FRT Construction and Equipped Quantum Linear Spaces

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Abstract
We show there exists a rigid monoidal category formed out by quantum spaces with an additional structure, such that FRT bialgebras and corresponding rectangular generalizations are its internal coEnd and coHom objects, respectively. This enable us to think of them as the coordinate rings of ‘quantum spaces of homomorphisms’ that preserve the mentioned structure. The well known algebra epimorphisms between FRT bialgebras and Manin quantum semigroups translate into ‘inclusions’ of the corresponding quantum spaces, as the space of endomorphisms of a metric linear space $V$ is included in $gl(V)$. Our study is mainly developed for quadratic quantum spaces, and later generalized to the conic case.

1 Introduction
Given a finite dimensional $k$-vector space $V$ and a linear endomorphism $R$ of $V \otimes V$, a universal bialgebra $A(R)$ can be constructed [1]. The assignment of $A(R)$ to each pair $(V, R)$ is known as FRT construction [2]. Every $A(R)$ is a generically non commutative quadratic algebra generated by a finite dimensional coalgebra, more precisely, by linearly independent coefficients of a multiplicative matrix $t$ [3]. This is why they are called quantum matrix bialgebras. Given a basis $\{v_i\}$ of $V$, coefficients of $t$ are identified with the elements $t^i_j = v^j \otimes v_i$ of $V^* \otimes V$, and the quadratic relations they must satisfy are sometimes written

$$R_{ijkl}^{st} t^s_k t^t_i - t^k_j t^l_m R_{ijkl}^{nm}; \quad i, j, n, m = 1, \ldots, \dim V,$$

being $R_{ijkl}^{st} \in k$ the coefficients of $R$ in the given basis.

Quantum matrix bialgebras are the ‘dual’ version of quantum universal enveloping algebras, such as Drinfeld-Jimbo [4] quantized Lie bialgebras $U_q(g)$.

1It is worth mentioning we are not asking for $R$ to be a Yang-Baxter operator. If this were the case, $R$ would indicate the so called $R$-matrix.
On the other hand, they are quotient of Manin quantum (semi)groups $\text{end}[\mathcal{V}]$, i.e. the internal coEnd objects of the monoidal category QA of quadratic algebras. In other words, there exists a bialgebra epimorphism $\text{end}[\mathcal{V}] \to A(\mathbb{R})$ in QA. The relationship between $\mathbb{R}$ and $\mathcal{V}$ will be discussed later. By now, let us write $\mathcal{V} \triangleright \mathbb{R}$ when they are related. Using geometric language, each object $\text{end}[\mathcal{V}]$ is interpreted as the coordinate ring of a non commutative algebraic variety, or quantum linear space, living in the opposite category $\text{QA}^{op}$. It represents the quantum semigroup of endomorphisms corresponding to the quantum space $\mathcal{V}^{op}$. Thus, epimorphism above gives rise to a monic $A(\mathbb{R})^{op} \to \text{end}[\mathcal{V}]^{op}$ enabling us to regard $A(\mathbb{R})$ as the coordinate ring of a quantum subspace of $\text{end}[\mathcal{V}]^{op}$. Of course, Manin construction also includes quantum spaces of homomorphisms $\text{hom}_{\mathbb{R}}[\mathcal{W}, \mathcal{V}]$, indicating by $\mathcal{W}$ a quadratic algebra generated by a vector subspace $\mathcal{W}$. They have as $\text{FRT}$ analogue the rectangular quantum matrix algebras $A(\mathbb{R} : \mathbb{S})$, in the sense that there exist algebra epimorphisms $\text{hom}_{\mathbb{R}}[\mathcal{W}, \mathcal{V}] \to A(\mathbb{R} : \mathbb{S})$ leading us to a geometric interpretation as described before. $\mathbb{S}$ denotes a linear map $\mathcal{W} \otimes \mathcal{W} \to \mathcal{W} \otimes \mathcal{W}$ such that $\mathcal{W} \triangleright \mathbb{S}$. Given a basis $\{w_i\}$ of $\mathcal{W}$, the algebras $A(\mathbb{R} : \mathbb{S})$ are generated by symbols $t_{ij}^l = w^l \otimes v_i \in \mathcal{W}^* \otimes \mathcal{V}$ satisfying

$$R_{ij}^{kl} s_{kl}^m t_{ij}^l = s_{ij}^{kl} s_{kl}^m t_{ij}^l; \quad i, j = 1, \ldots, \dim \mathcal{V}, \quad n, m = 1, \ldots, \dim \mathcal{W}. \quad (1)$$

These algebras were studied in detail in $[8]$, where $\mathbb{R}$ and $\mathbb{S}$ are Yang-Baxter operators of Hecke type.

This paper was mainly motivated by the following question induced by `inclusions' $A(\mathbb{R} : \mathbb{S})^{op} \to \text{hom}_{\mathbb{R}}[\mathcal{W}, \mathcal{V}]^{op}$: Do the quadratic algebras $A(\mathbb{R} : \mathbb{S})$ represent homomorphisms between quantum spaces supplied with some additional structure, i.e. spaces that are not characterized just by its respective coordinate rings?

