KODAIRA DIMENSIONS OF ALMOST COMPLEX MANIFOLDS I

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Abstract. This is the first of a series of papers, in which we study the plurigenera, the Kodaira dimension and more generally the Iitaka dimension on compact almost complex manifolds.

Based on the Hodge theory on almost complex manifolds, we introduce the plurigenera, Kodaira dimension and Iitaka dimension on compact almost complex manifolds. We show that plurigenera and the Kodaira dimension are birational invariants in almost complex category, at least in dimension 4, where a birational morphism is defined to be a degree one pseudoholomorphic map. However, they are no longer deformation invariants, even in dimension 4 or under tameness assumption. On the way to establish the birational invariance, we prove the Hartogs extension theorem in the almost complex setting by the foliation-by-disks technique.

Some interesting phenomena of these invariants are shown through examples. In particular, we construct non-integrable compact almost complex manifolds with large Kodaira dimensions. Hodge numbers and plurigenera are computed for the standard almost complex structure on the six sphere $S^6$, which are different from the data of a hypothetical complex structure.

Contents

1. Introduction 1
2. Notations 5
3. Hodge theory on almost complex manifolds 6
4. Bundle almost complex structure and Iitaka dimension 13
5. Birational invariants 14
6. Examples 18
   6.1. The Kodaira-Thurston surface 18
   6.2. 4-torus 21
   6.3. Non-integrable almost complex manifolds with large Kodaira dimension 23
7. The six sphere 28
8. Appendix 32
References 33

1. Introduction

The Iitaka dimension for a holomorphic line bundle $L$ over a compact complex manifold is a numerical invariant to measure the size of the space of holomorphic
sections. It could be equivalently defined as the growth rate of the dimensions of
the spaces $H^0(X, L^\otimes d)$, or the maximal image dimension of the rational map to
projective space determined by powers of $L$, or 1 less than the dimension of the
section ring of $L$. The Iitaka dimension of the canonical bundle $K_X$ of a compact
complex manifold $X$ is called its Kodaira dimension and $\dim \mathbb{C} H^0(X, K_X^\otimes d)$ is called
the $d$-th plurigenus.

The Kodaira dimension, plurigenera and the canonical section ring are birational
invariants. They play important roles in the study of complex manifolds. In par-
ticular, Kodaira dimension is used to give a rough birational classification of com-
plex manifolds. It is known that Kodaira dimension is a deformation invariant for
compact complex surfaces, although it is no longer true for complex (non-Kähler)
3-folds [32]. Siu [35], [36] shows that plurigenera, and thus also the Kodaira di-

mension, are invariant with respect to projective deformations of algebraic varieties.
Birkar-Cascini-Hacon-McKernan [2] shows that the canonical ring of a smooth pro-
jective variety is finitely generated, which implies that there is a unique canonical
model for every variety of general type.

The theory of complex manifolds lies in the more general framework of almost
complex manifolds. In [42], intersection theory of almost complex manifolds is
introduced. As a consequence, pseudoholomorphic degree one maps are considered
as birational morphisms in the almost complex category. An important step to
develop birational geometry for almost complex manifolds is to introduce and study
birational invariants.

In this series of papers, we will generalize the notions of Kodaira dimension,
plurigenera as well as the space of holomorphic $p$-forms to compact almost complex
manifolds. The crucial initial step is to have generalizations of holomorphic line
bundle and its holomorphic sections. We have two equivalent versions.

The first is from differential geometry. A pseudoholomorphic structure on a com-
plex vector bundle $E$ over an almost complex manifold $X$ is a differential operator
$\bar{\partial}_E$ acting on smooth sections which satisfies the Leibniz rule (Definition 3.2). In par-
ticular, the canonical bundle has a natural pseudoholomorphic structure inherited
from the almost complex structure on $X$. By Koszul-Malgrange theorem, $\bar{\partial}_E^2 = 0$
on a complex manifold is equivalent to that $\bar{\partial}_E$ is induced by a holomorphic structure
on $E$. Our generalized version of holomorphic sections is just the smooth sections
in the kernel of $\bar{\partial}_E$.

Denote the space of generalized holomorphic sections of $E$ to be $H^0(X, E)$. To
show that it is of finite dimension, we apply the method of Hodge theory. Hodge
theory is well developed on compact complex manifolds and on general compact
Riemannian manifolds. On the way of making sense of the counting of pseudoholo-
morphic sections and defining plurigenera for almost complex manifolds, we develop
the Hodge theory for Hermitian bundles over compact almost complex manifolds in
details and show the following theorem, see Proposition 3.8 and Corollary 3.9.
Theorem 1.1. Let $E$ be a complex vector bundle with a pseudoholomorphic structure over a compact almost complex manifold $(X, J)$. Then $H^0(X, E)$ is finite dimensional. In particular, $H^0(X, K^{\otimes m})$ is finite dimensional and an invariant of $J$.

This result gives us a good base to count pseudoholomorphic sections. In fact, we are able to define groups of harmonic forms $H^{p,q}(\bar{\partial}^E X, E)$ which give Dolbeault type cohomology groups when $q = 0$. The vector space $H^0(X, E)$ is simply the case of $p = q = 0$. When $E$ is the trivial bundle, the space of harmonic forms of type $(p,q)$ has been defined in [21]. Its dimension is denoted by $h^{p,q}$. The Problem 20 in Hirzebruch’s list [21], raised by Kodaira and Spencer, asks whether the dimensions are independent of the choice of the Hermitian structure. (Recently, this problem is answered negatively in Theorem 5.1 of [22].) Our discussion gives an affirmative answer to this problem when $q = 0$ or, by Serre duality (Proposition 3.7), $q = \dim_C X$.

The second description of pseudoholomorphic sections is more geometric. There are special almost complex structures on the total space of the complex vector bundle $E$, called bundle almost complex structures, introduced by de Bartolomeis-Tian in [13]. The authors show that there is a bijection between bundle almost complex structures and pseudoholomorphic structures on $E$. We further observe that a section in the kernel of a pseudoholomorphic structure $\bar{\partial}^E$ is exactly a pseudoholomorphic section with respect to the bundle almost complex structure $J$ corresponding to $\bar{\partial}^E$.

With these two equivalent descriptions understood, we are able to give our definition of $(E, J)$-genus and the definition of Iitaka dimension, as well as their special cases - the plurigenera and the Kodaira dimension - in the end of Section 4.

Definition 1.2. Let $E$ be a complex vector bundle with bundle almost complex structure $J$ over a compact almost complex manifold $(X, J)$. The $(E, J)$-genus is defined as

$$P_{E, J} := \dim_C H^0(X, (E, J)),$$

where $H^0(X, (E, J))$ denotes the space of $(J, J)$ pseudoholomorphic sections. The $m$th plurigenus of $(X, J)$ is defined to be $P_m(X, J) = \dim_C H^0(X, K^{\otimes m})$.

Let $L$ be a complex line bundle with bundle almost complex structure $J$ over $(X, J)$. The Iitaka dimension $\kappa^J(X, (L, J))$ is defined as

$$\kappa^J(X, (L, J)) = \begin{cases} -\infty, & \text{if } P_{L^{\otimes m}, J} = 0 \text{ for any } m \geq 0 \\ \limsup_{m \to \infty} \frac{\log P_{L^{\otimes m}, J}}{\log m}, & \text{otherwise.} \end{cases}$$

The Kodaira dimension $\kappa^J(X)$ is defined by choosing $L = K$ and $J$ to be the bundle almost complex structure induced by $\bar{\partial}$.

The advantage to have the second description is that the intersection theory of almost complex submanifolds developed by the second author in [42] can come into play. The theory works particularly well when the base manifold $(X, J)$ is of
dimension 4. In this situation, the zero locus of a pseudoholomorphic section is a $J$-holomorphic curve in the first Chern class of the complex bundle $E$. With this understood, the rich theory of pseudoholomorphic curves are in our armory.

As we mentioned above, the plurigenera and thus the Kodaira dimension are classical birational invariants. We certainly expect it is true for $P_m(X)$ and $\kappa^J(X)$. As suggested by Theorem 1.5 in [42], a degree one pseudoholomorphic map is the right notion of birational morphism in the almost complex setting, at least in dimension 4. The next result, as a combination of Theorems 5.3 and 5.5 confirms the birational invariance of plurigenera, Kodaira dimension and $h^{1,0}(X) := \dim_{\mathbb{C}} H^0(X, \Omega^1(\mathcal{O})).$

**Theorem 1.3.** Let $u : (X,J_X) \to (Y,J_Y)$ be a degree one pseudoholomorphic map between closed almost complex 4-manifolds. Then $P_m(X,J_X) = P_m(Y,J_Y)$ and thus $\kappa^J_X(X) = \kappa^J_Y(Y)$. Moreover, the Hodge number $h^{1,0}(X) = h^{1,0}(Y)$.

The most essential ingredient is to establish the desired Hartogs extension theorem in the almost complex setting, which certainly has its independent interest. It is only established in dimension 4 by the foliation-by-disks technique (see e.g. [37, 42]). It is Theorem 5.2 which we reproduce in the following.

**Theorem 1.4.** Let $(E,J)$ be a complex vector bundle with a bundle almost complex structure over the almost complex 4-manifold $(X,J_X)$, and $p \in X$. Then any section in $H^0(X \setminus p, (E,J)|_{X \setminus p})$ extends to a section in $H^0(X, (E,J))$.

The next step is to study the property of plurigenera under deformation of almost complex structures. For projective manifolds, the plurigenera are showed to be invariant under projective deformation by a famous work of Siu [35, 36]. On complex surfaces, Friedman-Qin [18] proved the Van de Ven conjecture which claims that the Kodaira dimension of compact complex surfaces is a differentiable invariant. It turns out that the plurigenera are also differentiable invariants on compact complex surfaces (see Theorems 9.4 and 9.5 in [1] and the reference therein). When the complex dimension is greater than 2, the works of Catanese-LeBrun [9] and Răsdeaconu [34] show that Van de Ven conjecture does not hold any more.

By virtue of our Hodge theoretic description of plurigenera, they are upper semi-continuous functions under smooth deformation. However, it is easy to see that the dimensions could jump. When we deform an integrable almost complex structure of a surface of general type, a generic perturbed almost complex structure does not admit any pseudoholomorphic curve, while as mentioned above the zero locus of a non-trivial pseudoholomorphic section of a pluricanonical bundle is a pseudoholomorphic curve in the class $mK$. This argument itself does not exclude the possibility of invariance when the canonical class is torsion. In Section 6, we construct some explicit deformations on Kodaira-Thurston surface and 4-torus, and show that the plurigenera, the Kodaira dimension and $h^{1,0}$ are not constant under smooth deformation even when the canonical class is trivial. Recently, Kodaira dimension is computed for some interesting families of almost complex structures in [10, 11] where the deformation property and the relation with the curvature of the canonical connection for an almost Kähler manifold are explored.
In Section 6, the relation between non-integrability of almost complex structures and the Kodaira dimension is also studied. We search the possible values of Kodaira dimension if the almost complex structure is non-integrable. By applying the Riemann-Roch formula and the almost complex Künneth formula, we prove the following result (Theorem 6.10).

**Theorem 1.5.** For every \( k \in \{-\infty, 0, 1, \cdots, n - 1\} \), \( n \geq 2 \), there are examples of compact \( 2n \)-dimensional non-integrable almost complex manifolds with \( \kappa^J = k \).

We expect that the above range of Kodaira dimension for non-integrable almost complex structures is optimal. In the second paper of the series, we will show that if the second version of the Kodaira dimension equals the complex dimension of the manifold, then the almost complex structure must be integrable.

In the last section of the paper, we compute the Hodge numbers, the plurigenera and the Kodaira dimension on the six sphere \( S^6 \). It is classically known that there exist almost complex structures on \( S^6 \) [16]. A standard construction is to use the cross product of \( \mathbb{R}^7 \) applying to the tangent space of \( S^6 \). Denote this standard almost complex structure by \( J \). In Theorem 7.1, we prove that

**Theorem 1.6.** For the standard almost complex structure \( J \) on \( S^6 \), the following hold: (1) \( h^{1,0} = h^{2,0} = h^{2,3} = h^{3,3} = 0 \); (2) \( P_m(S^6, J) = 1 \) for any \( m \geq 1 \) and \( \kappa^J = 0 \).

This calculation is somewhat surprising since it is generally believed that the Kodaira dimension of a hypothetical complex structure is \(-\infty\). Our plurigenera distinguish \( J \) from hypothetical complex structures on \( S^6 \), since for the latter \( P_1 = h^{3,0} = 0 \).

In a subsequent paper [12], we will interpret the Kodaira dimension through the pluricanonical map and discuss the significant geometric consequences. We will also investigate its comparison with the symplectic Kodaira dimension [29] on symplectic 4-manifolds. Some vanishing theorems on positively-curved almost Hermitian manifolds will be discussed.

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2. Notations

We start by fixing our notations and explain the natural pseudoholomorphic structure on the pluricanonical bundles. Let \((X, J)\) be a \(2n\)-dimensional almost complex manifold. The complexification of the cotangent bundle of \(X\) decomposes as \( T^*X \otimes \mathbb{C} = (T^*X)^{1,0} \oplus (T^*X)^{0,1} \) where \((T^*X)^{1,0}\) annihilates the subspace in \(TX \otimes \mathbb{C}\).
where $J$ acts as $-i$. A $(1, 0)$-form is a smooth section of $(T^*X)^{1,0}$; similarly a $(0, 1)$-form is a smooth section of $(T^*X)^{0,1}$. The splitting of the cotangent bundle induces a splitting of all exterior powers. Write $\Lambda^{p,q}X = \Lambda^p((T^*X)^{1,0}) \otimes \Lambda^q((T^*X)^{0,1})$. Then for any $r \geq 0$, we have the decomposition

$$\Lambda^rT^*X \otimes \mathbb{C} = \oplus_{p+q=r}\Lambda^{p,q}X.$$ 

Let $\pi^{p,q}$ be the projection to $\Lambda^{p,q}X$. A $(p, q)$-form is a smooth section of $\Lambda^{p,q}X$. The space of all such sections is denoted $\Omega^{p,q}(X) = \Gamma(X, \Lambda^{p,q}X)$.

The $\bar{\partial}$ and $\partial$ operator can be defined by:

$$\bar{\partial} = \pi^{p,q+1} \circ d : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$$

$$\partial = \pi^{p+1,q} \circ d : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X),$$

where $d$ is the exterior differential. Both $\bar{\partial}$ and $\partial$ satisfy the Leibniz rule, but in general $\bar{\partial}^2$ and $\partial^2$ may not be zero. They contain important information of almost complex structures. Applying $\bar{\partial}$ to $\Lambda^{p,0}X$ and in particular to $\mathcal{K} = \Lambda^n,0X$, we have

$$\bar{\partial} : \Lambda^{p,0}X \to \Lambda^{p,1}X \cong (T^*X)^{0,1} \otimes \Lambda^{p,0}X,$$

$$\partial : \mathcal{K} \to \Lambda^{n,1}X \cong (T^*X)^{0,1} \otimes \mathcal{K}.$$ 

Here we write $\mathcal{K}$ (or any vector bundle) in short for any smooth sections of $\mathcal{K}$ (the vector bundle). We can extend the $\bar{\partial}$ to an operator $\bar{\partial}_m : \mathcal{K}^\otimes m \to (T^*X)^{0,1} \otimes \mathcal{K}^\otimes m$ for $m \geq 2$ inductively by the product rule

$$\bar{\partial}_m(s_1 \otimes s_2) = \bar{\partial}s_1 \otimes s_2 + s_1 \otimes \bar{\partial}_m s_2.$$ 

It satisfies the Leibniz rule $\bar{\partial}_m(f s) = \bar{\partial}f \otimes s + f \bar{\partial}_m s$ for any section $s$ of $\mathcal{K}^\otimes m$ and $\Lambda^{p,0}X$.

**Definition 2.1.** The space of pseudoholomorphic sections of $\mathcal{K}^\otimes m$ is defined to be

$$H^0(X, \mathcal{K}^\otimes m) = \{ s \in \Gamma(X, \mathcal{K}^\otimes m) : \bar{\partial}_m s = 0 \}.$$ 

**Remark 2.2.** The space $H^0(X, \mathcal{K}^\otimes m)$ is categorical in the almost complex category. Indeed, if $(X', J')$ is another almost complex manifold which has a diffeomorphism $F : X' \to X$ to $(X, J)$ satisfying $dF \circ J' = J \circ dF$, we say that $(X', J')$ is pseudoholomorphic isomorphic to $(X, J)$. Then $F^*(\Omega^{p,q}(X)) = \Omega^{p,q}(X')$ and $F^* \circ \bar{\partial}_J = \bar{\partial}_{J'} \circ F^*$. So $s \in \Gamma(X, \mathcal{K}_J)$ satisfying $\bar{\partial}s = 0$ if and only if $\bar{\partial}_{J'} F^* s = 0$. Similar result holds on $\mathcal{K}_J^\otimes m$ where $F^*$ and $\bar{\partial}$ are replaced by an isomorphism $F^*_m$ and the operator $\bar{\partial}_m$. Therefore, $F$ induces an isomorphism $F^*_m : H^0(X, \mathcal{K}_J^\otimes m) \to H^0(X', \mathcal{K}_{J'}^\otimes m)$ for any $m \geq 1$.

