Abstract

We construct the complete effective chiral pion–nucleon Lagrangian in the presence of virtual photons to one loop. As an application, we consider strong and electromagnetic isospin violation for scattering of neutral pions off nucleons. We show that for the scattering lengths these isospin violating terms are of the same size as the purely hadronic ones. We also analyze isospin–violating effects for the $\sigma$–term. These can be as large as 10\% for the absolute value but are negligible for the shift to the Cheng–Dashen point.
In his seminal paper in 1977, Weinberg pointed out that reactions involving nucleons and neutral pions might lead to gross violations of isospin symmetry \[1\]. In particular, he argued that the mass difference of the up and down quarks can produce a 30% effect in the difference of the $\pi^0 p$ and $\pi^0 n$ S-wave scattering lengths while these would be equal in case of isospin conservation. This was later reformulated in more modern terminology \[2\]. To arrive at the abovementioned result, Weinberg considered Born terms and the dimension two symmetry breakers, one related to the sum of the quark masses (i.e. the so-called $\sigma$-term) and the other proportional to $m_u - m_d$. Bernard et al. \[3\] showed that at this order there are other isospin-conserving terms which make a precise prediction for the individual $\pi^0 p$ or $\pi^0 n$ scattering length very difficult. Nevertheless, Bernstein proposed to measure $a(\pi^0 p)$ in neutral pion photoproduction off protons \[4\] based on the observation that the phase of the electric dipole amplitude below the secondary $\pi^+ n$ threshold is related to $a(\pi^0 p)$ by unitarity (Fermi–Watson theorem) \[^4\]. With the advent of high-precision data on pionic hydrogen and deuterium at PSI \[^4\] and neutral pion photoproduction data from MAMI \[^4\] and SAL \[^4\], novel interest has been spurred in separating electromagnetic effects and trying to filter out isospin violating contributions. In addition, in the framework of some models it has been claimed that the presently available pion–nucleon data basis exhibits isospin violation of the order of a few percent \[^4\]. However, to really pin down isospin breaking due to the light quark mass difference, one needs a machinery that allows to simultaneously treat the electromagnetic and the strong contributions. The only framework known at present allowing to do just that is (baryon) chiral perturbation theory. It is based on a consistent power counting scheme which allows to order the various terms according to the number of derivatives and/or meson mass insertions. This defines the so-called chiral dimension corresponding to the number of derivatives and/or pion mass insertions, with the pertinent small parameters collectively denoted by $q$. For the pion–nucleon system, the leading term is of dimension one and loops start to contribute at two orders higher (in the heavy fermion formalism \[^4\] which we will use). In what follows, we will add to the well-known chiral effective pion–nucleon Lagrangian the terms including virtual photons to one loop order. For that, we construct the generating functional of two flavor heavy baryon chiral perturbation theory (HBCHPT) \[^4\] in the presence of virtual photons and perform the necessary renormalization. We also construct all finite terms up-to-and-including third order in the chiral dimension. Throughout, we assign to the electric charge a dimension one, based on the observation that $e^2 \sim M^2_\pi/(4\pi F_\pi)^2 \sim 1/10$ (with $M_\pi$ and $F_\pi$ the pion mass and decay constant, respectively). We remark that some of these terms have also been constructed by van Kolck based on a different power counting scheme \[^4\] and that Lebed and Luty have considered the SU(3) electromagnetic terms relevant to the analysis of the baryon masses \[^4\]. We then work out the scattering lengths $a(\pi^0 p)$ and $a(\pi^0 n)$ to third order in small momenta, generalizing Weinberg’s result and discuss the relative size of the isospin conserving and violating terms. We also consider the much discussed pion–nucleon $\sigma$-term and analyze the isospin-violating contributions. In this letter, we only work to third order in small momenta. As has become clear from studies concerning a broad variety of processes in the pion–nucleon system, one has to go to fourth order\[^4\].

[^4]: This idea was also mentioned by J.D. Bjorken, see \[^1\].
[^5]: A status report with emphasis on pion photoproduction is given in the talk \[^10\].
to achieve a high precision. The results presented here should therefore be considered indicative and need to be supplemented by higher order calculations.

