Periodicity and quark–antiquark static potential

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Abstract

Beyond the standard model, a static potential between quark pairs is obtained phenomenologically (QCD inspired), associated with the range of strong interaction, when the virtual exchange gluon squared momentum transfer has a periodicity for periodic boundary conditions of the quark–pair system enclosed by a constant volume, in the lowest order of the effective perturbed QCD (in which the gluon propagator is replaced by the effective gluon one). This potential includes a periodicity dependent effect, characterized by a finite face value of the periodicity $N$, in addition to the periodicity independent potential (the Coulomb type plus linear one). That periodicity dependent effect, dominant at short distance, is applied to an explanation of the top quark mass

$$ m_t = 8\pi m_\pi N^{\frac{1}{2}}, $$

whose numerically calculated results indicate approximately both upper and lower bounds of $m_t$

$$ 177 \text{ GeV} > m_t > 173 \text{ GeV} $$

for the range of strong interaction $L = 1.40 \text{ fm} (= m_\pi^{-1})$. 
1. Introduction

The range of strong interaction $L$ is phenomenologically considered as a constant $m_\pi^{-1}$, the Compton wave length of the pion, which is consistent with the experimental proton charge radius $[1]$

$$L = m_\pi^{-1} = \text{a constant}.$$ 

This may be a standard length of the quark pair system, which is available for determining the quark sizes, i.e., the quark masses (the Compton wave length).

In this system, a particle (quark pair) with its momentum $k_n$ (specified by $n$) is confined in the range $r_n$, where this $r_n$ satisfies the relation

$$r_n k_n = \frac{1}{2} \quad \text{(the position momentum uncertainty)} \quad (1)$$

for the static case. We are interested in a quark pair enclosed by a cubic volume $L^3$, with periodic boundary conditions for momenta $[k'_i = \frac{2 \pi}{L} \eta_i \quad (i = 1, 2, 3; \eta_i = 0, \pm 1, \pm 2, \cdots)]$. The above momentum points are finite, equal spacing and periodic for periodic boundary conditions of the quark pair system $[2]$. This description may be extended formally to the one of the squared momentum transfer in the Minkowski momentum space and hence the effective gluon propagator may be derived in the fashion, as seen in section 2 of this work. Then, the static potential between quark pairs in the quantum view may be obtained from the analyticity of the effective gluon propagator for periodic boundary conditions of the quark pair system, as seen in the succeeding sections. This static potential includes a periodicity dependent effect, characterized by a finite face value of the periodicity $N$, in addition to the periodicity independent potential (the Coulomb type plus linear one), where the periodicity dependency of potential is defined by the periodicity condition for the static case, as seen in sections 3 and 4. That periodicity dependent effect, dominant at short distance, may be used to determine the heavy quark masses.

Recently, the top quark of about 180 GeV was discovered at Fermi lab experimentally $[3]$. At present, it may be important for us to find a way in explaining the top quark mass for a further theory beyond the standard model $[4],[5]$. All known phenomenological potentials for quark pairs (purely phenomenological or QCD inspired) are the flavor independent ones, but by these potentials no one has been successful in explaining the top quark mass $[6],[7]$. The existence of the (heavy) top quark may suggest that there exists another kind of static potential between quark pairs which includes, e.g., a periodicity dependent effect, characterized by the finite face value of the periodicity, in addition to the periodicity independent potential (the Coulomb type plus linear one).

The aim of this work is to build up a phenomenological model (QCD inspired) beyond the standard model which enables us to derive a static potential between quark
pairs, including the periodicity dependent effect (for explaining the top quark mass) in addition to the periodicity independent potential (the Coulomb type plus linear one). Our attention is concentrated on the result, \( L = 8\pi R N^{\frac{1}{2}} \) (or \( m_t = 8\pi m_N N^{\frac{1}{2}} \)) which is the multiple of the top quark confined range \( R \) and the square root of the top quark confined range number (the finite face value of the periodicity) \( N^{\frac{1}{2}} \) excepting \( 8\pi \). This \( N \) is determined from the periodicity dependent effect for \( R \). Since \( L (= \frac{1}{\pi} m_t^{-1}) \) is a constant, \( R (= \frac{1}{\sqrt{\pi}}) \) is obtained from \( L = 8\pi R N^{\frac{1}{2}} \) if \( N \) is determined as mentioned above. Our above discussion is true for the \( l \)-quark pair, e.g., \( L = 8\pi R_l N_l^{\frac{1}{2}} \), where \( l = u(d), s, c, b \) and \( t \), respectively. This model is developed within the framework of the quark model with three generations.

