The second law of thermodynamics from concave energy in classical mechanics

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A recently proposed quantum mechanical criterion ‘concavity of energy’ for the second law of thermodynamics is studied also for classical particle systems confined in a bounded region by a potential with a time-dependent coupling constant. It is shown that the time average of work done by particles in a quench process cannot exceed that in the corresponding quasi-static process, if the energy is a concave function of the coupling constant. It is proven that the energy is indeed concave for a general confining potential with certain properties. This result implies that the system satisfies the principle of maximum work in the adiabatic environment as an expression of the second law of thermodynamics.

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I. INTRODUCTION

The impossibility to make any perpetual motion of second kind is an expression of the second law of thermodynamics. This impossibility is expected to be clarified within the framework of classical or quantum mechanics. Planck’s principle is well-known as a specific expression of this impossibility. This principle claims that a positive amount of work cannot be extracted from thermodynamic systems in an arbitrary cyclic operation in the adiabatic environment. Any realization of a perpetual motion of second kind implies the violation of this principle. Passivity is known as a sufficient condition on a quantum state for the impossibility of work extraction from quantum systems. We say that a quantum state is passive, if a positive amount of work cannot be extracted from this state as an initial state in any cyclic unitary evolution. Some mixed states, such as the Gibbs states are passive, but general pure states except the ground state are not passive. The Gibbs distributions of classical dynamical systems have a similar property to passive quantum states. The well-known Jarzynski equality for classical dynamical systems enables us to show that the amount of work done by a classical dynamical system with an initial Gibbs distribution in a general process cannot exceed that in the quasi-static process. This property of the Gibbs distribution corresponds to the principle of maximum work in an isothermal environment in thermodynamics.

On the other hand, there have been several purely mechanical studies on the work extraction by a physical evolution from an initial pure state in an adiabatic environment within the framework of quantum mechanics. From the viewpoint of the second law of thermodynamics, Kaneko, Ioda and Sagawa study Planck’s principle from the viewpoint of the eigenstate thermalization hypothesis (ETH) analogue in quantum spin systems. They give numerical evaluations of amounts of works from energy eigenstates of quantum spin systems in a composite cyclic process which consists of instantaneous changes of the interaction and leaving the system a while. They find the strong ETH like behavior only in non-integrable systems, where negative amounts of work are extracted from every energy eigenstate in an cyclic process. In integrable systems, however, they find only weak ETH behavior. The principle of maximum work in adiabatic environment is also well-known as an equivalent principle to Planck’s principle. This principle states that work done by the system in an arbitrary process cannot exceed the work in the corresponding quasi-static process.

In the present paper, we check a proposed new criterion to satisfy the principle of maximum work in the adiabatic environment also for classical particle systems. Since the second law is valid irrespective of working substance in thermodynamics, its validity should be checked for the ideal gas as a simple example. Despite the well-known fact that interactions and non-integrability of systems are essential for thermalization of mechanical systems, the impossibility of perpetual motion of second kind should be valid for free particles. We study a classical dynamics of non-interacting particles confined in bounded region of one dimensional space by a time-dependent confining potential. We prove the concavity of energy as a function of the coupling constant is necessary also for classical particle systems to satisfy the principle of maximum work. We evaluate difference of work between quench and quasi-static changes of the coupling. Since the quench change of the coupling constant is operated instantaneously, the lost energy of the particle in the quench process depends on the position of the particle when the
coupling is changed. On the other hand, the amount of work is unique in the quasi-static process. We employ the time average of the work over the oscillation period of the confined particle, and prove that this average work in the quench process cannot exceed the work in the quasi-static process. This agrees with the maximal work principle in the adiabatic environment as an expression of the second law of thermodynamics, which is equivalent to Planck’s principle.

II. AVERAGE OF PARTICLE ENERGY

Consider a particle with a unit mass in one dimensional space, and let $x \in \mathbb{R}$ be its position coordinate. This particle is confined in a bounded region by a confining potential $\lambda u(x) \geq 0$ with a coupling constant $\lambda \geq 0$. Assume that (1) $u'(x)$ and $u''(x)$ exist at any $x \in \mathbb{R}$, (2) $u'(x) \leq 0$ for $x < 0$ and $u'(x) \geq 0$ for $x > 0$, (3) $u(-x) = u(x)$. The equation of motion is given by

$$\ddot{x} = -\lambda u'(x).$$

The kinetic energy

$$K = \frac{1}{2} \dot{x}^2,$$

and potential gives the energy of the particle

$$E = K + \lambda u(x),$$

which is a constant of the motion if $\lambda$ is a constant. One can imagine that the particle’s motion becomes a periodic oscillation around the origin $x = 0$ for a constant $\lambda$.