2. To introduce virtual photons in the effective pion–nucleon field theory, consider first the nucleon charge matrix \( Q = e \text{ diag}(1, 0) \). We form the matrices

\[
Q_\pm = \frac{1}{2} (u Q u^\dagger \pm u^\dagger Q u) , \quad \hat{Q}_\pm = Q_\pm - \frac{1}{2} \langle Q_\pm \rangle ,
\]

where \( U(x) = u^2(x) \) collects the pions and \( \langle \ldots \rangle \) denotes the trace in flavor space. By construction, the \( \hat{Q}_\pm \) are traceless. Under chiral SU(2)_L \times SU(2)_R symmetry, the \( Q_\pm \) transform as

\[
Q_\pm \rightarrow K Q_\pm K^\dagger ,
\]

with \( K \) the compensator field representing an element of the conserved subgroup SU(2)_V. Furthermore, under parity \( (P) \) and charge conjugation \( (C) \) transformations, one finds

\[
P Q_\pm P^{-1} = \pm Q_\pm , \quad C Q_\pm C^{-1} = \pm Q^T_\pm ,
\]

where \( Q^T \) is the transposed of the matrix \( Q \). For physical processes, only quadratic combinations of the charge matrix \( Q \) (or, equivalently, of the matrices \( Q_\pm \)) can appear (see also ref. [15]). It is now straightforward to implement the (virtual) photons given in terms of the gauge field \( A_\mu \) in the effective pion–nucleon Lagrangian. Starting from the relativistic theory and decomposing the spinor fields into light (denoted \( N \)) and heavy components (velocity eigenstates), one can use path integral methods to integrate out the heavy components in a systematic fashion. The effective Lagrangian decomposes into a sum of terms with increasing chiral dimension (throughout, we use the notation and conventions of the review [16]),

\[
\mathcal{L}_{\pi N} = \sum_{n \geq 1} \mathcal{L}_{\pi N}^{(n)} = \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \mathcal{L}_{\pi N}^{(3)} + \ldots .
\]

In particular, to lowest order \( (n = 1) \)

\[
\mathcal{L}_{\pi N}^{(1)} = \bar{N} \left( i v \cdot \tilde{D} + g_A S \cdot \tilde{u} \right) N ,
\]

with

\[
\tilde{D}_\mu = D_\mu - i Q_+ A_\mu , \quad \tilde{u}_\mu = u_\mu - 2 Q_- A_\mu ,
\]

and \( v_\mu \) denoting the nucleons’ four–velocity, \( D_\mu = \partial_\mu + \Gamma_\mu \) the chiral covariant derivative, \( S_\mu \) the covariant spin–vector à la Pauli–Lubanski, \( u_\mu = i u^\dagger \nabla_\mu U u^\dagger \) and \( g_A \) the axial–vector coupling constant. Here

\[
\nabla_\mu U = \partial_\mu U - i (v_\mu + a_\mu + QA_\mu) U + i U (v_\mu - a_\mu + QA_\mu) ,
\]

is the generalized pion covariant derivative containing the external vector \( (v_\mu) \) and axial–vector \( (a_\mu) \) sources. Note that to leading order, no symmetry breaking appears in the
effective pion–nucleon field theory. At next order \((n = 2)\), we have (remember that we count \(e\) as a small momentum, \(\mathcal{O}(e) \sim \mathcal{O}(q)\))

\[
\mathcal{L}^{(2)}_{\pi N} = \mathcal{L}^{(2)}_{\pi N,\text{str}} + \mathcal{L}^{(2)}_{\pi N,\text{em}},
\]

\[
\mathcal{L}^{(2)}_{\pi N,\text{str}} = \bar{N} \left\{ \frac{1}{2m} (v \cdot D)^2 - \frac{1}{2m} D \cdot D + c_1 \langle \chi_+ \rangle + \left( c_2 - \frac{g_A}{8m} \right) (v \cdot u)^2 + c_3 u \cdot u + \ldots + c_5 \left( \chi_+ - \frac{1}{2} \langle \chi_+ \rangle \right) + \ldots \right\} N,
\]

\[
\mathcal{L}^{(2)}_{\pi N,\text{em}} = \bar{N} F^2 \left\{ f_1 \langle Q^2_+ - Q^2_- \rangle + f_2 \bar{Q}_+ \langle Q_+ \rangle + f_3 \langle Q^2_+ + Q^2_- \rangle + f_4 \langle Q_+ \rangle^2 \right\} N, \tag{10}
\]

