Slow dynamics for the dilute Ising model
in the phase coexistence region

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Abstract

In this paper we consider the Glauber dynamics for a disordered ferromagnetic Ising model, in the region of phase coexistence. It was conjectured several decades ago that the spin autocorrelation decays as a negative power of time [HFS7]. We confirm this behavior by establishing a corresponding lower bound in any dimensions \( d \geq 2 \), together with an upper bound when \( d = 2 \). Our approach is deeply connected to the Wulff construction for the dilute Ising model. We consider initial phase profiles with a reduced surface tension on their boundary and prove that, under mild conditions, those profiles are separated from the (equilibrium) pure plus phase by an energy barrier.

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1 Introduction and definitions

1.1 Introduction

For many years the Ising model and the corresponding Glauber dynamics have been a very active research field. In the 1990’s the asymptotics of the spectral gap of the Ising model with free boundary condition were connected to surface tension, see [Tho89, Mar99] and references therein. The inversion time of the infinite volume Ising model phase under a small field was then related to Wulff energies [Sch94, SS98]. In the last decade, precise estimates were achieved for the spectral gap and mixing time [BM02]. Recently impressive moves towards evidence of Lifshitz behavior and mean-curvature displacement of interfaces were achieved in [MT10, LS10, LMST10, CMST10].

The focus of the present paper is on the consequences of the presence of disorder on the dynamics, in the phase coexistence region. Since [Mar99] (and references therein) it has been known that dilution in the Ising model triggers slow, non-exponential relaxation to equilibrium in the Griffiths phase. Here we focus on the phase coexistence region, which means that, at equilibrium, the system can be either in the plus or the minus phase. This setting was considered already in [HF87] where heuristic discussions suggested that autocorrelation decays as a negative power of time. In the present paper we turn these heuristics into rigorous proofs. Previous stages of this project were the adaptation of the coarse graining and of the Wulff construction to the disordered setting, see [Wou08, Wou09] respectively.

Our main result is a lower bound on the autocorrelation (Theorem 2.2) which validates the heuristics of [HF87]. That is to say, when both the initial configuration includes a droplet of the minus phase and the surface tension on the boundary of the droplet is smaller than its quenched value, the system must cross an energy barrier before the droplet can disappear. Interestingly we show that the energy gap can be computed on continuous evolutions of the droplet, a slight improvement in comparison with the usual scheme of computing the bottleneck as the maximum gap in energy over all intermediate magnetization, as in [Mar99, BI04].

We also present in this work an upper bound on the autocorrelation when $d = 2$ (Theorem 2.4) together with some consequences of our estimates on the typical spectral gap and mixing times in finite volume (Theorems 2.5).

We would like to mention that, although is it the case here, we do not expect that dilution always slows down relaxation. Indeed, for the infinite volume dilute Ising system subject to a small positive external field, the dilution has a catalyst effect on the transition from the minus to the (equilibrium) plus phase, cf. [BGW12].

The organization of the paper is as follows: in the remaining part of the current Section, we define the dilute Ising model and the Glauber dynamics. We also introduce the necessary
tools and technical assumptions. Then in Section 2 we present our main results. Heuristics and proofs are given in Section 3.

1.2 The dilute Ising model

The canonical vectors of \( \mathbb{R}^d \) are denoted by \((e_i)_{i=1}^d\). For any \( x = \sum_{i=1}^n x_i e_i = (x_1, \ldots, x_d) \in \mathbb{R}^d \) we consider the following norms:

\[
\|x\|_1 = \sum_{i=1}^d |x_i|, \quad \|x\|_2 = \left( \sum_{i=1}^d x_i^2 \right)^{1/2} \quad \text{and} \quad \|x\|_\infty = \max_{i=1}^d |x_i|.
\]  

(1.1)

Given \( x, y \in \mathbb{Z}^d \) we say that \( x, y \) are nearest neighbors (which we denote \( x \sim y \)) if they are at Euclidean distance 1, i.e. if \( \|x - y\|_2 = 1 \). To any domain \( \Lambda \subset \mathbb{Z}^d \) we associate the edge sets

\[
E(\Lambda) = \{ \{x,y\} : x,y \in \Lambda \text{ and } x \sim y \} \quad \text{and} \quad E^w(\Lambda) = \{ \{x,y\} : x \in \Lambda, y \in \mathbb{Z}^d \text{ and } x \sim y \}.
\]  

(1.2)

We consider in this paper the dilute Ising model on \( \mathbb{Z}^d \) for \( d \geq 2 \). It is defined in two steps: first, the couplings between adjacent spins are represented by a random sequence \( J = (J_e)_{e \in E(\mathbb{Z}^d)} \) of law \( \mathbb{P} \), such that the \( (J_e)_{e \in E(\mathbb{Z}^d)} \) are independent, identically distributed in \([0,1]\) under \( \mathbb{P} \). For convenience we write \( J = [0,1]^{E(\mathbb{Z}^d)} \) the set of possible realizations of \( J \). Given \( \Lambda \subset \mathbb{Z}^d \) a finite domain, \( J \in J \) and a spin configuration \( \sigma \in \Sigma_\Lambda^+ = \{ \sigma : \mathbb{Z}^d \to \{\pm 1\} : \sigma_z = 1, \forall z \notin \Lambda \} \), we let

\[
H^J_\Lambda^+(\sigma) = - \sum_{e = \{x,y\} \in E^w(\Lambda)} J_e \sigma_x \sigma_y
\]  

(1.4)

the Hamiltonian with plus boundary condition on \( \Lambda \). The dilute Ising model on \( \Lambda \) with plus boundary condition, given a realization \( J \) of the couplings, is the probability measure \( \mu^J_\Lambda^+ \) on \( \Sigma_\Lambda^+ \) that satisfies

\[
\mu^J_\Lambda^+(\{\sigma\}) = \frac{1}{Z_{\Lambda,\beta}^J} \exp \left( -\frac{\beta}{2} H^J_\Lambda^+(\sigma) \right), \quad \forall \sigma \in \Sigma_\Lambda^+\]  

(1.5)

where \( \beta \geq 0 \) is the inverse temperature and \( Z_{\Lambda,\beta}^J \) is the partition function

\[
Z_{\Lambda,\beta}^J = \sum_{\sigma \in \Sigma_\Lambda^+} \exp \left( -\frac{\beta}{2} H^J_\Lambda^+(\sigma) \right).
\]  

(1.6)

Consider

\[
m_\beta = \lim_{N \to \infty} \mathbb{E} \mu^J_{\Lambda_N,\beta}^+ (\sigma_0)
\]  

(1.7)

the magnetization in the thermodynamic limit, where \( \hat{\Lambda}_N \) is the symmetric box \( \hat{\Lambda}_N = \{-N, \ldots, N\}^d \) and \( \mathbb{E} \) the expectation associated with \( \mathbb{P} \). When \( m_\beta > 0 \) the boundary condition has an influence even if it is arbitrary far away from the origin. In particular the decreasing limit \( \mu^J_- = \lim_N \downarrow \mu^J_{\hat{\Lambda}_N,\beta} \) does not coincide with \( \mu^J_\beta = \lim_N \uparrow \mu^J_{\hat{\Lambda}_N,\beta} \) (\( \mathbb{P} \)-almost surely). The two infinite volume measures \( \mathbb{E} \mu^J_\beta^+ \) and \( \mathbb{E} \mu^J_\beta^- \) are respectively the plus and minus phases.
It was shown in [ACCN87] that the dilute Ising model undergoes a phase transition at low temperature when the random interactions percolate. In our settings, this means that the critical inverse temperature

$$\beta_c = \inf \{ \beta \geq 0 : m_\beta > 0 \}, \quad (1.8)$$

which is never smaller than \( \beta_c^{\text{pure}} \) – the critical inverse temperature for the pure Ising model corresponding to \( J \equiv 1 \) – is finite if and only if \( P(J_e > 0) > p_c(d) \) where \( p_c(d) \) is the threshold for bond percolation on \( \mathbb{Z}^d \).

1.3 The Fortuin-Kasteleyn representation

The Ising model has a percolation-like representation which is very convenient for formulating two of the fundamental concepts for the study of equilibrium phase coexistence: renormalization and surface tension. We call the set of cluster configurations on \( \mathbb{Z}^d \), and for any \( \omega \in \Omega \) and \( E \subset \mathbb{Z}^d \) we call \( \omega_{|E} \) the restriction of \( \omega \) to \( E \), defined by

$$\langle \omega_{|E} \rangle_e = \begin{cases} \omega_e & \text{if } e \in E \\ 0 & \text{else.} \end{cases}$$

The set of cluster configurations on \( E \) is \( \Omega_E = \{ \omega_{|E} : \omega \in \Omega \} \). Given a parameter \( q \geq 1 \) and an inverse temperature \( \beta \geq 0 \), a realization of the random couplings \( J : E(\mathbb{Z}^d) \to [0,1] \), a finite edge set \( E \subset E(\mathbb{Z}^d) \) and a boundary condition \( \pi \in \Omega_E \) we consider the random cluster model \( \Phi_{E,\beta}^{J,\pi,q} \) on \( \Omega_E \) defined by

$$\Phi_{E,\beta}^{J,\pi,q}(\{\omega\}) = \frac{1}{Z_{E,\beta}^{J,\pi,q}} \prod_{e \in E} p_e^{\omega_e} (1 - p_e)^{1 - \omega_e} \times q^{C_E^\pi(\omega)}, \quad \forall \omega \in \Omega_E \quad (1.9)$$

where \( p_e = 1 - \exp(-\beta J_e) \), \( C_E^\pi(\omega) \) is the number of clusters of the set of vertices in \( \mathbb{Z}^d \) attained by \( E \) under the wiring \( \omega \cap \pi \) such that \( (\omega \cap \pi)_e = \max(\omega_e, \pi_e) \), and \( Z_{E,\beta}^{J,\pi,q} \) is the renormalization constant that makes of \( \Phi_{E,\beta}^{J,\pi,q} \) a probability measure.

For convenience we use the same notation for the probability measure \( \Phi_{E,\beta}^{J,\pi,q} \) and for its expectation. Most of the time we will take either \( \pi = f \), where \( f \) is the free boundary condition: \( f_e = 0, \forall e \in E^c \), or \( \pi = w \) where \( w \) is the wired boundary condition: \( w_e = 1, \forall e \in E^c \). When the parameters \( q \) and \( \beta \) are clear from the context we omit them. Given \( \mathcal{R} \) a compact subset of \( \mathbb{R}^d \) (usually a rectangular parallelepiped) we denote by \( \Phi_{\mathcal{R}}^{J,\pi} \) the measure \( \Phi_{E(\mathcal{R} \cap \mathbb{Z}^d)}^{J,\pi} \) on the cluster configurations on \( E(\mathcal{R} \cap \mathbb{Z}^d) \), where \( \mathcal{R} \) stands for the interior of \( \mathcal{R} \). In particular, for any \( g, h : \Omega \to \mathbb{R} \) the quantities \( \Phi_{\mathcal{R}_1}^{J,\pi}(g) \) and \( \Phi_{\mathcal{R}_2}^{J,\pi}(h) \) are independent under \( P \) when \( \mathcal{R}_1 \cap \mathcal{R}_2 = \emptyset \).

The connection between the dilute Ising model \( \mu_{A,\beta}^{J,+} \) and the random-cluster model was made explicit in [ES88]. Consider the joint probability measure

$$\Psi_{A,\beta}^{J,+}(\{\sigma, \omega\}) = \frac{1}{Z_{A,\beta}^{J,+}} \prod_{e \in E^w(\Lambda)} (p_e)^{\omega_e} (1 - p_e)^{1 - \omega_e}, \quad \forall (\sigma, \omega) \in \Sigma_{A}^{\pm} \times \Omega_{E(\Lambda)}$$

where \( p_e = 1 - \exp(-\beta J_e) \), \( \sigma \triangleleft \omega \) is the event that \( \sigma \) and \( \omega \) are compatible, namely that \( \omega_e = 1 \Rightarrow \sigma_x = \sigma_y, \forall e = \{x, y\} \in E^w(\Lambda) \), and \( Z_{A,\beta}^{J,+} \) is the corresponding normalizing factor.
i. The marginal of $\Psi_{\Lambda,\beta}^{J}$ on the variable $\sigma$ is the Ising model $\mu_{\Lambda}^{J}$.

ii. Its marginal on the variable $\omega$ is the random-cluster model $\Phi_{E(\Lambda),\beta}^{J,w,2}$ with wired boundary condition $w$ and parameter $q = 2$.

iii. Conditionally on $\omega$, the spin $\sigma$ of each connected component of $\Lambda$ for $\omega$ (from now on cluster) is constant, and equal to +1 if the cluster is connected to $\Lambda^c$. The spin of all clusters not touching $\Lambda^c$ are independent and equal to +1 with a probability $1/2$.

iv. Conditionally on $\sigma$, the edges are open (i.e. $\omega_e = 1$ for $e = \{x,y\}$) independently, with respective probabilities $p_e \delta_{\sigma_x,\sigma_y}$.

Furthermore, the distribution $\Phi_{E,\beta}^{J,\pi,q}$ increases with $\beta,J,\pi$, it satisfies the FKG inequality and the DLR equation, cf. [ACCN88].

1.4 Slab percolation

We say that slab percolation holds under $E \Phi_{E(\mathbb{Z}^d),\beta}^{J,f,q}$ when either

$$d \geq 3 \quad \text{and} \quad \exists H \in \mathbb{N}^*, \inf_{N \in \mathbb{N}^*} \inf_{x,y \in S_{N,H}} E \Phi_{S_{N,H},\beta}^{J,f,q}(x \leftrightarrow y) > 0,$$

or

$$d = 2 \quad \text{and} \quad \exists \kappa : \mathbb{N}^* \rightarrow \mathbb{N}^*, \lim_{N \rightarrow \infty} E \Phi_{S_{N,\kappa(N)}^+,\beta}^{J,f,q}(\exists \text{ a horizontal crossing for } \omega) > 0$$

where $S_{N,H} = \{1, \ldots, N\}^{d-1} \times \{1, \ldots, H\}$ is the slab of height $H$. The critical threshold for slab percolation is

$$\hat{\beta}_c = \inf \left\{ \beta \geq 0 : \text{slab percolation occurs under } E \Phi_{E(\mathbb{Z}^d),\beta}^{J,f,q} \right\}. \quad (1.10)$$

We believe that $\hat{\beta}_c$ and $\beta_c$ coincide, where $\beta_c$ is the critical inverse temperature for the dilute Ising model defined at (1.8). We also consider

$$\mathcal{N} = \left\{ \beta \geq 0 : \lim_{N \rightarrow \infty} E \Phi_{S_{N,\beta}^+,\beta}^{J,\pi,q} \neq \lim_{N \rightarrow \infty} E \Phi_{S_{N,\beta}^\pi,\beta}^{J,f,q} \right\}, \quad (1.11)$$

the set of inverse temperatures at which the infinite volume random media random cluster measure is not unique. It was shown in [Wou08], Theorem 2.3, that $\mathcal{N}$ is at most countable. Under the assumptions $\beta > \hat{\beta}_c$ and $\beta \notin \mathcal{N}$, one can use a renormalization procedure (Theorem 5.1 in [Wou08]) which gives a precise meaning to the notion of plus and minus phases and is hence a fundamental tool for the study of equilibrium phase coexistence.

