Topological Entropy for Arbitrary Subsets of Infinite Product Spaces

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Abstract

In this note a notion of generalized topological entropy for arbitrary subsets of the space of all sequences in a compact topological space is introduced. It is shown that for a continuous map on a compact space the generalized topological entropy of the set of all orbits of the map coincides with the classical topological entropy of the map. Some basic properties of this new notion of entropy are considered; among them are: the behavior of the entropy with respect to disjoint union, cartesian product, component restriction and dilation, shift mapping, and some continuity properties with respect to Vietoris topology. As an example, it is shown that any self-similar structure of a fractal given by a finite family of contractions gives rise to a notion of intrinsic topological entropy for subsets of the fractal. A generalized notion of Bowen’s entropy associated to any increasing sequence of compatible semimetrics on a topological space is introduced and some of its basic properties are considered. As a special case for $1 \leq p \leq \infty$ the Bowen $p$-entropy of sets of sequences of any metric space is introduced. It is shown that the notions of generalized topological entropy and Bowen $\infty$-entropy for compact metric spaces coincide.

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1 Introduction

The notion of topological entropy for continuous maps on compact spaces have been introduced by Adler, Konheim, and McAndrew in [1] as a topological version of the measure theoretic entropy in Ergodic Theory defined by Kolmogorov and Sinai [7]. Then Bowen [2] defined a notion of entropy for uniformly continuous maps on arbitrary metric spaces and showed [3] that his notion coincides with the topological entropy on compact metric spaces. Also, Bowen’s definition has been extended to uniformly continuous maps on uniform spaces by Hood [5]. For more details on the subject and various notions of entropy we refer the reader to [4].

The main aim of this note is to introduce a new generalized notion of topological entropy. Indeed we show that for any compact topological space $X$ there is a well-behaved

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and intrinsic notion of entropy for arbitrary subsets of the infinite product space \( X^\infty \). It will be clear that for any continuous map \( T : X \to X \) if \( \bar{T} \) denotes the subset of \( X^\infty \) of all orbits of \( T \) then our entropy value \( H(\bar{T}) \) coincides with the classical topological entropy of \( T \). This suggests that for an arbitrary noncontinuous map or set-valued map \( T \) we consider the value \( H(\bar{T}) \) as topological entropy of \( T \).

In Section 2 we consider the main definition of this note. In Section 3 we consider some basic properties of our notion of entropy. Among them are: the behavior of the entropy with respect to disjoint union, cartesian product, component restriction and dilation, shift mapping, and some continuity properties with respect to Victoris topology. We show that for any compact Hausdorff space (finite or infinite) any real number \( u \) with \( 0 \leq u \leq \log |X| \) appears as the entropy of a subset of \( X^\infty \). We also consider some examples and show that any self-similar structure of a fractal given by a finite family of contractions gives rise to a notion of intrinsic topological entropy for subsets of the fractal. In Section 4 we consider a Bowen’s notion of entropy for arbitrary subsets of any topological space endowed with an increasing sequence of compatible semimetrics and we consider some elementary properties of this new notion. In Section 5 as an special case of the notion given in Section 4 we introduce a notion of Bowen \( p \)-entropy \( (1 \leq p \leq \infty) \) for subsets of \( X^\infty \) where \( X \) is an arbitrary metric space. It will be clear that the Bowen \( \infty \)-entropy of \( \bar{T} \) for a continuous map \( T : X \to X \) coincides with the usual Bowen entropy of \( T \). We also show that Bowen \( \infty \)-entropy of closed subsets of \( X^\infty \) coincides with the generalized topological entropy given in Section 2 when \( X \) is a compact metric space.

**Notations.** For a topological space \( X \) we denote by \( X^\infty \) the space of all sequences \((x_n)_{n \geq 0}\) of elements of \( X \) endowed with the product topology.

### 2 The Main Definition

Let us begin by some conventions. For open covers \( U, V \) of a topological space \( X \) we write \( U \preceq V \) if every member of \( V \) is a subset of a member of \( U \). \( \bigvee_{i=1}^n U_i \) denotes the join of the finite family \( U_1, \ldots, U_n \) of open covers of \( X \) i.e. \( \bigvee_{i=1}^n U_i := \{ \bigcap_{i=1}^n U_i : U_i \in \mathcal{U} \} \). If \( \mathcal{V} \) is an open cover of a space \( X_i \) for \( i = 1, \ldots, n \) then the associated product cover is defined to be the open cover of \( \prod_i X_i \) given by \( \{ \prod V_i : V_i \in \mathcal{V}_i \} \).

Let \( S \) be a nonempty subset of a space \( X \). For an open cover \( \mathcal{U} = \{ U_i \}_{i \in I} \) of \( X \) we denote by \( N_{S/X}(\mathcal{U}) \) the minimum of the cardinals \( |J| \) where \( J \subseteq I \) is such that \( S \subseteq \bigcup_{i \in J} U_i \). We denote the value \( N_{X/X}(\mathcal{U}) \) simply by \( N(\mathcal{U}) \).

Topological entropy of continuous mappings \( [8] \) is defined as follows: Let \( T : X \to X \) be a continuous mapping. For any \( n \geq 0 \) denote by \( T^{-n}\mathcal{U} \) the open cover of \( X \) given by \( \{ T^{-i}(U_i) \}_{i \in I} \). Let

\[
    h(T, \mathcal{U}) := \limsup_{n \to \infty} \frac{1}{n} \log N(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}). \tag{1}
\]

It is well-known that the \( \limsup \) in (1) is a limit and is equal to \( \inf_n \frac{1}{n} \log N(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}) \). If \( X \) is compact then the topological entropy of \( T \) is defined to be the value

\[
    h(T) := \sup_{\mathcal{U}} h(T, \mathcal{U}) \tag{2}
\]

where the supremum is taken over all open covers \( \mathcal{U} \) of \( X \).
We generalize the classical definitions given by (1) and (2) as follows. For an open cover \( \mathcal{U} = \{U_i\}_{i \in I} \) of \( X \) and any integer \( n \geq 1 \) let \( \mathcal{U}^n \) denote the open cover of \( X \) given by

\[
\mathcal{U}^n := \left\{ U_{i_0} \times \cdots \times U_{i_{n-1}} \times X \times X \times \cdots \mid i_0, \ldots, i_{n-1} \in I \right\}.
\]

**Definition 2.1.** Let \( X \) be a compact space and \( S \) be a subset of \( X \). For any open cover \( \mathcal{U} \) of \( X \) the topological entropy of \( S \) relative to \( \mathcal{U} \) is defined to be

\[
H^t(S, \mathcal{U}) := \limsup_{n \to \infty} \frac{1}{n} \log N_{S/X}(\mathcal{U}^n).
\]

The topological entropy of \( S \) (as a subset of \( X \)) is defined by

\[
H^t_X(S) = H^t(S) := \sup_{\mathcal{U}} H^t(S, \mathcal{U})
\]

where the supremum is taken over all open covers \( \mathcal{U} \) of \( X \).