Consider the case that $\lambda(t)$ is a function of a time parameter $t \in \mathbb{R}$. In this case the energy $E_t$ is also a function of the time $t \in \mathbb{R}$. In the present paper, we compare work done by the particle in a quasi-static process and a quench process. First, we consider a quasi-static process. Assume an initial condition $(x(0), \dot{x}(0)) = (a_0, 0)$ with the coupling $\lambda(0) = \lambda_0$, and the initial energy is given by $E_0 = \lambda_0 u(a_0)$. At an arbitrary time $t > 0$, the amplitude $\alpha_t$ determines the energy $E_t = \lambda(t) u(\alpha_t)$. The quasi-static process is given by an infinitesimally slow change of the coupling constant $\lambda(t)$. Assume that the motion of the particle is a periodic oscillation around the origin $x = 0$ with an amplitude $\alpha_t$. For an arbitrary quasi-static process, the adiabatic invariant

$$I = 2 \int_{-\alpha_t}^{\alpha_t} \sqrt{2(\lambda(t) u(\alpha_t) - \lambda(t) u(x))} dx,$$

is known as a constant of motion for the slowly varying coupling constant $\lambda(t)$ and. Here, we regard the amplitude $\alpha(\lambda)$ as an implicit function of $\lambda$, such that the adiabatic invariant $I$ defined by

$$I = 2 \int_{-\alpha}^{\alpha} \sqrt{2\lambda u(\alpha) - 2\lambda u(x)} dx,$$

is a constant independent of $\lambda$ and define the energy

$$E(\lambda) = \lambda u(\alpha(\lambda)),$$

as a function of $\lambda$. Note that

$$\alpha_t = \alpha(\lambda(t)), \quad E_t = E(\lambda(t)).$$

Under the quasi-static change $\lambda_0 \rightarrow \lambda_1$, the change of energy $\Delta E = E(\lambda_1) - E(\lambda_0)$ of the particle gives the work $W = -\Delta E$ done by the particle in this process.

Next, to consider a quench process, define the period of the oscillating particle as a function of a coupling $\lambda$ by

$$T(\lambda) = 2 \int_{-\alpha}^{\alpha} \sqrt{2\lambda u(\alpha) - 2\lambda u(x)} dx.$$

Define a function $\tau$, which is a spending time of the particle in an interval $(a, b) \subset \mathbb{R}$ in the one dimensional space during the period $T(\lambda)$

$$\tau(a, b, \lambda) := \int_{a}^{b} \frac{2}{\sqrt{2E(\lambda) - \lambda u(x)}} dx,$$

as a function of $a, b$ and the coupling constant $\lambda$. Note that the period is represented in terms of $\tau$

$$T(\lambda) = \tau(-\alpha(\lambda), \alpha(\lambda), \lambda).$$

Let us regard the position coordinate $x$ of the particle as a random variable, and define the probability for $x \in (a, b)$ by a ratio of particle’s spending time in the interval $(a, b)$

$$P[x \in (a, b)] := \frac{\tau(a, b, \lambda)}{T(\lambda)}.$$

For a particle with an initial condition $(x(0), \dot{x}(0)) = (\alpha, 0)$ in a given potential $\lambda u(x)$, above definition of the probability gives the following probability density function $p(x, \alpha)$ of $x$

$$p(x, \alpha) := \frac{2}{T(\lambda) \sqrt{2(\lambda u(\alpha) - \lambda u(x))}}.$$

