Network Quantum Steering

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The development of large-scale quantum networks promises to bring a multitude of technological applications as well as shed light on foundational topics, such as quantum nonlocality. It is particularly interesting to consider scenarios where sources within the network are statistically independent, which leads to so-called network nonlocality, even when parties perform fixed measurements. Here we promote certain parties to be trusted and introduce the notion of network steering and network local hidden state (NLHS) models within this paradigm of independent sources. In one direction, we show how results from Bell nonlocality and quantum steering can be used to demonstrate network steering. We further show that it is a genuinely novel effect, by exhibiting unsteerable states that nevertheless demonstrate network steering, based upon entanglement swapping, yielding a form of activation. On the other hand, we provide no-go results for network steering in a large class of scenarios, by explicitly constructing NLHS models.

The quest to deepen our understanding of quantum theory and its seemingly counter-intuitive properties has lead to many fruitful avenues of research. In particular, the phenomenon of quantum correlations have enjoyed significant attention and developments, see e.g. [1–3].

Quantum correlations expose a rich structure when considered in scenarios with many parties. A case of particular interest is that of quantum networks, featuring a number of distant parties connected by several quantum sources. Significant further work is still required to reach a deeper theoretical understanding of these scenarios, whilst also keeping inline with experimental and technological developments towards quantum networks [4].

Recently, a generalisation of the concept of Bell locality [5] was proposed to tackle the question of quantum nonlocality in networks; see [6] for a recent review. The key idea is to consider the various sources in the network to be statistically independent [7–9]. This independence leads to non-convexity in the space of relevant correlations, undermining the use of pre-existing tools and creating a need for new approaches, both analytically [10–18] and numerically [19]. The network structure offers new interesting effects, such as the possibility to certify quantum nonlocality “without inputs” (i.e. a scenario where each party performs a fixed quantum measurement) [8, 9, 20–22]. Also, the use of non-classical measurements allows for novel forms of quantum nonlocal correlations that are genuine to networks [23]. In parallel, several works have explored the structure of quantum states assuming a certain underlying network structure [24–27].

In this work, motivated by the difficulty in characterising quantum networks both conceptually and computationally, we consider quantum network scenarios in which some of the parties are trusted while the others are untrusted. This naturally connects to the notion of quantum steering [28] (see [2, 3] for reviews) which captures quantum correlations in a scenario involving a trusted and an untrusted party. While the notion of multipartite steering has been previously considered [29, 30], our work explores a different direction, targeting the scenario of networks with independent sources.

Our main focus here will be on the simplest setting of a linear network with trusted endpoints and intermediate untrusted parties who each perform a fixed measurement. We begin by formalising the notions of network local hidden state (NLHS) models, and network steering. We then leverage standard steering and nonlocality scenarios to provide simple examples of network steering. Next, we outline a surprising effect in which two-way unsteerable states can demonstrate network steering through entanglement swapping, leading to a form of activation. Finally, we characterise some natural scenarios that always admit an NLHS model by identifying properties of the sources. We conclude by listing some promising future avenues for research.

Basic concepts. — We first briefly summarise the notion of steering, as it represents the basis of what is to follow.

In a (bipartite) steering scenario, one party performs measurements on a shared state \( \rho^{AB} \), which ‘steers’ the quantum state of the other particle. If Alice performs a set of measurements, labelled by \( x \), with outcomes \( a \), and corresponding POVM elements \( M_{a|x} \), then the collection of sub-normalised ‘steered states’ of Bob are \( \sigma_{a|x} := \text{Tr}_A(M_{a|x}^{A} \otimes I_B \rho^{AB}) \), where \( p(a|x) = \text{Tr}(\sigma_{a|x}) \) are the statistics of Alice’s measurements. The collection of sub-normalised states \( \{\sigma_{a|x}\}_{a,x} \) are commonly referred to as an assemblage [31]. If the assemblage can be explained by a local hidden state (LHS) model, of the form \( \sigma_{a|x} = \sum_{\lambda} p(\lambda) \rho(a|x, \lambda) \sigma_{\lambda} \), where \( \lambda \) is a hidden variable, distributed according to \( p(\lambda) \), \( \sigma_{\lambda} \) are ‘hidden states’ of Bob, and \( p(a|x, \lambda) \) are local ‘response functions’ of Alice, then we say that it has LHS form, or does not demonstrate steering [28]. If there exist measurements such that \( \sigma_{a|x} \) does not admit such an LHS decomposition, we say that the state \( \rho^{AB} \) is steerable from \( A \) to \( B \). If for all measurements we can never demonstrate steering with a
given state, we say it is unsteerable (from A to B) [32].

Network Steering.— We will now introduce our main new notion, that of network steering. Here, we have a collection of independent sources which distribute quantum states to a subset of parties. In the standard network nonlocality scenario all parties are assumed to be untrusted, and to perform ‘black-box’ measurements. Here, in contrast, inspired by the steering scenario, we will consider only a subset of the parties to be untrusted, and the remainder trusted. We will be interested in the (sub-normalised) states that are prepared for the trusted parties by the measurements of the untrusted parties. We refer to this general set-up as network steering.

