The Connected Components of the Projective Line over a Ring

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Abstract

The main result of the present paper is that the projective line over a ring $R$ is connected with respect to the relation “distant” if, and only if, $R$ is a GE$_2$-ring.

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1 Introduction

One of the basic notions for the projective line $\mathbb{P}(R)$ over a ring $R$ is the relation distant ($\triangle$) on the point set. Non-distant points are also called parallel. This terminology goes back to the projective line over the real dual numbers, where parallel points represent parallel spears of the Euclidean plane [4, 2.4].

We say that $\mathbb{P}(R)$ is connected (with respect to $\triangle$) if the following holds: For any two points $p$ and $q$ there is a finite sequence of points starting at $p$ and ending at $q$ such that each point other than $p$ is distant from its predecessor. Otherwise $\mathbb{P}(R)$ is said to be disconnected. For each connected component a distance function and a diameter (with respect to $\triangle$) can be defined in a natural way.

One aim of the present paper is to characterize those rings $R$ for which $\mathbb{P}(R)$ is connected. Here we use certain subgroups of the group $\text{GL}_2(R)$ of invertible $2 \times 2$-matrices over $R$, namely its elementary subgroup $E_2(R)$ and the subgroup $\text{GE}_2(R)$ generated by $E_2(R)$ and the set of all invertible diagonal matrices. It turns out that $\mathbb{P}(R)$ is connected exactly if $R$ is a GE$_2$-ring, i.e., if $\text{GE}_2(R) = \text{GL}_2(R)$.

Next we turn to the diameter of connected components. We show that all connected components of $\mathbb{P}(R)$ share a common diameter.

It is well known that $\mathbb{P}(R)$ is connected with diameter $\leq 2$ if $R$ is a ring of stable rank 2. We give explicit examples of rings $R$ such that $\mathbb{P}(R)$ has one of the following properties: $\mathbb{P}(R)$ is connected with diameter 3, $\mathbb{P}(R)$ is connected with diameter $\infty$, and $\mathbb{P}(R)$ is disconnected with diameter $\infty$. In particular, we show that there are chain geometries over disconnected projective lines.

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2 Preliminaries

Throughout this paper we shall only consider associative rings with a unit element 1, which is inherited by subrings and acts unitally on modules. The trivial case $1 = 0$ is not excluded. The group of invertible elements of a ring $R$ will be denoted by $R^*$.

Firstly, we turn to the projective line over a ring: Consider the free left $R$-module $R^2$. Its automorphism group is the group $\text{GL}_2(R)$ of invertible $2 \times 2$-matrices with entries in $R$. A pair $(a, b) \in R^2$ is called admissible, if there exists a matrix in $\text{GL}_2(R)$ with $(a, b)$ being its first row. Following [14, 785], the projective line over $R$ is the orbit of the free cyclic submodule $R(1, 0)$ under the action of $\text{GL}_2(R)$. So

$$\mathbb{P}(R) := R(1, 0)^{\text{GL}_2(R)}$$

or, in other words, $\mathbb{P}(R)$ is the set of all $p \leq R^2$ such that $p = R(a, b)$ for an admissible pair $(a, b) \in R^2$. As has been pointed out in [8, Proposition 2.1], in certain cases $R(x, y) \in \mathbb{P}(R)$ does not imply the admissibility of $(x, y) \in R^2$. However, throughout this paper we adopt the convention that points are represented by admissible pairs only. Two such pairs represent the same point exactly if they are left-proportional by a unit in $R$.

The point set $\mathbb{P}(R)$ is endowed with the symmetric relation distant ($\triangle$) defined by

$$\triangle := (R(1, 0), R(0, 1))^{\text{GL}_2(R)}. \tag{1}$$

Letting $p = R(a, b)$ and $q = R(c, d)$ gives then

$$p \triangle q \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(R).$$

In addition, $\triangle$ is anti-reflexive exactly if $1 \neq 0$.

The vertices of the distant graph on $\mathbb{P}(R)$ are the points of $\mathbb{P}(R)$, the edges of this graph are the unordered pairs of distant points. Therefore basic graph-theoretical concepts are at hand: $\mathbb{P}(R)$ can be decomposed into connected components (maximal connected subsets), for each connected component there is a distance function (dist($p, q$) is the minimal number of edges needed to go from vertex $p$ to vertex $q$), and each connected component has a diameter (the supremum of all distances between its points).

