Horizon states and the sign of their index in $\mathcal{N} = 4$ dyons

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Abstract: Classical single centered solutions of 1/4 BPS dyons in $\mathcal{N} = 4$ theories are usually constructed in duality frames which contain non-trivial hair degrees of freedom localized outside the horizon. These modes are in addition to the fermionic zero modes associated with broken supersymmetry. Identifying and removing the hair from the 1/4 BPS index allows us to isolate the degrees of freedom associated with the horizon. The spherical symmetry of the horizon then ensures that index of the horizon states has to be positive. We verify that this is indeed the case for the canonical example of dyons in type IIB theory on $K3 \times T^2$ and prove this property holds for a class of states. We generalise this observation to all CHL orbifolds, this involves identifying the hair and isolating the horizon degrees of freedom. We then identify the horizon states for 1/4 BPS dyons in $\mathcal{N} = 4$ models obtained by freely acting $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifolds of type IIB theory compactified on $T^6$ and observe that the index is again positive for single centred black holes. This observation coupled with the fact the 1/4 BPS index of single centred solutions without removal of the hair violates positivity indicates that there exists no duality frame in these models without non-trivial hair.
1 Introduction

Counting microscopic degrees of freedom for extremal black holes in string theory is a useful probe into aspects of quantum gravity [1]. For supersymmetric black holes, one should in principle be able identify the degrees of freedom both from the macroscopic solution as well as count them from the microscopic description of these black holes. The 1/4 BPS dyonic black holes in $\mathcal{N} = 4$ theory is a system which has been extensively studied in this context, see [2, 3] for reviews. The identification of the degrees of freedom is complicated by the fact that classical solutions of black holes are multi-centered and usually they contain hair degrees of freedom localized outside the horizon [4–6]. The microscopic analysis counts all these configurations together. Let us make this precise, let $d_{\text{micro}}(\vec{q})$ be the degeneracy or in the case of the extremal supersymmetric black holes the appropriate supersymmetric index evaluated from the microscopic description of a BPS state with charge $\vec{q}$. Similarly let $d_{\text{macro}}(\vec{q})$ be the corresponding macroscopic index. Then

$$d_{\text{macro}}(\vec{q}) = \sum_n \sum_{\sum_{i=1}^n \vec{q}_i + \vec{q}_{\text{hair}} = \vec{q}} \left( \prod_{i=1}^n d_{\text{hor}}(\vec{q}_i) \right) d_{\text{hair}}(\vec{q}_{\text{hair}}; \{ \vec{q}_i \})$$  (1.1)
Each term on the right hand side of (1.1) is the contribution to the index of say the $n$-centered black hole configuration. $d_{\text{hor}}(\vec{q}_i)$ is the contribution to the index from the horizon degrees of freedom with charge $q_i$ and $d_{\text{hair}}(\vec{q}_{\text{hair}}; \{\vec{q}_i\})$ is the index of the hair carrying total charge $\vec{q}_{\text{hair}}$ of a $n$-centered black hole whose horizons carry charges $\vec{q}_1, \cdots \vec{q}_n$. We expect

$$d_{\text{macro}}(\vec{q}) = d_{\text{micro}}(\vec{q}).$$

(1.2)

It would simplify matters if we can restrict our attention to single centred black hole configurations. Then (1.1) indicates that we would need to identify the hair to isolate the horizon degrees of freedom. Since we are dealing with $1/4$ BPS states in $\mathcal{N} = 4$ theories, which break 12 supersymmetries, the degeneracy $d(\vec{q})$ will refer to the index

$$B_6 = \frac{1}{6!} \text{Tr}((2J)^6(-1)^{2J}),$$

(1.3)

where $J$ is the component of the angular momentum in say the 3 direction. The factorized form of the Hilbert space corresponding to the hair degrees of freedom and the horizon degrees of freedom follows from the fact the these are well separated due the presence of an infinite throat [4].

The utility of identifying the horizon degrees of freedom lies in the fact that the horizon is spherically symmetric and therefore carries zero momentum $J = 0$. The index taken over the horizon states reduces to $(-1)^{2J}d_{\text{hor}} = d_{\text{hor}}$, where $d_{\text{hor}}$ is the total number of states associated with the horizon. Therefore the index of the horizon states must be a positive number. This leads to an important check on the microscopic counting and the equality (1.2). Once one determines the hair degrees of freedom for a given macroscopic black hole and factors them out of the index, what must remain is a positive number which counts the index of the horizon states. This argument clearly relies on what are the hair degrees of freedom and this in turn depends on the duality frame of the macroscopic solution. This prediction was tested in [7] with the assumption that there exists a frame in which the only hair degrees of freedom are the fermionic zero modes associated with the broken supersymmetry generators. For black holes in $\mathcal{N} = 8$ there is evidence towards this fact in [8, 9]. These authors worked in a frame in which the black hole configuration reduced to a system of only D-branes and showed the only hair degrees of freedom were the fermionic zero modes and the BPS configuration indeed had zero angular momentum. However such a frame has not yet been shown to exist for black holes in $\mathcal{N} = 4$ theory.

Given this situation, one way of proceeding is to evaluate the partition functions corresponding to the hair degrees of freedom and isolate the horizon degrees of freedom in a given frame. This has already been done in [5, 6], for $1/4$ BPS dyons in the type IIB frame, but a test of positivity of the index for the resulting horizon degrees of freedom has not been done. We perform this analysis in this paper and indeed
demonstrate that the $d_{\text{hor}}$ is indeed positive. This is quite remarkable as we will see, since factorizing the hair degrees of freedom naively seems to introduce terms with negative contributions to the index. We adapt the proof of [10] for configurations with magnetic charge $P^2 = 2$ and demonstrate that the index is positive. We then extend this observation to all the CHL models and to other orbifolds associated with Mathieu moonshine introduced in [11, 12].

In [11, 13] it was observed that for $\mathcal{N} = 4$ models obtained by freely acting $\mathbb{Z}_2, \mathbb{Z}_3$ orbifolds of type IIB on $T^6$, the index for single centered configurations after factorising the sign due to the fermionic zero modes did not obey the expectation $d_{\text{hor}}$ is positive \footnote{Please see tables 16, 17, 18 reproducing this observation}. But as the above discussion shows, a possible reason for this could be that the assumption that there exists a frame in which the fermionic zero modes are the only hair degrees of freedom might not be true. Therefore we re-examine this question in this paper. Following the same procedure used in the CHL models we isolate the hair degrees of freedom in the type IIB frame. Then on examining the sign of the index for single centered black holes we observe that $d_{\text{hor}}$ is positive.

The organisation of the paper is as follows. In the section 2 we briefly review the statements about the hair and the the partition function for the horizon degrees of freedom. for the 1/4 BPS dyonic black hole in type IIB compactified on $K3 \times T^2$. We then generalise this to all the CHL orbifolds as well as other orbifolds associated with Mathieu Moonshine. Finally we construct the partition function for the horizon states for the toroidal models obtained by freely acting $\mathbb{Z}_2, \mathbb{Z}_3$ orbifolds of type IIB on $T^6$. In section 3 we perform a consistency check on the $d_{\text{hor}}$ obtained. This check relies on the fact that the 5-dimensional BMPV black hole has the same near horizon geometry [5, 6]. Therefore $d_{\text{hor}}$ for the BMPV black hole should agree with that of the 1/4 BPS dyon. We show that this is indeed the case for all the examples. Finally in section 4, armed with the $d_{\text{hor}}$ for all the models we study the positivity of the index for single centered black holes for all the models. We have evaluated numerically the indices of horizon states for several charges in all the $\mathcal{N} = 4$ models for which dyon partition functions are known which confirm that the index is positive. We adapt the proof of [10] to show that the index is positive for charge configurations with $P^2 = 2$. Section 5 contains our conclusions.

\section{Horizon states for the 1/4 BPS dyon}

In this section we construct the partition function for the horizon states for 1/4 BPS dyons in $\mathcal{N} = 4$ compactifications. This is done by identifying the ‘hair’ degrees of freedom which are localized outside the horizon. Such a partition function for the horizon states was constructed for the canonical $\mathcal{N} = 4$ theory obtained by
compactifying type IIB string theory $K3 \times T^2$ in [5, 6] in the type IIB frame. We review this in section and then extend the analysis for other $\mathcal{N} = 4$ models.

The $\mathcal{N} = 4$ compactifications of interest are type IIB theory on $K3 \times T^2/\mathbb{Z}_N$ where $\mathbb{Z}_N$ acts as an automorphism $g'$ on $K3$ along with a shift of $1/N$ units on one of the circles of $T^2$. The action of $g'$ can be labelled by the 26 conjugacy classes of the Mathieu group $M_{23}$. The classes $pA$ with $p = 2, 3, 5, 6, 7, 8$ and the class $4B$ are called as Nikulin’s automorphism of $K3$. They were first introduced in [14, 15] as models dual to heterotic string theory with $\mathcal{N} = 4$ superymmetry but with gauge groups with reduced from the maximal rank of 28. All these compactifications admit $4$ models dual to heterotic string theory with $\mathcal{N} = 4$ superysmmetry but with gauge groups with reduced from the maximal rank of 28. All these compactifications admit $1/4$ BPS dyons, let $(Q, P)$ be the electric and magnetic charge vector of these dyons, then the $1/4$ BPS index $B_6$ is given by [16–20]

$$- B_6 = \frac{1}{N} (-1)^{Q+1} \int d\rho d\sigma dv \, e^{-\pi i (N\rho Q^2 + \sigma P^2 / N + 2\nu Q)} \frac{1}{\Phi_k(\rho, \sigma, v)},$$

(2.1)

where $C$ is a contour in the complex 3-plane defined by

$$\rho_2 = M_1, \quad \sigma_2 = M_2, \quad \tilde{\nu}_2 = -M_3,$$

$$0 \leq \rho_1 \leq 1, \quad 0 \leq \sigma_1 \leq N, \quad 0 \leq \tilde{\nu}_1 \leq 1.$$

Here $\rho = \rho_1 + i \rho_2, \sigma = \sigma_1 + i \sigma_2, v = \nu_1 + i \nu_2$ and $M_1, M_2, M_3$ are positive numbers, which are fixed and large and $M_3 << M_1, M_2$. The contour in (2.2) implies that we first expand in powers or $e^{2\pi i \rho}, e^{2\pi i \sigma}$ and at the end perform the expansion in $e^{2\pi iv}$.

