Students’ abstraction in solving system of linear equations with two variables

F Pangaribuan
Nommensen HKBP University, Jl. Sutomo No. 4A, Perintis, Medan, 20235, Indonesia
Email: firmanpangribfkuphn@gmail.com

Abstract. In teaching and learning process, the teacher usually guides students to follow the demonstrated procedure; however, the result of learning process sometimes shows that students cannot construct new knowledge. In constructing new knowledge, abstraction is important for students. The objective of this research is to describe students’ abstraction in solving system of linear equations with two variables problem. This research uses the qualitative method. The subjects are two female students from seventh grade of junior high school. These students have not yet studied the system of linear equations with two variables formally. The data were gained as a result of solving the contextual problem in student worksheet and from an interview with the students. The results of this study were that students used their prior knowledge of the linear equations with one variable formally. Then students could solve the system of linear equations with two variables in formal symbolic by abstraction. The students needed scaffolding while abstracting. In teaching and learning process, teachers need to facilitate students by abstraction.

1. Introduction
The system of linear equations with two variables (SLETV) is learned in second-year of junior high school. The basic competence is that students explain the solution of SLETV in a contextual problem. Students’ experience of resolving contextual issues becomes the path to achieve basic competencies in understanding the concept of solving SLETV in formal symbolic.

In teaching learning process, usually, the teacher guides students to follow the demonstrated procedure. However, the result sometimes shows that students cannot construct new knowledge for themselves. This is not in accordance with the curriculum, in which teaching and learning processes are expected to increase students’ thinking skills so they can demonstrate logical, critical, creative, and innovative thinking skill.

Knowledge is not given as a product that ready to consume by the students. Students must construct it for themselves [1]. By constructing, students are expected to think critically or creatively. Piaget noted that all new knowledge requires abstraction [2]; without abstraction, there will be no new knowledge.

To improve long-term conceptual learning, the framework formulated here suggests that the whole curriculum must be framed with an awareness of the abstraction process to produce thinkable concepts at every stage [3]. Understanding the concept must go through the process of abstraction step by step.

Constructing knowledge is an abstraction activity. Abstraction is the process in which the learner constructs new knowledge, by what we call vertical mathematization. It is the process of constructing new knowledge with vertical mathematization, through recognizing and gathered previously possessed
knowledge, and the knowledge identified and gathered must be relevant to the new knowledge to be constructed [4]. Vertical mathematization in realistic mathematics education is a process of increasing the complexity of a concept in the mathematical system itself. Horizontal mathematization is the process of transforming understanding from real situations to mathematical situations. In this research, the abstractions observed and those that appear in students' minds are three components: 1) recognizing, 2) building-with, and 3) constructing.

Abstraction process of solving the SLETV will be focused on observation: 1) recognizable element of knowledge that is the initial knowledge possessed by student in solving SLETV, 2) the aspect of knowledge that is strung together is a combination of recognized element, for example solving routine problem (not solving problem which new for the students), and last is 3) construct the new knowledge for the student which is the answer of the problem. In problem solving process, a new knowledge or concept formed is expressed in objective learning.

Some researchers stated that abstraction plays an important role in constructing mathematical knowledge. Dooley shows that student able to construct new knowledge about decimal fraction expansion and non-terminating decimals [5]. Ozmantar and Monaghan also demonstrate that student constructed a method to sketch the graphs of \(|f(|x|)|\) while \(f(x)\) is a linear equation [6]. They have demonstrated how one can abstract knowledge empirically. Unfortunately, they have not shown how students abstract in solving SLETV. This research aims to describe students’ construction of solution in SLETV problem with abstraction stage. Theoretically, this research adds more input to the theory of abstraction that can be used in building knowledge of SLETV material. Practically, could be the input on the learning management as alternative learning with abstraction theory on the SLETV.

2. Method
This research uses the qualitative method. The subjects of this research are two female students of seven grade of junior high school, and they have not studied formal SLETV material yet, but already understood the linear equation with one variable. Data collection was done by giving three SLETV problems in each worksheet that contains pictures. The three SLETV had been validated for students and shown that problems can be understood and can be solved by them. The results of the written answers were then being interviewed in depth, then the written and oral answers were validated by seeing the consistency of both the answers. The data collection was recorded by using the audio-visual recorder, then transcribed and analyzed. The conclusion is made by giving meaning and explanation to the valid data, which are in recognizing, building with, and constructing phases.

3. Results and discussion
The following description describes of how the abstraction takes places on solving three SLETV problems. In worksheet 1, the subjects are given a picture containing two packages that must be sold at a store. Package A includes 1 glass and 2 hats for Rp260,000 and packet B comprises two glasses and 2 hats for Rp280,000. Subjects are asked to determine the price for each item.

