Lattice super Yang-Mills: 
A virial approach to operator dimensions

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Abstract

The task of calculating operator dimensions in the planar limit of $\mathcal{N} = 4$ super Yang-Mills theory can be vastly simplified by mapping the dilatation generator to the Hamiltonian of an integrable spin chain. The Bethe ansatz has been used in this context to compute the spectra of spin chains associated with various sectors of the theory which are known to decouple in the planar (large-$N_c$) limit. These techniques are powerful at leading order in perturbation theory but become increasingly complicated beyond one loop in the 't Hooft parameter $\lambda = g_{YM}^2 N_c$, where spin chains typically acquire long-range (non-nearest-neighbor) interactions. In certain sectors of the theory, moreover, higher-loop Bethe ansätze do not even exist. We develop a virial expansion of the spin chain Hamiltonian as an alternative to the Bethe ansatz methodology, a method which simplifies the computation of dimensions of multi-impurity operators at higher loops in $\lambda$. We use these methods to extract previously reported numerical gauge theory predictions near the BMN limit for comparison with corresponding results on the string theory side of the AdS/CFT correspondence. For completeness, we compare our virial results with predictions that can be derived from current Bethe ansatz technology.

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1 Introduction

A two-dimensional parameter space emerges around the BMN limit of $\mathcal{N} = 4$ $SU(N_c)$ super Yang-Mills (SYM) theory and the dual pp-wave limit of IIB superstring theory on $AdS_5 \times S^5$. This space is parameterized on one axis by the perturbative gauge theory expansion in powers of the 't Hooft coupling $\lambda = g_{YM}^2 N_c$, and on the other by a string theory curvature expansion away from the Penrose limit in inverse powers of the string angular momentum $J$. The original comparison made by Berenstein, Maldacena and Nastase [1], which has sparked a number of direct tests of the AdS/CFT correspondence in recent years, lies at the intersection of the one-loop order ($O(\lambda)$) gauge theory prediction and the zeroth-order ($O(J^0)$) limit in the string curvature expansion. To explore a larger region of this duality landscape, a number of studies have pushed the string theory calculation to higher orders in the $1/J$ curvature expansion [2, 3, 4], while operator dimensions in the gauge theory have been computed to higher orders in $\lambda$ (see, eg., [5, 6, 7, 8, 9]).

The first $1/J$ curvature correction to the fully quantized string theory near the pp-wave limit of $AdS_5 \times S^5$ was studied for two-impurity string states in [2, 3, 4]. In this setting, the extended superconformal multiplet of the theory is a 256-dimensional multiplet built on a scalar primary (or “highest-weight”) state. This analysis was extended to the three-impurity, 4,096-dimensional supermultiplet in [10]. To test predictions of the AdS/CFT correspondence, it was necessary in the course of the latter study to obtain higher-impurity, higher-loop operator dimensions from the gauge theory near the BMN limit. This is hard to do by standard diagrammatic methods, but the problem has been drastically simplified by the recent discovery that, in certain sectors of the gauge theory, the dilatation operator can be mapped to the Hamiltonian of an integrable spin-chain system. Calculating operator dimensions is therefore equivalent to finding the eigenvalue spectrum of these spin chains and certain established techniques associated with integrable systems, most notably the Bethe ansatz, have proved useful in this context (for a general review of the Bethe ansatz method, see [11]). The utility of this approach in the setting of $\mathcal{N} = 4$ super Yang-Mills theory was first demonstrated by Minahan and Zarembo [12]. For operators with two $R$-charge impurities, the spin chain spectra can be computed exactly via the Bethe ansatz. For three or higher-impurity operators, however, the Bethe equations have (to the best of our knowledge) only been solved perturbatively near the limit of infinite chain length [12, 13, 14]. Furthermore, at higher-loop order in $\lambda$, the spin chain Hamiltonians typically acquire long-range or non-nearest-neighbor interactions for which a Bethe ansatz may not be available. For example, while the action of the spin chain Hamiltonian in the “closed $su(2|3)$” sector is known to three-loop order [7], the corresponding long-range Bethe ansatz is not known (though it may well exist). A long-range Bethe ansatz does exist for the particularly simple “closed $su(2)$” sector of the theory [3, 4], and our methods will provide a useful cross-check on these approaches to higher-order gauge theory anomalous dimensions.

To improve on the current limitations of Bethe ansatz techniques, we have developed a virial approach to the spin chain systems of $\mathcal{N} = 4$ super Yang-Mills theory. The generic spin-chain Hamiltonian acts on single-impurity pseudoparticles as a lattice Laplacian and
higher $N$-body interactions among pseudoparticles are suppressed relative to the one-body pseudoparticle energy by inverse powers of the lattice length $L$. Surprisingly, this expansion of the spin chain Hamiltonian is truncated at $O(L^{-3})$ in certain subsectors of the theory, allowing straightforward eigenvalue calculations that are exact in the chain length for operators with more than two $R$-charge impurities. Furthermore, since the goal is to eventually compare anomalous dimensions with $1/J$ energy corrections to corresponding string states near the pp-wave limit of $AdS_5 \times S^5$, and because the string angular momentum $J$ is proportional to the lattice length $L$, this virial expansion is precisely what is needed to devise a practical method for testing the AdS/CFT correspondence at any order in the gauge theory loop expansion for an arbitrary number of $R$-charge (or worldsheet) impurities.

The purpose of this paper is to provide a detailed description of these methods, and to compare previously derived numerical predictions [10] with corresponding predictions from Bethe ansatz technology near the BMN limit. We will focus on three particular closed sectors of the theory, each labeled by the subalgebra of the full superconformal algebra which characterizes the spin variables of the equivalent spin chain system. Specifically, there are two sectors spanned by bosonic operators and labeled by $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ subalgebras plus an $\mathfrak{su}(2|3)$ sector which includes fermionic operators. Section 2 is dedicated to an analysis of the bosonic $\mathfrak{su}(2)$ closed sector to three-loop order in $\lambda$. In section 3 we analyze an $\mathfrak{su}(1|1)$ subsector of the closed $\mathfrak{su}(2|3)$ system to three-loop order. The spin-chain Hamiltonian in the bosonic $\mathfrak{sl}(2)$ sector has previously been determined to one loop, and we analyze this system in section 4. We conclude in the final section with a discussion of future applications and directions of study.

## 2 The $\mathfrak{su}(2)$ sector

Single-trace operators in the closed $\mathfrak{su}(2)$ sector are constructed from two complex scalar fields of $\mathcal{N} = 4$ SYM, typically denoted by $Z$ and $\phi$. Under the $SO(6) \simeq U(1)_R \times SO(4)$ decomposition of the full $SU(4)$ $R$-symmetry group, the $Z$ fields are charged under the scalar $U(1)_R$ component and $\phi$ is a particular scalar field carrying zero $R$-charge. The basis of length-$L$ operators in the planar limit is constructed from single-trace monomials with $I$ impurities and total $R$-charge equal to $L - I$:

$$
\text{Tr}(\phi^I Z^{L-I}) , \quad \text{Tr}(\phi^{I-1} Z \phi Z^{L-I-1}) , \quad \text{Tr}(\phi^{I-2} Z \phi^2 Z^{L-I-1}) , \quad \ldots \\
$$

The statement that this sector of operators is “closed” means simply that the anomalous dimension operator can be diagonalized on this basis, at least to leading order in large $N_c$ [5,17].

The heart of the spin-chain approach is the proposition that there exists a one-dimensional spin system whose Hamiltonian can be identified with the large-$N_c$ limit of the anomalous dimension operator acting on this closed subspace of operators [12]. Since the anomalous dimensions are perturbative in the ’t Hooft coupling $\lambda$, it is natural to expand the $\mathfrak{su}(2)$ spin
chain Hamiltonian in powers of $\lambda$ as well:

$$H_{\text{su}(2)} = I + \sum_n \left( \frac{\lambda}{8\pi^2} \right)^n H^{(2n)}_{\text{su}(2)}. \quad (2.2)$$

Comparison with the gauge theory has shown that successive terms in the expansion of the Hamiltonian have a remarkably simple structure: the one-loop-order Hamiltonian $H^{(2)}_{\text{su}(2)}$ is built out of permutations of pairs of nearest-neighbor fields and, at $n$-th order, the Hamiltonian permutes among themselves fields which are at most $n$ lattice sites apart. This is a universal structure which leads to remarkable simplifications in the various closed sectors of the theory \cite{13}.

Beisert, Kristjansen and Staudacher \cite{5} have introduced the following useful notation for products of permutations acting on operators separated by an arbitrary number of lattice sites:

$$\{n_1, n_2, \ldots\} = \sum_{k=1}^L P_{k+n_1, k+n_1+1} P_{k+n_2, k+n_2+1} \cdots, \quad (2.3)$$

where $P_{i,j}$ simply exchanges fields on the $i^{th}$ and $j^{th}$ lattice sites on the chain. The upshot of the gauge theory analysis is that the equivalent spin-chain Hamiltonian for the $\text{su}(2)$ sector can be written in a rather compact form in terms of this notation. The result, correct to three-loop order, is (see \cite{5} for details)

$$H^{(2)}_{\text{su}(2)} = 2 \{\} - \{0\} \quad (2.4)$$

$$H^{(4)}_{\text{su}(2)} = 2(-4\{\} + 6\{0\} - (\{0, 1\} + \{1, 0\})) \quad (2.5)$$

$$H^{(6)}_{\text{su}(2)} = 4(15\{\} - 26\{0\} + 6(\{0, 1\} + \{1, 0\}) + \{0, 2\} - (\{0, 1, 2\} + \{2, 1, 0\})) \quad (2.6)$$

The form of the three-loop term $H^{(6)}_{\text{su}(2)}$ was first conjectured in \cite{5} based on integrability restrictions and BMN scaling; this conjecture was later corroborated by direct field-theoretic methods in \cite{7} (see also \cite{6} for relevant discussion on this point). Our goal is to develop practical methods for finding the eigenvalue spectrum of the spin-chain Hamiltonian for various interesting cases.

### 2.1 One-loop order

We start at one-loop order with $H^{(2)}_{\text{su}(2)}$ in eqn. (2.2), which provides a natural ‘position-space’ prescription for constructing matrix elements in an $I$-impurity basis of operators. As an explicit example, we consider first the basis of two-impurity operators of length $L = 8$:

$$\text{Tr}(\phi^2 Z^6) \quad \text{Tr}(\phi Z \phi Z^5) \quad \text{Tr}(\phi Z^2 \phi Z^4) \quad \text{Tr}(\phi Z^3 \phi Z^3). \quad (2.7)$$


It is easy to see that the one-loop Hamiltonian mixes the four elements of this basis according to the matrix

$$H_{\text{su}(2)}^{(2)} = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2\sqrt{2} \\ 0 & 0 & -2\sqrt{2} & 4 \end{pmatrix}. \tag{2.8}$$

This tri-diagonal matrix generalizes to arbitrary $L$ and it is simple to show that the two-impurity one-loop eigenvalues of $H_{\text{su}(2)}^{(2)}$ are given by the formula [16]

$$E_{\text{su}(2)}^{(2)} = 8 \sin^2 \left( \frac{\pi n}{L-1} \right) \quad n = 0, \ldots, n_{\text{max}} = \begin{cases} (L - 2)/2, & L \text{ even} \\ (L - 3)/2, & L \text{ odd} \end{cases}. \tag{2.9}$$

Although we defer our discussion of the Bethe ansatz approach until later in this section, we note that the two-impurity $\text{su}(2)$ Bethe equations for this spin chain [12] can be solved exactly and their solution agrees with eqn. (2.9). Note that if the denominator $L - 1$ were replaced by $L$, the above expression would agree with the usual lattice Laplacian energy for a lattice of length $L$. The difference amounts to corrections to the free Laplacian of higher order in $1/L$ and we will seek to understand the physical origin of such corrections in what follows.

