On the Circumradius Condition for Piecewise Linear Triangular Elements

Kenta Kobayashi · Takuya Tsuchiya

Received: date / Accepted: date

Abstract We discuss the error analysis of linear interpolation on triangular elements. We claim that the circumradius condition is more essential than the well-known maximum angle condition for convergence of the finite element method. Numerical experiments show that this condition is the best possible. We also point out that the circumradius condition is closely related to the definition of surface area.

Keywords linear interpolation · the circumradius condition · the finite element method · Schwarz’s example · the definition of surface area

Mathematics Subject Classification (2000) 65D05, 65N30, 26B15

1 Introduction

In numerical analysis, linear interpolation on triangular elements is one of the more fundamental conceptions. Specifically, as meshes become finer, it is an important tool in understanding why and how finite element approximations converge to an exact solution.

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. Suppose that we would like to solve the Poisson equation of finding $u \in H^1_0(\Omega)$ for a given $f \in L^2(\Omega)$ such that

$$-\Delta u = f \quad \text{in} \quad \Omega.$$  (1)
With a triangulation \( \tau \) of \( \Omega \), we define the FEM solution \( u_h \) by

\[
\int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx = \int_{\Omega} f v_h \, dx, \quad \forall v_h \in S_{\tau},
\]

where \( S_{\tau} \) is the set of all polynomials whose degree is at most 1 and

\[
S_{\tau} := \left\{ v_h \in C^0(\overline{\Omega}) \cap H^1_0(\Omega) \mid v_h|_K \in \mathcal{P}_1, \forall K \in \tau \right\}.
\]

Let \( K \subset \mathbb{R}^2 \) be a triangle with apices \( x_i, i = 1, 2, 3 \). We shall always consider \( K \) to be a closed set in \( \mathbb{R}^2 \). For a continuous function \( f \in C^0(\Omega) \), the linear interpolation \( I_K f \in \mathcal{P}_1 \) is defined by

\[
(I_K f)(x_i) = f(x_i), \quad i = 1, 2, 3.
\]

If \( f \in C^0(\overline{\Omega}) \), the linear interpolation \( I_K f \) is defined by \( (I_K f)|_K = I_K f \). Céa’s lemma claims that the error \( |u - u_h|_{1,2,\Omega} \) is estimated as

\[
|u - u_h|_{1,2,\Omega} \leq \inf_{v \in S_{\tau}} |u - v|_{1,2,\Omega} \leq |u - I_K u|_{1,2,\Omega} = \left( \sum_{K \in \tau} |u - I_K u|_{1,2,K}^2 \right)^{1/2}.
\]

Therefore, the interpolation error \( |u - I_K u|_{1,2,K} \) provides an upper bound of \( |u - u_h|_{1,2,\Omega} \).

It has been known that we need to impose a geometric condition to \( K \) to obtain an error estimation of \( |u - I_K u|_{1,2,K} \). We mention the following well-known results. Let \( h_K \) be the diameter of \( K \) and \( \rho_K \) be the maximum radius of the inscribed circle in \( K \).

- **The minimum angle condition.** Zlámal [20] (1968).
  Let \( \theta_0, 0 < \theta_0 < \pi/3 \), be a constant. If any angle \( \theta \) of \( K \) satisfies \( \theta \geq \theta_0 \) and \( h_K \leq 1 \), then there exists a constant \( C = C(\theta_0) \) independent of \( h_K \) such that

\[
\|v - I_K v\|_{1,2,K} \leq C h_K \|v\|_{2,2,K}, \quad \forall v \in H^2(K).
\]

- **The regularity (inscribed ball) condition,** see, for example, Ciarlet [4].
  Let \( \sigma > 0 \) be a constant. If \( h_K/\rho_K \leq \sigma \) and \( h_K \leq 1 \), then there exists a constant \( C = C(\sigma) \) independent of \( h_K \) such that

\[
\|v - I_K v\|_{1,2,K} \leq C h_K \|v\|_{2,2,K}, \quad \forall v \in H^2(K).
\]

- **The maximum angle condition.** Babuška-Aziz [1], Jamet [7] (1976).
  Let \( \theta_1, 2\pi/3 \leq \theta_1 < \pi \), be a constant. If any angle \( \theta \) of \( K \) satisfies \( \theta \leq \theta_1 \) and \( h_K \leq 1 \), then there exists a constant \( C = C(\theta_1) \) independent of \( h_K \) such that

\[
\|v - I_K v\|_{1,2,K} \leq C h_K \|v\|_{2,2,K}, \quad \forall v \in H^2(K).
\]

It is easy to show that the minimum angle condition is equivalent to the regularity condition [4] Exercise 3.1.3, p130]. Since its discovery, the maximum angle condition was believed to be the most essential condition for convergence of solutions of the finite element method.