For any (not necessarily continuous) mapping \( T : X \to X \) let \( \text{Gr} T \) denote the subset of \( X \) given by \( \{(T^n x)_{n \geq 0} : x \in X\} \). Similarly if \( T \) is a set-valued mapping on \( X \) let

\[
\text{Gr} T := \left\{ (x_n)_{n \geq 0} : x_0 \in X, x_{n+1} \in T x_n \right\} \subseteq X.
\]

**Proposition 2.2.** Let \( X \) be a compact space and \( T : X \to X \) be a continuous mapping. Then for any open cover \( \mathcal{U} \) of \( X \) we have \( h(T, \mathcal{U}) = H^t(\text{Gr} T, \mathcal{U}) \). Hence

\[
h(T) = H^t(\text{Gr} T).
\]

**Proof.** It follows from the easily checked identity \( N(\bigvee_{i=0}^{n-1} T^{-i} \mathcal{U}) = N_{\text{Gr} T/X}(\mathcal{U}^n) \).

Proposition 2.2 justifies that \( H^t_X(\text{Gr} T) \) is a natural generalization of the concept of topological entropy for any non-continuous (set-valued) mapping \( T : X \to X \).

3 Basic Properties of \( H^t_X \) and some Examples

In this section, we consider some elementary properties and examples of our generalized topological entropy. Throughout this section, \( X \) and \( Y \) denote compact Hausdorff spaces.

**Theorem 3.1.** For any finite subset \( S \) of \( X \) we have \( H^t(S) = 0 \).

**Proof.** For any open cover \( \mathcal{U} \) of \( X \) and every \( n \geq 0 \) we have \( N_{S/X}(\mathcal{U}^n) \leq |S| \) and hence \( H^t(S, \mathcal{U}) = 0 \). Thus \( H^t(S) = 0 \). \( \square \)

**Lemma 3.2.** Let \( S \subseteq X \). For open covers \( \mathcal{U} \preceq \mathcal{V} \) of \( X \) we have \( H^t(S, \mathcal{U}) \leq H^t(S, \mathcal{V}) \).

**Proof.** It follows immediately from the obvious inequality \( N_{S/X}(\mathcal{U}^n) \leq N_{S/X}(\mathcal{V}^n) \). \( \square \)

**Theorem 3.3.** For \( S \subseteq T \subseteq X \) we have

\[
H^t(S) \leq H^t(T).
\]

If \( S \) is finite then \( H^t(T \setminus S) = H^t(T) \).
Proof. Let \( \mathcal{U} \) be an open cover for \( X \). We have \( N_{\mathcal{S}/X}(\overline{\mathcal{U}}^n) \leq N_{\mathcal{T}/X}(\overline{\mathcal{U}}^n) \). This implies that 
\[ H^i(S,\mathcal{U}) \leq H^i(T,\mathcal{U}). \] Thus \( H^i(S) \leq H^i(T) \). Suppose now that \( S \) is finite. We have
\[ N_{\mathcal{T}/X}(\overline{\mathcal{U}}^n) \leq |S| + N_{(T\setminus S)/X}(\overline{\mathcal{U}}^n) \quad (n \geq 0). \]
Thus \( H^i(T,\mathcal{U}) \leq H^i(T \setminus S,\mathcal{U}) \). The proof is complete. \( \square \)

For a set \( S \) in a space we denote by \( \text{cls}(S) \) its closure.

**Lemma 3.4.** Let \( \mathcal{U}, \mathcal{V} \) be open coverings of \( X \) such that for every \( V \in \mathcal{V} \) there exists \( U \in \mathcal{U} \) with \( \text{cls}(V) \subseteq U \). Then for any \( S \subseteq X \) we have
\[
N_{\text{cls}(S)/X}(\mathcal{U}) \leq N_{\mathcal{S}/X}(\mathcal{V}).
\]

**Proof.** Suppose that \( V_1, \ldots, V_k \) in \( \mathcal{V} \) be such that \( S \subseteq \cup_{i=1}^k V_i \). Then \( \text{cls}(S) \subseteq \cup_{i=1}^k \text{cls}(V_i) \). Thus if \( U_i \in \mathcal{U} \) be such that \( \text{cls}(V_i) \subseteq U_i \) then \( \text{cls}(S) \subseteq \cup_{i=1}^k U_i \). This implies that \( N_{\text{cls}(S)/X}(\mathcal{U}) \leq k \). The proof is complete. \( \square \)

**Theorem 3.5.** For every \( S \subseteq \overline{X} \) we have
\[
H^i(\text{cls}(S)) = H^i(S).
\]

**Proof.** Let \( \mathcal{U} \) be an arbitrary open cover for \( X \). Since \( X \) is compact and Hausdorff there exists an open cover \( \mathcal{V} \) of \( X \) such that for every \( V \in \mathcal{V} \) there is \( U \in \mathcal{U} \) with \( \text{cls}(V) \subseteq U \). It follows that for every \( n \geq 1 \) the closure of every member of \( \overline{\mathcal{V}}^n \) is contained in a member of \( \overline{\mathcal{U}}^n \). Thus by Lemma 3.4 we have
\[
N_{\text{cls}(S)/X}(\overline{\mathcal{U}}^n) \leq N_{\mathcal{S}/X}(\overline{\mathcal{V}}^n).
\]
This implies that \( H^i(\text{cls}(S),\mathcal{U}) \leq H^i(S,\mathcal{V}) \) and hence \( H^i(\text{cls}(S),\mathcal{U}) \leq H^i(S) \). Thus we have \( H^i(\text{cls}(S)) \leq H^i(S) \). The reverse direction follows from Theorem 3.3. \( \square \)

**Theorem 3.6.** Let \( X \) be a discrete finite space and \( z \in X \). Let \( \tilde{X}_z \) denote the set of those sequences \( (x_n)_n \) in \( \overline{X} \) such that there exists \( n_0 \) with \( x_n = z \) for every \( n \geq n_0 \). Then
\[
H^i(\tilde{X}_z) = H^i(\overline{X}) = \log |X|.
\]
In particular for every \( S \subseteq \overline{X} \), \( H^i(S) \leq \log |X| \).

**Proof.** Let \( \mathcal{U} \) denote the maximal open cover of \( X \) with respect to \( \approx \) i.e. \( \mathcal{U} = \{\{x\}\}_{x \in X} \). It follows from Lemma 3.2 that \( H^i(\overline{X}) = H^i(\overline{X},\mathcal{U}) = \lim_{n \to \infty} \frac{1}{n} \log |X|^n = \log |X| \). The same proof shows that \( H^i(\tilde{X}_z) = \log |X| \). The last part follows from Theorem 3.3. \( \square \)

**Theorem 3.7.** Let \( X \) be infinite and for \( z \in X \) let \( \tilde{X}_z \subseteq \overline{X} \) be as in Theorem 3.6. Then
\[
H^i(\tilde{X}_z) = H^i(\overline{X}) = \infty.
\]

