Models of torsors over curves

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Abstract. Let \( R \) be a complete discrete valuation ring with fraction field \( K \) and with algebraically closed residue field. Let \( X \) be a faithfully flat \( R \)-scheme of finite type of relative dimension 1 and \( G \) be any affine \( K \)-group scheme of finite type. We prove that every \( G \)-torsor \( Y \) over the generic fibre \( X_\eta \) of \( X \) can be extended to a torsor over \( X' \) under the action of an affine and flat \( K \)-group scheme of finite type \( G' \) where \( X' \) is obtained by \( X \) after a finite number of Néron blowing ups. Moreover if \( G \) is finite and étale (resp. admits a finite and flat model) we find \( X' \) such that \( G' \) is finite and étale (resp. finite and flat) after, if necessary, extending scalars. We provide examples explaining the new techniques.

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1 Introduction

1.1 Aim and scope

Let \( S \) be a Dedekind scheme of dimension one and \( \eta = \text{Spec}(K) \) its generic point; let \( X \) be a scheme, \( f : X \to S \) a faithfully flat morphism of finite type and \( f_\eta : X_\eta \to \eta \) its generic fiber. Assume we are given a finite \( K \)-group scheme \( G \) and a \( G \)-torsor \( Y \to X_\eta \). So far the problem of extending the \( G \)-torsor \( Y \to X_\eta \) has consisted in finding a finite and flat \( S \)-group scheme \( G' \) whose generic fibre is isomorphic to \( G \) and a \( G' \)-torsor \( T \to X \) whose generic fibre is isomorphic to \( Y \to X_\eta \) as a \( G \)-torsor. Some solutions, from Grothendieck’s first ideas until nowadays, are known in some particular relevant cases that we briefly recall: Grothendieck proves that, possibly after extending scalars, the problem has a solution when \( S \) is the spectrum of a complete discrete valuation ring with algebraically closed residue field of positive characteristic \( p \), with \( X \) proper and smooth over \( S \) with geometrically connected fibres and \( p \nmid |G| \) (Exposé X); when \( S \) is the spectrum of a discrete valuation ring of residue characteristic \( p \), \( X \) is a proper and smooth curve over \( S \) then Raynaud suggests a solution, possibly after extending scalars, for \( |G| = p \) (§3); a similar problem has been studied by Saitō in [19] for formal curves of finite type and \( G = (\mathbb{Z}/p\mathbb{Z})_K \); when \( S \) is the spectrum of a discrete valuation ring \( R \) of mixed characteristic \((0,p)\) Tossici provides a solution, possibly after extending scalars, for \( G \) commutative when \( X \) is a regular scheme, faithfully flat over \( S \), with further assumptions on \( X \) and \( Y \) (Corollary 4.2.8). Finally in [3], §3.2 and §3.3 we provide a solution for \( G \) commutative, when \( S \) is a connected Dedekind scheme and \( f : X \to S \) is a smooth morphism satisfying additional assumptions (in this last case we do not need to extend scalars) and in [2] we deal with the case \( G \) solvable. However a general solution does not exist. Moreover we know that it can even happen that \( G \) does not admit a finite and flat model (see [15], Appendix B, Proposition B.2 for the positive equal characteristic case or [18], §3.4 for the mixed characteristic case). What is always true is that \( G \) admits at least an affine, quasi-finite (then of finite type, according to our conventions, see Appendix B, Proposition B.2 for the positive equal characteristic case or [18], §3.4 for the mixed characteristic case). Indeed \( G \) is isomorphic to a closed subgroup scheme of some \( GL_{n,K} \) (§3.4) then it is sufficient to consider its schematic closure in \( GL_{n,S} \). In this paper we study the problem of extending torsors in a much more general context, that is when \( G \) is only affine and of finite type. When in particular \( G \) is finite we will be able to apply our techniques to the following two cases: when \( G \) admits a finite and flat model over \( S \) and when \( G \) does not admit such a model. In this last case while looking for a \( G' \)-torsor \( Y' \to X \), model of \( Y \to X_\eta \), we will only ask \( G' \) to be affine, quasi-finite and flat: this approach is completely new and there are no other restrictions on \( G \). A particular attention will be given to the case when \( G \) is finite and smooth. In this paper we only consider the case of relative curves i.e. \( \text{dim}(X) = 2 \). The higher dimension case will be considered in a forthcoming paper.
1.2 Structure of the paper

In [2] we recall the definition of Néron blowing up which we will strongly use in this paper and then we also use it in order to Néron blow up torsors. This will provide a useful tool to build new torsors from old ones. As an application we will use this construction to describe all the torsors (cf. Proposition 2.7) under a particular quasi-finite group scheme with generic fibre of order \( p \) and special fibre of order 1, using the well known description for \( \mathbb{Z}/p\mathbb{Z} \)-torsors in positive characteristic \( p \).

We skip the description of §3.1 where we only provide a technical lemma which will be used in what follows. In §3.2 we prove some existence results for models of torsors. The main one is the following:

**Theorem 1.1.** (cf. Theorem 3.6) Let \( S \) be the spectrum of a complete discrete valuation ring \( R \) with algebraically closed residue field and with fraction field denoted by \( K \). Let \( X \) be an integral and regular curve, separated and faithfully flat over \( S \). Let \( G \) be an affine \( K \)-group scheme of finite type and \( f : Y \to X_\eta \) a \( G \)-torsor. Then there exist a \( R \)-valued section \( x \in X(R) \), an affine, finite type and flat \( R \)-group scheme \( G' \), model of \( G \), and a \( G' \)-torsor \( f' : Y' \to X' \) extending the given \( G \)-torsor \( Y \), where \( X' \) is obtained by \( X \) after a finite number of Néron blowing ups of \( X \) in \( x \in X \).

We emphasize that this result is considering a very general situation, as \( G \) is any affine algebraic \( K \)-group scheme with no other assumption. In particular when \( G \) is finite then \( G' \) is quasi-finite. Once we have a technique to find a model for a given finite \( G \)-torsor \( Y \to X_\eta \) we can push it to its limit in order to find the most beautiful possible model. For example we have already mentioned that \( G \) in general does not admit a finite and flat model over \( S \), but let us assume that it does admit such a finite and flat model, can we find a finite model for the torsor too? An affirmative answer is given in Corollary 3.7 thus providing a generalization of [2], Theorem 1.1 to any \( G \) finite. The second, more important, question concerns smoothness. Let us assume that \( Y \to X_\eta \) is a finite smooth \( G \)-torsor, is it possible to find a smooth model for it? Unfortunately it is well known that in many cases finite étale torsors do not admit finite étale models over \( X \); for example one can find an abelian scheme \( X \) and an integer \( m \) such...
that the multiplication by \( m \) map \( m_X : X \to X \) is generically smooth but not smooth over some closed points of \( S \). It is not difficult to see that this implies that all the other possible models of \( m_{X_s} : X_s \to X_s \) are not smooth neither. However if we accept to slightly modify the scheme \( X \) (in the sense of Theorem 1.1) then it becomes possible to find a smooth model. This is studied in Corollaries 3.8 and 3.9; here we recall the latter:

**Corollary 1.2.** (cf. Corollary 3.9) Let notations be as in Theorem 1.1. Let \( G \) be any étale finite group scheme and let \( f : Y \to X_\eta \) be a \( G \)-torsor. Then, up to finite extension of scalars, there exist a \( R \)-valued section \( x \in X(R) \) and a \( G' \)-torsor \( f' : Y' \to X' \) extending the given \( G \)-torsor \( Y \), for some finite étale group scheme \( G' \), where \( X' \) is obtained by \( X \) after a finite number of Néron blowing ups of \( X \) in \( x_s \in X_s \).

