Geometric Aspects of the Deformation Method

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At the classical level, redefinitions of the field content of a Lagrangian allow to rewrite an interacting model on a flat target space, in the form of a free field model (no potential term) on a curved target space. In the present work we extend the idea of the ‘deformation method’ introduced in [1], to show that it is possible to write an explicit correspondence between the metrics of the curved target spaces that arise in the free versions of distinct scalar field models. This is accomplished by obtaining an explicit relation between the map function linking the fields and the free models’ metrics. By considering complex and even quaternionic field models, we extend the procedure –initially proposed for models of a single scalar field– to systems with a content of two and four (despite constrained) real fields, respectively, widening the range of applicability. We also analyze supersymmetric models to illustrate more possibilities. In particular, we show how to relate a flat Minkowskian metric to a Fubini-Study space.

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I. INTRODUCTION

The ‘deformation method’ –introduced in [1] and explored in several works [2]– consist in a recipe to link two distinct classical scalar field models, in such a way that the corresponding solution spaces can be explicitly connected, by means of a map \( f \) called ‘deformation function’. This is accomplished whenever there is a specific relation between the potentials governing the dynamics of the fields. That relation can be obtained by construction and, in that sense, this method results much more efficient for finding new solutions than, for instance, the ‘trial orbit method’ of [7].

At the classical level, certain redefinitions of the field content of a model of interacting fields on a flat target space, governed by a scalar potential, allow to rewrite the Lagrangian density to get a free field model on a curved target space.

As it is well known, the holonomy group of a semi-Riemannian manifold \((M, g)\) at a point \( p \in M \) is defined as the group of parallel transports along loops based on \( p \). It provides a powerful tool to study the geometric structure of the manifold and to discuss, for instance, the existence of parallel sections of geometric vector bundles. In his seminal paper [3], Berger gave a list of possible holonomy groups of simply connected (semi-)Riemannian manifolds under the assumption that the group acts irreducibly on the tangent space at \( p \). The list comprised groups that only occur as holonomies in certain dimensions, such as the exceptional compact Lie group \( G2 \) (as the holonomy group of a 7-dimensional Riemannian manifold), or its non-compact real form \( G2(2) \subset SO(4,3) \) (as the holonomy group of a manifold with metric of signature \((4,3)\)). Precisely such properties of these manifolds allow the mentioned maps between them to certainly exist.

The main objective of the present work is to show how the idea of the deformation method can be applied to establish a correspondence or duality between the metrics describing the free versions of the related models. This can be accomplished by constructing an explicit relation between the metrics of the free models. This result allows the application of the deformation method to make explicit the relation between distinct geometrical models of interest in a wide range of contexts like Sugra, D-branes, etc.

In principle, the proposed relation can be attained straightforwardly for models containing a single real scalar field. However, considering complex and quaternionic field models, we can extend the procedure to systems with a content of two and four (despite constrained) real
fields, respectively, which enlarges the range of applicability of the results. Therefore, in the following, we will establish a connection between two models, described by an ‘original’ \( \mathcal{L} \) and a ‘deformed’ \( \tilde{\mathcal{L}} \) Lagrangian densities \[13\], and their corresponding (curved target space) free models \( \mathcal{L}_g \) and \( \tilde{\mathcal{L}}_g \), respectively). The link between these free models is implemented through the ‘deformation function’ \( f \). The general idea of these mappings is synthesized in the diagram (1).

\[
\begin{align*}
\mathcal{L} [V(\phi)] & \leftrightarrow \mathcal{L}_g [(g(\theta))] \\
\tilde{\mathcal{L}} [\tilde{V}(\phi)] & \leftrightarrow \tilde{\mathcal{L}}_g [(\tilde{g}(\tilde{\theta}))]
\end{align*}
\]

The article is organized as follows. In Section II we present the procedure applied to the simplest, real case, and after discussing the geometrical meaning of the mapping, we illustrate it by showing an explicit relation between the well known ‘\( \phi^4 \)’ and ‘sine-Gordon’ scalar field models. In Section III we apply the procedure to the complex case. In Section IV we extend further the applicability of the procedure, considering a Wess-Zumino model in a quaternionic space, developed in a previous work \[8\]. Then, in Section V we analyze the possibilities of bypassing the severe limitations of the deformation method by constructing a purely geometric deformation (with no defect-type solutions involved), considering supersymmetric models and working out the very important case of the Fubini-Study space. Finally, Section VI is devoted to summarize and discuss our results.

II. REAL FIELD MODEL

The model for an interacting real scalar field \( \phi \) on a bi-dimensional Minkowski spacetime can be described by means of a scalar potential \( V(\phi) \). The action and the corresponding static field equation read

\[
I = \int dx \left( \frac{1}{2} \dot{\phi}^2 - V(\phi) \right) \Rightarrow \ddot{\phi} - V'(\phi) = 0,
\]

where the dot indicates derivation with respect to the spacetime coordinate \( x \), and the prime with respect to the argument of the function.

We consider potentials presenting spontaneous symmetry breaking features, supporting defect-like solutions. In order to analyze physical (finite energy) solutions, the asymptotic
conditions, \( \phi \to v_{\pm} \) and \( \dot{\phi} \to 0 \) as \( x \to \pm \infty \), where \( V'(v_{\pm}) = 0 \), must be satisfied. Under such conditions, the static field equation in (1) can be integrated once to get
\[
\dot{\phi}^2 = 2(V(\phi) - V_0),
\]
where \( V_0 = V(v_{\pm}) \) is the ‘classical vacuum’ (value of the degenerate minima of the scalar potential).

**Free real model.** The free version of the model above is described on a curved target space with line element \( d\ell = g^2(\theta)d\theta^2 \). Assuming the metric \( g \) to be non degenerated, we write down the action and corresponding field equation for the free field \( \theta \),
\[
I_g = \frac{1}{2} \int dx \, g(\theta) \dot{\theta}^2 \quad \Rightarrow \quad \ddot{\theta} + \frac{1}{2} \ln(g(\theta))' \dot{\theta}^2 = 0 \quad (3)
\]

The equation above can be integrated once to get a first order equation
\[
\dot{\theta}^2 = C_0 g(\theta)^{-1} \quad (4)
\]
with \( C_0 \) an integration constant.

If these models (1) and (3) are two different descriptions of the same physics, we must make explicit the relation between the fields \( \phi \) and \( \theta \). Matching the first order equations (2) and (4), we establish the relations
\[
\theta = \pm \phi + C_1 \quad (5)
\]
\[
C_0 g^{-1}(\pm \phi + C_1) = 2 \left( V(\phi) - V_0 \right) \quad (6)
\]
where \( C_1 \) is a new integration constant.

