NORMAL FORM AND PARABOLIC DYNAMICS FOR QUADRATICALLY GROWING AUTOMORPHISMS OF FREE GROUPS

MARTIN LUSTIG AND KAIDI YE

Abstract. We present a normal form for outer automorphisms \( \phi \) of a non-abelian free group \( F_N \) which grow quadratically (measured through the maximal growth of conjugacy classes in \( F_N \) under iteration of \( \phi \)). In analogy to the known normal form for linearly growing automorphisms as efficient Dehn twist, our normal form for \( \phi \) is given in terms of a 2-level Dehn twist on a graph-of-groups \( G \) with \( \pi_1 G \cong F_N \), where a conjugacy class of \( F_N \) grows at most linearly if and only if it is contained in a vertex group of \( G \).

Our proof is based on earlier work of the second author \([13, 14, 15]\) and on a new cancellation result, which also allows us to show that the dynamics of the induced \( \phi \)-action on Outer space \( CV_N \) consists entirely of parabolic orbits, with limit points all assembled in the simplex \( \Delta_G \subset \partial CV_N \) determined by \( G \).

1. Introduction

Outer automorphisms of a free group \( F_N \) of finite rank \( N \geq 2 \) have received a lot of attention in the past 30 years, since the groundbreaking papers of Culler-Vogtmann [6] and Bestvina-Handel [3]. Much progress has been obtained, in particular from the attempt to mimic important known features from mapping classes and from the action of the mapping class group \( \text{Mod}_g \) on Teichmüller space \( T_g \). However, in many aspects automorphisms of free groups can be intrinsically more complicated than mapping classes, and the group \( \text{Out}(F_N) \) is less tractable and more mysterious than \( \text{Mod}_g \). In addition, the natural analogue of \( T_g \) with its canonical \( \text{Mod}_g \)-action, namely Outer space \( CV_N \) equipped with a canonical \( \text{Out}(F_N) \)-action, is not a manifold and hence immune towards all attempts to mimic directly the well developed analytic theory for \( T_g \).

One of the most obvious differences to \( \text{Mod}_g \) is that \( \text{Out}(F_N) \) contains elements which grow polynomially of degree \( d \geq 2 \). Here the growth function \( \text{Gr}_\phi(t) \) of an outer automorphism \( \phi \in \text{Out}(F_N) \) is given by considering, for any element \( w \in F_N \), the function \( \| \phi^t([w]) \|_A \), where \( A \) is any basis of \( F_N \), and \( \| [u] \|_A \) denotes the length of a cyclically reduced word in \( A^{\pm 1} \) which.

2000 Mathematics Subject Classification. Primary 20F, Secondary 20E, 57M.

Key words and phrases. Dehn twist, free group automorphism, quadratic growth, graph of groups.
represents the conjugacy class $[u] \subset F_N$. The choice of $A$ is immaterial if one is only interested in the type of the function $Gr_\varphi(t)$, which is taken as maximum over the above functions, for any $w \in F_N$. It is a well known consequence of [3] that $Gr_\varphi(t)$ is either an exponential function with growth rate given by a Perron-Frobenius transition matrix derived from $\varphi$, or else $Gr_\varphi(t)$ must be a polynomial of some degree $d \geq 0$.

Exponentially growing automorphisms $\varphi \in \text{Out}(F_N)$ have received on the whole more attention than polynomially growing ones, as they occur more frequently. In addition, polynomially growing such $\varphi$ are technically often harder to deal with, since there is less rigidity in the intrinsic structure of such automorphisms. Nevertheless, the only type of automorphism of $F_N$ for which a normal form was available so far are linearly growing automorphisms:

In two joint papers [4, 5] with Marshall Cohen the first author derived this normal form by exhibiting, for any linearly growing $\varphi \in \text{Out}(F_N)$, a certain type of graph-of-groups $\mathcal{G}$ with a marking isomorphism $\pi_1 \mathcal{G} \cong F_N$, together with a graph-of-groups automorphism $D : \mathcal{G} \to \mathcal{G}$, called an efficient Dehn twist (see section 2.3). The map $D$ induces for some integer $m \geq 1$ the outer automorphism $\hat{D} = \varphi^m$ on $\pi_1 \mathcal{G}$, and as such is unique up to graph-of-groups isomorphisms. It was later shown in joint work with S. Krstic and K. Vogtmann [7] that $\varphi$ itself is also induced by a graph-of-groups automorphism $R : \mathcal{G} \to \mathcal{G}$ with $R^m = D$.

In this paper the authors follow very much the same strategy for quadratically growing automorphisms: We define efficient 2-level Dehn twists as graph-of-groups automorphisms $H : \mathcal{G} \to \mathcal{G}$, but while for the above $D$ the induced local automorphisms on the vertex groups are the identity, the vertex group automorphisms of $H$ are themselves given through efficient Dehn twists. The conditions imposed by our Definition 4.4 ensure that $\hat{H}$ has always quadratic growth.

It follows from the results of [14, 15] that every quadratically growing $\varphi \in \text{Out}(F_N)$ has a positive power which can be represented by some 2-level Dehn twist, and through suitable modifications (see section 4) the latter can be made efficient. Much harder is the question about uniqueness, and again we follow closely the method employed by [4, 5]: There it has been shown that every Dehn twist automorphism induces on $CV_N$ an action with parabolic orbits, where the limit point of any orbit is contained in a simplex $\Delta_{\mathcal{G}}$ in the boundary $\partial CV_N$, which in turn is given by varying the edge lengths of the graph $\mathcal{G}$ on which the efficient Dehn twist $D$ had been defined.

Exactly the same is shown in our Theorem 6.3 quoted here for simplicity without the detailed information about the limit points. However, it should be noted that the proof in the quadratic growth case is substantially harder than in the linear case: We need to employ the concept of $H$-conjugacy (see section 3.1) to the correction terms of the edges of $\mathcal{G}$ (see Definition 2.6), and the proof given here uses crucially a growth result about the $H$-conjugacy
classes of those correction terms, proved previously by the second author in [14].

**Theorem 1.1.** Let \([\Gamma]\) be any point in Outer space \(CV_N\), given by a marked metric graph \(\Gamma\). Then for any automorphism \(\varphi \in \text{Out}(F_N)\), represented by an efficient 2-level Dehn twist \(H : \mathcal{G} \to \mathcal{G}\), the \(\varphi\)-orbit of \([\Gamma]\) is parabolic, with limit point contained in the interior of the simplex \(\Delta_{\mathcal{G}} \subset \partial CV_N\).

A first convergence result for the action of quadratically (or higher-degree polynomially) growing automorphism \(\varphi \in \text{Out}(F_N)\) on \(CV_N = CV_N \cup \partial CV_N\) has been obtained by M. Bestvina, M. Feighn and M. Handel's in Theorem 1.4 of [2].

From Theorem 1.1 we derive the desired uniqueness, thus justifying our terminology “normal form” (see Theorem 7.1). Related results have been obtained by M. Rodenhausen in [10] (see Remarks 3.1 and 4.6 below).

**Theorem 1.2.** Two efficient 2-level Dehn twists \(H : \mathcal{G} \to \mathcal{G}\) and \(H' : \mathcal{G}' \to \mathcal{G}'\) represent outer automorphisms \(\hat{H}\) and \(\hat{H}'\) of a free group \(F_N\) which are conjugate in \(\text{Out}(F_N)\) if and only if there exists a graph-of-groups isomorphism \(F : \mathcal{G} \to \mathcal{G}'\) which satisfies:

\[
\hat{H} = \hat{F}^{-1} \hat{H}' \hat{F}
\]

It turns out that the extension of the above normal form to roots of 2-level Dehn twists is easier than in the linear case, since contrary to that case, for 2-level Dehn twists \(H : \mathcal{G} \to \mathcal{G}\) the edge groups of \(\mathcal{G}\) are trivial. We obtain (see Theorem 7.3):

**Corollary 1.3.** (1) Every automorphism \(\varphi \in \text{Out}(F_N)\) with exponent \(m \geq 1\), such that \(\varphi^m\) is represented by an efficient 2-level Dehn twist \(H : \mathcal{G} \to \mathcal{G}\), can be represented by a graph-of-groups automorphism \(R : \mathcal{G} \to \mathcal{G}\).

(2) Two graph-of-groups automorphism \(R : \mathcal{G} \to \mathcal{G}\) and \(R' : \mathcal{G}' \to \mathcal{G}'\) as in part (1) represent outer automorphisms \(\hat{R}\) and \(\hat{R}'\) of a free group \(F_N\) which are conjugate in \(\text{Out}(F_N)\) if and only if there exists a graph-of-groups isomorphism \(F : \mathcal{G} \to \mathcal{G}'\) which satisfies:

\[
\hat{R} = \hat{F}^{-1} \hat{R}' \hat{F}
\]

There are a number of obvious algorithmic questions issuing from the above results; they will be answered in the forthcoming joint work [9].

**Acknowledgements:** Both authors would like to thank Arnaud Hilion for several helpful discussions.

2. **Preliminaries**

2.1. **Graphs-of-groups and their isomorphisms.**

In this subsection we set up the basic notation while recalling some fundamental facts about graph-of-groups and their isomorphisms. For more details on graph-of-groups we refer the reader to [1] [5] [8] [11].
Unless otherwise stated, a graph $\Gamma$ in this paper is finite, non-empty and connected. We denote the vertex set of $\Gamma$ by $V(\Gamma)$ and the set of oriented edges by $E(\Gamma)$. For any edge $e$ in $E(\Gamma)$ we denote by $\tau(e)$ its terminal vertex, and by $\overline{e}$ the edge with reversed orientation. Hence the initial vertex of $e$ is given by $\tau(\overline{e})$.

Our graph $\Gamma$ is non-oriented, but one can always choose an orientation of $\Gamma$, given as subset $E^+(\Gamma) \subseteq E(\Gamma)$ such that $E^+(\Gamma) \cup \overline{E^+(\Gamma)} = E(\Gamma)$ and $E^+(\Gamma) \cap \overline{E^+(\Gamma)} = \emptyset$, where $\overline{e} = \{\tau | e \in E^+(\Gamma)\}$.

**Definition 2.1.** A graph-of-groups $G$ is given by the following data: A graph $\Gamma$, a vertex group $G_v$ for each $v \in V(\Gamma)$, an edge group $G_e$ for each $e \in E(\Gamma)$, with $G_e = G_{\overline{e}}$, and an injective edge homomorphism $f_e : G_e \to G_{\tau(e)}$ for every edge $e$ of $\Gamma$.

Given a graph-of-groups $G$, we usually denote its underlying graph by $\Gamma(G)$, while the vertex set and edge set of $\Gamma(G)$ are denoted by $V(G)$ and $E(G)$ respectively. To each edge $e \in E(G)$ we abstractly associate a stable letter $t_e$.

If the underlying graph $\Gamma(G)$ consists of a single vertex only, then the graph-of-groups $G$ is sometimes called trivial.

**Definition 2.2.** (1) For any graph-of-groups $G$ the word group $W(G)$ is defined to be the free product of the vertex groups and the free group generated by all stable letters:

$$W(G) = \left( \bigast_{v \in V(\Gamma)} G_v \right) \ast F(\{t_e | e \in E(\Gamma)\})$$

(2) The path group $\Pi(G)$ is defined to be the quotient of $W(G)$ modulo the relations $t_\tau = t_\overline{e}^{-1}$ and $f_e(g) = t_e f_e(g) t_e^{-1}$ for all $e \in E(G)$ and $g \in G_e$.

**Remark 2.3.** (1) Since $\Pi(G)$ is vastly more important than $W(G)$, any word $W = r_0 t_1 r_1 \ldots r_{q-1} t_q r_q$, with $t_i = t_{e_i}$ for some $e_i \in E(G)$ and $r_i \in \bigast_{v \in V(\Gamma)} G_v$, though formally an element in $W(G)$, will always be understood as element in $\Pi(G)$ (unless explicitly stated otherwise). In particular, if $W'$ is a second such word, then $W = W'$ means that they are equal in $\Pi(G)$.

