum_{j=0}^{q} b_j f^j \) are two mutually prime polynomials in \( f \). If \( T(r, a_k) = S(r, f) \) and \( T(r, b_j) = S(r, f) \) for \( k = 0, 1, 2, \ldots, p \) and \( j = 0, 1, 2, \ldots, q \) and \( a_p \neq 0, b_q \neq 0 \), then \( T(r, R(f)) = \max\{p, q\} T(r, f) + S(r, f) \).

Lemma 3.4. [7] The coefficients \( a_0(\neq 0), a_1, \ldots, a_{n-1} \) of the differential equation \( f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f^{(1)} + a_0 f = 0 \) are polynomials if and only if all solutions of it are entire functions of finite order.
Lemma 3.5. Let \( f \) be a non-constant entire function and \( a_1, a_2 \) be two non-zero distinct finite numbers. If \( f \) and \( L(f) \) \((k \geq 1)\) share the set \( \{a_1, a_2\} \) CMW and \( T(r, h) \neq S(r, f) \), where

\[
  h = \frac{(L(f) - a_1)(L(f) - a_2)}{(f - a_1)(f - a_2)},
\]

then the following hold:

(i) \( \Psi \neq 0 \) and \( T(r, \Psi) = S(r, f) \), where

\[
  \Psi = \frac{(f^{(1)}h - L^{(1)}(f))(f^{(1)}h + L^{(1)}(f))}{(L(f) - a_1)(L(f) - a_2)}.
\]

(ii) \( T(r, L(f)) = N \left( r, \frac{1}{L(f) - a_j} \right) + S(r, f) \) for \( j = 1, 2 \).

(iii) \( m \left( r, \frac{1}{f - c} \right) = S(r, f) \), where \( c \neq a_1, a_2 \in \mathbb{C} \).

(iv) \( T(r, h) = m \left( r, \frac{1}{f - a_1} \right) + m \left( r, \frac{1}{f - a_2} \right) + S(r, f) \)

\[
  = m \left( r, \frac{1}{f^{(1)}} \right) + S(r, f) \leq m \left( r, \frac{1}{L(f)} \right) + S(r, f).
\]

(v) \( 2T(r, f) - 2T(r, L(f)) = m \left( r, \frac{1}{h} \right) + S(r, f) \).

Proof. Since \( f \) and \( L(f) \) share the set \( \{a_1, a_2\} \) CMW, \( N(r, h) + N(r, 1/h) = S(r, f) \). Now if \( \Psi \equiv 0 \), then \( h = \pm L^{(1)}(f)/f^{(1)} \). This implies that \( T(r, h) = S(r, f) \), which contradicts to our assumption. Therefore \( \Psi \neq 0 \).

Let \( z_0 \) be a zero of \( (L(f) - a_1)(L(f) - a_2) \) and \( (f - a_1)(f - a_2) \) of multiplicity \( p \geq 2 \). Then \( z_0 \) is a zero of \( (f^{(1)}h - L^{(1)})(f^{(1)}h + L^{(1)}) \) with multiplicity \( 2(p - 1) \geq p \). So, \( z_0 \) is not a pole of \( \Psi \).

From (3.3), we get

\[
  (L(f) - a_1)(L(f) - a_2) = h(f - a_1)(f - a_2).
\]

Differentiating (3.3), we obtain

\[
  L^{(1)}(f)(2L(f) - a_1 - a_2) = h^{(1)}(f - a_1)(f - a_2) + hf^{(1)}(2f - a_1 - a_2).
\]

Let \( z_0 \) be a simple zero of \( (L(f) - a_1)(L(f) - a_2) \) and \( (f - a_1)(f - a_2) \). Then

\[
  2L(f)(z_1) - a_1 - a_2 = \pm(2f(z_1) - a_1 - a_2).
\]

So from (3.6), we get

\[
  (h(z_1)f^{(1)}(z_1) - (L(f)(z_1))^2)(h(z_1)f^{(1)}(z_1) + (L(f)(z_1))^2) = 0.
\]

Hence from (3.4), we see that \( z_1 \) is not a pole of \( \Psi \). Since \( f \) and \( L(f) \) share the set \( \{a_1, a_2\} \) CMW, we obtain \( N(r, \Psi) = S(r, f) \).

By (3.3), we get

\[
  \frac{f^{(1)}h - L^{(1)}(f)}{L(f) - a_1} = \frac{f^{(1)}L(f)}{(f - a_1)(f - a_2)} - \frac{a_2f^{(1)}}{(f - a_1)(f - a_2)} = \frac{L^{(1)}(f)}{L(f) - a_1}.
\]
Since
\[
\frac{a_2 f^{(1)}}{(f - a_1)(f - a_2)} = \frac{1}{a_1 - a_2} \left( \frac{f^{(1)}}{f - a_1} - \frac{f^{(1)}}{f - a_2} \right),
\]
we get from (3.7) that
\[
m \left( r, \frac{f^{(1)} h - L^{(1)}(f)}{L(f) - a_1} \right) = S(r, f).
\]
Similarly,
\[
m \left( r, \frac{f^{(1)} h + L^{(1)}(f)}{L(f) - a_2} \right) = S(r, f).
\]
Therefore, from (3.4) we obtain
\[
m \left( r, \Psi \right) = S(r, f)
\]
and hence
\[
T \left( r, \Psi \right) = S(r, f),
\]
which is (i).

