Comment on energy level statistics in the mixed regime

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Abstract. We comment on the recent paper by Abul-Magd (J.Phys.A: Math.Gen. 29 (1996) 1) concerning the energy level statistics in the mixed regime, i.e. such having the mixed classical dynamics where regular and chaotic regions coexist in the phase space. We point out that his basic assumption on the additive property of the level-repulsion function $r(S)$ (conditional probability density) in the sense of dividing it linearly into the regular and chaotic part in proportion to the classical fractional phase space volumes $\rho_1$ and $\rho_2 = q$ is not justified, since among other things, it relies on the type of Berry’s ergodic assumption, which however is right only in a homogeneous ensemble of ergodic systems, but not in the neighbourhood of an integrable system. Thus his resulting distribution cannot be regarded as a theoretically well founded object. We point out that the semiclassical limiting energy level spacing distribution must be of Berry-Robnik (1984) type, and explain what transitional behaviour of the Brody-type (with fractional power-law energy level repulsion) we observe in the near semiclassical regime where effective $\hbar$ is not yet small enough. Thus we refer to the derivation, arguments and conclusions in our paper (Prosen and Robnik, J.Phys.A: Math.Gen. 26 (1994) 8059), and explain again the behaviour in this double transition region.

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In a recent paper (Abul-Magd 1996) offers a new theoretical energy level spacings distribution for quantal Hamiltonian systems whose classical dynamics is of the mixed type, i.e. such having regular regions of invariant tori coexisting in the phase space (and on the energy surface) with chaotic regions, a typical KAM scenario. In this comment we want to point out that his result is not theoretically well founded and is in fact erroneous, and has no other merit than mathematical simplicity, which however is of course not a sufficient condition for the scientific merit. Abul-Magd uses the famous Wigner surmise (Wigner 1956, Brody 1973, Brody et al 1981, Robnik 1984, Bohigas and Giannoni 1984), which by itself is a sound argument, but he makes an assumption about the conditional probability density \( r(S) \) (the so-called level repulsion function), in conjunction with the Berry-type argument on the ergodicity of quantal energy spectra in an ensemble of classically ergodic systems (Berry 1981, 1983, 1985), which is wrong in his context, because it is applied to the systems that are not ergodic but close to an integrable system (KAM type systems).

Quite generally, by knowing \( r(S) \) one gets the level spacings distribution \( P(S) \) at once as \( P(S) = r(S) \exp(- \int_0^S r(x) dx) \). For example, \( r(S) = 1 \) implies Poisson distribution, \( r(S) = \pi S/2 \) implies Wigner (2-D GOE), \( r(S) \propto S^3 \) implies Brody distribution and so on.

In order to derive \( r(S) \) Abul-Magd refers to the ergodicity argument by Berry (1981, 1983, 1985) where in calculating the \( P(S) \) at small \( S \) he replaces the average over the energy spectrum by the average over an ensemble of classically ergodic systems parametrized by at least two parameters (because the degeneracies, the diabolical points, have codimension 2), which is a very reasonable assumption indeed, and it immediately yields the linear level repulsion. However, as it has been pointed out by one of us (Robnik 1984) this ergodicity assumption cannot be applied in the neighbourhood of an integrable system, simply because there is no local uniformity in the parameter space, and as we approach (the coordinates/parameters of) the integrable system we see greater and greater density of degeneracy points: they are not uniformly distributed in the space \{parameter A x parameter B x energy\}.

Therefore, the ansatz ("the basic assumption") of Abul-Magd for \( r(S) = \ldots \)
\[ \rho_1 + \rho_2 S, \]  
where \( \rho_1 + \rho_2 = 1 \) (in his notation \( q \equiv \rho_1 \)), is not justified theoretically, but is just a guess. In fact it leads to a distribution function which is mathematically simple, normalized, but not its first moment, which is another deficiency of the model.

Further we claim with full theoretical justification that the correct ultimate semiclassical energy level spacings distribution is in fact Berry-Robnik (1984). (Similar thinking can be of course applied to other statistical measures, such as number variance and delta statistics, etc, see e.g. (Seligman and Verbaarschot 1985)). This assertion has a sound theoretical foundation. It is based on the picture in the quantum phase space (the Wigner functions of stationary eigenstates) in the strict semiclassical limit, \( \hbar \to 0 \), where we observe the condensation of states in volume elements of order \( (2\pi\hbar)^f \), where \( f \) is the number of degrees of freedom (see e.g. Robnik 1988, 1997), on classical invariant objects, which is the contents of the so-called principle of uniform semiclassical condensation. The prediction agrees with the rigorous results by Lazutkin (1981, 1991) on splitting the energy spectra and the eigenstates in regular and irregular levels/states (qualitatively predicted by Percival (1973)), in the special case of convex billiards with smooth boundaries. This has been analyzed also in (Li and Robnik 1995). We have at least two special but typical mixed dynamical systems for which we have demonstrated with a very great accuracy that the semiclassically limiting statistics is Berry-Robnik, namely in the quantized compactified standard map (on a torus) (Prosen and Robnik 1994a,b) and in the 2-D semiseparable oscillator (Prosen 1995, 1996). In both cases the quantally derived \( \rho_1 \) agrees with the classical one better than within 1%. It is interesting to note that in analyzing the numerical spectra we had to use the infinitely dimensional GOE statistics on chaotic component (Wigner distribution = 2-dim GOE was not good enough) in order to achieve perfect agreement between the numerical results and the best fitting Berry-Robnik distribution.

