Quite a Character:
The Spectrum of Yang-Mills on $S^3^*$

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October 2008

Abstract

We introduce a simple method to extract the representation content of the spectrum of a system with SU(2) symmetry from its partition function. The method is easily generalized to systems with SO(2,4) symmetry, such as conformal field theories in four dimensions. As a specific application we obtain an explicit generating function for the representation content of free planar Yang-Mills theory on $S^3$. The extension to $\mathcal{N} = 4$ super Yang-Mills is also discussed.

*Based on the Brown University Undergraduate Thesis Lie Algebras and $\mathcal{N} = 4$ Yang-Mills Theory by Taylor H. Newton ’08.
1 Introduction and Summary

Although the vast majority of work on gauge/string duality has been conducted within the solid framework of the AdS/CFT correspondence, the decades-old arguments of ’t Hooft and Polyakov suggest that general large $N$ gauge theories should admit dual reformulations as string theories. Indeed even the simplest large $N$ gauge theory, pure SU($N$) Yang-Mills, exhibits a string-like Hagedorn density of states when the theory is formulated on a sphere whose radius is small compared to the dynamically generated scale, where the partition function may be reliably computed in perturbation theory [1, 2] (Hagedorn behaviour in the supersymmetric case was first discussed in [3, 4, 5]).

The present work was motivated by [6], where the representation content of free large $N$ Yang-Mills theory on $S^3$, or equivalently the spectrum of particles in this theory’s as yet unknown string dual, was determined. Specifically, an explicit algorithm was presented for calculating the number $N_{[d, j_1, j_2]}$ of times that SO(2,4) representation $[d, j_1, j_2]$ appears in the spectrum of the theory. Here $d$ is an integer labelling the energy of a state (in units of $1/R$) and $j_1, j_2$ are the two spin quantum numbers arising from the SO(4) isometry group of $S^3$. Although explicit, the result of [6] is rather complicated. We present in eq. (29) a relatively simple explicit formula for the generating function

$$G(q, a, b) = \sum_{d=1}^{\infty} \sum_{j_1, j_2 \in \mathbb{N}/2} N_{[d, j_1, j_2]} q^d a^{j_1} b^{j_2}$$

of the spectrum degeneracies. As a check on our result, we display the expansion of our generating function through order $q^9$ in Table 1 and find perfect corroboration with the corresponding results displayed in [6].

The key step in our construction follows from a straightforward manipulation of SU(2) characters detailed in section 2 and extended to the four-dimensional conformal group SO(2,4) in section 3. In the final subsection of the paper we comment on the possibility of extending our approach to maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory. Partition functions for this theory have been studied extensively, with the most closely related work including [7, 8, 9, 10, 11]. Let us note however that in contrast to those papers, where partition functions counted numbers of (super)conformal primary states, here (as in [6]) we focus on the problem of counting the number
of (super)conformal primary representations in the spectrum. To elucidate this distinction we point out that to count the number of primary states in the SO(2,4) case would require including an additional factor $(2j_1+1)(2j_2+1)$ inside the summand of eq. (5).

2 A Simple Relation for SU(2) Characters

2.1 The Relation

We begin by considering a partition function

$$Z(a) = \text{Tr}[a^{J_3}]$$

(2)

for some system with a discrete spectrum exhibiting an SU(2) symmetry. Additional factors such as $x^H$ (where $H$ is any operator that commutes with $J_3$) may be included inside the trace without affecting the following arguments. A consequence of SU(2) symmetry is that the function $Z(a)$ must be expressible as a linear combination

$$Z(a) = \sum_{j \in \mathbb{N}/2} N_j \chi_{[j]}(a),$$

(3)

of non-negative integer coefficients $N_j$ and the characters $\chi_{[j]}$ of the irreducible representations of SU(2). We first restrict our attention to cases where this sum has only a finite number of nonzero terms. It is then clear from the explicit formula

$$\chi_{[j]}(a) = \sum_{m=-j}^{+j} a^m = \frac{a^{j+1/2} - a^{-j-1/2}}{a^{1/2} - a^{-1/2}} = \frac{\sin((j + 1/2)\theta)}{\sin(\theta/2)} \quad a = e^{i\theta}$$

