Alternative to Higgs and Unification

renormalization and essential singularity

Miyuki NISHIKAWA*)

Physics Department, University of Tokyo, Tokyo 113

Abstract

In this paper, we discuss the self-consistency condition for the spherical symmetric Klein-Gordon equation, and then discuss a natural possibility that gravity and weak coupling constants $g_G$ and $g_W$ may be defined after $g_{EM}$. In this point of view, gravity and the weak force are subsidiary derived from electricity. Particularly, $SU(2)_L \times U(1)$ unification is derived without assuming a phase transition. A possible origin of the Higgs mechanism is proposed. Each particle pair of the standard model is associated with the corresponding asymptotic expansion of an eigen function.

*) E-mail: nisikawa@hep-th.phys.s.u-tokyo.ac.jp
§1. Introduction

In usual dimensional counting, momentum has dimension one. But a function $f(x)$, when differentiated $n$ times, does not always behave like one with its power smaller by $n$. For example, this can occur in the neighborhood of $x = 0$ if the function $f(x)$ has an essential singularity at $x = 0, f(0) \to 0(x \to 0)$. Thus a dimension of momentum is such an operator that cannot be fixed unless the operand of the differential operator is explicit. This inevitable uncertainty may be essential in general theory of renormalization, including quantum gravity.

As an example, we first consider a one-dimensional Schrödinger equation, noting the two possible cases of singularities for a potential in section 2. In section 3 we construct the most general type of a singularity that is closed in usual operations. Then we classify the singularities for a potential by comparing the eigen function $y(x)$ with its second derivative, assuming that $y$ is $C^2$-class. The result crucially depends on the analytic property of the eigen function near its 0 point. In cases the eigen function has an essential singularity at $z = 0$ which is $C^2$-class when approached from the positive real axis, the eigen function does not satisfy the Lipschitz continuity condition. Then the solution is not always unique, which may be the origin of the gauge ambiguity important to quantum field theory. Section 4 is devoted to a philosophical discussion of this kind.

Section 5 is the extension to higher dimensions and the long distance limit. We consider the Klein-Gordon equation with a spherical symmetric $U(1)$ potential $A^\mu := (\phi(r), 0, 0, 0)$, assuming that the potential $\phi$ has at least one normalizable eigen function $R(r)$, which in turn creates another potential $\phi' \propto R$. There are 10 cases in the long distance limit.

Applying the results of the previous section, we can obtain theorems for the long distance limit in section 6. Particularly, we can prove $SU(2)_L \times U(1)$ unification without assuming a phase transition or the detailed Higgs mechanism. Section 7 is a proposal for the origin of the Higgs mechanism, making use of the asymptotic mathematical ambiguity of the expansions or degenerate eigen functions. In section 8 we try to deal with gravity within the framework of the standard model $+\alpha$ and redefine gravity as an integral constant of the laplacian. Furthermore, we associate each pair of the particles in the standard model with the corresponding asymptotic expansion.

§2. Possibility of singularity and domain of definition

For simplicity, let us first consider a one-dimensional Schrödinger equation

$$-y'' + V(x)y = Ey.$$  \hfill (2.1)
$y(x)$ is defined in $0 < x < x_0$, where $x_0$ is some positive constant. In fact, any $C^2$-class function $y(x)$ satisfies (2.1) if we take

$$V = y''/y, \quad E = 0.$$ (2.2)

Here the replacement of the constant $E \rightarrow E'$ is equivalent to $e\phi \rightarrow e\phi - E' + E$, so from now on we take $E = 0$. There are 2 possible cases for (2.2) to have a singularity at $x \rightarrow +0$:

(I) $y(x) \rightarrow 0$ for $x \rightarrow +0$,

(II) $y''$ does not converge for $x \rightarrow +0$.

For (I), the $V(x)$ in the L.H.S. of (2.2) is called a potential iff there exists at least one $C^2$-class eigen function $y(x)$ satisfying (2.1).

§3. Classification of the power of possible singularities

Now let us move the possible singularity to $x = 0$ by the redefinition of the origin and consider the behavior of $V$ as $x \rightarrow +0$. Let $y(z)$ be the natural analytic continuation of $y(x)$ (from the real axis) to the complex plane.

(CASE 1) $y(z)$ has no essential singularity at $z = 0$.

(a) If $y(z)$ can be Laurent expanded around $z = 0$ as

$$y = \sum_{n=k}^{\infty} a_n z^n, \quad a_k \neq 0,$$ (3.1)

then

$$\frac{y''}{y} = \frac{\sum_{n=k}^{\infty} a_n n(n - 1) z^{n-2}}{\sum_{n=k}^{\infty} a_n z^n} \rightarrow \begin{cases} \frac{a_d d(d - 1) z^{d-2-k}}{a_k} (0 \leq k) \\ k(k - 1)z^{-2} (k < 0) \end{cases},$$ (3.2)

with $d$ the lowest power such that $a_d \neq 0$ and $1 < d$ (if there is no such $d$, $a_d = 0$).

(b) When we replace the power of the finite number of terms in the type (a) expansion with an arbitrary real number, *):

$$\frac{y''}{y} \rightarrow \begin{cases} \frac{a_d d(d - 1) z^{d-2-k}}{a_k} \left( y = a_0 + a_1 z + a_d z^d \cdots \text{ or} \right) \\ k(k - 1)z^{-2} (k < 0) \text{ (otherwise)} \end{cases},$$ (3.3)  

* From now on, the expansion coefficients are all real except if mentioned, and the branch is chosen so that the function takes unique real value at $z \rightarrow +0$. More precisely, a branching point with the power of an irrational number is an essential singularity, but the difference is not important here.
where $a_d$ is the coefficient of the lowest power except for 0, 1. Thus far, the powers $\nu$ where
the potential can behave like $V \to x^\nu$ as $x \to +0$ are
for (I), $\nu = -2$; $-1 \leq \nu$,
for (II), $-2 \leq \nu < 0$.

(c) 
\[ y = \sum_{n=l}^{k} a_n (\log z)^n, \quad a_l \neq 0, \]  
(3.4)
if the above expansion is possible, then
\[ \frac{y''}{y} = \sum_{n=l}^{k} n a_n \{(n-1)(\log z)^{n-2} - (\log z)^{n-1}\} \to \frac{-k}{z^2 \log z}, \]  
(3.5)
where $\log z$ diverges as $z \to 0$, but for an arbitrary integer $n$, $z(\log z)^n$ tends to 0. So we can regard $\log z$ as ‘an infinitely small negative power’ $z^{-\epsilon} (\epsilon > 0)$. Then we can generalize type (b) expansion by the replacement of the finite number of terms
\[ a_n z^n \to z^n \sum_{m=l_n}^{k_n} a_{mn} (\log z)^m \quad (m \in R). \]  
(3.6)
This has the effect of
\[ \begin{cases} 
  z^{d-2-k} \to z^{d-2-k}(\log z)^m \quad (m \in R) \\
  z^{-2} \to z^{-2}/\log z 
\end{cases} \]  
(3.7)
in (3.3), i.e.,
\[ \text{for (I), } \nu = -2 (+\epsilon) ; \quad -1 \leq \nu. \]  
(3.8)
*) Let us call this type of expansion type (c). For type (c) expansions, We can define the index of power $k_y, \mu_y, \nu_y (z \to +0)$ as follows:
\[ y \to z^{k_y}, \quad y' \to z^{\mu_y}, \quad y'' \to z^{\nu_y}. \]  
(3.9)
Type (c) property is invariant under finite times of summations, subtractions, and differentiations.

(d) When we apply finite times of summations, subtractions, multiplications, divisions (by $\neq 0$), differentiations, and compositions (with the shape of $f(g(z))$, $0 \leq k_y, \quad g(+0) = +0$ where $f, g$ are type (c) expansions), $k_y, \mu_y, \nu_y$ can also be defined. As an arbitrary type (d)
expansion \( f(z) \) has a countable number of terms and a nonzero 'radius of convergence \(^*\) ', \( r \) where the expansion converges for \( 0 < |z| < r \), it can be written as

\[
f(z) = \sum_{n=0}^{\infty} f_n.
\]  

(3.10)

As the 'principal part' which satisfies \( k_{f_n} < 0 \) consists of finite number of terms, a type (d) expansion diverges or converges monotonically as \( z \to +0 \), so enables the expansion of (3.10) in the order of ascending powers. As the expansion is almost the same as that of type (c) (the only differences are the multiplications by \((\log z)^n\) for an infinite number of terms and the appearance of the terms like \(\log(z \log z)\)), the region of \( \nu_y \) remains.

