A NEW APPROACH TO GET SOLUTIONS FOR KIRCHHOFF-TYPE FRACTIONAL SCHRÖDINGER SYSTEMS INVOLVING CRITICAL EXPONENTS

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ABSTRACT. In this paper, we study the following Kirchhoff-type fractional Schrödinger system with critical exponent in \( \mathbb{R}^N \):

\[
\begin{align*}
(s-2)u + u &= \mu_1 |u|^{2^*_s-2}u + \beta \gamma |u|^{\alpha-2}u|v|^\beta + k|u|^{p-1}u, \\
(s-2)v + v &= \mu_2 |v|^{2^*_s-2}v + \beta \gamma |u|^{\alpha-2}v|v|^\beta + k|v|^{p-1}v,
\end{align*}
\]

where \((-\Delta)^s\) is the fractional Laplacian, \(0 < s < 1, N > 2s, 2^*_s = \frac{2N}{N-2s}\) is the fractional critical Sobolev exponent, \(\mu_1, \mu_2, \gamma, k > 0, \alpha + \beta = 2^*_s, 1 < p < 2^*_s - 1, a_i, b_i \geq 0, \text{ with } a_i + b_i > 0, \text{ for } i = 1, 2\). By using appropriate transformation, we first get its equivalent system which may be easier to solve:

\[
\begin{align*}
(-\Delta)^s u + u &= \mu_1 |u|^{2^*_s-2}u + \frac{\beta \gamma}{\alpha} |u|^{\alpha-2}u|v|^\beta + k|u|^{p-1}u, \quad x \in \mathbb{R}^N, \\
(-\Delta)^s v + v &= \mu_2 |v|^{2^*_s-2}v + \frac{\beta \gamma}{\alpha} |u|^{\alpha-2}v|v|^\beta + k|v|^{p-1}v, \quad x \in \mathbb{R}^N, \\
\lambda_1^s - a_1 \lambda_1^{N-2s} \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 dx &= 0, \quad \lambda_1 \in \mathbb{R}^+, \\
\lambda_2^s - a_2 \lambda_2^{N-2s} \int_{\mathbb{R}^N} |(-\Delta)^s v|^2 dx &= 0, \quad \lambda_2 \in \mathbb{R}^+.
\end{align*}
\]

Then, by using the mountain pass theorem, together with some classical arguments from Brézis and Nirenberg, we obtain the existence of solutions for the new system under suitable conditions. Finally, based on the equivalence of two systems, we get the existence of solutions for the original system. Our results give improvement and complement of some recent theorems in several directions.

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1. Introduction. In this paper, we consider the following Kirchhoff-type fractional Schrödinger system with critical exponents in \( \mathbb{R}^N \):

\[
\begin{cases}
(a_1 + b_1 \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 \, dx) (-\Delta)^s u + u = \mu_1 |u|^{2^*_s - 2} u + \frac{a_1}{2^*_s} |u|^{\alpha - 2} u|v|^{\beta} + k|u|^{p - 1} u, \\
(a_2 + b_2 \int_{\mathbb{R}^N} |(-\Delta)^s v|^2 \, dx) (-\Delta)^s v + v = \mu_2 |v|^{2^*_s - 2} v + \frac{b_2}{2^*_s} |u|^{\alpha} |v|^{\beta - 2} v + k|v|^{p - 1} v,
\end{cases}
\]

where \((-\Delta)^s\) is the fractional Laplacian, \(0 < s < 1\), \(2^*_s = 2N/(N - 2s)\) is a fractional critical Sobolev exponent, \(\mu_1, \mu_2, \gamma, k > 0\), \(N > 2s\), \(\alpha + \beta = 2^*_s\), \(1 < p < 2^*_s - 1\), \(a_i, b_i \geq 0\), with \(a_i + b_i > 0\), \(i = 1, 2\). The fractional Laplacian \((-\Delta)^s\) is defined by

\[
(-\Delta)^s u(x) = C(N, s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_{\varepsilon}(x)} \frac{u(x) - u(y)}{|x - y|^{N + 2s}} \, dy, \quad x \in \mathbb{R}^N,
\]

along functions \(u \in C_0^\infty(\mathbb{R}^N)\), here \(C(N, s)\) is a suitable positive normalizing constant, \(B_{\varepsilon}(x)\) denotes the ball of \(\mathbb{R}^N\) centered at \(x \in \mathbb{R}^N\) and radius \(\varepsilon > 0\). We refer to \([12, 22, 24, 25, 31, 33]\) for simple introduction to basic properties of the fractional Laplace operator and concrete applications based on variational methods. The interesting part of the paper is to find appropriate transformation to get the following equivalent system of (1), which is easier to solve

\[
\begin{cases}
(-\Delta)^s u + u = \mu_1 |u|^{2^*_s - 2} u + \frac{a_1}{2^*_s} |u|^{\alpha - 2} u|v|^{\beta} + k|u|^{p - 1} u, \quad x \in \mathbb{R}^N, \\
(-\Delta)^s v + v = \mu_2 |v|^{2^*_s - 2} v + \frac{b_2}{2^*_s} |u|^{\alpha} |v|^{\beta - 2} v + k|v|^{p - 1} v, \quad x \in \mathbb{R}^N, \\
\lambda_1^* - a - b_1 \frac{N - 2s}{2s} \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 \, dx = 0, \quad \lambda_1 \in \mathbb{R}^+, \\
\lambda_2^* - a - b_2 \frac{N - 2s}{2s} \int_{\mathbb{R}^N} |(-\Delta)^s v|^2 \, dx = 0, \quad \lambda_2 \in \mathbb{R}^+.
\end{cases}
\]

Recently, great attention has been paid to the study of problems involving fractional Laplace operators. This type of operator arises in a quite natural way in many different applications, such as, the thin obstacle problem, finance, phase transitions, anomalous diffusion, flame propagation and many others (see\([12, 23, 28]\) and references therein). In particular, nonlocal integro-differential operators arise naturally in the study of stochastic processes with jumps, and more precisely in Lévy processes. In this case, the fractional Laplacian operator can be viewed as the infinitesimal generator of radially symmetric stable processes in Lévy processes.

In fact, problem (1) is a fractional version of a model, the so-called Kirchhoff problems, introduced by Kirchhoff. To be more precise, Kirchhoff established a model given by the following:

\[
\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{p_0}{h^2} + \frac{E}{2L} \int_0^L \left( \frac{\partial u}{\partial x} \right)^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = f(x, t, u)
\]

Here, \(E\) is the Young modulus of the material, \(L\) is the length of the string, \(h\) is the area of the cross section, \(\rho\) denotes mass density and \(p_0\) is the initial tension, \(u\) is the displacement, \(f(t, x, u)\) is the external force. There are a large number of papers concerning the existence of solutions for Kirchhoff-type problems, so we do not intend to cite all, here we just refer the interested reader to \([21, 10]\) for some results on the Kirchhoff-type problems.

In the Laplacian setting, Wu and Zhou in \([30]\) considered the existence of nodal solutions of the following Kirchhoff problem

\[
- \left( a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \Delta u + u = g(u), \quad \text{in} \ \mathbb{R}^N,
\]
where \( a, b > 0 \) and \( N \geq 3 \). The authors first proved that (3) is equivalent to the following system with respect to \((u, \lambda)\):

\[
\begin{aligned}
- \Delta u + u &= g(u), \quad \text{in } \mathbb{R}^N, \\
\lambda - a - b \lambda^{\frac{N-2}{2}} \int_{\mathbb{R}^N} |\nabla u|^2 dx &= 0, \quad \lambda_1 \in \mathbb{R}^+.
\end{aligned}
\]

(4)

Then radial solutions of (3) with exact integer \( k \) nodes were obtained by dealing with the system (4) under some suitable hypotheses. Similarly, Zhou and Yang in [34] investigated problem (3) with \( g(u) = kf(u) + |u|^{p^*-2}u \) by transforming it into the corresponding equivalent system, and hence obtained the existence of solutions by mountain pass theorem without the restriction of the (AR) condition.

In the fractional Laplacian setting, there is an increasing interest in the study of the following Kirchhoff-type fractional Laplacian problem in recent years:

\[
\left(a + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{N}{2}} u|^2 dx \right) (-\Delta)^{s} u = f(u), \quad x \in \mathbb{R}^N.
\]

When the parameter \( b \) is sufficiently small, Ambrosio and Isernia [2] showed the existence of multiple radial solutions in the space \( H^s(\mathbb{R}^N) \) when \( f \) is subcritical and satisfying the Berestyki-Lions conditions. Liu et al. in [20] studied the existence of ground state solutions when \( f \) being critical growth. By using Ekeland’s variational principle and the mountain pass theorem, Pucci et al. in [24] considered the multiple solutions for nonhomogeneous Schrödinger–Kirchhoff type equations involving the fractional \( p \)-Laplacian in \( \mathbb{R}^N \). Fiscella and Valdinoci in [14] considered the following stationary Kirchhoff variational model with critical growth in bounded regular domains of \( \mathbb{R}^N \)

\[
\begin{aligned}
M \left(a + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{N}{2}} u|^2 dx \right) (-\Delta)^{s} u &= \lambda f(x, u) + |u|^{p^*-2}u, \quad x \in \Omega, \\
u &= 0, \quad x \in \mathbb{R}^N \setminus \Omega,
\end{aligned}
\]

and they proved the existence of non-negative solutions when \( M \) and \( f \) satisfy some suitable conditions. Pucci and Saldi in [26] investigated existence and multiplicity of nontrivial solutions by variational methods. By using the Pohozaev-Nehari manifold methods, Jeanjean’s monotonicity trick and the concentration-compactness principle, He and Zou in [18] studied the following Kirchhoff problems with critical growth and potential function

\[
\left(a + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{N}{2}} u|^2 dx \right) (-\Delta)^{s} u + V(x)u = |u|^{p^*-2}u + \mu |u|^{p-1}u, \quad x \in \mathbb{R}^N,
\]

and they showed that the existence of a positive ground state solution under suitable assumptions on \( V \). For more existence results for related nonlocal problems, we refer to [4, 5, 11, 13, 15, 16, 22] and the references therein.

Inspired by the above-mentioned works, in this paper we give a new idea to get solutions for Kirchhoff-type fractional Schrödinger equation involving critical exponent. By using a suitable transformation, we get a equivalent system, which is more easier to solve. Then, by using mountain pass theorem, together with Brézis and Nirenberg’s trick [3], we obtain the existence of solutions for equation (7) under some suitable conditions. Finally, we solve the algebra equation (8). When dealing with the nonlocal problem (1), we may meet several difficulties to overcome. The first difficulty is that the nonlinearity \( |u|^{p^*-2}u + \mu |u|^{p-1}u \) does not satisfy the global (AR) condition and hence it is more difficult to establish the boundedness of any \((PS)_c\) sequence. The second one is that the presence of the Kirchhoff term may cause the delicate difficulty when verifying the Palais-Smale condition, since
we don’t know whether the sequence \( (\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_n|^2 dx)^2 \) converges strongly to \( (\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx)^2 \) in \( H^s(\mathbb{R}^N) \). To overcome these difficulties, in this paper we will transform the Kirchhoff-type fractional problem (1) to an equivalent fractional algebra system without Kirchhoff term by using a suitable transformation. It is worth mentioning that the Pohozaev-Nehari manifold method does not work for coupled Kirchhoff fractional system. Indeed, for the fractional Laplacian system, it seems difficult to get the Pohozaev identity for coupled fractional system of Kirchhoff type because of the regularity of solutions. Thus, some classical approaches do not work for Kirchhoff-type coupled fractional system.

2. Statement of main results. Before we present the results for system, we explain the key ideas and main results for single equation. To this end, we first study the following single equation:

\[
(a + b \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx) (-\Delta)^s u + u = |u|^{2^*_s - 2} u + k |u|^{p-1} u, \quad x \in \mathbb{R}^N. \tag{5}
\]

To obtain the existence of solutions to (5), we transform it into its equivalent system with respect to \((u, \lambda)\)

\[
\begin{cases}
(-\Delta)^s u + u = |u|^{2^*_s - 2} u + k |u|^{p-1} u, \quad x \in \mathbb{R}^N, \\
\lambda^s - a - b \lambda^{N-2s} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx = 0, \quad \lambda \in \mathbb{R}^+. \tag{6}
\end{cases}
\]

We take two steps to get the existence of solutions for above system. Firstly, by mountain pass lemma, together with Brézis and Nirenberg’s trick [3] to overcome the lack of compactness for critical sobolev embedding, we get the existence of solutions for the following equation

\[
(-\Delta)^s u + u = |u|^{2^*_s - 2} u + k |u|^{p-1} u, \quad x \in \mathbb{R}^N. \tag{7}
\]

Then, we solve the algebra equation when \( u \) is known.

\[
\lambda^s - a - b \lambda^{N-2s} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx = 0, \quad \lambda \in \mathbb{R}^+. \tag{8}
\]

Hence we can get the existence of solutions for system (1) by the equivalence of (6) to (1). The energy functional associated with (7) is given by

\[
I(u) = \frac{1}{2} ||u||^2_{D^s(\mathbb{R}^N)} + \frac{1}{2} \int_{\mathbb{R}^N} u^2 - \frac{1}{2^*_s} \int_{\mathbb{R}^N} |u|^{2^*_s} dx - \frac{k}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx. \tag{9}
\]

Our main results for single equation are stated as follows:

**Theorem 2.1.** (i) If \(2s < N < 4s\), and \(p \in (1, \frac{6s-N}{N-2s})\), then (5) has a nontrivial solution for any \(a > 0, b > 0\) and \(k > 0\) is chosen sufficiently large.

(ii) If \(2s < N < 4s\), and \(p \in (\frac{6s-N}{N-2s}, \frac{N+2s}{N-2s})\), then (5) has a nontrivial solution for any \(a > 0, b > 0, k > 0\).