The Siegel modular form $\tilde{\Phi}(\rho, \sigma, v)$ transforming under $Sp(2, \mathbb{Z})$, or its subgroups for $N > 1$ admits an infinite product representation given by

$$\tilde{\Phi}_k(\rho, \sigma, v) = e^{2\pi i (\rho + \sigma / N + v)} \times$$

$$\prod_{r=0}^{N-1} \prod_{k \in \mathbb{Z}, l \in \mathbb{Z}} \prod_{j \in \mathbb{Z}, j \neq 0} (1 - e^{2\pi i (k' \sigma + l \rho + j v)}) \sum_{s=0}^{N-1} c^{(r, s)}(4kl - j^2).$$

(2.3)

The coefficients $c^{(r, s)}$ are determined from the expansion of the twisted elliptic genera for the various order $N$ orbifolds $g'$ of $K3$. The twisted elliptic genus of $K3$ is defined by

$$F^{(r, s)}(\tau, z) = \frac{1}{N} \text{Tr}_{RR, g'} \left[ (-1)^{F_{K3} + \tilde{F}_{K3}} \frac{q^s e^{2\pi i z F_{K3}} q^{-\tilde{F}_{K3} - \tilde{F}_0 + \tilde{z}}}{q^{-\tilde{F}_0 + \tilde{z} / 2\pi i}} \right],$$

(2.4)

$$= \sum_{j \in \mathbb{Z}, n \in \mathbb{Z} / N} c^{(r, s)}(4n - j^2) e^{2\pi i \nu r + 2\pi i j z}.$$

$$0 \leq r, s \leq N - 1.$$

The trace is performed over the Ramond-Ramond sector of the $\mathcal{N} = (4, 4)$ super conformal field theory of $K3$ with $(c, \bar{c}) = (6, 6)$, $F$ is the Fermion number and $j$ is
the left moving $U(1)$ charge of the $SU(2)$ $R$-symmetry of $K3$. The twisted elliptic genera for the $g'$ corresponding to conjugacy classes of $M_{23} \cup M_{24}$ have been evaluated in [11]. These take the form

\begin{align}
F^{(0,0)}(\tau, z) &= \alpha^{(0,0)}_{g'} A(\tau, z), \\
F^{(r,s)}(\tau, z) &= \alpha^{(r,s)}_{g'} A(\tau, z) + \beta^{(r,s)}_{g'} B(\tau, z),
\end{align}

where

\begin{align}
A(\tau, z) &= \frac{\theta_2^2(\tau, z)}{\theta_2^2(\tau, 0)} + \frac{\theta_3^2(\tau, z)}{\theta_3^2(\tau, 0)} + \frac{\theta_4^2(\tau, z)}{\theta_4^2(\tau, 0)}, \\
B(\tau, z) &= \frac{\eta^8(\tau)}{\eta^8(\tau) + \eta^8(\tau+\tau)}.
\end{align}

The coefficients $\alpha^{(r,s)}_{g'}$ in (2.5) are numerical constants, while $\beta^{(r,s)}_{g'}(\tau)$ are modular forms that transform under $\Gamma_0(N)$. For $g'$ corresponding to conjugacy classes of $M_{23}$, they can be read out from appendix E of [11]. For example, in the case of the $pA$ orbifolds with $p = 1, 2, 3, 5, 7$, they are given by [18].

\begin{align}
F^{(0,0)} &= \frac{8}{N} A(\tau, z), \\
F^{(0,s)} &= \frac{8}{(N+1)N} A(\tau, z) - \frac{2}{N+1} B(\tau, z) \mathcal{E}_N(\tau), \\
F^{(r,s)} &= \frac{8}{N(N+1)} A(\tau, z) + \frac{2}{N(N+1)} B(\tau, z) \mathcal{E}_N(\tau + k \frac{N}{N}), \\
\mathcal{E}_N(\tau) &= \frac{12i}{\pi(N-1)} \frac{\partial}{\partial \tau} [\ln \eta(\tau) - \ln \eta(N\tau)].
\end{align}

For $N$ composite corresponding to the classes $4B, 6A, 8A$, the strategy for construction of the twisted elliptic genus was first given in [21] and it was worked out explicitly for the $4B$ example \footnote{Suresh Govindarajan informed us that the authors of [21] also explicitly constructed all the sectors of the $6A$ and $8A$ twisted elliptic genera though it was not reported in the paper.}. The papers [22–24] contain the twining characters, $F^{(0,s)}$ and [25] also contains the strategy to construct the twisted elliptic genera for other conjugacy classes of $M_{23}$ and a Mathematica code for generating the elliptic genera.

The weight of the Siegel modular form $\Phi(\rho, \sigma, v)$ is given by

\begin{equation}
k = \frac{1}{2} \sum_{s=0}^{N-1} c^{(0,s)}(0). \tag{2.8}
\end{equation}

For the classes $pA, p = 1, 2, 3, 5, 7, 11$ we have

\begin{equation}
k = \frac{24}{p+1} - 2, \tag{2.9}
\end{equation}

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for 4B, 6A, 8A we have $k = 3, 2, 1$ respectively and for 14A, 15A $k = 0$.

Finally, as discussed in the introduction the study of horizon states would be much simpler if one could focus on single centered dyons. Such a system would have only one horizon. The choice of the contour chosen in (2.2) together with some kinematic constraints on charges such as (4.3) ensures that we are in the attractor region of the axion-dilaton moduli and the index given by (2.1) is that of single centred dyons [7, 26]. All the indices evaluated in this section paper is done using the contour (2.2).

2.1 The canonical example: $K3 \times T^2$

In the work of [5, 6] the hair modes of the 1/4 BPS dyonic black hole in type IIB theory compactified on $K3 \times T^2$ were constructed. Here we briefly review this construction. These modes were shown to be deformations localized outside the horizon and they preserved supersymmetry. Let us first recall that the dyonic black hole in 4-dimensions is constructed by placing the 5 dimensional BMPV black hole or the rotating D1-D5 system [27] in Taub-Nut space [28]. The Taub-Nut space has the geometry which at the origin is $R^4$ but at infinity it is $R^3 \times \tilde{S}$. The isometry along $S^1$ coincides with the angular direction the BMPV rotates. The hair modes arise from the collective modes of the D1-D5 system thought of as an effective string along say the $x^5$ and the time $t$ directions. Therefore these modes are oscillations of the effective string, they are left moving since they have to preserve supersymmetry. After allowing the fermionic zero modes associated with the 12 broken susy generators to saturate $(2J)^6/6!$ in the helicity trace given in (1.3), the non-trivial hair modes consist of

- 4 left moving fermionic modes arising from the deformations of the gravitino giving rise to the contribution

$$Z_{\text{hair:}1A}^{4d:f} = \prod_{l=1}^{\infty} (1 - e^{2\pi i l \rho})^4$$

(2.10)

- 3 left moving bosonic modes associated with the oscillation of the effective string in the 3 transverse directions $R^3$ as Taub-NUT is asymptotically $R^3 \times \tilde{S}^1$.

$$Z_{\text{hair:}1A}^{4d:b\perp} = \prod_{l=1}^{\infty} \frac{1}{(1 - e^{2\pi i l \rho})^3}$$

(2.11)

- 21 left moving bosonic modes, these arise from the deformations of the 21 anti-self-dual forms of type IIB on K3. These deformations involve 21 scalar functions folded with the 2 form $d\omega_{TN}$ on the Taub-Nut given by

$$\delta H^s = h^s(v)dv \wedge d\omega_{TN}, \quad v = t + x^5, \quad s = 1, \cdots 21$$

(2.12)

\footnote{It is easy to see from the heterotic frame that only left moving oscillations preserve supersymmetry.}
Counting these oscillations we obtain

\[ Z_{\text{4d:asd}}^{\text{hair:1A}} = \prod_{l=1}^{\infty} \frac{1}{(1 - e^{2\pi i l \rho})^{21}}. \]  

(2.13)

The 21 anti-self dual forms arise from compactifying the RR 4-form on the 19 anti-self dual 2 form of the $K3$ together with the NS 2-form and the RR 2-form of type IIB.

Note that in the partition function we labelled the chemical potential to count the oscillations by $\rho$, this is because exciting these left moving momentum modes correspond to exciting the electric charge of the dyon [19]. Now combining these partition functions we obtain

\[ Z_{\text{4d:asd}}^{\text{hair:1A}} = Z_{\text{4d:asd}}^{\text{hair:1A}} \times Z_{\text{4d:asd}}^{\text{hair:1A}} . \]

(2.14)

\[ = \prod_{l=1}^{\infty} (1 - e^{2\pi i (l \rho)})^{-20} . \]

The Bosonic hair partition function is given by

\[ Z_{\text{bosons}}^{\text{hair:1A}} = Z_{\text{hair:1A}}^{\text{4d:}\perp} \times Z_{\text{hair:1A}}^{\text{4d:asd}} = \frac{e^{2\pi i \rho}}{\eta^{24}(\rho)} . \]

(2.15)

this is identical to that of the counting the degeneracy of purely electric states in this model without the zero point energy. This observation will help in the generalizations to CHL models.

To obtain the partition function of horizon states we factor out the hair degrees of freedom resulting in

\[ Z_{\text{hor}} = \frac{1}{\Phi_{10}(\rho, \sigma, v)} Z_{\text{4d:asd}}^{\text{hair:1A}} . \]

(2.16)

The index for the horizon states can be then be obtained by extracting the Fourier coefficients using the expression given by

\[ d_{\text{hor}} = -(-1)^{Q-P} \int_{C} d\rho d\sigma dv e^{-\pi i (\rho Q^2 + \sigma P^2 + 2v Q P)} \frac{1}{\Phi_{10}(\rho, \sigma, v)} \prod_{l=1}^{\infty} (1 - e^{2\pi i (l \rho)})^{20} . \]

(2.17)

Here the contour $C$ is same as that defined in (2.2).

2.2 Orbifolds of $K3 \times T^2$

2A orbifold

Before we present the analysis for the most general orbifold, let us examine in detail the analysis for the 2A orbifold. In this case, the orbifold acts by exchanging 8 pairs of anti-self dual $(1,1)$ forms out of the 19 anti-self dual forms of $K3$ with the $1/2$ shift on $S^1$ [15]. Note that because of the $1/2$ shift, the natural unit of momentum on $S^1$ is $N = 2$. With this input we are ready to repeat the analysis for the partition function of the hair modes
• The 4 left moving fermionic modes arising from the deformations of the gravitino give rise to the contribution

\[ Z^4_{\text{hair:2A}} = \prod_{l=1}^{\infty} (1 - e^{4\pi i l \rho})^4. \] (2.18)

Note that due to the fact that the periodicity is now \( \frac{2\pi}{N} \), the unit of momentum is doubled.

• The 3 transverse bosonic deformations along \( R^3 \) of the effective string results in

\[ Z^4_{\text{hair:2A}} = \prod_{l=1}^{\infty} \frac{1}{(1 - e^{4\pi i l \rho})^3}. \] (2.19)

• The action of the orbifold projects out 8 anti-self dual forms. The analysis for \( 13 = 11 + 2 \).\(^4\) invariant anti-self dual forms proceeds as before except for the fact that the unit of momentum is 2

\[ Z^4_{\text{hair:2A}|\text{invariant}} = \prod_{l=1}^{\infty} \frac{1}{(1 - e^{4\pi i l \rho})^{13}}. \] (2.20)

Consider the following boundary conditions of the function \( h(s) \) in (2.12) for the 8 projected anti-self dual forms.

\[ h(v + \frac{2\pi}{N}) = -h(v), \quad N = 2. \] (2.21)

These deformation pick up sign when one move by 1/2 unit on \( S^1 \). The partition function corresponding to these modes is given by

\[ Z^4_{\text{hair:2A}|\text{twisted}} = \prod_{l=1}^{\infty} \frac{1}{(1 - e^{2\pi i (2l-1) \rho})^8}. \] (2.22)

Note that these modes are twisted for the circle of radius \( 2\pi / N, N = 2 \), they obey anti-periodic boundary conditions. However in supergravity periodicities are measured over the circle of radius \( 2\pi \) and they are periodic for this radius, therefore these modes can be counted as hair modes. Together, the contribution of the anti-self dual forms to the partition function is given by

\[ Z^4_{\text{hair:2A}} = Z^4_{\text{hair:2A}|\text{invariant}} \times Z^4_{\text{hair:2A}|\text{twisted}}, \] (2.23)

\[ = \prod_{l=1}^{\infty} \frac{1}{(1 - e^{4\pi i l \rho})^5} \prod_{l=1}^{\infty} \frac{1}{(1 - e^{2\pi i l \rho})^8} \]

\(^4\)The 2 arises from the anti-self dual component of the RR 2-form and the NS 2-form.
Now combining all the hair modes we obtain
\[
Z_{\text{hair:2A}}^{4d} = Z_{\text{hair:2A}}^{4d:f} \times Z_{\text{hair:2A}}^{4d:1} \times Z_{\text{hair:2A}}^{4d:asd}
\]
\[
= \prod_{l=1}^{\infty} (1 - e^{4\pi i l \rho})^{-4} (1 - e^{2\pi i l \rho})^{-8}.
\]

Observe that the partition function of the bosonic hair modes is given by
\[
Z_{\text{hair:2A}}^{4d:b} = Z_{\text{hair:2A}}^{4d:1} \times Z_{\text{hair:2A}}^{4d:asd},
\]
\[
= \prod_{l=1}^{\infty} (1 - e^{4\pi i l \rho})^{-8} (1 - e^{2\pi i l \rho})^{-8},
\]
\[
= \frac{e^{2\pi i \rho}}{\eta^8(2\rho) \eta^8(\rho)}.
\]

This is the partition function of the fundamental string in the $N = 2$ CHL orbifold of the heterotic theory with the zero point energy removed \[19, 29\].

