EFFECTIVE BOUNDARY CONDITIONS FOR THE HEAT EQUATION WITH THREE-DIMENSIONAL INTERIOR INCLUSION

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Abstract. This paper is motivated by a heat equation on a domain containing an inside layer, which is thin compared to the scale of the domain. Moreover, the thermal conductivity in the layer is drastically different from that in the other part. We study the effects of the layer by investigating the asymptotic behavior of the solution of the heat equation with Dirichlet condition imposed on the outer boundary, as the thickness of the layer shrinks. In this paper, particularly, we focus on the case where the domain is three dimensional, and derive the related “effective boundary conditions” (EBCs), which reveal the effects of the inclusion.

1. Introduction. We consider the physical problem of insulating an isotropically thermal conducting body by using a thin layer, which can work as an isotropic coating but with drastically different thermal conductivity. The multi-scales in size and the different thermal conductivity lead to the computational burden inevitably. Nonetheless, one way to avoid aforementioned difficulty, is to think of the thin layer as a thickness surface, on which we can get “effective boundary conditions” (EBCs). More often than not, these EBCs will be not only helpful in numerical computation, but will also give us a physical interpretation of the effects of layers.

In [2], it was studied for the EBCs of a heat equation subject to the Dirichlet boundary condition, when the layer is of interior inclusion with isotropically thermal conductivity in the 2-dimensional domain. The purpose of this paper is to generalize their results on the case that the domain and layer are both three dimensional, by deriving the effective boundary conditions as the thickness of the thin layer shrinks to zero. The problem can be described as follows. Let $\Omega_3$ be the thin layer inside the body $\Omega$ and consider the bounded domain $\Omega = \overline{\Omega_1} \cup \overline{\Omega_3} \cup \Omega_2 \subset \mathbb{R}^3$ (see Figure 1) with the following Dirichlet problem for the heat equation.

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For any fixed finite $T > 0$,
\[
\begin{cases}
  u_t - \nabla \cdot (A(x) \nabla u) = f(x, t), & (x, t) \in Q_T, \\
  u = 0, & (x, t) \in S_T, \\
  u = u_0, & (x, t) \in \Omega \times \{0\},
\end{cases}
\]
where $Q_T = \Omega \times (0, T)$ and $S_T = \partial \Omega \times (0, T)$ with the initial condition $u_0 \in L^2(\Omega)$ and $f \in L^2(Q_T)$; $A(x)$ is the thermal conductivity defined by
\[
A(x) = \begin{cases}
  k_1, & x \in \Omega_1 \\
  \sigma, & x \in \Omega_3 \\
  k_2, & x \in \Omega_2
\end{cases}
\]
Throughout the paper, we shall always assume that $\sigma$ is a positive function of $\delta$, mathematically, and that $k_1, k_2$ are two positive constants independent of $\delta > 0$.
Furthermore, both $\Omega$ and $\Omega_1$ are fixed and bounded with $C^2$-smooth boundaries $\partial \Omega$ and $\Gamma_1(= \partial \Omega_1)$, respectively, and like in [2][3], the thin layer $\Omega_3$ is uniformly thick with thickness $\delta$ as shown in Figure 1; $\Gamma_2$ approximates to $\Gamma_1$ as $\delta \to 0$.

As one method of avoiding the multi-scales, the derivation of EBCs was first mentioned by Carslaw and Jaeger in their classic book [1] in 1959 and was first used rigorously by Sanchez-Palencia [5] in 1974. Later, the elliptic problem was studied by Brezis, Caffarelli and Friedman [6] in 1980. Apart from the mentioned results, there are lots of follow-up works on Poisson equation and heat/diffusion equation [7, 8, 9, 10, 11, 12]; see also the review paper[13].

2. Asymptotic behavior of $u$ and EBCs.

2.1. Preliminaries. Define a map $F$ by
\[
\partial \Omega_1 \times [0, \delta] \ni x = F(p, r) = p + r n(p) \in \mathbb{R}^3
\]
where $p$ is the projection of $x$ on $\partial \Omega_1$; $n(p)$ is the unit outer normal vector at $p$ and $r$ is the distance from $x$ to $\partial \Omega_1$. If $\delta$ is small enough, then, by lemma 14.16 of [4], $F$ is a $C^1$ smooth diffeomorphism from $\partial \Omega_1 \times [0, \delta]$ to $\Omega_3$; Since $\partial \Omega_1$ is $C^2$ smooth, it can be covered by finitely many local charts with standard compatibility conditions. Using the local coordinate: $s = (s_1, s_2)$ in a typical chart on $\partial \Omega_1$,
\[
x = F(p, r) = F(s_1, s_2, r) \quad dx = [1 + 2H(s)r + \kappa(s)r^2]dsdr
\]
where $ds$ is the surface element; $H(s)$ and $\kappa(s)$ represent mean curvature and Gaussian curvature at $p$ on $\partial \Omega_1$ respectively. In the curvilinear coordinate, the Riemann metric matrix at $x$ induced from $\mathbb{R}^3$ is defined as $G(s, r)$, with elements
\[
g_{ij} = g_{ji} = \langle F_{s_i}, F_{s_j} \rangle_{\mathbb{R}^3}, i, j = 1, 2
\]
Let \(|G| = |\det G|\) and \(g^{ij}(s, r)\) be the elements of the inverse matrix of \(G\), denoted by \(G^{-1}\). Therefore, we have the following expressions for the derivative in the curvilinear coordinate:

