Markovian multiserver queues with staggered setup for data centers

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Abstract—Cloud computing is a new paradigm where a company makes money by selling computer resources including both software and hardware. The core part of cloud computing is data center where a huge number of servers are available. These servers consume a large amount of energy to run and to keep cool. Therefore, a reduction of a few percent of the power consumption means saving a large amount of money and the environment. In the current technology, an idle server still consumes about 60% of its peak. Thus, the only way to save energy is to turn off servers which are not processing a job. However, when there are some waiting jobs, we have to turn on the OFF servers. A server needs some setup time to be active during which it consumes energy but cannot process a job. Therefore, there exists a trade-off between power consumption and delay performance. In [7], [8], the authors analyze this tradeoff using an M/M/c queue with setup time for which they present a decomposition property by solving difference equations. In this paper, using an alternative simple approach, we obtain explicit expressions for partial generating functions, factorial moments and the joint stationary distribution of the number of active servers and that in the system. One of our main results is a new conditional decomposition formula showing that the number of waiting customers under the condition that all servers are busy has the same distribution with the sum of two independent random variables having a clear physical meaning.

I. INTRODUCTION

Cloud computing is a new paradigm where companies make money by providing computing service through the Internet. In cloud computing, users buy software and hardware resources from a provider and access to these resources through the Internet so they do not have to install and maintain by themselves. The core part of cloud computing is data center where there are a huge number of servers. The key issue for the management of data centers is to minimize the power consumption while keeping an acceptable service level for customers. It is reported that under the current technology an idle server still consumes about 60% of its peak processing jobs [3]. Thus, the only way to save power is to turn off idle servers. However, if the workload increases, OFF servers should be turned on to serve waiting customers. Servers need some setup time during which they consume energy but cannot process jobs. Therefore, customers have to wait a longer time in comparison with the case where the servers are always ON.

Although queues with setup time have been extensively investigated in the literature, most of papers deal with single server models [20], [22], [4], [5] where the service time follows a general distribution. Artalejo et al. [1] present a thorough analysis for multiserver queues with setup time where the authors consider the case in which at most one server can be in setup mode at a time. This policy is referred to as staggered setup in [8]. It is pointed out in [1] that the model belongs to a QBD class for which the rate matrix is explicitly obtainable. By solving difference equations, Artalejo et al. [1] derive an analytical solution where the stationary distribution is recursively obtained without any approximation. Recently, motivated by applications in data centers, multiserver queues with setup time have been extensively investigated in the literature. In particular, Gandhi et al. [8], [9], [10], [11], [12] analyze multiserver queues with setup time. They consider the M/M/c system with staggered setup and derive some closed form approximations for the ON/OFF policy where the number of servers the setup mode at a time is not limited. Gandhi et al. [10] extend their analysis to the case where a free server waits for a while before being shutdown.

In [7], [8], the authors show that the waiting time in the model with setup is decomposed to the sum of the setup time and the waiting time in the corresponding M/M/c system without setup time. This is equivalent to that the number of waiting customers can be decomposed to the sum of the number of waiting customers in the conventional M/M/c system and that arrives during the remaining setup time. The detailed analysis is provided in a technical report available online [6]. The authors first derive the joint stationary distribution using the difference equation technique which is also adopted by Artalejo et al. [1]. The solution in [8] is much simpler than that obtained by Artalejo et al. [1] using the same difference equation methodology.

In this paper, using a generating function approach, we derive a clear solution for all the partial generating functions for the joint stationary distribution of the number of active servers and that of customers in the system. The generating functions are explicitly obtained using recursive formulæ. Furthermore, we show that the partial generating functions are of rational forms allowing us to decompose them into the simplest forms. From the simplified form of the partial generating functions, we prove that the joint stationary distribution is a linear combination of geometric distributions. Our results conform to those derived in Artalejo et al. [1]. Furthermore, using the partial generating functions and the distributional Little’s law, we show that the waiting time distribution is a linear combination of exponential distributions. Furthermore, we derive recursive formulæ for partial factorial moments.
Although our expressions look different to those in Gandhi et al. [8], our numerical results confirm the decomposition property in Gandhi et al. [8]. From these points of view, our paper is a complement for the analyses of Artalejo et al. [11] and Gandhi et al. [8]. Another contribution is a conditional decomposition formula where the number of waiting customers under the condition that all \( c \) servers are busy is equal in distribution to the sum of two independent random variables with a clear physical meaning.

Some more related work are as follows. Mitrani [18], [17] considers a model for server farms with setup cost. The author considers the case where group of reserve servers are shutdown concurrently if the workload is lower than some lower threshold and is powered up simultaneously when the workload exceeds some upper threshold. Because of this simultaneous setup, the underlying Markov chain in [17] has a simple birth-and-death structure allowing closed form solutions. The author investigates the optimal lower and upper thresholds for the system. The same author [16] extends their analysis to the case where each customer has an exponentially distributed random timer exceeding which the customer abandons the system. Mazzucco and Mitrani [14] show that the theoretical results in [16], [17] fit that of real experiments. Schwartz et al. [19] consider a similar model to that in [16]. The main purpose of these papers is to find the optimal thresholds using which the group of reserve servers is turned ON or OFF. Mazzucco et al. [14] use the Erlang-C and Erlang-A formulae to approximate server farms with setup costs. For discussions on power consumption issue in data centers and cloud computing, we refer to [19], [3], [13], [14], [15], [2].

