Curvature vs degrees of freedom: The case of the critical 2+1 Hořava theory

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Abstract

We present the interesting case of the 2 + 1 nonprojectable Hořava theory formulated at the critical point, where it does not possess local degrees of freedom. The critical point is defined by the value of a coupling constant of the theory. We discuss how, in spite of the absence of degrees of freedom, the theory admits solutions with nonflat or nonconstant curvature. We consider the theory without cosmological constant and without terms of higher order derivatives, hence this is an effect that can be seen at the same order of 2 + 1 general relativity. We present an exact nonflat solution that is not asymptotically flat. The presence of solutions with nontrivial curvature seems to be related to the relaxing of the asymptotically flat condition. We discuss that there is no analogue of Newtonian potential in this theory, and a broad class of asymptotically flat geometries leads to the restriction that the only solutions that can be found among them are the flat ones.

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1 Introduction

In gravitation it is commonly accepted that the absence of propagating degrees of freedom leads to the triviality of the curvature, or at most, to constant curvature in the presence of a cosmological constant. Of course, this is the case of 2+1 general relativity. Nontrivial configurations are more likely associated to global effects, like the case of the Bañados-Teitelboim-Zanelli black hole in 2+1 general relativity with negative cosmological constant [1], but not to the presence of nontrivial or nonconstant local curvature.

Hořava gravity [2] provides a different theoretical framework. It is based on the assumption of a foliation of spacelike surfaces along a given direction of time. The symmetry of coordinate transformations which the theory is based on is given by those transformations that preserve the given foliation. A spacetime metric is not necessary as fundamental object in Hořava gravity. Nevertheless, the gravitational fields are taken from the Arnowitt-Deser-Misner (ADM) decomposition of general relativity. This facilitates the comparison between both theories. The underlying gauge symmetry of the Hořava theory leads to a quantum theory with improved behavior in the ultraviolet. This is due to the fact that terms of higher order in spatial derivatives can be incorporated in the action. Simultaneously, the unitarity of the theory can be safe since no terms of higher order in time derivatives are necessary to define the theory.

The best known formulation of the Hořava theory, both in its projectable and nonprojectable versions, propagates a physical degree of freedom additional to the ones of general relativity. Although this can be generically associated to the reduced symmetry group of the Hořava theory, actually this is not a unavoidable feature. Indeed, there is a critical case where the extra physical mode disappears. The kinetic term of the Lagrangian has the general form \( \sqrt{g}N(K_{ij}K^{ij} - \lambda (g^{ij}K_{ij})^2) \), where \( \lambda \) is an arbitrary coupling constant. The additional extra mode is eliminated when \( \lambda \) takes the critical value \( \lambda = 1/d \), where \( d \) is the spatial dimension. The canonical formulation showing this in \( d = 3 \) for the case of the nonprojectable theory can be found in Ref. [3]. The elimination of the extra mode in the critical case is a consequence of the dynamics, rather than the symmetry. Two additional second-class constraints arise in the critical case, which are not associated to gauge symmetries. These constraints eliminate the extra mode.

It is interesting to compare Hořava gravity with general relativity in \( d = 2 \) spatial dimensions. In particular we want to compare the large-distance effective action of the Hořava theory, which is a theory of only second order in derivatives, as general relativity. The generic formulation of the 2+1 Hořava theory possesses one scalar mode. But, borrowing the analysis done in 3+1 dimensions, one expect that the critical formulation of the Hořava theory in 2+1 dimensions does not propagate any mode at all, since the would-be single scalar mode should be eliminated by the second-class constraints. We show this rigorously in this paper. Therefore, the Hořava theory at the critical point \( \lambda = 1/2 \) in 2 spatial dimensions behaves like
2 + 1 general relativity in the sense that local degrees of freedom are absent.

Despite the fact that the effective theory is of second order in derivatives, there is no reason to expect that the vacuum field equations (without cosmological constant) lead to zero curvature, since the field equations of the Hořava theory are different to the ones of general relativity. The ADM fields enter in some combinations that do not belong to the spacetime curvature. Therefore, solutions of the effective theory that correspond to vacuum configurations (without external sources), actually can have nonzero curvature due to the presence of more terms than the curvature ones. This effect can be easily seen by using the equivalent formulation of the hypersurface orthogonal Einstein-aether theory [4], which is a generally-covariant theory but with breaking of the local Lorentz symmetry. This theory consists in the coupling of general relativity to a timelike unit vector field, called the aether. If the aether field is restricted to be hypersurface orthogonal, the resulting theory is physically equivalent to the large-distance effective theory of the nonprojectable Hořava theory [5, 6]. The gravitational field equations of the Einstein-aether theory incorporate the energy-momentum tensor of the aether field. Hence it is clear that, whenever solutions with nonzero aether energy-momentum tensor exist, these solutions have nontrivial spacetime curvature tensor. We remark that in the framework of the Hořava theory the aether energy-momentum tensor should not be interpreted as an external source, since its presence is the way of representing the intrinsically gravitational terms of the Hořava theory that are different to the terms of general relativity.