with \(\chi_+ = u^\dagger \chi u + u \chi^\dagger u\) and \(\chi = 2B_0 (s + ip)\) subsumes the external scalar and pseudoscalar sources. The external scalar source contains the quark mass matrix, \(s(x) = \text{diag}(m_u, m_d) + \ldots\) and \(B_0 = |\langle 0 | \bar{q} q | 0 \rangle|/F^2\) measures the strength of the spontaneous symmetry breaking. We assume \(B_0 \gg F_\pi\) (standard CHPT). Various remarks are in order. For the strong part, we have only displayed the terms of relevance to the calculations discussed later. In particular, the term \(\sim c_5\) is the only one which leads to an effect of the order \(m_d - m_u\). This is exactly the B–term in Weinberg’s notation \([2]\). All the others are isospin conserving. The values of the low–energy constants (LECs) have been determined in ref. \([17]\). For the electromagnetic terms, we have written down the minimal number allowed by all symmetries. Note that the last two terms in \(\mathcal{L}^{(2)}_{\pi N,\text{em}}\) are proportional to \(e^2 \bar{N} N\). This means that they only contribute to the electromagnetic nucleon mass in the chiral limit and are thus not directly observable. However, this implies that in the chiral two–flavor limit \((m_u = m_d = 0, m_s\) fixed), the proton is heavier than the neutron since it acquires an electromagnetic mass shift. Only in pure QCD \((e^2 = 0)\), this chiral limit mass is the same for the two particles. Since we work to order \(q^3\) in the chiral expansion, we can always use the physical nucleon mass, denoted \(m\), and do not need to bother about its precise value in the chiral limit. The numerical values of the electromagnetic LECs \(f_1\) and \(f_2\) will be discussed below. The normalization factor of \(F^2_\pi\) in the electromagnetic pion–nucleon Lagrangian is introduced so that the \(f_i\) have the same dimension as the strong LECs \(c_i\). The corresponding dimension two meson Lagrangian with virtual photons has been constructed in \([18]\) \([15]\),

\[
\mathcal{L}^{(2)}_{\pi \pi} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{\lambda}{2} (\partial_\mu A^\mu)^2 + \frac{F^2_\pi}{4} (\nabla_\mu U \nabla^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U) + C \langle Q U Q U^\dagger \rangle, \tag{11}
\]

with \(F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) the photon field strength tensor and \(\lambda\) the gauge–fixing parameter (from here on, we work in the Lorentz gauge \(\lambda = 1\)). It is important to stress that in ref. \([13]\), \(Q\) denotes the quark charge matrix. To make use of the nucleon charge matrix used here, we perform a transformation of the type \(Q \to Q + \alpha \mathbf{1}\), with \(\alpha\) a real parameter and observe that \(d\langle Q U Q U^\dagger \rangle/d\alpha \sim e^2 \mathbf{1}\), i.e. to this order the difference between the two charge matrices can completely be absorbed in an unobservable constant term. To use the higher order terms constructed in \([15]\), one would have to rewrite them in terms of the nucleon charge matrix. In this paper, however, we do not need these terms and thus do not consider them any further. Throughout, we work in the \(\sigma\)–model gauge for the pions. In that case, the last term in \(\mathcal{L}^{(2)}_{\pi \pi}\) leads only to a term quadratic in pion
fields. Consequently, the LEC \( C \) can be calculated from the neutral to charged pion mass difference since this term leads to \( (\delta M^2)_{em} = 2e^2C/F_\pi^2 \). This gives \( C = 5.9 \cdot 10^{-5} \text{GeV}^4 \).

Consider now the neutron-proton mass difference. It is given by a strong insertion \( \sim c_5 \) and an electromagnetic insertion \( \sim f_2 \),

\[
m_n - m_p = (m_n - m_p)_{str} + (m_n - m_p)_{em} = 4c_5B_0(m_u - m_d) + 2e^2F_\pi^2f_2 + \mathcal{O}(q^4). \tag{12}\]

Note that one–loop corrections to the mass difference only start beyond the order we are considering here. This can be traced back to the fact that the photonic self–energy diagram of the proton at order \( q^3 \) vanishes since it is proportional to \( \int d^4k [k^2v \cdot k]^{-1} \) and the hadronic \( \mathcal{O}(q^3) \) mass corrections are the same for the \( p \) and the \( n \). To the order we are working, the electromagnetic LEC \( f_2 \) can therefore be fixed from the electromagnetic proton mass shift, \( (m_n - m_p)_{em} = -(0.7 \pm 0.3) \text{MeV} \), i.e. \( f_2 = -(0.45 \pm 0.19) \text{GeV}^{-1} \). The strong contribution has been used in \( [17] \) to fix the LEC \( c_5 = -0.09 \pm 0.01 \text{GeV}^{-1} \).

Note that it is known that one–loop graphs with an insertion \( \sim m_d - m_u \) on the internal nucleon line, which in our counting appear at fourth order, can contribute sizeable to the strong neutron-proton mass difference \( [19] \). Such effects go, however, beyond the accuracy considered here but it underlines the need for a complete \( \mathcal{O}(q^4) \) calculation.