\[
\nabla u = u_n + \nabla_s u
\]

\[
\nabla_s u = \sum_{i,j=1,2} g^{ij}(s, r) u_{x_j} F_s(s, r), \quad \nabla_{\Gamma_1} u = \sum_{i,j=1,2} g^{ij}(s, 0) u_{x_j} p_s(s)
\]

\[
\nabla \cdot (A(x) \nabla u) = \frac{\sigma}{\sqrt{|G|}} (\sqrt{|G|} u_r) + \sigma \Delta_s u
\]

\[
\Delta_s u = \nabla_s \cdot \nabla_s u = \frac{1}{\sqrt{|G|}} \sum_{i,j=1,2} (\sqrt{|G|} g^{ij} u_{s_i})_{s_j}
\]

Now, we introduce several required Sobolev space. Firstly, let \(W^{1,0}_2(Q_T)\) be the space of \(L^2(Q_T)\) with first order weak derivatives in \(x\) also in \(L^2(Q_T)\); \(W^{1,1}_2(Q_T)\) is defined similarly but with the first order weak derivative in \(t\) also in \(L^2(Q_T)\). Moreover, \(W^{1,0}_2(Q_T)\) is the closure in \(W^{1,0}_2(Q_T)\) of \(C^\infty\) functions vanishing near \(\overline{S_T}\) and define \(W^{1,1}_2(Q_T)\) similarly; Finally, denote \(V^{1,0}_2(Q_T) = W^{1,0}_2(Q_T) \cap C([0, T]; L^2(\Omega))\).

**Definition 2.1.** A function \(u\) is said to be a weak solution of (1.1), if \(u \in V^{1,0}_2(Q_T)\) and for any \(\zeta \in W^{1,1}_2(Q_T)\) satisfying \(\zeta = 0\) at \(t = T\), it holds

\[
\mathcal{A}[u, \zeta] := -\int_{\Omega} u_0 \zeta(x, 0) dx + \int_{Q_T} (A(x) \nabla u \cdot \nabla \zeta - u\zeta_t - f\zeta) dx dt = 0. \tag{2.2}
\]

satisfying the following “transmission condition”:

\[
\left\{
\begin{array}{ll}
u_1 = u_\delta, & k_1 \nabla u_1 \cdot n_1 = \sigma \nabla u_\delta \cdot n_1, \quad \text{on} \quad \Gamma_1 \\
u_2 = u_\delta, & k_2 \nabla u_2 \cdot n_2 = \sigma \nabla u_\delta \cdot n_2, \quad \text{on} \quad \Gamma_2
\end{array}
\right.
\]

where \(u_1, u_\delta\) and \(u_2\) are the restrictions of \(u\) on \(\Omega_1 \times (0, T), \Omega_\delta \times (0, T)\) and \(\Omega_2 \times (0, T)\);

\(n_1\) and \(n_2\) are the outward unit normal vector of \(\Gamma_1\) and \(\Gamma_2\) respectively.

**2.2. Basic energy estimates.** We have the following basic energy estimates:

**Lemma 2.2.** Let arbitrary \(f \in L^2(Q_T)\) and \(u_0 \in L^2(\Omega)\). Then, any weak solution \(u\) of (1.1) satisfies the following inequalities:

\[
(i) \quad \max_{t \in [0, T]} \int_{\Omega} u^2(x, t) dx + \int_{Q_T} \nabla u \cdot A(x) \nabla u dx dt \leq C(T) \left( \int_{\Omega} u_0^2 dx + \int_{Q_T} f^2 dx dt \right);
\]

\[
(ii) \quad \max_{t \in [0, T]} \int_{\Omega} u \nabla \cdot A(x) \nabla u dx + \int_{Q_T} t u^2 dx dt \leq C(T) \left( \int_{\Omega} u_0^2 dx + \int_{Q_T} f^2 dx dt \right);
\]

**Proof.** It is very easy to prove above estimates using integration by parts. \(\square\)

**Lemma 2.3.** Assume \(\sigma\) is bounded from below by a positive constant independent of \(\delta > 0\). With conditions in lemma 2.2, for any fixed \(t_0 > 0\), \(u \in W^{2,1}_2(\Omega_\delta \times (t_0, T))\), \(h = 1, \delta, 2\).