(CASE 2) \( y(z) \) has an isolated essential singularity at \( z = 0 \). In complex analysis, a sequence of points can converge to any value depending on its approach to an essential singularity (with infinite order)\(^3\). But now that we deal with only the case along the real axis \( z \to +0 \), the limit is sometimes well defined. Let us study the following cases.

(e) When the following expansion is possible (type (e)): \( y = \pm e^{f(z)} \), where \( f \) is a type (d) expansion. We can define the finite values \( \mu_y, \nu_y \) by

\[
\begin{align*}
\frac{y'}{y} &= f' \to z^{\mu_y}, \quad \mu_y = k_f + \mu_f, \\
\frac{y''}{y} &= f'^2 + f'' \to z^{\nu_y}, \quad \nu_y \geq \min(2k_f + 2\mu_f, k_f + \nu_f).
\end{align*}
\]

Let us consider the region of \( \nu_y \). For \( k_f \geq 0 \) it is the same as for the type (d). For

\[
y = e^{az^k}, \quad a, k \in \mathbb{R}, \quad k \leq 0
\]

satisfies

\[
\frac{y''}{y} = a^2 k^2 z^{2k-2} + ak(k-1)z^{k-2} \to \begin{cases} z^{-2\pm \epsilon} \ (k = -\epsilon) \\ a^2 k^2 z^{2k-2} \ (k < 0) \end{cases},
\]

combination with type (c) case leads to the region of \( \nu_y \) being:

for (I), \( \nu_y \leq -2 + \epsilon \); \(-1 \leq \nu_y \),

for (II), an arbitrary negative number.

Let us then consider if we can fill the remaining ‘window’ of the region of \( \nu_y \) for (I),

\(-2 + \epsilon < \nu_y < -1.\)

(f) When we can write \( y = f_0 + \sum_{n=1}^{m}(\pm)e^{f_n} \), where \( f_n \) is of type (c), \( k_{f_n} < 0 \), and \((\pm)\) takes each of the signatures +- -. We can assume that each terms in \( \sum \) are ordered in the

\(^*\) The meaning of this term is different from the usual one because \( z = 0 \) can be a singularity point.
increasing absolute values for $z \to +0$. Because

$$e^{az^k} \to \begin{cases} 
  z^0 & (k \geq 0, \ a \neq 0) \\
  0 & (k < 0, \ a < 0) \\
  \infty & (k < 0, \ a > 0)
\end{cases}$$

and $y \to 0$ for (I),

$$y = \left( \sum_{n=0}^{\infty} \sum_{m=0}^{m_n} a_{nm} z^n (\log z)^m \right) + \sum_{n=1}^{l} \left( \sum_{k_n} \sum_{i=0}^{i_n} a_{ni} z^i (\log z)^j \right). \quad (3.12)$$

If the second term sum at the R.H.S. is not 0, we can write

$$k_1 < \cdots < k_1 < 0, \ a_{nk_n k_{ni}} < 0. \quad (3.13)$$

As $y$ is of $C^2$-class, the first term can be written as

$$y = a_{10} z + \sum_{n=2}^{\infty} \sum_{m=1}^{m_n} \cdots, \ m_2 = 0. \quad (3.14)$$

As

$$y'' \to \begin{cases} 
  (z^n (\log z)^m)' \to z^{n-m\epsilon - 2} & \text{The term such that } n - m\epsilon \text{ is the smallest} \\
  \{a_{nk_n k_{ni}} z^{k_n} (\log z)^{k_{ni}}\}^{-2} + \{a_{nk_n k_{ni}} z^{k_n} (\log z)^{k_{ni}}\}^{-2} \\
  \times e^{\sum_{i=k_n}^{\infty} \sum_{j=0}^{i_n} a_{ni} z^i (\log z)^j} & (\forall a_{nm} = 0)
\end{cases} \quad (3.15)$$

for $z \to +0$,

$$y'' \to \begin{cases} 
  z^{n-m\epsilon - 3} (a_{10} \neq 0 \text{ and } \exists a_{nm} \neq 0) \\
  z^{2k_n - 2k_n \epsilon - 2} e^{a_{nk_n k_{ni}} z^{k_n} (\log z)^{k_{ni}}} \to 0 & (a_{10} \neq 0 \text{ and } \forall a_{nm} = 0) \\
  z^{-2} (a_{10} = 0 \text{ and } \exists a_{nm} \neq 0) \\
  z^{2k_n - 2k_n \epsilon - 2} (a_{10} = 0 \text{ and } \forall a_{nm} = 0)
\end{cases} \quad (3.16)$$

The possible values of $\nu_y$ for (I) remain the same: $\nu_y \leq -2 + \epsilon$; $-1 \leq \nu_y$.

(g) Whole of the expansions obtained from type (f) expansions by finite times of summations, subtractions, multiplications, divisions (by $\neq 0$), differentiations, and compositions (with the shape of $f(g(z))$, $0 \leq k_g$, $g(+0) = +0$ where $f, g$ are type (f) expansions).

This type of expansion is very complicated compared to an ordinary Laurent expansion, but in any case has a countable number of terms and a nonzero ‘radius of convergence’ $r$.
where $y$ is analytic for $0 < |z| < r$. This can also be ordered partially in the ascending powers and we can write the first term explicitly, and so monotonically diverges or converges but never oscillates as $z \to +0$. Its general shape is the whole sum

$$ (1)_i + (2)_j + \cdots + (m)_k , \quad (3.17) $$

where

$$(1)_i := \left( \sum_{m_1, \ldots, m_{i_d} = -\infty}^{\infty} \sum_{n \in \{n\}_i} a_{inm_1 \cdots m_{i_d}} z^n (-\log z)^{m_1} (-\log(-z/\log z))^{m_2} \cdots (-\log(-z/(-\log(-z/\log \cdots z)))))^{m_{i_d}} \right)_i ,$$

$$(2)_{\pm j} := \sum_{i \in \{i\}_j} (\pm) e^{\pm(1)_i} ,$$

$$(3)_{\pm k} := \sum_{j \in \{j\}_k} (\pm) e^{\pm(2)_j} ,$$

$$\vdots \quad (3.18)$$

Here the $(\pm)$ in front of $e$ takes each of the signatures depending on each $i$ (or $j, k, \cdots$), while the $\pm$ on the shoulder of $e$ and in front of $j, k, \cdots$ takes the signature such that the coefficient of the first term in $\sum$ is of the same signature as $j$ after choosing the signatures. Each term is ordered in partially ascending powers with regards for any sums. The sum with index $n$ is performed according to the monotonically non-decreasing sequence of real numbers $\{n_i \} (-\infty < n_i)$ depending on $i$. In the same manner, the sum with index $i, j, \cdots$ is performed according to the finite, monotonically non-decreasing sequence $\{i_j \}, \{j_k \} \cdots$ of natural numbers. $m_1, \ldots, m_{i_d}$ take finite values, but they increase in correspondence with $n$ and grows $\to \infty$ as $n \to \infty$, and depend on $i$. $d_i$ is the maximal ‘depth’ of the composition of logs, or the number of logs, depending on $i$ and of finite value. ***)

As the sum of the shape of $(m)_i$ can always be represented as the exp of the infinite sum of the same shape,

$$(m)_i = (\pm)e^{(m)_0} , \quad (m)_0 := \log \left( \text{sum of the finite number of } e^{(m-1)\cdot s} \right)$$

$$= (m - 1)_1 + \log \left( 1 \pm e^{(m-1)_2} + \cdots \right) , \quad (3.19)$$

type (g) expansion can in fact be written in only ‘one term’ exp$(m)_{i+1}$.

*) Of course, the meaning is different from the usual one.

