Fast Computation of the Circular Map

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Abstract This paper presents a new numerical implementation of Koebe’s iterative method for computing the circular map of bounded and unbounded multiply connected regions of connectivity \( m \). The computational cost of the presented method is \( O(m^2n + mn \log n) \) where \( n \) is the number of nodes in the discretization of each boundary component. The accuracy and efficiency of the method presented are demonstrated by several numerical examples. These examples include regions with high connectivity, a region with close-to-touching boundaries, and a region with piecewise smooth boundaries.

Keywords Numerical conformal mapping · Generalized Neumann kernel · Koebe’s method

Mathematics Subject Classification Primary 30C30; Secondary 65R20

1 Introduction

Numerous canonical regions have been considered in the literature for conformal mapping of multiply connected regions in the extended complex plane \( \mathbb{C} \cup \{\infty\} \). Thirty-nine canonical slit regions have been catalogued by Koebe [21]. A novel method for computing the conformal mapping from bounded and unbounded multiply connected regions onto these 39 canonical regions has been presented in [27–30]. The method
has also been used to compute the conformal mapping onto the canonical region obtained by removing rectilinear slits from an infinite strip [32]. The method is based on a uniquely solvable boundary integral equation with the generalized Neumann kernel. Only the right-hand side of the integral equation is different from one canonical region to another. A fast method for solving the integral equation with the generalized Neumann kernel is given in [31,32]. For multiply connected regions of connectivity $m$, the method requires $O(mn \log n)$ operations where $n$ is the number of nodes in the discretization of each boundary component. The method presented in [28,32] can be used to map bounded and unbounded simply connected regions (i.e., $m = 1$) onto the unit disk and the exterior unit disk, respectively, in $O(n \log n)$ operations.

An important canonical region which has not been considered in [27–30,32] is the multiply connected circular region, i.e., a region all of whose boundaries are circles. The canonical multiply connected circular region is important from physical and computational points of view. For example, recently, analytic formulas for several problems in fluid mechanics are given for multiply connected circular regions [3–8,10]. Circular regions are also ideal regions for using Fourier series and FFT [2,11,38,39].

For computing the conformal mapping onto the canonical multiply connected circular region, the known numerical methods are only iterative methods [2,11,12,16–20,23–25,39,40,44]. Koebe’s iterative method is the first numerical method for computing the conformal mapping from multiply connected regions onto the circular region [20]. The method goes back to 1910 [20]. Koebe’s method can be used for bounded and unbounded multiply connected regions. For numerical implementations of Koebe’s method, see [16,23–25,44]. Other numerical methods for circular regions are Wegmann’s iterative method [39] and Fornberg’s iterative method [2,11] which can be used to compute the inverse conformal mapping from the circular region onto the multiply connected regions. These methods are based on using trigonometric interpolation and FFT. A comparison between these two methods is given in [2]. For multiply connected regions of connectivity $m$, if $n$-point trigonometric interpolation is used, the computational cost of these methods is $O(m^2 n^2)$ [2].

In this paper, we present a new numerical implementation of Koebe’s iterative method based on a boundary integral equation with the generalized Neumann kernel. The method provides us with the boundary values of the conformal mapping and its derivative. The values of the mapping function for interior points are calculated using the Cauchy integral formula. Cauchy integral formula can be also used to calculate the interior values of the derivative of the mapping function as well as the interior values of the inverse mapping function. The computational cost of the presented method is $O(m^2 n + mn \log n)$ where $m$ is the connectivity of the region and $n$ is the number of nodes in the discretization of each boundary component.

The remainder of this paper is organized as follows: the circle map is defined in Sect. 2. In Sect. 3, we present a fast numerical method for computing the conformal mapping of simply connected regions. This fast method with Koebe’s method will be used to compute the circular map of bounded and unbounded multiply connected regions in Sects. 4 and 5, respectively. The computational complexity of the method presented is discussed in Sect. 6. In Sect. 7, we present seven numerical examples. A short conclusion is given in Sect. 8.

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2 The Circular Map

Let $G$ be a multiply connected region of connectivity $m$ in the extended complex plane $\mathbb{C} \cup \{\infty\}$. The region $G$ can be bounded or unbounded. For bounded $G$, we assume that $\alpha$ is a given point in $G$. See Fig. 1 (left). If $G$ is unbounded, then we assume that $\infty \in G$ and $\alpha$ is a given point in the complement of $G$. See Fig. 2 (left). Let $G$ have the boundary

$$\Gamma = \partial G = \bigcup_{j=1}^{m} \Gamma_j$$

where $\Gamma_1, \ldots, \Gamma_m$ are closed smooth Jordan curves. The orientation of $\Gamma$ is such that $G$ is always on the left of $\Gamma$. The curve $\Gamma_j$ is parametrized by a $2\pi$-periodic twice continuously differentiable complex function $\eta_j(t)$ with non-vanishing first derivative $\eta_j'(t) \neq 0$ for $t \in J_j = [0, 2\pi]$. The total parameter domain $J$ is the disjoint union of the $m$ intervals $J_1, \ldots, J_m$. We define a parametrization of the whole boundary $\Gamma$ as the complex function $\eta$ defined on $J$ by

$$\eta(t) = \begin{cases} 
\eta_1(t), & t \in J_1, \\
\vdots \\
\eta_m(t), & t \in J_m.
\end{cases}$$

(1)

For bounded $G$, there exists a conformal mapping $w = \omega(z)$ from the bounded region $G$ onto a bounded circular region $\Omega$ (see Fig. 1 (right)). The external circle is the unit circle, i.e., its centre is $w_m = 0$ and its radius is $r_m = 1$. For the inner circles $C_j$ for $j = 1, 2, \ldots, m - 1$, the centres $w_j$ and the radii $r_j$ are unknown and should be determined. When the conformal mapping $w = \omega(z)$ is normalized by

$$\omega(\alpha) = 0, \quad \omega'(\alpha) > 0,$$

(2)

For bounded $G$, there exists a conformal mapping $w = \omega(z)$ from the bounded region $G$ onto a bounded circular region $\Omega$ (see Fig. 1 (right)). The external circle is the unit circle, i.e., its centre is $w_m = 0$ and its radius is $r_m = 1$. For the inner circles $C_j$ for $j = 1, 2, \ldots, m - 1$, the centres $w_j$ and the radii $r_j$ are unknown and should be determined. When the conformal mapping $w = \omega(z)$ is normalized by

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For bounded $G$, there exists a conformal mapping $w = \omega(z)$ from the bounded region $G$ onto a bounded circular region $\Omega$ (see Fig. 1 (right)). The external circle is the unit circle, i.e., its centre is $w_m = 0$ and its radius is $r_m = 1$. For the inner circles $C_j$ for $j = 1, 2, \ldots, m - 1$, the centres $w_j$ and the radii $r_j$ are unknown and should be determined. When the conformal mapping $w = \omega(z)$ is normalized by

$$\omega(\alpha) = 0, \quad \omega'(\alpha) > 0,$$

(2)
then the conformal mapping \( \omega \) as well as the circular region \( \Omega \) are uniquely determined by the region \( G \) \cite{18,39}.

For unbounded \( G \), there exists a conformal mapping \( w = \omega(z) \) from the unbounded region \( G \) onto an unbounded circular region \( \Omega \) (see Fig. 2 (right)). The centres \( w_j \) and the radii \( r_j \) of the circles \( C_j \) for \( j = 1, 2, \ldots, m \) are unknown and should be determined. When the conformal mapping \( w = \omega(z) \) is normalized by the condition near infinity

\[
\omega(z) = z + O \left( \frac{1}{z} \right),
\]

then the conformal mapping \( \omega \) as well as the circular region \( \Omega \) are uniquely determined by the region \( G \) \cite{2,11,14,18,39}.

For both bounded and unbounded \( G \), the region \( \Omega \) is called a circular region and the mapping function \( \omega \) is called the circular map of \( G \) \cite[p. 488]{18}. The centres and the radii of the circles are called the parameters of the circular region \( \Omega \).

Koebe’s iterative method is a classical method for computing the circular map for bounded and unbounded multiply connected regions \( G \). For bounded \( G \), each iteration \( k \) of Koebe’s method requires computing the conformal mapping from an unbounded simply connected region onto the exterior unit disk \( m - 1 \) times and computing the conformal mapping from a bounded simply connected region onto the unit disk once. For unbounded \( G \), each iteration \( k \) of Koebe’s method requires computing the conformal mapping from an unbounded simply connected region onto the exterior unit disk \( m \) times. The successive computational region becomes gradually more circular. In the process, we compute approximate values of the centres and the radii of the circles.