To compare gauge theory predictions with $1/J$ corrections to the three-impurity spectrum of the string theory on $AdS_5 \times S^5$, we need to determine the large-$L$ behavior of the three-impurity spin chain spectrum. We are primarily interested in systems with few impurities compared to the length of the spin chain and we expect that impurity interaction terms in the Hamiltonian will be suppressed by powers of the impurity density (i.e. inverse powers of the lattice length). This suggests that we develop a virial expansion of the spin chain Hamiltonian in which the leading-order term in $1/L$ gives the energy of free pseudoparticle states on the lattice (labeled by lattice momentum mode numbers as in the two-impurity spectrum eqn. (2.9)) and higher $1/L$ corrections come from $N$-body interactions described by vertices $V_N$. A reasonable guess about how the $N$-body interactions should scale with $1/L$ suggests that we can write the one-loop-order energy for $I$ impurities in the form

$$E(\{n_i\}) = I + \frac{\lambda}{2\pi^2} \sum_{i=1}^{I} \sin^2 \frac{n_i \pi}{L} + \sum_{N=2}^{2I} \frac{\lambda}{L^{2N-1}} V_{N-\text{body}}(n_1, \ldots, n_I) + \cdots, \tag{2.10}$$

where the leading-order contribution $I$ measures the naive dimension minus $R$-charge, the next term is the lattice Laplacian energy of $I$ non-interacting pseudoparticles and the $1/L$ corrections account for interactions between pseudoparticles (which may depend on the lattice momenta mode numbers $n_i$). In the many-body approach, one would try to derive such energy expressions by rewriting the Hamiltonian in terms of creation/annihilation operators $b_{n_i}$, $b_{n_i}^\dagger$ for the pseudoparticles (commuting or anticommuting as appropriate). The $N$-body
interaction vertex would generically be written in terms of the $b, b^\dagger$ as

$$V_N = \sum_{n_i, m_i} \delta_{n_1 + \ldots + n_N, m_1 + \ldots + m_N} f_N(\{n_i\}, \{m_i\}) \prod_{i=1}^N b_{n_i}^\dagger \prod_{i=1}^N b_{m_i},$$

(2.11)

where $f_N(\{n_i\}, \{m_i\})$ is some function of the lattice momenta and the Kronecker delta enforces lattice momentum conservation. One has to determine the functions $f_N$ by matching the many-body form of the Hamiltonian to exact spin chain expressions such as eqn. (2.4).

We will see that, once the Hamiltonian is in many-body form, it is straightforward to obtain a density expansion of the higher-impurity energy eigenvalues.

The discussion so far has been in the context of one-loop gauge theory physics, but the logic of the virial expansion should be applicable to the general case. To include higher-loop order physics we must do two things: a) generalize the functions $f_N(\{n_i\}, \{m_i\})$ defining the multi-particle interaction vertices to power series in $\lambda$ and b) allow the free pseudoparticle kinetic energies themselves to become power series in $\lambda$. We will be able to carry out the detailed construction of the higher-loop virial Hamiltonian in a few well-chosen cases. To match this expansion at $n$-loop order in $\lambda$ to the corresponding loop order (in the modified \('t Hooft coupling $\lambda' = g_{YM}^2 N_c / J^2$) in the string theory, we need to determine the Hamiltonian to $O(L^{-(2n+1)})$ in this virial expansion. (The first curvature correction to the pp-wave string theory at one loop, for example, appears at $O(\lambda' / J)$ or, in terms of gauge theory parameters, at $O(\lambda / L^3)$.) Auspiciously, it will turn out that this virial expansion in the $\text{su}(2)$ sector is truncated at small orders in $1/L$, allowing for simple eigenvalue calculations that are exact in $L$ (although perturbative in $\lambda$).

The first step toward obtaining the desired virial expansion is to recast the spin chain Hamiltonian $H_{\text{su}(2)}$, which is initially expressed in terms of permutation operators, in terms of a creation and annihilation operator algebra. We begin by introducing the spin operators

$$S^\pm = \frac{1}{2} (\sigma_x \pm i \sigma_y), \quad S^z = \frac{1}{2} \sigma_z,$$

(2.12)

where $\sigma$ are the Pauli matrices and $S^\pm_j, S^z_j$ act on a two-dimensional spinor space at the $j^{th}$ lattice site in the chain. In this setting the $Z$ and $\phi$ fields are understood to be modeled by up and down spins on the lattice. The nearest-neighbor permutation operator $P_{i,i+1}$ can be written in terms of spin operators as

$$P_{i,i+1} = S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ + 2 S_i^z S_{i+1}^z + \frac{1}{2},$$

(2.13)

and the one-loop Hamiltonian in eqn. (2.4) can be written as

$$H_{\text{su}(2)}^{(2)} = - \sum_{j=1}^L (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) - 2 \sum_{j=1}^L S_j^z S_{j+1}^z + \frac{1}{2}.$$

(2.14)

A Jordan-Wigner transformation can now be used to express the spin generators in terms of anti-commuting creation and annihilation operators (anti-commuting because each site
can be either unoccupied \((Z)\) or occupied once \((\phi)\). A pedagogical introduction to this technique can be found in [18]. The explicit transformation is
\[
S_j^+ = b_j^\dagger K(j) = K(j)b_j^\dagger \\
S_j^- = K(j)b_j = b_jK(j) \\
S_j^z = b_j^\dagger b_j - 1/2 , \tag{2.15}
\]
where the Klein factors
\[
K(j) = \exp\left(i\pi \sum_{k=1}^{j-1} b_k^\dagger b_k\right) \tag{2.16}
\]
serve to ensure that spin operators on different sites commute, despite the anticommuting nature of the \(b_j\). The functions \(K(j)\) are real, Abelian and, for \(j \leq k\),
\[
[K(j), S_k] = 0. \tag{2.17}
\]
The operators \(b_j\) and \(b_j^\dagger\) can therefore be written as
\[
b_j^\dagger = S_j^+ K(j) \quad b_j = S_j^- K(j) , \tag{2.18}
\]
and we easily verify that they satisfy the standard anticommutation relations
\[
\{b_j, b_k^\dagger\} = \delta_{jk} \quad \{b_j^\dagger, b_k^\dagger\} = \{b_j, b_k\} = 0 . \tag{2.19}
\]
Cyclicity on the lattice requires that \(S_{L+1} = S_1\), a condition which can be enforced by the following boundary condition on the creation and annihilation operators:
\[
b_{L+1} = (-1)^{L+1} b_1 \quad I \equiv \sum_{j=1}^{L} b_j^\dagger b_j , \tag{2.20}
\]
where the integer \(I\) counts the number of spin chain impurities. Since we are primarily interested in the three-impurity problem (for comparison with the corresponding string results reported in [10]), we will henceforth impose the boundary conditions in eqn. (2.20) for odd impurity number only. We can use all of this to re-express eqn. (2.14) in creation and annihilation operator language, with the result
\[
H_{\text{su}(2)}^{(2)} = \sum_{j=1}^{L} \left(b_j^\dagger b_j + b_{j+1}^\dagger b_{j+1} - b_j^\dagger b_{j+1} - b_{j+1}^\dagger b_j + 2 b_j^\dagger b_{j+1} b_j b_{j+1}\right) . \tag{2.21}
\]
Converting to momentum space via the usual Fourier transform
\[
b_j = \frac{1}{\sqrt{L}} \sum_{p=0}^{L-1} e^{-2\pi i j p / L} \tilde{b}_p \tag{2.22}
\]
yields

\[ H_{\text{su}(2)}^{(2)} = 4 \sum_{p=0}^{L-1} \sin^2 \left( \frac{\pi p}{L} \right) \tilde{b}_p^\dagger \tilde{b}_p + \frac{2}{L} \sum_{p,q,r,s=0}^{L-1} e^{2\pi i (q-s)} L \tilde{b}_p^\dagger \tilde{b}_q^\dagger \tilde{b}_s \delta_{p+q,r+s}. \] (2.23)

This is a rather standard many-body Hamiltonian: it acts on a Fock space of momentum eigenstate pseudoparticles, contains a one-body pseudoparticle kinetic energy term and a two-body pseudoparticle interaction (the latter having the critical property that it conserves the number of pseudoparticles). Note that the Hamiltonian terminates at two-body interactions, a fact which will simplify the virial expansion of the energy spectrum. This termination is a consequence of the fact that the one-loop Hamiltonian contains only nearest-neighbor interactions and that lattice sites can only be once occupied.

Because the pseudoparticle (or impurity) number is conserved by the interaction, three-impurity eigenstates of the Hamiltonian must lie in the space spanned by

\[ \tilde{b}_{k_1}^\dagger \tilde{b}_{k_2}^\dagger \tilde{b}_{k_3}^\dagger |L\rangle \quad k_1 + k_2 + k_3 = 0 \mod L, \] (2.24)

where the ground state \(|L\rangle\) is identified with the zero-impurity operator \(\text{Tr}(Z^L)\) and the condition of vanishing net lattice momentum arises from translation invariance on the spin chain (which in turn arises from the cyclicity of the single-trace operators in the operator basis). As a concrete example, the basis of three-impurity states of the \(L = 6\) \(\text{su}(2)\) spin chain is

\[ \tilde{b}_{0}^\dagger \tilde{b}_{1}^\dagger \tilde{b}_{5}^\dagger |L\rangle \quad \tilde{b}_{0}^\dagger \tilde{b}_{2}^\dagger \tilde{b}_{4}^\dagger |L\rangle \quad \tilde{b}_{1}^\dagger \tilde{b}_{2}^\dagger \tilde{b}_{3}^\dagger |L\rangle \quad \tilde{b}_{0}^\dagger \tilde{b}_{2}^\dagger \tilde{b}_{5}^\dagger |L\rangle , \] (2.25)

and the matrix elements of the Hamiltonian (2.23) in this basis are easily computed:

\[ H_{\text{su}(2)}^{(2)} = \begin{pmatrix}
\frac{1}{3} & -1 & \frac{1}{3} & \frac{1}{3} \\
-1 & 3 & -1 & -1 \\
\frac{1}{3} & -1 & \frac{19}{3} & \frac{1}{3} \\
\frac{1}{3} & -1 & \frac{1}{3} & \frac{19}{3}
\end{pmatrix}. \] (2.26)

The first-order perturbation theory corrections to the three-impurity operator anomalous dimensions are the eigenvalues of this matrix.