We focus primarily on a simple scenario, with \( n \) parties arranged in a line, where the endpoint parties are trusted, and intermediate parties are untrusted and each perform a single, fixed measurement. The simplest such scenario has three parties and two sources (see Fig. 1e), as in entanglement swapping [33]. Here the first two parties share a state \( \rho^{AB} \) and the second and third parties share a state \( \rho^{BC} \), and the central party performs a fixed measurement \( M_B^{BB'} \). The sub-normalised states prepared for A and C by this measurement are

\[
\sigma_b^{AC} = \text{Tr}_{BB'} \left( \left[ 1^A \otimes M_B^{BB'} \otimes I_C \right] \rho^{AB} \otimes \rho^{BC} \right),
\]

which occur with probability \( p(b) = \text{Tr}(\sigma_b^{AC}) \). We will refer to \( \{\sigma_b\}_b \) as a network assemblage.

In order to determine when this network assemblage demonstrates network steering we need to introduce the notion of a network local hidden state (NLHS) model, which takes the form

\[
\sigma_b^{AC} = \sum_{\beta, \gamma} p(\beta) p(\gamma) p(b|\beta, \gamma) \sigma^A_\beta \otimes \sigma^C_\gamma,
\]

where \( \beta \) and \( \sigma^A_\beta \) are the hidden variable and hidden states of the first source, \( \gamma \) and \( \sigma^C_\gamma \) those of the second source, and \( p(b|\beta, \gamma) \) the local response function of Bob. If there is no such model that can explain the network assemblage \( \sigma_b \), then we say it demonstrates network steering. Interestingly, whereas conventional quantum steering requires multiple measurements to be performed by the untrusted party, just as with network nonlocality, we shall see here that even a fixed measurement can suffice to demonstrate network steering.

We note first in (2) that each \( \sigma_b^{AC} \) is in fact separable. Thus the presence of entanglement in any single \( \sigma_b \) suffices to rule out an NLHS model, and therefore demonstrates network steering.

The above generalises in a natural way to the \( n \)-party line network depicted in Fig. 1f, with outcomes \( b_2, \ldots, b_{n-1} \). We explicitly include the straightforward generalisation of (1) and (2) in Appendix C, and see that the following observation holds generally:

**Observation 1.** For any linear network with trusted endpoints, the entanglement of a single \( \sigma_{b_2, \ldots, b_{n-1}} \) is sufficient to rule out an NLHS model, and thus demonstrate network steering.

For more general networks, we can represent them as undirected graphs, where each node is either untrusted or trusted, and the edges represent independent sources. If all the parties are untrusted, the quantity of interest is the observed statistics \( p(a, b, \ldots | x, y, \ldots) \). When at least one party is trusted this is replaced by some network assemblage \( \sigma_{a,b,\ldots | x,y,\ldots} \). A key observation that will prove useful is the following equivalence between networks, a generalisation from the network nonlocality case [9]:

**Observation 2.** Any network with an untrusted party \( A \) that has an input \( x \), received with probability \( p(x) \), and outcome \( a \), is equivalent to a network with an additional untrusted party \( A' \) who shares an additional source with \( A \), neither of whom now has an input. In this new network, the outcome of \( A' \) is \( x \), the old input of \( A \). The relation between the network assemblages in the first and second scenarios are \( p(x) \sigma_{a,x,\ldots}^{A'} = \sigma_{a,x,\ldots} \).

By virtue of the fact that quantum mechanics admits local tomography, we also note the following:

**Observation 3.** A trusted party connected to \( n \) independent sources can without loss of generality be replaced by \( n \) endpoint trusted parties, each connected to a single source.

This allows us, for example, to interpret linear networks as rings with a single trusted party – e.g. the four party linear network with trusted endpoints can also be viewed as the triangle network where one of the parties is trusted, as in Figs. 1c and 1d. This observation motivates our choice to focus our discussion on linear networks, which
we understand now to be relevant for more complex, non-linear networks. We detail further basic observations in Appendix C.

**Demonstrating Network Steering.** — We now begin our exploration of demonstrating network steering, and explain how and when steerable states will lead to network steering when placed in a network. We consider first the scenario of Fig. 1e. If one source distributes a state which is steerable in the standard steering scenario, then Observation 2 would seem to indicate that even if the second source distributes only separable states (which we will refer to as a separable source), it should still be possible to use this to encode ‘the input’ to the measurement, and thus demonstrate network steering. Here we make this intuition precise.

Consider network scenario depicted in Fig. 1b, with two untrusted parties without inputs steering a third, leading to a network assemblage $\sigma_{a,x}$. Here the NLHS condition reads

$$\sigma_{a,x} = \sum_{\beta,\gamma} p(\beta)p(\gamma) \ p(x|\beta) p(a|\beta,\gamma) \sigma_{\gamma}. \quad (3)$$

We can then observe the following:

**Claim 1.** If $\sigma_{a,x}$ has an NLHS model, then $\sigma_{a|x} := \sigma_{a,x}/p(x)$ has an LHS model, where $p(x) = \text{Tr} \sum_a \sigma_{a,x}$.

**Proof.** We can write (3) as

$$p(x) = \text{Tr} \sum_{a,x} \sigma_{a,x} = \sum_{\beta} p(\beta) \ p(x|\beta) \ p(a|\beta,\gamma).$$

This follows from [36], as for any state $|\psi> = (|01> - |10>)/\sqrt{2}$, and $\Lambda_\eta(\rho) = \eta \rho + (1 - \eta) |\gamma><\gamma|$, where $|\gamma> = |10>$. The result then follows. \qed

The above constructions of network steering relied on steering or nonlocality in standard scenarios. Here we show that network steering is possible even when using only (two-way) unsteerable states, which can be viewed as a form of activation. Note that this complements previous examples of activation of steering in the standard bipartite scenario [35].

