A note on exceptional unimodal singularities and K3 surfaces

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Abstract

We study the relation between the graded stable derived categories of 14 exceptional unimodal singularities and the derived categories of K3 surfaces obtained as compactifications of the Milnor fibers. As a corollary, we obtain a basis of the numerical Grothendieck group similar to the one given by Ebeling and Ploog [EP10].

1 Introduction

Let \( f \in \mathbb{C}[x, y, z] \) be a weighted homogeneous polynomial defining one of Arnold’s 14 exceptional unimodal singularities [Arn75]. The list of corresponding weight systems \((a, b, c; h) = \deg(x, y, z; f)\) is given in Table 1.1. The quotient ring \( R = \mathbb{C}[x, y, z]/(f) \) is the homogeneous coordinate ring of a weighted projective line \( \mathbf{X} \) in the sense of Geigle and Lenzing [GL87, Len94], and the Dolgachev number \( \delta = (\delta_1, \delta_2, \delta_3) \) of the singularity is defined as the orders of the isotropy groups of \( \mathbf{X} \).

On the other hand, one can choose a distinguished basis \((\alpha_i)_{i=1}^{\gamma_1+\gamma_2+\gamma_3}\) of vanishing cycles of \( f \) so that the Coxeter–Dynkin diagram is given by the diagram \( \hat{T}(\gamma_1, \gamma_2, \gamma_3) \) shown in Figure 1.1. The triple \( \gamma = (\gamma_1, \gamma_2, \gamma_3) \) defined in this way is called the Gabrielov number of the singularity. The strange duality discovered by Arnold [Arn75] states that the 14 exceptional unimodal singularities come in pairs \((f, \hat{f})\) such that the Dolgachev number of \( f \) is equal to the Gabrielov number of \( \hat{f} \) and vice versa. Pinkham [Pin77] and Dolgachev and Nikulin [Dol83, Nik79] gave an interpretation of the strange duality in terms of algebraic cycles and transcendental cycles of K3 surfaces.

Let \( g(x, y, z, w) \in \mathbb{C}[x, y, z, w] \) be a very general weighted homogeneous polynomial with \( \deg(x, y, z, w; g) = (a, b, c, 1; h) \) and \( S = \mathbb{C}[x, y, z, w]/(g) \) be the quotient ring. The Deligne–Mumford stack

\[
\mathcal{Y} = \text{Proj} S = [(g^{-1}(0) \setminus 0)/\mathbb{C}^\times]
\]

is a compactification of the Milnor fiber of \( f \). Here, the symbol \( \text{Proj} \) is used to indicate that \( \text{Proj} S \) is considered not as a scheme but as a stack.

The stable derived category of \( S \) is defined as the quotient category

\[
D^b_{\text{sing}}(\text{gr} S) = D^b(\text{gr} S)/D^b_{\text{perf}}(\text{gr} S)
\]

doing the bounded derived category \( D^b(\text{gr} S) \) of finitely generated \( \mathbb{Z} \)-graded \( S \)-modules by the full triangulated subcategory \( D^b_{\text{perf}}(\text{gr} S) \) consisting of bounded complexes of projective
| name  | \((a, b, c; h)\)   | \(\delta\) | \(\gamma\) | dual       |
|-------|-------------------|-----------|-----------|------------|
| \(E_{12}\) | \((6, 14, 21; 42)\) | \((2, 3, 7)\) | \((2, 3, 7)\) | \(E_{12}\) |
| \(E_{13}\) | \((4, 10, 15; 30)\) | \((2, 4, 5)\) | \((2, 3, 8)\) | \(Z_{11}\) |
| \(Z_{11}\) | \((6, 8, 15; 30)\) | \((2, 3, 8)\) | \((2, 4, 5)\) | \(E_{13}\) |
| \(E_{14}\) | \((3, 8, 12; 24)\) | \((3, 3, 4)\) | \((2, 3, 9)\) | \(Q_{10}\) |
| \(Q_{10}\) | \((6, 8, 9; 24)\) | \((2, 3, 9)\) | \((3, 3, 4)\) | \(E_{14}\) |
| \(Z_{12}\) | \((4, 6, 11; 22)\) | \((2, 4, 6)\) | \((2, 4, 6)\) | \(Z_{12}\) |
| \(W_{12}\) | \((4, 5, 10; 20)\) | \((2, 5, 5)\) | \((2, 5, 5)\) | \(W_{12}\) |
| \(Z_{13}\) | \((3, 5, 9, 18)\) | \((3, 3, 5)\) | \((2, 4, 7)\) | \(Q_{11}\) |
| \(Q_{11}\) | \((4, 6, 7; 18)\) | \((2, 4, 7)\) | \((3, 3, 5)\) | \(Z_{13}\) |
| \(W_{13}\) | \((3, 4, 8, 16)\) | \((3, 4, 4)\) | \((2, 5, 6)\) | \(S_{11}\) |
| \(S_{11}\) | \((4, 5, 6, 16)\) | \((2, 5, 6)\) | \((3, 4, 4)\) | \(W_{13}\) |
| \(Q_{12}\) | \((3, 5, 6; 15)\) | \((3, 3, 6)\) | \((3, 3, 6)\) | \(Q_{12}\) |
| \(S_{12}\) | \((3, 4, 5; 13)\) | \((3, 4, 5)\) | \((3, 4, 5)\) | \(S_{12}\) |
| \(U_{12}\) | \((3, 4, 4; 12)\) | \((4, 4, 4)\) | \((4, 4, 4)\) | \(U_{12}\) |

Table 1.1: 14 exceptional unimodal singularities

Figure 1.1: The diagram \(\hat{T}(\gamma_1, \gamma_2, \gamma_3)\)
modules \cite{Buc87, Hap01, Kra05, Orl09}. Since $S$ is Gorenstein with parameter zero, one has an equivalence
\[ \Psi_S : D^b_{\text{sing}}(\gr S) \tilde{\to} D^b \coh \sY \]
by Orlov \cite[Theorem 2.5]{Orl09}. The stable derived category of $R = S/(w)$ is defined similarly as $D^b_{\text{sing}}(\gr R) = D^b(\gr R)/D_{\text{perf}}(\gr R)$, and studied by Kajiura, Saito and Takahashi \cite{KST09} and Lenzing and de la Peña \cite{LdlP11}. Since $R$ is Gorenstein with parameter $-1$, one has a fully faithful functor
\[ \Psi_R : D^b \coh \sY_{\infty} \to D^b_{\text{sing}}(\gr R) \]
and a semiorthogonal decomposition
\[ D^b_{\text{sing}}(\gr R) = \langle \Psi_R(D^b \coh \sY_{\infty}), R/\mathfrak{m}_R \rangle \]
by a result of Orlov \cite{Orl09}, where $R/\mathfrak{m}_R$ is the residue field by the maximal ideal $\mathfrak{m}_R = (x, y, z)$ of the origin and $\sY_{\infty} := \Proj R$ is the divisor at infinity.

Let $\Phi_{\gr} : \gr R \to \gr S$ be the functor sending an $R$-module to the same module considered as an $S$-module by the natural projection $\varphi : S \to R$. Since $R$ is perfect as an $S$-module, the functor $\Phi_{\gr}$ sends a perfect complex of $R$-modules to a perfect complex of $S$-modules and induces the push-forward functor
\[ \Phi_{\text{sing}} : D^b_{\text{sing}}(\gr R) \to D^b_{\text{sing}}(\gr S) \]
studied in \cite{DM, PV}. Let further $\iota : \sY_{\infty} \hookrightarrow \sY$ be the inclusion.

**Theorem 1.1.** The composite functor
\[ \Psi_S \circ \Phi_{\text{sing}} \circ \Psi_R : D^b \coh \sY_{\infty} \to D^b \coh \sY \]
is isomorphic to the push-forward functor
\[ \iota_* : D^b \coh \sY_{\infty} \to D^b \coh \sY, \]
and the image of the residue field $R/\mathfrak{m}_R$ in $D^b_{\text{sing}}(\gr R)$ by $\Psi_S \circ \Phi_{\text{sing}}$ is isomorphic to the structure sheaf $\mathcal{O}_\sY[2]$ shifted by 2.

Let $Y$ be the minimal resolution of the coarse moduli space of $\sY$. The McKay correspondence as a derived equivalence \cite{KV00, BKR01} gives
\[ \Upsilon : D^b \coh \sY \tilde{\to} D^b \coh Y. \quad (1.1) \]
Recall that the numerical Grothendieck group $\mathcal{N}(Y)$ is the quotient of the Grothendieck group $K(Y)$ of $Y$ by the radical of the Euler form
\[ \chi ([\mathcal{E}], [\mathcal{F}]) = \sum_i (-1)^i \dim \Ext^i(\mathcal{E}, \mathcal{F}). \]
The integral cohomology ring
\[ H^\bullet(Y, \mathbb{Z}) = H^0(Y, \mathbb{Z}) \oplus H^2(Y, \mathbb{Z}) \oplus H^4(Y, \mathbb{Z}) \]
equipped with the Mukai pairing

\[(a_0, a_2, a_4), (b_0, b_2, b_4) = (a_2, b_2) - (a_0, b_4) - (a_4, b_0)\]

is called the Mukai lattice. For a coherent sheaf \(E\), its Mukai vector is defined by

\[v(E) = ch(E) \sqrt{td(Y)}.\]

Riemann–Roch theorem states that

\[\chi(E, F) = -(v(E), v(F)),\]

so that \(\mathcal{N}(Y)\) can be identified with the image of \(K(Y)\) in the Mukai lattice \(H^*(Y, \mathbb{Z})\) under the map \(v : K(Y) \rightarrow H^*(Y, \mathbb{Z})\).