(iii) If \(N = 4s\), there exists \(b_0 > 0\), such that, for any \(a > 0, k > 0\), then (5) has a solution when \(b < b_0\), and (1) has no solution when \(b \geq b_0\).

(iv) If \(N > 4s\), there exists \(a_0 > 0\), such that for any \(k > 0\), (5) has two solutions \(u_1\) and \(u_2\) if \(ab \frac{2s}{N+2s} < a_0\), (5) has a nontrivial solution \(u\) if \(ab \frac{2s}{N+2s} = a_0\) and (5) has no solution if \(ab \frac{2s}{N+2s} > a_0\).

Next, we give the results for the degenerate case, i.e., \(a = 0\).

**Theorem 2.2.** (i) If \(2s < N < 4s\), and \(p \in (1, \frac{6s-N}{N-2s})\), then (1) has a nontrivial solution for any \(b > 0\) and \(k > 0\) is chosen sufficiently large.
(ii) If \(2s < N < 4s\) and \(p \in \left(\frac{6s-N}{N-2s}, \frac{N+2s}{N-2s}\right)\), or \(N > 4s\), then \(1\) has a nontrivial solution for any \(b > 0, k > 0\).

(iii) If \(N = 4s\), there exists \(b_0 = \frac{1}{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} u|^2 dx} > 0\), such that, for any \(k > 0\), then \(1\) has infinite nontrivial solutions when \(b = b_0\), and \(1\) has no solution when \(b \neq b_0\).

To express our main results about equation \(7\), we give following conditions:

\[(E_1)\] \(N \geq 4s\), or \(2s < N < 4s\), and \(p \in \left(\frac{6s-N}{N-2s}, \frac{N+2s}{N-2s}\right)\).

\[(E_2)\] \(2s < N < 4s\), and \(p \in \left(\frac{6s-N}{N-2s}, \frac{N+2s}{N-2s}\right)\).

**Theorem 2.3.** If \((E_1)\) holds and \(k > 0\), or \((E_2)\) holds and \(k > 0\) is chosen sufficiently large, then equation \(7\) has at least one solution.

In the second part, we present the results for nonlinear Kirchhoff-type fractional Schrödinger system \((1)\). To get the existence of solutions to system \((1)\), we transform it into its equivalent system \((2)\) with respect \((u, v, \lambda_1, \lambda_2) \in \mathcal{H} \times \mathbb{R}^+ \times \mathbb{R}^+\), which is easier to solve. We first prove the existence of solutions for the following fractional systems:

\[
\begin{cases}
(-\Delta)^s u + u = \mu_1 |u|^{2^*-2}u + \frac{\mu_2}{\lambda_1}\left|u^{\alpha-2}u\right|^2\lambda_1^{\beta} + k|u|^{p-1}u, & x \in \mathbb{R}^N, \\
(-\Delta)^s v + v = \mu_2 |v|^{2^*-2}v + \frac{\mu_1}{\lambda_2}\left|v^{\alpha-2}v\right|^2\lambda_2^{\beta} + k|v|^{p-1}v, & x \in \mathbb{R}^N.
\end{cases}
\]

Then we solve the following algebra system when \((u, v)\) is known:

\[
\begin{cases}
\lambda_1^s - a_1 - b_1 \lambda_1^{\frac{N-2s}{N+2s}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} u|^2 dx = 0, & \lambda_1 \in \mathbb{R}^+, \\
\lambda_2^s - a_2 - b_2 \lambda_2^{\frac{N-2s}{N+2s}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} v|^2 dx = 0, & \lambda_2 \in \mathbb{R}^+.
\end{cases}
\]

Our main results for coupled Schrödinger system are the following:

**Theorem 2.4.** Under the condition of \(0 < \mu_1 \leq \mu_2\) and \(1 < \beta < 2\), or \(0 < \mu_2 \leq \mu_1\) and \(1 < \alpha < 2\), if \((E_1)\) holds and \(k > 0\), or \((E_2)\) holds and \(k > 0\) is chosen sufficiently large, then system \((10)\) has at least one nontrivial positive ground state solution.

**Theorem 2.5.** Under the conditions of \(0 < \mu_1 \leq \mu_2\) and \(1 < \beta < 2\), or \(0 < \mu_2 \leq \mu_1\) and \(1 < \alpha < 2\).

(i) If \(2s < N < 4s\), and \(p \in \left(\frac{6s-N}{N-2s}, \frac{N+2s}{N-2s}\right)\), then \((1)\) has at least one non-trivial solutions for any \(a_i > 0, b_i > 0, i = 1, 2\) and \(k > 0\) is chosen sufficiently large.

(ii) If \(2s < N < 4s\), and \(p \in \left(\frac{6s-N}{N-2s}, \frac{N+2s}{N-2s}\right)\), then \((1)\) has at least one non-trivial solutions for any \(a_i > 0, b_i > 0, i = 1, 2\).

(iii) If \(N = 4s\), there exists \(b_i^* = \frac{1}{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} u|^2 dx} > 0, b_2^* = \frac{1}{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} v|^2 dx} > 0\) such that, for any \(a_i > 0, i = 1, 2, k > 0\), then \((1)\) has at least one non-trivial solutions when \(b_1 < b_1^*\) and \(b_2 < b_2^*\), and \((1)\) has no solution when \(b_1 \geq b_1^*\) or \(b_2 \geq b_2^*\).

(iv) If \(N > 4s\), then \((1)\) has at least one non-trivial solutions if and only if \(a_1 b_1^{\frac{N-2s}{N}} \leq \alpha_1\) and \(a_2 b_2^{\frac{N-2s}{N}} \leq \alpha_2\), where

\[
\alpha_1 = \frac{N-4s}{N-2s} \left(\frac{2s}{(N-2s) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} u|^2 dx}\right)^{\frac{2s}{N-2s}},
\]

\[
\alpha_2 = \frac{N-4s}{N-2s} \left(\frac{2s}{(N-2s) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} v|^2 dx}\right)^{\frac{2s}{N-2s}}.
\]
Theorem 2.6. Under the condition of

\[
\alpha_2 = \frac{N - 4s}{N - 2s} \left( \frac{2s}{(N - 2s) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 dx} \right)^{\frac{s}{4s - N}}.
\]

Moreover, if \( a_1 b_1^{\frac{2s}{4s - N}} > \alpha_1 \) or \( a_2 b_2^{\frac{2s}{4s - N}} > \alpha_2 \), then (1) has no solution.

Finally we give the results for the degenerate case, i.e., \( a_i = 0, i = 1, 2 \).

Theorem 2.6. Under the condition of \( 0 < \mu_1 \leq \mu_2 \) and \( 1 < \beta < 2, \) or \( 0 < \mu_2 \leq \mu_1 \) and \( 1 < \alpha < 2 \).

(i) If \( 2s < N < 4s, \) and \( p \in (1, \frac{6s - N}{N - 2s}) \), then (1) has at least one non-trivial solutions for any \( b_i > 0, i = 1, 2 \) and \( k > 0 \) is chosen sufficiently large.

(ii) If \( 2s < N < 4s, \) and \( p \in (\frac{6s - N}{N - 2s}, \frac{N + 2s}{N - 2s}) \), or \( N > 4s, \) then (1) has at least one non-trivial solutions for any \( b_i > 0, \) \( i = 1, 2 \) and \( k > 0 \).

(iii) If \( N = 4s, \) there exists \( b_1^* = \frac{1}{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx} > 0, \) \( b_2^* = \frac{1}{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 dx} > 0, \)

such that, for any \( k > 0, \) then (1) has at least one non-trivial solutions when \( b_1 = b_1^* \) and \( b_2 = b_2^* \), (1) has no solution when \( b_1 \neq b_1^* \) or \( b_2 \neq b_2^* \).

Remark 1. In a sense, the above results in this paper extends and complements the corresponding results established in [7, 8], where the authors considered the special case \( s = 1, k = 0, a_i = 1, b_i = 0, \) \( \alpha = \beta, \) and extends and complements the results established in [18], in which the authors considered the scale equation, but here we employ different approaches.

Remark 2. We believe that for the classical Kirchhoff-type nonlinear Schrödinger system \((s = 1), \) by using the similar approaches as in this paper, we can get similar results as Theorems 2.1–2.6. Compared with the classical case, the fractional case is more difficult since we need to make complicated analysis and calculations. So the existence results for the classical case are not considered in this paper.

The paper is organized as follows. In Section 3, we introduce some preliminaries that will be used to prove our main results. In Section 4, we show Theorem 2.3 and Theorem 2.4. In Section 5, we give the proof of Theorem 2.1–2.2. Finally, Section 6 is devoted to the proof of Theorems 2.5–2.6.

3. Preliminaries. Let \( H^s(\mathbb{R}^N) \) be the Hilbert space of function in \( \mathbb{R}^N \) endowed with the standard inner product and norm

\[
\langle u, v \rangle = \int_{\mathbb{R}^N} ((-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}} v + uv) \, dx, \quad \|u\|_{H^s(\mathbb{R}^N)}^2 = \langle u, u \rangle.
\]

Let \( D_s(\mathbb{R}^N) \) be the Hilbert space defined as the completion of \( C_c^\infty(\mathbb{R}^N) \) with the inner product

\[
\langle u, v \rangle_{D_s(\mathbb{R}^N)} = \frac{C(N, s)}{2} \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|y - x|^{N + 2s}} \, dx \, dy
\]

and norm

\[
\|u\|_{D_s(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx = \frac{C(N, s)}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|y - x|^{N + 2s}} \, dx \, dy.
\]

Denote \( \mathcal{H} = H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \) and \( D = D_s(\mathbb{R}^N) \times D_s(\mathbb{R}^N), \) with the norm given by

\[
\|(u, v)\|_D^2 = \|u\|_{H^s(\mathbb{R}^N)}^2 + \|v\|_{H^s(\mathbb{R}^N)}^2 + \|u\|_{L^2(\mathbb{R}^N)}^2 + \|v\|_{L^2(\mathbb{R}^N)}^2,
\]

where \( \|(u, v)\|_D^2 = \|u\|_{D_s(\mathbb{R}^N)}^2 + \|v\|_{D_s(\mathbb{R}^N)}^2. \) Let \( \mathcal{H}_r = H^s_r(\mathbb{R}^N) \times H^s_r(\mathbb{R}^N), \) where
From [17] we know that \( H^s_\alpha(\mathbb{R}^N) \) is endowed with the \( H^s(\mathbb{R}^N) \) topology: \( \| \varphi \|_{H^s_\alpha(\mathbb{R}^N)} = \| \varphi \|_{H^s(\mathbb{R}^N)} \). Let \( S_s \) be the sharp imbedding constant of \( D_s(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N) \),

\[
S_s = \inf_{u \in D_s(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_{D_s(\mathbb{R}^N)}^2}{(\int_{\mathbb{R}^N} |u|^{2^*_s} dx)^{\frac{2^*_s}{2}}}.
\]

(14)

Define

\[
S_{\gamma} = \inf_{(u,v) \in D \setminus \{(0,0)\}} \frac{||(u,v)||_D^2}{(\int_{\mathbb{R}^N} (\mu_1 |u|^2 + \mu_2 |v|^2 + \gamma |u|^\alpha |v|^\beta) dx)^{\frac{2}{2}}}.
\]

(15)

From [17] we know that \( S_{\gamma} \) is attained by \( (U,V) \in D \) and from [9] \( S_s \) is attained in \( \mathbb{R}^N \) by \( \tilde{u}(x) = \kappa (\varepsilon^2 + |x-x_0|^2)^{-\frac{N-2s}{2}} \), where \( \kappa \neq 0 \in \mathbb{R}, \varepsilon > 0 \) are fixed constants and \( x_0 \in \mathbb{R}^N \).

The energy functional associated with (10) is given by

\[
E(u,v) = \frac{1}{2} \| (u,v) \|_D^2 + \frac{1}{2} \int_{\mathbb{R}^N} (u^2 + v^2) dx - \frac{1}{2} \int_{\mathbb{R}^N} (\mu_1 |u|^{2^*_s} + \mu_2 |v|^{2^*_s} + \gamma |u|^\alpha |v|^\beta) dx - \frac{k}{p+1} \int_{\mathbb{R}^N} (|u|^{p+1} + |v|^{p+1}) dx.
\]

(16)

Define the Nehari manifold

\[
\mathcal{M} = \left\{ (u,v) \in \mathcal{H} \setminus \{(0,0)\} : \| (u,v) \|_{D}^2 + \int_{\mathbb{R}^N} (u^2 + v^2) dx = \frac{1}{2} \int_{\mathbb{R}^N} (\mu_1 |u|^{2^*_s} + \mu_2 |v|^{2^*_s} + \gamma |u|^\alpha |v|^\beta) dx + k \int_{\mathbb{R}^N} (|u|^{p+1} + |v|^{p+1}) dx \right\}
\]

and

\[
a := \inf_{(u,v) \in \mathcal{M}} I^\infty(u,v) = \inf_{(u,v) \in \mathcal{M}} \left[ \frac{s}{N} \int_{\mathbb{R}^N} (\mu_1 |u|^{2^*_s} + \mu_2 |v|^{2^*_s} + \gamma |u|^\alpha |v|^\beta) dx + \left( \frac{1}{2} - \frac{1}{p+1} \right) k \int_{\mathbb{R}^N} (|u|^{p+1} + |v|^{p+1}) dx \right].
\]

(17)

In this section, we will first prove the existence of solutions for equation (7). For this, we will use the mountain pass theorem in [1] to prove Theorem 2.3. We first show that \( I(u) \) has a \((PS)_c\) sequence in \( H^s_\alpha(\mathbb{R}^N) \). Choose \( \varphi \in C^\infty_0(\mathbb{R}^N) \cap H^s_\alpha(\mathbb{R}^N) \) with \( \varphi \neq 0 \), then there exists \( t_0 > 0 \) such that \( I(t_0 \varphi) < 0 \) for all \( t \geq t_0 \). Take \( u_0 = t \varphi \) with \( t \geq t_0 \) large enough. Let

\[
\Gamma_1 = \left\{ \gamma \in \mathcal{C}([0,1], H^s_\alpha(\mathbb{R}^N)) : \gamma(0) = (0,0), \quad \gamma(1) = (u_0, v_0) \right\}.
\]

