Solving Atanassov’s I-fuzzy Linear Programming Problems Using Hurwicz’s Criterion

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**ABSTRACT**

Dubey et al. [40] have shown that solving an Atanassov’s I-fuzzy Linear Programming Problem represented by Atanassov’s I-fuzzy sets with linear membership and non-membership functions is equivalent to solving an appropriate fuzzy optimisation problem with piecewise linear S-shaped membership functions. The equivalence is established using Hurwicz optimism–pessimism criterion [38] and indeterminacy resolution in Atanassov’s I-fuzzy sets. Moreover, in case of convex break points in the piecewise linear membership function, the crisp counterpart of the equivalent optimisation problem involves binary variables. Here, in this paper we first convert the resulting fuzzy optimisation problem having convex break points into an equivalent fuzzy optimisation problem having concave break points on the lines of Inuiguchi et al. [34], before formulating its crisp equivalent. The advantage of this strategy is that the resulting crisp equivalent problem has no binary variables. Further, we also make use of the indeterminacy factor resolution principle to establish a duality relation which can be interpreted as a Atanassov’s I-fuzzy variant of the (crisp) weak duality theorem.

**KEYWORDS**

Atanassov's I-fuzzy sets; piecewise linear membership function; I-fuzzy goals; I-fuzzy decision; fonts; references; I-fuzzy duality

1. Introduction

Atanassov [1–3] integrated the notion of hesitancy degree in a fuzzy set by adding a new component which describes the degree of non-membership of an element in a given fuzzy set and called such a set as Atanassov’s I-fuzzy set or intuitionistic fuzzy set. While the definition of fuzzy set provides the degree of membership of an element in a given set and its non-membership degree is understood as one minus its membership degree, the definition of Atanassov’s I-fuzzy set provides more-or-less independent degrees of membership and non-membership of an element in a given set. The only requirement in latter is that the sum of two degrees is less than or equal to 1. As a result, an Atanassov’s I-fuzzy set exhibits characteristics of affirmation, negation and hesitation. For instance, in any confronting situation in decision making, beside support or positive response, objection or negative response, there could be an abstention which indicates hesitation or indeterminacy in response to the situation. For detailed description and properties for Atanassov’s...
I-fuzzy sets, we may refer to Atanasov [4], Szmidt and Kacprzyk [5] and other references cited therein.

The domain of Atanassov’s I-fuzzy set or intuitionistic fuzzy set is not devoid of its share of controversies (see [6, 7]). The nomenclature of intuitionistic fuzzy set itself was an issue of debate because same nomenclature had also been used in intuitionistic fuzzy logic, and the two differ in their mathematical structure and treatment. It obviously makes sense to avoid using same terminology for two different concepts. As suggested in [6] and [7], in this paper, Atanassov’s I-fuzzy set or intuitionistic fuzzy set is simply called as I-fuzzy set (IFS).

The I-fuzzy sets have been widely applied in real life decision-making problems. We may refer to [8] for clustering, medical diagnosis [9–18] for multi-criteria decision making, [19, 20] for pattern recognition and [21–24] for multi-attribute group decision-making problems.

Inspired by the celebrated work of Bellman and Zadeh [25] and Zimmermann [26], Angelov [27] studied an optimisation problem with I-fuzzy set and illustrated its application in a classical transportation problem. On the lines of Bector et al. [28], Aggarwal et al. [29, 30] studied duality for linear programming problems with I-fuzzy inequalities and discussed their application in zero sum matrix games. In all above cases, the membership and non-membership functions which define an I-fuzzy set were restricted to be linear.

Several attempts have been reported in the literature to study different preferences of decision maker by taking different types of membership functions. Hannan [31] interpolated fuzzy set defining goals and constraints by piecewise linear concave membership functions and solve the same with goal programming approach. Nakamura [32], Yang et al. [33] and Inuiguchi et al. [34] proposed different techniques to solve fuzzy optimisation problem with piecewise linear (quasi-concave) membership function. In Nakamura’s [32] approach, a subsidiary piecewise linear function was introduced, which separates the whole membership function into a finite number of concave and convex subfunctions. In this way, a piecewise linear membership function is expressed in terms of logical functions and the fuzzy optimisation problem is studied by solving finite number of subproblems. Yang et al. [33] considered piecewise linear S-shaped membership function and reformulate its crisp equivalent as an integer linear program having binary variables. This method was further developed by Li and Yu [35], Lin and Chen [36] and Chang [37]. As the presence of binary variables increases the computational burden on the solution process, Inuiguchi et al. [34] proposed a method to transform the piecewise linear membership function with convex break points into piecewise linear membership function with only concave break points, consequently getting an equivalent crisp optimisation model devoid of binary variables.

Motivated by the works of Hurwicz [38] and Yager [39], Dubey et al. [40] studied an I-fuzzy optimisation problem with I-fuzzy goals, by resolving the indeterminacy factor before applying the Bellman and Zadeh extension principle [25] to it. In this process of conversion of an I-fuzzy optimisation problem with I-fuzzy goals into a fuzzy optimisation problem using Hurwicz optimism–pessimism criterion [38], the associated membership functions come out to be piecewise linear S-shaped involving convex and concave break points. The convex break points give rise to binary variables in an equivalent crisp optimisation model.

The aim of this paper is to review Dubey et al. [40] approach to study an I-fuzzy optimisation problem with I-fuzzy goals. Recently, Inuiguchi et al. [34] proposed a method which
transform the piecewise linear membership function with convex break points into piecewise linear membership function with only concave break points. Here we use Inuiguchi et al. [34] approach to convert the fuzzy optimisation problem having convex break points into an equivalent fuzzy optimisation problem having concave break points. The advantage is that the resultant crisp equivalent of an I-fuzzy optimisation problem has no binary variables. This approach enabled us to propose a dual problem for I-fuzzy optimisation problem. Further, we establish a weak duality relation between them, which can be interpreted as an I-fuzzy version of the (crisp) weak duality theorem.

The paper is organised as follows. Section 2 presents the preliminary definitions related to I-fuzzy sets and its subsequent sections respectively review the decision-making criteria in an I-fuzzy environment and crisp equivalent of an I-fuzzy optimisation problem presented by Dubey et al. [40]. Section 3 details various steps of an algorithm for converting the resultant fuzzy optimisation problem of an I-fuzzy linear programming problem under consideration, having piecewise linear membership function with convex break points into an equivalent another fuzzy optimisation problem having piecewise linear concave break points. An example is presented to illustrate the algorithm. In Section 4, we study the duality for the same on the lines of resolving the indeterminacy first and establish the weak duality relation for the same. A numerical example is solved to support the theory. Section 5 is the concluding section.

2. Preliminaries

In this section, we first recall few notations and definitions from references [1–3, 39–41] with regard to IFS. Further, on the lines of Yager’s [39] comment on the Angelov’s [27] model for decision making in IFS, we briefly recall Dubey et al. [40] models in this regard.

**Definition 1 (IFS):** Let $X$ be a universal set. An IFS (an intuitionistic fuzzy set in [1]) $\tilde{A}$ in $X$ is described by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x), v_{\tilde{A}}(x)) \mid x \in X\}$ where $\mu_{\tilde{A}} : X \rightarrow [0, 1]$ and $v_{\tilde{A}} : X \rightarrow [0, 1]$ define respectively the degree of belongingness and the degree of non-belongingness of an element $x \in X$ to the set $\tilde{A}$ with $0 \leq \mu_{\tilde{A}}(x) + v_{\tilde{A}}(x) \leq 1$.

If $\mu_{\tilde{A}}(x) + v_{\tilde{A}}(x) = 1$, for all $x \in X$, then $\tilde{A}$ degenerates to a standard fuzzy set.

