The geometrical properties of parity and time reversal operators in two dimensional spaces

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Abstract

The parity operator \( P \) and time reversal operator \( T \) are two important operators in the quantum theory, in particular, in the \( PT \)-symmetric quantum theory. By using the concrete forms of \( P \) and \( T \), we discuss their geometrical properties in two dimensional spaces. It is showed that if \( T \) is given, then all \( P \) links with the quadric surfaces; if \( P \) is given, then all \( T \) links with the quadric curves. Moreover, we give out the generalized unbroken \( PT \)-symmetric condition of an operator. The unbroken \( PT \)-symmetry of a Hermitian operator is also showed in this way.

1 Introduction

Quantum theory is one of the most important theories in physics. It is a fundamental axiom in quantum mechanics that the Hamiltonians should be Hermitian, which implies that the values of energy are real numbers. However, non-Hermitian Hamiltonians are also studied in physics. One of the attempts is Bender’s \( PT \)-symmetric theory [1]. In this theory, Bender and his colleagues attributed the reality of the energies to the \( PT \)-symmetric property, where \( P \) is a parity operator and \( T \) is a time reversal operator. Since then, many physicists discussed the properties of \( PT \)-symmetric quantum systems [2]. It also has theoretical applications in quantum optics,

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quantum statistics and quantum field theory. Recently, Bender, Brody and Muller constructed a Hamiltonian operator $H$ with the property that if its eigenfunctions obey a suitable boundary condition, then the associated eigenvalues correspond to the nontrivial zeros of the Riemann zeta function, where $H$ is not Hermitian in the conventional sense, while $iH$ has a broken $\mathcal{PT}$-symmetry. This result may shed light on the new application of $\mathcal{PT}$-symmetric theory in discussing the Riemann hypothesis. It was discovered by Mostafazadeh that the $\mathcal{PT}$-symmetric case can be generalized to a more general pseudo-Hermitian quantum theory, and the generalized $\mathcal{PT}$-symmetry was also discussed. Smith studied the time reversal operator $T$ satisfying that $T^2 = -I$ and the corresponding $\mathcal{PT}$-symmetric quantum theory.

In this paper, by using the concrete forms of $\mathcal{P}$ and $\mathcal{T}$ in two dimensional spaces, we discuss their geometry properties. It is showed that if $\mathcal{T}$ is given, then all $\mathcal{P}$ links with the quadric surfaces; if $\mathcal{P}$ is given, then all $\mathcal{T}$ links with the quadric curves. Moreover, we give out the generalized unbroken $\mathcal{PT}$-symmetric condition of an operator $H$. The unbroken $\mathcal{PT}$-symmetry of a Hermitian operator is also showed in this way.

2 Preliminaries

In this paper, we only consider two dimensional complex Hilbert space $\mathbb{C}^2$. Let $L(\mathbb{C}^2)$ be the complex vector space of all linear operators on $\mathbb{C}^2$, $I$ be the identity operator on $\mathbb{C}^2$, $\bar{z}$ be the complex conjugation of complex number $z$.

An operator $T$ on $\mathbb{C}^2$ is said to be anti-linear if $T(sx_1 + tx_2) = \bar{s}T(x_1) + \bar{t}T(x_2)$. It is obvious that the composition of two anti-linear operators is a linear operator and the composition of an anti-linear operator and a linear operator is still anti-linear. Similar to linear operators, anti-linear operators can also correspond to a matrix with slightly different laws of operation.

A time reversal operator $T$ is an anti-linear operator which satisfies $T^2 = I$ or $T^2 = -I$. A parity operator $P$ is a linear operator which satisfies $P^2 = I$.

The Pauli operators will be used frequently in our discussions. Given the basis $\{e_i\}_{i=1}^2$ of $\mathbb{C}^2$, they are usually defined as follows:

\begin{align}
\sigma_1(x_1e_1 + x_2e_2) &= x_2e_1 + x_1e_2, \\
\sigma_2(x_1e_1 + x_2e_2) &= -ix_2e_1 + ix_1e_2, \\
\sigma_3(x_1e_1 + x_2e_2) &= x_1e_1 - x_2e_2.
\end{align}

To put it another way, the representation matrices of $\sigma_1, \sigma_2$ and $\sigma_3$ are:

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]
Pauli operators have the following useful properties \[12\]:

\[
\sigma_i \sigma_j = -\sigma_j \sigma_i = i \epsilon_{ijk} \sigma_k, \quad i \neq j, \\
\sigma_i^2 = I,
\]

where \(i, j, k \in \{1, 2, 3\}\), \(\epsilon_{ijk}\) is the Levi-Civita symbol:

\[
\epsilon_{ijk} = \begin{cases} 
\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \\
\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1, \\
0, \text{ otherwise.}
\end{cases}
\]

The well known commutation and anti-commutation relations are:

\[
\sigma_i \sigma_j - \sigma_j \sigma_i = 2 i \epsilon_{ijk} \sigma_k, \\
\sigma_i \sigma_j + \sigma_j \sigma_i = 2 \delta_{ij} I,
\]

where \(i, j, k \in \{1, 2, 3\}\) and \(\delta_{ij}\) is the Kronecker symbol.

Denote \(I\) by \(\sigma_0\), then \(\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}\) is a basis of \(L(\mathbb{C}^2)\). Moreover, an operator \(\sigma = t \sigma_0 + x \sigma_1 + y \sigma_2 + z \sigma_3 \in L(\mathbb{C}^2)\) is Hermitian if and only if the coefficients \(\{t, x, y, z\}\) are real numbers.

Given the basis \(\{e_i\}_{i=1}^2\) of \(\mathbb{C}^2\) and any vector \(x = \sum x_i e_i\), one can define an important anti-linear operator, namely the conjugation operator \(T_0\), by \(T_0(x) = \sum \overline{x_i} e_i\).

Similar to \(T_0\), one can define another important anti-linear operator \(\tau_0\) by

\[
\tau_0(x_1 e_1 + x_2 e_2) = -\overline{x_2} e_1 + \overline{x_1} e_2.
\]

Furthermore, define \(\tau_1 = \tau_0 \sigma_1, \tau_2 = \tau_0 \sigma_2, \tau_3 = \tau_0 \sigma_3\), that is, \(\tau_i\) is defined to be the composition of \(\tau_0\) and \(\sigma_i\). The anti-linear operators \(\{\tau_0, \tau_1, \tau_2, \tau_3\}\) forms a basis of the anti-linear operator space of \(\mathbb{C}^2\). This basis has the following properties \[10\]:

\[
\tau_0^2 = -I, \\
\tau_0 \sigma_i = -\sigma_i \tau_0 = \tau_i, \\
\tau_i \tau_0 = -\tau_0 \tau_i = \sigma_i, \\
\tau_i \tau_j = \sigma_i \sigma_j = i \epsilon_{ijk} \sigma_k \quad (i \neq j), \\
\tau_i \tau_j - \tau_j \tau_i = 2 i \epsilon_{ijk} \sigma_k,
\]

where \(i, j \in \{1, 2, 3\}\).

All the equations above can be verified by the using the definitions of Pauli operators and \(\tau_0\). However, for further use, we show that \(\tau_0 \sigma_i = -\sigma_i \tau_0 = \tau_i\) in detail. Consider \(\tau_2 = \tau_0 \sigma_2\). By \eqref{2.2}...
and (2.6), we have

\[
\begin{align*}
\tau_0 \sigma_2 (x_1 e_1 + x_2 e_2) &= ix_1 e_1 + ix_2 e_2, \\
\sigma_2 \tau_0 (x_1 e_1 + x_2 e_2) &= -ix_1 e_1 - ix_2 e_2.
\end{align*}
\]

Thus, \( \tau_0 \sigma_2 = -\sigma_2 \tau_0 = \tau_2 \). Along similar lines, one can verify that \( \tau_0 \sigma_i = -\sigma_i \tau_0 = \tau_i \) is also valid for \( \sigma_1 \) and \( \sigma_3 \).

Moreover, it follows from \( \tau_0 \sigma_i = -\sigma_i \tau_0 = \tau_i \) that \( \sigma_j \tau_i = \sigma_j \tau_0 \sigma_i = -\tau_0 \sigma_j \sigma_i \). Combining with (2.4) and (2.5), one can further obtain the following relations:

\[
\begin{align*}
\sigma_j \tau_i &= \tau_i \sigma_j = -ie_{ijk} \tau_k, \quad i \neq j, \quad (2.7) \\
\tau_i \sigma_i &= -\sigma_i \tau_i = \tau_0, \quad (2.8)
\end{align*}
\]

where \( i, j, k \in \{1, 2, 3\} \).

