Scale-free power spectrums in the delayed cosmology

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Abstract

The delayed cosmology [JCAP 02(2012)046] assumes that the evolution of geometries is delayed relative to that of matter and/or energies. This idea allows inflation to occur without inflaton fields or vacuum energies of any kind as drivings. We considered the production and evolution of primordial perturbations in this model. The result indicate that, with delaying, we could get a nearly scale-free power spectrum of perturbations starting from a radiation dominated early universe.

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1 Introduction

The scenario of inflation [1, 2] developed since 1980s may be the most successful part of modern cosmologies [3, 4]. It solves the horizon, flatness and other problems which are unavoidable in the conventional-cosmology. Most importantly, it provides an elegant mechanism for the production of primordial seeds of structures in the universe [3, 5]. On March 2014, the BICEP2 team announced their detection of primordial gravitational waves through the B-mode power spectrum [6] of cosmological microwave background radiation. If confirmed, this would be a strong evidence for the scenario of inflation.

In most of the existing models, see examples reviewed by [3, 4], inflations during the early universe is facilitated with the aid of some one or more scalar fields called inflaton. As far as we know, all this
Figure 1: By introducing a delay to the source term of Friedmann equation, inflation is obtained in the absence of extra scalar field, and is accompanied with a smooth graceful exit for a wide range of $\alpha$. In this figure the delay parameter is chosen as $\tau = 10t_{pl}$. The inflation lasts in the period $\tau < t < 2\tau$.

Kind of inflation models violate the strong energy conditions, some of them require some specifically designed mechanism to protect the universe from eternal inflation, and some of them require artificial choosing of initial values for the inflaton fields. What’s more, in the standard model of elementary particles, we have not found any scalar fields could play the role of inflatons.

Noticing that in many natural phenomenas, responses of the system are usually delayed relative to the driving forces, A.A.Sen et al. proposed [7] that, in Friedmann equation, similar delaying effects may also occur,

$$\frac{a'(t)}{a(t)} = \frac{1}{3} \rho(t - \tau).$$

(1)

where $a$, $\rho$, $t$ and $'$ denote the scale factor, energy density, cosmic time and derivatives respect to $t$ respectively, $\tau$ is the delaying. With this assumption and some rather general initial conditions, it is found that an early inflation and elegant exit from it would occur very naturally. Of course, to prevent any contradictions between the known observations and predictions following from Eq. (1), the parameter $\tau$ has to be fixed on the order of planck time $t_{pl} \sim O(10^{-43}s)$. A. A. Sen et al. discussed the initial condition of $a_0 < t < \tau = t_\alpha$ and the resulting evolution of the scale factor is as follows

$$a(t) = \begin{cases} 
\tau^\alpha \exp \left( \frac{3\tau(1+w)\alpha}{1-3(1+w)\alpha} \right), \tau < t < 2\tau \\
\text{graceful exit}, t < \infty
\end{cases}$$

(2)

see Figure 1 for illustrations. Following Ref. [7], we will call this idea as delayed cosmology, or sometimes, delayed inflation.

This is indeed an ingenious idea for the mechanism of inflation. It avoids the introduction of “professional” fields which is not in the known lists of particle physics as driving forces and solves the exit problem gracefully. Certainly, as a new kind of inflation mechanism, enabling the exponential growth of scale factor is not adequate. The more important question is, could this mechanism provide seeds of structures for the universe. As it is well known that, in the conventional models, the quantum fluctuations of inflaton fields after the pulling out of horizons by the accelerating expansion, become the seeds of structures for late time evolution. For a mechanisms without inflaton fields as the delayed cosmology, we could only depend on the quantum fluctuations of pre-inflation cosmic-contents (radia-
tion/matter) to provide such seeds of structures. The purpose of this paper is to consider the evolution of perturbations in the delayed inflation and calculate the power spectrum of primordial fluctuations.

The organization of this paper is as follows, this section is a brief introduction to the basic idea of delayed cosmologies. The next section calculates the power spectrum of primordial perturbations, including scalar and tensor ones, caused by the delayed inflation. The next next section numerically evolves the primordial fluctuations in the late time universe and obtain the power spectrum of matter distributions today. The last section is our conclusion.