3. To go beyond tree level, we have to construct the terms of order \( q^3 \). From the building blocks at our disposal, we can construct a minimal set of 20 independent terms which are compatible with all symmetries, following ref. \( [20] \). The number of possible terms is limited due to the fact that at least two charge matrices must appear. In particular, charge conjugation allows to sort out terms which would otherwise be allowed. Some of these terms are accompanied by finite LECs where as the others absorb the divergences appearing at one loop with LECs that are only finite after renormalization. We now calculate these divergences. For doing that, we construct the generating functional and extract the pertinent divergences making use of heat kernel techniques for elliptic Euclidean differential operators. We follow here the approach outlined in \( [21] \). However, due to the presence of the virtual photons, we have to expand the generating functional to second order in fluctuations around the classical solutions for the meson and photons fields, \( U = U^{cl} + i\xi u/F_\pi + \ldots \) and \( A_\mu = A^{cl}_\mu + \epsilon_\mu \), respectively. We consider only the the irreducible tadpole (\( \Sigma_1 \)) and self–energy (\( \Sigma_2 \)) graphs, leading to the one–loop functional at \( \mathcal{O}(q^3) \),

\[
Z_{tr}[s, p, v_\mu, a_\mu; R_v; A_\mu] = \int d^4x d^4x' d^4y d^4y' \bar{R}_v(x) A_{\mu}^{-1}(x, y) \times [\Sigma_1(y, y')\delta^{(4)}(y - y') + \Sigma_2(y, y')] A_{\mu}^{-1}(x', y') R_v(x') \tag{13}\]

where \( A_{\mu}^{-1} \) is the propagator for the nucleon field \( N \) in the presence of external fields and \( R_v \) is the corresponding velocity–projected nucleon source field. The extraction of the \( \Sigma_i \) in Euclidean space is straightforward. Consider first the tadpole graph:

\[
\Sigma_1 = \frac{i}{8} \tau_k [v \cdot \Sigma G - G v \cdot \Sigma]_{ki} \tau_i - \frac{3}{8} [\tau_i, Q_-] v_\mu G_{ij} \\
+ \frac{g_A}{8} [\tau_i, [\tau_j, S \cdot u]] G_{ij} + ig_A [\tau_i, Q_+] S_\mu G_{ij}, \tag{14}\]

\[
\Sigma_2 = \frac{i}{8} [\tau_i, [\tau_j, S \cdot u]] G_{ij} + ig_A [\tau_i, Q_+] S_\mu G_{ij}. \tag{15}\]

\[
\Sigma_3 = \frac{i}{8} [\tau_i, [\tau_j, S \cdot u]] G_{ij} + ig_A [\tau_i, Q_+] S_\mu G_{ij}. \tag{16}\]
with $\mathcal{G}$ the combined full pion $(G_{ij})$ and photon $(G_{\mu\nu})$ propagator,
\[
\mathcal{G} = [-\Sigma \Sigma + \Lambda]^{-1} = \begin{pmatrix} G_{ij} & G_{i\mu} \\ G_{\nu j} & G_{\mu\nu} \end{pmatrix},
\]
where $i, j = 1, 2, 3; \mu, \nu = 0, 1, 2, 3$. $\Sigma$ is the combined photon–pion covariant derivative,
\[
\Sigma = \partial_\mu 1 + Y_\mu,
\]
with
\[
Y^{AB}_\mu = \begin{pmatrix} -\frac{1}{2} \langle [\tau_i, \tau_j] \Gamma_\mu \rangle & \frac{F_\pi}{2} \langle Q^- \tau_i \rangle \delta_{\mu^}\beta \\ -\frac{F_\pi}{2} \langle Q^- \tau_j \rangle \delta_{\mu^}\alpha & 0 \end{pmatrix},
\]
where $A = (i, \alpha), B = (j, \beta)$ and $\Gamma_\mu$ is the chiral connection contained in the covariant derivative $D_\mu$, Eq.(3). Furthermore, $\Lambda$ is defined via
\[
\Lambda^{AB} = \begin{pmatrix} \sigma_{ij} & \gamma_{ij} \\ \gamma_{\alpha j} & \rho_{\alpha j} \end{pmatrix},
\]
with
\[
\sigma_{ij} = \frac{1}{8} \langle \{\tau_i, \tau_j\} \chi^+ \rangle - C F_\pi^2 \langle [\tau_i, Q^+] [\tau_j, Q^+] - [\tau_i, Q^-] [\tau_j, Q^-] \rangle + \frac{1}{8} \langle [\tau_i, u^-] [\tau_j, u^-] \rangle - F_\pi^2 \langle Q^- \tau_i \rangle \langle Q^- \tau_j \rangle \dd\alpha \beta,
\]
\[
\rho_{\alpha j} = \frac{3}{2} F_\pi^2 \langle Q^- \rangle \delta_{\alpha j}, \quad \gamma_{\alpha i} = -\frac{F_\pi}{2} \left[ i \langle u^- [\tau_i, Q^+] \rangle + \langle \tau_i [D_{\alpha}, Q^-] \rangle \right].
\]
Note that $\Lambda$ and its diagonal submatrices are symmetric under the interchange of the pertinent indices. The off–diagonal matrix–elements in $\mathcal{G}$, Eq.(15), describe (virtual) pion–photon transitions which can occur in the presence of outgoing pions or other external sources. Similarly, the self–energy graph leads to
\[
\Sigma_2 = \frac{1}{16} [v \cdot u, \tau_i] A_{(1)}^{-1} [v \cdot u, \tau_j] G_{ij} - Q_+ A_{(1)}^{-1} Q_+ v^- \int_\mu \mathcal{G}_{\mu\nu} \\
- 4 g^2 A^2 S_{\mu} A_{(1)}^{-1} S_{\nu} S_\mu S_\nu + 2 g_A [Q_+ v_\mu A_{(1)}^{-1} Q^- S_\nu + Q^- S_\mu A_{(1)}^{-1} v_\nu Q_+] \mathcal{G}_{\mu\nu} \\
- \frac{i}{4} [v \cdot u, \tau_i] A_{(1)}^{-1} (Q_+ v_\nu - 2 g_A Q^- S_\nu) \mathcal{G}_{ij} - \frac{i}{4} (Q_+ v_\nu - 2 g_A Q^- S_\nu) A_{(1)}^{-1} [v \cdot u, \tau_j] \mathcal{G}_{\mu j} \\
+ \frac{i}{4} g_A [v \cdot u, \tau_i] A_{(1)}^{-1} \mathcal{G}_{ij} S_\tau \sum_k \tau_k + \frac{i}{4} g_A \tau_k S_\tau \sum_k A_{(1)}^{-1} [v \cdot u, \tau_j] \mathcal{G}_{ij} \\
+ g_A (Q_+ v_\nu - 2 g_A Q^- S_\nu) A_{(1)}^{-1} \mathcal{G}_{ij} S_\tau \sum_k \tau_k - g_A^2 \tau_k S_\tau \sum_k A_{(1)}^{-1} \mathcal{G}_{ij} S_\tau \sum_k \tau_k \\
+ g_A \tau_k S_\tau \sum_k A_{(1)}^{-1} (Q_+ v_\nu - 2 g_A Q^- S_\nu) \mathcal{G}_{ij}.
\]
The functionals $\Sigma_{1,2}(y, y')$ are divergent in the coincidence limit $y \rightarrow y'$. Following Ecker [21], the divergences can be extracted in a straightforward manner. The electromagnetic part of the dimension three Lagrangian takes the form (after combining all