However, Hannukainen-Korotov-Křížek pointed out that “the maximum angle condition is not necessary for convergence of the finite element method” by showing
The circumradius condition

The circumradius condition is closely related to the definition of surface area. In the 19th century, people believed that surface area could be defined as the limit of the area of inscribed polygonal surfaces. In the 1880s, Schwarz and Peano independently presented their famous example that refutes this expectation. See [3], [14], [15]. We shall observe below that, in Schwarz’s example, the limit of the inscribed polygonal surfaces is equal to the area of the cylinder if and only if the circumradius of triangles converges to 0. We shall also show that the following theorem is a direct consequence of the circumradius condition.

Theorem 1 Let \( \Omega \subset \mathbb{R}^2 \) be a rectangular domain. Let \( \{ \tau_n \}_{n=1}^{\infty} \) be a sequence of triangulations of \( \Omega \). If \( f \in W^{2,1}(\Omega) \) and \( \{ \tau_n \}_{n=1}^{\infty} \) satisfies the circumradius condition, then we have

\[
\lim_{n \to \infty} \int_{\Omega} \sqrt{1 + |\nabla I_{\tau_n} f|^2} \, dx = \int_{\Omega} \sqrt{1 + |f|^2} \, dx.
\]

Let us summarize the notation and terminology to be used. The Lebesgue and Sobolev spaces on a domain \( \Omega \subset \mathbb{R}^2 \) are denoted by \( L^p(\Omega) \) and \( W^{m,p}(\Omega) \), \( m = 1, 2 \), \( 1 \leq p \leq \infty \). As usual, \( W^{1,\infty}(\Omega) \) is denoted by \( H^1(\Omega) \). The norms and seminorms of \( L^p(\Omega) \) and \( W^{m,p}(\Omega) \) are denoted by \( \| \cdot \|_{m,p,\Omega} \) and \( | \cdot |_{m,p,\Omega} \), \( m = 0, 1, 2 \), \( 1 \leq p \leq \infty \). For a polygonal domain \( \Omega \subset \mathbb{R}^2 \), a triangulation \( \tau \) is a set of triangles which satisfies the following properties: (recall that each \( K \) is a closed set.)

(i) \( \bigcup_{K \in \tau} K = \Omega \), and \( \text{int}K \cap \text{int}K' = \emptyset \) for any \( K, K' \in \tau \) with \( K \neq K' \).

(ii) If \( K \cap K' \neq \emptyset \) for \( K, K' \in \tau \), \( K \cap K' \) is either their apices or their edges.

For a triangulation \( \tau \), we define \( |\tau| := \max_{K \in \tau} \text{diam}K \).

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded Lipschitz domain. By Sobolev’s imbedding theorem, we have the continuous inclusion \( W^{2,p}(\Omega) \subset C^{1}(\Omega) \) for any \( p \in [1, \infty] \). Note that for \( p = 1 \) Morry’s inequality is not applicable and the inclusion \( W^{2,1}(\Omega) \subset C^{0}(\Omega) \) is not so obvious. We still can prove the critical inclusion by a manner similar to the proof of Lemma 4.3.4 in [2] with [17] Theorem 5, p.180.

2 Kobayashi’s formula, the circumradius condition, and Schwarz’s example

Recently, we made progress on the error analysis of linear interpolation on triangular elements. Liu-Kikuchi presented an explicit form of the constant \( C \) in the maximum
angle condition \[ \text{[11]} \]. Being inspired by Liu-Kikuchi’s result, Kobayashi, one of the authors, obtained the following remarkable result with the assistance of numerical validated computation \[ \text{[8], [9]} \].