In order to answer this question we encode Manin and $\text{FRT}$ constructions, reformulating and generalizing the latter, in the unifying language of rigid monoidal categories $[7]$. We show that bialgebras $A(\mathbb{R})$ can be seen as internal coEnd objects contained in certain rigid monoidal category $(\text{EQA}, \otimes)$, the equipped quantum spaces, formed out by pairs $\mathfrak{M} = (\mathcal{V} : \mathbb{R})$ with $\mathcal{V} \triangleright \text{QA}$ and $\mathcal{V} \triangleright \mathbb{R}$. More precisely, there exists a surjective embedding $U : \text{EQA} \to \text{QA} : (\mathcal{V} : \mathbb{R}) \mapsto \mathcal{V}$, and a related opposite $U^{op} : \text{EQA}^{op} \to \text{QA}^{op}$, such that for each pair $\mathfrak{M}$, the object $\text{hom}_{\mathbb{R}}[\mathfrak{M}, \mathfrak{M}] = \text{end}[\mathfrak{M}]$ is funtored to $A(\mathbb{R})$. In general, coHom objects $\text{hom}_{\mathbb{R}}[\mathfrak{M}, \mathfrak{M}]$, with $\mathfrak{M} = (\mathcal{V} : \mathbb{S})$, are sent to $A(\mathbb{R} : \mathbb{S})$. Moreover, from the general formalism of rigid monoidal categories follows existence and associativity properties of rectangular comultiplication maps defined in $[7]$, and also existence of algebra epimorphisms

$$\text{hom}_{\text{QA}}[\mathfrak{M}, \mathfrak{M}] \to \cup_{\text{hom}_{\text{EQA}}[\mathfrak{M}, \mathfrak{M}]}, \quad \forall \mathfrak{M}, \mathfrak{N} \in \text{EQA}, \quad (2)$$

from which previously mentioned 'inclusions' are deduced. We conclude, each algebra $A(\mathbb{R} : \mathbb{S})$ is the coordinate ring of the space $\text{hom}_{\mathbb{R}}[\mathfrak{M}, \mathfrak{M}]^{op} \in \text{EQA}^{op}$ of

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$^2$We are using the convention of ref. $[8]$ to evaluate the maps $\mathbb{R}$ and $\mathbb{S}$ appearing in the rectangular quantum matrix algebras, instead of the one used in $[7]$.
homomorphisms between spaces \( \mathfrak{M}^{op} \) and \( \mathfrak{N}^{op} \). Thus, such spaces are described by their respective coordinate rings \( U\text{hom}[\mathfrak{M}, \mathfrak{N}] = A(\mathbb{R} : S) \), \( U\mathfrak{M} \) and \( U\mathfrak{N} \) (given by quadratic algebras), and an additional data. We also show EQA is equivalent to a category whose objects are pairs \((V, \mathbb{R})\) as described above, in such a way that we can write \( (V; \mathbb{R}) \equiv (V, \mathbb{R}) \) if \( V \) is generated by \( V \). Hence, we are assigning a bialgebra \( A(\mathbb{R}) = U\text{end}[\mathfrak{N}] \) to each pair \((V, \mathbb{R})\) in a universal way.

2 Quantum linear spaces

In what follows \( \mathbb{k} \) indicates some of the numerics fields, \( \mathbb{R} \) or \( \mathbb{C} \). The usual tensor product on \( \mathbb{k}-\text{Alg} = \text{Alg} \) and \( \text{Vct}_\mathbb{k} = \text{Vct} \) (the categories of unital associative \( \mathbb{k} \)-algebras and of \( \mathbb{k} \)-vector spaces, respectively) is denoted by \( \otimes \). \( \text{Vct}_f \) indicates the full subcategory of \( \text{Vct} \) formed out by finite dimensional vector spaces.

Originally \( [5] \), Manin defined quantum spaces as opposite objects to quadratic algebras. The latter are pairs \((A_1, A)\), with \( A \in \text{Alg} \) generated by \( A_1 \) in \( \text{Vct}_f \), such that the canonical epimorphism \( A_1^\otimes \to A \) has as kernel a bilateral ideal algebraically generated by a subspace of \( A_1^{\otimes 2} \). As usual, \( A_1^\otimes = \bigoplus_{n \in \mathbb{N}_0} A_1^{\otimes n} \) denotes the tensor algebra of \( A_1 \) (being \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \)). To be more explicit, for every quadratic algebra \((A_1, A)\) there exists a subspace \( R \subset A_1^{\otimes 2} \) such that

\[
\ker [A_1^\otimes \to A] = I [R] = A_1^\otimes \cdot R \cdot A_1^\otimes.
\]

In general, we note by \( I [X] \subset A_1^\otimes \) the bilateral ideal generated by a set \( X \subset A_1^\otimes \).

For instance, each algebra \( A(\mathbb{R} : S) \) defines a quantum space

\[
A(\mathbb{R} : S) \equiv (W^* \otimes V, A(\mathbb{R} : S))
\]

The kernel of its related canonical epimorphism is generated by the elements given in Eq. \( \text{IV} \). The category QA, as mentioned before, has above pairs as objects and as arrows \((A_1, A) \to (B_1, B)\) algebra homomorphisms \( A \to B \) that preserve generating spaces, that is to say, \( A \to B \) restricted to \( A_1 \) defines a linear map \( A_1 \to B_1 \).

