Overlapping iterated function systems from the perspective of Metric Number Theory

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Abstract

In this paper we develop a new approach for studying overlapping iterated function systems. This approach is inspired by a famous result due to Khintchine from Diophantine approximation. This result shows that for a family of limsup sets, their Lebesgue measure is determined by the convergence or divergence of naturally occurring volume sums. For many parameterised families of overlapping iterated function systems, we prove that a typical member will exhibit similar Khintchine like behaviour. Families of iterated function systems our results apply to include those arising from Bernoulli convolutions, the \{0, 1, 3\} problem, and affine contractions with varying translation parameter. As a by-product of our analysis we obtain new proofs of well known results due to Solomyak on the absolute continuity of Bernoulli convolutions, and when the attractor in the \{0, 1, 3\} problem has positive Lebesgue measure.

For each $t \in [0, 1]$ we let $\Phi_t$ be the iterated function system given by

$$\Phi_t := \{ \phi_1(x) = \frac{x}{2}, \phi_2(x) = \frac{x + 1}{2}, \phi_3(x) = \frac{x + t}{2}, \phi_4(x) = \frac{x + 1 + t}{2} \}.$$ 

We include a detailed study of this family. We prove that either $\Phi_t$ contains an exact overlap, or we observe Khintchine like behaviour. Our analysis of this family shows that by studying the metric properties of limsup sets, we can distinguish between the overlapping behaviour of iterated function systems in a way that is not available to us by simply studying properties of self-similar measures.

Last of all, we introduce a property of an iterated function system that we call being consistently separated with respect to a measure. We prove that this property implies that the pushforward of the measure is absolutely continuous. We include several explicit examples of consistently separated iterated function systems.

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1 Introduction

Attractors generated by iterated function systems are among the first fractal sets a mathematician encounters. The familiar middle third Cantor set and the Koch curve can both be realised as attractors for appropriate choices of iterated function system. Attractors generated by iterated function systems have the property that they are equal to several scaled down copies of themselves. When these copies are disjoint, or satisfy some weaker separation assumption, then much can be said about the attractors metric and topological properties. However, when these copies overlap significantly the situation is much more complicated. Measuring how an iterated function system overlaps, and determining properties of the corresponding attractor, are two important problems that are occupying much current research (see for example [27, 28, 51, 52, 54, 61, 62]).
The purpose of this paper is to develop a new approach for measuring how an iterated function system overlaps. This approach is inspired by classical results from Diophantine approximation and metric number theory. One such result due to Khintchine demonstrates that for a class of limsup sets defined in terms of the rational numbers, their Lebesgue measure is determined by the convergence or divergence of naturally occurring volume sums (see [34]). Importantly this result provides a quantitative description of how the rational numbers are distributed within $\mathbb{R}$.

In this paper we study limsup sets that are defined using iterated function systems (for their definition see Section 1.3). We are motivated by the following goals:

1. We would like to determine whether it is the case that for a parameterised family of iterated function systems, a typical member will satisfy an appropriate analogue of Khintchine’s theorem.

2. We would like to answer the question: Does studying the metric properties of limsup sets allow us to distinguish between the overlapping behaviour of iterated function systems in a way that was not previously available?

3. We would like to understand how the metric properties of limsup sets relates to traditional methods for measuring how an iterated function system overlaps, such as the dimension and absolute continuity of self-similar measures.

In this paper we make progress with each of these goals. Theorem 2.2, Theorem 2.6, and Theorem 2.9 address the first goal. These results demonstrate that for many parameterised families of overlapping iterated function systems, it is the case that a typical member will satisfy an appropriate analogue of Khintchine’s theorem. To help illustrate this point, and to motivate what follows, we include here a result which follows from Theorem 2.2.

**Theorem 1.1.** For Lebesgue almost every $\lambda \in (1/2, 0.668)$, Lebesgue almost every $x \in [\frac{1}{1-\lambda}, \frac{1}{1+\lambda}]$ is contained in

$$\left\{ x \in \mathbb{R} : \left| x - \sum_{i=1}^{m} d_i \lambda^{i-1} \right| < \frac{1}{2^m \cdot m} \text{ for infinitely many } (d_i)_{i=1}^{m} \in \bigcup_{n=1}^{\infty} \{-1, 1\}^{n} \right\}.$$ 

Theorem 2.10 allows us to answer the question stated in our second goal in the affirmative. See the discussion in Section 2.3.1 for a more precise explanation. Theorem 2.15 addresses the third goal. It shows that if we are given some measure $\mathfrak{m}$, and our IFS satisfies a strong version of Khintchine’s theorem with respect to $\mathfrak{m}$, then the pushforward of $\mathfrak{m}$ must be absolutely continuous. Moreover, we demonstrate with several examples that this strong version of Khintchine’s theorem is not equivalent to the absolute continuity of the pushforward.

In the rest of this introduction we provide some more background to this topic, and introduce the limsup sets that will be our main object of study.

### 1.1 Attractors generated by iterated function systems

We call a map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ a contraction if there exists $r \in (0, 1)$ such that $|\phi(x) - \phi(y)| \leq r|x - y|$ for all $x, y \in \mathbb{R}^d$. We call a finite set of contractions an iterated function system or IFS for short. A well known result due to Hutchinson [29] states that given an IFS $\Phi = \{\phi_i\}_{i=1}^{l}$, then there exists a unique, non-empty, compact set $X$ satisfying

$$X = \bigcup_{i=1}^{l} \phi_i(X).$$

We call $X$ the attractor generated by $\Phi$. When an IFS satisfies $\phi_i(X) \cap \phi_j(X) = \emptyset$ for all $i \neq j$, or is such that there exists an open set $O \subset \mathbb{R}^d$, for which $\phi_i(O) \subset O$ for all $i$ and
\( \phi_i(O) \cap \phi_j(O) = \emptyset \) for all \( i \neq j \), then many important properties of \( X \) can be determined (see [19]). This latter property is referred to as the open set condition. Without these separation assumptions determining properties of the attractor can be significantly more complicated.

The study of attractors generated by iterated function systems is classical within fractal geometry. One of the most important problems in this field is to determine the metric properties of attractors generated by overlapping iterated function systems. To understand the properties of an attractor \( X \), in both the overlapping case and non-overlapping case, it is useful to study measures supported on \( X \). A particularly distinguished role is played by the measures described below, that are in a sense dynamically defined.

Let \( \pi : \{1, \ldots, l\}^N \to X \) be given by

\[
\pi((a_j)_{j=1}^\infty) := \lim_{n \to \infty} (\phi_{a_1} \circ \cdots \circ \phi_{a_n})(0).
\]

The map \( \pi \) is surjective and is also continuous when \( \{1, \ldots, l\}^N \) is equipped with the product topology. The sequence space \( \{1, \ldots, l\}^N \) comes with a natural left-shift map \( \sigma : \{1, \ldots, l\}^N \to \{1, \ldots, l\}^N \) defined via the equation \( \sigma((a_j)_{j=1}^\infty) = (a_{j+1})_{j=1}^\infty \). Given a finite word \( a = (a_1, \ldots, a_n) \), we associate its cylinder set

\[
[a] := \{(b_j) \in \{1, \ldots, l\}^N : b_1, \ldots, b_n = a_1, \ldots, a_n\}.
\]

We call a measure \( m \) on \( \{1, \ldots, l\}^N \) \( \sigma \)-invariant if \( m([a]) = m(\sigma^{-1}([a])) \) for all finite words \( a \). We call a probability measure \( m \) ergodic if \( \sigma^{-1}(A) = A \) implies \( m(A) = 0 \) or \( m(A) = 1 \). Given a measure \( m \) on \( \{1, \ldots, l\}^N \), we obtain the corresponding pushforward measure \( \mu \) supported on \( X \) using the map \( \pi \), i.e. \( \mu = m \circ \pi^{-1} \).

We define the dimension of a measure \( \mu \) on \( \mathbb{R}^d \) to be

\[
\dim \mu = \inf \{\dim_H(A) : \mu(A) > 0\}.
\]

Note that for any pushforward measure \( \mu \) we have \( \dim \mu \leq \dim_H(X) \). The problem of determining \( \dim_H(X) \) is often solved by finding a \( \sigma \)-invariant ergodic probability measure whose pushforward has dimension equal to some known upper bound for \( \dim_H(X) \). This approach is especially useful when the iterated function system is overlapping.

When studying attractors of iterated function systems, one of the guiding principles is that if there is no obvious mechanism preventing an attractor from satisfying a certain property, then one should expect this property to be satisfied. This principle is particularly prevalent in the many conjectures which state that under certain reasonable assumptions, the Hausdorff dimension of \( X \), and the Hausdorff dimension of dynamically defined pushforward measures supported on \( X \), should equal the value asserted by a certain formula. A particular example of this phenomenon is provided by self-similar sets and self-similar measures. We call a contraction \( \phi \) a similarity if there exists \( r \in (0, 1) \) such that \( |\phi(x) - \phi(y)| = r|x - y| \) for all \( x, y \in \mathbb{R}^d \). If an IFS \( \Phi \) consists of similarities then it is known that

\[
\dim_H(X) \leq \min\{\dim_S(\Phi), d\},
\]

(1.1)

where \( \dim_S(\Phi) \) the unique solution to \( \sum_{i=1}^l r_i^s = 1 \). Given a probability vector \( p = (p_1, \ldots, p_l) \), we let \( m_p \) denote the corresponding Bernoulli measure supported on \( \{1, \ldots, l\}^N \). If an IFS consists of similarities, then we define the self-similar measure corresponding to \( p \) to be \( \mu_p := m_p \circ \pi^{-1} \).

The measure \( \mu_p \) can also be defined as the unique measure satisfying the equation

\[
\mu_p = \sum_{i=1}^l p_i \cdot \mu_p \circ \phi_i^{-1}.
\]
For any self-similar measure $\mu_p$, we have the upper bound:

$$\dim \mu_p \leq \min \left\{ \frac{\sum_{i=1}^{l} p_i \log p_i}{\sum_{i=1}^{l} p_i \log r_i}, d \right\}. \quad (1.2)$$

For an appropriate choice of $p$, it can be shown that equality in (1.2) implies equality in (1.1). An important conjecture states that if an IFS consisting of similarities avoids certain degenerate behaviour, then we should have equality in (1.2) for all $p$, and therefore equality in (1.1) (see [27, 28]). In $\mathbb{R}$ this conjecture can be stated succinctly as: If an IFS does not contain an exact overlap, then we should have equality in (1.2) for all $p$. Recall that an IFS is said to contain an exact overlap if there exists two distinct words $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_m)$ such that

$$\phi_{a_1} \circ \cdots \circ \phi_{a_n} = \phi_{b_1} \circ \cdots \circ \phi_{b_m}.$$

In [27] and [28] significant progress was made with this conjecture. In particular in [27] it was shown that in $\mathbb{R}$, if strict inequality holds in (1.2) for some $p$, then

$$\lim_{n \to \infty} \frac{-\log \Delta_n}{n} = \infty,$$

where

$$\Delta_n := \min_{a \neq b \in \{1, \ldots, l\}^n} |(\phi_{a_1} \circ \cdots \circ \phi_{a_n})(0) - (\phi_{b_1} \circ \cdots \circ \phi_{b_n})(0)|.$$

Using this statement it can be shown that if the parameters defining our IFS are algebraic, and there are no exact overlaps, then equality holds in (1.2) for all $p$, and therefore also in (1.1).

In addition to expecting equality to hold typically in (1.2), it is expected that if

$$\frac{\sum p_i \log p_i}{\sum p_i \log r_i} > d,$$

and the IFS avoids certain obstacles, then $\mu_p$ will be absolutely continuous with respect to $d$-dimensional Lebesgue measure. A standard technique for proving an attractor has positive $d$-dimensional Lebesgue measure is by showing there is an absolutely continuous pushforward measure. Note that by a recent result of Simon and Vágó [57], it follows that the list of mechanisms leading to the failure of absolute continuity is strictly greater than the list of mechanisms leading to the failure of equality in (1.2).

The usual methods for gauging how an iterated function system overlaps are to determine whether the Hausdorff dimension of the attractor satisfies a certain formula, to determine whether the dimension of pushforwards of dynamically-defined measures satisfy a certain formula, and to determine whether these measures are absolutely continuous with respect to the Lebesgue measure. If an IFS did not exhibit the expected behaviour, then this would be indicative of something degenerate within our IFS that was either preventing $X$ from being well spread out within $\mathbb{R}^d$, or was forcing mass from the pushforward measure into some small subregion of $\mathbb{R}^d$. This method for gauging how an iterated function system overlaps has its limitations. If each of the expected behaviours described above occurs for two distinct IFSs within a family, then we have no method for distinguishing their overlapping behaviour. The approach put forward in this paper shows how we can still make a distinction (see the discussion in Section 2.3.1). As previously stated this approach is inspired by results from Diophantine approximation and metric number theory. We now take the opportunity to briefly recall some background from this area.
1.2 Diophantine approximation and metric number theory

Given \( \Psi : \mathbb{N} \to [0, \infty) \) we can define a limsup set defined in terms of neighbourhoods of rationals as follows. Let

\[
J(\Psi) := \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| \leq \Psi(q) \text{ for i.m. } (p, q) \in \mathbb{Z} \times \mathbb{N} \right\}.
\]

Here and throughout we use i.m. as a shorthand for infinitely many. If \( x \in J(\Psi) \) we say that \( x \) is \( \Psi \)-approximable. An immediate application of the Borel-Cantelli lemma implies that if \( \sum_{q=1}^{\infty} q \cdot \Psi(q) < \infty \), then \( J(\Psi) \) has zero Lebesgue measure. The following theorem due to Khintchine shows that a partial converse to this statement holds. This theorem motivates much of the present work.

**Theorem 1.2** (Khintchine [34]). If \( \Psi : \mathbb{N} \to [0, \infty) \) is decreasing and

\[
\sum_{q=1}^{\infty} q \cdot \Psi(q) = \infty,
\]

then Lebesgue almost every \( x \in \mathbb{R} \) is \( \Psi \)-approximable.

Results analogous to Khintchine’s theorem are ubiquitous in Diophantine approximation and metric number theory. We refer the reader to [8] for more examples.

By an example of Duffin and Schaeffer, it can be seen that it is not possible to remove the decreasing assumption from Theorem 1.2. Indeed in [15] they constructed a \( \Psi \) such that \( \sum_{q=1}^{\infty} q \cdot \Psi(q) = \infty \), yet \( J(\Psi) \) has zero Lebesgue measure. This gave rise to the conjecture stated below which has received much attention.

**Conjecture 1.3** (Duffin and Schaeffer). If \( \Psi : \mathbb{N} \to [0, \infty) \) satisfies

\[
\sum_{q=1}^{\infty} \varphi(q) \cdot \Psi(q) = \infty,
\]

then Lebesgue almost every \( x \in \mathbb{R} \) is \( \Psi \)-approximable.

Here \( \varphi \) is the Euler totient function. For some recent developments on this conjecture we refer the reader to [1, 9, 26].

By studying the Lebesgue measure of \( J(\Psi) \) for those \( \Psi \) satisfying \( \sum_{q=1}^{\infty} q \cdot \Psi(q) = \infty \), we obtain a quantitative description of how the rationals are distributed within the reals. The example of Duffin and Schaeffer demonstrates that there exists some interesting non-trivial interactions occurring between fractions of different denominator.

1.3 Two families of limsup sets

Before defining the limsup sets we study in this paper it is necessary to introduce some notation. In what follows we let

\[
\mathcal{D} := \{1, \ldots, l\}, \quad \mathcal{D}^* := \bigcup_{j=1}^{\infty} \{1, \ldots, l\}^j, \quad \mathcal{D}^N := \{1, \ldots, l\}^N.
\]

Given an IFS \( \Phi = \{\phi_i\}_{i \in \mathcal{D}} \) and \( \mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{D}^* \), let

\[
\phi_{\mathbf{a}} = \phi_{a_1} \circ \cdots \circ \phi_{a_n}.
\]

Let \( |\mathbf{a}| \) denote the length of \( \mathbf{a} \in \mathcal{D}^* \). If \( \Phi \) has attractor \( X \), then for each \( \mathbf{a} \in \mathcal{D}^* \) let

\[
X_\mathbf{a} := \phi_{\mathbf{a}}(X).
\]
1.3.1 The set $W_{\Phi}(z, \Psi)$

Given an IFS $\Phi, \Psi : D^* \to [0, \infty)$, and an arbitrary $z \in X$, we let

$$W_{\Phi}(z, \Psi) := \left\{ x \in \mathbb{R}^d : |x - \phi_a(z)| \leq \Psi(a) \text{ for i.m. } a \in D^* \right\}.$$ 

Throughout this paper we will always have the underlying assumption that $\Psi$ satisfies

$$\lim_{n \to \infty} \max_{a \in D^n} \Psi(a) = 0.$$ 

This condition guarantees $W_{\Phi}(z, \Psi) \subseteq X$.

The study of the metric properties of $W_{\Phi}(z, \Psi)$ will be one of the main focuses of this paper. Proceeding via analogy with Khintchine’s theorem, it is natural to wonder what metric properties of $W_{\Phi}(z, \Psi)$ are encoded in the volume sum:

$$\sum_{n=1}^{\infty} \sum_{a \in D^n} \Psi(a)^{\dim_H(X)}.$$ (1.3)

It is an almost immediate consequence of the definition of Hausdorff measure that if we have convergence in (1.3), then $\mathcal{H}^{\dim_H(X)}(W_{\Phi}(z, \Psi)) = 0$ for all $z \in X$. Given the results mentioned in the previous section, it is reasonable to expect that divergence in (1.3) might imply some metric property of $W_{\Phi}(z, \Psi)$ which demonstrates that a typical element of $X$ is contained in $W_{\Phi}(z, \Psi)$. A classification of those $\Psi$ for which divergence in (1.3) implies a typical element of $X$ is contained in $W_{\Phi}(z, \Psi)$ would provide a quantitative description of how the images of $z$ are distributed within $X$. This in turn provides a description of how the underlying iterated function system overlaps. This idea provides us with a new tool for describing the overlapping behaviour of iterated function systems. We refer the reader to Section 2.3.1 for further discussions which demonstrate the utility of this idea.

The question of whether divergence in (1.3) implies a typical element of $X$ is contained in $W_{\Phi}(z, \Psi)$ was studied previously by the author in [4, 3, 5]. Related work appears in [35, 44, 45]. In [3] the following theorem was proved:

**Theorem 1.4.** [3, Theorem 1.4] If $\Phi$ is a conformal iterated function system and satisfies the open set condition, then for any $z \in X$, if $\theta : \mathbb{N} \to [0, \infty)$ is a decreasing function and satisfies

$$\sum_{n=1}^{\infty} \sum_{a \in D^n} (\text{Diam}(X_a)\theta(n))^{\dim_H(X)} = \infty,$$

then $\mathcal{H}^{\dim_H(X)}$-almost every $x \in X$ is contained in $W_{\Phi}(z, \text{Diam}(X_a)\theta(|a|))$.

Note that for a conformal iterated function system it is known that the open set condition implies $0 < \mathcal{H}^{\dim_H(X)}(X) < \infty$ (see [39]). For the definition of a conformal iterated function system see Section 2.3. Note that an iterated function system consisting of similarities is automatically a conformal iterated function system. In [3, Theorem 6.1] it was also shown that if $\Phi$ is a conformal iterated function system and contains an exact overlap, then there exist many natural choices of $\Psi$ such that we have divergence in (1.3), yet $\dim_H(W_{\Phi}(z, \Psi)) < \dim_H(X)$. As such an exact overlap effectively prevents any Khintchine like behaviour.

In [4] and [5] the author studied the family of IFSs $\Phi_\lambda := \{ \lambda x, \lambda x + 1 \}$, where $\lambda \in (1/2, 1)$. For each element of this family the corresponding attractor is $[0, 1/\lambda]$. In [4] the author proved that if the reciprocal of $\lambda$ belongs to a special class of algebraic integers known as Garzia
numbers, then for a general class of \( \Psi \), divergence in (1.3) implies that for all \( z \in [0, \frac{1}{1-\lambda}] \), Lebesgue almost every \( x \in [0, \frac{1}{1-\lambda}] \) is contained in \( W_{\Phi, \lambda}(z, \Psi) \). For more on this result and Garsia numbers we refer the reader to Section \[9\] where this result is recovered using a different argument. The main result of \[3\] provides strong evidence to suggest that for a general class of \( \Psi \), for a typical \( \lambda \in (1/2, 1) \), we should expect that divergence in (1.3) implies that Lebesgue almost every \( x \in [0, \frac{1}{1-\lambda}] \) is contained in \( W_{\Phi, \lambda}(z, \Psi) \). A consequence of the main result of \[3\] is that for Lebesgue almost every \( \lambda \in (1/2, 0.668) \), for all \( z \in [0, \frac{1}{1-\lambda}] \), Lebesgue almost every \( x \in [0, \frac{1}{1-\lambda}] \) is contained in \( W_{\Phi, \lambda}(z, \log[a]/2m) \). Note that the results in \[4\] and \[5\] are phrased for \( z = 0 \) but can easily be adapted to the case of arbitrary \( z \in [0, \frac{1}{1-\lambda}] \).

1.3.2 The set \( U_\Phi(z, m, h) \)

Instead of studying the sets \( W_\Phi(z, \Psi) \) directly it is more profitable to study a related family of auxiliary sets. These sets are interesting in their own right and are defined in terms of a measure \( m \) supported on \( D^N \). Our approach doesn’t work for all \( m \) and we will require the following additional regularity assumption.

Given a probability measure \( m \) supported on \( D^N \), we say that \( m \) is slowly decaying if

\[
\text{ess inf}_{(a_j) \sim m} \inf_{k \in \mathbb{N}} \frac{m([a_1, \ldots, a_{k+1}])}{m([a_1, \ldots, a_k])} > 0.
\]

If \( m \) is slowly decaying we let

\[
c_m := \text{ess inf}_{(a_j) \sim m} \inf_{k \in \mathbb{N}} \frac{m([a_1, \ldots, a_{k+1}])}{m([a_1, \ldots, a_k])}.
\]

By definition \( c_m > 0 \) if \( m \) is slowly decaying. If \( m \) is slowly decaying, then for \( m \)-almost every \( (a_j) \in D^N \), we have

\[
\frac{m([a_1, \ldots, a_{k+1}])}{m([a_1, \ldots, a_k])} \geq c_m,
\]

for all \( k \in \mathbb{N} \). Examples of slowly decaying measures include Bernoulli measures, and Gibbs measures for Holder continuous potentials (see \[12\]). In fact any measure with the quasi-Bernoulli property is slowly decaying.

Given a slowly decaying probability measure \( m \), for each \( n \in \mathbb{N} \) we let

\[
L_{m,n} := \{ a \in D^* : m([a_1, \ldots, a_{|a|}]) \leq c_m^n < m([a_1, \ldots, a_{|a|-1}]) \}
\]

and

\[
P_{m,n} := \#L_{m,n}.
\]

The elements of \( L_{m,n} \) are disjoint and the union of their cylinders has full \( m \) measure. Importantly, by the slowly decaying property, the cylinders corresponding to elements of \( L_{m,n} \) have comparable measure up to a multiplicative constant. Note that when \( m \) is the uniform \((1/l, \ldots, 1/l)\) Bernoulli measure the set \( L_{m,n} \) is simply \( D^n \).

Given \( z \in X \) and a slowly decaying probability measure \( m \), we let

\[
Y_{m,n}(z) := \{ \phi_a(z) \} \in L_{m,n}.
\]

Obtaining information on how the elements of \( Y_{m,n}(z) \) are distributed within \( X \) for different values of \( n \) will occupy a large part of this paper.

Given a slowly decaying measure \( m \), an IFS \( \Phi, h : \mathbb{N} \rightarrow [0, \infty) \), and \( z \in X \), we can define a limsup set as follows. Let

\[
U_\Phi(z, m, h) := \left\{ x \in \mathbb{R}^d : |x - \phi_a(z)| \leq (m([a])h(n))^{1/d} \text{ for i.m. } a \in \bigcup_{n=1}^{\infty} L_{m,n} \right\}.
\]
Throughout this paper we will always assume that $m$ is non-atomic and $h$ is a bounded function. These properties ensure

$$U_{\Phi}(z, m, h) \subseteq X.$$ 

In this paper we study the metric properties of the sets $U_{\Phi}(z, m, h)$ for parameterised families of IFSs when the underlying attractor typically has positive $d$-dimensional Lebesgue measure. In which case, for the set $U_{\Phi}(z, m, h)$, the appropriate volume sum that we expect to determine the Lebesgue measure of $U_{\Phi}(z, m, h)$ is

$$\sum_{n=1}^{\infty} h(n).$$

It can be shown using the Borel-Cantelli lemma that if $\sum_{n=1}^{\infty} h(n) < \infty$, then $U_{\Phi}(z, m, h)$ has zero Lebesgue measure. For us the interesting question is: When does $\sum_{n=1}^{\infty} h(n) = \infty$ imply $U_{\Phi}(z, m, h)$ has positive or full Lebesgue measure?

The sets $U_{\Phi}(z, m, h)$ are easier to work with than then the sets $W_{\Phi}(z, \Psi)$. In particular we can use properties of the measure $m$ to aid with our analysis. As we will see, the sets $U_{\Phi}(z, m, h)$ can be used to prove results for the sets $W_{\Phi}(z, \Psi)$, but only under the following additional assumption.

Given a slowly decaying measure $m$ and $h : \mathbb{N} \to [0, \infty)$, we say that $\Psi$ is equivalent to $(m, h)$ if $\Psi(a) \approx (m([a])h(n))^{1/d}$ for all $a \in \cup_n L_{m,n}$. Here and throughout, for two real valued functions $f$ and $g$ defined on some set $S$, we write $f \asymp g$ if there exists a positive constant $C$ such that

$$C^{-1} \cdot g(x) \leq f(x) \leq Cg(x)$$

for all $x \in S$. As we will see, if $\Psi$ is equivalent to $(m, h)$ and $U_{\Phi}(z, m, h)$ has positive Lebesgue measure, then $W_{\Phi}(z, \Psi)$ will also have positive Lebesgue measure (see Lemma 3.7).

2 Statement of results

Before stating our theorems we need to define the entropy of a measure $m$ supported on $\mathcal{D}^N$ and introduce a class of functions that are the natural setting for some of our results.

For any $\sigma$-invariant measure $m$ supported on $\mathcal{D}^N$, we define the entropy of $m$ to be

$$h(m) := \lim_{n \to \infty} -\frac{1}{n} \sum_{a \in \mathcal{D}^n} m([a]) \log m([a]).$$

The entropy of a $\sigma$-invariant measure always exists.

Given a set $B \subset \mathbb{N}$, we define the lower density of $B$ to be

$$\underline{d}(B) := \liminf_{n \to \infty} \frac{\# \{1 \leq j \leq n : j \in B \}}{n},$$

and the upper density of $B$ to be

$$\overline{d}(B) := \limsup_{n \to \infty} \frac{\# \{1 \leq j \leq n : j \in B \}}{n}.$$

Given $\epsilon > 0$, let

$$H_\epsilon^* := \left\{ h : \mathbb{N} \to [0, \infty) : \sum_{n \in B} h(n) = \infty, \forall B \subseteq \mathbb{N} \text{ s.t. } \overline{d}(B) > 1 - \epsilon \right\}.$$

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and

\[ H_\epsilon := \left\{ h : \mathbb{N} \to [0, \infty) : \sum_{n \in B} h(n) = \infty, \forall B \subseteq \mathbb{N} \text{ s.t. } \overline{d}(B) > 1 - \epsilon \right\}. \]

We also define

\[ H^* := \bigcup_{\epsilon \in (0,1)} H^*_\epsilon \quad (2.1) \]

and

\[ H := \bigcup_{\epsilon \in (0,1)} H_\epsilon. \quad (2.2) \]

For any \( \epsilon > 0 \) we have \( H_\epsilon \subset H^*_\epsilon \). Therefore \( H \subset H^* \). It can be shown that \( H^* \) contains all decreasing functions satisfying \( \sum_{n=1}^{\infty} h(n) = \infty \). Most of the time we will be concerned with the class of functions \( H \). The class \( H^* \) will only appear in Theorem 2.10.

We say that a function \( \Psi : \mathcal{D}^* \to [0, \infty) \) is weakly decaying if

\[ \inf_{a \in \mathcal{D}^*} \min_{i \in \mathcal{D}} \frac{\Psi(ia)}{\Psi(a)} > 0. \]

Given a measure \( \mu \) supported on \( \mathcal{D}^\mathbb{N} \), we let

\[ \Upsilon_{\mu} := \{ \Psi : \mathcal{D}^* \to [0, \infty) : \Psi \text{ is weakly decaying and equivalent to } (\mu, h) \text{ for some } h \in H \}. \]

As we will see, the weakly decaying property will allow us to obtain full measure statements.

### 2.1 Parameterised families with variable contraction ratios

Let \( D := \{d_1, \ldots, d_l\} \) be a finite set of real numbers. To each \( \lambda \in (0,1) \), we associate the iterated function system

\[ \Phi_{\lambda,D} := \{ \phi_i(x) = \lambda x + d_i \}. \]

It is straightforward to check that the corresponding attractor for \( \Phi_{\lambda,D} \) is

\[ X_{\lambda,D} := \left\{ \sum_{j=0}^{\infty} d_j \lambda^j : d_j \in D \right\}, \]

and the projection map \( \pi_{\lambda,D} : \mathcal{D}^\mathbb{N} \to X_{\lambda,D} \) takes the form

\[ \pi_{\lambda,D}((a_j)_{j=1}^{\infty}) = \sum_{j=1}^{\infty} d_{a_j} \lambda^{j-1}. \]

To study this family of iterated function systems, it is useful to study the set \( \Gamma := D - D \) and the corresponding class of power series

\[ B_{\Gamma} := \left\{ g(x) = \sum_{j=0}^{\infty} g_j x^j : g_j \in \Gamma \right\}. \]

To each \( B_{\Gamma} \) we associate the set

\[ \Lambda(B_{\Gamma}) := \left\{ \lambda \in (0,1) : \exists g \in B_{\Gamma}, g \neq 0, g(\lambda) = g'(\lambda) = 0 \right\}. \]

In other words, \( \Lambda(B_{\Gamma}) \) is the set of \( \lambda \in (0,1) \) that can be realised as a double zero for a non-trivial function in \( B_{\Gamma} \). We let

\[ \alpha(B_{\Gamma}) = \inf \Lambda(B_{\Gamma}), \]
if $\Lambda(\mathcal{B}_T) \neq \emptyset$, and let $\alpha(\mathcal{B}_T) = 1$ otherwise.

These families of iterated function systems were originally studied by Solomyak in [59]. He was interested in the absolute continuity of self-similar measures. In particular, he was interested in the pushforward of the uniform $(1/l, \ldots, 1/l)$ Bernoulli measure. We denote this measure by $\mu_{\lambda,D}$. The main result of [59] is the following theorem.

**Theorem 2.1.** For Lebesgue almost every $\lambda \in (1/l, \alpha(\mathcal{B}_T))$, the measure $\mu_{\lambda,D}$ is absolutely continuous and has a density in $L^2(\mathbb{R})$.

Using Theorem 2.1, Solomyak proved the well known result that for Lebesgue almost every $\lambda \in (1/2, 1)$, the unbiased Bernoulli convolution is absolutely continuous and has a density in $L^2(\mathbb{R})$. As a by-product of our analysis, in Section 4 we give a short intuitive proof that for Lebesgue almost every $\lambda \in (1/2, 1)$, the unbiased Bernoulli convolution is absolutely continuous.

Instead of using the Fourier transform or by differentiating measures, as in [59] and [42], our proof makes use of the fact that self-similar measures are of pure type, i.e. they are either singular or absolutely continuous with respect to the Lebesgue measure. As a further by-product of our analysis, in Section 4 we recover another result of Solomyak from [59]. We prove that for Lebesgue almost every $\lambda \in (1/3, 2/5)$, the set

$$C_\lambda := \left\{ \sum_{j=0}^{\infty} d_j \lambda^j : d_j \in \{0, 1, 3\} \right\}$$

has positive Lebesgue measure. Interestingly our proof of this statement does not rely on showing that there is an absolutely continuous measure supported on this set. Instead we study a subset of this set, and show that for Lebesgue almost every $\lambda \in (1/3, 2/5)$, this set has positive Lebesgue measure.

For the families of iterated function systems introduced in this section, our main result is the following.