**Corollary 1.2.** There is a full exceptional collection \((S_\alpha)_{\alpha=0}^{\delta_1+\delta_2+\delta_3-1}\) in \(D^{\text{b}}_{\text{sing}}(\text{gr}\; R)\) the images \(E_\alpha = Y \circ \Psi_\alpha \circ \Phi_{\text{sing}}(S_\alpha)\) of which satisfy the following:

- The endomorphism dg algebra of \(\bigoplus_{\alpha=1}^{\delta_1+\delta_2+\delta_3-1} E_\alpha\) is the trivial extension of the endomorphism dg algebra of \(\bigoplus_{\alpha=1}^{\delta_1+\delta_2+\delta_3-1} S_\alpha\).
- The sequence \((E_\alpha)_{\alpha=0}^{\delta_1+\delta_2+\delta_3-1}\) is a spherical collection.
- The Coxeter–Dynkin diagram of the spherical collection is \(\widehat{T}(\delta_1, \delta_2, \delta_3)\).
- The spherical collection is a basis of the numerical Grothendieck group \(\mathcal{N}(Y)\).
- The spherical collection split-generates \(D^b\text{coh}\; Y\).

Recall that an object \(E\) is said to be spherical if \(\text{Hom}^i(E, E)\) is isomorphic to \(\mathbb{C}\) for \(i = 0, 2\) and zero otherwise [ST01 Definition 1.1]. A sequence of objects is called a spherical collection if each object is spherical. The definition of the trivial extension can be found in [Sei10], which is called the cyclic completion in [Seg08]. The endomorphism dg algebra of \(\bigoplus_{\alpha=0}^{\delta_1+\delta_2+\delta_3-1} E_\alpha\) is not the trivial extension of the endomorphism dg algebra of \(\bigoplus_{\alpha=0}^{\delta_1+\delta_2+\delta_3-1} S_\alpha\); otherwise, the derived category \(D^b\text{coh}\; Y\) will not depend on the defining equation of \(Y\). The spherical collection \((E_\alpha)_{\alpha=0}^{\delta_1+\delta_2+\delta_3-1}\) has the same properties as the collection given by Ebeling and Ploog [EP10].

Let \((Y, \bar{Y})\) be a pair of K3 surfaces obtained as compactifications of Milnor fibers of a dual pair of exceptional unimodal singularities. According to [Kob08 Theorem 4.3.9], such a pair can be realized as smooth anticanonical hypersurfaces in a pair \((X, \bar{X})\) of toric weak Fano manifolds associated with a polar dual pair \((\Delta, \Delta^*)\) of reflexive polytopes.

The \(A\)-model VHS \((H_{A, Z}, \nabla^A, \mathcal{F}_A, Q_A)\) associated with \(Y\) is an integral variation of pure and polarized Hodge structures of weight 2 in a neighborhood

\[U = \{ \beta + \sqrt{-1} \omega \in \text{NS}(Y) \otimes \mathbb{C} \mid \langle \omega, d \rangle \gg 0 \text{ for any non-zero } d \in \text{Eff}(Y) \}\]

of the large radius limit in the complexified Kähler moduli space of \(Y\). Here \(\text{NS}(Y) \subset \mathcal{N}(Y)\) is the Néron–Severi group of \(Y\), \(\text{Eff}(Y) \subset \mathcal{N}(Y)\) is the semigroup of effective curves on \(Y\), \(H_{A, Z}\) is the trivial local system on \(U\) with fiber \(\mathcal{N}(Y)\) and \(\nabla^A = d\) is the associated
trivial flat connection on $\mathcal{H}_A = H_{A,Z} \otimes \mathcal{O}_U$. The polarization $Q_A$ is given by the Mukai pairing, and the Hodge filtration is such that

$$\mathcal{U} = \exp(\beta + \sqrt{-1}\omega) = \left(1, (\beta + \sqrt{-1}\omega), \frac{1}{2}(\beta + \sqrt{-1}\omega)^2\right)$$

spans the $(2,0)$-part of $H_{A,C} = H_{A,Z} \otimes \mathbb{C}$.

The K3 surface $\tilde{Y}$ comes in a family $\tilde{\varphi} : \tilde{Y} \to \tilde{M}$ where $\tilde{M}$ is an algebraic torus of the same dimension as $U$. Let $U'$ be a neighborhood of the large complex structure limit in $\tilde{M}$ and $\tilde{U}$ be its universal cover. The local system $R^2\tilde{\varphi}_!\mathbb{Z}_2$ carries an integral variation of polarized mixed Hodge structures, and the $B$-model VHS $H_{B,Z}$ is defined as the pull-back to $\tilde{U}$ of the graded subquotient $gr^B R^2\tilde{\varphi}_!\mathbb{Z}_2$ of weight 2. There is a biholomorphic map $\varsigma : \tilde{U} \to U$ called the mirror map. Iritani has introduced certain subsystems $H_{A,Z}^{\text{amb}} \subset H_{A,Z}$ and $H_{B,Z}^{\text{vc}} \subset H_{B,Z}$ and given an isomorphism

$$\text{Mir}_Y : \varsigma^*(H_{A,Z}^{\text{amb}}, \nabla^A, \mathcal{F}_A, Q_A) \sim (H_{B,Z}^{\text{vc}}, \nabla^B, \mathcal{F}_B, Q_B)$$

of integral variations of pure and polarized Hodge structures [Iri, Theorem 6.9].

**Corollary 1.3.** One has $H_{A,Z}^{\text{amb}} = H_{A,Z}$ and $H_{B,Z}^{\text{vc}} = H_{B,Z}$, so that the isomorphism (1.2) gives an isomorphism

$$\text{Mir}_Y : \varsigma^*(H_{A,Z}, \nabla^A, \mathcal{F}_A, Q_A) \sim (H_{B,Z}, \nabla^B, \mathcal{F}_B, Q_B).$$

of integral variations of pure and polarized Hodge structures.

The equalities $H_{A,Z}^{\text{amb}} = H_{A,Z}$ and $H_{B,Z}^{\text{vc}} = H_{B,Z}$ fail in general for a K3 hypersurface in a smooth toric weak Fano variety. A typical counter-example is the case when $Y$ is the quartic surface in the projective space, where the class of a point does not belong to $H_{A,Z}^{\text{amb}}$ and only four times the class of a point does (cf. [Iri] Section 6.6). Hodge-theoretic mirror symmetry for the quartic surface is studied in detail by Hartmann [Har].

The organization of this paper is as follows: We prove Theorem 2.2 in Section 2, which is slightly more general than Theorem 1.1. In Section 3 we use an exceptional collection given by Lenzing and de la Pená [LdlP11] to prove Corollary 1.2. Variations of Hodge structures is discussed in Section 4.

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## 2 Push-forward in stable derived categories

Let $k$ be a field and $A = \bigoplus_{i \geq 0} A_i$ be a Noetherian graded $k$-algebra. We assume that $A$ is connected in the sense that $A_0 = k$, and write the maximal ideal as $m_A = \bigoplus_{i \geq 1} A_i$. The graded ring $A$ is said to be Gorenstein if $A$ has a finite injective dimension $n$ and

$$\mathbb{R}\text{Hom}_A(k, A) = k(a)[-n]$$
for some integer $a$, which is called the Gorenstein parameter of $A$. If $A = k[x_1, \ldots, x_n]/(f)$ for $\deg(x_1, \ldots, x_n; f) = (a_1, \ldots, a_n; h)$, then $A$ is Gorenstein with parameter $a = a_1 + \cdots + a_n - h$.

Let $\gr A$ be the abelian category of finitely generated $\mathbb{Z}$-graded right $A$-modules, and $\tor A$ be the full subcategory consisting of graded modules which are finite-dimensional over $k$. The quotient category $\gr A/\tor A$ will be denoted by $\qgr A$, which is equivalent to the abelian category of coherent sheaves on the quotient stack $\Proj A = [(\Spec A \setminus 0)/\mathbb{G}_m]$ by Serre’s theorem [Orl09, Proposition 2.16].

Let $D^b(\gr A)$ be the bounded derived category of $\gr A$. An object of $D^b(\gr A)$ is said to be perfect if it is quasi-isomorphic to a bounded complex of projective modules. The full subcategory of $D^b(\gr A)$ consisting of perfect complexes will be denoted by $D^\perf(\gr A)$. The quotient category

$$D^b_{\sing}(\gr A) = D^b(\gr A)/D^\perf(\gr A)$$

is called the bounded stable derived category of $\gr A$ [Buc87, Hap91, Kra05, Orl04].