In \( [10] \), Manin extended the concept to arbitrary finitely generated algebras, i.e. pairs \((A_1, A)\) as above, but without restrictions on their respective canonical epimorphisms \( A_1^\otimes \to A \). We shall indicate FGA the category formed out by these pairs. Its arrows are again algebra homomorphisms preserving the generating linear spaces. Thus, QA is a full subcategory of FGA. Note that arrows \( \alpha : (A_1, A) \to (B_1, B) \) in FGA are characterized by linear maps \( \alpha_1 : A_1 \to B_1 \) such that

\[
\alpha_1^\otimes (\ker [A_1^\otimes \to A]) \subset \ker [B_1^\otimes \to B].
\]

In QA, if \( R \) and \( S \) are the subspaces generating the respective kernels, last condition reads \( \alpha_1^\otimes (R) \subset S \). In \( [11] \) we study another full subcategory of FGA, namely CA, the conic algebras or conic quantum spaces. Its objects \((A_1, A)\) are such that \( A \) is a graded algebra and \( A_1 \) is its subspace of homogeneous elements of degree one, or equivalently, its related ideal (i.e. the kernel of its
canonical epimorphism) is a graded subalgebra of $A_1^\otimes$. Examples of them, beside quadratics, are the so called $m$-th quantum spaces, whose associated ideals are generated by a subspace of $A_1^\otimes m$, for some $m \geq 2$. The latter, in turn, form a full subcategory $C^m$ of $CA$, leading us to the full inclusions $C^m \subset CA \subset FGA$.

Of course, $QA = CA^2$.

The monoid we consider on these categories is the bifunctor $\circ$, given on objects by

$$(A_1, A) \circ (B_1, B) = (A_1 \otimes B_1, A \circ B),$$

with $A \circ B$ the subalgebra of $A \otimes B$ generated by $A_1 \otimes B_1$. On arrows, it assigns to $\alpha$ and $\beta$, with domains $(A_1, A)$ and $(B_1, B)$, respectively, the algebra morphism $\alpha \circ \beta = \alpha \otimes \beta|_{A \circ B}$. The unit object is $I = (k, k)$ in $FGA$ and $K = (k, k \otimes)$ in $CA$ and every $C^m$. Let us mention the forgetful functor $F : FGA \to Alg : (A_1, A) \mapsto A$ preserves the units, since $FI = k$, but is not monoidal. Nevertheless, it is easy to check that algebra inclusions $i_{A,B} : A \circ B \hookrightarrow A \otimes B$, related to quantum spaces $A = (A_1, A)$ and $B = (B_1, B)$, define a natural transformation $F \circ \to \otimes (F \times F)$. (Note that $F(A \circ B) = A \circ B$ and $F(A \otimes FB = A \otimes B)$.) That is to say, for any couple of arrows $\alpha, \beta \in FGA$, with $\alpha : A \to C$ and $\beta : B \to D$, the diagram

$$\begin{array}{ccc}
A \circ B & \xleftarrow{i_{A,B}} & A \otimes B \\
F(\alpha \circ \beta) & \downarrow & F\alpha \otimes F\beta \\
C \circ D & \xleftarrow{i_{C,D}} & C \otimes D
\end{array}$$

is commutative. When restricted to $CA$ and every $C^m$, the above natural transformation holds, but $F$ does not respects units, because $FK = k^\otimes$. However, the canonical projection $k^\otimes \to k$ defines epimorphisms $p_A : k^\otimes \otimes A \to k \otimes A$, with $F(A = A$, that make commutative the diagrams

$$\begin{array}{ccc}
k^\otimes & \xleftarrow{i_{k,A}} & k^\otimes \otimes A \\
& p_A & \to k \otimes A \\
& \searrow & \searrow \\
& A & \searrow
\end{array}$$

Indicating by $e$ the generator of $k$, then

$k^\otimes = k[e]$ and $p_A(e^n \otimes a) = e \otimes a$.

The isomorphisms $A \simeq k \otimes A$ and $A \simeq k^\otimes \otimes A$ are the functorial isomorphisms related to the left unital constraint in $Alg$ and $CA$, respectively. Of course, a diagram analogous to (5) but with $k$ on the right is also fulfilled.

There exist internal coHom objects on each one of this monoidal categories. For instance, for $A = (A_1, A)$ and $B = (B_1, B)$ in $CA$ (resp. $C^m$), they are given by graded algebras $hom[B, A]$ generated by $B_1^m \otimes A_1$ and constrained by homogeneous relations (resp. of $m$-th order). For more details, see [11].
3 The equipped quantum spaces

Consider $\mathcal{A} = (A_1, A) \in \text{QA}$ and a linear map $R : A_1^{\otimes 2} \to A_1^{\otimes 2}$.

**Definition 1** We say $R$ is **compatible** with $\mathcal{A}$, and use the shorthand notation $\mathcal{A} \vdash R$, if

$$\ker [A_1^{\otimes 2} \to A] = I [\text{Im} R].$$

A pair $(\mathcal{A}; R)$ such that $\mathcal{A} \vdash R$ will be called **equipped quantum space** (or equipped quadratic algebra) with **structure** $R$. If a morphism of quantum spaces $\alpha : \mathcal{A} \to \mathcal{B}$, with $\mathcal{A} \vdash R$ and $\mathcal{B} \vdash S$, satisfies $\alpha^{\otimes 2} R = S \alpha^{\otimes 2}$, we say that $\alpha$ preserves structures $R$ and $S$. The category formed out by equipped quantum spaces and structure preserving arrows will be denoted $\text{EQA}$. ■

From now on, we reserve the name **pair** only for equipped quantum spaces (in contrast to last section where we used it for ordinary ones).

A simple characterization of equipped quantum spaces is given by the following result.

