Critical Exponents of the $O(N)$-symmetric $\phi^4$ Model from the $\varepsilon^7$ Hypergeometric-Meijer Resummation

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Abstract

We extract the $\varepsilon$-expansion from the recently obtained seven-loop $g$-expansion for the renormalization group functions of the $O(N)$-symmetric model. The different series obtained for the critical exponents $\nu$, $\omega$ and $\eta$ have been resummed using our recently introduced hypergeometric-Meijer resummation algorithm. In three dimensions, very precise results have been obtained for all the critical exponents for $N = 0, 1, 2, 3$ and $4$. To shed light on the obvious improvement of the predictions at this order, we obtained the divergence of the specific heat critical exponent $\alpha$ for the XY model. We found the result $-0.01280$ compared to the famous experimental result of $-0.0127(3)$ from the specific heat of zero gravity liquid helium while the six-loop Borel with conformal mapping resummation result in literature gives the value $-0.007(3)$. For the challenging case of resummation of the $\varepsilon$-expansion series in two dimensions, we showed that our resummation results reflect an improvement to the previous six-loop resummation predictions.

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I. INTRODUCTION

Quantum field theory (QFT) offers a successful way to study critical phenomena in many physical systems [1–11]. It is universality that is behind the scene where different systems sharing the same symmetry properties follow the conjecture that they ought to behave in a similar manner at phase transition. So it is not strange to have a fluid possessing the same critical exponents like a magnetic one when both lie in the class of universality. The $O(N)$ vector model from scalar field theory has an infrared attractive fixed point and possesses the symmetry that can describe many physical systems at phase transitions. Near phase transitions, the theory is totally non-perturbative where in literature there exist many computational trends used to study the critical phenomena within the $O(N)$ vector model.

For the study of critical phenomena in the $O(N)$ vector model, Monte Carlo simulations have been used successfully and give precise results for the critical exponents [12–20]. Besides, bootstrapping the model in three dimensions has been accomplished recently and researchers succeeded to obtain precise results [21–26]. The nonperturbative renormalization group has been applied to the same model and gives accurate results too [27]. Apart from these non-perturbative methods, the oldest way to tackle the critical phenomena in QFT is resummation techniques applied to resum the divergent perturbation series associated with that model. The most traditional algorithm is Borel and its extensions which have been widely used in literature [3, 4, 7–9, 28–30]. In fact, the recent progress in obtaining higher orders of the perturbation series stimulates the need for the application of the resummation techniques to investigate the theory. Regarding that, the six-loop of the renormalization group functions has been recently obtained [29] and then the seventh order has been obtained too [31]. These orders are representing the renormalization group functions within the minimal subtraction regularization scheme in $D = 4 - \varepsilon$ dimensions.

The study of critical phenomena by finding an approximant to the perturbation series follows different routes. For instance, perturbative calculations at fixed $D$ dimensions [4, 28] are always giving better results specially in three dimensions. However, while exact results are known in two dimensions, the resummation of perturbation series did not give reliable results for some exponents [32, 34]. This point has been studied in Refs. [29, 33, 34] and it has been argued there that the reason behind this is thought to be the non-analyticity of the $\beta-$function at the fixed point. For the $\varepsilon$-expansion on the other hand, perturbation
series though possesses slower convergence [4, 29], might not suffer from non-analiticity issues like the $g$-series [35]. In view of the recent seven-loops ($g$-expansion) calculations [31], one thus can aim to get improved results from resumming the corresponding seventh order of $\varepsilon$ expansion in three dimensions ($\varepsilon = 1$) as well as get improved predictions for two dimensions $\varepsilon = 2$. A note to be mentioned here is that the most accurate renormalization group prediction for the exponent $\nu$ for the $XY$ model [28] (for instant) has a relatively large uncertainty. This makes it excluded from playing a role in the current $\lambda$-point dispute [47, 48]. Accordingly, higher order predictions from renormalization group is more than important.

In previous articles [36, 37], we introduced and applied the hypergeometric-Meijer resummation algorithm. What make our algorithm preferable is its simplicity and of having no arbitrary parameters like Borel algorithm and its extensions. Besides, it gives very competitive predictions when compared to the more sophisticated Borel with conformal mapping algorithm for instance. The algorithm has been applied successfully for the six-loop $\varepsilon$–series and for the seven-loop coupling series in Ref. [36]. For expansions in $4-\varepsilon$ dimensions, however, it is always believed that the $\varepsilon$–series has better convergence than the coupling-series [4]. In fact one can speculate about this by considering the large order behavior for both series. For the coupling series, the large order behavior includes the term $(-g_c)^n$ ($g_c \equiv$ critical coupling) while the $\varepsilon$–series has the term $(-\sigma \varepsilon)^n$ with $\sigma = \frac{3}{N+8}$. For $N = 1$ and in three dimensions (for instance), at the fixed point the $g$-series behaves as $(-0.47947)^n$ (from seven-loops calculations) while the $\varepsilon$–series behaves as $(-0.33333)^n$. So it is expected that the resummation of the $\varepsilon$–series has better convergence.

The recent resummation results of the six-loop $\varepsilon$-series [29, 36] gave accurate predictions for the critical exponents $\nu, \eta$ and $\omega$ for the $O(N)$-symmetric $\phi^4$ theory. However, the predictions of the relatively small exponents like divergence of specific heat exponent $\alpha$ are still far away from expected results. For the $XY$ model for instance, our hypergeometric-Meijer algorithm gives the result $\alpha = -0.00885$ [36] while Borel with conformal mapping result in Ref. [29] is $-0.007(3)$ and the resummation of seven-loop $g$-series in Ref. [36] predicts the value $-0.00860$. All of these predictions are all not close enough to the result of the famous experiment in Ref. [38]. In that reference, the measurement of the specific heat of liquid helium in zero gravity yields the result $-0.0127(3)$. Moreover, either the six-loop $\varepsilon$-expansion or the seven-loop $g$-expansion are not giving results that overlap with
Mote Carlo and conformal bootstrap results [48]. Accordingly, resumming the seven-loop $\varepsilon$-series represents an important point to monitor the improvement of the predictions of the critical exponents. With that in mind, our aim in this work is to first obtain the $\varepsilon$-series corresponding to the recent seven-loop coupling series for the $\beta, \gamma_m^2$ and $\gamma_\phi$ renormalization group functions and then apply our resummation algorithm to the series representing the critical exponents $\nu, \eta$ and $\omega$ for the $O(N)$--symmetric quantum field model.

The organization of this paper is as follows. In Sec.II, a brief description of the hypergeometric-Meijer resummation algorithm is introduced. We present in Sec.III the extracted seven-loop $\varepsilon$-expansion of the renormalization group functions. The resummation of the different $\varepsilon$-series representing the critical exponents $\nu, \eta$ and $\omega$ is presented in Sec.IV. In this section a comparison with predictions from other methods for $N = 0, 1, 2, 3$ and $4$ is listed in different tables for each $N$ individually. The study of the challenging two-dimensional case will follow in Sec.V. The last section in this paper (Sec.VI) is dedicated for the summary and conclusions.