***) The power is smaller when $m_1 + m_2 + \cdots + m_i$ is greater for the same $n$, and when it is also the same and $m_1$ is smaller, and when it is also the same and $m_2$ is smaller, ..., and so on.
Now, for the part of $i \leq 0$ in $(m)_i$, satisfying $0 \leq k(m)_i$, $\exp(m)_i$ can be written within the shape of $(m)_i$ as the composition of $e^z$ and $(m)_i$. Then we can write for (I)
\[
y = bz + \sum_{n=2}^{\infty} a_n z^n \sim + \sum_{i<0} (\pm)e^{-b_i z^{\infty}} + \sum_{j<0} (\pm)e^{-c_j z^{\infty}} + \sum_{k<0} (\pm)e^{-d_k z^{\infty}} \cdots,
\]
where $b_i, c_j, d_k, \cdots > 0$, $\sim$ represents the abbreviation of $\log z \sim$, and $\cdots$ the higher order terms. The power of $y''/y$ can be classified by whether $b = 0$ or not, and what is the first of $b_i, c_j, d_k, \cdots$ such that the corresponding term is not 0:
\[
y''/y \rightarrow \begin{cases} 
(\pm) z^{n-me-3} & (b \neq 0 \text{ and } 3a_n \neq 0, \ n-me \geq 2) \\
(\pm) 0 & (b \neq 0 \text{ and } \forall a_n = 0 \text{ and } 3b_i \text{ or } c_j \text{ or } d_k \cdots > 0) \\
+ze^{-2} & (b = 0 \text{ and } 3a_n \neq 0) \\
+ze^{2+2b-2} & (b = \forall a_n = 0 \text{ and } 3b_i > 0) \\
+\infty & (b = \forall a_n = \forall b_i = 0 \text{ and } 3c_j \text{ or } d_k \text{ or } \cdots > 0)
\end{cases},
\]
where $\forall b_i = 0$ means that there is no term in $\sum_{i<0}$.

After all, $\nu_y \leq -2 + \epsilon, \ -1 \leq \nu_y$ for (I), where $\epsilon$ represents the power like $\log z \sim$.

(h) It is unclear to me whether there are other cases. But we shall not discuss such cases further, for type (g) expansion is closed in usual operations, and thus is most general.

(CASE 3) $y(z)$ has a non-isolated essential singularity at $z = 0$.

(i) When we allow complex coefficients in (g). The discussion above is almost valid in this case, except that when $a$ is complex $e^{az}$ shows oscillatory behavior, and so $y$ is not monotonic as $z \rightarrow 0$ and generally has an accumulation point of poles or essential singularities, keeping us away from defining $k_y, \mu_y, \text{or } \nu_y$. For example,
\[
y = z^5 \sin(z^{-1})
\]
satisfies the condition of (I) and the term with the smallest power in $y$ cancels that of $y''$, yet higher order oscillation remains.

(j) It is unclear to me whether there are other cases. In such a case $\nu_y$ would not be clearly physical, even if defined. Therefore, we shall neglect such possibilities.

§4. Physical Explanation of the Result

The above result is not mathematically perfect, but shows that very wide types of functions such that closed in usual operations, only by satisfying the second order differential
equation, can restrict the behavior of the potential. Or physically, if there exists a wave function that can be applied to every point of the world, the point of nonzero charge should also be included in the domain, which determines the shape of a force. We will see this for more general case in the next section.

Notice that type (g) expansion is valid under the special rule that we must not decompose an exponential until the end of the calculation. Each expansion has several infinite series of different order. Having nonzero ‘radius of convergence’, it can be calculated as a usual function. Instead, near \( z = 0 \), if we do not obey the rule and try to calculate by extracting all the terms below a certain order, the result, even if finite, may depend on the arrangement of terms. (It is known in mathematics that an infinite series that does not converge absolutely does not always converge to a unique value.) This implies an interesting non-commutative property.

Notice also that the difficulties caused by point-like particles may be absent here. If we assume that the existence of an eigen function is more fundamental than that of a potential, there can be the region where the potential is not defined (where the eigen function is 0). Even if the analyticity of matter field is not a quantity distinguished by finite times of measurement, this inevitable ambiguity may be the origin of gauge uncertainty \(^4\).

§5. Extension to higher dimensions

We can extend the results to a spatial dimension \( N \) as follows. Let us consider a spherical-symmetric Klein-Gordon equation with a time-independent \( U(1) \) gauge potential \( A^\mu := (\phi(r), 0, 0, 0) \) (only the first time component is nonzero and the rest \( N - 1 \) components are 0),

\[
-\Delta y = \frac{(E - e\phi)^2 - m^2 c^4}{\hbar^2 c^2} y =: -V(r)y.
\]

(5.1)

For simplicity, we assume that the eigen function \( y \) is a \( N \)-dimensional spherical symmetric function \( R(r) \). For \( a = 0 \) and \( N \neq 1 \), (3.21) is clearly replaced by

\[
\frac{\Delta R(r)}{R(r)} = \frac{R''}{R} + \frac{N - 1}{r} \frac{R'}{R} \rightarrow \begin{cases} 
+ (N - 1)r^{-2} & (a \neq 0) \\
+ n(n + N - 2)r^{-2} & (a = 0 \text{ and } \exists a_n \neq 0) \\
+ (-ib_i)^2 r^{2i+2c-2} & (a = \forall a_n = 0 \text{ and } \exists b_i > 0) \\
+ \infty & (a = \forall a_n = \forall b_i = 0 \text{ and } \exists c_j \text{ or } d_k \text{ or } \cdots > 0)
\end{cases}.
\]

(5.2)
We can extend the results to \( r \to \infty \) case as follows. If we change the variable to \( z := \frac{1}{r} \) and assume that \( R(z) \) is \( C^2 \)-class (expanded as below)

\[
R = a + bz + \sum_{n=2}^{\infty} a_n z^n \sim \cdots + \sum_{i<0} (\pm) e^{-b_i z^{\cdot \cdot \cdot}} \cdots + \sum_{j<0} (\pm) e^{-c_j z^{\cdot \cdot \cdot}} \cdots + \sum_{k<0} (\pm) e^{-d_k z^{\cdot \cdot \cdot}} \cdots ,
\]

(5.2) is clearly replaced by

\[
\Delta R(r) = \frac{1}{R(z)} \left\{ \frac{dz}{dr} \frac{dR}{dz} + (N - 1) \frac{dz}{dr} \frac{dR}{dz} \right\} = z^4 \frac{R''(z)}{R(z)} - z^3 (N - 3) \frac{R'(z)}{R(z)}
\]

\[
\to \begin{cases} 
(3 - N) \frac{b}{a} z^3 & (a \neq 0 \text{ and } b \neq 0 \text{ and } N \neq 3) \\
(n - N + 2) a a_n z^{n+2} & \text{(or higher order)} \quad (a \neq 0 \text{ and } b = 0 \text{ and } 3a_n \neq 0 \text{ and } N \neq 3) \\
(n - 1) a a_n z^{n+2} & (a \neq 0 \text{ and } 3a_n \neq 0 \text{ and } N = 3) \\
(\pm) 0 & (a \neq 0 \text{ and } b = -\nu a_n = 0 \text{ and } 3b_i \text{ or } c_j \text{ or } d_k \text{ or } \cdots > 0) \\
(3 - N) z^2 & (a = 0 \text{ and } b \neq 0 \text{ and } N \neq 3) \\
(n - 1) a a_n z^{n+1} & (a = 0 \text{ and } b \neq 0 \text{ and } 3a_n \neq 0 \text{ and } N = 3) \\
(\pm) 0 & (a = b = -\nu a_n = 0 \text{ and } 3b_i \text{ or } c_j \text{ or } d_k \text{ or } \cdots > 0 \text{ and } N = 3) \\
(n - N + 2) n z^2 & \text{(or higher order)} \quad (a = b = 0 \text{ and } 3a_n \neq 0) \\
+(-ib_i)^2 z^{2i+2e+2} & (a = b = -\nu a_n = 0 \text{ and } 3b_i > 0) \\
+\infty & (a = b = -\nu a_n = -\nu b_i = 0 \text{ and } 3c_j \text{ or } d_k \text{ or } \cdots > 0)
\end{cases} 
\]