With the help of \( \{\sigma_i\} \) and \( \{\tau_i\} \), ones can determine the concrete forms of \( P \) and \( T \):

**Lemma 2.1.** Let \( P \) be a parity operator and \( T \) be a time reversal operator on \( \mathbb{C}^2 \). Then

(i). Either \( P = \pm I \) or \( P = \sum_{i=1}^{3} a_i \sigma_i \), where \( a_i \) satisfying \( \sum_{i=1}^{3} a_i^2 = 1 \). The latter case is referred to as the nontrivial \( P \). A nontrivial \( P \) has the following matrix representation:

\[
P = \begin{pmatrix}
a_3 & a_1 - ia_2 \\
a_1 + ia_2 & -a_3
\end{pmatrix}, \quad (2.9)
\]

(ii). \( T = e \sum_{i=0}^{3} c_i \tau_i \), where \( c_i \) are real numbers, if \( T^2 = I \), then \( c_1^2 + c_2^2 + c_3^2 - c_0^2 = 1 \); if \( T^2 = -I \), then \( c_1^2 + c_2^2 + c_3^2 - c_0^2 = -1 \), \( e \) is a unimodular complex number \( [10] \).

**Proof.** (i). Suppose \( P = \sum_{i=0}^{3} a_i \sigma_i \). According to the properties of Pauli operators, we have \( I = P^2 = (\sum_{i=0}^{3} a_i^2) I + 2a_0(a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3) \). Note that \( \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\} \) is a basis of \( L(\mathbb{C}^2) \), we conclude that \( \sum_{i=0}^{3} a_i^2 = 1 \) and \( a_0 a_1 = a_0 a_2 = a_0 a_3 = 0 \). If \( a_0 \neq 0 \), then \( a_1 = a_2 = a_3 = 0 \), which implies that \( P = \pm I \). If \( a_0 = 0 \), then the only constraint is \( \sum_{i=1}^{3} a_i^2 = 1 \) and the matrix takes the form in (2.9).

(ii). The proof can be found in \( [10] \). \( \square \)
Example 1. In (2.9), if we take $a_2 = 0$, $a_1, a_3$ are real numbers satisfying that $a_1^2 + a_3^2 = 1$, and denote $a_1$ by $\sin \alpha$, $a_3$ by $\cos \alpha$, then $P$ has the matrix representation $$
abla \begin{pmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{pmatrix} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ Thus $P$ is composed of a reflection and a rotation.

Example 2. In (2.9), if $a_1 = a_2 = 0$, $a_3 = 1$, then $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If $a_2 = a_3 = 0$, $a_1 = 1$, then $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. These two parity operators were widely used in [2].

3 The existence of $P$ commuting with $T$

In physics, it is demanded that $P$ and $T$ are commutative, that is, $PT = TP$. In finite dimensional spaces case, by using the canonical forms of matrices, one can show that if $T^2 = I$, then such $P$ always exists. In two dimensional case, we can prove it by utilizing Pauli operators.

Theorem 3.1. For each time reversal operator $T$, if $T^2 = I$, then there exists a nontrivial parity operator $P$ such that $PT = TP$. If $T^2 = -I$, then there is no $P$ commuting with $T$ except $P = \pm I$.

Proof. We will use the following well known equation frequently,

$$(\sigma \cdot A)(\sigma \cdot B) = (A \cdot B)I + i\sigma \cdot (A \times B), \quad (3.1)$$

where $A$ and $B$ are two vectors in $\mathbb{C}^3$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. The symbols $\cdot$ and $\times$ represent the dot and cross product of vectors, respectively.

(i). When $T^2 = I$.

Let $T = e^{\sum_{i=0}^3 c_i \tau_i}$ and $P = \sum_{i=1}^3 a_i \sigma_i$, as was given in Lemma 2.1

According to (2.7) and (2.8), $TP = PT$ is equivalent to

$$( -c_0 \sigma_0 + \sum_{j=1}^3 c_j \sigma_j)(\sum_{i=1}^3 a_i \sigma_i)\tau_0 = (\sum_{i=1}^3 a_i \sigma_i)(c_0 \sigma_0 - \sum_{j=1}^3 c_j \sigma_j)\tau_0. $$

Denote $f_i = Re(a_i)$, $b_i = Im(a_i)$, $\vec{f} = (f_1, f_2, f_3)$, $\vec{b} = (b_1, b_2, b_3)$ and $\vec{c} = (c_1, c_2, c_3)$. Utilizing (3.1) to expand the equation above, we have

$$(\vec{f} \cdot \vec{c})\sigma_0 - \sigma \cdot [\vec{b} \times \vec{c} + c_0 \vec{f}] = 0. \quad (3.2)$$
It follows that $\mathcal{T}\mathcal{P} = \mathcal{P}\mathcal{T}$ is equivalent to
\begin{align}
c_0\tilde{f} + \tilde{b} \times \tilde{c} &= 0, \quad (3.3) \\
\tilde{f} \cdot \tilde{c} &= 0. \quad (3.4)
\end{align}

