Generalised Poincaré series and embedded resolution of curves

Julio José Moyano-Fernández *

Institut für Mathematik, Universität Osnabrück
Email: jmoyanof@uni-osnabrueck.de

Abstract

The purpose of this paper is to extend the notions of generalised Poincaré series and divisorial generalised Poincaré series (of motivic nature) introduced by Campillo, Delgado and Gusein–Zade for complex curve singularities to curves defined over perfect fields, as well as to express them in terms of an embedded resolution of curves.

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1 Introduction

A. Campillo, F. Delgado and K. Kiyek introduced in 1994 a multivariable Poincaré series $P(t_1, \ldots, t_r)$ (from now on denoted by $P(t)$) associated with the valuations of the integral closure of a one–dimensional local Cohen–Macaulay ring (cf. [8, (3.8)]). For the case of valuations of a complex plane curve singularity, Campillo, Delgado and Gusein–Zade interpreted the Poincaré series $P(t)$ as an integral over the local ring of germs with respect to the Euler characteristic, expressing it also in terms of an embedded resolution of curves ([4], [5], [6]). This new approach allowed them also to define the Poincaré series $\hat{P}(t)$ associated with the extended semigroup of the curve singularity just by taking the integral over the extended semigroup (cf. [6]). The same philosophy can be applied to divisorial valuations to get a divisorial Poincaré series $P^D(t)$ of the divisorial value semigroup (i.e., the value semigroup arising by considering divisorial valuations), or even a semigroup divisorial Poincaré series $\hat{P}^D(t)$ if we take the divisorial extended semigroup of the singularity (see [10], [9]).

We can also take the generalised Euler characteristic instead of the classical one; it leads to the study of the motivic versions of the previous series, namely:

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the generalised Poincaré series \( P_g(t) \); the generalised semigroup Poincaré series \( \hat{P}_g(t) \); the generalised divisorial Poincaré series \( P^D_g(t) \); and the generalised divisorial semigroup Poincaré series \( \hat{P}^D_g(t) \). They all were studied in [7] for complex curve singularities.

An interesting question is how to translate these ideas in more general contexts than \( \mathbb{C} \). Some work in this direction has been already done (see [11] for Poincaré series over finite fields, and [14] for \( P^D(t) \) and \( P^D_g(t) \) over non–finite fields). Also, a related motivic zeta function was treated by W. Zúñiga and the author in [21]. Following this direction, we wish to investigate in the present paper the generalised Poincaré series \( P_g(t) \), the generalised divisorial Poincaré series \( P^D_g(t) \) and the generalised divisorial semigroup Poincaré series \( \hat{P}^D_g(t) \) attached to curves defined over perfect fields; in particular we want to express them—in this more general context—in terms of an embedded resolution of the curve. In the process we include some omissions in [7], which we will opportune indicate. As for prerequisites, the reader is expected to be familiar with the theory of two–dimensional regular local rings, especially with the concept of ideal transform. For a treatment of this topic we refer the reader to [15].

The paper is organised as follows. Sect. 2 establishes the terminology and contains a brief summary about embedded resolution of curves. Sect. 3 deals with the Grothendieck ring of the category of quasi–projective schemes of finite type over a perfect field \( k \), the generalised Euler characteristic and the definition of generalised Poincaré series, according to the exposition presented in [11]. The aim of Sect. 4 is to give a formula expressing the generalised Poincaré series in terms of an embedded resolution of curves. First, we define a semigroup homomorphism (the map \( \text{Init} \) of Subsect. 4.1) which will be a locally trivial fibration over each connected component of its image. We study also the preimage \( \text{Init}^{-1} (\varsigma) \) for every element \( \varsigma \) on the connected component of the image (Corollary (4.12)), and compute its (finite) codimension by means of the Hoskin–Deligne formula (see Subsect. 4.2; the explicit calculation is provided in Proposition (4.20)). Using all these data and applying Fubini’s formula to the map \( \text{Init} \) we get a description of the generalised Poincaré series in terms of the embedded resolution (Theorem (4.23)). Finally, in Sect. 5 we introduce both the generalised divisorial Poincaré series and the generalised divisorial semigroup Poincaré series of a curve defined over a perfect field and proceed with a similar study; the first series is described in Theorem (5.4), whereas the second one needs some extra work: we show in (5.6) the analogous of the map \( \text{Init} \) in this context; it is in fact an isomorphism (Lemma (5.8)). The final formula is given by Theorem (5.9). In the rest of the section we explain briefly some conventions, terminology and basic definitions to be used along this paper.

2 Preliminaries on blowing–ups and exceptional divisors

(2.1) Let \( R \) be a two–dimensional regular local ring having a maximal ideal \( m \) and a perfect residue field \( k_R \). Let \( f \in R \setminus \{0\} \) be a non–unit reduced element. We say that \( f \) defines a curve \( C_f \) (or simply \( C \) when no confusion can arise) on \( R \). The irreducible factors of \( f \), let us say \( f = f_1 \cdot \ldots \cdot f_r \), are called the
irreducible components of $C$. Sometimes we will refer to this fact expressing $C = C_1 \cup \ldots \cup C_r$. The curve is reduced (resp. analytically reduced) if $R/\mathfrak{f}R$ is reduced (resp., if $\bar{R}/\mathfrak{f}\bar{R}$ is reduced, where $\bar{R}$ stands for the $m$-adic completion of $R$).

(2.2) Let $C = C_1 \cup \ldots \cup C_r$ be an analytically reduced curve defined by the element $f = f_1 \cdot \ldots \cdot f_r \in R \setminus \{0\}$. Take the ring $R/\mathfrak{f}R$ and its integral closure $\bar{R}/\mathfrak{f}\bar{R}$. It defines finitely many discrete Manis valuations $v_i$ having $\bar{R}/\mathfrak{f}\bar{R}$ as discrete Manis valuation ring, for all $1 \leq i \leq r$ (cf. [15, Chap. I, (2.2)]). Note that the $v_1, \ldots, v_r$ are not valuations of the ring $R$ (the preimage of $\infty$ via $v_i$ is $f_i R$, for $1 \leq i \leq r$), but they allow us to define a multi-index filtration of the ring $R$ given by the ideals $J(n) := \{ z \in R \mid v_i(z) \geq n_i, \ 1 \leq i \leq r \}$, for $n := (n_1, \ldots, n_r) \in \mathbb{Z}^r$.

(2.3) The multiplicity intersection of two curves $C_f, C_g$ given by elements $f, g \in R \setminus \{0\}$ is just the multiplicity intersection of these elements:

$$(C_f \cdot C_g) = (f \cdot g)_R := \ell_R(R/(f, g)).$$

(2.4) A curve $C$ is said to be normal crossing if it is regular, or if $C$ has two regular components $C_1, C_2$, defined by two elements $f_1, f_2 \in R \setminus \{0\}$ resp., such that $\ell_R(R/(f_1 R, f_2 R)) = 1$. In this case, $\{f_1, f_2\}$ is a regular system of parameters of $R$.

(2.5) Let $X_0 := \text{Spec}(R)$ be a regular scheme of finite type over $k$ of dimension two; take a closed point $p_0 \in X_0$ and blow up at $p_0$ to get another two-dimensional regular scheme $X_1$. By repeating the process $s$ times we obtain in this way a finite sequence of blowing-ups

$$X := X_s \xrightarrow{\pi_s} X_{s-1} \xrightarrow{\pi_{s-1}} \ldots \xrightarrow{\pi_3} X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0,$$

where $\pi_i$ is the blow up at a closed point $p_{i-1} \in X_{i-1}$, for every $1 \leq i \leq s$. Write $\pi := \pi_s \circ \pi_{s-1} \circ \ldots \circ \pi_2 \circ \pi_1$. The exceptional divisor of $\pi$ is defined to be the reduced inverse image $\pi^{-1}(\{m\})$ of the maximal ideal $m$ of the ring $R$. It coincides with the union of the strict transforms of the exceptional divisors of each $\pi_j$, $1 \leq j \leq s - 1$, together with $\pi^{-1}_s(\{p_{s-1}\})$, i.e., it has $s$ different irreducible components, every two components meet transversally at one point, and no three components meet at a point. Each irreducible component of $E$ is isomorphic to a scheme $\text{Proj}(k_j[\pi, \bar{\pi}])$, $1 \leq j \leq s$, where $k_j$ is a finite extension of $k_R$ of degree $h_j$ (cf. [15, Chap. VII], [20, Sect. 2]).

(2.6) We will write $E_{i,j}$ for the exceptional divisor of $\pi_i$ as divisor of $X_i$, and we denote by $E_{i,j}$ (resp. $E_{i,j}^*$) the strict transform (resp. the total transform) of $E_{i,j}$ in $X_j$ by the morphism $X_j \rightarrow X_i$, for $j > i$. We denote by $E_i$ (resp. $E_i^*$) the strict (resp. total) transform $E_{i,s}$ (resp. $E_{i,s}^*$) by the morphism $X \rightarrow X_i$. Let $E$ be the subgroup of 1–cycles of $X$ of the form $\sum_{i=1}^a n_i E_i$, with $n_i \in \mathbb{Z}$. 

3
Both the set \( \{ E_i \} \) and \( \{ E_i^* \} \) are bases of \( E \). One has also a symmetric bilinear intersection form \( E \times E \rightarrow \mathbb{Z} \) given by intersecting cycles \( (A, B) \rightarrow \deg_X(A \cdot B) \). From the projection formula follows

\[
\deg_X(E_i^* \cdot E_j^*) = -\delta_{ij}h_i, \tag{\dag}
\]

where \( \delta_{ij} \) is the Kronecker’s delta (see [18, Theorem 9.2.12, p. 398]).

(2.7) Let \( p_i \in X_i, p_j \in X_j \) with \( (\pi_j \circ \pi_{j-1} \circ \ldots \circ \pi_1)(p_j) = p_i \), for \( j > i \). We say that \( p_j \) is proximate to \( p_i \), and denote it by \( p_j \succ p_i \), if and only if we have \( p_j \in E_{i,j-1} \) (set-theoretically). The basis change matrix from \( \{ E_i^* \} \) to \( \{ E_i \} \) is called the proximity matrix associated to \( \pi \). This matrix will be denoted by \( P_\pi \).