**Proof.** By lemma 2.2, for any small \(t_0 \in (0, T)\), \(u_t \in L^2(\Omega_\delta \times (t_0, T))\). We use the Galerkin method to prove the second order estimates. Then, let \(u_n(x, t) = \sum_{i=1}^n c_i(t) e_i(x), \) where \(e_i(x) \in H^1_0(\Omega)\) are the eigen-functions of the elliptic operator: \(-\nabla \cdot (A(x) \nabla)\) with Dirichlet boundary condition. By the standard interior \(H^2\) estimate in \(\Omega_\delta\), it suffices to prove the second order estimates near \(\partial \Omega_\delta\). Take a finite cover of \(\partial \Omega_\delta : \{ \cup B_{r_i}, \Phi \}\) and then straighten the boundary locally. Using the
similar technique in the proof of $H^2$ estimates for the general elliptic problem, we choose a test function with the cut-off function in spatial variable $x$ and consider the second difference quotient of $u_n$ in the direction parallel to the boundary. As for the term $(u_n)_{rr}$, the estimate results from the PDE in the curvilinear coordinate. By sending $n \to \infty$, we obtain the required second order estimates for $u$ and hence, we omit the details.

2.3. Results and Proof.

**Theorem 2.4.** Assume $\sigma$ is bounded from below by a positive constant independent of $\delta > 0$. Let $u$ be the unique weak solution of (1.1) with $f \in L^2(Q_T)$, and $u_0 \in L^2(\Omega)$. Then, as $\delta \to 0$, the following holds:

(i) If $\lim_{\delta \to 0} \sigma \delta = \alpha \in [0, \infty)$, then $u \to v$ weakly in $W^{1,0}_{2,0}(Q_T)$, strongly in $C([0, T]; L^2(\Omega))$, where $v$ is the weak solution of the effective model:

\[
\begin{align*}
\begin{cases}
  v_t - \nabla \cdot (A_0(x) \nabla v) = f(x, t), & (x, t) \in Q_T, \\
  v = 0, & (x, t) \in S_T, \\
  v = u_0, & x \in \Omega, t = 0,
\end{cases}
\end{align*}
\]

(2.3)

subject to the following effective boundary conditions on $\Gamma_1 \times (0, T)$:

\[ v_1 = v_2, \quad k_1 \frac{\partial v_1}{\partial n} - k_2 \frac{\partial v_2}{\partial n} = \alpha \Delta_{\Gamma_1} v \]  

(2.4)

where $v_1$ and $v_2$ are the restrictions of $v$ on $\Omega_1 \times (0, T)$ and $(\Omega \setminus \Omega_1) \times (0, T)$ respectively; $n$ is the unit outer normal vector of the surface $\Gamma_1$, and

\[ A_0(x) = \begin{cases}
  k_1, & x \in \Omega_1 \\
  k_2, & x \in \Omega \setminus \Omega_1
\end{cases} \]

(ii) If $\lim_{\delta \to 0} \sigma \delta = \infty$, then $u \to v$ weakly in $W^{1,0}_{2,0}(Q_T)$, strongly in $C([0, T]; L^2(\Omega))$, where $v$ is the weak solution of the effective model (2.3), subject to the following effective boundary condition on $\Gamma_1 \times (0, T)$:

\[ v_1 = v_2, \quad \nabla_{\Gamma_1} v = 0, \quad \int_{\Gamma_1} (k_1 \frac{\partial v_1}{\partial n} - k_2 \frac{\partial v_2}{\partial n}) ds = 0 \]  

(2.5)

**Remark 1.** (i): $\Delta_{\Gamma_1}$ is the Laplace-Beltrami operator on the surface $\Gamma_1$ and $\nabla_{\Gamma_1} v = 0$ indicates that $v$ is a constant in the spatial variable.

(ii): The above theorem does not cover the case of $\sigma \to 0$ as $\delta \to 0$. Fortunately, it was solved for the case of the Poisson problem in any number of dimensions in [6]. Their proofs can be transplanted to the heat equation, resulting in the following conclusion: suppose $\sigma \to 0$, as $\delta \to 0$ and $\lim_{\delta \to 0} \frac{\sigma \delta}{\pi} = \beta \in [0, \infty]$, then $u \to w$ in $C([0, T]; L^2(\Omega))$, where $w$ is the weak solution of (2.3), subject to the following effective boundary condition on $\Gamma_1 \times (0, T)$:

\[ k_1 \frac{\partial v_1}{\partial n} = \beta (v_2 - v_1), \quad k_1 \frac{\partial v_1}{\partial n} = k_2 \frac{\partial v_2}{\partial n}. \]  

(2.6)