We define the Doubly-Erased Werner (DEW) state as the two-qubit Werner state after both subsystems have undergone an identical erasure channel:

$$\rho_{\text{DEW}}(\eta, \omega) := \Lambda_\eta \otimes \Lambda_\omega \left( |\psi^\perp>\!\!\!<\psi^\perp| + (1 - \omega) |\eta><\eta| \right), \quad (7)$$

We can then observe the following:

**Claim 2.** If $\sigma_b$ has an NLHS model, then $\sigma_{b|x} := \text{Tr}_A([|x>|\otimes I^C]\sigma_b)$ has an NLHS model, for any measurement $M_b$.

**Proof.** When $\sigma_b$ has an NLHS model of the form (2), it follows that

$$\sigma_{b|x} = \sum_{\beta,\gamma} p(\beta)p(\gamma) \text{Tr}(M_b \sigma_{\beta}) p(x|\beta) \sigma_{\gamma}, \quad (5)$$

which is an NLHS model of the form (3), with $p(x|\beta) := \text{Tr}(M_b \sigma_{\beta})$. \qed

Putting this together, suppose that $\rho^{BC}$ is steerable, such that $\sigma_{b|x} := \text{Tr}_A([M_{b|x}\otimes I^C]\rho^{BC})$ demonstrates steering for some $M_{b|x}$. Let $\rho^{AB} = \sum_x \frac{1}{d} |x><x| \otimes |x><x|$ where $d$ is the number of measurements $x$, and $\{ |x> \}_x$ form an orthonormal basis, and $M_b = \sum_x |x><x| \otimes M_{b|x}$. The resulting network assemblage $\sigma_b$, from (1), is seen to be

$$\sigma_b = \sum_x \frac{1}{d} |x><x| \otimes \sigma_{b|x}. \quad (6)$$

Now, from the above claims we can see that this must demonstrate network steering. Indeed, if instead it had an NLHS model, then from Claim 2, $\sigma_{b|x} := \text{Tr}_A([|x>|\otimes I^C]\sigma_b) = \frac{1}{2} \sigma_{b|x}$ would have an NLHS model with $p(x) = 1/d$. Then, from Claim 1, $\sigma_{b|x}$ would have an LHS model, but by assumption it does not. This shows that all steerable states lead also to network steering when placed in a network with an appropriate separable state. Interestingly, this occurs even though $\sigma_b$ is separable.

**Simulated Algorithms** apply for showing that in the line with four parties from Fig. 1d, we can always demonstrate network steering when the central state is nonlocal, and the adjacent endpoint sources are suitable separable states, providing the inputs. We link this to use this to encode ‘the input’ to the measurement, and thus demonstrate network steering. Here we make this intuition precise.
Figure 2: Classifying the structure of some NLHS models. Green circles represent trusted parties, and red squares represent untrusted parties who perform a fixed measurement. (a) In the scenario of Fig. 1e, when one source is separable (SEP), this acts as an input to the adjacent measurements, and by taking the second source as unsteerable (UNS) then this always leads to an NLHS model. (b) Similar results hold in the “unwrapped” triangle scenario (Fig. 1d), where now sources can also be taken as local, (LOC). We expand and detail this further in Appendix A.

noted above, that the element $\sigma_{0,0,0}$ (corresponding to a successful swap in each case), will be proportional to the state $\rho_{\text{dep}}(\eta, \omega')$ with $\omega' > \frac{1}{3}$, and therefore entangled. From Observation 1, this precludes an NLHS model description, and therefore demonstrates network steering, even though each DEW state was unsteerable.

Simple NLHS models. — We finish our exploration by considering to what extent the properties of the quantum sources directly affect the possibility of an NLHS model. We will refer to a source as being separable, unsteerable or local if it is only capable of generating separable, unsteerable or local states respectively. As an illustrative example, in the three-party scenario of Fig. 1e if one source is separable and the other source is unsteerable (towards the trusted party), then for any fixed central measurement the network assemblage $\sigma_b$ will always be NLHS. Indeed taking $\rho^{AB} = \sum_\gamma p(\gamma) \sigma^A_\gamma \otimes \sigma^B_\gamma$, and inserting into (1) gives

$$\sigma_b = \sum_\gamma p(\gamma) \sigma^A_\gamma \otimes \text{Tr}_{B'}\left(M_b \otimes \mathbb{I}_C \right) \sigma^B_\gamma \otimes \rho^{B'C}.$$ (8)

Defining $M_{b|\gamma} := \text{Tr}_B\left(M_b[\sigma^B_\gamma \otimes \mathbb{I}_{B'}]\right)$ which form a set of valid measurement operators leads us to write

$$\sigma_b = \sum_\gamma p(\gamma) \sigma^A_\gamma \otimes \text{Tr}_{B'}\left[M_{b|\gamma} \otimes \mathbb{I}_C \right] \rho^{B'C}.$$ (9)

If $\rho^{B'C}$ is unsteerable from $B'$ to $C$, this allows us to extract a LHS model, yielding

$$\sigma_b = \sum_\gamma p(\gamma) \sigma^A_\gamma \otimes \left(\sum_\lambda p(\lambda) p(b|\lambda, \gamma) \sigma^C_\lambda\right)$$ (10)

$$= \sum_\gamma p(\gamma) p(b|\lambda) p(b|\lambda, \gamma) \sigma^A_\gamma \otimes \sigma^C_\lambda,$$ (11)

which is an NLHS model (2). Hence the combination of a separable and unsteerable source (to the trusted party) can never lead to network steering, as shown in Fig. 2a.

