Algebraic (2, 2)-transformation groups

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Abstract

In this paper we determine all algebraic transformation groups $G$, defined over an algebraically closed field $k$, which operate transitively, but not primitively, on a variety $\Omega$, provided the following conditions are fulfilled. We ask that the (non-effective) action of $G$ on the variety of blocks is sharply 2-transitive, as well as the action on a block $\Delta$ of the normalizer $G_\Delta$. Also we require sharp transitivity on pairs $(X, Y)$ of independent points of $\Omega$, i.e. points contained in different blocks.

Although classifications of imprimitive permutation groups appeared already at beginning of the last century (see [10]) and imprimitive actions play an important role in geometry, the corresponding literature is actually less well-developed than the one concerning primitive groups. For finite groups some classification has been done (see for instance [1], [5] and [11]). In [1] by using wreath products, the best-known construction principle to get imprimitive groups, a classification of finite imprimitive groups, acting highly transitively on blocks and satisfying conditions very common in geometry, is achieved.

The present paper arises with the aim to obtain classifications for infinite imprimitive groups belonging to well-studied categories. We start with an imprimitive algebraic group $G$, over an algebraically closed field $k$, operating on an algebraic variety $\Omega$ of positive dimension in such a way that the induced actions on the set $\overline{\Omega}$ of blocks and on a block $\Delta$ are both sharply 2-transitive. Moreover we ask the group to act sharply transitively on pairs of points lying in different blocks. The latter condition, frequently occurring in geometry (see for instance [2]), avoids a too general context. For the classification we do not need the group actions be bi-regular morphisms but we just ask that the orbit maps be separable morphisms. It turns out that $G$ is the semidirect product of a 3-dimensional unipotent connected group $G_u$ by a 1-dimensional connected torus $T$, both acting on the points of an affine plane over $k$ with a full set of parallel lines as the blocks.

There are two subgroups which play a fundamental role for the classification: the kernel $G_{\overline{\Omega}}$ of the representation on $\overline{\Omega}$ (the so-called inertia subgroup) and its stabilizer $G_{\overline{\Omega}}|_O$ of a fixed point $O$, which turns out to be even the point-wise stabilizer of the block containing $O$. There exists a $G$-invariant transversal $L$ of $G$ with respect to $G_{\overline{\Omega}}|_O$ which is essential for the classification. $L$ is a subgroup precisely if $G_u/\mathfrak{z}(G_u)$ is commutative, in such a case $G_u/\mathfrak{z}(G_u)$ is even a vector group. Fixing the structure of $L$, the classification (see the main theorem) depends on four (not necessarily independent) integer parameters which distinguish the isomorphism class of $G$. But if the char $k$ is positive, then for suitable values of the integer parameters it happens that $L$ could be both a vector group and a non-commutative group.
Throughout the paper, let $G$ denote an algebraic group defined over an algebraically closed field $k$, operating effectively on the points of a variety $\Omega$ of positive dimension. We assume that the orbit maps $g \mapsto g(X)$ are separable morphisms $G \to \Omega$ and $G$ acts transitively with a nontrivial system of imprimitivity $\Pi$. Moreover, putting

- the normalizer $G_{\Delta} := \{ g \in G : g(\Delta) = \Delta \}$ of $\Delta \in \Pi$,
- the centralizer $G_{|\Delta|} := \{ g \in G_{\Delta} : g(X) = X \ \forall X \in \Delta \}$ of $\Delta \in \Pi$,
- the inertia subgroup $G_{|\Pi|} := \{ g \in G : g(\Delta) = \Delta \ \forall \Delta \in \Pi \}$,

we require the following transivities:

1. $G_{\Delta}/G_{|\Delta|}$ acts sharply 2-transitively on $\Delta$,
2. $G/G_{|\Pi|}$ acts sharply 2-transitively on $\Pi$,
3. $G$ acts sharply transitively on $\Lambda := \{(X,Y) \in \Omega^2 : \Delta_X \neq \Delta_Y \}$, where $\Delta_x \in \Pi$ denotes the block containing $Z \in \Omega$.

We call such a triple $G = (G, \Omega, \Pi)$ a $(2,2)$-imprimitive algebraic group. Since the stabilizer of a point is not trivial, conditions 3 and 1 guaranty that the centre of $G$ consists just of the identity. Hence the algebraic group $G$ must be affine.

1. **Proposition:**
   i) Every block $\Delta \in \Pi$ is closed and $G_{\Delta} = G_{|\Pi|}G_X$ for any $X \in \Delta$;
   ii) the inertia subgroup $G_{|\Pi|}$ is closed.

**Proof:** Every block $\Delta \in \Pi$ is a constructible set as the union, for $X \in \Delta$, of two $G_X$-orbits, $\{X\}$ and $\Delta \setminus \{X\}$, so $\Delta$ is closed by Theorem 1.6 in [7]. Then $G_{|\Pi|}$ is the intersection of all closed subgroups $G_{\Delta}$. Finally $G_{\Delta} = G_{|\Pi|}G_X$ follows from the fact that the normal subgroup $G_{|\Pi|}$ acts transitively on $\Delta$. $\square$

2. **Remark:** As orbit maps are separable morphisms $G \to \Omega$, by the universal mapping property we may identify $\Omega$ with the homogeneous space $G/O$ for a fixed stabilizer $G_O = \{ g \in G : g(O) = O \}$, $O \in \Omega$. As well as, in view of Proposition 1, we may identify $\Pi$ with the homogeneous space $G/G_{\Delta}$.

3. **Proposition:** For all $X \in \Omega$ the centralizer $G_{|\Pi|}X = \{ g \in G_{|\Pi|} : g(X) = X \}$ is contained in $G_{|\Delta_X|}$ and $G_{|\Pi|} = G_{|\Pi|}X \times G_{|\Pi|}Y$ for any $(X,Y) \in \Lambda$.

