ANDERSON LOCALIZATION
FOR GENERIC DETERMINISTIC OPERATORS

VICTOR CHULAEVSKY

Abstract. We consider a class of ensembles of lattice Schrödinger operators with
deterministic random potentials, including quasi-periodic potentials with Diophan-
tine frequencies, depending upon an infinite number of parameters in an auxiliary
measurable space. Using a variant of the Multi-Scale Analysis, we prove Anderson
localization for generic ensembles in the strong disorder regime and establish an
analog of Minami-type bounds for spectral spacings.

1. Introduction. Formulation of the results.

In this paper, we study spectral properties of finite-difference operators, usually
called lattice Schrödinger operators (LSO), of the form
\[(H(\omega, \theta)f)(x) = \sum_{y: \|y-x\|_1=1} f(y) + gv(T^x \omega, \theta)f(x), \quad x, y \in \mathbb{Z}^d,\]
where \(v: \Omega \times \Theta \to \mathbb{R}\) is a measurable function on the direct product of the probability
space \((\Omega = T^\nu, \mathfrak{F}, P)\), endowed with the normalized Haar measure \(P\), and an auxiliary
probability space \((\Theta, \mathfrak{B}, \mu)\). \(T: \mathbb{Z}^d \times \Omega \to \Omega\) is an ergodic dynamical system with
discrete time \(\mathbb{Z}^d, d \geq 1\). Here \(\|x\|_1 = \sum_i |x_i|\). The function \(v\) will be referred to as the
hull of the ergodic potential \(V\). In an earlier work [9], we considered a particular case
where the hull \(v\) was discontinuous on the torus, for a.e. \(\theta \in \Theta\). In fact, the arguments
of [9] were based on a particular construction which required \(v\) to be discontinuous. In
the present paper, we show that methods of [9] can be adapted to the hulls of class
\(C^M(T^\nu)\) for any given \(M \geq 1\). The probability space \(\Theta\) plays the role of a parameter
space with measure \(\mu\) constructed in such a way that \(\mu\)-a.e. hull \(\omega \mapsto v(\omega, \theta)\) is suitable
for the Multi-Scale Analysis (MSA), so that the exponential spectral localization can
be established for \(\mu\)-a.e. ergodic ensemble of operators \(H(\cdot, \theta)\) (with \(g\) large enough).

Recall that in the works by Sinai [17] and by Fröhlich et al. [14] Anderson localization
was proven for the one-dimensional LSO with quasi-periodic potential of the form
\(V(x; \omega) = v(\omega + x\alpha), \omega \in T^1\), where the hull \(v: T^1 \to \mathbb{R}\) was assumed of class \(C^2(T^1)\)
with exactly two critical points, both non-degenerate. Later, it became clear that in
order to extend these techniques to more general hull functions and multi-dimensional
phase spaces, e.g., \(\Omega = T^\nu\) with \(\nu > 1\), it would be necessary to exclude an infinite
number of ‘degeneracies’ which cannot be described explicitly; cf. [10]. Here we show
that necessary regularizations, required in the course of the MSA procedure, can be
performed with the help of relatively simple probabilistic techniques.

Bourgain, Goldstein and Schlag developed earlier a different approach; see, e.g.,
[11,13]. Their method is based on the assumption of analyticity of the hull \(v: T^\nu \to \mathbb{R}\).
Auxiliary parameters allowing to avoid ”small denominators” in the MSA procedure
can be introduced in many ways. For example, it is quite natural to consider a series
with ‘random’ coefficients $a_n(\theta)$ relative to a probability space $(\Theta, \mathcal{B}, \mu)$

$$v(\omega, \theta) = \sum_{n \in \mathbb{Z}} a_n(\theta) \varphi_n(\omega)$$

and wavelet-like functions $\varphi_n$. It turns out, however, that the orthogonality properties are of little importance here. Earlier, we proposed in [6,7] a class of parametric families of deterministic potentials which we called "randelette expansions"; cf. Section 4.

An important class of examples is obtained by taking an ergodic action of the group $\mathbb{Z}^d$ on the torus $T^\nu$, $\nu \geq 1$, generated by quasi-periodic shifts

$$T_{\alpha_j} : \omega \mapsto \omega + \alpha_j, \quad \alpha_j \in T^\nu.$$  

Recently, Chan [5] proved the Anderson localization for single-frequency quasi-periodic operators with the hull $v$ of class $C^3(T^1)$, using a parameter exclusion technique which is different from presented in this paper.

Our main requirement for the dynamical system is the condition (1.1) of “uniformly slow” returns of any trajectory $\{T^x \omega, x \in \mathbb{Z}^d\}$ toward its starting point $\omega \in \Omega$.

The main result of the present paper, Theorem 1, is formulated in Section 1.4.

An interesting "by-product" of our approach is a Minami-type bound; cf. Section 5.

1.1. Requirements for the dynamical system. We assume that the underlying dynamical system $T$ on the phase space $\Omega$, endowed with a distance $d_\Omega(\cdot, \cdot)$, satisfies the following condition of uniformity slow returns:

$$d_\Omega(T^x \omega, T^y \omega) \geq 4C\|x - y\|^{-A}. \quad (1.1)$$

In this paper, we consider the case where $\Omega = T^\nu$, $\nu \geq 1$, and it is convenient to define the distance $d_\Omega(\omega', \omega'') \equiv d_{T^\nu}(\omega', \omega'')$ as follows:

$$d_{T^\nu}(\omega_1', \ldots, \omega_v'), (\omega_1'', \ldots, \omega_v'') = \max_{1 \leq i \leq v} d_{T^1}(\omega_i', \omega_i''),$$

where $d_{T^1}$ is the conventional distance on the unit circle $T^1$. The reason for the choice of the phase space $\Omega = T^\nu$ is that many parametric families of ensembles of potentials $V(x; \omega, \theta)$ and various dynamical systems can be made explicit in this case; in fact, the torus can be replaced by a compact Riemannian manifold of class $C^M$, $M \geq 1$. For the rotations of the torus $T^\nu$,

$$T^x \omega = \omega + x \alpha_1 + \cdots + x_d \alpha_d, \quad x \in \mathbb{Z}^d, \quad \alpha_j \in T^\nu, \quad 1 \leq j \leq d,$$

the USR property reads as a Diophantine condition for the frequency vectors $\alpha_j$. Recall that, owing to a well-known result by Gordon [15], a quasi-periodic operator with irrational frequency abnormally fast approximated by rational numbers may have no decaying solution to the problem $H \psi = E \psi$ (hence, no $L^2$-eigenfunction).

We also assume a polynomial bound on the rate of local divergence of trajectories (fulfilled for rotations of the torus, as well as for skew shifts):

$$d_\Omega(T^x \omega, T^y \omega) \leq C'|x|^{A'} d_\Omega(\omega, \omega'). \quad (1.2)$$

1 Although ergodicity per se is not required for the proof of localization, it follows from (USR) for rotations of the torus.
1.2. Geometrical notions and constructions.

We will consider lattice cubes $B_L(u) = \{ x \in \mathbb{Z}^d : \| x - u \| \leq L \}$, for which we define the internal boundary $\partial^{-} B_L(u) = \{ x : \| x - u \| < L \}$, external boundary composed of nearest neighbors $y \in \mathbb{Z}^d \setminus B_L(u)$ of $\partial^{-} B_L(u)$, and boundary $\partial B_L(u)$ composed of nearest-neighbor pairs $(x,y)$, with $x \in \partial^{-} B_L(u)$ and $y \in \partial^{+} B_L(u)$. Here and below, we use the max-norm for vectors $x \in \mathbb{R}^d$: $\| x \| := \max_{1 \leq i \leq d} |x_i|$, so that cubes $B_L(u)$ are actually balls of radius $L$ centered at $u$. For this reason, we will often refer to the “radius” (= $L$) of a cube $B_L(u)$. We will work with restrictions of the operator $H(\omega, \theta)$ to lattice cubes $B_{L_k}(u)$ with Dirichlet boundary conditions on $\partial^{+} B_{L_k}(u)$, where $L_0 > 2$ is a sufficiently large integer and $L_k$, $k \geq 1$, are defined by the recursion

$$L_k = [L_{k-1}]^\alpha + 1, \quad \alpha = 3/2. \tag{1.3}$$

Next, define a sequence of positive real numbers

$$\delta_k = g^{-a} e^{-4L_k^b}, \quad k \geq 0; \quad a = 1/2, \quad b = 1/2. \tag{1.4}$$

It is convenient to introduce also the scale length $L_{-1} = 0$ (cf., e.g., Definition 2.1). Cubes $B_0(u)$ are single-point sets: $\{ x : \| x - u \| \leq 0 \} \equiv \{ u \}$. The restriction of operator $H(\omega, \theta)$ on a $B_{L_{-1}}(u)$ is the operator of multiplication by $V(u; \omega, \theta)$.

The spectrum of operator $H_B(\omega, \theta)$ in a cube $B$ will be denoted by $\Sigma_{\omega,\theta}(B)$.

We will also use cubes $Q_r(\omega') \subset \mathbb{T}^\nu$ of the form

$$Q_r(\omega') = \{ \omega \in \Omega : \| \omega - \omega' \| \leq r \}, \quad r > 0.$$

We always assume that $g > 0$.

