A Five Dimensional Perspective on Many Particles in the Snyder basis of Double Special Relativity

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After a brief summary of Double Special Relativity (DSR), we concentrate on a five dimensional procedure, which consistently introduce coordinates and momenta in the corresponding four-dimensional phase space, via a Hamiltonian approach. For the one particle case, the starting point is a de Sitter momentum space in five dimensions, with an additional constraint selected to recover the mass shell condition in four dimensions. Different basis of DSR can be recovered by selecting specific gauges to define the reduced four dimensional degrees of freedom. This is shown for the Snyder basis in the one particle case. We generalize the method to the many particles case and apply it again to this basis. We show that the energy and momentum of the system, given by the dynamical variables that are generators of translations in space and time and which close the Poincaré algebra, are additive magnitudes. From this it results that the rest energy (mass) of a composite object does not have an upper limit, as opposed to a single component particle which does.

I. INTRODUCTION

Double Special Relativity (DSR), or Special Relativity with two invariants, arose originally [1] as an attempt to describe modified dispersion relations of particles, presumably originating as a low energy consequence of quantum gravity modifications of space-time, in a way consistent with a relativity principle; i.e. without the need of introducing a preferred coordinate system or ether [2].

One of the simplest examples is provided by the model of Magueijo and Smolin which is characterized by the modified dispersion relation [3]

\[
\frac{1}{(1 - \kappa p_0)²} \eta^{\mu\nu} p_\mu p_\nu = m^2 c^2,
\]

that reduces to the standard Lorentz case when \( \kappa \to 0 \). The model is constructed by deforming the Lorentz group algebra in momentum space, with generators

\[
L_{\mu\nu} = p_\mu \frac{\partial}{\partial p_\nu} - p_\nu \frac{\partial}{\partial p_\mu},
\]

in such a way that the rotation generators \( J^i = \epsilon^{ijk} L_{jk} \) are kept the same, while the boost generators are changed to

\[
K^i = L^i_0 + \kappa p^i \frac{\partial}{\partial p_0} = U^{-1}(p_0)L^i_0 U(p_0), \quad U(p_0) p_\mu = \frac{1}{(1 - \kappa p_0)} p_\mu.
\]

The above modification preserves the standard Lorentz algebra among \( J^i, K^j \) but the Lorentz transformations are now realized non-linearly in the form

\[
W[\omega^{\mu\nu}] = U^{-1}(p_0) \exp(\omega^{\mu\nu} L_{\mu\nu}) U(p_0).
\]

When applied to the momentum, the transformation for a boost in the z-direction produces

\[
p'_0 = \frac{1}{D} \gamma(p_0 - v p_z), \quad p'_z = \frac{1}{D} \gamma(p_z - v p_0), \quad p'_x = \frac{p_x}{D}, \quad p'_y = \frac{p_y}{D},
\]

\[
D = 1 + \kappa(\gamma - 1)p_0 - \kappa \gamma v p_z,
\]

which indeed preserve the dispersion relation (1). The main point of the transformation [5] is that preserves the energy \( E_0 = 1/\kappa \) under the corresponding transformations [3]. Also this energy is a maximum energy, as can be seen...
in the case of a massive elementary particle. The energy and the momentum of such particle in an arbitrary frame with velocity \( \mathbf{v} \) are given by

\[
E = \frac{m_0 \gamma}{1 + \frac{m_0 \gamma}{E_0}}, \quad \mathbf{p} = \frac{m_0 \gamma}{1 + \frac{m_0 \gamma}{E_0}} \mathbf{v}
\]

and we can verify that \( E(m_0 \gamma) \leq E_0 \). Usually \( E_0 \) is taken as the Planck energy \( E_P \). At first sight this produces a contradiction with the existence of composite macroscopic particles, which can certainly have energies much larger than \( E_P \). This is the so-called "soccer ball problem" and its resolution has to do with the fact that the energy in this model is not additive, due to the appearance of the non-linear transformations.

The idea of deforming the Poincaré algebra to produce alternative modified dispersion relations has been generalized under the assumption that the only accepted modification arises in the boost sector and that it is compatible with rotation invariance. Under these conditions, the most general modification is

\[
[K_i, p_0] = C p_i, \quad [K_i, p_j] = A \delta_{ij} + B p_i p_j + D \epsilon_{ijk} p_k,
\]

where the functions \( A, B, C, D \) depend only on \( p_0, \mathbf{p}^2 \) (scalars under rotation) and on the parameter \( \kappa \), in such a way that the standard Poincaré limit is recovered when \( \kappa \to 0 \). For example, the choice

\[
C = i, \quad D = 0, \quad B = -i \kappa, \quad A = i \left( \frac{1}{2\kappa} (1 - e^{-2p_0 \kappa}) + \frac{\kappa}{2} \mathbf{p}^2 \right),
\]

leads to the invariant dispersion relation

\[
2 \cosh(\kappa p_0) - 1 \frac{\kappa^2}{\kappa^2} - \mathbf{p}^2 e^{\kappa p_0} = m^2,
\]

together with explicit expressions for the modified transformations in momentum space. The above deformation defines what is called the bicrossproduct basis in the literature [4].

One of the main drawbacks of this approach is the lack of information regarding the coordinate space, which leads to ambiguities in the definition of the velocity of the particle. Also, one would like to have a classical version of the theory, in terms of a phase space endowed with a symplectic structure. This will be the subject of Sections 3 and 4.

II. THE FIVE DIMENSIONAL POINT OF VIEW

One of the challenges faced in DSR is the construction of an appropriate phase endowed with coordinates and momenta that allow for the description of events in inertial frames and which transformations laws include an additional universal invariant length (or energy) scale, besides the standard invariant light velocity. Leaning on the analogy that the formulation of Lorentz invariance in three dimensions, appropriate to the description of a point particle having three degrees of freedom, gets drastically simplified when going to four dimensions, a main road suggested in the case of DSR is to start from a curved five dimensional space. In the standard Lorentz case only one first class constraint is required to recover the three degrees of freedom, while in this case we will need two first class constraints to do the job.

