Minimax testing and quadratic functional estimation for circular convolution

Sandra Schluttenhofer *  Jan Johannes *
Ruprecht-Karls-Universität Heidelberg

Abstract

In a circular convolution model, we aim to infer on the density of a circular random variable using observations contaminated by an additive measurement error. We highlight the interplay of the two problems: optimal testing and quadratic functional estimation. Under general regularity assumptions, we determine an upper bound for the minimax risk of estimation for the quadratic functional. The upper bound consists of two terms, one that mimics a classical bias-variance trade-off and a second that causes the typical elbow effect in quadratic functional estimation. Using a minimax optimal estimator of the quadratic functional as a test statistic, we derive an upper bound for the nonasymptotic minimax radius of testing for nonparametric alternatives. Interestingly, the term causing the elbow effect in the estimation case vanishes in the radius of testing. We provide a matching lower bound for the testing problem. By showing that any lower bound for the testing problem also yields a lower bound for the quadratic functional estimation problem, we obtain a lower bound for the risk of estimation. Lastly, we prove a matching lower bound for the term causing the elbow effect in the estimation problem. The results are illustrated considering Sobolev spaces and ordinary or super smooth error densities.

Keywords: nonparametric test theory, nonasymptotic separation radius, minimax theory, inverse problem, circular data, deconvolution, quadratic functionals, goodness-of-fit

AMS 2000 subject classifications: primary 62G10; secondary 62G05, 62C20

1 Introduction

The statistical model. We consider a circular convolution model, where a random variable that takes values on the circle is observed contaminated by an additive error. The aim of this paper is to highlight the interplay of the two problems: optimal testing and quadratic functional estimation for its density. Identifying the circle with the unit interval $[0, 1)$, the observable random variable is

$$ Y = X + \varepsilon - \lfloor X + \varepsilon \rfloor, \quad (1.1) $$

where $X$ and $\varepsilon$ are independent random variables supported on the interval $[0, 1)$ and $\lfloor \cdot \rfloor$ denotes the floor-function. Let $X$ be distributed with the unknown density of interest $f$ and

*Institut für Angewandte Mathematik, MATHEMATIKON, Im Neuenheimer Feld 205, D-69120 Heidelberg, Germany, e-mail: {schluttenhofer|johannes}@math.uni-heidelberg.de
the error $\varepsilon$ with the known density $\varphi$. The density $g$ of the observable random variable $Y$ satisfies $g = f \circ \varphi$, where $\circ$ denotes circular convolution defined by

$$f \circ \varphi(y) = \int_0^1 f(y - s - \lfloor y - s \rfloor) \varphi(s) ds, \quad y \in [0, 1).$$

Hence, making inference on $f$ based on observations from $g$ is a deconvolution problem.

**Related literature.** Circular data, also called spherical, directional or wrapped (around the circumference of the unit circle), appears in various applications. For an in-depth review of many examples for circular data we refer the reader to Mardia [1972], Fisher [1995] and Mardia and Jupp [2009]. Let us only briefly mention two popular fields of application. Circular models are used for data with a temporal or periodic structure, where the circle is identified e.g. with a clock face (cp. Gill and Hangartner [2010]). Moreover, identifying the circle with a compass rose, directional data can also be represented by a circular model. Kerkyacharian et al. [2011] and Lacour and Ngoc [2014], for instance, investigated a circular model with multiplicative error. Nonparametric estimation in the additive error model (1.1) has amongst others been considered in Efromovich [1997], Comte and Taupin [2003] and Johannes and Schwarz [2013].

Quadratic functional estimation in direct models has received a lot of attention in the literature, let us only mention a few references. Bickel and Ritov [1988] and Birge and Massart [1995] establish minimax rates for the estimation of functionals of a density, where they discovered a typical phenomenon in quadratic functional estimation: the so-called elbow effect, which also appears in our results. It refers to a sudden change in the behaviour of the rates, as soon as the smoothness parameter crosses a critical threshold.

In a Gaussian sequence space model, which is closely related to our model, for instance, Laurent and Massart [2000], Laurent [2005] consider adaptive quadratic functional estimation via model selection, Cai and Low [2005] and Cai and Low [2006] derive minimax optimal estimators under Besov-type regularity assumptions. Collier et al. [2017] consider sparsity constraints. Quadratic functional estimation in an inverse Gaussian sequence space model is treated by Butucea and Meziani [2011] (known operator) and Kroll [2019] (partially unknown...
operator). For quadratic functional estimation for deconvolution on the real line we refer to Butucea [2007] and Chesneau [2011].

Concerning the testing task, in the literature there exist several definitions of rates and radii of testing in an asymptotic and nonasymptotic sense. The classical definition of an asymptotic rate of testing for nonparametric alternatives is essentially introduced in the series of papers Ingster [1993a], Ingster [1993b] and Ingster [1993c]. For fixed noise levels, two alternative definitions of a nonasymptotic radius of testing are typically considered. For prescribed error probabilities \( \alpha, \beta \in (0, 1) \), Baraud [2002], Laurent et al. [2012] and Marteau and Sapatinas [2017], amongst others, define a nonasymptotic radius of testing as the smallest separation radius \( \rho \) such that there is an \( \alpha \)-test with maximal type II error probability over the \( \rho \)-separated alternative smaller than \( \beta \). The definition we us in this paper – which is based on the sum of both error probabilities – is adapted e.g. from Collier et al. [2017]. The connection between quadratic functional estimation and testing has for example been used in Collier et al. [2017] (in a direct Gaussian sequence space model under sparsity), Kroll [2019] (in an indirect Gaussian sequence space model under regularity constraints) and Butucea [2007] (in a convolution model on the real line). Let us now introduce our setting.

**Quadratic functional estimation.** Denote by \( \mathcal{D} \) the subset of real probability densities in \( \mathcal{L}^2 := \mathcal{L}^2([0,1]) \), the Hilbert space of square-integrable complex-valued functions on \([0,1]\) equipped with its usual norm \( \|\cdot\|_{\mathcal{L}^2} \). Since we are interested in the estimation of the quadratic functional \( q^2(f) = \|f\|_{\mathcal{L}^2}^2 \) of \( f \), we assume throughout this paper that both \( f \) and \( \varphi \) (and, hence, \( g \)) belong to \( \mathcal{D} \). We also want to compare \( f \) to the density \( f_0 = 1_{[0,1]} \) of a uniform distribution by estimating their \( \mathcal{L}^2([0,1]) \)-distance \( q^2(f) = \|f - f_0\|_{\mathcal{L}^2} \). Since \( q^2(f) = \tilde{q}^2(f) - 1 \), these problems are equivalent and we will focus on the estimation of \( \tilde{q}^2(f) \). Let \( \{Y_k\}_{k \in \{1, \ldots, n\}} \) be \( n \) independent and identically distributed observations with density \( g \), i.e. the observations are given by

\[
Y_k \overset{iid}{\sim} g = f \odot \varphi, \quad k \in \{1, \ldots, n\}.
\]  

Denote by \( \mathbb{P}_f \) and \( \mathbb{E}_f \) the probability distribution and the expectation associated with the data (1.2), respectively. For a nonparametric class of functions \( \mathcal{E} \), we measure the accuracy of an estimator \( \tilde{q}^2 \), i.e. a measurable function \( \tilde{q}^2 : \mathbb{R}^n \to \mathbb{R} \), by its **maximal risk**

\[
r^2(\tilde{q}^2, \mathcal{E}) := \sup_{f \in \mathcal{E}} \mathbb{E}_f \left( \tilde{q}^2(f) - q^2(f) \right)^2
\]

and compare its performance to the **minimax risk of estimation**

\[
r^2(\mathcal{E}) := \inf_{\tilde{q}^2} r^2(\tilde{q}^2, \mathcal{E}),
\]

where the infimum is taken over all possible estimators. An estimator \( \tilde{q}^2 \) is called minimax optimal for the class \( \mathcal{E} \), if its maximal risk is bounded by the minimax risk \( r^2(\mathcal{E}) \) up to a constant.

**The testing task.** Based on the observations (1.2), we test the null hypothesis \( \{f = f_0\} \) against the alternative \( \{f \neq f_0\} \). To make the null hypothesis and the alternative distinguishable, we separate them in the \( \mathcal{L}^2 \)-norm. For a separation radius \( \rho \in \mathbb{R}_+ \), let us define \( \mathcal{L}_\rho^2 := \{f \in \mathcal{L}^2 : \|f\|_{\mathcal{L}^2} \geq \rho \} \), which is called the **energy condition**. For a nonparametric class of densities \( \mathcal{E} \), called the **regularity condition**, the testing problem can be written as

\[
H_0 : f = f_0 \quad \text{against} \quad H_1^\rho : f \in \mathcal{L}_\rho^2 \cap \mathcal{E}.
\]  

(1.3)
We measure the accuracy of a test $\Delta$, i.e., a measurable function $\Delta : \mathbb{R}^n \to \{0, 1\}$, by its maximal risk defined as the sum of the type I error probability and the maximal type II error probability over the $\rho$-separated alternative

$$R(\Delta \mid \mathcal{E}, \rho) := \mathbb{P}_f(\Delta = 1) + \sup_{f \in \mathcal{E} \cap \mathcal{E}} \mathbb{P}_f(\Delta = 0).$$

We are particularly interested in the smallest possible value of $\rho^2$ by which we need to separate the null and the alternative for them to be distinguishable. A value $\rho^2(\Delta, \mathcal{E})$ is called radius of testing of the test $\Delta$ over the alternative $\mathcal{E}$, if for all $\alpha \in (0, 1)$ there exist constants $\underline{A}_\alpha, \overline{A}_\alpha \in \mathbb{R}_+$ such that

(i) for all $A \geq \overline{A}_\alpha$ we have $R(\Delta \mid \mathcal{E}, A \rho(\Delta, \mathcal{E})) \leq \alpha$, \hspace{1cm} (upper bound)

(ii) for all $A \leq \underline{A}_\alpha$ we have $R(\Delta \mid \mathcal{E}, A \rho(\Delta, \mathcal{E})) \geq 1 - \alpha$. \hspace{1cm} (lower bound)

The difficulty of the testing problem can be characterized by the minimax risk

$$R(\mathcal{E}, \rho) := \inf_{\Delta} R(\Delta \mid \mathcal{E}, \rho)$$

where the infimum is taken over all possible tests. The value $\rho^2(\mathcal{E})$ is called minimax radius of testing, if for all $\alpha \in (0, 1)$ there exist constants $\underline{A}_\alpha, \overline{A}_\alpha \in \mathbb{R}_+$ such that

(i) for all $A \geq \overline{A}_\alpha$ we have $R(\mathcal{E}, A \rho(\mathcal{E})) \leq \alpha$, \hspace{1cm} (upper bound)

(ii) for all $A \leq \underline{A}_\alpha$ we have $R(\mathcal{E}, A \rho(\mathcal{E})) \geq 1 - \alpha$. \hspace{1cm} (lower bound)

If $\rho^2(\mathcal{E})$ is a radius of testing for the test $\Delta$, then the test is called minimax optimal.