**Theorem 2.2.** Let $D$ be a finite set of real numbers. The following statements are true:

1. Let $\mathbf{m}$ be a slowly decaying $\sigma$-invariant ergodic probability measure with $\mathfrak{h}(\mathbf{m}) > 0$ and $(a_j) \in D^\mathbb{N}$. For Lebesgue almost every $\lambda \in (e^{-\mathfrak{h}(\mathbf{m})}, \alpha(\mathcal{B}_T))$, for any $h \in H$ the set $U_{\Phi_{\lambda,D}}(\sum_{j=1}^{\infty} d_j a_j \lambda^{-j-1}, \mathbf{m}, h)$ has positive Lebesgue measure.

2. Let $\mathbf{m}$ be the uniform $(1/l, \ldots, 1/l)$ Bernoulli measure. For Lebesgue almost every $\lambda \in (1/l, \alpha(\mathcal{B}_T))$, for any $z \in X_{\lambda,D}$ and $h \in H$, the set $U_{\Phi_{\lambda,D}}(z, \mathbf{m}, h)$ has positive Lebesgue measure.

3. Let $\mathbf{m}$ be a slowly decaying $\sigma$-invariant ergodic probability measure with $\mathfrak{h}(\mathbf{m}) > 0$ and $(a_j) \in D^\mathbb{N}$. For Lebesgue almost every $\lambda \in (e^{-\mathfrak{h}(\mathbf{m})}, \alpha(\mathcal{B}_T))$, for any $\Psi \in \Upsilon_{\mathbf{m}}$ Lebesgue almost every $x \in X_{\lambda,D}$ is contained in $W_{\Phi_{\lambda,D}}(\sum_{j=1}^{\infty} d_j a_j \lambda^{-j-1}, \Psi)$.

4. Let $\mathbf{m}$ be the uniform $(1/l, \ldots, 1/l)$ Bernoulli measure. For Lebesgue almost every $\lambda \in (1/l, \alpha(\mathcal{B}_T))$, for any $z \in X_{\lambda,D}$ and $\Psi \in \Upsilon_{\mathbf{m}}$, Lebesgue almost every $x \in X_{\lambda,D}$ is contained in $W_{\Phi_{\lambda,D}}(z, \Psi)$.

To aid with our exposition we will prove in Section 4 the following corollary to Theorem 2.2.

**Corollary 2.3.** Let $D$ be a finite set of real numbers and $\mathbf{m}$ be a Bernoulli measure corresponding to the probability vector $(p_1, \ldots, p_l)$. Then for any $(a_j) \in D^\mathbb{N}$, for Lebesgue almost every $\lambda \in (\prod_{i=1}^{l} p_i, \alpha(\mathcal{B}_T))$, Lebesgue almost every $x \in X_{\lambda,D}$ is contained in the set

$$\left\{ x \in X : \left| x - \phi_{\mathbf{a}} \left( \sum_{j=1}^{\infty} d_j a_j \lambda^{-j-1} \right) \right| \leq \frac{\prod_{j=1}^{l} p_{a_j}}{|\mathbf{a}|} \right\}$$

for i.m. $\mathbf{a} \in D^*$. 

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In Section 4 we will apply these results to obtain more explicit statements in the setting of Bernoulli convolutions and the \{0,1,3\} problem.

Certain lower bounds for the transversality constant $\alpha(B_T)$ are known. Let $D$ be a finite set of real numbers and assume $d_j \neq d_k$ for all $j \neq k$, so

$$b(D) := \sup \left\{ \frac{|d_i - d_j|}{|d_k - d_i|} : k \neq i \right\} < \infty.$$ 

The proposition stated below provides a summary of the lower bounds obtained separately in [43], [46], and [55].

**Proposition 2.4.** Let $D$ be a finite set of real numbers and $b(D)$ be as above. Then the following statements are true:

- If $b(D) = 1$ then $\alpha(B_T) > 0.668$.
- If $b(D) = 2$ then $\alpha(B_T) = 0.5$.
- $\alpha(B_T) = (b(D) + 1)^{-1}$ whenever $b(D) \geq 3 + \sqrt{8}$
- $\alpha(B_T) \geq (b(D) + 1)^{-1}$ for all $D$.

### 2.2 Parameterised families with variable translations

Suppose \{A_i\}_{i=1}^l is a collection of $d \times d$ non-singular matrices each satisfying $\|A_i\| < 1$. Given a vector $t = (t_1, \ldots, t_l) \in \mathbb{R}^{ld}$ we can define an IFS to be the set of contractions

$$\Phi_t := \{\phi_i(x) = A_ix + t_i\}_{i=1}^l.$$ 

Unlike in the previous section where we obtained a family of iterated function systems by varying the contraction ratio, here we obtain a family by varying the translation parameter $t$. For each $t \in \mathbb{R}^{ld}$ we denote the attractor by $X_t$, and the corresponding projection map from $\mathcal{D}^N$ to $X_t$ by $\pi_t$. The attractor $X_t$ is commonly referred to as a self-affine set.

This family of iterated function systems was introduced by Falconer in [21], and subsequently studied by Solomyak in [58], and later Jordan, Pollicott, and Simon in [31]. For this family an important result is the following.

**Theorem 2.5** (Falconer [21], Solomyak [58]). Assume the $A_i$ satisfy the additional hypothesis that $\|A_i\| < 1/2$ for all $1 \leq i \leq l$. Then for Lebesgue almost every $t \in \mathbb{R}^{ld}$ the attractor $X_t$ satisfies:

$$\dim_H(X_t) = \dim_B(X_t) = \min\{\dim(A_1, \ldots, A_l), d\}.$$ 

Here $\dim(A_1, \ldots, A_l)$ is a quantity known as the affinity dimension. For its definition see [21]. Theorem 2.5 was originally proved by Falconer in [21] under the assumption $\|A_i\| < 1/3$ for all $1 \leq i \leq l$. This upper bound was improved to $1/2$ by Solomyak in [58]. The bound $1/2$ is known to be optimal (see [16, 56]). An analogue of Theorem 2.5 for measures was obtained by Jordan, Pollicott, and Simon in [31]. A recent result of Bárány, Hochman, and Rapaport, proved in [63], significantly improves upon Theorem 2.5. They proved that we have $\dim_H(X_t) = \dim_B(X_t) = \min\{\dim(A_1, \ldots, A_l), d\}$ under some very general assumptions on the $A_i$ and $t$. In particular, their result gives rise to many explicit examples where equality is satisfied.

Given $a = (a_1, \ldots, a_n) \in \mathcal{D}^*$, we let

$$A_a := A_{a_1} \circ \cdots \circ A_{a_n},$$
Corollary 2.7. Suppose there exists \( 1 \leq A \) denote the singular values of \( A \). The singular values of a non-singular matrix \( A \) are the positive square roots of the eigenvalues of \( AA^T \). Alternatively they are the lengths of the semi-axes of the ellipse \( A(B(0,1)) \). Given a \( \sigma \)-invariant ergodic probability measure \( m \), then there exists positive constants \( \lambda_1(m), \cdots, \lambda_d(m) \), such that for \( m \)-almost every \((a_j) \in D^N\) we have

\[
\lim_{n \to \infty} \frac{\log \alpha_k(A_{a_1,\ldots,a_n})}{n} = \lambda_k(m),
\]

for all \( 1 \leq k \leq d \). We call the numbers \( \lambda_1(m), \cdots, \lambda_d(m) \) the Lyapunov exponents of \( m \). The existence of Lyapunov exponents for \( \sigma \)-invariant ergodic measures \( m \) was established in [31].

The theorem stated below is our main result for this family of iterated function systems.

**Theorem 2.6.** Suppose \( ||A_i|| < 1/2 \) for all \( 1 \leq i \leq l \). Then the following statements are true:

1. Let \( m \) be a slowly decaying \( \sigma \)-invariant ergodic probability measure with \( h(m) > -(\lambda_1(m) + \cdots + \lambda_d(m)) \) and \((a_j) \in D^N\). For Lebesgue almost every \( t \in \mathbb{R}^d \), for any \( h \in H \) the set \( U_{\Phi_t}(\pi_t(a), m, h) \) has positive Lebesgue measure.

2. Let \( m \) be the uniform \((1/l, \ldots, 1/l)\) Bernoulli measure and suppose there exists \( A \) such that \( A_i = A \) for any \( 1 \leq i \leq l \). If \( \log l > -(\lambda_1(m) + \cdots + \lambda_d(m)) \), then for Lebesgue almost every \( t \in \mathbb{R}^d \), for any \( z \in X_t \) and \( h \in H \), the set \( U_{\Phi_t}(z, m, h) \) has positive Lebesgue measure.

3. Let \( m \) be a slowly decaying \( \sigma \)-invariant ergodic probability measure and \((a_j) \in D^N\). Suppose that \( h(m) > -(\lambda_1(m) + \cdots + \lambda_d(m)) \) and one of the following three properties are satisfied:
   - Each \( A_i \) is a similarity.
   - \( d = 2 \) and all the matrices \( A_i \) are equal.
   - All the matrices \( A_i \) are simultaneously diagonalisable.

Then for Lebesgue almost every \( t \in \mathbb{R}^d \), for any \( \Psi \in Y_m \) Lebesgue almost every \( x \in X_t \) is contained in \( W_{\Phi_t}(\pi_t(a), \Psi) \).

4. Let \( m \) be the uniform \((1/l, \ldots, 1/l)\) Bernoulli measure and suppose there exists \( A \) such that \( A_i = A \) for any \( 1 \leq i \leq l \). Suppose that \( \log l > -(\lambda_1(m) + \cdots + \lambda_d(m)) \) and one of the following three properties are satisfied:
   - \( A \) is a similarity.
   - \( d = 2 \).
   - The matrix \( A \) is diagonalisable.

Then for Lebesgue almost every \( t \in \mathbb{R}^d \), for any \( z \in X_t \) and \( \Psi \in Y_m \), Lebesgue almost every \( x \in X_t \) is contained in \( W_{\Phi_t}(z, \Psi) \).

The following corollary follows immediately from Theorem 2.6.

**Corollary 2.7.** Suppose there exists \( \lambda \in (0, 1/2) \) and \( O \in O(d) \) such that \( A_i = \lambda \cdot O \) for all \( 1 \leq i \leq l \). Then if \( \frac{\log l}{\log x} > d \), we have that for Lebesgue almost every \( t \in \mathbb{R}^d \), for any \( z \in X_t \), Lebesgue almost every \( x \in X_t \) is contained in the set

\[
\left\{ x \in \mathbb{R}^d : |x - \phi_a(z)| \leq \frac{l^{-|a|}}{|a|} \text{ for i.m. } a \in D^* \right\}.
\]
The assumption $\|A_i\| < 1/2$ appearing in Theorem 2.6 is necessary as the example below shows.

**Example 2.8.** Consider the iterated function system $\Phi_{\lambda,t_1,t_2} = \{\lambda x + t_1, \lambda x + t_2\}$, where $\lambda \in (1/2, 1)$ and $t_1, t_2 \in \mathbb{R}$. Whenever $t_1 \neq t_2$ we can apply a change of coordinates and identify this iterated function system with $\{\lambda x, \lambda x + 1\}$. For any $\epsilon > 0$, there exists $\lambda^* \in (1/2, 1/2 + \epsilon)$ such that $\{\lambda x, \lambda x + 1\}$ contains an exact overlap. Using this fact and our change of coordinates, it can be shown that $U_{\Phi_{\lambda^*,t_1,t_2}}(\pi (a_j), m, h)$ has zero Lebesgue measure when $m$ is the $(1/2, 1/2)$ Bernoulli measure and $h$ is any bounded function.

Even though Example 2.8 demonstrates the condition $\|A_i\| < 1/2$ is essential, the author expects Theorem 2.6 to hold more generally. In this paper we prove a random version of Theorem 2.6 which supports this claim. This random version is based upon the randomly perturbed self-affine sets studied in [31]. Our setup is taken directly from this paper.

Fix a set of matrices $\{A_i\}_{i=1}^l$ each satisfying $\|A_i\| < 1$, and a vector $t = (t_1, \ldots, t_l) \in \mathbb{R}^d$. We obtain a randomly perturbed version of the IFS $\{\phi_i(x) = A_ix + t_i\}$ in the following way. Suppose that $\eta$ is an absolutely continuous distribution with density supported on a disc $D$. The distribution $\eta$ gives rise to a random perturbation of $\phi_a$ via the equation

$$\phi_{y_a} := (\phi_{a_1} + y_{a_1}) \circ (\phi_{a_2} + y_{a_2}) \circ \cdots \circ (\phi_{a_n} + y_a),$$

where the coordinates of

$$(y_{a_1}, y_{a_2}, \ldots, y_a) \in D \times \cdots \times D$$

are i.i.d. with distribution $\eta$. For notational convenience we enumerate the errors using the natural numbers. Let $\rho : \mathcal{D}^* \to \mathbb{N}$ be an arbitrary bijection. We obtain a sequence of errors $y = (y_k)_{k=1}^\infty \in \mathcal{D}^\infty$ according to the rule

$$y_k := y_a \text{ if } \rho(a) = k.$$

Given $y \in \mathcal{D}^\infty$, we obtain a perturbed version of our original attractor defined via the equation

$$X_y := \bigcap_{n=1}^\infty \bigcup_{a \in \mathcal{D}^n} \phi_{y_a}^n(B),$$

where $B$ is some sufficiently large ball. We let $\pi_y : \mathcal{D}^\infty \to X_y$ be the projection map given by

$$\pi_y(a_j) := \lim_{n \to \infty} \phi_{y_{a_1},\ldots,a_n}^n(0).$$

On $\mathcal{D}^\infty$ we define the measure

$$P := \eta \times \cdots \times \eta \times \cdots .$$

We may now define our limsup sets for these randomly perturbed attractors. Given $y \in \mathcal{D}^\infty$, $(a_j) \in \mathcal{D}^\infty$, and $\Psi : \mathcal{D}^* \to [0, \infty)$, we define

$$W_y((a_j), \Psi) := \left\{ x \in \mathbb{R}^d : |x - \pi_y(a_j)| \leq \Psi(a) \text{ for i.m. } a \in \mathcal{D}^* \right\} .$$

Given a slowly decaying measure $m$, $y \in \mathcal{D}^\infty$, $(a_j) \in \mathcal{D}^\infty$, and $h : \mathbb{N} \to [0, \infty)$, we let

$$U_y((a_j), m, h) := \left\{ x \in \mathbb{R}^d : |x - \pi_y(a_j)| \leq (m([a])h(n))^{1/d} \text{ for i.m. } a \in \bigcup_{n=1}^\infty L_{m,n} \right\} .$$

The sets $W_y((a_j), \Psi)$ and $U_y((a_j), m, h)$ serve as our analogues of $W_\Psi(z, \Psi)$ and $U_\Psi(z, m, h)$ in this random setting. Note that here we have defined our limsup sets in terms of neighbourhoods.
of $\pi_Y(\mathbf{a}(a_j))$ rather than $\phi^n_Y(\pi_Y(\mathbf{a}(a_j)))$. In the deterministic setting considered above these quantities coincide. In the random setup it is not necessarily the case that $\pi_Y(\mathbf{a}(a_j)) = \phi^n_Y(\pi_Y(\mathbf{a}(a_j)))$. The theorem stated below is the random analogue of Theorem 2.6. It suggests that one should be able to replace the assumption $\|A_i\| < 1/2$ with some other reasonable conditions.

Theorem 2.9. Fix a set of matrices $\{A_i\}_{i=1}^t$ each satisfying $\|A_i\| < 1$ and $t \in \mathbb{R}^{ld}$. Then the following statements are true:

1. Let $\mathbf{m}$ be a slowly decaying $\sigma$-invariant ergodic probability measure with $h(\mathbf{m}) > -(\lambda_1(\mathbf{m}) + \cdots + \lambda_d(\mathbf{m}))$ and $(a_j) \in \mathcal{D}^\mathbb{N}$. For $\mathbf{P}$-almost every $\mathbf{y} \in \mathcal{D}^\mathbb{N}$, for any $h \in H$ the set $U_\mathbf{y}((a_j), \mathbf{m}, h)$ has positive Lebesgue measure.

2. Let $\mathbf{m}$ be the uniform $(1/l, \ldots, 1/l)$ Bernoulli measure and suppose there exists $A$ such that $A_i = A$ for all $1 \leq i \leq l$. If $\log l > -(\lambda_1(\mathbf{m}) + \cdots + \lambda_d(\mathbf{m}))$, then for $\mathbf{P}$-almost every $\mathbf{y} \in \mathcal{D}^\mathbb{N}$, for any $h \in H$, the set $U_\mathbf{y}((a_j), \mathbf{m}, h)$ has positive Lebesgue measure.

3. Let $\mathbf{m}$ be a slowly decaying $\sigma$-invariant ergodic probability measure with $h(\mathbf{m}) > -(\lambda_1(\mathbf{m}) + \cdots + \lambda_d(\mathbf{m}))$ and $(a_j) \in \mathcal{D}^\mathbb{N}$. For $\mathbf{P}$-almost every $\mathbf{y} \in \mathcal{D}^\mathbb{N}$, for any $\Psi$ equivalent to $(\mathbf{m}, h)$ for some $h \in H$ the set $W_\mathbf{y}((a_j), \Psi)$ has positive Lebesgue measure.

4. Let $\mathbf{m}$ be the uniform $(1/l, \ldots, 1/l)$ Bernoulli measure and suppose there exists $A$ such that $A_i = A$ for all $1 \leq i \leq l$. If $\log l > -(\lambda_1(\mathbf{m}) + \cdots + \lambda_d(\mathbf{m}))$, then for $\mathbf{P}$-almost every $\mathbf{y} \in \mathcal{D}^\mathbb{N}$, for any $(a_j) \in \mathcal{D}^\mathbb{N}$ and $\Psi$ equivalent to $(\mathbf{m}, h, \Psi)$ for some $h \in H$, the set $W_\mathbf{y}((a_j), \Psi)$ has positive Lebesgue measure.

The reason we cannot obtain the full measure statements from Theorem 2.6 in our random setting is because of how $X_\mathbf{y}$ is defined. In particular, $X_\mathbf{y}$ cannot necessarily be expressed as finitely many scaled copies of itself like in the deterministic setting. The proof of statements 3 and 4 from Theorem 2.6 rely on the fact that the underlying attractor satisfies the equation $X = \bigcup_{i=1}^t \Phi_i(X)$.

2.3 A specific family of IFSs

We now introduce a family of iterated function systems for which we can make very precise statements. To each $t \in [0, 1]$ we associate the IFS:

$$\Phi_t = \left\{ \phi_1(x) = \frac{x}{2}, \phi_2(x) = \frac{x + 1}{2}, \phi_3(x) = \frac{x + t}{2}, \phi_4(x) = \frac{x + 1 + t}{2} \right\}.$$ 

For each $\Phi_t$ the corresponding attractor is $[0, 1 + t]$. We denote the projection map from $\mathcal{D}^\mathbb{N}$ to $[0, 1 + t]$ by $\pi_t$. For this family of iterated function systems we will be able to replace the almost every statements appearing in Theorem 2.2 and Theorem 2.6 with something more precise. The reason we can make these stronger statements is because separation properties for $\Phi_t$ can be deduced from the continued fraction expansion of $t$. Recall that for any $t \in [0, 1] \setminus \mathbb{Q}$, there exists a unique sequence $(\zeta_m) \in \mathbb{N}^\mathbb{N}$ such that

$$t = \frac{1}{\zeta_1 + \frac{1}{\zeta_2 + \frac{1}{\zeta_3 + \cdots}}}.$$
We call the sequence \((\zeta_m)\) the continued fraction expansion of \(t\). Given \(t\) with continued fraction expansion \((\zeta_m)\), for each \(m \in \mathbb{N}\) we let
\[
\frac{p_m}{q_m} := \frac{1}{\zeta_1 + \frac{1}{\zeta_2 + \frac{1}{\zeta_3 + \ddots + \frac{1}{\zeta_m}}}}.
\]
We call \(p_m/q_m\) the \(m\)-th partial quotient of \(t\). We say that \(t\) is badly approximable if the integers appearing in the continued fraction expansion of \(t\) can be bounded from above.

The main result of this section is the following.

**Theorem 2.10.** Let \(\mathbf{m}\) be the uniform \((1/4, 1/4, 1/4, 1/4)\) Bernoulli measure. The following statements are true:

1. If \(t \in \mathbb{Q}\) then \(\Phi_t\) contains an exact overlap, and for any \(z \in [0, 1 + t]\) the set \(U_{\Phi_t}(z, \mathbf{m}, 1)\) has Hausdorff dimension strictly less than 1.

2. If \(t \notin \mathbb{Q}\), then there exists \(h : \mathbb{N} \to [0, \infty)\) depending upon the continued fraction expansion of \(t\), such that \(\lim_{n \to \infty} h(n) = 0\), and for any \(z \in [0, 1 + t]\) Lebesgue almost every \(x \in [0, 1 + t]\) is contained in \(U_{\Phi_t}(z, \mathbf{m}, h)\).

3. If \(t\) is badly approximable, then for any \(z \in [0, 1 + t]\) and \(h : \mathbb{N} \to [0, \infty)\) satisfying \(\sum_{n=1}^{\infty} h(n) = \infty\), we have that Lebesgue almost every \(x \in [0, 1 + t]\) is contained in \(U_{\Phi_t}(z, \mathbf{m}, h)\).

4. If \(t \notin \mathbb{Q}\) and is not badly approximable, then there exists \(h : \mathbb{N} \to [0, \infty)\) satisfying \(\sum_{n=1}^{\infty} h(n) = \infty\), yet \(U_{\Phi_t}(z, \mathbf{m}, h)\) has zero Lebesgue measure for any \(z \in [0, 1 + t]\).

5. Suppose \(t \notin \mathbb{Q}\) is such that for any \(\epsilon > 0\), there exists \(L \in \mathbb{N}\) for which the following inequality holds for \(M\) sufficiently large:
\[
\sum_{\substack{1 \leq m \leq M \\ \frac{m+1}{m+L}}} \log_2(\zeta_{m+1} + 1) \leq \epsilon M.
\]
Then for any \(z \in [0, 1 + t]\) and \(h \in H^*,\) Lebesgue almost every \(x \in [0, 1 + t]\) is contained in \(U_{\Phi_t}(z, \mathbf{m}, h)\).

6. Suppose \(\mu\) is an ergodic invariant measure for the Gauss map and satisfies
\[
\sum_{n=1}^{\infty} \mu\left(\left[\frac{1}{m+1}, \frac{1}{m}\right]\right) \log_2(m) < \infty.
\]
Then for \(\mu\)-almost every \(t\), we have that for any \(z \in [0, 1 + t]\) and \(h \in H^*,\) Lebesgue almost every \(x \in [0, 1 + t]\) is contained in \(U_{\Phi_t}(z, \mathbf{m}, h)\). In particular, for Lebesgue almost every \(t \in [0, 1]\), we have that for any \(z \in [0, 1 + t]\) and \(h \in H^*,\) Lebesgue almost every \(x \in [0, 1 + t]\) is contained in \(U_{\Phi_t}(z, \mathbf{m}, h)\).

We include the following corollary to emphasise the strong dichotomy that follows from statement 1 and statement 2 from Theorem 2.10.
Corollary 2.11. Either \( t \) is such that \( \Phi_t \) contains an exact overlap and for any \( z \in [0, 1 + t] \)

\[
\dim_H \left( \left\{ x \in [0, 1 + t] : |x - \phi_a(z)| \leq \frac{1}{4|a|} \text{ for i.m. } a \in D^* \right\} \right) < 1,
\]
or for any \( z \in [0, 1 + t] \) Lebesgue almost every \( x \in [0, 1 + t] \) is contained in

\[
\left\{ x \in [0, 1 + t] : |x - \phi_a(z)| \leq \frac{1}{4|a|} \text{ for i.m. } a \in D^* \right\}.
\]

Theorem 2.10 is stated in terms of the auxiliary sets \( U_{\Phi_t}(z, m, h) \) rather than in terms of \( W_{\Phi_t}(z, \Psi) \). Where here the underlying measure \( m \) is the uniform \((1/4, 1/4, 1/4, 1/4)\) Bernoulli measure. Note however that if \( \Psi : D^* \to [0, \infty) \) is a function that only depends upon the length of the word \( a \), then \( \Psi(a) = h(|a|)m(|a|) \) for some appropriate choice of \( h \). Combining this observation with the fact \( L_m = D^n \) for this choice of \( m \), it follows that \( W_{\Phi_t}(z, \Psi) = U_{\Phi_t}(z, m, h) \) for this choice of \( h \). Therefore Theorem 2.10 can be reinterpreted in terms of the sets \( W_{\Phi_t}(z, \Psi) \) when \( \Psi \) only depends on the length of the word.

2.3.1 New methods for distinguishing between the overlapping behaviour of IFSs

In this section we explain how Theorem 2.10 allows us to distinguish between iterated function systems in a way that is not available to us by simply studying properties of self-similar measures. We start this discussion by stating the following result that will follow from the proof of Theorem 2.10.

Theorem 2.12. Let \( t \in [0, 1] \). It is the case that either \( \Phi_t \) contains an exact overlap, or for infinitely many \( n \in \mathbb{N} \) we have

\[
|\phi_a(z) - \phi_{a'}(z)| \geq \frac{1}{8 \cdot 4^n},
\]

for any \( z \in [0, 1 + t] \) for distinct \( a, a' \in D^n \).

Theorem 2.12 effectively states that for this family of IFSs, we either have an exact overlap, or for infinitely many scales we exhibit the optimal level of separation. This level of separation can be seen to be optimal by the pigeonhole principle, which tells us that for any \( z \in [0, 1 + t] \) and \( n \in \mathbb{N} \), there must exist distinct \( a, a' \in D^n \) such that

\[
|\phi_a(z) - \phi_{a'}(z)| \leq \frac{1 + t}{4^n - 1}.
\]

Because of the strong dichotomy demonstrated by Theorem 2.12, we believe that this family of IFSs will serve as a useful toy model for other problems.

For a probability vector \( p = (p_1, p_2, p_3, p_4) \) we denote the corresponding self-similar measure for the IFS \( \Phi_t \) by \( \mu_{p,t} \). It follows from Theorem 2.12 and the work of Hochman [27, Theorem 1.1.] that the following theorem holds.

Theorem 2.13. Either \( \Phi_t \) contains an exact overlap, or for any probability vector \( p \) we have

\[
\dim \mu_{p,t} = \min \left\{ \frac{\sum_{i=1}^4 p_i \log p_i}{-\log 2}, 1 \right\}.
\]

The following theorem follows from the work of Shmerkin and Solomyak [54, Theorem A].

Theorem 2.14. For every \( t \in [0, 1] \) outside a set of Hausdorff dimension 0, we have that \( \mu_{p,t} \) is absolutely continuous whenever

\[
\frac{\sum_{i=1}^4 p_i \log p_i}{-\log 2} > 1.
\]
To apply Theorem A from [54] we have to check that a non-degenerate condition is satisfied. Checking this condition holds is straightforward in our setting so we omit the details.

It is known that the set of badly approximable numbers has Hausdorff dimension 1 and Lebesgue measure zero. Therefore, applying Theorem 2.13 and Theorem 2.14, it follows that there exists a badly approximable number \( t \), and some \( t' \) that is not badly approximable, such that for any probability vector \( p \) we have

\[
\dim \mu_{p, t} = \dim \mu_{p, t'} = \min \left\{ \frac{\sum_{i=1}^{4} p_i \log p_i}{-\log 2}, 1 \right\},
\]

and whenever

\[
\frac{\sum_{i=1}^{4} p_i \log p_i}{-\log 2} > 1
\]

the measures \( \mu_{p, t} \) and \( \mu_{p, t'} \) are both absolutely continuous. As such, the overlapping behaviour of \( \Phi_t \) and \( \Phi_{t'} \) are indistinguishable from the perspective of self-similar measures. However, we see from statement 3 and statement 4 from Theorem 2.10 that there exists \( h : \mathbb{N} \to [0, \infty) \) such that \( U_{\Phi_t}(z, m, h) \) has full Lebesgue measure for all \( z \in [0, 1 + t] \), and \( U_{\Phi_{t'}}(z, m, h) \) has zero Lebesgue measure for all \( z \in [0, 1 + t'] \). Therefore we see that by studying the metric properties of limsup sets we can distinguish between the overlapping behaviour of \( \Phi_t \) and \( \Phi_{t'} \). Studying the metric properties of limsup sets detects some of the finer details of how an iterated function system overlaps.

2.4 The CS property and absolute continuity.

We saw in the previous section that by studying IFSs using ideas from metric number theory, one can distinguish between IFSs in a way that is not available to us by simply studying pushforwards of Bernoulli measures. It is natural to wonder how Khintchine like behaviour relates to these measures. In this paper we show that there is a connection between a strong type of Khintchine like behaviour and the absolute continuity of these measures.

Given an IFS \( \Phi \) and a slowly decaying measure \( m \), we say that \( \Phi \) is consistently separated with respect to \( m \), or \( \Phi \) has the CS property with respect to \( m \), if there exists \( z \in X \) such that for any \( h : \mathbb{N} \to [0, \infty) \) satisfying

\[
\sum_{n=1}^{\infty} h(n) = \infty,
\]

the set \( U_{\Phi}(z, m, h) \) has positive Lebesgue measure. Using this terminology we see that statement 3 and statement 4 from Theorem 2.10 imply that an IFS \( \Phi_t \) has the CS property with respect to the \((1/4, 1/4, 1/4, 1/4)\) Bernoulli measure if and only if \( t \) is badly approximable. The use of the terminology consistently separated will become clearer in Section 6 (see Theorem 6.1). We prove the following result.

**Theorem 2.15.** For a slowly decaying \( \sigma \)-invariant ergodic probability measure \( m \), if \( \Phi \) has the CS property with respect to \( m \), then the pushforward of \( m \) is absolutely continuous.

We emphasise here that an IFS having the CS property with respect to \( m \) and the pushforward of \( m \) being absolutely continuous are not equivalent statements. There are many examples of \( m \) and \( \Phi \) such that the pushforward of \( m \) is absolutely continuous, yet \( \Phi \) does not have the CS property with respect to \( m \). In particular, for the family of IFSs \( \{\Phi_t\} \) studied in the previous section, it can be shown that the pushforward of the uniform \((1/4, 1/4, 1/4, 1/4)\) Bernoulli measure is absolutely continuous for any \( t \in [0, 1] \). However as remarked above, \( \Phi_t \) has the CS property with respect to this measure if and only if \( t \) is badly approximable. We include several explicit examples of consistently separated iterated function systems in Section 9.
2.5 Overlapping self-conformal sets

Theorems 2.2, 2.6 and 2.10 are stated in terms of parameterised families of overlapping IFSs where one would expect that for a typical member of this family the corresponding attractor would have positive Lebesgue measure. In Theorem 14 the attractor can have arbitrary Hausdorff dimension, but we assume that the underlying IFS satisfies some separation hypothesis. None of these results cover the case when the IFS is overlapping and the attractor is not expected to have positive Lebesgue measure. The purpose of this section is to fill this gap for IFSs consisting of conformal mappings. We recall some background on this class of IFS below.

Let \( V \subset \mathbb{R}^d \) be an open set, a \( C^1 \) map \( \phi : V \to \mathbb{R}^d \) is a conformal mapping if it preserves angles, or equivalently \( \Phi \) is a conformal mapping if the differential \( \phi' \) satisfies \( |\phi'(x)y| = |\phi'(x)||y| \) for all \( x \in V \) and \( y \in \mathbb{R}^d \). We call an IFS \( \Phi = \{\phi_i\}_{i=1}^N \) a conformal iterated function system on a compact set \( Y \subset \mathbb{R}^d \) if each \( \phi_i \) can be extended to an injective conformal contraction on some open connected neighbourhood \( V \) that contains \( Y \), and

\[
0 \leq \inf_{x \in V} |\phi_i'(x)| \leq \sup_{x \in V} |\phi_i'(x)| < 1.
\]

Throughout this paper we will assume that the differentials are Hölder continuous, i.e., there exists \( \alpha > 0 \) and \( c > 0 \) such that

\[
|\phi_i'(x)| - |\phi_i'(y)| \leq c|x - y|^\alpha
\]

for all \( x, y \in V \). If our IFS is a conformal iterated function system on some compact set, then we call the corresponding attractor \( X \) a self-conformal set. Self-conformal sets are a natural generalisation of self-similar sets.

