Kinetic modeling of economic markets with various saving propensities

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Abstract

In this paper, two kinetic models are proposed for a closed economic market of agents with homogeneous or inhomogeneous saving interests. For Model I, the trading volume depends on the average saving propensities of an arbitrary pair of agents in trade. For Model II, the transaction is governed by a stochastic parameter between the saving propensities of two traders. Besides, two sampling methods are introduced for the random selection of two agents in the iterative process. Specifically, Technique I is sampling with replacement and is easier to program. Technique II is sampling without replacement and owns a higher computing efficiency. There are slight differences between the stationary wealth distributions simulated by using the econophysics models and computational approaches. The accuracy and robustness of the models and methods are validated by typical numerical tests. Moreover, the impact of saving propensities of agents in two groups on the wealth distributions is studied, and the influence of proportions of agents is investigated as well. To quantitatively measure the wealth inequality, the Gini coefficients, Kolkata indices, and deviation degrees of all agents and two groups are simulated and analyzed in detail.

Keywords: Econophysics, Kinetic model, Wealth distribution, Wealth inequality, Saving propensity

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1. Introduction

Econophysics is a typical interdiscipline that applies theories and methods of statistical physics to solve problems in economics [1]. As a fundamental issue in econophysics and economics, the wealth distribution plays a key role in human society [2, 3]. Due to the great importance, a larger number of economists, mathematicians and physicists are devoted to the scientific research of wealth distributions [2, 4, 5]. In theory, a financial society is analogous to a physical system [5]. To be specific, the agent, wealth and average wealth per agent are equivalent to the particle, energy and temperature, respectively [6]. By analogy with physical models, econophysics modeling of complex social systems has attracted more and more attention [2, 6, 7]. In general, for a coarse-graining financial model, the laws of agent and money conservation are obeyed in a closed market during a certain stage [7]. Based on reasonable assumptions, a series of successful theoretical models and numerical simulations promise deep insights into the characteristics and mechanisms of wealth distributions [6, 7, 8, 9].

In statistical mechanics, an ideal gas system holds the Boltzmann-Gibbs law, \( p_i \propto \exp(\epsilon_i / (kT)) \), where \( p_i \) is the probability in state \( i \), \( \epsilon_i \) the corresponding energy, \( T \) the thermodynamic temperature, and \( k \) the Boltzmann’s constant [10]. Similarly, the wealth distribution also takes the Boltzmann-Gibb exponential form in a closed equilibrium system with arbitrary and random trades [8] (see Eq. 3). Besides, another famous empirical form for the wealth distribution is the Pareto power-law function, \( P(m) \propto m^{-\alpha} \), in terms of the Pareto index \( \alpha \) and money \( m \) [9]. Usually, the Boltzmann-Gibbs function fits well the low and middle ranges of wealth distribution [8, 10, 11], and the Pareto function is in line with the high range [9, 12, 13]. Although the classical economic theories are helpful for the study of financial markets, a deeper understanding of the dynamics of trading process often requires more versatile methodologies.

With the rapid development of computational science and technology, mu-
Numerical simulations provide a convenient tool for the analysis of economic issues. As an interdisciplinary approach, econophysics modeling of financial markets has made great success in recent decades \cite{6,14,15,16,17,18,19,20}. Early in 2000, the Chakraborti-Chakrabarti (CC) model was proposed for a closed economic system and the influence of saving propensities of the agents upon the equilibrium probability distribution of money was studied \cite{6}. In 2004, Chatterjee et al. presented a kinetic model for an economic market where the interest of saving varies from person to person \cite{14}. In 2017, Lima employed the anisotropic Ising model in an external field and with an ion single anisotropy term as a mathematical model for the financial dynamics \cite{15}. In 2018, Pinasco et al. proposed a model of wealth distribution considering both the simplicity of statistical physics models and the flexibility of evolutionary game theory \cite{16}. In 2020, Cui and Lin developed a two-dimensional lattice gas automaton (LGA) economic model of the income distribution under the conditions of the Matthew effect, income tax and charity \cite{17}. Next year, Cui and Lin further proposed a simple and efficient one-dimensional LGA for the wealth distribution of agents with or without saving propensities \cite{18}. In 2021, Zhou et al. introduced the wealth substitution rate into the collision kernel of the Boltzmann equation to investigate the wealth distribution \cite{19}. Recently, Paul et al. investigated the variations of income or wealth distribution of agents with saving propensities by using kinetic exchange models \cite{20}.

