Right-handed Neutrinos in F-theory Compactifications

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Abstract

F-theory is one of the frameworks where up-type Yukawa couplings of SU(5) unified theories are naturally generated. As charged matter fields have localized zero modes in F-theory, a study of flavor structure could be easier in F-theory than in Heterotic string theory. In a study of flavor structure in the lepton sector, however, an important role is played by right-handed neutrinos, which are not charged under the SU(5) unified gauge group. It is therefore solicited to find out what right-handed neutrinos are in F-theory compactifications and how their Majorana mass terms are generated together with developing a theoretical framework where effective Yukawa couplings involving both SU(5)-neutral and charged fields can be calculated. We find that the complex structure moduli chiral multiplets of F-theory compactifications are good candidates to be right-handed neutrinos, and that their Majorana masses are automatically generated in flux compactifications. The mass scale is predicted to be somewhat below the GUT scale, which is in nice agreement with the $\Delta m^2$ of the atmospheric neutrino oscillation through the see-saw mechanism. We also discuss various scenarios of solving the dimension-4 proton decay problem in supersymmetric F-theory compactifications, along with considering the consequences of those scenarios in the nature of right-handed neutrinos.
1 Introduction

The Standard Model of particle physics, including neutrino masses, have 20–22 parameters associated with flavor, depending on whether the neutrino masses are Dirac or Majorana. The flavor parameters constitute the dominant fraction of the parameters appearing in the effective action of the Standard Model and this calls for a better theoretical understanding. Super Yang–Mills theories in higher-dimensional spacetime realized in geometric compactifications of superstring theory can yield charged massless matter fields as well as their Yukawa interactions by further compactification down to 3+1 dimensions. The possible forms of trilinear couplings of the matter fields are determined by the Lie algebra of the microscopic super Yang–Mills theory, and a detailed pattern of the Yukawa couplings follows from geometry. Compactifications of superstring theory, therefore, have a chance to be a framework for understanding of the flavor physics.

If one takes SU(5)\textsubscript{GUT} unification seriously\textsuperscript{1}, then one has to wonder what higher dimensional super Yang–Mills theory would give rise to the up-type Yukawa couplings

\[
\Delta \mathcal{L} = \lambda^{(u)} \mathbf{10}^{ab} \mathbf{10}^{cd} H(5)^e \epsilon_{abcde}.
\]

This contraction by \(\epsilon_{abcde}\) arises from the structure constant of \(E_6\) Lie algebra and a \(E_{7,8}\) that contains \(E_6\). Heterotic \(E_8 \times E_8\) string theory, M-theory and F-theory are the candidates for such a theoretical framework\textsuperscript{1}.

In M-theory compactifications on manifolds with \(G_2\) holonomy and in F-theory compactifications on elliptic fibered Calabi–Yau 4-folds, the charged matter fields are localized in the internal spaces. This localized picture of matter fields enables us to discuss the flavor pattern of Yukawa matrices in rather intuitive ways. Explanations for hierarchical eigenvalues\textsuperscript{2} as well as generation structures in the quark sector\textsuperscript{3} have been achieved in phenomenological models by using higher-dimensional gauge theory with localized matter wavefunctions. An intuitive picture of localized matter fields and the mechanism of generating Yukawa couplings in M-theory compactifications has also been exploited in\textsuperscript{4}, to see that the up-type Yukawa matrix in\textsuperscript{4} tends to have suppressed diagonal entries in M-theory compactifications on manifolds with \(G_2\) holonomy. That is not in very good agreement with the real world. It is also worth studying flavor pattern of generic F-theory compactifications, exploiting the fact that charged matter fields have localized wavefunctions.

\textsuperscript{1} quark doublets and lepton doublets belong to different irreducible representations
It is by now known how many massless charged matter fields are in the low-energy spectrum, and how to determine their zero-mode wavefunctions, once a geometry for supersymmetric F-theory compactification is given [4, 5, 6, 7, 8, 9]. A prescription for how to calculate Yukawa couplings for three charged matter fields in F-theory compactifications has also been written down [7, 6, 10, 9]. A field theory formulation capturing a certain sector of dynamical degrees of freedom in F-theory [11, 6, 7] turns out to be very useful for this purpose [9].

Nevertheless, the Yukawa couplings involving only SU(5)$_{\text{GUT}}$-charged matter fields do not account for neutrino Yukawa couplings. In order to study flavor physics in the lepton sector, it is crucial that we have a theoretical formulation dealing with Yukawa couplings that involve neutral matter fields. Therefore, in this article, we develop the crucial issue of identifying the (SU(5)$_{\text{GUT}}$-neutral) right-handed neutrinos in the F-theory compactifications. We develop a general prescription of calculating the Yukawa couplings involving neutral fields, with a direct application for neutrino Yukawa couplings.

In section 2, we will discuss a possibility that complex structure moduli of F-theory compactifications are identified with chiral multiplets of right-handed neutrinos. Those moduli fields acquire large masses in flux compactifications. A main result of our paper is that the typical mass scale in flux compactifications, being below the GUT scale, is just about right for the mass scale of the right-handed neutrinos predicted by experimental data.

The neutral fields of unified theories such as right-handed neutrinos are better captured in generic F-theory compactifications as moduli fields of Calabi–Yau 4-fold compactifications. On the other hand, field theory models for local geometry of Calabi–Yau 4-folds are better tools in calculating Yukawa couplings such as neutrino Yukawa couplings, where both charged fields and these moduli fields are involved. We therefore explain in section 3 how to combine the two descriptions of F-theory compactifications to calculate Yukawa couplings involving neutral fields. As a digression, we discuss the cubic term of a neutral field in the next-to-minimal supersymmetric standard model in section 3.2.2.

The dimension-4 proton decay is a serious phenomenological problem in compactifications with low-energy supersymmetry. The absence of rapid proton decay implies that the complex structure moduli for our real world is somewhat special. Thus, right-handed neutrinos should be regarded as fluctuations of complex structure moduli from a special choice of complex structure moduli in our vacuum. In section 4, we present some scenarios solving the dimension-4 proton decay problem, and discuss the consequences in the physics of right-handed neutrinos.

We noticed that recent articles [12, 13, 14, 15, 16, 17] cover similar subjects.
\section{Singlet Masses from Moduli Stabilization}

The right-handed neutrinos $\bar{N}$ are not charged under SU(5)$_{\text{GUT}}$ unification group, and they are supposed to have trilinear couplings

$$\Delta \mathcal{L} = \lambda^{(\nu)}_{ij} \bar{N}_i l_j h_u + \text{h.c.}. \quad (2)$$

Here, $l_j$ are lepton doublets of the Standard Model, and $h_u$ the Higgs doublet. This is the only certainty from phenomenology. Therefore, in string phenomenology, any light degrees of freedom that are neutral under SU(5)$_{\text{GUT}}$ are qualified to be considered right-handed neutrinos, as long as they have the coupling (2).

Neutrino masses indicated by neutrino oscillation experiments are much smaller than the mass eigenvalues of the quarks and charged leptons of the Standard Model, this being a natural prediction of the see-saw mechanism. If right-handed neutrinos have mass terms,

$$\Delta \mathcal{L} = M_{ii'} \bar{\bar{N}}_i \bar{N}_{i'} + \text{h.c.}, \quad (3)$$

with the eigenvalues of $M_{ii'}$ much larger than the electroweak scale, then the Majorana masses of the left-handed neutrinos are generated, with their mass eigenvalues much smaller than the electroweak scale. From the measured value $\Delta m^2 \simeq 2 - 3 \times 10^{-3}$ eV$^2$ of the atmospheric neutrino oscillation [18], one can conclude that the lightest right-handed neutrino is not heavier than about

$$\frac{(v\lambda^{(\nu)})^2}{\sqrt{\Delta m^2}} = (\lambda^{(\nu)})^2 \times (5.5 - 6.7) \times 10^{14} \text{ GeV}. \quad (4)$$

Here, $v \simeq 174$ GeV is the Higgs vacuum expectation value (vev). The neutrino Yukawa couplings $\lambda^{(\nu)}$ should not be much larger than unity, otherwise the perturbative field theory description would immediately become invalid because of the renormalization group flow of the couplings $\lambda^{(\nu)}$. Thus, the mass scale of right-handed neutrinos is below the energy scale of gauge coupling unification $M_{\text{GUT}} \sim 10^{16}$ GeV (simply called the GUT scale) with a safe margin.

We will now address the issues:

- What are the right-handed neutrinos in string phenomenology, and how many of them are there?
- Where does the energy scale of the Majorana masses $M_{ii'}$ come from?
- Where do the neutrino Yukawa couplings (2) come from?
2.1 Complex Structure Moduli as Right-Handed Neutrinos

The compactification of F-theory to $\mathcal{N} = 1$ supersymmetry in 4-dimensions is described by a set of data, $(X, G^{(4)})$. $X$ is a Calabi–Yau 4-fold that is an elliptic fibration over a base 3-fold $B_3$:

$$\pi_X : X \rightarrow B_3.$$  \hspace{1cm} (5)

$G^{(4)} = dC^{(3)}$ is a 4-form flux on $X$, and $C^{(3)}$ is the 3-form potential in the language of eleven-dimensional supergravity. The 4-form flux $G^{(4)}$ has to take its value only in the $(2,2)$ component, and be primitive in order to leave an unbroken $\mathcal{N} = 1$ supersymmetry [19]. This class of compactifications has $h^{3,1}$ complex structure moduli, $h^{1,2}$ moduli associated with the configuration of $C^{(3)}$ that is not reflected in the field strength $G^{(4)} = dC^{(3)}$, and Kähler moduli chiral multiplets.

The complex structure moduli have interactions in the superpotential [20]

$$\Delta W = W_{GVW} = \int_X \Omega \wedge G^{(4)}. \hspace{1cm} (6)$$

In the absence of a flux $G^{(4)}$ on $X$, complex structure moduli parametrizing $\Omega$ would have remained massless (in the absence of supersymmetry breaking). Once a generic flux $G^{(4)}$ is introduced, however, potential is generated for the moduli, and the complex structure of $X$ dynamically sets itself to a minimum of the potential, so that $G^{(4)}$ has vanishing $(1,3)$ components. All the complex structure moduli generically have mass terms around such a minimum [21].

When the Calabi–Yau 4-fold $X$ is given by an equation $f = 0$ on a space with a set of local coordinates $(x_1, x_2, \cdots, x_5)$, then $\Omega$ has an expression

$$\Omega = \text{Res}_{f=0} \frac{dx_1 \wedge \cdots \wedge dx_5}{f} = \frac{dx_1 \wedge \cdots \wedge dx_4}{\partial f/\partial x_5} = \cdots. \hspace{1cm} (7)$$

Coefficients in the defining equation $f = 0$—collectively denoted by $a$—set the complex structure of $X$. The vacuum choice $\Omega(a = a_0)$ is a pure $(4,0)$-form when evaluated on a holomorphic coordinates corresponding to the vacuum value $a = a_0$. When $\Omega(a)$ is expanded in fluctuations from vacuum $\delta a \equiv a - a_0$, $\Omega(a)$ at order $O(\delta a)$ stays within $H^{(4,0)}(X; \mathbb{C}) \oplus H^{(3,1)}(X; \mathbb{C})$ with respect to the vacuum holomorphic coordinates at $a = a_0$ [22]. This is why non-vanishing flux $G^{(4)}$ in the $(1,3)$-component when evaluated at vacuum $a = a_0$ would have meant non-vanishing F-term vev $\langle \delta W/\delta a \rangle|_{a=a_0}$ (and supersymmetry breaking). The 4-form
Ω(a) at order $\mathcal{O}((\delta a)^2)$ remains within $H^{(4,0)}(X; \mathbb{C}) \oplus H^{(3,1)}(X; \mathbb{C}) \oplus H^{(2,2)}(X; \mathbb{C})$ evaluated the vacuum complex structure, and is expressed as a sum of the form

$$\Omega(a) = \Omega(a_0) + (k_a \Omega(a_0) + \chi_a) (\delta a)_a + (k_{ab} \Omega(a_0) + l_{ab} \chi_c + \psi_{ab}) (\delta a)_a (\delta a)_b. \quad (8)$$

Here, $\chi_a$'s are basis of $H^{(3,1)}(X_{a=a_0}; \mathbb{C})$, and $\psi_{ab}$ elements of $H^{(2,2)}(X_{a=a_0}; \mathbb{C})$. The Gukov–Vafa–Witten superpotential with generic flux $G^{(4)} \in H^{(2,2)}(X_{a=a_0})$ gives rise to the non-vanishing mass terms (quadratic term)

$$W_{GVW} = \int_X \langle \Omega \rangle \wedge \langle G^{(4)} \rangle + \left( \int_{X_{a=a_0}} \psi_{ab} \wedge \langle G^{(4)} \rangle \right) (\delta a)_a (\delta a)_b + \cdots \quad (9)$$

for the fluctuations of the complex structure moduli $(\delta a)$.

This is a standard story of flux compactification and stabilization of complex structure moduli in Type IIB string theory / F-theory. Now, as discussed at the beginning of this section, any light degrees of freedom that are neutral under $SU(5)_{GUT}$ have a chance to be identified with right-handed neutrinos. In this article the complex structure moduli are identified with the right-handed neutrinos so the moduli masses from the flux compactification immediately become the Majorana masses of right-handed neutrinos.

Note that the need for right-handed neutrinos in the see-saw scenario does not motivate phenomenologically an unbroken SO(10) symmetry in the effective theory in 3+1 dimensions, or its realization on a stack of coincident branes. What used to be right-handed neutrinos in the spinor representation of SO(10) GUT just becomes neutral vector-bundle/brane-configuration moduli of SU(5) GUT, as one can see immediately by following the Higgs cascade studied in [23, 24]. The idea of identifying right-handed neutrinos with vector bundle moduli in Heterotic string compactification with SU(5) GUT dates back (at least to our knowledge) to [25]. Under the duality between the Heterotic string and F-theory, all of vector bundle moduli and complex structure moduli in Heterotic string theory correspond to the complex structure moduli $H^{3,1}(X; \mathbb{C})$ (and $H^{1,2}(X; \mathbb{C})$, which we mention later) in F-theory compactification [26, 27, 24, 28, 29]. Since the vector bundle moduli in Heterotic string theory have the desired Yukawa couplings for neutrinos

$$\Delta W = \lambda \bar{5} 1 5, \quad (10)$$

we expect that the complex structure moduli in F-theory also have (at least qualitatively) the same interactions. Therefore, it is quite natural to identify the complex structure moduli with
right-handed neutrinos in F-theory compactification. Their Majorana masses are generated by the Gukov–Vafa–Witten superpotential, as discussed before.

In F-theory compactifications, the number of complex structure moduli, $h^{3,1}(X)$, is usually much larger than three, the number of “generations” of SU(5)$_{\text{GUT}}$-charged quarks and leptons. This is not a contradiction from phenomenological perspectives. All we know for sure is that at least two right-handed neutrinos are necessary in order to account for all the data of neutrino oscillation experiments in the see-saw mechanism [30]. There is no upper bound from phenomenology on the number of right-handed neutrinos. In fact, there is even an indication [31] that the number of right-handed neutrinos may be much larger than just three. Thus, a higher number of right-handed neutrinos possibly obtained from complex structure moduli is not problematic at all, and may even be a blessing in disguise.

2.2 Estimation of the Majorana Mass Scale

Let us now estimate the energy scale of the Majorana masses of right-handed neutrinos, assuming that the right-handed neutrinos are complex structure moduli, and that the Majorana masses derive from the superpotential (6, 9) in F-theory compactifications. We begin with a review of a similar problem in Type IIB orientifold compactifications.

In Type IIB string compactification on Calabi–Yau orientifolds on $B_3 = \tilde{B}/\mathbb{Z}_2$, the complex structure moduli acquire masses $m_{cs}$ with an order of magnitude estimate given by [32]

$$m_{cs}^2 \sim m_{KK}^6 l_s^4 = m_{KK} \times \left( \frac{l_s}{R_6} \right)^2. \quad \text{(11)}$$

Here, $m_{KK} = 1/R_6$ is the “Kaluza–Klein scale”, assuming that the Calabi–Yau 3-fold $\tilde{B}$ for an orientifold compactification of Type IIB string theory is almost isotropic and its radius characterized by a single parameter $R_6$. We define the string length $l_s$ as $l_s \equiv (2\pi/\sqrt{\alpha'})$.

This expression is understood as follows. The quantum fluctuations of the complex structure moduli fields correspond to $(0, 2)$-type fluctuations of the metric on the Calabi–Yau 3-fold. Simple dimensional reduction leads to kinetic terms

$$\Delta L_{\text{kin}} \sim \frac{1}{l_s^4 g_s^2} \int_{\tilde{B}} d^6 y R \sim M_{\text{Pl}}^2 |\partial \phi|^2 \sim \frac{R_6^6}{l_s^8 g_s^2} |\partial \phi|^2. \quad \text{(12)}$$

$\phi$ are complex scalar fields in the effective field theory on 3+1 dimensions and correspond to the complex structure moduli fields. Mass dimension of the $\phi$ fields is set to zero here. On
the other hand, such fluctuations of the metric change the complex structure of the Kähler manifold, and the imaginary-self-dual 3-form flux configuration in a vacuum is no longer imaginary-self-dual, costing potential energy. To obtain an estimate of the field strength of the 3-form flux \( G^{(3)} \equiv F^{(3)} - \tau H^{(3)} \), note that

\[
\frac{1}{l_s^4} \int_B H^{(3)} \wedge F^{(3)} = \frac{g_s}{2l_s^4} \int_B d^6 y \left| G^{(3)} \right|^2
\]

for imaginary-self-dual flux \( *G^{(3)} = iG^{(3)} \). Since the left-hand side of (13) is quantized, and hence so is the right-hand side, the typical value of the field strength will be of order

\[
\left\langle G^{(3)} \right\rangle \sim \frac{l_s^4}{R_6^3 \sqrt{g_s}}.
\]

Thus, one finds that the potential energy of the complex structure moduli field \( \phi \) is of order

\[
V_{cs}(\phi) \sim \frac{1}{l_s^4 g_s} \int_B d^6 y \left| G^{(3)} \right|^2 \sim \frac{1}{l_s^4 g_s} \times \text{fcn}(\phi).
\]

This is why the mass-square of canonically normalized \( \phi \) at a vacuum are typically of order

\[
m_{cs}^2 \sim \frac{1}{l_s^4 g_s} \times \frac{l_s^8 g_s^2}{R_6^6} = \frac{g_s l_s^4}{R_6^6}.
\]

When an F-theory compactification on a Calabi–Yau 4-fold \( X \) allows an interpretation as a Calabi–Yau orientifold \( B_3 = \tilde{B}/\mathbb{Z}_2 \) compactification of the Type IIB string theory, then the complex structure moduli of \( X \) consist of complex structure moduli of the Calabi–Yau 3-fold \( \tilde{B} \) and moduli describing the locus of D7-branes in \( \tilde{B} \), in addition to the axio-dilaton chiral multiplet \( \tau \). In a generic F-theory compactifications, which do not necessarily correspond to simple Calabi–Yau orientifold compactifications of Type IIB string theory, there is no distinction between the complex structure of \( \tilde{B} \) and the moduli of D7-brane

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\[ \text{2The } 1/\sqrt{g_s} \text{ dependence is missing in some literature papers. Interestingly, the estimate (14) corresponds to a naive geometric mean of } \left\langle F^{(3)} \right\rangle \sim l_s^2/R_6^3 \text{ and } \left\langle g_s^{-1} H^{(3)} \right\rangle \sim l_s^2/(R_6^3 g_s). \]

\[ \text{3In (12) and in all the rest of this article, all the lengths, volumes, Kaluza–Klein scales etc. are measured with a metric in the string frame. It is conventional in Type IIB orientifolds that the string frame metric } g^{(S)} \text{ and the Einstein frame metric } g^{(E)} \text{ in 10 dimensions are related by Weyl rescaling } g^{(S)} = e^{\phi/2} g^{(E)}, \text{ where the dilaton field } \phi \text{ includes both its vacuum value } \left\langle \phi \right\rangle = g_s \text{ and fluctuation from the vev, } (\phi - \left\langle \phi \right\rangle). \text{ All physical observables } M_{(d)} \text{ with mass dimension } d \text{ (except } l_s) \text{ in string frame and Einstein frame are related by } M_{(d)}^{(S)} = g_s \frac{d}{4} M_{(d)}^{(E)}. \text{ Thus, this dependence on the frame or convention of Weyl rescaling cancels as long as we talk of dimensionless ratios. } \left( m_{cs}/m_{KK} \right)^2 \sim (l_s/R_6^3)^4 \text{ in (11) is expressed in terms of } R_6 \text{ in the Einstein frame, and this is equivalent to } (g_s l_s^4)/(R_6^4)^4, \text{ which is the same as (16).} \]
configuration in $\tilde{B}$; we just have complex structure moduli of $X$ as a whole, and generically, all of these moduli are stabilized. The estimate (11) was obtained for Calabi–Yau orientifold compactifications of Type IIB string theory with D7-branes (but without D5-branes, O5-planes), which is a special subclass of F-theory compactifications. It is not immediately clear whether the result for the complex structure moduli of $\tilde{B}$ in a special subclass of F-theory compactifications is readily applied for all the complex structure moduli of $X$ of generic F-theory compactification. We therefore perform an analysis separately in the following for generic F-theory compactifications.