**Methodology.** We characterise both the minimax risk and the minimax radius in terms of the sample size $n$, the parameters of $\mathcal{E}$ and the error density $\varphi$. Our approach heavily depends on the properties of the Hilbert space $\mathcal{L}^2([0, 1])$ equipped with its usual inner product $\langle \cdot, \cdot \rangle$. Given the exponential basis $e_j$, $j \in \mathbb{Z}$ of $\mathcal{L}^2([0, 1])$, with $e_j(x) = \exp(2\pi i j x)$ for $x \in [0, 1)$, we denote the Fourier coefficients of $f \in \mathcal{L}^2([0, 1])$ by $f_j = \langle f, e_j \rangle$, $j \in \mathbb{Z}$. This leads to the discrete Fourier series expansion

$$f = \sum_{j \in \mathbb{Z}} f_j e_j; \hspace{1cm} (1.4)$$

where equality holds in $\mathcal{L}^2([0, 1])$. The nonparametric class of functions $\mathcal{E}$ is formulated in terms of the Fourier coefficients and characterises the regularity of the function. Let $R > 0$ and let $a = (a_j)_{j \in \mathbb{N}}$ be a strictly positive, monotonically non-increasing sequence. We assume that the density of interest $f$ belongs to the $\mathcal{L}^2$-ellipsoid

$$\mathcal{E}_a^R = \left\{ f \in \mathcal{D} : 2 \sum_{j \in \mathbb{N}} a_j^{-2} |f_j|^2 \leq R^2 \right\}. \hspace{1cm} (1.5)$$

Note that $f \in \mathcal{E}_a^R$ imposes conditions on all coefficients $f_j$, $j \in \mathbb{Z}$, since $|f_j|^2 = |f_{-j}|^2$, $j \in \mathbb{N}$, for all real-valued functions and, additionally, $f_0 = 1$ for all densities. The definition (1.5) is general enough to cover classes of ordinary and analytically smooth densities. Expanding $f$ and $f_0$ in the exponential basis as in (1.4) and applying Parseval’s Theorem yields a representation of the quadratic functional $q^2(f) = \|f - f_0\|_{\mathcal{L}^2}^2$ in their Fourier coefficients $q^2(f) = \sum_{j \in \mathbb{Z}} |f_j - f_{0,j}|^2 = \sum_{j \in \mathbb{Z}} |f_{-j} - f_{0,-j}|^2.
2 \sum_{j \in \mathbb{N}} |f_j|^2$. Moreover, by the circular convolution theorem we have $g = f \ast \varphi$ if and only if the Fourier coefficients satisfy $g_j = f_j \cdot \varphi_j$ for all $j \in \mathbb{Z}$. Here and subsequently, we assume that the Fourier coefficients of the noise density $\varphi$ are non-vanishing everywhere, i.e. $|\varphi_j| > 0$ for all $j \in \mathbb{Z}$. The quadratic functional can then be expressed as

$$q^2(f) = \sum_{j \in \mathbb{Z}} \left| g_j - \varphi_j f_j \right|^2 \left| \varphi_j \right|^2 = 2 \sum_{j \in \mathbb{N}} \left| g_j \right|^2.$$  

(1.6)

The only unknown quantities in (1.6) are the Fourier coefficients $g_j$, $j \in \mathbb{Z}$, of $g$, which can easily be estimated. Since for $j \in \mathbb{Z}$, $g_j = \langle g, e_j \rangle = E_f(e_j(-Y_1))$, a natural estimator is given by replacing the expectation with the empirical counterpart $\hat{g}_j = \frac{1}{n} \sum_{k=1}^{n} e_j(-Y_k)$. Inserting these estimators into the quadratic functional, however, generates a bias in every component. Since $|\hat{g}_j|^2 - \frac{1}{n-1} |g_j|^2$ is an unbiased estimator of the numerator $|g_j|^2$, for $j \in \mathbb{N}$, for each $k \in \mathbb{N}$ we consider the estimator

$$\hat{q}_k^2 := 2 \sum_{j=1}^{n} \left| \varphi_j \right|^{-2} \left( |\hat{g}_j|^2 - \frac{1 - |g_j|^2}{n-1} \right),$$  

(1.7)

which is an unbiased estimator of the truncated quadratic functional $q_k^2 := 2 \sum_{j \in \mathbb{N}} \left| g_j \right|^2$.

Using $\hat{q}_k^2$ (defined in (1.7)) as an estimator of the distance $\|f - f_o\|^2_{\mathbb{P}^2}$, we construct a test that, roughly speaking, compares the estimator to a multiple of its standard deviation. Precisely, for $k \in \mathbb{N}$ and a constant $C_\alpha$, we consider the test

$$\Delta_{\alpha,k} := \mathbb{I}\left\{ \hat{q}_k^2 \geq C_\alpha r_k^2 \right\} \quad \text{with} \quad r_k^2 := \frac{1}{n} \left\{ \sum_{0<|j| \leq k} \frac{1}{|\varphi_j|^2} \right\}^{1/2}.$$  

(1.8)

**Minimax results.** We show that for fixed $k$ the risk of the estimator $\hat{q}_k^2$ in (1.7) is bounded by

$$\rho_k^4 \lor r_o^4 \quad \text{with} \quad \rho_k^4 := \left\{ a_k^4 \lor \nu_k^4 \right\} \quad \text{and} \quad r_o^4 := \max_{m \in \mathbb{N}} \left\{ a_m^4 \land \frac{a_m^2}{n \left| \varphi_m \right|^2} \right\},$$  

(1.9)

up to a constant. The base level term $r_o^4$ is present for all $k \in \mathbb{N}$, whereas the term $\rho_k^4$, which represents a typical bias-variance trade-off, explicitly depends on the dimension parameter $k \in \mathbb{N}$ and can, thus, be optimised with respect to $k$. More precisely, choosing $\kappa_\alpha$ as a minimizer of $\rho_k^4$, the risk of $\hat{q}_{\kappa_\alpha}$ is up to a constant bounded by

$$\rho_{\kappa_\alpha}^4 \lor r_o^4 := \left\{ \min_{k \in \mathbb{N}} \rho_k^4 \right\} \lor r_o^4.$$  

(1.10)

The term $r_o^4$ causes the classical elbow effect in quadratic functional estimation, since it prevents the rate from being faster than parametric. The upper bound shows the expected behaviour: a faster decay of the Fourier coefficients of $\varphi$, i.e. a smoother error density, results in a slower rate. Therefore, we call the decay of $\{|\varphi_j|\}_{j \in \mathbb{N}}$ the *degree of ill-posedness* of the model (1.1). On the other hand, a faster decay of the Fourier coefficients of the density of interest $f$ yields a faster rate. We use the estimation upper bound to determine an upper bound for a radius of testing of the test $\Delta_{\alpha,k}$ defined in (1.8). For appropriately chosen $C_\alpha$, an upper bound for the radius of testing of $\Delta_{\alpha,k}$ is given by

$$\rho_k^2 = a_k^2 \lor \frac{1}{n} \sum_{0<|j| \leq k} \frac{1}{|\varphi_j|^2}.$$
which can again be optimised with respect to $k \in \mathbb{N}$. Again choosing $\kappa_*$ as the minimizer of $\rho_k^2$, the radius of testing of $\Delta_{a,\kappa_*}$ is of order

$$
\rho_*^2 = \min_{k \in \mathbb{N}} \left\{ a_k^2 \sqrt{\frac{1}{n} \sum_{0 < |j| \leq k} \frac{1}{|\varphi_j|^4}} \right\}.
$$

(1.11)

Interestingly, the term causing the elbow effect in the estimation case vanishes in the radius of testing. Roughly speaking, the densities that cause $r_0^4$ in (1.10) and, hence, the elbow effect, are difficult to estimate (since they have large energy), but easy to test (since they are far from the null). This observation is explicitly used in the proof of the testing upper bound.

Outline of the paper. The upper bound for the estimation risk and the radius of testing is derived in Section 2 and Section 3, respectively. Section 4 provides a matching lower bound for the testing problem. In Section 5 we first show that testing is faster than quadratic functional estimation if we correct for the missing square, formally $r^4(\mathcal{E}) \geq C \rho^2(\mathcal{E})$ for some $C > 0$. Using this connection between quadratic functional estimation and testing, we immediately obtain a lower bound for the estimation problem. It remains to prove an additional lower bound for the term $r_0^4$ in (1.10) that causes the elbow effect. Thus, we establish the order of both the minimax estimation rate and the minimax radius of testing. Technical results and their proofs are deferred to Appendix A.

## 2 Upper bound for the estimation risk

The next proposition presents an upper bound for the quadratic functional estimator defined in (1.7) for arbitrary $f \in \mathcal{D}$ and $k \in \mathbb{N}$. The key element of the proof is rewriting the estimator as a U-statistic and exploiting a well-known formula for its variance.