To any conformal IFS we associate the family of potentials \( f_s : \mathcal{D}^N \to \mathbb{R} \) given by

\[
f_s((a_j)) = s \cdot \log |\phi'_{a_1}(\pi(a_j))|.
\]

Where here \( s \in (0, \infty) \). We define the topological pressure of \( f_s \) to be

\[
P(f_s) := \sup \left\{ h(m) + \int f_s dm : m \text{ is } \sigma\text{-invariant} \right\}.
\]

For more on topological pressure and thermodynamic formalism we refer the reader to [12] and [20]. It can be shown that for any conformal IFS, there exists a unique value of \( s \) satisfying the equation \( P(f_s) = 0 \). We call this parameter the similarity dimension of \( \Phi \) and denote it by \( \dim_S(\Phi) \). When \( \Phi \) is a conformal IFS and satisfies the open set condition, it is known that \( \dim_H(X) = \dim_B(X) = \dim_S(\Phi) \). Importantly there exists a unique measure \( m_\phi \) such that

\[
h(m_\phi) + \int f_{\dim_S(\Phi)} dm_\phi = 0.
\]

The pushforward of the measure \( m_\phi \), which we denote by \( \mu_\phi \), is a particularly useful tool for determining metric properties of the attractor \( X \). In particular, when \( \Phi \) satisfies the open set condition it can be shown that \( \mu_\phi \) is equivalent to \( \mathcal{H}^{\dim_H(X)} |_X \) (see [39]). Note that when \( \Phi \) consists of similarities, i.e. \( \Phi = \{\phi_i(x) = r_iO_i(x + t_i)\}_{i=1}^N \), then \( m_\phi \) is simply the Bernoulli measure corresponding to the probability vector \( (r_1^s, \ldots, r_N^s) \), where \( s \) is the unique solution to the equation \( \sum_{i=1}^N r_i^s = 1 \).

Our main result for conformal iterated function systems is the following theorem.

**Theorem 2.16.** If \( \Phi \) is a conformal iterated function system, then for any \( z \in X \), if \( \theta : \mathbb{N} \to [0, \infty) \) is a decreasing function and satisfies

\[
\sum_{n=1}^\infty \sum_{a \in \mathcal{D}^n} (\operatorname{Diam}(X_a)\theta(n))^{\dim_S(\Phi)} = \infty,
\]

then \( \mu_\phi \)-almost every \( x \in X \) is an element of \( W_\Phi(z, \operatorname{Diam}(X_a)\theta(|a|)) \).
As stated above, when \( \Phi \) satisfies the open set condition then \( \mu_\Phi \) is equivalent to \( \mathcal{H}^{\dim_H(X)}|_X \), it follows therefore that Theorem 2.16 implies Theorem 1.4. For our purposes the real value of Theorem 2.16 is demonstrated in the following corollary.

**Corollary 2.17.** Let \( \Phi \) be a conformal iterated function system and suppose \( \dim \mu_\Phi = \dim_H(X) \). Then for any \( z \in X \), if \( \theta : \mathbb{N} \to [0, \infty) \) is a decreasing function and satisfies

\[
\sum_{n=1}^{\infty} \sum_{a \in \mathcal{D}^n} \left( \text{Diam}(X_a) \theta(n) \right)^{\dim_S(\Phi)} = \infty,
\]

then \( W_\Phi(z, \text{Diam}(X_a) \theta(|a|)) \) has Hausdorff dimension equal to \( \dim_H(X) \).

Corollary 2.17 effectively reduces the problem of determining the Hausdorff dimension of \( W_\Phi(z, \text{Diam}(X_a) \theta(|a|)) \) to determining whether \( \dim \mu_\Phi = \dim_H(X) \). Thankfully there are many results on the latter problem, and we can use these results together with Corollary 2.17 to deduce further statements. We mention here only one such statement for the sake of brevity. The following statement follows by combining Theorem 1.1 from [27] and Corollary 2.17.

**Corollary 2.18.** Assume \( d = 1 \) and \( \Phi \) consists solely of similarities. If

\[
\liminf_{n \to \infty} \frac{-\log \Delta_n}{n} < \infty,
\]

where

\[
\Delta_n := \min_{a \neq b \in \mathcal{D}^n} |\phi_a(0) - \phi_b(0)|,
\]

then for any \( z \in X \), if \( \theta : \mathbb{N} \to [0, \infty) \) is a decreasing function and satisfies

\[
\sum_{n=1}^{\infty} \sum_{a \in \mathcal{D}^n} \left( \text{Diam}(X_a) \theta(n) \right)^{\dim_S(\Phi)} = \infty,
\]

then \( W_\Phi(z, \text{Diam}(X_a) \theta(|a|)) \) has Hausdorff dimension equal to \( \dim_H(X) \).

### 2.6 Structure of the paper

The rest of the paper is arranged as follows. In Section 3 we prove some general results that will allow us to prove our main theorems. In Section 4 we prove Theorem 2.2, Theorem 2.6, and Theorem 2.9. In Section 5 we prove Theorem 2.10. Section 6 is then concerned with the proof of Theorem 2.15 and in Section 7 we prove Theorem 2.16. In Section 8 we apply the mass transference principle of Beresnevich and Velani to show how one can use our earlier results to deduce results on the Hausdorff measure and Hausdorff dimension of certain \( W_\Phi(z, \Psi) \) when

\[
\sum_{a \in \mathcal{D}^\ast} \Psi(a)^{\dim_H(X)} < \infty.
\]

In Section 9 we include some explicit examples to accompany our main theorems. We conclude with some general discussion and pose some open questions in Section 10.

### 3 Preliminary results

#### 3.1 A general framework

In this section we prove some useful preliminaries that will allow us to prove the main results of this paper. Throughout this section \( \Omega \) will denote a metric space equipped with some finite
Borel measure $\eta$, and $\tilde{X}$ will denote some compact subset of $\mathbb{R}^d$. For each $n \in \mathbb{N}$ we will assume that there exists a finite set of continuous functions $\{f_{l,n} : \Omega \to \tilde{X}\}_{l=1}^{R_n}$. For each $\omega \in \Omega$ we let

$$Y_n(\omega) := \{f_{l,n}(\omega)\}_{l=1}^{R_n}.$$ 

Before stating our general result we need to introduce some notation. Given $r > 0$ we say that $Y \subset \mathbb{R}^d$ is an $r$-separated set if $|z - z'| > r$, $\forall z, z' \in Y$ such that $z \neq z'$. Given a finite set $Y \subset \mathbb{R}^d$ and $r > 0$, we let

$$T(Y, r) := \sup\{\#Y' : Y' \subset Y \text{ and } Y' \text{ is an } r\text{-separated set}\}.$$ 

We call $Y' \subset Y$ a maximal $r$-separated subset if $Y'$ is $r$-separated and $\#Y' = T(Y, r)$. Clearly a maximal $r$-separated subset always exists. Given a finite set $Y$ and $r > 0$, we will denote by $S(Y, r)$ an arbitrary choice of maximal $r$-separated subset.

The proposition stated below is the main technical result of this section.

**Proposition 3.1.** Suppose the following properties are satisfied:

- There exists $\gamma > 1$ such that $R_n \asymp \gamma^n$.
- There exists $G : (0, \infty) \to (0, \infty)$ such that $\lim_{s \to 0} G(s) = 0$, and for all $n \in \mathbb{N}$ we have
  $$\eta(\Omega) - \int_{\Omega} \frac{T(Y_n(\omega), \frac{s}{R_n})}{R_n} d\eta(\omega) \leq G(s).$$

Then for $\eta$-almost every $\omega \in \Omega$, for any $h \in H$ the set

$$\{x \in \mathbb{R}^d : |x - f_{l,n}(\omega)| \leq \left(\frac{h(n)}{R_n}\right)^{1/d} \text{ for i.m. } (l, n) \in \{1, \ldots, R_n\} \times \mathbb{N}\}$$

has positive Lebesgue-measure.

Recall that the set of functions $H$ was defined in (2.2).

Given $c > 0, s > 0$, and $n \in \mathbb{N}$, we let

$$B(c, s, n) := \{\omega \in \Omega : \frac{T(Y_n(\omega), \frac{s}{R_n})}{R_n} > c\}.$$ 

The following lemma shows that under the hypothesis of Proposition 3.1, a typical $\omega \in \Omega$ is contained in $B(c, s, n)$ for a large set of $n$ for appropriate choices of $c$ and $s$. This lemma will play an important role in Section 4 when we recover results of Solomyak on the absolute continuity of Bernoulli convolutions, and on the Lebesgue measure of the attractor in the $\{0, 1, 3\}$ problem.

**Lemma 3.2.** Assume there exists $G : (0, \infty) \to (0, \infty)$ such that $\lim_{s \to 0} G(s) = 0$, and for all $n \in \mathbb{N}$ we have

$$\eta(\Omega) - \int_{\Omega} \frac{T(Y_n(\omega), \frac{s}{R_n})}{R_n} d\eta(\omega) \leq G(s).$$

Then

$$\eta\left(\bigcap_{c > 0, s > 0} \bigcup \{\omega : d(n : \omega \in B(c, s, n)) \geq 1 - c\}\right) = \eta(\Omega)$$
Proof. Observe that
\[ 0 \leq \frac{T(Y_n(\omega), \frac{\omega}{R_n^{1/d}})}{R_n} \leq 1 \]
for all \( \omega \in \Omega \) and \( n \in \mathbb{N} \). As a result of this inequality and our underlying assumption, for any \( c > 0, s > 0, \) and \( n \in N \), we have
\[ \eta(B(c, s, n)) + c \cdot \eta(B(c, s, n)^c) \geq \int_{\Omega} \frac{T(Y_n(\omega), \frac{s}{R(n)^{1/d}})}{R(n)} d\eta(\omega) \geq \eta(\Omega) - G(s). \]
This in turn implies
\[ \eta(B(c, s, n)) \geq (1 - c) \eta(\Omega) - G(s). \]
It follows that given \( \epsilon > 0 \), we can pick \( c > 0 \) and \( s > 0 \) independent of \( n \) such that
\[ \eta(B(c, s, n)) \geq \eta(\Omega) - \epsilon. \quad (3.1) \]
Applying Fatou’s lemma we have
\[ \int_{\Omega} d(\omega : \omega \in B(c, s, n)) d\eta = \int_{\Omega} \limsup_{N \to \infty} \frac{\#\{1 \leq n \leq N : \omega \in B(c, s, n)\}}{N} d\eta \]
\[ = \int_{\Omega} \limsup_{N \to \infty} \frac{\sum_{n=1}^{N} \chi_{B(c, s, n)}(\omega)}{N} d\eta \]
\[ \geq \limsup_{N \to \infty} \frac{\sum_{n=1}^{N} \int_{\Omega} \chi_{B(c, s, n)}(\omega) d\eta}{N} \]
\[ = \limsup_{N \to \infty} \frac{\sum_{n=1}^{N} \eta(B(c, s, n))}{N} \]
\[ \geq \eta(\Omega) - \epsilon. \quad (3.1) \]
Summarising the above, we have shown that for this choice of \( c \) and \( s \) we have
\[ \int_{\Omega} d(\omega : \omega \in B(c, s, n)) d\eta \geq \eta(\Omega) - \epsilon. \quad (3.2) \]
For the purpose of obtaining a contradiction, suppose
\[ \eta\left(\omega : d(\omega : \omega \in B(c, s, n)) \leq 1 - \sqrt{\epsilon}\right) > \sqrt{\epsilon}. \quad (3.3) \]
Then using the fact
\[ 0 \leq d(\omega : \omega \in B(c, s, n)) \leq 1 \]
for all \( \omega \in \Omega \), we have
\[ \int_{\Omega} d(\omega : \omega \in B(c, s, n)) d\eta \leq \eta(\omega : d(\omega : \omega \in B(c, s, n)) \leq 1 - \sqrt{\epsilon})(1 - \sqrt{\epsilon}) \]
\[ + \eta(\omega : d(\omega : \omega \in B(c, s, n)) > 1 - \sqrt{\epsilon}) \]
\[ = \eta(\omega : d(\omega : \omega \in B(c, s, n)) \leq 1 - \sqrt{\epsilon})(1 - \sqrt{\epsilon}) \]
\[ + \eta(\Omega) - \eta(\omega : d(\omega : \omega \in B(c, s, n)) \leq 1 - \sqrt{\epsilon}) \]
\[ = \eta(\Omega) - \sqrt{\epsilon} \eta(\omega : d(\omega : \omega \in B(c, s, n)) \leq 1 - \sqrt{\epsilon}) \]
\[ < \eta(\Omega) - \epsilon. \]
This contradicts (3.2). Therefore (3.3) is not possible and we have that for any \( \epsilon > 0 \), there exists \( c, s > 0 \) such that

\[
\eta\left( \omega : \bar{d}(n : \omega \in B(c,s,n)) > 1 - \sqrt{\epsilon} \right) \geq \eta(\Omega) - \sqrt{\epsilon} \tag{3.4}
\]

Equation (3.4) in turn implies that for any \( \epsilon > 0 \) we have

\[
\eta\left( \bigcup_{c,s>0} \{ \omega : \bar{d}(n : \omega \in B(c,s,n)) \geq 1 - \epsilon \} \right) = \eta(\Omega). \tag{3.5}
\]

One can see how (3.5) follows from (3.4) by first fixing \( \epsilon > 0 \) and then applying (3.4) for a countable collection of \( \epsilon_k \) strictly smaller that \( \epsilon \). Now intersecting over all \( \epsilon > 0 \), we see that (3.5) implies the desired equality:

\[
\eta\left( \bigcap_{\epsilon>0} \bigcup_{c,s>0} \{ \omega : \bar{d}(n : \omega \in B(c,s,n)) \geq 1 - \epsilon \} \right) = \eta(\Omega).
\]

To prove Proposition 3.1 and many other results in this paper, we will rely upon the following useful lemma.

**Lemma 3.3.** Let \((X, A, \mu)\) be a finite measure space and \(E_n \in A\) be a sequence of sets such that \(\sum_{n=1}^{\infty} \mu(E_n) = \infty\). Then

\[
\mu(\limsup_{n \to \infty} E_n) \geq \limsup_{Q \to \infty} \frac{(\sum_{n=1}^{Q} \mu(E_n))^2}{\sum_{n,m=1}^{Q} \mu(E_n \cap E_m)}.
\]

For a proof of Lemma 3.3 see either [25, Lemma 2.3] or [60, Lemma 5].

Proposition 3.1 will follow from the following proposition. This result will also be useful when it comes to proving some of our later results.

**Proposition 3.4.** Let \( \omega \in \Omega \) and \( h : \mathbb{N} \to [0, \infty) \). Assume the following properties are satisfied:

- There exists \( \gamma > 1 \) such that
  \[ R_n \asymp \gamma^n. \]
- There exists \( c > 0 \) and \( s > 0 \) such that
  \[ \sum_{n: \omega \in B(c,s,n)} h(n) = \infty. \]

Then

\[
\left\{ x \in \mathbb{R}^d : |x - f_{l,n}(\omega)| \leq \left( \frac{h(n)}{R_n} \right)^{1/d} \text{ for i.m. } (l,n) \in \{1, \ldots, R_n\} \times \mathbb{N} \right\}
\]

has positive Lebesgue measure.

**Proof of Proposition 3.4.** We split our proof into individual steps for convenience.

**Step 1. Replacing our approximating function.**

Let \( \omega \) and \( h \) be fixed, and \( c \) and \( s > 0 \) be as in the statement of the proposition. We claim that

\[
\sum_{n: \omega \in B(c,s,n)} \sum_{u \in S(Y_n(\omega)) \cap \mathbb{R}^{n/d}} \mathcal{L}\left( B\left( u, \left( \frac{h(n)}{R_n} \right)^{1/d} \right) \right) = \infty. \tag{3.6}
\]
This follows from our assumption
\[ \sum_{n : \omega \in B(c, s, n)} h(n) = \infty, \]
and the following:
\[
\sum_{n : \omega \in B(c, s, n)} \sum_{u \in S(Y_n(\omega), \frac{s}{R_n^{1/d}})} \mathcal{L}(B(u, \left( \frac{h(n)}{R_n^{1/d}} \right)^{1/d})) = \sum_{n : \omega \in B(c, s, n)} \sum_{u \in S(Y_n(\omega), \frac{s}{R_n^{1/d}})} \frac{h(n) \mathcal{L}(B(0, 1))}{R_n} \\
= \sum_{n : \omega \in B(c, s, n)} T(Y_n(\omega), \frac{s}{R_n^{1/d}}) \frac{h(n) \mathcal{L}(B(0, 1))}{R_n} \\
\geq c \mathcal{L}(B(0, 1)) \sum_{n : \omega \in B(c, s, n)} h(n) \\
= \infty.
\]

Let us now define
\[ g(n) := \min \left\{ \left( \frac{h(n)}{R_n^{1/d}} \right)^{1/d}, \frac{s}{3R_n^{1/d}} \right\}. \]

We claim that we still have
\[ \sum_{n : \omega \in B(c, s, n)} \sum_{u \in S(Y_n(\omega), \frac{s}{R_n^{1/d}})} \mathcal{L}(B(u, g(n))) = \infty. \]  \((3.7)\)

If \( g(n) = \frac{s}{3R_n^{1/d}} \) for finitely many \( n \in \mathbb{N} \), then \((3.6)\) would imply \((3.7)\). Suppose therefore that \( g(n) = \frac{s}{3R_n^{1/d}} \) for infinitely many \( n \in \mathbb{N} \). For such an \( n \) we would have
\[
\sum_{u \in S(Y_n(\omega), \frac{s}{R_n^{1/d}})} \mathcal{L}(B(u, g(n))) = T(Y_n(\omega), \frac{s}{R_n^{1/d}}) \frac{s^d \mathcal{L}(B(0, 1))}{3^d R_n^{1/d}} > \frac{cs^d \mathcal{L}(B(0, 1))}{3^d}.
\]

This lower bound is strictly positive and independent of \( n \). As such summing over it for infinitely many \( n \) guarantees divergence. Therefore \((3.7)\) holds.

Note that it follows from the definition of \( g \) that we will have proved our result if we can show that
\[
\mathcal{L} \left( \left\{ x \in \mathbb{R}^d : |x - f_{l,n}(\omega)| \leq g(n) \text{ for i.m. } (l, n) \in \{1, \ldots, R_n\} \times \mathbb{N} \right\} \right) > 0. \]  \((3.8)\)

**Step 2: Constructing our \( E_n \).**

Since \( g(n) \leq \frac{s}{3R_n^{1/d}} \) for all \( n \in \mathbb{N} \), it follows that for any distinct \( u, v \in S(Y_n(\omega), \frac{s}{R_n^{1/d}}) \), we must have
\[ B(u, g(n)) \cap B(v, g(n)) = \emptyset. \]  \((3.9)\)

For each \( n \) such that \( \omega \in B(c, s, n) \), let
\[ E_n = \bigcup_{u \in S(Y_n(\omega), \frac{s}{R_n^{1/d}})} B(u, g(n)). \]
We will show that
\[ \mathcal{L}\left( \{ x \in \mathbb{R}^d : x \in E_n \text{ for i.m. } n \in \mathbb{N} \text{ such that } \omega \in B(c,s,n) \} \right) > 0. \quad (3.10) \]

Equation (3.10) implies (3.8), so to complete our proof it suffices to show that (3.10) holds. It follows from (3.7) and (3.9) that
\[ \text{Equation (3.11) shows that our collection of sets } \{ E_n \}_{n \in \mathbb{N}} \text{ satisfies the hypothesis of Lemma 3.3.} \]

We record here for later use the following fact, for each \( n \in \mathbb{N} \text{ such that } \omega \in B(c,s,n), \) we have
\[ \mathcal{L}(E_n) \propto R_n g(n)^d. \quad (3.12) \]

Equation (3.12) follows from (3.9) and the fact that for each \( n \in \mathbb{N} \text{ such that } \omega \in B(c,s,n), \) we have
\[ c R_n \leq T\left( Y_n(\omega), \frac{s}{R_n^{1/d}} \right) \leq R_n. \]

**Step 3: Bounding** \( \mathcal{L}(E_n \cap E_m). \)

Assume \( n \) is such that \( \omega \in B(c,s,n), \) \( m \) is such that \( \omega \in B(c,s,m), \) and \( m \neq n. \) Fix \( u \in S(Y_n(\omega), \frac{s}{R_n^{1/d}}). \) We want to bound the quantity:
\[ \# \left\{ v \in S\left( Y_m(\omega), \frac{s}{R_m^{1/d}} \right) : B(u, g(n)) \cap B(v, g(m)) \neq \emptyset \right\}. \]

If \( g(m) \geq g(n), \) then every \( v \in S\left( Y_m(\omega), \frac{s}{R_m^{1/d}} \right) \) satisfying \( B(u, g(n)) \cap B(v, g(m)) \neq \emptyset \) must also satisfy \( B(v, g(m)) \subseteq B(u, 3g(m)). \) It follows therefore from (3.9) and a volume argument that
\[ \# \left\{ v \in S\left( Y_m(\omega), \frac{s}{R_m^{1/d}} \right) : B(u, g(n)) \cap B(v, g(m)) \neq \emptyset \right\} = O(1). \quad (3.13) \]

If \( g(m) < g(n), \) then every \( v \in S\left( Y_m(\omega), \frac{s}{R_m^{1/d}} \right) \) satisfying \( B(u, g(n)) \cap B(v, g(m)) \neq \emptyset \) must also satisfy \( B(v, g(m)) \subseteq B(u, 3g(n)). \) Since the elements of \( S\left( Y_m(\omega), \frac{s}{R_m^{1/d}} \right) \) are by definition separated by a factor \( \frac{1}{R_m^{1/d}}, \) it follows from a volume argument that
\[ \# \left\{ v \in S\left( Y_m(\omega), \frac{s}{R_m^{1/d}} \right) : B(u, g(n)) \cap B(v, g(m)) \neq \emptyset \right\} = O\left( \frac{(g(n))^d R_m}{s^d} + 1 \right). \quad (3.14) \]

Combining (3.13) and (3.14), we see that for any \( n \in \mathbb{N} \) such that \( \omega \in B(c,s,n), \) and \( m \neq n \) such that \( \omega \in B(c,s,m), \) we have
\[ \# \left\{ v \in S\left( Y_m(\omega), \frac{s}{R_m^{1/d}} \right) : B(u, g(n)) \cap B(v, g(m)) \neq \emptyset \right\} = O\left( \frac{(g(n))^d R_m}{s^d} + 1 \right). \quad (3.15) \]

We now use (3.15) to bound \( \mathcal{L}(E_n \cap E_m): \)
\[ \mathcal{L}(E_n \cap E_m) \leq \sum_{u \in S(Y_n(\omega), \frac{s}{R_n^{1/d}})} \mathcal{L}(B(u, g(n)) \cap E_m). \]
Focusing on the second term in (3.18), we have
\[
\leq \sum_{u \in S(Y_n(\omega), \frac{R_n}{R_m^{1/d}})} \mathcal{L}(0, g(m)) \# \left\{ v \in S(Y_n(\omega), \frac{s}{R_m^{1/d}}) : B(u, g(n)) \cap B(v, g(m)) \neq \emptyset \right\}
\]
\[
\overset{\text{(3.16)}}{=} \sum_{u \in S(Y_n(\omega), \frac{R_n}{R_m^{1/d}})} \mathcal{O}\left( g(m)^d \left( \frac{g(n)^d R_m}{s^d} + 1 \right) \right)
\]
\[
= \mathcal{O}\left( R_n g(m)^d \left( \frac{g(n)^d R_m}{s^d} + 1 \right) \right).
\]

Summarising the above, we have shown that for any \( n \in \mathbb{N} \) such that \( \omega \in B(c, s, n) \), and \( m \neq n \) such that \( \omega \in B(c, s, m) \), we have:
\[
\mathcal{L}(E_n \cap E_m) = \mathcal{O}\left( R_n g(m)^d \left( \frac{g(n)^d R_m}{s^d} + 1 \right) \right).
\]  
(3.16)

**Step 4. Applying Lemma 3.3.**

By Lemma 3.3 to prove that (3.10) holds, and to finish our proof, it suffices to show that
\[
\sum_{n,m=1}^{Q} \mathcal{L}(E_n \cap E_m) = \mathcal{O}\left( \left( \sum_{n=1}^{Q} \mathcal{L}(E_n) \right)^2 \right).
\]  
(3.17)

This we do below. We start by separating terms:
\[
\sum_{n,m=1}^{Q} \mathcal{L}(E_n \cap E_m) = \sum_{n,m=1}^{Q} \mathcal{L}(E_n) + 2 \sum_{m=2}^{Q} \sum_{n=1}^{m-1} \mathcal{L}(E_n \cap E_m).
\]  
(3.18)

Focusing on the first term on the right hand side of (3.18), we know that
\[
\sum_{n=1}^{\infty} \mathcal{L}(E_n) = \infty
\]
by (3.11). Therefore, for all \( Q \) sufficiently large we have
\[
\sum_{n=1}^{Q} \mathcal{L}(E_n) \geq 1.
\]

This implies that
\[
\sum_{n=1}^{Q} \mathcal{L}(E_n) = \mathcal{O}\left( \left( \sum_{n=1}^{Q} \mathcal{L}(E_n) \right)^2 \right).
\]  
(3.19)

Focusing on the second term in (3.18), we have
\[
\sum_{m=2}^{Q} \sum_{n=1}^{m-1} \mathcal{L}(E_n \cap E_m) \overset{\text{(3.16)}}{=} \mathcal{O}\left( \sum_{m=2}^{Q} \sum_{n=1}^{m-1} R_n g(m)^d \left( \frac{g(n)^d R_m}{s^d} + 1 \right) \right)
\]
\[
= \mathcal{O}\left( \sum_{m=2}^{Q} \sum_{n=1}^{m-1} R_n g(n)^d R_m g(m)^d \right).
\]

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\[ + \mathcal{O}\left( \sum_{m=2}^{Q} \sum_{n=1}^{m-1} R_n g(m)^d \right) \]

Focusing on the first term in the above, we see that
\[
\begin{align*}
\sum_{m=2}^{Q} \sum_{n=1}^{m-1} R_n g(m)^d R_m g(m)^d &= \sum_{m=2}^{Q} R_m g(m)^d \sum_{n=1}^{m-1} R_n g(n)^d \\
&\leq \left( \sum_{n=1}^{Q} R_n g(n)^d \right)^2 \\
&= \mathcal{O}\left( \sum_{n=1}^{Q} \mathcal{L}(E_n) \right)^2.
\end{align*}
\]

Focusing on the second term, we have
\[
\begin{align*}
\sum_{m=2}^{Q} \sum_{n=1}^{m-1} R_n g(m)^d &= \mathcal{O}\left( \sum_{m=2}^{Q} R_m g(m)^d \right) \\
&= \mathcal{O}\left( \sum_{m=1}^{Q} \mathcal{L}(E_m) \right) \\
&= \mathcal{O}\left( \left( \sum_{m=1}^{Q} \mathcal{L}(E_m) \right)^2 \right).
\end{align*}
\]

In the first equality above we used the assumption that \( R_n \asymp \gamma^n \), and therefore by properties of geometric series
\[
\sum_{n=1}^{m-1} R_n \leq \sum_{n=1}^{m-1} R_n = \mathcal{O}(R_m).
\]

This is the only point in the proof where we use the assumption \( R_n \asymp \gamma^n \).

Collecting the bounds obtained above, we see that
\[
\sum_{m=2}^{Q} \sum_{n=1}^{m-1} \mathcal{L}(E_n \cap E_m) = \mathcal{O}\left( \left( \sum_{n=1}^{Q} \mathcal{L}(E_n) \right)^2 \right). \tag{3.20}
\]

Substituting the bounds \( \text{(3.19)} \) and \( \text{(3.20)} \) into \( \text{(3.18)} \), we see that \( \text{(3.17)} \) holds as required. This completes our proof.

With Proposition 3.4 we can now prove Proposition 3.1.

Proof of Proposition 3.1. Let
\[
P := \bigcap_{\epsilon > 0} \bigcup_{c,s > 0} \{ \omega : d(n : \omega \in B(c,s,n)) \geq 1 - \epsilon \}.
\]
Fix $\omega \in P$. For any $h \in H$, by definition there exists $\epsilon > 0$ such that $h \in H_\epsilon$. It follows from the definition of $P$, that there exists $c, s > 0$ such that

$$\bar{d}(n : \omega \in B(c, s, n)) > 1 - \epsilon.$$ 

In which case, by the definition of $H_\epsilon$, we must have

$$\sum_{n : \omega \in B(c, s, n)} h(n) = \infty.$$ 

Applying Proposition [3.4] it follows that

$$\left\{ x \in \mathbb{R}^d : |x - f_{l,n}(\omega)| \leq \left( \frac{h(n)}{R_n} \right)^{1/d} \text{ for i.m. } (l, n) \in \{1, \ldots, R_n\} \times \mathbb{N} \right\}$$

has positive Lebesgue measure. Our result now follows since $\omega \in P$ was arbitrary and we know by Lemma [3.2] that $\eta(P) = \eta(\Omega)$.

### 3.1.1 Verifying the hypothesis of Proposition [3.1]

To prove Theorem 2.2, Theorem 2.6, and Theorem 2.9, we will apply Proposition 3.1. Naturally to do so we need to verify the hypothesis of Proposition 3.1. The exponential growth condition on the number of elements in our set will be automatically satisfied. Verifying the second integral condition is more involved. We will show that this integral condition holds via a transversality argument. Unfortunately the quantity $T(Y_n(\omega), \frac{s}{R_n^{1/d}})$ is not immediately amenable to transversality techniques. Instead we study the quantity:

$$R(\omega, s, n) := \left\{ (l, l') \in \{1, \ldots, R_n\}^2 : |f_{l,n}(\omega) - f_{l',n}(\omega)| \leq \frac{s}{R_n^{1/d}} \text{ and } l \neq l' \right\}.$$ 

The following lemma allows us to deduce the integral bound appearing in Proposition 3.1 from a similar bound for $R(\omega, s, n)$.

**Lemma 3.5.** Assume there exists $G : (0, \infty) \to (0, \infty)$ satisfying $\lim_{s \to 0} G(s) = 0$, such that for all $n \in \mathbb{N}$ we have

$$\int_{\Omega} \frac{\#R(\omega, s, n)}{R_n} d\eta \leq G(s).$$

Then for all $n \in \mathbb{N}$ we have

$$\eta(\Omega) - \int_{\Omega} \frac{T(Y_n(\omega), \frac{s}{R_n^{1/d}})}{R_n} d\eta \leq G(s).$$

**Proof.** Let

$$W(\omega, s, n) := \left\{ l \in \{1, \ldots, R_n\} : |f_{l,n}(\omega) - f_{l',n}(\omega)| > \frac{s}{R_n^{1/d}} \forall l' \neq l \right\}.$$ 

Since for any distinct $l, l' \in W(\omega, s, n)$, the distance between $f_{l,n}(\omega)$ and $f_{l',n}(\omega)$ is at least $\frac{s}{R_n^{1/d}}$, it follows that $W(\omega, s, n)$ is a $\frac{s}{R_n^{1/d}}$-separated set. Therefore

$$\#W(\omega, s, n) \leq T(Y_n(\omega), \frac{s}{R_n^{1/d}}).$$

(3.21)
Importantly we also have

$$\#W(\omega, s, n)^c = \#\{l \in \{1, \ldots, R_n\} \mid |f_{l,n}(\omega) - f_{l', n}(\omega)| \leq \frac{s}{R_n^{1/d}} \text{ for some } l' \neq l\} \leq \#R(\omega, s, n).$$

(3.22)

This follows because the map \( f : R(\omega, s, n) \to W(\omega, s, n)^c \) defined by \( f(l, l') = l \) is a surjective map.

Now suppose we have \( G : (0, \infty) \to (0, \infty) \) satisfying the hypothesis of our proposition. Then for any \( s > 0 \) and \( n \in \mathbb{N} \) we have

$$\eta(\Omega) = \int_\Omega \frac{\#W(\omega, s, n) + \#W(\omega, s, n)^c}{R_n} d\eta \leq \int_\Omega \frac{T(Y_n(\omega), \frac{s}{R_n^{1/d}})}{R_n} d\eta + \int_\Omega \frac{\#R(\omega, s, n)}{R_n} d\eta \leq \int_\Omega \frac{T(Y_n(\omega), \frac{s}{R_n^{1/d}})}{R_n} d\eta + G(s).$$

This implies

$$\eta(\Omega) - \int_\Omega \frac{T(Y_n(\omega), \frac{s}{R_n^{1/d}})}{R_n} d\eta \leq G(s).$$

3.1.2 The non-existence of a Khintchine like result

The purpose of this section is to prove the following proposition. It will be used in the proof of Theorem 2.10 and Theorem 2.15. It demonstrates that a lack of separation along a subsequence can lead to the non-existence of a Khintchine like result.

