Fixed Point Problems on Generalized Metric Spaces in Perov’s Sense

Liliana Guran 1, Monica-Felicia Bota 2,* and Asim Naseem 3

1 Department of Pharmaceutical Sciences, “Vasile Goldiş” Western University of Arad, L. Rebreanu Street, No. 86, 310048 Arad, Romania; lguran@uvvg.ro or gliliana.math@gmail.com
2 Department of Mathematics, Babeş-Bolyai University, Kogălniceanu Street No. 1, 400084 Cluj-Napoca, Romania
3 Department of Mathematics, GC University Lahore, Katchery Road, Lahore 54000, Pakistan; dr.asimnaseer@gcu.edu.pk

* Correspondence: bmonica@math.ubbcluj

Received: 9 March 2020; Accepted: 6 May 2020; Published: 22 May 2020

Abstract: The aim of this paper is to give some fixed point results in generalized metric spaces in Perov’s sense. The generalized metric considered here is the $w$-distance with a symmetry condition. The operators satisfy a contractive weakly condition of Hardy–Rogers type. The second part of the paper is devoted to the study of the data dependence, the well-posedness, and the Ulam–Hyers stability of the fixed point problem. An example is also given to sustain the presented results.

Keywords: fixed point; coupled fixed points; Perov space; generalized $w$-distance; Ulam–Hyers stability; well-posedness; data dependence

1. Introduction and Preliminaries

The well-known Banach contraction principle was extended by Perov in 1964 to the case of spaces endowed with vector-valued metrics. In [1], Perov introduced the concept of vector-valued metric as follows.

Let $X$ be a nonempty set. A mapping $\tilde{d}: X \times X \to \mathbb{R}^m$ where $\tilde{d} = \begin{pmatrix} d_1(x,y) & \cdots & d_m(x,y) \\ \vdots & \ddots & \vdots \\ d_m(x,y) & \cdots & d_1(x,y) \end{pmatrix}$ for every $m \in \mathbb{N}$ is called vector-valued metric on $X$ if the following properties are satisfied.

1. $\tilde{d}(x,y) \geq 0$ for all $x, y \in X$, and $\tilde{d}(x,y) = 0$ implies $x = y$;
2. $\tilde{d}(x,y) = \tilde{d}(y,x)$;
3. $\tilde{d}(x,y) \leq \tilde{d}(x,z) + \tilde{d}(z,y)$ for all $x, y, z \in X$.

In this case, the pair $(X, \tilde{d})$ is called a generalized metric space in Perov’s sense. Some examples of fixed points on the sense of vector-valued metric are given in [2–6]. Throughout this paper $\mathcal{M}_{m,m}(\mathbb{R}_+)$ will denote the set of all $m \times m$ matrices with positive elements. We also denote by $\Theta$ the zero $m \times m$ matrix and $0_{1 \times m} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ by $I$ the identity $m \times m$ matrix and $I_{1 \times m} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ and by $U$ the unity $m \times m$ matrix and $U_{1 \times m} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. If $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, then the symbol $A^T$ stands for the transpose matrix of $A$.

Recall that a matrix $A$ is said to be convergent to zero if and only if $A^n \to \Theta$ as $n \to \infty$.

Let us recall the following theorem, which is useful for the proof of the main result, see [7].
Theorem 1. Let $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$. The following assertions are equivalent.

(i) $A$ is a matrix convergent to zero;
(ii) $A^n \to \emptyset$ as $n \to \infty$;
(iii) The eigenvalues of $A$ are in the open unit disc, i.e., $|\lambda| < 1$, for each $\lambda \in \mathbb{C}$ with $\det(A - \lambda I) = 0$;
(iv) The matrix $I - A$ is non-singular and

$$(I - A)^{-1} = I + A + \ldots + A^n + \ldots;$$

(v) The matrix $I - A$ is non-singular and the matrix $(I - A)^{-1}$ has nonnegative elements.

In [8], one can find that the notion of K-metric, which is an extension of the Perov’s metric. Huang and Zhang reconsidered in [9] the notion of K-metric under the name cone metric.

Hardy and Rogers [10] proved in 1973 a generalization of Reich fixed point theorem. Having this as a starting point, many authors obtained fixed point results for Hardy–Rogers type operators.

Remark 1. Each metric is a $\bar{w}_0$-distance, but the reverse is not true.

For the following notations see I.A. Rus [21,22], I.A. Rus, A. Petruşel, A. Sîntămărian [23], and A. Petruşel [24].
**Definition 3.** Let \((X,d)\) be a metric space and \(f : X \to X\) be a single-valued operator. \(f\) is a weakly Picard operator (briefly WPO) if the sequence of successive approximations for \(f\) starting from \(x \in X\), \((f^n(x))_{n \in \mathbb{N}}\), converges, for all \(x \in X\) and its limit is a fixed point for \(f\).

If \(f\) is a WPO, then we consider the operator
\[
f^\infty : X \to X \text{ defined by } f^\infty(x) := \lim_{n \to \infty} f^n(x).
\]
Notice that \(f^\infty(X) = \text{Fix}(f)\).

**Definition 4.** Let \((X,d)\) be a metric space, \(f : X \to X\) be a WPO and \(c > 0\) be a real number. By definition, the single-valued operator \(f\) is \(c\)-weakly Picard operator (briefly \(c\)-WPO) if and only if the following inequality holds,
\[
d(x, f^\infty(x)) \leq cd(x, f(x)), \text{ for all } x \in X.
\]

For the theory of weakly Picard operators, for single-valued operators, see [21]. I.A. Rus gave in [22] the definition of Ulam–Hyers stability as follows.

**Definition 5.** Let \((X,d)\) be a metric space and \(f : X \to X\) be a single-valued operator. By definition, the fixed point equation
\[
x = f(x)
\]
is Ulam–Hyers stable if there exists a real number \(c_f > 0\) such that for each \(\epsilon > 0\) and each solution \(y^*\) of the inequality
\[
d(y, f(y)) \leq \epsilon
\]
there exists a solution \(x^*\) of Equation (1) such that
\[
d(y^*, x^*) \leq c_f \epsilon.
\]

**Remark 2.** If \(f\) is a \(c\)-weakly Picard operator, then the fixed point Equation (1) is Ulam–Hyers stable.

The Ulam stability of different functional type equations have been investigated by many authors (see [25–35]). We present in the first part of this paper some fixed point results in generalized metric spaces in Perov’s sense. The operator satisfies a contractive condition of Hardy–Rogers type. In the second part of the paper, we study the data dependence of the fixed point set. The well-posedness of the fixed point problem and the Ulam–Hyers stability are also studied.

2. Fixed Point Results

First, let us we recall the notion of generalized \(w\)-distance defined in [36] by L. Guran.

**Definition 6.** Let \((X, d)\) be a generalized metric space. The mapping \(\tilde{w} : X \times X \to \mathbb{R}_+^m\) is called generalized \(w\)-distance on \(X\) if it satisfies the following conditions.

