The response of the competitive balance model to the external field

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Abstract

The competitive balance model was proposed as an extension of the balance theory to address heterogeneities in real-world networks. In this model, different paradigms lead to form different friendship and enmity. As an example, friendship or enmity between countries can have a political or religious basis. The suggested Hamiltonian is symmetrical between paradigms. In this paper, we investigate the influence of the external field on the evolution of the network. We drive the mean-field solutions of the model and verify the accuracy of our analytical solutions by performing Monte-Carlo simulations. We observe that the external field breaks the symmetry of the system. The response of the system to the external field, depending on the temperature, is paramagnetic or ferromagnetic. Similar to the magnetic systems, susceptibility follows Curie’s law. We also observed a hysteresis behavior. Once communities are formed based on a certain paradigm, then they resist change.

Keywords: Structural balance theory, Competition, Mean-field, Signed heterogeneous networks

1. Introduction

Structural balance theory is a sociological model for describing the dynamics of friendship and enmity in signed networks. This model considers the 3-cycles in the network and states that a dyadic relation is influenced by third parties who are in contact with both of them. The model was originally introduced by Heider in 1946 \cite{1}. In recent years, the model has become popular to address a wide range of phenomena ranging from interpersonal...
relationships in society [2, 3], Online interactions through the Web [4, 5] to the formation of international coalitions [6, 7, 8, 9].

Throughout the years, many studies have introduced augmentations to Heider’s original formulation of balance theory [10, 11, 12, 13, 14, 15, 16]. In this regard, the Competitive balance model [17] has been proposed as an extension to the balance model to address the heterogeneity in relationships. To this end, this model includes two different types of friendship and enmity. As an example, on the international level, coalitions can form based on geographical interests or religious ones. The dynamic of this model has been studied in a symmetrical condition and in the absence of the external force [17]. In this paper, we study the dynamics of the competitive balance model in the presence of an external field.

In the Heider balance model, the relationship between members is either friendly or enmity, which is denoted with +1 and -1, respectively. Considering these two types of links, four types of triangles can form. A triangle is balanced if the product of its edges is positive, otherwise, it is unbalanced. The mathematical framework of balance theory for collections of more than three members was formulated by Cartwright and Harary [18]. They showed that a fully connected network is balanced if all the triangles in the network are balanced. In 2005, Antal et al. presented two discrete dynamical models to examine how an unbalanced network evolves toward a balanced state [19]. At the same time, Kulakowski et al. formulated continuous dynamic equations for the networks with continuous values of friendship and enmity [10].

Agent based models have become popular to model soci-economic systems [20, 21, 22, 23]. In recent years, increasing attention have been devoted to heterogeneities of real-world networks [24, 25, 26, 27]. Despite the success of balance theory to describe a wide range of phenomena, we still know that the real-world networks have more complexities [12, 13]. Recently, increasing attention has been devoted to possible extensions of the Heider balance model which might address complexities of the real world, see for example [28].

It is simplistic to assume that all relationships either friendship or enmity have the same origin. Various influential factors such as religion, politics, and culture can lead to the formation of different kinds of friendships and enmities. As an example, friendship and enmity relations between Middle Eastern countries have religious origins, but their relations with Western countries are based on political and economic interests. Thus, various factors are influential in the formation of coalitions. The competitive balance model
aims to enrich balance theory to cover such heterogeneity and highlight the
fact that there are conflicts of interest in socio-political networks [17]. In
this model, two different interests form two different types of friendship or
enmity. In fact, it is not enough to say that the two agents are friends or
enemies. You also need to specify the type of friendship or enmity based on
the interests of formation.

In the competitive balance model, real and imaginary numbers have been
utilized to distinguish between different types of relations. In other words,
dyadic relations besides ±1 can get values ±i that stand for friendship or en-
mity based on different interests. The proposed Hamiltonian is symmetrical
regarding both types of relations (real and imaginary). Also, the Hamilto-
nian is reduced to the Hamiltonian of balance theory in its original form if
all links are real or imaginary in a network.

The evolution of such networks has been studied in Ref. [17]. It has been
shown, while the edges switch their type (using Monte Carlo simulation) to
minimize the energy, the symmetry is spontaneously broken and in the end,
only one type of link (real or imaginary) prevails in the network. The role of
temperature in the model has been studied in [29]. It has been shown that
the model undergoes a first-order phase transition with temperature from a
homogenous ordered phase to a disordered phase.

