The Drinfeld–Kohno theorem for the superalgebra $\mathfrak{gl}(1|1)$

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Abstract

We revisit the derivation of Knizhnik–Zamolodchikov equations in the case of non-semisimple categories of modules of a superalgebra in the case of the generic affine level and representations parameters. A proof of existence of asymptotic solutions and their properties for the superalgebra $\mathfrak{gl}(1|1)$ gives a basis for the proof of existence of associator which satisfy braided tensor categories requirements. Braided tensor category structure of $U_h(\mathfrak{gl}(1|1))$ quantum algebra is calculated, and the tensor product ring is shown to be isomorphic to $\mathfrak{gl}(1|1)$ ring, for the same generic relations between the level and parameters of modules. We review the proof of Drinfeld–Kohno theorem for non-semisimple category of modules suggested by Geer (Adv Math 207:1–38, 2006) and show that it remains valid for the superalgebra $\mathfrak{gl}(1|1)$. Examples of logarithmic solutions of KZ equations are also presented.

Keywords

Knizhnik–Zamolodchikov equations · Quantum groups · Tensor categories

Mathematics Subject Classification

Primary 32G34; Secondary 17B37 · 18D10

1 Introduction

Drinfeld–Kohno (DK) theorem [1–5] states braided tensor equivalence between seemingly different categories of modules: on the one hand, quasitriangular quasi-Hopf universal enveloping algebra modules associated with a simple Lie algebra $\mathfrak{g}$ with associator and braiding defined through Knizhnik–Zamolodchikov (KZ) equation with quantum deformation parameter $h$ and on the other hand—of modules of quasitriangular Hopf $h$-quantized universal enveloping algebra associated with $\mathfrak{g}$. By this equivalence the quantization parameter $h$ of the latter algebra corresponds in the former to deformation parameter of associator which arises as a monodromy of KZ
equation solutions associated with the chosen representations category. This theorem
was proved by Drinfeld using series expansion in $h$ around zero and is valid for generic
values of this parameter, with excluded specific rational values. Later on, in the semi-
nal series of papers [6–9], this equivalence was addressed more generally by Kazhdan
and Lusztig. In [6,7] they showed that when $l \not\in \mathbb{Q}$ or when $l \in \mathbb{Q}$ but $l < -h^\vee$,
where $h^\vee$ is the dual Coxeter number of $g$, a certain category of affine level $l$ Lie
algebra $\hat{g}$ modules has a natural braided tensor category structure, and they proved
rigidity for most of these tensor categories in [9]. Kazhdan–Lusztig construction was
then extended by Finkelberg to rational positive level categories of affine modules in
[10]. In this affine algebraic context the KZ equation appears naturally, and Kazhdan
and Lusztig have proved [8,9] braided tensor equivalence of their category at level $l$
to the category of finite-dimensional modules over the quantum group $U_q(g)$ where
$q = e^{m((l+h^\vee)/2)}$, $m$ is the ratio of the squared length of the long roots of $g$ to the squared
length of the short roots.

The interest to this equivalence of representation categories was renewed in the
context of attempts to understand representation theory of logarithmic conformal field
theories [11,12] or of logarithmic vertex operator algebras (VOA)—their mathemati-
cally rigorous incarnation (see, e.g., [13] and references therein in for mathematically
oriented and [14] for physically oriented reviews). One of the main ingredients which
differ logarithmic VOA from rational ones is essential role played by reducible but
indecomposable modules. The current understanding of representation theory of log-
arithmetic VOA is far from being complete. Since set of intertwiner operators of VOA
satisfy KZ equations, analogs of DK theorem, and especially its extension to all the
values of deformation parameter, can add to understanding of the representation theory
of logarithmic VOAs. The VOA related to the affine Lie superalgebra $\hat{gl}(1|1)$ is one of
the archetypical examples of logarithmic VOAs [15]. This motivates to start from DK
theorem for this algebra for suitable category of representations, for generic values
of the affine level, with a hope to extend analysis of this example beyond the scope
of generic values, with further extension to logarithmic VOAs. The description of the
category of modules we consider and restrictions on their parameters corresponding
to situation of generic level (deformation parameter) will be given below.

Of course, the question about DK theorem for superalgebras was addressed before.
It turns out that direct copy of Drinfeld’s proof of DK theorem for Lie superalgebras
is impossible because of the obstacles explained in particular in [16]. Nevertheless
the author succeeded to prove DK theorem for the classical superalgebras applying
Etingof–Kazhdan approach to quantization [17–19] as a bridge for tensor equivalence.
We refer to [16] and references therein for details, which will be reviewed below.

The main object which makes the equivalence of categories explicit is the twist $\mathcal{F}$.
Its explicit, non-perturbative in $h$ construction in the case of simple Lie algebras is
difficult. Some attempt of such explicit construction for simple Lie algebras known to
us, without proofs that the constructions indeed implement full braided tensor equiva-

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The main result of the paper is Theorem 5. It claims that for the superalgebra \( gl(1|1) \) two non-semisimple categories of modules are braided tensor equivalent. The first category is the Drinfeld category \( \mathcal{D} \) with (equivalence classes of) the typical \( T_{e,n} \), atypical \( A_n \), and projective \( P_n \) modules as objects, such that the parameters \( e_i \) satisfy 
\[
\frac{e_i}{\kappa} \notin \mathbb{Z} \quad \text{and} \quad \frac{(e_i + e_j)}{\kappa} \notin \mathbb{Z}\setminus\{0\}
\]
for any pair of typical modules. The second category is the tensor category \( \mathcal{C}_\kappa \) of corresponding modules \( T^\kappa_{e,n}, A^\kappa_n, P^\kappa_n \) of quantum group \( U_\kappa(gl(1|1)) \).

The paper is organized as follows. In Sect. 2 we review the main steps of derivation of KZ equations, first in operator form for intertwining operators, then—for correlation functions of intertwiners. There is almost no difference in it compared to Lie algebra case when non-semisimple finite-dimensional modules are included. In Sect. 3 we define Drinfeld category \( \mathcal{D} \) for any Lie (super)algebra, and its tensor ring structure in the \( gl(1|1) \) case for three types of \( gl(1|1) \)-modules. The main part of this section is the proof of existence of associator in the \( gl(1|1) \) case with its standard properties, as well as the braiding. Section 4 defines the category \( \mathcal{C}_\kappa \) of corresponding \( U_\kappa(gl(1|1)) \) quantum group modules with its tensor product ring and other braided tensor category structures. Section 5 reviews different aspects of proof of equivalence of the two categories of modules. Some perspectives of continuation of this research are summarized in Sect. 6. Many technical details, such as bases of the representations, solutions of KZ equations, their asymptotic needed for the proof of associator existence, are collected in Appendix A 7. Similar technical information about the quantum group side, including the proof of the tensor product ring structure of the modules in specified bases, is found in Appendix B 8.

For the rest of the paper we make an important remark:

The proofs of statements and theorems cited below as known do not use the fact of algebra semisimplicity or semisimplicity of the category of its modules under consideration. The cases where it requires different proofs or leads to different results (like as in analysis of asymptotic solutions of KZ equations) are considered in details. Modifications of proofs related to the fact that we deal with superalgebra are trivial and do not change the cited statements of known theorems. The only needed modifications is in definition of \( \mathbb{Z}_2 \) graded commutator

\[
[A, B] = AB - (-1)^{p(A)p(B)} BA
\]

and the manipulations with tensor products

\[
(A \otimes B)(a \otimes b) = (-1)^{p(B)p(a)} Aa \otimes Bb
\]

where \( p(x) \) is the parity of the object \( x \). An exception from this general rule appears in tensor product decomposition of \( \mathbb{Z}_2 \) graded modules which sometimes involve parity reverse operator. (It will be explained in the proper cases below.)
2 Generic $\kappa$ KZ equation

Below we recall standard derivation of operator KZ equation for intertwiners of any affine algebra $\hat{\mathfrak{g}}$ with some remarks specifying the super-case, for affinization of any category of finite-dimensional $g$-modules (possibly indecomposable) at generic $\kappa$. By $\kappa$ we denote the inverse quantization parameter discussed above $\kappa = h^{-1} = h^\vee + k$, $h^\vee$ is dual Coxeter number and $k$ is the level of affine (super)algebra $\mathfrak{g}$. We also recall standard derivation of KZ equations for correlation functions. The fact that some of modules are indecomposable does not hamper to repeat the standard steps of derivation for generic $\kappa$ (see [21] Lecture 3 for a review). In the case of $\mathfrak{gl}(1|1)$ we have $h^\vee = 0$ and generic means generic values of $k$ which will be specified below.

2.1 Intertwining operators

We start from formulation of affine intertwiners of Tsuchia and Kanie [22], summarized in [21], Lecture 3, generalizing it to the non-semisimple $g$-modules. Let $\mathfrak{g}$ be a simple Lie (super)algebra over $\mathbb{C}$. Let $M_p$ be a finite-dimensional indecomposable (possibly reducible) $g$-module, $p$—some set of parameters which characterize the module. The module is weight: for any homogeneous vector $u \in M_p$, $\hbar u = \lambda u$ for some $\lambda_u \in \mathbb{C}$. We assume that Casimir element $\Omega$ of $U(\mathfrak{g})$ can act non-diagonally, and we decompose $\Omega = C_d + C_{nil}$ where $C_d$ acts diagonally with the same eigenvalue $\lambda_p$ on all the vectors of the module, and $C_{nil}$ is a nilpotent part of non-diagonal action: $(C_{nil})^n = 0$ for some nonnegative integer $n$. In the case of non-super-Lie algebras $M_p$ is assumed to be a highest weight module and $C_{nil} = 0$. We relax this requirement.

In what follows all commutations and tensor products are understood as $\mathbb{Z}_2$ graded for the case of superalgebras. We recall some notations and definitions related to affine Lie (super)algebraic modules. We consider triangular decomposition with respect to $\mathbb{Z}$-grading $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_{\geq 0} \oplus \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_{> 0}$, and induced $\hat{\mathfrak{g}}$-modules $M_{p,k} = \text{Ind}_{\hat{\mathfrak{g}}_{\geq 0}}^{\hat{\mathfrak{g}}_{> 0}} M_p$, for generic $k$, where the action of $\hat{\mathfrak{g}}_{> 0} = \mathfrak{g} \otimes \mathbb{C}[t]$ is trivial and the action of $\hat{\mathfrak{g}}_0 = \mathfrak{g} \oplus k\mathbb{C}$ is such that it is isomorphic to the action of $g$ for the first summand and is multiplication by $k$—for the second. The modules $M_{p,k}$ admit $\mathbb{Z}$-grading

$$M_{p,k} = \bigoplus_{n \geq 0} M_{p,k}[-n]$$

and $M_{p,k}[-n]$ is the eigenspace of affine Lie (super)algebra derivation $d$ with the eigenvalue $-n - \Delta_{p,k}$, $\Delta_{p,k}$ is the conformal dimension. The $M_{p,k}[0]$ is naturally a $g$-module isomorphic to $M_p$.

Recall that for simple Lie algebras we can restrict and define as generic $\kappa = k + h^\vee$ such that $k \notin \mathbb{Q}$. In our superalgebra case we mean by generic $k \in \mathbb{C}$ restricted by suitable constraints dependent on modules parameters, such that they guarantee that if $M_p$ is irreducible then $M_{p,k}$ is also irreducible, and if $M_p$ is of finite length with composition factors $L_{p_i}$, then $M_{p,k}$ is also of finite length with the composition factors $L_{p_i,k} = \text{Ind}_{\hat{\mathfrak{g}}_{\geq 0}}^{\hat{\mathfrak{g}}_{> 0}} L_{p_i}$. We will see below what restrictions on the level $k$ for the affine $\mathfrak{gl}(1|1)_k$ case it implies.
Another important kind of \( \widehat{\mathfrak{g}} \)-modules we need in order to define affine intertwiner is evaluation module. Let \( M_p \) be a \( \mathfrak{g} \)-module which admits weight decomposition with finite-dimensional weight spaces. For a nonzero complex number \( z \) and for any element \( x \otimes P(t) \) of \( \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \) where \( P \) is a polynom, we define its action \( x \otimes P(t) \cdot u = P(t) x u \), where \( u \in M_p \), and the central element of \( \widehat{\mathfrak{g}} \) acts trivially. After that one can extend such action on the whole \( \widehat{\mathfrak{g}} \) by replacing the space of action by a bigger one \( M_p \otimes z^{-\Delta} \mathbb{C}[z, z^{-1}] \). \(^1\)

With a standard extension of \( \widehat{\mathfrak{g}} \) to \( \widehat{\mathfrak{g}} \) by derivation \( d \), we define the action of the affine derivation as \( d = z \frac{d}{dz} \). Here \( \Delta \in \mathbb{C} \) is specified as module-dependent parameter (its conformal dimension). Now we have the generating function for \( \mathbb{Z} \)-graded modes \( v[n] = v \otimes t^n \in M_{p,k} \): \( v(z) = \sum_{n \in \mathbb{Z}} v[n] z^{-\Delta-n} \). We denote the vector space of such objects as \( M_p(z) \).

We are equipped now with all necessary ingredients for construction of affine intertwiner. Assume we can classify all \( \mathfrak{g} \)-homomorphisms of the form \( \varphi : M_{p_1} \rightarrow M_{p_2} \otimes M_p, g \in \mathfrak{g} \). We call them \( \mathfrak{g} \)-intertwiners and we want to lift them to \( \widehat{\mathfrak{g}} \)-intertwiners \( \Phi \). Our consideration will be restricted to the special class of intertwiners which preserve the superalgebra \( \mathbb{Z}_2 \) grading. We define a \( \widehat{\mathfrak{g}} \)-intertwiner as a homomorphism \( \Phi : M_{p_1,k} \rightarrow M_{p_0,k} \otimes M_p(z) \) which satisfies

\[
\Phi(z)x[n] = (x[n] \otimes 1 + z^n \cdot 1 \otimes x)\Phi(z)
\]

where \( z \) is a nonzero complex number. Here \( \widehat{\otimes} \) is understood as a completed tensor product consisting of infinite sums of tensor products of homogeneous vectors.

