SOME OPEN PROBLEMS AND CONJECTURES ON
SUBMANIFOLDS OF FINITE TYPE: REVISITED

BANG-YEN CHEN

Abstract. Submanifolds of finite type were introduced by the author during the late 1970s (cf. [C1-C4]). The first results on this subject had been collected in author's books [C4,C7]. A detailed survey on results on finite submanifolds up to 1996 was given in [C25]. Moreover, a list of ten open problems and three conjectures on submanifolds of finite type was published in [C18, 1981]. The main purpose of this article is thus to provide some updated information on the three conjectures listed in [C18].

1. Preliminaries

Let \( x : M \to \mathbb{E}^m \) be an immersion of an \( n \)-dimensional, connected manifold \( M \) into the Euclidean \( m \)-space \( \mathbb{E}^m \). With respect to the Riemannian metric \( g \) on \( M \) induced from the Euclidean metric of the ambient space \( \mathbb{E}^m \), \( M \) is a Riemannian manifold. Denote by \( \Delta \) the Laplacian operator of the Riemannian manifold \( (M, g) \). The immersion \( x \) is said to be of finite type if each component of the position vector field of \( M \) in \( \mathbb{E}^m \), also denoted by \( x \), can be written as a finite sum of eigenfunctions of the Laplacian operator, that is, if

\[
x = c + x_1 + x_2 + \ldots + x_k
\]  

(1.1)

where \( c \) is a constant vector, \( x_1, \ldots, x_k \) are non-constant maps satisfying \( \Delta x_i = \lambda_i x_i, \ i = 1, \ldots, k \). In particular, if all eigenvalues \( \{\lambda_1, \ldots, \lambda_k\} \) are mutually different, then the immersion \( x \) (or the submanifold \( M \)) is said to be of \( k \)-type. And the decomposition (1.1) is called the spectral decomposition (or the spectral resolution) of the immersion \( x \). In particular, if one of \( \{\lambda_1, \ldots, \lambda_k\} \) is zero, then the immersion is said to be of null \( k \)-type. Just like minimal submanifolds, submanifolds of finite type can be described by a spectral variation.
principle, namely as critical points of directional deformations (see, \[CDVV2, CDVV4, C27\] for details).

A submanifold is said to be of infinite type if it is not of finite type. It is clear that every submanifold of null \(k\)-type is non-compact. If \(x(M)\) is contained in a hypersphere \(S^{m-1}\) of \(E^m\), then the immersion \(x\) is said to be mass-symmetric in \(S^{m-1}\) if the constant vector \(c\) in the spectral decomposition (1.1) is the center of the hypersphere \(S^{m-1}\) in \(E^m\).

In terms of finite type submanifolds, a result of Takahashi \([Ta1]\) says that a submanifold of a Euclidean \(m\)-space \(E^m\) is of 1-type if and only if it is either a minimal submanifold of \(E^m\) or it is a minimal submanifold of a hypersphere of the Euclidean space.

Let \(M\) be a finite type submanifold whose spectral decomposition is given by (1.1). If we define a polynomial \(P\) by
\[
P(t) = \prod_{i=1}^{k}(t - \lambda_i),
\]
then \(P(\Delta)(x - c) = 0\). The polynomial \(P\) is called the minimal polynomial of the finite type submanifold \(M\). For an \(n\)-dimensional submanifold \(M\) of a Euclidean space, the mean curvature vector \(H\) of \(M\) satisfies
\[
\Delta x = -nH.
\]
From (1.3) we see that the minimal polynomial \(P\) of the \(k\)-type submanifold \(M\) also satisfies the condition \(P(\Delta)H = 0\). Conversely, if \(M\) is compact and if there exists a constant vector \(c\) and a nontrivial polynomial \(P\) such that \(P(\Delta)(x - c) = 0\) (or \(P(\Delta)H = 0\), then \(M\) is of finite type (see, \[C2, C4, C7\].) This characterization of finite type submanifolds in terms of the minimal polynomial plays an important role in the study of submanifolds of finite type.

If \(M\) is non-compact, then the existence of a nontrivial polynomial \(P\) such that that \(P(\Delta)H = 0\) does not imply that \(M\) is of finite type in general. However, if \(M\) is 1-dimensional or \(P\) is a polynomial of degree \(k\) which has exactly \(k\) distinct roots, then the existence of the polynomial \(P\) satisfying the condition \(P(\Delta)(x - c) = 0\) for constant
vector $c$ guarantees that $M$ is of finite type (in fact, it is of $k$-type with $k \leq \deg(P)$) (see, [CP]).

The following formula of $\Delta H$ obtained in [C2, C4, C7] plays an important role in the study of submanifolds of low type and also in the study of biharmonic submanifolds (cf. §5.1).

$$\Delta H = \Delta^D H + \sum_{i=1}^{n} h(e_i, A_H e_i) + 2 \tr(A_{DH}) + \frac{n}{2} \grad \langle H, H \rangle,$$  

(1.4)

where $\Delta^D$ is the Laplacian operator associated with the normal connection $D$, $h$ the second fundamental form, and $\{e_1, \ldots, e_n\}$ a local orthonormal frame of $M$. In particular, if $M$ is a hypersurface of a Euclidean space $\mathbb{E}^{n+1}$ (respectively, if $M$ is a hypersurface of the unit hypersphere of $\mathbb{E}^{n+2}$ centered at the origin), then formula (1.4) yields

$$\Delta H = (\Delta \alpha + \alpha \|h\|^2) \xi + 2 \tr(A_{DH}) + \frac{n}{2} \grad \langle H, H \rangle,$$  

(1.5)

(respectively,

$$\Delta H = (\Delta \alpha' + \alpha' \|h\|^2) \xi - n \langle H, H \rangle x + 2 \tr(A_{DH}) + \frac{n}{2} \grad \langle H, H \rangle,$$  

(1.6)

where $\alpha$ is the mean curvature and $\xi$ a unit normal vector of $M$ in $\mathbb{E}^{n+1}$ (respectively, $\alpha'$ is the mean curvature and $\xi$ a unit normal vector of $M$ in $S^{n+1}$.)

A similar formula of $\Delta H$ holds for submanifolds in pseudo-Euclidean spaces. Also, similar formulas of $\Delta H$ for submanifolds in the hyperbolic hypersurface $H^{m-1}$ and in the anti-de Sitter hypersurface $H^{m-1}_i$ also had been obtained in [C6, C9].

2. Submanifolds of Finite Type

2.1. Finite Type Hypersurfaces of Euclidean Space. The class of finite type submanifolds is very large. For instances, minimal submanifolds of a Euclidean space, minimal submanifolds of a hypersphere are of 1-type and compact homogeneous submanifolds, equivariantly immersed, are of finite type [C4] (see, also [De1,De2] and [Ta2] for irreducible case). But very few is know about the most elementary submanifolds of Euclidean space, namely hypersurfaces of a Euclidean
space, in particular, surfaces in Euclidean 3-space. So far no surfaces of finite type in $\mathbb{E}^3$ are known, other than minimal surfaces, circular cylinders and the spheres. Therefore, the following problem seems to be very interesting (cf. [C7, C11]).

**Problem 1.** Classify all finite type hypersurfaces in $\mathbb{E}^{n+1}$. In particular, classify all finite type surfaces in $\mathbb{E}^3$.

Related to this problem we recall that every compact 2-type hypersurface of $\mathbb{E}^{n+1}$ has non-constant mean curvature [CLn] (this had also been pointed out by Garay, see [CLn]). And if $n = 1$, this problem has a complete solution. In fact, it was proved in [C3, C4] that circles are the only finite type closed planar curves. It was pointed out in [C14] that lines are the only non-closed planar curves of finite type (in fact, it is the only null finite type planar curves) (cf. [CDVV] for the details.)

The first result concerning the classification of finite type surfaces in Euclidean 3-space was obtained in [C11] which stated that circular cylinders are the only tubes of finite type. In [G2] it shown that a cone in $\mathbb{E}^3$ is of finite type if and only if it is a plane. It was proved in [CDVV] that a ruled surface in $\mathbb{E}^3$ is of finite type if and only if it is a plane, a circular cylinder or a helicoid. Furthermore, [CD2] proved that spheres and circular cylinders are the only quadrics of finite type in $\mathbb{E}^3$. It seems that the only surfaces of finite type in $\mathbb{E}^3$ are minimal surfaces, spheres and circular cylinders. For compact finite type surfaces in $\mathbb{E}^3$, the author made the following conjecture in [C7, C18].