The rest of our paper is organized as follows. First we present the model in Section II. Section III is devoted to the detailed analysis where we derive explicit expressions for the partial generating function and the joint stationary distribution. In Section IV, we discuss the decomposition property for the queue length. Section V is devoted to our new result where we present a conditional decomposition formula. In Section VI, we provide extensive numerical results to validate the decomposability in Section IV and to show the performance of the system.

II. MODEL

We consider M/M/c queueing systems with staggered setup time. Customers arrive at the system according to a Poisson process with rate \( \lambda \). In this system, an idle server is shutdown immediately. If there are some waiting customers, OFF servers are turned on one by one. This is to prevent an unexpected increase in energy consumption. This model also fits for a manufacturing system context where all the servers are monitored by one administrator. In this case, the administrator turns on the OFF servers one by one. Furthermore, a server needs some setup time to be active so as to serve a waiting customer. We assume that the setup time follows the exponential distribution with mean \( 1/\alpha \). If a server finishes a job, this server picks a waiting customer if any. If there is not a waiting customer, the server in setup process and idle ones are turned off immediately. Let \( j \) denotes the number of customers in the system and \( i \) denotes the number of active servers. The number of servers in setup process is \( \min(j-i,1) \). Under these assumptions, the number of active servers is smaller than or equal to the number of customers in the system. It should be noted that in this model a server is in either BUSY or OFF or SETUP. We assume that every customer that enters the system receives service and departs. This means that there is no abandonment. We assume that the service time of jobs follows an exponential distribution with mean \( 1/\mu \).

III. ANALYSIS OF THE MODEL

A. Generating functions

Let \( C(t) \) and \( N(t) \) denote the number of busy servers and the number of jobs in the system at time \( t \), respectively. Under the assumptions made in Section II, it is easy to see that \( \{X(t) = (C(t), N(t)); t \geq 0\} \) forms a Markov chain in the state space

\[
\mathcal{S} = \{(i,j); j \in \mathbb{Z}_+, i = 0, 1, \ldots, \min(c, j)\},
\]

where \( \mathbb{Z}_+ = \{0, 1, \ldots\} \). See Fig. 1 for the transitions among states.

In this paper, we assume that \( \rho = \lambda/(c\mu) < 1 \) which is the necessary and sufficient condition for the stability of the Markov chain. In what follows, we assume that the Markov chain is ergodic. Under this ergodic condition, let

\[
\pi_{i,j} = \lim_{t \to \infty} \Pr(N(t) = i, C(t) = j), \quad (i,j) \in \mathcal{S},
\]

denote the stationary probability of state \((i, j)\).

The balance equations for states \((0, j) \quad (j \in \mathbb{N} = \{1, 2, \ldots\})\) read as follows.

\[
\lambda \pi_{0,j-1} = (\lambda + \alpha) \pi_{0,j}, \quad j \in \mathbb{N}.
\]

Let \( \Pi_0(z) = \sum_{j=0}^{\infty} \pi_{0,j} z^j \). Multiplying the above equation by \( z^j \) and adding over \( j \in \mathbb{N} \) yields,

\[
\lambda z \Pi_0(z) = (\lambda + \alpha) (\Pi_0(z) - \pi_{0,0}),
\]

or equivalently

\[
\Pi_0(z) = \frac{(\lambda + \alpha) \pi_{0,0}}{\lambda + \alpha - \lambda z}.
\]
The balance equation for state \((0, 0)\) is given by
\[
\lambda \pi_{0,0} = \mu \pi_{1,1}.
\]

This equation is also derived from the global balance between flows in and out the states \(\{(0, j); j \in \mathbb{Z}_+\}\). Indeed, we have
\[
\alpha(\Pi_0(1) - \pi_{0,0}) = \mu \pi_{1,1},
\]
leading to
\[
\pi_{1,1} = \frac{\alpha(\Pi_0(1) - \pi_{0,0})}{\mu} = \frac{\lambda}{\mu} \pi_{0,0}.
\]

Now, we shift to the case where there is one active server, i.e., \(i = 1\). We have
\[
(\lambda + \mu)\pi_{1,1} = \alpha \pi_{0,1} + \mu \pi_{1,2} + 2\mu \pi_{2,2}, \quad j = 1,
\]
\[
(\lambda + \mu + \alpha)\pi_{1,j} = \lambda \pi_{1,j-1} + \alpha \pi_{0,j} + \mu \pi_{1,j+1}, \quad j \geq 2.
\]

We define the generating for the states with \(i = 1\) as follows.
\[
\Pi_1(z) = \sum_{j=0}^{\infty} \pi_{1,j+1} z^j.
\]

\(\Pi_1(z)\) represents the generating function of the number of waiting customers while there is one active server.

Multiplying (2) by \(z^0\) and (3) by \(z^{j-1}\) and taking the sum over \(j \in \mathbb{N}\) yields,
\[
(\lambda + \mu + \alpha)\Pi_1(z) - \alpha \pi_{1,1} = \lambda z \Pi_1(z) + \frac{\alpha}{z} \left(\Pi_0(z) - \pi_{0,0}\right) + 2\mu \pi_{2,2},
\]
where
\[
f(z) = \frac{\alpha}{z} \left(\Pi_0(z) - \pi_{0,0}\right) + \frac{\mu}{z} (\Pi_1(z) - \pi_{1,1}) + 2\mu \pi_{2,2},
\]