On the base of aforementioned econophysics models \cite{14,18,20}, we propose two kinetic models and introduce two sampling techniques for the commercial transactions between agents with or without saving propensities. The rest of the paper is organized as follows. In Sec. 2, the economic transaction models and computational approaches are introduced in detail. In Sec. 3, the numerical robustness and reliability of the models are validated. Then we study the wealth distributions of two groups of agents with various saving propensities. Finally, conclusions are drawn in Sec. 5.
2. Transaction model

Let us consider a closed economic market, where the total amount of money $M$ and the number of agents $N \gg 1$ are constant during a certain period. Consequently, the mean money is $m_0 = M/N$. The agent $A_i$ is introduced to represent an individual or corporation, with the subscript $i = 1, 2, \ldots, N$. The agent $A_i$ possesses money $m_i$ and has a saving propensity $\lambda_i$ within the range $0 \leq \lambda_i \leq 1$. The wealth and saving interest vary from person to person. For an arbitrary pair of agents $i$ and $j$ in trade, and their money changes as follows

$$
\begin{align*}
    m'_i &= m_i - \Delta m, \\
    m'_j &= m_j + \Delta m.
\end{align*}
$$

where $\Delta m$ stands for the trading volume, $m_i$ and $m'_i$ denote the money of agent $i$ before and after transaction, respectively; $m_j$ and $m'_j$ represent the money of agent $j$ before and after transaction, respectively. For simplicity, no debt or profit is under consideration.

In fact, the key step is to determine the mathematical expression of $\Delta m$ in Eq. 1. The trading volume is expressed by

$$
\Delta m = (1 - \lambda_e) [m_i - \varepsilon (m_i + m_j)],
$$

where $\varepsilon$ is a random fraction uniformly distributed in the interval $0 \leq \varepsilon \leq 1$, and the parameter $\lambda_e$ depends on the saving propensities $\lambda_i$ and $\lambda_j$. In this work, we proposed two types of transaction models, i.e., Model I and Model II. To be specific, the parameter is $\lambda_e = (\lambda_i + \lambda_j)/2$ for Model I, and $\lambda_e$ is a stochastic number within the range, $\min (\lambda_i, \lambda_j) \leq \lambda_e \leq \max (\lambda_i, \lambda_j)$ for Model II.

It is clear that the agent $A_i$ is a “loser” and the other agent $A_j$ becomes the “winner” in the case of $\Delta m > 0$, and vice versa. Moreover, it can be found from Eq. 2 that the relation $m_i \geq \Delta m$ (or $m_j \geq \Delta m$) is satisfied when the agent $A_i$ (or $A_j$) is a “loser”. Consequently, the debt does not occur because the relations $m'_i \geq 0$ and $m'_j \geq 0$ are satisfied by Eqs. 1-2.
Moreover, how to choose two agents with random transaction is a key in the dynamic process of an economic market. In this paper, two sampling methods (i.e., Techniques I and II) are introduced for the random selection of two agents making transaction. To be specific, Technique I is sampling with replacement. Each agent has the equal chance of being selected, and a pair of agents are chosen from all samples randomly with the same probability at each iterative step in the loop of computer programming.

On the other hand, Technique II is sampling without replacement. Namely, we deliberately avoid choosing an agent of the sample more than once during one main loop that takes $N/2$ temporal steps. Specifically, one pair of agents $i$ and $j$ are randomly selected from $N$ agents at the first temporal step $t = 1$. Then at the second temporal step, $t = 2$, another two agents $i$ and $j$ are randomly selected from the remaining $N - 3$ samples. In succession, all agents have made a transaction once in the end of the main loop at $t = N/2$. Then the next loop begins when $t = N/2 + 1$, and ends until all the agents have made a deal at $t = N$. Figure 1 displays the sketch of the random selection process of Technique II.

It should be mentioned that both Techniques I and II can be adopted to
predict the stationary state of the economic market. Actually, the simulated relaxation processes in the evolution of the market are different by using the two sampling methods. Namely, the relaxation time is longer by using Technique I than Technique II, see Fig. 2 for more details.

Moreover, both Models I and II reduce to the CC model if the saving propensities of all agents are equal to each other. Besides, in Ref. 6, an arbitrary pair of agents $i$ and $j$ are chosen randomly from all agents $N$ to get engaged in a trade. In other words, Technique I is adopted for the CC model in Ref. 6. Consequently, in the case of $\lambda_i = \lambda_j$, Model I or II plus Technique I is equivalent to the CC model in Ref. 6, as shown in Fig. 3.