In this paper we carry out explicitly this analysis on the 2 + 1 nonprojectable Hořava theory defined at the critical point \( \lambda = 1/2 \). We study the degrees of freedom of the theory and the curvature yielded by the field equations. In the first part we develop a detailed canonical analysis, showing the self-consistency of the theory and the fact that it has no propagating degrees of freedom. In the second part we discuss how the field equations lead in general to nontrivial curvature. Although this seems to be a general result, still some restrictions occur, and these restrictions are similar to the case of 2 + 1 general relativity. In particular, we point out that there is no analogue of Newtonian force in this theory. Nevertheless, we look for an exact solution that exhibits the main feature that we want to highlight: it is a solution of a gravitational theory with no propagating degrees of freedom and simultaneously it has nontrivial curvature. We study this solution in the class of static configurations with rotational symmetry. We complete the analysis of the spacetime curvature with the formulation of the critical theory in the framework of the equivalent hypersurface-orthogonal Einstein-aether theory.
2 The absence of local degrees of freedom

2.1 Nonperturbative canonical formulation

We study the Hořava theory in 2 spatial dimensions. A foliation of spacelike surfaces along a given direction of time is assumed from the definition. We may use local coordinates \((t, \vec{x})\) on the foliation. The theory is defined in terms of the ADM variables \(N(t, \vec{x}), N_i(t, \vec{x})\) and \(g_{ij}(t, \vec{x})\). We consider the nonprojectable case where the lapse function \(N\) is allowed to depend on the time and the spatial point. We consider the effective action for large distances, which has a potential of \(z = 1\) order, according to the criterium of anisotropy introduced in Ref. [2]. The action of the purely gravitational theory (without sources) is

\[
S = \int dt d^2x \sqrt{g}N \left( G^{ijkl} K_{ij} K_{kl} + \beta R + \alpha a^i a^i \right). \tag{2.1}
\]

In this section we use the standard notation of Riemannian geometry to denote the objects associated to the two-dimensional spatial metric \(g_{ij}\). Thus, \(K_{ij}\) is the extrinsic curvature of the leaves,

\[
K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - 2\nabla_i N_j) \tag{2.2}
\]

\(K\) is its trace, \(K = g^{ij} K_{ij}\), and the dot stands for derivative with respect the time. The hypermatrix \(G^{ijkl}\) is defined by

\[
G^{ijkl} = \frac{1}{2} \left( g^{ik} g^{jl} + g^{il} g^{jk} \right) - \lambda g^{ij} g^{kl}. \tag{2.3}
\]

It contains the arbitrary constant \(\lambda\), which is the coupling constant of the kinetic term [2]. \(R\) is the spatial Ricci scalar. \(a_i\) is the FDiff-covariant vector \(a_i = \partial_i \ln N\) [7]. \(\lambda, \beta\) and \(\alpha\) are the independent coupling constants of the model.

From the general relation

\[
G^{ijkl} g_{kl} = (1 - 2\lambda) g^{ij}, \tag{2.4}
\]

it follows that for the value \(\lambda = 1/2\) the hypermatrix is not invertible. This critical condition has profound consequences on the dynamics of the theory; this is our case of interest in this paper.

We perform the Legendre transformation for the critical case \(\lambda = 1/2\) to cast the theory in its canonical formulation[2]. The analysis is parallel to the critical theory in three spatial dimensions (the kinetic-conformal nonprojectable theory in 3 spatial dimentions) [3]. The phase space is spanned by the conjugated pairs \((g_{ij}, \pi^{ij})\) and

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2 Two of us presented the canonical formulation for the \(2 + 1\) nonprojectable Hořava theory with \(\lambda \neq 1/2\) in Ref. [5].
\((N, P_N)\). The action does not depend on the time derivative of lapse function \(N\), hence we obtain the primary constraint
\[
P_N = 0. \tag{2.5}
\]
The momentum conjugate of the spatial metric obeys the relation
\[
\frac{\pi^{ij}}{\sqrt{g}} = G^{ijkl} K_{kl}. \tag{2.6}
\]
Due to (2.4), the trace of this yields another primary constraint, namely,
\[
\pi \equiv g_{ij} \pi^{ij} = 0. \tag{2.7}
\]
After the Legendre transformation, one integration by parts and the addition of the two primary constraints with Lagrange multipliers, we obtain the Hamiltonian (omitting boundary terms)
\[
H = \int d^2x \left[ \sqrt{g} N \left( \frac{\pi^{ij} \pi_{ij}}{g} - \beta R - \alpha a_i a^i \right) + N_i \mathcal{H}^i + \mu \pi + \sigma P_N \right]. \tag{2.8}
\]
In the process the momentum constraint \(\nabla_i \pi^{ij} = 0\) arises. We extend the definition of this constraint in order to get the generator of spatial diffeomorphisms on the full phase space \([9]\), hence we define the momentum constraint
\[
\mathcal{H}^i \equiv -2 \nabla_k \pi^{ik} + P_N \partial^i N = 0. \tag{2.9}
\]
We apply Dirac’s procedure to obtain the full set of constraints. We first impose the preservation in time of the primary constraint \(P_N\). This yields the Hamiltonian constraint
\[
\mathcal{H} \equiv \frac{\pi^{ij} \pi_{ij}}{\sqrt{g}} - \sqrt{g} \beta R + \alpha \sqrt{g}(2 \nabla_k a^k + \alpha a_k a^k) = 0. \tag{2.10}
\]
Next, the preservation of the \(\pi = 0\) constraint yields a new constraint which we denote by \(\mathcal{C}\). It is given by
\[
\mathcal{C} \equiv \frac{N}{\sqrt{g}} \pi^{ij} \pi_{ij} - \beta \sqrt{g} \nabla^2 N = 0. \tag{2.11}
\]
From this equation we could derive an intriguing condition on \(\beta\) if we assume that asymptotically flat configurations with some nonzero canonical momentum, hence with \(\pi^{ij} \pi_{ij} > 0\), exist in the space of solutions. Consider the set of configurations for which \(N = \mathcal{O}(r^{-1})\) at spatial infinity. This is in correspondence with asymptotic flatness. On this set, we may multiply Eq. (2.11) by \(N\), integrate by parts on the space and discard the boundary term. This leads to the integral equation
\[
-\beta \int d^2x \nabla_k N \partial^k N = \int d^2x \frac{\pi^{ij} \pi_{ij}}{\sqrt{g}} N^2. \tag{2.12}
\]
Since both integrands are nonnegative, the only way to have a consistent solution of this equation for nonzero canonical momentum is when $\beta < 0$. This bound could be a bit surprising, since in the $3 + 1$ Hořava theory $\beta$ is the squared speed of the tensorial waves, hence it must be positive. Actually, in this critical $2+1$ theory there is no place for such a comparison since there are no waves at all, as we show below. Another theory to compare with is the $2 + 1$ noncritical ($\lambda \neq 1/2$) Hořava theory, which has a propagating mode. The squared speed of this mode is proportional to $\beta^2$, hence it does not restrict the sign of $\beta$. In any case, the condition $\beta < 0$ has arisen as a requisite for asymptotically flat with nonzero canonical momentum configurations exist in the space of solutions, but, according to the analysis we develop in the rest of the paper from other points of view, it is not clear whether such (vacuum) configurations really exist in this theory. Therefore, we do not take this bound on $\beta$ as a definitive condition.