(5.4)

\[ ^* \text{Noting that } 2 \leq n \text{ and } i < 0, \text{ we conclude the potential } V(r) \text{ as } r \to \infty \text{ must be positive for } (N \leq 3 \text{ and } \nu = -2) \text{ or } -2 < \nu, \text{ where } \nu \text{ is the power of the potential } V \to r^\nu \text{ as } r \to \infty; \text{ can take both signs for other cases. There is no reason to assume that } R(z) \text{ is } C^2 \text{-class, but}
\]

\[ ^{\ast} \text{The line 2 includes the case } a \neq 0 \text{ and } b = 0 \text{ and } 3a_n \neq 0 \text{ and } N = n + 2 \neq 3, \text{ when } \frac{\Delta R(r)}{R(r)} \to (m - N + 2)m a_{n+2} z^{m+2} \text{ or the like, where } a_m \text{ is the term next to } a_n z^n. \]

\[ \text{The line 8 includes the case } a = b = 0 \text{ and } 3a_n \neq 0 \text{ and } N = n - 2, \text{ when } \frac{\Delta R(r)}{R(r)} \to (m - N + 2)m a_{n+2} z^{m-2} \text{ or the like, where } a_m \text{ is the term next to } a_n z^n. \]

\[ \text{In addition, line 2, 4, 7, 8 includes the Yukawa potential case, when } \phi \sim Z^l e^{-\frac{2\pi}{l} z} \text{ in (6.3), the only finite solutions are that of the footnote 6, i.e., } \frac{\Delta R(z)}{R(z)} \rightarrow \frac{d}{dz} \left( R(z) = \begin{cases} a + b z^n + d z^k e^{-\frac{b_i}{z}} + \cdots \quad (b = 0 \text{ if } N \neq n + 2) \\
a z^n + d z^k e^{-\frac{b_i}{z}} + \cdots \end{cases} \right), \text{ where } a, d \neq 0 \text{ and } b_i, i > 0 \text{ and } k = 2i - l. \text{ It is curious that there are some 'degenerate' eigen functions for the same asymptotic potential, even if not normalizable for } N < 4. \]

\[ \text{Finally, this and line 9 allow for a single term with pure imaginary } b_i, \text{ except for which an imaginary coefficient on the shoulder of the exponential leads to a non-physical oscillatory or imaginary potential. This is indeed the case for the Coulomb scattering of a photon.} \]
more natural normalizability condition that $R(r)$ is a $L^2$ function leads to small modification $N < 2n$ instead of $2 \leq n$ in (5.3) and so, $a = 0$ if $0 < N$ and R.H.S. of (5.4) is replaced by *)

$$\frac{\Delta R(r)}{R(r)} \to \begin{cases} 
(n + N - 2)n_z^2 \text{ (or higher order) } & (a \neq 0 \text{ and } 3a_n \neq 0 \text{ and } 0 < n) \\
0 & (a = 0 \text{ or } a_{2-N} \neq 0 \text{ and } \nu a_n = 0 \text{ for } n \neq 2 - N \text{ and } 3b_i \text{ or } c_j \text{ or } d_k \text{ or } \cdots > 0) \\
(n + N - 2)n z^{-2} \text{ (or higher order) } & (a = 0 \text{ and } 3a_n \neq 0 \text{ and } 0 < n) \\
(-ib_i)^2 z^{2i \pm 2+2} & (a = b = \nu a_n = 0 \text{ and } 3b_i > 0) \\
+ \infty & (a = b = \nu a_n = \nu b_i = 0 \text{ and } 3c_j \text{ or } d_k \text{ or } \cdots > 0) 
\end{cases} \quad (5.5).$$

In this case, the potential must be positive for $(\nu = -2 \text{ and } N < n + 2)$ or $-2 < \nu$. Notice that (5.2) for more general cases of $N$, $a$ can be obtained from (5.4) by the trivial replacement $N \to 4 - N$ and $z \to r$ with its power smaller by 4. Then, we conclude the potential $V(r)$ as $r \to +0$ must be positive for $(1 \leq N \text{ and } \nu = -2)$ or $\nu < -2$, where $\nu$ is the power of the potential $V \to r^\nu$ as $r \to +0$; can take both signs for other cases. If we assume $R(r)$ is $L^2$ instead of $C^2$, $-2n < N$ instead of $2 \leq n$, and so by renaming $a_1 := b$ the results are

$$\frac{\Delta R(r)}{R(r)} \to \begin{cases} 
(n + N - 2)n z^{2}n^{-2} \text{ (or higher order) } & (a \neq 0 \text{ and } 3a_n \neq 0 \text{ and } 0 < n) \\
0 & (a = 0 \text{ or } a_{2-N} \neq 0 \text{ and } \nu a_n = 0 \text{ for } n \neq 2 - N \text{ and } 3b_i \text{ or } c_j \text{ or } d_k \text{ or } \cdots > 0) \\
(n + N - 2)n z^{-2} \text{ (or higher order) } & (a = 0 \text{ and } 3a_n \neq 0 \text{ and } 0 < n) \\
(-ib_i)^2 z^{2i \pm 2+2} & (a = \nu a_n = 0 \text{ and } 3b_i > 0) \\
+ \infty & (a = \nu a_n = \nu b_i = 0 \text{ and } 3c_j \text{ or } d_k \text{ or } \cdots > 0) 
\end{cases} \quad (5.6).$$

**) Above results show that for a physical dimension $N = 1, 2, 3$, the sign of a potential $V$ must be positive for $\nu < -2 + \epsilon \ (r \to 0)$ and $-2 - \epsilon \leq \nu \ (r \to \infty)$, but can be negative for other cases.

*) It is impossible for the R.H.S. to be

$$\to \begin{cases} 
(n - 1)n z^{2}n^{-1} \text{ (or higher order) } & (a = 0 \text{ and } b \neq 0 \text{ and } 3a_n \neq 0 \text{ and } 1 < n \text{ and } N = 3) \\
0 & (a = 0 \text{ and } b \neq 0 \text{ and } \nu a_n = 0 \text{ and } 3b_i \text{ or } c_j \text{ or } d_k \text{ or } \cdots > 0 \text{ and } N = 3) 
\end{cases} ,$$

because $b$ appears. In addition, ‘higher order’ does not appear in case of $n \leq 2$ nor $N \leq 4$.

For the realistic $N = 3$ case, the weaker $L^2$ condition to allow logarithmic divergence is equivalent to $R(z)$ of $C^1$-class, with only difference that $1 < n$ instead of $2 \leq n$ in (5.2) and (5.3). ++

++) The line $1,3$ includes the special case $N = 2 - n$, when $\Delta \frac{R(r)}{R(r)} \to (m + N - 2)n z^{2}n^{-2}$, $(m + N - 2)n z^{2}n^{-2}$ or the like, respectively, where $a_m$ is the term next to $a_n r^n$ such that $m \neq 0$. In addition, ‘higher order’ does not appear in case of $n \leq -2$ nor $4 \leq N$.\v

\v

11
§6. Theorems for the long distance limit

Now we define the following conditions for later convenience. The first are normalization conditions naturally required for the long and distance limit of a boson or fermion free field. Clearly, **Normalization Conditions for Free fields** are for **Massive Boson field in the Long distance limit (NC-MBL)**,
\[ n \leq -\frac{N+1}{2}, \]
for **Massless Boson field in the Long distance limit (NC-0BL)**,
\[ n \leq \frac{1-N}{2}, \]
for **Massive Fermion field in the Long distance limit (NC-MFL)**,
\[ n \leq -\frac{N+1}{2}, \]
for **Massless Fermion field in the Long distance limit (NC-0FL)**,
\[ n \leq -\frac{N}{2}, \]
for **Boson field in the Short distance limit (NC-BS)**,
\[ \frac{1-N}{2} \leq n, \]
for **Fermion field in the Short distance limit (NC-FS)**,
\[ -\frac{N}{2} \leq n, \]
where the field behaves like \( r^n \) and = means logarithmic divergence, all of them with the **Exceptional rule for massless particles (NC-0Ex)** that an arbitrary constant can be added.

It’s unclear to me whether or not previous results are valid for the Dirac equation, but for a moment we keep away the validity or spin effects as a later discussion, and just describe the specific results given by application for fermions by writing in a [].

**Positive Potential Condition (PPC)**
The potential \( V(r) \) defined in section 2 is positive.
This is indeed satisfied for the non-relativistic approximation of a Klein-Gordon equation (5.1), if
\[
-V = \frac{(E - e\phi)^2 - m^2 c^4}{\hbar^2 c^2} \\
\approx \frac{2m}{\hbar^2}(E - mc^2 - e\phi) < 0,
\]
where \(|E - mc^2|, |e\phi| \ll mc^2\). In addition, (6.1) shows that PPC is strictly valid for massless bosons. Indeed, PPC means that the particle is in the bound state. A force is defined as \(-e\nabla\phi\). Thus from the previous result we can verify the following theorems for the long distance limit.