**Conjecture 1.** The only compact finite type surfaces in $\mathbb{E}^3$ are the spheres.

Some partial affirmative answers to this conjecture were given in [B, AGM, C11, DDV1, DPV, HV4, V2]. However, Conjecture 1 remains open.

It also seems to the author that hyperspheres are the only compact hypersurfaces of finite type in Euclidean space. However, this seems to be a quite difficult problem.
2.2. **Finite Type Hypersurfaces of Hyperspheres.** For finite type hypersurfaces of a hypersphere the following problem is also quite interesting.

**Problem 2.** Classify finite type hypersurfaces of a hypersphere in $\mathbb{E}^{n+2}$.

In contrast with hypersurfaces of finite type in Euclidean space, there exist many examples of 1-type hypersurfaces as well as many examples of (mass-symmetric) 2-type hypersurfaces in a hypersphere of $\mathbb{E}^{n+2}$. In fact, it was first proved in [8] that every isoparametric hypersurface of a hypersphere is either of 1-type or mass-symmetric and of 2-type. Since there are ample examples of non-minimal isoparametric hypersurfaces in a hypersphere, we have ample examples of mass-symmetric 2-type hypersurfaces in a hypersphere. It was proved that every 3-type spherical hypersurface has non-constant mean curvature ([15], [14]). Moreover, [CD1] proved that standard 2-spheres in $S^3$ and products of plane circles are the only finite type compact surfaces with constant Gauss curvature in $S^3$. From all the information we have, it seems to the author that there exist no surfaces of $k$-type in $S^3$ for any finite $k$ greater than 2.

At an international conference held at Berlin in the summer of 1990 the author had announced the following conjecture concerning finite type surface in a 3-sphere.

**Conjecture 2.** Minimal surfaces, standard 2-spheres and products of plane circles are the only finite type surfaces in $S^3$ (imbedded standardly in $\mathbb{E}^4$).

Some partial affirmative answers to this conjecture were obtained in [15], [CBG], [14], [FL], [Na].

**Remark 1.** This conjecture stays open.
3. 2-type Submanifolds

2-type submanifolds are the “simplest submanifolds” next to minimal submanifolds. In particular, 2-type submanifolds mass-symmetric spherical 2-type submanifolds, deserve special attention.

Mass-symmetric spherical 2-type submanifolds have some special properties. For instances, every mass-symmetric spherical 2-type submanifolds has constant mean curvature which is completely determined by its order \( C^4 \). Moreover, such a submanifold is pointwise orthogonal (in the sense of §4.)

3.1. Spherical 2-type hypersurfaces. The first classification theorem of 2-type surfaces of a hypersphere was obtained in \( [C3, C4] \) which says that a compact surface of a hypersphere \( S^3 \) in \( E^4 \) is the product of two plane circles with different radii if and only if it is mass-symmetric and of 2-type. \( [BG1] \) proved that the assumption on “mass-symmetry” can be removed. It was proved in \( [HV1] \) that the above classification of 2-type surfaces in \( S^3 \) indeed is of local nature which says that the same result still holds without the assumption of compactness.

For a 2-type hypersurface \( M \) of a hypersphere \( S^{n+1} \), it was proved in \( [CS] \) that every mass-symmetric 2-type hypersurface \( M \) of \( S^{n+1} \) has nonzero constant mean curvature in \( S^{n+1} \) and constant scalar curvature (this result is also of local nature, because the proof of this fact given in \( [CS] \) did not use compactness of \( M \) at all).

It was shown in \( [CBG] \) that a spherical 2-type hypersurface is mass-symmetric if and only if it has constant mean curvature. Locally, a hypersurface \( M \) of \( S^{n+1} \) is the product of two spheres \( S^p(r_1) \times S^{n-p}(r_2) \) with \( r_1^2 + r_2^2 = 1 \) and \( (r_1, r_2) \neq (\sqrt{\frac{p}{n}}, \sqrt{\frac{n-p}{n}}) \) if and only if \( M \) is of 2-type and it has at most two distinct principal curvatures \( [C16] \) (see, also \( [CBG] \) for compact case). Recently, \( [HV2] \) proved that every 2-type hypersurface of a hypersphere \( S^{n+1} \) is mass-symmetric and hence it has constant mean curvature and constant scalar curvature.
Conversely, it was known that each hypersurface of $S^{n+1}$ with nonzero constant mean curvature in $S^{n+1}$ and constant scalar curvature is mass-symmetric and of 2-type unless it is a portion of small hypersphere of $S^{n+1}$ (a result obtained in [CS] for compact case. For noncompact case, this fact is a consequence of formula (1.6) and Proposition 4.3 of [CP] (see [CL]) and this was also pointed out in [HV2] independently). For spherical 2-type hypersurfaces, the author had proposed in [C18] the following

**Problem 3.** Study and classify 2-type hypersurfaces in a hypersphere of $E^{n+2}$. In particular, classify 3-dimensional 2-type hypersurfaces of a hypersphere $S^4$ in $E^5$.

Since every spherical 2-type hypersurface is mass-symmetric, a result of [CS] implies that if either $M$ is a 3-dimensional 2-type Dupin hypersurface of $S^4$ or $M$ is a spherical 2-type Dupin hypersurface with at most 3 distinct principal curvatures, then the Dupin hypersurface is isoparametric. For a general Dupin hypersurfaces, the author asked in [C18] the following

**Problem 4.** When is a Dupin hypersurface $M$ of a hypersphere of finite type? When is a finite type Dupin hypersurface of a hypersphere isoparametric?

In contrast with the existence of many compact 2-type hypersurfaces in hyperspheres of a Euclidean space, there exits no compact 2-type hypersurface in the hyperbolic space $H^{n+1}$ (imbedded standardly in the Minkowski space-time $E_1^{n+2}$ by the equation $\langle x, x \rangle = -1, \ t > 0$), although there exist complete, non-compact 2-type hypersurfaces in $H^{n+1}$. (See [C9, C19] for more results on finite type submanifolds in pseudo-Euclidean spaces, in particular, on finite type submanifolds in hyperbolic spaces and in de Sitter space-times. For instances, 2-type surfaces in hyperbolic 3-spaces have been classified in [C19] and it is also known in [C19] that every space-like 2-type hypersurface of the de Sitter space-time $S^{n+1}_1$, imbedded standardly in $E^{n+2}_1$, has nonzero constant mean curvature and constant scalar curvature.)
3.2. 2-type Submanifolds of Codimension 2. It was proved in \[\text{[BC1]}\] that there exist no mass-symmetric surfaces in \(S^4\). So far there are no known examples of non-mass-symmetric 2-type surfaces in \(S^4\). In this respect, the author asked in \[\text{[C18]}\] the following

**Problem 5.** Do there exist (non-mass-symmetric) 2-type surfaces in \(S^4\)?

\[\text{[CL]}\] showed that a 2-type submanifold with parallel mean curvature vector is either spherical or of null 2-type. Related with this fact, the author had asked in \[\text{[C18]}\] the following

**Problem 6.** Is every \(n\)-dimensional non-null 2-type submanifold of \(\mathbb{E}^{n+2}\) with constant mean curvature spherical?

If the answer to Problem 6 is affirmative, then an \(n\)-dimensional 2-type submanifold of \(\mathbb{E}^{n+2}\) is spherical if and only if it is non-null and it has constant mean curvature.

3.3. 2-type Spherical Surfaces of Higher Codimension. There exist ample examples of 1-type (mass-symmetric) surfaces which lie fully in odd-dimensional spheres as well as in even-dimensional spheres. Although there exists abundant examples of mass-symmetric 2-type surfaces lying fully in odd-dimensional hyperspheres (cf. for instances, \[\text{[BC1, C4, G1, Ko, Mi]}\]), in contrast there exist no examples of mass-symmetric 2-type surfaces which lie fully in a hypersphere of a Euclidean space for any even codimension. Hence, it is natural to ask the following problem which is more general than Problem 5 (see, also \[\text{[Ko]}\]).

**Problem 7.** Do there exist 2-type surfaces which lie fully in an even-dimensional hypersphere of a Euclidean space? In particular, do there exist mass-symmetric 2-type surfaces which lie fully in an even-dimensional hypersphere?