Recall that in the case of simple non-superalgebras, for the picture of complete braided tensor category (BTC) structure, it is enough to consider affinization of finite-dimensional highest weight \( \mathfrak{g} \)-modules (sometimes called Weyl modules), and evaluation modules. We will do the same for superalgebras relaxing the condition that we affinize and build evaluation modules over highest weight irreducible modules: the \( \mathfrak{g} \)-modules are not necessarily highest weight, and may be reducible but indecomposable. Almost all the steps of intertwiners construction can be copied from the non-super-case. In particular the following assertion can be proved as a slight generalization to the superalgebra case of Theorem 3.1.1 of [21], which was originally proved in [22] in \( \mathfrak{sl}(2) \) case (Theorem 1, followed from Proposition 2.1 and Theorem 2.2)

**Proposition 1** Let \( \varphi : M_{p_1} \rightarrow M_{p_2} \otimes M_p \) be a \( \mathfrak{g} \)-homomorphism. Then for generic \( k \) there exists a unique \( \widehat{\mathfrak{g}} \)-intertwiner \( \Phi(z) : M_{p_1,k} \rightarrow M_{p_0,k} \otimes M_p(z) \) such that for every vector \( v \in M_{p_1,k}[0] \simeq M_{p_1} \) the zero degree component of \( \Phi(z)v \) is equal to \( \varphi v \).

We recall here the sketch of the proof. Because of the annihilation condition for \( w \in M_{p_1,k}[0] \simeq M_{p_1} \) by \( \mathfrak{g} \otimes \mathbb{C}[t] \) we have

\[
\Phi(z)w \in \left( M_{p_0,k} \otimes M_p(z) \right)^{\mathfrak{g} \otimes \mathbb{C}[t]} = \text{Hom}_{\mathfrak{g} \otimes \mathbb{C}[t]} \left( M_{p_0,k}, M_p(z) \right)
\]

\(^1\) Strictly speaking it requires to consider \( z \) as a formal variable \( z \) and restoration of status of complex variable requires subtle procedure worked out in vertex operator algebras formalism by Huang and Lepowsky, and later by Huang, Lepowsky and Zhang for logarithmic vertex operator algebras (see the list of references in [13]).
The contragredient module $M_{p_0,k}^*$ is freely generated over $\mathfrak{g} \otimes \mathbb{C}[t]$ for generic $k$; therefore the restriction map

$$\text{Hom}_{\mathfrak{g} \otimes \mathbb{C}[t]} \left( M_{p_0,k}^*, M_p(z) \right) \to \text{Hom}_{\mathbb{C}} \left( M_{p_0}^*, M_p \right) \simeq (\Pi) M_{p_0} \otimes M_p$$

is an isomorphism. ($\Pi$ means the parity inversion which sometimes is necessary in superalgebra case, see below.) Therefore $\Phi(z)w$ is uniquely defined by its zero grade component.

Then we define the homomorphic action of $\Phi(z)$ on any $Xw = u \in M_{p_1,k}$, where $X \in U(\hat{\mathfrak{g}})$, $X = \prod_i x_{n_i}^{(i)}$, is written in PBW basis, where $x^{(i)} \in \mathfrak{g}$ and $n_i \in \mathbb{Z}$. We define it inductively over each factor $x_{n_i}^{(i)}$ of $X$ by

$$\Phi(z)x_{n_i}^{(i)} = [x_{n_i}^{(i)} \otimes 1 + z^n(1 \otimes x_{n_i}^{(i)})]\Phi(z), \ n < 0$$

It defines an $\hat{\mathfrak{g}}$-intertwiner uniquely with an obvious property that it is a lifting of the $\mathfrak{g}$-intertwiner $\varphi$.

A subtle point of the above definition is that some more general and rigorous construction is needed in order to treat $z$ as a genuine complex variable and not just formal variable. This construction was elaborated in the seminal series of papers by Huang and Lepowsky—see the footnote above—in the framework of vertex operator algebras (VOA). In particular more general intertwiners of the form

$$\mathcal{Y}(\_, z) : W_1 \to \text{Hom}(W_2, W_3)[\log z]$$

are usually needed in logarithmic vertex operator superalgebra (VOSA) $V$ case, where $W_i$ are some $V$-modules. It is precisely relevant for our $\mathfrak{gl}(1|1)$ case, but we will continue to use the definition of intertwiners described above, without use of VOSA language. Despite the lack of proper rigorously we will continue to treat $z$ in our approach as complex variable, defining, where it is needed a branch of multivalued functions. In particular for what follows we chose $\ln z = \ln |z| + i \text{Arg}(z), -\pi < \text{Arg}(z) < \pi$.

An important remark is in order here. Recently, when a preliminary version of this paper was finished, an important progress was achieved in understanding of braided tensor category structure of the $\mathfrak{gl}(1|1)$ VOSA [23].

The next standard step is to extend this $\hat{\mathfrak{g}}$-homomorphism to $\tilde{\mathfrak{g}}$-homomorphism, where $\tilde{\mathfrak{g}}$ is the standard extension of $\hat{\mathfrak{g}}$ by affine derivation $d = -L_0$, with $L_m$ defined by Sugawara construction.

$$L_m = \frac{1}{2(k + h^\vee)} \sum_{a,b} \sum_{n \in \mathbb{Z}} B^{-1}_{ab} : J_n^a J^b_{m-n} :$$  \hspace{0.5cm} (2.3)
where $h^\vee$ is a dual Coxeter number of (super)algebra, and $B$–$\mathfrak{g}$-invariant (super)symmetric non-degenerated bilinear form. If we want to extend the intertwining homomorphism $\Phi(z)$ defined in Proposition 1 to $\bar{\mathfrak{g}}$-homomorphism we have to twist it. We define two twisted intertwiners: for $w \in M_{p_1,k}$

$$
\hat{\Phi}^g(z)w = (z^{L_0} \otimes z^{L_0}) \left( z^{-L_0} \Phi(z) wz^{L_0} \right) (z^{-L_0} \otimes z^{-L_0}), \quad (2.4)
$$

$$
\hat{\Phi}^g(z)w = (z^{L_0} \otimes 1) \left( z^{-L_0} \Phi(z) wz^{L_0} \right) (z^{-L_0} \otimes 1) \quad (2.5)
$$

They remain intertwiners with image in

$$
z^{-L_0} M_{p_0,k} z^{L_0} \otimes z^{-L_0} M_p z^{L_0} [z, z^{-1}]$$

and

$$
z^{-L_0} M_{p_0,k} z^{L_0} \hat{\otimes} M_p [z, z^{-1}],
$$

respectively. In the case of irreducible highest weight modules $M_{p_i}$ with highest weight $p_i$ these twists reduce to the standard scalar factors twists $\Phi^g(z) = \sum_n \Phi(n) z^{-n-\Delta}$, $\Delta = \Delta(p_1) - \Delta(p_0) - \Delta(p)$, and the same for $\hat{\Phi}^g$ with $\Delta(p_1) - \Delta(p_0) - \Delta(p)$, where $\Delta_i = \frac{\langle p_i, p_i + 2p \rangle}{2(k+h^\vee)}$. (The factor $z^{\Delta(p_0) + \Delta(p)}$ is moved to the definition of $\Phi^g(z)$ by the first and the last parenthesis factors.)

For the restricted dual $M_p$ and its evaluation module $M_p^*(z) \cong (M_p(z))^\ast$ which are assumed to be well defined, we can take any vector $u \in M_p^\ast$, define $\hat{\Phi}^g_u(z)w = \langle 1 \otimes u, \hat{\Phi}^g(z)w \rangle$, $w \in M_{p_1,k}$ and regard it as an operator $\hat{\Phi}^g_u(z) : M_{p_1,k} \to M_{p_0,k}$. Then the proof of the theorem [24,25] Theorem 2.1, about the operator form of KZ equation which says that

$$
(k + h^\vee) \frac{d}{dz} \hat{\Phi}^g_u(z) = \sum_{a \in B} : J_a(z) \hat{\Phi}^g_{au}(z) : \quad (2.6)
$$

(summation is over the basis $B$ of $\mathfrak{g}$) generalizes to the case of indecomposable modules $M_{p_i}$ actually without changes. Recall the proof.

Obviously the intertwining relation (2.2) is satisfied for $\hat{\Phi}^g(z)$ as well. Applying contravariant bilinear form in the space $M_p$ this relation can be written as

$$
[\hat{\Phi}^g_u(z), x[n]] = z^n \hat{\Phi}^g_{xu}(z)
$$

If we introduce currents $J_x^\pm(z)$ for any algebra element $x$

$$
J_x(z) = J_x^+(z) - J_x^-(z), \quad J_x^+(z) = \sum_{n<0} x[n] z^{-n-1}, \quad J_x^-(z) = -\sum_{n\geq0} x[n] z^{-n-1}
$$

\footnote{This construction can be modified in the case of non-semisimple (super)algebra. It acts as a scalar on simple modules, but sometimes acts non-diagonally on indecomposables, as for example in the case of $\mathfrak{gl}(1\vert 1)$.}
then in terms of these currents the last intertwining property takes the form

\[ [J^\pm_\chi(\xi), \tilde{\Phi}^g_u(z)] = \frac{1}{z - \xi} \tilde{\Phi}^g_u(z) \]  

(2.7)

(plus sign corresponds to $|\xi| < |z|$, and minus sign—to $|\xi| > |z|$). Now we write the $d$-invariance property of $\tilde{\Phi}^g_u(z)$:

\[ z \frac{d}{dz} \tilde{\Phi}^g_u(z) = -[d, \tilde{\Phi}^g_u(z)] \]

which is the same as

\[ z \frac{d}{dz} \tilde{\Phi}^g_u(z) = -[d, \tilde{\Phi}^g_u(z)] + z \frac{d}{dz} (1 \otimes z^{-L_0}) \tilde{\Phi}^g_u(z)(1 \otimes z^{L_0}) \]

We can continue by Sugawara construction

\[
\frac{B_{-1}^{a,b}}{2(k + h^\vee)} \left( \sum_{n \leq 0} [J^a[n] J^b[-n], \tilde{\Phi}^g_u(z)] + \sum_{n > 0} [J^a[-n] J^b[n], \tilde{\Phi}^g_u(z)] \right) \\
+ z \frac{d}{dz} (1 \otimes z^{-L_0}) \tilde{\Phi}^g_u(z)(1 \otimes z^{L_0}) \\
= \frac{B_{-1}^{a,b}}{2(k + h^\vee)} (2z J^+_b(z) \tilde{\Phi}^g_{au}(z) - 2z \tilde{\Phi}^g_{bu}(z) J^-_a(z) + J^+_b[0] \tilde{\Phi}^g_{au}(z) - \tilde{\Phi}^g_{bu}(z) J^-_a[0]) \\
+ z \frac{d}{dz} (1 \otimes z^{-L_0}) \tilde{\Phi}^g_u(z)(1 \otimes z^{L_0}) \\
= \frac{B_{-1}^{a,b}}{k + h^\vee} : J_a(z) \tilde{\Phi}^g_{bu}(z) : + \frac{B_{-1}^{a,b}}{2(k + h^\vee)} (J^+_b[0] \tilde{\Phi}^g_{au}(z) - \tilde{\Phi}^g_{bu}(z) J^-_a[0]) \\
+ z \frac{d}{dz} (1 \otimes z^{-L_0}) \tilde{\Phi}^g_u(z)(1 \otimes z^{L_0})
\]

The last two terms cancel because they can be written as

\[
\frac{1}{2(k + h^\vee)} \tilde{\Phi}^g_{Cu}(z) - \Delta(p) \tilde{\Phi}^g_u(z)
\]

where $C = B_{-1}^{a,b} J_a J_b$ is a Casimir element of the algebra $g$, (not to be confused with tensor Casimir defined in (2.15). This completes the proof.

Concluding this section about systematic definition of intertwining operators for affine Lie (super)algebra we can illustrate an important difference of a non-semisimple case from a semisimple one. Suppose we have a finite-dimensional $g$-module with non-semisimple action of the Casimir element $C$ which we can represent as $C = \lambda I + C_{\text{nil}}$, where $I$ acts as identity and $C_{\text{nil}}$ acts nilpotently: $C_{\text{nil}}^n = 0$ on each vector of the module.
Then the action of $z^a C$ where $z, \lambda \in \mathbb{C}$, on any vector $w$ of the module can be written as

$$z^C w = z^\lambda \sum_{m=1}^{n-1} \frac{(\ln z)^m}{m!} C_{\text{nil}}^m w$$

(2.8)

leading to a presence of logarithms (with a choice of a branch), which necessarily arises with non-semisimplicity in logarithmic vertex operator algebras and in logarithmic conformal field theories. A modification of this example for the operator like $z^{\Omega_{ij}}$ will appear below where the Casimir $C$ is replaced by quantum Casimir $\Omega_{ij}$, see (2.15) below.

### 2.2 KZ equation for correlation functions

The way of derivation of correlation functions KZ equation from operator KZ equation (2.6) first appeared in the seminal paper of Knizhnik and Zamolodchikov [25]. Later it was derived more rigorously in [24] and in textbooks like [21], Sect. 3.4. Possible non-semisimplicity of Lie (super)algebra $g$-modules does not lead to serious modifications in the derivation. We again recall the main steps of it.

In order to define correlation function consider the modules $M_{q_i,k}, i = 1, \ldots, N$, and $M_{p_i}, i = 1, \ldots, N + 1$. Let $\hat{\Phi}^i(z_i) : M_{p_i,k} \rightarrow M_{p_{i-1},k} \otimes M_{q_i,z_i^{\pm 1}}$ be an intertwiner as explained above, where $M_{q_i}(z_i)$ is evaluation module. We consider the homomorphism

$$\Psi(z_1, \ldots, z_N) = \left( \hat{\Phi}^1(z_1) \otimes 1 \ldots \otimes 1 \right) \ldots \left( 1 \otimes \cdots \otimes \hat{\Phi}^{N-1}(z_{N-1}) \otimes 1 \right) \times (1 \otimes \cdots \otimes \hat{\Phi}^N(z_N))$$

(2.9)

that maps $M_{pN,k} \rightarrow M_{p0,k} \otimes M_{q1} \otimes \cdots \otimes M_{qN}$.

This formula for homomorphism makes sense at least being understood as formal power series in $z_1, z_2, \ldots, z_N$ and their logarithms.