Note that, as \( x \to \pm \infty \), the solution reaches the minima, \( \phi \to v_{\pm} \), and the right hand side of (5) goes to zero, implying on a divergence of the metric in that points \( (g(\theta) \to v_{\pm} + C_1) \to \infty) \). That is, the asymptotic values of the field, which are critical points of the potential \( V \), are mapped (up to constants) to singular points of the target space of the free model. We will comment further on this behavior later.

**A. Real model deformation**

Following [1], the Langrangian density describing the model for the ‘deformed field’ \( \tilde{\phi} \) is
\[
\tilde{\mathcal{L}} = \frac{1}{2} \dot{\tilde{\phi}}^2 - \tilde{V}(\tilde{\phi}) \quad (7)
\]
The deformation prescription establishes a connection between the deformed and the undeformed potentials and solutions through the deformation function $f$

$$
\tilde{V}(\tilde{\phi}) = V(f(\tilde{\phi}))[f'(\tilde{\phi})]^{-2}, \quad \phi^K_x = f(\tilde{\phi}^K_x),
$$

where $\phi^K_x = \pm \phi(x)$ and $\tilde{\phi}^K_x = \pm \tilde{\phi}(x)$, are the corresponding kink–anti-kink solutions for the original and the deformed models, respectively.

The Lagrangian density describing the free (curved target space) version of this model is

$$
\tilde{\mathcal{L}}_g = \frac{1}{2} \tilde{g}(\tilde{\theta}) \dot{\tilde{\theta}}^2
$$

Now, using (5) and (6), we can write

$$
\tilde{\theta} = \pm \tilde{\phi} + \tilde{C}_1,
$$

$$
\tilde{C}_0 \tilde{g}^{-1}(\tilde{\theta}) = 2 \left( \tilde{V}(\tilde{\phi}) - \tilde{V}_0 \right)
= 2V(f(\tilde{\phi}))[f'(\tilde{\phi})]^{-2} - 2V_0 [f'(\tilde{v}_\pm)]^{-2},
$$

where we have used that $\tilde{V}_0 = \tilde{V}(\tilde{v}_\pm) = V(f(\tilde{v}_\pm))[f'(\tilde{v}_\pm)]^{-2}$. On the other hand, expression (11) can also be related to the free Lagrangian of the undeformed model since, from Eq. (5), we have

$$
2V(\phi) = C_0 g^{-1}(\pm \phi + C_1) - 2V_0.
$$

Thus, (11) can be rewritten as

$$
\tilde{C}_0 \tilde{g}^{-1}(\pm \tilde{\phi} + \tilde{C}_1) = C_0 g^{-1}(\pm f(\tilde{\phi}) + C_1) [f'(\tilde{\phi})]^{-2} - 2V_0 \left( [f'(\tilde{v}_\pm)]^{-2} - [f'(\tilde{\phi})]^{-2} \right).
$$

To simplify a bit this relation we can make some reasonable assumptions. First, we can choose degenerated minima of the potential ($V_0$) to be zero. Second, the constant factors $C_0$ and $\tilde{C}_0$ can be absorbed into the definitions of the metrics, $g$ and $\tilde{g}$. Finally we can take the constant $C_1$ ($\tilde{C}_1$) relating the fields $\theta$ and $\phi$ ($\tilde{\theta}$ and $\tilde{\phi}$) to be zero. Under these conditions, equation (13) assumes the extremely simple and suggestive form

$$
f'(\tilde{\phi})^2 = g(f(\tilde{\phi}))^{-1} \tilde{g}(\tilde{\phi}).
$$

This expression has a two-folded reading. On one side we have that, by choosing an arbitrary or convenient deformation function $f$, relation (14) takes an (undeformed, free) initial model, represented by $g$, to another (free, deformed) model related to the metric $\tilde{g}$. That is,

$$
\tilde{g}(\tilde{\phi}) = g(f(\tilde{\phi}))[f'(\tilde{\phi})]^2.
$$
The second possible interpretation result from taking [14] as a differential equation that, given two geometries (metrics) $g$ and $\tilde{g}$, allows to find the deformation function $f$ that connects the two free field models with the geometries in their target spaces. That leads to the implicit expression for $f$

$$
\int df \sqrt{g(f)} = \pm \int \tilde{d}\phi \sqrt{\tilde{g}(\tilde{\phi})}.
$$

(16)

It is important to remark that, as stressed before, the asymptotic values of the field solutions lead to singular points of the target space metric, as $g^{-1}(\phi_\pm^K(x) \rightarrow v_\pm) \rightarrow V(v_\pm) - V_0 = 0$ and, naturally, a similar behavior should be expected from the metric $\tilde{g}$ of the ‘free deformed model’ [16]. Thus, while degenerated minima of the original potential are mapped to degenerated minima of the deformed model, in the free model case, they are mapped to the same singular point of the metric, as they depend on the value of the potential calculated at the minima, which, by assumption, are degenerated. That is, topological sectors of the field models has their counterpart as disconnected regions on the target space, isolated by singularities.

**B. Geometrical interpretation of the deformation procedure**

As is well know, Sigma models can be generically described by an action of the form

$$
S = \int d\phi G_{ij}(\phi) * d\phi
$$

(17)
or, in the case of the Born-Infeld theoretical framework, as

$$
S = \int \sqrt{G(\phi)},
$$

(18)

where now the determinant of the metric over the target space is precisely the wedge product of the Maurer-Cartan forms. Classically and geometrically speaking, we can think $G_{ij}(\phi)$ as a metric on the target space ($T$).

Expression [16] remind us the invariance of the action $S$ under Diff($T$):

$$
\phi \rightarrow \phi' (\phi) \Rightarrow G_{ij}(\phi) \rightarrow G_{ij'} (\phi').
$$

(19)

This is reminiscent of the determinantal character of the Sigma model as measure, and means that the Sigma model is defined by an equivalence class of metrics.
The other important ingredient is the invariance under reparameterization implied by (16) due to the square root (Nambu-Goto/Barbashov-Chernikov action)

\[ \int df \sqrt{g(f)} = \pm \int d\phi \sqrt{\tilde{g}(\phi)}. \]

Defining a canonical basis \( \chi \) in the target space (where the functional quantities have explicit dependence) we have

\[ \tilde{g}(\chi) = g(f(\chi))[f']^2 \quad \text{and} \quad \tilde{g}(\chi) = g(\tilde{\phi}(\chi))[\tilde{\phi}']^2, \]

so, immediately, we obtain

\[ \frac{g(\tilde{\phi})}{g(f)} = \frac{[f']^2}{[\tilde{\phi}']^2} = \left( \frac{df}{d\phi} \right)^2 \]

As given in prescription (3) the function \( f \) connects the field solutions of two different models \( (\phi = f(\tilde{\phi}) \). Actually, as seen above, \( f \) simply connects different metrics related by the equivalence class of diffeomorphisms in the target plus reparameterization invariance.