The construction and diagonalization of the Hamiltonian matrix on the degenerate basis of three-impurity operators can easily be carried out for larger \(L\). The results of doing this\(^1\) for lattice sizes out to \(L = 40\) are displayed in figure 1. According to eqn. (2.10), we expect the eigenvalues of \(H_{\text{su}(2)}^{(2)}\) to scale for large \(L\) according to

\[ E_L(\{k_i\}) = \frac{\lambda}{L^2} E^{(1,2)}(\{k_i\}) + \frac{\lambda}{L^3} E^{(1,3)}(\{k_i\}) + O(\lambda L^{-4}). \] (2.27)

\(^1\)Using the position- or momentum-space formalism is purely a matter of convenience. In practice we have found that for all sectors the momentum-space treatment is computationally much more efficient. The large-\(L\) extrapolations of both methods can be checked against each other, and we of course find that they are in agreement.
The scaling coefficients $E^{(1,2)}_{\text{su}(2)}$ and $E^{(1,3)}_{\text{su}(2)}$ can easily be extracted from the data displayed in figure 1 by fitting the spectral curves to large-order polynomials in $1/L$ (a similar treatment was used in [19]). The results of this procedure are recorded for several low-lying levels in the spectrum (excluding zero eigenvalues) in table 1. As originally reported in [10], string

| $E^{(1,2)}_{\text{su}(2)}$ | $E^{(1,3)}_{\text{su}(2)}$ | $E^{(1,3)}_{\text{su}(2)} / E^{(1,2)}_{\text{su}(2)}$ | Lattice Momenta $(k_1, k_2, k_3)$ |
|--------------------------|--------------------------|--------------------------|--------------------------|
| $1 + 2.6 \times 10^{-9}$ | $2 - 4.9 \times 10^{-7}$ | $2 - 5.0 \times 10^{-7}$ | $(1, 0, -1)$ |
| $3 + 4.6 \times 10^{-9}$ | $7 - 8.8 \times 10^{-7}$ | $7/3 - 3.0 \times 10^{-7}$ | $(1, 1, -2)$ |
| $3 + 4.6 \times 10^{-9}$ | $7 - 8.8 \times 10^{-7}$ | $7/3 - 3.0 \times 10^{-7}$ | $(-1, -1, 2)$ |
| $4 + 6.0 \times 10^{-9}$ | $8 - 1.1 \times 10^{-6}$ | $2 - 2.9 \times 10^{-7}$ | $(2, 0, -2)$ |
| $7 + 3.2 \times 10^{-8}$ | $14 - 7.1 \times 10^{-6}$ | $2 - 1.0 \times 10^{-6}$ | $(1, 2, -3)$ |
| $7 + 3.2 \times 10^{-8}$ | $14 - 7.1 \times 10^{-6}$ | $2 - 1.0 \times 10^{-6}$ | $(-1, -2, 3)$ |
| $9 + 2.2 \times 10^{-7}$ | $18 - 5.1 \times 10^{-5}$ | $2 - 5.7 \times 10^{-6}$ | $(3, 0, -3)$ |
| $12 + 5.7 \times 10^{-5}$ | $28 + 3.8 \times 10^{-3}$ | $7/3 - 1.4 \times 10^{-3}$ | $(2, 2, -4)$ |
| $12 + 5.7 \times 10^{-5}$ | $28 + 3.8 \times 10^{-3}$ | $7/3 - 1.4 \times 10^{-3}$ | $(-2, -2, 4)$ |
| $13 - 5.6 \times 10^{-5}$ | $26 - 3.8 \times 10^{-3}$ | $2 + 1.3 \times 10^{-3}$ | $(1, 3, -4)$ |
| $13 - 5.6 \times 10^{-5}$ | $26 - 3.8 \times 10^{-3}$ | $2 + 1.3 \times 10^{-3}$ | $(-1, -3, 4)$ |

Table 1: Scaling limit of three-impurity $\text{su}(2)$ numerical spectrum at one loop in $\lambda$ theory makes the following simple predictions for the large-$L$ $\text{su}(2)$ expansion coefficients
\[
E_{\text{su}(2)}^{(1,3)} \text{ and } E_{\text{su}(2)}^{(1,2)}: \\
E_{\text{su}(2)}^{(1,2)} = \frac{\left(k_1^2 + k_2^2 + k_3^2\right)}{2} \quad k_1 + k_2 + k_3 = 0 \\
E_{\text{su}(2)}^{(1,3)}/E_{\text{su}(2)}^{(1,2)} = 2 \quad (k_1 \neq k_2 \neq k_3) \\
E_{\text{su}(2)}^{(1,3)}/E_{\text{su}(2)}^{(1,2)} = \frac{7}{3} \quad (k_1 = k_2, k_3 = -2k_1).
\tag{2.28}
\]

Note the slight annoyance that we must distinguish the case where all mode indices are unequal from the case where two indices are equal and different from the third. The last column of table 1 displays the choice of indices \(\{k_i\}\) that best fit each spectral series and the other columns display the deviation of the extrapolation coefficients from the string theory predictions of eqn. (2.28). As the lattice momenta increase, higher-order \(1/L\) corrections to the spectrum become stronger and more data will be required to maintain a given level of precision of the polynomial fit. This effect can be seen directly in the extrapolated eigenvalues in table 1. Nonetheless, it is clear from the table that the gauge theory match to the string theory prediction is extremely good.

We also note that the spectrum in table 1 exhibits a degeneracy of eigenstates whose momentum labels are related by an overall sign flip (a symmetry that is implemented on the operator basis by a parity operator \(P\) which reverses the ordering of all fields within the trace). This degeneracy among “parity pairs” of gauge theory operators was observed in [5], where it was shown that it arises as a consequence of integrability (which can, in turn, be used to constrain the form of the Hamiltonian at higher loop order [6]). See [20] for further discussion on the implications of this degeneracy.

To corroborate these results on the gauge theory side we turn to the one-loop Bethe ansatz for the Heisenberg spin chain. The Bethe ansatz for chains of spins in arbitrary representations of arbitrary simple Lie groups was developed some time ago [21] (see also [22] for an extension to supersymmetric spin chains) and applied only recently to the specific case of the dilatation operator of \(\mathcal{N} = 4\) SYM [12, 13]. In the notation of [13], the Bethe equations are expressed in terms of the so-called Bethe roots (or rapidities) \(u_i\) associated with the various impurity insertions in the single-trace ground state \(\text{Tr} \ Z_L^L\). In a one-dimensional dynamical interpretation, the impurities are pseudoparticle excitations and the roots parameterize in some fashion the lattice momenta of the pseudoparticles. The index \(i\) in the Bethe root \(u_i\) runs over the total number \(I\) of impurities. A second index \(q_i = 1, \ldots, 7\) is used to associate each of the \(I\) Bethe roots with a particular simple root of the \(\mathfrak{sl}(4|4)\) symmetry algebra associated with \(\mathcal{N} = 4\) SYM. The Bethe ansatz then takes the form (see [13] and references therein for further details)

\[
\left(\frac{u_i + \frac{i}{2} V_{q_i}}{u_i - \frac{i}{2} V_{q_i}}\right)^L = \prod_{j \neq i}^{I} \left(\frac{u_i - u_j + \frac{i}{2} M_{q_i,q_j}}{u_i - u_j - \frac{i}{2} M_{q_i,q_j}}\right),
\tag{2.29}
\]

where \(V_{q_i}\) denotes the \(q_i^{th}\) Dynkin coefficient of the spin representation and \(M\) is the Cartan matrix of the algebra. To be slightly more specific, if \(\alpha_{q_i}\) are the root vectors associated with
the nodes of the Dynkin diagram and $\mu$ is the highest weight of the spin representation, then
the Dynkin coefficient (for a bosonic algebra) is $V_{q_i} = 2 \alpha(q_i) \cdot \mu / (\alpha(q_i))^2$ and the elements of
the Cartan matrix are $M_{q_i, q_j} = 2 \alpha(q_i) \cdot \alpha(q_j) / (\alpha(q_j))^2$ (note that diagonal elements $M_{q_i, q_i} = 2$).
(For superalgebras see, eg., \cite{23,24}.) Furthermore, since the spin chain systems of interest
to us are cyclic and carry no net momentum (analogous to the level-matching condition in
the string theory), the Bethe roots $u_i$ are subject to the additional constraint
\begin{equation}
1 = \prod_{i} \left( \frac{u_i + \frac{\sqrt{2}}{2} V_{q_i}}{u_i - \frac{\sqrt{2}}{2} V_{q_i}} \right). \tag{2.30}
\end{equation}
Finally, having found a set of Bethe roots $u_i$ that solve the above equations, the corresponding
energy eigenvalue (up to an overall additive constant; see, eg., \cite{13}) is given by
\begin{equation}
E = \sum_{j=1}^{I} \left( \frac{V_{q_j}}{u_j^2 + \frac{1}{4} V_{q_j}^2} \right). \tag{2.31}
\end{equation}
In the current application all impurities are of the same type (i.e. carry the same Dynkin
label), so the index $q_i$ can be ignored. It is worth noting, however, that the Dynkin coefficient
$V_{q_i}$ can vanish, in which case the associated Bethe roots do not contribute directly to the
energy.

The Bethe equations are typically exactly soluble for the case of two identical impurities
(i.e. two Bethe roots $u_1, u_2$ associated with the same simple root of the algebra). The
two-impurity $su(2)$ Bethe equations, for example, yield solutions that reproduce the familiar
two-impurity anomalous dimension formula noted above in eqn. (2.9) (see \cite{12,13} for further
examples). For three and higher impurities, however, exact solutions are not known. Since
we are ultimately interested in comparing with string theory predictions at large values
of the $S^5$ angular momentum $J$, an alternate approach is to solve the Bethe equations
perturbatively in small $1/L$. Experience shows that, in the limit where we can neglect
interactions between excitations (or impurities), the Bethe roots are simply the inverse of
the conserved momentum carried by the impurities. With a little work, one can show that
the Bethe ansatz conditions, eqns. (2.29,2.30), can be solved order-by-order in a large-$L$
expansion:
\begin{equation}
u_i = \frac{1}{2\pi k_i} \left( L + A_i \sqrt{L} + B_i + \cdots \right), \tag{2.32}
\end{equation}
where $0 < k_i < L$ is the usual integer lattice momentum. The half-integer powers of $L$ may
or may not be present in eqn. (2.32); they are needed to deal with special kinematic situations
(such as when a pair of impurities have the same lattice momentum) where the integral
power expansion would be singular. The eigenvalues of the spin chain (or the anomalous
dimensions of the corresponding gauge theory operator) are then obtained as a power series
in $1/L$ by substituting the expansion of the Bethe roots into eqn. (2.31). This is the approach
introduced by Minahan and Zarembo for the $so(6)$ spin chain in \cite{12}. Since we wish to carry
out similar calculations at higher orders in $\lambda$, we will review this methodology at one-loop order for the specific case of three identical impurities in the $\mathfrak{su}(2)$ spin chain. (Since the $\mathfrak{su}(2)$ chain is a subsector of the $\mathfrak{so}(6)$ system studied in [12], the three-impurity Bethe ansatz predictions derived here are of course implied by the all-impurity $\mathfrak{so}(6)$ anomalous dimension formula derived in [12] at one loop.)