**pA orbifolds** $p = 2, 3, 5, 7$

The construction of the hair modes for the case of orbifolds of prime order, the method proceeds as discussed in detail for the 2A orbifold. In each case we need to count the number of 2-forms which are left invariant and which pick up phases and evaluate the partition function. The result for the bosonic hair modes is given by
\[
Z_{\text{hair: pA}}^{4d:b} = \prod_{l=1}^{\infty} \frac{1}{(1 - e^{2\pi i l \rho N})^{k+2} (1 - e^{2\pi i l \rho})^{k+2}},
\]
where
\[
k = \frac{24}{p + 1} - 2.
\]

Note that this is the partition of the states containing only the electric charges or the fundamental string without the zero point energy \[19\]. Now including the 4 fermionic deformations we obtain
\[
Z_{\text{hair: pA}}^{4d} = \prod_{l=1}^{\infty} (1 - e^{2\pi i (N l \rho)})^{-k} (1 - e^{2\pi i l \rho})^{-k} (1 - e^{2\pi i (N l \rho)})^4
\]
\[
= \prod_{l=1}^{\infty} (1 - e^{2\pi i (N l \rho)})^{-2k} \prod_{N \not| l} (1 - e^{2\pi i l \rho})^{-k+2}
\]

It is useful to rewrite this expression as follows
\[
Z_{\text{hair: pA}}^{4d} = \prod_{l=1}^{\infty} (1 - e^{2\pi i (N l \rho)})^{-\sum c^{(0,s)}(0) \prod_{N \not| l} (1 - e^{2\pi i l \rho})^{-\sum e^{2\pi i l / N} c^{(0,s)}(0)}
\]
\[
= \prod_{l \neq 0} (1 - e^{2\pi i l \rho})^{-\sum e^{-2\pi i l / N} c^{(0,s)}(0)}
\]
The sum is on the range of \( s = 0 \) to \( N - 1 \) and \( N \nmid l \) implies \( N \) does not divide \( l \).

The values of \( \sum_{s=0}^{N-1} e^{-2\pi isl/N} e^{(0,s)}(-b^2) \) for prime \( N \) are listed in table 1

| \( N \) | \( l \) | \(-b^2\) | \( \sum_{s=0}^{N-1} e^{-2\pi isl/N} e^{(0,s)}(-b^2) \) |
|-----|-----|-----|--------------------|
| \( p \) | \( N|l \) | 0 | \( 2k = \frac{48}{N+1} - 4 \) |
| \( N \nmid l \) | 0 | \( k + 2 = \frac{24}{N+1} \) |
| \( N \nmid l \) | -1 | 0 |

Table 1: Values of \( \sum_{s=0}^{N-1} e^{-2\pi isl/N} e^{(0,s)}(-b^2) \) for orbifolds of \( K3 \) with prime order \( (N = p) \)

**Orbifolds of composite order: 4B, 6A, 8A**

One can count the hair modes in a similar fashion as the orbifolds of prime order. The only difference would arise for the bosonic modes \( Z^{bosons}_{hair} \), which needs to be replaced by the fundamental string in these theories without the zero point energy. Including the 4 fermionic hairs, we see that the answer can be written in the same form as that seen for orbifolds with prime order.

\[
Z^{4d}_{hair:CHL} = \prod_{l=1}^{\infty} (1 - e^{2\pi i l(N \rho)}) - \sum_{N \nmid l} e^{2\pi i l(\rho)} - \sum_{N \nmid l} e^{-2\pi isl/N} e^{(0,s)(0)}.
\] (2.30)

The sum ranges from \( s = 0 \) to \( N - 1 \). This can be rewritten as

\[
Z^{4d}_{hair:CHL} = \prod_{l=1}^{\infty} (1 - e^{2\pi i l(\rho)}) - \sum e^{-2\pi isl/N} e^{(0,s)(0)}.
\] (2.31)

For the geometric CHL orbifolds, we list \( \sum_{s=0}^{N-1} e^{-2\pi isl/N} e^{(0,s)}(-b^2) \) for different \( N = 4, 6, 8 \) in table 2
| $N$ | $l$      | $-b^2$ | $\sum_{s=0}^{N-1} e^{-2\pi i s l/N} c^{(0,s)}(-b^2)$ |
|-----|----------|--------|--------------------------------------------------|
| 4   | $4|l$    | 0      | 6                                                |
|     |         | -1     | 2                                                |
|     | $2|l$, $4 \nmid l$ | 0      | 6                                                |
|     | $2 \nmid l$ | 0      | 4                                                |
| 6   | $6|l$    | 0      | 4                                                |
|     |         | -1     | 2                                                |
|     | $2|l$, $6 \nmid l$ | 0      | 4                                                |
|     | $3|l$, $6 \nmid l$ | 0      | 4                                                |
|     | $2 \nmid l$, $3 \mid l$ | 0      | 2                                                |
| 8   | $8|l$    | 0      | 2                                                |
|     |         | -1     | 2                                                |
|     | $2|l$, $4 \nmid l$ | 0      | 3                                                |
|     | $4|l$, $8 \nmid l$ | 0      | 4                                                |
|     | $2 \nmid l$ | 0      | 2                                                |

Table 2: Values of $\sum_{s=0}^{N-1} e^{-2\pi i s l/N} c^{(0,s)}(-b^2)$ for non-prime CHL orbifolds of $K3$. $\sum_{s=0}^{N-1} e^{-2\pi i s l/N} c^{(0,s)}(-1) = 0$ if $N \nmid l$ for any of these cases.
Using the data from table 2 we obtain

\[ Z_{\text{hor:4B}}^{4d} = \prod_{l=1}^{\infty} (1 - e^{2\pi i(4l\rho)})^4 (1 - e^{2\pi i(4l\rho)})^{-4} (1 - e^{2\pi i(3l\rho)})^{-2} (1 - e^{2\pi i(l\rho)})^{-4} \]

\[ = \prod_{l=1}^{\infty} (1 - e^{2\pi i(2l\rho)})^{-2} (1 - e^{2\pi i(l\rho)})^{-4} \]

\[ Z_{\text{hor:6A}}^{4d} = \prod_{l=1}^{\infty} (1 - e^{2\pi i(6l\rho)})^4 (1 - e^{2\pi i(6l\rho)})^{-2} (1 - e^{2\pi i(3l\rho)})^{-2} (1 - e^{2\pi i(l\rho)})^{-2} \]

\[ = \prod_{l=1}^{\infty} (1 - e^{2\pi i(2l\rho)}) (1 - e^{2\pi i(3l\rho)})^{-2} (1 - e^{2\pi i(l\rho)})^{-2} \]

\[ Z_{\text{hor:8A}}^{4d} = \prod_{l=1}^{\infty} (1 - e^{2\pi i(8l\rho)}) (1 - e^{2\pi i(6l\rho)})^{-2} (1 - e^{2\pi i(4l\rho)})^{-1} (1 - e^{2\pi i(l\rho)})^{-2} \]

\[ = \prod_{l=1}^{\infty} (1 - e^{2\pi i(2l\rho)})^{-1} (1 - e^{2\pi i(4l\rho)})(1 - e^{2\pi i(l\rho)})^{-2}. \]

**Horizon states**

We factor out the hair degrees of freedom to obtain the horizon states, this is given by

\[ Z_{\text{hor:CHL}}^{4d} = \frac{1}{\tilde{\Phi}_k(\rho, \sigma, \nu)} \prod_{l=1}^{\infty} (1 - e^{2\pi i(l\rho)}) \sum_s e^{-2\pi isl/N} e^{(0, s) (0)} \]

(2.33)

It is useful to use the product form of \( \tilde{\Phi}_k \) given in (2.3) to rewrite the partition function of the horizon states as follows

\[ Z_{\text{hor:CHL}}^{4d} = -e^{-2\pi i(\rho + \sigma/N + \nu)} \prod_{r=0}^{N-1} \prod_{k' \in \mathbb{Z}} (1 - e^{2\pi i(k' \rho + j \nu)}) \sum_s e^{-2\pi isl/N} e^{(r, s) (4k' \nu - j)} \]

\[ \times \prod_{l=1}^{\infty} (1 - e^{2\pi i(Nl\rho + \nu)})^{-2} \prod_{l=1}^{\infty} (1 - e^{2\pi i(Nl\rho - \nu)})^{-2} (1 - e^{-2\pi iv})^{-2} \]

(2.34)

\[ = -e^{-2\pi i(\rho + \sigma/N)} \prod_{r=0}^{N-1} \prod_{k' \in \mathbb{Z}} (1 - e^{2\pi i(k' \rho + j \nu)}) \sum_s e^{-2\pi isl/N} e^{(r, s) (4k' \nu - j)} \]

\[ \times \prod_{l=1}^{\infty} (1 - e^{2\pi i(Nl\rho + \nu)})^{-2} \prod_{l=1}^{\infty} (1 - e^{2\pi i(Nl\rho - \nu)})^{-2} (e^{\pi iv} - e^{-\pi iv})^{-2}. \]
This form of the horizon partition function is useful in the next section. The index for the horizon states is given by

\[
d_{\text{hor.CHL}} = (-1)^{Q-P} \int_C d\rho d\sigma dv \, e^{-\pi i (N_\rho Q^2 + \sigma P^2 / N + 2\pi Q \cdot P)} \frac{1}{\Phi_k(\rho, \sigma, v)} \times \\
\prod_{l=1}^{\infty} (1 - e^{2\pi i l(\rho)}) \sum_s e^{-2\pi i s l / N} c^{(0, s)}(0).
\]  \tag{2.35}

Non-geometric orbifolds: 11A, 14A, 15A, 23A

For completeness we note that we can extend the counting of hair modes to \( g' \) orbifolds of \( K3 \) where \( g' \) corresponds all the remaining conjugacy classes of \( M_{23} \). The CHL orbifolds also form a part of these, however the ones discussed in this section are non-geometric. The hair modes in these cases can also be written as:

\[
Z_{\text{hair}, g'}^{4d} = \prod_{l \neq 0} (1 - e^{2\pi i l(\rho)}) \sum e^{-2\pi i s l / N} c^{(0, s)}(0) \] \tag{2.36}