This paper is organized as follows: In section 2, an effective gluon propagator is derived as an average of the gluon propagator in the perturbed QCD on the finite and equal spacing (arbitrary) squared momentum transfers (periodicity recurrent points intervals) for periodic boundary conditions of the quark pair system. In section 3, the periodicity independent potential (the Coulomb type plus linear one) is obtained from the above fashion on the finite and equal spacing (arbitrary) squared momentum transfers (periodicity recurrent points intervals) under the periodicity condition for the static case. In section 4, the static potential between quark pairs, as done in the previous section, is derived when the gluon squared momentum transfers (periodicity recurrent points intervals) are finite, equal spacing (definite) and periodic for periodic boundary conditions of the quark pair system. This static potential includes the periodicity dependent effect in addition to the periodicity independent one. That periodicity dependent effect produces a relation \( m_t = 8\pi m_N N^{\frac{1}{2}} \) (or \( L = 8\pi R N^{\frac{1}{2}} \)). Section 5 is devoted to the explanation of the top quark mass \( m_t \), approximately \( 177 \text{ GeV} > m_t > 173 \text{ GeV} \), for the range of strong interaction \( L = 1.40 \ f_m (= \frac{1}{\pi} m_t) \) by the periodicity dependent effect for \( R \). Section 6 is left for the concluding remarks. In Appendix, the excitation probability in section 2 is derived. Throughout this paper, the particle size is expressed by the Compton wave length.

2. Effective Gluon Propagator

A static potential between a quark pair is a function of \( r \), where \( r \) is the distance between the center of the quark and the one of its antiquark, with the Compton wave length radius \( \frac{1}{2} r_n \) (\( r_n \equiv r_{\bar{n}} \) (the Compton wave length of anti-particle)), respectively. The closest distance between the quark pair is defined as

\[ r = \frac{1}{2} r_n + \frac{1}{2} r_n = r_n, \] (2)

which enables us to obtain the corresponding quark mass \( m_n \) (the Compton wave length) and assures there is no static potential inside this \( r_n \) (the closest distance).
This means the Compton wave length of a quark (fermion) equals to the closest
distance between the quark–pair (boson).

According to the field theory [4], if the quark pair at the closest distance, which
behaves like one bosonic particle, is enclosed by the cubic volume \( L^3 \), the plane wave
functions of this system, \( \exp(i\vec{k} \cdot \vec{r}) = \exp(i\vec{k}' \cdot \vec{r}) \) if \( k'_i = \frac{1}{2}k_i, \ r'_i = 2r_i \), form a
complete set and the periodic conditions show that \( k' = \frac{2\pi}{L}n \) with \( k = \frac{1}{2}k_i \). The choice of
\( r_i = \frac{1}{2}r'_i \) (and \( k_i = 2k'_i \)) may be adequate to the lattice point in the 3-dimensional
Euclidean space if \( r'_1 = r'_2 = r'_3 = L \). The above discussion may be extended formally
to the one in the 4-dimensional Euclidean space which is connected to the Minkowski
momentum space, \( k'_i = -ik'_0 \) (real)\[4].

Introducing such a lattice point in the 4-dimensional Euclidean space
\[
n' \equiv \eta'^2 = \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2,
\]
where \( \eta_4 = 0, \pm 1, \pm 2, \cdots \) and \( n' = 0, 1, 2, \cdots \) (by definition), respectively, the squared
momentum transfer in the Minkowski momentum space is
\[
-k^2 \equiv k_E^2 = k_1^2 + k_2^2 + k_3^2 + k_4^2 = \left( \frac{2\pi}{L} \right)^2 (\eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2)
= c^2n'(\frac{cL}{4\pi})^{-2} = c^2n'a_c^{-2} \geq 0 \text{ (} a_c \equiv cL(4\pi)^{-1} \text{)},
\]
for \(-k^2\). For the static case, we have the following periodicity condition
\[
-k^2 \equiv k_E^2 = c^2n' \quad \text{ (} n' = 0, 1, 2, \cdots \text{)},
\]
for \(-k^2\). For the static case, we have the following periodicity condition for \(|\vec{k}|^2\)
\[
a_c^2|\vec{k}|^2 = c^2n \quad \text{ (} n = 0, 1, 2, \cdots \text{)},
\]
where \(|\vec{k}|^2 = k_1^2 + k_2^2 + k_3^2 \) and \( n \equiv \eta'^2 = \eta_1^2 + \eta_2^2 + \eta_3^2(n' = n + n_3, n' = n, \text{ if } \eta_4 = 0, \text{)}
respectively. We call this \( n \) as the periodicity and the largest \( n = N \) as the finite face
value of the periodicity, vice versa.

Let us start with the gluon propagator in the perturbed QCD
\[
-iD_{\mu\nu}^{\alpha\beta}(k) = -i \left( \frac{\delta_{\alpha\beta}}{k^2} \right) \left[ g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right].
\]
For \( \xi = 1 \), instead of Eq.(3), we may start with
\[
-iD_{\mu\nu}^{\alpha\beta}(k) = -i \frac{\delta_{\alpha\beta}g_{\mu\nu}}{k^2}.
\]
In Eq.\( \text{(3)} \) or Eq.\( \text{(7)} \), \( k^2 \) in all denominators should be considered as \( k^2 + i\epsilon \ (\epsilon \to 0^+) \), because of the causal property of the gluon propagator. When the hadronic system (characterized in terms of both the squared momentum transfer and the finite length parameter \( a_c \)) is introduced, the intervals between the squared momentum transfer periodicity recurrent points are finite and equal spacing (arbitrary) as seen in Eq.\( \text{(4)} \).