### 3. Hodge theory on almost complex manifolds

In this section, we will define Dolbeault cohomology groups for a complex bundle associated with a pseudoholomorphic structure. We show that they are finite dimensional when $X$ is compact in Theorem [3.6]. As a consequence, $H^0(X, \mathcal{K}^\otimes m)$ is finite dimensional. We will follow the method of Hodge theory to define a formal adjoint operator of $\bar{\partial}_m$ and apply the elliptic theory.
Hodge theory is well developed on compact complex manifolds (see [20], [23], [39], [44]). Its extension on compact almost complex manifolds was initiated in [21] for \((p, q)\) forms. To derive our results, we will set up Hodge theory for \((p, q)\) forms with values in a Hermitian pseudoholomorphic bundle \(E\). One of the new observations is that for a holomorphic vector bundle \(E\) over a complex manifold, the \(\bar{\partial}\) operator on the holomorphic dual \(E^*\) coincides with the \((0, 1)\) part of a Hermitian connection on \(E\) when identifying \(E^*\) with \(\bar{E}\) by a Hermitian metric.

Choose a Riemannian metric \(g\) on \(X\) compatible with \(J\), namely \(g(Ju, Jv) = g(u, v)\) for any \(u, v \in TX\). Then \(g\) induces a Hermitian structure \(h\) on \(TX \otimes \mathbb{C}\) by \(h = g - i\omega\), where \(\omega(u, v) = g(Ju, v)\). \(g\) also induces Hermitian structures on the bundles \(\Lambda^{p,q}X\) for any \((p, q)\) which we still denote by \(h\). Explicitly, assume that \(\{e_i\}\) is a local unitary frame in \(TX \otimes \mathbb{C}\) and \(\{\phi_i\}\) is the unitary coframe. Assume \(\alpha = \{i_1, i_2, \ldots, i_p\}, \beta = \{j_1, j_2, \ldots, j_q\}\) is any ordered set of \((p, q)\) multiindices. Denote \(\phi_\alpha = \phi_{i_1} \wedge \phi_{i_2} \wedge \cdots \wedge \phi_{i_p}\), \(\phi_\beta = \phi_{j_1} \wedge \phi_{j_2} \wedge \cdots \wedge \phi_{j_q}\). Then \(h\) on \(\Lambda^{p,q}X\) is defined by letting \(\{\phi_\alpha \wedge \bar{\phi}_\beta\}\) be orthogonal and \(h(\phi_\alpha \wedge \bar{\phi}_\beta, \phi_\alpha \wedge \bar{\phi}_\beta) = 2^{p+q}\).

The \(*\) operator on an almost Hermitian manifold is the unique \(\mathbb{C}\)-linear operator

\[
*: \Lambda^{p,q}X \to \Lambda^{n-q,n-p}X
\]

satisfying

\[(1) \quad h(\varphi_1, \varphi_2) dV = \varphi_1 \wedge \bar{\varphi}_2\]

where \(dV\) is the volume form of \(g\) and \(\varphi_1, \varphi_2 \in \Lambda^{p,q}X\).

Using the unitary coframe \(\{\phi_i\}\), we can write out the \(*\) operator directly. Let \(\hat{\alpha}, \hat{\beta}\) be the ordered set of the complement multi-indices of \(\alpha, \beta\) in \(\{1, 2, \ldots, n\}\). As \(\omega = \frac{i}{2} \sum_{i=1}^{n} \phi_i \wedge \bar{\phi}_i\) and \(dV = \frac{\omega^n}{n!}\), if we define on the basis

\[(2) \quad *(\phi_\alpha \wedge \bar{\phi}_\beta) = 2^{(p+q-n)}(-i)^n \epsilon_{\alpha\beta\hat{\alpha}\hat{\beta}} \phi_\beta \wedge \bar{\phi}_{\hat{\alpha}}\]

where \(\epsilon_{\alpha\beta\hat{\alpha}\hat{\beta}}\) is the sign of permutation of

\[(i_1, \ldots, i_p, j_1, \ldots, j_q, \hat{i}_1, \ldots, \hat{i}_{n-q}, \hat{j}_1, \ldots, \hat{j}_{n-p}) \to (1, 1', 2, 2', \ldots, n, n'),\]

then the operator of (2) satisfies (1). By uniqueness it gives the \(*\) operator.

Define a global inner product on \(\Omega^{p,q}(X)\) by \(\langle \varphi_1, \varphi_2 \rangle = \int_X h(\varphi_1, \varphi_2) dV\) for \(\varphi_1, \varphi_2 \in \Omega^{p,q}(X)\). Let

\[\bar{\partial}^* = -* \partial * .\]

Then the following holds

\[\langle \bar{\partial} \varphi_1, \varphi_2 \rangle = \langle \varphi_1, \bar{\partial}^* \varphi_2 \rangle.\]
The proof is the same with the integrable case, because the Leibniz rule holds and \( \bar{\partial} = d \) acting on \( \Omega^{p,q} \). Indeed, by (1), we have
\[
\langle \bar{\partial} \varphi_1, \varphi_2 \rangle = \int_X h(\bar{\partial} \varphi_1, \varphi_2) dV = \int_X \bar{\partial} \varphi_1 \wedge \star \varphi_2
= \int_X \bar{\partial}(\varphi_1 \wedge \star \varphi_2) - (-1)^{p+q-1} \int_X \varphi_1 \wedge \bar{\partial} \star \varphi_2
= \int_X \varphi_1 \wedge \star (\bar{\partial} \star \varphi_2)
= \int_X h(\varphi_1, \bar{\partial} \star \varphi_2) dV = \langle \varphi_1, \bar{\partial} \star \varphi_2 \rangle,
\]
where we use the Stokes’ theorem in the third line.

The above discussion produces the formal dual operator of \( \bar{\partial} \) on \( \Omega^{p,q} \). Next we generalize it to any Hermitian bundle \( E \) with a pseudoholomorphic structure.

**Definition 3.1.** Let \( (E, h_E) \) be a Hermitian vector bundle over \( (X, J) \). A connection \( \nabla : \Gamma(X, E) \to \Gamma(X, (T^*X \otimes \mathbb{C}) \otimes E) \) is called a Hermitian connection if
\[
d(h_E(s_1, s_2)) = h_E(\nabla s_1, s_2) + h_E(s_1, \nabla s_2),
\]
for any two sections \( s_1, s_2 \) of \( E \).

**Definition 3.2.** A pseudoholomorphic structure on \( E \) is given by a differential operator \( \partial_E : \Gamma(X, E) \to \Gamma(X, (T^*X)^{0,1} \otimes E) \) which satisfies the Leibniz rule
\[
\partial_E(fs) = \partial f \otimes s + f \partial_E s
\]
where \( f \) is a smooth function and \( s \) is a section of \( E \).

Let \( E \) be a complex vector bundle over a complex manifold \( X \). Then by Koszul-Malgrange Theorem, the datum of a pseudoholomorphic structure \( \partial_E \) satisfying \( \partial_E^2 = 0 \) is equivalent to a holomorphic structure on \( E \). In particular, any pseudoholomorphic structure on a complex vector bundle over a Riemann surface \( S \) is holomorphic, since \( (T^*S)^{0,2} = 0 \).

Denote \( \nabla^{(1,0)}, \nabla^{(0,1)} \) the \( (1,0) \) and \( (0,1) \) components of \( \nabla \). We have

**Lemma 3.3.** For any Hermitian bundle \( (E, h_E) \) with a pseudoholomorphic structure \( \partial_E \), there is a unique Hermitian connection \( \nabla \) so that \( \nabla^{(0,1)} = \partial_E \).

The lemma is well known when \( J \) is integrable and should be known to experts for general \( J \) (see [13]). We include a proof for convenience of readers.

**Proof.** We first prove the existence. Assume that \( \{U_\alpha\} \) is an open chart covering of \( X \) with partition of unity \( \{\varphi_\alpha\} \) such that \( E|_{U_\alpha} \) is trivial. For any \( s \in \Gamma(X, E) \) and any connection \( \nabla \), \( \nabla s = \nabla \sum_\alpha \varphi_\alpha s = \sum_\alpha \nabla(\varphi_\alpha s) \). So we only need to define \( \nabla \) on \( U_\alpha \) locally.

Let \( \{s_i, 1 \leq i \leq N\} \) be a unitary frame of \( E \) on \( U_\alpha \). Using summation notation, denote \( \partial_E s_i = \theta_i^j s_j \), where \( \theta_i^j \in (T^*X)^{0,1} \). Let \( \{s'_i, 1 \leq i \leq N\} \) be another unitary
frame of $E$ with $s'_i = f^i_j s_j$. As $\sum_j f^i_j \bar{f}^j_k = \delta_{ik}$, $(f^{-1})^j_i = \bar{f}^j_i$. Denote $\bar{\partial}_E s'_i = (\theta')^i_j s'_j$.

We have

$$(\theta')^i_j = \sum_k (\bar{\partial}_k f^i_k + f^i_\theta^k_j) \bar{f}^k_j.$$ 

To define $\nabla$, let $\omega^i_j = \theta^i_j - \bar{\theta}^i_j$ and $\nabla \bar{s}_i = \omega^i_j s_j$. Similarly, for $\{s'_i\}$, let $(\omega')^i_j = (\theta')^i_j - (\bar{\theta}')^i_j$ and $\nabla s'_i = (\omega')^i_j s'_j$. If $\omega^i_j$ and $(\omega')^i_j$ satisfy the transition equation

$$(\omega')^i_j = \sum_k (df^k_i + f^i_\omega^k_j) \bar{f}^k_j,$$

they give a well defined connection. This follows by

$$\begin{align*}
(\omega')^i_j &= (\theta')^i_j - (\bar{\theta}')^i_j \\
&= \sum_k ((\bar{\partial}_k f^i_k + f^i_\theta^k_j) \bar{f}^k_j - (\bar{\partial}_k f^{\bar{k}}_j + f^{\bar{k}}_{\bar{\theta}}^j) f^k_j) \\
&= \sum_k (df^k_i + f^i_\omega^k_j) \bar{f}^k_j = \sum_k (df^k_i + f^i_\omega^k_j) \bar{f}^k_j
\end{align*}$$

where we use $\partial_0 (\sum_k f^k_i \bar{f}^k_j) = 0$ for the third equality. So $\nabla$ is independent of the frames. From the skew symmetry of $\nabla$, we know that it is a Hermitian connection compatible with $h_E$.

The uniqueness follows easily if we restrict $\nabla$ to the open chart above. □

**Remark 3.4.** Recall that the almost Chern connection \[17\](see also \[19\]) associated to $g$ is the unique connection $\nabla^c$ on the tangent bundle such that $\nabla^c J = \nabla^c g = 0$ and that the torsion $\Theta$ has vanishing $(1, 1)$ part. The $\bar{\partial}$ operator on $(T^*X)^{1,0}$ induces a natural pseudoholomorphic structure. It turns out that the unique Hermitian connection on $(T^*X)^{1,0}$ induced by $\bar{\partial}$ as in Lemma 3.3 equals $\nabla^c$. To see this, assume that $\nabla^c e_i = \omega^i_j e_j$ for a unitary frame $\{e_i\}$. By the first structure equation, the $i$th component of $\Theta$ is $\Theta^i = d\phi_i + \omega^i_j \wedge \phi_j$, where $\{\phi_i\}$ is the coframe. Also $\nabla^c$ acts on $(T^*X)^{1,0}$ by $\nabla^c \phi^i = -\omega^i_j \phi_j$. Then $\Theta^i$ has vanishing $(1, 1)$ part if and only if $\bar{\partial}_i \phi_i + (\omega^i_j)^{0,1} \phi_j = 0$ which is equivalent to $(\nabla^c)^{(0,1)} = \bar{\partial}$.

Suppose that $E$ is the pluricanonical bundle $K^{2m}_X$ with the induced pseudoholomorphic structure $\bar{\partial}_m$. Then the unique Hermitian connection on $K^{2m}_X$ induced by the Chern connection on $(T^*X)^{1,0}$ is just the Hermitian connection determined by $\bar{\partial}_m$ in Lemma 3.3.

Let $(E, h_E)$ be a Hermitian bundle with a pseudoholomorphic structure $\bar{\partial}_E$. We can define a unique dual pseudoholomorphic structure on $E^*$:

$$\bar{\partial}_{E^*} : \Gamma(X, E^*) \rightarrow \Gamma(X, (T^*X)^{0,1} \otimes E^*)$$

as follows. For any section $s^* \in \Gamma(X, E^*)$ and any section $s' \in \Gamma(X, E)$, let

$$\bar{\partial}_{E^*}(s^*)(s') = \bar{\partial}(s^*(s')) - s^*(\bar{\partial}_E(s')).$$

It is easy to verify that $\bar{\partial}_{E^*}$ satisfies the Leibniz rule, giving a pseudoholomorphic structure. On the other side, with the Hermitian structure $h_E$, there exists a natural complex linear isomorphism $E^* \cong \bar{E}$, where $\bar{E}$ is the conjugate bundle of $E$. Therefore, $\bar{\partial}_{E^*}$ induces a pseudoholomorphic structure on $\bar{E}$. By Lemma 3.3, there...
is a unique Hermitian connection $\nabla$ on $E$ determined by $\bar{\partial}_E$ and $h_E$. The conjugate of the $(1,0)$ part of $\nabla$ induces

$$\nabla^{(1,0)} : \bar{\partial}_E \rightarrow (T^* X)^{0,1} \otimes \bar{\partial}_E.$$

Define $\bar{\partial}_E = \nabla^{(1,0)}$. We prove

**Lemma 3.5.** By identifying $\bar{\partial}_E$ with $E^*$, $\bar{\partial}_E = \bar{\partial}_E^*$.  

**Proof.** Let $s \in \bar{\partial}_E$ and $s' \in E$. The inner product $h_E$ on $E$ induces a bilinear pairing between $E$ and $\bar{\partial}_E$ which we still denote by $h_E$. Then by (3),

$$\bar{\partial}_E(s', s) = h_E(\nabla^{0,1} s', s) + h_E(s', \nabla^{(1,0)} s) = h_E(\bar{\partial}_E s', s) + h_E(s', \bar{\partial}_E s).$$

Therefore, $\bar{\partial}_E$ satisfies the product rule (4). Then $\bar{\partial}_E = \bar{\partial}_E^*$.  

Next we extend the $\bar{\partial}_E$ operator to $\Lambda^{p,q} X \otimes E$ and $\bar{\partial}_E$ to $\Lambda^{r,s} X \otimes \bar{\partial}_E$ by

$$\bar{\partial}_E(\varphi \otimes u) = (\bar{\partial} \varphi) \otimes u + (-1)^{p+q} \varphi \wedge \bar{\partial}_E u$$

$$\bar{\partial}_E(\phi \otimes v) = (\bar{\partial} \phi) \otimes v + (-1)^{r+s} \phi \wedge \bar{\partial}_E v.$$  

Then there is a wedge pairing

$$\wedge : (\Lambda^{p,q} X \otimes E) \times (\Lambda^{r,s} X \otimes \bar{\partial}_E) \rightarrow \Lambda^{p+r,q+s} X$$

defined by $(\varphi \otimes u) \wedge (\varphi_2 \otimes v) = h_E(u, v) \varphi \wedge \varphi_2$. As before, in this situation, $h_E(u, v)$ is denoted to be the $C$-bilinear product between $E$ and $\bar{\partial}_E$. We have the Leibniz rule for the wedge pairing

$$\bar{\partial}_E(\varphi_1 \otimes u) \wedge (\varphi_2 \otimes v) = (\bar{\partial}_E \varphi_1) \otimes u + (-1)^{p+q} \varphi_1 \wedge \bar{\partial}_E u \wedge (\varphi_2 \otimes v)$$

$$= h_E(u, v) \bar{\partial}_E \varphi_1 \wedge \varphi_2 + (-1)^{2(p+q)} h_E(\bar{\partial}_E u, v) \wedge \varphi_1 \wedge \varphi_2$$

$$= h_E(u, v)(\bar{\partial}(\varphi_1 \wedge \varphi_2) - (-1)^{p+q} \varphi_1 \wedge \bar{\partial}_E \varphi_2)$$

$$+ (\bar{\partial} h_E(u, v) - h_E(u, \bar{\partial}_E v)) \wedge \varphi_1 \wedge \varphi_2$$

$$= \bar{\partial}((\varphi_1 \otimes u) \wedge (\varphi_2 \otimes v)) - (-1)^{p+q}(\varphi_1 \otimes u) \wedge \bar{\partial}_E(\varphi_2 \otimes v)$$

where we use (5) in the third line and $h(u, \bar{\partial}_E v) \wedge \varphi_1 \wedge \varphi_2 = (-1)^{p+q+r+s}(\varphi_1 \otimes u) \wedge (\varphi_2 \otimes \bar{\partial}_E v)$ for the fourth line.