**Proposition 1** The category $\text{EQA}$ is equivalent to one whose objects are pairs $(V, R)$, with $V \in \text{Vct}$ and $R : V^{\otimes 2} \to V^{\otimes 2}$ a linear map, and whose arrows $(V, R) \to (W, S)$ are linear homomorphisms $l : V \to W$ such that $l^{\otimes 2} R = S l^{\otimes 2}$.

**Proof:** The equivalence is defined by functors

$$f : (\mathcal{A}; R) \mapsto (A_1, R) \quad \text{and} \quad g : (V, R) \mapsto ((V, V^{\otimes} / I [\text{Im} R]); R). \quad (6)$$

On arrows, $f \alpha = \alpha_1$ and $g l$ is the extension of $l$ to an algebra homomorphism. If $l$ goes from $(V, R)$ to $(W, S)$, since

$$l^{\otimes} (I [\text{Im} R]) = W^{\otimes} \cdot l^{\otimes} (I [\text{Im} R]) \cdot W^{\otimes} = W^{\otimes} \cdot (I [\text{Im} R^{\otimes 2}) \cdot W^{\otimes} = W^{\otimes} \cdot (I [\text{Im} S l^{\otimes 2}) \cdot W^{\otimes} \subset W^{\otimes} \cdot I [\text{Im} S] \cdot W^{\otimes} = I [\text{Im} S],$$

then such extension is well defined.

Natural equivalence $f \circ g \simeq id$ is immediate. The functorial isomorphisms for equivalence $g \circ f \simeq id$ are given by the algebra isomorphisms $A \simeq A^{\otimes} / I [\text{Im} R]$, which are well defined provided $\ker [A_1^{\otimes} \to A] = I [\text{Im} R]$. ■

Because of this equivalence, we identify the objects of both categories. That is to say, we understand pairs $(A_1, R)$ and $(\mathcal{A}; R)$ as the same thing, indicating both categories by $\text{EQA}$.\(^3\) Since to deal with pairs $(V, R)$ is often easier than to deal with $(\mathcal{A}; R)$, we shall work out our constructions mainly in terms of the former. Naturally, they also provide a more direct contact to $\text{FRT}$ construction.\(^4\)

\(^3\)Note that the category $\mathcal{YB}$ defined in [6], formed out by pairs $(V, R)$ such that $R$ is a Yang-Baxter solution of $q$-Hecke type, is a full subcategory of $\text{EQA}$.\(^4\)
3.1 Products and duals

A monoidal structure and an involution can be attached to EQA in the following way. Let us consider the canonical algebra isomorphism \( \varphi_{V,W} \) between \([V \otimes W]^\otimes\) and

\[
V \otimes W \cong \bigoplus_{n \in \mathbb{N}_0} V^\otimes n \otimes W^\otimes n,
\]

the subalgebra of \( V^\otimes \otimes W^\otimes = \bigoplus_{n,m} V^\otimes n \otimes W^\otimes m \) generated by \( V \otimes W \). The restriction of \( \varphi_{V,W} \) to \((V \otimes W)^\otimes_2\), which we also denote \( \varphi_{V,W} \), is given by

\[
v \otimes w \otimes v' \otimes w' \mapsto v \otimes v' \otimes w \otimes w'; \quad \forall v, v' \in V, \ w, w' \in W.
\]

We define the bifunctor \( \boxtimes : EQA \times EQA \to EQA \) as

\[
(V, R) \times (W, S) \mapsto (V \otimes W, R \boxtimes S), \quad k \times l \mapsto k \boxtimes l \equiv k \otimes l,
\]

being

\[
R \boxtimes S \equiv \varphi_{V,W}^{-1}(R \otimes \mathbb{I} + \mathbb{I} \otimes S) \varphi_{V,W}
\]

(taking into account the restriction of \( \varphi_{V,W} \)). \( \mathbb{I} \) denotes the identity endomorphism of the corresponding vector spaces. Identifying \( V \otimes W \) and \([V \otimes W]^\otimes\) we shall write \( R \boxtimes S \approx R \otimes \mathbb{I} + \mathbb{I} \otimes S \). Straightforwardly, the bifunctor \( \boxtimes \) defines a symmetric monoidal structure with unit object \( \mathbb{R} = (\mathbb{K}, \otimes) \), being \( \otimes \) the null endomorphism of \( \mathbb{K}^\otimes \), i.e. \( \text{Im } \otimes = \{0\} \). The functorial isomorphisms \( \tau_{\mathbb{K}, W} : \mathbb{K} \boxtimes W \cong \mathbb{K} \boxtimes \mathbb{S} \) related to symmetry, with \( \mathbb{V} = (V, R) \) and \( \mathbb{W} = (W, S) \), are given by canonical flipping maps \( V \otimes W \rightarrow W \otimes V, v \otimes w \mapsto w \otimes v \). The ones related to unit are \( \ell_\mathbb{K} : v \in V \rightarrow e \otimes v \) and \( r_\mathbb{K} : v \in V \rightarrow v \otimes e \), indicating by \( e \) the generator of \( k \).

