Some explicit travelling-wave solutions of a perturbed sine-Gordon equation

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Abstract

We present in closed form some special travelling-wave solutions (on the real line or on the circle) of a perturbed sine-Gordon equation. The perturbation of the equation consists of a constant forcing term $\gamma$ and a linear dissipative term, and the equation is used to describe the Josephson effect in the theory of superconductors and other remarkable physical phenomena. We determine all travelling-wave solutions with unit velocity (in dimensionless units). For $|\gamma| \leq 1$ we find families of solutions that are all (except the obvious constant one) manifestly unstable, whereas for $|\gamma| > 1$ we find families of stable solutions describing each an array of evenly spaced kinks.

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1 Introduction and preliminaries

The scope of this communication is the determination in closed form of some special solutions of the class of partial differential equations

\[ \varphi_{tt} - \varphi_{xx} + \sin \varphi + \alpha \varphi_t + \gamma = 0 \quad x \in \mathbb{R}, \quad (1) \]

parametrized by constants \( \alpha > 0, \gamma \in \mathbb{R} \), more precisely the determination of the travelling-wave solutions \( \varphi(x, t) = g(x - vt) \) with velocity \( v = \pm 1 \).

This equation (here written in dimensionless units) has been used to describe with a good approximation a number of interesting physical phenomena, notably Josephson effect in the theory of superconductors [6], which is at the base [1] of a large number of advanced developments both in fundamental research (e.g. macroscopic effects of quantum physics, quantum computation) and in applications to electronic devices (see e.g. Chapters 3-6 in [2]), or more recently also the propagation of localized magnetohydrodynamic modes in plasma physics [9]. The last two terms are respectively a dissipative and a forcing one; the sine-Gordon equation (sGe) is obtained by setting them equal to zero.

The sGe describes also the dynamics of the continuum limit of a sequence of neighbouring heavy pendula constrained to rotate around the same horizontal \( x \)-axis and coupled to each other through a torque spring [8] (see fig. 1); \( \varphi(x, t) \) is the deviation angle from the lower vertical position at time \( t \) of the pendulum having position \( x \). One can model also the dissipative term \( -\alpha \varphi_t \) of (1) by immersing the pendula in a linearly viscous fluid, and the forcing term \( \gamma \) by assuming that a uniform, constant torque distribution is applied to the pendula. This mechanical analog allows a qualitative comprehension of the main features of the solutions, e.g. of their instabilities. The constant solutions of (1) exist only for \( |\gamma| \leq 1 \) and are, mod \( 2\pi \),

\[ \varphi^s(x, t) \equiv -\sin^{-1} \gamma, \quad \varphi^u(x, t) \equiv \sin^{-1} \gamma + \pi. \quad (2) \]

If \( |\gamma| < 1 \) the former is stable, the latter unstable, as they yield respectively local minima and maxima of the energy density

\[ h := \frac{\varphi^2}{2} + \frac{\varphi^2}{2} + \gamma \varphi - \cos \varphi. \quad \text{(3)} \]

In the mechanical analog they respectively correspond to configurations with all pendula hanging down or standing up. If \( \gamma = \pm 1 \) \( \varphi^s = \varphi^u = \mp \pi/2 \) (mod \( 2\pi \)), which is unstable because it is an inflection point for \( h \).

In [5] we have performed a detailed analysis of travelling-wave solutions of (1). We briefly recall the framework adopted there and some of the results. Without loss of generality we can and shall assume \( \gamma \geq 0 \); if originally \( \gamma < 0 \), we just need to replace \( \varphi \rightarrow -\varphi \). Moreover, space or time translations transform any solution into a two-parameter family of solutions, so one can choose any of them as the family representative element; for travelling-wave solutions this reduces to translation of the only independent variable. In agreement with the conventions adopted in [5], we specify our travelling-wave Ansatz as follows:

\[ \xi := \pm x - t, \quad \varphi(x, t) = g(\xi) - \pi. \quad (4) \]
Replacing the Ansatz in (1) one obtains the first order ordinary differential equation

$$\alpha g' = \gamma - \sin g.$$  \hspace{1cm} (5)

We have already recalled the constant solutions. If $g'$ is not identically zero, by integrating $d\xi = \alpha dg / (\gamma - \sin g)$ one finds

$$\xi - \xi_0 = \int_{\xi_0}^{\xi} d\xi = \alpha \int_{g_0}^{g} \frac{ds}{\gamma - \sin s}$$

separately in each interval in which $g'$ keeps its sign. This allows to determine the solution implicitly, namely the inverse $\xi(g)$.

