We consider the extension of static dimensional reduction to real-time. For a scalar field theory it is shown that in the high-temperature limit this leads to an effective classical theory. Quantum corrections to the leading classical behavior are determined by an effective action which can be calculated perturbatively in the background of the classical fields. Feynman rules for $\lambda \phi^4$-theory are given and the extension to SU($N$) gauge theories is outlined.

I. INTRODUCTION

Currently there is considerable interest in the behavior of low-momentum fields in a high-temperature plasma environment. For a bosonic field $\phi$ the objects of study are the time-dependent correlation functions for the soft modes

$$\langle \phi(t_1, \vec{p}_1) ... \phi(t_n, \vec{p}_n) \rangle \mid _{\vec{p} < \Lambda} \sim g^2 T$$

and one would like to derive an effective theory that governs the real-time dynamics of these correlation functions for the long-wave-length modes. In this paper we discuss the derivation of such an effective theory for scalar field theories and outline the extension to gauge theories.

The standard approach to deriving an effective theory for the soft modes is the introduction of an intermediate energy-scale $\Lambda << T$ that explicitly separates the hard modes with $|\vec{p}| > \Lambda$ from the soft modes with $|\vec{p}| < \Lambda$. The hard modes are integrated out as irrelevant degrees of freedom and the effective theory of the soft modes is approximated by a classical theory. However there are two problems: firstly, the results may depend on the scale $\Lambda$ and, secondly, such a cut-off breaks gauge invariance.

We will present here a different approach, without intermediate cutoff, in which the classical effective theory is constructed as a “natural” extension of static dimensional reduction to real-time. The essence is that the zero modes of static dimensional reduction are given the role of initial conditions for classical background fields satisfying effective equations of motion in the environment of the quantum fluctuations which are taken into account perturbatively. This method we will call real-time dimensional reduction.

II. REAL-TIME DIMENSIONAL REDUCTION

Our starting point is the standard formulation of real-time thermal field theory in which the temporal dependence of the fields has support on the contour depicted in fig. 1.

![FIG. 1. The time-contour C of thermal field theory.](image)

Equilibrium correlation functions may be obtained from the generating functional

$$Z[j] = \int D\phi D\pi \exp iS[\phi, \pi] + ij \cdot \phi$$

with $\phi(x)$ periodic over the entire contour $C$. The dot-notation in the source term is an abbreviation for the inner product $j \cdot \phi = \int_C dtd^3x j(x) \phi(x)$. Since we are interested in real-time correlation functions, we only allow for sources on the real-time part of the contour $C_{12} = C_1 \cup C_2$. We shall keep the integration over the momenta $\pi(x)$ explicit, since we want to end up with an effective classical theory with classical initial conditions on fields and conjugate momenta.

We now separate off the integration over the zero modes $\Phi = \Phi(\vec{x})$ and $\Pi = \Pi(\vec{x})$ according to...
where the operator $P$ projects out the zero static mode on the Euclidean part of the contour $C$

$$P\phi = iT \int_{C_3} dt \phi(t, \vec{x})$$

On the real-time part $C_{12}$ of the time-contour the path integration is unrestricted. This yields the important corollary that local symmetries on $C_{12}$ are respected in the path integral which is especially useful for gauge theories.

III. STATIC DIMENSIONAL REDUCTION

To gain some understanding of the formal manipulation performed in (3), let us consider for a moment a purely Euclidean theory with a time-contour running straight down from $t_i$ to $t_i - i\beta$, that is, the contour $C$ with $t_i = t_f$. In this case the field is periodic on $C_3$ and may be expanded in Matsubara modes

$$\phi(t_i - i\tau, \vec{x}) = \sum_n \phi_n(\vec{x}) e^{i\omega_n \tau}$$

with Matsubara frequency $\omega_n = 2\pi nT$. The operator $P$ then just projects out the zero Matsubara mode

$$P\phi = T \int_0^\beta d\tau \phi(t_i - i\tau, \vec{x}) = \phi_0(\vec{x}) = \Phi(\vec{x})$$

and similarly the zero momentum mode $P\pi = \Pi$. The path integral over the field in (3) simply factorizes into an integration over the zero modes and one over the non-zero Matsubara modes