Similar results follow in more complicated scenarios. In Fig. 2 (b) we give the three configurations which always lead to NLHS models in the (unwrapped) triangle scenario of Fig. 1d, and we give further generalisations for the line scenario of Fig. 1f in Appendix A. The main concept behind all of these results is that separable and unsteerable sources provide a form of input, allowing us to write down large classes of non-trivial NLHS models.

Conclusions. — We have introduced the notions of network steering and network local hidden state models. We discussed illustrative examples, and showed that the network scenario leads to a form of activation of steering. Finally, we have started a characterisation of NLHS models based solely upon properties of the sources. There are many fascinating and novel future questions to tackle.

First, it would be interesting to determine if either NLHS assemblages or the full set of network assemblages can be characterized via techniques based on semi-definite programming, using for instance the approach of [18]. A related direction is to further classify NLHS models based on the properties of the sources. For instance, consider four parties sharing separable, local and unsteerable sources, or five parties sharing separable, local, local, and separable sources. In neither of these cases do we currently know if network steering can arise or not.

Here we have focused primarily on the properties of the sources, but it would also be interesting to consider the measurements, and understand which of their properties (e.g. entanglement or incompatibility) are relevant for network steering. Future work could also consider the significance of our work for quantum repeaters [4], explore links with superactivation of quantum steering [35], or extend recent work on post-quantum steering [38] to this setting.

Finally, our initial motivation for this work was to attempt to gain clarity on network nonlocality problems, such as those in the triangle network. It is our hope that developing our framework further will lead to discovering novel nonlocal correlations, unique to networks.

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Appendix A: Network Steering and NLHS models on the line

1. The Simplest Scenario

In the main text we mainly discuss the scenario with three parties and trusted endpoints. Here we will extend and generalise this, first to four parties on the line, which we can also interpret as the triangle with a single trusted party. We then generalise our discussion to lines (equivalently, rings) of arbitrary length. To fix notation, Greek subscripts will denote random (hidden) variables (such as $\rho$), $\alpha$ or $\sigma$, or the measurement $M$. Subsystem labels will be denoted by superscripts, for example $\rho^{AB}$ is a quantum state on subsystems $A$ and $B$, and $M_{B'}^{BP}$ is a measurement on systems $B$ and $B'$ (with outcomes $b$) – here subsystems $B$ and $B'$ are implicitly assumed to belong to the same party.

Recalling the 3 party simple scenario, such a scenario is described through quantum mechanics as the existence of
quantum sources $\rho^{AB}, \rho^{B'C}$ and a fixed measurement on the central party $M^{BB'}_b$ such that the resulting set of states can be written as

$$\sigma^A_b = \text{Tr}_{BB'}\left(\left[1^A \otimes M^{BB'}_b \otimes 1^C\right] \rho^{AB} \otimes \rho^{B'C}\right).$$

(A1)

As in the main text, our definition of a NLHS model here is the existence of probability distributions $p(\alpha), p(\gamma)$ and $p(b|\alpha, \gamma)$, and normalised states $\sigma^A_\alpha, \sigma^C_\gamma$ such that

$$\sigma^A_{\alpha} = \sum_{\alpha, \gamma} p(\alpha)p(\gamma)p(b|\alpha, \gamma) \sigma^A_\alpha \otimes \sigma^C_\gamma.$$

(A2)

2. The Triangle Scenario

We can naturally extend this to the line with four parties and trusted endpoints, equivalently viewing this as the triangle network with a single trusted party (Figure 3).

Here the quantum description would be

$$\sigma_{b,c}^{AD} = \text{Tr}_{BB'C'C'}\left(\left[1^A \otimes M_b^{BB'} \otimes M_c^{CC'} \otimes 1^D\right] \rho^{AB} \otimes \rho^{B'C} \otimes \rho^{C'D}\right),$$

(A3)

and the network assemblage $\sigma_{b,c}^{AD}$ would admit an NLHS description if it could be written in the form

$$\sigma_{b,c}^{AD} = \sum_{\alpha, \beta, \gamma} p(\alpha)p(\beta)p(\gamma)p(a|\beta, \gamma)p(b|\alpha, \gamma) \sigma^A_\alpha \otimes \sigma^D_\beta.$$

(A4)