To be explicit, we list list of values of \( \sum_{s=0}^{N-1} e^{-2\pi i s l / N} c^{(0, s)}(-b^2) \) for different \( N \) for \( N = 11, 14, 15, 23 \) in table 3.
\[
\sum_{s=0}^{N-1} e^{-2\pi isl/N} e^{(0,s)} (-b^2)
\]

| \(N\) | \(l\) | \(-b^2\) | \(\sum_{s=0}^{N-1} e^{-2\pi isl/N} e^{(0,s)} (-b^2)\) |
|------|------|------|----------------------------------|
| 11   | 11| 0   | 0                                |
|      | 11| 0   | 2                                |
|      | 11| 0   | 2                                |
| 14   | 14| 0   | 0                                |
|      | 14| 0   | 2                                |
|      | 2| 0   | 2                                |
|      | 7| 0   | 2                                |
|      | 2| 0   | 1                                |
| 15   | 15| 0   | 0                                |
|      | 15| 0   | 2                                |
|      | 3| 0   | 2                                |
|      | 5| 0   | 1                                |
|      | 3| 0   | 1                                |
| 23   | 23| 0   | -2                               |
|      | 23| 0   | 2                                |
|      | 23| 0   | 1                                |

**Table 3**: Values of \(\sum_{s=0}^{N-1} e^{-2\pi isl/N} e^{(0,s)} (-b^2)\) for non-geometric orbifolds of \(K3\) where \(g' \in [M_{23}]\). \(\sum_{s=0}^{N-1} e^{-2\pi isl/N} e^{(0,s)} (-1) = 0\) if \(N \nmid l\) for any of these cases.
Using the results from table 3 we write:

\[
Z_{\text{hor:11A}}^{4d} = \prod_{l=1}^{\infty} (1 - e^{2\pi i(11l\rho)})^4 (1 - e^{2\pi i(l\rho)})^{-2} (1 - e^{2\pi i(11l\rho)})^{-2} \quad (2.37)
\]

\[
Z_{\text{hor:14A}}^{4d} = \prod_{l=1}^{\infty} (1 - e^{2\pi i(14l\rho)})^4 (1 - e^{2\pi i(14l\rho)})^{-1}
\]

\[
(1 - e^{2\pi i(2l\rho)})^{-1} (1 - e^{2\pi i(7l\rho)})^{-1} (1 - e^{2\pi i(l\rho)})^{-1} \quad (2.38)
\]

\[
Z_{\text{hor:15A}}^{4d} = \prod_{l=1}^{\infty} (1 - e^{2\pi i(15l\rho)})^4 (1 - e^{2\pi i(15l\rho)})^{-1}
\]

\[
(1 - e^{2\pi i(3l\rho)})^{-1} (1 - e^{2\pi i(5l\rho)})^{-1} (1 - e^{2\pi i(l\rho)})^{-1} \quad (2.39)
\]

\[
Z_{\text{hor:23A}}^{4d} = \prod_{l=1}^{\infty} (1 - e^{2\pi i(23l\rho)})^4 (1 - e^{2\pi i(23l\rho)})^{-1} (1 - e^{2\pi i(l\rho)})^{-1} \quad (2.40)
\]

The partition function of the horizon states in these models are given by the same expressions as in (2.34) with \( N \) replaced by the order of the conjugacy class and the coefficients \( c^{(r,s)} \) read out from the respective twisted elliptic genus. Let us conclude by writing the general formula for the horizon states as

\[
Z_{\text{hor: g'}}^{4d} = -e^{-2\pi i(\rho+\sigma/N)} \prod_{r=0}^{N-1} \prod_{k' \in \mathbb{Z} + r/N} \prod_{l \in \mathbb{Z}} \prod_{j \geq 0} (1 - e^{2\pi i(k'\sigma + l\rho + j\nu)}) - \sum_{s} e^{-2\pi is/N} c^{(r,s)}(4k'-j')
\]

\[
\times \prod_{l=1}^{\infty} (1 - e^{2\pi i(Nl\rho+v)})^{-2} \prod_{l=1}^{\infty} (1 - e^{2\pi i(Nl\rho-v)})^{-2} (e^{\pi iv} - e^{-\pi iv})^{-2} \quad (2.41)
\]

2.3 Toroidal orbifolds

In this section we construct the hair for \( \mathcal{N} = 4 \) theories obtained by freely acting \( \mathbb{Z}_2, \mathbb{Z}_3 \) involutions on \( T^6 \) [30]. Let us first briefly recall how these are constructed. In the type IIB frame, they are obtained by 4 of the co-ordinates together with a half shift along one of the \( S^1 \). The type IIA description of the theory is that of a freely acting orbifold with the action of \((-1)^{F_L}\) and a 1/2 shift along one of the circles of \( T^6 \). A similar compactification of order 3 given by a 2\( \pi /3 \) rotation along one 2D plane of \( T^4 \) and a \(-2\pi /3 \) rotation along another plus an 1/3 shift along one of the circles of \( T^2 \) was also discussed in [18]. We call these models \( \mathbb{Z}_2 \) and \( \mathbb{Z}_3 \) toroidal orbifolds.

One key property of these models to keep in mind which will be important is that the breaking of the 32 supersymmetries of type IIB to 16 is determined by the size of \( S^1 \). This was not the case for the orbifolds of \( K3 \times T^2 \), where supersymmetry was broken by the \( K3 \). For the toroidal models if the size of \( S^1 \) is infinite, the theory

\footnote{For details of these descriptions and the dyon configuration refer [18].}
effectively behaves as though the theory has 32 supersymmetries. We will use this fact to propose certain fermionic zero modes which were present for the CHL models will become singular at the horizon.

The dyon partition function for the toroidal models is given by \[31\]:

\[
\tilde{\Phi}_k(\rho, \sigma, v) = e^{2\pi i (\rho+v)} \prod_{b=0,1}^{N-1} \prod_{r=0}^{N-1} \prod_{k' \in \mathbb{Z}^+, l \in \mathbb{Z}} \left(1 - e^{2\pi i (k'\sigma + l\rho + jv)}\right)^{\sum_{s=0}^{N-1} e^{2\pi isl/N}} e^{-\frac{1}{4}(4k'l-j^2)}.
\]

The coefficients \(c^{(r,s)}\) are read out from the following twisted elliptic genus for \(\mathbb{Z}_2\) orbifold:

\[
\begin{align*}
F^{(0,0)} &= 0, \\
F^{(0,1)} &= \frac{8}{3} A(\tau, z) - \frac{4}{3} B(\tau, z) E_2(\tau), \\
F^{(1,0)} &= \frac{8}{3} A(\tau, z) + \frac{2}{3} B(\tau, z) E_2(\tau), \\
F^{(1,1)} &= \frac{8}{3} A(\tau, z) + \frac{2}{3} B(\tau, z) E_2\left(\frac{\tau+1}{2}\right).
\end{align*}
\]

The corresponding Siegel form of weight \(k = 2\) can be written as

\[
\tilde{\Phi}_2(\rho, \sigma, v) = \frac{\tilde{\Phi}_6^2(\rho, \sigma, v)}{\tilde{\Phi}_{10}(\rho, \sigma, v)},
\]

where \(\tilde{\Phi}_6\) is the weight 6 Siegel modular form associated with the order 2 CHL orbifold. For the \(\mathbb{Z}_3\) toroidal case the twisted elliptic genus is given by

\[
\begin{align*}
F^{(0,0)} &= 0, \\
F^{(0,s)} &= A(\tau, z) - \frac{3}{4} B(\tau, z) E_3(\tau), \\
F^{(r,rk)} &= A(\tau, z) + \frac{1}{4} B(\tau, z) E_3\left(\frac{\tau+k}{3}\right), \quad r = 1, 2.
\end{align*}
\]

The Siegel modular form associated with the \(\mathbb{Z}_3\) toroidal orbifold has weight \(k = 1\) and is given by

\[
\tilde{\Phi}_1(\rho, \sigma, v) = \frac{\tilde{\Phi}_4^{3/2}(\rho, \sigma, v)}{\tilde{\Phi}_{10}^{1/2}(\rho, \sigma, v)},
\]

where \(\tilde{\Phi}_4\) is the weight 4 Siegel modular form associated with the order 3 CHL orbifold.

Let us construct the hair modes and horizon states for these models.
$T^6/Z_2$ model

- Just as in the case of the CHL models, we have 4 left moving fermions. This gives rise to

$$Z^{4d:f}_{\text{hair: } T^6/Z_2} = \prod_{l=1}^{\infty} (1 - e^{2\pi i (2l-1)\rho})^4.$$ (2.47)

- The deformations corresponding to the motion of the effective string in the 3 transverse directions of $R^3 \times \tilde{S}^1$ of the Taub-Nut space together with the fluctuations of the anti-self dual forms can be determined easily by examining the partition function of the fundamental string in this theory and removing the zero point energy. This partition function was determined in [18], using this result we obtain

$$Z^{4d b}_{\text{hair: } T^6/Z_2} = \prod_{l=1}^{\infty} \left[ (1 - e^{2\pi i (2l-1)\rho})^8 (1 - e^{4\pi i \rho})^{-8} \right].$$ (2.48)

- Contribution of the zero modes: The quantum mechanics of the bosonic zero modes describing the motion of the D1-D5 system in the Taub-Nut result in the following partition function [19]

$$Z^{4d: \text{zeromodes}}_{\text{hair: } T^6/Z_2} = -e^{2\pi i v}(1 - e^{2\pi iv})^{-2}.$$ (2.49)

For orbifolds of $K3$, this contribution from the bosonic zero modes was cancelled by the zero modes of 4 fermions from the right moving sector carrying angular momentum $J = \pm \frac{1}{2}$ whose partition function is given by $-(e^{\pi iv} - e^{-\pi iv})^2$ [5]. However for the toroidal model, we propose that these zero modes do not form part of the hair. They are either singular at the horizon or they are not localized outside the horizon. This is possible, the fact that we are in a theory with 16 supersymmetries is tied to the the radius of $S^1$. Verification of this proposal would involve a detailed study of the zero mode wave functions which we leave for the future. However we will perform consistency checks of this proposal in section 4. by evaluating the index of the horizon states.

Thus the hair modes of the $Z_2$ toroidal model is given by

$$Z^{4d}_{\text{hair: } T^6/Z_2} = -(e^{\pi iv} - e^{-\pi iv})^{-2} \prod_{l=1}^{\infty} \left[ (1 - e^{2\pi i (2l-1)\rho})^8 (1 - e^{4\pi i \rho})^{-8} \right].$$ (2.50)

The partition function of the horizon states of this model are given by

$$Z^{4d}_{\text{hor: } T^6/Z_2} = \frac{1}{\Phi_2(\rho, \sigma, v) Z^{4d}_{\text{hair: } T^6/Z_2}}.$$ (2.51)

---

6One can also obtain this by counting the number of invariant 2-forms and the forms which pick up a phase as done in [20].
where \( \tilde{\Phi}_2(\rho, \sigma, v) \) is given in (2.44) or (2.42).

The toroidal model has another special feature, they admit Wilson lines along \( T^4 \) [31], their partition function is given by

\[
Z_{\text{Wilson; } T^4/Z_2} = \prod_{l=1}^{\infty} \left[ (1 - e^{2\pi i(2l-1)\rho + 2\pi iv})^2 (1 - e^{2\pi i(2l-1)\rho - 2\pi iv})^2 (1 - e^{2\pi i(2l-1)\rho})^{-4} \right]
\]

(2.52)

It is possible that the Wilson lines might also be part of the hair modes. In section 4 we will see that including the Wilson lines as hair modes instead of the bosonic zero modes given in (2.49) does not preserve the positivity of the index of the horizon states.