(2) This is justified by the following “normal form” in $\Pi(G)$:

Let $W = r_0 t_1 r_1 \ldots r_{q-1} t_q r_q$ and $W' = r'_0 t'_1 r'_1 \ldots r'_{q-1} t'_{q} r'_{q}$ be two words in $W(G)$. Then $W$ and $W'$ define the same element in $\Pi(G)$ if and only if $q = q'$, and if for any $k = 1, \ldots, q$ one has $t'_k = t_k$ and there exist elements $g_k \in G_{e_k}$ such that the equalities $r'_k = f_{e_k}(g_k) r_k f_{\tau(e_{k+1})}(g_{k+1})$ for $k \neq q$ as well as $r'_0 = f_{e_0}(g_1)$ and $r'_{q} = f_{e_q}(g_q)$ hold.

(3) As a consequence, the path length (or $G$-length) of any word $W = r_0 t_1 r_1 \ldots r_{q-1} t_q r_q$, given by

$$|W|_G = q,$$

is a well defined notion in $\Pi(G)$. If the context is unambiguous, we sometimes write $|W|$ for $|W|_G$. 


Definition 2.4. A word \( W = r_0t_1r_1...r_{q-1}t_qr_q \) in \( W(\mathcal{G}) \) is said to be

(1) **connected** if the sequence \( e_1e_2...e_q \) (for \( t_k = t_{e_k} \)) forms a connected path \( \gamma \), and if \( r_0 \in G_{\tau(\gamma_1)} \) and \( r_i \in G_{\tau(e_i)} \) for \( 1 \leq i \leq q \). In this case we write \( \tau(W) := \tau(\gamma) := \tau(e_q) \), and call it the terminal vertex of the word \( W \) or of the path \( \gamma \). Similarly, their initial vertex is given by \( \tau(\gamma_1) \).

(2) **closed connected** if it is connected and \( \tau(\gamma_1) = \tau(e_q) \). In order to specify the initial vertex we sometimes call \( W \) a closed connected word issued at \( \tau(\gamma_1) \).

(3) **reduced** if \( q = 0 \) and \( r_0 \neq 1 \), or if, in case \( q > 0 \), whenever \( t_i = t_{i+1}^{-1} \) for some \( 1 \leq i \leq q - 1 \) we have \( r_i \notin f_{e_i}(G_{e_i}) \).

(4) **cyclically reduced**: if it is reduced and if, in case \( q > 0 \) and \( t_1 = t_q^{-1} \), one has \( r_q \neq r_0 \notin f_{e_q}(G_{e_q}) \).

It follows from Remark 2.3 (2) that the terminology introduced in the last definition applies as well to the element in \( \Pi(\mathcal{G}) \) defined by the word \( W \) in \( W(\mathcal{G}) \).

Definition 2.5. For any graph-of-groups \( \mathcal{G} \) and any vertex \( v \in V(\mathcal{G}) \), we denote by \( \pi_1(\mathcal{G}, v) \) the fundamental group based at \( v \), which consists of all elements in \( \Pi(\mathcal{G}) \) that are represented by closed connected words issued at \( v \).

For distinct vertices \( v_1, v_2 \in V(\mathcal{G}) \), the fundamental groups \( \pi_1(\mathcal{G}, v_1) \) and \( \pi_1(\mathcal{G}, v_2) \) are conjugate in \( \Pi(\mathcal{G}) \). Sometimes, we write \( \pi_1(\mathcal{G}) \) when the base point does not make a difference.

Definition 2.6. Given two graphs-of-groups \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \), a graph-of-groups isomorphism \( H : \mathcal{G}_1 \rightarrow \mathcal{G}_2 \) consists of

(1) a graph isomorphism \( H_\Gamma : \Gamma(\mathcal{G}_1) \rightarrow \Gamma(\mathcal{G}_2) \),

(2) a group isomorphism \( H_v : G_v \rightarrow G_{H_\Gamma(v)} \) for each vertex \( v \in V(\mathcal{G}_1) \),

(3) a group isomorphism \( H_e = H_\tau : G_e \rightarrow G_{H_\tau e} \) for each edge \( e \in E(\mathcal{G}_1) \), and

(4) for every \( e \in E(\mathcal{G}_1) \) an element \( \delta(e) = \delta_H(e) \in G_{H_\tau e} \), called the correction term for \( e \), which satisfies:

\[
H_\tau(e)f_e = ad_e \delta_H(e) f_{H_\tau(e)} H_e
\]

Here and below we denote by \( ad_g \) the inner automorphism \( x \mapsto gxg^{-1} \).

The isomorphism \( H : \mathcal{G}_1 \rightarrow \mathcal{G}_2 \) induces an isomorphism \( H_* : \Pi(\mathcal{G}_1) \rightarrow \Pi(\mathcal{G}_2) \) defined on the generators by

(a) \( H_*(g) = H_v(g) \) for any \( g \in G_v \) and any \( v \in V(\mathcal{G}_1) \), and

(b) \( H_*(t_e) = \delta_H(t_{H_\tau(e)} \delta(e))^{-1} \) for any \( e \in E(\mathcal{G}_1) \).

For every \( v \in V(\mathcal{G}) \), the isomorphism \( H_* \) induces an isomorphism \( H_{*v} : \pi_1(\mathcal{G}_1, v) \rightarrow \pi_1(\mathcal{G}_2, H_\tau(v)) \). We denote by \( \tilde{H} \) the outer isomorphism induced by \( H_* \). Here we use the terminology introduced in 5: for arbitrary groups \( G_1 \) and \( G_2 \) an outer isomorphism \( \tilde{f} \) is an equivalence class of isomorphism,
where two isomorphism \( f : G_1 \to G_2 \) and \( f' : G_1 \to G_2 \) are equivalent if there is an element \( g \in G_2 \) such that \( f' = ad_g \circ f \) (for \( ad_g \) as defined above).

In the particular case where \( G_1 = G_2 \) the isomorphism \( H \) is called a graph-of-groups automorphism. If furthermore the automorphism \( H : G \to G \) induces the identity on the underlying graph \( \Gamma(G) \), then the group isomorphisms for edge groups and vertex groups are all automorphisms.

**Remark 2.7.** (1) Consider any group \( G \) and any automorphism \( \Phi \in \text{Aut}(G) \). As in [14] (compare also [10]), for any \( g \in G \) and \( s < t \in \mathbb{N} \), we define iterated products \( \Phi(s,t)(g) \) and \( \Phi(s,t)(g) \), given by

\[
\Phi(s,t)(g) = g\Phi(g)\Phi^2(g) \cdots \Phi^{t-1}(g),
\]

and

\[
\Phi(s,t)(g) = \Phi^s(g)\Phi^{s+1}(g) \cdots \Phi^{t-1}(g).
\]

(2) Note in particular that, in the case where \( H : G \to G \) is a graph-of-groups isomorphism which induces the identity on the underlying graph \( \Gamma(G) \), for any \( t \in \mathbb{N} \) the iteration of \( H^t \) on \( e \) gives (via the well known formulae for the composition of graph-of-groups isomorphism, see Remark 2.10 of [13]):

\[
H^t_e(e) = H^{t-1}_e(\delta(\tau)) \cdots H_e(\delta(\tau))\delta(\tau) \cdot e \cdot \delta(e)^{-1} H_e(\delta(e)^{-1}) \cdots H^{t-1}_e(\delta(e)^{-1})
\]

\[
= H^{t}_e(\delta(\tau)^{-1}) \cdot e \cdot H^{t}_e(\delta(e)^{-1}).
\]

### 2.2. Equivalences of graph-of-groups and their automorphisms.

The following statements are well known:

**Lemma 2.8** (Section 2.4 in [13]). Let \( G \) and \( G' \) be two graphs-of-groups which are identical everywhere, except that for some \( e \in E(G) \) and \( g \in G_{\tau(e)} \) one has \( f'_e = ad_g \circ f_e \).

Then there is a canonical graph-of-groups isomorphism \( H_g : G \to G' \) which is the identity on the underlying graph, on all vertex and edge groups, and has all correction terms equal to 1, except that \( \delta(e) = g^{-1} \).

**Lemma 2.9** ([13], Lemma 2.11). Let \( G, G' \) as well as \( e, g \) and \( H_g \) be as in Lemma 2.8 and let \( H : G \to G \) be any graph-of-groups automorphism.

Then there is a graph-of-groups automorphism \( H' : G' \to G' \) which coincides with \( H \) everywhere except that \( \delta_H(e) = H_{\tau(e)}(g)\delta_H(e)g^{-1} \), and \( H' \) is conjugate to \( H \) through \( H_g \):

\[
H' = H_g \circ H \circ H_g^{-1}
\]

**Lemma 2.10** ([7], Corollary 4.8). Let \( H : G \to G \) and \( H' : G \to G \) be two graph-of-groups automorphisms, let \( v \) be a vertex of \( G \) and \( g \) an element of the vertex group \( G_v \). Assume that \( H \) and \( H' \) agree everywhere, except that \( H'_v = ad_g H_v \) and for any edge \( e \) with terminal vertex \( \tau(e) = v \) one has \( \delta_{H'}(e) = g\delta_H(e) \). Then \( H \) and \( H' \) induce the same outer automorphism:

\[
\hat{H} = \hat{H'} : \pi_1 G \to \pi_1 G
\]
We also need to consider how “honest” automorphisms, rather than outer ones, behave under base point change. This turns out to be a rather tricky issue:

**Remark 2.11.** Let \( H : \mathcal{G} \to \mathcal{G} \) be a graph-of-groups automorphism, and let \( v \) and \( v' \) be two vertices of \( \mathcal{G} \) which are fixed by \( H_\Gamma \). For some word \( U \in \Pi(\mathcal{G}) \), based on a path with initial vertex \( v \) and terminal vertex \( v' \), we consider the isomorphism

\[
\theta_U : \pi_1(\mathcal{G}, v) \to \pi_1(\mathcal{G}, v), \quad \theta_U(W) :\to UW'U^{-1} [= ad_U(W')]
\]

and define \( H_{sv',U} := ad_U^{-1}H_s(U)H_{sv'} \), in order to obtain, for a change of base point without changing the induced automorphism:

\[
H_{sv',U} = \theta_U^{-1}H_{sv}\theta_U
\]

Indeed, one has for any \( W' \in \pi_1(\mathcal{G}, v') \) the equalities

\[
H_{sv',U}(W') = ad_U^{-1}H_s(U)H_{sv'}(W') = U^{-1}H_s(U)H_{sv'}(W')H_s(U^{-1})U = H_{sv'}(H_s^{-1}(U^{-1})U)H_{sv'}(W')H_s(U^{-1})H_s^{-1}(U)) = H_s(U^{-1})UW'U^{-1}H_s(U)) = U^{-1}H_{sv}(UW'U^{-1})U = ad_U^{-1}H_sU(\theta_U(W')) = \theta_U^{-1}H_{sv}\theta_U(W')
\]

**2.3. Dehn twists and efficient Dehn twists.**

Dehn twists defined by means of graphs-of-groups seem to have appeared in the literature independently from each other at various places, with a slight degree of variation in generality and in the set-up. We follow here closely the “original sources” \([4, 5, 7]\) and \([13]\); for an interesting alternative the reader is referred to \([8]\).

**Definition 2.12.** A *Dehn twist* \( D : \mathcal{G} \to \mathcal{G} \) is a graph-of-groups automorphism such that the graph isomorphism \( D_\Gamma \) as well as the group automorphisms \( D_e \) and \( D_v \), for any \( e \in E(\mathcal{G}) \) and any \( v \in V(\mathcal{G}) \), are all equal to the identity map. In addition, for any \( e \in E(\mathcal{G}) \) the correction term \( \delta(e) \in G_{\tau(e)} \) is contained in the centralizer \( C_e \) of \( f_e(G_e) \) in \( G_{\tau(e)} \).

If one has \( C_e = f_e(G_e) \) and \( G_e \) is free, as is the case if \( \pi_1 \mathcal{G} \cong F_N \) and \( G_e \neq \{1\} \), then there is an element \( \gamma_e \) in the center \( Z(G_e) \) of \( G_e \) such that \( \delta(e) = f_e(\gamma_e) \). In this case the *twistor* \( z_e \) of \( e \) is defined by \( z_e = \gamma \gamma_e^{-1} \). This yields \( z_e = z_e^{-1} \), and for \( z_e \neq 1 \) it follows that \( G_e \cong \mathbb{Z} \).