This result is particularly important because, as we have just recalled, it is sharp: in general we cannot do it if we do not modify \( X \). It is now reasonable to wonder whether a family of torsors (not just one) over \( X_\eta \) can be extended over the same \( X' \). To be more precise we state the following:

**Conjecture 1.3.** Let \( S \) be the spectrum of a complete discrete valuation ring \( R \) with algebraically closed residue field and with fraction field denoted by \( K \). Let \( X \) be an integral and regular curve, separated and faithfully flat over \( S \). Then there exist a \( S \)-scheme \( X' \), obtained by \( X \) after a finite number of Néron blowing ups \( X \) in of \( x_s \in X_s \) such that every \( X_\eta \) torsor over \( X_\eta \) can be extended to a quasi-finite torsor over \( X' \).

A positive answer can have interesting consequences on the study of the fundamental group scheme.

In §A.1 we will provide some simple examples with the purpose to easily explain how to extend torsors using our techniques.

**Acknowledgements** TBA ....

### 1.3 Notations and conventions

Let \( S \) be any scheme, \( X \) a \( S \)-scheme, \( G \) an affine (faithfully) flat \( S \)-group scheme and \( Y \) a \( S \)-scheme endowed with a right action \( \sigma : Y \times G \to Y \). A \( S \)-morphism \( p : Y \to X \) is said to be a \( G \)-torsor if it is affine, faithfully flat, \( G \)-invariant and the canonical morphism \( (\sigma, \text{pr}_Y) : Y \times G \to Y \times_X Y \) is an isomorphism. Let \( H \) be a flat \( S \)-group scheme and \( q : Z \to X \) a \( H \)-torsor; a morphism between two such torsors is a pair \((\beta, \alpha) : (Z, H) \to (Y, G)\) where \( \alpha : H \to G \) is a \( S \)-morphism of group schemes, and \( \beta : Z \to Y \) is a \( X \)-morphism of schemes such that the following diagram commutes

\[
\begin{array}{ccc}
Z \times H & \xrightarrow{\beta \times \alpha} & Y \times G \\
H\text{-action} \downarrow & & \downarrow \text{G-action} \\
Z & \xrightarrow{\beta} & Y
\end{array}
\]
(thus $Y$ is isomorphic to the contracted product $Z \times^H G$ through $\alpha$, cf. [5, III, §4, 3.2]). In this case we say that $Z$ precedes $Y$. Assume moreover that $\alpha$ is a closed immersion. Then $t$ is a closed immersion too and we say that $Z$ is a subtorsor of $Y$ (or that $Z$ is contained in $Y$, or that $Y$ contains $Z$).

Let $q \in S$ be any point. For any $S$-scheme $T$ we will denote by $T_q$ the fiber $T \times_S \text{Spec}(k(q))$ of $T$ over $q$. In a similar way for any $S$-morphism of schemes $v : T \to T'$ we will denote by $v_q : T_q \to T'_q$ the reduction of $v$ over $\text{Spec}(k(q))$. When $S$ is irreducible $\eta$ will denote its generic point and $K$ its function field $k(\eta)$. Any $S$-scheme whose generic fibre is isomorphic to $T_\eta$ will be called a model of $T_\eta$. Furthermore when $v_\eta$ is an isomorphism we will often say that $v$ is a model map. When $S$ is the spectrum of a discrete valuation ring then $s \in S$ will always denote the special point.

Throughout the whole paper a morphism of schemes $f : Y \to X$ will be said to be quasi-finite if it is of finite type and for every point $x \in X$ the fiber $Y_x := Y \times_X \text{Spec}(k(x))$ is a finite set. Let $S$ be any scheme and $G$ an affine $S$-group scheme. Then we say that $G$ is a finite (resp. quasi-finite/ finite type) $S$-group scheme if the structural morphism $G \to S$ is finite, (resp. quasi-finite/ of finite type). A $G$-torsor $f : Y \to X$ is said to be finite (resp. quasi-finite/ finite type) if $G$ is a flat $S$-group scheme which is moreover finite (resp. quasi-finite/ finite type). Of course when $S$ is the spectrum of a field a $S$-group scheme is quasi-finite if and only if it is finite.

2 Néron blowing ups and applications

In this section we recall the definition of Néron blowing up then we use it in order to Néron blow up torsors. This technique in practice provides a useful tool to build new torsors from old ones. As an application we will use this construction to describe all the torsors (cf. Proposition 2.7) under a particular quasi-finite group scheme with generic fibre of order $p$ and special fibre of order 1, using the well known description for some finite torsors of order $p$. Unless stated otherwise, from now till the end of section 2 we only consider the following situation:

**Notation 2.1.** We denote by $S$ the spectrum of a discrete valuation ring $R$ with uniformising element $\pi$ and with fraction and residue field respectively denoted by $K$ and $k$. As usual $\eta$ and $s$ will denote the generic and special point of $S$ respectively. Finally we denote by $X$ a faithfully flat $S$-scheme of finite type.

Hereafter we recall a well known result which we will use throughout all the paper:

**Proposition 2.2.** Let notations be as in 2.1 let $C$ be a closed subscheme of the special fibre $X_s$ of $X$ and let $\mathcal{I}$ be the sheaf of ideals of $\mathcal{O}_X$ defining $C$. Let $X' \to X$ be the blowing up of $X$ at $C$ and $u : X^C \to X$ denote its restriction to the open subscheme of $X'$ where $\mathcal{I} \cdot \mathcal{O}_X$ is generated by $\pi$. Then:

1. $X^C$ is a flat $S$-scheme, $u$ is an affine model map.
2. For any flat $S$-scheme $Z$ and for any $S$-morphism $v : Z \to X$ such that $v_k$ factors through $C$, there exists a unique $S$-morphism $v' : Z \to X^C$ such that $v = u \circ v'$.

Proof. Cf. [1], §3.2 Proposition 1 or [1], II, 2.1.2 (A). □

The morphism $X^C \to X$ (or sometimes only the scheme $X^C$) as in Proposition 2.2 is called the Néron blowing up of $X$ at $C$ and property 2 is often referred to as the universal property of the Néron blowing up.

When $F$ is a functor from the category of schemes over $S$ to the category of sets we denote, as usual, by $F_{fpqc}$ the sheaf in the fpqc-topology associated to $F$. Here we state a theorem, due to Raynaud, which will be used in Lemma 2.4:

**Theorem 2.3.** Let $T$ be any locally noetherian scheme, $Z$ a quasi-finite $T$-scheme, $G$ a flat $T$-group scheme acting on $Z$ such that the natural morphism $Z \times T G \to Z \times T Z$ is a closed immersion. Then the sheaf $(Z/G)_{fpqc}$ is representable and the representing scheme coincides with the ringed space $Z/G$. Furthermore the canonical morphism $p : Z \to Z/G$ is faithfully flat.

Proof. This result has been stated in [10], §5, Théorème 1 (v) and a proof can be found in [1], Appendice I, Théorème 7. The last assertion is just [10], §4, Proposition 2. □

**Lemma 2.4.** Let $k$ be any field. Let $T$ be an integral scheme of finite type over $\text{Spec}(k)$, $H$ a finite $k$-group scheme, $Z$ a $k$-scheme provided with a right $H$-action $\sigma : Z \times H \to Z$ and $g : Z \to T$ a $H$-invariant finite morphism (i.e. $g \circ \sigma = g \circ \text{pr}_Z$) such that the natural morphism $Z \times T H \to Z \times T Z$ is an isomorphism. Then $g : Z \to T$ is a $H$-torsor if $\dim(Z) = \dim(T)$.