2 The power spectrum of perturbations in the delayed inflation

2.1 The equations of motion for fluctuations

The basic idea of delayed cosmology/inflation is revising the Friedman equation so that the growth of the cosmic scale factor is delayed relative to the evolution of energy-densities. When we consider the fluctuation and structure formation questions, it is very natural to generalize this idea to the full Einstein equation, so that

$$G_{\mu\nu}(\vec{x}, t) = T_{\mu\nu}(\vec{x}, t - \tau).$$  \hspace{1cm} (3)

We will use the linearization of this equation to constrain the evolution of fluctuations in the delayed inflation.

According to the standard theory of cosmological perturbations \cite{8, 9, 10}, in the Newton + gravitational wave gauge, we could write the perturbed cosmic metric as

$$ds^2 = -(1 - 2\Phi)dt^2 + a^2[\delta_{ij}(1 + 2\Phi) + h_{ij}]dx^idx^j$$  \hspace{1cm} (4)

where $h_{ij}$ takes the form of

$$h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (5)

Substituting this perturbed metric into the delayed Einstein equation (3), taking the adiabatic assumption and first order approximations, we will get

$$\nabla^2 \Phi - 3\dot{\Phi} - 3H^2\Phi = \frac{1}{2}a^2\delta \rho_r$$  \hspace{1cm} (6)

$$D_i \left( \dot{\Phi} + \mathcal{H}\Phi \right) = -\frac{1}{2}a^2 (\rho_r + p_r) \nabla_i V$$  \hspace{1cm} (7)

$$\ddot{\Phi} + 3\dot{\mathcal{H}}\Phi + \left(2\mathcal{H} + H^2\right)\Phi = \frac{1}{2}a^2\delta \rho_r$$  \hspace{1cm} (8)

for scalar perturbation and

$$\ddot{h}_\lambda + 2\mathcal{H}\dot{h}_\lambda + k^2 h_\lambda = 0$$  \hspace{1cm} (9)

for tensor perturbation. In these equations, $\{\rho_r, p_r, \cdots\} \equiv \{\rho(t - \tau), p(t - \tau), \cdots\}$, $\mathcal{H} \equiv \frac{d\mathcal{H}}{dt}$, while various overdots symbol represent derivatives respect to the conformal time $\frac{d}{d\eta} = a \frac{d}{dt}$.

Through simple combinatorics among (6)-(8), we could eliminate the source term from the scalar
perturbation equations and get

\[ \ddot{\Phi} + 3H (1 + c_s^2) \dot{\Phi} + c_s^2 k^2 \Phi + \left[ 2 \dot{H} + (1 + 3c_s^2) H^2 \right] \Phi = 0 \]  

(10)

where \( c_s^2 = \frac{\partial p}{\partial \rho}_{\text{adiabatic}} = w \). This is a highly nontrivial result. It is different from the scalar perturbation equations in the conventional inflations driven by inflaton fields, but it tells us that the effects of delay enter the evolution of scalar perturbations only through the coefficient functions. This will greatly simplify our calculation of power spectrums in the delayed cosmology. While for the tensor perturbations, Eq. (9) in the delayed cosmology is almost the same as that in conventional inflation models driven by inflaton fields.

2.2 The quantization and power spectrum of perturbations

Now, let us follow the standard path of inflationary cosmologies, to consider the quantization of perturbations in the delayed inflation. For tensor perturbation, since its equation of motion is completely of the same form as conventional inflation models, we can directly borrow from them the expression of power spectrum \[10\]

\[ P_h(k) = \frac{2}{a^2} \left| u(k, \eta) \right|^2 \bigg|_{aH=k} = \frac{H^2}{k^3} |aH=k|. \]  

(11)

While for scalar perturbations, we will rewrite the perturbation equations \[10\] into the form of harmonic/near harmonic oscillator, so that \( \Phi \) could be quantized by language of creation and annihilation operators. In order to do so, we define

\[ \varphi = a^p \Phi, \]  

(12)

where \( p = \frac{3}{2}(1 + w) \). To get more beautiful analytical results, we now constrain ourselves to the initial conditions \( a_{0 < t < \tau} = c \cdot t^\alpha \) with \( \alpha \approx 0 \). As can be seen from Figure 2, this initial condition leads to exponential/near-exponential inflations, during which \( H \) are constants. However, even if \( \alpha \neq 0 \), delayed inflation could also occur, and the power of primordial perturbations could also be calculated, but more numerics is needed.