For the rest of this subsection we will assume for simplicity that \( D : \mathcal{G} \to \mathcal{G} \) is a *classical* Dehn twist, which means that all edge groups are infinite cyclic,
so that the outer automorphism $\hat{D}$ induced by $D$ is well defined by specifying the twistor $z_e$ of every edge $e$.

Indeed, the Dehn twist $D$ determines an automorphism $D_*$ on the path group $\Pi(\mathcal{G})$ which given on the generators as follows:

$$D_*(g) = g, \text{ for } g \in G_v, \, v \in V(\mathcal{G});$$

$$D_*(t_e) = t_ef_e(z_e), \text{ for } e \in E(\mathcal{G}).$$

The induced outer automorphism $\hat{D}$ of $\pi_1(\mathcal{G})$, as well as for any $v \in V(\mathcal{G})$ the induced automorphism $D_\ast_v$ of $\pi_1(\mathcal{G}, v)$ are called a Dehn twist automorphism. An automorphism $\varphi \in \text{Out}(F_N)$ is a Dehn twist automorphism if it is induced by some Dehn twist $D : \mathcal{G} \to \mathcal{G}$ via a suitable identification $F_N \cong \pi_1 \mathcal{G}$.

**Remark 2.13.** Not every lift $\Phi \in \text{Aut}(F_N)$ of a Dehn twist automorphism $\varphi \in \text{Out}(F_N)$ is itself represented by a Dehn twist automorphism, if we stick to a given identification $F_N \cong \pi_1 \mathcal{G}$, as there might not be a suitable vertex $v$ in $\mathcal{G}$.

**Definition 2.14.** Given a Dehn twist $D : \mathcal{G} \to \mathcal{G}$, with family of twistors $(z_e)_{e \in E(\mathcal{G})}$, any two edges $e_1$ and $e_2$ with common terminal vertex $v$ are said to be

1. *positively bonded*, if $f_{e_1}(z_{e_1}^{n_1})$ and $f_{e_2}(z_{e_2}^{n_2})$ are conjugate in $G_v$ for some $n_1, n_2 \geq 1$.
2. *negatively bonded*, if $f_{e_1}(z_{e_1}^{n_1})$ and $f_{e_2}(z_{e_2}^{n_2})$ are conjugate in $G_v$ for some $n_1 \geq 1$ and $n_2 \leq -1$.

For the rest of this subsection, we always assume that $\mathcal{G}$ is a graph-of-groups such that its fundamental group $\pi_1(\mathcal{G})$ is free and of rank $N \geq 2$.

**Definition 2.15.** A Dehn twist $D : \mathcal{G} \to \mathcal{G}$ is said to be efficient if the following conditions are satisfied:

1. **$\mathcal{G}$ is minimal:** if $v = \tau(e)$ is a valence-one vertex, then the edge homomorphism $f_e : G_e \to G_v$ is not surjective.
2. **There is no invisible vertex:** there is no valence-two vertex $v = \tau(e_1) = \tau(e_2)$ ($e_1 \neq e_2$) such that both edge maps $f_{e_i} : G_{e_i} \to G_v$ ($i = 1, 2$) are surjective.
3. **No proper power:** if $r^p \in f_e(G_e)$ ($p \neq 0$) then $r \in f_e(G_e)$, for all $e \in E(\Gamma)$.
4. **If $v = \tau(e_1) = \tau(e_2)$, then $e_1$ and $e_2$ are not positively bonded.**
5. **No unused edge:** for every $e \in E(\Gamma)$ the twistor satisfies $z_e \neq 1_{G_e}$ (or equivalently $\gamma_e \neq \gamma(\tau)$).

It has been shown in [5] that every Dehn twist can be transformed algorithmically into an efficient Dehn twist. Thus every Dehn twist automorphism can be represented by some efficient Dehn twist.

Efficient Dehn twists are useful because of the following uniqueness result:
Theorem 2.16 ([5], Theorem 1.1). Two efficient Dehn twists \( D : \mathcal{G} \to \mathcal{G} \) and \( D' : \mathcal{G}' \to \mathcal{G}' \) define outer automorphisms that are conjugate to each other if and only if there is a graph-of-groups isomorphism \( H : \mathcal{G} \to \mathcal{G}' \) with 
\[ \hat{D}' = \hat{H} \hat{D} \hat{H}^{-1}. \]

3. \( H \)-conjugation and 2-level graph-of-groups

Remark 3.1. Some of the material presented in this section seems to be in close proximity to work done by M. Rodenhausen in his thesis [10], in particular to his sections 4.4, 4.5, and 4.6.

3.1. \( H \)-conjugacy.

The following definition, applied to graph-of-groups automorphisms, turns out to play a crucial role in our context:

Definition 3.2. Let \( G \) be a group and \( \Phi : G \to G \) be an automorphism of \( G \). Then two elements \( g, g' \in G \) are \( \Phi \)-conjugate to each other, written \( g \simeq_\Phi g' \), if there exists \( h \in G \) such that 
\[ g' = h^{-1} g \Phi(h). \]

The set of all elements \( \Phi \)-conjugate to \( g \) will be denoted by \([g]_\Phi\) (where it easy to verify that \( \simeq_\Phi \) is an equivalence relation and hence \([g]_\Phi\) a coset of the latter).

An element \( g \in G \) is called \( \Phi \)-trivial if it is \( \Phi \)-conjugate to the neutral element \( 1 \in G \).

It follows directly from this definition that \( g \simeq_\Phi h \) if and only if \( g^{-1} \simeq_{\Phi^{-1}} h^{-1} \). However, note that \( \Phi \)-conjugacy and \( \Phi^{-1} \)-conjugacy do in general disagree.

We will now specialize to the case where \( G \) and \( \Phi \) are given in graph-of-groups language:

Let \( \mathcal{G} \) be a graph-of-groups, let \( v \) be a vertex of \( \mathcal{G} \), and let \( H : \mathcal{G} \to \mathcal{G} \) be a graph-of-groups automorphism, which throughout most of this subsection will act trivially on the underlying graph \( \Gamma(\mathcal{G}) \). Then \( H \) induces an automorphism \( H_{sv} : \pi_1(\mathcal{G}, v) \to \pi_1(\mathcal{G}, v) \), and the notions introduced in Definition 3.2 can be applied to \( H_{sv} \). However, the group \( \pi_1(\mathcal{G}, v) \) is canonically embedded in the ambient group \( \Pi(\mathcal{G}) \), and many issues, in particular those coming from a change of base point in \( \mathcal{G} \), can be much better understood there. It might thus be tempting to pass directly to the automorphism \( H_* : \Pi(\mathcal{G}) \to \Pi(\mathcal{G}) \), which after all restricts on the subgroup \( \pi_1(\mathcal{G}, v) \) to \( H_{sv} \), and to consider directly \( H_* \)-conjugacy in \( \Pi(\mathcal{G}) \). This, however, would lead to rather undesired phenomena:

Example 3.3. Let \( H : \mathcal{G} \to \mathcal{G} \) be a graph-of-groups isomorphism which acts as identity on \( \Gamma(\mathcal{G}) \) and on all vertex groups. We furthermore assume that \( \mathcal{G} \) has trivial edge groups. We specify \( \mathcal{G} \) and \( H \) as follows:
(1) Let \( \Gamma(\mathcal{G}) \) be the graph which consists of a single edge \( e \) and two distinct vertices \( v_1 = \tau(e) \neq \tau(\overline{e}) \), and set \( \delta(e) = a^{-1} \neq 1 \in G_{v_1}, \delta(\overline{e}) = b^{-1} \neq 1 \in G_v \). For \( t = t_e \) this gives \( H_v(t) = b^{-1}ta \).

We compute \( a \simeq_{H_v} ta(a^{-1}t^{-1}b) = b \), thus obtaining an example of two non-trivial reduced words \( a \) and \( b \) in \( \Pi(\mathcal{G}) \) which are \( H_v \)-conjugate to each other, but their underlying loops are distinct (since trivial loops at distinct vertices are distinct).

(2) We start out with all data as in example (1), except that \( a = 1 \). We then add a second edge \( e' \) and a third vertex \( v_2 \) different from \( v \) and \( v_1 \), with \( v_2 = \tau(e') \) and \( v = \tau(\overline{e'}) \). We set \( \delta(e') = 1 \in G_{v_2} \) and \( \delta(\overline{e'}) = c^{-1} \in G_v \), so that for \( t' = t_{e'} \) one has \( H_v(t') = c^{-1}t \).

As in example (1) we compute \( 1 \simeq_{H_v} t_1(t^{-1}b) = b \) and \( 1 \simeq_{H_v} t'1(t'^{-1}c) = c \), and thus \( b \simeq_{H_v} c \). Thus, if we further specify \( G_{v_1} = G_{v_2} = \{1\} \) and \( G_v \cong F_2 = \langle b, c \rangle \), then we see that \( H_v = \text{id}_{F_2} \), but the two generators \( b \) and \( c \) are \( H_v \)-conjugate.

Alternatively one could set \( G_v \cong \mathbb{Z} = \langle b \rangle \) and \( c = b^2 \), thus getting \( b \simeq_{H_v} b^2 \) for \( H_v = \text{id}_{\mathbb{Z}} \).

The reason for the “misbehavior” in the above Example 3.3 (2) comes from the fact that \( H_v \)-conjugacy classes do not inject into \( H_v \)-conjugacy classes (under the subgroup embedding): the \( H_v \)-trivial class has more than one preimage. We remedied this by the following definition, where we note that in the definition of a “reduced word” \( W \) (see Definition 2.4) the case \( W = 1 \in \Pi(\mathcal{G}) \) is excluded.

**Definition 3.4.** (1) Let \( W^*(\mathcal{G}) \) be defined as \( W(\mathcal{G}) \), except that, whenever the trivial element \( 1 \in \bigvee_{v \in V(\Gamma)} G_v \) appears in a reduced word \( W \) (possibly invisible, since suppressed when it occurs between two subsequent letters from \( \{ t_e \mid e \in E(\Gamma) \} \) then it has to be specified to which of the free factors \( G_v \) it belongs. We do this notationally by writing \( 1_v := 1_{G_v} \).

(2) Furthermore, if a word \( W \in W^*(\mathcal{G}) \) is *connected*, then we require in addition that for any syllable \( t_e1_v t_{e'} \) of \( W \) one has \( v = \tau(e) = \tau(\overline{e'}) \). In this case, however, since \( v \) is uniquely determined by \( e \) and \( e' \), one is again allowed to suppress \( 1_v \) and write again \( t_e t_{e'} \) for \( t_e1_v t_{e'} \).

(3) Accordingly, we say that a connected word \( W \in \Pi^*(\mathcal{G}) \) is a *reduced* if either \( W \in \Pi(\mathcal{G}) \setminus \{1\} \) and \( W \) is reduced, or else if \( W = 1_v \) for some vertex \( v \) of \( \mathcal{G} \). We carry over the terminology from Definition 2.4 (1) and (2) in the obvious way, where for \( W = 1_v \) the *terminal vertex* \( \tau(W) \) is given by \( v \), and \( W \) is *issued at* \( v \).

We’d like to give the reader a hint why the slightly bizarre definition of \( \Pi^*(\mathcal{G}) \) makes sense: What one really considers here are pairs \( (W, v) \), where \( W \) is an element of \( \Pi(\mathcal{G}) \) with underlying connected path \( \gamma \), given as reduced connected word, and \( v \) is its terminal vertex in \( \mathcal{G} \). However, since for \( |W| \geq 1 \) the underlying non-trivial path \( \gamma \) determines \( v = \tau(\gamma) \), writing \((W, v)\) for
QUADRATIC GROWTH AUTOMORPHISMS OF $F_N$

$W$ would just be uselessly cumbersome. The same reasoning extends to $|W| = 0$ as long as $W \in G_v \setminus \{1\} \subset \Pi(\mathcal{G})$. But for any pair $(1, v)$ the second coordinate becomes meaningful, and thus the reader is free to simply interpret $1_v$ as an abbreviation for the pair $(1, v)$.

