A Reduced Model of Noncommutative S-brane Spectrum

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Abstract

CFT construction of S-branes describing the rolling and bouncing tachyons is analyzed in the context of a $\theta$-noncommutative deformation of minisuperspace. Half s-brane and s-brane in the noncommutative minisuperspace are analyzed and exact analytic solutions, involving the noncommutative parameter $\theta$ and compatible with the boundary conditions at infinity, are found. Comparison with the usual commutative minisuperspace is finally performed.
1 Introduction

It is common knowledge that the presence of tachyonic modes in a quantum field theory signals the existence of an unstable vacuum. In the past, “annoying” tachyon states would be responsible for the discredit of their own embracing field theory which was then doomed to oblivion unless some cure could be implemented. In the case of string theory for instance, the mechanism credited for getting rid of this unappealing circumstance was the remarkable discovery of supersymmetry. Not surprisingly, it has been precisely string theory which has taught us over the past few years that there is also much to be learned in the study of the long time eluded tachyon. As a result we have witnessed an increasing interest on this reach and extensive field of research (for reviews on different subjects concerning tachyons, see [1]).

A particular case of a system with tachyonic excitations in string theory which later on grew in importance due to its relevance in time-dependent string backgrounds (see below) is that of a coincident D-brane anti D-brane pair. It was realized in [2] that in the NS sector ground state of this system survives the GSO projection giving rise to a complex scalar field with negative squared mass for Type II theory, this renders the system unstable. As shown by Sen [3] a tachyonic kink solution on this brane anti-brane pair can be identified with a stable non-BPS D-brane of the same theory in one lower dimension. On the other hand, he also showed how BPS D-branes can be associated with tachyonic kink solutions on non-BPS D-branes on one higher dimension, effectively uncovering a web of descent relations between BPS and non-BPS D-branes of Type II string theories, opening the way to the description of D-brane charges in terms of elements of K-theory [4].

The above kink solutions that interpolate between two minima of the tachyon potential acquired relevance in the context of time-dependent string backgrounds when Strominger and Gutperle [5] adapted Sen’s argument and showed that when the interpolation occurs in the timelike direction we are effectively describing objects which are localized on a spacelike hypersurface. These are known as S-branes or Spacelike branes and can be thought of as the description of the time-dependent processes involving the creation and subsequent decay of unstable D-branes in string theory. Also, because of their own nature, S-brane solutions in string and supergravity theories have attracted much attention towards the possible cosmological interpretations. For recent works on S-branes see [6] throughout [25], research in the cosmological aspects include [26] throughout [40].

Noteworthy, in [41] Sen explores the dynamics of the tachyon from a string field theory point of view. He showed that the worldsheet action of the boundary conformal
field theories associated with the classical time-dependent solutions describing the motion of unstable bosonic D-branes can be taken to be

\[ S = -\frac{1}{4\pi} \int_{\Sigma} d^{2}\sigma \partial^{a} \partial_{a} X_{\mu} \partial_{a} X_{\mu} + \lambda \int_{\partial\Sigma} d\tau \cosh X^{0}(\tau), \]  

(1.1)

(we use \(\alpha' = 1\)) and he gave an analogous treatment for the superstring in [42]. In the formal limit of weak coupling constant \(g_{s} \to 0\) the action (1.1) actually corresponds to a CFT construction of the spacelike brane. A related boundary interaction of

\[ \frac{\lambda}{2} \int_{\partial\Sigma} d\tau \exp(X^{0}), \]  

(1.2)

was discussed in [43] motivated in part by the resemblance of this system with the boundary Liouville theory which has been extensively studied before [44] (eventually this led to the construction of timelike boundary Liouville theory [25]). Following the usual terminology we will refer to (1.1) as the bouncing tachyon or s-brane and to (1.2) as the rolling tachyon or half s-brane.

Our main interest in this work will be focused on the minisuperspace approximation to these systems first introduced in this context by Strominger in [43] in analogy to the ordinary Liouville theory [45]. In these reduced models the approximation consists in considering only the zero-mode dependence of the boundary interactions. Recently a detailed analysis of the minisuperspace energy spectrum of these models was performed in [12] (for further work in the context of the minisuperspace approach, see [46]) and the main task in this note will be to extend their results to a deformed minisuperspace on which we shall introduce a noncommutative parameter between the minisuperspace variables in the spirit of [47]. The main motivation comes from an analogous treatment in the context of quantum gravity in reference [48] wherein it is introduced a noncommutativity between the variables of the reduced quantum cosmology model of the Kantowski-Sachs metric. In section 2 we will give a brief review of this ideas introducing a novel example of an exact solution to the noncommutative Wheeler-DeWitt equation for the standard massless scalar field minimally coupled. It is worth to mention that a somewhat different approach involving the minisuperspace formulation of quantum S-branes has been worked out and will appear elsewhere [49].

Nevertheless, we have to point out that the noncommutativity considered here in the context of tachyon dynamics is of a different nature than that of quantum cosmology minisuperspace models, since usually the minisuperspace variable \(x^{0}\) which parametrizes the dependence of the boundary interaction terms in the reduced forms of the actions above is indeed the timelike parameter, and we will assume a non-zero noncommutative

\[ 1 \text{For an approach to the full space noncommutative tachyon see for instance [50].} \]
parameter between the commutators \([x^\mu, x^\nu]\), including those of \(x^0\) with its spacelike counterparts. Actually in the literature it is often assumed that the noncommutativity is present only in the spacelike directions of the system under consideration, the reasons for this are fairly obvious in that common issues of causality have to be considered, in fact it is known that fundamental changes have to be implemented, for instance, in the formulation of ordinarily quantum mechanics if a noncommutative timelike parameter is included \[51\]. In spite of these facts, we will push on and continue with our assumptions since we believe that ideas of this kind deserve a closer examination, as have been shown in the past in other contexts\(^2\).