**Proposition 2.1 (Upper bound for the estimation risk).** For $n \geq 2$ and $k \in \mathbb{N}$ the estimator defined in (1.7) satisfies

$$
\mathbb{E}_f \left( \hat{q}_k^2 - q^2(f) \right)^2 \leq \left( \sum_{|j| > k} |f_j|^2 \right)^2 + \frac{c}{n^2} \sum_{0 < |j| \leq k} \frac{1}{|\varphi_j|^4} + \frac{c}{n} \sum_{0 < |j| \leq k} \frac{|f_j|^2}{|\varphi_j|^4}
$$

with $c := \|f \otimes \varphi\|_\infty = \|g\|_\infty := \sup_{x \in [0,1]} |g(x)|$.

**Proof of Proposition 2.1.** The bound follows from a classical bias-variance decomposition of the risk:

$$
\mathbb{E}_f \left( \hat{q}_k^2 - q^2(f) \right)^2 = \left( \sum_{|j| > k} |f_j|^2 \right)^2 + \text{var}_f \left( \hat{q}_k^2 \right).
$$

(2.2)

To bound the variance, we rewrite the estimator as a U-statistic

$$
\hat{q}_k = \frac{1}{n(n-1)} \sum_{l \neq m} \sum_{0 < |j| \leq k} \frac{e_j(-Y_l)e_j(Y_m)}{|\varphi_j|^2} =: \frac{1}{n(n-1)} \sum_{l \neq m} h(Y_l, Y_m) =: \frac{1}{2} U_n,
$$

where $h(y_1, y_2) := \sum_{0 < |j| \leq k} \frac{e_j(-y_1)e_j(y_2)}{|\varphi_j|^2}$ for $y_1, y_2 \in [0,1)$ and $U_n := \binom{n}{2}^{-1} \sum_{l \neq m} h(Y_l, Y_m)$. The kernel $h$ is symmetric and real-valued, i.e. $h(y_2, y_1) = h(y_1, y_2)$ equals its complex conjugate.
h(y_1, y_2). Let us define the function \( h_1 : [0, 1] \rightarrow \mathbb{C}, y \mapsto h_1(y) := E_f(h(y, Y_2)) \). By Lemma A on p. 183 in Serfling [2009], the variance of the U-statistic \( U_n \) is determined by

\[
\text{var}_f(U_n) = \left( \frac{1}{2} \right)^{n-1} 2(n-2)\xi_1 + \xi_2 \quad \text{with} \quad \xi_1 := \text{var}_f(h_1(Y_1)) \text{ and } \xi_2 := \text{var}_f(h(Y_1, Y_2)).
\]

Next, we bound the two terms \( \xi_1 \) and \( \xi_2 \). Since \( h_1(y) = E_f(h(y, Y_2)) = \sum_{0 < |j| \leq k} c_j e_j(-y) \), we obtain by Parseval's identity

\[
\xi_1 \leq \mathbb{E}_f |h_1(Y_1)|^2 \leq \|g\|_\infty \|h_1\|_{\mathcal{L}^2}^2 = \|g\|_\infty \sum_{0 < |j| \leq k} \frac{|f_j|^2}{|\varphi_j|^2}.
\]

Now consider the term \( \xi_2 \). It holds

\[
\xi_2 = \text{var}_f(h(Y_1, Y_2)) \leq \mathbb{E}_f |h(Y_1, Y_2)|^2 \leq \|g\|_\infty \int \int |h(y_1, y_2)|^2 \, dy_1 \, g(y_2) \, dy_2
\]

where

\[
\int |h(y_1, y_2)|^2 \, dy_1 = \sum_{0 < |j| \leq k} \frac{1}{|\varphi_j|^2} \int_0^1 e_j(y_2 - y_1) e_j(y_2 - y_1) \, dy_1 = \sum_{0 < |j| \leq k} \frac{1}{|\varphi_j|^4}
\]

and, hence,

\[
\int \int |h(y_1, y_2)|^2 \, dy_1 \, g(y_2) \, dy_2 = \sum_{0 < |j| \leq k} \frac{1}{|\varphi_j|^4} \int g(y_2) \, dy_2 = \sum_{0 < |j| \leq k} \frac{1}{|\varphi_j|^4}
\]

Finally, combining the bounds for \( \xi_1 \) and \( \xi_2 \) yields

\[
\text{var}_f(\hat{a}_k^2) = \frac{1}{4} \text{var}_f(U_n) = \frac{2(n-2)\xi_1 + \xi_2}{2n(n-1)} \leq \frac{\|g\|_\infty \sum_{0 < |j| \leq k} |f_j|^2 + \|g\|_\infty \sum_{0 < |j| \leq k} \frac{1}{|\varphi_j|^4}}{n^2} \quad (2.3)
\]

where we used that \( \frac{1}{(n-1)^2} \leq \frac{1}{n} \) for \( n \geq 2 \). Together with (2.2), this proves the assertion.

The upper bound in (2.1) depends on the quantity \( c = \|g\|_\infty \leq \|\varphi\|_\infty \), which is uniformly bounded for all \( f \in \mathcal{D} \) if \( \|\varphi\|_\infty < \infty \). By additionally exploiting the regularity condition (1.5), we obtain a uniform bound for the risk, valid for all \( f \in \mathcal{E}_n^R \).

**Corollary 2.2 (Uniform upper bound for the risk of estimation).** Consider \( \nu_k^4 \) and \( r_o^4 \) as defined in (1.8) and (1.9), respectively. For \( n, k \in \mathbb{N}, n \geq 2 \) the estimator defined in (1.7) satisfies

\[
\sup_{f \in \mathcal{E}_n^R} E_f \left( \hat{a}_k^2 - a^2(f) \right)^2 \leq c_1 a_k^4 \lor c_2 \nu_k^4 \lor c_3 r_o^4 \quad (2.4)
\]

with \( c_1 := 3R^4, \ c_2 := 3(\|\varphi\|_\infty + R^2), \ c_3 := 3\|\varphi\|_\infty R^2 \).

**Proof of Corollary 2.2.** We exploit the upper bound in (2.1). Since the sequence \( a \) is non-increasing, the first term on the right-hand side in (2.1) (the bias term) is bounded by

\[
\sum \frac{|f_j|^2}{|j| > k} \sum \frac{|f_j|^2}{|j| > k} a_{|j|}^2 \leq \sum \frac{|f_j|^2}{|j| > k} a_{|j|}^2 \leq R^2 a_k^2.
\]
To bound the second term on the right-hand side of (2.1), we bound each summand, i.e. for each $j \in \mathbb{N}$ we have $\frac{1}{n} |f_j|^2 \leq \frac{|f_j|^2}{a_j^2} \left(1 + \frac{1}{n|\varphi_j|^2} a_j^2\right)$ if $n|\varphi_j|^2 a_j^2 \geq 1$ and $\frac{1}{n} |f_j|^2 \leq \frac{R^2}{n^2|\varphi_j|^4}$ otherwise. Hence, we obtain a bound for the entire sum

$$\frac{1}{n} \sum_{0 < |j| \leq k} \frac{|f_j|^2}{|\varphi_j|^2} \leq \sum_{0 < |j| \leq k} \frac{|f_j|^2}{a_j^2} \left(1 + \frac{1}{n|\varphi_j|^2} a_j^2\right) + \frac{R^2}{n^2} \sum_{0 < |j| \leq k} \frac{1}{|\varphi_j|^4} \leq R^2 r_0^4 + R^2 \nu_k^4.$$

Combining both bounds yields the assertion. \hfill $\square$

**Remark 2.3 (Optimal choice of the dimension parameter).** The first two terms in the upper bound (2.4) depend on the dimension parameter $k \in \mathbb{N}$, whereas the last term $c_3 r_0^4$ does not. It plays the role of a base-level error, which causes the well-known elbow effect in quadratic functional estimation (cp. also Illustration 2.6 below). It can easily be seen that $r_0^4$ is always of order larger than $\frac{1}{n}$. In other words, no matter the choice of $k$ the estimation rate can never be faster than parametric. The first two terms, however, depend on $k \in \mathbb{N}$ and can therefore be optimised. We define the optimal dimension

$$\kappa_* = \min \left\{ k \in \mathbb{N} : a_k^4 \leq \frac{1}{n^2} \sum_{0 < |j| \leq k} \frac{1}{|\varphi_j|^4} \right\},$$

as the $k$ that achieves an optimal bias-variance trade-off. \hfill $\square$

**Theorem 2.4 (Upper bound for the minimax risk of estimation).** For $n \geq 2$ and $\kappa_*$ as in (2.5)

$$r^2(\hat{\mathcal{C}}^R_a) \leq r^2(\hat{\mathcal{C}}^R_{\kappa_*}) \leq C \left( \rho_*^4 \lor r_0^4 \right)$$

with $C := R^4 + \|\varphi\|_\infty + R^2 + \|\varphi\|_\infty R^2$. \hfill $\square$

**Proof of Theorem 2.4.** We apply Corollary 2.2 to $\hat{\mathcal{C}}^R_{\kappa_*}$ with $\kappa_*$ as in (2.5). \hfill $\square$

We now provide an additional upper bound for the variance of the estimator (1.7), which is used in the next section to derive an upper bound for the testing radius.