**Proof.** Let \( k \geq 2 \) be arbitrary and let \( x_1, \ldots, x_k \) be distinct elements of \( X \). There is an open cover \( \mathcal{U} \) of \( X \) such that every member of \( \mathcal{U} \) contains at most one of the \( x_1, \ldots, x_k \). We have
\[
H^i(\tilde{X}_z) \geq H^i(\tilde{X}_z,\mathcal{U}) \geq \lim_{n \to \infty} \frac{1}{n} \log k^n = \log k.
\]
Thus \( H^i(\tilde{X}_z) = \infty \). \( \square \)
Theorem 3.8. Let $Y$ be a (closed) subspace of $X$ and $S \subseteq \overline{Y}$. Then
\[ H^1_Y(S) = H^1_X(S). \]
Proof. Let $\mathcal{U} = \{U_i\}$ be an open cover for $X$. Then $\mathcal{U} \cap Y := \{U_i \cap Y\}$ is an open cover for $Y$ and we have $N_{S/\overline{Y}}(\mathcal{U}^n) = N_{S/\overline{Y}}(\mathcal{U} \cap Y^n)$). This implies that $H^1(S, \mathcal{U}) = H^1(S, \mathcal{U} \cap Y)$ and hence $H^1_X(S) \leq H^1_Y(S)$. Let $V = \{V_i\}$ be an open cover for $Y$. Let $U_i$ denote an open subset of $X$ with $V_i = U_i \cap Y$. Then $\mathcal{U} = \{U_i\} \cup \{X \setminus Y\}$ is an open cover for $X$ and we have $N_{S/\overline{Y}}(\mathcal{U}^n) = N_{S/\overline{Y}}(\mathcal{V}^n)$. This implies that $H^1(S, \mathcal{U}) = H^1(S, \mathcal{V})$ and hence $H^1_Y(S) \geq H^1_Y(S)$. The proof is complete. □

Here is another proof of Theorem 3.7. For any $k$ let $Y = \{x_1, \ldots, x_{k-1}, z\}$ be a subset of $X$ with $k$ elements. By Theorem 3.6 $H^1_Y(\tilde{Y}_z) = \log k$, by Theorem 3.8 $H^1_X(\tilde{Y}_z) = H^1_Y(\tilde{Y}_z)$, and by Theorem 3.3 $H^1_X(\tilde{X}_z) \geq H^1_Y(\tilde{Y}_z)$. Thus $H^1_X(\tilde{X}_z) = \infty$.

We now want to show that every positive real number less than $\log |X|$ appears as the entropy of a subset of $\overline{X}$. For this aim we need the following two lemmas.

Lemma 3.9. Suppose that $X$ is finite and $z \in X$. Let $r \leq 1$ be a positive real number. Suppose that there exist two sequences $(p_k)_k, (q_k)_k$ of natural numbers satisfying the following conditions:

\( (i) \ p_k \leq q_k, \ p_k < p_{k+1}, \text{ and } q_k < q_{k+1}. \)
\( (ii) \ (q_k+1 - q_k) \geq (p_{k+1} - p_k) \text{ and } \frac{p_k}{q_k} \leq \frac{p_{k+1}}{q_{k+1}}. \)
\( (iii) \ \sup_k \frac{p_k}{q_k} = r. \)

Then there exists a subset $S_r \subseteq \overline{X}$ such that $H^1(S_r) = r \log |X|$.

Proof. Let $S_r \subseteq \overline{X}$ be defined by
\[ S_r := \{z\} \times \cdots \{z\} \times X \times \cdots \times \{z\} \times \cdots \times \{z\} \times X \times \cdots \times \{z\} \times \cdots. \]

Let $\mathcal{U}$ be as in the proof of Theorem 3.6. Then $N_{S_r/\overline{X}}(\mathcal{U}^n) = |X|^{p_k}$. This shows that
\[ H^1(S_r) = H^1(S_r, \mathcal{U}) = \sup_k \frac{p_k}{q_k} \log |X| = r \log |X|. \]

□

Lemma 3.10. Suppose that $r$ is a positive real number with $r \leq 1$. Then there exist sequences $(p_k)_k, (q_k)_k$ satisfying the conditions (i)-(iii) of Lemma 3.9

Proof. For every two natural numbers $a, b$ and any rational number $s$ satisfying $\frac{a}{b} \leq s \leq 1$ it is easily verified that there exist natural numbers $a', b'$ such that
\[ a < a', \ b < b', \ (b' - b) \geq (a' - a), \ \frac{a'}{b'} = s. \] (3)

Let $(r_k)_k$ be a sequence of positive rational numbers such that $r_k \leq r_{k+1}$ and $\sup_k r_k = r$.

We define inductively the desired sequences $(p_k)_k, (q_k)_k$ as follows. Let $p_1, q_1$ be natural
numbers with \( \frac{a_k}{q_k} = r_1 \). Suppose that we have chosen natural numbers \( p_1, \ldots, p_{\ell}, q_1, \ldots, q_{\ell} \) such that the inequalities in (i) and (ii) of Lemma 3.9 are satisfied for \( k = 1, \ldots, \ell - 1 \). Then we let \( p_{\ell + 1} := a', q_{\ell + 1} := b' \) where \( a', b' \) satisfy (3) for \( a = p_\ell, b = q_\ell \), and \( s = r_{\ell + 1} \). The proof is complete.

**Theorem 3.11.** For any compact space \( X \), every number \( u \) with \( 0 \leq u \leq \log |X| \) appears as the topological entropy of a subset of \( X \).

**Proof.** First suppose that \( X \) is finite. Thus \( \log |X| < \infty \). Let \( r := u/\log |X| \). By Lemmas 3.9 and 3.10 there exists a set \( S_r \subseteq X \) such that \( H^t(S_r) = r \log |X| = u \). For infinite \( X \) by Theorem 3.7 it is enough to suppose that \( u < \infty \). Then the desired result is concluded from the result for a finite subspace \( Y \) of \( X \) with \( u \leq \log |Y| \) and Theorem 3.8.

**Theorem 3.12.** Let \( S \subseteq X, T \subseteq Y \). Let \( X \cup Y \) denote the disjoint union space of \( X \) and \( Y \). Consider \( S \cup T \) as a subset of \( X \cup Y \) in the obvious way. Then

\[
H^t_{X \cup Y}(S \cup T) = \max\{H^t_X(S), H^t_Y(T)\}.
\]

**Proof.** Let \( U, V \) be open covers respectively for \( X, Y \). Then \( U \cup V \) is an open cover for \( X \cup Y \). Note also that every open cover of \( X \cup Y \) is of this form. We have

\[
N_{S \cup T \subseteq X \cup Y}(U \cup V) = N_{U \subseteq X}(U^\alpha) + N_{V \subseteq Y}(V^\beta).
\]

Thus we have

\[
\log N_{S \cup T \subseteq X \cup Y}(U \cup V) \leq \max \left\{ \log N_{U \subseteq X}(U^\alpha) + \log N_{V \subseteq Y}(V^\beta) \right\}.
\]

This implies that \( H^t_{X \cup Y}(S \cup T) \leq \max\{H^t_X(S), H^t_Y(T)\} \). The reverse direction follows from Theorems 3.3 and 3.8.

We identify \( X \times Y \) with \( X \times Y \) in the obvious way via the mapping

\[
((x_n)_n, (y_n)_n) \mapsto (x_n, y_n)_n.
\]

**Theorem 3.13.** Let \( S \subseteq X, T \subseteq Y \). Then

\[
H^t_{X \times Y}(S \times T) \leq H^t_X(S) + H^t_Y(T).
\]

**Proof.** Let \( U, V \) be open covers respectively for \( X, Y \) and let \( U \times V \) denote the associated product cover. We have \( N_{S \times T \subseteq X \times Y}(U \times V) = N_{U \subseteq X}(U^\alpha)N_{V \subseteq Y}(V^\beta) \). This implies that

\[
H^t(S \times T, U \times V) \leq H^t(S, U) + H^t(T, V).
\]

(4)

On the other hand if \( W \) is an open cover for \( X \times Y \) then there exist open covers \( U, V \) for \( X, Y \) such that \( W \subseteq U \times V \) and hence by Lemma 3.2, \( H^t(S \times T, W) \leq H^t(S \times T, U \times V) \). Thus the inequality (4) implies the desired result.