Define

\[
c_1 := \inf_{\gamma \in \Gamma_1} \max_{t \in [0,1]} I(\gamma(t)).
\]

Set \( \mathcal{H}_r \) and \( \mathcal{D}_r \) as the following

\[
\mathcal{H}_r = \{(u,v) \in \mathcal{H} : u, v \text{ are radial}\},
\]

\[
\mathcal{D}_r = \{(u,v) \in \mathcal{D} : u, v \text{ are radial}\},
\]

with norm deduced from \( \mathcal{H} \) and \( \mathcal{D} \) respectively. Define the Nehari Manifold in \( \mathcal{H}_r \) as

\[
\mathcal{M}_r = \{(u,v) \in \mathcal{M} : u, v \text{ are radial}\}.
\]
Let
\[ \Gamma_2 = \{ \gamma \in C([0, 1], \mathcal{H}_r) : \gamma(0) = (0, 0), \quad \gamma(1) = (u_0, v_0) \}. \]

Define
\[ a_1 := \inf_{\gamma} \max_{t \in [0, 1]} E(\gamma(t)). \]

**Lemma 3.1.** There exists a Palais-Smale sequence \( \{u_n\} \subset H^s_\rho(\mathbb{R}^N) \) such that
\[ I(u_n) \to c_1 \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as} \quad n \to +\infty, \quad (18) \]
and there exists a Palais-Smale sequence \( \{(u_n, v_n)\} \subset \mathcal{H}_r \) such that
\[ E(u_n, v_n) \to a_1 \quad \text{and} \quad (1^\infty)'(u_n, v_n) \to 0 \quad \text{as} \quad n \to +\infty, \quad (19) \]

**Proof.** We only give the details of the proof for \( I(u) \), the proof of the system is similar. We first claim that \( I(u) \) possesses a mountain pass geometry around \((0, 0)\):

1. there exists \( \alpha, \rho > 0 \), such that \( I(u) > \alpha \) for all \( ||u||_{H^s_\rho(\mathbb{R}^N)} = \rho \);
2. there exists \( u_0 \in H^s_\rho(\mathbb{R}^N) \) such that \( ||u_0||_{H^s_\rho(\mathbb{R}^N)} > \rho \) and \( I(u_0) < 0 \).

Since
\[ I(u) = \frac{1}{2} ||u||^2_{H^s(\mathbb{R}^N)} + \frac{1}{2} \int_{\mathbb{R}^N} u^2 dx - \frac{1}{2s} \int_{\mathbb{R}^N} |u|^{2s} dx - \frac{k}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx \]
\[ \geq \frac{1}{2} ||u||^2_{H^s_\rho(\mathbb{R}^N)} - \frac{C_2}{2s} ||u||^{2s}_{H^s_\rho(\mathbb{R}^N)} - \frac{C_3}{p+1} \lambda ||u||^{p+1}_{H^s_\rho(\mathbb{R}^N)}. \]

Choose \( \rho > 0 \) sufficiently small, if \( ||(u, v)||^2_{H^s_\rho} = \rho \), then
\[ I(u) \geq \frac{1}{2} ||u||^2_{H^s_\rho(\mathbb{R}^N)} - \frac{C_2}{2s} ||u||^{2s}_{H^s_\rho(\mathbb{R}^N)} - \frac{C_3}{p+1} \lambda ||u||^{p+1}_{H^s_\rho(\mathbb{R}^N)} > 0. \]

(2) is obvious when we choose \( t \) large enough such that \( ||u_0||_{H^s_\rho(\mathbb{R}^N)} = t ||\varphi||_{H^s_\rho(\mathbb{R}^N)} > \rho \).

By the mountain pass theorem, for the constant \( 0 < c_1 := \inf_{\gamma \in \Gamma_1} \max_{t \in [0, 1]} I(\gamma(t)), \) there exists a \((PS)_{c_1}\) sequence \( \{u_n\} \subset H^s_\rho(\mathbb{R}^N) \) such that
\[ I(u_n) \to c_1 \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as} \quad n \to +\infty, \quad (20) \]
where
\[ \Gamma_1 = \{ \gamma \in C([0, 1], H^s_\rho(\mathbb{R}^N)) : \gamma(0) = (0), \quad \gamma(1) = (u_0) \}. \]
As desired. \( \square \)

Define
\[ a_2 := \inf_{\mathcal{H}_r \setminus \{(0, 0)\}} \max_{t > 0} E(tu, tv). \]

By the properties of symmetric radial decreasing rearrangement, we know a defined by \((17)\) is the same as the following:
\[ a := \inf_{(u, v) \in \mathcal{M}_r} E(u, v) = \inf_{(u, v) \in \mathcal{M}_r} \left[ \frac{s}{N} \int_{\mathbb{R}^N} \left( \mu_1 |u|^2 + \mu_2 |v|^2 + \gamma |u|^a |v|^b \right) dx + \left( \frac{1}{2} - \frac{1}{p+1} \right) k \int_{\mathbb{R}^N} (|u|^{p+1} + |v|^{p+1}) dx \right]. \]

Then, we have the following lemma.

**Lemma 3.2.** \( a_1 = a = a_2. \)

**Proof.** The proof is similar as the prove of Lemma 2.2 in [32], so we omit the details here. \( \square \)
To prove the existence of solutions for system (10), we consider a function \( f : \mathbb{R}^+ \rightarrow \mathbb{R} \) defined as follows:
\[
f(\tau) = \frac{1 + \tau^2}{(\mu_2 + \mu_1 \tau^2 + \gamma \tau^\alpha)^2}, \quad \tau \geq 0.\tag{21}
\]
Without loss of generality, we assume \( 0 < \mu_1 \leq \mu_2 \), then
\[
\lim_{\tau \to 0^+} f(\tau) = f(0) = \left( \frac{1}{\mu_2} \right)^2 \leq \lim_{\tau \to +\infty} f(\tau) = \left( \frac{1}{\mu_1} \right)^2.
\]
So there exists a minimal point \( 0 \leq \tau_{\text{min}} < +\infty \) such that
\[
f(\tau_{\text{min}}) = \min_{\tau \geq 0} f(\tau).\tag{22}
\]
Then, we give the following lemma.

**Lemma 3.3.** \( S_\gamma = f(\tau_{\text{min}})S_s \).

**Proof.** For any \( \varphi \in D_s(\mathbb{R}^N) \setminus \{0\} \), let \((u, v) = (\tau_{\text{min}} \varphi, \varphi)\), then by the definition of \( S_\gamma \), we have
\[
\frac{1 + \tau^2_{\text{min}}}{(\mu_2 + \mu_1 \tau^2_{\text{min}} + \gamma \tau^\alpha_{\text{min}})^2} \frac{\|\varphi\|^2_{D_s(\mathbb{R}^N)}}{(\int_{\mathbb{R}^N} |\varphi|^2dx)^2} \geq S_\gamma.
\]
Hence
\[
f(\tau_{\text{min}})S_s \geq S_\gamma.\tag{23}
\]
On the other hand, let \((u_n, v_n) \in \mathcal{D}\) be a minimizing sequence of \( S_\gamma \). Let \( z_n = t_n u_n \), where \( t_n = \left( \frac{\int_{\mathbb{R}^N} |v_n|^2dx}{\int_{\mathbb{R}^N} |u_n|^2dx} \right)^{\frac{1}{2}} > 0 \), then
\[
\int_{\mathbb{R}^N} |z_n|^2dx = \int_{\mathbb{R}^N} |v_n|^2dx.\tag{24}
\]
By (24) and the H"older inequality, we have
\[
\int_{\mathbb{R}^N} |z_n|^\alpha |v_n|^\beta dx \leq \int_{\mathbb{R}^N} |z_n|^2dx = \int_{\mathbb{R}^N} |v_n|^2dx.\tag{25}
\]
Therefore, by (24) and (25), we obtain
\[
\frac{\| (u_n, v_n) \|^2_{D_s(\mathbb{R}^N)}}{(\int_{\mathbb{R}^N} (\mu_1 |u_n|^2 + \mu_2 |v_n|^2 + \gamma |u_n|^\alpha |v_n|^\beta)dx)^2} \geq \frac{1}{(\mu_2 + \mu_1 t_n^{-2} + \gamma t_n^{-\alpha})^2} \frac{\| (\tau^{-1} z_n, v_n) \|^2_{D_s(\mathbb{R}^N)}}{(\int_{\mathbb{R}^N} |v_n|^2dx)^2} \geq \frac{1 + t_n^{-2}}{(\mu_2 + \mu_1 t_n^{-2} + \gamma t_n^{-\alpha})^2} \frac{\| v_n \|^2_{D_s(\mathbb{R}^N)}}{(\int_{\mathbb{R}^N} |v_n|^2dx)^2} \geq f(t_n^{-1})S_s \geq \min_{\tau > 0} f(\tau)S_s.
\]
Let \( n \to +\infty \), we see that
\[
S_\gamma \geq f(\tau_{\text{min}})S_s.\tag{26}
\]
Combining (23) with (26), we get
\[
S_\gamma = f(\tau_{\text{min}})S_s.
\]
This ends the proof. □
Consequently, we get for $q$

**Proof.** First we prove $t$.

So

Remark 3. By Lemma 3.3, we can see that $S_s$ has the minimizer $(\tau_{\min} U, U_r)$, where $\tau_{\min}$ satisfies (22) and $S_s$ is attained in $\mathbb{R}^N$ by $U_r$. Thus, the ground state solutions of Theorem 1.1 in [17] must have the above form.

Define

$$K_1 = \inf_{(u,0) \in M_r} E(u,0), \quad K_2 = \inf_{(0,v) \in M_r} E(0,v).$$

(27)

**Lemma 3.4.** Under the condition of $0 < \mu_1 \leq \mu_2$ and $1 < \beta < 2$, or $0 < \mu_2 \leq \mu_1$ and $1 < \alpha < 2$, if $(E_1)$ holds and $k > 0$, or $(E_2)$ holds and $k > 0$ is chosen sufficiently large, then

$$a < \min \left\{ K_1, K_2, \frac{s}{N} S^2 \right\}.$$

Proof. First we prove $a < \min \{K_1, K_2\}$. Indeed, it is easy to see that if $\mu_1 \leq \mu_2$, then $K_1 \leq K_2$. In order to show $a < \min \{K_1, K_2\}$, we only need to prove $a < K_1$.

Define a function $H : \mathbb{R}^2 \to \mathbb{R}$ by

$$H(t, \tau) = \Psi(tu_1, t\tau v_1),$$

where

$$\Psi(u, v) = \|(u, v)\|_{E_r}^2 + \int_{\mathbb{R}^N} (u^2 + v^2)dx$$

$$- \int_{\mathbb{R}^N} (\mu_1 |u|^2 + \mu_2 |v|^2 + \gamma |u|^{\alpha} |v|^\beta)dx - k \int_{\mathbb{R}^N} (|u|^{p+1} + |v|^{p+1})dx.$$

Since $H(1,0) = 0$ and $H_t(1,0) \neq 0$, then by the implicit function theorem, there exists $\delta > 0$ and a function $t(\tau) \in C^1(-\delta, \delta)$ such that

$$t(0) = 1, \quad t'(\tau) = -\frac{H_t(t, \tau)}{H_t(t, \tau)} \quad \text{and} \quad H(t(\tau), \tau) = 0, \quad \forall \tau \in (-\delta, \delta),$$

which implies that

$$(t(\tau) u_1, t(\tau) \tau v_1) \in M_r, \quad \forall \tau \in (-\delta, \delta).$$

Since $1 < \beta < 2$, by direct calculation, we have

$$\lim_{\tau \to 0} t'(\tau) = \frac{-\beta \gamma \int_{\mathbb{R}^N} |u_1|^{\alpha} |v_1|^\beta dx}{(2^* - 2) \int_{\mathbb{R}^N} \mu_1 |u_1|^{2^*} dx + k(p - 1) \int_{\mathbb{R}^N} |u_1|^{p+1} dx} < 0.$$

That is

$$t'(\tau) = \frac{-\beta \gamma \int_{\mathbb{R}^N} |u_1|^{\alpha} |v_1|^\beta dx}{(2^* - 2) \int_{\mathbb{R}^N} \mu_1 |u_1|^{2^*} dx + k(p - 1) \int_{\mathbb{R}^N} |u_1|^{p+1} dx} |\tau|^{-2} \tau (1 + o(1)) \quad \text{as} \ \tau \to 0.$$

So

$$t(\tau) = 1 - \frac{\gamma \int_{\mathbb{R}^N} |u_1|^{\alpha} |v_1|^\beta dx}{(2^* - 2) \int_{\mathbb{R}^N} \mu_1 |u_1|^{2^*} dx + k(p - 1) \int_{\mathbb{R}^N} |u_1|^{p+1} dx} |\tau|^{\beta} (1 + o(1)) \quad \text{as} \ \tau \to 0.$$