**Remark 3.5.** It is easy to see that with the above definitions the $H_\ast$-action on $\Pi(\mathcal{G})$ extends directly to a well defined $H_\ast$-action on $\Pi^\ast(\mathcal{G})$, through setting $H_\ast(1_v) := 1_{H_\ast(v)}$.

We can now go on to define:

**Definition 3.6.** Let $H : \mathcal{G} \to \mathcal{G}$ be a graph-of-groups automorphism which acts trivially on $\Gamma(\mathcal{G})$. Let $W, W' \in \Pi^\ast(\mathcal{G})$ be two closed connected words, issuing from vertices $v$ and $v'$ respectively. Then $W$ and $W'$ are $H$-conjugate, written $W \simeq_H W'$, if there exists a connected word $U$ with initial vertex $v$ and terminal vertex $v'$ such that one has:

$$W' = U^{-1}WH_\ast(U)$$

**Remark 3.7.** (1) It is easy to verify that $H$-conjugation is indeed an equivalence relation, where we denote the equivalence class of $W$ by $\left[ W \right]^H$.

(2) It follows directly, for closed connected words $W$ and $W'$ issuing from the same vertex $v$, that $W$ and $W'$ are $H$-conjugate if and only if they are $H_\ast(v)$-conjugate.

(3) Furthermore, for any two vertices $v$ and $v'$ and any “connecting word” $U$ as in Definition 3.6 if we use conjugation by $U$ in $\Pi(\mathcal{G})$ to identify $\pi_1(\mathcal{G}, v)$ and $\pi_1(\mathcal{G}, v')$, then $H_\ast(v)$-conjugacy and $H_\ast(v, U)$-conjugacy coincide (for $H_\ast(v, U)$ as defined in Remark 2.11).

(4) However, it is important to note that in the situation considered in (3) above, the notion of being “$H_\ast(v)$-trivial” and “$H_\ast(v)$-trivial” do not coincide: In general, $H$-conjugation of an element $W \simeq_{H_\ast} 1$ in $\pi_1(\mathcal{G}, v)$ will give an element $W' = U^{-1}WH_\ast(U) \in \pi_1(\mathcal{G}, v')$ which is not $H_\ast(v)$-trivial.

The notion of $H$-conjugacy enables us to perform $H$-reduction on a closed connected word $W \in \Pi(\mathcal{G})$, by $H$-conjugating it to a word $W'$ with $\left| W' \right| \mathcal{G} < \left| W \right| \mathcal{G}$. We say that $W$ is $H$-reduced if such a shortening of the length through $H$-conjugation is not possible.

**Remark 3.8 ([13], Remark 4.8).** (1) Let $W, W' \in \Pi^\ast(\mathcal{G})$ be two $H$-reduced closed connected words, with underlying closed paths $\gamma$ and $\gamma'$ respectively. If $W$ is $H$-conjugate to $W'$, and either $\gamma$ or $\gamma'$ is non-trivial, then it can be shown that $\gamma$ and $\gamma'$ must agree up to a cyclic permutation.

However, if both $\gamma$ and $\gamma'$ are trivial, this conclusion may fail, as shown in Example 3.3 (1).

(2) We say that a closed connected word $W \in \Pi^\ast(\mathcal{G})$ is $H$-zero if it is $H$-conjugate to some word $W'$ of $\mathcal{G}$-length $\left| W' \right| \mathcal{G} = 0$.

In other words, after $H$-reducing $W$ we obtain a word $W'$ which is based on a trivial loop. In general, however, the information on which vertex this
trivial loop starts and finishes depends on \(W'\) (and thus on \(W\)) and can not be changed without changing the \(H\)-conjugacy class.

3.2. 2-level graph-of-groups.

In the subsequent sections of this paper we will consider the situation where a graph-of-groups \(G\) is given by defining, for each vertex \(V\) of \(G\), the vertex group \(G_V\) through a (possibly trivial) local graph-of-groups \(G_V\):

\[G_V \cong \pi_1 G_V\]

To avoid confusion, we denote in this context the non-local vertices and edges by capital letters. For simplicity we restrict our attention to the only case which matters for this paper, namely where all edge groups \(G_E\) of \(G\) are trivial:

\[G_E = \{1\}\]

We now want to define a graph-of-groups automorphism \(H\) of \(G\), and we’d like to do this while minimizing the amount of technical data given though making sure that the outer automorphism \(\hat{H}\) is well defined. Again for simplicity, we assume that the underlying graph automorphism \(H_\Gamma\) equals to the identity map \(id_\Gamma(G)\).

In order to define such \(H : G \to G\), we need the following data:

1. For every vertex \(V\) of \(G\) we consider a local graph-of-groups automorphism \(H_V : G_V \to G_V\), for which we also assume \((H_V)_\Gamma = id_\Gamma(G_V)\).
2. For every edge \(E\) of \(G\) with terminal vertex \(V = \tau(E)\) let \(\delta^*(E) \in \Pi^*(G_V)\) be a closed connected word.
3. Furthermore, for every vertex \(V\) of \(G\) we choose a local base point \(v_V\) in \(G_V\), and for every edge \(E\) with \(\tau(E) = V\) a connecting word \(U_E \in \Pi^*(G_V)\), with underlying path \(\gamma_E\) which connects \(\tau(\delta^*(E))\) to \(v_V\).

We now specify the vertex groups \(G_V = \pi_1(G_V, v_V)\) and the correction terms \(\delta(E) = (H_V)_*(U_E^{-1})\delta^*(E)U_E\) and obtain thus a well defined graph-of-groups automorphism \(H : G \to G\) with underlying graph automorphism \(H_\Gamma = id_\Gamma(G)\), and with vertex group automorphisms given by \(H_V = (H_V)_* v_V\) for each vertex \(V\) of \(G\). The equality \(2.11\) from Definition \(2.6\) is automatically satisfied, since all edge groups \(G_E\) are trivial.

**Proposition 3.9.** The outer automorphism \(\tilde{H}\) on \(\pi_1 G\) induced by \(H\) depends only on the above data (1) and (2), but not on (3).

More precisely, if \(H' : G' \to G'\) is a second graph-of-groups automorphisms which agrees with \(H\) in (1) and (2), then there is a graph-of-groups isomorphism \(H_0 : G \to G'\) such that one has:

\[\tilde{H}' = \tilde{H}_0^{-1} \tilde{H} \tilde{H}_0\]

**Proof.** For any vertex \(V\) of \(\Gamma = \Gamma(G) = \Gamma(G')\) let \(v_V\) and \(v'_V\) be the local base points in \(G_V\) for \(G\) and \(G'\) respectively, given by condition (3) above. Similarly, for any \(E\) of \(\Gamma\) we denote the connecting words from (3) by \(U_E\)
and $U'_E$. The issuing correction terms for $H$ and $H'$ are given by $\delta(E) = (\mathcal{H}_V)_s(U^{-1}_E)\delta^*(E)U_E$ and $\delta'(E) = (\mathcal{H}_V)'_s(U'^{-1}_E)\delta^*(E)U'_E$ respectively. The conjugating graph-of-groups isomorphism $\tilde{H}_0$ is now constructed in three steps:

(1) First, choose for every vertex $V$ a word $U_V \in \Pi^*(\mathcal{G}_V)$ with underlying path that connects $v_V$ to $v'_V$, and consider the isomorphisms

$$\theta_{U_V} : \pi_1(\mathcal{G}_V, v_V) \to \pi_1(\mathcal{G}_V, v'_V), \quad W \mapsto U_V W U_V^{-1}$$

and

$$(\mathcal{H}_V)_{s v'_V U_V} = \theta_{U_V}^{-1}(\mathcal{H}_V)_{sv} \theta_{U_V}$$

from Remark 2.11. One then defines a graph-of-groups isomorphism $F : \mathcal{G} \to \mathcal{G}'$ with $F_V = \theta_{U_V}$ for every vertex $V$ and $\delta_F^*(E) = 1$ for every edge $E$ of $\mathcal{G}$, and obtains $H'' = FHF^{-1} : \mathcal{G}' \to \mathcal{G}'$ with $H''_V := (\mathcal{H}_V)_{sv'_V U_V}$ and (using the formulae from Remark 2.10 of [13])

$$\delta_{H''}(E) = \theta_{U_V}^{-1}(\delta(E)) = U_V^{-1}(\mathcal{H}_V)_s(U^{-1}_E)\delta^*(E)U_E U_V$$

for every edge $E$ with terminal vertex $V$.

(2) We now use Lemma 2.10 to define a graph-of-groups automorphism $H''' : \mathcal{G}' \to \mathcal{G}'$ with $\tilde{H}''' = \tilde{H}''$, such that for every vertex $V$ of $\mathcal{G}'$ one has $H'''_V = ad_{U_V^{-1}(\mathcal{H}_V)_{sv}}^{-1}(\mathcal{H}_V)'_s$ and for every edge $E$ of $\mathcal{G}'$ one has:

$$\delta_{H'''}(E) = (U_V^{-1}(\mathcal{H}_V)_s(U_V))^{-1}\delta_{H''}(E)$$

$$= [(\mathcal{H}_V)_s(U_V)^{-1} U_V][U_V^{-1}(\mathcal{H}_V)_s(U^{-1}_E)\delta^*(E)U_E U_V]$$

$$= (\mathcal{H}_V)_s(U_V)^{-1}(\mathcal{H}_V)_s(U^{-1}_E)\delta^*(E)U_E U_V$$

$$= (\mathcal{H}_V)_s(U^{-1}_V U^{-1}_E)\delta^*(E)U_E U_V$$

(3) Finally, one applies Lemma 2.9 to obtain a graph-of-groups automorphism $R' : \mathcal{G}' \to \mathcal{G}'$ which is the identity everywhere and conjugates $H'''$ to $H''' = R' H''' R'^{-1}$, such that the correction term $\delta_{H'''}(E)$ of any edge $E$ can take on an arbitrary value within its $(H''')^{-1}$-equivalence class. In particular one can choose $R'$ to obtain

$$\delta_{H'''}(E) = H'''_V((U_V^{-1} U^{-1}_E U'^{-1}_E)^{-1} \cdot \delta_{H''}(E) \cdot (U_V^{-1} U^{-1}_E U'^{-1}_E)$$

$$= (\mathcal{H}_V)'_s(U_V^{-1} U'^{-1}_E U')^{-1}[(\mathcal{H}_V)_s(U_V^{-1} U^{-1}_E)\delta^*(E)U_E U_V]U^{-1}_E U'^{-1}_E$$

$$= (\mathcal{H}_V)_{s v'_V U_V}^{-1} \delta^*(E) U'_E = \delta'(E)$$

and hence $H''' = H'$, which proves our claim. \hfill \Box

In the above proof we have actually shown something slightly stronger:

**Corollary 3.10.** The outer automorphism $\tilde{H}$, for $H$ as in Proposition 3.9, depends only on the $\mathcal{H}_V^{-1}$-conjugacy classes of the elements $\delta^*(E)$ and not on the representatives $\delta^*(E)$ themselves. \hfill \Box
The following definition turns out to be very useful:

**Definition 3.11.** We say that an edge $E$ of $G$, assumed as before to have trivial edge group $G_E = \{1\}$ and terminal vertex $V$, is locally zero if the correction term $\delta(E) \in G_V \cong \pi_1 G_V$ is given by an element $\delta^*(E) \in \Pi^*(G_V)$ which is $\mathcal{H}_V^{-1}$-zero (i.e. $\mathcal{H}_V^{-1}$-conjugate to an element of $G_V$-length 0, see Remark 3.8 (2)).

Let us finish this subsection by pointing out that it is fairly easy to give examples of automorphisms $\varphi, \varphi' \in \text{Out}(F_N)$ which are defined by graph-of-groups isomorphisms $H$ and $H'$ as in Proposition 3.9 respectively, where all edges are locally zero, with correction terms given by elements that are indeed “locally trivial” (i.e. any $\delta^*(E)$ is $\mathcal{H}_V^{-1}$-conjugate to the trivial element), but $\varphi$ and $\varphi'$ are not conjugate in $\text{Out}(F_N)$, since for some edge $E$ the element $\delta^*(E)$ is $\mathcal{H}_V^{-1}$-conjugate to $1_v$, while $\delta'^*(E)$ is $\mathcal{H}'_V^{-1}$-conjugate to $1_{v'}$ with $v \neq v'$.