For exponential/near-exponential type inflations, since \( H \) is approximately constant, the following
The approximation is valid
\[ \eta = \int_{a_e}^{a} \frac{da}{Ha^2} \approx \frac{1}{H} \int_{a_e}^{a} \frac{da}{a^2} \approx -\frac{1}{aH}, \]
according to which
\[ \frac{\dot{a}}{a} \approx -\frac{1}{\eta}, \quad \frac{\ddot{a}}{a} \approx \frac{2}{\eta^2}. \]
Substituting (12) and (14) to (10) we will have
\[ \ddot{\varphi} + \left[ \frac{\nu^2}{\eta^2} - \frac{3}{4} \left( 3w^2 + 4w + 1 \right) \frac{1}{\eta^2} \right] \varphi = 0. \]
After the definition
\[ \nu^2 \equiv \frac{9}{4} \left( w + \frac{2}{3} \right)^2, \]
we get the more terse form
\[ \ddot{\varphi} + \left( c_s^2 k^2 - \frac{\nu^2 - \frac{1}{2}}{\eta^2} \right) \varphi = 0. \]
Except for the definition of \( \nu^2 \), this is identically the same as equation of perturbation in the conventional inflation models. We can therefore borrow the idea from them \[11\], and write \( \varphi \) as a quantum operator
\[ \hat{\varphi}(\vec{k}, \eta) = v(\vec{k}, \eta) \hat{a}_k + v^*(\vec{k}, \eta) \hat{a}^+_k \]
The coefficient functions \( v(\vec{k}, \eta) \) and \( v^*(\vec{k}, \eta) \) satisfy equations completely the same as (17), the solution of which has asymptotics
\[ v = e^{i\left( \nu - \frac{1}{2} \right) \frac{\pi}{2} 2^{\nu - \frac{1}{2}} \Gamma(\nu) \Gamma\left( \frac{3}{2} \right)} \frac{1}{\sqrt{2c_s k}} (-c_s k \eta)^{\frac{1}{2} - \nu}, \text{ as } \frac{k}{aH} \to 0. \]
With these derivations, we can easily write down the variance of the quantized perturbation \( \varphi \) and \( \Phi \equiv a^{-p} \varphi \) as follows
\[ \langle \hat{\varphi}^\dagger (\vec{k}, \eta) \hat{\varphi}(\vec{k}', \eta) \rangle = \left| v(\vec{k}, \eta) \right|^2 (2\pi)^3 \delta^3(\vec{k} - \vec{k}'), \]
\[ \langle \hat{\Phi}^\dagger (\vec{k}, \eta) \hat{\Phi}(\vec{k}', \eta) \rangle = \frac{1}{a^{2p}} \left| v(\vec{k}, \eta) \right|^2 (2\pi)^3 \delta^3(\vec{k} - \vec{k}'), \]
\[ \equiv (2\pi)^3 P_\Phi(k) \delta^3(\vec{k} - \vec{k}'), \]
where
\[ P_\Phi(k) = \frac{1}{a^{2p}} \frac{2^{2\nu - 3} \Gamma(\nu) \Gamma\left( \frac{3}{2} \right)}{2c_s k} (-c_s k \eta)^{1 - 2\nu} \big|_{aH = k} \]
where \( a_e \) denotes the value of \( a \) when inflation end. Just as shown in \[7\] and reproduced in this paper, Figure II after the time \( t = 2\tau \) no more e-folds of inflation could be gained, i.e. inflation caused by delays exits at this point.

Obviously, in inflations causing by delaying instead of inflaton fields, the most natural form of
\[ \nu = \frac{1 + \delta + \epsilon}{1 - \epsilon} + \frac{1}{2}, \]
where \( \epsilon, \delta \) are the corresponding slow roll parameters.
dominating energy of the early universe is radiation, for which \( c_s^2 = w = 1/3, \nu = 3/2 \). As results

\[
a_e = c \cdot \tau^\alpha \exp \left( \frac{H_i \tau^{1-2\alpha}}{1-2\alpha} \right), \quad H_i \equiv \sqrt{\frac{\rho_0}{3}},
\]

(24)

\[
P_\Phi(k) = \frac{3\sqrt{3}H^2_{aH=k}}{2c^2\tau^\alpha e^{2H_1t-2\alpha/1-2\alpha}} \frac{1}{k^3},
\]

(25)

where we have added parameter \( c \) to ensure the dimensionless of scale factor. Neglecting the dependence of \( H \) on scales (through taking values at epoch \( aH = k \)), this exactly corresponds to a scale-free power spectrum. If for some unknown reason, the pre/during delay-inflation universe is dominated by matter instead of radiation, then \( w = 0 \). In this case, we could reasoning from Eq. (17) that \( P_\Phi \) is totally-independent of \( k \).

\[
P_\Phi(k) = \text{const}.
\]

(26)

This is obviously contradict with observations [5, 12].

