On the Escape of a Random Walk
From Two Pieces of a Tripartite Set

Michael Carlisle
Baruch College, CUNY

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Abstract

Let \( \{ A, B, C \} \) be a partition of a sample space \( \Omega \). For a random walk \( S_n = x + \sum_{j=1}^{n} X_j \) starting at \( x \in A \), we find estimates for the Green’s function \( G_{A \cup B}(x, y) \) and the hitting time \( E^x(T_C) \) for \( x, y \in A \cup B \), with interest in the case where \( C \) “separates” \( A \) and \( B \) in a sense (e.g., the probability of jumping from \( A \) to \( B \), or vice versa, before hitting \( C \), is small).

1 Green’s functions

Let \( S_n := x + \sum_{j=1}^{n} X_j \) be a random walk starting at \( x \) on a partitioned sample space \( \Omega = A \sqcup B \sqcup C \), i.e., for any \( x, y \in \Omega \), the one-step transition probability is, with \( P^x \) the probability measure of the random walk starting at \( x \),

\[
p_1(x, y) = P^x(S_1 = y).
\]

Define the first hitting time of \( S_n \) on a set \( B \) by

\[
T_B := \inf\{ k \geq 0 : S_k \in B \}.
\]
Spitzer, in [4], defines the \textit{truncated Green’s function}, for \( x, y \in A \) of a random walk from \( x \) to \( y \) before exiting \( A \) as the total expected number of visits to \( y \), starting from \( x \):

\[
G_A(x, y) := \mathbb{E}^x \left[ \sum_{j=0}^{\infty} 1_{\{S_j = y; j < T_{A^c}\}} \right] = \sum_{j=0}^{\infty} P^x(S_j = y; j < T_{A^c})
\]

(2)

and 0 if \( x \) or \( y \) \( \notin A \). An elementary result for any random walk (found, for example, in [4], or [2, Sect. 1.5]) is that, for \( x, y \in A \subset D \), there are more possible visits inside \( D \) than inside \( A \):

\[
G_A(x, y) \leq G_D(x, y).
\]

(3)

Starting at a point \( x \in A^c \), the \textit{hitting distribution} of \( A \) is defined as

\[
H_A(x, y) := P^x(S_{T_A} = y).
\]

(4)

The \textit{last exit decomposition} of a hitting distribution is based on the Green’s function: for \( A \) a proper subset of \( \Omega \), \( x \in A^c \), and \( y \in A \),

\[
H_A(x, y) = \sum_{z \in A^c} G_{A^c}(x, z)p_1(z, y).
\]

(5)

Simple lower bounds for the Green’s function \( G_{A \cup B} \), by (3), are obvious; for upper bounds for these cases, we examine excursions between \( A \) and \( B \) before hitting \( C \).

**Proposition 1.** For \( a, a' \in A \) and \( b, b' \in B \), with \( \theta_t \) the usual shift operators,

\[
T^*_B := \inf\{k > T_A : S_k \in B\} = T_A + T_B \circ \theta_{T_A},
\]

\[
T^*_A := \inf\{k > T_B : S_k \in A\} = T_B + T_A \circ \theta_{T_B},
\]

and defining

\[
\psi_a := \sum_{b' \in B} H_{B \cup C}(a, b') = P^a(T_B < T_C)
\]

(6)

\[
\sigma_b := \sum_{a' \in A} H_{A \cup C}(b, a') = P^b(T_A < T_C)
\]

(7)

\[
\rho_a := \sum_{b' \in B} H_{B \cup C}(a, b')\sigma_{b'} = P^a(T_B, T^*_A < T_C)
\]

(8)

\[
\phi_b := \sum_{a' \in A} H_{A \cup C}(b, a')\psi_{a'} = P^b(T_A, T^*_B < T_C),
\]

(9)
we have the Green’s function bounds
\[ G_A(a, a') \leq G_{A\cup B}(a, a') \leq G_A(a, a') + \frac{\rho_a}{1 - \rho_a} G_A(a', a') \] (10)
\[ G_B(b, b') \leq G_{A\cup B}(b, b') \leq G_B(b, b') + \frac{\phi_{b'}}{1 - \phi_{b'}} G_B(b', b') \] (11)
\[ 0 \leq G_{A\cup B}(a, b) \leq \min \left\{ \frac{\sigma_b}{1 - \rho_a} G_A(a, a), \frac{\psi_a}{1 - \phi_b} G_B(b, b) \right\} . \] (12)

Note that \( \psi_a \geq \rho_a \) for every \( a \in A \) and \( \sigma_b \geq \phi_b \) for every \( b \in B \).