A convergence proof and the rate of convergence for Koebe’s method can be found in \cite{12,18}. Let \( \omega^k \) be the approximate circular map obtained in the \( k \)th iteration of Koebe’s method. Then, we have the following theorem from \cite[p. 503]{18} and \cite[p. 148]{38}.
Theorem 1 For unbounded region $G$, there exists a constant $c > 0$ such that for $k = 1, 2, 3, \ldots$ and for all $\eta \in \Gamma$,

$$\|\omega^k(\eta) - \omega(\eta)\|_\infty \leq cp^k,$$

where

$$p = \hat{p}^4 \quad \text{and} \quad \hat{p} = \max_{1 \leq i, j \leq m, i \neq j} \frac{|r_i + r_j|}{|w_i - w_j|}.$$ (5)

It is clear that Koebe’s method always converges since $0 < p < 1$. However, the numerical results of this paper indicate that this value of $p$ is not the optimal estimate of the rate of convergence of Koebe’s iterative method. The actual convergence rate is much better than $p$.

The rate of convergence $p$ in the above theorem depends on prior knowledge of the parameters of the circular region $\Omega$. Recently, Andreev and McNicholl [1] have computed a convergence rate only from information about the original region $G$. However, computing the convergence rate given in [1] requires a lot of calculations so it will not be computed in this paper.

3 The Conformal Mapping of Simply Connected Regions

Let $S$ be a simply connected region in the extended complex plane $\mathbb{C} \cup \{\infty\}$. The region $S$ can be bounded or unbounded. For bounded $S$, we assume that $\alpha$ is a given point in $S$ (see Fig. 3 (left)). If $S$ is unbounded, then we assume that $\infty \in S$ and $\alpha$ is a given point in the complement of $S$ (see Fig. 4 (left)). The boundary $L = \partial S$ is assumed to be a closed smooth Jordan curve. The orientation of $L$ is such that $S$ is always on the left of $L$, i.e., $L$ is counterclockwise oriented for bounded $S$ and clockwise oriented for unbounded $S$.

In this section, we shall present a fast numerical method for computing the conformal mapping $\Phi$ from the bounded simply connected region $S$ onto the unit disk $D^+$ (see Fig. 3). The method can also be used for computing the conformal mapping $\Psi$ from the unbounded simply connected region $S$ onto the exterior unit disk $D^-$ (see Fig. 4). This method with Koebe’s iterative method will be used in Sects. 4 and 5 to compute the circular map of bounded and unbounded multiply connected regions.
3.1 The Generalized Neumann Kernel

The curve $L$ is parametrized by a $2\pi$-periodic twice continuously differentiable complex function $\zeta(t)$ with non-vanishing first derivative $\zeta'(t) \neq 0$ for $t \in [0, 2\pi]$. We define a complex-valued function $A$ on $L$ by

$$A(t) = \begin{cases} 
\zeta(t) - \alpha, & \text{if } S \text{ is bounded}, \\
1, & \text{if } S \text{ is unbounded}.
\end{cases}$$

(6)

The generalized Neumann kernel formed with $A$ and $\zeta$ is defined by [26, 41]

$$N(s, t) = \frac{1}{\pi} \text{Im} \left( \frac{A(s)}{A(t)} \frac{\zeta'(t)}{\zeta(t) - \zeta(s)} \right).$$

(7)

The kernel $N$ is continuous with

$$N(t, t) = \frac{1}{\pi} \left( \frac{1}{2} \text{Im} \frac{\zeta''(t)}{\zeta'(t)} - \text{Im} \frac{A'(t)}{A(t)} \right).$$

(8)

We also define a kernel

$$M(s, t) = \frac{1}{\pi} \text{Re} \left( \frac{A(s)}{A(t)} \frac{\zeta'(t)}{\zeta(t) - \zeta(s)} \right).$$

(9)

The kernel $M$ is singular. However, it can be written as

$$M(s, t) = -\frac{1}{2\pi} \cot \frac{s - t}{2} + M_1(s, t)$$

(10)

with a continuous kernel $M_1$ which takes on the diagonal the values

$$M_1(t, t) = \frac{1}{\pi} \left( \frac{1}{2} \text{Re} \frac{\zeta''(t)}{\zeta'(t)} - \text{Re} \frac{A'(t)}{A(t)} \right).$$

(11)
Thus, the integral operator

\[ N_\mu(s) = \int_0^{2\pi} N(s, t) \mu(t) \, dt, \quad s \in [0, 2\pi], \]  

(12)
is a Fredholm integral operator and the operator

\[ M_\mu(s) = \int_0^{2\pi} M(s, t) \mu(t) \, dt, \quad s \in [0, 2\pi], \]  

(13)
is a singular integral operator.

For more details on the generalized Neumann kernel, see [26–28,41,42]. The simply connected region \( S \) is a special case of the region considered in [28]. Thus, we have the following theorem from [28].

**Theorem 2** For a given 2\( \pi \)-periodic Hölder continuous function \( \gamma \), there exists a unique constant \( h \) and a unique 2\( \pi \)-periodic Hölder continuous function \( \phi \) such that

\[ A(t)f(\zeta(t)) = \gamma(t) + h + i\phi(t), \quad t \in [0, 2\pi], \]  

(14)
are boundary values of an analytic function \( f \) in \( G \) with \( f(\infty) = 0 \) for unbounded \( S \). The function \( \phi \) is the unique solution of the integral equation

\[ (I - N)\phi = -M\gamma \]  

(15)
and the constant \( h \) is given by

\[ h = [(I - N)\gamma - M\phi]/2. \]  

(16)

For the multiply connected region considered in [28], the function \( h \) in (16) is a piecewise constant function. For the above simply connected region \( S \), the function \( h \) in (16) is a constant function. So, we can find simple formulas for computing \( h \) instead of (16) (see e.g., [41, Sect. 11]). The function \( g \) defined on \( S \) for bounded \( S \) by

\[ g(z) = (z - \alpha)f(z) - h \]  

(17a)
and for unbounded \( S \) by

\[ g(z) = f(z) - h, \]  

(17b)
is analytic in \( S \) with \( g(\alpha) = -h \) for bounded \( S \) and \( G(\infty) = -h \) for unbounded \( S \). It follows from (14) and (17) that the boundary values of the function \( g \) are given by

\[ g(\zeta(t)) = \gamma(t) + i\phi(t), \quad t \in [0, 2\pi]. \]  

(18)
Thus, by the Cauchy integral formula, the value of the unknown constant \( h \) is given for bounded \( S \) by

\[ h = -g(\alpha) = -\frac{1}{2\pi i} \int_{\Gamma} \frac{\gamma + i\phi}{\eta - \alpha} \, d\eta. \]  

(19a)
For unbounded $S$, $\alpha$ is in the exterior of $S$ and $L$ is clockwise oriented. Thus, by the Cauchy integral formula [13, p. 2], the value of the unknown constant $h$ is given by

$$h = -g(\infty) = \frac{1}{2\pi i} \int_L \gamma + i\phi \frac{\eta - \alpha}{\eta} \, d\eta.$$  \hspace{1cm} (19b)

### 3.2 The Bounded Simply Connected Region

The boundary values of the conformal mapping $\Phi$ from the bounded simply connected region $S$ onto the unit disk $D^+$ are given by

$$\Phi(\xi(t)) = e^{i\theta(t)} \hspace{1cm} (20)$$

where $\theta(t)$ is the boundary correspondence function of the mapping function $\Phi$. The function $\theta(t) - t$ is $2\pi$-periodic function and $\theta'(t) > 0$ for all $t \in [0, 2\pi]$. By differentiating both sides of (20) with respect to the parameter $t$, we obtain

$$\zeta'(t)\Phi'(\xi(t)) = i\theta'(t)e^{i\theta(t)}. \hspace{1cm} (21)$$

With the normalization

$$\Phi(\alpha) = 0, \hspace{0.5cm} \Phi'(\alpha) > 0, \hspace{1cm} (22)$$

the mapping function $\Phi$ is unique. The function $w = \Phi(z)$ can be written as

$$\Phi(z) = c(z - \alpha)e^{(z - \alpha)f(z)}, \hspace{1cm} (23)$$

where $f$ is analytic function on $S$ and $c = \Phi'(\alpha) > 0$. Hence

$$\log \Phi(z) = \log c + \log(z - \alpha) + (z - \alpha)f(z). \hspace{1cm} (24)$$