1.5 Surface tension

Surface tension is another essential tool for the study of equilibrium phase coexistence. In the context of the dilute Ising model, it is a random quantity since it depends on the couplings $J$. We recall here some important definitions and results from [Wou09]. Let $S^{d-1}$ be the set of unit vectors of $\mathbb{R}^d$. Given $\mathbf{n} \in S^{d-1}$ we let

$$S_n = \left\{ \sum_{k=1}^{d-1} [-1/2, 1/2] u_k ; (u_1, \ldots, u_{d-1}, \mathbf{n}) \text{ is an orthonormal basis of } \mathbb{R}^d \right\},$$

Hausdorff measure. The $L^\infty$ in the following, see Corollary 1.9 therein. The set

$$\partial \mathcal{R} = \left\{ y \in \hat{\mathcal{R}} : \exists z \in \mathbb{Z}^d \setminus \hat{\mathcal{R}}, z \sim y \right\}.$$  

For any $\mathcal{R}$ as in (1.12) we decompose $\partial \hat{\mathcal{R}}$ into its upper and lower parts $\partial^+ \hat{\mathcal{R}} = \{ y \in \partial \hat{\mathcal{R}} : (y-x) \cdot n > 0 \}$ and $\partial^- \hat{\mathcal{R}} = \{ y \in \partial \hat{\mathcal{R}} : (y-x) \cdot n < 0 \}$. Then we call

$$\mathcal{D}_\mathcal{R} = \left\{ \omega \in \Omega : \partial^+ \hat{\mathcal{R}} \not\rightarrow \partial^- \hat{\mathcal{R}} \right\}$$  

the event of disconnection between the upper and lower parts of $\partial \hat{\mathcal{R}}$, and

$$\tau^J_\mathcal{R} = -\frac{1}{J} \log \Phi^{J\omega}_\mathcal{R}(\mathcal{D}_\mathcal{R}).$$  

the surface tension in $\mathcal{R}$. Surface tension is sub-additive and has a typical quenched value

$$\tau^q_\beta(n) = \lim_{N \to \infty} \tau^J_{\mathcal{R}_{0,N,\delta N}}(S,n) \text{ in } \mathbb{P}\text{-probability}$$  

that does not depend on $\delta > 0$ nor on $S \in \mathbb{S}_n$ (Theorem 1.3 in [Wou09]). It is positive for any $\beta > \hat{\beta}_c$ (Proposition 1.5 in the same reference). We denote by $J^{\min}$ and $J^{\max}$ the extremal values of the support of $J$. Since $\tau^J_\mathcal{R}$ increases with $J$, surface tension can be as low as $\tau^{\min}_\mathcal{R}$ that corresponds to the constant couplings $J \equiv J^{\min}$, and as large as $\tau^{\max}_\mathcal{R}$ when $J \equiv J^{\max}$. According to the convergence in (1.15), $\tau^{\min}_\mathcal{R}$ and $\tau^{\max}_\mathcal{R}$ also converge when $\mathcal{R} = \mathcal{R}_{0,N,\delta N}(S,n)$ with $N \to \infty$ and we call their respective limits $\tau^{\min}(n)$ and $\tau^{\max}(n)$. Surface tension can deviate from $\tau^q_\beta(n)$. Upper large deviations happen at a volume order (Theorem 1.4 in [Wou09]) and are irrelevant to surface phenomenon like phase coexistence. Lower deviations under $\tau > \tau^{\min}(n)$ occur according to the rate function

$$I_n(\tau) = \lim_{N \to \infty} -\frac{1}{N^{d-1}} \log \mathbb{P} \left( \tau^J_{\mathcal{R}_{0,N,\delta N}}(S,n) \leq \tau \right),$$  

see Theorem 1.6 in [Wou09]. The set

$$\mathcal{N}_I = \left\{ \beta \geq 0 : \exists n \in S^{d-1} \text{ and } r > 0 \text{ such that } I_{\beta,n}(\tau^q_\beta(n) - r) = 0 \right\}$$  

is at most countable, see Corollary 1.9 therein.

### 1.6 Magnetization profiles

In the following, $\mathcal{L}^d$ stands for the Lebesgue measure on $\mathbb{R}^d$ and $\mathcal{H}^{d-1}$ for the $d-1$ dimensional Hausdorff measure. The $L^1$-distance between two Borel measurable functions $u, v : [0,1]^d \to \mathbb{R}$ is

$$\|u - v\|_{L^1} = \int_{[0,1]^d} |u - v| d\mathcal{L}^d,$$
and the set \( L^1 \) is
\[
\left\{ u : [0,1]^d \to \mathbb{R} \text{ Borel measurable, } \|u\|_{L^1} < \infty \right\}.
\]

In order that \( L^1 \) be a Banach space for the \( L^1 \)-norm, we identify \( u : [0,1]^d \to \mathbb{R} \) with the class of functions \( \{ v : \|u - v\|_{L^1} = 0 \} \) that coincide with \( u \) on a set of full measure. We also denote by \( V(u, \delta) \) the neighborhood of radius \( \delta > 0 \) in \( L^1 \) around \( u \in L^1 \). Given a Borel set \( U \subset \mathbb{R}^d \), we call
\[
\mathbf{1}_U : x \in \mathbb{R}^d \mapsto \begin{cases} 1 & \text{if } x \notin U \\ -1 & \text{if } x \in U \end{cases}
\]
the phase profile corresponding to \( U \), and call \( \mathcal{P}(U) \) the perimeter of \( U \) (as defined in Chap. 3 of [AFP00]). The set of bounded variation profiles is
\[
\text{BV} = \left\{ u = \mathbf{1}_U : U \subset (0,1)^d \text{ is a Borel set and } \mathcal{P}(U) < \infty \right\}.
\]

Bounded variations profiles \( u = \mathbf{1}_U \in \text{BV} \) have a reduced boundary \( \partial^* u \) and an outer normal \( n^u : \partial^* u \to S^{d-1} \) with, in particular, \( \mathcal{H}^{d-1}(\partial^* u) = \mathcal{P}(U) \). As the outer normal \( n^u \) defined on \( \partial^* u \) is Borel measurable, we can consider integrals of the kind
\[
\mathcal{F}_q(u) = \int_{\partial^* u} \tau_q(n^u_x)d\mathcal{H}^{d-1}(x), \quad \forall u \in \text{BV}
\]
that define the quenched surface energy of a given profile. When \( u = \mathbf{1}_U \in \text{BV} \) we also denote by \( \mathcal{F}_q(U) \) the surface energy of \( u \).

### 1.7 Initial configuration and gap in surface energy

As described with further detail in the heuristics (Section 3.1), our strategy for controlling the Glauber dynamics is to start from some metastable initial configuration, from which a positive gap in surface energy must be overcome before the system can reach the pure plus phase. This metastable configuration is characterized, on the one side, by an initial phase profile \( u_0 \in \text{BV} \), and on the second side, by a reduced surface tension \( \tau^r \) on the boundary of \( u_0 \). So a so-called initial configuration has actually two microscopic counterparts. First, to \( u_0 \) will correspond a set of initial spin configurations for the Glauber dynamics, while to the reduced surface tension \( \tau^r \) we will associate a dilution event on the couplings \( J \).

Before we can define the set of initial configurations \( \text{IC} \) at (1.19), we need still a few more definitions.

**Definition 1.1.** We say that a profile \( u = \mathbf{1}_U \) is regular if

i. \( U \) is open and at positive distance from the boundary \( \partial[0,1]^d \) of the unit cube,

ii. \( \partial U \) is \( d-1 \) rectifiable and

iii. for small enough \( r > 0 \), \( [0,1]^d \setminus (\partial U + B(0,r)) \) has exactly two connected components.

We recall that \( E \subset \mathbb{R}^d \) is a \( d-1 \) rectifiable set if there exists a Lipschitz function mapping some bounded subset of \( \mathbb{R}^{d-1} \) onto \( E \) (Definition 3.2.14 in [Fed69]). It is the case in particular of the boundary of Wulff crystals (Theorem 3.2.35 in [Fed69]) and of bounded polyhedral sets.
It follows from Proposition 3.62 in \cite{AFP00} that any \( u = \chi_U \) regular belongs to BV and that \( \partial U = \partial^* u \) up to a \( H^{d-1} \)-negligible set. Finally, we call

\[
\text{IC} = \left\{ (u_0, \tau^r) \in \text{BV} \times \mathbb{C}([0,1]^d, \mathbb{R}) : \; u_0 \text{ is regular and there is } \varepsilon > 0 \text{ such that, for all } x \in \partial^* u_0, \tau^\min(n^u_x) + \varepsilon < \tau^r(x) \leq \tau^q(n^u_x). \right\}
\]  

(1.19)

Given an initial configuration \((u_0, \tau^r) \in \text{IC}\), we define the reduced surface energy as

\[
\mathcal{F}^r(u) = \int_{\partial^* u_0 \cap \partial^* u} \tau^r(x) dH^{d-1}(x) + \int_{\partial^* u_0 \setminus \partial^* u} \tau^q(n_x) dH^{d-1}(x)
\]  

(1.20)

which is obviously smaller than \( \mathcal{F}^q(u) \). The reduced surface energy of the initial phase profile \( u_0 \) is

\[
\mathcal{F}^r(u_0) = \int_{\partial^* u_0} \tau^r(x) dH^{d-1}(x)
\]  

(1.21)

while the cost of dilution is

\[
\mathcal{I}^r(u_0) = \int_{\partial^* u_0} I_{n_x}(\tau^r(x)) dH^{d-1}(x).
\]  

(1.22)

For any \( \varepsilon > 0 \), we call \( \mathcal{C}_\varepsilon(u_0) \) the set of sequences of phase profiles that evolve from \( u_0 \) to 1 (pure plus phase) by jumps with \( L^1 \) norm less than \( \varepsilon \), that is

\[
\mathcal{C}_\varepsilon(u_0) = \left\{ (v_i)_{i=0}^{k} : \; k \in \mathbb{N} ; \; v_0 = u_0 \text{ and } v_k = 1 \right\}
\]  

(1.23)

Finally we define the gap in surface energy or energy barrier for removing the droplet \( u_0 \), under the reduced surface tension \( \tau^r \): this is

\[
\mathcal{K}^r(u_0) = \lim_{\varepsilon \to 0^+} \inf_{v \in \mathcal{C}_\varepsilon(u_0)} \max_i \mathcal{F}^r(v_i) - \mathcal{F}^r(u_0).
\]  

(1.24)

1.8 The Glauber dynamics

The Glauber dynamics is characterized by a family of transition rates \( c^J(x, \sigma) \) at which the configuration \( \sigma \) changes to \( \sigma^x \) defined by

\[
\sigma^x_y = \begin{cases} 
\sigma_y & \text{if } y \neq x \\
-\sigma_y & \text{if } y = x.
\end{cases}
\]

In other words, \( c^J(x, \sigma) \) is also the rate at which the spin at \( x \) flips. We make standard assumptions on the transition rates, namely:

**Finite range** There exists \( r < \infty \), the range of interaction, such that \( c^J(x, \sigma) \) is independent of \( \sigma(y) \) when \( d(x,y) > r \), and of \( J_e \) when \( d(e,x) > r \).

**Rates are bounded** The rates are uniformly bounded from below and above: there are \( c_m, c_M \in (0, \infty) \) such that

\[
c_m \leq c^J(x, \sigma) \leq c_M, \quad \forall x \in \mathbb{Z}^d, J \in J, \sigma \in \Sigma.
\]
Detailed balance The rates satisfy the detailed balance condition: for all \( \sigma \in \Sigma \) and \( x \in \mathbb{Z}^d \), for all \( J \in \mathcal{J} \), the product
\[
c^J(x, \sigma) \times \exp \left( \frac{\beta}{2} \sum_{y \sim x} J_{\{x,y\}} \sigma_x \sigma_y \right)
\]
does not depend on \( \sigma_x \).

Translation invariance If, for some \( z \in \mathbb{Z}^d \) one has
\[
J'_e = J_{z+e}, \forall e \in E(\mathbb{Z}^d) \quad \text{and} \quad \sigma'_x = \sigma_{x+z}, \forall x \in \mathbb{Z}^d,
\]
then \( c^J(x, \sigma') = c^J(x + z, \sigma) \).

Attractivity Given any \( \sigma, \sigma' \in \Sigma \) with \( \sigma \leq \sigma' \), the equality \( \sigma_x = \sigma'_x \) implies
\[
\sigma'_x c^J(x, \sigma') \leq \sigma_x c^J(x, \sigma).
\]

Two important examples of the Glauber dynamics are the Metropolis dynamics, often used in computer simulations, for which
\[
c^J(x, \sigma) = \max \left( 1, \exp \left( -\beta \sum_{y \sim x} J_{\{x,y\}} \sigma_x \sigma_y \right) \right)
\]
and the heat-bath dynamics
\[
c^J(x, \tau) = \mu^J_{\{x\}} (\sigma_x = -\tau_x | \sigma_y = \tau_y, \forall y \neq x).
\]
Given the transition rates, one can proceed to a graphical construction of the dynamics, as follows: equip each site \( x \in \mathbb{Z}^d \) with a Poisson process valued on \( \mathbb{R}^+ \), with intensity \( c_M \) (we recall that \( c_M \) is a uniform bound on the rates of the dynamics). Consider now the time \( t \) growing from 0. When the Poisson process at \( x \) has a point at \( t \), flip the spin at position \( x \) with probability \( c^J(x, \sigma_t) / c_M \). Because the flip rates are bounded, the determination of \( \sigma_t(x) \) involves only finitely many sites and therefore the dynamics is well defined, even in the infinite domain \( \mathbb{Z}^d \).