\( T^6/Z_3 \) model

Performing the same analysis as done for the \( Z_2 \) orbifold we obtain the following partition function for the hair modes.

\[
Z_{4d/\text{hair; } T^6/Z_3} = -(e^{\pi iv} - e^{-\pi iv})^{-2} \prod_{l=1}^{\infty} \left[ (1 - e^{2\pi i(3l-1)\rho})^3 (1 - e^{2\pi i(3l-2)\rho})^3 (1 - e^{2\pi i(3l-1)\rho})^{-2} \right].
\]

(2.53)

The horizon states is given by

\[
Z_{4d/\text{hor; } T^6/Z_3} = \frac{1}{\tilde{\Phi}_1(\rho, \sigma, v) Z_{4d/\text{hair; } T^6/Z_3}^d},
\]

(2.54)

where \( \tilde{\Phi}_1 \) is given by (2.46) or (2.42). For reference we also provide the partition function of the Wilson lines in this model

\[
Z_{\text{Wilson; } T^4/Z_3} = \prod_{l=1}^{\infty} \left[ (1 - e^{2\pi i((3l-1)\rho + v)}) (1 - e^{2\pi i((3l-2)\rho + v)}) (1 - e^{2\pi i((3l-1)\rho - v)}) (1 - e^{2\pi i((3l-2)\rho - v)}) \right]
\]

(2.55)

From the expression for the Wilson lines and the infinite product representation given for \( \tilde{\Phi}_k \) given in (2.42) we obtain the following useful expression for the partition function for the horizon modes for both the toroidal orbifolds.

\[
Z_{4d/\text{hor; } T^6/Z_N} = e^{-2\pi iv} \prod_{r=0}^{N-1} \prod_{k \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} e^{-2\pi isl/N} e^{(r-s)(4kl-j^2)}
\]

(2.56)

\[
\times \prod_{l=1}^{\infty} \left[ (1 - e^{2\pi i(Nl+\rho+v)})^{-2} (1 - e^{2\pi i(Nl-\rho-v)})^{-2} \right] (e^{\pi iv} - e^{-\pi iv})^2 \times Z_{\text{Wilson; } T^4/Z_N}.
\]
3 Horizon states for the BMPV black hole

We now examine the BMPV black hole in 5 dimensions, that is the transverse space now does not have the Taub-Nut solution. The main reason for studying the problem in 5 dimensions is that the near horizon geometry of the BMPV black hole in 5 dimensions is same as the of the 1/4 BPS dyon in 4 dimensions. This implies that the partition function of the horizon states of these 2 systems should be identical. In this section we construct the partition function of the hair and the horizon states for the BMPV black hole in type IIB on $K^3 \times S^1 / g'$ as well as toroidal orbifolds of $T^5$. Here $g'$ corresponds to all the conjugacy classes of $M_{23}$.

3.1 Partition function of BMPV black holes

The partition function for these black holes in the canonical compactification $K^3 \times S^1$, was constructed in [5]. The same analysis can be extended to all the CHL models. The partition function receives contributions from the following sectors.

- The bound states of the D1-D5 system, this is given by the elliptic genus of the symmetric product of $K^3 / g'$. This contribution was evaluated in [19]. It is given by

$$\sum_{r=0}^{N-1} \prod_{k \in \mathbb{Z} + r/N, l \in \mathbb{Z}} \prod_{j \in \mathbb{Z}, k>0, l>0} \left(1 - e^{2\pi i (k\rho + lj + jv)}\right) \prod_{s \in \mathbb{Z}} e^{-2\pi i s l/N} c(r,s) (4kl - j^2).$$

(3.1)

- The centre of mass motion of the D1-D5 system in flat space. The degrees of freedom consist of 4 bosons and 4 fermions. 2 pairs of bosons carry the angular momentum $J = \pm 1$. [5].

$$Z_{\text{c.o.m}}^{5d} = \prod_{l=1}^{\infty} \left[ (1 - e^{2\pi i (Nl\rho + v)})^{-2} (1 - e^{-2\pi i (Nl\rho - v)})^{-2} (1 - e^{2\pi i Nl\rho})^{-4} \right].$$

(3.2)

Note that the only difference from the canonical model is that the unit of momentum on $S^1$ is $N$ due to the $1/N$ shift.

- 4 right chiral zero modes which contribute as $(-1)^{J} e^{2\pi J}$ which contribute in pairs with $J = \pm \frac{1}{2}$.

$$Z_{\text{zeromodes}}^{5d} = -(e^{\pi iv} - e^{-\pi iv})^2.$$

(3.3)

- A shift of $e^{-2\pi i \rho}$ to ensure to take into account of the difference in the electric charge measured at infinity and the horizon [5].
Combining all the sectors we obtain the following expression for the partition function for BMPV black hole for all orbifolds of $K^3 \times S^1$.

\[
Z_{5d}^{5d} = -e^{-2\pi i (\rho + \sigma/N)} \prod_{r=0}^{N-1} \prod_{k \in \mathbb{Z} + r/N, l \in \mathbb{Z}, j \in \mathbb{Z}, k > 0, l \geq 0} (1 - e^{2\pi i (k\rho + l\sigma + j\nu)}) - \sum_s e^{-2\pi isl/N} e^{(r,s)(4kl-j^2)}
\times \prod_{l=1}^{\infty} \left[ (1 - e^{2\pi i (Nl\rho + v)})^{-2} (1 - e^{2\pi i (Nl\rho - v)})^{-2} (e^{\pi iv} - e^{-\pi iv})^2 (1 - e^{2\pi i Nl\rho})^4 \right]
\]

Here the coefficients $c^{(r,s)}$ have to be read out from the twisted elliptic genus of $K^3$ by $g'$ corresponding to the conjugacy classes of $M_{23}$.

Using the counting of states for the dyon partition function done in [31] we can extend the analysis to the toroidal models. We present the analysis in some detail for the $T^6/\mathbb{Z}_2$ model. Here the contributions arise from the following:

- **The bound state of the D1-D5 system on the $T^4/\mathbb{Z}_2$ orbifold is given by**

\[
Z_{5d}^{5d} = \prod_{r=0}^{N-1} \prod_{k \in \mathbb{Z} + r/N, l \in \mathbb{Z}, j \in \mathbb{Z}, k > 0, l \geq 0} (1 - e^{2\pi i (k\rho + l\sigma + j\nu)}) - \sum_s e^{-2\pi isl/N} e^{(r,s)(4kl-j^2)}
\]

Here the coefficients $c^{(r,s)}$ are read out from the expansion of the functions given in (2.43).

- **The contribution of the Wilson lines on $T^4/\mathbb{Z}_2$ which is given by**

\[
Z_{\text{Wilson}:T^4/\mathbb{Z}_2}^{5d} = \prod_{l=1}^{\infty} \left[ (1 - e^{2\pi i (2l-1)\rho + 2\pi i v})^2 (1 - e^{2\pi i (2l-1)\rho - 2\pi i v})^2 (1 - e^{2\pi i (2l-1)\rho})^4 \right]
\]

(3.6)

- **The partition function corresponding to the centre of mass motion of the D1-D5 system in the transverse space**

\[
Z_{\text{c.o.m}}^{5d} = \prod_{l=1}^{\infty} \left[ (1 - e^{2\pi i (Nl\rho + v)})^{-2} (1 - e^{2\pi i (Nl\rho - v)})^{-2} (1 - e^{2\pi i Nl\rho})^4 \right], \quad N = 2.
\]

(3.7)

- **The contribution of the zero modes**

\[
Z_{\text{zeromodes}}^{5d} = -(e^{\pi iv} - e^{-\pi iv})^2.
\]

(3.8)
The shift in the electric charge accounted for by the factor $e^{-2\pi i \rho}$.

Combining all the contributions we obtain

$$Z_{T^5/Z_N}^{5d} = -e^{-2\pi i \rho} \prod_{k \in \mathbb{Z}+l/N, l \in \mathbb{Z}, j \in \mathbb{Z}, k \geq 0, l \geq 0} (1 - e^{2\pi i (k\sigma + l\rho + jv)}) - \sum_l e^{-2\pi i l/N} \prod_{r,s} (1 - e^{2\pi i \rho})^4 (e^{\pi iv} - e^{-\pi iv})^2 \times Z_{\text{Wilson}:T^4/Z_N}$$

$$N = 2.$$  \hfill (3.9)

The partition function of the BMPV black hole in the $T^5/Z_3$ is obtained model is given by the same expression as in (3.9) except that the coefficients $c^{(r,s)}$ must be read out from the functions given in (2.45) and $N \to 3$.

### 3.2 Orbifolds of $K3 \times S^1$

We now construct the hair modes in 5 dimensions for the $K3 \times S^1/g'$ where the quotient by $g'$ associated with any conjugacy classes of the Mathieu group $M_{23}$. The analysis proceeds identical to that done in [6], the only difference being that the unit of momentum on $S^1$ is $N$. Here we briefly state the contributions.

- The contribution of the 4 real left moving gravitino deformations of the BMPV black hole \textsuperscript{7}.

$$Z_{\text{hair: } g'}^{5d; f} = \prod_{l=1}^{\infty} (1 - e^{2\pi i lN\rho})^4.$$ \hfill (3.10)

- The contribution of the 8 real gravitino zero modes among the 12 modes due to broken supersymmetries which carry angular momentum $J = \pm \frac{1}{2}$.

$$Z_{\text{hair: } g'}^{5d; \text{zero modes}} = (e^{\pi iv} - e^{-\pi iv})^4.$$ \hfill (3.11)

Combining these contributions we obtain

$$Z_{\text{hair: } g'}^{5d} = (e^{\pi iv} - e^{-\pi iv})^4 \prod_{l=1}^{\infty} (1 - e^{2\pi i lN\rho})^4.$$ \hfill (3.12)

The partition function for the horizon states is given by

$$Z_{\text{hor: } g'}^{5d} = \frac{Z_{\text{hor: } g'}}{Z_{\text{hair: } g'}}.$$ \hfill (3.13)

Now comparing the horizon states of the 4d dyons from (2.41) and using (3.4) and (3.12) in (3.13) we can easily conclude

$$Z_{\text{hor: } g'}^{5d} = Z_{\text{hor: } g'}^{4d}.$$ \hfill (3.14)

\textsuperscript{7}The bosonic deformations were shown to be singular at the horizon in [6].
3.3 Toroidal models

For the toroidal models the contributions of the hair are as follows.

- The contribution of the 4 left moving gravitino modes which result in

\[
Z_{5d; f}^{5d; \text{hair}: T^5/Z_N} = \prod_{l=1}^{\infty} (1 - e^{2\pi i l N \rho})^4, \quad N = 2, 3.
\]

(3.15)

- The contribution of the zero modes. As we discussed earlier, supersymmetry in these models is tied to the radius of $S^1$. We propose that due to this, out of 8 gravitino zero modes arising from broken supersymmetries which has angular momentum $J = \pm \frac{1}{2}$, the wave functions of 4 of them either become singular at the horizon or they not localized outside the horizon. These 4 modes should not be counted as hair modes. Therefore the contribution of the zero modes in these models are given by

\[
Z_{5d; \text{zero modes}}^{5d; \text{hair}: T^5/Z_N} = -(e^{\pi iv} - e^{-\pi iv})^2.
\]

(3.16)

As we will see consistency checks for this proposal will be done in section (4).

