Neutrino Energy Loss from the Plasma Process
at all Temperatures and Densities

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Abstract

We present a unified approach which is accurate at all temperatures and densities for calculating the energy loss from a stellar plasma due to the plasma process, the decay of photons and plasmons into neutrino pairs. To allow efficient numerical calculations, an analytic approximation to the dispersion equations for photons and plasmons is developed. It is correct to order $\alpha$ in the classical, degenerate, and relativistic limits for all momenta $k$ and is correct at small $k$ for all temperatures and electron densities. Within the same approximations, concise expressions are derived for the transverse, longitudinal, and axial vector components of the neutrino emissivity.
The emission of neutrinos can be an important energy loss mechanism for very hot or dense stars. The collective effects of the stellar plasma can significantly alter the production rate of neutrinos. The most dramatic example is the “plasma process”, the decay of photons and plasmons into neutrino pairs, a process that owes its very existence to plasma effects. It was pointed out by Adams, Ruderman, and Woo [1] in 1963 that the plasma process could be the dominant energy loss mechanism for very hot and dense stars. It was recently shown that the formulae for the plasma process that have been used in all previous work are inaccurate at relativistic temperatures and electron densities [2], underestimating the emissivity by a factor as large as 3.185. The purpose of this paper is to provide a unified treatment of the plasma process that is accurate at all temperatures and densities and also allows efficient numerical calculations. We introduce a simple analytic approximation to the dispersion equations for photons and plasmons which becomes exact in the classical limit, the degenerate limit, and the relativistic limit, and interpolates smoothly between these limits. Within the same approximation, we obtain simple expressions for the transverse, longitudinal, and axial vector components of the neutrino emissivity from the plasma process. The derivation of the analytic dispersion equations for photons and plasmons is presented in Appendix A. The effective neutrino-photon interaction that is responsible for the plasma process is discussed in Appendix B and the decay rate of a photon or plasmon into a neutrino pair is calculated in Appendix C.

Because a plasma contains mobile charged particles, an electromagnetic wave propagating through the plasma consists of coherent vibrations of both the electromagnetic field and the density of charged particles. These coherent vibrations behave qualitatively differently from electromagnetic waves in the vacuum in that there are longitudinal waves as well as transverse waves, and they propagate at less than the speed of light. The quantization of the electromagnetic waves in a plasma gives rise to a spin-1 particle with 1 longitudinal and 2 transverse spin polarizations. It is common in the literature to refer to all 3 polarization states as “plasmons”, to emphasize that their dispersion relations depend on the properties of the plasma. The longitudinal and transverse modes are then awkwardly labelled “longitudinal plasmons” and “transverse plasmons”. While the longitudinal mode owes its very existence to the plasma, the transverse mode simply has its dispersion relation at low frequencies modified by the plasma. For this reason and for the sake of concise terminology, the
label “plasmon” will be reserved in this paper for the longitudinal mode, and the transverse modes will be called “photons” whether they are propagating in a plasma or in the vacuum.

The dispersion relations for photons and for plasmons depend on the temperature $T$ and the net density $n_e$ of electrons minus positrons. The general expression for the net electron density as a function of $T$ and the electron chemical potential $\mu$ is

$$n_e(T, \mu) = \frac{1}{\pi^2} \int_0^\infty dp \, p^2 \left( n_F(E) - \bar{n}_F(E) \right) \cdot EB$$

Throughout this paper, we use units in which $\hbar = c = k_B = 1$. The integral in (1) is over the momentum $p$ of electrons or positrons, $E = \sqrt{p^2 + m_e^2}$ is their energy, and $m_e$ is the electron mass. The Fermi distributions for electrons and positrons respectively are

$$n_F(E) = \frac{1}{e^{(E-\mu)/T} + 1}$$

and

$$\bar{n}_F(E) = \frac{1}{e^{(E+\mu)/T} + 1} \cdot \tag{3}$$

Given the net electron density $n_e$, the chemical potential is determined by inverting (1) to obtain $\mu(T, n_e)$ as a function of $T$ and $n_e$.

In the vacuum, photons cannot decay into neutrino pairs because they are massless. Their dispersion relation is $\omega^2 = k^2$, and the phase space available for the decay is proportional to $\omega^2 - k^2 = 0$. The qualitative behavior of the dispersion relations in a plasma is shown in Figure 1. The upper and lower solid curves are the dispersion relations $\omega_t(k)$ for photons and $\omega_{\ell}(k)$ for plasmons at a temperature of $T = 10^{11}$ K = 8.6 MeV and an electron density corresponding to $\rho/\mu_e = 10^{12}$ g/cm$^3$. (The quantity $\rho/\mu_e$ is the mass density of protons in the plasma and is related to the net electron density $n_e$ by $\rho/\mu_e = m_p n_e$, where $m_p$ is the proton mass.) As $k \to 0$, the dispersion relations $\omega_t(k)$ and $\omega_{\ell}(k)$ both approach the plasma frequency $\omega_p$, which is given by

$$\omega_p^2 = \frac{4\alpha}{\pi} \int_0^\infty dp \, \frac{p^2}{E} \left( 1 - \frac{1}{3} v^2 \right) \left( n_F(E) + \bar{n}_F(E) \right) \cdot \tag{4}$$
where $v = p/E$ is the velocity of the electrons or positrons. At large $k$, the behavior of the photon dispersion relation is $\omega_t(k)^2 \to k^2 + m_t^2$, where $m_t$ is the transverse photon mass:

$$m_t^2 = \frac{4\alpha}{\pi} \int_0^\infty dp \frac{p^2}{E} \left( n_F(E) + \bar{n}_F(E) \right). \quad (5)$$

Comparison with (4) reveals that the transverse mass lies in the range $\omega_p \leq m_t \leq \sqrt{3/2} \omega_p$.

As $k$ increases, the dispersion relation $\omega_t(k)$ for plasmons eventually crosses the light cone $\omega = k$ at a point $k_{max}$ given by

$$k_{max}^2 = \frac{4\alpha}{\pi} \int_0^\infty dp \frac{p^2}{E} \left( \frac{1}{v} \log \frac{1 + v}{1 - v} - 1 \right) \left( n_F(E) + \bar{n}_F(E) \right). \quad (6)$$

It satisfies $\omega_p \leq k_{max} < \infty$ and represents the maximum momentum for which a plasmon can propagate. As discussed in Appendix A.2, the expressions (4), (5), and (6) for $\omega_p$, $m_t$, and $k_{max}$ are correct to first order in the electromagnetic fine structure constant $\alpha \simeq 1/137.036$.

Since the photon dispersion relation satisfies $\omega_t(k) > k$ for all $k$ and the plasmon dispersion relation satisfies $\omega_t(k) > k$ for $k < k_{max}$, both photons and plasmons can decay into neutrino pairs.

The production rate of neutrino pairs from the plasma process is sensitive to the precise form of the photon and plasmon dispersion relations. In 1961, Tsytovich [3] wrote down general integral equations for the dispersion relations which include the effects of electrons in the plasma to first order in $\alpha$. In their pioneering work on the plasma process in 1963, Adams, Ruderman, and Woo [1] repeated the general integral equations, but they used the following simple dispersion relations in their numerical work:

$$\omega_t(k)^2 = \omega_p^2 + k^2, \quad 0 \leq k < \infty, \quad (7)$$

$$\omega_t(k)^2 = \omega_p^2, \quad 0 \leq k < \omega_p. \quad (8)$$

At $k = 0$, these dispersion relations have the correct value $\omega_p$ at all temperatures and electron densities. At nonzero $k$, they are accurate only at temperatures and densities where the electrons are nonrelativistic. In 1967, Baudet, Petrosian, and Salpeter [4] improved on
the dispersion relations (7) and (8) by including the first relativistic c orrection:

\[ \omega_t^2 = \omega_p^2 + k^2 + \frac{\omega_1^2 k^2}{\omega_t} , \quad 0 \leq k < \infty , \]  
(9)

\[ \omega_\ell^2 = \omega_p^2 + \frac{3\omega_1^2 k^2}{5 \omega_\ell^2} , \quad 0 \leq k < \sqrt{\omega_p^2 + 3\omega_1^2/5} , \]  
(10)

where \( \omega_1 \) is given by the integral

\[ \omega_1^2 = \frac{4\alpha}{\pi} \int_0^\infty dp \, \frac{p^2}{E} \left( \frac{5}{3} v^2 - v^4 \right) \left( n_F(E) + \bar{n}_F(E) \right) . \]  
(11)

It lies in the range \( 0 \leq \omega_1 \leq \omega_p \). At small \( k \), the dispersion relations (9) and (10) have the correct behavior to order \( k^2 \) at all temperatures and densities, but they are not accurate at large \( k \) if electrons are relativistic.

In the subsequent 24 years, the dispersion relations (7) for photons and (10) for plasmons were used in all calculations of the plasma process [3, 4, 5]. It was pointed out by Braaten [2] in 1991 that they could lead to significant errors at temperatures or densities where electrons become relativistic. In the relativistic limit, the correct dispersion relations are the solutions to the following transcendental equations:

\[ \omega_t^2 = k^2 + \omega_p^2 \frac{3\omega_1^2}{2k^2} \left( 1 - \frac{\omega_t^2 - k^2 \omega_t}{\omega_t^2} \log \frac{\omega_t + k}{\omega_t - k} \right) , \quad 0 \leq k < \infty , \]  
(12)

\[ \omega_\ell^2 = \omega_p^2 \frac{3\omega_1^2}{k^2} \left( \frac{\omega_\ell^2}{2k} \log \frac{\omega_\ell + k}{\omega_\ell - k} - 1 \right) , \quad 0 \leq k < \infty . \]  
(13)

These dispersion relations were first derived by Silin in 1960 using kinetic theory [8], and were rederived in 1982 by Klimov and by Weldon using thermal field theory methods [9]. Expanding the right sides of (12) and (13) in powers of \( k \) and using the fact that \( \omega_1 = \omega_p \) in the relativistic limit, one finds that they agree with (9) and (10) to order \( k^2 \). They differ significantly at large \( k \). For example, the transverse mass in the relativistic limit is \( m_t = \sqrt{3/2} \omega_p \), while (9) gives the value \( \sqrt{6/5} \omega_p \). Also, the maximum plasmon momentum approaches infinity in the relativistic limit, while (10) gives a value of \( \sqrt{8/5} \omega_p \) for \( k_{max} \).
In a subsequent paper by Itoh et al. [10], the energy loss from the plasma process was calculated for a relativistic plasma using the dispersion relations for a degenerate plasma at zero temperature. The use of the zero-temperature dispersion relations is valid only if $T$ is negligible compared to the electron Fermi energy. Since this condition is not satisfied by the hottest stellar plasmas, the calculation of Ref. [10] can be improved by properly taking into account the effects of temperature on the dispersion relations. The general equations for the dispersion relations at zero temperature were obtained in closed form by Jancovici in 1962 [11] and used in the calculations of Ref. [10], but they are too lengthy to reproduce here.