II. THE HYPERGEOMETRIC-MEIJER RESUMMATION ALGORITHM

To make the work self consistent, we summarize in this section the hypergeometric-Meijer resummation algorithm that was firstly introduced in Ref.[37] and then applied to the six-loop ($\varepsilon$-expansion) and seven-loop $g$-expansion in Ref.[36]. Now, consider a perturbation series of a physical quantity $Q$ for which the first $M + 1$ terms are known:

$$Q(x) \approx \sum_{0}^{M} k_i x^i.$$  

Assume that the asymptotic large-order behavior for the series is also known to be of the form:

$$c_n = \alpha n!(-\sigma)^n n^b \left(1 + O\left(\frac{1}{n}\right)\right), \quad n \rightarrow \infty.$$  

As shown in Ref.[37], the hypergeometric series \( \mathbf{pFp−2}(a_1, a_2, ..., a_p; b_1, b_2, ..., b_p−2; −\sigma x) \) can reproduce the same large-order behavior with constraint on its numerator and denominator parameters as:

$$\sum_{i=1}^{p} a_i = \sum_{i=1}^{p−2} b_i − 2 = b.$$  

4
So the hypergeometric series \(_pF_{p-2}(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots b_{p-2}; -\sigma x)\) possesses all the known features of the given series when matching order by order the first \(M + 1\) coefficients from the perturbation series in Eq. (1) with the first \(M + 1\) coefficients of the expansion of the hypergeometric function \(_pF_{p-2}(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots b_{p-2}; -\sigma x)\). This type of hypergeometric functions have the expansion:

\[
pF_p^{p-2}(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots b_{p-2}; 0)_\sigma = \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n)}{\Gamma(a_1)} \ldots \frac{\Gamma(a_p+n)}{\Gamma(a_p)} \frac{\Gamma(b_1+n)}{\Gamma(b_1)} \ldots \frac{\Gamma(b_{p-2}+n)}{\Gamma(b_{p-2})} (-\sigma x)^n.
\]

Once parametrized by matching with the given series, the divergent hypergeometric series is now known up to any order and can be resummed by using its representation in terms of the Meijer-G function of the form [39]:

\[
pF_q(a_1, a_2; \ldots a_p; b_1, \ldots b_q; x) = \prod_{k=1}^{q} \frac{\Gamma(b_k)}{\Gamma(a_k)} G_{1,p}^{1,q+1}(1-a_1, \ldots, 1-a_p, 1-b_1, \ldots, 1-b_q | x).
\]

Note that for \(M\) even, \(M\) equations are generated by matching with the available orders from the given perturbation series to solve for \(M = (2p - 2)\) unknown parameters in the hypergeometric function. In the odd \(M\) case, we employ the constraint in Eq. (3) to get \(M + 1\) equations to solve for the \(M + 1\) unknown parameters. In any case, we always need an even number of equations to determine the \(2p - 2\) unknown parameters.

To give an example, consider the lowest order approximant (two-loops) \(_2F_0(a_1, a_2; ; -\sigma x)\) when matched we get the results:

\[
-a_1 a_2 \sigma = k_1
\]

\[
\frac{1}{2} a_1 (1 + a_1) a_2 (1 + a_2) (-\sigma)^2 = k_2.
\]

These equations are solved for the unknown parameters \(a_1\) and \(a_2\) provided that the parameter \(\sigma\) is known from the large-order behavior. Then we use the Meijer G-function representation given by:

\[
_2F_0(a_1, a_2; ; -\sigma x) = \frac{2}{\Gamma(a_1) \Gamma(a_2)} G_{1,2}^{1,2}(1-a_1, 1-a_2 | 0, -\sigma x),
\]

to obtain an approximant for the quantity \(Q(x)\) in Eq. (1) for \(M = 2\).

For the \(M = 3\) approximant \(_3F_1(a_1, a_2, a_3; b_1; -\sigma x)\), we have the equations:
\[ -\frac{a_1a_2a_3}{b_1} \sigma = k_1, \]
\[ \frac{1}{2} \frac{a_1(1 + a_1)a_2(1 + a_2)a_3(1 + a_3)}{b_1(1 + b_1)} (-\sigma)^2 = k_2, \]
\[ \frac{1}{6} \frac{a_1(1 + a_1)(2 + a_1)a_2(1 + a_2)(2 + a_2)a_3(1 + a_3)(2 + a_3)}{b_1(1 + b_1)(2 + b_1)} (-\sigma)^3 = k_3, \]
\[ a_1 + a_2 + a_3 - b_1 - 2 = b, \]

(7)

to be solved for the four unknowns \(a_1, a_2, a_3\) and \(b_1\). Thus we get the approximation of \(Q(x)\) as:
\[ Q(x) \approx \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} G_{3,2}^{1,3} \left( \frac{1-a_1,1-a_2,1-a_3}{0,1-b_1} \right) - \sigma x. \]

(8)

Our aim in this work is to resum the \(\varepsilon^2\) series for the critical exponents of the \(O(N)\)-symmetric model. Up to the best of our knowledge, the \(\varepsilon\)-expansion for these exponents (for all \(N\) cases studied here and for the same exponents) is not available so far in literature. So in the following section, we shall extract them first from the recent seven-loop calculations in Ref.\[31\].

III. \(\varepsilon\)-EXPANSION FOR THE SEVEN-LOOP CRITICAL EXPONENTS OF THE \(O(N)\)-SYMMETRIC MODEL

For the \(O(N)\)-vector model, the Lagrangian density is given by:
\[ \mathcal{L} = \frac{1}{2} (\partial \Phi)^2 + \frac{m^2}{2} \Phi^2 + \frac{16\pi^2 g}{4!} \Phi^4, \]

(9)

where \(\Phi = (\phi_1, \phi_2, \phi_3, \ldots, \phi_N)\) is an \(N\)-component field. This Lagrangian obeys an \(O(N)\) symmetry where \(\Phi^4 = (\phi_1^2 + \phi_2^2 + \phi_3^2 + \ldots + \phi_N^2)^2\). In \(4 - \varepsilon\) dimensions within the minimal subtraction technique, Oliver Schnetz has obtained the seven-loops order \((g\)-expansion) for the renormalization group functions \(\beta, \gamma_{m^2}\) and \(\gamma_\phi \[31\]. Here \(\beta\) is the famous \(\beta\)-function that determines the flow of the coupling in terms of mass scale, \(\gamma_{m^2}\) is the mass anomalous dimension and \(\gamma_\phi\) represents the field anomalous dimension. In the following subsections, we list the corresponding seven-loop \(\varepsilon\)-expansion for each individual exponent for the cases \(N = 0, 1, 2, 3, 4\), respectively.
III.1. The seven-loop $\varepsilon$-expansion for self-avoiding walks ($N = 0$)

For $N = 0$, we have the results [31]:

$$
\beta (g) \approx -\varepsilon g + 2.667g^2 - 4.667g^3 + 25.46g^4 - 200.9g^5 + 2004g^6 - 23315g^7 + 303869g^8,$$
$$
\gamma_{\phi} (g) \approx 0.05556g^2 - 0.03704g^3 + 0.1929g^4 - 1.006g^5 + 7.095g^6 - 57.74g^7,
$$
$$
\gamma_{m^2} (g) \approx -0.6667g + 0.5556g^2 - 2.056g^3 + 10.76g^4 - 75.70g^5 + 636.7g^6 - 6080g^7.
$$