Let us define the contravariant functor \( \dagger : EQA \to EQA \),

\[
\dagger : (V, R) \mapsto (V^*, R^\dagger) \equiv (V^*, -R^*), \quad \dagger : l \mapsto l^\dagger \equiv l^*,
\]

being \( V^* \) the dual of \( V \), and \( R^* \) the transpose map w.r.t. the usual extension to \( V^\otimes_2 \) of the pairing between \( V \) and \( V^* \). It is clear that \( \dagger^2 = \dagger \) is naturally equivalent to \( \text{id}_{\text{EQA}} \). In particular, \( \mathbb{V}^{\dagger^1} \cong \mathbb{V}, \forall \mathbb{V} \in EQA \). The relation between \( \boxtimes \) and \( \dagger \) can be summarized by equations

\[
(\mathbb{V} \boxtimes \mathbb{W})^{\dagger^1} \cong \mathbb{V}^{\dagger^1} \boxtimes \mathbb{W}^{\dagger^1}, \quad \mathbb{R}^{\dagger^1} = \mathbb{R}.
\]

In terms of pairs \( (A; R), \boxtimes \) and \( \dagger \) are given by

\[
(A; R) \times (B; S) \mapsto (A \boxtimes B; R \boxtimes S), \quad \alpha \times \beta \mapsto \alpha \boxtimes \beta,
\]

and

\[
(A; R) \mapsto (A^\dagger; R^\dagger) \equiv ((A^*_1, A^*_{1}\otimes / \text{Im } R^*) ; -R^*), \quad \alpha \mapsto \alpha^\dagger,
\]

respectively, being \( A \boxtimes B \equiv \left( A_1 \otimes B_1, [A_1 \otimes B_1]^\otimes / \text{Im } [R \boxtimes S]\right) \). The arrows \( \alpha \boxtimes \beta \) and \( \alpha^\dagger \) are the extension of \( \alpha \otimes \beta \) and \( \alpha^\dagger \) to an algebra map. The unit object for \( \boxtimes \) is \((\mathbb{K}; \otimes)\).
3.2 The embedding $EQA \hookrightarrow QA$

Now, we study the relationship between $EQA$ and $QA$ as monoidal categories. There exists an obvious forgetful functor between these categories.

**Proposition 2** The function $(A; R) \mapsto A$ defines a surjective embedding $U : EQA \hookrightarrow QA$.

**Proof**: We just need to show the function is surjective, i.e., given $A \in QA$, there exists a compatible map $R$ such that $U(A; R) = A$. Let $I[R]$ be the ideal related to $A$. Consider a decomposition $A \oplus R = R \oplus R'$, with associated projections $P$ such that $\text{Im} P = R$. Since $I[R] = I[\text{Im} P]$, then $A \vdash P$ and the proposition follows. ■

On pairs $(V, R)$ the embedding is given by $(V, R) \mapsto (V, V \otimes / I[\text{Im} R])$ (see second part of Eq. 6). The surjectivity is up to isomorphisms in $QA$. The functor $U$ obviously preserves the unit objects, in fact

$$U = U(k, \mathcal{O}) = (k, k^\otimes / I[\text{Im} \mathcal{O}]) = (k, k^\otimes) = K,$$

but is not monoidal. Nevertheless,

**Proposition 3** There exist functorial epimorphisms $U(\mathcal{V} \boxtimes \mathcal{W}) \twoheadrightarrow U\mathcal{V} \circ U\mathcal{W}$, $\mathcal{V}, \mathcal{W} \in EQA$, defining a natural transformation $U \triangleright \circ (U \times U)$.

**Proof**: It is clear that, given pairs $\mathcal{V}$ and $\mathcal{W}$, we have

$$\text{Im } R \boxtimes S \approx \varphi_{V,W}(\text{Im } R \boxtimes S) = \text{Im } [R \otimes 1 + 1 \otimes S] \subset \text{Im } R \otimes W^\otimes + V^\otimes \otimes \text{Im } S,$$

and accordingly,

$$\varphi_{V,W}(I[\text{Im } R \boxtimes S]) \subset (I[\text{Im } R] \otimes W^\otimes + V^\otimes \otimes I[\text{Im } S]) \cap V \otimes W.$$

The first ideal in above inclusion is related to the quantum space $U(\mathcal{V} \boxtimes \mathcal{W})$, and the latter to $U\mathcal{V} \circ U\mathcal{W}$, since it defines the algebra

$$(V^\otimes / I[\text{Im } R]) \circ (W^\otimes / I[\text{Im } S]).$$

Note the corresponding algebras are quotient of $[V \otimes W]^\otimes$. Hence, for every couple $\mathcal{V}, \mathcal{W} \in EQA$, we have an epimorphism $p_{\mathcal{V},\mathcal{W}} : U(\mathcal{V} \boxtimes \mathcal{W}) \twoheadrightarrow U\mathcal{V} \circ U\mathcal{W}$.

By straightforward calculations, it can be checked commutativity of diagrams

$$\begin{align*}
U(\mathcal{V} \boxtimes \mathcal{W}) & \xrightarrow{p_{\mathcal{V},\mathcal{W}}} U\mathcal{V} \circ U\mathcal{W} \\
U(\alpha \boxtimes \beta) & \downarrow \quad U\alpha \circ U\beta \\
U(\mathcal{X} \boxtimes \mathcal{Y}) & \xrightarrow{p_{\mathcal{X},\mathcal{Y}}} U\mathcal{X} \circ U\mathcal{Y}
\end{align*}$$

(10)
for every couple of arrows $\alpha : \mathcal{U} \to \mathcal{X}$ and $\beta : \mathcal{W} \to \mathcal{Y}$ in EQA.

This result, together with the ones relating monoids $\circ$ and $\otimes$, will be useful in order to construct the rectangular comultiplications maps.