In this paper, we study the role of external forces in the model. We
analyze the response of the system to the external forces which break the
symmetry in favor of one of these two types. The effect of external forces
on the evolution of the Ising model has been widely studied in the litera-
ture, see for example [30, 31, 32]. It has been shown that below the critical
temperature the response of the systems to the external field depending on
the strength of the external field can have different regimes, see for example
[33]. It is interesting to know that though the Ising model has no hysteresis
at equilibrium when it comes to dynamic, it shows hysteresis, see for exam-
ple [34, 35] and references therein. Such a dynamical hysteresis has come in
handy to address the response of the economic networks to the stimulations.
Interestingly, such studies have provided correct predictions for the response
of the economies of the European Union and the United States to the eco-
nomic stimulations by governments during the 2009 recession [36, 37, 38].

In multi-state systems, the external field breaks the symmetry of the
system, so that one of the minimums is preferred over the others. We also
examine the hysteresis properties of the model. We show that the final state
of the network depends on the initial condition, temperature, and how the
field is applied to it, that is, whether the field is increasing or decreasing during the application. To find stable configurations of this model, we use two approaches: 1- mean-field approximation [39, 40, 41, 42, 43, 29], 2- Monte-Carlo simulation. Finally, we compare the result of these two approaches which are in good agreement with each other.

2. Model

In balance theory, each pair of agents are either friend or enemy which are labeled by +1 and −1 respectively. The energy of each triangle is defined as:

$$\sigma_{ij}\sigma_{jk}\sigma_{ki},$$  \hspace{1cm} (1)

where $\sigma_{ij}$ stands for the relation between agents $i$ and $j$. Then, the Hamiltonian of the network is given by

$$\mathcal{H}_0 = -\sum_{i>j>k}\sigma_{ij}\sigma_{jk}\sigma_{ki}.$$  \hspace{1cm} (2)

In competitive balance theory friendship and enmity have two different bases. So two forms of friendship or enmity exist in relation. In this respect, complex numbers have been utilized to model such complex forms of relationship. In the pair-wise relation, friendship is labeled either by +1 or $-i$ which stands for friendships based on the first interest or the second one. Enmity, as well, has two different labels denoted by $-1$ and $+i$. Now the energy of each triangle is defined as:

$$-Re(\sigma_{ij}\sigma_{jk}\sigma_{ki}) - Im(\sigma_{ij}\sigma_{jk}\sigma_{ki}),$$  \hspace{1cm} (3)

where $Re$ indicates the real part of the product and $Im$ indicates the imaginary part. By this definition, the energy of each triangle in the competitive balance model is either −1 or +1 which is similar to the Heider balance model. This energy shows if the triple relation is in tension or not. So the Hamiltonian of this network is

$$\mathcal{H}_0 = -Re(\sum_{i>j>k}\sigma_{ij}\sigma_{jk}\sigma_{ki}) - Im(\sum_{i>j>k}\sigma_{ij}\sigma_{jk}\sigma_{ki}),$$  \hspace{1cm} (4)

This definition guarantees symmetry between both forms of relation. Besides, the energy of a network reduces to the energy of regular balance if all links are either real or imaginary [17].
Evolution in the competitive balance model has been studied in Refs [17]. In such analysis, an ensemble of fully connected networks starts the evolution with random initial conditions where each link is randomly assigned a number from $\pm 1, \pm i$. Then, using the Monte Carlo method, the edges are randomly selected and updated to minimize the energy as defined in Eq (4). It is observed that during this evolution; the symmetry is broken and one form of relations dominates. In other words, homogeneity arises in the network so that the majority of links are either real or imaginary.

In this work, we examine the effect of an external field on the dynamic of competitive balance theory. The external field is supposed to influence the network in favor of one of the interests. To define interaction with the external field, we aim to impose the following restrictions:

1. The external field is applied to the links ($\sigma_{ij}$) of the network, not the triangles ($\sigma_{ij}\sigma_{jk}\sigma_{ki}$).
2. The external field supports either real or imaginary forms of relation, i.e. it does not matter if the relationship is friendship or enmity.
3. Hamiltonian should be real value; so, we use the quadratic form of $\sigma_{ij}$.

To satisfy the above-mentioned conditions, we add a term to Hamiltonian as

$$H = H_0 - h \sum_{i>j} \sigma_{ij}^2,$$

where $h$ represents the external field. If the value of $h$ is positive, then the field term in Eq. (5) i.e. $-h \sum_{i>j} \sigma_{ij}^2$ is reduced when links turn real. Conversely, if the value of $h$ is negative, then the field term decreases as the number of imaginary links increases. So, the external field breaks the symmetry in favor of one form of relations.

In the absence of an external field the only term for total energy comes from the energy of triangles which we call “mean-triangle-energy” $E_\triangle$:

$$E_\triangle = - \frac{Re(\sum \sigma_{ij}\sigma_{jk}\sigma_{ki}) + Im(\sum \sigma_{ij}\sigma_{jk}\sigma_{ki})}{N_\triangle},$$

where $N_\triangle$ is the total number of triangles in the network. This parameter indicates how close the network is to the balanced state. The value of $E_\triangle$ varies between $-1$ and $+1$. It is $-1$ when the network is in a balanced
state and all triangles are balanced. As the number of unbalanced triangles increases, the value of $E_{\Delta}$ grows, i.e., the network moves away from the balanced state.