The generalized topological entropy is *shift invariant*:
Theorem 3.14. For $S \subseteq \overline{X}$ and any $k \geq 1$ let $\mathcal{G}^k S \subseteq \overline{X}$ be obtained by $k$-times action of the shift map on $S$:

$$\mathcal{G}^k S := \left\{ (x_n)_{n \geq 0} \mid (y_0, \ldots, y_{k-1}, x_0, x_1, \ldots) \in S \text{ for some } y_0, \ldots, y_{k-1} \in X \right\}.$$ 

Then

$$H^t_X(\mathcal{G}^k S) = H^t_X(S).$$

Proof. Let $\mathcal{U}$ be an arbitrary open cover for $X$. Without loss of generality we may suppose that $|\mathcal{U}| < \infty$. Then

$$N_{\mathcal{E}^k S/\overline{X}}(\overline{\mathcal{U}^n}) \leq N_{\overline{S/\overline{X}}}^k(\overline{\mathcal{U}^{kn}}) \leq |\mathcal{U}|^k N_{\mathcal{E}^k S/\overline{X}}(\overline{\mathcal{U}^n}).$$

This implies that $H^t(\mathcal{G}^k S, \mathcal{U}) = H^t(S, \mathcal{U})$. Hence $H^t_X(\mathcal{G}^k S) = H^t_X(S)$. \hfill \square

Theorem 3.15. Let $S \subseteq \overline{X}$. For $k \geq 2$ we identify $\overline{X}^k$ with $\overline{X}$ in a obvious way via the mapping

$$((x_1^n, \ldots, x_k^n))_{n \geq 0} \mapsto (x_0^n, \ldots, x_0^k, x_1^1, \ldots, x_1^k, x_2^1, \ldots).$$

Then

$$H^t_X(S) = kH^t_X(S).$$

Proof. Let $\mathcal{U}$ be an open cover for $X$ and consider the associated product cover $\mathcal{U}^k$ of $X^k$. For $n \geq 1$ let $\left\lceil \frac{n}{k} \right\rceil$ denote the greatest integer less than or equal to $\frac{n}{k}$. It is easily verified that for every $n \geq k$,

$$N_{\overline{S/\overline{X}}}^k(\overline{\mathcal{U}^{\left\lceil \frac{n}{k} \right\rceil}}) \leq N_{\overline{S/\overline{X}}}^k(\overline{\mathcal{U}^{\left\lceil \frac{n}{k} \right\rceil + 1}}).$$

Thus we have

$$\frac{k}{n} \log N_{\overline{S/\overline{X}}}^k(\overline{\mathcal{U}^n}) \leq \frac{1}{\left\lceil \frac{n}{k} \right\rceil} \log N_{\overline{S/\overline{X}}}^k(\overline{\mathcal{U}^{\left\lceil \frac{n}{k} \right\rceil + 1}}).$$

This implies that

$$kH^t(S, \mathcal{U}) = \limsup_{n \to \infty} \frac{k}{n} \log N_{\overline{S/\overline{X}}}^k(\overline{\mathcal{U}^n}) \leq \limsup_{\ell \to \infty} \frac{1}{\ell} \log N_{\overline{S/\overline{X}}}^k(\overline{\mathcal{U}^{\ell + 1}}) = H^t(S, \mathcal{U}^k).$$

Similarly it is proved that $H^t(S, \mathcal{U}^k) \leq kH^t(S, \mathcal{U})$. Thus we have

$$H^t(S, \mathcal{U}^k) = kH^t(S, \mathcal{U}). \quad (5)$$

If $\mathcal{V}$ is an open cover for $X^k$ then there is an open cover $\mathcal{U}$ for $X$ such that $\mathcal{V} \preceq \mathcal{U}^k$ and hence by Lemma 3.2 $H^t_X(S, \mathcal{V}) \leq H^t_X(S, \mathcal{U}^k)$. Thus (5) implies $H^t_X(S) = kH^t_X(S)$. \hfill \square

Theorem 3.16. Let $S \subseteq \overline{X}$. For $k \geq 2$ let $\mathcal{R}^k S \subseteq \overline{X}$ denote the $k$-restriction of $S$:

$$\mathcal{R}^k S := \left\{ (x_{kn})_{n \geq 0} \mid (x_n)_{n \geq 0} \in S \right\}.$$

Then we have

$$H^t(\mathcal{R}^k S) \leq kH^t(S).$$
Proof. Let \( \mathcal{U} \) be an open cover for \( X \). We have \( N_{\mathcal{U}^k \cap X}(\mathcal{U}^n) \leq N_{S \cap X}(\mathcal{U}^{kn}) \). This implies that \( H^1(\mathcal{U}^k S, \mathcal{U}) \leq kH^1(S, \mathcal{U}) \). Thus \( H^1(\mathcal{U}^k S) \leq kH^1(S) \). \( \square \)

**Theorem 3.17.** Let \( S \subseteq \overline{X} \). For \( k \geq 2 \) let \( \mathcal{D}^k S \subseteq \overline{X} \) denote the \( k \)-dilation of \( S \):

\[
\mathcal{D}^k S := \{ (x_0, \ldots, x_k, x_{k+1}, \ldots) : (x_n)_{n \geq 0} \in S \}.
\]

Then

\[
H^1(\mathcal{D}^k S) = \frac{1}{k} H^1(S).
\]

**Proof.** Let \( \mathcal{U} \) be an open cover for \( X \). For every \( n \geq 1 \) and \( i = 0, \ldots, k-1 \) we have

\[
N_{\mathcal{D}^k S}(\mathcal{U}^{nk-i}) = N_{S}(\mathcal{U}^{n}).
\]

Thus for \( m = nk - i \) we have

\[
(k - \frac{i}{n})\frac{1}{m} \log N_{\mathcal{D}^k S}(\mathcal{U}^m) = \frac{1}{n} N_{S}(\mathcal{U}^{n}).
\]

This implies that \( H^1(S, \mathcal{U}) = kH^1(\mathcal{D}^k S, \mathcal{U}) \). Hence \( H^1(S) = kH^1(\mathcal{D}^k S) \). \( \square \)

By the following simple result we may find upper bounds for the entropy of a (not small) class of noncontinuous mappings. (See Example 3.26 below.)