An F-theory compactification on an elliptically fibered Calabi–Yau 4-fold $X$ is regarded as an M-theory compactification on the same $X$ in the limit the volume of the elliptic fiber is zero. We use the supergravity language of M-theory compactifications, to estimate the mass scale of the complex structure moduli fields. Let us first remind ourselves of dictionary between parameters in the duality of M-theory compactification on $T^2$ and Type IIB string theory on 10 dimensions. The $T^2$ compactification of 11-dimensional supergravity has two compactification parameters $\rho_\alpha$ and $\rho_\beta$, the size of the two edges of $T^2$, in addition to the unique theoretical parameter $l_{11}^0 \equiv (4\pi)\kappa_{11}^2$ appearing in the action of the 11-dimensional supergravity. The $T^2$ compactification of the M-theory is dual to the Type IIB string theory compactified on $S^1$. The Type IIB string theory has two theoretical parameters $l_s$ and $g_s$, and one compactification parameter, the circumference $R_3$ of the compact $S^1$. The dictionary between the two descriptions is

$$
\frac{i}{g_s} = \frac{i\rho_\beta}{\rho_\alpha}, \quad M_s^4 \equiv \frac{1}{g_s l_s^4} = \frac{\rho_\alpha^2}{l_{11}^0}, \quad R_3 = \frac{r_s^2}{\rho_\beta} = \frac{l_{11}^3}{\rho_\beta^3}.
$$

(17)

where $\rho \equiv \sqrt{\rho_\alpha \rho_\beta}$. The F-theory limit corresponds to $\rho \to 0$ and $l_{11} \to 0$ (relatively to the Kaluza–Klein radius or to the horizon size of the universe), while keeping $M_s^4 = \rho/l_{11}^0$ finite. In $T^2$-fibered compactification of 11-dimensional supergravity, the value of $\rho_\beta/\rho_\alpha$ may vary over the base manifold. In F-theory language, $i/g_s$ varies over the base space $B_3$. The two combinations, $M_s$ and $R_3$, however, depend only on $l_{11}$ and $\rho$, but not on $\rho_\beta/\rho_\alpha$, and remain constant over the base.

We now consider an M-theory compactification on a Calabi–Yau 4-fold $X$ that is an elliptic fibration $\pi : X \to B_3$ on a 3-fold $B_3$. Let the typical size of the base $B_3$ be $R_6$, and that of the elliptic fiber be $\rho$. Then the effective action of the complex structure moduli of $X$ in 3+1 dimensions is of the form

$$
R_3 \Delta \mathcal{L}_{3+1} = \Delta \mathcal{L}_{2+1}; \quad \Delta \mathcal{L}_{2+1} \sim \frac{R_6^2 \rho^2}{l_{11}^0} \times \left[ |\partial \phi|^2 - \langle G^{(4)} \rangle^2 \times \text{fcn}(\phi) \right].
$$

(18)
The overall factor $R_6^6 \rho^2 / l_{11}^9$ comes from integration over the real 8-dimensional manifold $X$, and $R_3$ is factored out from $\Delta L_{2+1}$ to obtain the effective action in the 3+1-dimensions. The overall factor, however, is irrelevant to the masses of the complex structure moduli fields $\phi$. The value of the field strength is directly relevant to the mass scale of the moduli; $\langle G^{(4)} \rangle$ is the typical value of the 4-form field strength $C^{(4)}_{R LMN}$, which is of mass dimension 1 because we treat the 3-form potential field $C^{(3)}_{LMN}$ as dimensionless. As noted in [33], the field strength of the 4-form is typically of order

$$\langle G^{(4)} \rangle \sim \frac{l_{11}^3}{R_8^3}$$  \hspace{1cm} (19)$$

in Calabi–Yau 4-fold compactification of M-theory; $R_8$ is the typical size of the real 8-dimensional manifold $X$. In F-theory compactifications that leave SO(3,1) unbroken Lorentz symmetry, only a limited class of components of the 4-form field strength $G^{(4)}$ is allowed. The vacuum value of $G^{(4)}$ can be introduced only in components with one leg in the elliptic fiber, and the remaining three legs in the base $B_3$ [34]. Therefore, we conclude that the moduli masses are typically of order

$$m_{cs} \sim \langle G^{(4)} \rangle \sim \frac{l_{11}^3}{R_6^3 \rho} \sim \frac{1}{R_6^3 M_s^2},$$  \hspace{1cm} (20)$$

where we used the dictionary \[17\] of the Type IIB Calabi–Yau orientifolds. \[4\] This estimate turns out to be the same as the result \[16\] of the Type IIB Calabi–Yau orientifolds.

Therefore, the Majorana masses of complex structure moduli (and hence of right-handed neutrinos) are in the energy scale just below Kaluza–Klein scale $1/R_6$, as we consider in a regime where $R_6$ is larger than the string length $l_s$. The SU(5) GUT symmetry can be broken by introducing U(1)$_Y$ flux on the locus of $A_4 \simeq$ SU(5) singularity [10, 35, 36] in which case, the GUT scale is identified with the Kaluza–Klein scale. Thus, the Majorana masses of the right-handed neutrinos are typically somewhat below the GUT scale. This is a perfect agreement with the phenomenological requirement that the Majorana masses should be below the GUT scale.

\[4\] The overall factor in the effective action in 3+1 dimensions becomes $R_6^6 M_s^8 \sim M_{Pl}^2$, and remains finite.

\[5\] In the estimation \[20\], there is no strong argument for taking $R_4^4 \sim R_6^3 \rho$, $R_6^3 \rho_3$ or $R_6^3 \rho$. Thus, the estimate \[20\] contains an uncertainty at least of order $g_s^{1/2}$, $g_s^{-1/2}$. In generic F-theory compactifications, however, the value of $g_s$ varies over $B_3$, and cannot stay much larger or less than unity over the entire $B_3$, because of the non-trivial SL(2,$Z$) monodromy. It will be smaller than unity in some region of $B_3$, and will be larger in other. Thus, despite the uncertainty and varying value of $g_s$, it will be safe to infer conclude that some of complex structure moduli will have masses below the mass scale \[20\].

\[6\] There is no extra $g_s$ dependence in the Type IIB Calabi–Yau orientifolds.

\[7\] See [37, 38] for Type IIB versions.
Let us refine the estimate of the mass scale a bit more, by making a distinction between the Kaluza–Klein scale and the GUT scale. In the language of Type IIB orientifold compactifications, three observables—the unified gauge coupling constant \( \alpha_{\text{GUT}} \equiv g_{\text{GUT}}^2/(4\pi) \), energy scale of gauge coupling unification \( M_{\text{GUT}} \) and the reduced Planck scale \( M_{\text{Pl}} = 1/\sqrt{8\pi G_N} \), are given by compactification parameters as follows:

\[
\begin{align*}
\frac{1}{\alpha_{\text{GUT}}} &= \frac{R_{\text{GUT}}^4}{g_s l_s^4} = (R_{\text{GUT}} M_*)^4 \approx 24, \\
M_{\text{GUT}} &= \frac{c}{R_{\text{GUT}}} \sim 10^{16} \text{ GeV}, \\
M_{\text{Pl}}^2 &= \frac{(4\pi) R_6^6}{g_s^2 l_s^8} = (4\pi) R_6^6 M_*^8 \approx (2.4 \times 10^{18} \text{ GeV})^2.
\end{align*}
\]

Since \( g_s \) and \( l_s \) come in in the three equations above only as a combination \( (g_s l_s^4) \equiv 1/M_*^4 \), we consider that all the three equations above perfectly makes sense in generic F-theory compactifications without an interpretation as Type IIB Calabi–Yau orientifold. The volume of the locus \( S \) of \( A_4 \) singularity is \( R_{\text{GUT}}^4 \), and the volume of \( B_3 \) is \( R_6^6 \). Thus, the energy scale of gauge coupling unification \( M_{\text{GUT}} \) should roughly be the same as \( 1/R_{\text{GUT}} \), but they can be different, for example by a factor \( c = 2\pi \) in case \( S = T^4 \). The numerical coefficient \( c \) can be regarded as a factor of order unity in general, ranging in between \( c \sim (1-2\pi) \). We assumed that there are no extra SU(5)_{\text{GUT}}-charged particles much below the unification scale other than the particles in the minimal supersymmetric standard model; otherwise, the value of \( \alpha_{\text{GUT}} \) should be a little larger. One should also note that the value of \( M_{\text{GUT}} \) is accompanied by uncertainty of the half order of magnitude or so, coming from unknown tree-level and 1-loop threshold corrections. The three observables \( \alpha_{\text{GUT}}, M_{\text{GUT}} \) and \( M_{\text{Pl}} \) are expressed in terms of the same number of fundamental parameters \( M_*, R_{\text{GUT}} \) and \( R_6 \) in (21–23). Thus, the values of these parameters can be determined from the values of the observables in our world, which are also given in (21–23).

8 The reduced Planck scale is defined as the coefficient in the Einstein–Hilbert term \( \mathcal{L} = (M_{\text{Pl}}^2/2)\sqrt{-g}R + \cdots \) in the effective action in the 3+1 dimensions. The reduced Planck scale is different from \( 1/\sqrt{G_N} \approx 1.2 \times 10^{19} \text{ GeV} \).

9 Here, we do not pay attention to a possible factor 2 associated with orientifold projection.

10 A beautiful study of GUT scale threshold corrections is found in [36]. See also [39, 35, 40] and [41].
Figure 1: Cartoon picture of $B_3$ in the homogeneous model (a) and in the tubular model (b). This is a reproduction of essentially the same figure in [10] just for readers’ convenience.

Following [10], we introduce a dimensionless parameter $\epsilon$:

$$\epsilon \equiv \left( \frac{R_{\text{GUT}}}{R_6} \right)^3 = \frac{\sqrt{4\pi} M_{\text{GUT}}}{\alpha_{\text{GUT}} c M_{\text{Pl}}} \sim 0.35 \times \left( \frac{M_{\text{GUT}}}{c \times 10^{16} \text{ GeV}} \right). \quad (25)$$

In a homogeneous model, where $B_3$ looks like Figure 1 (a), the 4-form field strength $\langle G^{(4)} \rangle$ in (19) becomes $l_{11}^3/(\rho R_6 R_{\text{GUT}}^{3-n})$ for some $n = 0, 1, 2, 3$. The unspecified parameter $n$ remains because some of the 4-form fluxes are associated with 4-cycles around $S$, others are not. Using $\epsilon = 1/3 = (R_{\text{GUT}}/R_6)$, the estimate of the masses of complex structure moduli is rewritten as

$$m_{\text{cs}} \sim M_{\text{GUT}} \times \frac{\sqrt{\alpha_{\text{GUT}}}}{c} \times (\epsilon^{1/3})^n, \quad (26)$$

where we have used (21, 22). All the factors $\sqrt{\alpha_{\text{GUT}}}$, $(1/c)$ and $(\epsilon^n)$ make the moduli mass smaller than the unification scale. In a tubular model where $B_3$ is like a product of $A_4$.

The two energy scales in the effective theory in 3+1 dimensions, $M_{\text{GUT}}$ and the Planck scale $1/\sqrt{G_N}$ have a hierarchy of order $M_{\text{GUT}} \sqrt{G_N} \sim 0.8 \times 10^{-3} \times (M_{\text{GUT}}/10^{16} \text{ GeV})$, and some people have taken it seriously. This apparent hierarchy in the effective theory, however, is expressed in terms of more fundamental parameters as

$$M_{\text{GUT}} \sqrt{G_N} \sim \frac{\alpha_{\text{GUT}}}{\sqrt{32\pi^2}} \times \left[ c(\epsilon^{1/3})^5 \right] = (2.3 \times 10^{-3}) \times [0.35 \times (M_{\text{GUT}}/10^{16} \text{ GeV})]. \quad (24)$$

The apparent hierarchy of three orders of magnitude in the effective theory is mostly due to the first factor, a combination of small value of $\alpha_{\text{GUT}}$ and some powers of $\pi$ in the denominator. This hierarchy has little to do with a moderately small parameter $\epsilon = (R_{\text{GUT}}/R_6)^3$, which directly parametrizes the degree of “decoupling”. It is therefore misleading to take the small value of $M_{\text{GUT}} \sqrt{G_N} \sim 10^{-3}$ as an indication of “decoupling”. Unlike in the original context of decoupling [12] where a large hierarchy between the electroweak scale and the Planck scale was discussed, the hierarchy under consideration here is not that large, and most of this hierarchy is accounted for by the first factor of (24).
singularity locus $S$ and a real 2-dimensional space, like in Figure 1 (b), $R_6^3 = R_{GUT}^2 R_\perp$, and one finds that $R_{GUT}/R_\perp = \epsilon^{\gamma=1}[10]$. Depending on the topological cycles $G^{(4)}$ are associated with, the field strength of $\epsilon^{\gamma=1}$ may vary as $l_1^3/(\rho R_\perp^2 R_{GUT}^{2-n})$ for $n = 0, 1, 2$ (and maybe 3). Thus the estimate of the moduli masses become

$$m_{cs} \sim M_{GUT} \times \sqrt{\frac{\alpha_{GUT}}{c}} \times (\epsilon^{\gamma=1})^n.$$  

(27)

In either one of the two models in Figure 1 we can safely conclude that there are complex structure moduli fields (right-handed neutrinos) whose Majorana masses are smaller than the GUT scale at least by one order of magnitude.

To recapitulate, we have four fundamental parameters, $l_1, \rho, R_6, R_{GUT}$ for this compactifications. As we consider the F-theory limit with $\rho \to 0$ and $l_1 \to 0$ (while keeping $l_1^3/\rho$ finite), there are only three relevant parameters. From these three parameters, six observable parameters can be derived: $M_{Pl}, \alpha_{GUT}$, the energy scale of gauge coupling unification $M_{GUT}$, right-handed neutrino masses $M_R = m_{cs}$ from flux compactification, Kaluza–Klein scale of bulk modes $M_{KK} (1/R_6$ or $1/R_\perp)$, and the energy scale $M_{str} \sim 1/l_s$ where stringy excitations appear. The last two of them, though, may not be practically observable, except that those mass scales may set some limitations on inflation models in F-theory compactifications. We have determined all the three relevant fundamental parameters $R_6, R_{GUT}$ and $M_*$ in (21–23) by using the known value of the three observable parameters $M_{GUT}, \alpha_{GUT}$ and $M_{Pl}$, and then the value of the remaining one observable $M_R = m_{cs}$ is predicted by (20). Although its upper bound has been indicated phenomenologically by the observed value of $\Delta m^2$ in neutrino oscillation, its value is theoretically independent of the GUT scale or Kaluza–Klein scale in effective field theory in general (except in individual models). In generic F-theory compactifications with fluxes, however, an upper bound of the energy scale of right-handed neutrinos is predicted, and what is more, the prediction implies that the phenomenological upper bound is fortunately satisfied.

3 Yukawa Couplings Involving Singlets

Now that we have seen the complex structure moduli of F-theory compactifications are good candidates for the right-handed neutrinos, let us study how we should compute their Yukawa couplings.

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12If the fundamental parameters $\rho$ and $l_1$ are kept small but non-zero, then there is another observable parameter, the $S^1$ radius $R_3/(2\pi)$ of the compactification of this 3+1 dimensional “Minkowski” space. Thus, the parameter counting argument for predictability in the following does not change.
Yukawa couplings of quarks and charged leptons are trilinear interactions where all the three fields are charged under the GUT gauge group SU(5)\textsubscript{GUT}. Such charged matter fields are associated with rank-1 enhancement of singularity \[24, 11\], and the Yukawa couplings with rank-2 enhancement \[7, 6, 8, 9\]. In the absence of microscopic formulation of F-theory, it would not be practical to study such Yukawa couplings by using the singular geometry of \(X\). Instead, the field theory formulation of \[11, 6, 7\] provides an approximate description of local geometry of \(X\), whether \(X\) is singular or not. Such field theory local models\[13\] were used to study the Yukawa couplings of charged matter fields \[10, 43, 9\].

Neutrino Yukawa couplings \(2, 10\), on the other hand, involve two SU(5)\textsubscript{GUT}-charged fields and one SU(5)\textsubscript{GUT}-neutral field. Such trilinear couplings also appear in the next-to-minimal supersymmetric standard model (NMSSM), \(\Delta W = \lambda SH_uH_d\). SU(5)\textsubscript{GUT}-neutral fields are, in general, identified with moduli of Calabi–Yau 4-fold \(X\), say, \(H^3,1(X;\mathbb{C})\) or \(H^{1,2}(X;\mathbb{C})\). As the field theory formulation has been very useful in studying Yukawa couplings, we will study in the following the way to include the moduli fields of Calabi–Yau 4-fold in the field theory language\[14\].

### 3.1 Capturing the Moduli Fields in the Field Theory Formulation

Suppose that a local geometry of \(X\) is given by an Weierstrass equation

\[
y^2 = x^3 + xf(z, u, v) + g(z, u, v),
\]

(28)

where \((x, y)\) are the coordinates of the elliptic fiber, \((u, v)\) are local coordinates of a divisor \(S\) in \(B_3\), and \(z\) a normal coordinate of \(S\) in \(B_3\). Let us further assume that the surface with \((x, y, z)\) coordinates have \(A_4\) singularity at \(z = 0\). \(S\) is identified with the locus of \(A_4\) singularity. Four vanishing 2-cycles \(C_A\) \((A = 1, 2, 3, 4)\) are in the \((x, y, z)\) plane, and have intersection form that is negative of the Cartan matrix of \(A_4\). That is the local condition for the SU(5)\textsubscript{GUT} theories.

Let us focus on a local region of \(S\). Through a projection \((x, y, z, u, v) \mapsto (u, v)\), \(X\) is now regarded locally as a fibration over \(S\). By focusing further on a region of \(X\) near the \(A_4\) singularity locus, the geometry of \(X\) in this local region may be regarded approximately as

\[13\] In this article, as in [9], we mean by a “field theory local model” a field theory that models a local geometry of \(X\); to be more precise, that is a field theory on a local patch \(U\) of \((A_4)\)-singularity locus \(S\) that provides an approximate description of physics associated with local geometry of \(X\) containing \(U \subset S \subset X\). Thus, the “local model” does not imply local geometry of \(X\) along the entire \(S\) that models the physics of visible elementary particles or engineering of the Standard Model.

\[14\] This study has been initiated by [6], but we will extend the results in [6].
an ALE fibration on \( S \), with non-vanishing 2-cycles \( C_P \) and the vanishing 2-cycles \( C_A \). When the intersection form of the space spanned by \( C_A \) and \( C_P \) is that of negative of Cartan matrix of Lie algebra \( \mathfrak{g} \), then a field theory local model can be constructed for this local region of \( X \), with the gauge group \( G \) \[6\] \[7\]. The structure group of a Higgs bundle on \( S \) has a structure group \( G' \), which is the commutant of the unbroken symmetry group \( G'' = \text{SU}(5)_{\text{GUT}} \) in \( G \). See \[9\] \[12\] for the use of Higgs bundle in F-theory compactifications; references on mathematics of Higgs bundle are also provided there.

### 3.1.1 Determination of the Field Theory Background

The compactification data \((X, G'(4))\) are encoded in the 4-form
\[
\Omega(a) = \frac{dx \wedge dz \wedge du \wedge dv}{y} = \frac{dx \wedge dz \wedge du \wedge dv}{\sqrt{x^3 + xf + g}},
\]
and the 3-form potential \( C^{(3)} \) on \( X \). Here, a collective denotes the coefficients in the defining equation \(28\). Using the (Poincare dual of the) 2-cycles \( C_A \) and \( C_P \) in the fiber direction, \( \Omega(a) \) and \( C^{(3)} \) can be expanded as
\[
\Omega(a) = \sum_P C_P \otimes \varphi^P + \cdots, \tag{30}
\]
\[
C^{(3)} = \sum_P C_P \otimes A^P + \cdots, \tag{31}
\]
where ellipses stand for a linear combination running over non-compact 2-cycles. Coefficients for the 2-cycles \( C_A \) should be dropped for a background configuration \((\Omega(a_0), \langle C^{(3)} \rangle)\), because we want to preserve the \( G'' = \text{SU}(5)_{\text{GUT}} \) symmetry \[16\]. A background configuration \((\Omega(a_0), \langle C^{(3)} \rangle)\) therefore determines \((\mathfrak{h}' \otimes \mathbb{C})\)-valued \((2,0)\)-form \( \varphi(a_0) \) and \( \mathfrak{h}' \)-valued 1-form \( \langle A \rangle \); here, \( \mathfrak{h}' \) is the Cartan subalgebra of the structure group \( \mathfrak{g}' \).

