On Diff($S^1$) Covariantization Of
Pseudodifferential Operator

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Abstract

A study of diff($S^1$) covariant properties of pseudodifferential operator of integer degree is presented. First, it is shown that the action of diff($S^1$) defines a hamiltonian flow defined by the second Gelfand-Dickey bracket if and only if the pseudodifferential operator transforms covariantly. Secondly, the covariant form of a pseudodifferential operator of degree $n \neq 0, \pm 1$ is constructed by exploiting the inverse of covariant derivative. This, in particular, implies the existence of primary basis for $W_{KP}^{(n)} (n \neq 0, \pm 1)$. 
I. Introduction

It has been known that the Lax formulation [1-3] of integrable systems provides a very useful method to construct W-type algebras. For example, the so called $W_0$ algebra[4] is nothing but the second Gelfand-Dickey bracket associated with the differential operator[5]

$$\Lambda_n = \partial^n + u_2 \partial^{n-1} + \ldots + u_n$$  \hspace{1cm} (1.1)

Recently, the study of the hamiltonian structures of the KP hierarchy[6-9] leads to the consideration of the Gelfand-Dickey brackets associated with the pseudodifferential operators of the form[10-13]

$$L_n = \partial^n + u_2 \partial^{n-1} + \ldots + u_p \partial^{n-p} + \ldots$$ \hspace{1cm} (1.2)

It is now known that the second Gelfand-Dickey bracket indeed defines a hamiltonian structure provided $n \neq 0$. Interestingly, it is possible to generalize further to the case when $n$ is a complex number $q$. This generalization leads to a large class of W-type algebras called $W^{(q)}_K\text{P}$[14].

The importance of these W-type algebras comes from the fact that they all contain a Virasoro subalgebra and a set of generators of spin higher than 2. For instance, for the $W_0$ algebra, which is associated with the differential operator (1.1), $u_2$ is the Virasoro generator and there is a differential algebra automorphism $\{u_j\}_{j \geq 3} \rightarrow \{w_j\}_{j \geq 3}$ such that $w_j$ is a primary generator of spin $j$[15-17]. The proof of the last statement relies on the equivalence of the diff($S^1$)-covariance and the Virasoro hamiltonian flow defined by the second Gelfand-Dickey bracket and on the possibility of covariantizing the differential operator. The set of generators $\{u_2, w_3, \ldots, w_n\}$ is called the primary basis for $W_n$. It is believed that, for $n \neq 0, \pm 1$, a primary basis for $W^{(n)}_K\text{P}$ should also exist[14,18]. However, to the best of our knowledge, an explicit proof is still lacking. It is the purpose of this paper to explicitly covariantize the pseudodifferential operators and thus provides a proof of the existence of primary basis.
We organize this paper as follows. In Sec. II, we collect basic definitions and concepts. In Sec. III, we prove the equivalence of the dif(S^1)-covariance and the Virasoro hamiltonian flow. In Sec. IV, the covariantization of the pseudodifferential operators is carried out. In Sec. V, the algebra W_{KP}^{(n)} with n ≥ 2 is studied and, in particular, the first few primary generators of spin higher than n are explicitly defined. Finally, we present discussions and conclusions in Sec. VI.

II. Basic Definitions

We consider the differential operator \( \partial = \frac{\partial}{\partial x} \). The powers of this operator satisfy

\[
\begin{align*}
\partial^0 &= 1 \\
\partial^i \partial^j &= \partial^{i+j} \\
\partial^i f &= \sum_{k=0}^{\infty} \binom{i}{k} f^{(k)} \partial^{i-k}
\end{align*}
\]  

(2.1)

where \( f^{(k)} \) is the k-th derivative of the function \( f(x) \) and \( \binom{i}{k} = \frac{i(i-1)\ldots(i-k+1)}{k!} \)

\[
\begin{align*}
\binom{i}{0} &= 1 \\
\binom{i}{k} &= \frac{i(i-1)\ldots(i-k+1)}{k!}
\end{align*}
\]  

(2.2)

Quite often we include a subscript (e.g. \( \partial_t = \frac{\partial}{\partial t} \)) to emphasize the particular choice of variable. Given a pseudodifferential operator

\[
A = \sum_{k=-\infty}^{n} a_k \partial^k
\]  

(2.3)

we define, respectively, its differential and integral parts as

\[
A_+ = \sum_{k=0}^{n} a_k \partial^k, \quad A_- = \sum_{-\infty}^{-1} a_k \partial^k
\]  

(2.4)

The residue and the trace of \( A \) are, respectively,

\[
\text{Res}(A) = a_{-1}
\]  

(2.5)
and
\[ Tr(A) = \int Res(A)dx = \int a_{-1}dx \] (2.6)

For the pseudodifferential operator given by (1.2), the hamiltonian flow, defined by the second Gelfand-Dickey bracket, generated by the functional
\[ F[u_i] = \int \epsilon_i(x)u_i(x)dx \] (2.7)

has the following operator form[3]
\[ \delta^{GD}_{\epsilon_i}L_n = (L_nV)_+L_n - L_n(VL_n)_+ \] (2.8)

where
\[ V = \partial^{-n+i-1}\epsilon_i(x) + \partial^{-n}q(x) \] (2.9)

with \( q(x) \) satisfying
\[ Res[L_n,V] = 0 \] (2.10)

(2.10) is the consistency condition arising from the constraint \( u_1 = 0 \). For reasons that will become clear later when \( i=2 \) (2.8) is called Virasoro hamiltonian flow.