1.3. Local Variation Bound. The random field $v : \Omega \times \Theta \rightarrow \mathbb{R}$ on $\Omega$ relative to the auxiliary probability space $(\Theta, \mathfrak{B}, \mu)$ is assumed to fulfill the following condition:

**LVB**: There exists a family of sigma-algebras $\mathfrak{B}(L, x) \subset \mathfrak{B}$, labeled by non-negative integers $L$ and lattice points $x$, such that for all $L$ and $x$

(i) all random variables $v(T^y \omega, \theta)$ with $y \in B_{L+}(x) \setminus \{ x \}$ are $\mathfrak{F} \times \mathfrak{B}(L,x)$-measurable;

(ii) the random variable $(\omega, \theta) \rightarrow v(T^x \omega, \theta)$ on $\Omega \times \Theta$ admits a bounded conditional probability density $p_{v,x}(\cdot | \mathfrak{F} \times \mathfrak{B}(L,x))$:

$$\| p_{v,x}(\cdot | \mathfrak{F} \times \mathfrak{B}(L,x)) \|_{\infty} \leq C'' L^B, \quad C'' \in (0, +\infty). \tag{1.5}$$

In other words, for any lattice cube $B_L(u)$, even with the phase point $\omega$ in $\Omega$ and all values $\{ v(y), y \in B_L(x), y \neq x \}$ fixed, there is ‘enough parametric freedom’ in the potential value $V(x, \omega, \theta)$ to guarantee absolute continuity of its (conditional) probability distribution.

To clarify the nature of the sigma-algebras $\mathfrak{B}(L, x)$, which are constructed explicitly in Section 4 for a particular class of hulls $v : \Omega \times \Theta \rightarrow \mathbb{R}$, note that there actually exist sigma-algebras $\mathfrak{B}_L \subset \mathfrak{B}$, labeled only by scales $L > 0$, such that conditional on $\mathfrak{B}_L$ and on $\mathfrak{F}$ (i.e., with $\omega$ fixed) all values of the potential $\{ V(x, \omega, \theta), x \in B_L(u) \}$ in any cube of radius $L$ are conditionally independent and admit individual (non-identical) conditional probability densities $p_{v, \omega}(\cdot | \mathfrak{F} \times \mathfrak{B}_L)$ uniformly bounded by $O(L^B)$. This property gives rise to Wegner-type and Minami-type bounds in finite cubes, although these bounds deteriorate as the size of the cube grows. Furthermore, the exponent $B$ of the power-law growth of $\| p_{v, \omega}(\cdot | \mathfrak{F} \times \mathfrak{B}_L) \|_{\infty}$ depends upon the rate of returns figuring in the assumption (USR) and becomes higher when the exponent $A$ in (1.1) grows.

Following [6, 7], we will call a random field satisfying (LVB) a regular grand ensemble. Examples of such ensembles are given in Section 4. It is worth mentioning that for
any $M \geq 1$ there exist quite natural ”grand ensembles” with all samples $v(\omega, \cdot)$ of class $C^M(\Omega)$. Moreover, there exist regular grand ensembles with discontinuous samples for which one can prove Anderson localization in strong disorder regime; cf. [3]. On the other hand, (LVB) says that the local interpolation problem for the field $v(\omega, \cdot)$ on $\Omega$, relative to $(\Theta, \mathfrak{B}, \mu)$, must not admit an exact solution. This explains why our approach does not allow to treat analytic hulls. So, from several points of view, it is complementary to the approach developed by Bourgain, Goldstein and Schlag.

Note also that if $v(\omega, \theta)$ is a regular grand ensemble of class $C^M$ and $W \subset C^M(\Omega)$ is an arbitrary ”background potential”, then the operators $H = \Delta + W(\omega) + gV(\omega, \theta)$ feature Anderson localization for sufficiently large $g$; in fact, the background potential $W : \Omega \to \mathbb{R}$ would play almost no role in our analysis.

In this paper, we make one more assumption: a uniform boundedness of the gradient of the sample hull functions $\omega \mapsto v(\omega, \theta)$:

$$\exists \tilde{C} < \infty \forall \theta \in \Theta \quad \|v(\cdot, \theta)\|_{C^1(T^*)} \leq \tilde{C}. \tag{1.6}$$

The construction of the grand ensemble with the help of ”randedelle expansions” in Section 4 guarantees (1.6). In a more general context, one would need to exclude samples with excessively large gradients. Clearly, if $\mu \{ \|v(\cdot, \theta)\|_{C^1(T^*)} < \infty \} = 1$, then the set of samples with large gradients must have a small $\mu$-measure.

It is readily seen that for the random variables $gv(x; \omega, \theta)$, Eqn (1.5) gives rise to

$$\|p_{gv,x}\|_\infty \leq C'' g^{-1} L^B. \tag{1.7}$$

We will denote by $\mathbb{P}^{\Omega \times \Theta} \{ \cdot \}$ the product measure $\mathbb{P} \times \mu$ on $\Omega \times \Theta$ and by $\mathbb{E}^{\Omega \times \Theta} \{ \cdot \}$ the respective expectation.

### 1.4. Main result.

**Theorem 1.** Consider a family of lattice Schrödinger operators in $l^2(\mathbb{Z}^d)$,

$$H(\omega, \theta) = \Delta + gV(x; \omega, \theta)$$

where $V(x; \omega, \theta) = v(T^x \omega, \theta)$. Suppose that the dynamical system $T : \mathbb{Z}^d \times \Omega \to \Omega$ and the function $v : \Omega \times \Theta \to \mathbb{R}$ satisfy the conditions (USR), (LVB) and (DIV) for some $A, B, C, C', C'' \in (0, \infty)$. For sufficiently large $g \geq g_0(A, B, C, C', C'')$, there exists a subset $\Theta_{\infty}(g) \subset \Theta$ of measure $\mu \{ \Theta_{\infty}(g) \} \leq 1 - c(A, B, C, C', C'') g^{-1/2}$ with the following property: if $\theta \in \Theta_{\infty}$, then for $\mathbb{P}$-a.e. $\omega \in \Omega$ the operator $H(\omega, \theta)$ has pure point spectrum, and for every eigenfunction $\psi_j \in l^2(\mathbb{Z}^d)$ there exist $u_j \in \mathbb{Z}^d$ and $L \in \mathbb{N}$ such that for all $x$ with $\|x - u_j\| \geq L$

$$|\psi_j(x)| \leq e^{-m|x-u_j|}$$

with $m = m(g, A, B, C, C', C'') > c_1(A, B, C, C', C'') \ln g > 0$.

**An informal outline of the proof.**

1. We use the general strategy of the MSA which requires, at each scale $L_k = L_0^k$, $\alpha > 1$, two kinds of estimates:
   - an eigenvalue concentration (EVC) bound for the probability of having two disjoint cubes of radius $L_k$ inside a larger cube $B_{L_{k+1}}(w)$ with spectra abnormally close to each other, at some energy $E$ ("$E$-resonant" cubes);
   - a bound for the probability to have at least $\nu + 2 \equiv \dim \Omega + 2$ cubes of radius $L_k$ inside a larger cube $B_{L_{k+1}}(w)$ in which the decay of the matrix elements of the resolvent is insufficient ("singular" cubes).
The EVC bound in our case is proven in two ways:

- for a large set of parameters $\theta \in \Theta$ and for any phase point $\omega \in \Omega$, the maximal number of simultaneously "resonant" cubes in $B_{L_{k+1}}(w)$ does not exceed $\nu + 1$;
- the $\mathbb{P}$-probability to have at least two "resonant" cubes in $B_{L_{k+1}}(w)$ is small (sufficient for the purposes of the MSA).

The first property rules out – in a deterministic way – an accumulation of resonant cubes, which is inevitable (albeit unlikely) in the case of random (e.g., IID) potentials. The second is necessary for the MSA; it is simpler to prove than the first one. Both of them are proven without scale induction.

It is worth noticing that at the initial scale $L_0$, under the assumption of large disorder, "non-resonant" cubes are also "non-singular". Therefore, the first property rules out an accumulation of "singular" cubes at least at the initial scale. This "sparseness" property is then to be proven inductively at all scales; see the proof of Lemma 3.5.

Finally, we modify the traditional MSA tactics, which delays the analysis of the eigenfunction decay until the last stage where the finite-volume bounds of Green functions are established at any scale. Namely, we make use of the Geometric Resolvent Inequality (GRI) for eigenfunctions at scale $L_k$ and derive from the "sparseness" of "singular" cubes and EVC estimates a lower bound on the probability of having all eigenfunctions exponentially decaying in a cube $B_{L_k}(u)$. The spectral localization in $\mathbb{Z}^d$ is proven in the usual way.

2. Scale induction

2.1. Resonances and tunneling.

**Definition 2.1.** Given a real number $E$, a cube $B_{L_k}(u)$, $k \geq 0$, is called

- $(E, \omega, \theta)$-non-resonant ($(E, \omega, \theta)$-NR) if
  \[ d(\Sigma_{\omega, \theta}(B_{L_k}(u)), E) \leq g\delta_k \equiv g^{1-a} e^{-4L_k^b}, \tag{2.1} \]
  and $(E, \omega, \theta)$-resonant ($(E, \omega, \theta)$-R), otherwise;
- $(E, \omega, \theta)$-completely non-resonant ($(E, \omega, \theta)$-CNR) if it does not contain $(E, \omega, \theta)$-resonant cubes of radius $\ell \geq L_{k-1}$ (including itself); otherwise, it is called $(E, \omega, \theta)$-partially resonant ($(E, \omega, \theta)$-PR);
- $(\omega, \theta)$-tunneling ($(\omega, \theta)$-T) if for some $E \in \mathbb{R}$ it contains two disjoint $(E, \omega, \theta)$-PR cubes of radius $\geq L_k^{1/4}$, and $(\omega, \theta)$-non-tunneling ($(\omega, \theta)$-NT), otherwise;
- $(\omega, \theta)$-multi-resonant ($(\omega, \theta)$-MR) if for some $E \in \mathbb{R}$ it contains at least $\nu + 2$ disjoint $(E, \omega, \theta)$-PR cubes of radius $L_{k-1}$; otherwise, it is called $(\omega, \theta)$-NMR.