3. Numerical validation

3.1. Relaxation process of the economic market

The first three problems to be resolved are as follows: (i) As the financial markets are continuously traded, whether a steady state could be reached after a sufficiently long period. (ii) Whether the kinetic models and sampling methods produce a similar steady state of the economic market. (iii) How the models and techniques affect the relaxation process of the economic market. For this purpose, numerical simulations are carried out in four different ways, i.e., Model I & Technique I, Model I & Technique II, Model II & Technique I, and Model II & Technique II, respectively.

Now, let us consider the following configuration. There are $N = 1000$ agents in a closed economic market, and each agent initially possesses the same money $m_i = 1$. Half agents have the same saving propensity $\lambda_A$, and the other traders own the saving propensity $\lambda_B$. Specifically, four cases of the saving propensities are considered: (a) $\lambda_A = \lambda_B = 0.1$, (b) $\lambda_A = 0.3$ & $\lambda_B = 0.7$, (c) $\lambda_A = 0.7$ & $\lambda_B = 0.3$, and (d) $\lambda_A = \lambda_B = 0.9$.

Figure 2 illustrates the Gini coefficients ($G_1$) versus time in the aforementioned four cases. Obviously, the Gini coefficient increases from zero to a steady value, and its change rate reduces gradually in the evolution. In other words, the
Figure 2: Evolution of Gini coefficients in four cases: (a) $\lambda_A = \lambda_B = 0.1$, (b) $\lambda_A = 0.3$ & $\lambda_B = 0.7$, (c) $\lambda_A = 0.7$ & $\lambda_B = 0.3$, and (d) $\lambda_A = \lambda_B = 0.9$. The insets show the corresponding profiles within the temporal range $0 \leq t \leq 4000$. The squares denote the simulation results of Model I & Technique I, the circles the simulation results of Model I & Technique II, the red lines the simulation results of Model II & Technique I, and the blue lines the simulation results of Model II & Technique II.
economic system reaches the steady state as there are continuous transactions after a long time.

Additionally, in Fig. 2, there is a remarkable agreement among the final steady values of the Gini coefficients obtained from the simulations by using Model I & Technique I, Model I & Technique II, Model II & Technique I, and Model II & Technique II. That is to say, Models (I and II) and Techniques (I and II) give a similar steady state of the economic market.

Moreover, it can be found from the insets in Fig. 2 that, the simulated Gini coefficients by using Model I & Technique I and Model II & Technique I coincide with each other. Similarly, there is a nice agreement between the numerical results from Model I & Technique II and Model II & Technique II. Namely, there are few differences between Models I and II in the relaxation process.

On the other hand, as shown in the insets in Fig. 2, the simulated Gini coefficients by using Model I & Technique I are lower than those by using Model I & Technique II. And there are similar differences between Model II & Technique I and Model II & Technique II. In fact, the economic market evolves faster in the way of Technique II than I. From a statistical point of view, to ensure that all agents have business dealings, it takes a longer time by using Technique I than II.

Furthermore, it is clear that, with the increasing of saving propensities $\lambda_A$ and $\lambda_B$, the steady values of the Gini coefficients decrease. To be specific, the steady Gini coefficients are about $G_{1S} = 0.443$, $0.285$, $0.285$, and $0.106$ in Figs. 2(a)-(d), respectively. That is to say, the inequality of income or wealth reduces as the saving propensities increase, which is reasonable.

Next, to have a quantitative study of the relaxation time, let us introduce a symbol $\tau$ that denotes the time instant when the Gini coefficient increases to 0.99 times the steady Gini coefficients, i.e., $G_1(\tau) = 0.99G_{1S}$. It can be found that the corresponding temporal steps are $\tau = 4660$, $4940$, $4940$, and $19830$ in Figs. 2(a)-(d), respectively. Because it takes a longer time for the market to reach the steady state as the agents transact with less money each time. Consequently, the
relaxation time becomes longer for greater saving propensities. The simulation results are reasonable as well.

In theory, the total money of all agents should be unchanged in the evolution of a closed economic market. To further validate the models and techniques, let us measure the amount of money in the process of economic transactions. Figure 3 displays the average money versus the time in the same four cases as Fig. 2. Obviously, the mean money remains unchanged during the evolution of economic transactions, as shown in Figs. 3 (a)-(d). That is to say, the principle of money conservation is obeyed by the econophysics models and sampling techniques.

3.2. Transaction without saving propensities

In theory, it is demonstrated by Drăgulescu and Yakovenko [8] that the income distributions of individual agents and two-earner families in the equilibrium state of an ideal free market read

\[ P_1 (m) = \frac{1}{m_0} \exp \left( -\frac{m}{m_0} \right), \quad \text{(3)} \]

and

\[ P_2 (m) = \frac{m}{m_0} \exp \left( -\frac{m}{m_0} \right), \quad \text{(4)} \]

respectively.