Now let $(E, h_E)$ be a Hermitian bundle. Define

$$* : \Lambda^{p,q} X \otimes E \rightarrow \Lambda^{n-q,n-p} X \otimes E$$

by $*(\varphi \otimes u) = *(\varphi) \otimes u$ for any $\varphi \in \Lambda^{p,q} X, u \in E$. From the definition we have

$$h_E(\varphi_1 \otimes u_1, \varphi_2 \otimes u_2) dV = \varphi_1 \otimes u_1 \wedge *(\varphi_2 \otimes u_2)$$

where $\varphi_1 \otimes u_1, \varphi_2 \otimes u_2 \in \Lambda^{p,q} X \otimes E, h_E$ denotes the inner product on $\Lambda^{p,q} X \otimes E$ and $dV$ is the volume form of $X$. The inner product on $\Gamma(X, \Lambda^{p,q} X \otimes E)$ is given by

$$\langle \varphi_1 \otimes u_1, \varphi_2 \otimes u_2 \rangle = \int_X h_E(\varphi_1 \otimes u_1, \varphi_2 \otimes u_2) dV.$$  

Define

$$\bar{\partial}_E = -* \nabla^{(1,0)} *.$$
We have
\[ \langle \bar{\partial}_E (\phi \otimes w), \varphi \otimes u \rangle = \int_X h_E (\bar{\partial}_E (\phi \otimes w), \varphi \otimes u) dV \]
\[ = \int_X \bar{\partial}_E (\phi \otimes w) \wedge \ast (\varphi \otimes u) \]
\[ = \int_X \bar{\partial}_E (\phi \otimes w) \wedge \ast (\varphi \otimes u) - (-1)^{p+q-1} (\phi \otimes w) \wedge \bar{\partial}_E (\ast (\varphi \otimes u)) \]
\[ = - \int_X (\phi \otimes w) \wedge (-1)^{p+q-1} \nabla^{(1,0)} \ast (\varphi \otimes u) \]
\[ = \langle \phi \otimes w, \bar{\partial}_E^* (\varphi \otimes u) \rangle \]
for \( \phi \otimes w \in \Lambda^{p,q-1} X \otimes E, \varphi \otimes u \in \Lambda^{p,q} X \otimes E \). We use \([7]\) in the third line and Stokes' theorem in the fourth line. So \( \bar{\partial}_E^* \) gives the formal adjoint of \( \bar{\partial}_E \).

Define the Laplacian
\[ \Delta_{\bar{\partial}_E} = \bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E. \]
As \( \langle \Delta_{\bar{\partial}_E} s, s \rangle = \langle \bar{\partial}_E s, \bar{\partial}_E s \rangle + \langle \bar{\partial}_E^* s, \bar{\partial}_E^* s \rangle \), \( \Delta_{\bar{\partial}_E} s = 0 \) if and only if \( \bar{\partial}_E s = 0 \) and \( \bar{\partial}_E^* s = 0 \). Denote the space of harmonic \((p,q)\) form section of \( E \) by
\[ H^{(p,q)}_{\bar{\partial}_E} (X, E) = \{ s \in \Gamma(X, \Lambda^{p,q} X \otimes E) \mid \Delta_{\bar{\partial}_E} s = 0 \}. \]
Assuming that \( X \) is compact, we get the following.

**Theorem 3.6.** Let \((X, J)\) be a compact almost complex manifold. Let \((E, h_E)\) be a Hermitian vector bundle over \( X \) with a pseudoholomorphic structure \( \bar{\partial}_E \). Then \( \Delta_{\bar{\partial}_E} \) is an elliptic differential operator and \( H^{(p,q)}_{\bar{\partial}_E} (X, E) \) is finite dimensional.

When \( E \) is the trivial bundle, this was pointed out in \([21]\).

**Proof.** We first prove the case when \( E \) is a trivial line bundle with \( \bar{\partial}_E = \bar{\partial} \). We shall show that \( \Delta_{\bar{\partial}} \) is elliptic at any point \( p \in X \). As it is a local property, it suffices to discuss in a coordinate chart \( U \). Let \((J_c, h_c)\) be the constant almost complex structure and Hermitian structure on \( U \) with \( J_c(p) = J(p), h_c(p) = h(p) \). \( J_c \) is isomorphic to the canonical complex structure of open set in \( \mathbb{C}^n \). Denote \( \bar{\partial}_c \) the \( \bar{\partial} \)-operator of \( J_c \) and \( \ast_c \) the operator corresponding to \( h_c \). We have
\[ (\bar{\partial}_c \varphi - \bar{\partial}_c \varphi) (p) = 0, \quad \ast_c (\varphi)(p) = \ast_c (\varphi)(p) \]
for any \( \varphi \in \Gamma(U, \Lambda^{p,q}) \). So \( \bar{\partial} \) and \( \bar{\partial}_c \) differ by a differential operator whose coefficients vanish at \( p \). As the principal symbol is only related to the highest degree differential, any operator from compositions of \( \bar{\partial}, \ast, \bar{\partial} \) would have the same principal symbol at \( p \) with the operators obtained by replacing them by \( \bar{\partial}_c, \ast_c, \bar{\partial}_c \). In particular, \( \Delta_{\bar{\partial}} \) has the same principal symbol with \( \Delta_{\bar{\partial}_c} \) at \( p \). The latter is the flat Laplacian on \( \mathbb{C}^n \) which is elliptic. Therefore \( \Delta_{\bar{\partial}} \) is elliptic at \( p \) and hence everywhere.

For a complex vector bundle \( E \). Let \( \{\partial_i\} \) be a local unitary coframe of \((T^*X)^{1,0}\) and \( \{u_\nu\} \) a unitary frame of \( E \). Using Einstein summation notation, any section \( s \)
of $\Lambda^{p,q}X \otimes E$ can be expressed as:

$$s = f^{\alpha\beta\nu}(\phi_\alpha \wedge \bar{\phi}_\beta) \otimes u_\nu,$$

where $(\alpha, \beta)$ runs over all multi-indices of type $(p, q)$ and $\nu$ runs over 1 to the dimension of $E$. Then by Leibniz rule,

$$\bar{\partial}_E s = \bar{\partial}(f^{\alpha\beta\nu}(\phi_\alpha \wedge \bar{\phi}_\beta) \otimes u_\nu) + (-1)^{p+q}(f^{\alpha\beta\nu}(\phi_\alpha \wedge \bar{\phi}_\beta) \otimes \bar{\partial}E u_\nu)$$

$$= \bar{\partial} f^{\alpha\beta\nu}(\phi_\alpha \wedge \bar{\phi}_\beta) \otimes u_\nu + f^{\alpha\beta\nu}(\partial \phi_\alpha \wedge \bar{\phi}_\beta) \otimes u_\nu + (-1)^{p+q} f^{\alpha\beta\nu}(\phi_\alpha \wedge \bar{\phi}_\beta) \otimes \bar{\partial}E u_\nu.$$

The calculation of the principal symbol relies only on the differential of $f^{\alpha\beta\nu}$ (see e.g. Page 113 in [27]). So any operator which has the same differential term on the coefficients $f^{\alpha\beta\nu}$ with $\bar{\partial}_E$ would have the same principal symbol too. Denote $\simeq$ to be the relation for operators with the same principal symbol. Let $\bar{\partial} \otimes id$ represent the differential operator determined by $\bar{\partial} \otimes id((f^{\alpha\beta\nu}(\phi_\alpha \wedge \bar{\phi}_\beta) \otimes u_\nu) := \bar{\partial}(f^{\alpha\beta\nu}(\phi_\alpha \wedge \bar{\phi}_\beta) \otimes u_\nu$ (it depends on choice of $\{u_\nu\}$ and is only defined locally). Let $\Delta_{\bar{\partial}} \otimes id$ be the operator given by $\Delta_{\bar{\partial}} \otimes id((f^{\alpha\beta\nu}(\phi_\alpha \wedge \bar{\phi}_\beta) \otimes u_\nu) := \Delta_{\bar{\partial}}(f^{\alpha\beta\nu}(\phi_\alpha \wedge \bar{\phi}_\beta) \otimes u_\nu)$. We have

$$\bar{\partial}_E \simeq \bar{\partial} \otimes id$$

We have shown above that $\Delta_{\bar{\partial}}$ is elliptic. Then $\Delta_{\bar{\partial}} \otimes id$ is elliptic. Therefore $\Delta_{\bar{\partial}_E}$ is elliptic.

The second part follows directly from the elliptic theory (e.g. Theorem 5.5, Corollary 5.7 in [27]). Explicitly, there is a Green operator $G$ together with the projection operator $H : \Omega^{p,q}(X, E) \to \mathcal{H}^{(p,q)}(X, E)$ such that $\Delta_{\bar{\partial}_E} \circ G + H = Id$. Also, $\Delta_{\bar{\partial}_E}$ and $G$ are both Fredholm operators and $\mathcal{H}^{(p,q)}(X, E)$ is finite dimensional.

The following Serre duality also holds on compact almost complex manifolds.

**Proposition 3.7.** For any $0 \leq p, q \leq n$, $\mathcal{H}^{(p,q)}(X, E) \cong (\mathcal{H}^{(n-p,n-q)}(X, E^*))^*$.  

**Proof.** The argument is essentially the same as the classical case (see for example Proposition 4.1.15 of [23]), except clarifying the operators on the bundle $E^*$. It suffices to show that the natural paring between $\mathcal{H}^{(p,q)}_{\bar{\partial}_E}(X, E)$ and $\mathcal{H}^{(n-p,n-q)}_{\bar{\partial}_E^*}(X, E^*)$ is nondegenerate. For any nonzero $s \in \mathcal{H}^{(p,q)}_{\bar{\partial}_E}(X, E)$, since $\bar{\partial}_E s = \bar{\partial}_E s = 0$, by Lemma 3.5 we have

$$\bar{\partial}_E^*(\bar{s}) = \bar{\partial}_E(\bar{s}) = \nabla((1, 0) \ast s) = 0,$$

and

$$\bar{\partial}_E^*(\bar{s}) = -\ast \bar{\partial}_E \ast (\bar{s}) = -(-1)^{(p+q)(n-p-q)} \ast \bar{\partial}_E s = 0.$$

So $\bar{s} \in \mathcal{H}^{(n-p,n-q)}_{\bar{\partial}_E^*}(X, E^*)$. As $\int_X s \wedge \bar{s} = \|s\|^2 \neq 0$, the non-degeneracy stands. □
the pseudoholomorphic structure on \( \Omega^p(E) \) coincides with the \( \bar{\partial}_E \) operator given by Equation (6). Define

\[
H^0(X, \Omega^p(E)) = \{ s \in \Gamma(X, \Omega^p(E)) = \Omega^{p,0}(X, E) : \bar{\partial}_E s = 0 \}.
\]

We have

**Proposition 3.8.** Let \( E \) be a complex vector bundle with a pseudoholomorphic structure over a compact almost complex manifold \( X \), then \( H^0(X, \Omega^p(E)) \) is finite dimensional for \( 0 \leq p \leq n \).

**Proof.** As \( \bar{\partial}^\ast_E = 0 \) on \( \Omega^{p,0}(X, E) \), \( \bar{\partial}_E s = 0 \) is equivalent to \( \Delta_{\bar{\partial}_E} s = 0 \). So

\[
H^0(X, \Omega^p(E)) = H^0_{\bar{\partial}E}(X, E),
\]

which is finite dimensional. \( \square \)

**Corollary 3.9.** \( H^0(X, \mathcal{K}^{\otimes m}) \) is finite dimensional.

**Proof.** Let \( E = \mathcal{K}^{\otimes m} \) with \( \bar{\partial}_E = \bar{\partial}_m \) and \( p = 0 \). Then it follows from Proposition 3.8. \( \square \)

Proposition 3.8 also implies that \( H^0_{\bar{\partial}E}(X, E) \) is independent of the Hermitian metric used to define \( \Delta_{\bar{\partial}_E} \). This should not hold for \( H^{(p,q)}_{\bar{\partial}E}(X, E) \) for \( q > 0 \). However, it is possible that the dimension of \( H^{(p,q)}_{\bar{\partial}E}(X, E) \) is independent of the defining Hermitian metric. When \( q = \dim_{\mathbb{C}} X \), Proposition 3.7 implies this is true. In general, the following question is a generalization of Problem 20 (Kodaira-Spencer) in Hirzebruch’s list [21].

**Question 3.10.** For which pairs \((p, q)\) with \( 0 < q < \dim_{\mathbb{C}} X \), do the dimensions \( \dim H^{(p,q)}_{\bar{\partial}E}(X, E) \) depend only on \( J \) and \( \bar{\partial}_E \)?

4. **Bundle almost complex structure and Iitaka dimension**

Recall that for any holomorphic vector bundle over a complex manifold, the total space is also a complex manifold so that any smooth section \( s \) is \( \bar{\partial} \) closed if and only if \( s \) induces a holomorphic map. For a complex vector bundle \( E \) over the almost complex manifold \( (X, J) \), a bundle almost complex structure as in [13] (here we use the rephrasing from [28]) is an almost complex structure \( J \) on \( TE \) so that

(i) the projection is \((J, J)\)-holomorphic,

(ii) \( J \) induces the standard complex structure on each fiber, i.e. multiplying by \( i \),

(iii) the fiberwise addition \( \alpha : E \times_X E \to E \) and the fiberwise multiplication by a complex number \( \mu : \mathbb{C} \times E \to E \) are both pseudoholomorphic.

Proposition 1.3 in [13] shows that a bundle almost complex structure \( J \) on \( E \) determines a pseudoholomorphic structure \( \bar{\partial}_J \), and the map \( J \mapsto \bar{\partial}_J \) is a bijection between the spaces of bundle almost complex structures and pseudoholomorphic structures on \( E \). Denote \( \bar{\partial}_J \) to be the pseudoholomorphic structure determined by a bundle almost complex structure \( J \). We have
Lemma 4.1. For any \( s \in \Gamma(X, E) \), \( \bar{\partial}_J s = 0 \) if and only if \( s \) is \( (J, J) \) holomorphic.

Proof. We apply a local argument. For each 2-dimensional \( J \)-invariant subspace \( P \) in \( T_pX \) at a point \( p \), we know there is a \( J \)-holomorphic disk \( D \) passing through \( p \) with the tangent plane \( P \). Then \( J \) is integrable on \( D \) and \( E|_D \) is a holomorphic bundle by dimension reason (see the argument after Definition 3.2). Restricted on \( D \), it is known that \( \bar{\partial}_J s = 0 \) is equivalent to that \( s \) is \( (J, J) \) holomorphic. Since both \( \bar{\partial}_J s = 0 \) and \( (J, J) \) holomorphic are local conditions and only depend on the complex direction, we can choose \( P \) to be any direction and obtain the general equivalence. \( \square \)

We call such a section \( s \) a pseudoholomorphic section of \((E, J)\). The above correspondence builds the bridge to the second author’s paper \([42]\) on the intersections of almost complex submanifolds, and is used frequently in this paper. In particular, when \( E \) is a complex line bundle over a 4-manifold \((X, J)\), the zero locus of a pseudoholomorphic section \( s \) is a \( J \)-holomorphic 1-subvariety in the class \( c_1(E) \) by Corollary 1.3 of \([42]\).

For any \((E, J)\) over a compact almost complex manifold \((X, J)\), define \( H^0(X, (E, J)) = \{ s \in \Gamma(X, E) : \bar{\partial}_J s = 0 \} \).

By Theorem 3.6, it is finite dimensional. The \((E, J)\)-genus of \( X \) is defined as \( P_{E, J} := \dim \mathbb{C} \ H^0(X, (E, J)) \). When there is no confusion of the choice of bundle almost complex structure \( J \), we will simply write it as \( P_E \). The bundle almost complex structure \( J \) on \( E \) induces a bundle almost complex structure on \( E^\otimes m \), which is also denoted by \( J \). Thus the notation \( P_{E^\otimes m, J} \) or simply \( P_{E^\otimes m} \) makes sense. When \( E = K \) is endowed with the standard bundle almost complex structure, the \( m \)th plurigenus of \((X, J)\) is defined to be \( P_m(X, J) = \dim \mathbb{C} \ H^0(X, K^\otimes m) \).

We are now ready to define the Iitaka dimension (and the Kodaira dimension).

Definition 4.2. The Iitaka dimension \( \kappa^J(X, (L, J)) \) of a complex line bundle \( L \) with a bundle almost complex structure \( J \) over a compact almost complex manifold \((X, J)\) is defined as

\[
\kappa^J(X, (L, J)) = \begin{cases} 
-\infty, & \text{if } P_{L^\otimes m, J} = 0 \text{ for any } m \geq 0 \\
\limsup_{m \to \infty} \frac{\log P_{L^\otimes m, J}}{\log m}, & \text{otherwise.}
\end{cases}
\]

The Kodaira dimension \( \kappa^J(X) \) is defined by choosing \( L = K \) and \( J \) to be the bundle almost complex structure induced by \( \bar{\partial} \).