This is not the only reason why the procedure works: it is achieved only by the fact that in all the cases treated, just one field is involved. What changes is the field of the numbers (real, complex, quaternions) where this unique quantity is defined.

Finally, notice that, from a perturbative quantum mechanical point of view, \( G_{ij}(X) \) is sometimes thought as the sum of an infinite number of coupling constants: \( G_{ij}(X) = G_{ij}^0 + G_{ij}^1 X^k + ... \) Theses constants do not preserve the invariances mentioned above, leading to a breaking of the full theory symmetry at this quantum perturbative level.

C. Example: Sine-Gordon as \( \phi^4 \) model deformation

Let us now illustrate the above results with a very simple example of application.

Consider the two very well known real scalar field models \( \phi^4 \) and sine-Gordon. As shown in [4], these two models can be mapped one into the other by the deformation method. This is accomplished by simply taking the \( \phi^4 \) model,

\[ V(\phi) = \frac{1}{2}(1 - \phi^2)^2, \]

which supports two \( \mathbb{Z}_2 \) kink solutions \( \phi^K_\pm(x) = \pm \tanh(x) \) and considering the deformation function

\[ f_\pm(\tilde{\phi}) = \pm \sin(\tilde{\phi}). \]
Using the prescription (18), it immediately take us to
\[ \mathcal{V}(\tilde{\phi}) = \frac{V(f(\tilde{\phi}))}{[f'(\tilde{\phi})]^2} = \frac{1}{2}(1 - \sin(\tilde{\phi}))^2, \]
which is the sine-Gordon potential, corresponding to fixing the parameters of the general 
expression \( V(\phi) = \alpha \cos(\beta \phi) + |\alpha| \) to the values \( \alpha = \frac{1}{4}, \beta = 2 \). The sine-Gordon solutions are then obtained by inverting the \( f \) function, namely
\[ \tilde{\phi}_k(x) = f_k^{-1}(\phi^K(x)) = \pm \arcsin(\tanh(x)) + k\pi, \]
where \( k \in \mathbb{Z} \) specifies the branch of the inverse of \( f \). This result is consistent with the general form of the sine-Gordon solutions, \( \phi_n(x) = 2 \arctan(e^x) + (2n + 1)\frac{\pi}{2} \), when taking the specific values of \( \alpha \) and \( \beta \), and \( n = k - 1 \), as can be easily checked.

Note that, while the kink solutions of the \( \phi^4 \) model connect the single topological sector defined by the pair of degenerated minima of the potential \((v_+ = +1 \text{ and } v_- = -1)\), the sine-Gordon model presents an infinite number of topological solutions, one for each topological sector, connecting adjacent minima \((k - 1/2)\pi \text{ and } (k + 1/2)\pi\) – see Fig. 1 in [4].

D. ‘Geometric’ deformation.

Let us now consider the free versions of the models. From relations (13) and (10) we have, for the \( \phi^4 \) model
\[ g_\lambda(\pm \phi + C_1) = \frac{C_0}{(1 - \phi^2)^2}, \]
and for the sine-Gordon model,
\[ \tilde{g}_{sG}(\pm \tilde{\phi} + \tilde{C}_1) = \frac{\tilde{C}_0}{\cos^2(\tilde{\phi})}, \]
with \( \theta = \pm \phi + C_1 \) and \( \tilde{\theta} = \pm \tilde{\phi} + \tilde{C}_1 \), respectively.

Now, following (16), the deformation function connecting these two free models is given by
\[ \int df \sqrt{g(f)} = \int df \frac{1}{1 - f^2} = \pm \int d\tilde{\phi} \frac{1}{\cos(\tilde{\phi})} = \pm \int d\tilde{\phi} \sqrt{\tilde{g}(\tilde{\phi})} \]
\[ \arctanh(f) = \ln(\sec(\tilde{\phi}) + \tan(\tilde{\phi})), \]
or, simply, \( f = \sin(\tilde{\phi}) \). That is, using the ‘geometric version’ of the deformation method, we have obtained exactly the same deformation function that connects the interacting version of the models given in \( [23] \).

It is interesting to note here the relation between the zeros of the potential and the periodic character of the singularities of the metric describing the free nonlinear sigma model in the solution above. That clearly indicates the existence of an interplay between the singularities at the geometrical level and the minima of the potential, or, better, between the topological sectors of the potential and the causally disconnected regions of the target space determined by the singularities. This fact may result relevant for some supergravity theories, that demand such a type of constructions — see, for instance, \( [10] \).

III. COMPLEX FIELD MODEL

Let us now consider a field model described by the action

\[
\mathcal{I} = \int dx \left( \frac{1}{2} \dot{\phi} \dot{\bar{\phi}} - V(\phi, \bar{\phi}) \right),
\]

where \( \phi \) is a complex scalar field, and \( V(\phi, \bar{\phi}) \) a scalar potential. It presents a dynamical equation that can be integrated to obtain a first order ordinary differential equation

\[
\dot{\phi} \left( \ddot{\phi} - \frac{\partial V}{\partial \phi} \right) = 0 \quad \Leftrightarrow \quad \frac{\dot{\phi}^2}{2} = V(\phi) + \text{const.}
\]

Adding this equation to its complex conjugate and integrating we get the first order equation

\[
\dot{\varphi} \dot{\bar{\varphi}} = V(\varphi, \bar{\varphi}) - V_0
\]

In some cases of interest (let say, bosonic sector of supersymmetric theories), the potential \( V \) can be put in terms of a ‘superpotential’ \( W \) \( (V(\varphi) = \frac{1}{2} W'(\varphi) W'(\varphi)) \), which leads to the first order equations

\[
\dot{\varphi} = e^{i\alpha} W'(\varphi), \quad \dot{\bar{\varphi}} = e^{-i\alpha} W'(\varphi),
\]

and relation \( (32) \) assumes the form

\[
\dot{\varphi} \dot{\bar{\varphi}} = W'(\varphi) W'(\varphi) - V_0
\]
**Free complex model.** The free action corresponding to the model \((30)\) above requires the introduction of a Kähler manifold with metric \(g_{z\bar{z}}\)

\[
\mathcal{I}_g = \frac{1}{2} \int dx^n g_{z\bar{z}} \dot{\theta}^z \dot{\bar{\theta}}^{\bar{z}}.
\]

(35)

Again, the dot indicates derivation with respect to the spatial coordinate \((\dot{\theta}^z = d\theta^z/dx)\).