Keeping these remarks in mind throughout the rest of this paper, we will introduce a deformation of the minisuperspace models above in sections 3 and 4 for the half s-brane and the s-brane respectively. We will see how the resulting effective equation for the corresponding energy spectrums have analytic solution without the need for an expansion in the noncommutativity parameter \(\theta\). The resulting eigenstates of the corresponding models acquire a shift in the timelike parameter proportional to \(\theta\). For the purpose of visualizing the effect of noncommutativity, wave packets with suitable weighing functions are constructed and analyzed for special cases. In general the \(\theta\)-effect shows up as an enhancement of the localization in the timelike direction of the wave packet, apart from an overall shift. Finally in section 5 we summarize and comment on our results.

2 Overview of the Noncommutative Wheeler-DeWitt Equation

Quantum Gravity is an old subject of theoretical physics. It grew in the late 1930’s from the desire to emulate the successes which the then new quantum field theory had had with regards the beginning of quantum electrodynamics. The feeling at that time was that only technical difficulties remained to be surpassed to achieve the desired goal. At present, a full description of a quantum theory of the gravitational field remains a (yet to be found) eagerly searched milestone of contemporary knowledge. The simplest model of quantum gravity describes the quantum behavior of 3-space regarding it as a time-varying geometrical object. The quantum states are usually chosen in the so called metric representation in which the wave function of the (entire) universe \(\Psi\) becomes a functional of the metric components \(g_{\mu\nu}\) and the momenta become functional differential operators.

\(^2\)For instance, the study of timelike dualities has proved worth of attention in the context of the search for a formulation of a dS/CFT duality \[52\].
with respect to the metric components. Consistency with the canonical constraints then leads to the well known Wheeler-DeWitt equation for the functional $\Psi$ which then depends solely on the spatial components $g_{ij}$ of the metric, and the space of all consistent 3-geometries is denominated superspace. Of course most of the analysis of the Wheeler-DeWitt equation deal with a reduced form of this superspace known as minisuperspace, in which almost all of the degrees of freedom of the gravitational field are ‘frozen out’. As our example of this minisuperspace techniques we shall consider metrics which in appropriate units read

$$ds^2 = N^2(t)dt^2 + a^2(t)d\Omega_3^2,$$

where $N(t)$ is known as the lapse function, $a(t)$ is the scale factor and $d\Omega_3^2$ is the metric of a three-sphere of unit radius. Actually the solutions of the Wheeler-DeWitt equation are independent of $N$ and $t$ since the wave function $\Psi(a, \phi)$ is a function solely of $a$ and possible matter fields denoted by $\phi$. We shall consider only a massless scalar field $\phi$ minimally coupled to the gravitational filed. In this case the Wheeler-DeWitt equation with the standard factor ordering prescription reads \[53\]:

$$a \frac{\partial}{\partial a} a \frac{\partial}{\partial a} - \frac{\partial^2}{\partial \phi^2} - a^4 \] \Psi(a, \phi) = 0. \tag{2.2}$$

Solutions to this equation are readily found and their interpretation range from wave functions corresponding to classical Lorentzian-Friedmann universes to wormholes for appropriate linear combinations satisfying certain boundary conditions \[53\].

The proposal of \[48\] was to introduce a noncommutativity between the minisuperspace variables, in this case $a$ and $\phi$, so as to mimic the spacetime noncommutativity motivated by string theory. With this in mind, we will assume a non-zero commutator between $a$ and $\phi$ is given by

$$[a, \phi] = i\theta, \tag{2.3}$$

where $\theta$ is the noncommutativity parameter.

The standard procedure now is to implement this noncommutativity in the the theory through a Moyal deformation of the usual product of functions. Thus the noncommutative Wheeler-DeWitt equation which corresponds to (2.2) reads

$$\left[a \frac{\partial}{\partial a} a \frac{\partial}{\partial a} - \frac{\partial^2}{\partial \phi^2} - a^4 \right] \star \Psi(a, \phi) = 0, \tag{2.4}$$

where all products of functions are with respect to the Moyal star product $\star$ defined by

$$f(a, \phi) \star g(a, \phi) := f(a, \phi)e^{i(\theta/2)\left((\overleftarrow{\partial_a \partial_\phi} - \overrightarrow{\partial_\phi \partial_a})\right)}g(a, \phi). \tag{2.5}$$
Thus for instance \((a \frac{\partial}{\partial a} a \frac{\partial}{\partial a}) \ast \Psi(a, \phi)\) stands for \(a \ast \frac{\partial}{\partial a} (a \ast \frac{\partial \Psi(a, \phi)}{\partial a})\). Now we make use of the identity [54]:

\[
V(a, \phi) \ast \Psi(a, \phi) = V(a + \frac{1}{2} i \theta \frac{\partial}{\partial \phi}, \phi - \frac{1}{2} i \theta \frac{\partial}{\partial a}) \Psi(a, \phi),
\]