It is easy to check that any entry \( p_{ij} \) of \( P_\pi \) is equal to 1 if \( i = j \), to -1 if \( j > i \) and 0 otherwise. The matrix of the intersection form given in (2.6) in the basis \( \{ E_i^* \} \) is \( -\Delta_\pi \), being \( \Delta_\pi \) the \( s \times s \)-diagonal matrix with entries the extension degrees \( h_1, \ldots, h_s \). Moreover, the matrix of the intersection form in the basis \( \{ E_i \} \) is \( N_\pi := -P_\pi \cdot \Delta_\pi \cdot P^t_\pi \), where \( P^t_\pi \) denotes the transpose of \( P_\pi \). The matrix \( N_\pi \) is called the intersection matrix (with respect to the basis \( \{ E_i \} \) associated with \( \pi \). We will write \( P, N \) and \( \Delta \) instead of \( P_\pi, N_\pi \) and \( \Delta_\pi \) whenever the blow–up given by \( \pi \) is clear from the context (cf. [20, Sect. 4]).

3 Integrals with respect to the generalised Euler characteristic

(3.1) Let \( \mathcal{V}_k \) be the category of quasi–projective schemes of finite type over a perfect field \( k \). The Grothendieck ring of \( \mathcal{V}_k \), denoted by \( K_0(\mathcal{V}_k) \), is defined to be the free Abelian group on isomorphism classes \( [X] \) of quasi–projective schemes \( X \) of finite type over \( k \) subject to the relations (i) \( [X_1] = [X_2] \) if \( X_1 \cong X_2 \) for \( X_1, X_2 \in \mathcal{V}_k \); (ii) \( [X] = [X \setminus Z] + [Z] \) for a closed subscheme \( Z \) of \( X \in \mathcal{V}_k \); and taking the fibred product as multiplication: (iii) \( [X_1] \cdot [X_2] = [X_1 \times_k X_2] \) for \( X_1, X_2 \in \mathcal{V}_k \). Notice that, if \( X_1 \) and \( X_2 \) are reduced, then \( X_1 \times_k X_2 \) is also reduced, because \( k \) is perfect. The neutral element of \( K_0(\mathcal{V}_k) \) with respect to the addition will be denoted by \( 0 \) and corresponds to the class \( [\emptyset] \) of the empty set. The Grothendieck ring \( K_0(\mathcal{V}_k) \) is commutative and with unit, the unit being the class of \( \text{Spec}(k) \). Let \( k[T] \) be the polynomial ring in one indeterminate \( T \) over the field \( k \). We denote \( \mathbb{A}^1_k := \text{Spec}(k[T]) \) and \( L := [\mathbb{A}^1_k] \), which is a non–zero divisor of \( K_0(\mathcal{V}_k) \) (see [22]); thus we can consider \( \{ L^n \}_{n \in \mathbb{N}} \) as a multiplicatively closed subset of \( K_0(\mathcal{V}_k) \), and localise with respect to \( L \) to define \( \mathcal{M}_k := K_0(\mathcal{V}_k)_L \).

(3.2) We can consider locally closed subspaces \( Y \) of \( X \in \mathcal{V}_k \) as schemes and therefore as elements of \( \mathcal{V}_k \) (cf. [16, Proposition 4.6.1., p. 273]). Hence the locally closed subsets have a natural image on the Grothendieck ring \( K_0(\mathcal{V}_k) \). We would like also to be able to speak about classes of constructible subsets (i.e., finite disjoint unions of locally closed subspaces) of elements of \( \mathcal{V}_k \). Then we recall the following result (see [12, Introduction]):
(3.3) Proposition: If $Y$ is a scheme of finite type over $k$, then the map $Y' \to [Y']$ from the set of closed subschemes of $Y$ to $K_0(U_k)$ extends uniquely to a map $Z \to [Z]$ from the set of constructible subsets of $Y$ to $K_0(U_k)$ satisfying $[Z \cup Z'] = [Z] + [Z'] - [Z \cap Z']$ for $Z, Z' \in U_k$.

Therefore, every constructible subset $Z$ of an element of $U_k$ has a well-defined image $[Z] \in K_0(U_k)$. Note the importance of constructible subsets of a scheme $X$: They are the smallest algebra of sets containing the closed sets for the Zariski topology. Moreover, the Grothendieck ring behaves also well with respect to trivial fibrations (see [1, p. 6]):

(3.4) Lemma: Let $f : Y \to X$ be a piecewise trivial fibration with constant fibre $Z$. This means that one can write $X = \bigsqcup X_i$ as a finite disjoint union of locally closed subsets $X_i$ such that over each $X_i$ one has $f^{-1}X_i \cong X_i \times Z$ and $f$ is given by the projection onto $X_i$. Then in $K_0(U_k)$, we have $[Y] = [X] \cdot [Z]$.

(3.5) The morphism $\phi : A^1_k \setminus \{0\} \times A^1_k \setminus \{0\} \to A^1_k \setminus \{0\}$ so that $\phi(a, b) = a \cdot b$ for all $a, b \in A^1_k \setminus \{0\}$ defines a group scheme which is called the multiplicative group and is denoted by $G_m$ (see [13, p. 324]). We define the class $[k^*]$ of the group of units $k^*$ of $k$ as the class $[G_m] \in K_0(U_k)$. Since $G_m \cong \text{Spec}(k[\frac{1}{x}, x])$, we get $[G_m] = [\text{Spec}(k[\frac{1}{x}, x])]$, and therefore $[k^*] = L - 1$.

(3.6) Let $C$ be a curve with equation $f$ and $O := R/fR$. Let $p$ be a non-negative integer and let $J^0_p$ be the space of $p$-jets over $O$, which is a $k_R$-vector space of finite dimension $d(p)$. Let us consider its projectivisation $P J^0_p$ and let us adjoin one point to this (that is, $P J^0_p = P J^0_p \cup \{\ast\}$ with $\ast$ representing the added point) in order to have a well-defined map $p_p : P O \to P^* J^0_p$. A subset $X \subset P O$ is said to be cylindric if there exists a constructible subset $Y \subset P J^0_p$ such that $X = \pi^{-1}_p(Y)$.

The generalised Euler characteristic $\chi_g(X)$ of a cylindric subset $X$ is the element $[Y] \cdot L^{-d(p)}$ in the ring $M_{kR}$, where $Y = \pi^{-1}_p(X)$ is a constructible subset of $P O$. Note that $\chi_g(X)$ is well-defined, because if $X = \pi^{-1}_q(Y')$, $Y' \subset P J^0_q$ and $p \geq q$, then $Y$ is a locally trivial fibration over $Y'$ and therefore $[Y] = [Y'] \cdot L^{d(p) - d(q)}$.

Let $\psi : P O \to G$ be a function with values in an Abelian group $G$ with countably many values. It is said cylindric if, for each $a \in G \setminus \{0\}$, the set $\psi^{-1}(a) \subset P O$ is cylindric. As it is defined in [3], the integral of a cylindric function $\psi$ over $P O$ with respect to the generalised Euler characteristic is

$$\int_{P O} \psi d\chi_g := \sum_{a \in G \setminus \{0\}} \chi_g(\psi^{-1}(a)) \cdot a,$$

if this sum makes sense in $M_{kR} \otimes_G^Z G$; in such a case, the function $\psi$ is said to be integrable.
Definition: The generalised Poincaré series of the multi-index filtration given by the ideals \( J(n) \) (see (2.2)) is the integral

\[
P_g(t_1, \ldots, t_r; L) := \int_{\mathbb{P}R} t_v(h) d\chi \in \mathcal{M}_{kR}
\]

where \( t_v(h) := t_1^{v_1(h)} \cdots t_r^{v_r(h)} \) is considered as a function on \( \mathbb{P}R \) with values in \( \mathbb{Z} \). Notice that the sub-index \( g \) in \( P_g(t) \) is just notation: It refers to integration with respect to the generalised Euler characteristic.

4 Generalised Poincaré series in terms of an embedded resolution

From now on, we assume that the residue field \( k_R \) is isomorphic to a perfect field \( K \) contained in \( R \), that is, we assume the existence of a perfect coefficient field \( K \) (it will be only needed to define the integral over \( \mathbb{P}R \)).

Let \( C \) be a curve with equation \( f \) satisfying the same hypothesis as in (2.2). Since the curve is analytically reduced, the total transform of \( C \) is a normal crossing curve at some point of a sequence of quadratic transforms ([15, Chap. VII, (8.13), p. 301]). Consequently, we get a finite sequence of blowing-ups

\[
X = X_s \xrightarrow{\pi_s} X_{s-1} \xrightarrow{\pi_{s-1}} \ldots \xrightarrow{\pi_3} X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \text{Spec}(R)
\]

with \( \pi = \pi_s \circ \pi_{s-1} \circ \ldots \circ \pi_2 \circ \pi_1 \), as in (2.5). Let \( E \) be the exceptional divisor of \( \pi \). For each \( 1 \leq i \leq s \), let \( \overset{\circ}{E}_i \) be the component \( E_i \) of the exceptional divisor of \( \pi \) minus the intersection points with all other components of the total transform of the curve. In the same way, we consider the set \( \overset{\bullet}{E}_i \), which is just the component \( E_i \) of the exceptional divisor minus the intersection points with other components \( E_j \) with \( j \neq i \). Note that both \( \overset{\circ}{E}_i \) and \( \overset{\bullet}{E}_i \) are quasi-projective schemes.

The goal in writing this section is to describe the integral defining the generalised Poincaré series (cf. Definition (3.7)) in terms of an embedded resolution of curves. First of all, we will distinguish three different types of points on the exceptional divisor \( E \): The intersection points among components of \( E \), the intersection points between components of \( E \) and the components of the strict transform of the curve, and the points in \( \overset{\circ}{E}_i \). By considering separately the set of the previous first two kind of points, as well as the symmetric product of the smooth components \( \overset{\circ}{E}_i \), we will define in Subsect. 4.1 a semigroup \( Y \) and a surjection

\[
\text{Init} : \mathbb{P}R^* \to Y
\]

from the projectivisation of \( R \setminus \{0\} \) to \( Y \) which will be in fact a semigroup homomorphism. The preimage of each point of \( Y \) under the map \( \text{Init} \) will give us an affine space whose codimension \( F(n) \) in \( \mathbb{P}R^* \) will be determined in Subsect. 4.2 in order to show finally an explicit formula for the generalised Poincaré series in terms of the embedded resolution in Subsect. 4.3.
4.1 Definition of the map \textit{Init.}

\textbf{(4.1) Notation:} Let be the set $R^* := \{ z \in R \mid v_i(z) < \infty, \, 1 \leq i \leq r \}$. For $g \in R^*$, let $\Gamma_g$ be the strict transform of the curve given by $g$. Since $E_i$ and $\Gamma_g$ have no common components, the set $E_i \cap \Gamma_g$ is finite for all $i \in \{1, \ldots, s\}$. Let $I_0 := \{ \sigma := (i_1, i_2) \in \{1, 2, \ldots, s\} \times \{1, 2, \ldots, s\} \mid i_1 < i_2, \, E_i \cap E_{i_2} \neq \emptyset \}$. For every $\sigma = (i_1, i_2) \in I_0$, let $P_{\sigma} := E_i \cap E_{i_2}$ and $a_{i_1} := i_1, i_2 \sigma := i_2$. Let $J_0 := \{1, 2, \ldots, r\}$. For $j \in J_0$, let $P_j := E_{i_1(j)} \cap \tilde{C_j}$, where $\tilde{C_j}$ is the strict transform of $\pi$ of the component $C_j$ of the curve $C$, and $E_{i_1(j)}$ is the component of the exceptional divisor of $\pi$ which intersects $\tilde{C_j}$.