Now in view of (3.3), we get from (3.4) that
\[
1 \frac{f^{(1)} h - L^{(1)}(f)}{L(f) - a_1} = \frac{1}{\Psi} \left( \frac{f^{(1)}}{f - a_1} - \frac{L^{(1)}(f)}{(L(f) - a_1)(L(f) - a_2)} \right).
\]
Therefore,
\[
m \left( r, \frac{1}{f^{(1)} h - L^{(1)}(f)} \right) = S(r, f).
\]
Similarly, we get
\[
m \left( r, \frac{1}{f^{(1)} h + L^{(1)}(f)} \right) = S(r, f).
\]
So we obtain
\[
m \left( r, \frac{1}{L(f) - a_1} \right) \leq m \left( r, \frac{f^{(1)} h - L^{(1)}(f)}{L(f) - a_1} \right) + m \left( r, \frac{1}{f^{(1)} h - L^{(1)}(f)} \right) = S(r, f)
\]
and
\[
m \left( r, \frac{1}{L(f) - a_2} \right) = S(r, f).
\]
Therefore,
\[
T \left( r, L(f) \right) = N \left( r, \frac{1}{L(f) - a_j} \right) + S(r, f),
\]
for \( j = 1, 2 \), which is (ii).

for \( c \neq a_1, a_2 \), we get from (3.7)
\[
\frac{f^{(1)} h - L^{(1)}(f)}{(L(f) - a_1)(f - c)} = \frac{f^{(1)} L(f)}{(f - c)(f - a_1)(f - a_2)} - \frac{a_2 f^{(1)}}{(f - c)(f - a_1)(f - a_2)} - \frac{L^{(1)}(f)}{(L(f) - a_1)(f - c)}.
\]
We note that
\[
\frac{a_2 f^{(1)}}{(f - c)(f - a_1)(f - a_2)} = \alpha \frac{f^{(1)}}{f - c} + \beta \frac{f^{(1)}}{f - a_1} + \gamma \frac{f^{(1)}}{f - a_2},
\]
where \( \alpha = \frac{a_2}{(a_1-c)(a_2-c)} \), \( \beta = \frac{a_2}{(c-a_1)(a_2-a_1)} \) and \( \gamma = \frac{a_2}{(c-a_2)(a_1-a_2)} \).

Therefore, we get
\[
m \left( r, \frac{f^{(1)}h - L^{(1)}(f)}{(f-c)(L(f) - a_1)} \right) = S(r, f).
\]

Since by (3.4),
\[
\frac{1}{f-c} = \frac{1}{\Psi(f-c)(L(f) - a_1)} \frac{f^{(1)}h + L^{(1)}(f)}{(L(f) - a_1)(L(f) - a_2)},
\]
we get
\[
m \left( r, \frac{1}{f-c} \right) = S(r, f),
\]
which is (iii). Since
\[
h = \frac{(L(f))^2 - (a_1 + a_2)L(f) + a_1a_2}{(f-a_1)(f-a_2)},
\]
we have
\[
T(r, h) = m(r, h) + S(r, f) \leq m \left( r, \frac{1}{(f-a_1)(f-a_2)} \right) + S(r, f)
\]
\[
\leq m \left( r, \frac{1}{f^{(1)}} \right) + m \left( r, \frac{f^{(1)}}{(f-a_1)(f-a_2)} \right) + S(r, f)
\]
\[
\leq m \left( r, \frac{1}{f^{(1)}} \right) + S(r, f).
\] (3.8)

Since
\[
\frac{\Psi}{f^{(1)}} = \frac{f^{(1)}(L(f))^2 - (a_1 + a_2)L(f) + a_1a_2}{(f-a_1)(f-a_2)} - \frac{L^{(1)}(f)}{f^{(1)}} \frac{L^{(1)}(f)}{(L(f) - a_1)(L(f) - a_2)},
\]
we get by (i) that
\[
m \left( r, \frac{1}{f^{(1)}} \right) \leq m \left( r, \frac{\Psi}{f^{(1)}} \right) + S(r, f)
\]
\[
\leq m \left( r, \frac{1}{(f-a_1)(f-a_2)} \right) + S(r, f).
\] (3.9)

Since \( \frac{1}{(f-a_1)(f-a_2)} \), we have by (ii) that
\[
m \left( r, \frac{1}{(f-a_1)(f-a_2)} \right) \leq T(r, h) + S(r, f).
\] (3.10)