We now apply this to the closed $\mathfrak{su}(2)$ sector where the Dynkin diagram has a single node, the Cartan matrix is $M_{\mathfrak{su}(2)} = 2$ and the Dynkin coefficient of the fundamental representation is $V_{\mathfrak{su}(2)} = 1$. Consequently, the Bethe equations (2.29, 2.30) reduce to

$$\left( \frac{u_i + i/2}{u_i - i/2} \right)^L = \prod_{j \neq i} \left( \frac{u_i - u_j + i}{u_i - u_j - i} \right)$$

$$(2.33)$$

$$1 = \prod_i \left( \frac{u_i + i/2}{u_i - i/2} \right).$$

$$(2.34)$$

With three or more pseudoparticle excitations, bound-state solutions can arise which satisfy the second equation (2.34). These solutions are characterized as having pseudoparticle states sharing the same lattice momenta (eg. $k_i = k_j$ for the $i^{th}$ and $j^{th}$ roots). The generic solutions to the Bethe equations can therefore be loosely divided into those which do or do not contain bound states. For three impurities with no bound states present ($k_1 \neq k_2 \neq k_3$), eqn. (2.34) states that $k_3 = -k_1 - k_2$. The strategy of [12] can then be used to obtain a systematic expansion of $\mathfrak{su}(2)$ Bethe roots in powers of $L^{-1}$, with the result

$$u_1 = \frac{L - 4}{2\pi k_1} + \frac{3k_1}{\pi (k_1 - k_2)(2k_1 + k_2)} + O(L^{-1})$$

$$u_2 = \frac{(L - 4)k_1^2 + (L - 4)k_1k_2 - 2L - 1 - k_2}{2\pi k_2(k_1^2 + k_1k_2 - 2k_2^2)} + O(L^{-1})$$

$$u_3 = -\frac{(L - 1)k_1^2 - (8 - 5L)k_1k_2 + 2L - 1 - k_2}{2\pi (k_1 + k_2)(2k_1 + k_2)(1 + k_2^2)} + O(L^{-1}).$$

$$(2.35)$$

Substituting these roots into the energy formula eqn. (2.30) gives the following expression for the anomalous dimension of the $\mathfrak{su}(2)$ three-impurity operator at one-loop:

$$E^{(2)}_{\mathfrak{su}(2)}(k_1, k_2) = \frac{8\pi^2}{L^3} \left( k_1^2 + k_1k_2 + k_2^2 \right) (L + 2) + O(L^{-4}) \quad (k_1 \neq k_2 \neq k_3).$$

$$(2.36)$$

This is in perfect agreement with the string theory results of eqn. (2.28) and, of course, the numerical gauge theory results in table 1. When a single bound state is present the Bethe roots must be altered. Taking, for example, $k_1 = k_2$, the cyclic constraint in eqn. (2.34) sets
\( k_3 = -2k_1 \), and the Bethe roots are
\[
\begin{align*}
  u_1 &= -7 + 3i\sqrt{L} + 3L + O(L^{-1/2}) \\
  u_2 &= -7 + 3i\sqrt{L} - 3L + O(L^{-1/2}) \\
  u_3 &= 4 - 3L + O(L^{-1/2}) .
\end{align*}
\] (2.37)

In this case the anomalous dimension is predicted to be
\[
E^{(2)}_\text{su}(2)(k_1) = \frac{8\pi^2}{L^3} k_1^2 (3L + 7) + O(L^{-4}) \quad (k_1 = k_2, \quad k_3 = -2k_1) ,
\] (2.38)

which is again in agreement with the results of eqn. (2.28) and table II (note that the fractional powers of \( L^{-1} \) have obligingly canceled out of the final expression for the energy).

### 2.2 Two and three-loop order

A similar analysis can be performed on the two-loop \( \text{su}(2) \) spin-chain Hamiltonian. As before, we use the Jordan-Wigner transformation restricted to an odd-impurity basis of operators to rewrite the two-loop Hamiltonian in terms of position-space fermionic oscillators, obtaining a result similar to eqn. (2.21):
\[
H^{(4)}_\text{su}(2) = \sum_{j=1}^{L} \left\{ -\frac{1}{2} \left[ b_{j+2}^\dagger b_j + b_j^\dagger b_{j+2} - 4 \left( b_{j+1}^\dagger b_j + b_j^\dagger b_{j+1} \right) \right] - 3 b_j^\dagger b_j - 4 b_{j+1}^\dagger b_{j+1} b_j b_{j+1} + b_{j+2}^\dagger b_{j+2} b_{j+1} + b_j^\dagger b_{j+1} b_j b_{j+2} + b_{j+1}^\dagger b_{j+2} b_j b_{j+2} \right\} .
\] (2.39)

Passing to momentum space, we obtain the two-loop analogue of eqn. (2.28):
\[
H^{(4)}_\text{su}(2) = -8 \sum_{p=0}^{L-1} \sin^4 \left( \frac{p\pi}{L} \right) b_p^\dagger b_p + 1 \sum_{p,q,r,s=0}^{L-1} \left( e^{\frac{2\pi i (p+q)}{L}} + e^{\frac{-2\pi i (p+s)}{L}} + e^{\frac{4\pi i (q-s)}{L}} - 4 e^{\frac{2\pi i (q-s)}{L}} \right) b_p^\dagger b_q^\dagger b_s^\dagger b_{p+q,r+s} .
\] (2.40)

Although the two-loop Hamiltonian includes “long-range” interactions among non-neighboring lattice sites, the momentum-space Hamiltonian conveniently terminates at two-body interaction terms. An equally important point is that, for fixed momenta \( p, q, \ldots \), the one-body (two-body) operators scale as \( L^{-4} (L^{-5}) \) for large \( L \) (the corresponding scalings for the
one-loop Hamiltonian were $L^{-2}$ ($L^{-3}$). This special relation between density scaling and power of coupling constant is critical for matching to string theory.

We deal with the problem of finding the eigenvalues of the combined one- and two-loop Hamiltonian via Rayleigh-Schrödinger perturbation theory: at each value of the lattice length $L$ we treat the one-loop operator $H^{(2)}_{su(2)}$ as a zeroth-order Hamiltonian and regard $H^{(4)}_{su(2)}$ as a first-order perturbation. The $O(\lambda^2)$ corrections to the spectrum of $H^{(2)}_{su(2)}$ are then found by taking expectation values of the perturbation $H^{(4)}_{su(2)}$ in the (numerically-determined) eigenvectors of $H^{(2)}_{su(2)}$. This is the recipe for non-degenerate first-order perturbation theory and we might worry that the previously-noted parity-pair degeneracy of the eigenvalues of $H^{(2)}_{su(2)}$ would force us to use the rules of degenerate perturbation theory. As discussed in [3, 10, 20], however, parity degeneracy can be traced to the existence of a higher Abelian charge which is conserved to at least three-loop order. This charge can be used to show that the formulas of non-degenerate perturbation theory can be used without modification. The basic observation is that conservation of the Abelian charge guarantees that the matrix element of $H^{(4)}_{su(2)}$ between two degenerate eigenstates of $H^{(2)}_{su(2)}$ with different eigenvalues of the higher Abelian charge vanishes: this eliminates the vanishing energy-denominator singularities that would otherwise invalidate the non-degenerate first-order perturbation theory formulas (and similar arguments apply to the higher-order cases).

Using this method, we have evaluated the $O(\lambda^2)$ corrections to the spectrum of anomalous dimensions for lattice sizes from $L = 6$ to $L = 40$. As before, we fit the spectral data to a power series in $1/L$ to read off the leading scaling coefficients of the low-lying eigenvalues. As mentioned in the discussion of the two-loop Hamiltonian (2.40), we expect the two-loop eigenvalues to have the following scaling behavior in $1/L$:

$$E^{(2)}_L(\{k_i\}) = \frac{\lambda^2}{L^4} E^{(2,4)}(\{k_i\}) + \frac{\lambda^2}{L^5} E^{(2,5)}(\{k_i\}) + O(\lambda^2 L^{-6}) \, .$$

(2.41)

The numerical data confirm that the eigenvalues scale at least as fast as $L^{-4}$. The resulting numerical values for the leading scaling coefficients of low-lying eigenvalues, $E^{(2,4)}_{su(2)}$ and $E^{(2,5)}_{su(2)}$, are presented in table 2. As originally reported in [10], string theory makes the following simple predictions for the two-loop large-$L$ expansion coefficients:

$$E^{(2,4)}_{su(2)} = -(k_1^2 + k_2^2 + k_3^2)/16 \quad k_1 + k_2 + k_3 = 0$$

$$E^{(2,5)}_{su(2)} / E^{(2,3)}_{su(2)} = 8 \quad (k_1 \neq k_2 \neq k_3)$$

$$E^{(2,5)}_{su(2)} / E^{(2,3)}_{su(2)} = \frac{76}{9} \quad (k_1 = k_2, k_3 = -2k_1) \, .$$

(2.42)

The low-lying levels in the table match the string theory predictions quite accurately and the decline in precision as one goes to higher energies is expected. As a consistency check we note that this time we have no freedom to choose the momenta $(k_1, k_2, k_3)$ associated with each state: they have been fixed in the one-loop matching exercise.

The three-loop $su(2)$ Hamiltonian (2.14) can be dealt with in a similar fashion. The position space operator version of this Hamiltonian is too long to record here, but its momentum
we therefore expect the leading scaling coefficients in the $O(\lambda^6)$ numerical spectrum at two loops in $\lambda$ space version is fairly compact:

$$H^{(6)}_{\text{su}(2)} = 32 \sum_{p=0}^{L-1} \sin^6 \left( \frac{p\pi}{L} \right) \tilde{b}_p \tilde{b}_p + \frac{1}{2L} \sum_{p,q,r,s=0}^{L-1} \left\{ -10 e^{\frac{2\pi i (q+r)}{L}} + e^{\frac{2\pi i (2q+r)}{L}} + e^{\frac{2\pi i (q+2r)}{L}} + e^{\frac{2\pi i (q-3s)}{L}} 
+ e^{\frac{2\pi i (2q-2r-3s)}{L}} + e^{\frac{2\pi i (3q-2r-3s)}{L}} + e^{\frac{2\pi i (q-r-3s)}{L}} - e^{\frac{2\pi i (q-2s)}{L}} - 10 e^{\frac{2\pi i (q-r-2s)}{L}} 
- e^{\frac{2\pi i (2q-s)}{L}} + e^{\frac{2\pi i (3q-s)}{L}} - e^{\frac{2\pi i (q-r-s)}{L}} + e^{\frac{2\pi i (2q-r-s)}{L}} + e^{\frac{2\pi i (q+2r-s)}{L}} + e^{\frac{2\pi i (q+3r-s)}{L}} \right\} \tilde{b}_p \tilde{b}_q \tilde{b}_r \tilde{b}_s \delta_{p+q+r+s}$$

$$+ \frac{1}{L^2} \sum_{p,q,r,s,t,u=0}^{L-1} \left\{ e^{\frac{2\pi i (q+3r-2t-3u)}{L}} + e^{\frac{2\pi i (q+2r-s-2t-3u)}{L}} \right\} \tilde{b}_p \tilde{b}_q \tilde{b}_r \tilde{b}_s \tilde{b}_t \tilde{b}_u \delta_{p+q+r+s+t+u} . \quad (2.43)$$