In the absence of the external field, the Hamiltonian is symmetrical and the value of energy cannot identify whether the number of real links dominates or the number of imaginary links. The second term in Hamiltonian Eq. 5 however, breaks the symmetry and can identify the dominant type of links. Since below the critical temperature for any given energy, the system has two different symmetrical equilibrium states, we borrow the concept of magnetization and call the following parameter the ”generalized magnetization” $M$:

$$M = \frac{1}{L} \sum_{i>j} \sigma_{ij}^2 = \frac{L_{re} - L_{im}}{L}, \tag{7}$$

where $L_{re}$ and $L_{im}$ are respectively the number of real and imaginary links and $L = (L_{re} + L_{im})$ is the total number of the links in the network. The value of $M$ ranges between -1 and 1.

In the following sections, we obtain the stable states of this model using the mean-field method. Statistical features of systems are seriously influenced by the dimension of the network [44, 45, 46, 47, 48, 49, 50]. Fully connected networks are in the mean-field universality class. So, in our work, we will examine our mean-field solution with the simulation in a fully connected network.

3. Mean-field solution

We consider a fully connected network with $N=50$ nodes. Though the solution does not depend on the size, in a fully connected network, the quantitative values such as the critical temperature depend on the size.

To start, we separate the share of $\sigma_{ij}$ from the rest of the Hamiltonian, i.e.,

$$\mathcal{H} = \mathcal{H}_{ij} + \mathcal{H}',$$  \tag{8}

in which $\mathcal{H}_{ij}$ is the sum of all terms in Hamiltonian that contain $\sigma_{ij}$ and $\mathcal{H}'$ includes the remaining terms. So

$$- \mathcal{H}_{ij} = Re \left[ \sigma_{ij} \sum_{k \neq i,j} \sigma_{jk} \sigma_{ki} \right] + Im \left[ \sigma_{ij} \sum_{k \neq i,j} \sigma_{jk} \sigma_{ki} \right] + h\sigma_{ij}^2. \tag{9}$$
By inserting the above relations in Eq. 10, we obtain:

\[
\langle \sigma_{ij} \rangle = \frac{1}{Z} \sum_G \sigma_{ij} e^{-\beta \mathcal{H}(G)} = \frac{1}{Z} \sum_{\{\sigma \neq \sigma_{ij}\}} e^{-\beta \mathcal{H}'} \sum_{\{\sigma_{ij}=\pm 1, \pm i\}} \sigma_{ij} e^{-\beta \mathcal{H}_{ij}}
\]

\[
= \frac{\sum_{\{\sigma \neq \sigma_{ij}\}} e^{-\beta \mathcal{H}'} \sum_{\{\sigma_{ij}=\pm 1, \pm i\}} \sigma_{ij} e^{-\beta \mathcal{H}_{ij}}}{\sum_{\{\sigma \neq \sigma_{ij}\}} e^{-\beta \mathcal{H}'} \sum_{\{\sigma_{ij}=\pm 1, \pm i\}} e^{-\beta \mathcal{H}_{ij}}} = \frac{\langle \sum_{\{\sigma_{ij}=\pm 1, \pm i\}} \sigma_{ij} e^{-\beta \mathcal{H}_{ij}} \rangle_G'}{\langle \sum_{\{\sigma_{ij}=\pm 1, \pm i\}} e^{-\beta \mathcal{H}_{ij}} \rangle_G'}.
\]  

(10)

\(\langle ... \rangle_G'\) means the ensemble average over other parts of the graph that do not contain \(\sigma_{ij}\). Now, we define \(p \equiv \langle \sigma_{ij} \rangle\).

As a mean-field approximation, we replace \(\sigma_{jk} \sigma_{ki}\) with its ensemble average \(q \equiv \langle \sigma_{jk} \sigma_{ki} \rangle\). So, Eq. 9 can be written as:

\[-\mathcal{H}_{ij} \approx Re [\sigma_{ij}(N-2)q] + Im [\sigma_{ij}(N-2)q] + h\sigma_{ij}^2.\]  

(11)

Since \(p\) and \(q\) are complex numbers, we name the real and imaginary part of them as follows:

\[
\begin{align*}
p_r &= Re(p) ; & p_i &= Im(p) \\
q_r &= Re(q) ; & q_i &= Im(q).
\end{align*}
\]  

(12)