**Proof:** $G_{|\Pi|}X$ acts (effectively and) sharply transitively on the block $\Delta_Y$, the centralizer $G_X\Delta_Y$ being trivial. If blocks contain finitely many points the order of $G_{|\Pi|}X$ is $|\Delta|$. In such a case $G_{|\Pi|}X$ operates non-effectively on $\Delta_X \setminus \{X\}$ with orbits of the same length $\theta$, since $G_{|\Pi|}X_{X,X'} = G_{|\Pi|} \cap G_{|\Delta_X|}$ for any $X' \in \Delta \setminus \{X\}$. But $\gcd(|\Delta| - 1, |\Delta|) = 1$ forces $\theta = 1$.

If blocks contain infinitely many points, $G_{|\Pi|}X$ acts on $\Delta_Y$ as the kernel of the Frobenius group $G_{\Delta_Y}/G_{|\Delta_Y|}$. So $G_{|\Pi|}X$ is a 1-dimensional connected unipotent group by [7] (Theorem 1.10), hence must act trivially on $\Delta_X$ by Proposition 1 in [8]. Therefore in any case $G_{|\Pi|}X < G_{|\Delta_X|}$ and this forces $G_{|\Pi|}X$ to be a normal subgroup of $G_{|\Pi|}$. The last claim follows from the sharply transitivity of $G$ on $\Lambda$. $\square$
4. Proposition:

a) If \( \Omega \) contains infinitely many blocks and every block contains infinitely many points;

b) \( G_\pi \) is the semidirect product of the 1-dimensional connected unipotent subgroup \( G_\pi \) by a 1-dimensional connected torus \( T \);

c) \( G/G_\pi \) is a 2-dimensional Frobenius algebraic group with complement \( \Omega \) isomorphic to \( G_\pi /X \) for any \( X \in \Omega \cap \Delta \).

d) For all \( \Delta \in \Omega \), \( G_\Delta /G_{\Delta_\pi} \) is a 2-dimensional Frobenius algebraic group whose 1-dimensional kernel is isomorphic to \( G_\pi /X \) for any \( X \in \Omega \cap \Delta \).

Proof: The group \( G_\pi /G_{\Delta_\pi} \) acts effectively and sharply transitively on \( \Omega \). Thus \( |\Omega| < \infty \) implies \( |\Delta_\pi| < \infty \) and \( \Omega \) would be of finite cardinality. So infinitely many blocks occur and the kernel of the Frobenius algebraic group \( G/G_\pi \) is a 1-dimensional connected unipotent group ([17], Theorems 1.8 and 1.10) with a 1-dimensional connected torus \( T \) as the complement \( G_{\Delta_\pi}/G_\pi \) isomorphic to \( G_{\pi}/\pi \) ([15], Proposition 1).

Finally the non-trivial factor group \( G_\pi /G_{\Delta_\pi} \) acts transitively on every block \( \Delta \), hence sharply transitive, the group \( G_{\Delta_\pi}/G_{\Delta_\pi} \) being primitive.

5. Proposition: \( G \) is a solvable connected affine group of dimension 4 and \( G \) is the semidirect product of its unipotent radical \( G_u \) by the torus \( T \). Moreover the centre \( z(G_u) \) of \( G_u \) is contained in \( G_\pi \) and for any \( X \in \Omega \) we have \( G_{\pi} \simeq z(G_u) \times G_{\pi}/X \).

Proof: As \( G_{\pi} \) is a 2-dimensional connected unipotent group by Propositions 4.d and 3 and \( G/G_{\pi} \) is a connected solvable 2-dimensional group by Proposition 4.c, the unipotent radical \( G_u \) has codimension 1 and acts transitively on \( \Omega \). We have \( z(G_u) < G_{\pi} \) since \( z(G_u) \) centralizes each \( G_{\pi}/X \). Finally \( z(G_u) \) is transitively on every block \( \Delta \), hence sharply transitive, the group \( G_{\Delta}/G_{\Delta_\pi} \) being primitive.

6. Remark: If we denote by \( g_u \) and \( g_y \) the images of \( g \in G \) under the projections \( G_u \rightarrow G_u \) and \( G_u \rightarrow T \), respectively, the mapping \( \pi : G \rightarrow G_u \) satisfies \( \pi(g) = g_uG_{\pi}/O \) turns out to be a separable morphism of algebraic varieties. The fibres of \( \pi \) are precisely the cosets \( gG_u \), so \( gG_u \rightarrow gG_{\pi}/O \) yields an isomorphism \( G/G_u \rightarrow G_u/G_{\pi}/O \). So we may take the homogeneous space \( G_u/G_{\pi}/O \) as \( \Omega \) and

\[
\left( g, hG_{\pi}/O \right) \mapsto ghg^{-1}G_{\pi}/O \quad (g \in G, h \in G_u)
\]

as the action of \( G_u \) on \( \Omega \) since \( (g_1g_2)_u = (g_1)_u(g_2)_u(g_1)_u^{-1} \). In particular \( \Omega \) is a 2-dimensional (irreducible affine) variety with

\[
\Omega = \bigcup_{g \in G_u} \Delta_{g(\pi)} = \bigcup_{g \in G_u} \psi(g)G_{\pi}/O.
\]

\[\Sigma\] Let \( G = U \times T \) be a semidirect product of an n-dimensional connected unipotent group \( U \) by a 1-dimensional connected torus \( T \). According to Serre [14], p. 172, the group \( U \) has a representation on the affine space \( k^n \) in such a way the subspaces

\[
U_i = \left\{ (x_1, \ldots, x_n) \in k^n : x_{i+1} = \ldots = x_n = 0 \right\}
\]

are normal subgroups of \( G \), the product is given by \( (x_1, \ldots, x_n)(y_1, \ldots, y_n) = (x_1 + y_1, \psi_1(x_2, \ldots, x_n, y_2, \ldots, y_n), \ldots, x_{n-1} + y_{n-1} + \psi_{n-1}(x_n, y_n), x_n + y_n) \).
for suitable polynomials $\psi_j \in k[x_j, \ldots, x_n, y_{j+1}, \ldots, y_n]$, and the automorphism of $U$ induced by an element $\tau \in T$ maps $(x_1, \ldots, x_n)$ to
\[
(a_1^\tau x_1 + \psi_1^{(\tau)}(x_2, \ldots, x_n), \ldots, a_n^\tau x_n)
\]
with $a_\tau \in k^\ast$, an element depending bi-regularly on $\tau$, the map $\psi_j^{(\tau)}$ a morphism $U_n/U_j \to U_j/U_{j-1}$ and $e_j$ a fixed integer.