**Corollary 2.5 (Upper bound for the variance).** Let $f_0 = \mathbf{1}_{[0,1]}$ and $f \in \mathcal{D}$. For $n, k \in \mathbb{N}, n \geq 2$ and $\nu_k^2$ as in (1.8) the estimator defined in (1.7) satisfies

$$\text{var}_{f_0}(\hat{q}_k^2) \leq \nu_k^4,$$

$$\text{var}_f(\hat{q}_k^2) \leq \|\varphi\|_\infty \cdot q_k^2(f) \nu_k^2 + \|\varphi\|_\infty \cdot \nu_k^2.$$  

**Proof of Corollary 2.5.** We use the bound (2.3) derived in the proof of Proposition 2.1. The first term on the right hand side can be bounded due to the Cauchy-Schwarz inequality by

$$\sum_{0 < |j| \leq k} \frac{|f_j|^2}{|\varphi_j|^2} \leq \left( \sum_{0 < |j| \leq k} \frac{|f_j|^4}{\varphi_j^4} \right)^{1/2} \left( \sum_{0 < |j| \leq k} \frac{1}{\varphi_j^2} \right)^{1/2} \leq \sum_{0 < |j| \leq k} \frac{|f_j|^2}{\varphi_j^2} \left( \sum_{0 < |j| \leq k} \frac{1}{\varphi_j^2} \right)^{1/2} = q_k^2(f)n\nu_k^2,$$

exploiting $\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}$ for any $x, y \geq 0$ in the last inequality. Combining this bound with $\|g\|_\infty \leq \|\varphi\|_\infty$ shows (2.8). Additionally, for $f = f_0 = \mathbf{1}_{[0,1]}$, and hence $g = \mathbf{1}_{[0,1]}$, we have $\|g\|_\infty = 1$ and $q_k^2(f) = 0$, which proves (2.7). \hfill $\square$
Illustration 2.6. Throughout the paper we illustrate the order of the estimation risk under the following typical smoothness and ill-posedness assumptions for the density of interest $f$ and the noise density $\varphi$, respectively. For two real-valued sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ we write $x_n \lesssim y_n$ if there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$, $x_n \leq cy_n$. We write $x_n \sim y_n$, if both $x_n \lesssim y_n$ and $y_n \lesssim x_n$. We call $y_n$ the order of $x_n$. Concerning the class $\mathcal{E}_a^R$ we distinguish two behaviours of the sequence $a$, namely the ordinary smooth case $a_j \sim j^{-s}$ for $s > 1/2$ where $\mathcal{E}_a^R$ corresponds to a Sobolev ellipsoid, and the super smooth case $a_j \sim \exp(-j^p)$ for $s > 0$, corresponding to a class of analytic functions. We also distinguish two cases for the regularity of the error density $\varphi$. For $p > 1/2$ we consider a mildly ill-posed model $|\varphi_j| \sim |j|^{-p}$ and for $p > 0$ a severely ill-posed model $|\varphi_j| \sim \exp(-|j|^p)$. Many examples of circular densities can be found in Chapter 3 of Mardia and Jupp [2009]. The table below presents the order of the upper bound for $r_*^2$ in (2.6), in Section 5 we provide a matching lower bound, thus establishing the rate-optimality of the upper bound. The derivations of the risk bounds can be found in Appendix A.3.

| $a_j$ (smoothness) | $|\varphi_j|$ (ill-posedness) | $\rho_*^4$ | $r_*^4$ | $r_*^2$ |
|-------------------|-------------------|---------|--------|--------|
| $j^{-s}$ | $|j|^{-p}$ | $n^{-\frac{8s}{4s+4p+1}}$ | $n^{-\frac{8s}{4s+4p+1}}$ if $s - p < 0$ | $n^{-\frac{8s}{4s+4p+1}}$ if $s - p \geq 0$ |
| $j^{-s}$ | $e^{-|j|^p}$ | $(\log n)^{-\frac{4s}{p}}$ | $(\log n)^{-\frac{4s}{p}}$ | $(\log n)^{-\frac{4s}{p}}$ |
| $e^{-j^p}$ | $|j|^{-p}$ | $n^{-2(\log n)^{\frac{4p}{s}}}$ | $n^{-1}$ | $n^{-1}$ |

\[ \Box \]

3 Upper bound for the radius of testing

In this section we derive an upper bound for the radius of testing of the task (1.3). We consider the test $\Delta_{a,k} = I\{\hat{q}^2_k \geq c \alpha \hat{\varphi}^2_k\}$ defined in (1.8), that is based on the estimator $\hat{q}_k^2$ in (1.7) of the distance $\|f_o - f\|_{L_2}^2$ to the null hypothesis.

Proposition 3.1 (Upper bound for the radius of testing of $\Delta_{a,k}$). Let $\alpha \in (0,1)$, $c := \|\varphi\|_\infty$ and $C_\alpha, \tilde{A}_\alpha \in \mathbb{R}_+$ be such that

\[ \frac{2C_\alpha + 1}{C_\alpha^2} c \leq \frac{\alpha}{2} \quad \text{and} \quad \frac{2C_\alpha + 1}{(\tilde{A}_\alpha - C_\alpha)^2} c \leq \frac{\alpha}{2}. \]  

(3.1)

Set $\tilde{A}_\alpha^2 := R^2 + \tilde{A}_\alpha^2$. Then, for all $A \geq \tilde{A}_\alpha$ and all $k \in \mathbb{N}$ we obtain

\[ \mathcal{R}(\Delta_{a,k} \mid \mathcal{E}_a^R, A \rho_k) \leq \alpha, \]  

(3.2)

i.e. $\rho_k^2$ is an upper bound for the radius of testing of $\Delta_{a,k}$.

Remark 3.2 (Choice of $C$ and $\tilde{A}_\alpha$). In particular, (3.1), and, hence, Proposition 3.1 is satisfied for $C_\alpha = 6\alpha^{-1}\|\varphi\|_\infty$ and $\tilde{A}_\alpha = C_\alpha + 2\alpha^{-1}\sqrt{12\|\varphi\|_\infty^2\alpha^{-1} + \|\varphi\|_\infty}$. \[ \Box \]

Proof of Proposition 3.1. We show that both the type I error probability and the type II error probability of the test (1.8) are bounded. Consider first the type I error probability.
Applying first Markov’s inequality and then the second inequality (2.7) from Corollary 2.5, we obtain
\[
\mathbb{P}_{f_a}(\Delta_{a,k} = 1) = \mathbb{P}_{f_a}(q_k^2 \geq C_a \nu_k^2) \leq \frac{\mathbb{E}_{f_a}(\hat{q}_k^2)^2}{C_a^2 \nu_k^4} = \frac{\text{var}_{f_a}(\hat{q}_k^2)}{C_a^2 \nu_k^4} \leq \frac{1}{C_a^2} \leq \frac{\alpha}{2},
\]
for all \( C_a \) satisfying (3.1), since \( ||\varphi||_{\infty} \geq 1 \). Next, we consider the type II error probability. Let \( f \) be contained in the \((\bar{\alpha}_a \rho_k)\)-separated alternative, i.e. \( f \in \mathcal{E}_a^R \) and \( q^2(f) \geq (\bar{\alpha}_a)^2 \rho_k^2 \). We expand
\[
\mathbb{P}_f(\Delta_{a,k} = 0) = \mathbb{P}_f(q_k^2 < C_a \nu_k^2) = \mathbb{P}_f(\hat{q}_k^2 - q_k^2(f) < C_a \nu_k^2 - q_k^2(f))
\]
and distinguish the following two cases for the density \( f \)

1. \( q_k^2(f) \geq 2C_a \nu_k^2 \), \hspace{1cm} \text{(easy to test)}
2. \( q_k^2(f) < 2C_a \nu_k^2 \), \hspace{1cm} \text{(difficult to test)}

**Case 1. (easy to test)** We have \( C_a \nu_k^2 - q_k^2(f) \leq -\frac{1}{2}q_k^2(f) \) and, therefore, due to Markov’s inequality
\[
\mathbb{P}_f(\Delta_{a,k} = 0) \leq \mathbb{P}_f(\hat{q}_k^2 - q_k^2(f) \leq -\frac{1}{2}q_k^2(f)) = \mathbb{P}_f(q_k^2(f) - \hat{q}_k^2 \geq \frac{1}{2}q_k^2(f)) \leq 4 \frac{\text{var}_f(\hat{q}_k^2)}{(q_k^2(f))^2}.
\]
On the one hand, by the case distinction, we have \( q_k^2(f) \geq 2C_a \nu_k^2 \), on the other hand we have \( \text{var}_f(\hat{q}_k^2) \leq c \nu_k^2 + \nu_k^4 \) with \( c = ||\varphi||_{\infty} \) due to (2.8) in Corollary 2.5. Hence,
\[
\mathbb{P}_f(\Delta_{a,k} = 0) \leq 4 \frac{c \nu_k^2(f) \nu_k^2 + \nu_k^4}{(q_k^2(f))^2} = 4 \left( \frac{c \nu_k^2}{q_k^2(f)} + \frac{\nu_k^4}{(q_k^2(f))^2} \right) \leq 4 \left( \frac{c}{2C_a^2} + \frac{1}{2C_a^2} \right) = \frac{2c}{C_a^2} \leq \frac{\alpha}{2}.
\]

**Case 2. (difficult to test)** Under the alternative exploiting \( q_2(f) = \sum_{0<|j|<\infty} |f_j|^2 \geq (\bar{\alpha}_a)^2 \rho_k^2 \) and \( \sum_{|j|>k} |f_j|^2 \leq a_k^2 R^2 \), it follows
\[
q_k^2(f) = q_k^2(f) - \sum_{|j|>k} |f_j|^2 \geq (\bar{\alpha}_a)^2 \nu_k^2 - a_k^2 R^2 = \bar{\alpha}_a^2 \nu_k^2 + a_k^2 R^2 - a_k^2 R^2 = \bar{\alpha}_a^2 \nu_k^2.
\]
Hence, due to Markov’s inequality, the type II error probability satisfies
\[
\mathbb{P}_f(\Delta_{a,k} = 0) = \mathbb{P}_f(\hat{q}_k^2 - q_k^2(f) \leq C_a \nu_k^2 - q_k^2(f)) \leq \mathbb{P}_f(\hat{q}_k^2 - q_k^2(f) \leq (C_a - \bar{\alpha}_a)^2 \nu_k^2) = \mathbb{P}_f(-\hat{q}_k^2 + q_k^2(f) \geq (-C_a + \bar{\alpha}_a)^2 \nu_k^2) \leq \frac{\text{var}_f(\hat{q}_k^2)}{(\bar{\alpha}_a^2 - C_a^2) \nu_k^4}.
\]
By (2.7) in Corollary 2.5, the case distinction and the choice of \( \bar{\alpha}_a \) in (3.1), it follows
\[
\mathbb{P}_f(\Delta_{a,k} = 0) \leq \frac{c q_k^2(f) \nu_k^2 + \nu_k^4}{(\bar{\alpha}_a^2 - C_a^2) \nu_k^4} \leq \frac{2cC_a + c}{(\bar{\alpha}_a^2 - C_a^2) ^2} \leq \frac{\alpha}{2}.
\]
Combining the last bound and (3.3), we obtain the assertion, which completes the proof. \( \square \)
From Proposition 3.1 with \( \kappa_* \) as in (2.5) and \( \rho_* \) as in (1.11), we immediately obtain the following corollary and, hence, omit the proof.