**Proposition 3.6.** Let \( \omega \in \Omega \) and suppose that for some \( s > 0 \) we have

$$\liminf_{n \to \infty} \frac{T(Y_n(\omega), \frac{s}{R_n^{1/d}})}{R_n} = 0.$$  

Then there exists \( h : \mathbb{N} \to [0, \infty) \) such that

$$\sum_{n=1}^\infty h(n) = \infty,$$

yet

$$\{ x : |x - f_{l,n}(\omega)| \leq \left( \frac{h(n)}{R_n} \right)^{1/d} \text{ for i.m. } (l, n) \in \{1, \ldots, R_n\} \times \mathbb{N} \}$$

has zero Lebesgue measure.

**Proof.** Let \( \omega \in \Omega \) and \( s > 0 \) be as above. By our assumption, there exists a strictly increasing sequence \((n_j)\) such that

$$T\left(Y_{n_j}(\omega), \frac{s}{R_{n_j}^{1/d}}\right) \leq \frac{R_{n_j}}{j^2}$$

(3.23)

for all \( j \in \mathbb{N} \). By the definition of a maximal \( s \cdot R_{n_j}^{-1/d} \)-separated set, we know that for each \( l \in \{1, \ldots, R_{n_j}\} \), there exists \( u \in S(Y_{n_j}(\omega), \frac{s}{R_{n_j}^{1/d}}) \) such that \( |u - f_{l,n_j}(\omega)| \leq s \cdot R_{n_j}^{-1/d} \). It follows that

$$\bigcup_{l \in \{1, \ldots, R_{n_j}\}} B(f_{l,n_j}(\omega), \frac{s}{R_{n_j}^{1/d}}) \subseteq \bigcup_{u \in S(Y_{n_j}(\omega), \frac{s}{R_{n_j}^{1/d}})} B\left(u, \frac{3s}{R_{n_j}^{1/d}}\right).$$

(3.24)
We now define our function $h : \mathbb{N} \to [0, \infty)$:

$$h(n) = \begin{cases} 
  s & \text{if } n = n_j \text{ for some } j \in \mathbb{N} \\
  0 & \text{otherwise}
\end{cases}$$

This function obviously satisfies

$$\sum_{n=1}^{\infty} h(n) = \infty.$$ 

By (3.24) and the definition of $h$, we see that

$$L \left( \left\{ x : |x - f_{l,n}(\omega)| \leq \left( \frac{h(n)}{R_n} \right)^{1/d} \text{ for i.m. } (l, n) \in \{1, \ldots, R_n\} \times \mathbb{N} \right\} \right) \leq L \left( \left\{ x : |x - u| \leq \left( \frac{3s}{R_n j} \right)^{1/d} \text{ for i.m. } u \in \bigcup_{j=1}^{\infty} S \left( Y_{n_j}(\omega), \frac{s}{R_n j} \right) \right\} \right). \quad (3.25)$$

So to prove our result it suffices to show that the right hand side of (3.25) is zero. This fact now follows from the Borel-Cantelli lemma and the following inequalities:

$$\sum_{j=1}^{\infty} \sum_{u \in S(Y_{n_j}(\omega), \frac{s}{R_n j})} L(B(u, \frac{3s}{R_n j}^{1/d})) = \sum_{j=1}^{\infty} T(Y_{n_j}(\omega), \frac{s}{R_n j^{1/d}}) \frac{(3s)^d L(B(0, 1))}{R_n j} \leq \sum_{j=1}^{\infty} \frac{(3s)^d L(B(0, 1))}{j^2} < \infty.$$

### 3.2 Full measure statements

The main result of the previous section was Proposition 3.1. This result provides sufficient conditions which allow us to conclude that for a parameterised family of points, almost surely each member of a class of limsup sets defined in terms of neighbourhoods of these points will have positive Lebesgue measure. We will eventually apply Proposition 3.1 to the sets $U_{\Psi}(z, m, h)$ defined in the introduction. Instead of just proving positive measure statements, we would like to be able to prove full measure results. The purpose of this section is to show how one can achieve this goal. Proposition 3.8 achieves this by imposing some extra assumptions on the function $\Psi$. Proposition 3.9 achieves this by imposing some stronger separation hypothesis.

The following lemma follows from Lemma 1 from [11]. It is a consequence of the Lebesgue density theorem.

**Lemma 3.7** ([11]). The following statements are true:

1. Let $(x_j)$ be a sequence of points in $\mathbb{R}^d$ and $(r_j), (r'_j)$ be two sequences of real numbers satisfying $r_j \asymp r'_j$. Then

$$L(x : x \in B(x_j, r_j) \text{ for i.m. } j) = L(x : x \in B(x_j, r'_j) \text{ for i.m. } j).$$

2. Let $B(x_j, r_j)$ be a sequence of balls in $\mathbb{R}^d$ such that $r_j \to 0$. Then

$$L(x : x \in B(x_j, r_j) \text{ for i.m. } j) = L \left( \bigcap_{0 < c < 1} \{ x : x \in B(x_j, cr_j) \text{ for i.m. } j \} \right).$$
Lemma 3.7 implies the following useful fact. If \( \Psi : D^* \to [0, \infty) \) is equivalent to \((m, h)\) and \( U_\Phi(z, m, h) \) has positive Lebesgue measure, then \( W_\Phi(z, \Psi) \) has positive Lebesgue measure. We will use this fact several times throughout this paper.

Lemma 3.7 will be used in the proof of the following proposition and in the proofs of our main theorems. Recall that we say that a function \( \Psi : D^* \to [0, \infty) \) is weakly decaying if

\[
\inf_{a \in D^*} \min_{i \in D} \frac{\Psi(ia)}{\Psi(a)} > 0.
\]

**Proposition 3.8.** The following statements are true:

1. Assume \( \Phi \) is a collection of similarities with attractor \( X \). If \( z \in X \) is such that \( \mathcal{L}(W_\Phi(z, \Psi)) > 0 \) for some \( \Psi \) that is weakly decaying, then Lebesgue almost every \( x \in X \) is containing in \( W_\Phi(z, \Psi) \).

2. Assume \( \Phi \) is an arbitrary IFS and there exists \( \mu \), the pushforward of a \( \sigma \)-invariant ergodic probability measure \( m \), satisfying \( \mu \sim \mathcal{L}|_X \). Then if \( z \in X \) is such that \( \mathcal{L}(W_\Phi(z, \Psi)) > 0 \) for some \( \Psi \) that is weakly decaying, then Lebesgue almost every \( x \in X \) is containing in \( W_\Phi(z, \Psi) \).

**Proof.** We prove each statement separately.

**Proof of statement 1.**

Let \( \Phi \) be an IFS consisting of similarities and suppose \( z \) and \( \Psi \) satisfy the hypothesis of the proposition. Let

\[
A := \bigcap_{0 < c < 1} W_\Phi(z, c\Psi).
\]

It follows from Lemma 3.7 that

\[
\mathcal{L}(A) = \mathcal{L}(W_\Phi(z, \Psi)) > 0.
\]

We claim that

\[
\mathcal{L} \left( \bigcup_{a \in D^*} \phi_a(A) \right) = \mathcal{L}(X). \tag{3.26}
\]

To see that (3.26) holds, suppose otherwise and assume

\[
\mathcal{L} \left( X \setminus \bigcup_{a \in D^*} \phi_a(A) \right) > 0.
\]

Moreover, let \( x^* \) be a density point of

\[
X \setminus \bigcup_{a \in D^*} \phi_a(A).
\]

Such a point has to exist by the Lebesgue density theorem.

Let \( (b_j) \in D^N \) be such that \( \pi((b_j)) = x^* \), and let \( 0 < r < Diam(X) \) be arbitrary. We let \( n(r) \in \mathbb{N} \) be such that

\[
Diam(X) \prod_{j=1}^{n(r)} r_{b_j} < r \leq Diam(X) \prod_{j=1}^{n(r)-1} r_{b_j}.
\]

The parameter \( n(r) \) satisfies the following:

\[
(\phi_{b_1} \circ \cdots \circ \phi_{b_{n(r)}})(X) \subseteq B(x^*, r), \tag{3.27}
\]
and
\[
\frac{r \cdot \min_{i \in D} r_i}{\text{Diam}(X)} \leq \prod_{j=1}^{n(r)} r_{b_j}.
\]

(3.28)

Using (3.27) and (3.28) we can now bound
\[
\mathcal{L}\left(B(x^*, r) \cap \left(X \setminus \bigcup_{a \in D^*} \phi_a(A)\right)\right).
\]

Observe
\[
\mathcal{L}\left(B(x^*, r) \cap \left(X \setminus \bigcup_{a \in D^*} \phi_a(A)\right)\right) \leq \mathcal{L}(B(x^*, r)) - \mathcal{L}((\phi_{b_1} \circ \cdots \circ \phi_{b_n(r)})(A))
\]
\[
= \mathcal{L}(B(0, 1)) r^d - \left(\prod_{j=1}^{n(r)} r_{b_j}\right) \mathcal{L}(A)
\]
\[
\leq \mathcal{L}(B(0, 1)) r^d - \frac{r^d \mathcal{L}(A) \min_{i \in D} r_i^d}{\text{Diam}(X)}
\]
\[
\leq \mathcal{L}(B(0, 1)) r^d \left(1 - \frac{\mathcal{L}(A) \min_{i \in D} r_i^d}{\mathcal{L}(B(0, 1)) \text{Diam}(X)}\right).
\]

Therefore
\[
\limsup_{r \to 0} \frac{\mathcal{L}(B(x^*, r) \cap \left(X \setminus \bigcup_{a \in D^*} \phi_a(A)\right))}{\mathcal{L}(B(x^*, r^d))} < 1.
\]

This implies that \(x^*\) cannot be a density point of
\[
X \setminus \bigcup_{a \in D^*} \phi_a(A),
\]

and we may conclude that (3.26) holds.

We will now show that
\[
\bigcup_{a \in D^*} \phi_a(A) \subseteq W_{\Phi}(z, \Psi).
\]

(3.29)

Let
\[
y \in \bigcup_{a \in D^*} \phi_a(A)
\]
be arbitrary. Let \((b_1, \ldots, b_k) \in D^*\) and \(v \in A\) be such that
\[
(\phi_{b_1} \circ \cdots \circ \phi_{b_k})(v) = y.
\]

Let
\[
d := \inf_{a \in D^*} \min_{i \in D} \frac{\Psi(ia)}{\Psi(a)}.
\]

(3.30)

Since \(\Psi\) is weakly decaying \(d > 0\).

Now suppose \(a \in D^*\) is such that
\[
|\phi_a(z) - v| \leq c \Psi(a),
\]

where
\[
c := d^k \max_{i \in D} \{r_i\}^{-k}.
\]

(3.31)
Then
\[
| (\phi_{b_1} \circ \cdots \circ \phi_{b_k} \circ \phi_a)(z) - y | = | (\phi_{b_1} \circ \cdots \circ \phi_{b_k} \circ \phi_a(z) - \phi_{b_1} \circ \cdots \circ \phi_{b_k}(v) | \\
= | \phi_a(z) - v | \prod_{j=1}^{k} r_{b_j} \\
\leq c \Psi(a) \prod_{j=1}^{k} r_{b_j} \\
\leq cd^{-k} \Psi((b_1, \ldots, b_k, a)) \prod_{j=1}^{k} r_{b_j} \\
\leq \Psi((b_1, \ldots, b_k, a)) \tag{3.30}
\]

It follows that for this choice of \( c \), whenever
\[
| \phi_a(z) - v | \leq c \Psi(a), \tag{3.32}
\]
we also have
\[
| (\phi_{b_1} \circ \cdots \circ \phi_{b_k} \circ \phi_a(z) - y | \leq \Psi((b_1, \ldots, b_k, a)). \tag{3.33}
\]

It follows from the definition of \( A \) that \( v \) has infinitely many solutions to (3.32), therefore \( y \) has infinitely many solutions to (3.33) and \( y \in W_\Phi(z, \Psi) \). It follows now from (3.26) and (3.29) that Lebesgue almost every \( x \in X \) is contained in \( W_\Phi(z, \Psi) \).

**Proof of statement 2.**
Let \( \Phi \) be an IFS and \( \mu \) be the pushforward of some \( \sigma \)-invariant ergodic probability measure \( m \). We assume that assume \( \mu \sim L|_X \). Let \( z \) and \( \Psi \) satisfy the hypothesis of our proposition. Let \( A \) be as in the proof of statement 1. It follows from our assumptions and Lemma 3.7 that \( L(A) > 0 \). Since \( \mu \sim L|_X \) we also have \( \mu(A) > 0 \). We will prove that
\[
\mu\left( \bigcup_{a \in D^*} \phi_a(A) \right) = 1. \tag{3.34}
\]
By our assumption \( m(\pi^{-1}(A)) = \mu(A) > 0 \). By the ergodicity of \( m \) we have
\[
m\left( \bigcup_{n=0}^{\infty} \sigma^{-n}(\pi^{-1}(A)) \right) = 1. \tag{3.35}
\]
Now observe that
\[
\mu\left( \bigcup_{a \in D^*} \phi_a(A) \right) = m\left( \pi^{-1}\left( \bigcup_{a \in D^*} \phi_a(A) \right) \right) \geq m\left( \bigcup_{n=0}^{\infty} \sigma^{-n}(\pi^{-1}(A)) \right).
\]
Therefore (3.35) implies (3.34). Since \( \mu \sim L|_X \) it follows that
\[
L\left( \bigcup_{a \in D^*} \phi_a(A) \right) = L(X). \tag{3.36}
\]
We now let
\[
y \in \bigcup_{a \in D^*} \phi_a(A)
\]
be arbitrary. Defining appropriate analogues of \((b_1, \ldots, b_k)\) and \(c\) as in the proof of statement 1, it will follow that \(y \in W_\Phi(z, \Psi)\). Therefore

\[
\bigcup_{a \in D^*} \phi_n(A) \subseteq W_\Phi(z, \Psi).
\]

Combining this fact with (3.36) completes the proof of statement 2.

Proposition 3.8 is a useful technique for determining full measure statements, but there is an additional cost as we require the function \(\Psi\) to be weakly decaying. The following proposition requires no extra condition on the function \(\Psi\), but does require some stronger separation assumptions.

**Proposition 3.9.** Suppose \(\Phi = \{A_i x + t_i\}\) is a collection of affine contractions and \(\mu\) is the pushforward of a Bernoulli measure \(\mathfrak{m}\). Assume that one of following properties are satisfied:

- \(\Phi\) consists solely of similarities.
- \(d = 2\) and all the matrices \(A_i\) are equal.
- All the matrices \(A_i\) are simultaneously diagonalisable.

Let \(z \in X\) and suppose that for some \(s > 0\) there exists a subsequence \((n_k)\) satisfying

\[
\lim_{k \to \infty} T(Y_{\mathfrak{m}, n_k}(z), \frac{s}{R_{\mathfrak{m}, n_k}}) = 1.
\]

Then \(\mu \sim \mathcal{L}|_X\), and for any \(h\) that satisfies

\[
\sum_{k=1}^{\infty} h(n_k) = \infty,
\]

we have that Lebesgue almost every \(x \in X\) is contained in \(U_\Phi(z, \mathfrak{m}, h)\).

The proof of Proposition 3.9 is more involved than Proposition 3.8 and will rely on the following technical result.

**Lemma 3.10.**

1. Let \(\mu\) be a self-similar measure. Then either \(\mu \sim \mathcal{L}|_X\) or \(\mu\) is singular.

2. Suppose \(\Phi = \{A_i x + t_i\}\) is a collection of affine contractions and one of following properties are satisfied:

   - \(d = 2\) and all the matrices \(A_i\) are equal.
   - All the matrices \(A_i\) are simultaneously diagonalisable.

   Then if \(\mu\) is the pushforward of a Bernoulli measure we have either \(\mu \sim \mathcal{L}|_X\) or \(\mu\) is singular.

3. Let \(\Phi\) be an arbitrary iterated function system and \(\mu\) be the pushforward of a \(\sigma\)-invariant ergodic probability measure \(\mathfrak{m}\). Then either \(\mu \ll \mathcal{L}\) or \(\mu\) is singular (i.e. \(\mu\) is of pure type).

**Proof.** A proof of statement 1 can be found in [41]. It makes use of an argument originally appearing in [37]. Statement 2 was proved in [53, Section 4.4.] using ideas of Guzman [24] and Fromberg [38].

We could not find a proof of statement 3 so we include one for completeness. Suppose that \(\mu\) is not singular, then by the Lebesgue decomposition theorem \(\mu = \mu_0 + \mu_1\) where \(\mu_0 \ll \mathcal{L}, \mu_1 \perp \mathcal{L}\),

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and \( \mu_0(X) > 0 \). Suppose that \( \mu \neq \mu_0 \). Then there exists \( A \) such that \( \mu_1(A) > 0 \). Since \( \mu_1 \perp \mathcal{L} \), we may assume without loss of generality that \( \mathcal{L}(A) = 0 \). Using the ergodicity of \( \mathfrak{m} \), it follows from an analogous argument to that used in the proof of statement 2 from Proposition 3.8 that 
\[
\mu(\bigcup_{a \in \mathcal{D}} \phi_a(A)) = 1.
\]
Therefore we must have \( \mu_0(\bigcup_{a \in \mathcal{D}} \phi_a(A)) > 0 \), and by absolute continuity 
\( \mathcal{L}(\bigcup_{a \in \mathcal{D}} \phi_a(A)) > 0 \). Since each \( \phi_a \) is a contraction, \( \mathcal{L}(A) = 0 \) implies that \( \mathcal{L}(\phi_a(A)) = 0 \) for all \( a \in \mathcal{D}^* \). This contradicts that \( \mathcal{L}(\bigcup_{a \in \mathcal{D}} \phi_a(A)) > 0 \). Therefore we must have \( \mu = \mu_0 \). \( \square \)

Only statement 1 and statement 2 from Lemma 3.10 will be needed in the proof of Proposition 3.9. Statement 3 is needed in the proof of the following result which we formulate as generally as possible.

**Proposition 3.11.** Let \( \mu \) be the pushforward of a slowly decaying \( \sigma \)-invariant ergodic probability measure \( \mathfrak{m} \). If for some \( z \in X \) and \( s > 0 \) we have

\[
\limsup_{n \to \infty} \frac{T(Y_{m,n}(z), \frac{s}{R_{m,n}})}{R_{m,n}} > 0,
\]

then \( \mu \ll \mathcal{L} \).

**Proof.** We start our proof by remarking that for any \( z \in X \),

\[
\mu = \lim_{n \to \infty} \sum_{a \in L_{m,n}} \mathfrak{m}(\{a\}) \cdot \delta_{\phi_a(z)}.
\]

(3.37)

Where convergence is meant with respect to the weak star topology. By our assumption, for some \( z \in X \) and \( s > 0 \), there exists a sequence \((n_k)\) and \( c > 0 \) such that

\[
\lim_{n \to \infty} \frac{T(Y_{m,n_k}(z), \frac{s}{R_{m,n_k}})}{R_{m,n_k}} > c
\]

(3.38)

for all \( k \). Define

\[
\mu'_{n_k} := \sum_{a \in L_{m,n_k}} \mathfrak{m}(\{a\}) \cdot \delta_{\phi_a(z)}
\]

\[
\phi_a(z) \in S(Y_{m,n_k}(z), \frac{s}{R_{m,n_k}})
\]

and

\[
\mu''_{n_k} := \sum_{a \in L_{m,n_k}} \mathfrak{m}(\{a\}) \cdot \delta_{\phi_a(z)}
\]

\[
\phi_a(z) \notin S(Y_{m,n_k}(z), \frac{s}{R_{m,n_k}})
\]

Then

\[
\sum_{a \in L_{m,n_k}} \mathfrak{m}(\{a\}) \cdot \delta_{\phi_a(z)} = \mu'_{n_k} + \mu''_{n_k}.
\]

Without loss of generality, we may assume by compactness that there exist two finite measures \( \nu' \) and \( \nu'' \) such that \( \lim_{k \to \infty} \mu'_{n_k} = \nu' \) and \( \lim_{k \to \infty} \mu''_{n_k} = \nu'' \). Therefore by (3.37) we have \( \mu = \nu' + \nu'' \). We will prove that \( \nu'(X) > 0 \) and \( \nu' \) is absolutely continuous with respect to the Lebesgue measure. Since \( \mu \) is either singular or absolutely continuous by Lemma 3.10 it will follow that \( \mu \ll \mathcal{L} \).

It follows from the definition of \( L_{m,n} \) that for any \( a, a' \in L_{m,n} \) we have

\[
\mathfrak{m}(\{a\}) \asymp \mathfrak{m}(\{a'\}).
\]

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This implies that for any $a \in L_{m,n}$ we have

$$m([a]) \asymp R_{m,n}^{-1}. \quad (3.39)$$

Using (3.38) and (3.39), we have that

$$\nu'_k(X) = \sum_{a \in L_{m,n}} m([a]) T(Y_{m,n_k}(z), \frac{s}{R_{m,n_k}}) \frac{\nu}{R_{m,n_k}} \asymp c \cdot R_{m,n_k} \cdot 1.$$ 

Therefore $\nu'(X) \geq \lim_{k \to \infty} \nu'_k(X) > 0$. Now we prove that $\nu'$ is absolutely continuous. Fix an arbitrary open $d$-dimensional cube $(x_1, x_1 + r) \times \cdots \times (x_d, x_d + r) \subset \mathbb{R}^d$, we have

$$\mu'_k((x_1, x_1 + r) \times \cdots \times (x_d, x_d + r)) = \sum_{a \in L_{m,n}} m([a])$$

$$\times \phi_a(z) \in S(Y_{m,n_k}(z), \frac{s}{R_{m,n_k}}) \cap (x_1, x_1 + r) \times \cdots \times (x_d, x_d + r)$$

$$= O \left( \frac{\# \{ \phi_a(z) \in S(Y_{m,n_k}(z), \frac{s}{R_{m,n_k}}) \cap (x_1, x_1 + r) \times \cdots \times (x_d, x_d + r) \}}{R_{m,n}} \right). \quad (3.40)$$

In the last line we used (3.39). Since the elements of $S(Y_{m,n_k}(z), \frac{s}{R_{m,n_k}})$ are separated by a factor $\frac{s}{R_{m,n}}$, it follows from a volume argument that we must have

$$\# \{ \phi_a(z) \in S(Y_{m,n_k}(z), \frac{s}{R_{m,n_k}}) \cap (x_1, x_1 + r) \times \cdots \times (x_d, x_d + r) \} = O \left( \frac{r^d R_{m,n}}{s^d} \right).$$

Substituting this bound into (3.40) we have

$$\mu'_k((x_1, x_1 + r) \times \cdots \times (x_d, x_d + r)) = O \left( \frac{r^d R_{m,n}}{s^d} \right) = O \left( \frac{r^d}{s^d} \right).$$

Letting $k \to \infty$, it follows that for any $d$-dimensional cube we have

$$\nu'((x_1, x_1 + r) \times \cdots \times (x_d, x_d + r)) = O \left( \frac{r^d}{s^d} \right).$$

Since $s$ is fixed $\nu'$ must be absolutely continuous. This completes our proof. \hfill \Box

As well as Proposition 3.11 being used in our proof of Proposition 3.9, it can be seen as a new tool for proving that measures are absolutely continuous. Proposition 3.11 can be used in conjunction with Lemma 3.2 and Lemma 3.5 to recover known results on the absolute continuity of measures within a parameterised family. We include one such instance of this in Section 4, where we recover the well known result due to Solomyak that for almost every $\lambda \in (1/2, 1)$, the unbiased Bernoulli convolution is absolutely continuous [59].

With these preliminary results we are now in a position to prove Proposition 3.9.

**Proof of Proposition 3.9.** Let $\Phi$ be an IFS satisfying one of our conditions and $\mu$ be the push-forward of a Bernoulli measure $m$. Let $z \in X$, $s > 0$, and $(n_k)$ satisfy the hypothesis of our
proposition. By an application of Proposition 3.11 we know that \( \mu \ll \mathcal{L} \). Moreover, by Lemma 3.10 we also know that \( \mu \sim \mathcal{L}|_X \).

To prove our result, it will be sufficient to show that

\[
\mathcal{L}\left( \left\{ x \in \mathbb{R}^d : |x - \phi_a(z)| \leq \left( \frac{h(n_k)}{R_{m,n_k}} \right)^{1/d} \text{ for i.m. } a \in \bigcup_{k=1}^{\infty} L_{m,n_k} \right\} \right) = \mathcal{L}(X),
\]

for any \( h \) satisfying

\[
\sum_{k=1}^{\infty} h(n_k) = \infty.
\]

It will then follow from Lemma 3.10 and the fact that \( m([a]) \asymp R_{m,n}^{-1} \) for \( a \in L_{m,n} \), that (3.41) implies that Lebesgue almost every \( x \in X \) is contained in \( U_\Phi(z, m, h) \) for any \( h \) satisfying (3.42).

Our proof of (3.41) will follow from a similar type of argument to that given in the proof of Proposition 3.4. Where necessary to avoid repetition we will omit certain details. Our strategy for proving (3.41) holds is to prove that for Lebesgue almost every \( y \in X \), there exists \( c_y > 0 \), such that for all \( r \) sufficiently small we have

\[
\mathcal{L}\left( B(y, 2r) \cap \left\{ x \in \mathbb{R}^d : |x - \phi_a(z)| \leq \left( \frac{h(n_k)}{R_{m,n_k}} \right)^{1/d} \text{ for i.m. } a \in \bigcup_{k=1}^{\infty} L_{m,n_k} \right\} \right) \geq c_y r^d.
\]

Importantly \( c_y \) will not depend upon \( r \). It follows by an application of the Lebesgue density theorem that (3.43) implies (3.41). As in the proof of Proposition 3.4, we split our proof of (3.43) into smaller steps.

**Step 1. Local information.**

We have already established that \( \mu \sim \mathcal{L}|_X \). Let \( d \) denote the Radon-Nikodym derivative \( d\mu/d\mathcal{L} \). For \( \mu \)-almost every \( y \) we must have \( d(y) > 0 \). It follows now by the Lebesgue differentiation theorem, and the fact that \( \mu \sim \mathcal{L}|_X \), that for Lebesgue almost every \( y \in X \) we have

\[
\lim_{r \to 0} \frac{\mu(B(y, r))}{\mathcal{L}(B(y, r))} = d(y) > 0.
\]

In what follows \( y \) is a fixed element of \( X \) satisfying this property. Let \( r^* \) be such that for all \( r \in (0, r^*) \), we have

\[
\frac{d(y)}{2} < \frac{\mu(B(y, r))}{\mathcal{L}(B(y, r))} < 2d(y).
\]

Now using that \( \mu \) is the weak star limit of the sequence of measures

\[
\mu_{n_k} := \sum_{a \in L_{m,n_k}} m([a]) \cdot \delta_{\phi_a(z)},
\]

together with (3.44), we can assert that for each \( r \in (0, r^*) \), for \( k \) sufficiently large we have

\[
\mu_{n_k}(B(y, r)) = \sum_{a \in L_{m,n_k}, \phi_a(z) \in B(y, r)} m([a]) \asymp d(y) r^d.
\]

By construction we know that each \( a \in L_{m,n_k} \) satisfies

\[
m([a]) \asymp R_{m,n_k}^{-1}.
\]
Therefore it follows from (3.45) that for all \( k \) sufficiently large
\[
\frac{\#\{\alpha \in L_{m,n_k} : \phi_\alpha(z) \in B(y,r)\}}{R_{m,n_k}} \asymp d(y)r^d. \tag{3.46}
\]
Let
\[
A(y,r,k) := \{ \phi_\alpha(z) : \phi_\alpha(z) \in S(Y_{m,n_k}(z), \frac{s}{R_{m,n_k}^{1/d}}) \cap B(y,r) \}.
\]
Since
\[
\lim_{k \to \infty} \frac{T(Y_{m,n_k}(z), \frac{s}{R_{m,n_k}^{1/d}})}{R_{m,n_k}} = 1,
\]
it follows from (3.46) that there exists \( K(r) \in \mathbb{N} \), such that for any \( k \geq K(r) \) we have
\[
\#A(y,r,k) \asymp R_{m,n_k}d(y)r^d. \tag{3.47}
\]
Equation (3.47) has established that for each \( k \geq K(r) \), there is a large separated set that is local to \( B(y,r) \).

We will prove that there exists \( c_y > 0 \) such that
\[
L \left( B(y,2r) \cap \left\{ x \in \mathbb{R}^d : |x - \phi_\alpha(z)| \leq \left( \frac{h(n_k)}{R_{m,n_k}} \right)^{1/d} \text{ for i.m. } \phi_\alpha(z) \in \bigcup_{k=K(r)}^{\infty} A(y,r,k) \right\} \right) \geq c_y r^d. \tag{3.48}
\]
Equation (3.48) implies (3.43). So to complete our proof it suffices to show that (3.48) holds.

For later use note that
\[
\sum_{k=K(r)}^{\infty} \sum_{\phi_\alpha(z) \in A(y,r,k)} L \left( B(\phi_\alpha(z), \frac{h(n_k)}{R_{m,n_k}}^{1/d}) \right) = \infty. \tag{3.49}
\]
Equation (3.49) is true because
\[
\sum_{k=K(r)}^{\infty} \sum_{\phi_\alpha(z) \in A(y,r,k)} L \left( B(\phi_\alpha(z), \frac{h(n_k)}{R_{m,n_k}}^{1/d}) \right) = \sum_{k=1}^{\infty} \#A(y,r,k) \frac{L(B(0,1))h(n_k)}{R_{m,n_k}} \asymp d(y)r^d L(B(0,1)) \sum_{k=K(r)}^{\infty} h(n_k) \equiv \infty.
\]

Step 2. Replacing our approximating function.
Let
\[
g(n_k) := \min \left\{ \left( \frac{h(n_k)}{R_{m,n_k}} \right)^{1/d}, \frac{s}{3R_{m,n_k}^{1/d}} \right\}.
\]
For each \( k \geq K(r) \) we define the set
\[
E_{n_k} := \left\{ B(\phi_\alpha(z), g(n_k)) : \phi_\alpha(z) \in A(y,r,k) \right\}.
\]
By construction the balls in \( E_{n_k} \) are disjoint. Therefore by (3.47), for each \( k \geq K(r) \)
\[
L(E_{n_k}) \asymp g(n_k)^d R_{m,n_k}d(y)r^d. \tag{3.50}
\]
By a similar argument to that given in the proof of Proposition 3.4, it follows from (3.49) that we have
\[
\sum_{k=K(r)}^{\infty} \mathcal{L}(E_{n_k}) = \infty.
\] (3.51)

Therefore we satisfy the assumptions of Lemma 3.3. We will use this lemma to show that
\[
\mathcal{L}\left(\limsup_{k \to \infty} E_{n_k}\right) \geq c_y r^d.
\] (3.52)

Since
\[
\limsup_{k \to \infty} E_{n_k} \subseteq B(y, 2r),
\]
because \(\lim_{k \to \infty} g(n_k) = 0\), we see that (3.52) implies (3.48). So verifying (3.52) will complete our proof.

**Step 3. Bounding** \(L(E_{n_k} \cap E_{n_k}')\).

By an analogous argument to that given in the proof of Proposition 3.4 we can show that for any \(\phi_a(z) \in A(y, r, k)\), we have
\[
\# \{ \phi_{a'}(z) \in A(y, r, k') : B(\phi_a(z), g(n_k)) \cap B(\phi_{a'}(z), g(n_{k'})) \neq \emptyset \} = O\left(\frac{(g(n_k)^d R_{m,n_k})}{s^d} + 1\right).
\]

Using this estimate and (3.47), it can be shown that for any distinct \(k, k' \geq K(r)\) we have
\[
\mathcal{L}(E_{n_k} \cap E_{n_k}') = O\left(d(y)r^d R_{m,n_k} g(n_{k'})^d \left(\frac{g(n_k)^d R_{m,n_k}}{s^d} + 1\right)\right). \tag{3.53}
\]

**Step 4. Applying Lemma 3.3**

Using (3.53), we can then replicate the arguments used in the proof of Proposition 3.4 to show that that
\[
\sum_{k,k'=K(r)}^{Q} \mathcal{L}(E_{n_k} \cap E_{n_k}') = O\left(d(y)r^d \left(\sum_{k=K(r)}^{Q} R_{m,n_k} g(n_k)^d\right)^2\right). \tag{3.54}
\]

We emphasise here that the underlying constants in (3.54) do not depend upon \(r\). By (3.50) we have
\[
\left(\sum_{k=K(r)}^{Q} \mathcal{L}(E_{n_k})\right)^2 \asymp r^{2d} d(y)^2 \left(\sum_{k=K(r)}^{Q} R_{m,n_k} g(n_k)\right)^2. \tag{3.55}
\]

Applying Lemma 3.3 in conjunction with (3.54) and (3.55) yields
\[
\mathcal{L}\left(\limsup_{k \to \infty} E_{n_k}\right) \geq c_y r^d,
\]
for some \(c_y > 0\) that does not depend upon \(r\). Therefore (3.52) holds and we have completed our proof.

\[\square\]

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4 Applications of Proposition 3.1

In this section we apply the results of Section 3 to prove Theorem 2.2, Theorem 2.6, and Theorem 2.9. We begin by briefly explaining why the exponential growth condition appearing in Proposition 3.1 will always be satisfied in our proofs.

