Recursive bi-orthogonalisation approach
and orthogonal projectors

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Abstract

An approach is proposed which, given a family of linearly independent functions, constructs the appropriate bi-orthogonal set so as to represent the orthogonal projector operator onto the corresponding subspace. The procedure evolves iteratively and it is endowed with the following properties: i) it yields the desired bi-orthogonal functions avoiding the need of inverse operations ii) it allows to quickly update a whole family of bi-orthogonal functions each time that a new member is introduced in the given set. The approach is of particular relevance to the approximation problem arising when a function is to be represented as a finite linear superposition of non orthogonal waveforms.

I. Introduction

Mathematical modelling of physical systems often involves the representation of functions as a linear superposition of waveforms. The formal theory of such representations is concerned with completeness and expansion properties of the corresponding waveforms. The most general setting for this purpose being provided by the frame theory. Although introduced early in the context of non-harmonic Fourier analysis [1], [2] it is only recently that the theory of frames has received great interest, as it was adopted to study wavelets and Gabor-like functions [3], [4], [5], [6]. Frames are characterised by being, in general, redundant (overcomplete). This implies that for a given frame the removal of some of its elements can still leave a frame. Otherwise the frame is said to be an exact one. In Hilbert spaces, to which our considerations will be restricted, an exact frame is equivalent to a Riesz basis. A further requirement on the nullity of the inner product between different elements amounts to an orthogonal basis.

The representation of functions within the frame structure has been the subject of much work during the last decade [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]. On the other hand, the problem of approximating functions through the linear superposition of a finite number of frame elements has been addressed less extensively (some discussion in the context of particular applications can be found in [15], [16], [17]). Here we would like to contribute to such a discussion by focusing on an approximation problem arising by considering a finite number of elements out of a Riesz basis. We believe it to be important to stress that approximations arising by truncating expansions given in terms of orthogonal waveforms and those given in terms of non-orthogonal ones are of quite different nature. By truncating an orthogonal expansion one obtains the approximation in the corresponding subspace which minimises the distance to the exact function under consideration. Nevertheless, an incomplete non-orthogonal expansion does not guarantee an approximation of the same quality. If one wishes to obtain an optimal approximation in the minimum distance sense by means of a non-orthogonal expansion, the coefficients of the corresponding linear combination should be re-calculated. In other words,
while by truncating terms in the representation of an identity operator which is given by orthogonal waveforms one obtains an orthogonal projector, an equivalent procedure in the non-orthogonal case does not leave an orthogonal projector.

Representations of the identity operator in term of non-orthogonal bases involves bi-orthogonal ones. Indeed, if the sequence $\alpha_n; n = 1, \ldots, \infty$ is a Riesz basis for a Hilbert space, then there exists a reciprocal basis $\tilde{\alpha}_n; n = 1, \ldots, \infty$ which is bi-orthogonal to the former, i.e. $\langle \tilde{\alpha}_m|\alpha_n \rangle = \delta_{m,n}$. The identity operator in the corresponding Hilbert space can, thereby, be expressed as

$$I = \sum_{n=1}^{\infty} \alpha_n \langle \tilde{\alpha}_n | \cdot \rangle$$

(1)

where $\langle \tilde{\alpha}_n | \cdot \rangle$ indicates that $I$ acts by performing inner products, as in $I f = \sum_{n=1}^{\infty} \alpha_n \langle \tilde{\alpha}_n | f \rangle$. Now, if the sum in (1) is truncated up to $N$ terms, then the approximation of a function $f$ that we obtain by computing $\sum_{n=1}^{N} \alpha_n \langle \tilde{\alpha}_n | f \rangle$ is not the best approximation of $f$ that one can obtain as a linear superposition of $N$ elements $\alpha_n; n = 1, \ldots, N$. This is tantamount to stating that the operator that is obtained by truncating the sum in (1) up to $N$ terms is not the orthogonal projector onto the subspace spanned by the elements $\alpha_n; n = 1, \ldots, N$. Indeed, as will be discussed in subsequent sections, the bi-orthogonality condition does not yield, per se, orthogonal projections. In order to construct an orthogonal projector operator the appropriate bi-orthogonal functions need to be computed.

Given a set of $N$ linearly independent functions spanning a subspace, say $S_N$, our goal is to develop an effective procedure for constructing the corresponding bi-orthogonal functions giving rise to the orthogonal projector onto $S_N$. An advantage of the algorithm we propose here being its capability of building up the desired bi-orthogonal functions avoiding the need of inverting operators. This feature allows to quickly adapt the whole set of bi-orthogonal functions each time that the corresponding subspace is enlarged by adding a new element to the given initial set. We assume that all these elements are linearly independent, however, in the event that this hypothesis fails to be true, the proposed algorithm provides a criterion for disregarding linearly dependent elements. Thereby the algorithm itself can be used for selecting subsets of linearly independent functions.