We shall study how the coefficient functions \( u_i \)'s transform under diffeomorphism \( x \rightarrow t \). The following terminologies are essential. A function \( f \) is called a primary of spin \( h \) if, under \( x \rightarrow t \), it transforms as
\[ f(t) = \left( \frac{dx}{dt} \right)^h f(t) \] (2.11)

We denote by \( F_h \) the space of all primaries of spin \( h \). A pseudodifferential operator \( \Delta \) is called a covariant operator if it maps from \( F_h \) to \( F_l \) for some \( h \) and \( l \). Symbolically, we denote
\[ \Delta : F_h \rightarrow F_l \] (2.12)

We shall see later that when \( \Delta = L_n \) its form fixes \( h \) and \( l \) simultaneously. It is not hard to see that (2.12) is equivalent to
\[ \Delta(t) = \phi^{-l}\Delta(x)\phi^h \] (2.13)
where $\phi(x) = \frac{dt}{dx}$.

We end this section by listing several elementary but useful properties of primaries and covariant operators.

(i) If $f \in F_h$ and $g \in F_l$, then $fg \in F_{h+l}$.

(ii) If $\Delta_1 : F_h \to F_k$ and $\Delta_2 : F_k \to F_l$, then $\Delta_2 \Delta_1 : F_h \to F_l$.

(iii) If $\Delta : F_h \to F_l$ and its inverse $\Delta^{-1}$ exists, then $\Delta^{-1} : F_l \to F_h$

The proof of (iii) is simple. We just note that inverting (2.13) gives

$$\Delta^{-1}(t) = \phi^{-h} \Delta^{-1}(x) \phi^l$$ \hspace{1cm} (2.14)

III. Diff($S^1$) Covariance and Virasoro Flow

Now we assume that the pseudodifferential operator $L_n$ can be covariantized, i.e.

$$L_n : F_h \to F_l \hspace{1cm} (3.1)$$

or, equivalently,

$$L_n(t) = \phi^{-l} L_n(x) \phi^h \hspace{1cm} (3.2)$$

for some $h$ and $l$. Quite obviously, not all values of $h$ and $l$ are possible. To determine these values, let us recall that $u_1(t)$ and $u_1(x)$ must both vanish. However, the relation

$$\partial_t = \phi^{-1}(x) \partial_x \hspace{1cm} (3.3)$$

suggests that the leading term $\partial^n_t$ could possibly contribute to the coefficient function $u_1(x)$. Indeed, for $n > 0$

$$\partial^n_t \phi^{-h} = \phi^{-1} \partial_x \ldots \phi^{-1} \partial_x \phi^{-h}$$

$$= \phi^{-(h+n)} \partial^n_x - [h + (h + 1) + \ldots + (h + n - 1)] \phi^{-(h+n+2)} \partial_x^{n-1} + \ldots \hspace{1cm} (3.4)$$

Hence, the covariance condition (3.2) requires

$$h + (h + 1) + \ldots + (h + n - 1) = nh + \frac{n(n+1)}{2} = 0 \hspace{1cm} (3.5)$$
and

\[ l = h + n \]  \hspace{1cm} (3.6)

It follows\[17]\]

\[ h = -\frac{n-1}{2}, \quad l = \frac{n+1}{2} \]  \hspace{1cm} (3.7)

For \( n < 0 \), the uses of

\[ \partial_t^{-1} = \partial_x^{-1} \phi \]  \hspace{1cm} (3.8)

also lead to the same conclusion. In other words, (3.7) is valid for all nonzero \( n \).

After imposing (3.7) the covariance condition (3.2) reads

\[ L_n(t) \phi_{n}^{\frac{n-1}{2}} = \phi_{\frac{n+1}{2}} L_n(x) \]  \hspace{1cm} (3.9)

which determines unambiguously how the coefficient functions \( u_i \)'s transform under diffeomorphisms. The transformation of \( u_2 \) is of particular importance as we shall see. Expanding the first two terms on the left hand side of (3.9) gives

\[ \partial_t^n \phi_{\frac{n-1}{2}} = \phi_{-\frac{n+1}{2}} \left[ \partial_x^n - \frac{n(n^2 - 1)}{12} \phi^2 \{\{x, t\}\} \partial_x^{n-2} \right] + \ldots \]  \hspace{1cm} (3.10)

and

\[ u_2(t) \partial_t^{n-2} \phi_{\frac{n-1}{2}} = \phi_{-\frac{n+3}{2}} u_2(x) \partial_x^{n-2} + \ldots \]  \hspace{1cm} (3.11)

where the schwartzian derivative \( \{\{x, t\}\} \) is defined as

\[ \{\{x, t\}\} = \frac{d^3 x}{dt^3} - \frac{3}{2} \left( \frac{d^2 x}{dt^2} \right)^2 \]  \hspace{1cm} (3.12)

Equating both sides of (3.9) now yields

\[ u_2(t) = u_2(x) \left( \frac{dx}{dt} \right)^2 + c_n \{\{x, t\}\} \]  \hspace{1cm} (3.13)

where

\[ c_n = \frac{n(n^2 - 1)}{12} \]  \hspace{1cm} (3.14)
We therefore see that \( u_2 \) does not transform as a primary of spin 2 but has an “anomalous” term proportional to the schwartzian derivative. Since \( u_2 \) has the same transformation law under diffeomorphisms as the energy-momentum tensor in the conformal field theory, we call it the Virasoro generator. We shall see in the next section that it is the presence of the anomalous term that enables us to decompose the functions \( u_i \)’s into primaries.

