Solitary wave solutions of nonlinear partial differential equations based on the simplest equation for the function $1/\cosh^n$

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Abstract

The method of simplest equation is applied for obtaining exact solitary traveling-wave solutions of nonlinear partial differential equations that contain monomials of odd and even grade with respect to participating derivatives. The used simplest equation is $f^2_\xi = n^2(f^2 - f^{(2n+2)/n})$. The developed methodology is illustrated on two examples of classes of nonlinear partial differential equations that contain: (i) only monomials of odd grade with respect to participating derivatives; (ii) only monomials of even grade with respect to participating derivatives. The obtained solitary wave solution for the case (i) contains as particular cases the solitary wave solutions of Korteweg-deVries equation and of a version of the modified Korteweg-deVries equation.

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1 Introduction

The methods of nonlinear dynamics, nonlinear time series analysis and theory of differential equations are highly actual because of their numerous applications in data analysis, theory of chaos, social dynamics, etc. [1] - [23]. Traveling wave solutions of the nonlinear partial differential equations are studied much in the last decades [24] - [30] as such waves exist in many natural systems [31] - [37]. In addition effective methods exist for obtaining exact traveling-wave solutions, e.g., the method of inverse scattering transform or the method of Hirota [38] - [41]. Many other approaches for obtaining exact special solutions of nonlinear PDEs have been developed in the recent years. Let us note here only the method of simplest equation and its version called modified method of simplest equation [42] - [47] as these methods are closely connected to the discussion below.

The method of simplest equation is based on a procedure analogous to the first step of the test for the Painlevé property [42], [43], [48] - [50]. In the version of the method called modified method of the simplest equation [44] - [47] this procedure is substituted by the concept for the balance equation. This version of the method of simplest equation has been successfully applied for obtaining exact traveling wave solutions of numerous nonlinear PDEs such as versions of generalized Kuramoto - Sivashinsky equation, reaction - diffusion equation, reaction - telegraph equation [44], [51] generalized Swift - Hohenberg equation and generalized Rayleigh equation [45], generalized Fisher equation, generalized Huxley equation [52], generalized Degasperis - Procesi equation and b-equation [53], extended Korteweg-de Vries equation [54], etc. [55], [56].

A short summary of the method of simplest equation we shall use below is as follows. First of all by means of an appropriate ansatz (for an example the traveling-wave ansatz) the solved nonlinear partial differential equation

\[ P(u, u_\xi, u_{\xi\xi}, \ldots) = 0 \]  

(1.1)

Then the solution \( u(\xi) \) is searched as some function of another function \( f(\xi) \). Often this function is a finite-series solution

\[ u(\xi) = \sum_{\mu=-\nu}^{\nu_1} p_\mu [f(\xi)]^\mu, \]  

(1.2)

where \( p_\mu \) are coefficients and \( f(\xi) \) is solution of simpler ordinary differential equation called simplest equation. Eq. (1.2) is substituted in Eq. (1.1) and let the result of this substitution be a polynomial of \( f(\xi) \). Then Eq. (1.2) is a
candidate for solution of Eq. (1.1) if all coefficients of the obtained polynomial of $f(\xi)$ are equal to 0. This condition leads to a system of nonlinear algebraic equations for the coefficients of the solved nonlinear PDE and for the coefficients of the solution. Any nontrivial solution of this algebraic system leads to a solution the studied nonlinear partial differential equation.

Below we shall consider traveling-wave solutions $u(x,t) = u(\xi) = u(\alpha x + \beta t)$, constructed on the basis of the simplest equation

$$f_\xi^2 = n^2 \left( f_x^2 - f_{x\xi}^{2\!\!n+2} \right), \quad (1.3)$$

where $n$ is arbitrary positive real number. The solution of this equation is $f(\xi) = \frac{1}{\cosh^n(x)}$. The text is organized as follows. In Sect. 2 we describe the main result which is formulated as a theorem. In section 3 we discuss several examples for solitary wave solutions obtained by the developed methodology. Several concluding remarks are summarized in Sect. 4.