Then, in view of (20), we obtain

$$i\theta(t) = \log c + \log(\xi(t) - \alpha) + A(t)f(\xi(t)). \hspace{1cm} (25)$$

Hence, the boundary values of the analytic function $f$ are given by

$$A(t)f(\xi(t)) = \gamma(t) + h + i(\mu(t) + \theta(t)) \hspace{1cm} (26)$$

where $h = -\log c$ and the real functions $\gamma$, $\mu$ are given by

$$\gamma(t) + i\mu(t) = -\log(\xi(t) - \alpha). \hspace{1cm} (27)$$

Hence, it follows from Theorem 2 that the unknown function $\phi = \mu(t) + \theta(t)$ is the unique solution of the integral equation (15) where the functions $\gamma$ and $\mu$ are given by (27). The unknown constant $h$ is given by (19a). Thus the boundary correspondence function $\theta$ is given by $\theta = \phi - \mu$ and the constant $c$ is given by $c = e^{-h}$.
3.3 The Unbounded Simply Connected Region

The boundary values of the mapping function $\Psi$ from the unbounded simply connected region $S$ onto the exterior unit disk $D^-$ satisfy

$$\Psi(\zeta(t)) = e^{-i\theta(t)}. \quad (28)$$

The function $\theta(t)$ is the boundary correspondence function of the mapping function $\Psi$ where $\theta(t) - t$ is $2\pi$-periodic and $\theta'(t) > 0$ for all $t \in [0, 2\pi]$. By differentiating both sides of (28) with respect to $t$, we obtain

$$\zeta'(t)\Psi'(\zeta(t)) = -i\theta'(t)e^{-i\theta(t)}. \quad (29)$$

With the normalization

$$\Psi(\infty) = \infty, \quad \Psi'(\infty) > 0, \quad (30)$$

the mapping function $\Psi$ is unique. The function $w = \Psi(z)$ can be written as

$$\Psi(z) = c(z - \alpha)e^{-f(z)}, \quad (31)$$

where $f$ is analytic function on $S$ with $f(\infty) = 0$ and $c = \Psi'(\infty) > 0$. Hence

$$\log \Psi(z) = \log c + \log(z - \alpha) - f(z). \quad (32)$$

Then, in view of (28), we obtain

$$-i\theta(t) = \log c + \log(\zeta(t) - \alpha) - A(t)f(\zeta(t)). \quad (33)$$

Hence, the boundary values of the function $f$ are given by

$$A(t)f(\zeta(t)) = \gamma(t) + h + i(\mu(t) + \theta(t)) \quad (34)$$

where $h = \log c$ and

$$\gamma(t) + i\mu(t) = \log(\zeta(t) - \alpha). \quad (35)$$

Thus, Theorem 2 implies that the function $\phi = \mu(t) + \theta(t)$ is the unique solution of the integral equation (15) where the functions $\gamma$ and $\mu$ are given by (35). The constant $h$ is given by (19b). Then, $\theta = \phi - \mu$ and $c = e^h$.

3.4 The Fast Numerical Method

In the numerical results presented in this paper, the integral equation (15) is solved via the MATLAB function $\text{FBIE}$ presented in [31]. The integral equation (15) is
discretized by the Nyström method with the trapezoidal rule with \( n \) equidistant collocation points,

\[
t_i = (i - 1) \frac{2\pi}{n}, \quad i = 1, 2, \ldots, n, \tag{36}
\]
to obtain an \( n \times n \) linear system which is solved by the MATLAB function \texttt{gmres}. Each iteration of the function \texttt{gmres} requires a matrix-vector product which is computed in \( O(n) \) operations by the MATLAB function \texttt{zfmm2dpart} in the MATLAB toolbox FMMLIB2D developed by Greengard and Gimbutas [15]. Computing the right-hand side of the integral equation makes use of the MATLAB functions \texttt{fft} and \texttt{ifft} which requires \( O(n \log n) \) operations. Hence, the complexity of solving the integral equation (15) is \( O(n \log n) \) operations.

For function \texttt{zfmm2dpart}, we assume that \( \text{iprec} = 5 \) which means that the tolerance of the FMM is \( 0.5 \times 10^{-15} \). For the function \texttt{gmres}, we choose the parameters \( \text{restart} = 10, \text{gmrestol} = 10^{-14}, \) and \( \text{maxit} = 10 \), which means that the GMRES method is restarted every 10 inner iterations, the tolerance of the GMRES method is \( 0.5 \times 10^{-14} \), and the maximum number of outer iterations of GMRES method is 10. See [15,31,32,37] for more details.

The GMRES method will converge significantly faster since the eigenvalues of the discretization matrix are clustered around 1. In fact, for simply connected regions and the function \( A \) defined by (6), the generalized Neumann kernel \( N \) has only real eigenvalues in the interval \([-1, 1]\) where \(-1\) is a simple eigenvalue [34]. Thus, for sufficiently large \( n \), the discretization matrix of the integral equation (15) has only real eigenvalues on the interval \((0, 2]\) with \(2\) as a simple eigenvalue and the other eigenvalues are clustered around \(1\) [33,34]. For a nearly circular region \( S \), the eigenvalues become very strongly clustered around \(1\). When \( L \) is the unit circle, the generalized Neumann kernel becomes

\[
N(s, t) = -\frac{1}{2\pi}
\]

which has only the eigenvalues 0 and \(-1\). Thus, the discretization matrix of the integral equation (15) has only two eigenvalues 2 and 1 where 2 is a simple eigenvalue and 1 has the algebraic multiplicity \( n - 1 \).

Solving the integral equation with generalized Neumann kernel (15) using the fast method presented in [31] does not require the second derivative \( \eta''(t) \) of the parametrization of the boundary. In view of (20), (21), (28), and (29), computing the boundary values of the mapping functions \( \Phi(\xi(t)), \Psi(\xi(t)) \) and their derivatives \( \Phi'(\xi(t)), \Psi'(\xi(t)) \) requires calculating the values of the boundary correspondence function \( \theta(t) \) and its first derivative \( \theta'(t) \) at the points (36). The values of the boundary correspondence function \( \theta(t) \) at the points (36) are calculated from the solution of the integral equation (15). Then, we represent the \(2\pi\)-periodic function \( \theta(t) - t \) on \([0, 2\pi]\) by the interpolating trigonometric polynomial of degree \(n/2\),

\[
\theta(t) - t = a_0 + \sum_{j=1}^{n/2} a_j \cos jt + \sum_{j=1}^{n/2-1} b_j \sin jt, \tag{37}
\]
that interpolates $\theta(t) - t$ at the $n$ equidistant points (36) (see [40, p. 364]). The coefficients $a_0, a_1, \ldots, a_n, b_1, \ldots, b_n$ are calculated by the FFT in $O(n \log n)$ operations. Then the function $\theta'(t) - 1$ is approximated by the trigonometric polynomials of degree $n/2$,

$$
\theta'(t) - 1 = a'_0 + \sum_{j=1}^{n/2} a'_j \cos j t + \sum_{j=1}^{n/2-1} b'_j \sin j t,
$$

where

$$
a'_0 = a'_n = 0, \quad a'_j = j b_j, \quad b'_j = -j a_j, \quad j = 1, 2, \ldots, n - 1.
$$

The values of the function $\theta'(t)$ at the $n$ points (36) are calculated by the FFT.