Variation of this action leads to the field equations

\[
\ddot{\theta}^w + g^{w\bar{z}} \left[ g_{z\bar{z},w} \dot{\theta}^w + g_{z\bar{z},\bar{w}} \dot{\bar{\theta}}^{\bar{w}} - g_{z\bar{z},w} \dot{\theta}^w \right] \dot{\bar{\theta}}^{\bar{z}} = 0
\]

(36)

Being \(g_{zw}\) the metric of a Kähler manifold, the last two terms in \((36)\) cancel, as it satisfies, in complex coordinates, that \(g_{z\bar{z}} = g_{\bar{z}z}\) and \(g_{aa} = g_{\bar{a}\bar{a}} = 0\). Then we obtain

\[
g_{z\bar{z}} \ddot{\theta}^z + g_{z\bar{z},w} \dot{\theta}^w \dot{\bar{\theta}}^{\bar{z}} = 0
\]

(37)

We can also drop the indices and put the expression in the simpler form

\[
\ddot{\theta} + \frac{\partial \ln g}{\partial \theta} \dot{\theta}^2 = 0
\]

(38)

This expression can be integrated to obtain the relation

\[
\dot{\theta} \dot{\bar{\theta}} = C_0 g^{-1}(\theta, \bar{\theta}),
\]

(39)

with \(C_0 \in \mathbb{C}\) an integration constant.

Comparing \((32)\) and \((39)\), we can write down the relations between the curved (free) and the flat (interacting) versions of the model

\[
\dot{\theta} \dot{\bar{\theta}} = \varphi \bar{\varphi}
\]

(40)

\[
C_0 g^{-1}(\theta, \bar{\theta}) = V(\varphi, \bar{\varphi}) - V_0,
\]

(41)

which are in complete analogy with relations \((5)\) and \((6)\), obtained in the real case.

Equation \((40)\) can be rewritten in terms of the real and imaginary components of the fields as \(\dot{\theta}_1^2 + \dot{\theta}_2^2 = \dot{\varphi}_1^2 + \dot{\varphi}_2^2\). The trivial solution is to take, as in the real case, \(\theta_1 = \varphi_1 + \mathcal{C}_1\) and \(\theta_2 = \varphi_2 + \mathcal{C}_2\) or, simply

\[
\theta = \varphi + \mathcal{C}
\]

(42)

with \(\mathcal{C} \in \mathbb{C}\), an integration constant.
A. Complex model deformation

The deformation procedure for complex field models was explored in Ref. [5]. The deformed Lagrangian density has the general form

\[ \tilde{\mathcal{L}} = \frac{1}{2} \partial^\mu \tilde{\varphi} \partial_\mu \tilde{\varphi} - \tilde{V}(\tilde{\varphi}, \bar{\tilde{\varphi}}) \]  

and describes the dynamics of the new complex field \( \tilde{\varphi} = \tilde{\varphi}_1 + i \tilde{\varphi}_2 \), related to the original field by a (at least) meromorphic function \( f = f(\varphi) \) such that

\[ \varphi = f(\tilde{\varphi}) = f_1(\tilde{\varphi}_1, \tilde{\varphi}_2) + i f_2(\tilde{\varphi}_1, \tilde{\varphi}_2), \]  

which, naturally, must fulfill Cauchy-Riemann conditions

\[ \frac{\partial f_1}{\partial \tilde{\varphi}_1} = \frac{\partial f_2}{\partial \tilde{\varphi}_2}, \quad \frac{\partial f_1}{\partial \tilde{\varphi}_2} = -\frac{\partial f_2}{\partial \tilde{\varphi}_1}. \]  

The dynamics governed by \( \mathcal{L} \) and \( \tilde{\mathcal{L}} \) are different, but we can define \( \tilde{V}(\tilde{\varphi}, \bar{\tilde{\varphi}}) \) and \( \tilde{W}(\tilde{\varphi}) \) by the equation – see [5] –,

\[ \tilde{V}(\tilde{\varphi}, \bar{\tilde{\varphi}}) = \frac{V(f(\tilde{\varphi}), f(\bar{\tilde{\varphi}}))}{|f'(f(\tilde{\varphi}))|^2} = \frac{1}{2} \tilde{W}'(\tilde{\varphi}) \bar{\tilde{W}}'(\bar{\tilde{\varphi}}) \]  

In this case, the first-order equations read

\[ \frac{d\tilde{\varphi}}{dx} = e^{i\alpha} \tilde{W}'(\tilde{\varphi}); \quad \frac{d\bar{\tilde{\varphi}}}{dx} = e^{-i\alpha} \bar{\tilde{W}}'(\bar{\tilde{\varphi}}) \]  

The \( BPS \) kink solutions for this system are obtained from the solutions of the undeformed first order equations by simply taking the inverse of the deformation function, that is,

\[ \tilde{\varphi}^K(x) = f^{-1}(\varphi^K(x)). \]  

B. ‘Geometric’ deformation

Now, starting with the free version of the deformed model

\[ \tilde{\mathcal{L}}_g = \frac{1}{2} \tilde{g}_{\hat{\varphi}\hat{\varphi}} \hat{\dot{\varphi}} \hat{\varphi} \]  

the relations (40) and (41), take the form

\[ \hat{\dot{\varphi}} \hat{\varphi} = \hat{\dot{\tilde{\varphi}}} \tilde{\varphi} \]  

\[ \tilde{C}_0 \tilde{g}^{-1}(\tilde{\theta}, \tilde{\varphi}) = \tilde{V}(\tilde{\varphi}, \bar{\tilde{\varphi}}) - \tilde{V}_0 \]
From prescription (46) we can rewrite (51) as

\[
\tilde{C}_0 \tilde{g}^{-1}(\tilde{\theta}, \bar{\tilde{\theta}}) = \frac{V(f(\tilde{\varphi}_\pm), f(\bar{\tilde{\varphi}}_\pm))}{|f'(\tilde{\varphi}_\pm)|^2} - \frac{V_0}{|f'(\tilde{\varphi}_\pm)|^2} \tag{52}
\]

Now, using (40), the above expression becomes

\[
\tilde{C}_0 \tilde{g}^{-1}(\tilde{\theta}, \bar{\tilde{\theta}}) = C_0 g^{-1}(\theta, \bar{\theta}) + V_0 - \frac{V_0}{|f'(\tilde{\varphi}_\pm)|^2} \tag{53}
\]

Finally, using relation (42) and letting constants aside, we obtain the ordinary differential equation

\[
f'(\tilde{\varphi})f''(\tilde{\varphi}) = |f'(\tilde{\varphi})|^2 = \frac{\tilde{g}(\tilde{\theta}, \bar{\tilde{\theta}})}{g(\theta, \bar{\theta})}, \tag{54}
\]

which implements the map between the metrics of the free complex models. This result is completely analogous to expression (14), obtained in the real case.