(4)

that $Z(a)$ admits a Laurent expansion in $a^{1/2}$ with only a finite number of nonzero coefficients. Since each $\chi_{[j]}(a)$ is symmetric under $a \to 1/a$, the partition function $Z(a)$ must have this symmetry as well, so the Laurent expansion must take the form

$$Z(a) = Z_0 + \sum_{k \in \mathbb{Z}^+/2} Z_k(a^k + a^{-k}) = Z_0 + 2 \sum_{k \in \mathbb{Z}^+/2} Z_k \cos(k\theta).$$

(5)
Our goal is to find a simple way to find all of the coefficients $N_j$ given some generic partition function $Z(a)$ of this form. Of course any individual coefficient $N_j$ may be obtained by exploiting orthogonality of the SU(2) characters,
\[
\int_0^{4\pi} \frac{\sin^2(\theta/2)}{2\pi} d\theta \, \chi_{[j]}(a)\chi_{[k]}(a) = \delta_{j,k},
\]
from which it follows that
\[
N_j = \int_0^{4\pi} \frac{\sin^2(\theta/2)}{2\pi} d\theta \, \chi_{[j]}(a)Z(a).
\]

In order to encapsulate all of the $N_j$ simultaneously we find it convenient to define a generating function $G(a)$ for the $N_j$,
\[
G(a) = \sum_{j \in \mathbb{N}/2} N_j a^j.
\]

Substituting eq. (7) into eq. (8) and using eqs. (4), (5) we find
\[
G(a) = \sum_{j \in \mathbb{N}/2} a^j \int_0^{4\pi} \frac{d\theta}{2\pi} \sin((j + 1/2)\theta)\sin(\theta/2) \left[ Z_0 + 2 \sum_{k \in \mathbb{Z}^+} Z_k \cos(k\theta) \right].
\]

Since the sum over $k$ involves (by assumption) only a finite number of non-zero terms we are free to perform the integral first, using the result
\[
\int_0^{4\pi} \frac{d\theta}{2\pi} \sin((j + 1/2)\theta)\sin(\theta/2) \cos(k\theta) = \frac{1}{2} \delta_{j,k} - \frac{1}{2} \delta_{j+1,k}, \quad k > 0,
\]
\[
= \delta_{j,0}, \quad k = 0.
\]

Thus we arrive at
\[
G(a) = \sum_{j \in \mathbb{N}/2} a^j \left[ Z_0 \delta_{j,0} + \sum_{k \in \mathbb{Z}^+} (\delta_{j,k} - \delta_{j+1,k}) Z_k \right].
\]

This equation, which is the central result of this section, may be succinctly summarized as
\[
G(a) = \{(1 - 1/a)Z(a)\}_{a \geq 0}
\]
where the notation $a^{\geq 0}$ indicates that one should calculate the Laurent expansion of the quantity in curly braces and then truncate that expansion by keeping only those terms with non-negative powers of $a$. 

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We have established eq. (12) by a rather complicated argument, but it is straightforward to check that it is correct by directly substituting eqs. (3) and (4), which gives

\[
\{(1 - 1/a)Z(a)\}_{a \geq 0} = \left\{ (1 - 1/a) \sum_{j \in \mathbb{N}/2} N_j \frac{a^{j+1/2} - a^{-j-1/2}}{a^{1/2} - a^{-1/2}} \right\}_{a \geq 0}
\]

\[
= \left\{ \sum_{j \in \mathbb{N}/2} N_j (a^j - a^{-j-1}) \right\}_{a \geq 0}
\]

\[
= \sum_{j \in \mathbb{N}/2} N_j a^j
\]

\[
= G(a), \quad (13)
\]

Although we assumed above that \(Z(a)\) had a finite Laurent expansion, to ease our way through the proof, the simple argument presented in eq. (13) demonstrates that eq. (12) holds more generally for formal power series.