(iii): The existence and uniqueness of the weak solution of (1.1) is standard. As for the weak solution of (2.3), it may not be well-known for the existence and uniqueness, along with the boundary conditions (2.4),(2.5) and (2.6). However, they all follow the existence and uniqueness by the abstract parabolic theory (Section 2, [2]) with the corresponding weak solutions defined; see also [15] and [16].
Proof of the theorem 2.4. For any fixed small $t_0 \in (0, T]$, according to Lemma 2.2, \( \{u\}_{\delta > 0} \) is bounded in \( W^{1, 0}_2(Q_T) \) and \( C([t_0, T]; H^1_0(\Omega)) \). By Banach-Eberlein theorem, \( u \) converges to some \( v \) weakly in \( C([t_0, T]; H^1_0(\Omega)) \) after passing to a subsequence of \( \delta \to 0 \). Together with the compactness of the embedding \( H^1(\Omega) \hookrightarrow L^2(\Omega) \), \( \{u\}_{\delta > 0} \) is precompact in \( L^2(\Omega) \). Moreover, the functions \( \{u\}_{\delta > 0}: t \in [t_0, T] \mapsto u(\cdot, t) \in L^2(\Omega) \) are equicontinuous for the boundedness of the term \( \int_{Q_T} tu^2 \, dx \, dt \).

Consequently, the generalized Arzela-Ascoli theorem indicates that after passing to a further subsequence of \( \delta \to 0 \), \( u \to v \) strongly in \( C([t_0, T]; L^2(\Omega)) \).

Now, it suffices to show the strong convergence in \( C([0, T]; L^2(\Omega)) \). Construct a sequence \( u^n_0 \in C^\infty(\Omega) \) by multiplying \( u_0 \) by cut-off functions in \( r \) variable such that \( \|\nabla u^n_0\|_{L^2(\Omega)} \leq C(n) \) independent of \( \delta \), with \( u^n_0 \) vanishing in \( \Omega_\delta \) and \( \|u_0 - u^n_0\|_{L^2(\Omega)} \leq \frac{1}{n} + \|u_0\|_{L^2(\Omega_\delta)} \). Then, we decompose \( u = u_1 + u_2 \), where \( u_1 \) is the unique weak solution of (1.1) with \( f = 0 \) and the initial value changed by \( u_0 - u^n_0 \), and \( u_2 \) is the unique weak solutions of (1.1) with the initial value changed by \( u^n_0 \). By energy estimates on the new PDEs, we have

\[
\|u_1(\cdot, t)\|_{L^2(\Omega)} \leq \|u_0 - u^n_0\|_{L^2(\Omega)} \leq \frac{1}{n} + \|u_0\|_{L^2(\Omega_\delta)}
\]

and

\[
\int_{Q_T} (u^n_2)^2 \, dx \, dt + \int_\Omega \nabla u_2 \cdot A(x) \nabla u_2 \, dx \\
\leq \int_{Q_T} f^2 \, dx \, dt + k_1 \int_{\Omega_1} |\nabla u^n_0|^2 \, dx + k_2 \int_{\Omega_2} |\nabla u^n_0|^2 \, dx =: C(f, u^n_0)
\]

Hence, for any \( t \in [0, t_0] \), it follows that

\[
\|u_2(\cdot, t) - u^n_0(\cdot)\|_{L^2(\Omega)}^2 = 2 \int_0^t \int_{\Omega} (u_2(x, t) - u^n_0(x))(u_2)_t \, dx \, dt \\
\leq 2 \left( \int_0^t \int_{\Omega} |u_2(x, t) - u^n_0(x)|^2 \right)^{\frac{1}{2}} \left( \int_0^t \int_{\Omega} (u_2)_t^2 \right)^{\frac{1}{2}} \, dx \, dt \\
\leq 2\sqrt{t_0} \max_{t \in [0, t_0]} \|u_2(\cdot, t) - u^n_0(\cdot)\|_{L^2(\Omega)} \cdot (C(f, u^n_0))^{\frac{1}{2}}
\]

Consequently, it turns out that

\[
\|u(\cdot, t) - u_0(\cdot)\|_{L^2(\Omega)} \leq \|u_1(\cdot, t)\|_{L^2(\Omega)} + \|u_2(\cdot, t) - u^n_0(\cdot)\|_{L^2(\Omega)} + \|u_0 - u_0(\cdot)\|_{L^2(\Omega)} \\
\leq \frac{1}{n} \|u_0\|_{L^2(\Omega_\delta)} + 2\sqrt{t_0} (C(f, u^n_0))^{\frac{1}{2}}
\]

which is arbitrary small for \( t_0 \) and \( \delta \) are small enough.