Arranging (4) we obtain,
\[
f_1(z)\Pi_1(z) = \alpha \Pi_0(z) + \alpha z \pi_{1,1} - \alpha \pi_{0,0} - \mu \pi_{1,1} + 2\mu z \pi_{2,2},
\]

for \(j \geq i + 1\). We define the partial generating function for the case of having \(i\) active servers as follows,
\[
\Pi_i(z) = \sum_{j=i}^{\infty} \pi_{i,j} z^{j-i}, \quad i = 2, 3, \ldots, c - 1.
\]

Multiplying (6) by \(z^0\) and (7) by \(z^{j-i}\) and adding over \(j = i, i+1, \ldots\), we obtain
\[
(\lambda + i\mu + \alpha)\Pi_i(z) - \alpha \pi_{i,i} = \lambda z \Pi_i(z) + \frac{\alpha}{z} \left(\Pi(z) - \pi_{i,i}\right) + \frac{\alpha}{z} (\Pi_{i-1}(z) - \pi_{i-1,i-1}) + (i + 1)\mu \pi_{i+1,i+1},
\]
or equivalently
\[
f_i(z)\Pi_i(z) - \alpha \pi_{i,i} = \left(i + 1\right)\mu z \pi_{i+1,i+1} - i\mu \pi_{i,i} + \alpha (\Pi_{i-1}(z) - \pi_{i-1,i-1}),
\]
where \(f_i(z) = (\lambda + \alpha)z - \lambda z^2 - i\mu\) since \(f_i(0) = -i\mu < 0\) and \(f_i(1) = \alpha > 0, 0 < \exists z < 1\) such that \(f_i(z) = 0\). We have
\[
\pi_{i+1,i+1} = \frac{(i\mu - \alpha \pi_{i,i})\pi_{i,i} + \alpha (\Pi_{i-1,i-1} - \Pi_{i-1}(z))}{(i + 1)\mu z_i},
\]

Putting \(z = z_i\) into (8), we obtain
\[
\pi_{i+1,i+1} = \frac{(i\mu - \alpha \pi_{i,i})\pi_{i,i} + \alpha (\Pi_{i-1,i-1} - \Pi_{i-1}(z))}{(i + 1)\mu z_i},
\]

Remark 2: At this point, we have expressed the generating functions \(\Pi_i(z)\) \((i = 0, 1, \ldots, c-1)\) and boundary probabilities \(\pi_{i,i}\) \((i = 0, 1, \ldots, c-1)\) in terms of \(\pi_{0,0}\).

Finally, we consider the case \(i = c\), i.e., all servers are active. Balance equations are given as follows,
\[
(\lambda + c\mu)\pi_{c,c} = \alpha \pi_{c-1,c} + c\mu \pi_{c,c+1},
\]
\[
(\lambda + c\mu)\pi_{c,j} = \alpha \pi_{c-1,j} + \lambda \pi_{c,j-1} + c\mu \pi_{c,j+1},
\]

We define the generating function for the case \(i = c\) as follows.
\[
\Pi_c(z) = \sum_{j=c}^{\infty} \pi_{j,c} z^{j-c}.
\]

Multiplying (10) by \(z^0\) and (11) by \(z^{j-c}\) and summing over \(j \geq c\), we obtain
\[
(\lambda + c\mu)\Pi_c(z) = \frac{\alpha}{z} (\Pi_{c-1}(z) - \pi_{c-1,c-1}) + \frac{c\mu}{z} (\Pi_c(z) - \pi_{c,c}) + \lambda \pi_c(z),
\]
or equivalently
\[
\Pi_c(z) = \frac{\alpha (\Pi_{c-1}(z) - \pi_{c-1,c-1}) - c\mu \pi_{c,c}}{(z-1)(c\mu - \lambda z)}.
\]
The numerator of $\Pi_c(z)$ vanishes at $z = 1$ due to the balance between the flows in and out the group of states $\{(c,j); j = c, c+1, \ldots\}$. Thus, applying l'Hôpital's rule yields

$$\Pi_c(1) = \frac{\alpha \Pi_{c-1}(1)}{c\mu - \lambda}. \quad (12)$$

**Remark 3:** It should be noted that we have expressed $\Pi_i(z)$ $(i = 0, 1, \ldots, c)$ in terms of $\pi_{0,0}$, which is uniquely determined by the following normalization condition:

$$\sum_{i=0}^{c} \Pi_i(1) = 1. \quad (13)$$

According to (12), in order to calculate $\Pi_c(1)$, we need $\Pi_{c-1}(1)$ which is recursively obtained by Theorem 3.1.

In Section III-B we show some simple recursive formulae for the partial factorial moments while Section III-C is devoted to the exact expressions of the joint stationary distribution.

**B. Factorial moments**

In this section, we derive simple recursive formulae for factorial moments. Because the generating function $\Pi_{0,0}(z)$ is given in a simple form, its derivatives at $z = 1$ are also explicitly obtained in a simple form.

**Theorem 3.1:** The first partial moments of the queue length is recursively calculated as follows.

$$\Pi_i'(1) = \Pi_i'(1) + \frac{\lambda - \alpha - i\mu}{\alpha} \Pi_i(1) + \frac{(i+1)\mu \pi_{i+1,i+1} + \alpha \pi_{i,i}}{\alpha}, \quad i = 1, 2, \ldots, c - 1, \quad (14)$$

where $\Pi_0(1) = 2\pi_{0,0}\lambda^2(\lambda + \alpha)/\alpha^3$. Furthermore, the $n$-th ($n \geq 2$) partial factorial moments are given by

$$\Pi_i^{(n)}(1) = \Pi_i^{(n)}(1) + \frac{\lambda n(n-1)\Pi_i^{(n-2)}(1) + n(\lambda - i\mu - \alpha)\Pi_i^{(n-1)}(1)}{\alpha}, \quad (15)$$

where $\Pi_0^{(n)}(1) = n!\pi_{0,0}\lambda^n(\lambda + \alpha)/\alpha^{n+1}$.