### 4. 2-level Dehn twists

From Definition 2.12 we see that any graph-of-groups automorphism $D : \mathcal{G} \rightarrow \mathcal{G}$, which is given through the identity map on the underlying graph $\Gamma(\mathcal{G})$, on any vertex group $G_V$ and on any edge group $G_e$, is a Dehn twist. Inspired by this observation, in [13] partial Dehn twists $H : \mathcal{G} \rightarrow \mathcal{G}$ have been defined, which differ from $D$ as before in that for some vertices of $\mathcal{G}$ the vertex group automorphisms are different from the identity map. A special case, already considered in [13], is given by the following:

**Definition 4.1.** A 2-level Dehn twist is given by a graph-of-groups $\mathcal{G}$ with trivial edge groups, and a graph-of-groups isomorphism $H : \mathcal{G} \rightarrow \mathcal{G}$ which induces the identity on the underlying graph $\Gamma(\mathcal{G})$ and which induces on every (possibly trivial) vertex group a Dehn twist automorphism or an inner automorphism.

To be more specific regarding the vertex group automorphisms, we want to use now the concept of a 2-level graph-of-groups as introduced in section 3.2 to describe a 2-level Dehn twist. As done there, we will denote vertices and edges of $\mathcal{G}$ by capital letters, while for all local graph-of-groups we use small letters. We first note a slightly technical point, which arises when $H$ as in Definition 4.1 is specified to an automorphism of a 2-level graph-of-groups as in section 3.2.

**Remark 4.2.** In Definition 4.1 at any vertex $V$ of $\mathcal{G}$ the local Dehn twist automorphism at $V$ is formally defined as element of $\text{Aut}(G_V)$ and not of $\text{Out}(G_V)$. From Remark 2.13 it may appear that this is a serious restriction, once the identification $G_V \cong \pi_1 G_V$ is fixed. However, it follows from Lemma 2.10 that this is not true, since one can adapt the correction terms of any edge $E$ with terminal vertex $V$ accordingly.
As has been recalled in section 2.3 every Dehn twist automorphism of a free group \( F_N \) can be represented by an efficient Dehn twist \( \tau : \Gamma \to \Gamma \), and the latter is unique (on the level of the induced outer automorphisms) up to conjugation with graph-of-groups isomorphisms, see Theorem 2.16. We will hence assume below always that any 2-level Dehn twist, conjugation with graph-of-groups isomorphisms, see Theorem 2.16. We will hence assume below always that any 2-level Dehn twist \( H : \Gamma \to \hat{\Gamma} \) comes for every vertex group \( G_V \) of \( \Gamma \) with an efficient Dehn twist \( \tau_V : \Gamma \to \Gamma \) and an identification \( G_V \cong \pi_1 \hat{\Gamma} \) such that \( \tau_V \) induces the outer automorphism defined by the vertex group automorphism \( H_V : G_V \to G_V \).

**Remark 4.3.** (1) In the above set-up, the suppression of a local base point \( v_V \in V(\Gamma_V) \) in the identification \( G_V \cong \pi_1 \Gamma_V \) (rather than specifying \( G_V \cong \pi_1 \Gamma_V, v_V \)), and naming only the condition \( \tau_V = \hat{\tau}_V \) (rather than specifying it to \( \tau_V = H_{v_V} \)), is not notational sloppiness, but rather has been done purposefully.

Indeed, we recall from Proposition 3.9 that for any edge \( E \) of \( \hat{\Gamma} \), say with terminal vertex \( V = \tau(E) \), in order to determine the outer automorphism \( \hat{\tau} \) induced by \( \tau \), it suffices to specify a word \( \delta^*(E) \in \Pi^* \Gamma(\hat{\Gamma}) \). Indeed, the \( D^{-1} \)-equivalence class of \( \delta^*(E) \) is sufficient, see Corollary 3.10 so that we will notationally not distinguish between the word \( \delta^*(E) \) and the element \( \delta(E) \) used in Proposition 3.9 to specify \( \tau \) and simply refer to either as the “correction term” of \( E \). Whenever the only case occurs where the difference between \( \Pi^* \Gamma(\hat{\Gamma}) \) and \( \Pi(\hat{\Gamma}) \) matters, i.e. if \( \delta(E) = 1 \), we will be careful and specify 1 to 1 for the appropriate vertex \( v \) of \( \Gamma_V \).

(2) We also recall (see Definition 3.11) that for the edge \( E \) the notion of being “locally zero” is well defined, without having specified the base point \( v_V \) for the graph-of-groups \( \Gamma_V \).

In the following definition the fact that \( \pi_1 \hat{\Gamma} \) is a free group is only assumed by practical reasons for this paper; the notion of efficient 2-level Dehn twists makes also sense for more general groups.

**Definition 4.4.** A 2-level Dehn twist \( H : \Gamma \to \hat{\Gamma} \) is called efficient if \( \pi_1 \hat{\Gamma} \) is free of finite rank \( N \geq 2 \) and if the following conditions are satisfied:

1. For any edge \( E \) of \( \hat{\Gamma} \) precisely one of the two, \( E \) or \( \overline{E} \), is locally zero (i.e. precisely one of the two correction terms, \( \delta(E) \in G_{\tau(E)} \) or \( \delta(E) \in G_{\tau(E)} \), is \( H^{-1} \)-conjugate to an element of \( \Gamma_V \)-length 0, for \( V = \tau(E) \) or \( V = \tau(\overline{E}) \) respectively).

   We say that \( E \) is forward oriented if \( E \) is not locally zero, and we define \( E^+(\Gamma(\hat{\Gamma})) = E^+(\hat{\Gamma}) \) to be the orientation on \( \Gamma(\hat{\Gamma}) \) which contains all such forward oriented edges.

2. For any two distinct forward oriented edges \( E \) and \( E' \) of \( \hat{\Gamma} \) with common terminal vertex \( V := \tau(E) = \tau(E') \), the correction terms \( \delta(E) \) and \( \delta(E') \) are not \( D^{-1} \)-conjugate.

Given any 2-level Dehn twist \( H_0 : \Gamma_0 \to \Gamma_0 \), we can iteratively transform \( \Gamma_0 \) and \( H_0 \) through intermediate 2-level Dehn twists \( H_1 : \Gamma_1 \to \Gamma_1 \), \( H_2 : \Gamma_2 \to \Gamma_2 \), \( \cdots \), \( H_n : \Gamma_n \to \Gamma_n \), \( n \geq 2 \), in order to determine their efficient counterparts, \( H'_1, \ldots, H'_n \), respectively, and make both of the edges \( E_j \) and \( E'_j \) locally zero, for \( j = 1, \ldots, n \), which leads to a sequence of \( \pi_1 \hat{\Gamma}_j \) free of finite rank, \( \bigcap_{j=1}^n \pi_1 \hat{\Gamma}_j \cong \pi_1 \Gamma(N) \) from Corollary 3.10.
$G_2 \to G_2$, etc, with canonical isomorphisms $\pi_1 G_j \cong \pi_1 G_{j+1}$ that induce $\bar{H}_j = \bar{H}_{j+1}$, to obtain after finitely many steps a 2-level Dehn twist $H_m : G_m \to G_m$ which is efficient.

The modifications, employed in this procedure to pass from $H_j : G_j \to G_j$ to $H_{j+1} : G_{j+1} \to G_{j+1}$, are all of one of the following four types:

1. Subdivide an edge $E$ by introducing a new vertex with trivial vertex group. Choose the correction terms on the subdivided edges to be trivial at the new vertex, and to coincide with $\delta(E)$ or $\delta(E')$ otherwise.

   This subdivision is in particular always done if both $E$ and $E'$ are not locally zero.

2. Contract an edge $E$, if both $E$ and $E'$ are locally zero, through a blow-up of the local graph-of-groups automorphisms $D_\tau(E)$ and $D_\tau(E')$ along $E$ as introduced in [13]. Subsequently make the resulting local Dehn twist on the new blown-up vertex group again efficient.

3. For any edge $E$ with with terminal vertex $V$ and correction term $\delta(E)$ which is not $D_V^{-1}$-zero, if there is any second edge $E' \neq E$, also with terminal vertex $V$, such that $\delta(E)$ is $D_V^{-1}$-conjugate to $\delta(E')$, one performs $D_V^{-1}$-conjugation on $\delta(E)$ to obtain $\delta(E) = \delta(E')$.

4. If two distinct edges $E$ and $E'$ with common terminal vertex $V$ have equal correction term $\delta(E) = \delta(E')$ which is not $D_V^{-1}$-zero, we first perform a subdivision of $E$ and $E'$ as in modification (1) above, so that we can now assume that $\delta(E) = \delta(E') = 1$ and $V_1 := \tau(E) \neq V_2 := \tau(E')$.

   We then fold $E$ onto $E'$ and identify $V_1$ and $V_2$ to get a new vertex $V'$ with trivial vertex group.

Any such modification does not increase the number of edges $E$ which are not locally zero. However, in the process of doing our modifications, one eventually decreases their number (through modifications (3) and (4)) until any two edges $E$ and $E'$ with common terminal vertex $V$ have correction terms in distinct $D_V^{-1}$-conjugacy classes. We then finish the procedure by applying iteratively the modification (2) finitely many times. This shows:

**Proposition 4.5.** For every 2-level Dehn twist with fundamental group $F_N$ there exists an efficient 2-level Dehn twist which defines the same outer automorphism of $F_N$.

In [9] we will give more details which will show that the procedure described here is in fact algorithmic.

**Remark 4.6.** Moritz Rodenhausen has obtained in his thesis [10] results that seem to come very close to what has been presented in this section, in a context that is more general (polynomial growth automorphisms of arbitrary degree), and in a language that is not far from ours, but in its technical
details sufficiently different to make a formal translation non-evident. In any case, it seems more than likely that the algorithmic device in his section 8 is strongly related to the above explained procedure, and in particular his Theorem 8.6 to our Proposition 4.5.

5. Cancellation bounds

5.1. Some basic cancellation facts on graph-of-groups.

Recall that for any graph-of-groups $\mathcal{G}$ and any reduced word

$$W = w_0t_1w_1 \ldots w_{r-1}t_rw_r \in \Pi(\mathcal{G})$$

we denote by $|W| := r$ the $\mathcal{G}$-length of $W$.

For any two reduced words $V, W \in \Pi(\mathcal{G})$ we say that $V$ cancels against $W$ in the product $VW$ if one has $|VW| = 0$.

Furthermore, we say that the cancellation in a family of products $W_1(t) \cdot W_2(t) \cdot \ldots \cdot W_m(t)$ is bounded, for reduced words $W_1(t), W_2(t), \ldots, W_m(t) \in \Pi(\mathcal{G})$, if there exists a constant $K \geq 0$ independent of the parameter $t$ such that:

$$|W_1(t)| + |W_2(t)| + \ldots + |W_m(t)| - |W_1(t)W_2(t)\ldots W_m(t)| \leq K$$

holds for any value of $t$.

Remark 5.1. For any graph-of-groups isomorphism $H : \mathcal{G} \to \mathcal{G'}$ the above notions are preserved under $H$. For example, if $V$ cancels against $W$ in the product $VW$, then $H(V)$ cancels against $H(W)$ in the product $H(VW)$.

Definition 5.2. For any graph-of-groups $\mathcal{G}$ we say that edges $e$ and $e'$ are bonded if they terminate at the same vertex $v = \tau(e) = \tau(e')$, and if there exist non-trivial elements $g_e \in G_e$ and $g_{e'} \in G_{e'}$ with images $f_e(g_e)$ and $f_{e'}(g_{e'})$ that are conjugate in the vertex group $G_v$.

To be specific, we call an element $h \in G_v$ with $f_e(g_e) = h f_{e'}(g_{e'}) h^{-1}$ a bond conjugator, while $g_e$ and $g_{e'}$ are called the edge bonders.

A path $e_1e_2\ldots e_r$ in $\mathcal{G}$ is a bonded path if for subsequent indices the edges $e_i$ and $e_{i+1}$ are bonded, with a family of edge bonders that satisfy $g_{e_i} = g_{e_{i+1}}$ for any $i = 2, \ldots, r - 1$.