**Theorem 2 (Kobayashi’s formula)** Let \( A, B, C \) be the lengths of the three edges of \( K \) and \( S \) be the area of \( K \). Define the constant \( C(K) \) by

\[
C(K) := \sqrt{\frac{A^2 B^2 C^2}{16S^2} - \frac{A^2 + B^2 + C^2}{30} - \frac{S^2}{5} \left( \frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2} \right)},
\]

then the following estimate holds:

\[
|v - I_Kv|_{1,2,K} \leq C(K)|v|_{2,2,K}, \quad \forall v \in H^2(K).
\]

Let \( R_K \) be the radius of the circumcircle of \( K \). From the formula \( R_K = \frac{ABC}{4S} \), we realize \( C(K) < R_K \) and obtain a corollary of Kobayashi’s formula.

**Corollary 3** For any triangle \( K \subset \mathbb{R}^2 \), the following estimate holds:

\[
|v - I_Kv|_{1,2,K} \leq R_K|v|_{2,2,K}, \quad \forall v \in H^2(K).
\]

Consider the isosceles triangle \( K \) depicted in Figure\[ \text{[1]} \] If \( 0 < h < 1 \) and \( \alpha > 1 \), then \( h^\alpha < h \) and the circumradius of \( K \) is \( h^\alpha / 2 + h^{2-\alpha} / 8 \). Hence, Kobayashi’s formula and its corollary yield that, if \( 1 < \alpha < 2 \) and \( |v|_{2,2,K} \) is bounded, the error \( |v - I_Kv|_{1,2,K} \) converges to 0, whereas the maximum angle approaches \( \pi \) as \( h \to 0 \). Therefore, the circumradius of a triangle is a more important indicator than its minimum and maximum angles. Without the assistance of numerical validated computation, the authors then proved for arbitrary \( p \in [1, \infty] \) the following theorem.

**Theorem 4 (The circumradius condition \[ \text{[10]} \])** For an arbitrary triangle \( K \) with \( R_K \leq 1 \), there exists a constant \( C_p \) independent of \( K \) such that the following estimate holds:

\[
\|v - I_Kv\|_{1,p,K} \leq C_p R_K|v|_{p,p,K}, \quad \forall v \in W^{2,p}(K), \quad 1 \leq p \leq \infty.
\]

Combining Céa’s lemma and Corollary \[ \text{[3]} \] or Theorem \[ \text{[4]} \] we immediately obtain the following estimation.

**Theorem 5** Let \( u \) be the exact solution of \[ \text{[1]} \] and \( u_h \) be the FEM solution of \[ \text{[2]} \]. Suppose that \( u \in H^2(\Omega) \). Then we have, for \( R_T \leq 1 \),

\[
\|u - u_h\|_{1,2,\Omega} \leq CR_T|u|_{2,2,\Omega}, \quad R_T := \max_{K \in \tau} R_K,
\]

where the positive constant \( C \) depends only on \( C_2 \) and \( \Omega \).
Note that it follows from Corollary 1 that $C_2 = 1$. However, proving this without using validated numerical computation is not easy.

The isosceles triangle in Figure 1 reminded the authors of Schwarz’s example. As is well understood, the length of a curve is defined as the limit of the length of the inscribed polygonal edges. Hence, one might think that the area of a surface could be defined in a similar manner. Actually, mathematicians in the 19th century believed that the area of surface is the limit of the areas of inscribed polygonal surfaces.

In the 1880s, Schwarz and Peano independently showed, however, that this definition does not work [3], [14], [15], [19]. Let $\Omega$ be a rectangle of height $H$ and width $2\pi r$. Let $m, n$ be positive integers. Suppose that this rectangle is divided into $m$ equal strips, each of height $H/m$. Each strip is then divided into isosceles triangles whose base length is $2\pi r/n$, as depicted in Figure 2. Then, the piecewise linear map $\varphi_\tau : \Omega \to \mathbb{R}^3$ is defined by “rolling up this rectangle” so that all vertexes are on the cylinder of height $H$ and radius $r$. Then, the cylinder is approximated by the inscribed polygonal surface which consists of $2mn$ congruent isosceles triangles. Because the height of each triangle is $\sqrt{\left(\frac{H}{m}\right)^2 + r^2(1 - \cos (\frac{\pi}{n}))^2}$ and the base length is $2r \sin (\frac{\pi}{n})$, the area $A_E$ of the inscribed polygonal surface is

$$A_E = 2mn r \sin \frac{\pi}{n} \sqrt{\left(\frac{H}{m}\right)^2 + r^2 \left(1 - \cos \frac{\pi}{n}\right)^2}$$

$$= 2\pi r \frac{\sin \frac{\pi}{n}}{\pi} \sqrt{H^2 + \frac{\pi^4 r^4}{4} \left(\frac{m}{n^2}\right)^2 \left(\frac{\sin \frac{\pi}{n}}{\frac{2\pi}{n}}\right)^4}.$$