Let $m$ be a slowly decaying probability measure supported on $\mathcal{D}$. We remark that each $a \in L_m, n$ satisfies

$m([a]) \asymp c_n^{-m}$.  

Recall that $c_m$ is defined in Section 1.3.2. Importantly the cylinders corresponding to elements of $L_m, n$ are disjoint, and we have $m(\bigcup_{a \in L_m, n} [a]) = 1$. It follows from these observations that

$R_m, n \asymp c^{-n}$.  

Similarly, if $\tilde{L}_m, n \subseteq L_m, n$ and $m(\bigcup_{a \in \tilde{L}_m, n} [a]) > d$ for some $d > 0$ for each $n$, then

$\# \tilde{L}_m, n =: \tilde{R}_m, n \asymp c^{-n}$.  

Where the underlying constants depend upon $d$ but are independent of $n$. In our proofs of Theorem 2.2, Theorem 2.6, and Theorem 2.9, it will be necessary to define an $\tilde{L}_m, n$ contained in $L_m, n$, whose union of cylinders has measure uniformly bounded away from zero. By the above discussion the exponential growth condition appearing in Proposition 3.1 will automatically be satisfied.

4.1 Proof of Theorem 2.2

Before proceeding with our proof of Theorem 2.2 we recall some useful results from [59].

**Lemma 4.1** (Lemma 2.1 [59]). For any $\epsilon_1 > 0$, there exists $\delta = \delta(\epsilon_1) > 0$ such that if $g \in B_\Gamma$, and $g(0) \neq 0$, then

$x \in (0, \alpha(B_\Gamma) - \epsilon_1], |g(x)| < \delta \implies |g'(x)| > \delta$.  

**Lemma 4.2.** Let $\epsilon_1 > 0$ and $\delta(\epsilon_1) > 0$ be as in Lemma 4.1. Then for any $\epsilon_2 > 0$ and $g \in B_\Gamma$ such that $g(0) \neq 0$, we have

$\mathcal{L}(\{\lambda \in (0, \alpha(B_\Gamma) - \epsilon_1] : |g(\lambda)| \leq \epsilon_2\}) = O(\epsilon_2)$.  

Where the underlying constant depends upon $\delta(\epsilon_1)$.

**Lemma 4.2** follows from the analysis given in Section 2.4. from [59]. Equipped with Lemma 4.1 and the results of Section 3 we can now prove Theorem 2.2.

**Proof of Theorem 2.2.** We treat each statement in this theorem individually. We start with the proof of statement 1.

**Proof of statement 1.**

Let us start by fixing $m$ a slowly decaying $\sigma$-invariant ergodic probability measure with $h(m) > 0$, and $(a_j) \in \mathcal{D}$. Let $\epsilon_1 > 0$ be arbitrary. We now choose $\epsilon_2 > 0$ sufficiently small so that we have

$e^{h(m) - \epsilon_2}(e^{-h(m)} + \epsilon_1) > 1$.  

(4.1)

By the Shannon-McMillan-Breiman theorem, we know that for $m$-almost every $a \in \mathcal{D}$ we have

$\lim_{n \to \infty} -\frac{\log m([a_1, \ldots, a_n])}{n} = h(m)$.  

(4.2)
It follows from [12] and Egorov’s theorem, that there exists $C = C(\epsilon_2) > 0$ such that
\[
m(a \in D^n : \frac{e^n(-b(m) - \epsilon_2)}{\alpha} \leq m([a_1, \ldots, a_n]) \leq Ce^n(-b(m) + \epsilon_2), \forall n \in \mathbb{N}) > 1/2.
\]

Let
\[
\tilde{L}_{m,n} := \left\{ a \in L_{m,n} : \frac{e^n(-b(m) - \epsilon)}{\alpha} \leq m([a_1, \ldots, a_n]) \leq Ce^n(-b(m) + \epsilon) \right\}
\]
and
\[
\tilde{R}_{m,n} := \#\tilde{L}_{m,n}.
\]
Since
\[
m\left( \bigcup_{a \in \tilde{L}_{m,n}} [a] \right) = 1,
\]
it follows from the above that
\[
m\left( \bigcup_{a \in \tilde{L}_{m,n}} [a] \right) > 1/2. \tag{4.3}
\]

By the discussion at the beginning of this section we know that $\tilde{R}_{m,n}$ satisfies the exponential growth condition of Proposition 3.1. It also follows from this discussion that
\[
\tilde{R}_{m,n} \asymp R_{m,n}. \tag{4.4}
\]
Recalling the notation used in Section 3, let
\[
R(\lambda, s, n) := \left\{ (a, a') \in \tilde{L}_{m,n} \times \tilde{L}_{m,n} : \left| \phi_a \left( \sum_{j=1}^{\infty} d_{a_j} \lambda^{j-1} \right) - \phi_{a'} \left( \sum_{j=1}^{\infty} d_{a_j} \lambda^{j-1} \right) \right| \leq \frac{s}{R_{m,n}} \text{ and } a \neq a' \right\}
\]
The main step in our proof of statement 1 is to show that
\[
\int_{e^{-b(m) + \epsilon_1}}^{e^{-b(m) - \epsilon_1}} \frac{\#R(\lambda, s, n)}{R_{m,n}} d\lambda = O(s). \tag{4.5}
\]
We will then be able to employ the results of Section 3 to prove our theorem. Our proof of (4.5) is based upon an argument of Benjamini and Solomyak [7], which in turn is based upon an argument of Peres and Solomyak [12].

**Step 1. Proof of (4.5).**

Observe the following:
\[
\int_{e^{-b(m) + \epsilon_1}}^{e^{-b(m) - \epsilon_1}} \frac{\#R(\lambda, s, n)}{R_{m,n}} d\lambda
\]
\[
= \int_{e^{-b(m) + \epsilon_1}}^{e^{-b(m) - \epsilon_1}} \frac{1}{R_{m,n}} \sum_{a, a' \in \tilde{L}_{m,n} \atop a \neq a'} \chi_{[-\frac{s}{R_{m,n}}, \frac{s}{R_{m,n}}]} \left( \phi_a \left( \sum_{j=1}^{\infty} d_{a_j} \lambda^{j-1} \right) - \phi_{a'} \left( \sum_{j=1}^{\infty} d_{a_j} \lambda^{j-1} \right) \right) d\lambda
\]
\[
= O \left( \tilde{R}_{m,n} \int_{e^{-b(m) + \epsilon_1}}^{e^{-b(m) - \epsilon_1}} \sum_{a, a' \in \tilde{L}_{m,n} \atop a \neq a'} \chi_{[-\frac{s}{R_{m,n}}, \frac{s}{R_{m,n}}]} \left( \phi_a \left( \sum_{j=1}^{\infty} d_{a_j} \lambda^{j-1} \right) - \phi_{a'} \left( \sum_{j=1}^{\infty} d_{a_j} \lambda^{j-1} \right) \right) m([a]) m([a']) d\lambda \right)
\]

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In the penultimate line we used that for any $a \in \hat{L}_{m,n}$ we have $m([a]) \asymp \hat{R}_{m,n}^{-1}$.

Note that for any distinct $a, a' \in \hat{L}_{m,n}$ we have

$$\phi_a \left( \sum_{j=1}^{\infty} d_{aj} \lambda^{j-1} \right) - \phi_{a'} \left( \sum_{j=1}^{\infty} d_{aj} \lambda^{j-1} \right) \in B_\Gamma.$$

Let $|a \land a'| := \inf \{ k : a_k \neq a'_k \}$. Then

$$\phi_a \left( \sum_{j=1}^{\infty} d_{aj} \lambda^{j-1} \right) - \phi_{a'} \left( \sum_{j=1}^{\infty} d_{aj} \lambda^{j-1} \right) = \lambda^{|a \land a'| - 1} g(\lambda),$$

for some $g \in B_\Gamma$ satisfying $g(0) \neq 0$. Therefore, for any distinct $a, a' \in \hat{L}_{m,n}$ we have

$$\int_{e^{-h(m)} + \epsilon_1}^{\alpha(B_\Gamma) - \epsilon_1} \chi_{\left[ e^{-h(m)}, \frac{s}{R_{m,n}} \right]} \left( \phi_a \left( \sum_{j=1}^{\infty} d_{aj} \lambda^{j-1} \right) - \phi_{a'} \left( \sum_{j=1}^{\infty} d_{aj} \lambda^{j-1} \right) \right) d\lambda$$

$$= \mathcal{L} \left( \lambda \in \left( e^{-h(m)} + \epsilon_1, \alpha(B_\Gamma) - \epsilon_1 \right) : \phi_a \left( \sum_{j=1}^{\infty} d_{aj} \lambda^{j-1} \right) - \phi_{a'} \left( \sum_{j=1}^{\infty} d_{aj} \lambda^{j-1} \right) \in \left[ - \frac{s}{R_{m,n}}, \frac{s}{R_{m,n}} \right] \right)$$

$$= \mathcal{L} \left( \lambda \in \left( e^{-h(m)} + \epsilon_1, \alpha(B_\Gamma) - \epsilon_1 \right) : \lambda^{|a \land a'| - 1} g(\lambda) \in \left[ - \frac{s}{R_{m,n}}, \frac{s}{R_{m,n}} \right] \right)$$

$$= \mathcal{L} \left( \lambda \in \left( e^{-h(m)} + \epsilon_1, \alpha(B_\Gamma) - \epsilon_1 \right) : g(\lambda) \in \left[ - \frac{s\lambda^{-|a \land a'|+1}}{R_{m,n}}, \frac{s\lambda^{-|a \land a'|+1}}{R_{m,n}} \right] \right)$$

$$\leq \mathcal{L} \left( \lambda \in \left( e^{-h(m)} + \epsilon_1, \alpha(B_\Gamma) - \epsilon_1 \right) : g(\lambda) \in \left[ - \frac{s(e^{-h(m)} + \epsilon_1)^{-|a \land a'|+1}}{R_{m,n}}, \frac{s(e^{-h(m)} + \epsilon_1)^{-|a \land a'|+1}}{R_{m,n}} \right] \right)$$

$$= \mathcal{L} \left( \frac{s(e^{-h(m)} + \epsilon_1)^{-|a \land a'|}}{R_{m,n}} \right).$$

Where in the last line we used Lemma 4.2. Summarising the above, we have shown that

$$\int_{e^{-h(m)} + \epsilon_1}^{\alpha(B_\Gamma) - \epsilon_1} \chi_{\left[ e^{-h(m)}, \frac{s}{R_{m,n}} \right]} \left( \phi_a \left( \sum_{j=1}^{\infty} d_{aj} \lambda^{j-1} \right) - \phi_{a'} \left( \sum_{j=1}^{\infty} d_{aj} \lambda^{j-1} \right) \right) d\lambda = \mathcal{O} \left( \frac{s(e^{-h(m)} + \epsilon_1)^{-|a \land a'|}}{R_{m,n}} \right)$$

(4.7)

Substituting (4.7) into (4.6) we obtain:

$$\int_{e^{-h(m)} + \epsilon_1}^{\alpha(B_\Gamma) - \epsilon_1} \frac{\# R(\lambda, s, n)}{R_{m,n}} d\lambda = \mathcal{O} \left( \hat{R}_{m,n} \sum_{a, a' \in \hat{L}_{m,n}} \frac{m([a])m([a'])}{a \neq a'} \frac{s(e^{-h(m)} + \epsilon_1)^{-|a \land a'|}}{R_{m,n}} \right)$$

$$= \mathcal{O} \left( s \sum_{a, a' \in \hat{L}_{m,n}} \frac{m([a])m([a'])}{a \neq a'} \frac{(e^{-h(m)} + \epsilon_1)^{-|a \land a'|}}{R_{m,n}} \right)$$

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Therefore by Lemma 3.5 we know that
\[
\sum_{k=1}^{\infty} e^{-k(h(m)-\epsilon_2)}(e^{-h(m)} + \epsilon_1)^{-k} < \infty.
\]
Therefore
\[
\int_{e^{-h(m)}+\epsilon_1}^{\alpha(B_T)-\epsilon_1} \frac{\#R(\lambda, s, n)}{R_{m,n}} d\lambda = O \left( s \sum_{a \in L_{m,n}} m([a]) \right) = O(s)
\] as required.

Step 2. Applying (4.5).

Combining (4.5) and Lemma 3.5 we obtain
\[
\mathcal{L}(e^{-h(m)} + \epsilon_1, \alpha(B_T) - \epsilon_1) - \int_{e^{-h(m)} + \epsilon_1}^{\alpha(B_T) - \epsilon_1} T \left( \phi_a(\sum_{j=1}^{\infty} d_{aj} \lambda^{j-1}) \right) a \in L_{m,n}, \frac{s}{R_{m,n}} = O(s).
\]

Therefore by Proposition 3.1 we know that for Lebesgue almost every \( \lambda \in [e^{-h(m)} + \epsilon_1, \alpha(B_T) - \epsilon_1] \), the set
\[
\left\{ x \in \mathbb{R}^d : |x - \phi_a(\sum_{j=1}^{\infty} d_{aj} \lambda^{j-1})| \leq \frac{h(n)}{R_{m,n}} \text{ for i.m. } a \in L_{m,n} \right\}
\]
has positive Lebesgue measure for any \( h \in H \). Since \( \epsilon_1 \) was arbitrary, we can assert that for Lebesgue almost every \( \lambda \in (e^{-h(m)}, \alpha(B_T)) \), for any \( h \in H \) the above set has positive Lebesgue measure. By (4.4) we know that \( R_{m,n} \sim R_{m,n} \). Which by the discussion given at the start of this section implies \( R_{m,n} \sim m([a]) \) for each \( a \in L_{m,n} \). Therefore by Lemma 3.1 we may conclude that for Lebesgue almost every \( \lambda \in (e^{-h(m)}, \alpha(B_T)) \), for any \( h \in H \) the set \( U_{\Phi_{\lambda,D}}(\sum_{j=1}^{\infty} d_{aj} \lambda^{j-1}, m, h) \) has positive Lebesgue measure.

Proof of statement 2.

We start our proof of this statement by remarking that since \( m \) is the uniform \((1/l, \ldots, 1/l)\) Bernoulli measure, we have \( L_{m,n} = \mathcal{D}^n \) for each \( n \in \mathbb{N} \). Since the words in \( \mathcal{D}^n \) have the same length and each similarity contracts by a factor \( \lambda \), it can be shown that
\[
\phi_a(z) - \phi_a(z') = \lambda^n (z - z'),
\]
for all \( a \in D^n \) for any \( z, z' \in X_{\lambda,D} \). Importantly this difference does not depend upon \( a \). Therefore the sets \( \{ \phi_a(z) \}_{a \in D^n} \) and \( \{ \phi_a(z') \}_{a \in D^n} \) are translates of each other. In which case

\[
T \left( \{ \phi_a(z) \}_{a \in D^n}, \frac{s}{2^n} \right) = T \left( \{ \phi_a(z') \}_{a \in D^n}, \frac{s}{2^n} \right)
\]

(4.9)

for any \( z, z' \in X_{\lambda,D} \).

Examining the proof of statement 1, the important step was obtaining (4.5) then applying Proposition 3.1. Examining the proof of Proposition 3.1, we see that (4.5) allows us to conclude that for any \( (a_j) \in D^N \), for Lebesgue almost every \( \lambda \in [1/l + \epsilon_1, \alpha(B_l) - \epsilon_1] \), given \( \epsilon > 0 \) we can pick \( c, s > 0 \) such that

\[
\frac{d}{n} \left( n : \frac{T \left( \{ \phi_a(z) \}_{a \in D^n}, \frac{s}{2^n} \} \geq c \right) > 1 - \epsilon. \right.
\]

(4.10)

If \( \lambda \) is such that (4.10) holds for a specific sequence \( (a_j) \in D^N \), then (4.9) implies that it must hold for all \( (a_j) \in D^N \) simultaneously. Therefore, we may assert that for Lebesgue almost every \( \lambda \in [1/l + \epsilon_1, \alpha(B_l) - \epsilon_1] \), given \( \epsilon > 0 \) we can pick \( c, s > 0 \), such that for any \( z \in X_{\lambda,D} \) we have

\[
\frac{d}{n} \left( n : \frac{T \left( \{ \phi_a(z) \}_{a \in D^n}, \frac{s}{2^n} \} \geq c \right) > 1 - \epsilon. \right.
\]

(4.11)

Again examining the proof of Proposition 3.1, we see that (4.11) implies that for Lebesgue almost every \( \lambda \in [1/l + \epsilon_1, \alpha(B_l) - \epsilon_1] \), for any \( z \in X_{\lambda,D} \) and \( h \in H \), the set

\[
\left\{ x \in \mathbb{R}^d : |x - \phi_a(z)| \leq \frac{h(n)}{2^n} \text{ for i.m. } a \in \bigcup_{n=1}^\infty D^n \right\}
\]

has positive Lebesgue measure. In other words, for Lebesgue almost every \( \lambda \in [1/l + \epsilon_1, \alpha(B_l) - \epsilon_1] \), for any \( z \in X_{\lambda,D} \) and \( h \in H \), the set \( U_{\Phi_{\lambda,D}}(z, m, h) \) has positive Lebesgue measure. Since \( \epsilon_1 \) was arbitrary we can conclude our result for Lebesgue almost every \( \lambda \in (1/l, \alpha(B_l)) \).

**Proof of statement 3.**

By statement 1 we know that for any \( (a_j) \in D^N \), for Lebesgue almost every \( \lambda \in (e^{-h(m)}, \alpha(B_l)) \), for any \( h \in H \) the set \( U_{\Phi_{\lambda,D}}(\sum_{j=1}^\infty d_j \lambda_j^{-1}, m, h) \) has positive Lebesgue measure. It follows therefore by Lemma 3.7 that for any \( (a_j) \in D^N \), for Lebesgue almost every \( \lambda \in (e^{-h(m)}, \alpha(B_l)) \), for any \( \Psi \) that is equivalent to \( (m, h) \) for some \( h \in H \), the set \( W_{\Phi_{\lambda,D}}(\sum_{j=1}^\infty d_j \lambda_j^{-1}, \Psi) \) has positive Lebesgue measure. Applying Proposition 3.8 we may conclude that for any \( (a_j) \in D^N \), for Lebesgue almost every \( \lambda \in (e^{-h(m)}, \alpha(B_l)) \), for any \( \Psi \in \Upsilon_m \) Lebesgue almost every \( x \in X_{\lambda,D} \) is contained in \( W_{\Phi_{\lambda,D}}(\sum_{j=1}^\infty d_j \lambda_j^{-1}, \Psi) \).

**Proof of statement 4.**

The proof of statement 4 is analogous to the proof of statement 3. The only difference is that instead of using statement 1 at the beginning we use statement 2.

We now explain how Corollary 2.3 follows from Theorem 2.2.

**Proof of Corollary 2.3** Let us start by fixing \( h : \mathbb{N} \rightarrow [0, \infty) \) to be \( h(n) = 1/n \). We remark that this function \( h \) is an element of \( H \). This can be proved using the well known fact

\[
\sum_{n=1}^N \frac{1}{n} \sim \log N.
\]
Let us now fix a Bernoulli measure $\mathbf{m}$ as in the statement of Corollary 2.3. Observe that for any $a \in \mathcal{D}^*$ we have

$$
\left(\min_{i \in \mathcal{D}} p_i \right) |a| \leq m(|a|) \leq \left(\max_{i \in \mathcal{D}} p_i \right) |a|
$$

(4.12)

Using (4.12) and the fact that each $a \in L_{m,n}$ satisfies $m(|a|) \asymp c_m^{-n}$, it can be shown that each $a \in L_{m,n}$ satisfies

$$|a| \asymp n.$$

This implies that for any $a \in L_{m,n}$ we have

$$
\prod_{j=1}^{|a|} |a_j| \asymp m(|a|) / n.
$$

In other words, the function $\Psi : \mathcal{D}^* \to [0, \infty)$ given by

$$
\Psi(a) = \prod_{j=1}^{|a|} |a_j| / |a|
$$

is equivalent to $(m, h)$ for our choice of $h$. One can verify that our function $\Psi$ is weakly decaying and therefore $\Psi \in \Upsilon_m$. Therefore by Theorem 2.2, for any $(a_j) \in \mathcal{D}^N$, for almost every $\lambda \in \prod_{i=1}^N p_i^\lambda (\mathcal{B}_\Gamma)$, Lebesgue almost every $x \in X_{\lambda,D}$ is contained in the set $W_{\Phi_{\lambda,D}}(\sum_{j=1}^\infty d_{a_j} \lambda^{j-1}, \Psi)$.

4.1.1 Bernoulli Convolutions

Given $\lambda \in (0,1)$ and $p \in (0,1)$, let $\mu_{\lambda,p}$ be the distribution of the random sum

$$
\sum_{j=0}^\infty \pm \lambda^j,
$$

where $+$ is chosen with probability $p$, and $-$ is chosen with probability $(1-p)$. When $p = 1/2$ we simply denote $\mu_{\lambda,1/2}$ by $\mu_\lambda$. We call $\mu_{\lambda,p}$ a Bernoulli convolution. When we want to emphasise the case when $p = 1/2$ we call $\mu_{\lambda}$ the unbiased Bernoulli convolution. Importantly, for each $p \in (0,1)$ the Bernoulli convolution $\mu_{\lambda,p}$ is a self-similar measure for the iterated function system $\{\lambda x - 1, \lambda x + 1\}$.

The study of Bernoulli convolutions dates back to the 1930s and to the important work of Jessen and Wintner [30], and Erdős [17, 18]. When $\lambda \in (0,1/2)$ then $\mu_{\lambda,p}$ is supported on a Cantor set and determining the dimension of $\mu_{\lambda,p}$ is relatively straightforward. When $\lambda \in (1/2, 1)$ the support of $\mu_{\lambda,p}$ is the interval $[\lambda - 1, \lambda + 1]$. Analysing a Bernoulli convolution for $\lambda \in (1/2, 1)$ is a more difficult task. The important problems in this area are:

- To classify those $\lambda \in (1/2, 1)$ and $p \in (0,1)$ such that

$$
\dim_H \mu_{\lambda,p} = \min \left\{ \frac{p \log p + (1-p) \log (1-p)}{\log \lambda}, 1 \right\}.
$$

(4.13)

- To classify those $\lambda \in (1/2, 1)$ and $p \in (0,1)$ such that $\mu_{\lambda,p} \ll \mathcal{L}$.

Initial progress was made on the second problem by Erdős in [17]. He proved that whenever $\lambda$ is the reciprocal of a Pisot number then $\mu_\lambda \perp \mathcal{L}$. This result was later improved upon in two papers by Alexander and Yorke [2], and Garsia [23], who independently proved that $\dim_H \mu_\lambda < 1$ when $\lambda$ is the reciprocal of a Pisot number. Garsia in [22] also provided an explicit class of algebraic
integers for which $\mu_\lambda \ll \mathcal{L}$. The next breakthrough came in a result of Solomyak \cite{solomyak2004} who proved that $\mu_\lambda \ll \mathcal{L}$ with a density in $L^2$ for almost every $\lambda \in (1/2, 1)$. His proof relied on studying the Fourier transform of $\mu_\lambda$. A simpler proof of this result was subsequently obtained by Peres and Solomyak in \cite{peres2005}. This proof relied upon a characterisation of absolute continuity in terms of differentiation of measures (see \cite{hochman2008}). Improvements and generalisations of this result appeared subsequently in \cite{Varju2011}, \cite{shmerkin2013}, and \cite{shmerkin2014}. Over the last few years dramatic progress has been made on the problems listed above. In particular, Hochman in \cite{hochman2008} proved that for a set $E$ of packing dimension 0, it is the case that if $\lambda \in (1/2, 1) \setminus E$ then we have equality in (4.13) for any $p \in (0, 1)$. Building upon this result, Shmerkin in \cite{shmerkin2014} proved that $\mu_\lambda \ll \mathcal{L}$ for every $\lambda \in (1/2, 1)$ outside of a set of Hausdorff dimension zero. This result was later generalised to the case of general $p$ by Shmerkin and Solomyak in \cite{shmerkin2014}. Similarly building upon the result of Hochman, Varju recently proved in \cite{Varju2020} that $\dim_H \mu_\lambda = 1$ whenever $\lambda$ is a transcendental number. Varju has also recently provided new explicit examples of $\lambda$ and $p$ such that $\mu_{\lambda,p} \ll \mathcal{L}$ (see \cite{Varju2021}).

Theorem \ref{thm:4.2} can be applied to the IFS $\{\lambda x - 1, \lambda x + 1\}$. In \cite{solomyak2004} Solomyak proved that $\alpha(\mathcal{B}(-1,0,1)) > 0.639$, this was subsequently improved upon by Shmerkin and Solomyak in \cite{shmerkin2014} who proved that $\alpha(\mathcal{B}(-1,0,1)) > 0.668 \ldots$. Using this information we can prove the following result.

**Theorem 4.3.** Let $\Psi : \mathcal{D}^* \to [0, \infty)$ be given by $\Psi(a) = \frac{1}{2^{a_1}|a|}$. Then for Lebesgue almost every $\lambda \in (1/2, 0.668)$, we have that for any $z \in \left[\frac{1}{1-\lambda}, \frac{1}{1-\lambda}\right]$, Lebesgue almost every $x \in \left[\frac{1}{1-\lambda}, \frac{1}{1-\lambda}\right]$ is contained in $W_{\Phi_{\lambda,(-1,1)}}(z, \Psi)$.

The proof of Theorem \ref{thm:4.3} is an adaptation of the proof of Corollary \ref{cor:2.3} and is therefore omitted.

As a by-product of our analysis we can recover the result of Solomyak that for almost every $\lambda \in (1/2, 1)$ the unbiased Bernoulli convolution is absolutely continuous. Our approach does not allow us to assert anything about the density. However our approach does have the benefit of being particularly simple and intuitive. Instead of relying on the Fourier transform, differentiation of measures, or the advanced entropy methods of Hochman \cite{hochman2008}, the proof given below appeals to the fact that $\mu_\lambda$ is of pure type and makes use of a decomposition argument due to Solomyak. For the sake of brevity, the proof below only focuses on the important features of the argument.

**Theorem 4.4 (Solomyak \cite{solomyak2004}).** For Lebesgue almost every $\lambda \in (1/2, 1)$ we have $\mu_\lambda \ll \mathcal{L}$.

**Proof.** We split our proof into individual steps.

**Step 1. Proof that $\mu_\lambda \ll \mathcal{L}$ for Lebesgue almost every $\lambda \in (1/2, 0.668).$**

Fix $(a_j) \in \mathcal{D}^N$. We know by our proof of Theorem \ref{thm:2.2} that for any $\epsilon_1 > 0$ we have

$$\mathcal{L}([1/2 + \epsilon_1, 0.668 - \epsilon_1]) - \int_{1/2+\epsilon_1}^{0.668-\epsilon_1} \frac{T(\{\phi_a(\sum_{j=1}^{\infty} a_j \lambda^{j-1})\}_{a \in \mathcal{D}^N}, \Psi)}{2^n} d\lambda = O(s). \quad (4.14)$$

Combining (4.14) with Lemma 3.2 we may conclude that

$$\mathcal{L}\left(\bigcap_{c>0, s>0} \{\lambda \in ([1/2+\epsilon_1, 0.668-\epsilon_1]) : d(n : \omega \in B(c, s, n)) \geq 1 - \epsilon\}\right) = \mathcal{L}([1/2 + \epsilon_1, 0.668 - \epsilon_1]). \quad (4.15)$$

Where here

$$B(c, s, n) = \left\{\lambda \in [1/2 + \epsilon_1, 0.668 - \epsilon_1] : \frac{T(\{\phi_a(\sum_{j=1}^{\infty} a_j \lambda^{j-1})\}_{a \in \mathcal{D}^N}, \Psi)}{2^n} \geq c\right\}.$$
In particular, \((4.15)\) implies that for Lebesgue almost every \(\lambda \in [1/2 + \epsilon_1, 0.668 - \epsilon_1]\), there exists \(c > 0\) and \(s > 0\) such that
\[
\limsup_{n \to \infty} \frac{T(\{a(\sum_{j=1}^{\infty} a_j \lambda^{j-1}\})_{a \in D^n, \frac{s}{2^n}})}{2^n} \geq c.
\]

Applying Proposition 3.11 it follows that for Lebesgue almost every \(\lambda \in [1/2 + \epsilon_1, 0.668 - \epsilon_1]\), the measure \(\mu_\lambda\) is absolutely continuous. Since \(\epsilon_1\) is arbitrary we know that for Lebesgue almost every \(\lambda \in (1/2, 0.668)\) the measure \(\mu_\lambda\) is absolutely continuous.

**Step 2. Proof that \(\mu_\lambda \ll \mathcal{L}\) for Lebesgue almost every \(\lambda \in (2^{-2/3}, 0.713)\).**

Let \(\eta_\lambda\) denote the distribution of the random sum
\[
\sum_{j=0}^{\infty} \pm \lambda^j,
\]
where each digit is chosen with probability 1/2. One can show that \(\mu_\lambda = \eta_\lambda * \nu_\lambda\) for some measure \(\nu_\lambda\). Since the convolution of an absolutely continuous measure with an arbitrary measure is still absolutely continuous, to prove \(\mu_\lambda \ll \mathcal{L}\) for Lebesgue almost every \(\lambda \in (2^{-2/3}, 0.713)\), it suffices to show that \(\eta_\lambda\) is absolutely continuous for Lebesgue almost every \(\lambda \in (2^{-2/3}, 0.713)\).

Importantly \(\eta_\lambda\) can be realised as the self-similar measure for the iterated function system
\[
\left\{ \rho_1(x) = \lambda^3 x + 1 + \lambda, \rho_2(x) = \lambda^3 x - 1 + \lambda, \rho_3(x) = \lambda^3 x + 1 - \lambda, \rho_4(x) = \lambda^3 x - 1 - \lambda \right\}
\]
and the uniform \((1/4, 1/4, 1/4, 1/4)\) Bernoulli measure. Because the translation parameter depends upon \(\lambda\), this family of iterated function systems does not immediately fall into the class considered by Theorem 2.2. However this distinction is only superficial, and one can adapt the argument used in the proof of \((4.14)\) to prove that for any \(\epsilon_1 > 0\) and \((a_j) \in \{1, 2, 3, 4\}^\mathbb{N}\), we have
\[
\mathcal{L}([2^{-2/3} + \epsilon_1, 0.713 - \epsilon_1]) - \int_{2^{-2/3} + \epsilon_1}^{0.713 - \epsilon_1} \frac{T(\{\rho_\lambda(a(\pi(a_j)))\}_{a \in \{1, 2, 3, 4\}^\mathbb{N}, \frac{s}{2^n}})}{4^n} d\lambda = O(s). \quad (4.16)
\]

The parameter 0.713 comes from [59] and is a lower bound for the appropriate analogue of \(\alpha(\mathcal{B}(-1, 0, 1))\) for the family of iterated function systems \(\{\rho_1, \rho_2, \rho_3, \rho_4\}\). Without going into details, it can be shown that appropriate analogues of Lemma 4.1 and Lemma 4.2 persist for this family of iterated function systems. These statements can then be used to deduce that \((4.16)\) holds. By the arguments used in step 1, we can use \((4.16)\) in conjunction with Lemma 3.2 and Proposition 3.11 to deduce that \(\eta_\lambda\) is absolutely continuous for Lebesgue almost every \(\lambda \in (2^{-2/3}, 0.713)\).