The recipe to extract the corresponding $\varepsilon$-expansion is direct where we solve the equation $\beta (g) = 0$ (fixed point) for the critical coupling $g_c$ as a function of $\varepsilon$ and then substitute in the equations for $\gamma_{\phi} (g_c)$ and $\gamma_{m^2} (g_c)$. Note that the critical exponents $\nu$ and $\eta$ are obtained from the relations $\nu = (2 + \gamma_{m^2} (g_c (\varepsilon)))^{-1}$ and $\eta (\varepsilon) = 2 \gamma_{\phi} (g_c (\varepsilon))$ while the correction to scaling exponent $\omega$ is given as $\omega = \beta' (g_c)$. In Ref. [40], the method of Lagrange inversion has been used to get the exact seven-loop $\varepsilon$-expansion coefficients and has been applied to the $N = 1$ case but there the series has been obtained for $\nu$ while here we list the series for $\nu^{-1}$.

However, here we will obtain the $\varepsilon$-series by solving the equation $\beta (g) = 0$ implicitly and then expand the implicit solution as a power series in $\varepsilon$ keeping only orders up $\varepsilon^7$. As we will see, our results are compatible with those obtained in Ref. [40] for $\eta$ and $\omega$ for $N = 1$. For $\nu^{-1}$, $\eta$ and $\omega$ for $N = 0, 1, 2, 3$ and 4, we found that our results are compatible with the five-loop results available in Ref. [4] and six-loop series (after proper scaling) in Ref. [29].

For $N = 0$ and after solving the equation $\beta (g(\varepsilon)) = 0$, we get the result:

$$
g_c = 0.37500\varepsilon + 0.24609\varepsilon^2 - 0.18043\varepsilon^3 + 0.36808\varepsilon^4 - 1.2576\varepsilon^5 + 5.0625\varepsilon^6 - 23.392\varepsilon^7, \quad (11)
$$

and

$$
\nu^{-1} = 2.0000 - 0.25000\varepsilon + 0.08594\varepsilon^2 + 0.11443\varepsilon^3 + 0.28751\varepsilon^4 + 0.95613\varepsilon^5 - 3.8558\varepsilon^6 + 17.784\varepsilon^7, \quad (12)
$$

$$
\eta = 0.015625\varepsilon^2 + 0.016602\varepsilon^3 - 0.0083675\varepsilon^4 + 0.026505\varepsilon^5 - 0.090730\varepsilon^6 + 0.37851\varepsilon^7, \quad (13)
$$

$$
\omega = 1.0000\varepsilon - 0.65625\varepsilon^2 + 1.8236\varepsilon^3 - 6.2854\varepsilon^4 + 26.873\varepsilon^5 - 130.01\varepsilon^6 + 692.10\varepsilon^7. \quad (14)
$$
III.2. The ε-expansion for Ising-like model (N = 1)

In this case the seven-loop $\beta$-function is presented in Ref. [31] as:

$$\beta(g) \approx -\varepsilon g + 3.000g^2 - 5.667g^3 + 32.55g^4 - 271.6g^5 + 2849g^6 - 34776g^7 + 474651g^8,$$

(15)

and

$$\gamma_{m^2}(g) \approx -g + 0.8333g^2 - 3.500g^3 + 19.96g^4 - 150.8g^5 + 1355g^6 - 13760g^7,$$

(16)

while

$$\gamma_{\phi}(g) \approx 0.08333g^2 - 0.06250g^3 + 0.3385g^4 - 1.926g^5 + 14.38g^6 - 124.2g^7.$$

(17)

Solving the equation $\beta(g) = 0$, we get the critical coupling as:

$$g_c = 0.3333\varepsilon + 0.20988\varepsilon^2 - 0.13756\varepsilon^3 + 0.26865\varepsilon^4 - 0.84368\varepsilon^5 + 3.1544\varepsilon^6 - 13.483\varepsilon^7.$$

Substituting this form in $\gamma_{m^2}(g_c)$ and keep orders up to $\varepsilon^7$ only we get:

$$\nu^{-1} = 2.0000 - 0.3333\varepsilon - 0.11728\varepsilon^2 + 0.12453\varepsilon^3 - 0.30685\varepsilon^4$$

$$+ 0.95124\varepsilon^5 - 3.5726\varepsilon^6 + 15.287\varepsilon^7.$$

(18)

Similarly, the forms for $\eta$ and $\omega$ can be obtained as:

$$\eta = 0.018519\varepsilon^2 + 0.018690\varepsilon^3 - 0.0083288\varepsilon^4 + 0.025656\varepsilon^5 - 0.081273\varepsilon^6 + 0.31475\varepsilon^6,$$

(19)

$$\omega = \varepsilon - 0.62963\varepsilon^2 + 1.61822\varepsilon^3 - 5.23514\varepsilon^4 + 20.7498\varepsilon^5 - 93.1113\varepsilon^6 + 458.742\varepsilon^7.$$

(20)

III.3. The ε-expansion for N = 2 (XY universality class)

For N = 2, the renormalization group functions are obtained in Ref. [31] as:

$$\beta \approx -\varepsilon g + 3.333g^2 - 6.667g^3 + 39.95g^4 - 350.5g^5 + 3845g^6 - 48999g^7 + 696998g^8,$$

(21)

$$\gamma_{m^2} \approx -1.333g + 1.111g^2 - 5.222g^3 + 31.87g^4 - 255.8g^5 + 2434g^6 - 26086g^7,$$

(22)

$$\gamma_{\phi} \approx 0.11111g^2 - 0.09259g^3 + 0.5093g^4 - 3.148g^5 + 24.71g^6 - 224.6g^7.$$

(23)
We extracted from these equations the following forms for $g_c$ and the exponents $\nu$, $\eta$ and $\omega$:

$$g_c = 0.30000\varepsilon + 0.18000\varepsilon^2 - 0.10758\varepsilon^3 + 0.20502\varepsilon^4 - 0.59124\varepsilon^5 + 2.0719\varepsilon^6 - 8.2614\varepsilon^7,$$  

(24)

$$\nu^{-1} = 2.0000 - 0.40000\varepsilon - 0.14000\varepsilon^2 + 0.12244\varepsilon^3 - 0.30473\varepsilon^4$$  

$$+ 0.87924\varepsilon^5 - 3.1030\varepsilon^6 + 12.419\varepsilon^7,$$  

(25)

$$\eta = 0.020000\varepsilon^2 + 0.019000\varepsilon^3 - 0.0078936\varepsilon^4 + 0.023209\varepsilon^5 - 0.068627\varepsilon^6 + 0.24861\varepsilon^7,$$  

(26)

and

$$\omega = \varepsilon - 0.60000\varepsilon^2 + 1.4372\varepsilon^3 - 4.4203\varepsilon^4 + 16.374\varepsilon^5 - 68.777\varepsilon^6 + 316.48\varepsilon^7.$$  

(27)

### III.4. The $\varepsilon$-expansion for the exponents $\nu$, $\omega$ and $\eta$ in the Heisenberg universality class ($N = 3$)