The paper is organised as follows: Section II introduces the notation and discusses the problem of constructing orthogonal projections by means of non-orthogonal waveforms. In section III an iterative procedure to compute bi-orthogonal functions giving rise to orthogonal projector operators is advanced. The algorithm is illustrated by computing the bi-orthogonal functions amounting to the orthogonal projector operator onto a subspace generated by a few Mexican hat wavelets.
II. Notation, background and preliminary considerations

A. Bi-orthogonal expansions

Adopting Dirac’s vector notation \cite{18} we represent an element \( f \) of a Hilbert space \( \mathcal{H} \) as a vector \(|f\rangle\) and its dual as \( \langle f|\). Given a set of \( \delta \)-normalised continuous orthogonal vectors \( \{|t\rangle; -\infty < t < \infty; \langle t|t'\rangle = \delta(t-t')\} \), the unity operator in \( \mathcal{H} \) is expressed

\[
\hat{I}_\mathcal{H} = \lim_{T \to \infty} \int_{-T}^{T} \langle t|t\rangle dt.
\]  

(2)

Thus, for all \(|f\rangle\) and \(|g\rangle \in \mathcal{H}\), by inserting \( \hat{I}_\mathcal{H} \) in \( \langle f|g\rangle \), i.e,

\[
\langle f|\hat{I}_\mathcal{H}|g\rangle = \lim_{T \to \infty} \int_{-T}^{T} \langle f|t\rangle \langle t|g\rangle dt
\]

(3)

one is led to a representation of \( \mathcal{H} \) in terms of the space of square integrable functions, with \( \langle t|g\rangle = g(t) \) and \( \langle g|t\rangle = \overline{g(t)} \), where \( \overline{g(t)} \) indicates the complex conjugate of \( g(t) \).

Let vectors \(|\alpha_n\rangle \in \mathcal{H}; n = 1, \ldots, \infty\) be a Riesz Basis for \( \mathcal{H} \). Then there exists a reciprocal basis \(|\tilde{\alpha}_n\rangle; n = 1, \ldots, \infty\) for \( \mathcal{H} \) to which the former basis is bi-orthogonal i.e., \( \langle \tilde{\alpha}_n|\alpha_m\rangle = \delta_{n,m} \). These two bases amount to a representation of the identity operator as given by:

\[
\hat{I} = \sum_{n=1}^{\infty} |\alpha_n\rangle \langle \tilde{\alpha}_n| \equiv \sum_{n=1}^{\infty} |\tilde{\alpha}_n\rangle \langle \alpha_n|.
\]

(4)

so that every \(|f\rangle \in \mathcal{H}\) can be expanded in the form

\[
|f\rangle = \sum_{n=1}^{\infty} |\alpha_n\rangle \langle \tilde{\alpha}_n|f\rangle \equiv \sum_{n=1}^{\infty} |\tilde{\alpha}_n\rangle \langle \alpha_n|f\rangle.
\]

(5)

The statements above concerning bi-orthogonal bases are well established results \cite{2}. Nevertheless, the corresponding approximations arising by considering a finite number of vectors \(|\alpha_n\rangle; n = 1, \ldots, N\) have thusfar received much less consideration. It is important to stress that some properties holding on truncation of an orthogonal basis, no longer hold in the bi-orthogonal case. Let us consider for instance that the sum in \( (4) \) is truncated up to \( N \) terms. Thus, rather than a representation of the identity operator in \( \mathcal{H} \) we obtain an operator, \( \hat{Q} \) say, given by

\[
\hat{Q} = \sum_{n=1}^{N} |\alpha_n\rangle \langle \tilde{\alpha}_n|.
\]

(6)

Unlike the case involving an orthogonal basis, \( \hat{Q} \) is not the orthogonal projector operator onto the subspace spanned by \( N \) vectors \(|\alpha_n\rangle; n = 1, \ldots, N\). The bi-orthogonality property of families \(|\alpha_n\rangle; n = 1, \ldots, N \) and \(|\tilde{\alpha}_n\rangle; n = 1, \ldots, N\) implies that \( \hat{Q}^2 = \hat{Q} \), hence \( \hat{Q} \) is indeed a projector. However, \( \hat{Q}^\dagger \) (the adjoint of \( \hat{Q} \)) is not equal to \( \hat{Q} \) and therefore \( \hat{Q} \) is not an orthogonal projector. As
a consequence, the approximation $|f_N\rangle$ of $|f\rangle$ that we obtain from:

$$|f_N\rangle = \hat{Q}f = \sum_{n=1}^{N} |\alpha_n\rangle \langle\alpha_n|f\rangle$$  \hspace{1cm} (7)$$
is not the best approximation of $|f\rangle$ that can be obtained as a linear superposition of $N$ vectors $|\alpha_n\rangle$. This being actually a major difference between orthogonal and bi-orthogonal expansions. Whilst the condition of orthogonality yields orthogonal projections by truncating expansions, the bi-orthogonality condition, per se, does not guarantee projections of such a nature. If one wishes to obtain orthogonal projections by means of bi-orthogonal families, then bi-orthogonal vectors specially devised for such a purpose must be constructed.