1. \(\tilde{w}(x,y) \leq \tilde{w}(x,z) + \tilde{w}(z,y)\), for every \(x, y, z \in X\);
2. \(\tilde{w}\) is lower semicontinuous with respect to the second variable;
3. For any \(\varepsilon := \begin{pmatrix} \varepsilon_1 & \cdots & \varepsilon_m \end{pmatrix} > 0\), there exists \(\delta := \begin{pmatrix} \delta_1 & \cdots & \delta_m \end{pmatrix} > 0\), such that \(\tilde{w}(z, x) \leq \delta\) and \(\tilde{w}(z, y) \leq \delta\) implies \(d(x, y) \leq \varepsilon\).
Examples of generalized \( w \)-distance and some of its useful properties are also given in [36] and [37]. In the same framework, let us give the definition of generalized \( w_0 \)-distance.

**Definition 7.** Let \((X, \tilde{d})\) be a generalized metric space. A mapping \( \tilde{w} : X \times X \to [0, \infty) \) is called generalized \( \tilde{w}_0 \)-distance if it is generalized \( w \)-distance on \( X \) with \( \tilde{w}(x, x) = 0 \) for every \( x \in X \).

Let us recall the following useful result.

**Lemma 1.** Let \((X, \tilde{d})\) be a generalized metric space, and let \( \tilde{w} : X \times X \to \mathbb{R}^m \) be a generalized \( w \)-distance on \( X \). Let \((x_n)\) and \((y_n)\) be two sequences in \( X \), let \( \alpha_n := \begin{pmatrix} \alpha_{n1} \\ \cdots \\ \alpha_{nm} \end{pmatrix} \in \mathbb{R}^m_+ \) and \( \beta_n := \begin{pmatrix} \beta_{n1} \\ \cdots \\ \beta_{nm} \end{pmatrix} \in \mathbb{R}^m_+ \) be two sequences such that \( \alpha_{n(i)} \) and \( \beta_{n(i)} \) converge to zero for each \( i \in \{1, 2, \ldots, m\} \). Let \( x, y, z \in X \). Then, the following assertions hold, for every \( x, y, z \in X \).

1. If \( \tilde{w}(x_n, y) \leq \alpha_n \) and \( \tilde{w}(x_n, z) \leq \beta_n \) for any \( n \in \mathbb{N} \), then \( y = z \).
2. If \( \tilde{w}(x_n, y_n) \leq \alpha_n \) and \( \tilde{w}(x_n, z) \leq \beta_n \) for any \( n \in \mathbb{N} \), then \((y_n)\) converges to \( z \).
3. If \( \tilde{w}(x_n, x_m) \leq \alpha_n \) for any \( n, m \in \mathbb{N} \) with \( m > n \), then \((x_n)\) is a Cauchy sequence.
4. If \( \tilde{w}(y, x_n) \leq \alpha_n \) for any \( n \in \mathbb{N} \), then \((x_n)\) is a Cauchy sequence.

Next, let us give the definition of single-valued weakly Hardy–Rogers type operator on generalized metric space in Perov’s sense.

**Definition 8.** Let \((X, \tilde{d})\) be a generalized metric space in Perov’s sense, \( \tilde{w} : X \times X \to \mathbb{R}^m_+ \) be a generalized \( w \)-distance, and \( f : X \to X \) be a single-valued operator. We say that \( f \) is a weakly Hardy–Rogers type operator if the following inequality is satisfied,

\[
\tilde{w}(f(x), f(y)) \leq A \tilde{w}(x, y) + B[\tilde{w}(x, f(x)) + \tilde{w}(y, f(y))] + C[\tilde{w}(x, f(y)) + \tilde{w}(y, f(x))],
\]

for all \( x, y \in X \) and \( A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+) \).

The first fixed point result of this paper is the following.

**Theorem 2.** Let \((X, \tilde{d})\) be a complete generalized metric space in Perov’s sense, \( \tilde{w} : X \times X \to \mathbb{R}^m_+ \) be a generalized \( w_0 \)-distance. Let \( f : X \to X \) be a single-valued weakly Hardy–Rogers type operator such that

(a) \( f \) is continuous;
(b) there exist matrices \( A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+) \) such that

(i) \( M = (I - (B + C))^{-1}(A + B + C) \) converges to \( \Theta \);
(ii) \( I - (B + C) \) is nonsingular and \( (I - (B + C))^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+) \);
(iii) \( I - (A + 2B + 2C) \) is nonsingular and \( [I - (A + 2B + 2C)]^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+) \).

Then, \( \text{Fix}(f) \neq \emptyset \). Moreover, if \( x^* = f(x^*) \), then \( \tilde{w}(x^*, x^*) = 0 \).

**Proof.** Fix \( x_0 \in X \). Let \( x_1 = f(x_0) \) and \( x_2 = f(x_1) \). Then, we have

\[
\tilde{w}(x_1, x_2) = \tilde{w}(f(x_0), f(x_1)) A \tilde{w}(x_0, x_1) + B[\tilde{w}(x_0, f(x_0)) + \tilde{w}(x_1, f(x_1))] + C[\tilde{w}(x_0, f(x_1)) + \tilde{w}(x_1, f(x_0))]
\]

\[
+ \tilde{w}(x_1, f(x_0)) = (A + B) \tilde{w}(x_0, x_1) + B[\tilde{w}(x_0, x_1) + \tilde{w}(x_1, x_2)] + C[\tilde{w}(x_0, x_1) + \tilde{w}(x_1, x_2)]
\]

\[
= (A + B) \tilde{w}(x_0, x_1) + B(\tilde{w}(x_1, x_2)) + C[\tilde{w}(x_0, x_1) + \tilde{w}(x_1, x_2)]
\]

\[
= (A + B + C) \tilde{w}(x_0, x_1) + (B + C) \tilde{w}(x_1, x_2).
\]

Then, we have \( [I - (B + C)] \tilde{w}(x_1, x_2) \leq (A + B + C) \tilde{w}(x_0, x_1) \).
We get the inequality
\[
\tilde{w}(x_1, x_2) \leq [I - (B + C)]^{-1}(A + B + C)\tilde{w}(x_0, x_1) = M\tilde{w}(x_0, x_1).
\] (3)

For the next step, we have
\[
\tilde{w}(x_2, x_3) = \tilde{w}(f(x_1), f(x_2))A\tilde{w}(x_1, x_2) + B[\tilde{w}(x_1, f(x_1)) + \tilde{w}(x_2, f(x_2))] + C[\tilde{w}(x_1, f(x_2))]
\]
\[
+ \tilde{w}(x_2, f(x_1))] = A\tilde{w}(x_1, x_2) + B[\tilde{w}(x_1, x_2) + \tilde{w}(x_2, x_3)] + C[\tilde{w}(x_1, x_3) + \tilde{w}(x_2, x_2)]
\]
\[
= (A + B)\tilde{w}(x_1, x_2) + B(\tilde{w}(x_2, x_3)) + C[\tilde{w}(x_1, x_2) + \tilde{w}(x_2, x_3)]
\]
\[
= (A + B + C)\tilde{w}(x_1, x_2) + (B + C)\tilde{w}(x_2, x_3).
\]

Then, we have \([I - (B + C)]\tilde{w}(x_2, x_3) \leq (A + B + C)\tilde{w}(x_1, x_2).

