Rankin-Cohen brackets for Calabi-Yau modular forms

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Abstract

For any positive integer \( n \), we introduce a quasi-homogeneous vector field \( D \) of degree 2 on a moduli space \( T \) of enhanced Calabi-Yau \( n \)-folds arising from the Dwork family. By Calabi-Yau quasi-modular forms for Dwork family we mean the elements of the graded \( \mathbb{C} \)-algebra \( \mathcal{M} \) generated by the components of a particular solution of \( D \), which are provided with natural weight. Using \( D \) we introduce the derivation \( \mathcal{D} \) and the Ramanujan-Serre type derivation \( \partial \) on \( \mathcal{M} \). We show that they are degree 2 differential operators and there exists a proper subspace \( \mathcal{M} \subset \mathcal{M} \), called the space of Calabi-Yau modular forms, which is closed under \( \partial \). Using the derivation \( \mathcal{D} \), we define the Rankin-Cohen brackets for Calabi-Yau quasi-modular forms and prove that the subspace generated by the positive weight elements of \( \mathcal{M} \) is closed under the Rankin-Cohen brackets.

1 Introduction

The proof of Fermat’s last theorem led to the celebrated modularity theorem, which states that elliptic curves over the field of rational numbers \( \mathbb{Q} \) are related with modular forms. Elliptic curves are 1-dimensional Calabi-Yau (CY) varieties, which makes it natural to ask whether a similar statement of modularity holds for higher dimensional CY varieties. This question persuaded mathematicians and theoretical physicists to the subject of modularity of CY manifolds which is one of the considerable present challenges of the modern algebraic number theory. Some relevant results can be found, for instance, in [Yui13] and the references therein. Yui in [Yui13] divides the modularity of CY varieties in arithmetic modularity and geometric modularity including (1) the modularity (automorphy) of Galois representations of CY varieties (or motives) defined over \( \mathbb{Q} \) or number fields, (2) the modularity of solutions of Picard-Fuchs differential equations of families of CY varieties, and mirror maps (mirror moonshine), (3) the modularity of generating functions of invariants counting certain quantities on CY varieties, and (4) the modularity of moduli for families of CY varieties. But so far, in a general context, even there is no unified formulation or statement of the modularity of CY varieties. Yamaguchi and Yau [YY04] in 2004 showed that the partition functions for the mirror quintic can be expressed in terms of finitely many generators of a differential ring, which somehow play the role of quasi-modular forms; then Alim and Lange [AL07] in 2007 generalized their results for arbitrary CY 3-folds. Movasati in [Mov17] says: "All the attempts to find an arithmetic modularity for mirror quintic have failed, and this might be an indication that maybe such varieties need a new kind of modular forms." In this way, he introduced CY
(quasi-)modular forms which somehow can be considered as a modern generalization of the classical quasi-modular forms (automorphic forms) theory. The present paper provides some evidences in favor of this generalization; namely, we introduce the space of CY quasi-modular forms $\tilde{M}$ for the Dwork family and furnish it with a Rankin-Cohen algebra structure. Then we find a proper subspace of $\tilde{M}$ which is closed under the Rankin-Cohen brackets. This can be considered as a generalization of the work of Zagier [Zag94] for the space of classical (quasi-)modular forms.

Movasati in [Mov12] used an algebraic method, called Gauss-Manin connection in disguise (GMCD), in a geometric framework and reencountered the Ramanujan [Ram16] vector field (system) $R_a$ (see (3.22)) on certain moduli of a family of enhanced elliptic curves (see (3.24) and (3.25)). It is known that the triple of Eisenstein series $(E_2, E_4, E_6)$ gives a solution of the Ramanujan system $R_a$. For the classical quasi-modular forms (automorphic forms) theory. The present paper provides a basis of the $n$-th algebraic de Rham cohomology $H^{n}_{dr}(X)$ which is compatible with the Hodge filtration of $H^{n}_{dR}(X)$ (see (3.18)) and its intersection form matrix is constant (see (3.19)). We showed that there exist a unique vector field $R = R_n$, called modular vector field, and regular functions $Y_i$, $1 \leq i \leq n - 2$, that satisfy certain equation involving the Gauss-Manin connection of the universal family of $T$ (see Theorem 3.1 and also [Nik15, Theorem 1.1] in a more general context). Due to [Mov12] we can say that the modular vector field $R$ is a generalization of the Ramanujan vector field $R_a$. For $n = 1, 2, 3, 4$ we found $q$-expansion of solution components of the modular vector field $R$ whose coefficients are surprisingly integers. Actually, for $n = 1, 2$, where $T$ is the moduli of enhanced elliptic curves and K3-surfaces, respectively, the solution components, as it was expected, are quasi-modular forms (see (3.36) and (3.37)). See also [Ali17] for similar computations. In the case $n = 3$, $R_3$ is explicitly computed in [Mov15] and it is verified that $Y_1$ is the Yukawa coupling introduced in [COGP91], which predicts the numbers of rational curves of various degrees on a general quintic three-fold. For $n = 4$, we computed the modular vector field $R_4$ explicitly in [MN21] and we observed that $Y_2 = Y_2$ is the same as 4-point function presented in [GMP95, Table 1, $d = 4$], and we computed the modular coordinate $z$ given in [KP08, §6.1] in terms of solution components of $R_4$. Unlike the cases $n = 1, 2$, for $n = 3, 4$ we believe that it is not possible to write the solution components of $R$ in terms of classical quasi-modular forms, since the coefficients of their $q$-expansions increase very rapidly. This leads us to think to another theory which generalizes the theory of quasi-modular forms, where the space generated by solution components of $R$ is the adequate candidate of the desired generalization.
One of the initial steps in the above-mentioned generalization is the correct assignment of weights to the components of a solution of \( R \). In order to do this, we recall another important property of the Ramanujan vector field \( Ra \). We can easily observe that the Lie algebra generated by the Ramanujan vector field \( Ra = \frac{1}{12} (t_1^2 - t_2) \frac{\partial}{\partial t_1} + \frac{1}{3} (t_1 t_2 - t_3) \frac{\partial}{\partial t_2} + \frac{1}{2} (t_1 t_3 - t_2^2) \frac{\partial}{\partial t_3} \), the radial vector field \( H = 2t_1 \frac{\partial}{\partial t_1} + 4t_2 \frac{\partial}{\partial t_2} + 6t_3 \frac{\partial}{\partial t_3} \) and the constant vector field \( F = -12 \frac{\partial}{\partial t_1} \) is isomorphic to the Lie algebra \( sl_2(\mathbb{C}) \) (remember that \( (t_1, t_2, t_3) = (E_2, E_4, E_6) \) is a solution of \( Ra \)). Note that \( \deg(E_2) = 2 \), \( \deg(E_4) = 4 \), \( \deg(E_6) = 6 \) and these integers appear as coefficients of the components of the vector field \( H \). Moreover, we have \( F(t_1) = -12 \), \( F(t_2) = F(t_3) = 0 \); indeed, if we consider \( F \) as a derivation on \( \mathcal{N} \), then \( \ker F = \mathcal{M} \). Our attention in \[Nik20\] was dedicated to the Lie algebra \( sl_2(\mathbb{C}) \) and we proved that for any \( n \), there are vector fields \( H \) and \( F \) on \( T = T_n \) such that along with the modular vector field \( \mathcal{R} \) generate a copy of \( sl_2(\mathbb{C}) \) in \( \mathfrak{X}(T) \) (see Theorem 3.4) (the notations \( H \) and \( F \) in the whole manuscript are used for the same vector fields given in Theorem 3.4). Furthermore, we observe that the vector field \( H \) can be written in the form \( \mathbf{H} = \sum_{j=1}^{n} w_j \partial_j \), where \( \mathbf{d} = \dim \mathbf{T} \), \( (t_1, t_2, \ldots, t_d) \) is a chart of \( \mathbf{T} \), which will be constructed in Subsection 3.1, and \( w_j \in \mathbb{Z}_{\geq 0}, \ j = 1, 2, \ldots, d \) (see (3.55)). These facts lead us to define \( \deg(t_j) := w_j \), \( j = 1, 2, \ldots, d \). By applying these weights, in Proposition 3.1 we show that for any positive integer \( n \) the modular vector field \( \mathcal{R} = \mathcal{R}_n \) is a quasi-homogeneous vector field of degree 2. If \( n = 1 \) or \( n \) is even, then \( \mathcal{F}(t_j) = 0 \), for all \( j \neq 2 \), and \( \mathcal{F}(t_2) \neq 0 \). But, if \( n \geq 3 \) is odd, then we observe that \( \mathcal{F}(t_j) = 0 \), for all \( j \neq 2, d \), and \( \mathcal{F}(t_2) \neq 0, \mathcal{F}(t_d) \neq 0 \) which will cause problems for our purposes in Section 5. To avoid these problems, we introduce another vector field \( \mathcal{D} \) (see Subsection 4), which coincides with \( \mathcal{R} \) when \( n = 1 \) or \( n \) is even, but for odd \( n \geq 3 \) it can be different from \( \mathcal{R} \). We prove that \( \mathcal{D} \) along with \( \mathcal{H} \) and the constant vector field \( (1 + \delta_2^3) \frac{\partial}{\partial t_2} \) forms a copy of \( sl_2(\mathbb{C}) \), where \( \delta_2^3 \) is the Kronecker delta, and \( \mathcal{D} \) is a quasi-homogeneous vector field of degree 2 in \( T \) (see Lemma 4.1 and Corollary 4.1). Now, suppose that \( t_j, \ j = 1, 2, \ldots, d \), is the component of a particular solution of \( \mathcal{D} \) associated with the coordinate chart \( t_j \) carrying the same weight, i.e., \( \deg(t_j) = w_j \). We define the space of CY quasi-modular forms for Dwork family as \( \mathcal{M} := \mathbb{C}[t_1, t_2, t_3, \ldots, t_d, \frac{1}{t_{n+2}(t_{n+2} - t_1^r - t_1^{r-1})}] \) and the space of CY modular forms for Dwork family as \( \mathcal{M} := \mathbb{C}[t_1, t_2, t_3, t_4, \ldots, t_d, \frac{1}{t_{n+2}(t_{n+2} - t_1^r - t_1^{r-1})}] \), where \( \tilde{t} \) is a product of a few number of \( t_j \)'s (see (3.24)) and the symbol \( \tilde{t}_2 \) means that the component \( t_2 \) is omitted, i.e., \( t_2 \notin \mathcal{M} \); indeed \( \mathcal{M} \) is a subspace of \( \mathcal{M} \), and \( \mathcal{M} = \mathcal{M}(t_2) \). For any \( n \), Remark 3.3 yields that \( \deg(t_2) = 2 \). In our approach \( t_2 \) plays the same role of the quasi-modular form \( E_2 \) in the theory of quasi-modular forms for \( SL_2(\mathbb{Z}) \), which gives sense to the definition of \( \mathcal{M} \) (recall that \( \tilde{M} = M(E_2) \)). Throughout by CY quasi-modular forms or CY modular forms we mean CY quasi-modular forms or CY modular forms for the Dwork family.

To motivate and explain better our main results, we recall again some known facts of classical theory of quasi-modular forms. It is well known that the derivative of a modular form is not necessarily a modular form. More precisely, for any positive integer \( r \) and any modular form \( f \in \mathcal{M}_r \) of weight \( r \) for \( SL_2(\mathbb{Z}) \), we know that \( f' \in \mathcal{M}_{r+2} \) is a quasi-modular form of weight \( r + 2 \) which is not necessarily modular. But the derivative \( f' \) can be corrected using the Ramanujan-Serre derivation \( \partial f = f' - \frac{1}{12} r E_2 f \) which yields \( \partial f \in \mathcal{M}_{r+2} \) (see (2.4) and (2.5)). Rankin in \[Ran56\] described some necessary conditions under which a polynomial in a given modular form and its derivatives is again a modular form. Cohen \[Coh77\] generalized the result of Rankin and for any non-negative integer \( k \), defined a bilinear operator \( F_k(\cdot, \cdot) \) and proved that for all \( f \in \mathcal{M}_r, \ g \in \mathcal{M}_s \) one gets
Later, Zagier in [Zag94] called these bilinear forms as Rankin-Cohen brackets and denoted them by $[\cdot, \cdot]_k$ (see (2.6)). Furthermore, he developed the theory of Rankin-Cohen algebras, which are briefly described in Section 2. The principal objective of this paper is to endow $\mathcal{M}$ and $\mathcal{M}$ with standard Rankin-Cohen and canonical Rankin-Cohen algebra structure, respectively. In order to do this we will need a degree 2 differential operator and a Ramanujan-Serre type derivation on $\mathcal{M}$ and $\mathcal{M}$, respectively. To this end, we observe that $D$ induces a differential operator on $\mathcal{M}$ which is denoted by $D$ (see (5.4)). It is not difficult to observe that the space of CY modular forms $\mathcal{M}$ is not closed under $D$, but by correcting the derivation $D$ we can define the Ramanujan-Serre type derivation $\partial$ (see (5.5)). In the following theorem we state the first main result of this work.

**Theorem 1.1.** Let $D$ and $\partial$ be the derivations defined in (5.4) and (5.5), respectively. Then the following hold.

1. The derivation $D$ is a degree 2 differential operator on $\mathcal{M}$.
2. The Ramanujan-Serre type derivation $\partial$ is a degree 2 differential operator on $\mathcal{M}$.

We emphasize that, due to Theorem 1.1, $\mathcal{M}$ is closed under $\partial$, and in particular for all integers $r$ we have $\partial: \mathcal{M}_r \to \mathcal{M}_{r+2}$. Using the derivation $D$, for any non-negative integers $k, s, r$ and any $f \in \mathcal{M}_r, g \in \mathcal{M}_s$, we define the $k$-th Rankin-Cohen bracket $[f, g]_{D, k}$ of CY quasi-modular forms in (5.26) and observe that $[f, g]_{D, k} \in \mathcal{M}_{r+s+2k}$. Indeed, $[\cdot, \cdot]_{D, k}$ provides $\mathcal{M}$ with a standard Rankin-Cohen algebra structure. Finally, in the following theorem we establish the second main, and more important, result of the present paper.

**Theorem 1.2.** For all non-negative integers $r, s, k$ and for any $f \in \mathcal{M}_r, g \in \mathcal{M}_s$ we have:

$$[f, g]_{D, k} \in \mathcal{M}_{r+s+2k}.$$ 

In the other words, Theorem 1.2 says that the space of CY modular forms of positive weight is closed under the Rankin-Cohen brackets of the CY quasi-modular forms, and hence we provide this space with a canonical Rankin-Cohen algebra structure. We prove Theorem 1.1 and Theorem 1.2 in Section 5. It is worth to mention that for various examples of CY modular forms of negative weight we used the computer and observed that their Rankin-Cohen brackets are again CY modular forms. Thus, we conjecture that the whole space of the CY modular forms $\mathcal{M}$ is closed under the Rankin-Cohen brackets.

This manuscript is organized as follows. In Section 2 we briefly review the relevant definitions and facts of [Zag94] which will be used in the rest of the text. Section 3 starts with a short summary of [MN21] and [Nik20] in Subsections 3.1 and 3.2 which constructs the foundation of the present research and also lets us to have a self contained manuscript. After that, in Subsection 3.3 we prove that the modular vector field $R$ is a quasi-homogeneous vector field of degree 2. We introduce the vector field $D$ in Section 4 and demonstrate the fundamental lemma. In Section 5 our main results are stated and proved. Namely, we define the concepts of: spaces of CY quasi-modular forms and CY modular forms, derivation $D$, Ramanujan-Serre type derivation $\partial$ and Rankin-Cohen brackets of the CY quasi-modular forms. We provide the proofs of Theorem 1.1 and Theorem 1.2 in this section. In various examples of the same section, for $n = 1, 2, 3, 4$, the derivations $D, \partial$ and Rankin-Cohen brackets of a few CY modular forms are explicitly calculated. Section 6 deals with the final remarks. In this section we state a conjecture which improves our results.
Acknowledgment. The initial inspiration of the present study came from a conversation between Hossein Movasati and Don Zagier, which was shared later with the author and others by Movasati. At that moment we did not succeed in solving the problem, because of the absence of some key points such as the correct weight of the CY (quasi-)modular forms and etc. After the work [Nik20], the author could find the missing points of the research and completed the present work. Because of this, the author would like to thank both Movasati and Zagier, in particular he is very grateful to Movasati for his helpful discussions and comments.

