Multiquark Cluster Form Factors In
the Relativistic Harmonic Oscillator Model

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Abstract

A QCD multiquark cluster system is studied in the relativistic harmonic oscillator potential model (RHOPM), and the electromagnetic form factors of the pion, proton and deuteron in the RHOPM are predicted. The calculated theoretical results are then compared with existing experimental data, finding very good agreement between the theoretical predictions and experimental data for these three target particles. We claim that this model can be applied to study QCD hadronic properties, particularly neutron properties, and to find six-quark cluster and/or nine-quark cluster probabilities in light nuclei such as helium $^3He$ and tritium $^3H$. This is a problem of particular importance and interest in quark nuclear physics.

Key words: multiquark cluster system form factors, Relativistic Harmonic Oscillator Potential Model, quarks, QCD.

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1 Introduction

Hadrons are particles which interact by the strong interaction. Hadrons have two classification: mesons and baryons. Mesons, with meson number $|M| = \frac{1}{\sqrt{3}} |q_\alpha \bar{q}_\beta|$, are made...
up of a quark and an antiquark, with $q$ denotes it’s quark state and $\alpha$ is the quantum numbers. Baryons, with baryon number $|B| = \sqrt{6} \varepsilon_{\alpha\beta\gamma} q_{\alpha} q_{\beta} q_{\gamma}$, are made up of three quarks. Apart from these, there is much more picture than this, the constituent quarks being surrounded by a cloud of gluons, the exchange particles for the strong force\[1\].

The strong force hold two or more quarks together, which formed hadron. Valence quarks determine the quantum numbers of hadrons, besides these, any hadron is made up of an indefinite number of sea quarks, antiquarks and gluons, which do not influence the quantum numbers of hadrons. Here, we investigate only valence quark cluster systems and do not consider the existence of sea quarks and gluons.

Our present understanding of hadrons as extended objects containing colored quarks and gluons suggests that a nucleus might not always behave as a simple collection of nucleons. Even in the loosely bound deuteron there is a few percent probability that the nucleons are separated by a distance less than their radius. In such a situation it seems reasonable that instead of talking of two clusters of three quarks one should speak of a single six-quark system\[2\]. Of course, if we were to decompose the six-quark system into clusters they could be either color singlets or octets\[3\]. A specific estimate of about 7% is obtained by theoretical models for the deuteron form factor\[4\].

In short, strongly interacting composite particles can be viewed as multiquark clusters. The deuteron is thus made up of six quarks if the proton and neutron are overlapping; in the same circumstances, $^3$He is a system of nine quarks.

Many studies have been done using conventional methods for the form factors of strong interaction composite particles. More recently, however, Refs.[5,6] study hadron form factors in perturbative QCD and QCD-inspired models. We work in the framework of a relative harmonic oscillator potential model (RHOPM), an N-valence quark cluster system where the quarks move in a relativistic harmonic oscillator potential.

## 2 Form factors of multiquark bound states

Closely following Ref.[7], we consider a system consisting of $N$ quarks moving in the field of a relativistic harmonic oscillator potential. The wave function has the form

$$
\Psi_N^{p}(x_1, x_2, \ldots, x_N) = \tilde{A} \Phi_N(x_1, x_2, \ldots, x_N)U_N^{N}(\vec{P})
$$

where $\tilde{A}$ is the quark antisymmetrization operator, $\Phi_N(x_1, x_2, \ldots, x_N)$ is the space-time wave function, and $U_N^{N}(\vec{P})$ is the spin wave function. We assume that the wave function $\Phi_N$ obeys the Klein-Gordon equation within a relativistic harmonic oscillator potential\[7\]

$$
\{ \sum_{i=1}^{N} p_i^2 + \kappa^2 \sum_{i<j}^{N \to N-1} (x_i - x_j)^2 \} \Phi_N(x_1, x_2, \ldots, x_N) = 0
$$

where $p_i = -i\partial/\partial x_i$ and $\kappa$ are, respectively, the 4-momentum and the oscillator parameter, $x_i$ is the 4-coordinate of the $i$-th quark. Let us assume isospin invariance and all quark masses are equal. After some derivation, one can represent Eq. (2) in the form

$$
(P^2 - M_p^2) \Phi_N^{q}(r_0, r_1, \ldots, r_{N-1}, P) = 0,
$$

$$
M_p^2 = -2\alpha_N a_{i\mu} a_{i\mu} + \text{const},
$$
\[ \alpha_N = \kappa N \sqrt{N} \]  

(5)

where \( P \) is the total momentum, \( M_P \) is the mass of the system, \( a^+_{i\mu} \) is particle creation operators, and \( a_{i\mu} \) is particle annihilation operators. With the Takabayashi condition[8], removing nonphysical oscillations, \( p^\mu a^+_{i\mu} \Phi_{Nq} = 0 \), one gets the solution

\[ \Phi_{Nq}(r_0, r_1, \ldots, r_{N-1}, P) = \left( \frac{\alpha_N}{\pi N} \right)^{N-1} \exp(\frac{\alpha_N}{2N} K^{\mu\nu} \sum_{i=1}^{N-1} r_{i\mu} r_{i\nu}) \]  