From (3.8), (3.9) and (3.10), we have
\[
T(r, h) = m \left( r, \frac{1}{f-a_1} \right) + m \left( r, \frac{1}{f-a_2} \right) + S(r, f) \leq m \left( r, \frac{1}{L(f)} \right) + S(r, f),
\]
which is (iv).
Keeping in view of (3.3), we get from (ii) and (iv) that
\[
2T(r, L(f)) = N \left( r, \frac{1}{L(f) - a_1} \right) + N \left( r, \frac{1}{L(f) - a_2} \right) + S(r, f)
\]
\[
= N \left( r, \frac{1}{(L(f) - a_1)(L(f) - a_2)} \right) + S(r, f)
\]
\[
= N \left( r, \frac{1}{h(f - a_1)(f - a_2)} \right) + S(r, f)
\]
\[
= 2T(r, f) - m \left( r, \frac{1}{f - a_1} \right) - m \left( r, \frac{1}{f - a_2} \right) + N \left( r, \frac{1}{h} \right) + S(r, f)
\]
\[
= 2T(r, f) - T(r, h) + N \left( r, \frac{1}{h} \right) + S(r, f).
\]

So, \(2T(r, f) - 2T(r, L(f)) = m \left( r, \frac{1}{h} \right) + S(r, f)\), which is (v). This completes the proof of the lemma.

\[\square\]

**Lemma 3.6.** Let \(f\) be a non-constant entire function and \(a_1, a_2\) be two distinct finite complex numbers. If \(f\) and \(L(f)\) share the set \(\{a_1, a_2\}\) CMW, then \(T(r, h) = S(r, f)\), where \(h\) is defined in Lemma 3.5.

**Proof.** Since \(f\) and \(L(f)\) share the set \(\{a_1, a_2\}\) CMW, we must have \(N(r, h) = S(r, f)\) and \(N(r, 1/h) = S(r, f)\). Assume on the contrary that \(T(r, h) \neq S(r, f)\). By Lemma 3.5 we know that \(\Psi \neq 0\) and \(T(r, \Psi) = S(r, f)\).

Differentiating (3.3), we get
\[
2L(f)L^{(1)}(f) - (a_1 + a_2)L^{(1)}(f) = (2ff^{(1)} - (a_1 + a_2)f^{(1)})h + h^{(1)}(f - a_1)(f - a_2). \tag{3.11}
\]

From (3.3) and (3.11), we obtain
\[
\frac{(2L(f) - (a_1 + a_2))L^{(1)}(f)}{(L(f) - a_1)(L(f) - a_2)} = \frac{(2f - (a_1 + a_2))f^{(1)}}{(f - a_1)(f - a_2)} + \frac{h^{(1)}}{h}.
\]

Squaring the above equation, we get
\[
\frac{((2L(f) - (a_1 + a_2))^2L^{(1)}(f))^2}{(L(f) - a_1)^2(L(f) - a_2)^2} = \frac{(2f - (a_1 + a_2))^2f^{(1)}}{(f - a_1)^2(f - a_2)^2} + \beta^2 + \frac{2\beta(2f - (a_1 + a_2))f^{(1)}}{(f - a_1)(f - a_2)},
\]

where \(\beta = h^{(1)}/h\).

Eliminating \(L^{(1)}(f)\) from (3.3), (3.4) and the above equation, we get
\[
\frac{(2L(f) - (a_1 + a_2))^2\Psi}{(L(f) - a_1)(L(f) - a_2)} = \frac{4(L(f) + f - (a_1 + a_2))(L(f) - f)(f^{(1)})^2}{(f - a_1)^2(f - a_2)^2} - \beta^2 - \frac{2\beta(2f - (a_1 + a_2))f^{(1)}}{(f - a_1)(f - a_2)}. \tag{3.12}
\]

Let \(z_0\) be a zero of \((f - a_1)(f - a_2)\) which is also a zero of \((L(f) - a_1)(L(f) - a_2)\). Since \(f\) and \(L(f)\) share the set \(\{a_1, a_2\}\) CMW and \(T(r, \beta) = S(r, f)\), almost all the poles of right hand side of (3.12) are simple, and hence it follows from the same
Combining the above two we have
\[ N \left( r, \frac{1}{L(f) - a_j} \right) = \mathcal{N} \left( r, \frac{1}{L(f) - a_j} \right) + S(r, f), \quad j = 1, 2. \]  
(3.13)

Differentiating (3.4), we get
\[ 2h^2 f^{(1)}(f^{(2)} + \beta f^{(1)}) - 2L^{(1)}(f)L^{(2)}(f) = \Psi^{(1)}(L(f) - a_1)(L(f) - a_2) + \Psi(2L(f) - (a_1 + a_2)L^{(1)}(f)). \]

Now eliminating \( h \) from the above equation by using (3.4), we get
\[ \left[ 2\Psi(f^{(2)} + \beta f^{(1)}) - f^{(1)} \Psi^{(1)} \right] (L(f) - a_1)(L(f) - a_2) = L^{(1)} \left[ 2f^{(1)}\Psi L - (a_1 + a_2)f^{(1)}\Psi - 2(\beta f^{(1)} + f^{(2)})L^{(1)} + 2f^{(1)}L^{(2)} \right]. \]  
(3.14)

From the above equation, we see that any simple zeros of \((L(f) - a_1)(L(f) - a_2)\) must be the zeros of \(2f^{(1)}\Psi L(f) - (a_1 + a_2)f^{(1)}\Psi - 2(\beta f^{(1)} + f^{(2)})L^{(1)}(f) + 2f^{(1)}L^{(2)}(f)\).