The gluon propagator (keeping \( \delta_{\alpha\beta}g_{\mu\nu} \) out) \( -k^{-2} \) is the inverse of the “squared momentum transfer operator \( -k^{2n} \)” in this gauge \( \text{[8]} \). This means that both the finite and equal spacing squared momentum transfer and the corresponding gluon propagator have the same excitation probability between the respective finite and equal spacing levels. It is to be stressed that the periodicity of the gluon squared momentum transfer may be represented by the one of \( -k^2 \).

The excitation probability between the gluon propagators \( (n' + 1)(-k^{-2}) \) and \( (n' + 2)(-k^{-2}) \) \( (n' = 0, 1, 2, \cdots) \) is \( \exp(a_c^2k^2) \) as shown in Appendix. This \( \exp(a_c^2k^2) \) \( (1 > \exp(a_c^2k^2) > 0 \ \text{for} \ 0 < -k^2 < \infty) \) is the most probable one according to the information theory \( \text{[9]} \). Thus for the finite and equal spacing periodic squared momentum transfer recurrent points intervals case, this model requires the average \( (-k^{-2}) \) as

\[
(-k^{-2})_{av} = \left( 1 \cdot (-k^{-2}) + 2 \cdot (-k^{-2}) \exp(a_c^2k^2) + \cdots \right.

\]

\[
+ \left( n' + 1 \right)(-k^{-2}) \exp(n'a_c^2k^2) + \cdots \right) \left( \sum_{n'=0}^{\infty} \exp(n'a_c^2k^2) \right)^{-1}
\]

\[
= \left( -k^{-2} \right) \sum_{n'=0}^{\infty} (n' + 1) \exp(n'a_c^2k^2) \left( \sum_{n'=0}^{\infty} \exp(n'a_c^2k^2) \right)^{-1}
\]

\[
= \left( -k^{-2} \right) \left\{ X \frac{\partial}{\partial X} \sum_{n'=0}^{\infty} X^{n'} + \sum_{n'=0}^{\infty} X^{n'} \right\} \left( \sum_{n'=0}^{\infty} X^{n'} \right)^{-1}
\]

\[
= \left( -k^{-2} \right)(1 - X)^{-1}
\]

(6)

instead of \( (-k^{-2}) \), owing to the periodicity condition for \( (-k^2) \), Eq.\( \text{(4)} \), where \( X = \exp(a_c^2k^2) \). Noting the original form of \( D_{\mu\nu}^{\alpha\beta}(k) \), we have the following effective gluon propagator

\[
-iD_{\mu\nu}^{\alpha\beta}(a_c, k) = \left( -i\delta_{\alpha\beta}k^{-2} \left( 1 - \exp(a_c^2k^2) \right)^{-1} \right) \left[ g_{\mu\nu} - (1 - \xi)k_\mu k_\nu k^{-2} \right],
\]

(7)

which behaves like the gluon propagator of the perturbed (the \( k^{-4} \) type \( \text{[11]} \)) QCD in the infinite (the zero) squared momentum transfer limit. If we put

\[
f(-a_c^2k^2) \equiv 1 - \exp(a_c^2k^2),
\]

(8)

this \( f(-a_c^2k^2) \) may be an important weight function describing the hadronic structure. In fact, since \( \lim_{-k^2 \to \infty} f(-a_c^2k^2) = 1 \) and \( \lim_{-k^2 \to 0} f(-a_c^2k^2) = \lim_{-k^2 \to 0} (-a_c^2k^2), \)
we have \( \lim_{-k^2 \to \infty} (-k^{-2}f^{-1}(-a_c^2k^2)) = \lim_{-k^2 \to \infty} (-k^{-2}) \) and \( \lim_{-k^2 \to 0} (-k^{-2}f^{-1}(-a_c^2k^2)) = \lim_{-k^2 \to 0} (a_c^{-2}k^{-4}) \). Finally, we may get

\[
-iD^{\alpha\beta}_{\mu\nu}(a_c, k) = \left(-i\delta^{\alpha\beta}k^{-2}f^{-1}(-a_c^2k^2)\right) \left[g_{\mu\nu} - (1 - \xi)k_{\mu}k_{\nu}k^{-2}\right].
\] (9)

For \( n' = N' \gg 1 \), Eq.(8) may be still applicable (approximately) to our calculation, since such \( n' \) is considered as continuous.