It contains at most three-body operators and a careful examination of terms shows that, for fixed momenta, the one-body operators scale as $L^{-6}$, the two-body operators as $L^{-7}$ and so on. We therefore expect the leading scaling coefficients in the $O(\lambda^3)$ eigenvalues to be $E^{(3,6)}_{\text{su}(2)}$ and $E^{(3,7)}_{\text{su}(2)}$, to use a by-now-familiar notation. To find the eigenvalues to this order, we continue with the Rayleigh-Schrödinger perturbation theory strategy: the $O(\lambda^3)$ correction to any eigenvalue is the sum of the matrix element of $H^{(6)}_{\text{su}(2)}$ in the appropriate eigenvector of $H^{(2)}_{\text{su}(2)}$ plus the second-order sum-over-states contribution of $H^{(4)}_{\text{su}(2)}$. These two pieces can easily be computed numerically from the explicit Hamiltonian operators at a fixed $L$. 

| $E^{(2,4)}_{\text{su}(2)}$ | $E^{(2,5)}_{\text{su}(2)}$ | $E^{(2,8)}_{\text{su}(2)} / E^{(2,4)}_{\text{su}(2)}$ | $(k_1, k_2, k_3)$ |
|-------------------------|-------------------------|----------------------------------|-----------------|
| $-0.25 - 4.6 \times 10^{-9}$ | $-2 + 8.0 \times 10^{-7}$ | $8 - 3.4 \times 10^{-6}$ | $(1, 0, -1)$ |
| $-2.25 - 1.4 \times 10^{-6}$ | $-19 + 2.6 \times 10^{-4}$ | $76/9 + 1.2 \times 10^{-4}$ | $(1, 1, -2)$ |
| $-2.25 - 1.4 \times 10^{-6}$ | $-19 + 2.6 \times 10^{-4}$ | $76/9 + 1.2 \times 10^{-4}$ | $(-1, -1, 2)$ |
| $-4 + 8.3 \times 10^{-7}$ | $-32 - 1.1 \times 10^{-4}$ | $8 + 3.0 \times 10^{-5}$ | $(2, 0, -2)$ |
| $-12.25 - 9.9 \times 10^{-6}$ | $-98 + 2.3 \times 10^{-3}$ | $8 - 2.0 \times 10^{-4}$ | $(1, 2, -3)$ |
| $-12.25 - 9.9 \times 10^{-6}$ | $-98 + 2.3 \times 10^{-3}$ | $8 - 2.0 \times 10^{-4}$ | $(-1, -2, 3)$ |
| $-20.25 + 3.2 \times 10^{-3}$ | $-161.4$ | $7.97$ | $(3, 0, -3)$ |
| $-36 - 2.8 \times 10^{-3}$ | $-304.6$ | $8.46$ | $(2, 2, -4)$ |
| $-36 - 2.8 \times 10^{-3}$ | $-304.6$ | $8.46$ | $(-2, -2, 4)$ |
| $-42.25 + 4.9 \times 10^{-3}$ | $-337.0$ | $7.97$ | $(1, 3, -4)$ |
| $-42.25 + 4.9 \times 10^{-3}$ | $-337.0$ | $7.97$ | $(-1, -3, 4)$ |
Parity degeneracy and conservation of the higher Abelian charge mentioned above continue to hold, and we can again use non-degenerate perturbation theory formulas to compute the eigenvalue corrections. We have generated numerical eigenvalue data for lattices from $L = 6$ to $L = 40$ and the large-$L$ scaling coefficients of the low-lying states extracted from those data are given in table 3. As is by now well-known, the detailed match to string theory breaks down at three-loop order, so there is no point in trying to match these results to string predictions. We were initially motivated to pursue a virial treatment of this problem because the more classic Bethe ansatz methods were not yet able to deal with non-local spin chains. Important progress has recently been made on the Bethe ansatz side and it may be useful to compare what can be done by each method (results should of course agree).

A modified Bethe ansatz for the $su(2)$ sector of the gauge theory, possibly incorporating all orders of higher-loop physics, has recently been proposed \[8, 9\].\(^2\) It is an instructive exercise, and a useful consistency check on this bold proposal, to verify that it reproduces the higher-loop scaling coefficients for three impurity anomalous dimensions that we have just computed by virial methods (and displayed in tables 1, 2 and 3). For completeness, we briefly summarize the new ansatz, referring the reader to \[9\] for a detailed account. In the new ansatz, the momenta $p_i$ of the excitations (closely related to the Bethe roots) become functions of $\lambda$ (as well as $L$ and mode numbers) and are determined by a modified version of eqns. (2.33,2.34):

$$e^{i L p_i} = \prod_{j \neq i} \frac{\varphi(p_i) - \varphi(p_j) + i}{\varphi(p_i) - \varphi(p_j) - i} \quad \sum_{i=1}^{L} p_i = 0 .$$

(2.44)

Dependence on $\lambda$ enters through the phase function $\varphi(p_i)$ which is defined in terms of the

| $E_{su(2)}^{(3,4)}$ | $E_{su(2)}^{(3,7)}$ | $E_{su(2)}^{(3,7)}/E_{su(2)}^{(3,6)}$ | $(k_1, k_2, k_3)$ |
|----------------------|----------------------|----------------------|----------------------|
| 0.1250 | 2.0003 | 16.003 | $(1, 0, -1)$ |
| 4.125 | 58.03 | 14.07 | $(1, 1, -2)$ |
| 4.125 | 58.03 | 14.07 | $(-1, -1, 2)$ |
| 7.999 | 128.2 | 16.03 | $(2, 0, -2)$ |
| 49.62 | 713.3 | 14.37 | $(1, 2, -3)$ |
| 49.62 | 713.3 | 14.37 | $(-1, -2, 3)$ |
| 91.15 | 1,454 | 15.96 | $(3, 0, -3)$ |
| 263.8 | 3,739 | 14.17 | $(2, 2, -4)$ |
| 263.8 | 3,739 | 14.17 | $(-2, -2, 4)$ |
excitation momenta \( p_i \) as follows:

\[
\varphi(p_i) = \frac{1}{2} \cot\left(\frac{p_i}{2}\right) \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p_i}{2}\right)} .
\]  

(2.45)

The energy eigenvalue corresponding to a particular root of these equations is given in terms of the excitation momenta \( p_i \) by the formula

\[
E_{\text{su}(2)} = \sum_{i=1}^{I} \frac{8\pi^2}{\lambda} \left( \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p_i}{2}\right)} - 1 \right) .
\]  

(2.46)

Finding exact solutions of these equations is even more difficult than before, but we can follow the previous strategy of developing an expansion in powers of \( 1/L \) about non-interacting impurities on an infinite lattice. This is achieved by expanding the excitation momenta \( p_i \) according to

\[
p_i = \frac{2\pi k_i}{L} + \sum_{n=1}^{8} \frac{p_i^{(n)}}{L^{n+2}} ,
\]  

(2.47)

where the integers \( k_i \) (subject to the cyclicity constraint \( \sum_i k_i = 0 \)) characterize the non-interacting state about which the expansion is developed. The appearance of half-integer powers of \( L^{-1} \) in this expansion is needed to accommodate bound-state solutions to the Bethe equations which arise when some of the momenta \( k_i \) are equal. Solutions to the Bethe equation (2.44) will determine the expansion coefficients \( p_i^{(n)} \) in terms of the mode numbers \( k_i \) and ultimately lead to expansions of the energies as power series in \( L^{-1} \), with coefficients that are functions of \( \lambda/L^2 \).

Explicit results for the \( L^{-1} \) expansion of gauge theory operators of arbitrary impurity number, derived by the above method, were presented in [15].\(^3\) As usual, expressions are different depending on whether all momenta are unequal or some subset of them are equal. For all mode numbers \( k_i \) unequal the \( I \)-impurity energy formula in [15] is

\[
E_{\text{su}(2)} = L - I + \sum_{i=1}^{I} \left( \sqrt{1 + \lambda' k_i^2} - \frac{\lambda'}{L - I} \frac{I k_i^2}{\sqrt{1 + \lambda' k_i^2}} \right) + \frac{\lambda'}{L - I} \sum_{i,j=1}^{I} \frac{2k_i^2 k_j}{k_i^2 - k_j^2} \left( k_j + k_i \sqrt{1 + \lambda' k_j^2} \frac{1 + \lambda' k_i^2}{1 + \lambda' k_j^2} \right) + O(L^{-2}) ,
\]  

(2.48)

where we have used \( \lambda' = \lambda/J^2 = \lambda/(L - I)^2 \) for convenience (\( J = L - I \) is the total \( R \)-charge). To compare with our virial results, we must further expand in \( \lambda \); expanding to first

\(^3\)It is important to note that the focus of this paper is a different Bethe ansatz, designed to match the spectrum of the string theory: the gauge theory Bethe ansatz results are derived for comparison purposes.
To compare with the virial results, one must again expand the energy in powers of particular case of three impurities with a two-excitation bound state (formula for the many cases in which subsets of momenta are equal but the solution for the $2$ and the analytic string formulas in eqns. (2.28,2.42). It is harder to write down a general These one- and two-loop coefficients match the numerical results presented in tables $1$ and $2$ and the analytic string formulas in eqns. (2.48,2.50). The three-loop coefficients obtained by expanding the energy formulas in eqns. (2.48,2.50)

\[ E^{(1,2)}_{\text{su}(2)} = k_1^2 + k_1 k_2 + k_2^2 \quad E^{(1,3)}_{\text{su}(2)} = 2(k_1^2 + k_1 k_2 + k_2^2) \]

\[ E^{(2,4)}_{\text{su}(2)} = -\frac{1}{4}(q^2 + q r + r^2)^2 \quad E^{(2,5)}_{\text{su}(2)} = -2(q^2 + q r + r^2)^2 . \] (2.49)

These one- and two-loop coefficients match the numerical results presented in tables $1$ and $2$ and the analytic string formulas in eqns. (2.28,2.42). It is harder to write down a general formula for the many cases in which subsets of momenta are equal but the solution for the particular case of three impurities with a two-excitation bound state ($k_1 = k_2 = n, \ k_3 = -2n$) was also presented in [13]:

\[ E_{\text{su}(2)} = L - 3 + 2\sqrt{1 + \lambda' n^2} + \sqrt{1 + \lambda' 4n^2} \]

\[ -\frac{\lambda' n^2}{L - 3} \left( \frac{1}{1 + \lambda' n^2} + \frac{6}{\sqrt{1 + \lambda' 4n^2}} + \frac{12}{\sqrt{1 + \lambda' 4n^2}} - \frac{8}{\sqrt{1 + \lambda' n^2} \sqrt{1 + \lambda' 4n^2}} \right) . \] (2.50)

To compare with the virial results, one must again expand the energy in powers of $\lambda$. Doing so yields the following one- and two-loop bound-state scaling coefficients:

\[ E^{(1,2)}_{\text{su}(2)} = 3 n^2 \quad E^{(1,3)}_{\text{su}(2)} = 7 n^2 \]

\[ E^{(2,4)}_{\text{su}(2)} = -\frac{9}{4} n^4 \quad E^{(2,5)}_{\text{su}(2)} = -19 n^4 . \] (2.51)

We easily verify that this agrees with numerical virial results to two-loop order.