**Step 3. Proof that \(\mu_\lambda \ll \mathcal{L}\) for Lebesgue almost every \(\lambda \in (1/2, 1)\).**

Since \((1/2, 1/\sqrt{2}) \subset (1/2, 0.668) \cup (2^{-2/3}, 0.713)\), we know by the two previous steps that for Lebesgue almost every \(\lambda \in (1/2, 1/\sqrt{2})\), the measure \(\mu_\lambda\) is absolutely continuous. For any \(\lambda \in (2^{-1/k}, 2^{-1/2k})\) for some \(k \geq 2\), we can express \(\mu_\lambda\) as \(\mu_{\lambda k} * \nu_\lambda\) for some measure \(\nu_\lambda\). Since for Lebesgue almost every \(\lambda \in (1/2, 1/\sqrt{2})\), the measure \(\mu_{\lambda k}\) is absolutely continuous, it follows that for Lebesgue almost every \(\lambda \in (2^{-1/k}, 2^{-1/2k})\), the measure \(\mu_{\lambda k}\) is also absolutely continuous. Since \(\mu_{\lambda k} = \mu_\lambda * \nu_\lambda\) it follows that \(\mu_\lambda\) is absolutely continuous for Lebesgue almost every \(\lambda \in (2^{-1/k}, 2^{-1/2k})\). Importantly the intervals \((2^{-1/k}, 2^{-1/2k})\) exhaust \((1/\sqrt{2}, 1)\). It follows therefore that \(\mu_\lambda\) is absolutely continuous for Lebesgue almost every \(\lambda \in (1/\sqrt{2}, 1)\). Our previous steps cover the interval \((1/2, 1/\sqrt{2})\), so we may conclude that \(\mu_\lambda\) is absolutely continuous for Lebesgue almost every \(\lambda \in (1/2, 1)\).
4.1.2 The \(\{0, 1, 3\}\) problem

Let \(\lambda \in (0, 1)\) and

\[
C_\lambda := \left\{ \sum_{j=0}^{\infty} a_j \lambda^j : a_j \in \{0, 1, 3\} \right\}.
\]

\(C_\lambda\) is the attractor of the IFS \(\{\lambda x, \lambda x + 1, \lambda x + 3\}\). When \(\lambda \in (0, 1/4)\) the IFS satisfies the strong separation condition and one can prove that \(\dim_H C_\lambda = \frac{\log 3}{-\log 3}\). When \(\lambda \geq 2/5\) the set \(C_\lambda\) is the interval \([0, \frac{2}{1-\lambda}]\). The two main problems in the study of \(C_\lambda\) are:

- Classify those \(\lambda \in (1/4, 1/3)\) such that \(\dim_H C_\lambda = \frac{\log 3}{-\log 3}\).
- Classify those \(\lambda \in (1/3, 2/5)\) such that \(C_\lambda\) has positive Lebesgue measure.

Initial progress on these problems was made by Pollicott and Simon in [46], Keane, Smorodinsky and Solomyak in [32], and Solomyak in [59]. In [46] it was shown that for Lebesgue almost every \(\lambda \in (1/4, 1/3)\) we have \(\dim_H C_\lambda = \frac{\log 3}{-\log 3}\). In [59] it was shown that for Lebesgue almost every \(\lambda \in (1/3, 2/5)\) the set \(C_\lambda\) has positive Lebesgue measure. It follows from the recent work of Hochman [27], and Shmerkin and Solomyak [54], that the set of exceptions for both of these statements has zero Hausdorff dimension.

In [59] it was shown that \(\alpha(B(\{0, \pm 1, \pm 2, \pm 3\})) > 0.418\). Using this information we can prove the following result.

**Theorem 4.5.** Let \(\Psi : \mathcal{D}^* \to [0, \infty)\) be given by \(\Psi(a) = \frac{1}{3^{|a|}}\). Then for Lebesgue almost every \(\lambda \in (1/3, 0.418)\), we have that for any \(z \in C_\lambda\), Lebesgue almost every \(x \in C_\lambda\) is contained in \(W_{\Phi_\lambda(0, 1, 3)}(z, \Psi)\).

Just like the proof of Theorem 4.3, the proof of Theorem 4.5 is an adaptation of the proof of Corollary 2.3 and is therefore omitted.

As stated above, in [59] it was shown that for Lebesgue almost every \(\lambda \in (1/3, 2/5)\) the set \(C_\lambda\) had positive Lebesgue measure. This was achieved by proving \(C_\lambda\) supported an absolutely continuous self-similar measure. To the best of the authors knowledge, all results establishing \(C_\lambda\) has positive Lebesgue measure for some \(\lambda \in (1/3, 2/5)\) do so by proving that \(C_\lambda\) supports an absolutely continuous self-similar measure. It is interesting therefore to note that our methods yield a simple proof of the fact stated above without any explicit mention of a measure. In the proof below, we instead construct a subset of \(C_\lambda\) that has positive Lebesgue measure for Lebesgue almost every \(\lambda \in (1/3, 2/5)\).

**Theorem 4.6 (Solomyak [59]).** For Lebesgue almost every \(\lambda \in (1/3, 2/5)\) the set \(C_\lambda\) had positive Lebesgue measure.

**Proof.** Taking \((a_j)\) to be the sequence consisting of all zeros in our proof of Theorem 2.2, so that \(\pi(a_j) = 0\) for all \(\lambda\), it can be shown that for any \(\epsilon_1 > 0\) we have

\[
\mathcal{L}([1/3 + \epsilon_1, 4/5 - \epsilon_1]) - \int_{1/3+\epsilon}^{4/5-\epsilon_1} T(\{\sum_{j=0}^{n-1} a_j \lambda^j\}_{a \in \{0, 1, 3\}^n}) \frac{3^n}{3^{3n}} = \mathcal{O}(s).
\]

Therefore by Lemma 3.2 we have

\[
\mathcal{L}\left(\bigcap_{\epsilon>0, \delta>0} \{\lambda \in ([1/3 + \epsilon_1, 4/5 - \epsilon_1]) : d(\omega) > B(c, s, n) \geq 1 - \epsilon_1\}\right) = \mathcal{L}([1/3 + \epsilon_1, 4/5 - \epsilon_1]).
\]
This implies that for Lebesgue almost every $\lambda \in [1/3 + \epsilon_1, 4/5 - \epsilon_1]$, there exists $c > 0$ and $s > 0$ such that for infinitely many $n \in \mathbb{N}$ we have
\[
\frac{T}{n} \left( \sum_{j=0}^{n-1} a_j \lambda^n \right) \leq \frac{c}{n^3} \geq c.
\] (4.18)

Let $\lambda' \in [1/3 + \epsilon_1, 4/5 - \epsilon_1]$ be a $\lambda$ satisfying (4.18) for infinitely many $n$. For any $n \in \mathbb{N}$ satisfying (4.18) we must also have
\[
\begin{align*}
\mathcal{L} \left( \left| x - \phi_0(0) \right| \leq \frac{s}{2 \cdot 3^n} \text{ for i.m. } a \in \{0, 1, 3\}^n \right) \\
= \mathcal{L} \left( \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{a \in \{0, 1, 3\}^n} \left( \sum_{j=0}^{n-1} a_j \lambda'^j - \frac{s}{2 \cdot 3^n} \sum_{j=0}^{n-1} a_j \lambda'^j + \frac{s}{2 \cdot 3^n} \right) \right) \\
= \lim_{N \to \infty} \mathcal{L} \left( \bigcup_{n=N}^{\infty} \bigcup_{a \in \{0, 1, 3\}^n} \left( \sum_{j=0}^{n-1} a_j \lambda'^j - \frac{s}{2 \cdot 3^n} \sum_{j=0}^{n-1} a_j \lambda'^j + \frac{s}{2 \cdot 3^n} \right) \right) \\
\geq cs.
\end{align*}
\] (4.19)

In which case it follows from (4.18) and (4.19) that
\[
\begin{align*}
\mathcal{L} \left( \left| x - \phi_0(0) \right| \leq \frac{s}{2 \cdot 3^n} \text{ for i.m. } a \in \{0, 1, 3\}^n \right) \\
\geq cs.
\end{align*}
\]

In the penultimate equality we used that Lebesgue measure is continuous from above. In the final inequality we used that there are infinitely many $n \in \mathbb{N}$ such that (4.18) holds, and therefore infinitely many $n \in \mathbb{N}$ such that (4.19) holds.

Since
\[
\left\{ x : \left| x - \phi_0(0) \right| \leq \frac{s}{2 \cdot 3^n} \text{ for i.m. } a \in \{0, 1, 3\}^n \right\} \subset C_{\lambda'},
\]
it follows $C_{\lambda'}$ has positive Lebesgue measure. Since $\lambda'$ was arbitrary, it follows that for Lebesgue almost every $\lambda \in [1/3 + \epsilon_1, 4/5 - \epsilon_1]$, the set $C_{\lambda}$ has positive Lebesgue measure. Since $\epsilon_1$ was arbitrary we can upgrade this statement and conclude that for Lebesgue almost every $\lambda \in (1/3, 4/5)$, the set $C_{\lambda}$ has positive Lebesgue measure.

\section{Proof of Theorem 2.6}

In this section we prove Theorem 2.6. Recall that in the setting of Theorem 2.6 we obtain a family of IFSs by first of all fixing a set of $d \times d$ non-singular matrices $\{A_i\}_{i=1}^d$ each satisfying $\|A_i\| < 1$. For any $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$ we then define $\Phi_t$ to be the IFS consisting of the contractions
\[
\phi_i(x) = A_i x + t_i.
\]

The parameter $t$ is allowed to vary. We denote the corresponding attractor by $X_t$ and the projection map from $D^\mathbb{N}$ to $X_t$ by $\pi_t$.

To prove Theorem 2.6 we will need a technical result due to Jordan, Pollicott, and Simon from [31]. It is rephrased for our purposes.
Lemma 4.7. \textit{[31, Lemma 7]} Assume that $\|A_i\| < 1/2$ for all $1 \leq i \leq l$ and let $U$ be an arbitrary open ball in $\mathbb{R}^{ld}$. Then for any two distinct sequences $a, b \in \mathcal{D}^N$ we have

$$L \left( \{ t \in U : |\pi_t(a) - \pi_t(b)| \leq r \} \right) = O \left( \frac{r^d}{\prod_{i=1}^{d} \alpha_i(A_{a_1,\ldots,a_{[a \wedge b] - 1}})} \right)$$

With Lemma 4.7 we are now in a position to prove Theorem 2.6.

Proof of Theorem 2.6. Let us start by fixing a set of $d \times d$ non-singular matrices $\{A_i\}_{i=1}^{l}$ such that $\|A_i\| < 1/2$ for all $1 \leq i \leq l$. We prove each statement appearing in this theorem individually.

Proof of statement 1

Instead of proving our result for Lebesgue almost every $t \in \mathbb{R}^{ld}$, it is sufficient to prove our result for Lebesgue almost every $t \in U$, where $U$ is an arbitrary ball in $\mathbb{R}^{ld}$. In what follows we fix such a $U$.

By the Shannon-McMillan-Breiman theorem, and the definition of the Lyapunov exponent, we know that for $m$-almost every $a \in \mathcal{D}^N$ we have

$$\lim_{n \to \infty} -\frac{\log m([a_1, \ldots, a_n])}{n} = h(m)$$

and for each $1 \leq i \leq d$

$$\lim_{n \to \infty} \frac{\log \alpha_i(A_{a_1,\ldots,a_n})}{n} = \lambda_i(m).$$

Applying Egorov’s theorem, it follows that for any $\epsilon > 0$, there exists $C > 0$ such that the set of $a \in \mathcal{D}^N$ satisfying

$$\frac{e^{n(-h(m)-\epsilon)}}{C} \leq m([a_1, \ldots, a_n]) \leq C e^{n(-h(m)+\epsilon)}$$

and

$$\frac{e^{n(\lambda_i(m)-\epsilon)}}{C} \leq \alpha_i(A_{a_1,\ldots,a_n}) \leq C e^{n(\lambda_i(m)+\epsilon)}$$

for each $1 \leq i \leq d$ for all $n \in \mathbb{N}$ has $m$-measure strictly larger than $1/2$. In what follows we will assume that $\epsilon$ has been picked to be sufficiently small so that we have

$$h(m) - \epsilon > -\lambda_1(m) - \cdots - \lambda_d(m) + d\epsilon. \quad (4.22)$$

Such an $\epsilon$ exists because of our underlying assumption $h(m) > -\lambda_1(m) - \cdots - \lambda_d(m)$.

Let

$$\tilde{L}_{m,n} = \{ a \in L_{m,n} : (4.20) \text{ and } (4.21) \text{ hold } \}$$

and

$$\tilde{R}_{m,n} := \# \tilde{L}_{m,n}.$$

It follows from the above that

$$m \left( \bigcup_{a \in \tilde{L}_{m,n}} [a] \right) > 1/2.$$

By the discussion given at the beginning of this section, we known $\tilde{R}_{m,n}$ satisfies the exponential growth condition of Proposition 3.1. It also follows from our construction that

$$\tilde{R}_{m,n} \asymp R_{m,n}. \quad (4.23)$$
Let us now fix \((a_j) \in D^N\) and let

\[
R(t, s, n) := \left\{ (a, a') \in \tilde{L}_{m,n} \times \tilde{L}_{m,n} : |\phi_a(\pi_t(a)) - \phi_a(\pi_t(a'))| \leq \frac{s}{\tilde{R}_{m,n}^{1/d}} \quad \text{and} \quad a \neq a' \right\}.
\]

Our goal now is to prove the bound:

\[
\int_U \frac{\#R(t, s, n)}{\tilde{R}_{m,n}} \, d\mathcal{L} = O(s^d). \quad (4.24)
\]

Repeating the arguments given in the proof statement 1 from Theorem 2.2, it can be shown that

\[
\int_U \frac{\#R(t, s, n)}{\tilde{R}_{m,n}} \, d\mathcal{L} = O \left( \tilde{R}_{m,n} \sum_{a, a' \in L_{m,n}} m([a])m([a'])\mathcal{L}(t \in U : |\pi_t(a(a_j)) - \pi_t(a'(a_j))| \leq \frac{s}{\tilde{R}_{m,n}^{1/d}}) \right).
\]

Applying the bound given by Lemma 4.7, we obtain

\[
\int_U \frac{\#R(t, s, n)}{\tilde{R}_{m,n}} \, d\mathcal{L} = O \left( \tilde{R}_{m,n} \sum_{a, a' \in L_{m,n}} m([a])m([a']) \frac{s^d}{\tilde{R}_{m,n} \prod_{i=1}^d \alpha_i(A_{a_1, \ldots, a_{|a|}; a'_1, \ldots, a'_{|a'|})^{-1}} \right)
\]

\[
= O \left( \sum_{a, a' \in L_{m,n}} m([a])m([a']) \frac{s^d}{\prod_{i=1}^d \alpha_i(A_{a_1, \ldots, a_{|a|}; a'_1, \ldots, a'_{|a'|})^{-1}} \right)
\]

\[
= O \left( s^d \sum_{a \in L_{m,n}} \frac{m([a])}{|a|} \sum_{k=1}^{d-1} \frac{m([a'])}{\prod_{i=1}^d \alpha_i(A_{a_1, \ldots, a_{k|-1}})} \right)
\]

We now substitute in the bounds provided by (4.20) and (4.21) to obtain

\[
\int_U \frac{\#R(t, s, n)}{\tilde{R}_{m,n}} \, d\mathcal{L} = O \left( s^d \sum_{a \in \tilde{L}_{m,n}} m([a]) \sum_{k=1}^{\frac{|a|}{d}} \frac{e^{c(h(m)+\epsilon)}}{e^{|\lambda_i(m)-\epsilon|}} \right)
\]

\[
= O \left( s^d \sum_{a \in \tilde{L}_{m,n}} m([a]) \sum_{k=1}^{\frac{|a|}{d}} \frac{e^{c|h(m)+\epsilon|}}{e^{|\sum_{i=1}^d \lambda_i(m)-d\epsilon|}} \right)
\]

\[
= O \left( s^d \sum_{a \in \tilde{L}_{m,n}} m([a]) \sum_{k=1}^{\infty} \frac{e^{c|h(m)+\epsilon|}}{e^{|\sum_{i=1}^d \lambda_i(m)-d\epsilon|}} \right)
\]

\[
= O \left( s^d \sum_{a \in \tilde{L}_{m,n}} m([a]) \right)
\]

\[
= O(s^d).
\]

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In our penultimate equality we used $[4.22]$ to assert that
\[ \sum_{k=1}^{\infty} e^{k(-h(m)+\epsilon)} \frac{e^{k\left(\sum_{i=1}^{d} \lambda_i(m) - d\right)}}{e^{k\left(\sum_{i=1}^{d} \lambda_i(m) - d\right)}} < \infty. \]

We have shown that $[4.24]$ holds. It follows now from $[4.24]$ and Lemma 3.5 that
\[ \mathcal{L}(U) - \int_{U} T\big(\{\phi_a(\pi_t(a_j))\}_{a \in L_{m,n}}; \frac{s}{\tilde{R}_{m,n}}\big) d\lambda = O(s^d). \] (4.25)

Therefore, by Proposition 3.1 we have that for Lebesgue almost every $t \in U$, the set
\[ \left\{ x \in X : |x - \phi_a(\pi_t(a_j))| \leq \left(\frac{h(n)}{\tilde{R}_{m,n}}\right)^{1/d} \text{ for i.m. } a \in \bigcup_{n=1}^{\infty} \tilde{I}_{m,n} \right\} \]
has positive Lebesgue measure for any $h \in H$. By $[4.28]$ we know that $\tilde{R}_{m,n} \asymp R_{m,n}$. Which by the discussion given at the beginning of this section implies $\tilde{R}_{m,n} \asymp \mu([a])$ for each $a \in L_{m,n}$. Combining this fact with Lemma 3.7 and the above, we can conclude that for Lebesgue almost every $t \in U$, for any $h \in H$ the set $U_{\tilde{\Phi}_t}(\pi_t(a_j), \mu, h)$ has positive Lebesgue measure.

**Proof of statement 2**

Under the assumptions of statement 2, it can be shown that for any $a \in D^n$ the difference $\phi_a(z) - \phi_a(z')$ is independent of $a$ for any $z, z' \in X$. Therefore $\{\phi_a(\pi_t(z))\}_{a \in D^n}$ is a translation of $\{\phi_a(\pi_t(z'))\}_{a \in D^n}$ for any $z, z' \in X$. The proof of statement 2 now follows by the same reasoning as that given in the proof of statement 2 from Theorem 2.2.

**Proof of statement 3**

As in the proof of statement 1, it suffices to show that statement 3 holds for Lebesgue almost every $t \in U$ where $U$ is an arbitrary ball. We know by statement 1 that for any $(a_j) \in D^N$, for Lebesgue almost every $t \in U$, the set $U_{\tilde{\Phi}_t}(\pi_t(a_j), \mu, h)$ has positive Lebesgue measure for any $h \in H$. Applying Lemma 3.7 it follows that for any $(a_j) \in D^N$, for Lebesgue almost every $t \in U$, the set $W_{\tilde{\Phi}_t}(\pi_t(a_j), \Psi)$ has positive Lebesgue measure for any $\Psi$ that is equivalent to $(\mu, h)$ for some $h \in H$. If each $A_i$ were a similarity then we could apply Proposition 3.8 to assert that for any $(a_j) \in D^N$, for Lebesgue almost every $t \in U$, for any $\Psi \in \Upsilon_m$ Lebesgue almost every $x \in X_t$ is contained in $W_{\tilde{\Phi}_t}(\pi_t(a_j), \Psi)$.

To prove statement 3 in the case when $d = 2$ and all the matrices are equal, and in the case when all the matrices are simultaneously diagonalisable, we will apply the second part of Proposition 3.8. We need to show that under either of these conditions, for Lebesgue almost every $t \in U$ the measure $\mu$, the pushforward of our $\mu$, is equivalent to $\mathcal{L}|_{X_t}$. Now let us assume our set of matrices satisfies either of these conditions. By (4.25) and Lemma 3.2 we know that
\[ \mathcal{L}\left(\bigcap_{\epsilon > 0} \bigcup_{c > 0} \{t \in U : \tilde{d}(n : t \in B(c, s, n)) \geq 1 - \epsilon\}\right) = \mathcal{L}(U). \]

In particular, this implies that for Lebesgue almost every $t \in U$, there exists some $c, s > 0$ such that
\[ T\big(\{\phi_a(\pi_t(a_j))\}_{a \in \tilde{I}_{m,n}}; \frac{s}{\tilde{R}_{m,n}}\big) > c \frac{\tilde{R}_{m,n}}{\tilde{R}_{m,n}} \]
for infinitely many $n \in \mathbb{N}$. By Proposition 3.11 it follows that $\mu \ll \mathcal{L}$ for Lebesgue almost every $t \in U$. By our hypothesis and Lemma 3.10 we can improve this statement to $\mu \sim \mathcal{L}|_{X_t}$.

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for Lebesgue almost every $t \in U$. Now applying Proposition 3.8 we can conclude that for any $(a_j) \in D^N$, for Lebesgue almost every $t \in U$, for any $\Psi \in \Upsilon_m$ Lebesgue almost every $x \in X_t$ is contained in $W_{\Phi_t}(\pi_t(a_j), \Psi)$.

**Proof of statement 4**
The proof of statement 4 is an adaptation of statement 3, where the role of statement 1 is played by statement 2.

The proof of Corollary 2.7 is analogous to the proof of Corollary 2.3 and is therefore omitted.

### 4.3 Proof of Theorem 2.9

The proof of Theorem 2.9 mirrors the proof of Theorem 2.6. As such we will only state the appropriate analogue of Lemma 4.7 and leave the details to the interested reader. The following lemma was proved in [31].

**Lemma 4.8.** [31, Lemma 6] Assume that $\|A_i\| < 1$ for all $1 \leq i \leq l$. For any two distinct sequences $a, b \in D^N$ we have

$$
P(\gamma \in D^N : |\pi_{\gamma}(a) - \pi_{\gamma}(b)| \leq r) = O\left(\frac{r^d}{\prod_{i=1}^{d} \alpha_i(A_{a_1,\ldots,a_{\lceil a \wedge b \rceil -1}})}\right)
$$

### 5 A specific family of IFSs

In this section we focus on the following family of IFSs:

$$
\Phi_t := \{\phi_1(x) = \frac{x}{2}, \phi_2(x) = \frac{x+1}{2}, \phi_3(x) = \frac{x+t}{2}, \phi_4(x) = \frac{x+1+t}{2}\}.
$$

Where $t \in [0, 1]$. We also fix $m$ throughout to be the uniform $(1/4,1/4,1/4,1/4)$ Bernoulli measure. To each $t \in [0, 1] \setminus \mathbb{Q}$ we associate the continued fraction expansion $(\zeta_m) \in \mathbb{N}^\infty$ and the corresponding sequence of partial quotients $(p_m/q_m)$. In this section we will make use of the following well known properties of continued fractions.

- For any $m \in \mathbb{N}$ we have

$$
\frac{1}{q_m(q_{m+1} + q_m)} < \left| t - \frac{p_m}{q_m} \right| < \frac{1}{q_m q_{m+1}}. \tag{5.1}
$$

- If we set $p_{-1} = 1, q_{-1} = 0, p_0 = 0, q_0 = 1$, then for any $m \geq 1$ we have

$$
p_m = \zeta_m p_{m-1} + p_{m-2} \tag{5.2}
$$

$$
q_m = \zeta_m q_{m-1} + q_{m-2}.
$$

- If $t$ is such that $(\zeta_m)$ is bounded, i.e. $t$ is badly approximable, then there exists $c_t > 0$ such that for any $(p, q) \in \mathbb{Z} \times \mathbb{N}$, we have

$$
\left| t - \frac{p}{q} \right| \geq \frac{c_t}{q^2}. \tag{5.3}
$$

- If $q < q_{m+1}$ then

$$
|qt - p| \geq |q_m t - p_m| \tag{5.4}
$$

for any $p \in \mathbb{Z}$.
For a proof of these properties we refer the reader to [13] and [14].

Let us now remark that for any \( a \in D^n \), there exists two sequences \((b_j), (c_j) \in \{0,1\}^n\) satisfying

\[
\phi_a(x) = \frac{x}{2^n} + \sum_{j=1}^{n} \frac{b_j}{2^j} + t \sum_{j=1}^{n} \frac{c_j}{2^j}.
\]  

(5.5)

Importantly for each \( a \in D^n \) the sequences \((b_j), (c_j)\) satisfying (5.5) are unique. What is more, for any \((b_j), (c_j) \in \{0,1\}^n\), there exists a unique \( a \in D^n \) such that \((b_j)\) and \((c_j)\) satisfy (5.5) for this choice of \( a \).

We separate our proof of Theorem 2.10 into individual propositions. Statement 1 from this theorem is contained in the following result.

**Proposition 5.1.** \( \Phi_t \) contains an exact overlap if and only if \( t \in \mathbb{Q} \). Moreover if \( t \in \mathbb{Q} \), then for any \( z \in [0,1+t] \), the set \( U_{\Phi_t}(z,m,1) \) has Hausdorff dimension strictly less than 1.

**Proof.** If \( \Phi_t \) contains an exact overlap then there exists distinct \( a, a' \in D^n \) such that \( \phi_a = \phi_{a'} \). By considering \( a a' \) and \( a' a \) if necessary, we can assume that \( |a| = |a'| \). Using (5.5) we see that the following equivalences hold:

There exists distinct \( a, a' \in D^n \) such that \( \phi_a = \phi_{a'} \).

\[\iff\] There exists \((b_j), (c_j), (b'_j), (c'_j) \in \{0,1\}^n\) such that \( \sum_{j=1}^{n} \frac{b_j}{2^j} + t \sum_{j=1}^{n} \frac{c_j}{2^j} = \sum_{j=1}^{n} \frac{b'_j}{2^j} + t \sum_{j=1}^{n} \frac{c'_j}{2^j} \)

and either \( (b_j) \neq (b'_j) \) or \( (c_j) \neq (c'_j) \).

\[\iff\] There exists \((b_j), (c_j), (b'_j), (c'_j) \in \{0,1\}^n\) such that \( \sum_{j=1}^{n} \frac{b_j - b'_j}{2^j} = t \sum_{j=1}^{n} \frac{c'_j - c_j}{2^j} \)

and either \( (b_j) \neq (b'_j) \) or \( (c_j) \neq (c'_j) \).

\[\iff\] There exists \((b_j), (c_j), (b'_j), (c'_j) \in \{0,1\}^n\) such that \( \sum_{j=1}^{n} 2^{n-j} (b_j - b'_j) = t \sum_{j=1}^{n} 2^{n-j} (c'_j - c_j) \)

and either \( (b_j) \neq (b'_j) \) or \( (c_j) \neq (c'_j) \).

\[\iff\] There exists \( 1 \leq p, q \leq 2^n - 1 \) such that \( p = qt \).

It follows from these equivalences that there is an exact overlap if any only if \( t \in \mathbb{Q} \).

We now prove the Hausdorff dimension part of our proposition. By the first part we know that \( t \in \mathbb{Q} \) if and only if \( \Phi_t \) contains an exact overlap. It follows from the presence of an exact overlap that if \( t \in \mathbb{Q} \), then there exists \( c > 0 \), such that for any \( z \in [0,1+t] \) we have

\[# \{ \phi_a(z) : a \in D^n \} = \mathcal{O}(4 - c)^n.\]  

(5.6)

For any \( z \in [0,1+t] \) and \( N \in \mathbb{N} \), the set of intervals

\[\{ [\phi_a(z) - 4^{-n}, \phi_a(z) + 4^{-n}] \}_{\phi_a(z) : a \in D^n \}
\]

forms a \( 2 \cdot 4^{-N} \) cover of \( U_{\Phi_t}(z,m,1) \). Now let \( s \in (0,1) \) be sufficiently large that

\((4 - c) < 4^s)\). It follows now that we have the following bound on the \( s \)-dimensional Hausdorff measure of \( U_{\Phi_t}(z,m,1) \)

\[\mathcal{H}^s(U_{\Phi_t}(z,m,1)) \leq \lim_{N \to \infty} \sum_{n=N}^{\infty} \sum_{\phi_a(z) : a \in D^n} \text{Diam}([\phi_a(z) - 4^{-n}, \phi_a(z) + 4^{-n}])^s\]  

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\[
\lim_{N \to \infty} O \left( \sum_{n=N}^{\infty} (4 - c)^n 4^{-ns} \right) = 0.
\]

In the last line we used that \((4 - c) < 4^s\) to guarantee \(\sum_{n=1}^{\infty} (4 - c)^n 4^{-ns} < \infty\). Therefore \(\dim_H(U_{\Phi_t}(z, m, 1)) \leq s\) for any \(z \in [0, 1 + t]\).

Adapting the proof of the first part of Proposition 5.1 we can show that the following simple lemma holds.

**Lemma 5.2.** Let \(t \in [0, 1]\), \(z \in [0, 1+t]\), and \(s > 0\). For \(n\) sufficiently large, there exists distinct \(a, a' \in D^n\) such that

\[
\phi_a(z) - \phi_{a'}(z) \leq \frac{s}{4^n}
\]

if and only if there exists \(1 \leq p, q \leq 2^n - 1\) such that

\[
|qt - p| \leq \frac{s}{2^n}.
\]

Lemma 5.2 will be used in the proofs of all the full measure statements in Theorem 2.10. It immediately yields the following proposition which corresponds to statement 3 from Theorem 2.10.

**Proposition 5.3.** If \(t\) is badly approximable, then for any \(z \in [0, 1 + t]\) and \(h : \mathbb{N} \to [0, \infty)\) satisfying \(\sum_{n=1}^{\infty} h(n) = \infty\), we have that Lebesgue almost every \(x \in [0, 1 + t]\) is contained in \(U_{\Phi_t}(z, m, h)\).

**Proof.** Since \(t\) is badly approximable, we know by (5.3) that there exists \(c_t > 0\) such that

\[
|qt - p| \geq \frac{c_t}{q}
\]

for all \((p, q) \in \mathbb{Z} \times \mathbb{N}\). Equation (5.7) implies that for any \(1 \leq p, q \leq 2^n - 1\) we have

\[
|qt - p| > \frac{c_t}{2^n}.
\]

Applying Lemma 5.2 we see that for any \(z \in [0, 1 + t]\), for all \(n\) sufficiently large, if \(a, a' \in D^n\) are distinct then

\[
\phi_a(z) - \phi_{a'}(z) > \frac{c_t}{4^n}.
\]

Therefore, for any \(z \in [0, 1 + t]\) we have

\[
S \left( \{ \phi_a(z) \}_{a \in D^n}, \frac{c_t}{4^n} \right) = \{ \phi_a(z) \}_{a \in D^n}
\]

for all \(n\) sufficiently large. Our result now follows by an application of Proposition 3.9.

For our other full measure statements a more delicate analysis is required. We need to identify integers \(n\) for which the set of images \(\{ \phi_a(z) \}_{a \in D^n}\) are well separated. This we do in the following two lemmas.

**Lemma 5.4.** Let \(s > 0\). For \(n\) sufficiently large, if \(n\) satisfies

\[
2sq_m \leq 2^n - 1 < q_m
\]

for some \(m\), then for any \(z \in [0, 1 + t]\) we have

\[
\phi_a(z) - \phi_{a'}(z) > \frac{s}{4^n},
\]

for distinct \(a, a' \in D^n\).
Proof. Fix $s > 0$. If $2^n - 1 < q_m$, then by (5.1) and (5.4), for all $1 \leq p, q \leq 2^n - 1$ we have

$$|qt - p| \geq |q_{m-1}t - p_{m-1}| \geq \frac{1}{2q_m}$$

If $2sq_m \leq 2^n - 1$ as well, then the above implies that for all $1 \leq p, q \leq 2^n - 1$ we have

$$|qt - p| \geq \frac{s}{2^n - 1} > \frac{s}{2^n}.$$ 

Applying Lemma 5.2 completes our proof. □

Lemma 5.4 demonstrates that if $2^n - 1$ is strictly less than but close to some denominator arising from the partial quotients of $t$, then at the $n$-th level we have good separation properties. The following lemma demonstrates a similar phenomenon but instead relies upon the digits appearing in the continued fraction expansion. To properly state this lemma we need to define the following sequence. Given $t$ with corresponding sequence of partial quotients $(p_m/q_m)$, we define the sequence of integers $(m_n)$ via the inequalities:

$$q_{m_n} \leq 2^n - 1 < q_{m_n+1}.$$ 

Lemma 5.5. Let $s > 0$. For $n$ sufficiently large, if $n$ is such that $\zeta_{m_n+1} \leq (3s)^{-1}$, then for any $z \in [0, 1 + t]$ we have

$$|\phi_a(z) - \phi_{a'}(z)| > \frac{s}{4^n}$$

for distinct $a, a' \in D^n$.

Proof. By (5.1), (5.2), and (5.4), we know that for any for any $1 \leq p, q \leq 2^n - 1$ we have

$$|qt - p| \geq |q_{m_n}t - p_{m_n}| \geq \frac{1}{(q_{m_n+1} + q_{m_n})} = \frac{1}{(\zeta_{m_n+1} + 1)q_{m_n} + q_{m_n-1}} > \frac{1}{3\zeta_{m_n+1}q_{m_n}}.$$ 

Now using our assumption $\zeta_{m_n+1} \leq (3s)^{-1}$, we may conclude that for any $1 \leq p, q \leq 2^n - 1$ we have

$$|qt - p| \geq \frac{s}{q_{m_n}} > \frac{s}{2^n}.$$ 

Applying Lemma 5.2 our result follows. □

With Lemma 5.4 and Lemma 5.5 in mind we introduce the following definition. We say that $n$ is a good $s$-level if either

$$2sq_m \leq 2^n - 1 < q_m$$

for some $m$, or if

$$\zeta_{m_n+1} \leq (3s)^{-1}.$$ 

It follows from Lemma 5.4 and Lemma 5.5 that if $n$ is a good $s$-level then

$$S \left( \{\phi_a(z)\}_{a \in D^n}, \frac{s}{4^n} \right) = \{\phi_a(z)\}_{a \in D^n}$$

for any $z \in [0, 1 + t]$. The following proposition implies statement 2 from Theorem 2.10.