Consider a quench process given by an instantaneous change $\lambda_0 \rightarrow \lambda_1$ of the coupling constant $\lambda$. If this instantaneous change occurs at time $t$, the change of the particle energy is given by $(\lambda_1 - \lambda_0) u(x(t))$. The work $-(\lambda_1 - \lambda_0) u(x(t))$ done by particle in this quench process depends on $t$. To evaluate the work done by the particle in the quench process, we employ the average of $u(x(t))$ over the oscillation period $T(\lambda_0)$ represented in terms of the integration over $x \in (-\alpha_0, \alpha_0)$

$$\bar{u}(\lambda_0) = \frac{2}{T(\lambda_0)} \int_{0}^{T(\lambda_0)} u(x(t)) dt$$

$$= \int_{-\alpha}^{\alpha} u(x) p(x, \alpha_0) dx$$

$$= \frac{2}{T(\lambda_0)} \int_{-\alpha}^{\alpha} \frac{u(x)}{\sqrt{2(2E(\lambda_0) - \lambda_0 u(x))}} dx.$$
Note that the average \( \bar{u}(\lambda) \) in this probability is given by the derivative \( E'(\lambda) \) of the function \( E(\lambda) \) defined by (11) and (15) implicitly

\[
E'(\lambda) = \bar{u}(\lambda).
\] (13)

This formula is a classical analogue to the Hellmann-Feynman theorem in quantum mechanics\cite{16,17}. Therefore, the average work done by the quench process \( \lambda_0 \to \lambda_1 \) is

\[
(\lambda_0 - \lambda_1)E'(\lambda_0).
\] (14)

If the quasi-static process gives the maximal work, we have the following inequality

\[
(\lambda_0 - \lambda_1)E'(\lambda_0) \leq E(\lambda_0) - E(\lambda_1).
\] (15)

This implies the concavity of the function \( E(\lambda) \).

Next, we consider \( N \) independent particles confined by the same confining potential. The mechanical state is described by a set \( x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \) of positions. The total energy \( E \) is given by a summation of energy over all particles with a set \( \alpha(\lambda) \) := \( (\alpha_1(\lambda), \alpha_2(\lambda), \ldots, \alpha_N(\lambda)) \) \( \in \mathbb{R}^N \) of amplitudes

\[
E(\alpha(\lambda)) := \sum_{n=1}^{N} E_n(\lambda),
\] (16)

where the energy of \( n \)-th particle is defined by \( E_n(\lambda) := \lambda u(\alpha_n(\lambda)) \). Since these particles are independent, the probability density function of \( x \in \mathbb{R}^N \) is given by a product of that of the position coordinate of each particle

\[
P(x, \alpha(\lambda)) := \prod_{n=1}^{N} p(x_n, \alpha_n(\lambda)).
\]

Then, the average work done by \( N \) particles in a quench process \( \lambda_0 \to \lambda_1 \) is

\[
(\lambda_0 -\lambda_1) \sum_{n=1}^{N} \bar{u}(\alpha_n(\lambda_0)) = (\lambda_0 - \lambda_1) \sum_{n=1}^{N} E_n'(\lambda).
\] (17)

The work in the corresponding quasi-static process is

\[
E(\alpha(\lambda_0)) - E(\alpha(\lambda_1)) = \sum_{n=1}^{N} (E_n(\lambda_0) - E_n(\lambda_1)).
\]

Therefore, the work in quench process cannot exceed that in the corresponding quasi-static process, if

\[
(\lambda_0 - \lambda_1)E_n'(\lambda) \leq E_n(\lambda_0) - E_n(\lambda_1),
\]

which is guaranteed by the concavity of each \( E_n(\lambda) \). We prove the concavity of a single particle energy in the following.

### III. Proof of Concavity

To prove the concavity of the function \( E(\lambda) \), first we show that increasing in \( \lambda \) of \( \tau(0,\alpha,\lambda) \) implies the concavity of the energy function \( E(\lambda) \) with \( E(\lambda) = \lambda u(\alpha) \). The derivative of the energy function is

\[
E'(\lambda) = \bar{u}(\lambda) = \frac{E(\lambda) - \bar{K}(\lambda)}{\lambda}.
\] (18)

where the average of the kinetic energy as a function of \( \lambda \) is defined by

\[
\bar{K}(\lambda) = \frac{\sqrt{2}}{T(\lambda)} \int_{-\alpha(\lambda)}^{\alpha(\lambda)} \sqrt{E(\lambda) - \lambda u(x)} dx = \frac{I}{2T(\lambda)}.
\] (19)

Note that the identities (18) and (19) give the second derivative \( E''(\lambda) \) of the energy function

\[
E''(\lambda) = \frac{-\bar{K}'(\lambda)}{\lambda} = \frac{IT'(\lambda)}{2T(\lambda)^2}.
\] (20)