Combining these contributions we obtain

\[
Z_{\text{hair: T}^5/Z_N}^{5d} = -(e^{\pi iv} - e^{-\pi iv})^2 \prod_{l=1}^{\infty} (1 - e^{2\pi i l N \rho})^4.
\]

(3.17)

The horizon partition function from the 5d perspective is given by

\[
Z_{\text{hor: T}^5/Z_N}^{5d} = \frac{Z_{\text{5d}}^{5d; \text{hair: T}^5/Z_N}}{Z_{\text{hor: T}^5/Z_N}^{5d}}.
\]

(3.18)

Comparing the 4d horizon partition function given in (2.56) and using (3.9) and (3.17) in (3.18) we see that

\[
Z_{\text{hor: T}^5/Z_N}^{5d} = Z_{\text{hor: T}^6/Z_N}^{4d}.
\]

(3.19)

4 The sign of the index for horizon states

In this section we will address the main goal of the paper. We observe that the index of horizon states is always positive.
4.1 Canonical example: $K3 \times T^2$

For the un-orbifolded model recall that the hair in 4d is given by

$$Z_{4d}^{4d} = \prod_{l=1}^{\infty} \frac{1}{1 - e^{2\pi i l \rho}}.$$  \hspace{1cm} (4.1)

The partition function of the horizon states is obtained by

$$Z_{\text{hor:1A}}^{} = \frac{1}{\Phi_{10}(\rho, \sigma, v) Z_{4d}^{4d}} = \prod_{l=1}^{\infty} \frac{1 - e^{2\pi i l \rho}}{\Phi_{10}(\rho, \sigma, v)}.$$  \hspace{1cm} (4.2)

It was observed in [7] that the index $-B_6$ or the Fourier coefficients of $1/\Phi_{10}$ extracted using the contour in (2.2) subject to the kinematic restrictions

$$Q \cdot P \geq 0, \quad Q \cdot P \leq Q^2, \quad Q \cdot P \leq P^2, \quad Q^2, P^2, (Q^2 P^2 - (Q \cdot P)^2) > 0 \quad (4.3)$$

were positive. The contour together with the above kinematic constraints ensures that the index counts single centred dyons. Further more [10] proved that the index of all single centered dyons with $P^2 = 2, 4$ is positive. These works assumed that there existed a frame in which the fermionic zero modes associated with broken supersymmetries were the only hair. We have seen that the type IIB frame the hair degrees of freedom is given by (4.1). Now naively it seems from the expression for the horizon states in (4.2) there are negative terms introduced due to the factor in the numerator and the observation of positivity seen in ([7]) and [10] might be violated once the hair in the type IIB frame is factored out. However we will show by adapting the proof of [10] that single centred dyons with $P^2 = 2$ do have positive index. For other values of charges we evaluate the index numerically, our results are presented in table 4. We observe that for single centered dyons the index is indeed positive.

| $(Q^2, P^2)$ | \(Q \cdot P\) | 0 | 1 | 2 | 3 | 4 |
|-------------|----------------|---|---|---|---|---|
| (2, 2)      | 28944          | 13863 | 1608 | 327 | 0 |
| (2, 4)      | 761312         | 406296 | 72424 | 6036 | −648 |
| (2, 6)      | 12324920       | 6095541 | 1423152 | 96619 | −13680 |
| (2, 8)      | 148800072      | 88006584 | 19366320 | 1152216 | −164244 |
| (4, 2)      | 272832         | 154236 | 28944 | 1836 | −648 |
| (4, 4)      | 12980224       | 8595680 | 2665376 | 406296 | 25760 |
| (4, 6)      | 333276712      | 235492308 | 85781820 | 16141380 | 1423152 |
| (6, 6)      | 6227822652     | 4771720755 | 2158667028 | 572268361 | 85781820 |

Table 4: Index of horizon states for $K3 \times T^2$, note that negative numbers have zero or negative values for $Q^2 P^2 - (Q \cdot P)^2$. 


Proof of positivity at $P^2 = 2$

We can do a Fourier expansion of $\frac{1}{\Phi_{10}(\tau,\sigma,z)}$ in terms of Jacobi forms.

\[
\frac{1}{\Phi_{10}(q,p,y)} = \sum_{m=-1}^{\infty} \psi_m(\tau,z) q^m, \quad q = e^{2\pi i \tau}, \quad p = e^{2\pi i \sigma}, \quad y = e^{2\pi iz}.
\] (4.4)

$\psi_m(\tau,z)\eta^{24}(\tau)$ is a weak Jacobi form of weight 2 and index $m$. In [32] it was shown that $\psi_m(\tau,z)$ admits the following decomposition

\[
\psi_m(\tau,z) = \psi^p_m(\tau,z) + \psi^F_m(\tau,z),
\] (4.5)

where, $\psi^F_m(\tau,z)$ has no poles in $z$. The polar part is given by an Appell-Lerch sum:

\[
\psi^p_m(\tau,z) = \frac{p_{24}(m+1)}{\eta^{24}(\tau)} A_{2,m}(\tau,z),
\] (4.6)

\[
A_{2,m}(\tau,z) = \sum_{s \in \mathbb{Z}} q^{ms^2 + sy^2ms + 1}(1 - q^s y^2)^2.
\]

At $P^2 = 2$ we have $m = 1$ and we can write

\[
\psi^F_1(\tau,z) = -\frac{3}{\Delta}(E_4 B(\tau,z) + 216 \mathcal{H}(\tau,z)).
\] (4.7)

We need to show that $\psi^h = -\frac{3}{q^{1/2}(1-q^2)^3}(E_4 B(\tau,z) + 216 \mathcal{H}(\tau,z))$ has the positivity property. Here $\mathcal{H}$ is the simplest Jacobi mock modular form defined by the Hurwitz-Kronecker class numbers

\[
\mathcal{H}(\tau,z) = \sum_{n=0}^{\infty} H(4n - j^2)q^n y^j.
\] (4.8)

The coefficients $H(n)$ are defined by

\[
H(n) = 0 \quad \text{for } n < 0,
\] (4.9)

\[
\sum_{n \in \mathbb{Z}} H(n)q^n = -\frac{1}{12} + \frac{1}{3} q^3 + \frac{1}{2} q^4 + q^7 + q^8 + q^{11} + \cdots
\] (4.10)

\[
\mathcal{H}(\tau,z) = \theta_3(2\tau,2z) h_0(\tau) + \theta_2(2\tau,2z) h_1(\tau).
\] (4.11)

We can write the weak Jacobi form $B(\tau,z)$ given in (2.6) as:

\[
B(\tau,z) = \frac{\theta_3^4(\tau,z)}{\eta^6} = \frac{1}{\eta^6} (\theta_3(2\tau)\theta_3(2\tau,2z) - \theta_3(2\tau)\theta_2(2\tau,2z))
\] (4.12)

\[\text{We use the variable } \tau \text{ instead of } \rho \text{ and } z \text{ in place of } v \text{ to keep consistency with previous work [13]}\]
where, \( \theta_2(\tau, z) = \sum_{n \in \mathbb{Z}} q^{\frac{(n+1/2)^2}{2}} y^{n+1/2} \) and \( \theta_3(\tau, z) = \sum_{n \in \mathbb{Z}} q^{n^2/2} y^n \) and \( y = e^{2\pi i z} \). So we see that even and odd powers of \( y \) are separated in \( \psi^F \) by the two theta functions. With this we can write \( \psi^F_1 \) and \( \psi^h_1 \) as follows:

\[
\psi^F_1 = \frac{3}{\Delta_4} \left( \theta_2(2\tau, 2z) \left( \frac{\theta_3(2\tau)}{\eta^6} E_4 - 216h_1(\tau) \right) - \theta_3(2\tau, 2z) \left( \frac{\theta_2(2\tau)}{\eta^6} E_4 + 216h_0(\tau) \right) \right)
\]

\[
\psi^h_1 = \frac{3}{\Delta_4} \left( \theta_2(2\tau, 2z) \left( \frac{\theta_3(2\tau)}{\eta^6} E_4 - 216h_1(\tau) \right) - \theta_3(2\tau, 2z) \left( \frac{\theta_2(2\tau)}{\eta^6} E_4 + 216h_0(\tau) \right) \right),
\]

\[(4.13)\]

where \( \Delta_4 = q \prod_{n=1}^{\infty} (1 - q^n)^4 \). We know the following results,

1. The Fourier coefficients in \( h_0(\tau) \) and \( h_1(\tau) \) are positive except for \( q^0 \) in \( h_0(\tau) \) \[10\].

2. All Fourier coefficients in the \( q \) expansion of \( \frac{\theta_2(2\tau)}{\eta^n} \) or \( \frac{\theta_3(2\tau)}{\eta^n} \) are positive.

3. \( E_4 = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \), where \( \sigma_3(n) \) is given by, \( \sum_{d \mid n} d^3 \).

Let us observe the expression: \( \left( \frac{\theta_2(2\tau)}{\eta^6} E_4 + 216h_0(\tau) \right) \). The only negative Fourier coefficient appears at \( q^0 \). We can prove the following lemma:

**Lemma 1.** For a function \( f(q) = -1 + \sum_{n=1}^{\infty} a(n) q^n \) having all positive \( a(n) \), the function \( \frac{f(q)}{\prod_{n=1}^{\infty} (1 - q^n)^k} \) has positive coefficients as long as \( a(1) > k \) and \( a(n + 1) > k \) for all \( n \in \mathbb{N} \).

**Proof.** We prove this for \( \frac{1}{(1-q)^k} \) and then the rest can be similarly proved by using \( q \to q^r \) and taking \( f_{r+1}(q) = \frac{f_r(q)}{(1-q^{r+1})^k} \). For \( f_2 \) the coefficient of \( q^1 \) is, \( a(1) - k > 0 \) and the coefficient of \( q^N \) for \( N > 1 \) is given by,

\[ - \binom{N+k-1}{N} + \binom{N+k-2}{N-1} a(1) + \binom{N+k-3}{N-2} a(2) \cdots > k. \]

Q.E.D

We can write

\[
\frac{1}{16} \left( \frac{\theta_2(2\tau)}{\eta^6} E_4 + 216h_0(\tau) \right) = -1 + \sum_{n=1}^{\infty} a(n) q^n.
\]

\[(4.14)\]

Here \( a(1) > 15\sigma(1) > 4 \). Hence the removal of hair degrees of freedom ensures positivity of \( -B_6 \) for the sector \( Q \cdot P = \text{even} \) when \( Q^2 \geq 0 \).

In the series associated with \( \theta_2(2\tau, 2z) \) in equation \( (4.13) \) the Fourier coefficient of \( q^{n-1/4} \) is bounded from below by,

\[
10\sigma_3(n) - 9H(4n - 1).
\]

Its positivity is ensured starting from \( n = 2 \) using the following bounds:
1. $\sigma_3(n) \geq n^3$,
2. $H(n) < n$ [10].