**Proof:** Differentiating (8), we obtain

$$[(\lambda + i\mu + \alpha)z - \lambda z^2 - i\mu] \Pi_i'(z) = - (\lambda + i\mu + \alpha - 2\lambda z)\Pi_i(z) + \lambda \Pi_i'(z) + \alpha \pi_{i,i} + (i+1)\mu \pi_{i+1,i+1}.$$ Substitute $z = 1$ into the above equation and arranging the result yields (14). Differentiating (8) for $n \geq 2$ times at $z = 1$ and arranging the result, we obtain (15).

**Theorem 3.2:** We have

$$\Pi_c^{(n)}(1) = \frac{\alpha \Pi_{c-1}^{(n+1)}(1) + \lambda n(n+1)\Pi_{c-1}^{(n-1)}(1)}{(n+1)(c\mu - \lambda)}, \quad n \in \mathbb{N}. \quad (16)$$

**Proof:** We have

$$(z - 1)(c\mu - \lambda)\Pi_c(z) = \alpha(z - \pi_{c-1,c-1} - c\mu \pi_{c,c}).$$

Differentiating this equation $n \geq 1$ times, we obtain

$$(z - 1)(c\mu - \lambda)\Pi_c^{(n)}(z) + n\Pi_c^{(n-1)}(z)(\lambda + c\mu - 2\lambda z) + n(n-1)\Pi_c^{(n-2)}(z)(-2\lambda) = \alpha \Pi_c^{(n-1)}(z),$$

where $\Pi_c^{(-1)}(z) = 0, \forall |z| < 1$. Arranging this equation leads to

$$\Pi_c^{(n)}(z) = \frac{A_n(z)}{(z - 1)(c\mu - \lambda)}.$$ Where $A_n(z) = \alpha \Pi_{c-1}^{(n)}(z) + \lambda n(n+1)\Pi_{c-1}^{(n-2)}(z) + n(2\lambda z - c\mu)\Pi_{c-1}^{(n-1)}.$

We observe inductively that both the denominator and numerator in the right hand side of (17) vanish at $z = 1$. Thus, applying L'Hôpital's rule yields,

$$\Pi_c^{(n)}(1) = \frac{\alpha \Pi_{c-1}^{(n+1)}(1) + \lambda n(n+1)\Pi_{c-1}^{(n-1)}(1) + n(\lambda - c\mu)\Pi_c^{(n)}(1)}{(c\mu - \lambda)},$$

leading to (16).

**Remark 4:** It should be noted that in order to obtain the $n$-th factorial moment $\Pi_i^{(n)}(1)$, we need to have the $(n+1)$-th factorial moment $\Pi_{i+1}^{(n+1)}(1)$. Fortunately, $\Pi_{c-1}^{(n+1)}(1)$ is expressed in terms of $\Pi_{c-1}^{(n+1)}(1)$ which is explicitly obtained for any $n$ according to Theorem 3.1.

**C. Joint stationary distribution**

In this section, we prove that the joint stationary distribution is a linear combination of multiple geometric distributions. To this end, we decompose the partial generating functions into the simplest forms from which we obtain exact expressions for the joint stationary distribution.

**Definition 3.3:** We define $\varphi_i$ $(i = 0, 1, \ldots, c - 1, c)$ as follows.

$$\varphi_0 = \frac{\lambda}{\lambda + \alpha}, \quad \varphi_i = \frac{\lambda \varphi_{i-1}}{i\mu}, \quad i = 1, 2, \ldots, c - 1, \quad \varphi_c = \frac{\lambda}{c\mu}.$$ \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm}

**Remark 5:** It is easy to see that $\varphi_i \neq \varphi_j$ (&forall; $i \neq j \in \{0, 1, \ldots, c - 1\}$). The expression for $\Pi_i(z)$ for the case $\exists k \in \{0, 1, \ldots, c - 1\}$ such that $\varphi_k = \varphi_c$ and for the case $\varphi_k \neq \varphi_c, \forall k \in \{0, 1, \ldots, c - 1\}$ are slightly different.

**Theorem 3.4:** The partial generating functions are decomp-
posed as follows.

\[
\Pi_0(z) = \pi_{0,0} \frac{a_{0,0}}{1 - \varphi_{0} z},
\]

\[
\Pi_1(z) = \pi_{1,1} \left( \frac{a_{1,0}}{1 - \varphi_{0} z} + \frac{a_{1,1}}{1 - \varphi_{1} z} \right),
\]

\[
\Pi_2(z) = \pi_{2,2} \left( \frac{a_{2,0}}{1 - \varphi_{0} z} + \frac{a_{2,1}}{1 - \varphi_{1} z} + \frac{a_{2,2}}{1 - \varphi_{2} z} \right),
\]

\[
\vdots
\]

\[
\Pi_{i-1}(z) = \pi_{i-1,c-1} \times \left( \frac{a_{c-1,0}}{1 - \varphi_{0} z} + \frac{a_{c-1,1}}{1 - \varphi_{1} z} + \ldots + \frac{a_{c-1,c-1}}{1 - \varphi_{c-1} z} \right),
\]

\[
\pi_{i,i} = \gamma_i \pi_{i-1,i-1}, \quad i = 1, 2, \ldots, c,
\]

and

\[
\Pi_i(z) = \frac{1}{\pi_c} \left( \sum_{j=0}^{c-1} a_{j,i} \left( \frac{1}{1 - \varphi_j z} \right) \right),
\]

where \(a_{i,j}\) (\(i = 0, 1, \ldots, c, j = 0, 1, \ldots, i\)), \(\tilde{a}_{c,c}, \tilde{a}_{c,k}\) and \(\gamma_i (i = 1, 2, \ldots, c)\) are exactly given and \(\pi_{0,0}\) is uniquely determined by the normalization condition [13].