2 Main result

We shall use the concept of grade of a monomial with respect to participating derivatives. Let us consider polynomials that are linear combination of monomials and the monomials contain product of terms consisting of product of powers of derivatives of some function $u(\xi)$ of different orders. This product of terms can be multiplied by a polynomial of $u$. Let a term from a monomial contains $k$th power of a derivative of $u$ of $l$th order. We shall call the product $kl$ grade of the term with respect to participating derivatives. The sum of these grades of all terms of the monomial will be called grade of the monomial with respect to participating derivatives. In the general case the polynomial will contain monomials of odd and even grades with respect to participating derivatives. There are two particular cases: (i) the polynomial contains monomials that are only of odd grades with respect to participating derivatives; (ii) the polynomial contains monomials that are only of even grades with respect to participating derivatives. Below we formulate a theorem about solitary wave solutions for the class of nonlinear PDEs that contain monomials of derivatives which order with respect to participating derivatives is even and monomials of derivatives which order with respect to participating derivatives is odd.

First of all we shall use the method of induction to prove an useful lemma and then we shall prove a theorem that will be our main result.

**Lemma.** Let $f(\xi)$ be a function that is solution of Eq. (1.3). Let us consider the function $F(\xi) = f^{(in+2j)/n}$ where $n$ is a positive real number and $i$ and $j$
are non-negative integer numbers. Then the odd derivatives of $F$ contain $f_\xi$ multiplied by expression(s) of the kind $f^{(kn+2j)/n}$ and the even derivatives of $F$ contain expressions of the kind $f^{(kn+2j)/n}$, where $k$ is some integer number.

**Proof.** Let us consider first the case $i = 0$. Then $F(\xi) = f^{2j/n}$. The first and the second derivatives of $F(\xi)$ are

$$F_\xi = \frac{2n}{j} f^{(2j-n)/n} f_\xi \quad (2.1)$$

$$F_{\xi \xi} = \frac{2j}{n} \left( f^{(2j-2n)/n} f_\xi^2 + \frac{2j}{n} f^{(2j-n)/n} f_{\xi \xi} \right) \quad (2.2)$$

$f_\xi^2$ can be substituted from Eq. (1.3) and from the same equation one obtains

$$f_{\xi \xi} = n^2 \left( f - \frac{n+1}{n} f^{(n+2)/n} \right) \quad (2.3)$$

As one can see $F_\xi$ contains $f_\xi$ multiplied by a term of the kind $f^{(kn+2j)/n}$, and the substitution of Eqs. (1.3) and (2.3) in Eq. (2.2) leads to the conclusion that the even derivative $F_{\xi \xi}$ contains only expressions of the kind $f^{(kn+2j)/n}$ (namely $f^{2j/n}$ and $f^{2(j+1)/n}$) as stated in the lemma. The calculations can be made further. The result is that the third derivative $F_{\xi \xi \xi}$ (which is an odd derivative) contains $f_\xi$ multiplied by a sum of expressions of the kind $f^{(kn+2j)/n}$. The fourth derivative $F_{\xi \xi \xi \xi}$ (which is an even derivative) contains only sum of expressions of the kind $f^{(kn+2j)/n}$, etc.

Let us now consider the derivative of $F$ of order $2q$ where $q$ is a natural number. According to the lemma we assume that this derivative (denoted as $F_{\xi}^{(2q)}$) contains only sum of expressions of the kind $f^{(kn+2j)/n}$. Then the obtaining the derivative $F_{\xi}^{(2q+1)}$ includes many operations similar to the operation of obtaining $F_\xi$ from Eq. (2.1). The result for $F_{\xi}^{(2q+1)}$ will be an expression consisting of $f_\xi$ multiplied by terms of the kind $f^{(kn+2j)/n}$. The next even derivative $F_{\xi}^{(2q+2)}$ will be obtained in a way similar to obtaining $F_{\xi \xi}$ from Eq. (2.2) and the relationship for $F_{\xi}^{(2q+2)}$ will contain only expressions of the kind $f^{(kn+2j)/n}$. This concludes the proof of the lemma for the case $i = 0$.