If $m, n \to \infty$, we observe

$$\lim_{m, n \to \infty} A_E = 2\pi r \sqrt{H^2 + \frac{\pi^4 r^4}{4} \lim_{m, n \to \infty} \frac{m}{n^2}},$$

in particular,

$$\lim_{m, n \to \infty} A_E = 2\pi r H \quad \text{if and only if} \quad \lim_{m, n \to \infty} \frac{m}{n^2} = 0.$$

1 The subscript ‘E’ of $A_E$ stands for ‘Elementary’.

Fig. 2 Schwarz’s example.
As we are now aware that the circumradius is an important factor, we compute the circumradius $R$ of the isosceles triangle in Schwarz's example. By a straightforward computation, we find that

$$R = \frac{H^2 + \pi^2/4 \left( \frac{m}{n} \right)^2 \left( \frac{\sin \pi/2}{n} \right)^2}{2 \sqrt{H^2 + \pi^2/4 \left( \frac{m}{n} \right)^2 \left( \frac{\sin \pi/2}{n} \right)^4}}$$

and immediately realize that

$$\lim_{m, n \to \infty} A_E \leq \frac{2\pi r H}{\pi} \iff \lim_{m, n \to \infty} \frac{m}{n} = 0 \iff \lim_{m, n \to \infty} R = 0. \quad (5)$$

This fact strongly suggests that the circumradius of triangles in a triangulation is essential for error estimations of linear interpolations.

With (5) in mind, we perform a numerical experiment similar to the one in [6]. Let $\Omega := (-1,1) \times (-1,1)$, $f(x,y) := a^2/(a^2 - x^2)^{3/2}$, and $g(x,y) := (a^2 - x^2)^{1/2}$ with $a := 1.1$. Then we consider the following Poisson equation: Find $u \in H^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial \Omega. \quad (6)$$

The exact solution of (6) is $u(x,y) = g(x,y)$ and its graph is a part of the cylinder. For a given positive integer $N$ and $\alpha > 1$, we consider the isosceles triangle with base length $h := 2/N$ and height $2/[2/h^\alpha] \approx h^\alpha$, as depicted in Figure 1. For comparison, we also consider the isosceles triangle with base length $h$ and height $h/2$ for $\alpha = 1$. We triangulate $\Omega$ with this triangle, as shown in Figure 3. The behavior of the error is given in Figure 3. The horizontal axis represents the mesh size measured by the maximum diameter of triangles in the meshes and the vertical axis represents the error associated with FEM solutions in $H^1$-norm. The graph clearly shows that the convergence rates worsen as $\alpha$ approaches 2.0. For $\alpha = 2.1$, the FEM solutions even diverge. We replot the same data in Figure 4, in which the horizontal axis represents the maximum of the circumradius of triangles in the meshes. Figure 4 shows convergence rates are almost the same in all cases if we measure these with the circumradius.

From the results of the numerical experiments, we draw the following conclusions: Suppose that we consider the Poisson equation (6).

- In our example, although the triangulation does not satisfy the maximum angle condition, the FEM solutions converge to the exact solution and the error behaves exactly as the estimation (4) predicts. If the triangulation does not satisfy the circumradius condition, the FEM solutions diverge even if meshes become finer with respect to the maximum diameter of the triangles. From this observation, we infer that, for convergence of the FEM solutions, the circumradius solution is more essential than the maximum angle condition and is the best possible as a geometric condition for triangulation.

- The numerical experiments in [6] show that, in certain combinations of an exact solution and triangulation, FEM solutions can converge to an exact solution, although triangulation does not satisfy the maximum angle condition. We notice
that their triangulations do not satisfy the circumradius condition either. Hence, the circumradius condition is not necessary for convergence of the finite element method.

These conclusions answer, partially but not completely, the question which Hannukainen-Korotov-Křížek posed. We infer from the numerical experiments that matching between exact solutions and geometry of triangulation seems important. Further and deeper understanding of how FEM solutions converge to an exact solution is strongly desired.