Proposition 5.6. If $t \notin \mathbb{Q}$, then there exists $h : \mathbb{N} \to [0, \infty)$ depending upon the continued fraction expansion of $t$, such that $\lim_{n \to \infty} h(n) = 0$, and for any $z \in [0, 1 + t]$ Lebesgue almost every $x \in [0, 1 + t]$ is contained in $U_{\phi_2}(z, m, h)$. 56
Proof. Fix \( t \notin \mathbb{Q} \) and let \( s = 1/8 \). For any \( m \in \mathbb{N} \), it follows from the definition that \( n \) is a good 1/8-level if \( n \) satisfies
\[
\frac{q_m}{4} \leq 2^n - 1 < q_m. \tag{5.8}
\]
For any \( m \) sufficiently large there is clearly at least one value of \( n \) satisfying (5.8). As such there are infinitely many good 1/8 levels. Now let \( h : \mathbb{N} \to [0, \infty) \) be a function satisfying \( \lim_{n \to \infty} h(n) = 0 \) and
\[
\sum_{n \text{ is a good 1/8-level}} h(n) = \infty.
\]
Now as remarked above, if \( n \) is a good 1/8-level, then
\[
S \left( \{ \phi_a(z) \}_{a \in \mathcal{D}^n}, \frac{1}{8}, 4^n \right) = \{ \phi_a(z) \}_{a \in \mathcal{D}^n}
\]
for any \( z \in [0, 1 + t] \). We may now apply Proposition 5.9 and conclude that for any \( z \in [0, 1 + t] \), Lebesgue almost every \( x \in [0, 1 + t] \) is contained in \( U_{\Phi_t}(z, m, h) \) for this choice of \( h \).

In the proof of Proposition 5.6 we showed that if \( t \notin \mathbb{Q} \), then for infinitely many \( n \in \mathbb{N} \) we have
\[
S(\{ \phi_a(z) \}_{a \in \mathcal{D}^n}, \frac{1}{8}, 4^n) = \{ \phi_a(z) \}_{a \in \mathcal{D}^n}.
\]
Theorem 2.12 now follows from this observation and Proposition 5.1.

The following proposition implies statement 5 from Theorem 2.10.

**Proposition 5.7.** Suppose \( t \notin \mathbb{Q} \) is such that for any \( \epsilon > 0 \), there exists \( L \in \mathbb{N} \) for which the following inequality holds for \( M \) sufficiently large:
\[
\sum_{1 \leq m \leq M} \frac{\log_2(\zeta_{m+1} + 1)}{q_m} \leq \epsilon M.
\]
Then for any \( z \in [0, 1 + t] \) and \( h \in H^* \), Lebesgue almost every \( x \in [0, 1 + t] \) is contained in \( U_{\Phi_t}(z, m, h) \).

**Proof.** Fix \( t \) satisfying the hypothesis of our proposition and \( h \in H^* \). By definition, there exists \( \epsilon > 0 \) such that for any \( B \subset \mathbb{N} \) satisfying \( d(B) \geq 1 - \epsilon \) we have
\[
\sum_{n \in B} h(n) = \infty. \tag{5.9}
\]
Now let us fix \( s \) to be sufficiently small so that
\[
\sum_{1 \leq m \leq 2N+2} \frac{\log_2(\zeta_{m+1} + 1)}{q_m} \leq \epsilon N \tag{5.10}
\]
for \( N \) sufficiently large.

We observe that if \( n \) is not a good \( s \)-level then by (5.2) we must have
\[
\frac{q_{m+1}}{q_m} > \frac{1}{3s}.
\]
Using (5.2) and an induction argument, one can also show that
\[
q_m \geq 2^{\frac{m-2}{2}}
\]

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for all $m \geq 1$. Combining these observations, it follows that if $1 \leq n \leq N$ and $n$ is not a good $s$-interval, then there exists $1 \leq m \leq 2N + 2$ such that $\frac{q_{m+1}}{q_m} \geq 1/3s$ and $q_m \leq 2^n - 1 < q_{m+1}$. As such we have the bound

$$
\#\{1 \leq n \leq N : n \text{ is not a good } s\text{-interval}\} \leq \sum_{1 \leq m \leq 2N+2} \#\{n : q_m \leq 2^n - 1 < q_{m+1}\}. \quad (5.11)
$$

By (5.12) we know that for any $m \in \mathbb{N}$ we have

$$
\#\{n : q_m \leq 2^n - 1 < q_{m+1}\} \leq \log_2(\zeta_{m+1} + 1).
$$

Substituting this bound into (5.11) and applying (5.10), we obtain

$$
\#\{1 \leq n \leq N : n \text{ is not a good } s\text{-interval}\} \leq \sum_{1 \leq m \leq 2N+2} \log_2(\zeta_{m+1} + 1) \leq \epsilon N
$$

for $N$ sufficiently large. It follows therefore that

$$
d(n : n \text{ is a good } s\text{-level}) \geq 1 - \epsilon.
$$

In which case, by (5.9) we have

$$
\sum_{n \text{ is a good } s\text{-level}} h(n) = \infty. \quad (5.12)
$$

We know that for a good $s$-level we have

$$
S\left(\{\phi_a(z)\}_{a \in D^n}, \frac{s}{4^n}\right) = \{\phi_a(z)\}_{a \in D^n},
$$

for all $z \in [0, 1 + t]$. Therefore combining (5.12) with Proposition 3.9 finishes our proof. \qed

**Proposition 5.8.** Suppose $\mu$ is an ergodic invariant measure for the Gauss map, and satisfies

$$
\sum_{m=1}^{\infty} \mu\left(\left[\frac{1}{m+1}, \frac{1}{m}\right]\right) \log_2(m+1) < \infty.
$$

Then for $\mu$-almost every $t$, we have that for any $z \in [0, 1 + t]$ and $h \in H^*$, Lebesgue almost every $x \in [0, 1 + t]$ is contained in $U_{\Phi_t}(z, m, h)$. In particular, for Lebesgue almost every $t \in [0, 1]$, we have that for any $z \in [0, 1 + t]$ and $h \in H^*$, Lebesgue almost every $x \in [0, 1 + t]$ is contained in $U_{\Phi_t}(z, m, h)$.

**Proof.** Let $\mu$ be a measure satisfying the hypothesis of our proposition. To prove the first part of our result we will show that $\mu$-almost every $t$ satisfies the hypothesis of Proposition 5.7.

Recall that the Gauss map $T : [0, 1] \setminus \mathbb{Q} \to [0, 1] \setminus \mathbb{Q}$ is given by

$$
T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.
$$

It is well known that the dynamics of the Gauss map and the continued fraction expansion of a number $t$ are intertwined. In particular, it is known that

$$
\zeta_{m+1} = \zeta \text{ if and only if } T^m(t) \in \left(\frac{1}{\zeta+1}, \frac{1}{\zeta}\right). \quad (5.13)
$$
By (5.2) we know that $\frac{q_{m+1}}{q_m} \geq L$ implies $\zeta_{m+1} \geq L - 1$. Using (5.13) and this observation we have that for any $t \notin \mathbb{Q}$

$$
\sum_{1 \leq m \leq M} \log_2 (\zeta_{m+1} + 1) \leq \sum_{1 \leq m \leq M} \chi_{(0, \frac{1}{L+1})}(T^m(t)) \log_2 f(T^m(t)) + 1.
$$

(5.14)

Where $f : (0, 1] \to \mathbb{N}$ is given by

$$
f(t) = N \text{ if } t \in \left(\frac{1}{N + 1}, \frac{1}{N}\right].
$$

By our assumptions on $\mu$, we know that for any $\epsilon > 0$ we can pick $L$ sufficiently large such that

$$
\sum_{m=L-1}^{\infty} \mu \left(\left[\frac{1}{m+1}, \frac{1}{m}\right]\right) \log_2 (m+1) < \epsilon.
$$

Assuming that we have picked such an $L$ sufficiently large, we know by the Birkhoff ergodic theorem that for $\mu$-almost every $t$ we have

$$
\lim_{M \to \infty} \frac{1}{M} \sum_{1 \leq m \leq M} \chi_{(0, \frac{1}{L+1})}(T^m(t)) \log_2 f(T^m(t)) + 1 = \int \chi_{(0, \frac{1}{L+1})} \log(f(t) + 1) d\mu(t)
$$

$$
= \sum_{m=L-1}^{\infty} \mu \left(\left[\frac{1}{m+1}, \frac{1}{m}\right]\right) \log_2 (m+1)
$$

$$
< \epsilon.
$$

Combining the above with (5.14) shows that $\mu$-almost every $t$ satisfies the hypothesis of Proposition 5.7. Applying Proposition 5.7 completes the first half of our proof.

To deduce the Lebesgue almost every part of our proposition we remark that the Gauss measure given by

$$
\mu_G(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} dx
$$

is an ergodic invariant measure for the Gauss map and is equivalent to the Lebesgue measure restricted to $[0, 1]$. One can easily check that $\mu_G$ satisfies

$$
\mu_G \left(\left[\frac{1}{m+1}, \frac{1}{m}\right]\right) = O \left(\frac{1}{m^2}\right),
$$

which clearly implies

$$
\sum_{m=1}^{\infty} \mu_G \left(\left[\frac{1}{m+1}, \frac{1}{m}\right]\right) \log(m+1) < \infty.
$$

Applying the first part of this proposition completes the proof.

The following proposition proves statement 4 from Theorem 2.10.

**Proposition 5.9.** Suppose $t \notin \mathbb{Q}$ is not badly approximable. Then there exists $h : \mathbb{N} \to [0, \infty)$ such that $\sum_{n=1}^{\infty} h(n) = \infty$, yet $U_{\Psi_1}(z, m, h)$ has zero Lebesgue measure for any $z \in [0, 1 + t]$. 
Proof. Let \( t \notin \mathbb{Q} \) and suppose \( t \) is not badly approximable. We will prove that there for some \( s > 0 \) we have
\[
\liminf_{n \to \infty} \frac{\mathcal{T}(\{\phi_a(z)\}_{a \in \mathcal{D}^n}, \frac{s}{4^n})}{4^n} = 0,
\]
for all \( z \in [0, 1 + t] \). Proposition 3.6 then guarantees for each \( z \in [0, 1 + t] \) the existence of a \( h \) satisfying \( \sum_{n=1}^{\infty} h(n) = \infty \), such that \( U_{\phi_t}(z, m, h) \) has zero Lebesgue measure. What is not clear from the statement of Proposition 3.6 is whether there exist \( h \) which satisfies this property simultaneously for all \( z \in [0, 1 + t] \). Examining the proof of Proposition 3.6 we see that the function \( h \) that is constructed only depends upon the speed at which \( T(\{\phi_a(z)\}_{a \in \mathcal{D}^n}, \frac{s}{4^n}) \)
converges to zero along a subsequence. Since
\[
T\left(\{\phi_a(z)\}_{a \in \mathcal{D}^n}, \frac{s}{4^n}\right) = T\left(\{\phi_a(z')\}_{a \in \mathcal{D}^n}, \frac{s}{4^n}\right)
\]
for any \( z, z' \in [0, 1 + t] \) and \( n \in \mathbb{N} \), it is clear that the speed of convergence to zero along any subsequence is independent of \( z \). Therefore the \( h \) constructed in Proposition 3.6 will work for all \( z \in [0, 1 + t] \) simultaneously. As such to prove our proposition it is sufficient to show that (5.15) holds for all \( z \in [0, 1 + t] \).

It also follows from (5.16) that to prove there exists \( s > 0 \) such that (5.15) holds for all \( z \in [0, 1 + t] \), it suffices to prove that there exists \( s > 0 \) such that (5.15) for a specific \( z \in [0, 1 + t] \).

As such let us now fix \( z = 0 \). It can be shown that for any \( n \in \mathbb{N} \) we have
\[
\{\phi_a(0)\}_{a \in \mathcal{D}^n} = \left\{ \frac{p + qt}{2^n} : 0 \leq p \leq 2^n - 1, 0 \leq q \leq 2^n - 1 \right\}.
\]

Since \( t \) is badly approximable, there exists a sequence \( (m_k) \) such that \( \zeta_{m_k+1} \geq k^3 \) for all \( k \in \mathbb{N} \). In which case, by (5.1) and (5.2) we know that
\[
|q_{m_k}t - p_{m_k}| \leq \frac{1}{k^3 q_{m_k}}
\]
for each \( k \in \mathbb{N} \). Without loss of generality we assume \( q_{m_k}t - p_{m_k} > 0 \) for all \( k \). This assumption will simplify some of our later arguments.

Define the sequence \( (n_k) \) via the inequalities
\[
2^{n_k} \leq k^2 q_{m_k} < 2^{n_k+1}.
\]
Consider the set of \( (p, q) \in \mathbb{N}^2 \) satisfying
\[
kp_{m_k} \leq p \leq 2^{n_k} - 1
\]
and
\[
0 \leq q \leq 2^{n_k} - 1 - kq_{m_k}.
\]
Note that for any \( (p, q) \in \mathbb{N}^2 \) satisfying (5.19) and (5.20) we have
\[
0 \leq p - ip_{m_k} \leq 2^{n_k} - 1
\]
and
\[
0 \leq q + iq_{m_k} \leq 2^{n_k} - 1
\]
for all \( 0 \leq i \leq k \).
Given \( k \in \mathbb{N} \) we let
\[
  z_1 = \inf \left\{ \frac{p + tq}{2^{nk}} : (p, q) \text{ satisfy } (5.19) \text{ and } (5.20) \right\}.
\]

Equations (5.17) and (5.18) imply that for all \( 0 \leq i \leq k \) we have
\[
z_1 + i(q_{mk} t - p_{mk}) \leq \left[ z_1, z_1 + \frac{k}{2^{nk}} k^3 q_{mk} \right] \subseteq \left[ z_1, z_1 + \frac{1}{4^{nk}} \right].
\]

Assume we have defined \( z_1, \ldots, z_l \), we then define \( z_{l+1} \) to be
\[
z_{l+1} = \inf \left\{ \frac{p + qt}{2^{nk}} : p + qt > z_l + \frac{1}{4^{nk}} \text{ and } (p, q) \text{ satisfy } (5.19) \text{ and } (5.20) \right\},
\]
assuming the set we are taking the infimum over is non-empty. By an analogous argument to that given above, it can be shown that for all \( 0 \leq i \leq k \) we have
\[
z_{l+1} + i(q_{mk} t - p_{mk}) \in \left[ z_{l+1}, z_{l+1} + \frac{1}{4^{nk}} \right].
\]

This process must eventually end yielding \( z_1, \ldots, z_{L(k)} \). By our construction, we known that if \( (p, q) \) satisfy (5.19) and (5.20), then there must exists \( 1 \leq l \leq L(k) \) such that
\[
  \frac{p + qt}{2^{nk}} \in \left[ z_l, z_l + \frac{1}{4^{nk}} \right].
\]

It also follows from our construction that each interval \( [z_l, z_l + 4^{-nk}] \) contains \( k + 1 \) distinct points of the form \( \frac{p + qt}{2^{nk}} \) where \( 0 \leq p \leq 2^{nk} - 1 \) and \( 0 \leq q \leq 2^{nk} - 1 \). Since there are only \( 4^{nk} \) such points we have
\[
  L(k) \leq \frac{4^{nk}}{k + 1}.
\]

We also have the bound
\[
  \# \{ (p, q) : \text{either (5.19) or (5.20) is not satisfied} \} = \mathcal{O}(2^{nk} k p_{mk} + 2^{nk} k q_{mk}).
\]

Now let \( S(\{\frac{p + tq}{2^{nk}}, \frac{1}{4^{nk}}\}) \) be a maximal \( 4^{-nk} \) separated subset of \( \{\frac{p + tq}{2^{nk}}\} \), or equivalently of \( \{\phi_a(0)\}_{a \in \mathcal{D}^{nk}} \). Then we have
\[
  \frac{T(\{\frac{p + tq}{2^{nk}}, \frac{1}{4^{nk}}\}, \frac{1}{4^{nk}})}{4^{nk}} = \frac{\# \{ (p, q) : (5.19) \text{ and } (5.20) \text{ are satisfied and } \frac{p + tq}{2^{nk}} \in S(\{\frac{p + tq}{2^{nk}}, \frac{1}{4^{nk}}\}) \}}{4^{nk}}
  \quad + \frac{\# \{ (p, q) : \text{either (5.19) or (5.20) is not satisfied and } \frac{p + tq}{2^{nk}} \in S(\{\frac{p + tq}{2^{nk}}, \frac{1}{4^{nk}}\}) \}}{4^{nk}}.
\]

If \( (p, q) \) satisfy (5.19) and (5.20), then as stated above \( \frac{p + tq}{2^{nk}} \in [z_l, z_l + \frac{1}{4^{nk}}] \) for some \( 1 \leq l \leq L(k) \). Clearly a \( 4^{-nk} \) separated set can only contain one point from each interval \( [z_l, z_l + \frac{1}{4^{nk}}] \). Therefore
\[
  \# \{ (p, q) : (5.19) \text{ and } (5.20) \text{ are satisfied and } \frac{p + tq}{2^{nk}} \in S(\{\frac{p + tq}{2^{nk}}, \frac{1}{4^{nk}}\}) \} \leq L(k).
\]

Substituting the bounds (5.21), (5.22), and (5.24) into (5.23), we obtain
\[
  \frac{T(\{\frac{p + tq}{2^{nk}}, \frac{1}{4^{nk}}\}, \frac{1}{4^{nk}})}{4^{nk}} = \mathcal{O}\left( \frac{1}{k + 1} + \frac{k p_{mk} + k q_{mk}}{2^{nk}} \right).
\]

Employing (5.18) and the fact \( q_{mk} \approx p_{mk} \), we obtain
\[
  \frac{T(\{\frac{p + tq}{2^{nk}}, \frac{1}{4^{nk}}\}, \frac{1}{4^{nk}})}{4^{nk}} = \mathcal{O}\left( \frac{1}{k} \right).
\]

Therefore
\[
  \lim_{k \to \infty} \frac{T(\{\phi_a(z)\}_{a \in \mathcal{D}^{nk}}, \frac{1}{4^{nk}})}{4^{nk}} = 0
\]
and our proof is complete. \( \square \)
6 Proof of Theorem 2.15

In this section we prove Theorem 2.15. We start with a reformulation of what it means for an IFS to be consistently separated with respect to a measure \( m \).

**Theorem 6.1.** Let \( m \) be a slowly decaying measure. An IFS has the CS property with respect to \( m \) if and only if there exists \( z \in X \) and \( s > 0 \) such that

\[
\liminf_{n \to \infty} \frac{T(\{\phi_a(z)\}_{a \in L_{m,n}, s R_{m,n}^1/d})}{R_{m,n}} > 0.
\]

**Proof.** Suppose that for any \( z \in X \) and \( s > 0 \) we have

\[
\liminf_{n \to \infty} \frac{T(\{\phi_a(z)\}_{a \in L_{m,n}, s R_{m,n}^1/d})}{R_{m,n}} = 0.
\]

Then by Proposition 3.6, Lemma 3.7 and the fact \( m([a]) \asymp R_{m,n} \), for any \( z \in X \) there exists \( h : \mathbb{N} \to [0, \infty) \) such that \( \sum_{n=1}^{\infty} h(n) = \infty \), yet \( U_\Phi(z, m, h) \) has zero Lebesgue measure. Therefore the IFS cannot satisfy the CS property with respect to \( m \). So we have shown the rightwards implication in our if and only if.

Now suppose that there exists \( z \in X \) and \( s > 0 \) such that

\[
\liminf_{n \to \infty} \frac{T(\{\phi_a(z)\}_{a \in L_{m,n}, s R_{m,n}^1/d})}{R_{m,n}} > 0.
\]

Then there exists \( c > 0 \) such that for all \( n \) sufficiently large we have

\[
\frac{T(\{\phi_a(z)\}_{a \in L_{m,n}, s R_{m,n}^1/d})}{R_{m,n}} > c.
\]

Combining the fact that \( R_{m,n}^{-1} \asymp m([a]) \) for \( a \in L_{m,n} \) together with Lemma 3.7 we see that Proposition 3.4 implies that the set \( U_\Phi(z, m, h) \) has positive Lebesgue measure for any \( h \) satisfying \( \sum_{n=1}^{\infty} h(n) = \infty \). Therefore our IFS satisfies the CS property with respect to \( m \) and we have proved the leftwards implication of our if and only if.

The reformulation of the CS property provided by Theorem 6.1 better explains why we used the terminology consistently separated to describe this property.

With the reformulation provided by Theorem 6.1 we can give a short proof of Theorem 2.15.

**Proof of Theorem 2.15.** Let \( m \) be a slowly decaying \( \sigma \)-invariant ergodic probability measure. Suppose that \( \mu \), the pushforward of \( m \), is not absolutely continuous. Then by Proposition 3.11 for any \( z \in X \) and \( s > 0 \) we have

\[
\lim_{n \to \infty} \frac{T(\{\phi_a(z)\}_{a \in L_{m,n}, s R_{m,n}^1/d})}{R_{m,n}} = 0.
\]

By Theorem 6.1 it follows that the IFS \( \Phi \) does not satisfy the CS property with respect to \( m \).
Proof of Theorem 2.16

In this section we prove Theorem 2.16. Recall that Theorem 2.16 relates to conformal iterated function systems. The parameter \( \dim_S(\Phi) \) is the unique solution to

\[
P(s \cdot \log |\phi'_{a_1}(\pi(\sigma(a_j)))|) = 0.
\]

Moreover, \( m_\Phi \) is the unique measure supported on \( D^\mathbb{N} \) satisfying

\[
h_{m_\Phi} + \int \dim_S(\Phi) \cdot \log |\phi'_{a_1}(\pi(\sigma(a_j)))| \, dm_\Phi = 0.
\]

To prove Theorem 2.16 we need to state some additional properties of the measure \( m_\Phi \):

- Let \( x \in X \) and \( (a_j) \) be such that \( \pi(a_j) = x \). Then for any \( r \in (0, \text{Diam}(X)) \), there exists \( N(r) \in \mathbb{N} \) such that
  \[
  X_{a_1, \ldots, a_{N(r)}} \subseteq B(x, r) \text{ and } \text{Diam}(X_{a_1, \ldots, a_{N(r)}}) \approx r. \tag{7.1}
  \]

- For any \( a \in D^* \) we have
  \[
  m_\Phi([a]) \propto \text{Diam}(X_a)^{\dim_S(X)}. \tag{7.2}
  \]

- For any \( a, b \in D^* \) we have
  \[
  m_\Phi([ab]) \propto m_\Phi([a])m_\Phi([b]). \tag{7.3}
  \]

- For any \( a, b \in D^* \) we have
  \[
  \text{Diam}(X_{ab}) \propto \text{Diam}(X_a)\text{Diam}(X_b). \tag{7.4}
  \]

- There exists \( \gamma \in (0, 1) \) such that
  \[
  m_\Phi([a]) = O(|a|^\gamma). \tag{7.5}
  \]

For a proof of these properties we refer the reader to [20], [39], and [49].

Before giving our proof we make an observation. Given \( \theta : \mathbb{N} \to [0, \infty) \) we have the following equivalences:

\[
\sum_{n=1}^{\infty} \sum_{a \in D^n} (\text{Diam}(X_a)\theta(n))^\dim_S(X) = \infty \iff \sum_{n=1}^{\infty} \theta(n)^{\dim_S(\Phi)} \sum_{a \in D^n} \text{Diam}(X_a)^{\dim_S(X)} = \infty
\]

\[
\iff \sum_{n=1}^{\infty} \theta(n)^{\dim_S(\Phi)} \sum_{a \in D^n} m_\Phi([a]) = \infty
\]

\[
\iff \sum_{n=1}^{\infty} \theta(n)^{\dim_S(\Phi)} = \infty.
\]

So the hypothesis of Theorem 2.16 can be restated in terms of the divergence of \( \sum_{n=1}^{\infty} \theta(n)^{\dim_S(\Phi)} \).

Proof of Theorem 2.16. We split our proof into individual steps.
Step 1. Lifting to $\mathcal{D}^N$.

Let us fix $z \in X$ and $\theta$ satisfying the hypothesis of our theorem. For any $a \in \mathcal{D}^*$ consider the ball

$$B(\phi_a(z), Diam(X_a)\theta(|a|)).$$

By (7.1) we know that there exists $N(a, \theta)$ such that

$$X_{a,z_1,\ldots,z_{N(a,\theta)}} \subseteq B(\phi_a(z), Diam(X_a)\theta(|a|))$$

and

$$Diam(X_{a,z_1,\ldots,z_{N(a,\theta)}}) \asymp Diam(X_a)\theta(|a|).$$

In what follows we let

$$a_\theta := a_{z_1, \ldots, z_{N(a,\theta)}}.$$

Equation (7.6) implies the following:

$$\mu(\Phi(W_{\Phi}(z, \theta)) = m_{\Phi}(\{b_j : \pi(b_j) \in W_{\Phi}(z, \theta))$$

$$\geq m_{\Phi}(\{b_j : (b_j) \in [a_\theta] \text{ for i.m. } a \in \mathcal{D}^*}.\)$$

To complete our proof it therefore suffices to show that

$$m_{\Phi}(\{b_j : (b_j) \in [a_\theta] \text{ for i.m. } a \in \mathcal{D}^*) = 1. \quad (7.8)$$

Note that we have

$$\sum_{n=1}^{\infty} \sum_{a \in \mathcal{D}^n} m_{\Phi}([a_\theta]) = \infty. \quad (7.9)$$

This follows because of our underlying divergence assumption and

$$\sum_{n=1}^{\infty} \sum_{a \in \mathcal{D}^n} m_{\Phi}([a_\theta]) \asymp \sum_{n=1}^{\infty} \sum_{a \in \mathcal{D}^n} Diam(X_{a_\theta})^{dim_S(\Phi)} \asymp \sum_{n=1}^{\infty} \sum_{a \in \mathcal{D}^n} (Diam(X_a)\theta(|a|))^{dim_S(\Phi)}.$$

Step 2. A density theorem for $\mathcal{D}^N$.

To prove (7.8) we will make use of a density argument. Since we are working in $\mathcal{D}^N$ we do not have the Lebesgue density theorem. Instead we have the statement: suppose $E \subset \mathcal{D}^N$ satisfies $m_{\Phi}(E) > 0$, then for $m_{\Phi}$-almost every $(c_j) \in E$ we have

$$\lim_{M \to \infty} \frac{m([c_1, \ldots, c_M] \cap E)}{m([c_1, \ldots, c_M])} = 1. \quad (7.10)$$

One can see that this statement holds using the results of Rigot [48]. In particular, we can equip $\mathcal{D}^N$ with a metric so that $m_{\Phi}$ is doubling measure. We can then apply Theorem 2.15 and Theorem 3.1 from [48]. Using (7.10), we see that to prove (7.8), it suffices to show that for any $(c_j) \in \mathcal{D}^N$, there exists $d > 0$ such that

$$\frac{m([c_1, \ldots, c_M] \cap \{(b_j) : (b_j) \in [a_\theta] \text{ for i.m. } a \in \mathcal{D}^*\})}{m([c_1, \ldots, c_M])} > d \quad (7.11)$$

for all $M$ sufficiently large. The rest of the proof now follows from a similar argument to that given by the author in [4]. The difference being here we are now working in the sequence space $\mathcal{D}^N$ rather than $\mathbb{R}^d$. We include the details for the sake of completion.
Step 3. Defining $E_n$ and verifying the hypothesis of Lemma 3.3
Let us fix $(c_j) \in \mathcal{D}^N$ and $M \in \mathbb{N}$. In what follows we let $c = (c_1, \ldots, c_M)$. For $n \geq M$ let
$$E_n := \{ [a] : a \in \mathcal{D}^n \text{ and } a_1 \ldots, a_M = c \},$$
and let
$$E := \limsup_{n \to \infty} E_n.$$
Note that
$$E \subseteq [c] \cap \{ (b_j) : (b_j) \in [a] \text{ for i.m. } a \in \mathcal{D}^* \}.$$
Therefore to prove (7.11), it is sufficient to prove that there exists $d > 0$ independent of $M$ such that
$$m_{\Phi}(E) > d m_{\Phi}([c]). \quad (7.12)$$
Note that
$$\sum_{n=M}^{\infty} m_{\Phi}(E_n) = \infty.$$
This follows from
$$\sum_{n=M}^{\infty} m_{\Phi}(E_n) = \sum_{n=M}^{\infty} \sum_{a \in \mathcal{D}^n, a_1 \ldots, a_M = c} m([a])$$
$$= \sum_{n=M}^{\infty} \sum_{b \in \mathcal{D}^{n-M}} m([cbz_1, \ldots, z_N(cb,\theta)])$$
$$\quad \times \sum_{n=M}^{\infty} \sum_{b \in \mathcal{D}^{n-M}} Diam(X_{cbz_1,\ldots,z_N(cb,\theta)\theta})^{\dim_S(\Phi)}$$
$$\quad \times \sum_{n=M}^{\infty} \sum_{b \in \mathcal{D}^{n-M}} (Diam(X_{cb})\theta(n))^{\dim_S(\Phi)}$$
$$\quad \times \sum_{n=M}^{\infty} \sum_{b \in \mathcal{D}^{n-M}} Diam(X_{c})^{\dim_S(\Phi)}(Diam(X_{b})\theta(n))^{\dim_S(\Phi)}$$
$$\quad \times Diam(X_{c})^{\dim_S(\Phi)} \sum_{n=M}^{\infty} \theta(n)^{\dim_S(\Phi)} \sum_{b \in \mathcal{D}^{n-M}} m_{\Phi}([b])$$
$$= Diam(X_{c})^{\dim_S(\Phi)} \sum_{n=M}^{\infty} \theta(n)^{\dim_S(\Phi)}$$
$$= \infty.$$
In the last line we made use of our underlying hypothesis and the equivalence stated before our proof. Importantly we see that the collection of sets $\{E_n\}_{n \geq M}$ satisfies the hypothesis of Lemma 3.3.

