AUTOMORPHISMS OF KRONROD-REEB GRAPHS OF MORSE FUNCTIONS ON 2-SPHERE

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Abstract. Let $M$ be a compact two-dimensional manifold, $f \in C^\infty(M, \mathbb{R})$ be a Morse function, and $\Gamma_f$ be its Kronrod-Reeb graph. Denote by $O(f) = \{f \circ h \mid h \in \mathcal{D}\}$ the orbit of $f$ with respect to the natural right action of the group of diffeomorphisms $\mathcal{D}$ on $C^\infty(M, \mathbb{R})$, and by $S(f) = \{h \in \mathcal{D} \mid f \circ h = f\}$ the corresponding stabilizer of this function. It is easy to show that each $h \in S(f)$ induces a homeomorphism of $\Gamma_f$. Let also $\mathcal{D}_{id}(M)$ be the identity path component of $\mathcal{D}(M)$, $S'(f) = S(f) \cap \mathcal{D}_{id}(M)$ be group of diffeomorphisms of $M$ preserving $f$ and isotopic to identity map, and $G_f$ be the group of homeomorphisms of the graph $\Gamma_f$ induced by diffeomorphisms belonging to $S'(f)$. This group is one of the key ingredients for calculating the homotopy type of the orbit $O(f)$.

Recently the authors described the structure of groups $G_f$ for Morse functions on all orientable surfaces distinct from 2-torus $T^2$ and 2-sphere $S^2$. The present paper is devoted to the case $M = S^2$. In this situation $\Gamma_f$ is always a tree, and therefore all elements of the group $G_f$ have a common fixed subtree $Fix(G_f)$, which may even consist of a unique vertex. Our main result calculates the groups $G_f$ for all Morse functions $f: S^2 \to \mathbb{R}$ whose fixed subtree $Fix(G_f)$ consists of more than one point.

1. Introduction

Let $M$ be a compact two-dimensional manifold and $\mathcal{D}(M)$ the group of diffeomorphisms of $M$. Then there exists a natural right action

$$\phi: C^\infty(M, \mathbb{R}) \times \mathcal{D}(M) \to C^\infty(M, \mathbb{R})$$

of this group on the space of smooth functions on $M$ defined by the formula $\phi(f, h) = f \circ h$. For $f \in C^\infty(M, \mathbb{R})$ denote by

$$S(f) = \{h \in \mathcal{D}(M) \mid f \circ h = f\}$$

its stabilizer with respect to the specified action.

Definition 1.1. Let $\mathcal{F}(M, \mathbb{R})$ be the subset of $C^\infty(M, \mathbb{R})$ consisting of maps $f: M \to \mathbb{R}$ such

(1) $f$ takes constant values on the connected components of the boundary $\partial M$ and has no critical points on $\partial M$;
(2) for each critical point $z$ of $f$ there are local coordinates $(x, y)$ in which $z = (0, 0)$ and $f(x, y) = f(z) + g_z(x, y)$, where $g_z: \mathbb{R}^2 \to \mathbb{R}$ is a homogeneous polynomial without multiple factors.

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Notice that every critical point of \( f \in \mathcal{F}(M, \mathbb{R}) \) is isolated.

A function \( f \in \mathcal{F}(M, \mathbb{R}) \) is called Morse, if \( \deg g_z = 2 \) for each critical point \( z \) of \( f \). In that case, due to Morse Lemma, one can assume that \( g_z(x, y) = \pm x^2 \pm y^2 \).

We will denote by \( \mathcal{M}(M, \mathbb{R}) \) the space of all Morse maps \( M \to \mathbb{R} \).

Homotopy types of stabilizers and orbits of Morse functions and functions from \( \mathcal{F}(M, \mathbb{R}) \) were studied in \([8], [9], [10], [1], [2], [3], [4], [5], [6]\).

Let \( f \in C^\infty(M, \mathbb{R}) \), \( \Gamma_f \) be a partition of the surface \( M \) into the connected components of level sets of this function, and \( p: M \to \Gamma_f \) be the canonical factor-mapping, associating to each \( x \in M \) the connected component of the level set \( f^{-1}(f(x)) \) containing that point.