Let

$$A = \frac{\gamma \int_{\mathbb{R}^N} |u_1|^{\alpha} |v_1|^\beta dx}{(2^* - 2) \int_{\mathbb{R}^N} \mu_1 |u_1|^{2^*} dx + k(p - 1) \int_{\mathbb{R}^N} |u_1|^{p+1} dx},$$

consequently, we get for $q \geq 1$

$$t^q(\tau) = 1 - qA |\tau|^{\beta} (1 + o(1)) \quad \text{as} \ \tau \to 0.$$
Thus, notice that $p > 1$ and $1 < \beta < 2$, for $\tau > 0$ small, 
\[
a < I^\infty(t(\tau)u_1, t(\tau)v_1) - \frac{1}{2} \Psi(tu_1, tv_1)
\]
\[
= \frac{s}{N} (t(\tau))^{2s} \int_{\mathbb{R}^N} (\mu_1 |u_1|^{2s} + \mu_2 |v_1|^{2s} + \gamma \tau^p |u_1|^\alpha |v_1|^\beta) dx
\]
\[+
\]
\[
\left(\frac{1}{2} - \frac{1}{p + 1}\right) \int_{\mathbb{R}^N} |u_1|^p dx
\]
\[+
\]
\[
\left(\frac{1}{2} - \frac{1}{p + 1}\right) \int_{\mathbb{R}^N} |v_1|^p dx
\]
\[\leq \frac{s}{N} \int_{\mathbb{R}^N} |u_1|^{2s} dx + \left(\frac{1}{2} - \frac{1}{p + 1}\right) k \int_{\mathbb{R}^N} |u_1|^p dx
\]
\[-
\]
\[
\left(\frac{sA^2_s}{N} \int_{\mathbb{R}^N} |u_1|^{2s} dx + \frac{(p - 1)kA}{2} \int_{\mathbb{R}^N} |u_1|^{p+1} dx - \frac{s\gamma}{N} \int_{\mathbb{R}^N} |u_1|^\alpha |v_1|^\beta dx \right) |\tau|^\beta
\]
\[+
\]
\[
o(|\tau|^\beta).
\]
Since, when $N > 2s$, for all $k > 0$, we have 
\[
\frac{sA^2_s}{N} \int_{\mathbb{R}^N} |u_1|^{2s} dx + \left(\frac{1}{2} - \frac{1}{p + 1}\right) k \int_{\mathbb{R}^N} |u_1|^p dx
\]
\[-
\]
\[
\left(\frac{sA^2_s}{N} \int_{\mathbb{R}^N} |u_1|^{2s} dx + \frac{(p - 1)kA}{2} \int_{\mathbb{R}^N} |u_1|^{p+1} dx - \frac{s\gamma}{N} \int_{\mathbb{R}^N} |u_1|^\alpha |v_1|^\beta dx \right) |\tau|^\beta > 0.
\]
If we choose $\tau$ small enough, we get 
\[
a < \frac{s}{N} \int_{\mathbb{R}^N} |u_1|^{2s} dx + \left(\frac{1}{2} - \frac{1}{p + 1}\right) k \int_{\mathbb{R}^N} |u_1|^p dx = K_1.
\]
Hence, 
\[
a < K_1 \leq K_2.
\]

Next, we show $a < \frac{s}{N} S^N_{2s}$. Firstly, from [9] we know that $S_s$ is attained in $\mathbb{R}^N$ by 
\[
U_0(x) = C(N, s) \left(\frac{1}{1 + |x|^2}\right)^{\frac{N - 2s}{2}},
\]
where $C(N, s)$ is chosen so that 
\[
|U_0(x)|^2_{D^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |U_0(x)|^{2s} dx = S^N_{2s}.
\]
By Lemma 3.3, $S_\gamma$ is achieved by $(U, V) \in D_s(\mathbb{R}^N) \times D_s(\mathbb{R}^N)$ with the form 
$(\tau_{min}U_0, U_0)$, where $f(\tau)$ gets its minimum at $\tau_{min}$. Take $\eta(x) \in C^\infty_0(\mathbb{R}^N, [0, 1])$ be a cut-off function such that $0 \leq \eta \leq 1$, $\eta = 1$ on $B(0, \delta)$ and $\eta = 1$ on $\mathbb{R}^N \setminus B(0, 2\delta)$. Let 
\[
u_\epsilon = \eta(x)\tau_{min}U_\epsilon(x), \quad v_\epsilon = \eta(x)U_\epsilon(x)
\]
and 
\[
U_\epsilon(x) = \epsilon^{-\frac{N - 2s}{2}} U_0(\frac{x}{\epsilon}).
\]
According to Proposition 21 and Proposition 22 in [27], it is easy to deduce that the following estimates hold true 
\[
\|v_\epsilon\|^2_{D^s(\mathbb{R}^N)} \leq \|U_0\|^2_{D^s(\mathbb{R}^N)} + o(\epsilon^{N - 2s}),
\]
\[
\int_{\mathbb{R}^N} |v_\epsilon|^2 dx \cong \begin{cases} 
C\epsilon^{2s} + o(\epsilon^{N - 2s}), & \text{if } N > 4s, \\
C\epsilon^{2s} \log(\frac{1}{\epsilon}) + o(\epsilon^{2s}), & \text{if } N = 4s, \\
C\epsilon^{N - 2s} + o(\epsilon^{2s}), & \text{if } N < 4s,
\end{cases}
\]
\[
\int_{\mathbb{R}^N} |v_\epsilon|^{2^*} \, dx = \int_{\mathbb{R}^N} |U_0|^{2^*} \, dx + o(\epsilon^N). 
\]

By the similar arguments as in Proposition 22 in [27], we can deduce that

\[
\int_{\mathbb{R}^N} |v_\epsilon|^{p+1} \, dx \cong \begin{cases} 
Ce^{N-\frac{N+2}{p}2(p+1)} + o(\epsilon^{\frac{N+2}{p}2(p+1)}), & \text{if } N > \frac{p+1}{p}2s, \\
Ce^{\frac{N}{2}s}(\log(\frac{1}{\epsilon})) + o(\epsilon^{\frac{N}{2}s}), & \text{if } N = \frac{p+1}{p}2s, \\
Ce^{N-\frac{N+2}{p}2(p+1)} + o(\epsilon^{N-\frac{N+2}{p}2(p+1)}), & \text{if } N < \frac{p+1}{p}2s.
\end{cases} 
\]

Let \( t_{u,v} \) be such that \( I^\infty(t_{u,v}u_\epsilon, t_{u,v}v_\epsilon) = \max_{t > 0} I^\infty(tu_\epsilon, tv_\epsilon) \), then

\[
t_{u,v} \left[ \left( \|(u_\epsilon, v_\epsilon)\|_{L^2}^2 + \int_{\mathbb{R}^N} (u_\epsilon^2 + v_\epsilon^2) \, dx \right) \right] 
= t_{u,v}(1 + \tau^2_{\min}) \left[ \left( \|v_\epsilon\|^2_{L^2} + \int_{\mathbb{R}^N} v_\epsilon^2 \, dx \right) \right] 
= t_{u,v}^{2s-1} \int_{\mathbb{R}^N} (\mu_1|u_\epsilon|^{2s} + \mu_2|v_\epsilon|^{2s} + \gamma |u_\epsilon|^\alpha |v_\epsilon|^{\beta}) \, dx + t_{u,v}^p k \int_{\mathbb{R}^N} |v_\epsilon|^{p+1} + |v_\epsilon|^{p+1} \, dx
\]

We claim that when \( \epsilon > 0 \) is small and \( k \int_{\mathbb{R}^N} |v_\epsilon|^{p+1} \, dx \leq 1 \), \( t_{u,v} \in (a_1, a_2) \) for some constants \( 0 < a_1 < a_2 \) independent of \( k \) and \( \epsilon \). Indeed, since \( k \int_{\mathbb{R}^N} |v_\epsilon|^{p+1} \, dx \to 0 \) as \( \epsilon \to 0 \), from (32) and \( k > 0 \), we know that

\[
t_{u,v}^{2s-2} \leq \frac{\left( \|(u_\epsilon, v_\epsilon)\|_{L^2}^2 + \int_{\mathbb{R}^N} V_{\infty}(u_\epsilon^2 + v_\epsilon^2) \, dx \right)}{\int_{\mathbb{R}^N} (\mu_1|u_\epsilon|^{2s} + \mu_2|v_\epsilon|^{2s} + \gamma |u_\epsilon|^\alpha |v_\epsilon|^{\beta}) \, dx}.
\]

Therefore, by (28)–(30), we can see that \( t_{u,v} < a_2 \) independent of \( k \) when \( \epsilon \) is small. For the other side of the inequality, from the formula on page 97 in [27],

\[
\int_{\mathbb{R}^{2N}} \frac{|v_\epsilon(x) - v_\epsilon(y)|^2}{|x - y|^{n+2s}} \, dxdy = \int_{B_1 \times B_1} \frac{|U_\epsilon(x) - U_\epsilon(y)|^2}{|x - y|^{n+2s}} \, dxdy 
+ 2 \int_{\Omega_1 \times \Omega_1} \frac{|U_\epsilon(x) - U_\epsilon(y)|^2}{|x - y|^{n+2s}} \, dxdy + O(\epsilon^{N-2s})
\]

\[
\geq \int_{B_1 \times B_1} \frac{|U_\epsilon(x) - U_\epsilon(y)|^2}{|x - y|^{n+2s}} \, dxdy + O(\epsilon^{N-2s})
= \int_{B_{\frac{1}{2}} \times B_{\frac{1}{2}}} \frac{|U_0(x) - U_0(y)|^2}{|x - y|^{n+2s}} \, dxdy + O(\epsilon^{N-2s})
\]

\[
\geq \frac{1}{2} \|U_0\|^2_{L^2(\mathbb{R}^N)}.
\]

Under the choice of \( k \int_{\mathbb{R}^N} |v_\epsilon|^{p+1} \, dx \leq 1 \), when \( \epsilon \) is small. From (32), we have

\[
(1 + \tau^2_{\min}) \left[ \|v_\epsilon\|^2_{L^2} + \int_{\mathbb{R}^N} v_\epsilon^2 \, dx \right] 
\leq t_{u,v}^{2s-2} \left( \mu_1\tau^2_{\min} + \mu_2 + \gamma\tau^\alpha_{\min} \right) \int_{\mathbb{R}^N} |v_\epsilon|^{2^*} \, dx + t_{u,v}^p \left( 1 + \tau^p_{\min} \right).
\]

Therefore, it is not hard to see that \( t_{u,v} \geq a_1 > 0 \) independent of \( \lambda \) when \( \epsilon \) is small.
Since $S_γ$ is achieved by $(U, V) \in D^1(\mathbb{R}^N) \times D^1(\mathbb{R}^N)$ with the form $(τ_{\min} U_0, U_0)$, where $f(τ)$ gets its minimum at $τ_{\min}$ and $S_γ = f(τ_{\min}) S_γ$. By the estimates in (28)–(31), we can deduce

\begin{align*}
P^∞(t_{u,v}, t_{u,v}) &= \frac{t_{u,v}^2}{2} \left[ ||(u_v, v)||_{D^1}^2 + \int_{\mathbb{R}^N} (u_v^2 + v^2) dx \right] \\
&- \frac{t_{u,v}^2}{2} \int_{\mathbb{R}^N} (μ_1 |u_v|^s + μ_2 |v|^s + γ |u_v|^{s-1} v^2) dx - \frac{t_{u,v}^{p+1}}{p+1} k \int_{\mathbb{R}^N} (|u_v|^{p+1} + |v|^{p+1}) dx \\
&\leq \max_{t \geq 0} \left\{ \frac{t^2}{2} \left[ ||(u_v, v)||_{D^1}^2 + \int_{\mathbb{R}^N} V_0 (u_v^2 + v^2) dx \right] \\
&- \frac{t_{u,v}^2}{2} \int_{\mathbb{R}^N} (μ_1 |u_v|^s + μ_2 |v|^s + γ |u_v|^{s-1} v^2) dx - \frac{t_{u,v}^{p+1}}{p+1} k \int_{\mathbb{R}^N} (|u_v|^{p+1} + |v|^{p+1}) dx \right\} \\
&\leq \frac{α}{N} \left[ ||(u_v, v)||_{D^1}^2 + \int_{\mathbb{R}^N} (u_v^2 + v^2) dx \right] \left[ \frac{(||(u_v, v)||_{D^1}^2 + f_{N,v} v^2 dx)}{f_{N,v} |v_v|^2 dx} \right] G^2 \frac{α}{2} \\
&- \frac{t_{u,v}^{p+1}}{p+1} k(1 + τ_{\min}) \int_{\mathbb{R}^N} |v_v|^{p+1} dx \\
&\leq \frac{α}{N} \left[ S_γ^2 + o(ε^{-2}) + f_{N,v} v^2 dx \right] \left[ S_γ^2 + o(ε^{-2}) \right] \frac{α}{2} \\
&- \frac{t_{u,v}^{p+1}}{p+1} k(1 + τ_{\min}) \int_{\mathbb{R}^N} |v_v|^{p+1} dx \\
&\leq \frac{α}{N} f(τ_{\min}) S_γ^2 + C_1 \int_{\mathbb{R}^N} v^2 dx + o(ε^{-2}) - \frac{t_{u,v}^{p+1}}{p+1} k(1 + τ_{\min}) \int_{\mathbb{R}^N} |v_v|^{p+1} dx \\
&\leq \left( \frac{α}{N} S_γ^2 + C_1 \right) \int_{\mathbb{R}^N} v^2 dx + o(ε^{-2}) - \frac{t_{u,v}^{p+1}}{p+1} k(1 + τ_{\min}) \int_{\mathbb{R}^N} |v_v|^{p+1} dx.
\end{align*}

Next, we show

\begin{align*}
\frac{α}{N} S_γ^2 + C_1 \int_{\mathbb{R}^N} v^2 dx + o(ε^{-2}) &\leq \frac{t_{u,v}^{p+1}}{p+1} k(1 + τ_{\min}) \int_{\mathbb{R}^N} |v_v|^{p+1} dx < \frac{α}{N} S_γ^2
\end{align*}

under suitable conditions.