Next, what we want to do is to show that \( v \) is a weak solution of the effective model with boundary condition (2.4), or (2.5). Then, by uniqueness, we can obtain the convergence without passing a subsequence. Notice that we have already taken care of the initial condition. So, we can take a test function \( \xi \in C^\infty(\overline{\Omega} \times (0, T]) \) with \( \xi = 0 \) on \( \partial \Omega \times (0, T) \) and \( \xi = 0 \) for \( t \in [0, \varepsilon] \cup [T - \varepsilon, T] \) for some small \( \varepsilon > 0 \). By the definition 2.1,

\[
A[u, \xi] := -\int_{Q_T} u_0 \xi_0 \, dx \, dt + k_1 \int_0^T \int_{\Omega_1} \nabla u \cdot \nabla \xi \, dx \, dt + k_2 \int_0^T \int_{\Omega_2} \nabla u \cdot \nabla \xi \, dx \, dt \\
+ \sigma \int_0^T \int_{\Omega_3} \nabla u \cdot \nabla \xi \, dx \, dt - k_2 \int_0^T \int_{\Omega_4} \nabla u \cdot \nabla \xi \, dx \, dt - \int_{Q_T} f \xi \, dx \, dt = 0
\]
By the Holder inequality and estimates in Lemma 2.2,

\[ \left| \int_0^T \int_{\Omega_s} \nabla u \cdot \nabla \xi dxdt \right| \leq \left( \int_{Q_T} |\nabla u|^2 dxdt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\Omega_s} |\nabla \xi|^2 dxdt \right)^{\frac{1}{2}} \leq O(\delta) \]

Since \( u \to v \) in \( W^{1,0}_2(Q_T) \) as \( \delta \to 0 \), it turns out that (2.7) can be rewritten as

\[ \mathcal{L}[v, \xi] := - \int_{Q_T} u_0 \xi_t dxdt + A_0 \int_0^T \int_{\Omega} \nabla v \cdot \nabla \xi dxdt - \int_{Q_T} f \xi dxdt \]

(2.8)

To investigate the asymptotic behavior of this term, we first consider the case

\[ \lim_{\delta \to 0} \sigma \delta = \alpha \in [0, \infty). \]

Using integration by parts,

\[ \sigma \int_0^T \int_{\Omega_s} \nabla u \cdot \nabla \xi dxdt = -\sigma \int_0^T \int_{\Omega_s} u \Delta \xi dxdt + \sigma \int_0^T \int_{\partial \Omega_s} u \frac{\partial \xi}{\partial n} ds dt \]  \tag{2.9} \]

The first term in the right hand side of (2.9) in the curvilinear coordinate is

\[
-\sigma \int_0^T \int_{\Gamma_1} \int_0^\delta \left| u(s, r, t)|\Delta \xi(s, r, t)(1 + 2H(s) r + \kappa(s) r^2)|\right| ds dr dt
\]

\[
= -\sigma \int_0^T \int_{\Gamma_1} \int_0^\delta [u(s, r, t) - u(s, 0, t)] F(s, r, t) ds dr dt
\]

\[
- \sigma \int_0^T \int_{\Gamma_1} \int_0^\delta u(s, 0, t)[F(s, r, t) - F(s, 0, t)] ds dr dt
\]

\[
- \sigma \int_0^T \int_{\Gamma_1} \int_0^\delta u(s, 0, t) F(s, 0, t) ds dr dt
\]

where \( F(s, r, t) := \Delta \xi(s, r, t)(1 + 2H(s) r + \kappa(s) r^2) \). By lemma 2.2 and the assumption on \( \sigma \), it holds

\[
\sigma \left| \int_0^T \int_{\Gamma_1} \int_0^\delta [u(s, r, t) - u(s, 0, t)] F(s, r, t) dr ds dt \right| \leq O(\sigma) \int_0^T \int_{\Gamma_1} \int_0^\delta |u(s, r, t) - u(s, 0, t)| dr ds dt
\]

\[
\leq O(\sigma) \int_0^T \int_{\Gamma_1} \int_0^\delta \int_0^r |u_\tau(s, \tau, t)| dr ds dt
\]

\[
\leq O(\sigma) \left( \int_0^T \int_{\Gamma_1} \int_0^\delta \int_0^r u_\tau^2(s, \tau, t) dr ds dt \right)^{\frac{1}{2}} \cdot \left( \int_0^T \int_{\Gamma_1} \int_0^\delta \int_0^r dr ds dt \right)^{\frac{1}{2}}
\]