From the above discussion, we easily see that it is the fact delaying effects enter the equation of perturbations only through coefficient functions that make our calculations possible. Obviously, any specific component of Eqs. (6)-(8) has no such good features, it emerges only in their combinations.

### 2.3 Tensor to scalar ratio and determination of model parameters

There are 4 parameters in the delayed inflation model to be determined through observations

- \( c, \alpha \) — related with the scale factor \( a \) of pre-inflation era through \( a_{0<t<\tau} = c \cdot t^\alpha \)
- \( \tau \) — the amount of delaying, \( G_{\mu\nu}(\vec{x}, t) = T_{\mu\nu}(\vec{x}, t-\tau) \)
- \( H_i \) — the initial value of Hubble parameter, \( H_i = \sqrt{\frac{\rho_0}{3}} \)

In Ref. [7], A. A. Sen et al discussed the constraints on \( \alpha \) and \( \tau \) imposed by the number of e-foldings \( 65 \leq N_e \). Their conclusion is, \( \tau \) is of order \( 10^2 \sim 10^3t_{pl} \), \( \alpha \) could be negative as well as positive in a rather broad range, see the Figure 2 of their paper. However, this conclusion is based on the condition that \( \rho_0 = 1 \), whose default unit is \( M_{pl}^4 \). In our paper we will relax this pre-assumption.

For simplicity, we will limit ourselves to the case \( \alpha = 0 \), but use the observation value of scalar perturbation magnitude, tensor-to-scalar ratio as well as the lower limit of the number of e-foldings to find the allowed range of parameter \( c, \tau, H_i \). According our calculations in the above subsection,

\[
\frac{P_h}{P_\Phi} = H^2_{aH=k} \cdot \frac{1}{k^3P_\Phi} = H^2_i \cdot \frac{1}{k^3P_\Phi}.
\]

(27)

The measurements of COBE satellite tells us that [13]

\[
k^3P_\Phi = \frac{50\pi^2}{9} \cdot (1.9 \times 10^{-5}) = 1.979 \times 10^{-8}.
\]

(28)

So, if the result of BICEP2 \( \frac{P_h}{P_\Phi} \approx 0.2 \) is reliable, then the relation (27) implies that \( H_i \) is of the same order of \( \sqrt{k^3P_\Phi} \),

\[
H_i = \sqrt{k^3P_\Phi \times \frac{P_h}{P_\Phi}} = 6.292 \times 10^{-5},
\]

(29)

the default unit is \( M_{pl} \). This happens to be the typical energy scale of many inflations on the market. Substituting the results (29) and (28) into Eq. (25) and setting \( H_{aH=k} = H_i \), which follows from
Figure 3: Numerical results after we re-adjust the parameters according to observations, which require that \( \tau = 10^6 t_{pl} \). All these new parameters function well in producing inflations and exits. And the approximation of (14) holds good.

\[ \alpha = 0, \text{ we will get} \]
\[
c \cdot e^{H_i \tau} = \sqrt{\frac{3\sqrt{3}H_i^2}{2k^3P_\Phi}} = \sqrt{\frac{3\sqrt{3}P_h}{2P_\Phi}} = 0.721.
\]

Finally, let us consider the constraint of e-folding numbers. Using (24), we know that in the delayed inflation \( \frac{a_{\text{end}}}{a_{\text{begin}}} = e^{H_i \tau} \). If we require that \( N_e \geq 65 \), then

\[ H_i \tau > 65. \]

This means that \( \tau \) is of order \( 10^6 t_{pl} \), somewhat out of our expectation. However, we verify in Figure 3 that these new choice of parameters works equally-well on producing inflation and exit, just as those chosen by A. A. Sen et al do in their original work.