3. Periodicity Independent Static Potential

To obtain the periodicity independent static potential between quark pairs, we have to use the effective gluon propagator Eq.(7), because of the periodicity condition, Eq.(4'). For the primitive form of static quark-antiquark potential in our model (based on the quantum views), we can start with the integral

\[
I = -(2\pi)^{-3} \int_0^\infty |\vec{k}|^2 d|\vec{k}| \int_0^1 d(\cos \theta) \int_0^{2\pi} d\phi \exp(i\vec{k} \cdot \vec{r})|\vec{k}|^{-2} \left(1 - \exp(-a_c^2|\vec{k}|^2)\right)^{-1},
\] (10)

where we have used Eq.(7) with \( \xi = 1 \) and have kept \( \delta_{\alpha\beta}g_{\mu\nu} \) out. This \( I \) may be reduced to the usual QCD case if \( a_c \to \infty \), i.e., \( L \to \infty \). After elementary calculations, including the exchange \( |\vec{k}| \to -|\vec{k}| \), we have

\[
I = -(2\pi r)^{-1}(2\pi i)^{-1} \int_{-\infty}^\infty d|\vec{k}| \exp(i\vec{k} |r|)|\vec{k}|^{-1} \left(1 - \exp(-a_c^2|\vec{k}|^2)\right)^{-1}.
\] (11)

This integral, which is analytic in the upper half plane of complex momentum (according to the Jordan’s lemma [11]), is rewritten as

\[
I = -(2\pi r)^{-1}(2\pi i)^{-1} \oint dz \exp(izr)z^{-1} \left(1 - \exp(-a_c^2z^2)\right)^{-1},
\] (12)

where \( z \equiv |\vec{k}| \exp(i\psi)(0 \leq \psi \leq \pi) \). In Eq.(12), a triple pole at \( z = 0 \) (in Eq.(4'), \( n = 0 \), i.e., the periodicity independent case) is enclosed within the contour integral in the upper half plane of complex momentum. Some residue calculations give us

\[
I = -(2\pi r)^{-1}(2\pi i)^{-1} \oint dz \exp(izr)z^{-1} \left(1 - \exp(-a_c^2z^2)\right)^{-1} \equiv \left(-(2\pi r)^{-1}\right) a_{-1},
\] (13)

where

\[
a_{-1} = \frac{1}{2} \left(1 - a_c^{-2}r^2\right) + (2r)(\pi a_c^2)^{-1}(\lambda^{-1})_{\lambda \to 0}.
\] (14)
Thus we have

\[ I = (4\pi)^{-1}(-r^{-1} + a_c^{-2}r) - (\pi^2 a_c^2)^{-1}(\lambda^{-1})_{\lambda \to 0}. \]  \hspace{1cm} (15)

In Eq. (15), the last term is infrared divergent (\( \lambda = 0 \)), but is independent of \( r \). So we will neglect this term throughout our discussion. After a normalization, \( \frac{4}{3} g_s^2 \), we find the following periodicity independent potential between quark pairs, in the lowest order

\[ V(a_c, r) = \frac{4}{3} g_s^2 I = \alpha_s(-r^{-1} + a_c^{-2}r), \]  \hspace{1cm} (16)

where \( \alpha_s = \frac{4}{3} \left( \frac{g_s^2}{4\pi} \right) \) and \( a_c \) is defined as in Eq. (3). This \( V(a_c, r) \) is just the periodicity independent potential, which is similar to the Coulomb type plus linear one [6].

If we take a pure phenomenological weight function

\[ f_p(-a_c^2 k^2) = -a_c^2 k^2 (1 - a_c^2 k^2)^{-1} \]  \hspace{1cm} (17)

instead of \( f(-a_c^2 k^2) \) in Eq. (8), then we have

\[ -k^{-2} f_p^{-1} (-a_c^2 k^2) = -k^{-2} + a_c^{-2} k^{-4} \]  \hspace{1cm} (18)

and from Eq. (18) we obtain the static quark-antiquark potential \( V_p(a_c, r) = \alpha_s(-r^{-1} + a_c^{-2}r) \) in the lowest order. Our weight function, Eq. (8) is rewritten as

\[ f(-a_c^2 k^2) = f_p(-a_c^2 k^2) + (1 - a_c^2 k^2)^{-1} \left\{ 1 - g(-a_c^2 k^2) \right\}, \]  \hspace{1cm} (19)

where \( g(-a_c^2 k^2) = (1 - a_c^2 k^2) \exp(a_c^2 k^2) \), \( \lim_{k^2 \to 0} g(-a_c^2 k^2) = 1 \) and \( \lim_{k^2 \to \infty} g(-a_c^2 k^2) = 0 \), respectively. Thus, it is found that \( \lim_{k^2 \to 0} f(-a_c^2 k^2) = \lim_{k^2 \to \infty} f_p(-a_c^2 k^2) = 0 \) and \( \lim_{k^2 \to \infty} f(-a_c^2 k^2) = \lim_{k^2 \to 0} f_p(-a_c^2 k^2) = 1 \), respectively.