The three-loop coefficients obtained by expanding the energy formulas in eqns. (2.48,2.50) are given by

\[ E^{(3,6)}_{\text{su}(2)} = \frac{1}{16} \left( 2 k_1^6 + 6 k_1^5 k_2 + 15 k_1^4 k_2^2 + 20 k_1^3 k_2^3 + 15 k_1^2 k_2^4 + 6 k_1 k_2^5 + 2 k_2^6 \right) \]

\[ E^{(3,7)}_{\text{su}(2)} = \frac{1}{4} \left( 8 k_1^6 + 24 k_1^5 k_2 + 51 k_1^4 k_2^2 + 62 k_1^3 k_2^3 + 51 k_1^2 k_2^4 + 24 k_1 k_2^5 + 8 k_2^6 \right) , \] (2.52)

for ($k_1 \neq k_2 \neq k_3$), and

\[ E^{(3,6)}_{\text{su}(2)} = \frac{33}{8} n^6 \quad E^{(3,7)}_{\text{su}(2)} = 58 n^6 . \] (2.53)

for the bound-state solution with ($k_1 = k_2 = n, \ k_3 = -2n$). The numerical values of these $O(\lambda^3)$ coefficients are tabulated for several low-lying states in the spectrum in table $4$. The correspondence with table $3$ which displays the three-loop expansion coefficients extracted from numerical diagonalization of the three-loop Hamiltonian, is good. At this order in the loop expansion higher-order $1/L$ corrections to the spectrum are more important (compared to the one- and two-loop cases), and the numerical extrapolation is less reliable (especially as the lattice momenta increase). The precision can always be improved by including data from larger lattices in the extrapolation. We emphasize that this discussion concerns the different methods of calculation of operator dimensions in the su(2) sector only. It seems to us to give useful further evidence that the long-range Bethe ansatz for the su(2) sector of the gauge theory [9] is exact.
Table 4: Three-impurity $\mathfrak{su}(2)$ spectrum from the long-range Bethe ansatz at three loops

| $E^{(3,0)}_{\mathfrak{su}(2)}$ | $E^{(3,7)}_{\mathfrak{su}(2)}$ | $E^{(3,7)}/E^{(3,0)}_{\mathfrak{su}(2)}$ | $(k_1, k_2, k_3)$ |
|-----------------|-----------------|-----------------|-----------------|
| 0.125           | 2               | 16              | (1, 0, −1)     |
| 4.125           | 58              | 14.06           | (1, 1, −2)     |
| 4.125           | 58              | 14.06           | (−1, −1, 2)    |
| 8               | 128             | 16              | (2, 0, −2)     |
| 49.625          | 713             | 14.37           | (1, 2, −3)     |
| 49.625          | 713             | 14.37           | (−1, −2, 3)    |
| 91.125          | 1, 458          | 16              | (3, 0, −3)     |
| 264             | 3, 712          | 14.06           | (2, 2, −4)     |
| 264             | 3, 712          | 14.06           | (−2, −2, 4)    |

3 A closed $\mathfrak{su}(1|1)$ subsector of $\mathfrak{su}(2|3)$

The three-impurity string theory analysis of [10] identified a fermionic sector of the theory which is diagonalized by string states composed of fermionic excitations projected onto particular four-dimensional subspaces (which transform in an $SU(2)^2 \times SU(2)^2$ notation as a $(2, 1; 2, 1)$ or $(1, 2; 1, 2)$ of $SO(4) \times SO(4)$) and symmetrized in their $SO(4) \times SO(4)$ indices. It was also shown that this three-impurity subsector of the theory decouples at all orders in $\lambda$.

On the gauge theory side this subsector corresponds to an $\mathfrak{su}(1|1)$ subgroup of the closed $\mathfrak{su}(2|3)$ sector studied by Beisert in [7, 17]. (Supersymmetric integrable $\mathfrak{su}(n|m)$ spin chains have previously been studied in certain condensed-matter applications; see, eg., [25].) In the present setting the fields of $\mathfrak{su}(2|3)$ consist of three complex scalars $\phi_a$ and two complex fermions $\psi_\alpha$. In the closed $\mathfrak{su}(1|1)$ subspace we restrict to a single scalar denoted by $Z$ and a single fermion labeled by $\psi$. Just as in the $\mathfrak{su}(2)$ sector, we use the fermionic position-space oscillators $b_j^\dagger$, $b_j$ to create or annihilate fermionic $\psi$ insertions in a ground state composed of $L$ scalars:

$$|L\rangle = \text{Tr}(Z^L) \quad b_j^\dagger |L\rangle = \text{Tr}(Z_1 \cdots Z_{j-1}\psi Z_{j+1} \cdots Z_L) .$$ (3.1)

In [7], Beisert gave the action of the Hamiltonian on the $\mathfrak{su}(2|3)$ spin chain to three-loop order. In the notation of [7], the action of the Hamiltonian on basis states can be represented in terms of special permutation operators denoted by

$$\left\{ \begin{array}{c} A_1 \ldots A_N \\ B_1 \ldots B_N \end{array} \right\} .$$

Beisert’s three-loop Hamiltonian was restricted in [7] to the bosonic sector, but the author has since provided us with the complete version.
which replace all occurrences of the upper sequence of fields $A_1 \ldots A_N$ in the trace by the lower sequence $B_1 \ldots B_N$. Restricting Beisert’s $\mathfrak{su}(2|3)$ Hamiltonian to the $\mathfrak{su}(1|1)$ subsector at one-loop order yields

$$H^{(2)}_{\mathfrak{su}(1|1)} = \left\{ \begin{array}{c} Z\psi \\ Z\psi \end{array} \right\} + \left\{ \begin{array}{c} \psi Z \\ \psi Z \end{array} \right\} - \left\{ \begin{array}{c} Z\psi \\ Z\psi \end{array} \right\} + 2\left\{ \begin{array}{c} \psi\psi \\ \psi\psi \end{array} \right\} .$$

(3.2)

In terms of the position-space oscillators of eqn. (3.1), the $\mathfrak{su}(1|1)$ Hamiltonian can be assembled by inspection and takes the form

$$H^{(2)}_{\mathfrak{su}(1|1)} = \sum_{j=1}^{L} \left( b_j^\dagger b_j + b_{j+1}^\dagger b_{j+1} - b_{j+1}^\dagger b_{j} - b_j^\dagger b_{j+1} \right) .$$

(3.3)

There are no higher-body interaction terms at this order in $\lambda$. This fact can be checked by computing

$$\langle L|b_{i+1}b_i(H^{(2)}_{\mathfrak{su}(1|1)})b_{i+1}^\dagger b_i^\dagger|L\rangle = 2 ,$$

(3.4)

which reproduces the two-body matrix element given by the last term in eqn. (3.2). In momentum space we obtain

$$H^{(2)}_{\mathfrak{su}(1|1)} = 4 \sum_{p=0}^{L-1} \sin^2 \left( \frac{p\pi}{L} \right) \tilde{b}_p^\dagger \tilde{b}_p.$$  

(3.5)

The two-loop $\mathfrak{su}(1|1)$ momentum-space Hamiltonian can be extracted in the same manner (the position-space version is too long to print here):

$$H^{(4)}_{\mathfrak{su}(1|1)} = -8 \sum_{p=0}^{L-1} \sin^4 \left( \frac{p\pi}{L} \right) \tilde{b}_p^\dagger \tilde{b}_p + \frac{1}{4L} \sum_{p,q,r,s=0}^{L-1} \left\{ e^{\frac{2\pi i(q-r)}{L}} + e^{\frac{2\pi i(2q-r)}{L}} - 4 e^{\frac{2\pi i(q-r)}{L}} \right\} \tilde{b}_p^\dagger \tilde{b}_q^\dagger \tilde{b}_r \tilde{b}_s \delta_{p+q,r+s} .$$

(3.6)
Finally, the complete three-loop Hamiltonian for this subsector is

\[ H_{\text{su}(1|1)}^{(6)} = 32 \sum_{p=0}^{L-1} \sin^6 \left( \frac{p\pi}{L} \right) \hat{b}_p^\dagger \hat{b}_p - \frac{1}{16} \sum_{p,q,r,s=0}^{L-1} e^{-\frac{6 \pi i (q-r)}{L}} \left\{ 2 e^{-\frac{2 \pi i (29q-29r)}{L}} + 2 e^{-\frac{2 \pi i (28q-28r)}{L}} \right. \]

\[ -4 e^{-\frac{2 \pi i (27q-27r)}{L}} + 37 e^{-\frac{2 \pi i (29q-29r)}{L}} - 6 e^{-\frac{2 \pi i (29q-27r)}{L}} + 8 e^{-\frac{2 \pi i (29q)}{L}} - \frac{56 \pi i (q-r)}{L} - 72 e^{-\frac{2 \pi i (29q-29r)}{L}} \]

\[ -6 e^{-\frac{2 \pi i (27q-29r)}{L}} - 40 e^{-\frac{2 \pi i (29q-27r)}{L}} - 40 e^{-\frac{2 \pi i (29q-28r-s)}{L}} - 4 e^{-\frac{2 \pi i (29q-27r+s)}{L}} \]

\[ +8 e^{-\frac{2 \pi i (29q-30r-2s)}{L}} + 2 e^{-\frac{2 \pi i (29q-27r-2s)}{L}} + 8 e^{-\frac{2 \pi i (29q-30r-2s)}{L}} \]

\[ \left. + \frac{1}{16} \sum_{p,q,r,s,t,u=0}^{L-1} \right\} \left\{ 2 e^{-\frac{2 \pi i (q+2r-3s-2t)}{L}} - e^{-\frac{2 \pi i (q+3r-3s-2t)}{L}} - 4 e^{-\frac{2 \pi i (q+2r-3s-t)}{L}} \right. \]

\[ - e^{-\frac{2 \pi i (2q+3r-3s-t)}{L}} + 8 e^{-\frac{2 \pi i (2q+2r-3t-2s)}{L}} + 2 e^{-\frac{2 \pi i (2q+3r-2s-t)}{L}} - 4 e^{-\frac{2 \pi i (q+2r-3s-2t-u)}{L}} \]

\[ + 2 e^{-\frac{2 \pi i (q+2r-3s-2t-u)}{L}} + 2 e^{-\frac{2 \pi i (q+2r-3s-2t-u)}{L}} \]

\[ - 4 e^{-\frac{2 \pi i (q+2r-3s-2t-u)}{L}} - 4 e^{-\frac{2 \pi i (q+2r-2s+u)}{L}} \]

\[ \left\{ \hat{b}_p^\dagger \hat{b}_q^\dagger \hat{b}_r^\dagger \hat{b}_s^\dagger \hat{b}_u \delta_{p+q+r+s+t+u} \right\}. \tag{3.7} \]

We note that \( H_{\text{su}(1|1)}^{(2)} \), \( H_{\text{su}(1|1)}^{(4)} \) and \( H_{\text{su}(1|1)}^{(6)} \) terminate at one-body, two-body and three-body interactions, respectively. This will permit us to obtain the exact \( L \)-dependence of successive terms in the \( \lambda \) expansion of energy eigenvalues.