Dirac’s procedure ends with the step of imposing the preservation of the secondary $H$ and $C$ constraints. These conditions generate elliptic differential equations for the Lagrange multipliers $\sigma$ and $\mu$. The two resulting equations are equivalent to the system

$$0 = \beta \nabla^2 \mu + 4 \alpha a_k \nabla^k \mu - \frac{2 \alpha}{N} \nabla_k (a^k \sigma) + \frac{\pi^{ij} \pi_{ij}}{g} \left( \frac{2 \alpha \sigma}{\beta} N - \left( 1 + \frac{2 \alpha}{\beta} \right) \mu \right) + 2 \left( 1 - \frac{\alpha}{\beta} \right) \frac{N^2 \pi_{ij}}{\sqrt{g}} a^i a^j, \quad (2.13)$$

$$0 = \beta \nabla^2 \sigma - \frac{\pi^{ij} \pi_{ij}}{g} (\sigma - N \mu) - 2 \beta N \frac{\pi_{ij}}{\sqrt{g}} \left( 2 \nabla^i a^j + \left( 1 - \frac{\alpha}{\beta} \right) a^i a^j \right). \quad (2.14)$$

In order to keep this system elliptic on $\mu, \sigma$ we must discard the possibility of zero $\beta$, ensuring in this way the consistency of the dynamics of the theory. Therefore, the set of constraints closes consistently. In summary, the nonreduced phase space of the nonprojectable Hořava theory in 2 spatial dimensions and formulated at the critical point $\lambda = 1/2$ is spanned by the variables $(g_{ij}, \pi^{ij})$ and $(N, P_N)$, which amount for eight functional degrees of freedom. The theory possesses the momentum constraint $H^i = 0$, which is a first-class constraint, and the second-class constraints $P_N = 0$, $\pi = 0$, $H = 0$ and $C = 0$. This gives a total of six constraints that must be imposed. Subtracting the two functional degrees of freedom corresponding to the gauge symmetry of spatial diffeomorphisms, we have that no physical, propagating, degree of freedom is left in the phase space. In other words, the reduced phase space of this theory has no dimension at all. With regards to the absence of local physical degrees of freedom, this theory behaves like 2+1 general relativity.
The equations of motion in the canonical formalism are
\[ \dot{N} = \sigma + N^k \nabla_k N, \]  
\[ \dot{g}_{ij} = 2N \frac{\pi_{ij}}{\sqrt{g}} + 2\nabla (g N_j) + \mu g_{ij}, \]  
\[ \dot{\pi}^{ij} = -\frac{N}{2\sqrt{g}} (4\pi^{k(i} \pi^{j)}_k - g^{ij} \pi^{kl} \pi_{kl}) - \frac{\alpha}{2} \sqrt{g} N \left( 2a^i a^j - g^{ij} a^k a^k \right) + \beta \sqrt{g} \left( \nabla^{ij} N - g^{ij} \nabla^2 N \right) - \mu \pi^{ij} - 2\nabla_k N (N^{(i} \pi^{j)k} + \nabla_k (N^k \pi^{ij})). \]  

To arrive at this form of the equations of motion we have considered the Hamiltonian only with the primary constraints added, as it is shown in Eq. (2.8). The secondary constraints \( H \) and \( C \) can also be added, but one finds that, with suitable boundary conditions, the solution for their corresponding Lagrange multipliers is zero, which is equivalent to drop these constraints out from the Hamiltonian (an extended discussion about this issue in the 3+1 theory can be found in [10]). Thus, the equations of motion (2.15) - (2.17) are the evolution equations of the theory on very general grounds.

Viewed as a problem of initial data, the absence of local degrees of freedom means that the constraints, together with the choice of a gauge to fix the freedom of performing spatial diffeomorphisms, determine the initial data completely. To hold this we assume that some boundary conditions are given. Once the initial data is determined by the constraints, the evolution equations (2.15) – (2.17) give its flow in time, but the freedom of changing functionally this initial data has been already fixed by the constraints and the gauge chosen. In the next section we will see how this works explicitly by means of a perturbative analysis.

### 2.2 Linearized version

It is important to have a way for solving the contraints explicitly and checking that no propagation of free data is allowed. This can be achieved conveniently in the linearized theory. The absence of fluctuations on the Minkowski background provides a clear example. The configuration that is the analogue of the Minkowski space in 2+1 dimensions (in Cartesian coordinates) is given by the setting \( N = 1, g_{ij} = \delta_{ij} \) and \( \pi^{ij} = 0 \), with multipliers \( N_i = \sigma = \mu = 0 \). This is an exact solution of all the constraints and the evolution equations of the theory shown in the previous section. We perform perturbations around this solution by means of the canonical variables

\[ N = 1 + n, \quad g_{ij} = \delta_{ij} + h_{ij}, \quad \pi^{ij} = p^{ij}. \]  

For concretness we assume the asymptotic decay \( h_{ij} \sim O(r^{-1}), p^{ij} \sim O(r^{-2}) \) and \( n \sim O(r^{-1}) \), where \( r = \sqrt{x^i x^j} \).