Consider a subspace of weight $\lambda_N$ of $M_{pN,k}[0]$, and subspace of weight $-\lambda_0$ of $M^*_{p0,k}[0]$. The object $\Psi(z_1, \ldots, z_N)_{\lambda_N}$ takes values in the space $M_{q_1} \otimes M_{q_2} \cdots \otimes M_{q_N} \otimes M_{p0}$. We can take a projection of it onto finite-dimensional invariant subspace of the weight $\lambda_N - \lambda_0$ in the $M_{p0}$ component of it $V = (M_{q_1} \otimes M_{q_2} \cdots \otimes M_{q_N})_{\lambda_N - \lambda_0}$. If we take $\lambda_N = \lambda_0$, then we get the $g$ invariant subspace $V^g$. This sort of projection of $\Psi$ on such a subspace, with some chosen $u_{N+1} \in M_{pN,k}[0], u_0 \in M_{p0,k}[0]$, is called a correlation function

$$\psi(z_1, \ldots, z_N) = \langle u_0, \Psi(z_1, \ldots, z_N)u_{N+1} \rangle$$

$$\psi(z_1, \ldots, z_N) \in (M_{q_1} \otimes M_{q_2} \cdots \otimes M_{q_N})^{\lambda_N - \lambda_0}$$

(2.10)

(the vector $\langle u_0 | \in M^*_{p0,k}[0]$) Taking into account the remark (2.8) we can say that $\psi$ here is defined as a formal power series: it belongs to

\[ \text{ Springer} \]
\[ \prod_i z_i^{-\Delta(p_i)+\Delta(p_{i-1})+\Delta(q_i)} (\ln z_i/z_{i-1})^{\eta_i} \mathbb{C}[\{z_1, \ldots, z_N\}]. \] Equivalently one can define correlation function as \( \mathbb{C} \)-valued if choosing \( u_i \in M_{q_i}, i = 1, \ldots, N \), we define
\[
\psi_{u_1,\ldots,u_{N+1}}(z_1,\ldots,z_N) = \langle u_0, \tilde{\Phi}^{g_i}_{u_1}(z_1) \ldots \tilde{\Phi}^{g_N}_{u_{N+1}}(z_N) u_{N+1} \rangle \in \mathbb{C}
\] (2.11)
In particular one can take \( M_{p_0,k} = M_{p_N,k} \) to be the scalar representation \( M_0 \), i.e., \( M_{p_0,k}, M_{p_N,k} \)—induced vacuum modules with the zero grade vector \( u_0 \), and define \( V \)-valued correlation function.
\[
\phi(z_1,\ldots,z_N) = \langle u_0, \Psi(z_1,\ldots,z_N) u_0 \rangle
\] (2.12)
Then \( \phi(z_1,\ldots,z_N) \in V^g \).

The main theorem proved in [25] for simple highest weight modules of (non-super) algebra claims the KZ equation on (2.10) in the form
\[
(k + h^\vee) \partial_i \psi = \left( \sum_{j \neq i=1}^{N} \frac{\Omega_{ij}}{z_i - z_j} + \frac{\Omega_{i,N+1}}{z_N} \right) \psi, \quad i = 1, \ldots, N + 1
\] (2.13)
Equivalent form of KZ equation can be obtained by adding one more formal variable \( z_{N+1} \) to the function \( \psi(z_1,\ldots,z_N) = \psi(z_1 - z_{N+1},\ldots,z_N - z_{N+1}) \), giving
\[
(k + h^\vee) \partial_i \psi = \left( \sum_{j \neq i=1}^{N+1} \frac{\Omega_{ij}}{z_i - z_j} \right) \psi, \quad i = 1, \ldots, N + 1
\] (2.14)
Here we denote tensor Casimir
\[
\Omega_{ij} = B^{-1}_{ab}(x^a)_i \otimes^s (x^b)_j
\] (2.15)
(the lower indices \( i, j \) indicate the spaces of the tensor product in \( V \) where the generators \( x^a \) act.) and \( z_{N+1} = 0 \). Recall that the vectors \( u_0 \in M_{p_0,k}[0] \) and \( u_{N+1} \in M_{p_N,k}[0] \) have grade 0. Here we use the super-tensor product which for two matrices \( A_{\alpha \gamma} \) and \( B_{\beta \delta} \) is defined as \( (A \otimes^s B)_{\alpha \beta}^{\gamma \delta} = (-1)^{\beta(\alpha + \gamma)} A_{\alpha \gamma} B_{\beta \delta} \), where the indices lifted to exponential of \( (-1) \) are parities of corresponding indices in \( \mathbb{Z}_2 \) graded vector spaces. The main difference compared to the usual non-superalgebras and irreducible finite-dimensional highest weight modules is that \( \Omega_{ij} \) can act now non-diagonally on the modules. In this sense they are not eigenvalue numbers but operators. With the assumption that \( u_{N+1} \) is the vector of scalar representation (at least in the sense described in the footnote) the last term in (2.13) disappears, and the equation we will deal with in what follows

---

3 In the superalgebras case it sometimes happens that a scalar representation appears only as a (part of) atypical module. By general tensor category “ideology” atypical modules should be replaced by their projective covers. But even then there is a “bottom” vector \( u_{N+1} \) in it satisfying \( gu_{N+1} = 0 \).
The proof of the theorem claiming (2.16) for correlation functions for superalgebras with non-semisimple modules is a copy of the proof in the case of simple modules over usual Lie algebras. The proof uses commutation relations (2.7) and the fact that $u_0, u_{N+1}$ are zero grade states.

Looking for solutions for $\psi \in V^g$ is not the only option. One can get a set of solutions when $\psi$ is projected onto some weight subspace $\psi \in V^\lambda$ of weight $\lambda$. Usually, when the spaces $M_{p_i}$ are highest weight ones $\mu_i$, the solutions with values in the space $(V^{n^+})^\lambda$ are considered. If $\lambda = \sum \mu_i - \mu, \mu = \sum n_i\alpha_i$, $\alpha_i \in Q^+$, the value $|\mu|$ is the level of the equation. Usually level one solutions for $N = 3$ already give solutions with a basis of hypergeometric functions. But in order to see such hypergeometric solutions in $V^g$, one has to take at least $N = 4$ correlation functions.

Important particular case of KZ equation when it becomes an ordinary differential equation, is the $N = 3$ case. As one can show (see, e.g., [21]), in this case any solution of KZ equation can be written as

$$\psi(z_1, z_2, z_3) = (z_1 - z_3)(\Omega_{12} + \Omega_{13} + \Omega_{23})/\kappa f\left(\frac{z_1 - z_2}{z_1 - z_3}\right)$$

where $f(z) \in V$ satisfies the differential equation

$$\kappa \partial_z f = \left(\frac{\Omega_{12}}{z} + \frac{\Omega_{23}}{z - 1}\right) f \quad (2.17)$$

For the irreducible modules $M_{q_1}, \ldots, M_{q_N}$ of highest/lowest weight there is a classification and explicit form of solutions of KZ equation for specified level of weights in root lattice grading. Level zero solution is always of the form

$$\psi_0(z_1, \ldots, z_N) = \psi_0(z_1, \ldots, z_N)v, \quad v = \mu_1 \otimes \mu_2 \cdots \otimes \mu_N,$$

$$\psi_0(z_1, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^{\mu_i \mu_j / 2\kappa}$$

Solutions of higher levels of KZ equations in the case of highest or lowest weight modules $M_{\lambda_i}$ at generic $\kappa$ one can obtain by the following procedure. (We consider highest weight modules). Define multivalued function

$$\phi_1(z_1, \ldots, z_N, t) = \prod_{i=1}^N (t - z_i)^{\mu_i / \kappa}$$

---

4 It will be interesting to find a direct way to obtain nonzero level solution from the zero level solutions ones, as it was done in non-super-case [26].
and fix a closed contour $C$ in $t$ complex plane not containing any of $z_i$, and having a continuous branch along $C$. Example of such contour is Pochhammer contour for two $z_a, z_b$. Existence and classification of such contours is known for semisimple case, but is a non-trivial question for non-semisimple case. Then a general level one solution $Ψ_1(z_1, \ldots, z_N)$ can be obtained as

$$Ψ_1(z_1, \ldots, z_N) = ψ_0(z_1, \ldots, z_N) \sum_{r=1}^{N} \left( \int_C dt \phi_1(z_1, \ldots, z_N, t) \frac{1}{t - z_r} \right) f_r v$$

(2.18)

where $v = v_1 \otimes \ldots \otimes v_N$ is the highest weights tensor product, and the step operator $f_r$ acts on the $r$th component of tensor product. The proof is by direct calculations. Explicit realization of this solution gives rise to integral representations of hypergeometric functions $2F_1$. Level $l$ solution can be similarly generated by integration of operator valued differential $l$-forms. The answer in this case is much more involved [26].

For the case of semisimple categories of finite-dimensional $g$-modules at generic level $κ$ the most important statement says that the monodromy of KZ equations give rise to braided tensor categories and that they are equivalent to the categories of specific quantum group representation. One of the ways to see it for generic level case was worked out by Schechtman and Varchenko [27] using the integral formulas of the KZ solutions by analysis of geometry of integration cycles. Can the same be done in the case of non-semisimple categories of $g$-modules when solutions involve logarithms? We are going to address this question elsewhere.

All the construction above treats $z_i$ as formal variables. There is a theorem proved for KZ equations in semisimple case that $ψ$ is an analytic function of $z_i$ in the region $|z_1| > |z_2| > \cdots > 0$. This analyticity should be modified in the non-semisimple case because of presence of logarithms in intertwiners mode expansions.

Consistency and $g$-invariance of KZ equation, as in semisimple case, follow from $g$-invariance of Casimir operator. It has an important practical application: in order to find the full set of independent KZ equations for a given correlation function one should find the basis of invariants of the space $V$ — the set of tensor product vectors annihilated by all the generators of $g$, and then project the equations on these vectors. One can find some examples of such calculations in “Appendix 7.3.” Explicit construction of tensor category structures of solutions of KZ equations requires calculations up to $N = 4$ — four-point correlation functions.

The final goal is investigation of monodromy properties of solutions of KZ equation. By this we mean the following. The system of KZ equations being consistent can be interpreted as a flat connection in the trivial vector bundle with the fiber $V$ over the configuration space $X_N = \{(z_1, z_2, \ldots, z_N) \in \mathbb{C}^N \mid z_i \neq z_j\}$. For any path $γ : [0, 1] \to X_N$ we denote by $M_γ$ the operator of holonomy along $γ$. It can be considered as an operator in $V$ and it depends only on homotopy class of $γ$, or as operator of analytic continuation along $γ$. From $g$-invariance of $Ω$ follows that for any $γ$ $M_γ : V \to V$ is a $g$-homomorphism. If $V$ is completely reducible, then it
means that $M_\gamma$ preserves subspace of singular vectors in $V$ and is uniquely defined by its action on this subspace.

3 Drinfeld category of $\mathfrak{gl}(1|1)$ modules

In this section we consider the KZ equation as an equation on functions

$$\psi(z_1,\ldots,z_N): \mathbb{C}^N\{\text{Diag}\} \to V[[\kappa^{-1}]]$$

where the set of points $\{\text{Diag}\} : z_i = z_j, i \neq j$ are removed from the domain $\mathbb{C}^N$. The functions are valued in $V[[\kappa^{-1}]]$, where $V = V_1 \otimes \cdots \otimes V_N$ is a tensor product of moduli representation spaces $V_i$ of the superalgebra $\mathfrak{gl}(1|1)$. We define abelian tensor Drinfeld supercategory $\mathcal{D}$ of subset of finite-dimensional non-semisimple $\mathfrak{gl}(1|1)$-moduli with all moduli homomorphisms as the category morphisms and construct its braided tensor category structure.

The superalgebra $\mathfrak{gl}(1|1)$ is the algebra of endomorphisms of the vector super-space $\mathbb{C}^{1|1}$ $\mathfrak{gl}(1|1) = \text{span}\{E, N, \psi^+, \psi^-\}$ with two-dimensional even $\mathfrak{gl}(1|1)_e = \text{span}\{E, N\}$ and two-dimensional odd $\mathfrak{gl}(1|1)_o = \text{span}\{\psi^+, \psi^-\}$ subspaces written in the superalgebra basis. The commutation relations of the algebra, explicit form of the set of typical modules $\{\text{Diag}\}$: $\text{Diag}_{\{z_1,\ldots,z_N\}}$ is a tensor product of

$$\mathcal{A}_n \otimes \mathcal{A}_{n'} = \mathcal{A}_{n+n'}, \quad \mathcal{A}_n \otimes \mathcal{T}_{e,n'} = \mathcal{T}_{e,n+n'}$$
$$\mathcal{T}_{e,n} \otimes \mathcal{T}_{e',n'} = \mathcal{T}_{e+e',n+n'+1/2} \oplus \mathcal{P} \mathcal{T}_{e+e',n+n'+1/2}$$
$$\mathcal{T}_{e,n} \otimes \mathcal{T}_{-e,n'} = \mathcal{P}_{n+n'}, \quad \mathcal{A}_n \otimes \mathcal{P}_{n'} = \mathcal{P}_{n+n'}$$
$$\mathcal{T}_{e,n} \otimes \mathcal{P}_n = \mathcal{P}_n \mathcal{T}_{e,n+n'} + \mathcal{P} \mathcal{T}_{e,n+n'} \otimes \mathcal{P}_n$$

The functor $\mathcal{P}$ for some modules on the right hand side denotes parity reversion of the $\mathbb{Z}_2$ grading of even and odd module subspaces. The obvious requirements on the set of parameters $n_i$ of the modules in the category similarly follow by closure of the tensor product decomposition. No other restrictions on the moduli parameters $n_i$ are imposed.
Of course there are infinitely many other finite-dimensional indecomposable \( g\ell(1|1) \)-modules, but our choice seems to be the minimal set of (isomorphism classes of) modules closed under the tensor product decomposition with a non-trivial braiding structure described below.

The indecomposable modules \( \mathcal{P}_n \) are called projective, because they are projective covers for \( A_n \). The typical modules \( T_{e,n} \) are their own projective covers. Indecomposable structure of the modules can be found in the same reference [15], Section 2.2. The modules of our category are finitely generated and are semisimple under the action of the even part of the superalgebra. Some properties of such categories of \( g\ell(m|n) \)-modules were reviewed in [28].

We see that one should include in the category the modules obtained by the parity change functor \( \Pi \). It means the above tensor rules should be completed by the copy of them with the obvious action of \( \Pi \), which we omit for brevity. All the statements below will be proved for the part of tensor ring (3.1) and are identical for its parity change analog. The standard parity for the modules is chosen in the following way. We assume the highest weight of the two-dimensional typical module \( \sigma \) to be Grassmann even, as well as the one-dimensional atypical module \( A_{n} \), and the top vector of the projective module \( \mathcal{P}_n \) (see “Appendix 7.2”) to be also even.