![Student’s solution before using variable.](image-url)
The subjects recognize the solution of the SLETVE in their own way after observing the given problem in the picture. Their first step is categorized as the recognition of the problem because the subjects complete it based on their experience and the subjects’ written as Figure 1. This step has not requested the use of variables yet.

**Figure 2.** Students’ work of linear equations with variable.  
**Figure 3.** Students solution in formal symbolic.

Subsequently, subjects are asked to write a mathematical model by giving the price of one glass is $x$ and the price of one hat is $y$. Subjects recognized the mathematical model of the given problem by changing the image or visual into a formal symbol form with the use of $x$ and $y$ variables. Initially, the subjects did not add the corresponding variable in the mathematical model but exposing the process in the figure. For example, package A becomes $x + y + y = x + 2y = 260,000$ and does not directly write $x + 2y = 260,000$. Similarly, with package B the subject writes $x + x + y + y = 2x + 2y = 280,000$. Likewise, the writing of equations is not written in one line, but into three lines and students write the linear equation with variables in Figure 2. In this solution, the subject had built-with the combination of using the model and the solution by himself. In this worksheet 1, the subject had not constructed a new knowledge, because there was no obstacle to the student in comprehending it. On the answers, the subject writes $x = (2x + 2y) - (x + 2y) = 280,000 - 260,000 = 20,000$, for package A, and the subject still writes $2y = (x + 2y) - x = 260,000 - 20,000$. This subject’s text indicates that algebra reduction is already recognized and students solution as formal symbolic in Figure 3.

The more complex problems are prepared in worksheet 2. The subjects were also given a picture containing 2 packages to be sold at a store. Package A contains 1 glass and 2 hats, its total price is Rp220,000 and package B contains 2 glasses and 3 hats for Rp360,000. Subjects are asked to determine the price for each item.

Problem worksheet 2 cannot be solved in the same way as in worksheet 1, because the structures are different. The subject does so by trial and error with the price of one hat and the price of one glass, then testing the price given by substituting the figure on the problem. The subject gets the price of one glass is Rp60,000, and a one hat price is Rp80,000. Another subject answered by forming a “new package” that is the difference between package B and A; the price of one hat and one glass are Rp140,000. This new package combines with package A to get the price of a hat. The price of package A (2 hats and 1 glass) is Rp220,000 along with “new package’s” price contained one hat, and one glass is Rp140,000. So the subjects get the answer of the price of one hat as the difference between the two packages. In changing the SLETVE into the form of the picture and its description into the form of a formal symbol, the subject can recognize the mathematical model by writing $x + y + y = x + 2y = 60,000 + 160,000 = 220,000$ of package A and $2x + 3y = 120,000 + 240,000 = 360,000$.

To find out the subject of how to “building-with” in solving problem 2, it is obtained through the subjects answer in section 2c “Write down your solution by using $x$ and $y$.” Building within this section is to combine the solution obtained by the subject through problem 2a and 2b that is the combination of the solution by the subject itself and the mathematical model of the given SLETVE. Solution 1b and 2b are about writing the mathematical model in a different way. The subject does not
write the price per unit while solving problem 1b since on problem 2b, the subject writes the prices x, 2y, 2x, and 3y. These prices are obtained through the answer on problem 2a, and the subject has written SLETV in formal symbol.

The subjects exposed his work by telling "Now we see package B is 360,000 and package A is 220,000. So, package B minus package A is 140,000. One hundred and forty is still the price of 1 glass and 1 hat. It is still the price of 1 glass and 1 hat. We want to find the price of each hat and glass. While in package A there is 1 glass and 2 hats that cost 220,000. One hundred forty thousand, sorry. So packet A is worth 220,000. It is subtracted by 140,000 is equal to 80. The price of 1 hat and 1 glass is 140,000 and the price of 2 hats and 1 glass is 220,000. Two hats and one glass less than one hat and one glass with one hat. One hat is 80,000". The subject then determines the price of a glass with the following exposure. "One hat is 80,000. So, if the two hats are 160,000. Two glasses and 1 glass are 220,000. So 1 glass is 220,000 minus 160,000 equals to 60,000 ". Exposure to this subject is already the construct of a new knowledge for him even though he did not finish it in a formal symbol. Particularly his ability constructs a new statement from two packets A and B, by taking the difference. The completion of this subject has also used elimination, i.e. eliminating the price of a glass to obtain the price of a hat. This subject will not determine the solution directly from the SLETV form of formal symbols with the reason that he has gotten it. In that solution, the subjects perceive variables on SLETV have meaning as contents of the original problem. His understanding of SLETV does not show meaningless variables.