**Definition 2 (Set theoretic operations in IFS):** Let $\tilde{A}$ and $\tilde{B}$ be two IFSs in $X$. Their standard union is an IFS $\tilde{C}$, $\tilde{C} = \tilde{A} \cup \tilde{B}$, and is defined as

$$\tilde{C} = \{(x, \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \min(v_{\tilde{A}}(x), v_{\tilde{B}}(x))) \mid x \in X\}. $$

Similarly their standard intersection is an IFS $\tilde{D}$, $\tilde{D} = \tilde{A} \cap \tilde{B}$, defined as

$$\tilde{D} = \{(x, \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \max(v_{\tilde{A}}(x), v_{\tilde{B}}(x))) \mid x \in X\}. $$

Further, the standard negation of $\tilde{A}$ is an IFS $\tilde{E}$, $\tilde{E} = \tilde{A}'$, defined as

$$\tilde{E} = \{(x, v_{\tilde{A}}(x), \mu_{\tilde{A}}(x)) \mid x \in X\}. $$
Definition 3 (Score function): Let $\tilde{A}$ be an IFS in $X$. Then the function $s_{\tilde{A}}(x)$ given by

$$s_{\tilde{A}}(x) = \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x), \quad x \in X,$$

is called the score function of $\tilde{A}$. It measures the degree of suitability with respect to a set of criteria represented by vague values.

Definition 4 (Measure of indeterminacy): Let $\tilde{A}$ be an IFS set in $X$. Then the value $\pi_{\tilde{A}}(x)$ given by

$$\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x), \quad x \in X,$$

is called the measure of indeterminacy or undecidedness of $\tilde{A}$ [2].

Note that the range of undecidedness is an interval $[\mu_{\tilde{A}}(x), 1 - \nu_{\tilde{A}}(x)]$ and the measure of indeterminacy is its length. Now using Hurwicz optimism–pessimism criterion [38], for a fixed $\lambda$, $\lambda \in [0, 1]$, an IFS $\tilde{A}$ will get transformed into a fuzzy set $\tilde{A}$ with membership function given by

$$f_{\tilde{A}}(\lambda, x) = (1 - \lambda)\mu_{\tilde{A}}(x) + \lambda(1 - \nu_{\tilde{A}}(x)), \quad x \in X.$$

Henceforth, we shall be calling this function as indeterminacy resolving function of $\tilde{A}$. The parameter $\lambda$ depicts the attitude of the decision maker when $\lambda = 0$ means indeterminacy is fully resolved in favour of membership (optimism) while $\lambda = 1$ means indeterminacy get resolved fully in negation or the non-membership function (pessimism).

2.1. Decision Making in I-fuzzy Environment

Let $X$ be any set. Let $G_l, l = 1, \ldots, r$, be the set of $r$ goals and $C_i, i = 1, \ldots, m$, be the set of $m$ constraints, each of which can be characterised by an IFS on $X$. Using the Bellman and Zadeh’s extension principle [25], Angelov [27] defined the I-fuzzy decision $\tilde{D} = (\tilde{G}_1 \cap \tilde{G}_2 \cap \cdots \cap \tilde{G}_r) \cap (\tilde{C}_1 \cap \tilde{C}_2 \cap \cdots \cap \tilde{C}_m)$ as an IFS, defined as $\tilde{D} = \{(x, \mu_{\tilde{D}}(x), \nu_{\tilde{D}}(x)) \mid x \in X\}$, where

$$\mu_{\tilde{D}}(x) = \min_{l,i} \left\{ \mu_{\tilde{G}_l}(x), \mu_{\tilde{C}_i}(x) \right\},$$

and

$$\nu_{\tilde{D}}(x) = \max_{l,i} \left\{ \nu_{\tilde{G}_l}(x), \nu_{\tilde{C}_i}(x) \right\}.$$

Let $s_{\tilde{D}}(x) = \mu_{\tilde{D}}(x) - \nu_{\tilde{D}}(x), \ x \in X$, be the score function of the IFS $\tilde{D}$. Then $\bar{x} \in X$ is an optimal decision in an I-fuzzy scenario if $s_{\tilde{D}}(\bar{x}) \geq s_{\tilde{D}}(x)$, for all $x \in X$, i.e. $s_{\tilde{D}}(\bar{x}) = \max_{x \in X} s_{\tilde{D}}(x)$. Let $\alpha$ and $\beta$ denote respectively the minimal degree of acceptance and the maximal degree of rejection of the I-fuzzy decision set $\tilde{D}$. The I-fuzzy decision problem of maximising the score function $s_{\tilde{D}}(x)$ of the I-fuzzy decision $\tilde{D}$ is transformed by Angelov [27] into the following
optimisation problem:
\[
\max (\alpha - \beta)
\]
subject to
\[
\mu_{G_l}(x) \geq \alpha, \quad l = 1, \ldots, r,
\]
\[
v_{G_l}(x) \leq \beta, \quad l = 1, \ldots, r,
\]
\[
\mu_{C_i}(x) \geq \alpha, \quad i = 1, \ldots, m,
\]
\[
v_{C_i}(x) \leq \beta, \quad i = 1, \ldots, m,
\]
\[
\alpha \geq \beta \geq 0, \quad \alpha + \beta \leq 1, \quad x \in X.
\]

Yager [39] pointed out certain demerits of this approach. For instance, if \( x \) and \( y \) are two alternatives with \( \mu_{\tilde{A}}(x) = 0.49, v_{\tilde{A}}(x) = 0.51 \) and \( \mu_{\tilde{A}}(y) = 0, v_{\tilde{A}}(y) = 0 \), then the score function approach selects \( y \) over \( x \) despite \( y \) having membership function value zero. Yager [39] therefore suggested an alternative view by resolving the indeterminacy using Hurwicz optimism–pessimism criterion [38]. Dubey et al. [40] implemented Yager’s [39] ideas to study optimisation problems involving interval uncertainty represented by IFSs. Let us describe their [40] decision-making approach in I-fuzzy environment.

Let \( \lambda \in [0, 1] \) be fixed, then the indeterminacy resolving function for a fuzzy set \( \tilde{D} \) associated with the I-fuzzy decision set \( \tilde{D} \) is
\[
f_{\tilde{D}}(\lambda, x) = \min_{l, i} \left\{ f_{\tilde{G}_l}(\lambda, x), f_{\tilde{C}_i}(\lambda, x) \mid x \in X \right\},
\]
where \( f_{\tilde{G}_l}(\lambda, x) \) and \( f_{\tilde{C}_i}(\lambda, x) \) are the indeterminacy resolving functions associated with the \( l \)th goal and \( i \)th constraint respectively. Then \( \tilde{x} \in X \) is an \textit{optimal decision}, if \( f_{\tilde{D}}(\lambda, \tilde{x}) \geq f_{\tilde{D}}(\lambda, x) \), for all \( x \in X \), i.e. \( f_{\tilde{D}}(\lambda, \tilde{x}) = \max_{x \in X} f_{\tilde{D}}(\lambda, x) \).

Hence solving an I-fuzzy linear programming problem with I-fuzzy goal is equivalent to solve the following optimisation problem:
\[
\max \alpha
\]
subject to
\[
f_{\tilde{G}_l}(\lambda, x) \geq \alpha, \quad l = 1, \ldots, r,
\]
\[
f_{\tilde{C}_i}(\lambda, x) \geq \alpha, \quad i = 1, \ldots, m,
\]
\[
0 \leq \alpha \leq 1, \quad x \in X.
\]

Next we describe an application of decision-making criterion for solving I-fuzzy linear programming with I-fuzzy goal.