Let now $i > 0$. The first derivative of $F(\xi)$ is

$$F_\xi = \frac{i n + 2j}{n} f^{(i n+2j)/n} f_\xi \quad (2.4)$$

This is an odd derivative and it contains $f_\xi$ and expression of the kind $f^{(kn+2j)/n}$ as stated in the lemma. The second derivative of $F(\xi)$ is

$$F_{\xi \xi} = \frac{in + 2j}{n} \left( f^{(i n+2j)/n} f_\xi^2 + \frac{in + 2j}{n} f^{(i n+2j)/n} f_{\xi \xi} \right) \quad (2.5)$$
\( f_\xi \) can be substituted from Eq. (1.3) and \( f_{\xi\xi} \) can be substituted from Eq. (2.3). The substitution of Eqs (1.3) and (2.3) in Eq. (2.5) leads to the conclusion that the even derivative \( F_{\xi\xi} \) contains only expressions of the kind \( f^{(in+2j)/n} \) (namely \( f^{(in+2j)/n} \) and \( f^{(in+2(j+1))/n} \)) as stated in the lemma. The calculations can be made further. The result is that the third derivative \( F_{\xi\xi\xi} \) (which is an odd derivative) contains \( f_\xi \) multiplied by a sum of expressions of the kind \( f^{(in+2j)/n} \). The fourth derivative \( F_{\xi\xi\xi\xi} \) (which is an even derivative) contains only sum of expressions of the kind \( f^{(in+2j)/n} \), etc.

Let us now consider the derivative of \( F \) of order \( 2q \) where \( q \) is a natural number. According to the lemma we assume that this derivative (denoted as \( F^{(2q)}_\xi \)) contains only sum of expressions of the kind \( f^{(in+2j)/n} \). Then the obtaining the derivative \( F^{(2q+1)}_\xi \) includes many operations similar to the operation of obtaining \( F_\xi \) from Eq. (2.1). The result for \( F^{(2q+1)}_\xi \) will be an expression consisting of \( f_\xi \) multiplied by terms of the \( f^{(in+2j)/n} \). Then the next even derivative \( F^{(2q+2)}_\xi \) be obtained in a way similar to obtaining \( F_{\xi\xi} \) from Eq. (2.2) and the relationship for \( F^{(2q+2)}_\xi \) will contain only expressions of the kind \( f^{(in+2j)/n} \). This concludes the proof of the lemma.

Now we are ready to formulate and prove our main result.

**Theorem.** Let \( P \) be a polynomial of the function \( u(x,t) \) and its derivatives. \( u(x,t) \) belongs to the differentiability class \( C^k \), where \( k \) is the highest order of derivative participating in \( P \). \( P \) can contain some or all of the following parts: (A) polynomial of \( u \); (B) monomials that contain derivatives of \( u \) with respect to \( x \) and/or products of such derivatives. Each such monomial can be multiplied by a polynomial of \( u \); (C) monomials that contain derivatives of \( u \) with respect to \( t \) and/or products of such derivatives. Each such monomial can be multiplied by a polynomial of \( u \); (D) monomials that contain mixed derivatives of \( u \) with respect to \( x \) and \( t \) and/or products of such derivatives. Each such monomial can be multiplied by a polynomial of \( u \); (E) monomials that contain products of derivatives of \( u \) with respect to \( x \) and derivatives of \( u \) with respect to \( t \). Each such monomial can be multiplied by a polynomial of \( u \); (F) monomials that contain products of derivatives of \( u \) with respect to \( x \) and mixed derivatives of \( u \) with respect to \( x \) and \( t \). Each such monomial can be multiplied by a polynomial of \( u \); (G) monomials that contain products of derivatives of \( u \) with respect to \( t \) and mixed derivatives of \( u \) with respect to \( x \) and \( t \). Each such monomial can be multiplied by a polynomial of \( u \); (H) monomials that contain products of derivatives of \( u \) with respect to \( x \), derivatives of \( u \) with respect to \( t \) and mixed derivatives of \( u \) with respect to \( x \) and \( t \). Each such monomial can be multiplied by a polynomial of \( u \).
Let us consider the nonlinear partial differential equation:

$$P = 0$$  \hspace{1cm} (2.6)