3 The circumradius condition and the definition of surface area

At the present time, the most general definition of surface area is that of Lebesgue. Let $\Omega := (a, b) \times (c, d) \subset \mathbb{R}^2$ be a rectangle and $\tau_n$ be a sequence of triangulation of $\Omega$ such that $\lim_{n \to \infty} |\tau_n| = 0$. Let $f \in C^0(\overline{\Omega})$ be a given continuous function. Let $f_n \in S_{\tau_n}$ be such that $\{f_n\}_{n=1}^\infty$ converges uniformly to $f$ on $\overline{\Omega}$. Note that the graph of $z = f_n(x, y)$ is a set of triangles and its area is defined as a sum of these triangular
Let $\Phi_f$ be the set of all such sequences $\{(f_n, \tau_n)\}_{n=1}^{\infty}$. Then the area $A_L(f) = A_L(f; \Omega)$ of the graph $z = f(x, y)$ is defined by

$$A_L(f) = A_L(f; \Omega) := \inf_{\{(f_n, \tau_n)\} \in \Phi_f} \liminf_{n \to \infty} A_E(f_n).$$

This $A_L(f)$ is called the surface area of $z = f(x, y)$ in the Lebesgue sense. For a fixed $f$, $A_L(f; \Omega)$ is additive and continuous with respect to the rectangular domain $\Omega$. Tonelli then presented the following theorem.

For a continuous function $f \in C^1(\Omega)$, we define $W_1(x)$, $W_2(y)$ by

$$W_1(x) := \sup_{\tau(y)} \sum_{i} |f(x, y_{i-1}) - f(x, y_i)|, \quad x \in (a, b),$$

$$W_2(y) := \sup_{\tau(x)} \sum_{j} |f(x_{j-1}, y) - f(x_j, y)|, \quad y \in (c, d),$$

where $\tau(y)$, $\tau(x)$ are subdivisions $c = y_0 < y_1 < \cdots < y_N = d$ and $a = x_0 < x_1 < \cdots < x_M = b$, respectively and ‘sup’ are taken for all such subdivisions. Then, a function $f$ has bounded variation in the Tonelli sense if

$$\int_a^b W_1(x)dx + \int_c^d W_2(y)dy < \infty.$$ 

Also, a function $f$ is called absolutely continuous in the Tonelli sense if, for almost all $y \in (c, d)$ and $x \in (a, b)$, the functions $g(x) := f(x, y)$ and $h(y) := f(x, y)$ are absolutely continuous on $(a, b)$ and $(c, d)$, respectively.

**Theorem 6 (Tonelli)** For a continuous function $f \in C^1(\Omega)$ defined on a rectangular domain $\Omega$, its graph $z = f(x, y)$ has finite area $A_L(f) < \infty$ if and only if $f$ has bounded variation in the Tonelli sense. If this is the case, we have

$$A_L(f) \geq \int_\Omega \sqrt{1 + f_x^2 + f_y^2} \, dx.$$

In the above inequality, the equality holds if and only if $f$ is absolutely continuous in the Tonelli sense.

For a proof of this theorem, see [16] Chapter V, pp.163–185.

We now give a proof of Theorem 1 in the following. Suppose that we have a sequence $\{\tau_n\}_{n=1}^{\infty}$ of triangulation of $\Omega$ which satisfies the circumradius condition (7). Suppose also that $f \in W^{2,1}(\Omega)$. At first, we notice $f$ is of bounded variation and absolutely continuous in the Tonelli sense. Let $\omega := \{x\} \times (c, d)$ for $x \in (a, b)$. We consider the trace operator $\gamma : W^{2,1}(\Omega) \to W^{1,1}(\omega)$ defined by $(\gamma f)(y) := f(x, y)$.