In addition, the cumulative fraction of individual agents \( x_1 \) and the cumulative share of money \( y_1 \) satisfy the following relation [8],

\[
\begin{align*}
    x_1 &= 1 - \exp \left( -\frac{m}{m_0} \right), \\
    y_1 &= x_1 - \frac{m}{m_0} \exp \left( -\frac{m}{m_0} \right).
\end{align*}
\quad \text{(5)}
\]

The cumulative fraction of two-earner families \( x_2 \) and the cumulative share of wealth \( y_2 \) are expressed by [8],

\[
\begin{align*}
    x_2 &= 1 - \left( 1 + \frac{m}{m_0} \right) \exp \left( -\frac{m}{m_0} \right), \\
    y_2 &= x_2 - \frac{1}{2} \frac{m^2}{m_0} \exp \left( -\frac{m}{m_0} \right).
\end{align*}
\quad \text{(6)}
\]

Next, to verify the kinetic models and sampling techniques, let us make a comparison between the abovementioned theoretical solutions and the numerical
Figure 3: Evolution of average money in four cases: (a) $\lambda_A = \lambda_B = 0.1$, (b) $\lambda_A = 0.3$ & $\lambda_B = 0.7$, (c) $\lambda_A = 0.7$ & $\lambda_B = 0.3$, and (d) $\lambda_A = \lambda_B = 0.9$. The squares denote the simulation results of Model I & Technique I, the circles the simulation results of Model I & Technique II, the red lines the simulation results of Model II & Technique I, and the blue lines the simulation results of Model II & Technique II.
It is clear in Fig. 4 that all simulation results of the two econophysics models and sampling methods are in nice agreement with the theoretical results. To be specific, as shown in Fig. 4 (a), the stationary money distribution function becomes the Boltzmann-Gibbs distribution when the agents do not save. It can be found in Fig. 4 (b) that the peak of the curve is located at $m/m_0 = 1$, namely, the probability of two-earner families achieves the maximum when the wealth of the family equals the average wealth of individual agents. Moreover, the simulation by using Model I & Technique I gives the Gini coefficients $G_1 = 0.495$ and $G_2 = 0.358$ for the individual agents and dual-earner families, respectively.
Compared with the exact solutions $G_1 = 1/2$ and $G_2 = 3/8$ in Ref. [8], the relative errors are 1% and 4.5%, respectively. In addition, Model II or Technique II gives similar results. The simulation results are satisfactory.

### 3.3. Transaction with saving propensities

The main advantage of the econophysics models is the capability of describing transaction with or without saving propensities. In the above part, it is demonstrated that the two kinetic models with either of the two computational methods are suitable for an ideal free market without saving propensities. Next, let us validate that the transaction with saving propensities can also be simulated by using the two models and approaches.

Figure 5 illustrates the money distributions of individual agents (a) and dual-earner families (b), and displays the cumulative shares of individual agents (c) and dual-earner families (d) in five cases of saving propensities, i.e., $\lambda_i = 0.0, 0.1, 0.4, 0.7, \text{ and } 0.9$, respectively. For each case, two agents who get engaged in a trade have the same saving propensities, hence both Models I and II reduce to the CC model [6]. For the sake of brevity, only Model I is employed in Fig.
and Model II and CC model give exactly the same simulation results (not shown here).

It is clear in Fig. 5 that the simulation results of Techniques I and II overlap each other. As shown in Fig. 5(a), the money distributions of individual agents $P_1$ change from the Boltzmann-Gibb form to the asymmetric Gaussian-like form with the increasing saving propensities $\lambda_i$. It is obvious in Fig. 5(b) that, there is a peak of the money distributions of dual-earner families $P_2$ for any value of $\lambda_i$, and the peak becomes higher for a larger saving propensity. Moreover, with the increasing $\lambda_i$, the probabilities of the individual agents or dual-earner families whose money is close to the respective average value gradually grow, and the peaks move rightwards and become thinner and higher.