Since $\frac{p+1}{p} 2s < 4s$ as $p > 1$, by the estimates in (29), (31) and boundedness of $t_{u,v}$, we deduce

\begin{align*}
C_1 \int_{\mathbb{R}^N} V_0 v^2 dx + o(ε^{-2}) &\leq \frac{t_{u,v}^{p+1}}{p+1} k(1 + τ_{\min}) \int_{\mathbb{R}^N} |v_v|^{p+1} dx \\
&\leq \left\{ \begin{array}{ll}
C_1 ε^{2s} - C_2 λ e^{N - \frac{N-2s}{p+1}} + o(ε^{-2}) &\text{if } N > 4s, \\
C_1 ε^{2s} | \ln ε | - C_2 λ e^{N - \frac{N-2s}{p+1}} + o(ε^{-2}) &\text{if } N = 4s, \\
C_1 ε^{N-2s} - C_2 k e^{N - \frac{N-2s}{p+1}} + o(ε^{-2}) + o(ε^{2s}) &\text{if } N > 4s, \\
C_1 ε^{N-2s} - C_2 k e^{N - \frac{N-2s}{p+1}} + o(ε^{-2}) + o(ε^{2s}) &\text{if } N = 4s, \\
C_1 ε^{N-2s} - C_2 k e^{N - \frac{N-2s}{p+1}} + o(ε^{2s}) &\text{if } N > 4s, \\
C_1 ε^{N-2s} - C_2 k e^{N - \frac{N-2s}{p+1}} + o(ε^{2s}) &\text{if } N = 4s,
\end{array} \right.
\end{align*}

for $2s < N < \frac{p+1}{p} 2s$. 

Thus, for $N \geq 4s$ or $2s < N < 4s$ and $p \in (\frac{8s-N}{2s}, \frac{N+2s}{2s})$, if we take $\epsilon > 0$ sufficiently small, then for all $k > 0$, we have $a < \frac{s}{N}S_2^\frac{N}{2}$. For $2s < N < 4s$ and $p \in (1, \frac{8s-N}{2s}]$, if we take $\epsilon > 0$ sufficiently small and $k$ sufficiently large, we have $a < \frac{s}{N}S_2^\frac{N}{2}$. This completes the proof of Lemma 3.4.

**Remark 4.** For the case of $\mu_1 \geq \mu_2 > 0$, in order to show $a < K_2$, we need the condition of $1 < \alpha < 2$.

Next, we prove the claim that $K_1$ and $K_2$ (see (27)) are well defined. Firstly, we consider the existence of positive ground state for single equation

$$(-\Delta)^s u + u = \mu_1 |u|^{2^*-2}u + \lambda |u|^{p-1}u, x \in \mathbb{R}^N$$

(33)

Accordingly, the energy functional is as follows:

$$I(u) = \frac{1}{2} \left\| u \right\|_{D_1(\mathbb{R}^N)}^2 + \frac{1}{2} \int_{\mathbb{R}^N} u^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^N} \mu_1 |u|^{2^*} dx - \frac{\lambda}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx.$$

**Remark 5.** $I(u) = E(u,0)$. It is clear that \{$u \in H^s(\mathbb{R}^N)$ | $(u,0) \in M$\} is the Nehari Manifold for $I(u)$ in $H^s(\mathbb{R}^N)$. It is also easy to see that $K_1 > 0$ is the mountain pass critical point of $I(u)$.

**Lemma 3.5.** Assume $(E_1)$ holds and $\lambda > 0$, or $(E_2)$ holds and $\lambda > 0$ is chosen sufficiently large, then

$$K_1 < \frac{s}{N} \mu_1 \frac{N-2s}{2s} S_s^\frac{N}{2}.$$

**Proof.** The proof is similar to the proof of second part of Lemma 3.4. Take $u_\epsilon(x) = \eta(x)U_{\epsilon}(x)$. Then there is a constant $t_\epsilon > 0$ bounded independent of $\lambda$ when $\epsilon > 0$ small and $\lambda \int_{\mathbb{R}^N} u_\epsilon^{p+1} dx \leq 1$, such that $(t_\epsilon u_\epsilon, 0) \in M_\epsilon$. Therefore

$$K_1 \leq E(t, u_\epsilon, 0)$$

$$= \frac{s}{2} \left[ \left\| u_\epsilon \right\|_{D_1(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} u_\epsilon^2 dx \right] - \frac{\epsilon^{2^*}}{2^*} \int_{\mathbb{R}^N} \mu_1 |u_\epsilon|^{2^*} dx - \frac{T_\epsilon^{p+1}}{p+1} \lambda \int_{\mathbb{R}^N} |u_\epsilon|^{p+1} dx$$

$$\leq \frac{s}{N} \left[ \left\| u_\epsilon \right\|_{D_1(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} u_\epsilon^2 dx \right] \left( \frac{\left\| u_\epsilon \right\|_{D_1(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} u_\epsilon^2 dx}{f_{\text{Nehari}}(u_\epsilon) \mu_1 |u_\epsilon|^{2^*} dx} \right)^{\frac{2^*}{2^*-1}} - \frac{T_\epsilon^{p+1}}{p+1} \lambda \int_{\mathbb{R}^N} |u_\epsilon|^{p+1} dx$$

$$= \frac{s}{N} \mu_1 \left( \frac{\left\| u_\epsilon \right\|_{D_1(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} u_\epsilon^2 dx}{f_{\text{Nehari}}(u_\epsilon) \mu_1 |u_\epsilon|^{2^*} dx} \right)^{\frac{2^*}{2^*-1}} - \frac{T_\epsilon^{p+1}}{p+1} \lambda \int_{\mathbb{R}^N} |u_\epsilon|^{p+1} dx.$$

By the same argument as in Lemma 3.4, we know that if $(E_1)$ holds and $\lambda > 0$, or $(E_2)$ holds and $\lambda > 0$ is chosen sufficiently large, then

$$K_1 < \frac{s}{N} \mu_1 \frac{N-2s}{2s} S_s^\frac{N}{2}.$$

This finishes the proof.

**Proposition 1.** If $(E_1)$ holds and $\lambda > 0$, or $(E_2)$ holds and $\lambda > 0$ is chosen sufficiently large, then (33) has a positive ground state solution.

**Proof.** We just need to work in the space $H^s(\mathbb{R}^N)$. Since $K_1$ is a mountain pass critical value of $I(u)$ in $H^s(\mathbb{R}^N)$. There exists a sequence $u_n \in H^s(\mathbb{R}^N)$, such that

$$I(u_n) \to K_1, \quad I'(u_n) \to 0 \text{ as } n \to +\infty.$$
It is easy to see that $u_n$ is bounded in $H^s(\mathbb{R}^N)$. By the Sobolev imbedding theorem, there exists $u \in H^s(\mathbb{R}^N)$ such that

$$
\begin{cases}
u_n \rightharpoonup u, \text{ weakly in } H^s(\mathbb{R}^N), \\
u_n \rightarrow u, \text{ strongly in } L^p(\mathbb{R}^N), \text{ for } 2 < p < 2^*_s, \\
u_n \rightarrow u, \text{ a.e. } \mathbb{R}^N.
\end{cases}
$$

Then, we have $I(u) = 0$. Let $w_n = u_n - u$. By the Brézis-Lieb lemma, we can get

$$I(u_n) = I(u) + \frac{1}{2} \left( \|w_n\|_{D_s(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} w_n^2 \, dx \right) - \frac{1}{2s} \int_{\mathbb{R}^N} \mu_1 |w_n|^{2^*_s} \, dx + o(1),$$

$$I'(u_n) = I'(u) + \|w_n\|_{D_s(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} w_n^2 \, dx - \int_{\mathbb{R}^N} \mu_1 |w_n|^{2^*_s} \, dx + o(1)$$

$$= \|w_n\|_{D_s(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} w_n^2 \, dx - \int_{\mathbb{R}^N} \mu_1 |w_n|^{2^*_s} \, dx + o(1).$$

If $\|w_n\|_{D_s(\mathbb{R}^N)} \rightarrow 0$ as $n \rightarrow +\infty$, then $u_n \rightarrow u$ strongly in $H^s(\mathbb{R}^N)$. If $\|w_n\|_{D_s(\mathbb{R}^N)} \rightarrow \ell > 0$ as $n \rightarrow +\infty$, from

$$S_s \mu_1 \frac{N-2s}{N} \leq \frac{\|w_n\|_{D_s(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} w_n^2 \, dx}{\left( \int_{\mathbb{R}^N} \mu_1 |w_n|^{2^*_s} \, dx \right)^{\frac{N-2s}{2s}}}$$

and

$$I'(u) = 0, \quad I(u) \geq 0, \quad \langle I'(u_n), u_n \rangle = o(1),$$

we have

$$K_1 = I(u) + \frac{S}{N} \left[ \|w_n\|_{D_s(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} w_n^2 \, dx \right] + o(1)$$

$$\geq \frac{S}{N} \left[ \|w_n\|_{D_s(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} w_n^2 \, dx \right] + o(1)$$

$$\geq \frac{S}{N} \left( S_s \mu_1 \frac{N-2s}{N} \right)^{\frac{N}{2s}} = \frac{S}{N} \frac{N}{S_s \mu_1} \frac{N-2s}{2s}.$$

Let $n \rightarrow +\infty$, then we get a contradiction to Lemma 3.5. Therefore, $u_n \rightarrow u$ strongly in $H^s(\mathbb{R}^N), I(u) = K_1$, $u$ is a ground state solution of (33).

Next we show $u$ is positive. Since

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^2 - |u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy$$

$$= 2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)||u(y)| - u(x)u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \geq 0,$$

hence,

$$\|u\|_{D_s(\mathbb{R}^N)}^2 \leq \|u\|_{D'_s(\mathbb{R}^N)}.$$

Now, for the ground state solution $u$ of (33), we have

$$\|u\|_{D_s(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} |u|^2 \, dx \leq \|u\|_{D_s(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} u^2 \, dx$$

$$= \int_{\mathbb{R}^N} \mu_1 |u|^{2^*_s} \, dx + \lambda \int_{\mathbb{R}^N} |u|^{p+1} \, dx.$$
This implies that there exists $t_0 \in (0, 1]$ such that $t_0|u| \in \mathbb{M}_{s}^{\infty}$. Therefore,

$$K_1 = I(u) = \inf_{(u,0) \in \mathbb{M}_{s}^{\infty}} E(u, 0) \leq I^{\infty}(t_0 |u|, 0)$$

$$= st_0^{2s} N \int_{\mathbb{R}^N} \mu_1 |u|^{2s} dx + \left( \frac{1}{2} - \frac{1}{p+1} \right) \lambda t_0^{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx$$

$$\leq I^{\infty}(u, 0) = K_1 \text{ as } 0 < t_0 \leq 1.$$  

Thus, we conclude that

$$t_0 = 1, \quad \|u\|_{D_{s}^{2}(\mathbb{R}^N)} = \|u\|_{D_{s}^{2}(\mathbb{R}^N)}.$$  

From (34) we know that $u(x)$ can not change sign. We may assume that $u$ is non-negative. By the strong maximum principle for the fractional Laplacian (see Proposition 2.17 in [28]), we obtain that $u$ is positive. This ends the proof.  

\[\square\]

**Lemma 3.6.** (i) Equation (5) has at least one nontrivial solution $v \in H^{s}(\mathbb{R}^N)$ if and only if system (6) has at least one nontrivial solution $(u, \lambda) \in H^{s}(\mathbb{R}^N) \times \mathbb{R}^{+}$.

(ii) System (1) has at least one nontrivial solution $(u, v) \in \mathcal{H}$ if and only if system (2) has at least one nontrivial solution $(u, v, \lambda_1, \lambda_2) \in \mathcal{H} \times \mathbb{R}^{+} \times \mathbb{R}^{+}$.

**Proof.** On the one hand, if (5) has a solution $v \in H^{s}(\mathbb{R}^N)$, then

$$(a + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 dx) (-\Delta)^{s} v + v = |v|^{2^*-2} v + k |v|^{p-1} v, \quad x \in \mathbb{R}^N.$$  

Letting $\lambda^s = a + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 dx$ and $u(x) = v(\lambda^\frac{1}{2} x) = v(y)$, then we have

$$\lambda^s = a + b \lambda^{\frac{N-2s}{2}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx$$

and

$$(-\Delta)^{s} u(x) + u(x) = \lambda^s (-\Delta)^{s} v(y) + v(y)$$

$$= \left( a + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 dx \right) (-\Delta)^{s} v(y) + v(y)$$

$$= |v(y)|^{2^*-2} v(y) + k |v|^{p-1} v(y)$$

$$= |u(x)|^{2^*-2} u(x) + k |u(x)|^{p-1} u(x),$$

which implies that $(u, \lambda) \in H^{s}(\mathbb{R}^N) \times \mathbb{R}^{+}$ is a solution of (6).

On the other hand, if (6) has a solution $(u, \lambda) \in H^{s}(\mathbb{R}^N) \times \mathbb{R}^{+}$, then

$$(-\Delta)^{s} u + u = |u|^{2^*-2} u + k |u|^{p-1} u, \quad x \in \mathbb{R}^N,$$

and

$$\lambda^s = a + b \lambda^{\frac{N-2s}{2}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx, \quad \lambda \in \mathbb{R}^{+}.$$
Let \( v(x) = u(\lambda_{-\frac{1}{2}}x) = u(y) \), we have

\[
(a + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}}v|^{2}dx) (-\Delta)^{s}v(x) + v(x) \\
= \lambda^{-s} \left( a + b \lambda^{\frac{N-2s}{2s}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}}u|^{2}dx \right) (-\Delta)^{s}u(y) + u(y) \\
= |u(y)|^{2^{*}-2}u(y) + k|u(y)|^{p-1}u(y) \\
= |v(x)|^{2^{*}-2}v(x) + k|v(x)|^{p-1}v(x),
\]

which implies that \( v \) is a solution of (5).