Jancovici also gave simplified equations for the dispersion relations:

$$\omega^2 = k^2 + \omega_p^2 \frac{3\omega_l^2}{2v_F^2 k^2} \left( 1 - \frac{\omega^2 - v_F^2 k^2}{\omega_l^2} \frac{\omega_l + v_F k}{\omega_l - v_F k} \log \frac{\omega_l + v_F k}{\omega_l - v_F k} \right), \quad 0 \leq k < \infty , \quad (14)$$

$$\omega^2 = \omega_p^2 \frac{3\omega_l^2}{2v_F^2 k^2} \left( \frac{\omega_l}{2v_F k} \log \frac{\omega_l + v_F k}{\omega_l - v_F k} - 1 \right), \quad 0 \leq k < k_{\max} , \quad (15)$$

where $v_F = p_F/E_F$ is the Fermi velocity, $E_F$ is the Fermi energy, and $p_F = \sqrt{E_F^2 - m_e^2}$ is the Fermi momentum. The maximum plasmon momentum is

$$k_{\max} = \left[ \frac{3}{v_F^2} \left( \frac{1}{2v_F} \log \frac{1 + v_F}{1 - v_F} - 1 \right) \right]^{1/2} \omega_p . \quad (16)$$

The dispersion equations (14) and (15) have been rederived using thermal field theory methods [12]. In Ref. [11], these simple dispersion equations were only claimed to be valid for $k \ll p_F$ and $\omega \ll E_F$. However, as shown in Appendix A.2, they are in fact valid for all $k$. They take into account correctly all effects of electrons in the plasma to first order in $\alpha$. Thus they could have been used in the calculations of Ref. [10] without any loss of accuracy. Expanding the right side of (14) and (15) for small $k$ and using the fact that $\omega_1 = v_F \omega_p$ in the degenerate limit, we see that they agree with (9) and (10) to order $k^2$. In the relativistic limit where $v_F \to 1$, (14) and (15) reduce to the relativistic dispersion relations (12) and (13).

Aside from the numerical complications of solving integral equations, the general dispersion relations for photons and plasmons given in Ref. [3] also suffer from a technical
difficulty in that \( \omega_t(k) \) and \( \omega_\ell(k) \) become complex-valued if the temperature or the Fermi energy is sufficiently high. The imaginary parts appear when \( \omega_p \) exceeds the threshold \( 2m_e \) for decay of a photon into an electron-positron pair in the vacuum. This threshold is unphysical, because the rest mass of an electron in a relativistic plasma is significantly greater than in the vacuum. The effects of the plasma on the electron and photon dispersion relations are such that the decay of a photon or plasmon into an \( e^+e^- \) pair is always forbidden by energy and momentum conservation \[13\]. The problem of complex-valued dispersion relations persists in the degenerate limit, although the threshold is increased to \( \omega > E_F + m_e \), corresponding to the production of a positron at rest and an electron at the Fermi surface. The effects of the unphysical process \( \gamma \rightarrow e^+e^- \) may be numerically small, but it would be preferable to have dispersion relations from which they are absent altogether.

In Appendix A.2, we derive dispersion relations for photons and plasmons which receive no contributions from the unphysical process \( \gamma \rightarrow e^+e^- \). This is achieved without any sacrifice in accuracy: all effects from electrons and positrons in the plasma are still included to first order in \( \alpha \). The resulting expressions (4), (5), and (6) for the plasma frequency, the transverse photon mass, and the maximum plasmon momentum are much simpler than, but just as accurate as, the corresponding expressions that follow directly from the general integral equations. The dispersion relations remain real-valued at all temperatures and electron densities. Their behavior at small \( k \) is consistent with (9) and (10), they reduce to (12) and (13) in the relativistic limit, and they reduce to (14) and (15) in the degenerate limit.

The dispersion equations given in Appendix A.2 are integral equations involving 1-dimensional integrals over the momenta of electrons and positrons. Only in the classical limit, the degenerate limit, and the relativistic limit can the integrals be evaluated analytically. At intermediate temperatures and electron densities, calculating the dispersion relation \( \omega_t(k) \) or \( \omega_\ell(k) \) requires finding, for each value of \( k \), the zero in \( \omega \) of an \( \omega \)-dependent integral. The numerical solution of such a complicated dispersion equation is too inefficient for many applications, such as calculating the energy loss from the plasma process. We have therefore developed an approximation to these dispersion relations that is remarkably accurate at all temperatures and electron densities, but is simple enough to be used for prac-
tical calculations. The derivation of these dispersion relations is presented in Appendix A.4. They are correct to order $\alpha$ in the classical limit, the degenerate limit, and the relativistic limit, and they provide a smooth interpolation to intermediate temperatures and electron densities. They are also correct to order $k^2$ at small $k$ for all temperatures and electron densities.

Our approximate dispersion relations are as simple as the degenerate dispersion relations given in (14) and (15). Given the plasma frequency (4) and the frequency $\omega_1$ given by (11), we define a parameter $v_*$:

$$v_* = \frac{\omega_1}{\omega_p}.$$  

(17)

It lies in the range $0 \leq v_* \leq 1$, and has the intuitive interpretation of a typical velocity of electrons in the plasma. Our approximate dispersion relations $\omega_t(k)$ and $\omega_\ell(k)$ are obtained by solving the following equations which depend on $v_*$:

$$\omega_t^2 = k^2 + \omega_p^2 \frac{3\omega_t^2}{2v_*^2k^2} \left(1 - \frac{\omega_t^2 - v_*^2k^2}{\omega_t^2} \frac{\omega_t + v_*k}{\omega_t - v_*k} \log \frac{\omega_t + v_*k}{\omega_t - v_*k} \right), \quad 0 \leq k < \infty,$$  

(18)

$$\omega_\ell^2 = \omega_p^2 \frac{3\omega_\ell^2}{v_*^2k^2} \left(\frac{\omega_\ell + v_*k}{2v_*k} \log \frac{\omega_\ell + v_*k}{\omega_\ell - v_*k} - 1 \right), \quad 0 \leq k < k_{max}.$$  

(19)

The maximum plasmon momentum is

$$k_{max} = \left[\frac{3}{2v_*} \left(\frac{1}{2v_*} \log \frac{1 + v_*}{1 - v_*} - 1\right)\right]^{1/2} \omega_p.$$  

(20)

The transverse photon mass is

$$m_t = \left[\frac{3}{2v_*^2} \left(1 - \frac{1 - v_*^2}{2v_*} \log \frac{1 + v_*}{1 - v_*}\right)\right]^{1/2} \omega_p.$$  

(21)

These expressions satisfy $2m_t^2 + (1 - v_*^2)k_{max}^2 = 3\omega_p^2$. The dispersion equations (18) and (19) are correct to order $\alpha$ for all $k$ in three limiting cases that are studied in Appendix A.3. In the classical limit, (17) gives $v_* = \sqrt{5T/m_e}$. The dispersion equations (18) and (19), when expanded to first order in $v_*^2$, agree with (8) and (10). In the relativistic limit, $v_* = 1$ and
(18) and (19) reduce to the relativistic dispersion equations (12) and (13). In the degenerate limit, \( v_* \) is equal to the Fermi velocity \( v_F \) and the dispersion equations reduce to (14) and (15). Our approximate dispersion relations are also correct to order \( k^2 \) at all temperatures and electron densities. Expanding the right side of (19) and (18) in powers of \( k \), we find that to order \( k^2 \) they reduce to (9) and (10). At large \( k \), the dispersion equations (18) and (19) are correct to order \( \alpha \) only in the classical, degenerate, and relativistic limits. For example, the expressions (20) and (21) for \( k_{\text{max}} \) and \( m_t \) are not identically equal to the general expressions (6) and (5), but the differences are found empirically to be remarkably small.