**Proof:** We prove by mathematical induction. Indeed, it follows from (11) that Theorem 3.4 is true for \(i = 0\) where \(a_{0,0} = 1\). Assuming that Theorem 3.4 is true for \(i - 1\), i.e.,

\[
\Pi_{i-1}(z) = \pi_{i-1,i-1} \sum_{j=0}^{i-1} \frac{a_{i-1,j}}{1 - \varphi_j z}, \quad i = 1, 2, \ldots, c,
\]

we derive the decomposition result for \(\Pi_i(z)\). First of all, we prove that \(\pi_{i,i} = \gamma_i \pi_{i-1,i-1}\) for some given \(\gamma_i\). Indeed, the global balance between \(\{(i - 1, j); j = i - 1, i, \ldots\}\) and \(\{(i, j); j = i, i + 1, \ldots\}\) yields,

\[
i \mu \pi_{i,i} = \alpha (\Pi_{i-1}(1) - \pi_{i-1,i-1})
\]

\[
= \alpha \pi_{i-1,i-1} \sum_{j=0}^{i-1} \frac{a_{i-1,j}}{1 - \varphi_j} - 1,
\]

leading to

\[
\gamma_i = \frac{\alpha}{\mu} \sum_{j=0}^{i-1} \frac{a_{i-1,j}}{1 - \varphi_j} - 1.
\]

It should be noted that we have used the mathematical induction to calculate \(\Pi_{i-1}(1)\).

Substituting (2) into (8) and arranging the result yields,

\[
\Pi_i(z) = \frac{\alpha (z_i \Pi_{i-1}(z) - z \Pi_{i-1}(z_i))}{i \mu (z - z_i)(1 - \varphi_z z)} + \frac{\alpha \pi_{i-1,i-1} + 1}{i \mu (1 - \varphi_z z)}.
\]

We show that \(z = z_i\) is a removable singular point. Indeed, from the assumption of the mathematical induction, we have

\[
\Pi_{i-1}(z) = \pi_{i-1,i-1} \sum_{j=0}^{i-1} \frac{a_{i-1,j}}{1 - \varphi_j z}.
\]

On the other hand, we have

\[
\frac{z_i}{1 - \varphi_j z} \frac{1}{1 - \varphi_j z} = \frac{(z - z_i) \varphi_j (z + z_i) - 1}{(1 - \varphi_j z)(1 - \varphi_j z_i)}
\]

\[
= \frac{z - z_i}{1 - \varphi_j z} \left( \frac{\varphi_j z_i}{1 - \varphi_j z_i} - 1 \right),
\]

and

\[
\frac{1}{(1 - \varphi_j z)(1 - \varphi_j z_i)} \frac{\varphi_j - z_i}{\varphi_j - z_i} = \frac{\varphi_j - z_i}{1 - \varphi_j z} + \frac{\varphi_j - 1}{1 - \varphi_j z}.
\]

Substituting these three expressions into (19) and arranging the result, we obtain

\[
\Pi_i(z) = \pi_{i,i} \sum_{j=0}^{i} \frac{a_{i,j}}{1 - \varphi_j z},
\]

where \(a_{i,j}\) are given in terms of \(a_{i-1,j}\) and \(\gamma_i\) as follows.

\[
a_{i,j} = \frac{\alpha \varphi_j^2 z_i a_{i,j-1}}{i \mu \gamma_i (1 - \varphi_j z_i)(\varphi_j - \varphi_i)}, \quad j = 0, 1, \ldots, i - 1,
\]

\[
a_{i,i} = 1 + \frac{\alpha}{i \mu \gamma_i} \sum_{j=0}^{i-1} a_{i-1,j} \left( \frac{\varphi_j - \gamma_i}{(\varphi_j - \varphi_i) - 1} \right).
\]

Next, we derive formula for \(\Pi_c(z)\). We express \(\Pi_c(z)\) in terms of \(\Pi_{c-1}(z)\) as follows. We consider the case \(\varphi_i \neq \varphi_c\) (\(i = 0, 1, \ldots, c - 1\)).

\[
\Pi_c(z) = \pi_c \frac{\alpha (\Pi_{c-1}(z) - \Pi_{c-1}(1))}{(z - 1)(1 - \varphi_c z)}
\]

\[
= \alpha \pi_c \frac{\Pi_{c-1}(z) - \Pi_{c-1}(1)}{(1 - \varphi_c z)} - \frac{1}{1 - \varphi_c z}
\]

\[
= \alpha \pi_c \frac{\sum_{j=0}^{c-1} a_{c-1,j} \varphi_j}{1 - \varphi_j} \frac{1}{1 - \varphi_j} + \frac{\varphi_c}{1 - \varphi_c z} \frac{1}{1 - \varphi_c z}
\]

\[
= \pi_c \frac{\varphi_j}{1 - \varphi_j} \frac{1}{1 - \varphi_j} + \frac{\varphi_c}{1 - \varphi_c z} \frac{1}{1 - \varphi_c z},
\]

implying the announced result in Theorem 3.4 where

\[
a_{c,j} = \frac{\alpha \varphi_j^2 a_{c-1,j}}{i \mu \gamma_c (1 - \varphi_j)(\varphi_j - \varphi_c)}, \quad j = 0, 1, \ldots, c - 1,
\]

\[
a_{c,c} = \frac{\alpha}{i \mu \gamma_c} \sum_{j=0}^{c-1} \left( \frac{a_{c-1,j} \varphi_j^2}{1 - \varphi_j} \right).
\]