#6 As an excellent check, we recover the results of ref. [23] when switching off the virtual photons.
finite terms with the ones obtained after renormalization)
\[ \mathcal{L}^{(3)}_{\pi N, \text{em}} = \sum_{i=1}^{20} g_i \tilde{N} \mathcal{O}_i N, \]  
(21)

with the \( \mathcal{O}_i \) monomials in the fields of dimension three. The low–energy constants \( g_i \) absorb the divergences in the standard manner,
\[ g_i = \kappa_i L + g_i^{\mu}(\mu), \]  
(22)
\[ L = \frac{\mu^{d-4}}{16\pi^2} \left\{ \frac{1}{d-4} - \frac{1}{2} \left[ \ln(4\pi) + \Gamma'(1) + 1 \right] \right\}, \]  
(23)

with \( \mu \) the scale of dimensional regularization and \( d \) the number of space–time dimensions. The explicit expressions for the operators \( \mathcal{O}_i \) and their \( \beta \)–functions \( \kappa_i \) are collected in table 1. The \( g_i^{\mu}(\mu) \) are the renormalized, finite and scale–dependent low–energy constants. These can be fixed by data or have to be estimated with the help of some model. They obey the standard renormalization group equation. To arrive at the terms in the table, we have used the relation
\[ [D_\mu, Q_{\pm}] = -\frac{i}{2} [u_\mu, Q_{\pm}] + c^\pm_\mu \]
\[ c^\pm_\mu = \frac{1}{2} \left\{ u(\partial_\mu Q - i[v_\mu - a_\mu, Q])u^\dagger \pm u^\dagger(\partial_\mu Q - i[v_\mu + a_\mu, Q])u \right\}. \]  
(24)
Figure 1: Graphs contributing to isospin violation in $\pi^0$–proton scattering. Solid and dashed lines denote nucleons and pions, in order. The heavy dot and the box refer to the em counterterm at order $e^2$, i.e. the term proportional to $C$, and the dimension two strong insertion $\sim m_u - m_d$, respectively. Diagram a) has previously been considered by Weinberg [1].

