LIE GROUP CLASSIFICATION A GENERALIZED COUPLED (2+1)-DIMENSIONAL HYPERBOLIC SYSTEM

Ben Muatjetjeja

Department of Mathematics, Faculty of Science, University of Botswana
Private Bag 22, Gaborone, Botswana
Department of Mathematical Sciences, Material Science Innovation and Modelling Focus Area
North-West University, Mafikeng Campus, Private Bag X 2046
Mmabatho, 2735, Republic of South Africa

Dimpho Millicent Mothibi

Department of Mathematical Sciences, Sol Plaatje University
Private Bag X5008, Kimberley 8300, Republic of South Africa

Chaudry Masood Khalique

International Institute for Symmetry Analysis and Mathematical Modelling
Department of Mathematical Sciences, North-West University
Mafikeng Campus, Private Bag X2046, Mmabatho 2735, Republic of South Africa
College of Mathematics and Systems Science Shandong University of Science and Technology
Qingdao, Shandong, 266590, China

Abstract. In this paper we perform Lie group classification of a generalized coupled (2+1)-dimensional hyperbolic system, viz., $u_{tt} - u_{xx} - u_{yy} + f(v) = 0$, $v_{tt} - v_{xx} - v_{yy} + g(u) = 0$, which models many physical phenomena in nonlinear sciences. We show that the Lie group classification of the system provides us with an eleven-dimensional equivalence Lie algebra, whereas the principal Lie algebra is six-dimensional and has several possible extensions. It is further shown that several cases arise in classifying the arbitrary functions $f$ and $g$, the forms of which include, amongst others, the power and exponential functions. Finally, for three cases we carry out symmetry reductions for the coupled system.

1. Introduction. The blow up problem for positive solutions of parabolic and hyperbolic problems with reaction terms of local and nonlocal type involving a variable exponent was studied in [9]. Parabolic problems appear in many branches of applied mathematics and can be used to model, for example, chemical reactions, heat transfer and population dynamics (see [9] and references therein). Escobedo and Herrero [1] extended the work of [9] and studied the system of equations

$$u_t - \Delta u = u^p,$$
$$v_t - \Delta v = v^q,$$

(1)

where $p, q$ are arbitrary constants and investigated the boundedness and blow-up of its solutions. The uniqueness and global existence of solutions of the system
(1) was studied in [2]. Recently, the authors of [4] considered nonlinear parabolic and hyperbolic systems with variable exponents and obtained results concerning the existence and blow-up property of solutions.

Inspired by the works done in [1, 2, 4], more recently the authors of [7] studied the coupled (2+1)-dimensional hyperbolic system

\[ \begin{align*}
  u_{tt} - u_{xx} - u_{yy} + \alpha v^q &= 0, \\
  v_{tt} - v_{xx} - v_{yy} + \beta u^p &= 0,
\end{align*} \]

(2)

where \( q, p, \alpha \) and \( \beta \) are non-zero constants. A complete Noether symmetry classification was carried out in [7] and it was shown that four main cases arose in the Noether classification with respect to the standard Lagrangian. The conservation laws were also constructed for the cases which admitted Noether point symmetries.

In this work we consider the generalization of the system (2), namely

\[ \begin{align*}
  u_{tt} - u_{xx} - u_{yy} + f(v) &= 0, \\
  v_{tt} - v_{xx} - v_{yy} + g(u) &= 0,
\end{align*} \]

(3)

where \( f(v) \) and \( g(u) \) are nonzero arbitrary functions of their respective arguments.

The aim of this work is to perform Lie group classification of the system (3).

2. Equivalence transformations. An equivalence transformation (see for example [5]) of (3) is an invertible transformation involving the independent variables \( t, x, y \) and the dependent variables \( u \) and \( v \) that map (3) into itself. The vector field

\[ Y = \xi^1(t, x, y, u, v) \frac{\partial}{\partial t} + \xi^2(t, x, y, u, v) \frac{\partial}{\partial x} + \xi^3(t, x, y, u, v) \frac{\partial}{\partial y} + \eta^1(t, x, y, u, v) \frac{\partial}{\partial u} + \eta^2(t, x, y, u, v) \frac{\partial}{\partial v} + \mu^1(t, x, y, u, v, f, g) \frac{\partial}{\partial f} + \mu^2(t, x, y, u, v, f, g) \frac{\partial}{\partial g}, \]