\section*{2.2. I-fuzzy Linear Programming: A Review}

Let \( \mathbb{R}^n \) denote the \( n \)-dimensional Euclidean space and \( \mathbb{R}_+^n \) be its non-negative orthant. Let \( c \in \mathbb{R}^n, b \in \mathbb{R}^m \) and \( A \in \mathbb{R}^{m \times n} \). Consider a general model for I-fuzzy linear programming problem (IFLP) in which the aspiration level \( Z_0 \) for the objective function (I-fuzzy goal) is indicated by the decision maker.
(IFLP) Find \( x \in \mathbb{R}^n \) such that

\[
\begin{align*}
  c^T x & \succsim^I Z_0, \\
  A_ix & \preclim^I b_i, \quad i = 1, \ldots, m, \\
  x & \geq 0.
\end{align*}
\]

Here \( \succsim^I \) and \( \preclim^I \) are the I-fuzzy versions of the fuzzy symbols \( \succsim \) and \( \preclim \) respectively and have interpretation of ‘essentially greater than or equal to’ and ‘essentially less than or equal to’ in I-fuzzy sense. We may recall Aggarwal et al. [30] and Dubey et al. [40] to understand the meaning of an I-fuzzy inequality of type \( x \succsim^I a \).

Let \( p_i, q_i, 0 < q_i < p_i, i = 0, 1, \ldots, m, \) denote the tolerances associated respectively with the acceptance and the rejection of \( m+1 \) constraints in (IFLP) with \( i = 0 \) stands for \( c^T x \geq Z_0 \). Let \( f_0(\lambda, c^T x) \) and \( f_i(\lambda, A^T_ix), i = 1, \ldots, m, \) be their indeterminacy resolving functions. In [40], in absence of any information on attitude of the decision maker, authors took \( \lambda = \frac{1}{2} \) only. Then \( f_0(\frac{1}{2}, c^T x) \) and \( f_i(\frac{1}{2}, A^T_ix), i = 1, 2, \ldots, m, \) are respectively written as \( f_0(c^T x) \) and \( f_i(A^T_ix), i = 1, 2, \ldots, m. \) As we have to define the membership and non-membership functions for each I-fuzzy inequality, two approaches have been given in [40] to define it. We discuss these approaches as follows.

2.2.1. Pessimistic Approach

The decision maker has the pessimistic attitude in acceptance. A complete rejection of a criterion does not amount to fully accepting it. In this case, the indeterminacy resolving functions associated with the \( m+1 \) constraints of (IFLP) are described as follows:

\[
f_0(c^T x) = \begin{cases} 
0, & c^T x \leq Z_0 - p_0, \\
\frac{p_0 + q_0}{2p_0} \left(1 + \frac{c^T x - Z_0}{p_0}\right), & Z_0 - p_0 \leq c^T x \leq Z_0 - p_0 + q_0, \\
\frac{c^T x - Z_0}{2p_0}, & Z_0 - p_0 + q_0 \leq c^T x \leq Z_0, \\
1, & c^T x \geq Z_0,
\end{cases}
\]

and for \( i = 1, \ldots, m \)

\[
f_i(A^T_ix) = \begin{cases} 
1, & A^T_ix \leq b_i, \\
\frac{b_i - A^T_ix}{2p_i}, & b_i \leq A^T_ix \leq b_i + p_i - q_i, \\
\frac{p_i + q_i}{2q_i} \left(1 + \frac{b_i - A^T_ix}{p_i}\right), & b_i + p_i - q_i \leq A^T_ix \leq b_i + p_i, \\
0, & A^T_ix \geq b_i + p_i.
\end{cases}
\]

Here \( f_0(.) \) and \( f_i(.), \ i = 1, \ldots, m, \) are the piecewise linear functions and have concave break points (i.e. two adjacent pieces of a function together make a concave function). Applying Yang et al. [33] approach, Dubey et al. [40] introduced the following equivalent problem for
\[\text{(IFLP):} \]

\[
\begin{align*}
\max & \quad \alpha \\
\text{subject to} & \quad f_{01}(c^T x) \geq \alpha, \\
& \quad f_{02}(c^T x) \geq \alpha, \\
& \quad f_i(A_i x) \geq \alpha, \quad i = 1, \ldots, m, \\
& \quad x \geq 0, \alpha \in [0, 1].
\end{align*}
\]

2.2.2. Optimistic Approach

The decision maker has an optimistic attitude in rejection; a complete zero acceptance does not amount to full rejection. Therefore the decision maker has a liberal view for rejection. On defining the membership and non-membership functions, the indeterminacy resolving function for \(m+1\) constraints of \((IFLP)\) are described as follows:

\[
f_0(c^T x) = \begin{cases} 
0, & c^T x \leq Z_0 - p_0 - q_0, \\
\frac{c^T x - (Z_0 - p_0 - q_0)}{2(p_0 + q_0)}, & Z_0 - p_0 - q_0 \leq c^T x \leq Z_0 - p_0, \\
\frac{1}{2} + \frac{(c^T x - Z_0)}{2p_0 + q_0}, & Z_0 - p_0 \leq c^T x \leq Z_0, \\
1, & c^T x \geq Z_0,
\end{cases}
\]

and for \(i = 1, \ldots, m,\)

\[
f_i(A_i x) = \begin{cases} 
1, & A_i x \leq b_i, \\
\frac{1}{2} + \frac{(b_i - A_i x)}{2p_i + q_i}, & b_i \leq A_i x \leq b_i + p_i, \\
\frac{b_i + p_i + q_i - A_i x}{2(p_i + q_i)}, & b_i + p_i \leq A_i x \leq b_i + p_i + q_i, \\
0, & A_i x \geq b_i + p_i + q_i.
\end{cases}
\]

Here \(f_0(c^T x)\) and \(f_i(A_i x), \ i = 1, \ldots, m,\) are piecewise linear S-shaped functions with convex-type break points as depicted in Figure 1 and Figure 2, respectively. By employing Yang et al. [33] approach, Dubey et al. [40] presented the following equivalent problem of \((IFLP)\) involving binary variables \(\delta_i, i = 0, 1, \ldots, m.\)

\[
\begin{align*}
\max & \quad \alpha \\
\text{subject to} & \quad f_{01}(c^T x) + M\delta_0 \geq \alpha, \\
& \quad f_{02}(c^T x) + M(1 - \delta_0) \geq \alpha, \\
& \quad f_{i1}(A_i x) + M\delta_i \geq \alpha, \quad i = 1, \ldots, m,
\end{align*}
\]
Figure 1. Indeterminacy resolving function of goal in optimistic approach.

\[
\begin{align*}
  f_0(c^T x) \\
  f_0 \\
  z_0 - p_0 - q_0 \\
  z_0 - p_0 \\
  z_0 \\
  c^T x
\end{align*}
\]

Figure 2. Indeterminacy resolving function of constraint in optimistic approach.

\[
\begin{align*}
  f_i(A_i^T x) \\
  f_1 \\
  f_2 \\
  0 \\
  b_i \\
  b_i + p_i \\
  b_i + p_i + q_i \\
  A_i^T x
\end{align*}
\]

Figure 1. Indeterminacy resolving function of goal in optimistic approach.

\[
\begin{align*}
  f_0(c^T x) \\
  f_0 \\
  z_0 - p_0 - q_0 \\
  z_0 - p_0 \\
  z_0 \\
  c^T x
\end{align*}
\]

Figure 2. Indeterminacy resolving function of constraint in optimistic approach.

\[
\begin{align*}
  f_i(A_i^T x) \\
  f_1 \\
  f_2 \\
  0 \\
  b_i \\
  b_i + p_i \\
  b_i + p_i + q_i \\
  A_i^T x
\end{align*}
\]

\[
\begin{align*}
  f_{i2}(A_i x) + M(1 - \delta_i) & \geq \alpha, \quad i = 1, \ldots, m, \\
  x & \geq 0, \quad \alpha \in [0, 1], \quad \delta_i \in \{0, 1\}, \quad i = 0, 1, \ldots, m,
\end{align*}
\]

where \(M\) is a large positive real number. Let \((x^*, \alpha^*, \delta_i^*, i = 0, 1, \ldots, m)\) be an optimal solution of this formulation. Then \(x^*\) is identified as an optimal solution of (IFLP) and \(\alpha^*\) is interpreted as the highest common degree up to which all constraints as well as the aspiration level \(Z_0\) of the decision maker is met.