We calculate Eq. 11 for different value of \(\sigma_{ij}\):

\[
\begin{align*}
-\mathcal{H}_{ij}(\sigma_{ij} = +1) &= (N-2)(q_r + q_i) + h \\
-\mathcal{H}_{ij}(\sigma_{ij} = -1) &= -(N-2)(q_r + q_i) + h \\
-\mathcal{H}_{ij}(\sigma_{ij} = +i) &= (N-2)(q_r - q_i) - h \\
-\mathcal{H}_{ij}(\sigma_{ij} = -i) &= -(N-2)(q_r - q_i) - h.
\end{align*}
\]  

(13)

By inserting the above relations in Eq. 10, we obtain:

\[
p(q_r, q_i; N, \beta, h) = \frac{e^{\beta h} \sinh (\beta(N-2)(q_r + q_i)) + i e^{-\beta h} \sinh (\beta(N-2)(q_r - q_i))}{e^{\beta h} \cosh (\beta(N-2)(q_r + q_i)) + e^{-\beta h} \cosh (\beta(N-2)(q_r - q_i))}.\]  

(14)

In the same way, we compute \(q\):

\[
q(q_r, q_i; N, \beta, h) \equiv \langle \sigma_{jk} \sigma_{ki} \rangle = \frac{1}{Z} \sum_G \sigma_{jk} \sigma_{ki} e^{-\beta \mathcal{H}(G')} = \frac{\langle \sum_{\{\sigma_{jk, \sigma_{ki}=\pm 1, \pm i}\}} \sigma_{jk} \sigma_{ki} e^{-\beta \mathcal{H}_{jk}} \rangle_{G''}}{\langle \sum_{\{\sigma_{jk, \sigma_{ki}=\pm 1, \pm i}\}} e^{-\beta \mathcal{H}_{jk}} \rangle_{G''}},
\]  

(15)
Figure 1: Graphical representation of solutions of Eq. 17 for different values of $h$ and $T$. Respectively, black and green curves indicate solutions of $f(q_r, q_i; N, \beta, h) - q_r = 0$ and $g(q_r, q_i; N, \beta, h) - q_i = 0$. The intersections of these two curves are desired solutions. Blue points represent the stable solutions and red points represent the unstable solutions.
where $\langle \ldots \rangle_{G''}$ is ensemble average over all configurations which do not contain $\sigma_{jk}$ and $\sigma_{ki}$. After calculation we end up with:

$$q_{MF} \approx \frac{F(p, q; N, \beta, h)}{G(p, q; N, \beta, h)}.$$

(16)

where details of the calculation and the explicit form of $F$ and $G$ have been provided in Appendix A. By calculating the real and imaginary part of $q$, we obtain two self-consistency equations:

$$\begin{cases}
q_r = \text{Re} \left[ \frac{F(q, p; N, \beta, h)}{G(q, p; N, \beta, h)} \right] \equiv f(q_r, q_i; N, \beta, h) \\
q_i = \text{Im} \left[ \frac{F(q, p; N, \beta, h)}{G(q, p; N, \beta, h)} \right] \equiv g(q_r, q_i; N, \beta, h).
\end{cases}$$

(17)

Numerical solutions of both above equations are plotted separately in Fig. 1 within the allowed range of $q_r, q_i \in [-1, 1]$. Simultaneous solutions of these two equations are the points where two curves cross each other.

The number of solutions depends on the model’s free parameters i.e. $h$ and $T$. As we can see in Fig. 1 and Fig. 2, we have more than one solution for some $h$ and $T$. Each solution is a pair of $(q_r, q_i)$. The stability and instability of the solutions are determined through the perturbation of solution and their recursive update through the Eq. 17; i.e. through analyzing whether the fixed points are attractive or repulsive [29, 43].

As the diagrams on the left column of Fig. 1 shows, in the absence of the external field ($h = 0$) the solutions are symmetric to positive and negative of $q_r$ where the blue points represent the stable solutions and the red ones represent the unstable ones. On the right column of Fig. 1 the external field has a positive value ($h = 10$), and as we expect the symmetry of the graphs disappears. As a result, there is no solution in the range $q_r < 0$ for $T = 10, 15, 20$ in $h = 10$. There are three solutions in the $q_r \geq 0$ region so that two of them are stable and the other one is unstable. For the negative external field, everything becomes reversed and as a result there is no solution in the range $q_r > 0$. Since the behavior of the system concerning the positive and negative fields is perfectly symmetric, we only plot the diagrams of $h > 0$ in Fig. 1.

Fig. 2 illustrates phase diagram in $h$ and $T$ space. We achieve this figure by computing the number of solutions over the different values of external field and temperature. As the figure shows phase transition occurs in
Figure 2: The number of numeric solutions over the different values of external field $h$ and temperature $T$.