4 Rectangular quantum matrix algebras

Now, the central result. We shall show the following theorem later, in a more general context.

**Theorem 4** The monoidal category $(\text{EQA}, \boxtimes, \mathcal{K})$ is rigid, and has $\dagger$ as duality functor. For every $\mathcal{V} = (\mathcal{V}, R)$ in EQA, the evaluation and coevaluation arrows, $\text{ev}_{\mathcal{V}} : \mathcal{V}^\dagger \boxtimes \mathcal{V} \to \mathcal{K}$ and $\text{coev}_{\mathcal{V}} : \mathcal{K} \to \mathcal{V} \boxtimes \mathcal{V}^\dagger$, respectively, are given by the corresponding maps for $\mathcal{V}$ in the rigid monoidal category $(\text{Vct}_{f}, \otimes, k)$. ■

We can define the internal coHom object related to a couple $\mathcal{W}, \mathcal{V} \in \text{EQA}$ as $\text{hom}[\mathcal{W}, \mathcal{V}] = \mathcal{W}^\dagger \boxtimes \mathcal{V}$, and take

$$\delta_{\mathcal{V}, \mathcal{W}} = \tau_{\mathcal{W}, \text{hom}[\mathcal{W}, \mathcal{V}]} (\text{coev}_{\mathcal{W}} \otimes I) \ell_{\mathcal{V}} : \mathcal{V} \to \text{hom}[\mathcal{W}, \mathcal{V}] \boxtimes \mathcal{W}$$

as the (left) coevaluation arrow. Its well known universality property says: given $\mathcal{H} \in \text{EQA}$ and $\varphi : \mathcal{V} \to \mathcal{H} \boxtimes \mathcal{W}$, there exists a unique morphism $\alpha : \text{hom}[\mathcal{W}, \mathcal{V}] \to \mathcal{H}$ making commutative the diagram

$$\begin{array}{ccc}
\mathcal{V} & \xrightarrow{\varphi} & \mathcal{H} \\
\downarrow{\delta_{\mathcal{V}, \mathcal{W}}} & & \downarrow{\text{hom}[\mathcal{W}, \mathcal{V}] \boxtimes \mathcal{W} \xrightarrow{\alpha \otimes I} \mathcal{H} \boxtimes \mathcal{W}} \\
\text{hom}[\mathcal{W}, \mathcal{V}] \boxtimes \mathcal{W} & \xrightarrow{} & \mathcal{H} \boxtimes \mathcal{W}
\end{array}$$

From (11) and general properties of monoidal categories follow the existence of comultiplication

$$\text{hom}[\mathcal{W}, \mathcal{V}] \to \text{hom}[\mathcal{V}, \mathcal{U}] \boxtimes \text{hom}[\mathcal{W}, \mathcal{U}], \forall \mathcal{U}, \mathcal{V}, \mathcal{W} \in \text{EQA},$$

(12)

given by $\Delta_{\mathcal{V}, \mathcal{W}} = \tau_{\text{hom}[\mathcal{V}, \mathcal{U}], \text{hom}[\mathcal{W}, \mathcal{U}]} (I \boxtimes \text{coev}_{\mathcal{U}} \otimes I) (I \otimes \ell_{\mathcal{V}})$, and counit arrows

$$\varepsilon_{\mathcal{V}} = \text{ev}_{\mathcal{V}} : \text{end}[\mathcal{V}] \to \mathcal{K}, \forall \mathcal{V} \in \text{EQA}. \quad (13)$$

Coevaluations are particular comultiplications. Indeed, since $\text{hom}[\mathcal{K}, \mathcal{V}] = \mathcal{V}$, $\forall \mathcal{V} \in \text{EQA}$, it can be seen that $\delta_{\mathcal{V}, \mathcal{V}} = \Delta_{\mathcal{V}, \mathcal{V}, \mathcal{K}}$. On the other hand, if $\mathcal{U} = \mathcal{V} = \mathcal{W}$, $\Delta_{\mathcal{V}} = \Delta_{\mathcal{V}, \mathcal{V}, \mathcal{K}}$ and $\varepsilon_{\mathcal{V}}$ gives $\text{end}[\mathcal{V}]$ a coalgebra structure in EQA, and $\delta_{\mathcal{V}} = \delta_{\mathcal{V}, \mathcal{V}}$ makes $\mathcal{V}$ an $\text{end}[\mathcal{V}]$-corepresentation in the same category.
We add that arrows $12$ and $13$ satisfy usual associativity and unit constraints, expressed by commutativity of the following diagrams\footnote{Of course, we have a similar commutative diagram where $\mathcal{R}$ is on the right.}:

\[
\begin{array}{ccc}
\text{hom} \mathcal{X} \otimes \text{hom} \mathcal{X} & \rightarrow & \text{hom} \mathcal{Y} \otimes \text{hom} \mathcal{X} \\
\downarrow & & \downarrow \\
\text{hom} \mathcal{W} & \rightarrow & \text{hom} \mathcal{Y} \otimes \text{hom} \mathcal{W}
\end{array}
\]

(14)

\[
\begin{array}{ccc}
\mathcal{R} \otimes \text{hom} \mathcal{W} & \rightarrow & \text{end} \mathcal{W} \otimes \text{hom} \mathcal{W} \\
\downarrow & & \downarrow \\
\text{hom} \mathcal{W} & \rightarrow & \text{hom} \mathcal{W}
\end{array}
\]