Similarly, by utilizing (3.1) and Lemma 2.1, the contraints $\mathcal{P}^2 = I$ and $\mathcal{T}^2 = I$ can be reduced to the equations as follows,
\begin{align}
\tilde{f} \cdot \tilde{b} &= 0, \quad (3.5) \\
\|\tilde{f}\|^2 - \|\tilde{b}\|^2 &= 1, \quad (3.6) \\
\|\tilde{c}\|^2 - c_0^2 &= 1. \quad (3.7)
\end{align}

Thus, the problem of finding a parity operator $\mathcal{P}$ commuting with $\mathcal{T}$ reduces to finding the vectors $\tilde{f}$ and $\tilde{b}$ satisfying (3.3) – (3.6).

If $c_0 = 0$, then we can choose $\tilde{b} = 0$ and a unit vector $\tilde{f}$ orthogonal to $\tilde{c}$. Thus all the conditions (3.3) – (3.6) are satisfied.

If $c_0 \neq 0$. Let $\tilde{b}$ be a vector such that $\tilde{b}$ is orthogonal to $\tilde{c}$ and $\|\tilde{b}\| = |c_0|$. Moreover, take $\tilde{f} = \frac{1}{c_0}(\tilde{c} \times \tilde{b})$. Direct calculations show that such vectors $\tilde{f}$ and $\tilde{b}$ satisfy (3.3) – (3.6), which completes the proof of the existence of $\mathcal{P}$.

(ii). When $\mathcal{T}^2 = -I$.

The equation (3.7) is replaced by the following:
\[\|\tilde{c}\|^2 - c_0^2 = -1. \quad (3.8)\]

Thus $c_0 \neq 0$. On the other hand, it follows from (3.3) that
\[\tilde{f} = \frac{1}{c_0}(\tilde{c} \times \tilde{b}). \quad (3.9)\]

Substituting (3.8) and (3.9) into (3.6), we have $\|\tilde{f}\|^2 - \|\tilde{b}\|^2 = 1 < -\frac{1}{c_0^2}\|\tilde{b}\|^2$, which is a contradiction. Thus, when $\mathcal{T}^2 = -I$, there is no $\mathcal{P}$ commuting with $\mathcal{T}$ except $\mathcal{P} = \pm I$.

\[\square\]

**Remark 3.1.** When the space is $\mathbb{C}^4$, although $\mathcal{T}^2 = -I$, one can find nontrivial $\mathcal{P}$ commuting with $\mathcal{T}$ [6].
4 The geometrical properties of $\mathcal{P}$ and $\mathcal{T}$

Theorem 4.1. Let $\mathcal{T}$ be a time reversal operator satisfying $\mathcal{T}^2 = I$. The set of parity operators $\mathcal{P}$ commuting with $\mathcal{T}$ correspond uniquely to a hyperboloid in $\mathbb{R}^3$.

Proof. As was mentioned above, the determination of $\mathcal{P}$ is equivalent to finding out $\tilde{f}$ and $\tilde{b}$ satisfying $(3.3) - (3.6)$. Now consider $\tilde{m} = \tilde{f} + \tilde{b}$. We shall prove that all the $\tilde{m}$ form a hyperboloid.

To this end, construct a new coordinate system by taking the direction of $\tilde{c}$ as that of the $X'$ axis. The $Y' - Z'$ plane is perpendicular to $\tilde{c}$ and contains the origin point of $\mathbb{R}^3$. Assume that $\tilde{m} = (x', y', z')$ in the new $X'Y'Z'$ coordinate system.

(i). If $c_0 = 0$, then it follows from $(3.3) - (3.5)$ that $\tilde{b}$ is proportional to $\tilde{c}$ and that $\tilde{f}$ is orthogonal to both $\tilde{c}$ and $\tilde{b}$. Thus, in the new $X'Y'Z'$ coordinate system,

$$\tilde{b} = (x',0,0),$$

$$\tilde{f} = (0,y',z').$$

On the other hand, equation $(3.6)$, namely $\|\tilde{f}\|^2 - \|\tilde{b}\|^2 = 1$, implies that

$$y'^2 + z'^2 - x'^2 = 1. \hspace{1cm} (4.1)$$

It is apparent that one pair of $\tilde{f}$ and $\tilde{b}$ correspond to one point $\tilde{m} = (x', y', z')$, and vice versa. In addition, $(4.1)$ represents a hyperboloid in $\mathbb{R}^3$.