The simplest case is to start with a de Sitter space, which is defined as a four dimensional surface embedded in a five dimensional flat momentum space according to [4]

\[
\eta_{MN} P^M \eta^P N + \kappa^2 = 0.
\]

Here \( M, N = 0, 1, 2, 3 \) and \( \eta_{MN} = \text{diag}(1, -1, -1, -1, -1) \). The set of transformations that leave the above surface invariant is the de Sitter group \( SO(4, 1) \), which algebra is given by

\[
[l_{MN}, l_{PQ}] = \eta_{MP} l_{QN} + \eta_{MQ} l_{NP} + \eta_{NP} l_{MQ} + \eta_{NQ} l_{PM}.
\]

In this way, \( \kappa \) is interpreted as the invariant energy. The generators \( l_{PQ} \) have the following matrix realization

\[
l_{PQ} \big|_N = \delta_P^M \eta_{QN} - \delta_Q^M \eta_{PN}.
\]

The generators act in momentum space as

\[
P^M \to P^M = \exp(\theta_{PQ} l_{PQ}) \big|_N P^N.
\]
Using the relation
\[
\{ P^M, L_{PQ} \} = \left( \frac{\partial P^M}{\partial \theta_{PQ}} \right)_{\theta=0},
\]  
(15)
to define the group action in momentum space through the bracket \{\}, one readily obtains
\[
\{ P^M, L_{NQ} \} = \delta^M_N P_Q - \delta^M_Q P_N.
\]  
(16)

Also, it can be shown that the brackets \{L_{MN}, L_{PQ}\} inherit the algebra (12). In the following it proves convenient to make the splitting \{M\} = \{0, 4; \mu = 0, 1, 2, 3\}. The notation is
\[
L_k = \frac{1}{2} \epsilon_{kmn} L_{lm}, \quad B_i = L_{0i}, \quad D_\mu = L_{\mu 4}.
\]  
(17)

One consequence of this five-dimensional starting point is the appearance of five additional symmetries \(P_4, D_\mu\), besides the usual ten symmetries (3 rotations + 3 boosts + 4 translations) which describe the transformations among inertial frames. A possible interpretation of such extra symmetries will be given in Section 4.

Starting from this five-dimensional perspective there are at least two roads to construct the required phase space: (1) to search for a realization of the physical coordinates \(x^\mu\) as combinations of appropriate generators of the de Sitter group in momentum space and (2) to build DSR as a constrained theory in this five-dimensional space. We present a brief review of the first method in this section and develop the second in the following sections, including the one and many particles cases.

Approach (1) is rooted in the work of Snyder [5], who was the first in introducing a fundamental invariant length by starting from the curved momentum space (11). Later, Kowalski-Glikman showed that the Snyder construction can be generalized to include all deformations described in Eq. (8) [6]. The basic idea is to define a map
\[
x^\mu = x^\mu (P^M, L_{PQ}, \kappa), \quad p_\nu = p_\nu (P^M, \kappa),
\]  
(18)

which recovers the phase space version of a given deformed algebra together with the corresponding invariant dispersion relation. Two typical examples of this procedure are the construction of (i) the Snyder basis and (ii) the bicrossproduct basis, which correspond to the choices
\[
x_\mu = -\frac{D_\mu}{\kappa}, \quad p_\mu = \kappa \frac{P^\mu}{P_4},
\]  
(19)

and
\[
x_0 = -\frac{D_0}{\kappa}, \quad x_i = \frac{1}{\kappa} (B_i + D_i), \quad p_0 = \kappa \ln \left( \frac{P_4 - P_0}{\kappa} \right), \quad p_i = \frac{\kappa P_i}{P_0 - P_4},
\]  
(20)

respectively.

As we can see, this method provides no criteria to single out some specific choice which could be subsequently subjected to experimental/observational verification. Also, it does not provide any natural definition of the velocity.

### III. THE ONE-PARTICLE CASE

As we showed in the previous sections, it is possible to introduce an invariant length (or energy) in the transformations that connect inertial frames by starting from a momentum space with constant curvature. Nevertheless, the definition of the corresponding phase space, together with that of the particle velocity remains an open problem due to the many possibilities that arise. In this and the following sections we discuss a method which provides a unified version of the many alternatives already present. In complete analogy with the formulation of the relativistic particle in four dimensions, the basic idea is to view a DSR particle as arising from a constrained system in five dimensions, defined through a first order action which includes the concepts of coordinates and velocities from the very beginning. Since the initial phase space has now ten degrees of freedom, we will require two first class constraints (as opposed to one first class constraint in the relativistic case) in order to recover the final three degrees of freedom in coordinate space. As we will explain in the following, the different basis previously discussed, and many others, arise in this formulation as the result of different gauge fixings for one of the first class constraints. This method, applied to the one-particle case, has been previously discussed in Ref. [9].
Our starting point is the five-dimensional action

\[ S = \int d\tau \left( \dot{X}^\mu \eta_{MN} P^N - \Lambda H_{5d} - \lambda H_{4d} \right), \]  

(21)

where \( \Lambda \) and \( \lambda \) are Lagrange multipliers. Here \( \tau \) is the proper time and \( \dot{A} = dA/d\tau \). The constraints are

\[ H_{5d} = P^\mu P_\mu + \kappa^2, \quad H_{4d} = P^\mu P_\mu - m^2, \]  

(22)

where we have chosen \( H_{4d} \) as the four-dimensional mass shell condition for a particle with mass \( m \). This constraint can be more conveniently written as

\[ H_{4d} = P^4 - M, \quad M = \sqrt{m^2 + \kappa^2}, \quad P^4 = \sqrt{P^\mu P_\mu + \kappa^2}. \]  

(23)

Introducing a small change in notation (\( X_\mu = z_4, \quad P^4 = \xi^4 \)) the initial action is now written as

\[ S = \int d\tau \left( z_4 \xi^4 + \dot{X}_\mu P^\mu - \Lambda \left( \xi_4 \xi^4 + P_\mu P^\mu + \kappa^2 \right) - \lambda \left( \xi^4 - \sqrt{m^2 + \kappa^2} \right) \right). \]  

(24)