Let $\mathcal{D}$ be a $k$-linear triangulated category and $\mathcal{N} \subset \mathcal{D}$ be a full triangulated subcategory. The right orthogonal to $\mathcal{N}$ is the full subcategory $\mathcal{N}^\perp \subset \mathcal{D}$ consisting of objects $M$ satisfying $\Hom(\mathcal{N}, M) = 0$ for any $N \in \mathcal{N}$. The left orthogonal $\perp \mathcal{N}$ is defined similarly. The subcategory $\mathcal{N}$ is said to be right admissible if the embedding $I : \mathcal{N} \hookrightarrow \mathcal{D}$ has a right adjoint functor $Q : \mathcal{D} \rightarrow \mathcal{N}$. Left admissibility is defined similarly as the existence of a left adjoint functor, and $\mathcal{N}$ is said to be admissible if it is both right and left admissible. A subcategory $\mathcal{N}$ is right admissible if and only if for any $X \in \mathcal{D}$, there exists a distinguished triangle $N \rightarrow X \rightarrow M \rightarrow N[1]$ with $N \in \mathcal{N}$ and $M \in \mathcal{N}^\perp$. Such a triangle is unique up to isomorphism, and one has $Q(X) = N$ in this case. If $\mathcal{N}$ is right admissible, then the quotient category $\mathcal{D}/\mathcal{N}$ is equivalent to $\mathcal{N}^\perp$. Analogous statements also hold for left admissible subcategories. A sequence $(\mathcal{N}_1, \ldots, \mathcal{N}_n)$ of triangulated subcategories in a triangulated category $\mathcal{D}$ is called a weak semiorthogonal decomposition if there is a sequence $\mathcal{D}_1 = \mathcal{N}_1 \subset \mathcal{D}_2 \subset \cdots \subset \mathcal{D}_n = \mathcal{D}$ of left admissible subcategories such that $\mathcal{N}_i$ is left orthogonal to $\mathcal{D}_{i−1}$. A weak semiorthogonal decomposition will be denoted by

$$\mathcal{D} = \langle \mathcal{N}_1, \ldots, \mathcal{N}_n \rangle.$$

An object $E$ of $\mathcal{D}$ is exceptional if $\Ext^i(E, E) = 0$ for $i \neq 0$ and $\Hom(E, E)$ is spanned by the identity morphism. An exceptional collection is a sequence $(E_1, \ldots, E_n)$ of exceptional objects such that $\Ext^i(E_j, E_l) = 0$ for any $i$ and any $1 \leq \ell < j \leq n$. A full triangulated subcategory generated by an exceptional collection is always admissible [Bon89, Theorem 3.2].

For an integer $i$, let $\gr A_{\geq i}$ be the full abelian subcategory of $\gr A$ consisting of graded modules $M$ such that $M_e = 0$ for $e < i$. Let further $\mathcal{S}_{\geq i}$ and $\mathcal{P}_{\geq i}$ be the full triangulated subcategories of $D^b(\gr A)$ generated by graded torsion modules $A/m_A(e)$ for $e \leq −i$ and graded free modules $A(e)$ for $e \leq −i$ respectively. By [Orl09, Lemma 2.4], the subcategories $\mathcal{S}_{\geq i}$ and $\mathcal{P}_{\geq i}$ are right and left admissible, respectively, in $D^b(\gr A_{\geq i})$, and let $\mathcal{D}^A_i$ and $\mathcal{T}^A_i$ be their right and left orthogonal subcategories. It follows that one has weak semiorthogonal decompositions

$$D^b(\gr A_{\geq i}) = \langle \mathcal{D}^A_i, \mathcal{S}^A_{\geq i} \rangle,$$

$$D^b(\gr A_{\geq i}) = \langle \mathcal{P}^A_{\geq i}, \mathcal{T}^A_i \rangle,$$

(2.1) (2.2)
where $D^A_t$ is equivalent to the quotient category $D^b(\text{gr } A_{\geq i})/S^A_i$ which in turn is equivalent to $D^b(qgr A)$, and $T^A_i$ is equivalent to the quotient category $D^b(\text{gr } A_{\geq i})/P^A_{\geq i}$ which in turn is equivalent to $D^b_{\text{sing }}(\text{gr } A)$. In addition, one has a semiorthogonal decomposition

$$T^A_0 = \langle A/m_A, A/m_A(-1), \ldots, A/m_A(a+1), D^A_{-a} \rangle$$

(2.3)

if $a \leq 0$ by Orlov [Orl09, Equation (12)].

The semiorthogonal decomposition (2.3) can be rephrased as

$$T^A_0 = \langle D^A_0, A/m_A, A/m_A(-1), \ldots, A/m_A(a+1) \rangle.$$  

(2.4)

Indeed, one has the semiorthogonal decomposition

$$D^b(\text{gr } A) = \langle S^A_{\leq i}, D^b(\text{gr } A_{\geq i}) \rangle$$

for any $i \in \mathbb{Z}$ by [Orl09, Equation (7)], which gives

$$D^b(\text{gr } A_{\geq 0}) = \langle A/m_A, \ldots, A/m_A(a+1), D^b(\text{gr } A_{\geq -a}) \rangle.$$  

On the other hand, one has

$$D^b(\text{gr } A_{\geq -a}) = \langle D^A_{-a}, S^A_{-a} \rangle$$

by (2.1), so that

$$D^b(\text{gr } A_{\geq 0}) = \langle A/m_A, \ldots, A/m_A(a+1), D^A_{-a}, S^A_{-a} \rangle.$$  

(2.5)

By comparing

$$D^b(\text{gr } A_{\geq 0}) = \langle D^A_0, S^A_{\geq 0} \rangle = \langle D^A_0, A/m_A, \ldots, A/m_A(a+1), S^A_{\geq -a} \rangle$$

with (2.3) and (2.5), one obtains (2.4).

Let $\varphi : S \to R$ be a morphism of graded connected Gorenstein rings, and $\Phi_{\text{gr }} : \text{gr } R \to \text{gr } S$ be the exact functor which sends an $R$-module to the same module considered as an $S$-module via $\varphi$. The functor $\Phi_{\text{gr }}$ sends finite-dimensional $R$-modules to finite-dimensional $S$-modules, and induces an exact functor $\Phi_{\text{qgr }} : qgr R \to qgr S$.

If $R$ has finite projective dimension as an $S$-module, then $\Phi_{\text{gr }}$ sends perfect complexes of $R$-modules to perfect complexes of $S$-modules, and induces a functor $\Phi_{\text{sing }} : D^b_{\text{sing }}(\text{gr } R) \to D^b_{\text{sing }}(\text{gr } S)$ of stable derived categories.

Now assume that $S$ is a graded connected Gorenstein ring with parameter $a_S$, and $w \in S_d$ is a homogeneous element of degree $d > 0$ which is not a zero divisor. Then the exact sequence

$$0 \to S(-d) \overset{w}{\to} S \to R \to 0$$

is a locally free resolution of the quotient ring $R = S/(w)$ as a graded $S$-module, and $R$ is a graded connected Gorenstein ring with parameter $a_R = a_S - d$ (see e.g. [GW78, Proposition 2.2.10]). We write the natural projection as $\varphi : S \to R$, which induces the functor $\Phi_{\text{gr }} : \text{gr } R \to \text{gr } S$ as above. In this case, we have the following:

**Lemma 2.1.** If $X \in D^b(\text{gr } R)$ satisfies $\mathbb{R}\text{Hom}_R(R/m_R(i), X) = 0$ for any $i \leq 0$, then $\Phi_{\text{gr }}(X) \in D^b(\text{gr } S)$ satisfies $\mathbb{R}\text{Hom}_S(S/m_S(i), \Phi_{\text{gr }}(X)) = 0$ for any $i \leq 0$. 

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The quotient category \( \mathcal{D} \)

\textbf{Theorem 2.2.} Let \( S \) be a graded connected Gorenstein ring with Gorenstein parameter \( a_S = 0 \), and \( R = S/(w) \) be the quotient ring defined by a homogeneous element \( w \in S_1 \) of degree one which is not a zero divisor. Then the composite functor

\[ \Phi \]

\[ \mathcal{O} \]

\[ \text{is isomorphic to the functor } \Phi_{\text{qgr}} : D^b qgr R \to D^b qgr S, \text{ and the image of } R/m_R \in \mathcal{T}_0^R \text{ by} \]

\[ \mathcal{T}_0^R \sim \mathcal{D}_{\text{sing}}^b (gr R) \xrightarrow{\Phi_{\text{qgr}}} \mathcal{D}_{\text{sing}}^b (gr S) \sim \mathcal{T}_0^S = \mathcal{D}_0^S \sim D^b (qgr S) \]

is isomorphic to \( \mathcal{O}[\dim S - 1] \), where \( \mathcal{O} \) is the image of the free module \( S \in gr S \) by the projection \( gr S \to qgr S \) and \( [\dim S - 1] \) is the shift in the derived category.