(4)

is the generator of the equivalence group for (3) provided it is admitted by the extended system [6, 8]

\[ \begin{align*}
  u_{tt} - u_{xx} - u_{yy} + f(v) &= 0, \quad v_{tt} - v_{xx} - v_{yy} + g(u) = 0, \\
  f_t = f_x = f_y = f_u = 0, \quad g_t = g_x = g_y = g_v = 0.
\end{align*} \]

(5)

(6)

The prolonged operator of (4) for the extended system (5)-(6) is given by

\[ \tilde{Y} = Y^{[2]} + \omega^1 \frac{\partial}{\partial f_t} + \omega^2 \frac{\partial}{\partial f_x} + \omega^3 \frac{\partial}{\partial f_y} + \omega^4 \frac{\partial}{\partial g_t} + \omega^5 \frac{\partial}{\partial g_x} + \omega^6 \frac{\partial}{\partial g_y} + \omega^7 \frac{\partial}{\partial g_v}, \]

(7)

\[ \omega^1 = \omega^2 = \omega^3 = \omega^4 = \omega^5 = \omega^6 = \omega^7 = \omega^8 = \cdots. \]

(8)

where \( Y^{[2]} \) is the second-prolongation of (4) given by

\[ \begin{align*}
  Y^{[2]} &= Y + \zeta^1 \frac{\partial}{\partial u_t} + \zeta^2 \frac{\partial}{\partial u_x} + \zeta^3 \frac{\partial}{\partial u_y} + \zeta^4 \frac{\partial}{\partial v_t} + \zeta^5 \frac{\partial}{\partial v_x} + \zeta^6 \frac{\partial}{\partial v_y} \\
  &+ \zeta^7 \frac{\partial}{\partial u_{tt}} + \zeta^8 \frac{\partial}{\partial u_{xx}} + \zeta^9 \frac{\partial}{\partial u_{yy}} + \zeta^{10} \frac{\partial}{\partial v_{tt}} + \zeta^{11} \frac{\partial}{\partial v_{xx}} + \zeta^{12} \frac{\partial}{\partial v_{yy}} + \cdots.
\end{align*} \]
Here the variables $\zeta$'s and $\omega$'s are defined by

\[
\begin{align*}
\zeta_t^1 &= D_t(\eta^1) - u_t D_v(\xi^1) - u_x D_x(\xi^2) - u_y D_v(\xi^3), \\
\zeta_x^1 &= D_x(\eta^1) - u_t D_x(\xi^1) - u_x D_x(\xi^2) - u_y D_x(\xi^3), \\
\zeta_y^1 &= D_y(\eta^1) - u_t D_y(\xi^1) - u_x D_y(\xi^2) - u_y D_y(\xi^3), \\
\zeta_t^2 &= D_t(\eta^2) - v_t D_t(\xi^1) - v_x D_x(\xi^2) - v_y D_t(\xi^3), \\
\zeta_x^2 &= D_x(\eta^2) - v_t D_x(\xi^1) - v_x D_x(\xi^2) - v_y D_x(\xi^3), \\
\zeta_y^2 &= D_y(\eta^2) - v_t D_y(\xi^1) - v_x D_y(\xi^2) - v_y D_y(\xi^3), \\
\zeta_{tt} &= D_t(\zeta_t^1) - u_{tt} D_v(\xi^1) - u_{tx} D_x(\xi^2) - u_{ty} D_t(\xi^3), \\
\zeta_{xx} &= D_x(\zeta_x^1) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2) - u_{xy} D_x(\xi^3), \\
\zeta_{yy} &= D_y(\zeta_y^1) - u_{ty} D_y(\xi^1) - u_{xy} D_y(\xi^2) - u_{yy} D_y(\xi^3), \\
\zeta_{tt} &= D_t(\zeta_t^2) - v_{tt} D_t(\xi^1) - v_{tx} D_x(\xi^2) - v_{ty} D_t(\xi^3), \\
\zeta_{xx} &= D_x(\zeta_x^2) - v_{tx} D_x(\xi^1) - v_{xx} D_x(\xi^2) - v_{xy} D_x(\xi^3), \\
\zeta_{yy} &= D_y(\zeta_y^2) - v_{ty} D_y(\xi^1) - v_{xy} D_y(\xi^2) - v_{yy} D_y(\xi^3),
\end{align*}
\]