We will now consider how NLHS models can naturally arise by considering properties of the three sources. If the central source is separable, i.e. $\rho^{B'C} = \sum_\gamma p(\gamma)\sigma^B_\gamma \otimes \sigma^C_\gamma$. Inserting this into Equation (A3) yields

$$\sigma_{b,c}^{AD} = \text{Tr}_{BB'C'C'}\left(\left[1^A \otimes M_b^{BB'} \otimes M_c^{CC'} \otimes 1^D\right] \rho^{AB} \otimes \rho^{B'C} \otimes \rho^{C'D}\right)$$

$$= \sum_\gamma p(\gamma) \text{Tr}_{BB'C'C'}\left(\left[1^A \otimes M_b^{BB'} \otimes M_c^{CC'} \otimes 1^D\right] \rho^{AB} \otimes \sigma^B_\gamma \otimes \sigma^C_\gamma \otimes \rho^{C'D}\right)$$

$$= \sum_\gamma p(\gamma) \text{Tr}_{BB'}\left(\left[M_b^{BB'} \otimes 1^A\right] \rho^{AB} \otimes \sigma^B_\gamma\right) \otimes \text{Tr}_{CC'}\left(\left[M_c^{CC'} \otimes 1^D\right] \sigma^C_\gamma \otimes \rho^{C'D}\right)$$

$$= \sum_\gamma p(\gamma) \text{Tr}_{B}\left(\left[M^{B}_{b|\gamma} \otimes 1^A\right] \rho^{AB}\right) \otimes \text{Tr}_{C'}\left(\left[M^{C}_{c|\gamma} \otimes 1^D\right] \rho^{C'D}\right),$$

(A8)
where we defined $M_{bb_\gamma}^B := \text{Tr}_B \left( M_b^{BB'} I^B \otimes \sigma_{\gamma}^{B'} \right)$ and $M_{c_\gamma}^C := \text{Tr}_C \left( M_{CC'}^{CC} \sigma_{\gamma}^{C} \otimes I^C \right)$ as valid sets of measurements.

Then if $\rho^{AB}$ and $\rho^{C'D}$ are unsteerable towards $A$ and $D$ respectively (but possibly entangled: see the main text and [3]), we can extract a local hidden state (LHS) model to obtain

$$
\sigma_{bc}^{AD} = \sum_\gamma p(\gamma) \left( \sum_\alpha p(\alpha) p(b|\alpha, \gamma) \sigma_\alpha^A \right) \otimes \left( \sum_\beta p(\beta) p(c|\beta, \gamma) \sigma_\beta^D \right) 
$$

(A9)

$$
= \sum_{\alpha,\beta,\gamma} p(\alpha)p(\beta)p(\gamma) p(b|\alpha,\beta) p(c|\beta,\gamma) \sigma_\alpha^A \otimes \sigma_\beta^D 
$$

(A10)

which has exactly the same form as the NLHS condition in Equation (A4). Therefore taking $\rho^{AB}$ as separable and $\rho^{AB}$ and $\rho^{C'D}$ unsteerable towards $A$ and $D$ respectively, we will always arrive at an NLHS model, for any intermediate measurements $M_{bb_\gamma}^B$ and $M_{c_\gamma}^C$.

Similarly suppose now that the source $\rho^{AB} = \sum_\alpha p(\alpha) \sigma_\alpha^A \otimes \sigma_\beta^B$ is separable. Then we find

$$
\sigma_{bc} = \sum_\alpha p(\alpha) \sigma_\alpha^A \otimes \text{Tr}_{BB'CC'} \left( \left[ M_{bb_\gamma}^{BB'} \otimes M_{c_\gamma}^{CC'} \right] \sigma_\alpha^B \otimes \rho^{BC'} \otimes \rho^{C'D} \right) 
$$

(A11)

If $\rho^{C'D}$ is also separable, and $\rho^{BC'}$ is local (in the Bell nonlocality sense, see the main text and [1]), we get

$$
\sigma_{bc} = \sum_{\alpha,\beta} p(\alpha)p(\beta) \text{Tr}_{BB'CC'} \left( \left[ M_{bb_\gamma}^{BB'} \otimes M_{c_\gamma}^{CC'} \right] \sigma_\alpha^B \otimes \rho^{BC'} \otimes \sigma_\beta^{C'} \right) \sigma_\alpha^A \otimes \sigma_\beta^D 
$$

(A12)

$$
= \sum_{\alpha,\beta} p(\alpha)p(\beta) \text{Tr}_{BB'CC'} \left( \left[ M_{bb_\gamma}^{BB'} \otimes M_{c_\gamma}^{CC'} \right] \rho^{BC'} \right) \sigma_\alpha^A \otimes \sigma_\beta^D 
$$

(A13)

$$
= \sum_{\alpha,\beta,\gamma} p(\alpha)p(\beta)p(\gamma) p(b|\alpha,\beta,\gamma) \sigma_\alpha^A \otimes \sigma_\beta^D 
$$

(A14)

where in the final line we extracted a local hidden variable (LHV) model using the locality of $\rho^{BC'}$. Hence taking the central source $\rho^{BC'}$ as local, and the adjacent sources as separable will also always lead to an NLHS model, for any measurements.