$$\int D\phi D\pi \exp (P\phi - \Phi) \delta (P\pi - \Pi)$$

and we end up with the effective static theory

$$Z = \int D\Phi D\Pi \exp iS[\Phi, \Pi] + iW_{DR}[\Phi]$$

where the effective action $W_{DR}$ is defined by

$$\exp iW_{DR}[\Phi] = \int D\phi_{n\neq 0} \exp iS[\Phi; \phi_{n\neq 0}]$$

with $S[\Phi; \phi_{n\neq 0}]$ the action of the non-zero Matsubara modes in a static background field $\Phi$. For scalar fields the effective action does not depend on the static momenta, since momenta appear only linearly and quadratically in the action. We recognize (3) and (4) as the standard formulation of static dimensional reduction.

The effective action $W_{DR}[\Phi]$ gives the corrections of the non-zero Matsubara modes to the classical action.

Since the non-zero Matsubara modes have masses of the order of the temperature, no IR problems occur when the non-zero Matsubara modes are integrated out. This implies that the effective action has the property that it is in principle perturbatively calculable.

IV. EFFECTIVE ACTION

Returning now to the real-time case we introduce two classical fields $\phi_{cl}, \pi_{cl}$ as background fields on the entire contour

$$\phi(t, x) \rightarrow \phi(t, \vec{x}) + \phi_{cl}(t, \vec{x})$$

$$\pi(t, \vec{x}) \rightarrow \pi(t, \vec{x}) + \pi_{cl}(t, \vec{x})$$

On the Euclidean branch of the contour the classical fields are chosen to be constant in time and equal to the zero modes

$$\phi_{cl}(t, \vec{x}) = \Phi(\vec{x}), \quad \pi_{cl}(t, \vec{x}) = \Pi(\vec{x})$$

On the real-time part of the contour the classical fields will satisfy effective equations of motion (to be specified later on) with initial conditions $\Phi, \Pi$.

Performing the shift in (3), we can cast the generating functional in the form of a path integral over the initial state

$$Z[j] = \int D\Phi D\Pi \exp iS_{cl} + iW[\phi_{cl}, \pi_{cl}; J] + ij \cdot \phi_{cl}$$

with classical action $S_{cl} = S[\phi_{cl}, \pi_{cl}]$. The quantum corrections are contained in the effective action

$$\exp iW[\phi_{cl}, \pi_{cl}; J] = \int D\phi D\pi \exp (P\phi) \delta (P\pi)$$

where we used that the classical fields on $C_3$ are equal to the zero modes to simplify the delta functions

$$\delta (P(\phi + \phi_{cl}) - \Phi) = \delta (P\phi)$$

The action $S[\phi_{cl}, \pi_{cl}; \phi, \pi]$ describes the quantum fields in a background of the classical fields. By definition it contains only the terms of quadratic-and-higher order in the quantum fields $\phi, \pi$. By demanding $\pi_{cl} = \partial_t \phi_{cl}$ on $C_{12}$ the term linear in $\pi$ has been made to vanish, and the linear term in $\phi$ has been absorbed in the source $J = j + \delta_\phi S|_{\phi=\phi_{cl}}$.

V. PROPAGATOR

We consider the example of a $\lambda\phi^4$-model and split the action of the quantum fluctuations in (13) into a free and interaction part:

$$S[\phi_{cl}, \pi_{cl}; \phi, \pi] = S_0[\phi, \pi] + S_I[\phi_{cl}; \phi]$$
with the interactions given by
\[ S_t[\phi_{cl}; \phi] = \frac{1}{4} \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{3!} \frac{\partial^4 \phi}{\partial x^4} + \frac{1}{4!} \frac{\partial^6 \phi}{\partial x^6} \]  

(16)

The dashed lines denote the classical background fields and the solid lines the quantum fields.