**Theorem 3.18.** Let \( T : X \to X \) be a (not necessarily continuous) mapping and \( K \) be a nonempty proper subset of \( X \) such that \( K, X \setminus K \) are invariant under \( T \); i.e., \( T(K) \subseteq K, T(X \setminus K) \subseteq X \setminus K \). For a fixed \( z \in K \) let the mapping \( T_{K,z} : X \to X \) be defined by \( T_{K,z}(x) = T(x) \) if \( x \in X \setminus K \) and otherwise \( T_{K,z}(x) = z \). Then

\[
H^1(\text{Gr} T_{K,z}) \leq H^1(\text{Gr} T).
\]

**Proof.** Straightforward. \( \square \)

**Theorem 3.19.** Let \( \phi : X \to Y \) be a continuous map and \( S \subseteq \overline{X} \). Let \( \phi S \subseteq \overline{Y} \) denote the subset \( \{ (\phi x_n) : (x_n) \in S \} \). Then for any open cover \( \mathcal{V} \) of \( Y \), \( H^1(S, \phi^{-1} \mathcal{V}) = H^1(\phi S, \mathcal{V}) \). Thus we have

\[
H^1_\phi(S) \leq H^1_X(S).
\]

If \( \phi \) is a homeomorphism then

\[
H^1_\phi(S) = H^1_X(S).
\]

**Proof.** Straightforward. \( \square \)

Note that the inequality in Theorem 3.19 can not be equality in general: If \( X \) is infinite and \( Y \) is a singleton space then \( H^1(\overline{X}) = \infty \) and \( H^1(\phi \overline{X}) = 0 \).

**Theorem 3.20.** Let \( S, S' \subseteq \overline{X} \). Then

\[
H^1(S \cup S') = \max\{H^1(S), H^1(S')\}.
\]

**Proof.** By Theorem 3.12, \( H^1_{\overline{X}}(S \cup S') = \max\{H^1(S), H^1(S')\} \). The canonical mapping \( \phi : X \cup X \to X \) is continuous and \( \phi(S \cup S') = S \cup S' \). Thus it follows from Theorem 3.19 that \( H^1(S \cup S') \leq \max\{H^1(S), H^1(S')\} \). The reverse follows from Theorem 3.3. \( \square \)
For any topological space $Z$ let $\text{CL}(Z)$ denote the set of all nonempty closed subsets of $Z$. Recall that the upper Vietoris topology on $\text{CL}(Z)$ is the smallest topology containing all the subsets of $\text{CL}(Z)$ of the form $\{K \in \text{CL}(Z) : K \subseteq U\}$ where $U \subseteq Z$ is open.

**Theorem 3.21.** Let $(S_\lambda)_\lambda$ be a net in $\text{CL}(\overline{X})$ converging to a closed subset $S$ of $\overline{X}$ in upper Vietoris topology. Let $\mathcal{U}$ be an open cover for $X$. Suppose that for every $\lambda$ the $\limsup_n$ in the definition of $H^t(S, \mathcal{U})$ (given by (2.1)) is equal to $\inf_n$ (e.g. for every $\lambda$ there is a continuous mapping $T_\lambda : X \to X$ such that $S_\lambda = \text{Gr}(T_\lambda)$). Then

$$\limsup_\lambda H^t(S_\lambda, \mathcal{U}) \leq H^t(S, \mathcal{U}).$$

**Proof.** Without loss of generality suppose that $\mathcal{U}$ is finite and let $\mathcal{U} = \{U_1, \ldots, U_\ell\}$. For a fixed $n \geq 1$ let $k = N_{S/X}(U_n)$. For $i = 1, \ldots, k$ and $j = 0, \ldots, n - 1$ there is an index $q^i_j$ with $1 \leq q^i_j \leq \ell$ such that

$$S \subseteq V := \bigcup_{i=1}^k U_{q^i_0} \times \cdots \times U_{q^i_{n-1}} \times X \times X \times \cdots$$

Since $V$ is open there is $\lambda_n$ such that for every $\lambda \geq \lambda_n$, $S_\lambda \subseteq V$ and hence $N_{S_\lambda/X}(\overline{U^n}) \leq k$. This implies that

$$H^t(S_\lambda, \mathcal{U}) \leq \frac{1}{n} \log N_{S_\lambda/X}(\overline{U^n}) \leq \frac{1}{n} \log k = \frac{1}{n} \log N_{S/X}(\overline{U^n}) \quad (\lambda \geq \lambda_n).$$

The desired result follows from the above inequality. \qed

Another easy result related to Vietoris topology is as follows:

**Theorem 3.22.** Let $(Z_n)_{n \geq 0}$ be a sequence in $\text{CL}(X)$ converging to a singleton $\{z\}$ in upper Vietoris topology. Then for $S := \prod_{n=0}^\infty Z_n$ we have $H^t_X(S) = 0$.

**Proof.** Straightforward. \qed

The following is a simple generalization of [8, Theorem 7.6]. (If $X$ is endowed with a compatible metric then for an open cover $\mathcal{U}$ of $X$, $\text{diam}(\mathcal{U})$ denotes the supremum of diameters of members of $\mathcal{U}$.)

**Theorem 3.23.** Let $(X, d)$ be a compact metric space and $S \subseteq \overline{X}$. Suppose that $(\mathcal{U}_i)_i$ is a sequence of open covers of $X$ such that $\lim_{i \to \infty} \text{diam}(\mathcal{U}_i) = 0$. Then

$$H^t(S) = \lim_{i \to \infty} H^t(S, \mathcal{U}_i).$$

**Proof.** Suppose that $H^t(S) < \infty$. Let $\epsilon > 0$ be arbitrary and let $\mathcal{V}$ be an open cover for $X$ such that $H^t(S) < H^t(S, \mathcal{V}) + \epsilon$. Suppose that $\delta > 0$ is a Lebesgue number for $\mathcal{V}$ and $i_0$ be such that for every $i \geq i_0$, $\text{diam}(\mathcal{U}_i) < \delta$. Then for $i \geq i_0$ we have $\mathcal{V} \preceq \mathcal{U}_i$ and hence by Lemma [8, 3.2] $|H^t(S) - H^t(S, \mathcal{U}_i)| < \epsilon$. The proof in the case $H^t(S) = \infty$ is similarly. \qed

We end this section by three funny (!) examples.
Example 3.24. For an integer \( k \geq 2 \) we may identify the unit interval \([0, 1]\) with a subset of \( X \) where \( X := \{0, \ldots, k-1\} \) via the map that associates to any number \( r \in [0, 1] \) the lower representation of \( r \) in the basis \( X \). (Thus for instance in the case that \( k = 2 \) for \( \frac{1}{3} \) we choose the representation \((0111\cdots)\) rather than \((1000\cdots)\).) Thus for every subset of \( S \subseteq [0, 1] \) we may consider \( H^X_\gamma(S) \) as a measure of intrinsic chaos of \( S \). Similar to the proof of Theorem 3.6 it is easily shown that the entropy of rationals in \([0, 1]\) as well as irrationals is equal to \( \log k \). In the case \( k = 3 \) the entropy of the Cantor set is \( \log 2 \).

Any labeling of points of a space \( F \) by the sequences of points of another space \( X \) gives rise to an entropy theory for subsets of \( F \) as we saw in Example 3.25. Here is another example of this sort from Fractal Theory:

Example 3.25. Let \( F \) denote the self-similar set induced by an \( N \)-tuple \((f_1, \ldots, f_N)\) of contractions on a complete metric space \( Z \). Then as it is shown in [5, Theorem 1.2.3] the \( N \)-tuple \((f_1, \ldots, f_N)\) induces also a canonical continuous surjection \( \pi : X \to F \) where \( X := \{1, \ldots, N\} \). Thus for any subset \( S \) of \( F \) we may consider the value \( H^X_\gamma(\pi^{-1}S) \) as the entropy of \( S \).