and

\[
\begin{align*}
\omega_t^1 &= \bar{D}_t(\mu^1) - f_t \bar{D}_t(\xi^1) - f_x \bar{D}_x(\xi^2) - f_y \bar{D}_t(\xi^3) - f_u \bar{D}_u(\eta^1), \\
\omega_x^1 &= \bar{D}_x(\mu^1) - f_t \bar{D}_x(\xi^1) - f_x \bar{D}_x(\xi^2) - f_y \bar{D}_x(\xi^3) - f_u \bar{D}_u(\eta^1), \\
\omega_y^1 &= \bar{D}_y(\mu^1) - f_t \bar{D}_y(\xi^1) - f_x \bar{D}_y(\xi^2) - f_y \bar{D}_y(\xi^3) - f_u \bar{D}_u(\eta^1), \\
\omega_t^2 &= \bar{D}_t(\mu^2) - g_t \bar{D}_t(\xi^1) - g_x \bar{D}_x(\xi^2) - g_y \bar{D}_t(\xi^3) - g_u \bar{D}_u(\eta^1), \\
\omega_x^2 &= \bar{D}_x(\mu^2) - g_t \bar{D}_x(\xi^1) - g_x \bar{D}_x(\xi^2) - g_y \bar{D}_x(\xi^3) - g_u \bar{D}_u(\eta^1), \\
\omega_y^2 &= \bar{D}_y(\mu^2) - g_t \bar{D}_y(\xi^1) - g_x \bar{D}_y(\xi^2) - g_y \bar{D}_y(\xi^3) - g_u \bar{D}_u(\eta^1), \\
\omega_v^2 &= \bar{D}_v(\mu^2) - g_t \bar{D}_v(\xi^1) - g_x \bar{D}_v(\xi^2) - g_y \bar{D}_v(\xi^3) - g_u \bar{D}_v(\eta^1),
\end{align*}
\]

respectively, where

\[
\begin{align*}
D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} + \cdots, \\
D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v} + \cdots, \\
D_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial v} + \cdots,
\end{align*}
\]

are the usual total differentiation operators and

\[
\begin{align*}
\bar{D}_t &= \frac{\partial}{\partial t} + f_t \frac{\partial}{\partial f} + g_t \frac{\partial}{\partial g} + \cdots, \\
\bar{D}_x &= \frac{\partial}{\partial x} + f_x \frac{\partial}{\partial f} + g_x \frac{\partial}{\partial g} + \cdots, \\
\bar{D}_y &= \frac{\partial}{\partial y} + f_y \frac{\partial}{\partial f} + g_y \frac{\partial}{\partial g} + \cdots,
\end{align*}
\]
\[ D_u = \frac{\partial}{\partial u} + f_u \frac{\partial}{\partial f} + g_u \frac{\partial}{\partial g} + \cdots, \]
\[ D_v = \frac{\partial}{\partial v} + f_v \frac{\partial}{\partial f} + g_v \frac{\partial}{\partial g} + \cdots \]

are the new total differentiation operators for the extended system. The application of the operator (7) and the invariance conditions of system (5)-(6), after some lengthy calculations, leads to the following equivalence generators:

\[ Y_1 = \frac{\partial}{\partial t}, \ Y_2 = \frac{\partial}{\partial x}, \ Y_3 = \frac{\partial}{\partial y}, \ Y_4 = x \frac{\partial}{\partial t} + \frac{\partial}{\partial x}, \ Y_5 = y \frac{\partial}{\partial t} + \frac{\partial}{\partial y}, \]
\[ Y_6 = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}, \ Y_7 = u \frac{\partial}{\partial u} + f \frac{\partial}{\partial f}, \ Y_8 = v \frac{\partial}{\partial v} + g \frac{\partial}{\partial g}, \]
\[ Y_9 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - 2f \frac{\partial}{\partial f} - 2g \frac{\partial}{\partial g}, \ Y_{10} = \frac{\partial}{\partial u}, \ Y_{11} = \frac{\partial}{\partial v}, \]