2.2 Restricting to Non-Negative Powers

In this section we suggest a simple method to explicitly implement the step of restricting to non-negative powers of a Laurent expansion, which plays a crucial role in our result (12).

We first consider a function \(f(z)\) with Laurent expansion

\[
f(z) = \sum_{n=-\infty}^{\infty} c_n z^n. \quad (14)
\]

The basic identity we need involves the contour integral

\[
\frac{1}{2\pi i} \oint_C dz \frac{z^n}{z-a}, \quad (15)
\]

where \(n\) is an integer and \(C\) is a contour around the origin with radius greater than \(|a|\). For \(n \geq 0\) the result is \(a^n\), by Cauchy’s integral formula, while for \(n < 0\) the result of integration is zero. Therefore, in terms of the notation introduced in eq. (12) we find

\[
\{f(a)\}_{a \geq 0} = \frac{1}{2\pi i} \oint_C dz \frac{f(z)}{z-a}. \quad (16)
\]
This analysis is valid for a function $f(z)$ whose Laurent expansion has only integer powers of $z$. In order to apply this to an SU(2) partition function $Z(a)$ we need to modify it to allow half-integer powers as well. This is easily accomplished by taking $a \rightarrow a^2$ before performing the contour integral, and then $a \rightarrow \sqrt{a}$ to restore $a$ afterwards. Also including the factor $(1 - 1/a)$ which appears in eq. (12), we find that our result (12) may be recast as the contour integral

$$G(a) = \{(1 - 1/a)Z(a)\}_{a \geq 0} = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz \frac{1 - 1/z^2}{z - \sqrt{a}} Z(z^2),$$

where $\mathcal{C}$ is a contour around the origin with radius greater than $|a|^{1/2}$.

3 Extension to SO(2,4)

In this section our goal will be to extend the result of section 2 to the conformal group in four dimensions, SO(2,4). For a conformally invariant theory we can consider the partition function

$$Z(q, a, b) = \text{Tr}[q^H a^{J_3} b'^{J'_3}]$$

where $(H, J'_3, J_3)$ are simultaneously commuting generators of the maximal compact subgroup SO(2)×SU(2)×SU(2) of SO(2,4). In the application to Yang-Mills theory on $S^3$ discussed below, $H$ will be the Hamiltonian and the $J_3$’s will be generators of the isometry group of $S^3$.

Conformal symmetry guarantees that the partition function may be expressed as a linear combination

$$Z(q, a, b) = \sum_{d, j_1, j_2} N_{[d, j_1, j_2]} \chi_{[d, j_1, j_2]}(q, a, b)$$

of the characters of irreducible representations of SO(2,4). The non-negative integer coefficient $N_{[d, j_1, j_2]}$ counts the number of times the representation $[d, j_1, j_2]$ appears in the spectrum. Our goal is to find a simple way to obtain, from a given $Z(q, a, b)$, the generating function (1) for these multiplicities $N_{[d, j_1, j_2]}$.

The analysis is somewhat complicated by the fact that the irreducible representations $[d, j_1, j_2]$ of SO(2,4) come in a couple of different varieties (see
for example \cite{12}). The generic (“long”) multiplet has \( d > j_1 + j_2 + 2 \) and character

\[
\chi_{[d,j_1,j_2]}(q, a, b) = \frac{q^d \chi_{[j_1]}(a) \chi_{[j_2]}(b)}{(1 - qx_1)(1 - qx_2)(1 - qx_3)(1 - qx_4)},
\]

where

\[
x_1 = \sqrt{ab}, \quad x_2 = \sqrt{a/b}, \quad x_3 = \sqrt{b/a}, \quad x_4 = 1/\sqrt{ab}.
\]

Let us first consider for simplicity the case of a theory which has only long multiplets in its spectrum. Then, we can substitute eq. (20) into eq. (19), multiply both sides by \( \prod (1 - qx_i) \), and apply the result (12) separately for each SU(2) to arrive at the following simple result

\[
G(q, a, b) = \left\{ (1 - 1/a)(1 - 1/b)Z(q, a, b) \prod_{i=1}^{4} (1 - qx_i) \right\}_{a \geq 0, b \geq 0}.
\]

Here, the subscript \( a \geq 0, b \geq 0 \) is an instruction to perform a Laurent expansion of the quantity in curly braces in the variables \( a \) and \( b \), and then to discard any terms containing a negative power of \( a \) or \( b \).