Still taking $\rho^{AB}$ as separable as in Equation (A11), if instead now $\rho^{BC'}$ and $\rho^{C'D}$ are unsteerable towards $C$ and $D$ respectively, we have

$$
\sigma_{bc} = \sum_\alpha p(\alpha) \sigma_\alpha^A \otimes \text{Tr}_{CC'} \left( \left[ M_{c_\gamma}^{CC'} \otimes I_D \right] \text{Tr}_{BB'} \left( \left[ M_{bb_\gamma}^{BB'} \otimes I_C \right] \sigma_\alpha^B \otimes \rho^{BC'} \right) \otimes \rho^{C'D} \right) 
$$

(A15)

$$
= \sum_\alpha p(\alpha) \sigma_\alpha^A \otimes \text{Tr}_{CC'} \left( \left[ M_{c_\gamma}^{CC'} \otimes I_D \right] \text{Tr}_{BB'} \left( \left[ M_{bb_\gamma}^{BB'} \otimes I_C \right] \rho^{BC'} \right) \otimes \rho^{C'D} \right) 
$$

(A16)

$$
= \sum_\alpha p(\alpha) \sigma_\alpha^A \otimes \text{Tr}_{CC'} \left( \left[ M_{c_\gamma}^{CC'} \otimes I_D \right] \sum_\gamma p(\gamma)p(b|\alpha,\gamma) \sigma_\gamma^C \otimes \rho^{C'D} \right) 
$$

(A17)

$$
= \sum_{\alpha,\gamma} p(\alpha)p(\gamma) \sigma_\alpha^A \otimes p(b|\alpha,\gamma) \text{Tr}_{C} \left( \left[ M_{c_\gamma}^{CC'} \otimes I_D \right] \rho^{C'D} \right) 
$$

(A18)

$$
= \sum_{\alpha,\gamma,\beta} p(\alpha)p(\beta)p(\gamma) p(b|\alpha,\beta,\gamma) \sigma_\alpha^A \otimes \sigma_\beta^D 
$$

(A19)

also leading to a NLHS model.

To summarise, ordering the sources as {\( \rho^{AB}, \rho^{BC'}, \rho^{C'D} \)} and denoting SEP as the set of separable states, LOC as the set of Bell-local states, and UNSTEER\( \rightarrow \) as the set of unsteerable states (in an appropriate direction) we have that {SEP, LOC, SEP}, {\( \leftarrow \) UNSTEER, SEP, UNSTEER\( \rightarrow \)}, {SEP, UNSTEER\( \rightarrow \), UNSTEER\( \rightarrow \)} and (by symmetry) {\( \leftarrow \) UNSTEER, \( \leftarrow \) UNSTEER, SEP} all admit NLHS models, for any measurements. This is captured in Figure 5b.

Recalling that there exist entangled yet unsteerable states, and steerable yet Bell-local states (that is SEP \( \subset \) UNS\( \rightarrow \), \( \subset \) LOC) demonstrates that these models are indeed non-trivial. Indeed network steering is truly a novel phenomena, and fully characterising the resources needed to demonstrate it is an open and fascinating new research question.
3. General Line/Ring Networks

We can generalise this to an arbitrary line network with trusted endpoints (Figure 4), which again could be interpreted as a ring network with a single trusted party.

For $n$ parties, here an observed set of states would be described by

$$\sigma_{b_2,\ldots,b_{n-1}}^{A_1\ldots A_n} = \text{Tr} \left[ A_2 A_3 \cdots A_{n-1} A_n \right] \left( \mathbb{1}^{A_1} \otimes M_{b_2}^{A_2} \otimes \cdots \otimes M_{b_{n-1}}^{A_{n-1}} \otimes \mathbb{1}^{A_n} \right) \rho^{A_1 A_2} \otimes \rho^{A_2 A_3} \otimes \cdots \otimes \rho^{A_{n-1} A_n}.$$  \hfill (A20)

The NLHS condition here generalises to

$$\sigma_{b_2,\ldots,b_{n-1}}^{A_1\ldots A_n} = \sum_{\lambda_1,\ldots,\lambda_{n-1}} p(\lambda_1) \cdots p(\lambda_{n-1}) \times \rho^{A_1 A_2} \otimes \rho^{A_2 A_3} \otimes \cdots \otimes \rho^{A_{n-1} A_n}.$$  \hfill (A21)

We first remark that as stated in the main text, $\sum_b \sigma_{b_2,\ldots,b_{n-1}}$ is a product state for any $b_i$, and the entanglement of a single $\sigma_{b_2,\ldots,b_{n-1}}$ suffices to demonstrate network steering, being incompatible with Equation (A21).

From the previous calculations for the line with four parties (Equations (A5) - (A15)), we see more generally how taking certain sources as separable can introduce natural sufficient conditions on the other sources to result in an NLHS model overall. For example if a single source is separable, then taking all other sources as unsteerable (in the direction away from this source) leads to an overall NLHS model for a line of any length – this is a generalisation from the above Equations (A15) to (A19).

As a small example, we discuss the scenario in Figure 5c marked with (†). The separable source second from the left provides an input to the adjacent sources, from which arises natural steering assemblages such as $\text{Tr} \left( M_{b\lambda}^{B_1} \otimes I^{B'} \rho_{BB'} \right)$. If these adjacent sources are steerable in the appropriate direction, we can extract an LHS model, whose corresponding state assemblages can act as an input to the next party. As the parties second and third from the right now receive effective inputs, the relevant condition on the second source from the right to admit a local model is of locality. Therefore taking the sources as described would lead to an overall NLHS model for any measurements performed. These type of arguments would hold more generally for arbitrary linear network structures.

Appendix B: Entanglement Swapping of Doubly-Erased Werner States

Here we will elaborate on and detail more closely how Doubly-Erased Werner (DEW) states can demonstrate network steering, despite being two way-unsteerable. Recall that the Erasure channel is given by

$$\Lambda_\eta(\rho) = \eta \rho + (1 - \eta) \text{tr}(\rho) |d\rangle\langle d|.$$

For example, this channel acting on a qubit state would result in a qutrit state, where now the original qubit state $\rho$ is viewed as being embedded in the $\{ |0\rangle, |1\rangle \}$ subspace, and loss of the system is represented by the $|2\rangle$ state.