The author would like to point out that the answer to Problem 7 is negative if the mass-symmetric spherical 2-type surface is one of the following surfaces:

- a stationary surface \[\text{[BC1]},\]
• a topological 2-sphere \([Ko1]\),
• a surface with constant Gauss curvature \([Mi]\),
• a flat Chen surface \([G1]\).

4. Linearity Independent Submanifolds.

The notion of linearly independent immersions and linearly independent submanifolds were defined as follows \([C17]\):

Let \(x : M \to \mathbb{E}^m\) be a \(k\)-type isometric immersion whose spectral decomposition is given by (1.1). Denote by \(E_i\) the subspace of \(\mathbb{E}^m\) spanned by \(\{x_i(p), p \in M\} (i \in \{1, \ldots, k\})\). The immersion \(x\) (or the submanifold \(M\)) is said to be linearly independent if the subspaces \(E_1, \ldots, E_k\) are linearly independent. And the immersion \(x\) (or the submanifold) is said to be orthogonal if the subspaces \(E_1, \ldots, E_k\) are mutually orthogonal.

Clearly, every orthogonal immersion is a linearly independent immersion and every 1-type immersion is an orthogonal immersion. There exist many examples of orthogonal immersions and abundant examples of linearly independent immersions which are not orthogonal. In fact every \(k\)-type curve lying fully in \(\mathbb{E}^{2k}\) and every null \(k\)-type curve lying fully in \(\mathbb{E}^{2k-1}\) are linearly independent curves, but \(W\)-curves are the only orthogonal curves in a Euclidean space.

For a linearly independent immersion \(x : M \to \mathbb{E}^m\) and a point \(p \in M\) one has the notion of the adjoint hyperquadric \(Q_p\) at \(p\) (cf. \([C17]\).) If submanifold \(M\) lies in one of the adjoint hyperquadrics \(Q_p\) \((p \in M)\), then all of the adjoint hyperquadrics \(Q_p\) \((p \in M)\) are the same adjoint hyperquadric. This common adjoint hyperquadric is called the adjoint hyperquadric of the linearly independent immersion \(x\).

It was shown in \([C17]\) that if \(x : M \to \mathbb{E}^m\) is a linearly independent immersion of a compact manifold \(M\), then the submanifold \(M\) is contained in its adjoint hyperquadric if and only if the submanifold \(M\) is spherical (with an appropriate center). Furthermore, a non-minimal, linearly independent immersion \(x : M \to \mathbb{E}^m\) is orthogonal if and only if \(M\) is immersed by \(x\) as a minimal submanifold of the adjoint hyperquadric \([C17]\) (for some special orthogonal immersions of compact
manifolds, see also [HV3]). As a consequence it follows that every orthogonal immersion of a compact manifold is spherical. Moreover, one may also prove that every compact homogeneous submanifold, equivariantly immersed in $\mathbb{E}^m$, is orthogonal and hence it is immersed as a minimal submanifold in its adjoint hyperquadric ( [C17]). Linearly independent hypersurfaces of a Euclidean space are hyperspheres, minimal hypersurfaces or spherical hypercylinders ( [CP, HV4, CDVV2]). (See also [G3] and [DPV] for some special cases.) By applying the classification theorem of 2-type curves in Euclidean space obtained in [CDV], we may conclude that the only linearly independent curves of codimension 2 in a Euclidean space are circles, lines and circular helices. As we already know, there are abundant examples of linearly independent curves of codimension 3 in Euclidean space.

In views of these, the author had proposed the following two problems concerning linearly independent immersions.

**Problem 8.** Study and classify linearly independent 2-type immersions.

**Problem 9.** Study and classify linearly independent submanifolds of codimension 2.

Let $x : M \to \mathbb{E}^m$ be a $k$-type isometric immersion whose spectral decomposition is given by (1.1). The immersion is said to be **pointwise linearly independent** (respectively, **pointwise orthogonal**) if, for each point $p \in M$, the $k$ vectors $x_1(p), \ldots, x_k(p)$ are linearly independent (respectively, are orthogonal.)

The class of pointwise linearly independent submanifolds and pointwise orthogonal submanifolds are much wider than the class of linearly independent submanifolds and orthogonal submanifolds. For example, every mass-symmetric spherical 2-type submanifold is pointwise orthogonal, although it is not orthogonal in general. (This follows from the definition of mass-symmetric spherical 2-type submanifolds and Theorem 4.1 of p.274 of [C4] (see [Mi]).)
One may study similar problems for pointwise linearly independent submanifolds and for pointwise orthogonal submanifolds. However, these seem to be quite difficult.

5. Biharmonic submanifolds.

Let \( x : M \to \mathbb{E}^m \) be an isometric immersion. As we mentioned in Preliminaries, the position vector of \( M \) in \( \mathbb{E}^m \) satisfies
\[
\Delta x = -nH.
\] (5.1)

Formula (5.1) implies that the immersion is minimal if and only if the immersion is harmonic, that is, \( \Delta x = 0 \). An isometric immersion \( x : M \to E^m \) is called biharmonic if we have
\[
\Delta^2 x = 0, \quad \text{that is,} \quad \Delta H = 0.
\] (5.2)

It is obvious that minimal immersions are biharmonic.

In [C18] the author had asked the following simple geometric question (see also [CI1]).

**Problem 10.** Other than minimal submanifolds of \( \mathbb{E}^m \), which submanifolds of \( \mathbb{E}^m \) are biharmonic?

The study of biharmonic submanifolds was initiated by the author in the middle of 1980s in his program of understanding the finite type submanifolds in Euclidean spaces; also independently by G.-Y. Jiang [J] for his study of Euler-Lagrange’s equation of bienergy functional in the sense of Eells and Lemaire.

The author showed in 1985 that biharmonic surfaces in \( \mathbb{E}^3 \) are minimal (unpublished then, also independently by Jiang [J]). This result was the starting point of I. Dimitric’s work on his doctoral thesis at Michigan State University (cf. [Di0]). In particular, Dimitric extended author’s unpublished result to biharmonic hypersurfaces of \( \mathbb{E}^m \) with at most two distinct principal curvatures [Di0]. In his thesis, Dimitric also proved that every biharmonic submanifold of finite type in \( \mathbb{E}^m \) is minimal. Another extension of this result on biharmonic surfaces was given by T. Hasanis and T. Vlachos in [HV5] (see also [Dei]). They proved that biharmonic hypersurfaces of \( \mathbb{E}^4 \) are minimal.
The author made in [C18] the following Biharmonic Conjecture.

**Conjecture 3**: The only biharmonic submanifolds of Euclidean spaces are the minimal ones.

A biharmonic map is a map \( \phi : (M, g) \to (N, h) \) between Riemannian manifolds that is a critical point of the bienergy functional:

\[
E^2(\phi, D) = \frac{1}{2} \int_D ||\tau_\phi||^2 \ast 1
\]

for every compact subset \( D \) of \( M \), where \( \tau_\phi = \text{trace}_g \nabla d\phi \) is the tension field \( \phi \). The Euler-Lagrange equation of this functional gives the biharmonic map equation (see [J])

\[
\tau^2_\phi := \text{trace}_g (\nabla^\phi \nabla^\phi - \nabla^\phi_{\nabla^M}) \tau_\phi - \text{trace}_g R^N(d\phi, \tau_\phi) d\phi = 0,
\]

where \( R^N \) is the curvature tensor of \((N, h)\). Equation (5.4) states that \( \phi \) is a biharmonic map if and only if its bi-tension field \( \tau^2_\phi \) vanishes.

Let \( M \) be an \( n \)-dimensional submanifold of a Euclidean \( m \)-space \( \mathbb{E}^m \). If we denote by \( \iota : M \to \mathbb{E}^m \) the inclusion map of the submanifold, then the tension field of the inclusion map is given by \( \tau_\iota = -\Delta \iota = -nH \) according to Beltrami’s formula. Thus \( M \) is a biharmonic submanifold if and only if \( n\Delta H = -\Delta^2 \iota = -\tau^2_\iota = 0 \), i.e., the inclusion map \( \iota \) is a biharmonic map.