7. **Lemma:** Let $n \geq 2$. Then for any $\tau \in T$ the morphism $\varphi_n^{(\tau)}$ yields a group homomorphism $U_n/U_{n-1} \to U_{n-1}/U_{n-2}$. Moreover we may take as $\psi_{n-1}$

- the zero polynomial, if $U_n/U_{n-2}$ is a vector group,
- $b \sum_{i=1}^{p-1} \frac{1}{p} \psi_n^{(\tau)}(x_i y_n^{(p-1)p^r})$, if $U_n/U_{n-2}$ is a group homomorphism $U_n/U_{n-1} \to U_{n-1}/U_{n-2}$ and $e_{n-1} = e_n \deg(\psi_{n-1})$.

**Proof:** We may take as $\psi_{n-1}(x_n, y_n)$ (see for instance Lemma 7.1 in [6])

\[
\begin{align*}
0, & \quad \text{if } U_n/U_{n-2} \text{ is a vector group,} \\
- b \sum_{i=1}^{p-1} \frac{1}{p} \psi_n^{(\tau)}(x_i y_n^{(p-1)p^r}), & \quad \text{if } U_n/U_{n-2} \text{ is Abelian but not a vector group,} \\
- b x_n y_n^{p^r}, & \quad \text{if } U_n/U_{n-2} \text{ is not commutative,}
\end{align*}
\]

for some non-negative integers $r, s$ with $r < s$ and a non-zero scalar $b$, that may assumed 1 thanks to the isomorphism
\[
(\ldots, x_{n-1}, x_n)U_{n-2} \cong (\ldots, bx_{n-1}, x_n)U_{n-2}.
\]

Now the fact that $\tau$ operates on $U_n$ as an automorphism group implies that the co-boundary
\[
\delta^1(\varphi_n^{(\tau)})(x_n, y_n) = \varphi_n^{(\tau)}(y_n) - \varphi_n^{(\tau)}(x_n) + \varphi_n^{(\tau)}(x_n)
\]

is one of the following

\[
\begin{align*}
0, & \quad \text{if } U_n/U_{n-2} \text{ is a vector group;} \\
- b \sum_{i=1}^{p-1} \frac{1}{p} \psi_n^{(\tau)}(x_i y_n^{(p-1)p^r}), & \quad \text{if } U_n/U_{n-2} \text{ is a group homomorphism } U_n/U_{n-1} \to U_{n-1}/U_{n-2}. \\
\end{align*}
\]

In the latter case the fact that $\psi_{n-1}$ is not a co-boundary forces each $a_\tau$ to be a root of the polynomial $T e_n \deg(\psi_{n-1}) - T e_n - 1$ and this forces the condition $e_{n-1} = e_n \deg(\psi_{n-1})$. As a consequence $\delta^1(\varphi_n^{(\tau)})$ must be in any case the zero polynomial, which means that $\varphi_n^{(\tau)}$ yields a group homomorphism $U_n/U_{n-1} \to U_{n-1}/U_{n-2}$. \qed

8. **Remark:** It follows from [3] that the action of a 1-dimensional torus on a 2-dimensional connected unipotent group $U$ may be given by diagonal $(2 \times 2)$-matrices with entries in $k$. The following lemma, which generalizes both the lemma on p. 109 in [12] and Corollary 2.9 in [7], shows that this can be done without destroying the group structure of $U$.

9. **Lemma:** Let $\varphi_n^{(\tau)} = \cdots = \varphi_n^{(\tau)} = 0$ and assume $\varphi_1^{(\tau)}$ is a group homomorphism $U_n/U_{n-1} \to U_1$. Then there exists a bi-regular section $\sigma : U_n/U_{n-1} \to U_n$ such that $\sigma(x_n U_{n-1}) = (f(x_n), 0, \ldots, 0, x_n)$ with $\delta^1(f) = 0$ and $\sigma(U_n/U_{n-1})$ invariant under $T$. 

Proof: We may suppose \( \varphi^{(r)} \in k[x_n] \) with \( \varphi^{(r)}(x_n) = \sum_{i \in I, j \in J} c_{ij} a_i^r x_n^j \) for some finite sets \( I \) and \( J \) of integers with
\[
I = \begin{cases} 
\{1\}, & \text{if } \text{char } k = 0; \\
a \text{ finite set of } p\text{-powers}, & \text{if } \text{char } k = p > 0.
\end{cases}
\]
The product \( \tau_1 \tau_2 \) of two elements of \( T \) gives
\[
\varphi^{(r_1 r_2)}(x_n) = a_i^{r_1} \varphi^{(r_2)}(x_n) + \varphi^{(r_1)}(a_n^r x_n),
\]
hence for each \( i \in I \)
\[
\sum_{j \in J} c_{ij} a_i^r \tau_2 = \sum_{j \in J} c_{ij} \left( a_i^{r_1} a_2^j + a_i^{r_2} a_2^j \right).
\]
By comparing we infer that just \( c_{i, e_1} \) and \( c_{i, e_n} \) can occur as nonzero entries. So
\[
c_{i, e_1} a_i^{r_2} + c_{i, e_n} a_i^{r_2} = c_{i, e_1} \left( a_i^{r_1} a_2^j + a_i^{r_2} a_2^j \right) + c_{i, e_n} \left( a_i^{r_1} a_2^j + a_i^{r_2} a_2^j \right),
\]
or \( c_{i, e_1} + c_{i, e_n} = 0 \). Therefore \( \varphi^{(r)}(x_n) = \sum_{i \in I} c_{i, e_1} \left( a_i^{r_1} - a_i^{r_2} \right) x_n^i \) and
\[
\left\{ - \sum_{i \in I} c_{i, e_1} x_n^i, 0, \ldots, 0, x_n \right\} : x_n \in k
\]
turns out to be \( T \)-invariant with \( \delta^1 : \sum_{i \in I} c_{i, e_1} x_n^i \mapsto 0 \). □

10. Lemma: Let \( n \geq 3 \). Assume the centralizer \( \mathfrak{C}_{U_{n-1}}(v) \) of \( v \) in \( U_{n-1} \) satisfies the condition \( \mathfrak{C}_{U_{n-1}}(v) = U_{n-2} \) mod \( U_{n-3} \) for all \( v \in M \). Then the automorphism \( \rho_v \) of \( U_{n-1}/U_{n-3} \) induced by conjugation by \( v \) maps
\[
\left( \ldots, x_{n-2}, x_{n-1}, 0 \right) U_{n-3} \mapsto \left( \ldots, x_{n-2} + u^k x_{n-1}, x_{n-1}, 0 \right) U_{n-3}
\]
with \( h \) and \( k \) \( p \)-powers if \( \text{char } k = p > 0 \), \( h = k = 1 \) otherwise.