$T_c = 17.7$ for $h = 0$. The figure shows that above the transition temperature, there is one solution that corresponds to the unbalanced coexistence region where both balanced and unbalanced triangles and both real and imaginary links are found in almost equal ranges in the network.

The region below the transition temperature shows that there is more than one solution. In this region, one of the interests (real or imaginary) dominates the relations. So, the non-zero value for $q_r$ could be a stable solution. This shows the occurrence of a symmetry-breaking transition.

Now, the mean value of $M$ can be calculated:

$$
\langle M \rangle = \langle \sigma_{ij}^2 \rangle = \frac{1}{Z} \sum_G \sigma_{ij}^2 e^{-\beta H(G)} =
$$

\[
\frac{\sum_{\{\sigma \neq \sigma_{ij}\}} e^{-\beta H'} \sum_{\{\sigma_{ij}=\pm 1, \pm i\}} \sigma_{ij}^2 e^{-\beta H_{ij}}}{\sum_{\{\sigma \neq \sigma_{ij}\}} e^{-\beta H'} \sum_{\{\sigma_{ij}=\pm 1, \pm i\}} e^{-\beta H_{ij}}} =
\]

\[
eq \frac{e^{2\beta h} \cosh \left( \beta (N-2) (q_r + q_i) \right) - \cosh \left( \beta (N-2) (q_r - q_i) \right)}{e^{2\beta h} \cosh \left( \beta (N-2) (q_r + q_i) \right) + \cosh \left( \beta (N-2) (q_r - q_i) \right)}. \tag{18}
\]
In the end, for calculating the mean value of $E_\Delta$ we need to drive $r \equiv \langle \sigma_{ij} \sigma_{jk} \sigma_{ki} \rangle$ (see Appendix B):

$$\langle E_\Delta \rangle = Re(r) + Im(r).$$

By placing mean-field solutions (each pair of $(q_r, q_i)$) in Eq. 18 and Eq. 19, we obtain $M$ and $E_\Delta$. For stable pairs of $(q_r, q_i)$, we obtain stable $M$ and $E_\Delta$. In the same way, we obtain unstable $M$ and $E_\Delta$ from unstable pairs of $(q_r, q_i)$.

In addition to analytical solutions, simulation results for $M$ and $E_\Delta$ has been depicted in Fig. 3 and Fig. 4. As seen, there is a good agreement between simulation and the mean-field solutions. We evaluate the uncertainty of the value calculated over the various simulations; the results show that the simulations can distinguish between the two branches (stable solutions). Only in the case of Fig. 3b the curves are close in the error bars. Still, in them, the curves are distinct by their Standard deviation.
4. Response to the external field

We perform simulations on a fully connected network with $N = 50$ nodes. The links of the network can take four values $-1, +1, -i, +i$. The network evolves via Monte-Carlo simulation. In each update step, one link is randomly selected and converted to one of three other types with equal chance. This update will be accepted if the total energy of the network decreases, otherwise, the conversion will be accepted with probability $p = exp(-\Delta E/kT)$, where $T$ indicates the network temperature and $\Delta E = E_2 - E_1$ is the energy difference between after and before conversion. The process continued until the system relaxed.

The result of simulations is investigated for three different initial conditions: (1) a balanced network in which all links are $+1$. (2) a balanced network in which all links are $-i$. (3) an unbalanced network in which all links are randomly given a value from $\pm 1, \pm i$.

Fig. 3 represents the value of the generalized magnetization for two temperatures: $T = 15$ (below $T_c$) and $T = 25$ (above $T_c$). Fig. 4 represents the mean-triangle-energy for different values of the external field. Results are the average of an ensemble of 100 realization.

As the results show, a system with the competitive balance Hamiltonian
has hysteresis. In other words, over the evolution, the values of macro variables depend on the initial conditions. Such an observation is similar to the Heider balance theory [39].

In the absence of the external field, the Hamiltonian is symmetrical with both paradigms. An external field breaks the symmetry and predominates one of the paradigms. As depicted in Fig. 3 the non-zero value of the generalized magnetization is a consequence of the external field. Above the critical temperature, the system lives in a paramagnetic phase. Below the critical temperature, a discrete transition occurs.

The value of mean-triangle-energy as a function of the external field for the two different temperatures has been depicted in Fig. 4. As it can be seen, the level of energy is different for the two temperatures. For $T = 15 < T_c$, the value of mean-triangle-energy is close to $−1$ which corresponds to a balanced network. For $T = 25$ which is above the critical temperature it is observed that while for strong external fields, the absolute value of magnetization is close to one, the level of mean-triangle-energy is close to zero. In the magnetic systems, as the strength of the external field grows, spins align and as the result, the value of energy declines. In this system, however, a value close to one for the generalized magnetization only means that one of the paradigms has dominated the system. The energy curve clears the fact that the tension is still high in the network. Unlike the Ising model, the competitive balance model in the paramagnetic phase, though a strong external field manages to break symmetry and lead total energy to its minimum, it fails to lead the system to the minimum of mean-triangle-energy. In other words, in the competitive balance model, the dominance of one paradigm does not mean a reduction of tension. Fig. 4 illustrates such differences clearly.