2 Rankin-Cohen algebra

In this section we recall the important facts and terminologies of [Zag94] which are necessary for the present paper. Let $\mathcal{M} = \bigoplus_{r \geq 0} \mathcal{M}_r$ and $M = \bigoplus_{r \geq 0} M_r$, respectively, be the graded algebras of quasi-modular forms and of modular forms, where $\mathcal{M}_r := \mathcal{M}_r(\mathbb{SL}_2(\mathbb{Z}))$ and $M_r := M_r(\mathbb{SL}_2(\mathbb{Z}))$, respectively, are the spaces of quasi-modular forms and of modular forms of weight $r$ for $\mathbb{SL}_2(\mathbb{Z})$. It is well known that $\mathcal{M} = \mathbb{C}[E_2, E_4, E_6]$ and $M = \mathbb{C}[E_4, E_6]$, where $E_2, E_4, E_6$ are Eisenstein series given as:

\[(2.1) \quad E_{2j}(q) = 1 + b_j \sum_{k=1}^{\infty} \sigma_{2j-1}(k)q^k \text{ with } \sigma_i(k) = \sum_{d|k} d^i, \quad (b_1, b_2, b_3) = (-24, 240, -504).\]

Note that $E_4$ and $E_6$ are modular forms of weight 4 and 6, respectively, while $E_2$ is a quasi-modular form of weight 2 which is not modular. The triple $(E_2, E_4, E_6)$ satisfies the system of ordinary differential equations

\[(2.2) \quad \text{Ra} : \begin{cases} t'_1 = \frac{1}{12}(t_1^2 - t_2) \\ t'_2 = \frac{1}{7}(t_1 t_2 - t_3) \\ t'_3 = \frac{1}{7}(t_1 t_3 - t_2^2) \end{cases}\]

which is known as the Ramanujan relations between Eisenstein series, and from now on we call it the Ramanujan vector field. Note that here $t'_j = \frac{\partial t_j}{\partial t}$, $t = e^{2\pi i \tau}$ and $\tau \in \mathbb{H} := \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$. The Rankin vector field $\text{Ra} = t'_1 \frac{\partial}{\partial t_1} + t'_2 \frac{\partial}{\partial t_2} + t'_3 \frac{\partial}{\partial t_3}$, together with two vector fields $H = 2t_1 \frac{\partial}{\partial t_1} + 4t_2 \frac{\partial}{\partial t_2} + 6t_3 \frac{\partial}{\partial t_3}$ and $F = -12 \frac{\partial}{\partial t_3}$ forms a copy of $\frak{sl}_2(\mathbb{C})$; this follows from the fact that $[\text{Ra}, F] = H$, $[H, \text{Ra}] = 2\text{Ra}$, $[H, F] = -2F$, where $[\ , \ ]$ refers to the Lie bracket of vector fields. We know that if $f \in \mathcal{M}_r$ is a modular form of weight $r$, then $f'$ is not necessarily a modular form. If instead of the usual derivation, we use the so-called Ramanujan-Serre derivation $\partial$ given by

\[(2.3) \quad \partial f = f' - \frac{1}{12} r E_2 f,\]

then $\partial f$ is a modular form of weight $r + 2$. After substituting $(t_1, t_2, t_3)$ by $(E_2, E_4, E_6)$ in the Ramanujan vector field (2.2), for any non-negative integer $r$ and any $f \in \mathcal{M}_r$, we get $f' = D f$ where the differential operator $D$ on $\mathcal{M} = \mathbb{C}[E_2, E_4, E_6]$ is given as follows:

\[(2.4) \quad D : \bar{\mathcal{M}}_r \rightarrow \bar{\mathcal{M}}_{r+2}; \quad f' = D f = \frac{E_2^2 - E_4}{12} \frac{\partial f}{\partial E_2} + \frac{E_2 E_4 - E_6}{3} \frac{\partial f}{\partial E_4} + \frac{E_2 E_6 - E_4^2}{2} \frac{\partial f}{\partial E_6},\]

\[5\]
which is a degree 2 differential operator. Therefore, for any \( f \in \mathcal{M}_r \) since \( \frac{\partial f}{\partial E_2} = 0 \), we can express the Ramanujan-Serre derivation (2.3) as follows:

\[
\partial f = -\frac{E_6}{3} \frac{\partial f}{\partial E_4} - \frac{E_4^2}{2} \frac{\partial f}{\partial E_6},
\]

from which we get that the Ramanujan-Serre derivation \( \partial \) kills the terms which include \( E_2 \). Zagier [Zag94] in 1994, based on the works of Rankin [Ran56] and Cohen [Coh77], for any non-negative integer \( k \) introduced the \( k \)-th Rankin-Cohen bracket \([\cdot, \cdot]_k\) defined as follows:

\[
[f, g]_k := \sum_{i+j=k} (-1)^i \binom{k + r - 1}{i} \binom{k + s - 1}{j} f^{(i)} g^{(j)}, \quad f \in \mathcal{M}_r \text{ and } g \in \mathcal{M}_s,
\]

where \( f^{(j)} \) and \( g^{(j)} \) refer to the \( j \)-th derivative of \( f \) and \( g \) with respect to the derivation given in (2.4). It was proven by Cohen that \([f, g]_k \in \mathcal{M}_{r+s+2k}\). Note that the 0-th bracket is considered as usual multiplication, i.e. \([f, g]_0 = fg\). We list some algebraic properties of the Rankin-Cohen brackets given in [Zag94] below, in which we assume \( f \in \mathcal{M}_r \), \( g \in \mathcal{M}_s \) and \( h \in \mathcal{M}_l \):

\[
[f, g]_k = (-1)^k [g, f]_k, \quad \forall k \geq 0,
\]

\[
[[f, g], h]_0 = [f, [g, h]]_0, \quad \forall k \geq 0,
\]

\[
[f, 1]_0 = [1, f]_0 = f, \quad [f, 1]_k = [1, f]_k = 0, \quad \forall k > 0,
\]

\[
[[f, g], h]_1 = [[g, h], f] + [[h, f], g] = 0,
\]

\[
[[f, g], h]_1 + [[g, h], f] + [[h, f], g] = 0,
\]

\[
l([[f, g], h]_1 + 2[[g, h], f] + [[h, f], g])_0 = 0,
\]

\[
[[f, g], h]_1 = [[g, h], f]_0 - [[h, f], g]_1,
\]

\[
(r + s + l)([[f, g], h]_1 + 2[[g, h], f] + [[h, f], g])_0 = (r + 1)(l + 1) [[f, g], h]_2_0
\]

\[
+ (r + 1)(r + s + 1) [[g, h], f]_2_0 + (s + 1)(r + s + 1) [[h, f], g]_0_0
\]

\[
(r + s + l + 1) ([[f, g], h]_2 + s + 1)(r + s + 1) [[f, g], h]_2_0
\]

\[
- (r + 1)(r + s + 1) [[g, h], f]_1 - (s + 1)(r + s + 1) [[h, f], g]_0_2
\]

\[
[[f, g], h]_1 = [[g, h], f]_2 - [[h, f], g]_2 + [[g, h], f]_0 - [[h, f], g]_0.
\]

Zagier defined a Rankin-Cohen algebra over a field \( k \) (of characteristic zero) as a graded \( k \)-vector space \( M = \bigoplus_{r \geq 0} M_r \) with \( M_0 = k \cdot 1 \) and \( \dim_k M_r \) finite for all \( r \), together with bilinear operations \([\cdot, \cdot]_k : M_r \otimes M_s \to M_{r+s+2k}\), \( r, s, k \geq 0 \), which satisfy (2.7)-(2.17) and all the other algebraic identities satisfied by the Rankin-Cohen brackets given in (2.6). A basic example of Rankin-Cohen algebras can be constructed as follows, and for future uses we state it as a remark.

**Remark 2.1.** Let \( M \) be a commutative and associative graded algebra with unit over the field \( k \) together with a derivation \( D \) of degree 2, i.e. \( D : M_r \to M_{r+2} \) for all integers \( r \geq 0 \).

Given \( f \in \mathcal{M}_r \) and \( g \in \mathcal{M}_s \), for any non-negative integer \( k \) define the Rankin-Cohen bracket \([f, g]_{D,k}\) as follows:

\[
[f, g]_{D,k} := \sum_{i+j=k} (-1)^i \binom{k + r - 1}{i} \binom{k + s - 1}{j} f^{(i)} g^{(j)} \in M_{r+s+2k},
\]
where \( f^{(j)} = D^j f \) and \( g^{(j)} = D^j g \) are the \( j \)-th derivative of \( f \) and \( g \) with respect to the derivation \( D \). Then \((M, [\cdot, \cdot]_{D,k})\) is a Rankin-Cohen algebra which is called the standard Rankin-Cohen algebra.

For example, \((\tilde{M}, [\cdot, \cdot]_{D,k})\) is a standard Rankin-Cohen algebras. Hence, \((M, [\cdot, \cdot]_{D,k})\) is a sub Rankin-Cohen algebra of \((\tilde{M}, [\cdot, \cdot]_{D,k})\), but it is not a standard Rankin-Cohen algebras, since \( M \) is not closed under \( D \). We can relate \((M, [\cdot, \cdot]_{D,k})\) with another bilinear form which is defined using the Ramanujan-Serre derivation \( \partial \). This fact, in a more general version, is given in the following proposition, and since a part of its proof will be needed, we summarize the proof and for more details the reader is referred to the given Ref.

**Proposition 2.1.** ([Zag94] Proposition 1)] Let \( M \) be a commutative and associative graded \( k \)-algebra with \( M_0 = k \cdot 1 \) together with a derivation \( \partial \) of degree 2 on \( M \), and let \( \Lambda \in M_4 \). For any \( k \geq 0 \) define brackets \([\cdot, \cdot]_{\partial,\Lambda,k}\) by

\[
[f, g]_{\partial,\Lambda,k} = \sum_{i+j=k} (-1)^j \binom{k+r-1}{i} \binom{k+s-1}{j} f^{(j)} g^{(i)} \in M_{r+s+2k},
\]

where \( f \in M_r \), \( g \in M_s \), and \( f^{(j)} \in M_{r+2j} \), \( g^{(i)} \in M_{s+2i} \) are defined recursively as follows

\[
f^{(j+1)} = \partial f^{(j)} + j(j+r+1)f^{(j-1)}, \quad g^{(i+1)} = \partial g^{(i)} + i(i+s+1)g^{(i-1)},
\]

with initial conditions \( f^{(0)} = f \), \( g^{(0)} = g \). Then \((M, [\cdot, \cdot]_{\partial,\Lambda,k})\) is a Rankin-Cohen algebra.

**Sketch of proof.** The only way is to embed \((M, [\cdot, \cdot]_{\partial,\Lambda,k})\) into a standard Rankin-Cohen algebra \((R, [\cdot, \cdot]_{D,k})\) for some larger \( R \) with derivation \( D \). Indeed, it is taken \( R = M[\lambda] := M \otimes_k k[\lambda] \), where \( \lambda \notin M_2 \) has degree 2, and the derivation \( D \) is defined on the generators of \( R \) as follows:

\[
D(f) = \partial(f) + k\lambda f \in R_{k+2}, \quad \text{for any } f \in M_k, \quad \text{and } D(\lambda) = \lambda + \lambda^2 \in R_4,
\]

which can be extended uniquely as a derivation on \( R \). Then, for any \( k \geq 0 \) and any \( f \in M_r \), \( g \in M_s \), for all \( r, s \in \mathbb{Z}_{\geq 0} \), we have:

\[
[f, g]_{D,k} = [f, g]_{\partial,\Lambda,k} \quad (\text{see the proof of [Zag94] Proposition 1}).
\]

This completes the proof, since \( M \) is obviously closed under the brackets \([\cdot, \cdot]_{\partial,\Lambda,k}\).

A Rankin-Cohen algebra \((M, [\cdot, \cdot]_{\Lambda})\) is called canonical if its brackets are given as in Proposition 2.1 for some derivation \( \partial \) of degree 2 on \( M \) and some element \( \Lambda \in M_4 \), i.e., \([\cdot, \cdot]_{\Lambda} = [\cdot, \cdot]_{\partial,\Lambda,k} \). For example, \((M, [\cdot, \cdot]_{\Lambda})\) is a canonical Rankin-Cohen algebra with the Ramanujan-Serre derivation \( \partial \) and \( \Lambda = \frac{1}{12} E_4 \).

### 3 GMCD for the Dwork family

In Subsections 3.1 and 3.2 we first recall some relevant facts and terminologies from [MN21, Nik20], and for more details one is referred to the same references. Then, we will observe some new important results in Subsection 3.3 which will be used in the subsequent section. In this manuscript for any positive integer \( n \) we fix the notation \( m := \frac{n+1}{2} \) if \( n \) is odd, and \( m := \frac{n}{2} \) if \( n \) is even.
3.1 Moduli spaces and modular vector field $R$

This subsection is based on [MN21]. Let $W_z$, for $z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, be an $n$-dimensional hypersurface in $\mathbb{P}^{n+1}$ given by the so-called Dwork family:
\[
f_z(x_0, x_1, \ldots, x_{n+1}) := x_0^{n+2} + x_1^{n+2} + x_2^{n+2} + \cdots + x_{n+1}^{n+2} - (n + 2)x_0x_1x_2 \cdots x_{n+1} = 0.
\]
$W_z$ represents a family of CY $n$-folds. The group $G := \{(\zeta_0, \zeta_1, \ldots, \zeta_{n+1}) \mid \zeta_i^{n+2} = 1, \zeta_0\zeta_1 \cdots \zeta_{n+1} = 1\}$, acts canonically on $W_z$ as
\[
(\zeta_0, \zeta_1, \ldots, \zeta_{n+1}).(x_0, x_1, \ldots, x_{n+1}) = (\zeta_0x_0, \zeta_1x_1, \ldots, \zeta_{n+1}x_{n+1}).
\]

We obtain the variety $X = X_z$, $z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, by desingularization of the quotient space $W_z/G$ (for more details see [GMP95]). From now on, we call $X = X_z$ the mirror variety\footnote{The reason for this name is that due to argument given in [GP90], the family $X_z$ belongs to the mirror parameter space.} which is also a CY $n$-fold. It is known that $\dim(H^n_{dR}(X)) = n + 1$ and all Hodge numbers $h^{ij}, i + j = n$, of $X$ are one.