It will be useful to have some specific examples in mind. In F-theory compactifications with Heterotic dual, \( X \to S \) can be regarded as a K3-fibration globally over \( S \). The K3-fiber has 22 topological 2-cycles, 8 of which have intersection form that is the negative of the Cartan matrix of \( E_8 \). We can choose a basis \( C_A \ (A = 1, 2, 3, 4) \) and \( C_P \ (P = \tilde{8}, 6, 7, -\theta) \), so that their intersection form is described by the Dynkin diagram in Figure \[2\] \( G'' = \text{SU}(5)_{\text{GUT}} \)

\[^{15}\] The coefficient of \( du \wedge dv \) transforms like a section of \( \mathcal{O}(K_S) \), because \( dx/y \) transforms as sections of \( \mathcal{O}(K_{B_3}) = \mathcal{O}(K_S) \otimes N_{S|B_3}^{-1} \), and \( dz \) as those of \( N_{S|B_3} \).

\[^{16}\] \( \langle C^{(3)} \rangle \) can be introduced in the \( C_A \) components as well, in order to break the \( \text{SU}(5)_{\text{GUT}} \) symmetry to the Standard Model gauge symmetry.
We take the 2-cycles $C^I (I = 1, \ldots, 6)$ as $S^1$-fibrations over intervals from $z = z_I$ to $z = z_{I+1}$. The intersection form of $C^I$'s is negative of Cartan matrix of $A_6$, and hence the gauge group of the field theory model for this local geometry is $G = \text{SU}(7)$. $C_A$ ($A = 1, 2, 3, 4$) have vanishing size, whereas the others $C_{5,6}$ have non-vanishing sizes, and hence $G'' = \text{SU}(5)_{\text{GUT}}$ generated by $C_A$'s is the unbroken symmetry, and the structure group is $G' = \text{SU}(2) \times \text{U}(1)$. The period integral over these 2-cycles determine the (2,0)-form field backgrounds $\varphi^P (a_0)$. The period integral over the 2-cycles $C_I$ can be carried out explicitly, and

$$\int_{C_I} \Omega = -2\pi i (z_{I+1} - z_I) \, du \wedge dv.$$  

(35)
The background (2,0)-form configuration \([9]\)

\[
\varphi(a_0) = 2\pi i \text{diag}(0, \cdots, 0, z_-(u, v), z_+(u, v)) \, du \wedge dv
\]

(36)

reproduces the period integral above for positive roots \(G_I\)’s.

We have provided an explicit procedure of obtaining \((\varphi, A)\) background for a given set of data \((X, G^{(4)})\). Although the dictionary between the deformation of ADE singularity and Cartan vev of \(\varphi\) has been known in principle, some ambiguity remained. The deformation parameters of ADE singularity are certainly identified with \((\mathfrak{h} \otimes \mathbb{C})/W\), where \(\mathfrak{h}\) is the Cartan subalgebra of the corresponding ADE Lie algebra, and \(W\) its Weyl group, but the dictionary still has an overall scaling ambiguity \([14]\). The overall scaling ambiguity comes in at each point \((u, v)\) in the base. There is a discrepancy between the choice of \(\varphi\) in \([9]\) and one in \([45]\), and it comes exactly from the point-wise overall scaling ambiguity (see the appendix for more). The explicit procedure above, which uses the period integral, fixes this ambiguity.

Enhancing the rank of the gauge group by 1 is the minimal choice of the field theory local models for regions along matter curves, but the gauge group should be chosen at least by rank-2 higher for regions containing codimension-3 singularities where singularity types are enhanced by rank-2. These choices are minimal, and one can choose larger gauge groups of the field theory model larger, by maintaining higher order terms in the local defining equation of the geometry. It is a matter of the level of approximation how many higher order terms are taken into account.

In the two examples above, we set the background \(\varphi\) field configuration in the diagonal entries, fixing all the gauge transformation non-commuting with \(\langle \varphi \rangle\). The gauge symmetry of the field theory local models remains (when \(\varphi\) is set diagonal) only in the SU(5)\(_{\text{GUT}}\) subgroup (and some U(1) factors for now). Since all other gauge degrees of freedom were redundant from the beginning, and were even absent in the Calabi–Yau 4-fold description of F-theory compactification, we never need to introduce such gauge degrees of freedom in any descriptions.

### 3.1.2 Moduli (Massless) Fluctuations

Although we have so far only talked about the vacuum configuration \(\varphi^P(a_0) = \langle \varphi^P \rangle\) in \([32, 36]\), the same process defines a set of corresponding \(\varphi^P(a)\) for any deformed complex structure moduli \(a = \delta a + a_0\) (where \(\delta a\) need not be infinitesimal). Because of the procedure above, the resulting \(\varphi^P(a)\)’s determines a field configuration \(\varphi(a)\) – a (2,0)-form on \(S\) – taking
its value in the Cartan part of the structure group $G'$. Thus, the deformation of the complex structure of $X$ of an F-theory compactification, $a$, corresponds to those of $(2,0)$-form $\varphi$ on $S$ in the Cartan part of $G' \subset G$, not the entire $G'$-adj. representation. Similarly, (31) defines a Cartan-valued 1-form field $A$.

This may seem odd at first sight, because bosonic fields $(A, \varphi)$ and fermionic fields $(\eta, \psi, \chi)$ are introduced for all the roots of the gauge group $G$, except for the first component in the decomposition

$$\text{Res}_{(G')_G}^G \text{G-adj.} \rightarrow (\text{adj.}, 1) + (1, \text{adj.}) + \oplus_i (U_i, R_i). \quad (37)$$

In the $(\text{adj.}, 1)$ component, only the Cartan part was restored from metric and 3-form potential by the reduction procedure (30, 31). The entire dynamical fields other than those in the Cartan part are missing. This is not surprising, however, because only the Cartan part is obtained by dimensional reduction of metric and $C^{(3)}$ in geometric engineering of an ADE gauge group, and all other dynamical degrees of freedom for the roots of the ADE gauge group originate from M2-branes wrapping on vanishing topological 2-cycles. “W-bosons” corresponding to blown-up/deformed 2-cycles become massive, and can be integrated out of an effective theory. Since all the $\langle \varphi \rangle$ eigenvalues (e.g. in (32, 36)) in $\mathfrak{g}'$ are different generically, all the 2-cycles for the roots in $G'$-adj. are blown up, and the corresponding fields become massive. In the field theory language, this is understood as Higgs mechanism by the Higgs vev $\langle \text{adj.}(\varphi) \rangle$. Thus, apart from a neighborhood of branch loci, where two eigenvalues of $\langle \varphi \rangle$ become degenerate, we do not need to keep the dynamical fields for the roots of $\mathfrak{g}'$ in an effective theory on $S$.

The BPS condition for the field configuration is

$$\omega \wedge F - \frac{|\alpha|^2}{2} [\varphi, \overline{\varphi}] = 0, \quad F^{(0,2)} = 0, \quad \overline{\partial} A \varphi = 0, \quad (38)$$

and zero-mode equations are obtained by imposing the same set of conditions for the background field configuration $(\langle A \rangle_m, \langle \varphi \rangle)$ plus infinitesimal deformation $(\psi_m, \chi)$. This is applied to any irreducible components in (37), and the $(\text{adj.}, 1)$ component is not an exception. The zero mode equations for the Cartan part of $(\text{adj.}, 1)$ are

$$\omega \wedge d\psi^{\text{tot}} = 0, \quad \overline{\partial} \psi = 0, \quad \overline{\partial} \chi = 0, \quad (39)$$

Here, $d = \partial + \overline{\partial}$ and $\psi^{\text{tot}}$ is a 1-form $\psi + \overline{\psi}$. We understand that $\langle A \rangle^{(0,1)/(1,0)}, \langle \varphi \rangle$ and $\langle \overline{\varphi} \rangle$ can be made diagonal at generic points on $S$ (other than at branch loci) which lead to the

\[\text{17} \text{ See [8] for details of the conventions. } \alpha \in \mathbb{C} \text{ is a coefficient associated with the ambiguity in the normalization of } \varphi \text{ field.}\]
simplification of the zero-mode equation. These equations tell us that the fluctuations in \( \chi = \delta \varphi \) and \( \psi = \delta A^{(0,1)} \) can be chosen independently. Furthermore, remembering that there is a monodromy around branch loci, and that fields are twisted by the Weyl group of \( G' \), the Cartan-valued fields (which means that there are rank-\( g' \) components) \( \psi \) and \( \chi \) on \( S \) are better understood as single-valued fields \( \psi \) and \( \chi \) on a spectral surface \( C_V \) of a certain representation. Thus, the zero-mode equations (39) implies that the zero modes from the \((\text{adj}, 1)\) sector are characterized as

\[
\chi = \delta \varphi \in H^0(C_V; K_{C_V}), \quad \text{and} \quad \psi = \delta A^{(0,1)} \in H^{0,1}(C_V; \mathbb{C}) \simeq H^1(C_V; \mathcal{O}_{C_V}), \tag{40}
\]

because \( \chi \) is a holomorphic \((2,0)\)-form, and \( \psi_{\text{tot}} \) a harmonic 1-form. Note the argument so far does not provide justification for the holomorphicity of \( \chi \) or harmonic nature of \( \psi \) at the ramification locus of the spectral surface \( C_V \) over \( S \), as the eigenvalues of \( \langle \varphi \rangle \) have been assumed to be non-degenerate so far. Thus, the characterization of the moduli (40) should not be regarded as something derived already in the field theory formulation of F-theory. But (40) is the same as the characterization of vector bundle moduli in Heterotic string compactification: moduli of the spectral surface and Wilson lines on the spectral surface. Thus, we consider that the argument above as quite reasonable. In short, moduli characterized as \( H^{3,1}(X; \mathbb{C}) \) globally in \( X \) are captured as \( H^0(C_V; K_{C_V}) \) in local models, and the \( H^{1,2}(X; \mathbb{C}) \) moduli in global picture are treated as \( H^1(C_V; \mathbb{C}) \) in field theory local models.

Let us digress for a moment to provide a little mathematical characterization of the moduli fields (40) as a whole. The following observation is theoretically interesting on its own, but it also turns out to be useful later in this article in understanding the moduli zero modes in a limit of complex structure where the spectral surface \( C_V \) is singular. Let us first note this relation:

\[
0 \rightarrow H^1(C_V; \mathcal{O}_{C_V}) \rightarrow \text{Ext}^1(i_{C_V*}\mathcal{O}, i_{C_V*}\mathcal{O}) \rightarrow H^0(C_V; K_{C_V}) \rightarrow H^2(C_V; \mathcal{O}_{C_V}), \tag{41}
\]

---

18 When we are referring to matter curves, Higgs/vector bundles, spectral surfaces and other things associated with an irreducible component \((U_i, R_i)\), it is sometimes convenient to use \( R_i \) as a label, but \( U_i \) is more useful in other situations. We therefore introduce the following convention in this article: when a representation of the unbroken symmetry is used as a subscript, we put it as \( \mid_{(U_i)} \), but when we use a representation of the structure group, the subscript becomes just \( \mid_{U_i} \).

19 To our knowledge, configuration of \( \varphi, \bar{\varphi} \) and gauge fields \( A^{(0,1)/(1,0)} \) around the ramification locus has not been discussed in sufficient details in the literature so far.

20 A slight difference is, though, that only a subspace of the moduli of the spectral surface is actually the moduli of the vector bundle in Heterotic string theory; the subspace is characterized as the kernel of \( H^0(S; R^3\pi, \text{adv}) \rightarrow H^2(S; R^0\pi, \text{adv}) \). We have not figured out where this discrepancy comes from. We will come back to this issue shortly.
where \( i_{C_V} : C_V \hookrightarrow \mathbb{K}_S \), and \( \mathbb{K}_S \) is the total space of the canonical bundle \( K_S \) of \( S \). See [46, 47] for an almost similar discussion in Type IIB language. The extension group picks up only the kernel of the last map from the moduli of the spectral surface \( H^0(C_V; K_{C_V}) \), just as in the Heterotic dual. Thus, the extension group above can be regarded as the characterization of the moduli fields. Since it does not matter if we replace both of \( \mathcal{O}_{C_V} \)'s in \( \text{Ext}^1(i_{C_V*}\mathcal{O}_{C_V}, i_{C_V*}\mathcal{O}_{C_V}) \) by a line bundle on \( C_V \) simultaneously, we take
\[
(\text{adj.}, \mathbf{1}) : \quad \text{Ext}^1(i_{C_V*}\mathcal{N}_V, i_{C_V*}\mathcal{N}_V)
\]
as the characterization of the moduli fields. \( \mathcal{V} \equiv i_{C_V*}\mathcal{N}_V \) is the Higgs sheaf\footnote{See [9, 12] for the use of Higgs sheaf in F-theory compactifications. The notion of Higgs sheaf was already introduced to physics in the context of Type IIB compactifications [17, 18].} of the F-theory compactification.

The characterization of the massless modes from the \((\text{adj.}, \mathbf{1})\) component \((42)\) goes completely in parallel with those of charged matter fields \([6, 7, 8, 9]\)
\[
(1, \text{adj.}) : \quad \text{Ext}^1(i_{\sigma*}\mathcal{L}, i_{\sigma*}\mathcal{L}),
\]
\[
(U_i, R_i) : \quad \text{Ext}^1(i_{\sigma*}\rho_{R_i}(\mathcal{L}), i_{C_{U_i}*}\mathcal{N}_{U_i}).
\]
All of these matter fields are described by extension groups in \( \mathbb{K}_S \), and now so is the moduli fields in the \((\text{adj.}, \mathbf{1})\) component. All of these zero-modes are given interpretations as if they were matters arising from intersection of a pair of D7-branes, although completely generic F-theory compactifications are being discussed here.

It is true at the moment that \((42)\) is nothing more than \((40)\). One should note, however, that we have arrived at \((40)\) by assuming implicitly that the spectral surface is smooth. When the spectral surface becomes singular, it is not that the zero modes split into fluctuations of \( \varphi \) and those of \( A_m \). The expression \((42)\), however, still remains an appropriate way to characterize the moduli zero modes, as we will see some examples in section 4. The expression \((42)\) also makes it possible to understand the Yukawa couplings of \( H^{1,2}(X; \mathbb{C}) \) zero modes and \( H^{3,1}(X; \mathbb{C}) \) zero modes in an “unified” way, as we see shortly.

3.2 Yukawa Couplings involving Singlets

3.2.1 Neutral-Charged-Charged Yukawa Couplings

Now that we have understood how to deal with the moduli fields of Calabi–Yau 4-fold compactification within the field theory local models of \([11, 6, 7, 9]\), we can use the field theory
local models to study the Yukawa couplings involving the moduli fields. The mode expansion

$$\Phi = \sum_I \chi_I^{(2,0)} \phi_I(x, \theta), \quad A = \sum_I \psi_I^{(0,1)} \phi_I(x, \theta)$$

(45)

with chiral multiplets $\phi_I(x, \theta)$ of $N = 1$ supersymmetry in 3+1 dimensions and their mode functions $(\psi_I, \chi_I)$ is inserted in the superpotential \[22\]

$$\Delta W_{DW-BHV} = \int_S \text{tr} (\Phi \wedge F),$$

(46)

and the overlap integration of the mode functions over $S$ yields the coupling constants of the chiral multiplets.

Let us first study how the Yukawa couplings of the form (10) are generated. Neutrino Yukawa couplings and the $SH_uH_d$ interaction of the next-to-minimal theory are in this form.

The SU(5)$_{GUT}$ singlet field in the Yukawa couplings (10) may be fluctuations from the vacuum in $H^{1,2}(X; \mathbb{C})$, or in $H^{3,1}(X; \mathbb{C})$. In the former case, the Yukawa coupling $\lambda$ is calculated by an overlap integration

$$\lambda \sim 2 \int_S \text{tr} \left( \chi_U \wedge \psi_{\text{adj.}(U)} \wedge \psi_{\text{adj.}(U)} \right),$$

(47)

where $(\psi_U, \chi_U)$ is the zero mode wavefunction for a massless chiral multiplet from the $(U, \bar{5})$ irreducible component of $G' \times \text{SU}(5)_{GUT}$, and $(\psi_{\bar{10}}, \chi_{\bar{10}})$ is for the $(\bar{10}, \bar{5})$ irreducible component. $\psi_{\text{adj.}(U)}$ is the wavefunction for the $(\text{adj.}, 1)$ component. In local models with gauge group $G = \text{SU}(6)$, for example, $U$ is a line bundle whose structure group is the commutant of SU(5)$_{GUT}$ within this $G = \text{SU}(6)$ (possibly along with U(1)$_Y$). When a local model with $G = E_8$ is chosen, then $U = \lambda^2 \bar{5}$. The zero mode wavefunctions satisfy a relation $(\psi_{\text{adj.}(U)}, \chi_{\text{adj.}(U)}) = (-\psi_U, \chi_U)$ (e.g., [9]) when we make an approximation that the 4-form flux is ignored, and hence the factor 2 follows. Unless the singlet wavefunction $\psi_{\text{adj.}(U)}$ is always pointing towards the normal direction of the matter curve of SU(5)$_{GUT}$-5 + $\bar{5}$ representations, this trilinear coupling does not vanish. This overlap integration has contributions from everywhere along the matter curve $\bar{c}(5)$.

---

22 This superpotential can be regarded as a non-Abelian extension of the Gukov–Vafa–Witten superpotential [6].

23 We did not pay attention to the overall normalization of the superpotential [4, 6, 115], or in the reduction [88, 114]. Reference 119 may be useful when determining the superpotential.

24 This is a good approximation in the large Kaluza–Klein radius limit.
The overlap integration (47) is carried out over the surface \( S \), but there might be an alternative expression where the overlap integration is done over the matter curve. Notice that the matter fields in the SU(5)\(_{\text{GUT}}\) - 5 and \( \bar{5} \) representations are characterized by \( [6, 7, 8] \):

\[
f(5) \in H^0(\tilde{c}(5); \mathcal{L}(5) \otimes K_{\tilde{c}(5)}^{1/2}), \quad f(\bar{5}) \in H^0(\tilde{c}(\bar{5}); \mathcal{L}(\bar{5}) \otimes K_{\tilde{c}(\bar{5})}^{1/2}),
\]

and hence \( f(5) \cdot f(\bar{5}) \) is in \( H^0(\tilde{c}(\bar{5}); K_{\tilde{c}(\bar{5})}) \); here, \( \mathcal{L}(\bar{5}) \) is a line bundle for the fields in \( \bar{5} \) representation, coming from the \( C^{(3)} \) potential. A Wilson line \( \psi_{\text{adj}}(U) \in H^1(\tilde{C}_{U}; \mathbb{C}) \) on \( \tilde{C}_U \) also defines a Wilson line \( \psi(0,1) \) on the covering matter curve \( \tilde{c}_U = \tilde{c}(5) \). An overlap integration

\[
\int_{\tilde{c}(5)} f^{(1/2,0)}(5) \cdot \psi^{(0,1)}(\bar{5}) \cdot f^{(1/2,0)}(\bar{5})
\]

is well-defined, because the integrand can be regarded as a \((1,1)\) form on \( \tilde{c}(\bar{5}) \); all of \( f(5) \), \( f(\bar{5}) \) and the Wilson line \( \psi_{\text{adj}}(U) \) are objects on the same spectral surface \( \tilde{C}_U \), and taking a product as above is a natural operation. Although we have not shown that (47) is the same as the new expression (49), if they are the same, the latter expression will allow us to short-cut the process of calculation, and to save time in computing the wavefunctions \( (\psi_U, \chi_U) \) from \( f(\bar{5}) \) and carrying out overlap integration on \( S \) numerically.

Changing the vev of \( C^{(3)} \) on \( X \) infinitesimally by \( H^{1,2}(X; \mathbb{C}) \) corresponds to turning on an infinitesimally small vev in the SU(5)\(_{\text{GUT}}\) singlet chiral multiplets \( 1 \)'s in the Yukawa couplings (10). Thus, if a pair of vector-like 5 + \( \bar{5} \) states are in the low-energy spectrum, and if they have a Yukawa coupling with a \( H^{1,2}(X; \mathbb{C}) \) moduli, then the vev of \( C^{(3)} \) is chosen in our vacuum is such that the 5 + \( \bar{5} \) pair happens to appear in the low-energy spectrum.

Yukawa couplings of the form (10) with a singlet from \( H^{3,1}(X; \mathbb{C}) \), on the other hand, are calculated by an overlap integral:

\[
\lambda = -2 \int_S \text{tr} (\psi_U \wedge \chi_{\text{adj}}(U) \wedge \psi_{\overline{U}}) .
\]

The \((0,1)\)-form valued wavefunctions \( \psi_U \) and \( \psi_{\overline{U}} \) differ only by a phase \((-1)\) in the approximation of ignoring the 4-form fluxes. Thus, the overlap integration vanishes at the level of this approximation. The Yukawa couplings of this type may, therefore, be suppressed at least by some positive powers of a ratio \((l_s/R_{\text{GUT}}) < 1\). The \( H^{3,1}(X; \mathbb{C}) \) moduli fields under

\[25\] Since both \( H^{1,2}(X; \mathbb{C}) \) and \( H^{3,1}(X; \mathbb{C}) \) moduli fields can have Yukawa couplings of the form (10), kinetic mixing between the two types of the moduli fields is generated at 1-loop level. At quantitative level, however, the mixing terms may still be loop-suppressed.
consideration here correspond to the deformation of spectral surface in Heterotic dual, and it was shown in [50] that this type of singlets has a Yukawa coupling with $5-\bar{5}$ pairs in Heterotic compactifications, so that the deformation of the spectral surface change the number of massless states. It is hard to believe that the same couplings vanish in F-theory compactifications. This overlap integration in F-theory description is not necessarily localized at codimension-3 singularity points either.

The neutral-charged-charged Yukawa couplings (47) and (50) are given an alternative expression. A product

$$\text{Ext}^1(i_{\sigma^*}O_S, i_{CU+\mathcal{N}_U}) \times \text{Ext}^1(i_{CU+\mathcal{N}_U}, i_{CU+\mathcal{N}_U}) \times \text{Ext}^1(i_{\sigma^*}O_S, i_{\sigma^*}O_S) \to \text{Ext}^3(i_{\sigma^*}O_S, i_{\sigma^*}O_S)$$

(51)

is well-defined, and here, both $H^0(C_U; K_{CU}) (H^{3,1}(X; \mathbb{C}))$ moduli and $H^1(C_U; \mathbb{C}) (H^{1,2}(X; \mathbb{C}))$ are treated at once. Because

$$\text{Ext}^3(i_{\sigma^*}O_S, i_{\sigma^*}O_S) \simeq H^2(S; K_S) = H^{2,2}(S; \mathbb{C}) \simeq \mathbb{C},$$

(52)

the product above returns a complex number.