Next, we prove system (1) and system (2) are equivalent. On the one hand, if (1) has a solution \((u, v) \in \mathcal{H}\), then

\[
\begin{cases}
    (a_1 + b_1 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}}u|^{2}dx) (-\Delta)^{s}u + u = \mu_1 |u|^{2^{*}-2}u + \frac{\alpha^2}{2s} |u|^{\alpha-2}u|v|^{\beta} + k|u|^{p-1}u, \\
    x \in \mathbb{R}^N, \\
    (a_2 + b_2 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}}v|^{2}dx) (-\Delta)^{s}v + v = \mu_2 |v|^{2^{*}-2}v + \frac{\beta^2}{2s} |u|^{\alpha}|v|^{\beta-2}v + k|v|^{p-1}v, \\
    x \in \mathbb{R}^N.
\end{cases}
\]

Letting

\[
\lambda_1^* = a_1 + b_1 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}}u|^{2}dx \text{ and } U(x) = u(\lambda_{\frac{1}{2}}^* x) = u(y),
\]

\[
\lambda_2^* = a_2 + b_2 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}}v|^{2}dx \text{ and } V(x) = v(\lambda_{\frac{1}{2}}^* x) = v(y),
\]

then we have

\[
\lambda_1^* = a_1 + b_1 \lambda_1^{\frac{N-2s}{2s}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}}U|^{2}dx,
\]

\[
\lambda_2^* = a_2 + b_2 \lambda_2^{\frac{N-2s}{2s}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}}V|^{2}dx,
\]

and

\[
(-\Delta)^{s}U(x) + U(x) = \lambda_1^* (-\Delta)^{s}u(y) + u(y)
= \left( a_1 + b_1 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}}u|^{2}dx \right) (-\Delta)^{s}u(y) + u(y)
= \mu_1 |u(y)|^{2^{*}-2}u(y) + \frac{\alpha^2}{2s} |u(y)|^{\alpha-2}u|v(y)|^{\beta} + k|u(y)|^{p-1}u(y)
= \mu_1 |U(x)|^{2^{*}-2}U(x) + \frac{\alpha^2}{2s} |U(x)|^{\alpha-2}U(x)|V(x)|^{\beta} + k|U(x)|^{p-1}U(x),
\]

and

\[
(-\Delta)^{s}V(x) + V(x) = \lambda_2^* (-\Delta)^{s}v(y) + v(y)
= \left( a_2 + b_2 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}}v|^{2}dx \right) (-\Delta)^{s}v(y) + v(y)
= \mu_2 |v(y)|^{2^{*}-2}v(y) + \frac{\beta^2}{2s} |u(y)|^{\alpha}|v(y)|^{\beta-2}v(y) + k|v(y)|^{p-1}v(y)
= \mu_2 |V(y)|^{2^{*}-2}V(y) + \frac{\beta^2}{2s} |U(y)|^{\alpha}|V(y)|^{\beta-2}V(y) + k|V(y)|^{p-1}V(y),
\]

which implies that \((U, V, \lambda_1, \lambda_2) \in \mathcal{H} \times \mathbb{R}^+ \times \mathbb{R}^+\) is a solution of (2).
On the other hand, if (2) has a solution \( (u, v, \lambda_1, \lambda_2) \in \mathcal{H} \times \mathbb{R}^+ \times \mathbb{R}^+ \), then
\[
\begin{cases}
(-\Delta)^s u + u = \mu_1 |u|^{2^* - 2} u + \frac{\alpha \gamma}{2s} |u|^{\alpha - 2} u |v|^\beta + k |u|^{p-1} u, & x \in \mathbb{R}^N, \\
(-\Delta)^s v + v = \mu_2 |v|^{2^* - 2} v + \frac{\beta \gamma}{2s} |u|^{\alpha} |v|^{\beta - 2} v + k |v|^{p-1} v, & x \in \mathbb{R}^N,
\end{cases}
\]
(35)
and
\[
\begin{cases}
\lambda_1^s - a_1 - b_1 \lambda_1^{\frac{N-2s}{N-2}} \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 dx = 0, & \lambda_1 \in \mathbb{R}^+, \\
\lambda_2^s - a_2 - b_2 \lambda_2^{\frac{N-2s}{N-2}} \int_{\mathbb{R}^N} |(-\Delta)^s v|^2 dx = 0, & \lambda_2 \in \mathbb{R}^+.
\end{cases}
\]
(36)
Let
\[
U(x) = u(\lambda_1^{-\frac{1}{s}} x) = u(y) \quad V(x) = v(\lambda_2^{-\frac{1}{s}} x) = V(y),
\]
by (35) and (36), we have
\[
\begin{align*}
& a_1 + b_1 \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 dx \left( (-\Delta)^s U(x) + U(x) \right) \\
& = \lambda_1^{1-s} \left( a_1 + b_1 \lambda_1^{\frac{N-2s}{N-2}} \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 dx \right) \left( (-\Delta)^s u(y) + u(y) \right) \\
& = \mu_1 |u(y)|^{2^* - 2} u(y) + \frac{\alpha \gamma}{2s} |u(y)|^{\alpha - 2} u(y) |v(y)|^\beta + k |u(y)|^{p-1} u(y) \\
& = \mu_1 |U(x)|^{2^* - 2} U(x) + \frac{\alpha \gamma}{2s} |U(x)|^{\alpha - 2} U(x) |V(x)|^\beta + k |U(x)|^{p-1} U(x),
\end{align*}
\]
and
\[
\begin{align*}
& a_2 + b_2 \int_{\mathbb{R}^N} |(-\Delta)^s v|^2 dx \left( (-\Delta)^s V(x) + V(x) \right) \\
& = \lambda_2^{1-s} \left( a_2 + b_2 \lambda_2^{\frac{N-2s}{N-2}} \int_{\mathbb{R}^N} |(-\Delta)^s v|^2 dx \right) \left( (-\Delta)^s v(y) + v(y) \right) \\
& = \mu_2 |v(y)|^{2^* - 2} v(y) + \frac{\beta \gamma}{2s} |u(y)|^{\alpha} |v(y)|^{\beta - 2} v(y) + k |v(y)|^{p-1} v(y) \\
& = \mu_2 |V(y)|^{2^* - 2} V(y) + \frac{\beta \gamma}{2s} |U(y)|^{\alpha} |V(y)|^{\beta - 2} V(y) + k |V(y)|^{p-1} V(y),
\end{align*}
\]
which implies that \( (U, V) \in \mathcal{H} \) is a solution of (1). This completes the proof. \( \Box \)

4. Proof of Theorems 2.3–2.4. In this section, we first sketch our idea of the proof of Theorem 2.4. It is well known that the Sobolev embedding \( H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N) \) are not compact for \( 2 \leq p \leq 2^*_s \). In order to overcome the lack of compactness, we work in the space of radial functions \( \mathcal{H}_r = H^s_+(\mathbb{R}^N) \times H^s_+(\mathbb{R}^N) \), where
\[
H^s_+(\mathbb{R}^N) = \{ \varphi \in H^s(\mathbb{R}^N) : \varphi \text{ is radial} \}
\]
and \( H^s_+(\mathbb{R}^N) \) is endowed with the \( H^s(\mathbb{R}^N) \) topology: \( \| \varphi \|_{H^s(\mathbb{R}^N)} = \| \varphi \|_{H^s_+(\mathbb{R}^N)} \). Let
\[
D^s_+(\mathbb{R}^N) = \{ \varphi \in D_+(\mathbb{R}^N) : \varphi \text{ is radial} \}.
\]
Furthermore, we can show that the critical value \( a \) of the energy functional for system (10) is strictly less than \( \frac{\alpha \gamma}{2s} S^{N/(2s)}_1 \) and \( (PS)_a \) condition is satisfied. Then by the standard arguments, we can obtain a ground state solution for system (10).

By Lemma 3.1, there exists a bounded sequence \( \{(u_n, v_n)\} \subset H^s_+(\mathbb{R}^N) \) such that
\[
I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \text{ as } n \to +\infty.
\]
(37)
So, by Sobolev imbedding theorem, there exists $u \in H^s_\gamma(\mathbb{R}^N)$ such that
\[
\begin{cases}
  u_n \rightharpoonup u, \quad \text{weakly in } H^s_\gamma(\mathbb{R}^N), \\
  u_n \rightarrow u, \quad \text{strongly in } L^p(\mathbb{R}^N), \text{ for } 2 < p < 2^*_s, \\
  u_n \rightarrow u, \quad \text{a.e. } \mathbb{R}^N.
\end{cases}
\]
Then, we have
\[I'(u) = 0.\] (38)

Let $w_n = u_n - u$. We claim
\[
I(u_n) = I(u) + \frac{1}{2} ||w_n||^2_{D_s(\mathbb{R}^N)} + \frac{1}{2} \int_{\mathbb{R}^N} w_n^2 dx - \frac{1}{2s} \int_{\mathbb{R}^N} |w_n|^{2s} dx + o(1), \quad (39)
\]
and
\[
\langle I'(u_n), (u_n) \rangle = \langle I'(u), (u) \rangle + \|w_n\|_{D_s(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} w_n^2 dx - \int_{\mathbb{R}^N} |w_n|^{2s} dx + o(1). \quad (40)
\]

By Brézis-Lieb Lemma in [6], there holds
\[
\int_{\mathbb{R}^N} |w_n|^{2s} dx = \int_{\mathbb{R}^N} |u_n|^{2s} dx - \int_{\mathbb{R}^N} |u|^{2s} dx + o_n(1),
\]
\[
\int_{\mathbb{R}^N} |\sigma_n|^{2s} dx = \int_{\mathbb{R}^N} |v_n|^{2s} dx - \int_{\mathbb{R}^N} |v|^{2s} dx + o_n(1),
\]
\[
\int_{\mathbb{R}^N} |w_n|^2 dx = \int_{\mathbb{R}^N} |u_n|^2 dx - \int_{\mathbb{R}^N} |u|^2 dx + o_n(1),
\]
\[
\int_{\mathbb{R}^N} |\sigma_n|^2 dx = \int_{\mathbb{R}^N} |v_n|^2 dx - \int_{\mathbb{R}^N} |v|^2 dx + o_n(1),
\]
\[
\int_{\mathbb{R}^N} |u_n|^{p+1} dx = \int_{\mathbb{R}^N} |w_n|^{p+1} dx + \int_{\mathbb{R}^N} |u|^{p+1} dx + o_n(1)
\]
\[= \int_{\mathbb{R}^N} |u|^{p+1} dx + o_n(1) \quad \text{as } \int_{\mathbb{R}^N} |w_n|^{p+1} dx \to 0,
\]
\[
\int_{\mathbb{R}^N} |v_n|^{p+1} dx = \int_{\mathbb{R}^N} |\sigma_n|^{p+1} dx + \int_{\mathbb{R}^N} |v|^{p+1} dx + o_n(1)
\]
\[= \int_{\mathbb{R}^N} |v|^{p+1} dx + o_n(1) \quad \text{as } \int_{\mathbb{R}^N} |\sigma_n|^{p+1} dx \to 0.
\]

It is easy to see that
\[
||w_n||^2_{D_s(\mathbb{R}^N)} = ||u_n||^2_{D_s(\mathbb{R}^N)} - ||u||^2_{D_s(\mathbb{R}^N)} + o_n(1).
\]

Thus, by above equalities, we obtain the identity (39) and (40). If $||w_n||^2_{D_s(\mathbb{R}^N)} \to 0$ as $n \to +\infty$, then $u_n \to u$ strongly in $H^s_\gamma(\mathbb{R}^N)$.

Assume $||w_n||^2_{D_s(\mathbb{R}^N)} = l > 0$ as $n \to +\infty$, then by (39), (40) and the results in Lemma 3.5 for $\mu = 1$, we have
\[
c = I(u) + \frac{s}{N} \left[ ||w_n||^2_{D_s(\mathbb{R}^N)} + \int_{\mathbb{R}^N} |w_n|^2 dx \right] + o_n(1)
\]
\[\geq \frac{s}{N} \left[ ||w_n||^2_{D_s(\mathbb{R}^N)} + \int_{\mathbb{R}^N} |w_n|^2 dx \right] + o_n(1).
\]
By (40) and the definition of $S_s$, we have

$$S_s \leq \frac{||w_n||^2_{D_s(\mathbb{R}^N)} + \int_{\mathbb{R}^N} w_n^2 dx}{(\int_{\mathbb{R}^N} |w_n|^{2^*_s} dx)^{\frac{1}{2^*_s}}} \leq \left[||w_n||^2_{D_s(\mathbb{R}^N)} + \int_{\mathbb{R}^N} w_n^2 dx\right]^\frac{1}{2^*_s}.$$  

Thus,

$$c \geq \frac{s}{N} S_s^N.$$  

This contradicts the results in Lemma 3.5 for $\mu = 1$. Therefore

$$u_n \to u$$ strongly in $H^s_\ast(\mathbb{R}^N).$$

As a result, $u$ is a critical point of $I(u)$ and

$$I(u) = c$$ and $I'(u) = 0.$

Thus, $u \neq 0$ is a nontrivial solution.

Next, we prove Theorem 2.4. We divide the proof into two steps. First, we prove the existence of ground state solutions for system (10), then we claim there exists a positive ground state solution.