If $\gamma \leq 1$, as $g$ approaches respectively $\sin^{-1}\gamma$ or $\pi - \sin^{-1}\gamma$ (mod. $2\pi$) the denominator of the integrand goes to zero (linearly if $\gamma < 1$, quadratically if $\gamma = 1$) while keeping the same sign, and therefore the integral diverges, implying that the corresponding $\xi$ respectively goes either to $\pm\infty$, or to $\mp\infty$ [5]. In either case the range of $\xi(g)$ is the whole $\mathbb{R}$, implying that by taking the inverse one obtains $g(\xi)$ already in all the domain. If $\gamma > 1$ the denominator of the integrand is positive for all $s \in \mathbb{R}$, so that the solution $g$ is strictly monotonic and linear-periodic, i.e. the
sum of a linear and a periodic function, and
\[ g(\xi + \Xi) = g(\xi) + 2\pi, \quad \Xi := \alpha \int_0^{2\pi} \frac{ds}{\gamma - \sin s}. \]  

(6)

Denoting as \( \tilde{\varphi}^\pm \) the corresponding solutions with \( \xi := \pm x - t \), by (4) this implies
\[ \tilde{\varphi}^\pm(x + \Xi, t) = \tilde{\varphi}^\pm(x, t) \pm 2\pi. \]  

(7)

This behaviour is illustrated in fig. 2 by a picture of the corresponding configuration for the mechanical model of fig. 1.

\( \tilde{\varphi}^\pm \) can be interpreted also as solutions of (1) on a circle of length \( L = m\Xi \), for any \( m \in \mathbb{N} \). The integer \( m \) parameterizes different topological sectors: in the \( m \)-th the pendula chain twists around the circle \( m \) times.

2 Explicit travelling-wave solutions with unit velocity

The purpose of this work is to determine in closed form the travelling-wave solutions (4) just described. We first transform eq. (5), with the help of the identities (18), into
\[ 4\alpha \frac{F'}{1+F^2} = \gamma - \frac{4F(1-F^2)}{(1+F^2)^2} \]
by looking for \( g \) in the form \( g = 4 \tan^{-1} F \) and then into
\[ 2\alpha y' = 2y + \gamma(1+y^2) \]  

(8)

by looking for \( F \) in the form \( F = y + \sqrt{1+y^2} \). Note that diverging of \( |y| \) at some point \( \xi_0 \) does not affect the continuity (and smoothness) of \( g \) at \( \xi_0 \), even if the right limit is \( \infty \) and the left one is \( -\infty \), or viceversa: \( y \to \pm \infty \) respectively implies \( F \to \infty, 0 \) whence \( g \to 0 \mod 2\pi \) in either case, which is compatible with a continuous \( g \).

Below we solve for \( y(\xi) \) explicitly. Putting all redefinitions together, we shall find solutions \( \varphi \) through the formula
\[ \varphi^\pm(x, t) = 4 \tan^{-1} \left[ y(\pm x - t) + \sqrt{1+y^2(\pm x - t)} \right] - \pi. \]  

(9)

Only if \( \gamma \leq 1 \) the solutions \( y_\pm = -\gamma^{-1} \pm \sqrt{\gamma^{-2} - 1} \) of the second degree equation \( y^2 + y2/\gamma + 1 = 0 \) are real and therefore give (real) constant solutions \( y(\xi) \equiv y_\pm \) of (8), whence the already mentioned constant solutions \( \varphi^s, \varphi^u \) of (1). For nonconstant solutions (8) is equivalent to
\[ d\xi = \frac{2\alpha}{\gamma} \frac{dy}{\gamma y^2 + \frac{2}{\gamma} y + 1}. \]  

(10)

separately in each interval where \( y' \) keeps its sign. The discussion of (10) depends now on the value of the discriminant \( \Delta = 4/\gamma^2 - 4 \) of the equation \( y^2 + y2/\gamma + 1 = 0 \).
If $\gamma < 1$, then $\Delta > 0$, $y_{\pm}$ are real and different and (10) can be written as

$$
d\xi = \frac{2\alpha}{\gamma} \frac{dy}{(y - y_+)(y - y_-)} = \frac{\alpha}{\sqrt{1 - \gamma^2}} \left[ \frac{dy}{y - y_+} - \frac{dy}{y - y_-} \right],
$$

which is integrated to give the two families of solutions

$$
y_1(\xi) = \frac{y_+ + y_- e^{A(\xi - \xi_0)}}{1 + e^{A(\xi - \xi_0)}}, \quad y_2(\xi) = \frac{y_+ - y_- e^{A(\xi - \xi_0)}}{1 - e^{A(\xi - \xi_0)}},
$$