A bonded loop is analogously defined, with index $i$ understood cyclically modulo $r$.

Any trivial path or trivial loop is formally defined to be bonded.

Lemma 5.3. Let $\mathcal{G}$ be a graph-of-groups, and let $V, W \in \Pi(\mathcal{G})$ be reduced words with common underlying path which is not bonded, and which terminates in some vertex $v$. Assume that $V$ cancels against $W^{-1}$ in the product $VW^{-1}$. Then for no $u \in G_v \setminus \{1\}$ the element $Vu$ cancels against $W^{-1}$ in the product $VuW^{-1}$.

Proof. Let $V = w_0t_1w_1 \ldots w_{r-1}t_rw_r$ be the given reduced word, based on a path $e_1 \ldots e_r$. From the assumption that $V$ cancels against $W^{-1}$ in the product $VW^{-1}$ we know that $W = u'V = u'w_0t_1w_1 \ldots w_{r-1}t_rw_r$ for some
u' \in G_\tau(\pi_1)$. Hence, if $Vu$ cancels against $W^{-1}$ in the product $VuW^{-1}$, then from the normal form for reduced words in $\Pi(G)$ (see Remark 2.3) we know that for $k = 1, \ldots, r$ there exist elements $u_k \in G_e = G_\tau$, such that $f_{e_k}(u_k) = w_k f_{r_{k+1}}(u_{k+1}) w_k^{-1}$ for all $k = 1, \ldots, r-1$, and $f_{e_r}(u_r) = u$. From the assumption $u \neq 1$ it follows hence that $u_k \neq 1$ for all indices $k$. But then the path $e_1 \ldots e_r$ is bonded, which contradicts our assumption. \qed

**Lemma 5.4.** Let $D: G \to G$ be an efficient Dehn twist, with $\pi_1 G = F_N$ and $N \geq 2$. Then no non-trivial loop in $G$ is bonded.

*Proof.* This is a direct consequence of Definition 5.2 and the observation that a non-trivial bonded loop leads directly to commuting elements in $\pi_1 G$ which are not powers of each other, as one is based on the loop, and the other one is represented by the edge bonder, which can be “shifted” around the loop.

However, this is impossible as by assumption $\pi_1 G$ is a free group $F_N$ if rank $N \geq 2$. \qed

**Lemma 5.5.** Let $D: G \to G$ be an efficient Dehn twist, with $\pi_1 G = F_N$ and $N \geq 2$, and let $V, W \in \Pi(G)$ be reduced words based on non-trivial loops which are proper powers. If $V$ cancels against $W^{-1}$ in $VW^{-1}$, then for no $t \neq 0$ the element $V$ cancels against $D^t(W^{-1})$ in $VD^t(W^{-1})$.

*Proof.* As in the above proof of Lemma 5.4 we have reduced words $V = w_0 t_1 w_1 \ldots w_{r-1} t_r w_r$ and $W = u'V = u'w_0 t_1 w_1 \ldots w_{r-1} t_r w_r$ for some $u' \in G_\tau(\pi_1)$, both based on a path $e_1 \ldots e_r$ which is assumed to be closed. We compute $D^t(W) = u'w_0 t_1 f_1(z_1)^t w_1 \ldots w_{r-1} t_r f_r(z_r)^t w_r$, where we use the convention $t_k := t_{e_k}$ and $z_k := z_{e_k}$.

Thus, if $V$ cancels against $D^t(W^{-1})$ in $VD^t(W^{-1})$, then one obtains iteratively for all $k = 1, \ldots, r-1$ that $w_k f_{r_{k+1}}(z_{k+1})^t w_k^{-1}$ is equal to $f_k(z_k)^m k$, for some $m_k \in t\mathbb{Z}$, since by the definition of an efficient Dehn twist both, the generators of the cyclic subgroups $f_{e_k}(G_{e_k})$ and $w_k f_{r_{k+1}}(G_{e_{k+1}}) w_k^{-1}$ are not proper powers in their ambient vertex group $G_\tau(e_k)$. Furthermore we see iteratively that all $m_k$ must have the same sign as $t$, as otherwise $e_k$ and $e_{k+1}$ were positively bonded, contradicting the definition of an efficient Dehn twist. In particular the exponents can never add up to 0.

Thus any subsequent edges in the path $e_1 \ldots e_r$ are bonded. But by the “proper powers” assumption on $e_1 \ldots e_r$ in the statement of our proposition, this path must run more than once around some non-trivial loop, which hence must also be bonded. But this contradict Lemma 5.4. \qed

### 5.2. Application to iterated $D$-products.

Recall from section 2.2 that for any group automorphism $F: G \to G$ and any integer $t \geq 0$ we use the following notation:

$$F^{(t)}(g) := gF(g)F^2(g) \ldots F^{t-1}(g)$$

and

$$F^{(s,t)}(g) := F^s(g)F^{s+1}(g) \ldots F^{t-1}(g)$$
for any exponents $s < t$.

**Proposition 5.6.** Let $D : G \to G$ be an efficient Dehn twist, and let $W_1, W_2 \in \Pi(G)$ be reduced words based on non-trivial loops.

Assume that for some $U \in \Pi(G)$ and $t \geq 0$ the cancellation (with respect to $G$-length) in the family of products

$$D_s(t)(W_1) \cdot D_s(U) \cdot (D_s(t)(W_2))^{-1}$$

is unbounded. Then $W_1$ and $W_2$ are $D$-conjugate to each other. More precisely, one has:

$$U^{-1}W_1D_s(U) = W_2$$

**Proof.** From the claimed statement we observe that one can assume without loss of generality that $W_1$ and $W_2$ are $D$-reduced, and that $U$ (and hence any $D_s(t)(U)$) is reduced. Thus the unboundedness hypothesis of the cancellation in (5.1) implies that there is a reduced product decomposition $U = U_1U_2$ such that $D_s(t)(U_1)$ cancels completely against the end of $D_s(t)(W_1)$, and $D_s(U_2^{-1})$ against the end of $D_s(t)(W_2)$. Hence, through possibly replacing $W_1$ by $U_1W_1D_s(U_1^{-1})$ and $W_2$ by $U_2^{-1}W_2D_s(U_2)$, we can furthermore assume the $U = 1$. By doing these replacements iteratively, where at each step the replacement $U_1'W_1D_s(U_1'^{-1})$ is taken such that $U_1'$ is an initial subword of $U_1$ of $G$-length $|U_1'| \leq |U_1|$ (and analogously for a terminal subword $U_2'$ of $U_2'$) we see that these replacements will preserve the property that both, $W_1$ and $W_2$ are $D$-reduced.

By checking the $G$-lengths $m = |W_1|$ and $n = |W_2|$, unbounded cancellation in the products

$$P(t) := D_s(t)(W_1) \cdot (D_s(t)(W_2))^{-1}$$

implies that for some sufficiently large $t$ the suffix $D_s(t-n,t)(W_1)$ of $D_s(t)(W_1)$ cancels against $D_s(t-m,t)(W_2)^{-1}$, and subsequently $D_s(t-2n,t-n)(W_1)$ against $D_s(t-2m,t-m)(W_2)^{-1}$, and so on. However, since $D_s(t-n,t)(W_1) = D_s^n(D_s(t-2n,t-n)(W_1))$ and $D_s(t-m,t)(W_2) = D_s^m(D_s(t-2m,t-m)(W_2))$, by Lemma 5.5 this implies $m = n$, or equivalently, $|W_1| = |W_2|$.

This implies that $D_s(t-1)(W_1)$ cancels against $D_s(t-1)(W_2)^{-1}$, and $D_s(t-2)(W_1)u$ cancels against $D_s(t-2)(W_2)^{-1}$, for $u := D_s(t-1)(W_1)(D_s(t-1)(W_2)^{-1})$ of $G$-length $|u| = 0$. Since $D_s(t-1)(W_1) = D((D_s(t-2)(W_1))$ and $D_s(t-1)(W_2) = D((D_s(t-2)(W_2))$, it follows (see Remark 5.3) that $D_s(t-2)(W_1)$ cancels against $D_s(t-2)(W_2)^{-1}$, so that Lemma 5.3 (applicable by Lemma 5.4) implies $u = 1$. Hence we obtain indeed $D_s(t-1)(W_1) = D_s(t-1)(W_2)$ and thus $W_1 = W_2$.  

**5.3. Limit lengths.**

For any basis $A = \{a_1, \ldots, a_N\}$ of $F_N$ and any $w \in F_N$ we denote by $|w|_A$ the length of the reduced word in $A^{\pm 1} = \{a_1, a_1^{-1}, \ldots, a_N, a_N^{-1}\}$ representing $w$, and by $\|w\|_A$ the length of any cyclically reduced word representing the conjugacy class $[w]$. 

\[ QUADRATIC GROWTH AUTOMORPHISMS OF F_N \]
Below one considers, for any graph-of-groups $\mathcal{G}$ and any vertex $v$ of $\mathcal{G}$, an identification $\pi_1(\mathcal{G}, v) = F_N$, and we choose any basis $A$ of $F_N$. For any edge $e$ of $\mathcal{G}$, any element $z \in G_e$ and any connected word $W \in \Pi(\mathcal{G})$ starting at $v$ and terminating at $\tau(e)$, we will write for simplicity $\|z\|_A$ instead of $\|Wf_e(z)W^{-1}\|_A$ or $\|Wt_e^{-1}f_e(z)t_eW^{-1}\|_A$, which would be formally correct, but clearly the conjugacy class of $Wf_e(z)W^{-1} = Wt_e^{-1}f_e(z)t_eW^{-1}$ depends only on $z$.

The following length estimate has been shown in $[14]$. A slightly weaker but much related result is given by Proposition 6.19 of $[10]$.

**Proposition 5.7.** Let $D: \mathcal{G} \to \mathcal{G}$ be an efficient Dehn twist with twistors $z_e$ for any edge $e$ of $\mathcal{G}$, and let $v$ be a vertex of $\mathcal{G}$. For the identification $\pi_1(\mathcal{G}, v) \cong F_N$ we denote by $D \in \text{Aut}(F_N)$ the automorphism induced by $D$. Let $W = w_0t_1w_1 \ldots w_{r-1}t_rw_r$ be a $D$-reduced word in $\Pi(\mathcal{G})$ based on a non-trivial closed connected path $\gamma = e_1e_2 \ldots e_r$, and let $V \in \Pi(\mathcal{G})$ and $g \in F_N$ be such that $V^{-1}WD(V)$ is an element in $\pi_1(\mathcal{G}, v)$ which represents $g$. Then one has, for any basis $A$ of $F_N$:

$$\lim_{t \to \infty} \frac{|D(t)(g)|_A}{t^2} = \frac{1}{2} \sum_{i=1}^r \|z_{e_i}\|_A.$$

**Remark 5.8.** (1) In the present version of $[14]$ the equality from Proposition 5.7 is not stated precisely as given here. However, in the proof of Proposition 6.5 of $[14]$ all arguments are given, except that in the last paragraph the estimation for the upper bound has to be taken slightly more sharply.

(2) Also, the cancellation arguments from the proof of Proposition 6.5 in $[14]$ apply as well to a basis $A$ of any larger free group $F_N$ which contains $\pi_1(\mathcal{G}, v)$ as free factor.

**Remark 5.9.** If in Proposition 5.7 the path $\gamma$ is trivial, then $W = w_0$ has length $|W| = 0$ and belongs to one of the vertex groups of $\mathcal{G}$, which are element-wise fixed by the induced automorphism $D_a$. Hence $D(t)(W) = W^t$, so that $|D(t)(g)|_A$ grows at most linearly in $t$. On the other hand, for such $\gamma$ the sum $\sum_{i=1}^r \|z_{e_i}\|_A$ is taken over the empty set and hence equal to 0, so that the conclusion of Proposition 5.7 holds in this case as well.

We now use the crucial Proposition 5.7 and combine it with the cancellation results from the previous subsection, to obtain:

**Proposition 5.10.** Let $D: \mathcal{G} \to \mathcal{G}$ be an efficient Dehn twist with twistors $(z_e)_{e \in E(\mathcal{G})}$, and let $v$ be a vertex of $\mathcal{G}$. For the identification $\pi_1(\mathcal{G}, v) \cong F_N$ let $D \in \text{Aut}(F_N)$ the automorphism induced by $D$. Let $A$ be any basis of a free group $F_M$ which contains $F_N$ as free factor.