Assuming that the effective action can be calculated perturbatively, to be checked afterwards, we have
\[ \exp iW[\phi_{cl}; J] = \exp iS_t[\phi_{cl}; -i\delta_J] \exp iW_0[J] \]  

(17)

with \( W_0[J] \) the effective action of the free theory. As it turns out for a scalar \( \phi^4 \)-theory the effective action does not depend on the classical momentum. The free effective theory can be calculated by a Gaussian integration over the fields, but with a constraint on the integration enforced by the \( \delta \)-functions in \([13]\). The result is \([14]\)
\[ \exp iW_0[J] = \exp -\frac{1}{2} J \cdot \Delta_C \cdot J \]  

(18)

where the propagator of the quantum fluctuation
\[ \Delta_C(t - t'; \vec{k}) = D_C(t - t'; \vec{k}) - S_C(t - t'; \vec{k}) \]  

(19)

is equal to the thermal propagator \( D_C \) with a subtrac-
tion \( S_C \) due to the integration constraints. The thermal propagator on the contour has the standard form \([10]\)
\[ D_C(t - t'; \vec{k}) = \frac{1}{2\omega_k} \left[ \Theta_C(t - t')e^{-i\omega_k(t - t')} \right. \]
\[ + \Theta_C(t' - t)e^{i\omega_k(t - t')} \]
\[ + n(\omega_k) \left( e^{-i\omega_k(t - t')} + e^{i\omega_k(t - t')} \right) \]  

(20)

with energy \( \omega_k^2 = \vec{k}^2 + m^2 \) and Bose-Einstein distribution \( n(\omega) = \left(e^{\beta \omega} - 1\right)^{-1} \), whereas the subtraction on the various parts of the contour is given by the expressions
\[ D_C(t - t'; \vec{k}) = \begin{cases} \frac{T}{\omega_k} \cos \omega_k(t - t') & t, t' \in C_{12} \\ \frac{T}{\omega_k} \cos \omega_k(t_{in} - t') & t \in C_3, t' \in C_{12} \\ \frac{T}{\omega_k} \cos \omega_k(t - t_{in}) & t \in C_{12}, t' \in C_3 \\ \frac{T}{\omega_k} & t, t' \in C_3 \end{cases} \]  

(21)

On \( C_{12} \) we recognize \( S_C \) as the classical propagator and on \( C_3 \) as the zero-mode propagator. The two contributions connecting the vertical branch \( C_3 \) of the contour with the real-time branch \( C_{12} \) depend on the initial time \( t_{in} \). In thermal field theory it is shown that in the limit \( t_{in} \rightarrow -\infty \), the two horizontal real-time contours \( C_{12} \) decouple from the Euclidean branch \( C_3 \), provided that one introduces an infinitesimal damping coefficient \([10]\). However, we have to keep \( t_{in} \) finite. Then obviously the contributions of \( C_3 \) and \( C_{12} \) do not separate and the system retains a memory of the initial state. How this works out in the equations of motion is discussed in Appendix A.

Let us consider the IR limit \( \omega_k/T \rightarrow 0 \) of the propagator of the quantum fields \([19]\). The dominant contribution to the thermal propagator comes from the thermal part of \([20]\), since \( n(\omega_k) \rightarrow T/\omega_k \). Therefore, in the high-temperature limit the thermal propagator reduces to
\[ D_C(t_1 - t_2; \vec{k}) \rightarrow S_C(t_1 - t_2; \vec{k}) \quad (\omega_k \rightarrow 0) \]  

(22)

when we ignore terms \( 1/\omega_k \) compared to \( T/\omega_k^2 \).

In particular on \( C_3 \) the propagator of the quantum fluctuations is just the propagator of static dimensional reduction, namely the propagator of the non-static Matsubara modes
\[ \Delta_C(-i\tau, \vec{k}) = T \sum_{n \neq 0} e^{\frac{i\omega_n \tau}{\omega_k^2}} \]  

(23)

Consequently, the effective action will contain as a subset also all diagrams of the Euclidean dimensionally reduced action.