To calculate the energy loss from the plasma process, we require the photon and plasmon dispersion relations \( \omega_t(k) \) and \( \omega_\ell(k) \), the corresponding residue factors \( Z_t(k) \) and \( Z_\ell(k) \), and the axial polarization function \( \Pi_A(\omega, k) \) evaluated at the photon dispersion relation \( \omega = \omega_t(k) \). General expressions for the residue factors and for \( \Pi_A(\omega, k) \) are given in Appendices A and B. The use of these general expressions in calculations such as the energy loss is very cumbersome. Fortunately, the methods used to derive the approximate dispersion relations (18) and (19) can also be used to derive compact analytic expressions for \( Z_t, Z_\ell, \) and \( \Pi(\omega_t, k) \) that are correct to order \( \alpha \) in the classical limit, the degenerate limit, and the relativistic limit and are also correct for small \( k \) at all temperatures and electron densities. As shown in Appendix A.4, the transverse and longitudinal residue factors can be approximated by

\[
Z_t(k) = \frac{2 \omega_t^2(\omega_t^2 - v_*^2 k^2)}{3 \omega_p^2 \omega_t^2 + (\omega_t^2 + k^2)(\omega_t^2 - v_*^2 k^2) - 2 \omega_t^2(\omega_t^2 - k^2)} ,
\]

(22)

\[
Z_\ell(k) = \frac{2(\omega_\ell^2 - v_*^2 k^2)}{3 \omega_p^2 - (\omega_\ell^2 - v_*^2 k^2)} .
\]

(23)

As shown in Appendix B, the axial polarization function \( \Pi(\omega, k) \) evaluated at \( \omega = \omega_t(k) \) can be approximated by

\[
\Pi_A(\omega_t, k) = \omega_A \frac{\omega_t^2 - k^2}{\omega_t^2 - v_*^2 k^2} \frac{3 \omega_p^2 - 2(\omega_t^2 - k^2)}{\omega_p^2} .
\]

(24)
Its behavior as $k \to 0$ is $\Pi_A(\omega_t, k) \to \omega_A k$, where the coefficient $\omega_A$ is

$$\omega_A = \frac{2\alpha}{\pi} \int_0^\infty dp \frac{p^2}{E^2} \left( 1 - \frac{2}{3} v^2 \right) \left( n_F(E) - \bar{n}_F(E) \right). \tag{25}$$

The emissivity $Q$ of a plasma is the rate of energy loss per unit volume. To calculate the emissivity from the plasma process, one must first calculate the rates $\Gamma_t(k)$ and $\Gamma_\ell(k)$ for the decay of a photon and a plasmon of momentum $k$ into a $\nu\bar{\nu}$ pair. These rates are calculated in Appendix C under the assumption that the number density of neutrinos and antineutrinos remains negligible. The emissivity is then obtained by integrating over the phase space of the photon or plasmon, weighted by the number density and by the energy. Summing over the 3 polarization states of photons and plasmons and over the neutrino types $\nu = \nu_e, \nu_\mu, \nu_\tau$, the total emissivity is

$$Q = \sum_\nu \int \frac{d^3k}{(2\pi)^3} \left( 2 n_B(\omega_t(k)) \omega_t(k) \Gamma_t(k) + n_B(\omega_\ell(k)) \omega_\ell(k) \Gamma_\ell(k) \right). \tag{26}$$

The number densities of photons and plasmons are given by the Bose distribution:

$$n_B(\omega) = \frac{1}{e^{\omega/T} - 1}. \tag{27}$$

The total emissivity $Q$ can be separated into the vector ($Q_T$) and axial vector ($Q_A$) components of the photon contribution and the plasmon contribution ($Q_L$):

$$Q_T = \sum_\nu C_V^2 \frac{G_F^2}{96\pi^4\alpha} \int_0^{k_{\text{max}}} dk \ k^2 \ Z_t(k) \left( \omega_t(k)^2 - k^2 \right)^3 n_B(\omega_t(k)),$$  \tag{28}

$$Q_A = \sum_\nu C_A^2 \frac{G_F^2}{96\pi^4\alpha} \int_0^{k_{\text{max}}} dk \ k^2 \ Z_\ell(k) \left( \omega_\ell(k)^2 - k^2 \right) \Pi_A(\omega_t(k), k) n_B(\omega_t(k)),$$  \tag{29}

$$Q_L = \sum_\nu C_V^2 \frac{G_F^2}{96\pi^4\alpha} \int_0^{k_{\text{max}}} dk \ k^2 \ Z_\ell(k) \left( \omega_\ell(k)^2 - k^2 \right)^2 n_B(\omega_\ell(k)). \tag{30}$$

The coefficients $C_V$ and $C_A$ depend on the neutrino type and are given in Appendix B. The combinations that arise in the plasma process are

$$\sum_\nu C_V^2 = \frac{3}{4} - 2 \sin^2 \theta_W + 12 \sin^4 \theta_W \approx 0.911 , \tag{31}$$
\[ \sum_{\nu} C_{\nu}^{2} = \frac{3}{4}. \]  

(32)

The formulas for \( Q_T \) and \( Q_A \) were first correctly given up to the factor of \( \sum C_{\nu}^{2} \) in Ref. [14]. The correct values of \( C_V \) were first included in Ref. [5]. The formula (30) for \( Q_L \) differs from that given in Refs. [2] and [10] because the definition of the longitudinal residue factor \( Z_\ell(k) \) differs by a factor of \( (\omega_\ell^2 - k^2)/\omega_\ell^2 \). The formula (29) for \( Q_A \) supercedes that given in Ref. [15], in which the factor \( Z_t(k) \) was omitted and the photon was assumed to satisfy the simple dispersion relation (7).

The important momentum scales in the emissivity integrals appearing in (28), (29), and (30) are the plasma frequency \( \omega_p \) (which enters into the dispersion relations, the residue factors, and \( \Pi_A(\omega_t, k) \)), and the temperature \( T \) (which occurs in the Bose distribution). The integrals can be simplified in the limiting cases \( T >> \omega_p \) and \( \omega_p << T \). We first consider the high temperature limit \( T >> \omega_p \). The integral (28) for the transverse emissivity is dominated by momenta \( k \) of order \( T \). Since \( T >> \omega_p \), the photon dispersion relation can be approximated by \( \omega_t^2 = k^2 + m_t^2 \). We can therefore replace the factors of \( \omega_t(k)^2 - k^2 \) in (28) by \( m_t^2 \), and set \( \omega_t = k \) everywhere else in the integrand. The integral over \( k \) can then be evaluated analytically in terms of the Riemann zeta function: \( \zeta_3 \simeq 1.202057 \). The result is

\[ Q_T \to \left( \sum_{\nu} C_{\nu}^{2} \right) \frac{G_F^2}{96\pi^4\alpha} \frac{4\zeta_3 m_t^6}{T^3}. \]  

(33)

The integral (29) for the axial vector emissivity is also dominated by momenta \( k \) of order \( T \). Applying similar approximations as were used for the transverse emissivity, it can also with some effort be evaluated analytically:

\[ Q_A \to \left( \sum_{\nu} C_{A\nu}^{2} \right) \frac{G_F^2}{96\pi^4\alpha} 2 \left[ \frac{3}{v_s^2} \left( \frac{1}{2} \log \frac{1 + v_s}{1 - v_s} - 1 \right) \right]^2 m_t^6 \omega_A^2 T \log \frac{2T}{m_t}. \]  

(34)

Up to logarithmic factors, the axial vector emissivity is suppressed relative to \( Q_T \) by a factor of \( \omega_A^2/T^2 \). In the limit \( T >> \omega_p \), the integral (30) for the longitudinal emissivity is dominated by momenta \( k \) of order \( \omega_p \). This is obvious in the nonrelativistic limit, because the upper limit \( k_{\max} \) is of order \( \omega_p \). In the relativistic limit where \( k_{\max} \) approaches infinity, the factors
\(Z_\ell(k)\) and \((\omega_\ell(k)^2 - k^2)^2\) both approach zero very rapidly for \(k >> \omega_p\), providing a cutoff on the integral of order \(\omega_p\). If \(k\) is of order \(\omega_p\), then \(\omega_\ell(k)\) is also of order \(\omega_p\). Using \(\omega_\ell(k) << T\), the Bose factor in (30) can be simplified to \(n_B(\omega_\ell) \rightarrow T/\omega_\ell\). The integral can still not be calculated analytically, but by dimensional analysis it must be proportional to \(\omega_p^8 T\). The longitudinal emissivity then has the form

\[
Q_L \rightarrow \left( \sum_{\nu} C^2_V \right) \frac{G_F^2}{96\pi^4\alpha} A(v_*) \omega_p^8 T .
\]  

(35)

The coefficient \(A(v_*)\) varies slowly from 8/105 in the nonrelativistic limit \((v_* \rightarrow 0)\) to 0.349 in the relativistic limit \((v_* \rightarrow 1)\) \[2\]. The longitudinal emissivity is suppressed relative to the transverse emissivity by a factor of \(\omega_p^2/T^2\).

We next consider the low temperature limit \(T << \omega_p\). The integrals in (28) and (29) are dominated by momenta \(k << \omega_p\). The Bose distribution can therefore be approximated by a Gaussian in \(k\):

\[
n_B(\omega_\ell(k)) \rightarrow e^{-\omega_p/T} \exp(-\omega''(0)k^2/2T), \quad \text{where} \quad \omega''(0) = (1 + v_2^2/5)/\omega_p.
\]

Everywhere else in the integrands, we can set \(\omega_\ell = \omega_p\) and ignore \(k\) relative to \(\omega_p\). The integrals can then be evaluated analytically, with the results

\[
Q_T \rightarrow \left( \sum_{\nu} C^2_V \right) \frac{G_F^2}{96\pi^4\alpha} \sqrt{2\pi} \left( 1 + \frac{1}{5}v_*^2 \right)^{-3/2} \omega_p^{15/2} T^{3/2} e^{-\omega_p/T} ,
\]

(36)

\[
Q_A \rightarrow \left( \sum_{\nu} C^2_A \right) \frac{G_F^2}{96\pi^4\alpha} \sqrt{6\pi} \left( 1 + \frac{1}{5}v_*^2 \right)^{-5/2} \omega_p^{9/2} \omega_A^{5/2} T^{5/2} e^{-\omega_p/T} .
\]

(37)

The axial vector emissivity is suppressed relative to \(Q_T\) by a factor of \(\omega_A^2 T/\omega_p^3\). A similar approximation can be applied to the longitudinal emissivity provided that \(T << v_*^2 \omega_p\). The Bose distribution in (30) can then be approximated by a Gaussian in \(k\): \(n_B(\omega_\ell(k)) \rightarrow e^{-\omega_p/T} \exp(-\omega''(0)k^2/2T), \) where \(\omega''(0) = (3/5)v_*^2/\omega_p\). The expression for the emissivity then reduces to

\[
Q_L \rightarrow \left( \sum_{\nu} C^2_V \right) \frac{G_F^2}{96\pi^4\alpha} \sqrt{\frac{\pi}{2}} \left( \frac{3}{5}v_*^2 \right)^{-3/2} \omega_p^{15/2} T^{3/2} e^{-\omega_p/T} .
\]

(38)

At relativistic electron densities \((v_* \rightarrow 1)\), the longitudinal emissivity is smaller than the transverse emissivity only by a factor of \(\sqrt{2}\)
In Figure 2, the components $Q_T$, $Q_A$, and $Q_L$ of the emissivity are shown as a function of the proton mass density $\rho/\mu_e = m_p n_e$ at a temperature $T = 10^{11}$ K = 8.6 MeV. Also shown as dotted lines are the corresponding emissivities calculated using the 0-temperature dispersion relations as in Ref. [10]. While they give the same results at high densities, there are significant discrepancies at the lowest densities considered in Figure 2. The shapes of the dispersion relations at $T = 0$ and $T = 10^{11}$ K are very similar, since the electrons are relativistic over the entire range of densities shown. The discrepancies in Figure 2 therefore arise primarily from the difference in the value of the plasma frequency. At the highest densities shown, $\omega_p$ is determined primarily by the Fermi energy, so the 0-temperature dispersion relations provide a good approximation. At the lowest densities shown, $\omega_p$ is determined primarily by the temperature. The emissivities $Q_T$ and $Q_L$ therefore become almost independent of the density, as can be seen in Figure 2.