Using (3) we obtain the inequality
\[
\tilde{w}(x_2, x_3) \leq [I - (B + C)]^{-1}(A + B + C)\tilde{w}(x_1, x_2) = M\tilde{w}(x_1, x_2) \leq M^2\tilde{w}(x_0, x_1).
\] (4)

By induction we obtain a sequence \((x)_n \in X\), with \(x_n = f(x_{n-1})\) such that
\[
\tilde{w}(x_n, x_{n+1}) \leq M^n\tilde{w}(x_0, x_1),
\] (5)

with \(M \in \mathcal{M}_{m,m}(\mathbb{R}_+)\) and \(n \in \mathbb{N}\).

We will prove next that \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence, by estimating \(\tilde{w}(x_m, x_n)\), for every \(m, n \in \mathbb{N}\) with \(m > n\).
\[
\tilde{w}(x_m, x_n) \leq \tilde{w}(x_m, x_{n+1}) + \tilde{w}(x_{n+1}, x_{n+2}) + \ldots + \tilde{w}(x_{m-1}, x_m)
\]
\[
\leq M^n(\tilde{w}(x_0, x_1)) + M^{n+1}(\tilde{w}(x_0, x_1)) + \ldots + M^{m-1}(\tilde{w}(x_0, x_1))
\]
\[
\leq M^n(I + M + M^2 + \ldots + M^{m-n-1})(\tilde{w}(x_0, x_1)) \leq M^n(I - M)^{-1}\tilde{w}(x_0, x_1)).
\]

Note that \((I - M)\) is nonsingular since \(M\) is convergent to zero. This implies
\[
\lim_{n \to \infty} w(x_n, x_m) \leq \lim_{n \to \infty} M^n(I - M)^{-1}\tilde{w}(x_0, x_1)) \quad \frac{d}{\to} 0_{1 \times m}.
\]

By Lemma 1 (3) the sequence \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence.

By (a) we have \(\tilde{w}(f(x_{n-1}), f(x_n)) \quad \frac{d}{\to} 0_{1 \times m}\) as \(n \to \infty\). As \((X, d)\) is complete, there exists \(x^* \in X\) such that \(\lim_{n \to \infty} x_n \quad \frac{d}{\to} x^*\) as \(n \to \infty\). From the continuity of \(f\), it follows that \(x_{n+1} = f(x_n) \quad \frac{d}{\to} f(x^*)\) as \(n \to \infty\). By the uniqueness of the limit, we get \(x^* = f(x^*)\), that is, \(x^*\) is a fixed point of \(f\). Then \(Fix(f) \neq \emptyset\).

Let \(x^* \in X\) such that \(x^* = f(x^*)\). Then, we have
\[
\tilde{w}(x^*, x^*) = \tilde{w}(f(x^*), f(x^*)) \leq A\tilde{w}(x^*, x^*)
\]
\[
+ B[\tilde{w}(x^*, f(x^*)) + \tilde{w}(x^*, f(x^*))] + C[\tilde{d}(x^*, f(x^*)) + \tilde{d}(x^*, f(x^*))]
\]
\[
= A\tilde{w}(x^*, x^*) + 2B\tilde{w}(x^*, x^*) + 2C\tilde{w}(x^*, x^*).\]

(6)

This implies \([I - (A + 2B + 2C)]\tilde{w}(x^*, x^*) \leq 0_{1 \times m}\). By hypothesis (iii) we get \(\tilde{w}(x^*, x^*) = 0_{1 \times m}\). □

We can replace the continuity condition on the operator \(f\) and we obtain the following fixed point theorem.
Theorem 3. Let \( (X, \tilde{d}) \) be a complete generalized metric space in Perov’s sense and \( \tilde{w} : X \times X \to \mathbb{R}_+^m \) be a generalized \( \omega_0 \)-distance. Let \( f : X \to X \) be a single-valued weakly Hardy–Rogers type operator such that the following conditions are satisfied,

(a) \( \inf \{ \tilde{w}(x, y) + \tilde{w}(x, f(x)) : x \in X \} > 0 \);

(b) there exist matrices \( A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+) \) such that:

(i) \( M = (I - (B + C))^{-1}(A + B + C) \) converges to \( \Theta \);

(ii) \( I - (B + C) \) is nonsingular and \( (I - (B + C))^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+) \);

(iii) \( I - (A + 2B + 2C) \) is nonsingular and \( [I - (A + 2B + 2C)]^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+) \).

Then \( \text{Fix}(f) \neq \emptyset \). Moreover, if \( x^* = f(x^*) \), then \( w(x^*, x^*) = 0 \).

Proof. Following the same steps as in the previous theorem, Theorem 2, we have the estimation

\[
\tilde{w}(x_n, x_m) \leq M^n(I - M)^{-1}\tilde{w}(x_0, x_1)
\]

with \( M \in \mathcal{M}_{m,m}(\mathbb{R}_+) \) and \( n \in \mathbb{N} \).

By Lemma 1 (3), the sequence \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence. As \((X, \tilde{d})\) is complete, there exists \( x^* \in X \) such that \( x_n \overset{\tilde{d}}{\to} x^* \). Let \( n \in \mathbb{N} \) be fixed. Then, as \( (x_m)_{m \in \mathbb{N}} \overset{\tilde{d}}{\to} x^* \) and \( \tilde{w}(x_m, \cdot) \) is lower semicontinuous, we have

\[
\tilde{w}(x_n, x_m) \leq \liminf_{m \to \infty} \tilde{w}(x_n, x_m) \leq M^n(I - M)^{-1}\tilde{w}(x_0, x_1).
\]

Assume that \( x^* \neq f(x^*) \). Then, for every \( x \in X \), by hypothesis (a) we have

\[
0 < \inf \{ \tilde{w}(x, x^*) + \tilde{w}(x, f(x)) : x \in X \} \leq \inf \{ \tilde{w}(x_n, x^*) + \tilde{w}(x_n, x_{n+1}) : n \in \mathbb{N} \}
\]

\[
\leq \inf \{ M^n(I - M)^{-1}\tilde{w}(x_0, x_1) + M^n\tilde{w}(x_0, x_1) \} = 0.
\]

This is a contradiction. Therefore \( x^* = f(x^*) \), so \( \text{Fix}(f) \neq \emptyset \). For the proof of the last part of this theorem we use the same steps as is the previous theorem, Theorem 2.

Further we give a more general fixed point result concerning this new type of operators.

Theorem 4. Let \((X, \tilde{d})\) be a complete generalized metric space in Perov’s sense, \( \tilde{w} : X \times X \to \mathbb{R}_+^m \) be a generalized \( \omega_0 \)-distance, and \( f : X \to X \) be a single-valued weakly Hardy–Rogers type operator. There exist matrices \( A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+) \) such that

(i) \( M = (I - (B + C))^{-1}(A + B + C) \) converges to \( \Theta \);

(ii) \( I - (B + C) \) is nonsingular and \( (I - (B + C))^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+) \);

(iii) \( I - (A + 2B + 2C) \) is nonsingular and \( [I - (A + 2B + 2C)]^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+) \).