From \([22]\) and \([23]\) we see that the propagator of the quantum fluctuations is better IR-behaved than the thermal propagator. Hence the important conclusion is that a perturbative evaluation of the effective action is possible without encountering the severe IR divergences of the full thermal theory. In other words: the proposed splitting of the classical and quantum degrees of freedom isolates the IR sensitive part of the theory in the classical part of the theory.

\[ \]  

VI. EQUATIONS OF MOTION

Let us return to the generating functional \([12]\). The classical and effective action in the integrand depend on the classical fields \( \phi_{cl} \), \( \pi_{cl} \) which were defined both on \( C_{12} \) and \( C_3 \). The contributions of these two separate contour segments have a completely different interpretation. This difference can be made more explicit by choosing the classical background field \( \phi_{cl} \) equal on the upper and lower real-time branch
\[ \phi_{cl}(t_1, \vec{x}) = \phi_{cl}(t_2, \vec{x}) \quad t_1 = t \in C_1, \quad t_2 = t \in C_2 \]  

(24)

Recalling also \([11]\), we see that the two contributions of the classical action on the upper and lower time-
branch cancel, and that we are left with \( iS[\phi_{cl}, \pi_{cl}] = -\beta H[\Phi, \Pi] \), which is the classical thermal weight in terms of the Hamiltonian. In a similar fashion, the effective action may be split into two parts
with an influence action $W_{IF}$ that vanishes at zero source and $W_{DR}$ the effective action of static dimensional reduction \[9\], which depends solely on the initial field $\Phi$.

The generating functional may then be written

$$Z[j] = \int D\Phi D\Pi \exp \left( -\beta H[\Phi, \Pi] + iW_{DR}[\Phi] \right) \exp \left( iW_{IF}[\phi_{cl}; J] + ij \cdot \phi_{cl} \right)$$

where we have separated the exponent into the effective thermal weight factor and a second factor which is the effective source term for the real-time correlation functions.

The expression above is still quite general as no approximation has been made. All complications of the interaction between hard and soft modes are contained in the influence functional. However, this functional may be calculated perturbatively and we are only looking for an effective theory that contains the leading high-temperature behavior. Moreover, we still have not made a unique choice for the classical field on $C_{12}$. An obvious choice is to equate the mean classical field with the full mean field including quantum fluctuations

$$\delta_j W_{IF}[\phi_{cl}; J]|_{j=0} = 0$$

This may be translated into the equation of motion

$$\delta_{\phi} \Gamma[\phi]|_{\phi=\phi_{cl}} = 0$$

with $\Gamma[\phi]$ the Legendre transform of $W_{IF}[\phi_{cl}; J]$ with $\phi_{cl} = 0$ on $C_{12}$. If we now confine ourselves to the leading contribution, this reduces to

$$\delta_{\phi} \Gamma_{HTL}[\phi]|_{\phi=\phi_{cl}} = 0$$

with $\Gamma_{HTL}[\phi]$ the hard thermal loop (HTL) effective action in the low-momentum limit. It is noteworthy to mention that large contributions from soft modes on internal lines of Feynman diagrams that arise in thermal theories are not present in the effective action, since the IR-sensitive part of the thermal propagator is subtracted in \[14\]. As a consequence, the leading contributions do come solely from the hard thermal loops.

Since correlation functions are evaluated at zero-source, the leading contribution to statistical correlation functions is given by the classical contribution, that is, the influence function in \[23\] can be ignored in this calculation. However, if we calculate, for instance, the expectation value of the commutator of fields, the classical contribution is zero (classical fields commute). Then the leading HTL contribution has to be obtained from the influence action by source differentiation.

In scalar $\phi^4$-theory, the only hard thermal loop contribution is the tadpole contribution to the thermal mass $m_{th}^2 = m^2 + \lambda T^2/24$. When calculating this contribution we encounter a linear divergence, that may be set to zero if we use the dimensional regularization scheme. However, as a general rule, UV-divergences do not disappear but act as counter terms for the same divergences occurring in the classical theory. This general feature is easily understood from the fact that the full quantum theory is renormalizable. Hence, all classical divergences must be exactly neutralized by corresponding counter terms from the quantum corrections.

Because we only have to take into account the thermal mass, the HTL equations of motion \[22\] for $\lambda \phi^4$-theory simply read

$$(\partial_t^2 - \vec{\nabla}^2 + m_{th}^2)\phi_{cl}(t, \vec{x}) + \frac{1}{3!} \lambda \phi_{cl}^3(t, \vec{x}) = 0$$

with initial conditions

$$\phi_{cl}(t_{in}, \vec{x}) = \Phi(\vec{x}); \quad \pi_{cl}(t_{in}, \vec{x}) = \Pi(\vec{x})$$

So in combination with \[26\] we have obtained a classical statistical theory, with the classical field satisfying the equation of motion \[50\] and thermal averaging over the initial conditions. The effect of the initial conditions on the equation of motion is studied further in appendix A.