\section*{C. Interesting relation with SUSY models}

Consider a simple superspace action supporting a Kähler potential \( K \) for a number of chiral complex fields \( \varphi^i \)

\[
\mathcal{I}_K = \int dx^4 d\Theta^4 K(\varphi^i, \bar{\varphi}^i, \Theta). \tag{55}
\]

On performing the \( \Theta \) integrations we end up with the bosonic part

\[
\mathcal{I}_K = -\frac{1}{2} \int dx^4 g_{ij}(\varphi) \partial_\mu \varphi^i \partial^\mu j^i \tag{56}
\]

where the metric \( g_{ij} \) for the Kähler manifold with potential \( K(\varphi^i, \bar{\varphi}^i) \), is given by

\[
g_{ij} = \frac{\partial}{\partial \varphi^i} \frac{\partial}{\partial j^i} K(\varphi, \bar{\varphi}) = \partial_i \partial_j K(\varphi, \bar{\varphi}) \tag{57}
\]

This is the target space metric for a free model. In the case of a single complex field, we can construct the link between the model above and a new one by solving

\[
f'(\tilde{\varphi})f''(\tilde{\varphi}) = \frac{\partial_i \partial_j \tilde{K}(\tilde{\varphi}, \bar{\tilde{\varphi}})}{\partial_i \partial_j K(\varphi, \bar{\varphi})}, \tag{58}
\]

which is nothing but relation (54) written in terms of \( K \).

We will explore further this result later, in Section V.
IV. QUATERNIONIC FIELD MODEL

In a previous work [8], we have shown genuine BPS solutions for the quaternionic Wess-Zumino model (WZ), in the context of hyper-Kähler structures. This is a standard choice however, it is valid to work with the WZ superpotential since it is the basic prototype for any analysis involving hypercomplex quaternionic structures, appearing in several areas of modern theoretical physics.

In general, domain walls are co-dimension one solutions, that separate the spacetime into regions corresponding to different vacua. In the simplest case, a domain wall is supported by a gauge potential that couples to its world volume and the field strength of this gauge potential is dual to a cosmological constant. In a more general setting, with nontrivial couplings to scalar fields, this cosmological constant appears as an extremum of the potential term in the Lagrangian. The resulting solution describes then a flow towards an extremum and, if the potential possesses several extrema, the solution may interpolate between them.

Here, we will analyze a domain wall-like solution, analog to the one obtained in [12], developing a connection between solutions obtained in our previous work [8].

A. Hyperkahler domain wall solution

As starting point, consider a single quaternionic field governed by the Generalized Quaternionic Lagrangian density (GQL) of the form – see [8] –

\[ \mathcal{L} = \frac{1}{2} \Pi q \Pi q - \frac{1}{2} |W'(q)|^2, \tag{59} \]

where the Cauchy-Fueter operator is given by \( \Pi \equiv \hat{\partial}_0 - \hat{i}^k \partial_k \) and \( \partial_k \equiv \partial/\partial x^k \), while \( \hat{i}^0 = \mathbb{I} \), and \( \hat{i}^k \ (k = 1, 2, 3) \) obey the standard quaternionic algebra. Latin indices \( j, k \) run from 1 to 3 and the Einstein’s summation convention is adopted. \( Sc \) and \( Vec \) denote the scalar and vector parts of the quaternionic expressions, while the prime indicates derivative with respect to the argument of the function.

The quaternionic Wess-Zumino (WZ) superpotential takes the form

\[ W'(q) = n - q^N = n - (q_0 + i^k q_k)^N, \quad N \in \mathbb{Z} \tag{60} \]

In principle \( n \in \mathbb{H} \) but, through this work, we will take \( n \in \mathbb{C} \) or in its subgroups.
The potential for the $N = 2$ case of our WZ model takes the form

$$V = \frac{1}{2}(n - q^2)(n - q^2)$$

$$= \frac{1}{2}n^2 - n(q_0^2 - q_3^2) + \frac{1}{2}(q_0^2 + q_3^2)^2$$

Fig. [IV A] shows the form of potential $V$ as function of two of the quaternionic components ($q_0$ and $q_3$)

![Figure 1: Quaternionic potential $V(q)$ ($n = 1$).](image)

The corresponding vacuum (minima) manifold is described by the set of the $N$-roots of $n$ in the field of the quaternions, i.e. $S^2$ spheres. The first order equation, $\Pi q = \overline{W'(q)}$, for our WZ potential (60) reads

$$\frac{dq}{dx} = n - \overline{q}^N = n - (q_0 - i^k q_k)^N$$

This expression, with $x$ identified below, arises from the relation between the left regular superpotential $W(q)$ and the BPS conditions.

**Case $N = 2$ (non-commutative) BPS solution.** We present here a new BPS quaternionic solution (not worked out in [8]), for the case $N = 2$ with the fields on a quaternionic base manifold (non-commutative spacetime equivalent) as target space. This is obtained by identifying the spacetime spatial coordinate $x$ with one of the complex directions of the quaternionic manifold. In this case we take $x \rightarrow \hat{i}_3 X^3$ (i.e. $x \in SU(2)$) and, consequently, $\Pi = -\hat{i}_3 \partial_3$. As a consequence of this choice, the spacetime assumes the structure $S_1 \otimes O(3) \sim S_1 \otimes SU(2)$. 
The first order equation (62) takes the form
\[
\frac{dq}{dX_3} = n - (q_0^2 - q_i^2 - 2 \hat{i}^k q_k q_0), \quad n \in \mathbb{Z}.
\] (63)

Breaking equation (63) into its $\text{Sc}$ and $\text{Vec}$ parts, we obtain the system of equations
\[
\begin{cases}
\frac{dq_0}{dX_3} = -2q_0 q_3, \\
\frac{dq_1}{dX_3} = n - q_0^2 + q_i^2, \\
\frac{dq_3}{dX_3} = -2q_0 q_2, \\
\frac{dq_2}{dX_3} = 2q_0 q_1
\end{cases}
\] (64)