Caddeo, Montaldo and Oniciuc [CMO2] proved that every biharmonic surface in the hyperbolic 3-space \( H^3(-1) \) of constant curvature \(-1\) is minimal. They also proved that biharmonic hypersurfaces of \( H^n(-1) \) with at most two distinct principal curvatures are minimal [CMO1]. Based on these, Caddeo, Montaldo and Oniciuc made in [CMO1] the following generalized biharmonic conjecture.

*Every biharmonic submanifold of a Riemannian manifold with non-positive sectional curvature is minimal.*

The study of biharmonic submanifolds is nowadays a very active research subject. In particular, since 2000 biharmonic submanifolds have been receiving a growing attention and have become a popular subject of study with many progresses.
5.1. Recent developments on my original biharmonic conjecture. Let \( x : M \to \mathbb{E}^m \) be an isometric immersion of a Riemannian \( n \)-manifold \( M \) into a Euclidean \( m \)-space \( \mathbb{E}^m \). Then \( M \) is biharmonic if and only if it satisfies the following fourth order strongly elliptic semi-linear PDE system (see, for instance, [C2, C3, C4, C27])

\[
\begin{aligned}
\Delta^D H + \sum_{i=1}^{n} \sigma(A_H e_i, e_i) &= 0, \\
n \nabla \langle H, H \rangle + 4 \text{ trace } A_{DH} &= 0,
\end{aligned}
\]

where \( \Delta^D \) is the Laplace operator associated with the normal connection \( D \), \( \sigma \) the second fundamental form, \( A \) the shape operator, \( \nabla \langle H, H \rangle \) the gradient of the squared mean curvature, and \( \{e_1, \ldots, e_n\} \) an orthonormal frame of \( M \).

An immersed submanifold \( M \) in a Riemannian manifold \( N \) is said to be properly immersed if the immersion is a proper map, i.e., the preimage of each compact set in \( N \) is compact in \( M \).

The total mean curvature of a submanifold \( M \) in a Riemannian manifold is given by \( \int_M |H|^2 \, dv \).

Denote by \( K(\pi) \) the sectional curvature of a given Riemannian \( n \)-manifold \( M \) associated with a plane section \( \pi \subset T_pM, p \in M \). For any orthonormal basis \( e_1, \ldots, e_n \) of the tangent space \( T_pM \), the scalar curvature \( \tau \) at \( p \) is defined to be \( \tau(p) = \sum_{i<j} K(e_i \wedge e_j) \).

Let \( L \) be a subspace of \( T_pM \) of dimension \( r \geq 2 \) and \( \{e_1, \ldots, e_r\} \) an orthonormal basis of \( L \). The scalar curvature \( \tau(L) \) of \( L \) is defined by

\[
\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r.
\]

For an integer \( r \in [2, n - 1] \), the \( \delta \)-invariant \( \delta(r) \) of \( M \) is defined by (cf. [?], [C26, C27])

\[
\delta(r)(p) = \tau(p) - \inf \{\tau(L)\}, \quad (5.5)
\]

where \( L \) run over all \( r \)-dimensional linear subspaces of \( T_pM \).

For any \( n \)-dimensional submanifold \( M \) in \( \mathbb{E}^m \) and any integer \( r \in [2, n - 1] \), the author proved the following general sharp inequality (cf.
A submanifold in $\mathbb{E}^m$ is called $\delta(r)$-ideal if it satisfies the equality case of \((5.6)\) identically. Roughly speaking ideal submanifolds are submanifolds which receive the least possible tension from its ambient space (cf. \([C26, C11]\)).

A hypersurface of a Euclidean space is called weakly convex if it has non-negative principle curvatures.

It follows immediately from the definition of biharmonic submanifolds and Hopf’s lemma that every biharmonic submanifold in a Euclidean space is non-compact.

The following provides an overview of some affirmative partial solutions to my original biharmonic conjecture.

- Biharmonic surfaces in $\mathbb{E}^3$ (B.-Y. Chen \([C18, C27]\) and G. Y. Jiang \([J]\)).
- Biharmonic curves (I. Dimitric \([Di0, Di1]\)).
- Biharmonic hypersurfaces in $\mathbb{E}^4$ (T. Hasanis and T. Vlachosin \([HV5]\)) (a different proof by F. Defever \([Def]\)).
- Spherical submanifolds (B.-Y. Chen \([C18]\)).
- Biharmonic hypersurfaces with at most 2 distinct principle curvatures (I. Dimitric \([Di0]\)).
- Biharmonic submanifolds of finite type (I. Dimitric \([Di0, Di1]\)).
- Pseudo-umbilical biharmonic submanifolds (I. Dimitric \([Di1]\)).
- Biharmonic submanifolds which are complete and proper (Akh tagawa and Maeta \([AM]\)).
- Biharmonic properly immersed submanifolds (S. Maeta \([Ma2]\)).
- Biharmonic submanifolds satisfying the decay condition at infinity

$$\lim_{\rho \to \infty} \frac{1}{\rho^2} \int_{f^{-1}(B_\rho)} |H|^2 dv = 0,$$

where $f$ is the immersion, $B_\rho$ is a geodesic ball of $N$ with radius $\rho$ (G. Wheeler \([Wh]\)).
• Submanifolds whose $L^p$, $p \geq 2$, integral of the mean curvature vector field satisfies certain decay condition at infinity (Y. Luo [Lu03]).

• $\delta(2)$-ideal and $\delta(3)$-ideal biharmonic hypersurfaces (B.-Y. Chen and M. I. Munteanu [CMu]).

• Weakly convex biharmonic submanifolds (Y. Luo in [Luo1]).

In [Ou1], Y.-L. Ou constructed examples to show that my original biharmonic conjecture cannot be generalized to the case of biharmonic conformal submanifolds in Euclidean spaces.

Remark 2. Conjecture 3 remains open.

Remark 3. Conjecture 3 is false if the ambient Euclidean space were replaced by a pseudo-Euclidean space. The simplest examples are constructed by Chen and Ishikawa in [CI1]. For instance, we have the following.

Example. Let $f(u, v)$ be a proper biharmonic function, i.e. $\Delta f \neq 0$ and $\Delta^2 f = 0$. Then

$$x(u, v) = (f(u, v), f(u, v), u, v)$$

defines a biharmonic, marginally trapped surface in the Minkowski 4-space $\mathbb{E}_4^4$ with the Lorentzian metric $g_0 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2$.

Here, by a marginally trapped surface, we mean a space-like surface in $\mathbb{E}_4^4$ with light-like mean curvature vector field.

It was proved in [CI1] that the biharmonic surfaces defined by (5.7) are the only biharmonic, marginally trapped surfaces in $\mathbb{E}_4^4$.

5.2. Recent developments on generalized biharmonic conjecture. Let $M$ be a submanifold of a Riemannian manifold with inner product $\langle \cdot, \cdot \rangle$, then $M$ is called $\epsilon$-superbiharmonic if

$$\langle \Delta H, H \rangle \geq (\epsilon - 1)|\nabla H|^2,$$

where $\epsilon \in [0, 1]$ is a constant. For a complete Riemannian manifold $(N, h)$ and $\alpha \geq 0$, if the sectional curvature $K^N$ of $N$ satisfies

$$K^N \geq -L(1 + \text{dist}_N(\cdot, q_0)^2)^\frac{\alpha}{2},$$
for some $L > 0$ and $q_0 \in N$, then we call that $K^N$ has a polynomial growth bound of order $\alpha$ from below.

There are also many affirmative partial answers to the generalized Chen’s biharmonic conjecture. The following provides a brief overview of the affirmative partial answers to this generalized conjecture.