where the product of functions in the rhs is the usual product of functions. Upon substitution of this expression on (2.4) the advantage of the chosen model becomes apparent since no \(\phi\)-terms arise in the resultant equation, only derivatives with respect to \(\phi\). This makes the resultant equation separable. Choosing the \(\phi\)-dependence of the harmonic solution \(\Psi(a, \phi) = e^{ik\phi} \psi(a)\), the resulting ordinary differential equation for the \(a\)-dependence is:

\[
\psi''(a) + \frac{2}{2a - k\theta} \psi'(a) + \frac{16k^2 - (k\theta - 2a)^4}{4(2a - k\theta)^2} \psi(a) = 0,
\]

where prime’s denote differentiation with respect to \(a\). The solutions to this equation are modified Bessel functions of the first kind \(I_n(z)\) of imaginary order \(n\), however we prefer to analyze linear combinations of this solutions which are well behaved (finite) for large \(a\), thus our complete solutions are of the form:

\[
\Psi_{\pm}^k(a, \phi) = e^{ik\phi} K_{\pm \frac{i}{2} k} \left(\frac{1}{8} (k\theta - 2a)^2\right),
\]

where \(K_n(z)\) are the modified Bessel functions of the second kind. Following [48], we construct the special gaussian weighted wave packet

\[
\Omega(a, \phi) = \mathcal{N} \int_{-\infty}^{\infty} e^{-\tau(k - \eta)^2} \Psi_{\pm}^k(a, \phi) dk,
\]

so as to be able to visualize the effect of the noncommutativity parameter \(\theta\) in the shape of the probability distribution for this packet. Here \(\mathcal{N}\) is a normalization constant and \(\tau\) and \(\eta\) are adjustable parameters. The integral is performed numerically and in figure 1 we show a comparison between the case of zero noncommutative parameter \((\theta = 0)\) and the non-zero one. There we show a slice of the probability distribution \(|\Omega|^2\) as a function of \(a\) for the value \(\phi = 0\). As can be seen from the figure, the most notable effect of noncommutativity is to induce the formation of a local maximum apart from the absolute maximum encounter in the \(\theta = 0\) case. This is an example of the phenomenon discovered in [48] which was interpreted there as the possibility of tunneling among different states of the universe.

This completes our review of noncommutative quantum gravity and we now proceed to implement some of these ideas to the context of the bouncing and rolling tachyons.
The Half S-brane

For the half s-brane the worldsheet action is (1.1) but boundary interaction (1.2) is given by:

\[ S = -\frac{1}{4\pi} \int_{\Sigma_2} d^2 \sigma \partial^\alpha X^\mu \partial_\alpha X_\mu + \frac{\lambda}{2} \int_{\partial \Sigma_2} d\tau \exp(X^0). \] (3.1)

As it was mentioned before, the minisuperspace approximation concerns only the zero modes of the fields. After the usual mode expansion of the open string the effective action is given by

\[ S = \int d\tau \left\{ -\frac{1}{4} \dot{x}^\mu \dot{x}_\mu + (N - 1) + \lambda \exp x^0 \right\}, \] (3.2)

where it is assumed that \( x^\mu = x^\mu(\tau) \), \( x^\mu \) being the zero modes of the embedding fields \( X^\mu(\sigma, \tau) \), and \( N - 1 \) is the oscillator contribution. To proceed further, it is useful to examine the hamiltonian constraint, this can be cast in the form of a Klein-Gordon type equation for the minisuperspace wave function \( \psi(x) \):

\[ [\partial^\mu \partial_\mu - \lambda \exp x^0 - (N - 1)]\psi(x) = 0. \] (3.3)

At this point we introduce commutation relations for the minisuperspace variables as follows

\[ [x^\mu, x^\nu] = i\theta^{\mu\nu}, \] (3.4)

where \( \theta^{\mu\nu} \) is antisymmetric and for simplicity we will take it to be constant: \( \theta^{\mu\nu} = \theta \varepsilon^{\mu\nu} \) for \( \mu < \nu \). As is well known, we can implement this noncommutativity through a deformation of the minisuperspace product of functions by using the Moyal product:

\[ f(x) * g(x) = f(x) \exp \left\{ -\frac{i}{2} \sum_{j,k} \theta^{jk} \hat{\partial}_j \hat{\partial}_k \right\} g(x) \] (3.5)

(we use Greek letters to denote lorentzian indices raised and lowered with metric \( \eta^{\mu\nu} \) and Latin letters for euclidean indices, here \( j, k = 0, 1, \ldots, d - 1 \) as well as \( \mu, \nu \) where \( d \) is the
spacetime dimension and $\theta^{jk}$ is defined as above.) In this manner, the noncommutative version of (3.3) becomes

$$[\partial^\mu \partial_\mu - \lambda \exp x^0 - (N - 1)] * \psi(x) = 0. \quad (3.6)$$

For our purposes, we rephrase the star product in terms of the well known formula [54]

$$f(x) * g(x) = f(x^j - \frac{i}{2} \theta^{jk} \partial_k) g(x). \quad (3.7)$$

By making use of this expression, Eq. (3.6) becomes

$$\left( \partial^\mu \partial_\mu - (N - 1) - \lambda \exp \left( x^0 - \frac{i}{2} \theta \sum_{k \neq 0} \partial_k \right) \right) \psi(x) = 0. \quad (3.8)$$