We call \( P^J \) the law of the Markov process \( (\sigma(t))_{t \geq 0} \) associated to this dynamics, and \( P^J_{\sigma(0)} \) the law conditioned on the initial configuration \( \sigma(0) \). It is convenient to introduce the semi-group \( T^J \) defined by
\[
[T^J(t)f](\sigma) = E^J_{\sigma}(f(\sigma(t))). \tag{1.25}
\]
The detailed balance condition makes the generator
\[
(L^Jf)(\sigma) = \sum_{x \in \mathbb{Z}^d} c^J(x, \sigma)(f(\sigma^x) - f(\sigma)) \tag{1.26}
\]
self-adjoint in \( L^2(\mu^J_{\mathcal{F}^+}) \), and ensures that the Gibbs measure \( \mu^J_{\mathcal{F}^+} \) is reversible for the dynamics. One way of quantifying the approach to equilibrium of the dynamics in infinite volume is
therefore the averaged autocorrelation

\[ A^\lambda(t) = \mathbb{E} \left( \left[ \text{Var}_{\mu^J}(T^J(t)\pi_0) \right]^\lambda \right) \]

\[ = \mathbb{E} \left( \left[ \int_{\Sigma} \left[ (T^J(t)\pi_0)(\rho) - \mu^J(\sigma_0) \right]^2 d\mu^J(\rho) \right]^\lambda \right) \]

\[ = \mathbb{E} \left\| T^J(t)\pi_0 - \mu^J(\sigma_0) \right\|_{L^2(\mu^J)}^{2\lambda} \]

where \( \pi_0 : \Sigma \rightarrow \mathbb{R} \) is the function that, to the spin configuration \( \sigma \), associates \( \pi_0(\sigma) = \sigma_0 \), and \( \lambda > 0 \) is an arbitrary positive number.

Although our work aims primarily at describing the asymptotics of the averaged autocorrelation, we also derive some upper bounds on the relaxation and mixing times in finite volume. To this aim, we introduce the Dirichlet form

\[ \mathcal{E}^J_\Lambda(f, f) = \frac{1}{2} \sum_{\sigma, x} \mu^J(\sigma)c^J(x, \sigma)(f(\sigma^x) - f(\sigma))^2 \]

and the spectral gap

\[ \text{gap } L^J_\Lambda = \inf \left\{ \mathcal{E}^J_\Lambda(f, f) : f \text{ with } \text{Var}^J_\Lambda(f) \neq 0 \right\}. \]  

The inverse of the spectral gap is the relaxation time

\[ \mathcal{T}^J_\Lambda = 1/ \text{gap } L^J_\Lambda. \]

One fundamental property of the spectral gap is that, for any \( f \),

\[ \text{Var}^J_\Lambda(T^J_\Lambda(t)f) \leq \exp \left( -2t/\mathcal{T}^J_\Lambda \right) \text{Var}^J_\Lambda(f) \]

where \( T^J_\Lambda \) is the semi-group corresponding to the Glauber dynamics restricted to \( \Lambda \) with boundary condition \( \rho \).

Finally, we define the total variation distance between two probability measures \( \mu \) and \( \nu \), as

\[ \|\mu - \nu\|_{TV} = \sup_A \mu(A) - \nu(A). \]

The mixing time is

\[ \mathcal{T}^{J_\Lambda}_\text{mix} = \inf \left\{ t > 0 : \sup_{\sigma_0 : \sigma_0 = \rho \text{ on } \Lambda^c} \left\| P^J_\sigma(t) - \mu^J_\Lambda \right\|_{TV} \leq e^{-1} \right\}. \]

Given any function \( f \), we have

\[ \sup_{\sigma_0 : \sigma_0 = \rho \text{ on } \Lambda^c} \left| \mathcal{E}^J_\sigma(f(\sigma_t)) - \mu^J_\Lambda(f) \right| \leq \exp \left( - \left[ t/\mathcal{T}^{J_\Lambda}_\text{mix} \right] \right) \|f\|_{\infty}. \]
2 Main Results

2.1 Slow dynamics in infinite volume

The main result of the paper is a rigorous lower bound on the averaged autocorrelation which confirm some of the claims of the foreseeing paper [HF87]. We recall that the set of (metastable) initial configurations IC is defined at (1.19), while the surface energy $F^r$, the cost of initial dilution $I^r(u_0)$ and the gap in surface energy $K^r(u_0)$ associated to $(u_0, \tau^r) \in IC$ are defined at (1.20), (1.22) and (1.24), respectively. Now we define the exponent

$$X_\lambda = \inf_{(u_0, \tau^r) \in IC: K^r(u_0) > 0} \frac{T^r(u_0) + \lambda F^r(u_0)}{K^r(u_0)}$$

(2.1)

for any $\lambda \geq 0$. When no initial configuration $(u_0, \tau^r) \in IC$ leads to a positive surface energy gap $K^r(u_0) > 0$ we adopt the convention that $X_\lambda = +\infty$. Under mild conditions the exponent $X_\lambda$ is finite and even bounded from above:

**Proposition 2.1.** Assume that $\tau^{\text{min}}(n) < \tau^q(n)$ for all $n \in S^{d-1}$. Then:

1. For all $(u_0, \tau^r) \in IC$ such that the boundary of $u_0$ is $C^1$, the gap in surface energy $K^r(u_0)$ is strictly positive.

2. If $0 < \mathbb{P}(J_e = 0) < 1 - p_c(d)$, for every $\lambda \geq 0$ there is $C > 0$ such that, for any $\beta$ large enough,

$$X_\lambda \leq \frac{C}{\beta}.$$  

(2.2)

Note that the above proposition is a corollary of Theorem 3.21 which is to be found, together with its proof, in Section 3.4.1.

Now we present our main result, which relates the decay of the autocorrelation of the infinite volume Glauber dynamics in the plus phase with the exponent $X_\lambda$ (we recall that $\mathcal{N}$ and $\mathcal{N}_I$ are defined respectively at (1.11) and (1.17)).

**Theorem 2.2.** For any $\beta > \beta_c$ such that $\beta \notin \mathcal{N} \cup \mathcal{N}_I$, for any $\lambda \geq 1$, for any $\delta > 0$, for any $t > 0$ large enough,

$$A^\lambda(t) \geq t^{-X_\lambda - \delta}.$$  

(2.3)

**Remark 2.3.** In general we have not been able to compute $K^r(u_0)$ nor $X_\lambda$. Note however that we give another formulation of $K^r(u_0)$ in Theorem 3.25. This alternate formulation is the key for computing $K^r(u_0)$ in two particular cases, see Sections 3.4.3 and 3.4.4. These computations also point out the fact that the constraint of continuous evolution in the definition of $K^r(u_0)$ at (1.24) makes, in some cases, the gap in surface energy bigger than if we only take into account the constraint of continuous evolution of the overall magnetization (see Lemma 3.29).

Theorem 2.2 above was established some time ago during the PhD Thesis [Wou07] directed by Thierry Bodineau. Later on the author received the indications by Fabio Martinelli of how a corresponding upper bound could be established. The result is complementary to the former Theorem and confirms that the autocorrelation indeed decays as a power of time when $d = 2$. Furthermore, when $0 < \mathbb{P}(J_e = 0) < 1 - p_c(d)$, the exponent in both the upper and the lower bounds match up to a multiplicative constant as $\beta \to \infty$. 

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Theorem 2.4. Assume that $d = 2$. For any $\lambda > 0$, there is $c > 0$ such that, for all $\beta > \hat{\beta}_c$, for all $t > 0$ large enough,

$$A^\lambda(t) \leq t^{-c/\beta}. \quad (2.4)$$

2.2 Slow dynamics in finite volume

Our strategy for providing a lower bound on the autocorrelation also yields lower bounds on typical relaxation and mixing times in finite volume. Given $d \geq 2$, the law of interactions $\mathbb{P}$ and the inverse temperature $\beta$, we call

$$\kappa = \frac{d}{X_0} \sup_{(u_0, \tau^r) \in \text{IC}} \frac{K^r(u_0)}{\mathcal{T}(u_0)} = d/X_0.$$

We recall that the relaxation time $\mathcal{T}_{\text{rel}}$ was defined at (1.29), while the mixing time $\mathcal{T}_{\text{mix}}$ was defined at (1.31).

Theorem 2.5. Assume $d \geq 2$ and $\beta > \hat{\beta}_c$ with $\beta \notin N \cup N_I$. For any $\delta > 0$,

$$\lim_N \mathbb{P} \left( \mathcal{T}_{\text{rel}}^{J_{N,N,+}} \geq N^{\kappa-\delta} \right) = \lim_N \mathbb{P} \left( \mathcal{T}_{\text{mix}}^{J_{N,N,+}} \geq N^{\kappa-\delta} \right) = 1.$$

It is instructive to compare this bound with the asymptotics of the relaxation time in the pure Ising model. When $d = 2$, $\mathcal{T}_{\text{rel}}^{J_{1,1,N,+}} = N$ apart from logarithmic corrections, see [BM02]. For dilute models with $0 < \mathbb{P}(J_e = 0) < 1 - p_c(d)$, Proposition 2.1 states that $\kappa \geq c\beta$ for $\beta$ large enough: this is another illustration of the fact that dilution makes the relaxation time much larger.

3 Proofs

3.1 Heuristics and organization of the proofs

The object of Section 3.2 is the proof of the upper bound on the autocorrelation when the dimension is $d = 2$ (Theorem 2.4). The strategy for the proof, which we owe to Fabio Martinelli, relies on a uniform lower bound on the spectral gap in a square box with uniform plus boundary condition (cf. Theorem 6.4 in [Mar99] or (3.2) below). Some extra but classical work is then required to use that estimate for the infinite volume Glauber dynamics started from the plus phase.

Section 3.3 concentrates the most complex part of the paper and is dedicated to the proofs of Theorems 2.2 and 2.5. The corresponding heuristics are derived from [HFS7], where it was already suggested that dilution and reduction of surface tension, which has a surface cost, could trigger metastability for initial spin configurations corresponding to the minus phase in the region surrounded by the diluted surface (i.e., with our notation, the region where $u_0 \equiv -1$, up to an appropriate scaling). In the present work, we have formalized that idea of reducing the surface tension along an initial contour with the set of initial configurations $\text{IC}$ defined at (1.19). The concept of energy barrier has lead to the definition of the gap in surface energy $K^r(u_0)$ at (1.24). The reader will note that we have introduced, in that concept of energy barrier, the
requirement that evolution of the phase profile is (almost) continuous in the $L^1$ norm (with initial configuration $u_0$ and final configuration $1$, corresponding to uniform plus phase). Once these concepts are defined, remains a substantial work for relating the concepts and definitions to the physical phenomena. This is done as follows. First, in Sections 3.3.1 and 3.3.2 we define a microscopic counterpart for the $(u_0, \tau^r)$ dilution, the event $\mathcal{G}((R_i)_{i=1..n})$, which depends only on the interaction strength $J_e$. We prove that this event has the expected probability (the cost for dilution) and also we establish, conditionally on $\mathcal{G}((R_i)_{i=1..n})$, upper and lower bounds on phase coexistence. In Section 3.3.3 we introduce a certain bottleneck set, that corresponds with the phase profiles with the highest cost along all relaxation paths from $u_0$ to $1$. Then we express the probability, under the equilibrium measure, that the actual phase profile is in the bottleneck set in terms of its reduced surface energy, and in turn we relate the infimum of the reduced surface energy in the bottleneck set to the energy barrier $K^r(u_0)$. In Section 3.3.4 we relate the former estimates on the equilibrium measure to the Glauber dynamics and establish an important result on the dynamics in a finite box (namely, Proposition 3.18). That Proposition states that, conditionally on the event of dilution, the average magnetization remains significantly different from the plus phase magnetization (i.e. a droplet of minus phase remains), with a probability corresponding to the reduced surface energy, until a time determined by the energy barrier. We conclude the proof of Theorem 2.2 and Theorem 2.5 in Section 3.3.5. For the proof of the first Theorem we relate the autocorrelation to the evolution of the magnetization in a finite volume.

Finally, Section 3.4 is concerned with geometrical estimates. By decomposing the difference between the initial droplet and the current droplet into many small droplets, we are able to use the assumption that the initial droplet has a smooth boundary and prove that the energy barrier is strictly positive. We also prove that, under appropriate assumptions, the exponent $\lambda_\alpha$ is bounded by $C/\beta$ (Proposition 2.1). Then in Section 3.4.2 we derive another formulation of the energy barrier, which shows that the energy barrier can be computed assuming not only the $L^1$ continuity of the droplet evolution with time, but also the $\mathcal{H}^{d-1}$ continuity of the part of the droplet surface that corresponds with the initial droplet contour (where dilution takes place). We establish that alternative and informative formulation by providing an interpolation between any two phase profiles that is continuous for both volume and surface measures, with the additional property that along the interpolation, the surface energy does not exceed the maximum of the surface energy among the two interpolated phase profiles. Finally, in Sections 3.4.3 and 3.4.4 we use that alternative formulation to compute the energy barrier in specific cases. These computations put in evidence the fact that the energy barrier, as defined by $K^r(u_0)$, is higher than the energy barrier computed with a constraint on the overall magnetization.

### 3.2 Upper bound on the autocorrelation

Here we give the proof of Theorem 2.4. As said above, the scheme of proof was suggested by Fabio Martinelli. We call

$$f^J_t(\sigma) = (T^J(t)\pi_0)(\sigma)$$
$$f^J_{\Lambda,t}(\sigma) = (T^J_{\Lambda,t}(t)\pi_0)(\sigma)$$
$$m^J = \mu^{J,+}(\sigma_0)$$
therefore $f^J_t(\sigma)$ is the mean value of the spin at the origin under the Glauber dynamics performed until time $t$, initiated with the configuration $\sigma$, while $f^J_{\Lambda,t}(\sigma)$ is the corresponding quantity for the dynamics restricted to $\Lambda$, with plus boundary condition on $\Lambda^c$. Using these notations we can write the averaged autocorrelation $A^\lambda(t)$ defined at (1.27) as

$$A^\lambda(t) = \mathbb{E}\left( \left[ \text{Var}_{\mu^J,+}(f^J_t) \right]^\lambda \right). \quad (3.1)$$

**Lemma 3.1.** One has

$$\frac{1}{2} \text{Var}_{\mu^J,+}(f^J_t) \leq f^J_t(+) - m^J$$

where $+$ means the constant plus spin configuration.