By Lemma 3.1, there exists a sequence $\{(u_n, v_n)\} \subset \mathcal{H}_r$ such that

$$E(u_n, v_n) \to a_1$$ and $E'(u_n, v_n) \to 0$ as $n \to +\infty.$  

(41)

We claim $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}_r$. For $n$ large enough, by Sobolev’s imbedding theorem, we have

$$a_1 + o(1)(||u_n, v_n||) = E(u_n, v_n) - \frac{1}{p + 1} (E'(u_n, v_n), (u_n, v_n))$$

$$\geq \left(\frac{1}{2} - \frac{1}{p + 1}\right) \left[||u_n, v_n||^2_{D_r} + \int_{\mathbb{R}^N} (u_n^2 + v_n^2) dx\right]$$

$$\geq \left(\frac{1}{2} - \frac{1}{p + 1}\right) ||u, v||^2_{\mathcal{H}_r}.$$  

Consequently, $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}_r$. Since $a_1 > 0$, $\{(u_n, v_n)\}$ is not $(0,0)$ for $n$ large. So, by the Sobolev imbedding theorem, there exists $(u, v) \in \mathcal{H}_r$ such that

$$\begin{cases} (u_n, v_n) \rightharpoonup (u, v), \quad \text{weakly in} \quad \mathcal{H}_r, \\ (u_n, v_n) \to (u, v), \quad \text{strongly in} \quad L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N), \text{for } 2 < p < 2^*_s, \\ (u_n, v_n) \to (u, v), \quad \text{a.e.} \quad \mathbb{R}^N. \end{cases}$$

Then, we have

$$\langle E'(u, v), (u, v) \rangle = 0.$$  

(42)

Let $w_n = u_n - u$ and $\sigma_n = v_n - v$. We claim

$$E(u_n, v_n) = \int_{\mathbb{R}^N} (\mu_1 |w_n|^{2^*_s} + \mu_2 |\sigma_n|^{2^*_s} + \gamma |w_n|^\alpha |\sigma_n|^\beta) dx + o(1),$$

(43)

and

$$\langle E'(u_n, v_n), (u_n, v_n) \rangle = \langle E'(u, v), (u, v) \rangle + ||(w_n, \sigma_n)||^2_{D_r} + \int_{\mathbb{R}^N} (w_n^2 + \sigma_n^2) dx$$

$$- \int_{\mathbb{R}^N} (\mu_1 |w_n|^{2^*_s} + \mu_2 |\sigma_n|^{2^*_s} + \gamma |w_n|^\alpha |\sigma_n|^\beta) dx + o(1).$$

(44)
By means of the well-known Brézis-Lieb lemma, there holds
\[
\int_{\mathbb{R}^N} |w_n|^2 \, dx = \int_{\mathbb{R}^N} |u_n|^2 \, dx - \int_{\mathbb{R}^N} |u|^2 \, dx + o(1),
\]
\[
\int_{\mathbb{R}^N} |\sigma_n|^2 \, dx = \int_{\mathbb{R}^N} |v_n|^2 \, dx - \int_{\mathbb{R}^N} |v|^2 \, dx + o(1),
\]
\[
\int_{\mathbb{R}^N} |w_n|^2 \, dx = \int_{\mathbb{R}^N} |u_n|^2 \, dx - \int_{\mathbb{R}^N} |u|^2 \, dx + o(1),
\]
\[
\int_{\mathbb{R}^N} |\sigma_n|^2 \, dx = \int_{\mathbb{R}^N} |v_n|^2 \, dx - \int_{\mathbb{R}^N} |v|^2 \, dx + o(1),
\]
\[
\int_{\mathbb{R}^N} |u_n|^{p+1} \, dx = \int_{\mathbb{R}^N} |w_n|^{p+1} \, dx + \int_{\mathbb{R}^N} |u|^{p+1} \, dx + o(1)
\]
\[
= \int_{\mathbb{R}^N} |u|^{p+1} \, dx + o(1) \quad \text{as} \quad \int_{\mathbb{R}^N} |w_n|^{p+1} \, dx \to 0,
\]
\[
\int_{\mathbb{R}^N} |v_n|^{p+1} \, dx = \int_{\mathbb{R}^N} |\sigma_n|^{p+1} \, dx + \int_{\mathbb{R}^N} |v|^{p+1} \, dx + o(1)
\]
\[
= \int_{\mathbb{R}^N} |v|^{p+1} \, dx + o(1) \quad \text{as} \quad \int_{\mathbb{R}^N} |\sigma_n|^{p+1} \, dx \to 0.
\]
Since
\[
\|(w_n, \sigma_n)\|^2_{\mathcal{D}_r} = \|(w_n, v_n)\|^2_{\mathcal{D}_r} - \|(u, v)\|^2_{\mathcal{D}_r} + o(1),
\]
and from [19, Lemma 2.1], we also get
\[
\int_{\mathbb{R}^N} |w_n|^\alpha |\sigma_n|^\beta \, dx = \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \, dx - \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \, dx + o(1). \tag{45}
\]
Thus, by the above equalities, we obtain the identities (43) and (44).

If \( \|(w_n, \sigma_n)\|^2_{\mathcal{D}_r} \to 0 \) as \( n \to +\infty \), then \((u_n, v_n) \to (u, v)\) strongly in \( \mathcal{H}_r \).

Assume that \( \|(w_n, \sigma_n)\|^2_{\mathcal{D}_r} = l > 0 \) as \( n \to +\infty \), then by (43), (44) and Lemma 3.2, we have
\[
a = E(u, v) + \frac{s}{N} \left[ \|(w_n, \sigma_n)\|^2_{\mathcal{D}_r} + \int_{\mathbb{R}^N} (|w_n|^2 + |\sigma_n|^2) \, dx \right] + o(1)
\]
\[
\geq \frac{s}{N} \left[ \|(w_n, \sigma_n)\|^2_{\mathcal{D}_r} + \int_{\mathbb{R}^N} (|w_n|^2 + |\sigma_n|^2) \, dx \right] + o(1).
\]

By (44) and the definition of \( S_\gamma \), we have
\[
S_\gamma \leq \frac{\|(w_n, \sigma_n)\|^2_{\mathcal{D}_r} + \int_{\mathbb{R}^N} (w_n^2 + \sigma_n^2) \, dx}{\left( \int_{\mathbb{R}^N} (\mu_1 |w_n|^2 + \mu_2 |\sigma_n|^2 + \gamma |w_n|^\alpha |\sigma_n|^\beta) \, dx \right)^{\frac{1}{2}}}
\]
\[
\leq \left[ \|(w_n, \sigma_n)\|^2_{\mathcal{D}_r} + \int_{\mathbb{R}^N} (w_n^2 + \sigma_n^2) \, dx \right]^{\frac{2}{N}}.
\]

Thus,
\[
a \geq \frac{s}{N} S_\gamma^{\frac{N}{2}}.
\]

This contradicts Lemma 3.4. Therefore, we get
\( (u_n, v_n) \to (u, v) \) strongly in \( \mathcal{H}_r \).

As a result, \((u, v)\) is a critical point of \( E(u, v) \) and by Lemma 3.2, we have
\[ E(u, v) = a \quad \text{and} \quad E'(u, v) = 0. \]
Thus, \((u, v)\) is a ground state solution. Next, we show that the ground state solution is not the type of \((u, 0)\) and \((0, v)\).

Assume by contradiction that \((u, 0)\) is a ground state solution of (10), then we prove Theorems 2.1–2.2.

5. Proof of Theorems 2.1–2.2. In this section, we first give some existence results for the algebra equation (8), then we prove Theorems 2.1–2.2.

Proposition 2. (i) If \(2s < N < 4s\), then for any \(a, b > 0\), (8) has a solution \(\lambda\) and \((u, \lambda)\) is a solution of (6).

(ii) If \(N = 4s\), then there exists \(b_0 = \frac{1}{\int_{\mathbb{R}^N} |(-\Delta)^s u|^2 dx} > 0\), such that, for any \(a > 0\), then (8) has a solution when \(b < b_0\), and (8) has no solution when \(b \geq b_0\).
(iii) If \( N > 4s \), then there exists \( \alpha_0 = \frac{N-4s}{N-2s} \left( \frac{2s}{(N-2s) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx} \right)^{\frac{2s}{N-2s}} > 0 \), such that (8) has two solution \( \lambda_1 \) and \( \lambda_2 \) if \( ab\frac{2s}{N-2s} < \alpha_0 \), (8) has a unique solution \( \lambda_u \) if \( ab\frac{2s}{N-2s} = \alpha_0 \) and (8) has no solution if \( ab\frac{2s}{N-2s} > \alpha_0 \).

Proof. We define a function

\[ f_u(\lambda) = \lambda^s - a - b\lambda^{\frac{N-2s}{2}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx, \quad \forall \lambda \in \mathbb{R}^+; \]

**Case 1:** \( 2s < N < 4s \). By the definition of \( f_u(\lambda) \), we know that \( \lim_{\lambda \to +\infty} f_u(\lambda) \to +\infty \) for any \( a, b > 0 \). Since \( f_u(\lambda) < 0 \) for \( \lambda \in (0, a^\frac{1}{2}) \), there exists \( \lambda > a^\frac{1}{2} \) such that \( f_u(\lambda) = 0 \). Thus, \( (u, \lambda) \) is a solution of (6).

**Case 2:** \( N = 4s \). Since \( f_u(\lambda) = (1-b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx)\lambda^s - a \), let \( b_0 = \frac{1}{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx} \).

If \( b < b_0 \), then \( \lambda = \left( \frac{\frac{a}{1-b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx}}{b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx} \right)^{\frac{1}{2}} \) is a solution of \( f_u(\lambda) = 0 \) and \( (u, \lambda) \) is a solution of (6). If \( b \geq b_0 \), then (6) has nontrivial solution.

**Case 3:** \( N > 4s \). It is easy to know that \( \lim_{\lambda \to +\infty} f_u(\lambda) \to -\infty \) for any \( a, b > 0 \) and \( f_u(\lambda) < 0 \) for any \( \lambda \in (0, a^\frac{1}{2}) \), for each \( u \neq 0 \), we have

\[ \frac{df_u(\lambda)}{d\lambda} = \lambda^{s-1} \left( 8\lambda^s - b N - \frac{2s}{2} \lambda^{\frac{N-2s}{2}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right). \]

It is easy to see that \( f_u(\lambda) \) has a unique maximum point

\[ \lambda_u = \left( \frac{2s}{b(N-2s) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx} \right)^{\frac{2}{N-2s}} > 0 \]

and

\[ \max_{\lambda \in \mathbb{R}^+} f_u(\lambda) = f_u(\lambda_u) = \frac{N-4s}{N-2s} \left( \frac{2s}{b(N-2s) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx} \right)^{\frac{2}{N-2s}} - a. \]

Let

\[ \alpha_0 = \frac{N-4s}{N-2s} \left( \frac{2s}{(N-2s) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx} \right)^{\frac{2}{N-2s}}. \]

If \( ab\frac{2s}{N-2s} < \alpha_0 \), there exists \( \lambda_1 \in (0, \lambda_u) \), \( \lambda_2 \in (\lambda_u, +\infty) \), such that \( (u, \lambda_1) \), \( (u, \lambda_2) \) solve (6). If \( ab\frac{2s}{N-2s} = \alpha_0 \), then \( (u, \lambda_u) \) solve (6). If \( ab\frac{2s}{N-2s} > \alpha_0 \), then (6) has nontrivial solution.

For the degenerate case, i.e. \( a = 0 \), we give the following existence results.

**Proposition 3.**

(i) If \( 2s < N < 4s \), or \( N > 4s \), then for any \( b > 0 \), (8) has a solution.

(ii) If \( N = 4s \), there exists \( b_0 = \frac{1}{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx} > 0 \), such that (8) has a solution when \( b = b_0 \), and (8) has no solution when \( b \neq b_0 \).

Proof. We consider a function

\[ f_u(\lambda) = \lambda^s - b\lambda^{\frac{N-2s}{2}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx, \quad \forall \lambda \in \mathbb{R}^+; \]

**Case 1:** \( 2s < N < 4s \). By the definition of \( f_u(\lambda) \), it is easy to see that for any \( b > 0 \), there exists \( \lambda = (b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx)^{\frac{2s}{N-2s}} \) such that \( f_u(\lambda) = 0 \). Thus, \( (u, \lambda) \)
is a solution of (6).

**Case 2:** \( N = 4s \). Since \( f_u(\lambda) = (1 - b \int_{\mathbb{R}^N} |(-\Delta)u|^2 dx)\lambda^s \), let \( b_1 = \frac{1}{\int_{\mathbb{R}^N} |(-\Delta)u|^2 dx} \), if \( b = b_0 \), then for any \( \lambda > 0 \), \( f_u(\lambda) = 0 \), which implies that (6) has infinite nontrivial solutions. If \( b \neq b_0 \), then (6) has nontrivial solution.

**Case 3:** \( N > 4s \). By the definition of \( f_u(\lambda) \), it is easy to see that for any \( b > 0 \), there exists \( \lambda = \left( \frac{1}{b \int_{\mathbb{R}^N} |(-\Delta)u|^2 dx} \right)^{\frac{s}{2-s}} \) such that \( f_u(\lambda) = 0 \). Thus, \((u, \lambda)\) is a solution of (6). \( \square \)

**Proof of Theorems 2.5–2.6.** Theorems 2.1–2.2 can be derived by Proposition 2, Lemma 3.6 and Proposition 3. \( \square \)

6. **Proof of Theorems 2.5–2.6.** Firstly, we give the existence results for algebra system (11).

**Proposition 4.** (i) If \( 2s < N < 4s \), for any \( a, b > 0 \), there exists \( \lambda_1 < a^\frac{s}{2-s}, \lambda_2 > a^\frac{s}{2-s} \), then (11) has a solution \((\lambda_1, \lambda_2)\).

(ii) If \( N = 4s \), there exists \( b_1^* = \frac{1}{\int_{\mathbb{R}^N} |(-\Delta)^2 u|^2 dx} > 0, b_2^* = \frac{1}{\int_{\mathbb{R}^N} |(-\Delta)^2 v|^2 dx} > 0 \) such that, for any \( a_i > 0, i = 1, 2 \), then (11) has a solution when \( b_1 < b_1^* \) and \( b_2 < b_2^* \). (11) has no solution when \( b_1 \geq b_1^* \) or \( b_2 \geq b_2^* \).