Moreover, it can be seen in Figs. 5 (c) and (d) that the ranges of cumulative shares of money are $0 \leq L_1 \leq 1$ for individual agents and $0 \leq L_2 \leq 1$ for dual-earner families. Namely, for both individual agents and dual-earner families, the cumulative shares of money rise from zero to one. With the increasing saving propensities, the curves of the cumulative shares of money become close to the line of perfect equality, and the curvature decrease. Obviously, Figs. 5 (c) and (d) are consistent with Figs. 5 (a) and (b), and all simulation results are reasonable. Additionally, it can be found in Figs. 5(a)-(d) that the wealth distributions are functions of the saving propensities, i.e., $P_1 = P_1(\lambda_A, \lambda_B)$, $P_2 = P_2(\lambda_A, \lambda_B)$, $L_1 = L_1(\lambda_A, \lambda_B)$, and $L_2 = L_2(\lambda_A, \lambda_B)$, which can be used for the definition of deviation degree in Eq. 7.

To further test the numerical robustness and reliability of the models and techniques, let us carry out simulations with different numbers of agents $N$, values of average money $m_0$, and temporal steps $t$. Figures 6 (a)-(d) display the stationary money distributions of individual agents simulated by using Model I & Technique I, Model I & Technique II, Model II & Technique I, and Model II & Technique II, respectively. The lines, squares, circles, and triangles stand for the simulation results with $(N, m_0, t) = (500, 1, 3000)$, $(400, 1, 3000)$, $(500, 1, 5000)$, and $(500, 5, 3000)$, respectively. Clearly, all numerical results with various parameters are in nice agreement with each other in each plot. Besides,
those simulation results given by the kinetic models and sampling techniques in the four plots are similar to each other. In other words, the simulation results are independent of the numbers of the agents, total amount of money, and temporal steps for Model I, Model II, Technique I and Technique II, respectively. It is demonstrated that the models and techniques own a high numerical robustness and reliability.

4. Numerical investigation

In this section, let us study the features of the stationary wealth distributions of agents with or without saving propensities in the economic society. To this end, we assume that there are two groups $A$ and $B$ in a financial market, see Fig. 7. The numbers of agents in the two groups are $N_A$ and $N_B$. The ranges of saving propensities of agents in groups $A$ and $B$ are $0 \leq \lambda_A \leq 1$ and $0 \leq \lambda_B \leq 1$, respectively. After plenty of transactions, the commercial system reaches the steady state.

4.1. Impact of saving propensities

First of all, the influence of saving propensities upon the wealth distribution is investigated. For convenience, the numbers of agents in groups $A$ and $B$ are
An economic market

Group A
Group B

\[
\begin{align*}
N_A &\quad N_B \\
0 \leq \lambda_A &\leq 1 & 0 \leq \lambda_B &\leq 1
\end{align*}
\]

\[N_A + N_B = 1000\]

Figure 7: Sketch of two groups of agents in an economic market.

Figure 8: Contours of Gini coefficients in the space of saving propensities \(\lambda_A\) and \(\lambda_B\): (a) Gini coefficients of the whole economic system, (b) Gini coefficients of group A, and (c) Gini coefficients of group B.

set as \(N_A = N_B = 500\), and the saving propensities of agents in the two groups increase from zero to one. To quantitatively study the relationship between saving propensities and wealth inequality, three useful parameters (i.e., the Gini coefficient \(G\), Kolkata index \(k\), and deviation degree \(\Delta\)) are employed in this part.

Figure 8 depicts the Gini coefficients versus the saving propensities \(\lambda_A\) and \(\lambda_B\). Figures 8(a)-(c) show the Gini coefficients of the whole economic system, group A, and group B, respectively. The values of the Gini coefficients are labelled in the legend. On the whole, the contours of Gini coefficients decrease with the increasing of \(\lambda_A\) and/or \(\lambda_B\). In each plot, the minima and maxima of saving propensities are 0 and 0.5, respectively. That is to say, the wealth
distribution tends to be even as the saving propensities increases. Additionally, the contours are symmetrical about the diagonal line in Fig. 8 (a), and are asymmetric about the diagonal line in Fig. 8 (b) or (c).

Apart from the Gini coefficient, the Kolkata index is another important parameter to measure the wealth inequality. Next, let us study the Kolkata indices when the economic system reaches the steady state. Figure 9 illustrates the Kolkata indices versus the saving propensities $\lambda_A$ and $\lambda_B$. It can be seen in Figs. 9 (a)-(c) that the range of the Kolkata indices is from 0.5 to 0.68, and the Kolkata indices become smaller for greater saving propensities $\lambda_A$ and/or $\lambda_B$. Similar to the Gini coefficient, the contours of Kolkata indices are also symmetrical about the diagonal line in Fig. 9 (a), and are asymmetric about the diagonal line in Fig. 9 (b) or (c).