Proof: As \( U_{n-1}/U_{n-2} \leq \mathfrak{s}(U_{n-2}/U_{n-3}) \), we have
\[
\rho_v : \left( \ldots, x_{n-2}, x_{n-1}, 0 \right) U_{n-3} \mapsto \left( \ldots, x_{n-2} + \sigma(u, x_{n-1}), x_{n-1}, 0 \right) U_{n-3}
\]
for some additive polynomial \( \sigma \in k[ u, x_{n-1}] \), which turns out to be monomial because \( \mathfrak{C}_{U_{n-1}}(v) = U_{n-2} \) mod \( U_{n-3} \) forces \( \rho_v \) to act fixed-point freely on \( U_{n-2}/U_{n-3} \). Thus \( \sigma(u, x_{n-1}) = cu^k x_{n-1}^k \) for some integers \( h, k \) and scalar \( c \in k^* \) that we may assume 1, up to the isomorphism \( \left( \ldots, x_{n-2}, x_{n-1}, x_{n} \right) U_{n-3} \mapsto \left( \ldots, x_{n-2}, c^{-h} x_{n-1}, x_{n} \right) U_{n-3} \). Clearly the integers \( h \) and \( k \) have to satisfy the claimed conditions. □

In the remaining part of the paper we ask the torus \( T \) to act sharply transitively on \( U_n/U_{n-1} \). This means
\[
e_n = \begin{cases} 
1, & \text{if } \text{char } k = 0; \\
a \text{ } p\text{-power}, & \text{if } \text{char } k = p > 0.
\end{cases}
\]

§3. Now we go back to the the \((2,2)\)-imprimitive algebraic group \( G = (G, \Omega, \Omega) \). This section is devoted to the case where the 2-dimensional factor group \( G_u/\mathfrak{s}(G_u) \) is commutative.

11. Proposition: \( G_u/\mathfrak{s}(G_u) \) is a vector group.
Proposition: Assume $\sigma$ is a group homomorphism $\sigma : g \mapsto [g, x]$ of the commutator subgroup of $G_n$ into $\ker G$, then each $\ker G_n$ is a connected subgroup of $G_n$ with dimension $\geq 2$. So $(g, x, y) = 1$ require that $\ker G_n$ is the commutator subgroup of $G_n$, hence that each commutator morphism $\sigma_g : x \mapsto [g, x]$ is a group homomorphism $G \to \ker G$, whose kernel must have dimension $\geq 2$. So $(g, x, y) = 1$, a contradiction since $\bigcap_{g \in G_n} \ker G_n = \ker G_n$. □

12. Proposition: There exists a $T$-invariant normal subgroup $L$ of $G_n$ containing the centre $\mathfrak{z}(G_n)$ and $G_n = L \times G_n$. Proof: By [12] (Lemma on p. 109) the $T$-invariant subgroup $G_T$ of $G_n$ has a $T$-invariant complement, say $L/\mathfrak{z}(G_n)$ for some $T$-invariant normal subgroup $L$ of $G_n$ containing $\mathfrak{z}(G_n)$. □

According to the notation of §2 we may take $U_1 = \mathfrak{z}(G_n), U_2 = G_n, U_3 = G_n$. In addition we may choose

$$G_{[\pi]} = \{0, x_2, 0 : x_2 \in k\},$$

the subgroup $G_{[\pi]}_O$ being $T$-invariant. Observing that the normal subgroup $L$ of $G$ is not contained in $U_2$, we may also put

$$L = \{(x_1, 0, x_3) : x_1, x_3 \in k\}.$$ 

Thus the product $(x_1, 0, x_3)(y_1, 0, y_3)$ of two elements of $L$ is given by

$$(x_1 + y_1 + \beta(x_3, y_3), 0, x_3 + y_3)$$

and by Lemma 7 we may take

$$\beta(x_3, y_3) = \begin{cases} 0, & \text{if } L \text{ is a vector group,} \\ \sum_{i=1}^{r-s} \frac{1}{p}(\frac{r}{s}) \frac{x_1^i y_3^{p-i}}{y_3^i}, & \text{if } L \text{ is commutative and exponent } p^2, \\ x_3^p, & \text{if } L \text{ is not commutative,} \end{cases}$$

for some nonnegative integers $r, s$ with $r < s$. Besides an element $v = (0, 0, u) \in L$ moves the block $\Delta_0$ to a different block $\Delta_{v(O)}$ (Remark 6), so $\nu$ centralizes no element in $G_{[\pi]}_O$, the intersection $G_{[\pi]}_O \cap G_{[\pi]}_L(O)$ being trivial. Then Lemma 10 applies and, up to the isomorphism $(x_1, x_2, x_3) \mapsto (x_1, e^{\frac{\pi}{2}x_2}, x_3)$, we may claim

13. Proposition: The product $(x_1, x_2, x_3)(y_1, y_2, y_3)$ in $G_n$ may be defined through

$$\left(x_1 + y_1 + \frac{h_1}{2}. x_2^2 + \beta(x_3, y_3), x_2 + y_2, x_3 + y_3\right),$$

where $\beta$ is given by (2) and each exponent $h_i$ is a $p$-power in case $\kappa = p > 0$, $h_i = 1$ otherwise. □

As we observed in Remark 8, there is no loss of generality if we assume the action of the torus $T$ on the affine plane $L$ given by diagonal $(2 \times 2)$--matrices. But $G_{[\pi]}_O$ occurs as a further $T$-invariant subgroup of dimension 1, so the diagonal action of each $\tau \in T$ extends to the whole group $G_n$ via

$$(x_1, x_2, x_3) \mapsto (a_2^x x_1, a_2^x x_2, a_2^x x_3).$$
The value of the exponent $c_3$ was given by (1), whereas the possible relationship occurring between $e_1$ and $e_3$ was stated in Lemma 7. Now by imposing that $\tau$ is a group homomorphism we find

$$e_1 = e_2 h_2 + e_3 h_3$$

with $h_i$ arising from the product of $G_u$ given in Proposition 13.