For $n = 1$ the positivity still holds as $H(3) = 1/3$. So the complete $q$ series expansion of $\left( \frac{\theta_3(2\tau)}{\eta^6} E_4 - 216 h_1(\tau) \right)$ contains no negative Fourier coefficient. This could also be seen from the Fourier expansion of $\left( \frac{\theta_3(2\tau)}{\eta^6} E_4 - 216 h_1(\tau) \right)$,

$$\left( \frac{\theta_3(2\tau)}{\eta^6} E_4 - 216 h_1(\tau) \right) = q^{-1/4} (1 + 176q + \cdots). \quad (4.15)$$

This ensures the positivity of $-B_6$ for $Q \cdot P = \text{odd}$ and hence for $\psi_1^h$ as expected for $P^2 = 2$.

### 4.2 Orbifolds of $K3 \times T^2$

For the $2A$ orbifold we extract the index of single centred dyons by using the contour in (2.2) together with the following kinematic constraints on the charges [7].

$$Q^2 > 0, \; P^2 > 0, \; Q \cdot P \geq 0, \; P^2 Q^2 - (Q \cdot P)^2 > 0, \quad (4.16)$$

$$2Q^2 \geq Q \cdot P, \; P^2 \geq Q \cdot P, \; P^2 + 2Q^2 \geq 3Q \cdot P.$$

The index of the horizon states for the $2A$ orbifold is given in table 5.

| $(Q^2, \; P^2)$ \ \ $\backslash$ $Q \cdot P$ | 0   | 1   | 2   | 3   | 4   |
|----------------|-----|-----|-----|-----|-----|
| (1, 2)         | 580 | 176 | -2  | 0   | 0   |
| (1, 4)         | 5504| 1856| 32  | 0   | 0   |
| (1, 6)         | 41476| 16200| 996| 52  | 0   |
| (1, 10)        | 1293256| 589200| 63556| 2752| -104|
| (2, 2)         | 1312| 576 | 48  | 0   | 0   |
| (2, 4)         | 16896| 8640| 1280| 64  | 0   |
| (3, 2)         | 9708| 4696| 580 | 52  | 0   |

**Table 5**: Index of horizon states for the $2A$ orbifold of $K3$

The kinematic constraints on the charges for the $3B$ orbifold so that the dyons are single centered are given by

$$\{Q^2, \; P^2, P^2 Q^2 - (Q \cdot P)^2 \} > 0 \; Q \cdot P \geq 0, 3Q^2 \geq Q \cdot P, \; P^2 \geq Q \cdot P; \quad (4.17)$$

$$2P^2 + 3Q^2 \geq 5Q \cdot P, \; P^2 + 6Q^2 \geq 5Q \cdot P, \; 2P^2 + 6Q^2 \geq 7Q \cdot P.$$

The index for the horizon states is then obtained using contour (2.2) and is listed in table 6.
For an orbifold of order \( N > 3 \) there are infinite set of constraints for the charges to ensure that the index corresponds to single centered dyons [7]. However we see as long as the norms of electric and magnetic charges are positive and \( Q \cdot P \geq 0 \) together with \( Q^2P^2 - (Q \cdot P)^2 > 0 \), the index \(-B_6\) remains positive for the orbifolds of \( K3 \) (see the tables 7-15). These orbifolds maybe geometric like that of CHL or even non-geometric where \( g' \in [M_{23}] \).

| \((Q^2, P^2)\) | \(Q \cdot P\) | 0 | 1 | 2 | 3 | 4 |
|----------------|-------------|---|---|---|---|---|
| \((2/3, 2)\)   | 216         | 27| 0 | 0 | 0 | 0 |
| \((2/3, 4)\)   | 1548        | 342| 0 | 0 | 0 | 0 |
| \((2/3, 6)\)   | 8532        | 2430| 54| 0 | 0 | 0 |
| \((4/3, 2)\)   | 540         | 216| 0 | 0 | 0 | 0 |
| \((4/3, 4)\)   | 5820        | 2698| 136| 0 | 0 | 0 |
| \((2, 2)\)     | 1728        | 621| 54| 0 | 0 | 0 |
| \((2, 6)\)     | 204264      | 117837| 23400| 765| 0 | 0 |
| \((2, 8)\)     | 1440288     | 896670| 216540| 13932| 54 | 0 |

**Table 6:** Index of horizon states for the 3A orbifold of \( K3 \)

| \((Q^2, P^2)\) | \(Q \cdot P\) | 0 | 1 | 2 | 3 | 4 |
|----------------|-------------|---|---|---|---|---|
| \((1/2, 2)\)   | 64          | 8 | 0 | 0 | 0 | 0 |
| \((1/2, 4)\)   | 288         | 80| 0 | 0 | 0 | 0 |
| \((1/2, 6)\)   | 1088        | 464| 24| 0 | 0 | 0 |
| \((1, 2)\)     | 96          | 48| 0 | 0 | 0 | 0 |
| \((1, 4)\)     | 464         | 480| 16| 0 | 0 | 0 |
| \((3/2, 4)\)   | 640         | 1680| 160| 0 | 0 | 0 |
| \((3/2, 6)\)   | 3958        | 11448| 2026| 38 | 0 | 0 |
| \((3/2, 22)\)  | 232188670   | 421276388| 228036842| 43979890| 2695862|

**Table 7:** Index of horizon states for the 4B orbifold of \( K3 \)

| \((Q^2, P^2)\) | \(Q \cdot P\) | 0 | 1 | 2 | 3 | 4 |
|----------------|-------------|---|---|---|---|---|
| \((2/5, 2)\)   | 44          | 1 | 0 | 0 | 0 | 0 |
| \((2/5, 4)\)   | 220         | 20| 0 | 0 | 0 | 0 |
| \((2/5, 6)\)   | 880         | 125| 0 | 0 | 0 | 0 |
| \((4/5, 2)\)   | 88          | 16| 0 | 0 | 0 | 0 |
| \((4/5, 4)\)   | 560         | 160| 0 | 0 | 0 | 0 |
| \((6/5, 6)\)   | 8360        | 3755| 310| 0 | 0 | 0 |
| \((6/5, 8)\)   | 37394       | 18720| 2202| 16 | 0 | 0 |

**Table 8:** Index of horizon states for the 5A orbifold of \( K3 \)
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\((Q^2, P^2)\) & \(Q \cdot P\) & 0 & 1 & 2 & 3 & 4 \\
\hline
\((1/3, 2)\) & 24 & 1 & 0 & 0 & 0 \\
\((1/3, 4)\) & 92 & 12 & 0 & 0 & 0 \\
\((1/3, 6)\) & 318 & 49 & 0 & 0 & 0 \\
\((2/3, 2)\) & 44 & 10 & 0 & 0 & 0 \\
\((2/3, 4)\) & 236 & 68 & 0 & 0 & 0 \\
\((1, 6)\) & 2702 & 1201 & 100 & 0 & 0 \\
\((1/3, 34)\) & 15836220 & 6614053 & 409414 & 1789 & -14 \\
\hline
\end{tabular}
\caption{Index of horizon states 6A orbifold of \(K3\)}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\((Q^2, P^2)\) & \(Q \cdot P\) & 0 & 1 & 2 & 3 & 4 \\
\hline
\((2/7, 2)\) & 18 & 0 & 0 & 0 & 0 \\
\((2/7, 4)\) & 72 & 3 & 0 & 0 & 0 \\
\((2/7, 6)\) & 240 & 18 & 0 & 0 & 0 \\
\((4/7, 2)\) & 30 & 3 & 0 & 0 & 0 \\
\((4/7, 4)\) & 150 & 31 & 0 & 0 & 0 \\
\((6/7, 8)\) & 5580 & 2304 & 0 & 0 & 0 \\
\((2/7, 40)\) & 46940778 & 18696804 & 1139238 & 4689 & -18 \\
\hline
\end{tabular}
\caption{Index of horizon states for the 7A orbifold of \(K3\)}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\((Q^2, P^2)\) & \(Q \cdot P\) & 0 & 1 & 2 & 3 & 4 \\
\hline
\((1/4, 2)\) & 12 & 0 & 0 & 0 & 0 \\
\((1/4, 4)\) & 40 & 2 & 0 & 0 & 0 \\
\((1/4, 6)\) & 124 & 10 & 0 & 0 & 0 \\
\((1/2, 2)\) & 20 & 2 & 0 & 0 & 0 \\
\((1/2, 4)\) & 88 & 16 & 0 & 0 & 0 \\
\((3/4, 4)\) & 176 & 52 & 0 & 0 & 0 \\
\((3/4, 6)\) & 708 & 248 & 6 & 0 & 0 \\
\((1/4, 46)\) & 37469836 & 15088039 & 845410 & 2491 & -10 \\
\hline
\end{tabular}
\caption{Index of horizon states for the 8A orbifold of \(K3\)}
\end{table}

It is interesting to see that the index for horizon states even in non-geometric orbifolds of \(K3\) retains positivity of the index in the domain \(NQ^2 \geq Q \cdot P, P^2 \geq Q \cdot P, Q^2P^2 - (Q \cdot P)^2 > 0\).
| $(Q^2, P^2)$ \ $Q \cdot P$ | 0  | 1  | 2  | 3  | 4  |
|----------------------------|----|----|----|----|----|
| $(2/11, 2)$                 | 6  | 0  | 0  | 0  | 0  |
| $(2/11, 4)$                 | 18 | 0  | 0  | 0  | 0  |
| $(2/11, 6)$                 | 50 | 1  | 0  | 0  | 0  |
| $(4/11, 2)$                 | 8  | 0  | 0  | 0  | 0  |
| $(4/11, 4)$                 | 32 | 4  | 0  | 0  | 0  |
| $(6/11, 8)$                 | 592| 172| 2  | 0  | 0  |
| $(6/11, 10)$                | 1568| 527| 16 | 0  | 0  |

**Table 12:** Index of horizon states for the 11A orbifold of $K3$

| $(Q^2, P^2)$ \ $Q \cdot P$ | 0  | 1  | 2  | 3  | 4  |
|----------------------------|----|----|----|----|----|
| $(1/7, 2)$                 | 3  | 0  | 0  | 0  | 0  |
| $(1/7, 4)$                 | 7  | 0  | 0  | 0  | 0  |
| $(1/7, 6)$                 | 18 | 0  | 0  | 0  | 0  |
| $(2/7, 2)$                 | 4  | 0  | 0  | 0  | 0  |
| $(2/7, 4)$                 | 14 | 1  | 0  | 0  | 0  |
| $(3/7, 8)$                 | 163| 45 | 0  | 0  | 0  |
| $(3/7, 10)$                | 390| 116| 2  | 0  | 0  |
| $(4/7, 10)$                | 774| 329| 14 | 0  | 0  |

**Table 13:** Index of horizon states for the 14A orbifold of $K3$

| $(Q^2, P^2)$ \ $Q \cdot P$ | 0  | 1  | 2  | 3  | 4  |
|----------------------------|----|----|----|----|----|
| $(2/15, 2)$                | 3  | 0  | 0  | 0  | 0  |
| $(2/15, 4)$                | 6  | 0  | 0  | 0  | 0  |
| $(2/15, 6)$                | 15 | 0  | 0  | 0  | 0  |
| $(4/15, 2)$                | 3  | 1  | 0  | 0  | 0  |
| $(4/15, 4)$                | 10 | 4  | 0  | 0  | 0  |
| $(2/5, 8)$                 | 125| 31 | 0  | 0  | 0  |
| $(2/5, 10)$                | 277| 80 | 1  | 0  | 0  |
| $(8/15, 10)$               | 527| 227| 9  | 0  | 0  |

**Table 14:** Index of horizon states for the 15A orbifold of $K3
Table 15: Index of horizon states for the $23A$ orbifold of $K3$

| $(Q^2, P^2)$ | $Q \cdot P$ | 0 | 1 | 2 | 3 | 4 |
|--------------|-----------|---|---|---|---|---|
| $(2/23, 2)$  |           | 1 | 0 | 0 | 0 | 0 |
| $(2/23, 4)$  |           | 2 | 0 | 0 | 0 | 0 |
| $(2/23, 6)$  |           | 5 | 0 | 0 | 0 | 0 |
| $(4/23, 2)$  |           | 14| 2 | 0 | 0 | 0 |
| $(4/23, 4)$  |           | 28| 4 | 0 | 0 | 0 |
| $(6/23, 8)$  |           | 87| 36| 4 | 0 | 0 |
| $(6/23, 10)$ |           | 144| 57| 6 | 0 | 0 |

4.3 Toroidal orbifolds

In [13] we have seen that positivity of index for single centred dyons was violated for the toroidal models. For completeness we have reproduced some of the indices evaluated in [13] in tables 16, 17, 18.