We introduce the transverse-longitudinal decomposition

\[ h_{ij} = \left( \delta_{ij} - \frac{\partial_i \partial_j}{\Delta} \right) h^T + \partial_i h_{ij}, \]  

where \( \Delta = \delta_{ij} - \partial_i \partial_j \).
where $\Delta$ is the two-dimensional flat Laplacian, and similarly for $p^{ij}$. We impose the transverse gauge $\partial_i h_{ij} = 0$, under which the longitudinal sector of the metric is eliminated, $h_i = 0$.

The linearized momentum constraint (2.9) becomes $\partial_i p^{ij} = 0$, hence $p_i = 0$, whereas the linearized version of the $\pi = 0$ constraint eliminates the transverse scalar mode, $p^T = 0$. Thus, in the linearized theory the canonical momentum is completely frozen. Constraint $\mathcal{C}$, given in Eq. (2.11), becomes $\beta \Delta n = 0$. Since in this perturbative analysis we are considering $n = \mathcal{O}(r^{-1})$, the solution for $n$ is $n = 0$. Finally, the Hamiltonian constraint, given in Eq. (2.10), becomes $\beta \Delta h^T = 0$, and since we consider $h^T = \mathcal{O}(r^{-1})$, this constraint fixes $h^T = 0$. Therefore, all the perturbations of the canonical variables are frozen by the constraints and the choice of a gauge condition, no fluctuations of the background are allowed.

Since all perturbations of the canonical variables are frozen (and the background is static), the evolution equations (2.15) – (2.16) yield no relevant information, except for the fact that they fix the possible fluctuations of the Lagrange multipliers. This includes the shift vector since we have already fixed the gauge symmetry of spatial diffeomorphisms. Let us see this explicitly. We suppose that $N_1$, $\sigma$ and $\mu$ are variables of first order in perturbations. At linear order, Eq. (2.15) fixes $\sigma = 0$, whereas Eq. (2.17) yields no information. The linearized equation (2.16) has three different components, which are equations for the three unknowns $N_1$, $N_2$ and $\mu$. Written in matrix form, these equations are

$$\begin{pmatrix} 2\partial_1 & 0 & 1 \\ 0 & 2\partial_2 & 1 \\ \partial_2 & \partial_1 & 0 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ \mu \end{pmatrix} = 0. \quad (2.20)$$

By multiplying this equation from the left with

$$\frac{1}{2} \begin{pmatrix} \partial_1 & -\partial_1 & 2\partial_2 \\ -\partial_2 & \partial_2 & 2\partial_1 \\ 2\partial_2^2 & 2\partial_1^2 & -4\partial_1\partial_2 \end{pmatrix}, \quad (2.21)$$

we obtain

$$\Delta \begin{pmatrix} N_1 \\ N_2 \\ \mu \end{pmatrix} = 0. \quad (2.22)$$

By assuming that the Lagrange multipliers decay fast enough at infinity, we find that the only solution is $N_1 = N_2 = \mu = 0$. Therefore, also the Lagrange multipliers are frozen in the linearized theory.

### 3 Nontrivial curvature

We want to contrast the previously shown feature that this theory has no local degrees of freedom with its (spatial and spacetime) curvature, according to the field
equations. The curvature can be conveniently study in the Lagrangian formulation, with the action \(2.1\). The equations of motion derived from the action \(2.1\) at the critical point \(\lambda = 1/2\) are

\[
K_{ij}K_{ij} - \frac{1}{2}K^2 + \beta R + \alpha a_i a^i - 2\alpha \frac{\nabla^2 N}{N} = 0, \quad (3.1)
\]

\[
G^{ijkl}\nabla_j K_{kl} = 0, \quad (3.2)
\]

\[
\frac{1}{\sqrt{g}} \frac{\partial}{\partial t} \left( \sqrt{g}G^{ijkl}K_{kl} \right) + 2G^{klm(i}N_l(j)K_{tm)} - G^{ijkl}\nabla_n(N^nK_{kl})
+ 2N(K^i_kK^{jk} - \frac{1}{2}K^2K_{ij}) - \frac{1}{2}Ng^{ij}G^{klmn}K_{kl}K_{mn}
\]

\[
- \beta \left( \nabla^{ij}N - g^{ij}\nabla^2 N \right) + \alpha N \left( a^i a^j - \frac{1}{2}g^{ij}a_k a^k \right) = 0. \quad (3.3)
\]

### 3.1 Newtonian force

We may ask whether there is a place for a Newtonian force in this theory. To address this question we couple the theory to a massive particle at rest and consider that a configuration analogue to the Newtonian potential of general relativity must approach the flat space at infinity, that is, it must be asymptotically flat. Therefore, we require that all the ADM variables are asymptotically flat. In the Appendix we show the extension of the vacuum field equations \(3.1\) – \(3.3\) to the case where there is a massive particle. We consider that the dynamics of the particle has a relativistic form (although this is not a crucial choice since, at the end, the particle is considered at rest). We may impose the gauge \(N_i = 0\) for this analysis. The dynamics of the particle is parametrized by the embedding fields \(q^0, q^i\), which are functions of particle’s worldline. In the equations of motion \(A.3\) – \(A.7\), we consider that the particle and its gravitational field (the \(N\) and \(g^{ij}\) fields) are static. A suitable choice for the location of the particle is \(q^0 = t, q^i = 0\), which means that the particle remains at the origin. Under these settings, Eq. \(A.5\), which is the analogue of the space-space components of the Einstein equations, becomes

\[
\beta \left( \nabla^{ij}N - g^{ij}\nabla^2 N \right) - \frac{\alpha}{N} \left( \nabla^i N \nabla^j N - \frac{1}{2}g^{ij}\nabla_k N \nabla^k N \right) = 0. \quad (3.4)
\]