The transformation laws for other coefficient functions can be worked out in the same spirit. But the calculations are too tedious to be carried out here. However, when we consider the infinitesimal form of diffeomorphisms the corresponding transformation laws are quite manageable. Let

\[
t = x - \epsilon(x)
\]

then within linear approximation

\[
\phi(x) = \frac{dt}{dx} = 1 - \epsilon'(x)
\]

From (3.16) it is quite easy to show that (3.3) and (3.8) are equivalent to

\[
\partial^\pm_1 = \partial^\pm_1 + [\partial^\pm_1, \epsilon] \partial_x \tag{3.17}
\]

As a matter of fact, we can show easily by induction that (3.17) leads to

\[
\partial^i_t = \partial^i_x + [\partial^i_x, \epsilon] \partial_x \tag{3.18}
\]

for any integer \( i \). Since

\[
u_i(t) = u_i(x - \epsilon(x)) + \delta \epsilon u_i(x) \tag{3.19}
\]

we have

\[
L_n(t) = L_n(x - \epsilon(x)[\partial_x, L_n(x)] + [L_n(x), \epsilon(x)] \partial_x + \delta \epsilon L_n(x) \tag{3.20}
\]

On the other hand,

\[
(1 - \epsilon'(x))^{\frac{n+1}{2}} L_n(x)(1 - \epsilon'(x))^{-\frac{n-1}{2}} = L_n(x) + \frac{n+1}{2} \epsilon'(x)L_n(x) + \frac{n-1}{2} L_n(x) \epsilon'(x) \tag{3.21}
\]
Hence, (3.9), (3.20) and (3.21) together imply

$$\delta \epsilon L_n(x) = \frac{n+1}{2} \epsilon'(x) L_n(x) + \frac{n-1}{2} L_n(x) \epsilon'(x) + \epsilon(x)[\partial_x, L_n(x)] - [L_n(x), \epsilon(x)] \partial_x \quad (3.22)$$

(3.22) is a consequence of the covariance condition, which summaries the transformation laws of $u_i$'s in the infinitesimal form.

Now we like to show that the transformation law (3.22) is nothing but the hamiltonian flow (2.8) generated by the functional

$$F_2[u_2] = \int \epsilon(x) u_2(x) dx \quad (3.23)$$

First, we solve (2.10) for the above functional. The result is

$$V = \partial_x^n \left( \frac{n-1}{2} \right) \epsilon'(x) + \partial_x^{n+1} \epsilon(x) \quad (3.24)$$

Simple algebras then give

$$(L_n V)_+ = \frac{n+1}{2} \epsilon'(x) + \epsilon(x) \partial_x$$

$$(V L_n)_+ = -\frac{n-1}{2} \epsilon'(x) + \epsilon(x) \partial_x \quad (3.25)$$

Substituting (3.25) into (2.8) we obtain

$$\delta^{GD}_\epsilon L_n = \left[ -\frac{n+1}{2} \epsilon'(x) + \epsilon(x) \partial_x \right] L_n(x) - L_n(x) \left[ -\frac{n-1}{2} \epsilon'(x) + \epsilon(x) \partial_x \right]$$

$$= \frac{n+1}{2} \epsilon'(x) L_n(x) + \frac{n-1}{2} L_n(x) \epsilon'(x) + \epsilon(x) \partial_x L_n(x) - L_n(x) \epsilon(x) \partial_x \quad (3.26)$$

$$= \frac{n+1}{2} \epsilon'(x) L_n(x) + \frac{n-1}{2} L_n(x) \epsilon'(x) + \epsilon(x) [\partial_x, L_n(x)] + [\epsilon(x), L_n(x)] \partial_x$$

Note that (3.26) completely agrees with (3.22). We therefore have proved that the infinitesimal form of the diff($S^1$) covariance of $L_n$ is equivalent to the hamiltonian flow, defined by the second Gelfand-Dickey bracket, generated by the functional given by (3.23).

From (3.26) we deduce with the help of (3.13)

$$\delta^{GD}_\epsilon u_2(x) = \int \{u_2(x), u_2(y)\}^{GD}_\epsilon \epsilon(y) dy$$

$$= u'_2(x) \epsilon(x) + 2u_2(x) \epsilon'(x) + c_n \epsilon'''(x) \quad (3.27)$$
or, equivalently,

\[ \{u_2(x), u_2(y)\}_{GD} = [u'(x) + 2u_2(x)\partial_x + c_n\partial_x^3]\delta(x - y) \quad (3.28) \]

(3.28) is the well-known Virasoro algebra. Because of (3.28) we have called the hamiltonian flow generated by the functional (3.23) the Virasoro hamiltonian flow or simply Virasoro flow.