Let $U, V$ and $W$ be reduced words in $\pi_1(\mathcal{G}, v) \subset \Pi(\mathcal{G})$, and let $e_1e_2 \ldots e_r$ and $e_1'e_2' \ldots e_r'$ be the (possibly trivial) loops on which $D$-reduced words are based that are $D$-conjugate to $V$ and $W$ respectively. If these two loops are non-trivial, we assume furthermore that $U^{-1}VD_w(U) \neq W$. 


Then one has, for \( g_1, g_2, h \in F_N \) representing \( V, W \) and \( U \) respectively:

\[
\lim_{t \to \infty} \frac{|D^{(t)}(g_1)D^{(t)}(h)(D^{(t)}(g_2))^{-1}|_A}{t^2} = \frac{1}{2} \sum_{i=1}^{r} \|z_{e_i}\|_A + \sum_{i=1}^{r'} \|z_{e'_i}\|_A
\]

**Proof.** From the assumed inequality \( U^{-1}VD_\ast(U) \neq W \) and from Proposition 5.6 we deduce that the cancellation in the products \( D^{(t)}(V) \cdot D^{(t)}(U) \cdot (D^{(t)}(W))^{-1} \), and hence in the products \( D^{(t)}(g_1) \cdot D^{(t)}(h) \cdot (D^{(t)}(g_2))^{-1} \), is bounded independently of \( t \). Furthermore, \( D^{(t)}(h) \) grows at most linearity in \( t \). Hence, both, the possible cancellation in the products as well as the factor \( D^{(t)}(h) \) can be neglected when taking the limit quotient modulo \( t^2 \). Thus the desired equality follows directly from Proposition 5.7 and Remark 5.9. \( \square \)

**Remark 5.11.** The set-up given in the first paragraph of Proposition 5.10 will be encountered frequently in the next section, as local graph-of-groups isomorphism. We hence want to formalize slightly some of the above:

1. Let \( g \in F_N \cong \pi_1(\mathcal{G}, v) \), and let \( V = w_0t_1w_1 \ldots w_{r-1}t_rw_r \) be a connected, closed and \( D \)-reduced word in \( \Pi(\mathcal{G}) \) based on a (possibly trivial) path \( \gamma = e_1e_2 \ldots e_r \), and let \( U \in \Pi(\mathcal{G}) \) be such that \( W := U^{-1}VD(U) \) is an element in \( \pi_1(\mathcal{G}, v) \) which represents \( g \). We observe that the limit quotient for \( g \) considered in Proposition 5.7 does not depend on \( W \), but only on \( V \), or more precisely, on the \( D \)-conjugacy class of \( W \).

2. We thus define for any closed connected word \( W \in \Pi(\mathcal{G}) \) the \( D \)-length of the \( D \)-conjugacy class \( [W]_D \) by

\[
\|[W]_D\|_A := \frac{1}{2} \sum_{i=1}^{r} \|z_{e_i}\|_A
\]

where \( e_1e_2 \ldots e_r \) is the path in \( \Gamma(\mathcal{G}) \) underlying any \( D \)-reduced word \( V \) which is \( D \)-conjugate to \( W \).

3. Since the \( D \)-conjugacy class of \( W \) consists of all inverses of the \( D^{-1} \)-conjugacy class of \( W^{-1} \) (see Definition 5.2 and the subsequent paragraph), and since the twistors for \( D^{-1} \) are given by the inverses \( z_e^{-1} \) of the twistors \( z_e \) of \( D \), which satisfy furthermore \( \|z_e^{-1}\| = \|z_e\| \), one has:

\[
\|[W^{-1}]_{D^{-1}}\|_A = \|[W]_D\|_A
\]

6. **Parabolic dynamics on Outer space**

Let \( H : \mathcal{G} \to \mathcal{G} \) be an efficient 2-level Dehn twist as in section 4 and let \( W = w_0t_1w_1 \ldots w_{q-1}t_qw_q \) be a reduced word in \( \Pi(\mathcal{G}) \). Then from Remark 2.7 (2) we know that for any \( t \in \mathbb{Z} \) one has

\[
H^t_k(W) = w_0(t)t_1w_1(t) \ldots w_{q-1}(t)t_qw_q(t),
\]

with

\[
w_k(t) = H^t_k(\delta^{-1}_k)H^t_k(w_k)H^t_k(\delta^{-1}_k)\delta^{-1}_k \]

(6.1)
for any $k = 0, \ldots, q$. Here we formally define $\delta_0 = \delta_{q+1} = 1$, and otherwise adopt the convention that $\delta_k$ denotes the correction term of the edge $E_k$ corresponding to the stable letter $t_k$, while $\bar{\delta}_k$ denotes the correction term of $\bar{E}_k$. Furthermore, $H_k$ denotes the vertex group isomorphism $H_{\tau(E_k)}$. We also define

\begin{equation}
(6.2) \quad w_{q,0}(t) := w_q(t)w_0(t) = H_q^{(t)}(\delta_q^{-1})H_q^i(w_qw_0)H_q^{(t)}(\bar{\delta}_1^{-1})^{-1}
\end{equation}

**Proposition 6.1.** Let $H : G \to \mathcal{G}$ be an efficient 2-level Dehn twist which induces on $\pi_1 G \cong F_N$ the automorphism $\varphi \in \text{Out}(F_N)$. For any $g \in F_N$ let $W = w_0t_1w_1 \ldots w_{q-1}t_qw_q$ be a cyclically reduced word in $\Pi(\mathcal{G})$ which represents the conjugacy class $[g] \subset F_N$. Then, for any basis $A$ of $F_N$, we have

\[
\lim_{t \to \infty} \frac{\|\varphi^t([g])\|_A}{t^2} = \sum_{i=1}^q \|\delta_i\|_{D_i} \|A\,
\]

where for any stable letter $t_i$ in $W$ we denote by $\delta_i$ the correction term $\delta(E_i)$ or $\delta(\bar{E}_i)$ for the edge $E_i$ associated to $t_i$, according to which of the two, $E_i$ or $\bar{E}_i$, is not locally zero. By $D_i$ we denote here the efficient Dehn twist which represents $H_{\tau(E_i)}$ or $H_{\tau(\bar{E}_i)}$ respectively.

**Proof.** We consider the cyclically reduced word $W$ which represents the conjugacy class $[g]$ and its images $H^t(W)$ representing $\varphi^t([g])$, for increasing exponents $t \geq 0$ (or, similarly, for decreasing exponents $t \leq 0$). Since the vertex groups of $\mathcal{G}$ are free factors of $\pi_1 G$ and of $\Pi(\mathcal{G})$, any cancellation between adjacent elements of distinct vertex groups, or between distinct conjugates of the same vertex group, are a priori bounded, for any chosen basis $A$ of $F_N$. Hence we can consider separately the growth of each "syllable" $t_kw_kt_{k+1}$ in $W$ under iteration of $H$, which has been described above in (6.1) by the elements $w_k(t)$ (while the stable letters stay constant and can hence be ignored when passing to the limit quotient by $t^2$). Since $W$ is furthermore assumed to be cyclically reduced, we can also similarly consider the "cyclic syllable" $t_qw_qw_0t_0$ and its $H$-iterations given through $w_{q,0}(t)$ in (6.2), to obtain:

\[
\lim_{t \to \infty} \frac{\|\varphi^t([g])\|_A}{t^2} = \left( \sum_{k=1}^{q-1} \lim_{t \to \infty} \frac{|w_k(t)|_A}{t^2} \right) + \lim_{t \to \infty} \frac{|w_{q,0}(t)|_A}{t^2}
\]

For each of the $w_k(t)$ we can apply Proposition 5.10, which gives as limit quotient $\|\delta(E_k)^{-1}\|_{D_{\tau(E_k)}} \|A\| + \|\delta(\bar{E}_{k+1})^{-1}\|_{D_{\tau(\bar{E}_{k+1})}} \|A\|$, which by Remark 5.11 (3) is equal to $\|\delta(E_k)\|_{D_{\tau(E_k)}} \|A\| + \|\delta(\bar{E}_{k+1})\|_{D_{\tau(\bar{E}_{k+1})}} \|A\|$. Similarly, for $w_{q,0}(t)$ we obtain the limit quotient $\|\delta(E_q)\|_{D_{\tau(E_q)}} \|A\| + \|\delta(\bar{E}_1)\|_{D_{\tau(\bar{E}_1)}} \|A\|$. We note here that the hypothesis $U^{-1}V D_{\tau}(U) \neq W$ from Proposition 5.10 is satisfied since for $E_k \neq \bar{E}_{k+1}$ the correction terms $\delta(E_k)$ and $\delta(\bar{E}_{k+1})$ are not $D_{\tau(E_k)}$-conjugate, by the definition of an efficient 2-level twist (see Definition 4.4). For $E_k = \bar{E}_{k+1}$ we use the hypothesis that $W$ is reduced. The
analogous arguments apply to $E_q$ and $\overline{E}_1$, since $W$ is furthermore cyclically reduced. We thus have:

$$\lim_{t \to \infty} \frac{\| \varphi^t([g]) \|_A}{t^2} = \left( \sum_{k=1}^{q-1} \| [\delta(E_k)]_{D_{\tau}(E_k)} \|_A + \| [\delta(\overline{E}_{k+1})]_{D_{\tau}(E_k)} \|_A \right)$$

$$+ \| [\delta(E_q)]_{D_{\tau}(E_q)} \|_A + \| [\delta(\overline{E}_1)]_{D_{\tau}(E_q)} \|_A$$

$$= \sum_{i=1}^q \| [\delta(E_i)]_{D_{\tau}(E_i)} \|_A + \| [\delta(\overline{E}_i)]_{D_{\tau}(E_i)} \|_A$$

Since from the definition of an efficient 2-level Dehn twist we know that for each edge $E$ of $\mathcal{G}$ precisely one of $E$ or $\overline{E}$ is locally zero (and hence $\delta(E)$ or $\delta(\overline{E})$ has zero $D_V$-length, for $V = \tau(E)$ or $V = \tau(\overline{E})$ respectively), the last sum gives precisely the desired result. \qed

Remark 6.2. For any length function $\| \cdot \|$ on the conjugacy classes of $F_N$ which is induced by a length function that is quasi-isometric to the one given by any basis $A$ of $F_N$, the equation

$$\lim_{t \to \infty} \frac{\| \varphi^t([g]) \|}{t^2} = \sum_{i=1}^q \| \delta_i \|_{D_i}$$

analogous to the result in Proposition 6.1 stays valid, since the linear quasi-isometry constants disappear in the limit when considering the quotient by $t^2$, and the previously used cancellation arguments apply to $\| \cdot \|$ as well.

Remark 6.2 applies in particular to translation length functions $\| \cdot \|_\Gamma$ on a metric tree $\overline{\Gamma}$ given as universal covering of a metric graph $\Gamma$ equipped with a marking isomorphism $\theta : \pi_1 \Gamma \to F_N$. Such length functions define, after passing to the projective class $[\Gamma]$, a point in Outer space $CV_N$. For background on Outer space $CV_N$ and its “Thurston compactification” $\overline{CV}_N = CV_N \cup \partial CV_N$ we refer the reader to [12].