We start with 12 coordinates: \( z_4, X_\mu, \xi^4, P^\mu, \Lambda, \lambda; \) together with their respective momenta \( \Pi^0, \Pi_\mu, \Pi^\mu, \Pi_\lambda, \Pi_\lambda \); satisfying the standard Poisson brackets for canonical variables. The primary constraints are

\[ \Pi_4^4 - \xi^4 \approx 0, \quad \Pi_\mu^0 - P^\mu \approx 0, \quad \Pi_\mu^4 \approx 0, \quad \Pi_\lambda \approx 0, \quad \Pi_\lambda \approx 0. \]  

(25)

The extended Hamiltonian is

\[ H = \Lambda \left( \xi_4 \xi^4 + P_\mu P^\mu + \kappa^2 \right) + \lambda \left( \xi^4 - \sqrt{m^2 + \kappa^2} \right) + a_4 \left( \Pi_4^4 - \xi^4 \right) + b_\mu \left( \Pi_\mu^0 - P^\mu \right) + d_4 \Pi_\mu^4 + e_\mu \Pi_\mu ^0 + g \Pi_\lambda + h \Pi_\lambda. \]  

(26)

where \( a_4, b_\mu, d_4, e_\mu, g, h \) are arbitrary functions. The conservation of the primary constraints fixes some arbitrary functions

\[ \{ \Pi_4^4 - \xi^4, H \} \approx 0 \rightarrow d_4 = 0, \quad \{ \Pi_\mu^0 - P^\mu, H \} \approx 0 \rightarrow e_\mu = 0, \]  

\[ \{ \Pi_\mu^4, H \} \approx 0, \quad \rightarrow a_4 = 2\Lambda \xi^4 - \lambda, \quad \{ \Pi_\mu ^0, H \} \approx 0 \rightarrow b^\mu = 2\Lambda P^\mu. \]  

(27)

and also provides the following secondary constraints

\[ \{ \Pi_\lambda, H \} = \xi_4 \xi^4 + P_\mu P^\mu + \kappa^2 \approx 0, \quad \{ \Pi_\lambda, H \} = \xi^4 - \sqrt{m^2 + \kappa^2} \approx 0. \]  

(29)

The secondary constraints turn out to be automatically conserved. In this way the Hamiltonian results

\[ H = \Lambda \left( \xi_4 \xi^4 + P_\mu P^\mu + \kappa^2 \right) + \lambda \left( \xi^4 - \sqrt{m^2 + \kappa^2} \right) + \left( 2\Lambda \xi - \lambda \right) \left( \Pi_4^4 - \xi^4 \right) + 2\Lambda P_\mu \left( \Pi_\mu^0 - P^\mu \right) + g \Pi_\lambda + h \Pi_\lambda. \]  

(30)

The above constraints can be further classified into four first class constraints

\[ \Pi_\lambda \approx 0, \quad \Pi_\lambda \approx 0, \]  

\[ 2 \left( \xi_4 \Pi_4^4 + p_\mu \Pi_\mu ^0 \right) - \left( \xi_4 \Pi_\mu ^0 + p_\mu p^\mu \right) + \kappa^2 \approx 0, \]  

\[ \Pi_4^4 - \sqrt{m^2 + \kappa^2} \approx 0 \]  

(31)

(32)

(33)

and ten second class constraints

\[ \Pi_4^4 - \xi^4 \approx 0, \quad \Pi_\mu^0 - P^\mu \approx 0, \quad \Pi_\mu ^4 \approx 0, \quad \Pi_\mu ^0 \approx 0. \]  

(34)

We can now verify the count of degrees of freedom (DOF) in coordinate space, which is

\[ \# DOF = \frac{1}{2} (2 \times 12 - 2 \times 4 - 10) = 3, \]  

(35)

as expected. After imposing strongly the second class constraints, we are left with the Hamiltonian

\[ H = \Lambda \left( \xi_4 \xi^4 + P_\mu P^\mu + \kappa^2 \right) + \lambda \left( \xi^4 - \sqrt{m^2 + \kappa^2} \right) + g \Pi_\lambda + h \Pi_\lambda, \]  

(36)
together with the first class constraints
\[ \Pi_A \approx 0, \quad \Pi_\lambda \approx 0, \quad H_{5d} = \xi_4\xi^4 + P_\mu P^\mu + \kappa^2 \approx 0, \quad H_{4d} = \xi^4 - \sqrt{m^2 + \kappa^2} \approx 0. \]  

The Dirac brackets among the remaining phase space variables turn out to be identical with the original Poisson brackets. In an abuse of notation, we do not label with additional indexes the resulting Dirac brackets that appear in each step of the calculation. The non-zero values are
\[ \{ z_4, \xi^4 \} = 1, \quad \{ X_\mu, P^\nu \} = \delta^\nu_\mu, \quad \{ \Lambda, \Pi_\lambda \} = 1, \quad \{ \lambda, \Pi_\lambda \} = 1. \]  

Before dealing with particular cases, we state some general requirements to implement such procedure. Since we have four first class constraints we will require to add four additional constraints \( \chi^1, \chi^2, \chi^3, \chi^4 \), which must have zero Poisson bracket with the Hamiltonian, and be such that the whole set of eight constraints is now second class.

We will make use of the iterative method of sequentially fixing the gauge, calculating in each step the resulting Dirac brackets. First we eliminate the variables \( \Lambda, \lambda, \Pi_\lambda, \Pi_\lambda \). To this end we take
\[ \bar{\chi}^1 = \Lambda - \tilde{\Lambda} (z_4, X_\mu, \xi^4, P^\mu) \approx 0, \quad \bar{\chi}^2 = \lambda - \tilde{\lambda} (z_4, X_\mu, \xi^4, P^\mu) \approx 0. \]  

The time evolution \( d\bar{\chi}^{1,2}/d\tau = 0 \) fixes the arbitrary functions \( g \) and \( h \). The remaining canonical variables are \( z_4, X_\mu, \xi^4, P^\mu \) with non zero Dirac brackets
\[ \{ z_4, \xi^4 \} = 1, \quad \{ X_\mu, P^\nu \} = \delta^\nu_\mu \]  

and Hamiltonian
\[ H = \tilde{\Lambda} (\xi_4\xi^4 + P_\mu P^\mu + \kappa^2) + \tilde{\lambda} (\xi^4 - \sqrt{m^2 + \kappa^2}), \]  

together with the constraints
\[ H_{5d} = \xi_4\xi^4 + P_\mu P^\mu + \kappa^2 \approx 0, \quad H_{4d} = \xi^4 - \sqrt{m^2 + \kappa^2} \approx 0, \quad \chi^1 \approx 0, \quad \chi^2 \approx 0. \]  