For bounded region $S$, by obtaining the functions $\theta$ and $\theta'$, we obtain the boundary values of the mapping function $\Phi$ and its derivative $\Phi'$. The functions $\Phi$ and $\Phi'$ are analytic in $S$. Hence, the values of the functions $\Phi(z)$ and $\Phi'(z)$ for interior points $z \in S$ can be computed by the Cauchy integral formula

$$
\Phi(z) = \frac{1}{2\pi i} \int_{L} \frac{\Phi(\zeta)}{\zeta - z} \, d\zeta,
$$

$$
\Phi'(z) = \frac{1}{2\pi i} \int_{L} \frac{\Phi'(\zeta)}{\zeta - z} \, d\zeta.
$$

For unbounded region $S$, obtaining the functions $\theta$ and $\theta'$ yields the boundary values of the mapping function $\Psi$ and its derivative $\Psi'$. The functions $\frac{\Psi(z)}{z - \alpha}$ and $\Psi'(z)$ are analytic in $S$ and have the same value $c = e^h$ at $\infty$ where the constant $h$ is given by (19b). By the Cauchy integral formula, the values of the functions $\Psi(z)$ and $\Psi'(z)$ for interior points $z \in S$ can be computed by [13, p. 2]

$$
\Psi(z) = (z - \alpha) \left[ c + \frac{1}{2\pi i} \int_{L} \frac{\Psi(\zeta)}{\zeta - \alpha} \frac{1}{\zeta - z} \, d\zeta \right],
$$

$$
\Psi'(z) = c + \frac{1}{2\pi i} \int_{L} \frac{\Psi'(\zeta)}{\zeta - z} \, d\zeta.
$$

For the computational cost of computing the boundary values of the mapping functions $\Phi, \Psi$ and their derivatives $\Phi', \Psi'$: solving the integral equation (15) via the MATLAB function FBIE requires two FMMs and three FFTs for computing the right-hand side of the integral equation, and one FMM for each iteration of the GMRES method. Computing the function $\theta'$ requires two FFTs. Thus, the method requires two FMMs, five FFTs, and one FMM for each iteration of the GMRES method. Since one application of the FMM requires $O(n)$ operations and one application of the FFT requires $O(n \log n)$ operations, the complexity of computing the functions $\theta$ and $\theta'$ is $O(n \log n)$ operations. Thus, the complexity of computing the boundary values of the mapping functions $\Phi, \Psi$ and their derivatives $\Phi', \Psi'$ is $O(n \log n)$ operations.
4 Koebe’s Method for Bounded Multiply Connected Regions

In this section, based on the results of the previous section, a numerical implementation of Koebe’s iterative method for computing the circular map \( w = \omega(z) \) of the bounded multiply connected region \( G \) (see Fig. 1) will be described. We shall present a method for computing the boundary values of the circular map,

\[ \xi(t) = \omega(\eta(t)), \quad t \in J, \]

the boundary values of its derivative,

\[ \xi'(t) = \eta'(t)\omega'(\eta(t)), \quad t \in J, \]

and the parameters \( w_j, r_j, j = 1, 2, \ldots, m - 1 \), of the circular region \( \Omega \) where \( \eta(t) \) is the parametrization of the boundary \( \Gamma \) of \( G \) and is given by (1).

4.1 Initializations

At the beginning, we set

\[ C_{i,0}^0 = \Gamma_i \quad \text{for} \quad i = 1, 2, \ldots, m. \]

The curve \( C_{i,0}^0 \) is parametrized by \( \xi_{i,0}^0(t) \) which is defined by

\[ \xi_{i,0}^0(t) = \eta_i(t), \quad t \in J_i, \quad i = 1, 2, \ldots, m. \]

Hence

\[ \frac{d}{dt} \xi_{i,0}^0(t) = \eta_i'(t), \quad t \in J_i, \quad i = 1, 2, \ldots, m. \]

We assume also that \( w_{i,0}^0 \) is a given point inside \( C_{i,0}^0 \) for \( i = 1, \ldots, m - 1 \) and \( w_m^0 = \alpha \). Hence, initial values of the boundary values of the circular map are given by

\[ \omega^0(\eta(t)) = \begin{cases} \xi_{1,0}^0(t), & t \in J_1, \\ \vdots \\ \xi_{m,0}^0(t), & t \in J_m. \end{cases} \]

4.2 Iterations

For \( k = 1, 2, 3, \ldots \), where \( k \) denotes the iteration number, we repeat the following three steps:
4.2.1 Step I: The Internal Curves

For \( j = 1, 2, \ldots, m - 1 \), let \( \Psi_{k,j} \) be the conformal mapping from the exterior region of the curve \( C_{j}^{k-1,j-1} \) onto the exterior unit disk. Then

\[
C_{j}^{k-1,j} = \Psi_{k,j}(C_{j}^{k-1,j-1})
\]

is the unit circle. The curve \( C_{j}^{k-1,j} \) is parametrized by \( \xi_{j}^{k-1,j}(t) \) which is defined as the boundary values of the conformal mapping \( \Psi_{k,j} \), i.e.,

\[
\xi_{j}^{k-1,j}(t) = \Psi_{k,j}(\xi_{j}^{k-1,j-1}(t)), \quad t \in J_{j}.
\]

The derivative of the function \( \xi_{j}^{k-1,j}(t) \) is given by

\[
\frac{d}{dt} \xi_{j}^{k-1,j}(t) = \Psi_{k,j}'(\xi_{j}^{k-1,j-1}(t)) \frac{d}{dt} \xi_{j}^{k-1,j-1}(t), \quad t \in J_{j}.
\]

The boundary values of the mapping function \( \Psi_{k,j} \) and its derivative \( \Psi_{k,j}' \) can be computed using the method presented in Sect. 3.3.

The smooth Jordan curves \( C_{i}^{k-1,j-1}, i = 1, 2, \ldots, m, i \neq j \), are external to the curve \( C_{j}^{k-1,j-1} \). Thus, the function \( \Psi_{k,j} \) maps the curves \( C_{i}^{k-1,j-1} \) onto smooth Jordan curves

\[
C_{i}^{k-1,j} = \Psi_{k,j}(C_{i}^{k-1,j-1}), \quad i = 1, 2, \ldots, m, \quad i \neq j,
\]

external to \( C_{j}^{k-1,j} \). For \( i = 1, 2, \ldots, m \) such that \( i \neq j \), the curve \( C_{i}^{k-1,j} \) is parametrized by

\[
\xi_{i}^{k-1,j}(t) = \Psi_{k,j}(\xi_{i}^{k-1,j-1}(t)), \quad t \in J_{i},
\]

where

\[
\frac{d}{dt} \xi_{i}^{k-1,j}(t) = \Psi_{k,j}'(\xi_{i}^{k-1,j-1}(t)) \frac{d}{dt} \xi_{i}^{k-1,j-1}(t), \quad t \in J_{i}.
\]

Since \( \xi_{i}^{k-1,j-1}(t), t \in J_{i} \), are in the exterior region of \( C_{j}^{k-1,j-1} \), then the values of the function \( \Psi_{k,j} \) and its first derivative at the points \( \xi_{i}^{k-1,j-1}(t) \), i.e., \( \Psi_{k,j}(\xi_{i}^{k-1,j-1}(t)) \) and \( \Psi_{k,j}'(\xi_{i}^{k-1,j-1}(t)) \), can be computed using the Cauchy integral formula as explained in (41) and (42).

Finally, we set

\[
w_{j}^{k-1,j} = 0.
\]
For $i = 1, 2, \ldots, m$, $i \neq j$, the point $w_{i}^{k-1,j-1}$ inside the curve $C_{i}^{k-1,j-1}$ in the exterior region of the curve $C_{j}^{k-1,j-1}$ will be mapped by the function $\Psi_{k,j}$ into the point $\Psi_{k,j}(w_{i}^{k-1,j-1})$ inside the curve $C_{i}^{k-1,j}$ in the exterior of the circle $C_{j}^{k-1,j}$. The value of the function $\Psi_{k,j}$ at the point $w_{i}^{k-1,j-1}$ can be computed by the Cauchy integral formula. We define

$$w_{i}^{k-1,j} = \Psi_{k,j}(w_{i}^{k-1,j-1}), \quad i = 1, 2, \ldots, m, \ i \neq j.$$  

### 4.2.2 Step II: The External Curve

Let $\Phi_{k}$ be the conformal mapping from the interior region of the curve $C_{m}^{k-1,m-1}$ onto the unit disk. Then

$$C_{m}^{k-1,m} = \Phi_{k}(C_{m}^{k-1,m-1})$$

is the unit circle. The curve $C_{m}^{k-1,m}$ is parametrized by $\xi_{m}^{k-1,m}(t)$ which is defined as the boundary values of the conformal mapping $\Phi_{k}$, i.e.,

$$\xi_{m}^{k-1,m}(t) = \Phi_{k}(\xi_{m}^{k-1,m-1}(t)), \quad t \in J_{m}.$$  

The derivative of the function $\xi_{m}^{k-1,m}(t)$ is given by

$$\frac{d}{dt} \xi_{m}^{k-1,m}(t) = \Phi'_{k}(\xi_{m}^{k-1,m-1}(t)) \frac{d}{dt} \xi_{m}^{k-1,m-1}(t), \quad t \in J_{m}.$$  

The boundary values of the mapping function $\Phi_{k}$ and its derivative $\Phi'_{k}$ can be computed using the method presented in Sect. 3.2.