- Biharmonic hypersurfaces in the hyperbolic 3-space $H^3(-1)$ (Caddeo, Montaldo and Oniciuc [CMO1]).
- Biharmonic hypersurfaces in $H^4(-1)$ (Balmuş, Montaldo and Oniciuc [BMO4]).
- Pseudo-umbilical biharmonic submanifolds of $H^m(-1)$ (Caddeo, Montaldo and Oniciuc [CMO1]).
- Biharmonic hypersurfaces of $H^{n+1}(-1)$ with at most two distinct principal curvatures (Balmuş, Montaldo and Oniciuc [BMO1]).
- Totally umbilical biharmonic hypersurfaces in Einstein spaces (Y.-L. Ou [Ou2]).
- Biharmonic hypersurfaces with finite total mean curvature in a Riemannian manifold of non-positive Ricci curvature (Nakauchi and Urakawa [NU1]).
- Biharmonic submanifolds with finite total mean curvature in a Riemannian manifold of non-positive sectional curvature (Nakauchi and Urakawa [NU2]).
- Complete biharmonic hypersurfaces $M$ in a Riemannian manifold of non-positive Ricci curvature whose mean curvature vector satisfies $\int_M |H|^\alpha dv < \infty$ for some $\epsilon > 0$ with $1 + \epsilon \leq \alpha < \infty$ (S. Maeta [Ma4]).
- Biharmonic properly immersed submanifolds in a complete Riemannian manifold with non-positive sectional curvature whose sectional curvature has polynomial growth bound of order less than 2 from below (S. Maeta [Ma3]).
- Complete biharmonic submanifolds with finite bi-energy and energy in a non-positively curved Riemannian manifold (N. Nakauchi, H. Urakawa and S. Gudmundsson [NUG]).
• Complete oriented biharmonic hypersurfaces $M$ whose mean curvature $H$ satisfying $H \in L^2(M)$ in a Riemannian manifold with non-positive Ricci tensor (Alias, García-Martínez and Rigoli [AGR]).

• Compact biharmonic submanifolds in a Riemannian manifold with non-positive sectional curvature (G.-Y. Jiang [J] and S. Maeta [Ma4]).

• $\epsilon$-superbiharmonic submanifolds in a complete Riemannian manifolds satisfying the decay condition at infinity

$$\lim_{\rho \to \infty} \frac{1}{\rho^2} \int_{f^{-1}(B_\rho)} |H|^2 dv = 0,$$

where $f$ is the immersion, $B_\rho$ is a geodesic ball of $N$ with radius $\rho$ (G. Wheeler [Wh]).

• Complete biharmonic submanifolds (resp., hypersurfaces) $M$ in a Riemannian manifold of non-positive sectional (resp., Ricci) curvature whose mean curvature vector satisfies $\int_M |H|^p dv < \infty$ for some $p > 0$ (Y. Luo [Luo2]).

• Complete biharmonic submanifolds (resp., hypersurfaces) in a Riemannian manifold whose sectional curvature (resp., Ricci curvature) is non-positive with at most polynomial volume growth (Y. Luo [Luo2]).

• Complete biharmonic submanifolds (resp., hypersurfaces) in a negatively curved Riemannian manifold whose sectional curvature (resp., Ricci curvature) is smaller that $-\epsilon$ for some $\epsilon > 0$ (Y. Luo [Luo2]).

• Proper $\epsilon$-superharmonic submanifolds $M$ with $\epsilon > 0$ in a complete Riemannian manifold $N$ whose mean curvature vector satisfying the growth condition

$$\lim_{\rho \to \infty} \frac{1}{\rho^2} \int_{f^{-1}(B_\rho)} |H|^{2+a} dv = 0,$$

where $f$ is the immersion, $B_\rho$ is a geodesic ball of $N$ with radius $\rho$, and $a \geq 0$ (Luo [Luo2]).

On the other hand, it was proved by Y.-L. Ou and L. Tang in [OuT] that the generalized Chen’s biharmonic conjecture is false in general
by constructing foliations of proper biharmonic hyperplanes in a 5-
dimensional conformally flat space with negative sectional curvature.

Further counter-examples were constructed in [LO] by T. Liang and
Y.-L. Ou.

5.3. Two related biharmonic conjectures. Now, I present two bi-
harmonic conjectures related to my original biharmonic conjecture.

**Biharmonic Conjecture for Hypersurfaces:** *Every bihar-
monic hypersurface of Euclidean spaces is minimal.*

The global version of my original biharmonic conjecture can be found,
for instance, in [AM, Ma4].

**Global Version of Chen’s biharmonic Conjecture:** *Every complete biharmonic submanifold of a Euclidean space is minimal.*

6. Null 2-type submanifolds.

From the definition of null 2-type submanifolds and formula (1.3), it
follows that the mean curvature vector $H$ of a null 2-type submanifold
satisfies the following simple condition:

$$\Delta H = cH,$$

for some non-zero constant $c$. In fact beside biharmonic submanifolds,
null 2-type submanifolds and 1-type submanifolds are the only sub-
manifolds of a Euclidean space whose mean curvature vector satisfies
condition (5.3) for some constant $c$ (Lemma 1 of [C14]). Hence, null
2-type submanifolds (together with biharmonic submanifolds, minimal
submanifolds of Euclidean space and minimal submanifolds of hyper-
sphere) are the special class of submanifolds which can be characterized
by the simple geometric condition (5.3). Since the author had classified
biharmonic surfaces and null 2-type surfaces in $\mathbb{E}^3$, the classification
of surfaces in $\mathbb{E}^3$, satisfying condition (5.3) had been done in the 1980s.

From the classification of finite type planar curves, we know that
there exist no null 2-type curves in a plane. Furthermore it was proved
in [C14] that null 2-type curves in Euclidean spaces are circular heli-
ces in a Euclidean 3-space with nonzero torsion. For null 2-type sur-
faces, first, we know in [C13] that circular cylinders are the only null
2 type surfaces in Euclidean 3-space. Moreover, null 2-type surfaces in a Euclidean 4-space are helical cylinders if they have constant mean curvature [C14]. The author doesn’t know whether every null 2-type surface in a Euclidean 4-space has constant mean curvature. Also it is easy to see that formula (1.5) implies that hyperplanes, hyperspheres, and hypersurfaces of null 2-type are the only hypersurfaces of a Euclidean space which have constant mean curvature and constant scalar curvature.

As a generalization of [C13], [FL] used the same method of [C13] to study null 2-type hypersurfaces with at most two distinct principal curvatures, (see also [FGL2] for null 2-type conformally flat hypersurfaces of dimension $\neq 3$).

For null 2-type submanifolds of codimension 2, the author proposed in [C18] the following

**Problem 11.** Study and classify null 2-type submanifolds. In particular, classify all null 2-type surfaces in 4-dimensional Euclidean space and in 4-dimensional pseudo-Euclidean spaces.

### 7. Finite Type Submanifolds in Homogeneous Spaces

Let $N$ be a compact connected Riemannian homogeneous manifold with irreducible isotropy action. Let $G$ be the identity component of the group of all isometries of $N$. Then $G$ is a compact Lie group which acts on $N$ transitively. For each positive eigenvalue $\lambda$ of the Laplacian operator $\Delta$ we denote by $m_\lambda$ the multiplicity of $\lambda$. Let $\phi_1, \ldots, \phi_{m_\lambda}$ be an orthonormal basis of the eigenspace $V_\lambda$ with eigenvalue $\lambda$. We define a map $x : M \to \mathbb{E}^{m_\lambda}$ by

$$x(p) = c(\phi_1(p), \ldots, \phi_{m_\lambda}(p)),$$

where $c$ is a positive constant.

For a suitable positive constant $c$, $x$ defines an isometric 1-type immersion of $N$ into $\mathbb{E}^{m_\lambda}$. If $\lambda$ is the $i$-th positive eigenvalue of $\Delta$, then the immersion $x$ is called the $i$-th standard immersion of $N$. For example, if $N$ is a standard $n$-sphere, the first standard immersion is in fact
the standard imbedding of the $n$-sphere as a standard hypersphere of $\mathbb{E}^{n+1}$.

Similar to the studies of finite type submanifolds of a hypersphere of Euclidean space, one may consider the following

**Problem 12.** Let $N$ be an irreducible compact homogeneous manifold immersed in a Euclidean space $\mathbb{E}^N$ by its first standard immersion $\phi$ and $M$ a submanifold of $N$. When $M$ is of finite type in $\mathbb{E}^N$ via $\phi$? In particular, when $M$ is of 1- or 2-type in $\mathbb{E}^N$ via $\phi$?

When $N$ is a projective space $FP^m$ over a field $F = R, C$ or $H$ equipped with a standard Riemannian metric, this problem has been studied in $[BC2]$, $[BU]$, $[C4]$, $[C5]$, $[GR]$, $[MR]$, $[Ros1]$, $[Ros2]$, $[Ros3]$, $[UD1]$, $[Di1]$, $[Di2]$, $[Di3]$ among others. See $[BN]$ for $N$ to be either the real Grassmannian $G^R(p, q)$ or the space $U(n)/O(n)$. More recent results were obtained in $[Di4]$, $[Di5]$, $[Di6]$.