**Proof.** The attractivity of the dynamics implies that $f^J_t(\sigma) \leq f^J_t(\sigma^\times)$ for any $\sigma$. Then:

$$\begin{align*}
\frac{1}{2} \text{Var}_{\mu^J,+}(f^J_t) & = \frac{1}{4} \int \int (f^J_t(\sigma) - f^J_t(\eta))^2 \, d\mu^J,+(\sigma) \, d\mu^J,+(\eta) \\
& = \frac{1}{2} \int \int (f^J_t(\sigma) - f^J_t(\eta))^2 \, d\mu^J,+(\sigma) \, d\mu^J,+(\eta) \\
& \leq \int \int (f^J_t(\sigma) - f^J_t(\eta)) \, d\mu^J,+(\sigma) \, d\mu^J,+(\eta) \\
& \leq \text{attractivity} \int (f^J_t(\sigma) - f^J_t(\eta)) \, d\mu^J,+(\eta).
\end{align*}$$

And the last term equals $f^J_t(+) - m^J$. \hfill \Box

We apply then standard controls on the spectral gap:

**Lemma 3.2.** There exist positive and finite constants $C_1, C_2, C_3$ such that, for all $J \in J$, all $t > 0$ and $\beta > 0$,

$$f^J_{\Lambda_N,t}(+) - f^J_{\Lambda_N,t}(-) \leq e^{C_1 \beta N^d - C_2 t N^{-d} e^{-C_2 \beta N^{d-1}}}. \quad (3.2)$$

**Proof.** The difference $f^J_{\Lambda_N,t}(+) - f^J_{\Lambda_N,t}(-)$ is not larger than twice the $\| \cdot \|_\infty$ norm of $f^J_{\Lambda_N,t}$, which according to (3.15) in [Mar99] does not exceed

$$\| f^J_{\Lambda_N,t} \|_\infty \leq \left[ \inf_{\sigma} \mu^J_{\Lambda_N}(\sigma) \right]^{-1/2} \exp \left( -t \text{gap}(L^J,\Lambda_N,+) \right) \| \pi_0 \|_{L^2(\mu^J_{\Lambda_N})}. $$

The conclusion comes then from the general lower bound

$$\text{gap}(L^J,\Lambda_N,+) \geq c_m(2N + 1)^{-d} e^{-C_2 \beta N^{d-1}}, \quad (3.2)$$

cf. Theorem 6.4 in [Mar99]. \hfill \Box

An easy consequence of monotonicity is the next Lemma:
Lemma 3.3. For any $J \in \mathcal{J}$, one has

$$f_{J}^{t}(-) - m^{J} \leq \mu_{\Lambda}^{J+}(\sigma_{0}) - m^{J} \leq 2\mu_{\Lambda}^{J+}(C_{\Lambda}^{c})$$

where $C_{\Lambda}$ is the event that there exists a contour of plus spins in $\Lambda$, around the origin.

Proof. The inequality $f_{J}^{t}(-) \leq \mu_{\Lambda}^{J+}(\sigma_{0})$ is immediate. Then we remark that, conditionally on $C_{\Lambda}$, the expectation of $\sigma_{0}$ under $\mu^{J}$ is at least $\mu_{\Lambda}^{J+}(\sigma_{0})$. Otherwise it is at least $-1$. Hence

$$m^{J} \geq \mu_{\Lambda}^{J+}(C_{\Lambda}) \times \mu_{\Lambda}^{J+}(\sigma_{0}) + \mu_{\Lambda}^{J+}(C_{\Lambda}^{c}) \times (-1) \geq \mu_{\Lambda}^{J+}(\sigma_{0}) - 2\mu_{\Lambda}^{J+}(C_{\Lambda}^{c}) .$$

\[\square\]

The average of $\mu_{\Lambda}^{J+}(C_{\Lambda}^{c})$ decays exponentially fast with the size of $\Lambda$ under the slab percolation assumption when $d = 2$ (see (1.10) for the definition of $\hat{\beta}_{c}$). When $\beta$ is large, we can also give quantitative estimates.

Lemma 3.4. Assume $d = 2$ and $\beta > \hat{\beta}_{c}$, then there is $c > 0$ such that, for $N$ large enough,

$$\mathbb{E}\mu_{\Lambda}^{J+}(C_{\Lambda}^{c}) \leq \exp(-cN). \quad (3.3)$$

Proof. We use the FK representation of the spin model and the renormalization framework of [Wou08]. The event $C_{\Lambda_{N}}$ is realized with conditional probability one if both of the following occur in the edge configuration:

i. There is an infinite $\omega$-open path issued from $\hat{\Lambda}_{N/2}$

ii. There is a surface that lies in $\hat{\Lambda}_{N} \setminus \hat{\Lambda}_{N/2}$ for which all points are $\omega$-connected.

Since $d = 2$, the second point reduces to finding a circuit of open edges inside $\hat{\Lambda}_{N} \setminus \hat{\Lambda}_{N/2}$. We can cover $\hat{\Lambda}_{N} \setminus \hat{\Lambda}_{N/2}$ with 16 blocks of side-length $N/2$. When all these blocks are good in the sense of Theorem 2.1 in [Wou08], the second point is realized, and this occurs with a probability $1 - \exp(-cN)$ as we assume that $\beta > \hat{\beta}_{c}$. Similarly, the first point is realized as well with a probability $1 - \exp(-cN)$ and (3.3) follows. \[\square\]

Remark 3.5. In larger dimensions, under the assumption that $\mathbb{P}(J_{e} \geq \varepsilon) = 1$ for some $\varepsilon > 0$, one can use Peierls estimates to prove that $\mathbb{E}\mu_{J}^{J+}(C_{\Lambda_{N}}^{c}) \leq \exp(-c\beta N)$ for some $c > 0$, for any $\beta$ large. Still, this is not useful for generalizing Theorem 2.4 as can be seen in the final optimization below.

Now we give the proof of Theorem 2.4.

Proof. According to Lemmas 3.1, 3.2, 3.3 and to the inequality $f_{t}^{J}(+) \leq f_{\Lambda_{N},t}^{J}(+)$ due to the monotonicity of the dynamics, we have

$$\text{Var}_{\mu_{J}^{+}}(f_{t}^{J}) \leq 2e^{C_{1}\beta N^{d} - C_{3}tN^{-d} - C_{2}\beta N^{d-1}} + 4\mu_{J}^{+}(C_{\Lambda_{N}}^{c}).$$
We recall the assumption that \( d = 2 \) and \( \beta > \hat{\beta}_c \). According to Lemma 3.4 there is \( c > 0 \) such that, for all \( N \) large enough:

\[
\mathbb{E} \mu_{J^+} \left( C_{\Lambda_N}^c \right) \leq \exp(-cN),
\]

hence Markov’s inequality implies that

\[
\mathbb{P} \left( \mu_{J^+} \left( C_{\Lambda_N}^c \right) \geq \exp(-cN/2) \right) \leq \exp(-cN/2).
\] (3.4)

Now we define \( N \) by the relation

\[
C_2 \beta N^{d-1} = (1 - \delta) \log t.
\] (3.5)

For large enough \( t \), we have

\[
\text{Var}_{\mu(t)} \left( f_{J^+} \right) \leq 4 \left( \exp(-cN/2) + \mu_{J^+} \left( C_{\Lambda_N}^c \right) \right).
\]

Combining this with (3.1) and (3.4) gives

\[
A^\lambda(t) \leq 8^4 \left[ \exp(-c\lambda N/2) + \exp(-cN/2) \right]
\]

which, according to the assumption that \( d = 2 \) and to the definition (3.5) of \( N \), establishes the claim (2.4).

3.3 Lower bound on the autocorrelation

The subject of the present Section is the proof of Theorems 2.2 and 2.5. It is organized as follows. In Section 3.3.1 we define the notion of covering of the border of a magnetization profile by rectangles. Then, in Section 3.3.2 we define the event of dilution according to some initial profile \((u_0, \tau^r) \in \text{IC}\) and prove that it has the expected probability. We show how previous result from [Wou09] apply for the probability of phase coexistence, given the event of dilution. In Section 3.3.3 we show that phase profiles evolve continuously in \( L^1 \) and relate this property to the bottleneck and to the gap in free energy \( K^r(u_0) \). Finally in 3.3.5 we conclude the proof of Theorems 2.2 and 2.5.

3.3.1 Covering of the boundary of phase profiles

As in the work [Wou09], the coverings of the boundary of macroscopic phase profile are a fundamental tool for relating the macroscopic shape of the magnetization to the microscopic spin system. The definition that we present here is more restrictive than in [Wou09] since it takes into account the set IC, together with a new parameter \( \gamma \) (fourth line in (v) below).

Definition 3.6. Let \((u_0 = \chi_{U_0}, \tau^r) \in \text{IC}\) and \( u \in \text{BV} \), together with \( \delta, \gamma > 0 \). We say that a rectangular parallelepiped \( \mathcal{R} \subseteq [0,1]^d \) is \((u_0, \tau^r, \delta, \gamma)-\text{adapted to } u \) at \( x \in \partial^* u \) if:

i. If \( n = n^u_x \) is the outer normal to \( u \) at \( x \), there are \( S \in S_n \) and \( h \in (0, \delta) \) such that, either \( \mathcal{R} \subseteq (0,1)^d \) (we say that \( \mathcal{R} \) is interior) and

\[
\mathcal{R} = x + hS + \left[ \pm \delta h \right] n,
\]

either \( \mathcal{R} \cap \partial [0,1]^d \neq \emptyset \) (we say that \( \mathcal{R} \) is on the border) and \( x \in \partial [0,1]^d \), \( n \) is also the outer normal to \([0,1]^d\) at \( x \) and

\[
\mathcal{R} = x + hS + \left[ -\delta h, 0 \right] n.
\]
ii. We have
\[
\mathcal{H}^{d-1}(\partial^* u \cap \partial \mathcal{R}) = 0,
\]
\[
\left| 1 - \frac{1}{h^{d-1}} \mathcal{H}^{d-1}(\partial^* u \cap \mathcal{R}) \right| \leq \delta,
\]
and
\[
\left| \tau^q(n) - \frac{1}{h^{d-1}} \int_{\partial^* u \cap \mathcal{R}} \tau^q(n^u) d\mathcal{H}^{d-1} \right| \leq \delta.
\]
iii. If \( \chi : \mathbb{R}^d \to \{\pm1\} \) is the characteristic function of the half-space above the center of \( \mathcal{R} \), namely
\[
\chi(z) = \begin{cases} +1 & \text{if } (z-x) \cdot n \geq 0, \\ -1 & \text{else} \end{cases}, \quad \forall z \in \mathbb{R}^d,
\]
then
\[
\frac{1}{2h^d} \int_{\mathcal{R}} |\chi - u| d\mathcal{H}^d \leq \delta.
\]
iv. If \( \mathcal{R} \cap \partial[0,1]^d \neq \emptyset \), then \( \mathcal{R} \) does not intersect \( \partial U_0 \).
v. If \( \mathcal{R} \) intersects \( \partial U_0 \), then \( x \in \partial U_0 \) and
\[
\left| I_{\tau^r(x)} - \frac{1}{h^{d-1}} \int_{\partial^* u \cap \mathcal{R}} I_{\tau^r(z)} d\mathcal{H}^{d-1}(z) \right| \leq \delta;
\]
and
\[
\left| I_{\tau^r(x) - \gamma} - \frac{1}{h^{d-1}} \int_{\partial^* u \cap \mathcal{R}} I_{\tau^r(z) - \gamma} d\mathcal{H}^{d-1}(z) \right| \leq \delta.
\]
vi. If \( \mathcal{R} \) intersects \( \partial U_0 \), then the enlarged volume
\[
\mathcal{R}' = \mathcal{R} + B(0,2\sqrt{dh^2}) = \left\{ z \in \mathbb{R}^d : d(z, \mathcal{R}) \leq 2\sqrt{dh^2} \right\}
\]
satisfies
\[
\frac{1}{h^{d-1}} \mathcal{H}^{d-1}(\partial^* u_0 \cap \mathcal{R} \setminus \mathcal{R}) \leq \delta. \quad (3.6)
\]
Note that conditions i to iii above mean exactly that \( \mathcal{R} \) is \( \delta \)-adapted to \( \partial^* u \) at \( x \in \partial^* u \) in the sense of Definition 3.1 of [Won09]. Also, by \( \Delta \) we mean the symmetric difference.

**Definition 3.7.** Let \((u_0, \tau^r) \in IC \) and \( u \in BV, \delta, \gamma > 0 \). A finite sequence \((\mathcal{R}_i)_{i=1...n}\) of disjoint rectangular parallelepipeds included in \([0,1]^d\) is said to be a \((u_0, \tau^r, \delta, \gamma)\)-covering for \( \partial^* u \) if each \( \mathcal{R}_i \) is \((u_0, \tau^r, \delta, \gamma)\)-adapted to \( u \) and if
\[
\mathcal{H}^{d-1}(\partial^* u \setminus \bigcup_{i=1}^n \mathcal{R}_i) \leq \delta. \quad (3.7)
\]
The main result of this Section is:

**Proposition 3.8.** Let \((u_0, \tau^r) \in \text{IC} \text{ and } u \in \text{BV}, \text{ together with } \gamma, \delta > 0.\) There exists a \((u_0, \tau^r, \delta, \gamma)\)-covering for \(\partial^* u.\)

**Proof.** As the proof follows a classical argument, we will only give here the main steps in the proof. First, we claim that, for any \(u, u_0 \in \text{BV}, \) for \(H^{d-1}\) almost all \(x \in \partial^* u \cap \partial^* u_0,\)

\[
\lim_{r \to 0^+} \frac{1}{r^{d-1}} \mathcal{H}^{d-1}((\partial^* u \Delta \partial^* u_0) \cap B(x, r)) = 0.
\]

This can be proven by applying, for instance, the Besicovitch derivation Theorem (Theorem 2.22 in [AFP00]) to the Borel measurable function

\[
f : x \in \mathbb{R}^d \mapsto \begin{cases} 1 & \text{if } x \in \partial^* u_0 \\ 2 & \text{else, } \end{cases}
\]

with the result that for \(H^{d-1}\)-almost all \(x \in \partial^* u \cap \partial^* u_0,\)

\[
\lim_{r \to 0^+} \frac{1}{r^{d-1}} \int_{\partial^* u \cap B(x, r)} f d\mathcal{H}^{d-1} = f(x)
\]

where \(\alpha_{d-1} = \mathcal{H}^{d-1}(\{x \in B(0, 1) : x \cdot e_d = 0\}).\) Therefore,

\[
\lim_{r \to 0^+} \frac{1}{\alpha_{d-1} r^{d-1}} \left[ \frac{\mathcal{H}^{d-1}((\partial^* u \cap B(x, r))}{\alpha_{d-1} r^{d-1}} + \frac{\mathcal{H}^{d-1}((\partial^* u \setminus \partial^* u_0) \cap B(x, r))}{\alpha_{d-1} r^{d-1}} \right] = 1.
\]

As the first term goes to 1 already as \(r \to 0,\) we conclude that

\[
\lim_{r \to 0^+} \frac{1}{\alpha_{d-1} r^{d-1}} \mathcal{H}^{d-1}((\partial^* u \setminus \partial^* u_0) \cap B(x, r)) = 0
\]

for \(H^{d-1}\)-almost all \(x \in \partial^* u \cap \partial^* u_0.\) The claim follows as \(u\) and \(u_0\) play a symmetric role.