### 3.2.2 Trilinear Yukawa Couplings among Neutral Fields

It is also of phenomenological interest whether the singlet chiral multiplet $S$ of the next-to-minimal supersymmetric Standard Model (NMSSM) has trilinear coupling $\Delta W = \kappa S^3$ or not. If this coupling is of order unity, then that is an ordinary NMSSM. If it is small, then an accidental $U(1)$ global Peccei–Quinn symmetry exists, and there exists a light pseudo Goldstone boson in the spectrum [51]. The Higgs mass bound from the LEP experiment is relaxed in the presence of such a pseudo Goldstone boson, and the Higgs detection strategy at the LHC should be may also have to be different [52]. Such a light boson may also be relevant to some cosmic ray signals [53].

Following the arguments above, we consider the $H^{1,2}(X; \mathbb{C})$ moduli and $H^{3,1}(X; \mathbb{C})$ moduli as the candidates for such a singlet field $S$. If $S$ is a $H^{3,1}(X; \mathbb{C})$ moduli, then such trilinear couplings may appear from the cubic term of $(\delta a)$ in the expansion in (2); the 4-form $\Omega(a = \delta a + a_0)$ up to the order of $(\delta a)^3$ remains within $H^{1,0}(X; \mathbb{C}) \oplus H^{3,1}(X; \mathbb{C}) \oplus H^{2,2}(X; \mathbb{C}) \oplus H^{1,3}(X; \mathbb{C})$ in the vacuum complex structure, and can be expressed as

$$\Omega(a) = \Omega(a_0) + \cdots + (k_{abc} \Omega(a_0) + \tilde{l}_{abc}^d \chi_d + \tilde{\psi}_{abc} + \tilde{l}_{abcd} \tilde{\chi}^d(\delta a)_a(\delta a)_b(\delta a)_c + \mathcal{O}((\delta a)^4),$$

(53)

where $\tilde{\psi}_{abc} \in H^{2,2}(X_{a=a_0}; \mathbb{C})$, $\tilde{\chi}^d$ form a basis of $H^{1,3}(X|_{a=a_0}; \mathbb{C})$, and ellipses stand for the terms in (8) that are linear or quadratic in the fluctuations in the complex structure moduli
The trilinear coupling for the chiral multiplets $S_a$ corresponding to $(\delta a)_a$ is given by

$$\Delta W = \lambda_{abc} S_a S_b S_c; \quad \lambda_{abc} = \int_X \tilde{\psi}_{abc} \wedge \langle G^{(4)} \rangle.$$  \hspace{1cm} (54)

Generically, this coupling does not vanish. It should be reminded, though, that the moduli fields from $H^{3,1}(X; \mathbb{C})$ generically obtain masses in the presence of 4-form fluxes, and the energy scale of their masses in section 2 is quite large. Without a special reason, such moduli fields are not expected to appear at low energy scale such as the electroweak scale.

If the singlet field $S$ is from $H^{1,2}(X; \mathbb{C})$ moduli, on the other hand, it has a good reason to not have a mass term. This class of moduli originates from the 3-form potential field $C^{(3)}$, and the gauge symmetry shifting $C^{(3)}$ by an exact 3-form prevents the moduli of this class from having arbitrary form of superpotential, other than (63) and its non-Abelian extension (46). The zero modes of $C^{(3)}$ have the Yukawa couplings (47, 49), because the charged matter fields originate from M2-branes, which carry electric charges of $C^{(3)}$.

As we have already noted in footnote 23, the moduli chiral multiplets $\phi_{(3,1)}$ from $H^{3,1}(X; \mathbb{C})$ and the chiral multiplets $\phi_{(1,2)}$ from $H^{(1,2)}(X; \mathbb{C})$ tend to have kinetic mixing of the form

$$\Delta K_{4D,\text{eff.}} = \left( \phi_{(3,1)}^\dagger, \phi_{(1,2)}^\dagger \right) \left( \begin{array}{cc} 1 & \epsilon^* \\ \epsilon & 1 \end{array} \right) \left( \begin{array}{c} \phi_{(3,1)} \\ \phi_{(1,2)} \end{array} \right);$$  \hspace{1cm} (55)

where the mixing coefficients $\epsilon$ may be loop-suppressed. This kinetic term and the mass term

$$\Delta W_{\text{eff.}} = \frac{1}{2} \left( \phi_{(3,1)}, \phi_{(1,2)} \right) \left( \begin{array}{cc} M_R & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{c} \phi_{(3,1)} \\ \phi_{(1,2)} \end{array} \right)$$  \hspace{1cm} (56)

should be diagonalized simultaneously, in order to identify the mass-eigen-basis with diagonal kinetic terms. It is done by

$$\left( \begin{array}{c} \phi_{(3,1)} \\ \phi_{(1,2)} \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ -\epsilon & 1 \end{array} \right) \left( \begin{array}{c} \hat{\phi}_{(3,1)} \\ \hat{\phi}_{(1,2)} \end{array} \right),$$  \hspace{1cm} (57)

where $\hat{\phi}_{(3,1)}$ and $\hat{\phi}_{(1,2)}$ forms the basis we want. The effective superpotential in the $(\hat{\phi}_{(3,1)}, \hat{\phi}_{(1,2)})$ basis is obtained by substituting (57) to the effective superpotential written in terms of $(\phi_{(3,1)}, \phi_{(1,2)})$. Because $\hat{\phi}_{(1,2)}$ fields only pick up interactions that $\phi_{(1,2)}$ have, and because (54) of $\phi_{(3,1)}$ fields is the only cubic term of the SU(5)$_{\text{GUT}}$ singlets, the $\hat{\phi}_{(1,2)}$ fields are not involved in any cubic interactions among singlets in the superpotential. This observation may well be taken as a prediction that singlet chiral multiplets appearing in the NMSSM do not have trilinear terms, apart from those possibly generated by M5-brane instantons with highly suppressed coefficients.
The $H^{1,2}(X; \mathbb{C})$ moduli fields not only have trilinear couplings to vector-like pairs in the SU(5)$_{\text{GUT}}$ visible sector on $S \subset B_3$, but also to any vector-like pairs $\Psi^+ \bar{\Psi}$ on other irreducible pieces of the discriminant. Thus, the singlet field $S$ of the NMSSM may have an effective superpotential of the form

$$\Delta W = \lambda SH_uH_d + \xi S\Psi\bar{\Psi},$$

(58)

which is exactly the theory considered in [53].

4 Dimension-4 Proton Decay Problem Revisited

Up to know, we have only imposed the conditions that the chiral multiplets of right-handed neutrinos are characterized as SU(5)$_{\text{GUT}}$ singlets that have Yukawa couplings of the form (10). This is, however, not all that is known from phenomenology.

So far, we have not introduced an $R$-parity or anything that replaces it. Thus, in generic F-theory compactifications that result in supersymmetric extensions of the Standard Model, we should expect the dimension-4 proton decay operators

$$\Delta W = \mathbf{5} \ \mathbf{10} \ \mathbf{5} = \lambda'' D \bar{D} \bar{U} + \lambda' D Q \bar{L} + \lambda L \bar{E} \bar{L}$$

(59)

to be generated with unsuppressed couplings. That is a totally unacceptable phenomenologically. The absence of dimension-4 proton decay operators implies that the complex structure for F-theory compactification cannot be just generic, but somewhat special. Moduli fields in our vacuum must be fluctuations from such a special choice of complex structure, and chiral multiplets of right-handed neutrinos are among those fluctuations. The special choice of complex structure at our vacuum may have some special implications in physics of right-handed neutrinos.

Whether the right-handed neutrinos have localized wavefunctions or not, and whether their Majorana masses and Yukawa couplings are localized or not are certainly part of questions of phenomenological interest. As we will see in this section, the answer to these questions are indeed different, for various mechanisms that solve the dimension-4 proton decay problem.

4.1 $R$-parity

Imposing $R$-parity is arguably the most conventional way to get rid of dimension-4 proton decay operators from the low-energy effective theory. If the Calabi–Yau 4-fold $X$ and a 4-form flux $G^{(4)}$ on it has a $\mathbb{Z}_2$ symmetry, then the symmetry manifests itself in the effective theory.
By assumption, the $A_4$ singularity locus $S$ is mapped to itself by the $\mathbb{Z}_2$ transformation, and matter curves of various representations to themselves. Note that we consider $(X, G^{(4)})$ that has a $\mathbb{Z}_2$ symmetry in this scenario; it is not that we take a quotient of $(X, G^{(4)})$ by the $\mathbb{Z}_2$ symmetry.

Because of the $\mathbb{Z}_2$ invariance of the matter curves $\bar{c}(R)$ and the sheaves $\mathcal{F}(R)$ on them, the $\mathbb{Z}_2$ transformation also acts on the zero modes—vector spaces of holomorphic sections $H^{p=0,1}(\bar{c}(R); \mathcal{F}(R))$. The zero modes in a given representation $R$ of $SU(5)_{GUT}$, therefore, split into a direct sum of $\mathbb{Z}_2$-odd part and $\mathbb{Z}_2$-even part. We need to assume that

$$h_-^0(\bar{c}(10); \mathcal{F}(10)) = 3, \quad h_-^0(\bar{c}(5); \mathcal{F}(3\subset 5)) = 3, \quad h_-^1(\bar{c}(5); \mathcal{F}(2\subset 5)) = 0,$$

$$h_+^0(\bar{c}(10); \mathcal{F}(10)) = 0, \quad h_+^0(\bar{c}(5); \mathcal{F}(3\subset 5)) = 0, \quad h_+^1(\bar{c}(5); \mathcal{F}(2\subset 5)) = 1, \quad h_+^1(\bar{c}(5); \mathcal{F}(2\subset 5)) = 1,$$

where $\pm$ denotes $\mathbb{Z}_2$-even/odd part of the cohomology groups, and that all other cohomology groups vanish, because we need three copies of $(Q, \bar{U}, \bar{E})$-like chiral multiplets and $(\bar{D}, L)$-like chiral multiplets that are odd under the $R$-parity (matter parity), and we need one pair of Higgs doublets that are even under the $\mathbb{Z}_2$ symmetry. Note that the Higgs doublets and the $(\bar{D}, L)$-type fields are all localized along the same matter curve, $\bar{c}(5)$, and yet the curve $\bar{c}(5)$ does not have to split into two (or more) irreducible pieces. Codimension-3 singularity points for $A_4 \rightarrow E_6$ enhancement and those for $A_4 \rightarrow D_6$ enhancement are also mapped to the codimension-3 singularity points of the same type by the $\mathbb{Z}_2$ transformation. Yukawa couplings of the form $[59]$ will be generated at each codimension-3 singularity point of $A_4 \rightarrow D_6$ enhancement, but they vanish as a result of $\mathbb{Z}_2$-odd nature of the wavefunctions of the three relevant chiral multiplets after contributions from all the codimension-3 singularity points of this type are summed up. The Yukawa couplings for down-type quarks and charged leptons, on the other hand, will remain non-zero, because the $\mathbb{Z}_2$ symmetry does not ensure cancellation of all the contributions to the Yukawa couplings of this type.

The chiral multiplets of right-handed neutrinos correspond to $H^{3,1}_-(X; \mathbb{C})$, because only the $\mathbb{Z}_2$-odd ones have trilinear couplings suitable for the neutrino Yukawa couplings $\Delta W = H_u \bar{N}L$. The Majorana mass terms of the complex structure moduli fields $H^{3,1}(X; \mathbb{C})$ in $[9]$ consist of $H^{3,1}_+(X; \mathbb{C})-H^{3,1}_+(X; \mathbb{C})$ mass terms and $H^{3,1}_+(X; \mathbb{C})-H^{3,1}_+(X; \mathbb{C})$ mass terms, but the mass terms $[9]$ do not have a mixed $H^{3,1}_+(X; \mathbb{C})-H^{3,1}_+(X; \mathbb{C})$ mass term because of the $\mathbb{Z}_2$ symmetry of $(X, G^{(4)})$. The $H^{3,1}_-(X; \mathbb{C})-H^{3,1}_-(X; \mathbb{C})$ mass terms in $[9]$ are identified with the Majorana masses of the right-handed neutrinos.

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$^{26}$ $H^{1,2}_-(X; \mathbb{C})$ moduli fields also qualify, if the neutrino masses are Dirac.
In this scenario, the neutrino Yukawa couplings are localized along the matter curve $\tilde{c}(\tilde{8})$ of SU(5)$_{\text{GUT}}$-5 + 5 representations, but not further localization is expected along the curve. The Majorana mass terms of the right-handed neutrinos come from the Gukov–Vafa–Witten superpotential, and are not localized anywhere in the Calabi–Yau 4-fold, but come from the entire bulk of the geometry.

The story developed in sections 2 and 3 holds without any modification in the $\mathbb{Z}_2$ symmetry scenario. Apart from imposing (assuming) a $\mathbb{Z}_2$ symmetry in $(X, G^{(4)})$, one does not have to do anything non-trivial. The $\mathbb{Z}_2$-odd part of the moduli fields automatically have neutrino Yukawa couplings, and the $H^{3,1}_\chi(X; \mathbb{C})$ moduli, in particular, have Majorana masses from flux compactification. The Majorana mass scale is just that of the right-handed neutrinos to account for the low-energy neutrino masses through the see-saw mechanism. There was nothing non-trivial in the scenario above, yet this is still quite remarkable.

Traditionally in field theory model building in 3+1 dimensions, one tends to consider that it is not an easy task to generate the Majorana masses of right-handed neutrinos, if the gauge group is larger than SU(5)$_{\text{GUT}}$ at high-energy (or at microscopic level). In SO(10) GUT models, for example, there is an option to introduce an exotic SO(10)-126 field that is $\mathbb{Z}_2$-even. Its vev does not break the $\mathbb{Z}_2$ symmetry. Majorana mass terms are obtained if an effective theory has the following renormalizable coupling,

$$\Delta W = \langle 126_+ \rangle \ 16_-. \ 16_-.$$  \hfill (60)

The situation does not change very much when the gauge group at the microscopic level is the SU(5)$_{\text{GUT}} \times U(1)_\chi$ subgroup of SO(10). The 126 field is replaced by an SU(5)$_{\text{GUT}}$ singlet that carries +10 units of the U(1)$_\chi$ charge\footnote{It is conventional to set the normalization of U(1)$_\chi$ charge so that 10 carries $Q_\chi = -1$, $\tilde{D}, L$ has $Q_\chi = +3$ units of the U(1)$_\chi$ charge, $H(5)$ [resp. $\tilde{H}(\tilde{5})$] $Q_\chi = +2$ [resp. $Q_\chi = -2$] units, and right-handed neutrinos $\bar{N}$ carry $Q_\chi = -5$ units.}. An equally popular alternative is to introduce a less exotic field in the SO(10)-16 representation that is $\mathbb{Z}_2$-even. Majorana mass terms of right-handed neutrinos are available, if an effective theory has the following non-renormalizable interactions:

$$\Delta W = \langle \bar{16}_+ \rangle \langle \bar{16}_+ \rangle \ 16_- 16_-. \hfill (61)$$

The origin of those fields and their interactions, as well as the $\mathbb{Z}_2$-parity assignment are usually not questioned in field theory model building and such assumptions are just introduced by hand for phenomenological convenience. In string phenomenology, however, these issues are “the questions” to be addressed. All necessary fields and interactions should be
derived from known sources. Since charged matter fields of SU(5) GUT models originate from breaking of a symmetry $G$ larger than SU(5)$_{\text{GUT}}$, for example, in Heterotic $E_8 \times E_8$ string compactifications (with supergravity approximation), intersecting D-brane systems, and in deformation of singularity in M/F-theory compactifications [11], a larger symmetry $G$ exists at this microscopic level in these compactifications, and the origin of the necessary fields and interactions have been an issue with no clear answer.

To see why/how this issue was overcome in generic F-theory compactifications without any difficulties, let us remind ourselves of the following. As reviewed in section 2 of [9], physics of SU(5)$_{\text{GUT}}$ GUT models in generic F-theory compactifications is described by a patchwork of gauge theories. The locus of $A_4$ singularity $S$ is covered by various patches, and individual patches have their own gauge groups containing SU(5)$_{\text{GUT}}$. In patches that contain the matter curve $\bar{c}_{(10)}$, the gauge group should be $G = \text{SO}(10) \supset \text{SU}(5)$_{\text{GUT}} or larger. If a patch in $S$ contains the matter curve $\bar{c}_{(5)}$, then the gauge group in this patch has to be $G = \text{SU}(6)$ or larger. The gauge group in a patch containing a codimension-3 singularity point should be larger than SU(5)$_{\text{GUT}}$ at least by rank 2, that is, $G$ is at least $E_6$, SO(12) or SU(7), because 2 independent topological 2-cycles collapse at the codimension-3 singularity point. The gauge groups in these field theory local models can be chosen even larger for higher level of approximation. The field theory local models defined on various patches on $S$ may have different gauge groups $G$ with different ranks, but they can be glued together within an overlap of two neighboring patches, because a gauge theory with lower rank gauge group can provide reasonably well approximation when one of 2-cycles becomes large.

An important point is that there is no common “microscopic gauge group” $G$ containing $G' \times \text{SU}(5)$_{\text{GUT}} over the entire $A_4$ singularity locus $S$ in generic F-theory compactifications.\footnote{F-theory compactifications with Heterotic dual are an exception, because a unique gauge group $G = E_8$ and the structure group $G' = \text{SU}(5)$_{\text{str}} are found.} The absence of the common structure group $G'$ means that the right-handed neutrinos and anything that effectively becomes the Majorana mass parameters cannot be characterized as something in definite representation of the “common structure group $G'$” (or of the “microscopic gauge group $G$”), and all the theoretical constraints associated with the representation theory of $G'$ or $G$ have gone. In generic F-theory compactifications, no clear line is drawn between the fields in the gauge theory sector and bulk gravity sector, as opposed to the (dual of) Heterotic string compactifications with supergravity approximation, or to the Type IIB Calabi–Yau orientifold compactifications. Complex structure moduli $H^{3,1}(X; \mathbb{C})$ of bulk gravity can also be regarded locally as a part of Cartan subspace in local models on $S$ with...
$G = \text{SU}(6)$ gauge groups. This class of SU(5)$_{\text{GUT}}$-neutral fields can be identified with right-handed neutrinos, because they have a natural SU(6) gauge interactions that can be identified with the neutrino Yukawa couplings, and yet, the gauge invariance of $G = \text{SU}(6)$ constrains the interactions of these right-handed neutrinos only in a local geometry, and does not prevent them from having the Majorana mass terms in the Gukov–Vafa–Witten superpotential. This is the heart of the trick that solve the long-standing problem in generic F-theory compactifications.

4.2 Reducible Limit of Spectral Surface

Some other solutions to the dimension-4 proton decay problem in F-theory have already been discussed in the literature; the $\mathbb{Z}_2$ symmetry scenario in section 4.1 is not the first one. In sections 4.2–4.4, we will elaborate on these alternative scenarios, and discuss phenomenological consequences in these scenarios. In section 4.2, we will work on a scenario where spectral surfaces are reducible, and an extra U(1) symmetry is used to bring the dimension-4 proton decay operators under control, instead of the $\mathbb{Z}_2$ symmetry.

Defining equation of local geometry of Calabi–Yau 4-fold $X$ is given by

$$y^2 = x^3 + (a_5 xy + a_4 zx^2 + a_3 z^2 y + a_2 z^3 x + a_0 z^5) + (A_5 zy + A_4 z^2 x^2 + A_3 z^3 y + A_2 z^4 x + A_0 z^6) + \cdots$$

for generic F-theory compactifications that leave SU(5) GUTs. $(x, y)$ are the coordinates of the elliptic fiber of $X$. The discriminant of this elliptic fibration is given by

$$\Delta = z^5 \left[ \frac{1}{16} a_5^2 P^{(5)} + \frac{z}{16} a_5^7 \left\{ 12 \left( a_4 + \frac{a_5 A_5}{2} \right) P^{(5)} - a_5^2 \tilde{R}^{(5)} \right\} + \mathcal{O}(z^2) \right],$$

where

$$P^{(5)} = a_0 a_5^2 - a_2 a_5 a_3 + a_4 a_3^2,$$

and the definition of $\tilde{R}^{(5)}$ is given later in this article. An irreducible component $S$ of the discriminant locus is found along $z = 0$; and hence $z$ is the normal coordinate of $S$ in $B_3$. We will take a set of local coordinates $(u, v)$ on a local patch of $S$. $a_r(u, v)$ and $A_r(u, v)$ ($r = 0, 2, \cdots, 5$) are locally regarded as functions on $S$, and ellipses stand for terms higher order in the series expansion of $z$. $a_r$’s, $A_r$’s and all others are regarded globally as holomorphic sections of appropriate line bundles; defining a divisor $t$ (or equivalently $\eta$) on $S$ through $c_1(N_{S|B_3}) \equiv t \equiv (6K_S + \eta)$, relevant line bundles are

$$a_r \in \Gamma(S; \mathcal{O}(rK_S + \eta)) = \Gamma(S; \mathcal{O}(-(6 - r)K_S + t)), \quad A_r \in \Gamma(S; \mathcal{O}(-(6 - r)K_S)).$$
In the first line of (62), \((a_5xy + \cdots + a_0z^6)\) part closely resembles the defining equation of a spectral surface of an SU(5) vector bundle in Heterotic \(E_8 \times E_8\) string compactification on an elliptic fibered Calabi–Yau 3-fold \(Z\),

\[
5\sigma + \text{div}(a_5xy + a_4x^2 + a_3y + a_2x + a_0).
\] (66)

Here, \(\sigma\) is the zero section of the elliptic fibration

\[
\pi_Z : Z \to S,
\] (67)

and \(a_r (r = 0, 2, \cdots, 5)\) are regarded as holomorphic sections of line bundles on \(S\) given as in (65) for some divisor \(\eta\) of \(S\). This resemblance is at the heart of the duality between Heterotic string and F-theory compactifications \([54, 55, 6, 8, 9]\). Although not all of F-theory compactifications have a Heterotic dual, physics along the \(A_4\) singularity locus \(S\) at \(z = 0\) for generic F-theory compactifications is still qualitatively similar to that of Heterotic string compactification.