Let
\[ \Psi_1 = \frac{2f^{(1)}\Psi L(f) - (a_1 + a_2)f^{(1)}\Psi - 2(\beta f^{(1)} + f^{(2)})L^{(1)}(f) + 2f^{(1)}L^{(2)}(f)}{(f - a_1)(f - a_2)}. \]  
(3.15)

Since \( f \) and \( L(f) \) share the set \( \{a_1, a_2\} \) CMW and “almost all” the zeros of \((L(f) - a_1)(L(f) - a_2)\) are simple, we must have \( N(r, \Psi_1) = S(r, f) \).

On the hand, by the lemma of logarithmic derivative, it can be easily seen that \( m(r, \Psi_1) = S(r, f) \). Hence, \( T(r, \Psi_1) = S(r, f) \).

We now consider the following two cases:

**Case 1:** \( \Psi_1 \neq 0 \). Then it follows from (3.15) that
\[ 2T(r, f) = T(r, (f - a_1)(f - a_2)) + S(r, f) \]
\[ = m(r, (f - a_1)(f - a_2)) + S(r, f) \]
\[ \leq m(r, (f - a_1)(f - a_2)\Psi_1) + m \left( r, \frac{1}{\Psi_1} \right) + S(r, f) \]
\[ \leq m(r, f^{(1)}) + m(r, L(f)) + T(r, \Psi_1) + S(r, f) \]
\[ \leq T(r, f) + T(r, L(f)) + S(r, f). \]

Therefore, \( T(r, f) \leq T(r, L(f)) + S(r, f) \).

Since \( L(f) \) is a linear differential polynomial in \( f \), we get
\[ T(r, L(f)) \leq T(r, f) + S(r, f). \]

Combining the above two we have \( T(r, f) = T(r, L(f)) + S(r, f) \).

By Lemma 3.3 (ii) and (3.13), we get
\[ 2T(r, f) = \mathcal{N} \left( r, \frac{1}{f - a_1} \right) + \mathcal{N} \left( r, \frac{1}{f - a_2} \right) + S(r, f), \]
which implies that
\[ m \left( r, \frac{1}{f - a_1} \right) + m \left( r, \frac{1}{f - a_2} \right) = S(r, f). \]
Thus from (3.3), the lemma of logarithmic derivative and the above observation, we get

\[ T(r, h) = N(r, h) + m(r, h) \]

\[ \leq m \left( r, \frac{(L(f))^2}{(f-a_1)(f-a_2)} \right) + m \left( r, \frac{a_1 a_2}{(f-a_1)(f-a_2)} \right) + 2m \left( r, \frac{L(f)}{(f-a_1)(f-a_2)} \right) + S(r, f) = S(r, f). \]

i.e., \( T(r, h) = S(r, f) \), which contradicts to our assumption.

**Case 2:** \( \Psi_1 \equiv 0 \). Then from (3.14) and (3.15), we obtain

\[ \frac{\Psi^{(1)}}{\Psi} = 2 \left( \frac{h^{(1)}}{h} + \frac{f^{(2)}}{f^{(1)}} \right). \]

Integrating above, we get

\[ (h f^{(1)})^2 = c \Psi, \quad (3.16) \]

where \( c \) is a non-zero constant.

It follows from (3.4) and (3.10) that

\[ (L^{(1)}(f))^2 = -((L(f))^2 - (a_1 + a_2)L(f) + (a_1 a_2 - c))\Psi \]

\[ = -(L(f) - d_1)(L(f) - d_2)\Psi, \]

where \( d_1 \) and \( d_2 \) are two complex constants. If \( d_1 \neq d_2 \), then by the lemma of logarithmic derivative, we get

\[ m \left( r, \frac{1}{L^{(1)}(f)} \right) = m \left( r, \frac{-L^{(1)}(f)}{(L(f) - d_1)(L(f) - d_2)} \right) = S(r, f). \]

Therefore,

\[ m \left( r, \frac{1}{(f-a_1)(f-a_2)} \right) \leq m \left( r, \frac{L^{(1)}(f)}{(f-a_1)(f-a_2)} \right) + m \left( r, \frac{1}{L^{(1)}(f)} \right) = S(r, f). \]

Hence keeping in view of the above, we get from (3.3) and the lemma of logarithmic derivative

\[ T(r, h) = N(r, h) + m(r, h) \]

\[ \leq m \left( r, \frac{(L(f))^2}{(f-a_1)(f-a_2)} \right) + m \left( r, \frac{L(f)}{(f-a_1)(f-a_2)} \right) + S(r, f) \]

\[ = S(r, f), \]

which contradicts to our assumption.