We can distinguish three types of points belonging to each irreducible component $E_i$ of the exceptional divisor of $\pi$, namely: points of type $P_{\sigma}$ for some $\sigma \in I_0$, points of type $P_j$ for some $j \in J_0$ and smooth points of the exceptional divisor.

\textbf{(4.2) Points of type $P_{\sigma}$.} They are defined to be the closed intersection points $E_i \cap E_{i_2}$ when $\sigma = (i_1, i_2)$, for $i_1, i_2 \in \{1, 2, \ldots, s\}$ and $i_1 < i_2$. Take such a point $P_{\sigma}$, for $\sigma \in I_0$. Choose local coordinates $x_\sigma, y_\sigma$ at $P_{\sigma}$ so that the components of the total transform of the curve are the coordinate lines: $E_i(\sigma) := \{ y_\sigma = 0 \}, E_{i_2}(\sigma) := \{ x_\sigma = 0 \}$. We will denote the local ring at $P_{\sigma}$ by $R_{\sigma}$, and its residue field by $k_{\sigma}$.

The quotient ring $R_{\sigma}/x_\sigma R_{\sigma}$ (resp. $R_{\sigma}/y_\sigma R_{\sigma}$) is a discrete valuation ring with associated valuation $\omega_{x_\sigma}$ (resp. $\omega_{y_\sigma}$). We consider the canonical maps

$$
\varphi_{x_\sigma} : R_{\sigma} \rightarrow R_{\sigma}/x_\sigma R_{\sigma}
$$

$$
\varphi_{y_\sigma} : R_{\sigma} \rightarrow R_{\sigma}/y_\sigma R_{\sigma}
$$

Let $g \in R^*$, and denote by $\gamma_g$ the equation at $P_{\sigma}$ of the strict transform $\Gamma_g$ of $g$ on $X$ (which must be different from 0); for $\sigma \in I(g) := \{ \sigma \in I_0 \mid P_{\sigma} \in \Gamma_g \}$, we define

$$
n'_{\sigma}(g) := \omega_{x_\sigma}(\gamma_g \mod x_\sigma R_{\sigma})
$$

$$
n''_{\sigma}(g) := \omega_{y_\sigma}(\gamma_g \mod y_\sigma R_{\sigma}).
$$

Note that $n'_{\sigma}(g) \neq \infty$ and $n''_{\sigma}(g) \neq \infty$, since $\gamma_g \notin (x_\sigma)$ and $\gamma_g \notin (y_\sigma)$. Note also that $\varphi_{x_\sigma}(y_\sigma)$ (resp. $\varphi_{y_\sigma}(x_\sigma)$) is a uniformising parameter of the discrete valuation ring $R_{\sigma}/x_\sigma R_{\sigma}$ (resp. $R_{\sigma}/y_\sigma R_{\sigma}$). Therefore we have

$$
\varphi_{x_\sigma}(\gamma_g) = \alpha_{\sigma} \cdot \varphi_{x_\sigma}(y_\sigma)^{n'_{\sigma}(g)}
$$

$$
\varphi_{y_\sigma}(\gamma_g) = \beta_{\sigma} \cdot \varphi_{y_\sigma}(x_\sigma)^{n''_{\sigma}(g)},
$$

where $\alpha_{\sigma}$ is a unit of $R_{\sigma}/x_\sigma R_{\sigma}$ and $\beta_{\sigma}$ is a unit of $R_{\sigma}/y_\sigma R_{\sigma}$. We can also consider the image $a_{\sigma}$ of $\alpha_{\sigma}$ and the image $b_{\sigma}$ of $\beta_{\sigma}$ in the residue field $k_{\sigma}$ of $R_{\sigma}$, which are both different from 0. The elements $\alpha_{\sigma}$, $\beta_{\sigma}$, $a_{\sigma}$ and $b_{\sigma}$ depend only on the choice of the equations $x_\sigma$ and $y_\sigma$ and also of the equation $\gamma_g$ chosen for the strict transform of the ideal $gR$. However, it is easy to check that the ratio $\frac{a_{\sigma}}{b_{\sigma}} \in k^*_\sigma$ is independent of the generator of the principal ideal $\gamma_g S_{\sigma}$. We will denote $\lambda_{\sigma}(g) := \frac{a_{\sigma}}{b_{\sigma}}.$
**4.3 Points of type $P_j$.** They are the points in the intersection between $E_i$ and the strict transform of the $j$–th irreducible component of the curve. Let us take a point $P_j$, for $j \in J_0$ and choose local coordinates $x_j, y_j$ in a neighbourhood of $P_j$ in such a way that the components of the total transform $\pi^{-1}(C)$ of the curve $C$ are $E_{i(j)} = \{ y_j = 0 \}$ and $\bar{C}_j = \{ x_j = 0 \}$, and the set $\{x_j, y_j\}$ is a regular system of parameters of the local ring $R_j := \mathcal{O}_{X, P_j}$. We will use $k_j$ to denote the residue field of $R_j$. Let $g \in R^*$. Analogously as done in (4.2), for each element $j$ in $J(g) := \{ j \in J_0 \mid P_j \in \Gamma_g \}$, we define the natural numbers

$$\overline{n}'_j(g) = \omega_{x_j}(\gamma_j \text{ mod } x_j R_j) \neq \infty$$

$$\overline{n}''_j(g) = \omega_{y_j}(\gamma_j \text{ mod } y_j R_j) \neq \infty,$$

as well as two elements $\alpha_j, \beta_j \in R_j$, whose images $a_j, b_j$ in the residue field $k_j$ of $R_j$ are different from 0 and uniquely determined modulo $x_j$ and $y_j$. We also set $\mu_j(g) := \frac{\alpha_j}{\beta_j} \in k_j^*$. 

**4.4 Notation:** Let $S$ be a quadratic transform of $R$, let $f \in R$. The strict transform of $f$ in $S$ will be denoted by $(fR)^S$. This concept will play an important role in (4.5), Lemma (4.9), as well as in (5.6) and Lemma (5.7). For a recent account of the theory we refer the reader to [15]. See also [20, Sect. 2].

**4.5 Points of type “smooth”.** They are the points belonging to $\bar{E}_i$, for $1 \leq i \leq s$. Since the points on $E_i$, $1 \leq i \leq s$ are closed, if we take $g \in R^*$ and $\Gamma_g$ the strict transform of the curve given by $g$, then we can count the number of intersection points between $\Gamma_g$ and $\bar{E}_i$ with multiplicities as

$$n_i(g) = \sum_{P \in E_i \cap \Gamma_g} ((gR)^{S_P} \cdot (mR)^{S_P}), \text{ for } i \in \{1, \ldots, s\}.$$ 

**4.6** We want to get a space $Y$ and a map between $\mathbb{P}R^*$ and $Y$ so that the preimage of any point of $Y$ is an affine space of finite codimension, and such a map is a locally trivial fibration over each connected component of $Y$. This will allow us to apply certain integration rules to the map in order to connect it with Definition (3.7).

For any (topological) space $\mathcal{X}$, and $m \in \mathbb{N}$, we define the $m$–th symmetric power of $\mathcal{X}$ to be $\mathcal{S}^m \mathcal{X} := \mathcal{X}^m/S_m$, where $S_m$ is the group of permutations of $m$ elements.

**4.7 Notation:** Consider the sets $I_0$ and $J_0$. For $I \subset I_0$ and $J \subset J_0$, let

$$\mathcal{N}_{I,J} := \{ n := (n_i, n'_\sigma, n''_\sigma, \overline{n}_j, \overline{n}'_j) \mid n_i \geq 0, 1 \leq i \leq s; n'_\sigma > 0, n''_\sigma > 0, \sigma \in I; \overline{n}_j > 0, \overline{n}'_j > 0, j \in J \}. $$
For each \( n \in \mathbb{N} \), we define
\[
Y_n := \prod_{i=1}^{s} S^{n_i} \hat{E}_i \times \prod_{\sigma \in I} k_\sigma^* \times \prod_{j \in J} k_j^*,
\]
with \( S^{n_i} \hat{E}_i \) being the \( n_i \)-th symmetric power of \( \hat{E}_i \), \( k_\sigma^* \) the residue field at \( P_\sigma \) minus 0 corresponding to the element \( \sigma \) of \( I \), and \( k_j^* \) the residue field minus zero at \( P_j \) corresponding to the element \( j \) of \( J \). Setting
\[
Y := \bigcup_{I \subseteq I_0} \bigcup_{J \subseteq J_0} Y_n \tag{†}
\]
as in [7, p. 203], it is easily checked that \( Y \) can be endowed with a structure of semigroup.

