Let us consider a number of spaces: \( \text{Riem} \), the set of all riemannian three-metrics on a given compact manifold; \( \text{superspace} \), obtained from \( \text{Riem} \) by identifying as three-geometries all three-metrics that are related by coordinate transformations; and \( \text{conformal superspace} \), \( (\text{CS}) \), obtained from superspace by identifying all three-geometries whose metrics are related by conformal rescalings \( g_{ab} \rightarrow \omega^2 g_{ab} \) where \( \omega \) is an arbitrary strictly positive function. One can regard the conformal factor as a fourth ‘coordinate’ on three-space. We also consider \( \text{restricted conformal superspace} \), \( (\text{CS}^*) \), in which only conformal three-geometries with the same volume are identified. In canonical general relativity \( (1) \) one is given a pair \( \{g_{ab}, \pi^{ab}\} \) where \( g_{ab} \) is a riemannian three-metric and \( \pi^{ab} \), the conjugate momentum, is a symmetric three-tensor-density. These must satisfy the four constraints

\[
\pi_{ab} = 0; \quad gR = \pi^{ab}\pi_{ab} - \frac{1}{2}(\text{tr}\pi)^2, \quad (1)
\]

where \( R \) is the scalar curvature of \( g_{ab} \).

There are twelve degrees of freedom per space point in the pair \( \{g_{ab}, \pi^{ab}\} \) but three of them represent the three coordinates and we also need to include the four constraints. Hence the initial data really have five degrees of freedom. Four represent the true gravitational degrees of freedom while the fifth is kinematical and represents the freedom to imbed the spacelike slice in the spacetime.

York has cogently argued \( (2) \) that the four dynamical degrees of freedom are coded into the conformal geometry of the spacelike slice and that the configuration space of gravity should be conformal superspace. This almost works in general relativity. The six components of the metric reduce to two when one subtracts off one conformal and three coordinate degrees of freedom. Also any symmetric tensor, \( A_{ab} \), can be uniquely decomposed \( (3) \)

\[
A_{ab} = A_{ab}^{TT} + (LW)_{ab} + \theta g_{ab} = A_{ab}^{TT} + (KW)_{ab} + \bar{\theta} g_{ab} \quad (2)
\]

where \( (LW)_{ab} = W_{a; b} + W_{b; a} - \frac{3}{2} W_{k; b} g_{a b} \) is the conformal killing form of a vector \( W^{k_i} \); \( (KW)_{ab} = W_{a; b} + W_{b; a} \) is the killing form of the same vector; \( 3\theta = \text{tr}A = g_{ab}A^{ab} \); and \( \bar{\theta} = \theta - \frac{3}{2} W^{k_i}. A^{TT} \) is both tracefree and divergencefree. \( TT \) tensors are conformally covariant: if \( A^{TT}_{ab} \) is \( TT \) with respect to a given metric \( g_{ab} \), then \( \omega^{-2} A^{TT}_{ab} \) is \( TT \) with respect the conformally related metric \( g_{ab}' = \omega^2 g_{ab} \). Thus any \( TT \) tensor represents a tangent vector in conformal superspace and clearly has two degrees of freedom per space point. In general relativity the standard approach is to choose the trace of the extrinsic curvature as the fifth degree of freedom. This breaks the conformal invariance.

We intend to take the conformal structure seriously and construct a new theory of gravity. We find a conformally invariant action which gives a measure (a ‘metric’) on conformal superspace. The solutions of the theory will be a family of unique geodesics in the configuration space. Since we start with a Jacobi action, the geodesic will be a parametrised curve. The parameter itself has no intrinsic meaning. It is remarkable how closely these curves in conformal superspace match curves in superspace which represent solutions of the Einstein equations.