As we will show in the following, some of the previously discussed DSR basis can be obtained by choosing different gauge fixings for the constraint \( H_{5d} \). The strategy is the following. Since we want to impose strongly \( H_{5d} \) and \( \chi^1 \) we require that
\[ \{ H_{5d}, \chi^1 \} = C(X, P) \neq 0, \]  

with
\[ \{ H, \chi^1 \} = \tilde{\Lambda} C + \tilde{\lambda} \{ H_{4d}, \chi^1 \} \approx 0, \]  

which determines the relation between \( \tilde{\Lambda} \) and \( \tilde{\lambda} \). Taking into account the constraints \( H_{5d} \) and \( \chi^1 \) we next define an invertible coordinate transformation
\[ (\xi_4, P_\mu, z^4, X^\mu) \rightarrow (p_\mu, H_{5d}, x^\mu, \chi^1), \]  

where
\[ x^\mu = x^\mu(\xi_4, P_\mu, z^4, X^\mu), \quad p_\mu = p_\mu(\xi_4, P_\mu, z^4, X^\mu), \]  

which are functions of the five-dimensional phase space, are what we define as the physical four dimensional phase space coordinates. We restrict ourselves to the case when
\[ \{ H_{5d}, x^\mu \} = \{ H_{5d}, p_\mu \} = \{ \chi^1, x^\mu \} = \{ \chi^1, p_\mu \} = 0, \]  

so that \( x^\mu \) and \( p_\mu \) are invariant under the gauge transformations generated by \( H_{5d} \) and \( \chi^1 \). In this way, they become observable, whose dynamics will be determined by the remaining constraint \( H_{4d} \) in the next step of the procedure. A first consequence of (47) is that the resulting Dirac bracket in this step,
\[ \{ A, B \}_{DB} = \{ A, B \} - \{ A, \chi^1 \} \frac{1}{C} \{ H_{5d}, B \} + \{ A, H_{5d} \} \frac{1}{C} \{ \chi^1, B \}, \]  

with
\[ \{ A, \chi^1 \} = C(X, P) \neq 0. \]
is equal to the previous bracket in the case of the dynamical four dimensional variables \( x^\mu \) and \( p_\mu \). Moreover, given any function \( A(x_4, p_4, z^4, X^\mu) \) having zero Poisson brackets with \( H_{5d} \) and \( \chi^1 \), it will depend only upon \( x^\mu \) and \( p_\mu \) after making the inverse transformation (49). This can be seen by calculating

\[
0 = \{ A, H_{5d} \} = \frac{\partial A}{\partial x^\mu} \{ x^\mu, H_{5d} \} + \frac{\partial A}{\partial p_\mu} \{ p_\mu, H_{5d} \} + \frac{\partial A}{\partial \chi^1} \{ \chi^1, H_{5d} \} = \frac{\partial A}{\partial \chi^1} \{ \chi^1, H_{5d} \},
\]

\[
0 = \{ A, \chi^1 \} = \frac{\partial A}{\partial x^\mu} \{ x^\mu, \chi^1 \} + \frac{\partial A}{\partial p_\mu} \{ p_\mu, \chi^1 \} + \frac{\partial A}{\partial H_{5d}} \{ H_{5d}, \chi^1 \} = -\frac{\partial A}{\partial H_{5d}} \{ \chi^1, H_{5d} \}.
\]

Since we require (43), the conclusion is that \( \partial A/\partial H_{5d} \) and \( \partial A/\partial \chi^1 \) are zero. In this way we confirm that the variables \( x^\mu \) and \( p_\mu \) allow to completely specify any observable function in the constraint surface determined by \( H_{5d} \) and \( \chi^1 \), which is where the dynamics occurs.

### A. The Snyder coordinates

In order to recover this basis we choose

\[
\chi^1_S = X^\mu P_\mu + z^4 \xi_4 - T.
\]

We can directly verify that \( C = 2\kappa^2 \neq 0 \) and that the condition (44) reduces to \( 2\Lambda \kappa^2 = \tilde{\lambda} \xi_4 \). The Snyder coordinates are subsequently defined as

\[
p_\mu = \kappa \frac{P_\mu}{\xi_4}, \quad x_\mu = \frac{1}{\kappa} (X_\mu \xi_4 - z_4 P_\mu),
\]

which satisfy the condition (47). The inversion of the above definition produces

\[
P_\mu = \frac{p_\mu}{\Theta}, \quad \xi_4 = \frac{\kappa}{\Theta}, \quad z_4 = \frac{x_\mu p_\mu - T}{\kappa \Theta}, \quad X^\mu = \Theta x^\mu + \frac{p_\mu}{\kappa \Theta} (x_\alpha p_\alpha - T), \quad \Theta = \sqrt{1 - p_\alpha p_\alpha / \kappa^2}.
\]

The bracket algebra for the Snyder coordinates is calculated from the already established five-dimensional algebra, via the Eqs. (52), and yields

\[
\{ x_\mu, x_\nu \} = -\frac{1}{\kappa^2} (x_\mu p_\nu - x_\nu p_\mu), \quad \{ x_\mu, p_\nu \} = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{\kappa^2}, \quad \{ p_\mu, p_\nu \} = 0.
\]

The Hamiltonian is

\[
H = \tilde{\lambda} \left( \frac{C}{\sqrt{\kappa^2 - p_\alpha p_\alpha}} - M \right) = \tilde{\lambda} H_{4d}.
\]

The constraint \( H_{4d} \) reduces to

\[
H_{4d} = p^\alpha p_\alpha - \kappa^2 \left( 1 - \frac{\kappa^2}{M^2} \right),
\]

which is interpreted as the mass shell condition for a relativistic particle with mass \( m \)

\[
m = \kappa \sqrt{1 - \frac{\kappa^2}{M^2}}.
\]