Proof. By assumption $Z$ is a $H$-torsor over $Z/H$ (apply Theorem 2.3 to $Z \to T$ endowed with the $H_T$-action) so that $g$ factors through $Z/H$, thus we only need to prove that the natural morphism $i : Z/H \to T$ is an isomorphism. To prove that it is a monomorphism we proceed as follows: first we observe that

$$Z \times T Z \simeq Z \times Z/H \times T Z^H \times Z/H Z$$

hence the pullback $(Z \times T Z) \times (Z/H \times \times Z/H) Z/H$, through the diagonal, is easily seen to be isomorphic to $Z \times Z/H Z/H \times Z/H Z$; but the latter is isomorphic to $Z \times T H$ so we have the cartesian square

$$
\begin{array}{ccc}
Z \times T Z & \xrightarrow{u} & Z \times T H \\
\downarrow & & \downarrow \\
Z/H \times T Z/H & \xrightarrow{\Delta} & Z/H
\end{array}
$$

where the vertical arrows are faithfully flat and $u$ is an isomorphism. This implies that $\Delta$ is an isomorphism too (by faithfully flat descent), hence $i : Z/H \to T$ is a monomorphism. Now, $Z/H \to T$ is separated, $Z \to T$ is proper.
and $Z \to Z/H$ is surjective then $Z/H \to T$ is proper (12, Ch. 3, Proposition 3.16, (f)) so in particular a closed immersion (7, Corollaire 18.12.6), but $T$ is integral and $\dim(Z/H) = \dim(T)$ hence $Z/H \simeq T$. \hfill \Box

Now we are going to explain how to Néron blow up torsors:

**Lemma 2.5.** Let $G$ be a quasi-finite and flat $S$-group scheme and $H$ a closed subgroup scheme of $G_s$. Assume moreover that $X_s$ is integral. Let $Y$ be a $G$-torsor and $Z$ a $H$-torsor over $X_s$, subtorsor of $Y_s \to X_s$. If $(Y^Z)_s \to X_s$ is finite then $Y^Z \to X$ is a $G^H$-torsor and the model map $Y^Z \to Y$ is a morphism of torsors.

**Proof.** From the universal property of Néron blowing ups we first obtain an action of $G^H$ on $Y^Z$: indeed $Y^Z \times G^H \to Y \times G \to Y$ (the last morphism is the action of $G$ on $Y$) specially factors through $Z$, whence a morphism $Y^Z \times G^H \to Y^Z$ that gives the desired action. Under this action $Y^Z \to X$ is $G^H$-invariant, then we have a natural morphism $Y^Z \times G^H \to Y^Z \times_X Y^Z$. Moreover $Y^Z \times_X Y^Z \to Y \times_X Y$ specially factors through $Z \times H$ then we obtain a morphism $Y^Z \times_X Y^Z \to Y^Z \times G^H$ and consequently $Y^Z \times G^H \cong Y^Z \times_X Y^Z$. Hence by Lemma 2.4 $(Y^Z)_s \to X_s$ is a $(G^H)_s$-torsor thus in particular $Y^Z \to X$ is faithfully flat (by the critère de platitude par fibres, 8, Théorème 11.3.10) and hence a $G^H$-torsor, by Theorem 2.3. \hfill \Box

We will understand in [A.1] Example A.3 why it is necessary to require that $(Y^Z)_s \to X_s$ is finite. The importance of the previous construction is that we can build new torsors from old ones. In order to use this construction we need the special fibre of our given torsor to properly contain some other torsors. This happens, for instance, when the special fibre is trivial, like in the following example:

**Example 2.6.** Assume $R$ has positive characteristic $p$. Let $X := \text{Spec}(R[x])$ be the affine line over $R$. Then

$$Y := \text{Spec}(R[x,y]/(y^p - y - \pi x))$$

is a non trivial $(\mathbb{Z}/p\mathbb{Z})_R$-torsor (13, III, Proposition 4.12), with special fibre

$$Y_s = \text{Spec}(k[x,y]/(y^p - y))$$

which is a trivial $(\mathbb{Z}/p\mathbb{Z})_k$-torsor. It is then clear that $X_s$ is a subtorsor of $Y_s$ and we can blow up $Y$ at $X_s$ following Lemma 2.5 hence getting a $M$-torsor where $M$ is obtained after Néron blowing up of $(\mathbb{Z}/p\mathbb{Z})_R$ at $\{1\}_k = \text{Spec}(k)$, closed subgroup scheme of $(\mathbb{Z}/p\mathbb{Z})_k$, so that $M = (\mathbb{Z}/p\mathbb{Z})_R^{(1)} = \text{Spec}(R[y]/(\pi^{p-1} y^p - y))$; indeed $M = \text{Spec}(R[M])$ where $R[M] := R[x, \pi^{-1} x]/(x^p - x) = R[y]/(\pi^{p-1} y^p - y)$ where we have set $y = \pi^{-1} x$. It is flat as the Néron blowing up is always flat, quasi-finite, but clearly not finite. In a similar way $Y^{X_s} = \text{Spec}(R[x,y]/(\pi^{p-1} y^p - y - x))$ then we obtain a quasi-finite $M$-torsor.

\footnote{The notion of subtorsor has been introduced in 16}
In a very similar way we obtain the description of $M$-torsors over an affine scheme:

**Proposition 2.7.** Assume $R$ has positive characteristic $p$. Let $X := \text{Spec}(A)$ be affine over $R$ with $X_s$ integral. Let $M := \text{Spec}(R[x]/(\pi^{p-1}x^p - x))$ be the $R$-group scheme defined in Example 2.6. Then any $M$-torsor over $X$ is isomorphic to a torsor of the form

$$Y := \text{Spec}(A[y]/(\pi^{p-1}y^p - y + a))$$

for some $a \in A$.

**Proof.** As in Example 2.6 if we start from any $(\mathbb{Z}/p\mathbb{Z})_R$-torsor $\text{Spec}(A[y]/(y^p - y + \pi a))$ and we Néron blow it up in $\text{Spec}(A_k) \hookrightarrow Y_s$ we obtain the equation $\text{Spec}(A[y]/(\pi^{p-1}y^p - y + a))$ which is a $M$-torsor. On the other hand if we start from a $M$-torsor $Y$ over $X$ then one can consider the contracted product $Y \times^M (\mathbb{Z}/p\mathbb{Z})_R$ which is a $(\mathbb{Z}/p\mathbb{Z})_R$-torsor $Z$ with trivial special fibre, so in particular $Y$ is easily seen to be the Néron blowing up of $Z$ in $X_s$, hence, as we have just observed, it is isomorphic to $\text{Spec}(A[y]/(\pi^{p-1}y^p - y + a))$.

$\blacksquare$

## 3 Extension of torsors

### 3.1 A useful lemma

Let $T$ be any scheme; following [6] (11.6) we associate to any locally free sheaf $V$ of rank $n$ the $GL_{n,T}$-torsor $\text{Isom}_{O_T}(\mathcal{O}_T^{\oplus n}, V) \to T$ thus obtaining a bijective map between isomorphism classes of locally free sheaves of rank $n$ over $T$ and isomorphism classes of $GL_{n,T}$-torsors over $T$ (cf. for instance [6] (11.6.2) or [5] III, §4, n° 2, 2.1). It is an exercise to prove that this construction base changes correctly (i.e. if $i : T' \to T$ is a morphism of schemes then $i^*(\text{Isom}_{O_T}(\mathcal{O}_T^{\oplus n}, V)) \simeq \text{Isom}_{O_{T'}}(\mathcal{O}_{T'}^{\oplus n}, i^*(V))$ as $GL_{n,T'}$-torsors). We state a last lemma which will conclude the section:

**Lemma 3.1.** Let $S$ be a Dedekind scheme with function field $K$ and $X \to S$ a faithfully flat morphism of finite type with $X$ regular and integral of dimension 2. For any vector bundle $V$ on $X_\eta$ there exists a vector bundle $W$ on $X$ such that $W|_{X_\eta} \simeq V$. Moreover for any $GL_{n,K}$-torsor $Z \to X_\eta$ there exists $GL_{n,S}$-torsor $Z' \to X$ extending it.