where $A := (\sqrt{1 - \gamma^2})\alpha^{-1}$ and $\xi_0$ is an integration constant. One easily checks that $y'_1, y'_2$ (and therefore also $g'_1, g'_2$) are respectively negative-, positive-definite; and that $\lim_{\xi \to \pm \infty} y_i(\xi) = y_{\mp}$ for both $i = 1, 2$. Using formulae (23-24) shown in the Appendix we thus find

$$
\lim_{\xi \to \infty} F[y_i(\xi)] = F(y_-) = \tan \theta,
\quad \lim_{\xi \to -\infty} F[y_i(\xi)] = F(y_+) = \tan \left( \frac{\pi}{4} - \theta \right).
$$

for both $i = 1, 2$, and mod $2\pi$ on one side a strictly decreasing $g_1(\xi)$ with

$$
\lim_{\xi \to -\infty} g_1 = \pi - \sin^{-1} \gamma,
\quad \lim_{\xi \to \infty} g_1 = \sin^{-1} \gamma,
$$

and on the other a strictly increasing $g_2(\xi)$ with

$$
\lim_{\xi \to -\infty} g_2 = \pi - \sin^{-1} \gamma,
\quad \lim_{\xi \to \infty} g_2 = 2\pi + \sin^{-1} \gamma.
$$

As already noted, the singularity of $y_2$ at $\xi = \xi_0$ does not affect the continuity (and smoothness) of $g_2$. Correspondingly, mod $2\pi$

$$
\lim_{x \to \pm \infty} \varphi^\pm_1 = -\sin^{-1} \gamma \equiv \varphi^s,
\quad \lim_{x \to \pm \infty} \varphi^\pm_2 = -\pi + \sin^{-1} \gamma \equiv \varphi^u, \quad \varphi^\pm_1 = \pi + \sin^{-1} \gamma \equiv \varphi^u,
$$

therefore $\varphi^\pm_1, \varphi^\pm_2$ are unstable solutions, as noted in [3].

If $\gamma = 1$, then $\Delta = 0$, $y_{\pm} = -1$ and (10) can be written as

$$
d\xi = 2\alpha \frac{dy}{(y + 1)^2} = -2\alpha \frac{1}{y + 1},
$$

which is integrated to give

$$
y(\xi) = - \left[ 1 + \frac{2\alpha}{\xi - \xi_0} \right].
$$

(14)

This implies, with the help of (25),

$$
\lim_{\xi \to \pm \infty} y(\xi) = -1, \quad \Rightarrow \quad \lim_{\xi \to \pm \infty} F[y(\xi)] = \sqrt{2} - 1 = \tan \frac{\pi}{8},
$$

5
whereas again the singularity of \( y \) at \( \xi = \xi_0 \) does not affect the continuity of \( g \). As \( y' \), and therefore also \( F' \), \( g' \), are positive-definite, one finds mod \( 2\pi \)
\[
\lim_{\xi \to -\infty} g = \frac{\pi}{2}, \quad \lim_{\xi \to \infty} g = \frac{5\pi}{2}
\]
and, correspondingly,
\[
\lim_{x \to \pm \infty} \varphi^\pm = -\frac{\pi}{2}, \quad \lim_{x \to \pm \infty} \varphi^\pm = \frac{3\pi}{2}; \quad (15)
\]
also these \( \varphi^\pm \) are unstable, as noted in [5].

Finally, if \( \gamma > 1 \), then \( \Delta < 0 \), \( y_\pm \) are complex conjugate and the denominator of (10) does not vanish for any value of \( y \). Setting \( w := \left( y_\gamma + 1 \right) / \sqrt{\gamma^2 - 1} \) (10) can be written as
\[
d\xi = \frac{2\alpha}{\sqrt{\gamma^2 - 1 + w^2}} \frac{dw}{1+w^2},
\]
which is integrated to give \( \xi - \xi_0 = 2\alpha \tan^{-1} w / \sqrt{\gamma^2 - 1} \), whence
\[
y(\xi) = -\gamma^{-1} + \frac{1}{\sqrt{1-\gamma^{-2}}} \tan \left[ \frac{\sqrt{\gamma^2 - 1}}{2\alpha} (\xi - \xi_0) \right], \quad (16)
\]
where \( \xi_0 \) is an integration constant. This is a periodic function with period
\[
\Xi := \frac{2\pi\alpha}{\sqrt{\gamma^2 - 1}}, \quad (17)
\]
and the latter is also the period occurring in (6). In fact, choosing \( \xi_0 = 0 \) for simplicity, we see that as \( \xi \) varies from \(-\Xi/2\) to \( \Xi/2 \), \( y(\xi) \) varies from \(-\infty\) to \( \infty \), \( F(\xi) \) varies from 0 to \( \infty \), \( g(\xi) \) varies from 0 to \( 2\pi \). By continuity of \( g \) (which again is not affected by the singularity of \( y \) at \( \xi = \Xi(k+1/2) \) \( \in \mathbb{Z} \)), we thus find the behaviour (6). The corresponding solutions \( \varphi^\pm \) fulfill (7), describe arrays of evenly-spaced kinks (see fig. 2) moving with velocity \( \pm 1 \) and are stable [5] (see also [7, 3]).