**Corollary 3.3 (Upper bound for the minimax radius of testing).** Under the conditions of Proposition 3.1 for all \( A \geq A_{\alpha} \) we obtain

\[
R(\mathcal{E}_{a}^{R}, A\rho_{\star}) \leq R(\Delta_{\alpha, \kappa_{\star}}| \mathcal{E}_{a}^{R}, A\rho_{\star}) \leq \alpha,
\]

i.e. \( \rho_{\star}^{2} \) is an upper bound for the minimax radius of testing.

**Illustration 3.4.** We illustrate the order of the upper bound for the radius of testing \( \rho_{\star}^{2} \) derived in Corollary 3.3 under the typical smoothness and ill-posedness assumptions introduced in Illustration 2.6. Comparing the next table with Illustration 2.6, we emphasize that there is no elbow effect. The derivation of the bounds is similar to the ones in Illustration 2.6 and is thus omitted.

| Order of the minimax radius of testing \( \rho_{\star}^{2} \) |  
|---|---|---|  
| \( a_{j} \) (smoothness) | \( |\varphi_{j}| \) (ill-posedness) | \( \rho_{\star}^{2} \)  
| \( j^{-s} \) | \( |j|^{-p} \) | \( n^{-\frac{4s+4p+1}{2s}} \)  
| \( j^{-s} \) | \( e^{-|j|^{p}} \) | \( (\log n)^{-\frac{2s}{p}} \)  
| \( e^{-j^{s}} \) | \( |j|^{-p} \) | \( n^{-1}(\log n)^{\frac{4p+1}{2s}} \)  

□

4 Lower bound for the radius of testing

In this section we prove a matching lower bound for the radius of testing. The proof is inspired by Assouad’s cube technique (see Tsybakov [2009], Chapter 2.7 for an explanation of the technique in the estimation case), where the testing risk is reduced to a distance between probability measures. It requires the construction of \( 2^{\kappa_{\star}} \) candidates (called hypotheses) in the class \( \mathcal{E}_{a}^{R} \), which are vertices on a hypercube. Roughly speaking, they are constructed such that they are statistically indistinguishable from the null \( f_{0} \), while having largest possible \( L^{2} \)-distance.

**Proposition 4.1 (Lower bound for the radius of testing).** Assume \( a := 2 \sum_{j \in \mathbb{N}} a_{j}^{2} < \infty \) and let \( \eta \in (0, 1] \) satisfy

\[
\left( a_{n_{\alpha}}^{2} \lor \nu_{n_{\alpha}}^{2} \right) \eta \leq \left( a_{n_{\alpha}}^{2} \land \nu_{n_{\alpha}}^{2} \right) \eta \leq \left( a_{n_{\alpha}}^{2} \lor \nu_{n_{\alpha}}^{2} \right) \eta.  
\]

(4.1)

For \( \alpha \in (0, 1) \) define \( A_{\alpha}^{2} := \eta \left( R^{2} \land \sqrt{\log(1 + 2\alpha^{2})} \land a^{-1} \right) \). Then, for all \( A \leq A_{\alpha} \)

\[
R(\mathcal{E}_{a}^{R}, A\rho_{\star}) \geq 1 - \alpha,
\]

i.e. \( \rho_{\star}^{2} \) is a lower bound for the minimax radius of testing.

**Proof of Proposition 4.1.** Reduction Step. To prove a lower bound for the testing radius we reduce the risk of a test to a distance between probability measures. Denote \( \mathbb{P}_{0} := \mathbb{P}_{f_{0}} \) and let
\(\mathbb{P}_1\), specified below, be a mixing measure over the \(A_\alpha\rho_*\)-separated alternative. The minimax risk can then be lower bounded by applying a classical reduction argument as follows

\[
\mathcal{R}(\mathcal{E}_\alpha^R, A_\alpha\rho_*) \geq \inf_\Delta \left( \mathbb{P}_0(\Delta = 1) + \mathbb{P}_1(\Delta = 0) \right) = 1 - TV(\mathbb{P}_0, \mathbb{P}_1) \geq 1 - \sqrt{\frac{\chi^2(\mathbb{P}_0; \mathbb{P}_1)}{2}}.
\]

where TV denotes the total variation distance and \(\chi^2\) the \(\chi^2\)-divergence. The last inequality follows e.g. from Lemma 2.5 combined with (2.7) in Tsybakov [2009].

**Definition of the mixtures.** On the alternative, we mix the Fourier coefficients uniformly over the vertices of a hypercube. Consider \(f \in \mathcal{E}_\alpha^R \cap \mathcal{L}_2^2\), with coefficients \(f_0 = 1, f_j = 0\) for \(|j| > \kappa_*\) and

\[
f_j := \frac{\sqrt{\zeta} \eta \rho_*}{\sqrt{\sum_{0 \leq |l| \leq \kappa_*} |\varphi_l|^2}} |\varphi_j|^2 \quad \text{for } 0 < |j| \leq \kappa_*
\]

with \(\zeta = R^2 \wedge \sqrt{\log(1 + 2\alpha^2)} \wedge a^{-1}\). For a sign vector \(\tau \in \{\pm\}^{\kappa_*}\), we define \(f^\tau \in \mathcal{E}_\alpha^R \cap \mathcal{L}_2^2\), through its Fourier coefficients \(f_0^\tau = 1, f_j^\tau = \tau_j f_j\) for \(0 < |j| \leq \kappa_*\) and \(f_j^\tau = 0\) otherwise. The quadratic functionals \(q^2(f^\tau) = q^2(f)\) and \(q_{\rho_*}^2(f^\tau) = q_{\rho_*}^2(f)\), \(k \in \mathbb{N}\) are invariant under \(\tau\). The resulting mixing measure is given by \(\mathbb{P}_1 := 2^{-\kappa_*} \sum_{\tau \in \{\pm\}^{\kappa_*}} \mathbb{P}_{f^\tau}\). Summarizing, \(f^\tau\) satisfies:

(a) \(\sum_{j \in \mathbb{Z}} |f_j|^2 < \infty\), for all \(\tau \in \{\pm\}^{\kappa_*}\), by construction. \hfill (\in \mathcal{L}^2)

(b) \(f_j^\tau = f^{-\tau}_j\), for all \(\tau \in \{\pm\}^{\kappa_*}\), by construction. \hfill (real-valued)

(c) \(f_0^\tau = 1\), for all \(\tau \in \{\pm\}^{\kappa_*}\), by construction. \hfill (normalized to 1)

(d) \(\sum_{|j| > 0} |f_j^\tau|^2 \leq 1\), for all \(\tau \in \{\pm\}^{\kappa_*}\), since

\[\sum_{|j| > 0} |f_j^\tau|^2 \leq \sum_{|j| > 0} a_j^2 \leq 1, \]

by the Cauchy-Schwarz inequality, since \(\sum_{|j| > 0} a_j^2 \leq \sqrt{\sum_{|j| > 0} a_j^2} \leq 1\), \(\zeta \leq 1\), \(\zeta \leq 1\), \(\zeta \leq 1\), \(\zeta \leq 1\), \(\zeta \leq 1\), \(\zeta \leq 1\), \(\zeta \leq 1\).

where the second last inequality follows as in (e).

(e) \(f \in \mathcal{E}_\alpha^R\), i.e. \(2 \sum_{j \in \mathbb{N}} a_j^2 |f_j|^2 \leq R^2\), by the monotonicity of \(a\), since

\[2 \sum_{j \in \mathbb{N}} a_j^2 |f_j|^2 \leq \sum_{0 < |l| \leq \kappa_*} \sum_{0 < |j| \leq \kappa_*} \sum_{0 \leq |l| \leq \kappa_*} |\varphi_l|^2 |\varphi_j|^2 \leq \zeta \leq R^2.
\]

(f) \(f \in \mathcal{L}_2^2\), i.e. \(q_{\kappa_*}(f) \geq A_\alpha\rho_*\), since

\[q_{\rho_*}^2(f) = \sum_{0 < |l| \leq \kappa_*} \sum_{0 \leq |j| \leq \kappa_*} |\varphi_l|^2 |\varphi_j|^2 \leq \zeta \leq R^2.
\]

(g) \(n^2 \sum_{0 < |j| \leq \kappa_*} |f_j|^4 |\varphi_j|^4 \leq \log(1 + 2\alpha^2)\), since

\[n^2 \sum_{0 < |j| \leq \kappa_*} |f_j|^4 |\varphi_j|^4 \leq \zeta^2 \leq \log(1 + 2\alpha^2).
\]

The conditions (a)-(d) guarantee that the vertices are densities, (e) and (f) guarantee that the vertices lie in the alternative.

**Bound of the \(\chi^2\)-divergence.** We apply Lemma A.2 in the appendix and obtain

\[
\chi^2 \left( \frac{1}{2\kappa_*} \sum_{\tau \in \{\pm\}} \mathbb{P}_{f^\tau}, \mathbb{P}_0 \right) \leq \exp \left( 2n^2 \sum_{j=1}^{\kappa_*} |g_j|^4 \right) - 1 = \exp \left( n^2 \sum_{0 < |j| \leq \kappa_*} |f_j|^4 |\varphi_j|^4 \right) - 1
\]

Hence, (g) guarantees that the induced distance between the mixing measure and the null is negligible. Combined with the reduction step, it follows \(\mathcal{R}(\mathcal{E}_\alpha^R, A_\alpha\rho_*) \geq 1 - \alpha\). \qed
Remark 4.2 (Conditions on $\eta$ and $a$). Proposition 4.1 involves the value $\eta$ satisfying (4.1), which depends on the joint behaviour of the sequences $\{a_j\}_{j \in \mathbb{N}}$ and $\{\varphi_j\}_{j \in \mathbb{Z}}$ and essentially guarantees an optimal balance of the bias and the variance term in the dimension $\kappa$. For all the typical smoothness and ill-posedness assumptions considered in Illustration 3.4 an $\eta$ exists such that (4.1) holds uniformly for all $n \in \mathbb{N}$. The additional assumption $a = 2 \sum_{j \in \mathbb{N}} a_j^2 < \infty$ in Proposition 4.1 is needed to ensure that the candidate densities constructed in the reduction scheme of the proof are indeed densities. This assumption is in particular satisfied for the typical smoothness classes introduced in Illustration 2.6. For Sobolev-type alternatives, i.e. $a_j \sim j^{-2s}$, $j \in \mathbb{N}$ it is satisfied as soon as $s > 1/2$, for analytic alternatives, i.e. $a_j \sim \exp(-j^s)$, $j \in \mathbb{N}$ it is satisfied for all positive $s$.