\[
\leq O(\delta^3/2) \sigma \left( \int_0^T \int_{\Gamma_1} \int_0^\delta \int_0^r u_\tau^2(s, \tau, t) dr ds dt \right)^{\frac{1}{2}} \leq O(\delta^3/2) \sigma
\]
By the lemma 2.3 and trace theorem, we have
\[
\sigma \left| \int_0^T \int_{\Gamma_1} \int_0^\delta u(s, 0, t)[F(s, r, t) - F(s, 0, t)]drdsdt \right|
\leq \sigma \int_0^T \int_{\Gamma_1} \int_0^\delta |u(s, 0, t) r F_r(s, r, t)|drdsdt
\leq O(\sigma) \int_0^T \int_{\Gamma_1} \int_0^\delta |u(s, 0, t) r|drdsdt
\leq O(\sigma) \left( \int_0^T \int_{\Gamma_1} \int_0^\delta u^2(s, 0, t)dsdrdt \right)^{\frac{1}{2}} \cdot \left( \int_0^T \int_{\Gamma_1} \int_0^\delta r^2 dsdrdt \right)^{\frac{1}{2}}
\leq O(\sigma \delta^2) \left( \int_0^T \int_{\Gamma_1} \int_0^\delta u^2(s, 0, t)dsdrdt \right)^{\frac{1}{2}} \leq O(\delta^2 \sigma)
\]
Furthermore, the last term above can be rewritten as
\[
- \sigma \int_0^T \int_{\Gamma_1} \int_0^\delta u(s, 0, t)F(s, 0, t)
= - \sigma \delta \int_0^T \int_{\Gamma_1} u(s, 0, t)[\xi_{rr} + 2H \xi_r + \Delta_{\Gamma_1} \xi](s, 0, t)dsdt
\]
Therefore, it turns out that
\[
- \sigma \int_0^T \int_{\Omega_0} u \Delta \xi dt = - \alpha \int_0^T \int_{\Gamma_1} u(s, 0, t)[\xi_{rr} + 2H \xi_r + \Delta_{\Gamma_1} \xi](s, 0, t)dsdt + O(\delta) \tag{2.10}
\]
Similarly, the second term in the right hand side of (2.9) is
\[
\sigma \int_0^T \int_{\partial\Omega_0} \frac{\partial \xi}{\partial n} dsdt = \sigma \int_0^T \left( \int_{\Gamma_1} + \int_{\Gamma_2} \right) \frac{\partial \xi}{\partial n} dsdt
= - \sigma \int_0^T \int_{\Gamma_1} u(s, 0, t) \xi_r (s, 0, t)dsdt + \sigma \int_0^T \int_{\Gamma_1} u(s, \delta, t) \xi_r (s, \delta, t)[1 + 2H \delta + \kappa \delta]dsdt
\]
\[
= \sigma \int_0^T \int_{\Gamma_1} [u(s, \delta, t) \xi_r (s, \delta, t) - u(s, 0, t) \xi_r (s, 0, t)]dsdt
\]
\[
+ \sigma \int_0^T \int_{\Gamma_1} u(s, \delta, t) \xi_r (s, \delta, t)[2H \delta + \kappa \delta]dsdt
= \sigma \int_0^T \int_{\Gamma_1} [u(s, \delta, t) - u(s, 0, t)] \xi_r (s, \delta, t)dsdt
\]
\[
+ \sigma \int_0^T \int_{\Gamma_1} u(s, 0, t)[\xi_r (s, \delta, t) - \xi_r (s, 0, t)]dsdt
= \sigma \delta^2 \int_0^T \int_{\Gamma_1} u(s, \delta, t) \xi_r (s, \delta, t) \kappa(s)dsdt + \sigma \delta \int_0^T \int_{\Gamma_1} u(s, \delta, t) \xi_r (s, \delta, t)2H dsdt
\]
By the lemma 2.2 and 2.3, it follows with the trace theorem and condition (2.1)

$$
\sigma \int_0^T \int_{\Gamma_1} [u(s, \delta, t) - u(s, 0, t)] \xi_r(s, \delta, t) dsdt = \sigma \int_0^T \int_{\Gamma_1} \int_0^\delta u_r(s, \delta, t) \xi_r(s, \delta, t) dr dsdt \\
\leq \sigma \int_0^T \int_{\Gamma_1} \int_0^\delta [u_r(s, \delta, t) - u_r(s, 0^+, t)] \xi_r(s, \delta, t) + \sigma \delta \int_0^T \int_{\Gamma_1} \int_0^\delta u_r(s, 0^+, t) \xi_r(s, \delta, t) \leq \sigma \int_0^T \int_{\Gamma_1} \int_0^\delta u_r(s, \tau, t) \xi_r(s, \delta, t) d\tau dsdt + \sigma \int_0^T \int_{\Gamma_1} k_1 u_r(s, 0^-, t) \xi_r(s, \delta, t) dsdt \\
\leq \sigma^{\frac{1}{2}} \left( \int_\varepsilon^T \int_{\Gamma_1} \int_0^\delta \int_0^\delta \sigma u_r^2(s, \tau, t) d\tau dsdt \right)^{\frac{1}{2}} \cdot \left( \int_\varepsilon^T \int_{\Gamma_1} \int_0^\delta \int_0^\delta \xi_r^2(s, \delta, t) d\tau dsdt \right)^{\frac{1}{2}} + O(\delta) \left( \int_\varepsilon^T \int_{\Gamma_1} \int_0^\delta u_r^2(s, 0^-, t) dsdt \right)^{\frac{1}{2}} \\
\leq O(\sigma^{\frac{1}{2}} \delta^{\frac{1}{2}}) \left( \int_\varepsilon^T \int_{\Gamma_1} \int_0^\delta \int_0^\delta \sigma u_r^2(s, \tau, t) d\tau dsdt \right)^{\frac{1}{2}} + O(\delta) \left( \int_\varepsilon^T \int_{\Gamma_1} \int_0^\delta u_r^2(s, 0^-, t) dsdt \right)^{\frac{1}{2}} \\
\leq O(\sigma^{\frac{1}{2}} \delta^{\frac{1}{2}}) + O(\delta) \left( \int_\varepsilon^T \int_{\Gamma_1} \int_0^\delta \int_0^\delta \xi_r^2(s, \delta, t) d\tau dsdt \right)^{\frac{1}{2}} \leq O(\delta) \\
$$