In 1962 Baierlein, Sharp and Wheeler (BSW) \( (4) \) constructed a Jacobi action for G.R. It was of the form

\[
I = \int d\lambda \sqrt{g} \sqrt{\tilde{R}} \sqrt{T} d^3x, \quad (3)
\]

where the ‘kinetic energy’ \( T \) is

\[
T = (g^{ab} g^{cd} - g^{oc} g^{cd}) \left( \frac{\partial g_{ab}}{\partial \lambda} - (KW)_{ab} \right) \left( \frac{\partial g_{cd}}{\partial \lambda} - (KW)_{cd} \right). \quad (4)
\]

This action reproduces the standard Einstein equations in the thin-sandwich form with lapse \( N = \sqrt{T/4R} \).

Barbour and Bertotti \( (5) \) realised that this action could be constructed naturally by a ‘best matching’ procedure. One picks two nearby metrics \( g_{ab} \) and \( g_{ab} + \delta g_{ab} \) and tries
to measure a separation between them while allowing for an arbitrary coordinate transformation on the second metric and simultaneously using the ‘potential energy’ term $\sqrt{R}$ as a weighting. Hence one minimizes the action over all vectors $W^a$. In fact, BSW actually had two coordinate transformations: one generated by $W^a$, the other which is implemented by the action being a geometric scalar.

The simplest conformalization of the BSW action is

$$I = \int d\lambda \sqrt{\sqrt{g} \sqrt{R - \frac{8\nabla^2\phi}{\phi}}} \sqrt{T} d^3x, \tag{5}$$

where $V(\phi) = \int \phi^6 \sqrt{g} d^3x$ and the new $T$ is

$$T = (g^{ac} g^{bd} - A g^{ab} g^{cd}) \left( \frac{\partial g_{cd}}{\partial \lambda} - (LW)_{cd} - \theta g_{cd} \right), \tag{6}$$

with $A$ an (as yet) arbitrary constant. We do not restrict ourselves to the DeWitt form but rather allow a more general supermetric in the theory. The denominator must be chosen to be of the same degree in $\phi$ as the numerator so that one cannot make the action vanish by a simple scaling on $\phi$. Other similar conformally invariant actions can be found by changing the power of $R$ in the numerator and appropriately changing the denominator.

As in Ref. [3] we compare two nearby metrics; however, in addition to the coordinate transformations, the new scalar, $\theta$, in the kinetic energy allows us to make an arbitrary conformal rescaling between the slices while the second function, $\phi$, allows an overall conformal rescaling.

Both the numerator and denominator are conformally invariant. Choose an arbitrary positive function $\omega(x^i, \lambda)$ and consider the following mapping

$$g'_{ab} = \omega^4 g_{ab}, \quad \frac{\partial g'_{ab}}{\partial \lambda} = \omega^4 \frac{\partial g_{ab}}{\partial \lambda} + 4\omega^3 \frac{\partial \omega}{\partial \lambda} g_{ab},$$

$$\phi' = \frac{\phi}{\omega}, \quad W'^a = W^a, \quad \theta' = \theta + 4\omega^{-1} \frac{\partial \omega}{\partial \lambda}. \tag{7}$$

This gives

$$R' - \frac{8\nabla^2\phi'}{\phi'} = \omega^{-4} (R - \frac{8\nabla^2\phi}{\phi}),$$

$$(L'W')'_{ab} = \omega^4 (LW)_{ab}, \quad T' = T,$$

$$\sqrt{g''} \phi'' \sqrt{R' - \frac{8\nabla'^2\phi''}{\phi''}} \sqrt{T'} = \sqrt{g'\phi'} \sqrt{R - \frac{8\nabla^2\phi}{\phi}} \sqrt{T}. \tag{8}$$

We find the constraints of the theory by varying with respect to $\phi$, $W$, and $\theta$. All the equations will be conformally invariant. Only the TT part of $\partial g/\partial \lambda$ contributes to the action; however, after we solve the constraints, $\partial g/\partial \lambda - (LW) - \theta g$ will not be just the TT part of $\partial g/\partial \lambda$, it will have a vector part arising from the fact that the potential energy is not constant.