A generic conformal theory also has “short” multiplets in its spectrum. These come in two varieties, one with character

\[
\chi_{[d,j_1,j_2]} = \chi_{[d,j_1,j_2]} - \chi_{[d+1,j_1-1/2,j_2-1/2]}
\]

for \( d = j_1 + j_2 + 2 \) and \( j_1 j_2 > 0 \), and the second with character

\[
\chi_{[d,j_1,j_2]} = \chi_{[d,j_1,0]} - \chi_{[d+1,j_1-1/2,1/2]} + \chi_{[d+2,j_1-1,0]}
\]

for \( d = j_1 + 1 \) and \( j_1 \geq 0 \) (or the same with the two SU(2)’s interchanged). If one were to naively apply eq. (22) to a partition function \( Z \) containing short multiplets, then some of the integer coefficients in the resulting generating function \( G \) would come out too small. In the applications considered below we will first identify all terms arising from short multiplets and then explicitly add back the necessary terms to compensate for the missing contributions to the generating function. An entirely analogous operation has been applied to very closely related manipulations on partition functions in \cite{7, 8, 9, 10, 11}. 

\[
7
\]
As an example of how this works in practice, let us consider a theory with many long multiplets and a single short multiplet of type \(24\) with \(J_1 = 1\). In this case the desired generating function \(G(q, a, b)\) may be calculated as

\[
\left\{(1 - 1/a)(1 - 1/b) \left[ Z(q, a, b) - \chi_{[2,1,0]}(q, a, b) \right] \prod_{i=1}^{4} (1 - qx_i) \right\}_{a \geq 0, b \geq 0} + q^2 a. \tag{25}
\]

That is, we first subtract the “offending” short representation \([2, 1, 0]\) from the partition function \(Z\), leaving a partition function with only long multiplets, enabling eq. (22) to be applied. Finally we add back to \(G(q, a, b)\) the term \(+q^2a\) which counts the multiplet \([2, 1, 0]\). A simple calculation then reveals that (25) is equivalent to

\[
\left\{(1 - 1/a)(1 - 1/b)Z(q, a, b) \prod_{i=1}^{4} (1 - qx_i) \right\}_{a \geq 0, b \geq 0} + (\sqrt{ab} - q)q^3. \tag{26}
\]

The conclusion of this analysis is that the result (22) may still be used to calculate the generating function \(G\) for a partition function \(Z\) containing the short multiplet \([2, 1, 0]\) as long as the compensating factor \(+ (\sqrt{ab} - q)q^3\) is added afterward. A similar analysis may be performed to find the necessary “compensating factor” for all other short multiplets.

4 Applications to Yang-Mills Theory on \(S^3\)

The method described in section 3 can be applied to any partition function \(Z(q, a, b)\) with \(SO(2,4)\) symmetry. We will now focus our attention on two particular examples of such theories. The first is pure Yang-Mills theory on \(S^3\) and the second is the maximally supersymmetric \(\mathcal{N} = 4\) Yang-Mills theory on \(S^3\). Both theories will be considered in the planar limit, i.e. with gauge group \(U(N)\) in the limit of infinite \(N\), and with coupling constant \(\lambda = 0\).