**Definition 2.2.** Given real numbers $E$ and $m > 0$, a cube $B_{L_k}(u)$ is called

- $(E, m, \omega, \theta)$-non-singular ($(E, m, \omega, \theta)$-NS) if
  \[ \max_{\|x-u\| \leq L_{k-1}} \sum_{(y,y') \in \partial B_{L_k}(u)} \left| G_{B_{L_k}(u)}(x, y; E; \omega, \theta) \right| \leq e^{-\gamma(m, L_k)L_k}, \tag{2.2} \]
  where
  \[ \gamma(m, L) := m(1 + L^{-1/8}) > m, \tag{2.3} \]

\(^2\)Recall that $\nu$ is the dimension of the phase space $\Omega = \mathbb{T}^\nu$. 
We prove Lemma 2.1 in Appendix, using the properties (L VB expansion (cf. Section 4), the Wegner-type bound was proven in our earlier work \[6\].

**Lemma 2.2.** For Wegner-type bounds

**Lemma 2.3.** Fix a point \(\omega\) and consider the event of the form

\[
\mathcal{R}(k, \omega') = \left\{ \theta : \exists \text{ disjoint cubes } B_{R_j}(v^{(j)}) \subset B_{L_k}(u^{(j)}) \subset B_{L_k}(0), 1 \leq j \leq J_{\nu}, \right.
\]

with \(R_j \in [L_{k-1}, L_k]\) and such that for \(j = 2, \ldots, J_{\nu}\),

\[
d \left( \Sigma_{\omega', \theta}(B_{R_j}(v^{(j)})), \Sigma_{\omega', \theta}(B_{R_j}(v^{(1)})) \right) \leq 4g\delta_k \}
\]

Then

\[
\mu \left\{ \mathcal{R}(k, \omega') \right\} \leq C_1 L_k^{J_{\nu}(3d+3)B+1} \delta_k^{J_{\nu}-1}.
\] (2.8)

**Proof.** Fix an arbitrary cubes \(B_{R_i}(u^{(i)}) \subset B_{L_k}(u^{(j)})\) as in \(\mathcal{R}(k, \omega')\) and consider sigma-algebras \(\mathcal{B}_j := \mathbb{B}(L_k^d, u^{(j)})\), \(1 \leq j \leq J_{\nu}\), figuring in the condition (LVB).

Observe that, by construction, for each \(j \geq 2\), all eigenvalues of operators \(H_{B_{R_i}(u^{(i)})}\) with \(i < j\) are \(\mathbb{F} \times \mathbb{B}(L_k^d, u^{(j)})\)-measurable, since \(B_{L_k}(u^{(i)}) \subset B_{L_k}(0) \setminus B_{L_k}(u^{(j)})\). So, conditioning on \(\mathbb{F} \times \mathbb{B}(L_k^d, u^{(j)})\) fixes \(\omega\) and spectra of \(H_{B_{R_i}(u^{(i)})}, i < j\). With \(\omega = \omega'\) fixed, all spectra become functions of \(\theta \in \Theta\). Let

\[
B_i = B_{R_i}(u^{(i)}), \quad 1 \leq i \leq J_{\nu},
\]

\[
\Sigma_i = \Sigma_{\omega', \theta} = B_{R_i}(u^{(i)})\), \quad 1 \leq i \leq J_{\nu},
\]

\[
D_i = D_i(\omega') = \left\{ \theta : d(\Sigma_1, \Sigma_i) \leq 4g\delta_k \right\}, \quad 2 \leq i \leq J_{\nu},
\]
and denote by $\mathfrak{B}_{<j}$ the sigma-algebra generated by sigma-algebras $\{\mathfrak{B}_i, 1 \leq i < j\}$. Then for every $2 \leq j \leq J_\nu$, we can write, using the inequality $R_i \leq L_k$:

$$
\mu \left\{ D_j \mid \mathfrak{B}_{<j} \right\} \leq (2L_k + 1)^{2d} \max_{\lambda^{(j)}_{\Sigma_j}, \lambda^{(1)}_{\Sigma_1}} \mu \left\{ \left| \lambda^{(j)}_\Sigma - \lambda^{(1)}_\Sigma \right| \leq 4g\delta_k \mid \mathfrak{B}_{<j} \right\}. 
$$

As was noticed, for fixed $\omega'$, eigenvalues $\theta \mapsto \lambda^{(j)}_\Sigma(\omega', \theta)$ are $\mathfrak{B}_{<j}$-measurable; we work with these random variables $\lambda^{(1)}_\Sigma(\omega', \cdot)$ and denote by $\mu_{j-1} \{ \cdot \}$ the conditional measure $\mu \left\{ \cdot \mid \mathfrak{B}_{<j} \right\}$. It suffices to bound the probability $\mu_{j-1} \left\{ \left| \lambda^{(j)}_\Sigma - E \right| \leq 4g\delta_k \right\}$ for any fixed $E \in \mathbb{R}$. This can be done with the help of the conventional Wegner bound. Indeed, consider operator $H_{B_j}$ in the cube $B_j$. For the Wegner bound to apply, it suffices that for each point $x \in B_j$, the random variable $gV(x; \omega, \theta)$ admit a bounded probability density, conditional on all other values of the potential $\{gV(y; \omega, \theta), y \in B_j \setminus \{x\}\}$.

Recall that owing to the assumption ((LVB), (i)), all values $V(y; \omega, \theta)$ with $x \neq y \in B_{L_k}(u^{(j)})$ are $(\mathfrak{F} \times \mathfrak{B}(L_k^d, x))$-measurable. Moreover, by assumption ((LVB), (ii)) (cf. (1.5), (1.7)) the random variable $gV(x; \omega, \theta)$, conditional on $\mathfrak{F} \times \mathfrak{B}(L_k^d, x)$, does indeed admit a probability density bounded by $C^{\nu}g^{-1}L_k^{B}$. As a result, for some $C_2 < \infty$,

$$
\mu_{j-1} \left\{ \left| \lambda^{(j)}_\Sigma - E \right| \leq 4g\delta_k \right\} \leq C_2 L_k^{B} \delta_k
$$

and since the number of pairs $(\lambda^{(j)}_\Sigma, \lambda^{(1)}_\Sigma)$ is bounded by $(2R_j + 1)^d(2R_1 + 1)^2 \leq 9L_k^{2d}$,

$$
\text{ess sup} \mu \left\{ D_j \mid \mathfrak{B}_{<j} \right\} \leq C_3 L_k^{2d+B} \delta_k.
$$

Next, one can re-write $\mu \{ \cap_{i \leq j} D_i \}$ as follows:

$$
\mathbb{E}^{(\theta)} \left[ \mathbb{E}^{(\theta)} \left[ \prod_{1 \leq i \leq j} 1_{D_i} \mid \mathfrak{B}_{<j} \right] \right] \leq \mu \left\{ \cap_{1 \leq i < j} D_i \right\} \text{ess sup} \mu \left\{ D_j \mid \mathfrak{B}_{<j} \right\} \quad (2.9)
$$

(here $\mathbb{E}^{(\theta)} \left[ \cdot \right]$ is the expectation relative to $(\Theta, \mathfrak{B}, \mu)$), and by induction

$$
\mu \{ \cap_{i \leq J_\nu} D_i \} \leq C_4 \left( L_k^{2d+B} \right)^{J_\nu - 1} \delta_k^{J_\nu - 1}.
$$

The total number of families $\{B_{L_k}(v^{(j)}) \subset B_{L_k}(u^{(j)}) \subset B_{L_k}(0)\}$ and arbitrary $R_j \leq L_k$ is bounded by $\frac{1}{l_\nu} L_k^{J_\nu(d+1)}$, so that for $L_0$ large enough we obtain

$$
\mu \{ \mathcal{R}(\omega', k) \} \leq L_k^{(3d+B+1)J_\nu} \delta_k^{J_\nu - 1}.
$$

This completes the proof.

**Corollary 2.1.** Let $N_k \geq 1$ be an integer, $k \geq 0$; set $r_k = 1/(2N_k)$ and cover the torus $T^d$ by the cubes $Q_r(i)(\omega_i), 1 \leq i \leq (N_k)^{\nu} = (2r_k)^{-\nu}$, with centers $\omega_i$ of the form $\omega_i = ((2l_1 + 1)r_k, \ldots, (2l_\nu + 1)r_k), l_1, \ldots, l_\nu \in [0, N_k - 1] \cap \mathbb{Z}$.

Using notations of Lemma 2.2, introduce the event

$$
\mathcal{N}_k = \Theta \setminus \bigcup_{1 \leq i \leq (N_k)^{\nu}} \mathcal{R}(k, \omega_i). \quad (2.10)
$$

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3As was pointed out in Section 1.8 for grand ensembles constructed in Section 4 there actually exist sigma-algebras $\mathfrak{B}_L \subset \mathfrak{B}$ such that, conditional on $\mathfrak{F} \times \mathfrak{B}_L$, all values $v(x; \omega, \theta)$ with $y \in B_L(x)$ and any $x$ are independent and admit individual conditional densities bounded by $O(L^B)$.