4.2.1 SO(10) Scenario

In Heterotic \(E_8 \times E_8\) string compactification, the structure group of a rank-5 vector bundle \(V_5\) may be reduced from SU(5) to its subgroup SU(4) \(\times\) U(1)\(\chi\), and \(V_5 = U_4 \oplus L_\chi\), where \(U_4\) and \(L_\chi\) are rank-4 and rank-1 bundles, respectively. This can be achieved, for example, by setting \(a_5 = 0\) in (66). The spectral surface \(C_{V_5} = C_{(10)}\) becomes reducible then:

\[
[4\sigma + \text{div}(a_4x^2 + \cdots a_0)] + \sigma.
\] (68)

\(U(1)_\chi\) symmetry transformation commutes with the structure group SU(4) \(\times\) U(1)\(\chi\), and hence remains as a global symmetry of the effective action. One can do the same thing in generic F-theory compactifications by setting \(a_5 = 0\) in (62) \([1]\). In this limit, the spectral surface of the Higgs bundle for fields in the SU(5)\(_{\text{GUT}}\)-10 representation, \(C_{(10)}\), becomes reducible:

\[
C_{(10)} \to C_{(16)} + \sigma, \quad [a_5 + a_4\xi + a_3\xi^2 + \cdots = 0] \to [a_4 + a_3\xi + \cdots = 0] + [\xi = 0],
\] (69)

where \(\sigma\) is the zero section of the canonical bundle \(\pi_{K_S} : K_S \to S\), and \(\xi\) is the coordinate of the rank-1 fiber vector space of the canonical bundle \(K_S\).

Now, with \(a_5 = 0\), the discriminant \(\Delta\) begins with an \(O(z^7)\) term, and we have a split \(D_5\) singularity along \(S\). There is an SO(10) gauge theory defined globally on \(S\). The SO(10) symmetry can be broken down to SU(5)\(_{\text{GUT}}\) \(\times\) U(1)\(\chi\) or \(G_{SM} \times U(1)_\chi\) \([G_{SM} \equiv SU(3)_C \times\)
SU(2)_L \times U(1)_Y] subgroup by turning on fluxes. The massless vector fields may or may not become massive through the St"uckelberg interactions, but the associated global U(1)\_\chi symmetry remains unbroken (unless spontaneous symmetry breaking is triggered; we will come back to this issue in section 4.4). The dimension-4 proton decay operators are not allowed in the presence of unbroken U(1)\_\chi symmetry, because the operators (59) have \(Q_\chi = +5 \neq 0\) units of the U(1)\_\chi charge (see footnote 27). Note that we do not need a Heterotic dual for this scenario to make sense; in other words, we do not need a rank-5 Higgs bundle defined globally on \(S\). The essence here is to have a U(1)\_\chi \subset SO(10) gauge symmetry on \(S\), which is characterized also as a reducible limit of the spectral surface \(C_{(10)}\). This characterisation, \(a_5 = 0\), involves information only of local geometry along \(S\), and fully generic F-theory compactifications can be considered (regardless of whether Heterotic dual exists or not), as long as this condition is satisfied.

Let us now give some thoughts on the candidates of the right-handed neutrinos in this scenario. We now consider a limit where the spectral surface \(C_{(10)}\) consists (at least) of two irreducible components. Along the intersection of the two components, the spectral surface \(C_{(10)}\) is singular. Thus, we have to reconsider the argument in section 3, where the spectral surface was implicitly assumed to be smooth.

Suppose that the Higgs bundle \((\varphi, V)\) [i.e. Higgs sheaf \(V\)] is decomposed into a direct sum \((\varphi_1 \oplus \varphi_2, V_1 \oplus V_2)\) [i.e., \(V = i_{C_1}\_\ast N_1 \oplus i_{C_2}\_\ast N_2\)]. Instead of \(H^0(C_{(10)}; K_{C_{(10)}})\) and \(H^1(C_{(10)}; \mathbb{C})\) in (10), we find that \(\text{Ext}^1(V, V)\) in (12) is the suitable characterization of the moduli fields, when the spectral surface consists of two irreducible pieces \(C_1 + C_2\). The extension group can be decomposed as follows in such a reducible limit:

\[
\text{Ext}^1(V, V) = \text{Ext}^1(i_{1\_\ast N_1}, i_{1\_\ast N_1}) + \text{Ext}^1(i_{2\_\ast N_2}, i_{2\_\ast N_2}) + \text{Ext}^1(i_{1\_\ast N_1}, i_{2\_\ast N_2}) + \text{Ext}^1(i_{2\_\ast N_2}, i_{1\_\ast N_1}).
\] (70)

The last two components are essentially localized along the intersection of the two surfaces, \(C_1 \) and \(C_2\). Irreducible pieces of the Higgs sheaf \(i_{C_i}\_\ast N_i\) \((i = 1, 2)\) on \(\mathbb{K}_S\) can be treated as if they were 7-branes in Calabi–Yau orientifolds; the spectral surfaces \(C_i\) play the role of the supports of the “7-branes” and \(N_i\) the “gauge bundles” on them. The localized pieces of the moduli are like “open strings” connecting the “7-branes”.

For the case of practical interest, we consider a limit where the Higgs sheaf for the SU(5)\_GUT-10 representation fields \(V_{(10)}\) becomes reducible:

\[
V_{(10)} \rightarrow i_{C_{(10)}\_\ast N_{(10)}} + i_\sigma L_\chi.
\] (71)
\(\mathcal{L}_\chi\) is the line bundle on \(S\) whose structure group is \(U(1)_\chi\). The two localized components of the moduli become

\[
\text{Ext}^1(i_{\sigma*}\mathcal{L}_\chi^{-Q_\chi}, i_{C(16)*}\mathcal{N}(16)) = H^0(\bar{c}_{(16)}; \mathcal{N}(16) \otimes \mathcal{L}_\chi^{-Q_\chi} \otimes K_S)
= H^0(\bar{c}_{(16)}; (\mathcal{L}(16) \otimes \mathcal{L}_\chi^{-Q_\chi}) \otimes \mathcal{O}(K_S + r_{(16)}/2)),
\]
where \(\mathcal{L}(16)\) is defined by \(\mathcal{N}(16) = \mathcal{O}(r_{(16)}/2) \otimes \mathcal{L}(16)\), and \(r_{(16)} \equiv K_{C(16)} - \pi^*_C K_S = C(16) - \pi^*_C K_S\) is the ramification divisor of the projection \(\pi_C : C(16) \to S\); \(\bar{c}_{(16)}\) is regarded as the intersection of the two irreducible pieces of the spectral surface, \(\sigma \cdot C(16)\), although it is also seen as the matter curve of \(SO(10)\) models. The \(SU(5)_{\text{GUT}}\)-neutral component in the \(SO(10)\) 16 representation has \(Q_\chi = -5\) units of the \(U(1)_\chi\) charge. If there are such chiral multiplets \(\chi(72)\) [denoted by \(\chi\text{-}N\)] and their vector-like partners \(\chi(73)\) [denoted by \(\chi\text{-}N^\dagger\)], then their products \(\chi\text{-}NN^\dagger\) are sections of a line bundle

\[
\mathcal{O}(2K_S + r_{(16)}(\bar{c}_{(16)}) = \mathcal{O}(K_S + C(16))|_{\bar{c}_{(16)}} = \mathcal{O}(5K_S + \eta)|_{\bar{c}_{(16)}};
\]

in the last step of the equation above, we used the fact that all the terms \(a_4, a_3\xi\) etc. appearing in the defining equation of \(C(16)\) are sections of \(\mathcal{O}(4K_S + \eta)\). Using the exact sequence

\[
0 \to \mathcal{O}_S(K_S) \to \mathcal{O}_S(K_S + C(16)) \to i_{\bar{c}_{(16)*}}\mathcal{O}(K_S + C(16)) \to 0
\]
and its long exact sequence

\[
0 \to H^0(S; K_S) \to H^0(S; \mathcal{O}(5K_S + \eta)) \to H^0(\bar{c}_{(16)}; \mathcal{O}(K_S + C(16))) \to H^1(S; K_S)
\]
one finds that the space of \(\chi\text{-}NN^\dagger\), the third term in \(\text{(75)}\), is identified with the second term, if \(h^{2,0}(S) = h^{0,1}(S) = 0\). Thus, non-vanishing expectation values of \(\chi\text{-}NN^\dagger\) correspond to non-vanishing \(a_5 \in \Gamma(S; \mathcal{O}(5K_S + \eta))\), and to recombination of the two irreducible pieces \(\sigma\) and \(C(16)\) of the spectral surface.\(^{29}\)

At the reducible limit of the spectral surface, the \((D,L)\)-type matter is on the matter curve \(\bar{c}_{(16)}\) of \(SO(10)\) models \((a_4 = 0)\), and the \(H_u \subset H(5)\) chiral multiplet is on the matter curve \(\bar{c}_{\text{vect.}}\) \((a_3 = 0)\).

Thus, the neutrino Yukawa couplings are likely to be localized at the intersection points of the two matter curves, \((a_3, a_4) = (0, 0)\). A trilinear interaction

\[
\Delta W = 16\text{ 16 vect.}
\]

\(^{29}\)The observation so far improves a similar argument made earlier in [1].
is generated at this type of codimension-3 singularity points \[7, 9\]. The moduli chiral multiplets \(\overline{N}\) in (72) are also regarded as the ordinary “right-handed neutrinos” in the spinor representation of SO(10), and these \(\overline{N}\) fields do have the neutrino Yukawa couplings with \(L\) and \(H_u\) as a part of the interactions above. Thus, the fields (72) are regarded indeed as the right-handed neutrinos.

SU(5)\(_{\text{GUT}}\)-neutral fields coming from \(\text{Ext}^1(i_{C(\mathbb{16})}, \mathcal{N}(\mathbb{16})>_5, i_{C(\mathbb{16})}, \mathcal{N}(\mathbb{16})>_8)\) (or equivalently, the bulk \(H^{3,1}\) and \(H^{1,2}\) moduli), on the other hand, are neutral under the \(U(1)_{\chi}\) symmetry. Since \(H_u \subset H(\mathbb{5})\) and \(L \subset (\bar{D}, L)\) have \(Q_{\chi} = +2\) and \(Q_{\chi} = +3\) units of the \(U(1)_{\chi}\) charges, \(U(1)_{\chi}\)-neutral fields cannot have Yukawa couplings with these two fields. Therefore, only (72) moduli can be identified with right-handed neutrinos in this scenario.

4.2.2 SU(6) Scenario

An alternative to the \(a_5 = 0\) scenario (or SO(10) scenario) is to consider another factorization limit of the spectral surface

\[
5\sigma + \text{div}(a_5xy + a_4zx^2 + a_3y + a_2x + a_0) \\
\rightarrow [2\sigma + \text{div}(p_2x + p_0)] + [3\sigma + \text{div}(q_3y + q_2x + q_0)]
\]

(78)

in Heterotic language. This is to consider a limit where the structure group of rank-5 vector bundle \(V_5\) becomes \(SU(3) \times SU(2) \times U(1)_{\tilde{q}}\), and \(V_5 = U_3 \oplus U_2\), where \(U_3\) and \(U_2\) are rank-3 and rank-2 bundles, respectively. In F-theory language, this is to require that for a pair of divisors \(\eta'\) and \(\eta''\) on \(S\) that satisfy \(\eta' + \eta'' = \eta\), there exist

\[
p_r \in \Gamma(S; \mathcal{O}(rK_S + \eta')) \quad (r = 0, 2), \quad q_r \in \Gamma(S; \mathcal{O}(rK_S + \eta'')) \quad (r = 0, 2, 3),
\]

(79)

and \(a_r\) \((r = 0, 2, 3, 4, 5)\) are given by

\[
a_0 = p_0q_0, \quad a_2 = p_0q_2 + p_2q_0, \quad a_4 = p_2q_2, \quad a_3 = p_0q_3, \quad a_5 = p_2q_3.
\]

(80)

No conditions are imposed on other sections such as \(A_r\)’s in (62) \[\text{[36]}\]. Under this choice of complex structure of \(X\), the defining equation of the \(\bar{c}_{(5)}\) matter curve of SU(5)\(_{\text{GUT}}\) models vanish identically on \(S\):

\[
P^{(5)} = p_0p_2q_3(p_2q_0 - (p_2q_0 + p_0q_2) + p_0q_2) = 0,
\]

(82)
and the discriminant $\Delta$ begins at the order of $z^6$. Split $A_5$ singularity is along the surface $S$, and an SU(6) gauge theory is globally defined on $S$. The SU(6) symmetry may be broken down to SU(5)$_{\text{GUT}} \times U(1)_{\tilde{q}}$ or $G_{S\text{M}} \times U(1)_{\tilde{q}}$ by turning on fluxes. Since the U(1)$_{\tilde{q}}$ symmetry is defined globally on $S$, and commutes with the structure group of the Higgs bundle in F-theory compactifications, it remains as a global symmetry of low-energy effective theory unless it is spontaneously broken by a vev of a field with non-vanishing U(1)$_{\tilde{q}}$ charge. The dimension-4 proton decay operators are absent when this global symmetry remains unbroken, just like in the SO(10) ($a_5 = 0$) scenario. The essence here is the U(1)$_{\tilde{q}}$ gauge symmetry on $S$, nothing else in the bulk of Calabi–Yau 4-fold $X$. Thus, the SU(6) scenario can be considered in fully generic F-theory compactifications, regardless of whether Heterotic dual exists or not.

Moduli (SU(5)$_{\text{GUT}}$-neutral) fields are decomposed as in (70) in this scenario as well. Here, the spectral surface $C(5)$ for the fields in SU(5)$_{\text{GUT}}$-5 representation reduces to

$$C(5) \rightarrow C(6) + C(\wedge^2 6) + \sigma,$$

(83)

and localized moduli are found on the curve $\bar{c}(6)$ given by the intersection$^\Box$ of $C(6)$ and $\sigma$. Let us denote the SU(5)$_{\text{GUT}}$-singlet chiral multiplets in the SU(6)-6 representation as $\overline{N}$, and those in the SU(6)-6 representation as $\overline{N}$. In this scenario, the up-type Higgs $H_u$ comes from the $H(5) \subset \text{adj}$ of SU(6) in the bulk of $S$, and lepton doublet $L \subset (\overline{D}, L)$ chiral multiplets are localized along the matter curve $\bar{c}(6)$ $^\Box$. The curve-curve-bulk Yukawa couplings

$$\Delta W = \bar{6} \text{ adj.} 6 \supset L H_u \overline{N}$$

(85)

is generated all along the curve $\bar{c}(6)$ $^\Box$, and this Yukawa couplings and the chiral multiplets $\overline{N}$ can be identified with the neutrino Yukawa couplings and the right-handed neutrinos in

$\text{(83)}$

Although the intersection of $\sigma$ and $C(\bar{6})$ can be regarded as that of two irreducible pieces in $\text{SU}(5)$, this intersection can also be regarded as the matter curve of SU(6)-6 representation in SU(6) unified theories. The discriminant is given by

$$\Delta = \frac{z^6}{16} p_2 q_1^2 S(6K_S + 3\eta' + 2\eta'') + O(z^7)$$

(84)

under the choice of complex structure in $\text{SU}(5)$ $^\Box$. Here, $S(6K_S + 3\eta' + 2\eta'')$ is a section of $O(6K_S + 3\eta' + 2\eta'')$, and is given by $p_{0,2}, q_{0,2,3}$ and $A_{0,2,3,4,5}$; its explicit form is so messy that we do not write it here, but the zero locus of $S(6K_S + 3\eta' + 2\eta'')$ is the matter curve $\bar{c}(6)$. The matter curve $\bar{c}(\wedge^2 6)$ is given by $(p_2 = 0) \in |2K_S + \eta'|$, while $(q_1 = 0) \in |3K_S + \eta''|$ is the matter curve $\bar{c}(\wedge^2 6)$. All the terms in $S(6K_S + 3\eta' + 2\eta'')$ contain either $p_2$ or $q_3$, and hence the curve $\bar{c}(6)$ (where $(\overline{D}, L)$-type zero modes are localized) pass through all the intersection points of the matter curves $\bar{c}(\wedge^2 6)$ (where SU(5)$_{\text{GUT}-10}$ fields are localized) and $\bar{c}(\wedge^2 6)$ (where SU(5)$_{\text{GUT}-H(5)}$ is supported). This means that the down-type / charged lepton Yukawa couplings are not necessarily suppressed; see $^\Box$ for more details.
this scenario. Just like in the SO(10) \((a_5 = 0)\) scenario, singlets from \(H^{3,1}\) or \(H^{1,2}\) that are not associated with the singularity of the spectral surface do not have appropriate \(U(1)_{\tilde{q}}\) charges, and hence cannot have trilinear couplings for neutrino Yukawas. Thus, they are not identified with the right-handed neutrinos.

### 4.2.3 Majorana masses

Both the SO(10) and SU(6) scenarios above leave an unbroken global \(U(1)\) symmetry. Such a \(U(1)\) symmetry is a powerful and reliable way to make sure that the dimension-4 proton decay operators are absent, but it is actually too powerful. Right-handed neutrinos in the two scenarios above are charged under the global \(U(1)_\chi\) or \(U(1)_{\tilde{q}}\) symmetry, and hence they cannot have Majorana mass terms. Without relying upon the see-saw mechanism of right-handed neutrinos, one has to resort to the see-saw mechanism of higgsino/wino in the (bilinear) R-parity violating scenario \([56, 57]\), or to assume that the neutrino Yukawa couplings are somehow sufficiently small; \(\lambda^{(\nu)} < 10^{-12}\). Although the \(H^{3,1}\) moduli irrelevant to the intersection of the spectral surfaces still have Majorana mass terms from \([9]\), they cannot have neutrino Yukawa couplings because the \(U(1)\) charge do not match.

One may not exclude a possibility that the Majorana masses are generated for fields like \([72]\) through M5-brane instanton effects. M5-brane instantons are much like D3-brane instantons in Type IIB string theory, and a recent review article is available \([58]\). Usually such an amplitude involves a small exponential factor, coming from the volume of a divisor of \(B_3\) that “D3-branes” are wrapped. Since the right-handed neutrinos are localized along a curve in \(S\) in either one of these SO(10) or SU(6) scenarios, the Kaluza–Klein scale of \(S\) and \(B_3\), that is, \(R_{\text{GUT}}\) and \(R_6\), set the scale of the volume of the divisor, unless there are some collapsed divisors in \(B_3\). As we will see in section 4.3.1 such an exponential factor is too small, when the volume is estimated by using \(R_{\text{GUT}}\) and \(R_6\). Thus, moduli of geometry have to be tuned so that \(B_3\) has a divisor with small volume, in order to generate large enough Majorana masses of the right-handed neutrinos.

### 4.3 Reducible Limit of \(SU(5)_{\text{GUT}}\)-5 Matter Curve

#### 4.3.1 Just the reducible limit of the matter curve

It was suggested in \([10, 36]\) that the dimension-4 proton decay problem may be solved by considering a reducible limit of the matter curve \(\tilde{c}_{(5)}\):

\[
\tilde{c}_{(5)} = \tilde{c}_{(DL)} + \tilde{c}_{(H)}.
\]

(86)
The idea is that the curve \( \bar{c}(\bar{DL}) \) supports only the three chiral multiplets of matter \((\bar{D}, L)\), and the up-type and down-type Higgs doublets are supported in the other curve \( \bar{c}(H) \). At all the codimension-3 singularity points for the enhancement \( A_4 \to D_6 \), the matter curve \( \bar{c}(5) \) forms a double point, and the idea is to assume that the two branches of the matter curve passing through the double point correspond to \( \bar{c}(\bar{DL}) \) and \( \bar{c}(H) \). Mathematically, this assumption is equivalent to factorization of the defining equation of the curve \( \bar{c}(5) \) as follows:

\[
P(5) = a_0 a_5^2 - a_2 a_5 a_3 + a_4 a_3^2 = (p_0 a_5 - p_2 a_3)(q_0 a_5 - q_2 a_3).
\]

Here, \( p_r \in \Gamma(S; \mathcal{O}(rK_S + \eta')) \) \((r = 0, 2)\), \( q_r \in \Gamma(S; \mathcal{O}(rK_S + \eta'')) \) \((r = 0, 2)\) and the divisors \( \eta' \) and \( \eta'' \) on \( S \) satisfy \( \eta' + \eta'' = \eta \). As noted in \([36]\), this condition is actually equivalent to the condition \((80)\) alone without \((81)\), and hence can be regarded as a generalization of the SU(6) scenario in section 4.2.

The assumptions in section 4.2 surely get rid of dimension-4 proton decay operators, but the unbroken U(1) symmetry were too powerful because they forbid the Majorana mass terms of right-handed neutrinos altogether at perturbative level. The condition \((80)\) alone is certainly more general, and the Majorana masses of right-handed neutrinos may be generated.