**B. Building Orthogonal Projections**

Let us consider a subspace, $S_N$ of $\mathcal{H}$ spanned by $N$ linearly independent vectors $|\alpha_n\rangle$; $n = 1, \ldots, N$. Hence, every $|f\rangle \in \mathcal{H}$ can be decomposed in the form

$$|f\rangle = |f_{S_N}\rangle + |f_{S_N}^\perp\rangle,$$  \hspace{1cm} (8)$$
where $|f_{S_N}\rangle \in S_N$, i.e., $|f_{S_N}\rangle = \sum_{n=1}^{N} c_n|\alpha_n\rangle$ for some coefficients $c_n$; $n = 1, \ldots, N$. Now, in order for $|f_{S_N}\rangle$ to be the orthogonal projection of $|f\rangle$ onto $S_N$ we should require that $|f_{S_N}^\perp\rangle$ belongs to $S_N^\perp$, the orthogonal complement of $S_N$ in $\mathcal{H}$. This implies that $|f_{S_N}^\perp\rangle$ should be orthogonal to every vector $|\alpha_n\rangle$; $n = 1, \ldots, N$ and we are thus led to the following equations:

$$\langle \alpha_k |f\rangle = \sum_{n=1}^{N} c_n \langle \alpha_k |\alpha_n\rangle ; \hspace{1cm} k = 1, \ldots, N.$$  \hspace{1cm} (9)$$
Let $|n\rangle$; $n = 1, \ldots, N$ be the standard basis for $C^N$, i.e, $\langle m |n\rangle = \delta_{m,n}$ so that vector $|c\rangle \in C^N$ can be expressed as $|c\rangle = \sum_{n=1}^{N} \langle n |c\rangle |n\rangle = \sum_{n=1}^{N} c_n |n\rangle$ and equations (9) can be re-written as

$$\langle \alpha_k |f\rangle = \hat{F} |c\rangle$$  \hspace{1cm} (10)$$
where operator $\hat{F}$ is given by

$$\hat{F} = \sum_{n=1}^{N} |\alpha_n\rangle \langle n|.$$  \hspace{1cm} (11)$$
Since the adjoint $\hat{F}^\dagger$ of $\hat{F}$ is

$$\hat{F}^\dagger = \sum_{n=1}^{N} |n\rangle \langle \alpha_n|,$$  \hspace{1cm} (12)$$the left-hand-side of equation (10) happens to give the components of vector $\hat{F}^\dagger |f\rangle \in C^N$, whereas on the right-hand-side we find the components of a vector $\hat{F}^\dagger \hat{F} |c\rangle \in C^N$. Thus, these equations can be recast in the form

$$\hat{F}^\dagger |f\rangle = \hat{F}^\dagger \hat{F} |c\rangle.$$  \hspace{1cm} (13)$$
Since vectors $|\alpha_n\rangle$; $n = 1, \ldots, N$ are linearly independent operator $\hat{F}^\dagger \hat{F}$ has an inverse and $|c\rangle$ is readily obtained as

$$|c\rangle = (\hat{F}^\dagger \hat{F})^{-1} \hat{F}^\dagger |f\rangle.$$  \hfill (14)

The corresponding components $c_n = \langle n|c\rangle$ being

$$c_n = \langle n|c\rangle = \langle n| (\hat{F}^\dagger \hat{F})^{-1} \hat{F}^\dagger |f\rangle = \langle \tilde{\alpha}^N_n|f\rangle,$$  \hfill (15)

with vectors $\langle \tilde{\alpha}^N_n|$ given by

$$\langle \tilde{\alpha}^N_n| = \langle n| (\hat{F}^\dagger \hat{F})^{-1} \hat{F}^\dagger.$$  \hfill (16)

Let us consider now the eigenvalue equation for the positive operator $\hat{F}^\dagger \hat{F}$

$$\hat{F}^\dagger \hat{F} |\eta_n\rangle = \lambda_n |\eta_n\rangle \quad ; \quad n = 1, \ldots, N$$  \hfill (17)

where $\lambda_n > 0$ and the corresponding eigenvectors $|\eta_n\rangle \in C^N$ satisfy the condition $\langle \eta_k|\eta_n\rangle = \delta_{k,n}$.

It readily follows that the orthonormal vectors $|\phi_n\rangle = \frac{\hat{F}|\eta_n\rangle}{\sqrt{\lambda_n}} \in \mathcal{H}$ are eigenvectors of the positive self-adjoint operator

$$\hat{G} = \hat{F}^\dagger \hat{F} = \sum_{n=1}^{N} |\alpha_n\rangle \langle \alpha_n|$$  \hfill (18)

with associated eigenvalues $\lambda_n$; $n = 1, \ldots, N$. Hence, the spectral decomposition of $\hat{G}$ is

$$\hat{G} = \sum_{n=1}^{N} |\phi_n\rangle \lambda_n \langle \phi_n|$$  \hfill (19)

and that of its inverse, $\hat{G}^{-1}$, is

$$\hat{G}^{-1} = \sum_{n=1}^{N} |\phi_n\rangle \frac{1}{\lambda_n} \langle \phi_n|.$$  \hfill (20)

Moreover, since $\text{Range}(\hat{G}) \equiv S_N$, the orthogonal projector operator onto $S_N$ can be expressed as

$$\hat{P} = \hat{G}^{-1} \hat{G} = \hat{G} \hat{G}^{-1} = \sum_{n=1}^{N} |\phi_n\rangle \langle \phi_n|$$  \hfill (21)

We are now in a position to provide an alternative form for vectors $\langle \tilde{\alpha}^N_n|$ given in (16).