Therefore, \( d_1 = d_2 = (a_1 + a_2)/2 = d \), say. Hence,

\[ (L^{(1)}(f))^2 = -\Psi(L(f) - d)^2. \quad (3.17) \]

From (3.3), (3.16) and (3.17), we get

\[ (L(f) - d)(L(f) - a_1)(L(f) - a_2) = c_2(f-a_1)(f-a_2)\Psi_2, \quad (3.18) \]

where \( c_2 \) is a non-zero constant satisfying \( c_2^2 = -c \) and \( \Psi_2 = L^{(1)}(f)/f^{(1)} \).
From (3.16), it can be easily seen that \( N(r, 1/f^{(1)}) = S(r, f) \). Therefore, \( N(r, \Psi_2) = S(r, f) \). On the other hand, by the lemma of logarithmic derivative, we have \( m(r, \Psi_2) = S(r, f) \), and hence \( T(r, \Psi_2) = S(r, f) \).

Since \( \Psi_2 \neq 0 \), it follow from (3.18) that
\[
3T(r, L(f)) = 2T(r, f) + S(r, f). \tag{3.19}
\]

Let
\[
\Psi_3 = \frac{L^{(1)}(f)}{L(f) - d}. \tag{3.20}
\]

Then from (3.17), we get \( \Psi_3^2 = -\Psi \). Hence, \( T(r, \Psi_3) = S(r, f) \) and \( \Psi_3 = 0 \).

Now from (3.3) and (3.11), we get
\[
(f - d)hf^{(1)} = (L(f) - d)^2\Psi_3 - \frac{1}{2}\beta(L(f) - a_1)(L(f) - a_2). \tag{3.21}
\]

By (3.16), we get
\[
T(r, hf^{(1)}) = S(r, f).
\]

Therefore, by (3.21), we obtain
\[
T(r, f) = T(r, L(f)), \text{ or } T(r, f) = 2T(r, L(f)) + S(r, f) \tag{3.22}
\]
according as when \( \Psi_3 = \beta/2 \) or not.

Combining (3.19) and (3.22), we get \( T(r, f) = S(r, f) \), which is a contradiction. Hence \( T(r, h) = S(r, f) \). This completes the proof of the lemma.

\[\square\]

**Lemma 3.7.** Let \( f \) be a non-constant entire function and \( a_1, a_2 \) be two non-zero distinct finite numbers. If \( f^k \) and \( M(f) \) \((k \geq 1)\) share the set \( \{a_1, a_2\} \) CMW and \( T(r, h_1) \neq S(r, f) \), where
\[
h_1 = \frac{(M(f) - a_1)(M(f) - a_2)}{(f^k - a_1)(f^k - a_2)}. \tag{3.23}
\]
then the following fold:

(i) \( \Phi \neq 0 \) and \( T(r, \Phi) = S(r, f) \), where
\[
\Phi = \frac{(f^k)^{(1)}h_1 - (M(f))^{(1)}h_1 + (M(f))^{(1)}}{(M(f) - a_1)(M(f) - a_2)}. \tag{3.24}
\]

(ii) \( T(r, M(f)) = N \left(r, \frac{1}{M(f) - a_j}\right) + S(r, f) \) for \( j = 1, 2 \).

(iii) \( m \left(r, \frac{1}{f^k - c}\right) = S(r, f) \), where \( c \neq a_1, a_2 \in \mathbb{C} \).

(iv)
\[
T(r, h_1) = m \left(r, \frac{1}{f^k - a_1}\right) + m \left(r, \frac{1}{f^k - a_2}\right) + S(r, f)
\]
\[
= m \left(r, \frac{1}{f^k}\right) + S(r, f) \leq m \left(r, \frac{1}{M(f)}\right) + S(r, f).
\]

(v) \( 2\lambda T(r, f) - 2T(r, M(f)) = m \left(r, \frac{1}{h_1}\right) + S(r, f) \).
Proof. The proof of this lemma can be carried out in a similar manner as done in the proof of Lemma 3.5. So, we omit the details. □

Lemma 3.8. Let \( f \) be a non-constant entire function and \( a_1, a_2 \) be two distinct finite complex numbers. If \( f^λ \) and \( M(f) \) share the set \( \{a_1, a_2\} \) CMW, then \( T(r, h_1) = S(r, f) \), where \( h_1 \) is defined in Lemma 3.7.