As in the \( \text{su}(2) \) sector, we can use non-degenerate perturbation theory to extract the \( L^{-1} \) scaling coefficients of the \( \text{su}(1|1) \) eigenvalue spectrum up to three loops in \( \lambda \). The scaling coefficients extrapolated from numerical diagonalization of lattices up to \( L = 40 \) are recorded for one-loop, two-loop and three-loop orders in tables 5, 6 and 7, respectively. The same increase in leading power of \( L^{-1} \) with corresponding order in \( \lambda \) that was noted in the \( \text{su}(2) \) sector is found here as well (we use the same notation for the scaling coefficients as before in order to keep track of these powers). It should also be noted that, because the impurities in this sector are fermions symmetricized on all group indices, the lattice momenta of all pseudoparticles must be different. The string theory results of [10] amount to the following predictions for the one-loop and two-loop scaling coefficients:

\[ E_{\text{su}(1|1)}^{(1,2)} = (k_1^2 + k_1 k_2 + k_2^2) \quad E_{\text{su}(1|1)}^{(1,3)} = 0 \]

\[ E_{\text{su}(1|1)}^{(2,4)} = \frac{1}{4} (k_1^2 + k_1 k_2 + k_2^2)^2 \quad E_{\text{su}(1|1)}^{(2,5)} = -(k_1^2 + k_1 k_2 + k_2^2)^2. \tag{3.8} \]

The agreement of these predictions with the data in tables 5 and 6 is excellent (with the usual caveat that data on larger and larger lattices is required to maintain a fixed precision as one goes to higher and higher energy levels).

The scaling limit of the three-loop ratio \( E_{\text{su}(1|1)}^{(3,7)} / E_{\text{su}(1|1)}^{(3,6)} \) is recorded for the first few low-lying states in the spectrum in table 7. These values are in disagreement with the corresponding three-loop predictions from the string theory as can be seen by comparing with the
To apply the general Bethe ansatz equation of eqn. (2.29), we note that the su(1|1) and E results of [10]. Given the well-established three-loop disagreement between the string and gauge theory in the su(2) sector, however, this disagreement in the su(1|1) subsector is not unexpected.

The extrapolated gauge theory results in eqn. [3.8] for the one-loop coefficients $E^{(1,3)}_{su(1|1)}$ and $E^{(1,2)}_{su(1|1)}$ should be checked against the predictions of the general one-loop Bethe ansatz [12, 13] applied to the su(1|1) sector (as far as we know, no higher-loop Bethe ansatz is available here). To apply the general Bethe ansatz equation of eqn. [2.29], we note that the

| $E^{(1,2)}_{su(1|1)}$ | $E^{(1,3)}_{su(1|1)}$ | $E^{(1,3)}_{su(1|1)}/E^{(1,2)}_{su(1|1)}$ | $(k_1, k_2, k_3)$ |
|----------------------|----------------------|---------------------------------|------------------|
| $1 + 1.3 \times 10^{-10}$ | $-1.9 \times 10^{-8}$ | $-1.9 \times 10^{-8}$ | $(1, 0, -1)$ |
| $4 - 1.0 \times 10^{-7}$ | $1.8 \times 10^{-5}$ | $4.6 \times 10^{-6}$ | $(2, 0, -2)$ |
| $7 - 2.5 \times 10^{-7}$ | $4.4 \times 10^{-5}$ | $6.3 \times 10^{-6}$ | $(1, 2, -3)$ |
| $7 - 2.5 \times 10^{-7}$ | $4.4 \times 10^{-5}$ | $6.3 \times 10^{-6}$ | $(-1, -2, 3)$ |
| $9 - 3.9 \times 10^{-7}$ | $7.9 \times 10^{-5}$ | $8.7 \times 10^{-6}$ | $(3, 0, -3)$ |
| $13 - 4.0 \times 10^{-6}$ | $8.2 \times 10^{-4}$ | $6.3 \times 10^{-5}$ | $(1, 3, -4)$ |
| $13 - 4.0 \times 10^{-6}$ | $8.2 \times 10^{-4}$ | $6.3 \times 10^{-5}$ | $(-1, -3, 4)$ |
| $16 - 2.0 \times 10^{-5}$ | $4.1 \times 10^{-3}$ | $2.6 \times 10^{-4}$ | $(4, 0, -4)$ |
| $19 - 3.5 \times 10^{-5}$ | $7.3 \times 10^{-3}$ | $3.8 \times 10^{-4}$ | $(2, 3, -5)$ |
| $19 - 3.5 \times 10^{-5}$ | $7.3 \times 10^{-3}$ | $3.8 \times 10^{-4}$ | $(-2, -3, 5)$ |

Table 5: Scaling limit of one-loop numerical spectrum of three-impurity su(1|1) subsector

| $E^{(2,4)}_{su(1|1)}$ | $E^{(2,5)}_{su(1|1)}$ | $E^{(2,5)}_{su(1|1)}/E^{(2,4)}_{su(1|1)}$ | $(k_1, k_2, k_3)$ |
|----------------------|----------------------|---------------------------------|------------------|
| $-0.25$ | $-0.99999$ | $3.99995$ | $(1, 0, -1)$ |
| $-4.00006$ | $-15.990$ | $3.998$ | $(2, 0, -2)$ |
| $-12.251$ | $-48.899$ | $3.992$ | $(1, 2, -3)$ |
| $-12.251$ | $-48.899$ | $3.992$ | $(-1, -2, 3)$ |
| $-20.25$ | $-80.89$ | $3.995$ | $(3, 0, -3)$ |
| $-42.25$ | $-168.2$ | $3.98$ | $(1, 3, -4)$ |
| $-42.25$ | $-168.2$ | $3.98$ | $(-1, -3, 4)$ |
| $-64.00$ | $-254.6$ | $3.98$ | $(4, 0, -4)$ |
| $-90.26$ | $-359.3$ | $3.98$ | $(2, 3, -5)$ |
| $-90.26$ | $-359.8$ | $3.99$ | $(-2, -3, 5)$ |

Table 6: Scaling limit of two-loop numerical spectrum of three-impurity su(1|1) subsector
Table 7: Scaling limit of three-loop numerical spectrum of three-impurity $\mathfrak{su}(1|1)$ fermionic subsector

$\mathfrak{su}(1|1)$ Dynkin diagram is just a single fermionic node: the Cartan matrix is empty and the single Dynkin label is $V_{\mathfrak{su}(1|1)} = 1$ \cite{23, 24}. We therefore obtain the simple one-loop Bethe equation

$$\left( \frac{u_i + \frac{i}{2}}{u_i - \frac{i}{2}} \right)^L = 1 .$$

(3.9)

Rather remarkably, eqn. (3.9) can be solved exactly for arbitrary impurity number! The general $\mathfrak{su}(1|1)$ Bethe roots are

$$u_i = \frac{1}{2} \cot \left( \frac{k_i \pi}{L} \right)$$

(3.10)

and the energy eigenvalues computed from eqn. (2.31) are

$$E_{\mathfrak{su}(1|1)} = 4 \sum_{i=1}^{L} \sin^2 \left( \frac{\pi k_i}{L} \right) ,$$

(3.11)

with the usual condition $\sum k_i \equiv 0 \mod L$ from eqn. (2.30). This is just the sum of free lattice Laplacian energies and clearly matches the energies one would obtain from the one-loop $\mathfrak{su}(1|1)$ Hamiltonian of eqn. (3.5) (since the latter has no interaction terms). No expansion in $1/L$ was necessary in this argument, but it is straightforward to expand the energies in $1/L$ and verify the numerical results obtained in table 5 and eqn. (3.8).

4 The $\mathfrak{sl}(2)$ sector

As noted in \cite{10}, integrable $\mathfrak{sl}(2)$ spin chains have previously been the subject of several studies involving, among other interesting problems, high-energy scattering amplitudes in
non-supersymmetric QCD (see, eg., [26] and references therein). The \( \mathfrak{sl}(2) \) closed sector of \( \mathcal{N} = 4 \) SYM was studied in [17], and the spin chain Hamiltonian in this sector is presently known to one loop in \( \lambda \).

The constituent fields in this sector are \( SO(6) \) bosons \( Z \) carrying a single unit of \( R \)-charge \( Z = \phi_5 + i\phi_6 \), and each lattice site on the \( \mathfrak{sl}(2) \) spin chain is occupied by a single \( Z \) field acted on by any number of the spacetime covariant derivatives \( \mathcal{D} \equiv \mathcal{D}_1 + i\mathcal{D}_2 \). The total \( R \)-charge of a particular operator is therefore equal to the lattice length \( L \), and an \( I \)-impurity operator basis is spanned by single-trace operators carrying all possible distributions of \( I \) derivatives among the \( L \) lattice sites:

\[
\text{Tr} \left( \mathcal{D}^I Z Z Z^{L-1} \right), \quad \text{Tr} \left( \mathcal{D}^{I-1} Z \mathcal{D} Z Z^{L-2} \right), \quad \text{Tr} \left( \mathcal{D}^{I-1} Z \mathcal{D} Z Z Z^{L-3} \right), \ldots \quad (4.1)
\]

The integer \( I \) counts the total number of derivatives in the operator and, since any number of impurities can occupy the same lattice site, one can think of \( n \) derivative insertions at the \( i \)th lattice site as \( n \) bosonic oscillator excitations at the \( i \)th lattice position:

\[
(a^\dagger_i)^n |L\rangle \sim \text{Tr} \left( Z_i^{n-1} \mathcal{D}^n Z Z^{L-i} \right), \ldots \quad (4.2)
\]

The ground state \( |L\rangle \) is represented by a length \( L \) chain with no derivative insertions: \( |L\rangle = \text{Tr} \left( Z^L \right) \).