\textbf{Proof.} The equivalence \( D^b (qgr R) \sim \mathcal{D}_0^R \) is inverse to the composition

\[ \mathcal{D}_0^R \to \langle \mathcal{D}_0^R, S^R_{\geq 0} \rangle = D^b (gr R_{\geq 0}) \to D^b (gr R_{\geq 0})/S^R_{\geq 0} \cong D^b (gr R)/D^b (tor R) = D^b (qgr R), \]

and the equivalence \( \mathcal{D}_0^S \sim D^b (qgr S) \) is defined similarly. Let \( \Phi_\mathcal{D} : \mathcal{D}_0^R \to \mathcal{D}_0^S \) be the functor defined as the composition

\[ \mathcal{D}_0^R \to D^b (gr R_{\geq 0}) \xrightarrow{\Phi_{\text{qgr}}} D^b (gr S_{\geq 0}) \to \mathcal{D}_0^S, \]

where the last arrow is the left adjoint functor to the embedding

\[ \mathcal{D}_0^S \hookrightarrow D^b (gr S_{\geq 0}) = \langle \mathcal{D}_0^S, S^S_{\geq 0} \rangle. \]

We show that the following diagram commutes up to natural isomorphism:

\[
\begin{align*}
\begin{array}{c}
D^b (qgr R) \\
\downarrow \cong \\
\mathcal{D}_0^R
\end{array}
\xrightarrow{\Phi_{\text{qgr}}} 
\begin{array}{c}
D^b (qgr S) \\
\uparrow \cong \\
\mathcal{D}_0^S
\end{array}
\end{align*}
\]

(2.6)

The quotient category \( D^b (qgr R) \) has the same set of objects as \( D^b (gr R) \), and only morphisms are different. The functor \( \Phi_{\text{qgr}} \) sends an object \( X \in D^b (qgr R) \), which is a complex of \( R \)-modules, to the same complex, considered as an object of \( D^b (qgr S) \). The equivalence \( D^b (gr R_{\geq 0})/S^R_{\geq 0} \cong D^b (gr R)/D^b (tor R) \) allows one
to assume that $X$ is an object of $D^b(\text{gr } R_{\geq 0})$. The functor $D^b(\text{qgr } R) \xrightarrow{\sim} D_0^R$ sends the object $X$ to an object $M \in D_0^R$ which fits in a distinguished triangle

$$N \to X \to M \to N[1] \quad (2.7)$$

with $N \in S_{\geq 0}^R$. The image $\Phi_D(M) \in D_0^S$ fits in the distinguished triangle

$$N' \to M \to \Phi_D(M) \to N'[1] \quad (2.8)$$

where $M$ is considered as an object of $D^b(\text{gr } S_{\geq 0})$ and $N'$ belong to $D^b(\text{gr } S_{\geq 0})$ as a complex of $S$-modules, both $N$ and $N'$ are isomorphic to the zero object in $D^b(\text{qgr } S)$ and exact sequences (2.7) and (2.8) give the isomorphisms $X \xrightarrow{\sim} M \xrightarrow{\sim} \Phi_D(M)$ in $D^b(\text{qgr } S)$. This isomorphism is natural since all constructions above are functorial, and the commutativity of (2.6) is proved.

One can similarly define the functor $\Phi_T : T_0^R \to T_0^S$ as the composition

$$T_0^R \hookrightarrow \langle D_{\text{gr } R_{\geq 0}}, T_0^R \rangle = D^b(\text{gr } R_{\geq 0}) \xrightarrow{\phi_{\text{gr}}} D^b(\text{gr } S_{\geq 0}) \to T_0^S$$

and prove that the following diagram also commutes up to natural isomorphism:

$$\begin{array}{ccc}
D^b_{\text{sing }}(\text{gr } R) & \xrightarrow{\phi_{\text{sing}}} & D^b_{\text{sing }}(\text{gr } S) \\
\downarrow \phi_T & & \downarrow \phi_T \\
T_0^R & \xrightarrow{\Phi_T} & T_0^S
\end{array} \quad (2.9)$$

To prove the first statement in Theorem 2.2, it remains to see that the following diagram commutes, where $I : D_0^R \hookrightarrow T_0^R$ is the embedding coming from the semiorthogonal decomposition in (2.4):

$$\begin{array}{ccc}
D_0^R & \xrightarrow{I} & T_0^R \\
\downarrow \phi_T & & \downarrow \phi_T \\
D_0^S & = & T_0^S
\end{array} \quad (2.10)$$

This comes from the fact that for any object $X \in D_0^R \subset D^b(\text{gr } R_{\geq 0})$, the object $\Phi_{\text{gr }}(X) \in D^b(\text{gr } S_{\geq 0})$ is right orthogonal to $S_{\geq 0}^S$ by Lemma 2.1, and hence belongs to $D_0^S = T_0^S$;

$$\Phi_D(X) = \Phi_T \circ I(X) = \Phi_{\text{gr }}(X).$$

For the second statement, note that the image of $R/\mathfrak{m}_R \in T_0^R$ by the composition

$$T_0^R \xrightarrow{\sim} D^b_{\text{sing }}(\text{gr } R) \xrightarrow{\phi_{\text{sing}}} D^b_{\text{sing }}(\text{gr } S)$$

is $S/\mathfrak{m}_S$. Its image by the equivalence

$$D^b_{\text{sing }}(\text{gr } S) \simeq D^b(\text{gr } S_{\geq 0}) / \mathcal{P}_{\geq 0}^S \xrightarrow{\sim} T_0^S$$

is the object $N \in T_0^S$ which fits in the distinguished triangle

$$N \to S/\mathfrak{m}_S \to M \to N[1]$$
with \( M \in \mathcal{P}^S_{\geq 0} \). Since \( S \) is Gorenstein with parameter zero, one has

\[
\text{Hom}(S/m_S, S(i)) = \begin{cases} 
  k[-\dim S] & i = 0, \\
  0 & \text{otherwise}, 
\end{cases}
\]

which shows that the cone \( N = \text{Cone}(S/m_S[-1] \to S[\dim S - 1]) \) belongs to \( T^S_0 \) and satisfies the desired property with \( M = S[\dim S] \). It is clear that \( S/m_S[-1] \in \mathcal{D}^b(\text{gr} A_{\geq 0}) \) goes to \( 0 \in \mathcal{D}^b(\text{gr} A) \) and \( S[\dim S - 1] \in \mathcal{D}^b(\text{gr} A_{\geq 0}) \) goes to \( \mathcal{O}[\dim S - 1] \in \mathcal{D}^b(\text{gr} A) \), so that \( N \in T^S_0 \subseteq \mathcal{D}^S_0 \subseteq \mathcal{D}^b(\text{gr} A_{\geq 0}) \) goes to \( \mathcal{O}[\dim S - 1] \in \mathcal{D}^b(\text{gr} A) \), and Theorem 2.2 it proved.

### 3 Spherical collections on K3 surfaces

Let \( X \) be the weighted projective line with weight \( p = (p_1, p_2, p_3) \) in the sense of Geigle and Lenzing [GL87]. The abelian category \( \text{coh} X \) of coherent sheaves on \( X \) is equivalent by Serre’s theorem [GL87, Section 1.8] to the quotient category \( \text{gr} T/\text{tor} T \) of the abelian category \( \text{gr} T \) of finitely generated \( L \)-graded \( T \)-modules by the full subcategory \( \text{tor} T \) consisting of torsion modules. Here \( L \) is the abelian group of rank one generated by four elements \( \vec{x}_1, \vec{x}_2, \vec{x}_3, \) and \( \vec{c} \) with relations \( p_1 \vec{x}_1 = p_2 \vec{x}_2 = p_3 \vec{x}_3 = \vec{c} \), and \( T = \mathbb{C}[x_1, x_2, x_3]/(x_1^{p_1} + x_2^{p_2} + x_3^{p_3}) \) is an \( L \)-graded ring of Krull dimension two. Let

\[
(\mathcal{P}_a)_{a=1}^{p_1+p_2+p_3-1} = (\mathcal{O}, U_1^{(1)}, \ldots, U_1^{(p_1-1)}, U_2^{(p_2-1)}, U_3^{(p_3-1)}, \mathcal{O}(-\vec{c} - \vec{c}[1]))
\]

be the full strong exceptional collection given by Lenzing and de la Penâ [LdlP11, Proposition 3.9], where \( U_1^{(j)} \) are defined by

\[
U_i^{(j)} = \text{coker}(\mathcal{O}(-(p_i - 1)\vec{x}_i) \hookrightarrow \mathcal{O}((-p_i + 1 + j)\vec{x}_i)),
\]

and \( \vec{c} = \vec{c} - \vec{x}_1 - \vec{x}_2 - \vec{x}_3 \in L \) is the dualizing element [GL87, Theorem 2.2]. Let \((S_\beta)_{\beta=1}^{p_1+p_2+p_3-1}\) be the right dual collection to \((\mathcal{P}_a)_{a=1}^{p_1+p_2+p_3-1}\), which is characterized by the property

\[
\dim \text{Hom}(\mathcal{P}_a, S_\beta) = \delta_{a,p_1+p_2+p_3-\beta},
\]

and given explicitly as

\[
(S_\beta)_{\beta=1}^{p_1+p_2+p_3-1} = (\mathcal{O}(-\vec{c}[2], \mathcal{O}(-\vec{x}_1)[1], S_1^{p_1-2}, \ldots, S_1^{(1)}, \ldots, \mathcal{O}(-\vec{x}_3)[1], S_3^{p_3-2}, \ldots, S_3^{(1)}, \mathcal{O}),
\]

where

\[
S_i^{(j)} = \text{coker}(\mathcal{O}(-(p_i - j - 2)\vec{x}_i) \hookrightarrow \mathcal{O}(-(p_i - j - 1)\vec{x}_i)).
\]