**Remark 1:** We may note that solving a fuzzy optimisation problem with piecewise linear membership function having convex break points is equivalent to solving a crisp optimisation problem having binary variables and the number of zero-one variables in the equivalent problem is equal to the number of intersections between convex piece and concave piece of the functions. On the other hand, solving a fuzzy optimisation problem with piecewise linear membership function having concave break points is equivalent to solving a crisp optimisation problem but with no binary variables. The presence of binary variables will increase the computational complexity as well as difficulty in studying duality theory for (IFLP). Our aim in this paper is to change (IFLP) problem having piecewise linear membership function with convex break points into a fuzzy optimisation problem having piecewise linear membership function with concave break points only. We follow Inuiguchi et al. [34]
algorithm to perform the same. Therefore it essentially amounts to study optimistic point of view of decision maker and model thereof.

3. 1-fuzzy Linear Programming: Proposed Model

In this section, we first present the Inuiguchi et al. [34] technique to convert the functions $f_0(c^T x)$ and $f_i(A_ix)$, $i = 1, \ldots, m$, associated with $m+1$ constraints of (IFLP) in optimistic approach, to concave piecewise linear functions. To implement the algorithm, we have to first write all $m+1$ constraints of (IFLP) either in $\geq_{IF}$ or in $\leq_{IF}$ form. For this, let $-A_i = \hat{A}_i$ and $-b_i = \hat{b}_i$, $(i = 1, \ldots, m)$. Then (IFLP) is equivalent to the following problem:

(IFLP) Find $x \in \mathbb{R}^n$ such that

$\begin{align*}
    c^T x &\geq_{IF} Z_0, \\
    \hat{A}_i x &\geq_{IF} \hat{b}_i, \quad i = 1, \ldots, m, \\
    x &\geq 0.
\end{align*}$

3.1. Algorithm to Convert Quasi-concave Piecewise Function into Concave Piecewise Function

The algorithm in [34] involves the following steps:

Step 1: Arrange $\frac{\lambda q_i}{(p_i + q_i)}$, $i = 0, 1, \ldots, m$, in ascending order and re-named them as $r_j$, $j = 2, \ldots, m+2$.

Let $r_1 = 0$ and $r_{m+3} = 1$, $f_0^{-1}(r_1) = Z_0 - p_0 - q_0$ and $f_i^{-1}(r_1) = \hat{b}_i - p_i - q_i$ for $i = 1, \ldots, m$. Further $f_0^{-1}(r_{m+3}) = Z_0$ and $f_i^{-1}(r_{m+3}) = \hat{b}_i$ for $i = 1, \ldots, m$. Now for $i = 0, 1, \ldots, m$ and $j = 1, \ldots, m+3$, define $v_j = f_i^{-1}(r_j)$.

Step 2: Set $\delta'_1 = 1$ and for $j = 1, \ldots, m+1$, calculate

$$\delta'_{j+1} = \delta'_j \min_{0 \leq i \leq m} \left( \frac{v_{j+2} - v_{j+1}}{v_{j+1} - v_j} \right).$$

Step 3: For $j = 1, \ldots, m+2$, compute $\delta_j = \frac{\delta'_j}{\sum_j \delta'_j}$.

Step 4: For $i = 0, 1, \ldots, m$ and $j = 1, \ldots, m+3$, find

$$f'_i(v_j) = \begin{cases} 0, & j = 1, \\ \sum_{k=1}^{j-1} \delta_k, & j \geq 2, \end{cases}$$

and

$$f'_i(x) = \begin{cases} 0, & x \leq v_1, \\ \min_{1 \leq j \leq m+2} \left( k'_j (x - v_j) + f'_i(v_j) \right), & v_1 \leq x \leq v_{m+3}, \\ 1, & x \geq v_{m+3}, \end{cases}$$
where for all $i = 0, 1, \ldots, m$ and $j = 1, \ldots, m + 2$,

$$k'_i = \frac{\delta_j}{v'_{j+1} - v'_i}.$$ 

The above procedure, when applied on $f_0(c^T x)$ and $f_i(\hat{A}_i x) i = 1, \ldots, m$, transformed them into concave piecewise linear functions. Subsequently, problem (IFLP), set-up in an optimistic scenario is equivalent to the following program:

\[(IFCP) \quad \max \alpha \]
subject to
\[
f'_0(\lambda, c^T x) \geq \alpha, \\
f'_i(\lambda, \hat{A}_i x) \geq \alpha, \quad i = 1, \ldots, m, \\
x \geq 0, \alpha \in [0, 1],
\]
i.e.

\[(IFCP) \quad \max \alpha \]
subject to
\[
\frac{\delta_j}{v_{j+1}^0 - v_0^0} (c^T x - v_0^j) + f'_0(v_0^j) \geq \alpha, j = 1, \ldots, m + 2, \\
\frac{\delta_j}{v'_{j+1} - v'_i} (\hat{A}_i x - v'_i) + f'_i(v'_i) \geq \alpha, \quad i = 1, \ldots, m, \quad j = 1, \ldots, m + 2, \\
x \geq 0, \quad \alpha \in [0, 1].
\]

Let $(\bar{x}, \bar{\alpha})$ be an optimal solution of this crisp formulation. Then $\bar{x}$ is identified as an optimal solution of (IFLP) and $\bar{\alpha}$ is interpreted as the highest common degree up to which all constraints as well as the aspiration level $Z_0$ of the decision maker is met.

**Remark 2:** For $q_0 = q_1 = q_2 = \cdots = q_m = 0$, problem (IFLP) will subsume to fuzzy optimisation problem studied by Bector et al. [28]. In this case, the index $j$ for $2 \leq j \leq m + 2$ is redundant and also $\delta_1 = 1$. Following the various steps of the algorithm, we have

$$v_0^1 = Z_0 - p_0, \quad v_0^2 = Z_0,$$

and

$$v_i^1 = \hat{b}_i - p_i, \quad v_i^2 = \hat{b}_i, \quad i = 0, 1, \ldots, m.$$
Therefore \( f'_i(v_i) = 0, \quad i = 0, 1, \ldots, m \), and (IFCP) reduces to the following problem:

\[
\begin{align*}
\max \alpha \\
\text{subject to} \\
1 + \frac{c^T x - z_0}{p_0} \geq \alpha, \\
1 - \frac{A_i x - b_i}{p_i} \geq \alpha, \quad i = 1, \ldots, m, \\
x \geq 0, \quad \alpha \in [0, 1],
\end{align*}
\]

which is the crisp formulation of fuzzy LPP studied by Bector et al. [28].