Then, $\gamma$ is a bounded linear operator and it is easy to see that

$$\sum_{\tau(y)} |(\gamma f)(y_{i-1}) - (\gamma f)(y_i)| \leq \int_c^d |(\gamma f)'(y)| \, dy, \quad W_1(x) \leq \int_c^d |f_y(x, y)| \, dy.$$
Similarly, we obtain $W_2(y) \leq \int_\Omega |f(x,y)| \, dx$ and Fubini’s theorem implies that $f$ has bounded variation in the Tonelli sense. We show that $f$ is absolutely continuous in the Tonelli sense in exactly the same manner. Therefore, Tonelli’s theorem implies

$$A_L(f) = \int_\Omega \sqrt{1 + f_x^2 + f_y^2} \, dx < \infty.$$  

For the piecewise linear interpolation $I_n f$, we have

$$A_E(I_n f) = A_L(I_n f) = \int_\Omega \sqrt{1 + (I_n f)_x^2 + (I_n f)_y^2} \, dx.$$  

Hence, $|A_L(f) - A_E(I_n f)|$ is estimated as

$$|A_L(f) - A_E(I_n f)| \leq \int_\Omega \left| \sqrt{1 + f_x^2 + f_y^2} - \sqrt{1 + (I_n f)_x^2 + (I_n f)_y^2} \right| \, dx$$

$$\leq \int_\Omega \frac{|f_x + (I_n f)_x| (f_x - (I_n f)_x) + (f_y + (I_n f)_y) (f_y - (I_n f)_y)|}{\sqrt{1 + f_x^2 + f_y^2 + 1 + (I_n f)_x^2 + (I_n f)_y^2}} \, dx$$

$$\leq |f - I_n f|_1 \cdot \| I_n f \|_{2, \Omega} \rightarrow 0 \quad \text{as} \quad R_n \rightarrow 0,$$

because

$$\frac{|f_x + (I_n f)_x|}{\sqrt{1 + f_x^2 + f_y^2 + 1 + (I_n f)_x^2 + (I_n f)_y^2}} \leq 1,$$

$$\frac{|f_y + (I_n f)_y|}{\sqrt{1 + f_x^2 + f_y^2 + 1 + (I_n f)_x^2 + (I_n f)_y^2}} \leq 1.$$

Thus, Theorem 7 is proved.

Note that, from Schwarz’s example, Theorem 7 is the best possible with respect to the geometric condition for triangulation. At this point, one might be tempted to define the surface area using the circumradius condition in the following way:

**Definition 7** Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. Suppose that a sequence $\{\tau_n\}$ of triangulation of $\Omega$ satisfies the circumradius condition. Then, for a continuous function $f \in C^0(\Omega)$, the area $A_{CR}(f)$ of the surface $z = f(x,y)$ is defined by

$$A_{CR}(f) := \lim_{n \to \infty} A_E(I_n f).$$

**Theorem 7** claims that, for $f \in W^{2,1}(\Omega)$, $A_{CR}(f)$ is well-defined and $A_{CR}(f) = A_L(f)$. It is not clear whether or not $A_{CR}(f)$ is well-defined and satisfies $A_{CR}(f) = A_L(f)$ in more general functional spaces. Therefore, we present the following problem.

**Problem 8** Let $\Omega \subset \mathbb{R}^2$ be a rectangular domain and $f \in C^0(\Omega)$. 
(1) Suppose that \( \{\tau_n\}^\infty_{n=1} \) and \( \{\mu_n\}^\infty_{n=1} \) are two triangulation sequences of \( \Omega \) which satisfy the circumradius condition. Then, prove or disprove whether

\[
\lim_{n \to \infty} A_E(I_{\tau_n}f) = \lim_{n \to \infty} A_E(I_{\mu_n}f).
\]

(2) Suppose that \( A_{CR}(f) \) is well-defined in the sense of (1). Then, prove or disprove whether \( A_{CR}(f) = A_L(f) \) for any \( f \in C^0(\Omega) \) including cases \( A_{CR}(f) = A_L(f) = \infty \).

4 Concluding remarks — History repeats itself

We have shown that the circumradius condition is more essential than the maximum angle condition for convergence of FEM solutions. Also, we have pointed out a close relationship between the circumradius condition and the definition of surface area. In concluding, we draw readers’ attention to the similarity of two histories. After Schwarz and Peano found their counter example, mathematicians naturally tried to find a proper definition of surface area. The authors are unfamiliar with the history behind that quest. Instead, we suggest that readers look at [15, Capter I] from which we mention the following remarks.