Then \( \text{Fix}(f) \neq \emptyset \). Moreover, if \( x^* = f(x^*) \), then \( w(x^*, x^*) = 0 \).

Proof. Following the same steps as in Theorem 2, we get the estimation

\[
\tilde{w}(x_n, x_m) \leq M^n(I - M)^{-1}\tilde{w}(x_0, x_1)
\]

with \( M \in \mathcal{M}_{m,m}(\mathbb{R}_+) \) and \( n \in \mathbb{N} \).

By Lemma 1 (3) the sequence \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence; since \((X, \tilde{d})\) is complete there exists \( x^* \in X \) such that \( x_n \overset{\tilde{d}}{\to} x^* \).

Let \( n \in \mathbb{N} \) be fixed. Then, as \( (x_m)_{m \in \mathbb{N}} \overset{\tilde{d}}{\to} x^* \), \( \tilde{w}(x_m, \cdot) \) is lower semicontinuous and letting \( n \to \infty \) we have

\[
\tilde{w}(x_n, x^*) \leq \liminf_{n \to \infty} \tilde{w}(x_n, x_m) \leq M^n(I - M)^{-1}\tilde{w}(x_0, x_1) \overset{\tilde{d}}{\to} 0_{1 \times m}.
\]
Let $f(x^*) \in X$. By triangle inequality and using (6) we obtain
\[
\tilde{w}(x_n, f(x^*)) = \tilde{w}(x_n, x^*) + \tilde{w}(x^*, f(x^*)) \leq \tilde{w}(x_n, x^*) + \tilde{w}(f(x^*), f(x^*)) \\
\leq M^n(I - M)^{-1}\tilde{w}(x_0, x_1) + [I - (A + 2B + 2C)]\tilde{w}(x^*, x^*) \overset{d}{=} 0_{1 \times m}.
\]
(11)

Using Lemma 1(1), by Equations (10) and (11), we get $x^* = f(x^*)$. Then, $\text{Fix}(f) \neq \emptyset$.

For the last part of the proof we use the same steps as in Theorem 2. \qed

Another fixed point result concerning the single-valued weakly Hardy–Rogers operators in generalized metric space is the following.

**Theorem 5.** Let $(X, \tilde{d})$ be a complete generalized metric space in Perov’ sense, $\tilde{w} : X \times X \rightarrow \mathbb{R}^n_+$ be a generalized $w_0$-distance and $f : X \rightarrow X$ be a single-valued Hardy–Rogers type operator. Suppose that all the hypothesis of Theorem 2 hold. Then, we have

1. Fix$(f) \neq \emptyset$.
2. There exists a sequence $(x_n)_{n \in \mathbb{N}} \in X$ such that $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$ and converge to a fixed point of $f$.
3. $\tilde{d}(x_n, x^*) \leq M^n\tilde{d}(x_0, x_1)$, where $x^* \in \text{Fix}(f)$.

**Example 1.** Let $X = \mathbb{R}^2$ be a normed linear space endowed with the generalized norm $\tilde{d}$ defined by
\[
\tilde{d}(x, y) = \left( \begin{array}{c} \|x_1 - y_1\| \\ \|x_2 - y_2\| \end{array} \right)
\]
and $\tilde{w}$ a generalized $w_0$-distance defined by $\tilde{w}(x, y) = \left( \begin{array}{c} \|y_1\| \\ \|y_2\| \end{array} \right)$, for each $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an operator given by
\[
f(x, y) = \begin{cases} \frac{4}{5}x + \frac{6}{5}y - 1, & \text{for } (x, y) \in \mathbb{R}^2, \text{ with } x \leq 5; \\ \frac{4}{5}x + \frac{6}{5}y - 1, & \text{for } (x, y) \in \mathbb{R}^2, \text{ with } x > 5. 
\end{cases}
\]

We take $f(x, y) = (f_1(x, y), f_2(x, y))$ where $f_1(x, y) = \begin{cases} \frac{4}{5}x + \frac{6}{5}y - 1, & \text{for } (x, y) \in \mathbb{R}^2, \text{ with } x \leq 5; \\ \frac{4}{5}x + \frac{6}{5}y - 1, & \text{for } (x, y) \in \mathbb{R}^2, \text{ with } x > 5. 
\end{cases}
and f_2(x, y) = \begin{cases} \frac{6}{5}y - 1, & \text{for } (x, y) \in \mathbb{R}^2, \text{ with } x \leq 5; \\ \frac{6}{5}y - 1, & \text{for } (x, y) \in \mathbb{R}^2, \text{ with } x > 5. \end{cases}$

Next, we show that weakly Hardy–Rogers type condition takes place.

Let $A = \begin{pmatrix} \frac{4}{5} & \frac{6}{5} \\ \frac{6}{5} & \frac{4}{5} \end{pmatrix}$.

Case 1. If $1 \leq x_1, x_2, y_1, y_2 \leq 5$ we have
\[
\tilde{w}(f(x), f(y)) = \left( \begin{array}{c} \|f_1(y_1, y_2)\| \\ \|f_2(y_1, y_2)\| \end{array} \right) = \left( \begin{array}{c} \|\frac{4}{5}y_1 + \frac{6}{5}y_2 - 1\| \\ \|0 \cdot y_1 + \frac{6}{5}y_2 - 1\| \end{array} \right) \leq \left( \begin{array}{c} \frac{4}{5}\|y_1\| + \frac{6}{5}\|y_2\| - 1 \\ 0 \cdot \|y_1\| + \frac{6}{5}\|y_2\| - 1 \end{array} \right)
\]
\[
\leq \left( \begin{array}{c} \frac{4}{5} \\ 0 \end{array} \right) \left( \begin{array}{c} \|y_1\| \\ \|y_2\| \end{array} \right) = A\tilde{w}(x, y).
\]

Case 2. If $x_1, x_2, y_1, y_2 > 5$ we have
\[
\tilde{w}(f(x), f(y)) = \left( \begin{array}{c} \|f_1(y_1, y_2)\| \\ \|f_2(y_1, y_2)\| \end{array} \right) = \left( \begin{array}{c} \|\frac{1}{2}y_1 + \frac{1}{2}y_2 - 1\| \\ \|0 \cdot y_1 + \frac{1}{2}y_2\| \end{array} \right) \leq \left( \begin{array}{c} \frac{1}{2}\|y_1\| + \frac{1}{2}\|y_2\| - 1 \\ 0 \cdot \|y_1\| + \frac{1}{2}\|y_2\| \end{array} \right)
\]
\[
\leq \left( \begin{array}{c} \frac{1}{2} \\ 0 \end{array} \right) \left( \begin{array}{c} \|y_1\| \\ \|y_2\| \end{array} \right) < \left( \begin{array}{c} \frac{4}{5} \\ 0 \end{array} \right) \left( \begin{array}{c} \|y_1\| \\ \|y_2\| \end{array} \right) = A\tilde{w}(x, y).
Case 3. For other choices of $x_1, x_2, y_1, y_2$ we have
\[ \tilde{\omega}(f(x), f(y)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \leq \begin{pmatrix} \frac{4}{5} & \frac{6}{5} \\ 0 & \frac{6}{5} \end{pmatrix} \begin{pmatrix} ||y_1|| \\ ||y_2|| \end{pmatrix} = A \tilde{\omega}(x, y). \]

Thus, the weakly Hardy–Rogers type condition is satisfied for $A = \begin{pmatrix} \frac{4}{5} & \frac{6}{5} \\ 0 & \frac{6}{5} \end{pmatrix}$ and $B = C = \Theta$ or $B + C = \Theta$.