Secondly, we recall that the set of all elementary matrices

$$B_{12}(t) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{21}(t) := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \quad \text{with} \quad t \in R \tag{2}$$

generates the elementary subgroup $E_2(R)$ of $\text{GL}_2(R)$. The group $E_2(R)$ is also generated by the set of all matrices

$$E(t) := \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} = B_{12}(1) \cdot B_{21}(-1) \cdot B_{12}(1) \cdot B_{21}(t) \quad \text{with} \quad t \in R, \tag{3}$$

since $B_{12}(t) = E(-t) \cdot E(0)^{-1}$ and $B_{21}(t) = E(0)^{-1} \cdot E(t)$. Further, $E(t)^{-1} = E(0) \cdot E(-t) \cdot E(0)$ implies that all finite products of matrices $E(t)$ already comprise the group $E_2(R)$.

The subgroup of $\text{GL}_2(R)$ which is generated by $E_2(R)$ and the set of all invertible diagonal matrices is denoted by $G_E_2(R)$. By definition, a $G_2$-ring is characterized by $\text{GL}_2(R) = G_E_2(R)$; see, among others, [10, 5] or [18, 114].
3 Connected Components

We aim at a description of the connected components of the projective line $\mathbb{P}(R)$ over a ring $R$. The following lemma, although more or less trivial, will turn out useful:

**Lemma 3.1** Let $X' \in \text{GL}_2(R)$ and suppose that the $2 \times 2$-matrix $X$ over $R$ has the same first row as $X'$. Then $X$ is invertible if, and only if, there is a matrix

$$M = \begin{pmatrix} 1 & 0 \\ s & u \end{pmatrix} \in \text{GE}_2(R)$$

such that $X = MX'$.

**Proof:** Given $X'$ and $X$ then $XX'^{-1} = \begin{pmatrix} 1 & 0 \\ s & u \end{pmatrix} =: M$ for some $s, u \in R$. Further, $X = MX'$ is invertible exactly if $u \in R^*$. This in turn is equivalent to (4). □

Here is our main result, where we use the generating matrices of $\text{E}_2(R)$ introduced in (3).

**Theorem 3.2** Denote by $C_\infty$ the connected component of the point $R(1,0)$ in the projective line $\mathbb{P}(R)$ over a ring $R$. Then the following holds:

(a) The group $\text{GL}_2(R)$ acts transitively on the set of connected components of $\mathbb{P}(R)$.

(b) Let $t_1, t_2, \ldots, t_n \in R$, $n \geq 0$, and put

$$(x,y) := (1,0) \cdot E(t_n) \cdot E(t_{n-1}) \cdots E(t_1).$$

Then $R(x,y) \in C_\infty$ and, conversely, each point $r \in C_\infty$ can be written in this way.

(c) The stabilizer of $C_\infty$ in $\text{GL}_2(R)$ is the group $\text{GE}_2(R)$.

(d) The projective line $\mathbb{P}(R)$ is connected if, and only if, $R$ is a $\text{GE}_2$-ring.

**Proof:** (a) This is immediate from the fact that the group $\text{GL}_2(R)$ acts transitively on the point set $\mathbb{P}(R)$ and preserves the relation $\triangle$.

(b) Every matrix $E(t_i)$ appearing in (5) maps $C_\infty$ onto $C_\infty$, since $R(0,1) \in C_\infty$ goes over to $R(1,0) \in C_\infty$. Therefore $R(x,y) \in C_\infty$.