For $\rho^{AB}$ a two-qubit state, a result from [36] states that $\Lambda_\eta \otimes \mathbb{1}^{AB}$ is unsteerable from Alice to Bob (for arbitrary measurements) if

$$\max_x \left[ (1 - 3\eta)|a \cdot x| + \frac{3\eta}{2} (1 + (a \cdot x)^2) + ||T x|| \right] \leq 1.$$  \hfill (B2)
where \( a \) is Alice’s local Bloch vector, \( T \) is the bipartite correlation matrix with entries \( T = \text{Tr}(\sigma_i \otimes \sigma_j) \) for \( \sigma_i \) the Pauli matrices, and the maximisation is over unit vectors \( \mathbf{x} \) in \( \mathbb{R}^3 \). For \( \rho^{AB} = \rho_W(\omega) = \omega |\psi^-\rangle\langle\psi^-| + (1 - \omega)\mathbb{I}/4 \) the Werner state, we have \( a = 0 \) and \( T = \text{diag}(-\omega, -\omega, -\omega) \) and this condition becomes

\[
\eta \leq \frac{2}{3}(1 - \omega). \tag{B3}
\]

Now as \( \mathbb{I}^A \otimes \Omega^B[\rho^{AB}] \) is unsteerable from Alice to Bob for any channel \( \Omega \) if \( \rho^{AB} \) is unsteerable from Alice to Bob [37], we have that the the Doubly-Erased Werner (DEW) state

\[
\rho_{\text{DEW}}(\eta, \omega) := \Lambda_\eta \otimes \Lambda_\eta \left( \omega |\psi^-\rangle\langle\psi^-| + (1 - \omega)\mathbb{I}/4 \right) \tag{B4}
\]

is unsteerable in both directions for \( \eta \leq \frac{2}{3}(1 - \omega) \).

We now detail the full calculation of entanglement swapping for Doubly-Erased Werner (DEW) states. Expanding out the DEW state gives

\[
\Lambda_\eta \otimes \Lambda_\eta \rho_W(\omega) = \eta^2 \rho_W(\omega) + \eta(1 - \eta)\frac{\mathbb{I}}{2} \otimes |2\rangle\langle 2| + \eta(1 - \eta)\frac{\mathbb{I}}{2} \otimes |2\rangle\langle 2| + (1 - \eta)^2 |2\rangle\langle 2| \otimes |2\rangle\langle 2|. \tag{B5}
\]

Now consider entanglement swapping with projector \( |\psi^-\rangle\langle\psi^-| \) (on the \{0\}, \{1\} subspace) onto two DEW states. We can write this as

\[
\text{Tr}_{BB'} \left( \mathbb{I}^A \otimes |\psi^-\rangle\langle\psi^-|^{BB'} \otimes \mathbb{I}^C \left[ \Lambda_\eta \otimes \Lambda_\eta \rho_W(\omega)^{AB} \right] \otimes \left[ \Lambda_\eta \otimes \Lambda_\eta \rho_W(\omega)^{BC} \right] \right) \tag{B6}
\]

\[
= \text{Tr}_{BB'} \left( \mathbb{I} \otimes |\psi^-\rangle\langle\psi^-| \otimes \mathbb{I} \left[ \eta^2 \rho_W(\omega) + \eta(1 - \eta)\frac{\mathbb{I}}{2} \otimes |2\rangle\langle 2| + \eta(1 - \eta)\frac{\mathbb{I}}{2} \otimes |2\rangle\langle 2| + (1 - \eta)^2 |2\rangle\langle 2| \otimes |2\rangle\langle 2| \right] \otimes^2 \right). \tag{B7}
\]
Note that any term with \( \langle \psi^- | \) acting on a \(| 2 \rangle \) subspace vanishes, so we can simplify this to

\[
\text{Tr}_{BB'}\left( 1 \otimes |\psi^- \rangle \otimes 1 \left[ \eta^3 \rho_W(\omega) \otimes \rho_W(\omega) + \eta^3 (1 - \eta) \rho_W(\omega) \otimes \frac{\| 2 \rangle \otimes | 2 \rangle}{2} \right] \\
+ \eta^3 (1 - \eta) | 2 \rangle \langle 2 | \otimes \frac{\| 2 \rangle \otimes | 2 \rangle}{2} \rho_W(\omega) + \eta^2 (1 - \eta)^2 | 2 \rangle \langle 2 | \otimes \frac{\| 2 \rangle}{2} \otimes \frac{\| 2 \rangle}{2} \right) \right) \tag{B9}
\]

\[
= \frac{1}{4} \left( \eta^4 \rho_W(\omega^2) + \eta^3 (1 - \eta) \frac{\| 2 \rangle \otimes | 2 \rangle}{2} \right) \\
+ \eta^3 (1 - \eta) | 2 \rangle \langle 2 | \otimes \frac{\| 2 \rangle}{2} + \eta^2 (1 - \eta)^2 | 2 \rangle \langle 2 | \otimes | 2 \rangle \right) \tag{B10}
\]

\[
= \frac{\eta^2}{4} \Lambda_\eta \otimes \Lambda_\eta \rho_W(\omega^2) \tag{B11}
\]

\[
= \frac{\eta^2}{4} \rho_{\text{DEW}}(\eta, \omega^2). \tag{B12}
\]

In lines (B9) - (B10) we used the fact that entanglement swapping of two Werner states leads to another Werner state with the product of the visibilities.