**Proposition 1:** Vectors $\langle \tilde{\alpha}^N_n|$, which are defined as

$$\langle \tilde{\alpha}^N_n| = \langle n| (\hat{F}^\dagger \hat{F})^{-1} \hat{F}^\dagger,$$  \hfill (22)

can be expressed as:

$$\langle \tilde{\alpha}^N_n| = \langle \alpha_n| \hat{G}^{-1}$$  \hfill (23)

**Proof:** Replacing $(\hat{F}^\dagger \hat{F})^{-1}$ in (16) by its spectral decomposition $\sum_{k=1}^{N} |\eta_k\rangle \frac{1}{\lambda_k} \langle \eta_k|$ we have:

$$\langle \tilde{\alpha}^N_n| = \langle n| (\hat{F}^\dagger \hat{F})^{-1} \hat{F}^\dagger = \sum_{k=1}^{N} \langle n|\eta_k\rangle \frac{1}{\lambda_k} \langle \eta_k| \hat{F}^\dagger.$$  \hfill (24)
Accordingly, given \( k \) and \( |\phi_k\rangle \) from (24) we obtain:

\[
|\hat{\alpha}_n^n\rangle = \sum_{k=1}^{N} \langle n|\hat{F}^\dagger|\phi_k\rangle \frac{1}{\lambda_k} |\phi_k\rangle = \langle n|\hat{F}^\dagger\hat{G}^{-1}. \tag{25}
\]

On the other hand, from (22) we gather that \( \langle n|\hat{F}^\dagger = \langle \alpha_n| \) and the proof is concluded \( \Box \)

**Proposition 2:** The reciprocal families \( |\hat{\alpha}_n^n\rangle ; n = 1,\ldots,N \) and \( |\alpha_n\rangle ; n = 1,\ldots,N \) are bi-orthogonal families.

**Proof:** It is straightforward by writing \( \langle \hat{\alpha}_n^n| = \langle n|\hat{F}^\dagger\hat{F}^{-1}\hat{F}^\dagger|\alpha_k\rangle = \hat{F}|\alpha_k\rangle \). Indeed,

\[
\langle \hat{\alpha}_n^n|\alpha_k\rangle = \langle n|\hat{F}^\dagger\hat{F}^{-1}\hat{F}|k\rangle = \langle n|k\rangle = \delta_{n,k} \tag{26}
\]

The next proposition shows that the families \( |\hat{\alpha}_n^n\rangle ; n = 1,\ldots,N \) and \( |\alpha_n\rangle ; n = 1,\ldots,N \) provide us with a representation of the orthogonal projection operator onto \( S_N \).

**Proposition 3:** Vectors \( |\hat{\alpha}_n^n\rangle ; n = 1,\ldots,N \) and \( |\alpha_n\rangle ; n = 1,\ldots,N \) amount to a representation of the orthogonal projection operator onto \( S_N \) as given by

\[
\hat{P} = \sum_{n=1}^{N} |\alpha_n\rangle \langle \hat{\alpha}_n^n| \equiv \sum_{n=1}^{N} |\alpha_n\rangle \langle \hat{\alpha}_n^n| \tag{27}
\]

**Proof:** It immediately follows by using (18) in (21)

\[
\hat{P} = \hat{G}\hat{G}^{-1} = \sum_{n=1}^{N} |\alpha_n\rangle \langle \alpha_n|\hat{G}^{-1} = \sum_{n=1}^{N} |\alpha_n\rangle \langle \hat{\alpha}_n^n| \\
= \hat{G}^{-1}
\]

\[
\hat{G} = \sum_{n=1}^{N} |\alpha_n\rangle \langle \alpha_n| = \sum_{n=1}^{N} \langle \hat{\alpha}_n^n| \langle \alpha_n| \delta_{n,k} \tag{28}
\]

It is clear at this point that if we introduce a new vector \( |\alpha_{N+1}\rangle \) in the initial set, then in order to represent an orthogonal projector onto the enlarged subspace all the corresponding bi-orthogonal vectors \( |\hat{\alpha}_n^{N+1}\rangle ; n = 1,\ldots,N + 1 \) must be re-calculated. The above discussion implies that the calculation of these vectors involves the computation of an inverse operator. Our goal is to avoid the need for this calculation. In the next section we propose a recursive procedure that enables us to compute bi-orthogonal vectors endowed with the desired properties.