Endow \( \Gamma_f \) with the factor topology with respect to the mapping \( p \): so a subset \( A \subset \Gamma_f \) will be regarded as open if and only if its inverse image \( p^{-1}(A) \) is open in \( M \). Then \( f \) induces the function \( \hat{f}: \Gamma_f \to \mathbb{R} \), such that \( f = \hat{f} \circ p \).

It is well known, that if \( f \in \mathcal{F}(M, \mathbb{R}) \), then \( \Gamma_f \) has a structure of a one-dimensional CW-complex called the Kronrod-Reeb graph, or simply the graph of \( f \). The vertices of this graph correspond to critical connected components of level sets of \( f \) and connected components of the boundary of the surface. By the edge of \( \Gamma_f \) we will mean an open edge, that is, a one-dimensional cell.

Denote by \( \mathcal{H}(\Gamma_f) \) the group of homeomorphisms of \( \Gamma_f \). Notice that each element of the stabilizer \( h \in S(f) \) leaves invariant each level set of \( f \), and therefore induces a homeomorphism \( \rho(h) \) of the graph of \( f \), so that the following diagram is commutative:

\[
\begin{array}{ccc}
M & \xrightarrow{p} & \Gamma_f \\
\downarrow{h} & & \downarrow{\rho(h)} \\
M & \xrightarrow{\hat{f}} & \mathbb{R}
\end{array}
\]

Moreover, the correspondence \( h \mapsto g(h) \) is a homomorphism of groups

\[ \rho: S(f) \to \mathcal{H}(\Gamma_f). \]

Let also \( D_{id}(M) \) be the path component of the identity map \( id_M \) in \( D(M) \). Put

\[ S'(f) = S(f) \cap D_{id}(M) \quad \text{and} \quad G_f = \rho(S'(f)). \]

Thus, \( G_f \) is the group of automorphisms of the Kronrod-Reeb graph of \( f \) induced by diffeomorphisms of the surface preserving the function and isotopic identity.

**Remark 1.2.** Since \( \hat{f} \) is monotone on edges of \( \Gamma_f \), it is easy to show that \( G_f \) is a finite group. Moreover, if \( g(E) = E \), for some \( g \in G \) and an edge \( E \) of the graph \( \Gamma_f \), then \( g(x) = x \) for all \( x \in E \).

Since \( G_f \) is finite and \( \rho \) is continuous, it follows that \( \rho \) reduces to an epimorphism

\[ \rho_0: \pi_0 S'(f) \to G_f, \]

of the group \( \pi_0 S'(f) \) path components of \( S'(f) \) being an analogue of the mapping class group for \( f \)-preserving diffeomorphisms.

Algebraic structure of the group \( \pi_0 S'(f) \) of connected components of \( S'(f) \) for all \( f \in \mathcal{F}(M, \mathbb{R}) \) on orientable surfaces \( M \) distinct from 2-torus and 2-sphere is described in \([11]\), and the structure of its factor group \( G_f \) is investigated in \([7]\). These groups play
an important role in computing the homotopy type of the path component $\mathcal{O}_f(f)$ of the orbit of $f$, see also [8], [9], [1], [2], [3].

The purpose of this note is to describe the groups $G_f$ for a certain class of smooth functions on 2-sphere $S^2$.

The main result Theorem 1.4 reduces computation of $G_f$ to computations of similar groups for restrictions of $f$ to some disks in $S^2$. As noted above the latter calculations were described in [7].

First we recall a variant of the well known fact about automorphisms of finite trees from graphs theory.

Lemma 1.3. Let $\Gamma$ be a finite contractible one-dimensional CW-complex («a topological tree»), $G$ be a finite group of its cellular homeomorphisms, and $\text{Fix}(G)$ be the set of common fixed points of all elements of the group $G$. Then $\text{Fix}(G)$ is either a contractible subcomplex or consists of a single point belonging to some edge $E$ an open 1-cell), and in the latter case there exists $g \in G$ such that $g(E) = E$ and $g$ changes the orientation of $E$.