which yields the eleven-parameter equivalence group given by

\[ Y_1 : \ \tilde{t} = a_1 + t, \ \tilde{x} = x, \ \tilde{y} = y, \ \tilde{u} = u, \ \tilde{v} = v, \ \tilde{f} = f, \ \tilde{g} = g, \]
\[ Y_2 : \ \tilde{t} = t, \ \tilde{x} = a_2 + x, \ \tilde{y} = y, \ \tilde{u} = u, \ \tilde{v} = v, \ \tilde{f} = f, \ \tilde{g} = g, \]
\[ Y_3 : \ \tilde{t} = t, \ \tilde{x} = x, \ \tilde{y} = a_3 + y, \ \tilde{u} = u, \ \tilde{v} = v, \ \tilde{f} = f, \ \tilde{g} = g, \]
\[ Y_4 : \ \tilde{t} = a_4 x + t, \ \tilde{x} = a_4 t + x, \ \tilde{y} = y, \ \tilde{u} = u, \ \tilde{v} = v, \ \tilde{f} = f, \ \tilde{g} = g, \]
\[ Y_5 : \ \tilde{t} = a_5 y + t, \ \tilde{x} = x, \ \tilde{y} = a_5 t + y, \ \tilde{u} = u, \ \tilde{v} = v, \ \tilde{f} = f, \ \tilde{g} = g, \]
\[ Y_6 : \ \tilde{t} = t, \ \tilde{x} = x - a_6 y, \ \tilde{y} = a_6 x + y, \ \tilde{u} = u, \ \tilde{v} = v, \ \tilde{f} = f, \ \tilde{g} = g, \]
\[ Y_7 : \ \tilde{t} = t, \ \tilde{x} = x, \ \tilde{y} = y, \ \tilde{u} = u e^{a_7}, \ \tilde{v} = v, \ \tilde{f} = f e^{a_7}, \ \tilde{g} = g, \]
\[ Y_8 : \ \tilde{t} = t, \ \tilde{x} = x, \ \tilde{y} = y, \ \tilde{u} = u, \ \tilde{v} = v e^{a_8}, \ \tilde{f} = f, \ \tilde{g} = g e^{a_8}, \]
\[ Y_9 : \ \tilde{t} = t e^{a_9}, \ \tilde{x} = x e^{a_9}, \ \tilde{y} = y e^{a_9}, \ \tilde{u} = u, \ \tilde{v} = v, \ \tilde{f} = f e^{-2a_9}, \ \tilde{g} = g e^{-2a_9}, \]
\[ Y_{10} : \ \tilde{t} = t, \ \tilde{x} = x, \ \tilde{y} = y, \ \tilde{u} = u + a_{11}, \ \tilde{v} = v, \ \tilde{f} = f, \ \tilde{g} = g, \]
\[ Y_{11} : \ \tilde{t} = t, \ \tilde{x} = x, \ \tilde{y} = y, \ \tilde{u} = u, \ \tilde{v} = v + a_{11}, \ \tilde{f} = f, \ \tilde{g} = g. \]

The composition of the above transformations gives

\[ \tilde{t} = a_1 + a_4 x + a_5 y + t e^{a_9}, \]
\[ \tilde{x} = a_2 + a_4 t - a_6 y + x e^{a_9}, \]
\[ \tilde{y} = a_3 + a_5 t + a_6 x + y e^{a_9}, \]
\[ \tilde{u} = e^{a_7} (u + a_{10}), \]
\[ \tilde{v} = e^{a_8} (v + a_{11}), \]
\[ \tilde{f} = e^{a_9 - 2a_9} f, \]
\[ \tilde{g} = e^{a_9 - 2a_9} g, \]

which are the equivalence transformations.

3. **Principal Lie algebra.** According to Lie’s theory the system of partial differential equations (PDEs) (3) is invariant under the group with generator

\[ \Gamma = \xi^1 (t, x, y, u, v) \frac{\partial}{\partial t} + \xi^2 (t, x, y, u, v) \frac{\partial}{\partial x} + \xi^3 (t, x, y, u, v) \frac{\partial}{\partial y} \]
\[ + \eta^1 (t, x, y, u, v) \frac{\partial}{\partial u} + \eta^2 (t, x, y, u, v) \frac{\partial}{\partial v} \]

(10)
if and only if
\[ \Gamma^{[2]} \left( u_{tt} - u_{xx} - u_{yy} + f(v) \right) \bigr|_{(3)} = 0, \quad \Gamma^{[2]} \left( v_{tt} - v_{xx} - v_{yy} + g(u) \right) \bigr|_{(3)} = 0, \]