The formulas for the dispersion relations and the neutrino emissivity that have been presented above were derived for a plasma of electrons and positrons in a uniform positively charged background that cancels the net charge of the electrons and positrons. In a stellar plasma, the cancelling charge is actually provided by protons and heavier ions. These charged particles will also give contributions both to the electromagnetic polarization functions and to the effective photon-neutrino interaction that add to those of the electrons and positrons. The effects of protons and heavier ions are negligible for stellar plasmas as long as they are nonrelativistic. At the highest densities shown in Figure 2, the protons are relativistic and their effects are significant. At $T = 10^{11}$ K and $\rho/\mu_e = 10^{14}$ g/cm$^3$, electrons and protons are degenerate with both having a Fermi momentum of 238 MeV. Their contributions to $\omega_p^2$ are proportional to their Fermi velocities, which is 1 for electrons and 0.246 for protons. Thus the protons increase the plasma frequency by about 12%. The resulting effect on the emissivity will be much larger, since it scales like a large power of the plasma frequency. The modifications to the formulas for the emissivities that are required in order to take into account the effects of protons are given in Appendix C.

We have developed a unified approach for calculating the neutrino energy loss from the plasma process at all temperatures and densities. We have introduced compact equations for the dispersion relations for photons and plasmons that are correct to order $\alpha$ for all $k$ in
the classical, degenerate, and relativistic limits and are also correct to order $k^2$ at small $k$ for all temperatures and electron densities. Compact expressions were also obtained for the other quantities that are required to calculate the transverse, longitudinal, and axial vector components of the neutrino emissivity. Most previous numerical studies of the energy loss from the plasma process have concentrated for simplicity on the case where the neutrino density remains negligible. In supernova explosions, neutrinos become trapped inside a neutrinosphere and the neutrino emissivity can be suppressed by Pauli blocking effects. Our approach should allow the efficient numerical investigation of such effects. It would also be useful to extend our approach to other neutrino emission processes, to axion emission, and to other particle physics processes that can play an important role in astrophysics over enormous ranges of temperatures and densities. The investigation of all of these processes would benefit from a unified treatment that remains accurate even under extreme variations of temperature and density.

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A Photon and plasmon dispersion relations

A.1 Dispersion relations to order $\alpha$

The effects of a plasma on the propagation of photons and plasmons is determined by the electromagnetic polarization tensor $\Pi^{\mu\nu}(K)$. If interactions with electrons and positrons are taken into account to leading order in the electromagnetic coupling constant $\alpha$, the polarization tensor is

$$\Pi^{\mu\nu}(K) = 16\pi\alpha \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E} \left( n_F(E) + \bar{n}_F(E) \right) \times \frac{P \cdot K (P^\mu K^\nu + K^\mu P^\nu) - K^2 P^\mu P^\nu - (P \cdot K)^2 g^{\mu\nu}}{(P \cdot K)^2 - (K^2)^2/4},$$

(39)

where $K^\mu = (\omega, \vec{k})$, $P^\mu = (E, \vec{p})$, $K^2 = \omega^2 - k^2$, and $P \cdot K = E\omega - \vec{p} \cdot \vec{k}$. The integral is over the momentum $\vec{p}$ of electrons and positrons with energy $E = \sqrt{p^2 + m_e^2}$. The polarization tensor satisfies $K_\mu \Pi^{\mu\nu}(K) = 0$, which is a consequence of gauge invariance. The transverse and longitudinal polarization functions $P_{i_t}(\omega, k)$ and $P_{i_\ell}(\omega, k)$ are

$$P_{i_t}(\omega, k) = \frac{1}{2} \left( \delta^{ij} - \hat{k}^i \hat{k}^j \right) \Pi^{ij}(\omega, \vec{k}),$$

(40)

$$P_{i_\ell}(\omega, k) = \Pi^{00}(\omega, \vec{k}).$$

(41)

They are related to the standard transverse and longitudinal dielectric functions $\epsilon_t$ and $\epsilon_\ell$ by $\epsilon_t = 1 - P_{i_t}/\omega^2$ and $\epsilon_\ell = 1 - P_{i_\ell}/k^2$.

In order to construct the effective propagator $D^{\mu\nu}(\omega, \vec{k})$ for the electromagnetic field, it is necessary to choose a gauge. The most convenient choice for treating plasma effects is the Coulomb gauge defined by $\vec{\nabla} \cdot \vec{A} = 0$. In Coulomb gauge, the nonzero components of the effective propagator are

$$D^{00}(\omega, \vec{k}) = \frac{1}{k^2 - P_{i_\ell}(\omega, k)},$$

(42)
\[ D^{ij}(\omega, \mathbf{k}) = \frac{1}{\omega^2 - k^2 - P_i(\omega, k)} \left( \delta^{ij} - \hat{k}^i \hat{k}^j \right). \] (43)

The dispersion relations \( \omega_\ell(k) \) for photons and \( \omega_\ell(k) \) for plasmons are the locations of the poles in the effective propagator:

\[ D^{00}(\omega, \mathbf{k}) \rightarrow \frac{\omega_\ell(k)^2}{k^2} \frac{Z_\ell(k)}{\omega^2 - \omega_\ell(k)^2} \text{ as } \omega \rightarrow \omega_\ell(k), \] (44)

\[ D^{ij}(\omega, \mathbf{k}) \rightarrow \frac{Z_\ell(k)}{\omega^2 - \omega_\ell(k)^2} \left( \delta^{ij} - \hat{k}^i \hat{k}^j \right) \text{ as } \omega \rightarrow \omega_\ell(k). \] (45)

The locations of the poles are independent of the choice of gauge \([16]\).

The equations (44) and (45) also define the residue functions \( Z_\ell(k) \) and \( Z_t(k) \), which determine the strength with which a plasmon or photon of momentum \( k \) couples to an electromagnetic current:

\[ Z_\ell(k) = \left[ 1 - \frac{\partial P_i}{\partial \omega^2}(\omega_\ell(k), k) \right]^{-1}, \] (46)

\[ Z_t(k) = \frac{k^2}{\omega_\ell(k)^2} \left[ - \frac{\partial P_i}{\partial \omega^2}(\omega_\ell(k), k) \right]^{-1}. \] (47)

The longitudinal residue \( Z_\ell(k) \) defined by (47) differs by a factor of \((\omega_\ell^2 - k^2)/\omega_\ell^2\) from that which was used in Refs. \([2]\) and \([10]\). The residue of a pole in \( \omega^2 \) of \( D^{\mu\nu}(\omega, k) \) can be identified as \( \epsilon^\mu(k) \epsilon^\nu(\bar{k})^\ast \), where \( \epsilon^\mu(k) \) is the polarization 4-vector for the appropriate propagating mode. These modes are conveniently labelled by the helicity \( \lambda \), which is the component of the angular momentum in the direction of \( \mathbf{k} \): \( \lambda = 0 \) for plasmons, \( \lambda = \pm 1 \) for photons. From (44) and (45), we identify the polarization 4-vectors to be

\[ \epsilon^\mu(\mathbf{k}, \lambda = 0) = \frac{\omega_\ell(k)}{k} \sqrt{Z_\ell(k)} (1, 0)^\mu, \] (48)

\[ \epsilon^\mu(\mathbf{k}, \lambda = \pm 1) = \sqrt{Z_\ell(k)} \left( 0, \bar{\epsilon}_{\pm}(\bar{k}) \right)^\mu, \] (49)

where \( \bar{\epsilon}_{\pm}(\bar{k}) \) and \( \bar{\epsilon}_{-}(\bar{k}) \) are orthogonal to \( \bar{k} \) and normalized so that \( \bar{\epsilon}_{\pm}(\bar{k}) \cdot \bar{\epsilon}_{\pm}(\bar{k})^\ast = 1 \). The polarization 4-vectors (48) and (49) satisfy the Coulomb gauge constraint

\[ \bar{k} \cdot \bar{\epsilon}(k, \lambda) = 0. \] (50)
They are used in Appendix C to calculate the decay rate of a photon or plasmon into neutrino pairs.

The dispersion relations $\omega_t(k)$ for photons and $\omega_\ell(k)$ for plasmons are the solutions to the equations

$$\omega_t(k)^2 = k^2 + P_i(\omega_t(k), k), \quad (51)$$

$$\omega_\ell(k)^2 = \frac{\omega_\ell(k)^2}{k^2} P_i(\omega_\ell(k), k). \quad (52)$$

If the complete order-$\alpha$ expression (39) for the polarization tensor is used to calculate these dispersion relations, they become complex-valued when the temperature or electron density is large enough that the plasma frequency $\omega_p$ exceeds $2m_e$. The imaginary part is proportional to the decay rate for a photon or plasmon into an electron-positron pair: $\gamma \rightarrow e^+e^-$. These imaginary parts are unphysical, because the plasma effects that give the photon a nontrivial dispersion relation also change the energy-momentum relation for the electron. These corrections are such that the decay $\gamma \rightarrow e^+e^-$ is always forbidden by energy and momentum conservation [13]. The unphysical effects from this forbidden process also reveal themselves in the analytic behavior of the real parts of the dispersion relations. Thus they can not be eliminated simply by defining the dispersion relations $\omega_t(k)$ and $\omega_\ell(k)$ to be the real parts of the poles in the effective propagator.