5. Birational invariants

As suggested by the results in \([42]\), degree 1 pseudoholomorphic maps are the right notion of birational morphism in the almost complex category. We define two almost complex manifolds \( M \) and \( N \) to be birational to each other if there are almost complex manifolds \( M_1, \cdots, M_{n+1} \) and \( X_1, \cdots, X_n \) such that \( M_1 = M \) and
$M_{n+1} = N$, and there are degree one pseudoholomorphic maps $f_i : X_i \to M_i$ and $g_i : X_i \to M_{i+1}$, $i = 1, \ldots, n$.

The next natural step is to find birational invariants. In birational geometry, there are many important birational invariants, including the fundamental group, the Hodge numbers $h^{p,0}$, the plurigenera and in particular the Kodaira dimension. As shown in Theorem 1.5 of [42], $X = M\#k\mathbb{CP}^2$ when there is a degree one pseudoholomorphic map $\phi : X \to M$ between 4-dimensional almost complex manifolds. Hence, the fundamental group is apparently also birationally invariant in the almost complex category.

We will show in this section that the almost complex Kodaira dimension $\kappa^J$, plurigenera $P_m$ and Hodge numbers $h^{p,0}$ are birational invariants for 4-dimensional almost complex manifolds.

We first show that plurigenera and Kodaira dimension for almost complex 4-manifolds are non-increasing under pseudoholomorphic maps of non-zero degree.

Lemma 5.1. Let $u : (X, J) \to (Y, J_Y)$ be a surjective pseudoholomorphic map between closed almost complex 2n-manifolds. Then $P_m(X, J) \geq P_m(Y, J_Y)$. Hence, $\kappa^J(X) \geq \kappa^{J_Y}(Y)$.

Proof. Pullback of sections defines $u_m^* : H^0(Y, K_Y^{\otimes m}) \to H^0(X, K_X^{\otimes m})$ for all $m \geq 1$. Combining the argument of Theorem 5.5 and the result of Theorem 3.8 in [42], we know that the singularity subset $S_u$ has finite $(2n-2)$-dimensional Hausdorff measure. (Theorem 1.4 of [42] shows that $S_u$ supports a $J$-holomorphic 1-subvariety when $n = 2$.)

For any $s \in H^0(Y, K_Y^{\otimes m})$, if $u_m^*(s) = 0$, then the restriction $u_m^*(s)|_{X\setminus S_u} = 0$ would imply $s|_{Y\setminus u(S_u)} = 0$. Since $s$ is smooth and $Y \setminus u(S_u) = Y$, we know $s = 0$. Hence $u_m^*$ is injective, which implies the inequalities. \qed

For any pseudoholomorphic structure $\bar{\partial}_E$ of a complex vector bundle $E$, it also induces a pseudoholomorphic structure on $E|_D$ for any non-compact embedded $J$-holomorphic curve $D \subset X$. By Koszul-Malgrange theorem, it is holomorphic. Since $D$ is Stein, by Oka’s principle, any holomorphic bundle is isomorphic to the product $D \times \mathbb{C}^k$.

Let $s$ be a smooth section of $E$ over a compact almost complex manifold $X$. Then for any point $x \in X$ and a $J$-holomorphic disk $D$ passing through it, we could write $s|_D$ as a vector valued complex function $s' : D \to \mathbb{C}^k$. In fact, $s'$ is the composition of $s$ with the projection from $D \times \mathbb{C}^k$ to $\mathbb{C}^k$. Since the projection is holomorphic, $s$ is $(J, \bar{\partial})$-holomorphic if and only if $s'$ is holomorphic. In other words, $s'$ is a holomorphic function. Later, we will simply write $s$ instead of $s'$ by abuse of notation.

Since there is no local complex coordinate system for a general almost complex manifold, we use the $J$-fiber-diffeomorphism [37] to play such a role.

We start with any point $x \in M$, and want to choose an open neighborhood $U$ of $x$. Without loss of generality, as in [37,42], we can assume the almost complex structure
$J$ is on $\mathbb{C}^2$ with coordinates $(w_0, w_1)$. It agrees with the standard almost complex structure $J_0$ at the origin, but typically nowhere else. Let $\Omega$ be a non-degenerate 2-form which is compatible with $J$ and $\Omega|_x = \frac{i}{2}(dw_0 \wedge d\overline{w}_0 + dw_1 \wedge d\overline{w}_1)$.

Denote a family of holomorphic disks $D_w := \{(\xi, w) | |\xi| < \rho\}$, where $w \in D$. What we get from [37], mainly Lemma 5.4, is a diffeomorphism $f : D \times D \rightarrow \mathbb{C}^2$ onto its image $U$, where $D \subset \mathbb{C}$ is the disk of radius $\rho$, such that:

- For all $w \in D$, $f(D_w)$ is a $J$-holomorphic submanifold containing $(0, w)$.
- For all $w \in D$, $\text{dist}((\xi, w); f(\xi, w)) \leq z \cdot \rho \cdot |\xi|$. Here $z$ depends only on $\Omega$ and $J$.
- For all $w \in D$, the derivatives of order $m$ of $f$ are bounded by $z_m \cdot \rho$, where $z_m$ depends only on $\Omega$ and $J$.

We call such a diffeomorphism $J$-fiber-diffeomorphism. We have freedom to choose the “direction” of these disks by rotating the Gaussian coordinate system. As in [3][37][42], we are also able to choose the center $f(0 \times D)$ to be $J$-holomorphic.

With these preparations, we are able to derive the following version of Hartogs’s extension theorem for almost complex manifolds.

**Theorem 5.2.** Let $(E, J)$ be a complex vector bundle with a bundle almost complex structure over the almost complex 4-manifold $(X, J)$, and $p \in X$. Then any section in $H^0(X \setminus p, (E, J)|_{X \setminus p})$ extends to a section in $H^0(X, (E, J))$.

**Proof.** Near $p$, as in [42], we choose a $J$-fiber-diffeomorphism of a neighborhood $U$ of $p$, $f : D \times D \rightarrow U$, such that $f(0 \times D)$ and $f(D \times w), \forall w \in D$ are embedded $J$-holomorphic disks. By possibly shrinking $U$, our complex vector bundle $(E, J)$ could be trivialized such that each section of it (on a subset of $U$) restricts to $f(D_w)$ and $f(0 \times D)$ are complex vector valued functions. We can achieve it by first choosing the trivialization along $f(0 \times D)$ and then fiberwise along each $f(D_w)$.

Let $s \in H^0(X \setminus p, (E, J)|_{X \setminus p})$. By choosing the above trivialization and the previous discussion, $s$ is a vector valued holomorphic function along each $D_w$ when $w \neq 0$, $(D \setminus \{0\}) \times 0$ and $0 \times (D \setminus \{0\})$. We use Cauchy integration formula to define

$$a_j(z_2) = \frac{1}{2\pi i} \int_{|\xi| = \rho} \frac{s(\xi, z_2)}{\xi^{j+1}} d\xi, \quad \forall j \in \mathbb{Z}.$$  

It is a smooth (vector valued) function and $a_0(z_2) = s(0, z_2)$ when $z_2 \neq 0$. Hence, in particular, $a_0(z_2)$ is holomorphic on $D \setminus \{0\}$. We let $a_j(0) = \lim_{z_2 \to 0} a_j(z_2)$. Since for fixed $z_2 \neq 0$, $s(\xi, z_2)$ is holomorphic for $\xi \in D$, we know $a_{-j}(z_2) = 0$ for $j > 0$. By the continuity of $s$, we know $a_{-j}(0) = 0, \forall j > 0$. Hence, $s(\xi, 0) = \sum_{j = 0}^{\infty} a_j(0)\xi^j = \sum_{j = 0}^{\infty} a_j(0)\xi^j$ is also holomorphic at $\xi = 0$ with value $a_0(0)$ at $\xi = 0$. In particular,

$$a_0(0) = \frac{1}{2\pi i} \int_{|\xi| = \rho} \frac{s(\xi, 0)}{\xi} d\xi.$$  

Since $a_0(z_2) = s(0, z_2)$ is holomorphic when $z_2 \neq 0$, the partial derivative

$$\frac{\partial}{\partial z_2} a_0(z_2) = \frac{1}{2\pi i} \int_{|\xi| = \rho} \frac{\partial}{\partial z_2} \frac{s(\xi, z_2)}{\xi} d\xi = 0.$$  

This extends to $z_2 = 0$ since $s$ is smooth. Hence, $a_0(z_2) = s(0, z_2)$ is also holomorphic at $z_2 = 0$. 

To summarize, what we have proved in the above is that the extensions of holomorphic functions \(s(0, z_2)\) and \(s(z_1, 0)\) on \(0 \times (D \setminus \{0\})\) and \((D \setminus \{0\}) \times 0\) to \((0, 0)\) have the same value \(a_0(0)\), and are holomorphic at both disks \(0 \times D\) and \(D_0\). As in [42], we can choose the center \(f(0 \times D)\) of the \(J\)-fiber diffeomorphism (that transverse to \(f(D_0)\)) to be \((0,0)\) any given \(J\)-holomorphic disk. Moreover, we can also choose a family of disks passing through \(p\) whose complex tangent directions at \(p\) form a disk around a given direction \(\kappa\) in \(\mathbb{C}P^1\). Moreover, each of them is the \(D_0\) fiber of a \(J\)-fiber diffeomorphism (see Lemma 5.8 of [37] or Lemma 3.10 of [42]). Since \(\mathbb{C}P^1\) is compact, we can choose finite many such families such that their union covers a neighborhood of \(p\), and their tangent directions cover \(\mathbb{C}P^1\).

We choose \(J\)-fiber diffeomorphisms around \(p\) whose fiber passes through \(p\) lying in the above union of families and take the center of the foliation to be either \(f(0 \times D)\) or \(f(D_0)\) in our above construction. By this process, we know all the disks in the above union of families have the same extended value at \(p\), and are holomorphic at all the directions. Hence, our section \(s \in H^0(X, (E, J)|_{X,Y})\) is extended over \(p\) to a section in \(H^0(X, (E, J))\).

We are ready to show the Kodaira dimension \(\kappa^J\) is a birational invariant for almost complex 4-manifolds.

**Theorem 5.3.** Let \(u : (X, J) \to (Y, J_Y)\) be a degree one pseudoholomorphic map between closed almost complex 4-manifolds. Then \(P_m(X, J) = P_m(Y, J_Y)\) and thus \(\kappa^J(X) = \kappa^{J_Y}(Y)\).

**Proof.** First, by Lemma 4.1, we know that any element in \(H^0(X, \mathcal{K}^\otimes m_X)\) is \((J, J_Y)\)-holomorphic where \(J_Y\) is the bundle almost complex structure corresponding to \(\tilde{\partial}_m\).

By Lemma 5.1, we only need to show \(u^*_m\) is surjective. By Theorem 1.5 of [42], we know there is a finite set \(Y_1 \subset Y\) such that

\[
u : X \setminus u^{-1}(Y_1) \to Y \setminus Y_1
\]

is a diffeomorphism. For \(\sigma \in H^0(X, \mathcal{K}^\otimes m_X)\), we could pull it back by \(u^{-1}|_{Y \setminus Y_1}\) to get \((u^{-1})^*(\sigma) \in H^0(Y \setminus Y_1, \mathcal{K}^\otimes m_{Y \setminus Y_1})\). By Theorem 5.2, we could extend point-by-point over \(Y_1\) to get a unique element in \(H^0(Y, \mathcal{K}^\otimes m_Y)\).

Hence, \(u^*_m\) is surjective and we complete the proof. □

There are also other birational invariants. Since \(\Lambda^{p,1}X \cong (T^*X)^{0,1} \otimes \Lambda^{p,0}X\), the \(\bar{\partial}\) operator induces a natural pseudoholomorphic structure on \(\Lambda^{p,0}X\) for \(0 \leq p \leq n\). By Proposition 3.8, \(H^0(X, \Omega^p_X) := H^0(X, \Omega^p(X)) = \mathcal{H}^{p,0}_{\bar{\partial}}(X)\) is finite dimensional. We denote \(h^{p,0}(X) := \dim \mathcal{H}(X, \Omega^p(X))\). For a pseudoholomorphic map \(u : (X, J) \to (Y, J_Y)\) between closed almost complex 2n-manifolds and any \(0 \leq p \leq n\), pullback of sections defines \(u^* : H^0(Y, \Omega^p_Y) \to H^0(X, \Omega^p_X)\). When \(u\) is surjective, by the same argument as Lemma 5.1, we have

**Lemma 5.4.** Let \(u : (X, J) \to (Y, J_Y)\) be a surjective pseudoholomorphic map between closed almost complex 2n-manifolds. Then \(u^* : H^0(Y, \Omega^p_Y) \to H^0(X, \Omega^p_X)\) is injective and \(h^{p,0}(X) \geq h^{p,0}(Y)\) for any \(0 \leq p \leq n\).
We can also show that \( h_{p,0} \) are birational invariants in dimension 4. In fact, the only one which does not follow from Theorem 5.3 is \( h^{1,0} \).

**Theorem 5.5.** Let \( u : (X,J) \to (Y,J_Y) \) be a degree one pseudoholomorphic map between closed almost complex 4-manifolds. Then \( h^{p,0}(X) = h^{p,0}(Y) \) for any \( 0 \leq p \leq 2 \).

**Proof.** First, by Lemma 4.1 we know that any element in \( H^0(X,\Omega^p_X) \) is \((J,J^-J^\circ)\)-holomorphic where \( J^-J^\circ \) is the bundle almost complex structure on \( \Lambda^p,0 \) corresponding to the pseudoholomorphic structure induced by \( \bar{\partial} \).

By Lemma 5.4 we only need to show \( u^* \) is surjective. By Theorem 1.5 of [42], we know there is a finite set \( Y_1 \subset Y \) such that

\[
u : X \setminus u^{-1}(Y_1) \to Y \setminus Y_1
\]

is a diffeomorphism. For \( \sigma \in H^0(X,\Omega^p_X) \), we could pull it back by \( u^{-1}|_{Y\setminus Y_1} \) to get \((u^{-1})^*(\sigma) \in H^0(Y \setminus Y_1,\Omega^p_{Y \setminus Y_1})\), which are the pseudoholomorphic sections of \( \Lambda^p,0(Y) \) over \( Y \setminus Y_1 \). By Theorem 5.2 we could extend it point-by-point over \( Y_1 \) to get a unique element in \( H^0(Y,\Omega^p_Y) \). Hence, \( u^* \) is surjective and we complete the proof.

We would like to remark that the dimension of the \( J \)-anti-invariant cohomology \( H_J(X,\mathbb{R}) \) defined in [30] is also a birational invariant as shown in Theorem 1.2 of [3].

### 6. Examples

In this section, we give some explicit examples on the calculation of the almost complex plurigenera, the Kodaira dimension \( \kappa_J \) and \( h^{1,0} \). As we have seen in Section 5 all of them are birational invariant on 4-manifolds. However, different from the integrable case, they are no longer deformation invariants. This is easy to see by deforming an integrable almost complex structure of a surface of general type as shown in the introduction. This argument does not quite extend to the case when the canonical class is torsion. Our first two examples study such explicit deformations on Kodaira-Thurston surface and 4-torus.

In Section 6.3 we show that there are examples of compact 2n-dimensional non-integrable almost complex manifolds with Kodaira dimension \( \{-\infty, 0, 1, \cdots, n-1\} \) for \( n \geq 2 \) (Theorem 6.10).

#### 6.1. The Kodaira-Thurston surface

Consider the Kodaira-Thurston surface \( X = S^1 \times (\Gamma \setminus \text{Nil}^3) \), where \( \text{Nil}^3 \) is the Heisenberg group

\[
\text{Nil}^3 = \{ A \in GL(3,\mathbb{R}) | A = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \}
\]

and \( \Gamma \) is the subgroup in \( \text{Nil}^3 \) consisting of element with integer entries, acting by left multiplication (see [38]). \( X \) is homogeneous and has trivial tangent and cotangent
bundle. An invariant frame of the tangent bundle is given by
\[
\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \frac{\partial}{\partial z},
\]
where \( t \) is the coordinate of \( S^1 \). The corresponding dual invariant coframe is given by
\[
dt, dx, dy, dz - xdy.
\]
For any \( a \neq 0 \in \mathbb{R} \), define the almost complex structures \( J_a \) by:
\[
J_a(\frac{\partial}{\partial t}) = \frac{\partial}{\partial x}, J_a(\frac{\partial}{\partial x}) = -\frac{\partial}{\partial t}, J_a(\frac{\partial}{\partial y} + x \frac{\partial}{\partial z}) = a \frac{\partial}{\partial z}, J_a(\frac{\partial}{\partial z}) = -\frac{a}{4}(\frac{\partial}{\partial y} + x \frac{\partial}{\partial z}).
\]
As \((T^*X)^{1,0}\) is spanned by \( \phi_1 = dt + idx, \phi_2 = dy + i(a(dz - xdy)) \), any section of \( \mathcal{K} \) can be written as \( s = f \phi_1 \wedge \phi_2 \). Since
\[
d\phi_2 = -iadx \wedge dy = -\frac{a}{4}(\phi_1 \wedge \bar{\phi}_2 - \bar{\phi}_1 \wedge \phi_2 - \bar{\phi}_1 \wedge \bar{\phi}_2 + \phi_1 \wedge \phi_2),
\]
we have \((d\phi_2)^{0,2} = \frac{a}{4} \phi_1 \wedge \bar{\phi}_2 \neq 0\), which implies the Nijenhuis tensor is nonzero. By the Newlander-Nirenberg theorem \[33\], we know \( J_a \) is not integrable.