Let us observe that the above relation leads to \( m = \kappa \) as the maximum mass accessible to such elementary particle. The equation of motions are

\[
\frac{dp_\mu}{d\tau} = \{ p_\mu, H \} = 0, \quad \frac{dx_\mu}{d\tau} = \{ x_\mu, H \} = \frac{\tilde{\lambda} p_\mu}{\sqrt{\kappa^2 - p_\alpha p_\alpha}}.
\]

leading to the velocity (\( c = 1 \))

\[
v^i = \frac{dx^i}{dt} = \frac{dx^i}{d\tau} \frac{d\tau}{dx^0} = \frac{p^i}{\bar{p}^0}.
\]
IV. THE MULTIPARTICLE CASE IN SNYDER COORDINATES

In this section we proceed in analogy with the one particle case in order to build a consistent DSR formulation for the case of \( N \) particles. In the previous case we enlarged the original phase space to five dimensions and imposed the constraint that the momentum sector has constant curvature \( \kappa \), corresponding to a de Sitter space. Also we required that the particle satisfied a four dimensional mass shell condition designed to recover the three DOF in coordinate space. In this way we have introduced two invariant constants: the curvature \( \kappa \) and the light velocity \( c \).

The natural generalization for the \( N \) particles case is to start from a configuration space \( \{X^a, z_a\}, a = 1, 2, \ldots, N \), with dimension \( 4N + 1 \). The additional coordinate \( z_a \) is taken to be spacelike. Here, each subset \( \{X_a^a\} \) describes the position of particle \( a \) having a mass \( m_a \). The momentum space is labeled by \( \left(P_a^\mu, \xi^a\right) \) and the mass shell condition \( P_a^\mu P_a^\mu = m_a^2 \) defines the universal speed of light \( c = 1 \). There is an additional condition that constrains the momentum space to an hypersurface with constant curvature \( \kappa \).

Let us consider the first order Lagrangian

\[
L = \dot{z}_a \xi^4 + \sum_{a=1}^N \dot{X}_a^a P_a^\mu - \Lambda \left( \xi_a \xi^4 + \sum_{a=1}^N P_a^\mu P_a^\mu + \kappa^2 \right) - \sum_a \lambda_a \left( P_a^\mu P_a^\mu - m_a^2 \right),
\]

(60)

with \( P_a^\mu = (P^0)^2 - P^i P^i \) for each particle. From the de Sitter constraint and the dispersion relations we can write

\[
\xi^4 = \sqrt{\kappa^2 + \sum_a m_a^2} \equiv M > 0.
\]

(61)

As a simplification, we replace the constraint for the first particle (\( a = 1 \)) by (61), in such a way that we start from

\[
L = \dot{z}_a \xi^4 + \sum_{a=1}^N \dot{X}_a^a P_a^\mu - \Lambda \left( \xi_a \xi^4 + \sum_{a=1}^N P_a^\mu P_a^\mu + \kappa^2 \right) - \sum_{b=2}^N \lambda_b \left( P_b^\mu P_b^\mu - m_b^2 \right). \tag*{(62)}
\]

The \( 9N + 3 \) coordinates are \( z_4, \xi^4, x^\mu_a, P_a^\mu, \lambda, \lambda_b \). The calculation of the canonical momenta produces the following primary constraints

\[
\Pi^i_z - \xi^4 \approx 0, \quad \Pi^a_{X^a} - P_a^\mu \approx 0, \quad \Pi_{\xi^4} \approx 0, \quad \Pi_{P_a^\mu} \approx 0, \quad \Pi_{\lambda} \approx 0, \quad \Pi_{\lambda_b} \approx 0. \tag*{(63)}
\]

The extended Hamiltonian is

\[
H = \Lambda \left( \xi_a \xi^4 + \sum_{a=1}^N P_a^\mu P_a^\mu + \kappa^2 \right) + \lambda \left( \xi^4 - M \right) + \sum_{b=2}^N \lambda_b \left( P_b^\mu P_b^\mu - m_b^2 \right) + a_4 \left( \Pi^j_z - \xi^4 \right) + \sum_{a=1}^N b_4^a \left( \Pi^a_{X^a} - P_a^\mu \right) + d_4 \Pi_{\xi^4} + \sum_{a=1}^N e_4^a \Pi_{P_a^\mu} + g \Pi_{\lambda} + h \Pi_{\lambda_b} + \sum_{a=1}^N j_a \Pi_{\lambda_a}, \tag*{(64)}
\]

with \( a_4, b_4^a, d_4, e_4^a, g, h, j_a \) being arbitrary functions. The conservation of the primary constraints fixes some of the arbitrary functions

\[
\{ \Pi^j_z, H \} \approx 0 \rightarrow d_4 = 0, \quad \{ \Pi^a_{X^a}, P_a^\mu, H \} \approx 0 \rightarrow e_4^a = 0, \tag*{(66)}
\]

\[
\{ \Pi_{\xi^4}, H \} \approx 0 \rightarrow a^4 = 2 \Lambda \xi^4 - \lambda, \quad \{ \Pi_{P_a^\mu}, H \} \approx 0 \rightarrow b_a^\mu = 2 \Lambda P_a^\mu + 2 \lambda_a P_a^\mu \tag*{(67)}
\]

and also generates the following secondary constraints

\[
\{ \Pi_{\lambda}, H \} \approx 0 \rightarrow \xi_a \xi^4 + \sum_{a=1}^N P_a^\mu P_a^\mu + \kappa^2 \equiv H_{D+1} \approx 0, \quad \{ \Pi_{\lambda_b}, H \} \approx 0 \rightarrow P_b^\mu P_b^\mu - m_b^2 \approx 0. \tag*{(68)}
\]