Second, we remark that more generally, all the absolute values in point (v) of Definition 3.7, and also left-hand side of inequality (3.6), have a zero limit as \(h \to 0\) for \(H^{d-1}\)-almost all \(x \in \partial^* u \cap \partial U_0.\) This is a consequence of the strong form of the Besicovitch derivation theorem (Theorem 5.52 in [AFP00]).

The two above facts enable, like in the proof of Theorem 3.4 in [Won09], the use of the Vitali covering Theorem (cf. Definition 13.2 and Theorem 13.3 in [Cer06]) to conclude to the existence of the desired covering.

### 3.3.2 The event of dilution

The purpose of this Section is to define the event of (microscopic) dilution and to prove several important properties about phase coexistence, given the event of dilution. We recall a notation from [Won09]; when \(\mathcal{R}\) is a rectangle and \(N > 0,\) we let

\[
\mathcal{R}^N = N \mathcal{R} + z_N(\mathcal{R})
\]

where \(z_N(\mathcal{R}) \in (-1/2, 1/2]^d\) is such that the center of \(\mathcal{R}^N\) belongs to \(\mathbb{Z}^d.\)
**Definition 3.9.** Let \((u_0, \tau^r) \in \text{IC}, \delta, \gamma > 0\) and let \((\mathcal{R}_i)_{i=1}^n\) be a \((u_0, \tau^r, \delta, \gamma)\)-covering of \(\partial^* u_0\). The event of dilution on this covering is

\[
G((\mathcal{R}_i)_{i=1}^n) = \left\{ J \in \mathcal{J} : \tau^r_{\mathcal{R}_i}^J \leq \tau^r(x_i), \forall i = 1 \ldots n \right\}.
\] (3.8)

The reader will check that the event of dilution affects only the random variables \(J_e\) with \(e\) at distance at most \(N \delta\) from the boundary \(N \partial^* u_0\).

The first property of the event of dilution is that it happens at the expected rate. Namely,

**Lemma 3.10.** Let \((u_0, \tau^r) \in \text{IC}\) and \(\xi > 0\). For \(\delta > 0\) small enough, for \(\gamma > 0\) arbitrary, if \((\mathcal{R}_i)_{i=1}^n\) is a \((u_0, \tau^r, \delta, \gamma)\)-covering of \(\partial^* u_0\), then

\[
P(G((\mathcal{R}_i)_{i=1}^n)) \geq \exp\left(-N^{d-1} (\mathcal{I}^r(u_0) + \xi)\right)
\]

for any \(N\) large enough.

**Proof.** We recall that \(\mathcal{I}^r(u_0)\) was defined at (1.22), see also (1.16). Since the \(\mathcal{R}_i^N\) are disjoint for large enough \(N\), we have

\[
\lim inf_{N \to \infty} \frac{1}{N^{d-1}} \log P(G((\mathcal{R}_i)_{i=1}^n)) = \lim inf_{N \to \infty} \frac{1}{N^{d-1}} \log P\left(\tau^r_{\mathcal{R}_i}^J \leq \tau^r(x_i)\right)
\]

\[
= - \sum_{i=1}^n h_i^{d-1} I_{n_i}(\tau^r(x_i))
\]

in view of (1.16). Note the role played by the assumption \(\tau^r(x) > \tau_{\min}(nx_0)\) in the definition of IC at (1.19). The properties of the covering (Definition 3.7 and point (iii) in Definition 3.6) imply the claim for \(\delta > 0\) small enough.

Then we show that the dilution has the expected impact on the probability for phase coexistence. This is expressed with the two complementary Propositions 3.11 and 3.12. We have to recall here the definition of the magnetization profile

\[
\mathcal{M}_K : [0,1]^d \to [-1,1]
\]

\[
x \mapsto \frac{1}{K^d} \sum_{z \in \Lambda_N \cap \Delta_i(x)} \sigma_z
\]

where \(K \in \mathbb{N}^*\) is the mesoscopic scale and

\[
i(x) = \left\lfloor \frac{Nx_1}{K} \right\rfloor, \ldots, \left\lfloor \frac{Nx_d}{K} \right\rfloor \right\rfloor \quad \text{and} \quad \Delta_i = Ki + \{1, \ldots, K\}^d.
\] (3.10)

Hence, unless \(x\) is too close to the border of \([0,1]^d\), \(\mathcal{M}_K(x)\) is the magnetization in a block of side-length \(K\) that contains \(Nx\). Theorem 5.7 in [Wou08] provides a strong stochastic control on \(\mathcal{M}_K\) when \(\beta > \beta_c\). In particular, when \(K\) is large enough, at every \(x\) the probability that \(\mathcal{M}_K(x)\) is close to either \(m_\beta\) or \(-m_\beta\) is close to one under the averaged measure \(E \mu_{\Lambda_N}^J\). The event \(\mathcal{M}_K/m_\beta \in \mathcal{V}(u_0, \varepsilon)\) means therefore that the system is close to plus (resp. minus) phase at \(Nx\) when \(u_0(x) = 1\) (resp. \(u_0(x) = -1\)).
Proposition 3.11. Assume that $\beta > \hat{\beta}_c$ and $\beta \notin \mathcal{N}$. Let $(u_0, \tau^r) \in \text{IC}$ and $\varepsilon, \xi > 0$. Let $\delta, \gamma > 0$ and consider $(\mathcal{R}_i)_{i=1..n}$ a $(u_0, \tau^r, \delta, \gamma)$-covering of $\partial^* u_0$. Then, if $\delta > 0$ is small enough (and $\gamma$ arbitrary), for $K$ large enough,

$$
\lim_{N \to \infty} \mathbb{P} \left( \mu_{L_N}^J \left( \frac{M_{\beta} \mathcal{N}}{m_{\beta}} \in \mathcal{V}(u_0, \varepsilon) \right) \geq \exp \left( -N^{d-1} (\mathcal{F}^r(u_0) + \xi) \right) \right) = 1.
$$

**Proof.** We use the notations of Proposition 3.9 in [Wou09]. We recall that $D_{U_0}^{N, \delta}$ is the event of $\omega$-disconnection around $N\partial^* u_0$ and that $\mathcal{E}_{U_0}^{N, \delta}$ is the set of edges close to $N\partial^* u_0$. We let then

$$
F^J_N = \inf_{\pi \in D_{U_0}^{N, \delta}} \Phi_{\Lambda_N}^{J, w} \left( \frac{M_{\beta} \mathcal{N}}{m_{\beta}} \in \mathcal{V}(u_0, \varepsilon) \middle| \omega = \pi \text{ on } \mathcal{E}_{U_0}^{N, \delta} \right).
$$

Proposition 3.9 in [Wou09] and the definition of the covering imply

i. For $\delta > 0$ small enough,

$$
\liminf_{N \to \infty} \inf_{J \in \mathcal{G}((\mathcal{R}_i)_{i=1..n})} \frac{1}{N^{d-1}} \log \Phi_{\Lambda_N}^{J, w} (D_{U_0}^{N, \delta}) \geq -F^r(u_0) - \xi/2. \quad (3.11)
$$

ii. For $\delta > 0$ small enough, for $K$ large enough,

$$
\lim_{N \to \infty} \mathbb{P} \left( F^J_N < \frac{1}{3} \right) = 0. \quad (3.12)
$$

The definition of $F^J_N$ yields

$$
\mu_{L_N}^J \left( \frac{M_{\beta} \mathcal{N}}{m_{\beta}} \in \mathcal{V}(u_0, \varepsilon) \right) \geq F^J_N \Phi_{\Lambda_N}^{J, w} (D_{U_0}^{N, \delta})
$$

hence, using (3.11) we obtain: for large enough $N$,

$$
\mathbb{P} \left( \mu_{L_N}^J \left( \frac{M_{\beta} \mathcal{N}}{m_{\beta}} \in \mathcal{V}(u_0, \varepsilon) \right) \geq \exp \left( -N^{d-1} (\mathcal{F}^r(u_0) + \xi) \right) \middle| \mathcal{G}((\mathcal{R}_i)_{i=1..n}) \right) \geq \mathbb{P} \left( F^J_N \geq \frac{1}{3} \right) \mathcal{G}((\mathcal{R}_i)_{i=1..n})
$$

Yet, the variable $F^J_N$ is independent of the $J_e$ with $e \in \mathcal{E}_{U_0}^{N, \delta}$. Thus it is as well independent of the dilution $\mathcal{G}((\mathcal{R}_i)_{i=1..n})$, and (3.12) yields the conclusion. \hfill \square

We also recall from [Wou09] the definition of the compact set

$$
\text{BV}_a = \{ u = \chi_U \in \text{BV} : \mathcal{P}(U) \leq a \} \quad (3.13)
$$

where $\mathcal{P}(U)$ is the perimeter of $U$.

**Proposition 3.12.** Assume $\beta > \hat{\beta}_c$ and $\beta \notin \mathcal{N} \cup \mathcal{N}_1$. Let $(u_0, \tau^r) \in \text{IC}$, $a > 0$, $\xi > 0$. There exists $\gamma > 0$ such that, for any $u \in \text{BV}_a$ there is $\varepsilon > 0$ such that, for any $\delta_0 > 0$ small enough, for any $(u_0, \tau^r, \delta_0, \gamma)$-covering $(\mathcal{R}_i)_{i=1..n}$ of $\partial^* u_0$, for any $K$ large enough,

$$
\lim_{N \to \infty} \mathbb{P} \left( \mu_{L_N}^{J, +} \left( \frac{M_{\beta} \mathcal{N}}{m_{\beta}} \in \mathcal{V}(u, \varepsilon) \right) \leq \exp \left( -N^{d-1} (\mathcal{F}^r(u) - \xi) \right) \middle| \mathcal{G}((\mathcal{R}_i)_{i=1..n}) \right) = 1.
$$
The proof of Proposition 3.12 is based on Proposition 3.11 in [Wou09] that relates the probability \( \mu_{N}^{J\Lambda}(M_{K}/m_{\beta} \in V(u, \varepsilon)) \) to the \( L^1 \)-notion of surface tension

\[
\tilde{\tau}_{N,R}^{J,\delta,K} = -\frac{1}{(hN)^{d-1}} \log \sup_{\sigma \in \Sigma_{N,R}} \mu_{N}^{J,\delta}(\|M_{K}/m_{\beta} - \chi\|_{L^1(R)} \leq 2\delta L^d(R)) \tag{3.14}
\]

together with Propositions 3.12 and 3.13 in [Wou09] that compare the two definitions of surface tensions \( \tilde{\tau}_{N,R}^{J,\delta,K} \) and \( \tau_{N,R}^{J} \). Before we complete the proof of Proposition 3.12 we examine the typical value of \( \tilde{\tau}_{N,R}^{J,\delta,K} \) given the event of dilution.

**Lemma 3.13.** Assume \( \beta > \beta_{c} \) and \( \beta \notin N \cup N_{1} \). Let \((u_{0}, \tau^{r}) \in IC \) and \( \gamma > 0 \). For \( \delta > 0 \) small enough, the following holds: for any \( u \in BV \) and any \( R \) that is \((u_{0}, \tau^{r}, \delta, \gamma)\)-adapted to \( \partial^{*}u \) at \( x \in \partial^{*}u \cap \partial^{*}u_{0} \), for any \((u_{0}, \tau^{r}, \delta, \gamma)\)-covering \((R_{i})_{i=1...n} \) of \( \partial^{*}u_{0} \) such that \( \delta_{0} \in (0, h^{2}) \), for any \( K \) large enough, then

\[
\lim_{N \to \infty} \frac{1}{N^{d-1}} \log P \left( \frac{z_{N,R}^{J,\delta,K}}{c_{d,\delta} + c_{d,\delta}} \leq \gamma \bigg| \mathcal{G}((R_{i})_{i=1...n}) \right) < 0
\]

where \( c_{d,\delta} = c_{d} \delta + c_{d,\delta} \) is the sum of the constants that appear in Propositions 3.12 and 3.13 in [Wou09], and is arbitrary small as \( \delta \to 0 \).