These equations are coupled in pairs, so we can obtain an explicit solution by just putting $q_2 = q_1 = 0$. The system (64) reduces then to
\[
\frac{dq_0}{dX_3} = -2q_3 q_0, \quad \frac{dq_3}{dX_3} = n - q_0^2 + q_3^2.
\] (65)

An interesting result arises if we note that system (65) can be reduced to a Liouville type equation. In fact, making the substitution $q_0 \equiv e^{2\alpha} = Y^2$, and putting $\frac{dY}{dX_3} \equiv Y'$, we obtain
\[
Y'' = Y^5 - nY \quad \Rightarrow \quad \left[(Y')^2 - \frac{1}{3}Y^6 + nY^2\right]' = 0,
\] (66)

from which we obtain an ‘energy equation’,
\[
\frac{dY}{\sqrt{C + \frac{1}{3}Y^6 - nY^2}} = \pm dX_3.
\] (67)

The above implicit relation is integrable but not invertible in general. However, in the important particular case $C = 0$, that is, the momentum map, on-shell or surface constraint, the equation is fully solvable and invertible. Thus, we obtain a non-commutative $N = 2$ $BPS$ solution of the form
\[
q(X_3) = (\sqrt{3}n) \text{Sech} \left(2i\sqrt{n}X_3 - \beta_n\right) + \hat{i}^3(i\sqrt{n}) \text{Tanh} \left(2i\sqrt{n}X_3 - \beta_n\right)
\] (68)

with $\beta_n = 2 \text{Ln} \left(3n e^{i\alpha \sqrt{3}}\right)$, where $i$ is the imaginary unit of the field of complex numbers $\mathbb{C}$ and $c_0$ an arbitrary (complex) constant.

It is worth noting here that, depending on the values of the constants $c_0$ and $n$ (and therefore $\beta_n$), solution (68) can switch between hyperbolic and trigonometric characters. In the context of quaternionic $BPS$ structures developed in [8], hyperbolic (negative $n$) case of solution (68) can be interpreted as a domain wall centered at $X_3|_{0} = \beta_n/2$; showing the typical defect behavior [17] – see figure 2. Naturally, this defect-like behavior is lost in the trigonometric case.
As a result of the geometrical structure of the quaternionic solution \([68]\), the potential \([61]\) for the \(N = 2\) case of our WZ model takes the form

\[
V[q(X_3)] = 2n^2 \text{Sech}^2(\Phi) \left( 4 \text{Sech}^2(\Phi) - 3 \right)
\]  

(69)

where \(\Phi = 2i\sqrt{n}X_3 - \beta_n\).

For the sake of illustration, let us particularize the parameters \(n\) and \(c_0\) in order to get \(\beta_n = 0\). Then, the profile of the potential \([69]\) in terms of the base space variable \(X_3\) take the different forms depicted in Fig.3.
B. Quaternionic Model Deformation

Analogously to the cases analyzed in the previous sections, our generalized quaternionic Lagrangian (59), can be explicitly connected to a free sigma model.

**Free quaternionic model.** As stated in [8], the connection (harmonic map) between the models can be reduced to the relation

\[ V = [\det(g_{ab})]^{-1} \equiv U^{-1}, \]  

(70)

where \( g_{ab} \) is the metric associated to the hyper-Kähler manifold (target space) of the free model and \( U \) is related to the line element as shown below. Thus, the metric determinant in the \( N = 2 \) case reads

\[
\det (g_{ab}) = \frac{1}{\frac{1}{2} [n - (q_0^2 - q_3^2)]^2 + 2q_0^2q_3^2} = \frac{1}{2n^2} \frac{\cosh^2(\Phi)}{1 - 4 \tanh^2(\Phi)}
\]  

(71)

where in the last equality we have made explicit the base space coordinate via solution (68), put in terms of \( \Phi = 2i\sqrt{n}X_3 - \beta_n \).

**Geometry of the wall: the line element.** In order to make the corresponding geometrical analysis, let us introduce the obvious transformation \( \Theta = \tanh \Phi \), which allows passing from hyperbolic/trigonometric to polynomial expressions. Then solution (68) and potential (69) take the form

\[
q(\Theta) = \hat{i}^0 (\sqrt{3n})\sqrt{1 - \Theta^2} + \hat{i}^3 (i\sqrt{n}) \Theta
\]

\[
V[\Theta] = 2n^2 (1 - \Theta^2) \left(1 - 4\Theta^2\right) = [\det(g_{ab})]^{-1}
\]

(72)

(73)

Now, taking into account the specific form of the line element in the hyper-Kähler and quaternionic cases we can write

\[
ds^2 = U^{-1} dq_0 \otimes dq_0 + U dq_3 \otimes dq_3
\]

\[
= 2n^2 (1 - \Theta^2) \Theta^2 \left[U^{-1} \hat{\Theta} \otimes \hat{\Theta} + \left(\frac{1 - \Theta^2}{3\Theta^2}\right) U \hat{\Theta} \otimes \hat{\Theta}\right]
\]

(74)

(75)

That is,

\[
g_{00} = -24n^4 \Theta^2 (1 - \Theta^2)^2 (1 - 4\Theta^2) , \quad g_{33} = \frac{2 (1 - \Theta^2)}{(1 - 4\Theta^2)}
\]

(76)
C. Geometric deformation

As a final illustration, let us propose the problem of finding a relation (deformation function) connecting our hyper-Kähler wall solution (68) to the solution obtained by Arai, Nitta and Sakai in [12]—see also [13]—, where the geometry specified by metric determinant

\[ g_{\text{ANS}} \equiv \det(g_{ab})|_{\text{ANS}} = \frac{\mu^2}{1 - Q_3^2} \]  

(77)

is related to the solution

\[ Q_3(y) = \tanh(\mu(y + y_0)) , \]  

(78)

One way to compare (78) to our wall solution

\[ q_3(X_3) = (i\sqrt{n})\Theta = (i\sqrt{n}) \tanh \left( \frac{2i\sqrt{n}(X_3 - \frac{\beta_n}{2i\sqrt{n}})}{\Theta} \right) , \]  

(79)

is to write both expression in the same coordinate basis. This is accomplished by adjusting the coefficients as: \( 2i\sqrt{n} = \mu \) and \( y_0 = i\beta_n/(2\sqrt{n}) \), and then making the identification

\[ X_3 \leftrightarrow y \quad \text{and} \quad Q_3 \leftrightarrow \Theta = \frac{1}{i\sqrt{n}}q_3. \]  

(80)