(15)

Consider pairs $\mathcal{V} = (V, R)$ and $\mathcal{W} = (W, S)$ in EQA, and take basis $\{v_i\}$ and $\{w_i\}$ of $V$ and $W$, respectively. The image under $U$ of the internal coHom object

\[
\text{hom} \mathcal{W} \otimes \text{hom} \mathcal{V} = W^* \otimes V^* \subseteq \mathcal{V}
\]

is a quadratic algebra generated by $W^* \otimes V$ and obeying relations $I[\text{Im} S^\dagger \otimes R]$. Writing $t^n \otimes v, t^m \otimes w, s^{nm}$, straightforwardly

\[
\text{Im} S^\dagger \otimes R = \text{span} \left[t^n_i \otimes w \otimes t^m_j \otimes v - t^n_i \otimes v \otimes t^m_j \otimes w, s^{nm}ight]_{i,j,n,m}.
\]

(16)

Comparing $1$ with $10$, we have $U_{\text{hom}} \mathcal{W} \otimes \mathcal{V} = A(R : S)$. In particular, $U_{\text{end}} \mathcal{W} = A(R)$. Thus, the algebras $A(R : S)$ (resp. bialgebras $A(R)$) are the coordinate ring of an equipped quantum space with structure $S^\dagger \otimes R$ (resp. $R^\dagger \otimes R$), representing the space of homomorphisms from $\mathcal{W}^\text{op}$ to $\mathcal{V}^\text{op}$.

### 4.1 Rectangular comultiplication and counit maps

Now, we are going to construct rectangular comultiplication and counit maps defined in $\mathcal{R}$ for algebras $A(R : S)$. This can be done in steps below:

1. Apply the functor $U$ to comultiplications given in $12$ to obtain the maps $U_{\text{hom}} \mathcal{W} \otimes \mathcal{V} \rightarrow U(\text{hom} \mathcal{U} \otimes \text{hom} \mathcal{W} \otimes \mathcal{U})$.

2. Compose it with the functorial epimorphism $U(\text{hom} \mathcal{U} \otimes \text{hom} \mathcal{W}) \rightarrow U_{\text{hom}} \mathcal{U} \otimes U_{\text{hom}} \mathcal{W}$.
3. Apply the forgetful functor $F : (A_1, A) \mapsto A$ to this composition. This gives us an algebra homomorphism

$$FU_{\text{hom}} [W, V] \rightarrow F (U_{\text{hom}} [U, W] \circ U_{\text{hom}} [W, U]).$$

4. Finally, compose the latter with the functorial inclusion

$$F (U_{\text{hom}} [U, W] \circ U_{\text{hom}} [W, U]) \hookrightarrow FU_{\text{hom}} [U, W] \otimes FU_{\text{hom}} [W, U].$$

The resulting maps are precisely the arrows $A (R : S) \rightarrow A (R : T) \otimes A (T : S)$ defined in [6]. For counits:

1. Apply $FU$ to the counit arrow $\varepsilon_W : \text{end} [W] \rightarrow k$ given in (13), to obtain the arrow $A (R) \rightarrow k^\otimes$.

2. Compose to surjection $k^\otimes \rightarrow k$.

The algebra homomorphism $A (R) \rightarrow k$ we obtain, together with comultiplication $A (R) \rightarrow A (R) \otimes A (R)$ gives $A (R)$ a bialgebra structure. In fact, diagrams (14) and (15), properly combined with (4), (5) and (10), lead us to associativity and unit constraints for arrows

$$A (R : S) \rightarrow A (R : T) \otimes A (T : S), \quad A (R) \rightarrow k.$$

Summing up, we have constructed square and rectangular quantum matrices as internal coHOM objects in the rigid category of equipped quantum spaces, giving a generalization of $FRT$ construction in the scenario of Manin quantum groups.

### 4.2 The inclusions $U_{\text{hom}} [W, V]^{op} \hookrightarrow \text{hom} [UW, UV]^{op}$

Let us show there exist epis $\text{hom} [UW, UV] \rightarrow A (R : S)$, with $\text{hom} [UW, UV]$ the internal coHOM object in QA related to quantum spaces $UW$ and $UV$. As we have claimed in the introduction, this is an indication that quadratic algebras $A (R : S)$ represent homomorphisms between structurally richer spaces, w.r.t. coHOM objects in QA.

Consider the evaluation map $\delta_{W, V} : W \rightarrow \text{hom} [W, V] \boxtimes V$ in EQA. In the previously given basis $\{v_i\}$ and $\{w_i\}$ this map is defined by the assignment $v_i \mapsto t^i_j \otimes w_j$, putting again $t^i_j = w_j \otimes v_i$. It is functored by $U$ to an arrow $UW \rightarrow U (\text{hom} [W, V] \boxtimes V)$ which, composed to the functorial epi

$$U (\text{hom} [W, V] \boxtimes V) \rightarrow U_{\text{hom}} [W, V] \circ UW,$$

defines in QA another arrow $\varphi : UW \rightarrow U_{\text{hom}} [W, V] \circ UW$, also defined by $v_i \mapsto t^i_j \otimes w_j$. From universality of internal coHOM objects in QA (see Equation (11)), there exists a unique arrow

$$\alpha : \text{hom} [UW, UV] \rightarrow U_{\text{hom}} [W, V]$$
such that \( \varphi = (\alpha \circ I) \delta_{U_2, U_2} \), being \( \delta_{U_2, U_2} \) the coevaluation associated to \( \text{hom}_\mathcal{U}(U_2, U_2) \in \mathcal{QA} \). Recalling that (since \( U_2 \) and \( U_2 \) are quadratic algebras generated by \( W \) and \( V \), resp.) \( \text{hom}_\mathcal{U}(U_2, U_2) \) is generated by \( W^* \otimes V \) and \( \delta_{U_2, U_2}(u_i) = t_i^j \otimes w_j \), then \( \alpha \) is the identity on generators. Consequently, due to \( U \text{hom}_\mathcal{U}(U_2, U_2) \) is also generated by \( W^* \otimes V \), \( \alpha \) is an algebra epimorphism.