§4. Assume now the factor group $G_u/G_u$ to be not commutative. This requires $\text{char} k = p > 0$ and we are going to see that even $p > 2$ holds.

Referring to the notation of §2 we may take again $U_3 = G_u$, $U_2 = G[\overline{\pi}]$, $U_1 = G(\overline{\pi})$ and

$$G[\overline{\pi}]_O = \{ (0, x_2, 0) : x_2 \in k \}.$$

Also, by Lemma 7,

$$\psi_2 : (x_3, y_3) \mapsto x_3^{p^m} y_3^m,$$

for some integer $p$-powers $p^m$ and $p^n$ such that $m < n$. Furthermore, looking at Remark 6, we see that an element $v = (0, 0, x_3)$ moves the block $\Delta_v(\overline{\pi})$ to a different block $\Delta_v(\overline{\pi})$. So $v$ does not centralize any element of $G[\overline{\pi}]_O$ because the intersection $G[\overline{\pi}]_O \cap G[\overline{\pi}]_{v(\overline{\pi})}$ is assumed to be trivial. So Lemma 10 applies and, up to an isomorphism, we may assume that the automorphism induced on $G[\overline{\pi}]$ by an element $(0, 0, x_3)$ maps

$$(y_1, y_2, 0) \mapsto (y_1 + y_2^2 x_3^3, y_2, 0)$$

for suitable integer $p$-powers $h_i = y_i^3$, $i = 2, 3$. If we represent $G_u$ as a non-central extension of the vector group $G[\overline{\pi}]$ by $G_u/G[\overline{\pi}]$, using the cross section $(x_1, x_2, x_3) G[\overline{\pi}] \mapsto (0, 0, x_3)$, the product $(x_1, x_2, x_3)(y_1, y_2, y_3)$ of two elements in $G_u$ can also be given by

$$(x_1 + y_1 + y_2^2 x_3^{h_3}, x_2 + y_2 + x_3^{p^m} y_3^m, x_3 + y_3)$$

with $\beta$ in $k[x_3, y_3]$ such that $\beta(0, y_3) = \beta(x_3, 0) = 0$ and $G_u$ is determined by taking

$$\psi_1(x_1, x_2, y_1, y_2) = y_2^2 x_3^3 + \beta(x_3, y_3).$$

Now associative law forces the polynomial

$$\delta^2(\beta)(z_1, z_2, z_3) = \beta(z_1, z_2) + \beta(z_1 + z_2, z_3) - \beta(z_2, z_3) - \beta(z_1, z_2 + z_3)$$

to be

$$\delta^2(\beta)(z_1, z_2, z_3) = z_1^{p^m} \frac{z_2^{p^2 + m}}{z_3^{p^2 + n}}$$

and we can state

14. Proposition: A necessary and sufficient condition in order that $G_u$ can be constructed as an extension of $G_u$ by a non-commutative connected unipotent group is that there exists a polynomial $\beta \in k[x_3, y_3]$ satisfying (5) with $\beta(0, y_3) = \beta(x_3, 0) = 0$. In such a case we may take $\psi_1(x_1, x_2, y_2, y_3) = y_2^2 x_3^3 + \beta(x_3, y_3)$. □

The crucial question now is under what conditions such a polynomial $\beta$ there exists. Using a universal property of the operator $\delta^2$ we have

$$\sum_{\pi \in S_n} \text{sign}(\pi) \delta^2(\beta)(z_{\pi(1)}, z_{\pi(2)}, z_{\pi(3)}) = 0$$
and this, in view of \((5)\), is equivalent to
\[
l_3 - l_2 = m, \quad \text{or} \quad l_3 - l_2 = n. \tag{6}\]

Assume now char \(k = 2\) and \(l_2 + m > 0\) and denote by \(\beta_i\) the homogeneous component of \(\beta\) of degree \(i\). As in our case the operator \(\delta^2\) is additive, \((5)\) says that \(\delta^2(\beta_i) = \delta^2(\beta_k)\), where \(k = 2^{l_2}(2^m + 2^n)\) with either \(q = m\), or \(q = n\) according as whether \(l_3 - l_2 = m\), or \(l_3 - l_2 = n\). Let
\[
\beta_k(y_1, y_2) = \sum_{i=0}^k a_i y_1^{k-i} y_2^i.
\]

Then \((5)\) becomes
\[
\sum_{i=0}^k a_i (z_2^{k-i} y_1^i + (z_1 + z_2)^{k-i} z_2^i + z_2^{k-i} y_1^i + z_1^{k-i}(z_2 + z_3)^i) = z_1^{2^m + q} z_2^{2^{l_2 + m}} z_3^{2^n + q}.
\]

Deriving this identity with respect to \(z_1\) and evaluating at \((0, y_1, y_2)\) we obtain
\[
a_{k-1} y_1^{k-1} + \frac{\partial}{\partial y_1} \beta_k(y_1, y_2) + a_{k-1}(y_1 + y_2)^{k-1} = 0, \tag{7}
\]

whereas deriving with respect to \(z_2\) and evaluating at \((y_1, y_2, 0)\) we get
\[
a_1(y_1 + y_2)^{k-1} + a_1 y_2^{k-1} + \frac{\partial}{\partial y_2} \beta_k(y_1, y_2) = 0. \tag{8}
\]