The renormalization group functions can be generated using the maple package in Ref.[31]
as:

$$\beta (g) \approx -\varepsilon g + 3.667g^2 - 7.667g^3 + 47.65g^4 - 437.6g^5 + 4999g^6 - 66243g^7 + 978330g^8,$$  

(28)

$$\gamma_\phi \approx 0.1389g^2 - 0.1273g^3 + 0.6993g^4 - 4.689g^5 + 38.44g^6 - 365.9g^7,$$  

(29)

and

$$\gamma_{m^2} (g) \approx -1.667g + 1.389g^2 - 7.222g^3 + 46.64g^4 - 394.9g^5 + 39506 - 44412g^7.$$  

(30)

From these functions one can obtain the following forms for the critical quantities $g_c$, $\nu$, $\eta$ and $\omega$ as:

$$g_c = 0.27273\varepsilon + 0.15552\varepsilon^2 - 0.086255\varepsilon^3 + 0.16154\varepsilon^4 - 0.42963\varepsilon^5 + 1.4210\varepsilon^6 - 5.3221\varepsilon^7,$$  

(31)

$$\nu^{-1} = 2.0000 - 0.45455\varepsilon - 0.15590\varepsilon^2 + 0.11507\varepsilon^3 - 0.29360\varepsilon^4$$  

$$+ 0.78994\varepsilon^5 - 2.6392\varepsilon^6 + 9.9452\varepsilon^7,$$  

(32)

$$\eta = 0.020661\varepsilon^2 + 0.018399\varepsilon^3 - 0.0074495\varepsilon^4 + 0.020383\varepsilon^5 - 0.057024\varepsilon^6 + 0.19422\varepsilon^7,$$  

(33)

and

$$\omega = -0.0000 - 0.57025\varepsilon^2 + 1.2829\varepsilon^3 - 3.7811\varepsilon^4 + 13.182\varepsilon^5 - 52.204\varepsilon^6 + 226.02\varepsilon^7.$$  

(34)
III.5. The seven-loop $\varepsilon$-expansion for the $O(4)$-symmetric case

For $N = 4$, we have the seven-loops $g$-series as:

$$
\beta(g) \approx -\varepsilon g + 4.000 g^2 - 8.667 g^3 + 55.666 g^4 - 533.0 g^5 + 6318 g^6 - 86768 g^7 + 1.326 \times 10^6 g^8,
$$

(35)

$$
\gamma_{m^2}(g) \approx -2.000 g + 1.667 g^2 - 9.500 g^3 + 64.39 g^4 - 571.9 g^5 + 5983 g^6 - 70240 g^7,
$$

(36)

and

$$
\gamma_{\phi} \approx 0.1667 g^2 - 0.1667 g^3 + 0.9028 g^4 - 6.563 g^5 + 55.93 g^6 - 555.2 g^7.
$$

(37)

Our prediction for the corresponding $\varepsilon-$expansion for the critical coupling and exponents are of the form:

$$
g_c = 0.25000 \varepsilon + 0.13542 \varepsilon^2 - 0.070723 \varepsilon^3 + 0.13030 \varepsilon^4 - 0.32185 \varepsilon^5 + 1.0099 \varepsilon^6 - 3.5738 \varepsilon^7,
$$

(38)

$$
\nu^{-1} = 2.0000 - 0.5 \varepsilon - 0.16667 \varepsilon^2 + 0.105856 \varepsilon^3 - 0.278661 \varepsilon^4 + 0.702167 \varepsilon^5 - 2.23369 \varepsilon^6 + 7.97005 \varepsilon^7,
$$

(39)

$$
\eta = 0.020833 \varepsilon^2 + 0.017361 \varepsilon^3 - 0.0070852 \varepsilon^4 + 0.017631 \varepsilon^5 - 0.047363 \varepsilon^6 + 0.15219 \varepsilon^7,
$$

(39)

and

$$
\omega = \varepsilon - 0.541667 \varepsilon^2 + 1.15259 \varepsilon^3 - 3.27193 \varepsilon^4 + 10.8016 \varepsilon^5 - 40.5665 \varepsilon^6 + 166.256 \varepsilon^7.
$$

(40)

It is well known that the $\varepsilon-$series is divergent and has an asymptotic large-order behavior of the type shown in Eq.(2) where $\sigma = -3\frac{N}{N+8}$, $b_{\nu-1} = 4 + \frac{N}{2}$, $b_\eta = 3 + \frac{N}{2}$ and $b_\omega = 5 + \frac{N}{2}$. Accordingly, the suitable hypergeometric approximant is of the form $_{p}F_{p-2}(a_1, a_2, ..., a_p; b_1, b_2, ... b_{p-2}; -\sigma \varepsilon)$. In the following section, we list the three dimensional ($\varepsilon=1$) resummation results for $N = 0,1,2,3,4$ cases for the exponents $\nu, \eta$ and $\omega$ within the $O(N)$-symmetric $\phi^4$ model.

IV. HYPERGEOMETRIC-MEIJER RESUMMATION OF THE $\varepsilon^7$ PERTURBATION SERIES

In this section we present the resummation results for the $\varepsilon^7$-series but numerical values in this section are for $\varepsilon = 1$. For the series representing $\nu^{-1}$ in Eqs. 12, 18, 25, 32, 38; the suitable approximant is $2_{5}F_{3}(a_1, ... a_5; b_1, b_2, b_3; -\sigma \varepsilon)$ with the corresponding Meijer-G function.
representation:
\[ \nu^{-1} \approx \frac{\prod_{k=1}^{3} \Gamma (b_k)}{\prod_{k=1}^{5} \Gamma (a_k)} G_{5,4}^{1,5} \left( \begin{array}{c} 1 - a_1, \ldots, 1 - a_5 \\ 0, 1 - b_1, 1 - b_2, 1 - b_3 \end{array} \right| - \sigma \varepsilon \right). \quad (41) \]

However, sometimes the determined parameters include values near the singular points in the \( \Gamma \)-functions (negative integers) defining the coefficients of the hypergeometric functions. In this case, approximate solutions though accurate in determining the parameters can lead to appreciable change from the actual value of the hypergeometric function. In that case, we resort to resummation of the subtracted series instead of the original one. This technique is well known in resummation methods \[4, 30, 36\].

For the series representing the critical exponent \( \omega \) in Eqs.\( (14, 20, 27, 34, 40) \), the suitable approximant is:
\[ \omega \approx {}_5 F_3(a_1, \ldots, a_5; b_1, b_2, b_3; -\sigma \varepsilon) - 1, \quad (42) \]
while based on the type of series given for the exponent \( \eta \) in Eqs.\( (13, 19, 26, 33, 39) \), it can be easily shown that the suitable hypergeometric approximant takes the form:
\[ \eta \approx {}_4 F_2(a_1, \ldots, a_4; b_1, b_2; -\sigma \varepsilon) - 1 - \frac{a_1a_2a_3a_4}{b_1b_2}(-\sigma \varepsilon). \quad (43) \]

In case there is no solution found for the set of equations that determine the parameters, one resort to successive subtractions which can lead to another approximant \[36\]. So, in the following subsections, we will list the resummation results for the seven-loop perturbation series where for each case we list the suitable approximants.