For the case \(\{\exists k \in \{0, 1, \ldots, c - 1\} | \varphi_k = \varphi_c\}\), \(\Pi_c(z)\) is decomposed as follows.

\[
\Pi_c(z) = \pi_{c,c} \left( \sum_{j=0}^{c-1} \frac{a_{c,j}}{1 - \varphi_j z} + \tilde{a}_{c,c} \varphi_c + \tilde{a}_{c,k} \right)
\]
where
\[
\hat{a}_{c,c} = \frac{\alpha}{\epsilon \mu \gamma_c} \sum_{j=0}^{c-1} \frac{ae-c-1-j}{j} \varphi_j \varphi_c, \\
\hat{a}_{c,k} = \frac{\alpha}{\epsilon \mu \gamma_c} \frac{ae-c-1-k}{1-\varphi_k},
\]

**Corollary 3.5:** We have
\[
\pi_{i,j} = \pi_{i,i} \sum_{j=0}^{i} a_{i,j} \varphi_j^{j-i}, \quad j \geq i, \quad i = 0, 1, \ldots, c-1.
\]
Furthermore,
\[
\pi_{c,j} = \pi_{c,c} \sum_{j=0}^{c} a_{c,j} \varphi_j^{j-c}, \quad j \geq c,
\]
if \( \varphi_i \neq \varphi_c, \forall i \in \{0, 1, \ldots, c-1\}, \) while
\[
\pi_{c,c} = \pi_{c,c}
\]

\[\times \sum_{j=0}^{c-1} \left( a_{c,j} \varphi_j^{j-c} + \hat{a}_{c,c} \varphi_j^{j-c} + \hat{a}_{c,k}(j-c+1) \varphi_j^{j-c} \right), \]
\[j \geq c, \quad \{ \exists k \in \{0, 1, \ldots, c-1\} \mid \varphi_k = \varphi_c \}.
\]

**Remark 6:** We observe that our expressions for the joint stationary distribution conform to those derived by Artalejo et al. \( \Pi \) using the difference equation approach.

**D. Waiting time distribution**

Let \( W \) and \( W(s) \) denote the waiting time and its LST, respectively. Also let \( f(x) \) denote the probability density of \( W \), i.e.
\[
f(x) = \frac{d \Pr(W < x)}{dx}.
\]

**Theorem 3.6:** i) \( \varphi_k \neq \varphi_c, \forall k \in \{0, 1, \ldots, c-1\} \), we have
\[
f(x) = \sum_{i=0}^{c} \frac{\lambda a_i}{\lambda - \lambda \varphi_i} \beta_i e^{-\beta_i x},
\]
where
\[
\beta_i = \frac{\lambda - \lambda \varphi_i}{\varphi_i}, \quad a_j = \sum_{i=j}^{c} a_{i,j} \pi_{i,i},
\]
ii) \( \exists k \in \{0, 1, \ldots, c-1\} \mid \varphi_k = \varphi_c \), we have
\[
f(x) = \sum_{i=0}^{c} \frac{\lambda a_i}{\lambda - \lambda \varphi_i} \beta_i e^{-\beta_i x} + \hat{a}_{c,k} x e^{-\beta_c x},
\]
where
\[
\beta_i = \frac{\lambda - \lambda \varphi_i}{\varphi_i}, \quad a_j = \sum_{i=j}^{c} a_{i,j} \pi_{i,i}, \quad j \neq k, c,
\]
and
\[
a_k = \sum_{i=k}^{c-1} \pi_{i,i} a_{i,j}, \quad a_c = \hat{a}_{c,c}.
\]

**Proof:** First we prove case i). Indeed, the generating function of the number of waiting customers in the system as follows.
\[
\Pi(z) = \sum_{i=0}^{c} \Pi_i(z) = \sum_{i=0}^{c} \frac{a_i}{1 - \varphi_i z},
\]
where \( a_j = \sum_{i=j}^{c} a_{i,j} \pi_{i,i} (j = 0, 1, \ldots, c) \). Let \( W(s) \) denote the LST of the waiting time in the queue. According to the distributional Little’s law, we have the following relation.
\[
\Pi(z) = W(\lambda - \lambda z),
\]
or equivalently,
\[
W(s) = \Pi(1 - s/\lambda) = \sum_{i=0}^{c} \frac{\lambda a_i}{\lambda - \lambda \varphi_i + \varphi_i s} = \sum_{i=0}^{c} \frac{\lambda a_i}{\lambda - \lambda \varphi_i + \varphi_i s + \beta_i},
\]
Inverting the LST \( W(s) \), we obtain the waiting time distribution \( 21 \). Using the same argument, we obtain \( 22 \). The proof of ii) is proceeded by the same manner.

**IV. DECOMPOSITION OF QUEUE LENGTH**

In this section is devoted to the decomposition property of the queue length where we show the single server system in Section IV-A and discuss the multiserver model in Section IV-B.