(iii) If \( N > 4s \), there exists \( \alpha_1 \) and \( \alpha_2 \) defined in (12) and (13) respectively, such that

- if \( a_1 b_1^{\frac{2s}{2-s}} < \alpha_1 \) and \( a_2 b_2^{\frac{2s}{2-s}} < \alpha_2 \), then (11) has four solutions \((\lambda_1^1, \lambda_2^1), (\lambda_1^2, \lambda_2^2), (\lambda_1^3, \lambda_2^3), (\lambda_1^4, \lambda_2^4)\);
- if \( a_1 b_1^{\frac{2s}{2-s}} = \alpha_1 \) and \( a_2 b_2^{\frac{2s}{2-s}} < \alpha_2 \), then (11) has two solutions \((\lambda_1^1, \lambda_2^1)\) and \((\lambda_1^2, \lambda_2^2)\);
- if \( a_1 b_1^{\frac{2s}{2-s}} < \alpha_1 \) and \( a_2 b_2^{\frac{2s}{2-s}} = \alpha_2 \), then (11) has two solutions \((\lambda_1^1, \lambda_2^1)\) and \((\lambda_1^2, \lambda_2^2)\);
- if \( a_1 b_1^{\frac{2s}{2-s}} = \alpha_1 \) and \( a_2 b_2^{\frac{2s}{2-s}} = \alpha_2 \), then (11) has unique solution \((\lambda_1^1, \lambda_2^1)\);
- if \( a_1 b_1^{\frac{2s}{2-s}} > \alpha_1 \) or \( a_2 b_2^{\frac{2s}{2-s}} > \alpha_2 \), then (11) has no solution.

**Proof.** We define two functions \( f_u(\lambda_1), g_v(\lambda_2) \) as following

\[
\begin{align*}
f_u(\lambda_1) &= \lambda_1^s - a_1 - b_1 \lambda_1^{\frac{s}{2-s}} \int_{\mathbb{R}^N} |(-\Delta)^2 u|^2 dx, \quad \forall \lambda_1 \in \mathbb{R}^+, \\
g_v(\lambda_2) &= \lambda_2^s - a_2 - b_2 \lambda_2^{\frac{s}{2-s}} \int_{\mathbb{R}^N} |(-\Delta)^2 v|^2 dx, \quad \forall \lambda_2 \in \mathbb{R}^+,
\end{align*}
\]

**Case 1:** \( 2s < N < 4s \). By the definition of \( f_u(\lambda_1), g_v(\lambda_2) \), we know that

\[
\lim_{\lambda_1 \to +\infty} f_u(\lambda_1) \to +\infty, \text{ for any } a_1, b_1 > 0, \quad \lim_{\lambda_2 \to +\infty} g_v(\lambda_2) \to +\infty \text{ for any } a_2, b_2 > 0.
\]

Since \( f_u(\lambda_1) < 0 \) for any \( \lambda_1 \in (0, a_1^\frac{1}{s}) \), \( g_v(\lambda_2) < 0 \) for any \( \lambda_2 \in (0, a_2^\frac{1}{s}) \), so by the intermediate value theorem there exists \( \lambda_1 > a_1^\frac{1}{s}, \lambda_2 > a_2^\frac{1}{s} \) such that \( f_u(\lambda_1) = 0 \) and \( g_v(\lambda_2) = 0 \), Thus, (11) has a solution \((\lambda_1, \lambda_2)\).
Then we make the following discussions:

and  

Therefore, we can deduce that 

g(\lambda_2) = (1 - b_2 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 dx) \lambda_2^s - a_2,

let  

\[ b_1^* = \frac{1}{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx} > 0, \quad b_2^* = \frac{1}{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 dx} > 0, \]

we know that if \( b_1 < b_1^* \), then \( \lambda_1 = \left( \frac{a_1}{1 - b_1 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx} \right)^{\frac{1}{2}} \) is the unique solution of \( f_u(\lambda_1) = 0 \) and if \( b_2 < b_2^* \), then \( \lambda_2 = \left( \frac{a_2}{1 - b_2 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 dx} \right)^{\frac{1}{2}} \) is the unique solution of \( g_v(\lambda_2) = 0 \). Meanwhile, we know that \( f_u(\lambda_1) = 0 \) has no solution if \( b_1 \geq b_1^* \) and \( g_v(\lambda_2) = 0 \) has no solution if \( b_2 \geq b_2^* \). Thus (11) has unique solution when \( b_1 < b_1^* \) and \( b_2 < b_2^* \), (11) has no solution when \( b_1 \geq b_1^* \) or \( b_2 \geq b_2^* \).

**Case 3:** \( N > 4s \). It is easy to know that

\[
\lim_{\lambda_1 \to +\infty} f_u(\lambda_1) \to -\infty \quad \text{for any } \lambda_1 \in (0, a_1^{\frac{1}{N}}), \quad \lim_{\lambda_2 \to +\infty} g_v(\lambda_2) \to -\infty \quad \text{for any } \lambda_2 \in (0, a_2^{\frac{1}{N}}).
\]

For each \( u \neq 0, \ v \neq 0 \), we have

\[
\frac{df_u(\lambda_1)}{d\lambda_1} = \lambda_1^{-1} \left[ sl_1^s - b_1 \frac{N - 2s}{2} \lambda_1^\frac{N - 2s}{N - s} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right], \quad \frac{dg_v(\lambda_2)}{d\lambda_2} = \lambda_2^{-1} \left[ sl_2^s - b_2 \frac{N - 2s}{2} \lambda_2^\frac{N - 2s}{N - s} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 dx \right].
\]

Therefore, we can deduce that \( f_u(\lambda_1) \) has a unique maximum point

\[
\lambda_{1,u} = \left( \frac{2s}{b_1(N - 2s) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx} \right)^{\frac{N - 2s}{N - s}} > 0,
\]

and \( g_v(\lambda_2) \) has a unique maximum point

\[
\lambda_{2,v} = \left( \frac{2s}{b_2(N - 2s) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 dx} \right)^{\frac{N - 2s}{N - s}} > 0.
\]

Meanwhile,

\[
\max_{\lambda_1 \in \mathbb{R}^+} f_u(\lambda_1) = f_u(\lambda_{1,u}) = \frac{N - 4s}{N - 2s} \left( \frac{2s}{b_1(N - 2s) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx} \right)^{\frac{2s}{N - s}} - a_1 = \alpha_1 b_1^{-\frac{2s}{N - s}} - a_1.
\]

\[
\max_{\lambda_2 \in \mathbb{R}^+} g_v(\lambda_2) = g_v(\lambda_{2,v}) = \frac{N - 4s}{N - 2s} \left( \frac{2s}{b_2(N - 2s) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 dx} \right)^{\frac{2s}{N - s}} - a_2 = \alpha_2 b_2^{-\frac{2s}{N - s}} - a_2.
\]

Then we make the following discussions:
Proof. Proposition 5.

If \( a_1 b_1^{2s/N} < \alpha_1 \), there exists \( \lambda_1^1 \in (0, \lambda_{1,u}) \), \( \lambda_1^2 \in (\lambda_{1,u}, +\infty) \), such that \( f_\alpha(\lambda_1^1) = 0 \) and \( f_\alpha(\lambda_1^2) = 0 \). If \( a_1 b_1^{2s/N} = \alpha_1 \), then \( f_\alpha(\lambda_{1,u}) = 0 \). If \( a_1 b_1^{2s/N} > \alpha_1 \), then \( f_\alpha(\lambda_1) = 0 \) has no solutions.

If \( a_2 b_2^{2s/N} < \alpha_2 \), there exists \( \lambda_2^1 \in (0, \lambda_{2,v}) \), \( \lambda_2^2 \in (\lambda_{2,v}, +\infty) \), such that \( g_\alpha(\lambda_2^1) = 0 \) and \( g_\alpha(\lambda_2^2) = 0 \). If \( a_2 b_2^{2s/N} = \alpha_2 \), then \( g_\alpha(\lambda_{2,v}) = 0 \). If \( a_2 b_2^{2s/N} > \alpha_1 \), then \( g_\alpha(\lambda_2) = 0 \) has no solutions.

Therefore, for the case \( N > 4s \), there exists \( \alpha_1 > 0 \), \( \alpha_2 > 0 \) such that

- if \( a_1 b_1^{2s/N} < \alpha_1 \) and \( a_2 b_2^{2s/N} < \alpha_2 \), then \( (11) \) has four solutions \( (\lambda_1^1, \lambda_2^1), (\lambda_1^1, \lambda_2^2), (\lambda_1^2, \lambda_2^1) \) and \( (\lambda_1^2, \lambda_2^2) \);
- if \( a_1 b_1^{2s/N} = \alpha_1 \) and \( a_2 b_2^{2s/N} < \alpha_2 \), then \( (11) \) has two solutions \( (\lambda_{1,u}, \lambda_2^1) \) and \( (\lambda_{1,u}, \lambda_2^2) \);
- if \( a_1 b_1^{2s/N} < \alpha_1 \) and \( a_2 b_2^{2s/N} = \alpha_2 \), then \( (11) \) has two solutions \( (\lambda_1^1, \lambda_{2,v}) \) and \( (\lambda_1^2, \lambda_{2,v}) \);
- if \( a_1 b_1^{2s/N} = \alpha_1 \) and \( a_2 b_2^{2s/N} = \alpha_2 \), then \( (11) \) has unique solution \( (\lambda_{1,u}, \lambda_{2,v}) \);
- if \( a_1 b_1^{2s/N} > \alpha_1 \) or \( a_2 b_2^{2s/N} > \alpha_2 \), then \( (11) \) has no solution.

The proof is thus complete. \( \square \)

Similarly, from the proof of Proposition 3, we have the following Proposition 5.

**Proposition 5.**

(i) If \( 2s < N < 4s \), or \( N > 4s \), then for any \( b_i > 0 \), \( i = 1, 2 \), let \( \lambda_1 = (b_1 \int_{\mathbb{R}^N} |(-\Delta)u|^2 dx)^{\frac{2}{N-2s}} \), \( \lambda_2 = (b_2 \int_{\mathbb{R}^N} |(-\Delta)v|^2 dx)^{\frac{2}{N-2s}} \), then \( (\lambda_1, \lambda_2) \) is a solution of \( (11) \).

(ii) If \( N = 4s \), then there exists \( b* = \frac{1}{\int_{\mathbb{R}^N} |(-\Delta)^*u|^2 dx} > 0 \), \( b* = \frac{1}{\int_{\mathbb{R}^N} |(-\Delta)^*v|^2 dx} > 0 \) such that \( (11) \) has a solution when \( b_1 = b_1^* \) and \( b_2 = b_2^* \), and \( (11) \) has no solution when \( b_1 \neq b_1^* \) or \( b_2 \neq b_2^* \).

**Proof.** We consider two functions \( f_u(\lambda_1), g_\alpha(\lambda_2) \) as following

\[
\begin{align*}
    f_u(\lambda_1) &= \lambda_1^N - b_1 \lambda_1^{N-2s} \int_{\mathbb{R}^N} |(-\Delta)^*u|^2 dx, \quad \forall \lambda_1 \in \mathbb{R}^+, \\
    g_\alpha(\lambda_2) &= \lambda_2^N - b_2 \lambda_2^{N-2s} \int_{\mathbb{R}^N} |(-\Delta)^*v|^2 dx, \quad \forall \lambda_2 \in \mathbb{R}^+,
\end{align*}
\]

**Case 1:** \( 2s < N < 4s \). By the definition of \( f_u(\lambda_1), g_\alpha(\lambda_2) \), it is easy to see that for any \( b_i > 0 \), \( i = 1, 2 \), there exists unique

\[
\lambda_1 = \left( b_1 \int_{\mathbb{R}^N} |(-\Delta)^*u|^2 dx \right)^{\frac{2}{N-2s}}, \quad \lambda_2 = \left( b_2 \int_{\mathbb{R}^N} |(-\Delta)^*v|^2 dx \right)^{\frac{2}{N-2s}},
\]

such that

\[
    f_u(\lambda_1) = 0, \quad g_\alpha(\lambda_2) = 0.
\]

Thus, \( (\lambda_1, \lambda_2) \) is a solution of \( (11) \) for the degenerate case, i.e. \( a_i = 0, i = 1, 2 \).

**Case 2:** \( N = 4s \). Consider

\[
    f_u(\lambda_1) = (1 - b_1 \int_{\mathbb{R}^N} |(-\Delta)^*u|^2 dx) \lambda_1^s, \quad g_\alpha(\lambda_2) = (1 - b_2 \int_{\mathbb{R}^N} |(-\Delta)^*v|^2 dx) \lambda_2^s.
\]

Let

\[
    b_1^* = \frac{1}{\int_{\mathbb{R}^N} |(-\Delta)^*u|^2 dx}, \quad b_2^* = \frac{1}{\int_{\mathbb{R}^N} |(-\Delta)^*v|^2 dx},
\]

...
if $b_1 = b'_1$, then for any $\lambda_1 > 0$, $f_u(\lambda_1) = 0$, if $b_2 = b'_2$, then for any $\lambda_2 > 0$, $g_v(\lambda_2) = 0$, if $b_1 \neq b'_1$, then $f_u(\lambda_1) = 0$ has no solutions, if $b_2 \neq b'_2$, then $g_v(\lambda_2) = 0$ has no solutions. Thus (11) has solutions when $b_1 = b'_1$ and $b_2 = b'_2$, (11) has no solution when $b_1 \neq b'_1$ or $b_2 \neq b'_2$.

**Case 3:** $N > 4s$. By the definition of $f_u(\lambda_1)$, $g_v(\lambda_2)$, it is easy to see that for any $b > 0$, there exists $\lambda_1 = \left( \frac{1}{b f_{\mathbb{R}^N}(\lambda_1)} \right)^{\frac{2}{2-4s}}$, $\lambda_2 = \left( \frac{1}{b_2 f_{\mathbb{R}^N}(\lambda_2)} \right)^{\frac{2}{2-4s}}$, such that $f_u(\lambda_1) = 0$, $g_v(\lambda_2) = 0$.  

**Proof of Theorems 2.5–2.6.** Theorems 2.5–2.6 can be easily derived from Propositions 4–5, Lemma 3.6 and Theorem 2.4.

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