The smooth Jordan curves $C_{i}^{k-1,m-1}, i = 1, 2, \ldots, m - 1$, are internal to the curve $C_{m}^{k-1,m-1}$. Thus, the function $\Phi_{k}$ maps the curves $C_{i}^{k-1,m-1}$ onto smooth Jordan curves

$$C_{i}^{k-1,m} = \Phi_{k}(C_{i}^{k-1,m-1}), \quad i = 1, 2, \ldots, m - 1,$$

internal to $C_{m}^{k-1,m}$. For $i = 1, 2, \ldots, m - 1$, the curve $C_{i}^{k-1,m}$ is parametrized by

$$\xi_{i}^{k-1,m}(t) = \Phi_{k}(\xi_{i}^{k-1,m-1}(t)), \quad t \in J_{i},$$

where

$$\frac{d}{dt} \xi_{i}^{k-1,m}(t) = \Phi'_{k}(\xi_{i}^{k-1,m-1}(t)) \frac{d}{dt} \xi_{i}^{k-1,m-1}(t), \quad t \in J_{i}.$$  

Since $\xi_{i}^{k-1,m-1}(t), t \in J_{i}, i = 1, 2, \ldots, m - 1$, are in the interior region of $C_{m}^{k-1,m-1}$, then the values of the function $\Phi_{k}$ and its derivative at the points $\xi_{i}^{k-1,m-1}(t)$, i.e.,
\( \Phi_k(\xi_i^{k-1,m-1}(t)) \) and \( \Phi'_k(\xi_i^{k-1,m-1}(t)) \), can be computed using the Cauchy integral formula as explained in (39) and (40).

Then, we set

\[
w_m^{k-1,m} = 0.
\]

For \( i = 1, 2, \ldots, m - 1 \), the point \( w_i^{k-1,m-1} \) inside the curve \( C_i^{k-1,m-1} \) in the interior region of the curve \( C_m^{k-1,m-1} \) is mapped by the function \( \Phi_k \) into the point \( \Phi_k(w_i^{k-1,m-1}) \) inside the curve \( C_j^{k-1,m} \) in the interior of the circle \( C_j^{k-1,m} \). We define

\[
w_i^{k-1,m} = \Phi_k(w_i^{k-1,m-1}), \quad i = 1, 2, \ldots, m - 1,
\]

where \( \Phi_k(w_i^{k-1,m-1}) \) is computed by the Cauchy integral formula.

4.2.3 Step III: Update and Conditions of Convergence

Let \( w = \omega^k(z) \) be the approximate circular map obtained in the \( k \)th iteration. Then the boundary values of \( \omega^k \) are given by

\[
\omega^k(\eta(t)) = \begin{cases} 
\xi_1^{k-1,m}(t), & t \in J_1, \\
\vdots \\
\xi_m^{k-1,m}(t), & t \in J_m.
\end{cases}
\] (43)

By obtaining the boundary values of \( \omega^k \), we test the convergence of the method. We stop the iteration if

\[
\|\omega^k - \omega^{k-1}\|_\infty < \varepsilon \quad \text{or} \quad k > \text{Max},
\] (44)

where \( \varepsilon \) is a given tolerance and \( \text{Max} \) is the maximum number of iterations allowed.

If the condition (44) is not satisfied, we set

\[
C_i^{k,0} = C_i^{k-1,m}, \quad i = 1, \ldots, m.
\]

For \( i = 1, 2, \ldots, m \), the curve \( C_i^{k,0} \) is parametrized by

\[
\xi_i^{k,0}(t) = \xi_i^{k-1,m}(t), \quad t \in J_i.
\]

Then, we set \( k = k + 1 \) and repeat Steps I–III.

If the method converges, then we consider the bounded multiply connected region bounded by the circles \( C_1^{k-1,m}, \ldots, C_m^{k-1,m} \), as the circular region \( \Omega \). The boundaries of \( \Omega \) are then given by

\[
C_i = C_i^{k-1,m}, \quad i = 1, \ldots, m.
\]
The centre $w_i$ and the radius $R_i$ of the circle $C_i$ are approximated by

$$w_i = w_i^{k-1,m}, \quad r_i = \frac{\sum_{j=1}^{n} |\xi_i^{k-1,m}(t_j) - w_i^{k-1,m}|}{n}, \quad i = 1, \ldots, m.$$  

The boundary values of the approximate circular map $\omega^k$ in (43) are considered as approximations of the boundary values of the circular map $\omega$. The boundary $C = \bigcup_{i=1}^{m} C_i$ of $\ominus_1$ is parametrized by the function

$$\xi(t) = \omega(\eta(t)) = \omega^k(\eta(t)), \quad t \in J.$$  

(45)

The derivative of the function $\xi(t) = \omega(\eta(t))$ with respect to $t$ can be computed by differentiating both sides of (43).

4.3 The Interior Points

Since $\omega$ is analytic in the region $G$, thus once we obtain its boundary values from (45), we can compute the values of $w = \omega(z)$ at interior points $z \in G$ using the Cauchy integral formula,

$$w = \omega(z) = \frac{1}{2\pi i} \int_C \frac{\omega(\eta(t))}{\eta(t) - z} \eta'(t) \, dt.$$  

Since the derivative $\omega'(\eta(t))$ is known, we can use the Cauchy integral formula to find the values of $\omega'(z)$ for interior points $z \in G$.

4.4 The Inverse Circular Map

The inverse circular map $\omega^{-1}$ is analytic in the circular region $\Omega$. Since the boundary $C$ is parameterized by $\xi(t), t \in J$, the values of $z = \omega^{-1}(w)$ at interior points $w \in \Omega$ can be computed using the Cauchy integral formula

$$z = \omega^{-1}(w) = \frac{1}{2\pi i} \int_C \frac{\omega^{-1}(\xi)}{\xi - w} \, d\xi = \frac{1}{2\pi i} \int_J \frac{\omega^{-1}(\xi(t))}{\xi(t) - w} \xi'(t) \, dt,$$  

(46)

where $\omega^{-1}(\xi(t)) = \eta(t)$.

5 Koebe’s Method for Unbounded Multiply Connected Regions

Based on the results of Sect. 3, this section presents a numerical implementation of Koebe’s iterative method for computing the circular map $w = \omega(z)$ of the unbounded multiply connected region $G$ (see Fig. 2). We present a method for computing the
boundary values of the circular map,

\[ \xi(t) = \omega(\eta(t)), \quad t \in J, \]

the boundary values of its derivative,

\[ \xi'(t) = \eta'(t)\omega'(\eta(t)), \quad t \in J, \]

and the parameters \( w_j, r_j, j = 1, 2, \ldots, m \), of the circular region \( \Omega \).

The details are similar to the bounded case presented in the previous section.

5.1 Initializations

At the beginning, we set

\[ C_i^{0,0} = \Gamma_i \quad \text{for} \quad i = 1, 2, \ldots, m. \]

The curve \( C_i^{0,0} \) is parametrized by \( \xi_{i}^{0,0}(t) \) where

\[ \xi_{i}^{0,0}(t) = \eta_i(t), \quad \frac{d}{dt}\xi_{i}^{0,0}(t) = \eta_i'(t), \quad t \in J_i, \quad i = 1, 2, \ldots, m. \]

We assume that \( w_i^{0,0} \) is a given point inside \( C_i^{0,0} \) for \( i = 1, \ldots, m \). Thus, initial values of the boundary values of the circular map are given by

\[ \omega^0(\eta(t)) = \begin{cases} \xi_{i}^{0,0}(t), & t \in J_1, \\ \vdots \\ \xi_{m}^{0,0}(t), & t \in J_m. \end{cases} \]

5.2 Iterations

For \( k = 1, 2, 3, \ldots \), where \( k \) denotes the iteration number, we shall repeat the following three steps:

5.2.1 Step I: The Curves

For \( j = 1, 2, \ldots, m \), let \( \Psi_{k,j} \) be the conformal mapping from the exterior region of the curve \( C_j^{k-1,j-1} \) onto the exterior unit disk. Then

\[ C_j^{k-1,j} = \Psi_{k,j}(C_j^{k-1,j-1}) \]

\[ \text{Springer} \]
is the unit circle. The curve $C_{j}^{k-1}$ is parametrized by

$$\xi_{j}^{k-1}(t) = \Psi_{k,j}(\xi_{j}^{k-1,j-1}(t)), \quad t \in J_j.$$  

The smooth Jordan curves $C_{i}^{k-1,j-1}, i = 1, 2, \ldots, m, i \neq j$, are external to the curve $C_{j}^{k-1,j-1}$. Thus, the function $\Psi_{k,j}$ maps the curves $C_{i}^{k-1,j-1}$ onto smooth Jordan curves external to $C_{j}^{k-1,j}$. The curve $C_{i}^{k-1,j}$ is parametrized by

$$\xi_{i}^{k-1,j}(t) = \Psi_{k,j}(\xi_{i}^{k-1,j-1}(t)), \quad t \in J_i, \quad i = 1, 2, \ldots, m, \ i \neq j.$$  

The derivative of the function $\xi_{i}^{k-1,j}(t), i = 1, 2, \ldots, m$, can be computed from the derivative of the mapping function $\Psi_{k,j}$ as in Sect. 4.