We observe now that with the decomposition in plane waves:

$$\psi(x) = \exp (i p \cdot x) \phi(x^0), \quad (3.9)$$

the above equation actually separates, yielding

$$\left( \partial^2_{x^0} + \lambda e^{\theta^k_{\cdot} e^{x^0} + \omega^2} \right) \phi(x^0) = 0, \quad (3.10)$$

where $\omega^2 = p^2 + N - 1$ and $k = \sum_{j \neq 0} p_j$. In this way we are led to identify the induced hamiltonian due to noncommutativity as the operator

$$H_{rt} = \partial^2_{x^0} + \lambda e^{\theta^k_{\cdot} e^{x^0} + \omega^2}. \quad (3.11)$$

We now proceed to analyze $H_{rt}$ in usual terms. This problem was essentially solved in reference [55] (see also [56]) motivated by different reasons than the present one and for clarity of the exposition we will review the arguments there in some detail. The first thing we ought to state clear is the domain $D(H_{rt})$ of definition of this hamiltonian. The natural choice is to require $H_{rt}$ to be an operator defined on the Hilbert space of square integrable functions on the whole real line with the standard inner product of functions. Thus $x^0$ takes values on the interval $(-\infty, \infty)$ and the state vectors on this Hilbert space are functions $\psi$ such that

$$||\psi||^2 = \langle \psi | \psi \rangle = \int_{-\infty}^{\infty} \psi^* (x^0) \psi(x^0) dx^0 < \infty. \quad (3.12)$$

For future convenience we shall consider the more general class of hamiltonians

$$H = \partial^2_{x^0} + V(x^0), \quad (3.13)$$

where $V(x^0)$ is a real valued smooth function of $x^0$. We will see that for the potentials $V$ that we need to deal with the resultant hamiltonians defined with such a domain as
above are not self-adjoint and we will have to find appropriate self-adjoint extensions of
them. At this point it is useful to make the change of variable
\[ t = e^{x^0}, \]
upon which the hamiltonian reads
\[ H = t \frac{d}{dt} \frac{d}{dt} + V(t). \]  
(3.15)

Now the state vectors are defined on the interval \( t \in (0, \infty) \) with scalar product
\[ \langle \psi | \phi \rangle = \int_0^\infty \frac{dt}{t} \psi^*(t) \phi(t), \]  
(3.16)
where the functions have compact support on \((0, \infty)\). Note the induced measure on
the inner product, because of this, functions belonging to the domain of \( H \) must vanish
as \( t \) approaches to zero. Suppose that \( \psi, \phi \in D(H) \), since \( V \) is real-valued, with a
straightforward partial integration we can show that
\[ \langle \phi | H \psi \rangle = \langle H \phi | \psi \rangle \]  
(3.17)
From this expression we can find a suitable domain in which \( H \) becomes a symmetric
operator, namely
\[ D(H) := \left\{ \psi | \psi \in L^2(0, \infty), H \psi \in L^2(0, \infty), \lim_{t \to 0^+, \infty} t \frac{d\psi(t)}{dt} = 0 \right\}. \]  
(3.18)
On the other hand suppose that \( \psi \) belongs to the domain of the adjoint operator \( H^* \), then
for all \( \phi \in D(H) \) we have
\[ \langle t \frac{d}{dt} \frac{d}{dt} \phi + V(t) \phi | \psi \rangle = \langle \phi | H^* \psi \rangle. \]  
(3.19)
If we further assume that the functions in the domain of \( H^* \) are well behaved (finite) as \( t \)
approaches 0 or \( \infty \), then because of the properties of the functions that belong to \( D(H) \)
we must have
\[ \langle \phi | t \frac{d}{dt} \frac{d}{dt} \psi + V(t) \psi \rangle = \langle \phi | H^* \psi \rangle. \]  
(3.20)
Thus \( H^* \) as a differential operator acting on the functions of its domain has the same
form as \( H \). This is very useful in the computation of the deficiency indices \((n_+, n_-)\) of
the symmetric operator \( H \), which are the dimensions of the kernel \( \mathcal{K}_\pm \) of the respective
operators \((H^* \pm i)\). First note that \( H \) commutes with complex conjugation since \( V \) is
real, this tells us that its deficiency indices are equal \( n_+ = n_- := n \), and therefore \( H \) is
self-adjoint or admits self-adjoint extensions depending of whether or not \( n \) is equal to
zero.

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In order to calculate \( n \) we have to particularize the discussion to \( H_{rt} \) defined by Eq. (3.11). The solutions of the equations \( H_{rt} \psi = \pm i \psi \) are Bessel functions of the first kind of complex order, two independent solutions for each equation. However for each sign there is only one square integrable solution, this can be checked easily from the asymptotics of the Bessel functions. Thus the respective kernel subspaces are

\[
\mathcal{K}_+ = \left\{ \beta J_{2 \sqrt{1 - \omega^2}} \left[ 2 \sqrt{\lambda} e^{\frac{k_0}{4} t^2} \right] | \beta \in \mathbb{C} \right\}, \tag{3.21}
\]

\[
\mathcal{K}_- = \left\{ \beta J_{2 \sqrt{1 + \omega^2}} \left[ 2 \sqrt{\lambda} e^{\frac{k_0}{4} t^2} \right] | \beta \in \mathbb{C} \right\}, \tag{3.22}
\]

and therefore the deficiency indices of \( H_{rt} \) are \((1, 1)\), confirming that it is not self-adjoint.