Next, to describe the difference of the wealth distributions of individual agents with and without saving propensities, we introduce the definition of deviation degree as follows:

$$\Delta = \frac{1}{2} \int_0^\infty |f - f^{eq}| dm,$$

where $f = P_1(\lambda_A, \lambda_B)$ denotes the wealth distribution of individual agents with or without saving propensities, and $f^{eq} = P_1(0,0)$ stands for the wealth distribution without saving propensities. In theory, the realm is $0 \leq \Delta \leq 1$, and
Figure 10 plots the deviation degrees versus the saving propensities $\lambda_A$ and $\lambda_B$. Figures 10 (a)-(c) illustrate the deviation degrees of the whole economic system, group A, and group B, respectively. Clearly, the deviation degree ranges from zero to one in each plot. In contrast to the Gini coefficients and Kolkata indices, the deviation degree become large for high saving propensities $\lambda_A$ and/or $\lambda_B$. Similar to Gini coefficients and Kolkata indices, the contours of deviation degrees are also symmetrical about the diagonal line in Fig. 10 (a), and are asymmetric about the diagonal line in Fig. 10 (b) or (c).

It should be pointed out that the Gini coefficients, Kolkata indices, deviation degrees are functions of the saving propensities $\lambda_A$ and $\lambda_B$, as shown in Figs. 8 - 10. The three parameters describe the wealth inequality from different perspectives. It can be found that the wealth inequality is weakened as the saving propensities $\lambda_A$ and/or $\lambda_B$ rise from zero to one. In addition, the contours for groups A and B are the same if the axes $\lambda_A$ and $\lambda_B$ are transposed in Figs. 8 - 10 because groups A and B own the same numbers of agents.
4.2. Impact of proportions of agents

In fact, the proportions of agents with different saving propensities make a significant impact on the wealth distribution in the realistic society. Next, we probe the effects of proportions of agents with different saving propensities in groups $A$ and $B$. Without loss of generality, the total number of agents is fixed as $N = N_A + N_B = 1000$. The proportions of the agents in the two groups are adjustable. Let us introduce the symbol, $F_A = N_A/N$, which stands for the proportion of agents in group $A$. Hence the proportion of agents in group $B$ is $F_B = 1 - F_A$. Then, the values of $F_A$ and $F_B$ can be adjusted from zero to one and from one to zero, respectively. Moreover, for brevity, only three typical cases with different saving propensities ($\lambda_A$, $\lambda_B$) = (0.0, 1.0), (0.2, 0.8), and (0.4, 0.6), are under consideration in this subsection. In other words, the sum of the saving propensities of the two groups is fixed as $\lambda_A + \lambda_B = 1$. Other interesting financial circumstances are beyond this paper.

Figure 11 delineates the Gini coefficients (a), Kolkata indices (b), and deviation degrees (c) versus the proportion of agents in group $A$ when the commercial market reaches the steady state. From Figs. 11(a)-(c), the following points can be obtained.

(i) On the whole, with the increasing number of agents in group $A$, both the
Figure 12: The Gini coefficients (a), Kolkata indices (b), and deviation degrees (c) versus proportion of agents in group A. The saving propensities are $\lambda_A = 0.2$ and $\lambda_B = 0.8$. The lines with squares, circles and triangles represent the agents of all, group A and group B, respectively.

Gini coefficients and Kolkata indices increase monotonously, while the deviation degrees decrease monotonously. The Gini coefficients of group A (B) are higher (lower) than that of all agents under the condition $0 < F_A < 1$. Besides, the Kolkata indices own similar features, while the deviation degrees show opposite tendencies.

(ii) As shown in Fig. 12 (a), the Gini coefficients of all agents, group A, and group B become close to the minimum $G = 0.27$ as the proportion $F_A$ approaches zero, and are near the maximum $G = 0.5$ as $F_A$ tends towards one. Similarly, the range of the Kolkata indices is $0.597 \leq k \leq 0.682$ in Fig. 12 (b). On the contrary, the deviation degrees tend to the maximum $\Delta = 0.35$ as $F_A \rightarrow 0$, and the minimum $\Delta = 0$ as $F_A \rightarrow 1$.

(iii) The patterns of Gini coefficients, Kolkata indices, and deviation degrees of all agents show smooth curves, while these parameters in group A or B have a inflection point located at the abscissa $F_A = 0.5$, i.e., $N_A = N_B$. Specifically, the slope of the Gini coefficient in groups A for $N_A < N_B$ is larger than that for $N_A < N_B$. On the contrary, the slope of the Gini coefficient in groups B increases significantly at the connection point $F_A = 0.5$. Furthermore, the Kolkata indices (deviation degrees) have similar (opposite) variations.