As char \(k = 2\), \(\frac{\partial}{\partial y_1} \beta_k(y_1, y_2)\) and \(\frac{\partial}{\partial y_2} \beta_k(y_1, y_2)\) are polynomials in \(y_1^2\) and \(y_2^2\), respectively, the identities \((7)\) and \((8)\) force \(a_{k-1} = a_1 = 0\), hence
\[
\frac{\partial}{\partial y_1} \beta_k(y_1, y_2) = \frac{\partial}{\partial y_2} \beta_k(y_1, y_2) = 0
\]

and this yields \(a_i = 0\) for all odd \(i\). Thus we may do the substitution \((z_1, z_2, z_3) \mapsto (z_1^2, z_2, z_3^2)\), hence \((z_1, z_2, z_3) \mapsto (z_1^{2^m + q}, z_2^{2^{l_2 + m}}, z_3^{2^n + q})\) by iterating the process. So \((5)\) turns into
\[
\delta^2(\gamma)(z_1, z_2, z_3) = z_1^{2^m + q} z_2 z_3^{2^n - q} \tag{9}
\]

with \(\gamma(y_1^{2^m + q}, y_2^{2^{l_2 + m}}) = \beta(y_1, y_2)\). Let \(\gamma_i\) be the homogeneous component of degree \(t := 1 + 2^m - q + 2^n - q\) of \(\gamma\) and let \(\gamma_i(y_1, y_2) = \sum_{i=0}^t b_{i} y_1^{t-i} y_2^i\). Then \((9)\) says that
\[
\sum_{i=0}^t b_{i} \left(z_1^{t-i} z_2^i + (z_1 + z_2)^{t-i} z_1^i + z_2^{t-i} z_3^i + z_1^{t-i}(z_2 + z_3)^i\right) = z_1^{2^m + q} z_2 z_3^{2^n - q}.
\]

Likewise above we obtain
\[
\begin{cases}
    b_{t-1} y_1^{t-1} + \frac{\partial}{\partial y_1} \gamma_i(y_1, y_2) + b_{t-1}(y_1 + y_2)^{t-1} = (1 - \epsilon) y_1 y_2^{2^n - q}, \\
    b_1(y_1 + y_2)^{t-1} + b_1 y_2^{t-1} + \frac{\partial}{\partial y_2} \gamma_i(y_1, y_2) = 0,
\end{cases}
\]

where either \(\epsilon = 0\), or \(\epsilon = 1\) according as whether \(q = m\), or \(q = n\). So Euler’s identity says that \(t \gamma_i(y_1, y_2)\) is the polynomial
\[
b_{t-1} y_1^{t-1}(y_1 + y_2)^{t-1} + (1 - \epsilon) y_1 y_2^{2^n - q} + b_1 y_2(y_1 + y_2)^{t-1} + y_2^{t-1}
\]
or the polynomial
\[ b_{l-1} \left( y_1^{1+2^n-m} y_2^{e-m} + y_1^{1+2^n-m} y_2^{2^n-m} + y_1 y_2^{2^n-m+2^e-m} + (1-e) y_1^2 y_2^{a-n-m} \right) + \]
\[ +b_1 \left( y_1^{a-n-m+2^e-m} y_2 + y_1^{2^n-m} y_2^{1+2^e-m} + y_1^{2^n-m} y_2^{1+2^e-m} \right). \]

Let \( q = m \). Then we have the polynomial identity
\[ (b_{l-1} + b_1) \left( y_1^{1+2^n-m} y_2^{e-m} + y_1^{1+2^n-m} y_2^{2^n-m} + y_1 y_2^{2^n-m+2^e-m} + (1-e) y_1^2 y_2^{a-n-m} \right) = 0 \]
which asks \( b_{l-1} = b_1 = 0 \) and, consequently, \( \frac{\partial}{\partial y_1} \gamma_t(y_1, y_2) = y_1 y_2^{a-n-m} \), a contradiction. Let \( q = n \). Then
\[ \gamma_t(y_1, y_2) = b_{l-1} y_1 y_2^{a-n-m} + b_1 y_1^{2^n-m+2^e-m} y_2^1 \]
and \( \delta_1(\gamma_t)(x_1, x_2, x_3) = 0 \). This contradicts (9) and \( p \neq 2 \) follows.

Actually, if \( p \neq 2 \) the polynomials
\[ \beta(x_3, y_3) = \begin{cases} \frac{1}{2} x_3^{p^{l_3+m}} y_3^{l_2+n} & \text{if } l_3 - l_2 = m; \\
\frac{1}{2} x_3^{p^{l_3+m}} y_3^{l_2+m} + \frac{1}{2} x_3^{p^{l_2+m}} y_3^{2^e} & \text{if } l_3 - l_2 = n. \end{cases} \] (10)
satisfy the conditions required in Proposition 14. Any other polynomial satisfying the conditions of Proposition 14 differs from (10) for a co-cycle satisfying the conditions required in Proposition 14. Any other polynomial \( \varphi \) depending only on \( x_3 \) because \( G_{[T]} \) is \( T \)-invariant. By imposing that \( \tau \) operates as a group homomorphism we obtain first
\[ e_2 = e_3 (p^n + p^n) \quad \text{and} \quad e_1 = e_3 h_2 + e_3 h_3 = e_3 (p^{l_3} + p^{l_2+m} + n) \] (11)
but also
\[ a_1^{e_1} \beta(x_3, y_3) - \beta(a_1^{e_1} x_3, a_1^{e_1} y_3) + a_1^{e_1} \kappa(x_3, y_3) - \kappa(a_1^{e_1} x_3, a_1^{e_1} y_3) = \delta_1(\varphi^{(\tau)})(x_3, y_3), \]
or
\[ a_1^{e_1} \kappa(x_3, y_3) - \kappa(a_1^{e_1} x_3, a_1^{e_1} y_3) = \delta_1(\varphi^{(\tau)})(x_3, y_3), \] (12)
because \( e_1 = e_3 \deg \beta \) in view of (11). Since \( e_3 \) is a \( p \)-power and \( p > 2 \), the integer \( e_1 \) can be, by (11), neither a \( p \)-power, nor the sum of two \( p \)-powers. Thus Theorem 4.6 in [4] guarantees that \( \kappa \) is a co-boundary, i.e. \( \kappa = \delta_1(g) \) for some polynomial \( g \in k[T] \), that may be eliminated using the substitution \( x_1 \mapsto x_1 - g(x_3) \). Such a replacement yields \( \delta_1(\varphi^{(\tau)})(x_3, y_3) = 0 \), i.e. \( \varphi^{(\tau)} \) is additive, and we may assume the action of \( T \) given by diagonal matrices, as Lemma 9 claims.
§ 5. Now we collect all information achieved in the previous sections and classify $G$ according to the structure of the transversal $L$. With the aid of Remark 6 we can state:

15. **Main Theorem:** Every $(2, 2)$-imprimitive algebraic group $G = (G, \Omega, \Pi)$ can be constructed on the affine variety $k^3 \times k^*$ as follows:

- define the unipotent radical $G_u$ on the affine space $k^3$ through the product

  $$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1 + \psi_1(x_3, y_2, y_3), x_2 + y_2 + \psi_2(x_3, y_3), x_3 + y_3),$$

  where either

  \[ \psi_2(x_3, y_3) = 0 \text{ and } \psi_1(x_3, y_2, y_3) = y_2^h_3 x_3 + \beta(x_3, y_3), \]

  or

  \[ \psi_2(x_3, y_3) = x_3^m y_3^n, \]

  with $p = \text{char } k > 2$ and $m, n$ non-negative integers such that $m < n$, and $\psi_1(x_3, y_2, y_3)$ as above with

  \[ \beta(x_3, y_3) = \begin{cases} 
  \frac{1}{2} y_3^p l_3^3 y_3^{p_2 + m}, & \text{if } l_3 - l_2 = m; \\
  x_3^{p_3 + p_2 + m} y_3^{l_3} + \frac{1}{2} x_3^{p_2 + m} y_3^{l_3}, & \text{if } l_3 - l_2 = n;
  \end{cases} \]

- leave $a \in k^*$ operate on $k^3$ via

  $$(u_1, u_2, u_3) \mapsto (a^{e_1} u_1, a^{e_2} u_2, a^{e_3} u_3)$$

  where

  - $e_1 = e_2 h_2 + e_3 h_3$, but also $e_1 = e_3 \text{deg } \beta$ if $\beta$ is not the zero polynomial;
  - $e_2 = e_3 \frac{\text{deg } \beta - h_2}{h_3}$, if $\beta$ is not the zero polynomial;
  - $e_3$ is a positive integer $p$-power in case $k = p > 0$, $e_3 = 1$ otherwise;

- identify $\Omega$ with the affine plane $k^2$ with the parallel lines $y = k$ and the set of blocks. Then a transformation $(u_1, u_2, u_3, a) \in G$ moves the point $(x, y) \in \Omega$ to the point

  $$(u_1 + a e_2 h_2 + e_3 h_3 x + \psi_1(u_3, 0, a^{e_3} y), u_3 + a^{e_3} y)) \quad \Box$$

The canonical representation of $G$ given through the main theorem depends on the polynomial $\beta$ as well as on the integer parameters $e_2, e_3, h_2, h_3$, though $h_2$ and $h_3$ could already be determined by $\beta, e_2$ and $e_3$. Labelling $G$ as $G^{(e_2, e_3, h_2, h_3)}$, we ask whether an isomorphism

$$\Phi : G^{(e_2, e_3, h_2, h_3)} \rightarrow G^{(e_2', e_3', h_2', h_3')}$$

between two $(2, 2)$-imprimitive algebraic groups with different parameters exists. Of course we may assume the same sets of points and blocks for both groups, so $\Phi$ is
a pair $(\Phi_1, \Phi_2)$ with $\Phi_1$ a group isomorphism $G_\beta^{(e_2, e_3, h_2, h_3)} \rightarrow G_\beta^{(e_2', e_3', h_2', h_3')}$ and $\Phi_2 : k^2 \rightarrow k^2$ a bijective morphism of the affine plane $k^2$ transforming horizontal lines into horizontal lines such that

$$\Phi_2(g(P)) = \Phi_1(g)(\Phi_2(P)) \quad (g \in G_\beta^{(e_2, e_3, h_2, h_3)}, P \in k^2).$$

As $G_u$ is transitive on $\Omega$, up to inner automorphisms we may assume that $\Phi_2$ leaves the point $O = (0, 0)$ of $\Omega$ fixed, hence the line $y = 0$ stable. Then the stabilizer of $O$, as well as the normalizer and centralizers of $\Delta_G$ correspond; in particular

$$\Phi_1((0, u_2, 0)) = (b_2 u_2, 0) \quad (b_2 \in k^*),$$

$$\Phi_1((u_1, 0, 0)) = (b_1 u_1, 0) \quad (b_1 \in k^*),$$

and, moreover,

$$\Phi_1((0, 0, u_3)) = (f_2(u_3), f_2(u_3), b_3 u_3) \quad (b_3 \in k^*),$$

$$\Phi_2((x, y)) = (b_1 x + f_1(y), b_3 y),$$

for suitable polynomials $f_j \in k[T]$ such that

$$\delta^j(f_2)(x_3, y_3) = b_2 \psi_2(x_3, y_3) - \psi_2'(b_3 x_3, b_3 y_3) \quad (x_3, y_3 \in k),$$

$$\delta^j(f_1)(x_3, y_3) = b_1 \psi_1(x_3, 0, y_3) - \psi_1'(b_3 x_3, f_2(y_3), b_3 y_3) \quad (x_3, y_3 \in k).$$