**IV.1. Seven-loop (\( \varepsilon^7 \)) Resummation results for self-avoiding walks (\( N = 0 \))**

For the reciprocal of the critical exponent \( \nu \) where the seven-loop order is represented by Eq.\( (12) \), the suitable approximant as mentioned above is \( {}_5 F_3(a_1, \ldots, a_5; b_1, b_2, b_3; -\sigma \varepsilon) \) which has eight unknown parameters. In matching the first seven orders of the expansion of this approximant (order by order) by the seven orders available for the corresponding series, one still in a need to employ the constraint:
\[ \sum_{i=0}^{5} a_i - \sum_{j=0}^{3} b_i - 2 = b_{\nu-1} = 4. \]

This constraint serves as the needed 8th equation for the determination of the eight unknown parameters. After solving the set of eight equations, we obtain the results: \( a_1 = -3.68209, \ldots \).
\[ a_2 = 10.5985 + 10.5125i, \quad a_3 = a_2^*, \quad a_4 = -1.18266, \quad a_5 = -0.0115771 \text{ and } b_1 = 13.3112, \quad b_2 = 0.688263, \quad b_3 = -3.67874. \] Accordingly, we get:

\[ \nu^{-1} \approx \frac{\prod_{k=1}^{3} \Gamma (b_k)}{\prod_{k=1}^{5} \Gamma (a_k)} \left( \begin{array}{c} 1-a_1,...,1-a_5 \\ 0.1-b_1,1-b_2,1-b_3 \end{array} \right) \frac{3}{8} \varepsilon \right).

This gives the result \( \nu = 0.58770 \). This result is compatible with the recent conformal bootstrap result \( \nu = 0.5877(12) \) in Ref.[26] and the result \( \nu = 0.5875970(4) \) from Monte Carlo simulations in Ref.[15].

For the critical exponent \( \omega \) represented by the series in Eq.(14), it can be approximated by using the hypergeometric function \( {}_5F_3(a_1,...a_5;b_1,b_2,b_3;\sigma\varepsilon) - 1 \), where the eight unknown parameters are determined from employing the seven order s in the given perturbation series as well as the constraint:

\[
\sum_{i=0}^{i=5} a_i - \sum_{j=0}^{j=3} b_i - 2 = 5.
\]

This gives the results \( a_1 = -2.75963, a_2 = 15.6885 - 8.23088i, a_3 = a_2^*, a_4 = -0.884248, a_5 = -0.137379 \) and \( b_1 = -2.74456, b_2 = 23.9408, b_3 = -0.600514 \). Accordingly, we get the approximation \( \omega = 0.84977 \). In fact, the calculations from Monte Carlo simulations in Ref.[15] gives the result \( \omega = \frac{4}{\nu} = 0.899(12) \) while the six-loop Borel with conformal mapping resummation[29] turns the result \( \omega = 0.841(13) \).

For the critical exponent \( \eta \), it has the seven-loop perturbation series in Eq.(12) and the suitable approximant is:

\[
\eta \approx {}_4F_2(a_1,...a_4; b_1, b_2; -\sigma\varepsilon) - 1 - \frac{a_1a_2a_3a_4}{b_1b_2}(-\sigma\varepsilon),
\]

where \( a_1 = 16.5429, a_2 = -2.59734, a_3 = -3.54461, a_4 = 0.0000969353 \) and \( b_1 = 0.348504, b_2 = -3.7142 \). This parametrization gives the result \( \eta \approx 0.03068. \) The bootstrap calculation gives the result \( \eta = 2\Delta_\phi - 1 = 0.0282(4) \)[26] and the Monte Carlo result is \( \eta = 0.031043(3) \)[14,29].

For the comparison with the predictions from different other methods, our results for the three exponents are listed again in table I.

**IV.2. Seven-loop Resummation results for Ising-like universality class (N = 1)**

The perturbation series for critical exponent \( \nu(\nu^{-1}) \) of the Ising-like model up to \( \varepsilon^7 \) is given by Eq.(18). The suitable approximant for this order is \( 2 - \frac{1}{3} \varepsilon {}_4F_4(a_1,...a_4; b_1, b_2; -\sigma\varepsilon) \)
TABLE I: The seven-loop ($\epsilon$-expansion) hypergeometric-Meijer ($\epsilon^7$; HM) resummation results for the exponents $\nu$, $\eta$ and $\omega$ of the self-avoiding walks model ($N=0$). Here we compare with our results for $\epsilon^6$ ($\epsilon^6$; HM) and seven-loop $g$-expansion (HMg) from Ref. [36]. The recent predictions of what is called self-consistent (SC) resummation algorithm introduced in Ref. [46] is also listed in the table. Besides, we list results from conformal bootstrap (CB) calculations [26], Monte Carlo simulation (MC) for $\nu$ from Ref. [14, 29] and $\eta$ from Ref. [15]. Also the predictions of the the resummation of six-loop series using Borel with conformal mapping (BCM) algorithm ($\epsilon^6$) from Ref. [29] and five-loop ($\epsilon^5$) from same reference is included.

| Method   | $\nu$      | $\eta$      | $\omega$    |
|----------|------------|-------------|-------------|
| $\epsilon^7$; HM: This work | 0.58770 | 0.03068 | 0.84977 |
| $\epsilon^6$: HM | 0.58744 | 0.03034 | 0.85559 |
| 7L: HMg | 0.58723 | 0.03129 | 0.85650 |
| SC | 0.5874(2) | 0.0304(2) | 0.846(15) |
| CB | 0.5877(12) | 0.0282(4) | —           |
| MC | 0.5875970(4) | 0.031043(3) | 0.899(12) |
| $\epsilon^6$: BCM | 0.5874(3) | 0.0310(7) | 0.841(13) |
| $\epsilon^5$: BCM | 0.5873(13) | 0.0314(11) | 0.835(11) |

which predicts the value $\nu \approx 0.62998$. To get an idea about how accurate this result is, we list her results from recent non-perturbative methods like the Monte Carlo simulations which gives the result $\nu = 0.63002(10)$ [12], the recent non-perturbative renormalization group (NPRG) method [27] which turns the result $\nu = 0.63012(16)$ as well as the recent conformal bootstrap result $\nu = 0.62999(5)$ in Ref. [23]. In view of these non-perturbative calculations and in looking at table II one can realize that the seven-loop resummation results add to the improvement of the six-loop results either in Ref. [36] or Ref. [29]. In fact, this is a general trend in all of the seven-loop calculations in this work. Note that, our prediction for $\nu$ is overlapped by the error bars of the three non-perturbative methods mentioned above.