A reducible limit of the matter curve is a weaker condition than the reducible limit of the spectral surface. But now we do not necessarily have such a U(1) symmetry in the effective theory, and it is not absolutely clear whether the dimension-4 proton decay operators are absent in the effective theory. We therefore study in this section 4.3 whether the reducible limit of the matter curve \((86)\) is a sufficient condition for the absence of the dimension-4 proton decay operators.

It is important to note that reducible limit of the matter curve \((86)\) does not immediately imply that the sheaf cohomology on the curve also splits as in

\[
H^0(\bar{c}(5); \mathcal{F}(5)) \to H^0(\bar{c}(\bar{DL}); \mathcal{F}(\bar{DL})) \oplus H^0(\bar{c}(H); \mathcal{F}(H)).
\]

For this splitting to take place, one needs to make sure in the reducible limit of the curve that the sheaf \( \mathcal{F}(5) \) also becomes

\[
\mathcal{F}(5) \to \mathcal{i}_{(\bar{DL})*} \mathcal{F}(\bar{DL}) \oplus \mathcal{i}_{(H)*} \mathcal{F}(H),
\]

\(31\) An option of taking a further reducible limit of \( \bar{c}(H) \to \bar{c}(Hu) + \bar{c}(Hd) \) has been discussed as a solution to the dimension-5 proton decay problem in \([10]\). Our discussion in this section \([43]\) is applied; in such a limit, mostly to \( \bar{c}(\bar{DL}) \) and \( \bar{c}(Hd) \). See also discussion at the end of section \([43]\) Note also that the dimension-5 proton decay problem is not as serious in such string compactifications as in GUT models on 3+1 dimensions; this is because not much is known about the Yukawa couplings involving Kaluza–Klein colored Higgsinos \([59, 10]\).
where

\[ i_{(DL)} : \mathcal{C}_{(DL)} \hookrightarrow \mathcal{C}(\bar{5}), \quad i_{(H)} : \mathcal{C}(\bar{5}) \hookrightarrow \mathcal{C}(\bar{5}). \tag{90} \]

Sections of a line bundle \( \mathcal{F}_{(DL)} \) [resp. \( \mathcal{F}_{(H)} \)] form a rank-1 fiber at a given point of the matter curve \( \mathcal{C}_{(DL)} \) [resp. \( \mathcal{C}(\bar{5}) \)], but rank of the the sheaf \( \mathcal{F}(\bar{5}) \) suddenly jumps up to 2 at the intersection point of the two curves in the case of \((89)\). The question is whether this condition is realized automatically at the reducible limit of the matter curve \((86)\).

Let us call \((p_0:a_5 - p_2:a_3) = 0\) piece as \(\mathcal{C}_{(DL)}\), and \((q_0:a_5 - q_2:a_3) = 0\) as \(\mathcal{C}(\bar{5})\). The two curves surely intersect at points in \(S\) where \((a_5,a_3) = (0,0)\). That is where \(A_4\) singularity is enhanced to \(D_6\). We know that the condition \((89)\) is satisfied there, at least at the level of analyses in \([8, 9]\). The other type of intersection points are found where

\[ p_0 : p_2 = q_0 : q_2 = a_3 : a_5, \tag{91} \]

but \((a_5,a_3) \neq (0,0)\). Since \((91)\) consists of two conditions, \((91)\) is satisfied at finite number of isolated points on \(S\) generically. Is the condition \((89)\) satisfied at this type of codimension-3 singularities?

Let us study this problem by constructing a field theory model of local geometry around a point of this type. The singularity of local geometry is observed better in a new set of coordinates,

\[
\tilde{x} = x + \frac{a_3}{a_5}z^2 + \left( \frac{2}{a_5} \right)^2 \left( \frac{a_2}{2} - \frac{a_3a_4}{a_5} + \frac{1}{4}(a_5A_3 - a_3A_5) \right)z^3, \tag{92}
\]

\[
\tilde{y} = y - \frac{1}{2}\left( (a_5 + A_5z)x + a_3z^2 + A_3z^3 \right). \tag{93}
\]

In this new set of local coordinates, the defining equation \((62)\) becomes

\[
\tilde{y}^2 = \frac{a_5^2}{4} \tilde{x}^2 + \frac{P(5)}{a_5^4}z^5 - \frac{\tilde{R}(5)}{a_5^3}z^6 + \mathcal{O}(z^7) + \mathcal{O}(z^7)\tilde{x}^2 + \mathcal{O}(z^4)\tilde{x}. \tag{94}
\]

\(P(5)\) is defined as before, and \(\tilde{R}(5)\) is given by

\[ \tilde{R}(5) \equiv \left( a_2 - \frac{2a_3a_4}{a_5} \right)^2 + \left( a_2 - \frac{2a_3a_4}{a_5} \right)(a_5A_3 - a_3A_5) - a_5^2A_4 - a_5^2y^2_*, \tag{95} \]

where \(y^2_* = x^2_* + A_2x_* + A_0\) and \(x_* = -(a_3/a_5)\). The last four terms of \((94)\) drop under the scaling

\[ (\tilde{x}, \tilde{y}, z) = (\lambda^3x_0, \lambda^3y_0, \lambda z_0) \quad \lambda \to 0, \tag{96} \]
because their weights are higher than 6. The equation (94) describes a
decomposition of $A_5$
singularity surface in the space with $(\hat{x}, \hat{y}, z)$ local coordinates. The undeformed singularity
is $A_4$ at a generic point on $S$, but it is enhanced to $A_5$ on either one of the matter curves $\tilde{c}(D_L)$ and $\tilde{c}(H)$, because $P^{(5)} = (p_0a_5 - p_2a_3)(q_0a_5 - q_2a_3)$ vanishes. At the intersection points
of the two curves, the first two terms of $\tilde{R}^{(5)}$ vanish, because

\[(a_2a_5 - 2a_3a_4) = p_2(q_0a_5 - q_2a_3) + q_2(p_0a_5 - p_2a_3). \tag{97}\]

The last two terms in $\tilde{R}^{(5)}$, however, do not have a reason to vanish at such intersection points, and hence $\tilde{R}^{(5)} \neq 0$ generically. The singularity in the $(\hat{x}, \hat{y}, z)$ space remains $A_5$
at this type of intersection points, without being enhanced to $A_6$. The absence of further enhancement can also be seen easily from the discriminant (63). The coefficient of the $z^6$ term becomes $-a_5^4 \tilde{R}^{(5)}/16 \rightarrow a_5^4(a_5^2A_4 + a_5^2y_+^2)/16$, which is the same as the last two terms of $\tilde{R}^{(5)}$.

It does not vanish generically, and hence the discriminant is not enhanced to $z^7$ at this type of codimension-3 singularity point. It is not that we failed to find an appropriate set of local coordinates and/or scaling limit to detect enhanced singularity, because the discriminant remains $\Delta \sim z^6$; the singularity remains $A_5$.

Local geometry around this type of intersection points, therefore, can be modelled by an SU(6) gauge theory. Let us take a set of local coordinates $(u, v)$ on $S$, so that

\[u \equiv \frac{p_0}{p_2} - \frac{a_3}{a_5}, \quad v \equiv \frac{q_0}{q_2} - \frac{a_3}{a_5}. \tag{98}\]

The background field value of $\varphi$ is chosen as

\[\langle \varphi \rangle \propto \text{diag}(0, \cdots, 0, uv \, du \land dv). \tag{99}\]

\[32\] In the language of elliptic fibered compactification of Heterotic string, the behavior of $\varphi$ is understood as follows. The stable degeneration limit of F-theory compactifications corresponds to a limit of Heterotic compactifications where supergravity approximation is good, and we restrict our attention to this region of the moduli space. In this limit, $A_4$ is small, and $f_0 = A_2$ and $g_0 = A_0$ determine the complex structure of elliptic fiber $y^2 = x^3 + f_0 x + g_0$ of Heterotic compactification. The spectral surface (65) determines five points in a given elliptic fiber. For small $(u, v)$, the $(x, y)$ coordinates of the two among the five points behave as

\[(x, y) \simeq (x_*, \pm y_*) + \frac{p_2q_2}{a_5} \frac{uv}{y_+^2} \left(1, \pm \frac{3x_+^2 + f_0}{2y_+} \right),\]

and three other points are determined by $(a_5/p_2q_2)y + x - x_* = 0$ on the elliptic curve. Thus, the group-law sum of the two points $p_\pm = (x, y)_\pm$ become $\xi_{p_+} \oplus \xi_{p_-} \simeq (p_2q_2/a_5)uv/y_+^2$; $\xi \simeq x/y$ is the local coordinate of the elliptic fiber near the infinity point. The addition theorem of elliptic functions is used here. Values of $\xi$ of nine other points of the form $p_j \oplus p_k$ are not close to zero. Thus, the spectral surface $C_{\xi_+ \xi_-}$ has only one smooth layer $\xi \simeq (p_2q_2/a_5)uv/y_+^2$ near the zero locus $\xi = 0$, and nine other layers are far away from $\xi = 0$.  

37
The zero modes in the SU(5)\textsubscript{GUT-5} representation originate from the $1 \times 5$ lower-left block in the $6 \times 6$ matrix representation of SU(6). An important point is that the zero mode wavefunction $(\psi, \chi)$ is in a single component representation of the background; 

$$\rho(\langle \varphi \rangle) \propto uv \, du \wedge dv$$

acts on the zero mode wavefunction as a $1 \times 1$ matrix. Zero mode wavefunctions $(\psi, \chi)$ will approximately be Gaussian $e^{-|u|^2}$ near the matter curve $\bar{c}_{(DL)}$ ($u = 0$), and will be Gaussian $e^{-|v|^2}$ near the matter curve $\bar{c}_{(H)}$ ($v = 0$), but the wavefunctions on the both branches cannot take different values at the intersection point. Both $(\psi, \chi)|_{\bar{c}_{(DL)}}$ along $(0, \forall v)$ on $\bar{c}_{(DL)}$ and $(\psi, \chi)|_{\bar{c}_{(H)}}$ along $(\forall u, 0)$ on $\bar{c}_{(H)}$ should approach the same value $(\psi, \chi)|_{(u,v)=(0,0)}$ at the intersection point. A line bundle $\mathcal{N}(\bar{5})$ is on the smooth spectral surface given by $\xi \propto uv$, and the sheaf $\mathcal{F}(\bar{5})$ on the curve $\bar{c}_\bar{5} \equiv \bar{c}_{(DL)} + \bar{c}_{(H)}$ is a restriction of a line bundle $\mathcal{N}(\bar{5}) \otimes K_S$ on $\bar{c}_\bar{5}$ given by $\xi = 0$ on the spectral surface. The sheaf $\mathcal{F}(\bar{5})$ remains strictly rank-1 at any points of the curve $\bar{c}_\bar{5}$, and the rank does not jump up to 2 at the intersection point. The zero modes in this case are not regarded as locally free fluctuations on $\bar{c}_{(DL)}$ and locally free fluctuations on $\bar{c}_{(H)}$ that are mutually independent locally, but the wavefunctions on the two curves are constrained to have the same value at the intersection points.

Thus, the reducible limit of the matter curve alone does not guarantee that the zero modes of the $(\bar{D}, L)$ type and Higgs type split into the two distinct curves. Therefore, we conclude that taking the reducible limit of the matter curve alone does not help remove the dimension-4 proton decay operators.

### 4.3.2 tuning more parameters

With a little more tuning of coefficients of the defining equation (62), the coefficient of the $z^6$ term ($\propto a_3^2 A_4 + a_5^2 y_2^2$) can be made vanish at each of such intersection points. The number of such points is given topologically by

$$\left( 5K_S + \eta + \eta' \right) \cdot \left( 5K_S + \eta + \eta'' \right) - \left( 5K_S + \eta \right) \cdot \left( 3K_S + \eta \right).$$

There may be many of them, and the same number of complex structure moduli may have to be tuned by hand, yet one might accept this tuning in order to avoid the dimension-4 proton decay.

Now the field theory local model is an SU(7) gauge theory. By rescaling the local coordinates $(u, v)$ if necessary, the defining equation of the spectral surface $C(\bar{5})$ can be made

$$A\xi^2 + (u + v)\xi + uv = 0,$$
and the $\xi = 0$ locus on $C_{(\mathbf{5})}$, that is $uv = 0$, is the matter curve $\tilde{c}_{(\mathbf{5})}$. $A$ is a coefficient that we do not specify; rescaling of local coordinates $(u,v)$ and $\xi$ cannot absorb this $A$. Local geometry of $X$ is given by $\tilde{y}^2 - \tilde{x}^2 \simeq z^5(Az^2 + (u+v)z + uv)$, and two topologically independent 2-cycles in the $(\tilde{x}, \tilde{y}, z)$ space vanish on $u = 0$ ($\tilde{c}_{(\mathbf{D}L)}$) and $v = 0$ ($\tilde{c}_{(H)}$) respectively. Thus, there are two extra topologically independent vanishing 2-cycles at the intersection point $(u,v) = (0,0)$, and it may be possible that the independent degrees of zero modes are locally associated with these independent 2-cycles. If this is proved to be true then the splitting of the matter fields in $SU(5)_{GUT-\mathbf{5}}$ representation as desired in (88, 89) may follow, as classification of zero-mode degrees of freedom.

It should be noted, however, that the spectral surface $C_{(\mathbf{5})}$ given by (101) is still a single irreducible surface unless $A = 1$. The matter curve is reducible, but the spectral surface is not. As we have learnt in [9], zero-mode wavefunctions of charged matter fields are regarded as single component $(\psi, \chi)$ wavefunctions on the spectral surface. Even for a zero-mode whose wavefunction can be characterized by a holomorphic section on the curve $u = 0$ [resp. $v = 0$] in $C_{(\mathbf{5})}$, we cannot imagine that the wavefunction $(\psi, \chi)$ is absolutely zero in any open subset of $C_{(\mathbf{5})}$; if it were, then it would be zero everywhere on $C_{(\mathbf{5})}$. Thus, zero modes that may be classified as “$H^0(\tilde{c}_{(\mathbf{D}L)}; \mathcal{F}_{(\mathbf{D}L)})$” should also have non-vanishing wavefunctions along the $\tilde{c}_{(H)}$ curve (and vice versa).

If we are to phrase this statement in language of field theory local models, that will be as follows. One can choose $SU(7)$ as the gauge group of field theory local models around points satisfying (91). Along the matter curve $\tilde{c}_{(H)}$ [resp. matter curve $\tilde{c}_{(\mathbf{D}L)}$] away from the $(u,v) = (0,0)$ points, however, field theory models with an $SU(6)_H \subset SU(7)$ [resp. $SU(6)_M \subset SU(7)$] gauge group can provide a reasonable approximation. An argument in the previous paragraph means that zero modes that may be classified as “$H^0(\tilde{c}_{(\mathbf{D}L)}; \mathcal{F}_{(\mathbf{D}L)})$” also have non-vanishing single component $(\psi, \chi)$ wavefunction in the $SU(6)_H$ gauge theory along the curve $\tilde{c}_{(H)}$.

Now, it is at the codimension-3 singularity points $(a_3, a_5) = (0,0)$ that the trilinear couplings for the dimension-4 proton decay (59) are potentially generated. $SO(12)$ gauge theories can be chosen as field theory local models of geometry around this type of singularity. Two branches of the matter curve $\tilde{c}_{(\mathbf{5})}$ pass through this type of points, and under the assumption (87), the two branches correspond to $\tilde{c}_{(\mathbf{D}L)}$ and $\tilde{c}_{(H)}$ respectively. Field theory local models with $SU(6)_H \subset SO(12)$ [resp. $SU(6)_M \subset SO(12)$] gauge group can provide a

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33 As the spectral surface (101) is singular at $(\xi, u, v) = (0,0,0)$, structure of the sheaves $\mathcal{N}_{(\mathbf{5})}$ and $\mathcal{F}_{(\mathbf{5})}$ is not obvious there.
good approximation along the matter curve \( \bar{c}(H) \) [resp. \( \bar{c}(\bar{D}L) \)] away from the \((a_3, a_5) = (0, 0)\) points. Any zero modes in the \(SU(5)_{GUT}-\bar{5} \) representation are assigned two-component \((\psi, \chi)\) wavefunctions in the \(SO(12)\) local models, and one of the two components correspond to the single component \((\psi, \chi)\) wavefunction in the \(SU(6)_H \supset SU(5)_{GUT}\) gauge theory, and the other component to the single component wavefunction in the \(SU(6)_M \supset SU(5)_{GUT}\) gauge theory. The trilinear couplings (59) are calculated by overlap integration in the \(SO(12)\) gauge theory, and the overlap integration picks up both of the two components. Thus, the couplings would vanish if zero modes that may be classified as “\(H^0(\bar{c}(\bar{D}L); \mathcal{F}(\bar{D}L))\)” had vanishing wavefunction in the \(SU(6)_H\) gauge theory. We have learnt in the previous paragraph, however, that this is not the case. Therefore, the trilinear couplings are indeed generated, even after tuning the complex structure of a 4-fold \(X\) so that \((a_3^2 A_4 + a_5^2 y_2^* )\) vanish at all the points satisfying (91).

Of course the value of the \((\bar{D}, L)\)-like wavefunction may well be small in the \(SU(6)_H\) gauge theory along \(\bar{c}(H)\) far away from the points satisfying (91). The question is whether such a mixing of the wavefunction is small enough to satisfy a phenomenological constraint,

\[
\sqrt{\lambda'' \lambda'} \lesssim 10^{-13},
\]  

which comes from experimental lower bounds on the proton lifetime. To make an estimate of the size of the trilinear couplings generated in this scenario, we make a conservative assumption that the zero-mode wavefunctions decrease as in Gaussian profile far away from the points satisfying (91). Then, the wavefunction of the \((\bar{D}, L)\)-like matter fields along the \(\bar{c}(H)\) branch at a \((a_5, a_3) = (0, 0)\) point is expected to be of order

\[
e^{- \left( \frac{4}{d} \right)^2} > e^{- \left( \frac{R_{GUT}}{\sqrt{\frac{g_s}{g_{*3}^2}} \frac{1}{M_4^4}} \right)^2} = e^{- \sqrt{\frac{\rho_{GUT}}{g_s A_4^2}} \frac{1}{M_4^4}} \approx 10^{-2.1 \left( \frac{1}{M_4^4} \right)^2} \approx 10^{-13} \frac{1}{(3M_4^4)^2} \approx \frac{1}{2\pi M_4} \approx 10^{-84} \frac{1}{(2\pi M_4)^2}.
\]  

(103)

Here, \(d\) is the width of the Gaussian profile, \(L\) is a distance to a point where \((a_5, a_3) = (0, 0)\), which is always smaller than \(R_{GUT}\). Since the size of \(S\) is finite, not infinitely large, wavefunctions cannot be smaller than the value given above; the ratio between the Kaluza–Klein radius and the width of localized wavefunctions sets the smallest possible value of trilinear couplings (c.f. [3]). For a rough estimate at the first try, we have used \((R_{GUT} M_4)^2\) for \((R_{GUT}/d)^2\), and we know its value from (21). Given the three crude estimates above, which correspond to \(d \sim 1/M_4, 1/(\sqrt{2\pi} M_4)\) and \(1/(2\pi M_4)\), it is therefore crucial to know the relation between \(d\) and \(1/M_4\) more precisely, to see if this scenario can be a viable solution to the dimension-4 proton decay problem.
Let us suppose that the local defining equation is
\[ y^2 \simeq x^2 + z^5(fu + z + \cdots), \tag{104} \]
where local coordinates \( u, z \) are made dimensionless by some unit length \( l_* \), and \( f \) is a dimensionless numerical coefficient. This equation describes a geometry near a matter curve at \( u = 0 \). To be a little more general, we can think of a spectral surface given by
\[ fu + \xi + \cdots = 0, \tag{105} \]
where the fiber coordinate \( \xi \) of \( \mathbb{K}_S \) is also dimensionless. It is natural to imagine that the dimensionless coefficient \( f \) is of order unity, although it should be an issue to be confirmed ultimately by flux compactification. Since the geometry in (104) allows for an interpretation as an intersecting D7–D7 system, we can determine the \( 2\alpha\varphi_{12} \) field vev for (104), without an ambiguity in the normalization. Using the fact that the mass of an open string state stretching a distance \( D \) is
\[ m = \frac{1}{2\pi\alpha'} D, \tag{106} \]
we find\(^{34}\) that \( 2\alpha\varphi_{12} = (4\pi\alpha')^{-1}fu' \), where \( u' \equiv ul_* \) is the local coordinate with a physical mass dimension \(-1\) restored. Corresponding Gaussian wavefunction is \( e^{-(4\pi\alpha')^{-1}fu'|^2} \). The typical width parameter, therefore, turns out to be \( d \sim \sqrt{4\pi\alpha'} \). If we further make a crude approximation\(^{35} \) \( g_s \sim \mathcal{O}(1) \), then \( d \sim 1/(\sqrt{\pi}M_* ) \), and\(^{36} \)
\[ \left( \frac{R_{\text{GUT}}}{d} \right)^2 \sim \pi (R_{\text{GUT}}M_*)^2 = \frac{\pi}{\sqrt{\alpha_{\text{GUT}}}} \simeq \frac{1}{0.06} \simeq 15.. \tag{107} \]
Thus, we have a lower bound on the \( R \)-parity violating trilinear couplings:
\[ \lambda, \lambda', \lambda'' \gtrsim 10^{-7}. \tag{108} \]
Given the so many crude approximations we have made (especially \( g_s \sim \mathcal{O}(1) \) and \( f \sim \mathcal{O}(1) \)), the inconsistency between the phenomenological limit (102) and the lower bound (108) should not be taken as an argument excluding the scenario in this section 4.3.2. One should also keep in mind, however, that the lower bound (108) is based on an inequality \( L < R_{\text{GUT}} \) that is virtually never saturated, and furthermore, wavefunctions do not always damp as fast as in the Gaussian profile; see \cite{9} and the appendix of this article.