Proof. The proof of this lemma is essentially can be done in a similar manner as Lemma 3.6. So, we omit the details. □

4. Proof of the main results

Proof of Theorem 2.1. Let \( 2^n \) be the principal branch of \( \log h \), where \( h \) is defined as in Lemma 3.5. Then by Lemma 3.6, we obtain

\[
T(r, e^n) = \frac{1}{2} T(r, h) + S(r, f) = S(r, f).
\]

Also (3.3) can be written as

\[
(L(f) − a_1)(L(f) − a_2) = e^{2n}(f − a_1)(f − a_2).
\]

(4.1)

And so

\[
GH = \left( \frac{a_1 − a_2}{2} \right)^2 (e^{2n} − 1),
\]

(4.2)

where

\[
G = e^n f - \frac{a_1 + a_2}{2} e^n + L(f) - \frac{a_1 + a_2}{2}
\]

and

\[
H = e^n f - \frac{a_1 + a_2}{2} e^n - L(f) + \frac{a_1 + a_2}{2}.
\]

If \( e^{2n} = 1 \), then from (4.1), we get

\[
(f − L(f))(f + L(f) − a_1 − a_2) = 0,
\]

which implies that either \( f = L(f) \), or \( f + L(f) = a_1 + a_2 \).

Now suppose that \( e^{2n} \neq 1 \). Since \( f \) is entire we get \( N(r, G) + N(r, H) = S(r, f) \), and so, from (4.2), we get \( N(r, 1/G) + N(r, 1/H) = S(r, f) \). Therefore,

\[
T\left(r, \frac{G^{(j)}}{G}\right) + T\left(r, \frac{H^{(j)}}{H}\right) = S(r, f),
\]

(4.3)

where \( j = 1, 2, \ldots, k \).

Suppose \( f^{(1)} = bL(f) \). Then using the condition (1.1), the lemma of logarithmic derivative, and the first fundamental theorem of Nevalinna, it is easily seen that \( T(r, b) = S(r, f) \).

From the definition of \( G \) and \( H \) it follows that

\[
G + H = e^n(2f − a_1 − a_2)
\]

(4.4)

and

\[
G − H = 2L(f) − a_1 − a_2 = 2bf^{(1)} − a_1 − a_2,
\]

(4.5)

where \( bλ = 1 \) and \( T(r, λ) = S(r, f) \) as \( T(r, b) = S(r, f) \).
Eliminating \( f \) and \( f'^{(1)} \), from (4.4) and (4.5), we get
\[
\left( e^n + \lambda \eta'^{(1)} - \lambda \frac{G'^{(1)}}{G} \right) G + \left( \lambda \eta'^{(1)} - e^n - \lambda \frac{H'^{(1)}}{H} \right) H + b(a_1 + a_2) = 0. \tag{4.6}
\]
Now eliminating \( H \) from (4.2) and (4.6), we obtain
\[
\Phi_1 G^2 + \Phi_2 G + \Phi_3 = 0, \tag{4.7}
\]
where
\[
\Phi_1 = e^n + \lambda \eta'^{(1)} - \lambda \frac{G'^{(1)}}{G}, \tag{4.8}
\]
\[
\Phi_2 = \lambda \eta'^{(1)} - e^n - \lambda \frac{H'^{(1)}}{H} \left( \frac{a_1 - a_2}{2} \right)^2 (e^{2n} - 1), \tag{4.9}
\]
\[
\Phi_3 = \lambda (a_1 + a_2). \tag{4.10}
\]
If \( \Phi_1 \not\equiv 0 \) or \( \Phi_2 \not\equiv 0 \), then by Lemma 3.2, we see from (4.7) that \( T(r, G) = S(r, f) \), and therefore from (4.4), we get \( T(r, f) = S(r, f) \), which is a contradiction. Therefore, \( \Phi_1 = \Phi_2 = 0 \). Then from (4.7), we get \( \Phi_3 = 0 \). This implies that
\[
e^n + \lambda \eta'^{(1)} - \lambda \frac{G'^{(1)}}{G} = 0, \tag{4.11}
\]
\[
\lambda \eta'^{(1)} - e^n - \lambda \frac{H'^{(1)}}{H} = 0, \tag{4.12}
\]
\[
a_1 + a_2 = 0. \tag{4.13}
\]
Adding (4.11) and (4.12), we get
\[
\frac{G'^{(1)}}{G} + \frac{H'^{(1)}}{H} = 2\eta'^{(1)},
\]
and so by integration, we have
\[
GH = c_0 e^{2n}, \tag{4.14}
\]
where \( c_0 \) is a non-zero constant.

Now from (4.2), (4.13) and (4.14), we get \( e^{2n} = A \), where \( A \) is a constant.

From (4.4), (4.5) and (4.13), we get
\[
\left( \sqrt{A} - \sum_{j=1}^{k} b_j G^{(j)} \right) G^2 = \left( \sqrt{A} + \sum_{j=1}^{k} b_j H^{(j)} \right) B, \tag{4.15}
\]
where \( B = (a_1 - a_2)^2 / 4(A - 1) \), constant.

If \( \sqrt{A} - \sum_{j=1}^{k} b_j G^{(j)} / G \not\equiv 0 \), then from (4.4) and (4.15), we get \( T(r, G) = S(r, f) \) and so from (4.4), we get \( T(r, f) = S(r, f) \). Therefore, from (4.4), we get \( T(r, f) = S(r, f) \), which is a contradiction. Hence we have \( \sum_{j=1}^{k} b_j G^{(j)} - \sqrt{A} G = 0 \) and \( \sum_{j=1}^{k} b_j H^{(j)} + \sqrt{A} H = 0 \). This implies by Lemma 3.2 that \( G \) and \( H \) are of finite order. Also from (4.14), we see that \( G \) and \( H \) do not assume the value 0.