(4.8) Recall that \( R^* := \{ z \in R \mid v_i(z) < \infty, \ 1 \leq i \leq r \} \). Consider the quotient \( R^* / \sim \), where \( \sim \) denotes the following equivalence relation: For two elements \( a, b \in R^* \), we say that \( a \sim b \) if there exists an element \( u \in k_R \cong K \) such that \( a = ub \). The set \( \mathbb{P}R^* := R^* / \sim \) can be endowed with a structure of semigroup just considering the multiplication of functions. Define a semigroup homomorphism
\[
\text{Init} : \mathbb{P}R^* \to Y = \bigcup_{I \subseteq I_0} \bigcup_{J \subseteq J_0} Y_n
\]
between \( \mathbb{P}R^* \) and \( Y \), in which the image \( \text{Init}(g) \) of any element \( g \in \mathbb{P}R^* \) is an element of the connected component \( Y_n \) of \( Y \) corresponding to \( I(g), J(g) \) and with \( \underline{n} = (n_i(g), n_\sigma'(g), n_\sigma''(g), n_j'(g), n_j''(g)) \), which is defined in every factor (connected component) of \( Y_n \) as follows:

- in \( S^{n_i} \hat{E}_i \), \( 1 \leq i \leq s \), \( \text{Init}(g) \) is represented by the set of intersection points between \( \Gamma_g \) and \( \hat{E}_i \) counted with multiplicities;
- in \( k_\sigma^* \), \( \sigma \in I(g) \), \( \text{Init}(g) \) is represented by the quotient \( \frac{a_\sigma(g)}{b_\sigma(g)} \);
- in \( k_j^* \), \( j \in J(g) \), \( \text{Init}(g) \) is represented by the quotient \( \frac{a_j(g)}{b_j(g)} \).

We take the opportunity to prove the following key result (it was already omitted for the complex case in [7]).

(4.9) Lemma: The map \( \text{Init} \) is surjective.

Proof. Let us fix \( I \subset I_0 \), \( J \subset J_0 \), \( \underline{n} \in \mathcal{N}_{I,J} \). Consider an element
\[
y_{\underline{n}} = ((\{ P_{\sigma i} \}_{j=1}^{n_i}; \ i = 1, \ldots, s), (\lambda_\sigma; \sigma \in I), (\mu_j; j \in J)) \in Y_{\underline{n}}.
\]
We will construct \( g \in R^* \) such that \( \text{Init}(g) = y_{\underline{n}} \) step by step, in fact it suffices to show the surjectivity in each factor of the space \( Y_{\underline{n}} \).
For a point \( P \in \tilde{E}_i \), Lemma 3.1.11 of [19] (see also [20, Sect. 6]) ensures the existence of an irreducible element \( gp \in R \) such that the strict transform of \( gpR \) in \( X \) is smooth and transversal to the exceptional divisor \( E \) at the point \( P \in \tilde{E}_i \), i.e., \( (gpR)^{S_Q} = S_Q \) for all \( Q \neq P \) (\( S_Q \) denotes the local ring of \( X \) at the point \( Q \in E \)) and \( (gpR)^{S_P} = \gamma_P S_P \) so that \( \{ x_P, \gamma_P \} \) is a parameter system of \( S_P \) (here \( x_P \) is a local equation for \( E_i \) at \( P \); i.e., \( x_P S_P = (m_R R)^{S_P} \)). If we take

\[
\left\{(P_{ij})^{n_{ij}} \mid i = 1, \ldots, s\right\} \in \prod_{i=1}^{s} S^{n_{ij}} \tilde{E}_i,
\]

then it is easily seen that \( g^{(1)} = \prod_{i,j} g_{ij} \in R \) defines an element of \( R \) such that \( I(g^{(1)}) = \langle J(g^{(1)}) \rangle = \emptyset \), and moreover \( n_i(g^{(1)}) = n_i \) for \( 1 \leq i \leq s \).

Let be now \( \sigma \in I, n'_\sigma, n''_\sigma > 0 \) and \( \lambda_\sigma \in k^*_\sigma \). Let us consider the discrete valuation \( \omega_{x_\sigma} \) (resp. \( \omega_{y_\sigma} \)) associated with the ring \( R_\sigma / x_\sigma R_\sigma \) (resp. \( R_\sigma / y_\sigma R_\sigma \)) (Recall that \( \{ x_\sigma, y_\sigma \} \) are the equations of the components of the exceptional divisor at the point \( P_\sigma = E_{i_1(\sigma)} \cap E_{i_2(\sigma)} \)). Let \( u_\sigma \in R_\sigma \) be a unit such that \( u_\sigma + m_\sigma = \lambda_\sigma \in k^*_\sigma \) and let us take \( \gamma_1 = x_\sigma + u_\sigma, \gamma_2 = y_\sigma + x_\sigma = 1 \in R_\sigma \). Since \( \{ y_\sigma, \gamma_1 \} \) is a regular system of parameters, we have that \( \gamma_1 R_\sigma \cap R \) is a prime ideal of height 1, therefore it is principal and generated by \( g_1^\sigma \in R \) whose strict transform in \( R_\sigma \) is just \( \gamma_1 R_\sigma \) (see [20, Prop. 6.2]). Identically we can argue with \( \gamma_2^\sigma \) to get an irreducible element \( g_2^\sigma \in R \) with \( (g_2^\sigma R)^{R_\sigma} = \gamma_2^\sigma R_\sigma \). Let us denote \( g^\sigma = g_1^\sigma \cdot g_2^\sigma \) and \( \gamma^\sigma = \gamma_1 \cdot \gamma_2^\sigma \). Notice that \( \gamma^\sigma \) is the strict transform of \( g^\sigma \). Then we have

\[
\varphi_{x_\sigma}(\gamma^\sigma) = \varphi_{x_\sigma}(u_\sigma) \cdot \varphi_{x_\sigma}(y_\sigma)^{n''_\sigma}
\]

\[
\varphi_{y_\sigma}(\gamma^\sigma) = \varphi_{y_\sigma}(x_\sigma)^{n''_\sigma},
\]

and since \( \varphi_{x_\sigma}(u_\sigma) \) is a unit, we have \( n'_\sigma(g^\sigma) = n'_\sigma \) and \( n''_\sigma(g^\sigma) = n''_\sigma \). Moreover, \( a_\sigma = \lambda_\sigma, b_\sigma = 1 \) and therefore \( \frac{a_\sigma}{b_\sigma} = \lambda_\sigma \in k^*_\sigma \).

Thus let \( g^{(2)} = \prod_{\sigma \in I} g^\sigma \in R \). From the previous construction, it follows that \( n_i(g^{(2)}) = 0 \) for all \( i = 1, \ldots, s \); \( J(g^{(2)}) = \emptyset \); \( I(g^{(2)}) = I \) and moreover, for every \( \sigma \in I \), we have \( n'_\sigma(g^{(2)}) = n'_\sigma, n''_\sigma(g^{(2)}) = n''_\sigma \) and \( \lambda_\sigma(g^{(2)}) = \lambda_\sigma \in k^*_\sigma \).

For every \( j \in J \) (and consequently for \( J \)) we can proceed analogously to the previous case in order to get an element \( g^{(3)} \in R \) such that \( n_i(g^{(3)}) = 0 \) for all \( i = 1, \ldots, s \); \( I(g^{(3)}) = \emptyset \); \( J(g^{(3)}) = J \) and \( \mu_j(g^{(3)}) = \mu_j \) and \( \mu_j(g^{(3)}) = \mu_j \in k^*_j \).
As a consequence, we obtain an element \( g = g^{(1)} \cdot g^{(2)} \cdot g^{(3)} \) satisfying
\[
\text{Init}(g) = \left( \{ (P_{i,j})_{j=1}^n \colon i = 1, \ldots, s \}, (\lambda_s; \sigma \in I), (\mu_j; j \in J) \right) \in \mathbb{P}.
\]

\((4.10)\) **Notation:** Take into account that \( X \) is an irreducible integral scheme, and \( E_i \) is an irreducible closed subset of \( X \) for every \( 1 \leq i \leq s \). Consider \( (E_i, \mathcal{O}_{E_i}) \subset (X, \mathcal{O}_X) \) with reduced structural sheaf. Let \( \eta_i \) be the generic point of \( E_i \). We consider the local ring \( \mathcal{O}_{X,E_i} \) of \( X \) along \( E_i \), which is nothing but the ring \( \mathcal{O}_{X,E_i} \). Then there exists an affine open subset \( U \) in \( X \) with \( \eta_i \in U \). Let be the ring \( A := \Gamma(U, \mathcal{O}_X) \). Then the generic point \( \eta_i \) corresponds to an ideal \( p_i \in \text{Spec}(A) \), and therefore \( \mathcal{O}_{X,E_i} = A_{p_i} \). It is easy to check that \( A_{p_i} \) is a discrete valuation ring, with associated discrete valuation \( w_i \), for \( 1 \leq i \leq s \).

\((4.11)\) **Proposition:** For \( g, g' \in R^* \), then \( \text{Init}(g) = \text{Init}(g') \) if and only if \( g = \zeta g' + h \), where \( \zeta \in k_R \setminus \{0\} \), \( w_i(g) = w_i(g') \) for all \( 1 \leq i \leq s \) and \( v_j(h) > v_j(g') \) for all \( j \in \{1, \ldots, r\} \).

**Proof.** Let us take a point \( q \in E_i \cap U \neq \emptyset \). The point \( q \) corresponds to a prime ideal \( q \in \text{Spec}(A) \) with \( q \supset p_i \). Therefore we have \( \mathcal{O}_{X,q} = A_q \) and \( \mathcal{O}_{X,E_i} = A_{p_i} = (A_q)_{p_i,A_q} \). Note that the field of rational functions of \( X \) (resp. of \( E_i \)) is \( K(X) := \text{Quot}(\mathcal{O}_{X,E_i}) = \text{Quot}(R) \) (resp. \( K(E_i) := A_{p_i}/p_iA_{p_i} \)).

We consider \( g, g' \in R \), the morphism \( \rho : X \to U \) and the induced ring homomorphism \( \theta_i : \Gamma(U, \mathcal{O}_X) \to \Gamma(U \cap E_i, \mathcal{O}_{E_i}) \), for every \( 1 \leq i \leq s \). Then \( \rho(g) \in \Gamma(U, \mathcal{O}_X) \) and \( \theta_i(\rho(g)) \in \Gamma(U \cap E_i, \mathcal{O}_{E_i}) \). Denote by \( \{ \theta_i(\rho(g)) \} \) the class of \( \theta_i(\rho(g)) \) as a rational function on \( E_i \). Let \( i_i : E_i \to X \) be the inclusion map. Define \( \bar{g} := g \circ \pi \) and \( \bar{g}' := g' \circ \pi \) in \( \Gamma(X, \mathcal{O}_X) \). Then \( \bar{g} \circ i_i = [\theta_i(\rho(g))] \), \( \bar{g}' \circ i_i = [\theta_i(\rho(g'))] \) and we can associate with \( \bar{g} \circ i_i \) (resp. \( \bar{g}' \circ i_i \)) a divisor
\[
\text{div}(\bar{g} \circ i_i) = \sum_{P \in E_i} (\Gamma_g \cdot E_i)_P \cdot P
\]
(resp. \( \text{div}(\bar{g}' \circ i_i) = \sum_{P \in E_i} (\Gamma_{g'} \cdot E_i)_P \cdot P \)), where \( \Gamma_g \) and \( \Gamma_{g'} \) are the strict transforms of \( g \) and \( g' \) by \( \pi \), respectively.