Manifestly tori fixing the point $O$ correspond under $\Phi_1$; in particular we have $\Phi_1(T_\beta^{(e_2, e_3, h_2, h_3)}) = T_\beta^{(e_2', e_3', h_2', h_3')}$ since tori are conjugated under $G_u$. This means

$$(u_1, u_2, u_3) \Phi_1^*(\tau) = (a_{+}^{e_3} u_1, a_{+}^{e_3} u_2, a_{+}^{e_3} u_3),$$

with $\varepsilon = \pm 1$. The identity $\Phi_1((0, 0, u_3)^\tau) = (\Phi_1((0, 0, u_3))^\tau)$ and the first part of (14) yield $\varepsilon = 1$, $e_3 = e_3'$ and $f_j(a_{+}^{e_3} u_3) = a_{+}^{e_3} f_j(u_3)$, $j = 1, 2$, whereas $\Phi_1((u_1, u_2, 0)^\tau) = (\Phi_1((u_1, u_2, 0))^\tau)$ and (13) give $e_1 = e_1'$ and $e_2 = e_2'$. So the polynomials $f_j$ must be monomials and consequently, in case $f_j \neq 0$,

$$e_j = e_j \deg(f_j) \quad (j = 1, 2).$$

Therefore $f_j(T) = d_j T^\frac{e_j}{d_j}$, $d_j \in k$, $j = 1, 2$. Furthermore imposing the condition $\Phi_1((0, 0, u_3) \Phi_1((0, u_2, 0)) = \Phi_1((0, 0, u_3)(0, u_2, 0))$ we obtain $b_2^{h_2} b_3^{h_3} u_3^{h_3} v_2^{h_2} = b_1^{h_3} v_2^{h_2}$, i.e. $(h_2, h_3) = (h_2, h_3)$ and

$$b_1 = b_2^{h_2} b_3^{h_3}.$$  

(17)

So the first step is achieved:

**16. Proposition:** Let $G_\beta^{(e_2', e_3', h_2', h_3')}$ and $G_\beta^{(e_2, e_3, h_2, h_3)}$ isomorphic as algebraic permutation groups. Then

$$(e_2', e_3', h_2', h_3') = (e_2, e_3, h_2, h_3).$$

Theorem 4.6 in [4] says that the first of (15) occurs precisely if

$$\delta^j(f_2) = \psi_2 - \psi_2' = 0.$$  

(18)
Also the fact that \( e_1 = e_3 \deg(\beta) \) if \( \beta \) is not the zero polynomial confines matters to examine the case where \( \text{char } k = p > 0, \beta = 0 \) and either \( \beta'(x,y) = \sum_{i=1}^{p-1} \frac{1}{x_i} x_i^{t_i} y^{(p-1)t_i} \) or \( \beta'(x,y) = x^{p} y^{3} \); by (16) we have \( \deg \beta' = \frac{p}{p+1} = \deg f_1 \) in case \( d_1 \neq 0 \). Then the second identity of (15) turns into

\[
\delta^1(f_1)(x,y) = -\beta'(b_1 x_3, b_3 y_3) - b_3^{r_3} x_3 f_2(y_3)^{h_2}
\]  

and again Theorem 4.6 in [4] excludes the possibility that \( f_2 \) is the zero polynomial. Then \( f_2 \) is an additive monomial by (18) and (16) forces \( e_2 \) to be a \( p \)-power. Thus, in view of the main theorem, both \( e_1 \) and \( \deg \beta' \), are the sum of two \( p \)-powers. So just the following two possibilities can occur: either \( \beta'(x,y) = x_3^{p} y_3^{3} \), or \( \text{char } k = 2 \) and \( \beta'(x,y) = x_3^{p} y_3^{3} \). Thus the main theorem gives either \( e_2 b_2 + e_3 h_3 = e_3 (p^r + p^s) \), or \( e_2 b_2 + e_3 h_3 = e_3 2^{r+s+1} \), which means that the pair of \( p \)-powers \((h_2, h_3)\) is one of the following

1. \((h_2, h_3) = (\frac{p}{p+1}, p^s)\);
2. \((h_2, h_3) = (\frac{p}{p+1}, p^s, p^r)\);
3. \((h_2, h_3) = (\frac{p}{p+1}, 2^s, 2^r)\).

As the right side of (19) must be a co-boundary, (20.1) gives, (20.2) and (20.3), lead respectively to

1. \( d_1 = b_3^{r_3} p^s + p^r = b_3^{r_3} 2^s p^r \), hence \( d_2 = b_3^{r_3} \); 
2. \( d_1 = 0 \) and \( b_3^{r_3} p^s + p^r = -b_3^{r_3} 2^s p^r \), hence \( d_2 = -b_3^{r_3} \);
3. \( b_3^{r_3+1} = b_3^{r_3} 2^s 2^r \), hence \( d_2 = b_3^{r_3} \).

Now it is straightforward calculation to verify that, for any \( b_1, b_2, d_3 \in k \), the maps

\[
\begin{align*}
G_0^{(e_2, e_3, \frac{p}{p+1}, p^s, p^r)} & \rightarrow G_0^{(e_2, e_3, \frac{p}{p+1}, p^s, p^r)}, \\
\left((u_1, u_2, u_3, a) \rightarrow (b_2^{r_2} b_3^{r_3} u_1 + (b_3 u_3) p^s + p^r, b_2 u_2 + (b_3 u_3) 2^s, b_3 u_3, a) \right) \\
G_0^{(e_2, e_3, \frac{p}{p+1}, p^r, p^s)} & \rightarrow G_0^{(e_2, e_3, \frac{p}{p+1}, p^r, p^s)}, \\
\left((u_1, u_2, u_3, a) \rightarrow (b_2^{r_2} b_3^{r_3} u_1, b_2 u_2 - (b_3 u_3) 2^s, b_3 u_3, a) \right) \\
G_0^{(e_2, e_3, \frac{p}{p+1}, 2^r, 2^s)} & \rightarrow G_0^{(e_2, e_3, \frac{p}{p+1}, 2^r, 2^s)}, \\
\left((u_1, u_2, u_3, a) \rightarrow (b_2^{r_2} b_3^{r_3} u_1 + d_1 u_3^{r_3+1}, b_2 u_2 + (b_3 u_3) 2^s, b_3 u_3, a) \right)
\end{align*}
\]

are group isomorphisms in correspondence to the values (20.1) of the pair of \( p \)-powers \((h_2, h_3)\). Manifestly such isomorphisms supply isomorphisms for the associated permutation groups. Summing up we have

17. Theorem: The integer parameters \( e_2, e_3, h_2, h_3 \) and the polynomial \( \beta \) determine uniquely the isomorphism class of the \( (2,2) \)-imprimitive algebraic group \( G \), except the cases where the pair \((h_2, h_3)\) takes one of the (integer) values (20.1) which produces the corresponding isomorphisms (21.1). \( \square \)
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