**Appendix**

We first recall the trigonometric identities
\[
\sin 2\alpha = \frac{2\tan \alpha}{1 + \tan^2 \alpha}, \quad \cos 2\alpha = \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha} \quad \Rightarrow \quad \sin 4\alpha = \frac{4\tan \alpha (1 - \tan^2 \alpha)}{(1 + \tan^2 \alpha)^2}. \quad (18)
\]

Given \( \gamma \in [0, 1] \), let \( \theta := \frac{1}{2} \sin^{-1} \gamma \in [0, \frac{\pi}{8}] \). Then \( \gamma = \sin 4\theta \), \( \sqrt{1-\gamma^2} = \cos 4\theta \) and, using the bisection formulae,
\[
\sqrt{1 + \sqrt{1 - \gamma^2}} = \sqrt{1 + \cos 4\theta} = \sqrt{2} \cos 2\theta, \quad (19)
\]
\[
\sqrt{1 - \sqrt{1 - \gamma^2}} = \sqrt{1 - \cos 4\theta} = \sqrt{2} \sin 2\theta, \quad (20)
\]
Figure 2: Photographs of the mechanical model of fig. 1: evenly spaced array of kinks (After A. C. Scott [8], courtesy of A. Barone, see [1])

whence in turn

$$\sqrt{2} - \sqrt{1+\sqrt{1-\gamma^2}} = \sqrt{2}(1-\cos 2\theta) = 2\sqrt{2}\sin^2 \theta,$$

(21)

$$\sqrt{2} - \sqrt{1-\sqrt{1-\gamma^2}} = \sqrt{2}(1-\sin 2\theta) = \sqrt{2} \left[1-\cos \left(\frac{\pi}{2}-2\theta\right)\right] = 2\sqrt{2}\sin^2 \left(\frac{\pi}{4}-\theta\right).$$

(22)

Hence, using also the sinus duplication formula, we end up with

$$F(y_+) = \frac{\sqrt{1-\sqrt{1-\gamma^2}}}{\gamma} \left[\sqrt{2} - \sqrt{1-\sqrt{1-\gamma^2}}\right] = \frac{4\sin 2\theta \sin^2 \left(\frac{\pi}{4}-\theta\right)}{\sin 4\theta}$$

$$= \frac{2\sin^2 \left(\frac{\pi}{4}-\theta\right)}{\cos 2\theta} = \frac{2\sin^2 \left(\frac{\pi}{4}-\theta\right)}{\sin \left(\frac{\pi}{2}-2\theta\right)} = \frac{\sin \left(\frac{\pi}{4}-\theta\right)}{\cos \left(\frac{\pi}{4}-\theta\right)} = \tan \left(\frac{\pi}{4}-\theta\right),$$

(23)

$$F(y_-) = \frac{\sqrt{1+\sqrt{1-\gamma^2}}}{\gamma} \left[\sqrt{2} - \sqrt{1+\sqrt{1-\gamma^2}}\right] = \frac{4\cos 2\theta \sin^2 \theta}{\sin 4\theta} = \frac{2\sin^2 \theta}{\sin 2\theta} = \tan \theta.$$ 

(24)

If we choose $\gamma = 1$ in (21) and use the sinus duplication formula we find as another consequence

$$\sqrt{2} - 1 = 2\sqrt{2}\sin^2 \left(\frac{\pi}{8}\right) = \frac{2\sin^2 \left(\frac{\pi}{8}\right)}{\sin \left(\frac{\pi}{4}\right)} = \frac{\sin \frac{\pi}{8}}{\cos \frac{\pi}{8}} = \tan \frac{\pi}{8}.$$ 

(25)

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