Therefore, let us assume that \( G = e^P \) and \( H = e^Q \), where \( P, Q \) are polynomials of degree \( p \) and \( q \), respectively. Differentiating \( j \) times, we obtain \( G^{(j)} = P_j e^P \) and \( H^{(j)} = Q_j e^Q \), where \( P_j \) and \( Q_j \) are polynomials of degree \( (p - 1)j \) and \( (q - 1)j \), respectively.
respectively. Since \( \sum_{j=1}^{k} b_j G^{(j)} = \sqrt{AG} \) and \( \sum_{j=1}^{k} b_j H^{(j)} = \sqrt{AH} \), we have \( p = q = 1 \). Hence in view of (4.14), we may write \( G = 2d_1 e^{cz} \) and \( H = 2d_2 e^{-cz} \), where \( c, d_1, d_2 \) are non-zero constants.

Now from (4.13) and (4.11), we get
\[
f = c_1 e^{cz} + c_2 e^{-cz},
\]
where \( c_1 = d_1 / \sqrt{A} \) and \( c_2 = d_2 / \sqrt{A} \).

Differentiating (4.16), we have
\[
f^{(j)} = \frac{c_1 c^j e^{2cz} + c_2 (-c)^j}{e^{cz}},
\]
where \( j = 1, 2, \ldots, k \). Therefore,
\[
L(f) = \sum_{j=1}^{k} b_j (c_1 c^j e^{2cz} + c_2 (-c)^j).
\]
Again from (4.5) and (4.13), we get
\[
L(f) = \frac{\sqrt{A}(c_1 e^{2cz} - c_2)}{e^{cz}}.
\]
Comparing (4.17) and (4.18), we obtain
\[
b_1 c + b_2 c^2 + \cdots + b_k c^k = \sqrt{A}
\]
and
\[
-b_1 c + b_2 c^2 - \cdots + (-1)^k b_k c^k = -\sqrt{A}.
\]
from (4.19) and (4.20), it is clear that \( A = (b_1 c + b_3 c^3 + \cdots + b_k c^k)^2 \), where \( k \) is an odd positive integer.

Now from (4.2) and (4.13), we see that \( 4d_1 d_2 = a_1^2 (A-1) \) and so
\[
4c_1 c_2 A = a_1^2 (A-1),
\]
where \( A = (b_1 c + b_3 c^3 + \cdots + b_k c^k)^2 \), where \( k \) is an odd positive integer. This completes the proof of the Theorem 2.1.

\[\square\]

**Proof of Theorem 2.2** Let \( 2\xi \) be the principal branch of \( \log h_1 \), where \( h_1 \) is defined as in (3.23) Then by Lemma 3.8 we obtain
\[
T(r, e^{\xi}) = \frac{1}{2} T(r, h_1) + S(r, f) = S(r, f).
\]

Also (3.23) can be written as
\[
(M(f) - a_1)(M(f) - a_2) = e^{2\xi}(f - a_1)(f - a_2),
\]
and so
\[
G_1 H_1 = \left( \frac{a_1 - a_2}{2} \right)^2 (e^{2\xi} - 1),
\]
where
\[
G_1 = e^{\xi} f^\lambda - \frac{a_1 + a_2}{2} e^{\xi} + M(f) - \frac{a_1 + a_2}{2}
\]
and
\[
H_1 = e^{\xi} e^\lambda - \frac{a_1 + a_2}{2} e^{\xi} - M(f) + \frac{a_1 + a_2}{2}.
\]
If $e^{2\xi} \equiv 1$, then from (4.21), we get

$$(f^\lambda - M(f))(f^\lambda + M(f) - a_1 - a_2) = 0,$$

which implies that either $f^\lambda = M(f)$, or $f^\lambda + M(f) = a_1 + a_2$.

Now suppose that $e^{2\xi} \not\equiv 1$. Since $f$ is entire we get $N(r, G_1) + N(r, H_1) = S(r, f)$, and so, from (4.22), we get $N(r, 1/H_1) + N(r, 1/G_1) = S(r, f)$. Therefore,

$$T \left( r, \frac{G_1^{(j)}}{G_1} \right) + T \left( r, \frac{H_1^{(j)}}{H_1} \right) = S(r, f), \quad (4.23)$$

where $j = 1, 2, \ldots, k$.

Suppose $(f^\lambda)^{(1)} = b_1 M(f)$. Then using the condition (2.2), the Lemma of logarithmic derivative, and the first fundamental theorem of Nevanlinna, it is easily seen that $T(r, b_1) = S(r, f)$.

From the definition of $G_1$ and $H_1$ it follows that

$$G_1 + H_1 = e^\xi (2f^\lambda - a_1 - a_2) \quad (4.24)$$

and

$$G_1 - H_1 = 2M(f) - a_1 - a_2 = 2\mu(f^\lambda)^{(1)} - a_1 - a_2, \quad (4.25)$$

where $b\mu = 1$ and so $T(r, \mu) = S(r, f)$ as $T(r, b) = S(r, f)$.