The total morphism algebra of the collection \((\mathcal{P}_a)_{a=1}^{p_1+p_2+p_3-1}\) is isomorphic to the path algebra of the quiver shown in Figure 3.1, where two dotted arrows represent two relations. In terms of quiver representations, \( \mathcal{P}_a \) are projective modules and \( S_\alpha \) are simple modules, and one has

\[
\dim \text{Hom}'(S_\alpha, S_\beta) = \begin{cases} 
  \delta_{a,\beta} & i = 0, \\
  \#(\text{solid arrows from } \beta \text{ to } \alpha) & i = 1, \\
  \#(\text{dotted arrows from } \beta \text{ to } \alpha) & i = 2.
\end{cases}
\]
Let $\mathcal{K}$ be the total space of the canonical bundle of $X$. Since the collection $(\mathcal{S}_\alpha)_{\alpha=1}^{p_1+p_2+p_3-1}$ is full, the push-forward $(\iota_*\mathcal{S}_\alpha)_{\alpha=1}^{p_1+p_2+p_3-1}$ generates the derived category $D^b\text{coh}_X\mathcal{K}$ of coherent sheaves on $\mathcal{K}$ supported on the image of the zero section $\iota : X \to \mathcal{K}$.

**Theorem 3.1** (Segal [Seg08, Theorem 4.2], Ballard [Bal, Proposition 4.14]). Let $\mathcal{S}$ be an object of $D^b\text{coh}X$ and $\iota_*\mathcal{S}$ be the push-forward of $\mathcal{S}$ along the zero-section. Then the endomorphism dg algebra of $\iota_*\mathcal{S}$ is the trivial extension of the endomorphism dg algebra of $\mathcal{S}$.

It follows that

$$\text{Hom}^i(\iota_*\mathcal{S}_\alpha, \iota_*\mathcal{S}_\beta) = \text{Hom}^i(\mathcal{S}_\alpha, \mathcal{S}_\beta) \oplus \text{Hom}^{2-i}(\mathcal{S}_\beta, \mathcal{S}_\alpha)^\vee,$$

so that

$$
\chi(\iota_*\mathcal{S}_\alpha, \iota_*\mathcal{S}_\beta) = \begin{cases}
2 & \text{if } \alpha = \beta, \\
-1 & \text{if } \alpha \text{ and } \beta \text{ are connected by a solid arrow}, \\
2 & \text{if } \alpha \text{ and } \beta \text{ are connected by two dotted arrows}, \\
0 & \text{otherwise}
\end{cases}
$$

(3.1)

for $1 \leq \alpha, \beta \leq p_1 + p_2 + p_3 - 1$.

Let $\mathcal{Y}$ be a very general hypersurface of degree $h$ in $\mathbb{P}(a, b, c, 1)$, where $(a, b, c; h)$ is a weight system in Table 1.1. The divisor $\mathcal{Y}_\infty = \{w = 0\} \subset \mathcal{Y}$ at infinity is a weighted projective line whose weight is given by the Dolgachev number of the singularity; $(p_1, p_2, p_3) = (\delta_1, \delta_2, \delta_3)$. Note that the formal neighborhood of $\mathcal{Y}_\infty$ in $\mathcal{Y}$ at infinity is a weighted projective line whose weight is given by the Dologachev number of the singularity; $(p_1, p_2, p_3) = (\delta_1, \delta_2, \delta_3)$. It suffices (see e.g. [CM03, Theorem 1.6]) to show $H^1(\mathcal{T}_X \otimes (\mathcal{N}^\vee_{X/K})^\nu) = 0$ and $H^1((\mathcal{N}^\vee_{X/K})^\nu) = 0$ for any $\nu \geq 1$, which easily follows from the fact that both the tangent sheaf $\mathcal{T}_X$ and the conormal sheaf $\mathcal{N}^\vee_{X/K}$ are isomorphic to $\mathcal{O}_X(-\vec{\omega})$. We fix such an isomorphism, which induces an equivalence

$$D^b\text{coh}_X\mathcal{K} \cong D^b\text{coh}_{\mathcal{Y}_\infty}\mathcal{Y}
$$

(3.2)
of triangulated categories. Since
\[ \text{Hom}^i(\mathcal{O}_Y, \iota_* S_\alpha) \cong H^*(\iota_* S_\alpha) \cong H^*(S_\alpha) \cong \text{Hom}^* (S_{p_1+p_2+p_3-1}, S_\alpha), \]
one has
\[ \dim \text{Hom}^i(O_Y[1], \iota_* S_\alpha) = \delta_{i1} \delta_{\alpha, p_1+p_2+p_3-1}, \quad (3.3) \]
so that the Euler form on the spherical collection \((O_Y[1], \iota_* S_1, \ldots, \iota_* S_{p_1+p_2+p_3-1})\) is identical to the spherical collection in Figure 3.3 given by Ebeling and Ploog [EP10].

**Lemma 3.2.** The spherical collection
\[ (O_Y, \iota_* S_1, \ldots, \iota_* S_{p_1+p_2+p_3-1}) \]
split-generates \(D^b \text{coh} Y\).

**Proof.** The line bundle \(O_Y(-kY_\infty)\) is contained in the full triangulated subcategory of \(D^b \text{coh} Y\) generated by the above spherical collection for any \(k \in \mathbb{N}\), since the cokernel of the inclusion \(O_Y(-kY_\infty) \hookrightarrow O_Y\) is supported on \(Y_\infty\) and hence contained in \(\text{coh}_{Y_\infty} Y\). For any coherent sheaf \(E\), there is a surjection
\[ \varphi_0 : O_Y(-n_0Y_\infty)^{\oplus k_0} \rightarrow E \]
for sufficiently large \(n_0\) and \(k_0\) (i.e. the hyperplane section \(Y_\infty\) is ample). Let \(E_1 = \ker \varphi_0\) be the kernel of this morphism. Then there is a surjection
\[ \varphi_1 : O_Y(-n_1Y_\infty)^{\oplus k_1} \rightarrow E_1 \]
for sufficiently large \(n_1\) and \(k_1\), and one can set \(E_2 = \ker \varphi_1\). By repeating this process, one obtains a distinguished triangle
\[ E_{k+1}[k] \rightarrow F \rightarrow E \xrightarrow{[+1]} E_{k+1}[k + 1], \]
where \(E_{k+1}\) is a coherent sheaf and
\[ F = \left\{ O_Y(-n_kY_\infty)^{\oplus m_k} \xrightarrow{\varphi_k} O_Y(-n_{k-1}Y_\infty)^{\oplus m_{k-1}} \xrightarrow{\varphi_{k-1}} \cdots \xrightarrow{\varphi_0} O_Y(-n_0Y_\infty)^{\oplus k_0} \right\} \]
for any \(k \geq 0\). Since \(Y\) is smooth, the homological dimension of \(\text{coh} Y\) is equal to the dimension of \(Y\), and this triangle splits for \(k > \dim Y\). It follows that any coherent sheaf is a direct summand of a complex of locally free sheaves contained in the full triangulated subcategory of \(D^b \text{coh} Y\) generated by \((S_\beta)^{p_1+p_2+p_3-1}\), and Lemma 3.2 is proved. \(\square\)

Let \(Y\) be the minimal resolution of the coarse moduli space of \(Y\). It can be realized as an anticanonical K3 hypersurface in a toric weak Fano manifold \(X\) [Ko08]. It contains the Milnor fiber as an open subset, and the complement consists of chains of \((-2)\)-curves intersecting as in Figure 3.2. It follows that the transcendental lattice of \(Y\) is isomorphic to the Milnor lattice of \(Y\). By the McKay correspondence as a derived equivalence [KV00, BKR01], one has an equivalence
\[ \Upsilon : D^b \text{coh} Y \xrightarrow{\sim} D^b \text{coh} Y \quad (3.4) \]
of triangulated categories. Set \(E_0 = O_Y[1]\) and \(E_\alpha = \Upsilon \circ \iota_* (S_\alpha)\) for \(\alpha = 1, \ldots, p_1+p_2+p_3-1\).
The numerical Grothendieck group $\mathcal{N}(Y)$ is spanned by $\bigl([E_{\alpha}]\bigr)_{\alpha=0}^{p_1+p_2+p_3-1}$ and isomorphic to the lattice $\hat{T}(p_1, p_2, p_3)$.