**References**

[AM] Akutagawa, K. and Maeta, S., *Biharmonic properly immersed submanifolds in Euclidean spaces*, Geom. Dedicata 164 (2013), 351–355.

[AGR] Alías, L. J., García-Martínez, S. C. and Rigoli, M., Biharmonic hypersurfaces in complex Riemannian manifolds, Pacific J. Math. 263 (2013), 1–12.

[AGM] Arroyo, J.; Garay, O. J.; Mencía, J. J., On a family of surfaces of revolution of finite Chen-type, Kodai Math. J. 21 (1998), no. 1, 73–80.

[AKP] Arvanitoyeorgos, A., Kaimakamis, G. and Palamourdas, D., Chen’s conjecture and generalizations, Symposium on the Differential Geometry of Submanifolds, Proceedings of the symposium held at the Université de Valenciennes, 109–112, 2007.

[BB1] Baikoussis, C. and Blair, D. E., 2-type integral surfaces in $S^5$, Tokyo J. Math. 14 (1991), 345–356.

[BBCD] Baikoussis, C., Blair, D. E., Chen, B.Y. and Defever, F., Hypersurfaces of restricted type in Minkowski space, Geom. Dedicata 62 (1996), 318–332.

[BCV1] Baikoussis, C., Chen, B.Y. and Verstraelen, L., Ruled surfaces and tubes with finite type Gauss maps, Tokyo J. Math. 16 (1993), 341–349.

[BDEV] Baikoussis, C., Defever, F., Embrechts, P. and Verstraelen, L., On the Gauss map of the cyclides of Dupin, Soochow J. Math. 19 (1993), 417–428.
[BDKV] Baikoussis, C., Defever, F., Koufogiorgos, T. and Verstraelen, L., Finite type immersions of flat tori into Euclidean spaces, Proc. Edinburgh Math. Soc. (2) 38 (1995), 413–420.

[BMO1] Balmuš, A., Montaldo, S. and Oniciuc, C., Classification results for biharmonic submanifolds in spheres, Israel J. Math. 168 (2008), 201–220.

[BMO2] Balmuš, A., Montaldo, S. and Oniciuc, C., Classification results and new examples of proper biharmonic submanifolds in spheres, Note Mat. 1 (2008), suppl. no. 1, 49-61.

[BMO3] Balmuš, A., Montaldo, S. and Oniciuc, C., Properties of biharmonic submanifolds in spheres, J. Geom. Symmetry Phys. 17 (2010), 87–102.

[BMO4] Balmuš, A., Montaldo, S. and Oniciuc, C., Biharmonic hypersurfaces in 4-dimensional space forms, Math. Nachr. 283 (2010), 1696–1705.

[BMO5] Balmuš, A., Montaldo, S. and Oniciuc, C., New results toward the classification of biharmonic submanifolds in $S^n$, An. St. Univ. Ovidius Constanța, 20 (2012), no. 2, 89–114.

[B] Barros, M., There exist no 2-type surfaces in $E^3$ which are images under stereographic projection of minimal surfaces in $S^3$, Ann. Global Anal. Geom. 10 (1992), no. 3, 219–226.

[BC1] Barros, M. and Chen, B. Y., Stationary 2-type surfaces in a hypersphere, J. Math. Soc. Japan 39 (1987), 627-648.

[BC2] Barros, M. and Chen, B. Y., Spherical submanifolds which are of 2-type via the second standard immersion of the sphere, Nagoya Math. J. 108 (1987), 77–91.

[BG] Barros, M. and Garay, O. J., 2-type surfaces in $S^3$, Geometriae Dedicata 24 (1987), 329–336.

[BG1] Barros, M. and Garay, O. J., On submanifolds with harmonic mean curvature, Proc. Amer. Math. Soc. 123 (1996), 545–2549.

[BU] Barros, M. and Urbano, F., Spectral geometry of minimal surfaces in the sphere, Tohoku Math. J. 39 (1987), 575–588.

[BN] Brada, D. and Niglio, L., Connected compact minimal Chen–type–1 submanifolds of Grassmannian manifold, Bull. Soc. Math. Belg. Sér. B. 44 (1992), 299–310.

[CMO1] Caddeo, R., Montaldo, S. and Oniciuc, C., Biharmonic submanifolds of $S^3$, Internat. J. Math. 12 (2001), 867–876.

[CMO2] Caddeo, R., Montaldo, S. and Oniciuc, C., Biharmonic submanifolds in spheres, Israel J. Math. 130 (2002), 109–123.

[CMP] Caddeo, R., Montaldo, S. and Piu, P., On biharmonic maps, Contemp. Math. 288 (2001), 286–290.
C1] Chen, B. Y., On the total curvature of immersed manifolds, IV: Spectrum and total mean curvature, Bull. Inst. Math. Acad. Sinica 7 (1979), 301–311.

C2] Chen, B. Y., On the total curvature of immersed manifolds, VI: Submanifolds of finite type and their applications, Bull. Inst. Math. Acad. Sinica 11 (1983), 309–328.

C3] Chen, B. Y., On submanifolds of finite type, Soochow J. Math., 9 (1983), 65-81.

C4] Chen, B. Y., Total Mean Curvature and Submanifolds of Finite Type, World Scientific Publisher, 1984.

C5] Chen, B. Y., On the first eigenvalue of Laplacian of compact minimal submanifolds of rank one symmetric spaces, Chinese J. Math., 11 (1983), 259–273.

C6] Chen, B. Y., Finite type submanifolds in pseudo-Euclidean spaces and applications, Kodai Math. J., 8 (1985), 358–374.

C7] Chen, B. Y., Finite Type Submanifolds and Generalizations, University of Rome, Rome, 1985.

C8] Chen, B. Y., 2-type submanifolds and their applications, Chinese J. Math. 14 (1986), 1–14.

C9] Chen, B. Y., Finite type pseudo-Riemannian submanifolds, Tamkang J. Math. 17 (1986), 137–151.

C10] Chen, B. Y., Some estimates of total tension and their applications, Kodai Math. J., 10 (1987), 93–101.

C11] Chen, B. Y., Surfaces of finite type in Euclidean 3-space, Bull. Soc. Math. Belg. Ser. B, 39 (1987), 243–254.

C12] Chen, B. Y., Mean curvature of 2-type spherical submanifolds, Chinese J. Math., 16 (1988), 1–9.

C13] Chen, B. Y., Null 2-type surfaces in $E^3$ are circular cylinders, Kodai Math. J. 11 (1988), 295–299.

C14] Chen, B. Y., Null 2-type surfaces in Euclidean space, Algebra, Analysis and Geometry (1988), 1–18.

C15] Chen, B. Y., 3-type surfaces in $S^3$, Bull. Soc. Math. Belg. Sér. B 42 (1990), 379–381.

C16] Chen, B. Y., Local rigidity theorems of 2-type hypersurfaces in a hypersphere, Nagoya Math. J. 122 (1991), 139–148.

C17] Chen, B. Y., Linearly independent, orthogonal, and equivariant immersions, Kodai Math. J. 14 (1991), 341–349.