Hysteresis- Since the system has hysteresis, we aimed to figure out the hysteresis diagram. The result has been depicted in Fig. 5. As it can be seen $M$ does not have a single value for a given $h$ and its value depends on its history. This means that once communities form and a paradigm dominate, it is not easy to change the situation and the system resists changes. The impact of temperature on hysteresis loops have been depicted in Fig. 5. As the temperature grows, hysteresis declines. Since the phase transition is discrete, hysteresis vanishes above the critical temperature, where we call it a paramagnetic phase.

The Curie’s law- Another interest is investigating the impact of temperature on susceptibility. Susceptibility which indicates the sensitivity of the generalized magnetization to the applied field $h$ is defined as $\chi = \frac{\partial \langle M \rangle}{\partial h}$. 
Similarities with the magnetic systems raise interest in the study of Curie’s law in the paramagnetic phase. Fig. 6 represents the result. As it can be seen susceptibility has a linear relation to the inverse of temperature.

5. Conclusion

The competitive balance model is a generalization of structural balance theory that takes into account heterogeneity and conflict of interest. The point is that in real-world networks more than one paradigm forms friendships and enmities, which compete with each other. As an example, while in the Middle East religion is the major factor to form relations between countries, in the West the political viewpoint or commercial interest might be more important. The model incorporates two different types of links, with each link type having a positive or negative sign.

In this paper, we analyzed the influence of the external field on the competitive balance model. We presented a mean-field solution of the model in the presence of the external field. Then, we evaluated the results of the Monte-Carlo simulation. We observe that there is a good agreement between the analytical solution and the simulation results. Our results show that the external field breaks the symmetry and leads its favorite paradigm (real or imaginary) to dominate the network. In the ferromagnetic phase (below $T_c$), the tension decreases and the network achieves a balanced state. But in
Figure 6: (a) Generalized magnetization for temperatures far above transition temperature $T \gg T_c$ and under the small external fields $h$, the slope of the line corresponding to each temperature indicates susceptibility $\chi(T)$. (b) Susceptibility against the inverse of temperature, the Curie’s law is detected.

6. Appendix

Appendix A. Mean-Field solution for two-body term $q$

At first step, we separate all terms that contain $\sigma_{jk}$ or $\sigma_{ki}$:

$$\mathcal{H} = \mathcal{H}_{\wedge jki} + \mathcal{H}''$$  \hspace{1cm} \text{(A.1)}
$$-\mathcal{H}_{jk_i} = \text{Re} \left[ \sigma_{jk} \sum_{\ell \neq i,j,k} \sigma_{j \ell} \sigma_{i k} \right] + \text{Im} \left[ \sigma_{jk} \sum_{\ell \neq i,j,k} \sigma_{j \ell} \sigma_{i k} \right] + \text{Re} \left[ \sigma_{ki} \sum_{\ell \neq i,j,k} \sigma_{ki} \sigma_{\ell i} \right]$$
$$+ \text{Im} \left[ \sigma_{ki} \sum_{\ell \neq i,j,k} \sigma_{ki} \sigma_{\ell i} \right] + \text{Re} (\sigma_{ij} \sigma_{jk} \sigma_{ki}) + \text{Im} (\sigma_{ij} \sigma_{jk} \sigma_{ki}) + h(\sigma_{jk}^2 + \sigma_{ki}^2)$$
$$\approx \text{Re} [\sigma_{jk}(N-3)q] + \text{Im} [\sigma_{jk}(N-3)q] + \text{Re} [\sigma_{ki}(N-3)q] + \text{Im} [\sigma_{ki}(N-3)q]$$
$$+ \text{Re} (\sigma_{ij} \sigma_{jk} \sigma_{ki}) + \text{Im} (\sigma_{ij} \sigma_{jk} \sigma_{ki}) + h(\sigma_{jk}^2 + \sigma_{ki}^2)$$

(A.2)

Different modes that these two links can take are

$$-\mathcal{H}_{jk_i} (\sigma_{jk} = 1, \sigma_{ki} = 1) = 2(N-3)(q_r + q_i) + p_r + p_i + 2h$$
$$-\mathcal{H}_{jk_i} (\sigma_{jk} = -1, \sigma_{ki} = -1) = -2(N-3)(q_r + q_i) + p_r + p_i + 2h$$
$$-\mathcal{H}_{jk_i} (\sigma_{jk} = 1, \sigma_{ki} = -1) = -(p_r + p_i) + 2h \rightarrow (\times 2)$$
$$-\mathcal{H}_{jk_i} (\sigma_{jk} = -1, \sigma_{ki} = 1) = 2(N-3)(q_r - q_i) - (p_r + p_i) - 2h$$
$$-\mathcal{H}_{jk_i} (\sigma_{jk} = -1, \sigma_{ki} = -1) = 2(N-3)(q_r + q_i) - (p_r + p_i) - 2h$$