To construct the self-adjoint extensions we need to find the partial isometries \( U \) from a set \( \mathcal{I}(U) \subseteq \mathcal{K}_+ \) into \( \mathcal{K}_- \). An operator \( U \) is an isometry if for all \( \psi \) belonging to the Hilbert space in question we have \( ||U\psi|| = ||\psi|| \). It is a partial isometry if it is an isometry when restricted to states not belonging to its kernel. To construct the possible operators \( U \) we first give the states that span \( \mathcal{K}_\pm \), to this end we make use of the relation

\[
\int_a^b \frac{dz}{z} J_k(z) J_l(z) = \frac{1}{k^2 - l^2} \left[ z \left( J_k(z) \frac{d}{dz} J_l(z) - J_l(z) \frac{d}{dz} J_k(z) \right) \right] |^b_a. \tag{3.23}
\]

For \( a \to 0 \) and \( b \to \infty \) we can profit from the asymptotics expressions for the Bessel functions

\[
J_k(z) \rightarrow_{z \to 0} \sqrt{\frac{2}{\pi z}} \cos \left[ z - \frac{\pi}{2} \left( k + \frac{1}{2} \right) \right], \tag{3.24}
\]

\[
J_k(z) \rightarrow_{z \to 0} \frac{1}{\Gamma(k+1)} \left( \frac{z}{2} \right)^k, \tag{3.25}
\]

\[
Y_k(z) \rightarrow_{z \to \infty} \sqrt{\frac{2}{\pi z}} \sin \left[ z - \frac{\pi}{2} \left( k + \frac{1}{2} \right) \right], \tag{3.26}
\]

\[
Y_k(z) \rightarrow_{z \to 0} - \frac{1}{\pi} \frac{\Gamma(k)}{\left( \frac{z}{2} \right)^k}, \quad \text{Re} k > 0 \tag{3.27}
\]

where \( k \in \mathbb{C}, \arg |z| < \pi \) and where we have written the expressions for the asymptotic behavior of the Bessel function of the second kind \( Y_k(z) \) for future use. Using this expressions we find

\[
\int_0^\infty \frac{dz}{z} J_{k^*(z)} J_l(z) = -\frac{2}{\pi} \frac{1}{(k^*+l)(k^*-l)} \sin \left[ \frac{\pi}{2} (k^*-l) \right], \tag{3.28}
\]

where we have used \( J_{k^*}(z) = J_{k^*}(z) \) for \( z \in \mathbb{R}^+ \). In particular for \( k = l \) we have

\[
||J_k(z)||^2 = \frac{1}{2\text{Re}(k)} \frac{\sinh(\pi\text{Im}(k))}{\pi\text{Im}(k)}. \tag{3.29}
\]

With this preliminaries we can write the normalized states \( \phi_{\pm} \) which span \( \mathcal{K}_\pm \) respectively

\[
\phi_+(t) = \mathcal{N} J_{2 \sqrt{1 - \omega^2}} \left[ 2 \sqrt{\lambda} e^{\frac{k_0}{4} t^2} \right] \tag{3.30}
\]
and
\[ \phi_-(t) = N J_{2\sqrt{i + \omega}} \left[ 2\sqrt{\lambda e^{\frac{it}{\lambda} t^2} t^2} \right], \]
where \( N \) is the normalization factor of both states since it turns out to be the same due to the fact that \( 2\sqrt{i + \omega} = (2\sqrt{i - \omega})^* \), namely
\[ N := \left( \text{Re}(2\sqrt{i - \omega}) \frac{\pi \text{Im}(2\sqrt{i - \omega})}{\sinh(\pi \text{Im}(2\sqrt{i - \omega}))} \right)^{1/2}. \] (3.32)

Now it is easy to see that the only isometry \( U \) between \( K_+ \) into \( K_- \) can be defined through its action over \( \phi_+ \) as \( U\phi_+ \rightarrow e^{2\pi i \nu} \phi_- \), with \( \nu \) a real parameter in the interval \([0, 1)\). In particular it is a partial isometry and since the extensions of operators are in one to one correspondence with partial isometries the general theory of linear operators asserts that the only extensions that \( H_{rt} \) does admit are then
\[ H_\nu = i \frac{d}{dt} t \frac{d}{dt} + \lambda e^{\frac{it}{\lambda} t} + \omega, \] (3.33)
with domain
\[ D(H_\nu) := \{ \phi + \beta \phi_+ + \beta e^{2\pi i \nu} \phi_- | \phi \in D(H_{rt}), \beta \in \mathbb{C} \}. \] (3.34)

Here \( D(H_{rt}) \) is the domain of the symmetric operator \( H_{rt} \) as defined in (3.18). Thus we have a one-parameter family of extensions of our original operator, labeled by the real number \( \nu \). Now, since the domain of the partial isometry \( U \) has been taken to be all of \( K_+ \), the deficiency indices of these extensions \( H_\nu \) are \( n_\pm(H_\nu) = n_\pm(H_{rt}) - \dim(K_+) = 0 \), which shows that we have achieved to construct a one-parameter family of self-adjoint extensions of \( H_{rt} \).