(6)

and

\[ \Phi_N(x_1, x_2, \ldots, x_N) = \exp[ip_{\mu} X_{\mu}] \Phi_{Nq}(r_0, r_1, \ldots, r_{n-1}, P) \]  

(7)

where \( K^{\mu\nu} = \eta^{\mu\nu} - 2p^\mu p^\nu / P^2 \). From Eq. (6), then, the N-quark cluster wave function \( \Phi_{Nq} \), with the subsidiary condition formulated by Takabayasi condition, can be written explicitly \( (n = N - 1) \) as:

\[ \Phi_{Nq} = \left( \frac{\alpha_N}{\pi N} \right)^n \exp(\frac{\alpha_N}{2N} (\eta^{\mu\nu} - 2p^\mu p^\nu / M^2_{Nq}) \sum_{i=1}^{n} r^i_{\mu} r^i_{\nu}) \]  

(8)

where the plane wave part for the center of mass coordinate has been dropped. It is well known that the wave function \( \Phi_{Nq} \) in Eq. (8) is characterized by the Lorentz contraction effect.

Transforming the non-relativistic spin wave function in the rest frame. Then the spin wave function \( U^N(\vec{P}) \) can be given by[9]

\[ U^N(\vec{P}) = B(\vec{P}) U^N(0) \]  

(9)

\[ U^N(0) = \begin{pmatrix} \chi \\ 0 \end{pmatrix} \]

\[ B(\vec{P}) = \exp[\frac{b}{2|P|} \rho_1 (\vec{P} \cdot \vec{\sigma})] = \exp[\rho_1 bH] \]  

(10)

where \( \chi \) is the non-relativistic spin function, \( H = (\vec{P} \cdot \vec{\sigma}) / 2|\vec{P}| \), \( b = \cosh^{-1} p_0 / M_P \) and \( \vec{\sigma} = \sum_{i=1}^{N} \vec{\sigma}^i \), with \( \vec{\sigma}^i \) is the Pauli matrices for the \( i \)-th quark.

According to Ref.[7], the electromagnetic action can be written as

\[ I_{em} = \int \prod_{i=1}^{N} dx_i \sum_{k} j_{k\mu}(x_1, x_2, \ldots, x_N) A_{\mu}(x_k) \equiv \int dX J^N_{\mu}(X) A_{\mu}(X) \]  

(11)
with
\[ j_{k\mu}(x_1, x_2, \ldots, x_N) = -i\Psi^N_{p'} Ne_k[g_{E}(q^2) \nabla_{x_{k\mu}} + ig_{M}(q^2)\sigma^k_{\mu\nu}(\nabla_{x_{k\nu}} + \nabla_{x_{k\nu}})]\Psi^N_p. \] (12)

In Eq. (12), \( \Psi^N_{p,(p')} \) is the initial (final) wave function as given in Eq.(1), \( e_k \) and \( \sigma^k_{\mu\nu} \) are, respectively, the charge and the spin matrices of the \( k \)-th quark, and \( \sigma^k_{ij} = \epsilon_{ijl}\sigma^k_l \), \( \sigma^k_{i4} = \sigma^k_4 = \rho_1\sigma^k_i \). \( g_{E}(q^2) \) and \( g_{M}(q^2) \) are quark charge and magnetic form factors, and \( q = p' - p \). Putting the wave function \( \Phi^N \) from Eq. (7) into Eqs. (11) and (12), and computing the integrals, the matrix elements of the effective current, \( J^N_{\mu}(0) \) can be given as:
\[ \langle p's' | J^N_{\mu}(0) | ps \rangle = \frac{I^N(q^2)}{\sqrt{2p_0p_0'}} \sum_{k=1}^{N} (\bar{U}^N_{s'}(p')\Gamma_{k,\mu}U^N_{s}(p)), \] (13)

where
\[ \Gamma_{k,\mu} = e_k[(p_\mu + p'_\mu)I_N(q^2)g_{E}(q^2) - iNg_{M}(q^2)e^{\mu}_{\nu}q^\nu]. \] (14)

Where \( I^N(q^2) \) and \( I_N(q^2) \) have the form:
\[ I^N(q^2) = \frac{1}{(1 + q^2/2M^2_{Nq})^{N-1}}\exp[-\frac{N - 1}{4\alpha_N} \frac{q^2}{1 + q^2/2M^2_{Nq}}], \] (15)
\[ I_N(q^2) = \frac{1 + Nq^2/2M^2_{Nq}}{1 + q^2/2M^2_{Nq}}. \] (16)