Now the metric determinants explicit the distinct curvatures of the solutions

\[ g_{\text{ANS}} = -\frac{4n}{1 - \Theta^2} \quad \text{and} \quad g = \frac{1}{2n^2 (1 - \Theta^2) (1 - 4\Theta^2)}. \]  

(81)

Then,

\[ (F'(\Theta))^2 = \frac{g(\Theta)}{g_{\text{ANS}}(\Theta)} = \frac{-1}{8n^3 (1 - 4\Theta^2)} \]  

(82)

from which we have

\[ F(\Theta) = -\frac{\sqrt{2}}{8n^{3/2}} \ln \left( \sqrt{n} \left( 2\Theta + \sqrt{4\Theta^2 - 1} \right) \right) + C_1. \]  

(83)

In order to get the function mapping one solution into the other, that is, \( f \) such that \( \Theta = f(Q_3) \), we use once again relation (21). Then, under conditions (80), we can write the derivative of the deformation function as

\[ (f'(Q_3))^2 = \frac{g_{\text{ANS}}(Q_3(X_3))}{g(f(Q_3))} = -8n^3 \frac{(1 - f(Q_3)^2) (1 - 4f(Q_3)^2)}{1 - Q_3^2}. \]  

(84)
Integrating this expression we get, besides the trivial constant solutions $f(Q_3) = \pm 1, \pm \frac{1}{2}$,

$$\Theta = f(Q_3) = i\sqrt{n} \text{ Sn} \left[ 2\sqrt{2}n^{3/2} \ln \left( \sqrt{n} \left( Q_3 + \sqrt{(Q_3^2 - 1)} \right) \right) + c_1 \right]$$

(85)

This is the function mapping the solutions of the two models $(q_3 = i\sqrt{n}\Theta = i\sqrt{n}f(Q_3))$ obtained by our geometric approach to the deformation method.

$$\Theta = \text{Sin} \left[ \frac{1}{2} \text{EllipticF} \left( \frac{1}{4} \text{Sin}^{-1}(2f) \right) \right]$$

(86)

V. FUBINI-STUDY SPACE DEFORMATION

Let us now extrapolate the results obtained in section III C to a case not involving defect-like solutions, that is, in which we are not worried about connecting the asymptotic values of the solutions of the original and deformed models. In particular, in connection with our previous results [8] we will analyze the Fubini-Study case.

A. Fubini-Study space from a complex field model.

One important (and usually disregarded) rôle of the Fubini-Study metric arises in the context of Quantum Mechanics. The Hilbert space description of quantum theory is generically complex, and, in order to compute transition probabilities, a (complex) scalar product is defined. This scalar product defines an Euclidean geometry. However, another geometry also emerges in this setting: the Fubini-Study geometry, which arises in the following way.

As the superposition principle is assumed to be valid in quantum processes, the theory must be linear, and two linear dependent vectors represent the same physical state. Restricting to unit vector reduces the redundancy to phase factors. Consequently, two curves on the unit vectors space—a curve could be, for instance a piece of a solution of the Schrödinger equation—differing only in a phase, describe the same set of physical states. In this context, the ‘Fubini-Study length’ corresponds to the minimal length a curve of states can assume. The requirement of the minimal length on the set of vector states induces a geometry, represented by the ‘Fubini-Study metric’ [14]. Imposing the condition that the Euclidean and the Fubini-Study lengths coincide piecewise in the curve (parallel transport condition) defines the geometric (or Berry) phase, which corresponds to the difference between an arbitrary
solution states (closed) curve and the one obtained imposing the minimal length condition. More precisely, its initial and final points will differ in a phase factor: the ‘geometric phase’.

The spaces described above are realized by Kähler Manifolds (complex manifold $M$ with a Hermitian metric $g$ and a fundamental closed 2-form $\omega$), with group structure $\mathbb{CP}^n = S^{2n+1} / S^1$. The Hermitian metric components are given by

$$h_{ij} = \frac{(1 + |z|^2) \delta_j^i - z_i z_j}{(1 + |z|^2)^2} = \partial_i \partial_j K(z, \bar{z})$$

This metric can be derived from the Kähler potential

$$K(z, \bar{z}) = \ln(1 + \delta_i^j z_i z_j).$$

Consider now the Lagrangian density

$$\tilde{L}_K = \frac{1}{2} \partial^\mu \tilde{\varphi} \partial_\mu \tilde{\varphi} - \tilde{V}(\tilde{\varphi}, \bar{\tilde{\varphi}}) = \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \left(1 + |\varphi|^2\right)^2$$

Looking at the potential term, it is straightforward to conclude that the target space of the free version of the model corresponds to a Fubini-Study space, with a Kähler potential as in (88).

Putting the field in its polar representation, $\tilde{\varphi} \equiv r e^{i\alpha}$, Lagrangian (89) takes the form

$$\tilde{L}_K = \frac{1}{2} \left[(\partial_\mu r \partial^\mu r) + (\partial^\mu \alpha \partial_\mu \alpha) r^2\right] - \left(1 + r^2\right)^2$$

The corresponding Euler-Lagrange equations are

$$\partial^\mu \left(r^2 \partial_\mu \alpha\right) = 0$$

$$\partial^\mu \partial_\mu r - (\partial_\mu \alpha \partial^\mu \alpha) r + 4r \left(1 + r^2\right)^2 = 0$$

Equation (91) above expresses the existence of the constant of motion

$$L_\mu = r^2 \partial_\mu \alpha = \text{const.}$$

Substituting $L_\mu$ in the field equation for $r$, (92), we have

$$\partial^\mu \partial_\mu r - \frac{L_\mu L_\mu}{r^3} + 4r \left(1 + r^2\right)^2 = 0$$
The simplest case to be considered in order to obtain a solution for this equation is the ‘momentum map’ \((L_\mu = 0)\). In that case, after covariant elimination, we arrive to the solution
\[
r = - \text{Re} \left[ i \left( 1 - \sqrt{1 + 2c} \right) \text{sn} \left( iw_\mu x^\mu, \frac{1 + \sqrt{1 + 2c}}{1 - \sqrt{1 + 2c}} \right) \right],
\]
where \(c\) and \(w_\mu\) are scalar and vector constants.