### A.2 Removing effects of electron pair production

To eliminate the unphysical effects of the forbidden decay $\gamma \rightarrow e^+e^-$, we exploit the fact that there is a separation of momentum scales between the particles whose propagation is significantly modified by the plasma and those that are responsible for the plasma corrections. For example, in the relativistic limit, the dominant contributions to the electromagnetic polarization tensor in (39) come from electrons and positrons with momentum $p$ of order the temperature $T$ or the Fermi momentum $p_F$, whichever is larger. However plasma corrections to the electron propagator are significant only for momenta that are smaller by a factor of $\sqrt{\alpha}$. In the nonrelativistic limit, plasma corrections to the electron propagator are always negligible. This justifies the use of the vacuum energy-momentum relation $E = \sqrt{p^2 + m_e^2}$.
in calculating $\Pi^{\mu\nu}$.

The separation of momentum scales also allows us to simplify the expression for the polarization tensor by dropping the term $(K^2)^2/4$ in the denominator. The physical interpretation is that this corresponds to calculating plasma corrections using the forward scattering amplitudes for electrons and positrons in the vacuum \[17\]. The mathematical approximation that is required is

$$|\omega^2 - k^2| << 2E|\omega - \vec{v} \cdot \vec{k}|,$$

(53)

where $\vec{v} = \vec{p}/E$ is the velocity of the electron or positron. We first discuss the case where $k$ is of order $\omega_p$. Since the dispersion relations $\omega_t$ and $\omega_\ell$ are then also of order $\omega_p$, the inequality (53) requires $\omega_p << E$. This is always satisfied in the nonrelativistic limit where $E$ is of order $m_e$. In the relativistic limit, the inequality $\omega_p << E$ fails to be satisfied only in a region of the electron or positron phase space whose contribution to the integral in (39) is down by a factor of $\omega_p^2/T^2$ in the high temperature limit and a factor of $\omega_p^2/E_F^2$ in the degenerate limit. In either case, the contribution is suppressed by a factor of $\alpha$. We next discuss the case in which $k$ is much greater than $\omega_p$. The solutions to the dispersion relations always have $\omega^2 - k^2$ of order $\omega_p^2$. Except in the relativistic limit, the inequality (53) then reduces to $\omega_p^2/k << E$, which is even more easily satisfied than the condition $\omega_p << E$ discussed above. In the relativistic limit, the inequality (53) may fail for electrons with velocity $\vec{v}$ within an angle of order $\omega_p/k$ of the momentum $\vec{k}$, but this region of phase space gives a contribution to the integral in (39) that is suppressed by at least a factor of $\alpha$. Thus the effects of the $(K^2)^2$ term in the denominator of (39) are always at most comparable to the order $\alpha^2$ corrections to the polarization tensor. It can therefore be dropped without any loss in accuracy.

Dropping the $(K^2)^2$ term in the denominator of (39) eliminates the effects of the unphysical process $\gamma \rightarrow e^+e^-$, so that the dispersion relations for photons and plasmons remain real-valued at all temperatures and densities. It also results in far simpler expressions for the transverse and longitudinal polarization functions defined in (40) and (41). The
angular integrals can be evaluated analytically, and the polarization functions reduce to

\[ P_{it}(\omega, k) = \frac{4\alpha}{\pi} \int_0^\infty dp \frac{p^2}{E} \left( \frac{\omega^2}{k^2} - \frac{\omega^2 - k^2}{k^2} \frac{\omega}{2vk} \log \frac{\omega + vk}{\omega - vk} \right) \left( n_F(E) + \bar{n}_F(E) \right), \quad (54) \]

\[ P_{it}(\omega, k) = \frac{4\alpha}{\pi} \int_0^\infty dp \frac{p^2}{E} \left( \frac{\omega}{vk} \log \frac{\omega + vk}{\omega - vk} - 1 - \frac{\omega^2 - k^2}{\omega^2 - v^2 k^2} \right) \left( n_F(E) + \bar{n}_F(E) \right). \quad (55) \]

Inserting (54) and (55) into (51) and (52) and solving these equations numerically, one finds that the dispersion relation \( \omega_t(k) \) remains real-valued for all \( k \) and \( \omega_t(k) \) remains real-valued for those values of \( k \) that satisfy \( \omega_t(k) > k \), corresponding to the timelike propagation of a plasmon. As \( k \to 0 \), both dispersion relations approach the plasma frequency, which is given by (4). At large \( k \), the behavior of the photon dispersion relation is governed by the transverse photon mass given by (5). The plasmon dispersion relation crosses the line \( \omega = k \) at the point \( k_{max} \) given by (6).

A.3 Limiting cases

The polarization functions (54) and (55) can be evaluated analytically in three limits: the classical limit, the degenerate limit, and the relativistic limit. This allows the equations (51) and (52) for the dispersion relations to be written in closed form. We also obtain analytic expressions for the plasma frequency (4) and the residue functions (46) and (47). For completeness, we also give in each case the solution of (1) for the chemical potential \( \mu \) as a function of the net electron density \( n_e \).

**Classical limit.** In the classical limit, the plasma is nonrelativistic \( (T << m_e) \) and nondegenerate \( (m_e - \mu >> T) \). The Fermi distribution for electrons can be approximated by the Boltzmann distribution \( e^{(\mu - E)/T} \), and contributions from positrons can be ignored. The net electron density, including the first correction proportional to \( T/m_e \), is

\[ n_e(T, \mu) = e^{(\mu - m_e)/T} \frac{1}{\sqrt{2\pi^3}} \left( m_e T \right)^{3/2} \left( 1 + \frac{15}{8} \frac{T}{m_e} \right). \quad (56) \]

This can be solved trivially for the chemical potential \( \mu(T, n_e) \) as a function of the temper-
ature and the net electron density. The plasma frequency $\omega_p$ is

$$\omega_p^2 = e^{(\mu-m_e)/T} \sqrt{\frac{8}{\pi}} \alpha (m_e T^3)^{1/2} \left(1 - \frac{5}{8} \frac{T}{m_e}\right). \quad (57)$$

Using (56) to eliminate $\mu$ in favor of the net electron density, the expression (57) reduces to

$$\omega_p^2 = \frac{4\alpha n_e}{m_e} \left(1 - \frac{5}{2} \frac{T}{m_e}\right). \quad (58)$$

The polarization functions, including the first correction proportional to $T/m_e$, are

$$\Pi_t(\omega, k) = \omega_p^2 \left(1 + \frac{k^2 T}{\omega^2 m_e}\right), \quad (59)$$

$$\Pi_\ell(\omega, k) = \omega_p^2 \left(\frac{k^2}{\omega^2} + 3 \frac{k^4 T}{\omega^4 m_e}\right). \quad (60)$$

The resulting dispersion equations are

$$\omega_t^2 = k^2 + \omega_p^2 \left(1 + \frac{k^2 T}{\omega_t^2 m_e}\right), \quad 0 \leq k < \infty, \quad (61)$$

$$\omega_\ell^2 = \omega_p^2 \left(1 + 3 \frac{k^2 T}{\omega_\ell^2 m_e}\right), \quad 0 \leq k < \sqrt{1 + 3T/m_e \omega_p}. \quad (62)$$

These are identical to the dispersion equations (9) and (10) with $\omega_1 = \sqrt{5T/m_e \omega_p}$. The transverse mass for the photon is $m_t = \sqrt{1 + T/m_e \omega_p}$. The residue factors are

$$Z_t(k) = 1 - \frac{\omega_p^2 T}{\omega_t^4 m_e}, \quad (63)$$

$$Z_\ell(k) = \frac{\omega_p^2}{\omega_\ell^2} \left(1 - 6 \frac{k^2 T}{\omega_\ell^4 m_e}\right). \quad (64)$$

**Degenerate limit.** The degenerate limit is the limit of low temperature $T << \mu - m_e$. In the limit $T = 0$, The Fermi distribution for electrons reduces to a step function: $n_F(E) = 1$ for $E < \mu$ and $n_F(E) = 0$ for $E > \mu$. The net electron density is

$$n_e(T = 0, \mu) = \frac{1}{3\pi^2} \left(\mu^2 - m_e^2\right)^{3/2}. \quad (65)$$
The chemical potential at $T = 0$ is called the Fermi energy: $\mu = E_F$. The solution to (65) for the Fermi momentum $p_F \equiv \sqrt{E_F^2 - m_e^2}$ as a function of $n_e$ is

$$p_F = \left(3\pi^2 n_e\right)^{1/3}.$$  

(66)

The plasma frequency $\omega_p$ reduces in the $T = 0$ limit to

$$\omega_p^2 = \frac{4\alpha}{3\pi} p_F^2 v_F,$$  

(67)

where the Fermi velocity is $v_F = p_F/E_F$. The polarization functions are

$$\Pi_t(\omega, k) = \omega^2 p_T^2 \frac{3\omega^2}{2v_F^2 k^2} \left(1 - \frac{\omega^2 - v_F^2 k^2}{\omega^2} \frac{\omega}{2v_F k} \log \omega + v_F k \right),$$  

(68)

$$\Pi_\ell(\omega, k) = \omega^2 p_F^2 \frac{3}{v_F^2} \left(\frac{\omega}{2v_F k} \log \frac{\omega + v_F k}{\omega - v_F k} - 1\right).$$  

(69)

The dispersion equations reduce to (14) and (15). The transverse photon mass is

$$m_t = \left[\frac{3}{2v_F^2} \left(1 - \frac{1 - v_F^2}{2v_F} \log \frac{1 + v_F}{1 - v_F}\right)\right]^{1/2} \omega_p.$$  

(70)

The inverses of the residue factors are

$$Z_t^{-1} = 1 - \frac{3}{2} \frac{\omega_p^2}{v_F^2 k^2} \left(\frac{3}{2} - \frac{3\omega_t^2 - v_F^2 k^2}{2\omega_t^2} \frac{\omega_t}{2v_F k} \log \frac{\omega_t + v_F k}{\omega_t - v_F k}\right),$$  

(71)

$$Z_\ell^{-1} = \frac{3}{2} \frac{\omega_p^2}{v_F^2 k^2} \left(\frac{\omega_\ell^2}{\omega_\ell^2 - v_F^2 k^2} - \frac{\omega_\ell}{2v_F k} \log \frac{\omega_\ell + v_F k}{\omega_\ell - v_F k}\right).$$  

(72)

The dispersion equations (14) and (15) can be used to eliminate the logarithms from (71) and (72). The resulting algebraic expressions for $Z_t$ and $Z_\ell$ have the form (72) and (23), with $v_* = v_F$.