(A.3)

By substituting above relations into Eq. [15] we attain following equation:

$$\langle \sigma_{jk} \sigma_{ki} \rangle \approx F(p, q; N, \beta, h)$$
$$G(p, q; N, \beta, h),$$

(A.4)

where

$$F(p, q; N, \beta, h) = e^{\beta[2(N-3)(q_r + q_i) + p_r + p_i + 2h]} + e^{\beta[2(N-3)(q_r + q_i) + p_r + p_i + 2h]} - 2e^{\beta[(p_r + p_i) + 2h]}$$
$$- e^{\beta[2(N-3)(q_r - q_i) - (p_r + p_i) - 2h]} - e^{\beta[2(N-3)(-q_r + q_i) - (p_r + p_i) - 2h]} + 2e^{\beta[p_r + p_i + 2h]}$$
$$+ 2ie^{\beta[2(N-3)q_r + p_r - p_i]} - 2ie^{\beta[2(N-3)q_r - p_r + p_i]} - 2ie^{\beta[-2(N-3)q_r - p_r + p_i]}$$
$$+ 2ie^{\beta[-2(N-3)q_r + p_r - p_i]},$$

(A.5)
\[ G(p, q; N, \beta, h) = e^{\beta[2(N-3)(q_r+q_i)+p_r+p_i+2h]} + e^{\beta[-2(N-3)(q_r+q_i)+p_r+p_i+2h]} + 2e^{\beta[-(p_r+p_i)+2h]} + e^{\beta[2(N-3)(q_r-q_i)-(p_r+p_i)-2h]} + e^{\beta[2(N-3)(-q_r+q_i)-(p_r+p_i)-2h]} + 2e^{\beta[p_r+p_i-2h]} + 2e^{\beta[2(N-3)q_r+p_r-p_i]} + 2e^{\beta[2(N-3)q_i-p_r+p_i]} + 2e^{\beta[-2(N-3)q_i-p_r+p_i]} + 2e^{\beta[-2(N-3)q_r+p_r-p_i]} \]  
\[ (A.6) \]
Appendix B. Mean-Field solution for three body interactions

Similar to the previous sections, at first, we must separate the sentences that contain links $\sigma_{ij}$, $\sigma_{jk}$, $\sigma_{ki}$:

$$\mathcal{H} = \mathcal{H}_{\Delta_{ijk}} + \mathcal{H}''$$  \hspace{1cm} (B.1)

$$r \equiv \langle \sigma_{ij}\sigma_{jk}\sigma_{ki} \rangle = \frac{1}{Z} \sum_{G} \sigma_{ij}\sigma_{jk}\sigma_{ki} e^{-\beta \mathcal{H}(G)}$$

$$= \frac{\sum_{\{\sigma \neq \sigma_{ij},\sigma_{jk},\sigma_{ki}\}} e^{-\beta \mathcal{H}''} \sum_{\{\sigma_{ij},\sigma_{jk},\sigma_{ki}=\pm1,\pm i\}} \sigma_{ij}\sigma_{jk}\sigma_{ki} e^{-\beta \mathcal{H}_{\Delta_{ijk}}}}{\sum_{\{\sigma \neq \sigma_{ij},\sigma_{jk},\sigma_{ki}\}} e^{-\beta \mathcal{H}''} \sum_{\{\sigma_{ij},\sigma_{jk},\sigma_{ki}=\pm1,\pm i\}} e^{-\beta \mathcal{H}_{\Delta_{ijk}}}}$$  \hspace{1cm} (B.2)

$$-\mathcal{H}_{\Delta_{ijk}} = \text{Re} \left[ \sigma_{ij} \sum_{\ell \neq i,j,k} \sigma_{i\ell} \sigma_{\ell j} \right] + \text{Im} \left[ \sigma_{ij} \sum_{\ell \neq i,j,k} \sigma_{i\ell} \sigma_{\ell j} \right] + \text{Re} \left[ \sigma_{jk} \sum_{\ell \neq i,j,k} \sigma_{j\ell} \sigma_{\ell k} \right]$$