(11)

where \( \Gamma^{[2]} \) denotes the second prolongation of the generator (10) and the symbol \( \bigr|_{(3)} \) means that it is evaluated on system (3). As the \( \xi \)'s and \( \eta \)'s do not depend on any derivatives of \( u \) and \( v \), the determining equations (11) split with respect to the derivatives of \( u \) and \( v \), yielding the following overdetermined system of thirty-one linear PDEs:

\[
\begin{align*}
\xi_1^1 &= 0, \xi_1^2 = 0, \xi_2^2 = 0, \xi_3^3 = 0, \xi_3^3 = 0, \eta_1^1 = 0, \eta_1^1 = 0, \\
\eta_1^2 &= 0, \eta_1^3 = 0, \eta_2^1 = 0, \eta_2^1 = 0, \\
\xi_2^1 &= 0, \xi_2^2 = 0, \xi_3^3 = 0, \xi_3^3 = 0, \eta_1^2 = 0, \eta_1^2 = 0, \\
\eta_2^2 &= 0, \eta_2^2 = 0, \eta_2^2 = 0, \eta_2^2 = 0, \\
\xi_3^1 &= 0, \xi_3^1 = 0, \xi_3^1 = 0, \xi_3^1 = 0, \eta_1^3 = 0, \eta_1^3 = 0, \\
\eta_3^1 &= 0, \eta_3^1 = 0, \eta_3^1 = 0, \eta_3^1 = 0, \\
\xi_3^2 &= 0, \xi_3^2 = 0, \xi_3^2 = 0, \xi_3^2 = 0, \eta_1^3 = 0, \eta_1^3 = 0, \\
\xi_3^3 &= 0, \xi_3^3 = 0, \xi_3^3 = 0, \xi_3^3 = 0, \eta_1^3 = 0, \eta_1^3 = 0, \\
\eta_3^2 &= 0, \eta_3^2 = 0, \eta_3^2 = 0, \eta_3^2 = 0, \\
\xi_3^1 &= 0, \xi_3^1 = 0, \xi_3^1 = 0, \xi_3^1 = 0, \eta_1^3 = 0, \eta_1^3 = 0, \\
\eta_3^3 &= 0, \eta_3^3 = 0, \eta_3^3 = 0, \eta_3^3 = 0, \\
\xi_3^2 &= 0, \xi_3^2 = 0, \xi_3^2 = 0, \xi_3^2 = 0, \eta_1^3 = 0, \eta_1^3 = 0, \\
\xi_3^3 &= 0, \xi_3^3 = 0, \xi_3^3 = 0, \xi_3^3 = 0, \eta_1^3 = 0, \eta_1^3 = 0, \\
\eta_3^3 &= 0, \eta_3^3 = 0, \eta_3^3 = 0, \eta_3^3 = 0.
\end{align*}
\]

(12)

Solving the above system for arbitrary \( f \) and \( g \), we find that the system (3) admits the six-dimensional Lie algebra spanned by

\[
\begin{align*}
\Gamma_1 &= \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}, \quad \Gamma_3 = \frac{\partial}{\partial y}, \quad \Gamma_4 = x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x}, \\
\Gamma_5 &= y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y}, \quad \Gamma_6 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y},
\end{align*}
\]

which is the principal Lie algebra of the system (3).

4. Lie group classification. Solving the system (12), we obtain the following classifying relations:

\[
(\alpha v + \beta) f'(v) + \gamma f(v) + \delta = 0, \\
(\theta u + \lambda) g'(u) + \varphi g(u) + \omega = 0,
\]

where \( \alpha, \beta, \gamma, \delta, \theta, \lambda, \varphi \) and \( \omega \) are constants. Using the equivalence transformations obtained in Section 2, this classifying relation is invariant under the equivalence transformations (9) if

\[
\begin{align*}
\bar{\alpha} &= \alpha, \quad \bar{\gamma} = \gamma, \quad \bar{\beta} = e^{\alpha x}(\beta - a_{11}^\alpha), \quad \bar{\delta} = \delta e^{\alpha t - 2a_9}, \quad \bar{\theta} = \theta, \quad \bar{\varphi} = \varphi, \\
\bar{\lambda} &= e^{\alpha t}(\lambda - a_{10} \theta), \quad \bar{\omega} = \omega e^{\alpha t - 2a_9}.
\end{align*}
\]

These classifying relations lead to twelve cases for the functions \( f \) and \( g \) and for each case we also provide the associated extended Lie point symmetries. 