Set \( \psi := \frac{\bar{g}'}{\bar{g}} \in \mathcal{O}_{X,E_i} \), for every \( 1 \leq i \leq s \). The condition \( \text{Init}(g) = \text{Init}(g') \) means, in particular, that \( \text{div}(\bar{g} \circ i_i) = \text{div}(\bar{g}' \circ i_i) \), and then the restriction of \( \psi \) to \( E_i \), i.e., the quotient \( \frac{[\theta_i(\rho(g))]}{[\theta_i(\rho(g'))]} \), is a regular function on \( E_i \). Similarly, since the irreducible components \( C_j \) of the strict transform of the curve \( C \) are smooth, we can also show that the restriction of \( \psi \) to each \( C_j \) is a regular function on \( C_j \), for every \( 1 \leq j \leq r \).

Moreover, regular functions defined over projective lines are constant, so the function \( \psi \) is constant on each component \( E_i \) of the exceptional divisor. In fact, as \( E_i = \text{Proj}(k_i, \mathcal{O}) \) for some finite field extension \( k_i \supseteq k_R \) (according to Sect. 2), we have
\[
\frac{[\theta_i(\rho(g))]}{[\theta_i(\rho(g'))]} = \zeta_i \in k_i \setminus \{0\}, 1 \leq i \leq s.
\]
Thus the value of \( \lambda \) is such that

\[ \text{there exist } \gamma, \zeta \text{ that } \phi \| \gamma \| \zeta \text{ is just } \frac{\varphi_{x\sigma} (\lambda)}{\varphi_{y\sigma} (\mu)} = \frac{\varphi_{x\sigma} (\lambda)}{\varphi_{y\sigma} (\mu)} = \frac{\varphi_{y\sigma} (\mu_2)}{\varphi_{x\sigma} (\lambda_1)} \iff \zeta_{11} = \zeta_{12}. \]

Furthermore, taking into account that \( E_1 \cup \ldots \cup E_s \) is a connected set, we obtain that \( \psi \) is a regular function on the total transform \( \pi^{-1}(C) \), and it is equal to \( \zeta \neq 0 \) on the exceptional divisor, which must be an element belonging to \( k_R \), because there is (at least) one \( E_i \) which is a scheme \( \text{Proj}(kR[\tau, \eta]) \).

Now we can take the function \( h := g - \zeta g' \). For each \( 1 \leq i \leq s \), we consider the canonical homomorphism

\[ \phi_i : \mathcal{A}_{p_i} \rightarrow \mathcal{A}_{p_i} / p_i \mathcal{A}_{p_i} = K(E_i) = k_i \]

The quotient \( g \) is \( \mathcal{O}_{X, n_i} = \mathcal{O}_{X, E_i} = \mathcal{A}_{p_i} \), and therefore \( \phi_i \left( \frac{g}{g'} \right) = \zeta \neq 0 \), then

\[ \frac{g}{g'} - \zeta \in \ker (\phi_i) = p_i \mathcal{A}_{p_i} \text{ and } 0 < w_i \left( \frac{g}{g'} - \zeta \right) = w_i \left( \frac{g - \zeta g'}{g'} \right) = w_i(g - \zeta g') - w_i(g'). \]
Hence \( w_i(h) > w_i(g) = w_i(g') \) for all \( 1 \leq i \leq s \). We can repeat the same argument for the valuations \( v_j, 1 \leq j \leq r \); the intersection \( C_j \cap \text{Spec}(A) \) is not empty, and so there exists an ideal \( q_j \in \text{Spec}(A) \) which is the generic point of \( C_j \) and \( \mathcal{O}_{X,C_j} = A_{q_j} \), which is a discrete valuation ring with associated discrete valuation \( v_j \), for \( 1 \leq j \leq r \). Then it is enough to take the homomorphism

\[
\phi_j : A_{q_j} \longrightarrow A_{q_j}/m(A_{q_j}),
\]

where \( m(A_{q_j}) \) is the maximal ideal of the local ring \( A_{q_j} \).

Conversely, if \( g = \zeta g' + h \) with \( \zeta \in k_R \setminus \{0\} \), \( w_i(g - \zeta g') > w_i(g) = w_i(g') \) for \( 1 \leq i \leq s \) and \( v_j(g - \zeta g') > v_j(g) = v_j(g') \) for all \( 1 \leq j \leq r \), then the function \( \psi \) is regular on every component \( E_i \) of \( E \) for \( 1 \leq i \leq s \) and on every component \( \tilde{C}_j \) of the strict transform of the curve, and therefore \( \psi|_{E} \equiv \zeta \).

Hence \( \text{div}(g' \circ \iota_i) = \text{div}(g \circ \iota_i) \) for \( 1 \leq i \leq s \) and the intersection points of the strict transforms \( \Gamma_{g'} \) and \( \tilde{C}_j \) with each component \( E_i \) of \( E \) coincide (counting multiplicities). For \( 1 \leq j \leq r \), we also have

\[
\left( \Gamma_{g'} \cdot \tilde{C}_j \right)_{P_j} = v_j(g) - w_{i,(j)}(g) = v_j(g') - w_{i,(j)}(g') = \left( \Gamma_g \cdot \tilde{C}_j \right)_{P_j}.
\]

Moreover, for points of type \( P_\sigma \) (the same holds for points of type \( P_\lambda \)) and following previous notations, the value of \( \frac{[q_{\sigma,(g)}]}{[q_{\sigma,(g')}] \mid P_\sigma} \) at \( P_\sigma \) is \( \zeta = \frac{\phi_\sigma(\lambda)}{\phi_\sigma(\lambda')} = \frac{\gamma_\sigma}{\delta_\sigma}, \)
but it is also \( \zeta = \frac{b_\sigma}{c_\sigma}, \) then \( \frac{\gamma_\sigma}{\delta_\sigma} = \frac{b_\sigma}{c_\sigma}. \)

\[\text{(4.12) Corollary: For any point } \zeta \in Y_n \subset Y, \text{ the preimage } \text{Init}^{-1}(\zeta) \text{ consists of an affine space given by a point } g \text{ plus the ideal } I_\zeta \text{ where} \]

\[I_\zeta = \{z \in R^* \mid w_i(z) \geq w_i(\underline{w}), 1 \leq i \leq s; v_j(z) \geq v_j(\underline{w}), 1 \leq j \leq r \}.\]

This space has finite codimension \( F(n) \) in \( \mathbb{PR}^* \). Furthermore, over each connected component \( Y_n \) of \( Y \), the map \( \text{Init} \) is a locally trivial fibration.

\[\text{4.2 Computation of the codimension } F(n).\]

\[\text{(4.13) Let } g \in R^* \text{ and } w_i \text{ the divisorial valuation corresponding to the component } E_i \text{ of } \pi, \text{ for all } 1 \leq i \leq s. \text{ Set } \underline{w}(g) := (w_1(g), \ldots, w_s(g)). \text{ We know that the value } w_i(g) \text{ is the multiplicity of the lifting } g \circ \pi \text{ of } g \text{ along } E_i, \text{ for all } 1 \leq i \leq s. \text{ Let } D(g) \text{ be the total transform by } \pi \text{ of the curve defined by the ideal } gR; \text{ i.e. } D(g) \text{ is the divisor in } X \text{ given by} \]

\[D(g) = \sum_{i=1}^{s} w_i(g)E_i + \Gamma_g,\]

where \( \Gamma_g \) is the strict transform of the curve defined by \( gR \). We have now

\[0 = \deg_X(D(g) \cdot E_i) = \deg_X(\Gamma_g \cdot E_i) + \sum_{j=1}^{s} w_j(g) \deg_X(E_i \cdot E_j)\]
and so $\hat{n}_i(g) := \deg_X(\Gamma_g \cdot E_i) = -\sum_{j=1}^s w_j(g)n_{ij}$, being $N = (n_{ij})$ the intersection matrix (see (2.7)). Thus we have that

$$w(g) := (w_1(g), \ldots, w_s(g)) = (\hat{n}_1(g), \ldots, \hat{n}_s(g)) \cdot M,$$

where $M := -N^{-1}$. The above formula provides the precise relation between the values $w(g) := (w_1(g), \ldots, w_s(g))$ and $\hat{w}(g) := (\hat{n}_1(g), \ldots, \hat{n}_s(g))$.

Let us now fix $\hat{n} = (\hat{n}_1, \ldots, \hat{n}_s) \in \mathbb{Z}_{\geq 0}$ and the corresponding divisor $A = \sum_{j=1}^s w_iE_i$, where $w = (w_1, \ldots, w_s) = \hat{n} \cdot M$. Let $J^D(w)$ be the divisorsial ideal of $R$ defined by

$$J^D(w) = \{ z \in R^* \mid w_i(z) \geq w_i, \ 1 \leq i \leq s \}$$

and $h^D(w) = \dim_{k_R} R/J^D(w)$.

The next proposition is known as the Hoskin–Deligne formula:

\begin{equation}
\textbf{(4.14) Proposition:} \text{ Let } \mathbb{K} = \sum_{i=1}^s E_i^* \text{ be the canonical divisor on } X \text{ and consider } A = \sum_{i=1}^s \alpha_iE_i^* \text{ be the expression of the divisor } A \text{ on the basis } \{E_i^*\}. \text{ Then}
\end{equation}

$$h^D(w) = \frac{1}{2} \sum_{i=1}^s h_i\alpha_i(\alpha_i + 1).$$

\textbf{Proof.} The genus formula gives us that

$$h^0(O_A) = h^D(w) = -\frac{\deg_X(A \cdot A) + \deg_X(A \cdot \mathbb{K})}{2}.$$

Now, by Equation (1) in (2.6) we deduce

$$\deg_X(A \cdot A) = -\sum_{i=1}^s \alpha_i^2h_i$$

$$\deg_X(A \cdot \mathbb{K}) = -\sum_{i=1}^s \alpha_ih_i.$$

Then the codimension of the ideal $J^D(w)$ is equal to $\frac{1}{2} \sum_{i=1}^s h_i\alpha_i(\alpha_i + 1)$. □

The point is now to compute the intersection degrees $\deg_X(A \cdot A)$ and $\deg_X(A \cdot \mathbb{K})$ in terms of the matrix $M$ and $\hat{n}_1, \ldots, \hat{n}_s$.