To have a deeper understanding of the impact of proportions of agents on the
wealth distribution, we make a comparison of simulations with various saving propensities. Figures 12 and 13 illustrate the simulation results with saving propensities \((\lambda_A, \lambda_B) = (0.2, 0.8)\) and \((0.4, 0.6)\), respectively. It can be found that Figs. 11 - 13 have the following three similarities.

(i) Both Gini coefficients and Kolkata indices increase monotonously, and the deviation degrees decrease monotonously, as the proportion of agents in group A becomes large. Meanwhile, in the case of \(0 < F_A < 1\), both Gini coefficients and Kolkata indices of group A (B) are higher (lower) than that of all agents, while the deviation degrees show opposite tendencies.

(ii) All lines of Gini coefficients, Kolkata indices and deviation degrees of all agents, group A and group B converge at the starting point \((F_A = 0)\) and endpoint \((F_A = 1)\). To be specific, the starting (end) points correspond to the minima (maxima) of Gini coefficients and Kolkata indices, and the maxima (minima) of deviation degrees.

(iii) The lines of Gini coefficients, Kolkata indices or deviation degrees of all agents are smooth in the whole range, while each curve for groups A or B has a inflection point located at the abscissa \(F_A = 0.5\), and the changing trends across the inflection point are similar for various saving propensities. For example,
the left limit of the slope of the Gini coefficient of group $A$ is higher than its right limit at the inflection point. The Kolkata indices (deviation degrees) have identical (opposite) characteristics.

In addition to aforementioned similarities, there are some differences among Figs. 11 - 13.

(i) The ranges (as well as the minima and maxima) of the Gini coefficients, Kolkata indices or deviation degrees are different under different conditions of saving propensities. For example, the ranges of the Gini coefficients are $0.27 \leq G \leq 0.5$, $0.15 \leq G \leq 0.40$, and $0.23 \leq G \leq 0.32$ in Figs. 11 - 13, respectively. So do the Kolkata indices and deviation degrees.

(ii) The variations of Gini coefficients, Kolkata indices or deviation degrees are different for diverse saving propensities $\lambda_A$ and $\lambda_B$. Moreover, the slope changes of Gini coefficients, Kolkata indices or deviation degrees around the connection point are different. For instance, the left (right) limits of the slopes of the Gini coefficients of group $A$ are different with each other in Figs. 11 - 13.

5. Conclusion

In this work, we propose two types of transaction models for a closed economic market, where the total amount of money and the number of agents are fixed, and the saving propensities of agents can be either homogeneous or inhomogeneous. For Model I, the trading results are a function of the mean saving propensities of two agents who get engaged in a trade. For Model II, the business is controlled by a random parameter between the saving propensities of two traders. Additionally, two sampling methods are introduced for the random selection of two agents making transaction. Specifically, Technique I is sampling with replacement and is easier to program due to its simplicity. Technique II is sampling without replacement and takes less time to produce the equilibrium wealth distribution because of its high computing efficiency.

To demonstrate the kinetic models and computational approaches, we measure both nonequilibrium and equilibrium states of the financial systems where
two groups of agents have the same or different saving propensities. On the one hand, the relaxation processes are similar for Models I and II, and different for Techniques I and II. On the other hand, there are slight differences between the steady wealth distributions obtained from the two kinetic models and two computational techniques. In the evolution of the economic market, the law of money conservation is always obeyed. Furthermore, under the condition without saving propensities, the simulation results coincide with the theoretical solutions of the wealth probabilities or Lorenz curves. Besides, in the case with or without saving propensities, our models and techniques give identical simulation data to those of the CC method. In addition, the numerical robustness and reliability are verified by using different numbers of agents, values of average money, and temporal steps.

Moreover, we investigate the influence of saving propensities upon the wealth distribution. The contours of Gini coefficients, Kolkata indices and deviation degrees of all agents and two groups are plotted in the space of saving propensities $\lambda_A$ and $\lambda_B$. On the whole, the Gini coefficients and Kolkata indices decrease with the increasing of $\lambda_A$ and/or $\lambda_B$, while the deviation degrees become large for high saving propensities $\lambda_A$ and/or $\lambda_B$. That is to say, the wealth inequality is weakened as the saving propensities become large. Finally, we probe the effects of proportions of agents with different saving propensities in two groups. Three cases of saving propensities are simulated under the condition $\lambda_A < \lambda_B$. It is found that both Gini coefficients and Kolkata indices increase monotonously, and the deviation degrees decrease monotonously, as the proportion of agents in group $A$ becomes large. The ranges and variations of the Gini coefficients, Kolkata indices or deviation degrees are different for various saving propensities.