Since the second term is traceless, the trace of this equation yields

\[
\nabla^2 N = 0. \quad (3.5)
\]

Imposing the asymptotic condition \(N\big|_\infty = 1\), which is appropriated for the Newtonian case, we have that the only solution to this equation is \(N = 1\) everywhere. Since a nonconstant \(-N^2\) would be the analogue of the Newtonian potential, we have that there is no Newtonian force in this theory. Hence, the situation is the same as in \(2 + 1\) general relativity. Indeed, the rest of the analysis for solving the
remaining field equations is parallel to general relativity, and it is also the same 
in the noncritical 2 + 1 Hořava gravity \[11\] (the solution is the same for \( \lambda \neq 1/2 \) and \( \lambda = 1/2 \)). Equation \( (A.4) \) is automatically solved. Particle’s equations of motion, given in Eqs. \( (A.6) - (A.7) \), are automatically solved considering that \( N = 1 \). There remains the analogue of the time-time component of the Einstein equations, Eq. \( (A.3) \). It takes the form

\[
\sqrt{g} R = \frac{2\kappa m}{\beta} \delta^{(2)}(x^i). \tag{3.6}
\]

This equation was solved in \[12\]. Its solution, in polar coordinates, is

\[
ds^2 = r^{\frac{-\kappa m}{\pi\kappa}} (dr^2 + r^2 d\theta^2). \tag{3.7}
\]

The generic geometry is a flat cone with a singularity at the origin (other geometries are possible in the space of parameters \[12, 13\]).

### 3.2 Exact vacuum solutions

We evaluate Eqs. \( (3.1) - (3.3) \) for the case of static configurations and considering the gauge fixing condition \( N_i = 0 \). Equation \( (3.2) \) is automatically solved. Equation \( (3.3) \) takes exactly the form given in \( (3.4) \), with trace \( (3.5) \). Unlike the previous section, in this analysis we do not impose boundary conditions, hence we continue on solving the equations without fixing \( N \) yet. We put together the resulting Eqs. \( (3.1) \) and \( (3.3) \), simplified by \( (3.5) \),

\[
\beta R + \alpha a_k a^k = 0, \tag{3.8}
\]

\[
\beta \nabla_{ij} N - \alpha N \left( a_i a_j - \frac{1}{2} g_{ij} a_k a^k \right) = 0. \tag{3.9}
\]

This is the system of equations that must be solved for static configurations (the trace of Eq. \( (3.9) \) reproduces Eq. \( (3.5) \)).

We introduce the ansatz of a static spatial metric with rotational symmetry, which in polar coordinates is

\[
ds^2 = f^{-1}(r) dr^2 + r^2 d\theta^2, \tag{3.10}
\]

and we assume that the lapse function depends only on the radius, \( N = N(r) \). Under this ansatz, Eq. \( (3.8) \) and the two diagonal components of the Eq. \( (3.9) \) take the form

\[
\beta \frac{f'}{rf} - \alpha \left( \frac{N'}{N} \right)^2 = 0, \tag{3.11}
\]

\[
\beta \frac{N''}{N} + \frac{\beta f' N'}{2 f N} - \alpha \frac{1}{2} \left( \frac{N'}{N} \right)^2 = 0, \tag{3.12}
\]

\[
\frac{N'}{N} \left( \frac{\alpha N'}{N} + \frac{2\beta}{r} \right) = 0. \tag{3.13}
\]
The off-diagonal component of Eq. (3.9) vanishes identically. We will see that Eq. (3.12) is implied by Eqs. (3.11) and (3.13), hence the system is consistent. Equation (3.13) has only two solutions. One is $N' = 0$, hence $N = \text{constant}$, which inserted in Eq. (3.11) gives also $f = \text{constant}$. Equation (3.12) is automatically solved under these conditions. This is a completely flat configuration. The other possibility is the vanishing of the second factor in (3.13),

$$\frac{\alpha N'}{N} + \frac{2\beta}{r} = 0.$$  

(3.14)

This condition requires, by consistency, that $\alpha \neq 0$. The solution for the lapse function $N$ is obtained by the direct integration of (3.14), and then the solution for $f$ by the integration of (3.11). In the integration of $N$, a multiplicative integration constant arises; it has no physical meaning since it can be absorbed by re-scaling the time parameter of the foliation. We put this integration constant equal to 1 for simplicity. Instead, the integration constant that arises by integrating $f$ has physical relevance, we denote it by $r_0$. In this way we get the exact vacuum solution

$$N(r) = r^{-2\beta/\alpha}, \quad f(r) = \left(\frac{r}{r_0}\right)^{4\beta/\alpha}.$$  

(3.15)

Equation (3.12) is solved by this configuration. Functions $N$ and $f$ are singular at the spatial infinity. The singularity could be 0 or $\infty$ depending on the sign of $\beta/\alpha$. Thus, by relaxing the boundary conditions we have found the additional solution (3.15).