This indicates that the positive semi-definiteness of \( \bar{K}'(\lambda) \) or \( -T'(\lambda) \) implies the concavity of the energy function \( E(\lambda) \). Intuitively, it is obvious that \( \bar{K}(\lambda) \) is monotonically increasing, if \( E(\lambda) \) is monotonically increasing. Since the change \( \Delta E = E(\lambda_1) - E(\lambda_0) \) by a quasi-static operation \( \lambda_0 \to \lambda_1 \) is shared by \( \Delta \bar{K} = \bar{K}(\lambda_1) - \bar{K}(\lambda_0) \) and \( \Delta \bar{U} = \lambda_1 \bar{u}(\lambda_1) - \lambda_0 \bar{u}(\lambda_0) \)

\[
\Delta E = \Delta \bar{K} + \Delta \bar{U},
\]

then, all signs of \( \Delta E, \Delta \bar{K} \) and \( \Delta \bar{U} \) should be the same.

Let us prove the concavity of the energy function \( E(\lambda) \). To this end, we prove the non-positivity of derivative \( T'(\lambda) \) on \( \lambda \). This derivative has an apparent divergence at \( x = \alpha \) in the following naive calculation

\[
\frac{1}{2} T'(\lambda) = \frac{d}{d\lambda} \tau(0,\alpha(\lambda),\lambda)
\]

\[
= \frac{\sqrt{2} \alpha'(\lambda)}{E(\lambda) - \lambda u(\alpha)} - \frac{1}{\sqrt{2}} \int_{0}^{\alpha(\lambda)} \frac{E'(\lambda) - u(x)}{(E(\lambda) - \lambda u(x))^2} dx,
\]

since \( E(\lambda) = \lambda u(\alpha(\lambda)) \). To regularize the integration at the boundary \( x = \alpha(\lambda) \), we decompose \( \tau(0,\alpha(\lambda),\lambda) \) into the following three terms

\[
\tau(0,\alpha(\lambda),\lambda) = \tau(0,c,\lambda) + \tau(c,\alpha(\lambda) - \epsilon,\lambda) + \tau(\alpha(\lambda) - \epsilon,\alpha(\lambda),\lambda),
\] (22)

where \( 0 < c < \alpha(\lambda) - \epsilon < \alpha(\lambda) \), and \( u(c) < \bar{u}(\lambda) \) with \( E(\lambda) = \lambda u(\alpha(\lambda)) \). We show that each term is monotonically decreasing. Note the following relation

\[
E'(\lambda) = u(\alpha(\lambda)) + \lambda u'(\alpha(\lambda)) \alpha'(\lambda).
\] (23)

The the derivative of the first term in (23) is obviously non-positive

\[
\frac{d}{d\lambda} \tau(0,c,\lambda) = -\frac{1}{\sqrt{2}} \int_{0}^{c} \frac{\bar{u}(\lambda) - u(x)}{(E(\lambda) - \lambda u(x))^2} dx.
\] (24)
The derivative of the second term in (23) is
\[ \frac{d}{d\lambda} \tau(c, \alpha(\lambda) - \epsilon, \lambda) \]
\[ = \sqrt{2} \alpha'(\lambda) \frac{E'(\lambda) - u(c)}{\sqrt{E(\lambda) - \lambda u(\alpha(\lambda) - \epsilon)}} - \frac{1}{\sqrt{2}} \int_c^{\alpha(\lambda) - \epsilon} \frac{E'(\lambda) - u(x)}{(E(\lambda) - \lambda u(x))^{1/2}} \, dx. \]  
(25)
(26)
(27)

An integration by parts in the second term gives
\[ \frac{1}{\sqrt{2}} \int_c^{\alpha(\lambda) - \epsilon} \frac{E'(\lambda) - u(x)}{(E(\lambda) - \lambda u(x))^{1/2}} \, dx \]
\[ = \frac{1}{\sqrt{2}} \int_c^{\alpha(\lambda) - \epsilon} \frac{E'(\lambda) - u(x)}{(E(\lambda) - \lambda u(x))^{1/2}} \, \lambda u'(x) \, dx \]
\[ = \left[ \frac{\sqrt{2}}{(E(\lambda) - \lambda u(x))^{1/2}} \frac{E'(\lambda) - u(x)}{\lambda u'(x)} \right]_c^{\alpha(\lambda) - \epsilon} \]
\[ - \int_c^{\alpha(\lambda) - \epsilon} \frac{\sqrt{2}}{(E(\lambda) - \lambda u(x))^{1/2}} \frac{d}{dx} \frac{E'(\lambda) - u(x)}{\lambda u'(x)} \, dx. \]  
(28)
(29)
(30)
(31)
(32)