Furthermore, one could use the relation
\[
[\nabla^\mu, u_\mu] = \frac{i}{2} \chi - \frac{i}{4} \langle \chi \rangle + \frac{AC}{F^2} [Q_+, Q_-] + O(q^4),
\]
(25)
to rewrite some of the terms tabulated. In fact, this has been done by Ecker [21] for the case without virtual photons ($e^2 = 0$). We prefer not to do this and therefore Ecker’s operator $O_8^{\text{str}} = [\chi_-, v \cdot u]$ in our basis reads $O_8^{\text{str}} = -2[\langle i\nabla^\mu, u_\mu \rangle, v \cdot u]$. A few remarks concerning the operators given in table 1 are in order. First, $O_7$ and $O_8$ only lead to an electromagnetic renormalization of $g_A$ and their effects can thus completely be absorbed in the physical value of the axial–vector coupling constant. The operators $O_{17,...,20}$ are only of relevance for processes with external axial–vector fields (or low–energy manifestations of $Z^0$–exchange) and $O_{13,...,16}$ can be eliminated by use of the nucleon equations of motion, i.e. they are not relevant for reactions involving on–shell nucleons. The general effective pion–nucleon Lagrangian with virtual photons constructed here allows now to systematically investigate the influence of isospin–breaking due to the quark mass difference $m_u - m_d$ and the dual effects from electromagnetism. Of particular importance are the processes $\pi N \rightarrow \pi N$, $\gamma N \rightarrow \pi N$ and $\pi N \rightarrow \pi \pi N$ since a large body of precise low–energy data exists which can be analysed within the framework outlined. Here, we concentrate on one particular reaction, namely elastic neutral pion scattering off nucleons.

4. We are now in the position to evaluate the isospin violating corrections to the $\pi^0 p$ scattering length. To one–loop order, we consider the isospin–conserving (IC) tree and loop graphs already evaluated in [3], the isospin–violating strong tree graph $\sim m_u - m_d$ [1] (cf. fig.1a) and the isospin–violating loop graphs (cf. fig.1b,c). For the latter two, isospin violation comes in solely through the charged to neutral pion mass difference, as indicated by an insertion $\sim C$ on the internal pion lines. The rescattering type of diagram (fig.1b) is well known from neutral pion photoproduction, where it leads to the cusp effect in the electric dipole amplitude [22].

Evaluating all diagrams, the resulting expression reads
\[
a(\pi^0 p) = \frac{1}{4\pi} \left(1 + \frac{M_{\pi^0}}{m_p}\right)^{-1} \left(a_{\text{IC}}^{(2)} + a_{\text{IV}}^{(2)} + a_{\text{IC}}^{(3)} + a_{\text{em}}^{(3)} \right),
\]
(26)
with
\[ a^{(2)}_{IC} = \frac{2M_{\pi}^2}{F_\pi^2} \left( c_2 + c_3 - 2c_1 - \frac{g_A^2}{8m} \right) , \] 
\[ a^{(2)}_{IV} = -\frac{2c_5}{F_\pi^2} B_0 (m_u - m_d) , \] 
\[ a^{(3)}_{IC} = \frac{3 g_A^2 M_{\pi}^3}{64 \pi F_\pi^4} , \] 
\[ a^{(3)}_{em} = -\frac{M_{\pi}^2}{8\pi F_\pi^2} \sqrt{M_{\pi}^2 - M_{\pi^0}^2} + \frac{3g_A^2 M_{\pi}^2}{32 \pi F_\pi^4} (M_{\pi^0} - M_{\pi^0}) . \]

In Weinberg’s analysis, only dimension two terms were considered \[1\]. He calculated the Born piece and the \( \sigma \)-term, i.e. the contributions \( \sim g_A^2 \) and \( \sim c_1 \) in \( a^{(2)}_{IC} \) as well as \( a^{(2)}_{IV} \). We recover, of course, these particular terms once we account for the differences in notation. For the neutron case, one finds a similar result. Most interestingly, the difference between \( a(\pi^0 p) \) and \( a(\pi^0 n) \) is given entirely by the dimension two term \( \sim m_u - m_d \), thus Weinberg’s result
\[ a(\pi^0 p) - a(\pi^0 n) = \frac{1}{4\pi} \left( 1 + \frac{M_{\pi^0}}{m_p} \right)^{-1} \left( -\frac{4c_5}{F_\pi^2} B_0 (m_u - m_d) \right) + \mathcal{O}(q^4) , \]
is only affected at next-to-next-to-leading order, \( \mathcal{O}(q^4) \). Let us now analyse the various pieces contributing to the neutral pion–proton scattering length. Using as input \( F_\pi = 92.4 \text{ MeV} \), \( g_A = 1.29 \) (from the Goldberger–Treiman relation with \( g_{\pi N} = 13.05 \) which is equivalent to using \( g_A = 1.26 \) and explicitly keeping the dimension three operator appearing in the Goldberger–Treiman discrepancy), \( M_{\pi^0} = 134.97 \text{ MeV} \), \( M_{\pi^+} = 139.57 \text{ MeV} \), \( m_p = 938.27 \text{ MeV} \) and the best fit values from \[17\], \( c_1 = -0.91 \text{ GeV}^{-1} \), \( c_2 = 3.25 \text{ GeV}^{-1} \) and \( c_3 = -5.16 \text{ GeV}^{-1} \) (see also the discussion below), we find\[17\]
\[ a^{(2)}_{IC} = -1.33 \text{ GeV}^{-1} , \] 
\[ a^{(2)}_{IV} = +0.12 \text{ GeV}^{-1} , \] 
\[ a^{(3)}_{IC} = +0.84 \text{ GeV}^{-1} , \] 
\[ a^{(3)}_{em} = -0.30 \text{ GeV}^{-1} . \]