**Proof.** The numerical Grothendieck group $\mathcal{N}(Y)$ is generated by the class $[\mathcal{O}_Y]$ of the structure sheaf, the Néron–Severi group $\text{NS}(Y)$, and the class $[\mathcal{O}_p]$ of a skyscraper sheaf. The structure of $\text{NS}(Y)$ for very general $Y$ is well studied (see e.g. [Bel02]), and generated by the irreducible components of the divisor $E = E_\infty \cup \bigcup_{i=1}^{3} \bigcup_{j=1}^{p_i-1} E_{i,j}$ at infinity. Both the structure sheaves of irreducible components of $E$ and a skyscraper sheaf $\mathcal{O}_p$ on $E$ belong to $D^b\text{coh}Y$, which is equivalent to $D^b\text{coh}_\infty Y$ by the functor $\Upsilon$. Since $D^b\text{coh}_\infty Y$ is generated by $(\iota_* S_{\alpha})_{\alpha=1}^{p_1+p_2+p_3-1}$, the collection $([E_{\alpha}])_{\alpha=1}^{p_1+p_2+p_3-1}$ generates $\text{NS}(Y)$ and $[\mathcal{O}_p]$, so that the collection $([E_{\alpha}])_{\alpha=0}^{p_1+p_2+p_3-1}$ generates $\mathcal{N}(Y)$. Since $\text{rank}\mathcal{N}(Y) = p_1 + p_2 + p_3$, the collection $([E_{\alpha}])_{\alpha=0}^{p_1+p_2+p_3-1}$ is a basis of $\mathcal{N}(Y)$. It is clear from (3.1) and (3.3) that $\mathcal{N}(Y)$ is isomorphic to $\hat{T}(p_1, p_2, p_3)$ as a lattice, and Proposition 3.3 is proved.

It is an interesting problem to see if the collection $([E_{\alpha}])_{\alpha=0}^{p_1+p_2+p_3-1}$ can be related to the collection of Ebeling and Ploog [EP10] shown in Figure 3.3 by an autoequivalence of $D^b\text{coh}Y$.

### 4 Variations of Hodge structures

We discuss Hodge-theoretic aspects of mirror symmetry [AM97, Dol96, Mor97, KKP08, In] for K3 surfaces associated with exceptional unimodal singularities in this section. Take a dual pair $((a, b, c; h), (\check{a}, \check{b}, \check{c}; \check{h}))$ of weight systems associated with exceptional unimodal singularities appearing in Table 1.1 and let $(\mathcal{Y}, \check{\mathcal{Y}})$ be a pair of very general hypersurfaces in $\mathbb{P}(a, b, c, 1)$ and $\mathbb{P}(\check{a}, \check{b}, \check{c}, 1)$ of degrees $h$ and $\check{h}$ respectively. Let further $(Y, \check{Y})$ be the minimal models of $(\mathcal{Y}, \check{\mathcal{Y}})$, which are smooth K3 surfaces. The transcendental lattice of $Y$ is isomorphic to the Milnor lattice of the exceptional unimodal singularity associated
with the weight system \((a, b, c; h)\), and the transcendental lattice of \(\tilde{Y}\) is isomorphic to the Milnor lattice of the dual singularity associated with \((\tilde{a}, \tilde{b}, \tilde{c}; \tilde{h})\).

Let \(N \cong \mathbb{Z}^3\) be a free abelian group of rank three and \(M = \text{Hom}(N, \mathbb{Z})\) be the dual group. Recall that the *fan polytope* of a fan is defined as the convex hull of primitive generators of one-dimensional cones of the fan. According to Kobayashi [Kob08, Theorem 4.3.9], there is a pair \((\Sigma, \tilde{\Sigma})\) of unimodular fans in \(N_\mathbb{R} = N \otimes \mathbb{R}\) and \(M_\mathbb{R} = M \otimes \mathbb{R}\) satisfying the following:

- The fan polytopes \((\Delta, \tilde{\Delta})\) of \((\Sigma, \tilde{\Sigma})\) are reflexive and polar dual to each other.
- There is an embedding \(\iota : Y \hookrightarrow X\) as an anti-canonical hypersurface in the toric variety \(X = X_\Sigma\) associated with the fan \(\Sigma\). Similarly, there is an anti-canonical embedding \(\iota : \tilde{Y} \hookrightarrow \tilde{X}\) into the toric variety \(\tilde{X} = X_{\tilde{\Sigma}}\) associated with the fan \(\tilde{\Sigma}\).
- The embedding \(\iota : Y \hookrightarrow X\) induces an isomorphism \(\iota^* : \text{NS}(X) \to \text{NS}(Y)\) of the Néron–Severi groups.

To be more precise, Kobayashi [Kob08, Theorem 4.3.9.(6)] states that the ranks of \(\iota^* \text{NS}(X)\) and \(\text{NS}(Y)\) are equal, although it is not difficult to check that \(\iota^*\) is an isomorphism by a case-by-case analysis.

Let \(\{b_1, \ldots, b_m\} \subset N\) be the set of generators of one-dimensional cones of the fan \(\Sigma\). One has the *fan sequence*

\[
0 \to \mathbb{L} \to \mathbb{Z}^m \xrightarrow{\beta} N \to 0
\]

and the *divisor sequence*

\[
0 \to M \xrightarrow{\beta^*} (\mathbb{Z}^m)^* \to \mathbb{L}^* \to 0
\]

where \(\beta\) sends the \(i\)th coordinate vector to \(b_i\) and

\[
\text{Pic}(X) \cong H^2(X; \mathbb{Z}) \cong \mathbb{L}^*.
\]
Set $\mathcal{M} = L^* \otimes \mathbb{C}^\times$ and $\tilde{T} = M \otimes \mathbb{C}^\times$ so that one has the exact sequence

$$1 \to \tilde{T} \to (\mathbb{C}^\times)^m \to \mathcal{M} \to 1.$$ 

The uncompactified mirror $\tilde{Y}_\alpha$ of the very general anticanonical hypersurface $Y \subset X$ is defined by

$$\tilde{Y}_\alpha = \{ y \in \tilde{T} | W_\alpha(y) = \sum_{i=1}^m \alpha_i y^i = 1 \}$$

where $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{C}^\times)^m$. The closure $\tilde{Y}$ of $\tilde{Y}_\alpha$ in $\tilde{X}$ for general $\alpha$ is a smooth anti-canonical K3 hypersurface, which is the compact mirror of $Y$. Let $\tilde{\varphi} : \tilde{Y} \to (\mathbb{C}^\times)^m$ be the second projection from

$$\tilde{Y} = \{(y, \alpha) \in \tilde{T} \times (\mathbb{C}^\times)^m | W_\alpha(y) = 1 \}.$$ 

The quotient of the family $\tilde{\varphi} : \tilde{Y} \to (\mathbb{C}^\times)^m$ by the free $\tilde{T}$-action

$$t \cdot (y, (\alpha_1, \ldots, \alpha_m)) = (t^{-1}y, (t^b \alpha_1, \ldots, t^{b_m} \alpha_m))$$

will be denoted by $\tilde{\varphi} : \tilde{Y} \to \mathcal{M}$ where $\mathcal{M} = (\mathbb{C}^\times)^m/\tilde{T}$. Choose an integral basis $p_1, \ldots, p_r$ of $L^* \cong \text{Pic} X$ such that each $p_i$ is nef. This gives the corresponding coordinate $q_1, \ldots, q_r$ on $\mathcal{M} = L^* \otimes \mathbb{C}^\times$. Let $U' \subset \mathcal{M}$ be a sufficiently small neighborhood of $q_1 = \cdots = q_r = 0$ so that the closure $\tilde{Y}$ of $\tilde{Y}_\alpha$ in $\tilde{X}$ is smooth for $q_1 \cdots q_r \neq 0$, and $\tilde{U}$ be the universal cover of $U'$. The B-model VHS $(H_{B,Z}, \nabla^B, \mathcal{F}_B^r, Q_B)$ on $\tilde{U}$ consists of the pull-back $H_{B,Z}$ of the local system $\varphi^* R^1 \tilde{\varphi}_* \mathcal{G}$, the Gauss–Manin connection $\nabla^B$ on $\mathcal{H}_B = H_{B,Z} \otimes \mathcal{O}_U$, the Hodge filtration $\mathcal{F}_B^r$, and the polarization $Q_B$ given by

$$Q_B(\omega_1, \omega_2) = \int_{\tilde{Y}_\alpha} \omega_1 \cup \omega_2.$$ 

The subsystem of $H_{B,Z}$ consists of vanishing cycles of $W_\alpha$ will be denoted by $H_{B,Z}^{\text{nc}}$.

On the A-model side, let

$$H^\bullet_{\text{amb}}(Y; \mathbb{C}) = \text{Im}(\iota^* : H^\bullet(X; \mathbb{C}) \to H^\bullet(Y; \mathbb{C}))$$

be the subspace of $H^\bullet(Y; \mathbb{C})$ coming from the cohomology classes of the ambient toric variety, and set

$$U = \{ \sigma = \beta + \sqrt{-1} \omega \in H^2_{\text{amb}}(Y; \mathbb{C}) | \langle \omega, d \rangle \gg 0 \text{ for any non-zero } d \in \text{Eff}(Y) \}.$$ 

where $\text{Eff}(Y)$ is the semigroup of effective curves. This open subset $U$ is considered as a neighborhood of the large radius limit point. The surjectivity of $\iota^* : \text{NS}(X) \to \text{NS}(Y)$ implies that $U$ here coincides with $U$ given in Section 1. Let $(\sigma^i)_{i=1}^r$ be the coordinate on $H^2_{\text{amb}}(Y; \mathbb{C})$ dual to the basis $(p_i)_{i=1}^r$, $\sigma = \sum_{i=1}^r \sigma^i p_i$.