The problem in worksheet 3 is more complex than in worksheet 2. Package A contains one glass and two hat price is Rp300,000 and Package B contains four glasses and one hat price is Rp360,000. Problem 2 can be solved by two stages, first by subtracting one package from another, then eliminating one variable, such as package A is subtracted with package B. In worksheet 3, it could not be solved similar like in worksheet 2.

The subject recognizes this SLETV problem but cannot recognize the solution yet. The subject recognizes the purpose of the problem in a formal symbol. A sample of the subject written "A package: \( x + y + y = x + 2y = 300,000 \) and \( B \) package: \( x + x + x + y = 3x + y = 525,000 \). The subject identified elements to be “built-with” in this SLETV solution process, even though the subject has not found the solution. For example, a combination of problems in the form of a picture and their descriptions and combined with a formal symbol. Also, it found elements are built-with between integer operations and meaning in the picture.

The subject experienced an obstacle while solving problem 3. Their inability to show the answer shows this obstacle. Subject who tries to determine the price of a glass and the price of a hat did not find the answer. It shows that this problem is more complex than the previous problem. Many subjects try to subtract package A from package B or vice versa but failed in interpretation.

To overcome the obstacle of the subject found in solving the problem, the interviewer scaffolded subject by giving a question as follows. “Can it be solved if there are some similar packages?”. The subject could describe his solution by making 3 similar package A, solved it similar to the answer written in answer sheet 2, and found the price of glass is 150,000 that is \( x = 150 \) and the price of a hat is 75,000, that is \( y = 75 \).

The subject can solve the problem after the researcher made scaffolding. The interviewer did not provide direct help, but with that question, the subject became triggered to construct his knowledge by completing a new knowledge of SLETV.

The problem given to the subjects is a problem they can image or a problem in contextual form, which expected they attempt to solve the problem packet in SLETV. It is possible if the problem is not related to real life problem, or in the form of a formal symbol, the subject will not be motivated to solve it, because subjects do not see the problem for what benefits. Through a given contextual problem, subjects have informal knowledge directed toward formal knowledge. This corresponds to one of the contents of the theory of realistic mathematics education that the child's learning begins with a real situation.
On the solution of the subject using the x and y symbols, x is the price of a single glass, and y is the price of a hat. The variable x and y is a variable meaning, i.e. a value, and it is not just a meaningless letter. Although the subject often calls a glass while pointing or writing the letter x and one hat pointing the letter y. Subject’ understanding of these letters accords with the first stage of understanding of variables according to Kucheman [5], that variables are a letter that will be determined by its value.

New knowledge is constructed by the subject while solving SLETV in worksheet 2. Usually, the SLETV solution is generally doubled package A so that it gets 2 glasses and 4 hats, then calculates the difference of package A duplicated with package B to obtain the price of one hat. The activity of the subject as demanded by the Curriculum 2013 is student-centered learning so that students find themselves new concepts. This finding is in line with the opinion of Glasersfeld [2] that the knowledge given is not in the finished form ready-made, but the student must construct that knowledge for himself. The activity of this subject is an abstraction of knowledge possessed formally and informally, so that constructed a new knowledge for student self. This is in accordance with Piaget [2] that all knowledge is based on abstraction.

In solving the problem SLETV number 3, subjects’ experience obstacles, the procedures they use on problem number 1 or number 2 cannot be used to solve the problem in number 3. Ozmantar and Monaghan [7] said that student is not easy to construct new knowledge; the teacher needs to do scaffolding. In order for subjects to construct this new knowledge, the researcher provides minimal assistance, by asking "Can it be solved if there are some similar A package?" This question is a trigger for the subject to expose his potential to solve SLETV problems. In accordance with Vygotsky’s theory that one has actual abilities and potential abilities and the area between these two abilities is expressed by the Zone of Proximal Development [8]. In order to the subject to arrive at this potential skill, a more mature person such as a peer or teacher needs to provide scaffolding.

4. Conclusion
The SLETV problem is presented in the form of a story or contextual problem with the picture, so students can recognize and even may feel challenged to solve it. Through a given contextual problem, subjects have informal knowledge directed toward formal knowledge. Students recognize the knowledge they have through their own understanding and write symbols informally. Students use their prior knowledge in a contextual and linear equation of one variable formally then directed by formal symbols, so that students can find the formal symbol (mathematical model) of the SLETV problem. Students need support in constructing new knowledge through solving SLETV problems. For further research, observations of students performing abstractions need to be done on more complex SLETV topics. In the learning process of SLETV, student-centered method with minimal guidance needs to be implemented so that learning is more meaningful.

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