Before ending this section we simply remark that the above proof of equivalence applies without any change to the case when \( L_n \) is just a differential operator.

IV. Covariantization of \( L_n \)

In this section we shall covariantize the pseudodifferential operator \( L_n \). The key ingredient is the concept of covariant derivative. Given a diffeomorphism \( x \rightarrow v(x) \), we define

\[ b(x) = \frac{d^2 v}{dx^2} \quad (4.1) \]

It is a simple matter to check that under the diffeomorphism \( x \rightarrow t \) this function transforms as

\[ b(t) = b(x)\left(\frac{dx}{dt}\right) + \frac{d^2 x}{dt^2}\left(\frac{dx}{dt}\right)^{-1} \quad (4.2) \]

One recognizes immediately that (4.2) is the transformation law for an anomalous spin-1 primary. With the function \( b \) we can define the covariant derivative as

\[ D_k = \partial_x - kb(x) \quad (4.3) \]

For later uses, we also define

\[ D^l_k = D_{k+l-1}D_{k+l-2} \ldots D_k \quad (l \geq 0) \quad (4.4) \]

Using (4.2) and (3.3) we can easily show that under diffeomorphism

\[ D^l_k(t) = \phi^{-k-l}D^l_k(x)\phi^k \quad (4.5) \]
where, as usual $\phi = \frac{dt}{dx}$. (4.5) means that the operator $D^l_k$ maps from $F_k$ to $F_{k+l}$. In other words, (4.3) and (4.4) really define a series of covariant operators in the sense of the definitions in Sec. II. These covariant derivatives have been used in ref.[17] to covariantize the differential operators. Since we are dealing with pseudodifferential operators, we need something more. Obviously, what we need are the inverses of these covariant operators, which have the following expressions

$$D^{-1}_k \equiv (\partial_x - kb)^{-1} = \partial_x^{-1} + k\partial_x^{-1}b\partial_x^{-1} + k^2\partial_x^{-1}b\partial_x^{-1}b\partial_x^{-1} + \ldots$$

$$D^{-l}_{k+l} \equiv (D^l_k)^{-1} = D^{-1}_{k+1}D^{-1}_{k+2} \ldots D^{-1}_{k+l} \quad (l \geq 1)$$

(4.6)

One should note that the subscript always denotes the spin of the domain space. From Property (iii) in Sec. II it follows that these inverses are again covariant derivatives.

For a given $L_n$ we shall choose $v(x)$ such that in this particular coordinate[17]

$$u_2(v) = 0$$

(4.7)

By (3.13) it is equivalent to choosing $b$ to satisfy

$$b'(x) - \frac{1}{2}b(x)^2 = \langle \{v, x\} \rangle = \frac{u_2(x)}{c_n}$$

(4.8)

Clearly, such a choice is possible only when $c_n$ does not vanish. Hence, we shall restrict ourselves to the cases $n \neq 0, \pm 1$. Even when such a $b$ exists it is not unique. We can replace $b$ by $b + \delta b$ as long as the schwartzian derivative $\langle \{v, x\} \rangle$ is kept fixed. It amounts to requiring

$$\delta b' = b\delta b,$$

which has two more useful equivalent forms

$$[\partial_x - (k + 1)b]\delta b = \delta b(\partial_x - kb)$$

(4.9)

and

$$[\partial_x - (k + 1)b]^{-1}\delta b = \delta b(\partial_x - kb)^{-1}$$

(4.10)
When $\delta b$ is constrained by (4.9) or (4.10) we can easily derive the following useful formula

$$\delta_b D^l_k = -\frac{l(l+2k-1)}{2} \delta b D^{l-1}_k \quad l = 0, \pm 1, \pm 2, \ldots$$ (4.11)

From the covariance condition (3.9) it can be easily seen that the differential part and the integral part of $L_n$ transform independently; i.e.

$$(L_n)_{\pm}(t) = \phi^{-\frac{n+1}{2}}(L_n)_{\pm}(x)\phi^{-\frac{n-1}{2}}$$ (4.12)

Thus, both parts should be covariantized separately. The covariantization of $(L_n)_{+}(n \geq 2)$ has been done in ref. [17] with the result

$$(L_n)_{+} = \Delta_2^{(n)}(u_2) + \sum_{k=3}^{n} \Delta_k^{(n)}(w_k, u_2)$$ (4.13)

where $w_k$ is a primary of spin $k$,

$$\Delta_2^{(n)}(u_2) = D^{n-\frac{n+1}{2}} = (\partial_x - \frac{n-1}{2} b)(\partial_x - \frac{n-3}{2} b) \ldots (\partial_x + \frac{n-1}{2} b)$$ (4.14)

$$\Delta_k^{(n)}(w_k, u_2) = \sum_{l=0}^{n-k} a^{(n)}_{k,l} [D^l_k w_k] D^{n-k-l-\frac{n-1}{2}}$$ (4.15)

and

$$a^{(n)}_{k,l} = \frac{\binom{k+l-1}{l} \binom{n-k}{l} \binom{2k+l-1}{l}}{\binom{k}{l}}$$ (4.16)