**Proof.** Let us denote by $j : X_\eta \to X$ the natural open immersion. First of all we observe that there exists a coherent sheaf $\mathcal{F}$ on $X$ such that $j^*(\mathcal{F}) \simeq V$ (cf. for instance [11], II, ex. 5.15). Then $\mathcal{F}^{\vee\vee}$, i.e. the double dual of $\mathcal{F}$, is a coherent reflexive sheaf. That $j^*(\mathcal{F}^{\vee\vee}) \simeq V$ follows from the well known fact that $j^*(\mathcal{F}^{\vee}) \simeq j^*(\mathcal{F})^{\vee} \simeq V$ (see for instance the proof of [10], Proposition 1.8). Since $\text{dim}(X) = 2$ then we set $W := \mathcal{F}^{\vee\vee}$ which is a vector bundle by [10], Corollary 1.4 and this is the first assertion. The second claim is a restatement of the first one following by previous discussion (and observing that a $GL_{n,K}$-torsor is the same as $GL_{n,X_\eta}$-torsor). $\blacksquare$

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3.2 Models of torsors

Unless stated otherwise, from now till the end of section 3.2 we only consider the following situation:

Notation 3.2. Let $S$ be a trait, i.e. the spectrum of a discrete valuation ring $R$ with uniformising element $\pi$, with fraction and residue field denoted by $K$ and $k$ respectively. We denote by $X$ a separated faithfully flat $S$-scheme of finite type.

Lemma 3.3 is the main ingredient of Theorem 3.5.

Lemma 3.3. Let notations be as in 3.2 where we assume $X = \text{Spec}(A)$ to be affine and provided with a section $x \in X(R)$. Let $G$ be an affine $K$-group scheme of finite type, $Y = \text{Spec}(B)$ a $K$-scheme and $f : Y \to X_{\eta}$ a $G$-torsor pointed in $y \in Y(K)$ lying over $x_{\eta}$. We need the following technical assumption:

• we fix an embedding $G \hookrightarrow \text{GL}_{d,K}$ and we consider the contracted product $Z := Y \times_{G} \text{GL}_{d,K}$; we assume that $Z \to X_{\eta}$ is a trivial $\text{GL}_{d,K}$-torsor (i.e. $Z \simeq \text{GL}_{d,X_{\eta}}$).

Then there exist a $G'$-torsor $f' : Y' \to X'$ extending the given $G$-torsor $Y$, where $G'$ is the closure of $G$ in $\text{GL}_{d,R}$ and $X'$ is obtained by $X$ after a finite number of Néron blowing ups of $X$ in $x_{s} \in X_{s}$.

Proof. By assumption $X = \text{Spec}(A)$ is an affine scheme over $S = \text{Spec}(R)$ and we denote by $X_{\eta} = \text{Spec}(A_{K})$ its generic fibre. The point $x$ corresponds to a $R$-ring morphism $\alpha : A \to R$ which, tensoring by $K$ over $R$, gives the $K$-morphism $\alpha_{K} : A_{K} \to K$, corresponding to $x_{\eta}$. Since we are assuming that $Y$ has a $K$-rational point $y : \text{Spec}(K) \to Y$ over $x_{\eta}$ then in particular $Y_{x_{s}} = \text{Spec}(B \otimes_{A_{K}} K) \simeq G$ and if we set $C := B \otimes_{A_{K}} K$ we can assume $G = \text{Spec}(C)$. Hence $C$ is a quotient of $B$ and we have the following commutative diagrams:

$$
\begin{array}{ccc}
C & \xrightarrow{q} & B \\
\downarrow{\alpha_{K}} & & \downarrow{q} \\
K & \xrightarrow{id_{C} \otimes q} & C \otimes_{K} C
\end{array}
$$

(1)

where $\Delta_{C}$ is the comultiplication of the $K$-Hopf algebra $C$ and $\rho_{B}$ is the coaction induced by the (right) action of $\sigma : Y \times G \to Y$ thus giving $B$ a structure of (left) comodule over $C$. Finally $q$ is the morphism induced by the closed immersion $G \hookrightarrow Y$ and we will denote by $\varepsilon_{C} : C \to K$ and $S_{C} : C \to C$, respectively, the counit and the coinverse morphisms of $C$. Now consider the surjective morphism of $A_{K}$-algebras induced by the closed immersion of $Y$ into the trivial $\text{GL}_{d,X_{\eta}}$-torsor:

$$
u : A_{K}[y_{11}, \ldots, y_{dd}, 1/\det[y_{ij}]] \to B
$$



\begin{footnote}[ootnotesize] {A $\text{GL}_{d,K}$-torsor over $X_{\eta}$ (affine or not) is always locally trivial for the Zariski topology: this is clear after §3.1.}
\end{footnote}
then if we identify $B$ with the quotient $A_K[y_{11}, ..., y_{dd}, 1/det[y_{ij}]]$ by $\ker(u)$ and we take, via $\alpha_K$, the tensor product over $K$, we obtain

$$B = \frac{A_K[y_{11}, ..., y_{dd}, 1/det[y_{ij}]]}{f_1, ..., f_s}$$

$$C = \frac{K[y_{11}, ..., y_{dd}, 1/det[y_{ij}]]}{\alpha_K \times (f_1), ..., \alpha_K \times (f_s)}$$

For each $i = 1, ..., s$ we assume that the polynomials $f_i$ have coefficients in $A$ (simply chasing denominators). Consequently the $\alpha_K \times (f_i)$ have coefficients in $R$.

From the comultiplication on $Z$ (i.e. $\Delta_Z(y_{ij}) = \sum_{r=1}^{d} y_{ir} \otimes y_{rj}$) we deduce:

$$\rho_B(y_{ij}) = \sum_{r=1}^{d} x_{ir} \otimes y_{rj}$$

and consequently (this is well known, [21], §3.4)

$$\Delta_C(x_{ij}) = \sum_{r=1}^{d} x_{ir} \otimes x_{rj}.$$  \hspace{1cm} (4)

Applying to the latter the equality $(\varepsilon_C \otimes id) \Delta_C = id$ and comparing coefficients we get

$$\varepsilon_C(x_{ij}) = \delta_{ij} \text{ (the Kronecker symbol).}$$  \hspace{1cm} (5)

Moreover recalling that $\Delta_C(S_C, id) = \varepsilon_C$ we obtain

$$\delta_{ij} = \sum_{r=1}^{d} S_{x_{ir}}x_{rj}$$

thus $S_C(x_{sr})$ is the $(s,r)$-th entry $(s$-th row, $r$-th column) in the $d \times d$ matrix $[x_{ij}]^{-1}$. In particular $\Delta_C(1/(det[x_{ij}])) = 1/(det[x_{ij}]) \otimes 1/(det[x_{ij}])$, since $\Delta_C(det[x_{ij}]) = det[x_{ij}] \otimes det[x_{ij}]$.