### III. Recursive bi-orthogonal approach

From the discussion of the previous section we are in a position to assume the existence of bi-orthogonal families of vectors in \( \mathcal{H} \) giving rise to orthogonal projections on the subspace they span. Accordingly, given \( k \) vectors \( |\alpha_n\rangle ; n = 1,\ldots,k \) spanning a subspace \( V_k \) let us denote \( |\hat{\alpha}_n^k\rangle ; n = 1,\ldots,k \) to the corresponding bi-orthogonal family yielding a representation of the orthogonal projector operator onto \( V_k \) i.e.,

\[
\hat{P}_k = \sum_{n=1}^{k} |\hat{\alpha}_n^k\rangle \langle \alpha_n| = \sum_{n=1}^{k} |\alpha_n\rangle \langle \hat{\alpha}_n^k| \tag{29}
\]
If we add now a new element $|\alpha_{k+1}\rangle$ to the previous family, in order to represent the orthogonal projector onto the the enlarged subspace $V_{k+1}$, we need to compute a new bi-orthogonal family $|\tilde{\alpha}_{n}^{k+1}\rangle; n = 1, \ldots, k + 1$. The operator $\hat{P}_{k+1}$ in terms of the new family is:

$$\hat{P}_{k+1} = \sum_{n=1}^{k+1} |\tilde{\alpha}_{n}^{k+1}\rangle\langle \alpha_{n}| = \sum_{n=1}^{k+1} |\alpha_{n}\rangle\langle \tilde{\alpha}_{n}^{k+1}|. \quad (30)$$

Now, $V_{k+1}$ is the the direct sum $V_{k+1} = V_{k} \oplus |\alpha_{k+1}\rangle$. Hence, the orthogonal projector operator onto $V_{k+1}$ can be decomposed in the fashion:

$$\hat{P}_{k+1} = \hat{P}_{k} + \hat{P}_{k}^{\perp} \quad (31)$$

where $\hat{P}_{k}$ is given in (29) and $\hat{P}_{k}^{\perp}$ is the orthogonal projector onto the subspace $V_{k}^{\perp}$ denoting the orthogonal complement of $V_{k}$ in $V_{k+1}$. Consequently, $V_{k}^{\perp}$ is spanned by the single element $|\psi_{k+1}\rangle$, which is obtained by removing from $|\alpha_{k+1}\rangle$ its component in $V_{k}$, i.e.

$$|\psi_{k+1}\rangle = |\alpha_{k+1}\rangle - \hat{P}_{k} |\alpha_{k+1}\rangle. \quad (32)$$

Hence $\hat{P}_{k}^{\perp}$ is obtained from $|\psi_{k+1}\rangle \langle \psi_{k+1}|$ as

$$\hat{P}_{k}^{\perp} = \frac{|\psi_{k+1}\rangle \langle \psi_{k+1}|}{|||\psi_{k+1}\rangle||^2} \quad (33)$$

and we can explicitly re-write (31) in the form

$$\sum_{n=1}^{k+1} |\tilde{\alpha}_{n}^{k+1}\rangle\langle \alpha_{n}| = \sum_{n=1}^{k} |\tilde{\alpha}_{n}^{k}\rangle\langle \alpha_{n}| + \frac{|\psi_{k+1}\rangle \langle \psi_{k+1}|}{|||\psi_{k+1}\rangle||^2} \quad (34)$$

or, equivalently,

$$|\tilde{\alpha}_{k+1}^{k+1}\rangle\langle \alpha_{k+1}| + \sum_{n=1}^{k} |\tilde{\alpha}_{n}^{k+1}\rangle\langle \alpha_{n}| = \sum_{n=1}^{k} |\tilde{\alpha}_{n}^{k}\rangle\langle \alpha_{n}| + \frac{|\psi_{k+1}\rangle \langle \psi_{k+1}|}{|||\psi_{k+1}\rangle||^2}. \quad (35)$$

By taking the inner product of both sides of (35) with $|\tilde{\alpha}_{n}^{k}\rangle; n = 1, \ldots, k$ we obtain:

$$|\tilde{\alpha}_{k+1}^{k+1}\rangle\langle \alpha_{k+1}| + |\tilde{\alpha}_{n}^{k+1}\rangle\langle \alpha_{n}| = |\tilde{\alpha}_{n}^{k}\rangle\langle \alpha_{n}|; \quad n = 1, \ldots, k. \quad (36)$$

On the other hand, by taking the inner product of both sides of (35) with $|\psi_{k+1}\rangle$ we have

$$|\tilde{\alpha}_{k+1}^{k+1}\rangle\langle \alpha_{k+1}| \psi_{k+1} \rangle = |\psi_{k+1}\rangle. \quad (37)$$

The last two equations yield the recurrent formula that was our goal to find:

$$|\tilde{\alpha}_{n}^{k+1}\rangle = |\tilde{\alpha}_{n}^{k}\rangle - \frac{|\psi_{k+1}\rangle \langle \alpha_{k+1}| |\tilde{\alpha}_{n}^{k}\rangle}{|||\psi_{k+1}\rangle||^2}; \quad n = 1, \ldots, k \quad (38)$$

$$|\tilde{\alpha}_{k+1}^{k+1}\rangle = \frac{|\psi_{k+1}\rangle \langle \alpha_{k+1}| \psi_{k+1} \rangle}{|||\psi_{k+1}\rangle||^2} = |\psi_{k+1}\rangle. \quad (38)$$
We show now that these vectors indeed satisfy the desired properties. The next theorem is in order.