Hence, we have shown: For every couple \( \mathcal{U}, \mathcal{V} \) of equipped quantum space, the quantum space \( \text{hom}_\mathcal{U}(U_2, U_2)^{op} \) ‘contains’ \( U \text{hom}_\mathcal{U}(U_2, U_2)^{op} \) as a subspace.

5 The equipped conic quantum spaces

Given a finite dimensional \( k \)-vector space \( V \in \text{Vct}_f \), consider the degree cero homogeneous linear endomorphisms of \( V^o \), i.e.

\[
R \in \text{End}_{\text{Vct}}[V^o] \text{ such that } R(V^{o \otimes n}) \subseteq V^{\otimes n}.
\]

Of course, each map \( R \) is defined by a family \( \{R_n\}_{n \in \mathbb{N}_0} \) of linear maps \( R_n : V^{\otimes n} \to V^{\otimes n} \). In terms of these endomorphisms, all above constructions can be repeated word by word in the category \( \mathcal{CA} \). That is to say, we can define equipped conic quantum spaces as pairs \( (A, R) \), \( A \in \mathcal{CA} \), such that \( A \vdash R \), i.e. \( \ker [A^\otimes \to A] = I [\text{Im } R] \). We just must change the defining condition for morphisms \( (A, R) \to (B, S) \) by \( \alpha_1^\otimes R = S \alpha_1^\otimes \), or \( \alpha_1^\otimes R_n = S_n \alpha_1^\otimes n \) for all \( n \in \mathbb{N}_0 \). Let us call \( \mathcal{ECA} \) the related category. It can be shown, the category of equipped conic quantum spaces \( \mathcal{ECA} \) is rigid w.r.t. the monoidal structure \( \boxtimes \), and has \( \dagger \) as duality functor.

Theorem 5 The category of equipped conic quantum spaces is rigid w.r.t. the monoidal structure \( \boxtimes \), and has \( \dagger \) as duality functor.

Proof: Consider an object \( \mathcal{U} = (V, R) \). We define the evaluation and the coevaluation morphisms,

\[
eval_{\mathcal{U}} : \mathcal{U}^\dagger \boxtimes \mathcal{U} \to R \quad \text{and} \quad \coeval_{\mathcal{U}} : R \to \mathcal{U} \boxtimes \mathcal{U}^\dagger,
\]
as the usual pairing \( v \otimes v' \mapsto \langle v, v' \rangle e \) and coevaluation of \( V \) and \( V^* \), respectively, being \( e \) the generator of \( k \). We must show that they are effectively arrows in \( \mathcal{ECA} \).
Note that the tensor product map $ev^\otimes_\mathcal{G}$ defines the algebra homomorphism $V^* \circ V \rightarrow k^\otimes,$

$$u \otimes v \in V^* \otimes V \otimes V \rightarrow \langle u, v \rangle e^n \in k^\otimes,$$

where we use $\varphi_{V^*, V}$ to identify the algebras $V^* \circ V$ and $[V^* \otimes V]^\otimes.$ By direct calculations (and from the very definition of $R^\dagger$), it can be seen

$$ev^\otimes_\mathcal{G} R^\dagger \boxtimes R = 0 = \mathcal{O} ev^\otimes_\mathcal{G}.$$

To show the analogous equation for the coevaluation map, let us first introduce some notation. Let $\{v_i\}$ be a basis of $V.$ Construct for each $n \in \mathbb{N}$ a basis $\{v_{R}^n\}$ of $V^\otimes_n,$ being $R = (r_1, ..., r_n)$ a multi-index with $1 \leq r_k \leq \dim V, \forall k = 1, ..., n,$ in such a way that $v_R = v_{r_1} \otimes ... \otimes v_{r_n}.$ Consider also the basis $\{v_R^*\}$ of $V^* \otimes V^\otimes_n.$

In these terms, the algebra homomorphism

$$v^n \in k^\otimes \mapsto v_R \otimes v_R^* \in V \circ V^*$$

(sum over repeated (multi)indices is understood) coincides with the map $coev^\otimes_\mathcal{G}.$

To see that equation

$$R \boxtimes R^\dagger coev^\otimes_\mathcal{G} = coev^\otimes_\mathcal{G} \mathcal{O}$$

holds, we just have to prove $R \otimes I - I \otimes R^*$ evaluated on $v_R \otimes v^R$ is equal to cero, i.e.

$$[R (v_R) \otimes v^R - v_S \otimes R^* (v^S)] = 0.$$

Writing $R (v_R) = R^S_R v_S,$ we have for the transpose $R^* (v^S) = v^R R^S_R,$ and consequently the left member of equation above is identically cero. So, the theorem is proven. ■

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