In addition, we have

$$
\sigma \int_0^T \int_{\Gamma_1} u(s, 0, t) \xi_r(s, \delta, t) - \xi_r(s, 0, t) dsdt \\
= \sigma \delta \int_0^T \int_{\Gamma_1} u(s, 0, t) \frac{\xi_r(s, \delta, t) - \xi_r(s, 0, t)}{\delta} dsdt \\
= \sigma \delta \int_0^T \int_{\Gamma_1} u(s, 0, t) \xi_{r\tau}(s, 0, t) dsdt \\
\leq O(\sigma \delta) \int_0^T \int_{\Gamma_1} u(s, 0, t) \xi_{r\tau}(s, 0, t) dsdt \\
\leq O(\sigma \delta) \left( \int_0^T \int_{\Omega_1} \left| \nabla u \right|^2 + |D^2 u|^2 \right) dsdt \leq O(\sigma \delta) + o(1) \text{ as } \delta \to 0 \\
$$

Finally, we are left with the only remaining term:

$$
\sigma \delta \int_0^T \int_{\Gamma_1} u(s, \delta, t) \xi_r(s, \delta, t) 2H(s) dsdt \\
= \sigma \delta \int_0^T \int_{\Gamma_1} [u(s, \delta, t) - u(s, 0, t)] \xi_r(s, \delta, t) 2H(s) dsdt \\
+ \sigma \delta \int_0^T \int_{\Gamma_1} u(s, 0, t) \xi_r(s, \delta, t) 2H(s) dsdt \\
$$
As a consequence, it turns out that the second term in the right hand side of (2.7) is a positive and arbitrary small. Thus, the test function can be chosen as

\[\alpha \int_0^T \int_{\Gamma_1} u(s, 0, t)[\xi_{rr}(s, 0, t) + 2H(s)\xi_r(s, 0, t)]dsdt + O(\delta)\]

Combined with (2.10), we have

\[-\alpha \int_0^T \int_{\Gamma_1} u(s, 0, t)\Delta_{\Gamma_1} \xi(s, 0, t)dsdt + O(\delta)\]

\[\rightarrow -\alpha \int_0^T \int_{\Gamma_1} v(s, 0, t)\Delta_{\Gamma_1} \xi(s, 0, t)dsdt \quad \text{as} \quad \delta \rightarrow 0\]

Therefore, we obtain

\[\mathcal{L}[v, \xi] = -\alpha \int_0^T \int_{\Gamma_1} v(s, 0, t)\Delta_{\Gamma_1} \xi(s, 0, t)dsdt \quad (2.11)\]

as \(\delta \rightarrow 0\).

To show that \(v\) is a weak solution of (2.3) together with boundary condition (2.4), we are left to prove \(v \in L^2((0, T); H^1(\Gamma_1))\) by the definition of weak solution from Section 2 in [2]. The subtle point is to have the following estimate

\[\int_0^T \int_{\Gamma_1} |\nabla_{\Gamma_1} v(s, 0, t)|^2dsdt < \infty\]

By the trace theorem, we have \(v \in L^2((t_0, T); H^1(\Gamma_1))\) for arbitrary small \(t_0 > 0\) for \(v \in W^{1,0}_2(\Omega_1 \times (t_0, T))\). We now construct a new function \(\overline{v}\), by extending the restriction of \(v\) on \(\Omega_1 \times (0, T)\) with \(\overline{v} \in W^{1,0}_2(\Omega \times (t_0, T))\), and

\[||\overline{v}||_{W^{1,0}_2(\Omega \times (t_0, T))} \leq C||v||_{W^{1,0}_2(\Omega_1 \times (t_0, T))} \quad (2.12)\]

where \(C\) is a constant independent of \(t_0\). Then, take two smooth cut-off functions \(\omega(t)\) and \(z(x)\) in \(t\) and \(x\), respectively, such that \(0 \leq \omega(t) \leq 1\), vanishing for \(t \in [0, \epsilon] \cup [T - \epsilon, T]\) with \(\omega = 1\) in \([2\epsilon, T - 2\epsilon]\), and \(0 \leq z(x) \leq 1\), vanishing in \(\{x \in \Omega_1 | \text{dist}(x, \Gamma_1) > \overline{\epsilon}\}\), with \(z(x) = 1\) in \(\Omega_1\) and \(|\nabla z| \leq O(1/\overline{\epsilon})\), where \(\epsilon\) and \(\overline{\epsilon}\) are positive and arbitrary small. Thus, the test function can be chosen as \(\overline{v}\omega(t)z(x)\) by the density theorem, which lead to