Therefore we have full confidence in the correctness of the asymptotic far semiclassical limit (\( \hbar \to 0 \)) of spectral statistics.

However, before reaching the ultimate semiclassical limit, in a regime which we call the near semiclassical limit, we find phenomenologically significant and to some extent universal statistical behaviour of energy spectra, espe-
cially in 2-D billiards and elsewhere (Prosen and Robnik 1993, 1994a,b). Namely, we typically observe the fractional power law level repulsion, \( P(S) \propto S^\beta \), where the exponent \( \beta \) can be anything between 0 and 1 for \( \text{OE} \) statistics, or \( \beta \in [0, 2] \) in case of \( \text{UE} \) (broken antiunitary symmetries, or more generally, complex representations, see (Robnik 1986, Leyvraz, Schmit and Seligman 1996, Keating and Robbins 1997, Dobnikar 1996, Robnik and Dobnikar 1997). It is now qualitatively understood that this statistics is another manifestation of quantum (dynamical) localization, i.e. the localization of quantum eigenstates related to the classical dynamics. In KAM systems we have a theory on \( \beta \), where we derive (Prosen and Robnik 1994b) the scaling law \( \beta = \text{const.} \hbar \), for sufficiently small \( \hbar \). Furthermore, we have shown that the fractional power-law regime with the given \( \beta \) should be observed for spacings \( S \) within the interval \( [\exp(-1/\beta), 1] \). Using the above scaling estimates, we see that the fractional power-law level repulsion is \textit{not} observed in the exponentially small interval \( [0, \exp(-\text{const.}/\hbar)] \). So, when the effective Planck constant goes to zero, \( \hbar \to 0 \), this interval becomes exponentially small and practically invisible, because there are usually not enough objects there. The estimate agrees perfectly well with the prediction by Berry and Robnik (1984) that there is such a exponentially small region at small \( S \) due to the tunneling phenomena. Therefore, the picture is now fully consistent, and it remains to explain what behaviour do we predict theoretically in this exponentially small region.

We know from the elementary thinking that in this region the \( P(S) \) must behave linearly \( P(S) \propto S \), which cannot be predicted semiclassically (Robnik 1986, Berry 1991, Robnik and Salasnich 1997) but only quantally (Robnik 1987). The reason is that for very small spacings the quantum degenerate 2-dim perturbation theory must be ultimately sufficient, which was demonstrated and argued in (Robnik 1987). Indeed, if one increases the dimensionality of such a model ("Poissson + GOE"), one finds the same linear level repulsion law for 3-dim and 4-dim (Izrailev 1993) and for higher dimensions (Prosen 1993). This quantum mechanical picture explains the linear level repulsion region, which is exponentially small.

The question then is to explain how - in this doubly transitional regime: mixed dynamics, and transition from near to far semiclassics - the Brody-like behaviour goes over into Berry-Robnik behaviour, as the \( \hbar \) tends to
0. For this we have no global quantitative theory, except for the more or less local features described above. Schematically we show this in figure 1. We also show in figure 2, schematically, the Brody-like distribution and the Berry-Robnik distribution, with the indicated (and schematically exaggerated) exponentially small region of linear level repulsion (the purely quantum regime). In practice, with actual spectra, it is almost impossible to detect the exponentially small region, and indeed this has not been observed until now in any specific system.

The existence of the fractional power-law level repulsion and Brody-like behaviour is definitely connected with the existence of (dynamical) localization, which is a topic of current research (Frahm and Shepelyanski 1997, Casati and Prosen 1997, Robnik et al 1997). Moreover, in ergodic systems, but with very slow diffusion, we also observe dynamical localization (Prosen and Robnik 1994b, which gives rise to the Brody-like behaviour, with fractional power-law level repulsion, but here the picture is much more complicated, and $\beta$ must tend to 1, rather than 0, as $\hbar \to 0$.

Finally, in regard of phenomenological formulae, we suggest that the so-called BRB-distribution (Berry-Robnik-Brody, see (Prosen and Robnik 1994b)), which is a two-parameter distribution function, is the best, because it has the theoretical foundation in the sense that it takes into account the division of the classical phase space (parameter $\rho_1$), and the localization of the chaotic states on the (subset of the) chaotic regions (whose measure is $\rho_2 = 1 - \rho_1$), captured by the level repulsion parameter $\beta$. Indeed, in our own work (Prosen and Robnik 1994b) we have confirmed the agreement in billiard systems and in mappings, and a similar success is reported in the context of theoretical nuclear spectra by Lopac, Paar and Brant (1996).

In conclusion, we propose that there is no place for other limiting semi-classical energy level statistics than Berry-Robnik (1984), in systems with mixed classical dynamics (KAM type systems), while in the transition regime there is evidence and substantial understanding that outside the exponentially small region of linear level repulsion due to tunneling, there is the fractional power-law level repulsion and Brody-like behaviour with exponent $\beta = \text{const.}\hbar$, which goes to zero when $\hbar$ goes to zero, thereby going over to the Berry-Robnik distribution. We have explained why the basic assumption
of Abul-Magd (1996) is not justified and therefore any significant agreement of his results with high-quality spectral data cannot be expected.

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Figure captions

**Figure 1:** We show the schematic diagramme of the doubly-transition region: from integrable to ergodic classical dynamics and from near- (not very small $\hbar$) to far- semiclassics (sufficiently small $\hbar$).

**Figure 2:** We show schematically two examples of the Brody-like level spacings distribution (with higher maximum) and Berry-Robnik type, but in both cases indicated the exponentially small (but here exaggerated) regime of linear level repulsion (see text).