On the other hand, any graph-of-groups $\mathcal{G}$ with marking isomorphism $\pi_1 \mathcal{G} \cong F_N$ defines a simplex $\Delta_{\mathcal{G}}$ in the boundary $\partial CV_N$ of $CV_N$ (or in $CV_N$, if all vertex groups of $\mathcal{G}$ are trivial), where a point in $\Delta_{\mathcal{G}}$ is given by defining an edge length $L(E) \geq 0$ for any edge $E$ of $\mathcal{G}$, with $L(E) = L(\overline{E})$. Thus, for $E^+(\mathcal{G})$ as given in Definition 4.1 (2), the Bass-Serre tree $T_{\mathcal{G},(T(E))_{E \in E^+(\mathcal{G})}}$ associated to the point $[\mathcal{G},(L(e))_{E \in E^+(\mathcal{G})}] \in \Delta_{\mathcal{G}}$ becomes a metric simplicial tree, equipped canonically with an action of $F_N$ by isometries. We obtain:

**Theorem 6.3.** Let $[\Gamma]$ be any point in Outer space $CV_N$, given by a marked metric graph $\Gamma$. Then for any automorphism $\varphi \in Out(F_N)$, represented by an efficient 2-level Dehn twist $D : \mathcal{G} \to \mathcal{G}$, the $\varphi$-orbit of $[\Gamma]$ is parabolic, with limit point contained in the interior of the simplex $\Delta_{\mathcal{G}} \subset \partial CV_N$. More precisely, one has:

$$\lim_{t \to \pm \infty} \varphi^t([\Gamma]) = [\mathcal{G},([\delta(E)]_{D_{\tau}(E)} \|_\Gamma)_{E \in E^+(\mathcal{G})}]$$
Proof. This is a direct consequence of Proposition [6.1] where for the “interior point” claim we observe that for any point \([\Gamma]\) in \(CV_N\) any non-trivial conjugacy class has non-zero \(\tilde{\Gamma}\)-length, so that for any edge \(E\) which is not locally zero one has \(\|\delta(E)\|_{D_\tau(E)}\|_{\tilde{\Gamma}} > 0\).

\(\Box\)

Remark 6.4. The “interior points” statement in Theorem 6.3 is useful, since if two marked graph-of-groups \(G\) and \(G'\) give rise to distinct simplexes \(\Delta_G \neq \Delta_G'\) in \(CV_N\), then \(\Delta_G\) and \(\Delta_G'\) can not intersect in points that are interior in both \(\Delta_G\) and \(\Delta_G'\).

This follows from the fact that for any interior point \([G, (L(e))_{E \in E^+(G)}] \in \Delta_G\) all edges \(e\) have length \(L(e) > 0\), so that the associated tree \(T_{(G, (L(E))_{E \in E^+(G)})}\), after forgetting the metric, is \(F_N\)-equivariantly homeomorphic to the Bass-Serre tree \(T_G\) associated to \(G\).

7. NORMAL FROM FOR QUADRATICALLY GROWING AUTOMORPHISMS

From the geometric result in the previous section we can now derive that efficient 2-level Dehn twists constitute indeed a normal from for the induced outer automorphisms:

**Theorem 7.1.** Two efficient 2-level Dehn twists \(H : G \rightarrow G\) and \(H' : G' \rightarrow G'\) represent outer automorphisms \(\tilde{H}\) and \(\tilde{H}'\) of a free group \(F_N\) which are conjugate in \(\text{Out}(F_N)\) if and only if there exists a graph-of-groups isomorphism \(F : G \rightarrow G'\) which satisfies:

\[\tilde{H} = \tilde{F}^{-1} \tilde{H}' \tilde{F}\]

*Proof.* The “if” direction is obvious. To show the “only if” direction we note that any conjugating automorphism \(\psi \in \text{Out}(F_N)\), with \(\tilde{H} = \psi^{-1} \tilde{H}' \psi\), must map any \(\tilde{H}\)-orbit of the \(\text{Out}(F_N)\)-action on \(CV_N\) to an \(\tilde{H}'\)-orbit, and hence the limit point of the former to the limit point of the latter. It follows from Remark 6.4 that \(\psi\) maps the limit simplex \(\Delta_G\) from Theorem 6.3 to the analogous simplex \(\Delta_{G'}\).

In particular, the automorphism \(\psi\) maps the center point of \(\Delta_G\), defined by setting all edge lengths equal to 1, to the analogously defined center point of \(\Delta_{G'}\). Hence \(\psi\) conjugates the \(F_N\)-action on the non-metric Bass-Serre tree \(T_G\) to that on the analogous tree \(T_{G'}\), thus inducing a graph-of-groups isomorphism \(F : G \rightarrow G'\) which satisfies \(\tilde{F} = \psi\). This last conclusion is a standard fact for graph-of-groups, see for instance Lemma 4.5 of [5] or Proposition 4.4 of [1].

\(\Box\)

Remark 7.2. The natural group action of \(\text{Out}(F_N)\) on Outer space \(CV_N\) is a right action, but of course it can be canonically transformed into a left action by setting \(\varphi \cdot [\Gamma] := [\Gamma] \cdot \varphi^{-1}\). The watchful reader will notice that in either case, in the above setting, the conjugating automorphism \(\psi\) maps \(\tilde{D}\)-orbits to a \(\tilde{D}'\)-orbits.
In [15] iterated Dehn twists of level \( k \geq 1 \) have been introduced as certain iteratively defined graph-of-groups automorphisms \( H : \mathcal{G} \to \mathcal{G} \). For \( k = 1 \) one obtains ordinary Dehn twists, and for \( k = 2 \) this notion agrees with what is called here “2-level Dehn twists”. It has been shown in Proposition 1.1 of [15] that every polynomially growing automorphism \( \varphi \in \text{Out}(F_N) \) has a positive power which is represented by an iterated Dehn twist of some level \( k \geq 1 \).

Furthermore, it can be shown that any iterated Dehn twist \( H : \mathcal{G} \to \mathcal{G} \) of level \( k \geq 2 \) represents an automorphism \( \varphi \) which either can be represented by an iterated Dehn twist of level \( k - 1 \), or else there is a conjugacy class in \( F_N \) which has polynomial growth of degree precisely equal to \( k \), under iteration of \( \varphi \). This has been shown in [14] for \( k = 2 \); a proof for \( k \geq 3 \) is obtained by analogous arguments, except that it becomes easier since the graph-of-groups in question for \( k \geq 3 \) have trivial edge groups (compare also [10], Proposition 4.21). Hence every quadratically growing automorphism \( \varphi \in \text{Out}(F_N) \) has a positive power which is represented by an efficient 2-level Dehn twist \( H : \mathcal{G} \to \mathcal{G} \).

The following extension of Theorem 7.1 gives hence a normal form for any quadratically growing automorphism of \( F_N \).

**Corollary 7.3.** (1) Every automorphism \( \varphi \in \text{Out}(F_N) \) with exponent \( m \geq 1 \) such that \( \varphi^m \) is represented by an efficient 2-level Dehn twist \( H : \mathcal{G} \to \mathcal{G} \) can be represented by a graph-of-groups automorphism \( R : \mathcal{G} \to \mathcal{G} \).

(2) Two graph-of-groups automorphism \( R : \mathcal{G} \to \mathcal{G} \) and \( R' : \mathcal{G}' \to \mathcal{G}' \) as in part (1) represent outer automorphisms \( \hat{R} \) and \( \hat{R}' \) of a free group \( F_N \) which are conjugate in \( \text{Out}(F_N) \) if and only if there exists a graph-of-groups isomorphism \( F : \mathcal{G} \to \mathcal{G}' \) which satisfies:

\[
\hat{R} = \hat{F}^{-1} \hat{R}' \hat{F}
\]

**Proof.** (1) From the hypothesis \( \varphi^m = \hat{H} \) we see that \( \varphi \) permutes the \( \hat{H} \)-orbits in \( CV_N \), and hence also their limit points. But since by Theorem 6.3 all of the latter are contained in the interior of the simplex \( \Delta_G \), it follows from Remark 6.4 that \( \varphi \) must map \( \Delta_G \) to itself, and hence fix its center point. Hence the claim follows by the exactly the same argument as given in the second paragraph of the proof of Theorem 7.1.

(2) As in the proof of Theorem 7.1 and of part (1) above, any automorphism \( \psi \in \text{Out}(F_N) \) with \( \hat{R} = \psi^{-1} \hat{R}' \psi \) must map \( H \)-orbits in \( CV_N \) to \( H' \)-orbits, where \( H \) and \( H' \) are positive powers of \( R \) and \( R' \) respectively which are efficient 2-level Dehn twists. As in the proof of Theorem 7.1 we deduce the existence of the desired graph-of-groups isomorphism \( F \), and hence the “only if” part of the claim. The “if” part is again obvious. \( \square \)

**Remark 7.4.** It is well known (see for instance section 2 of [13]) that the relationship between graphs-of-groups and their isomorphisms on one hand, and simplicial trees with equivariant homeomorphisms on the other, though
conceptually very appealing, is on a technical level not quite as smooth as one would wish. The “fault” lies entirely on the graph-of-groups side, which is technically loaded with data that are non-uniquely determined by the corresponding Bass-Serre trees.

For example, if $T$ is a simplicial tree with $G$-action, and $\mathcal{G}$ and $\mathcal{G}'$ two graph of groups with $\pi_1 \mathcal{G} = \pi_1 \mathcal{G}' = G$ and $G$-equivariant identifications $T = T_\mathcal{G} = T_{\mathcal{G}'}$, then there is indeed a graph-of-groups isomorphism $F : \mathcal{G} \to \mathcal{G}'$ which induces on the fundamental group the above identification.

Furthermore, if $\tilde{H} : T \to T$ a homeomorphism which commutes with an automorphism $\Phi : G \to G$ (in the sense that for any $g \in G$ one has $\tilde{H} g = \Phi(g) \tilde{H} : T \to T$), then $\tilde{H}$ descends to graph-of-groups automorphisms $H : \mathcal{G} \to \mathcal{G}$ and $H' : \mathcal{G}' \to \mathcal{G}'$ which satisfy:

$$\tilde{H} = \tilde{F}^{-1} \tilde{H}' \tilde{F}$$

However, whether $F$ (or $F$, $H$ and $H'$) can be chosen so that one obtains actually

$$H = F^{-1} H' F,$$

remains a question to which even under natural additional assumptions (like minimality of $T$) an answer seems in general not to be known.

References

[1] H. Bass, *Covering theory for graphs of groups*. Journal of Pure and Applied Algebra, Vol. 89 (1993), no. 12, pp. 3-47

[2] M. Bestvina, M. Feighn, and M. Handel, *The Tits alternative for Out($F_n$). II. A Kolchin type theorem*. Ann. of Math. (2) 161 (2005), no. 1, 1–59

[3] M. Bestvina, and M. Handel, *Train tracks and automorphisms of free groups*. Ann. of Math. (2) 135 (1992), no. 1, 1–51

[4] M. M. Cohen and M. Lustig, *Very small group actions on $R$-trees and Dehn twist automorphisms*. Topology, Vol. 34 (1995), no. 3, pp. 575-617

[5] M. M. Cohen and M. Lustig, *The conjugacy problem for Dehn twist automorphisms of free groups*. Commentarii Mathematici Helvetici, Vol. 74 (1999), no. 2, pp. 179-200

[6] M. Culler, K. Vogtmann, *Moduli of graphs and automorphisms of free groups*. Invent. Math. 84 (1986), no. 1, 91–119

[7] S. Krstic, M. Lustig and K. Vogtmann, *An equivariant Whitehead algorithm and conjugacy for roots of Dehn twist automorphisms*. Proceedings of the Edinburgh Mathematical Society, (2) Vol. 44 (2001), no. 1, pp. 117-141

[8] G. Levitt, *Automorphisms of hyperbolic groups and graphs of groups*, Geom. Dedic. 114 (2005), 49–70

[9] M. Lustig and K. Ye, *Algorithmic results for quadratically growing automorphisms of $F_X$*, in preparation.

[10] M. Rodenhausen, *Centralizers of polynomially growing automorphisms of free groups*. PhD thesis, Bonn 2013

[11] J-P. Serre, *Trees*. Springer, 1980.

[12] K. Vogtmann, *Automorphisms of Free Groups and Outer Space*. Geometriae Dedicata 94 (2002), 1–31

[13] K. Ye, *Quotient and blow-up of automorphisms of graph of groups*. arXiv: 1512.00542.
[14] K. Ye, *Partial Dehn twists of free groups relative to local Dehn twists - A dichotomy.*
    arXiv: 1605.04479
[15] K. Ye, *When is a polynomially growing automorphism of $F_n$ geometric?*
    arXiv: 1605.07390

Aix Marseille Université, CNRS, Centrale Marseille, I2M UMR 7373, 13453 Marseille, France

E-mail address: Martin.Lustig@univ-amu.fr
E-mail address: deloresye@gmail.com