The momentum conjugate to $g_{ab}$, found by varying the action with respect to $\partial g/\partial \lambda$, is

$$\pi^{ab} = \frac{\sqrt{g} \phi^4 \sqrt{R - \frac{8\nabla^2\phi}{\phi}}}{\sqrt{V(\phi)}} \left( g^{ac} g^{bd} - A g^{ab} g^{cd} \right) \left( \frac{\partial g_{cd}}{\partial \lambda} - (LW)_{cd} - \theta g_{cd} \right). \tag{9}$$

Note that we go from a displacement to a direction in Riem because $\sqrt{T}$ is the norm of the $\partial g/\partial \lambda$ term and so

$$\pi^{ab} \pi_{ab} - \frac{A}{3A - 1} (tr \pi)^2 = \frac{g \phi^8}{V(\phi)^2} \left( R - \frac{8\nabla^2\phi}{\phi} \right), \tag{10}$$

which is a reparametrisation identity arising directly from Eq. (11). When we vary the action w.r.t. $W^a$ and $\theta$ we get the striking result that $\pi^{ab}$ is TT and thus a direction in conformal superspace. In other words

$$\pi^{ab}_{;b} = 0, \quad tr \pi = 0. \tag{11}$$

This shows that the coefficient $A$ in the supermetric can be set to zero; the kinetic energy is positive; and the reparametrisation identity reduces to

$$\pi^{ab} \pi_{ab} \equiv \frac{g \phi^8}{V(\phi)^2} \left( R - \frac{8\nabla^2\phi}{\phi} \right). \tag{12}$$

We next consider the variation w.r.t. $\phi$. Effectively we are minimizing the BSW action on a fixed metric and further minimizing it by making an overall conformal transformation as defined by Eq. (13). The minimizing $\phi$ is the conformal factor that brings us to the minimizing metric. The equation is

$$\sqrt{\frac{\sqrt{g} \phi^4 \sqrt{R - \frac{8\nabla^2\phi}{\phi}}}{\sqrt{V(\phi)}}} - \frac{\nabla^2 \sqrt{\frac{\sqrt{g} \phi^4 \sqrt{R - \frac{8\nabla^2\phi}{\phi}}}{\sqrt{V(\phi)}}}}{\sqrt{V(\phi)}} = \frac{C \phi^5}{\sqrt{V(\phi)}}. \tag{13}$$

This is a conformally invariant eigenvalue equation. The eigenvalue $C$ arises from the variation of the denominator and it is what prevents $\phi \equiv 0$ from being a solution. We call this the energy norm equation.

Implementing $tr \pi = 0$ is trivial: one subtracts off the trace of $\partial g/\partial \lambda$. The energy norm equation and setting $\pi^{ab}_{;b} = 0$ are more difficult: one finds a set of four nonlinear coupled equations. These are analogous to the thin sandwich equations of general relativity [3]. We assume that they can be solved for a range of $\{g_{ab}, \partial g_{ab}/\partial \lambda\}$.

The problem of solving these equations can be avoided. We construct the hamiltonian version of conformal gravity and it is, as in general relativity, better posed than the thin sandwich version. The initial data now consist of a metric and a TT tensor density, $\{g_{ab}, V(\phi)^{\frac{1}{2}} \pi^{ab}_{TT} \}$, these should be thought of as a point and direction in conformal superspace. Since $\pi^{ab}$ (and not $\partial g_{ab}/\partial \lambda$) is now the basic variable the reparametrisation identity, Eq. (14), becomes
an equation and is solved for a positive \( \phi = \phi_s \). This \( \phi_s \) is substituted into the energy norm equation, Eq.\((13)\), which, in turn, is solved for \( T \). We can (if we wish) use the solution, \( \phi_s \), of the reparametrisation equation as a conformal factor to simplify matters. We call the system in this ‘simplest’ state the best-matched representation. The solution of the reparametrisation identity, in the best-matched representation, following Eq.\((7)\), is \( \phi' = \phi_s / \omega = \phi_s / \phi_s \equiv 1 \). Thus the energy norm equation, in the best matched representation, reduces to

\[
\sqrt{\frac{T}{R}} R - \nabla^2 \sqrt{\frac{T}{R}} = \frac{C}{\sqrt{g}},
\]

with the eigenvalue \( C \) satisfying \( C = \int \sqrt{gRT} d^3 x \), the numerator of the Lagrangian. This equation is homogeneous and thus the solution has an undetermined overall scale factor. It is this scale factor which allows the global reparametrisation of the solution curve in configuration space.