The momentum map \(L_\mu = 0\) introduces a degeneration of the solution \(\tilde{\varphi} = \tilde{\varphi} = r\), which results real, continuous and periodical, showing the typical behavior of \(\text{sn}\) – see Fig. 4.

\[
\text{Figure 4: Momentum map case (} \alpha = 0 \text{) } r(x) \text{ solution.}
\]

In the general case \((L_\mu \neq 0)\), the norm of the scalar field takes the form
\[
r = - \text{Re} \left\{ \frac{1}{2} \text{sn} \left( \frac{i \sqrt{4 + L_\mu L^\mu} w_\mu x^\mu}{2\sqrt{2}}, \frac{4 - L_\mu L^\mu}{4 + L_\mu L^\mu} \right) \right\}
\times \sqrt{4 + L_\mu L^\mu - 4 \left( \text{sn} \left( \frac{i \sqrt{4 + L_\mu L^\mu} w_\mu x^\mu}{2\sqrt{2}}, \frac{4 - L_\mu L^\mu}{4 + L_\mu L^\mu} \right) \right)^2}. \quad (97)
\]
This solution, depicted in Fig. 5(a), is periodic with compact support on the spacetime coordinates (for example, identifying \(x\) with the radial coordinate of a cylindrical spacetime).

The phase of the general solution can also be explicitly determined however, the expression is cumbersome and we will just mention here that it presents a highly oscillating behavior near the origin of coordinates, and stabilizes quickly to \(\alpha \to \sqrt{L_\mu L^\mu}\) as moving away, as shown in Fig. 5(b).

### B. Fubini-Study space as flat space deformation

We have shown an explicit complex field solution in connection with a Fubini-Study space. Consider now a one dimensional \((n = 1)\) flat complex space, described by a flat metric \(g_{\text{flat}}\).
From relation (41), the corresponding Lagrangian density with a potential reads

$$\mathcal{L}_\mathcal{K} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi, \bar{\varphi}) = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - m^2,$$

where, as before, we take $C_0 = 1$ and $V = V_0 = m^2 = \text{const.}$.

Using the precedent results, we can explicitly link model (98) to a Fubini-Study space, which corresponds to the target space of the free version of the Lagrangian (89),

$$\tilde{\mathcal{L}}_\mathcal{K} = \frac{1}{2} \partial_\mu \tilde{\varphi} \partial^\mu \tilde{\varphi} - \tilde{V}(\tilde{\varphi}, \bar{\tilde{\varphi}}) = \frac{1}{2} \partial_\mu \tilde{\varphi} \partial^\mu \tilde{\varphi} - (1 + |\tilde{\varphi}|^2)^2 \tilde{g}_{\text{flat}}^{-1} \tilde{g}_{\text{Fubini-Study}}$$

(99)

Again, we are choosing $\tilde{C}_0 = 1$. Now, as stated in (54), the fields $\varphi$ and $\tilde{\varphi}$, can be related by a function $f$ satisfying

$$|f'(\tilde{\varphi})|^2 = \left| \frac{d\varphi}{d\tilde{\varphi}} \right|^2 = \frac{\tilde{g}_{\text{Fubini-Study}}}{\tilde{g}_{\text{flat}}} = \frac{m^2}{(1 + |\tilde{\varphi}|^2)^2}.$$  

(100)

Therefore,

$$|d\varphi| = \frac{m}{1 + |\tilde{\varphi}|^2} |d\tilde{\varphi}|,$$

(101)

which directly leads to the relation

$$\varphi \bar{\varphi} = f(\varphi) \bar{f}(\varphi) = m \ln \left(1 + \varphi \bar{\varphi} \right)^2 = K(\varphi, \bar{\varphi}).$$  

(102)

That is, the deformation function relating the original and the deformed models is precisely the Kähler potential (88).
In order to check the relation above against the results of previous sections, we can easily reconstruct the deformed potential going backwards and writing

\[
\tilde{V}(\tilde{\phi}, \tilde{\psi}) = \frac{V(f(\tilde{\phi}), f(\tilde{\psi}))}{|f'(\tilde{\phi})|^2} = m^2 \left[ \frac{m^2}{(1 + |\tilde{\phi}|^2)^2} \right]^{-\frac{1}{2}} = (1 + |\tilde{\phi}|^2)^2
\]

(103)

This confirms that, at least for the present case, the procedure can be applied in a purely geometric context, that is, when there are no defect solutions associated.

VI. CONCLUDING REMARKS

In the present article we have considered a technique for generating solutions, known as deformation method. Differently from all the previous works on this subject, here we analyze the procedure under a geometrical point of view, in order to get a better understanding of the meaning and limitations of deformation method. Thus, the idea was applied to the metric field of the equivalent free models, making connection between different solutions and also generating new ones.

As main a result, we conclude that the deformation method is unavoidable limited to systems with higher symmetries and can be consistently implemented only in the case of single-field models. This is due mainly to the very restrictive constraint imposed by preservation of BPS conditions.

As a way of expanding the applicability of the procedure, we considered systems with multiple field components. That is, real ($\mathbb{R}$), complex ($\mathbb{C}$) and also quaternionic ($\mathbb{H}$) one-dimensional field models were analyzed. All cases where illustrated by constructing explicit maps between different geometries.

The present work is part of a larger investigation, centering efforts in finding complex and quaternionic BPS solutions. In that direction, we have shown a new quaternionic domain wall solution obtained by directly performing a geometrical map. For constructing domain wall solutions the method is highly suitable due do the geometries related in the transformations. However, our pure geometrical analysis put in clear evidence the limitations of the deformation procedure as tool for generating solutions, as it is the simplest harmonic map between the metrics (of the geometrical Lagrangians associated to different dynamical models).
Thus, in more involved geometries-field models, the use of pure geometrical approach to the procedure presented here is mandatory in order to preserve BPS conditions, SUSY structure or gauge transformations (as was clearly illustrated by the mapping of the Fubini-Study geometry in Section V). Differently from the ‘standard’ mapping of references [2], which does not assures the preservation of such symmetries, we expect that the geometrical approach could be extended to a $N = 2$ gauged supergravity theory in four dimensions. This should be possible given that a hyper-Kähler manifold $\mathcal{M}_H$ is a quaternionic space in which, as shown in [8], and also in Sec. IV A stable BPS solutions can be consistently constructed.

Another interesting potential feature of the procedure is the possibility of establishing a parallel between models with modified kinetic term (‘k-fields’) and models described by Lagrangians of standard form. These idea is under exploration and will be presented elsewhere.

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[15] Along the whole article, the tilde indicate the fields on the ‘deformed side’.

[16] Note, however, that the behavior of $f'$ could perhaps affect the divergence order of $\tilde{g}$, as can be seen from (15).

[17] Note that the pure imaginary profile of the scalar component of the solution ($q_0(X_3)$) is represented in Fig. 2 in the same graphic than the real $q_3(X_3)$ component.

[18] We think it is worth mention here that in [9] it was worked out the deformation of (the bosonic sector of) a Wess-Zumino complex field model. In that case, the deformation function turned out to be identified with the superpotential of the model.