### 3.2. Numerical Illustration 1

In this section, we present a simple illustrative example to solve I-fuzzy optimisation problem by implementing the above-described algorithm. Consider the following optimisation problem Find \((x_1, x_2) \in \mathbb{R}^2\) such that

\[
\begin{align*}
x_1 + x_2 &\gtrless_{IF} 7, \\
4x_1 - x_2 &\gtrless_{IF} 10, \\
x_1 + 2x_2 &\gtrless_{IF} 8, \\
5x_1 + 2x_2 &\gtrless_{IF} 20, \\
x_1, x_2 &\geq 0,
\end{align*}
\]

where the aspiration level and tolerances associated with I-fuzzy constraints are given by \(Z_0 = 7, p_0 = 3, q_0 = 2.5\) and \(p_1 = 3, q_1 = 2, p_2 = 4, q_2 = 3, p_3 = 5, q_3 = 4.\)

Re-writing it such that all constraints are in \(\gtrless_{IF}\) form (IFLP) Find \((x_1, x_2) \in \mathbb{R}^2\) such that

\[
\begin{align*}
x_1 + x_2 &\gtrless_{IF} 7, \\
-4x_1 + x_2 &\gtrless_{IF} -10, \\
-x_1 - 2x_2 &\gtrless_{IF} -8, \\
-5x_1 - 2x_2 &\gtrless_{IF} -20, \\
x_1, x_2 &\geq 0.
\end{align*}
\]

The indeterminacy resolving function in optimistic scenario associated with the goal and constraints are

\[
f_0(\lambda, c^T x) = \begin{cases} 
0, & x_1 + x_2 \leq 1.5, \\
\frac{\lambda(x_1 + x_2 - 1.5)}{5.5}, & 1.5 \leq x_1 + x_2 \leq 4, \\
1 + (x_1 + x_2 - 7) \frac{3 + 2.5(1 - \lambda)}{16.5}, & 4 \leq x_1 + x_2 \leq 7, \\
1, & x_1 + x_2 \geq 7,
\end{cases}
\]
and

\[ f_1(\lambda, \hat{A}_1x) = \begin{cases} 
0, & -4x_1 + x_2 \leq -15, \\
\frac{\lambda(-4x_1 + x_2 + 15)}{5}, & -15 \leq -4x_1 + x_2 \leq -13, \\
1 + (-4x_1 + x_2 + 10) \frac{3 + 2(1 - \lambda)}{15}, & -13 \leq -4x_1 + x_2 \leq -10, \\
1, & -4x_1 + x_2 \geq -10.
\end{cases} \]

\[ f_2(\lambda, \hat{A}_2x) = \begin{cases} 
0, & -x_1 - 2x_2 \leq -15, \\
\frac{\lambda(-x_1 - 2x_2 + 15)}{7}, & -15 \leq -x_1 - 2x_2 \leq -12, \\
1 + (-x_1 - 2x_2 + 8) \frac{4 + 3(1 - \lambda)}{28}, & -12 \leq -x_1 - 2x_2 \leq -8, \\
1, & -x_1 - 2x_2 \geq -8.
\end{cases} \]

\[ f_3(\lambda, \hat{A}_3x) = \begin{cases} 
0, & -5x_1 - 2x_2 \leq -29, \\
\frac{\lambda(-5x_1 - 2x_2 + 29)}{9}, & -29 \leq -5x_1 - 2x_2 \leq -25, \\
1 + (-5x_1 - 2x_2 + 8) \frac{5 + 4(1 - \lambda)}{45}, & -25 \leq -5x_1 - 2x_2 \leq -20, \\
1, & -5x_1 - 2x_2 \geq -20.
\end{cases} \]

Note that all four functions have convex break points. Let us solve the above I-fuzzy optimisation problem for \( \lambda = 0.5 \). On following various steps of the algorithm, we have

**Step 1:**

\[ r_1 = 0, \quad r_2 = 0.20, \quad r_3 = 0.2142, \quad r_4 = 0.2222, \quad r_5 = 0.2272, \quad r_6 = 1, \]

and

\[ v_0^1 = 1.5, \quad v_0^2 = 3.70, \quad v_0^3 = 3.909, \]
\[ v_0^4 = 3.944, \quad v_0^5 = 4, \quad v_0^6 = 7, \]
\[ v_1^1 = -15, \quad v_1^2 = -13, \quad v_1^3 = -12.94, \]
\[ v_1^4 = -12.91, \quad v_1^5 = -12.89, \quad v_1^6 = -10, \]
\[ v_2^1 = -15, \quad v_2^2 = -12.198, \quad v_2^3 = -12, \]
\[ v_2^4 = -11.95, \quad v_2^5 = -11.93, \quad v_2^6 = -8, \]
\[ v_3^1 = -29, \quad v_3^2 = -25.399, \quad v_3^3 = -25.14, \]
\[ v_3^4 = -25, \quad v_3^5 = -24.96, \quad v_3^6 = -20. \]

**Step 2:** Put \( \delta'_1 = 1 \), and obtain \( \delta'_2, \delta'_3, \delta'_4 \) and \( \delta'_5 \) as

\[ \delta'_2 = \delta'_1 \times \min_{0 \leq i \leq 3} \left( \frac{v_i^3 - v_i^2}{v_i^3 - v_i^1} \right) = 0.03, \]
\[ \delta'_3 = \delta'_2 \times \min_{0 \leq i \leq 3} \left( \frac{v_i^4 - v_i^3}{v_i^4 - v_i^2} \right) = 0.00502, \]
\[ \delta'_4 = \delta'_3 \times \min_{0 \leq i \leq 3} \left( \frac{v^5_i - v^4_i}{v^4_i - v^3_i} \right) = 0.001434, \]
\[ \delta'_5 = \delta'_4 \times \min_{0 \leq i \leq 3} \left( \frac{v^6_i - v^5_i}{v^5_i - v^4_i} \right) = 0.07681. \]

**Step 3:** Normalise \( \delta'_i, i = 1, 2, 3, 4 \) to obtain \( \delta_1 = 0.8982, \delta_2 = 0.02694, \delta_3 = 0.0045, \delta_4 = 0.00128 \) and \( \delta_5 = 0.06899. \)

**Step 4:** Therefore for \( i = 0, 1 \) and \( j = 1, 2, 3, 4, 5, \)

\[ f'_i(v'_j) = \begin{cases} 0, & j = 1, \\ \sum_{k=1}^{j-1} \delta_i, & j \geq 2. \end{cases} \]

On putting the values, we have

\[ f'_0(v'_0) = 0, f'_0(v'_2) = 0.8982, f'_0(v'_3) = 0.9251, \]
\[ f'_0(v'_4) = 0.9296, f'_0(v'_5) = 0.9309, f'_0(v'_6) = 1, \]

and
\[ f'_1(v'_1) = 0, f'_1(v'_2) = 0.8982, f'_1(v'_3) = 0.92514, \]
\[ f'_1(v'_4) = 0.9296, f'_1(v'_5) = 0.93092, f'_1(v'_6) = 1, \]
\[ f'_2(v'_2) = 0, f'_2(v'_3) = 0.8982, f'_2(v'_3) = 0.92514, \]
\[ f'_2(v'_4) = 0.9296, f'_2(v'_5) = 0.93092, f'_2(v'_6) = 1, \]
\[ f'_3(v'_3) = 0, f'_3(v'_3) = 0.8982, f'_3(v'_3) = 0.92514, \]
\[ f'_3(v'_4) = 0.9296, f'_3(v'_5) = 0.93092, f'_3(v'_5) = 1. \]