We search for solutions of this equation of the kind

$$u(\xi) = \gamma f(\xi), \quad \xi = \alpha x + \beta t.$$ 

$\gamma$ is a parameter and $f(\xi)$ is a solution of the simplest equation $f_\xi^2 = n^2 f^2 - n^2 f^{\frac{2n+2}{n}}$ where $n$ is a real positive number. The substitution of this solution in (2.6) leads to a relationship $R$ of the kind

$$R = \sum_{i=0}^{N_1} \sum_{j=0}^{M_1} C_{ij} f((\text{i}n+\text{j})^{2}/n) + f_\xi \sum_{k=-N_2}^{M_2} \sum_{l=0}^{l=M_2} D_{kl} f((\text{k}n+\text{l})^{2}/n),$$  \hspace{1cm} (2.7)

where $N_1, N_2, N^*_2, M_1$ and $M_2$ are natural numbers depending on the form of the polynomial $P$. The coefficients $C_{ij}$ and $D_{kl}$ depend on the parameters of Eq. (2.6) and on $\alpha, \beta, \gamma$. The sum $\sum_{i=0}^{N_1} \sum_{j=0}^{M_1} C_{ij} f((\text{i}n+\text{j})^{2}/n)$ consists of terms of the kind $C_p^* f(\xi)^{\alpha_p}$ where $\alpha_p$ is some number and $p = 1,\ldots$. The sum $\sum_{k=0}^{N_2} \sum_{l=0}^{l=M_2} D_{kl} f((\text{k}n+\text{l})^{2}/n)$ consists of terms of the kind $D_q^* f(\xi)^{\beta_q}$ where $\beta_q$ is some number and $q = 1,\ldots$. The coefficients $C_p^*$ and $D_q^*$ depend on the parameters of Eq. (2.6) and on $\alpha, \beta, \gamma$. Then any nontrivial solution of the algebraic system

$$C_p^* = 0; \quad D_q^* = 0,$$  \hspace{1cm} (2.8)

leads to a solitary wave solution of the nonlinear PDE (2.7).

Proof. Let $f(\xi)$ be a solution of the nonlinear ODE $f_\xi^2 = n^2 f^2 - n^2 f^{\frac{2n+2}{n}}$. According to the lemma above the higher derivatives of $f(\xi)$ contain terms of the kind $f((\text{i}n+\text{j})^{2}/n)$ or $f_\xi$ multiplied by terms of the kind $f((\text{k}n+\text{l})^{2}/n)$. The substitution of these derivatives in the solved nonlinear PDE $P = 0$ will reduce the solved equation to a relationship of the kind (2.7). In order to obtain a solution of Eq. (2.7) we have to solve the system of equations (2.8), i.e., a system of nonlinear algebraic equations for $\alpha, \beta, \gamma$ and the parameters participating in $P$. Any nontrivial solution of the last system of nonlinear algebraic equation leads to a solution of Eq. (2.6) of the kind $u = \gamma f(\xi)$ where $f(\xi)$ is solution of the simplest equation (1.3) (note that $n$ is an arbitrary real positive number).