4Recall that all samples $V(\cdot, \theta)$ are assumed to be smooth functions on $\Omega = T^d$, so for every $\omega' \in \Omega$ and $x \in \mathbb{Z}^d$ the value $V(T^d \omega', \cdot)$ is a well-defined random variable on $\Theta$. 
Then

\[ \mu \{ \Theta \setminus \mathcal{N}_k \} \leq C_5(\nu, d) L_k J_{-\nu}^{(3d+B+1)} r_k^{-\nu} \delta_k^{\nu+1}. \tag{2.11} \]

\textbf{Proof.} It suffices to apply Lemma 2.3 to each of the \((N_k)^\nu = (2r_k)^{-\nu}\) centers \(\omega_i\). \(\square\)

Now set

\[ r_k = \delta_k^{1+\frac{1}{\nu}}, \quad k \geq 0, \tag{2.12} \]

and observe that

\[ r_k^{-\nu} \delta_k^{\nu+1} = \delta_k^{\nu} \frac{2\delta_k}{2\nu} + \nu + 1 = \delta_k^{1/2}, \]

so that the bound (2.11) takes the form

\[ \mu \{ \Theta \setminus \mathcal{N}_k \} \leq C_5(\nu, d) L_k J_{-\nu}^{(3d+B+1)} \delta_k^{1/2}. \tag{2.13} \]

\textbf{Lemma 2.4.} For all \(\theta \in \mathcal{N}_k, E \in \mathbb{R}, u \in \mathbb{Z}^d\) and any \(\omega \in \Omega\) there are at most \(\nu + 1\) pairwise disjoint lattice cubes \(B_{L_k}(u^{(j)}) \subset B_{L_k}(u)\) which are \((E, \omega, \theta)\)-PR.

\textbf{Proof.} Since the gradient of the function \(\omega \mapsto v(\omega; \theta)\) is bounded (cf. (1.6)), we have

\[ \forall \theta \in \Theta \quad \sup_{\omega \in Q_{t_k}(\omega)} |gv(\omega; \theta) - gv(\omega_k; \theta)| \leq C_6(\nu) g \operatorname{diam}(Q_{t_k}(\omega_i)) \leq C_6(\nu) gr_k \]

and, therefore, for any \(u \in B_{L_k}^2(0)\), for \(\delta_k\) small enough (so that \(r_k = \delta_k^{1+\frac{1}{\nu}} < \delta_k\))

\[ \sup_{\omega \in Q_{t_k}(\omega)} \|H_{B_{L_k}(u)}(\omega, \theta) - H_{B_{L_k}(u)}(\omega_k; \theta)\| \leq C_6(\nu) gr_k \leq \frac{1}{2} g \delta_k \tag{2.14} \]

By construction of \(\mathcal{N}_k\), for any \(E \in \mathbb{R}\), if a cube \(B_{L_k}(u^{(1)})\) is \((E, \omega, \theta)\)-R, then there are at most \(\nu\) pairwise disjoint cubes \(B_{L_k}(u^{(j)})\), \(2 \leq j \leq \nu + 1\), disjoint also with \(B_{L_k}(u)\) and such that if a cube \(B_{L_k}(w)\) is disjoint with the collection \(\{B_{L_k}(u^{(j)}), 1 \leq j \leq \nu + 1\}\), then for any center \(\omega_i, 1 \leq i \leq N_k\), and any cubes \(B_R(v) \subset B_{L_k}(w), B_{R_1}(v^{(1)}) \subset B_{L_k}(u^{(1)})\) with \(R, R_1 \in \{L_{k-1}, L_k\}\)

\[ d(\Sigma(B_R(v), \omega_i; \theta), \Sigma(B_{R_1}(v^{(1)}), \omega_i; \theta)) \geq 4g \delta_k. \tag{2.15} \]

Pick any \(\omega \in \Omega\) and let \(Q_{t_k}(i') \subset T^\nu, i' = i'(\omega)\), be the cube containing \(\omega\). Taking into account (2.14–2.15), we can write

\[ \begin{align*}
    &d(\Sigma_{\omega, \theta}(B_R(v)), \Sigma_{\omega, \theta}(B_{R_1}(v^{(1)}))) \\
    &\geq d(\Sigma_{\omega, \theta}(B_R(v)), \Sigma_{\omega, \theta}(B_{R_1}(v^{(1)}), \omega_i; \theta)) \\
    &- d(\Sigma_{\omega, \theta}(B_R(v)), \Sigma_{\omega, \theta}(B_{R_1}(v^{(1)}), \Sigma_{\omega, \theta}(B_{R_1}(v^{(1)}))) \\
    &\geq 4g \delta_k - 2 \cdot \frac{1}{2} g \delta_k > 2g \delta_k.
\end{align*} \]

Therefore, for any \(E \in \mathbb{R}\) and any \(\omega \in \Omega\)

\[ \min \{ d(\Sigma_{\omega, \theta}(B_R(v)), E), d(\Sigma_{\omega, \theta}(B_{R_1}(v^{(1)})), E) \} > g \delta_k. \]

As a result, there is no collection of more than \(\nu + 1\) pairwise disjoint \((E, \omega, \theta)\)-PR cubes of radius \(L_k\) with centers in \(B_{L_k}(0)\). \(\square\)

\textbf{Proof of Lemma 2.2.} It suffices to notice that, by Lemma 2.3 for every \(\theta \in \mathcal{N}_k\) and any \(\omega \in \Omega\), any cube \(B_{L_k}(u)\) is NMR (cf. Definition 2.1), and by Corollary 2.1 with the convention (2.12), for \(L_k\) large enough,

\[ \mu \{ \Theta \setminus \mathcal{N}_k \} \leq C_5(\nu, d) L_k J_{-\nu}^{(3d+B+1)} \delta_k^{1/2} \leq g^{-a/2} e^{-L_k^b}. \]
3. MSA for grand ensembles of deterministic operators

3.1. Initial scale bounds.

**Lemma 3.1.** Let \( m > 0 \) and \( B_{L_0}(u) \) be an \((E, \omega, \theta)\)-NR cube, i.e.,
\[
\| (\Sigma_{\omega, \theta}(B_{L_0}(u)), E) \| \geq g \delta_0 = g^{1-a} e^{-4L_0^\alpha}, \tag{3.1}
\]
If \( g \delta_0 > 2d + 4de^{4\gamma(m, L_0)} \) and \( L_0 \) is large enough then \( B_{L_0}(u) \) is \((E, m, \omega, \theta)\)-NS.

**Proof.** By min-max principle applied to the operators \( gV(\omega, \theta) \) and \( H(\omega, \theta) = gV(\omega, \theta) + \Delta \), with \( \| \Delta \| \leq 2d \), the assumption of the Lemma implies that
\[
\| (\Sigma_{\omega, \theta}(B_{L_0}(u)), E) \| \geq g \delta_0 - 2d \geq 4de^{4\gamma(m, L_0)} =: \eta > 2.
\]
By Combes–Thomas estimate combined with \((3.1)\), the Green functions obey
\[
|G(x, y; E)| \leq \frac{2}{\eta} \exp \left( -\frac{1}{2} \left( \ln \frac{\eta}{4d} \right) \| x - y \|_1 \right) < e^{-2\gamma(m, L_0)\|x-y\|},
\]
since \( \eta > 2 \) and \( \|x\|_1 \geq \max_i |x_i| = \|x\| \). For \( L_0 \) large enough, this implies \((2.2)\). \( \square \)

3.2. Collections of singular cubes. Define the subsets \( T_k \subset \Theta \) of the form
\[
T_k = \{ \theta \mid \exists \{ \omega : B_{L_k}(u) \text{ is } (\omega, \theta)\text{-T } \} \geq \delta_k^{1/2} \}
\]
(recall that ",(\omega, \theta)\text{-T}" stands for "(\omega, \theta)\text{-tunneling}", cf. Definition \(2.1\)) and
\[
\Theta_k = \bigcap_{l \leq k} (\Lambda_l \setminus T_l), \quad k = 0, 1, \ldots, \tag{3.2}
\]
Next, introduce the following statement, relative to the scale \( L_k \), \( k \geq 0 \):

**Sparse(k):** For any \( \theta \in \Theta_k \), all \( \omega \in \Omega \), any \( E \in \mathbb{R} \) and any lattice cube \( B_{L_k^1}(u) \) there exist at most \( J_\nu - 1 = \nu + 1 \) pairwise disjoint cubes \( B_{L_k}(u^{(i)}) \subset B_{L_k^1}(u) \) which are \((E, m, \omega, \theta)\)-S.

**Remark 3.1.** The property \((\text{Sparse}(k))\) implies, in particular, that for any \((\omega, \theta) \in \Omega \times \Theta_k \), any cube \( B_{L_{k+1}}(u) \subset B_{L_k^1}(u) \) must be \((m, \omega, \theta)\)-good (cf. Definition \(2.2\)).