The structure of braided tensor category \( (\mathcal{D}, \times, 1, \lambda, \rho, \sigma) \) is defined as follows. The bifunctor \( \mathcal{D} \times \mathcal{D} \to \mathcal{D} \) is the tensor product of the modules that was described above. The unit object of \( \mathcal{D} \) is \( 1 = A_0 \) is simple and as follows from (3.1) the functorial isomorphisms \( \lambda : 1 \otimes U \xrightarrow{\sim} U, \rho : U \otimes 1 \xrightarrow{\sim} U \) are trivial. Below we will define and prove the existence of invertible associator—functorial isomorphism \( \alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \) for any triple of objects \( X, Y, Z \in \text{Obj}(\mathcal{D}) \). This isomorphism is defined using asymptotic solutions of KZ equations. The braiding \( \sigma : X \otimes Y \to Y \otimes X \) of any two objects is defined by \( \sigma = P e^{i \pi \Omega_{12}/\kappa} \) where \( P \) is graded permutation. The prove of coherence theorem for associator, i.e., pentagon and triangle relations for monoidal structure, becomes standard after the explicit construction of associator, as well as the proof of hexagon relation for braiding.

First we briefly recall the monodromy structure and asymptotic solutions of KZ equations for semisimple category of modules. We follow and recapitulate the main steps presented in [29], Section 2. The system of KZ equations can be interpreted as a flat connection in a trivial vector bundle with a fiber \( V = V_1 \otimes \cdots \otimes V_N, V_i \) are objects of \( \mathcal{D} \), over the configuration space \( X_{N} = \{ (z_1, \ldots, z_N) \in \mathbb{C}^N | z_i \neq z_j \} \). For any path \( \gamma : [0, 1] \to X_{N} \) one denotes by \( M_{\gamma} : V \to V \) the operator of holonomy along \( \gamma \), which can be considered as analytic continuation of KZ equation solutions \( \psi(z_1, \ldots, z_N) \) along \( \gamma \). \( M_{\gamma} \) is \( \mathfrak{g} \)-homomorphism since the tensor Casimir operator \( \Omega \) of KZ equation is \( \mathfrak{g} \)-invariant. Operator \( M_{\gamma} \) with \( \gamma(0) = \gamma(1) = z^0 = (z^0_1, \ldots, z^0_N) \) is called the monodromy operator. We have such \( M_{\gamma} \) as a monodromy representation of the fundamental group \( \pi_1(X_{N}, z^0) \) in \( V \). The dependence on the base point \( z^0 \) can be eliminated by conjugation, because \( X_{N} \) is connected. But the fundamental group \( \pi_1(X_{N}) \) is well known—it is \( P B_{N} \)—pure braid group. Moreover, one can construct the homomorphism of braid group \( B_{N} \to \pi_1(X_{N}/S_{N}) \) where \( S_{N} \) is the symmetric group: if we choose the \( z^0 \) such that \( z^0_i \in \mathbb{R} \) and \( z^0_1 > z^0_2 > \cdots > z^0_N \) then the action of \( b_i \) generator of \( B_{N} \) on \( z^0 \) corresponds to transposition of \( z^0_i \) and \( z^0_{i+1} \) (say, \( z^0_{i+1} \) and \( z^0_i \) is graded permutation. The prove of coherence theorem for associator, i.e., pentagon and triangle relations for monoidal structure, becomes standard after the explicit construction of associator, as well as the proof of hexagon relation for braiding.

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$z_i^0$ exchange their locations such that $z_i^0$ passes above $z_{i+1}^0$). For a fixed base point $z^0$ a loop $\gamma$ in $X_N/S_N$ can be considered as an element of $B_N$. Then we can lift it to a path in $X_N$ defining the operator $\hat{M}_\gamma = \sigma M_\gamma : V \to V^\sigma$, where $\sigma \in S_N$ is the image of $\gamma$ under the map $B_N \to S_N$ and $V^\sigma = V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(N)}$. For example, for the $\gamma$ which exchanges $z_i^0$ and $z_{i+1}^0$ we will have $\hat{M}_\gamma^\pm(z^0) = \hat{M}_{\gamma+1}^\pm$. The fact that the operators $\hat{M}_\gamma^\pm$ called half monodromy operators satisfy the equations

$$\hat{M}_\gamma^\pm \hat{M}_\gamma^\mp = I,$$
$$\hat{M}_\gamma^\pm \hat{M}_{\gamma+1}^\pm \hat{M}_\gamma^\mp = \hat{M}_{\gamma+1}^\pm \hat{M}_\gamma^\mp \hat{M}_{\gamma+1}^\mp$$

follows from the relation $\gamma \gamma_{i+1} \gamma_i = \gamma_i \gamma_{i+1} \gamma_i$ in the fundamental group of $X_N/S_N$.

The (half)monodromy operators being independent on the choice of base point can be calculated with a specific choice of it. One of the convenient choices of the base point is $z^0 : z_1^0 \gg z_2^0 \gg \cdots \gg z_N^0$. We will need also another choice of the base point for $N = 3$ correlation function below. We fix the region $D \subset X_N$, $D = \{ z = (z_1, \ldots, z_N) \in \mathbb{R}^N \mid z_1 > \cdots > z_N \}$. There is an isomorphism between the space of $V$-valued solutions $\Gamma_f(D, V_{KZ})$ of the KZ equation in the region $D$ and $V$: for any $z \in D$ the solution $\psi(z)$ is this isomorphism. It is useful to make the following change of variables.

$$u_i = \frac{z_i - z_{i+1}}{z_i - z_{i+1}}, \quad i = 2, \ldots, N - 1$$
$$u_1 = z_1 - z_2, \quad u_N = z_1 + \cdots + z_N$$

(3.2)

All $u_i$ are positive on $D$. One can see that $(z_1, \ldots, z_N) \to (u_1, \ldots, u_N)$ is one to one map with inverse polynomial mapping; therefore any analytic function $f(z)$ on $D$ can be considered as analytic function of $u$ on some subset $D_u \subset \mathbb{C}^N$—the image of the mapping, and closure of $D_u$ contains the origin. The change of variables (3.2) is chosen so that if we have a curve $z(t)$ such that $z(t) \to 0$ when $t \to 0$, then the condition $z_i(t)/z_{i+1}(t) \to \infty$ for $i = 1, \ldots, N - 1$, implies $u_i(t) \to 0$ for $i = 1, \ldots, N$.

We can define now the limit $\lim_{z_1 \gg \cdots \gg z_N} f(z) = v$ as a vector which satisfies $\lim_{u_i \to 0} f(u) = v$ for $i = 1, \ldots, N$ with $f$ being written in terms of new variables $u_i$.

We define the asymptotic $f \sim \phi_1(z)v$ of a smooth vector valued function $f(z)$ as the $z_1 \gg \cdots \gg z_N$ limit of $f$ in $D$, for some scalar function $\phi_1(z)$ and a vector $v \in V$, if

$$f(z) = \phi_1(z)(v + o(z))$$

(3.3)

where $o(z)$ considered as a $V$-valued function of $u$ in some neighborhood of the origin is regular and $o(u = 0) = 0$. We will sometimes put $z_N = 0$. If $f$ is translation invariant, then $\lim_{z_1 \gg \cdots \gg z_N} f(z) = \lim_{z_1 \gg \cdots \gg 0} f(z)$.

Another region we need is $D_0(z) : z_1 - z_2 \ll z_2 - z_3 \ll \cdots \ll z_{N-1} - z_N$ and as above we define the asymptotic of a function $f(z)$ in the region $D_0(z)$ as $f \sim \phi_0(z)v$. 
if \( f(z) = \phi_0(z)(v + o(z)) \) where \( o(z) \) considered as a \( V \)-valued function of \( u \) in some neighborhood of the point \( u_i \rightarrow \infty \).

The special case important for the proof of associator existence is \( N = 3 \). The KZ equation takes the form of ordinary differential equation in one variable. In terms of the variables (3.2) \( u_1 = z_1 - z_2, u_2 = \frac{z_1 - z_3}{z_1 - z_2}, u_3 = z_1 + z_2 + z_3 \) the KZ equations look like

\[
\begin{align*}
\kappa \partial_{u_1} \psi & = \frac{\Omega_{12} + \Omega_{13} + \Omega_{23}}{u_1} \psi \\
\kappa \partial_{u_2} \psi & = \left( \frac{\Omega_{12}}{u_2 + 1} + \frac{\Omega_{23}}{u_2} \right) \psi \\
\kappa \partial_{u_3} \psi & = 0
\end{align*}
\]

We introduce the function \( f \) defined by\(^5\)

\[
\psi(z_1, z_2, z_3) = (z_1 - z_3)^{\Omega_{12} + \Omega_{13} + \Omega_{23}}/\kappa f \left( \frac{z_1 - z_2}{z_1 - z_3} \right)
\]

Using the fact that all \( \Omega_{ij} \) commute with \( \Omega_{12} + \Omega_{13} + \Omega_{23} \) one can see by direct calculation that \( f = u_1^{-(\Omega_{12} + \Omega_{13} + \Omega_{23})/\kappa} \psi \) depends only on \( x = \frac{1}{u_2 + 1} \) and is \( u_1, u_3 \) independent. Thus we get one ODE for the \( V \)-valued function \( f(x) \)

\[
\kappa \partial_x f(x) = \left( \frac{\Omega_{12}}{x} + \frac{\Omega_{23}}{x - 1} \right) f(x)
\]

The asymptotic regions \( D_0(z), D_1(z) \) correspond to \( x \rightarrow 0 \) and \( x \rightarrow 1 \), respectively. The existence of asymptotic solutions of KZ equation as they are defined above is the main tool for the proof of existence of associator.

**Theorem 1** Let \( V = V_1 \otimes V_2 \otimes V_3 \) where \( \{V_i\} \)—any combination from the set \( \{\mathcal{A}, \mathcal{P}, \mathcal{T}\} \). If \( e_i \notin \mathbb{Z} \) and \( e_1 + e_2 \notin \mathbb{Z}\backslash\{0\} \) in the case \( V_i = \mathcal{T} \), \( i = 1, 2 \) in \( V \), then for every eigenvector \( v \in V \) of \( \Omega_{12} \) there exists unique asymptotic solution of (3.5) around 0 corresponding to \( v \) and this correspondence gives isomorphism \( \phi_0 : \Gamma_f(D, V_{KZ}) \rightarrow V \).

**Proof** The proof is based on straightforward linear algebra manipulations which we moved to Appendix A. We apply Lemma 1 or 2 (see Appendix A), considering all possible 6 combinations (up to a permutation) of \( V_1, V_2 : \mathcal{T}_{e_1,n_1} \otimes \mathcal{T}_{e_2,n_2}, \mathcal{T}_{e_1,n_1} \otimes \mathcal{P}_{n_2}, \mathcal{P}_{n_1} \otimes \mathcal{P}_{n_2}, \mathcal{T}_{e_1,n_1} \otimes \mathcal{A}_{n_2}, \mathcal{P}_{e_1,n_1} \otimes \mathcal{A}_{n_2}, \mathcal{A}_{n_1} \otimes \mathcal{A}_{n_2} \). The explicit form of the function solution \( \phi(x) \) is not important at this point, but one can find it in Appendix A. All we have to do is to check, case by case, the applicability of Lemmas 1, 2. Isomorphism to the space \( \Gamma_f(D, V_{KZ}) \) of KZ solution follows by linearity. The following data is obtained by direct diagonalization of \( \Omega_{12} \) on the basis of \( V_1 \otimes V_2 \).

1. \( \mathcal{T}_{e_1,n_1} \otimes \mathcal{T}_{e_2,n_2} \).

\(^5\) This function is well defined with the choice of the branch of logarithm fixed above because the operators \( \Omega_{ij} \) acting in the space \( V \) have nilpotent non-diagonalizable part.
When \( e_2 + e_1 \notin \mathbb{Z} \), there are no Jordan blocks and the eigenvalues are \( \lambda_1 = \delta_{12}^{++} \), \( \lambda_2 = \delta_{12}^{-} \), with two eigenvectors for each of them. Here and below \( \delta_{ij}^{\alpha\beta} = e_i e_j + e_i (n_j + \beta/2) + e_j (n_i + \alpha/2) \). The difference \( \lambda_1 - \lambda_2 = e_1 + e_2 \notin \mathbb{N} \) and by Lemma 1 there are four asymptotic solutions for four different eigenvectors.

When \( e_2 + e_1 = 0 \), there is one eigenvalue \( e_1 (n_2 - n_1) - e_1^2 \) with two eigenvectors without Jordan block and two other ones with Jordan block of size 2. By Lemma 2 there are four asymptotic solutions.

We cannot prove existence of asymptotic solutions using Lemma 1 in the case \( e_2 + e_1 \in \mathbb{Z} \setminus \{0\} \), but this case, from the perspective of affine Lie superalgebra, exactly corresponds to what we call non-generic case of representations [14].

2. \( T_{e_1,n_1} \otimes P_{n_2} \)

The set of eigenvalues are \( \lambda_1 = e_1 (n_2 - 1) \) and \( \lambda_2 = e_1 (n_2 + 1) \) with the difference \( 2e_1 \notin \mathbb{N} \). Each of them correspond to two eigenvectors without Jordan block and one Jordan block of size 2. By Lemmas 1, 2 there are asymptotic solutions for each eigenvector.

3. \( P_{n_1} \otimes P_{n_2} \)

There is one eigenvalue \( \lambda = 0 \) with the following structure of eigenvectors: there are 3 Jordan blocks of rank 2, one Jordan block of rank 3 and 7 eigenvectors without Jordan block structure. Again the condition \( \lambda + \mathbb{N} \) is not an eigenvalue is satisfied, therefore by Lemmas 1, 2 there are asymptotic solutions corresponding to each eigenvector.

4. \( T_{e_1,n_1} \otimes A_{n_2} \)

There is one eigenvalue \( \lambda = e_1 n_2 \) with two different eigenvectors without a Jordan block. Lemma 1 is applicable.

5. \( P_{n_1} \otimes A_{n_2} \)

There is one eigenvalue \( \lambda = 0 \) with four different eigenvectors without a Jordan block. Lemma 1 is applicable.

6. \( A_{n_1} \otimes A_{n_2} \)

There is one eigenvalue \( \lambda = 0 \) with one eigenvector. Lemma 1 is applicable.

\[ \square \]

**Theorem 2** The same claim as in Theorem 1, with the same restrictions on the parameters of typical modules \( T \) appearing as \( V_i, i = 2, 3 \) in \( V \), is valid for existence and uniqueness of asymptotic solutions of KZ equation (3.5) around \( x = 1 \).