**Theorem 1**
For the higher or smaller spatial dimension $N \neq 3$, a massless boson [or fermion] that behaves like $\sim \frac{1}{r}$ ($r \to \infty$) can not feel a dominant $\frac{1}{r^2}$-like long range force.  

**Proof**

For $N < 3$, this is proven by taking $E = m = 0$ in (6.1) and comparing with the line 5 of (5.4). The former violates PPC, which contradicts the latter condition that $V(+\infty)$ must be positive for $N < 3$. For $3 < N$, this is proven just because nonzero $b$ in the line 5 breaks NC-0BL and for $3 < N$, that is a weaker condition than NC-MBL, NC-0FL, NC-0FL.

A static spherical symmetric electric field like $\sim \frac{1}{r}$ is of course experimentally observed and therefore, a photon behaves like $A^\mu = (\phi(r), 0, 0, 0)$, $\phi(r) \sim \frac{1}{r}$. An interesting corollary of the above theorem is that, if $N > 3$, a photon can not feel gravity and there is no gravitational lens! Some people might suspect that we can not always take $E = 0$ because it means a virtual photon for $e\phi > 0$, but at least in situation a photon is bounded by the potential and another photon is not bounded, we can always take $E = 0$ and thus we dare say

**Theorem 2**

For $N > 3$, if a charged static spherical symmetric black hole can exist, the electric field decreases more rapidly than $\frac{1}{r^{N-1}}$.

**Proof**

Even a black hole has its gravitational potential $\sim \frac{1}{r}$ at distance, for its density is finite and spherical symmetric, and obeys Gauss' law and $\frac{1}{r^2}$-law experimentally, and gravity is always attractive force. A black hole is of course such a matter that even a light can not escape, therefore a bound state exist for a photon. Then, we can apply theorem 1. Strictly speaking, there might be exceptions for the theorem, in which the black hole potential is deviated from the $\frac{1}{r}$-law because of the presence of another long range force that a photon can feel. In the standard model of particles, this is not the case, for no other long range force (gluons nor a photon) couple with a photon. Noting that if the asymptotic $\frac{1}{r}$-law of gravity holds, from (5.1),

\[
V(r) = \frac{m^2c^4 - E^2 + 2eE\phi - (e\phi)^2}{\hbar^2c^2} \\
\rightarrow \begin{cases} 
(1) & m^2c^4 - E^2 \quad (E^2 \neq m^2c^4) \\
(2) & 2eE\phi \sim \frac{1}{r} \quad (E^2 = m^2c^4 \neq 0) \\
(3) & (e\phi)^2 \sim \frac{1}{r^2} \quad (E^2 = m^2c^4 = 0)
\end{cases}.
\]

Thus the only ways for the massless photon to allow such a black hole is the lines 8,9 of (5.4) \(^*\), i.e.,

\(^*\) For $N < 3$ (and also for $N = 3$ if we do not allow logarithmic divergence), this can also be derived from NC-0BF without assuming the $\frac{1}{r}$-law of gravity. For [fermions and] massive bosons, more severe than
(line 8.) This is (3) of (6.3), where the asymptotic $\frac{1}{r^2}$-law of gravity exactly holds and the photon remains massless, but the electric field behave as if away from a polarized matter with no electric charge as a whole. The former requires that no density is present at distance. Gauss’ law is geometric and valid in presence of gravity, therefore the latter requires real existence of the charge to cancel that of the black hole. Up to now all the particles with electric charge are massive, therefore the cancellation must be due to the electric charge density distributed in a finite region. From (6.3) and NC-0BL and (5.4), such a ‘medium range’ force can be felt dominant only by such fields that behave like $(b = 0$ and $\frac{N-1}{2} \leq n)$. (For other particles, Notice that in quantum mechanics, even a particle in a empty metal sphere can ‘feel the outer world’. Or

(line 9.) The electric field vanish exponentially, and $V(r)$ survive slowly than $\frac{1}{r^2}$. For each case of (6.3), (1) $i = -1$ (2) $i = -\frac{1}{2}$ (3) no $i$ allowed. Thus only possible cases are, either the photon ‘becomes massive’ (i.e., $m \neq 0$) or otherwise $E \neq 0$. It is an interesting possibility that a massive static photon, that is not bounded because violating PPC, can create $e^{\sqrt{r}}$-type electric field in the former case and even a massless photon can create Yukawa-type electric field in the latter case. Normalization condition is automatically satisfied for these exponentially vanishing solutions. Such a Yukawa-type electric field can be felt iff by a $b = 0$ massless boson [or fermion] satisfying the normalization condition. There is yet another possibility that asymptotic $\frac{1}{r^2}$-law of gravity changes to survive more slowly, because of the long tail of nonzero density the black hole is accompanied with. In this case, $i$ can take some negative value $\neq -1$ iff $(E^2 = m^2c^4$ and $-1 < i < 0)$ or $i < -1$, when the photon create neither long range nor Yukawa-type but rapidly vanishing electric field.

In addition, from (5.4), this is the only case for gluons to make $\phi \sim r$ potential at distance, when $i = -2$ regardless of $m, E$.

**Theorem 3**

For $N = 3$, a massless boson [or fermion] vanishes more rapidly than $\sim \frac{1}{r^2}$ ($r \to \infty$), if it feels a $\frac{1}{r^2}$-like dominant long range force.

**Proof**

This is proven by taking $m = 0$ in (6.3) and comparing with the line 8, 9 of (5.4), for they are the only cases for $\phi \sim \frac{1}{r^2}$ ($r \to \infty$) to exist.

Thus, Theorem 2 and its proof hold also for $N = 3$, only by adding the last of (line 8) <and (line 9)> the following sentences: ‘except for the line 6 < and 7> of (5.4), in which logarithmic divergence appears.
only logarithmic divergence appears for a massless boson that feels the photon which behaves like \( b \neq 0 \). But it causes a self contradiction to identify the massless boson as a photon feeling itself’.

An interesting corollary of Theorem 1, 3 is

**Corollary 1**

A massless gauge boson that feels a dominant \( \frac{1}{r^2} \)-like long range force can not create a long range force.

This is a bit strange, for a photon can not feel \( \frac{1}{r} \)-like potential of gravitational lens. Maybe \( \frac{1}{r} \)-rule of electric field is only approximation in presence of gravity, or \( \frac{1}{r^2} \)-rule of gravity is only approximation in presence of electric field, or the coexistence of a graviton and a photon leads to a contradiction in present theory and gravity should be derived from other forces. But this corollary well accounts for the properties of the standard model, for if a gluon or a glueball had an electric charge, it must be ‘massive’, and if a photon had a color, the photon must be ‘massive’, provided an isolated gluon or a glueball could be observed. In addition, a weak boson has an electric charge (this is followed by the experimental fact that an electron is suddenly created and comes out in \( \beta \)-decay), and then it must be ‘massive’, regardless of the Higgs mechanism. Thus we come to

**Corollary 2**

If a photon or graviton is massless and the self interactions of \( W^\pm \) bosons are not so strong as to create a long range force which vanish more slowly than \( \frac{1}{r^2} \), then the \( W^\pm \) bosons can not create a long range force. In the same way, if a graviton is massless, the glue-balls (if exist) and pions with weak enough self interaction can not create a long range force, and even if a graviton is massive and a photon is massless, the electrically charged glueballs (if exist) and pions with weak enough self interaction can not.

If the standard model particles are to be unified some day in such a manner that a photon is massless and a graviton has an electric charge, then the graviton must be ‘massive’. In the same way, if a graviton is massless, then the photon must be ‘massive’. Conversely, if a graviton is massless and a photon or gluon is massive, then the photon or gluon must be ‘massive’ (this is tautology).