5 Lower bound for the estimation risk

In this section we first explore the connection between quadratic functional estimation and testing. Every estimator for the functional $q^2(f) = \|f_o - f\|^2_{\mathcal{F}^2}$ can be used to construct a test by rejecting the null as soon as the estimated value of the quadratic functional exceeds a certain threshold. The next proposition shows how this connection can be formalized in terms of the minimax risk and the minimax radius.

**Proposition 5.1 (Testing is faster than quadratic functional estimation).** Let $\alpha \in (0, 1)$, $\mathcal{E} \subseteq \mathcal{L}^2$ be a class of functions and $\rho^2(\mathcal{E})$ a minimax radius of testing with $\mathcal{A}_\alpha$ as in the lower bound definition. Then, the minimax risk of estimation satisfies

$$r^2(\mathcal{E}) \geq (1 - \alpha) \frac{\mathcal{A}_\alpha^2}{8} \cdot \rho^4(\mathcal{E}).$$

**Proof of Proposition 5.1.** Let $\hat{q}^2$ be any estimator of $q^2(f)$. Define the test $\Delta := \mathbb{1}\{q^2 \geq \rho/2\}$ with $\rho = \mathcal{A}_\alpha \rho(\mathcal{E})$. We convert the mean squared error into the sum of type I and type II error probabilities, i.e. the testing risk, by applying Markov’s inequality. Keeping in mind that $q^2(f_o) = 0$, we have

$$r^2(\hat{q}^2, \mathcal{E}) = \sup \mathbb{E}_f \left( \hat{q}^2 - q^2(f_o) \right)^2 \geq \frac{1}{2} \left( \mathbb{E}_f \left( \hat{q}^2 \right)^2 + \sup \mathbb{E}_f \left( \hat{q}^2 - q^2(f) \right)^2 \right)$$

$$\geq \frac{\rho^4}{8} \left( \mathbb{P}_{f_o} \left( \hat{q}^2 \geq \frac{\rho^2}{2} \right) + \sup \mathbb{P}_f \left( q^2(f) - \hat{q}^2 \geq \frac{\rho^2}{2} \right) \right)$$

$$\geq \frac{\rho^4}{8} \left( \mathbb{P}_{f_o} \left( \hat{q}^2 \geq \frac{\rho^2}{2} \right) + \sup \mathbb{P}_f \left( \hat{q}^2 \leq \frac{\rho^2}{2} \right) \right) = \frac{\rho^4}{8} \mathcal{R}(\Delta \mid \mathcal{E}, \mathcal{A}_\alpha \rho(\mathcal{E})).$$

Since $\hat{q}^2$ is arbitrary and by definition $\mathcal{R}(\mathcal{E}, \mathcal{A}_\alpha \rho(\mathcal{E})) \geq 1 - \alpha$, we obtain the result. 

Recall that the upper bound for the risk of estimation in (2.4) is of order $\rho^4 \vee r^4_\circ$. There are two possible scenarios, either the risk is governed by the term $\rho^4_\circ = \min_k \{a^4_k \vee \nu^4_k\}$ or by the baseterm $r^4_\circ = \max_m \{a^4_m \left( 1 \wedge \frac{1}{n a^2_n \|f_m\|} \right) \}$. We separate lower bounds for these two cases. The lower bound in the first case is an immediate consequence of Proposition 5.1 combined with Proposition 4.1 and we omit its proof.
Corollary 5.2 (First lower bound for the risk of estimation). Let $\eta \in (0, 1]$ satisfy (4.1). Then, for all $n \geq 2$

$$r^2(\mathcal{E}_n^R) \geq \frac{\eta^2 (R^4 \wedge \log(3/2))}{16} \min_{k \in \mathbb{N}} \left\{ a_k^4 \vee \frac{1}{n^2} \sum_{0 \leq |j| \leq k} \frac{1}{|\varphi|^4} \right\}.$$ 

In contrast to the lower bound proved in Proposition 4.1, the proof of the next proposition only requires the construction of two candidate densities.

Proposition 5.3 (Second lower bound for the risk of estimation). For all $n \geq 2$ we have

$$r^2(\mathcal{E}_n^R) \geq \left( \frac{1}{64} \wedge \frac{R^4}{16} \right) \max_{m \in \mathbb{N}} \left\{ a_m^4 \left( 1 \wedge \frac{1}{n a_m^2 |\varphi_m|} \right) \right\}.$$ 

Proof of Proposition 5.3. Reduction Step. Denoting by $\mathbb{Q}_f$ the measure with density $f \odot \varphi$, the measure $\mathbb{P}_f$ associated with the observations equals the $n$-fold product measure of $\mathbb{Q}_f$. Let $f^+, f^- \in \mathcal{D}$ (to be specified below) with associated $\mathbb{P}_{f^+}$, $\mathbb{P}_{f^-}$ and quadratic functionals $p^2 = q^2(f^+)$ and $q^2 = q^2(f^-)$. Denote by $h(\mathbb{P}_{f^+}, \mathbb{P}_{f^-}) = \int \sqrt{d\mathbb{P}_{f^+} d\mathbb{P}_{f^-}}$ the Hellinger affinity between the two measures $\mathbb{P}_{f^+}$ and $\mathbb{P}_{f^-}$. We apply the reduction scheme in Lemma A.3 and obtain

$$r^2(\mathcal{E}_n^R) \geq \frac{1}{8} h^2(\mathbb{P}_{f^+}, \mathbb{P}_{f^-})(p^2 - q^2)^2.$$ 

(5.1)

Using the tensorization property of the Hellinger affinity and the definition of the Hellinger distance (cp. for instance Tsybakov [2009], p. 83), it follows $h(\mathbb{P}_{f^+}, \mathbb{P}_{f^-}) = \left( h(\mathbb{Q}_{f^+}, \mathbb{Q}_{f^-}) \right)^n = \left( 1 - \frac{1}{2} H^2(\mathbb{Q}_{f^+}, \mathbb{Q}_{f^-}) \right)^n$. Denoting $g^+ := f^+ \odot \varphi$, we will ensure that $g^- \geq \frac{1}{2}$ and $\|g^+ - g^-\|_{\mathcal{L}^2} \leq 1$. Hence,

$$H^2(\mathbb{Q}_{f^+}, \mathbb{Q}_{f^+}) = \int \frac{(g^+(x) - g^-(x))^2}{\left( \sqrt{g^+(x)} + \sqrt{g^-(x)} \right)^2} dx \leq 2 \left\| g^+ - g^- \right\|_{\mathcal{L}^2}^2$$

and by Bernoulli’s inequality $h^2(\mathbb{P}_{f^+}, \mathbb{P}_{f^-}) \geq 1 - 2n \|g^+ - g^-\|_{\mathcal{L}^2}^2$. From (5.1) it follows

$$r^2(\mathcal{E}_n^R) \geq \frac{1}{8} (p^2 - q^2)^2 \left( 1 - 2n \left\| f^+ \odot \varphi - f^- \odot \varphi \right\|_{\mathcal{L}^2}^2 \right).$$

(5.2)

Construction of the hypotheses $f^+$, $f^-$. Let $\tau \in \{\pm\}$ and let $m$ be arbitrary. Define the Fourier coefficients of the hypotheses $f^\tau$, $\tau \in \{\pm\}$ by

$$f^\tau_j \begin{cases} 1 & \text{if } j = 0 \\ (1 + \xi) Ca_m & \text{if } j = \pm m \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_j^- \begin{cases} 1 & \text{if } j = 0 \\ (1 - \xi) Ca_m & \text{if } j = \pm m \\ 0 & \text{otherwise} \end{cases}$$

with $C := \frac{1}{2} \wedge \frac{R}{\sqrt{8}}$ and $\xi := 1 \wedge \frac{1}{na_m^2 |\varphi_m|}$. Then, the hypotheses $f^\tau$, $\tau \in \{\pm\}$ satisfy the following conditions:

1. $f^\tau \in \mathcal{D}$,

(a) $\sum_{j \in \mathbb{Z}} f_j^\tau |j|^2 < \infty$, by construction,
(b) \( f_j^+ = \overline{f_j} \); by construction, \((\text{real-valued})\)
(c) \( f_0^- = 1 \); by construction. \((\text{normalized to 1})\)
(d) \( \sum_{|j| > 0} |f_j^+| = 2(1 + \xi)Ca_m \leq 2 \cdot 2Ca_m \leq 4C \leq 1 \) \((\text{positive})\)
(e) \( \sum_{|j| > 0} |f_j^-| |\varphi_j| = 2(1 - \xi)Ca_m |\varphi_m| \leq 2C \leq \frac{1}{2} \) \((\text{bounded from below})\)

2. \( f^r \in \mathcal{E}_a^R \)

(f) \( 2 \sum_{j \in \mathbb{N}} a_j^{-2} |f_j^+|^2 = 2a_m^{-2}(1 + \xi)^2C^2a_m^2 \leq 8C^2 \leq R^2 \) \((\text{smoothness})\)

3. We have \( q^2(f^r) = \sum_{|j| > 0} (f_j^r)^2 = 2(1 + \xi)^2a_m^2 \), therefore

(g) \( \|f^+ \otimes \varphi - f^- \otimes \varphi\|_{\mathcal{E}_a^R}^2 = 4C^2\xi^2a_m^2 |\varphi_m|^2 \leq 4C^2\frac{1}{n} \leq \frac{1}{4n} \) \((\text{separation})\)