Suppose $f : S^2 \to \mathbb{R}$ belongs to $\mathcal{F}(M, \mathbb{R})$. Then it is easy to show that $\Gamma_f$ is a tree, i.e., a finite contractible one-dimensional CW-complex, and by Remark 1.2 $G_f$ is a finite group of cellular homeomorphisms of $\Gamma_f$. Therefore, for $G_f$, the conditions of Lemma 1.3 are satisfied. Note that according to Remark 1.2 the second case of Lemma 1.3 is impossible, and hence $G_f$ has a fixed subtree.

In this paper we consider the case when the fixed subtree of the group $G_f$ contains more than one vertex, i.e. has at least one edge.

Let us also mention that $\mathcal{D}_d(S^2)$ coincides with the group $\mathcal{D}(S^2)$ of diffeomorphisms of the sphere preserving orientation, [12]. Therefore $S(f)$ consists of diffeomorphisms of the sphere preserving the function $f$ and the orientation of $S^2$.

Theorem 1.4. Let $f \in \mathcal{F}(M, \mathbb{R})$. Suppose that all elements of the group $G_f$ have a common fixed edge $E$. Let $x \in E$ be an arbitrary point and $A$ and $B$ be the closures of the connected components of $S^2 \setminus p^{-1}(x)$. Then

1. $A$ and $B$ are 2-disks being invariant with respect to $S(f)$;
2. the restrictions $f|_A \in \mathcal{F}(M, \mathbb{R})$ and $f|_B \in \mathcal{F}(M, \mathbb{R})$;
3. the map $\phi : G_f \to G_f|_A \times G_f|_B$ defined by the formula

$$\phi(\gamma) = (\gamma|_A, \gamma|_B)$$

is an isomorphism of groups.

Proof. (1) By assumption $x$ belongs to the open edge $E$. Therefore $p^{-1}(x)$ is a regular connected component of some level set of the function $f$, that is, a simple closed curve. Then, by Jordan Theorem, $p^{-1}(x)$ divides the sphere into two connected components whose closures are homeomorphic to two-dimensional disks. Consequently, $A$ and $B$ are two-dimensional disks.

Let us show that $A$ and $B$ are invariant with respect to $S_f$, i.e., $h(A) = A$ and $h(B) = B$ for each $h \in S(f)$. Denote

$$\Gamma_A = p(A) \quad \Gamma_B = p(B).$$

Then

$$\Gamma_A \cup \Gamma_B = \Gamma \quad \Gamma_A \cap \Gamma_B = \{x\}.$$
By definition, $\rho(h)(x) = x$, whence $\rho(h)$ either preserves both $\Gamma_A$ and $\Gamma_B$ or interchange them. We claim that

$$\rho(h)(\Gamma_A) = \Gamma_A \quad \rho(h)(\Gamma_B) = \Gamma_B.$$ 

Indeed suppose $\rho(h)(\Gamma_A) = \Gamma_B$. Since $\rho(h)$ is fixed on $E$, it follows that

$$\rho(h)(\Gamma_A \cap E) = \Gamma_A \cap E,$$

whence

$$\rho(h)(\Gamma_A \cap E) = \rho(h)(\Gamma_A) \cap \rho(E) = \Gamma_B \cap E \neq \Gamma_A \cap E,$$

which contradicts to our assumption. Thus $\Gamma_A$ and $\Gamma_B$ are invariant with respect to the group $G_f$.

Now we can show that $A$ and $B$ are also invariant with respect to $h$. By virtue of the commutativity of the diagram (1.1) $\rho(h)(p(y)) = p(h(y))$ for all $y \in \Gamma$. In particular:

$$p(h(A)) = \rho(h)(p(A)) = \rho(h)(\Gamma_A) = \Gamma_A.$$

Therefore, $h(A) = p^{-1}(\Gamma_A) = A$. The proof for $B$ is similar. Thus, $A$ and $B$ are invariant with respect to $S'(f)$.

(2) Notice that the function $f$ takes a constant value on the simple closed curve $p^{-1}(x)$ being a common boundary of disks $A$ and $B$, and does not contain critical points of $f$. Therefore, the restrictions $f|_A, f|_B$ satisfy the conditions 1) and 2) the Definition 1.1, and so they belong to $\mathcal{F}(M, \mathbb{R})$ and $\mathcal{F}(M, \mathbb{R})$ respectively.