The ambient A-model VHS $(\mathcal{H}_A', \nabla^{A'}, \mathcal{F}^r_A, Q_A)$ consists ($\text{Iri}$ Definition 6.2, cf. also [CK99 Section 8.5]) of the locally free sheaf $\mathcal{H}_A' = H^\bullet_{\text{amb}}(Y) \otimes \mathcal{O}_U$, the Dubrovin connection

$$\nabla^{A'} = d + \sum_{i=1}^r (p_i \sigma) \sigma^i : \mathcal{H}_A \to \mathcal{H}_A \otimes \Omega^1_U,$$
the Hodge filtration
\[ F_A^p = H^4_{\text{amb}}(Y) \otimes O_U, \]
and the Mukai pairing
\[ Q_A : \mathcal{H} \otimes \mathcal{H} \to O_U \]
which is symmetric and \( \nabla^{A'} \)-flat. Let \( L_Y(\sigma) \) be the fundamental solution of the quantum differential equation, that is, the \( \text{End}(H^*_{\text{amb}}(Y; \mathbb{C})) \)-valued functions satisfying
\[ \nabla^{A'} L_Y(\sigma) = 0, \quad i = 1, \ldots, r \]
and \( L_Y(\sigma) = \text{id} + O(\sigma) \). Since \( Y \) is a K3 surface, the quantum cup product \( \circ_\sigma \) coincides with the ordinary cup product, and the fundamental solution is given by
\[ L(\sigma) = \exp(-\sigma). \]

Let \( H'_{A,C} = \text{Ker} \nabla^{A'} \) be the \( \mathbb{C} \)-local system associated with \( \nabla^{A'} \) and define the integral local subsystem \( H'_{A,Z} \subset H'_{A,C} \) as
\[ H'_{A,Z} = \left\{ L_Y \left( \text{ch}(E) \sqrt{\text{td}(X)} \right) \mid E \in K(Y) \right\}. \]

Since \( L \) is a fundamental solution, the morphism
\[ L : \mathcal{H}_A(U) \to \mathcal{H}_A'(U) \]
of \( O_U \)-modules is flat (i.e. \( \nabla^{A'} L - L \nabla^A = 0 \)) and induces an isomorphism \( H_A \sim H'_A \) of \( \mathbb{C} \)-local systems. This isomorphism is compatible with Hodge filtrations since the generator \( e^\sigma \) of \( \mathcal{F}^2 \) goes to 1 in \( \mathcal{F}^2 \). It preserves the polarizations since \( L \) is an isometry of the Mukai lattice, and it is obvious from the definition that \( L \) preserves the integral structures. The local system \( H^\text{amb}_{A,Z} \) is defined as the local subsystem of \( H'_{A,Z} \) corresponding to \( \mathcal{N}(Y)^{\text{amb}} = \{ \iota^* E \mid E \in \mathcal{N}(X) \} \subset \mathcal{N}(Y) \).

Let \( u_i \in H^2(X; \mathbb{Z}) \) be the Poincaré dual of the toric divisor corresponding to the one-dimensional cone \( \mathbb{R} \cdot b_i \in \Sigma \) and \( v = u_1 + \cdots + u_m \) be the anticanonical class. Givental’s \( I \)-function is defined as the series
\[ I_{X,Y}(q,z) = e^{p \log q/z} \sum_{d \in \text{Eff}(X)} q^d \prod_{k=1}^m (v + k z) \prod_{j=1}^m (u_j + k z), \]
which is a multi-valued map from \( \hat{U}' \) (or a single-valued map from \( \hat{U} \)) to the classical cohomology ring \( H^\bullet(X; \mathbb{C}[z^{-1}]) \). Givental’s \( J \)-function is defined by
\[ J_Y(\tau, z) = L_Y(\tau, z)^{-1}(1) = \exp(\tau/z). \]

If we write
\[ I_{X,Y}(q,z) = F(q) + \frac{G(q)}{z} + \frac{H(q)}{z^2} + O(z^{-3}), \]

\[ 16 \]
then Givental’s mirror theorem \cite{Giv96, Giv98, CG07} states that

$$\text{Euler}(\omega_X^{-1}) \cup I_{X,Y}(q, z) = F(q) \cdot \iota_* J_Y(\varsigma(q), z)$$

where \(\text{Euler}(\omega_X^{-1}) \in H^2(X; \mathbb{Z})\) is the Euler class of the anticanonical bundle of \(X\), and the \textit{mirror map} \(\varsigma(q) : \hat{U} \to H^2_{\text{amb}}(Y; \mathbb{C})\) is defined by

$$\varsigma(q) = \iota^* \left( \frac{G(q)}{F(q)} \right).$$

The relation between \(\tau = \varsigma(q)\) and \(\sigma = \beta + \sqrt{-1}\omega\) is given by \(\tau = 2\pi \sqrt{-1}\sigma\), so that \(\Im(\sigma) \gg 0\) corresponds to \(\exp(\tau) \sim 0\). The functions \(F(q), G(q)\) and \(H(q)\) satisfy the Gelfand–Kapranov–Zelevinsky hypergeometric differential equations, and give periods for the B-model VHS \((\mathcal{H}_B, \nabla^B, \mathcal{F}_B^\bullet, Q_B)\). The isomorphism of integral structures is due to Iritani:

**Theorem 4.1** (Iritani \cite[Theorem 6.9]{Iri}). There is an isomorphism

$$\text{Miry} : \iota^* (H^\text{amb}_{A,Z}, \nabla^A, \mathcal{F}_A^\bullet, Q_A) \xrightarrow{\sim} (H^\text{we}_{B,Z}, \nabla^B, \mathcal{F}_B^\bullet, Q_B)$$

of integral variations of pure and polarized Hodge structures.

The following lemma concludes the proof of Corollary 1.3

**Lemma 4.2.** One has equalities

$$H^\text{amb}_{A,Z} = H_{A,Z}$$

and

$$H^\text{we}_{B,Z} = H_{B,Z}$$

of integral local systems.

**Proof.** To prove the equality \(H^\text{amb}_{A,Z} = H_{A,Z}\), it suffices to show that the map \(\iota^* : \mathcal{N}(X) \to \mathcal{N}(Y)\) between the numerical Grothendieck groups is surjective. First note that \(\text{NS}(Y) = \iota^* \text{NS}(X)\) by our choice of \(X\) at the beginning of this section. It is easy to see from Figure 3.2 that one can choose a pair \((D, E)\) of divisors on \(Y\) such that their intersection \(D \cdot E\) is a point. Take a pair \((\tilde{D}, \tilde{E})\) of divisors on \(X\) such that \(\iota^* \tilde{D} = D\) and \(\iota^* \tilde{E} = E\). Then one has \(\iota^* (\tilde{D} \cdot \tilde{E}) = D \cdot E\), so that the class of a point also belongs to \(\iota^* \mathcal{N}(X)\). The class of the structure sheaf clearly belongs to \(\iota^* \mathcal{N}(X)\) since \(\iota^* \mathcal{O}_X = \mathcal{O}_Y\), and the equality \(H^\text{amb}_{A,Z} = H_{A,Z}\) is proved.

For the equality \(H^\text{we}_{B,Z} = H_{B,Z}\), first note that the fiber of \(H_{B,Z} = \text{gr}_2^W H^2_c(\hat{Y}_\alpha; \mathbb{Z})\) at \(\alpha \in \hat{U}\) is the weight 2 part \(\text{gr}_2^W H^2_c(\hat{Y}_\alpha; \mathbb{Z})\) of the cohomology group of \(\hat{Y}_\alpha\) with compact support. Let \(D = \hat{Y} \setminus \hat{Y}_\alpha\) be the divisor at infinity in the smooth compactification \(\hat{Y}\) of \(\hat{Y}_\alpha\). The long exact sequence

$$\cdots \to H^1(D; \mathbb{Z}) \to H^2(\hat{Y}, D; \mathbb{Z}) \to H^2(\hat{Y}; \mathbb{Z}) \to H^2(D; \mathbb{Z}) \to \cdots$$

associated with the pair \((\hat{Y}, D)\) shows that the weight 2 part of \(H^2_c(\hat{Y}_\alpha; \mathbb{Z}) \cong H^2(\hat{Y}, D; \mathbb{Z})\) is the kernel of \(H^2(\hat{Y}; \mathbb{Z}) \to H^2(D; \mathbb{Z})\). This is equal to the transcendental lattice of \(\hat{Y}\). 

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for very general $\alpha$, which is well known \cite{Pin77,Dol83,Nik79} to be isomorphic to $\hat{T}(\gamma)$, where $\gamma$ is the Gabrielov number of the corresponding exceptional unimodal singularity.