The isomorphism given by $Y \times G \xrightarrow{\sim} Y \times X^{\mu} Y, (y, g) \mapsto (y, yg)$ gives rise to the isomorphism

$$\Psi : B \otimes A_K \otimes A_K \rightarrow C \otimes B \quad y_{ij} \otimes y_{rs} \mapsto \rho(y_{ij})(1 \otimes y_{rs})$$  \hspace{1cm} (6)

We are going to describe $\Psi^{-1}$. Since of course $\Psi^{-1}(1 \otimes y_{ij}) = (1 \otimes y_{ij})$ it only remains to compute $\Psi^{-1}(x_{ij} \otimes 1)$. We claim that

$$\Psi^{-1}(x_{ij} \otimes 1) = \sum_{r=1}^{d} y_{ir} \otimes H(y_{rj})$$

where, for all $(r,s) \in \{1, ..., d\}^2$, $H(y_{rs})$ denotes the $(s,r)$-th entry $(s$-th row, $r$-th column) in the $d \times d$ matrix $[y_{ij}]^{-1}$. Indeed

$$\Psi \left( \sum_{r=1}^{d} y_{ir} \otimes H(y_{rj}) \right) = \sum_{r=1}^{d} \rho(y_{ir})(1 \otimes H(y_{rj})) = \sum_{r=1}^{d} \left( \sum_{s=1}^{d} x_{is} \otimes (y_{sr}H(y_{rj})) \right) =$$

$$= \left( \sum_{r=1}^{d} x_{ir} \otimes H(y_{rj}) \right) \otimes y_{ij}.$$
\[
= \sum_{s=1}^{d} \left( x_{is} \otimes \sum_{r=1}^{d} (y_{sr} H(y_{rj})) \right) = \sum_{s=1}^{d} (x_{is} \otimes \delta_{sj}) = x_{ij} \otimes 1.
\]

Now it is important to observe that \( H(y_{rj}) = P \left( y_{11}, \ldots, y_{dd}, \frac{1}{\det(y_{ij})} \right) \in \mathbb{Z} \left[ y_{11}, \ldots, y_{dd}, \frac{1}{\det(y_{ij})} \right] \) so in particular it has coefficients in \( R \). So let us set

\[
B' := \frac{A[y_{11}, \ldots, y_{dd}, 1/\det(y_{ij})]}{f_1, \ldots, f_s} \quad (7)
\]

In order for \( \text{Spec}(B') \) to be a torsor over \( \text{Spec}(A) \) we need indeed \( B' \) to be \( A \)-faithfully flat, so we divide the reminder of the proof in two steps: in the first we explain that if \( B' \) is \( A \)-faithfully flat then \( \text{Spec}(B') \) is a \( \text{Spec}(C') \)-torsor over \( \text{Spec}(A) \), where \( C' := B' \otimes_A R \); in the second we will describe how to always reduce to this situation up to Néron blowing up the scheme \( X \) in \( x_s \), the special fibre of the \( R \)-valued point \( x \in X(R) \):

**Step 1:** let us assume that \( B' \) is \( A \)-faithfully flat: in this case \( \text{Spec}(B') \) is the schematic closure of \( Y \) in \( GL_{d,X} \) so one can proceed geometrically to observe that is actually a \( \text{Spec}(C') \)-torsor. Here we however do algebraic considerations: so \( C' = R[\frac{x_{11}, \ldots, x_{dd}, 1}{\det(x_{ij})}, \ldots, \frac{x_{ij}}{\det(x_{ij})}] \) is \( R \)-flat and it becomes a Hopf algebra over \( R \) when provided with the comultiplication given by the restriction of \( \Delta \) to \( C' \) (\( \text{Spec}(C') \) is, in this case, the \( R \)-group scheme obtained as the schematic closure of \( G \) into \( GL_{d,R} \)):

\[
\Delta_{C'} : C' \to C' \otimes C' \quad x_{ij} \mapsto \sum_{r=1}^{d} x_{ir} \otimes x_{rj}
\]

the coinverse given by

\[
S_{C'} : C' \to C' \quad x_{ij} \mapsto S_{C'}(x_{ij})
\]

where \( S_{C'}(x_{rs}) \) denotes the \((s, r)\)-th entry in the matrix \( [x_{ij}]^{-1} \), and finally the counity given by

\[
\varepsilon_{C'} : C' \to R \quad x_{ij} \mapsto \delta_{ij}.
\]

Moreover \( B' \) acquires a structure of (left) comodule over \( C' \) when provided with the coaction given by

\[
\rho_{B'} : B' \to C' \otimes_R B' \quad y_{ij} \mapsto \sum_{r=1}^{d} y_{ir} \otimes y_{rj}.
\]

Furthermore the natural morphism

\[
\Psi' : B' \otimes_A B' \to C' \otimes B' \quad y_{ij} \otimes y_{rs} \mapsto \rho_{B'}(y_{ij})(1 \otimes y_{rs}). \quad (8)
\]
has an inverse given by

$$
\Psi'^{-1}: C' \otimes B' \xrightarrow{\sim} B' \otimes_A B' \quad x_{ij} \otimes y_{uv} \mapsto \sum_{r=1}^d y_{ir} \otimes (H(y_{jr})y_{uv}) \tag{9}
$$

and it is thus an isomorphism. Setting $G' := \text{Spec}(C')$ and $Y' := \text{Spec}(B')$ then $G'$ is a $R$-flat group scheme of finite type acting on $Y'$ such that $Y' \to X$ is a $G'$-invariant morphism. Finally inverting arrows in (8) and (9) we obtain the desired isomorphism

$$
Y' \times G' \xrightarrow{\sim} Y' \times_X Y'
$$

so, by definition, $Y' \to X$ is a $G'$-torsor.

**Step 2:** when $B'$ is not $A$-faithfully flat we Néron blow up $X$: First, $A$ being of finite type over $R$, we can write $A = R[t_1, ..., t_r]/u_1(t_1, ..., t_r), ..., u_m(t_1, ..., t_r)$; so we rewrite in a useful way equations (2):

$$
B = \frac{K[t_1, ..., t_r, y_{11}, ..., y_{dd}, 1/det[y_{ij}]]}{u_1, ..., u_m, f_1, ..., f_s} \tag{10}
$$

where the $u_i = u_i(t_1, ..., t_r), i = 1, ..., m$ and $f_n = f_n(t_1, ..., t_r, y_{11}, ..., y_{dd}, 1/det[y_{ij}]), n = 1, ..., s$ are polynomials with coefficients in $K$. Chasing denominators if necessary we can assume that these polynomials have coefficients in $R$ with at least one coefficient with valuation equal to 0. Since $X$ is affine we can also assume, up to a translation, that the point $x \in X(R)$ is the origin so that for $C$ we obtain the following description:

$$
C = \frac{K[x_{11}, ..., x_{dd}, 1/det[x_{ij}]]}{\alpha_{K*}(f_1), ..., \alpha_{K*}(f_s)} \tag{11}
$$

and moreover for every $n = 1, ..., s$, $f_n(t_1, ..., t_r, y_{11}, ..., y_{dd}, 1/det[y_{ij}])$ can be rewritten as

$$
\alpha_{K*}(f_n)(y_{11}, ..., y_{dd}, 1/det[y_{ij}]) + \sum_{l=1}^{L_n} v_{nl}(y_{11}, ..., y_{dd}, 1/det[y_{ij}])g_{nl}(t_1, ..., t_r) \tag{12}
$$

for $L_n \in \mathbb{N}$, where $v_{nl}$ and $g_{nl}$ are polynomials with coefficients in $R$, by the above assumption, and $g_{nl}(0, ..., 0) = 0$. Hence we write $B'$ as follows

$$
B' = \frac{R[t_1, ..., t_r, y_{11}, ..., y_{dd}, 1/det[y_{ij}]]}{u_1, ..., u_m, f_1, ..., f_s} \tag{13}
$$