On the other hand let $r \in C_\infty$. Then there exists a sequence of points $p_i = R(a_i, b_i) \in \mathbb{P}(R)$, $i \in \{0, 1, \ldots, n\}$, such that

$$R(1,0) = p_0 \triangle p_1 \triangle \ldots \triangle p_n = r.$$  \quad (6)

Now the arbitrarily chosen admissible pairs $(a_i, b_i)$ are “normalized” recursively as follows:
First define $(x_{-1}, y_{-1}) := (0, -1)$ and $(x_0, y_0) := (1, 0)$. So $p_0 = R(x_0, y_0)$. Next assume that we already are given admissible pairs $(x_j, y_j)$ with $p_j = R(x_j, y_j)$ for all $j \in \{0, 1, \ldots, i-1\}$, $1 \leq i \leq n$. From Lemma 3.1, there are $s_i \in R$ and $u_i \in R^*$ such that

$$\begin{pmatrix} x_{i-1} & y_{i-1} \\ a_i & b_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s_i & u_i \end{pmatrix} \begin{pmatrix} x_{i-1} & y_{i-1} \\ -x_{i-2} & -y_{i-2} \end{pmatrix}.$$  \quad (7)
By putting \( x_i := u_{i-1}a_i, y_i := u_{i-1}b_i \), and \( t_i := u_{i-1}s_i \) we get
\[
\begin{pmatrix}
  x_i & y_i \\
  -x_{i-1} & -y_{i-1}
\end{pmatrix} = E(t_i) \cdot \begin{pmatrix}
  x_{i-1} & y_{i-1} \\
  -x_{i-2} & -y_{i-2}
\end{pmatrix}
\]
and \( p_i = R(x_i, y_i) \). Therefore, finally, \((x_n, y_n)\) is the first row of the matrix
\[
G' := E(t_n) \cdot E(t_{n-1}) \cdots E(t_1) \in E_2(R),
\]
and \( r = R(x_n, y_n) \).

(c) As has been noticed at the end of Section 2, the set of all matrices (3) generates \( E_2(R) \). This together with (b) implies that \( E_2(R) \) stabilizes \( C_\infty \). Further, \( R(1,0) \) remains fixed under each invertible diagonal matrix. Therefore \( GE_2(R) \) is contained in the stabilizer of \( C_\infty \).

Conversely, suppose that \( G \in GL_2(R) \) stabilizes \( C_\infty \). Then the first row of \( G \), say \((a, b)\), determines a point of \( C_\infty \). By (5) and (9), there is a matrix \( G' \in E_2(R) \) and a unit \( u \in R^* \) such that \((a, b) = (1,0) \cdot (uG')\). Now Lemma 3.1 can be applied to \( G \) and \( uG' \in GE_2(R) \) in order to establish that \( G \in GE_2(R) \).

(d) This follows from (a) and (c). \( \square \)

From Theorem 3.2 and (9), the connected component of \( R(1,0) \in \mathbb{P}(R) \) is given by all pairs of \((1,0) \cdot E_2(R)\) or, equivalently, by all pairs of \((1,0) \cdot GE_2(R)\). Each product (5) gives rise to a sequence
\[
(x_i, y_i) = (1,0) \cdot E(t_i) \cdot E(t_{i-1}) \cdots E(t_1), \quad i \in \{0, 1, \ldots, n\},
\]
which in turn defines a sequence \( p_i := R(x_i, y_i) \) of points with \( p_0 = R(1,0) \). By putting \( p_n = : r \) and by reversing the arguments in the proof of (b), it follows that (6) is true. So, if the diameter of \( C_\infty \) is finite, say \( m \geq 0 \), then in order to reach all points of \( C_\infty \) it is sufficient that \( n \) ranges from 0 to \( m \) in formula (5).

By the action of \( GL_2(R) \), the connected component \( C_p \) of any point \( p \in \mathbb{P}(R) \) is \( GL_2(R) \)-equivalent to the connected component \( C_\infty \) of \( R(1,0) \) and the stabilizer of \( C_p \) in \( GL_2(R) \) is conjugate to \( GE_2(R) \). Observe that in general \( GE_2(R) \) is not normal in \( GL_2(R) \). Cf. the example in 5.7 (c). All connected components are isomorphic subgraphs of the distant graph.