The coefficients given by (4.16) are determined by requiring $\Delta_k^{(n)}$’s to depend on $b$ through the schwartzian derivative; i.e.

$$\delta_b \Delta_k^{(n)} = 0$$ (4.17)

when $\delta b$ is subjected to (4.9). To covariantize $(L_n)_{-}$, we naturally consider the covariant pseudodifferential operators

$$\Delta_{n+k}^{(n)}(w_k, u_2) = \sum_{l=0}^{\infty} a^{(n)}_{n+k,l} [D^{l}_{n+k} w_{n+k}] D^{n-k-l-\frac{n+1}{2}} \quad k \geq 1$$ (4.18)
where \( a_{n+k,0}^{(n)} = 1 \). Requiring the dependence on \( b \) only through the schwartzian derivative now leads to the recursion relation

\[
a_{n+k,l}^{(n)} = -\frac{(k + l - 1)(n + k + l - 1)}{l(2n + 2k + l - 1)} a_{n+k,l-1}^{(n)} \quad (4.19)
\]

The solution to (4.19) is

\[
a_{n+k,l}^{(n)} = (-1)^l \frac{(k + l - 1)}{l} \frac{(n + k + l - 1)}{l} \frac{2n + 2k + l - 1}{l} \quad (4.20)
\]

We have therefore obtained the desired covariant form

\[
(L_n)_- = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} a_{n+k,l}^{(n)} [D_{n+k}^l w_{n+k}] D_{-\frac{n-1}{2}}^{-k-l} \quad (4.21)
\]

Working out explicitly the right hand side of (4.21) will yield the decompositions of the form

\[
u_{n+k} = w_{n+k} + G_{n+k}(w_{n+k-1}, \ldots, w_{n+1}, u_2) \quad (4.22)
\]

where \( G_{n+k} \) is a differential polynomial in \( u_2, w_{n+1}, \ldots, w_{n+k-1} \) and their derivatives. Inverting (4.22) gives the definitions of primaries in terms of coefficient functions

\[
w_{n+k} = u_{n+k} + H_{n+k}(u_{n+k-1}, \ldots, u_{n+1}, u_2) \quad (4.23)
\]

where \( H_{n+k} \) is again a differential polynomial. We thus conclude that the primaries of spin higher than or equal to \( n+1 \) can be defined from the coefficient functions. Combining this with the result of Sec. III we deduce that a primary basis for the algebra \( W_{KP}^{(n)}(n \geq 2) \) indeed exists.

Next, we consider \( L_{-n} \) with \( n \geq 2 \). Following the same steps we obtain

\[
L_{-n} = \Delta_2^{(-n)}(u_2) + \sum_{k=3}^{\infty} \Delta_k^{(-n)}(w_k, u_2), \quad (4.24)
\]
where
\[ \Delta_2(-n)(u_2) = \frac{1}{\partial_{\nu}^{-n}} = D_{n+1}^{-n} \] (4.25)
\[ \Delta_k(-n)(w_k, u_2) = \sum_{k=3}^{\infty} a_{k,l}^{(-n)} [D_k w_k] D_{n+1}^{-n-k-l} \] (4.26)
and
\[ a_{k,l}^{(-n)} = (-1)^l \left( \frac{k + n + l - 1}{l} \right) \left( \frac{k + l - 1}{l} \right) \left( \frac{2k + l - 1}{l} \right) \] (k ≥ 3) (4.27)

The conclusion is the same. For \( n \geq 2 \), \( L_{-n} \) can be covariantized and therefore the algebra \( W_{K,F}^{(-n)} \) has a primary basis.

Several remarks are in order. First, one sees that (4.16), (4.20) and (4.27) can be summarized by a single formula
\[ a_{k,l}^{(n)} = (-1)^l \left( \frac{k - n + l + 1}{l} \right) \left( \frac{k + l - 1}{l} \right) \left( \frac{2k + l - 1}{l} \right) \] (k ≥ 3, l ≥ 0) (4.28)

Note, in particular, that \( a_{k,l}^{(n)} \) given by (4.28) vanishes when \( n \geq k \geq 3 \) and \( l > n - k \). This is consistent with the fact that the right hand side of (4.15) is just a finite summation.

Secondly, using
\[ D_{p+q}^k = \sum_{l=0}^{\infty} \left( \frac{k}{l} \right) [D_p^l w_p] D_q^{k-l} \] (4.29)
which is the covariant analogue of (2.1), we observe that for \( |n| \geq 2 \)
\[ \Delta_2^{(n)}(u_2) \Delta_k^{(-n)}(w_k, -u_2) \Delta_2^{(n)}(u_2) = \sum_{l=0}^{\infty} b_{k,l}^{(n)} [D_k^l w_k] D_{n+1}^{n-k-l} \] (4.30)
where
\[ b_{k,l}^{(n)} = \sum_{p=0}^{\infty} \left( \frac{n}{p} \right) a_{k,l-p}^{(n)} \quad (a_{k,l}^{(n)} = 0 \text{ if } l < 0) \] (4.31)