Proof. Let us note two particular cases connected to the values $n = 1$ and $n = 2$. For the case $n = 1$ the simplest equation (1.3) becomes

$$f_\xi^2 = f^2 - f^4$$  \hspace{1cm} (2.9)
and its solution is \( f(\xi) = \frac{1}{\cosh(x)} \). In this case \( R \) from Eq. (2.7) can be reduced to
\[
R = \sum_{i=0}^{N} C_i f(\xi)^i + f_{\xi} \sum_{j=0}^{M} D_j f(\xi)^j,
\]
and the system of nonlinear algebraic equations becomes \( C_i = 0, D_j = 0; i = 0, \ldots, j = 0, \ldots \). The other particular case is \( n = 2 \). Here the simplest equation becomes
\[
f_{\xi}^2 = 4(f^2 - f^3)
\]
and the solution is \( f(\xi) = \frac{1}{\cosh^2(x)} \). For this particular case \( R \) again can be reduced to the relationship of the kind (2.10) and the described methodology leads to the solitary wave solutions of many famous water-waves equations such as the Korteweg-deVries equation, Boussinesq equation, Degasperis-Procesi equation, etc. [56].

3 Examples

We shall consider examples of nonlinear partial differential equations that contain monomials only of odd and even grade with respect to participating derivatives.

3.1 Case of a nonlinear partial differential equation that contains monomials only of odd grades with respect to participating derivatives

Let us consider as an example the equation
\[
a F^p \frac{\partial^p F}{\partial x^\mu} + b F^q \frac{\partial^q F}{\partial t^\nu} + c F^r \frac{\partial F}{\partial x} = 0
\]
where \( a, b, p, q, \mu, \) and \( \nu \) are parameters. We search for a solution of the kind \( F = \gamma f \) where \( \gamma \) is a parameter and \( f(x, t) = f(\xi); \xi = \alpha x + \beta t \) is solution of the simplest equation (1.3). The substitution of \( F \) in Eq. (3.1) leads to the following equation for \( f(\xi) \)
\[
\alpha^\mu \gamma^{p+1} a f^p \frac{df}{d\xi^\mu} + \beta^\nu \gamma^{q+1} b f^q \frac{df}{d\xi^\nu} + \alpha c \gamma^{r+1} f^r \frac{df}{d\xi} = 0
\]
The most simple case of nonlinear equation that contains only odd derivatives is \( \mu = 1, \nu = 3 \) or \( \mu = 3, \nu = 1 \).
3.1.1 Case $\mu = 1$, $\nu = 3$

In this case the substitution of Eq.(1.3) in Eq.(3.2) leads to the relationship

$$\alpha \gamma + 1 \alpha f + \beta^3 \gamma + n^2 b f q - \beta^3 \gamma + (n+1)(n+2) b f q + \alpha c \gamma + 1 \alpha f = 0 \quad (3.3)$$

The relationship (3.3) contains several powers of the function $f$. In order to obtain the system of the nonlinear algebraic relationships we have to perform the balance procedure from the modified method of simplest equation. As a result we have two possibilities: (i) $p = q + 2/n$, $r = q$; (ii) $r = q + 2/n$, $p = q$. For the case (i) the equation (3.1) becomes

$$(a F^{2/n} + c) \frac{\partial F}{\partial x} + b \frac{\partial^3 F}{\partial t^3} = 0 \quad (3.4)$$

We obtain a system of two nonlinear algebraic equations. The solution of this system is

$$\gamma = \left[ -\frac{(n+1)(n+2)}{n^2} \frac{c}{a} \right]^{n/2}; \quad \alpha = -\frac{\beta^3 b m^2}{c} \quad (3.5)$$

and the corresponding solution of the equation (3.4) is

$$F_n(x, t) = \frac{\left[ -\frac{(n+1)(n+2)}{n^2} \frac{c}{a} \right]^{n/2}}{\cosh^n \left[ -\frac{n^2 \beta^3 b c}{c} x + \beta t \right]} \quad (3.6)$$