4. Static Potential with Periodicity Dependent Effect

The periodicity condition, Eq. (11) or Eq. (11) shows that the intervals between the gluon squared momentum transfer periodicity recurrent points are finite and equal spacing (arbitrary). If the constant \( c \) is chosen as

\[ c = \left( 2\pi \right)^{\frac{1}{2}}, \]  \hspace{1cm} (20)

then we have the periodic condition

\[ -a^2 k^2 = 2\pi n' \quad (n' = 0, 1, 2, \cdots) : a = 2^{-1}(2\pi)^{-\frac{1}{2}} L, \]  \hspace{1cm} (21)
which means that the intervals between points of $-k^2 = \frac{2\pi}{q^2} n'$ are finite, equal spacing and periodic for periodic boundary conditions of the quark pair system. For the static case, we have the following periodic condition

$$a^2 |\vec{k}|^2 = 2\pi n \quad (n = 0, 1, 2, \cdots : a = 2^{-1}(2\pi)^{-\frac{1}{2}} L). \quad (21')$$

The periodic condition for the static case, Eq.(21') indicates that we have other simple poles at $z = ((2\pi n)^{\frac{1}{2}} a^{-1})(\pm 1 + i) 2^{-\frac{1}{2}}$ (in Eq.(14), $n = 1, 2, \cdots, N$ (the finite face value of the periodicity), i.e., the periodicity dependent cases) in addition to the triple pole at $z = 0$ (the periodicity independent case) in the integral, Eq.(14) of the upper half of the complex momentum. Hence we have a different type of static potential between quark pairs from the periodicity independent one, Eq.(16).

In fact, for the static case, when $c = (2\pi)^{\frac{1}{2}}$, we have

$$a^2 z^2 = a^2 |\vec{k}|^2 \exp(2i\psi) = 2n\pi(\cos 2\psi + i \sin 2\psi).$$

If we put $\psi = \frac{1}{4} \pi$ or $\psi = \frac{3}{4} \pi$, then we get

$$\exp(-a^2 z^2) = \exp(-2n\pi(\pm i)) = 1. \quad (22)$$

To obtain the static potential between quark pairs, we have to be careful in calculating the integral $I$, which includes simple poles at $z = ((2\pi n)^{\frac{1}{2}} a^{-1})(\pm 1 + i) 2^{-\frac{1}{2}}$ (the periodicity dependent cases) in addition to the triple pole at $z = 0$ (the periodicity independent case). Then we have

$$I = (4\pi)^{-1} \left\{ (-r^{-1} + a^{-2} r) + (-r^{-1})\pi^{-1} \sum_{n=1}^{N} n^{-1} \exp(-\kappa_n r) \sin(\kappa_n r) \right\}$$

$$= (4\pi)^{-1} \left\{ -r^{-1} \left(1 + P(a, r, N) + a^{-2} r \right) \right\}, \quad (23)$$

where $\kappa_n = (\pi n)^{\frac{1}{2}} a^{-1}$ and

$$P(a, r, N) = \pi^{-1} \sum_{n=1}^{N} n^{-1} \exp(-\kappa_n r) \sin(\kappa_n r). \quad (24)$$

In Eq.(23), $N$ must be finite for the Cauchy's integral formula is applied to the calculation of $I$. [11],[12]. This $N$ may be considered as the finite face value of the periodicity. And the finiteness of the fundamental constituent at $n \leq N < \infty$ as

$$r_n \geq r_N > 0$$
is equivalent to the one of the fundamental constituent at \( n \leq N < \infty \) as

\[ m_n \leq m_N < \infty. \]

The \( P(a, r, N) \), Eq.(24) is the periodicity dependent effect, with \( \lim_{a \to \infty} P(a, r, N) = 0 \) and \( \lim_{r \to 0} P(a, r, N) = 0 \). Now, this \( P(a, r, N) \) is rewritten as

\[
P(a, r, N) = \pi^{-1} \sum_{n=1}^{N} P_n(a, r, n),
\]

where

\[
P_n(a, r, n) = n^{-1} \exp(-\kappa_n r) \sin(\kappa_n r).
\]

Also, \( P_n(a, r, n) \), Eq.(26) satisfies the following differential equation

\[
\left( \frac{d^2}{dr^2} + 2n^\frac{3}{2} \frac{d}{dr} + 2n \right) P_n(a, r, n) = 0 \quad (n = 1, 2, \ldots, N),
\]

where \( r \) represents \((\pi^\frac{3}{2} a^{-1}) r\).

Finally, we obtain a static potential between quark pairs over all ranges in the lowest order

\[
V(a, r, N) = \frac{4}{3} g_s^2 I
\]

\[
= \alpha_s \left\{ -r^{-1}(1 + P(a, r, N)) + a^{-2} r \right\}.
\]

5. Explanation of Top Quark Mass

We are interested in an explanation of the top quark mass by the periodicity dependent effect in this model. It is to be noted that \( V(a, r, N) \), Eq.(28) is dependent of the coupling constant \( \alpha_s \), but \( P(a, r, N) \), Eq.(24) is irrelevant to \( \alpha_s \), dominant at short distance and periodicity dependent. The periodicity dependent effect in \( V(a, r, N) \) is originated in \( P(a, r, N) \). So our attention is paid to \( P(a, r, N) \) in the explanation of the top quark mass.