\(^{34}\)See the appendix of [9] for details on the normalization convention.

\(^{35}\)See footnote [5].

\(^{36}\)This parameter \( (d/R_{\text{GUT}})^2 \sim \sqrt{\alpha_{\text{GUT}}}/\pi \) may also sets the hierarchical scale of flavor physics (c.f. [43]).
If this scenario is phenomenologically acceptable, then the neutrino Yukawa couplings are generated around the codimension-3 singularity points at the points satisfying (91), because that is where the wavefunctions of lepton doublets $L \subset (\bar{D}, L)$ and $H_u \subset H(5)$ are not small. Right-handed neutrinos are identified either with $H^3,^1$ or $H^{1,2}$ in this scenario. The Yukawa couplings can be calculated by the SU(7) gauge theory local model, and the wavefunctions of right-handed neutrinos can be dealt with as in the prescription given in section 3. Majorana masses are generated for the $H^3,^1$ moduli from flux compactification, as we have explained in section 2.

4.3.3 yet another limit of reducible spectral surface

In order to safely remove the mixing of the wavefunction altogether between the $H_d \subset \bar{H}(5)$-like matter and $L \subset (\bar{D}, L)$-like matter fields, one should consider a limit where the spectral surface $C_{(\bar{5})}$ is reducible:

$$C_{(\bar{5})} = C_{(DL)} + C_{(H)},$$

where the $\xi = 0$ loci of $C_{(DL)}$ and $C_{(H)}$ become the matter curves $\bar{c}_{(DL)}$ and $\bar{c}_{(H)}$ in (86), respectively. The zero-mode wavefunctions of the $L \subset (\bar{D}, L)$-like matter becomes absolutely zero on the $C_{(H)}$ piece of $C_{(\bar{5})}$ in this case, and hence the couplings of the dimension-4 proton decay operators vanish.

In the SU(7) gauge theory which models the local geometry around points satisfying (91), this factorisation of the spectral surface is realized by further tuning complex structure so that $A = 1$ in (101). The two pieces $C_{(DL)} (\xi + u = 0)$ and $C_{(H)} (\xi + v = 0)$ intersect along $u = v = -\xi$, and form a double curve singularity. Since the gauge group of this local model is $A_N$-type, a natural Type IIB interpretation exists; this double curve singularity is nothing but D7–D7 intersection.

Now remember that the $H^{3,1}$ and $H^{1,2}$ moduli fields are captured as $\text{Ext}^1(i_*\mathcal{N}_{(10)}, i_*\mathcal{N}_{(10)})$ in $\mathbb{K}_S$ in the field theory local models, where $i_*\mathcal{N}_{(10)}$ [resp. $\mathcal{N}_{(10)}$] is the Higgs sheaf [resp. line bundle supported on the spectral surface $C_{(10)}$] for the fields in the SU(5)$_{\text{GUT}}$–10 representation. The double curve singularity in the spectral surface $C_{(\bar{5})}$ (other than those in the local models around codimension-3 singularity points of $A_1 \to D_6$ enhancement) for the

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37 The factorization limit of the spectral surface (109) here does not have an easy interpretation within the $E_8$ gauge theory in the dual Heterotic language. This is not a reducible limit of a vector bundle in an $E_8$; it is easy to see this because the tuning required in section 4.3.2 is to set $y_0^2 = x_3^3 + f_0 x_2 + g_0 = 0$ at the points satisfying (91), and this condition involves not just moduli of the spectral surface, but also the complex structure parameter $f_0$ and $g_0$ of the elliptic fibration.
fields in the SU(5)-$\bar{5}$ representation indicates that the spectral surface $C_{(10)}$ for the fields in the SU(5)$_{\text{GUT}}$-10 representation also has a double curve singularity. $C_{(10)}$ is not irreducible, but it also splits into $C_1 + C_2$. Then, the Higgs sheaf $\mathcal{V} = i_*\mathcal{N}_{(10)}$ also splits as $i_1\mathcal{N}_1 + i_2\mathcal{N}_2$, and the moduli fields also split as in (70). The last two components in (70) are localized at the intersection of the two pieces, or intuitively, the D7–D7 intersection curve. In the absence of ramification of the spectral surface, we purely have an SU(7) field theory local model. Open string interaction generates neutrino Yukawa couplings for the one of the last two components above, which correspond to the off-diagonal pieces in the 2 by 2 block of the 7 by 7 matrix of SU(7). We have thus arrived at the picture assumed in [10].

It is not obvious whether there is a global unbroken U(1) symmetry as an explanation for the absence of dimension-4 proton decay in this scenario with a factorized spectral surface. This is a crucial question because the Majorana masses of the right-handed neutrinos are forbidden as long as such an unbroken U(1) symmetry exists. At least, in the local field theory models with SO(12) and SU(7) gauge groups, the spectral surface for the fields in the SU(5)-$\bar{5}$ representation does not ramify, and one can find U(1) symmetry transformations in these gauge theories, where the lepton doublets and $H_d$ have distinct charges under the U(1)'s.\textsuperscript{38} It is not obvious, however, whether one should maintain such a U(1) symmetry in all the field theory local models of the patches covering the $A_4$ singularity surface $S$; the factorization limit of the spectral surface (109) is sufficient in removing all the dimension-4 proton decay operators, and it is not clear if the factorization limit immediately implies the existence of a U(1) symmetry in the effective theory. If it does not, then we do not strictly need a symmetry.\textsuperscript{39}

Even more controversial is whether the factorization limit (109) is well-defined. The spectral surface of Higgs bundle is defined in F-theory compactifications only in field theory local models. One can choose field theory local models with SU(6) or SO(10) gauge groups along the matter curves, and local models can be chosen with gauge groups SU(7), SO(12) and $E_6$ at codimension-3 singularity points. These choices, however, are just a the minimal choice preserving essential features of the local geometries. For higher level of approximation, local models can be replaced by gauge theories with higher-rank gauge groups. For example,

\textsuperscript{38}Whether an associated U(1) gauge symmetry remains massless and anomaly free in low-energy effective theory is yet another (and often global) issue, and we will not discuss here.

\textsuperscript{39}It certainly goes against a common sense of field theory model building to claim that certain operators are absent without an explanation in terms of symmetry, but such things may or may not happen in string theory. We do not have any arguments in favor of or against such a mechanism of vanishing couplings without a symmetry reason.
in F-theory compactifications with Heterotic dual, one can choose $E_8$ gauge theory as local models at any patches of $S$ (or even globally on $S$), not just the minimal rank-1 or rank-2 extension of the common $SU(5)_{GUT}$ over $S$. Even in F-theory compactifications with Heterotic dual, however, it is a good approximation to cut out the rank-5 Higgs bundle from the rest only in the stable degeneration limit. If the complex structure moduli are not necessarily in this limit, then taking just the rank-5 part into the field theory formulation on $S$ and discarding all the rest is not a systematically justified approximation. How can one extend the gauge group of the local models beyond $E_8$ to achieve a higher level of approximation within the field theory formulation? The situation is essentially the same in generic F-theory compactifications; field theory local models can capture local geometry of $X$ near the $A_4$ singularity surface $S$, but the field theory formulation does not offer a systematic way (order by order) to capture the entire geometry of $X$ for higher level of approximation. The field theory formulation can capture only a local geometry that is approximately a deformed ADE singularity fibered over a local patch of $S$. The whole geometry of $X$ is compact and is not an ALE fibration on $S$. Since the spectral surfaces of the Higgs bundles can be defined only within the field theory local models, the reducibility (factorization) of the spectral surfaces can also be defined order by order in this approximation, which will never be able to cover the entire geometry of $X$. At this moment, we do not have a clear idea how to define the factorization limit rigorously. For the real-world physics, however, the constraints from phenomenology always leave a room for very small couplings for the dimension-4 proton decay operators. Thus, it may be an option to enforce factorization limit in gauge theory local models with higher-rank gauge groups so that sufficiently high level of approximation is achieved, and trilinear couplings as small as $10^{-13}$ can be discussed.

Before closing this section 4.3, we comment on a variation of the scenario that has been discussed so far. We have discussed this scenario along the line of (86), where neither $H_u \subset H(5)$ nor $H_d \subset \bar{H}(\bar{5})$ are localized in the irreducible curve $\bar{c}(\bar{D}L)$. As a solution to the dimension-4 proton decay problem, however, only the distinction between $H_d$ and the three lepton doublets is essential. Thus, $H_u$ may originate from the same curve as the lepton

\[\text{The factorization (reducibility) of the spectral surface in the SU(7) local models is the same as the factorization of the discriminant locus. Thus, it might seem at first sight that the factorization condition of the spectral surface can be replaced by the factorization condition of the discriminant locus. But these conditions are actually totally different. As explained in section 4.3 of [9], the spectral surface for the SU(5)$_{GUT}$-5 representation fields consists of two irreducible pieces in the SO(12) local models around the codimension-3 singularities with enhanced $D_6$ singularity, whereas the corresponding discriminant locus consists of a single irreducible piece.}\]
doublets. That is, we can consider another reducible limit,
\[ \bar{c}(\bar{5}) \rightarrow \bar{c}(DLHu) + \bar{c}(Hd), \quad C(\bar{5}) \rightarrow C(DLHu) + C(Hd). \]  

This is theoretically possible; what was discussed in [61] is essentially the same thing from theoretical perspectives. In this new factorization scenario, neutrino Yukawa couplings are generated just as in section 3 and the $H^{3,1}(X; C)$ moduli have Majorana masses, just as in section 2. There is nothing to worry about the Majorana masses in the absence of possible protection by a $U(1)$ symmetry. Neutrino Yukawa couplings have contributions all along the curve $\bar{c}(DLHu)$, and the Majorana masses come from the entire bulk of $B_3$. The $\Delta W = SH_uH_d$ interaction of the NMSSM, on the other hand, is localized at the points satisfying (91) in this factorization limit, if such a massless singlet chiral multiplet $S$ exists in the spectrum.

Phenomenology of supersymmetry breaking terms in the scenarios in this section 4.3 is beyond the scope of this article.

### 4.4 $R$-parity Violating Scenarios

An unbroken $U(1)$ symmetry is powerful in removing the dimension-4 proton decay operators, but it also forbids the Majorana mass terms of right-handed neutrinos. That would be the executive summary of section 4.2 and may be also of the $C(DL)+C(HuHd)$ splitting scenario in section 4.3. If we could find a field $\phi$ with even units of charge of an unbroken $U(1)$ symmetry, then an unbroken $Z_2$ symmetry would be found after spontaneous breaking of the $U(1)$ by a vev of the field $\langle \phi \rangle$, but one still has to find how trilinear couplings $\Delta W = \phi NN$ like (60) would be generated. We have not found a way to discover such fields and such couplings.

If one throws away the $U(1)$ symmetry altogether and just impose a $Z_2$ symmetry from the beginning, then that is the $Z_2$ parity scenario in section 4.1.

Actually there is a caveat in this argument, however. Suppose that we begin with a compactification that leaves an unbroken $U(1)$ symmetry. This $U(1)$ symmetry can be broken spontaneously by vev’s only of chiral multiplets with, for example, positive $U(1)$ charges. Let us denote such chiral multiplets as $\phi_+$. We assume that all the fields with negative charges under the $U(1)$ symmetry do not have non-vanishing vev’s. Suppose that right-handed neutrinos $\bar{N}$ have a negative $U(1)$ charge, while the $U(1)$ charge of the dimension-4 proton decay operators $\bar{5} \ 10 \ 5$ is positive. Then the Majorana masses of right-handed

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41 The factorization condition of the spectral surface may be relaxed to the level of tuning in section 4.3.2.

42 A caveat in this argument is mentioned at the end of section 4.2.
neutrinos are allowed by the spontaneously broken symmetry, because the effective Majorana mass parameter $M_R$ can involve the vev’s with positive U(1) charges, $M_R \sim \langle \phi_+ \rangle^n$. On the other hand, the dimension-4 proton decay operators are still forbidden by the broken symmetry, because of the absence of chiral multiplet vev’s with negative U(1) charges. Supersymmetric D-term condition can be satisfied for this U(1) symmetry, because the vev’s of the positively charge field can balance against a negative Fayet–Iliopoulos parameter $\xi$ proportional to $\omega \wedge F$.

This scenario was proposed in [1], and studied in detail in [5, 9]. Study in [1, 5, 9] was done mostly in language of Heterotic string compactification. Although two of the authors (RT and TW) made an effort to provide a description of this scenario in F-theory language as well in [1], various theoretical aspects of F-theory compactifications and the duality between Heterotic string and F-theory were not as clearly understood back then as they are now, and the translation from Heterotic description to F-theory description was not completed there. With a better theoretical understanding of F-theory compactification, we now provide a little better version.

In the language of Heterotic $E_8 \times E_8$ string compactification on a Calabi–Yau 3-fold $Z$, the key idea of [1] was to use a rank-5 vector bundle $V_5$ with a structure

\begin{align*}
0 \to L_\chi \to V_5 \to U_4 \to 0 & \quad \text{or} \quad (111) \\
0 \to U_2 \to V_5 \to U_3 \to 0. & \quad (112)
\end{align*}

Here, $L_\chi$ [resp. $U_2$] is a rank-1 [resp. rank-2] sub-bulde of $V_5$, whose structure group is $SU(5)_{str} \subset E_8$. Zero mode chiral multiplets in the $SU(5)_{GUT}$-5 representation are $H^1(Z; \wedge^2 V_5)$ in general Heterotic string compactifications, but for $V_5$ with such a sub-bundle as above, a subspace

$$H^1(Z; L \otimes V_5) \subset H^1(Z; \wedge^2 V_5)$$

or

$$H^1(Z; U_2 \otimes V_5) \subset H^1(Z; \wedge^2 V_5)$$

(113)

is well-defined, and this subspace is identified with the $(\bar{D}, L)$-type matter fields; the $H_d \subset \bar{H}(\bar{5})$ field, on the other hand, is regarded as a generic element of $H^1(Z; \wedge^2 V_5)$. Chiral multiplets $10 = (Q, \bar{U}, \bar{E})$ are identified with $H^1(Z; U_2) \subset H^1(Z; V_5)$ in the case (112). All of the down-type/charged lepton Yukawa couplings and the dimension-4 proton decay operators originate from the product

$$H^1(Z; \wedge^2 V_5) \times H^1(Z; V_5) \times H^1(Z; \wedge^2 V_5) \to H^3(Z; \wedge^5 V_5) = H^{0,3}(Z; \mathbb{C})$$

(114)

\[43\] This selection rule is applied to, and only to, renormalizable operators in low-energy effective theories below the Kaluza–Klein scale. See [62, 59] for the discussion.
in the Heterotic string superpotential
\[
\Delta W_{\text{Het}} = \int_Z \Omega \wedge \text{tr} \left( AA + \frac{2}{3} AAA \right).
\]  
(115)

The product vanishes when both of the \( H^1(Z; \wedge^2 V_5) \) elements are in the subspace (113), and hence the dimension-4 proton decay operators are absent. See [1, 59] for more about this scenario.

The extension structures (111, 112) can be regarded as spontaneous breaking of \( U(1) \) symmetries. A rank-5 vector bundle \( L_\chi \oplus U_4 \) [resp. \( U_2 \oplus U_3 \)] has a structure group \( SU(4) \times U(1)_\chi \subset SU(5)_{\text{str}} \) [resp. \( SU(2) \times SU(3) \times U(1)_{\tilde{q}_7} \)]. The structure group in both cases has a \( U(1) \) factor. The rank-5 bundle \( V_5 \) is unstable, if the Fayet–Iliopoulos parameter \( \xi \propto \int_Z \omega \wedge \omega \wedge F \) of the \( U(1) \) symmetry does not vanish. In the case (111), for example, we assume that \( \xi_\chi \) is negative, and chiral multiplets \( \mathbf{N}^\chi \in H^1(Z; L_\chi \otimes U_4) \subset H^1(Z; \text{adj}(V_5)) \) with positive \( U(1)_\chi \) charge \( Q_\chi \) absorb the Fayet–Iliopoulos parameter. This is why \( L_\chi \) remains a well-defined sub-bundle, but \( U_4 \) does not.

Heterotic string compactification has an F-theory dual, when the Calabi–Yau 3-fold \( Z \) is an elliptic fibration
\[
\pi_Z : Z \to S
\]  
(116)
over a complex surface \( S \). In the following, we start from a Heterotic compactification on such a Calabi–Yau 3-fold \( Z \) with a vector bundle \( V_5 \) constructed from a pair of vector bundle \( (L_\chi, U_4) \) or \( (U_2, U_3) \) as above, and find out its F-theory dual description.

Bilinear \( R \)-parity violation [56] is generated in the scenario explained above [59]. An order of magnitude estimate of the bilinear \( R \)-parity violating parameters was given in [59], using weakly coupled Heterotic string compactification. \( R \)-parity violating decay of gravitino dark matter has been discussed as one of the possible explanations of the recent cosmic ray anomalies [63].

4.4.1 sub-bundle with vanishing first Chern class in the fiber

An F-theory dual description of this scenario becomes quite different, depending on whether \( c_1(L_\chi) = -c_1(U_4) \) [resp. \( c_1(U_2) = -c_1(U_3) \)] vanishes in the elliptic fiber direction of (116), or it is strictly negative [1]. Let us begin with the case with vanishing first Chern classes in the fiber direction.

\[44\] Not all of vector bundles with sub-bundles may admit such an interpretation. The following discussion, therefore, should be taken for granted only for bundles that can be constructed that way.
In this case, bundles $U_{1,2}$ ($U_1 \equiv L_\chi$ hereafter) and $U_{4,3}$ may be given by Fourier–Mukai transform separately. The spectral data, $(C_k, \mathcal{N}_k)$ ($k = 1, 2, 4, 3$), for Heterotic compactification can readily be used for the spectral data of Higgs bundles in F-theory compactification [9]. The case $V_5 = U_4 \oplus U_1$ (with a vanishing Fayet–Iliopoulos parameter $\xi_\chi$ even on the base space $S$) then corresponds to the SO(10) scenario in section 4.2 and $V_5 = U_3 \oplus U_2$ to the SU(6) scenario in section 4.2.

Discussion in section 4.2 corresponds to Higgs bundles in F-theory compactification where both $\omega \wedge F$ and $[\varphi, \varphi]$ vanish in the the first one of the BPS conditions (38). More general, however, is Higgs bundles where the first condition is satisfied as a combination $\omega \wedge F - |\alpha|^2 [\varphi, \varphi] / 2$, but not separately. This corresponds to non-vanishing Fayet–Iliopoulos parameter for the $U(1)_\chi [U(1)_{\tilde{q}}]$ symmetry; non-vanishing Fayet–Iliopoulos parameter is equivalent to $\omega \wedge F \neq 0$. The Fayet–Iliopoulos parameter triggers spontaneous breaking of the $U(1)$ symmetry; if the parameter is negative [resp. positive], then chiral multiplets with positive [resp. negative] $U(1)$ charge have tachyonic masses, and develop non-vanishing expectation values, which makes the $[\varphi, \varphi]$ term non-zero. In the end, stable minimum with vanishing D-term potential is equivalent to a $(A, \varphi)$ field configuration on $S$ satisfying the BPS conditions.

In the SO(10) scenario, for example, only the positively charged chiral multiplets $\overline{N}^c$ need to develop non-zero vev’s, in order to cancel the negative Fayet–Iliopoulos parameter, and $\langle \overline{N} \rangle$ remains zero. The $\langle \varphi \rangle$ configuration now have non-zero off-diagonal entries in the $5 \times 5$ matrix representation as in (32), but not in a way the spectral surface is affected. As discussed in section 4.2, only the vev of $\langle \overline{N}N^c \rangle$ modify the spectral surface; $\langle \overline{N} \rangle$ alone do not. The spectral surface only extracts information associated with eigenvalues of $\langle \varphi \rangle$, but symmetry breaking pattern in Higgs bundle is not always encoded only by the eigenvalues of $\langle \varphi \rangle$.

In the SO(10) scenario, a rank-1 bundle $L_\chi$ remains a $\varphi$-invariant sub–Higgs-bundle of the rank-5 Higgs bundle on $S$. In the SU(6) scenario, a rank-2 sub-Higgs-bundle remains $\varphi$-invariant. The rank-5 Higgs bundle in these scenarios are, so to speak, constructed by the parabolic construction on $S$; see section 5 of [28] for the parabolic construction of vector bundles on elliptic fibered Calabi–Yau manifolds $Z$.

Majorana mass terms of right-handed neutrinos can be generated in such a vacuum as

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45See also [47], where nilpotent Higgs vev is discussed in Type IIB compactifications.
long as an effective theory has a non-renormalizable term
\[ \Delta W = \overline{N} N N \overline{N}. \] (117)

Either \( \overline{N} \) or \( N \) develop non-vanishing expectation values to balance the non-vanishing Fayet–Iliopoulos parameter \( \xi \propto \omega \wedge F \), and then the counter part acquire Majorana masses \([1, 59]\). This term also lifts the D-flat direction of the U(1) symmetry \([64]\).