As all the hypothesis of Theorem 3 hold, $f$ has a fixed point and it is easy to check that $x = f(x) = (f_1(x), f_2(x))$, where $x = (1, 1)$.

Next, let us give some common fixed point results.

**Theorem 6.** Let $(X, \tilde{d})$ be a complete generalized metric space in Perov’s sense, $\tilde{\omega} : X \times X \rightarrow \mathbb{R}^m_+$ be a generalized $\omega$-distance, and $f, g : X \rightarrow X$ be two continuous single-valued weakly Hardy–Rogers type operators. There exist matrices $A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ such that

(i) $I - (B + C)$ is nonsingular and $(I - (B + C))^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$;

(ii) $M = (I - (B + C))^{-1}(A + B + C)$ converges to $\Theta$.

Then, $f$ and $g$ have a common fixed point $x^* \in X$.

**Proof.** (1) Let $x_0 \in X$. We consider $(x_n)_{n \in \mathbb{N}}$ the sequence of successive approximations for $f$ and $g$, defined by
\[ x_{2n+1} = f(x_{2n}), n = 0, 1, ... \]
\[ x_{2n+2} = g(x_{2n+1}), n = 0, 1, ... \]

Then, we have
\[ \tilde{\omega}(x_{2n}, x_{2n+1}) = \tilde{\omega}(x_{2n+1}, x_{2n+2}) \leq A \tilde{\omega}(x_{2n+1}, f(x_{2n})) + B \tilde{\omega}(x_{2n}, f(x_{2n})) + C \tilde{\omega}(x_{2n}, g(x_{2n+1})). \]

By the same argument as above, we get
\[ \tilde{\omega}(x_{2n+1}, x_{2n+2}) \leq A \tilde{\omega}(x_{2n+1}, f(x_{2n+1})) + B \tilde{\omega}(x_{2n}, f(x_{2n})) + C \tilde{\omega}(x_{2n}, g(x_{2n})). \]

Further, we obtain $\tilde{\omega}(x_n, x_{n+1}) \leq M \tilde{\omega}(x_0, x_1)$ for each $n \in \mathbb{N}$.

Following the same steps as in the proof of Theorem 2 we estimate $\tilde{\omega}(x_n, x_m)$, for every $m, n \in \mathbb{N}$ with $m > n$.
\[ \tilde{\omega}(x_n, x_m) \leq \tilde{\omega}(x_n, x_{n+1}) + \tilde{\omega}(x_{n+1}, x_{n+2}) + ... + \tilde{\omega}(x_{m-1}, x_m) \leq M^n(\tilde{\omega}(x_0, x_1)) + M^n(\tilde{\omega}(x_0, x_1)) + ... + M^{n-1}(\tilde{\omega}(x_0, x_1)) \leq M^n(I + M + M^2 + ... + M^{m-n-1})(\tilde{\omega}(x_0, x_1)) \leq M^n(I - M)^{-1}(\tilde{\omega}(x_0, x_1)). \]
Note that \((I - M)\) is nonsingular since \(M\) is convergent to \(\Theta\). Using Lemma 1 (3) the sequence 
\((x_n)_{n\in \mathbb{N}}\) is a Cauchy sequence.

Using the lower semicontinuity of the generalized \(\omega\)-distance, by relation (8) we have \(\tilde{\omega}(x_n, x^* ) \xrightarrow{d} 0_{1 \times m}\) as \(n \to \infty\). Then, we have \(\tilde{\omega}(x_{2n}, x^*) \xrightarrow{d} 0_{1 \times m}\) as \(n \to \infty\). By the continuity of \(f\) it follows \(x_{2n+1} = f(x_{2n}) \xrightarrow{d} f(x^*)\) as \(n \to \infty\). By the uniqueness of the limit we get \(x^* = f(x^*)\).

By \(\tilde{\omega}(x_n, x^*) \xrightarrow{d} 0_{1 \times m}\) as \(n \to \infty\) we have that \(\tilde{\omega}(x_{2n+1}, x^*) \xrightarrow{d} 0_{1 \times m}\) as \(n \to \infty\). By the continuity of \(g\) it follows \(x_{2n+2} = g(x_{2n+1}) \xrightarrow{d} g(x^*)\) as \(n \to \infty\). By the uniqueness of the limit we get \(x^* = g(x^*)\).

Then, \(x^*\) is a common fixed point for \(f\) and \(g\). \(\square\)

By replacing the continuity condition for the mappings \(f\) and \(g\), we can state the following result.

**Theorem 7.** Let \((X, \tilde{d})\) be a complete generalized metric space in Perov’s sense, \(\tilde{\omega} : X \times X \to \mathbb{R}^m_+\) be a generalized \(\omega\)-distance, and \(f, g : X \to X\) be two single-valued Hardy–Rogers type operators. There exist matrices \(A, B, C \in \mathcal{M}_{m,m}(\mathbb{R}_+)\) such that

\[(i) \quad I - (B + C)\text{ is nonsingular and } (I - (B + C))^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+);
(ii) \quad I - (A + 2B + 2C)\text{ is nonsingular and } [I - (A + 2B + 2C)]^{-1} \in \mathcal{M}_{m,m}(\mathbb{R}_+);
(iii) \quad M = (I - (B + C))^{-1}(A + B + C)\text{ converges to } \Theta.\]

Then \(f\) and \(g\) have a common fixed point \(x^* \in X\).

**Proof.** (1) As in the proof of the previous theorem, Theorem 6, for \(x_0 \in X\) we consider \((x_n)_{n\in \mathbb{N}}\) the sequence of successive approximations for \(f\) and \(g\), defined by

\[x_{2n+1} = f(x_{2n}), \quad n = 0, 1, \ldots\]
\[x_{2n+2} = g(x_{2n+1}), \quad n = 0, 1, \ldots\]

We define the sequence \((x_n)_{n\in \mathbb{N}} \in X\) such that

\[\tilde{\omega}(x_{2n+1}, x_{2n+2}) \leq (I - (B + C))^{-1}(A + B + C)\tilde{\omega}(x_{2n}, x_{2n+1}) = M\tilde{\omega}(x_{2n}, x_{2n+1}).\]

Further, we obtain \(\tilde{\omega}(x_n, x_{n+1}) \leq M^n \tilde{d}(x_0, x_1)\) for each \(n \in \mathbb{N}\).