(ii). If $c_0 \neq 0$, then it follows from $(3.7)$ that $\tilde{c} = (\sqrt{1 + c_0^2}, 0, 0)$ in the $X'Y'Z'$ coordinate system. In addition, suppose $\tilde{b} = (x_0, y_0, z_0)$ in the $X'Y'Z'$ coordinate system. By $(3.9)$, we have

$$\tilde{f} = \frac{1}{c_0}(\tilde{c} \times \tilde{b}) = \frac{\sqrt{1 + c_0^2}}{c_0}(0, -z_0, y_0).$$

Substituting $\tilde{b}$ and $\tilde{f}$ into $(3.6)$, we have

$$\frac{1}{c_0^2}(y_0^2 + z_0^2) - x_0^2 = 1. \hspace{1cm} (4.2)$$

Note that $x_0 = x', y_0 = \frac{\lambda x' + y'}{1 + \lambda^2}, z_0 = \frac{z' - \lambda y'}{1 + \lambda^2}$, where $\lambda = \frac{\sqrt{1 + c_0^2}}{c_0}$. Thus, one pair of $\tilde{f}$ and $\tilde{b}$ correspond to one point $\tilde{m} = (x', y', z')$, and vice versa. Moreover, it follows from $(4.2)$ that

$$\frac{1}{1 + 2c_0^2}(y'^2 + z'^2) - x'^2 = 1. \hspace{1cm} (4.3)$$

That is, all the $\tilde{m}$ form a hyperboloid.

\[\Box\]

Theorem 4.2. Let $\mathcal{P}$ be a nontrivial parity operator and let us consider the time reversal operators of the form $\mathcal{T} = \sum_{i=0}^{3} c_i \tau_i$ commuting with $\mathcal{P}$. All the points $\tilde{c} = (c_1, c_2, c_3)$ form an ellipse. The length of the semi-major axis is $\|\tilde{f}\|$ and the length of the semi-minor axis is $1$.
Proof. By (3.4) and (3.5), we know that both $\tilde{b}$ and $\bar{c}$ are orthogonal to $\tilde{f}$.

Construct a new $X'Y'Z'$ coordinate system by taking the direction of $\tilde{f}$ as that of the $Z'$ axis and the direction of $\tilde{b}$ as that of the $X'$ axis (If $\tilde{b} = 0$, take any vector orthogonal to $\tilde{f}$ as the direction vector of the $X'$ axis). Then we have $\tilde{b} = (x, 0, 0)$, $\tilde{f} = (0, 0, z)$ and $\bar{c} = (c'_1, c'_2, c'_3)$ in the $X'Y'Z'$ coordinate system. Now the conditions (3.3) \(\rightarrow\) (3.7) will reduce to

\[
\begin{align*}
x c'_3 &= 0, \quad \text{(4.4)} \\
x c'_2 + c_0 z &= 0, \quad \text{(4.5)} \\
z c'_3 &= 0, \quad \text{(4.6)} \\
z^2 - x^2 &= 1, \quad \text{(4.7)} \\
(c'_1)^2 + (c'_2)^2 + (c'_3)^2 - (c_0)^2 &= 1. \quad \text{(4.8)}
\end{align*}
\]

Note that (4.7) ensures that $z \neq 0$. Thus, (4.4) and (4.6) imply that $c'_3 = 0$, $\bar{c} = (c'_1, c'_2, 0)$. In addition, it follows from (4.5) that $c_0 = -\frac{x}{z} c'_2$. Substituting $c'_3 = 0$, $c_0 = -\frac{x}{z} c'_2$ and (4.7) into (4.8), we have

\[
(c'_1)^2 + \left(\frac{c'_2}{z}\right)^2 = 1. \quad \text{(4.9)}
\]

This is an equation of ellipse. Moreover, since $|z| = \|\tilde{f}\| > 1$, the length of the semi-major axis is $\|\tilde{f}\|$ and the length of the semi-minor axis is 1.

\[\square\]

In the following theorem, we only consider the $T$ with real coefficients.

**Theorem 4.3.** Let $T_1$, $T_2$ be two time reversal operators, $T_1 \neq \pm T_2$. If there exist two nontrivial parity operators $P_1$ and $P_2$ such that $P_i$ commutes with $T_1$ and $T_2$ simultaneously, then $P_1 = \pm P_2$.