**Theorem 1:** Let $|\alpha_n\rangle; n = 1, \ldots, k + 1$ be a set of linearly independent vectors spanning a subspace $V_{k+1}$. Then vectors $|\tilde{\alpha}_{n+1}\rangle; n = 1, \ldots, k + 1$ constructed as prescribed in (38), by considering $|\psi_1\rangle = |\alpha_1\rangle$, satisfy the following properties:

a) are bi-orthogonal to vectors $|\alpha_n\rangle; n = 1, \ldots, k + 1$ i.e.,

$$\langle \alpha_m | \tilde{\alpha}_{n+1} \rangle = \delta_{m,n}$$

b) provide a representation of the orthogonal projection operator onto $V_{k+1}$, i.e.,

$$\hat{P}_{k+1} = \sum_{n=1}^{k+1} |\alpha_n\rangle \langle \alpha_n |$$

(39)

Both the proofs of a) and b) are achieved by induction. The corresponding steps are to be found in Appendix A.

**Corollary 1:** The coefficients $c_{n+1}^{k+1}$ of the linear expansion

$$\sum_{n=1}^{k+1} c_{n+1}^{k+1} |\alpha_n\rangle$$

(40)

which approximates an arbitrary vector $|f\rangle \in \mathcal{H}$ at best in a minimum distance sense, can be recursively obtained as:

$$c_{n+1}^{k+1} = c_n - \langle \tilde{\alpha}_{n+1} | \alpha_{k+1} \rangle \frac{\langle \psi_{k+1} | f \rangle}{||\psi_{k+1}||^2}; \quad n = 1, \ldots, k$$

$$c_{k+1}^{k+1} = \frac{\langle \psi_{k+1} | f \rangle}{||\psi_{k+1}||^2},$$

(41)

with $c_1 = \frac{\langle \alpha_1 | f \rangle}{||\alpha_1||^2}$.

**Proof:** The proof results from the fact that the unique vector in $V_{k+1}$ which minimises the distance to $|f\rangle$ is obtained as $\hat{P}_{k+1}|f\rangle$. Indeed, let $|f'\rangle$ be an arbitrary vector in $V_{k+1}$ and let us write it as $|f'\rangle = |f'\rangle + \hat{P}_{k+1}|f\rangle - \hat{P}_{k+1}|f\rangle$. If we calculate the squared distance $|||f\rangle - |f'\rangle||^2$, since $(|f\rangle - \hat{P}_{k+1}|f\rangle)$ is orthogonal to every vector in $V_{k+1}$, we have

$$|||f\rangle - |f'\rangle||^2 = |||f\rangle - \hat{P}_{k+1}|f\rangle + |f'\rangle - \hat{P}_{k+1}|f\rangle||^2 = |||f\rangle - \hat{P}_{k+1}|f\rangle||^2 + |||f'\rangle - \hat{P}_{k+1}|f\rangle||^2$$

from where we gather that the distance is minimised if $|f'\rangle \equiv \hat{P}_{k+1}|f\rangle$. Hence (41) readily follows by using (38) in (39) and identifying $c_{n+1}^{k+1}$ with $\langle \tilde{\alpha}_{n+1} | f \rangle$ for $n = 1, \ldots, k + 1$.

We illustrate the proposed approach by the following example: Let us consider $N = 5$ elements $|\alpha_n\rangle; n = 1, \ldots, 5$ whose functional representations are given by the following shifted Mexican hat wavelets

$$\alpha_n(t) = \langle t | \alpha_n \rangle = \frac{2}{\sqrt{3\pi t^4}} e^{-\frac{1}{2} \left( \frac{t-n+1}{t} \right)^2} \left( 1 - \left( \frac{t-n+1}{t} \right)^2 \right)$$

(42)

\[DRAFT\]
Figure 1 plots the wavelet $\alpha_1$. Figure 2 plots the functions $\tilde{\alpha}_1^1$ (thin line), $\tilde{\alpha}_1^3$ (dotted line) and $\tilde{\alpha}_1^5$ (thick line) which are involved in the representation of the orthogonal projectors onto the subspaces spanned, respectively, by the sets $\{\alpha_1\}$, $\{\alpha_1, \alpha_2, \alpha_3\}$, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$

IV. Conclusions

An iterative procedure, for constructing bi-orthogonal functions yielding orthogonal projectors, has been proposed. The outcome being relevant to the approximation problem concerning the representation of a function as a linear supposition of waveforms. The proposed method avoids the computation of inverse operations and allows us to update the whole set of bi-orthogonal functions when the dimension of the subspace is increased. This feature is, thereby, of great importance in situations in which the waveforms are to be selected from a large number of possible ones [19]. It is appropriate to stress that if the available waveforms are linearly dependent, the proposed algorithm allows one to select subsets of linearly independent ones. This is to be achieved simply by disregarding those vector $|\alpha_{k+1}\rangle$ yielding corresponding vectors $|\psi_{k+1}\rangle$ (given in (32)) of zero norm.