4. (h) \( \|f^+ \otimes \varphi - f^- \otimes \varphi\|_{\mathcal{E}_a^R} \leq 4C^2a_m \left(1 \wedge \frac{1}{na_m |\varphi_m|^2}\right) \) for all \( m \in \mathbb{N} \), which proves the assertion.

\[ \square \]

A  Appendix

A.1  Auxiliary results for proving lower bounds of testing

**Lemma A.1 (Switching sums and products on cubes).** For \( k \in \mathbb{N} \) let \( J_j^+, J_j^- \), \( j \in \{1, \ldots, k\} \) be real numbers. Then,

\[ \frac{1}{2k} \sum_{\tau \in \{\pm\}^k} \prod_{j=1}^k J_j^{\tau_j} = \prod_{j=1}^k \frac{J_j^- + J_j^+}{2}. \]

**Proof of Lemma A.1.** The proof is by induction on \( k \). The base case \( k = 1 \) follows immediately. For the induction step, assume \( \frac{1}{2k} \sum_{\tau \in \{\pm\}^k} \prod_{j=1}^k J_j^{\tau_j} = \prod_{j=1}^k \frac{J_j^- + J_j^+}{2} \). Then,

\[ \frac{1}{2k+1} \sum_{\tau \in \{\pm\}^{k+1}} \prod_{j=1}^{k+1} J_j^{\tau_j} = \frac{1}{2k+1} \left( \left( \sum_{\tau \in \{\pm\}^k} \prod_{j=1}^k J_j^{\tau_j} \right) \cdot J_{k+1}^+ + \left( \sum_{\tau \in \{\pm\}^k} \prod_{j=1}^k J_j^{\tau_j} \right) \cdot J_{k+1}^- \right) \]

\[ = \frac{1}{2} \left( J_{k+1}^+ + J_{k+1}^- \right) \left( \frac{1}{2k} \sum_{\tau \in \{\pm\}^k} \prod_{j=1}^k J_j^{\tau_j} \right) = \frac{1}{2} \left( J_{k+1}^+ + J_{k+1}^- \right) \prod_{j=1}^k \frac{J_j^- + J_j^+}{2} = \prod_{j=1}^{k+1} \frac{J_j^- + J_j^+}{2}, \]

where the induction assumption was used in the second last step. \( \square \)

**Lemma A.2 (\( \chi^2 \)-divergence for mixtures over hypercubes).** Let \( k \in \mathbb{N} \). For \( \tau \in \{\pm\}^k \) and \( \theta \in \ell^2(\mathbb{N}) \) we define the coefficients \( \theta^r = (\theta_j^r)_{j \in \mathbb{Z}} \) and functions \( f^r \in \mathcal{L}^2 \) by setting

\[ \theta_j^r := \begin{cases} \tau_j \theta_j & |j| \in \{1, \ldots, k\} \\ 1 & j = 0 \\ 0 & |j| > k \end{cases} \quad \text{and} \quad f^r := \sum_{j=-k}^{k} \theta_j^r e_j = 1_{[0,1]} + \sum_{0<|j|\leq k} \theta_j^r e_j. \]

15
Assuming $f^\tau \in \mathcal{D}$ for each $\tau \in \{\pm\}$, we consider the mixing measure $\mathbb{P}_\mu$ with probability density
\[
\frac{1}{\mathbb{P}_\mu} \sum_{\tau \in \{\pm\}^k} \prod_{i=1}^n f^\tau(z_i), z_i \in [0, 1], i \in \{1, \ldots, n\},
\]
and denote $\mathbb{P}_0 = \mathbb{P}_{f_\phi}$. Then, the $\chi^2$-divergence satisfies
\[
\chi^2(\mathbb{P}_\mu, \mathbb{P}_0) \leq \exp \left(2n^2 \sum_{j=1}^k \theta_j^4\right) - 1.
\]

Proof of Lemma A.2. Recall that $\chi^2(\mathbb{P}_\mu, \mathbb{P}_0) = \mathbb{E}_0 \left(\frac{d\mathbb{P}_\mu}{d\mathbb{P}_0}(Z_1, \ldots, Z_n)\right)^2 - 1$ for random variables $(Z_j)_{j \in \mathbb{N}}$ with marginal density $f_\phi = 1_{[0, 1]}$ under $\mathbb{P}_0$. Let $z_1, \ldots, z_n \in [0, 1)$, then the likelihood ratio becomes
\[
\frac{d\mathbb{P}_\mu}{d\mathbb{P}_0}(z_1, \ldots, z_n) = \frac{1}{2^k} \sum_{\tau \in \{\pm\}^k} \prod_{i=1}^n f^\tau(z_i).
\]

Squaring, taking the expectation under $\mathbb{P}_0$ and exploiting the independence yields
\[
\mathbb{E}_0 \left(\frac{d\mathbb{P}_\mu}{d\mathbb{P}_0}(Z_1, \ldots, Z_n)\right)^2 = \left(\frac{1}{2^k}\right)^2 \sum_{\eta, \tau \in \{\pm\}^k} (\mathbb{E}_0 (f^\tau(Z_1)f^n(Z_1)))^n.
\]

Let us calculate $\mathbb{E}_0 (f^\tau(Z_1)f^n(Z_1)) = \int_{[0,1]} f^\tau(z)f^n(z)dz = 1 + 2 \sum_{j=1}^k \theta_j^4 \theta_j^n$, where the last equality is due to the orthonormality of $(\epsilon_j)_{j \in \mathbb{Z}}$ and the symmetry of $\theta^\tau$ and $\theta^n$. Applying the inequality $1 + x \leq \exp(x)$, which holds for all $x \in \mathbb{R}$, we obtain
\[
\mathbb{E}_0 (f^\tau(Z_1)f^n(Z_1)) \leq \left(1 + 2 \sum_{j=1}^k \theta_j^4 \theta_j^n\right) \leq \exp \left(2 \sum_{j=1}^k \theta_j^4 \theta_j^n\right) = \prod_{j=1}^k \exp \left(2\theta_j^4 \theta_j^n\right).
\]

Hence,
\[
\mathbb{E}_0 \left(\frac{d\mathbb{P}_\mu}{d\mathbb{P}_0}(Z_1, \ldots, Z_n)\right)^2 \leq \left(\frac{1}{2^k}\right)^2 \sum_{\eta, \tau \in \{\pm\}^k} \prod_{j=1}^k \exp \left(2n\theta_j^4 \theta_j^n\right),
\]

where we can apply Lemma A.1 to the $\eta$-summation with $J_j^\eta = \exp \left(2n\theta_j^4 \theta_j^n\right)$ and obtain
\[
\mathbb{E}_0 \left(\frac{d\mathbb{P}_\mu}{d\mathbb{P}_0}(Z_1, \ldots, Z_n)\right)^2 \leq \left(\frac{1}{2^k}\right)^2 \sum_{\tau \in \{\pm\}^k} \left(\prod_{j=1}^k \exp \left(-2n\theta_j \theta_j^n\right) + \exp \left(2n\theta_j \theta_j^n\right)\right).
\]

Again applying Lemma A.1 now to the $\tau$-summation with $J_j^\tau = \frac{\exp(-2n\theta_j) + \exp(2n\theta_j)}{2}$ yields
\[
\mathbb{E}_0 \left(\frac{d\mathbb{P}_\mu}{d\mathbb{P}_0}(Z_1, \ldots, Z_n)\right)^2 \leq \prod_{j=1}^k \frac{\exp(-2n\theta_j^2) + \exp(2n\theta_j^2) + \exp(2n\theta_j^2) + \exp(-2n\theta_j^2)}{4}
\]
\[
= \prod_{j=1}^k \left(\frac{\exp(-2n\theta_j^2) + \exp(2n\theta_j^2)}{2}\right) = \prod_{j=1}^k \cosh \left(2n\theta_j^2\right).
\]

Since $\cosh(x) \leq \exp(x^2/2)$, we obtain
\[
\mathbb{E}_0 \left(\frac{d\mathbb{P}_\mu}{d\mathbb{P}_0}(Z_1, \ldots, Z_n)\right)^2 \leq \prod_{j=1}^k \exp \left(2n^2\theta_j^4\right) = \exp \left(2n^2 \sum_{j=1}^k \theta_j^4\right),
\]

which completes the proof. \qed
A.2 Auxiliary results for proving lower bounds of estimation

Lemma A.3 (Reduction scheme for the estimation risk). For densities \( f^+, f^- \in \mathcal{E} \subseteq \mathcal{D} \) we have

\[
\inf_{q} \sup_{f \in \mathcal{E}} \mathbb{E}_f \left( \hat{q}^2 - q^2(f) \right)^2 \geq \frac{1}{8} h^2(\mathbb{P}_{f^+}, \mathbb{P}_{f^-})(q^2(f^+) - q^2(f^-))^2, \tag{A.1}
\]

where \( h(\mathbb{P}_{f^+}, \mathbb{P}_{f^-}) \) denotes the Hellinger affinity between \( \mathbb{P}_{f^+} \) and \( \mathbb{P}_{f^-} \).