**Remark 3.2.** The exponent 4 in \( L_k^1 \) is quite arbitrary; replacing it by any larger \( D > 0 \) would not affect main arguments, but only modify technical constants. We chose the exponent 4 > \( \alpha \) simply to stress that a cube of size much larger than \( L_{k+1} = L_k^s \) contains a limited number of simultaneously singular cubes of radius \( L_k \). We believe that an optimal scale should be exponential or sub-exponential in \( L_k \).

**Lemma 3.2.** The condition \((\text{Sparse}(0))\) is fulfilled for sufficiently large \( g \).

**Proof.** Consider an arbitrary cube \( B_{L_0}(u) \) and a number \( E \in \mathbb{R} \). By construction of the set \( \Theta_0 \), for any \( \omega \in \Omega \) there is a collection \( \mathcal{S}(E) \) of at most \( J_\nu - 1 \) cubes \( B_{L_0}(u^{(i)}) \subset B_{L_0}(u) \) such that any cube \( B_{L_0}(v) \) disjoint from \( \mathcal{S}(E) \) is \((E, m, \omega, \theta)\)-NR. By \(\text{Lemma 3.1}\) if \( g^{1/2} \delta_0 > 2d + 4de^{2\gamma(m, L_0)L_0} \), then such cubes \( B_{L_0}(v) \) must also be \((E, m, \omega, \theta)\)-NS. \( \square \)

**Lemma 3.3.** If for some \( \omega \in \Omega \), \( \theta \in \Theta \) and \( E \in \mathbb{R} \) a cube \( B_{L_k} \) is \((m, \omega, \theta)\)-good and \((E, \omega, \theta)\)-CNR, then it is also \((E, m, \omega, \theta)\)-NS.

**Proof.** The claim follows directly from \(\text{Lemma 3.2}\), its idea goes back to \(13, 12\). \( \square \)

It is convenient to re-formulate \(\text{Lemma 3.3}\) in the following way.
Lemma 3.4. Assume the property \((\text{Sparse}(k))\), and let \(\theta \in \Theta_k\). If a cube \(B_{L_{k+1}}(u)\) is \((E,\omega,\theta)\)-CNR, for some \(E \in \mathbb{R}\) and \(\omega \in \Omega\), then it is also \((E,m,\omega,\theta)\)-NS.

Proof. As was pointed out in Remark 3.4 if \(\theta \in \Theta_k\), then by assumed property \((\text{Sparse}(k))\) any cube \(B_{L_{k+1}}(u)\) is \((m,\omega,\theta)\)-good. Further, \(B_{L_{k+1}}(u)\) is assumed to be \((E,\omega,\theta)\)-CNR, so Lemma 3.3 implies that it is \((E,m,\omega,\theta)\)-NS. \(\Box\)

3.3. Scale induction.

Lemma 3.5. Statement \((\text{Sparse}(k))\) implies \((\text{Sparse}(k+1))\).

Proof. Pick any \(\theta \in \Theta_{k+1} \subset \mathcal{N}_{k+1}\) and any \(E \in \mathbb{R}\). By construction of the set \(\mathcal{N}_{k+1}\), for any \(\omega \in \Omega\) there exists a collection \(\mathcal{R}_{k,u}(E,\omega)\) of at most \(J_j - 1 = \nu + 1\) cubes \(B_{L_{k+1}}(u^{(j)}) \subset B_{L_{4j}}(u)\) such that any cube \(B_{L_{k+1}}(v)\) disjoint with \(\mathcal{R}_{k,u}(E,\omega)\) must be \((E,\omega,\theta)\)-CNR. Further, by assumption \((\text{Sparse}(k))\), for any \(\omega \in \Omega\) the cube \(B_{L_{k+1}}(v)\) cannot contain \(J_j\) or more disjoint \((E,m,\omega,\theta)\)-S cubes of radius \(L_k\), and by Lemma 3.3 it must be \((E,m,\omega,\theta)\)-NS. Therefore, any cube \(B_{L_{k+1}}(v) \subset B_{L_{k+1}}(u)\) disjoint with \(\mathcal{R}_{k,u}(E,\omega)\) is \((E,m,\omega,\theta)\)-NS; this proves the assertion \((\text{Sparse}(k+1))\). \(\Box\)

Since the validity of \((\text{Sparse}(0))\) is established in Lemma 3.2 we come, by induction, to the following conclusion:

Theorem 2. For \(g\) large enough, the condition \((\text{Sparse}(k))\) holds true for all \(k \geq 0\).

3.4. Localization of eigenfunctions in finite cubes.

Definition 3.1. Given a sample \(v(\omega,\theta)\), a cube \(B_{L_k}(u)\) is called \((m,\omega,\theta)\)-localized \(((m,\omega,\theta) - \text{Loc})\) if for any eigenfunction \(\psi_j\) of operator \(H_{B_{L_k}}(\omega,\theta)\) and any points \(x, y \in B_{L_k}(u)\) with \(\|x - y\| \geq L_k^{7/8}\)

\[|\psi_j(x)\psi_j(y)| \leq e^{-\gamma(m,L_{k-1})\|x-y\|},\]  

(3.3)
otherwise, it is called \((m,\omega,\theta)\)-non-localized \(((m,\omega,\theta)-\text{NLoc})\).

Set \(\Theta_\infty = \Theta_\infty(g) = \bigcap_{k \geq 0} \Theta_k\).

Theorem 3. (A) \(\mu \{ \Theta_\infty(g) \} \geq 1 - C_7g^{-a/2}\);  
(B) for any \(\theta \in \Theta_\infty\) and any \(k \geq 0\),

\[\mathbb{P}\{ \omega : B_{L_k}(u) \text{ is } (m,\omega,\theta)-\text{NLoc} \} \leq g^{-a/2}e^{-L_k^{b}}.\]  

(3.4)

Proof. The first assertion follows directly from Corollary 2.1 combined with assertion (B) of Lemma 2.1 (cf. Eqn 2.6). Further, let \(\theta \in \Theta_\infty \subset \Theta_k\). Then, by assertion (B) of Lemma 2.1 either \(B_{L_k}(u)\) is \((\omega,\theta)\)-tunneling, or it is \(m-\text{Loc}\), so that, owing to Lemma 2.1 we have

\[\mathbb{P}\{ B_{L_k}(u) \text{ is } (m,\omega,\theta)-\text{NLoc} \} \leq \mathbb{P}\{ B_{L_k}(u) \text{ is } (\omega,\theta)-\text{T} \} \leq g^{-a/2}e^{-L_k^{b-1}}.\]  

\(\Box\)
3.5. Spectral localization: Proof of Theorem 1. Fix any \( \theta \in \Theta_\infty \) and let \( \psi \) be a nontrivial, polynomially bounded solution of equation \( H(\omega, \theta) \psi = E \psi. \) There exists a point \( \hat{u} \) where \( \psi(\hat{u}) \neq 0 \) and, as a result, there exists an integer \( k_\omega \) such that for all \( L \geq L_{k_\omega} \) the cube \( B_L(\hat{u}) \) is \( (E, m, \omega, \theta)-S \): otherwise, the \( (E, m, \omega, \theta)-NS \) property would imply, for arbitrarily large \( L > 0 \),
\[
|\psi(\hat{u})| \leq O(L^e) e^{-mL} \xrightarrow{L \to \infty} 0.
\]

Let
\[
\Omega_j^* = \{ \omega : \forall k \geq j \text{ the cube } B_{L_k}(\hat{u}) \text{ is } (\omega, \theta)-NT \}.
\]

Since \( \theta \in \Theta_\infty \) and \( \Theta_\infty \cap T_\theta = \emptyset \), we have \( \mathbb{P} \{ \omega : B_{L_k}(\hat{u}) \text{ is } (\omega, \theta)-T \} \leq g^{-a/2} e^{-L_k^a} \), so it follows from Borel-Cantelli lemma that for \( \mathbb{P} \)-a.e. \( \omega \) there exists \( k_1 = k_1(\omega) \) such that \( \omega \in \Omega_{k'_1} \). Fix such an element \( \omega \) and set \( k_2(\omega) = \max\{k_0, k_1(\omega)\} \). From this point on, we will analyze the behavior of function \( \psi \) at distances \( \geq 3L_{k_2} \) from \( \hat{u} \).

Introduce annuli \( A_k = B_{3L_{k+1}}(\hat{u}) \setminus B_{3L_k}(\hat{u}) \), \( k \geq k_2 \), and let \( x \in A_k \). Set \( R := \|x - \hat{u}\| - 2L_{k-1} - 1 \) and consider an arbitrary cube \( B_{L_{k-1}}(y) \subset B_R(x) \). Since \( B_{3L_{k+1}}(\hat{u}) \) is \( (\omega, \theta)-NT \) and \( 3L_{k+1} < L_{k-1} \), either \( B_{L_{k-1}}(\hat{u}) \) or \( B_{L_{k-1}}(y) \) must be \( (E, \omega, \theta)-CNR \) (cf. Definition 2.1).