### 4 Generalized Chain Geometries

If \( K \subset R \) is a (not necessarily commutative) subfield, then the \( K \)-sublines of \( \mathbb{P}(R) \) give rise to a **generalized chain geometry** \( \Sigma(K, R) \); see [7]. In contrast to an ordinary chain geometry (cf. [14]) it is not assumed that \( K \) is in the centre of \( R \). Any three mutually distant points are on at least one \( K \)-chain. Two distinct points are distant exactly if they are on a common \( K \)-chain. Therefore each \( K \)-chain is contained in a unique connected component. Each connected component \( C \) together with the set of \( K \)-chains entirely contained in it defines an incidence structure \( \Sigma(C) \). It is straightforward to show that the automorphism group of
the incidence structure $\Sigma(K, R)$ is isomorphic to the wreath product of $\text{Aut} \Sigma(C)$ with the symmetric group on the set of all connected components of $\mathbb{P}(R)$.

If $\Sigma(K, R)$ is a chain geometry then the connected components are exactly the maximal connected subspaces of $\Sigma(K, R)$ [14, 793, 821]. Cf. also [15] and [16].

An $R$-semilinear bijection of $R^2$ induces an automorphism of $\Sigma(K, R)$ if, and only if, the accompanying automorphism of $R$ takes $K$ to $u^{-1}Ku$ for some $u \in R^*$. On the other hand, if $\mathbb{P}(R)$ is disconnected then we cannot expect all automorphisms of $\Sigma(K, R)$ to be semilinearly induced. More precisely, we have the following:

**Theorem 4.1** Let $\Sigma(K, R)$ be a disconnected generalized chain geometry, i.e., the projective line $\mathbb{P}(R)$ over $R$ is disconnected. Then $\Sigma(K, R)$ admits automorphisms that cannot be induced by any semilinear bijection of $R^2$.

**Proof:** (a) Suppose that two semilinearly induced bijections $\gamma_1, \gamma_2$ of $\mathbb{P}(R)$ coincide on all points of one connected component $C$ of $\mathbb{P}(R)$. We claim that $\gamma_1 = \gamma_2$. For a proof choose two distant points $R(a, b)$ and $R(c, d)$ in $C$. Also, write $\alpha$ for that projectivity which is given by the matrix \(
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\). Then $\beta := \alpha \gamma_1 \gamma_2^{-1} \alpha^{-1}$ is a semilinearly induced bijection of $\mathbb{P}(R)$ fixing the connected component $C_\infty$ of $R(1,0)$ pointwise. Hence $R(1,0)$, $R(0,1)$, and $R(1,1)$ are invariant under $\beta$, and we get

$$R(x, y)^\beta = R(x^\zeta u, y^\zeta u) \text{ for all } (x, y) \in R^2$$

with $\zeta \in \text{Aut}(R)$ and $u \in R^*$, say. For all $x \in R$ the point $R(x, 1)$ is distant from $R(1,0)$; so it remains fixed under $\beta$. Therefore $x = u^{-1}x^\zeta u$ or, equivalently, $x^\zeta u = ux$ for all $x \in R$. Finally, $R(x, y)^\beta = R(ux, uy) = R(x, y)$ for all $(x, y) \in R^2$, whence $\gamma_1 = \gamma_2$.

(b) Let $\gamma$ be a non-identical projectivity of $\mathbb{P}(R)$ given by a matrix $G \in \text{GE}_2(R)$, for example, $G = B_{12}(1)$. From Theorem 3.2, the connected component $C_\infty$ of $R(1,0)$ is invariant under $\gamma$. Then

$$\delta : \mathbb{P}(R) \to \mathbb{P}(R) : \begin{cases} p \mapsto p^\gamma & \text{for all } p \in C_\infty \\
p \mapsto p & \text{for all } p \in \mathbb{P}(R) \setminus C_\infty \end{cases}$$

is an automorphism of $\Sigma(K, R)$. The projectivity $\gamma$ and the identity on $\mathbb{P}(R)$ are different and both are linearly induced. The mapping $\delta$ coincides with $\gamma$ on $C_\infty$ and with the identity on every other connected component. There are at least two distinct connected components of $\mathbb{P}(R)$. Hence it follows from (a) that $\delta$ cannot be semilinearly induced. \(\square\)

If a cross-ratio in $\mathbb{P}(R)$ is defined according to [14, 1.3.5] then four points with cross-ratio are necessarily in a common connected component. Therefore, the automorphism $\delta$ defined in (11) preserves all cross-ratios. However, cross-ratios are not invariant under $\delta$ if one adopts the definition in [4, 90] or [14, 7.1] which works for commutative rings only. This is due to the fact that here four points with cross-ratio can be in two distinct connected components.