**Relativistic limit.** The relativistic limit is the limit of either high temperature $T >> m_e$ or high density $\mu >> m_e$. In the limit $m_e = 0$, the net electron density is

$$n_e(T, \mu) = \frac{1}{3\pi^2} \mu \left(\mu^2 + \pi^2 T^2\right).$$  

(73)
The solution for the chemical potential as a function of \( n_e \) is

\[
\mu(T, n_e) = \left( \sqrt{\left( \frac{1}{2} p_F^2 \right)^2 + \left( \frac{\pi^2 T^2}{3} \right)^2} + \frac{1}{2} p_F^2 \right)^{1/3} - \left( \sqrt{\left( \frac{1}{2} p_F^2 \right)^2 + \left( \frac{\pi^2 T^2}{3} \right)^2} - \frac{1}{2} p_F^2 \right)^{1/3}.
\]

(74)

where \( p_F = (3\pi^2 n_e)^{1/3} \). The plasma frequency is

\[
\omega_p^2 = \frac{4\alpha}{3\pi} \left( \mu^2 + \frac{1}{3} \pi^2 T^2 \right).
\]

(75)

The polarization functions are

\[
\Pi_t(\omega, k) = \frac{3\omega_p^2}{2k^2} \left( 1 - \frac{\omega^2 - k^2}{\omega^2} \frac{\omega + k}{2k} \log \frac{\omega}{\omega - k} \right),
\]

(76)

\[
\Pi_\ell(\omega, k) = 3 \omega_p^2 \left( \frac{\omega}{2k} \log \frac{\omega + k}{\omega - k} - 1 \right).
\]

(77)

The dispersion equations reduce to (12) and (13). The maximum plasmon momentum is infinite and the transverse mass is \( m_t = \sqrt{3/2} \omega_p \). The inverses of the residue factors are

\[
Z_t^{-1} = 1 - \frac{3}{2} \frac{\omega_p^2}{k^2} \left( \frac{3}{2} - \frac{3\omega_p^2 - k^2}{2\omega_p^2} \frac{\omega_t + k}{2\omega_t - k} \log \frac{\omega_t}{\omega_t - k} \right),
\]

(78)

\[
Z_\ell^{-1} = \frac{3}{2} \frac{\omega_p^2}{k^2} \left( \frac{\omega_\ell^2}{\omega_\ell^2 - k^2} - \frac{\omega_\ell + k}{2k} \log \frac{\omega_\ell}{\omega_\ell - k} \right).
\]

(79)

The dispersion equations (12) and (13) can be used to eliminate the logarithms from (78) and (79). The resulting algebraic expressions for \( Z_t \) and \( Z_\ell \) have the form (22) and (23) with \( v_* = 1 \).

### A.4 Analytic Approximation

The dispersion relations (51) and (52) are integral equations and solving them numerically can be computationally time-intensive. It is therefore desirable to have analytic approximations to these equations. To derive such equations, we begin from the expressions (54) and
For the polarization functions and integrate by parts:

\[
P_{i\ell}(\omega, k) = -\frac{4\alpha}{\pi} \int_0^{\infty} dp \frac{p^3}{E} \left\{ \frac{1}{2v^2} \left( \frac{\omega^2 - \omega^2v^2k^2}{k^2} - \frac{\omega + vk}{2vk} \log\frac{\omega + vk}{\omega - vk} \right) \right\} \frac{d}{dp}\left( n_F(E) + \bar{n}_F(E) \right),
\]

(80)

\[
P_{i\ell}(\omega, k) = -\frac{4\alpha}{\pi} \int_0^{\infty} dp \frac{p^3}{E} \left\{ \frac{1}{v^2} \left( \frac{\omega}{2vk} \log\frac{\omega + vk}{\omega - vk} - 1 \right) \right\} \frac{d}{dp}\left( n_F(E) + \bar{n}_F(E) \right).
\]

(81)

If the integrals over the momentum \( p \) of the electrons and positrons were dominated by a single velocity \( v_* \), then the factors in the curly brackets could be evaluated at \( v = v_* \) and pulled outside the integral. This is in fact the case in the classical limit, the relativistic limit, and the degenerate limit. In the classical limit, all the electrons have velocities \( v \) near 0. In the relativistic limit, electrons and positrons all have velocities \( v_* = 1 \). In the degenerate limit, the factor \( dn_F/dp \) is sharply peaked at the Fermi velocity \( v_F \). After pulling the expressions in curly brackets out of the integrands in (80) and (81), the remaining integrals over \( p \) are proportional to the square of plasma frequency. This can be seen by using integration by parts on (4):

\[
\omega_p^2 = -\frac{4\alpha}{3\pi} \int_0^{\infty} dp \frac{p^3}{E} \frac{d}{dp}\left( n_F(E) + \bar{n}_F(E) \right).
\]

(82)

The resulting analytic expressions for the polarization functions are

\[
P_{i\ell}(\omega, k) = \omega_p^2 \frac{3}{2v_*^2} \left( \frac{\omega^2}{k^2} - \frac{\omega^2 - v^2k^2}{k^2} \frac{\omega}{2v_k} \log\frac{\omega + v_*k}{\omega - v_*k} \right),
\]

(83)

\[
P_{i\ell}(\omega, k) = \omega_p^2 \frac{3}{v_*^2} \left( \frac{\omega}{2v_*k} \log\frac{\omega + v_*k}{\omega - v_*k} - 1 \right).
\]

(84)

Inserting these expressions into (51) and (52), we obtain the dispersion equations (18) and (19).

For suitable choices of the velocity \( v_* \), the expressions (83) and (84) will be accurate in the classical, degenerate, and relativistic limits. The parameter \( v_* \) can also be chosen so that they are correct at small \( k \) for all temperatures and electron densities. At small \( k \), the
general expressions (54) and (55) for the polarization functions reduce to

\[ P_i(\omega, k) \approx \frac{4\alpha}{\pi} \int_0^\infty dp \frac{p^2}{E} \left\{ \left( 1 - \frac{v^2}{3} \right) + \left( \frac{v^2}{3} - \frac{v^4}{5} \right) \frac{k^2}{\omega^2} \right\} \left( n_F(E) + \bar{n}_F(E) \right) , \] (85)

\[ P_{ii}(\omega, k) \approx \frac{4\alpha}{\pi} \int_0^\infty dp \frac{p^2}{E} \left\{ \left( 1 - \frac{v^2}{3} \right) \frac{k^2}{\omega^2} + \left( v^2 - \frac{3v^4}{5} \right) \frac{k^4}{\omega^4} \right\} \left( n_F(E) + \bar{n}_F(E) \right) . \] (86)

Expanding out the analytic expression (83) for the transverse polarization function in powers of \( k \), we find that it agrees with (85) to order \( k^2 \) provided that we take \( v_* = \omega_1/\omega_p \), where \( \omega_1 \) is given in (11). We find also that the analytic expression (84) for the longitudinal polarization function agrees with (86) to order \( k^4 \) for the same value of \( v_* \).

We can also obtain compact analytic expressions for the residue factors \( Z_t \) and \( Z_\ell \). Inserting (83) and (84) into the formulas (46) and (47), we obtain

\[ Z_t^{-1} = 1 - \frac{3}{2} \frac{\omega_p^2}{v_s^2k^2} \left( 3 - \frac{3\omega_t^2 - v_s^2k^2}{2\omega_t^2} \frac{\omega_t}{2v_sk} \log \frac{\omega_t + v_sk}{\omega_t - v_sk} \right) , \] (87)

\[ Z_\ell^{-1} = \frac{3}{2} \frac{\omega_p^2}{v_s^2k^2} \left( \frac{\omega_\ell^2}{\omega_\ell^2 - v_s^2k^2} - \frac{\omega_\ell}{2v_sk} \log \frac{\omega_\ell + v_\ell k}{\omega_\ell - v_\ell k} \right) . \] (88)

The dispersion equations (18) and (19) can be used to eliminate the logarithms from (87) and (88), resulting in the simple algebraic expressions (22) and (23).