Note that \( h \) in the statement of Lemma 3.13 refers to the largest dimension of \( R \), see (i) in Definition 3.6

![Figure 1: The scale of dilution \( \delta_{0} \).](image)

**Proof.** (Lemma 3.13). We consider \((R_{i})_{i=1...n} \) a \((u_{0}, \tau^{r}, \delta_{0}, \gamma)\)-covering for \( \partial^{*}u_{0} \). Thanks to the product structure of \( P \), for large enough \( N \) the surface tension \( \tilde{\tau}_{N,R}^{J,\delta,K} \) is independent of the \( \tau_{R_{i}}^{J} \) such that \( R_{i} \cap R = \emptyset \). Hence, for large enough \( N \), the conditional probability

\[
P \left( \frac{z_{N,R}^{J,\delta,K}}{c_{d,\delta} + c_{d,\delta}} \leq \gamma \bigg| \mathcal{G}((R_{i})_{i=1...n}) \right)
\]

equals

\[
P \left( \frac{z_{N,R}^{J,\delta,K}}{c_{d,\delta} + c_{d,\delta}} \leq \gamma \bigg| \tau_{R_{i}}^{J} \leq \tau^{r}(x_{i}), \forall i : R_{i} \cap R \neq \emptyset \right),
\]
which is not larger than
\[
p_N = \frac{\mathbb{P} \left( z^{J,\delta,K}_{NR_i} \leq \tau^r(x) - c'_{d,\delta} - \gamma \right) \mathbb{P} \left( \tau^I_{R_i} \leq \tau^r(x_i), \forall i : R_i \cap \mathcal{R} \neq \emptyset \right) }{\mathbb{P} \left( \tau^I_{R_i} \leq \tau^r(x_i), \forall i \in \mathcal{R}_i \cap \mathcal{R} = \emptyset \right)}.
\]
According to the definition of \( c'_{d,\delta} \), to Propositions 3.12 and 3.13 in [Wou09] (here we use the assumption that \( \beta > \hat{\beta}_c \) and \( \beta \notin \mathcal{N} \)) and to the definition of \( I_n \) at (1.16), for large enough \( K \) we have
\[
\limsup_{N \to \infty} \frac{1}{N^{d-1}} \log p_N \leq -h^{d-1} I_n (\tau^r(x) - \gamma) + \sum_{i: R_i \cap \mathcal{R} \neq \emptyset} h_i^{d-1} I_{n_i} (\tau^r(x_i)).
\]
We will show that the right-hand side of (3.15) is negative, for small enough \( \delta > 0 \) and \( \delta_0 \leq h^2 \). Thanks to (vi) in Definition 3.6, one sees (Figure 1) that the right-hand side of (3.15) is not larger, for \( \delta \leq 1/2 \) and \( \delta_0 \leq h^2 \), than
\[
\int_{\partial^* u_0 \cap \mathcal{R}} [I_{n_z} (\tau^r(z)) - I_{n_z} (\tau^r(z) - \gamma)] d\mathcal{H}^{d-1}(z) + M \delta h^{d-1}
\]
where
\[
M = 3 + 2 \sup_{x \in \partial^* u_0} I_{n_x} (\tau^r(x))
\]
is finite thanks to the definition of IC at (1.19). Now we give an upper bound on the integral. Since \( I_n \) is convex, its slope is non-increasing and therefore
\[
\int_{\partial^* u_0 \cap \mathcal{R}} [I_{n_z} (\tau^r(z)) - I_{n_z} (\tau^r(z) - \gamma)] d\mathcal{H}^{d-1}(z) \leq - \int_{\partial^* u_0 \cap \mathcal{R}} I_{n_z} (\tau^q(n_z) - \gamma) d\mathcal{H}^{d-1}(z) \leq - (1/2) h^{d-1} \inf_{n \in S^{d-1}} I_n (\tau^q(n) - \gamma).
\]
Now we show that \( \inf_{n \in S^{d-1}} I_n (\tau^q(n) - \gamma) > 0 \). If not, we can extract a converging sequence \( n_k \to n \) in the compact set \( S^{d-1} \) with \( I_{n_k} (\tau^q(n_k) - \gamma) \to 0 \) as \( k \to \infty \). The Fenchel-Legendre transform \( \tau^\lambda(n) = \inf_{\tau > \tau_{\min}(n)} \{ \lambda \tau + I_n(\tau) \} \) satisfies therefore
\[
\tau^\lambda(n_k) \leq \lambda \tau^q(n_k) - \lambda \gamma + o_{k \to \infty} (1), \quad \forall \lambda > 0.
\]
Since \( \tau^\lambda \) and \( \tau^q \) are continuous (Proposition 2.5 in [Wou09]), we obtain in the limit \( \tau^\lambda(n) \leq \lambda \tau^q(n) - \lambda \gamma \), and therefore, by duality of the Fenchel-Legendre transform \( I_n (\tau^q(n) - \gamma) = 0 \), which contradicts the assumption \( \beta \notin \mathcal{N}_l \). The claim follows for any \( \delta < 1/2 \) with \( \delta < \inf_{n \in S^{d-1}} I_n (\tau^q(n) - \gamma) / (2M) \).

\textbf{Proof.} (Proposition 3.12). Let \( \gamma < \xi/(2a) \). We take \( \delta > 0 \) small enough so that we can use Lemma 3.13 with \( c'_{d,\delta} \leq \gamma \). Let \( u \in \mathbb{B}_\epsilon \) and consider \( (\mathcal{R}^u_i)_{i=1...n(u)} \) a \( (u_0, \tau^r, \delta, \gamma) \)-covering of \( \partial^* u \). Proposition 3.11 in [Wou09] states that
\[
\frac{1}{N^{d-1}} \log \mu_{\lambda,N}^+ \left( M_K m_\beta \in \mathcal{V}(u, \epsilon) \right) \leq - \sum_{i=1}^{n(u)} (h^{d-1}_i \gamma I_{n_i} (\tau^r(x_i))).
\]
for $\varepsilon > 0$ small enough. According to the definitions of $\gamma$ and $\delta$ and to the properties of the coverings, provided $\delta$ was chosen small enough (still independently of $u \in BV_a$) we have

$$
\sum_{i=1}^{n(u)} (h_i^u)^{d-1} (\tau^f(x_i) - c_d' - \gamma) \geq \mathcal{F}^f(u) - \xi.
$$

Finally when we take $\delta_0 < \min_{i=1}^{n(u)} (h_i^u)^2$ and $K$ large enough, Lemma 3.13 concludes the proof.

Finally we remark that dilution has little influence on the overall magnetization

$$
m_{\Lambda} = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sigma_x.
$$

**Proposition 3.14.** Assume that $\beta > \hat{\beta}_c$ and $\beta \notin \mathcal{N}$. Let $(u_0, \tau^f) \in \text{IC}$ and $\varepsilon > 0$. Let $\delta, \gamma > 0$ and consider $(R_i)_{i=1,...,n}$ a $(u_0, \tau^f, \delta, \gamma)$-covering of $\partial^* u_0$. Let $N > 0$ and $E_N = E(N \cup \mathbb{Z}^d \setminus \{0\})$.

Then, if $\delta > 0$ is small enough,

$$
\lim_N P \left( \mu_{J, +}^{e_{\Lambda_N}^+} \geq m_{\beta} - \varepsilon, \forall e \in \mathcal{E}_N \right) = 1.
$$

Note that $\mu_{J, +}^{e_{\Lambda_N}^+}$ increases with every $J_e$, therefore the condition that $J_e = 0, \forall e \in \mathcal{E}_N$ is the worse condition that one can consider.

**Proof.** The proof is based on the renormalization procedure established in Theorem 5.10 in [Wou08]. As it is similar to that of Proposition 3.9 in [Wou09] we only sketch the argument. We cover $E(N \cup \mathbb{Z}^d \setminus \mathcal{E}_N)$ with blocks with side-length $L_N = [\sqrt{N}]$. To these blocks is associated a family of independent random variables $\varphi_i \in \{-1, 0, 1\}$ (independent of the $J_e, e \in \mathcal{E}_N$), the local $\varepsilon$-phase, that is zero with probability less than $e^{-c\sqrt{N}}$. Write $\mu_{J, +}^{e_{\Lambda_N}^+} = \lim_M \mu_{\Lambda_M}^{J, +} (m_{\Lambda_N})$ and take some finite $M$. A simple Peierls estimate shows that, given the condition $J_e = 0, \forall e \in \mathcal{E}_N$, the $\varepsilon$-local phase associated to any block in $\Lambda_N$ is $+1$ with a probability going to one as $N \to \infty$, uniformly in $N$. As described in Theorem 5.10 in [Wou08], this event implies that

$$
m_{\Lambda_N} \geq (m_{\beta} - \varepsilon) \left( \frac{1}{1 - \sum_{i=1}^{n} \text{Vol}(R_i)} \right).
$$

Therefore we just need to take $\varepsilon > 0$ small enough (named $\delta$ in Theorem 5.10 in [Wou08]) and $\delta > 0$ small enough so that $\sum_{i=1}^{n} \text{Vol}(R_i) \simeq \delta H^{d-1}(\partial^* u_0)$ is negligible.

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3.3.3 The bottleneck

Here we focus on the bottleneck in the dynamics. We recall that the set $C_\varepsilon(u_0)$, the set of sequences of profiles that evolve from $u_0$ to 1 with jumps in $L^1$-norm less than $\varepsilon$, was introduced at (1.23). Given $v = (v_i)_{i=0...k} \in C_\varepsilon(u_0)$, we call

$$\arg\max\varepsilon(v) = \min\left\{i : \inf_{u \in V(v, \varepsilon)} F^r(u) = \max_{l \in [k]} \inf_{u \in V(v_l, \varepsilon)} F^r(u)\right\}.$$ 

Now, given $(u_0, \tau^r) \in IC$, we define the $\varepsilon$-bottleneck set as

$$B_\varepsilon(u_0, \tau^r) = \bigcup_{v \in C_\varepsilon(u_0)} V(v_{\arg\max\varepsilon(v)}, \varepsilon). \quad (3.17)$$

When $(u_0, \tau^r)$ are clear from the context, we simply write $B_\varepsilon$ for $B_\varepsilon(u_0, \tau^r)$. Note that our motivation for the above definition is that the $\varepsilon$-enlargement of $B_\varepsilon$ is $V(B_\varepsilon, \varepsilon) = \bigcup_{v \in C_\varepsilon(u_0)} V(v_{\arg\max\varepsilon(v)}, 2\varepsilon)$, a fact that helps in the proof of Lemma 3.17 below.

Now we state three Lemmas related to the bottleneck set. Lemma 3.15 gives the asymptotics of the probability that the phase profile $M_K/m_\beta$ belongs to $B_\varepsilon$. In Lemma 3.16 we show that $F^r$ is lower semi-continuous, a requisite for the proof of Lemma 3.17 in which we show how the gap in surface energy $K^r(u_0)$ defined at (1.24) is related to the $\varepsilon$-enlargement of $B_\varepsilon(u_0, \tau^r)$.

**Lemma 3.15.** Assume $\beta > \hat{\beta}_c$ and $\beta \notin \mathcal{N} \cup \mathcal{N}_I$. Let $(u_0, \tau^r) \in IC$ and $\xi, \varepsilon > 0$. There exists $\gamma > 0$ such that, for any $(u_0, \tau^r, \delta, \gamma)$-covering $(R_i)_{i=1...n}$ of $\partial^* w_0$ with $\delta > 0$ small enough, for any $K$ large enough,

$$\lim_{N} \mathbb{P}\left(\mu_{\Lambda_N}^{J_+} \left(\frac{M_K}{m_\beta} \notin B_\varepsilon\right) \leq \exp\left(-N^{d-1} \left(\inf_{v(B_\varepsilon, \varepsilon)} F^r - \xi\right)\right) \big| \mathcal{G}((R_i)_{i=1...n})\right) = 1.$$ 

Proof. The exponential tightness property (Proposition 3.15 in [Won09]) tells that there exists $C > 0$ such that, for every $\varepsilon' > 0$, for any $K$ large enough,

$$\limsup_{N} \frac{1}{N^{d-1}} \log \mathbb{E} \mu_{\Lambda_N}^{J_+} \left(\frac{M_K}{m_\beta} \notin \mathcal{V}(BV_\alpha, \varepsilon')\right) \leq -C \alpha$$

(the set $BV_\alpha$ was defined at (3.13)). An immediate application of Markov’s inequality shows that

$$\frac{1}{N^{d-1}} \log \mu_{\Lambda_N}^{J_+} \left(\frac{M_K}{m_\beta} \notin \mathcal{V}(BV_\alpha, \varepsilon')\right) \leq -\frac{C \alpha}{2}$$

with a probability at least $1 - \exp(-C \alpha N^{d-1}/3)$, for large $N$. Since the surface cost of $\mathcal{G}((R_i)_{i=1...n})$ is bounded, we have therefore, for any $\varepsilon' > 0$, for any $a$ large enough, for $K$ large enough:

$$\lim_{N} \mathbb{P}\left(\mu_{\Lambda_N}^{J_+} \left(\frac{M_K}{m_\beta} \notin \mathcal{V}(BV_\alpha, \varepsilon')\right) \leq \exp\left(-\frac{C \alpha}{2} N^{d-1}\right) \big| \mathcal{G}((R_i)_{i=1...n})\right) = 1. \quad (3.18)$$

Now we take some $\xi, \varepsilon > 0$ and $a > 0$ large. We take for $\gamma > 0$ the one given by Proposition 3.12 (it does not depend on $\varepsilon$), and for any $u \in BV_\alpha$ we denote by $\varepsilon_{\xi}(u) > 0$ the parameter $\varepsilon$
covering \( \delta > 0 \). Combining (3.18) with (3.19) proves the Lemma, provided that \( a > R \).