For the fixed \( n \), the periodic condition for the static case, Eq.(21) gives a relation

\[
a|\vec{k}|_n \equiv a k_n = (2\pi)^{\frac{3}{2}} n^{\frac{1}{2}} \quad (|\vec{k}|_n \equiv k_n),
\]

and taking the position momentum uncertainty, Eq.(1), we obtain

\[
a = 2(2\pi)^{\frac{3}{2}} r_n n^{\frac{1}{2}} = 2(2\pi)^{\frac{3}{2}} R N^{\frac{1}{2}}
\]
or

\[ L = 8\pi r_n n_{1 \times}^3 = 8\pi R N_{1 \times}^3, \]

where \( R \equiv r_N \) is the shortest of \( r_n \) (the closest distance between the quark pair) and \( N \) the largest of \( n \) (the finite face value of the periodicity), respectively. For \( L = m_\pi^{-1} \), from Eq.(30) we get

\[ L^{-1} = m_\pi = (8\pi)^{-1} R^{-1} N_{1 \times}^{-\frac{1}{2}} \]

and

\[ m_t = 8\pi m_\pi n_{\bar{t}}^{\frac{1}{2}} \quad (N \equiv n_t, R = m_t^{-1}), \]

where \( N \) may be numerically determined from \( P(a, R, N) \) in Eq.(28) \[13\].

The correspondences among \( r, n \) and \( \kappa_n R \) are as follows:

| \( r \) | \( R \) (\( \equiv r_N \)) | \( \sim \) | \( r_n \) | \( \sim \) | \( r_1 \) (\( \equiv (8\pi)^{-1} L \)) |
|-------|-----------------|--------|--------|--------|---------------------|
| \( n \) | \( N \) | \( \sim \) | \( n \) | \( \sim \) | 1 |
| \( \kappa_n R \) | \((2 \cdot 2)^{-\frac{1}{2}}\)^{-1} | \( \sim \) | 

\((2 \cdot 2)^{-\frac{1}{2}}\)^{-1}(\frac{n}{N})^{\frac{1}{2}} \sim (2 \cdot 2)^{-\frac{1}{2}}(\frac{1}{N})^{\frac{1}{2}}

From Eqs.(30),(31) and Table 1, we have the following numerical results (including one of the top quark), for the range of strong interaction \( L = 1.40 \) \( fm \) (= \( m_\pi^{-1} \)) as shown in Table 2.

| \( m_t \) (GeV) | \( u(d) \) | \( s \) | \( c \) | \( b \) | \( t \) |
|-----------------|-------|-------|-------|-------|-------|
| 0.3             | 0.5   | 1.35  | 5.0   | 5.08  | 173.43 |
| 176.23          |       |       |       |       |       |
| \( R_t \) (fm)  | 0.6667| 0.4000| 0.1481| 0.0400| 0.0394 |
|                 | 1.153\times 10^{-3} | 1.135\times 10^{-3} |       |       |       |
| \( N_t \)       | 0.0836| 0.1393| 0.3761| 1.3925| 1.4142 |
|                 | 48.31 | 49.09 |       |       |       |
| \( L(8\pi)^{-1} \) (fm) | 0.0557 | 0.0557 | 0.0557 | 0.0557 | 0.0557 |
| \( L \) (fm)    | 1.40  | 1.40  | 1.40  | 1.40  | 1.40  |
|                 |       |       |       |       |       |
| \( N_t \)       | 0.0070| 0.0194| 0.1415| 1.9371| 2 2334 |
|                 | 2410  |       |       |       |       |
| \( P(N_t) \) \[13\] | 0     | 0     | 0     | 0.1000| 0.1850 |
|                 | 0.1850|       |       |       |       |

\( m_t = R_t^{-1} \), \( 1 \) \( \text{GeV}^{-1} = 0.2 \) \( fm \), \( m_\pi \approx \frac{1}{2} \) \( \text{GeV} \), \( P(a, R_t, N_t) \) \( \equiv P(N_t) \) and ( ) means a calculated value.

In this table, the \( P(N_t) \) is evaluated by the integer value \( N_b = 2 \) instead of \( N_{t0} = 1.9371 \) (\( \sigma N_b = (\frac{19371}{2}) \cdot 2 \)) and \( P(N_t) \)'s for \( u(d) \), \( s \) and \( c \) quarks may be regarded as zero, since \( N_t < 1 \). In this model, \( N_t \geq 1 \) (Eq.(24)). Table 3 is concerned with the numerically calculated values \( P(N) \) at \( N \). When \( P(N_t) = 0.1850 \) (\( N_t = 2410 \)) may approximately be suitable for the upper bound of \( m_t \), the lower bound of \( m_t \) may be obtained in such a way that \( P(N_{t0}) = 0.1850 \) (\( N_{t0} = \sigma N_t = 2334 \), where \( \sigma \) is defined as \( \sigma = \frac{N_{t0}}{N_b} = \frac{N_{t0}}{N_t} \), in accordance with the \( b \) quark case. For the same periodicity dependent effect on \( m_t \), between the upper and lower bounds of the top quark mass, the \( P(N_t) = P(N_{t0}) = 0.1850 \) have been chosen (see Eq(28)).
Table 3