**B Effective photon-neutrino interaction**

The decay of a photon or plasmon into neutrino pairs proceeds through an effective photon-neutrino interaction. This effective interaction arises from the electromagnetic coupling of a photon to electrons or positrons in the plasma, together with the weak interaction coupling of the electron or positron to a neutrino pair. It can be summarized by an effective vertex \( \Gamma^{\alpha\mu} \) for the interaction of the photon field \( A_\mu \) with the neutrino current \( \bar{\nu}\gamma_\alpha(1 - \gamma_5)\nu \):

\[ \Gamma^{\alpha\mu}(\omega, \hat{k}) = \frac{G_F}{\sqrt{2}} \frac{1}{\sqrt{4\pi\alpha}} \left( C_V P_i(\omega, k) \left( \frac{1}{\omega} \frac{\hat{k}^\alpha}{k} \right) ^\alpha \left( 1, \frac{\omega}{k} \right)^\mu \right) + g^{ij} \left[ C_V P_i(\omega, k) \left( \delta^{ij} - \hat{k}^i\hat{k}^j \right) + C_A \Pi_A(\omega, k) \left( ei^{ijm}\hat{k}^m \right) \right] g_{jn} , \] (89)
where $K^\mu = (\omega, \vec{k})$ is the 4-momentum of the photon. This effective vertex satisfies the identity $\Gamma^{\alpha \mu} K_\mu = 0$, which guarantees the gauge invariance of the interaction. The functions $P_i(\omega, k)$ and $P_{i\ell}(\omega, k)$ in (89) are the transverse and longitudinal electromagnetic polarization functions defined in (40) and (41). The axial polarization function calculated to leading order in $\alpha$ is

$$
\Pi_A(\omega, k) = 8\pi \alpha \frac{K^2}{k} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E} \left( n_F(E) - \bar{n}_F(E) \right) \frac{P \cdot K \omega - K^2E}{(P \cdot K)^2 - (K^2)^2/4},
$$

(90)

where $K^2 = \omega^2 - k^2$ and $P \cdot K = E\omega - \vec{p} \cdot \vec{k}$. The coefficients $C_V$ and $C_A$ depend implicitly on the neutrino type. The vector coefficients are

$$
C_V = 2\sin^2 \theta_W + \frac{1}{2} \quad \text{for} \quad \nu_e,
$$

(91)

$$
C_V = 2\sin^2 \theta_W - \frac{1}{2} \quad \text{for} \quad \nu_\mu, \nu_\tau,
$$

(92)

where $\theta_W$ is the weak mixing angle: $\sin^2 \theta_W \simeq 0.226$. The axial vector coefficients are

$$
C_A = \frac{1}{2} \quad \text{for} \quad \nu_e,
$$

(93)

$$
C_A = -\frac{1}{2} \quad \text{for} \quad \nu_\mu, \nu_\tau.
$$

(94)

The axial polarization function (90) agrees with the expression used in Ref. [15], provided that the nonrelativistic dispersion relation (7) is used to set $K^2 = \omega^2_p$.

For frequencies $\omega$ greater than $2m_e$, the expression (90) for the axial polarization function has an imaginary part that arises from the production of $e^+e^-$ pairs. As discussed in Appendix A, this unphysical behavior can be eliminated without any loss of accuracy by dropping the term $(K^2)^2/4$ in the denominator of (90). The resulting expression for $\Pi_A(\omega, k)$, when evaluated on the photon dispersion relation $\omega = \omega_t(k)$, is still correct to leading order in $\alpha$. Having dropped the $(K^2)^2/4$ term in the denominator of (90), the axial polarization function reduces to

$$
\Pi_A(\omega, k) = 2\pi \frac{\omega^2 - k^2}{\omega} \int_0^\infty dp \frac{p^2}{E^2} \left( \log \frac{\omega + vk}{\omega - vk} - \frac{\omega^2 - k^2}{\omega^2 - v^2k^2} \right) \left( n_F(E) - \bar{n}_F(E) \right).
$$

(95)
Its behavior for small $k$ is $\Pi_A(\omega, k) \to \omega_A k$, where $\omega_A$ is given in (25).

The expression (95) for the axial polarization function can be evaluated analytically in the classical limit, the degenerate limit, and the relativistic limit:

**Classical limit.** At leading order in $T/m_e$, the axial polarization function reduces to

$$\Pi_A(\omega, k) = e^{(\mu-m_e)/T} \sqrt{\frac{2}{\pi}} \alpha \left( \frac{T^3}{m_e} \right)^{1/2} \frac{k(\omega^2 - k^2)}{\omega^2}. \tag{96}$$

Eliminating the chemical potential and using the expression (57) for the plasma frequency in the classical limit, (96) simplifies further to

$$\Pi_A(\omega, k) = \frac{\omega_p^2}{2m_e} \frac{k(\omega^2 - k^2)}{\omega^2}. \tag{97}$$

**Degenerate limit.** At $T = 0$, the axial polarization function reduces to

$$\Pi_A(\omega, k) = \frac{2\alpha}{\pi} p_F \frac{\omega^2 - k^2}{k} \left( \frac{\omega + v_F k}{2v_F k} \log \frac{\omega + v_F k}{\omega - v_F k} - 1 \right). \tag{98}$$

After setting $\omega = \omega_t(k)$, the logarithm can be eliminated using the dispersion relation (14). The resulting expression for $\Pi_A(\omega_t, k)$ is (24) with $v_* = v_F$ and $\omega_A = (2\alpha/3\pi)p_F v_F^2$.

**Relativistic limit.** For $m_e = 0$, the axial polarization function reduces to

$$\Pi_A(\omega, k) = \frac{2\alpha}{\pi} \mu \frac{\omega^2 - k^2}{k} \left( \frac{\omega + k}{2k} \log \frac{\omega + k}{\omega - k} - 1 \right). \tag{99}$$

After setting $\omega = \omega_t(k)$, the logarithm can be eliminated using the dispersion relation (12). The resulting expression for $\Pi_A(\omega_t, k)$ is (24) with $v_* = 1$ and $\omega_A = (2\alpha/3\pi)\mu$.

We can derive an analytic approximation to the axial polarization function using the same methods that were applied to the transverse and longitudinal polarization functions in Appendix A. Starting with the expression (95) and integrating by parts, we obtain

$$\Pi_A(\omega, k) = -\frac{2\alpha}{\pi} \frac{\omega^2 - k^2}{k} \int_0^\infty dp \frac{p^3}{E^2} \left\{ \frac{1}{v^2} \left( \frac{\omega + v k}{2v k} \log \frac{\omega + v k}{\omega - v k} - 1 \right) \right\} \frac{d}{dp} \left( n_F(E) - \tilde{n}_F(E) \right). \tag{100}$$
In the classical, degenerate, and relativistic limits, the integral is dominated by a single momentum $v_\ast$. The factor in curly brackets can therefore be evaluated at $v_\ast$ and pulled outside the integral. The remaining integral over $p$ is proportional to $\omega_A$, which after applying integration by parts to the expression in (25) can be written in the form

$$\omega_A = -\frac{2\alpha}{3\pi} \int_0^\infty dp \frac{p^3}{E^2} \frac{d}{dp} \left( n_F(E) - n_{\bar{F}}(E) \right). \tag{101}$$

Our final expression for the axial polarization function is

$$\Pi_A(\omega, k) = \omega_A \omega^2 - k^2 \frac{3}{v_\ast^2} \frac{\log(\omega + v_\ast k)}{\omega - v_\ast k} - 1. \tag{102}$$

This expression is not only correct for all $k$ in the classical, degenerate, and relativistic limits, but it has the correct behavior for small $k$ at all temperatures and electron densities. The axial polarization function $\Pi_A(\omega_I, k)$ evaluated at the photon dispersion relation can be further simplified by using the dispersion relation (19) to eliminate the logarithm in (102), resulting in the analytic expression given in (24).

C Decay rate of photon or plasmon

In this Appendix, we calculate the rates $\Gamma_t(k)$ and $\Gamma_l(k)$ for a photon or plasmon of momentum $k$ to decay into neutrino pairs. We begin by writing down the matrix element $\mathcal{M}$ for the decay into a neutrino and antineutrino of momenta $\vec{p}_1$ and $\vec{p}_2$:

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} \left( \Gamma^{\alpha\mu} \varepsilon_\mu(\vec{k}, \lambda) \right) \bar{u}(\vec{p}_1) \gamma_\alpha(1 - \gamma_5) v(\vec{p}_2), \tag{103}$$

where $\varepsilon_\mu(\vec{k}, \lambda)$ is the polarization 4-vector given in (49) for photons or (48) for plasmons, $\Gamma^{\alpha\mu}$ is the effective photon-neutrino vertex given in (89), and $\bar{u}(\vec{p}_1)$ and $v(\vec{p}_2)$ are the spinors for the neutrino and antineutrino. The matrix element must be squared and integrated over the phase space of the neutrino and antineutrino:

$$\Gamma_\lambda(k) = \frac{1}{2\omega_\lambda(k)} \int \frac{d^3p_1}{(2\pi)^3} \frac{1}{2p_1} \int \frac{d^3p_2}{(2\pi)^3} \frac{1}{2p_2} (2\pi)^4 \delta^4(P_1 + P_2 - K) |\mathcal{M}|^2. \tag{104}$$
The subscript $\lambda$ on $\Gamma$ and on $\omega$ is either $t$, corresponding to helicity $\lambda = \pm 1$, or $\ell$, corresponding to helicity $\lambda = 0$. In the delta function, $P_1 = (p_1, \vec{p}_1)$, $P_2 = (p_2, \vec{p}_2)$, and $K = (\omega_\lambda(k), \vec{k})$ are the 4-momenta of the neutrino, antineutrino, and photon or plasmon, respectively. In the expression (104), we have assumed for simplicity that the number density of neutrinos and antineutrinos is negligible. Otherwise the phase space integrals in (104) must be weighted by appropriate Pauli blocking factors. The only dependence in the matrix element (103) on the momenta $\vec{p}_1$ and $\vec{p}_2$ is in the spinor factor. After multiplying the spinor factor by its complex conjugate, it can be expressed in the form of a Lorentz tensor:

$$\bar{u}(\vec{p}_1)\gamma^\alpha(1 - \gamma_5)v(\vec{p}_2)\bar{v}(\vec{p}_2)\gamma^\beta(1 - \gamma_5)u(\vec{p}_1) = 8 \left( P_1^\alpha P_2^\beta + P_2^\alpha P_1^\beta - P_1 \cdot P_2 g^{\alpha\beta} - i\epsilon^{\alpha\beta\mu\nu} P_1^\mu P_2^\nu \right).$$  \hspace{1cm} (105)