We may build a spacetime curvature for this last solution if we assume that a spacetime metric is built with the ADM fields, that is, $(3)g_{00} = -N^2$ and $(3)g_{ij} = g_{ij}$. The main point is that the spacetime curvature is not zero for this solution. Indeed, the only nonzero component of the Ricci tensor is $(3)R_{rr}$, which, together with the Ricci scalar, are

$$(3)R_{rr} = -\frac{4\beta}{\alpha} r^{-2}, \quad (3)R = R = -\frac{4\beta}{\alpha} r^{4\beta/\alpha-2}.$$  

(3.16)

3.3 The critical hypersurface orthogonal 2+1 Einstein-aether theory

The Einstein-aether theory is the coupling of general relativity with a timelike unit vector $u_\mu$, called the aether field. The Lagrangian of the aether field is of second order in derivatives of $u_\mu$. When the aether field is restricted to be hypersurface orthogonal, it can be solved in terms of a scalar function $T$ whose gradient is timelike, in the way

$$u_\mu = \frac{\partial_\mu T}{(-\partial_\alpha T \partial^\alpha T)^{1/2}}.$$  

(3.17)
By substituting this form of the aether field at the level of the definitive action, a gravitational theory coupled to a scalar field with broken local Lorentz symmetry is obtained. It is known as the khronometric theory [14] and the $T$-theory [6]. This theory is physically equivalent to the $z = 1$ truncation of the nonprojectable Hořava theory, which is defined by the action (2.1), including the critical case in the space of coupling constants. The action of the $T$-theory is

$$S_T = \int d^3x \sqrt{-g} \left( R - M^{\alpha\beta\gamma\delta} \nabla_\alpha u_\gamma \nabla_\beta u_\delta \right),$$

where the hypermatrix is defined by

$$M^{\alpha\beta\gamma\delta} = c_1 g^{\alpha\beta} g^{\gamma\delta} + c_2 g^{\alpha\gamma} g^{\beta\delta} + c_3 g^{\alpha\delta} g^{\beta\gamma} + c_4 u^\alpha u^\beta g^{\gamma\delta},$$

and the $c_{1,2,3,4}$ are coupling constants. To define the $T$-theory it is understood that the expression (3.17) is substituted in the above action. Without this restriction the action (3.18) defines the (unrestricted) Einstein-aether theory. One can get the $z = 1$ truncation of the nonprojectable Hořava theory written in its original variables if the gauge condition $T = t$, which fixes the freedom of transforming the time coordinate, is imposed. In this case, the coupling constants used in the definition of the Hořava theory (2.1) arise as the following combinations of the Einstein-aether coupling constants (in the $T$-theory in general only three coupling constants are independent),

$$\beta = \frac{1}{1 - c_1 - c_3}, \quad \lambda = \beta(1 + c_2), \quad \alpha = \beta(1 + c_4).$$

From these relations we see that the critical condition for the degrees of freedom, $\lambda = 1/2$, expressed in terms of the Einstein-aether coupling constants, is

$$c_2 = -\frac{1}{2}(1 + c_1 + c_3).$$

Thus, in the critical $T$-theory, only two combinations of the coupling constants $c_{1,2,3,4}$ are independent.

The field equations of the $T$-theory have been presented in [15, 16, 17]. We write the Einstein equations corresponding to the action (3.18) in the form

$$R_{\mu\nu} = T_{\mu\nu} - g_{\mu\nu} T^\alpha_{\alpha},$$

where the energy-momentum tensor of the $T$ field is

$$T^{\mu\nu} = -\nabla_\alpha \left( J^{[\mu|\alpha u|\nu]} - J^{[\alpha|u|\nu]} - J^{[\mu|u|\alpha]} \right) - c_1 \left( \nabla^\alpha u^\mu \nabla_\alpha u^\nu - \nabla^\mu u^\alpha \nabla^\nu u_\alpha \right) + \left( u_\alpha \nabla_\beta J^{[\gamma\beta|a]} - c_4 a^\alpha a^\alpha \right) u^\mu u^\nu - c_4 a^\mu a^\nu - \frac{1}{2} g^{\mu\nu} M^{\alpha\beta\gamma\delta} \nabla_\alpha u_\gamma \nabla_\beta u_\delta + 2 \left( \delta^\mu_\lambda u^\nu + u^\mu u^\nu u_\lambda \right) \left( \nabla_\gamma J^{[\gamma\lambda] - c_4 a_\gamma \nabla^\lambda u^\gamma} \right),$$

(3.23)
with
\[ a^\mu \equiv u^\rho \nabla_\rho u^\mu, \quad J^{\mu\nu} \equiv c_1 \nabla^{\mu} u^{\nu} + c_2 g^{\mu\nu} \nabla_\rho u^\rho + c_3 \nabla^{\nu} u^{\mu} + c_4 u^\mu a^\nu. \tag{3.24} \]

We recall that in these expressions \( u_\mu \) must be substituted by (3.17). The variation of the action (3.18) with respect to the scalar field \( T \) leads to the equation
\[ \nabla_\rho \left( \left( \delta_\lambda^\rho + u_\lambda u^\rho \right) \nabla_\alpha \left( M^{\alpha\beta\lambda\delta} \nabla_\beta u_\delta \right) \right) = 0. \tag{3.25} \]

This equation is implied by the Einstein equations (3.22), see [6], hence we do not need to consider it anymore.