**Proof** We will prove this for (10) and (12) (the proof for (11) matches (10)’s proof). By (2), for \( a, a' \in A \),
\[ G_{A\cup B}(a, a') = \sum_{i=0}^{\infty} P^a(S_i = a', i < T_C) \]
\[ = \sum_{i=0}^{\infty} [P^a(S_i = a', i < T_C, i < T_B) + P^a(S_i = a', T_B < i < T_C)] \]
\[ = G_A(a, a') + \sum_{i=0}^{\infty} P^a(S_i = a', T_B < i < T_C). \] (13)

Since \( a' \in A \), once the walk enters \( B \) it must return to \( A \) before hitting \( a' \) again. By splitting and switching sums and applying the strong Markov property at \( T_B \),
\[ G_{A\cup B}(a, a') = G_A(a, a') + \sum_{i=0}^{\infty} \sum_{b \in B} P^a(S_{T_B = b} = a, S_i = a', T_B < i < T_C) \]
\[ = G_A(a, a') + \sum_{b \in B} H_{B\cup C}(a, b)G_{A\cup B}(b, a'). \] (14)

We now switch from (10) to (12): for \( G_{A\cup B}(b, a') \), with \( b \in B \) and \( a' \in A \), decomposing over \( A \), and using the strong Markov property at \( T_A \),
\[ G_{A\cup B}(b, a') = \sum_{i=0}^{\infty} P^b(S_i = a', i < T_C) \]
\[ = \sum_{i=0}^{\infty} \sum_{a'' \in A} P^b(S_i = a', T_A \leq i < T_C, S_{T_A} = a'') \]
\[ = \sum_{a'' \in A} H_{A\cup C}(b, a'')G_{A\cup B}(a'', a'). \] (15)

We thus have a recurrence relation between (10) and (12).
By the strong Markov property at $T_{a'}$, we have the upper bound
\[ G_A(a'',a') = P^{a''}(T_{a'} < T_A) G_A(a',a') \leq G_A(a',a') \] (16)
which yields, by (7) (for $A \cup B$ instead of $A$),
\[ G_{A \cup B}(b,a') = \sum_{a'' \in A} H_{A \cup C}(b,a'') G_{A \cup B}(a'',a') \leq \sigma_b G_{A \cup B}(a',a'). \] (17)
Combining (14), (17), and (8) gives us
\[ G_{A \cup B}(a,a') = G_A(a,a') + \sum_{b \in B} H_{B \cup C}(a,b) G_{A \cup B}(b,a') \leq G_A(a,a') + \sum_{b \in B} H_{B \cup C}(a,b) \sigma_b \] (18)
\[ = G_A(a,a') + G_{A \cup B}(a',a') \rho_a. \] In particular, (18) gives us
\[ G_{A \cup B}(a',a') \leq \frac{G_A(a',a')}{1 - \rho_a}. \] (19)
(19) used again in (18) yields (10). Proving (11) similarly, (11) and (19) applied to (17) yields (12).

2 Hitting times

We now find the expected time of hitting the set $C$, starting from $A$, in terms of hitting $B \cup C$. Lower bounds are simple: just tack the other set on for a quicker hitting time. The upper bounds will require a recursive excursion treatment similar to the proof of Proposition 1.

Proposition 2. For $a \in A$ and $b \in B$, defining via (6) and (7),
\[ f_A := \sup_{a \in A} E^a(T_{B \cup C}), \quad f_B := \sup_{b \in B} E^b(T_{A \cup C}), \quad \psi := \sup_{a \in A} \psi_a, \quad \sigma := \sup_{b \in B} \sigma_b, \] (20)
we have the expected hitting time bounds
\[ E^a(T_{B \cup C}) \leq E^a(T_C) \leq E^a(T_{B \cup C}) + \psi_a \left[ \frac{f_B + \sigma f_A}{1 - \psi \sigma} \right]. \] (21)
\[ E^b(T_{A \cup C}) \leq E^b(T_C) \leq E^b(T_{A \cup C}) + \sigma_b \left[ \frac{f_A + \psi f_B}{1 - \psi \sigma} \right]. \] (22)