We are now ready to determine the eigenstates of our self-adjoint operator \( H_\nu \). Let us denote by \( \Omega \) the eigenvalues of \( H_\nu \)
\[ H_\nu \phi(t) = \Omega \phi(t). \] (3.35)
The solutions to this differential equation are Bessel functions \( J_{2\sqrt{\Omega - \omega}} \left[ 2\sqrt{\lambda e^{\frac{it}{\lambda} t^2} t^2} t^2 \right] \) and \( Y_{2\sqrt{\Omega - \omega}} \left[ 2\sqrt{\lambda e^{\frac{it}{\lambda} t^2} t^2} t^2 \right] \). However only the Bessel functions of the first kind are square integrable on the interval \((0, \infty)\), provided \( \Omega > \omega \), as can be checked explicitly from the asymptotic expressions above. Of these, only certain values of \( \Omega \) will render states that belong to the domain of \( H_\nu \). To determine these values we use the fact that it is necessary for \( \phi \) to fulfill the condition
\[ \langle H_\nu^* \phi | \psi \rangle = \langle \phi | H_\nu \psi \rangle \] (3.36)
for all \( \psi \in D(H_\nu) \). An equivalent form of this expression is given by
\[ \left[ t \psi \frac{d}{dt} \phi^* - t \phi^* \frac{d}{dt} \psi \right] \bigg|_0^\infty = 0. \] (3.37)
In particular for the linear combination $\psi = \phi_+ + e^{2\pi i\nu} \phi_-$ and using the asymptotic expressions for $J_k$ given as above we obtain the relation

$$e^{2\pi i\nu} \sin \left[ \pi \left( \sqrt{-i - \omega^2} - \sqrt{\Omega - \omega^2} \right) \right] + \sin \left[ \pi \left( \sqrt{i - \omega^2} - \sqrt{\Omega - \omega^2} \right) \right] = 0. \quad (3.38)$$

The real and imaginary parts of this expression are the same equation, solving for $\Omega$ we obtain

$$\sqrt{\Omega - \omega^2} = \kappa + n, \quad (3.39)$$

where $n$ is an integer and

$$\kappa = \text{Re}(\sqrt{i - \omega^2}) - \frac{1}{\pi} \tan^{-1} \left[ \frac{\cos(2\pi \nu) - 1}{\sin(2\pi \nu)} \tanh(\pi \text{Im}(\sqrt{i - \omega^2})) \right]. \quad (3.40)$$

In summary, the spectrum of $H_\nu$ exists for $\Omega > \omega^2$, it is discrete and the eigenvalues $\Omega(n, \nu)$ are parametrized by a real number $\nu \in [0, 1)$. The normalized eigenstates are

$$u_n(x^0) = (2(\kappa + n))^{1/2} J_{2(\kappa + n)} \left[ 2\sqrt{\lambda e^{\frac{1}{2}(x^0 + \frac{\kappa \theta}{2})} \right] , \quad (3.41)$$

where we have returned to the original coordinate $x^0$.

We see that the parameter $\theta$ induces a shift in the minisuperspace variable $x^0$ proportional to $k$. The full minisuperspace eigenfunctions are then $\psi_n(x) = \exp(\imath \mathbf{p} \cdot \mathbf{x}) u_n(x^0)$. In order to visualize more clearly the effect of noncommutativity, we construct wave-packets by summing over momentum states assuming that $k$ is, say $p_1$ and the rest of the $p_j$ vanish. Denoting by $\Psi(x^0, x^1)$ the full eigenfunctions under these assumptions, the formal structure of the wave-packet under consideration is

$$\Phi(x^0, x^1) = \int_{-\infty}^{\infty} dk e^{-\alpha(k - \beta)^2} \Psi(x^0, x^1), \quad (3.42)$$

where we have introduced a gaussian weighting function with parameters $\alpha$ and $\beta$. The integral is performed numerically keeping the order $2(\kappa + n)$ of the Bessel functions a positive real number, since $\kappa$ depends on $k$ through $\omega^2$ and we have to fulfill the condition $\Omega > \omega^2$. For simplicity we take $\nu = 0$, in such case $\kappa$ remains in the interval $(0, 1/\sqrt{2})$ as $k$ varies along the whole real line, as is easily seen from the expression \textsuperscript{3}. In figure 2 we show a comparison between the ordinary ($\theta = 0$) and the noncommutative case of the probability density $|\Phi|^2$ in the $(x^0, x^1)$ subspace. We observe an overall shift in the $x^0$ direction of the absolute maximum of the wave-packet, which is accompanied by a large enhancement of the height of the peak of the packet.

\textsuperscript{3}Here we are taking into account only the positive square root, and also we neglect the term $N - 1$ so that actually $\omega^2 = k^2$. 

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4 The S-brane

As mention in the introduction, the worldsheet action for the bouncing tachyon is given by equation (1.1). The corresponding minisuperspace noncommutative hamiltonian $H_{\text{bt}}$ is easily shown to be

$$ H_{\text{bt}} = \partial_{x^0}^2 + 2\lambda \cosh \left( x^0 + \frac{1}{2} \theta k \right) + \omega^2. \quad (4.1) $$

Since the formal arguments in the previous section apply straightforwardly to this hamiltonian, we shall be rather brief. Thus $H_{\text{bt}}$ becomes a symmetric operator if we choose for its domain of definition the following one

$$ D(H_{\text{bt}}) := \left\{ \psi | \psi \in L^2(-\infty, \infty), H\psi \in L^2(-\infty, \infty), \lim_{t \to -\infty, \infty} \frac{d\psi(x^0)}{dx^0} = 0 \right\}. \quad (4.2) $$