**Proof** The proof is based on Lemma 3 (see Appendix A) that replaces Lemmas 1, 2 in the proof of Theorem 1.

As we see, there are specific cases \( 2e_1 \in \mathbb{Z} \) and \( e_1 + e_2 \in \mathbb{Z} \setminus \{0\} \) for parameters of typical representations when we are not able to guarantee the existence and uniqueness of asymptotic solutions by Lemmas 1, 2, 3. We notice that for affine \( \widehat{gl}(1|1) \) (where the we always can put \( \kappa = k = 1 \)) these cases correspond to reducibility of the induced affine modules, and as we said above, we exclude these cases in the process of derivation of KZ equation.

**Proposition 2** With the restrictions on the parameters of typical modules as in Theorem 1 there is an isomorphisms of the spaces

\[ \alpha_{1,2,3} : (V_1 \otimes V_2) \otimes V_3 \longrightarrow \Gamma_f(D, V_{KZ}) \longrightarrow V_1 \otimes (V_2 \otimes V_3) \quad (3.6) \]
which will serve the associator in the Drinfeld tensor category.

**Proof** The first isomorphism $\phi_0 : (V_1 \otimes V_2) \otimes V_3 \rightarrow \Gamma_f(D, V_{KZ})$ is defined by the correspondence between the eigenvectors of $\Omega_{12}$ in $V$ and asymptotic solutions of KZ equation (3.5) around $x = 0$ established by Theorem 1. The second isomorphism $\phi_1^{-1} : \Gamma_f(D, V_{KZ}) \rightarrow V_1 \otimes (V_2 \otimes V_3)$ is the inverse of the isomorphism $\phi_1$ established by Theorem 2. 

**Remark 1** One can easily see that the associator (3.6) is trivial (equal to 1) when one of the spaces $V_i$, $i = 1, 2, 3$ is one-dimensional, as for example in the cases 4, 5, 6 of the proof of Theorem 1.

**Theorem 3** For any quadruple of objects $V_i$, $i = 1, \ldots, 4$ in the $\mathfrak{gl}(1|1)$ Drinfeld category $\mathcal{D}$, with the restrictions on the parameters of typical modules $e_i \notin \mathbb{Z}$, $e_i + e_j \notin \mathbb{Z} \setminus \{0\}$ for any pair $\mathcal{T}_{e_i, n_j}$, $\mathcal{T}_{e_j, n_j}$, the isomorphism $\alpha_{1,2,3}$ (3.6) satisfies pentagon equation $(V_1 \otimes V_2) \otimes V_3 \otimes V_4 \rightarrow V_1 \otimes (V_2 \otimes V_3 \otimes V_4)$

$$\alpha_{id_1 \otimes 2,3,4} \circ \alpha_{1,2 \otimes 3,4} \circ \alpha_{1,2,3 \otimes 1d_4} = \alpha_{1,2,3 \otimes 4} \circ \alpha_{1 \otimes 2,3,4} \quad (3.7)$$

The proof is based on decomposition of pentagon diagram into triangle ones, and each triangle is a commutative diagram which includes as a part the isomorphism (3.6). The proof uses only the fact of existence and uniqueness of invertible associator irrespectively of details of its construction from asymptotic solutions. We refer to the books [30], p.25, or [31], p.545 for details of the proof, which is independent on concrete form of asymptotic solutions but only on the fact of their existence. 

Recall the standard derivation of braiding $\sigma_{X,Y}$ from half-monodromy of KZ solutions (See [31] Section 16.2 and original references therein.) Since the solution of KZ equations for $N = 2$ is a function of difference $z_2 - z_1$, one can represent the braid group $B_2$ generator $\sigma_{12}$ which swaps $z_1$ and $z_2$, $z_1, z_2 \in D \subset \mathbb{C}^2$ by the loop contour $\vec{z}(s) = (z_1(s), z_2(s))$, $z_{1,2}(s) = a + be^{t_2 s}$, $a = (z_1 + z_2)/2$, $b = (z_1 - z_2)/2$, parametrized by $s \in [0, 1]$. It satisfies $\vec{z}(0) = z_1$, $\vec{z}(1) = z_2$. A pull back of the KZ $N = 2$ equation written for a one form $dw$ along this contour leads to the equation

$$\frac{dw}{ds} = \frac{\Omega_{12}}{\kappa} w(s) \quad (3.8)$$

with the solution

$$w(s) = e^{\frac{\Omega_{12}}{\kappa} s} w(0) \quad (3.9)$$

As before the exponent is understood here as classical series $\sum \left( \frac{\Omega_{12}}{\kappa} s \right)^n \frac{1}{n!}$, which converges on $\text{Aut}(V_1 \otimes V_2)$ because of the nilpotency of non-diagonal part of $\Omega_{12}$ acting on any tensor product of vectors. Therefore if we put $s = 1$ in the last equation, we get the monodromy representation of braid group

$$\rho_{N=2}(\sigma_{12})(v_1 \otimes v_2) = Pe^{\frac{\Omega_{12}}{\kappa}}(v_1 \otimes v_2) \quad (3.10)$$
It is straightforward now to generalize this representation of braiding through half-monodromy of KZ solution to $N > 2$.

$$
\rho_N(\sigma_{i,i+1})(v_1 \otimes \cdots \otimes v_N) = P_{i,i+1} e^{\frac{\Omega_{i,i+1}}{\kappa}}(v_1 \otimes \cdots \otimes v_N) \quad (3.11)
$$

**Theorem 4** For any triple of objects $V_1, V_2, V_3$ in the Drinfeld category $\mathcal{D}$ with the restrictions on parameters of $V_i = \mathcal{T}_{e_i, n_i}$ as above, associator $\alpha_{1,2,3}$ and braiding $\sigma_{1,2} : V_i \otimes V_j \longrightarrow V_j \otimes V_i$, $\sigma_{1,2} = P \exp(i\pi \Omega_{12}/\kappa)$ where $P$ is super-permutation of spaces, satisfy the hexagon relation $(V_1 \otimes V_2) \otimes V_3 \longrightarrow V_2 \otimes (V_3 \otimes V_1)$

$$
\alpha_{2,3,1} \circ \sigma_{1,2,3} \circ \alpha_{1,2,3} = (Id_2 \otimes \sigma_{1,3}^\pm) \circ \alpha_{1,2,3} \circ (\sigma_{1,2}^\pm \otimes Id_3) \quad (3.12)
$$

Moreover the half monodromy operators $\tilde{M}_i$ acting on $V_1 \otimes (V_2 \otimes V_3)$ defined above coincide with $\alpha_{1,2,3}^{-1} \sigma_{12} \alpha_{1,2,3}$.

The existence of the universal form of the representation of braiding $(3.10), (3.11)$ allows to apply the same proof as in the case of semisimple categories. We refer to [31], p.547 for details of the proof, which follows [4,5].

There is an interesting explicit representation of the associator written in terms of $P$-exponential. It was suggested by Drinfeld and a proof that this is indeed an associator can be found in [32]

$$
\alpha_{1,2,3} = \lim_{t \rightarrow 0} t^{-\Omega_{23}/\kappa} P \exp\left(\frac{1}{\kappa} \int_{t}^{1-t} \left(\frac{\Omega_{12}}{z} + \frac{\Omega_{23}}{z-1}\right) dz\right) t^{\Omega_{12}/\kappa} \quad (3.13)
$$

Unfortunately even in the case of $\mathfrak{gl}(1|1)$ superalgebra an explicit calculation of this expression is hard and leads to a complicated series and interesting algebraic structure [33] which we will not discuss here.

Braided tensor structure of this category is standard for modules category of quasitriangular Hopf algebra: trivial unit object, trivial associator and unit morphisms, and braiding morphisms $\sigma_{V,W} = PR_{V,W}$ where $P$ is super-permutation. The proof is standard and does not refer to any particular data and we refer to textbooks, for example, to [31]. For the correspondence with the Drinfeld category we mention the functorial isomorphism $\beta_{X,Y,Z}^\pm : X \otimes (Y \otimes Z) \rightarrow Y \otimes (X \otimes Z)$ defined by

$$
\beta_{X,Y,Z}^\pm = \alpha(\sigma_{XY}^\pm \otimes Id_Z)\alpha^{-1} \quad (3.14)
$$

It satisfies

$$
\beta_{X,Y,Z}^\pm \beta_{Y,X,Z}^{\mp} = Id \quad (3.15)
$$

Then the functorial isomorphisms

$$
\beta_{12}^\pm = \beta_{X,Y,Z \otimes U}^\pm : X \otimes (Y \otimes (Z \otimes U)) \rightarrow Y \otimes (X \otimes (Z \otimes U))
$$
\[ \beta_{23}^\pm = 1d_X \otimes \beta_{Y,Z,U}^\pm : X \otimes (Y \otimes (Z \otimes U)) \rightarrow X \otimes (Z \otimes (Y \otimes U)) \quad (3.16) \]

satisfy the relation

\[ \beta_{12}^\pm \beta_{23}^\pm \beta_{12}^\pm = \beta_{23}^\pm \beta_{12}^\pm \beta_{23}^\pm \quad (3.17) \]

We can summarize the construction of Drinfeld category by the following proposition based on Theorems 1, 2, 3, 4.

**Proposition 3** The category \( \mathcal{D} \) of typical, atypical and projective \( gl(1|1) \)-modules with the restrictions on typicals with \( e_i \notin \mathbb{Z}, (e_i + e_j)/\kappa \notin \mathbb{Z}\{0\} \) is braided tensor category with the structures as described above.

With these structures category \( \mathcal{D} \) of \( gl(1|1) \)-modules will be considered as category of modules of the algebra denoted by \( A_{g,\Omega}, (g = gl(1|1)) \).

### 4 Category \( C_K \) of \( U_h(gl(1|1)) \)-modules

We denote \( i\pi \kappa^{-1} = h \). The structure of quasitriangular \( h \)-adic Hopf superalgebra \( A = U_h(gl(1|1)), \kappa \in \mathbb{R}^\times \), is defined by the following commutation relations of its generators \( \psi^\pm, N, E \)

\[
\{\psi^+, \psi^-\} = 2 \sinh(hE)
\]

\[
[N, \psi^\pm] = \pm \psi^\pm, (\psi^+)^2 = (\psi^-)^2 = 0, [E, X] = 0 \text{ } \forall X \in U_h(gl(1|1))
\]

where \( \exp(\pm Eh) \) is understood as its Taylor series around \( h = 0 (\kappa = \infty) \). The Hopf algebra structure is defined as follows. Coproduct

\[
\Delta(E) = E \otimes 1 + 1 \otimes E, \quad \Delta(N) = N \otimes 1 + 1 \otimes N
\]

\[
\Delta(\psi^+) = \psi^+ \otimes e^{Eh/2} + e^{-Eh/2} \otimes \psi^+, \quad \Delta(\psi^-) = \psi^- \otimes e^{Eh/2} + e^{-Eh/2} \otimes \psi^-
\]

(4.1)

counit

\[
\epsilon(E) = \epsilon(N) = \epsilon(\psi^\pm) = 0
\]

(4.2)

and antipode

\[
\gamma(E) = -E, \quad \gamma(N) = -N, \quad \gamma(\psi^+) = -e^{Eh/2} \psi^+, \quad \gamma(\psi^-) = -\psi^- e^{-Eh/2}
\]

(4.3)

The algebra \( U_h(gl(1|1)) \) is quasitriangular. One can choose the universal R-matrix \( \overline{R} : A \otimes A \rightarrow A \otimes A \) in the form

\[
\overline{R} = \exp[h(E \otimes E + E \otimes N + N \otimes E)](1 - e^{Eh/2} \psi^+ \otimes e^{-Eh/2} \psi^-)
\]

(4.4)
It satisfies the standard quasitriangular Hopf algebra relations

\[
\begin{align*}
R\Delta(X) &= \Delta^{op}(X)R, \quad \forall X \in A \\
(\Delta \otimes Id)R &= R_{13}R_{23}, \\
(Id \otimes \Delta)R &= R_{13}R_{12},
\end{align*}
\]  

(4.5)

As any quasitriangular Hopf superalgebra \( U_h(\mathfrak{gl}(1|1)) \) induces braided tensor category structure on the category of finite-dimensional modules provided the latter is closed under the tensor product functor.

**Proposition 4** Restrictions on \( \kappa \) and parameters \( e \) of typical modules \( e_i \not\in \mathbb{Z}, e_i + e_j \not\in \mathbb{Z}\{0\} \) are enough for the category \( \mathcal{C}_\kappa \) of (equivalence classes of) the modules \( \mathcal{T}_{e,n}, \mathcal{P}_n, \mathcal{A}_n^\kappa \) to form a tensor product ring isomorphic to the tensor product ring (3.1) of the modules \( \mathcal{T}_{e,n}, \mathcal{P}_n, \mathcal{A}_n \). (See Appendix B 8 for definition of the tensor category \( \mathcal{C}_\kappa \) in a specified basis.)

We check this by direct calculation in Appendix B 8 using explicit basis of three types of representations. It is shown that with the restrictions on parameters mentioned in the theorem the same tensor product decomposition works in the quantum case, and the tensor rings are isomorphic.

## 5 Proof of braided tensor equivalence

The main result of this paper is the following theorem.

**Theorem 5** The categories of modules \( \mathcal{D} \) and \( \mathcal{C}_\kappa \) with the restrictions on objects of typical modules \( e_i/\kappa \not\in \mathbb{Z}, e_i/\kappa + e_j/\kappa \not\in \mathbb{Z}\{0\} \) are braided tensor equivalent categories.

Since our proof of this theorem follows [16], we have change the approach to KZ equation to a more general one used in [16]. Instead of the KZ equation (2.16) for correlation functions \( \psi \) built on intertwiners of \( \mathfrak{g} \)-modules consider the equation—we will call it KZ\( _g \)—one can consider KZ equation for superalgebra valued element \( \omega \in (U(\mathfrak{g}))^{\otimes N}[[h]] \) of the form

\[
\frac{1}{h} \partial_i \omega = \sum_{j \neq i}^N \frac{\Omega_{ij}}{z_i - z_j} \omega, \quad i = 1, \ldots, N
\]  

(5.1)

This gives rise to the topologically free quasitriangular quasi-Hopf superalgebra \( A_{g,\Omega} \) with the braiding defined as \( \phi^h \Omega \) and the coassociator defined by the monodromy of solutions of the equation (5.1). We refer to the standard description of this algebra in [1] for non-super-case and to straightforward generalization for the super-case [16] Section 4. The Drinfeld category \( \mathcal{D} \) is a category of topologically free modules over \( A_{g,\Omega} \).
The equivalence partner for the algebra $A_{g,\Omega}$ is the Drinfeld–Jimbo $h$-adic quantum superalgebra $U_h(g)$. Its structure in our specific case was described in the previous section.