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References

[1] R. N. Mantegna, H. E. Stanley, Introduction to econophysics: correlations and complexity in finance, Cambridge university press, Cambridge, 1999.

[2] D. Acemoglu, S. Johnson, J. A. Robinson, Reversal of fortune: Geography and institutions in the making of the modern world income distribution, Q. J. Econ. 117 (4) (2002) 1231–1294. doi:10.1162/003355302320935025.

[3] D. Ludwig, V. M. Yakovenko, Physics-inspired analysis of the two-class income distribution in the USA in 1983–2018, Philos. T. Roy. Soc. A. 380 (2224) (2022) 20210162. doi:10.1098/rsta.2021.0162.

[4] N. E. Aktaev, K. Bannova, Mathematical modeling of probability distribution of money by means of potential formation, Physica A 595 (2022) 127089. doi:10.1016/j.physa.2022.127089.

[5] S. Cordier, L. Pareschi, G. Toscani, On a kinetic model for a simple market economy, J. Stat. Phys. 120 (1) (2005) 253–277. doi:10.1007/s10955-005-5456-0.

[6] A. Chakraborti, B. K. Chakrabarti, Statistical mechanics of money: how saving propensity affects its distribution, Eur. Phys. J. B 17 (1) (2000) 167–170. doi:10.1007/s100510070173.

[7] H. Quevedo, M. N. Quevedo, Income distribution in the Colombian economy from an econophysics perspective, Cuadernos de Economía 35 (2016) 691 – 707. doi:10.15446/cuad.econ.v35n69.44876.

[8] A. Drăgulescu, V. M. Yakovenko, Evidence for the exponential distribution of income in the USA, Eur. Phys. J. B 20 (4) (2001) 585–589. doi:10.1007/PL00011112.

23
[9] B.-H. F. Cardoso, S. Goncalves, J. R. Iglesias, Wealth distribution models with regulations: Dynamics and equilibria, Physica A 551 (2020) 124201. doi:10.1016/j.physa.2020.124201

[10] J. W. Gibbs, Elementary Principles in Statistical Mechanics, Charles Scribner’s Sons, New York, 1902.

[11] Y. Tao, X. Wu, T. Zhou, W. Yan, Y. Huang, H. Yu, B. Mondal, V. M. Yakovenko, Exponential structure of income inequality: evidence from 67 countries, J. Econ. Interact. Coord. 14 (2) (2019) 345–376. doi:10.1007/s11403-017-0211-6

[12] M. Nirei, W. Souma, A two factor model of income distribution dynamics, Rev. Income Wealth 53 (3) (2007) 440–459. doi:10.1111/j.1475-4991.2007.00242.x

[13] M. Newby, A. Behr, M. S. Feizabadi, Investigating the distribution of personal income obtained from the recent US data, Econ. Modell. 28 (3) (2011) 1170–1173. doi:10.1016/j.econmod.2010.12.006

[14] A. Chatterjee, B. K. Chakrabarti, S. Manna, Pareto law in a kinetic model of market with random saving propensity, Physica A 335 (1-2) (2004) 155–163. doi:10.1016/j.physa.2003.11.014

[15] L. Lima, Modeling of the financial market using the two-dimensional anisotropic Ising model, Physica A 482 (2017) 544–551. doi:10.1016/j.physa.2017.04.090

[16] J. P. Pinasco, M. Rodriguez Cartabia, N. Saintier, A game theoretic model of wealth distribution, Dyn. Games Appl. 8 (4) (2018) 874–890. doi:10.1007/s13235-018-0240-3

[17] L. Cui, C. Lin, Lattice–Gas–Automaton Modeling of Income Distribution, Entropy 22 (7) (2020) 778. doi:10.3390/e22070778
[18] L. Cui, C. Lin, A simple and efficient kinetic model for wealth distribution with saving propensity effect: Based on lattice gas automaton, Physica A 561 (2021) 125283. doi:10.1016/j.physa.2020.125283

[19] X. Zhou, K. Xiang, R. Sun, The Study of a Wealth Distribution Model with a Linear Collision Kernel, Math. Probl. Eng. 2021 (2021) 2142876. doi:10.1155/2021/2142876

[20] S. Paul, S. Mukherjee, B. Joseph, A. Ghosh, B. K. Chakrabarti, Kinetic exchange income distribution models with saving propensities: inequality indices and self-organized poverty level, Philos. T. Roy. Soc. A. 380 (2224) (2022) 20210163. doi:10.1098/rsta.2021.0163