Case 1. \( f(v) \) and \( g(u) \) arbitrary but not of the form in Cases 2-12 given below.
In this case, we obtain the principal Lie algebra

\[ \Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}, \quad \Gamma_3 = \frac{\partial}{\partial y}, \quad \Gamma_4 = x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x}, \]

\[ \Gamma_5 = y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y}, \quad \Gamma_6 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}. \]

**Case 2.** \( f(v) = nv + \sigma \) and \( g(u) = mu + \theta \), where \( n, \sigma, m \) and \( \theta \) are constants

This case extends the principal Lie algebra by four Lie point symmetries, namely

\[ \Gamma_7 = \frac{\partial}{\partial u}, \]

\[ \Gamma_8 = nv \frac{\partial}{\partial u} + (mu + \theta) \frac{\partial}{\partial v}, \]

\[ \Gamma_9 = nu \frac{\partial}{\partial u} + (nv + \sigma) \frac{\partial}{\partial v}, \]

\[ \Gamma_{10} = nH \frac{\partial}{\partial u} + (H_{yy} + H_{xx} - H_{tt}) \frac{\partial}{\partial v}, \]

where \( H(t, x, y) \) is any solution of the PDE

\[ 2H_{ttyy} + 2H_{ttxx} - 2H_{xxxy} - H_{yyyy} - H_{tt} - H_{xx} + mnH - mC_1 - m\sigma C_4 - n\theta C_8 = 0 \]

and \( C_1, C_4, C_8 \) are arbitrary constants.

**Case 3.** \( f(v) = \alpha v^n \) and \( g(u) = \theta u^m \), where \( \alpha, n, \theta \) and \( m \) are constants

We have four subcases.

**Case 3.1.** \( n \neq \pm 1, m \neq \pm 1, mn \neq 1 \)

The principal Lie algebra is extended by one Lie point symmetry

\[ \Gamma_{11} = (mn - 1) \left( t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) - 2(n + 1)u \frac{\partial}{\partial u} - 2(m + 1)v \frac{\partial}{\partial v}. \]

**Case 3.2.** \( n = m = -1 \)

This subcase extends the principal Lie algebra by two Lie point symmetries, viz.,

\[ \Gamma_{12} = u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v}, \]

\[ \Gamma_{13} = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2v \frac{\partial}{\partial v}. \]

**Case 3.3.** \( mn = 1 \), where \( m \) and \( n \) are non-zero constants

Here the principal Lie algebra extends by one Lie point symmetry

\[ \Gamma_{14} = u \frac{\partial}{\partial u} + mv \frac{\partial}{\partial v}. \]

**Case 3.4.** \( n = 5 \) and \( m = 5 \)

In this subcase the principal Lie algebra extends by the following four Lie point symmetries:

\[ \Gamma_{15} = 2t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} - u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v}, \]

\[ \Gamma_{16} = 2y \frac{\partial}{\partial t} + 2xy \frac{\partial}{\partial x} + (t^2 - x^2 + y^2) \frac{\partial}{\partial y} - wy \frac{\partial}{\partial u} - vy \frac{\partial}{\partial v}, \]
\[ \Gamma_{17} = 2xt \frac{\partial}{\partial t} + (t^2 + x^2 - y^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y} - ux \frac{\partial}{\partial u} - vx \frac{\partial}{\partial v}, \]
\[ \Gamma_{18} = (t^2 + x^2 + y^2) \frac{\partial}{\partial t} + 2tx \frac{\partial}{\partial x} + 2ty \frac{\partial}{\partial y} - ut \frac{\partial}{\partial u} - vt \frac{\partial}{\partial v}. \]

**Case 4.** \( n = -1 \) and \( g(u) \) is arbitrary
This case extends the principal Lie algebra by one Lie point symmetry
\[ \Gamma_{19} = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2v \frac{\partial}{\partial v}. \]

**Case 5.** \( f(v) \) is arbitrary and \( m = -1 \)
Here the principal Lie algebra extends by one Lie point symmetry
\[ \Gamma_{20} = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2u \frac{\partial}{\partial u}. \]

**Case 6.** \( f(v) = \alpha e^{nv} \) and \( g(u) = \theta e^{mu} \), where \( \alpha, n, \theta \) and \( m \) are constants
This case extends the principal Lie algebra by one Lie point symmetry
\[ \Gamma_{21} = mn \left( t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) - 2(n + 1) \frac{\partial}{\partial u} - 2m \frac{\partial}{\partial v}. \]

**Case 7.** \( f(v) = \alpha v^n \) and \( g(u) = \theta e^{mu} \), where \( \alpha, n, \theta \) and \( m \) are constants
This case extends the principal Lie algebra by one Lie point symmetry
\[ \Gamma_{22} = mn \left( t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) - 2(nu) \frac{\partial}{\partial u} - 2(m + 1) \frac{\partial}{\partial v}. \]