From the above remarks it is expected that this technique should be of assistance in a broad range of problems concerning mathematical modelling of physical systems.

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Appendix

A. Proof of Theorem 1

We show first that vectors $|\tilde{\alpha}_{n+1}^k\rangle$ arising from $|\psi_1\rangle = |\alpha_1\rangle$ by the recursive formula:

$$
|\tilde{\alpha}_{n+1}^k\rangle = |\alpha_{n+1}^k\rangle - \frac{|\psi_{k+1}\rangle}{||\psi_{k+1}||^2} \langle \alpha_{k+1}^k | \alpha_n^k \rangle ; \quad n = 1, \ldots, k
$$

$$
|\tilde{\alpha}_{k+1}^k\rangle = \frac{|\psi_{k+1}\rangle}{||\psi_{k+1}||^2}
$$

(A.1)

are bi-orthogonal to vectors $|\alpha_n\rangle$; $n = 1, \ldots, k+1$, i.e., they satisfy the relation

$$
\langle \alpha_m^l | \tilde{\alpha}_{n+1}^k \rangle = \delta_{m,n} ; \quad n = 1, \ldots, k+1 ; \quad m = 1, \ldots, k+1
$$

(A.2)

For $k = 0$ the relation holds because $|\tilde{\alpha}_1^1\rangle = \frac{|\alpha_1\rangle}{||\alpha_1||}$ and therefore $\langle \alpha_1^1 | \tilde{\alpha}_1^1 \rangle = \frac{\langle \alpha_1^1 | \alpha_1^1 \rangle}{||\alpha_1||^2} = 1$. Assuming that for $k+1 = l$ it is true that

$$
\langle \alpha_m^l | \tilde{\alpha}_{n+1}^l \rangle = \delta_{m,n} ; \quad n = 1, \ldots, l+1 ; \quad m = 1, \ldots, l+1
$$

(A.3)

we shall prove that

$$
\langle \alpha_m^l | \tilde{\alpha}_{n+1}^l \rangle = \delta_{m,n} ; \quad n = 1, \ldots, l+1 ; \quad m = 1, \ldots, l+1.
$$

(A.4)

In order to prove this we need to discriminate between four different situations concerning the values of indices $m$ and $n$.

I) $m = 1, \ldots, l$ and $n = 1, \ldots, l$

For this situation $\langle \alpha_m^l | \psi_{l+1} \rangle = 0$. Hence, using (A.3) and (A.4) we have:

$$
\langle \alpha_m^l | \tilde{\alpha}_{n+1}^l \rangle = \langle \alpha_m^l | \tilde{\alpha}_{n+1}^l \rangle - \frac{\langle \alpha_m^l | \psi_{l+1} \rangle}{|| \psi_{l+1} ||^2} \langle \alpha_{l+1}^l | \tilde{\alpha}_{n+1}^l \rangle = \delta_{m,n}
$$

(A.5)

II) $m = l+1$ and $n = 1, \ldots, l$

Thus

$$
\langle \alpha_{l+1}^l | \tilde{\alpha}_{n+1}^l \rangle = \langle \alpha_{l+1}^l | \tilde{\alpha}_{n+1}^l \rangle - \frac{\langle \alpha_{l+1}^l | \psi_{l+1} \rangle}{|| \psi_{l+1} ||^2} \langle \alpha_{l+1}^l | \tilde{\alpha}_{n+1}^l \rangle = \langle \alpha_{l+1}^l | \tilde{\alpha}_{n+1}^l \rangle - \frac{\langle \alpha_{l+1}^l | \psi_{l+1} \rangle}{|| \psi_{l+1} ||^2} \langle \alpha_{l+1}^l | \tilde{\alpha}_{n+1}^l \rangle
$$

(A.6)

since $|| \psi_{l+1} ||^2 = \langle \alpha_{l+1}^l | \psi_{l+1} \rangle$.

III) $m = l+1$ and $n = l+1$

This implies

$$
\langle \alpha_{l+1}^l | \tilde{\alpha}_{l+1}^l \rangle = \frac{\langle \alpha_{l+1}^l | \psi_{l+1} \rangle}{|| \psi_{l+1} ||^2} = \frac{|| \psi_{l+1} ||^2}{|| \psi_{l+1} ||^2} = 1
$$

(A.7)
IV) \( m = 1, \ldots, l \) and \( n = l + 1 \)

In this case \( \langle \alpha_m | \psi_{l+1} \rangle = 0 \). Hence

\[
\langle \alpha_m | \tilde{\alpha}^{l+1}_k \rangle = \frac{\langle \alpha_m | \psi_{l+1} \rangle}{\| \psi_{l+1} \|^2} = 0
\]  