There exists a bounded positive-valued function \( f_1(\epsilon) \leq C_1 \sqrt{\epsilon} \) with a positive constant \( C_1 \) independent of \( \epsilon \), such that (31) is
\[ \left[ \frac{\sqrt{2}}{(E(\lambda) - \lambda u(x))^{1/2}} \frac{E'(\lambda) - u(x)}{\lambda u'(x)} \right]_c^{\alpha(\lambda) - \epsilon} \]
\[ = -\frac{\sqrt{2}}{(E(\lambda) - \lambda u(c) - \epsilon))^{1/2}} \lambda u'(\alpha(\lambda) - \epsilon) \]
\[ \frac{\sqrt{2}}{(E(\lambda) - \lambda u(c))^{1/2}} \lambda u'(c) \]
\[ = \sqrt{E'(\lambda) - u(c(\alpha(\lambda) - \epsilon))} + f_1(\epsilon) \]
\[ - \frac{\sqrt{2}}{(E(\lambda) - \lambda u(c))^{1/2}} \frac{E'(\lambda) - u(c)}{\lambda u'(c)}. \]  
(33)
(34)
(35)
(36)
(37)

Therefore this and (24) give
\[ \frac{d}{d\lambda} \tau(c, \alpha(\lambda) - \epsilon, \lambda) \]
\[ = -f_1(\epsilon) + \frac{\sqrt{2}}{(E(\lambda) - \lambda u(c))^{1/2}} \frac{E'(\lambda) - u(c)}{\lambda u'(c)} \]
\[ + \int_c^{\alpha(\lambda) - \epsilon} \frac{\sqrt{2}}{(E(\lambda) - \lambda u(x))^{1/2}} \frac{d}{dx} \frac{E'(\lambda) - u(x)}{\lambda u'(x)} \, dx. \]  
(38)
(39)

This gives another expression
\[ \frac{d}{d\lambda} \tau(c, \alpha(\lambda) - \epsilon, \lambda) \]
\[ = -f_1(\epsilon) + \frac{\sqrt{2}}{(E(\lambda) - \lambda u(c))^{1/2}} \frac{E'(\lambda) - u(c)}{\lambda u'(c)} \]
\[ + \int_c^{\alpha(\lambda) - \epsilon} \frac{\sqrt{2}}{(E(\lambda) - \lambda u(x))^{1/2}} \frac{d}{dx} \frac{E'(\lambda) - u(x)}{\lambda u'(x)} \, dx. \]  
(40)

This is negative semi-definite, since \( u(c) \leq u(c') \). Finally, we show that the third term in (23) is infinitesimal. An integration by parts in the third term in (23) gives
\[ \tau(\alpha(\lambda) - \epsilon, \alpha(\lambda), \lambda) = \int_c^{\alpha(\lambda) - \epsilon} \frac{\sqrt{2}}{(E(\lambda) - \lambda u(x))^{1/2}} \, dx \]
\[ = \int_c^{\alpha(\lambda) - \epsilon} \frac{\sqrt{2}}{(E(\lambda) - \lambda u(x))} \frac{d}{dx} \frac{E'(\lambda) - u(x)}{\lambda u'(x)} \, dx \]
\[ = \int_c^{\alpha(\lambda) - \epsilon} \left[ \frac{2\sqrt{2}(E(\lambda) - \lambda u(x))}{(-\lambda u'(x))^2} \right]^{\alpha(\lambda) - \epsilon} \frac{d}{dx} \frac{E'(\lambda) - u(x)}{\lambda u'(x)} \, dx. \]  
(41)
(42)
(43)
(44)
(45)
(46)
The derivative of the above in $\lambda$ is

$$
\frac{d}{d\lambda} \tau(\alpha(\lambda) - \epsilon, \alpha(\lambda), \lambda)
= \sqrt{2(E(\lambda) - \lambda u(\alpha(\lambda) - \epsilon))} \frac{\partial}{\partial \lambda} \lambda u'(\alpha(\lambda) - \epsilon)
+ \sqrt{2} \frac{E'(\lambda) - u(\alpha(\lambda) - \epsilon) - \lambda u'(\alpha(\lambda) - \epsilon) \alpha'(\lambda)}{\lambda u'(\alpha(\lambda) - \epsilon)}
\times \int_{\alpha(\lambda) - \epsilon}^{\alpha(\lambda)} \left[ \frac{2 \sqrt{2(E(\lambda) - \lambda u(x))}}{\lambda^2} - \frac{2 E'(\lambda) - 2u(x)}{\lambda \sqrt{2(E'(\lambda) - \lambda u(x))}} \right]
\times \frac{d}{dx} \frac{1 - (u'(\alpha(\lambda) - \epsilon))'}{u'(\alpha(\lambda) - \epsilon)}.
$$