**A. Single server**

We consider the single server case. The partial generating functions are given as follows.
\[
\Pi_0(z) = \frac{(1 - \rho)\alpha}{\lambda + \alpha - \lambda z}, \quad \Pi_1(z) = \frac{(1 - \rho)\lambda \alpha}{(\mu - \lambda z)(\lambda + \alpha - \lambda z)}.
\]
Let \( \Pi(z) \) denote the generating function of the number of waiting customers. We have
\[
\Pi(z) = \Pi_0(z) + \Pi_1(z) = (1 - \rho) \left( 1 + \frac{\rho}{1 - \rho z} \right) \frac{\alpha}{\lambda + \alpha - \lambda z}.
\]
It should be noted that
\[
(1 - \rho) \left( 1 + \frac{\rho}{1 - \rho z} \right)
\]
and
\[
\frac{\alpha}{\lambda + \alpha - \lambda z}
\]
represent the generating function of the number of waiting customers in the corresponding M/M/1 queue without setup time and that of customers arriving in the remaining setup time, respectively. Thus, we have
\[
L^d = L_1 + L_2,
\]
where the \( L \) is the queue length of the current model while \( L_1 \) and \( L_2 \) represent the queue length of the conventional M/M/1 queue and the number of customers that arrive during the remaining setup time.
B. Multiserver

In this section, we numerically investigate the decomposability of the queue length. In particular we answer the question: does equation (24) hold?

\[ L = L_1 + L_2, \]  

(24)

where \( L_1 \) is the queue length of the M/M/c without setup time and \( L_2 \) is the number of customers that arrive to the queue during the remaining setup time.

The generating function for the number of waiting customers in the conventional M/M/c queueing system is given by \( 1 - C(cp, c) + C(cp, c)(1 - \rho)/(1 - \rho z) \) where \( \rho = \lambda/(cp) = \varphi_c \) and \( C(cp, c) \) is the Erlang C formula for the waiting probability in the conventional M/M/c system without setup time. Therefore, if the decomposition result is established the generating function of the number of waiting customers in the system with setup time \( \Pi(z) \) must be given by the following formula.

\[ \Pi(z) = \frac{\alpha}{\alpha + \lambda - \lambda z} \left( 1 - C(cp, c) + C(cp, c) \frac{1 - \rho}{1 - \rho z} \right). \]

(25)

In [8] the authors state that the decomposition property is held for the model meaning that (25) is true. We verify this property by numerical experiments in Section VII. In Section VI we show a conditional decomposition for the conditional number of waiting customers.

V. CONDITIONAL DECOMPOSITION

We have derived the following result.

\[ \Pi_c(z) = \frac{\alpha(\pi_{c-1}(z) - \pi_{c-1,c-1}) - c\mu \pi_{c,c}}{(z - 1)(c\mu - \lambda z)}, \]

\[ \Pi_c(1) = \frac{\alpha \Pi'_{c-1}(1)}{c\mu - \lambda}. \]

Let \( Q^{(c)} \) denote the conditional queue length given that all \( c \) servers are busy, i.e.,

\[ \Pr(Q^{(c)} = i) = \Pr(N = i + c \mid C = c), \]

where \( N \) and \( C \) are the number of customers in the system and that of busy servers in the steady state, respectively. Let \( P_c(z) \) denote the generating function of \( Q^{(c)} \). It is easy to see that

\[ P_c(z) = \frac{\Pi_c(z)}{\Pi_c(1)} = \frac{\alpha(\Pi_{c-1}(z) - \pi_{c-1,c-1}) - c\mu \pi_{c,c}}{\alpha \Pi'_{c-1}(1)(z - 1)(c\mu - \lambda z)} = \frac{\Pi_{c-1}(z) - \Pi_{c-1}(1)}{\Pi'_{c-1}(1)(z - 1)} 1 - \rho z \]

\[ = \frac{\Pi_{c-1}(z) - \Pi_{c-1}(1)}{\Pi'_{c-1}(1)(z - 1)} 1 - \rho z \]

\[ = \sum_{j=1}^{\infty} \pi_{c-1,c-1+j}(z^j - 1) 1 - \rho z \]

\[ = \sum_{j=1}^{\infty} \pi_{c-1,c-1+j} \sum_{i=0}^{j-1} z^i 1 - \rho z \]

\[ = \sum_{i=0}^{\infty} \left( \sum_{j=i+1}^{\infty} \pi_{c-1,c-1+j} \right) z^i 1 - \rho z \]

where we have used \( c\mu \pi_{c,c} = \alpha(\Pi_{c-1}(1) - \pi_{c-1,c-1}) \) in the second equality.

It should be noted that \( (1 - \rho)/(1 - \rho z) \) is the generating function of the number of waiting customers in the conventional M/M/c system without setup time (denoted by \( Q^{(c)}_{ON-IDLE} \)) under the condition that \( c \) servers are busy.

We give a clear interpretation for the generating function:

\[ \sum_{i=0}^{\infty} \left( \sum_{j=i+1}^{\infty} \pi_{c-1,c-1+j} \right) z^i 1 - \rho z \]

For simplicity, we define

\[ p_{c-1,i} = \frac{\sum_{j=i+1}^{\infty} \pi_{c-1,c-1+j} \Pi'_{c-1}(1)}{\Pi'_{c-1}(1)}, \quad i \in \mathbb{Z}_+. \]

We have

\[ \sum_{j=i+1}^{\infty} \pi_{c-1,c-1+j} = \Pr(N - C > i \mid C = c - 1) \Pr(C = c - 1). \]

Thus, we have

\[ p_{c-1,i} = \frac{\Pr(N - C > i \mid C = c - 1)}{\Pr(N - C \mid C = c - 1)}. \]

It should be noted that \( N - C \) is the number of waiting customers. Thus, the discrete random variable with the distribution \( p_{c-1,i} \) \( (i = 0, 1, 2, \ldots) \) is the residual life time of the number of waiting customers under the condition that \( c - 1 \) servers are busy. Let \( Q_{res} \) denote this random variable.