Proof. Lemma 3.17. For any \( u \in BV \) and \( \epsilon > 0 \) small enough, for any \( (u_0, \tau^r, \delta, \gamma) \)-covering \( (R_i)_{i=1}^n \) of \( \partial^* u_0 \) and any \( K \) large enough,

\[
\lim_{N} \mathbb{P} \left( \frac{\mu_{\lambda,N}}{m_{\lambda}} \left( \frac{M_K}{m_{\lambda}} \in B_{\epsilon} \cap V(BV, \epsilon) \right) \leq n \exp \left( -N^{d-1} \left( \min_{i \in n; u_i \in V(BV, \epsilon)} F^r(u_i) - \xi \right) \right) \right) = 1. \tag{3.19}
\]

Combining (3.19) with (3.18) proves the Lemma, provided that \( a > 0 \) was chosen large enough.

\[
\text{Lemma 3.16. For any } (u_0, \tau^r) \in IC, \text{ the functional } F^r \text{ defined at } (1.20) \text{ is lower semi-continuous.}
\]

Proof. We show the lower semi-continuity as an application of the covering Proposition (Proposition 3.8). Let \( u \in BV \) and \( \delta, \gamma > 0 \), and consider a \( (u_0, \tau^r, \delta, \gamma) \)-covering \( (R_i)_{i=1}^n \) for \( \partial^* u \). Since the \( R_i \) are disjoint, for any \( v \in BV \) we have

\[
F^r(v) \geq \sum_{i=1}^n \int_{R_i \cap \partial^* u_0 \cap \partial^* v} \tau^r(x) dH^{d-1}(x) + \int_{(R_i \setminus \partial^* u_0) \cap \partial^* v} \tau^q(n^v_x) dH^{d-1}(x)
\]

\[
\geq \sum_{i: R_i \cap \partial^* u_0 \neq \emptyset} \int_{R_i \setminus \partial^* v} \tau^r(x) dH^{d-1}(x) + \sum_{i: R_i \cap \partial^* u_0 = \emptyset} \int_{R_i \cap \partial^* v} \tau^q(n^v_x) dH^{d-1}(x)
\]

since \( \tau^r(x) \leq \tau^q(n_x) \). Thanks to the lower semi-continuity of the surface energy in open sets (Chapter 14 in [Cer06]), the quantities \( H^{d-1}(R_i \setminus \partial^* v) \) and \( \int_{R_i \cap \partial^* v} \tau^q(n^v_x) dH^{d-1}(x) \) become not smaller than their value at \( u \) when \( v \) converges to \( u \) in \( L^1 \) norm. Hence,

\[
\lim_{\epsilon \to 0} \inf_{v \in V(u, \epsilon)} F^r(v) \geq \sum_{i: R_i \cap \partial^* u_0 \neq \emptyset} (1 - \delta)h^{d-1}_i \inf_{x \in R_i} \tau^r + \sum_{i: R_i \cap \partial^* u_0 = \emptyset} (1 - \delta)h^{d-1}_i \tau^q(n^v_x)
\]

which is arbitrary close to \( F^r(u) \) for small \( \delta \), thanks to the uniform continuity of \( \tau^r \).

\[
\text{Lemma 3.17. For any } (u_0, \tau^r) \in IC,
\]

\[
K^r(u_0) = \lim_{\epsilon \to 0^+} \inf_{u \in V(BV, (u_0, \tau^r), \epsilon)} F^r(u) - F^r(u_0). \tag{3.20}
\]
Proof. We prove first that $\mathcal{K}^r(u_0)$ is larger or equal to the right-hand side. Let $v \in \mathcal{C}_\varepsilon(u_0)$. Then,

$$
\max_i \mathcal{F}^r(v_i) \geq \max_i \inf_{v \in \mathcal{V}(v_i, 2\varepsilon)} \mathcal{F}^r(v_i) \geq \inf_{u \in \mathcal{V}(\mathcal{B}_\varepsilon(u_0, \tau^r), \varepsilon)} \mathcal{F}^r(u)
$$

and, when we optimize over $v \in \mathcal{C}_\varepsilon(u_0)$ and let $\varepsilon \to 0$ we obtain the inequality

$$
\mathcal{K}^r(u_0) + \mathcal{F}^r(u_0) \geq \lim_{\varepsilon \to 0} \inf_{u \in \mathcal{V}(\mathcal{B}_\varepsilon(u_0, \tau^r), \varepsilon)} \mathcal{F}^r(u).
$$

Now we prove the opposite inequality. Given any $\varepsilon > 0$, there is $u \in \mathcal{V}(\mathcal{B}_\varepsilon(u_0, \tau^r), \varepsilon)$ such that

$$
\mathcal{F}^r(u) \leq \inf_{w \in \mathcal{V}(v_i, 2\varepsilon)} \mathcal{F}^r(w) + \varepsilon. \tag{3.21}
$$

According to the definition of $\mathcal{B}_\varepsilon(u_0, \tau^r)$ at (3.17), there is $v = (v_i)_{i=0..k}$ in $\mathcal{C}_\varepsilon(u_0)$, that is an \( \varepsilon \)-continuous evolution such that $v_0 = u_0$, $v_k = \mathbf{1}$, that satisfies

$$
\mathcal{F}^r(u) \geq \max_{i=0}^k \inf_{w \in \mathcal{V}(v_i, 2\varepsilon)} \mathcal{F}^r(w). \tag{3.22}
$$

Now, for each $i = 0, \ldots, k$ we consider $v_{i+1}^r \in \mathcal{V}(v_i, 2\varepsilon)$ such that

$$
\mathcal{F}^r(v_{i+1}^r) \leq \inf_{w \in \mathcal{V}(v_i, 2\varepsilon)} \mathcal{F}^r(w) + \varepsilon. \tag{3.23}
$$

We let also $v'_0 = u_0$ and $v'_{k+2} = \mathbf{1}$. Clearly, the evolution $v' = (v'_i)_{i=0..k+2}$ is $5\varepsilon$-continuous. Therefore, we have

$$
\inf_{v'' \in \mathcal{C}_{5\varepsilon}(u_0)} \max_i \mathcal{F}^r(v''_i) \leq \max_{i=0}^{k+2} \mathcal{F}^r(v'_i).
$$

The maximum in the right-hand side does not occur at $i = k + 2$ since $\mathcal{F}^r(\mathbf{1}) = 0$. We also have the bound

$$
\mathcal{F}^r(v'_0) - \mathcal{F}^r(v'_1) \leq \mathcal{F}^r(u_0) - \inf_{w \in \mathcal{V}(u_0, 2\varepsilon)} \mathcal{F}^r(w)
$$

therefore,

$$
\inf_{v'' \in \mathcal{C}_{5\varepsilon}(u_0)} \max_i \mathcal{F}^r(v''_i) \leq \max_{i=1}^{k+1} \mathcal{F}^r(v'_i) + \left( \mathcal{F}^r(u_0) - \inf_{w \in \mathcal{V}(u_0, 2\varepsilon)} \mathcal{F}^r(w) \right)
$$

$$
\leq \max_{i=0}^k \inf_{w \in \mathcal{V}(v_i, 2\varepsilon)} \mathcal{F}^r(w) + \varepsilon + \left( \mathcal{F}^r(u_0) - \inf_{w \in \mathcal{V}(u_0, 2\varepsilon)} \mathcal{F}^r(w) \right)
$$

$$
\leq \mathcal{F}^r(u) + \varepsilon + \left( \mathcal{F}^r(u_0) - \inf_{w \in \mathcal{V}(u_0, 2\varepsilon)} \mathcal{F}^r(w) \right)
$$

$$
\leq \inf_{w \in \mathcal{V}(\mathcal{B}_\varepsilon(u_0, \tau^r), \varepsilon)} \mathcal{F}^r(w) + 2\varepsilon + \left( \mathcal{F}^r(u_0) - \inf_{w \in \mathcal{V}(u_0, 2\varepsilon)} \mathcal{F}^r(w) \right)
$$

where the second line is due to the definition of $v'_{i+1}$ at (3.23), the third one to (3.22) and the last one to (3.21). The lower semi-continuity of $\mathcal{F}^r$ (Lemma 3.16) imply that the last term goes to 0 as $\varepsilon \to 0$. Therefore taking $\varepsilon \to 0$ ends the proof.
3.3.4 Intermediate formulation of the metastability

The aim of this Section is the proof of an intermediate and useful formulation of the metastability:

**Proposition 3.18.** Assume \( \beta > \beta_c \) and \( \beta \notin N \cup N_I \). Let \((u_0, \tau^r) \in 1C, \xi > 0 \) and \( \varepsilon > 0 \) small enough. Then, there exists \( \gamma > 0 \) such that, for any \( \delta > 0 \) small enough, for any \((u_0, \tau^r, \delta, \gamma)\)-covering \((R_i)_{i=1,...,n}\) of \( \partial^*u_0 \),

\[
\liminf_N \inf_{t \in K_N} \mathbb{P} \left( \mu^{I,+}_{j,\Lambda_N} \left( T^{I,+}_{\Lambda_N}(t) m_{\Lambda_N} \leq m_\beta - 2\varepsilon \right) \geq e^{-N^{d-1}(\tau^r(u_0)+\xi)} \right) \mathcal{G} ((R_i)_{i=1,...,n}) = 1
\]

where \( K_N = \exp(N^{d-1}(K^r(u_0) - \xi)) \).

Important keys for the proof of Proposition 3.18 are Lemmas 3.15 and 3.17 about the bottleneck of the dynamics, together with Lemma 3.19 below on the so-to-say “continuous evolution” of the magnetization profile.

**Lemma 3.19.** Let \( \beta > 0 \) and \( \varepsilon > 0 \). There is \( c > 0 \) such that, for any \( N \) large enough,

\[
\sup_{t \in [0, \varepsilon/(4c_M)]} \sup_J \mathbb{P}^{I,\Lambda_N,+,+} \left( \| \mathcal{M}_K(\sigma(t)) - \mathcal{M}_K(\sigma(0)) \|_{L^1} \geq \varepsilon \right) \leq \exp \left( -cN^d \right)
\]

where \( c_M \) is a uniform upper bound on the rates of the Glauber dynamics.

**Proof.** (Lemma 3.19.) The \( L^1 \) distance \( \| \mathcal{M}_K(\sigma(t)) - \mathcal{M}_K(\sigma(0)) \|_{L^1} \) is bounded by \( 2/N^d \) times the number of jumps of the Glauber dynamics. These jumps occur at a rate bounded by \( c_M \). Therefore,

\[
\mathbb{P}^{I,\Lambda_N,+,+} \left( \| \mathcal{M}_K(\sigma(t)) - \mathcal{M}_K(\sigma(0)) \|_{L^1} \geq \varepsilon \right) \leq \mathbb{P} \left( X \geq \frac{\varepsilon N^d}{2} \right)
\]

where \( X \) is a Poisson variable with parameter \( c_M t N^d \leq \varepsilon N^d/4 \). Cramér’s Theorem imply the claim. \( \square \)

**Proof.** (Proposition 3.18.) In view of Proposition 3.11 it suffices to prove that

\[
\liminf_N \inf_{t \in K_N} \mathbb{P} \left( p^J \leq \frac{1}{2} e^{-N^{d-1}(\tau^r(u_0)+\xi)} \right) \mathcal{G} ((R_i)_{i=1,...,n}) = 1 \tag{3.24}
\]

where

\[
p^J = \mu^{I,+}_{j,\Lambda_N} \left( \mathcal{M}_K/m_\beta \in \mathcal{V}(u_0, \varepsilon) \text{ and } T^{I,+}_{\Lambda_N}(t)m_{\Lambda_N} > m_\beta - 2\varepsilon \right).
\]

So we focus on the proof of (3.24). We remark that, for any initial configuration \( \rho \),

\[
\mathbb{P}^{I,\Lambda_N,+(+)}(m_{\Lambda_N}(\sigma(t)) \geq m_\beta - 3\varepsilon) < \varepsilon \Rightarrow (T^{I,+}_{\Lambda_N}(t)m_{\Lambda_N})(\rho) < m_\beta - 2\varepsilon
\]

since \((T^{I,+}_{\Lambda_N}(t)m_{\Lambda_N})(\rho) = E^{I,\Lambda_N,+(+)}(m_{\Lambda_N}(\sigma(t))) \) and \( m_{\Lambda_N}(\sigma) \leq 1 \), for any \( \sigma \). Therefore

\[
p^J \leq \mu^{I,+}_{j,\Lambda_N} \left( \mathcal{M}_K/m_\beta \in \mathcal{V}(u_0, \varepsilon) \text{ and } \mathbb{P}^{I,\Lambda_N,+(+)}(m_{\Lambda_N}(\sigma(t)) \geq m_\beta - 3\varepsilon) \geq \varepsilon \right)
\]

\[
\leq \frac{1}{\varepsilon} \mathbb{P}^{I,\Lambda_N,+(+)} \left( \mathcal{M}_K(\sigma(0))/m_\beta \in \mathcal{V}(u_0, \varepsilon) \text{ and } m_{\Lambda_N}(\sigma(t)) \geq m_\beta - 3\varepsilon \right) \tag{3.25}
\]
where the second inequality is a consequence of Markov’s inequality. Now we fix \( k \in \mathbb{N}^* \) and call
\[
C = \left\{ \left\| \frac{\mathcal{M}_K}{m_{\beta}} \left( \sigma \left( \frac{(i+1)t}{k} \right) \right) - \frac{\mathcal{M}_K}{m_{\beta}} \left( \sigma \left( \frac{it}{k} \right) \right) \right\|_{L^1} \leq \varepsilon, \forall i < k \right\}
\]
the event of continuity, and
\[
D_a = \left\{ \forall i < k, \exists v_i \in BV_a : \frac{\mathcal{M}_K}{m_{\beta}} \left( \sigma \left( \frac{it}{k} \right) \right) \in \mathcal{V}(v_i, \varepsilon) \right\}
\]
the event that the magnetization profile is close to BV at any time \( it/k \). Then we remark that
\[
\begin{align*}
\text{Indeed, the definition of } D_a \text{ yields a sequence } v = (v_i)_{i=0}^k. \text{ We can take } v_0 = u_0 \text{ since we know that } \mathcal{M}_K(\sigma(0))/m_{\beta} \in \mathcal{V}(u_0, \varepsilon). \text{ This sequence is } 3\varepsilon \text{-continuous according to the properties of the } v_i \text{ and to the definition of } C. \text{ Finally, we have}
\end{align*}
\]
\[
\|1 - v_k\|_{L^1} = \int_{[0,1]^d} (1 - v_k(x)) d\mathcal{L}^d(x)
\leq \int_{[0,1]^d} (1 - \mathcal{M}_K(\sigma(t))/m_{\beta}) d\mathcal{L}^d(x) + \varepsilon
\leq 1 - m_{\Lambda_N} (\sigma_t)/m_{\beta} + 2\varepsilon
\leq 5\varepsilon/m_{\beta}
\]
and therefore \( \mathcal{M}_K(\sigma(t))/m_{\beta} \in \mathcal{V}(1, 6\varepsilon/m_{\beta}) \). So we can fix \( v_k = 1 \), which makes the evolution \( 8\varepsilon/m_{\beta} \)-continuous.