Let us show that cube \( B_{L_{k-1}}(\hat{u}) \) cannot be \( (E, \omega, \theta)-CNR. \) Indeed, since \( \theta \in \Theta_\infty \) by assumption, the cube \( B_{L_{k-1}}(\hat{u}) \) contains less than \( J_\nu \) disjoint \( (E, m)-S \) cubes of radius \( L_{k-2} \). Combined with \( (E, \omega, \theta)-CNR \) property, this would imply, by virtue of Lemma 5.2 that \( B_{L_{k-1}}(\hat{u}) \) is \( (E, m)-NS, \) which contradicts the choice of the scale \( L_{k_2}. \) Therefore, \( B_{L_{k-1}}(y) \) is \( (E, \omega, \theta)-CNR. \)

Using again the assumption \( \theta \in \Theta_\infty \), we see that every cube \( B_{L_{k-1}}(y) \subset B_R(x) \) contains less than \( J_\nu \) disjoint \( (E, m)-S \) cubes of radius \( L_{k-2} \) and is \( (E, \omega, \theta)-CNR \). By Lemma 5.2 all cubes \( B_{L_{k-1}}(y) \) are \( (E, m)-NS, \) so the same lemma implies that the cube \( B_R(x) \) itself is \( (E, m)-NS. \) Therefore, we can write, with the convention \( \ln 0 = -\infty, \)
\[
\ln |\psi(x)| \|x - \hat{u}\| \leq -\gamma(m, R) R \leq -m(1 + \|x - \hat{u}\|^{-1/8}) \|\|x - \hat{u}\| - 3L_{k-1}\| \leq -m.
\]

4. Examples of regular grand ensembles

4.1. ”Randelette” expansions. Following [6,7], consider a function \( v : \Omega \times \Theta \to \mathbb{R} \) given by a series of the form
\[
v(\omega, \theta) = \sum_{n=0}^{\infty} a_n \sum_{k=1}^{K_n} \theta_{n,k} \varphi_{n,k}(\omega)
\]
where the family of random variables \( \{\theta_{n,k}, n \in \mathbb{N}, k \in [1, K_n]\} \) on \( (\Theta, \mathcal{B}, \mu) \) is IID with bounded common probability density \( \rho_\theta \). Two particular choices are technically convenient: the uniform distribution on \( [0, 1] \), where \( \rho_\theta(t) = 1_{[0,1]}(t) \), and the standard Gaussian distribution \( \mathcal{N}(0, 1) \). Below we assume that \( \rho_\theta(t) = 1_{[0,1]}(t) \).

The functions \( \varphi_{n,k} \) with the same value of \( n \) (referred to as the \( n \)-th generation) are supposed to have a uniformly bounded overlap of their supports: for some \( K' \in \mathbb{N}, \)
\[
\sup_{n \geq 0} \sup_{\omega \in \Omega} \text{card}\{k : \omega \in \text{supp } \varphi_{n,k}\} \leq K'.
\]
Following [6], we will call these functions ronderelettes, and the representation (4.1) will be called a randelette expansion.
In order to obtain samples \( v(\cdot, \theta) \) of class \( C^M \), \( M \geq 1 \), the functions \( \varphi_{n,k} \), have to be assumed of class \( C^M (\Omega) \). In the case of the uniform distribution, the random variables \( \theta_{n,k} \) (‘siblings amplitudes’) are bounded, so that the convergence of the series \([11]\) is encoded in the decay properties of the ‘generation amplitudes’ \( a_n, \ n \geq 0 \).

On the other hand, the random field of the form \([11]\) has to fulfill the condition (LVB), and it is clear that an excessively rapid decay of amplitudes \( a_n \) can destroy the local variation bound. We will show that for every smoothness class \( C^M \), one can find an acceptable compromise between these two opposite requirements: convergence of the series \([11]\) in \( C^M \) and the ‘local freedom’ condition (LVB).

4.2. An example of \( C^1 \)-randerelles on \( T^1 \). The general structure of randelette expansions, as well as the term ”randelette”, is clearly inspired by wavelets (ondelettes, in French). However, the orthogonality issues are of little importance here, and the finite-overlap condition, serving as a substitute of orthogonality, is more than sufficient for applications to Wegner-type estimates and to localization theory.

Consider the following function on \( \mathbb{R} \):

\[
\Phi(t) = \frac{t^2}{2} 1_{[0,1)}(t) + \left( 1 - \frac{(t-2)^2}{2} \right) 1_{[1, 2]}(t) + 1_{[2, +\infty)}(t).
\]

By direct inspection, one can check that \( \Phi \in C^1(\mathbb{R}) \) and \( \|\Phi\|_{C^1(\mathbb{R})} = 1 \). Similarly, the function \( t \mapsto \Phi(12 - t) \) has unit \( C^1(\mathbb{R}) \)-norm, and so does the product \( \Phi(t) = \phi(t)\phi(12 - t) \), which vanish outside the interval \((0, 2^4)\) and equals 1 on the segment \([2, 14]\). Further, define a sequence of scaled functions

\[
\Phi_n(t) = \Phi(2^n t), \quad \|\Phi_n\|_{C^1(\mathbb{R})} = 2^n,
\]

with \( \text{supp} \Phi_n = [0, 2^{1-n}] \), and their shifts

\[
\varphi_{n,k}(t) = \Phi_n(t - k), \quad k \in \mathbb{Z},
\]

with \( \text{supp} \varphi_{n,k} = \left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right] \). Using the natural projection \( \mathbb{R} \to \mathbb{R}/\mathbb{Z} = T^1 \), one can consider \( \varphi_{n,k} \) as functions on the unit circle \( T^1 \), and it is clear that

- the family \( \{ \varphi_{n,k}, 1 \leq k \leq 2^n \} \) has a bounded overlap \( (K_n \leq 2^4) \),
- each point \( t \in T^1 \) is covered by a segment on which at least one of the functions \( \varphi_{n,k} \) identically equals 1.

Form now a randelette expansion \([1.1]\) with \( a_n = e^{cn} \). Since \( \|\varphi_{n,k}\|_{C^1} = O(2^n) \), we see that for \( c \) large enough the series \([1.1]\) converges uniformly in \( C^1(T^1) \), regardless of the values of the random coefficients \( \theta_{n,k} \) (all of them are bounded by \( 1 \)). So, we obtain an example of a smooth randelette expansion on the one-dimensional torus \( T^1 \).

4.3. Randerelles of class \( C^M \). An adaptation of the previously described construction to the case where the series \([1.1]\) is to be of class \( C^M \) is fairly straightforward. Indeed, for any \( M \geq 1 \) there exists a \( C^M \)-function \( \Phi : \mathbb{R} \to \mathbb{R} \) equal to 1 on an interval \([1/4, 3/4]\) and vanishing outside \([0, 1]\); it can be easily constructed explicitly, e.g., with the help of the so-called B-splines (convolutions of indicator functions of finite intervals). Then for the functions \( \varphi_{n,k}(t) = \Phi(2^n(t-k)) \) we have

\[
\|\varphi_{n,k}\|_{C^M(\mathbb{R})} \leq 2^n \|\Phi\|_{C^M(\mathbb{R})} \leq \text{Const} \ e^{-cMn}.
\]

Therefore, for \( c > 0 \) large enough and \( a_n = e^{-cn} \) the randelette expansion \([1.1]\) converges uniformly in \( C^M(T^1) \), regardless of the values of the coefficients \( \theta_{n,k} \) which are bounded by 1.
A similar construction can be extended to the torus $\mathbb{T}^\nu$, $\nu \geq 1$, by taking the "mother" randelette as the tensor product of its one-dimensional counterparts:

$$\Phi(t_1, \ldots, t_\nu) = \Phi(t_1) \cdots \Phi(t_\nu)$$

and then defining scaled and translated functions $\varphi_{n,k}(t_1, \ldots, t_\nu)$. Again, the randelette expansion with functions $\varphi_{n,k} \in C^M(\mathbb{T}^\nu)$ converges in $C^M(\mathbb{T}^\nu)$, when the generation amplitudes have the form $a_n = e^{-cn}$, with sufficiently large $c > 0$ proportional to $M$.

4.4. **Validity of the Local Variation Bound.** Let us show that the randelette expansions of an arbitrary smoothness class $C^M$, with amplitudes $a_n = e^{-cn}$ and arbitrarily large $c > 0$ satisfy (LVB) for some $B = B(M) \in (0, +\infty)$. This is the central point of our construction, allowing to apply the MSA approach to deterministic operators and to replace a complicated differential-geometric analysis of "small denominators", appearing in the course of scaling procedure, by simpler probabilistic arguments.

Given a positive integer $N$, the series (4.1) can be re-written as follows:

$$v(\omega, \theta) = \sum_{n=0}^{N-1} a_n \sum_{k=1}^{K_n} \theta_{n,k} \varphi_{n,k}(\omega) + \sum_{n=N}^{\infty} a_n \sum_{k=1}^{K_n} \theta_{n,k} \varphi_{n,k}(\omega) = S_N(\omega, \theta) + \sum_{n=N}^{\infty} a_n \sum_{k=1}^{K_n} \theta_{n,k} \varphi_{n,k}(\omega)$$

where $S_N$, considered as a random variable on $\Theta$, is measurable with respect to the sigma-algebra $\mathcal{B}_N$ generated by random coefficients $\{\theta_{n,k}, n < N, 1 \leq k \leq K_n\}$, while the remaining series in the RHS is $\mathcal{B}_N$-independent (again, as a random variable on $\Theta$). Introduce the sigma-algebra $\mathcal{B}_N^\Omega = \mathcal{F} \times \mathcal{B}_N$. Conditional on $\mathcal{B}_N^\Omega$, all functions $(\omega, \theta) \mapsto \varphi_{n,k}(\omega)$ become non-random, as well as the functions $(\omega, \theta) \mapsto \theta_{n,k}$ with $n < N$, while the random variables $(\omega, \theta) \mapsto \theta_{n,k}$ with $n \geq N$ are $\mathcal{B}_N^\Omega$-independent.