3 Late time evolutions

Some people may worry that delays as large as \( \tau \sim 10^6 t_{pl} \) may cause noticeable effects on the late time evolution of the universe, for instance, the structure formation processes. However, from the viewpoint of post-inflationary cosmologies, this is in fact an almost negligible period of time. To show that this is indeed the case, let us consider the evolution of matter/energy fluctuations in the late time of \( \Lambda \)-Cold Dark Matter model in this section. The basic idea of this section is just an exercise from [10] and is essentially using numerics to search solutions of the following equation array

\[
\dot{\Theta}_0 + \frac{k}{3} \Theta_1 = -\Phi, \quad \text{(32)}
\]

\[
\dot{\Theta}_1 + \frac{k}{3} \Theta_0 = -\frac{k}{3} \Phi, \quad \text{(33)}
\]

\[
\dot{\delta} + ikv = -3\Phi, \quad \text{(34)}
\]

\[
\dot{v} + \frac{\dot{a}}{a}v = ik\Phi, \quad \text{(35)}
\]
Figure 4: Power spectrum of matter distributions predicted by the inflation and ΛCDM cosmology with delays. The left hand side assumes that the early universe is dominated by radiation \( w = \frac{1}{3} \) and delayed appropriately, so that \( P_\Phi \propto k^{-3} \). The right hand side assumes that the early universe is dominated by matter \( w = 0 \). In this case, although delay mechanism cause inflation, the primordial power spectrum \( P_\Phi \) is a constant independent of \( k \).

where \( \Theta_0 \) and \( \Theta_1 \) are the zeroth and first moment of cosmic background radiation, \( \delta \) and \( v \) are those of matters, including both dark matter and baryons, \( \Phi, k \) and \( a \) are respectively scalar-perturbation, wave number and scale factors of the late time universe. \( \Phi \) and \( a \) satisfy equations

\[
k^2 \Phi + 3H \left( \dot{\Phi} + H \Phi \right) = \frac{1}{2} a^2 [ \rho \delta + 4 \rho_r \Theta_0 ] \tag{36}
\]

\[
\left( \frac{a'(t)}{a(t)} \right)^2 = \frac{H_0^2}{3} \left[ \frac{\rho_m (t - \tau)}{\rho_{tot}} + \frac{\rho_r (t - \tau)}{\rho_{tot}} + \frac{\rho_\Lambda}{\rho_{tot}} \right] \tag{37}
\]

Note again that in these equations, overdot represent derivatives respective to the conformal time \( \frac{d}{d \eta} \), while ‘, to the physical time \( \frac{d}{dt} = \frac{d}{d \eta}. \)

With the early time approximation and the primordial power spectrum \((25)-(26)\) as input

\[
\Phi(\eta_i) = 2\Theta_0 = \sqrt{P_\Phi} \tag{38}
\]

\[
\delta(\eta_i) = 3\Theta_0 \tag{39}
\]

\[
\Theta_1(\eta_i) = 0 \tag{40}
\]

\[
v(\eta_i) = 0 \tag{41}
\]

we get from Eqs \((32)-(37)\) the power spectrum of matters in the universe today, \( P_\delta \equiv |\delta_k|^2_{\text{today}} \). The result is illustrated in Figure 4. In this figure, we have two comments. The first is, only the radiation dominating early universe with delays could give late time power spectrum of matter distributions consistent with observations \([14, 15, 16]\). The second is, the delay of \( 10^6 t_{pl} \) is negligible in the late time evolutions.
4 Conclusion

This paper calculated the power spectrum of primordial perturbations, including both scalar and tensor types, produced during the inflations driven by delays. The result indicate that, in a delayed cosmology without inflaton field, if the early universe is radiation dominated, an appropriate period of delaying between the geometry and energy not only cause inflations with enough number of e-foldings and graceful exit, it also produces a nearly scale-free power spectrum of perturbations with the ratio of tensor to scalar perturbations be tuneable in almost arbitrarily broad range. If the early universe is dominated by non-relativistic matter, then inflation is possible but the result power spectrum is strong scale-dependent. Using observations of COBE and BICEP2, we determined the key parameters of the model. $\tau \approx 10^6 t_{pl}$, $H_i = H_e \approx 10^{-5} M_{pl}$. In the last section, we add delays to the $\Lambda$CDM and numerically evolve the primordial perturbation in the late time universe and get power spectrum of matter distributions today. The result is perfectly-consistent with observations.

As discussions, we comment here that, i) although the delay mechanism could drive the early time inflation, it could not alleviate our thirst of dark energies in late time evolutions. ii) in the delayed inflation, the scale factor $a$ is continuous but not smooth at the point $t = \tau$. Revealing effects of this feature on observations is a very valuable work for futures iii) according to some current idea $[17]$, cosmic magnetic field may also be the relic of inflations. It would be very interesting for discussion if the delayed cosmology could also accommodate this possibility. iv) if inflations are really caused by delay, then we need not any inflaton fields or exotic dark energies to dominate the early universe. The big bang just begins from the radiation time. As a result, we also need no mechanism of reheating. This should be a great simplification of cosmologies.

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