Thus our decomposition result is summarized as follows.

\[ Q^{(c)} \overset{d}{=} Q^{(c)}_{ON-IDLE} + Q_{res}. \]

Remark 7: Tian et al. [21], [23] obtain a similar result for a multiserver model with vacation. However, the random variable with the distribution \( p_{c-1,i} \) here is not given a clear physical meaning in [21], [23].

VI. NUMERICAL EXAMPLES

In all numerical experiments we fix \( \mu = 1 \).
A. Queue length decomposition

In this section, we show the decomposition property for the queue length of our system. We fix the number of servers by $c = 10$.

1) Mean number of waiting customers: In this section, we show the mean number of waiting customers against the arrival rate. In Fig. 2, we show the mean number of waiting customers $E[L]$ and its corresponding decomposition version $E[L_1] + E[L_2]$ against $\lambda$ for $\alpha = 0.1$ and 0.01. In these two cases, we observe that

$$E[L] = E[L_1] + E[L_2],$$

implying the decomposition property for the mean.

2) Coefficient of variation: Next, in Fig. 3 we show the coefficient of variation against $\lambda$ for $\alpha = 0.1$ and 0.01, which correspond to that the mean setup times are 10 and 100 times of the mean service time, respectively. We observe that the coefficients of variation for $L$ and $L_1 + L_2$ are the same meaning that the decomposition property is true.

B. Power consumption vs. traffic intensity

Fig. 4 shows the total power consumption against the offered traffic load $\lambda/\mu = c\rho$ where the number of servers is fixed to $c = 10$. The cost per unit time for each state: SETUP, ON and IDLE of a server is set as follows: $C_{setup} = 2, C_{run} = 1$ and $C_{idle} = 0.6$. The power consumption of our system with staggered setup is given by

$$P_{ON/OFF-staggered} = C_{setup}(1 - \sum_{i=0}^{c-1} \pi_{i,i} - \Pi_{c}(1)) + C_{run}c\rho,$$

where $c\rho = \lambda/\mu$ is the mean number of running servers. We plot four curves corresponding to the cases $\alpha = 0.1, 1, 10$ and 100. For comparison, we also plot the curves for the conventional M/M/c queue under the same setting. It should be noted that in the conventional M/M/c system, an idle server is not turned off. As a result, the cost for power consumption is given by

$$P_{ON/IDLE} = C_{run}c\rho + C_{idle}(c - c\rho).$$

We observe in Fig. 4 that a curve of the conventional M/M/c system crosses the corresponding one of the M/M/c-Staggered at some points $\rho_a$ which is increases with the setup rate $\alpha$. This suggests that the shorter the mean setup time, the longer the range in which the M/M/c-Staggered outperforms the conventional M/M/c-ON/IDLE system. In the other words,
if $\rho < \rho_\alpha$, M/M/c-Staggered outperforms the conventional M/M/c-ON/IDLE system from a power consumption point of view, while the latter is more power saving than the former under a relatively high load, i.e., $\rho > \rho_\alpha$.

C. Power consumption vs. setup rate

In this section, we investigate the influence of the setup rate $\alpha$ on the power consumption and the mean waiting time. Fig. 5 presents the power consumption against $\alpha$. We plot three curves for the cases: $\rho = 0.5$, $0.7$ and $0.9$, respectively. For a comparison purpose, we also draw the power consumption for the corresponding ON/IDLE cases. It should be noted that the power consumption in ON/IDLE policy does not depend on $\alpha$

We observe in Fig. 5 that for the case $\rho = 0.5$, the staggered setup policy outperforms the ON/IDLE policy for any $0.01 < \alpha < 100$. We also observe that for the case $\rho = 0.9$, the ON/IDLE policy always outperforms the staggered setup policy for all $0.01 < \alpha < 100$. This implies that when the traffic load is low, the staggered setup policy is always better for any setup rate while it is better to keep all the servers on for the case of heavy load.

For the case $\rho = 0.7$, we observe that there exists some $\alpha_0$ such that the ON/IDLE is superior over the staggered setup policy for $\alpha < \alpha_0$ while the staggered setup policy is more power saving when $\alpha > \alpha_0$. This is equivalent to say that we should apply staggered setup policy and the ON/IDLE policy for the case of fast enough setup time and that of slow enough setup time, respectively.

VII. Conclusion

In this paper, we have considered the M/M/c queueing system with setup time where only one server can be in setup at a time. A server is turned off immediately after serving a job and there is no waiting customer. If there are some waiting customers, the idle servers are turned on one by one. Using a generating function approach, we have obtained exact expressions for the partial generating functions. We also have obtained recursive formulae for computing the factorial moments of the number of waiting customers. Explicit expressions have shown that the joint stationary distribution is a linear combination of geometric distribution while the waiting time distribution is a linear combination of exponential distributions.

Our numerical results have confirmed the decomposition property derived in [2]. We have also shown a conditional decomposition property where the number of waiting customers under the condition that all servers are busy is equal in distribution to that of the conventional M/M/c queue without setup time and a random variable which represents the residual life time of the number of waiting customers under the condition that $c - 1$ servers are busy.

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