We shall give examples of disconnected (generalized) chain geometries in the next section.
5 Examples

There is a widespread literature on (non-)GE$_2$-rings. We refer to [1], [9], [10], [11], [12], [13], and [18]. We are particularly interested in rings containing a field and the corresponding generalized chain geometries.

Remark 5.1 Let $R$ be a ring. Then each admissible pair $(x, y) \in R^2$ is unimodular, i.e., there exist $x', y' \in R$ with $xx' + yy' = 1$. We remark that

$$(x, y) \in R^2 \text{ unimodular} \Rightarrow (x, y) \text{ admissible}$$

is satisfied, in particular, for all commutative rings, since $xx' + yy' = 1$ can be interpreted as the determinant of an invertible matrix with first row $(x, y)$. Also, all rings of stable rank 2 [19, 293] satisfy (12); cf. [19, 2.11]. For example, local rings, matrix rings over fields, and finite-dimensional algebras over commutative fields are of stable rank 2. See [13, 4.1B], [19, §2], [20], and the references given there.

The following example shows that (12) does not hold for all rings: Let $R := K[X, Y]$ be the polynomial ring over a proper skew field $K$ in independent central indeterminates $X$ and $Y$. There are $a, b \in K$ with $c := ab - ba \neq 0$. From

$$(X + a)(Y + b)c^{-1} - (Y + b)(X + a)c^{-1} = 1,$$

the pair $(X + a, -(Y + b)) \in R^2$ is unimodular. However, this pair is not admissible: Assume to the contrary that $(X + a, -(Y + b))$ is the first row of a matrix $M \in \text{GL}_2(R)$ and suppose that the second column of $M^{-1}$ is the transpose of $(v_0, w_0) \in R^2$. Then

$$P := \{(v, w) \in R^2 \mid (X + a)v - (Y + b)w = 0\} = (v_0, w_0)R.$$

On the other hand, by [17, Proposition 1], the right $R$-module $P$ cannot be generated by a single element, which is a contradiction.

Examples 5.2 (a) If $R$ is a ring of stable rank 2 then $\mathbb{P}(R)$ is connected and its diameter is $\leq 2$ [14, Proposition 1.4.2]. In particular, the diameter is 1 exactly if $R$ is a field and it is 0 exactly if $R = \{0\}$.

As has been pointed out in [2, (2.1)], the points of the projective line over a ring $R$ of stable rank 2 are exactly the submodules $R(t_2t_1 + 1, t_2)$ of $R^2$ with $t_1, t_2 \in R$. Clearly, this is just a particular case of our more general result in Theorem 3.2 (b).

Conversely, if an arbitrary ring $R$ satisfies (12) and $\mathbb{P}(R)$ is connected with diameter $\leq 2$, then $R$ is a ring of stable rank 2 [14, Proposition 1.1.3].

(b) The projective line over a (not necessarily commutative) Euclidean ring $R$ is connected, since every Euclidean ring is a GE$_2$-ring [13, Theorem 1.2.10].

Our next examples are given in the following theorem:

Theorem 5.3 Let $U$ be an infinite-dimensional vector space over a field $K$ and put $R := \text{End}_K(U)$. Then the projective line $\mathbb{P}(R)$ over $R$ is connected and has diameter 3.
**Proof:** We put $V := U \times U$ and denote by $\mathcal{G}$ those subspaces $W$ of $V$ that are isomorphic to $V/W$. By [5, 2.4], the mapping

$$\Phi : \mathbb{P}(R) \to \mathcal{G} : R(\alpha, \beta) \mapsto \{(u^\alpha, u^\beta) \mid u \in U\}$$

(13)

is bijective and two points of $\mathbb{P}(R)$ are distant exactly if their $\Phi$-images are complementary. By an abuse of notation, we shall write $\text{dist}(W_1, W_2) = n$, whenever $W_1, W_2$ are $\Phi$-images of points at distance $n$, and $W_1 \triangle W_2$ to denote complementary elements of $\mathcal{G}$. As $V$ is infinite-dimensional, $2 \dim W = \dim V = \dim W$ for all $W \in \mathcal{G}$.