Finally, we set

$$w_{j}^{k-1,j} = 0.$$  

For $i = 1, 2, \ldots, m, i \neq j$, the point $w_{i}^{k-1,j-1}$ inside the curve $C_{i}^{k-1,j-1}$ in the exterior region of the curve $C_{j}^{k-1,j-1}$ will be mapped by the function $\Psi_{k,j}$ into the point

$$w_{i}^{k-1,j} = \Psi_{k,j}(w_{i}^{k-1,j-1}), \quad i = 1, 2, \ldots, m, \ i \neq j,$$  

inside the curve $C_{i}^{k-1,j}$ in the exterior of the circle $C_{j}^{k-1,j}$.

5.2.2 Step II: Normalization

The function $\hat{\omega}^{k}$ with the boundary values

$$\hat{\omega}^{k}(\eta(t)) = \begin{cases} \xi_{1}^{k-1,m}(t), & t \in J_1, \\ \vdots \\ \xi_{m}^{k-1,m}(t), & t \in J_m, \end{cases}$$  

(47)

is the conformal mapping from the region $G$ onto the exterior region of the curves $C_{i}^{k-1,m}, i = 1, \ldots, m$. However, the function $\hat{\omega}^{k}$ does not satisfy the normalization (3). The function $\hat{\omega}^{k}$ has the expansion near $\infty$,

$$\hat{\omega}^{k}(z) = bz + c_0 + c_1z^{-1} + c_2z^{-2} + \cdots$$
with a positive real constant $b$. Since $\alpha$ is in the exterior of $G$, then the constants $b$ and $c_0$ can be computed by [13, p. 2]

$$b = -\frac{1}{2\pi i} \int_{\Gamma} \frac{\hat{\omega}_k(\eta)}{\eta - \alpha} \frac{d\eta}{\eta - \alpha}.$$

$$c_0 = -\frac{1}{2\pi i} \int_{\Gamma} \frac{[\hat{\omega}_k(\eta) - b\eta]}{\eta - \alpha} d\eta.$$

Define a function $\psi^k$ by

$$\psi^k(z) = \frac{z - c_0}{b}.$$

Then the function $\omega^k$ defined by

$$\omega^k(\eta(t)) = \psi^k \circ \hat{\omega}^k$$

is the conformal mapping from the region $G$ onto the exterior region of the curves

$$C^k_{i-1,m+1} = \psi^k(C^k_{i-1,m}), \quad i = 1, \ldots, m,$$

and satisfies the normalization (3). The curve $C^k_{i-1,m+1}$ is parametrized by

$$\xi^k_{i-1,m+1}(t) = \psi^k(\xi^k_{i-1,m}(t)), \quad i = 1, \ldots, m.$$

Then, we set

$$w^k_{i-1,m+1} = \psi^k(w^k_{i-1,m}), \quad i = 1, \ldots, m.$$

### 5.2.3 Step III: Update and Conditions of Convergence

The boundary values of the approximate circular map $w = \omega^k(z)$ are given by (48). We stop the iteration if

$$\|\omega^k - \omega^{k-1}\|_{\infty} \leq \epsilon \text{ or } k = \text{Max},$$

where $\epsilon$ is a given tolerance and Max is the maximum number of iterations allowed.

If the condition (44) is not satisfied, we set

$$C^k_{i,0} = C^k_{i-1,m+1}, \quad i = 1, \ldots, m.$$

For $i = 1, 2, \ldots, m$, the curve $C^k_{i,0}$ is parametrized by

$$\xi^k_{i,0}(t) = \xi^k_{i-1,m+1}(t), \quad t \in J_i.$$
Then, we set \( k = k + 1 \) and repeat Steps I–III.

If the method converges, then we consider the bounded multiply connected region bounded by the circles \( C_1^{k-1,m+1}, \ldots, C_m^{k-1,m+1} \), as the circular region \( \Omega \). The boundaries of \( \Omega \) are then given by

\[
C_i = C_i^{k-1,m+1}, \quad i = 1, \ldots, m.
\]

The centre \( w_i \) and the radius \( R_i \) of the circle \( Ci \) are approximated by

\[
w_i = w_i^{k-1,m+1}, \quad R_i = \sum_{j=1}^{n} \frac{|\xi_i^{k-1,m+1}(t_j) - w_i^{k-1,m+1}|}{n}, \quad i = 1, \ldots, m.
\]

We consider also the boundary values of the approximate circular map \( \omega^k \) in (48) as approximation of the boundary values of the circular map \( \omega \), i.e., the function

\[
\xi(t) = \omega(\eta(t)) = \omega^k(\eta(t)), \quad t \in J,
\]

is the parametrization of the boundary \( C = \bigcup_{i=1}^{m} C_i \) of \( \Omega \). The derivative of the function \( \xi(t) = \omega(\eta(t)) \) with respect to \( t \) can be computed from (47) and (48).

5.3 The Interior Values

Since \( \omega \) is analytic in the region \( G \) with the normalization (3) and \( \alpha \) is in the exterior of \( G \), the function \( \frac{\omega(z)}{z - \alpha} \) is analytic in \( G \) and its value at \( \infty \) equals to 1. Thus, once we obtain the boundary values of \( \omega \) from (48) and (50), we can compute the values of \( w = \omega(z) \) at interior points \( z \in G \) using the Cauchy integral formula [13, p. 2]

\[
\omega(z) = (z - \alpha) \left[ 1 + \frac{1}{2\pi i} \int_J \frac{\omega(\eta(t))}{\eta(t) - \alpha} - \frac{1}{z} \frac{\eta'(t)}{\eta(t) - \alpha} \, dt \right]. \tag{51}
\]

Since \( \omega \) is analytic in the region \( G \) with the normalization (3), the function \( \omega' \) is analytic in \( G \) with \( \omega'(\infty) = 1 \). In view of (50), the boundary values of the derivative of the function \( \omega \) can be obtained by differentiating both sides of (48) and (47). Thus, we can compute the values of \( \omega'(z) \) at interior points \( z \in G \) using the Cauchy integral formula [13, p. 2]

\[
\omega'(z) = 1 + \frac{1}{2\pi i} \int_J \frac{\omega'(\eta(t))}{\eta(t) - z} \eta'(t) \, dt. \tag{52}
\]

5.4 The Inverse Circular Map

The inverse circular map \( \omega^{-1} \) is analytic in the circular region \( \Omega \) with a simple pole at \( \infty \). The normalization (3) implies that the inverse function \( \omega^{-1} \) satisfies [43]

\[
\lim_{w \to \infty} \frac{\omega^{-1}(w)}{w} = 1. \tag{53}
\]
Let \( \hat{\alpha} \) be a given point in the exterior of \( \Omega \), then function \( \omega^{-1}(w) \) is analytic in \( \Omega \) and its value at \( \infty \) equals to 1. Thus, by the Cauchy integral formula, we can compute the values of \( z = \omega^{-1}(w) \) at interior points \( w \in \Omega \) using [13, p. 2]

\[
z = \omega^{-1}(w) = (w - \hat{\alpha}) \left[ 1 + \frac{1}{2\pi i} \int_{J} \frac{\omega^{-1}(\xi(t))}{\xi(t) - \hat{\alpha}} \frac{1}{\xi(t) - w} \xi'(t) \, dt \right],
\]

where \( \omega^{-1}(\xi(t)) = \eta(t) \).