**Case 8.** \( f(v) = \alpha e^{nv} \) and \( g(u) = \theta u^m \), where \( \alpha, n, \theta \) and \( m \) are constants
This case extends the principal Lie algebra by one Lie point symmetry
\[ \Gamma_{23} = mn \left( t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) - 4nu \frac{\partial}{\partial u} - 2(mn v + m \sigma + n v + \sigma) \frac{\partial}{\partial v}. \]

**Case 9.** \( f(v) = nv + \sigma \) and \( g(u) = \theta u^m \), where \( n, \sigma, \theta \) and \( m \) are constants with \( m \neq n \neq 1 \)
In this case the principal Lie algebra extends by one Lie point symmetry
\[ \Gamma_{24} = n(m - 1) \left( t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) - 4nu \frac{\partial}{\partial u} - 2(mn v + m \sigma + n v + \sigma) \frac{\partial}{\partial v}. \]

**Case 10.** \( f(v) = \alpha v^n \) and \( g(u) = mu + \theta \), where \( \alpha, n, m \) and \( \theta \) are constants with \( m \neq n \neq 1 \)
This case extends the principal Lie algebra by one Lie point symmetry
\[ \Gamma_{25} = m(n - 1) \left( t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) - 2(mnu + n \sigma + mu + \sigma) \frac{\partial}{\partial u} - 4mv \frac{\partial}{\partial v}. \]

**Case 11.** \( f(v) = nv + \sigma \) and \( g(u) = \theta e^{mu} \), where \( n, \sigma, \theta \) and \( m \) are constants
This case extends the principal Lie algebra by one Lie point symmetry
\[ \Gamma_{26} = mn \left( t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) - 4n \frac{\partial}{\partial u} - 2m(nv + \sigma) \frac{\partial}{\partial v}. \]

**Case 12.** \( f(v) = \alpha e^{nv} \) and \( g(u) = mu + \theta \), where \( \alpha, n, m \) and \( \theta \) are constants
This case extends the principal Lie algebra by one Lie point symmetry
\[ \Gamma_{27} = mn \left( t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) - 2n(mu + \theta) \frac{\partial}{\partial u} - 4m \frac{\partial}{\partial v}. \]

5. **Symmetry reductions of system (3).** In this section, we present a few symmetry reductions of system (3) using some of the symmetries derived in Section 3 [3]. We first consider Case 1 and use the rotational symmetry \( \Gamma_6 \) to perform symmetry reduction of system (3). The associated Lagrange system of \( \Gamma_6 \) is
\[
\frac{dt}{0} = \frac{dx}{y} = \frac{dy}{-x} = \frac{du}{0} = \frac{dv}{0},
\]
yields the invariants \( z = t, r = x^2 + y^2 \) and the group-invariant solution of system (3) is \( u(t, x, y) = F(z, r), v(t, x, y) = G(z, r) \), where \( F(z, r) \) and \( G(z, r) \) satisfy
\[
\begin{align*}
F_{zz} - 4rF_{rr} - 4F_r + f(G) &= 0, \\
G_{zz} - 4rG_{rr} - 4G_r + g(F) &= 0. \\
\end{align*}
\]
(13)

As a second example of symmetry reduction of system (3) we consider Case 3.1 with \( f(v) = \alpha v^n \) and \( g(u) = \theta u^m \) and use the rotational symmetry \( \Gamma_5 \). In this case the associated Lagrange system of \( \Gamma_5 \) is
\[
\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{0} = \frac{du}{0} = \frac{dv}{0},
\]
which gives the four invariants \( z = t, r = t^2 - y^2, u = F(z, r), v = G(z, r) \). Following the same procedure as above, system (3) reduces to
\[
\begin{align*}
F_{zz} - 4rF_{rr} - 4F_r - \alpha G^m &= 0, \\
G_{zz} - 4rG_{rr} - 4G_r - \theta F^m &= 0. \\
\end{align*}
\]
(14)

System (14) admits two symmetries, namely
\[
R_1 = (1 - mn)z \frac{\partial}{\partial z} + 2(1 - mn)r \frac{\partial}{\partial r} + 2(1 + m)G \frac{\partial}{\partial G} + 2(1 + n)F \frac{\partial}{\partial F}, \quad R_2 = \frac{\partial}{\partial z}. 
\]

Using the scaling symmetry \( R_1 \), we obtain the three invariants
\[
s = \frac{z}{\sqrt{r}}, \quad \phi(s) = F^{\frac{1+m}{nm-1}}, \quad \psi(s) = G^{\frac{1+m}{nm-1}}. 
\]