We denote by $S$ the moduli of the pairs $(X, \alpha_1)$, where $X$ is an $n$-dimensional mirror variety and $\alpha_1$ is a holomorphic $n$-form on $X$. We know that the family of mirror varieties $X_z$ is a one parameter family and the $n$-form $\alpha_1$ is unique, up to multiplication by a constant, therefore $\dim(S) = 2$. Analogous to the construction of $X_z$, let $X_{t_1, t_{n+2}}$, $(t_1, t_{n+2}) \in \mathbb{C}^2 \setminus \{(t_1^{n+2} - t_{n+2})t_{n+2} = 0\}$, be the mirror variety obtained by the quotient and desingularization of the CY $n$-folds given by
\[
f_{t_1, t_{n+2}}(x_0, x_1, \ldots, x_{n+1}) := t_{n+2}x_0^{n+2} + x_1^{n+2} + x_2^{n+2} + \cdots + x_{n+1}^{n+2} - (n + 2)t_1x_0x_1x_2 \cdots x_{n+1} = 0.
\]

We fix two $n$-forms $\eta$ and $\omega_1$ in the families $X_z$ and $X_{t_1, t_{n+2}}$, respectively, such that in the affine space $\{x_0 = 1\}$ are given as follows:
\[
\eta := \frac{dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{n+1}}{df_z}, \quad \omega_1 := \frac{dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{n+1}}{df_{t_1, t_{n+2}}}.
\]

Any element of $S$ is in the form $(X_z, a\eta)$ where $a$ is a non-zero constant. The pair $(X_z, a\eta)$ can be identified by $(X_{t_1, t_{n+2}}, \omega_1)$ as follows:
\[
(X_z, a\eta) \mapsto (X_{t_1, t_{n+2}}, \omega_1), \quad (t_1, t_{n+2}) = (a^{-1}, za^{-(n+2)}),
\]
\[
(X_{t_1, t_{n+2}}, \omega_1) \mapsto (X_z, t_1^{-1}\eta), \quad z = \frac{t_{n+2}}{t_1^{n+2}}.
\]

Hence, $(t_1, t_{n+2})$ construct a chart for $S$; in the other words
\[
S = \text{Spec}(\mathbb{C}[t_1, t_{n+2}, \frac{1}{(t_1^{n+2} - t_{n+2})t_{n+2}}]),
\]
and the morphism $X \to S$ is the universal family of $(X, \alpha_1)$. Let $\nabla : H^n_{dR}(X/S) \to \Omega^1_S \otimes_{\mathbb{C}} H^n_{dR}(X/S)$ be the Gauss-Manin connection of the two parameter family of varieties $X/S$. We define the $n$-forms $\omega_i, \ i = 1, 2, \ldots, n+1$, as follows
\[
\omega_i := (\nabla \frac{\partial}{\partial t_1})^{i-1}(\omega_1),
\]
in which \( \frac{\partial}{\partial t} \) is considered as a vector field on the moduli space \( S \). Then \( \omega := \{ \omega_1, \omega_2, \ldots, \omega_{n+1} \} \) forms a basis of \( H^{n}_{\text{dR}}(X) \) which is compatible with its Hodge filtration, i.e.,

\[
\omega_i \in F^{n+1-i} \setminus F^{n+2-i}, \quad i = 1, 2, \ldots, n + 1,
\]

where \( F^i \) is the \( i \)-th piece of the Hodge filtration of \( H^{n}_{\text{dR}}(X) \). We can write the Gauss-Manin connection of \( X/S \) in the basis \( \omega \) as follows

\[
\nabla \omega = B \omega, \quad \omega = (\omega_1 \quad \omega_2 \quad \ldots \quad \omega_{n+1})^T.
\]

If we denote by \( B[i,j] \) the \((i,j)\)-th entry of the Gauss-Manin connection matrix \( B \), then we obtain:

\[
B[i,i] = -\frac{i}{(n+2)t_{n+2}} dt_{n+2}, \quad 1 \leq i \leq n,
\]

\[
B[i,i+1] = \frac{t_1}{(n+2)t_{n+2}} dt_{n+2}, \quad 1 \leq i \leq n,
\]

\[
B[n+1,j] = \frac{S_2(n+2,j)t_1^n}{2t_{n+2}^n - t_{n+2}} dt_1 + \frac{S_2(n+2,j)t_1^{j+1}}{(n+2)t_{n+2}(t_{n+2}^n - t_{n+2})} dt_{n+2}, \quad 1 \leq j \leq n,
\]

\[
B[n+1,n+1] = \frac{S_2(n+2,n+1)t_1^{n+1}}{2t_{n+2}^{n+1} - t_{n+2}} dt_1 + \frac{n(n+1)t_{n+2}^{n+2} + (n+1)t_{n+2}^n}{(n+2)t_{n+2}(t_{n+2}^{n+1} - t_{n+2})} dt_{n+2},
\]

where \( S_2(r,s) \) is the Stirling number of the second kind defined by

\[
S_2(r,s) := \frac{1}{s!} \sum_{i=0}^{s} (-1)^i \binom{s}{i} (s-i)^r,
\]

and the rest of the entries of \( B \) are zero. For any \( \xi_1, \xi_2 \in H^{n}_{\text{dR}}(X) \), in the context of the de Rham cohomology, the intersection form of \( \xi_1 \) and \( \xi_2 \), denoted by \( \langle \xi_1, \xi_2 \rangle \), is given as

\[
\langle \xi_1, \xi_2 \rangle := \frac{1}{(2\pi i)^n} \int_X \xi_1 \wedge \xi_2,
\]

which is a non-degenerate \((-1)^n\)-symmetric form. We obtain

\[
\langle \omega_i, \omega_j \rangle = 0, \quad \text{if} \quad i + j \leq n + 1,
\]

\[
\langle \omega_1, \omega_{n+1} \rangle = (-n + 2)^n \frac{c_n}{t_1^{n+2} - t_{n+2}}, \quad \text{where} \quad c_n \quad \text{is a constant},
\]

\[
\langle \omega_j, \omega_{n+2-j} \rangle = (-1)^{j-1} \langle \omega_1, \omega_{n+1} \rangle, \quad \text{for} \quad j = 1, 2, \ldots, n + 1.
\]

On account of these relations, we can determine all the rest of \( \langle \omega_i, \omega_j \rangle \)'s in a unique way. If we set \( \Omega = \Omega_n := (\langle \omega_i, \omega_j \rangle)_{1 \leq i, j \leq n+1} \) to be the intersection form matrix in the basis \( \omega \), then we have

\[
d\Omega = B\Omega + \Omega B^T.
\]

For any positive integer \( n \) by moduli space \( \mathcal{T} = \mathcal{T}_n \) of enhanced mirror varieties we mean the moduli of the pairs \( (X, [\alpha_1, \cdots, \alpha_n, \alpha_{n+1}]) \), where \( X \) is an \( n \)-dimensional mirror variety and \( \{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\} \) constructs a basis of \( H^{n}_{\text{dR}}(X) \) satisfying the properties

\[
\alpha_i \in F^{n+1-i} \setminus F^{n+2-i}, \quad i = 1, \cdots, n, n + 1,
\]
and

\[(3.19)\quad [(\alpha_i, \alpha_j)]_{1 \leq i, j \leq n+1} = \Phi_n.\]

Here $\Phi = \Phi_n$ is the following constant $(n + 1) \times (n + 1)$ matrix:

\[(3.20)\quad \Phi_n := \begin{pmatrix} 0_m & J_m \\ -J_m & 0_m \end{pmatrix} \text{ if } n \text{ is odd, and } \Phi_n := J_{n+1} \text{ if } n \text{ is even,}\]

where by $0_k, k \in \mathbb{N}$, we mean a $k \times k$ block of zeros, $J_1 = 1$ and

\[(3.21)\quad J_k := \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}, \text{ for } k > 1.\]

In [MN21] the universal family $\pi : X \to T$ together with the global sections $\alpha_i, i = 1, \cdots, n + 1$, of the relative algebraic de Rham cohomology $H^n_{\text{dR}}(X/T)$ was constructed, and in its main theorem we observed that:

**Theorem 3.1.** ([MN21] Theorem 1.1) There exist a unique vector field $R = R_n \in \mathfrak{X}(T)$, and unique regular functions $Y_i \in \mathcal{O}_T, 1 \leq i \leq n - 2$, such that:

\[(3.22)\quad \nabla_R \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \\ \alpha_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathcal{Y}_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \mathcal{Y}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mathcal{Y}_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \\ \alpha_{n+1} \end{pmatrix},\]

and $Y\Phi + \Phi Y^r = 0$.

Here $\mathcal{O}_T$ refers to the $\mathbb{C}$-algebra of regular functions on $T$, and $\nabla_R$ stands for the algebraic Gauss-Manin connection

\[\nabla : H^n_{\text{dR}}(X/T) \to \Omega^1_T \otimes \mathcal{O}_T H^n_{\text{dR}}(X/T),\]

composed with the vector field $R \in \mathfrak{X}(T)$, in which $\Omega^1_T$ refers to the $\mathcal{O}_T$-module of differential 1-forms on $T$. We call $R$ as modular vector field attached to Dwork family. Moreover, we found that:

\[(3.23)\quad d = d_n := \dim(T) = \begin{cases} \frac{(n+1)(n+3)}{4} + 1, & \text{if } n \text{ is odd} \\ \frac{n(n+2)}{4} + 1, & \text{if } n \text{ is even} \end{cases}.\]

The above theorem is the key tool of GMCD. In the GMCD viewpoint, the vector field $Ra$ given in (2.2), up to multiplying the coordinates by constants $(t_1, t_2, t_3) = (12t_1, 12t_2, \frac{13}{8}t_3)$, is the unique vector field that satisfies

\[(3.24)\quad \nabla_{Ra} \alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \alpha,\]
where $\alpha = (\alpha_1 \alpha_2)^{tr}$ and $\nabla$ is the Gauss-Manin connection of the universal family of elliptic curves

$$y^2 = 4(x - t_1)^3 - t_2(x - t_1) - t_3, \quad \alpha_1 = \left[\frac{dx}{y}\right], \quad \alpha_2 = \left[\frac{x dx}{y}\right], \quad \text{with} \quad 27t_3^2 - t_2^3 \neq 0. \quad (3.25)$$

We can generalize the notion of the Ramanujan-Serre derivation $2.5$ and the Rankin-Cohen bracket $2.6$ for the modular vector fields $R = R_n$ using an analogous procedure explained for the Ramanujan vector field $Ra$, which will be treated in Section 5.

Next we are going to present a chart for the moduli space $T$. In order to do this, let $S = (s_{ij})_{1 \leq i, j \leq n+1}$ be a lower triangular matrix, whose entries are indeterminates $s_{ij}$, $i \geq j$ and $s_{11} = 1$. We define

$$\left( \begin{array}{ccccc} \alpha_1 & \alpha_2 & \ldots & \alpha_{n+1} \end{array} \right)^{tr}_\alpha = S \left( \begin{array}{ccccc} \omega_1 & \omega_2 & \ldots & \omega_{n+1} \end{array} \right)^{tr}_\omega,$$

which implies that $\alpha$ forms a basis of $H^1_{dR}(X)$ compatible with its Hodge filtration. We would like that $(X, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}])$ be a member of $T$, hence it has to satisfy $(\langle \alpha_i, \alpha_j \rangle)_{1 \leq i, j \leq n+1} = \Phi$, from what we get the following equation

$$S \Omega S^{tr} = \Phi. \quad (3.26)$$

Using this equation we can express $d_0 := \frac{(n+2)(n+1)}{2} - d - 2$ numbers of parameters $s_{ij}$’s in terms of other $d - 2$ parameters that we fix them as independent parameters. For simplicity we write the first class of parameters as $t_1, t_2, \ldots, t_{d_0}$ and the second class as $t_2, t_3, \ldots, t_{n+1}, t_{n+3}, \ldots, t_d$. We put the independent parameters $t_i$ inside $S$ according to the following rule which is not canonical: $t_i$’s are written in $S$ from left to right and top to bottom in the entries $(i, j)$ for $i + j < n + 2$ if $n$ is even and $i + j \leq n + 2$ if $n$ is odd. The position of $t_i$’s inside $S$ can be chosen arbitrarily. For instance, for $n = 1, 2, 3, 4, 5$ we have:

$$\left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ t_2 & 0 & 0 & 0 \\ t_4 & t_5 & 0 & 0 \\ t_7 & t_8 & t_9 & 0 \\ t_{11} & t_{12} & t_{13} & t_{14} \end{array} \right), \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ t_2 & t_3 & 0 & 0 \\ t_4 & t_5 & t_6 & 0 \\ t_8 & t_9 & t_{10} & t_{11} \\ t_{13} & t_{14} & t_{15} & t_{16} \end{array} \right), \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ t_2 & t_3 & t_4 & 0 \\ t_5 & t_6 & t_7 & t_8 \\ t_{10} & t_{11} & t_{12} & t_{13} \\ t_{16} & t_{17} & t_{18} & t_{19} \end{array} \right) \cdot \left( \begin{array}{cccc} t_1 & 0 & 0 & 0 \\ t_2 & t_3 & 0 & 0 \\ t_4 & t_5 & t_6 & 0 \\ t_7 & t_8 & t_9 & t_{10} \\ t_{11} & t_{12} & t_{13} & t_{14} \end{array} \right).$$

Note that we have already used $t_1, t_{n+2}$ as coordinate system of $S$. In particular we find:

$$s_{(n+2-i)(n+2-i)} = \frac{(-1)^{n+i+1} t_{n+2}^{n+2} - t_{n+2}^n}{c_n(n+2)^n s_{ii}}, \quad 1 \leq i \leq m. \quad (3.27)$$

In this way, $t := (t_1, t_2, \ldots, t_d)$ forms a chart for the moduli space $T$, and in fact

$$T = \text{Spec}(\mathbb{C}[t_1, t_2, \ldots, t_d, \frac{1}{t_{n+2}(t_{n+2} - t_{n+2}^2)}]), \quad (3.28)$$

$$\mathcal{O}_T = \mathbb{C}[t_1, t_2, \ldots, t_d, \frac{1}{t_{n+2}(t_{n+2} - t_{n+2}^2)}]. \quad (3.29)$$

Here, $\tilde{t}$ is the product of $m - 1$ independent parameters which are located in the main diagonal of $S$. From now on, we alternately use either $s_{ij}$’s, or $t_i$’s and $t_j$’s to refer the entries of $S$. If we denote by $A$ the Gauss-Manin connection matrix of the family $X/T$ written in the basis $\alpha$, i.e., $\nabla \alpha = A \alpha$, then we calculate $A$ as follows:

$$A = (dS + S \cdot B) S^{-1}. \quad (3.30)$$
If for any vector field \( E \in \mathfrak{X}(T) \) we define the \emph{Gauss-Manin connection matrix} attached to \( E \) as \((n + 1) \times (n + 1)\) matrix \( A_E \) given by:

\[
(3.31) \quad \nabla_{E\alpha} = A_{E\alpha},
\]

then from \((3.30)\) we obtain:

\[
(3.32) \quad \dot{S}_E = A_E S - S B(E),
\]

where \( \dot{S}_E = dS(E) \) and \( \dot{x} := dx(E) \) is the derivative of the function \( x \) along the vector field \( E \) in \( T \). Note that equalities corresponding to \((1, 1)\)-th and \((1, 2)\)-th entries of \((3.32)\) give us respectively \( t_1 \) and \( t_{n+2} \), and any \( t_i, 1 \leq i \leq d, i \neq 1, n + 2 \), corresponds to only one \( \dot{s}_{jk}, 1 \leq j, k \leq n + 1 \). In the following remarks we recall some useful results deduced from the proof of Theorem \((3.1)\) in \[MN21\] §7.

**Remark 3.1.** We obtain the functions \( Y_i \)'s given in \((3.22)\) as follows: if \( n \) is odd, then

\[
(3.33) \quad Y_i = -Y_{n-(i+1)} = \frac{s^{22} s^{(i+1)(i+1)}}{s^{(i+2)(i+2)}}, \quad i = 1, 2, \ldots, \frac{n-3}{2},
\]

\[
(3.34) \quad Y_{n+1} = (-1)^{\frac{3n+3}{2}} c_n (n+2)^n \frac{s^{22} s^{n+1}}{t_1^{n+2} - t_{n+2}},
\]

and if \( n \) is even, then

\[
(3.35) \quad Y_i = -Y_{n-(i+1)} = \frac{s^{22} s^{(i+1)(i+1)}}{s^{(i+2)(i+2)}}, \quad i = 1, 2, \ldots, \frac{n-2}{2}.
\]

**Remark 3.2.** Let \( E \in \mathfrak{X}(T) \). If \( \nabla_{E\alpha} = 0 \) for any \((X, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}]) \in T\), then \( E = 0 \).

We finish this subsection with the following example.