One should notice that the sign of \( u_2 \) in the expression of \( \Delta_k^{(-n)} \) has been reversed in order to assure that the same central charge \( c_n \) is used in (4.8). Since \( b_{k}^{(n)} = a_{k}^{(n)} = 1 \) and since
the right hand side of (4.30) must depend on \( b \) only through the schwartzian derivative as the left hand side does, we deduce

\[
a_{k,l}^{(n)} = b_{k,l}^{(n)} = \sum_{p=0}^{\infty} \binom{n}{p} a_{k,l-p}^{(-n)}
\]

and

\[
\Delta_{k}^{(n)}(w_k, u_2) = \Delta_{2}^{(n)}(u_2) \Delta_{k}^{(-n)}(w_k, -u_2) \Delta_{2}^{(n)}(u_2)
\]

(4.32) is not a obvious relation. We did explicitly verify it for a few simple cases even though we have not devised a direct proof. Finally, we like to remark that the covariant operator \( \Delta_{k}^{(n)} \) has been chosen to depend linearly on \( w_k \) and its derivative. In general, we can add terms which are multilinear in \( w_j \)’s and their derivatives. Quite clearly, adding

\[
\sum_{l_1, \ldots, l_p=0}^{\infty} a_{k_1, \ldots, k_p;l_1, \ldots, l_p}^{(n)} [D_{k_1}^{l_1} w_{k_1}] \cdots [D_{k_p}^{l_p} w_{k_p}] D_{n-k_1-\ldots-k_p-l_1-\ldots-l_p}^{n-1}
\]

where \( k_i \geq 3 \) and \( k_1 + \ldots + k_p = k \), to \( \Delta_{k}^{(n)} \) and choosing the coefficients properly would just change the form of the differential polynomials in (4.22) and (4.23). However, as one can see easily that the choice of the set of coefficients is not unique now, we will not discuss any further. Nevertheless, we will come back to this point when we try to explicitly decompose \( u_k \)’s into a differential polynomial in primaries and their derivatives in the next section.

V. Primaries in \( W_{k+2}^{(n)} \)

In this section we shall use the Virasoro hamiltonian flow (3.26) to decompose explicitly, for \( n \geq 2 \), the coefficient functions \( u_{n+1}, \ldots, u_{n+4} \) into differential polynomials in \( w_{n+1}, \ldots, w_{n+4} \) and \( u_2 \). There are two purposes for these calculations. First, we recall that when \( n > 0 \) the coefficient functions \( \{ u_j \}_{j \geq n+1} \) generate a diff(\( S^1 \)) submodule. However, (4.23) shows that in defining the primaries \( w_{n+k} \)’s we must introduce \( u_2 \) which belongs to the diff(\( S^1 \)) submodule generated by \( \{ u_2, \ldots, u_n \} \). Hence, it is worth while to do the decompositions directly from the Virasoro hamiltonian flow in order to see why \( u_2 \) must
appear. Secondly, these explicit expressions will be compared with those following from (4.21) to serve as an independent verification of our results.

With some straightforward algebras we obtain from (3.26)

\[
\delta \epsilon u_{n+p} = \epsilon u'_{n+p} + (n+p)\epsilon' u_{n+p} + \sum_{k=1}^{p-1} \left[ \frac{n-1}{2} \left( \frac{-k}{p-k} \right) - \left( \frac{-k}{p-k+1} \right) \right] \epsilon^{(p-k+1)} u_{n+k} \tag{5.1}
\]

The first four transformation laws from (5.1) are

\[
\begin{align*}
\delta \epsilon u_{n+1} &= \epsilon u'_{n+1} + (n+1)\epsilon' u_{n+1} \tag{5.2} \\
\delta \epsilon u_{n+2} &= \epsilon u'_{n+2} + (n+2)\epsilon' u_{n+2} - \frac{n+1}{2} \epsilon'' u_{n+1} \tag{5.3} \\
\delta \epsilon u_{n+3} &= \epsilon u'_{n+3} + (n+3)\epsilon' u_{n+3} - (n+2)\epsilon'' u_{n+2} + \frac{n+1}{2} \epsilon''' u_{n+1} \tag{5.4} \\
\delta \epsilon u_{n+4} &= \epsilon u'_{n+4} + (n+4)\epsilon' u_{n+4} - \frac{3(n+3)}{2} \epsilon'' u_{n+3} + \frac{(3n+5)}{2} \epsilon''' u_{n+2} \\
&\quad - \frac{n+1}{2} \epsilon''' u_{n+1} \tag{5.5}
\end{align*}
\]

It is an immediate consequence of (5.2) that \( u_{n+1} \) is a primary of spin \( n+1 \). We thus write

\[
u_{n+1} = w_{n+1} \tag{5.6}
\]

Next, we define

\[
w_{n+2} = u_{n+2} + \alpha_2 w'_{n+1} \tag{5.7}
\]

Using (5.2), (5.3) and (5.6) we find

\[
\begin{align*}
\delta \epsilon w_{n+2} &= \epsilon w'_{n+2} + (n+2)\epsilon' w_{n+2} + (\alpha_2 - \frac{1}{2})(n+1)\epsilon'' w_{n+1} \tag{5.8}
\end{align*}
\]