where the $f_n$ are as in equation (12). We can assume that $B'$ is $R$-flat (otherwise we can add other polynomials $f_{s+1}, ..., f_{s'}$ in $R[t_1, ..., t_r, y_{11}, ..., y_{dd}, 1/det[y_{ij}]]$ cutting the $R$-torsion, thus making it the only $R$-flat quotient of $A[y_{11}, ..., y_{dd}, 1/det[y_{ij}]]$ which is isomorphic to $B$ after tensoring with $K$ over $R$, by [7] Lemme 2.8.1.1;
this makes $\text{Spec}(B')$ to be the closure of $Y$ into $GL_{d,X})$. Finally we set $C' := B' \otimes_A R$ which is as follows

$$C' = \frac{R[x_{11}, \ldots, x_{dd}, 1/\det[x_{ij}]]}{\alpha_K, (f_1), \ldots, \alpha_K, (f_s)}$$

and it is not $R$-flat in general (it can happen for example that the coefficients of the $\alpha_K, (f_n)$ have all positive valuation, see for instance Example [x3]). Now, let $e \in \mathbb{N}$ be a positive integer, we Néron blow up $e$ times $X$ in $x_s$, the special fibre of the point $x \in X(R)$ that we are assuming to be the origin (as $x$ factors through $X^{e*}$, by the universal property of the Néron blowing up, it makes sense to iterate the Néron blowing up in $x_s$; this is also clear by the resulting equations below). This is equivalent to the following construction: we set

$$t'_\gamma := \pi^{-e}t_\gamma, \quad \gamma = 1, \ldots, r$$

and

$$A' := \frac{R[t_1, \ldots, t_r, t'_1, \ldots, t'_r]}{u_1'(t'_1, \ldots, t'_r), \ldots, u_m'(t'_1, \ldots, t'_r), \pi^e t'_1 - t_1, \ldots, \pi^e t'_r - t_r}$$

where $u'_i$ is obtained by $u_i$ replacing $t_\gamma$ with $\pi^e t'_\gamma$ and dividing it by a suitable power of $\pi$ so that the resulting polynomial has coefficients in $R$ with at least one with valuation zero. If we call $X' := \text{Spec}(A')$ then $X'$ is the desired Néron blowing up of $X$ in $x_s \in X_s$ $e$ times. In a similar way from $B'$ we obtain the $R$-flat algebra $B''$

$$\frac{R[t_1, \ldots, t_r, t'_1, \ldots, t'_r, y_{11}, \ldots, y_{dd}, 1/\det[y_{ij}]]}{u_1, \ldots, u_m, \{\alpha_K, (f_n) + \sum_{l=1}^n v_n g_{nl}\}_{n=1, \ldots, s}, \pi^e t'_1 - t_1, \ldots, \pi^e t'_r - t_r}$$

where we have first obtained $g'_n$ by $g_{nl}$ replacing $t_\gamma$ with $\pi^e t'_\gamma$ and then we have divided by a suitable power of $\pi$ the polynomials $\alpha_K, (f_n) + \sum_{l=1}^n v_n g_{nl}'$ thus obtaining $\{\alpha_K, (f_n) + \sum_{l=1}^n v_n g_{nl}'\}$ which now have coefficients in $R$ with at least one with valuation zero. We set $Y'' := \text{Spec}(B'')$ (it thus coincides, by construction, with the only closed subscheme of $Y' \times_X X'$ which is $R$-flat and generically isomorphic to $Y$) and $Y_s' = \text{Spec}(C'')$ where

$$C'' = \frac{R[x_{11}, \ldots, x_{dd}, 1/\det[x_{ij}]]}{\{\alpha_K, (f_1)\}_{j=1, \ldots, s}, \ldots, \{\alpha_K, (f_s)\}_{j=1, \ldots, s}}.$$

As $Y''_s$ contains the schematic closure of $y$ in $GL_{d,R}$ (which is, indeed, the unity of $GL_{d,R}$), then $Y_s''$ is surjective over $\text{Spec}(R)$. For a sufficiently big $e$, the exponent of $\pi$ in the equations $t'_\gamma = \pi^{-e}t_\gamma$, we have

$$Y_s'' = \text{Spec} \left( \frac{k[t'_1, \ldots, t'_r, y_{11}, \ldots, y_{dd}, 1/\det[y_{ij}]])}{u'_1, \ldots, u'_m, \{\alpha_K, (f_1)\}_{j=1, \ldots, s}, \ldots, \{\alpha_K, (f_s)\}_{j=1, \ldots, s}} \right)$$

where $\{\ldots\}$ means reduction to $k$; so $Y_s''$ is isomorphic to $\text{Spec}(C'')_s \times_k X'_s$ and thus faithfully flat over $X'_s$. By the already mentioned
critère de platitude par fibres it follows that $Y'' \to X'$ is faithfully flat too, which is what we wanted (and consequently $\text{Spec}(C'') \to \text{Spec}(R)$ is flat). We stress that $\text{Spec}(C'')$ coincides with $G'$, the schematic closure of $G$ in $GL_{d,R}$.

This concludes the proof.

\textbf{Remark 3.4.} In Lemma 3.3 it is clear that when $G$ is finite $G'$ is only quasi-finite, in general, and not necessarily finite.

We now state and prove the main Theorem of the paper:

\textbf{Theorem 3.5.} Let notations be as in 3.2 where we moreover ask $R$ to be complete and $k$ algebraically closed. Furthermore we assume $X$ to be an integral and regular relative curve. Let $G$ be an affine $K$-group scheme of finite type and $f : Y \to X_\eta$ a $G$-torsor. Then, possibly after a finite extension of scalars, there exist a $R$-valued section $x \in X(S)$, a finite type and flat $S$-group scheme $G'$, model of $G$, and a $G'$-torsor $f' : Y' \to X'$ extending the given $G$-torsor $Y$, where $X'$ is obtained by $X$ after a finite number of Néron blowing ups of $X$ in $x_s \in X_s$.

\textbf{Proof.} We fix an embedding $G \hookrightarrow GL_{d,K}$, and we call $G'$ the closure of $G$ in $GL_{d,R}$. We consider the $GL_{d,K}$-torsor $Z := Y \times^G GL_{d,K}$. By Lemma 3.1 there exists a $GL_{d,R}$-torsor $Z'$ extending $Z$.

We take an open affine covering $\{U_i\}_{i \in I}$ of $X$ such that $Z'|_{U_i} := Z' \times_X U_i$ is a trivial $GL_{d,R}$-torsor for every $i \in I$ and of course we can take $I$ a finite set. We take all those $U_i$ which surject onto $S$ and we consider their intersection $U := \bigcap_{j \in J} U_j$, $J \subseteq I$ which surjects onto $S$ too. Since $k$ is algebraically closed then there exists a point $x_s \in U_s(k)$ and since $R$ is Henselian $x_s$ can be lifted to a point $x \in U(S)$. We denote by $Y_i$ the fibre product $Y \times_{X_\eta} U_{i,\eta}$, which is thus a $G$-torsor. Let us assume that there exist a $K$-rational point $y \in Y_{x_s}(K)$ with the property that $y \in Y_j(K)$ for every $j \in J$ (this certainly happens on a finite extension $K \subset K'$, then one consider the integral closure of $R$ in $K'$). Now we observe that if $X^{x_s}$ is the Néron blowing up of $X$ in $x_s$ then the Néron blowing up

\footnote{By this we mean of absolute dimension 2.}
$U_{x_s}^x$ of $U_j$ in $x_s$ is isomorphic to $X^{x_s} \times_X U_j$ (simply using the universal property of the Néron blowing up, cf. for instance Proposition 2.22); in particular the $U_{x_s}^x$ form an affine open covering of $X^{x_s}$. As $x$ factors through $X^{x_s}$ (again by the universal property of the Néron blowig up) it makes sense to iterate the Néron blowing up in $x_s$. Hence since $J$ is finite then after an appropriate finite number of Néron blowing up of $X$ in $x_s$ we obtain a model map $X' \to X$, an affine open covering $\{U'_i\}_{i \in I}$ of $X'$ (the $\{U'_j\}_{j \in J}$, each containing $x$, plus the remaining $U'_i = U_i$, if any, entirely contained in $X_{\eta}$) such that for every $j \in J$ the $G$-torsor $Y_j$ over $U_{j, \eta}$ can be extended (this is Lemma 3.3. over $U'_j$ to a $G'$-torsor $Y'_j \to U'_j$). But $Y'_j$ is nothing but the closure of $Y_j$ in $Z' \times_X U'_j$ hence if we denote by $Y''$ the closure of $Y$ in $Z'' := Z' \times_X X'$, it is a $G'$-torsor obtained gluing together all the $G'$-torsors $Y'_j \to U'_j$ (all $j \in J$) and the remaining $G$-torsors $Y_i$ for $i \in I \setminus J$.