Moreover, we have
\[
\bar{\partial}(\phi_1 \wedge \phi_2) = -\phi_1 \wedge \bar{\partial}\phi_2 = \frac{a}{4} \phi_1 \wedge (\phi_1 \wedge \bar{\phi}_2 - \bar{\phi}_1 \wedge \phi_2)
\]
\[
= \frac{a}{4} \bar{\phi}_1 \wedge \phi_1 \wedge \phi_2.
\]
So \( \bar{\partial}s = 0 \) if and only if
\[
(9) \quad \bar{\partial}f + \frac{a}{4} f \bar{\phi}_1 = 0.
\]
Let \( w = t + ix, \frac{\partial}{\partial w} = \frac{1}{2}(\frac{\partial}{\partial t} + i \frac{\partial}{\partial x}) \) and \( V = \frac{1}{2}((\frac{\partial}{\partial y} + x \frac{\partial}{\partial z}) + i \frac{1}{a} \frac{\partial}{\partial z}) \). Then \( \frac{\partial}{\partial w}, V \) are dual vectors of \( \phi_1, \bar{\phi}_2 \). From \[9\] we have
\[
(10) \quad \frac{\partial f}{\partial w} + \frac{a}{4} f = 0
\]
\[
(11) \quad V(f) = 0.
\]
Let \( f = f_1 + if_2 \), where \( f_1, f_2 \) are smooth real functions on \( X \). From \[11\] we get that \( \bar{V}Vf = 0 \) where \( \bar{V} \) is the conjugate of \( V \). As \( \bar{V}V = \frac{1}{4}((\frac{\partial}{\partial y} + x \frac{\partial}{\partial z})^2 + (\frac{1}{a} \frac{\partial}{\partial z})^2) \), we obtain
\[
(12) \quad \frac{\partial^2 f_1}{\partial y^2} + 2x \frac{\partial^2 f_1}{\partial y \partial z} + (x^2 + \frac{1}{a^2}) \frac{\partial^2 f_1}{\partial z^2} = 0,
\]
\[
(13) \quad \frac{\partial^2 f_2}{\partial y^2} + 2x \frac{\partial^2 f_2}{\partial y \partial z} + (x^2 + \frac{1}{a^2}) \frac{\partial^2 f_2}{\partial z^2} = 0
\]
Consider the fibration \( \rho : X \to T^2 = \mathbb{R}^2/\mathbb{Z}^2 \) given by
\[
\rho([t, x, y, z]) = [t, x].
\]
The fiber of \( \rho \) is a torus with coordinate \((y, z)\). \[12\], \[13\] is strictly elliptic without zero order term when viewing \( f \) as a function of \( y, z \). As the fiber is compact, by the maximum principle \( f \) is constant in each fiber. We can push down \( f \) to a function
on the base $T^2$ with $(t, x)$ coordinate. To solve the equation (10) on $T^2$, consider the Fourier series

$$F(f) = \sum_{(k,l) \in \mathbb{Z}^2} f_{k,l} e^{2\pi i (kt+lx)}, \quad f_{k,l} = \int_{T^2} f(t, x) e^{-2\pi i (kt+lx)} dt dx.$$ 

For smooth function $f$, $f = 0$ if and only if $f_{k,l} = 0, \forall (k,l) \in \mathbb{Z}^2$ by the completion of the series $\{e^{2\pi i (kt+lx)}\}$. Apply $F$ to (10), we get

$$\sum_{(k,l) \in \mathbb{Z}^2} (a^4 + \pi (ik - l)) f_{k,l} e^{2\pi i (kt+lx)} = 0.$$ 

If $a \notin 4\pi \mathbb{Z}$, then $\frac{a}{4} + \pi (k - il) \neq 0$ for any $(k,l) \in \mathbb{Z}^2$. So $f_{k,l} = 0$ and $f = 0$. If $a = 4l\pi$ for some $l \in \mathbb{Z}\{0\}$, then $f = Ce^{2\pi ilx}$ are the solutions. Therefore we get

$$P_1(X, J_a) = \begin{cases} 
0, & a \notin 4\pi \mathbb{Z} \\
1, & a \in 4\pi \mathbb{Z} 
\end{cases}.$$ 

For $m \geq 2$, assume that $s = f(\phi_1 \wedge \phi_2)^{\otimes m}$ is a holomorphic section of $K^{\otimes m}$. Then

$$\bar{\partial}_m s = (\bar{\partial} f + \frac{ma}{4} \bar{f}\bar{\phi}_1)(\phi_1 \wedge \phi_2)^{\otimes m} = 0.$$ 

The same computation from above shows that $f$ is constant on $(y, z)$ and satisfies

$$\frac{\partial f}{\partial \bar{w}} + \frac{ma}{4} f = 0.$$ 

Using Fourier transform, we get that if $a \notin \frac{4}{m} \pi \mathbb{Z}$, then $f = 0$; if $a = \frac{4m}{m}$ for some $l \in \mathbb{Z}\{0\}$, then $f = Ce^{2\pi ilx}$. So

$$P_m(X, J_a) = \begin{cases} 
0, & a \notin \frac{4}{m} \pi \mathbb{Z} \\
1, & a \in \frac{4}{m} \pi \mathbb{Z} 
\end{cases}.$$ 

To compute $h^{1,0}(X)$, assume that $\gamma = g_1 \phi_1 + g_2 \phi_2 \in H^0(X, \Omega_X)$. As $d\phi_1 = 0$ and $\bar{\partial}\phi_2 = -\frac{a}{4} \phi_1 \wedge \bar{\phi}_2 + \frac{a}{4} \bar{\phi}_1 \wedge \phi_2$, from $\bar{\partial} \gamma = 0$ we obtain

$$\bar{\partial} g_1 + \frac{a}{4} g_2 \bar{\phi}_2 = 0 \tag{15}$$

$$\bar{\partial} g_2 - \frac{a}{4} g_2 \bar{\phi}_1 = 0 \tag{16}$$

(16) is in the same form with (9). So we have $V(g_2) = 0$ and then $g_2$ is independent of $y, z$. (15) is equivalent to

$$\frac{\partial g_1}{\partial \bar{w}} = 0 \tag{17}$$

$$V(g_1) + a \frac{g_2}{4} = 0 \tag{18}$$

From (17) we have that $g_1$ is independent of $t, x$. As $g_2$ is independent of $y, z$, composing $V$ to (18) we get that $VV(g_1) = 0$. Therefore, $g_1$ is a constant. Returning to (18) we get that $g_2 = 0$. Therefore $\gamma = c\phi_1$ for some constant $c$ and $h^{1,0}(X) = 1$. 


In conclusion, we have

**Proposition 6.1.** For any \(a \neq 0 \in \mathbb{R}\), there is a non-integrable almost complex structure \(J_a\) on \(X = S^1 \times (\mathbb{R}/\mathbb{Z})\) such that \(h^{1,0}(X) = 1\) and

\[
\kappa^{J_a}(X) = \begin{cases} 
-\infty, & a \notin \pi\mathbb{Q} \\
0, & a \in \pi\mathbb{Q}\setminus\{0\}
\end{cases}
\]

**Proof.** As \(\bigcup_{m \in \mathbb{Z}^+} \frac{4}{m} \pi \mathbb{Z} = \pi\mathbb{Q}\), we have that if \(a \notin \pi\mathbb{Q}\), \(P_m = 0\) for all \(m\); if \(a \in \pi\mathbb{Q}\), then \(P_m = 1\) for some \(m\). \(\square\)

If we choose \(a = 4\pi, 2\pi, \frac{4}{3}\pi, \cdots, \frac{4}{n}\pi, \cdots\), then the first nonzero plurigenera are \(P_1, P_2, P_3, \cdots, P_n, \cdots\). Therefore they are not birationally equivalent though \(\kappa^J = 0\).

**Remark 6.2.** Let \(J\) be the almost complex structure given by

\[
\begin{align*}
J(\frac{\partial}{\partial t}) &= \frac{\partial}{\partial z}, \\
J(\frac{\partial}{\partial x}) &= \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}.
\end{align*}
\]

Then \(J\) is integrable and induces the usual complex structure on \(X\). In this case, \(K\) is holomorphically trivial with a closed section \((dt + i(dz - xdy))\wedge(dy + idx)\). So \(P_m(X, J) = 1\) for any \(m \geq 1\) and \(\kappa^J(X) = 0\).

From (14), we see that both plurigenera and Kodaira dimension are not deformation invariant. However, we still have upper semi-continuity. Assume that \(\Delta\) is an open set in \(\mathbb{C}\) and \(\{J(t), t \in \Delta\}\) is a family of almost complex structures on a compact smooth manifold, depending smoothly on \(t\). Let \(P_m(t), h^{p,0}(t)\) be the \(m\)-th plurigenus and \((p,0)\) Hodge number of \(J(t)\). We have

**Proposition 6.3.** \(P_m(t)\) and \(h^{p,0}(t)\) are upper semi-continuous function of \(t\).

**Proof.** As all sections in \(H^0(X, K(t)^{\otimes m})\) and \(H^0(X, \Omega^p(t))\) are exactly the harmonic sections, by the properties of elliptic operators (Theorem 4.3 in [26], see also [14]), \(P_m(t)\) and \(h^{p,0}(t)\) are upper semi-continuous. \(\square\)

6.2. 4-torus. We offer another example on the four torus. Consider the four torus \(X = T^4 = \mathbb{R}^4/\mathbb{Z}^4\) with coordinates \((x_1, x_2, x_3, x_4)\). We study the almost complex structure \(J\) introduced in [8] given by

\[
J = \begin{pmatrix} 0 & -1 & \alpha & \beta \\ 1 & 0 & -\beta & \alpha \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.
\]

Assume that \(\alpha, \beta\) are any two real smooth functions on \(T^4\) satisfying \(\frac{\partial^2(\beta + i\alpha)}{\partial x_1^2} + \frac{\partial^2(\beta + i\alpha)}{\partial x_2^2} \neq 0\) in a dense open set. For example, \(\alpha = \cos 2\pi(x_1 + x_2), \beta = \sin 2\pi(x_1 + \cdots)\).
$x_2$). Direct computation shows that $J$ is integrable if and only if $\alpha, \beta$ are independent of $x_1, x_2$ (see [8]). Therefore, $J$ is not integrable by our assumption. Let

$$\phi_1 = dx_1 + i(dx_2 - \alpha dx_3 - \beta dx_4), \quad \phi_2 = dx_3 - idx_4.$$  

Then $(T^*X)^{1,0}$ is spanned by $\phi_1, \phi_2$. Assume that $s = f\phi_1 \wedge \phi_2$ is a smooth section of $\mathcal{K}$. Let $w = x_1 + ix_2, \frac{\partial}{\partial w} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}), \bar{\partial}s = 0$ if and only if

$$\bar{\partial}f + \frac{1}{2}\frac{\partial(\beta + i\alpha)}{\partial w} f\phi_2 = 0. \quad (19)$$

It is equivalent to

$$\frac{\partial f}{\partial w} = 0 \quad (20)$$

and

$$\frac{\partial f}{\partial x_3} + \alpha \frac{\partial f}{\partial x_2} - i(\frac{\partial f}{\partial x_4} + \beta \frac{\partial f}{\partial x_2}) + \frac{\partial(\beta + i\alpha)}{\partial w} f = 0. \quad (21)$$

As $T^4$ is compact, from (20) we get that $f$ is constant in the $(x_1, x_2)$ direction. Then (21) become

$$\frac{\partial f}{\partial x_3} - i\frac{\partial f}{\partial x_4} + \frac{\partial(\beta + i\alpha)}{\partial w} f = 0. \quad (22)$$

Apply $\frac{\partial}{\partial w}$ to (22) to get

$$\frac{\partial^2(\beta + i\alpha)}{\partial w \partial \bar{w}} f = 0. \quad (23)$$

By the assumption of $\alpha, \beta$, we have $f = 0$ and $s = 0$. Similarly, for $\mathcal{K}^\otimes m, m \geq 2$, if $s = f(\phi_1 \wedge \phi_2)^\otimes m$ is holomorphic, then

$$\bar{\partial}f + \frac{m}{2}\frac{\partial(\beta + i\alpha)}{\partial w} f\phi_2 = 0.$$  

The same argument gives that $s = 0$. Therefore, $P_m(X, J) = 0, m \geq 1$ and $\kappa^J(X) = -\infty$.

For $h^{1,0}$, assume that $\gamma = g_1\phi_1 + g_2\phi_2 \in H^0(X, J).$ Then from $\bar{\partial}\gamma = 0$ we get that

$$\bar{\partial}g_1 + \frac{1}{2}\frac{\partial(\beta + i\alpha)}{\partial w} g_1\phi_2 = 0 \quad (24)$$

$$\bar{\partial}g_2 + \frac{1}{2}\frac{\partial(\beta - i\alpha)}{\partial w} g_1\phi_1 = 0. \quad (25)$$

(24) is the same with (19), so we get that $g_1 = 0$. Putting it to (25), we deduce that $g_2$ is a constant. So $\gamma = c\phi_2$ and $h^{1,0} = 1$.

For any $t = t_1 + it_2 \in \mathbb{C}$, let

$$J(t) = \begin{pmatrix} 0 & -1 & t_1\alpha & t_2\beta \\ 1 & 0 & -t_2\beta & t_1\alpha \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. $$
$J(0)$ is the standard complex structure on $T^4$ and $J(1 + i) = J$. By the above calculation, for any $m \geq 1$, $t \in \mathbb{C}$, we have

$$P_m(t) = \begin{cases} 0, t \neq 0 & \text{, } \ \ \ \ \ \ \ h^{1,0}(t) = \begin{cases} 1, t \neq 0 & \text{, } \ \ \ \ \ \ \ 2, t = 0 & \text{.} \end{cases} \end{cases}$$

This gives an example where the plurigenera, the Kodaira dimension and $h^{1,0}$ are not constant under smooth deformation even when $K = 0$ for $t = 0$.

6.3. Non-integrable almost complex manifolds with large Kodaira dimension. Although a generic almost complex structure does not have any pseudoholomorphic curve, which forces Kodaira dimension to be $-\infty$ or 0, we still have interesting non-integrable examples with large Kodaira dimension. In this subsection, we give examples of non-integrable almost complex structures on $2n$-manifolds with Kodaira dimension lying among $-\infty, 0, 1, \ldots, n - 1$. First, we construct non-integrable almost complex 4-manifolds with $\kappa^J = 1$.

Let $S$ be a compact Riemann surface with genus $g \geq 2$. We shall define a non-integrable almost complex structure on $X = T^2 \times S$. Denote the two projections by

$$\pi_1 : T^2 \times S \rightarrow T^2, \pi_2 : T^2 \times S \rightarrow S.$$ Assume that $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ has coordinate $(x, y)$. Then $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ is a global frame on $T^2$. The tangent bundle of $X$ has a splitting $TX = TT^2 \times TS$. Let $J_S$ be the complex structure on $S$ with local holomorphic coordinate $w$ and $h = h(w)$ be a smooth real nonconstant function on $S$. $h$ is pulled back by $\pi_2$ to be a function on $X$ (we still denote it by $h$ which is constant on $(x, y)$ direction). Define an almost complex structure on $X$ by

$$J(\frac{\partial}{\partial x}) = -h \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, J(\frac{\partial}{\partial y}) = -(1 + h^2) \frac{\partial}{\partial x} + h \frac{\partial}{\partial y}$$

$$J|_{TS} = J_S$$

Then $J^2 = -id$ and $(TX)^{1,0} = \{ V, \frac{\partial}{\partial w} \}$, where $V = \frac{\partial}{\partial x} + i(h \frac{\partial}{\partial x} - \frac{\partial}{\partial y})$. As

$$[V, \frac{\partial}{\partial w}] = -i \frac{\partial h}{\partial w} \frac{\partial}{\partial x} = -i \frac{\partial h}{2 \partial w} (V + \bar{V}),$$

$J$ is not integrable by Newlander-Nirenberg’s theorem since $\frac{\partial h}{\partial w} \neq 0$. Then $J(dx) = -(hdx + (1 + h^2)dy)$. Let $\alpha = dx + i(hdx + (1 + h^2)dy)$. Locally

$$(T^*X)^{1,0} = \langle \alpha, dw \rangle \text{ and } \mathcal{K}^m_J = \langle \alpha \wedge dw \rangle$$

for any $m \geq 1$. There is an embedding

$$\pi^*_S : \Gamma(S, \mathcal{K}^m_S) \rightarrow \Gamma(X, \mathcal{K}^m_J)$$

given by $\pi^*_S(\gamma) = (\alpha)^m \wedge \gamma$ for any $\gamma \in \Gamma(S, \mathcal{K}^m_S)$. Define the $(0, 1)$ form $\beta = \frac{-i(h^{m+1})}{2(h-1)} \bar{\partial}h$. We get

**Lemma 6.4.** $\pi^*_S(\gamma) \in H^0(X, \mathcal{K}^m_J)$ if and only if $\bar{\partial} \gamma + m \beta \wedge \gamma = 0$. 