We are going to verify the theorem in terms of $\mathcal{G}$: So let $W_1, W_2 \in \mathcal{G}$. Put $Y_{12} := W_1 \cap W_2$ and choose $Y_{23} \leq W_2$ such that $W_2 = Y_{12} \oplus Y_{23}$. Then $W_1 \cap Y_{23} = \{0\}$ so that there is a $W_3 \in \mathcal{G}$ through $Y_{23}$ with $W_1 \triangle W_3$. By the law of modularity,

$$W_2 \cap W_3 = (Y_{23} + Y_{12}) \cap W_3 = Y_{23} + (Y_{12} \cap W_3) = Y_{23}.$$

Finally, choose $Y_{14} \leq W_1$ with $W_1 = Y_{12} \oplus Y_{14}$ and $Y_{34} \leq W_3$ with $W_3 = Y_{23} \oplus Y_{34}$. Hence we arrive at the decomposition

$$V = Y_{14} \oplus Y_{12} \oplus Y_{23} \oplus Y_{34}.$$  (14)

As $W_2 \in \mathcal{G}$, so is also $W_4 := Y_{14} \oplus Y_{34}$. Now there are two possibilities:

Case 1: There exists a linear bijection $\sigma : Y_{14} \to Y_{23}$. We define $Y := \{v + \sigma^v \mid v \in Y_{14}\}$. Then $Y_{14}$, $Y_{23}$, and $Y$ are easily seen to be mutually complementary subspaces of $Y_{14} \oplus Y_{23}$. Therefore, from (14),

$$V = Y_{14} \oplus Y_{12} \oplus Y \oplus Y_{34} = Y \oplus Y_{12} \oplus Y_{23} \oplus Y_{34},$$

(15)

i.e., $W_1 \triangle (Y \oplus Y_{34}) \triangle W_2$. So $\text{dist}(W_1, W_2) \leq 2$.

Case 2: $Y_{14}$ and $Y_{23}$ are not isomorphic. Then $\dim Y_{12} = \dim W_1$, since otherwise $\dim Y_{12} < \dim W_1 = \dim W_2$ together with well-known rules for the addition of infinite cardinal numbers would imply

$$\dim W_1 = \max\{\dim Y_{12}, \dim Y_{14}\} = \dim Y_{14},$$

$$\dim W_2 = \max\{\dim Y_{12}, \dim Y_{23}\} = \dim Y_{23},$$

a contradiction to $\dim Y_{14} \neq \dim Y_{23}$.

Likewise, it follows that $\dim Y_{34} = \dim W_3$. But this means that $Y_{12}$ and $Y_{34}$ are isomorphic, whence the proof in case 1 can be modified accordingly to obtain a $Y \leq Y_{12} \oplus Y_{34}$ such that $W_1 \triangle W_3 \triangle (Y \oplus Y_{14}) \triangle W_2$. So now $\text{dist}(W_1, W_2) \leq 3$.

It remains to establish that in $\mathcal{G}$ there are elements with distance 3: Choose any subspace $W_1 \in \mathcal{G}$ and a subspace $W_2 \leq W_1$ such that $W_1/W_2$ is 1-dimensional. With the previously introduced notations we get $Y_{12} = W_2$, $\dim Y_{14} = 1$, $Y_{23} = \{0\}$, $Y_{34} = W_3 \in \mathcal{G}$, and $W_4 = Y_{14} \oplus W_3$. As before, $V = W_2 \oplus W_4$ and from $\dim W_2 = 1 + \dim W_2 = \dim W_1 = \dim W_3 = 1 + \dim W_3 = \dim W_4$ we obtain $W_2, W_4 \in \mathcal{G}$. By construction, $\text{dist}(W_1, W_2) \neq 0, 1$. Also, this distance cannot be 2, since $W \triangle W_1$ implies $W \neq W_2 \neq V$ for all $W \in \mathcal{G}$.