Example 3.26. Let \( T : [0, 1] \to [0, 1] \) be defined by \( T(x) = x \) if \( x \) is irrational and \( T(x) = 0 \) if \( x \) is rational. Since the topological entropy of the identity map on \([0, 1]\) is zero it follows immediately from Theorem 3.18 that the generalized topological entropy of \( T \) is also zero i.e. \( H^1(\text{Gr}T) = 0 \).

4 Entropy Induced by a Sequence of Semimetrics

In this section we generalize Bowen’s notion of topological entropy of continuous mappings for subsets of a space with a sequence of semimetrics. We begin by some notations. Let \( X \) be a topological space and let \( \gamma \) be a compatible semimetric on \( X \) i.e. \( \gamma : X \times X \to [0, \infty) \) is a function satisfying (i) \( \gamma(x, x) = 0 \), (ii) \( \gamma(x, y) = \gamma(y, x) \), (iii) \( \gamma(x, z) \leq \gamma(x, y) + \gamma(y, z) \), and (iv) the topology of \( \gamma \) (induced by the family of open balls as the basis) is contained in the original topology of \( X \). For every compact subset \( K \) of \( X \) and any \( \epsilon > 0 \) we let \( N^d(K, \gamma, \epsilon) \) (resp. \( N^\text{id}(K, \gamma, \epsilon) \)) denote the smallest number \( k \) for which there exists a set \( \{x_1, \ldots, x_k\} \) of elements of \( X \) (resp. of \( K \)) such that for every \( x \in K \), \( \gamma(x, x_i) < \epsilon \) for some \( i = 1, \ldots, k \). Note that since \( K \) is \( \gamma \)-compact \( k \) exists. We let \( N^s(K, \gamma, \epsilon) \) denote the largest number \( \ell \) for which there is a subset \( \{y_1, \ldots, y_\ell\} \) of \( K \) with \( \gamma(y_i, y_j) \geq \epsilon \) for \( i \neq j \). Again since \( K \) is \( \gamma \)-compact \( \ell \) exists. In the above, the superscripts \( d \), \( \text{id} \), and \( s \) of \( N \) stand respectively for density, intrinsic density, and separateness. We have the following easily verified lemma. The proof is omitted.

Lemma 4.1. Let \( X, K, \gamma \) be as above. The following statements are satisfied.

(i) For \( 0 < \epsilon < \epsilon' \) we have

\[
N^d(K, \gamma, \epsilon') \leq N^d(K, \gamma, \epsilon), \quad N^\text{id}(K, \gamma, \epsilon') \leq N^\text{id}(K, \gamma, \epsilon), \quad N^s(K, \gamma, \epsilon') \leq N^s(K, \gamma, \epsilon).
\]

(ii) For every \( \epsilon > 0 \), \( N^\text{id}(K, \gamma, 2\epsilon) \leq N^d(K, \gamma, \epsilon) \leq N^\text{id}(K, \gamma, \epsilon) \) and

\[
N^s(K, \gamma, 2\epsilon) \leq N^\text{id}(K, \gamma, \epsilon) \leq N^s(K, \gamma, \epsilon).
\]
Proposition 4.2. Let sequence of compatible semimetrics on $X$ $\epsilon$ This shows that the first $\lim$ $B$

Proof. By Lemma 4.1(i) if $\epsilon$ then $N_{K/X}(U)$. Now (6) follows from (7) and (8).

Definition 4.3. Let $X$ be as above. Let $(\gamma_n)_{n \geq 1}$ be a (not necessarily increasing) sequence of compatible semimetrics on $X$. Then

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N^d(K, \gamma_n, \epsilon) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N_{id}(K, \gamma_n, \epsilon)$$

(6)

$$= \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N^s(K, \gamma_n, \epsilon).$$

This shows that the first $\lim_{\epsilon}$ in (6) exists an equals to $sup_{\epsilon}$. Similarly it is proved that the other two lims in (6) exist and are sups. By Lemma 4.1(ii) we have

$$\frac{1}{n} \log N^d(K, \gamma_n, 2\epsilon) \leq \frac{1}{n} \log N^d(K, \gamma_n, \epsilon) \leq \frac{1}{n} \log N_{id}(K, \gamma_n, \epsilon).$$

(7)

$$\frac{1}{n} \log N^s(K, \gamma_n, 2\epsilon) \leq \frac{1}{n} \log N^d(K, \gamma_n, \epsilon) \leq \frac{1}{n} \log N^s(K, \gamma_n, \epsilon).$$

(8)

Now (6) follows from (7) and (8).

Theorem 4.4. Let $\Gamma = (\gamma_n)_n, \Gamma' = (\gamma'_n)_n$ be increasing sequences of compatible semimetrics on $X$ and $S, S' \subseteq X$. The following statements hold.
(i) If $\Gamma$ is a constant sequence (i.e. $\gamma_n = \gamma$ for every $n$) then $H^B_1(S) = 0$.

(ii) For a constant $c > 0$ let $c\Gamma$ denote the sequence $(c\gamma_n)_n$. Then $H^B_{c\Gamma}(S) = H^B_1(S)$.

(iii) If $S \subseteq S'$ and $\gamma_n \leq \gamma'_n$ for every $n$ then $H^B_1(S) \leq H^B_{\Gamma'}(S)$.

(iv) If there is a constant $c > 0$ such that $\gamma_n \leq c\gamma'_n$ for every $n$ then $H^B_1(S) \leq H^B_{\Gamma'}(S)$.

(v) If $\Gamma$ and $\Gamma'$ are equi strongly-uniform-equivalent i.e. there exist constants $c, c' > 0$ such that $c'\gamma'_n \leq \gamma_n \leq c\gamma'_n$ for every $n$, then $H^B_1(S) = H^B_{\Gamma'}(S)$.

(vi) If $\sup_n \gamma_n$ is a compatible semimetric on $X$ then $H^B_1(S) = 0$.

Proof. (i) is trivial. (ii) follows from $N^\text{id}(K, c\gamma_n, \epsilon) = N^\text{id}(K, \gamma_n, \epsilon)$. (iii) follows from Lemma 4.1(v). (iv) follows from (ii) and (iii). (v) follows from (iv). (vi) follows from (i) and (iii).

**Theorem 4.5.** Let $\Gamma, \Gamma', S$ be as in Theorem 4.4.

(i) Suppose that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $n$ and every $x, x' \in X$ if $\gamma'_n(x, x') < \delta$ then $\gamma_n(x, x') < \epsilon$. Then $H^B_1(S) \leq H^B_{\Gamma'}(S)$.

(ii) Suppose that $\Gamma$ and $\Gamma'$ are equi uniformly-equivalent i.e. for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $n$ and every $x, x' \in X$ if $\gamma'_n(x, x') < \delta$ then $\gamma_n(x, x') < \epsilon$, and if $\gamma_n(x, x') < \delta$ then $\gamma'_n(x, x') < \epsilon$. Then $H^B_1(S) = H^B_{\Gamma'}(S)$.