Fix an integer $L > 1$ and points $u \in \mathbb{Z}^d, x, y \in B_L(u)$ with $x \neq y$. Since $|x - y| \leq 2L$, the condition (USR) implies that for any $\omega \in \Omega$

$$d(T^x \omega, T^y \omega) \geq C(2L)^{-A}.$$

As a result, the points $T^x \omega, T^y \omega$ are separated by the supports of all functions $\varphi_{n,k}$ with $n \geq N$, provided that

$$\max_{n \leq N} \max_k \text{diam supp } \varphi_{n,k} < \frac{1}{2} C(2L)^{-A}.$$

By construction of the functions $\varphi_{n,k}$, we have $\text{diam supp } \varphi_{n,k} \leq 2^{1-n}$, so that the above requirement is fulfilled for

$$N \geq N(L, A, C) := a(A, C) \ln L + b(A, C)$$

(4.3) with $a(A, C) = \frac{A}{\ln 2}, b(A, C) = A + 5 - \frac{\ln C}{\ln 2}$. We see that for $N$ starting from $\lfloor N(L, A, C) \rfloor + 1 = O(\ln L)$, no pair of phase points $T^x \omega, T^y \omega$ with $x, y \in B_L(u)$ can be covered by the support of the same function $\varphi_{n,k}$ with $n \geq N$. Therefore, no random variable $\theta_{n,k}$ with $n \geq N$ can affect two distinct values of the random potential $gV(x; \omega, \theta), gV(y; \omega, \theta)$ in any cube of radius $L$. Conditional on $\mathcal{B}_N^\Omega$, all values \(\{gV(x; \omega, \theta), x \in B_L(u)\}\) become (conditionally!) independent. Conditioning further on all $V(y; \omega, \theta)$ with $y \neq x$ does not change the conditional distribution of $V(x; \omega, \theta)$, so it suffices to examine the conditional probability density of $V(x; \omega, \theta)$ given $\mathcal{B}_N^\Omega$.
The latter does exist, since, by construction of the randelette expansion, for every \( n \geq 1 \), every point of the torus, including \( T^x \omega \), is covered by an interval where some function \( \varphi_{n,k} \) with \( k = k(x,n,\omega) \), equals 1. Therefore,

\[
V(x;\omega,\theta) = v(T^x \omega,\theta) = S_N(\omega,\theta) + a_n \theta_{n,k(x,n,\omega)} \cdot 1 + \xi(\omega,\theta),
\]

where \( S_N(\omega,\theta) \) is \( B_\omega^N \)-measurable and \( \xi \) is a sum of random variables conditionally independent of \( \theta_{n,k(x,n,\omega)} \). Since \( \xi \) is (conditionally) independent of \( \theta_{n,k(x,n,\omega)} \), their sum admits a probability density given by the convolution of the probability density of \( \theta_{n,k(x,n,\omega)} \) with the probability distribution of \( \xi \); this operation does not increase the sup-norm of the density.

Finally, the random variable \( a_n \theta_{n,k(x,n,\omega)} \) is uniformly distributed in \([0,a_n]\), so that its probability density is bounded by \( a_n^{-1} e^{cn} \). Setting \( n = N(L,A,C) = a \ln L + b \), we get an upper bound on the conditional probability density of \( V(x;\omega,\theta) \) of the form

\[
\|p_x(\cdot|B_\omega^N)\|_\infty \leq e^{\text{Const} \ln L} \leq L^B, \quad B = B(A,C) \in (0,+\infty).
\]

5. On Minami-type bounds for generic deterministic operators

As was shown in [10], [4], the spectral spacings \( |E_j - E_i| \) of a random LSO in a cube \( B_L(u) \) are positive with probability one, provided that the random potential \( V(\cdot;\omega) \) is an IID random field with bounded marginal probability density \( \rho \). Specifically, for any bounded interval \( I \subset \mathbb{R} \),

\[
P \{ \text{tr} \Pi_I(H_{B_L(u)}(\omega)) \geq J \} \leq \frac{\left(\frac{\pi \rho(\infty)}{\infty}\right)^J}{J!} |I|^J.
\]

A direct inspection of the proofs evidences that the requirement of independence of the potential field can be substantially relaxed: given an integer, \( J \geq 2 \), it suffices that, for any collection of \( J \) pairwise distinct points \( \mathcal{X}_J = \{x_1,\ldots,x_J\} \), the joint conditional probability distribution of the random variables \( V(x_1;\omega),\ldots,V(x_J;\omega) \) admits a bounded conditional probability density, given all values \( \{V(y;\omega),y \in B_L(u) \setminus \mathcal{X}_J\} \).

In other words, main results of [10], [4] can be re-formulated in the following way.

**Proposition 4** (Cf. [10], [4]). Assume that the random field \( V : \mathbb{Z}^d \times \Omega \to \mathbb{R} \) fulfills the following condition: for any cube \( B_L(u) \) and any subset \( \mathcal{X}_J = \{x_1,\ldots,x_J\} \), card \( \mathcal{X}_J = J \), with fixed \( J \geq 1 \), the joint conditional probability distribution of the vector \( (V(x_1;\omega),\ldots,V(x_J;\omega)) \), given \( \{V(y;\omega),y \in B_L(u) \setminus \mathcal{X}_J\} \), admits a bounded probability density \( \rho(t_1,\ldots,t_J) \leq C_\rho \). Then for any bounded interval \( I \subset \mathbb{R} \),

\[
P \{ \text{tr} \Pi_I(H_{B_L(u)}(\omega)) \geq J \} \leq \frac{\left(\frac{\pi C_\rho}{\infty}\right)^J}{J!} |I|^J.
\]

**Theorem 5.** Consider a regular randelette expansion of the form (4.1). For sufficiently large \( B' \in (0,\infty) \), any finite interval \( I \subset \mathbb{R} \), any \( L > 0 \) and some \( C_{10} \in (0,\infty) \)

\[
P^{\Omega \times \Theta} \{ \text{tr} \Pi_I(H_{B_L(u)}(\omega,\theta)) \geq J \} \leq C_{10} L^{B'} |I|^J.
\]

**Proof.** Re-write (4.1) as follows:

\[
v(\omega,\theta) = S_N(\omega,\theta) + \sum_{n=N}^\infty a_n \sum_{k=1}^{K_n} \theta_{n,k} \varphi_{n,k}(\omega)
\]

and set \( N = N(L,A,C) = a(A,C) \ln L + b(A,C) \) with \( a(A,C) = \frac{\Delta}{m^2} \), \( b(A,C) = A + 5 - \frac{\ln C}{m^2} \). Then any pair of phase points \( T^x \omega, T^y \omega \) with \( x \neq y \) is separated by
the supports of functions \( \{ \varphi_{n,k}, 1 \leq k \leq K(n) \} \) with any \( n \geq N(L, A, C) \). Therefore, conditional on all \( \theta_{n,k} \) with \( n \leq N(L, A, C) - 1 \) and with \( n \geq N(L, A, C) + 1 \), the values of the potential \( V(T^x; \omega, \theta) \) at different points \( x \in B_L(u) \) become independent and have uniform distributions in (different) intervals of length \( a_N^{-1} \). Now the claim follows directly from Proposition \( \text{A} \). \( \square \)

**APPENDIX**

5.1. **Proof of Lemma 2.1 (A)** Fix a cube \( B_{L_k}(u) \). We will seek first a bound for the LHS of (2.4) conditional on sigma-algebra \( \mathcal{B}_E \) generated by random variables \( (L, \omega, \theta) \mapsto \omega \) on \( \Omega \times \Theta \). By assumption (USR), if \( \omega \in \Omega \) and \( x \in B_{L_k}(u) \) are fixed, all phase points \( \{ T^x \omega, x \not= y \in B_{L_k}(u) \} \) lie outside the cube \( Q_{L_k}(\omega) \subset \Omega \).

Furthermore, conditional on \( \omega \) and on \( \mathcal{B}(L_k, x) \), all values of the potential \( gV(y; \omega, \theta) \) with \( y \in B_{L_k}(u) \setminus \{ x \} \) become non-random (measurable with respect to the condition), while the remaining random value \( gV(x; \omega, \theta) \) admits a conditional probability density bounded by \( C''L_k^b \) \( g^{-1} \), owing to assumption (LVB). Applying the conventional Wegner bound (see, e.g., [11]) to this conditional measure, we can write

\[
\mathbb{P}_{\Omega \times \Theta} \{ (\omega, \theta) : d(\Sigma_{\omega,\theta}(B_{L_k}(u)), E) \leq s \mid \mathcal{B}_E, \mathcal{B}(L_k, x) \} \leq C_8 \cdot L_k^d C''L_k^b g^{-1} \text{s}
\]

yielding

\[
\mathbb{P}_{\Omega \times \Theta} \{ (\omega, \theta) : d(\Sigma_{\omega,\theta}(B_{L_k}(u)), E) \leq s \}
= \mathbb{P}_{\Omega \times \Theta} \left[ \mathbb{P}_{\Omega \times \Theta} \{ (\omega, \theta) : d(\Sigma_{\omega,\theta}(B_{L_k}(u)), E) \leq s \mid \mathcal{B}_E, \mathcal{B}(L_k, x) \} \right]
\leq C_9 \cdot L_k^d L_k^b g^{-1} s .
\]