### 6 Computational Complexity

In each iteration of Koebe’s iterative method, we compute the boundary values of \( m \) conformal mappings of simply connected regions (i.e., \( \Psi_{k,1}, \ldots, \Psi_{k,m-1}, \) and \( \Psi_{k,m} \) or \( \Phi_{k} \)) and their first derivatives. As described in Sect. 3, computing the boundary values of these \( m \) conformal mappings requires \( O(mn \log n) \) operations. For each of these \( m \) conformal mappings, it is required to use the Cauchy integral formula for \( m - 1 \) times to compute the value of the mapping function at \( n \) points. Thus, in each iteration of Koebe’s method, we use the Cauchy integral formula to compute the value of an analytic function at \( n \) points for \( m(m - 1) \) times. The MATLAB function \( \text{zfmm2dpart} \) [15] has been used in [31,32] to develop a fast and accurate method to compute the Cauchy integral formula at \( n \) points in \( O(n) \) operations. Thus, each iteration of Koebe’s method requires additional \( O(m^{2}n) \) operations for computing the Cauchy integral formula. Hence, the computational cost of each iteration of Koebe’s method is \( O(m^{2}n + mn \log n) \) which implies that the computational cost of the presented method is \( O(m^{2}n + mn \log n) \).

### 7 Numerical Examples

We consider seven numerical examples. In the first example, we consider an example with known exact solution. In Examples 2 and 3, we consider examples from [39,40]. A region with close-to-touching boundaries is given in Example 4. In Examples 5 and 6, we consider regions with high connectivity. Finally, we consider a region with piecewise smooth boundaries in Example 7.

For Koebe’s iterations, we iterate until

\[
\| \omega^{k} - \omega^{k-1} \|_{\infty} < 0.5 \times 10^{-13},
\]

where \( \omega^{k} \) is the approximate circular map obtained in the \( k \)th iteration of Koebe’s method. The method converges after few iterations when the boundaries are well separated. The number of iterations increases for regions with close-to-touching boundaries.

The exact circular map \( \omega \) is known for Example 1. For Examples 2–7, the exact circular maps are unknown. However, we shall consider the approximate circular maps obtained with \( n = 8,192 \) as the exact circular maps \( \omega \). Hence, for all examples, we
shall present the maximum error norm

\[ \| \omega^k - \omega \|_{\infty}. \]

For unbounded \( G \), the rate of convergence \( p \) given in Theorem 1 will be computed. The numerical results obtained show that this rate \( p \) is not the optimal estimate of the rate of convergence of Koebe’s method. We shall find that the actual rate of convergence \( q \) is much better than \( p \). We shall estimate the value of \( q \) from the computed errors \( \| \omega^k - \omega^{k-1} \|_{\infty} \) and \( \| \omega^k - \omega \|_{\infty} \). The rate of convergence \( q \) will be estimated for all examples.

For the direct mapping \( w = \omega(z) \), we plot the images of horizontal and vertical lines in the \( z \)-plane. For the inverse mapping \( z = \omega^{-1}(w) \), we plot the images of radial lines and circles in the \( w \)-plane.

**Example 1** We consider an example with known exact circular map from [2]. The inverse circular map \( z = \omega^{-1}(w) \) that maps the circular region \( \Omega \) exterior to two circles with centres 0, 2.5 and radii 1, 0.5, respectively, in the \( w \)-plane onto an unbounded doubly connected region \( G \) exterior to curves \( \Gamma_1, \Gamma_2 \) in the \( z \)-plane is given by

\[ \omega^{-1}(w) = f_5(f_4(f_3(f_2(f_1(w))))). \] (55)

where

\[ f_1(w) = \frac{w - a}{aw - 1}, \quad f_2(w) = \beta w, \quad f_3(w) = w + \frac{1}{w}, \quad f_4(w) = \frac{\beta}{\beta^2 + 1} w, \]

and

\[ a = \frac{7 + 2\sqrt{6}}{5}, \quad \beta = 30. \]

The function \( f_5 \) is given by

\[ f_5(w) = \frac{C_1 w + C_2}{w + C_3}, \]

where

\[ C_1 = \frac{a^4 - \beta^2}{a(a^2 - \beta^2)}, \quad C_2 = \frac{-3a^2\beta^2 + a^2 - \beta^4 - 3\beta^2}{a^2\beta^2 + a^2 - \beta^2 - \beta^4}, \quad C_3 = \frac{a^2 + \beta^2}{a(\beta^2 + 1)}. \]

The original region \( G \), the image of \( G \), the circular region \( \Omega \), and the inverse image of \( \Omega \) are shown in Fig. 5.

The successive error \( \| \omega^k - \omega^{k-1} \|_{\infty} \) and the maximum error \( \| \omega^k - \omega \|_{\infty} \) vs. the number of iterations \( k \) for \( k = 1, 2, \ldots, 20 \) are shown in Fig. 6. Figure 6 shows also the lines \( cp^k \) and \( cq^k \) where the rate \( p = 0.1296 \) is calculated from (5) and the value of the rate \( q \) is estimated from the computed errors \( \| \omega^k - \omega^{k-1} \|_{\infty} \) and \( \| \omega^k - \omega \|_{\infty} \) to be \( q = 0.00012. \)
Fast Computation of the Circular Map

Fig. 5 The original region $G$ (top, left), the image of $G$ (top, right), the circular region $\Omega$ (bottom, left), the inverse image of $\Omega$ (bottom, right), obtained for Example 1 with $n = 128$ after 5 iterations

Fig. 6 The errors for Example 1 obtained with $n = 128$ where $p = 0.1296$ and $q = 0.00012$

The largest eigenvalue $\lambda_1$, the second largest eigenvalue $\lambda_2$, and the smallest eigenvalue $\lambda_n$ obtained with $n = 128$ for each of the boundaries $\Gamma_1$ and $\Gamma_2$ are shown in Fig. 7. The other $n - 3$ eigenvalues are in the interval $[\lambda_n, \lambda_2]$. When $k$ increases, $\lambda_1 \approx 2$ and $\lambda_n \approx \lambda_2 \approx 1$. Thus, the other eigenvalues are also approximately equal to 1. Figure 7 shows also the condition number of the coefficient matrices of the linear systems and the number of GMRES iterations for each of the boundaries $\Gamma_1$ and $\Gamma_2$.

Numerical computing of the inverse circular map $\omega^{-1}$ for this example using Wegmann’s and Fornberg’s methods has been given in [2]. As in [2], we compute in Table 1 the maximum numerical errors DF, DZ, and DR for the boundary values of the circular
map, the centres, and the radii, respectively. It is clear from Table 1 and from [2, Tab. 1–4] that the accuracy of the method presented is almost the same as the accuracy of Wegmann’s and Fornberg’s methods.

Example 2 In this example, we calculate the circular map $w = \omega(z)$ that maps a bounded region $G$ bounded by three ellipses in the $z$-plane onto a bounded circular region $\Omega$ in the $w$-plane. The same example has been considered in [40, Ex. 19] for computing $\omega^{-1}$ but with different normalizations. The inner ellipses $\Gamma_1$ and $\Gamma_2$ are parametrized by

$$
\Gamma_1 : \eta_1(t) = -0.1 + 0.5i + 0.3 \cos t - 0.2i \sin t,
$$

$$
\Gamma_2 : \eta_2(t) = +0.1 - 0.3i + 0.2 \cos t - 0.4i \sin t,
$$

Table 1 Discretization errors for Example 1

| $n$ | DF    | DZ    | DR    |
|-----|-------|-------|-------|
| 16  | 5.5(-07) | 3.6(-07) | 1.5(-07) |
| 32  | 3.0(-11) | 8.0(-13) | 7.4(-13) |
| 64  | 1.2(-13) | 7.1(-14) | 5.0(-14) |
| 128 | 1.2(-13) | 7.3(-14) | 4.6(-14) |
| 256 | 1.2(-13) | 6.7(-14) | 5.2(-14) |
for $0 \leq t \leq 2\pi$. The external boundary $\Gamma_3$ is the inverted ellipse parametrized by

$$\Gamma_3 \quad \eta_3(t) = \sqrt{1 - (1 - \rho^2) \cos^2 t} \ e^{i t}, \quad \rho = 0.5.$$ 

Figure 8 shows the original region $G$, the image of $G$, the circular region $\Omega$, and the inverse image of $\Omega$. The successive error $\|\omega^k - \omega^{k-1}\|_\infty$ and the maximum error $\|\omega^k - \omega\|_\infty$ vs. the number of iteration $k$ for $k = 1, 2, \ldots, 20$ are shown in Fig. 9. Since $G$ is bounded, only the line $cq^k$ is shown in Fig. 9 where the value of the rate $q$ is estimated to be $q = 0.049$. 