Now for $n = 1$ and for $n = 2$ we obtain the solitary wave solutions

$$F_1(x, t) = \frac{\left[ -\frac{6 c^5}{a} \right]^{1/2}}{\cosh \left[ -\frac{3 \beta b c}{c} x + \beta t \right]}; \quad F_2(x, t) = \frac{\left[ -\frac{6 c^5}{a} \right]}{\cosh^2 \left[ -\frac{4 \beta b c}{c} x + \beta t \right]}, \quad (3.7)$$

for $n = 10$ and for $n = 1/4$ we obtain the solitary wave solutions

$$F_{10}(x, t) = \frac{\left[ \frac{123 c^5}{100 a} \right]^{5}}{\cosh^{10} \left[ -\frac{100 \beta b c}{c} x + \beta t \right]}; \quad F_{1/4}(x, t) = \frac{\left[ -\frac{45 c^5}{a} \right]^{1/8}}{\cosh^{1/4} \left[ -\frac{\beta b c}{10c} x + \beta t \right]} \quad (3.8)$$

Note that $n$ can be arbitrary positive real number. For an example for $n = 2.22$ the solitary wave solution of Eq.(3.4) is

$$F_{2.22}(x, t) = \frac{\left[ -2.75716265 \frac{c}{a} \right]^{1.11}}{\cosh^{2.22} \left[ -\frac{4.9284 \beta b c}{c} x + \beta t \right]} \quad (3.9)$$
Let us now consider the case (ii). Eq. (3.1) becomes

\[(a + c F^{n/2}) \frac{\partial F}{\partial x} + b \frac{\partial^3 F}{\partial t^3} = 0\] (3.10)

which is equation of the same kind as Eq. (3.4). In this case the balance procedure leads again to a system of two nonlinear algebraic equations. The solutions of this system is

\[
\gamma = \left[- \frac{(n+1)(n+2)}{n^2} \frac{a}{c} \right]^{n/2}; \quad \alpha = -\frac{\beta^2 b n^2}{a} \] (3.11)

and in this case the solution becomes

\[
F(n) (x, t) = \left[- \frac{(n+1)(n+2)}{n^2} \frac{a}{c} \right]^{n/2} \cosh \left[ \frac{-n^2 \beta^2 b x + \beta t}{a^2} \right] \] (3.12)

### 3.1.2 Case \( \mu = 3, \nu = 1 \)

Let us now consider the case \( \mu = 3, \nu = 1 \). The substitution of the form of \( F \) and the derivatives of \( f \) into Eq. (3.2) leads to the relationship

\[
\alpha^3 a n^2 \gamma^{p+1} f^p - \alpha^3 a (n+1)(n+2) \gamma^{p+1} f^{p+2/n} + \beta \gamma^{q+1} b f^q + \alpha c \gamma^{r+1} f^r = 0 \] (3.13)

There are two possibilities: (i) \( q = p + 2/n, r = p \); (ii) \( r = p + 2/n, q = p \). For the case (i) Eq. (3.1) becomes

\[
a \frac{\partial^3 F}{\partial x^3} + b F^{2/n} \frac{\partial F}{\partial t} + c \frac{\partial F}{\partial x} = 0 \] (3.14)

The obtained system of nonlinear algebraic equations has a solution

\[
\alpha = \pm \left(- \frac{c}{a n^2}\right)^{1/2}; \quad \beta = \mp \left(- \frac{c}{a n^2}\right)^{3/2} \frac{a}{b} (n+1)(n+2) \gamma^{-2/n} \] (3.15)

and the solitary wave solution of Eq. (3.4) is

\[
F(x, t) = \gamma \cosh^n \left[ \pm \left(- \frac{c}{a n^2}\right)^{1/2} x \mp \left(- \frac{c}{a n^2}\right)^{3/2} \frac{a}{b} (n+1)(n+2) \gamma^{-2/n} t \right] \] (3.16)