The one-loop \( \mathfrak{sl}(2) \) spin chain Hamiltonian (corresponding to the dilatation operator in this sector) was constructed in [17] and was defined by its action on basis states rather than directly expressed as an operator:

\[
H_{\mathfrak{sl}(2)}^{(2)} = \sum_{j=1}^{L} H_{j,j+1}^{\mathfrak{sl}(2)} ,
\]

\[
H_{1,2}^{\mathfrak{sl}(2)} (a^\dagger_1)^j (a^\dagger_2)^{n-j} |L\rangle = \sum_{j'=0}^{n} \left[ \delta_{j=j'} (h(j) + h(n-j)) - \frac{\delta_{j \neq j'}}{|j-j'|} \right] (a_1^\dagger)^j (a_2^\dagger)^{n-j} |L\rangle \quad (4.3)
\]

(where \( h(n) = 1 + \ldots + 1/n \) are the harmonic numbers). In other words, \( H_{\mathfrak{sl}(2)}^{(2)} \) is a sum over the position-space Hamiltonian \( H_{j,j+1}^{\mathfrak{sl}(2)} \) which acts on the \( j \)th and \((j+1)\)th (neighboring) lattice sites; the action of \( H_{j,j+1}^{\mathfrak{sl}(2)} \) can be summarized by the explicit form given for \( H_{1,2}^{\mathfrak{sl}(2)} \) above. Since it is only defined by its action on the state \( (a_1^\dagger)^j (a_2^\dagger)^{n-j} |L\rangle \), it is difficult to immediately translate \( H_{\mathfrak{sl}(2)}^{(2)} \) to momentum space. However, it is possible to expand it in powers of fields and use eqn. (4.3) to iteratively determine the expansion coefficients. The virial argument furthermore tells us that higher powers in the fields will determine higher powers of \( L^{-1} \) in the expansion of the energy. For our current purposes, it suffices to know the Hamiltonian expanded out to terms of fourth order in the fields and this truncation of
the Hamiltonian can easily be constructed by inspection:

\[
H_{\text{sl}(2)}^{(2)} = -\sum_{j=1}^{L} \left[ (a_{j+1}^\dagger - 2a_j^\dagger + a_{j-1}^\dagger)(a_j - \frac{1}{2}a_j^2a_j^\dagger) + \frac{1}{4}(a_{j+1}^2 - 2a_j^2 + a_{j-1}^2)a_j^2 \right] + \cdots
\]  

(4.4)

Transformation to momentum space gives

\[
H_{\text{sl}(2)}^{(2)} = \sum_{p=0}^{L-1} 4 \sin^2 \frac{p\pi}{L} \tilde{a}_p^\dagger \tilde{a}_p + \frac{1}{L} \sum_{p,q,r,s=0}^{L-1} \delta_{p+q,r+s} \left( -\sin^2 \frac{p\pi}{L} - \sin^2 \frac{q\pi}{L} + \sin^2 \frac{(p+q)\pi}{L} \right) \tilde{a}_p^\dagger \tilde{a}_q^\dagger \tilde{a}_r \tilde{a}_s + \cdots
\]  

(4.5)

This Hamiltonian acts on an \( I \)-impurity Fock space spanned by the generic states

\[
\tilde{a}_{k_1}^\dagger \tilde{a}_{k_2}^\dagger \tilde{a}_{k_3}^\dagger \cdots |L\rangle ,
\]  

(4.6)

with lattice momenta labeled by \( k_i = 0, \ldots, L - 1 \), and subject to the constraint \( \sum_i k_i = 0 \) mod \( L \). Numerically diagonalizing this Hamiltonian on a range of lattice sizes, we obtain data from which we extract the numerical predictions for the one-loop coefficients \( E_{\text{sl}(2)}^{(1,2)} \) and \( E_{\text{sl}(2)}^{(1,3)} \) presented in table 8. String theory makes the following predictions [10] for the scaling coefficients

\[
E_{\text{sl}(2)}^{(1,2)} = (k_1^2 + k_1k_2 + k_2^2) \quad E_{\text{sl}(2)}^{(1,3)} / E_{\text{sl}(2)}^{(1,2)} = -2 \quad k_1 \neq k_2 \neq k_3
\]

\[
E_{\text{sl}(2)}^{(1,2)} = 3n^2 \quad E_{\text{sl}(2)}^{(1,3)} / E_{\text{sl}(2)}^{(1,2)} = -7/3 \quad k_1 = k_2 = n, k_3 = -2n ,
\]  

(4.7)

and we can easily verify that the agreement with table 8 is excellent.

The extrapolated predictions can again be checked against those of the corresponding one-loop Bethe ansatz equations. In the \( \text{sl}(2) \) sector the highest weight is \(-1/2\): the Dynkin diagram therefore has coefficient \( V_{\text{sl}(2)} = -1 \) and the Cartan matrix is \( M_{\text{sl}(2)} = 2 \). The Bethe equations (2.29,2.30) thus reduce to

\[
\left( \frac{u_i - i/2}{u_i + i/2} \right)^L = \prod_{j \neq i} \left( \frac{u_i - u_j + i}{u_i - u_j - i} \right) \]  

(4.8)

\[
1 = \prod_i \left( \frac{u_i - i/2}{u_i + i/2} \right).
\]  

(4.9)

Apart from a crucial minus sign, this is identical to the \( \text{su}(2) \) Bethe equation (2.31). In the
absence of bound states, eqn. (4.8) is satisfied by the following Bethe roots:

Using eqn. (2.31), we obtain

\[
\begin{align*}
E^{(1,2)}_{\text{sl}(2)} & = 1 + 1.2 \times 10^{-9} - 2 - 3.1 \times 10^{-7} - 2 - 3.1 \times 10^{-7} (1, 0, -1) \\
3 - 7.6 \times 10^{-9} & = -7 + 1.9 \times 10^{-6} -7/3 + 6.3 \times 10^{-7} (1, 1, -2) \\
3 - 7.6 \times 10^{-9} & = -7 + 1.9 \times 10^{-6} -7/3 + 6.3 \times 10^{-7} (-1, -1, 2) \\
4 - 2.8 \times 10^{-7} & = -8 + 6.9 \times 10^{-6} -2 + 1.7 \times 10^{-6} (2, 0, -2) \\
7 - 2.9 \times 10^{-7} & = -14 + 7.1 \times 10^{-5} -2 + 1.0 \times 10^{-5} (1, 2, -3) \\
7 - 2.9 \times 10^{-7} & = -14 + 7.1 \times 10^{-5} -2 + 1.0 \times 10^{-5} (-1, -2, 3) \\
9 - 4.1 \times 10^{-7} & = -18 + 1.0 \times 10^{-4} -2 + 1.0 \times 10^{-5} (3, 0, -3) \\
12 + 8.4 \times 10^{-7} & = -28 - 1.5 \times 10^{-4} -7/3 - 1.2 \times 10^{-5} (2, 2, -4) \\
12 + 8.4 \times 10^{-7} & = -28 - 1.5 \times 10^{-4} -7/3 - 1.2 \times 10^{-5} (-2, -2, 4) \\
13 - 7.0 \times 10^{-6} & = -26 + 1.7 \times 10^{-3} -2 + 1.3 \times 10^{-4} (1, 3, -4) \\
13 - 7.0 \times 10^{-6} & = -26 + 1.7 \times 10^{-3} -2 + 1.3 \times 10^{-4} (-1, -3, 4) \\
16 - 1.4 \times 10^{-6} & = -32 + 3.9 \times 10^{-4} -2 + 2.4 \times 10^{-5} (4, 0, -4) \\
19 - 7.5 \times 10^{-6} & = -38 + 2.2 \times 10^{-3} -2 + 1.1 \times 10^{-4} (2, 3, -5) \\
19 - 7.5 \times 10^{-6} & = -38 + 2.2 \times 10^{-3} -2 + 1.1 \times 10^{-4} (-2, -3, 5) \\
21 - 3.4 \times 10^{-6} & = -42 + 8.8 \times 10^{-4} -2 + 4.2 \times 10^{-5} (1, 4, -5) \\
21 - 3.4 \times 10^{-6} & = -42 + 8.8 \times 10^{-4} -2 + 4.2 \times 10^{-5} (-1, -4, 5)
\end{align*}
\]

Table 8: Scaling limit of numerical spectrum of three-impurity sl(2) sector at one-loop in \( \lambda \) absence of bound states, eqn. (4.8) is satisfied by the following Bethe roots:

\[
\begin{align*}
   u_1 &= -\frac{2(1 + L)k_1^2 - (4 + L)k_1k_2 - (4 + L)k_2^2}{2\pi k_1(k_2^2 + k_1k_2 - 2k_1^2)} + O(L^{-1}) \\
   u_2 &= -\frac{2(1 + L)k_2^2 - (4 + L)k_1k_2 - (4 + L)k_1^2}{2\pi k_2(k_1^2 + k_1k_2 - 2k_2^2)} + O(L^{-1}) \\
   u_3 &= -\frac{2(1 + L)k_1^2 + (8 + 5L)k_1k_2 + 2(1 + L)k_2^2}{2\pi(k_1 + k_2)(2k_1 + k_2)(k_1 + 2k_2)} + O(L^{-1}) . \quad (4.10)
\end{align*}
\]

Using eqn. (2.31), we obtain

\[
E^{(2)}_{\text{sl}(2)}(k_1, k_2) = \frac{\lambda}{L^3} \left( k_1^2 + k_1k_2 + k_2^2 \right) (L - 2) + O(L^{-4}) \quad (k_1 \neq k_2 \neq k_3) . \quad (4.11)
\]
For the bound state characterized by $k_1 = k_2 = n$ and $k_3 = -2n$, the Bethe roots are

\[
\begin{align*}
  u_1 &= \frac{7 - 3\sqrt{L} + 3L}{6\pi n} + O(L^{-1/2}) \\
  u_2 &= \frac{7 + 3\sqrt{L} + 3L}{6\pi n} + O(L^{-1/2}) \\
  u_3 &= -\frac{4 + 3L}{12\pi n} + O(L^{-1/2}),
\end{align*}
\]

with spin chain energy

\[
E_{\text{su}(2)}^{(2)}(n) = \frac{\lambda n^2}{L^3} (3L - 7) + O(L^{-4}) \quad (k_1 = k_2 = n, \ k_3 = -2n).
\]

These results again agree with the numerical results in table 5 (and match the corresponding three-impurity string theory predictions in [10]).

5 Conclusions

We have demonstrated that the virial expansion of the $\mathcal{N} = 4$ SYM spin chain Hamiltonian for small impurity number provides a simple and reliable method for computing exact anomalous dimensions of multi-impurity operators at small scalar $R$-charge (chain length) and estimating with great precision the near-BMN scaling behavior of these dimensions as the $R$-charge becomes large. The latter application, which is suited to direct comparison of gauge theory predictions with corresponding results on the string side of the AdS/CFT correspondence, works well for three-impurity operators to three-loop order in $\lambda$ in the $\text{su}(2)$ sector (the order to which the $\text{su}(2)$ Hamiltonian is known definitively). Specifically, the numerical predictions from the virial approach for the near-BMN scaling coefficients ($E_{\text{su}(2)}^{(1,2)}$, $E_{\text{su}(2)}^{(1,3)}$, $E_{\text{su}(2)}^{(2,4)}$, $E_{\text{su}(2)}^{(2,5)}$, $E_{\text{su}(2)}^{(3,6)}$, and $E_{\text{su}(2)}^{(3,7)}$) match corresponding results from the $\text{su}(2)$ long-range Bethe ansatz to three-loop order, and agree with near-plane-wave string theory predictions to two loops (the disagreement with string theory at three loops is by now an expected outcome in these studies). We also find convincing agreement near the BMN limit between the virial approach and the Bethe ansatz results at one-loop order in the closed $\mathfrak{sl}(2)$ and $\mathfrak{su}(1|1)$ subsectors. As a side result we have found in the $\mathfrak{su}(1|1)$ sector an exact (in chain length) agreement between the Bethe ansatz and the virial expansion for one-loop operator dimensions with arbitrary impurity number (this was only possible because the Bethe equations can be solved exactly in this subsector for any number of impurities). There are currently no higher-loop Bethe ansätze for the $\mathfrak{sl}(2)$ and $\mathfrak{su}(1|1)$ systems, however, so in this sense our numerical predictions go beyond the current state of Bethe ansatz technology (see [27] for further developments of higher-loop gauge theory physics in non-$\mathfrak{su}(2)$ sectors). It would be very interesting to find a general long-range Bethe equation appropriate for $\mathcal{N} = 4$ SYM at higher loop-order in $\lambda$, both for comparison with string predictions and with the virial approach studied here.
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