C18] Chen, B. Y., Some open problems and conjectures on submanifolds of finite type, Soochow J. Math. 17 (1991), 169–188.
[C19] Chen, B. Y., Submanifolds of finite type in hyperbolic spaces. Chinese J. Math. 20 (1992), no. 1, 521.
[C20] Chen, B. Y., Some pinching and classification theorems for minimal submanifolds, Arch. Math. 60 (1993), 568–578.
[C21] Chen, B. Y., Classification of tensor product immersions which are of 1–type, Glasgow Math. J. 36 (1994), 255–264.
[C22] Chen, B. Y., Some classification theorems for submanifolds in Minkowski space-time, Archiv der Math. 62 (1994), 177–182.
[C23] Chen, B. Y., Tubular hypersurfaces satisfying a basic equality, Soochow J. Math. 20 (1994), 569–586.
[C24] Chen, B. Y., Submanifolds in de Sitter space-time satisfying $\Delta H = \lambda H$, Isral J. Math. 89 (1995), 373–391.
[C25] B.-Y. Chen, A report on submanifolds of finite type, Soochow J. Math., 22 (1996), 117–337.
[C26] Chen, B. Y., Some new obstruction to minimal and Lagrangian isometric immersions, Japan. J. Math. 26 (2000), 105–127.
[C27] Chen, B.-Y., Pseudo-Riemannian geometry, δ-invariants and applications, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.
[CBG] Chen, B. Y., Barros, M. and Garay, O. J., Spherical finite type hypersurfaces, Algebras, Groups and Geometries 4 (1987), 58–72.
[CDVV] Chen, B. Y., J. Deprez, F. Dillen, L. Verstraelen and Vrancken, L., Finite type curves, Geometry and Topology of Submanifolds 2 (1990), 76–110.
[CDV1] Chen, B. Y., Deprez, J. and Verheyen, P., Immersions, dans un espace euclidien, d’un espace symétrique compact de rang un à géodésiques simples, C. R. Acad. Sc. Paris 304 (1987), 567-570.
[CDV2] Chen, B. Y., Deprez, J. and Verheyen, P., Immersions with many circular geodesics, Geometry and Topology of Submanifolds 4 (1992), 111–132.
[CDV3] Chen, B. Y., Deprez, J. and Verheyen, P., Immersions with geodesics of 2–type, Geometry and Topology of Submanifolds 4 (1992), 87–110.
[CDV4] Chen, B. Y. Deprez, J. and Verheyen, P., A note on the centroid set of compact symmetric spaces, Geometry and Topology of Submanifolds 4 (1992), 3–10.
[CD1] Chen, B. Y. and Dillen, F., Surfaces of finite type and constant curvature in the 3-sphere, C. R. Math. Rep. Acad. Sci. Canada 12 (1990), 47-49.
[CD2] Chen, B. Y. and Dillen, F., Quadrics of finite type, J. Geometry 38 (1990), 16–22.
[CD1] Chen, B. Y., F. Dillen and Song, H., Quadric hypersurfaces of finite type, Colloq. Math. 63 (1992), 145–152.
[CDV] Chen, B. Y., Dillen, F. and Verstraelen, L., Finite type space curves, Soochow J. Math. 12 (1986), 1–10.
[CDVV1] Chen, B. Y., Dillen, F., Verstraelen, L. and Vrancken, L., Ruled surfaces of finite type, Bull. Austral. Math. Soc. 42 (1990), 447–453.

[CDVV2] Chen, B. Y., Dillen, F., Verstraelen, L. and Vrancken, L., A variational minimal principle characterizes submanifolds of finite type, C.R. Acad. Sc. Paris 317 (1993), 961–965.

[CDVV3] Chen, B. Y., Dillen, F., Verstraelen, L. and Vrancken, L., Submanifolds of restricted type, J. of Geometry 46 (1993), 20–32.

[CDVV4] Chen, B. Y., Dillen, F., Verstraelen, L. and Vrancken, L., A variational minimal principle and its applications, Kyungpook Math. J. 35 (1995), no. 3, Special Issue, 435–444.

[CDVV5] Chen, B. Y., Dillen, F., Verstraelen, L. and Vrancken, L., Compact hypersurfaces determined by a spectral variational principle, Kyushu J. Math. 49 (1995), 103–121.

[CI1] Chen, B. Y. and Ishikawa, S., Biharmonic surfaces in pseudo-Euclidean spaces, Mem. Fac. Sci. Kyushu Univ. A 45 (1991), 323–347.

[CI2] Chen, B. Y. and Ishikawa, S., Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces, Kyushu J. Math. 52 (1998), 167–185.

[CI3] Chen, B. Y. and Ishikawa, S., On classification of some surfaces of revolution of finite type, Tsukuba J. Math. 17 (1993), 287–298.

[CK] Chen, B. Y. and Kuan, W. E., The cubic representation of a submanifold, Beiträge zur Algebra und Geometrie 35 (1994), 55-66.

[CL] Chen, B. Y. and Li, S. J., 3-type hypersurfaces in a hypersphere, Bull. Soc. Math. Belg. Sér. B 43 (1991), 135–141.

[CLu] Chen, B.Y. and Lue, H. S., Some 2-type submanifolds and applications, Ann. Fac. Sc. Toulouse Math. Ser. V, 9 (1988), 121–131.

[CMN1] Chen, B.Y., Morvan, J.-M. and Nore, T., Énergie, tension et ordre des applications à valeurs dans un espace euclidien, C. R. Acad. Sc. Paris 301 (1985), 123–126.

[CMN2] Chen, B.Y., Morvan, J.-M. and Nore, T., Energy, tension and finite type maps, Kodai Math. J. 9 (1986), 406–418.

[CMu] Chen, B. Y. and Munteanu, M. I., Biharmonic ideal hypersurfaces in Euclidean spaces, Differential Geom. Appl. 31 (2013), 1–16.

[CN] Chen, B.Y. and Nagano, T., Harmonic metrics, harmonic tensors, and Gauss maps, J. Math. Soc. Japan 36 (1984), 295–313.

[CO] Chen, B. Y. and Ogiue, K., On totally real submanifolds, Trans. Amer. Math. Soc. 193 (1974), 257–266.

[CP] Chen, B.Y. and Petrovic, M., On spectral decomposition of immersions of finite type, Bull. Austral. Math. Soc. 44 (1991), 117–129.

[CPi] Chen, B.Y. and Piccinni, P., Submanifolds with finite type Gauss map, Bull. Austral. Math. Soc. 44 (1987), 161–186.
[CS1] Chen, B.Y. and Song, H., Null 2-type surfaces in Minkowski space–time, Algebras, Groups and Geometries 6 (1989), 333–352.

[CS2] Chen, B.Y. and Song, H., Null 2-type surfaces in Minkowski space-time, II, Atti Acad. Peloritana Pericolanti Cl. Sci. Fis. Mat. Natur., 68 (1989), 1–12.

[CV] Chen, B.Y. and Verstraelen, L., Laplace Transformations of Submanifolds, Centre for Pure and Applied Differential Geometry (PADGE), vol. 1. Brussels-Leuven, 1995.

[Def] Defever, F., Hypersurfaces of $\mathbb{R}^4$ with harmonic mean curvature vector, Math. Nachr. 196 (1998), 61–69.

[DDV1] Defever, F., Deszez, R. and Verstraelen, L., The compact cylcides of Dupin and a conjecture of B.-Y. Chen, J. Geometry 46 (1993), 33–38.

[DDV2] Defever, F., Deszez, R. and Verstraelen, L., The Chen–type of noncompact cylcides of Dupin, Glasgow Math. J. 36 (1994), 71–75.

[De1] Deprez, J., Immersions of finite type of compact homogeneous Riemannian manifolds, Doctoral Thesis, Katholieke Universiteit Leuven, 1988.

[De2] Deprez, J., Immersions of finite type, Geometry and Topology of Submanifolds 2 (1990), 111–133.

[DDV] Deprez, J., Dillen, F. and Vrancken, L., Finite type curves on quadrics, Chinese J. Math. 18 (1990), 95–121.

[Dil1] Dillen, F., Ruled submanifolds of finite type, Proc. Amer. Math. Soc. 114 (1992), 795–798.

[DPV] Dillen, F., J. Pas, and Verstraelen, L., On surfaces of finite type in Euclidean 3-space, Kodai Math. J. 13 (1990), 10–21.

[Di0] I. Dimitrić, Quadric representation and submanifolds of finite type, Ph.D. Thesis, MSU, 1989.

[Di1] Dimitric, I., Submanifolds of $E^m$ with harmonic mean curvature vector, Bull. Inst. Math. Acad. Sinica 20 (1992), 53–65.

[Di2] Dimitric, I., Quadric representation of a submanifold and spectral geometry, Proc. Symp. Pure Math. 54, Part 3 (1993), 155–168.

[Di3] Dimitric, I., Quadric representation of a submanifold, Proc. Amer. Math. Soc. 114 (1992), 201–210.

[Di4] Dimitric, I., 1-type submanifolds of non-Euclidean complex space forms. Bull. Belg. Math. Soc. Simon Stevin 4 (1997), no. 5, 673–684.

[Di5] Dimitric, I., CR-submanifolds of $HP^n$ and hypersurfaces of the Cayley plane whose Chen-type is 1, Kyungpook Math. J. 40 (2000), 407–429.

[Di6] Dimitric, I., Low-type submanifolds of real space forms via the immersions by projector, Differential Geom. Appl. 27 (2009), no. 4, 507–526.