The Hamiltonian results

\[
H = \Lambda H_{D+1} + \lambda H_4 + \sum_{b=2}^N \lambda_a \left( P_b^\mu P_b^\mu - m_b^2 \right) + (2 \Lambda \xi^4 - \lambda) \left( \Pi_z^4 - \xi^4 \right) + 2 \sum_{a=1}^N \Lambda P_{a\mu} \left( \Pi_{X^a}^\mu - P_a^\mu \right) + 2 \sum_{b=2}^N \lambda_b P_{b\mu} \left( \Pi_b^\mu - P_b^\mu \right) + g \Pi_{\lambda} + h \Pi_{\lambda_b} + \sum_{a=1}^N j_a \Pi_{\lambda_a} \tag*{(69)}
\]
with the constraints given in (63) together with (68). The first class constraints are

\[ \Pi_\lambda \approx 0, \quad \Pi_{\lambda b} \approx 0, \quad \Pi_\chi \approx 0, \quad \Pi^4 - M \approx 0, \]
\[ \frac{1}{2} H_{D+1} \equiv \xi_4 \pi_4^2 + \sum_a P_{\mu}^a \Pi_{X_a}^\mu + \kappa^2 \approx 0, \quad 2P_{b \mu} \Pi_{\mu}^b - P_{a \mu} P_{\mu}^b - m_b^2 \approx 0, \quad b = 2, 3, ..., N, \] (70)

while the remaining second class constraints are

\[ \Pi_\xi^2 - \xi^4 \approx 0, \quad \Pi_{X_a}^\mu - P_{a \mu} \approx 0, \quad \Pi_\chi^2 \approx 0, \quad \Pi_{\mu}^a \approx 0. \] (71)

We have 9N + 3 coordinates with 8N + 2 second class constraints and 2N + 2 first class constraints. The count of the DOF is

\[ \# \text{DOF} = \frac{1}{2} (2(9N + 3) - 2(2N + 2) - (8N + 2)) = 3N, \] (72)

which corresponds to the N particles considered.

A. Imposing the constraints strongly

As in the previous section, we impose the constraints in a stepwise mode. Again, in an abuse of notation, in many occasions we will not introduce an additional notation for the modified Dirac brackets that arise at each step, unless some confusion arises. First we start with the second class constraints

\[ \phi_1 = \Pi_\xi^4 - \xi^4, \quad \phi_2 = \Pi_\xi^4 = 0, \quad \phi_{\mu}^{\alpha a} = \Pi_{X_a}^\mu - P_{a \mu}, \quad \phi_{\mu}^{\alpha a} = \Pi_{\mu}^a, \] (73)

which we use to eliminate the variables

\[ \Pi_\xi^4, \Pi_\xi^4, \Pi_{X_a}^\mu, \Pi_{\mu}^a. \] (74)

The non-zero brackets are

\[ \{ \phi_1, \phi_2 \} = -1, \quad \{ \phi_3^{\alpha a}, \phi_4^{\alpha c} \} = -\eta^{\mu \sigma} \delta^{ac}, \quad \{ \phi_5^{\alpha b}, \phi_6^{\alpha b} \} = -\eta^{\mu \nu}, \] (75)

and one can verify that the (Dirac) brackets for the variables that remain after having eliminated those indicated in (74) are equal to the initial Poisson brackets. In the restricted hypersurface determined by the second class constraints we have the Hamiltonian

\[ H = \Lambda H_{D+1} + \lambda H_4 + \sum_{b=2}^{N} \lambda_b \left( P_{\mu}^b P_{\mu}^b - m_b^2 \right) + g\Pi_\Lambda + h\Pi_\chi + \sum_{a=1}^{N} J_a \Pi_{\lambda a}, \] (76)

plus the first class constraints in (70). After the imposition of the second class constraints the first class constraints now read

\[ \Pi_\Lambda \approx 0, \quad \Pi_{\lambda b} \approx 0, \quad \Pi_\chi \approx 0, \quad H_4 \equiv \xi^4 - M \approx 0, \]
\[ \frac{1}{2} H_{D+1} \equiv \xi_4 \pi_4^2 + \sum_a P_{\mu}^a \Pi_{X_a}^\mu + \kappa^2 \approx 0, \quad P_{b \mu} P_{\mu}^b + m_b^2 \approx 0, \] (77)

B. Fixing the gauge in the first class constraints

Let us denote the gauge fixing conditions by \( \chi_\Lambda, \chi_{\lambda b}, \chi_\Lambda, \chi_{H_{D+1}}, \chi_{P_a} \) respectively. Taking into account the auxiliary character of the variables \( \Lambda, \lambda, \lambda_b \), we demand that the gauge fixing constraints depend upon them according to the following scheme

\[ \{ \chi_\Lambda, \Pi_\Lambda \} = \{ \chi_{\Lambda b}, \Pi_{\Lambda b} \} = 0, \quad \{ \chi_\Lambda, \Pi_{\lambda b} \} = \{ \chi_{\Lambda b}, \Pi_{\lambda b} \} = 0, \quad \{ \chi_{\lambda b}, \Pi_\Lambda \} = \{ \chi_{\lambda b}, \Pi_\Lambda \} = 0, \]
\[ \{ \chi_{H_{D+1}}, \Pi_\Lambda \} = \{ \chi_{H_{D+1}}, \Pi_\Lambda \} = 0, \quad \{ \chi_{\xi}, \Pi_\Lambda \} = \{ \chi_{\xi}, \Pi_\Lambda \} = 0, \quad \{ \chi_{P_a}, \Pi_\Lambda \} = \{ \chi_{P_a}, \Pi_\Lambda \} = 0. \] (78)
Using the above prescription, in order to eliminate the auxiliary variables \( \Lambda, \lambda, \lambda_b \) we choose
\[
\chi_\Lambda = \Lambda - \tilde{\Lambda} \left( z^4, \xi^4, X^\mu_a, P^\mu_a \right), \quad \chi_\lambda = \lambda - \tilde{\lambda} \left( z^4, \xi^4, X^\mu_a, P^\mu_a \right) \quad \chi_{\lambda b} = \lambda_b - \tilde{\lambda}_b \left( z^4, \xi^4, X^\mu_a, P^\mu_a \right).
\] (79)

Demanding conservation of the above constraints we determine the arbitrary functions appearing at this level are the same ones of the previous level, in the case of the remaining variables. Also we are left with the following constraints only
\[
H_{D+1} \approx 0, \quad H_4 = \xi^4 - M \approx 0, \quad P^\mu_b p^\mu_b - m_b^2 \approx 0,
\] (81)
\[
\chi_{H_{D+1}} \approx 0, \quad \chi_{H_4} \approx 0, \quad \chi_{P_b} \approx 0.
\] (82)

Let us consider now the pair \( (H_{D+1}, \chi_{H_{D+1}}) \). In order to fix the gauge, we must have
\[
\{ H_{D+1}, \chi_{H_{D+1}} \} = C(\xi^4, z^4, P^\mu_a, X^\mu_a) \neq 0.
\] (83)