**Example 3.1.** In \[MN21\] for \( n = 1, 2 \) we found the modular vector fields \( R_1, R_2 \), respectively, as follows:

\[
(3.36) \quad R_1 : \begin{cases} \dot{t}_1 = -t_1 t_2 - 9(t_1^3 - t_3) \\ \dot{t}_2 = 81 t_1 (t_1^3 - t_3) - t_2^2 \\ \dot{t}_3 = -3 t_2 t_3 \end{cases}, \quad R_2 : \begin{cases} \dot{t}_1 = t_3 - t_1 t_2 \\ \dot{t}_2 = 2 t_2^3 - \frac{1}{2} t_2^2 \\ \dot{t}_3 = -2 t_2 t_3 + 8 t_1^3 \\ \dot{t}_4 = -4 t_2 t_4 \end{cases},
\]

where by \( \dot{t}_j \) in \( R_1 \) we mean \( \dot{t}_j = 3 \cdot q \cdot \frac{\partial}{\partial q} \), and in \( R_2 \) we mean \( \dot{t}_j = -\frac{1}{3} \cdot q \cdot \frac{\partial}{\partial q} \), and furthermore in \( R_2 \) we have the polynomial equation \( t_2^3 = 4(t_1^4 - t_4) \). For a complex number \( \tau \) with \( \text{Im} \tau > 0 \), if we set \( q = e^{2\pi i \tau} \), then we obtained the following solutions of \( R_1 \) and \( R_2 \) respectively:

\[
(3.37) \quad \begin{cases} t_1(q) = \frac{1}{3}(2\theta_3(q^2)\theta_3(q^6) - \theta_3(-q^2)\theta_3(-q^6)), \\ t_2(q) = \frac{1}{8}(E_2(q^2) - 9E_2(q^6)), \\ t_3(q) = \frac{\eta^3(q^3)}{\eta(q)}, \end{cases} \quad \begin{cases} \frac{10}{3} t_1(q^3) = \frac{1}{2} \left( \theta_3^4(q^2) + \theta_3^4(q^6) \right), \\ \frac{10}{3} t_2(q^3) = \frac{1}{2} \left( E_2(q^2) + 2E_2(q^6) \right), \\ \frac{10}{3} t_3(q^3) = \eta^8(q) \eta^6(q^2), \end{cases}
\]

in which \( \eta \) and \( \theta_3 \)'s are the classical eta and theta series given as follows:

\[
(3.38) \quad \eta(q) = q^{1/2} \prod_{k=1}^{\infty} (1 - q^k), \quad \theta_2(q) = \sum_{k=-\infty}^{\infty} q^{\frac{1}{2}(k+1)^2}, \quad \theta_3(q) = 1 + 2 \sum_{k=1}^{\infty} q^{k^2}. 
\]
3.2 AMSY-Lie algebra and $\mathfrak{sl}_2(\mathbb{C})$ Lie algebra

In this subsection we give a summary of the main results of [Nik20]. For any positive integer $n$ the algebraic group:

$$(3.39) \quad G = G_n := \{g \in \text{GL}(n+1, \mathbb{C}) \mid g \text{ is upper triangular and } g^{tr} \Phi g = \Phi\},$$

acts on the moduli space $T$ from the right, and its Lie algebra:

$$(3.40) \quad \text{Lie}(G) = \{g \in \text{Mat}(n+1, \mathbb{C}) \mid g \text{ is upper triangular and } g^{tr} \Phi + \Phi g = 0\},$$

is a $d - 1$ dimensional Lie algebra with the canonical basis consisting of $g_{ab}$'s, $1 \leq a \leq m$, $a \leq b \leq 2m+1-a$, given as follows: if $n$ is odd, then

$$g_{ab} = (g_{kl})_{(n+1) \times (n+1)}, \quad \text{where} \quad \begin{cases} g_{ab} = 1, & g_{(n+2-b)(n+2-a)} = -1, \text{when } b \leq m, \\ g_{ab} = g_{(n+2-b)(n+2-a)} = 1, & \text{when } b \geq m+1, \\ \text{and the rest of the entries of } g_{ab} \text{ are zero.} \end{cases}$$

and if $n$ is even, then:

$$g_{ab} = (g_{kl})_{(n+1) \times (n+1)}, \quad \text{such that} \quad \begin{cases} g_{ab} = 1, & g_{(n+2-b)(n+2-a)} = -1, \\ \text{and the rest of the entries of } g_{ab} \text{ are zero.} \end{cases}$$

The following theorem was proved in [Nik20].

**Theorem 3.2.** ([Nik20 Theorem 1.2]) For any $g \in \text{Lie}(G)$, there exists a unique vector field $R_g \in \mathfrak{X}(T)$ such that:

$$(3.43) \quad A_{R_g} = g^{tr},$$

i.e., $\nabla_{R_g} \alpha = g^{tr} \alpha$.

This theorem yields that the Lie algebra generated by $R_g$'s, $1 \leq a \leq m$, $a \leq b \leq 2m+1-a$, in $\mathfrak{X}(T)$ with the Lie bracket of the vector fields is isomorphic to $\text{Lie}(G)$ with the Lie bracket of the matrices. Hence, we use $\text{Lie}(G)$ alternately either as a Lie subalgebra of $\mathfrak{X}(T)$ or as a Lie subalgebra of $\text{Mat}(n+1, \mathbb{C})$.

By $\text{AMSY-Lie algebra}[\Phi]$ we mean the $\mathcal{O}_T$-module generated by $\text{Lie}(G)$ and the modular vector field $R$ in $\mathfrak{X}(T)$. In what follows, $\delta_j^k$ denotes the Kronecker delta, $\varrho(n) = 1$ if $n$ is an odd integer, and $\varrho(n) = 0$ if $n$ is an even integer, $Y_j$'s, $1 \leq j \leq n-2$, are the functions given in Theorem 3.1 and besides them we let $Y_0 = -Y_{n-1} := 1$. The following theorem determines the Lie bracket of $\Phi$, which was demonstrated in [Nik20].

**Theorem 3.3.** ([Nik20 Theorem 1.3]) The following hold:

$$(3.44) \quad [R, R_{g_{11}}] = R,$$

$$(3.45) \quad [R, R_{g_{21}}] = -R,$$

$$(3.46) \quad [R, R_{g_{aa}}] = 0, \quad 3 \leq a \leq m,$$

$$(3.47) \quad [R, R_{g_{ab}}] = \Psi_1^{ab}(Y) R_{g_{(a+1)b}} + \Psi_2^{ab}(Y) R_{g_{a(b-1)}}; \quad 1 \leq a \leq m, \quad a+1 \leq b \leq 2m+1-a,$$

where

$$\begin{align*}
\Psi_1^{ab}(Y) &:= (1 + \varrho(n) \delta_{a+b}^{2m} - \delta_{a+b}^{2m+1})Y_{a-1}, \\
\Psi_2^{ab}(Y) &:= (1 - 2\varrho(n) \delta_{b}^{m+1})Y_{n+1-b}.
\end{align*}$$

\footnote{The AMSY-Lie algebra was discussed for the first time in [AMSY16] for non-rigid compact CY 3-folds, and in [AV21] it is established for mirror elliptic K3 surfaces. Note that the AMSY-Lie algebra is called Gauss-Manin Lie algebra by authors of [AV21].}
If \( n = 1, 2 \), then we see that \( \mathfrak{S} \) is isomorphic to \( \mathfrak{sl}_2(\mathbb{C}) \). In general, for \( n \geq 3 \) we have a copy of \( \mathfrak{sl}_2(\mathbb{C}) \) as a Lie subalgebra of \( \mathfrak{S} \) which contains the modular vector field \( \mathbf{R} \) and we state it in the following theorem from Ref. [Nik20].

**Theorem 3.4.** ([Nik20] Theorem 1.3) Let us define the vector fields \( \mathbf{H} \) and \( \mathbf{F} \) as follows:

1. if \( n = 1 \), then \( \mathbf{H} := -\mathbf{R}_{g_{11}} \) and \( \mathbf{F} := \mathbf{R}_{g_{12}} \),
2. if \( n = 2 \), then \( \mathbf{H} := -2\mathbf{R}_{g_{11}} \) and \( \mathbf{F} := 2\mathbf{R}_{g_{12}} \),
3. if \( n \geq 3 \), then \( \mathbf{H} := \mathbf{R}_{g_{22}} - \mathbf{R}_{g_{11}} \) and \( \mathbf{F} := \mathbf{R}_{g_{12}} \).

Then the Lie algebra generated by the vector fields \( \mathbf{R}, \mathbf{H}, \mathbf{F} \) in \( \mathfrak{S} \subset \mathfrak{X}(\mathbb{T}) \) is isomorphic to \( \mathfrak{sl}_2(\mathbb{C}) \); indeed we get:

\[
\left[ \mathbf{R}, \mathbf{F} \right] = \mathbf{H}, \quad \left[ \mathbf{H}, \mathbf{R} \right] = 2\mathbf{R}, \quad \left[ \mathbf{H}, \mathbf{F} \right] = -2\mathbf{F}.
\]

According to Theorem 3.4 if \( n = 1, 2 \), then \( \mathfrak{S} \) is isomorphic to \( \mathfrak{sl}_2(\mathbb{C}) \) (see Example 5.1), and for \( n \geq 3 \) the Lie subalgebra of \( \mathfrak{S} \) generated by \( \mathbf{R}, \mathbf{H} := \mathbf{R}_{g_{22}} - \mathbf{R}_{g_{11}} \) and \( \mathbf{F} := \mathbf{R}_{g_{12}} \) is isomorphic to \( \mathfrak{sl}_2(\mathbb{C}) \). Using the equalities corresponding to \( (1, 1) \)-th and \( (1, 2) \)-th entries of (3.32) for the vector fields \( \mathbf{R}_{g_{ab}} \)'s we obtain the diagonal matrix \( \mathbf{B}(\mathbf{R}_{g_{11}}) = \text{diag}(1, 2, \ldots, n+1) \) and the null matrices \( \mathbf{B}(\mathbf{R}_{g_{ab}}) = 0 \), for \( 1 \leq a \leq m, a \leq b \leq 2m+1-a, b \neq 1 \) (see [Nik20] § 4.4). Due to these facts and again (3.32), we can find \( \hat{\mathbf{S}}_{\mathbf{R}_{g_{ab}}} \)'s, and consequently we obtain \( \mathbf{R}_{g_{ab}} \)'s. In particular, knowing that \( \hat{\mathbf{S}}_{\mathbf{H}} = \hat{\mathbf{S}}_{\mathbf{R}_{g_{22}}} - \hat{\mathbf{S}}_{\mathbf{R}_{g_{11}}} \), we get \( dt_1(\mathbf{H}) = t_1, dt_{n+2}(\mathbf{H}) = (n+2)t_{n+2} \), and hence

\[
\hat{\mathbf{S}}_{\mathbf{H}} = \begin{pmatrix}
0 & 2s_{22} & 0 & 0 & \cdots & 0 & 0 & 0 \\
s_{31} & 2s_{32} & 3s_{33} & 0 & \cdots & 0 & 0 & 0 \\
s_{41} & 2s_{42} & 3s_{43} & 4s_{44} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
s_{(n-1)1} & 2s_{(n-1)2} & 3s_{(n-1)3} & 4s_{(n-1)4} & \cdots & (n-1)s_{(n-1)(n-1)} & 0 & 0 \\
s_{n1} & 2s_{n2} & 3s_{n3} & 4s_{n4} & \cdots & ns_{n(n-1)} & (n-1)s_{nn} & 0 \\
2s_{n1} & 3s_{n2} & 4s_{n3} & 5s_{n4} & \cdots & ns_{n(n-1)} & (n+1)s_{n(n+1)} & (n+2)s_{n(n+1)(n+1)}
\end{pmatrix}.
\]

Thus, for an even integer \( n \geq 5 \) we get:

\[
\mathbf{H} = t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2} + 3t_3 \frac{\partial}{\partial t_3} + \sum_{i=4}^{d-1} w_i t_i \frac{\partial}{\partial t_i} + (n+2)t_{n+2} \frac{\partial}{\partial t_{n+2}} + \frac{n+2}{2} t_{d+1} \frac{\partial}{\partial t_{d+1}},
\]

(3.52) \[
\mathbf{F} = \frac{\partial}{\partial t_2},
\]

with \( t_{d+1}^2 = \frac{s_{n+2}^2}{c_n(n+2)} = \frac{(-1)^{\frac{n}{2}}}{c_n(n+2)}(t_{n+2}^2 - t_{n+2}) \) (see (3.27)), and for an odd integer \( n \geq 5 \) we obtain:

\[
\mathbf{H} = t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2} + 3t_3 \frac{\partial}{\partial t_3} + \sum_{i=4}^{d-3} w_i t_i \frac{\partial}{\partial t_i} + (n+2)t_{n+2} \frac{\partial}{\partial t_{n+2}} + t_{d-1} \frac{\partial}{\partial t_{d-1}} + 2t_d \frac{\partial}{\partial t_d},
\]

(3.54) \[
\mathbf{F} = \frac{\partial}{\partial t_2} - t_{d-2} \frac{\partial}{\partial t_d}.
\]

In both equations (3.51) and (3.53) we have \( w_i = k \) if \( t_i = s_{jk} \) for some \( 1 \leq j, k \leq n+1 \), i.e., \( w_i \) is the number of the column of the entry \( t_i \). Note that \( \mathbf{H} \) and \( \mathbf{F} \) have been computed.
Let us attach to any \( F \) and \( H \) founded above for the cases \( n \geq 5 \). Hence, in general we can write \( H \) as:

\[
(3.55) \quad H = \sum_{i=1}^{d} w_i t_i \frac{\partial}{\partial t_i},
\]

where \( w_i \)'s are non-negative integers.

**Remark 3.3.**

1. If \( n = 1 \), then \( w_1 = 1, w_2 = 2, w_3 = 3 \).

2. If \( n = 2 \), then \( w_1 = 2, w_2 = 2, w_4 = 8 \).

3. If \( n = 3 \), then \( w_1 = 1, w_2 = 2, w_3 = 3, w_4 = 0, w_5 = 5, w_6 = 1, w_7 = 2 \).

4. If \( n \geq 4 \) is an even integer, then \( w_1 = 1, w_2 = 2, w_3 = 3, w_{n+2} = n + 2, w_d = 0 \).

5. If \( n \geq 5 \) is an odd integer, then \( w_1 = 1, w_2 = 2, w_3 = 3, w_{n+2} = n + 2, w_{d-2} = 0, w_{d-1} = 1, w_d = 2 \).

3.3 \( R \) as a quasi-homogeneous vector field

Let us attach to any \( t_i \) in \( \mathcal{O}_T \) the weight \( \deg(t_i) = w_i \), in which the non-negative integers \( w_i \)'s are given in (3.55). Recall that a vector field \( E = \sum_{j=1}^{d} E_j \frac{\partial}{\partial t_j} \in \mathcal{X}(T) \), with \( E_j \in \mathcal{O}_T \), is said to be quasi-homogeneous of degree \( d \) if for any \( 1 \leq j \leq d \) we have \( \deg(E_j) = w_j + d \). Hence, on account of (3.51), (3.52), (3.53), (3.54) and Remark 3.3 the vector fields \( H \) and \( F \) are quasi-homogeneous of degree 0 and \(-2\), respectively. The vector field \( H \) is also known as the radial vector field. Moreover, in the following proposition we show that \( R \) is a quasi-homogeneous vector field as well.