The equation for the time derivative of \( tr\pi \) in general relativity is

\[
\frac{\partial}{\partial t} \left( \frac{tr\pi}{\sqrt{g}} \right) = 2RN + \frac{N(tr\pi)^2}{g} - 2\nabla^2 N + \left( \frac{tr\pi}{\sqrt{g}} \right) N^a.
\]

Thus the equation for the lapse function which generates constant mean curvature slices at \( tr\pi = 0 \) is

\[
RN - \nabla^2 N = C_1,
\]

essentially the same as Eq.\((14)\). In the best-matched representation the reparametrisation identity, Eq.\((12)\), looks just like the hamiltonian constraint of general relativity at maximal expansion if we multiply the conformal gravity momentum by \( V^{\frac{T}{R}} \). Further, if we compare Eq.\((8)\) to the definition of the momentum in canonical general relativity, it is clear that the natural relationship is \( 2N = \sqrt{T/R} \), just as in BSW.

The lagrangian equation (or the second hamiltonian equation) is \( \partial \pi^{ab} / \partial \lambda = \partial L / \partial g_{ab} \). Thus the dynamical equations in the best-matched representation are

\[
\frac{\partial g_{ab}}{\partial \lambda} = \left( \sqrt{\frac{T}{gR}} V^{\frac{T}{R}} \pi_{ab} \right),
\]

\[
V^{\frac{T}{R}} \frac{\partial \pi_{ab}}{\partial \lambda} = \frac{1}{2} \sqrt{gRT} g^{ab} - \frac{1}{2} \sqrt{\frac{gT}{R}} R_{ab} + \frac{1}{2} \left( \sqrt{\frac{gT}{R}} \right)^{\alpha\beta} \left( \frac{gT}{R} \right)^{ab} - \frac{1}{2} \nabla^2 \sqrt{\frac{gT}{R}} g^{ab} - \sqrt{\frac{T}{gR}} \pi^{ac} \pi_{bc} - \sqrt{\frac{\sqrt{gT}}{3V}} g_{ab}.
\]

The initial data consist of a pair \( \{ g_{ab} , \pi^{ab} \} \) which satisfy the three constraints

\[
\pi_{ab} ; b = 0, \quad tr\pi = 0, \quad V^{\frac{T}{R}} \pi^{ab} \pi_{ab} = gR,
\]

and the function \( \phi \) must satisfy Eq.\((14)\). It is easy to show that the evolution equations preserve the constraints. Therefore they generate a unique curve in \( \text{Riem} \) which stays in the best-matched representation. We can add to each of the evolution equations a Lie derivative with respect to an arbitrary shift. This will give us a curve in superspace that remains in the best-matched representation. This is a representative of our desired curve in CS. If we substitute \( \{ g , \partial g / \partial \lambda \} \) from this curve into the original action, Eq.\((7)\), the best-matching procedure will give us \( \phi = 1, W^a = 0, \theta = 0 \).

It is remarkable that a scale-free theory nevertheless leads, through best-matching minimization, to a metric with scale. It was shown in \( \[6\] \) how local proper time emerges through best-matching on superspace. Now local lengths emerge from scale-free best-matching on conformal superspace. The determination of the full metric via the Lichnerowicz equation, which is essentially our Eq.\((12)\), as the final step in the York procedure has usually been regarded as a useful construct rather than something fundamental. Our work shows that it is natural and inevitable in conformal gravity.