Proof. Assume locally that \( \gamma = f(w)dw^{\otimes m} \). Then \( \pi_2^*(\gamma) = (\alpha)\otimes \gamma = f(\alpha \wedge dw)\otimes \gamma \). We have
\[
\bar{\partial}_f(\alpha \wedge dw) = (d\alpha \wedge dw)^{2,1} \\
= \frac{i}{2} \frac{\partial h}{\partial \bar{w}} (1 - \frac{2h(h+i)}{1+h^2}) \bar{w} \wedge \alpha \wedge dw \\
= -\frac{i(h+i)}{2(h-i)} \frac{\partial h}{\partial \bar{w}} \bar{w} \wedge \alpha \wedge dw.
\]
Let \( b = \frac{-i(h+i)}{2(h-i)} \frac{\partial h}{\partial \bar{w}} \) and \( \beta = \frac{-i(h+i)}{2(h-i)} \bar{\partial} h \). As \( f \) depends only on \( w \), we have
\[
\bar{\partial}_f(\pi_2^*(\gamma)) = (\frac{\partial f}{\partial \bar{w}} + mbf)dw(\alpha \wedge dw)^{\otimes m}.
\]
So \( \bar{\partial}_f(\pi_2^*(\gamma)) = 0 \) is equivalent to \( \frac{\partial f}{\partial \bar{w}} + mbf = 0 \) which gives \( \bar{\partial} \gamma + m\beta \wedge \gamma = 0 \). \( \square \)

Denote
\[
H^0_h(S, \mathcal{K}_S^{\otimes m}) = \{ \gamma \in \Gamma(S, \mathcal{K}_S^{\otimes m}), \bar{\partial} \gamma + m\beta \wedge \gamma = 0 \}.
\]
When \( h = 0 \), then \( \beta = 0 \) and the group is the ordinary holomorphic pluricanonical section group of \( \mathcal{K}_S^{\otimes m} \). From Lemma 6.4 we get an injective map
\[
\pi_2^*: H^0_h(S, \mathcal{K}_S^{\otimes m}) \rightarrow H^0(X, \mathcal{K}^{\otimes m}).
\]
We can compute the dimension of \( H^0_h(S, \mathcal{K}_S^{\otimes m}) \) explicitly when \( m > 1 \). Notice that the operator \( \bar{\partial}_h = \bar{\partial} + m\beta \wedge \_ \) satisfies the Leibniz rule and then gives a deformed holomorphic structure of \( \mathcal{K}_S^{\otimes} \) as \( \dim S = 1 \). For the holomorphic line bundle \( (\mathcal{K}_S^{\otimes m}, \bar{\partial}_h) \),
\[
\deg(\mathcal{K}_S^{\otimes m}, \bar{\partial}_h) = \deg(\mathcal{K}_S^{\otimes m}, \bar{\partial}) = 2m(g - 1)
\]
by the deformation invariance of \( c_1 \). Also,
\[
H^0(S, \mathcal{K}_S \otimes (\mathcal{K}_S^{\otimes m}, \bar{\partial}_h)^*) = 0
\]
for \( m > 1 \) since the degree is negative. Applying the Riemann-Roch formula to \( (\mathcal{K}_S^{\otimes m}, \bar{\partial}_h) \), we have
\[
\dim H^0_h(S, \mathcal{K}_S^{\otimes m}) = \dim H^0_h(S, \mathcal{K}_S^{\otimes m}) - \dim H^0(S, \mathcal{K}_S \otimes (\mathcal{K}_S^{\otimes m}, \bar{\partial}_h)^*) \\
= \deg(\mathcal{K}_S^{\otimes m}, \bar{\partial}_h) - g + 1 \\
= (2m - 1)(g - 1)
\]
(26)
for \( m > 1 \). When \( m = 1 \), as \( \deg(\mathcal{K}_S \otimes (\mathcal{K}_S, \bar{\partial}_h)^*) = 0 \), we have \( \dim H^0(S, \mathcal{K}_S \otimes (\mathcal{K}_S, \bar{\partial}_h)^*) \leq 1 \). Applying the Riemann-Roch formula we obtain
\[
g - 1 \leq \dim H^0_h(S, \mathcal{K}_S) \leq g.
\]
Next, we show that \( \pi_2^* \) is surjective.

Lemma 6.5. For any \( s \in H^0(X, \mathcal{K}^{\otimes m}) \), \( s = \pi_2^*(\gamma) \) for some \( \gamma \in H^0_h(S, \mathcal{K}_S^{\otimes m}) \).
Then invariance of the Chern classes, we have $c$ generalized to other cases (Theorem 6.7 below).

Assume that $s = g(\alpha \wedge dw)^m$ locally. As $\partial \alpha = \beta \wedge \alpha$, we have $\partial_m s = (\partial g + mg \beta)(\alpha \wedge dw)^m$. So $\partial g + mg \beta = 0$, which is equivalent to

$$V(g) = 0$$

(28)

$$\frac{\partial g}{\partial \bar{w}} + m g d\bar{w} = 0.$$  

Using the same technique in example 6.1, from (27) we get that $g$ is independent of $x, y$. So we can define $\gamma = g(dw)^\otimes m \in H^0(S, K_S^\otimes m)$, and $s = \pi_2^*(\gamma)$.

The second approach is more topological. Define a deformation

$$J_t(\frac{\partial}{\partial x}) = -(th) \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, J_t(\frac{\partial}{\partial y}) = -(1 + t^2 h^2) \frac{\partial}{\partial x} + (th) \frac{\partial}{\partial y}$$

$$J_t|_{TS} = J_S, 0 \leq t \leq 1.$$  

Then $J_1 = J$ and $J_0$ is the product complex structure on $X$. By the homotopy invariance of the Chern classes, we have $c_1(K_J) = c_1(K_{J_0}) = (2g - 2)[T^2]$, where $[T^2]$ is the cohomology class of the fiber of $\pi_2$. Also, each fiber $T^2$ is a $J$-holomorphic curve by definition.

Let $z_0 = (t_0, w_0)$ be any point in $X$ where $t_0 \in T^2, w_0 \in S$ and $s \in H^0(X, K_S^\otimes m)$ is a nontrivial section. First assume that $s(z_0) = 0$. By Lemma 4.1, $s$ induces a pseudoholomorphic map. Therefore, $s^{-1}(0)$ supports a pseudoholomorphic 1-subvariety in $X$ (Corollary 1.3 in [42]). By the positive intersection of pseudoholomorphic curves (see [42]), either $T^2 \times \{w_0\} \subset s^{-1}(0)$ or $T^2 \times \{w_0\}$ has positive intersection with $s^{-1}(0)$. As

$$[s^{-1}(0)] = m \cdot c_1(K_J) = m(2g - 2)[T^2]$$

and $[T^2] \cdot [T^2] = 0$, the latter case cannot be possible. So $s|_{T^2 \times \{w_0\}} = 0$.

Next, assume that $s(z_0) \neq 0$. Denote $H^1_h(S, K_S^\otimes m - \{w_0\})$ the first sheaf cohomology group of the tensor bundle $(K_S^\otimes m, \partial_h) \otimes (\{w_0\})$. By the Kodaira vanishing theorem, when $m > 1$,

$$H^1_h(S, K_S^\otimes m - \{w_0\}) = 0$$

as deg$(K_S^{\otimes (m-1)}, \partial_h) \geq 2$. From the exact sequence

$$0 \rightarrow K_S^\otimes m - \{w_0\} \rightarrow K_S^\otimes m \rightarrow K_S^\otimes m|_{w_0} \rightarrow 0,$$

we get the exact sequence of cohomology groups (20):

$$0 \rightarrow H^0_h(S, K_S^\otimes m - \{w_0\}) \rightarrow H^0_h(S, K_S^\otimes m) \rightarrow H^0_h(S, K_S^\otimes m|_{w_0}) \rightarrow H^1_h(S, K_S^\otimes m - \{w_0\}) = 0.$$  

Therefore, there is a $\tilde{\gamma} \in H^0_h(S, K_S^\otimes m)$ such that $\tilde{\gamma}(w_0) \neq 0$ when $m > 1$. Then $\pi_2^*(\tilde{\gamma})(z_0) \neq 0$. Since $s(z_0) \neq 0$, there is some $k \neq 0$ such that $(s - k \pi_2^*(\tilde{\gamma}))(z_0) = 0$. As $s - k \pi_2^*(\tilde{\gamma}) \in H^0(X, K_S^\otimes m)$, by the same argument as in the first case,

$$(s - k \pi_2^*(\tilde{\gamma}))|_{T^2 \times \{w_0\}} = 0.$$  

So $s = k \pi_2^*(\tilde{\gamma})$ on $T^2 \times \{w_0\}$. 

\textbf{Proof.} We offer two different proofs. The first follows from direct calculation. The second proof applies the results of intersection theory built in [42], which can be generalized to other cases (Theorem 6.7 below).
Therefore, in either case, \( s \) is constant on the fiber of \( \pi_2 \). Then we can push down \( s \) through \( \pi_2 \) to get a section \( \gamma \in \Gamma(S, \mathcal{K}_S^{\otimes m}) \) such that \( s = \pi_2^*(\gamma) \). By Lemma 6.4, \( \gamma \in H^*_h(S, \mathcal{K}_S^{\otimes m}) \).

Combining Lemma 6.4, Lemma 6.5 and (26), we have

**Proposition 6.6.** \( \pi^*_h : H^0_h(S, \mathcal{K}_S^{\otimes m}) \rightarrow H^0(X, \mathcal{K}^{\otimes m}) \) is an isomorphism. Therefore, \( P_m(X, J) = \dim H^0_h(S, \mathcal{K}_S^{\otimes m}) = (2m-1)(g-1) \) for \( m > 1 \), \( g-1 \leq P_1(X, J) \leq g \) and \( \kappa^J(X) = 1 \).

To compute the \( h^{1,0} \) of \( X \), assume that \( \tau \in H^0(X, \Omega_X) \). Locally write \( \tau = g_1 \alpha + g_2 dw \). From \( \bar{\partial} \tau = 0 \) we get

\[
\bar{\partial}g_1 + g_1 \beta = 0 \\
\bar{\partial}g_2 = 0.
\]

From (29) we get that

\[
\bar{V}(g_1) = 0 \\
\bar{\partial}g_2 = 0,
\]

which implies that \( g_2 \) is constant. Therefore \( \tau = g_1 \alpha + cdw \), with \( g_1 \in H^0_h(S, \mathcal{O}) \). As \( h^{1,0}(S) = g \), we obtain

\[
g \leq h^{1,0}(X) \leq g + 1,
\]

The case \( h^{1,0}(X) = g+1 \) corresponds to \( \dim H^0_h(S, \mathcal{O}) = 1 \) which implies that \( (\mathcal{O}, \bar{\partial}_h) \) is holomorphic trivial. The case \( h^{1,0}(X) = g \) corresponds to \( \dim H^0_h(S, \mathcal{O}) = 0 \).

We can generalize the calculation to the case where \( X \) admits a smooth pseudoholomorphic elliptic fibration.

**Theorem 6.7.** If \( (X^4, J) \) admits a smooth pseudoholomorphic elliptic fibration over a Riemann surface of genus greater than 1 with \( J \) tamed, then \( \kappa^J = 1 \).

**Proof.** Let \( \pi : X \rightarrow S \) be the pseudoholomorphic elliptic fibration. By [37], the canonical class \( K \) is represented by \( J \)-holomorphic 1-subvariety \( \Theta \). For the fiber class \( T \), we have \( T \cdot T = 0 \). Hence \( K \cdot T = 0 \) by adjunction formula. By positivity of intersection, any component of \( \Theta \) is contained in a fiber. Since each fiber is smooth, we have \( K = bT \). On the other hand, any section of \( \mathcal{K}_X \) pushed down to a section of \( \mathcal{K}_S \) by integrate out the fiber. Hence \( K = (2g-2)T \).

In other words, as complex bundles, \( \pi^*(\mathcal{K}_S^{\otimes m}) = \mathcal{K}_X^{\otimes m} \). We notice that \( \bar{\partial}_m \) maps \( \pi^*\Gamma(S, \mathcal{K}_S^{\otimes m}) \) to \( \pi^*\Gamma(S, \mathcal{K}_S^{\otimes m} \otimes T^*S) \). We thus denote \( \bar{\partial}_m \pi^*(f \gamma) = \pi^*(\bar{\partial}_m(\gamma)) \) where
\[ \bar{\partial}_n \text{ is an operator mapping } \Gamma(S,K^{\otimes m}_S) \text{ to } \Gamma(S,K^{\otimes m}_S \otimes T^*S). \] Hence, for any smooth function \( f \) on \( S \), by the Leibniz rule of \( \bar{\partial}_m \), we have

\[ \pi^*(\bar{\partial}_n(f\gamma)) = \bar{\partial}_m\pi^*(f\gamma) = \bar{\partial}\pi^*f \wedge \pi^*\gamma + \pi^*f \cdot \bar{\partial}_m\pi^*\gamma = \pi^*(\bar{\partial}_n f \wedge \gamma + f\bar{\partial}_n\gamma). \]

That is to say \( \bar{\partial}_n \) also satisfies the Leibniz rule and hence it is a pseudoholomorphic structure on \( K^{\otimes m}_S \). Since \( S \) is a Riemann surface, it defines a holomorphic structure on it. To summarize, \( \pi^*(\gamma) \in H^0(X,K^{\otimes m}) \) if and only if \( \gamma \in H^0(S,K^{\otimes m}_S) \), where \( H^0(S,K^{\otimes m}_S) = \{ \gamma \in \Gamma(S,K^{\otimes m}_S), \bar{\partial}_n\gamma = 0 \} \).

Since any section \( s \in H^0(X,K^{\otimes m}) \) would have zero locus a \( J \)-holomorphic 1-subvariety in class \( mK = m(2g - 2)T \), Lemma 6.5 (or the argument in the first paragraph) still applies and we know any section \( s \in H^0(X,K^{\otimes m}) \) is of the form \( \pi^*(\gamma) \) for some \( \gamma \in H^0(S,K^{\otimes m}_S) \). Therefore, \( P_m(X,J) = \dim H^0(S,K^{\otimes m}_S) = (2m - 1)(g - 1) \) for \( m > 1 \) and \( \kappa^J(X) = 1 \).

We remark that the only place we use tameness is that it guarantees the existence of pseudoholomorphic 1-subvariety in the (pluri)canonical class.

In fact, the examples in Section 6.1 (as well as Section 6.2) are smooth pseudo-holomorphic elliptic fibrations over \( T^2 \). In these cases, \( (K^{\otimes m}_S, \bar{\partial}_n) \) are holomorphic line bundle of degree 0. These bundles are holomorphically trivial if and only if \( P_m = 1 \).

With those 4-manifolds with \( \kappa^J = 1 \), we can construct more non-integrable almost complex manifolds with large Kodaira dimensions. First, we derive the Künneth formula for pluricanonical sections of almost complex manifolds. For two almost complex manifolds \( (X_1,J_1) \) and \( (X_2,J_2) \), the product map \( J_1 \times J_2 \) induces an almost complex structure on \( X_1 \times X_2 \). We have

**Proposition 6.8.** \( P_m(X_1 \times X_2,J_1 \times J_2) = P_m(X_1,J_1)P_m(X_2,J_2) \) for \( m \geq 1 \).

**Proof.** We apply the harmonic theory in section 3 to derive the formula, similar to the argument in the integrable case (see [20]). Let

\[ p_1 : X_1 \times X_2 \rightarrow X_1, \quad p_2 : X_1 \times X_2 \rightarrow X_2 \]

be the two projections. We have \( K_{X_1 \times X_2} = p_1^*(K_{X_1}) \wedge p_2^*(K_{X_2}) \). Choose Hermitian metrics \( g_1 \) and \( g_2 \) on \( X_1, X_2 \) respectively. Then \( g_1 \times g_2 \) gives a Hermitian metric on \( X_1 \times X_2 \). A form \( \phi \in \Gamma(X_1 \times X_2,K_{X_1 \times X_2}) \) is called decomposable if \( \phi = p_1^*(\phi_1) \wedge p_2^*(\phi_2) \). Similar arguments as those in Page 104 in [20] show that the decomposable smooth forms are dense in the Hilbert space \( L^2(X_1 \times X_2,K_{X_1 \times X_2}) \).