Proof of Lemma A.3. Let \( \hat{q}^2 \) be any estimator and denote \( \mathbb{P}_+ := \mathbb{P}_{f^+}, \mathbb{P}_- := \mathbb{P}_{f^-} \) and \( q^2 = q^2(f^+), \hat{q}^2 = q^2(f^-) \). We have

\[
h(\mathbb{P}_+, \mathbb{P}_-) = \int \sqrt{\text{d}\mathbb{P}_+ \text{d}\mathbb{P}_-} = \int \frac{|\hat{q}^2 - \mathbb{P}_+|}{\sqrt{|q^2 - \mathbb{P}_-|^2}} \sqrt{\text{d}\mathbb{P}_+ \text{d}\mathbb{P}_-}
\leq \left( \int \frac{|\hat{q}^2 - q^2|^2}{|q^2 - \mathbb{P}_-|^2} \text{d}\mathbb{P}_+ \right)^{\frac{1}{2}} \left( \int \text{d}\mathbb{P}_- \right)^{\frac{1}{2}} + \left( \int \frac{|\hat{q}^2 - \mathbb{P}_+|^2}{|q^2 - \mathbb{P}_-|^2} \text{d}\mathbb{P}_- \right)^{\frac{1}{2}} \left( \int \text{d}\mathbb{P}_+ \right)^{\frac{1}{2}}
\leq 2 |q^2 - \mathbb{P}_+|^{-1} \left( \mathbb{E}_{f^+} (\hat{q}^2 - q^2)^2 + \mathbb{E}_{f^-} (\hat{q}^2 - q^2)^2 \right)^{\frac{1}{2}}.
\]

Therefore,

\[
\sup_{f \in \mathcal{E}} \mathbb{E}_f \left( \hat{q}^2 - q^2(f) \right)^2 \geq \frac{1}{2} \left( \mathbb{E}_{f^+} (\hat{q}^2 - q^2)^2 + \mathbb{E}_{f^-} (\hat{q}^2 - q^2)^2 \right) \geq \frac{h^2(\mathbb{P}_+, \mathbb{P}_-)}{8} \left( q^2 - \mathbb{P}_+ \right)^2,
\]

which completes the proof. \( \square \)

A.3 Calculations for the risk bounds in Illustration 2.6

We determine the order of the terms \( r_4 \) and \( \rho_4^2 \) in (2.6) for each of the three combinations in Illustration 2.6 and determine the dominating term. Let \( m_* = \max \{m \in \mathbb{N} : a_m^4 \geq \frac{a_m^2}{n|\mathcal{P}_m|} \} \).

1. (ordinary smooth - mildly ill-posed) Consider first \( \rho_4^4 \) defined in (1.9). The variance term \( \nu_k^4 = \frac{1}{n^2} \sum_{0 < |j| \leq k} \frac{1}{|\mathcal{P}_j|} \sim \frac{1}{n^2} \sum_{0 < |j| \leq k} |j|^{4p} \) is of order \( \frac{1}{n^2} k^{4p+1} \) and the bias term \( a_k^4 \) is of order \( k^{-4s} \). Hence, the optimal \( \kappa_* \) satisfies \( \kappa_*^{-4s} \sim \frac{1}{n^2} a_k^{4p} \) and thus \( \kappa_* \sim n^{\frac{2}{4p+4s+1}} \), which yields an upper bound of order \( r_*^4 \sim \kappa_*^{-4s} \sim n^{-\frac{2}{4s+4p+1}} \). For the base level \( r_4^4 = \max_{m \in \mathbb{N}} \left\{ a_m^4 \wedge \frac{a_m^2}{n|\mathcal{P}_m|} \right\} \), the term \( \frac{a_m^2}{n|\mathcal{P}_m|} \sim \frac{1}{n} m^{2(p-s)} \) is monotonically increasing in \( m \) for \( p - s > 0 \) and monotonically non-increasing otherwise. Let \( p - s > 0 \), then \( m_* \) satisfies \( m_*^{-4s} \sim \frac{1}{n} m_*^{-2(s-p)} \) and is thus of order \( m_* \sim n^{\frac{2}{s-p}} \). Therefore, \( r_4^4 \sim n^{-\frac{s}{s-p}} \) is negligible compared with \( \rho_4^4 \). Let \( p - s \leq 0 \), then both \( a_m^4 \) and \( \frac{a_m^2}{n|\mathcal{P}_m|} \) are non-increasing. The maximum of their minimum is attained at \( m_* = 1 \), which yields \( r_4^4 \sim \frac{1}{n} \). Hence, \( r_4^4 \) is of larger order than \( \rho_4^4 \) for \( s - p > \frac{1}{4} \) only.

2. (ordinary smooth - severely ill-posed) Consider first \( \rho_4^4 \) defined in (1.9). The variance term \( \nu_k^4 = \frac{1}{n^2} \sum_{0 < |j| \leq k} \frac{1}{|\mathcal{P}_j|} \sim \frac{1}{n^2} \sum_{0 < |j| \leq k} \exp(|j|^{4p}) \) is of order \( \frac{1}{n^2} \exp(4k^p) \) and the bias term \( a_k^4 \) is of order \( k^{-4s} \). Hence, the optimal \( \kappa_* \) satisfies \( \kappa_*^{-4s} \sim \frac{1}{n^2} \exp(4k_*^p) \) and thus \( \kappa_* \sim \log(n^2/b_n)^{\frac{1}{p}} \).
3. (super smooth - mildly ill-posed) Consider first $\rho^4_*$ defined in (1.9). The term
\[ \frac{1}{n^2} \sum_{0<j<k} \frac{1}{|j|^{4p}} \] is of order $\frac{1}{n^2} k^{4p+1}$, whereas the bias term $a_k^4$ is of order $\exp(-4k^s)$. Hence, the optimal $\kappa_*$ satisfies $\exp(-4\kappa_*) \sim \frac{1}{n^2} k^{4p}$ and thus $\kappa_* \sim \log(n/b_n)\frac{1}{2}$ with $b_n \sim \log(n)\frac{1}{n^2}$, which yields an upper bound of order $r^2_* \sim \frac{1}{n^2} k^{4p+1} \sim \frac{1}{n} \log(n)^{4\nu+1}$. Considering the base level $r^4_* = c_3 \max_{m \in \mathbb{N}} \left\{ a^4_m \wedge \frac{a^2_m}{n|\varphi_m|^2} \right\}$, the term $\frac{a^2_m}{n|\varphi_m|^2} \sim \frac{m^{2s}}{n} \exp(2m^p)$ is monotonically decreasing in $m$. Hence, $m_*$ satisfies $m_*^{-4s} \sim \frac{1}{n^2} m_*^{4s} \exp(2m^p)$ and thus $m_* \sim \log(n/b_n)\frac{1}{n^2}$. Therefore, $r^4_* \sim \log(n)\frac{1}{n^2}$ is of the same order as $\rho^4_*$.

References

Y. Baraud. Non-asymptotic minimax rates of testing in signal detection. *Bernoulli*, 8(5): 577–606, 2002.

P. J. Bickel and Y. Ritov. Estimating integrated squared density derivatives: sharp best order of convergence estimates. *Sankhyā: The Indian Journal of Statistics, Series A*, pages 381–393, 1988.

L. Birge and P. Massart. Estimation of integral functionals of a density. *The Annals of Statistics*, 23(1):11–29, 02 1995.

C. Butucea. Goodness-of-fit testing and quadratic functional estimation from indirect observations. *The Annals of Statistics*, 35(5):1907–1930, 2007.

C. Butucea and K. Meziani. Quadratic functional estimation in inverse problems. *Statistical Methodology*, 8(1):31–41, 2011.

T. T. Cai and M. G. Low. Nonquadratic estimators of a quadratic functional. *The Annals of Statistics*, 33(6):2930–2956, 2005.

T. T. Cai and M. G. Low. Optimal adaptive estimation of a quadratic functional. *The Annals of Statistics*, 34(5):2298–2325, 2006.

C. Chesneaux. On adaptive wavelet estimation of a quadratic functional from a deconvolution problem. *Annals of the Institute of Statistical Mathematics*, 63(2):405–429, 2011.

O. Collier, L. Comminges, and A. B. Tsybakov. Minimax estimation of linear and quadratic functionals on sparsity classes. *The Annals of Statistics*, 45(3):923–958, 2017.

F. Comte and M.-L. Taupin. *Adaptive density deconvolution for circular data*. 2003.

S. Efromovich. Density estimation for the case of supersmooth measurement error. *Journal of the American Statistical Association*, 92(438):526–535, 1997.

N. I. Fisher. *Statistical analysis of circular data*. Cambridge University Press, 1995.
J. Gill and D. Hangartner. Circular data in political science and how to handle it. *Political Analysis*, 18(3):316–336, 2010.

Y. Ingster. Asymptotically minimax hypothesis testing for nonparametric alternatives I. *Mathematical Methods of Statistics*, 2(2):85–114, 1993a.

Y. Ingster. Asymptotically minimax hypothesis testing for nonparametric alternatives II. *Mathematical Methods of Statistics*, 2(2):171–189, 1993b.

Y. Ingster. Asymptotically minimax hypothesis testing for nonparametric alternatives III. *Mathematical Methods of Statistics*, 2(2):249–268, 1993c.

J. Johannes and M. Schwarz. Adaptive circular deconvolution by model selection under unknown error distribution. *Bernoulli*, 19(5A):1576–1611, 2013.

G. Kerkyacharian, T. M. Pham Ngoc, and D. Picard. Localized spherical deconvolution. *The Annals of Statistics*, 39(2):1042–1068, 04 2011.

M. Kroll. Rate optimal estimation of quadratic functionals in inverse problems with partially unknown operator and application to testing problems. *ESAIM: Probability and Statistics*, 23:524–551, 2019.

C. Lacour and T. M. P. Ngoc. Goodness-of-fit test for noisy directional data. *Bernoulli*, 20(4):2131–2168, 2014.

B. Laurent. Adaptive estimation of a quadratic functional of a density by model selection. *ESAIM: Probability and Statistics*, 9:1–18, 2005.

B. Laurent and P. Massart. Adaptive estimation of a quadratic functional by model selection. *The Annals of Statistics*, 28(5):1302–1338, 2000.

B. Laurent, J.-M. Loubes, and C. Marteau. Non asymptotic minimax rates of testing in signal detection with heterogeneous variances. *Electronic Journal of Statistics*, 6:91–122, 2012.

K. V. Mardia. *Statistics of directional data*. Academic press, 1972.

K. V. Mardia and P. E. Jupp. *Directional statistics*, volume 494. John Wiley & Sons, 2009.

C. Marteau and T. Sapatinas. Minimax goodness-of-fit testing in ill-posed inverse problems with partially unknown operators. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, 53(4):1675–1718, 2017.

R. J. Serfling. *Approximation theorems of mathematical statistics*, volume 162. John Wiley & Sons, 2009.

A. B. Tsybakov. *Introduction to Nonparametric Estimation*. Springer New York, 2009.