Proof We will prove (21) (the proof of (22) is the same). First, decompose $T_C$ along the two possibilities for $T_{B \cup C}$. Recall that $T_{B \cup C} = T_C \iff T_C < T_B$. By the strong Markov
property at \( T_B \),
\[
E^a(T_C) = E^a(T_C \mathbf{1}_{T_B \cup C = T_C}) + E^a(T_C \mathbf{1}_{T_B \cup C = T_B}) \\
\leq E^a(T_B \cup C) + \sum_{b \in B} H_{B \cup C}(a, b) E^b(T_C). \tag{23}
\]

Likewise, for \( b \in B \),
\[
E^b(T_C) \leq E^b(T_{A \cup C}) + \sum_{a' \in A} H_{A \cup C}(b, a') E^{a'}(T_C). \tag{24}
\]

By combining (23) and (24), recursing on itself, keeping the first couple terms in terms of \( a \), and maximizing the rest via (6), (7), and (20), we get
\[
E^a(T_C) \leq E^a(T_B \cup C) + \sum_{b \in B} H_{B \cup C}(a, b) \left( E^b(T_{A \cup C}) + \sum_{a' \in A} H_{A \cup C}(b, a') [f_A + \psi(f_B + \sigma[\ldots])] \right),
\]
which is bounded by
\[
E^a(T_C) \leq E^a(T_B \cup C) + \psi_a \left( f_B + \sigma[f_A + \psi(f_B + \sigma[\ldots])] \right) = E^a(T_B \cup C) + \psi_a(f_B + \sigma f_A) \frac{1}{1 - \psi \sigma}. \tag{25}
\]

3 Hitting distributions

If \( y \in A \subset D \), then for \( x \in D^c \subset A^c \), we have by (23) the monotonicity result
\[
H_A(x, y) = \sum_{z \in A^c} G_A(x, z)p_1(z, y) \geq \sum_{z \in D^c} G_D(x, z)p_1(z, y) = H_D(x, y) \tag{25}
\]
and the subset hitting time relations (assuming a recurrent random walk)
\[
P^x(T_A = T_D) = \sum_{z \in A} H_D(x, z); \\
P^x(T_A \neq T_D) = P^x(T_A > T_D) = \sum_{z \in D \setminus A} H_D(x, z). \tag{26}
\]

(25) and (26) hint at a relationship between the hitting distributions of two sets \( C \) and \( C \cup A \). We find a bound on this relationship. Let \( b \in B \) and \( c \in C \). By (25) with
$D = C \cup A$, there is a probability $p(b, c, C, A)$ such that

$$H_C(b, c) = H_{C\cup A}(b, c) + p(b, c, C, A).$$

(27)

To bound $p(b, c, C, A)$, we rewrite using the definition of $H_C(b, c)$ and decompose along the event \{$T_C < T_A\}$ (whose probability is $1 - \sigma_b$ in \([7]\)):

$$H_C(b, c) = P^b(S_{T_C} = c) = P^b(S_{T_C} = c, T_C < T_A) + P^b(S_{T_C} = c, T_A < T_C);$$

$$H_{C\cup A}(b, c) = P^b(S_{T_{C\cup A}} = c) = P^b(S_{T_{C\cup A}} = c, T_C < T_A) + P^b(S_{T_{C\cup A}} = c, T_A < T_C).$$

Note that

$$P^b(S_{T_C} = c, T_C < T_A) = P^b(S_{T_{C\cup A}} = c, T_C < T_A)$$

and

$$S_{T_{C\cup A}} = c \in C \implies T_C < T_A,$$

so clearly $P^b(S_{T_{C\cup A}} = c, T_A < T_C) = 0$ and we get the simple bound

$$p(b, c, C, A) = P^b(S_{T_C} = c, T_A < T_C) \leq P^b(T_A < T_C) = \sigma_b.$$

(28)

If $C$ is a set that “separates” $A$ and $B$ in some sense (e.g., if the probability distribution of the random walk is based on distance, and $C$ separates $A$ and $B$ into components), then $\sigma_b$ being small reflects the small difference between $H_C$ and $H_{C\cup A}$ (in that it is very likely, starting in $B$, to hit $C$ before $A$).

Note also that $p(C, A)$ is not symmetric; e.g., $p(A, C) = 1 - p(C, A) = 1 - \sigma_b$.

References

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