Eliminating $f^\lambda$ and $(f^\lambda)^{(1)}$ from (4.24) and (4.25), we obtain

$$\left( e^\xi + \mu \xi^{(1)} - \mu \frac{G_1^{(1)}}{G_1} \right) G_1 + \left( \mu \xi^{(1)} - e^\xi - \mu \frac{H_1^{(1)}}{H_1} \right) H_1 + b_1(a_1 + a_2) = 0. \quad (4.26)$$

Now eliminating $H_1$ from (4.22) and (4.26), we obtain

$$\chi_1 G^2 + \chi_2 G + \chi_3 = 0, \quad (4.27)$$

where

$$\chi_1 = e^\xi + \mu \xi^{(1)} - \mu \frac{G_1^{(1)}}{G_1}, \quad (4.28)$$

$$\chi_2 = \mu \xi^{(1)} - e^\xi - \mu \frac{H_1^{(1)}}{H_1} \left( \frac{a_1 - a_2}{2} \right)^2 (e^{2\xi} - 1), \quad (4.29)$$

$$\chi_3 = \mu (a_1 + a_2). \quad (4.30)$$

If $\chi_1 \not\equiv 0$ or $\chi_2 \not\equiv 0$, then by Lemma 3.2 we get from (4.27) that $T(r, G_1) = S(r, f)$, and so from (4.22), we get $T(r, H_1) = S(r, f)$. So, from (4.24), we get $T(r, f) = S(r, f)$, which is a contradiction. Therefore, $\chi_1 = \chi_2 = 0$. Then from (4.27), we get $\chi_3 = 0$. This implies that

$$e^\xi + \mu \xi^{(1)} - \mu \frac{G_1^{(1)}}{G_1} = 0, \quad (4.31)$$

$$\mu \xi^{(1)} - e^\xi - \mu \frac{H_1^{(1)}}{H_1} = 0, \quad (4.32)$$

$$a_1 + a_2 = 0. \quad (4.33)$$
Adding (4.31) and (4.32), we get
\[ \frac{G_1^{(1)}}{G_1} + \frac{H_1^{(1)}}{H_1} = 2\xi^{(1)}, \]
and so by integration, we have
\[ G_1H_1 = c_0^*e^{2\xi}, \quad (4.34) \]
where \( c_0^* \) is a non-zero constant.

Now from (4.22), (4.33) and (4.34), we get \( e^{2\xi} = A \), where \( A \) is a constant.

From (4.24), (4.25) and (4.33), we get
\[ \left( \frac{\sqrt{A} - G_1^{(1)}}{G_1} \right) G_1^2 = - \left( \frac{\sqrt{A}}{\mu} - \frac{H_1^{(1)}}{H_1} \right) B, \]
where \( B = (a_1 - a_2)^2/4(A - 1), \) constant.

If \( \sqrt{A}/\mu - G_1^{(1)}/G_1 \neq 0 \), then from (4.23) and (4.34), we get \( T(r, G_1) = S(r, f) \) and so from (4.34), we get \( T(r, H_1) = S(r, f) \). Therefore, from (4.24), we get \( T(r, f) = S(r, f) \), which is a contradiction. Hence we must have \( \mu G_1^{(1)} - \sqrt{A}G_1 = 0 \) and \( \mu H_1^{(1)} - \sqrt{A}H_1 = 0 \). This implies by Lemma 3.4 that \( G_1 \) and \( H_1 \) are of finite order. Also from (4.34), we see that \( G_1 \) and \( H_1 \) do not assume the value 0.

Therefore, we may assume that \( G_1 = e^P \) and \( H_1 = e^Q \), where \( P, Q \) are polynomials of degree \( p \) and \( q \), respectively.

Differentiating once, we get \( G_1^{(1)} = P^{(1)}e^P \) and \( H_1^{(1)} = Q^{(1)}e^Q \). Therefore, \( P^{(1)} \) and \( Q^{(1)} \) are polynomials of degree \((p - 1)\) and \((q - 1)\), respectively. Since \( \mu G_1^{(1)} = \sqrt{A}G_1 \) and \( \mu H_1^{(1)} = \sqrt{A}H_1 \), we have \( p = q = 1 \). Hence in view of (4.34), we may write \( G_1 = 2d_1e^{cz} \) and \( H = 2d_2e^{-cz} \), where \( c, d_1, d_2 \) are non-zero constants.

Now from (4.24), (4.25) and (4.33), we get
\[ f^\lambda = c_1e^{cz} + c_2e^{-cz}, \]
\[ M(f) = \frac{\sqrt{A}(c_1e^{2cz} - c_2)}{e^{cz}}, \]
where \( c_1 = d_1/\sqrt{A} \) and \( c_2 = d_2/\sqrt{A} \). This completes the proof of the theorem.

\[ \square \]

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