Proof. (i) Let $\epsilon > 0$ and suppose that $\delta > 0$ satisfies in the condition stated above. For every compact $K$ in $X$ we have $N^g(K, \gamma_n, \epsilon) \leq N^g(K, \gamma'_n, \delta)$ and hence

$$\limsup_{n \to \infty} \frac{1}{n} \log N^g(K, \gamma_n, \epsilon) \leq \limsup_{n \to \infty} \frac{1}{n} \log N^g(K, \gamma'_n, \delta) \leq H^B_{\Gamma'}(K).$$

This implies that $H^B_1(K) \leq H^B_{\Gamma'}(K)$. Hence $H^B_1(S) \leq H^B_{\Gamma'}(S)$. (ii) follows from (i).

Note that (i) and (ii) of Theorem 4.5 are extensions of (iv) and (v) of Theorem 4.4.

**Theorem 4.6.** Let $X, \Gamma, S$ be as in Theorem 4.4. Suppose that $S_1, \ldots, S_k$ are closed subsets of $X$ such that $S \subseteq \bigcup_{i=1}^k S_i$. Then

$$H^B_1(S) \leq \max_{i=1,\ldots,k} H^B_1(S_i). \quad (9)$$

Proof. Let $L, L_1, \ldots, L_k$ be compact subsets of $X$ with $L \subseteq \bigcup_{i=1}^k L_i$. For $\epsilon > 0$ we have

$$N^g(L, \gamma_n, \epsilon) \leq \sum_{i=1}^k N^g(L_i, \gamma_n, \epsilon). \quad (10)$$

Let $(n_\ell)$ be a subsequence of natural numbers such that

$$\lim_{\ell \to \infty} \frac{1}{n_\ell} \log N^g(L, \gamma_{n_\ell}, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log N^g(L, \gamma_n, \epsilon). \quad (11)$$
Then (10) shows that there exist some \(i_0\) with \(1 \leq i_0 \leq k\) and a subsequence \((n_{i_0})_j\) of \((n_i)_\ell\) such that \(\sum_{i=1}^{k} N^s(L_i, \gamma_{n_{i_0}}, \epsilon) \leq kN^s(L_{i_0}, \gamma_{n_{i_0}}, \epsilon)\). Thus by (11) we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log N^s(L, \gamma_n, \epsilon) \leq \limsup_{j \to \infty} \frac{1}{n_{i_0}} (k + \log N^s(L_{i_0}, \gamma_{n_{i_0}}, \epsilon))
\]

\[
\leq \limsup_{n \to \infty} \frac{1}{n} \log N^s(L_{i_0}, \gamma_n, \epsilon)
\]

\[
\leq \max_{i=1,...,k} \limsup_{n \to \infty} \frac{1}{n} \log N^s(L_i, \gamma_n, \epsilon)
\]

Letting \(\epsilon \to 0\) in (12) we find that \(H^B_\Gamma(L) \leq \max_{i=1,...,k} H^B_\Gamma(L_i)\). Now (9) follows from the latter inequality by letting \(L_i = L \cap S_i\) where \(L \subseteq S\).

**Theorem 4.7.** Let \(\Gamma = (\gamma_n)_n, \Sigma = (\sigma_n)_n\) be increasing sequences of compatible semimetrics on topological spaces \(X, Y\) respectively. Let \(\gamma_n \times \sigma_n\) be the semimetric on \(X \times Y\) given by \(((x, y), (x', y')) \mapsto \max(\gamma_n(x, x'), \sigma_n(y, y'))\) and denote by \(\Gamma \times \Sigma\) the sequence \((\gamma_n \times \sigma_n)_n\). Then for every \(S \subseteq X\) and \(T \subseteq Y\) we have

\[H^B_{\Gamma \times \Sigma}(S \times T) \leq H^B_\Gamma(S) + H^B_\Sigma(T).\]

**Proof.** Let \(K, L\) be compact subspaces with \(K \subseteq S, L \subseteq T\). For every \(n\) and \(\epsilon > 0\) we have

\[N^s(K \times L, \gamma_n \times \sigma_n, \epsilon) = N^s(K, \gamma_n, \epsilon)N^s(L, \sigma_n, \epsilon).\]

It follows that

\[H^B_{\Gamma \times \Sigma}(K \times L) \leq H^B_\Gamma(K) + H^B_\Sigma(L) \leq H^B_\Gamma(S) + H^B_\Sigma(T).\] (13)

Now suppose that \(M \subseteq S \times T\) is a compact subspace of \(X \times Y\). Then \(M \subseteq \text{pr}_X(M) \times \text{pr}_Y(M)\) where \(\text{pr}_X, \text{pr}_Y\) denote the canonical projections from \(X \times Y\) onto \(X\) and \(Y\). Note that \(\text{pr}_X(M)\) and \(\text{pr}_Y(M)\) are compact and also \(\text{pr}_X(M) \subseteq S\) and \(\text{pr}_Y(M) \subseteq T\). It follows from Lemma 4.4(v) that \(H^B_{\Gamma \times \Sigma}(M) \leq H^B_{\Gamma \times \Sigma}(\text{pr}_X(M) \times \text{pr}_Y(M))\). Thus (13) implies that

\[H^B_{\Gamma \times \Sigma}(M) \leq H^B_\Gamma(S) + H^B_\Sigma(T).\]

The proof is complete. \(\square\)

## 5 Bowen p-Entropy of Subsets of Infinite Product Spaces

The Bowen entropy [2][8] of continuous mappings on metric spaces is defined as follows. Let \((X, d)\) be a metric space and let \(T : X \to X\) be a continuous map. For every \(n \geq 1\) let \(d_n\) be the metric on \(X\) defined by

\[d_n(x, x') := \max_{i=0,...,n-1} d(T^i(x), T^i(x')).\] (14)

The topology induced by \(d_n\) coincides with the original topology of \(X\) and \(d_n \leq d_{n+1}\). Thus \(\Gamma := (d_n)_n\) is an increasing sequence of compatible metrics on \(X\). For any compact subset \(K\) of \(X\) the Bowen entropy of \(T\) with respect to \(K\) is denoted by \(h_d(T, K)\) and defined to be the value \(H^B_\Gamma(K)\) as described in Section 4. The Bowen entropy of \(T\) is denoted by \(h_d(T)\) and defined to be the value \(H^B_\Gamma(X)\).

Now we define a notion of Bowen entropy for subsets of infinite product spaces:
Thus with $\Gamma := (d_{14})$ we have compatible semimetric on $X$.

Let $\Delta_p := (d_{n,p})_n$ be an increasing sequence of compatible semimetric on $X$. For any subset $S$ of $X$ we call the value

$$H_{d,p}^B(S) := H_{d,n}^B(S) \quad (1 \leq p \leq \infty)$$

the Bowen $p$-entropy of $S$ with respect to $d$.