Note that \( n \) is arbitrary finite positive real number.
For the case (ii) Eq. (3.1) becomes

\[ a \frac{\partial^3 F}{\partial x^3} + b \frac{\partial F}{\partial t} + cF^{n/2} \frac{\partial F}{\partial x} = 0 \]  \hspace{1cm} (3.17)

and the obtained system of nonlinear algebraic equations has a solution

\[ \alpha = \pm \left[ \frac{c\gamma^{2/n}}{a(n+1)(n+2)} \right]^{1/2} ; \beta = \mp \frac{an^2}{b} \left[ \frac{c\gamma^{2/n}}{a(n+1)(n+2)} \right]^{3/2} \]  \hspace{1cm} (3.18)

The solitary wave solution of Eq. (3.17) is

\[ F(x, t) = \gamma \cosh^n \left[ \pm \left[ \frac{c\gamma^{2/n}}{a(n+1)(n+2)} \right]^{1/2} x \mp \frac{an^2}{b} \left[ \frac{c\gamma^{2/n}}{a(n+1)(n+2)} \right]^{3/2} t \right] \]  \hspace{1cm} (3.19)

Let us consider several particular cases of Eq. (3.17). For \( n = 2 \) Eq. (3.17) contains as particular case the Korteweg-deVries equation and the solution (3.18) is reduced to the famous \( \cosh^2 \) solitary wave solution. For \( n = 1 \) one of the the solutions of Eq. (3.17) is

\[ F(x, t) = \gamma \cosh \left[ \frac{c\gamma^{2/n}}{6a} \right]^{1/2} x - \frac{a}{b} \left[ \frac{c\gamma^{2/n}}{6a} \right]^{3/2} t \]  \hspace{1cm} (3.20)

For \( n = 1/2 \) Eq. (3.17) contains as particular case the modified Korteweg-deVries equation and one of the solutions of (3.19) is reduced to

\[ F(x, t) = \gamma \cosh^{1/2} \left[ \frac{4c\gamma^{4/15a}}{15a} \right]^{1/2} x - \frac{a}{4b} \left[ \frac{4c\gamma^{4/15a}}{15a} \right]^{3/2} t \]  \hspace{1cm} (3.21)

### 3.2 Case of a nonlinear partial differential equation that contains monomials only of even grade with respect to participating derivatives

Let us now consider the case of even derivatives for the equation

\[ aF^p \frac{\partial^\mu F}{\partial x^\mu} + bF^q \frac{\partial^\nu F}{\partial t^\nu} + cF^{r} + dF^{s} = 0 \]  \hspace{1cm} (3.22)

Let us discuss the case \( \mu = 4, \nu = 2 \). We need the following relationship

\[ f_{\xi\xi\xi\xi} = n^4 f - 2n(n+1)(n^2 + 2n + 2)f^{(n+2)/n} + n(n+1)(n+2)(n+3)f^{(n+4)/n} \]  \hspace{1cm} (3.23)
We remember that we search for a solution of the kind \( F(\xi) = \gamma f(\xi) \) where \( f \) is solution of the simplest equation (1.3). The application of the balance procedure from the modified method of simplest equation leads to the relationships \( q = p + 2/n, \ r = p + 1 \) together with one of the possibilities: \( s = p + (n+2)/n \) or \( s = p + (n+4)/n \). Let us consider the case \( q = p + 2/n, \ s = p + (n+2)/n, \ r = p + 1 \). The equation (3.22) becomes

\[
a \frac{\partial^4 F}{\partial x^4} + b \frac{\partial^2 F}{\partial t^2} + cF + d\frac{1}{n} = 0 \tag{3.24}
\]

Taking into account the forms of \( F \) and \( f \) we can reduce Eq. (3.24) to the following system of nonlinear ordinary differential equations

\[
(b\beta^2 n^2 + d)\gamma^{2/n} - 2a\alpha^4 n(n + 1)(n^2 + 2n + 2) = 0 \\
-\beta^2 b\gamma^{2/n} + a\alpha^4 (n + 3)(n + 2) = 0 \\
\alpha^4 n^4 + c = 0 \tag{3.25}
\]