Let $\nu_i$ (resp. $\bar{\nu}_i$) be the number of components $E_j$ of $E$ intersecting $E_i$ for $j \neq i$ (resp. this number plus the number of strict transforms of the curve $C$ intersecting $E_i$), but counting this number of components as many times as the degree of the extension $h_i = [k_i : k_R]$ says. Notice also that $\bar{\nu}_i = \deg_X(E_i \cdot (\sum_{j \neq i} E_j)) = \sum_{j \neq i} n_{ij}$.

\begin{equation}
\textbf{(4.15) Lemma:} \text{ For } i = 1, \ldots, s, \text{ let } \varepsilon_i := 2h_i - \bar{\nu}_i. \text{ Then we have}
\end{equation}

$$\deg_X(A \cdot \mathbb{K}) = \hat{n} \cdot 1^t - \hat{n} \cdot M \cdot \varepsilon^t.$$
Proof. For every $1 \leq i \leq s$, we claim that

$$\deg_X(E_i \cdot E_i) = -2h_i - \deg_X(K \cdot E_i). \quad (\ast)$$

In fact, let $P = (p_{ij})_{1 \leq i,j \leq s}$ be the proximity matrix of $\pi$ (cf. (2.7)). From the definitions of intersection matrix (see again (2.7)) and the divisor $K$, we have that $\deg_X(E_i \cdot E_i) = -\sum_{j=1}^s p_{ij}^2 h_i$ and also $\deg_X(K \cdot E_i) = -\sum_{j=1}^s p_{ij} h_i$, therefore $\deg_X(E_i \cdot E_i) + \deg_X(K \cdot E_i) = -\sum_{j=1}^s h_i p_{ij} (p_{ij} + 1)$. For $1 \leq j \leq s$, $p_{ij}(p_{ij} + 1) = 0$ if and only if $p_{ij} = 0$ or $p_{ij} = -1$, which occurs whenever $i \neq j$. Then, for every $1 \leq i \leq s$ we have

$$\deg_X(E_i \cdot E_i) + \deg_X(K \cdot E_i) = -h_i(p_{ii}^2 + p_{ii}) = -h_i(1 + 1) = -2h_i$$

and Equation $(\ast)$ has been shown. From this follows

$$\sum_{j=1}^s p_{ij} h_i = -\deg_X(K \cdot E_i) = 2h_i + \deg_X(E_i \cdot E_i), \quad 1 \leq i \leq s.$$

On the other hand, the definition of the intersection matrix implies

$$N \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = (\deg_X(E_1 \cdot E_1) + \nu_1) \vdots (\deg_X(E_s \cdot E_s) + \nu_s)$$

If we write $\mathbf{1} := (1,1,\ldots,1)$ and $\mathbf{\varepsilon} := (\varepsilon_1,\ldots,\varepsilon_s)$, with $\varepsilon_i := 2h_i - \nu_i$, for $1 \leq i \leq s$, then we obtain

$$N \cdot \mathbf{1}^t + \mathbf{\varepsilon}^t = P \cdot \Delta \cdot \mathbf{1}^t,$$

where $\Delta$ is the diagonal matrix defined in (2.7). Therefore

$$-\deg_X(A \cdot K) = (\alpha_1,\ldots,\alpha_s) \cdot \Delta \cdot (1,\ldots,1)^t$$

$$= \nu \cdot P \cdot \Delta \cdot (1,\ldots,1)^t$$

$$= \widehat{\nu} \cdot M \cdot P \cdot \Delta \cdot \mathbf{1}^t$$

$$= \widehat{\nu} \cdot M \cdot (N \cdot \mathbf{1}^t + \mathbf{\varepsilon}^t)$$

$$= -\widehat{\nu} \cdot \mathbf{1}^t + \widehat{\nu} \cdot M \cdot \mathbf{\varepsilon}^t.$$

\(\text{(4.16) Remark:}\) Let $\beta_i$ be the number of components in $\{E_j : j \neq i\}$ such that $E_i \cap E_j \neq \emptyset$. Then $\nu_i = h_i \beta_i$, and so $\varepsilon_i = h_i(2 - \beta_i)$. In the complex case $2 - \beta_i$ is the Euler characteristic of the space $\pi E_i = E_i \setminus \bigcup_{j \neq i} E_j$; thus in some sense $\varepsilon_i$ may be interpreted as the “Euler characteristic” of $\pi E_i$.

\(\text{(4.17) Lemma:}\) We have

$$-\deg_X(A \cdot A) = \widehat{\nu} \cdot M \cdot \widehat{\nu}^t.$$
Proof. Since $M = -N^{-1}$, $\mathbf{w} = \mathbf{n} \cdot M$, and $M$ is a symmetric matrix, we obtain

$$-\deg_X(A \cdot A) = (\alpha_1, \ldots, \alpha_s) \cdot \Delta \cdot (\alpha_1, \ldots, \alpha_s)^t$$

$$= (\mathbf{w} \cdot P) \cdot \Delta \cdot (\mathbf{w} \cdot P)^t$$

$$= -\mathbf{w} \cdot N \cdot \mathbf{w}^t$$

$$= -\mathbf{n} \cdot M \cdot N \cdot (\mathbf{n} \cdot M)^t$$

$$= \mathbf{n} \cdot M^t \cdot \mathbf{n}^t$$

$$= \mathbf{n} \cdot M \cdot \mathbf{n}^t,$$

as desired.

(4.18) Corollary: The codimension of the ideal $J^D(\mathbf{w})$ is

$$h^D(\mathbf{w}) = \frac{1}{2} \left( \sum_{i=i'}^{s} m_{i'\sigma} \hat{n}_i \hat{n}_{i'} + \sum_{i=1}^{s} \hat{n}_i \cdot \left( \sum_{i'=1}^{s} m_{i'\sigma} (2h_{i'} - \nu_{i'}^{\bullet}) - 1 \right) \right).$$

(4.19) Notation: Let $I \subset I_0$, $J \subset J_0$, $\mathbf{n} \in \mathcal{N}_{I,J}$ and $\mathbf{y} \in Y_{\mathbf{n}}$. Recall that $\text{Init}^{-1}(\mathbf{y})$ is an affine space in $\mathbb{P}R^*$ of finite codimension $F(\mathbf{n})$. Let $\mathbf{u} = (n_i, n'_\sigma, n''_\sigma, \tilde{n}'_j, \tilde{n}''_j) ; 1 \leq i \leq s, \sigma \in I, j \in J$. By considering $\mathbf{n} \in \mathbb{Z}_{\geq 0}$ the element with entries

$$\hat{n}_i := n_i + \sum_{\sigma \in I, i_1(\sigma) = i} n'_\sigma + \sum_{\sigma \in I, i_2(\sigma) = i} n''_\sigma + \sum_{j \in J, i_1(j) = i} \tilde{n}'_j,$$

we define

$$w(\mathbf{u}) := \hat{n} \cdot M$$

$$v_j(\mathbf{u}) := w_{i_1(j)}(\mathbf{u}) + \tilde{n}''_j \cdot h_{i_1(j)}$$

for $\mathbf{u} \in \mathcal{N}_{I,J}$ and $1 \leq j \leq r$. Note that $v_j(\mathbf{u})$ is nothing but the order of a function $g$ such that $\text{Init}(g) \in Y_{\mathbf{n}}$ on the component $C_j$ of the curve, $w_i(\mathbf{u})$ is the multiplicity along the component $E_i$ of the exceptional divisor of the lifting of a function $g$ such that $\text{Init}(g) \in Y_{\mathbf{n}}$, and $\tilde{n}'_j = \tilde{n}'_j(g)$.

(4.20) Proposition: We have

$$F(\mathbf{n}) = \frac{1}{2} \left( \sum_{i,i'=1}^{s} m_{i'i} \tilde{n}_i \tilde{n}_{i'} + \sum_{i=1}^{s} \tilde{n}_i \cdot \left( \sum_{i'=1}^{s} m_{i'i} (2h_{i'} - \nu_{i'}^{\bullet}) + (2h_{i'} - 1) \right) \right)$$

$$+ \sum_{j \in J} \tilde{n}''_j \cdot h_j.$$
Proof. The codimension \( F(n) \) is equal to the codimension in \( R \) of the ideal
\[
I_n = \{ z \in R^s \mid w_i(z) > w_i(n), \ 1 \leq i \leq s; \ v_j(z) > v_j(n), \ 1 \leq j \leq r \}
\]
minus 1 by Corollary (4.12). Making additional blow–ups at the intersection points of the strict transforms of the curve \( C \) with the exceptional divisor reduces our problem to the case \( J = \emptyset \). Then we have to compute the codimension \( h^D(w(n) + 1) \) – 1, with
\[
h^D(w) := \dim k_R R/J^D(w)
\]
for \( w = (w_1, \ldots, w_s) \in \mathbb{Z}^s \) and \( J^D(w) := \{ z \in R^s \mid w_i(z) \geq w_i, \ 1 \leq i \leq s \} \). By Proposition (4.14), Lemma (4.15) and Lemma (4.17), we get
\[
h^D(w(n)) = -\frac{1}{2} (\deg_X (A \cdot A) + \deg_X (A \cdot I))
\]
\[
= \frac{1}{2} \left( \hat{a} \cdot M \cdot \hat{a} - \hat{a} \cdot n + \hat{a} \cdot M \cdot n \right).
\]
The above formula cannot be applied to compute \( h^D(w + 1) \) because in general \( w + 1 \) is not of the form \( \hat{a} \cdot M \) (i.e. does not belong to the semigroup defined by the divisorial valuations considered). Of course, in such a case the computation follows in the same way. The map \( \text{Init} \) induces a fibration
\[
\mathbb{P} J^D(w(n)) \supset Z \longrightarrow \prod_{i=1}^s S^{\hat{n}_i} E_i = W,
\]
where \( Z = \text{Init}^{-1}(W) \) is an open subset of \( \mathbb{P} J^D(w(n)) \). The fibre is the set of elements in \( Z \) whose image via the map \( \text{Init} \) coincides (i.e., the map \( \text{Init} \) applied to a non–zero element \( g \in \mathbb{P} R \) with \( I(g) = \emptyset \); this set is by Proposition (4.11) equal to \( J^D(w(n) + 1) \), and then
\[
h^D(w(n) + 1) = 1 + h^D(w(n)) + \dim k_R \left( \prod_{i=1}^s S^{\hat{n}_i} E_i \right);
\]
hence
\[
F(n) = h^D(w(n)) + \dim k_R \left( \prod_{i=1}^s S^{\hat{n}_i} E_i \right) = h^D(w(n)) + \sum_{i=1}^s \hat{n}_i h_i
\]
and the formula follows straightforward. \( \blacksquare \)