For the exponent $\eta$, the seven-loop perturbation series is given by Eq. (19) while the suitable approximant is:

$$\eta \approx F_2\left(a_1, \ldots a_4; b_1, b_2 ; -\sigma \epsilon \right) - \frac{a_1 a_2 a_3 a_4}{b_1 b_2} (-\sigma \epsilon),$$

which gives the result $\eta \approx 0.03586$ compared to Monte Carlo simulation result of $\eta = 0.03627(10)$, NPRG result $\eta = 0.0361(11)$ [27] and conformal bootstrap calculation of $\eta = 0.03631(3)$ [23].
For the correction to scaling exponent \( \omega \), the up to seven-loop order of perturbation series is given by Eq. (20). The approximant used for that order is \( _5F_3(a_1, ... a_5; b_1, b_2, b_3; -\sigma \varepsilon) - 1 \) which turns the value \( \omega \approx 0.82359 \). NPRG method gives the result \( \omega = 0.832(14) \) while conformal bootstrap has the result \( \omega = 0.8303(18) \) and Monte Carlo simulations predicts the value \( \omega = 0.832(6) \). To get an idea about the improvement of the resummation results, in Table II we list the six-loop result from our previous work. Comparison with the predictions from more different methods is also listed in the same table.

**TABLE II:** The seven-loop (\( \varepsilon \)-expansion) hypergeometric-Meijer resummation results (\( \varepsilon^7 \); HM) for the exponents \( \nu, \eta \) and \( \omega \) of the Ising-like model (\( N = 1 \)). Here we compare with our results from Ref. 36 for \( \varepsilon^6 \) (\( \varepsilon^6 \); HM) and seven-loop \( g \)-expansion (HMg) from the same reference. The recent SC resummation results are listed for comparison. Also we list conformal bootstrap calculations from Ref. 23 and Monte Carlo simulation (MC) from Ref. 12. In this table also, we list the six-loop (\( \varepsilon^6 \)) resummation results from the Borel with conformal mapping (BCM) from Ref. 21 and five-loops (\( \varepsilon^5 \)) from same reference. The recent results from the non-perturbative renormalization group (NPRG) method is listed last.

| Method       | \( \nu \)     | \( \eta \)     | \( \omega \)   |
|--------------|---------------|---------------|----------------|
| \( \varepsilon^7 \); HM: This work | 0.62998       | 0.03586       | 0.82359       |
| \( \varepsilon^6 \); HM     | 0.62937       | 0.03545       | 0.82929       |
| 7L: HMg       | 0.62934       | 0.03684       | 0.82790       |
| SC            | 0.6296(3)     | 0.0355(3)     | 0.827(13)     |
| CB            | 0.62999(5)    | 0.03631(3)    | 0.8303(18)    |
| MC            | 0.63002(10)   | 0.03627(10)   | 0.832(6)      |
| \( \varepsilon^6 \); BCM    | 0.6292(5)     | 0.0362(6)     | 0.820(7)      |
| \( \varepsilon^5 \); BCM    | 0.6290(20)    | 0.0366(11)    | 0.818(8)      |
| NPRG          | 0.63012(16)   | 0.0361(11)    | 0.832(14)     |

**IV.3. Seven-loop Resummation results for XY universality class (\( N = 2 \))**

For \( N = 2 \), the perturbation series up to \( \varepsilon^7 \) for the critical exponent \( \nu \) is given by Eq. (25) while our suitable approximant is \( 2 - 0.4 \varepsilon _4F_2(a_1, ... a_4; b_1, b_2; -\sigma \varepsilon) \). The parametrization for that approximant yields the result \( \nu = 0.67093 \). In fact, this result is so impressive as it yields the value \( \alpha = -0.012801 \) for the specific heat singularity exponent extracted from
the hyper-scaling relation \( \alpha = 2 - 3\nu \). The microgravity experiment on the other hand gives the result \( \alpha = -0.0127(3) \). What makes our prediction for \( \alpha \) so impressive is that the six-loop resummation result using Borel with conformal mapping in Ref. [29] gives the value \( \alpha = -0.007(3) \). In using our algorithm to resum the same six-loop series in Ref. [30] we get the value \( \alpha = -0.00885 \) while in the same reference the seven-loop \( g \)-expansion gives the result \( \alpha = -0.00860 \). This shows that our resummation result, which is overlapped by the uncertainty in the experimental result, reinforces the expectation that the \( \varepsilon \)-series has better convergence than the \( g \)-series. Besides, the resummation of seventh order in this work clearly improves the previous \( \varepsilon^6 \) resummation results in Refs. [29, 30].

To compare with other methods for the \( \nu \) exponent, we mention that Monte Carlo result is \( \nu = 0.67183(18) \) [20], NPRG yields the prediction \( \nu = 0.6716(6) \) [27] while the recent conformal bootstrap prediction is \( \nu = 0.6719(11) \) [25].

For the critical exponent \( \eta \), the seven-loop hypergeometric-Meijer resummation result is \( \eta = 0.03775 \) using the approximant \( \varepsilon(4F_2(a_1, \ldots a_4; b_1, b_2; -\sigma\varepsilon) - 1) \). The NPRG method predicted the value \( \eta = 0.0380(13) \) [27], Monte Carlo simulations in Ref. [20] gives the result \( \eta = 0.03853(48) \) and conformal bootstrap has the prediction \( \eta = 0.03852(64) \) [25].

For the resummation of the critical exponent \( \omega \) given by Eq. (34), we used the approximant \( 5F_3(a_1, \ldots a_5; b_1, b_2, b_3; -\sigma\varepsilon) - 1 \) which yields the result \( \omega \approx 0.801541 \). The result \( \omega = 0.789 \) has been shown using high-precision Monte Carlo calculations [20] and the prediction of conformal bootstrap calculations yields the result \( \omega = 0.811(10) \) [24, 29] while the recent NPRG result is \( \omega = 0.791(8) \) [27].

In table III we list predictions from more different methods for the three exponents beside our predictions.

### IV.4. Seven-loop Resummation results for Heisenberg universality class ( \( N = 3 \))

For the case of the Heisenberg universality class ( \( N = 3 \)), the reciprocal of the critical exponent \( \nu \) has the \( \varepsilon^7 \) perturbation series given by Eq. (32) which is resummed via the approximant \( 2 - 0.454545\varepsilon \). The parametrization of this approximant leads to the prediction \( \nu = 0.70954 \). The conformal bootstrap calculations gives the value \( \nu = 0.7121(28) \) while the Monte Carlo simulations in Ref. [13] gives the result \( \nu = 0.7116(10) \) and the NPRG method has the value \( \nu = 0.7114(9) \) [27].
TABLE III: The hypergeometric-Meijer ($\varepsilon^7$; HM) resummation results for the exponents $\nu, \eta$ and $\omega$ of the $O(2)$-symmetric model. The recent SC results results are taken from Ref. [46]. Also, we compare with our results from Ref. [36] for $\varepsilon^6$ ($\varepsilon^6$: HM) and seven-loop $\varepsilon$-expansion (HMg). Other predictions are listed from conformal bootstrap calculations for $\nu$ and $\eta$, while $\omega$ result is taken from Ref. [24, 29] and MC calculations from Ref. [20]. The six-loop BCM resummation ($\varepsilon^6$) from Refs. [29] and the five-loops ($\varepsilon^5$) from same reference. In the last row we add the NPRG results up to $O(\partial^4)$.