We now study the N-quark bound state and write down it’s form factor
\[ F_{Nq}(q^2) = \int \Phi^*_{Nq}(r^1, \ldots; P_F)\exp[-iq\sum_{i=1}^{n} u^i_{r^1}]\Phi_{Nq}(r^1, \ldots; P_f) \times d^4r^1 \ldots d^4r^n, \] (17)

where \( u^i_{r} \) is the first component of eigenvector \( \mathbf{u}^i \), obey the normalization condition
\[ \sum_{i=1}^{n} |u^i_{r}|^2 = \frac{n}{N}, \] (18)

where \( n = N - 1 \). After some derivation using the operator \( a^i_{r\mu} = \frac{1}{\sqrt{2\alpha_N}}(\sqrt{N}p^i_{r\mu} - i\epsilon^{i\mu}_{N} r^i_{\mu}), \) where \( \alpha_N = N^{3/2}k \), and Eq. (17) takes the form
\[ F_{Nq}(Q^2) = \frac{1}{[1 + (Q^2/2M^2_{Nq})]^n}\exp[-\frac{n}{4\alpha_N} \frac{Q^2}{1 + (Q^2/2M^2_{Nq})}], \] (19)

where \( Q^2 = -q^2 \).
For the two-quark cluster of the pion \((N = 2)\), the form factor can be written \((n = N - 1 = 1)\) as 

\[
F_\pi(Q^2) = \left[1 + \frac{Q^2}{2M^2_\pi}\right]^{-1} \exp \left[-\frac{1}{4\alpha_\pi} \frac{Q^2}{1 + \frac{Q^2}{2M^2_\pi}}\right].
\] (20)

Similarly, for the nucleon three-quark cluster \((N = 3)\), the form factor can be expressed as 

\[
F_N(Q^2) = \left[1 + \frac{Q^2}{2M^2_N}\right]^{-2} \exp \left[-\frac{1}{2\alpha_N} \frac{Q^2}{1 + \frac{Q^2}{2M^2_N}}\right].
\] (21)

For the six-quark cluster system of deuteron, once the distance between the proton and the neutron is less than their radius, the form factor of six-quark cluster can be given as 

\[
F_D(Q^2) = \frac{1}{[1 + (Q^2/2M^2_D)]^5} \exp[-\frac{5}{4\alpha_D} \frac{Q^2}{1 + (Q^2/2M^2_D)}].
\] (22)

In Section 3, we compare our present theoretical results for the proton, pion and deuteron form factors with the experimental data.

3 Comparison with experimental data

Fig. 1 and Fig. 2 show our present calculated electromagnetic form factors for the proton and pion respectively, compared with the corresponding experimental data[10 – 12]. As Figs. 1 and 2 show, there is very good agreement between the theoretical and experimental data.

Fig. 3 shows the fit to the deuteron scalar form factor \(A(Q^2)\) at high energies, where it should be possible to predict the six-quark cluster probability in the deuteron wave function. Our predicted result for the deuteron electromagnetic form factor \(F_{6q}(Q^2) sin^2(\theta)\) is approximately identical to the deuteron scalar form factor \(A(Q^2)\) of the Rosenbluth separation at high energies, where \(sin^2(\theta)\) is the probability of the six-quark cluster component in the deuteron. Therefore, comparing the theoretically calculated result \(F_{6q}(Q^2) sin^2(\theta)\) with the experimental data for \(A(Q^2)\), we can get the value of \(sin^2(\theta)\). This is an interesting and important issue in modern nuclear physics and hadron physics.

4 Conclusions

In this paper, we studied the electromagnetic form factors of multiquark clusters in the RHOPM. Based on the belief that strongly interacting composite particles are made up of valence quarks, and assuming quarks move individually within the relativistic harmonic oscillator potential, we have calculated the electromagnetic form factor of the proton, the pion and the deuteron in the RHOPM. This model gives a fairly good simple description
of these three particle structures provided only one arbitrary parameter, $g_E(q^2) = 1.0$, is applied. Agreement with the corresponding experimental data is very good for all three particles.

The study of electromagnetic form factors of hadrons and nuclei has been a long-standing physical problem, on which much research work has already been published [13]. However, our theoretical investigations give not only a simple analytical expression of electromagnetic form factors of multiquark cluster systems, which is very useful for practical investigations, but also the results have pointed out a way to find six-quark and nine-quark cluster probabilities in nuclei. For example, comparing our calculated form factor for the deuteron $F_{6q}(Q^2) sin^2(\theta)$ at high energies with the Rosenbluth separation form factor $A(Q^2)$ gives the deuteron six-quark cluster probability, $sin^2(\theta)$, to be about 7%, since $F_D(Q^2) = F_{np}(Q^2) cos^2(\theta) + F_{6q}(Q^2) sin^2(\theta)$ and at high energies $F_{6q}(Q^2) sin^2(\theta) = A(Q^2)$. This model can easily be extended to other mesons and baryons, as well as any system with a number of quarks larger than three, e.g. light nuclei such as $^3He$ and $^3H$. Needless to say, finding six-quark and/or nine-quark probabilities in many-body nucleon systems is an important and interesting issue in nuclear physics which is helpful for the development of quark nuclear physics.

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Figure 1: $Q^2$-dependence of nucleon form factor $F_N(Q^2)$ and comparison with experimental data from Ref.[10].

Figure 2: $Q^2$-dependence of pion form factor $F_\pi(Q^2)$ and comparison with experimental data from Ref.[11].
Figure 3: $Q^2$-dependence of deuteron form factor $F_D(Q^2)$ and comparison with experimental data from Ref.[12].