Boundedness of High Order Commutators of Riesz Transforms Associated with Schrödinger Type Operators

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Received 27 August 2017; Accepted (in revised version) 8 June 2018

Abstract. Let $\mathcal{L}_2 = (-\Delta)^2 + V^2$ be the Schrödinger type operator, where $V \neq 0$ is a nonnegative potential and belongs to the reverse Hölder class $RH_q$, for $q > n/2$, $n \geq 5$. The higher Riesz transform associated with $\mathcal{L}_2$ is denoted by $R = \nabla^2 \mathcal{L}_2^{-1/2}$ and its dual is denoted by $R^* = \mathcal{L}_2^{-1/2} \nabla^2$. In this paper, we consider the $m$-order commutators $[b^m, R]$ and $[b^m, R^*]$, and establish the $(L^p, L^q)$-boundedness of these commutators when $b$ belongs to the new Campanato space $\Lambda^\theta_\beta(\rho)$ and $1/q = 1/p - m\beta/n$.

Key Words: Schrödinger operator, Campanato space, Riesz transform, commutator.

AMS Subject Classifications: 42B25, 35J10, 42B35

1 Introduction

In this paper, we consider the Schrödinger type operator

$$\mathcal{L}_2 = (-\Delta)^2 + V^2 \text{ on } \mathbb{R}^n, \quad n \geq 5,$$

where $V$ is nonnegative, $V \neq 0$, and belongs to the reverse Hölder class $RH_q$ for some $q \geq n/2$, i.e., there exists a constant $C$ such that

$$\left( \frac{1}{|B|} \int_B V(y)^q dy \right)^{1/q} \leq \frac{C}{|B|} \int_B V(y) dy$$

for every ball $B \subset \mathbb{R}^n$.

The higher Riesz transform associated with $\mathcal{L}_2$ is defined by $R = \nabla^2 \mathcal{L}_2^{-1/2}$, and its dual is defined by $R^* = \mathcal{L}_2^{-1/2} \nabla^2$. The $L^p$-boundedness of the higher Riesz transforms

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In this paper, we are interested in the boundedness of
\[ \| \mathcal{R} f \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)}. \]
If \( p' < p < \infty \), then for all \( f \in L^p(\mathbb{R}^n) \),
\[ \| \mathcal{R}^s f \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)}. \]

As in [2], for a given potential \( V \in RH_q \) with \( q > n/2 \), we define the auxiliary function
\[ \rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq 1 \right\}, \quad x \in \mathbb{R}^n. \]
It is well known that \( 0 < \rho(x) < \infty \) for any \( x \in \mathbb{R}^n \).

Let \( \theta > 0 \) and \( 0 < \beta < 1 \), in view of [3], the new Campanato class \( \Lambda^\theta_\beta(\rho) \) consists of the locally integrable functions \( b \) such that
\[ \frac{1}{|B(x,r)|^{1+\beta/n}} \int_{B(x,r)} |b(y) - b_B| dy \leq C \left( 1 + \frac{r}{\rho(x)} \right)^\theta \]
for all \( x \in \mathbb{R}^n \) and \( r > 0 \). A seminorm of \( b \in \Lambda^\theta_\beta(\rho) \), denoted by \( |b|_{\Lambda^\theta_\beta(\rho)} \), is given by the infimum of the constants in the inequalities above.

Note that if \( \theta = 0 \), \( \Lambda^0_\beta(\rho) \) is the classical Campanato space; If \( \beta = 0 \), \( \Lambda^\theta_0(\rho) \) is exactly the space \( BMO_\rho(\rho) \) introduced in [4].

We denote by \( \mathcal{K} \) and \( \mathcal{K}^s \) the kernels of \( \mathcal{R} \) and \( \mathcal{R}^s \), respectively. Let \( b \) be a locally integrable function, \( m \) be a positive integer. The \( m \)-order commutators generated by higher Riesz transform and \( b \) are defined by
\[ [b^m, \mathcal{R}]f(x) = \int_{\mathbb{R}^n} \mathcal{K}(x,y)(b(x) - b(y))^m f(y)dy \]
and
\[ [b^m, \mathcal{R}^s]f(x) = \int_{\mathbb{R}^n} \mathcal{K}^s(x,y)(b(x) - b(y))^m f(y)dy. \]
In this paper, we are interested in the boundedness of \([b^m, \mathcal{R}]\) and \([b^m, \mathcal{R}^s]\) on Lebesgue space when \( b \) belongs to the new Campanato class \( \Lambda^\theta_\beta(\rho) \). The main result of this paper is as follows.

**Theorem 1.1.** Suppose \( V \in RH_q \) with \( n/2 < q_1 < n \), \( 1/p_1 = 2/q_1 - 2/n \), \( p'_1 = p_1/(p_1 - 1) \). Let \( 0 < \beta < 1 \), and let \( b \in \Lambda^\theta_\beta(\rho) \). If \( p'_1 < p < \infty \), then for all \( f \in L^p(\mathbb{R}^n) \),
\[ \| [b^m, \mathcal{R}] f \|_{L^q(\mathbb{R}^n)} \leq C \| [b^m] \|^m \| f \|_{L^p(\mathbb{R}^n)}, \]
where \( 1/q = 1/p - m\beta/n \).