**Proposition 3.1.** The modular vector field \( R \) is a quasi-homogeneous vector field of degree 2 on \( T \).

**Proof.** Due to Example 5.1 the affirmation is valid for \( n = 1, 2, 3, 4 \). Hence we suppose that \( n \geq 5 \). First note that in the proof of Theorem 3.2 (see [Nik20 § 4.1]) it is verified that the equations \( S_{\Omega} S_{\Omega} = \Phi \) and \( \dot{S}_g = A_g S - S B(g) \) are compatible for any \( g \in \text{Lie}(G) \). In particular, it holds for \( g = H \). This implies that the degree of any entry \( s_{jk} \) of \( S \), \( 2 \leq j \leq n + 1, 1 \leq k \leq j \), is equal to the integer multiple of \( s_{jk} \) in the matrix \( \dot{S}_H \), which is stated in (3.50). If we set \( R = \sum_{i=1}^{d} i \frac{\partial}{\partial t_i} \), then \( \dot{i}_i \)'s follow from

\[
(3.56) \quad \dot{S}_R = Y S - S B(R).
\]

More precisely, from the equalities corresponding to (1,1)-th and (1,2)-th entries of (3.56) we obtain:

\[
(3.57) \quad \dot{i}_1 = s_{22} - t_1 s_{12} \quad \& \quad \dot{i}_{n+2} = -(n+2)s_{21} t_{n+2}.
\]

These equalities and (3.9)-(3.12) imply:

\[
\begin{align*}
-\frac{k}{(n+2)t_{n+2}} dt_{n+2} (R) &= ks_{21}, \quad 1 \leq k \leq n, \\
\left( dt_1 - \frac{t_1}{(n+2)t_{n+2}} dt_{n+2} \right) (R) &= s_{22}, \\
\left( -S_2(n+2,j) t_1 \right) dt_1 \frac{1}{t_1^n + 2 - t_{n+2}} + \frac{S_2(n+2,j) t_1^{j+1}}{(n+2) t_{n+2} (t_1^{n+2} - t_{n+2})} dt_{n+2} (R) &= \frac{-S_2(n+2,j) t_1^{j+1} s_{22}}{t_1^{n+2} - t_{n+2}},
\end{align*}
\]

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\[
\left( -S_2(n+2,n+1)t_{n+1}^{n+1} \right) dt_1 + \frac{n(n+1)t_{n+2}^{n+2} + (n+1)t_{n+2}^{n+2}dt_{n+2}}{(n+2)t_{n+2}(t_{n+2}^{n+2} - t_{n+2})} \] (R)
\[
= (n+1)s_{21} - \frac{(n+1)(n+2)}{2} \frac{t_{n+2}^{n+1}s_{22}}{t_{n+2}^{n+2} - t_{n+2}}.
\]

Note that in the above last equality we used the fact that \( S_2(n+2,n+1) = \frac{(n+1)(n+2)}{2} \).

Therefore:

\[
B(R) = \begin{pmatrix}
    s_{21} & s_{22} & 0 & 0 & \cdots & 0 \\
    s_{21} & s_{22} + 2s_{22} & s_{22} & 0 & \cdots & 0 \\
    s_{21} & s_{22} + 2s_{22} & s_{22} + 3s_{22} & s_{22} & \cdots & 0 \\
    s_{21} & s_{22} + 2s_{22} & s_{22} + 3s_{22} & s_{22} + 4s_{22} & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    s_{21} & s_{22} + 2s_{22} & s_{22} + 3s_{22} & s_{22} + 4s_{22} & \cdots & s_{21}
\end{pmatrix},
\]

hence, \( SB(R) \) equals (3.58)

\[
\begin{pmatrix}
    s_{21} & s_{22} & 0 & 0 & \cdots & 0 \\
    s_{21} & s_{22} + 2s_{22} & s_{22} & 0 & \cdots & 0 \\
    s_{21} & s_{22} + 2s_{22} & s_{22} + 3s_{22} & s_{22} & \cdots & 0 \\
    s_{21} & s_{22} + 2s_{22} & s_{22} + 3s_{22} & s_{22} + 4s_{22} & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    s_{21} & s_{22} + 2s_{22} & s_{22} + 3s_{22} & s_{22} + 4s_{22} & \cdots & s_{21}
\end{pmatrix},
\]

in which:

\[
SB(R)[n+1,1] = s_{(n+1)}s_{21} - \frac{S_2(n+2,1)t_1s_{22}s_{(n+1)}s_{(n+1)}}{t_{n+2}^{n+2} - t_{n+2}},
\]

\[
SB(R)[n+1,j] = s_{(n+1)}j - s_{22} + js_{(n+1)}j s_{21} - \frac{S_2(n+2,j)t_j s_{22}s_{(n+1)}s_{(n+1)}}{t_{n+2}^{n+2} - t_{n+2}}, \quad 2 \leq j \leq n,
\]

\[
SB(R)[n+1,n+1] = s_{(n+1)}n + s_{(n+1)}s_{(n+1)} \left( (n+1)s_{21} - \frac{(n+1)(n+2)}{2} \frac{t_{n+2}^{n+1}s_{22}}{t_{n+2}^{n+2} - t_{n+2}} \right).
\]

Observe that

\[
(3.59) \quad YS = \begin{pmatrix}
    s_{21} & s_{22} & 0 & 0 & \cdots & 0 \\
    Y_1 s_{31} & Y_1 s_{32} & Y_1 s_{33} & 0 & \cdots & 0 \\
    Y_2 s_{41} & Y_2 s_{42} & Y_2 s_{43} & Y_2 s_{44} & \cdots & 0 \\
    Y_{n-2} s_{n+1} & Y_{n-2} s_{n+2} & Y_{n-2} s_{n+3} & Y_{n-2} s_{n+4} & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    Y_{n-2} s_{n+1} & Y_{n-2} s_{n+2} & Y_{n-2} s_{n+3} & Y_{n-2} s_{n+4} & \cdots & Y_{n-2} s_{n+1}
\end{pmatrix},
\]

and (3.58)-(3.55) imply that \( \deg(Y_1) = \deg(Y_{n-2}) = 3 \) and \( \deg(Y_j) = 2, \quad 2 \leq j \leq n-3 \). If we denote the \((i,j)\)-th entry of \( \hat{S}_R \) by \( \hat{S}_R[i,j] \), then (3.56), (3.58) and (3.59) yield

\[
\deg(\hat{S}_R[i,j]) = \deg(s_{ij}) + 2, \quad 2 \leq i \leq n+1, \quad 1 \leq j \leq i,
\]

which complete the proof. \( \square \)

**Remark 3.4.** Using the matrix \( \hat{S}_R = YS - SB(R) \) computed in the proof of the above proposition we can encounter the modular vector field \( R \) explicitly for any \( n \geq 5 \).

We can prove Proposition 3.1 in a simpler way, but we cannot compute \( Ra \) explicitly in this case. Below we give this proof as well.

**Another proof of Proposition 3.1** Since \( H = \sum_{j=1}^d w_j \frac{\partial}{\partial t_j} \) and \( \deg(t_j) = w_j \), we can easily observe that for a given \( f \in \mathcal{O}_T \) we have \( H(f) = kf \), for a \( k \in \mathbb{Z} \), if and only if \( f \)
is a quasi-homogeneous element of degree $k$ of $\mathcal{O}_\mathcal{T}$. According to $[\mathcal{H}, \mathcal{R}] = 2\mathcal{R}$, for any quasi-homogeneous element $f \in \mathcal{T}$ of degree $k \in \mathbb{Z}$ we have:

$$[\mathcal{H}, \mathcal{R}](f) = 2\mathcal{R}(f) \Rightarrow H(\mathcal{R}(f)) - \mathcal{R}(H(f)) = 2\mathcal{R}(f) \Rightarrow H(\mathcal{R}(f)) = (k + 2)\mathcal{R}(f),$$

which implies $\mathcal{R}(f)$ is a quasi-homogeneous element of degree $k + 2$, and this is equivalent to say that $\mathcal{R}$ is a quasi-homogeneous vector field of degree 2. □

The following lemma is useful for the future use.

**Lemma 3.1.** If we write

$$\mathcal{R} = \sum_{j=1}^{d} \mathcal{R}^j(t_1, t_2, \ldots, t_d) \frac{\partial}{\partial t_j}, \quad \text{with} \quad \mathcal{R}^j \in \mathcal{O}_\mathcal{T},$$

and define

$$\Lambda(t_1, t_2, \ldots, t_d) := \begin{cases} -\frac{1}{2}R^2(t_1, t_2, \ldots, t_d) - \frac{1}{4}t_2^2, & \text{if } n = 2; \\ -R^2(t_1, t_2, \ldots, t_d) - t_2^2, & \text{if } n \neq 2, \end{cases}$$

(3.60)

then $\deg(\Lambda) = 4$ and $\frac{\partial \Lambda}{\partial t_2} = 0$.

**Proof.** For $n = 1, 2, 3, 4$ the modular vector field $\mathcal{R}$ has been explicitly stated in Example 5.1 and one can easily check the truth of the statement. For $n \geq 5$ the component $\mathcal{R}^2$ of the modular vector field $\mathcal{R}$ corresponds to the $(2, 1)$-th entry of the matrix $S_R = YS - S \mathcal{B}(R)$ computed in the proof of Proposition 3.1 that yields:

$$\mathcal{R}^2(t_1, t_2, \ldots, t_d) = Y_1 t_4 - t_2^2, \quad \text{(note that } t_2 = s_{21} \text{ and } t_4 = s_{31}).$$

From (3.33) and (3.35) we get $\gamma_1 = \frac{s_{22}}{s_{33}} = \frac{t_4}{t_6}$, which implies:

$$\mathcal{R}^2(t_1, t_2, \ldots, t_d) = \frac{t_4^2 t_1}{t_6} - t_2^2.$$

Hence, for $n \geq 5$ we obtain $\Lambda = -\frac{t_4^2 t_1}{t_6}$ and the proof is complete. □

### 4 Vector field $\mathcal{D}$

Remember from Section 1 that the Ramanujan vector field $\mathcal{R}a = \frac{1}{12}(t_1^2 - t_2) \frac{\partial}{\partial t_1} + \frac{1}{3}(t_1 t_2 - t_3) \frac{\partial}{\partial t_3} + \frac{1}{2}(t_1 t_3 - t_2^2) \frac{\partial}{\partial t_2}$ along with $H = 2t_1 \frac{\partial}{\partial t_1} + 4t_2 \frac{\partial}{\partial t_2} + 6t_3 \frac{\partial}{\partial t_3}$ and $F = -12 \frac{\partial}{\partial t_1}$ generates $\mathfrak{s_l}(\mathbb{C})$ and we have $F(t_1) = -12 \neq 0$, $F(t_2) = F(t_3) = 0$. In the case of modular vector field, we observed in Theorem 3.3 that the Lie algebra generated by $\mathcal{R}, \mathcal{H}, \mathcal{F}$ is isomorphic to $\mathfrak{s_l}(\mathbb{C})$. According to Example 5.1, (3.52) and (3.54), we find the vector field $\mathcal{F}$ for any positive integer $n$ as follows:

(4.1) $\quad \mathcal{F} = (1 + \delta_2^n) \frac{\partial}{\partial t_2}, \quad \text{if } n = 1 \text{ or } n \text{ is even},$

(4.2) $\quad \mathcal{F} = \frac{\partial}{\partial t_2} - t_4 \frac{\partial}{\partial t_7}, \quad \text{if } n = 3,$

(4.3) $\quad \mathcal{F} = \frac{\partial}{\partial t_2} - t_{d-2} \frac{\partial}{\partial t_d}, \quad \text{if } n \geq 5 \text{ is odd}.$
For any \( n \) we have \( F(t_2) = 1 + \delta_2^n \neq 0 \). If \( n = 1 \) or \( n \) is even, then \( F(t_j) = 0 \) for all \( 1 \leq j \leq d \) and \( j \neq 2 \). But if \( n \geq 3 \) is odd, then, besides \( F(t_2) \neq 0 \), we have \( F(t_7) = -t_4 \neq 0 \), when \( n = 3 \), and \( F(t_d) = -t_{d-2} \neq 0 \), when \( n \geq 5 \). These will cause problems for our purposes in Section 5 when \( n \geq 3 \) is odd. To overcome these problems we have the following two options.

**Option 1.** In this option we change the chart of \( T \), but \( R, H, F \) stay the same. If for any positive odd integer \( n \geq 3 \) we set:

\[
\tilde{t}_d := \begin{cases} 
t_7 + t_2t_4, & \text{if } n = 3; \\
t_d + t_2t_{d-2}, & \text{if } n \geq 5 \text{ is odd}; 
\end{cases}
\]

then it is easy to observe that \( F(\tilde{t}_d) = 0 \). According to (3.53), we observe that \( H(\tilde{t}_d) = 2 \), which means \( \deg(\tilde{t}_d) = 2 \). If instead the chart \((t_1, t_2, \ldots, t_d)\) for \( T \), we use the chart \((t_1, t_2, \ldots, t_{d-1}, \tilde{t}_d)\), and set \( D := R = \sum_{j=1}^{d} R^j \frac{\partial}{\partial t_j} \), then we have:

\[
D = \sum_{j=1}^{d-1} D^j \frac{\partial}{\partial t_j} + D^d \frac{\partial}{\partial \tilde{t}_d}, \quad \text{where } D^j = R^j, \ j = 1, 2, \ldots, d - 1 \text{ and } D^d = R(\tilde{t}_d),
\]

\[
H = \sum_{j=1}^{d-1} w_j t_j \frac{\partial}{\partial t_j} + 2 \tilde{t}_d \frac{\partial}{\partial \tilde{t}_d}, \quad \text{where } w_j, \ j = 1, 2, \ldots, d - 1, \text{ are as before},
\]

\[
F = \frac{\partial}{\partial \tilde{t}_2}.
\]

For simplicity, from now on, we denote \( \tilde{t}_d \) also by \( t_d \), but we remember that whenever we use \( D \) of this option we consider the chart \((t_1, t_2, \ldots, t_{d-1}, \tilde{t}_d)\), and \( D = R, H, F \) does not change.

**Option 2.** In this option we change the modular vector field \( R \), and consequently \( F \), but the chart of \( T \) and \( H \) remain the same. For any positive integer \( n \) we define:

\[
D := R + t_2 \left( [R, (1 + \delta_2^n) \frac{\partial}{\partial \tilde{t}_2}] - H \right).
\]

Note that if \( n = 1 \) or \( n \) is even, then \( (1 + \delta_2^n) \frac{\partial}{\partial \tilde{t}_2} = F \), which implies \( D = R \). But, if \( n \geq 3 \) is odd, then \( D \) is different from \( R \). In the next lemmas we show that \( D \) is a quasi-homogeneous vector field of degree 2, and along with \( H \) and \( (1 + \delta_2^n) \frac{\partial}{\partial \tilde{t}_2} \) forms a copy of \( \mathfrak{s}l_2(\mathbb{C}) \).

**Remark 4.1.**

1. The vector field \( D \) defined in **Option 1** is different from the one defined in **Option 2** whenever \( n \geq 3 \) is odd. The reason for denoting them using the same notation is that the main results of this paper which will be proved in Section 5 hold for both of them, and to avoid repeating the theorems and their proofs we use the same notation.

2. Since in **Option 1** we have \( D = R \), we can find the \( q \)-expansion of their solution components (at least for \( n = 1, 2, 3, 4 \)) and we know interesting facts about them, some of which were mentioned in Section 4. But, we do not know solutions of \( D \) given in **Option 2** when \( n \geq 3 \) is odd. In particular, the author tried to find the \( q \)-expansion of a solution of it for \( n = 3 \), but he did not succeed and it seems that the solutions of \( D \) in these cases does not have \( q \)-expansion around \( \infty \).
Next we state the fundamental lemma of this work, which will be used to prove Theorem 4.1. First, we recall that if we have two vector fields \( V = \sum_{j=1}^{d} V^j \frac{\partial}{\partial t_j} \) and \( W = \sum_{j=1}^{d} W^j \frac{\partial}{\partial t_j} \), then

\[
[V, W] = VW - WV = \sum_{j=1}^{d} \left( V(W^j) - W(V^j) \right) \frac{\partial}{\partial t_j}.
\]

**Lemma 4.1. (Fundamental lemma)** The vector field \( D \) is a quasi-homogeneous vector field of degree 2 in the AMSY-Lie algebra \( \mathfrak{g} \) that satisfies:

\[
[D, (1 + \delta_2^0) \frac{\partial}{\partial t_2}] = H.
\]

**Proof.** If \( D \) is the one given in **Option 1**, then \( D = R \) and \( F = (1 + \delta_2^0) \frac{\partial}{\partial t_2} \). Hence, due to Proposition 3.4 and Theorem 3.4, the lemma holds.