The theorem proved in [2], which can be modified to the superalgebra case at hand, claims that if two topological algebras $A_{g,\Omega}$, $U_h(g)$ are gauge equivalent (we will explain what it means below), then the categories of their topologically free modules of finite rank are braided tensor equivalent. Therefore it is enough for us to show gauge equivalence of the two superalgebras.

The plan of this section is the following. We start with recalling the standard proof of the gauge equivalence in the case of simple Lie algebras which one can find in Drinfeld’s paper [3] and explain why it is in general not applicable in the case of superalgebras. After that we explain the details of Geer’s proof [16] of gauge equivalence which avoids the points of Drinfeld’s proof problematical for superalgebras, but applicable for superalgebras of types $A - G$. At the end we argue why a proof found by Geer for classical superalgebras of types $A - G$ works also for $gl(1|1)$ case.

The Drinfeld’s proof.

We recall a proof of braided tensor equivalence of $U_{ih}(g)$ and $A_{g,\Omega}$ for $g$—non-supersuper Lie algebra [3], (see also the Section 16 of [31]). This proof is based on the proof of existence of the invertible element $F_h \in (U(g) \otimes U(g))[[h]]$ which implements the twist of the structures of the algebra $U(g)[[h]]$ to the structures of $A_{g,\Omega}$. The algebra $U_{ih}(g)$ is isomorphic as $\mathbb{C}[[h]]$ algebra to $U(g)[[h]]$. First, one obtains the algebra $(U(g) \otimes U(g))[[h]]$ from $U(g)[[h]]$ by application of the composite homomorphism

$$\tilde{\Delta}_h : U(g)[[h]] \rightarrow U_{ih}(g) \rightarrow \Delta \rightarrow U_{ih}(g) \otimes U_{ih}(g) \rightarrow (U(g) \otimes U(g))[[h]]$$

If one requires that $\tilde{\Delta}_h = \Delta(\text{mod} h)$ where $\Delta$ is the usual comultiplication in $U(g)$, then using the fact that $H^1(g, U(g) \otimes U(g)) = 0$ for simple Lie algebras, one gets that there must exist $F_h \in (U(g) \otimes U(g))[[h]]$ such that

$$F_h \equiv 1 \otimes 1 \text{(mod } h)$$

and

$$F_h^{-1} \tilde{\Delta}(x) F_h = \tilde{\Delta}_h(x), \forall x \in U(g) \tag{5.2}$$

Let the image of the universal R-matrix $\overline{R}$ of $U_{ih}(g) \cong U(g)[[h]]$ in $(U(g) \otimes U(g))[[h]]$ under $\tilde{\Delta}_h$ be $\overline{R}$. The quasitriangular Hopf algebra $U(g)[[h]]$ with trivial coassociator, the coproduct $\tilde{\Delta}_h$ and the R-matrix $\overline{R}$ can now be twisted by the element $F_h$, giving quasitriangular quasi-Hopf algebra $U(g)[[h]]$ with different comultiplication, different R-matrix and non-trivial coassociator. We would like them to be the same as of the algebra $A_{g,\Omega}$, i.e., $\Delta$—the trivial coproduct of $U(g)$. The standard properties of quasitriangular quasi-Hopf algebras are used to prove that all three structures can fit to the required ones of $A_{g,\Omega}$ using the existing twist element $F_h$. This element implements what we called above the gauge equivalence. Explicitly the twist equations are

$$(\epsilon \otimes id) F_h = (id \otimes \epsilon) F_h = 1 \tag{5.3}$$

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\[ F_h^{-1} \Delta(x) F_h = \Delta(x) \]  
(5.4)

\[ (F_h^{-1})_{21}^{-1} R_{12}(F_h)_{12} = R_{12}, \]  
(5.5)

\[ (F_h)_{23}(1 \otimes \Delta)(F_h) \alpha.[(F_h)_{12}(\Delta \otimes 1)(F_h)]^{-1} = 1 \otimes 1 \otimes 1 \]  
(5.6)

Therefore the braided equivalence prove is reduced to a proof of existence of invertible \( F_h \) which satisfies Eqs. (5.4)–(5.6). Equation (5.6) is the most important one. However explicit solution of Eqs. (5.4)–(5.6) is a very hard problem, which requires an explicit form of associator. All we are able to do in this context is to prove its existence, in a way described above. One of the problems to repeat these arguments of twist \( F_h \) existence for a superalgebra case is that the vanishing of the first cohomology \( H^1(g, U(g) \otimes U(g)) = 0 \) used above does not hold in general for superalgebras, in particular for \( g = gl(1|1) \), (see for example [34]). One should look for a way which avoids the cohomology vanishing arguments. One of such ways was suggested by N. Geer [16] by superalgebra modification of quantization procedure suggested by Etingof and Kazhdan (EK) [17–19].

**The Geer’s proof.**

We start from a concise outline and the main steps of the proof in [16] and then provide some details. The EK construction [17] includes two algebras—the algebra \( H = U_h(D(g)) \), the quantization of quantum double, and the algebra \( U_h(g) \). The latter is a quantum double which in general admits a non-trivial \( h \)-adic topology. Both quantizations are generalized in Sections 5, 6, 7 for the types \( A - G \) superalgebras in [16] and shown to be equivalent. The main features which make this way of quantization effective is commutativity with quantum double, by construction, of the first quantization, and functoriality of the second (Section 8 of [16]).

The next step of the proof ([16], Section 9) is isomorphism of the two equivalent EK quantizations \( U_h(g) \) to the standard Drinfeld–Jimbo quantization \( U^D h(A_g) \). The proof follows [19] where the assertion was proved for non-super-Kac–Moody algebras case but works for finite Lie algebras as well. The superalgebra case requires explicit check of additional Serre relations typical for the most of the quantum superalgebras of types \( A - G \).

The final steps of the proof are in Section 10. (All the references below are to the sections and equations of the paper [16].) If there is a gauge isomorphism \( \alpha \) between two quasitriangular quasi-superbialgebras, then it induces braided tensor equivalence between their modules (Theorem 47). Using the previous results on quantization of double with explicit form of the twist (eq. (32)) one leads to the conclusion that \( U_h(g) \) is a gauge twist \( (A_g, \Omega) F \) of \( A_g, \Omega \). The collection of these assertions finally leads to the required conclusion that the categories of topologically free \( A_g, \Omega \) and \( U_h(g) \) finite-dimensional modules are braided tensor equivalent (Theorem 48).

Now we explain some details of the proof steps described above and point out specific features of these steps in our \( gl(1|1) \) case at the end.

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6 The complete list of relevant sequel of their papers is longer, but the others will not be used in our discussion below.

7 In [17] the \( U_h(g) \) quantization was built to be applied to infinite-dimensional Lie algebras, when the quantization \( H \) doesn’t cannot be applied, but both quantizations work for finite-dimensional Lie algebras.
The superalgebra \( A_{\Omega, \kappa} \) is topologically free quasitriangular quasi-Hopf superalgebra built from \( g \), and the Drinfeld category of modules \( \mathcal{D}_g \) is braided tensor category of its modules with the structures based on the KZ equation, as described above.

Let \( g_+ \) be a finite-dimensional superbialgebra and \( g = D(g_+) \) be its double. In Section 5, following [17], Verma modules \( M_{\pm} = U(g) \otimes_{U(g_{\pm})} c_{\pm} \) over \( g \) are used in order to construct forgetful functor \( F \) from the Drinfeld category \( \mathcal{D}_g \) to the category of topologically free \( \mathbb{C}[[h]] \)-modules \( A \):

\[
F(V) = \text{Hom}_{\mathcal{D}_g}(M_+ \otimes M_-, V) \quad (5.7)
\]

Theorem 12 asserts that it is a tensor functor. More precisely, there exists a family of isomorphisms \( \mathcal{J}_{V, W}, V, W \in \mathcal{D}_g \) such that

\[
\mathcal{J}_{U \otimes V, W} \circ (\mathcal{J}_{V, V} \otimes 1) = \mathcal{J}_{U, V \otimes W} \circ (1 \otimes \mathcal{J}_{V, W}) \quad (5.8)
\]

namely

\[
\mathcal{J}_{V, W}(v \otimes w) = (v \otimes w) \circ \alpha_{1,2,34}^{-1}(1 \otimes \alpha_{2,3,4}) \circ \beta_{23} \circ (1 \otimes \alpha_{2,3,4}) \circ \alpha_{1,2,34} \circ (i_+ \otimes i_-) \quad (5.9)
\]

Here \( i_{\pm} \) is a coproduct defined on the highest (lowest) weights of the Verma modules as \( i_{\pm}(v_\pm) = v_\pm \otimes v_\pm \) and \( \beta \) is the morphism given by \( \tau e^{\Omega \kappa/2} \). Theorem 12 with a proof copied from [17] asserts that \( \mathcal{J}_{V, W} \) together with \( F \) is a tensor functor. The functor \( F \) can be thought of as a forgetful functor \( F(V) : V \to \text{Hom}_{\mathcal{D}_g}(U(g), V) \). Being a tensor functor it induces a bialgebra structure on the target. Moreover, it induces superbialgebra structure on \( U(g)[[h]] \) and give rise to a Hopf algebra \( H \) with structure isomorphic to a twist \( F \in U(g)^{\otimes 2}[[h]] \) determined by \( \mathcal{J}_{V, W} \) (eq. 32) of the usual structure of \( U(g)[[h]] \). Its R-matrix \( R = (F^{\text{op}})^{-1} e^{\Omega \kappa/2} F \). This \( R \) is polarized, i.e., \( R \in U_h(g_+) \otimes U_h(g_-) \). The final assertion of this part (Theorem 17) is that \( H \) is a quantization of superbialgebra \( g \). Two important features of this construction are that \( U_h(g_{\pm}) \) are closed under coproduct and that this quantization commutes with taking the double: \( D(U_h(g_{\pm})) \cong U_h(g_{\pm}) \otimes U_h(g_-) = H \) (Corollary 23). We refer to the Section 5 of [16] for details of this part of the proof steps.

The construction of this first EK quantization can be preserved in our \( gl(1|1) \) case. The Verma modules in our notations are isomorphic to the typical modules \( T_{e,n} \). The atypical modules are the quotients of \( T_{0,n} \), and the projectives \( P_n \) can be identified in this construction with \( M_+ \otimes M_- \cong T_{e,n_1} \otimes T_{-e,n_2}, e \neq 0, n_1 + n_2 = n \).

The second EK quantization is in a sense “dual” to the first EK quantization. For a finite-dimensional superbialgebra with a discrete topology (given by inverse limit of finite-dimensional topological superspaces) its modules are topological superspaces. For such topological modules of Drinfeld double one defines the dual Drinfeld category \( \mathcal{D}'_g \) with these modules as objects and morphisms \( \text{Hom}_{\mathcal{D}'_g}(V, W) = \text{Hom}_g(V, W)[[h]] \). One can also dualize \( i_{\pm} \) the maps \( i_{\pm} \). Similarly functor \( \overline{F} : \mathcal{D}'_g \rightarrow \mathcal{A}' \) from dual Drinfeld category to a symmetric tensor category \( \mathcal{A}' \) of \( \mathbb{C}[[h]] \) modules with continuous maps as morphisms, can be defined now by

\( \mathcal{A} \)}}
$F(V) = \text{Hom}_{\mathcal{D}_{g}}(M_{-}, M_{+}^{*} \otimes V)$. Similarly to the first EK quantization Theorem 26 asserts that together with the isomorphism

$$J_{V,W}(v \otimes w) = (i_{+}^{*} \otimes 1 \otimes 1) \circ \alpha_{1,2,34}^{-1}(1 \otimes \alpha_{1,2,3,4})$$
$$\circ \beta_{23}^{-1} \circ (1 \otimes \alpha_{2,3,4}) \circ \alpha_{1,2,34} \circ (v \otimes w) \circ i_{-} \quad (5.10)$$

it defines a tensor structure on $\mathcal{F}$. Further steps of the second EK quantization are parallel to the first one, similarly leading to the Hopf algebra $\mathcal{H}$ which is a quantization of $\mathfrak{g}$ (Theorem 28). Using Proposition 9.7 of [17] it is proved that there is an isomorphism of Hopf superbialgebras $\mathcal{H}$ and $\bar{\mathcal{H}}$.

The following theorems summarize the previous constructions of this step.

Theorem 33: There exists a functor from the category of finite-dimensional superbialgebra $\mathfrak{g}$ over $\mathbb{C}$ and the category of quantum universal enveloping superalgebra over $\mathbb{C}[[h]]$ such that $\mathfrak{g}$ is mapped to $U_{h}(\mathfrak{g})$ which is the second EK quantization.

Theorem 34: There exists a functor from the category of quasitriangular finite-dimensional superbialgebra $(\mathfrak{g}, r)$ over $\mathbb{C}$, where $r$ is classical R-matrix, to the category of quasitriangular quantum universal enveloping superalgebra $(U_{h}(\mathfrak{g}), R)$ which is the first EK quantization.

Functoriality is the main feature in the proof of Theorem 35: There is an isomorphism of the first and the second EK quantizations of quasitriangular superbialgebra as Hopf algebras.

And at last Theorem 37: The quantization of a finite-dimensional superbialgebra commutes with taking the double $D(U_{h}(\mathfrak{g})) \cong U_{h}(D(\mathfrak{g}))$.

Next step (Section 9) is to show that (both of) EK quantizations are isomorphic to the Drinfeld–Jimbo quantization. Following [19] it requires to show that the EK quantization is given by the desired generators and relations. The main effort in this part is to prove the additional quantum Serre-type relations which appear in the type $A-G$ superalgebras. Since there are no such additional relations, for $\mathfrak{gl}(1|1)$ superalgebra we omit this details.

The conclusive steps which lead to the main Theorem 48 were already described above and do not require details.

We remark that the Geer’s proof has no any restriction explicitly related to semisimplicity of superalgebra and to semisimplicity of the category of its finite-dimensional modules therefore can be applied to non-semisimple superalgebra $\mathfrak{gl}(1|1)$. The only feature one should be careful about is that the tensor functors $F, \bar{F}$ between categories are well defined for $\mathfrak{gl}(1|1)$ and their construction covers the set of objects of our category of $\mathfrak{gl}(1|1)$ modules. We argued above why it is the case.

This concludes the proof of our Theorem 5 claiming braided tensor equivalence of the Drinfeld category $\mathcal{D}$ and the category $\mathcal{C}_{k}$ of Drinfeld–Jimbo quantized superalgebra $\mathfrak{gl}(1|1)$.