(16)

It is often not comfortable to extend scalars, so we provide this second version of the previous result:

**Theorem 3.6.** Let notations be as in §3.2 where we moreover ask $R$ to be complete and $k$ algebraically closed. Furthermore we assume $X$ to be an integral and regular relative curve. Let $G$ be an affine $K$-group scheme of finite type and $f : Y \to X_{\eta}$ a $G$-torsor. Then there exist a $R$-valued section $x \in X(R)$, a finite type and flat $R$-group scheme $G'$, model of $G$, and a $G'$-torsor $f' : Y' \to X'$ extending the given $G$-torsor $Y$, where $X'$ is obtained by $X$ after a finite number of Néron blowing ups of $X$ in $x_s \in X_s$.

**Proof.** We repeat the proof of Theorem 3.5 (so many details will be omitted; like before $Z''$ will be a $GL_{d,R}$-torsor over $X'$ whenever $X'$ will intervene); however we assume that there exist no $K$-rational point $y \in Y_{x_s}(K)$ with the property that $y \in Y_j(K)$ for every $j \in I$. As mentioned this becomes however true after a finite extension of scalars $K \subset K'$; now we denote by $R'$ the integral closure of $R$ in $K'$, which is still a complete discrete valuation ring with residue field $k$. So
by previous discussion our problem can be solved over $X_{R'} := X \times_R R'$ i.e. we can find a $R'$-scheme $X'$ (obtained as finite number of Néron blowing ups of $X_{R'}$ in $x_s = (x_{R'})_s$) such that the closure $\overline{Y'}$ of $Y \times_K K'$ into $Z'' \times_X X'$ is a $G' \times_R R'$-torsor extending the $G \times_K K'$-torsor $Y \times_K K'$. After a suitable finite number of Néron blowing ups of $X$ in $x_s$ we obtain a scheme $X''$ such that, pulling back over $R'$, $X''_{R'} \to X_{R'}$ factors through $X'_{R'}$ (again by the universal property of the Néron blowing up). By faithfully flat descent ($R \to R'$ is faithfully flat) the closure $\overline{Y}$ of $Y$ into $Z'' \times_X X''$ is faithfully flat over $X''$ because $\overline{Y} \times_{X''} X''_{R'}$ is faithfully flat over $X''_{R'}$ ($\overline{Y} \times_{X''} X''_{R'}$ is isomorphic indeed to the pull back $\overline{Y} \times_X X''_{R'}$). So in particular $\overline{Y} \to X''$ is a $G'$-torsor, as desired. The following figure is given to help the reader to better understand our proof in the case where $X'' = X^{x_s}$ and $X' = X^{x_s}$:

\[\begin{tikzpicture}[scale=0.8]
  \draw (0,0) node[below] {$\Spec(K')$} to (3,0) node[below] {$\Spec(R')$} to (3,3) node[above] {$X'$} to (0,3) node[above] {$(X'')_{x_s}$} to (0,0);
  \draw (3,0) to (0,0);
  \draw (0,3) to (0,0);
  \draw (3,3) to (3,0);
  \draw (3,3) to (0,3);
  \draw (0,0) to (0,-3) node[below] {$\Spec(K)$} to (3,-3) node[below] {$\Spec(R)$} to (3,0);
  \draw (0,-3) to (0,0);
  \draw (3,-3) to (3,0);
  \draw (0,0) to (0,-3);
  \draw (3,0) to (3,-3);
  \draw (0,3) to (0,0);
  \draw (3,3) to (3,0);
\end{tikzpicture}\]

In order to find a model $G'$ of the group scheme $G$, in Theorem 3.6 we took any embedding of $G$ into $GL_{n,R}$ and the model $G'$ automatically arose as the schematic closure of $G$ into $GL_{n,S}$. If we change the embedding (maybe modifying $n$ too) we obtain a new model $G''$. However in some cases we can also do the contrary: let for example assume that $G$ admits finite flat models over $\Spec(R)$ and let $G'$ be any of them. Then we chose a closed immersion $u : G' \hookrightarrow GL_{n,R}$ (it is easy to adapt the proof of [21], §3.4 to the case of finite and flat group schemes over $S$) and we fix in this way a closed immersion $u_n : G \hookrightarrow GL_{n,R}$ which is not arbitrary anymore and we know that the schematic closure of $G$ into $GL_{n,R}$ is $G'$. So we have the following

**Corollary 3.7.** Let notations be as in Theorem 3.6. Let us assume that $G$ is finite and admits a finite and flat model $G'$ over $R$. Then there exist a $R$-valued section $x \in X(R)$ and a $G'$-torsor $f' : Y' \to X'$ extending the given $G$-torsor $Y$, where $X'$ is obtained by $X$ after a finite number of Néron blowing ups of $X$ in $x_s \in X_s$. 

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Proof. This follows from previous discussion and Theorem 3.6.

So we first chose the model $G'$ and then we find, as a model of the given torsor over $X_\eta$, a torsor under the action of $G'$. Of course the map $\lambda : X' \to X$ varies with the model $G'$. Now we know that in many cases (but not always) the $K$-finite group scheme $G$ does admit a finite and flat model and we also know that if moreover $G$ is étale we can often find an étale model for it. Let us analyse these cases:

**Corollary 3.8.** Let notations be as in Theorem 3.6. Let $\Gamma$ be an abstract finite group and let $\Gamma_K$ and $\Gamma_S$ be the finite constant group schemes over $K$ and $S$ respectively associated to it. Let $f : Y \to X_\eta$ be a $\Gamma_K$-torsor. Then there exist a $R$-valued section $x \in X(R)$ and a $\Gamma_S$-torsor $f' : Y' \to X'$ extending the given $\Gamma_K$-torsor $Y$, where $X'$ is obtained by $X$ after a finite number of Néron blowings of $X$ in $x_s \in X_s$.

**Proof.** This is a particular case of Corollary 3.4.

Finally as a consequence of Corollary 3.8 we have the following

**Corollary 3.9.** Let notations be as in Theorem 3.6. Let $G$ be any étale finite group scheme and let $f : Y \to X_\eta$ be a $G$-torsor. Then, up to finite extension of scalars, there exist a $R$-valued section $x \in X(R)$ and a $G'$-torsor $f' : Y' \to X'$ extending the given $G$-torsor $Y$, where $X'$ is obtained by $X$ after a finite number of Néron blowings of $X$ in $x_s \in X_s$.

**Proof.** The statement follows from Corollary 3.8 and the fact that every étale finite group scheme $G$ becomes constant after some finite field extension $K \hookrightarrow K'$.

As mentioned in the introduction this is certainly false if we do not modify $X$ and this is why the famous Grothendieck’s specialization morphism (cf. [9], X) fails to be injective in general (cf. also Appendix A.1 to see examples where we construct a smooth model after Néron blowing up $X$). So Corollary 3.7 is certainly sharp.