Let us consider a cosmology which satisfies the vacuum Einstein equations and goes from a big bang to a big crunch. This will have a moment of maximum expansion. At this point we will have initial data for both general relativity and conformal gravity using the relationship \( \pi^{ab}_{CG} = V^{\frac{\sqrt{T}}{R}} \pi^{ab}_{GR} \). Let us propagate the Einstein initial data in the constant mean curvature gauge and the conformal gravity data in the best-matched representation. We find that initially \( N \) is proportional to \( \sqrt{T/R} \) and can arrange

\[
\left[ \frac{\partial g_{ab}}{\partial t} \right]_{GR} = \left[ \frac{\partial g_{ab}}{\partial \lambda} \right]_{CG} ; V^{\frac{T}{R}} \left[ \frac{\partial \pi^{ab}}{\partial t} \right]_{GR} = \left[ \frac{\partial \pi^{ab}}{\partial \lambda} \right]_{CG}.
\]

We cannot maintain this matching at higher orders, because the GR momentum develops a trace, but we can certainly match to the next order by using a conformal rescaling to move the conformal gravity curve out of the best-matched representation.

It is easy to work out the hamiltonian. It is

\[
H = \frac{\sqrt{T} V(\phi) \frac{T}{R}}{2\sqrt{T} \phi^3} \left[ \pi^{ab}_{GR} - \frac{g\phi^2 (R - \frac{\sqrt{T} \phi^3}{3V})}{V(\phi) \phi} \right] - 2W_{a:b} \pi^{ab} - \delta tr\pi.
\]

It is the sum of the three constraints with lagrange multipliers. We have four quantities without associated momenta, \( W^{a} , \theta , T , \) and \( \phi \). When we vary w.r.t. the first three we get the three constraints, when we vary w.r.t. \( \phi \) we get the energy norm equation, Eq.\((13)\).

This analysis works in the case where the manifold is compact without boundary. In the asymptotically flat
case we cannot use the volume as denominator. The obvious solution is to use the numerator as the action and to control the conformal factor by the requirement that $\phi \to 1$ at infinity. This means that the energy norm equation is no longer an eigenvalue equation, and Eq. (21) becomes

$$\nabla^2 \sqrt{\frac{T}{R}} - \sqrt{\frac{T}{R}} R = 0. \quad (22)$$

This is the maximal slicing equation. Thus solutions of the vacuum Einstein equations in the maximal gauge (as curves in superspace) agree exactly with solutions of the conformally invariant equations in the best matched representation. Therefore conformal gravity should pass all the standard tests. The Hamiltonian will look just like Eq. (21) except that the $V(\phi)$ is omitted. This is non-differentiable and the usual surface terms will have to be added to control the integration by parts $\xi$. The positive energy theorem $\parallel$ continues to hold.

There are solutions of the vacuum Einstein equations which are not linked to solutions of the conformal equations. These are the solutions in GR which do not have a maximal slice such as cosmological solutions which expand forever. The reparametrisation identity demands that the scalar curvature be positive. This severely restricts the possible topologies $\parallel$. Thus we have some form of topological censorship.

Applying the standard canonical quantization procedure to this conformal theory is quite straightforward: the reparametrisation identity converts into a Wheeler-DeWitt equation and we get a time independent Schrödinger equation which gives us a probability distribution on conformal superspace. The other constraints act on the wavefunction to guarantee both coordinate independence and conformally invariance. The time independence also is natural: there is no time in the classical theory so why should there be one in the quantum theory? Finally, the supermetric is positive definite: there are none of the negative energy modes that bedevil the theory. This work has been partially supported by the Forbairt grant SC/96/750. We wish to thank Domenico Giulini, Ted Jacobson, Claus Kiefer, Karel Kuchař, and Lee Smolin for helpful comments.

ACKNOWLEDGMENTS

This work has been partially supported by the Forbairt grant SC/96/750. We wish to thank Domenico Giulini, Ted Jacobson, Claus Kiefer, Karel Kuchař, and Lee Smolin for helpful comments.

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