The adjoint operator $H_{\text{bt}}^*$ takes the same differential form as $H_{\text{bt}}$ and we now seek the kernel subspaces $\mathcal{K}_\pm$ of the corresponding operators ($H_{\text{bt}}^* \pm i$). The solutions of the equations $H_{\text{bt}}^* \psi = \pm i \psi$ are known as Mathieu functions $se_\nu(z, \lambda)$ and $ce_\nu(z, \lambda)$ of imaginary argument (sometimes called modified Mathieu functions). For each sign there are two linearly independent solutions, both of which are square integrable as we shall see later on. Therefore the relevant subspaces are

$$ \mathcal{K}_+ = \left\{ c_1 \phi_1^+(x^0) + c_2 \phi_2^+(x^0) | c_1, c_2 \in \mathbb{C} \right\}, \quad (4.3) $$

$$ \mathcal{K}_- = \left\{ c_1 \phi_1^-(x^0) + c_2 \phi_2^-(x^0) | c_1, c_2 \in \mathbb{C} \right\}. \quad (4.4) $$

where

$$ \phi_1^+(x^0) = se_{4(i-\omega^2)} \left( -\frac{1}{2} i \left( x^0 + \frac{1}{2} \theta k \right), 4\lambda \right), \quad (4.5) $$

$$ \phi_2^+(x^0) = ce_{4(i-\omega^2)} \left( -\frac{1}{2} i \left( x^0 + \frac{1}{2} \theta k \right), 4\lambda \right). \quad (4.6) $$

Figure 2. Wave packet for the half S-brane. Here we take $n = 0$ and the order of the eigenstates $2\kappa$ is kept a positive real number. The values of the remaining parameters are $\nu = 0$ and $\alpha = \beta = 1$. 
and
\[ \phi_1^-(x^0) = se_{4(-i-\omega^2)} \left( -\frac{1}{2}i \left( x^0 + \frac{1}{2} \theta k \right), 4\lambda \right), \] (4.7)
\[ \phi_2^-(x^0) = ce_{4(-i-\omega^2)} \left( -\frac{1}{2}i \left( x^0 + \frac{1}{2} \theta k \right), 4\lambda \right), \] (4.8)
with the resultant deficiency indices of \( H_{bt} \) being \((n_+, n_-) = (2, 2)\). From this we see that in order to construct self-adjoint extensions of the operator \( H_{bt} \) we need to find all the possible partial isometries from \( K_+ \) into \( K_- \) such that their domain of definition has dimension 2. We therefore choose the domain of definition \( I(U) \subseteq K_+ \) of a possible isometry \( U \) to be \( K_+ \) itself, then it is easy to see that we only have two possible families of partial isometries, which can be defined as
\[ U_1 \left( \frac{1}{\|\phi_1\|} \phi_1^+ \right) \rightarrow e^{2\pi i\nu_1} \left( \frac{1}{\|\phi_1\|} \phi_1^- \right), \] (4.9)
and
\[ U_2 \left( \frac{1}{\|\phi_2\|} \phi_2^+ \right) \rightarrow e^{2\pi i\nu_2} \left( \frac{1}{\|\phi_2\|} \phi_2^- \right), \] (4.10)
with \( \nu_1 \) and \( \nu_2 \) real parameters in the interval \([0, 1]\). Moreover, it turns out that the two resultant self-adjoint extensions of \( H_{bt} \) associated with \( U_1 \) and \( U_2 \) are really equivalent, since we can readily check that their domains of definition \( D(H_{U_1}) \) and \( D(H_{U_2}) \) are the same set. In this way, the only self-adjoint extension of \( H_{bt} \) is given by
\[ H_\nu = \partial_{x^0}^2 + 2\lambda \cosh \left( x^0 + \frac{1}{2} \theta k \right) + \omega^2, \] (4.11)
with domain
\[ D(H_\nu) := \left\{ \phi + c_1 \frac{1}{\|\phi_1\|} \phi_1^+ + c_2 \frac{1}{\|\phi_2\|} \phi_2^+ + e^{2\pi i\nu} \left( c_1 \frac{1}{\|\phi_1\|} \phi_1^- + c_2 \frac{1}{\|\phi_2\|} \phi_2^- \right) \mid \phi \in D(H_{bt}), c_1, c_2 \in \mathbb{C} \right\}. \] (4.12)

Now the two linearly independent solutions of the eigenvalue equation \( H_\nu \phi(x^0) = \Omega \phi(x^0) \) are \( se_{4(\Omega-\omega^2)} \left( -\frac{1}{2}i \left( x^0 + \frac{1}{2} \theta k \right), 4\lambda \right) \) and \( ce_{4(\Omega-\omega^2)} \left( -\frac{1}{2}i \left( x^0 + \frac{1}{2} \theta k \right), 4\lambda \right) \), of these, only those which satisfy the condition
\[ \left[ \psi \frac{d}{dx^0} \phi^* - \phi^* \frac{d}{dx^0} \psi \right] \bigg|_{-\infty}^{\infty} = 0, \] (4.13)
for all \( \psi \in D(H_\nu) \), will be elements of the domain of \( H_\nu \) and therefore eigenfunctions of it. At this point it is useful to obtain expressions for the asymptotic behavior of the
Mathieu functions as $x^0 \to \pm \infty$. A quick argument which we shall use was suggested in [12], we start by noticing that the operator (4.1) has the following asymptotics

$$H_{bt} \rightarrow_{x^0 \to \pm \infty} \partial_x^2 + \lambda e^{\pm \frac{i}{2} k} e^x + \omega^2$$  (4.14)

where

$$\xi := \begin{cases} 
  x^0, & \text{for } x^0 > 0 \\
  -x^0, & \text{for } x^0 < 0
\end{cases}$$  (4.15)