(A.8)

From I) II) III) and IV) we conclude that

\[
\langle \alpha_m | \tilde{\alpha}^{k+1}_n \rangle = \delta_{m,n} \quad ; \quad n = 1, \ldots, k+1 \quad ; \quad m = 1, \ldots, k+1
\]

Now we prove that vectors \( |\tilde{\alpha}^{k+1}_n\rangle \); \( n = 1, \ldots, k+1 \) given in (A.1) provide a representation of the orthogonal projection operator onto \( V_{k+1} \) as given by:

\[
\hat{P}_{k+1} = \sum_{n=1}^{k+1} |\tilde{\alpha}^{k+1}_n\rangle \langle \alpha_n| = \hat{P}_{k+1}^\dagger = \sum_{n=1}^{k+1} |\alpha_n\rangle \langle \tilde{\alpha}^{k+1}_n| 
\]  

(A.9)

We prove that \( \sum_{n=1}^{k+1} |\alpha_n\rangle \langle \tilde{\alpha}^{k+1}_n| \) is the orthogonal projector onto \( V_{k+1} \) by proving

i) \( \sum_{n=1}^{k+1} |\alpha_n\rangle \langle \tilde{\alpha}^{k+1}_n| g \rangle = |g\rangle \); \( \forall \; |g\rangle \in V_{k+1} \) and

ii) \( \sum_{n=1}^{k+1} |\alpha_n\rangle \langle \tilde{\alpha}^{k+1}_n| g^\perp \rangle = 0 \); \( \forall \; |g^\perp\rangle \in V^\perp_{k+1} \).

i) is actually a consequence of the bi-orthogonality condition (A.2). Because if \( |g\rangle \in V_{k+1} \) it can be expressed as \( |g\rangle = \sum_{n=1}^{k+1} c_n |\alpha_n\rangle \) for some coefficients \( c_n \); \( n = 1, \ldots, k+1 \), hence \( \sum_{n=1}^{k+1} |\alpha_n\rangle \langle \tilde{\alpha}^{k+1}_n| g \rangle = \sum_{n=1}^{k+1} c_n |\alpha_n\rangle = |g\rangle \).

We prove ii) by induction. For \( k+1 = 1 \) we have \( \hat{P}_1 |g^\perp\rangle = |\alpha_1\rangle \langle \alpha_1| |g^\perp\rangle = 0 \); \( \forall \; |g^\perp\rangle \in V^\perp_1 \), since \( \langle \alpha_1 | g^\perp \rangle = 0 \) for all \( |g^\perp\rangle \in V^\perp_1 \).

Let us assume that

\[
\hat{P}_k |g^\perp\rangle = \sum_{n=1}^{k} |\alpha_n\rangle \langle \tilde{\alpha}^{k}_n| g^\perp \rangle = 0 \quad ; \quad \forall \; |g^\perp\rangle \in V^\perp_k
\]  

(A.10)

and prove that it is then true that

\[
\hat{P}_{k+1} |g^\perp\rangle = \sum_{n=1}^{k+1} |\alpha_n\rangle \langle \tilde{\alpha}^{k+1}_n| g^\perp \rangle = 0 \quad ; \quad \forall \; |g^\perp\rangle \in V^\perp_{k+1}
\]  

(A.11)

Indeed,

\[
\sum_{n=1}^{k+1} |\alpha_n\rangle \langle \tilde{\alpha}^{k+1}_n| g^\perp \rangle = |\alpha_{k+1}\rangle \langle \psi_{k+1} | g^\perp \rangle + \hat{P}_k |g^\perp \rangle - \hat{P}_{k+1} |\alpha_{k+1}\rangle \langle \psi_{k+1} | g^\perp \rangle = 0.
\]  

(A.12)

The zero value of the above equation holds from the fact that \( \langle \alpha_n | g^\perp \rangle = 0 \); \( n = 1, \ldots, k+1 \) for all \( |g^\perp\rangle \in V^\perp_{k+1} \) and \( \hat{P}_k |g^\perp\rangle = 0 \) by hypothesis (A.10).

Since \( \hat{P}_{k+1} = \sum_{n=1}^{k+1} |\alpha_n\rangle \langle \tilde{\alpha}^{k+1}_n| \) is an orthogonal projector it is true that \( \hat{P}_{k+1} = \hat{P}^\dagger_{k+1} = \sum_{n=1}^{k+1} |\tilde{\alpha}^{k+1}_n\rangle \langle \alpha_n| \)

so that the proof is completed □
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Figure 1: Mexican hat wavelet $\alpha_1$
Figure 2: Functions $\tilde{\alpha}_1^1$ (thin line), $\tilde{\alpha}_1^3$ (dotted line) and $\tilde{\alpha}_1^5$ (thick line).
Figure Captions

**Figure 1:** Mexican hat wavelet $\alpha_1$.

**Figure 2:** Functions $\tilde{\alpha}_1$ (thin line), $\tilde{\alpha}_3$ (dotted line) and $\tilde{\alpha}_5$ (thick line).