The membership functions are subsequently given as follows:

\[ f'_0(c^T x) = \begin{cases} 0, & x_1 + x_2 \leq 1.5, \\ \min (0.4082x_1 + 0.4802x_2 - 0.61240, \\ 0.1288x_1 + 0.1288x_2 + 0.42127 \\ 0.02300x_1 + 0.02300x_2 + 0.8388), & 1.5 \leq x_1 + x_2 \leq 7, \\ 1, & 7 \geq x_1 + x_2, \end{cases} \]

and

\[ f'_1(\hat{A}_1 x) = \begin{cases} 0, & -4x_1 + x_2 \geq -15, \\ \min (-1.7964x_1 + 0.4491x_2 + 6.7365, \\ -0.60x_1 + 0.15x_2 + 2.88614 \\ -0.256x_1 + 0.064x_2 + 1.7584 \\ -0.09548x_1 + 0.02387x_2 + 1.2385), & -15 \leq -4x_1 + x_2 \leq -10, \\ 1, & -4x_1 + x_2 \geq -10. \end{cases} \]
The equivalent optimisation program of \((I\!F\!L\!P)\) is

\[
\max \alpha \\
\text{subject to}
\]

\[
\begin{align*}
0.4082x_1 + 0.4802x_2 - 0.61240 & \geq \alpha, \\
0.1288x_1 + 0.1288x_2 + 0.42127 & \geq \alpha, \\
0.02300x_1 + 0.02300x_2 + 0.8388 & \geq \alpha, \\
-1.7964x_1 + 0.4491x_2 + 6.7352 & \geq \alpha, \\
-0.60x_1 + 0.15x_2 + 2.86614 & \geq \alpha, \\
-0.256x_1 + 0.064x_2 + 1.75584 & \geq \alpha, \\
-0.09548x_1 + 0.02387x_2 + 1.2385 & \geq \alpha, \\
-0.3196x_1 - 0.6392x_2 + 4.794 & \geq \alpha, \\
-0.1417x_1 - 0.2834x_2 + 2.6266 & \geq \alpha, \\
-0.09x_1 - 0.18x_2 + 2.00514 & \geq \alpha, \\
-0.064x_1 - 0.128x_2 + 1.6944 & \geq \alpha, \\
-0.01755x_1 - 0.0351x_2 + 1.14034 & \geq \alpha, \\
-1.2475x_1 - 0.499x_2 + 7.2334 & \geq \alpha, \\
-0.526x_1 - 0.2104x_2 + 3.5710 & \geq \alpha, \\
-0.161x_1 - 0.0644x_2 + 1.7348 & \geq \alpha, \\
-0.0695x_1 - 0.0278x_2 + 1.27801 & \geq \alpha,
\end{align*}
\]

\(x_1, x_2 \geq 0, \quad \alpha \in [0, 1].\)

The optimal solution is \(x_1^* = 3.078, \ x_2^* = 3.041\) with \(\alpha^* = 0.9795.\)
Now, if we solve the same problem by Yang et al. [33] approach as solved by Dubey et al. [40], we get the following equivalent program:

\[
\begin{align*}
\text{max} & \quad \alpha \\
\text{subject to} & \quad x_1 + x_2 + 11M\delta_0 - 1.5 \geq 11\alpha, \\
& \quad 8.5x_1 + 8.5x_2 + 33M - 33M\delta_0 - 26.5 \geq 33\alpha, \\
& \quad -4x_1 + x_2 + 5M\delta_1 + 15 \geq 5\alpha, \\
& \quad -4x_1 + x_2 + 10M - 10M\delta_1 + 15 \geq 10\alpha, \\
& \quad -11x_1 - 22x_2 + 56M\delta_2 + 144 \geq 56\alpha, \\
& \quad x_1 - 2x_2 + 14M - 14M\delta_2 + 15 \geq 14\alpha, \\
& \quad -70x_1 - 28x_2 + 90M\delta_3 + 370 \geq 90\alpha, \\
& \quad -5x_1 - 2x_2 + 18M - 18M\delta_3 + 29 \geq 18\alpha, \\
& \quad x_1, x_2 \geq 0, \alpha \in [0, 1], \\
& \quad \delta_0, \delta_1, \delta_2, \delta_3 \in (0, 1).
\end{align*}
\]

Here \(M\) is a large positive real number. The optimal solution is \(x_1^* = 3.0760, x_2^* = 3.040, \alpha^* = 0.772, \delta_0 = 1, \delta_2 = 0, \delta_3 = 0\). It is to be noted that \(\bar{\alpha} = 0.954 > \alpha^* = 0.772\).

4. I-fuzzy Linear Programming Duality

In this section, we aim to study I-fuzzy linear programming duality via indeterminacy resolving function approach. Aggarwal et al. [30] studied the same by using score function and Angelov’s approach [27]. In the literature the following dual to (IFLP) is proposed.

\text{(IFLD)} \quad \text{Find} \quad w \in \mathbb{R}^m \quad \text{such that}

\[
\begin{align*}
\hat{b}^T w & \gtrless^Iw_0, \\
A_j^T w & \gtrless^I c_j, \quad j = 1, \ldots, n, \\
w & \geq 0,
\end{align*}
\]

where \(W_0\) is an aspiration level for the dual objective and has a similar interpretation as in Bector and Chandra [28]. Let us write the above problem (IFLD) in the form in which all the inequalities are in \(\gtrless^I\) form.

\text{(IFLD)} \quad \text{Find} \quad w \in \mathbb{R}^m \quad \text{such that}

\[
\begin{align*}
\hat{b}^T w & \gtrless^I \hat{W}_0, \\
A_j^T w & \gtrless^I c_j, \quad j = 1, \ldots, n, \\
w & \geq 0,
\end{align*}
\]

where \(-b = \hat{b}\) and \(-W_0 = \hat{W}_0\). Let \(t_j, s_j, 0 < s_j < t_j, j = 0, 1, \ldots, n\), denote the tolerances associated respectively with the acceptance and the rejection of \(n+1\) constraints in (IFLD).
Let us denote the indeterminacy resolving functions by \( g_0(\eta, b^T w) \) and \( g_j(\eta, A_j^T w), \) \( j = 1, \ldots, n \), for goal and constraints respectively. Therefore

\[
g_0(\eta, b^T w) = \begin{cases} 
0, & \hat{b}^T w \leq \hat{W}_0 - t_0 - s_0, \\
\frac{\eta(\hat{b}^T w - (\hat{W}_0 - t_0 - s_0))}{(t_0 + s_0)}, & \hat{W}_0 - t_0 - s_0 \leq \hat{b}^T w \leq \hat{W}_0 - t_0, \\
1 + (\hat{b}^T w - \hat{W}_0) \left( \frac{t_0 + (1 - \eta)s_0}{t_0(t_0 + s_0)} \right), & \hat{W}_0 - t_0 \leq \hat{b}^T w \leq \hat{W}_0, \\
1, & \hat{b}^T w \geq \hat{W}_0,
\end{cases}
\]

and for \( j = 1, \ldots, n \),

\[
g_j(\eta, A_j^T w) = \begin{cases} 
0, & A_j^T w \leq c_j - t_j - s_j, \\
\frac{\eta(A_j^T w - (c_j - t_j - s_j))}{(t_j + s_j)}, & c_j - t_j - s_j \leq A_j^T w \leq c_j - t_j, \\
1 + (A_j^T w - c_j) \left( \frac{t_j + (1 - \eta)s_j}{t_j(t_j + s_j)} \right), & c_j - t_j \leq A_j^T w \leq c_j, \\
1, & A_j^T w \geq c_j,
\end{cases}
\]

\((IFCD)\) \[\max \beta\]

subject to

\[
\frac{\xi_i}{u_{0}^{i+1} - u_{0}^{i}} (\hat{b}^T w - u_{0}^{i}) + g_0(u_{0}^{i}) \geq \beta, \quad i = 1, \ldots, n + 2,
\]

\[
\frac{\xi_i}{u_{j}^{i+1} - u_{j}^{i}} (A_j^T w - u_{j}^{i}) + g_j(u_{j}^{i}) \geq \beta, \quad j = 1, \ldots, n, \quad i = 1, \ldots, n + 2,
\]

\[w \geq 0, \quad \beta \in [0, 1].\]

where \( a_i \) is the \( i \)th number in the ascending re-arrangement of break points \( \eta s_j/(t_j + s_j) \), \( j = 0, 1, \ldots, n \) for \( i = 2, \ldots, n + 2 \). Further let \( a_1 = 0, a_{n+3} = 1 \).