**Proof.** Let $T_1 = \sum_{i=0}^{3} c_i^{(1)} T_i$, $T_2 = \sum_{i=0}^{3} c_i^{(2)} T_i$. Denote $\tilde{c}^{(1)} = (c_1^{(1)}, c_2^{(1)}, c_3^{(1)})$ and $\tilde{c}^{(2)} = (c_1^{(2)}, c_2^{(2)}, c_3^{(2)})$.

(i) If $c_0^{(1)} \neq 0$ and $c_0^{(2)} = 0$.

Suppose that $P$ commute with $T_i$ simultaneously. By (3.3), we have $\tilde{c}^{(2)} \times \tilde{b} = 0$. It follows that $\tilde{b} = mt \tilde{c}^{(2)}$. On the other hand, (3.3) implies that $\tilde{f} = \frac{1}{c_0^{(1)}} (\tilde{c}^{(1)} \times \tilde{b})$. Thus, $\tilde{f} = \frac{m}{c_0^{(1)}} (\tilde{c}^{(1)} \times \tilde{c}^{(2)})$.

Substituting $\tilde{f}$ and $\tilde{b}$ into (3.6), then we have

\[
m^2 \left( \frac{1}{c_0^{(1)}} \tilde{c}^{(1)} \times \tilde{c}^{(2)} \right)^2 - \|\tilde{c}^{(2)}\|^2 = 1.
\]

The equation has at most two real roots, which are opposite to each other. Thus, there exist at most two parity operators $P$ and $-P$ commuting with $T_i$ simultaneously.
Thus, we have $\mathcal{C}_k$, hence $t$ where

If $P_T$ respectively. Moreover, suppose that both operators only have two directions, which are opposite to each other. Thus, there exist at most two parity operators $P$ and $-P$ commuting with $T_i$ simultaneously.

(v). If $c_i^{(1)} \neq 0$, $c_i^{(2)} \neq 0$ and $\tilde{c}^{(1)} \neq t\tilde{c}^{(2)}$.

Let $P_1$ and $P_2$ be two parity operators, which are determined by $(f^{(1)}, \tilde{b}^{(1)})$ and $(f^{(2)}, \tilde{b}^{(2)})$ respectively. Moreover, suppose that both $P_1$ and $P_2$ commute with $T_i$ simultaneously.

By (3.3), we have $\tilde{c}^{(1)} \times \tilde{b} = \tilde{c}^{(2)} \times \tilde{b} = 0$. However, since $\tilde{c}^{(1)} \neq t\tilde{c}^{(2)}$, we have $\tilde{b} = 0$. Thus (3.6) implies that $\|\tilde{f}\| = 1$. Moreover, (3.4) implies that $\tilde{f}$ is orthogonal to both $\tilde{c}^{(1)}$ and $\tilde{c}^{(2)}$. So $\tilde{f}$ can only have two directions, which are opposite to each other. Thus, there exist at most two parity operators $P$ and $-P$ commuting with $T_i$ simultaneously.

Note that (i) – (v) contain all the situations, which completes the proof. \[\square\]

If we denote $\text{com}(T) = \{P|P T = T P, P^2 = I\}$, then the following corollary can be obtained.

**Corollary 4.1.** If $\mathcal{T}_1 = \sum_{i=0}^{3} c_i^{(1)} \tau_i$, $\mathcal{T}_2 = \sum_{i=0}^{3} c_i^{(2)} \tau_i$ are two time reversal operators, $T_{ij}^2 = I, j = 1, 2$. Then $\text{com}(\mathcal{T}_1) = \text{com}(\mathcal{T}_2)$ if and only if for each $i$, $c_i^{(1)} = e \epsilon c_i^{(2)}$, where $e$ is a unimodular coefficient.
5 \( PT \)-symmetric operators and unbroken \( PT \)-symmetric condition

A linear operator \( H \) is said to be \( PT \)-symmetric if \( HPT = PT H \). As is known, in standard quantum mechanics, the Hamiltonians are assumed to be Hermitian such that all the eigenvalues are real and the evolution is unitary. In the \( PT \)-symmetric quantum theory, Bender replaced the Hermiticity of the Hamiltonians with \( PT \)-symmetry. However, the \( PT \)-symmetry of a linear operator does not imply that its eigenvalues must be real. Thus, Bender introduced the unbroken \( PT \)-symmetric condition. The Hamiltonian \( H \) is said to be unbroken \( PT \)-symmetric if there exists a collection of eigenvectors \( \Psi_i \) of \( H \) such that they span the whole space and \( PTP_i = \Psi_i \).