As shown above, the perturbative approach to the effective classical theory, amounts to a resummation of the thermal mass into the classical propagator, while leaving the quantum propagator unmodified. This is similar to the resummation scheme of \[13\] for static quantities, where the thermal mass is resummed in the zero-mode propagator.

**VII. SU(N) GAUGE THEORY**

The method developed in this paper to study the IR-behavior of time-dependent fields, has an obvious application in non-Abelian gauge theories, which have a non-trivial IR structure \[14\]. The reasoning can be extended almost unchanged to derive an effective classical statistical theory for gauge fields \[13\]. The final result for the generating functional (in the Coulomb gauge) is

$$Z[j] = \int D\mathcal{A}^\mu D\mathcal{E}^j D\mathcal{C} D\mathcal{D} \exp \left( -\beta H[\mathcal{A}, \mathcal{E}] + iS_{gh}[\mathcal{C}, \mathcal{C}, \mathcal{A}] + iW_{DR} \right) \exp \left( iW_{IF}[\mathcal{A}_{cl}, E_{cl}; J] + ij \cdot A_{cl} \right)$$

with $\mathcal{A}, \mathcal{E}$ the static gauge fields, $\mathcal{C}, \mathcal{C}$ the ghost zero modes, and $S_{gh}$ the ghost action. We have extracted the effective classical theory that gives the leading-order contributions in the low-momentum limit. Subleading corrections in $g$ or $|p|/T$ are given by the quantum corrections contained in the effective actions $W_{IF}, W_{DR}$. Although subleading these contributions may be important to provide counter terms for the classical divergences, as in the static case \[13\].

4
The classical fields have to be determined from the HTL equation of motion, but calculated with the gauge propagators equivalent to \( \lambda \phi \)

\[
[D_{cl}^{\mu}, F_{\nu\mu}^{cl}](x) = 3\omega_p^2 \int \frac{d\Omega}{4\pi} \nu_\mu \int_{-\infty}^{t} dt' U_{cl}(x, x') v_\nu F_{\nu\mu}^{cl}(x')
\]

(33)

with \( \omega_p^2 = N g^2 T^2 / 6 \) the plasmon frequency. The angular integration is over the direction of \( \vec{v} \), \( |\vec{v}| = 1 \). Furthermore, \( x' = (t', x - \vec{v}(t - t')) \) and \( U_{cl}(x, x') = P \exp \left( -i g \int_0 dz A_\mu^c(z) \right) \), with \( \gamma \) a straight line from \( x \) to \( x' \), is the parallel transporter. The initial conditions are

\[
A_\mu^c(t_{in}, \vec{x}) = A_\mu^c(\vec{x}) ; \quad F_{\nu\mu}^{cl}(t_{in}, \vec{x}) = \epsilon_\nu(\vec{x})
\]

(34)

The field \( A_\mu^c \) at times prior to \( t_{in} \) on the r.h.s. of (33) should be taken constant in time and equal to the initial condition. This prescription takes into account the correlations between the initial conditions and the fields at later times, as explained in Appendix A.

Equation (33) is the well known HTL equation of motion, derived in a kinetic approach by Blazoiot and Iancu [19], where asymptotic initial conditions are considered. We have initial conditions at a finite time \( t_{in} \) over which a statistical average has to be taken. This implies that the physics at the scale \( g^2 T \) is still present in the effective theory.

The SU(\( N \)) effective theory has two obvious applications. The study of the IR sector of the theory, for instance by deriving an effective theory for the non-perturbative modes, that may be studied on the lattice [20]. And secondly to study the effect of IR effects on perturbative modes, that may be studied on the lattice.

The introduction of such a cut-off scale \( \Lambda \) can be inferred from its UV behavior. Since \( \Lambda \) is larger than the typical classical energy-scale the divergences in the theory without cut-off get replaced by powers of \( \Lambda / m_{th} \) or \( \log \Lambda / m_{th} \) in \( \lambda \phi^4 \)-theory. So the problem of introducing counter terms to cancel all \( \Lambda \)-dependencies, is the same problem of introducing counter terms for the divergences as \( \Lambda \to \infty \). It has been argued [21] that in a leading order calculation of IR-sensitive quantities the non-HTL contributions that depend on \( \Lambda \) may be neglected for \( \Lambda \) in the range \( gT \ll \Lambda \ll T \) (in gauge theories). Since these non-HTL contributions are suppressed by powers of the coupling constant or the classical energy scale over the temperature.