**(B)** Consider disjoint cubes \( B_\ell(x'), B_\ell(x'') \subset B_{L_k}(u), \ell \geq L_{k-1}, \) and let \( x \in B_{L_{k-1}}(x') \). Observe that conditioning on \( \mathcal{B}(L_k, x) \), used in the previous argument, fixes not only the values \( gV(y; \omega, \theta) \) with \( y \in B_\ell(x') \setminus \{ x \} \), but all values with \( y \in B_{L_k}(u) \setminus \{ x \} \). This includes the sample of the potential in the cube \( B_\ell(x'') \) disjoint from \( B_\ell(x') \). Therefore, conditional on \( \mathcal{B}(L_k, x) \), the spectrum \( \Sigma_{\omega,\theta}(B_\ell(x'')) \) also becomes non-random. If \( B_\ell(x') \) and \( B_\ell(x'') \) are \( (E, \omega, \theta) \)-R for some \( E \), then

\[
d(\Sigma_{\omega,\theta}(B_\ell(x')), \Sigma_{\omega,\theta}(B_\ell(x''))) \leq 2g\delta_k = 2g|V|e^{-4L_{k-1}^b} .
\]

By assertion (A), for each of \( (2\ell + 1)^d \) eigenvalues \( E_j'' \) of operator \( H_{B_\ell(x'')}(\omega, \theta) \) we can write

\[
\mathbb{P}_{\Omega \times \Theta} \{ (\omega, \theta) : d(\Sigma_{\omega,\theta}(B_\ell(x')), E_j'') \leq 2g\delta_k \} \leq C_9 L_k^d L_k^b g^{-1} \cdot g\delta_k .
\]

Now the bound (2.5) follows from (5.2) by Chebyshev’s inequality, since the number of all pairs \( x', x'' \in B_{L_k}(u) \) is bounded by \( (2L_k + 1)^{2d}/2 \), and \( \ell \leq L_k \) takes less than \( L_k \) possible values. \( \square \)

5.2. **“Radial descent” bounds.** In [8], we introduced the following notion.

**Definition 5.1.** Consider a set \( \Lambda \subset \mathbb{Z}^d \) and a bounded function \( f : \Lambda \to \mathbb{C} \). Let \( \ell \geq 1 \) be an integer and \( q > 0 \). Function \( f \) is called \((\ell, q)\)-subharmonic in \( \Lambda \) if for any \( u \) with \( d(u, \partial \Lambda) \geq \ell \), we have

\[
|f(u)| \leq q \max_{y : \|y-u\| \leq \ell+1} |f(y)|
\]

Function \( f \) is called \((\ell, q, S)\)-subharmonic, with \( S \subset \Lambda \), if for any \( u \in R := \Lambda \setminus S \) the bound (5.3) holds, while for any \( x \in S \) with \( d(x, \partial \Lambda) \geq \ell \)

\[
|f(x)| \leq q \max_{y : \|x-y\| \leq r(x) + \ell} |f(y)|,
\]

where \( r(x) \) is the distance from \( x \) to \( \partial S \). **Definitions 5.1** and 5.2 are consistent due to the following lemma.
where

\[ r(x) = \min\{r \geq l + 1 : \Lambda_{r-\ell}(x) \setminus \Lambda_{r-\ell}(x) \subset \mathcal{R}\} \tag{5.5} \]

provided that the set of values \( r \) in the RHS is non-empty. In all other cases, no specific upper bound on \( |f(x)| \) is assumed.

**Lemma 5.1** (Cf. \[8\]). Let \( f \) be an \((\ell, q, \mathcal{S})\)-subharmonic function on \( B_{L}(u) \). Suppose that \( \mathcal{S} \) can be covered by a collection of cubes \( Q_{1}, \ldots, Q_{K} \) with \( \sum_{i} \text{diam} Q_{i} \leq W \). Then

\[ |f(u)| \leq q^{(L-W)/\ell} M(f, B). \]

The motivation for the above definition comes from the following observations. Consider a pair of cubes \( B_{\ell}(u) \subset B_{L}(x_{0}) \). If \( B_{\ell}(u) \) is \((E, m)\)-NS, then the Geometric Resolvent Identity,

\[ G_{B_{L}(x_{0})}(x, y) = \sum_{(w, w') \in \partial B_{\ell}(u)} G_{B_{\ell}(u)}(u, w) G_{B_{L}(x_{0})}(w', y), \ y \not\in B_{\ell}(u), \]

implies that function \( f : x \mapsto G_{B_{L}(x_{0})}(x, y; E) \) satisfies, with \( q = e^{-\gamma(m, \ell)\ell} \),

\[ |f(u)| \leq q \max_{w : ||u-w||=\ell} |f(w)|. \]

Consider a more general case where \( B_{L}(x_{0}) \) contains at most \( K \) cubes, \( \{B_{\ell}(u_{j}), 1 \leq j \leq K\} =: \mathcal{S} \), such that any cube \( B_{\ell}(u) \) disjoint with \( \mathcal{S} \) is \((E, m)\)-NS. Define \( \mathcal{S} \) as the \((\ell + 1)\)-neighborhood of \( \mathcal{S} \) in \( B_{L}(x_{0}) \), \( \mathcal{R} := B_{L}(x_{0}) \setminus \mathcal{S} \), and let \( r(x) \) be defined for \( x \in \mathcal{S} \) as in \[8\]. It is clear that \( \mathcal{S} \) can be covered by a collection of cubes \( B_{\ell}(u_{j}) \) with \( W := \sum_{i} 2r_{i} \leq K(2\ell + 1) + \ell + 1 \). Assume that \( B_{L}(x_{0}) \) is \( E\)-CNR (hence, any cube \( B_{R}(x) \) with \( R \geq \ell \) is \( E-R \)), and pick \( x \in \mathcal{S} \). Applying the GRI twice, we get

\[ |f(x)| \leq O(\ell^{l-1}) e^{\ell L_{k}} \max_{||u-x||=\ell} |f(w)| \leq e^{-m\ell-\ell^{7/8}+4\ell^{7/8}+O(\ell^{l})} \max_{||w-x||=\ell} |f(w)| \leq e^{-m(\ell^{7/8})} \]

if \( \ell \) is large enough. In other words, \( f \) is \((\ell, q, \mathcal{S})\)-subharmonic with \( q = e^{-m(\ell^{7/8})} \).

Similarly one can treat the eigenfunction correlators \( f : (x, y) \mapsto \psi_{j}(x)\psi_{j}(y) \).

A direct application of Lemma 5.1 leads to the following statement which can be considered as a variant of a well-known technical result going back to papers \[13, 12\].

**Lemma 5.2** (Cf. \[8\]). Fix an integer \( K \geq 1 \) and suppose that for any \( E \in \mathbb{R} \) a cube \( B_{L}(u) \) contains no collection of \( K \) pairwise disjoint \((E, m)\)-NS cubes of radius \( L_{k-1} \).

(A) If for some \( E \in \mathbb{R} \), the cube \( B_{L}(u) \) is \((E, \omega, \theta)\)-CNR then for \( L_{0} \) large enough, it is \((E, m)\)-NS.

(B) If the cube \( B_{L}(u) \) is \((\omega, \theta)\)-NT, then for \( L_{0} \) large enough, it is \( m\)-localized.

**Proof.** The assertion (A) follows directly from Lemma 5.1 applied to the function \( f : x \mapsto G_{B_{L}(u)}(u, x; E) \). To prove (B), we make use of the subharmonicity of the functions \( \psi_{j}(x_{1})\psi_{j}(x_{2}) \) both in \( x_{1} \) and \( x_{2} \); here \( \psi_{j} \) is an eigenfunction with eigenvalue \( E_{j} \). More precisely, \((\omega, \theta)\)-NT property allows, for any \( E_{j} \), to exclude some cube \( B_{L_{k-1}}(u) \) such that any cube \( B_{L_{k-1}}(u) \) disjoint with it is \((E_{j}, \omega, \theta)\)-NR. With \( x_{1}, x_{2} \) fixed, set \( r_{i} = \min(0, ||x_{i}-w|| - 2L_{k-1} - 2) \), \( i = 1, 2 \), so that \( r_{1} + r_{2} \geq L_{k} - O(L_{k-1}) \).

If \( r_{1} > 0 \), then no cube \( B_{L_{k-1}}(y) \subset B_{r_{1}+L_{k-1}}(x_{1}) \) is \((E_{j}, \omega, \theta)\)-R, and one can apply Lemma 5.1 to the function \( f : x_{i} \mapsto \psi_{j}(x_{i})\psi_{j}(x_{2}) \) and obtain

\[ |\psi_{j}(x_{i})| \leq e^{-m(1+L_{k-1}^{7/8})} (r_{i} - O(L_{k-1})). \]
If both \( r_1 > 0 \) and \( r_2 > 0 \), then (B) follows by a straightforward calculation from

\[
|\psi_j(x_1)| \psi_j(x_2)| \leq e^{-m\left(1 + \frac{1}{2}L_{k-1}^{-1/8}\right)(r_2 + r_2 - O(L_{k-1}))},
\]

taking into account that \( r_1 + r_2 \geq L_k^{7/8} - O(L_{k-1}) \), and if one of the radii \( r_i \), \( i = 1, 2 \), is zero, the claim follows from the bound on the remaining value \( |\psi_j(x_{2-i})| \). \( \square \)

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Département de Mathématiques, Université de Reims, Moulin de la Housse, B.P. 1039, 51687 Reims Cedex 2, France, E-mail: victor.tchoulaevski@univ-reims.fr