Our aim is to evaluate the field equations on the critical condition (3.21). Although this can be done directly in the form that we have presented the energy-momentum tensor, the resulting expression can be a bit obscure since there are still redundant coupling constants when the notation of the \( c_{1,2,3,4} \) is used, even after substituting the critical condition (3.21). However, the expressions become clear when we consider a specific ansatz for the spacetime metric and impose the gauge condition \( T = t \), allowing that the two independent coupling constants of the Hořava theory at the critical point, \( \beta \) and \( \alpha \), emerge naturally. We consider the case of a spacetime metric with orthogonality between the time direction and the spatial section, but otherwise arbitrary. Thus, this example represents a broad class of spacetime metrics. It is given by
\[ ds^2 = -N^2(t, \vec{x}) dt^2 + \Omega^2(t, \vec{x}) \left( dx^2 + dy^2 \right). \tag{3.26} \]

Without loss of generality, we have written the spatial section in an explicit conformally flat form. According to (3.23), the components of the energy-momentum tensor become
\[ (T_{\mu\nu} - g_{\mu\nu} T_\alpha^\alpha)_{tt} = \frac{2}{N \Omega} \left( N \ddot{\Omega} - \dot{N} \dot{\Omega} \right), \tag{3.27} \]
\[ (T_{\mu\nu} - g_{\mu\nu} T_\alpha^\alpha)_{ti} = -\frac{\partial_i \ddot{\Omega}}{\Omega} + \frac{\dot{\Omega} \partial_i \Omega}{\Omega^2} + \frac{\partial_i N \dot{\Omega}}{N \Omega}, \tag{3.28} \]
\[ (T_{\mu\nu} - g_{\mu\nu} T_\alpha^\alpha)_{ij} = \frac{\delta_{ij}}{N^2} \left( \Omega \ddot{\Omega} + \dot{\Omega}^2 - \frac{\dot{N}}{N} \Omega \dot{\Omega} + \frac{\alpha}{\beta} N \Delta N \right) - \frac{\alpha}{\beta} \frac{\partial_i N \partial_j N}{N^2}, \tag{3.29} \]

where we have used the relations (3.20) between the coupling constants of the Einstein-aether theory and the ones of the Hořava theory. Our aim with these expressions is to show that, given a specific solution of the field equations (3.22), its curvature in general is given by the formulas (3.27) – (3.29). We remark that the ansatz (3.26) includes the static configurations with rotational symmetry that we study in Section 3.2, where we find a solution with nonzero curvature. On
the solution (3.15), fixing $r_0 = 1$ for simplicity, we may perform the coordinate transformation

$$r = (\nu \ln(x^2 + y^2))^{1/2 \nu}, \quad \tan \theta = \frac{y}{x}, \quad \nu \equiv -\frac{\beta}{\alpha}. \quad (3.30)$$

Depending on the sign of $\nu$, the domain of the transformation must be restricted such that one can have $\nu \ln(x^2 + y^2)$ positive. For each sign of $\nu$ there is range of validity in $(x, y)$. With this transformation the solution (3.15) acquires the form (3.26), with the new lapse function and conformal factor

$$N = \nu \ln(x^2 + y^2), \quad \Omega = \frac{(\nu \ln(x^2 + y^2)^{1/2 \nu}}{\sqrt{x^2 + y^2}}. \quad (3.31)$$

The spacetime Ricci scalar in this coordinates takes the form

$$R = \frac{4\nu}{(\nu \ln(x^2 + y^2))^{1/2 + 2}}. \quad (3.32)$$

Therefore, the configuration (3.31) is an example that nonflat solutions of the Einstein equations (3.22) with the energy-momentum tensor (3.27) – (3.29) do exist.

We can explore further on other possible solutions yielded by the ansatz (3.26). To pose the Einstein equations (3.22), we equate the Ricci tensor of the metric (3.26) to the energy-momentum tensor (3.27) – (3.29). The

$$\Delta N = 0, \quad (3.33)$$

$$\frac{\Delta \Omega}{\Omega} - \frac{\partial_k \Omega \partial_k \Omega}{\Omega^2} - \frac{\partial_x N \partial_x \Omega - \partial_y N \partial_y \Omega}{N \Omega} + \frac{\partial^2_{xx} N}{N} - \frac{\alpha}{\beta} \left( \frac{\partial_x N}{N} \right)^2 = 0, \quad (3.34)$$

$$\frac{\Delta \Omega}{\Omega} - \frac{\partial_k \Omega \partial_k \Omega}{\Omega^2} + \frac{\partial_x N \partial_x \Omega - \partial_y N \partial_y \Omega}{N \Omega} + \frac{\partial^2_{yy} N}{N} - \frac{\alpha}{\beta} \left( \frac{\partial_y N}{N} \right)^2 = 0, \quad (3.35)$$

$$\frac{\partial_x N \partial_y \Omega + \partial_y N \partial_x \Omega}{N \Omega} - \frac{\partial^2_{xy} N}{N} + \frac{\alpha}{\beta} \frac{\partial_x N \partial_y N}{N^2} = 0. \quad (3.36)$$

The time-space components are automatically solved. We highlight some relevant facts. The first one is that all time derivatives disappear from the Einstein equations. This is in agreement with the fact that in this theory there is no wave equation for any propagating mode. The second one is that Eq. (3.33) has reduced to the condition of harmonicity on the lapse function. This condition arises when we study static configurations in Section 3.2. Now we see that it is a more general feature of the class of orthogonal metrics given by (3.26).

We can combine Eqs. (3.33) – (3.36) with boundary conditions. If we assume that $N$ goes fast enough to the fixed value $N = 1$ at spatial infinity, which is a requisite for asymptotic flatness, we have that the only solution of (3.33) is $N = 1$.
everywhere. After this result is inserted in Eqs. (3.34) – (3.36), only one equation remains to be solved, namely,

$$\Omega \Delta \Omega - \partial_k \Omega \partial_k \Omega = 0.$$  \hspace{1cm} (3.37)

Suppose that also $\Omega$ goes to a fixed value at spatial infinity fast enough such that we can ignore boundary terms when we integrate by parts in Eq. (3.37). Then, the spatial integration of the Eq. (3.37) yields

$$-2 \int d^2 x \partial_k \Omega \partial_k \Omega = 0.$$  \hspace{1cm} (3.38)

The only solution of this equation is $\partial_i \Omega = 0$, hence $\Omega = \text{constant}$, with the value of the constant given at infinity. Therefore, if $N$ and $\Omega$ satisfy the asymptotic condition that they go fast enough to fixed values, the only solution found in the class of orthogonal metrics (3.26) is the totally flat configuration. Of course, this result does not forbids that other solutions arise with different asymptotic behavior, like the exact solution (3.31).