4.1 Pure Yang-Mills Theory

As mentioned in the introduction, this theory has a dimensionless coupling constant \(\lambda = \Lambda R\) where \(R\) is the radius of the \(S^3\) and \(\Lambda\) is the dynamically
generated scale. When \( \lambda = 0 \), the partition function of the theory may be calculated exactly \([1]\),

\[
Z(q, a, b) = \text{Tr}[q^H R a J_3 b J'_3] = - \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log [1 - z(q^n, a^n, b^n)].
\] (27)

Here \( H \) is the Hamiltonian of the theory on \( S^3 \), which at \( \lambda = 0 \) may be identified with the dilatation operator on \( \mathbb{R}^4 \), \( J_3 \) and \( J'_3 \) are generators of the two SU(2) symmetry groups, and \( \phi(n) \) is the Euler totient function which counts the number of positive integers less than or equal to \( n \) which are coprime to \( n \). Finally, \( z(q, a, b) \) is given by

\[
z(q, a, b) = 1 + \frac{(q - q^3)(x_1 + x_2 + x_3 + x_4) + q^4 - 1}{1 - qx_1}(1 -qx_2)(1 - qx_3)(1 - qx_4)
\] (28)

in terms of the \( x_i \) defined in eq. (21). Before proceeding we remark that the partition function for the theory with gauge group SU(\( N \)) rather than U(\( N \)) may be obtained trivially by adding \(-z(q, a, b)\) to eq. (27).

After a relatively simple analysis of the possible short multiplets, we find that the desired generating function for pure Yang-Mills theory is

\[
G(q, a, b) = \left\{ (1 - 1/a)(1 - 1/b)Z(q, a, b) \prod_{i=1}^{4}(1-qx_i) \right\}_{a \geq 0, b \geq 0} + C(q, a, b)
\] (29)

where \( Z \) is given in eq. (27) and the compensating factor

\[
C(q, a, b) = \frac{\sqrt{ab}(1 + \sqrt{ab}q + (a^2 + b^2)q^2)q^5}{1 - abq^2} + 2(\sqrt{ab} - q)q^3
\] (30)

corrects for the short multiplets as explained in the previous section. Here the first term accounts for the infinite tower of short multiplets of type (23) while the second term accounts for the two short multiplets (\([2, 1, 0]\) and \([2, 0, 1]\)) of type (24).

Formula (29) is the central result of this paper. The generating function \( G(q, a, b) \) encapsulates the number of times the SO(2,4) representation \([d, j_1, j_2]\) appears in the spectrum of free Yang-Mills theory on \( S^3 \). We can read off the individual \( N_{[d, j_1, j_2]} \) for some of the lowest lying states by performing a series expansion in \( q \). The coefficient of \( q^d \) is, for each integer \( d \), a polynomial \( p_d \) in \( a^{1/2} \) and \( b^{1/2} \). Through energy level \( d = 10 \) we have
calculated the representation content, as shown in Table 1. Our results are in perfect agreement with those of [6]. In contrast to the explicit, but rather complicated formula presented there, we find that the $N_{d,j_1,j_2}$ may be rather simply extracted from eq. (29).

| $d$ | $p_d(a, b)$ |
|-----|-------------|
| 1   | 0           |
| 2   | $a + b$     |
| 3   | 0           |
| 4   | $2 + a^2 + ab + b^2$ |
| 5   | $a^{3/2}b^{3/2}$ |
| 6   | $2 + 2a + a^2 + 2b + 2ab + a^2b + a^3b + ab^2 + a^2b^2 + b^3 + ab^3$ |
| 7   | $4a^{3/2}b^{3/2} + 2a^{5/2}b^{5/2} + 4a^{7/2}b^{7/2} + 4a^{5/2}b^{5/2} + 2a^{5/2}b^{5/2} + 2a^{7/2}b^{7/2} + 2a^{3/2}b^{3/2} + a^{5/2}b^{5/2}$ |
| 8   | $6 + 4a + 5a^2 + a^3 + 2a^4 + 4b + 10ab + 7a^2b + 5a^3b + a^4b + 5b^2 + 7ab^2 + 8a^2b^2 + 3a^3b^2 + a^4b^2 + b^3 + 5ab^3 + 3a^2b^3 + a^3b^3 + 2b^4 + ab^4 + a^2b^4$ |
| 9   | $14a^{3/2}b^{3/2} + 20a^{5/2}b^{5/2} + 15a^{7/2}b^{7/2} + 6a^{7/2}b^{7/2} + 20a^{7/2}b^{7/2} + 28a^{3/2}b^{3/2} + 18a^{5/2}b^{5/2} + 7a^{7/2}b^{7/2} + 2a^{9/2}b^{9/2} + 15a^{1/2}b^{5/2} + 18a^{5/2}b^{5/2} + 7a^{7/2}b^{7/2} + 2a^{9/2}b^{9/2} + 15a^{1/2}b^{5/2} + 18a^{3/2}b^{5/2} + 12a^{5/2}b^{5/2} + 4a^{7/2}b^{7/2} + 6a^{1/2}b^{7/2} + 7a^{3/2}b^{7/2} + 4a^{5/2}b^{7/2} + a^{7/2}b^{7/2} + 2a^{3/2}b^{3/2}$ |