Summarizing, we have checked that all the steps of the proof of braided tensor equivalence in [16] can be applied to the superalgebra $\mathfrak{gl}(1|1)$. It is based on the twist (5.9), which exists and is unique, at least on the categories of the solutions of KZ equations we consider. Unfortunately the formula (5.9) for twist is not practically useful in explicit calculations because it requires to know the explicit form of associator.
6 Outlook

The proved braided tensor equivalence of non-semisimple categories of $A_{\Omega, \kappa}$ and $U_h(\mathfrak{g})$ modules at generic values of $\kappa$ is a preliminary step toward an understanding of relation between corresponding modules for non-generic values of $\kappa$. In this case the problem actually becomes about a correspondence between the categories of modules of logarithmic vertex operator superalgebra $V(\mathfrak{gl}(1\mid 1), \kappa)$ and quantum group $U_q(\mathfrak{gl}(1\mid 1))$. Despite a big progress done in understanding of this correspondence in the last years for non-superalgebra case, the situation with superalgebras remains, to our knowledge, unclear. Recall that in the known cases of such correspondence for non-superalgebras the relevant second partner of the correspondence is restricted quantum group, or in the case of logarithmic VOA, unrolled restricted quantum group $[35,36]$. As we mentioned in Subsection 2.1 an essential progress has been achieved recently in $[23]$ in understanding of vertex tensor category structure of $V(\mathfrak{gl}(1\mid 1), \kappa)$ for any $\kappa$ including non-generic $\kappa$. It would be interesting to understand what is the quantum group partner for $V(\mathfrak{gl}(1\mid 1), \kappa)$—modules category for non-generic values of $\kappa$. On a VOA part of the correspondence a rigorous construction of intertwining operators for vertex operator superalgebras at non-generic $\kappa$ is an important first step (for non-superalgebras it has been recently done in $[37]$). Another hard problem is to understand practical applicability of vertex tensor categories structures (see $[13]$ and references therein) in concrete cases of superalgebras $[38]$.

Another interesting problem is a logarithmic generalization of the way to construct all the solutions of KZ equations for correlation function including non-semisimple finitely generated modules, by an integration operator as in (2.18) from some minimal set of basic solutions. It is natural to expect as a result a sort of logarithmic deformations of hypergeometric functions structures discovered in $[27]$.

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7 Appendix A

In this Appendix we collect some data about $\mathfrak{gl}(1\mid 1)$ and details of solutions of its KZ equations.

7.1 Asymptotic solutions of KZ equation

Lemma 1 If there is an eigenvector (not generalized) $v$ of $\Omega_{12}$ with eigenvalue $\lambda$, and there are no eigenvalues of $\Omega_{12}$ such that $\lambda + n\kappa, n \in \mathbb{N}$, then there exists unique asymptotic solution around $x = 0$

$$f(x) = x^{\lambda/\kappa}(v + o(x)), \quad \lim_{x \to 0} o(x) = 0$$
Proof  By not generalized eigenvector we mean that \( v \) is not a member of a Jordan block. We check existence and uniqueness of asymptotic solution of the form
\[
f(x) = x^{\lambda/\kappa} (v + xv_1 + x^2v_2 + \cdots), \quad o(v) = \sum_{n=1}^{\infty} x^n v_n \tag{7.1}
\]
with some perhaps infinite set of vectors \( v_n \). After the substitution of it into the left hand side of Eq. (3.5) we get
\[
\text{lhs} = x^{\lambda/\kappa} \left[ \lambda x^{-1} v + (\lambda + \kappa) v_1 + x(\lambda + 2\kappa)v_2 + x^2(\lambda + 3\kappa)v_3 + \cdots \right] \tag{7.2}
\]
On the right hand side we rewrite in the vicinity of \( x = 0 \) as
\[
\frac{\Omega_{12}}{x} - \Omega_{23}(1 + x + x^2 + \cdots)
\]
and now we act by it onto (7.1):
\[
\text{rhs} = x^{\lambda/\kappa} \left( \frac{\Omega_{12}}{x} - \Omega_{23}(1 + x + x^2 + \cdots) \right) (v + xv_1 + x^2v_2 + \cdots)
\]
\[
= x^{\lambda/\kappa} \left[ \lambda vx^{-1} + (\Omega_{12}v_1 - \Omega_{23}v) + (\Omega_{12}v_2 - \Omega_{23}v_1 - \Omega_{23}v)x + (\Omega_{12}v_3 - \Omega_{23}v_2 - \Omega_{23}v_1)x^2 + \cdots \right] \tag{7.3}
\]
Now we compare the multipliers of the same powers of \( x \) in (7.2) and (7.3) and get the infinite set of equations
\[
x^0 : (\Omega_{12} - (\lambda + \kappa)I) v_1 = \Omega_{23} v,
\]
\[
x^1 : (\Omega_{12} - (\lambda + 2\kappa)I) v_2 = \Omega_{23} v_1 + \Omega_{23} v,
\]
\[
x^2 : (\Omega_{12} - (\lambda + 3\kappa)I) v_3 = \Omega_{23} v_2 + \Omega_{23} v_1,
\]
\[
\ldots
\]
They can be solved one after another. Indeed, the right hand side of the first equation is a known vector, \( \det[\Omega_{12} - (\lambda + \kappa)I] \neq 0 \) because \( \lambda + \kappa \) is not an eigenvalue of \( \Omega_{12} \). Therefore the first equation has a unique solution \( v_1 \). The same arguments can now be applied to the second equation: \( \Omega_{23} v_1 \) is now a known vector. We can solve the second equation for \( v_2 \), which is possible because \( \det[\Omega_{12} - (\lambda + 2\kappa)I] \neq 0 \), for \( \lambda + 2\kappa \) is not an eigenvalue of \( \Omega_{12} \) and so on. Thus we find uniquely each vector \( v_i \) by this recurrent procedure, which proves the statement. We do not discuss the convergency question of the infinite sum of vectors in \( o(v) \) because we prove only the existence of asymptotic expansion. \( \square \)

The case of a Jordan block requires more general ansatz. The operator \( x^{\Omega_{12}/\kappa} \) is a well-defined operator on any finite-dimensional representation space on which \( \Omega_{12} \) acts nilpotently. In this case the operator
\[ x^{\Omega_{12}/\kappa} = \sum_{i=0}^{n} \frac{(\ln x)^i}{i!} \frac{\Omega_{12}^i}{\kappa^i} \]  (7.5)

where \( n \) is the degree of nilpotency of \( \Omega_{12} \). Then we can reformulate the lemma in the following way.

**Lemma 2** If there is a Jordan block of \( \Omega_{12} \) with eigenvalue \( \lambda \) with the set of eigenvectors \( v^{(i)}, i = 0, \ldots, n-1 \), \( \Omega_{12} v^{(i)} = \lambda v^{(i)} + v^{(i-1)} \), \( v^{(n-1)} = 0 \) and there are no eigenvalues of \( \Omega_{12} \) such that \( \lambda + n \kappa, n \in \mathbb{N} \), then there exist \( n \) asymptotic solutions around \( x = 0 \) of the form

\[ f_i(x) = x^{\lambda/\kappa} (v^{(i)} (\ln x)^i + \kappa^{-1} v^{(i-1)} (\ln x)^{i-1} + o^{(i)}(x)), \]

\[ \lim_{x \to 0^+} o^{(i)}(x) = 0, \quad i = 0, \ldots, n-1 \]  (7.6)

**Proof** To make the presentation more clear we put \( \kappa = 1 \) and prove the statement for the case of rank \( n = 2 \) Jordan block. With a more lengthy formulas the same proof can be repeated for \( n > 2 \). The claim of the lemma for \( f_0(x) \) becomes identical to the claim of Lemma 1, with the same proof and the same form of the vector \( o^{(0)}(x) = x v_1 + x^2 v_2 + \cdots \). Now we prove the lemma for \( f_1(x) \). We show existence and uniqueness of \( v_j, u_j, j = 1, 2, \ldots \) such that

\[ f_1(x) = x^{\Omega_{12}} (v^{(1)} \ln x + v^{(0)} + o^{(1)}(x)), \]

\[ o^{(1)}(x) = \sum_{j=1}^{\infty} v_j x^j \ln x + \sum_{j=1}^{\infty} u_j x^j \]  (7.7)

First we prove existence of the vectors \( v_j \). We substitute this ansatz for \( o^{(1)}(x) \) into the KZ equation (3.5). We see that the terms proportional to \( \ln x/x \) and \( 1/x \) cancel. Using the same expansion in powers of \( x \) of the term \( \Omega_{23}/(x-1) \) in before and extracting the terms containing \( \ln x \) we get the equations

\[ \ln x : (\Omega_{12} - (\lambda + 1)Id) v_1 = \Omega_{23} v^{(0)}, \]

\[ x \ln x : (\Omega_{12} - (\lambda + 2)Id) v_2 = \Omega_{23} (v^{(0)} + v_1) \]

\[ \ldots \]  (7.8)

As before we can solve these equations for \( v_1, v_2, \ldots \) sequentially because \( \lambda + n, n \geq 1 \) is not an eigenvalue of \( \Omega_{12} \) and the right hand side of these equations are known vectors. After we found \( v_i \) s we do the same extracting on both hand side of KZ equation the terms which are not proportional to \( \ln x \). We get

\[ x : (\Omega_{12} - (\lambda + 1)Id) u_1 = \Omega_{23} v^{(1)} + v_1, \]

\[ x^2 : (\Omega_{12} - (\lambda + 2)Id) u_2 = \Omega_{23} v^{(1)} + v_2 + \Omega_{23} u_1, \]

\[ \ldots \]  (7.9)
By the same reasons as before the equations can be uniquely solved sequentially for $u_i$. This completes the proof. $\square$

In the same way we can prove similar statements about existence of unique asymptotic solutions of the 3.5 equation around $x = 1, x < 1$.

**Lemma 3** If there is an eigenvector $v$ of $\Omega_{23}$ with eigenvalue $\lambda$, and there are no eigenvalues of $\Omega_{23}$ such that $\lambda + n\kappa, n \in \mathbb{N}$, then there exists unique asymptotic solution around $x = 1$ of the form

$$f(x) = (1 - x)^{-\lambda/\kappa} (v + o(x)), \quad \lim_{x \to 1^-} o(x) = 0 \quad (7.10)$$

in the case this eigenvector is not a member of a Jordan block. For the case of Jordan block of the size $n$ the $n$ asymptotic solutions are of the form

$$f_i(x) = (1 - x)^{-\lambda/\kappa} (v^{(i)}(\ln(1 - x))^i + \kappa^{-1} v^{(i-1)}(\ln(1 - x))^{i-1} + o^{(i)}(x)), \quad \lim_{x \to 1^-} o^{(i)}(x) = 0, \quad i = 0, \ldots, n - 1$$

Proof is the same as for Lemmas 1, 2.

**Corollary 1** If the above restriction conditions on the parameters of typical modules are satisfied, an equivalent form of asymptotic solutions of (3.5) around $x = 0$ is

$$f(x) = x^{\Omega_{12}/\kappa} (v_b + o(v)) \quad (7.11)$$

where $v_t$ is the same as $v$ in the case when there are no Jordan block structure for the action of $\Omega_{12}$, and $v_b$ is the bottom vector $v^{(n-1)}$ when there is a Jordan block of size $n$ for the action of $\Omega_{12}$.

Proof In the case without Jordan block this is just change of notations. In the case when there is Jordan block of size $n$, we split $\Omega_{12} = \Omega_{12}^{d} + \Omega_{12}^{nil}$ into diagonal and nilpotent parts and write $x^{\Omega_{12}/\kappa} = x^{\Omega_{12}^{d}/\kappa} \sum_i \frac{1}{i!} \left( \frac{\Omega_{12}^{nil}}{\kappa} \ln x \right)^i$. The action of it on the bottom vector of the set of generalized eigenvectors of $\Omega_{12}$ will generate the sum of vectors proportional to $(\ln x)^i v^{(i)}$ where $v^{(i)}$ are the same as in (7.6). Therefore the representation (7.6) is related to the expansion (7.11) by a change of basis of solutions of KZ equation. $\square$

This corollary enables to use without changes the standard proofs of BTC structure of category of $\mathfrak{gl}(1|1)$-modules with associator and braiding defined through the KZ solutions and their monodromies.

### 7.2 Basis for $\mathfrak{gl}(1|1)$ and its modules

The $\mathfrak{gl}(1|1)$ generators are $E, N, \psi^\pm$ with commutation relations $[N, \psi^\pm] = \pm \psi^\pm$, $\{\psi^+, \psi^-\} = E$ and $E$ is central. (Maybe some other choice of basis will be more...
convenient?) Chevalley involution can be chosen as \( \omega(E) = -E, \omega(N) = -N, \omega(\psi^{\pm}) = \pm \psi^{\mp} \) and produces the dual representation. The basis for typical representation \( \mathcal{T}_{e,n} \) of \( gl(1|1) \) can be chosen as

\[
N = \begin{pmatrix} n + 1/2 & 0 \\ 0 & n - 1/2 \end{pmatrix}, \quad E = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, \quad \psi^+ = \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix}, \quad \psi^- = \begin{pmatrix} 0 & 0 \\ e & 1 \end{pmatrix}
\]

(7.12)

The basis for weights of module \( \mathcal{T}_{e,n} \) is \( u = \uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) (even highest weight), and \( v = \downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) (odd), and for dual module \( \mathcal{T}^*_{e,n} - u^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) (odd lowest weight), and \( v^* = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \) (even). For one-dimensional atypical representation \( A_n \) there is one vector \( v_0 \) with the action of the algebra generators \( \psi^+ v_0 = \psi^- v_0 = 0, N v_0 = n v_0 \). The algebra action on it explicitly:

\[
N \cdot \uparrow = (n + 1/2) \uparrow, \quad N \cdot \downarrow = (n - 1/2) \uparrow, \quad \psi^+ \cdot \uparrow = \psi^- \cdot \downarrow = 0,
\]

\[
\psi^- \cdot \uparrow = \downarrow, \quad \psi^+ \cdot \downarrow = e \uparrow
\]

(7.13)

For four-dimensional atypical representation \( \mathcal{P}_n \) one can choose

\[
N = \begin{pmatrix} n + 1 & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n & 0 \\ 0 & 0 & 0 & n - 1 \end{pmatrix}, \quad \psi^+ = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \psi^- = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix},
\]

\[
E = 0 \times Id_4
\]

(7.14)

And the weights of the module

\[
u_1 = t = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_1 = r = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = l = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad u_2 = b = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}
\]

(7.15)

the even vectors are \( u_{1,2} \), the odd \( v_{1,2} \). This module is self-dual. The algebra action on it

\[
N \cdot t = nt, \quad N \cdot r = (n + 1)r, \quad N \cdot l = (n - 1)l, \quad N \cdot b = nb,
\]

\[
\psi^+ \cdot t = r, \quad \psi^+ \cdot r = b, \quad \psi^+ \cdot l = \psi^- \cdot t = l, \quad \psi^- \cdot r = -b, \quad \psi^- \cdot l = \psi^- \cdot b = 0,
\]

(7.16)

We will use the following choice of Casimir element

\[
\Omega = NE + EN + \psi^- \psi^+ - \psi^+ \psi^- + E^2
\]

(7.17)

and its tensor analog

\[
\Omega_{ij} = N_i \otimes E_j + E_i \otimes N_j + \psi^-_i \otimes \psi^+_j - \psi^+_i \otimes \psi^-_j + E_i \otimes E_j
\]

(7.18)
where the lower indices denote the spaces where the generator acts.

\[ gl(1|1) \text{ commutation relations} \]

\[ [N_r, E_s] = rk\delta_{r+s}, \quad [N_r, \psi^\pm_s] = \pm\psi^\pm_{r+s}, \quad \{\psi^+_r, \psi^-_s\} = E_{r+s} + rk\delta_{r+s} \quad (7.19) \]

One can rescale generators in such a way that \( k \) will become 1 (if it is not 0), but we will keep it. The generic \( k \) will mean \( e/k \notin \mathbb{Z} \) for all the modules involved into correlation function, as well as for all the modules appearing in tensor product decomposition.