This completes the proof. □
If $K$ is a proper skew field, then $K$ can be embedded in $\text{End}_K(U)$ in several ways [6, 17]; each embedding gives rise to a connected generalized chain geometry. (In [6] this is just called a “chain geometry”.) If $K$ is commutative, then $\text{End}_K(U)$ is a $K$-algebra and $x \mapsto x \text{id}_U$ is a distinguished embedding of $K$ into the centre of $\text{End}_K(U)$. In this way an ordinary connected chain geometry arises; cf. [14, 4.5. Example (4)].

Our next goal is to show the existence of chain geometries with connected components of infinite diameter.

**Remark 5.4** If $R$ is an arbitrary ring then each matrix $A \in \text{GE}_2(R)$ can be expressed in standard form

$$A = \text{diag}(u, v) \cdot E(t_1) \cdot E(t_2) \cdot \ldots \cdot E(t_n),$$

where $u, v \in R^*$, $t_1, t_2, t_3, \ldots, t_{n-1} \in R \setminus (R^* \cup \{0\})$, and $t_1, t_2 \neq 0$ in case $n = 2$ [10, Theorem (2.2)]. Since $E(0)^2 = \text{diag}(-1, -1)$, each matrix $A \in \text{GE}_2(R)$ can also be written in the form (16) subject to the slightly modified conditions $u, v \in R^*$, $t_1, t_2, t_3, \ldots, t_{n-1} \in R \setminus (R^* \cup \{0\})$, and $n \geq 1$. We call this a modified standard form of $A$.

Suppose that there is a unique standard form for $\text{GE}_2(R)$. For all non-diagonal matrices in $\text{GE}_2(R)$ the unique representation in standard form is at the same time the unique representation in modified standard form. Any diagonal matrix $A \in \text{GE}_2(R)$ is already expressed in standard form, but its unique modified standard form reads $A = -A \cdot E(0)^2$. Therefore there is also a unique modified standard form for $\text{GE}_2(R)$.

By reversing these arguments it follows that the existence of a unique modified standard form for $\text{GE}_2(R)$ is equivalent to the existence of a unique standard form for $\text{GE}_2(R)$.

**Theorem 5.5** Let $R$ be a ring with a unique standard form for $\text{GE}_2(R)$ and suppose that $R$ is not a field. Then every connected component of the projective line $\mathbb{P}(R)$ over $R$ has infinite diameter.

**Proof:** Since $R$ is not a field, there exists an element $t \in R \setminus (R^* \cup \{0\})$. We put

$$q_m := R(c_m, d_m) \text{ where } (c_m, d_m) := (1, 0) \cdot E(t)^m \text{ for all } m \in \{0, 1, \ldots\}. \quad (17)$$

Next fix one $m \geq 1$, and put $n-1 := \text{dist}(q_0, q_{m-1}) \geq 0$. Hence there exists a sequence

$$p_0 \triangle p_1 \triangle \ldots \triangle p_{n-1} \triangle p_n$$

such that $p_0 = q_0$, $p_{n-1} = q_{m-1}$, and $p_n = q_m$. Now we proceed as in the proof of Theorem 3.2 (b): First let $p_i = R(a_i, b_i)$ and put $(x_i, y_i) := (0, -1)$, $(x_0, y_0) := (1, 0)$. Then pairs $(x_i, y_i) \in R^2$ and matrices $E(t_i) \in E_2(R)$ are defined in such a way that $p_i = R(x_i, y_i)$ and that (8) holds for $i \in \{1, 2, \ldots, n\}$. It is immediate from (8) that a point $p_i$, $i \geq 2$, is distant from $p_{i-2}$ exactly if $t_i \in R^*$. Also, $p_i = p_{i-2}$ holds if, and only if, $t_i = 0$. We infer from (8) and $\text{dist}(p_i, p_j) = |i - j|$ for all $i, j \in \{0, 1, \ldots, n - 1\}$ that

$$\begin{pmatrix} x_n & y_n \\ -x_{n-1} & -y_{n-1} \end{pmatrix} = E(t_n) \cdot E(t_{n-1}) \cdot \ldots \cdot E(t_1), \quad (19)$$

8
where $t_i \in R \setminus \{R^* \cup \{0\}\}$ for all $i \in \{2, 3, \ldots, n - 1\}$. On the other hand, by (17) and 
\((c_{m-1}, d_{m-1}) = (0, -1) \cdot E(t)^m\), there are $v, v' \in R^*$ with
\[
\begin{pmatrix}
    x_n & y_n \\
    -x_{n-1} & -y_{n-1}
\end{pmatrix} = \text{diag}(v, v') \cdot E(t)^m.
\] (20)

From Remark 5.4, the modified standard forms (19) and (20) are identical. Therefore, $n = m$, 
\(\text{dist}(q_0, q_{m-1}) = m - 1\), and the diameter of the connected component of $q_0$ is infinite.