Again for \( i = 1, \ldots, n + 3 \) and \( j = 0, 1, \ldots, n \)

\[u_{j}^{i} = g_j^{-1}(a_i).\]

Now compute, for \( i = 1, \ldots, n + 1 \),

\[\xi_{i+1}^{'} = \xi_{i}^{'} \min_{0 \leq j \leq n} \left( \frac{u_{j}^{i+2} - u_{j}^{i+1}}{u_{j}^{i+1} - u_{j}^{i}} \right)\]

and for \( i = 1, \ldots, n + 2 \), \( \xi_i = \xi_i^{'} / \sum_i \xi_i^{'} \).

Hence for \( j = 0, 1, \ldots, n \), and \( i = 1, \ldots, n + 3 \), find

\[
g_j(u_{j}^{i}) = \begin{cases} 
0, & i = 1, \\
\sum_{h=1}^{i-1} \xi_{h}, & i \geq 2.
\end{cases}
\]
Let \((\bar{w}, \bar{\beta})\) be an optimal solution of this crisp formulation. Then \(\bar{w}\) is identified as an optimal solution of \((IFLD)\) and \(\bar{\beta}\) is interpreted as the highest common degree up to which all the constraints as well as the aspiration level \(W_0\) is met.

The following lemma connecting \((IFCP)\) and \((IFCD)\) can be interpreted as a \(I\)-fuzzy variant of the (crisp) weak duality theorem.

**Lemma 1:** Let \((x, \alpha)\) and \((w, \beta)\) be feasible solutions for \((IFCP)\) and \((IFCD)\) respectively. Then for all \(i = 1, \ldots, n + 2\) and \(j = 1, \ldots, m + 2\)

\[
(i) \quad -\sum_i w_i v_i^j - \sum_j x_j u_j^j \geq \sum_i w_i \frac{\alpha - f_i^j(v_i^j)}{k_i^j} + \sum_j x_j \frac{\beta - g_j^j(u_j^j)}{h_j^j}
\]

\[
(ii) \quad (c^T x + \tilde{b}^T w) - (v_0^j + u_0^j) \geq \frac{\alpha - f_0^j(v_0^j)}{k_0^j} + \frac{\beta - g_0^j(u_0^j)}{h_0^j}.
\]

**Proof:** (i) As \((x, \alpha)\) is feasible to \((IFCP)\), then

\[
\frac{\delta_j}{v_j^{i+1} - v_0^j} (\bar{A}_i x - v_i^j) + f_i^j(v_i^j) \geq \alpha, \quad j = 1, \ldots, m + 2.
\]

Then

\[-\bar{A}_i x - v_i^j \geq \frac{\alpha - f_i^j(v_i^j)}{k_i^j}, \quad i = 1, \ldots, m.
\]

It implies

\[-x^T \bar{A}^T w - \sum_i w_i v_i^j \geq \sum_i w_i \frac{\alpha - f_i^j(v_i^j)}{k_i^j}, \quad i = 1, \ldots, m.
\]

Similarly from dual, as \((w, \beta)\) is feasible to \((IFCD)\), then

\[
\frac{\lambda_i}{u_i^{j+1} - u_j^j} (\bar{A}_i^T w - u_i^j) + g_j^j(u_j^j) \geq \beta, \quad i = 1, \ldots, n + 2,
\]

or

\[\bar{A}_i^T w - u_i^j \geq \frac{\beta - g_j^j(u_j^j)}{h_j^j}.
\]

It implies

\[w^T \bar{A} x - \sum_j x_j u_j^j \geq \sum_j x_j \frac{\beta - g_j^j(u_j^j)}{h_j^j}.
\]

But \(w^T \bar{A} x = x^T \bar{A}^T w\), therefore the above inequalities yield

\[-\sum_i w_i v_i^j - \sum_j x_j u_j^j \geq \sum_i w_i \frac{\alpha - f_i^j(v_i^j)}{k_i^j} + \sum_j x_j \frac{\beta - g_j^j(u_j^j)}{h_j^j}.
\]
(ii) Again as \((x, \alpha)\) is feasible to \((IFCP)\), then
\[
\frac{\delta_j}{v_0^{j+1} - v_0} (c^T x - v_0^j) + f_0^j(v_0^j) \geq \alpha, \quad j = 1, \ldots, m + 2, \\
c^T x - v_0^j \geq \frac{\alpha - f_0^j(v_0^j)}{k_0^j}, \quad j = 1, \ldots, m + 2.
\]

Similarly from dual, as \((x, \beta)\) is feasible to \((IFCD)\), then
\[
\hat{b}^T w - u_0^j \geq \frac{\beta - g_0^j(u_0^j)}{h_0^j}.
\]

On adding, we get
\[
(c^T x + \hat{b}^T w) - (v_0^j + u_0^j) \geq \frac{\alpha - f_0^j(v_0^j)}{k_0^j} + \frac{\beta - g_0^j(u_0^j)}{h_0^j} \\
\quad j = 1, \ldots, m + 2, \quad i = 1, \ldots, n + 2.
\]

\[\blacksquare\]

**Remark 3:** For \(s_0 = s_1 = s_2 = \ldots = s_n = 0\), problem \((IFLD)\) will subsume to fuzzy optimisation problem studied by Bector et al. [28]. In this case, the index \(i\), for \(2 \leq i \leq n + 2\) is redundant and also \(\xi = 1\). Following the various steps of the algorithm, we have
\[
u_0^1 = W_0 - t_0, \quad u_0^2 = \hat{W}_0,
\]
and
\[
u_0^j = c_j - t_j, \quad u_0^j = c_j, \quad j = 1, \ldots, n.
\]
Therefore \(g_0^j(u_0^j) = 0\forall j = 0, 1, \ldots, n\) and \((IFCD)\) will reduce to
\[
\begin{align*}
\max \beta \\
\text{subject to} \\
\frac{(\hat{b}^T w - \hat{W}_0 + t_0)}{t_0} \geq \beta, \\
\frac{(A_j^T w - c_j + t_j)}{t_j} \geq \beta, \quad j = 1, \ldots, n, \\
w \geq 0, \quad \beta \in [0, 1],
\end{align*}
\]
which is the crisp formulation of fuzzy dual studied by Bector et al. [28].

**4.1. Numerical Illustration 2**

In order to illustrate the duality theory, we continue with the same example as given in Section 3.2 as primal problem. Let \(W_0 = 6\). Then the associated dual problem \((IFLD)\) is as follows:
Find \( w \in \mathbf{R}^3 \) such that

\[
10w_1 + 8w_2 + 20w_3 \preceq^I F 6, \\
4w_1 + w_2 + 5w_3 \succeq^I 1, \\
-w_1 + 2w_2 + 2w_3 \succeq^I 1, \\
w \geq 0.
\]