By the way, from the argument of footnote 6, we can verify also

**Theorem 4**

If a boson [or fermion] feels a dominant Yukawa-type potential, then the eigen function of the particle is not \( L^2 \). Particularly, such a boson [or fermion] must be massless for \( N < 5 \); can be massive for \( 5 \leq N \).
proof
This is because if we take \( E = mc^2 \) and Yukawa-type potential \( \phi \sim Ze^{-\frac{b_0}{z^l}} \) in (6.3), the only finite solutions are that of the footnote 6, i.e., \( \Delta R(z) \rightarrow \frac{d}{a}(b_i)^2 z' e^{-\frac{b_0}{z^l}} R(z) = \begin{cases} a + bz^n + dz^k e^{-\frac{b_i}{z^l}} + \cdots \quad (b = 0) \\ az^n + dz^{k+n} e^{-\frac{b_i}{z^l}} + \cdots \quad (N = 2) \end{cases} \)
where \( a, d \neq 0 \) and \( b_i, i > 0 \) and \( k = 2i - l \), which vanish no more rapidly than \( r^{2-N} \), to be compared with NC-MBL, NC-0BL, [NC-MFL, NC-0FL], and NC-0Ex. Notice that for \( N = 3 \), a logarithmic divergence inevitably appears even for massless boson in the lower case, and therefore this case is impossible for \( 3 < N \).

It is surely a severe condition for realistic physics and thus

**Corollary 3**
A short range force must be always dominated by a longer range force, or otherwise \( N < 5 \) and not felt by a massive boson [and fermion], or otherwise \( 5 \leq N \),
where the term ‘longer’ used to notice that any force that survives more slowly than any Yukawa potential is allowed.

From now on, we take \( N = 3 \) and concentrate on the self-consistent conditions for the standard model. Then, if we do not take account of gravity as the dominant force,

**Corollary 4**
A boson with a charge of a short range force must have another charge of a longer range force, or otherwise must be massless.

Thus, the unification of \( U(1) \times SU(2) \) can be proved without assuming experimental results. The corollary well accounts for the property of \( W^\pm \) [and quarks and charged leptons] which are massive and have both \( U(1) \) and \( SU(2) \) charge. An equivalent proposition that ‘a particle with no charge of any long range force must not have a charge of any short range force, or otherwise must be massless’ is satisfied for a \( Z^0 \) and a photon and a \( \pi^0 \) [and almost for neutrinos]. Only if we can neglect gravity····.

§7. Origin of Higgs mechanism

Now, remember that a massless free spin 1 boson is always identified with a photon and creates a \( \frac{1}{\epsilon} \)-like long range force which is identified with electric field, and gravity couples equivalently to all matter\(^{8}\). Suppose that a massive boson has a charge. If we do not take account of gravity as the dominant force, and if it is the charge of a short range force, then from **Corollary 4** the boson has a charge of a longer range force. Thus, we can naturally assume that a ‘boson with a charge of a short range force’ has a charge of a long range force,
say, electric charge, and call it $W^+$, for a photon of course exists and creates a long range force. Then, from the CPT-theorem, its anti-particle $W^-$ also exists. And this is the source of the short range force, therefore must be identified with the ‘photon that feels a $\frac{1}{r}$ force and can create a Yukawa-type electric field’ previously occurred in the case (1) of (6·3), for the Gauss’ law is also valid for the field of the ‘reshaped photon’ and gives the real charge distribution. Notice that in this case, there is no way to distinguish whether or not the ‘reshaped photon’ is massive, for the $V(r)$ becomes the constant $m^2c^4 - E^2$ asymptotically, and a massless photon with a positive energy $E$ is equivalent to an energy-less photon with the mass $mc^2 := iE$. Thus any photon to feel the same asymptotic nonzero $V(\infty)$ can create the same Yukawa-type electric field and therefore can be taken to be massless. Indeed, the Klein-Gordon equation for a massless photon with the $\frac{1}{r}$-potential $\phi_G$ (of gravity) can be solved from (5.4) to give

$$\frac{\Delta R(r)}{R(r)} = -\frac{(E_\gamma - \phi_G)^2}{\hbar^2 c^2} \sim m^2 - 2mn\frac{1}{r} + n(n-1)\frac{1}{r^2} = -\frac{(E_W - \phi_G - e\phi_{EM})^2 - M_W^2 c^4}{\hbar^2 c^2},$$  \[ R(r) \sim e^{-mr}r^n \] \hspace{1cm} (7.1)

This is in fact the definition of the mass, i.e., for a massless and charge-less photon to feel gravity, some universal unit for the mass is needed. Here we define the unit by the fine structure constant $\alpha := \frac{e^2}{\hbar c}$. Then, we can write $\phi_G := \frac{e^2 M}{r}$, where $\mathcal{M}$ is some constant proportional to the mass the photon feels. In the same way, the electric potential that the $W^+$ feels can be written as $\phi_{EM} := \frac{eQ}{r}$, where $Q$ is some constant proportional to the electric charge the $W^+$ feels. Thus, the initial energy $E_\gamma$ of the massless photon feeling the ‘universal potential of gravity’ $\phi_G$ is equal to the potential that the identified $W^+$ boson feels, i.e., asymptotically $E_\gamma^2 = E_W^2 - M_W^2 c^4$ with $E_W$ and $M_W$ the energy and mass of the $W^+$ boson respectively. From the line 9 of (5.4) with $i = -1$, $E_W$ is equal to $b_{-1} = -m^2$, which in turn creates ‘reshaped photon’ potential $\phi_W \sim e^{-mr}r^n$. In the low energy limit of the photon $E_\gamma = 0$, this means that a ‘stopped photon’ is just a static electric field.

This is the origin of the Higgs mechanism. Therefore, from (7.1)
for $W^\pm$. In the same way,

$$
\begin{align*}
\alpha(M)E_Z &= -mn\hbar c \\
(m\hbar c)^2 &= -E_\gamma^2 = M_Z^2c^4 - E_Z^2 \\
\Rightarrow\quad -E_\gamma^2 &= (m\hbar c)^2 \\
&= \frac{M_Z^2c^4}{1 + \frac{\alpha(M)^2}{\hbar^2c^2}}
\end{align*}
$$

with $e^{\eta^2} := \{\alpha(M)\}^2$ (7.3)

for $Z^0$. But, wait, the mass of $W^\pm$ must always be pure imaginary! Something is wrong · · ·

§8. Unification

Let’s keep away the problem of imaginary mass, and consider the original eigen function of the photon. The $\frac{1}{r^2}$-rule of gravity may not be altered much, for it is always attractive. Then, from Theorem 3 the photon can not create an exactly $\frac{1}{r^2}$-like long range force. But it can create almost $\frac{1}{r^2}$-like medium range force, by taking $n = 1 + \epsilon$ in the line 8 of (5-4), without violating (NC-0BL). Maybe a radical, but not so contradictory solution is just to expand a most general eigen function $R(z)$ in a shape like (5-3)

$$
R(z) = a + bz + \sum_{n=1}^{\infty} a_n z^n \sim \cdots + \sum_{i<0} (+) e^{-b_i z^i} \sim \cdots \\
+ \sum_{j<0} (+) e^{-z_j} \sim \cdots + \sum_{k<0} (+) e^{-e_{kz} e_k^{z_k}} \sim \cdots \cdots \\
$$

(8.1)

* and assume the shape of a graviton $G(z)$ as the first infinite sum part of (8-1) with $a = 0 \neq b$ and no $i, j, k, \cdots$. Then, we can always take $0 < b$ for a graviton without losing generality, and redefine the gravity potential by $\phi_G := g_G G(z)$, where $g_G$ is the positive coupling constant of gravity. Then, the gravity becomes by definition attractive. Let’s consider a virtual world in which only gravitons exist. Because $\phi_G$ is universal to any particles, it must also be satisfied for the graviton. Therefore,

$$
\frac{\Delta G(z)}{G(z)} = \frac{(E_G - \phi_G)^2}{\hbar^2c^2}.
$$

(8.2)

This is a self-consistent condition that resembles the Einstein equation in a sense, but from Theorem 3 there is no solution for (8-2) to create a $\frac{1}{r^2}$-like long range force, and (8-2) leads to $b = 0$. This indicates that a graviton must be accompanied with other field, say