Now we suppose that \( D \) is the one defined in **Option 2**. If \( n = 1, 2, 3, 4 \), then \( R, F, H \) are given explicitly in Example 5.1 and one can easily find that the affirmations hold. If \( n \geq 5 \) is even, then \( D = R \) and \( F = \frac{\partial}{\partial t_2} \), which yield the results. Suppose that \( n \geq 5 \) is odd. Then by applying (3.32) to \( R_{g_1} \) and \( R_{g_1(n+1)} \) we obtain \( R_{g_1} = \frac{\partial}{\partial t_{d-2}} + t_2 \frac{\partial}{\partial t_d} \) and \( R_{g_1(n+1)} = \frac{\partial}{\partial t_2} \). Therefore, the relation (3.47) yields:

\[
[R, \frac{\partial}{\partial t_d}] = [R, R_{g_1(n+1)}] = R_{g_1} = \frac{\partial}{\partial t_{d-2}} + t_2 \frac{\partial}{\partial t_d}.
\]

If we write \( R = \sum_{j=1}^{d} R_j \frac{\partial}{\partial t_j} \), then Remark 3.4 yields \( R^{d-2} = -t_d - t_2 t_{d-2} \), from which we get:

\[
[R, t_d - 2 \frac{\partial}{\partial t_d}] = R(t_d - 2) \frac{\partial}{\partial t_d} + t_d - 2[R, \frac{\partial}{\partial t_d}]
\]

\[\equiv R^{d-2} \frac{\partial}{\partial t_d} + t_d - 2 \frac{\partial}{\partial t_{d-2}} + t_2 t_{d-2} \frac{\partial}{\partial t_d}
\]

\[= t_d - 2 \frac{\partial}{\partial t_{d-2}} - t_d \frac{\partial}{\partial t_d}.
\]

Due to (3.51) we have \( \frac{\partial}{\partial t_2} = F + t_d - 2 \frac{\partial}{\partial t_d} \), hence

\[
D = R + t_2 \left( [R, \frac{\partial}{\partial t_2}] - H \right) = R + t_2 \left( [R, F + t_d - 2 \frac{\partial}{\partial t_d}] - H \right)
\]

\[= R + t_2 t_d - 2 \frac{\partial}{\partial t_d} - t_2 t_d \frac{\partial}{\partial t_d}.
\]

Note that in the last equality of the above equation we used (4.11) and the fact that \([R, F] = H\). Thus,

\[
[D, \frac{\partial}{\partial t_2}] = [R, \frac{\partial}{\partial t_2}] + [t_2 t_d - 2 \frac{\partial}{\partial t_d} - 2 \frac{\partial}{\partial t_d}] - [t_2 t_d \frac{\partial}{\partial t_d} - 2 \frac{\partial}{\partial t_d}]
\]

\[= [R, \frac{\partial}{\partial t_2}] - t_d - 2 \frac{\partial}{\partial t_{d-2}} + t_2 t_d \frac{\partial}{\partial t_d}
\]

\[= [R, \frac{\partial}{\partial t_2}] - t_d - 2 \frac{\partial}{\partial t_{d-2}} + t_2 t_d \frac{\partial}{\partial t_d} \equiv [R, F] = H.
\]
We know that $R$ is quasi-homogeneous of degree 2 and $\text{deg}(t_2) = 2$, hence \((4.12)\) implies that $D$ is quasi-homogeneous of degree 2. In order to get $D \in \mathcal{G}$, first observe that
$$
\frac{\partial}{\partial t_2} = F + t_{d-2} \frac{\partial}{\partial t_d} \in \mathcal{G},
$$
which yields $D \in \mathcal{G}$, and the proof is complete. $\square$

**Corollary 4.1.** The Lie algebra generated by the vector fields $D$, $H$ and $(1 + \delta^2_n)\frac{\partial}{\partial t_2}$ in the AMSY-Lie algebra $\mathcal{G} \subset \mathfrak{x}(T)$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

**Proof.** It suffices to show that $[D, \frac{\partial}{\partial t_2}] = H$, $[H, D] = 2D$, $[H, \frac{\partial}{\partial t_2}] = -2\frac{\partial}{\partial t_2}$. The truth of the first bracket is guaranteed by Lemma 4.1, and the last bracket follows from a simple computation after using \((3.51)\) or \((3.53)\), and \((4.8)\). To demonstrate the second bracket $[H, D] = 2D$, the same argument given in the proof of Lemma 4.1 works perfectly when $D$ is the vector field given in **Option 1**, and when $D$ is the one defined in **Option 2** for the cases $n = 1, 2, 3, 4$ or for even integers $n \geq 5$. For odd integers $n \geq 5$, we first use \((4.12)\) to obtain:
$$
[H, D] = [H, R] + [H, t_2 t_{d-2} \frac{\partial}{\partial t_{d-2}} - t_2 t_d \frac{\partial}{\partial t_d}],
$$
Then the statement follows from the fact $[H, R] = 2R$ given in Theorem 3.4 and using \((4.8)\) for $H$ stated in \((3.53)\). $\square$

## 5 Rankin-Cohen algebras for CY (quasi-)modular forms

Let us suppose that $t_1, t_2, \ldots, t_d$ denote the components of a solution of the vector field $D$, where $D$ is the vector field given either in **Option 1** or in **Option 2** of Section 4. The reader should take care to differ the notations $t_1, t_2, \ldots, t_d$ which stand for solution components of $D$ from the notations $t_1, t_2, \ldots, t_d$ which are used for the coordinate charts of $T$. Nevertheless, any solution component $\hat{t}_1$ is associated with the coordinate $t_1$. We define the **space of CY quasi-modular forms** $\mathcal{M}$ and the **space of CY modular forms** $\tilde{\mathcal{M}}$, respectively, as follows:

\begin{align}
(5.1) \quad \tilde{\mathcal{M}} & := \mathbb{C}[t_1, t_2, t_3, \ldots, t_d, \frac{1}{t_{n+2}(t_{n+2} - t_1^{n+2})}] , \\
(5.2) \quad \mathcal{M} & := \mathbb{C}[t_1, \hat{t}_2, t_3, t_4, \ldots, t_d, \frac{1}{t_{n+2}(t_{n+2} - t_1^{n+2})}] ,
\end{align}

in which $\hat{t}$ is associated with $\check{t}$ given in \((3.28)\) or \((3.29)\), and the symbol $\hat{t}$ means that the component $t_2$ is omitted, i.e., $t_2 \notin \mathcal{M}$. Indeed, we have $\tilde{\mathcal{M}} = \mathcal{M}[[t_2]]$ and in our generalization the CY quasi-modular form $t_2$ has the role of the quasi-modular form $E_2$ in the theory of quasi-modular forms. Let us attach to any solution component $t_i$, $1 \leq i \leq d$, the weight $\text{deg}(t_i) = w_i$, in which the non-negative integers $w_i$’s are given in \((3.55)\). For any integer $r \in \mathbb{Z}$ we define $\mathcal{M}_r$ and $\tilde{\mathcal{M}}_r$ to be the $\mathbb{C}$-vector spaces generated by \( \{ f \in \tilde{\mathcal{M}} | \text{deg}(f) = r \} \) and \( \{ f \in \mathcal{M} | \text{deg}(f) = r \} \), respectively. Note that any constant in $\mathbb{C}$ is considered as a weight zero CY (quasi-)modular form. Therefore, elements of $\tilde{\mathcal{M}}_r$ and $\mathcal{M}_r$ are CY quasi-modular forms and CY modular forms of weight $r$, respectively. In
particular, \( t_2 \) is a CY quasi-modular form of weight 2, see Remark 3.3 and the other \( t_j \)'s, \( 1 \leq j \leq d \) and \( j \neq 2 \), are CY modular forms of weight \( w_j \). In particular we have:

\[
\tilde{\mathcal{M}} = \bigoplus_{r \in \mathbb{Z}} \tilde{\mathcal{M}}_r \quad \text{and} \quad \mathcal{M} = \bigoplus_{r \in \mathbb{Z}} \mathcal{M}_r.
\]

Thus, \( \tilde{\mathcal{M}} \) and \( \mathcal{M} \) are commutative and associative graded algebras on \( \mathbb{C} \).

**Notation 5.1.** From now on \( \mathcal{R}, \mathcal{H} \) and \( \mathcal{F} \) refer to the differential operators on \( \mathcal{M} \) induced by the vector fields \( R, H \) and \( F \), respectively, in which we substitute the coordinate chart \( t_j \), \( 1 \leq j \leq d \), by the solution component \( t_j \) and \( \frac{\partial}{\partial t_j} \) by the partial derivation \( \frac{\partial}{\partial t_j} \). For example, if \( R = \sum_{j=1}^d R^j(t_1, t_2, \ldots, t_d) \frac{\partial}{\partial t_j} \), with \( R^j(t_1, t_2, \ldots, t_d) \in \mathcal{O}_T \), then \( \mathcal{R} = \sum_{j=1}^d R^j(t_1, t_2, \ldots, t_d) \frac{\partial}{\partial t_j} \). We consider the Lie bracket of the such obtained differential operators the same as the Lie bracket of the associated vector fields. Hence, due to Theorem 3.4 we get:

\[
[\mathcal{R}, \mathcal{F}] = \mathcal{H}, \quad [\mathcal{H}, \mathcal{R}] = 2\mathcal{R}, \quad [\mathcal{H}, \mathcal{F}] = -2\mathcal{F}.
\]

We recall that, for an integer \( d \), a degree \( d \) differential operator \( D \) on \( \mathcal{M} \), denoted by \( D : \mathcal{M}_* \to \mathcal{M}_{*+d} \), is a differential operator that satisfies \( D(\mathcal{M}_r) \subseteq \mathcal{M}_{r+d} \) for any positive integer \( r \). Indeed, if we can write \( D = \sum_{j=1}^d D^j \frac{\partial}{\partial t_j} \), with \( D^j \in \mathcal{M} \), then \( D \) has degree \( d \) provided \( \deg(D^j) - w_j = d \) for any \( 1 \leq j \leq d \). A degree \( d \) differential operator on \( \mathcal{M} \) is defined analogously.

**Definition 5.1.** We define the derivation \( \mathcal{D} \) on \( \mathcal{M} \) to be the differential operator induced by the vector field \( D \). In fact, \( \mathcal{D} \) is as follows:

\[
\mathcal{D} := \begin{cases} 
\mathcal{R} + t_2(\mathcal{R}, (1 + \delta_2^3) \frac{\partial}{\partial t_2}) - \mathcal{H}, & \text{if } D \in \text{Option 2, and } n \geq 3 \text{ is odd}; \\
\mathcal{R}, & \text{otherwise};
\end{cases}
\]

By the Ramanujan-Serre type derivation \( \partial \) on \( \mathcal{M} \) we mean the differential operator that on the generators of \( \mathcal{M} \) is defined as follows:

\[
\partial f := \mathcal{D} f + (1 - \frac{1}{2} \delta_2^3)rt_2 f, \quad \forall f \in \mathcal{M}_r \text{ and } \forall r \in \mathbb{Z}.
\]

We would like that the derivation \( \mathcal{D} \) and the Ramanujan-Serre type derivation \( \partial \) behave the same as the usual derivation (2.4) and the Ramanujan-Serre derivation (2.3) of the classical quasi-modular form theory, respectively. In the following example we state the derivations \( \mathcal{D} \) and \( \partial \) explicitly for \( n = 1, 2, 3, 4 \).

**Example 5.1.** In [Nik20] we found \( R, H, F \) explicitly for \( n = 1, 2, 3, 4 \). In these cases, we obtain the derivation \( \mathcal{D} \) and the Ramanujan-Serre type derivation \( \partial \) as follows:

- \( n = 1 \).

\[
\begin{align*}
R &= (-t_1t_2 - 9(t_1^2 - t_3)) \frac{\partial}{\partial t_1} + (81t_1(t_1^2 - t_3) - t_3^2) \frac{\partial}{\partial t_2} + (-3t_2t_3) \frac{\partial}{\partial t_3}, \\
H &= t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2} + 3t_3 \frac{\partial}{\partial t_3}, \\
F &= \frac{\partial}{\partial t_2}.
\end{align*}
\]
By definition, the vector field \((5.7)\) implies \(\deg(t_1) = 1, \deg(t_2) = 2\) and \(\deg(t_3) = 3\). Since \([R, F] = H\), we observe that:

\[
\begin{align*}
\mathcal{D} &= \mathcal{R}, \\
\partial &= -9(t_1^3 - t_3)\frac{\partial}{\partial t_1} + (81t_1(t_1^3 - t_3) + t_2^2)\frac{\partial}{\partial t_2}.
\end{align*}
\]

If we let \(\partial\) acts just on \(\mathcal{M}\), then we get:

\[
\partial = -9(t_1^3 - t_3)\frac{\partial}{\partial t_1}.
\]

- **\(n = 2\).**

\[
\begin{align*}
R &= (t_3 - t_1t_2)\frac{\partial}{\partial t_1} + (2t_1^2 - \frac{1}{2}t_2^2)\frac{\partial}{\partial t_2} + (-2t_2t_3 + 8t_1^3)\frac{\partial}{\partial t_3} + (-4t_2t_4)\frac{\partial}{\partial t_4}, \\
H &= 2t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2} + 4t_3\frac{\partial}{\partial t_3} + 8t_4\frac{\partial}{\partial t_4}, \\
F &= 2\frac{\partial}{\partial t_2},
\end{align*}
\]

where the polynomial equation \(t_3^2 = 4(t_1^2 - t_4)\) holds among \(t_i\)’s. From \((5.12)\) we get \(\deg(t_1) = 2, \deg(t_2) = 2, \deg(t_3) = 4\) and \(\deg(t_4) = 8\). Hence, due to \((5.4)\) and \((5.5)\) we find:

\[
\begin{align*}
\mathcal{D} &= \mathcal{R}, \\
\partial &= t_3\frac{\partial}{\partial t_1} + (2t_1^2 + \frac{1}{2}t_2^2)\frac{\partial}{\partial t_2} + 8t_3^3\frac{\partial}{\partial t_3}.
\end{align*}
\]

In the case that \(\partial\) is considered on \(\mathcal{M}\) we have:

\[
\partial = t_3\frac{\partial}{\partial t_1} + 8t_3^3\frac{\partial}{\partial t_3}.
\]

- **\(n = 3\).**

\[
\begin{align*}
R &= (t_3 - t_1t_2)\frac{\partial}{\partial t_1} + \frac{t_3^2t_4 - 5^4t_2^3(t_5^5 - t_5)}{5^4(t_1^5 - t_5)}\frac{\partial}{\partial t_2} \\
&\quad + \frac{t_3^2t_6 - 3 \times 5^4t_2t_3(t_5^5 - t_5)}{5^4(t_1^5 - t_5)}\frac{\partial}{\partial t_3} + (-t_2t_4 - t_7)\frac{\partial}{\partial t_4} \\
&\quad + (-5t_2t_5)\frac{\partial}{\partial t_5} + (5^5t_1^3 - t_2t_6 - 2t_3t_4)\frac{\partial}{\partial t_6} + (-5^4t_1t_3 - t_2t_7)\frac{\partial}{\partial t_7}, \\
H &= t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2} + 3t_3\frac{\partial}{\partial t_3} + 5t_5\frac{\partial}{\partial t_5} + t_6\frac{\partial}{\partial t_6} + 2t_7\frac{\partial}{\partial t_7}, \\
F &= \frac{\partial}{\partial t_2} - t_4\frac{\partial}{\partial t_7},
\end{align*}
\]