(3) We should prove that the map $\phi: G_f \to G_{f|_A} \times G_{f|_B}$ defined by formula

$$\phi(\gamma) = (\gamma|_{\Gamma_A}, \gamma|_{\Gamma_B})$$

is an isomorphism.

First we will show that $\phi$ is correctly defined. Let $\gamma \in G_f = \rho(S'(f))$, that is, $\gamma = \rho(h)$, where $h$ is a diffeomorphism of the sphere preserving the function $f$ and isotopic to the identity.

We claim that $h|_A \in S'(f|_A) = S(f|_A) \cap D_{id}(A)$. Indeed, for each point $x \in A$ we have that:

$$f(x) = f|_A(x) = f|_A(h|_A(x)) = f|_A(h(x)) = f(h(x)),$$

which means that $h|_A \in S(f|_A)$.

Moreover, since $h$ preserves the orientation of the sphere, it follows that $h|_A$ preserves the orientation of the disk $A$, and therefore by [12], $h|_A \in D_{id}(A)$. Thus $\gamma|_{\Gamma_A} \in G_{f|_A}$.

Similarly $\gamma|_{\Gamma_B} \in G_{f|_B}$, and so $\phi$ is well defined.

Let us now verify that $\phi$ is an isomorphism of groups, that is, a bijective homomorphism. Let $\delta, \omega \in G_f$. Then

$$\phi(\delta \circ \omega) = (\delta \circ \omega|_{\Gamma_A}, \delta \circ \omega|_{\Gamma_B}) =$$

$$= (\delta|_{\Gamma_A}, \delta|_{\Gamma_B}) \circ (\omega|_{\Gamma_A}, \omega|_{\Gamma_B}) =$$

$$= (\delta|_{\Gamma_A} \circ \omega|_{\Gamma_A}, \delta|_{\Gamma_B} \circ \omega|_{\Gamma_B}) =$$

$$= (\delta \circ \omega|_{\Gamma_A}, \delta \circ \omega|_{\Gamma_B}),$$

so $\phi$ is a homomorphism.

Let us show that $ker \phi = \{ \text{id}_f \}$. Indeed, suppose $\gamma \in ker \phi$, that is $\gamma|_{\Gamma_A} = \text{id}_{\Gamma_A}$ and $\gamma|_{\Gamma_B} = \text{id}_{\Gamma_B}$. Then $\gamma$ is fixed on $\Gamma_A \cup \Gamma_B = \Gamma$, and hence it is the identity map.
Surjectivity of $\phi: G_f \to G_{f|A} \times G_{f|B}$ is implied by the following simple lemma whose proof we leave to the reader.

**Lemma 1.5.** Suppose $f: D^2 \to \mathbb{R}$ belongs to the space $\mathcal{F}(M, \mathbb{R})$. Then for arbitrary $\alpha \in G_f$, there exists $a \in \mathcal{S}(f)$ fixed near the boundary $\partial D^2$ and such that $\alpha = \rho(a)$. □

Let $(\alpha, \beta) \in G_{f|A} \times G_{f|B}$, then by Lemma 1.5 there exist $a \in \mathcal{S}(f|A)$ and $b \in \mathcal{S}(f|B)$ fixed near $\partial A = \partial B = p^{-1}(x)$ and such that $\alpha = \rho_A(a)$ and $\beta = \rho_B(b)$. Define $h$ by the following formula:

$$h = \begin{cases} a(x), & x \in A, \\ b(x), & x \in B. \end{cases}$$

Then, $h$ is a diffeomorphism of the sphere, preserving the function and orientation, whence $h \in \mathcal{S}(f)$.

Moreover if we put $\gamma = \rho(h) \in G_f$, then $\gamma|_{r_A} = \rho(h|_{A}) = \alpha$ and $\gamma|_{r_B} = \rho(h|_{B}) = \beta$. In other words, $\phi(\gamma) = (\gamma|_{r_A}, \gamma|_{r_B}) = (\alpha, \beta)$, i.e., $\phi$ is surjective and therefore an isomorphism. □

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