From this we see that the solutions of the differential operator $H_{bt}$ must have the same asymptotics as the solutions of the differential operators in the rhs of equation (4.14) as $\xi \to \infty$, in particular

$$se_{q^2} \left( -\frac{1}{2} i \left( x^0 + \frac{1}{2} \theta k \right), \lambda \right) \rightarrow_{x^0 \to \pm \infty} Y_q \left[ \sqrt{\lambda} e^{\pm \frac{i}{2} (\xi \pm \frac{1}{2} \theta k)} \right]$$  (4.16)

and

$$ce_{q^2} \left( -\frac{1}{2} i \left( x^0 + \frac{1}{2} \theta k \right), \lambda \right) \rightarrow_{x^0 \to \pm \infty} J_q \left[ \sqrt{\lambda} e^{\pm \frac{i}{2} (\xi \pm \frac{1}{2} \theta k)} \right],$$  (4.17)

where $\xi$ is defined as above. Now, for our purposes, it suffices to take

$$\psi = \phi_1^+ + \phi_2^+ + e^{2\pi i \nu} \left( \phi_1^- + \phi_2^- \right)$$  (4.18)

and

$$\phi = se_{4(\Omega - \omega^2)} \left( -\frac{1}{2} i \left( x^0 + \frac{1}{2} \theta k \right), 4\lambda \right) + ce_{4(\Omega - \omega^2)} \left( -\frac{1}{2} i \left( x^0 + \frac{1}{2} \theta k \right), 4\lambda \right)$$  (4.19)

and, upon substitution in (4.13) and making use of the above asymptotics together with the corresponding ones for the Bessel functions, we are able to show that condition (4.13) is satisfied automatically for any value of $\Omega$, which of course means that the spectrum of the s-brane is continuous.

Summing up, the full minisuperspace eigenfunctions of the s-brane are given by

$$\psi_{\Omega}(x) = \exp \left( i \mathbf{p} \cdot \mathbf{x} \right) se_{4(\Omega - \omega^2)} \left( -\frac{1}{2} i \left( x^0 + \frac{1}{2} \theta k \right), 4\lambda \right)$$  (4.20)

and

$$\psi_{\Omega}(x) = \exp \left( i \mathbf{p} \cdot \mathbf{x} \right) ce_{4(\Omega - \omega^2)} \left( -\frac{1}{2} i \left( x^0 + \frac{1}{2} \theta k \right), 4\lambda \right).$$  (4.21)

with $\Omega$ being the real eigenvalue.

### 5 Summary and Conclusions

In this paper we have studied some consequences of the noncommutative deformation of the minisuperspace approach of the CFT construction of S-branes describing the bouncing
and rolling tachyon. This discussion is done in the spirit of Ref. [48] where for the case of quantum cosmology an exact solution to the Moyal deformation of Wheeler-DeWitt equation for the Kantowski-Sachs metric was found.

Similarly to the quantum cosmology case, for the CFT approach of the bouncing and rolling tachyon we find exact solutions of the corresponding hamiltonian constraint given by the Klein-Gordon equation in the minisuperspace, written in terms of elementary functions with the argument modified by the noncommutativity parameter $\theta$. We found how this resulting effective equation for the corresponding energy spectrums have analytic exact solution without the need for an expansion in the noncommutativity parameter $\theta$. The resulting eigenstates of the corresponding models acquire a shift in the timelike parameter proportional to $\theta$. For the purpose of visualizing the effect of noncommutativity, wave packets with suitable weighing functions are constructed and analyzed for especial cases. In general the $\theta$-effect shows up as an enhancement of the localization in the timelike direction of the wave packet, apart from an overall shift.

We would like to mention that comparison with the results of the commutative case treated in reference [12] is not directly applicable, since the authors’ approach to the subject involves, in particular for the $s$-brane, imposing independent boundary conditions at the far past and at the far future, which traduces itself in having two independent parameters ($\nu_\pm$ instead of our $\nu$ in equation (4.12)). This in no way implies that our results in the $\theta \to 0$ limit are contradictory, rather they are complementary since it is well known that general symmetric operators can have several independent self-adjoint extensions, differing only in subtle changes in the physical system being described, and this usually has to do with the chosen boundary conditions. Our approach to the subject has been performed through the slightly more formal procedures of reference [55], and as it was to be expected, in the commutative limit our results agree with those there, where applicable.

In a different order of ideas, as is evident from the discussion of the half $s$-brane in the main body of the text, the effect of the introduction of the noncommutative parameter $\theta$ can be naively accounted for with the substitution $\lambda \to e^{\frac{i}{2}k\theta}\lambda$, in the commutative action, where $k$ is defined as in equation (3.10). Thus, in principle, we could try to analyze the known results of the commutative case, with the naive substitution above. The possibility of having a slight wider window of options due to the enlarged parameter space with the inclusion of $\theta$, is an interesting motivation for pursuing this idea. Finally, it would be interesting to give a derivation of the noncommutative minisuperspace from a CFT description of bouncing and rolling tachyon by including the background $B$-field term, in a similar spirit of how noncommutativity is usually derived from string theory.
Progress along these lines will be posted elsewhere.

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