Following the same steps as in the proof of Theorem 6 we estimate \(\tilde{\omega}(x_n, x_m)\), for every \(m, n \in \mathbb{N}\) with \(m > n\) and we get \(\tilde{\omega}(x_n, x_{n+1}) \leq M^n (I - M)^{-1}\tilde{\omega}(x_0, x_1)\).

Note that \((I - M)\) is nonsingular since \(M\) is convergent to \(\Theta\). By Lemma 1 (3), the sequence \((x_n)_{n\in \mathbb{N}}\) is a Cauchy sequence. Using the lower semicontinuity of the generalized \(\omega\)-distance, by relation (8), we have \(\tilde{\omega}(x_n, x^*) \xrightarrow{d} 0_{1 \times m}\), as \(n \to \infty\). By (11) we have \(\tilde{\omega}(x_n, f(x^*)) \xrightarrow{d} 0_{1 \times m}\), as \(n \to \infty\). Then, using Lemma 1 (2), we get \(x^* = f(x^*)\).

Let us show that \(g(x^*) = x^*\). Then, by the definition of Hardy–Rogers type operators we have

\[\tilde{\omega}(x^*, g(x^*)) = \tilde{d}(f(x^*), g(x^*)) \leq A\tilde{\omega}(x^*, x^*) + B[\tilde{\omega}(x^*, f(x^*)) + \tilde{\omega}(x^*, g(x^*))] + C[\tilde{\omega}(x^*, g(x^*)) + \tilde{\omega}(x^*, f(x^*))].\]

Then, we get

\[\tilde{\omega}(x^*, g(x^*)) \leq (I - (B + C))^{-1}(A + B + C)\tilde{\omega}(x^*, x^*).\]

By (6) we get \(\tilde{\omega}(x^*, g(x^*)) = 0_{1 \times m}\).
Let \(g(x^*) \in X\). By triangle inequality and using (12) we obtain

\[\tilde{\omega}(x_0, g(x^*)) = \tilde{\omega}(x_0, x^*) + \tilde{\omega}(x^*, g(x^*)) \leq M^n (I - M)^{-1}\tilde{\omega}(x_0, x_1) + 0_{1 \times m} \xrightarrow{d} 0_{1 \times m}.\]
Using (8) and (13), by Lemma 1 (2), we obtain \( x^* = g(x^*) \). Then \( x^* \) is a common fixed point for \( f \) and \( g \).

**Remark 3.** In the case of common fixed points, the generalized \( \bar{w} \)-distance must not necessarily be a generalized \( \bar{w}_0 \)-distance.

### 3. Ulam–Hyers Stability, Well-Posedness, and Data Dependence of Fixed Point Problem

We begin this section with the extension of Ulam–Hyers stability for fixed point equation for the case of single-valued operators on generalized metric space in Perov’s sense. Then, let us recall the definition of weakly Ulam–Hyers stability.

**Definition 9.** Let \( (X, \tilde{d}) \) be a metric space, \( \tilde{w} : X \times X \to \mathbb{R}_+^m \) be a generalized \( \tilde{w} \)-distance, and \( f : X \to X \) be an operator. By definition, the fixed point equation

\[
x = f(x)
\]

is weakly Ulam–Hyers stable if there exists a real positive matrix \( N \in M_{m,m}(\mathbb{R}^+) \) such that, for each \( \varepsilon > 0 \) and each solution \( y^* \) of the inequality

\[
\tilde{w}(y, f(y)) \leq \varepsilon I_{1 \times m}
\]

there exists a solution \( x^* \) of the Equation (14) such that

\[
\tilde{d}(y^*, x^*) \leq N \varepsilon I_{1 \times m}.
\]

**Theorem 8.** Let \( (X, \tilde{d}) \) be a generalized metric space in Perov’s sense, \( \tilde{w} : X \times X \to \mathbb{R}_+^m \) be a generalized \( \tilde{w}_0 \)-distance and \( f : X \to X \) be a single-valued Hardy–Rogers type operator defined in (9). There exist matrices \( A, B, C \in M_{m,m}(\mathbb{R}^+) \) such that

(i) \( N = M^m(I - M)^{-1} \) is nonsingular and \( N = M^m(I - M)^{-1} \in M_{m,m}(\mathbb{R}_+) \), where \( M = (I(B + C))^{-1}(A + B + C) \) converges to \( \Theta \);

(ii) \( I - (A + 2B + 2C) \) is nonsingular and \( [I - (A + 2B + 2C)]^{-1} \in M_{m,m}(\mathbb{R}^+) \);

(iii) \( I - P^2 \) is nonsingular and \( I - P^2 \in M_{m,m}(\mathbb{R}_+) \) where \( P = [I - (A + C)]^{-1} \in M_{m,m}(\mathbb{R}_+) \).

Then, the fixed point Equation (14) is weakly Ulam–Hyers stable.

**Proof.** Let \( \delta I_{1 \times m} > 0 \) such that \( \tilde{w}(x_0, x_1) \leq \delta I_{1 \times m} \) for every \( x_0, x_1 \in X \) with \( x_1 = f(x_0) \). Let \( \text{Fix}(f) = \{x^*\} \) and \( u^* \in X \) be a solution of Equation (14). Then, \( \tilde{w}(u^*, f(u^*)) \leq \varepsilon I_{1 \times m} \). By the definition of the weakly Hardy–Rogers type operator we obtain

\[
\tilde{w}(x^*, u^*) \leq \tilde{w}(f(x^*), f(u^*)) \leq A\tilde{w}(x^*, u^*) + B[\tilde{w}(x^*, f(x^*)) + \tilde{w}(u^*, f(u^*))] + C[\tilde{w}(x^*, f(x^*)) + \tilde{w}(u^*, u^*)]
\]

\[
+ \tilde{w}(u^*, f(x^*)) = A\tilde{w}(x^*, u^*) + B[\tilde{w}(x^*, x^*) + \tilde{w}(u^*, u^*)] + C[\tilde{w}(x^*, u^*) + \tilde{w}(u^*, x^*)]
\]

(16)

\[
= (A + C)\tilde{w}(x^*, u^*) + B[\tilde{w}(x^*, x^*) + \tilde{w}(u^*, u^*)] + C\tilde{w}(u^*, x^*).
\]

By (6) we get

\[
\tilde{w}(x^*, x^*) = \tilde{w}(f(x^*), f(x^*)) \leq (A + 2B + 2C)\tilde{w}(x^*, x^*) \quad \text{and}
\]

\[
\tilde{w}(u^*, u^*) = \tilde{w}(f(u^*), f(u^*)) \leq (A + 2B + 2C)\tilde{w}(u^*, u^*)
\]

Using hypothesis (ii) we get \( \tilde{w}(x^*, x^*) = \tilde{w}(u^*, u^*) = 0_{1 \times m} \).