A remark: the structure of all modules for non-generic \( k \) for \( \widehat{gl}(1|1) \) and their tensor product decomposition is of course well known, but the KZ for this case and its solutions is another (next...) problem.

Conformal dimension of Virasoro primary field \( h = e(n + \frac{e_2}{2}). \)

We are going to find basis for invariants of level zero KZ equations for \( N = 2, 3, 4. \) Recall that level zero equations in the case of \( gl(1|1) \) mean that \( \sum e_i = 0 \), if typical representations are involved in correlation function. In addition the invariants can be classified according to the \( N \)-grading of the space of states \( V \) of correlation function.

### 7.3 Examples of solutions of KZ equation for correlation functions

In this section we collect examples of explicit form of KZ \( N = 2, 3 \) solutions on the space of \( gl(1|1) \) invariant functions. This class of solutions is the most interesting in the context of KZ equations for correlation functions of intertwining operators of affine Lie superalgebra \( gl(1|1)^\vee \). Similar calculations have been done in the paper [39].

1. \( N = 2 \)

There is one invariant for \( TT \) correlation function in the basis described above \( I_{-1}^{PP} = r b - b r \), and the list of invariants for \( PP \) correlation function is

\[ I_{-1}^{PP} = r b - b r \]
\[ I_{0,1}^{PP} = t b + r l - l r + b t \]
\[ I_{0,2}^{PP} = b b \]
\[ I_{1}^{PP} = l b - b l \]

The first subindex denotes the value of \( n_1 + n_2 \). (Recall that it is not an eigenvalue of \( N \) acting on the tensor product state. The latter is 0 for g-invariant correlation function.) Projection of KZ \( N = 2 \) equation onto this basis gives an ODE with solutions

\[ f(z_1, z_2) = [A(z_1 - z_2)^{\delta_{ij}/k}] I_{0}^{TT}, \quad \delta_{ij} = n_1 e_j + n_j e_i + e_i e_j \quad (7.21) \]

for \( T_{e_1 n_1} T_{-e_1 n_2} \) correlation function (\( A \) is a constant), and solutions

\[ f(z_1, z_2) = \text{const} \times I_{\pm 1}^{PP} \], for \( n_1 + n_2 = \pm 1 \]
\[ f(z_1, z_2) = A I_{0,2}^{PP} + (2A\kappa^{-1} \ln(z_1 - z_2) + B) I_{0,1}^{PP} \] for \( n_1 + n_2 = 0 \) (7.22)
where $A, B$ are constants. This is an example of logarithms in correlation functions of logarithmic vertex operator algebras.

2. $N = 3$

There are two invariants for $TTT$ correlation in the same notations as above

\[
I_{-1/2}^{TTT} = (\uparrow\downarrow\downarrow + \uparrow\uparrow\uparrow + \downarrow\uparrow\uparrow)
I_{+1/2}^{TTT} = (e_1 \uparrow\downarrow\downarrow - e_2 \downarrow\uparrow\uparrow + e_3 \downarrow\uparrow\uparrow),
\]

(7.23)

(Of course $e_1 + e_2 + e_3 = 0$.) Invariants of $TTT$ correlations are

\[
I_{-1}^{TTT} = \uparrow\uparrow b - \uparrow\downarrow r - \downarrow\uparrow r
I_{0,1}^{TTT} = e_1(\uparrow\uparrow l + \uparrow\downarrow t + \downarrow\uparrow t) + \uparrow\downarrow b + \downarrow\downarrow r
I_{0,2}^{TTT} = \uparrow\downarrow b + \downarrow\uparrow b
I_{1}^{TTT} = e_1(\uparrow\downarrow l + \downarrow\uparrow l) + \downarrow\downarrow b
\]

(7.24)

and the list of invariants of $PPP$ correlations are

\[
I_{-2}^{PPP} = rrb - rbr + brr
I_{-1,1}^{PPP} = trb - tbr - rrl - rbt + lrr + btr
I_{-1,2}^{PPP} = rtb + rrl - rlr + rbt - btr - brr
I_{-1,3}^{PPP} = rbb - brb
I_{-1,4}^{PPP} = rbb - bbr
I_{0,1}^{PPP} = btb + brl - blr + bbt
I_{0,2}^{PPP} = bbb
I_{1,1}^{PPP} = tlb - tbl - rll + llr - lbt + blt
I_{1,2}^{PPP} = ltb + lrl - llr + lbt - btl - btl
I_{1,3}^{PPP} = lbb - bbl
I_{1,4}^{PPP} = blb - bbl
I_{2}^{PPP} = llb - lbl + bll
\]

(7.25)

Projection of KZ equation in the form (3.5) onto these bases gives systems of ODEs with the following solutions. If the space of invariants with fixed first subindex, i.e., fixed sum of $n_1+n_2+n_3$, is one-dimensional equal to $I$, then the solution for correlation function in all three cases can be written as

\[
f(x) = Ax^{\alpha/\kappa} (1 - x)^{\beta/\kappa} I
\]

(7.26)

where $A \in \mathbb{C}$ is a constant, and $\alpha, \beta$ are eigenvalues of $\Omega_{12}, \Omega_{23}$ acting on $I$, respectively.
In the $TT\mathcal{P}$ case with $n_1 + n_2 + n_3 = 0$ solution contains logarithms:

$$f(x) = Ax^{5/2}/\kappa (1 - x)^{5/3}/\kappa \left[ I_{0,1}^{TT\mathcal{P}} + \left( B + \frac{e_1}{\kappa} \ln(1 - x) \right) I_{0,2}^{TT\mathcal{P}} \right]$$

(7.27)

In the $\mathcal{P}\mathcal{P}\mathcal{P}$ case with $n_1 + n_2 + n_3 = 0$ the solution is trivial

$$f(x) = AI_{0,1}^{\mathcal{P}\mathcal{P}\mathcal{P}} + BI_{0,2}^{\mathcal{P}\mathcal{P}\mathcal{P}}, \ A, B \in \mathbb{C}$$

(7.28)

But in the case $n_1 + n_2 + n_3 = \pm 1$ there are logarithms in the solutions:

$$f^\pm(x) = A^\pm I_{\pm,1,1}^{\mathcal{P}\mathcal{P}\mathcal{P}} + B^\pm I_{\pm,1,2}^{\mathcal{P}\mathcal{P}\mathcal{P}}$$

$$+ \left( C_3^\pm + \frac{A^\pm - B^\pm}{\kappa} \ln x + \frac{B^\pm - 2A^\pm}{\kappa} \ln(1 - x) \right) I_{\pm,1,3}^{\mathcal{P}\mathcal{P}\mathcal{P}}$$

$$+ \left( C_4^\pm + \frac{B^\pm}{\kappa} \ln x + \frac{A^\pm - B^\pm}{\kappa} \ln(1 - x) \right) I_{\pm,1,3}^{\mathcal{P}\mathcal{P}\mathcal{P}}$$

(7.29)

where $A^\pm, B^\pm, C_{3,4}^\pm$ are constants.

Another interesting problem is structure of solutions of KZ equations on a wider $N$-graded spaces, not necessarily invariants of $\mathfrak{gl}(1|1)$. We will address this problem elsewhere.

8 Appendix B

Here we will describe the basis and tensor product decomposition of $U_h(\mathfrak{gl}(1|1))-\text{modules}$ and will prove the Proposition 3.

We will choose $i\pi \kappa^{-1} = h$ and consider real $\kappa$. We use the following matrix basis for the three types of $U_h(\mathfrak{gl}(1|1))-\text{modules} \ T_{e,n}^\kappa, A_{n}^\kappa, \mathcal{P}_{n}^\kappa$ included into $C_{\kappa}$, as the basis for construction of tensor ring. For $T_{e,n}^\kappa$

$$E = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, \quad N = \begin{pmatrix} n + 1/2 & 0 \\ 0 & n - 1/2 \end{pmatrix}, \quad \psi^+ = \begin{pmatrix} 0 & 2 \sinh(eh) \\ 0 & 0 \end{pmatrix}, \quad \psi^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

with the vectors of the module

$$|e, n\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ (even)}, \quad |e, n - 1\rangle = \psi^-|e, n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ (odd)}$$

and for four-dimensional module we choose

$$N = \begin{pmatrix} n + 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & n & 0 \\ 0 & 0 & 0 & n - 1 \end{pmatrix}, \quad \psi^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 - e^{eh} & 0 \\ 0 & 0 & 0 & e^{eh} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$
\[
\psi^- = \begin{pmatrix}
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
-e^{-h} & 0 & 0 & 0 \\
0 & e^{-h} & -1 & 0
\end{pmatrix}, \quad E = 0 \times \text{Id}_4,
\]

The coordinates of the vectors of the four-dimensional vector space of this representation are graded as in (7.15). Let us note that there are many other matrix presentations of \( \mathcal{P}_n^k \) module which can contain some more free numerical parameters.

**Proof of Proposition 3** With these basis we can consider decomposition of tensor product of this set of three types of modules using the coproduct (4.1) and show that under some suitable assumptions on parameters of modules they form a ring. The cases

\[
\mathcal{A}_n^k \otimes \mathcal{A}_{n'}^k = \mathcal{A}_{n+n'}^k, \quad \mathcal{A}_n^k \otimes \mathcal{T}_{e,n'}^k = \mathcal{T}_{e,n+n'}^k, \quad \mathcal{A}_n^k \otimes \mathcal{P}_{e,n'}^k = \mathcal{P}_{e,n+n'}^k
\]

are obvious. More interesting are the remaining three cases.

Consider \( \mathcal{T}_{e,n}^k \otimes \mathcal{T}_{e,n'}^k \). The calculations of tensor product decomposition of two \( U_{lh}(\mathfrak{gl}(1|1)) \)-modules \( \mathcal{T}_{e_1,n_1}^k \otimes \mathcal{T}_{e_2,n_2}^k \) are completely parallel to the same calculations for \( \mathfrak{gl}(1|1) \)-modules. \( \mathcal{T}_{e,n}^k \) has two states—the highest weight \( v_1 = | \uparrow \rangle \) Grassmann even and \( v_2 = \psi^- v_1 = | \downarrow \rangle \)—Grassmann odd. We can start from two vectors \( w_2 = \alpha_2 | \uparrow \rangle \otimes | \downarrow \rangle + \beta_2 | \downarrow \rangle \otimes | \uparrow \rangle \) and \( u_1 = \alpha_1 | \uparrow \rangle \otimes | \downarrow \rangle + \beta_1 | \downarrow \rangle \otimes | \uparrow \rangle \) with constraint \( \alpha_2 = -\beta_1 \beta_2 / \alpha_1 \) which guarantees their orthogonality. We consider \( w_2 \) as highest weight of a grading reversed module, i.e., \( \Delta(\psi^+) w_2 = 0 \). It gives \( \beta_2 = -2 \alpha_2 e^{-he_1} \sinh(e_2 h) \). And we consider \( u_1 \) as lowest weight module, with Grassmann even highest weight. It means \( \Delta(\psi^-) u_1 = 0 \), which gives \( \beta_1 = \alpha_1 e^{he_2} \). Then one can easily check that corresponding lowest weight module of the first (grading reversed) module is \( 2 \alpha_1 \sinh((e_1 + e_2) h) | \downarrow \rangle \otimes | \downarrow \rangle \), and highest weight of the second module is \( \alpha_2 \sinh((e_1 + e_2) h) / \sinh(e_1 h) | \uparrow \rangle \otimes | \uparrow \rangle \). We see that conditions \( \sinh((e_1 + e_2) h) \neq 0, \sinh(e_1 h) \neq 0 \), which mean \( e_1/k \notin \mathbb{Z}, (e_1 + e_2)/k \notin \mathbb{Z} \setminus \{0\} \) are sufficient for decomposition

\[
\mathcal{T}_{e_1,n_1}^k \otimes \mathcal{T}_{e_2,n_2}^k = \mathcal{T}_{e_1+e_2,n_1+n_2+1/2}^k \oplus \mathcal{T}_{e_1+e_2,n_1+n_2-1/2}^k
\]

In the case \( e_1 + e_2 = 0 \) one can check that any vector of the form \( |t\rangle = \alpha | \uparrow \rangle \otimes | \downarrow \rangle + \beta | \downarrow \rangle \otimes | \uparrow \rangle \) with \( \alpha \neq e^{he_1} \beta \) serves as the \( |t\rangle \)-vector in the basis of the \( \mathcal{P}_{n_1+n_2}^k \) module of four vectors of the tensor product \( \mathcal{T}_{e_1,n_1}^k \otimes \mathcal{T}_{e_2,n_2}^k \). We see that the tensor product ring composed of the \( U_{lh}(\mathfrak{gl}(1|1)) \)-modules \( \mathcal{A}_n^k, \mathcal{T}_{e,n}^k, \mathcal{P}_n^k \) is the same as the tensor product ring of the category \( \mathcal{C}_k \) composed of \( \mathcal{A}_n^k, \mathcal{T}_{e,n}^k, \mathcal{P}_n^k \) for restriction on parameters the same as in Proposition 3. \( \square \)

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