| Method      | $\nu$  | $\eta$  | $\omega$  |
|-------------|--------|---------|-----------|
| $\varepsilon^7$; HM: This work | 0.67093 | 0.03775 | 0.801541  |
| $\varepsilon^6$: HM | 0.66962 | 0.03733 | 0.80580   |
| 7L: HMg | 0.66953 | 0.03824 | 0.80233   |
| SC        | 0.6706(2) | 0.0374(3) | 0.808(7)  |
| CB         | 0.6719(11) | 0.03852(64) | 0.811(10) |
| MC         | 0.67183(18) | 0.03853(48) | 0.789     |
| $\varepsilon^6$: BCM | 0.6690(10) | 0.0380(6) | 0.804(3)  |
| $\varepsilon^5$: BCM | 0.6687(13) | 0.0384(10) | 0.803(6)  |
| NPRG       | 0.6716(6) | 0.0380(13) | 0.791(8)  |

The series up to $\varepsilon^7$ for the exponent $\eta$ is given in Eq. (33) where it has been approximated using $\varepsilon_4F_2(a_1, ... a_4; b_1, b_2 ; -\sigma\varepsilon) - 1)$. This approximation yields the prediction $\eta = 0.03765$ compared to $\eta = 0.0386(12)$ from bootstrap calculations in Ref. [25], $\eta = 0.0376(13)$ from NPRG in Ref. [27], and $\eta = 0.0378(3)$ predicted by Monte Carlo simulations in Ref. [13].

The seven-loop $\varepsilon$-expansion for the exponent $\omega$ has been obtained in the previous section in Eq. (34). This series has been resummed through the use of the hypergeometric approximant $\varepsilon_4F_2(a_1, ... a_4; b_1, b_2 ; -\sigma\varepsilon)$ which gives the result $\omega = 0.768812$. For comparison, we list here the value $\omega = 0.791(22)$ from conformal bootstrap calculations [24, 29], $\omega = 0.773$ from Monte Carlo simulations [17] and $\omega = 0.769(11)$ from NPRG method [27].

In table IV we list more results from other methods to make it clear that the hypergeometric-Meijer resummation algorithm though simple is competitive to other more sophisticated algorithms and methods.
TABLE IV: The seven-loop ($\varepsilon^7$) hypergeometric-Meijer resummation for the exponents $\nu$, $\eta$ and $\omega$ of the $O(3)$-symmetric model. The results are compared with our results of six-loop resummation ($\varepsilon^6$: Hm) and seven-loop resummation of the $g$-series (HMg) from Ref. [36]. Then we list the SC resummation results from Ref. [46]. The recent results from conformal bootstrap calculations are listed also where the values of $\nu$ and $\eta$ are taken from Ref. [25] while $\omega$ from Refs [24, 29]. For MC simulations $\omega$ is taken from Ref. [17] while $\nu$ and $\eta$ are taken from Ref. [13]. The six-loop BCM resummation is taken from Ref. [24] and five-loops from same reference. As in all of above tables, we list in the last row the very recent calculations from NPRG method [27] (up to $O(\partial^4)$).

| Method        | $\nu$  | $\eta$  | $\omega$ |
|---------------|--------|---------|----------|
| $\varepsilon^7$: HM: This work | 0.70954 | 0.03765 | 0.76881 |
| $\varepsilon^6$: HM | 0.70722 | 0.037301 | 0.79272 |
| 7L: HMg       | 0.70810 | 0.03795 | 0.78683 |
| SC            | 0.70944(2) | 0.0373(3) | 0.794(4) |
| CB            | 0.7121(28) | 0.0386(12) | 0.791(22) |
| MC            | 0.7116(10) | 0.0378(3) | 0.773 |
| $\varepsilon^6$: BCM | 0.7059(20) | 0.0378(5) | 0.795(7) |
| $\varepsilon^5$: BCM | 0.7056(16) | 0.0382(10) | 0.797(7) |
| NPRG          | 0.7114(9) | 0.0376(13) | 0.769(11) |

IV.5. Seven-loop Resummation results for the $N = 4$ case

Similar to the above cases, the seven-loop perturbation series for the exponent $\nu$ has been obtained in the previous section in Eq. [35]. As in the previous $N = 0$ case, we resummed it using the approximant $\, _2F_3(a_1, ... a_5; b_1, b_2, b_3; -\sigma \varepsilon)$ which gives the result $\nu \approx 0.74391$. This result is compatible with the NPRG prediction of $\nu = 0.7478(9)$ in Ref. [27]. Also the conformal bootstrap result is $\nu = 0.751(3)$ from Ref. [24] and Monte Carlo simulations gives the result $\nu = 0.750(2)$ [13].

For the critical exponent $\eta$ with perturbative result in Eq. [39], we used the approximant $\, _2F_2(a_1, ... a_4; b_1, b_2; -\sigma \varepsilon) - 1$ which yields the result $\eta = 0.03642$. The NPRG result is $\eta = 0.0360(12)$ [27] and Monte Carlo simulations for that case gives the result $\eta = 0.0365(3)$ [13] while recent bootstrap calculations gives the value $0.0378(32)$ [22].

For the critical exponent $\omega$ represented by Eq. [40], we used the approximant
which yields the result \( \omega = 0.75285 \) compared to NPRG result of \( \omega = 0.761(12) \) \cite{27} while the result of Monte Carlo simulation in Ref.\cite{17} is \( \omega = 0.765 \) and \( \omega = 0.817(30) \) from conformal bootstrap calculations \cite{24, 29}.

**TABLE V:** The seven-loop (\( \epsilon^7 \)) hypergeometric-Meijer resummation for the exponents \( \nu \), \( \eta \) and \( \omega \) of the \( O(4) \)-symmetric model compared to our previous six-loop results and seven-loop resummation of the \( g \) series in Ref.\cite{36}. Results from conformal bootstrap calculations \cite{24, 29} for \( \nu \) and \( \omega \), while \( \eta \) from Ref.\cite{22} are listed. Besides, MC simulations for \( \omega \) is taken from Ref.\cite{17} while \( \nu \) and \( \eta \) are from Ref.\cite{13}. The six-loop BCM resummation (\( \epsilon^6 \)) is taken from Ref.\cite{29} and five-loops (\( \epsilon^5 \)) from same reference. NPRG results up to \( O(\partial^4) \) \cite{27} are shown in the last row.

| Method       | \( \nu \)      | \( \eta \)      | \( \omega \)   |
|--------------|-----------------|-----------------|----------------|
| \( \epsilon^7 \): HM: This work | 0.74391 | 0.03642 | 0.75285 |
| \( \epsilon^6 \): HM     | 0.74151 | 0.03621 | 0.76793 |
| 7L: HMg      | 0.750935 | 0.03740 | 0.80325 |
| SC           | 0.7449(4)  | 0.0363(2) | 0.7863(9) |
| CB           | 0.751(3)   | 0.0378(32) | 0.817(30) |
| MC           | 0.750(2)   | 0.0360(3) | 0.765(30) |
| \( \epsilon^6 \): BCM     | 0.7397(35) | 0.0366(4) | 0.794(9) |
| \( \epsilon^5 \): BCM     | 0.7389(24) | 0.0370(9) | 0.795(6) |
| NPRG         | 0.7478(9)   | 0.0360(12) | 0.761(12) |