Let \( t_0 = 4, s_0 = 2.5, t_1 = 4, s_1 = 1.5, t_2 = 3, s_2 = 0.5 \) be the tolerances associated with the acceptance and the rejection of the above I-fuzzy inequalities. Consider the membership and the non-membership functions of \((iFLD)\) with given tolerances and after resolving the indeterminacy for goal as well as for constraints we have the following:

\[
g_0(\eta, b^T w) = \begin{cases} 
0, & -10w_1 - 8w_2 - 20w_3 \leq -12.5, \\
\eta(12.5 - 10w_1 - 8w_2 - 20w_3), & -12.5 \leq -10w_1 - 8w_2 - 20w_3 \leq -10, \\
\frac{6.5}{1 + (6 - 10w_1 - 8w_2 - 20w_3)(6.5 - 2.5\eta)}, & -10 \leq -10w_1 - 8w_2 - 20w_3 \leq -6, \\
1, & -10w_1 - 8w_2 - 20w_3 \geq -6,
\end{cases}
\]

\[
g_1(\eta, A_1^T w) = \begin{cases} 
0, & 4w_1 + w_2 + 5w_3 \leq -4.5, \\
\frac{5.5}{1 + (4w_1 + w_2 + 5w_3 - 1)(5.5 - 1.5\eta)}, & -4.5 \leq 4w_1 + w_2 + 5w_3 \leq -3, \\
\frac{22}{1,} & 4w_1 + w_2 + 5w_3 \geq 1, \\
\end{cases}
\]

and

\[
g_2(\eta, A_2^T w) = \begin{cases} 
0, & -w_1 + 2w_2 + 2w_3 \leq -2.5, \\
\frac{3.5}{1 + (-w_1 + 2w_2 + 2w_3 - 1)(3.5 - 0.5\eta)}, & -2.5 \leq -w_1 + 2w_2 + 2w_3 \leq -2, \\
1, & -w_1 + 2w_2 + 2w_3 \geq 1.
\end{cases}
\]

Let us take \( \eta = 0.5 \). On following various steps of the algorithm to convert convex break points into concave once, we have

**Step 1:**

\[
\begin{align*}
u_0^1 &= -12.5, & u_0^2 &= -11.57, & u_0^3 &= -10.72, & u_0^4 &= -10, & u_0^5 &= -6, \\
u_1^1 &= -4.5, & u_1^2 &= -3.69, & u_1^3 &= -3, & u_1^4 &= -2.74, & u_1^5 &= 1, \\
u_2^1 &= -2.5, & u_2^2 &= -2, & u_2^3 &= -1.79, & u_2^4 &= -1.61, & u_2^5 &= 1.
\end{align*}
\]
Step 2: Set $\xi_1' = 1$ and obtain $\xi_i'$ for $i = 2, 3, 4$ as

$$
\xi_2' = \xi_1' \times \min_{0 \leq j \leq 2} \left( \frac{u_j^3 - u_j^2}{u_j^2 - u_j^1} \right) = 0.42,
$$

$$
\xi_3' = \xi_2' \times \min_{0 \leq j \leq 2} \left( \frac{u_j^4 - u_j^3}{u_j^3 - u_j^2} \right) = 0.1582,
$$

$$
\xi_4' = \xi_3' \times \min_{0 \leq j \leq 2} \left( \frac{u_j^5 - u_j^4}{u_j^4 - u_j^3} \right) = 0.8788.
$$

Step 3: Normalise $\xi_j'$, $j = 1, 2, 3, 4$, to obtain $\xi_1 = 0.4070$, $\xi_2 = 0.1709$, $\xi_3 = 0.0643$, $\xi_4 = 0.3576$.

Step 4: On substituting the values, we have

$$
g_0'(u_1^0) = 0, \quad g_0'(u_2^0) = 0.4070, \quad g_0'(u_3^0) = 0.5779, \quad g_0'(u_4^0) = 0.6422, \quad g_0'(u_5^0) = 1,
$$

$$
g_1'(u_1^1) = 0, \quad g_1'(u_2^1) = 0.4070, \quad g_1'(u_3^1) = 0.5779, \quad g_1'(u_4^1) = 0.6422, \quad g_1'(u_5^1) = 1,
$$

$$
g_2'(u_1^2) = 0, \quad g_2'(u_2^2) = 0.4070, \quad g_2'(u_3^2) = 0.5779, \quad g_2'(u_4^2) = 0.6422, \quad g_2'(u_5^2) = 1,
$$

and

$$
g_0'(b^T w) = \begin{cases} 
0, & -10w_1 - 8w_2 - 20w_3 \leq -12.5, \\
\min(0.4376(-10w_1 - 8w_2 - 20w_3 + 12.5), & 0.2010(-10w_1 - 8w_2 - 20w_3 + 11.57), \\
+0.4070, & -12.5 \leq -10w_1 - 8w_2 - 20w_3 \leq -6, \\
0.08930(-10w_1 - 8w_2 - 20w_3) + 1.535, & -10w_1 - 8w_2 - 20w_3 \geq -6, \\
1, & 4w_1 + w_2 + 5w_3 \leq -4.5, \\
\min(0.5024(4w_1 + w_2 + 5w_3 + 4.5), & 0.2476(4w_1 + w_2 + 5w_3 + 3.69) + 0.4070, \\
0.0956(4w_1 + w_2 + 5w_3 + 9.041), & -4.5 \leq 4w_1 + w_2 + 5w_3 \leq 1, \\
1, & 4w_1 + w_2 + 5w_3 \geq 1, \\
0, & -w_1 + 2w_2 + 2w_3 \leq -2.5, \\
\min(0.814(-w_1 + 2w_2 + 2w_3 + 2.5), & 0.3572(-w_1 + 2w_2 + 2w_3 + 1.79) + 0.5779, \\
0.1370(-w_1 + 2w_2 + 2w_3 + 0.8627), & -2.5 \leq -w_1 + 2w_2 + 2w_3 \leq 1, \\
1, & -w_1 + 2w_2 + 2w_3 \geq 1.
\end{cases}
$$
The equivalent linear programming corresponding to I-fuzzy dual problem is

\[
\begin{align*}
\text{max} & \quad \beta \\
\text{subject to} & \quad 4.376w_1 + 3.5w_2 + 8.752w_3 + \beta \geq 5.4073, \\
& \quad 2.01w_1 + 1.608w_2 + 4.02w_3 + \beta \geq 2.7332, \\
& \quad 0.894w_1 + 0.7152w_2 + 1.788w_3 + \beta \geq 1.5362, \\
& \quad 2.0096w_1 + 0.5024w_2 + 2.512w_3 - \beta \geq -2.2611, \\
& \quad 0.9892w_1 + 0.2473w_2 + 1.2365w_3 - \beta \geq -1.3198, \\
& \quad 0.38244w_1 + 0.09561w_2 + 0.47805w_3 - \beta \geq -0.90418, \\
& \quad -0.814w_1 + 1.628w_2 + 1.628w_3 - \beta \geq -2.035, \\
& \quad -0.3572w_1 + 0.7144w_2 + 0.7144w_3 - \beta \geq -1.21732, \\
& \quad -0.1370w_1 + 0.274w_2 + 0.274w_3 - \beta \geq -0.86278,
\end{align*}
\]

\[w_1, w_2, w_3, \beta \geq 0, \quad \beta \leq 1.\]

An optimal solution is \(\overline{w}_1 = 0.11, \overline{w}_2 = 0.55, \overline{w}_3 = 0.0\), with \(\overline{\beta} = 1.0\). With these values and the values obtained in numerical illustration 1 (see Section 3.1), we can easily verify the two inequalities of I-fuzzy weak duality theorem (Lemma 1).

5. Conclusion

In this paper, we solved an I-fuzzy optimisation problem with I-fuzzy goal. Using Hurwicz’s optimism–pessimism criterion, indeterminacy resolving function and Inuijuchi et al. method, we have shown that solving such problems is equivalent to solving crisp linear optimisation problems with no binary variable. Further we proposed dual problem on the lines of Bector et al. [28] and establish a relation which can be interpreted as an I-fuzzy version of the weak duality for fuzzy linear optimisation. The proposed work may be further enhanced to study I-fuzzy linear programming problem with I-fuzzy goal and I-fuzzy parameters. To compare two I-fuzzy numbers, the ranking method in I-fuzzy environment suggested by [17, 22, 42, 43] may be utilised in an appropriate manner. In future, one can study two person zero sum matrix game and Bi-matrix game in I-fuzzy environment by resolving indeterminacy first.

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