Taylor’s theorem for a differentiable function $f$ implies that there exists $\theta \in (0, 1)$, such that

$$
f(\alpha(\lambda) - \epsilon) = f(\alpha(\lambda)) - \epsilon f'(\alpha(\lambda) - \theta \epsilon).
$$

Therefore, there exists a positive number $C_2$ independent of $\epsilon$ such that the absolute value of the above is bounded by

$$
\left| \frac{d}{d\lambda} \tau(\alpha(\lambda) - \epsilon, \alpha(\lambda), \lambda) \right| = |f_2(\epsilon)| \leq C_2 \sqrt{\epsilon}.
$$

Finally, we have

$$
\frac{d}{d\lambda} \tau(0, \alpha(\lambda), \lambda)
= \frac{d}{d\lambda} \tau(0, c, \lambda) + \frac{d}{d\lambda} \tau(c, \alpha(\lambda) - \epsilon, \lambda)
+ \frac{d}{d\lambda} \tau(\alpha(\lambda) - \epsilon, \alpha(\lambda), \lambda)
= \frac{1}{\sqrt{2}} \int_0^c \frac{\bar{u}'(\lambda) - u(x)}{(E(\lambda) - \lambda u(x))^2} dx
+ \frac{\sqrt{2} E'(\lambda) - u(c)}{\lambda u'(c)}
+ \int_{\alpha(\lambda) - \epsilon}^{\alpha(\lambda)} \left[ \frac{\sqrt{2}}{(E(\lambda) - \lambda u(x))^2} - \frac{\sqrt{2}}{(E(\lambda) - \lambda u(c'))^2} \right]
\times \frac{\bar{u}(\lambda) - u(c)}{\lambda u'(\epsilon)}
\times \frac{u(\alpha(\lambda) - \epsilon) - \bar{u}(\lambda)}{\lambda u'(\alpha(\lambda) - \epsilon)} \sqrt{\frac{\sqrt{2}}{(E(\lambda) - \lambda u(c'))^2}}.
$$

Since $u(\epsilon) \leq \bar{u}(\lambda) \leq u(\epsilon') \leq u(\alpha)$ and $\epsilon > 0$ is arbitrary, the above estimates give

$$
\frac{d}{d\lambda} \tau(0, \alpha(\lambda), \lambda) \leq 0.
$$

This and the relation (20) imply that the energy function $E(\lambda)$ is concave. Therefore the work in the quench process $\lambda_0 \rightarrow \lambda_1$ cannot exceed that in the quasi-static process as given by the inequality (15).

IV. DISCUSSIONS

First, we summarize our results. We discuss classical dynamics of non-interacting particles confined by a time-dependent confining potential. The energy function can be regarded as a function of the coupling constant, such that the adiabatic invariant is constant of motion with infinitesimally slow change of the coupling constant. We have shown that the work done by the particles in a quench process cannot exceed that in quasi-static process, if the energy is a concave function of the coupling constant. We have proven the concavity in a classical single particle in a general potential, then also the energy of non-interacting particles is concave.

Next, we discuss a sequential process and its convergence to the quasi-static process in classical dynamics. Consider a composite process which consists of quench and waiting operations. Let $(\lambda_l)_{l=0,1,2,\ldots,L}$ be a sequence of coupling constants, and consider $l$-th process which consists of quench operation $\lambda_{l-1} \rightarrow \lambda_l$ and leaving the system during a time interval $(t_{l-1}, t_l)$. If the waiting time $t_{l-1} - t_{l-2}$ in the $(l-1)$-th process is sufficiently long, the work done by the particle in the quench process $\lambda_{l-1} \rightarrow \lambda_l$ is given by $(\lambda_{l-1} - \lambda_l) \bar{u}(\lambda_{l-1}) = (\lambda_{l-1} - \lambda_l) E'(\lambda_{l-1})$. Then, the total amount of the work done by the particle in this sequential process is