Step 4. Bounding $\sum_{n,m=M}^{Q} m_{\Phi}(E_n \cap E_m)$
To apply Lemma 3.3 we need to show that the following bound holds:
$$\sum_{n,m=M}^{Q} m_{\Phi}(E_n \cap E_m) = O\left( m_{\Phi}([c]) \left( \sum_{n=M}^{Q} \theta(n)^{\dim_S(\Phi)} + \left( \sum_{n=M}^{Q} \theta(n)^{\dim_S(\Phi)} \right)^2 \right) \right). \quad (7.13)$$
Let \( a \in \mathcal{D}^a \) be such that \( a_1, \ldots, a_M = c \) and \( m \geq M \). As a first step in our proof of \((7.13)\) we will bound

\[
m_\Phi([a_\theta] \cap E_m).
\]

There are two cases that naturally arise, when \( m > |a| + N(a, \theta) \) and when \( |a| < m \leq |a| + N(a, \theta) \). Let us consider first the case \(|a| < m \leq |a| + N(a, \theta)\). If \(|a| < m \leq |a| + N(a, \theta)\) then there is at most one \( a' \in \mathcal{D}^m \) such that

\[
[a_\theta] \cap [a'_\theta] \neq \emptyset.
\]

Moreover this \( a' \) must equal \( a z_1, \ldots, z_{m-n} \). This gives us the bound:

\[
m_\Phi([a_\theta] \cap E_m) = m_\Phi([a_\theta] \cap [a'_\theta])
\leq m_\Phi([a'_\theta])
\leq Diam(X_{a'_\theta})^{\dim_S(\Phi)}
\leq (Diam(X_{a'})\theta(m))^{\dim_S(\Phi)}
\leq (Diam(X_a)Diam(X_{z_1, \ldots, z_{m-n}})\theta(m))^{\dim_S(\Phi)}
\leq m(|a|)m(|z_1, \ldots, z_{m-n}|)\theta(m)^{\dim_S(\Phi)}
\leq m(|a|)m(|z_1, \ldots, z_{m-n}|)\theta(n)^{\dim_S(\Phi)}
\leq O\left(m(|a|)\theta(n)^{\dim_S(\Phi)} \gamma_{m-n}\right).
\]

In the penultimate line we used that \( \theta \) is decreasing. Thus we have shown that if \(|a| < m \leq |a| + N(a, \theta)\) then

\[
m_\Phi([a_\theta] \cap E_m) = O\left(m(|a|)\theta(n)^{\dim_S(\Phi)} \gamma_{m-n}\right).
\]

(7.14)

We now consider the case where \( m > |a| + N(a, \theta) \). In this case, if \( a' \in \mathcal{D}^m \) and

\[
[a_\theta] \cap [a'_\theta] \neq \emptyset,
\]

we must have

\[
a'_1, \ldots, a'_{|a|+N(a,\theta)} = a_\theta.
\]

Using this observation we obtain:

\[
m_\Phi([a_\theta] \cap E_m) = \sum_{a'_1, \ldots, a'_{|a|+N(a,\theta)} = a_\theta} m_\Phi([a'_\theta])
\leq \sum_{b' \in \mathcal{D}^{m-n-N(a,\theta)}} Diam(X_{a_\theta}b'z_1, \ldots, z_{N(a_\theta)b', \theta})^{\dim_S(\Phi)}
\leq \sum_{b' \in \mathcal{D}^{m-n-N(a,\theta)}} (Diam(X_{a_\theta}b')\theta(m))^{\dim_S(\Phi)}
\leq (Diam(X_{a_\theta})\theta(m))^{\dim_S(\Phi)} \sum_{b' \in \mathcal{D}^{m-n-N(a,\theta)}} Diam(X_{b'})^{\dim_S(\Phi)}
\leq (Diam(X_{a_\theta})\theta(m))^{\dim_S(\Phi)} \sum_{b' \in \mathcal{D}^{m-n-N(a,\theta)}} m_\Phi([b'])
\]

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Thus we have shown that if \( m > |a| + N(a, \theta) \) then
\[
\text{m}_\Phi([a] \cap E_m) \asymp \sum_{n,M} \text{m}_\Phi([a] \cap E_m) \asymp m_\Phi([a] \cap E_m) \times m_\Phi([a] \cap E_m) .
\] (7.15)

Combining (7.14) and (7.15) we obtain the bound
\[
\sum_{n,M} \text{m}_\Phi([a] \cap E_m) = \mathcal{O} \left( \sum_{n,M} \text{m}_\Phi([a] \cap E_m) \right) .
\] (7.16)

Importantly this bounds holds for all \( m > n \).

Applying (7.16) we obtain:
\[
\sum_{n,M} \text{m}_\Phi([a] \cap E_m) = \mathcal{O} \left( \sum_{n,M} \text{m}_\Phi([a] \cap E_m) \right) .
\] (7.17)

We now analyse each term in (7.17) individually. Repeating the arguments given at the end of Step 3, we can show that
\[
\sum_{n,M} \text{m}_\Phi([a] \cap E_m) \asymp \sum_{n,M} \text{m}_\Phi([a] \cap E_m) \asymp m_\Phi([a] \cap E_m) .
\] (7.18)

Focusing on the second term in (7.17) we obtain:
\[
\sum_{n,M} \text{m}_\Phi([a] \cap E_m) \asymp \sum_{n,M} \text{m}_\Phi([a] \cap E_m) \asymp m_\Phi([a] \cap E_m) .
\] (7.19)

In the last line we used that \( \gamma \in (0,1) \) so \( \sum_{n,M} \text{m}_\Phi([a] \cap E_m) \) can be bounded above by a constant independent of \( n \) and \( Q \).
We now focus on the third term in (7.17):

\[
\sum_{n=M}^{Q-1} \sum_{a_1, \ldots, a_M = c} \sum_{m=n+1}^{Q} m_\Phi([a]) \theta(n)^{\dim_S(\Phi)} \theta(m)^{\dim_S(\Phi)} \\
\times m_\Phi([c]) \sum_{n=M}^{Q-1} \sum_{b \in D^{n-M}} \sum_{m=n+1}^{Q} m_\Phi([b]) \theta(n)^{\dim_S(\Phi)} \theta(m)^{\dim_S(\Phi)} \\
= m_\Phi([c]) \sum_{n=M}^{Q-1} \theta(n)^{\dim_S(\Phi)} \sum_{b \in D^{n-M}} m_\Phi([b]) \sum_{m=n+1}^{Q} \theta(m)^{\dim_S(\Phi)} \\
\leq m_\Phi([c]) \left( \sum_{n=M}^{Q} \theta(n)^{\dim_S(\Phi)} \right)^2
\] (7.20)

Substituting (7.18), (7.19), and (7.20) into (7.17) we obtain

\[
\sum_{n,m=M}^{Q} m_\Phi(E_n \cap E_m) = \mathcal{O} \left( m_\Phi([c]) \left( \sum_{n=M}^{Q} \theta(n)^{\dim_S(\Phi)} + \left( \sum_{n=M}^{Q} \theta(n)^{\dim_S(\Phi)} \right)^2 \right) \right).
\]

Therefore (7.13) holds.

**Step 5. Applying Lemma 3.3.**

Since \( \sum_{n=M}^{\infty} \theta(n)^{\dim_S(\Phi)} = \infty \) there exists \( Q \) such that \( \sum_{n=M}^{Q} \theta(n)^{\dim_S(\Phi)} > 1 \). Therefore for \( Q \) sufficiently large we have

\[
\sum_{n=M}^{Q} \theta(n)^{\dim_S(\Phi)} < \left( \sum_{n=M}^{Q} \theta(n)^{\dim_S(\Phi)} \right)^2.
\] (7.21)

By the arguments given at the end of Step 3 we know that

\[
\sum_{n=M}^{Q} m_\Phi(E_n) \asymp m_\Phi([c]) \sum_{n=M}^{Q} \theta(n)^{\dim_S(\Phi)}.
\]

It follows therefore from (7.13) and (7.21) that there exists some \( d > 0 \) independent of \( M \) such that

\[
\limsup_{Q \to \infty} \frac{\left( \sum_{n,M}^{Q} m_\Phi(E_n) \right)^2}{\sum_{n,m,M}^{Q} m_\Phi(E_n \cap E_m)} \geq \limsup_{Q \to \infty} \frac{d \cdot \left( m_\Phi([c]) \sum_{n=M}^{Q} \theta(n)^{\dim_S(\Phi)} \right)^2}{m_\Phi([c]) \left( \sum_{n=M}^{Q} \theta(n)^{\dim_S(\Phi)} \right)^2} = d m_\Phi([c]).
\]

Applying Lemma 3.3 it follows that

\[
m_\Phi(\limsup_{n \to \infty} E_n) \geq d m_\Phi([c]).
\]

This implies (7.12) and completes our proof.
8 Applications of the mass transference principle

The main results of this paper give conditions ensuring a limsup set of the form $W_\Phi(z, \Psi)$ or $U_\Phi(z, m, h)$ has positive of full Lebesgue measure. For these results it is necessary to assume that some appropriate volume sum diverges. If the relevant volume sum converged, then the limsup set in question would automatically have zero Lebesgue measure by the Borel-Cantelli lemma. It is still an interesting problem to determine the metric properties of a limsup set when the volume sum converges. Thankfully there a powerful tool for determining the size of a limsup set when the volume sum converges. This tool is known as the mass transference principle and is due to Beresnevich and Velani [10]. We provide a brief account of this technique below.

We say that a set $X \subset \mathbb{R}^d$ is Ahlfors regular if

$$\mathcal{H}^{\dim_H(X)}(X \cap B(x, r)) \asymp r^{\dim_H(X)}$$

for all $x \in X$ and $0 < r < \text{Diam}(X)$. Given $s > 0$ and a ball $B(x, r)$, we define

$$B^s := B(x, r^{s/\dim_H(X)}).$$

The theorem stated below is a weaker version of a statement proved in [10]. It is sufficient for our purposes.

**Theorem 8.1.** Let $X$ be Ahlfors regular and $(B_j)$ be a sequence of balls with radii converging to zero. Let $s > 0$ and suppose that for any ball $B$ in $X$ we have

$$\mathcal{H}^{\dim_H(X)}(B \cap \limsup_{j \to \infty} B_j^s) = \mathcal{H}^{\dim_H(X)}(B).$$

Then, for any ball $B$ in $X$

$$\mathcal{H}^s(B \cap \limsup_{j \to \infty} B_j) = \mathcal{H}^s(B).$$

Theorem 8.1 can be applied in conjunction with Theorem 2.2, Theorem 2.6, and Theorem 2.10 to prove many Hausdorff dimension results for the limsup sets $W_\Phi(z, \Psi)$ and $U_\Phi(z, m, h)$ when the appropriate volume sum converges. We simply have to restrict to a subset of the parameter space where we know that the corresponding attractor will always be Ahlfors regular. For the sake of brevity we content ourselves with the following statement for the family of iterated function systems studied in Section 5. This statement is a consequence of Theorem 8.1 and Theorem 2.10.

**Theorem 8.2.** Suppose $t \notin \mathbb{Q}$, then for any $z \in [0, 1 + t]$ and $s > 0$ we have

$$\dim_H(W_\Phi(z, 4^{-|a|(1+s)}) = \frac{1}{1 + s}$$

and

$$\mathcal{H}^{1/s}(W_\Phi(z, 4^{-|a|(1+s)})) = \infty.$$

9 Examples

The purpose of this section is to provide some explicit examples to accompany the main results of this paper.
9.1 IFSs satisfying the CS property

Here we provide two classes of IFSs that satisfy the CS property with respect to some measure $m$. These IFSs will have contraction ratios lying in a special class of algebraic integers known as Garsia numbers. A Garsia number is a positive real algebraic integer with norm $\pm 2$ whose Galois conjugates all have modulus strictly greater than 1. Examples of Garsia numbers include $\sqrt{2}$ for any $n \in \mathbb{N}$, and $1.76929\ldots$, the appropriate root of $x^3 - 2x^2 - 2 = 0$. The lemma below is due to Garsia [22], for a short proof see [3].

**Lemma 9.1.** Let $\lambda$ be the reciprocal of a Garsia number. Then there exists $s > 0$ such that for any two distinct $a, a' \in \{-1, 1\}^n$ we have

$$\left| \sum_{j=0}^{n-1} a_j \lambda^j - \sum_{j=0}^{n-1} a'_j \lambda^j \right| > \frac{s}{2^n}.$$ 

**Example 9.2.** Let $m$ be the $(1/2, 1/2)$ Bernoulli measure and for each $\lambda \in (1/2, 1)$, let the corresponding IFS be

$$\Phi_\lambda := \{ \phi_{-1}(x) = \lambda x - 1, \phi_1(x) = \lambda x + 1 \}.$$ 

For any $a, a' \in \{-1, 1\}^n$ and $z \in [\frac{-1}{1-\lambda}, \frac{1}{1-\lambda}]$, it can be shown that

$$\phi_a(z) - \phi_{a'}(z) = \sum_{j=0}^{n-1} a_j \lambda^j - \sum_{j=0}^{n-1} a'_j \lambda^j.$$ 

Therefore by Lemma 9.1, if $\lambda$ is the reciprocal of a Garsia number, for any $z \in [\frac{-1}{1-\lambda}, \frac{1}{1-\lambda}]$ and distinct $a, a' \in \{-1, 1\}^n$, we have

$$|\phi_a(z) - \phi_{a'}(z)| > \frac{s}{2^n}.$$ 

It follows that for any $z \in [\frac{-1}{1-\lambda}, \frac{1}{1-\lambda}]$ we have

$$S(\{\phi_a(z)\}_{a \in \{-1, 1\}^n}, \frac{s}{2^n}) = \{\phi_a(z)\}_{a \in \{-1, 1\}^n}$$ 

for all $n \in \mathbb{N}$. Applying Proposition 3.9 we see that for any $z \in [\frac{-1}{1-\lambda}, \frac{1}{1-\lambda}]$ and $h : \mathbb{N} \to [0, \infty)$ satisfying $\sum_{n=1}^{\infty} h(n) = \infty$, we have that Lebesgue almost every $x \in [\frac{-1}{1-\lambda}, \frac{1}{1-\lambda}]$ is contained in $U_{\Phi_\lambda}(z,m,h)$. Therefore if $\lambda$ is the reciprocal of a Garsia number, then the IFS $\Phi_\lambda$ has the CS property with respect to $m$. This fact is a consequence of the main result of [4]. The proof given there relied upon certain counting estimates due to Kempton [33]. The argument given in the proof of Proposition 3.9 doesn’t rely on any such counting estimates. Instead we make use of the fact that the Bernoulli convolution is equivalent to the Lebesgue measure and is expressible as the weak star limit of weighted Dirac masses supported on elements of the set $\{\phi_a(z)\}_{a \in \{-1, 1\}^n}$.

**Example 9.3.** Let $m$ be the $(1/4, 1/4, 1/4, 1/4)$ Bernoulli measure and let our IFS be

$$\Phi_{\lambda_1, \lambda_2} := \{ \phi_1(x, y) = (\lambda_1 x + 1, \lambda_2 y + 1), \phi_2(x, y) = (\lambda_1 x + 1, \lambda_2 y - 1), \phi_3(x, y) = (\lambda_1 x - 1, \lambda_2 y + 1), \phi_4(x, y) = (\lambda_1 x - 1, \lambda_2 y - 1) \},$$ 

where $\lambda_1, \lambda_2 \in (1/2, 1)$. For each $\Phi_{\lambda_1, \lambda_2}$ the corresponding attractor is $[\frac{-1}{1-\lambda_1}, \frac{1}{1-\lambda_1}] \times [\frac{-1}{1-\lambda_2}, \frac{1}{1-\lambda_2}]$. If both $\lambda_1$ and $\lambda_2$ are reciprocals of Garsia numbers, then it follows from Lemma 9.1 that for some $s > 0$, for any $z \in [\frac{-1}{1-\lambda_1}, \frac{1}{1-\lambda_1}] \times [\frac{-1}{1-\lambda_2}, \frac{1}{1-\lambda_2}]$, we have

$$|\phi_a(z) - \phi_{a'}(z)| > \frac{s}{2^n}.$$
for distinct $a, a' \in \{1, 2, 3, 4\}^n$. Therefore

$$S\left(\{\phi_a(z)\}_{a \in \{1, 2, 3, 4\}^n}, \frac{S}{2^n}\right) = \{\phi_a(z)\}_{a \in \{-1, 1\}^n}$$

for any $z \in \left[\frac{1}{1 - \lambda_1}, \frac{1}{1 - \lambda_2}\right] \times \left[\frac{1}{1 - \lambda_2}, \frac{1}{1 - \lambda_1}\right]$ for all $n \in \mathbb{N}$.

Note that $d = 2$ and each of our contractions have the same matrix part. Applying Proposition 3.9, we see that for any $z \in \left[\frac{1}{1 - \lambda_1}, \frac{1}{1 - \lambda_2}\right] \times \left[\frac{1}{1 - \lambda_2}, \frac{1}{1 - \lambda_1}\right]$ and $h : \mathbb{N} \to [0, \infty)$ satisfying $\sum_{n=1}^{\infty} h(n) = \infty$, we have that Lebesgue almost every $x \in \left[\frac{1}{1 - \lambda_1}, \frac{1}{1 - \lambda_2}\right] \times \left[\frac{1}{1 - \lambda_2}, \frac{1}{1 - \lambda_1}\right]$ is contained in $U_{\Phi_{\lambda_1, \lambda_2}}(z, m, h)$. Therefore when $\gamma_1, \gamma_2$ are both reciprocals of Garsia numbers, the IFS $\Phi_{\lambda_1, \lambda_2}$ satisfies the CS property with respect to $m$.

It is perhaps also worth mentioning that by Proposition 3.9, if both $\gamma_1$ and $\gamma_2$ are reciprocals of Garsia numbers, then the pushforward of $m$ is absolutely continuous. To the best of the authors knowledge, there is no place in the literature that gives an explicit example of an absolutely continuous pushforward for an overlapping affine iterated function system.

9.2 The non-existence of Khintchine like behaviour without exact overlaps

In [3] the author asked whether the only mechanism preventing an IFS from observing some sort of Khintchine like behaviour was the presence of exact overlaps. The example below, which is based upon Example 1.2 from [28], shows that there are other mechanisms preventing Khintchine like behaviour.

Example 9.4. Pick $t^* \in (0, 2/3)$ so that the IFS

$$\Phi_{t^*} := \left\{ \phi_1(x) = \frac{x}{3}, \phi_2(x) = \frac{x + 1}{3}, \phi_3(x) = \frac{x + 2}{3}, \phi_4(x) = \frac{x + t^*}{3} \right\}.$$ 

does not contain an exact overlap. Now consider the following IFS acting on $\mathbb{R}^2$:

$$\Phi'_{t^*} := \left\{ \phi'_1(x, y) = (x/3, y/3), \phi'_2(x, y) = ((x + 1)/3, y/3), \phi'_3(x, y) = ((x + t^*)/3, y/3), \phi'_4(x, y) = (x/3, (y + 2)/3), \phi'_5(x, y) = ((x + 1)/3, (y + 2)/3), \phi'_6(x, y) = (x/3, y/3), \phi'_7(x, y) = ((x + t^*)/3, (y + 2)/3) \right\}.$$ 

The attractor $X$ for $\Phi'_{t^*}$ is $[0, 1] \times C$, where $C$ is the middle third Cantor set. Therefore $\dim_H(X) = 1 + \frac{\log 2}{\log 3}$. Since $\Phi_{t^*}$ did not contain an exact overlap, it follows that $\Phi'_{t^*}$ also does not contain an exact overlap.

Let $\gamma \approx 0.279$ be such that

$$8\gamma^{1 + \frac{\log 2}{\log 3}} = 1.$$ 

So in particular we have

$$\sum_{n=1}^{\infty} \sum_{a \in D^n} \gamma^{n\left(1 + \frac{\log 2}{\log 3}\right)} = \infty. \quad \text{(9.1)}$$

If it were the case that our IFS exhibited Khintchine like behaviour, then with (9.1) in mind, at the very least we would expect that there exists $z \in X$ such that the set

$$W := \left\{ (x, y) \in \mathbb{R}^2 : |(x, y) - \phi'_a(z)| \leq \gamma^{|a|} \text{ for i.m. } a \in \bigcup_{n=1}^{\infty} \{1, \ldots, 8\}^n \right\}$$

has Hausdorff dimension equal to $1 + \frac{\log 2}{\log 3}$. We now show that in fact $\dim_H(W) < 1 + \frac{\log 2}{\log 3}$. 

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Let
\[
\Phi'' := \left\{ \phi_1''(y) = \frac{y}{3}, \phi_2''(y) = \frac{y + 2}{3} \right\}.
\]

Clearly \( \Phi'' \) has the middle third Cantor set as its attractor. We now make the simple observation that if \((x, y) \in \mathbb{R}^2\) satisfies \(|(x, y) - \phi_a'(z)| \leq \gamma^n\) for some \(a \in \{1, \ldots, 8\}^n\) for \(z = (z_1, z_2)\), then \(|y - \phi_a''(z_2)| \leq \gamma^n\) for some \(a \in \{1, 2\}^n\). This means that if \(|(x, y) - \phi_a'(z)| \leq \gamma^n\) for some \(a \in \{1, \ldots, 8\}^n\), then \((x, y)\) must be contained in one of \(2^n\) horizontal strips of height \(2\gamma^n\) and width 1. Such a strip can be covered by \(C(1/\gamma)^n\) balls of diameter \(\gamma^n\) for some \(C > 0\) independent of \(n\). It follows that the set of \((x, y) \in \mathbb{R}^2\) satisfying \(|(x, y) - \phi_a'(z)| \leq \gamma^n\) for some \(a \in \{1, 2, \ldots, 8\}^n\), can be covered by \(C(2/\gamma)^n\) balls of diameter \(\gamma^n\). For each \(n\) let \(U_n\) be such a collection of balls. By construction, for any \(N \in \mathbb{N}\) the set \(\bigcup_{n \geq N} \{B \in U_n\}\) is a \(\gamma^N\) cover of \(W\).

Now let
\[
s > \frac{\log \gamma - \log 2}{\log \gamma} \approx 1.542. \tag{9.2}
\]

Then
\[
\mathcal{H}^s(W) \leq \lim_{N \to \infty} \sum_{n = N}^{\infty} \sum_{B \in U_n} \text{Diam}(B)^s \leq \lim_{N \to \infty} \sum_{n = N}^{\infty} C(2/\gamma)^n \cdot \gamma^{sn} = 0.
\]

In the final equality we used \((\ref{eq:7.2})\) to guarantee \(\sum_{n = 1}^{\infty} C(2/\gamma)^n \cdot \gamma^{sn} < \infty\). We have shown that \(\mathcal{H}^s(W) = 0\) for any \(s > \frac{\log n - \log 2}{\log \gamma}\). Therefore \(\dim_H(W) \leq \frac{\log 3 - \log 2}{\log \gamma}\). Since \(\frac{\log 3 - \log 2}{\log \gamma} \approx 1.542\) and \(1 + \frac{\log 2}{\log 3} \approx 1.631\), we have \(\dim_H(W) < 1 + \frac{\log 2}{\log 3}\) as required.

Note that this example can easily be generalised to demonstrate a similar phenomenon when the underlying attractor has positive Lebesgue measure.

10 Final discussion and open problems

A number of problems and questions naturally arise from the results of this paper. The first and likely most difficult question is the following:

- Can one derive general, verifiable conditions for an IFS under which we can conclude it exhibits Khintchine like behaviour?

This question seems to be very difficult and appears to be out of reach of our current methods. As such it seems that a more reasonable immediate goal would be to prove results for general parameterised families of iterated function systems. One can define a parameterised family of iterated function systems in the following general way. Suppose that \(U\) is an open subset of \(\mathbb{R}^k\), and for each \(u \in U\) we have an IFS given by
\[
\Phi_u := \{ \phi_{i,u}(x) = A_i(u)(x) + t_i(u) \}_{i=1}^l.
\]

Where for each \(1 \leq i \leq l\) we have \(A_i : U \to GL(d, \mathbb{R}) \cap \{ A : ||A|| < 1 \}\) and \(t_i : U \to \mathbb{R}^d\). For each \(u \in U\) we denote the attractor corresponding to this iterated function system by \(X_u\). We would like to be able to describe what, if any, Khintchine like behaviour is observed for \(\Phi_u\) for a typical \(u \in U\). The methods of this paper do not extend to this general a setting and only work when some transversality condition is assumed. We expect that the conjecture stated below holds under some weak assumptions on the functions \(A_i\) and \(t_i\).

For a \(\sigma\)-invariant ergodic probability measure \(m\), and a fixed \(u \in U\), we denote the corresponding Lyapunov exponents by \(\lambda_1(m, u), \ldots, \lambda_d(m, u)\).
Conjecture 10.1. Let $m$ be a slowly decaying $\sigma$-invariant ergodic probability measure and suppose that $h(m) > -(\lambda_1(m, u) + \cdots + \lambda_d(m, u))$ for Lebesgue almost every $u \in U$. Then the following statements hold:

- For Lebesgue almost every $u \in U$, for any $z \in X_u$ and $h \in H^*$, Lebesgue almost every $x \in X_u$ is contained in $U_{\Phi_u}(z, m, h)$.
- For Lebesgue almost every $u \in U$, for any $z \in X_u$, there exists $h : \mathbb{N} \to [0, \infty)$ such that $\sum_{n=1}^{\infty} h(n) = \infty$, yet $U_{\Phi_u}(z, m, h)$ has zero Lebesgue measure.

Much of the analysis of this paper was concerned with the sequence

$$
\left( \frac{T(\{\phi_a(z)\}_{a \in L_{m,n}} \cdot \frac{s}{R_{m,n}})}{R_{m,n}} \right)_{n=1}^{\infty},
$$

where $z \in X$ and $m$ is some slowly decaying $\sigma$-invariant ergodic probability measure. In fact each of our main results was obtained by deriving some quantitative information about the values this sequence takes for typical values of $n$. The behaviour of this sequence provides another useful method for measuring how an IFS overlaps. For the parameterised families considered above, we conjecture that the statement below is true under some weak assumptions on the maps $A_i$ and $t_i$.

Conjecture 10.2. Let $m$ be a slowly decaying $\sigma$-invariant ergodic probability measure and suppose that $h(m) > -(\lambda_1(m, u) + \cdots + \lambda_d(m, u))$ for Lebesgue almost every $u \in U$. Then for Lebesgue almost every $u \in U$, for any $z \in X_u$, for $s$ sufficiently small we have

$$
0 = \liminf_{n \to \infty} \frac{T(\{\phi_a(z)\}_{a \in L_{m,n}} \cdot \frac{s}{R_{m,n}})}{R_{m,n}} < \limsup_{n \to \infty} \frac{T(\{\phi_a(z)\}_{a \in L_{m,n}} \cdot \frac{s}{R_{m,n}})}{R_{m,n}} = 1.
$$

One of the interesting ideas to arise from this paper is the notion of an IFS satisfying the CS property with respect to a measure $m$. Proceeding via analogy with Theorem 2.10 we expect that given a measure $m$, it is the case that within a parameterised family of IFSs the CS property will not typically be satisfied with respect to $m$. Indeed if Conjecture 10.2 were true then this statement would follow from Proposition 8.6. That being said, we still expect that for a parameterised family of IFSs, it will often be the case that there exists a large subset of the parameter space where the IFS does satisfy the CS property with respect to $m$. We conjecture that the statement below is true under some weak assumptions on the maps $A_i$ and $t_i$.

Conjecture 10.3. Let $m$ be a slowly decaying $\sigma$-invariant ergodic probability measure and suppose that $h(m) > -(\lambda_1(m, u) + \cdots + \lambda_d(m, u))$ for Lebesgue almost every $u \in U$. Then there exists $U' \subset U$ such that $\dim_H(U') = k$, and for any $u \in U'$ the IFS $\Phi_u$ satisfies the CS property with respect to $m$.

Theorem 2.10 supports the validity of Conjectures 10.1, 10.2, and 10.3. Theorem 2.15 states that satisfying the CS property with respect to $m$ implies the pushforward $\mu$ is absolutely continuous. The CS property appears to only be satisfied in exceptional circumstances. As such it is natural to wonder whether there exists a more easily verifiable condition phrased in terms of limsup sets, which implies the absolute continuity of $\mu$. We pose the following question:

- Let $\mu$ be the pushforward of a measure $m$. What is the smallest class of functions, such that if for some $z \in X$ the set $U_{\Phi}(z, m, h)$ has positive Lebesgue measure for all $h$ belonging to this class, then $\mu$ will be absolutely continuous?
Much of the work presented in this paper is inspired by the classical theorem of Khintchine stated as Theorem 1.2 in our introduction. Along with Khintchine’s theorem, one of the first results encountered in a course on Diophantine approximation is the following result due to Dirichlet.

**Theorem 10.4** (Dirichlet). For any \( x \in \mathbb{R} \) and \( Q \in \mathbb{N} \), there exists \( 1 \leq q \leq Q \) and \( p \in \mathbb{Z} \) such that

\[
|x - \frac{p}{q}| < \frac{1}{qQ}.
\]

Therefore, for any \( x \in \mathbb{R} \) there exists infinitely many \((p, q) \in \mathbb{Z} \times \mathbb{N}\) satisfying

\[
|x - \frac{p}{q}| < \frac{1}{q^2}.
\]

For us the interesting feature of Dirichlet’s theorem lies in the fact that it is a statement for all \( x \in \mathbb{R} \). In our setting it is obvious that for any IFS \( \Phi \), for any \( z \in X \) we have

\[
X = \left\{ x \in \mathbb{R}^d : |x - \phi_a(z)| \leq \text{Diam}(X_a) \text{ for i.m. } a \in \mathcal{D}^* \right\}.
\]  

(10.2)

The results of this paper demonstrate that for many overlapping IFSs, given a \( z \in X \), then Lebesgue almost every point in \( X \) can be approximated by images of \( z \) infinitely often at a scale decaying to zero at an exponentially faster rate than \( \text{Diam}(X_a) \). See for example Theorem 2.10 where Lebesgue almost every point can be approximated at the scale \( 4^{-|a|} \), yet \( \text{Diam}(X_a) = 2^{-|a|} \). With Theorem 10.4 in mind, it is natural to wonder whether there exists conditions under which (10.2) can be improved upon.

- Can one construct an IFS for which there exists \( s > 1 \) such that

\[
X = \left\{ x \in \mathbb{R}^d : |x - \phi_a(z)| \leq \text{Diam}(X_a)^s \text{ for i.m. } a \in \mathcal{D}^* \right\}.
\]

Alternatively one could ask whether there exists \( s > 1 \) such that these sets differ by a finite or countable set.

We remark here that for the family of IFSs \( \{\lambda x - 1, \lambda x + 1\} \), it can be shown that there exists \( \lambda \in (1/2, 0.668) \) and \( z \in \left[\frac{1}{1-\lambda}, \frac{1}{1-\lambda}\right] \), such that Lebesgue almost every \( x \in \left[\frac{1}{1-\lambda}, \frac{1}{1-\lambda}\right] \) can be approximated by images of \( z \) at the scale \( 2^{-|a|} \), yet there exists a set of positive Hausdorff dimension within \( \left[\frac{1}{1-\lambda}, \frac{1}{1-\lambda}\right] \) that cannot be approximated by images of \( z \) at a scale better than \( \lambda^{|a|} \). For more details on this example we refer the reader to the discussion at the end of [4].

We conclude now by emphasising one of the technical difficulties that is present within this paper that is not present within similar works on this topic. In many situations, if \( \mu = \mu' \ast \mu'' \), and we have some method for measuring how evenly distributed a measure is with \( \mathbb{R}^d \) (examples of methods of measurement include: absolute continuity, entropy, and \( L^q \) dimension), then often \( \mu \) will be at least as evenly distributed as \( \mu' \) with respect to this method of measurement. One may in fact see a strict increase in how evenly distributed \( \mu \) is with respect to this method of measurement (see for example [27, 52]). A useful feature of the pushforward of Bernoulli measures is they are often equipped with some sort of convolution structure. In many papers this convolution structure and the above idea described above can be exploited to obtain results (see for example [27, 50, 52, 54, 59, 61]). Within this paper, the relevant method for measuring how evenly distributed a measure is, is to study the sequence given by (10.1). On a technical level, one of the main difficulties for us is that this method of measurement does not behave well under convolution. This is easy to see with an example. Let \( m \) be the \((1/2, 1/2)\) Bernoulli
measure and let our IFS be \( \{ \phi_1(x) = \frac{x}{2}, \phi_2(x) = \frac{x+1}{2} \} \). For this IFS the attractor is \([0, 1]\). We denote the pushforward of \( m \) by \( \mu' \). It is easy to see that for any \( z \in [0, 1] \) and \( n \in \mathbb{N} \), we have

\[
T(\{ \phi_a(z) \}_{a \in \{1,2\}^n, \frac{1}{2^n}}) = 1.
\]

(10.3)

So \( \mu' \) exhibits an optimal level of separation. Now let \( t \in (0, 1) \cap \mathbb{Q} \) and consider the IFS \( \{ \phi_1(x) = \frac{x}{2}, \phi_2(x) = \frac{x+1}{2}, \phi_3(x) = \frac{x+t}{2}, \phi_4(x) = \frac{x+1+t}{2} \} \). For this IFS the attractor is \([0, t]\). We denote the pushforward of \( m \) for this IFS by \( \mu'' \). It is easy to see that for \( \mu'' \) we also have the optimal level of separation described by (10.3). Consider the measure \( \mu = \mu' \ast \mu'' \). This measure is simply the pushforward of the \((1/4, 1/4, 1/4, 1/4)\) Bernoulli measure with respect to the IFS

\[ \{ \phi_1(x) = \frac{x}{2}, \phi_2(x) = \frac{x+1}{2}, \phi_3(x) = \frac{x+t}{2}, \phi_4(x) = \frac{x+1+t}{2} \}, \]

i.e. the IFS studied in Theorem 2.10. Examining the proof of Proposition 5.1, we see that for any \( t \in (0, 1) \cap \mathbb{Q} \), there exists \( c > 0 \) such that for any \( z \in [0, 1+t] \) and \( s > 0 \) we have

\[
T \left( \{ \phi_a(z) \}_{a \in \{1,2,3,4\}^n, \frac{s}{4^n}} \right) = \mathcal{O}((4-c^n)).
\]

(10.4)

Equation (10.4) demonstrates that we no longer have the strong separation properties that we saw earlier for our two measures \( \mu' \) and \( \mu'' \). We have in fact seen that after convolving \( \mu' \) and \( \mu'' \) there is a drop in how evenly distributed the resulting measure is within \( \mathbb{R} \). One could view this failure to improve under convolution as a consequence of how sensitive our method of measurement is to exact overlaps.

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