In the absence of Pauli blocking factors, the integral over the phase space of the neutrino and antineutrino in (104) can be carried out analytically, with the result

$$\int \frac{d^3p_1}{(2\pi)^3} \frac{1}{2p_1} \int \frac{d^3p_2}{(2\pi)^3} \frac{1}{2p_2} (2\pi)^4 \delta^4(P_1 + P_2 - K) \bar{u}(\vec{p}_1)\gamma^\alpha(1 - \gamma_5)v(\vec{p}_2)\bar{v}(\vec{p}_2)\gamma^\beta(1 - \gamma_5)u(\vec{p}_1) = \frac{1}{3\pi} \left( K^\alpha K^\beta - K^2 g^{\alpha\beta} \right), \hspace{1cm} (106)$$

where $K^2 = w^2_\lambda - k^2$. Since the effective vertex $\Gamma^{\alpha\mu}$ satisfies the identity $K^\alpha \Gamma^{\alpha\mu} = 0$, only the $K^2 g^{\alpha\beta}$ term in (106) contributes to the decay rate (104). The decay rate then reduces to

$$\Gamma_\lambda(k) = - \frac{G_F^2}{12\pi} \frac{\omega_\lambda(k)^2 - k^2}{\omega_\lambda(k)} \left( \Gamma^{\alpha\mu} \epsilon_\mu(\vec{k}, \lambda) \right) \left( \Gamma_{\alpha\beta} \epsilon^\beta(\vec{k}, \lambda) \right)^*.$$

(107)

To complete the calculation of the decay rate, the effective vertex $\Gamma^{\alpha\mu}$ in (89) must be contracted with the appropriate polarization 4-vector $\epsilon^\mu(\vec{k}, \lambda)$ and the energy $\omega$ must be evaluated at the corresponding dispersion relation $\omega_\ell(k)$ or $\omega_\ell(k)$. For the plasmon, only the first term in (89) contributes. By the plasmon dispersion equation (52), $\Pi_\ell(\omega, k)$ can be replaced by $k^2$ and the contraction of $\Gamma^{\alpha\mu}$ with the polarization vector (48) reduces to

$$\Gamma^{\alpha\mu}(\omega_\ell(k), \vec{k}) \epsilon_\mu(\vec{k}, 0) = C_V \frac{G_F}{\sqrt{8\pi\alpha}} \sqrt{Z_\ell(k)} \frac{\omega_\ell(k)}{k} \left( 1, \frac{\omega}{k} \right)^\alpha.$$

(108)
For the photon, it is the last term in the effective vertex (89) that contributes. By the photon dispersion equation (51), \( \Pi_t(\omega, k) \) can be replaced by \( \omega_t(k)^2 - k^2 \) and the contraction of \( \Gamma^{\alpha\mu} \) with the polarization vector (49) reduces to

\[
\Gamma^{\alpha\mu}(\omega_t(k), \vec{k}) \epsilon_\mu(\vec{k}, \pm 1) = \frac{G_F}{\sqrt{8\pi\alpha}} \sqrt{Z_t(k)} \left( C_V \left( \omega_t(k)^2 - k^2 \right) \left( 0, \vec{e}_\pm(\vec{k}) \right)^\alpha - C_A \Pi_A(\omega_t(k), k) \left( 0, i\vec{k} \times \vec{e}_\pm(\vec{k}) \right)^\alpha \right). \tag{109}
\]

Squaring the expression (109) and (108) and inserting into (107), the decay rates of the photon and plasmon into neutrino pairs reduce to

\[
\Gamma_t(k) = \frac{G_F^2}{48\pi^2\alpha} Z_t(k) \frac{\omega_t(k)^2 - k^2}{\omega_t(k)} \left( C_V^2 \left( \omega_t(k)^2 - k^2 \right)^2 + C_A^2 \Pi_A(\omega_t(k), k)^2 \right). \tag{110}
\]

\[
\Gamma_\ell(k) = C_V^2 \frac{G_F^2}{48\pi^2\alpha} Z_\ell(k) \omega_\ell(k) \left( \omega_\ell(k)^2 - k^2 \right)^2. \tag{111}
\]

The expressions (111) and (110) for the decay rates of photons and plasmons include only the effects of electrons and positrons in the plasma. It is straightforward to include also the effects of protons provided that the plasma frequency remains small compared to 700 MeV, so that form factor effects can be neglected. The effects of protons on the dispersion relations and on the effective neutrino-photon vertex must both be included. The expressions (110) and (111) for the decay rates are replaced by

\[
\Gamma_t(k) = \frac{G_F^2}{48\pi^2\alpha} Z_t \frac{\omega_t^2 - k^2}{\omega_t} \left( C_V P_{t_t}^{(e)}(\omega_t, k) - h_V P_{t_t}^{(p)}(\omega_t, k) \right)^2 \right) + \left( C_A \Pi_A^{(e)}(\omega_t, k) - h_A \Pi_A^{(p)}(\omega_t, k) \right)^2 \right) \right), \tag{112}
\]

\[
\Gamma_\ell(k) = \frac{G_F^2}{48\pi^2\alpha} Z_\ell \frac{\omega_\ell^2 - k^2}{k^4} \left( C_V P_{t_\ell}^{(e)}(\omega_\ell, k) - h_V P_{t_\ell}^{(p)}(\omega_\ell, k) \right)^2, \tag{113}
\]

where \( P_{t_t}^{(e)} \), \( P_{t_\ell}^{(e)} \), and \( \Pi_A^{(e)} \) are the contributions to the transverse, longitudinal, and axial polarization functions from electrons, while \( P_{t_t}^{(p)} \), \( P_{t_\ell}^{(p)} \), and \( \Pi_A^{(p)} \) are the corresponding
contributions from protons. The proton terms have the same form as the electron terms (83), (84), and (102), except that the parameters $\omega_p$, $v^* = \omega_1/\omega_p$, and $\omega_A$ are calculated using the integrals (4), (11), and (25) with the electron mass $m_e$ replaced by the proton mass $m_p$. The coefficients $h_V$ and $h_A$ that describe the interactions of the proton with neutrinos are

\[
h_V = -2\sin^2\theta_W + \frac{1}{2} \quad \text{for } \nu_e, \nu_\mu, \nu_\tau,
\]

\[
h_V = \frac{1}{2}g_A \quad \text{for } \nu_e, \nu_\mu, \nu_\tau,
\]

where $g_A \approx 1.26$. The dispersion equations (51) and (52) must also be modified to include the effects of protons:

\[
\omega_t^2 = k^2 + P_i^{(e)}(\omega_t, k) + P_i^{(p)}(\omega_t, k),
\]

\[
\omega_\ell^2 = \frac{\omega_t^2}{k^2} \left( P_i^{(e)}(\omega_\ell, k) + P_i^{(p)}(\omega_\ell, k) \right).
\]

The residue factors $Z_t$ in (112) and $Z_\ell$ in (113) are then given by the expressions (87) and (88), except that in addition to the electron term on the right side, there is also a proton term of the same form but with appropriate values for the parameters $\omega_p$ and $v^*$. 

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References

[1] J.B. Adams, M.A. Ruderman, and C.-H. Woo, *Phys. Rev.* **129**, 1383 (1963); C.L. Inman and M.A. Ruderman, *Phys. Rev.* **140**, 1025 (1965).

[2] E. Braaten, *Phys. Rev. Lett.* **66**, 1655 (1991).

[3] V.N. Tsytovich, *Sov. Phys. JETP* **13**, 1249 (1961).

[4] G. Baudet, V. Petrosian, and E.E. Salpeter, *Astrophys. J.* **150**, 979 (1967).

[5] D.A. Dicus, *Phys. Rev. D* **6**, 941 (1972).

[6] H. Munakata, Y. Kohyama, and N. Itoh, *Astrophys. J.* **296**, 197 (1986); N. Itoh, T. Adachi, M. Nakagawa, Y. Kohyama, and H. Munakata, *Astrophys. J.* **339**, 354 (1989); (erratum in *Astrophys. J.* **360**, 741 (1990)).

[7] P.J. Schinder, D.N. Schramm, P.J. Wiita, S.H. Margolis, and D.L. Tubbs, *Astrophys. J.* **313**, 531 (1987).

[8] V.P. Silin, *Sov. Phys. JETP* **11**, 1136 (1960).

[9] V.V. Klimov, *Sov. Phys. JETP* **55**, 199 (1982); H.A. Weldon, *Phys. Rev. D* **26**, 1394 (1982).

[10] N. Itoh, H. Mutoh, A. Hikjita, and Y. Kohyama, *Astrophys. J.* **395**, 622 (1992).

[11] B. Jancovici, *Nuovo Cim.* **25**, 428 (1962).

[12] T. Altherr, E. Petitgirard, and T. del Rio Gaztelurrutia, Annecy preprint ENSLAPP-A-412/92 (December 1992).

[13] E. Braaten, *Astrophys. J.* **392**, 70 (1992).

[14] V.N. Tsytovich, *Sov. Phys. JETP* **18**, 816 (1964); M.H. Zaidi, *Nuovo Cim.* **40**, 502 (1965).

[15] Y. Kohyama, N. Itoh, and H. Munakata, *Astrophys. J.* **310**, 815 (1986).
[16] R. Kobes, G. Kunstatter, and A. Rebhan, Phys. Rev. Lett. 64, 2992 (1990).

[17] G. Barton, Ann. Phys. (N.Y.) 200, 271 (1990); E. Braaten, Northwestern preprint NUHEP-TH-92-22 (October 1992).
Figure Captions

1. Dispersion relations $\omega(k)$ for photons (upper solid curve) and plasmons (lower solid curve) at temperature $T = 10^{11}$ K and proton mass density $\rho/\mu_e = 10^{12}$ g/cm$^3$.

2. Transverse (T), longitudinal (L), and axial vector (A) components of the neutrino emissivity (in units of erg/s/cm$^3$) as a function of the proton mass density $\rho/\mu_e$ (in units of g/cm$^3$) at the temperature $T = 10^{11}$ K (solid curves). Also shown are the corresponding emissivities calculated with the 0-temperature dispersion relations (dashed curves).