The dependence on the cut-off \( \Lambda \) of the classical theory can be inferred from its UV behavior. Since \( \Lambda \) is larger than the typical classical energy-scale the divergences in the theory without cut-off get replaced by powers of \( \Lambda / m_{th} \) or \( \log \Lambda / m_{th} \) (in \( \lambda \phi^4 \)-theory). So the problem of introducing counter terms to cancel all \( \Lambda \)-dependencies, is the same problem of introducing counter terms for the divergences as \( \Lambda \to \infty \). It has been argued [21] that in a leading order calculation of IR-sensitive quantities the non-HTL contributions that depend on \( \Lambda \) may be neglected for \( \Lambda \) in the range \( gT \ll \Lambda \ll T \) in gauge theories. Since these non-HTL contributions are suppressed by powers of the coupling constant or the classical energy scale over the temperature.
APPENDIX A: INITIAL CORRELATIONS

The effective action $\Gamma[\phi]$ in (28) depends on the zero mode $\Phi$. Therefore one may expect that in the equations of motion the zero mode will also occur, describing the correlations between the initial conditions and the classical field at later times. As an example let us consider the linear equations of motion

$$\left(\partial_t^2 + \omega_k^2\right)\phi_{cl}(t, \vec{k}) + \int_C dt^\prime \Sigma(t - t^\prime, \vec{k})\phi_{cl}(t^\prime, \vec{k}) = 0 \quad (A1)$$

with the time integral over the entire contour. Since the classical field is equal on the forward and backward time-branches, the real-time part of the self-energy entering the equations of motion is given by the retarded self-energy

$$\int_C dt^\prime \Sigma(t - t^\prime, \vec{k})\phi_{cl}(t^\prime, \vec{k}) = \int_{t_{cl}}^{t_{in}} dt^\prime \Sigma_R(t - t^\prime, \vec{k})\phi_{cl}(t^\prime, \vec{k})$$

$$+ \int_{C_3} dt^\prime \Sigma(t - t^\prime, \vec{k})\phi_{cl}(t^\prime, \vec{k}) \quad (A2)$$

The second term on the r.h.s. gives the correlations with the initial field. Since $t_{in}$ is finite this term cannot be dropped. However, we may rewrite it in a more convenient form. For this we deform the contour $C_3$ to first run back to some earlier time $t_{in}$ then down the Euclidean path $C_3'$ to $t_{in} - i\beta$ and finally straight back to $t_{in} - i\beta$. Using the assumption of analyticity of the self-energy in the complex time plane and periodicity, we arrive at

$$\int_{C_3} dt^\prime \Sigma(t - t^\prime, \vec{k}) = \int_{t_{in}}^{t_{in}} dt^\prime \Sigma_R(t - t^\prime, \vec{k})$$

$$+ \int_{C_3'} dt^\prime \Sigma(t - t^\prime, \vec{k}) \quad (A3)$$

Introducing an infinitesimal damping rate, the decoupling theorem of thermal field theory states that the second term on the r.h.s. of (A3) may be dropped in the limit $t_{in} \to -\infty$.

Returning to the equation of motion $\Gamma[\phi]$ we find

$$(\partial_t^2 + \omega_k^2)\phi_{cl}(t, \vec{k}) + \int_{-\infty}^{t_{in}} dt^\prime \Sigma_R(t - t^\prime, \vec{k})\phi_{cl}(t^\prime, \vec{k}) = 0$$

(A4)

where it should be understood that the classical field before $t_{in}$ appearing in the memory kernel equals the initial value: $\phi_{cl}(t, \vec{k}) = \Phi(\vec{k})$ for $t < t_{in}$. Physically this means that the system is frozen in the initial state from the infinite past till the initial time $t_{in}$ where a disturbance is applied. This result can easily be generalized to non-linear equations of motion with higher-point vertex functions.