$$+ \text{Im} \left[ \sigma_{jk} \sum_{\ell \neq i,j,k} \sigma_{j\ell} \sigma_{\ell k} \right] + \text{Re} \left[ \sigma_{ki} \sum_{\ell \neq i,j,k} \sigma_{k\ell} \sigma_{\ell i} \right] + \text{Im} \left[ \sigma_{ki} \sum_{\ell \neq i,j,k} \sigma_{k\ell} \sigma_{\ell i} \right]$$

$$+ \text{Re} (\sigma_{ij}\sigma_{jk}\sigma_{ki}) + \text{Im} (\sigma_{ij}\sigma_{jk}\sigma_{ki}) + h(\sigma_{ij}^2 + \sigma_{jk}^2 + \sigma_{ki}^2)$$

$$\approx \text{Re} [\sigma_{ij}(N-3)q] + \text{Im} [\sigma_{ij}(N-3)q] + \text{Re} [\sigma_{jk}(N-3)q] + \text{Im} [\sigma_{jk}(N-3)q]$$

$$+ \text{Re} [\sigma_{ki}(N-3)q] + \text{Im} [\sigma_{ki}(N-3)q] + \text{Re} (\sigma_{ij}\sigma_{jk}\sigma_{ki}) + \text{Im} (\sigma_{ij}\sigma_{jk}\sigma_{ki})$$

$$+ h(\sigma_{ij}^2 + \sigma_{jk}^2 + \sigma_{ki}^2)$$  \hspace{1cm} (B.3)

$$r_r = \text{Re}(r), \quad r_i = \text{Im}(r),$$  \hspace{1cm} (B.4)

$$\langle E_\Delta \rangle = r_r + r_i.$$  \hspace{1cm} (B.5)

So we have:

$$\langle E_\Delta \rangle \overset{MF}{=} \frac{V(q;N,\beta,h)}{W(q;N,\beta,h)},$$  \hspace{1cm} (B.6)
where:

\[
V(q; N, \beta, h) = -3e^{\beta[-(N-3)(3q_r-q_i)+1-h]} - e^{\beta[-3(N-3)(q_r-q_i)+1-3h]} - 3e^{\beta[-(N-3)(q_r+3q_i)+1+h]} \\
- 3e^{\beta[-(N-3)(q_r+q_i)+1+3h]} - 6e^{\beta[-(N-3)(q_r-q_i)+1+3h]} - 3e^{\beta[(N-3)(q_r-3q_i)+1-h]} \\
+ 3e^{-\beta[(N-3)(q_r-3q_i)+1+h]} - 3e^{\beta[(N-3)(q_r-q_i)+1-3h]} - 6e^{\beta[(N-3)(q_r+q_i)+1-h]} \\
+ 6e^{-\beta[(N-3)(q_r+q_i)+1+h]} - 3e^{\beta[(N-3)(3q_r+q_i)+1+h]} - e^{\beta[(N-3)(q_r+3q_i)+1+3h]} \\
+ 6e^{-\beta[(N-3)(q_r-3q_i)-1+h]} + 3e^{\beta[(N-3)(3q_r-q_i)-1+h]} + 3e^{-\beta[-(N-3)(q_r+q_i)-1-3h]} \\
+ e^{\beta[-3(N-3)(q_r+q_i)-1+3h]} + 3e^{-\beta[-(N-3)(q_r-q_i)-1-3h]} \tag{B.7}
\]

\[
W(q; N, \beta, h) = 3e^{\beta[-(N-3)(3q_r-q_i)+1-h]} + e^{\beta[-3(N-3)(q_r-q_i)+1-3h]} + 3e^{\beta[-(N-3)(q_r+3q_i)+1+h]} \\
+ 3e^{\beta[-(N-3)(q_r+q_i)+1+3h]} + 6e^{\beta[-(N-3)(q_r-q_i)+1+3h]} + 3e^{\beta[(N-3)(q_r-3q_i)+1-h]} \\
+ 3e^{-\beta[(N-3)(q_r-3q_i)+1+h]} + 6e^{\beta[(N-3)(q_r-q_i)+1-3h]} + 6e^{\beta[(N-3)(q_r+q_i)+1-h]} \\
+ 6e^{-\beta[(N-3)(q_r+q_i)+1+h]} + 3e^{\beta[(N-3)(3q_r+q_i)+1+h]} + e^{\beta[(N-3)(q_r+3q_i)+1+3h]} \\
+ 6e^{-\beta[(N-3)(q_r-3q_i)-1+h]} + 3e^{\beta[(N-3)(q_r+q_i)-1+3h]} + 3e^{-\beta[(N-3)(q_r+3q_i)-1+h]} \\
+ e^{\beta[(N-3)(3q_r-q_i)-1-3h]} + 3e^{\beta[-(N-3)(q_r+q_i)-1-3h]} + 3e^{-\beta[-(N-3)(q_r-q_i)-1-3h]} \tag{B.8}
\]

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