The total scattering length comes out to be \( a(\pi^0 p) = -8.8 \cdot 10^{-3}/M_\pi \). Notice the large cancellations between the isospin–conserving terms of dimension two and three, which make a precise knowledge of the LECs \( c_{1,2,3} \) necessary to sharpen this prediction. For this set of \( c_i \), \( a(\pi^0 p) \) decomposes as
\[ a(\pi^0 p) = a(\pi^0 p)_{\text{str,IC}} + a(\pi^0 p)_{\text{str,IV}} + a(\pi^0 p)_{\text{em}} = (-4.8 - 1.1 - 2.9) \cdot 10^{-3}/M_\pi , \]
which shows that the isospin–violating terms are of the same size as the isospin conserving ones. In particular, the electromagnetic correction is bigger than the one due to the quark mass difference and it is dominated by the rescattering graph (fig. 1b and first term in Eq.\[33\]). We also find that \( a(\pi^0 p) - a(\pi^0 n) = -2.2 \cdot 10^{-3}/M_\pi \) and thus the strong isospin violation is 25% in the difference. This agrees with Weinberg’s finding \[1\]. His numbers are different from ours because his isospin–conserving strong contribution is not the same
for the reasons mentioned above and also, he used SU(3) arguments to estimate $c_5$, which leads to a somewhat larger value than the one we use. However, the isospin violating effects might even be bigger. In the world of perfect isospin symmetry, the isospin–even scattering length $a^+$ is given exactly by the terms $a^+_{1C}$. This hadronic contribution has recently been measured at PSI by combining the strong interaction shift measurements of pionic hydrogen and deuterium, $a^+ = (0.5) \cdot 10^{-3} M^{-1}_\pi$. This number is, however, not consistent with $\pi N$ partial wave analysis, the value obtained in the SM95 solution of the VPI group is $a^+ = (-3.0) \cdot 10^{-3} M^{-1}_\pi[23]$. For that reason, in the fit which determined the $c_i$, $a^+$ was allowed to vary in the wide band between $-10 \ldots +10 \cdot 10^{-3} M^{-1}_\pi[17]$. A recent calculation to third order in small momenta for pion scattering off deuterium, based on the formalism developed in [27], leads to $a^+ = -(2.6 \pm 0.5) \cdot 10^{-3} M^{-1}_\pi[28].$

Of course, one can and should extend these considerations to elastic scattering of charged pions and charge exchange reactions. However, for these processes the isospin–conserving terms are much larger and also the effects of virtual photons are more dramatic. Nevertheless, in the light of the recent claims that the low–energy pion–nucleon scattering data basis shows isospin violation of the order of 5% [8][9], this issue has to be pursued. The framework presented here should allow to perform these calculations in a systematic fashion.

5. Furthermore, the $\sigma$–term analysis might also be strongly affected by isospin–violating effects (as already stressed by Weinberg). Here, we reconsider it to third order in small momenta. The scalar form factor is defined as the expectation value of the QCD quark mass term in a proton (neutron) state (note that due to the isospin–breaking, we have to work in the basis of physical states),

$$\sigma_p(t) = \langle p'|m_u \bar{u}u + m_d \bar{d}d|p \rangle,$$

with $|p\rangle$ a proton state of momentum $p$ and $t = (p' - p)^2$ the invariant momentum transfer squared. A similar definition applies for the neutron. The precise way of how to calculate this matrix element in HBCHPT is spelled out in ref. [12]. The $\sigma$–term is simply the scalar form factor at $t = 0$. Rewriting the quark mass term as a sum of an isoscalar and isovector component, one recovers the standard definition of the $\sigma$–term in the limit $m_u = m_d = \hat{m}$. To $O(q^3)$, the scalar form factor receives contributions from dimension two tree level insertions $\sim c_1$ and $\sim c_5$ as well as from finite one–loop graphs. In these, isospin violation comes in through the neutral to charged pion mass difference. Altogether, we find

$$\sigma_p(t) = -4M^2_{\pi^0}c_1 - 2B_0(m_u - m_d) c_5 - \frac{g_A^2 M^2_{\pi^0}}{32\pi F^2_{\pi}} (2M^2_{\pi^+} + M^2_{\pi^0})$$