By (16) we obtain

\[
\tilde{w}(x^*, u^*) \leq [I - (A + C)]^{-1}C\tilde{w}(u^*, x^*).
\]

(18)
By the definition of the weakly Hardy–Rogers type operator we get
\[ \tilde{w}(u^*, x^*) \leq [I - (A + C)]^{-1}C\tilde{w}(x^*, u^*) \]
and using (18) we obtain
\[ \tilde{w}(x^*, u^*) \leq ([I - (A + C)]^{-1}C)^2\tilde{w}(x^*, u^*) = P^2\tilde{w}(x^*, u^*). \] (19)

Then, \((I - P^2)\tilde{w}(x^*, u^*) \leq 0_{1 \times m}\). By hypothesis \((iii)\) we get \(\tilde{w}(x^*, u^*) = 0_{1 \times m}\).

Let \(x_n \in X\) such that, by Equations (8) and (19) we have
\[ \tilde{w}(x_n, x^*) \leq M^m(I - M)^{-1}\tilde{w}(x_0, x_1) \leq N\delta I_{1 \times m} \text{ and} \]
\[ \tilde{w}(x_n, u^*) \leq \tilde{w}(x_n, x^*) + \tilde{w}(x^*, u^*) \leq M^m(I - M)^{-1}\tilde{w}(x_0, x_1) + 0_{1 \times m} \leq N\delta I_{1 \times m}. \] (20)

Then, using the definition of generalized \(w\)-distance, there exists \(\epsilon I_{1 \times m} > 0_{1 \times m}\) such that
\[ \tilde{d}(x^*, u^*) \leq \epsilon I_{1 \times m} \leq NeI_{1 \times m}. \]

Then, the fixed point Equation (14) is weakly Ulam–Hyers stable.

The following result assures the well-posedness of the fixed point problem with respect to the generalized \(w_0\)-distance \(\tilde{w}\).

**Theorem 9.** Let \((X, \tilde{d})\) be a generalized metric space in Perov’s sense, \(\tilde{w} : X \times X \to \mathbb{R}^m_+\) be a generalized \(w_0\)-distance, and \(f : X \to X\) be a single-valued Hardy–Rogers type operator defined in Equation (8). If all the hypothesis of Theorem 2 (respectively, 3 and 4) are satisfied, the fixed point Equation (14) is well-posed with respect to the generalized \(w_0\)-distance \(\tilde{w}\), i.e., if \(\text{Fix}(f) = \{x^*\}\) and \(x_n \in \mathbb{N}\), with \(n \in \mathbb{N}\), such that \(\tilde{w}(x_n, f(x_n)) \to 0_{1 \times m}\) as \(n \to \infty\), then \(x_n \to x^*\) as \(n \to \infty\).

**Proof.** Let \(x^* \in \text{Fix}(f)\) and let \((x)_n \in X\) such that \(\tilde{w}(x_n, f(x_n)) \xrightarrow{d} 0_{1 \times m}\) as \(n \to \infty\). That means \(\tilde{w}(x_{n-1}, x_n) \xrightarrow{d} 0_{1 \times m}\) as \(n \to \infty\).

By the lower semicontinuity of the generalized \(w\)-distance, using (8) we have
\[ \tilde{w}(x_{n-1}, x^*) \leq \liminf_{m \to \infty} \tilde{w}(x_n, x_m) \leq M^m(I - M)^{-1}\tilde{w}(x_0, x_1) \xrightarrow{d} 0_{1 \times m}. \]

Then, using Lemma 1 (3) we get \(x_n \xrightarrow{d} x^*\) as \(n \to \infty\). □

The next theorem presents a data dependence result.

**Theorem 10.** Let \((X, \tilde{d})\) be a generalized metric space in Perov’s sense, \(\tilde{w} : X \times X \to \mathbb{R}^m_+\) be a generalized \(w_0\)-distance, and \(f_1, f_2 : X \to X\) be single-valued operators, which satisfy the following conditions,

(i) for \(A, B, C, M \in \mathcal{M}_{m \times m}(\mathbb{R}_+)\) with \(M = [I - (B + C)]^{-1}(A + B + C)\) a matrix convergent to \(\Theta\) such that, for every \(x, y \in X\) and \(i \in \{1, 2\}\), we have:
\[ \tilde{w}(f_i(x), f_i(y)) \leq A\tilde{w}(x, y) + B[\tilde{w}(x, f_i(x)) + \tilde{w}(y, f_i(y))] + C[\tilde{w}(x, f_i(y)) + \tilde{w}(y, f_i(x))]; \]

(ii) there exists \(\eta > 0\) such that \(\tilde{w}(f_1(x), f_2(x)) \leq \eta I, \) for all \(x \in X.\)

Then, for \(x_1^* = f_1(x^*)\) there exists \(x_2^* = f_2(x_2^*)\) such that \(\tilde{d}(x_1^*, x_2^*) \leq (I - M)^{-1}\eta I_{1 \times m};\) (respectively, for \(x_2^* = f_2(x_2^*)\) there exists \(x_1^* = f_1(x_1^*)\) such that \(\tilde{w}(x_2^*, x_1^*) \leq (I - M)^{-1}\eta I_{1 \times m}.\)
Proof. As in the proof of Theorem 2 (respectively, Theorem 3) we construct the sequence of successive approximations \((x_n)_{n \in \mathbb{N}} \in X \) of \(f_2\) with \(x_0 = x_1\) and \(x_1 = f_2(x_1)\) having the property \(\tilde{w}(x_n, x_{n+1}) \leq M^n \tilde{w}(x_0, x_1)\), where \(M = [1 - (B + C)]^{-1}(A + B + C)\).

If we consider the sequence \((x_n)_{n \in \mathbb{N}} \in X\) converges to \(x_2\), we have \(x_2 = f(x_2)\). Moreover, for each \(n, p \in \mathbb{N}\) we have \(\tilde{w}(x_n, x_{n+p}) \leq M^p (I - M)^{-1} \tilde{w}(x_0, x_1)\).

Letting \(p \to 0\) we get \(\tilde{w}(x_n, x_2) \leq I(I - M)^{-1} \tilde{w}(x_0, x_1)\).

Choosing \(n = 0\) we get \(\tilde{w}(x_0, x_2) \leq I(I - M)^{-1} \tilde{w}(x_0, x_1)\) and using above the notations we get our conclusion \(\tilde{w}(x_0, x_2) \leq (I - M)^{-1} \eta I_{X_M}\). \(\square\)

4. Conclusions

The purpose of this paper is to establish some fixed point results in generalized metric spaces in Perov’s sense. The generalized metric considered here is the \(w\)-distance, for which the symmetry condition is not satisfied. The operators satisfy a contractive weakly condition of Hardy–Rogers type. The second part of the paper is devoted to the study of the data dependence, as well as the well-posedness and the Ulam–Hyers stability of the fixed point problem. In order to prove our main results we had to impose a symmetry condition for the \(w\)-distance. The results presented in this paper generalize some recent ones.