**Conclusions**

We have seen that the critical condition on the nonprojectable Hořava theory in two spatial dimensions leads to somewhat unexpected features. We have studied the large-distance effective action, which has no higher order derivatives, such that these effects are presents at the level of second order in derivatives, the same order as general relativity. We have shown, by means of a rigorous canonical analysis, that this formulation of the Hořava theory has no propagating degrees of freedom. Nevertheless, we have seen that it admits exact vacuum solutions with nontrivial curvature. We have presented an specific nonflat solution, which is static and with rotational symmetry.

Although this is the general result, the theory still shares remarkable features with $2 + 1$ general relativity. Flat solutions persist, in the sense that we have found them by assuming the standard boundary condition for isolated sources, which is the condition of asymptotic flatness. We have seen that an asymptotically flat Newtonian potential that would be generated by a particle at rest does not exist, like $2 + 1$ general relativity (the solution in general is the same flat cone of $2 + 1$ general relativity). More generally, boundary conditions seems to play a key role in the arising of nonflat solutions. In the equivalent framework of the hypersurface orthogonal Einstein-aether theory, formulated in the critical condition, we have considered an ansatz for the spacetime metric limiting the geometry only by the orthogonality between the time direction and the spatial section. In this setting we have found again that the condition of asymptotic flatness leads exclusively to flat solutions. On the other hand, the nonflat exact solution we have found is not asymptotically flat. Indeed, the lapse function and the radial component of the
spatial metric are singular at the spatial infinity. Thus, we are led to think that it could be a general relationship between the asymptotically flat condition and the exclusiveness of the flat solutions. It would be interesting to elucidate if the condition of asymptotic flatness leads always to flat solutions, without assuming any previous restriction on the geometry. If this is proved to be true, then the class of asymptotically flat solutions of the $2+1$ nonprojectable Hořava theory at the critical point is the same class of solutions of $2+1$ general relativity. Leaving this issue aside, we have seen that the theory admits nonflat solutions that are not asymptotically flat.

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**A The coupling to a massive particle**

We assume that the dynamics of the particle is governed by an action of relativistic nature. Hence we assume that the action of the particle is proportional to the length of its worldline, embedded in a spacetime ambient. The ambient is taken from a solution of the Hořava theory. We choose the time coordinate $t$ of the ambient foliation to parameterize the worldline of the particle. The mechanics of the particle is characterized by the embedding fields $q^0 = q^0(t)$ and $q^i = q^i(t)$, which define the position of the particle in the foliation. Thus, the combined system critical $2+1$ Hořava gravity–point particle is given by the action

\[
S = \frac{1}{2\kappa} \int dt d^2x \sqrt{g} N \left( K_{ij} K^{ij} - \frac{1}{2} K^2 + \beta R + \alpha a_k a^k \right) - m \int dt \sqrt{L}, \tag{A.1}
\]

where

\[
L = (N^2 - N_k N^k) \left( \dot{q}^0 \right)^2 - 2 N_k \dot{q}^0 \dot{q}^k - g_{kl} \dot{q}^k \dot{q}^l, \tag{A.2}
\]

$m$ is the mass of the particle and $\kappa$ is a coupling constant. $L$ is the squared line element of the particle evaluated on the background of the ADM variables, and these variables are evaluated at the position of the particle in $L$.

The field equations of the ADM variables are obtained by varying the action \[2.1\] with respect to them. We shown the resulting equations directly evaluated on
the gauge $N_i = 0$, 

\[ K^{ij}K_{ij} - \frac{1}{2}K^2 + \beta R + \alpha a_i a^i - 2\alpha \nabla^2 N = 2\kappa m \frac{\langle q^0 \rangle^2 N}{{\sqrt g}L} \delta^{(d)}(x^k - q^k), \quad (A.3) \]

\[ G^{ijkl}\nabla_j K_{kl} = -\frac{\kappa m}{{\sqrt g}L} \dot q^0 \dot q^i \delta^{(d)}(x^m - q^m), \quad (A.4) \]

\[ \frac{1}{\sqrt g} \frac{\partial}{\partial t} \left( \sqrt g G^{ijkl} K_{kl} \right) + 2N(K^i_k K^{j}{}^k - \frac{1}{2}K K^{ij}) - \frac{1}{2}Ng^{ij}G^{klmn}K_{kl}K_{mn} \]

\[ - \beta \left( \nabla^i N - g^{ij} \nabla^2 N \right) + \frac{\alpha}{N} \left( \nabla^i N \nabla^j N - \frac{1}{2}g^{ij} \nabla_k N \nabla^k N \right) \]

\[ = \frac{\kappa m}{{\sqrt g}L} \dot q^i \dot q^j \delta^{(d)}(x^m - q^m). \quad (A.5) \]

In the $N_i = 0$ gauge we have that $K_{ij} = \dot g_{ij}/2N$. The equations of motion corresponding to the variations of the coordinates of the particle are

\[ q'''' + \Gamma_k^i q'' q' + \frac{1}{2} \nabla^2 N^2 \langle q^{0r} \rangle^2 = 0, \quad (A.6) \]

\[ \frac{2}{\sqrt L} \frac{d}{dt} N^2 q^{0r} + \partial_0 N^2 \langle q^{0r} \rangle^2 - \partial_0 g_{ij} q^i q^j = 0, \quad (A.7) \]

where the prime means

\[ \psi' \equiv \frac{1}{\sqrt L} \frac{\partial \psi}{\partial t}. \quad (A.8) \]

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