We obtain \(\deg(t_1) = 1, \deg(t_2) = 2, \deg(t_3) = 3, \deg(t_4) = 0, \deg(t_5) = 5, \deg(t_6) = 1, \deg(t_7) = 2\). For \(D\) given in **Option 1** remember that we substitute the coordinate \(t_7\) by:

\[
t_7 := t_7 + t_2t_4.
\]
from which we obtain:

\[ R(\vec{t}_7) = -5^4 t_1 t_3 + \frac{t_3^3 t_4^2}{5^4(t_1^5 - t_5)} - 2t_2 \vec{t}_7. \]

Note that in this case \( t_7 \) is the component of a solution of \( \mathcal{D} \) associated with coordinate \( \vec{t}_7 \). Hence, we get the derivation \( \mathcal{D} \) on \( \mathcal{M} \) as follows:

\[
\mathcal{D}_s = \mathcal{D} = (t_3 - t_2 t_1) \frac{\partial}{\partial t_1} + \left( \frac{t_3^3 t_4}{5^4(t_1^5 - t_5)} - t_2^2 \right) \frac{\partial}{\partial t_2} + \left( \frac{t_3 t_6}{5^4(t_1^5 - t_5) - 3t_2 t_3} \right) \frac{\partial}{\partial t_3} - t_7 \frac{\partial}{\partial t_4} - 5t_2 t_5 \frac{\partial}{\partial t_5} + \left( 5^5 t_1^3 - 2t_3 t_4 - t_2 t_6 \right) \frac{\partial}{\partial t_6} + \left( -5^4 t_1 t_3 + \frac{t_3^3 t_4}{5^4(t_1^5 - t_5)} - 2t_2 \vec{t}_7 \right) \frac{\partial}{\partial t_7},
\]

and we obtain \( \partial \) on \( \mathcal{M} \) as follows:

\[
\partial = t_3 \frac{\partial}{\partial t_1} + \frac{t_3^3 t_6}{5^4(t_1^5 - t_5)} \frac{\partial}{\partial t_3} - t_7 \frac{\partial}{\partial t_4} + (5^5 t_1^3 - 2t_3 t_4) \frac{\partial}{\partial t_6} - \left( 5^4 t_1 t_3 - \frac{t_3^3 t_4}{5^4(t_1^5 - t_5)} \right) \frac{\partial}{\partial t_7}.
\]

If \( \mathcal{D} \) is the vector field given in option 2, then we get \( \mathcal{D} : \mathcal{M} \rightarrow \mathcal{M} \) as follows:

\[
\mathcal{D}_s = \mathcal{D} = (t_3 - t_2 t_1) \frac{\partial}{\partial t_1} + \left( \frac{t_3^3 t_4}{5^4(t_1^5 - t_5)} - t_2^2 \right) \frac{\partial}{\partial t_2} + \left( \frac{t_3 t_6}{5^4(t_1^5 - t_5) - 3t_2 t_3} \right) \frac{\partial}{\partial t_3} - t_7 \frac{\partial}{\partial t_4} - 5t_2 t_5 \frac{\partial}{\partial t_5} + \left( 5^5 t_1^3 - 2t_3 t_4 - t_2 t_6 \right) \frac{\partial}{\partial t_6} + \left( -5^4 t_1 t_3 - 2t_2 \vec{t}_7 \right) \frac{\partial}{\partial t_7},
\]

and we obtain \( \partial : \mathcal{M} \rightarrow \mathcal{M} \) as follows:

\[
(5.19) \quad \partial = t_3 \frac{\partial}{\partial t_1} + \frac{t_3^3 t_6}{5^4(t_1^5 - t_5)} \frac{\partial}{\partial t_3} - t_7 \frac{\partial}{\partial t_4} + (5^5 t_1^3 - 2t_3 t_4) \frac{\partial}{\partial t_6} - 5^4 t_1 t_3 \frac{\partial}{\partial t_7}.
\]

\( \bullet \) \( n = 4. \)

\[
(5.20) \quad R = \left( t_3 - t_1 t_2 \right) \frac{\partial}{\partial t_1} + \frac{6^2 t_3^2 t_4 t_8 - t_1^6 t_2^2 + t_3^2 t_6}{t_1^6 - t_6} \frac{\partial}{\partial t_2} + \frac{6^2 t_3^2 t_5 t_8 - 3t_1^6 t_2 t_3 + 3t_2 t_3 t_6}{t_1^6 - t_6} \frac{\partial}{\partial t_3} + \frac{-6^2 t_3^2 t_7 t_8 - t_1^6 t_2 t_4 + t_2^2 t_4 t_6}{t_1^6 - t_6} \frac{\partial}{\partial t_4} + \frac{6^2 t_3^2 t_5 t_8 - 4t_1^6 t_2 t_5 - 2t_3^2 t_6}{t_1^6 - t_6} \frac{\partial}{\partial t_5} + \frac{2(t_1^6 - t_6)}{2 \times 6^2} \frac{\partial}{\partial t_6} + \frac{6^2 t_1^2 - t_1^2}{2 \times 6^2} \frac{\partial}{\partial t_7} + \frac{-3t_1^6 t_2 t_8 + 3t_1^5 t_3 t_5 + 3t_2 t_4 t_5}{t_1^6 - t_6} \frac{\partial}{\partial t_8},
\]

\[
(5.21) \quad H = t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2} + 3t_3 \frac{\partial}{\partial t_3} + t_4 \frac{\partial}{\partial t_4} + 2t_5 \frac{\partial}{\partial t_5} + 6t_6 \frac{\partial}{\partial t_6} + 3t_8 \frac{\partial}{\partial t_8},
\]

\[
(5.22) \quad F = \frac{\partial}{\partial t_2},
\]

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where the equation \( t_5^2 = 36(t_1^6 - t_6) \) holds among \( t_i \)'s. Analogous to the previous cases we have \( \text{deg}(t_1) = 1, \text{deg}(t_2) = 2, \text{deg}(t_3) = 3, \text{deg}(t_4) = 1, \text{deg}(t_5) = 2, \text{deg}(t_6) = 6, \text{deg}(t_7) = 0, \text{deg}(t_8) = 3 \). Due to (5.4) we find:

\[
D = R
\]

and (5.5) yields the Ramanujan-Serre type derivation on \( \mathcal{M} \) as follows:

\[
\partial = t_3 \frac{\partial}{\partial t_1} + \frac{6 - t_3^2 t_5}{t_1^6 - t_6} \frac{\partial}{\partial t_3} - \frac{6 - t_3^2 t_7}{t_1^6 - t_6} \frac{\partial}{\partial t_4} + \frac{6 - t_3^2 t_5 - 2 t_1^6 t_3 t_4 + 5 t_1^4 t_3 t_8 + 2 t_3 t_4 t_6}{2(t_1^6 - t_6)} \frac{\partial}{\partial t_5} + \frac{6 - t_3^2 - t_1^2}{2 \times 6 - t_1^6} \frac{\partial}{\partial t_7} + \frac{3 t_3^2 t_8}{t_1^6 - t_6} \frac{\partial}{\partial t_8}.
\]

**Remark 5.1.**

1. If we look closely to all cases stated in Example 5.1 we find out that the derivation \( \mathcal{D} \) and the Ramanujan-Serre type derivation \( \partial \) have degree 2. Besides these, the Ramanujan-Serre type derivation \( \partial \) sends any element of \( \mathcal{M} \) to another element of \( \mathcal{M} \). More precisely, the same as what we mentioned for the Ramanujan-Serre derivation given in (2.5), in all the above cases we observe that for any \( f \in \mathcal{M} \) the term \((1 - \frac{1}{2} \delta_n^2)t_2^2 f \) in (5.5) kills all the terms including \( t_2 \) in \( \mathcal{D} f \) which implies \( \partial f \in \mathcal{M}_{r+2} \), and consequently \( \mathcal{M} \) is closed under \( \partial \). All these facts hold for any positive integer \( n \) which are stated in Theorem 1.1.

2. In Example 5.1 we stated the derivation \( \mathcal{D} \) explicitly in the cases \( n = 1, 2, 3, 4 \). For \( n \geq 5 \), due to the proof of Lemma 4.1, we can state \( \mathcal{D} \) explicitly as follows:

   - if \( \mathcal{D} \) is the vector field given in Option 2 and \( n \geq 5 \) is odd, then \( \mathcal{D} = R + t_3 \frac{\partial}{\partial t_1} + \frac{t_2 t_3 - 2 \delta_{n-2}}{\partial t_2} - t_2 \frac{\partial}{\partial t_3} \),
   - otherwise, we have \( \mathcal{D} = R \).

Now we are in the situation that we can present the proof of Theorem 1.1.

**Proof of Theorem 1.1**

1. Due to Lemma 4.1 the proof is straightforward, since the differential operator \( \mathcal{D} \) is induced by the vector field \( \mathcal{D} \) which is a quasi-homogeneous vector field of degree 2.

2. First note that according to Remark 3.3 we always have \( \text{deg}(t_2) = w_2 = 2 \). Hence, from part 1 and (5.5) we deduce that \( \partial \) is a degree 2 differential operator. To prove that for all \( f \in \mathcal{M} \) we get \( \partial f \in \mathcal{M} \), it is enough to observe that for all integers \( r \)
and for all $f \in \mathcal{M}_r$ we have $\partial f \in \mathcal{M}_{r+2}$, which is equivalent to:

$$
\partial t_j \in \mathcal{M}_{w_j + 2}, \ \forall j \neq 2, \iff (1 + \delta^0_1) \frac{\partial}{\partial t_2} (\partial t_j) = 0, \ \forall j \neq 2,
$$

$$
\iff (1 + \delta^0_1) \frac{\partial}{\partial t_2} (\partial t_j + (1 - \delta^0_2)w_j t_2 t_j) = 0, \ \forall j \neq 2,
$$

$$
\iff (1 + \delta^0_1) \frac{\partial}{\partial t_2} (\partial t_j) = -w_j t_j, \ \forall j \neq 2,
$$

$$
\iff \sum^d_{j=1} (1 + \delta^0_2) \frac{\partial}{\partial t_2} (\partial t_j) \frac{\partial}{\partial t_j} = - \sum^d_{j=1} w_j t_j \frac{\partial}{\partial t_j} = -H,
$$

$$
\iff [(1 + \delta^0_2) \frac{\partial}{\partial t_2}, \partial] = -H,
$$

$$
\iff [\partial, (1 + \delta^0_2) \frac{\partial}{\partial t_2}] = H.
$$

The last affirmation is valid due to Lemma 4.1 which completes the proof.

Next, to use Proposition 2.1, we need the CY quasi-modular forms of positive weight. Hence, we consider the spaces of CY quasi-modular forms $\mathcal{M}^>0$ and CY modular forms $\mathcal{M}^>0$ of positive weight as follows:

$$
\mathcal{M}^>0 := \bigoplus_{r \geq 0} \mathcal{M}^>r, \quad \mathcal{M}^>0 := \bigoplus_{r \geq 0} \mathcal{M}_r,
$$

in which we suppose that $\mathcal{M}^0 = \mathcal{M}_0 = \mathbb{C}$. Thus, the space of CY quasi-modular forms of positive weight $\mathcal{M}^>0$ is a commutative and associative graded algebra with unit over the field $\mathbb{C}$ together with the derivation $\partial : \mathcal{M}^>0 \to \mathcal{M}^>2$ of degree 2. Therefore, due to Remark 2.1, $\mathcal{M}^>0, [\cdot, \cdot]_{\partial, s}$ is a standard Rankin-Cohen, and hence a Rankin-Cohen algebra. We call $[\cdot, \cdot]_{\partial, s}$ the Rankin-Cohen bracket for CY quasi-modular forms, and for any non-negative integers $k, r, s$ it is defined as

$$
[f, g]_{\partial, k} := \sum_{i+j=k} (-1)^j \binom{k+r-1}{i} \binom{k+s-1}{j} f^{(j)} g^{(i)}, \ \forall f \in \mathcal{M}_r, \ \forall g \in \mathcal{M}_s,
$$

where $f^{(j)} = \partial^j f$ and $g^{(j)} = \partial^j g$ refer to the $j$-th derivative of $f$ and $g$ under $\partial$, respectively. It is evident that $[f, g]_{\partial, k} \in \mathcal{M}_{r+s+2k}$. Next, we demonstrate Theorem 1.2 which shows that the space of CY modular forms of positive weight $\mathcal{M}^>0$ is closed under the Rankin-Cohen bracket for CY quasi-modular forms given in (5.26).

**Proof of Theorem 1.2**

The idea of the proof is to use Proposition 2.1 and its proof. To this end, first note that according to the part 2 of Theorem 1.1 the Ramanujan-Serre type derivation $\partial : \mathcal{M}^>0 \to \mathcal{M}^>2$ is a degree 2 differential operator. If we set $\Lambda = \Lambda(t_1, t_2, \ldots, t_d)$, where $\Lambda$ is given in Lemma 3.1, then the same lemma yields $\Lambda \in \mathcal{M}_4$. Therefore, from Proposition 2.1 we get that $(\mathcal{M}^>0, [\cdot, \cdot]_{\partial, \Lambda, k})$, where the $k$-th bracket $[\cdot, \cdot]_{\partial, \Lambda, k}$, $k \geq 0$, is given by (2.19), is a canonical Rankin-Cohen algebra. On the other hand, by letting $\lambda = \frac{1}{2} \delta^0_2 - 1$ $t_2$, from (5.5) we obtain

$$
\partial f = \partial f + r \lambda f, \ \forall f \in \mathcal{M}_r.
$$

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Furthermore, if we write $\mathcal{D} = \sum_{j=1}^{d} D^j \frac{\partial}{\partial t^j}$, with $D^j \in \mathcal{M}$, then

$$\mathcal{D}(\lambda) = \left(\frac{1}{2} \delta_2^\lambda - 1\right) \mathcal{D}(t^2) = \left(\frac{1}{2} \delta_2^\lambda - 1\right) D^2 .$$

Considering $\mathcal{R} = \sum_{j=1}^{d} R^j \frac{\partial}{\partial t^j}$, with $R^j \in \mathcal{M}$, the part 2 of Remark 5.1 yields $D^2 = R^2$. This fact along with (5.28) and (3.60) implies:

$$\mathcal{D}(\lambda) = \Lambda + \lambda^2 .$$

The relations (5.27) and (5.29) show that (2.21) is satisfied. Hence, from the proof of Proposition 2.1 we obtain $[\cdot, \cdot]_{\partial, \Lambda, \ast} = [\cdot, \cdot]_{\mathcal{D}, \ast}$ (see (2.22)). Finally, since $\mathcal{M}^{\ast > 0}$ is closed under $[\cdot, \cdot]_{\partial, \Lambda, \ast}$, we conclude that $\mathcal{M}^{\ast > 0}$ is closed under $[\cdot, \cdot]_{\mathcal{D}, \ast}$, and this finishes the proof of the theorem. $\square$

In particular, Theorem 1.2 implies that $(\mathcal{M}^{\ast > 0}, [\cdot, \cdot]_{\mathcal{D}, \ast})$ is a sub Rankin-Cohen algebra of $(\mathcal{M}^{\ast > 0}, [\cdot, \cdot]_{\mathcal{R}, \ast})$.

**Corollary 5.1.** The Rankin-Cohen bracket for CY quasi-modular forms $[\cdot, \cdot]_{\mathcal{D}, \ast}$ endows $\mathcal{M}^{\ast > 0}$ with a canonical Rankin-Cohen algebra structure.