4.3 Explicit formula.

(4.21) The generalised Euler characteristic satisfies the Fubini rule (see for instance [23, §3, pp. 128–129]). Because of Corollary (4.12) and Lemma (3.4), we can apply Fubini’s formula to the map \( \text{Init} \) in order to get
\[
P_y(t_1, \ldots, t_r; \mathcal{L}) = \int_Y L^{-F(w)} \mathcal{L}^{\mathcal{L}} d\chi_g = \sum_{I \subseteq I_0} \sum_{n \in N_{t,J}} L^{-F(w)} [Y_{\Delta}] \cdot L^{\mathcal{L}} \cdot \mathcal{L}^{\mathcal{L}}. \quad (**)
\]
By (4.7) we have \( Y_n = \prod_{i=1}^s S^{n_i} \hat{E}_i \times \prod_{\sigma \in I} k_\sigma^* \times \prod_{j \in J} k_j^* \), and the class of \( Y_n \) in the Grothendieck ring is
\[
[Y_n] = \prod_{i=1}^s [S^{n_i} \hat{E}_i] \cdot \prod_{\sigma \in I} [k_\sigma^*] \cdot \prod_{j \in J} [k_j^*],
\]
where \([k_\sigma^*] = [k_\sigma] - 1 = [\text{Spec}(k_\sigma)] \mathbb{L} - 1\) and \([k_j^*] = [k_j] - 1 = [\text{Spec}(k_j)] \mathbb{L} - 1\) (cf. (3.5)), and \([S^{n_i} \hat{E}_i]\) is computed in the following lemma.

(4.22) Lemma: Preserving notations as above, we have
\[
[S^{n_i} \hat{E}_i] = [\text{Spec}(k_i)]^{n_i} \mathbb{L}^{n_i} \sum_{l=0}^{\min\{n_i, \nu_i - 1\}} (-1)^l [\text{Spec}(k_i)]^{-l} \binom{\nu_i - 1}{l} \mathbb{L}^{-l}.
\]

Proof. Proceeding as in [17, Theorem 1, p. 51], since \([\hat{E}_i] = [\text{Spec}(k_i)] \mathbb{L} + 1 - \hat{\nu}_i\), we obtain
\[
\sum_{l=0}^{\infty} [S^l \hat{E}_i] t^l = (1 - t)^{-[\hat{E}_i]}
= (1 - t)^{-([\text{Spec}(k_i)] \mathbb{L} + 1 - \hat{\nu}_i)}
= (1 - t)^{-[\text{Spec}(k_i)] \mathbb{L} (1 - t)^{\nu_i - 1}}
= \sum_{l=0}^{\infty} [\text{Spec}(k_i)]^{l} \mathbb{L}^l (1 - t)^{\nu_i - 1}.\]

Therefore
\[
[S^{n_i} \hat{E}_i] = \sum_{l=0}^{\min\{n_i, \nu_i - 1\}} (-1)^l [\text{Spec}(k_i)]^{n_i - l} \binom{\nu_i - 1}{l} \mathbb{L}^{-l}.
\]

Once the classes in the Grothendieck ring have been computed, by Equation (**) in (4.21) we can conclude the description of the generalised Poincaré series in terms of an embedded resolution:

(4.23) Theorem:
\[
P_{\sigma}(t_1, \ldots, t_r; \mathbb{L}) = \sum_{l \leq l_0} \sum_{J \subset J_0} \sum_{n \in N_{l, J}} \prod_{i=1}^s [\text{Spec}(k_i)]^{n_i} S_I(\mathbb{L}) \cdot S_J(\mathbb{L}) \cdot \prod_{\sigma \in I} [\text{Spec}(k_\sigma)]^{n_\sigma}
 \times \prod_{i=1}^s \left( \sum_{l=0}^{\min\{n_i, \nu_i - 1\}} (-1)^l [\text{Spec}(k_i)]^{-l} \binom{\nu_i - 1}{l} \mathbb{L}^{-l} \right)^{n_i},
\]
where \(S_I(\mathbb{L}) := \prod_{\sigma \in I} ([\text{Spec}(k_\sigma)] \mathbb{L} - 1)\) and \(S_J(\mathbb{L}) := \prod_{J \in J} ([\text{Spec}(k_J)] \mathbb{L} - 1)\).
Notice that $F(n)$ has been already computed in Proposition (4.20) in terms of the intersection matrix. This result generalises Theorem 1 of [7], which holds only for complex curve singularities. Indeed, the formula turns out to be easier for the totally rational case, i.e. if all field extensions have degree one (in other words, $h_\sigma = h_j = 1$ for all $\sigma \in I$ and $j \in J$):

(4.24) Corollary: (Campillo, Delgado, Gusein–Zade) Assume the ring $R$ to be totally rational. We have

$$P_g(t_1, \ldots, t_r; L) = \sum_{I \subseteq I_0} \sum_{J \subseteq J_0} L^{\sharp(I)+\sharp(J)+\sum_{i=1}^s n_i - F(n)} (1 - L^{-1})^{\sharp(I)+\sharp(J)} \times \prod_{i=1}^s \left( \sum_{l=0}^{\min\{n_i, \nu_i - 1\}} (-1)^l \left( \frac{\nu_i - 1}{l} \right) L^{-l} \right)^{L^{\sharp(n)}}.$$

(4.25) Remark: The specialisation $L \to 1$ gives the connection with the classical Poincaré series $P(L)$:

$$P_g(t_1, \ldots, t_r; 1) = \sum_{I \subseteq I_0} \sum_{J \subseteq J_0} L^{\sharp(n)} = \int_Y L^{\sharp(n)} d\chi = P(L).$$

## 5 Divisorial Poincaré series and resolution

Let assume the ring $R$ to have a perfect coefficient field $K$ along this section.

(5.1) Definition: A divisorial valuation of $R$ is a discrete valuation of rank one (with group of values $\mathbb{Z}$) of the field of fractions of $R$ with $R \cap \mathfrak{m}_\sigma = \mathfrak{m}$ if $R_\sigma$ is the valuation ring with maximal ideal $\mathfrak{m}_\sigma$, and with transcendence degree of the residual extension $R_\sigma/\mathfrak{m}_\sigma : R/\mathfrak{m}$ equal to 1.

(5.2) Notation: Let $D := \{w_1, \ldots, w_s\}$ be a finite set of divisorial valuations of $R$, and let $W_i$ denote the discrete valuation ring associated with $w_i$ for all $1 \leq i \leq s$. The divisorial value semigroup associated with $D$ is the additive sub-semigroup $S_D$ of $\mathbb{Z}_{\geq 0}$ given by

$$S_D := \{ w(z) := (w_1(z), \ldots, w_s(z)) \mid z \in R \setminus \{0\} \}. $$

For $I \subseteq I_0$ consider the set

$$\mathcal{N}_I^D := \{ \underline{n} := (n_i, n'_\sigma, n''_\sigma) \mid n_i \geq 0, 1 \leq i \leq s; n'_\sigma > 0, n''_\sigma > 0, \sigma \in I \};$$

(the super–index $D$ of $\mathcal{N}_I^D$ and other next notations refers to the word “divisorial”). Consider the ideals $J_D^+(\underline{n}) := \{ z \in R \mid \underline{n}(z) \geq \underline{n} \}$ already defined in (4.13). We introduce a definition similar to (3.7) in the divisorial context (cf. [9]):

19
Definition: The generalized divisorial Poincaré series of the multi-index filtration given by the ideals $J_{D(w)}$ is defined to be the integral

$$P_{D}^{g}(t_{1},\ldots,t_{s};L) := \int_{P_{R}L^{W(z)}}d\chi_{g} \in \mathcal{M}_{k_{R}}.$$ 

We can express this divisorial series in terms of an embedded resolution of curves with a similar argument as that used in Theorem (4.23):

(5.4) Theorem:

$$P_{D}^{g}(t_{1},\ldots,t_{s};L) = \sum_{I \subset I_{0}} \sum_{J \subset J_{0}} \sum_{n \in \mathbb{N}^{D_{I}}} \frac{1}{\prod S_{I}(\mathbb{L})} \cdot \prod_{i=1}^{s} [\text{Spec}(k_{i})]^{n_{i}} \times$$

$$\times \prod_{i=1}^{s} \left( \frac{\min\{n_{i},\nu_{i}-1\}}{\sum_{l=0}^{\nu_{i}-1} (-1)^{l} [\text{Spec}(k_{i})]^{-l} \left( \nu_{i} - 1 \right)^{l} \mathbb{L}^{-l}} \right) t^{w_{I}(n)}.$$ 

where $S_{I}(\mathbb{L}) := \prod_{\sigma \in I} ([\text{Spec}(k_{\sigma})] \mathbb{L} - 1)$ and, for $n_{I} \in \mathbb{N}^{D_{I}}$ we denote

- $\hat{n}_{i} := n_{i} + \sum_{\sigma \in I_{1}(\sigma) = i} n'_{\sigma} + \sum_{\sigma \in I_{2}(\sigma) = i} n''_{\sigma}$

and the codimension $F_{D}^{I}(n_{\hat{I}})$ is equal to

$$\frac{1}{2} \left( \sum_{i,j = 1}^{s} m_{ij} \hat{n}_{i} \hat{n}_{j} + \sum_{i = 1}^{s} \hat{n}_{i} \cdot \left( \sum_{i' = 1}^{s} m_{ii'} (2h_{i'} - \hat{\nu}_{i'}) \right) \right).$$

Proof. We proceed as in Theorem (4.23), but taking the space

$$Y^{D} := \bigcup_{I \subset I_{0}} \bigcup_{n \in \mathbb{N}^{D_{I}}} Y_{n}^{D_{I}},$$

with $Y_{n}^{D_{I}} := \prod_{i=1}^{s} S_{i}^{n_{i}} E_{i}^{*} \times \prod_{\sigma \in I_{1}} k_{\sigma}^{*}$, instead of the space $Y$ defined in Equation (†) of (4.7).