$$
\sum_{l=1}^{L} (\lambda_{l-1} - \lambda_l) E'(\lambda_{l-1}).
$$

Let us consider the limit $L \rightarrow \infty$. If $\lim_{L \rightarrow \infty} \lambda_L = \lambda_\infty$ converges and the interval $\sup_l |\lambda_{l-1} - \lambda_l| \rightarrow 0$, the total amount of work converges to

$$
\lim_{L \rightarrow \infty} \sum_{l=1}^{L} (\lambda_{l-1} - \lambda_l) E'(\lambda_{l-1}) = - \int_{\lambda_0}^{\lambda_\infty} E'(\lambda) d\lambda = E(\lambda_0) - E(\lambda_\infty),
$$

which is identical to the work in the quasi-static process $\lambda_0 \rightarrow \lambda_\infty$. Note that $t_L - t_0$ diverges as $L \rightarrow \infty$ to obtain the quasi-static process as this limit. We consider that a general process with a finite operation speed can be constructed by a certain limit of a suitable sequential operation with a finite $t_L - t_0$. On the other hand, no waiting time $t_l - t_{l-1} = 0$ implies that the
process $\lambda_{l-1} \to \lambda_l \to \lambda_{l+1}$ becomes a single quench process, and the amount of work in this process is given by $(\lambda_{l-1} - \lambda_l + 1)\bar{u}(\lambda_{l-1})$.

Here, we comment on generalizations of our arguments. In the argument of classical dynamics, the condition on the confining potential can be relaxed. The following conditions on $u(x)$

$$\int_{0}^{c} \frac{\bar{u}(\lambda) - u(x)}{(E(\lambda) - \lambda u(x))^2} \, dx \geq 0, \quad u(c) \leq u(c')$$

are sufficient for the validity of the inequality to guarantee the concavity of $E(\lambda)$. Confining potential $u(x)$ in wider class satisfies the above condition, and amount of work in quasi-static operation $\lambda_0 \to \lambda_1$ is larger than that in the quench operation. Although we have assumed $u(-x) = u(x)$ for simplicity, the argument can be easily extend to general functions.

Finally, we discuss Maxwell’s demon in this system. The maximal work $W_{\text{max}}(\lambda_0 \to \lambda_1)$ done by the particle in the quench operation $\lambda_0 \to \lambda_1$ is given by

$$W_{\text{max}}(\lambda_0 \to \lambda_1) = \begin{cases} (\lambda_0 - \lambda_1)u(\alpha(\lambda_0)) & (\lambda_0 > \lambda_1) \\ 0 & (\lambda_0 \leq \lambda_1) \end{cases}.$$

This is larger than the work

$$W_{\text{qs}}(\lambda_0 \to \lambda_1) = E(\lambda_0) - E(\lambda_1),$$
in quasi-static operation $\lambda_0 \to \lambda_1$ as the maximal work in thermodynamics, since $E(\lambda)$ is convex and

$$E'(\lambda_0) = \bar{u}(\lambda_0) \leq u(\alpha(\lambda_0)).$$

For $\lambda_0 > \lambda_1$, $W_{\text{max}}(\lambda_0 \to \lambda_1) = (\lambda_0 - \lambda_1)u(\alpha(\lambda_0))$ is extracted by a quench expansion $\lambda_0 \to \lambda_1$ when the particle is at $x = \pm\alpha(\lambda_0)$, and for $\lambda_0 \leq \lambda_1$, no work $W_{\text{max}}(\lambda_0 \to \lambda_1) = 0$ is extracted by the quench compression at the origin $x = 0$. Note that the difference

$$\Delta W := W_{\text{max}}(\lambda_0 \to \lambda_1) - W_{\text{qs}}(\lambda_0 \to \lambda_1),$$
between the maximal quench work and the quasi-static work is

$$\Delta W = \begin{cases} \lambda_0 u(\alpha(\lambda_0)) - \lambda_1 u(\alpha(\lambda_1)) & (\lambda_0 > \lambda_1) \\ \lambda_0 u(\alpha(\lambda_0)) - \lambda_1 u(\alpha(\lambda_1)) & (\lambda_0 \leq \lambda_1) \end{cases}.$$

If Maxwell’s demon can control each potential confining the corresponding particle, the total amount of work exceeds the work in the quasi-static process.

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