By Theorem 3.2 (a), all connected components of $\mathbb{P}(R)$ have infinite diameter. □

Remark 5.6 Let $R$ be a ring such that $R^* \cup \{0\}$ is a field, say $K$, and suppose that we have
a degree function, i.e. a function $\text{deg} : R \to \{\infty\} \cup \{0, 1, \ldots\}$ satisfying
\[
\text{deg}a = -\infty \quad \text{if, and only if, } a = 0,
\text{deg}a = 0 \quad \text{if, and only if, } a \in R^*,
\text{deg}(a + b) \leq \max\{\text{deg}a, \text{deg}b\},
\text{deg}(ab) = \text{deg}(a) + \text{deg}(b),
\]
for all $a, b \in R$. Then, following [10, 21], $R$ is called a $K$-ring with a degree function.

If $R$ is a $K$-ring with a degree function, then there is a unique standard form for $\text{GE}_2(R)$ [10, Theorem (7.1)].

Examples 5.7 (a) Let $R$ be a $K$-ring with a degree-function such that $R \neq K$. From
Remark 5.6 and Theorem 5.5, all connected components of the projective line $\mathbb{P}(R)$
have infinite diameter.

The associated generalized chain geometry $\Sigma(K, R)$ has a lot of strange properties. For example, any two distant points are joined by a unique $K$-chain. However, we do not enter a detailed discussion here.

(b) Let $K[X]$ be the polynomial ring over a field $K$ in a central indeterminate $X$. From
(a) and Example 5.2 (b), the projective line $\mathbb{P}(K[X])$ is connected and its diameter is
infinite. On the other hand, if $K$ is commutative then $K[X]$ has stable rank 3 [20, 2.9];
see also [3, Chapter V, (3.5)]. So there does not seem to be an immediate connection
between stable rank and diameter.

(c) Let $R := K[X_1, X_2, \ldots, X_m]$ be the polynomial ring over a field $K$ in $m > 1$ independent central indeterminates. Then, by an easy induction and by [10, Proposition (7.3)],
\[
A_n := \left(\begin{array}{cc}
    1 + X_1X_2 & X_1^2 \\
    -X_2^2 & 1 - X_1X_2
\end{array}\right)^n = \left(\begin{array}{cc}
    1 + nX_1X_2 & nX_1^2 \\
    -nX_2^2 & 1 - nX_1X_2
\end{array}\right)
\] (21)
is in $\text{GL}_2(R) \setminus \text{GE}_2(R)$ for all $n \in \mathbb{Z}$ that are not divisible by the characteristic of $K$. Also, the inner automorphism of $\text{GL}_2(R)$ arising from the matrix $A_1$ takes $B_{12}(1) \in E_2(R)$ to a matrix that is not even in $\text{GE}_2(R)$; see [18, 121–122]. So neither $E_2(R)$ nor $\text{GE}_2(R)$ is a normal subgroup of $\text{GL}_2(R)$.
We infer that the projective line over $R$ is not connected. Further, it follows from (21) that the number of right cosets of $\text{GE}_2(R)$ in $\text{GL}_2(R)$ is infinite, if the characteristic of $K$ is zero, and $\geq \text{char} \ K$ otherwise. From Theorem 3.2, this number of right cosets is at the same time the number of connected components in $\mathbb{P}(R)$. Even in case of $\text{char} \ K = 2$ there are at least three connected components, since the index of $\text{GE}_2(R)$ in $\text{GL}_2(R)$ cannot be two. From (a), all connected components of $\mathbb{P}(R)$ have infinite diameter.

So, for each commutative field $K$, we obtain a disconnected chain geometry $\Sigma(K, R)$, whereas for each skew field $K$ a disconnected generalized chain geometry arises.

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