By making use of these invariants, system (14) becomes
\[
\begin{align*}
(mn - 1)^2 s^{2\lambda_1} - s^{2\gamma_1} \phi'' + (mn - 1)(mn + 4n + 3)s^{2\beta_1} \phi' + (4(n + 1)^2)s^{2\gamma_1} \phi \\
+ \alpha(mn - 1)^2 s^{2\gamma_1} \psi'' &= 0, \\
(mn - 1)^2 s^{2\lambda_2} - s^{2\gamma_2} \psi'' + (mn - 1)(mn + 4n + 3)s^{2\beta_2} \psi' + (4(n + 1)^2)s^{2\gamma_2} \psi \\
+ \alpha(mn - 1)^2 s^{2\gamma_2} \phi'' &= 0. \\
\end{align*}
\]
(15)

Thus the group-invariant solution of system (3) is \( u = F(z, r), v = G(z, r) \), where \( (F, G) \) and \( (\phi, \psi) \) are any solution of system (14) and (15) respectively with
\[
\begin{align*}
\lambda_1 &= \frac{2(n - 1 + 2mn)}{mn - 1}, \quad \gamma_1 = \frac{2(m + 1)n}{mn - 1}, \quad \beta_1 = \frac{2n - 1 + 3mn}{mn - 1}, \\
\lambda_2 &= \frac{2(m - 1 + 2mn)}{mn - 1}, \quad \gamma_2 = \frac{2(n + 1)m}{mn - 1}, \quad \beta_2 = \frac{2m - 1 + 3mn}{mn - 1}. \\
\end{align*}
\]

Another general group-invariant solution of system (3), will be derived from Case 7 with the generator \( \Gamma_4 \). After some straightforward but lengthy computations, we
derive the invariants $z = t, r = t^2 - x^2, u = G(z, r), v = H(z, r)$. Invoking these invariants, system (3) reduces to

$$
G_{zz} - 4rG_{rr} - 4G_r - \alpha H^n = 0,
H_{zz} - 4rH_{rr} - 4H_r - \theta e^{mg} = 0,
$$

(16)

where $G, H$ satisfy system (16). System (16) possess two generators, namely

$$
Q_1 = mnz \frac{\partial}{\partial z} + 2mnr \frac{\partial}{\partial r} - 2(1 + n)G \frac{\partial}{\partial G} - 2nH \frac{\partial}{\partial H},
Q_2 = \frac{\partial}{\partial z}.
$$

Employing symmetry $Q_1$ and follow the same procedure as above, we obtain three invariants, namely

$$
k = \frac{z}{\sqrt{r}}, \quad \Phi(k) = \frac{mnG + \ln(r) + n\ln(r)}{nm}, \quad \Upsilon(k) = Hr^{\frac{1}{n}},
$$

where $m$ and $n$ are non-zero constants. Using these invariants, system (16) transforms to

$$
(k^2 - 1)\Phi'' + k\Phi' + \alpha \Upsilon^n = 0,
n^2(k^3 - k^2)\Upsilon'' + n(n + 4)k^4\Upsilon' + (4\Upsilon + \theta n^2 e^{mg})k^3 = 0.
$$

(17)

Consequently, $u = G, v = H$ is a general group-invariant solution of system (3) where $(G, H)$ and $(\Phi, \Upsilon)$ satisfy system (16) and (17) respectively with

$$
\lambda_3 = \frac{2(n + 1)}{n}, \lambda_4 = \frac{3n + 2}{n}, \lambda_5 = \frac{2(2n + 1)}{n}.
$$

It is worth mentioning that by following the aforementioned procedure, one can derive more group-invariant solutions of the generalized coupled hyperbolic system (3).

6. Conclusion. In this paper, we have used the Lie group analysis to perform Lie group classification of the generalized coupled (2+1)-dimensional hyperbolic system (3). We showed that the system admitted eleven-dimensional equivalence Lie algebra. The functional forms of the arbitrary parameters were specified via the classical method of group classification and these included the power, exponential and linear functions. The six-dimensional principal Lie algebra was also obtained and several possible extensions of it were presented. Thereafter symmetry reductions of system (3) were performed for some cases.

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E-mail address: Ben.Muatjetjeja@mopipi.ub.bw
E-mail address: Dimpho.Mothibi@spu.ac.za
E-mail address: Masood.Khalique@nwu.ac.za