**V. TWO-DIMENSIONAL HYPERGEOMETRIC-MEIJER RESUMMATION**

In two dimensions or equivalently \( \epsilon = 2 \), there are two main differences from the three dimensional case. The first is that for \( N \geq 2 \), there is no broken-symmetry phase \cite{33}. For the other difference, since \( \epsilon = 2 \) is a large value and the strong-coupling asymptotic behavior of the \( O(N) \) symmetric model is not known yet, one expects a slower convergence of the resummation of the perturbation series. For the \( g \) expansion, it has been argued that the \( \beta \) function is not analytic at the fixed point \cite{33, 34, 41} which in turn slows the convergence down too. The effect of the non-analiticity of the \( \beta \) function is higher in two dimensions. This leads to inaccurate predictions for critical exponents from the \( g \) expansion in two dimensional case \cite{34}. Accordingly, testing the resummation algorithm for the \( \epsilon = 2 \)
case offers an interesting point about the capability of the $\varepsilon$-expansion to predict reliable results for that case. Apart from inaccurate resummation results from the $g$-expansion as well as previous results of the $\varepsilon$-expansion that needs more improvement, exact values for the two dimensional critical exponents are known and thus can be used to test the reliability of any approximating method.

For $N = 0$, our resummation result for the critical exponent $\nu$ is 0.75148 while the exact result is assumed to be $\frac{3}{4}$ [42] and the recent Borel with conformal mapping resummation for six-loop yields the result $\nu = 0.741(4)$ [29]. For the critical exponent $\omega$, our prediction is 1.9554 while the exact value is 2 [42, 43] and the recent six-loop resummation in Ref. [29] gives the result 1.90(25). For $\eta$, we get the value 0.18955 while the exact result is $(\frac{5}{24}) \approx 0.20833...$ [42] and the six-loop resummation (BCM) result is 0.201(25) [29]. One can realize that our predictions show a clear improvement for the previous resummation results in literature specially for the exponents $\nu$ and $\omega$. For the exponent $\eta$, our resummation result though very reasonable is in fact less accurate than the other two exponents. The reason behind this is that the perturbation series for $\eta$ in Eq. (13) starts from the $\varepsilon^2$ order and thus the corresponding hypergeometric approximant is parametrized by fewer number of parameters than the case for the other exponents. One more perturbative term is supposed to give a very accurate result.

For Ising-like case ($N = 1$), we obtained the result $\nu = 0.98499$ compared to the well known exact result $\nu = 1$ [44] while BCM result for six loops gives the value $\nu = 0.952(14)$. For $\omega$ we get the result 1.7202 while the exact value is $\omega = 1.75$ [45] and BCM result is 1.71(9). Our prediction for $\eta$ is 0.22277 while the exact value is 0.25 [42] and the six-loop BCM resummation result is 0.237(27).

From the calculations above and those listed in table VI, it is clear that the seventh order improves the six-loop calculations within our algorithm for all the exponents. Even, in comparison to the BCM six-loop calculations, our algorithm improves the predictions for the exponents $\nu$ and $\omega$. For the critical exponent $\eta$ however, the prediction is less accurate due to the reason mentioned above.
TABLE VI: The seven-loop ($\varepsilon^7$) hypergeometric-Meijer resummation for the exponents $\nu$, $\eta$ and $\omega$ for the Self-avoiding walks ($N = 0$) and the Ising-like model ($N = 1$) in two dimensions ($\varepsilon = 2$). For comparison with other predictions, we list the results from the same algorithm but for six loops, BCM results \[29\] for six loops too. Exact results for $N = 1$ for $\nu$ and $\eta$ are obtained in the seminal article in Ref.\[44\] and $\omega$ from Ref.\[45\]. For $N = 0$, exact values for $\nu$, $\eta$ and $\omega$ are conjectured in Ref.\[42\].

| N  | $\nu$     | $\eta$    | $\omega$    | Method                      |
|----|-----------|-----------|-------------|-----------------------------|
| 0  | 0.75148   | 0.18955   | 1.9554      | $\varepsilon^7$; HM: This work |
|    | 0.74274   | 0.18064   | 2.04477     | $\varepsilon^6$: HM         |
|    | 0.741(4)  | 0.201(25) | 1.90(25)    | $\varepsilon^6$: BCM        |
|    | 0.75      | 0.20833...| 2           | Exact                       |
| 1  | 0.98499   | 0.22277   | 1.7202      | $\varepsilon^7$; HM: This work |
|    | 0.954827  | 0.21163   | 1.80486     | $\varepsilon^6$: HM         |
|    | 0.952(14) | 0.237(27) | 1.71(9)     | $\varepsilon^6$: BCM        |
|    | 1         | 0.25      | 1.75        | Exact                       |

VI. SUMMARY AND CONCLUSIONS

The recent calculations of the seven-loop renormalization group functions of the $O(N)$-symmetric field theory ($g$-expansion) have motivated us to generate the corresponding $\varepsilon$-expansion for the critical exponents $\nu$, $\eta$ and $\omega$. Scaling relations can lead to other different critical exponents like the specific heat singularity exponent where it is given by $\alpha = 3 - D\nu$ with $D = 4 - \varepsilon$, the Euclidean space-time dimension. Getting the seven-loop order of the $\varepsilon$-expansion is important toward the improvement of the previous six-loop resummation results \[29, 36\]. In this work, we used our hypergeometric-Meijer algorithm \[36, 37\] to resum the up to $\varepsilon^7$ series for the critical exponents from the $O(N)$-symmetric $\phi^4$ theory for $N = 0, 1, 2, 3, 4$. The resummation results has shown clear improvement for the previous six-loop results. The most reflecting quantity for the improvement of the six-loop results is the specific heat critical exponent of the $XY$ model. Taking into account that the result $\alpha = -0.0127(3)$ from zero-gravity experiment in Ref.\[38\], the BCM six-loop result from ref.\[29\], which is $\alpha = -0.007(3)$ as well as our six-loop resummation in Ref.\[36\] that gives $\alpha = -0.00886$, one can easily realize the discrepancy between expected and so far calculated
results. Even the resummation of the seven-loop $g$-expansion gives the result $\alpha = -0.00859$ which in turn is still far away from the expected result. In view of our seven-loop result $\alpha = -0.012801$ in this work and the mentioned previous results, one can claim that the resummation of the seven-loop $\varepsilon$-expansion in this work is more than important.

While the predictions of the renormalization group at fixed dimensions gives accurate results in three-dimensions [4, 28], the story is different for the two dimensional cases. In two dimensions, the renormalization group at fixed dimensions gives inaccurate results especially for the critical exponents of small values [32, 34]. The reason behind this is the nonanalyticity of the $\beta$-function at the fixed point [29, 33, 34]. The $\varepsilon$-expansion on the other hand might not suffer from this problem [35]. We tested our resummation results in two dimensions and found an overall improvements to our six-loop resummation results in Ref.[36].

Our algorithm while simple gives astonishing results for the critical exponents which are competitive to the results from more sophisticated resummation algorithms, numerical methods as well as conformal field theory. This puts it among the preferred resummation algorithms applied to different problems in physics. A note to be mentioned here is that this work (up to the best of knowledge) represents the first resummation results for the $\varepsilon^7$ series in literature.

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