### Appendix

**A.1 Construction of a model: some examples**

Hereafter we construct explicitly models of torsors using the techniques of Lemma 3.3. We chose very simple examples because the only purpose here is to give very easy applications of the cumbersome proof given. Notations are as in Notation 2.1 where $R$ has positive characteristic $p$. Let us set $X := \text{Spec}(R[x])$. By [14], III, Proposition 4.12 we know that $Y := \text{Spec}\left(\frac{K[y,z]}{z^p-z-\gamma y}\right)$ is a non trivial $(\mathbb{Z}/p\mathbb{Z})_K$-torsor over $X_\eta$ pointed over the origin of the affine line $\text{Spec}(K[x])$, $\gamma \in \mathbb{Z}$. The morphism $Y \to X_\eta$ is of course given by $K[x] \to \to
Thus defining an embedding \((\frac{K[y,z]}{z^2 - z \cdot y})\), \(x \mapsto y\). Thus \(G = Y_{x_0} = \text{Spec} \left( \frac{K[z]}{z - x_0} \right)\). In the following examples we will discuss what happens for several values of \(\gamma\).

**Example A.1.** The easiest case is when \(\gamma = 0\) where clearly \(\text{Spec} \left( \frac{R[y,z]}{z^2 - z \cdot y} \right)\) is a \((\mathbb{Z}/p\mathbb{Z})_R\)-torsor, model of the given one and we will not spend more time on it.

**Example A.2.** If \(\gamma > 0\) then we observe, again, that \(\text{Spec} \left( \frac{R[y,z]}{z^2 - z \cdot y} \right)\) is a \((\mathbb{Z}/p\mathbb{Z})_R\)-torsor, finite model of \(Y\) with trivial special fibre. As in Example 2.6 we can Néron blow up the torsor \(\text{Spec} \left( \frac{R[y,z]}{z^2 - z \cdot y} \right)\) in \(\text{Spec}(k[z])\), \(\gamma\) times, in order to find the model \(\text{Spec} \left( \frac{R[y,z]}{z^2 - z \cdot y} \right)\), which is quasi-finite but not finite. One observes that it is not possible to blow up any longer.

So far we have not used, because not really necessary, Lemma 3.3, however it will be interesting to use it in the next case:

**Example A.3.** Here we only consider the case \(p = 2\). If \(\gamma < 0\) we argue as follows: we set \(C := \frac{K[x]}{x^2 + x}\) and \(B := \frac{K[y,z]}{z^2 - z \cdot y}\), the coaction \(\rho : B \to C \otimes_K B\) being given by

\[
\rho(1) = 1 \otimes 1, \quad \rho(z) = 1 \otimes z + x \otimes 1.
\]

Let us choose for \(C\) the \(K\)-basis \(1, \pi^\alpha z > 0\) (for any \(\alpha \in \mathbb{N}\) such that \(2\alpha + \gamma \geq 0\)) thus defining an embedding \((\mathbb{Z}/2\mathbb{Z})_K \hookrightarrow \text{GL}_{2,K}\) given by (just follow [21], §3.4)

\[
K[x_{11}, ..., x_{22}, 1/det[x_{ij}]] \rightarrow \frac{K[x]}{x^2 + x}
\]

sending

\[
x_{11} \mapsto 1, \ x_{21} \mapsto 0, \ x_{12} \mapsto \pi^\alpha x, \ x_{22} \mapsto 1, \ \frac{1}{det[x_{ij}]} \mapsto 1
\]

hence the morphism of torsors \(Y \to \text{GL}_{2,X_n}\) is given by

\[
K[y, z_{11}, ..., z_{22}, 1/det[z_{ij}]] \rightarrow \left( \frac{K[y, z]}{z^2 + z \cdot \pi^\gamma y} \right)
\]

sending

\[
z_{11} \mapsto 1, \ z_{21} \mapsto 0, \ z_{12} \mapsto \pi^\alpha z, \ z_{22} \mapsto 1, \ \frac{1}{det[z_{ij}]} \mapsto 1
\]

thus

\[
B \simeq \frac{K[y, z_{11}, ..., z_{22}]}{z^2_{12} + \pi^\alpha z_{12} + \pi^{2\alpha + \gamma} y, z_{11} + 1, z_{21}, z_{22} + 1}
\]

and we set

\[
B' := \frac{R[y, z_{11}, ..., z_{22}]}{z^2_{12} + \pi^\alpha z_{12} + \pi^{2\alpha + \gamma} y, z_{11} + 1, z_{21}, z_{22} + 1} \simeq \frac{R[z_{12}, y]}{z^2_{12} + \pi^\alpha z_{12} + \pi^{2\alpha + \gamma} y},
\]

which makes sense as \(2\alpha + \gamma \geq 0\), and \(C' := B' \otimes_A R \simeq \frac{R[x_{12}]}{z_{12} + \pi^\alpha z_{12}}\), which is \(R\)-flat (hence \(\text{Spec}(C')\), for \(\alpha\) varying, are the \(R\)-group schemes defined in [12].

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§3.2). It is interesting to notice at this point that if $\gamma$ is odd and $\alpha = (1 - \gamma)/2$ then the model is given by $Y' := \text{Spec} \left( \frac{R[z_{12},y]}{z_{12} + \pi(z_{12} + y)} \right)$ which has a trivial special fibre but when we try to Néron blow it up (following Lemma 2.5) we do not find a torsor over $X$; indeed the condition “$(Y'/Z)_s \to X_s$ is finite” (same notations as in Lemma 2.5) is not satisfied.

So far we always found a model over $X$. Let us now see, however, what can happen if in Example A.3 we take a different embedding for $G$:

Example A.4. So let us consider the case $p = 2$ and $\gamma = -1$ and let us choose for $C$ the $K$-basis : $< 1, x >$. Arguing as in Example A.3 we obtain, as a model for $B$, the $A$-algebra

$$B' \cong \frac{R[y, z_{11}, \ldots, z_{22}]}{\pi z_{12}^2 + \pi z_{12} + y} \cong \frac{R[y, z_{12}]}{\pi z_{12}^2 + \pi z_{12} + y}$$

but $B' \otimes_A R$, through $x \in X(R)$, gives $\frac{R[x_2]}{\pi x_2^2 + \pi x_2}$, which is not $R$-flat. So here we proceed Néron blowing up $\text{Spec}(A)$ in $x_s = \text{Spec}(k)$ obtaining $X' = \text{Spec}(A') = \text{Spec}(\frac{R[x_2]}{\pi x_2^2 + \pi x_2})$ and, as suggested by the proof of Lemma 3.3 we replace, in $B'$, $y$ with $\pi w$ (dividing by $\pi$ where necessary) thus obtaining $Y'' = \text{Spec}(B'') = \frac{R[y, z_{12}, w]}{z_{12}^2 + z_{12} + w}$ which is now a $(\mathbb{Z}/2\mathbb{Z})_R$-torsor over $X'$. So with this choice for the embedding $(\mathbb{Z}/2\mathbb{Z})_K \hookrightarrow GL_2_K$ we have needed to Néron blow up $X$ even if, as we have seen, this does not imply that $Y$ cannot be extended over $X$. This last example also shows how the process of Néron blowing up allows us to find a smooth model for a torsor; the reader has certainly observed that over $X$ it is not possible to find a smooth model for $\gamma = -1$. It is also interesting to notice that, this time details are left to the reader, if we Néron blow up $X'$ again we obtain a $(\mathbb{Z}/2\mathbb{Z})_R$-torsor with trivial special fibre: this is what we expected after the proof of Lemma 3.3.

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