Figure 8 The original region $G$ (top, left), the image of $G$ (top, right), the circular region $\Omega$ (bottom, left), the inverse image of $\Omega$ (bottom, right), obtained for Example 2 with $n = 128$ after 11 iterations

Figure 9 The errors for Example 2 obtained with $n = 128$ where $q = 0.049$
Fig. 10 The eigenvalues for $\Gamma_1$ (left), $\Gamma_2$ (middle), and $\Gamma_3$ (right), the condition number, and the number of GMRES iterations for Example 2.
The largest eigenvalue \( \lambda_1 \), the second largest eigenvalue \( \lambda_2 \), and the smallest eigenvalue \( \lambda_n \) obtained with \( n = 128 \) for each of the boundaries \( \Gamma_1 \), \( \Gamma_2 \), and \( \Gamma_3 \) are shown in Fig. 10. When \( k \) increases, \( \lambda_1 \approx 2 \) and \( \lambda_n \approx \lambda_2 \approx 1 \). Thus, the other \( n-3 \) eigenvalues are also approximately equal to 1. Figure 10 shows also the condition number of the coefficient matrices of the linear systems and the number of GMRES iterations for each of the boundaries \( \Gamma_1 \), \( \Gamma_2 \), and \( \Gamma_3 \).

Example 3 In this example, we calculate the circular map \( w = \omega(z) \) which maps an unbounded region \( G \) exterior to four ellipse in \( z \)-plane onto an unbounded circular region \( \Omega \) in the \( w \)-plane. This example has been considered in [39, Ex. 2] for computing \( \omega^{-1} \). The boundaries are parametrized by

\[
\begin{align*}
\Gamma_1 : \eta_1(t) &= 1.5 + \cos t - 0.8i \sin t, \\
\Gamma_2 : \eta_2(t) &= 1.2i + 0.8 \cos t - 0.6i \sin t, \\
\Gamma_3 : \eta_3(t) &= -1.5 + 0.5 \cos t - 0.8i \sin t, \\
\Gamma_4 : \eta_4(t) &= -1.5i + \cos t - 0.8i \sin t, \quad 0 \leq t \leq 2\pi.
\end{align*}
\]

The numerical results are shown in Figs. 11, 12.
Fig. 12 The errors for Example 3 obtained with $n = 128$ where $p = 0.50692$ and $q = 0.014$.

Fig. 13 The original region $G$ (top, left), the image of $G$ (top, right), the circular region $\Omega$ (bottom, left), the inverse image of $\Omega$ (bottom, right), obtained for Example 4 with $n = 1.024$ after 36 iterations.

**Example 4** In this example, we make the ellipses in the Example 3 much thinner and closer together. The boundaries are parametrized by

$$
\Gamma_1 : \eta_1(t) = 0.7 - 0.20i + 2 \cos t - 0.5i \sin t,
$$
$$
\Gamma_2 : \eta_2(t) = 0.55i + 1.35 \cos t - 0.2i \sin t,
$$
Fig. 14 The errors for Example 4 obtained with $n = 1.024$ where $p = 0.9815$ and $q = 0.46$.

\[ \Gamma_3 : \eta_3(t) = -1.5 + 0.15 \cos t - 0.75i \sin t, \]
\[ \Gamma_4 : \eta_4(t) = -0.95i + 2 \cos t - 0.2i \sin t, \]
for $0 \leq t \leq 2\pi$. The numerical results are shown in Figs. 13, 14.

Example 5 In this example, we compute the circular map from a bounded multiply connected region of connectivity 100 bounded by 100 ellipses. The numerical results are shown in Figs. 15, 16.
Fig. 16: The errors for Example 5 obtained with $n = 1,024$, where $q = 0.0165$ (left, middle) and the number of GMRES iterations for each boundary $\Gamma_j$, $j = 1, 2, \ldots, 100$, for the iterations $1, 2, 3, 8, 20$ (right).
Example 6  In this example, we compute the circular map from an unbounded multiply connected region of connectivity 103 bounded by 103 ellipses. The numerical results are shown in Figs. 17, 18.

Example 7  In this example, we compute the circular map from a bounded multiply connected region of connectivity 45. The boundaries $\Gamma_1, \ldots, \Gamma_{14}$ are circles, the boundaries $\Gamma_{15}, \ldots, \Gamma_{23}$ are piecewise smooth curves with one corner, the boundaries $\Gamma_{24}, \ldots, \Gamma_{32}$ are piecewise smooth curves with two corners, and the boundaries $\Gamma_{33}, \ldots, \Gamma_{45}$ are piecewise smooth curves with four corners. We assume that the corner points are not cusps and that the tangent vector of the boundary has only the first kind discontinuity at these corner points. The tangent vector at a corner point is defined to be the right tangent vector, the left tangent vector, or their average. The integral equation (15) is valid only at off-corner points [35]. Using the fact that the constant functions are eigenfunctions of $N$ with eigenvalue $-1$ [27,28], singularity subtraction [36] can be used to write the integral equation in the form

$$2\phi(s) - \int_0^{2\pi} N(s, t) [\phi(t) - \phi(s)] \, dt = -\int_0^{2\pi} M(s, t) \gamma(t) \, dt, \quad t \in [0, 2\pi],$$

which is valid for all points on $L$ even for the corner points [31,35]. The integral equation (56) is then discretized by the trapezoidal rule with a graded mesh [22] which is based on substituting a new variable in such a way that the derivative of the new integrand vanishes at the corner points. Thus, the discontinuity in the integrand in (56) is removed. See [22,31,35,36] for more details. By solving the integral equation,
Fig. 18. The errors for Example 6 obtained with $n = 512$ where $p = 0.6625$ and $q = 0.0166$ (left, middle) and the number of GMRES iterations for each boundary $\Gamma_j$, $j = 1, 2, \ldots, 103$, for the iterations 1, 2, 3, 8, 20 (right).
we obtain the values of the function $\theta$ for all $t \in [0, 2\pi]$. Then, for bounded $G$, we obtain the values of the mapping functions $\Phi$ for all points on $L$ even for the corner points from (20). Similarly, for unbounded $G$, we obtain the values of the mapping functions $\Psi$ for all points on $L$ even for the corner points from (28). The numerical results obtained with the grading parameter $p = 3$ are shown in Figs. 19, 20.

8 Conclusions

This paper presented a fast, efficient and easy-to-program numerical implementation of Koebe’s iterative method to compute the circular map of bounded and unbounded multiply connected regions. Several numerical examples were presented. It becomes apparent from the numerical results presented that the actual convergence rate of Koebe’s method is much better than the convergence rate given in (5).

The computational cost of the method presented is $O(m^2n + mn \log n)$. But, the constant of this $O(m^2n + mn \log n)$ could be large because $O(m^2n + mn \log n)$ is the computational cost of each iteration of the presented method. Hence, the computational cost of the present method is higher than the computational cost $O(mn \log n)$ of the method presented in [27–30,32] for computing the conformal mapping onto the canonical slit regions (see Fig. 21).
The errors for Example 7 obtained with $n = 4,096$ where $q = 0.033$ (left, middle) and the number of GMRES iterations for each boundary $\Gamma_j$, $j = 1, 2, \ldots, 45$, for the iterations 1, 2, 3, 8, 20 (right).
Recently, analytic formulas for several problems in fluid mechanics are given for multiply connected circular regions. These analytic formulas are described in terms of the Schottky–Klein prime function associated with the circular regions [3–8, 10]. So, the presented method with these analytic formulas provide us a method for solving these problems in general multiply connected regions. In addition, the method presented can be used to compute the conformal mapping, its derivative, and its inverse. Thus, this method will be useful particularly for fluid mechanics problems which requires determining the conformal mapping and its derivative (see e.g., [3, 7, 8]).

Further, the Schottky–Klein prime function has been used to obtain analytic formulas for the conformal mappings from multiply connected circular regions onto Koebe’s first category of canonical slits regions [9]. These analytic formulas can be combined with the method presented to obtain a method for computing the conformal mapping from general multiply connected regions onto Koebe’s first category of canonical slits regions. However, the computational cost of the combined method will be much higher than the computational cost of the direct method presented in [28, 32].

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