Note that this result might still change a bit since a more sophisticated treatment of Doppler–broadening for the width of the pionic hydrogen has to be performed. Also, the PSI–ETHZ group did not yet quote a value for $a^+$. We rather used their figure combining the published H and d level shift results to get the band given. If one combines their numbers from the pionic hydrogen shift and width measurements, one gets a negative value for $a^+$. This agrees with a novel pionic hydrogen measurement at PSI leading to $a^+ = (-8\ldots0) \cdot 10^{-3} M^{-1}_\pi[24].$
\[ + \frac{g_\pi^2 M_{\pi^0}^2}{128 \pi F_\pi^2} \left\{ 2(t - 2M_{\pi^0}^2) \int_0^1 dx \left[ M_{\pi^+}^2 + tx(x - 1) \right]^{-1/2} \\
+ (t - 2M_{\pi^0}^2) \int_0^1 dx \left[ M_{\pi^0}^2 + tx(x - 1) \right]^{-1/2} \right\} , \quad (36) \]

\[ \sigma_n(t) = \sigma_p(t) + 4B_0 (m_u - m_d) c_5 . \quad (37) \]

The scalar form factor at the Cheng–Dashen point \( t = 2M_{\pi^0}^2 \) can be easily worked out from this using

\[ \int_0^1 dx \left[ M^2 + tx(x - 1) \right]^{-1/2} = \frac{1}{\sqrt{t}} \ln \frac{2M + \sqrt{t}}{2M - \sqrt{t}} . \quad (38) \]

The difference \( \sigma_p(2M_{\pi^0}^2) - \sigma_p(0) \) is entirely given by the one–loop contributions since the tree insertions are momentum–independent. For the numerical analysis, we use the same parameters as above. We note that there is currently some controversy about the precise value of the \( \sigma \)–term at the Cheng–Dashen point, which might change some of the values quoted here. It is, however, straightforward to update these once the controversy has been settled. We find (for the isospin–conserving piece, we use the charged pion mass)

\[ \sigma_p(0) = \sigma_p^{IC}(0) + \sigma_p^{IV}(0) = 47.2 \text{ MeV} - 3.9 \text{ MeV} = 43.3 \text{ MeV} , \quad (39) \]

which means that the isospin–violating terms reduce the proton \( \sigma \)–term by \( \sim 8\% \). The electromagnetic effects are again dominating the isospin violation since the strong contribution is just half of the strong proton–neutron mass difference, 1 MeV. Furthermore, one gets \( \sigma_p(2M_{\pi^0}^2) - \sigma_p(0) = 7.5 \text{ MeV} \), which differs from the result in the isospin limit (7.9 MeV) by 5\% and is by about a factor two too small when compared to the dispersive analysis of ref. \([29]\). This small difference of 0.4 MeV is well within the uncertainties related to the so–called remainder at the Cheng–Dashen point \([30]\).

6. To summarize, we have constructed the complete chiral effective pion–nucleon Lagrangian including the effects of virtual photons to third order in small momenta (i.e. in the one–loop approximation). Counting the electric charge as a small momentum, there are in total 4 and 20 terms contributing to the electromagnetic Lagrangian at dimension two and three, respectively (some of these do not appear in observables, e.g. at dimension two one has effectively two terms). As an application, we have considered isospin violation in elastic \( \pi^0 \)–nucleon scattering. We have sharpened Weinberg’s time–honoured calculation \([1]\) by extending it to third order in small momenta. In addition to the sizeable isospin violation from the quark mass difference, there is an even bigger effect due to virtual photons hidden in the pion mass difference. Weinberg’s prediction concerning the difference of the S-wave scattering lengths \( a(\pi^0 p) - a(\pi^0 n) \) is, however, not affected at third order. We have also considered the scalar form factor and shown that the proton \( \sigma \)–term is reduced by about 8\% due to isospin violation. The effect in the difference \( \sigma_p(2M_{\pi^0}^2) - \sigma_p(0) \) is well within the theoretical uncertainties related to scalar meson exchange, strangeness effects and so on (for details, see ref. \([30]\)). In a forthcoming publication, we will present results also for the channels involving charged pions and thus clarify the origin of the isospin violation claimed to be seen in low–energy pion–nucleon scattering. Ultimately, these calculations should be carried out at fourth order in small momenta which will require some work.
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