### 5.1 Examples of Rankin-Cohen brackets of CY modular forms

We know that the modular discriminant is given by $\Delta = \frac{1}{1728}(E_4^3 - E_6^2)$, which is related with the discriminant $t_3^3 - 27t_3^2$ of the family of elliptic curves stated in (3.25). One can easily compute (or find in [Zag94]) the following examples of Rankin-Cohen brackets (2.6) of modular forms:

$$(5.30) \quad [E_4, E_6]_1 = -3456\Delta, \quad [E_4, E_6]_2 = 0, \quad [E_4, E_4]_2 = 4800\Delta, \quad [E_6, E_6]_2 = -21168E_4\Delta, \quad [\Delta, \Delta]_2 = -13E_4\Delta^2 .$$

Note that for any (quasi-)modular form or any CY (quasi-)modular form $f$ of non-negative weight $r$ and any integer $k \geq 0$ it is evident by definition that :

$$[f, f]_{2k+1} = 0 \quad \text{or} \quad [f, f]_{\mathcal{D}, 2k+1} = 0 .$$

For any positive integer $n$, the discriminant of the Dwork family (3.1) is given by the polynomial $t_{n+2}(t_1^{n+2} - t_{n+2})$. Hence, in the rest of this section for any $n$ we fix the notation $\Delta := t_{n+2}(t_1^{n+2} - t_{n+2})$. Next, we compute a few examples of Rankin-Cohen brackets (5.26) of CY modular forms for $n = 1, 2, 3, 4$, which are motivated by examples given in (5.30).

- $n = 1$. In this case we found $t_1, t_2, t_3$ in the first list of (3.37) and we have $\Delta = t_3(t_1^3 - t_3)$. The Rankin-Cohen brackets are calculated as follows:

  $$(5.32) \quad [t_1, t_3]_{\mathcal{D}, 1} = 27\Delta, \quad [t_1, t_3]_{\mathcal{D}, 2} = 729t_3^2\Delta, \quad [t_1, t_1]_{\mathcal{D}, 2} = 324\Delta, \quad [t_3, t_3]_{\mathcal{D}, 2} = -2916t_3^2\Delta, \quad [\Delta, \Delta]_{\mathcal{D}, 2} = -5103t_3^2\Delta^2 .$$

Before passing to the next case, we express the combinations of $t_1, t_2, t_3$ which appeared in the right hand side of the above relations in terms of eta and theta functions that seem to us interesting. These relations are obtained thanks to [OEL64] and one...
can find out more about them by seeing the corresponding pages and references given there. By comparing the coefficients of $t_1$ with [OEI64 A004016] we find:

\[(5.33)\]

\[t_1 = \frac{1}{3}(\theta_3(q)\theta_3(q^3) + \theta_2(q)\theta_2(q^3)),\]

and for $t_1^2$ and $t_1^4$ the reader is referred to [OEI64 A008653] and [OEI64 A008655], respectively. After computing the $q$-expansion of $\Delta$, from [OEI64 A007332] we get:

\[(5.34)\]

\[\Delta = \frac{1}{24}\eta^6(q)\eta^6(q^3),\]

and on account of [OEI64 A136747] we get:

\[(5.35)\]

\[t_1^2\Delta = \frac{1}{243}\eta^6(q)\eta^4(q^3)(\eta^3(q) + 9\eta^3(q^9))^2.\]

The equations (5.33), (5.34) and (5.35) yield:

\[(5.36)\]

\[3t_1 = \theta_3(q)\theta_3(q^3) + \theta_2(q)\theta_2(q^3) = \frac{\eta^3(q) + 9\eta^3(q^9)}{\eta(q^3)}.\]

- $n = 2$. Here $t_1, t_2, t_4$ are stated in the second list of (5.37). We know that $\Delta = t_1(t_1^4 - t_4)$, and we obtain:

\[(5.37)\]

\[\begin{align*}
[t_1, t_4]_{g, 1} &= -8t_3t_4, \\
[t_1, t_4]_{g, 2} &= 192t_1^3t_4, \\
[t_1, t_1]_{g, 2} &= 36t_1^4 - 9t_3^2 = 36t_1, \\
[t_2, t_4]_{g, 2} &= -576t_1^2t_4^3, \\
[\Delta, \Delta]_{g, 2} &= -1088t_1^2t_4(t_1^4 + 8t_4)\Delta.
\end{align*}\]

Note that in the third bracket of (5.37) we used the fact that $t_3^2 = 4(t_1^4 - t_4)$, which also implies:

\[(5.38)\]

\[t_1, t_4]_{g, 1} = 64t_1^3t_4^2 = 256t_4^2\Delta.\]

- $n = 3$. If $D$ is the vector field given in the Option 1, then one can find the $q$-expansion of $t_1, t_2, \ldots, t_7$ in [Mov15]. For $D$ given in Option 2 we can not find $q$-expansion of solution components of $D$ around $\infty$, because if we suppose that $\dot{t}_j = a\frac{\partial t_j}{\partial q}$, for a constant $a \in \mathbb{C}$, then we find $a = 0$. In this case we have $\Delta = t_5(t_5^6 - t_3)$, and we calculate the Rankin-Cohen brackets, for $D$ induced by $D$ given in both Option 1 and Option 2, as follows:

\[(5.39)\]

\[\begin{align*}
[t_1, t_5]_{g, 1} &= -5t_3t_5, \\
[t_1, t_5]_{g, 2} &= \frac{-4t_1^3t_4t_5 + 3t_1^3t_5t_6}{125(t_1^6 - t_5)}, \\
[t_1, t_1]_{g, 2} &= \frac{-2500t_1^2(t_5^6 - t_5) - 2t_1^3t_5(t_1t_4 - t_6)}{625(t_1^6 - t_5)}, \\
[t_5, t_5]_{g, 2} &= \frac{-6t_1^3t_4t_5^2}{25(t_1^6 - t_5)}, \\
[\Delta, \Delta]_{g, 2} &= \frac{t_1^2t_5^2}{25}(t_5^3(-20625t_1^5 - 55000t_5 + 22t_1t_3t_6) - 44t_3t_4(t_1^6 - t_5)).
\end{align*}\]

- $n = 4$. Here, the first 7 coefficients of the $q$-expansions of $t_1, t_2, \ldots, t_7, t_8$ are given in [MN21 Table 2]. We get $\Delta = t_6(t_6^6 - t_6)$ and hence:

\[(5.40)\]

\[\begin{align*}
[t_1, t_6]_{g, 1} &= -6t_3t_6, \\
[t_1, t_6]_{g, 2} &= \frac{-9t_1^3t_4^2t_6 + 7t_1^3t_5t_6t_8}{12(t_1^6 - t_6)}, \\
[t_1, t_1]_{g, 2} &= \frac{-72t_1^2(t_6^6 - t_6) - t_1^2t_5t_8(t_1t_4 - t_5)}{18(t_1^6 - t_6)}, \\
[t_6, t_6]_{g, 2} &= \frac{-7t_1^3t_4^2t_8}{t_1^6 - t_6}, \\
[\Delta, \Delta]_{g, 2} &= t_3^2t_5^2(t_1^4(-1404t_1^6 - 4680t_6 + 26t_1t_5t_8) - 52t_4t_8(t_1^6 - t_6)).
\end{align*}\]
The relations given in (3.57) yield $D_t t_1 = t_3 - t_1 t_2$ and $D t_{n+2} = -(n+2) t_2 t_{n+2}$ for any integer $n \geq 3$, from which we conclude the following expected result (see (5.39) and (5.40)):

\[(5.41) [t_1, t_{n+2}]_{\mathcal{G},1} = -(n+2) t_3 t_{n+2}, \quad \forall n \geq 3.\]

Another interesting point that we observe in the above examples is that in all the cases $n = 1, 2, 3, 4$ the bracket $[\Delta, \Delta]_{\mathcal{G},2}$ is expressed as a polynomial in terms of $t_1, t_2, \ldots, t_d$, and we expect that this happens for higher dimensions as well.

It is also worth to point out that for any CY (quasi-)modular form $f$ of weight $r$, the second Rankin-Cohen bracket $[f, f]_{\mathcal{G},2}$ provides a second order differential equation which is satisfied by $f$. More precisely, from (5.26) we obtain:

\[(5.42) [f, f]_{\mathcal{G},2} = 6 f D^2 f - 9 (D f)^2,\]

which implies that $f$ satisfies the second order ODE:

\[(5.43) 6 y D^2 y - 9 (D y)^2 = [f, f]_{\mathcal{G},2}.\]

For example, if $n = 1$, then from the third bracket of (5.32) we get that the function

\[t_1 = \frac{1}{3} (2 \theta_3(q^2) \theta_3(q^6) - \theta_3(-q^2) \theta_3(-q^6)) = \frac{1}{3} (\theta_3(q) \theta_3(q^3) + \theta_2(q) \theta_2(q^3)) = \frac{\eta^3(q) + 9 \eta^3(q^9)}{3 \eta(q^3)},\]

satisfies the following second order ODE:

\[(5.44) 2 \ddot{y} - 3 \dot{y}^2 = 4 \eta^6(q) \eta^6(q^3),\]

in which $\dot{y} = 3q \frac{\partial u}{\partial q} = \frac{3}{2 \pi i} \frac{du}{dt}$.

6 Final remarks

One of weak points of Theorem 1.2 is that we are just considering the CY modular forms of positive weight. If we look closely to the definition of $\tilde{\mathcal{M}}$ and $\mathcal{M}$ given in (5.1) and (5.2), respectively, we observe that they contain non-constant elements of weight zero and elements of negative weight. For example for $n = 3$, the element $t_4 \in \mathcal{M}$ is a non-constant element of weight zero and $t_5 (t_5 - t_3) \in \mathcal{M}$ is an element of weight $-10$. Thus, in general it is not necessarily valid that $\tilde{\mathcal{M}}_0 = \mathcal{M}_0 = \mathbb{C}$; indeed, $\tilde{\mathcal{M}}_0$ and $\mathcal{M}_0$ are generated by $\mathbb{C} \cup \{ f \in \tilde{\mathcal{M}} \mid \deg(f) = 0 \}$ and $\mathbb{C} \cup \{ f \in \mathcal{M} \mid \deg(f) = 0 \}$, respectively. We can consider the definition of the Rankin-Cohen bracket (5.26) for elements of negative weight as well, and hence we can endow $\tilde{\mathcal{M}}$ with a Rankin-Cohen algebra structure. Using the computer we observed that the Rankin-Cohen brackets of all examined CY modular forms of negative weight are again CY modular forms, in the cases $n = 1, 2, 3, 4$, but we could not prove theoretically the assertion that the space of CY modular forms $\tilde{\mathcal{M}}$ is closed under the Rankin-Cohen bracket (5.26). We believe to the truth of this assertion, but our main difficulty in carrying out its proof is the use of Proposition 2.1, where the weight of non-constant elements of the graded algebra are considered positive. This led us to the following conjecture.
Conjecture 1. The proposition holds if the graded algebra \( M \), besides elements of positive weight, also contains elements of negative weight or non-constant elements of weight zero. In the other words, if \( M = \bigoplus_{k \in \mathbb{Z}} M_k \), in which it is not necessary that \( M_0 = k.1 \).

In the above conjecture by constant elements we mean the elements of the field \( k \). If we want to prove Conjecture in an analogous way to the proof of Zagier given for Proposition 1, the unsolved part is the equality \((2.22)\). Once we prove Conjecture we can prove that the space of CY modular forms \( \mathcal{M} \) is closed under the Rankin-Cohen brackets \((5.26)\).

Since the CY 3-folds are more important in the literature, we state the Gauss-Manin connection matrix of \( D \) for \( n = 3 \) here:

\[
(6.1) \quad A_D = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & Y_1 & 0 \\
t_2t_4 & 0 & 0 & -1 \\
-t_2(t_2t_4 + t_7) & t_2t_4 & 0 & 0
\end{pmatrix},
\]

in which \( Y_1 = \frac{t_3^3}{5^4(t_5^3 - t_5)} \). Note that, due to Theorem 3.1, the Gauss-Manin connection matrix of \( R \) is as follows:

\[
(6.2) \quad A_R = \begin{pmatrix}
0 & 0 & Y_1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

in which we also have \( Y_1 = \frac{t_3^3}{5^4(t_5^3 - t_5)} \). If we substitute the solutions of \( R \) in \( Y_1 \), then we get the Yukawa coupling. It would be very interesting, and maybe helpful, if one can find out the (physical) interpretation of the non-zero part of the lower triangle of the matrix \( A_D \) stated in \((6.1)\).

References

[Ali17] Murad Alim. Algebraic structure of \( tt^* \) equations for Calabi-Yau sigma models. *Commun. Math. Phys.*, 353(3):963–1009, 2017.

[AL07] Murad Alim and Jean Dominique Lange. Polynomial Structure of the (Open) Topological String Partition Function. *JHEP*, 0710:045, 2007.

[AMSY16] Murad Alim, Hossein Movasati, Emanuel Scheidegger, and Shing-Tung Yau. Gauss-Manin connection in disguise: Calabi-Yau threefolds. *Commun. Math. Phys.*, 344(3):889–914, 2016.

[AV21] Murad Alim and Martin Vogrin. Gauss-Manin Lie algebra of mirror elliptic K3 surfaces. *Mathematical Research Letters*, 28(3):637–663, 2021.

[BCOV93] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa.. Holomorphic anomalies in topological field theories. *Nuclear Phys. B*, 405(2-3):279–304, 1993.

[COGP91] Philip Candelas, Xenia C. de la Ossa, Paul S. Green, and Linda Parkes. A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. *Nuclear Phys. B*, 359(1):21–74, 1991.
[Coh77] H. Cohen. Sums involving the values at negative integers of L–functions of quadratic characters. *Math. Ann.*, 217:81–94, 1977.

[CS17] Henry Cohen and Fredrik Strömberg. Modular Forms: A Classical Approach. *American Mathematical Society, Providence, Rhode Island*, 2017.

[GP90] B. R. Greene and M. R. Pless. Duality in Calabi-Yau moduli space. *Nuclear Physics B.*, 338(1):15–37, 1990.

[GMP95] Brian R. Greene, David R. Morrison, and M. Ronen Pless. Mirror manifolds in higher dimension. *Comm. Math. Phys.*, 173:559–598, 1995.

[KP08] A. Klemm and R. Pandharipande. Enumerative geometry of Calabi-Yau 4-folds. *Commun. Math. Phys.*, 281(3):621–653, 2008.

[Mov12] Hossein Movasati. Quasi modular forms attached to elliptic curves, I. *Annales Mathématique Blaise Pascal*, 19:307–377, 2012.

[Mov15] Hossein Movasati. Modular-type functions attached to mirror quintic Calabi-Yau varieties. *Math. Zeit.*, 281, Issue 3, pp. 907-929(3):907–929, 2015.

[Mov17] Hossein Movasati. Gauss-Manin connection in disguise: Calabi-Yau modular forms. *International Press, Somerville, Massachusetts, U.S.A, and Higher Education Press, Beijing, China*, 2017.

[MN21] Hossein Movasati and Younes Nikdelan. Gauss-Manin Connection in Disguise: Dwork-Family. *J. Differential Geometry*, 119: 73-98, 2021.

[Nik15] Younes Nikdelan. Darboux–Halphen–Ramanujan vector field on a moduli of Calabi–Yau manifolds. *Qual. Theory Dyn. Syst.*, 14(1):71–100, 2015.

[Nik20] Younes Nikdelan. Modular vector fields attached to Dwork family: $\mathfrak{sl}_2(\mathbb{C})$ Lie algebra. *Moscow Math. J.*, 20(1):127–151, 2020.

[OEI64] The OEIS Foundation. The On-line Encyclopedia of Integer Sequences. [http://oeis.org/], 1964.

[Ram16] S. Ramanujan. On certain arithmetical functions. *Trans. Cambridge Philos. Soc.*, 22:159–184, 1916.

[Ran56] R. A. Rankin. The construction of automorphic forms from the derivatives of a given form. *Indian Math. Soc.*, 20:103–116, 1956.

[YY04] Satoshi Yamaguchi and Shing-Tung Yau. Topological string partition functions as polynomials. *JHEP*, 07:047, 2004.

[Yui13] Noriko Yui. Modularity of Calabi–Yau Varieties: 2011 and Beyond. In: Laza R., Schütt M., Yui N. (eds) Arithmetic and Geometry of K3 Surfaces and Calabi–Yau Threefolds. Fields Institute Communications, vol 67. pp 101-139, Springer, New York, NY, 2013.

[Zag94] D. Zagier, Modular forms and differential operators. *Proceedings Mathematical Sciences*, 104(1):57–75, 1994.