It was shown that for a \( PT \)-symmetric Hamiltonian \( H \), its eigenvalues are all real if and only if \( H \) is unbroken \( PT \)-symmetric \(^2\). In two dimensional space case, this condition has a much simpler description and an important illustrative example. That is, if \( P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( T = \tau_0 \), \( H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix} \), then \( H \) is unbroken \( PT \)-symmetric if \( s^2 \geq r^2 \sin^2 \theta \) \(^2\).

In the following part, we shall give the unbroken \( PT \)-symmetry condition for general \( PT \)-symmetric operators. To this end, we need the following proposition.

Proposition 5.1. If \( H \) is a \( PT \)-symmetric operator, then it has four real parameters. Moreover, if \( H = h_0\sigma_0 + h_1\sigma_1 + h_2\sigma_2 + h_3\sigma_3 \) is written in terms of Pauli operators, then

\[
\begin{align*}
\text{Im}(h_0) &= 0, \quad (5.1) \\
\text{Re}(h_1)\text{Im}(h_1) + \text{Re}(h_2)\text{Im}(h_2) + \text{Re}(h_3)\text{Im}(h_3) &= 0. \quad (5.2)
\end{align*}
\]

Proof. It is apparent that \( PT \) is also a time reversal operator. Thus we can assume that \( PT = \sum_{j=0}^3 c_j\tau_j \). Now the condition \( PT H = HPT \) is equivalent to

\[
\left( \sum_{j=0}^3 c_j\tau_0\sigma_j \right) \left( \sum_{i=0}^3 h_i\sigma_i \right) = \left( \sum_{i=0}^3 h_i\sigma_i \right) \left( \sum_{j=0}^3 c_j\tau_0\sigma_j \right).
\]

According to (3.1), this equation can be reduced to

\[
c_0(h_0 - h_0) + \sum_{i=1}^3 c_i(h_0 - \overline{h}_0)\sigma_i + \sum_{i=1}^3 c_i(h_i + \overline{h}_i) + i\sigma \cdot \left[ \overline{\sigma} \times (\overline{h} - \overline{h}) \right] - \sum_{i=1}^3 c_i(h_i + \overline{h}_i)\sigma_i = 0,
\]

where \( \overline{h} = (h_1, h_2, h_3) \) and \( \overline{h} = (\overline{h}_1, \overline{h}_2, \overline{h}_3) \).
The equation above is equivalent to

\[ \text{Im}(h_0) = 0, \quad (5.3) \]
\[ \sum_{i=1}^{3} c_i \text{Re}(h_i) = 0, \quad (5.4) \]
\[ \bar{c} \times \text{Im}(h) - c_0 \text{Re}(h) = 0, \quad (5.5) \]

where \( \text{Re}(h) = (\text{Re}(h_1), \text{Re}(h_2), \text{Re}(h_3)) \) and \( \text{Im}(h) = (\text{Im}(h_1), \text{Im}(h_2), \text{Im}(h_3)) \).

(i). When \( c_0 \neq 0 \). It follows (5.5) that \( \text{Re}(h) = \frac{1}{c_0}(\bar{c}\times \text{Im}(h)) \). Thus, the four parameters \( \text{Im}(h)_1, \text{Im}(h)_2, \text{Im}(h)_3 \) and \( \text{Re}(h_0) \) determine \( H \).

Note that (5.1) is the same as (5.3). On the other hand, \( \text{Re}(h) = \frac{1}{c_0}(\bar{c}\times \text{Im}(h)) \) implies that \( \text{Re}(h) \cdot \text{Im}(h) = 0 \). Thus, (5.2) is also valid.

(ii). When \( c_0 = 0 \). (5.5) implies that \( \text{Im}(h) = t\bar{c} \). Thus, we only need one real parameter \( t \) to determine \( \text{Im}(h) \). (5.4) implies that \( \text{Re}(h) \) should be orthogonal to \( \bar{c} \). Hence two parameters are needed. With \( \text{Re}(h_0) \), we have four parameters altogether.