To make \( w_{n+2} \) a primary we must set

\[
\alpha_2 = \frac{1}{2}, \tag{5.9}
\]

which then gives the decomposition

\[
u_{n+2} = w_{n+2} - \frac{1}{2} w'_{n+1} \tag{5.10}
\]
Now let
\[ w_{n+3} = u_{n+3} + \alpha_3 w_{n+2}' + \beta_3 w_{n+1}'' \]  
(5.11)
The transformation law for \( w_{n+3} \) reads
\[
\delta_\epsilon w_{n+3} = \epsilon w_{n+3}' + (n + 3)\epsilon' w_{n+3} + (\alpha_3 - 1)(n + 2)\epsilon'' w_{n+2} \\
+ [\beta_3(2n + 3) + \frac{1}{2}(n + 2)]\epsilon'' w_{n+1}' + (\beta_3 + \frac{1}{2})(n + 1)\epsilon''' w_{n+1} 
\]  
(5.12)
Clearly, no choice of \( \alpha_3 \) and \( \beta_3 \) can simultaneously make the last three terms on the right hand side of (5.12) vanish. In other words, the primary \( w_{n+3} \) can not be defined as a differential polynomial in \( u_{n+1}, u_{n+2} \) and \( u_{n+3} \). However, we observe that \( \epsilon''' \) is proportional to the anomalous term in \( \delta_\epsilon u_2 \). This suggests a way out. We choose \( \alpha_3 \) and \( \beta_3 \) to make the first two terms vanish and then add a term \( \gamma_3(u_2 w_{n+1}) \) to the right hand side of (5.11) to take care of the last term. Indeed, from the transformation law
\[
\delta_\epsilon(u_2 w_{n+1}) = \epsilon(u_2 w_{n+1})' + (n + 3)\epsilon'(u_2 w_{n+1}) + c_n\epsilon''' w_{n+1} 
\]  
(5.13)
we see that the addition of the new term would simply add \( c_n\gamma_3\epsilon''' w_{n+1} \) to the right hand side of (5.12). Hence, the choice
\[
\alpha_3 = 1, \quad \beta_3 = -\frac{n + 2}{2(2n + 3)}, \quad \gamma_3 = -\frac{(n + 1)^2}{2c_n(2n + 3)} 
\]  
(5.14)
makes \( w_{n+3} \) a real primary and thus the decomposition of \( u_{n+3} \) is
\[
u_{n+3} = w_{n+3} - w_{n+2}' + \frac{n + 2}{2(2n + 3)}w_{n+1}'' + \frac{(n + 1)^2}{2c_n(2n + 3)}u_2 w_{n+1} 
\]  
(5.15)
This analysis shows that when \( n = 1 \) the primary \( w_{n+3} = w_4 \) can not be defined due to the fact that \( c_n = c_1 = 0 \) (equivalently, \( u_2 \) does not transform anomalously)[18]. It amounts to saying that the algebra \( W_{KP}^{(1)} \), which the second hamiltonian structure of the KP hierarchy, has no primary basis.

Finally, we consider
\[
w_{n+4} = u_{n+4} + \alpha_4 w_{n+3}' + \beta_4 w_{n+2}'' + \gamma_4 w_{n+1}''' + \mu_4 u_2 w_{n+2} + \\
\xi_4 u_2 w_{n+1}' + \eta_4 u_2 w_{n+1}' 
\]  
(5.16)
whose transformation law reads

\[
\delta \epsilon_{w_{n+4}} = \epsilon' w_{n+4} + (n+4) \epsilon' w_{n+4} + \epsilon''(A w_{n+3} + B w'_{n+2} + C w''_{n+1} + D u_2 w_{n+1})
\]

\[
\epsilon'''(E w_{n+2} + F w'_{n+1}) + G \epsilon''' w_{n+1}
\]

where

\[
A = (\alpha_4 - \frac{3}{2})(n+3)
\]

\[
B = (2n + 5)\beta_4 + \frac{3(n+3)}{2}
\]

\[
C = (3n + 6)\gamma_4 - \frac{3(n+2)(n+3)}{4(2n+3)}
\]

\[
D = 2\eta_4 + (n+1)\xi_4 - \frac{3(n+3)(n+1)^2}{4cn(2n+3)}
\]

\[
E = (n+2)\beta_4 + c_n \mu_4 + \frac{3n+5}{2}
\]

\[
F = (3n+4)\gamma_4 + c_n \xi_4 - \frac{3n+5}{4}
\]

\[
G = (n+1)\gamma_4 + c_n \eta_4 - \frac{(n+1)}{2}
\]

Demanding \(A = B = C = D = E = F = G = 0\) gives seven equations for only six unknowns. If we drop \(D = 0\) for a moment and solve the other six equations, we get

\[
\alpha_4 = \frac{3}{2}, \quad \beta_4 = -\frac{3(n+3)}{2(2n+5)}, \quad \gamma_4 = \frac{n+3}{4(2n+3)}
\]

\[
\mu_4 = -\frac{(n+1)(3n+7)}{2cn(2n+5)},
\]

\[
\eta_4 = \xi_4 = \frac{3(n+1)^2}{4cn(2n+3)}
\]