\[\int_{Q_T} v_1 \overline{v}\omega(t)z(x)dxdt + k_1 \int_0^T \int_{\Omega_1} |\nabla v|^2\omega(t)dxdt + k_2 \int_{\Omega_1} \omega(t)\nabla v \cdot \nabla(\overline{v}(x))dxdt \]

\[+ \alpha \int_0^T \int_{\Gamma_1} |\nabla_{\Gamma_1} v(s, 0, t)|^2\omega(t)dsdt = \int_{Q_T} \overline{v}\omega(t)z(x)dxdt \quad (2.13)\]

For small \(\epsilon\) and \(\overline{\epsilon}\), the fact that all terms without \(\nabla_{\Gamma_1} v\) and \(v_1\) are bounded is a consequence of the fact that \(v \in W^{1,0}_2(Q_T)\) and (2.12). Sending \(\overline{\epsilon} \rightarrow 0, \epsilon \rightarrow 0\), we have the first term

\[\int_0^T \int_{\Omega_1} v_1 u dx dt = \frac{1}{2} \left( \int_{\Omega_1} v_1 ^2(x, T) dx - \int_{\Omega_1} u_0 ^2(x) \right) < \infty\]
which indicates
\[ \alpha \int_0^T \int_{\Gamma_1} |\nabla_{\Gamma_1} v(s, 0, t)|^2 dsdt < \infty \]

From now on, we consider the case of \( \sigma \delta \to \infty \). Since \( v \in W^{1,0}_2(Q_T) \), this implies that \( v_1 = v_2 \) on \( \Gamma_1 \) in the sense of trace. Divided by \( \alpha \) in both sides of (2.11), we have
\[ \int_0^T \int_{\Gamma_1} v(s, 0, t) \Delta_{\Gamma_1} \xi(s, 0, t) dsdt = 0 \]
If we take special test function \( \xi = v(s, 0, t) \), then \( \nabla_{\Gamma_1} v(s, 0, t) = 0 \), which means \( v \) is a constant on \( \Gamma_1 \) in the spatial variable. With further assumption \( \xi = m(t) \) on \( \Gamma_1 \), using the transmission condition (2.1), the only term left in (2.7) is
\[
\sigma \int_0^T \int_{\Omega_s} \nabla u \cdot \nabla \xi dxdt \\
= -\sigma \int_0^T \int_{\Omega_s} \xi \Delta u dxdt + \sigma \int_0^T \int_{\partial\Omega_s} \xi \frac{\partial u}{\partial n} dsdt \\
= -\sigma \int_0^T \int_{\Omega_s} (\xi - m(t)) \Delta u dxdt - \sigma \int_0^T \int_{\Omega_s} m(t) \Delta u dxdt + \sigma \int_0^T \int_{\partial\Omega_s} \xi \frac{\partial u}{\partial n} dsdt \\
= -\sigma \int_0^T \int_{\Omega_s} (\xi - m(t)) \Delta u dxdt + \sigma \int_0^T \int_{\Omega_s} (\xi - m(t)) \frac{\partial u}{\partial n} dsdt \\
\leq \left( \int_{\Omega_s} (\xi - m(t))^2 dxdt \right)^{\frac{1}{2}} \left( \int_{\Omega_s} (u_t - f)^2 dxdt \right)^{\frac{1}{2}} \\
+ \sigma \int_0^T \int_{\Gamma_1} (\xi(s, \delta, t) - m(t)) u_r(s, \delta^-, t)[1 + 2H\delta + \kappa \delta^2] dsdt \\
\leq O((\delta^{1/2}) + k_2 \int_{\Gamma_1} (\xi(s, \delta, t) - m(t)) u_r(s, \delta^+, t)[1 + 2H\delta + \kappa \delta^2] dsdt \\
\leq O(\delta^{1/2}) + O(\delta) \left( \int_{\Omega_s} u_r^2(s, \delta^+, t) dsdt \right)^{\frac{1}{2}} \\
\leq O(\delta^{1/2}) + O(\delta) \left( \int_{\Omega} |D^2 u|^2 dxdt \right)^{\frac{1}{2}} \to 0, \quad \text{as} \quad \delta \to 0
\]
where the first inequality holds by the PDE and the last step holds by the trace theorem. Hence in (2.7), as \( \delta \to 0 \), we have
\[ \mathcal{L}[v, \xi] = \int_{Q_T} (-v\xi_t + A_0(x)\nabla v \cdot \nabla \xi - f \xi) dxdt = 0 \]
which implies the last boundary condition. Thus, we have already show that \( v \) is a weak solution of (2.3) with the boundary condition (2.5), which completes the whole proof. \( \square \)
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