Denote \( \Delta_{J_1}, \Delta_{J_2} \) the Laplacian operators associated to \( \bar{\partial}_{J_1}, \bar{\partial}_{J_2} \) as given in [8]. By the definition, they are both semi-positive operators. Let \( \varphi_1, \varphi_2, \ldots \), be the eigenforms of \( \Delta_{J_1} \) in \( \Gamma(X_1,K_{X_1}) \) with eigenvalues \( \lambda_1, \lambda_2, \ldots \) and \( \psi_1, \psi_2, \ldots \), be the eigenforms of \( \Delta_{J_2} \) in \( \Gamma(X_2,K_{X_2}) \) with eigenvalues \( \mu_1, \mu_2, \ldots \). Then \( \lambda_i \geq 0, \mu_i \geq 0 \) for any \( i \). Let \( \Delta_{J_1 \times J_2} \) be the Laplacian operator associated to \( J_1 \times J_2 \) and \( g_1 \times g_2 \). From the definition, we directly get \( \Delta_{J_1 \times J_2} = \Delta_{J_1} + \Delta_{J_2} \). Also,

\[ \Delta_{J_1 \times J_2}(p_1^*(\varphi_i) \wedge p_2^*(\psi_j)) = (\lambda_i + \mu_j)p_1^*(\varphi_i) \wedge p_2^*(\psi_j). \]
So we have $\Delta_{J_1 \times J_2}(p_1^*(\varphi_i) \wedge p_2^*(\psi_j)) = 0$ if and only if $\lambda_i = \mu_j = 0$. As $\{\varphi_i\}$, $\{\psi_j\}$ give Hilbert bases for $L^2(X_1, K_{X_1})$ and $L^2(X_2, K_{X_2})$ respectively, $\{p_1^*(\varphi_i) \wedge p_2^*(\psi_j)\}$ gives a Hilbert basis of $L^2(X_1 \times X_2, K_{X_1 \times X_2})$ by the density of decomposable forms. Therefore, we get $\text{Ker}(\Delta_{J_1 \times J_2}) = \langle p_1^*(\varphi_i) \wedge p_2^*(\psi_j) \rangle$ with $\lambda_i = \mu_j = 0$. Namely, $H^0(X_1 \times X_2, K_{X_1 \times X_2}) = H^0(X_1, K_{X_1}) \otimes H^0(X_2, K_{X_2})$.

This shows that $P_1(X_1 \times X_2, J_1 \times J_2) = P_1(X_1, J_1)P_1(X_2, J_2)$. Similar argument gives that $H^0(X_1 \times X_2, K_{X_1 \times X_2}^m) = H^0(X_1, K_{X_1}^m) \otimes H^0(X_2, K_{X_2}^m)$ and $P_m(X_1 \times X_2, J_1 \times J_2) = P_m(X_1, J_1)P_m(X_2, J_2)$ for any $m > 1$. \qed

From the definition of Kodaira dimension we have

**Corollary 6.9.** $\kappa^{i_1 \times J_2}(X_1 \times X_2) = \kappa^{i_1}(X_1) + \kappa^{J_2}(X_2)$ for any two compact almost complex manifolds $(X_1, J_1)$, $(X_2, J_2)$.

**Theorem 6.10.** There are examples of compact $2n$-dimensional non-integrable almost complex manifolds with Kodaira dimension lying among $\{-\infty, 0, 1, \ldots, n - 1\}$ for $n \geq 2$.

**Proof.** By taking direct products of the Kodaira-Thurston surface with copies of two torus $T^2$, we can get compact $2n$-manifolds with non-integrable almost complex structure and $\kappa^J = -\infty$ or 0.

By taking direct products of the 4-manifold $X = T^2 \times S$ as in Proposition 6.6 with copies of 2-torus $T^2$ or the Riemann surface $\Sigma$ with $g > 1$, we get compact $2n$-manifolds with non-integrable almost complex structures and $\kappa^J = 1, 2, \ldots, n - 1$. \qed

7. The Six Sphere

By a result of Borel and Serre [4], the only spheres which admit almost complex structures are $S^2$ and $S^6$. The standard way to construct an almost complex structure on $S^6$ is to use the cross product of $\mathbb{R}^7$ applying to the tangent space of $S^6$. In this section, we will compute the Hodge numbers, the plurigenera and Kodaira dimension of the standard almost complex structure. Our method is to consider $S^6$ as a homogeneous space of the exceptional Lie group $G_2$ and apply an explicit real representation of the Lie algebra $\mathfrak{g}_2$.

First, we review some definitions following [7]. Let $e_1, e_2, \ldots, e_7$ be the standard basis of $\mathbb{R}^7$ and $e^1, e^2, \ldots, e^7$ be the dual basis. Denote $e^{ijk}$ the wedge product $e^i \wedge e^j \wedge e^k$ and define

$$\Phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}.$$ 

Then $\Phi$ induces a unique bilinear mapping, the cross product: $\times : \mathbb{R}^7 \times \mathbb{R}^7 \longrightarrow \mathbb{R}^7$ by $(u \times v) \cdot w = \Phi(u, v, w)$, where $\cdot$ is the Euclidean metric on $\mathbb{R}^7$. It follows that $u \times v = -v \times u$ and

$$(u \times v) \cdot u = 0.$$ (33)
Also, further discussion (see [7]) shows that
\[ u \times (u \times v) = (u \cdot v)u - (u \cdot u)v. \] (34)

We remark that the cross product \( \times \) differs from the cross product induced by Cayley’s multiplication table of octonions, though they are isomorphic. For example, here \( e_1 \times e_6 = e_7 \).

The six sphere \( S^6 = \{ u \in \mathbb{R}^7, u \cdot u = ||u|| = 1 \} \). The tangent space at \( u \in S^6 \) is \( T_u S^6 = \{ v \in \mathbb{R}^7 | u \cdot v = 0 \} \). Let \( J_u = u \times \_ \) be the cross product operator of \( u \). Then by (33), (34), \( J_{e^1} (T^u S^6) \subset T^u S^6 \) and \( J^2_u = -id \) on \( T^u S^6 \). In particular, when \( u = e_1 \), we have
\[ J_{e^1}(e_2) = e_3, \quad J_{e^1}(e_3) = -e_2, \quad J_{e^1}(e_4) = e_5, \]
\[ J_{e^1}(e_5) = -e_4, \quad J_{e^1}(e_6) = e_7, \quad J_{e^1}(e_7) = -e_6. \] (35)

Let \( J = \{ J_u, u \in S^6 \} \). Then \( J \) gives an almost complex structure on \( S^6 \) which is the standard almost complex structure we consider. It is shown [15] [17] that \( J \) is not integrable since the Nijenhuis tensor of \( J \) is nowhere vanishing.

On the other side, denote
\[ G_2 = \{ g \in GL(7, \mathbb{R}) | g^* (\Phi) = \Phi \}, \] (36)
where \( G_2 \) is the simple Lie group of type \( G_2 \) which is compact, connected and simply connected with real dimension 14 (see [5]). \( G_2 \) preserves the inner product \( \cdot \) and the cross product \( \times \) and acts transitively on \( S^6 \). Let \( G_2 \times S^6 \rightarrow S^6 \) be the transitive action and \( p : G_2 \rightarrow S^6 \) the induced map given by \( p(g) = g(e_1) \). The map \( p \) is a submersion with \( p^{-1}(e_1) = \{ g \in G_2 | g(e_1) = e_1 \} \cong SU(3) \). This makes \( G_2 \) into a principal right \( SU(3) \) bundle over \( S^6 \).

Next, we give the explicit representation of the Lie algebra \( g_2 \) of \( G_2 \) and define a left invariant almost complex structure on it. Let \( \epsilon_{ijk} \) be skew-symmetric unit indices such that \( \Phi = \frac{1}{6!} \epsilon_{ijk} e^{ijk} \). For example, \( \epsilon_{123} = -\epsilon_{132} = \epsilon_{231} = 1 \). By the characterization in [6] (section 2.5 there), a skew-symmetric matrix \( A = (a_{jk}) \) is in \( g_2 \) if and only if
\[ \sum_{j,k=1}^{7} \epsilon_{ijk} a_{jk} = 0 \] (37)
for all \( 1 \leq i \leq 7 \). These give a system of seven linear equations. For example, the first equation reads
\[ a_{23} + a_{45} + a_{67} = 0. \]
Let \( \vec{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6, \vec{y} = (y_1, y_2, \cdots, y_6) \in \mathbb{R}^8 \). Solving (37) gives that a general element \( A \) in \( g_2 \subset \mathfrak{gl}(7, \mathbb{R}) \) has the following form

\[
A = \{\vec{x}, \vec{y}\} := \begin{pmatrix}
0 & x_1 & -x_2 & x_3 & -x_4 & x_5 & -x_6 \\
-x_1 & 0 & y_1 & -x_6 + y_4 & x_5 + y_3 & x_4 - y_6 & -x_3 - y_5 \\
x_2 & -y_1 & 0 & -y_3 & y_4 & y_5 & -y_6 \\
-x_3 & x_6 - y_4 & y_3 & 0 & -y_1 + y_2 & -x_2 - y_8 & x_1 - y_7 \\
x_4 & -x_5 - y_3 & -y_4 & y_1 - y_2 & 0 & y_7 & -y_8 \\
-x_5 & -x_4 + y_6 & -y_5 & x_2 + y_8 & -y_7 & 0 & -y_2 \\
x_6 & x_3 + y_5 & y_6 & -x_1 + y_7 & y_8 & y_2 & 0
\end{pmatrix}.
\]

Here \( \{\cdot, \cdot\} \) denotes an operation \( \{\cdot, \cdot\} : \mathbb{R}^6 \times \mathbb{R}^8 \to g_2 \) whose definition is stated above. The above expression is chosen so that it suits our later discussion on \( S^6 \).

Denote \( \vec{\alpha}_i, i = 1, \cdots, 6 \), the \( i \)-th unit vector in \( \mathbb{R}^6 \) and \( \vec{\beta}_j, j = 1, \cdots, 8 \), the \( j \)-th unit vector in \( \mathbb{R}^8 \). Define \( f_i = \{\vec{\alpha}_i, \vec{0}\}, h_j = \{\vec{0}, \vec{\beta}_j\} \). For example, \( f_1 = \{(1, 0, 0, 0, 0, 0, 0, 0), \vec{0}\}, h_2 = \{0, (0, 1, 0, 0, 0, 0, 0, 0)\} \). Then \( \{f_i, h_j; 1 \leq i \leq 6, 1 \leq j \leq 8\} \) forms a basis of \( g_2 \). The Lie brackets between \( f_i \) and \( h_j \) are computed in the appendix. Let

\[
m = \text{span}\{f_1, \cdots, f_6\}, \quad \mathfrak{h} = \text{span}\{h_1, \cdots, h_8\}.
\]

Then \( g_2 = m \oplus \mathfrak{h}, \ [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \) and \( \mathfrak{h} \cong su(3) \). A Cartan subalgebra of \( g_2 \) is given by the span of \( h_1, h_2 \). The corresponding decomposition of \( g_2 \) into root spaces can be also calculated. For the projection \( p : G_2 \to S^6 \), we have

\[
\text{ker } dp = \mathfrak{h}, \quad dp(f_i) = (-1)^i e_{i+1}.
\]

Define an almost complex structure \( \tilde{J} \) on \( g_2 \) by

\[
\tilde{J}(f_1) = -f_2, \quad \tilde{J}(f_3) = -f_4, \quad \tilde{J}(f_5) = -f_6,
\]

\[
\tilde{J}(h_1) = -h_2, \quad \tilde{J}(h_3) = -h_4, \quad \tilde{J}(h_5) = -h_6, \quad \tilde{J}(h_7) = -h_8.
\]

\( \tilde{J} \) induces a left invariant almost complex structure on \( G_2 \) which is still denoted by \( \tilde{J} \). By (35), the following holds at \( 1_{G_2} \),

\[
(38) \quad dp \circ \tilde{J} = J \circ dp.
\]

Since both \( \tilde{J} \) and \( J \) are \( G_2 \)-invariant, (38) holds globally on \( G_2 \). In other words, \( p \) is a \((\tilde{J}, J)\)-pseudoholomorphic map.

With the construction of \( \tilde{J} \), we prove the following

**Theorem 7.1.** For the standard almost complex structure \( J \) on \( S^6 \), \( h^{1,0} = h^{2,0} = h^{2,3} = h^{1,3} = 0, P_m(S^6, J) = 1 \) for any \( m \geq 1 \) and \( \kappa^3 = 0 \).

We would like to thank several people, including Huijun Fan, Valentino Tosatti, Jiaping Wang and Bo Yang for encouraging us to proceed the calculation.
Proof. Compute the plurigenera \( P_m(S^6, J) \) first. Denote \((T^*S^6)^{1,0}\) the bundle of \((1,0)\) forms on \(S^6\) and \(p^*\) the pull back map of forms. As \(p\) is \((J, J)\)-pseudoholomorphic, we have \(p^*((T^*S^6)^{1,0}) \subset (T^*G_2)^{1,0} = (g_2)^{1,0}\). \(p^*\) is injective since \(p\) is a submersion. Denote \(\{f^i, h^j\}\) the basis in \(g_2^\star\), dual to \(\{f_i, h_j\}\). Then \((g_2^\star)^{1,0}\) is generated by \(\{\phi^1, \cdots, \phi^7\}\), where

\[
\begin{align*}
\phi^1 &= f^1 - if^2, \quad \phi^2 = f^3 - if^4, \quad \phi^3 = f^5 - if^6, \\
\phi^4 &= h^1 - ih^2, \quad \phi^5 = h^3 - ih^4, \quad \phi^6 = h^5 - ih^6, \quad \phi^7 = h^7 - ih^8.
\end{align*}
\]

As \([h, h] \subset h\), using the Lie brackets in the appendix, we get

\[
\begin{align*}
df^1 &= -f^2 \wedge h^1 - 2f^3 \wedge f^6 + f^3 \wedge h^4 - 2f^4 \wedge f^5 - f^4 \wedge h^3 - f^5 \wedge h^6 + f^6 \wedge h^5, \\
df^2 &= f^3 \wedge h^3 + f^4 \wedge h^4 - f^5 \wedge h^5 - f^6 \wedge h^6 + f^1 \wedge h^1, \\
df^3 &= -f^1 \wedge h^4 + 2f^4 \wedge f^6 + f^2 \wedge f^5 - f^2 \wedge h^3 + f^4 \wedge h^4 - f^4 \wedge h^2 - f^5 \wedge h^8 + f^6 \wedge h^7, \\
df^4 &= f^1 \wedge f^5 + f^1 \wedge h^3 - f^2 \wedge h^4 - f^3 \wedge h^5 + f^3 \wedge h^2 - f^5 \wedge h^7 - f^6 \wedge h^8, \\
df^5 &= -f^1 \wedge f^4 + f^1 \wedge h^0 - f^2 \wedge f^3 + f^2 \wedge h^5 + f^3 \wedge h^8 + f^4 \wedge h^7 + f^6 \wedge h^2, \\
df^6 &= -2f^1 \wedge f^3 - f^1 \wedge h^5 + f^2 \wedge h^6 - f^3 \wedge h^7 + f^4 \wedge h^8 - f^5 \wedge h^2.
\end{align*}
\]

Therefore, the definition of \(\bar{\partial}\) gives

\[
\bar{\partial}\phi^1 = -\frac{i}{2} \phi^1 \wedge \phi^4 - i \phi^2 \wedge \phi^5 + i \phi^3 \wedge \phi^6, \\
\bar{\partial}\phi^2 = -\frac{i}{2} \phi^1 \wedge \phi^3 - \frac{1}{2} \phi^2 \wedge \phi^4 + i \phi^5 \wedge \phi^7, \\
\bar{\partial}\phi^3 = \frac{i}{2} \phi^1 \wedge \phi^2 - \frac{i}{2} \phi^2 \wedge \phi^1 + \frac{1}{2} \phi^3 \wedge \phi^4.
\]

Then

\[
\bar{\partial}(\phi^1 \wedge \phi^2 \wedge \phi^3) = \bar{\partial}\phi^1 \wedge \phi^2 \wedge \phi^3 - \phi^1 \wedge \bar{\partial}\phi^2 \wedge \phi^3 + \phi^1 \wedge \phi^2 \wedge \bar{\partial}\phi^3 = 0.
\]

By the arguments in [7] (equation (2.11) in [7]), \(\phi^1 \wedge \phi^2 \wedge \phi^3\) induces a nowhere-vanishing \(G_2\)-invariant \((3,0)\)-form \(\Phi\) on \(S^6\). As \(p\) is pseudoholomorphic and \(p^*\) is injective, \(\bar{\partial}\Phi = 0\).

Assume \(s \in H^0(S^6, K_J)\), then \(s = f\Phi\), where \(f\) is a smooth function on \(S^6\). From \(\bar{\partial}s = 0\) we get that \(\bar{\partial}f = 0\). Since \(S^6\) is compact, the maximum principle gives that \(f\) is a constant. Therefore, \(P_1(S^6, J) = h^{0,0} = 1\) with \(\Phi\) being a generator. Similarly, we get \(P_m(S^6, J) = 1\) for \(m \geq 2\), with \(\Phi^m\) being a generator of \(H^0(S^6, K_J^{3m})\). So \(\kappa^1 = 0\).