| N   | 1       | 2       | 3       | 5       | 10      | 100     | 1000    | 2000    | 2334    | 2410    | ∞       |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| P(N)| 0.0784  | 0.1000  | 0.1123  | 0.1263  | 0.1422  | 0.1725  | 0.1833  | 0.1847  | 0.1850  | 0.1850  | 0.1884  |

\[ P(N) = \pi^{-1} \sum_{n=1}^{N} n^{-1} \exp \left( -\frac{1}{2 \cdot 2^\pi} \left( \frac{n}{N} \right)^\frac{1}{2} \right) \sin \frac{1}{2 \cdot 2^\pi} \left( \frac{n}{N} \right)^\frac{1}{2}, \quad RN^\frac{1}{2} = \frac{L}{8\pi}. \]

From Table 2, it may be found that \( b \) and \( t \) quarks, belonging to the third generation of the quark model, have the periodicity dependent effect between these quark pairs, and that the larger \( N_l \) is, the smaller \( \Delta P(N_l) \) is, where \( \Delta P(N_l) \) means an increase of \( P(N_l) \) in \( N_l \). We adopt \( P(N_t) = 0.1850 \) for the numerically calculated values \( P(N_{t0}) = 0.1850 \) (\( N_{t0} = 2334 \)) and \( P(N_t) = 0.1850 \) (\( N_t = 2410 \)) as illustrated in Table 2. Finally, the upper and lower bounds of the top quark mass have the same order periodicity dependent effect as in Table 2. Thus, from Table 2 we have approximately

\[ 177 \text{ GeV} > m_t > 173 \text{GeV} \quad (32) \]

for the range of strong interaction \( L = 1.40 \text{fm} \left(= m_n^{-1}\right) \).

6. Concluding Remarks

In this paper, the phenomenological model for quark pairs (QCD inspired) beyond the standard model has been developed within the framework of the quark model with three generations. The closest distance between the quark pair \( r_i \), Eq.(2) is related to the quark mass \( m_l = (r_i^{-1}) \) (by definition). This is realized by the constant \( L \left(= m_n^{-1}\right) \) and the finite face value of the periodicity \( n_l \) as

\[ m_l = 8\pi m_n n_l^\frac{1}{2} \quad \text{(or} \quad L = 8\pi r_l n_l^\frac{1}{2} \text{)}, \quad (30') \]

where \( l \) is one of \( n \)'s such as \( l = u(d), \; s, \; c, \; b \) and \( t \), respectively. In this model, Eq. \( (30') \) is valid for \( n_l \geq 1 \). However, this equation is still valid for \( n_l < 1 \), numerically. So we may accept this equation for all \( n_l \) \((0 < n_l < \infty)\), as mentioned above. The flavor in the quark model with three generations may have such periodicity in this model as one of the flavor attributes. The periodicity independence (or the periodicity dependency) of the static potential between quark pairs may correspond to the flavor independence (or the flavor dependency) of it. This may suggest that there exists a corresponding symmetry or a corresponding broken symmetry (a source of the heavy quark masses) of it.

The momentum of one quark pair, enclosed by the constant cubic volume \( L^3 \), is finite, equal spacing and periodic for periodic boundary conditions of the quark pair.
system. The static potential between such quark pair in the quantum view has been derived, in our fashion, as follows:

\[
V(a, r, N) = \alpha_s \left\{ -\frac{1}{r} (1 + P(a, r, N)) + a^{-2} r \right\} 
\]

\[
\left( \alpha_s = \frac{4}{3} \left( \frac{g_s^2}{4\pi} \right), a = 2^{-1}(2\pi)^{-\frac{1}{2}} L \right)
\]

where the periodicity dependent effect \( P(a, r, N) \) is

\[
P(a, r, N) = \pi^{-1} \sum_{n=1}^{N} n^{-1} \exp(-\kappa_n r) \sin \kappa_n r
\]

\[
(\kappa_n = (\pi n)^{\frac{1}{2}} a^{-1}).
\]

The top quark confined range number (the finite face value of the periodicity) \( N \) is numerically determined from \( P(a, r, N) \) at \( r = R \), where \( R \) is the top quark confined range. Thus from Eq.(30) we obtain

\[
m_t = 8\pi m_\pi N^{\frac{1}{2}},
\]

where \( m_t = R^{-1} \) and \( m_\pi = L^{-1} \), respectively. It is to be noted that the factor \( (8\pi m_\pi) \) in Eq.(31) (or \( r_t^{-1} = \left( \frac{L}{8\pi} \right)^{-1} \) in Table 1) is another constant in this system.

Finally, our numerical results of the top quark mass in Table 2, approximately

\[
177 \text{ GeV} > m_t > 173 \text{ GeV}
\]

for the range of strong interaction \( L = 1.40 \text{ fm} \) (\( = m_\pi^{-1} \)), are consistent with the experiments at Fermi Laboratory [16], [17].