In this case, (5.2) follows from the fact \( \text{Im}(h) = t\bar{c} \) and the equation (5.4). 

\[ \square \]

**Theorem 5.1.** If \( H \) is a \( \mathcal{PT} \)-symmetric operator and \( \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \) is the representation matrix of \( H \), then \( H \) is unbroken if and only if

\[ (\text{Re}(h_{11} + h_{22}))^2 - 4\text{Re}(h_{11}h_{22} - h_{12}h_{21}) \geq 0. \]

**Proof.** Let \( \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \) be the matrix of \( H \), \( \lambda \) be an eigenvalue of \( H \), then

\[ \lambda^2 - (h_{11} + h_{22})\lambda + h_{11}h_{22} - h_{12}h_{21} = 0. \quad (5.6) \]

On the other hand, rewrite \( H = h_0 \sigma_0 + h_1 \sigma_1 + h_2 \sigma_2 + h_3 \sigma_3 \). It follows from (5.1) and (5.2) that

\[ \text{Im}(h_{11} + h_{22}) = 2\text{Im}(h_0) = 0, \]
\[ \text{Im}(h_{11}h_{22} - h_{12}h_{21}) = -\text{Re}(h_1)\text{Im}(h_1) - \text{Re}(h_2)\text{Im}(h_2) - \text{Re}(h_3)\text{Im}(h_3) = 0. \]

The two equations above imply that

\[ -\text{Im}(h_{11} + h_{22})\lambda + \text{Im}(h_{11}h_{22} - h_{12}h_{21}) = 0. \quad (5.7) \]

Substitute (5.7) into (5.6). Now the equation (5.6) reduces to

\[ \lambda^2 - \text{Re}(h_{11} + h_{22})\lambda + \text{Re}(h_{11}h_{22} - h_{12}h_{21}) = 0, \quad (5.8) \]
According to (5.8), \( \lambda \) is a real number, that is, \( H \) is unbroken \( \mathcal{PT} \)-symmetric, if and only if
\[
(\text{Re}(h_{11} + h_{22}))^2 - 4\text{Re}(h_{11}h_{22} - h_{12}h_{21}) \geq 0.
\] (5.9)

**Remark 5.1.** Note that when the equality is valid in (5.9), \( H \) may be non-diagonalisable in general. In this case, the space \( \mathbb{C}^2 \) is actually spanned an eigenvector \( \psi_1 \) satisfying \( (H - \lambda_0 I)\psi_1 = 0 \) and a generalized eigenvector \( \psi_2 \) satisfying \( (H - \lambda_0 I)^2\psi_2 = 0 \), where \( \lambda_0 = \frac{1}{2}\text{Re}(h_{11} + h_{22}) \) is the eigenvalue.

**Remark 5.2.** Note that Bender’s unbroken \( \mathcal{PT} \)-symmetric condition in [2] is a special case of (5.9). To see this, let \( H = \begin{pmatrix} r e^{i\theta} & s \\ s & r e^{-i\theta} \end{pmatrix} \), we have
\[
\text{Re}(h_{11}) = \text{Re}(h_{22}) = r \cos \theta,
\]
\[
\text{Re}(h_{11}h_{22} - h_{12}h_{21}) = r^2 - s^2.
\]
Then (5.9) holds iff \( s^2 \geq r^2 \sin^2 \theta \).

**Remark 5.3.** If \( H \) is a Hermitian operator, then it is also unbroken \( \mathcal{PT} \)-symmetric. Usually, this can be shown by using canonical forms. However, in \( \mathbb{C}^2 \), it also follows from direct calculation.

In fact, since \( H = h_0\sigma_0 + h_1\sigma_1 + h_2\sigma_2 + h_3\sigma_3 \) is Hermitian, each \( h_i \) is a real number. Now we only need to find real coefficients \( c_0, c_1, c_2 \) and \( c_3 \) such that \( c_0^2 + c_1^2 + c_2^2 - c_3^2 = 1 \) and equations (5.3) – (5.5) hold. Take \( c_0 = 0 \) and \( c_1, c_2, c_3 \) are such real numbers that \( c_1\text{Re}(h_1) + c_2\text{Re}(h_2) + c_3\text{Re}(h_3) = 0 \) and \( c_1^2 + c_2^2 + c_3^2 = 1 \). Let \( \mathcal{PT} = \sum_{i=0}^{3} c_i \tau_i \). It is apparent that \( (\mathcal{PT})^2 = I \) and \( H \) is \( \mathcal{PT} \)-symmetric. Moreover, if we rewrite the Hermitian matrix as \( H = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \), then \( \text{Re}(h_{11} + h_{22})^2 - 4\text{Re}(h_{11}h_{22} - h_{12}h_{21}) = 4a^2 - 4(a^2 - |b|^2) = 4|b|^2 \geq 0 \) holds, so \( H \) is also unbroken.

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