Next, we compute the Hodge numbers \(h^{1,0}\) and \(h^{2,0}\). Assume that \(s \in H^{1,0}(S^6)\). Then \(p^*\) is in the span space of \(\{\phi^1, \phi^2, \phi^3\}\), satisfying \(\bar{\partial}(p^*s) = 0\). Let \(p^*s = k_1\phi^1 + k_2\phi^2 + k_3\phi^3\), where \(k_i\) are smooth functions on \(G_2\). From (39) we get that

\[
\bar{\partial}k_3 = ik_1\phi^6 + ik_2\phi^7 + \frac{1}{2}k_3\phi^4.
\]
Let \( X_i, 1 \leq i \leq 7, \) be the dual complex vector of \( \phi^i. \) Namely, \( X_1 = \frac{1}{2} (f_1 + if_2), \cdots, X_7 = \frac{1}{2} (h_7 + ih_8). \) From the Appendix, the following Lie brackets hold

\[
[X_1, X_2] = -i X_3 + \frac{i}{2} \bar{X}_5, \quad [X_3, X_5] = \frac{i}{2} h_1, \quad [X_3, \bar{X}_3] = \frac{i}{2} h_2.
\]

Equation (40) gives us that

\[
\bar{X}_1(k_3) = \bar{X}_2(k_3) = \bar{X}_3(k_3) = \bar{X}_5(k_3) = 0.
\]

From (41), we have \( X_3(k_3) = 0 \) and \( h_1(k_3) = 0. \) Then by the last relation in (41), \( h_2(k_3) = 0. \) So \( \bar{X}_4(k_3) = 0. \) Evaluate \( \bar{X}_4 \) to (40) we get that \( k_3 = 0. \) Then (40) directly gives that \( k_1 = k_2 = 0. \) Therefore, \( p^* s = 0. \) As \( p^* \) is injective, we get \( s = 0. \)

Hence, \( H^{1,0}(S^6) = 0 \) and \( h^{1,0} = 0. \)

To calculate \( h^{2,0}, \) assume that \( \sigma \in H^{2,0}(S^6). \) Then \( p^* \sigma \) satisfies \( \bar{\partial}(p^* \sigma) = 0. \) Let \( p^* \sigma = l_1 \phi^1 \wedge \phi^2 + l_2 \phi^2 \wedge \phi^3 + l_3 \phi^3 \wedge \phi^1, \) where \( l_i \) are smooth functions on \( G_2. \) From (39) we get

\[
\bar{\partial}(\phi^1 \wedge \phi^2) = \frac{1}{2} \phi^1 \wedge \phi^2 \wedge \bar{\phi}^4 + i \phi^2 \wedge \phi^3 \wedge \bar{\phi}^6 - i \phi^1 \wedge \phi^3 \wedge \bar{\phi}^7,
\]

\[
\bar{\partial}(\phi^2 \wedge \phi^3) = \frac{i}{2} \phi^1 \wedge \phi^3 \wedge \bar{\phi}^3 - \frac{i}{2} \phi^2 \wedge \phi^3 \wedge \bar{\phi}^4 + \frac{i}{2} \phi^1 \wedge \phi^2 \wedge \bar{\phi}^2,
\]

\[
\bar{\partial}(\phi^3 \wedge \phi^1) = -\frac{i}{2} \phi^1 \wedge \phi^2 \wedge \bar{\phi}^1 + \frac{1}{2} \phi^1 \wedge \phi^3 \wedge \bar{\phi}^4 - i \phi^2 \wedge \phi^3 \wedge \bar{\phi}^5.
\]

So \( \bar{\partial}(p^* \sigma) = 0 \) gives that

\[
\bar{\partial} l_2 = -il_1 \bar{\phi}^6 + \frac{i}{2} l_2 \bar{\phi}^4 + il_3 \bar{\phi}^5.
\]

Then \( \bar{X}_1(l_2) = \bar{X}_2(l_2) = \bar{X}_3(l_2) = \bar{X}_7(l_2) = 0. \) By the Appendix, the following hold

\[
[X_1, X_7] = -\frac{i}{2} h_2, \quad [X_1, h_2] = i X_7, \quad [X_2, X_3] = -if_1 - \frac{i}{2} h_7.
\]

From (43), we have \( h_2(l_2) = 0 \) and \( X_7(l_2) = 0, \) which implies \( h_7(l_2) = h_3(l_2) = 0. \) Putting it to the last equation in (43), we have \( f_1(l_2) = 0. \) As \( \bar{X}_1(l_2) = 0, \) then \( 0 = [f_1, X_1](l_2) = -\frac{i}{2} (h_1 + h_2)(l_2). \) So \( h_1(l_2) = 0 \) and then \( \bar{X}_4(l_2) = 0. \) Evaluating in (42), we derive \( l_1 = l_2 = l_3 = 0. \) So \( p^* \sigma = 0. \) By the injectivity, \( \sigma = 0. \) Therefore, \( H^{2,0}(S^6) = 0 \) and \( h^{2,0} = 0. \)

By the Serre duality (Proposition 3.7), we have \( h^{1,3}(S^6) = h^{2,3}(S^6) = 0. \)

On the other hand, for a hypothetical complex structure on \( S^6, \) \( P_1 = h^{3,0} = 0. \) The key point is a \( \partial \)-closed \((3, 0)\) form is also \( d \)-closed on a complex 3-fold.

8. Appendix

Direct calculation gives the following Lie brackets of \( g_2: \)
\[ [f_1, f_2] = h_1 + h_2, \quad [f_1, f_3] = 2f_0, \quad [f_1, f_4] = f_5, \quad [f_1, f_5] = -f_4, \]
\[ [f_1, f_6] = -2f_3, \quad [f_1, h_1] = -(f_2 - h_8), \quad [f_1, h_2] = -h_8, \]
\[ [f_1, h_3] = -(f_4 + h_6), \quad [f_1, h_4] = f_3, \quad [f_1, h_5] = f_6, \]
\[ [f_1, h_6] = -f_5 + h_3, \quad [f_1, h_7] = 0, \quad [f_1, h_8] = h_2, \]
\[ [f_2, f_3] = f_5 - h_3, \quad [f_2, f_4] = -h_4, \quad [f_2, f_5] = -f_3 + h_5, \]
\[ [f_2, f_6] = h_6, \quad [f_2, h_1] = f_1 + h_7, \quad [f_2, h_2] = -h_7, \]
\[ [f_2, h_3] = f_3 - h_5, \quad [f_2, h_4] = f_4, \quad [f_2, h_5] = -f_5 + h_3, \]
\[ [f_2, h_6] = -f_6, \quad [f_2, h_7] = h_2, \quad [f_2, h_8] = 0 \]
\[ [f_3, f_4] = h_2, \quad [f_3, f_5] = h_8, \quad [f_3, f_6] = 2f_1, \quad [f_3, h_1] = f_4 + h_6, \]
\[ [f_3, h_2] = -f_4, \quad [f_3, h_3] = -(f_2 - h_8), \quad [f_3, h_4] = -f_1, \quad [f_3, h_5] = 0, \]
\[ [f_3, h_6] = -(h_1 + h_2), \quad [f_3, h_7] = f_6, \quad [f_3, h_8] = -f_5, \]
\[ [f_4, f_5] = 2(f_1 + h_7), \quad [f_4, f_6] = h_8, \quad [f_4, h_1] = h_5 - f_3, \]
\[ [f_4, h_2] = f_3, \quad [f_4, h_3] = f_1 + h_7, \quad [f_4, h_4] = -f_2, \]
\[ [f_4, h_5] = -(h_1 + h_2), \quad [f_4, h_6] = 0, \quad [f_4, h_7] = -f_5, \quad [f_4, h_8] = -f_6, \]
\[ [f_5, f_6] = -h_2, \quad [f_5, h_1] = h_4, \quad [f_5, h_2] = f_6, \]
\[ [f_5, h_3] = 0, \quad [f_5, h_4] = -h_1, \quad [f_5, h_5] = f_2 - h_8, \]
\[ [f_5, h_6] = f_1 + h_7, \quad [f_5, h_7] = f_4, \quad [f_5, h_8] = f_3, \]
\[ [f_6, h_1] = h_3, \quad [f_6, h_2] = -f_5, \quad [f_6, h_3] = -h_1, \quad [f_6, h_4] = 0, \]
\[ [f_6, h_5] = -f_1, \quad [f_6, h_6] = f_2, \quad [f_6, h_7] = -f_3, \quad [f_6, h_8] = f_4, \]
\[ [h_1, h_2] = 0, \quad [h_1, h_3] = -2h_4, \quad [h_1, h_4] = 2h_3, \quad [h_1, h_5] = -h_6, \]
\[ [h_1, h_6] = h_5, \quad [h_1, h_7] = h_8, \quad [h_1, h_8] = -h_7, \]
\[ [h_2, h_3] = h_4, \quad [h_2, h_4] = -h_3, \quad [h_2, h_5] = -h_6, \quad [h_2, h_6] = h_5, \]
\[ [h_2, h_7] = -2h_8, \quad [h_2, h_8] = 2h_7, \quad [h_3, h_4] = -2h_1, \quad [h_3, h_5] = -h_8, \]
\[ [h_3, h_6] = h_7, \quad [h_3, h_7] = -h_6, \quad [h_3, h_8] = h_5, \]
\[ [h_4, h_5] = -h_7, \quad [h_4, h_6] = -h_8, \quad [h_4, h_7] = h_5, \quad [h_4, h_8] = h_6, \]
\[ [h_5, h_6] = -2(h_1 + h_2), \quad [h_5, h_7] = -h_4, \quad [h_5, h_8] = -h_3, \]
\[ [h_6, h_7] = h_3, \quad [h_6, h_8] = -h_4, \quad [h_7, h_8] = -2h_2. \]

References

[1] W. Barth, K. Hulek, C. Peters, A. Van de Ven, Compact Complex Surfaces, 2nd ed. Springer, Berlin, Heidelberg (2004).
[2] C. Birkar, P. Cascini, C. Hacon, J. McKernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. 23 (2010), no. 2, 405–468.
[3] L. Bonthonne, W. Zhang, J-holomorphic curves from closed J-anti-invariant forms, arXiv:1808.09356, to appear in Comm. Anal. Geom.
[4] A. Borel, J. P. Serre, *Groupes de Lie et puissances réduites de Steenrod*, Amer. J. Math. 75 (1953), 409–448.
[5] R. Bryant, *Metrics with Exceptional Holonomy*, Ann. of Math. Second Series, Vol. 126, No. 3 (Nov., 1987), pp. 525–576.
[6] R. Bryant, *Some remarks on G2-structures*, Proc. Gökova Geometry-Topology Conference (2005), 75–109.
[7] R. Bryant, S.-S. Chern’s study of almost-complex structures on the six-sphere, Internat. J. Math. 32 (2021), no. 12, Paper No. 2140006, 15 pp.
[8] K.F. Chan, S. Karigiannis, C.C. Tsang, *Cohomologies on almost complex manifolds and the ∂∂̄-lemma*, Asian J. Math. 23 (2019), 561–584.
[9] F. Catanese, C. LeBrun, *On the scalar curvature of Einstein manifolds*, Math. Res. Lett., 4 (1997), 843–854.
[10] A. Cattaneo, A. Nannicini, A. Tomassini, *Kodaira dimension of almost Kähler manifolds and curvature of the canonical connection*, Annali di Mat. Pura ed Appl., 199 (5), 1815–1842 (2020).
[11] A. Cattaneo, A. Nannicini, A. Tomassini, *On Kodaira dimension of almost complex 4-dimensional solvmanifolds without complex structures*, Internat. J. Math. 32 (2021), no. 10, Paper No. 2150075, 41 pp.
[12] H. Chen, W. Zhang, *Kodaira dimensions of almost complex manifolds II*, arXiv:2004.12825, to appear in Comm. Anal. Geom.
[13] P. de Bartolomeis, G. Tian, *Stability of complex vector bundles*, J. Diff. Geom. 43 (1996), no. 2, 231–275.
[14] R. Friedman, Z. Qin, *The smooth of invariance of the Kodaira dimension of a complex surface*, Math. Res. Lett. 1, 369–376 (1994).
[15] B. Eckmann, A. Frölicher, *Sur l’intégrabilité des structures presque-complexes*, C.R. Acad. Sci. Paris vol. 232 (1951) pp. 2284–2286.
[16] C. Ehresmann, *Sur les variétés presque-complexes*, Proceedings of the International Congress of Mathematicians, 1950, vol. II, Amer. Math. Society, 1952, pp. 412–419.
[17] C. Ehresmann, P. Libermann, *Sur les structures presque hermitiennes isotropes*, C. R. Acad. Sci. Paris vol. 232 (1951) pp. 1281–1283.
[18] R. Friedman, Z. Qin, *The smooth of invariance of the Kodaira dimension of a complex surface*, Math. Res. Lett. 1, 369–376 (1994).
[19] P. Gauduchon, *Hermitian connections and Dirac operator*, Boll. Un. Mat. Ital. B 11 (1997), no. 2, suppl., 257–288.
[20] P. Griffiths, J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics. Wiley-Interscience, New York, 1978.
[21] F. Hirzebruch, *Some problems on differentiable and complex manifolds*, Ann. of Math. (2) 60, (1954), 213–236.
[22] T. Holt, W. Zhang, *Harmonic forms on the Kodaira-Thurston manifold*, Adv. Math. 400 (2022), Paper No. 108277, 30 pp.
[23] D. Huybrechts, *Complex geometry*, Universitext. Springer-Verlag, Berlin, 2005. xii+309 pp. ISBN: 3-540-21290-6.
[24] S. Iitaka, *On D-dimensions of algebraic varieties*, J. Math. Soc. Japan 23 (1971), 356–373.
[25] S. Iitaka, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 76, Springer–Verlag, New York–Berlin, 1982. An introduction to birational geometry of algebraic varieties; North-Holland Mathematical Library, 24.
[26] K. Kodaira, J. Morrow, *Complex manifolds*, Holt, Rinehart and Winston, 1971.
[27] H. B. Lawson, M-L. Michelsohn, *Spin Geometry*, Princeton University Press. (1989).
[28] L. Lempert, R. Szöke, *The tangent bundle of an almost complex manifold*, Canad. Math. Bull. 44 (2001), no. 1, 70–79.
[29] T.J. Li, *Symplectic 4-manifolds with Kodaira dimension zero*, J. Diff. Geom. 74 (2006), no. 2, 321–352.
[30] T. J. Li, W. Zhang, *Comparing tamed and compatible symplectic cones and cohomological properties of almost complex manifolds*, Comm. Anal. Geom. 17 (2009), no. 4, 651–683.

[31] T. J. Li, W. Zhang, *J-holomorphic curves in a nef class*, Int. Math. Res. Not. (2015) Vol. 2015 12070-12104.

[32] I. Nakamura, *Complex parallelisable manifolds and their small deformations*, J. Diff. Geom. 10 (1975), 85–112.

[33] A. Newlander, L. Nirenberg, *Complex analytic coordinates in almost complex manifold*, Ann. of Math. Second Series. 65 (3) (1957): 391–404.

[34] R. Rădulescu, *The Kodaira dimension of diffeomorphic Kähler threefolds*, Proc. Amer. Math. Soc. 134 (2006), no.12, 3543–3553.

[35] Y.-T. Siu, *Invariance of plurigenera*, Invent. Math. 134, 661–673 (1998).

[36] Y.-T. Siu, *Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semi-positively twisted plurigenera for manifolds not necessarily of general type*, Complex geometry (Gottingen, 2000), Berlin: Springer-Verlag, pp. 223–277.

[37] C. Taubes, *SW $\Rightarrow$ Gr: from the Seiberg-Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. 9 (1996), no. 3, 845–918.

[38] V. Tosatti, B. Weinkove, *The Calabi-Yau equation on the Kodaira-Thurston manifold*, J. Inst. Math. Jussieu 10 (2011), no.2, 437–447.

[39] C. Voisin, *Hodge Theory and Complex Algebraic Geometry (I)*, Cambridge University Press, (2007).

[40] W. Zhang, *Geometric structures, Gromov norm and Kodaira dimensions*, Adv. Math. 308 (2017), 1–35.

[41] W. Zhang, *The curve cone of almost complex 4-manifolds*, Proc. Lond. Math. Soc. (3) 115 (2017), no. 6, 1227–1275.

[42] W. Zhang, *Intersection of almost complex submanifolds*, Camb. J. Math. 6 (2018), no. 4, 451–496.

[43] W. Zhang, *Moduli space of $J$-holomorphic subvarieties*, Selecta Math. (N.S.) 27 (2021), no. 2, Paper No. 29, 44 pp.

[44] F. Zheng, *Complex differential geometry*, AMS/IP Studies in Advanced Mathematics, 18. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2000.

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