**Acknowledgements**

The author would like to thank the particle physics group, Institute of Theoretical Physics, SNU and Professors I. T. Cheon, H. W. Lee, and M. Rho (Saclay) for their kind discussions. Also, he is much indebted to Professor N. Z. Cho for numerical calculation.
Appendix

A derivation of the excitation probability, \( P(-a_c^2 k^2) = \exp(a_c^2 k^2) \)

Consider a particle characterized by a dimensionless scalar quantity \(-a_c^2 k^2 \equiv x(0 < x < \infty)\), associated with both the (finite and equal spacing) squared momentum transfer \(-k^2(0 < -k^2 < \infty)\) in our sense and the finite length parameter \(a_c\). Let \(P(x)\) be a probability that such a particle is excited by \(-k^2\) between \(n'(-k^2)\) and \((n' + 1)(-k^2)\) \((n' = 0, 1, 2, \cdots)\) without suffering any change. It is to be noted that the change \(\delta(-k^2)\) in \(-k^2\) corresponds to the change \(\delta x\) in the probability variable \(x\), as \(-k^2\) does to \(x\) (where \(\int_0^\infty P(x)dx = 1\) and \(\int_0^\infty P(x)d(-k^2) = a_c^{-2}\)). Of course \(P(0) = 1\), since a particle has no chance of any changing in \(-k^2 \to 0\). On the other hand, \(P(x)\) decreases as \(-k^2\) increases without suffering any change. Finally, \(P(x) \to 0\) as \(-k^2 \to \infty\).

For an infinitesimal dimensionless scalar quantity \(\delta x\), from the above consideration on \(P(x)\), we have

\[
P(x + \delta x) \equiv P(x) - P(x)\delta x \quad \text{(physically),} \quad (A.1)
\]

which is equivalent to

\[
P(x) - P(x + \delta x) = P(x)\delta x. \quad (A.2)
\]

From (A.2), we have

\[
P(x) - [P(x) + \frac{\delta P(x)}{\delta x} \cdot \delta x] = P(x)\delta x. \quad (A.2')
\]

Hence

\[
\frac{\delta P(x)}{P(x)} = -\delta x \quad (A.3)
\]

or

\[
P(x) = \gamma \exp(-x). \quad (A.4)
\]

Here the integration constant \(\gamma\) can be determined by the condition \(P(0) = 1\). Thus, one obtain \(\gamma = 1\) and

\[
P(-a_c^2 k^2) = \exp(a_c^2 k^2) \quad \left(0 < \exp(a_c^2 k^2) < 1 \text{ for } 0 < -k^2 < \infty\right), \quad (A.5)
\]

where \(a_c\) is determined so that our static quark-antiquark potential, Eq.(28) fits experimental data at short or long distances.
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[12] \[ \pi^{-1} \lim_{r \to 0} \sum_{n=1}^{N} P_n(a, r, n) = \pi^{-1} \sum_{n=1}^{N} \lim_{r \to 0} P_n(a, r, n) = P(a, 0, N) = 0 \text{ for the finite } N. \]

[13] For \( N \gg 1 \), we may consider \( n \) as continuous. Then, we have

\[
P(a, R, N) = \pi^{-1} \sum_{n=1}^{N} n^{-1} \exp(-\kappa_n R) \sin \kappa_n R
\]

\[
= \pi^{-1} \sum_{n=1}^{N} N^{-1} (n^{-1} N) \exp \left( -2^{-1} \cdot 2^{-\frac{1}{2}} \left( \frac{n}{N} \right)^\frac{1}{2} \right) \sin \left( 2^{-1} \cdot 2^{-\frac{1}{2}} \left( \frac{n}{N} \right)^\frac{1}{2} \right)
\]

\[
\xrightarrow{N \to \infty} \pi^{-1} \int_{0}^{1} dx \, x^{-1} \exp(-2^{-1} \cdot 2^{-\frac{1}{2}} x^\frac{1}{2}) \sin(2^{-1} \cdot 2^{-\frac{1}{2}} x^\frac{1}{2})
\]

\[
= 2\pi^{-1} \int_{0}^{1} dy \, y^{-1} \exp(-2^{-1} \cdot 2^{-\frac{1}{2}} y) \sin(2^{-1} \cdot 2^{-\frac{1}{2}} y)
\]

\[
= 0.1884 \text{ numerically } (P(a, R, N) < 0.1884),
\]
where

\[ \kappa_n R = (\pi^{\frac{1}{2}}a^{-1})RN^{\frac{1}{2}}(\frac{n}{N})^{\frac{1}{2}} \]
\[ = (\pi^{\frac{1}{2}}a^{-1})(2^{-1}(2\pi)^{-\frac{1}{2}})a(\frac{n}{N})^{\frac{1}{2}} \]
\[ = 2^{-1} \cdot 2^{-\frac{1}{2}}(\frac{n}{N})^{\frac{1}{2}} \]

and \( x = y^2 \), respectively.

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