One possible solution of the system (3.25) is

\[
\alpha = \left(\frac{-ca^3}{an}\right)^{1/4} \\
\beta = \left[\frac{(n^2 + 5n + 6)d}{bn(n^3 + n^2 + 2n + 4)}\right]^{1/2} \\
\gamma = \left[-\frac{c(n^3 + n^2 + 2n + 4)}{dn^3}\right]^{n/2} \tag{3.26}
\]

and the corresponding solitary wave solution is

\[
F(x,t) = \left[-\frac{c(n^3 + n^2 + 2n + 4)}{dn^3}\right]^{n/2} \cosh^n \left\{\left(\frac{-ca^3}{an}\right)^{1/4} x + \left[\frac{(n^2 + 5n + 6)d}{bn(n^3 + n^2 + 2n + 4)}\right]^{1/2} t\right\} \tag{3.27}
\]

where \( n \) can be arbitrary positive finite real number.

The second possibility is \( q = p + 2/n \) and \( s = p + (n+4)/n \). In this case the equation (3.22) becomes

\[
a \frac{\partial^4 F}{\partial x^4} + b \frac{\partial^2 F}{\partial t^2} + cF + d\frac{1}{n} = 0 \tag{3.28}
\]
Taking into account that $F = \gamma f$ and the simplest equation for $f$ we can reduce Eq. (3.28) to the system of nonlinear algebraic equations

\[-bn\beta^2(n + 1)\gamma^{2/n} + d\gamma^{4/n} + a\alpha^4n(n + 3)(n + 2)(n + 1) = 0 \]
\[-\frac{1}{2}\beta^2b\gamma^{2/n} + a\alpha^4(n + 1)(n^2 + 2n + 2) = 0 \]
\[a\alpha^4n^4 + c = 0 \]  

(3.29)

One possible solution of this system is

\[\alpha = \frac{1}{n} \left( -\frac{c}{a} \right)^{1/4} \]
\[\beta = \left[ -\frac{cd^{1/2}(n + 1)(n^3 + n^2 + 2n + 4) + n(n^3 + 6n^2 + 11n + 6)}{b n^3(n + 1)[-c(n + 1)(n^3 + n^2 + 2n + 4)]^{1/2}} \right]^{1/2} \]
\[\gamma = \left[ -\frac{c(n + 1)(n^3 + n^2 + 2n + 4)}{dn^4} \right]^{n/4} \]  

(3.30)

and the corresponding solitary wave solution is

\[F(x, t) = \left[ -\frac{c(n + 1)(n^3 + n^2 + 2n + 4)}{dn^4} \right]^{n/4} / \cosh^n \left\{ \frac{1}{n} \left( -\frac{c}{a} \right)^{1/4} x + \right.\]
\[\left. -\frac{cd^{1/2}(n + 1)(n^3 + n^2 + 2n + 4) + n(n^3 + 6n^2 + 11n + 6)}{b n^3(n + 1)[-c(n + 1)(n^3 + n^2 + 2n + 4)]^{1/2}} \right\}^{1/2} t \]  

(3.31)

4 Concluding Remarks

In this article we have continued our research from [56] on the methodology connected to the method of simplest equation for obtaining exact solutions of nonlinear partial differential equations. We have formulated a theorem about a solitary wave solution of kind $1/\cosh^n(\xi)$, $\xi = \alpha x + \beta t$ that may help us to find solitary wave solutions of a large class of nonlinear partial differential equations. The developed methodology is applied to two classes of nonlinear PDEs. We note that the methodology can be applied also to more complicated nonlinear PDEs, e.g., to equations containing together monomials of odd and even grades with respect to participating derivatives. Let us note that the limits of the applicability of the methodology will be reached when
the size of the system of nonlinear algebraic equations becomes large. Then
the number of parameters participating in the solved equation and in the so-
lution can become smaller than the number of equations. Another problem
may arise if some of the algebraic equations have nonlinearity of high order
and because of this an analytical solution is impossible to be obtained. But
as we have shown the methodology is effective and leads to exact solitary
wave solutions to many nonlinear partial differential equations.

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