Let \( S \subset \mathbb{R}^3 \) be a general parametric surface. If there exists a Lipschitz map \( \varphi : \Omega \to \mathbb{R}^3 \) defined on a domain \( \Omega \subset \mathbb{R}^2 \) such that \( S = \varphi(\Omega) \), \( S \) is called rectifiable. Let \( \Omega \) be a rectangle and \( \varphi : \Omega \to \mathbb{R}^3 \) be a rectifiable surface. Suppose that we have a sequence \( \{\tau_n\}^\infty_{n=1} \) of triangulation of \( \Omega \) such that \( |\tau_n| \to 0 \) as \( n \to \infty \). Then, the rectifiable surface \( \varphi \) has linear interpolations \( I_{\tau_n}\varphi \). Rademacher showed [12, 13] that if \( \{\tau_n\}^\infty_{n=1} \) satisfies the minimum angle condition we have \( \lim_{n \to \infty} A_E(I_{\tau_n}\varphi) = A_L(\varphi) \). Then, Young showed [18] that if \( \{\tau_n\}^\infty_{n=1} \) satisfies the maximum angle condition we have \( \lim_{n \to \infty} A_E(I_{\tau_n}\varphi) = A_L(\varphi) \). See also the comment by Fréchet [5] on Young’s result.

This means that the minimum and maximum angle conditions were already found about 50 years before they were rediscovered by FEM exponents. This is an interesting example of the proverb History repeats itself.

Acknowledgements The first author is supported by Inamori Foundation and JSPS Grant-in-Aid for Young Scientists (B) 22740059. The second author is partially supported by JSPS Grand-in-Aid for Scientific Research (C) 22440139 and Grand-in-Aid for Scientific Research (B) 23340023.

References

1. Babuška, I., Aziz, A.K.: On the angle condition in the finite element method, SIAM J. Numer. Anal. 13, 214–226 (1976)
2. Brenner, S.C., Scott, L.R.: The Mathematical Theory of Finite Element Methods, 3rd edition, Springer, 2008.
3. Cesari, K.: Surface Area, Princeton Univ. Press, 1956.
4. Ciarlet, P.G.: The Finite Element Methods for Elliptic Problems, North Holland, 1978, reprint by SIAM 2008.
5. Fréchet, M.: Note on the area of a surface, in Records of proceedings at meetings, Proc. London Math. Soc., 24, page xlviii (1926) doi:10.1112/plms/s2-24.1.1-s
6. Hannukainen, A., Korotov, S., Krýzek, M.: The maximum angle condition is not necessary for convergence of the finite element method, Numer. Math., 120, 79–88 (2012)
7. Jamet, P.: Estimations d’erreur pour des éléments finis droits presque degeneres, R.A.I.R.O. Anal. Numer., 10, 43–61 (1976)
8. Kobayashi, K.: On the interpolation constants over triangular elements (in Japanese), RIMS Kokyuroku, 1733, 58-77 (2011)
9. Kobayashi, K.: Remarkable upper bounds for the interpolation error constants on the triangles, in preparation
10. Kobayashi, K., Tsuchiya, T.: A Babuška-Aziz type proof of the circumradius condition, submitted, arXiv:1306.2997
11. Liu, X., Kiruchi, F.: Analysis and estimation of error constants for $P_0$ and $P_1$ interpolations over triangular finite elements, J. Math. Sci. Univ. Tokyo, 17, 27–78 (2010)
12. Rademacher, H.: Über partielle und totale differenzierbarkeit von funktionen mehrerer variablen und über die transformation der doppellintegrale, Math. Ann., 79, 340–359 (1919) doi 10.1007/BF01498415
13. Rademacher, H.: Über partielle und totale differenzierbarkeit von funktionen mehrerer variablen II, Math. Ann., 81, 52–63 (1920) doi 10.1007/BF01563620
14. Radó, T.: Length and Area, American Mathematical Society, 1948.
15. Radó, T.: On the Problem of Plateau, Chelsea, 1951.
16. Saks, S.: Theory of the Integral, Warszawa-Lwów, 1937, reprint by Dover 2005.
17. Stein, E. L.: Singular integrals and differentiability properties of functions, Princeton University Press (1970).
18. Young, W.H.: On the triangulation method of defining the area of a surface, Proc. London Math. Soc., 19, 117–152 (1921) doi:10.1112/plms/s2-19.1.117
19. Zames, F.: Surface area and the cylinder area paradox, The College Mathematics Journal, 8, 207–211 (1977) This article is downloadable from http://mathdl.maa.org/mathDL/22
20. Zlámal, M.: On the finite element method, Numer. Math., 12, 394–409 (1968)