[EL1] Eells, J. and Lemaire, L., A report on harmonic maps, Bull. London Math. Soc. 10 (1978), 1–68.
[EL2] Eells, J. and Lemaire, L., Another report on harmonic maps, Bull. London Math. Soc. 20 (1988), 385–524.

[FGL1] Ferrandez, A., Garay, O. J. and P. Lucas, Finite type ruled manifolds shaped on spherical submanifolds, Arch. fur Math., 57 (1991), 97–104.

[FGL2] Ferrandez, A., Garay, O.J. and P. Lucas, On a certain class of conformally flat Euclidean hypersurfaces, Global differential geometry and global analysis (Berlin, 1990), 48–54, Lecture Notes in Math., 1481, Springer, Berlin, 1991.

[FL] Ferrandez, A. and Lucas, P., Null finite type hypersurfaces in space forms. Kodai Math. J. 14 (1991), no. 3, 406–419.

[G1] Garay, O.J., Spherical Chen surfaces which are mass-symmetric and of 2-type, J. Geometry 33 (1988), 39–52.

[G2] Garay, O.J., Finite type cones shaped on spherical submanifolds, Proc. Amer. Math. Soc. 104 (1988), 868–870.

[G3] Garay, O.J., An extension of Takahashi's theorem, Geom. Dedicata 34 (1990), 105–112.

[GR] Garay, O.J. and Romero, A., An isometric embedding of the complex hyperbolic space in a pseudo-Euclidean space and its application to the study of real hypersurfaces, Tsukuba J. Math., 14 No. 2. (1990)

[HV1] Hasanis, T. and Vlachos, T., A local classification of 2-type surfaces in $S^3$, Proc. Amer. Math. Soc. 122 (1991), 533–538.

[HV2] Hasanis, T. and Vlachos, T., Spherical 2-type hypersurfaces, J. Geometry 40 (1991), 82–94.

[HV3] Hasanis, T. and Vlachos, T., Hypersurfaces of $E_n^{n+1}$ satisfying $\Delta x = Ax + B$, J. Austral. Math. Soc. A. 53 (1992), 377–384.

[HV4] Hasanis, T. and Vlachos, T., Surfaces of finite type with constant mean curvature, Kodai Math. J. 16 (1993), 244–352.

[HV5] Hasanis, T. and Vlachos, T., Hypersurfaces in $E^4$ with harmonic mean curvature vector field, Math. Nachr. 172 (1995), 145–169.

[Ho1] Houh, C.-S., Null 2–type surfaces in $E^2_1$ and $S^3_1$, Algegra, Analysis and Geometry (1988), 19–37.

[Ho2] Houh, C.-S., Rotation surfaces of finite type, Algebras, Groups and Geometries 7 (1990), 199–209.

[HL] Houh, C.-S. and Li, S. J., Generalized Chen submanifolds, J. Geometry 48 (1993), 144–156.

[Is] Ishikawa, S., Classification problems of finite type submanifolds and biharmonic submanifolds, Doctoral Thesis, Kyushu University 1992.

[J] Jiang, G. Y., 2-harmonic isometric immersions between Riemannian manifolds, Chinese Ann. Math. Ser. A, 7 (1986), 130–144.
[L] Li, S. J., Null 2-type Chen surfaces, Glasgow Math. J. 37 (1995), 233–242.

[LO] Liang, T and Ou, Y.-L., Biharmonic hypersurfaces in a conformally flat space, Results Math. DOI 10.1007/s00025-012-0299-x, 2013.

[Luo1] Luo, Y., Weakly convex biharmonic hypersurfaces in Euclidean spaces are minimal, arXiv: 1305.71981v1, 2013.

[Luo2] Luo, Y., On biharmonic submanifolds in non-positively curved manifolds and $\epsilon$-superbiharmonic submanifolds, arXiv:1306.6069v1, 2013.

[Lu03] Luo, Y., On the Willmore energy and Chen’s conjecture, preprint, 2013.

[Ko] Kotani, M., A decomposition theorem of 2-type immersions, Nagoya Math. J. 118 (1990), 55–66.

[Ma1] Maeta, S., $k$-harmonic maps into a Riemannian manifold with constant sectional curvature, Proc. Amer. Math. Soc. 140 (2012), 1635–1847.

[Ma2] Maeta, S., Biminimal properly immersed submanifolds in the Euclidean spaces, J. Geom. Phys. 62 (2012), 2288–2293.

[Ma3] Maeta, S., Biminimal properly immersed submanifolds in complete Riemannian manifold of non-positive curvature, arXiv:1208.0473v2, 2012.

[Ma4] Maeta, S., Biharmonic maps from a complete Riemannian manifold into a non-positively curved manifold, arXiv:1305.7065v1, 2013.

[MU] Maeta, S. and Urakawa, H., Biharmonic Lagrangian submanifolds in Kähler manifolds, Glasgow Math. J. 55 (2013), 465–480.

[MR] Martinez, A. and Ros, A., On real hypersurfaces of finite type of $CP^m$, Kodai Math. J., 7 (1984), 304–316.

[Mi] Miyata, Y., 2-type surfaces of constant curvature in $S^n$, Tokyo J. Math. 11 (1988), 157–204.

[Na] Nagatomo, Y., Finite type hypersurfaces of a sphere, Tokyo J. Math. 14 (1991), 85–92.

[NU1] Nakauchi, N. and Urakawa, H., Biharmonic hypersurfaces in a Riemannian manifold with non-positive Ricci curvature, Ann. Glob. Anal. Geom. 40 (2011), 125–131.

[NU2] Nakauchi, N. and Urakawa, H., Biharmonic submanifolds in a Riemannian manifold with non-positive curvature, Results Math. 63 (2013), 467–471.

[NUG] Nakauchi, N., Urakawa, H. and S. Gudmundsson, Biharmonic maps into a Riemannian manifold of non-positive curvature, Geom. Dedicata, DOI 10.1007/s10711-013-9854-1, 2013.

[Og] Ogiso, K., Some recent topics in the theory of submanifolds, Sugaku Expositions 4 (1991), 21–41.

[Ou1] Ou, Y.-L., On conformal biharmonic immersions, Ann. Global Anal. Geom. 36 (2009), no. 2, 133–142.
Ou, Y.-L., Biharmonic hypersurfaces in Riemannian manifolds, Pacific J. Math. 248 (2010) 217–232.

Ou, Y.-L., Some constructions of biharmonic maps and Chen’s conjecture on biharmonic hypersurfaces, J. Geom. Phys. 62 (2012), 751–762.

Ou, Y.-L. and Tang, L. The generalized Chen’s conjecture on biharmonic submanifolds is false, Michigan Math. J. 61 (2012), 531–542.

Rouxel, B., Chen submanifolds, Geometry and Topology of Submanifolds 6 (1994), 185–198.

Ros, A., Spectral geometry of CR-minimal submanifolds in the complex projective space, Kodai Math. J. 6 (1983), 88–99.

Ros, A., On spectral geometry of Kaehler submanifolds, J. Math. Soc. Japan 36 (1984), 433–448.

Ros, A., Curvature pinching and eigenvalue rigidity for minimal submanifolds, Math. Z., 191 (1986), 537-548.

Sasahara, T., A class of biminimal Legendrian submanifolds in Sasaki space forms, to appear in Math. Nachr.

Takahashi, T., Minimal immersions of Riemannian manifolds, J. Math. Soc. Japan 18 (1966), 380–385.

Takahashi, T., Isometric immersions of Riemannian homogeneous manifolds, Tsukuba J. Math. 12 (1988), 231-233.

Udagawa, S., Bi-order real hypersurfaces in complex projective space, Kodai Math. J., 10 (1987), 182-196.

Verstraelen, L., On submanifolds of finite Chen type and of restricted type, Results in Math. 20 (1991), 744-755.

Verstraelen, L., Curves and surfaces of finite Chen type , Geometry and Topology of Submanifolds 3 (1991), 304–314.

Wallach, N.R., Minimal immersions of symmetric spaces into spheres, in Symmetric Spaces, M. Dekker (1972), 1–40.

Wheeler, G., Chen’s conjecture and $\epsilon$-superbiharmonic submanifolds of Riemannian manifolds, Intern. J. Math. 24(2013), no. 4, 1350028 (6 pages).