* The only deference is that the first infinite sum can start from $z^{1+\epsilon}$, which means to assume $C^1$-class $\phi(z)$ instead of $C^2$, but causes no problem for a moment.
photon. From above discussion, we can naturally assume the shape of a photon $A(z)$ as the first infinite sum part of (8.1) with $a = b = 0$ (may start from $z^{1+i}$) and no $i, j, k, \cdots$. Then, we can redefine the electric potential and gravity potential by $\phi_{EM} := g_{EM}A(z)$ and $\phi_G := g_{EM}bz$, where $g_{EM}$ is the positive coupling constant of electricity. But in this case, there is no way to make $\phi_{EM}$ always positive, as it depends on the relative sign of $a_n$ to 0 < $b$ in $G(z)$. With these redefinition, a graviton and a photon suit well for (5.4) and (7.1). Let’s abandon (8.2) and interpret $G(z)$ as a virtual field of the first term $bz$. Then, the only equation for $A(z)$ to satisfy is

$$\frac{\Delta A(z)}{A(z)} = -\frac{(E_\gamma - \phi_G)^2}{\hbar^2c^2},$$

where $\phi_G = g_{EM}bz$. (8.3)

This has a consistent solution for $E_\gamma = 0$ as in the (line 8) of (5.4). Let’s come back again to (7.1). Exactly $\frac{1}{r}$-like potential comes only from the first $g_{EM}\frac{b}{z}$ term in this equation. For $W^\pm$ to make a short range force, the solution $W^\pm(z)$ must be the (line 9) case of (5.4). For almost all the unbounded states, $Mc^2 < E_W$. Then, from the previous discussion we must take $i = -1, |m| = b_i = E_\gamma = \sqrt{E_W^2 - M^2c^4} > 0$ in the (line 9). From footnote 6 this ‘$W^\pm$ boson with imaginary mass’ is just a free photon. Notice that $E_\gamma$ can always be taken positive by definition and negative energy means just a complex conjugate. Let’s denote these eigen functions of unbounded photons (those were not included in the previous expansion because of imaginary coefficients) as $A^\pm(z)$, and previous one (with 0 energy) $A^0(z)$. Contrastingly, for a bounded state $E_W < Mc^2$, $i = -1, m = |b_i| = |E_\gamma| = \sqrt{M^2c^4 - E_W^2} > 0$ and this $W^\pm$ boson is really massive. But for $M$ to be real, only pure imaginary $E_\gamma$ is allowed, and $m$ must be positive to satisfy normalization conditions. With this $i$, $W^\pm(z)$ is identified to be the second infinite sum appears in the general expansion (8.1) **). Experimentally, $M_W < M_Z$. Then from (7.2) and (7.3) $Q < -2M$ or $0 < Q$, the latter of which is in good accordance with the interpretation that gravity is a virtual force induced by electric polarization. Suppose that there are two metal balls, one is neutral and the other with positive or negative electric charge. Then, in both case the two balls will attract each another by the induced surface charge. Then, what is the meaning of $M$ and $Q$? well, say, the mass and the charge induced on the surface of

---

\[ (8.1) \text{is closed in this shape of iterative expansion and no } i, j, k, \cdots \text{ appears.} \]

\[ **) \text{This expansion is also closed in itself if we take } i = -1, \text{ in such a sense that no other } i \text{ appears in the iterative expansion } W^\pm(z) = \sum_{k=1,2,3,\cdots}^{\infty} \sum_{l_k} e^{-k^2z^l_k}, \text{ except if we consider the special case } E_W = Mc^2 \text{ when a half integer } i = -\frac{1}{2} \text{ must also be included. If we must assume (8.1) to be Taylor expanded in } r, \text{ we can avoid other eccentric exponential decay with a non-integer } i = -1 \text{ or appearance of } j, k, \cdots. \]
the universe, for they are universal constants and every matter is bounded by gravity in the universe. Isn’t it an interesting idea?

Then, let’s consider the eigenfunction \( W^\pm(z) \) and \( Z^0(z) \). The only difference of them are that \( W^\pm \) can feel electricity but \( Z^0 \) can not. Therefore in a theory that includes no graviton like the standard model, from Theorem 4 a massive \( Z^0 \) boson must be written as
\[
Z^0(z) = b_Z z + d_Z z^{1-n^c} e^{-\frac{m}{E}} + \cdots
\]
where \( b_Z, d_Z \neq 0 \), while \( W^\pm(z) = \sum_{n=1}^{\infty} a_{W,n} z^n + \cdots + d_W \sum_{k=1,2,3} z^{1-n^c} e^{-k \frac{m}{E}} + \cdots \), where \( a_{W,n}, d_W \neq 0 \) and \( n^c, n^c' \) are the n s which satisfy (7.2) and (7.3) respectively. But from (7.1) this is a virtual shape of a massless photon only feeling gravity and with energy \( E_\gamma := \pm imhc \). Above discussion is valid only if \( E_\gamma \) can take a specific pure imaginary value. This is artificially accomplished by the redefinition of \( \phi_G \), i.e., taking \( G(z) \to G(z) \pm i \frac{mhc}{ge} \) in (8.3). In the standard model, neutrino mass must exactly be 0 because their eigen functions must be
\[
\nu(z) = a_{\nu} + d_{\nu} z^{-n^c} e^{-\frac{m}{E}} + \cdots
\]
to avoid divergence. Strictly speaking, \( Z^0(z) \) must also have this shape, i.e.,
\[
Z^0(z) = a_Z + d_Z z^{-n^c} e^{-\frac{m}{E}} + \cdots
\]
(\( b =^= a_n = 0 \) because it can not feel electric field) and then
\[
W^\pm(z) = a_W + \sum_{j=1}^{\infty} a_{W,j} z^j + \cdots + d_W z^{-n^c} e^{-\frac{m}{E}} + \cdots
\]
(\( ^3 a_{W,j} \neq 0 \) because it can feel electric field),
where \( n \) is the solution of (7.2) with \( \phi_G = g_{EM}bz \)
to avoid logarithmic divergence and thus must be massless. The only way to allow massive \( Z^0 \) is then to subtract \( a_Z \) from every particles that feel weak force by using the constant ambiguity of a potential. This requires Higgs vacuum energy \( < v > \neq 0 \). Then, for \( Z^0 \) and \( W^\pm \) to feel weak force of the same strength, \( d_W = d_Z \) and we must replace \(-n \to 1 - n \) in the term \( z^{-n} e^{-\frac{m}{E}} + \cdots \). Then, at last gauge bosons [and neutrinos and charged leptons (and quarks)] in the standard model can be unified (or decomposed) in the following form: *)

\[
R(z) = A^0(z) + A^\pm(z) + Z^0(z) + W^\pm(z) + \nu(z) + L^\pm(z) (+ G(z) + Q^\pm(z)),
\]