Conservation in time of the constraint \( \chi_{H_{D+1}} \) produces the following relation
\[
- \dot{\lambda} C + \dot{\lambda} \{ \chi_{H_{D+1}}, H_4 \} + \sum_b \dot{\lambda}_b \{ \chi_{H_{D+1}}, P^\mu_b p_b^\mu \} = 0.
\] (84)

Next we introduce the invertible coordinate transformation
\[
(\xi^4, P^\mu_b, z^4, X_{b\mu}) \rightarrow (p^\mu_a, x^\mu_a, H_{D+1}, \chi_{H_{D+1}}),
\] (85)
with
\[
\{ H_{D+1}, x^\mu_a \} = \{ H_{D+1}, p^\mu_a \} = \{ \chi_{H_{D+1}}, x^\mu_a \} = \{ \chi_{H_{D+1}}, p^\mu_a \} = 0.
\] (86)

The Dirac brackets for the surviving variables, arising after this gauge fixing, remain the same as the previous ones. Imposing these two constraints in the Hamiltonian, and after the redefinition \( \tilde{\lambda} = \tilde{\lambda}(\xi^4 + M) \) we get
\[
H = \tilde{\lambda} \left( (\xi^4)^2 - M^2 \right) + \sum_b \lambda_b \left( p^\mu_b p_b^\mu - m_b^2 \right) = \tilde{\lambda} \left( \sum_a p^\mu_a p_a^\mu + \kappa^2 - M^2 \right) + \sum_b \lambda_b \left( p^\mu_b p_b^\mu - m_b^2 \right),
\] (87)
\[
= \tilde{\lambda} \left( \sum_a p^\mu_a p_a^\mu - m_a^2 \right) + \sum_b \lambda_b \left( p^\mu_b p_b^\mu - m_b^2 \right),
\] (88)
\[
= \tilde{\lambda} \left( \sum_a \left( p^\mu_a p_a^\mu - m_a^2 \right), \quad a = 1, \ldots, N, \quad \tilde{\lambda}_1 = \tilde{\lambda}.
\] (89)

At this stage, the model is invariant under \( SO(3,1) \).

C. The Snyder basis

To determine this basis we choose the gauge fixing
\[
\chi_{H_{D+1}} = \frac{\kappa}{\xi^4} P^\mu_a - T,
\] (90)
which produces \( C = 2\kappa^2 \) from Eq. (87). The new coordinates in phase space are
\[
p^\mu_a = \kappa \frac{P^\mu_a}{\xi^4}, \quad x^\mu_a = \frac{1}{\kappa} \left( X^\mu_a \xi_4 - z_4 p^\mu_a \right),
\] (91)
having zero brackets with \( H_{D+1} \) and \( \chi_{H_{D+1}} \). Solving for \( x^a_\mu, p^a_\mu \) results in

\[
p^a_\mu = \kappa^2 \frac{P^a_\mu}{(\xi^4)^2},
\]

(89)

\[
\sum_a p^a_\mu = \frac{\kappa^2}{(\xi^4)^2} \sum_a P^a_\mu = \frac{\kappa^2}{(\xi^4)^2} \sum_a m^2_a = \kappa^2 \frac{\kappa^2 - (\xi^4)^2}{(\xi^4)^2},
\]

(90)

\[
\xi^4 = \frac{\kappa}{\sqrt{1 + \sum_a p^a_\mu / \kappa^2}}.
\]

(91)

\[
P^a_\mu = \frac{\xi^4 p^a_\mu}{\kappa} = \frac{p^a_\mu}{\sqrt{1 + \sum_c p^c_\mu / \kappa^2}},
\]

(92)

\[
\sum_a x^a_\mu p^a_\mu = -X^a_\mu p^a_\mu = \frac{z_4}{\xi^4} - T + \left( (\xi^4)^2 - \sum_a m^2_a \right) \frac{z_4}{\xi^4} = -T + \frac{\kappa^2}{\xi^4} z_4,
\]

(93)

\[
z_4 = \frac{\xi^4}{\kappa^2} \left( \sum_a x^a_\mu p^a_\mu + T \right) = \frac{\sum_a x^a_\mu p^a_\mu + T}{\xi^4} \sqrt{1 + \sum_c p^c_\mu / \kappa^2},
\]

(94)

\[
X^a_\mu = \left( \frac{1}{\sqrt{1 + \sum_a p^a_\mu / \kappa^2}} x^a_\mu + \frac{1}{\kappa} \frac{\sum_a x^a_\mu p^a_\mu + T}{\xi^4} \right).
\]

(95)

The brackets are calculated from the definitions [88] yielding

\[
\{ x^a_\mu, x^c_\nu \} = -\frac{1}{\kappa^2} \left( x^a_\mu p^c_\nu - p^c_\mu x^a_\nu \right), \quad \{ p^a_\mu, p^c_\nu \} = \eta_{\mu \nu} \delta^{ac} - \frac{p^a_\mu p^c_\nu}{\kappa^2}.
\]

(96)

Let us observe that they reproduce the corresponding brackets [53] for the one particle case.

In terms of the new variables, the Hamiltonian is, after some redefinitions of the arbitrary functions \( \tilde{\lambda}_a \) (which we denote by the same letter),

\[
H = \sum_a \tilde{\lambda}_a \left( p^a_\mu p^a_\mu - m^2_a \right) = \sum_a \tilde{\lambda}_a \left( \frac{\kappa^2 p^a_\mu p^a_\mu}{\kappa^2 - \sum_c p^c_\mu / \kappa^2} - m^2_a \right),
\]

(97)

\[
H = \sum_a \tilde{\lambda}_a \left( p^a_\mu p^a_\mu - m^2_a \left( 1 - \sum_c p^c_\mu / \kappa^2 \right) \right).
\]

We also use the following relations

\[
\sum_a p^a_\mu / \kappa^2 = \frac{1}{(\xi^4)^2} \sum_a P^a_\mu = \left( \sum_a p^a_\mu / \kappa^2 + 1 \right) \sum_a m^2_a / \kappa^2,
\]

\[
\left( 1 - \sum_c m^2_c / \kappa^2 \right) \sum_a p^a_\mu / \kappa^2 = \sum_a m^2_a / \kappa^2,
\]

\[
\sum_a p^2_\mu / \kappa^2 = \kappa^2 \frac{\sum_a m^2_a / \kappa^2}{\kappa^2 - \sum_c m^2_c / \kappa^2}.
\]