Table 16: The index $d(Q, P)$ for the $Z_2$ toroidal orbifold some low lying values of $Q^2, P^2$ with $Q \cdot P = 0$. 

| $Q^2$ \ $P^2$ | 2  | 4  | 6  | 8  |
|---------------|----|----|----|----|
| 1             | -224 | -1248 | 1728 | 95104 |
| 2             | 1152 | 18240 | 233984 | 2432544 |
| 3             | -3392 | -10320 | 542976 | 12103360 |
| 4             | -11520 | 200736 | 4575744 | 86712256 |
| 5             | -30336 | -55424 | 12914944 | 412163328 |
| 6             | 83968 | 1544832 | 61928448 | 2013023104 |
| 7             | -202560 | -179022 | 175358304 | 8292093664 |
| 8             | 496512 | 9480000 | 638922240 | 32998944096 |
| 9             | -1118496 | -155232 | 1735394112 | 119618619520 |
| 10            | 2521600 | 49523928 | 5364983808 | 415768863360 |
### Table 17: The index $d(Q, P)$ for the $\mathbb{Z}_2$ toroidal orbifold some low lying values of $Q^2, P^2$ with $Q \cdot P = 1$.  

| $Q^2$ \ $\backslash$ $P^2$ | 2   | 4   | 6   | 8       |
|--------------------------|-----|-----|-----|---------|
| 1                        | 96  | 1968| 22528| 190047  |
| 2                        | -256| 840 | 70912| 1127672 |
| 3                        | 1376| 34656| 728256 | 11046139|
| 4                        | -3840| 16632| 2497408 | 61486056|
| 5                        | 13152| 343152| 13144832 | 348876305|
| 6                        | -33536| 171152| 42058240 | 1603241304|
| 7                        | 92928| 2476752| 162898624 | 7016918625|
| 8                        | -220672| 1265256| 480911872 | 27503872048|
| 9                        | 540416| 14545584| 1556616646 | 102315259287|
| 10                       | -1204992| 7558560| 4271142656 | 354800345088|

### Table 18: The index $d(Q, P)$ for the $\mathbb{Z}_2$ toroidal orbifold some low lying values of $Q^2, P^2$ with $Q \cdot P = 2$.  

| $Q^2$ \ $\backslash$ $P^2$ | 2     | 4     | 6       | 8       |
|--------------------------|-------|-------|---------|---------|
| 1                        | 0     | -12   | -224    | -1248   |
| 2                        | 64    | 2502  | 43264   | 491904  |
| 3                        | -224  | 2432  | 191168  | 3805600 |
| 4                        | 1152  | 43392 | 1440256 | 30853488|
| 5                        | -3392 | 33720 | 5363680 | 171782688|
| 6                        | 11520| 414336| 24533248| 893029504|
| 7                        | -30336| 302400| 80281536| 3063098880|
| 8                        | 83968| 2926080| 287831552| 16432260672|
| 9                        | -202560| 2049968| 851816352| 62214237440|
| 10                       | 496512| 16919712| 2627695616| 222752294016|

### Positivity of the horizon states for toroidal models

The indices in tables 16, 17, 18 were obtained under the assumption that there exists a frame in which the fermionic zero modes associated with broken supersymmetries are the only hair. In (2.50) and (2.53) we have proposed the partition function for the hair degrees of freedom in the type IIB frame for the $\mathbb{Z}_2, \mathbb{Z}_3$ toroidal orbifolds respectively. We evaluate the indices of horizon states in the following tables (19-24) and observe that they are all positive for single centered dyons.
\[ Q^2 \quad \| \quad P^2 \]

|   | 2   | 4   | 6   | 8   |
|---|-----|-----|-----|-----|
| 1 | 832 | 14816 | 158848 | 1283902 |
| 2 | 3840 | 101008 | 1425920 | 14471264 |
| 3 | 14624 | 556176 | 10273024 | 129971582 |
| 4 | 48128 | 2588336 | 62037760 | 971443680 |
| 5 | 143424 | 10594400 | 325402624 | 6254176746 |
| 6 | 394112 | 39145344 | 1521266688 | 35582718576 |
| 7 | 1016080 | 133122060 | 6465235840 | 182481593350 |
| 8 | 2480512 | 422430736 | 25355844096 | 85661245280 |
| 9 | 5786240 | 1264061344 | 92844570752 | 372638152610 |
| 10 | 12968576 | 3595680768 | 320340466176 | 1517055578876 |

Table 19: Index of horizon states for the $\mathbb{Z}_2$ orbifold of $T^6$ for $Q \cdot P = 0$.

\[ Q^2 \quad \| \quad P^2 \]

|   | 2   | 4   | 6   | 8   |
|---|-----|-----|-----|-----|
| 1 | 480 | 9012 | 98784 | 811166 |
| 2 | 2496 | 69328 | 1001472 | 10329280 |
| 3 | 9888 | 403448 | 7664064 | 98689790 |
| 4 | 33664 | 1946480 | 48074496 | 766539920 |
| 5 | 102272 | 8155848 | 258619232 | 5063997322 |
| 6 | 286208 | 30667504 | 1231379200 | 29352001136 |
| 7 | 747456 | 105699406 | 5306269024 | 152656500694 |
| 8 | 1847040 | 339109664 | 7737446688 | 3180401982114 |
| 9 | 4350816 | 1024054008 | 270248202752 | 13043376086768 |
| 10 | 9841408 | 2935991504 | 720448202752 | 13043376086768 |

Table 20: Index of horizon states for the $\mathbb{Z}_2$ orbifold of $T^6$ for $Q \cdot P = 1$.

\[ Q^2 \quad \| \quad P^2 \]

|   | 2   | 4   | 6   | 8   |
|---|-----|-----|-----|-----|
| 1 | 96 | 1880 | 21056 | 178660 |
| 2 | 640 | 21312 | 329728 | 3577216 |
| 3 | 2992 | 151056 | 3115712 | 42306045 |
| 4 | 11008 | 813280 | 22062720 | 371908566 |
| 5 | 35840 | 3696900 | 128569280 | 2665839255 |
| 6 | 105472 | 14554120 | 647882496 | 16372365048 |
| 7 | 288192 | 52296704 | 2913889600 | 88924896642 |
| 8 | 738560 | 173535528 | 11950263808 | 436628175032 |
| 9 | 1798688 | 539132792 | 45385181120 | 1969579830259 |
| 10 | 4187008 | 1583791144 | 161466383616 | 826279311120 |

Table 21: Index of horizon states for the $\mathbb{Z}_2$ orbifold of $T^6$ for $Q \cdot P = 2$. 
We now enumerate the consistency checks we have done for the proposal for the hair modes in the $T^6/Z_2$ toroidal model given in (2.50).

1. If we do not include the zero modes $-e^{2\pi i v}(1 - e^{2\pi i v})^{-2}$ as part of the hair partition function in $T^6/Z_2$, then we observe the violation of positivity in index for $P^2 = 6, Q^2 = 1, Q \cdot P = 2$ and $P^2 = 6, Q^2 = 2, Q \cdot P = 3$. The indices for these dyonic charges are $-224$ and $-256$ respectively. These charges are within the kinematic domain defined by (4.16).

2. If we include the contribution of the Wilson lines given in (2.52) as part of the hair partition function and remove the contribution of the zero modes $-e^{2\pi i v}(1 - e^{2\pi i v})^{-2}$, we find violations in positivity of the index. This can be observed at $P^2 = 6, Q^2 = 1, Q \cdot P = 2, P^2 = 6, Q^2 = 2, Q \cdot P = 3, P^2 = 4, Q^2 = 4, Q \cdot P = 3$, the indices are given by $-64, -64, -4$ respectively.
These two observations show that we certainly need to include the contribution of the zero modes $-e^{2\pi i v}(1 - e^{2\pi i v})^{-2}$ as part of the hair partition function which is consistent with our proposal. It would be interesting to prove this by studying the wave function of the gravitino zero modes in the toroidal models.

A very similar analysis holds true for $T^6/Z_3$. The index of horizon states obtained by considering the proposal given in (2.50) for the hair partition function is positive as shown in the subsequent tables. We have also repeated the consistency checks we mentioned earlier for the $Z_2$ orbifold in this case with the same conclusions.

| $(Q^2, P^2)$ \ $Q \cdot P$ | 0   | 1     | 2       | 3  | 4   |
|---------------------------|-----|-------|---------|----|-----|
| (2/3, 2)                  | 162 | 90    | 9       | 0  | 0   |
| (2/3, 4)                  | 1944| 1134  | 162     | 0  | 0   |
| (2, 6)                    | 14598| 8748 | 1149    | 0  | 0   |
| (4/3, 2)                  | 540 | 324   | 72      | 0  | 0   |
| (4/3, 4)                  | 8856| 5724  | 1458    | 54 | 0   |
| (2, 2)                    | 1566| 1008  | 243     | 18 | 0   |
| (2, 4)                    | 34344| 23652| 7290    | 810| 0   |
| (2, 6)                    | 402972| 286734| 98613  | 13614| 249 |

Table 24: Index of horizon states for the $T^6/Z_3$ orbifold

5 Conclusions

We have constructed the horizon partition function of the 1/4 BPS dyonic black hole in $\mathcal{N} = 4$ theories obtained by compactifying type IIB on orbifolds of $K3 \times T^2$. We then observed that the index of the horizon states of single centred black holes are all positive. We adapted the proof of [10] and showed that the index of the horizon partition function of single centred dyons with $P^2 = 2$ remains positive.

For the toroidal models we propose that the hair modes are given by (2.50) and (2.50). We showed the index of horizon states with this proposal is positive and performed consistency checks. As mentioned earlier it would be interesting to study the wave function of the zero modes of the gravitino in the toroidal models to check the proposal in (2.50) and (2.53). In [13] it was noticed that that the index of single centred dyons in these models were not positive when one assumed that the only hair modes are the Fermionic zero modes associated with broken supersymmetry generators. Since hair modes are frame dependent, the observations in this paper indicates that there is possibly no duality frame for these models which contains only the Fermionic zero modes as the hair. It will be interesting to verify this explicitly by an study similar to that done in [8, 9] for the $\mathcal{N} = 8$ theory.
The observation that the index of horizon states in the canonical compactification on $K3 \times T^2$ is positive is worth further study. It should be possible to extend the proof of [10] to higher values of $P^2$.

**Note Added:** As this work was nearing completion, we became aware of the work done in [33]. The analysis of the hair modes done for the CHL orbifolds of $K3$ in section 2 and 3 overlaps with parts of [33].

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