(5.5) Let us now consider the extended semigroup $\hat{S}_{D}$ coming from the divisorial filtration $\{J_{D(w)}\}$ (cf. [9]) and the Poincaré series

$$\hat{P}_{D}^{g}(t_{1},\ldots,t_{s};L) := \int_{\hat{S}_{D}} t^{w_{D}(h)}d\chi_{g}.$$ 

(5.6) The next goal is to express this Poincaré series in terms of an embedded resolution. First, we define a semigroup homomorphism $\Pi : Y^{D} \rightarrow \mathbb{F}\hat{S}_{D}$ in the following way: For $y \in Y^{D}$, $y$ is represented by a set of smooth closed points of $E = \bigcup_{i=1}^{s} E_{i}$ with $n_{i}$ points $Q_{1}^{i},\ldots,Q_{n_{i}}^{i}$ on the component $E_{i}$. Let $Q^{i}$ be a point of $E_{i}$. Assume $E_{i}$ to have a local equation $\{x_{i} = 0\}$ at $Q^{i}$. Let us
take an element \( \gamma_{Q_i} \) meeting \( E_i \) at \( Q_i \) transversally, and let \( R_{Q_i} := \mathcal{O}_{X,Q_i} \). By [19, Lemma 3.1.11] (see also [20, Section 6]), there exists an element \( g_{Q_i} \in R \) such that its strict transform at \( Q_i \) is \( \tilde{\gamma}_{Q_i} \). Recall now the existence of a the one–to–one correspondence between the components of the exceptional divisor and the set \( D \) (see [15, Chap. VII]). If we consider the transform of \( g_{Q_i} \) in the discrete valuation ring \( W_i \) corresponding to the component \( E_i \) (cf. (5.2)), let us say \( \tilde{g}_{Q_i} \in W_i \) such that its strict transform at \( Q_i \) is \( \tilde{\gamma}_{Q_i} \). By definition, \( \Pi(y) \in \mathbb{P}\hat{S}_D \) is represented by the element \((v(g),\alpha(g)) \in \hat{S}_D\), where \( g = \prod_{i=1}^s \prod_{j=1}^{t_i} g_{Q_{ij}} \). But it is independent of the chosen representant \( \tilde{\gamma}_{Q_i} \).

By definition, \( \Pi(y) \) belongs to the projectivisation \( \mathbb{P}\hat{S}_D \) of the divisorial extended semigroup \( \hat{S}_D \) does not depend on the choice of the curves \( \gamma_P \), with \( P \) a point at \( E_i \) for any \( i \in \{1,\ldots,s\} \).

**Proof.** Let \( \tilde{\gamma}_P \) be a transversal element to \( E \) (a curvette) at a closed point \( P \in E_i \) for some \( i \in \{1,\ldots,s\} \) coming from an element \( g'_P \in R \) and let \( g' = \prod_{i=1}^s \prod_{j=1}^{t_i} g'_{Q_{ij}} \). Let \( \tilde{g} = g \circ \pi \) and \( \tilde{g}' = g' \circ \pi \) be the liftings of the functions \( g \) and \( g' \) to the space \( X \) of the resolution. We can set the function \( \psi = \tilde{\frac{g}{g}} \), whose restriction to every component \( E_i \) of \( E \) is regular, and therefore constant (since the \( E_i \) are projective lines). A similar argument as that in the proof of Proposition (4.11) shows that \( \psi \) is constant on \( E \) and equal to some \( \alpha \in k_R \). It implies that \( \psi(g') = \psi(\alpha g) = \psi(g) \) and \( \alpha(g') = \alpha \cdot \alpha(g) \).

**(5.8) Lemma:** The semigroup homomorphism \( \Pi \) is an isomorphism.

**Proof.** It is a straight consequence of Proposition (4.11).

**(5.9) Theorem:** If we denote \( \frac{\mathbf{m}^r}{\mathbf{m}^r} := (1 - \frac{\mathbf{m}^r}{\mathbf{m}^r})(1 - \frac{\mathbf{m}^r}{\mathbf{m}^r}) \), then the following equalities hold:

\[
\hat{P}_g(t_1,\ldots,t_s;L) = \chi_g(\mathbb{P}\hat{S}_D) = \int_{\mathbb{P}\hat{S}_D} \mathbf{m}^r d\chi_g = \frac{\prod_{\sigma \in I_0} \left( \frac{\mathbf{m}^r}{\mathbf{m}^r} \right)^{h_0} + \left( \frac{\mathbf{m}^r}{\mathbf{m}^r} \right)^{h_0 - 1} (\text{Spec}(k_\sigma))L - 1) \mathbf{m}^r}{\prod_{i=1}^s (1 - \mathbf{m}^r)(1 - \text{Spec}(k_i)L\mathbf{m}^r)}.
\]
Proof. By Lemma (5.8), we have

\[
\int_{\mathfrak{p}S_D} t^{\mathfrak{p}} d\chi_g = \int_D t^{\mathfrak{p}} d\chi_g = \sum_{I \in \mathcal{L}} \sum_{n \in N^n_D} [Y_h^D] \cdot \delta^{n}_D
\]

\[
= \sum_{I \in \mathcal{L}} \sum_{n \in N^n_D} \prod_{\sigma \in \mathcal{L}} ([\text{Spec}(k_\sigma)]_{\mathfrak{p}} - 1) \prod_{i=1}^{s} [S^n, E^i] \cdot \hat{t} \sum_{i=1}^{n_i} \mu^i_{n_i}.
\]

where

\[
A(t) := \sum_{i=1}^{s} \prod_{n \geq 0} [S^n, E^i] \cdot \hat{t} \sum_{i=1}^{n_i} \mu^i_{n_i},
\]

\[
B(t) := \sum_{I \in \mathcal{L}} \prod_{\sigma \in \mathcal{L}} ([\text{Spec}(k_\sigma)]_{\mathfrak{p}} - 1) \sum_{n_\sigma > 0} \mu^i_{n_\sigma} \cdot \mu^{\sigma}_n.
\]

Concerning \(A(t)\) we refer to [17, Theorem 1] to get

\[
A(t) = \prod_{i=1}^{s} \left( \sum_{n=0}^{\infty} [S^n, E^i] \cdot t^n \mu^i_n \right) = \prod_{i=1}^{s} (1 - t^{\mu^i_n})^{-1}.
\]

Since \([P^1_{k_i}] = [\text{Spec}(k_i)]_{\mathfrak{p}} + 1\) and \((1 - t)^{-1} \cdot (1 - t)^{-1} = (1 - t)^{-1}\) \((\text{cf. [17]}))\), we obtain

\[
(1 - t^{\mu^i_n})^{-1} = (1 - t^{\mu^i_n})^{-1} [\text{Spec}(k_i)]_{\mathfrak{p}} \cdot (1 - t^{\mu^i_n})^{-1}
\]

\[
= (1 - [\text{Spec}(k_i)]_{\mathfrak{p}}^{-1} t^{\mu^i_n})^{-1} \cdot (1 - t^{\mu^i_n})^{-1}.
\]

On the other hand, by the equality

\[
(1 - t^{\mu^i_n})^{-1} = \prod_{\sigma \in \mathcal{L}} (1 - t^{\mu^i_{1, \sigma}}) h_\sigma (1 - t^{\mu^i_{2, \sigma}}) h_\sigma,
\]

one deduces

\[
A(t) = \prod_{i=1}^{s} \frac{1}{(1 - t^{\mu^i_n}) (1 - [\text{Spec}(k_i)]_{\mathfrak{p}}^{-1} t^{\mu^i_n})} \cdot \prod_{\sigma \in \mathcal{L}} ((1 - t^{\mu^i_{1, \sigma}}) (1 - t^{\mu^i_{2, \sigma}})) h_\sigma.
\]

The factor \(B(t)\) is equal to

\[
B(t) = \sum_{I \in \mathcal{L}} \prod_{\sigma \in \mathcal{L}} ([\text{Spec}(k_\sigma)]_{\mathfrak{p}} - 1) \prod_{\sigma \in \mathcal{L}} \frac{\mu^i_{1, \sigma}}{1 - t^{\mu^i_{1, \sigma}}} \cdot \frac{\mu^i_{2, \sigma}}{1 - t^{\mu^i_{2, \sigma}}}
\]

\[
= \prod_{\sigma \in \mathcal{L}} \left( 1 + ([\text{Spec}(k_\sigma)]_{\mathfrak{p}} - 1) \frac{\mu^i_{1, \sigma}}{1 - t^{\mu^i_{1, \sigma}}} \cdot \frac{\mu^i_{2, \sigma}}{1 - t^{\mu^i_{2, \sigma}}} \right)
\]

\[
= \prod_{\sigma \in \mathcal{L}} \left( (1 - t^{\mu^i_{1, \sigma}})(1 - t^{\mu^i_{2, \sigma}}) + ([\text{Spec}(k_\sigma)]_{\mathfrak{p}} - 1) \mu^i_{1, \sigma} \mu^i_{2, \sigma} \right)
\]

\[
= \prod_{\sigma \in \mathcal{L}} \left( (1 - t^{\mu^i_{1, \sigma}})(1 - t^{\mu^i_{2, \sigma}}) + (1 - t^{\mu^i_{1, \sigma}}) t^{\mu^i_{2, \sigma}} \right).
\]
Denoting \(l_{m_\tau} := (1 - l_{m_1 (\tau)})(1 - l_{m_2 (\tau)})\), we get

\[
A(\tau) \cdot B(\tau) = \frac{\prod_{\sigma \in I_0} (l_{m_\tau})^{h_{\sigma} - 1} + (l_{m_\tau})^{h_{\sigma}} + (\text{Spec}(k_{\sigma}) | L - 1)l_{m_1 (\tau)}l_{m_2 (\tau)} \prod_{i=1}^{\ell}(1 - l_{m_i})}{\prod_{i=1}^{\ell}(1 - l_{m_i}) (1 - |\text{Spec}(k_i)| L_{m_i})}.
\]

(5.10) Corollary: (Campillo, Delgado, Gusein–Zade) If the ring \(R\) is totally rational, then we have

\[
\hat{\beta}^D_{g}(t_1, \ldots, t_\ell; L) = \frac{\prod_{\sigma \in I_0} (1 - l_{m_1 (\tau)} - l_{m_2 (\tau)} + l_{m_1 (\tau)}l_{m_2 (\tau)} \prod_{i=1}^{\ell}(1 - l_{m_i})}{\prod_{i=1}^{\ell}(1 - l_{m_i}) (1 - |\text{Spec}(k_i)| L_{m_i})}.
\]

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