Two independent origins of the effective interaction \([117]\) have been discussed \([1, 65, 59]\) in language of Heterotic string compactification. One of them \([65, 59]\) is that the interaction \([117]\) is generated in effective theory, when Kaluza–Klein modes are integrated out of a field theory description. The superpotential \([115]\) contains interactions not only of Kaluza–Klein zero modes, but also Kaluza–Klein mass terms (the mass parameter coming from the derivative \( d \)) and trilinear interactions involving the Kaluza–Klein modes.

\[ W = \sum_I M_I \Phi_I \Phi_I + \sum_I \lambda_I \overline{N} \Phi_I \overline{N} + \cdots. \] (118)

Non-renormalizable terms like \([117]\) are generated in the effective theory, when the Kaluza–Klein modes \( \Phi_I \) are integrated out (c.f. \([25]\));

\[ \Delta W = \left( \sum_I \frac{\lambda_I^2}{M_I} \right) \overline{N} N N \overline{N}. \] (119)

There is no reason to doubt that the same thing happens in F-theory compactifications, because the superpotential for F-theory \([46]\) is essentially the same as that of the Heterotic string compactification \([115]\); only difference is that the F-theory superpotential \([46]\) has Kaluza–Klein modes on \( S \) alone, not on the Kaluza–Klein modes on Calabi–Yau 3-folds, but the Kaluza–Klein modes on \( S \) should be enough in generating interactions like \([117]\). Thus, the Majorana mass terms are generated for the right-handed neutrinos in F-theory compactifications also in the SO(10) and SU(6) scenarios with a non-vanishing Fayet–Iliopoulos parameter.

The mass scale of the Majorana mass is given approximately by
\[ M_R \sim \frac{(\lambda \langle N \rangle)^2}{M_{KK}} \sim \frac{\lambda^2}{M_{KK}} \frac{\xi}{M_{KK}}, \] (120)

where \( \xi \) is the Fayet–Iliopoulos parameter. The Fayet–Iliopoulos parameter has been calculated in Type IIB Calabi–Yau orientifold compactifications. Applying the result of \([66]\) naively
for a case a Calabi–Yau 3-fold $\tilde{B}$ is like $C \times S$, and $\text{vol}(\tilde{B}) = \text{vol}(C) \times \text{vol}(S) = R_1^2 R_{\text{GUT}}^4$ (like in the tubular model of Fig. 1(b)), and restoring proper dimensionality and $g_s$ dependence, we obtain

$$\xi \sim M_{\text{Pl}}^2 \frac{1}{\pi} \frac{g_s l_s^4}{R_{\text{GUT}}^2 R_1^2}. \quad (121)$$

Since this expression comes in a combination $g_s l_s^4 = 1/M_{\text{Pl}}^4$, which remains constant (relatively to $l_{11}$) everywhere in $B_3$ in generic F-theory compactifications, we dare to use this expression for F-theory compactifications that are not necessarily Type IIB Calabi–Yau orientifolds.

This estimate of the Fayet–Iliopoulos parameter is simplified by using (21–23) as

$$\xi \sim 4M_{\text{Pl}}^4 R_{\text{GUT}}^2 \sim \frac{4}{\alpha_{\text{GUT}}} \frac{M_{\text{GUT}}^2}{c^2}. \quad (122)$$

This result perfectly agrees with the Heterotic result in [1], up to a factor of $O(1)$ that we did not care about here. This result does not depend on $R_1$ or on the geometry in the direction transverse to $S$. Canonically normalized zero modes $|N| \ [\text{resp. } |\overline{N}|]$ develop vev’s of order $\sqrt{\xi}$, meaning that the original Higgs bundle that corresponds to $U_4 \oplus U_1$ [resp. $U_3 \oplus U_2$] receives an order-one correction to become a Higgs bundle with the extension structure as in [111, 112]. An estimate of the Majorana masses of right-handed neutrinos is obtained by plugging the estimate of $\xi$ in (120). Typical value of the trilinear couplings $\lambda_I$ in (118) are of order $g_{\text{GUT}}$ with suppression factors coming from overlap integration of normalized wavefunctions (c.f [3] for more details). The overlap integration tends to be smaller in the trilinear couplings like those in (118) (c.f. [59]), because the overlap integral involves two almost “flat wavefunctions” for two zero modes and one “higher Fourier mode” for the Kaluza–Klein states. Two $g_{\text{GUT}}$ coming from $\lambda^2$ cancel those in $1/\alpha_{\text{GUT}}$ in $\xi$, and the mass scale of right-handed neutrinos in this scenario is somewhere around the GUT scale with a suppression factor coming from the overlap integrals. This result fits very well with phenomenological expectation [4].

The other known mechanism for generating (117) is the world-sheet instanton effect in the language of Heterotic string compactification [64, 1]. Although such world-sheet instanton effects are known to cancel for certain class of (0,2) Heterotic compactifications, there may be other choices of geometries other than in such a class, and one does not have to rule this possibility out. A world-sheet instanton contribution from a curve $\Sigma$ in the base surface $S$ in Heterotic theory corresponds in F-theory [67] to M5-brane (Euclidean D3) instanton contribution from a divisor of $B_3$ that is projected on to the same curve of $S$ in the $\mathbb{P}^1$.
fibration
\[ \pi_{B_3} : B_3 \to S. \]  

An exponential suppression factor associated with this non-perturbative effect is of order
\[ \exp \left( -\frac{\text{vol}(\Sigma)}{2\pi \alpha'} \right) \bigg|_{\text{Het}} = e^{-\left[(2\pi)M_4^2R_{\text{GUT}}^2R_2^2\right]} \bigg|_F = e^{-\frac{\alpha_{\text{GUT}}}{\epsilon}} \simeq 10^{-660}, \]  
and is too small to be relevant for phenomenology. However, this estimate is very crude, and does not take account of a possibility that there may be a collapsed divisor in \( B_3 \). Thus, this estimate does not completely exclude a possibility that the effective interactions (117) are generated by M5-brane instanton effects.

We have so far discussed this \( R \)-parity violating scenario in F-theory compactification, by starting from a class of Heterotic string compactifications, and translating into F-theory language using the duality. Rank-5 vector bundles in Heterotic compactification on a Calabi–Yau 3-fold \( Z \) are translated into rank-5 Higgs bundles on a base 2-fold \( S \) for F-theory compactification. The extension structure of the vector bundles (and hence the subspace structure of the zero modes) in Heterotic theory is carried over to F-theory compactification as the extension structure of the Higgs bundles (and as the subspace structure of the zero modes). This way of understanding, however, raises a question whether this scenario is possible only in F-theory compactification with Heterotic dual. In F-theory compactification with a Heterotic dual, a Higgs bundle with a fixed rank can be defined globally on \( S \). Generic F-theory compactifications, however, have field theory local models only locally on \( S \), and physics associated with \( S \) (that is, GUT physics) has to be recovered by gluing those local models together. Since the nature of gluing process is at most approximate, there is a concern that the notion of sub-Higgs-bundle may not be well-defined globally on \( S \). If so, that would be a problem, given the severe constraint on the couplings of the dimension-4 proton decay operators.

In each local field theory model of a generic F-theory compactification, however, there is a well-defined U(1) symmetry: U(1)\( _\chi \) in the SO(10) scenario, and U(1)\( _{\tilde{q}} \) in the SU(6) scenario. The absence of the dimension-4 proton decay operators is guaranteed by the U(1) symmetry broken only by positively [resp. negatively] charged fields, and hence the \( R \)-parity violating scenario in this section \( 4.4.1 \) is available not just for F-theory compactifications with Heterotic dual. \footnote{The estimate of the suppression factor of M5-brane instanton effect can be a little more moderate in generic F-theory compactifications, as \((1/\epsilon^{\gamma=1})^2 \) in (124) may be replaced by \((1/\epsilon^{\gamma=1/3})^2 \) in the homogeneous}
4.4.2 sub-bundle with non-vanishing first Chern class in the fiber

Let us now study the F-theory dual description of the scenario explained at the beginning of this section in the case the first Chern class in the elliptic fiber direction does not vanish. For stability of $V_5$, $c_1(U_{1,2})|_{T^2} = -c_1(U_{4,3})|_{T^2}$ is negative. We will consider a region of the moduli space where the Kähler class of the $T^2$ fiber is smaller than those of the base $S$. This is where the Heterotic–F theory duality with 16 SUSY charges can be promoted to the duality with smaller number of SUSY charges adiabatically.

Since the bundles $U_{1,2}$ and $U_{4,3}$ have non-vanishing first Chern classes in the fiber direction, they are not given by spectral cover construction. It is thus non-trivial to see even such a thing as whether the charged matter fields are localized in the F-theory dual description. Let us begin with addressing this question.

Because $c_1(U_{1,2})$ in the fiber direction is negative, and $c_1(U_{4,3})|_{T^2}$ positive, $R^0\pi_Z U_{1,2}$ and $R^1\pi_Z U_{4,3}$ vanish. Since $V_5$ restricted on a fiber becomes a flat bundle, and is non-trivial generically, $R^0\pi_Z V_5 = 0$. Thus, this exact sequence follows:

$$0 \rightarrow R^0\pi_Z U_{4,3} \rightarrow R^1\pi_Z U_{1,2} \rightarrow R^1\pi_Z V_5 \rightarrow 0. \quad (125)$$

The support of the sheaves $R^0\pi_Z U_{4,3}$ and $R^1\pi_Z U_{1,2}$ is the entire $S$, but the support of $R^1\pi_Z V_5$ can be a curve in $S$. The map from $R^0\pi_Z U_{4,3}$ to $R^1\pi_Z U_{1,2}$ is to multiply a global section of $R^1\pi_Z \text{adj} V_5 = R^1\pi_Z (\mathcal{U}_{4,3} \otimes U_{1,2})$, which describes how $U_{4,3}$ are extended by $U_{1,2}$ to become $V_5$ in each fiber. Remember that the moduli of this global holomorphic section is identified with that of the spectral surface, when $U_{4,3}$ and $U_{1,2}$ are bundles $\mathcal{W}_{4,3}$ and $\mathcal{W}_{1,2}$ constructed in and $\mathcal{W}_{k} \oplus \mathcal{W}_{5-k}^* (k = 4, 3)$ is minimally unstable. The support of $R^1\pi_Z V_5$ is where the extension of $U_{4,3}$ in the $T^2$ fiber by $U_{1,2}$ becomes less non-trivial.

Massless matter fields in the SU(5)GUT–10 representation are identified with $H^1(Z; V_5) \simeq H^0(S; R^1\pi_Z V_5)$. Using the exact sequence, one finds a following long exact sequence where $H^0(S; R^1\pi_Z V_5)$ is in:

$$0 \rightarrow H^0(S; R^0\pi_Z U_{4,3}) \rightarrow H^0(S; R^1\pi_Z U_{1,2}) \rightarrow H^0(S; R^1\pi_Z V_5) \rightarrow H^1(S; R^0 U_{4,3}) \rightarrow H^1(S; R^1\pi_Z U_{1,2}). \quad (126)$$

This does not make a practical difference, though, as the exponential suppression factor remains extremely small. Section 5 of explains in detail how to construct a vector bundle $V_5$ on an elliptic fibered Calabi–Yau manifold, using a vector bundle $\mathcal{W}_k$ for $U_{4,3}$ and $\mathcal{W}_{5-k}^*$ for $U_{1,2}$. The bundles $\mathcal{W}_k$ and $\mathcal{W}_{5-k}^*$ satisfy $c_1(\mathcal{W}_k)|_{T^2} = -c_1(\mathcal{W}_{5-k}^*)|_{T^2} = 1$.

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Thus, $H^1(Z;V_5) \simeq H^0(S;R^1\pi_{Z\ast}V_5)$ has a subspace

$$\text{Coker}(H^0(S;R^0\pi_{Z\ast}U_{4,3}) \rightarrow H^0(S;R^1\pi_{Z\ast}U_{1,2})), \quad (127)$$

and the quotient by this subspace is

$$\text{Ker}(H^1(S;R^0\pi_{Z\ast}U_{4,3}) \rightarrow H^1(S;R^1\pi_{Z\ast}U_{1,2})). \quad (128)$$

The vector space of the zero modes, $H^0(S;R^1\pi_{Z\ast}V_5)$, has a subspace structure like (113). A chain complex (125) is regarded as the essence of this subspace structure in F-theory language.

Similar argument can be repeated for the zero modes in the SU(5)$_{\text{GUT}}$ and -5 representations. The subspace structure of the zero modes (113) does follow for these representations. The $(D,L)$-type fields are identified with this subspace. Since the trilinear couplings (59) vanish at least in Heterotic string compactifications with supergravity approximation, the same should be true in the stable degeneration limit of F-theory compactifications that have Heterotic dual.

Massless moduli fields coming from $H^1(S;R^0\pi_{Z\ast}U_4 \otimes U_1^{-1})$ have trilinear couplings with $L \subset (D,L) \in H^1(Z;U_1 \otimes V_5)$ and $H_u \subset H(5) \in H^1(Z;\wedge^2U_4)$ in the case with the structure (111). Thus, they are identified with the right-handed neutrinos $\overline{N}$. Moduli fields $H^0(S;R^1\pi_{Z\ast}U_4 \otimes U_1)$, denoted as $\overline{N}$, do not have the couplings to be identified with the neutrino Yukawa couplings. In the case with the structure (112), on the other hand, $H^0(S;R^1\pi_{Z\ast}U_2 \otimes U_3)$ is identified with the right-handed neutrinos $\overline{N}$ (11).

Many issues are still beyond the scope of this article. We have not discussed whether the dimension-4 proton decay operators are still protected in moduli space not necessarily at the stable degeneration limit. Such issues as how important the world-sheet instanton effects would become in small $T^2$-fiber limit (in Heterotic language) or how to define the subspace structure more rigorously despite the “approximate” nature of the field theory formulation of F-theory remain totally unaddressed. More detailed study of the structure of (125–128) and that for $R^1\pi_{Z\ast} \wedge^2 V_5$ is also desirable.

\footnote{A sequence of sheaves on $S$ similar to (125) is derived for $R^1\pi_{Z\ast} \wedge^2 V_5$, following the same line of argument as for $R^1\pi_{Z\ast}V_5$. However, we have no idea how to construct something like a principal-bundle version of such sequences for different representations.}
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A Branch Cut, Orbifold and 2-Form VEV

Non-Abelian gauge theories arising from F-theory compactification are described conveniently by using the field theory on 8-dimensions introduced by [11, 6, 7]. Local geometry of Calabi–Yau 4-fold \(X\) and base 3-fold \(B_3\) containing a divisor \(S\), the discriminant locus for the non-Abelian gauge fields in 3+1-dimensions, is encoded as the choice of gauge group in a local region of \(S\) and a field vev in the field theory that models the local geometry [11, 6, 7, 9]. Such filed theories constructed in local patches of \(S\) are glued together to reproduce all the information encoded in the geometry of \(X\) and \(B_3\). It is of crucial importance, therefore, to properly translate the geometry into the field vev of field theory local models.

The choice of the field background for F-theory compactification on a Calabi–Yau 3-fold (that is, the base manifold is a 2-fold, \(S\) is a curve and low-energy effective theory is on 5+1-dimensions) is discussed in [11]. The codimension-1 locus where 1-form field \(\varphi\) on \(S\) vanishes is called matter locus. Nothing else to discuss. Upon compactification to 3+1 dimensions, however, one more extra complication appears [9]. Local geometry of \(X\) around \(S\) is regarded as fibered space with the fiber being a surface with (partially) deformed ADE singularity, but the topological 2-cycles in the fiber have monodromies, and these monodromies introduce branch cuts and monodromies under the Weyl group in the field theory. Branch locus and matter locus are both codimension-1 loci in a surface \(S\), and hence their intersections become points. These points are codimension-3 in \(B_3\). It is inevitable to have these codimension-3 singularity points in compactification down to 3+1 dimensions. How to choose the vev of \(\varphi\) field around such codimension-3 singularity points actually still remains a bit of an issue.

The choice of \(\varphi\) vev around codimension-3 singularities was one of the main issues in [9]. Reference [45] further noted that the field theory local modes on \(S\) with branch cuts and the Weyl-group monodromies has equivalent descriptions on the covering space \(\tilde{S}\) of \(S\). The latter description does not involve branch cuts or twists, and the former description is obtained by taking a quotient of the latter. The \(\varphi\) field vev configuration in [9] and that of [45] are largely the same under this identification, but there still is a difference. This is why
the choice of $\varphi$ for a given geometry around codimension-3 singularity points still remains an issue. As we will note at the end of this appendix, this difference in $\langle \varphi \rangle$ leads directly to difference in phenomenology. Thus, this is not just an academic issue.

Let us first briefly review the relation between the field theory local models with branch cuts [9] and the description on the covering space [45]. We choose simplest case for illustrative purpose: a local geometry given by

$$y^2 = x^2 + (z^2 + 2uz + v + u^2)z^N,$$

(129)

where $(u, v)$ are local coordinates of $S$. This local geometry can be modelled by a field theory on a local region of $S$ with SU$(N + 2)$ gauge group [9], and it was claimed in [9] that the field vev $\langle \varphi \rangle = \langle \varphi_{uv} \rangle du \wedge dv$ should be chosen as

$$\langle \varphi_{uv} \rangle |_{2 \times 2} = \text{diag}(\tau_+, \tau_-); \quad \tau_{\pm} = -u \pm \sqrt{-v}.$$

(130)

$\tau_{\pm}$ above corresponds to the two roots of $(z^2 + 2uz + v + u^2) = 0$. The branch locus is $v = 0$. The spectral surface is

$$\xi^2 + 2u\xi + v + u^2 = 0.$$

(131)

Because the monodromy around the branch locus $v = 0$ is the Weyl reflection $Z_2 = \mathfrak{S}_2$ of the SU$(2) \subset$ SU$(N + 2)$ structure group, one can describe the the same field theory in the covering space $\tilde{S}$; the field theory with branch locus above can be regarded as a $Z_2$ orbifold of a theory on the covering space [45]. Let the local coordinates of $\tilde{S}$ be $(s, t)$, and the vev was chosen in [45] as

$$\langle \varphi \rangle |_{2 \times 2} = \text{diag}(s, t) \, ds \wedge dt.$$

(132)

This background has a $Z_2$ symmetry transformation acting on $\tilde{S}$ as exchange of the coordinates $s$ and $t$, accompanied by the Weyl-group transformation

$$\varphi_{st} = \begin{pmatrix} s \\ t \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. $$

(133)

Thus, one can think of a quotient of the free SU$(N + 2)$ gauge theory on $\tilde{S}$ under the $Z_2$ symmetry transformation. To see the relation between the description on the covering space $\tilde{S}$ and the quotient space $S$, it is convenient to take the following coordinates on $\tilde{S}$:

$$u = -\frac{1}{2}(s + t), \quad \bar{v} = \frac{1}{2}(s - t).$$

(134)

It is certainly a concern whether “twisted sectors” have to be introduced or not. Nothing is known about this issue up until now, however.
The $Z_2$ transformation flips the sign of $\tilde{v}$, while $u$ remains invariant, and hence a point $(u, \tilde{v})$ in $\tilde{S}$ is sent to $(u, v) = (u, -\tilde{v}^2)$ in $S$ under the $Z_2$ quotient map. The filed vev (132) becomes

$$\langle \phi \rangle_{2\times2} = \text{diag}(\tau_+, \tau_-) 2du \wedge d\tilde{v} = \text{diag}(\tau_+, \tau_-) du \wedge dv \left( \frac{-1}{\sqrt{-v}} \right).$$ (135)

This $\phi$ vev configuration is almost the same as the one in (130), but differs by a factor $v^{-1/2}$. For a given point $(u,v)$ in $S$, the difference is the overall scaling between $\mathfrak{h}/W \otimes \mathbb{C}$-valued $\text{diag}(\tau_+, \tau_-)$ and $\text{diag}(\tau_+, \tau_-)/\sqrt{-v}$, which cannot be determined by the dictionary in [44].

We have discussed in the main text how to obtain the 2-form field vev $\langle \phi \rangle$ from the 4-form $\Omega$ on the original Calabi–Yau 4-fold. Discussion leading to (34, 36) revealed that (130) is the right choice. The deformation parameters of ADE singularities are regarded as sections of $\mathfrak{h}/W \otimes K_S$, and hence the deformation parameters should be identified with the coefficients of the holomorphic top form made out of the local coordinates of $S$, not with those of the top form made of the local coordinates of the covering space $\tilde{S}$.

The choice of the background configuration of $\phi$ is relevant to phenomenology, because it determines the asymptotic behavior of the zero mode wavefunctions away from the matter curves. If the background were (132), we knew that the zero mode wavefunctions would be $e^{-|s|^2}$ and $e^{-|t|^2}$ in the covering space $\tilde{S}$, and they become a doublet wavefunction $(e^{-|u+\sqrt{-v}|^2}, e^{-|u-\sqrt{-v}|^2})$ on the $Z_2$ quotient $S$. In an asymptotic region where $|u|$ remains small, but $|v|$ becomes large, this zero mode wavefunction would decrease as $e^{-|v|}$, not as fast as in the Gaussian profile $e^{-|v|^2}$. It turns out, however, that (130) is the right choice for the $\phi$ field background, and the wavefunction falls as $e^{-|v|^{3/2}}$ in the asymptotic region, which is a little faster than the $e^{-|v|}$ fall off, but still slower than in the Gaussian profile. As one can see in the discussion around (103), how fast zero mode wavefunctions fall off in regions away from the matter curves is an important issue in phenomenology. Such a difference in the wavefunction profile in the asymptotic region also affects the phenomenological analysis of flavor pattern in [3, 31] as well.

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