Definition 5.1 is a generalization of the original definition of Bowen entropy:

**Proposition 5.2.** Let $T : (X, d) \to (X, d)$ be a continuous map. Then

$$h_d(T) = H_{d,\infty}^B(\text{GrT})$$

**Proof.** Let $\phi : X \to X$ denote the canonical map associated to $T$ given by $x \mapsto (T^n(x))_{n \geq 0}$. Then $\phi$ is a homeomorphism from $X$ onto $\text{GrT}$. For every $n \geq 1$ by the notations as in (14) we have

$$d_n(x, x') = d_{n,\infty}(\phi(x), \phi(x')) \quad (x, x' \in X).$$

Thus with $\Gamma := (d_n)_n$ we may identify the pair $(X, \Gamma)$ with the pair $(\text{GrT}, \Delta_\infty)$ under $\phi$. Then we have

$$h_d(T) = H_{d}^B(X) = H_{d,\infty}^B(\text{GrT}) = H_{d,\infty}^B(\text{GrT}).$$

Following Proposition 5.2 we define Bowen $p$-entropy of $T : (X, d) \to (X, d)$ by

$$h_{d,p}(T) := H_{d,p}^B(\text{GrT}) \quad (1 \leq p \leq \infty).$$

(15)

**Theorem 5.3.** Let $(X, d)$ be a metric space and $S \subseteq X$. If $1 \leq p \leq q \leq \infty$ then

$$H_{d,\infty}^B(S) \leq H_{d,q}^B(S) \leq H_{d,p}^B(S) \leq H_{d,1}^B(S).$$

**Proof.** It is clear that $d_{n,\infty} \leq d_{n,p}$. Thus by Theorem 4.3(iii) we have $H_{d,\infty}^B(S) \leq H_{d,p}^B(S)$. Suppose that $1 \leq p \leq q < \infty$. Let $\epsilon \in (0,1)$ be arbitrary and $n \geq 1$ be fixed. Suppose that $(x_k)_k, (x'_k)_k \in X$ are such that

$$d_{n,p}((x_k)_k, (x'_k)_k) < \epsilon^p.$$

Then we have

$$\sum_{i=0}^{n-1} (d(x_i, x'_i))^q \leq \sum_{i=0}^{n-1} (d(x_i, x'_i))^p < \epsilon^q.$$

Hence $d_{n,q}((x_k)_k, (x'_k)_k) < \epsilon$. Now it follows from Theorem 4.3(i) that $H_{d,q}^B(S) \leq H_{d,p}^B(S)$. \qed
We do not yet know if the notion $h_{d,p}$ for $p \neq \infty$ is useful or if it has some advantages rather than the classical notion $h_d$. But it is clear that in order to have a correct notion of entropy in the case $1 \leq p < \infty$ we must put some conditions on $(X,d)$ to guaranty at least the property $h_{d,p}(\text{id}_X) = 0$ or equivalently (by Theorem 5.3) $h_{d,1}(\text{id}_X) = 0$. Here is one such a condition:

**Theorem 5.4.** Suppose that $(X,d)$ is a metric space with the property that: For every compact subset $K$ of $X$ there exists a constant $\alpha \geq 1$ such that

$$\limsup_{\epsilon \to 0} \epsilon^{\alpha} N^s(K, d, \epsilon) < \infty. \quad (16)$$

Then $h_{d,1}(\text{id}_X) = 0$.

**Proof.** First of all observe that we must show that for any compact $K \subseteq X$,

$$\lim \limsup_{n \to \infty} \frac{1}{n} \log N^s(K, nd, \epsilon) = 0.$$

Let $\epsilon > 0$ be arbitrary and fixed. There is a bound $M > 0$ such that for large enough $n$,

$$N^s(K, d, \frac{\epsilon}{n}) \leq M \frac{n^\alpha}{\epsilon^\alpha}.$$

Thus we have

$$\limsup_{n \to \infty} \frac{1}{n} \log N^s(K, nd, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log N^s(K, d, \frac{\epsilon}{n}) \leq \alpha \log n + \log M - \alpha \log \epsilon \leq 0.$$

The proof is complete. \[\Box\]

We remark that the quantities similar to the one given by $\limsup$ in (16) appear in Dimension Theory of metric spaces. The reader may easily convince him/herself that in the case that $X$ is the Euclidean space $\mathbb{R}^q$ and $d$ is the usual Euclidean metric (or any other strongly uniform equivalent metric), $(16)$ is satisfied for any compact subset $K$ of $\mathbb{R}^q$ with $\alpha = q$.

The following result is a generalization of [8, Theorem 7.4].

**Theorem 5.5.** Let $X$ be a space with two uniformly equivalent metrics $d,d'$, i.e. for every $\epsilon > 0$ there exists $\delta > 0$ such that if $d(x,x') < \delta$ then $d'(x,x') < \epsilon$, and such that if $d'(x,x') < \delta$ then $d(x,x') < \epsilon$. Then for every $S \subseteq X$ we have

$$H^B_{d,\infty}(S) = H^B_{d',\infty}(S).$$

**Proof.** It is easily seen that $\Delta_{\infty} = (d_n,\infty)_n$ and $\Delta'_{\infty} = (d'_n,\infty)_n$ are equi uniformly-equivalent. Thus the desired result follows from Theorem 4.5 \[\Box\]

We have the following trivial corollary of Theorem 5.5.

**Corollary 5.6.** If $X$ is compact then $H^B_{d,\infty}(S)$ only depends on $S$ and the topology of $d$. 15
Theorem 5.7. Let \( X \) be a space with two strongly uniform equivalent metrics \( d, d' \), i.e. there exist constants \( c, c' > 0 \) such that \( c'd' \leq d \leq cd' \). Then for every \( S \subseteq \overline{X} \) we have

\[
H_{d,p}^B(S) = H_{d',p}^B(S) \quad (1 \leq p \leq \infty).
\]

Proof. It is easily verified that \( \Delta_p = (d_{n,p})_n \) and \( \Delta'_p = (d'_{n,p})_n \) are equi strongly-uniform-equivalent. Thus the desired result follows from Theorem 4.4(v).

Now we show that as in the classical case \[8, Theorem 7.8\] Bowen \( \infty \)-entropy and generalized topological entropy coincide in compact metric spaces.

Theorem 5.8. Let \((X, d)\) be a compact metric space and \( L \subseteq \overline{X} \) be closed. Then

\[
H_{d,\infty}^B(L) = H_X^1(L).
\]

Proof. Let \((U_k)_k\) be a sequence of open covers of \( X \) such that \( \text{diam}(U_k) < \frac{1}{k} \) for every \( k \). It follows from (ii), (iii) and (iv) of Lemma 4.11 that

\[
N_{L/X}(U^n_k) \leq N^S(L, d_{n,\infty}, \frac{1}{2k}) \leq N_{L/X}(U^n_{2k}).
\]

This implies that

\[
H_X^1(L, U_k) \leq \limsup_{n \to \infty} \frac{1}{n} \log N^S(L, d_{n,\infty}, \frac{1}{2k}) \leq H_X^1(L, U_{2k})
\]

Now let \( k \to \infty \). Then the desired result is concluded from Theorem 3.23.

Note that the above result contains the result stated in Corollary 5.6 for closed subsets.

Theorem 5.9. Let \((X, d)\) be a finite metric space. Then for any \( S \subseteq \overline{X} \) and every \( p \geq 1 \),

\[
H_{d,p}^B(S) = H_{d,\infty}^B(S).
\]

Proof. It follows from the obvious fact that if \( \varepsilon < \min \{d(x, x') : x, x' \in X, x \neq x'\} \) then for any (compact) subset \( K \) of \( X \), every \( n \geq 1 \), and every \( p \geq 1 \) we have

\[
N^S(K, d_{n,p}, \varepsilon) = N^S(K, d_{n,\infty}, \varepsilon).
\]

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