Author Contributions: Conceptualization, L.G. and M.-F.B.; methodology, A.N.; software, L.G.; validation, L.G., M.-F.B. and A.N.; formal analysis, M.-F.B.; investigation, L.G.; resources, A.N.; data curation, A.N.; writing—original draft preparation, L.G.; writing—review and editing, M.-F.B., L.G.; visualization, A.N.; supervision, L.G.; project administration, L.G., M.-F.B.; funding acquisition, M.-F.B. All authors have read and agreed to the published version of the manuscript.

Funding: The second author wish to thanks Babes-Bolyai University, Cluj-Napoca, Romania, for the financial support.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Perov, A.I. On Cauchy problem for a system of ordinary differential equations. Priklizhen. Metody Reshen. Differ. Uravn. 1964, 2, 115–134. (In Russian)
2. Bucur, A.; Guran, L.; Petrusel, A. Fixed points for multivalued operators on a set endowed with vector-valued metrics and applications. Fixed Point Theory 2009, 10, 19–34.
3. Filip, A.D.; Petru¸sel, A. Fixed point theorems on spaces endowed with vector-valued metrics. Fixed Point Theory Appl. 2010, 2010, 281381. [CrossRef]
4. Safia, B.; Fateh, E.; Abdelkrimz, A. Fixed point theory on spaces with vector-valued metrics and applications. J. Math. Anal. Appl. 2017, 46, 457–464. [CrossRef]
5. O’Regan, D.; Precup, R. Continuation theory for contractions on spaces with two vector-valued metrics. Appl. Anal. 2003, 82, 131–144. [CrossRef]
6. O’Regan, D.; Shahzad, N.; Agarwal, R.P. Fixed point theory for generalized contractive maps on spaces with vector-valued metrics. Fixed Point Theory Appl. 2007, 6, 143–149.
7. Rus, I.A. The theory of a metrical fixed point theorem: Theoretical and applicative relevances. Fixed Point Theory 2008, 9, 541–559.
8. Zabreiko, P.P. K-metric and K-normed linear spaces: Survey. Collect. Math. 1997, 48, 825–859.
9. Huang, L.G.; Zhang, X. Cone metric spaces and fixed point theorems of contractive mappings. J. Math. Anal. Appl. 2007, 332, 1468–1476. [CrossRef]
10. Hardy, G.E.; Rogers, A.D. A generalisation of fixed point theorem of Reich. Canad. Math. Bull. 1973, 16, 201–208. [CrossRef]
11. Kada, O.; Suzuki, T.; Takahashi, W. Nonconvex minimization theorems and fixed point theorems in complete metric spaces. Math. Jpn. 1996, 44, 381–391.
12. Suzuki, T.; Takahashi, W. Fixed points theorems and characterizations of metric completeness. Topol. Methods Nonlinear Anal. J. Juliusz Schauder Cent. 1996, 8, 371–382. [CrossRef]
13. García-Falset, J.; Llorens-Fuster, E. Diametrically contractive mappings with respect to a \(w\)-distance. J. Nonlinear Conv. Anal. 2016, 17, 1975–1984.
14. Guran, L.; Latif, A. Fixed point theorems for multivalued contractive operators on generalized metric spaces. *Fixed Point Theory* 2015, 16, 327–336.
15. Latif, A.; Albar, W.A. Fixed point results in complete metric spaces. *Demonstr. Math.* 2008, 41, 1129–1136.
16. Latif, A.; Abdou, A.A.N. Fixed point results for generalized contractive multimaps in metric spaces. *Fixed Point Theory Appl.* 2009, 2009, 432130. [CrossRef]
17. Latif, A.; Abdou, A.A.N. Multivalued generalized nonlinear contractive maps and fixed points. *Nonlinear Anal.* 2011, 74, 1436–1444. [CrossRef]
18. Mongkolkeha, C.; Cho, Y.J. Some coincidence point theorems in ordered metric spaces via w-distances. *Carpathian J. Math.* 2018, 34, 207–214.
19. Takahashi, W.; Wong, N.C.; Yao, J.C. Fixed point theorems for general contractive mappings with w-distances in metric spaces. *J. Nonlinear Conv. Anal.* 2013, 14, 637–648.
20. Du, W.S. Fixed point theorems for generalized Hausdorff metrics. *Int. Math. Forum* 2008, 3, 1011–1022.
21. Rus, I.A. *Generalized Contractions and Applications*; Cluj University Press: Cluj-Napoca, Romania, 2001.
22. Rus, I.A. Remarks on Ulam stability of the operatorial equations. *Fixed Point Theory* 2009, 10, 305–320.
23. Petrușel, A.; Rus, I.A.; Săntăiano, A. Data dependence of the fixed point set of multivalued weakly Picard operators. *Nonlinear Anal.* 2003, 52, 1947–1959.
24. Petrușel, A. Multivalued weakly Picard operators and applications. *Sci. Math. Jpn.* 2004, 1, 1–34.
25. Brzdek, J.; Popa, D.; Xu, B. The Hyers-Ulam stability of nonlinear recurrences. *J. Math. Anal. Appl.* 2007, 335, 443–449. [CrossRef]
26. Brzdek, J.; Popa, D.; Xu, B. Hyers-Ulam stability for linear equations of higher orders. *Acta Math. Hungr.* 2008, 120, 1–8. [CrossRef]
27. Bota-Boriceanu, M.F.; Petrușel, A. Ulam–Hyers stability for operatorial equations. *Analele Univ. Al. I. Cuza Iasi* 2011, 57, 65–74. [CrossRef]
28. Hyers, D.H. On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. USA* 1941, 27, 222–224. [CrossRef]
29. Hyers, D. H.; Isac, G.; Rassias, T. *Stability of Functional Equations in Several Variables*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 1998.
30. Jung, S.M.; Lee, K.S. Hyers-Ulam stability of first order linear partial differential equations with constant coefficients. *Math. Ineq. Appl.* 2007, 10, 261–266.
31. Lazăr, V. L. Ulam–Hyers stability for partial differential inclusions. *Electron. J. Qual. Theory Differ. Equ.* 2012, 21, 1–19. [CrossRef]
32. Petru, T.P.; Petrușel, A.; Yao, J.C. Ulam–Hyers stability for operatorial equations and inclusions via nonself operators. *Taiwan. J. Math.* 2011, 15, 2195–2212. [CrossRef]
33. Popa, D. Hyers-Ulam stability of the linear recurrence with constant coefficients. *Adv. Differ. Equ.* 2005, 2, 101–107.
34. Rus, I.A. Ulam stability of ordinary differentialial equations. *Studia Univ. Babeș-Bolyai Math.* 2009, 54, 125–133.
35. Ulam, S.M. *Problems in Modern Mathematics*; John Wiley and Sons: New York, NY, USA, 1964.
36. Guran, L. A multivalued Perov-type theorems in generalized metric spaces. *Creat. Math. Inform.* 2008, 17, 412–419.
37. Guran, L. Ulam–Hyers stability of fixed point equations for single-valued operators on KST spaces. *Creat. Math. Inform.* 2012, 21, 41–47.