*) Usage of these equations: If you want to calculate the two-body problem of \( X \) and \( Y \), use the reduced mass \( \frac{M_X M_Y}{M_X + M_Y} \) for the last term of a potential \( V \), and the particles obey the Klein Gordon equation such that both particles feel the same potential. Then, the eigen function for this system can be obtained from \( X(z), Y(z) \) such that both expansions contain the same order terms. Notice that previous discussions and symmetry requirement lead to \( b \sim M \sim M_X + M_Y \), when the Newtonian potential and its Schwarzschild correction can naturally be derived by scaling \( E \) by the unit \( \frac{M_X M_Y}{M_X + M_Y} \). \( A^0(z) \) must start from just \( n = n_{\text{min}} = 2 \) for a massive \( W^\pm \) to stop. \( a_{W,n} \) and \( a_{L,n} \) and \( a_{Q,n} \) must start from at least \( 2 + \epsilon \) to avoid logarithmic divergence.
\[ A^0(z) := \sum_{n=n_{\text{min}}}^{\infty} a_{An} z^n \sim \ldots \text{ and} \]
\[ A^\pm(z) := \sum_{k=1,2,3,\ldots} \sum_{l_k=1}^{\infty} d_{Aklk} e^{\pm ik^2 c^2 / \hbar^2} z^{l_k} \text{ satisfy} \]
\[ \Delta A(z) \over A(z) = -\left( \frac{\pm E_\gamma - \phi_G}{\hbar^2 c^2} \right)^2 \text{ with} \]
\[ E_\gamma = \begin{cases} 
\text{im} \hbar c \text{ (for } A^0(z) \text{)} \\
(\text{im} + \omega) \hbar c \text{, where } \omega \text{ is any positive number (for } A^\pm(z) \text{)} \end{cases} \]
\[ Z^0(z) := (a_{Z} - bz) \left( \delta_{E_{Z}M_{Z}} \pm \sum_{k=1,2,3,\ldots} \sum_{l_k=1}^{\infty} d_{Zklk} e^{-k \frac{m}{\hbar^2} z^{l_k}} + \ldots \right) \text{ and} \]
\[ \nu^0(z) := a_{\nu} \left( \delta_{E_{\nu}M_{\nu}} \pm \sum_{k=1,2,3,\ldots} \sum_{l_k=1}^{\infty} d_{\nu klk} e^{-k \frac{m}{\hbar^2} z^{l_k}} + \ldots \right) \text{ and} \]
\[ W^\pm(z) := a_{W} \left( \pm 1 \text{ or } \sum_{n=2}^{\infty} a_{Wn} z^n \sim \delta_{E_{W}M_{W}} + \sum_{k=1,2,3,\ldots} \sum_{l_k=1}^{\infty} d_{Wklk} e^{-k \frac{m}{\hbar^2} z^{l_k}} + \ldots \right) \text{ and} \]
\[ L^\pm(z) := a_{L} \left( \pm 1 \text{ or } \sum_{n=2}^{\infty} a_{Ln} z^n \sim \delta_{E_{L}M_{L}} + \sum_{k=1,2,3,\ldots} \sum_{l_k=1}^{\infty} d_{Lklk} e^{-k \frac{m}{\hbar^2} z^{l_k}} + \ldots \right) \text{ and} \]
\[ Q^\pm(z) := a_{Q} \left( \pm 1 \text{ or } \sum_{n=2}^{\infty} a_{Qn} z^n \sim \delta_{E_{Q}M_{Q}} + \sum_{k=1,2,3,\ldots} \sum_{l_k=1}^{\infty} d_{Qklk} e^{-k \frac{m}{\hbar^2} z^{l_k}} + \ldots \right) \]
\[ \text{or } \sum_{k'=1,2,3,\ldots}^{\infty} \sum_{l'_k=1}^{\infty} f_{Qk'l'} e^{-k' \frac{m}{\hbar^2} z^{l'_k}} + \ldots \text{ satisfy} \]
\[ \Delta Z^0(z) \over Z^0(z) = -\left( \frac{\pm E_{Z} - \phi_G - \vec{I}_{Z} \cdot \vec{\phi}_{W}}{\hbar^2 c^2} \right)^2 - M_{Z}^2 c^4 \text{ and} \]
\[ \Delta \nu(z) \over \nu(z) \approx -\left( \frac{\pm E_{\nu} - \phi_G - \vec{I}_{\nu} \cdot \vec{\phi}_{W}}{\hbar^2 c^2} \right)^2 - M_{\nu}^2 c^4 \text{ and} \]
\[ \Delta W^\pm(z) \over W^\pm(z) = -\left( \frac{\pm E_{W} - \phi_G - e \phi_{EM} - \vec{I}_{W} \cdot \vec{\phi}_{W}}{\hbar^2 c^2} \right)^2 - M_{W}^2 c^4 \text{ and} \]
\[ \Delta L^\pm(z) \over L^\pm(z) \approx -\left( \frac{\pm E_{L} - \phi_G - e \phi_{EM} - \vec{I}_{L} \cdot \vec{\phi}_{W}}{\hbar^2 c^2} \right)^2 - M_{L}^2 c^4 \text{ and} \]
\[ \Delta Q^\pm(z) \over Q^\pm(z) \approx -\left( \frac{\pm E_{Q} - \phi_G - q \phi_{EM} - \vec{I}_{Q} \cdot \vec{\phi}_{W} \pm C_{G} \phi_{S}}{\hbar^2 c^2} \right)^2 - M_{Q}^2 c^4 \text{, where} \]
\[ \phi_G := g_{EM} b \omega + \sqrt{\frac{mc}{\hbar}} \text{ and} \]
\[ \phi_{EM} := g_{EM} A^0(z) \text{ and} \]
\[ \vec{\phi}_{W} := g_{EM} (W^+(z) \delta_{E_{W}M_{W}} - a_{W}, W^-(z) \delta_{E_{W}M_{W}} - a_{W}, Z^0(z) \delta_{E_{Z}M_{Z}} - a_{Z}) \text{ and} \]
\[ d := (d_{W+1}, d_{W-1}, d_{Z11}), \quad T_{3} A_{Z} g_{EM} := \text{im} \hbar c, \]
\[ M_Z^2 c^4 - E_Z^2 = (m\hbar c)^2 = M_W^2 - E_W^2 \] and

\[ M_Z \frac{\vec{I}_Z \cdot \vec{d}}{d Z} = (m\hbar c)^2 = M_W \frac{\vec{I}_W \cdot \vec{d}}{d W} \approx M_\nu \frac{\vec{I}_\nu \cdot \vec{d}}{d \nu} \approx M_L \frac{\vec{I}_L \cdot \vec{d}}{d L} \approx M_Q \frac{\vec{I}_Q \cdot \vec{d}}{d Q} \] and

\[ \frac{n(n-1)}{eM_W} = 2 \frac{g_{EM}}{\hbar^2 c^2} a_{A2} \approx \frac{n(n-1)}{qM_Q}, \quad \text{where} \]

\[ a_{Wn} z^n \sim \text{is the first term for the stopped } W^\pm, \text{ etc.,} \]

\[ G^\pm (z) := \sum_{n=-1}^{\infty} a_{Gn} z^n + \cdots + \sum_{k'=1}^{\infty} \sum_{l'=1}^{\infty} f_{Gk'l'} e^{-k'm} z^{k'} \sim + \cdots \text{ satisfies} \]

\[ \frac{\Delta G^\pm(z)}{G^\pm(z)} = -\frac{(\pm E_G - \phi_G - \vec{C}_G \cdot \vec{\phi}_S)^2}{\hbar^2 c^2}, \text{ where} \]

\[ \vec{\phi}_S := g_S \vec{G}(z), \quad \left( \frac{g_S \vec{C}_G \cdot \vec{a}_{G-1}}{\hbar c} \right)^2 = (2M)^2, \quad -(\frac{g_{EM} b}{\hbar c})^2 = n_{\text{min}}(n_{\text{min}} - 1). \] (8.5)

Thus, the God made the light at first, or a man can, by defining all the coupling constants after \( g_{EM} \) etc.

### §9. Conclusion

In this paper we classified possible singularities of a potential for the spherical symmetric Klein-Gordon equation, assuming that a potential \( V \) has at least one \( C^2 \)-class eigen function. The result crucially depends on the analytic property of the eigen function near its 0 point. Above analysis indicates that possible shapes of the potential and the eigen functions of particles are restricted by the consistency condition of this simple model. Then we discussed a natural possibility that gravity and weak coupling constants \( g_G \) and \( g_W \) are defined after \( g_{EM} \). In this point of view, gravity and the weak force are subsidiary derived from electricity. The fact that the iterative solution inevitably includes several infinite series with different order in one expansion may be the origin of the non-commutative gauge invariance.

### Acknowledgments

I am grateful to Izumi Tsutsui and Toyohiro Tsurumaru for useful discussion. This paper is partially motivated by some implication given by Tsutomu Kambe and Kazuo Fujikawa. I also appreciate Syu Kato and my family Yuko, Takeo, Megumi, and Masahide for spiritual support. This work is partly supported by Ikuei-kai grant. Finally, I thank Maya Y for her respectable life.
References
1) M.B. Pour-El and J.I. Richards, *Computability in analysis and physics* (Springer, New York, 1989).
2) R. Penrose, *The emperor’s new mind: concerning computers, minds, and the laws of physics* (Oxford University Press, Oxford, 1989).
3) L.V. Ahlfors, *Complex analysis: an introduction to the theory of analytic functions of one complex variable* (McGraw-Hill, New York, 1953).
4) J. Simon, Phys. Rev. **D41** (1990), 3720.
5) M. Abe, N. Nakanishi, Rev. Mod. Phys. Lett. **A10** (1995), 1501.
6) L.I. Schiff, *Quantum Mechanics* (McGraw-Hill, Tokyo, 1968).
7) S. Weinberg, *THE QUANTUM THEORY OF FIELDS. VOL. 1,2* (Cambridge, UK, 1995)
8) S. Weinberg, Phys. Rev. **133** (1964), B1318; Phys. Rev. **134** (1965), B882; Phys. Rev. **138** (1965), B988; Phys. Rev. **181** (1969), 1893; Phys. Rev. Lett. **19** (1967), 1264.
9) M. Nishikawa, preprint hep-th/0110095; hep-th/0202086.