Substituting (5.18) back into (5.17.5) we find that \(D = 0\) is indeed satisfied. Therefore, with the coefficients given by (5.18), (5.16) defines a primary. The decomposition for \(u_{n+4}\) is then

\[
u_{n+4} = w_{n+4} - \frac{3}{2} w'_{n+3} + \frac{3(n+3)}{2(2n+5)} w''_{n+2} - \frac{n+3}{4(2n+3)} w'''_{n+1}
\]

\[
+ \frac{(n+1)(3n+7)}{2cn(2n+5)} u_2 w'_{n+2} - \frac{3(n+1)^2}{4cn(2n+3)} (u_2 w_{n+1})'
\]

We should know that the decomposition (5.19) is by no means unique. A redefinition like

\[
w_{n+4} \rightarrow w_{n+4} + w_3 w_{n+1}
\]
is certainly allowed. Redefinitions of this sort amount to introducing terms bilinear in \( w_i \)'s and their derivatives in the decompositions. As we remarked in Sec. IV., more generally, terms which are multilinear in \( w_i \)'s and their derivatives can be introduced. \( n + 4 \) is the lowest spin where the arbitrariness of this type shows up.

Now we like to compare the above explicit decomposition formulae with those resulting form (4.21). The needed formulae are

\[
D_k w_k = w'_k - kbw_k \\
D_k^2 w_k = w''_k - (2k + 1)bw'_k - kb'_w + k(k + 1)b^2 w_k \\
D_k^3 w_k = w'''_k - 3kbw'' - (3k + 1)b'w'_k + (13k^2 + 6k + 2)b^2 w'_k \\
- kb''w_k + k(3k + 4)bb'w_k - k(k + 1)(k + 2)b^3 w_k
\]

and

\[
D_k^{-1} = \partial_x^{-1} + (k - 1)b\partial_x^{-2} - [(k - 1)b' - (k - 1)^2 b^2]\partial_x^{-3} \\
+ [(k - 1)b'' - 3(k - 1)^2 b b' + (k - 1)^3 b^3] \partial_x^{-4} + \ldots \\
D_k^{-2} = \partial_x^{-2} + (2k - 1)b\partial_x^{-3} + [-(3k - 4)b' + (3k^2 - 9k + 7)b^2] \partial_x^{-4} + \ldots \\
D_k^{-3} = \partial_x^{-3} + 3(k - 2)b\partial_x^{-4} + \ldots \\
D_k^{-4} = \partial_x^{-4} + \ldots
\]

After some algebras we obtain

\[
\Delta_{n+1}^{(n+1)}(w_{n+1}, u_2) = w_{n+1} \partial_x^{-1} - \frac{1}{2} w'_{n+1} \partial_x^{-2} + \left[ \frac{n + 2}{2(2n + 3)} w''_{n+1} + \frac{(n + 1)^2}{2c_n(2n + 3)} u_2 w_{n+1} \right] \partial_x^{-3} \\
+ \left[ - \frac{n + 3}{4(2n + 3)} w'''_{n+1} - \frac{3(n + 1)^2}{4c_n(2n + 3)} (u_2 w_{n+1})' \right] \partial_x^{-4} + \ldots
\]

\[
\Delta_{n+2}^{(n)}(w_{n+2}, u_2) = w_{n+2} \partial_x^{-2} + \left[ \frac{3(n + 3)}{2(2n + 5)} w''_{n+2} + \frac{(n + 1)(3n + 7)}{2c_n(2n + 5)} u_2 w_{n+2} \right] \partial_x^{-4} + \ldots
\]

\[
\Delta_{n+3}^{(n)}(w_{n+3}, u_2) = w_{n+3} \partial_x^{-3} - \frac{3}{2} w'_{n+2} \partial_x^{-4} + \ldots
\]

\[
\Delta_{n+4}^{(n)}(w_{n+4}, u_2) = w_{n+4} \partial_x^{-4} + \ldots
\]

It is a simple matter to check that (5.23) completely agrees with (5.6), (5.10), (5.15) and (5.19). This completes our comparison.
VI. Discussions and Conclusions

We have shown that when \( n \neq 0, \pm 1 \) the pseudodifferential operators given by (1.2) can be covariantized, that is, the coefficient functions can be decomposed into differential polynomials in primaries and their derivatives. This therefore gives a proof for the existence of primary basis for the corresponding algebra \( W_{KP}^{(n)} \). For \( n \geq 2 \) we have worked out the decompositions of \( \{u_j\}_{n+4 \geq j \geq n+1} \) from their transformation laws under Virasoro hamiltonian flow. The results completely agree with those from \( \text{diff}(S^1) \) covariantization of \( L_n \).

Two possible generalizations of the constructions in this paper are worth mentioning. First, we may consider the pseudodifferential operator of a complex degree \( q \). For such a case it is necessary to define an object like \( D^q_k \), which is a covariant operator with a complex power. Once this is done, an analogous construction then can be carried out. Secondly, we may consider the super-pseudodifferential operators in which the derivative \( \partial_x \) is replaced by the super-derivative \( D = \partial_\theta + \theta \partial_x \) in the (1, 1) superspace and the coefficient functions \( \{u_i\} \) by the superfields \( \{U_i = v_i + \theta w_i\} \). Works in these two directions are now in progress.

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