(98)

The final expression is

\[
H = \sum_a \tilde{\lambda}_a \left( p^a_\mu p^a_\mu - m^2_a \left( \frac{\kappa^2}{\kappa^2 - \sum_c m^2_c / \kappa^2} \right) \right).
\]

(99)

D. Symmetries

Since the constraint that defines the Snyder basis is Lorentz invariant in four dimensions we next verify that the resulting theory respects such symmetry. We start from the original four-dimensional Lorentz generators for each
particle $L_{\mu\nu}^a = X_{\mu}^a P_{\nu}^a - X_{\nu}^a P_{\mu}^a$, which maintain the same form when written in terms of the physical variables

$$L_{\mu\nu}^a = x_{\mu}^a p_{\nu}^a - x_{\nu}^a p_{\mu}^a. \hspace{1cm} (100)$$

Using the brackets $\{\}$ we can calculate the corresponding algebra which yields

$$\{L_{\mu
u}^a, L_{\sigma\tau}^b\} = \delta^{ab} (\eta_{\sigma\nu} (x_{\mu}^a p_{\tau}^b - x_{\tau}^a p_{\nu}^b) + \eta_{\nu\tau} (x_{\mu}^a p_{\sigma}^b - x_{\sigma}^a p_{\nu}^b)
+ \eta_{\nu\sigma} (x_{\mu}^a p_{\tau}^b - x_{\tau}^a p_{\nu}^b)), \hspace{1cm} (101)$$

$$\{L_{\mu\nu}^a, p_{\sigma}^b\} = \delta^{ab} (-\eta_{\sigma\nu} p_{\mu}^a + \eta_{\nu\sigma} p_{\mu}^b), \hspace{1cm} \{p_{\mu}^a, p_{\nu}^b\} = 0. \hspace{1cm} (102)$$

Next we sum over $a$ and $b$, obtaining

$$\{L_{\mu\nu}, L_{\sigma\tau}\} = \eta_{\sigma\nu} L_{\mu\tau} - \eta_{\mu\tau} L_{\sigma\nu} - \eta_{\tau\nu} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\tau\nu}, \hspace{1cm} (103)$$

$$\{L_{\mu\nu}, p_{\sigma}\} = \eta_{\nu\sigma} P_{\mu} - \eta_{\mu\sigma} P_{\nu}, \hspace{1cm} \{p_{\mu}, p_{\nu}\} = 0, \hspace{1cm} (104)$$

where $L_{\mu\nu} = \sum a L_{\mu\nu}^a$, $p_{\mu} = \sum a p_{\mu}^a$. This is precisely the Poincaré algebra.

### E. The equations of motion

They are

$$\dot{x}_{\mu}^a = \{x_{\mu}^a, H\} = 2 \left(-\tilde{\lambda}_a + \sum_b \frac{\tilde{\lambda}_b p_{\mu}^b}{\kappa^2}\right) p_{\mu}^a. \hspace{1cm} (105)$$

Again, using

$$p_{\mu}^a = \kappa^2 \frac{p_{\mu}^a}{(\xi^4)^2} = \kappa^2 \frac{m_{\mu}^a}{(\xi^4)^2}, \hspace{1cm} (\xi^4)^2 = \kappa^2 - \sum_{a} m_{\mu}^a, \hspace{1cm} p_{\mu}^2 = \frac{\kappa^2 m_{\mu}^a}{\kappa^2 - \sum_{c} m_{\mu}^c} \hspace{1cm} (106)$$

we finally obtain

$$\dot{x}_{\mu}^a = 2 \left(-\tilde{\lambda}_a + \sum_b \frac{\tilde{\lambda}_b m_{\mu}^b}{\kappa^2 + \sum_{c} m_{\mu}^c}\right) p_{\mu}^a. \hspace{1cm} (107)$$

The remaining equations are

$$\dot{p}_{\mu}^a = \{p_{\mu}^a, H\} = 0, \rightarrow \dot{x}_{\mu}^a = 0. \hspace{1cm} (108)$$

To obtain the velocity $v_{\mu}^b$ as a function of time, we have to eliminate the arbitrary functions $\tilde{\lambda}_a$ which is simply done by defining

$$v_{\mu}^b = \frac{p_{\mu}^b}{P_{\mu}^0}. \hspace{1cm} (109)$$

The expressions for the energy and momenta of each particle, which here we assume to be additive, written in terms of the velocities are

$$P_{\mu}^0 = \frac{\kappa m_a}{\sqrt{\kappa^2 + \sum_c m_c^2}} \frac{1}{\sqrt{1 - v^2}}, \hspace{1cm} P_{\mu}^a = \frac{\kappa m_a}{\sqrt{\kappa^2 + \sum_c m_c^2}} \frac{v_{\mu}}{\sqrt{1 - v^2}} \hspace{1cm} (110)$$

The rest mass of each particle, that is its energy for $v = 0$, is

$$m_{(a)0} = m_a \sqrt{1 + \frac{\sum_c m_c^2}{\kappa^2}}. \hspace{1cm} (111)$$

The total energy, when a bunch of particles moves with the same velocity, can be used to define the rest mass of the corresponding system, obtaining

$$M_0 = \frac{\sum a m_a}{\sqrt{1 + \frac{\sum c m_c^2}{\kappa^2}}}. \hspace{1cm} (112)$$
Let us consider $N$ identical particles ($m_a = m$). In this case we have the following two limits

\begin{align}
(1) \quad N m^2 & \gg \kappa^2 : \quad M_0 = \frac{N m}{\sqrt{1 + N m^2 / \kappa^2}} \simeq \sqrt{N \kappa}, \quad (113) \\
(2) \quad N m^2 & \ll \kappa^2 : \quad M_0 \simeq N m \left(1 - \frac{N m^2}{2 \kappa^2}\right). \quad (114)
\end{align}

The above is an important result which means that, in this model, point like particles have masses $m_a \leq \kappa$, while composite systems ($N$ particles) admit masses $M_0 \geq \kappa$.

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