On the other hand, the local system $H^{\text{vc}}_{\mathcal{B},\mathbb{Z}}$ is isomorphic to $H^{\text{amb}}_{\mathcal{A},\mathbb{Z}}$ by Theorem 4.1, which is isomorphic to $\hat{T}(\delta)$ by Proposition 3.3 where $\delta$ is the Dolgachev number of the singularity associated with $Y$. Since the pair $(Y,\hat{Y})$ comes from a strange dual pair of exceptional unimodal singularities, one has $\gamma = \delta$. It follows that the determinants of the Gram matrices of the generators of $H_{\mathcal{B},\mathbb{Z}}$ and $H^{\text{vc}}_{\mathcal{B},\mathbb{Z}}$ are the same. Since $H^{\text{vc}}_{\mathcal{B},\mathbb{Z}}$ is a sublattice of $H_{\mathcal{B},\mathbb{Z}}$, this implies $H^{\text{vc}}_{\mathcal{B},\mathbb{Z}} = H_{\mathcal{B},\mathbb{Z}}$ and the lemma is proved. \hfill \qed

\textbf{References}

\begin{itemize}
  \item \textbf{[AM97]} Paul S. Aspinwall and David R. Morrison, \textit{String theory on K3 surfaces}, Mirror symmetry, II, AMS/IP Stud. Adv. Math., vol. 1, Amer. Math. Soc., Providence, RI, 1997, pp. 703–716. MR 1416354 (97i:81128)
  \item \textbf{[Arn75]} V. I. Arnol'd, \textit{Critical points of smooth functions}, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, Canad. Math. Congress, Montreal, Que., 1975, pp. 19–39. MR 0431217 (55 #4218)
  \item \textbf{[Bal]} Matthew Robert Ballard, \textit{Sheaves on local Calabi-Yau varieties}, arXiv:0801.3499.
  \item \textbf{[Bel02]} Sarah-Marie Belcastro, \textit{Picard lattices of families of K3 surfaces}, Comm. Algebra \textbf{30} (2002), no. 1, 61–82. MR 1880661 (2003d:14048)
  \item \textbf{[BKR01]} Tom Bridgeland, Alastair King, and Miles Reid, \textit{The McKay correspondence as an equivalence of derived categories}, J. Amer. Math. Soc. \textbf{14} (2001), no. 3, 535–554 (electronic). MR MR1824990 (2002f:14023)
  \item \textbf{[Bon89]} A. I. Bondal, \textit{Representations of associative algebras and coherent sheaves}, Izv. Akad. Nauk SSSR Ser. Mat. \textbf{53} (1989), no. 1, 25–44. MR MR992977 (90i:14017)
  \item \textbf{[Buc87]} Ragnar-Olaf Buchweitz, \textit{Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings}, Available from https://tspace.library.utoronto.ca/handle/1807/16682, 1987.
  \item \textbf{[CG07]} Tom Coates and Alexander Givental, \textit{Quantum Riemann-Roch, Lefschetz and Serre}, Ann. of Math. (2) \textbf{165} (2007), no. 1, 15–53. MR 2276766 (2007k:14113)
  \item \textbf{[CK99]} David A. Cox and Sheldon Katz, \textit{Mirror symmetry and algebraic geometry}, Mathematical Surveys and Monographs, vol. 68, American Mathematical Society, Providence, RI, 1999. MR 1677117 (2000d:14048)
  \item \textbf{[CM03]} César Camacho and Hossein Movasati, \textit{Neighborhoods of analytic varieties}, Monografías del Instituto de Matemática y Ciencias Afines [Monographs of the Institute of Mathematics and Related Sciences], vol. 35, Instituto de Matemática y Ciencias Afines, IMCA, Lima, 2003. MR 2010707 (2004j:32008)
\end{itemize}
[DM] Tobias Dyckerhoff and Daniel Murfet, *Pushing forward matrix factorisations*, arXiv:1102.2957.

[Dol83] Igor Dolgachev, *Integral quadratic forms: applications to algebraic geometry (after V. Nikulin)*, Bourbaki seminar, Vol. 1982/83, Astérisque, vol. 105, Soc. Math. France, Paris, 1983, pp. 251–278. MR 728992 (85f:14036)

[Dol96] I. V. Dolgachev, *Mirror symmetry for lattice polarized K3 surfaces*, J. Math. Sci. 81 (1996), no. 3, 2599–2630, Algebraic geometry, 4. MR 1420220 (97i:14024)

[EP10] Wolfgang Ebeling and David Ploog, *McKay correspondence for the Poincaré series of Kleinian and Fuchsian singularities*, Math. Ann. 347 (2010), no. 3, 689–702. MR 2640048

[Giv96] Alexander Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices (1996), no. 13, 613–663. MR 1408320 (97e:14015)

[Giv98] , *A mirror theorem for toric complete intersections*, Topological field theory, primitive forms and related topics (Kyoto, 1996), Progr. Math., vol. 160, Birkhäuser Boston, Boston, MA, 1998, pp. 141–175. MR 1653024 (2000a:14063)

[GL87] Werner Geigle and Helmut Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 265–297. MR 915180 (89b:14049)

[GW78] Shiro Goto and Keiichi Watanabe, *On graded rings. I*, J. Math. Soc. Japan 30 (1978), no. 2, 179–213. MR 494707 (81m:13021)

[Hap91] Dieter Happel, *On Gorenstein algebras*, Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991), Progr. Math., vol. 95, Birkhäuser, Basel, 1991, pp. 389–404. MR 1112170 (92k:16022)

[Har] Heinrich Hartmann, *Period- and mirror-maps for the quartic K3*, arXiv:1101.4601.

[Iri] Hiroshi Iritani, *Quantum cohomology and periods*, arXiv:1101.4512.

[KKP08] L. Katzarkov, M. Kontsevich, and T. Pantev, *Hodge theoretic aspects of mirror symmetry*, From Hodge theory to integrability and TQFT tt*-geometry, Proc. Sympos. Pure Math., vol. 78, Amer. Math. Soc., Providence, RI, 2008, pp. 87–174. MR 2483750 (2009j:14052)

[Kob08] Masanori Kobayashi, *Duality of weights, mirror symmetry and Arnold’s strange duality*, Tokyo J. Math. 31 (2008), no. 1, 225–251. MR 2426805 (2010f:32022)

[Kra05] Henning Krause, *The stable derived category of a Noetherian scheme*, Compos. Math. 141 (2005), no. 5, 1128–1162. MR 2157133 (2006e:18019)
[KST09] Hiroshige Kajiura, Kyoji Saito, and Atsushi Takahashi, *Triangulated categories of matrix factorizations for regular systems of weights with $\epsilon = -1$*, Adv. Math. **220** (2009), no. 5, 1602–1654. MR MR2493621

[KV00] M. Kapranov and E. Vasserot, *Kleinian singularities, derived categories and Hall algebras*, Math. Ann. **316** (2000), no. 3, 565–576. MR MR1752785 (2001h:14012)

[LdlP11] Helmut Lenzing and José A. de la Peña, *Extended canonical algebras and Fuchsian singularities*, Math. Z. **268** (2011), no. 1-2, 143–167. MR 2805427 (2012d:16048)

[Len94] H. Lenzing, *Wild canonical algebras and rings of automorphic forms*, Finite-dimensional algebras and related topics (Ottawa, ON, 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 424, Kluwer Acad. Publ., Dordrecht, 1994, pp. 191–212. MR 1308987 (95m:16008)

[Mor97] David R. Morrison, *Mathematical aspects of mirror symmetry*, Complex algebraic geometry (Park City, UT, 1993), IAS/Park City Math. Ser., vol. 3, Amer. Math. Soc., Providence, RI, 1997, pp. 265–327. MR 1442525 (98g :14044)

[Nik79] V. V. Nikulin, *Integer symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 1, 111–177, 238. MR 525944 (80j:10031)

[Orl04] D. O. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, Tr. Mat. Inst. Steklova **246** (2004), no. Algebr. Geom. Metody, Svyazi i Prilozh., 240–262. MR MR2101296

[Orl09] Dmitri Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, Progr. Math., vol. 270, Birkhäuser Boston Inc., Boston, MA, 2009, pp. 503–531. MR 2641200 (2011c:14050)

[Pin77] Henry Pinkham, *Singularités exceptionnelles, la dualité étrange d’Arnold et les surfaces $K - 3$*, C. R. Acad. Sci. Paris Sér. A-B **284** (1977), no. 11, A615–A618. MR 0429876 (55 #2886)

[PV] Alexander Polishchuk and Arkady Vaintrob, *Matrix factorizations and singularity categories for stacks*, arXiv:1011.4544.

[Seg08] Ed Segal, *The $A_\infty$ deformation theory of a point and the derived categories of local Calabi-Yaus*, J. Algebra **320** (2008), no. 8, 3232–3268. MR MR2450725 (2009k:16016)

[Sei10] Paul Seidel, *Suspending Lefschetz fibrations, with an application to local mirror symmetry*, Comm. Math. Phys. **297** (2010), no. 2, 515–528. MR 2651908

[ST01] Paul Seidel and Richard Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108** (2001), no. 1, 37–108. MR MR1831820 (2002e:14030)
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