Table 1: The spectrum of pure Yang-Mills theory on $S^3$ at zero coupling for energy levels $d = 1$ through $d = 10$ (in units of the inverse radius of $S^3$). The coefficient of $a^{d_1}b^{d_2}$ in the polynomial $p_d(a, b)$ is the number of times that the SO(2,4) irreducible representation $[d, j_1, j_2]$ appears in the spectrum. These numbers were obtained in [6] by completely dissimilar means.

### 4.2 $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

Maximally supersymmetric Yang-Mills theory is exactly conformal for any value of the coupling constant. The partition function of the free ($\lambda = 0$) supersymmetric theory on $S^3$ is given by the same formula (27) as above,
but with a modified formula for \( z \) which now takes the form

\[
z(q, a, b) = 1 + \frac{(q - q^3)(6 + x_1 + x_2 + x_3 + x_4) + q^4 - 1 + 4q^{3/2}(1 - q)y}{(1 - qx_1)(1 - qx_2)(1 - qx_3)(1 - qx_4)},
\]

where the \( x_i \) are still given by eq. (21) and

\[
y = \sqrt{a} + \frac{1}{\sqrt{a}} + \sqrt{b} + \frac{1}{\sqrt{b}}.
\]

The first \( \mathcal{O}(\lambda) \) correction to the free partition function was calculated in [13]. Although we could apply eq. (22), with an appropriate compensating term to account for short multiplets, the result would be only partially satisfying. The reason is that the SO(2,4) conformal symmetry is only part of the much larger PSU(2,2|4) symmetry group of the supersymmetric theory. A single irreducible representation of PSU(2,2|4) contains numerous irreducible SO(2,4) representations (see [14] as well as the encyclopedic reference [15]), each of which would be counted separately if we were to apply eq. (22). In order to exploit the full symmetry we would like to consider a finer partition function

\[
Z(q, a, b, y_1, y_2, y_3) = \text{Tr}[q^{HR} a^{J_3} b^{J'_3} y_1^{R_1} y_2^{R_2} y_3^{R_3}],
\]

which includes three chemical potentials for the SO(6) R-symmetry, and then find the corresponding generating function

\[
G(q, a, b, y_1, y_2, y_3) = \sum N_{[d, j_1, j_2, s_1, s_2, s_3]} q^d a^{j_1} b^{j_2} y_1^{s_1} y_2^{s_2} y_3^{s_3}
\]

where \( N_{[d, j_1, j_2, s_1, s_2, s_3]} \) counts the number of irreducible representations with quantum numbers \([d, j_1, j_2, s_1, s_2, s_3]\). Although these degeneracies could certainly be computed on a case-by-case basis using the orthogonality of characters, in order to find a simple formula along the lines of eq. (22) it would first be necessary to somehow generalize the analysis of section 2 from SU(2) to SO(6). We leave this intriguing problem open for future work.

Acknowledgments

M. S. is grateful to A. Volovich for helpful comments. The work of M. S. is supported by the US National Science Foundation under grant PHY-0638520 and by the US Department of Energy under contract DE-FG02-91ER40688.
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