Hence entanglement swapping of two DEW states leads to another DEW state with the product of the original Werner visibilities. As discussed in the main text, the DEW state \( \rho_{\text{DEW}}(\eta, \omega^2) \) is entangled for \( \omega > \frac{1}{3} \), and so by choosing appropriate parameters we can witness network steering in a line of arbitrary length by entanglement swapping these unsteerable-yet-entangled states. Therefore network steering is a fundamentally different phenomenon to conventional quantum steering.

**Appendix C: Further Observations**

Here we detail two basic observations relating to general network scenarios that are not discussed in the main text.

1. **Endpoint sources between untrusted nodes with no inputs can be taken to be separable.**

We claim that in the following scenario (Figure 6), we can take \( \rho_{AB} \) to be separable without loss of generality.

\[A \quad \rho_{AB} \quad B\]

![Figure 6](image)

Suppose the overall state (or probabilities, if all other nodes are untrusted) is

\[
\sigma_{a,b,...} = \text{Tr}_{AB...} \left( M_a^A \otimes M_b^B \otimes ... \left[ \rho_{AB} \otimes ... \right] \right) \tag{C1}
\]

\[
= \text{Tr}_{B...} \left( M_b^B \otimes ... \left[ \text{Tr}_A \left( M_a \rho_{AB} \right) \otimes ... \right] \right) \tag{C2}
\]

Now set \( \rho'_{AB} \) as

\[
\rho'_{AB} = \sum_{a'} |a' \rangle \langle a'| \otimes \text{Tr}_A \left( M_{a' \rho_{AB}} \right) \tag{C3}
\]

which is normalised. Then also set

\[
N_a = |a \rangle \langle a| \tag{C4}
\]
This gives

\[ \text{Tr}_{AB...} \left( N^A_a \otimes M^B_b \otimes \cdots \left[ \rho'_{AB} \otimes \cdots \right] \right) = \text{Tr}_{B...} \left( M^B_b \otimes \cdots \left[ \text{Tr}_A \left( \rho_{AB} \right) \otimes \cdots \right] \right) \quad (C5) \]

as before, reproducing the same assemblage/probabilities using a separable state (and projective measurement \( M_a \)).

2. NLHS models on the line can be reproduced with separable states and separable measurements.

First recall that a measurement on subsystems \( A \) and \( B \) is said to be separable if each effect \( M^{AB}_x \) can be written as a sum of tensor products, that is for each \( M^{AB}_x \) there exist \( A_i \) and \( B_i \) such that

\[ M^{AB}_x = \sum_i A_i \otimes B_i. \]

If a measurement is not separable it is said to be joint. We prove the statement for the general line network, which has quantum model and NLHS as defined in Equations (A20) and (A21) respectively. We can rewrite the NLHS model as

\[ \sigma_{A_1 A_2 \cdots A_n b_1 b_2 \cdots b_{n-1}} = \sum_{\lambda_2, \ldots, \lambda_{n-2}} p(\lambda_2) \cdots p(\lambda_{n-2}) p(b_3 | \lambda_2, \lambda_3) \cdots p(b_{n-2} | \lambda_{n-3}, \lambda_{n-2}) \sigma_{A_1}^{A_2} \otimes \sigma_{A_{n-1}}^{A_n} \rho_{A_1 A_2} \otimes \cdots \otimes \rho_{A_{n-1} A_{n-1}} \rho_{A_{n-1} A_n} \quad (C6) \]

by setting \( \sigma_{A_1}^{A_2} = \sum_{\lambda_1} p(\lambda_1) \rho_{A_1 A_2} | \lambda_1 \rangle \langle \lambda_1 | \) and similarly for \( \sigma_{A_{n-1}}^{A_n} \). These standard LHS assemblages can be prepared with separable states and commuting measurements \([39, 40]\). We can set \( \rho_{A_1 A_2} \) and \( \rho_{A_{n-1} A_n} \) to be these separable states in question, and set all other sources according to the separable states

\[ \rho_{A_1 A_{i+1}} = \sum_{\lambda_i} p(\lambda_i) | \lambda_i \rangle \langle \lambda_i |. \quad (C7) \]

We then set the first measurement \( M_{b_2 A_2}^{A_2 A_3} \) to be the separable measurement \( \sum_{\lambda_2} N_{b_2} | \lambda_2 \rangle \langle \lambda_1 | \), where \( N_{b_2} \) are the commuting ones mentioned above, and similarly for the last measurement \( M_{b_{n-1} A_{n-1}}^{A_{n-1} A_n} \). Then all other measurements can be defined via

\[ M_{b_i}^{A_i A_{i+1}} = \sum_{\lambda_{i-1}, \lambda_i} p(b | \lambda_{i-1}, \lambda_i) | \lambda_{i-1} \lambda_i \rangle \langle \lambda_{i-1} \lambda_i |, \quad (C8) \]

which are separable. We can then see that inserting these expressions into (A20) would yield the desired NLHS assemblage in (A21).