Now we conclude the proof of (3.24). As a consequence of (3.25) and (3.26) we have the inequality
\[
\varepsilon p^J \leq p^J_{\mu_{\Lambda_N}} \left( \frac{\mathcal{M}_K}{m_{\beta}} (\sigma (\frac{it}{k})) \in \mathcal{V}(v_i, \varepsilon) \right) + p^J_{\mu_{\Lambda_N}} (C_c \cup D_a^c)
\]
and therefore, according to the invariance of \( \mu_{\Lambda_N}^J \) for the Glauber dynamics, and to the definition (3.17) of the bottleneck set \( B_{\varepsilon}(u_0, \tau') \),
\[
p^J \leq p^J_1 + p^J_2 + p^J_3
\]
where
\[
\begin{align*}
p^J_1 &= \frac{k+1}{\varepsilon} \mu_{\Lambda_N} \left( \frac{\mathcal{M}_K}{m_{\beta}} \in B_{8\varepsilon/m_{\beta}} \right)
\end{align*}
\]
\[
\begin{align*}
p^J_2 &= \frac{k+1}{\varepsilon} \mu_{\Lambda_N} \left( \frac{\mathcal{M}_K}{m_{\beta}} \not\in \mathcal{V}(BV_a, \varepsilon) \right)
\end{align*}
\]
\[
\begin{align*}
p^J_3 &= \frac{k}{\varepsilon} \mu_{\Lambda_N} \left( \left\| \frac{\mathcal{M}_K}{m_{\beta}} \left( \sigma \left( \frac{t}{k_{\tau'}} \right) \right) - \mathcal{M}_K(\sigma(0)) \right\|_{L^1} \geq \varepsilon \right)
\end{align*}
\]
Finally we bound each contribution separately. First, according to Lemmas 3.15 and 3.17 we have
\[
\lim_{N} \mathbb{P} \left( \frac{M}{m_{\beta}} \left( \frac{M}{m_{\beta}} \in B_{\frac{\delta}{m_{\beta}}} \right) \leq e^{-N^{d-1}(K^{r}(u_{0})+F^{r}(u_{0})-2\xi)} \right| \mathcal{G}((R_{i})_{i=1...n}) \right) = 1
\]
for \( \varepsilon > 0 \) small enough, for some \( \gamma > 0 \), and for any \((u_{0}, r, \delta, \gamma)\)-covering \((R_{i})_{i=1...n}\) of \(\partial^{*}u_{0}\) with \(\delta > 0\) small enough, for any \(K\) large enough. So for
\[
k = k_{N} = \left[ \exp(N^{d-1}(K^{r}(u_{0}) - \xi/2)) \right]
\]
we have, under the same conditions,
\[
\lim_{N} \mathbb{P} \left( p_{1}^{J} \leq \frac{1}{6} e^{-N^{d-1}(F^{r}(u_{0})-\xi)} \right| \mathcal{G}((R_{i})_{i=1...n}) \right) = 1. \tag{3.28}
\]
Then, the exponential tightness property (see (3.18) above) implies that, for \(a > 0\) large enough,
\[
\lim_{N} \mathbb{P} \left( p_{2}^{J} \leq \frac{1}{6} e^{-N^{d-1}(F^{r}(u_{0})-\xi)} \right| \mathcal{G}((R_{i})_{i=1...n}) \right) = 1 \tag{3.29}
\]
and finally Lemma 3.19 implies that, for large \(N\), uniformly over \(J\) and \(t \leq K_{N}\),
\[
p_{3}^{J} \leq \exp(-cN^{d}). \tag{3.30}
\]
Summing the last three displays yields (3.24).

\section{3.3.5 Proofs of Theorems 2.2 and 2.5}
Here we conclude the proof of Theorems 2.2 and 2.5. We state one more Lemma that relates the averaged autocorrelation \(A^{\lambda}(t)\) defined at (1.27) to the dynamics of the overall magnetization \(m_{\Lambda_{N}}\) defined at (3.16).

\textbf{Lemma 3.20.} For any \(\varepsilon > 0\), \(N \in \mathbb{N}^{+}\) and \(\lambda \geq 1\) one has
\[
A^{\lambda}(t) \geq \varepsilon^{2\lambda} \mathbb{E} \left[ 1_{\{\mu^{J,+}(m_{\Lambda_{N}}) \geq m_{\beta} - \varepsilon\}} \times \left( \mu^{J,+}_{\Lambda_{N}} \left( T^{J,+}_{\Lambda_{N}}(t)m_{\Lambda_{N}} \leq m_{\beta} - 2\varepsilon \right) \right) \right] \tag{3.31}
\]

\textbf{Proof.} Minkowski’s inequality implies, for any \(\nu \geq 1\), that
\[
\frac{1}{N^{d}} \sum_{x \in \Lambda_{N}} \left[ \int_{\Sigma} \left| T^{J}(t) \pi_{x} - \mu^{J,+}_{\Lambda_{N}} \right|^{\nu} d\mu^{J,+} \right]^{1/\nu} \geq \left[ \int_{\Sigma} \left| T^{J}(t)m_{\Lambda_{N}} - \mu^{J,+}(m_{\Lambda_{N}}) \right|^{\nu} d\mu^{J,+} \right]^{1/\nu}
\]
where \(\pi_{x}: \Sigma \rightarrow \mathbb{R}\) is the function which associates, to the spin configuration \(\sigma \in \Sigma\), the spin at \(x, \sigma(x)\). Taking \(\nu = 1/\lambda\), the translation invariance of \(\mathbb{E}\) and of the Glauber dynamics implies that
\[
A^{\lambda}(t) \geq \mathbb{E} \left[ \left( \int_{\Sigma} \left| T^{J}(t)m_{\Lambda_{N}} - \mu^{J,+}(m_{\Lambda_{N}}) \right|^{2} d\mu^{J,+} \right)^{1/\lambda} \right].
\]

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Hence, for any $\varepsilon > 0$ we have
\[ A^\lambda(t) \geq \varepsilon^2 \mathbb{E} \left[ 1\{\mu^{J,+}(m_{\Lambda_N}) \geq m_\beta - \varepsilon\} \times (\mu^{J,+}(T^J(t)m_{\Lambda_N} \leq m_\beta - 2\varepsilon))^\lambda \right] \]
and we conclude by using the attractivity of the Glauber dynamics:
\[ \mu^{J,+}(T^J(t)m_{\Lambda_N} \leq m_\beta - 2\varepsilon) \geq \mu^{J,+}(T^{J,\lambda}(t)m_{\Lambda_N} \leq m_\beta - 2\varepsilon) \]
\[ \geq \mu^{J,+}(T^{J,\lambda}(t)m_{\Lambda_N} \leq m_\beta - 2\varepsilon) \]
(3.32)

Proof. (Theorem 2.2) Let $\delta > 0$ and $\lambda \geq 1$. We assume that $X^\lambda < \infty$, otherwise there is nothing to prove. We fix $(u_0, \tau^r) \in IC$ and $\xi \in (0, K^r(u_0))$ such that
\[ I^r(u_0) + \xi + \lambda(F^r(u_0) + \xi)(1 + \xi) \leq X^\lambda + \delta/2. \]
Then, for any $t > 0$ we call $N(t)$ the smallest integer $N$ such that $t \leq \exp(Nd - 1((K^r(u_0) - \xi))$.

According to Lemma 3.20 to Propositions 3.14 and 3.18 to Lemma 3.10 for any $\varepsilon > 0$ small enough we can find $\delta, \gamma > 0$ such that, provided that $N(t)$ is large enough,
\[ A^\lambda(t) \geq \frac{\varepsilon^\lambda}{4} \exp \left( -\lambda N(t)^{d-1}(F^r(u_0) + \xi) - N(t)^{d-1}(I^r(u_0) + \xi) \right). \]
The definition of $N(t)$ implies finally that
\[ A^\lambda(t) \geq \frac{\varepsilon^\lambda}{4} \exp \left( -(\log t)^{\lambda(F^r(u_0) + \xi)} + (I^r(u_0) + \xi) \right) \left( \frac{N(t)}{N(t) - 1} \right)^{d-1} \]
for $t$ large enough, and this gives the claim as $N(t) \to +\infty$. □

Proof. (Theorem 2.5) Let $\delta > 0$ and assume that $\kappa > 0$. There exists $(u_0, \tau^r) \in IC$ and $\xi > 0$ such that
\[ \frac{K^r(u_0) - \xi}{I^r(u_0) + \xi} \leq \kappa - \delta/2. \]
Given $N \in \mathbb{N}^*$ large, we define an intermediate side-length $M = M(N)$ as the largest integer $M$ such that
\[ N^d \geq \exp(M^{d-1}(I^r(u_0) + 2\xi)). \]
First, we prove that the event of dilution, on scale $M$, occurs with large probability inside $\Lambda_M$. More precisely, we consider $L = M + \lfloor \sqrt{M} \rfloor$ and
\[ X_N = \{0, \ldots, [N/L] - 1\}^d. \]
We also consider $\delta > 0$ and a $\delta$-covering for $\partial u_0$, and for any $x \in X$ we call
\[ G^x_M = \{ \text{dilution occurs in } Lx + \Lambda_M \} \]
the $Lx$-translate of $G_M$. According to Lemma 3.10 we have, for $\delta$ small enough,

\[
\mathbb{P}\left( \bigcap_{x \in X} (G_M^x)^c \right) \leq \left( 1 - \exp(-M^{d-1} (\mathcal{T}(u_0) + \xi)) \right)^{[N/L]^d} \\
\leq \exp\left( - \left[ \frac{N}{L} \right]^d \exp(-M^{d-1} (\mathcal{T}(u_0) + \xi)) \right) \\
\leq \exp\left( - \left( \frac{1}{C \log N} \right)^d \exp(M^{d-1} \xi) \right) \\
\leq \exp\left( - N \frac{d \kappa^c_\ell (u_0) + \xi)}{\log N} \right)
\]

(3.33)

where we use the definition of $M$ and the fact that $L \leq C \log N$, for some $C < \infty$. We call

\[ D_x = G_x \cap \bigcap_{z < x} (G_z)^c \]

the event that $x$ is the smallest box for which dilution occurs, where $X$ is ordered according to the lexicographic order. According to Proposition 3.18 and to the attractivity of the dynamics, the conditional probability

\[
\mathbb{P}\left( \mu_{\Lambda_N}^{J_+} \left( \mathcal{T}_{\Lambda_N}^{J_+} \left( \exp(M^{d-1} (\mathcal{K}(u_0) - \xi)) \right) \right) m_{Lx+\Lambda_M} \leq m_\beta - 2 \varepsilon \mid D_M^x \right) \geq \exp \left( - M^{d-1} (\mathcal{F}(u_0) + \xi) \right)
\]

goes to 1 as $N \to \infty$ (and thus $M$). On the other hand, Proposition 3.14 and the monotonicity of the system imply that, as well, the conditional probability

\[
\mathbb{P}\left( \mu_{\Lambda_N}^{J_+} \left( m_{Lx+\Lambda_M} \right) \geq m_\beta - \varepsilon \mid D_M^x \right)
\]

go to 1 as $N \to \infty$. Therefore, we have shown that the $\mathbb{P}$-probability that there exists $x \in X_N$ with both

\[
\mu_{\Lambda_N}^{J_+} \left( \mathcal{T}_{\Lambda_N}^{J_+} \left( \exp(M^{d-1} (\mathcal{K}(u_0) - \xi)) \right) \right) m_{Lx+\Lambda_M} \leq m_\beta - 2 \varepsilon \\
\mu_{\Lambda_N}^{J_+} \left( m_{Lx+\Lambda_M} \right) \geq m_\beta - \varepsilon
\]

go to one as $N \to \infty$. Taking $m_{Lx+\Lambda_M}$ as a test function in (1.30) proves the lower bound on the relaxation time. For obtaining the lower bound on the mixing time one can consider $f = m_{Lx+\Lambda_M}$ in (1.32), and $\sigma_0$ any initial configuration for which $T_{\Lambda_N}^{J_+} \left( \exp(M^{d-1} (\mathcal{K}(u_0) - \xi)) \right) m_{Lx+\Lambda_M} \leq m_\beta - 2 \varepsilon$, which is possible as this event has positive probability.

### 3.4 The geometry of relaxation

The first part of this Section is dedicated to the proof of Proposition 2.1. Then we show that the gap in surface energy can be computed on more restrictive evolutions, and finally we compute $\mathcal{K}(u_0)$ for two simple initial configurations.

#### 3.4.1 A lower bound on the gap in surface energy

The statement of theorems 2.2 and 2.5 is non-empty only if $\mathcal{K}(u_0)$ happens to be strictly positive for some initial profiles $(u_0, \tau') \in \text{IC}$. We state the next Theorem, which has Proposition 2.1 as a Corollary:
Theorem 3.21. Let \((u_0, \tau^r) \in I_C\). Assume that the boundary of \(u_0\) is \(C^1\). There exists a non-decreasing function \(H : \mathbb{R}^+ \to \mathbb{R}^+\), with \(H(\delta) > 0\), \(\forall \delta > 0\), that depends only on \((u_0, \tau^r)\), such that

\[
K^r(u_0) \geq \sup_{n \in S^{d-1}} \tau^q(n) \times H \left( \inf_{x \in \partial^* u_0} \tau^q(n_{x}^{u_0}) - \tau^r(x) \right) \max(\sup_{n \in S^{d-1}} \tau^q(n), 1),
\]

(3.34)

Proof. (Proposition 2.1). When \(\tau_{\min}(n) < \tau^q(n)\) for all \(n \in S^{d-1}\) and the boundary of \(u_0\) is \(C^1\), it follows at once from (3.34) that \(K^r(u_0) > 0\). Now we address the proof of (2.2). We discuss first the consequences of the assumption \(\mathbb{P}(J_e = 0) > 0\). When \(J = 0\) along a section of a rectangle \(\mathcal{R}\) the surface tension in that rectangle is \(\tau_{\mathcal{R}}^d = 0\), thus for all \(\varepsilon > 0\), all \(\beta > 0\),

\[
I_n(\varepsilon) \leq -\|n\|_1 \log \mathbb{P}(J_e = 0).
\]

(3.35)

Now we consider the initial configuration defined by \(u_0 = \chi_{B(z_0,1/4)}\) and \(\tau^r(x) = \min(1, \tau^q(n_{x}^{u_0})/2), \forall x \in \partial^*u_0\)

where \(z_0 = (1/2, \ldots, 1/2)\). According to (3.35) there is \(C < +\infty\) not depending on \(\beta\) such that the cost of dilution \(I^r(u_0)\) defined at (1.22) satisfies \(I^r(u_0) \leq C\). We also have \(\mathcal{F}^r(u_0) \leq C'\) where \(\mathcal{F}^r(u_0)\) is the initial surface energy defined at (1.20), and \(C'\) is the perimeter of \(u_0\). So we conclude already that the numerator \(\mathcal{I}^r(u_0) + \lambda \mathcal{F}^r(u_0)\) in the definition (2.1) of \(\lambda_\beta\) is bounded by a constant which does not depend on \(\beta\). Finally we recall that, thanks to the assumption that \(\mathbb{P}(J_e = 0) < 1 - p_c(d)\) and to Proposition 2.13 in [Wou09],

\[
\lim_{\beta \to \infty} \inf_{n \in S^{d-1}} \frac{\tau^q(n)}{\beta} > 0,
\]

(3.36)

and therefore the inequality (3.34) implies that, for some \(c > 0\) and for all \(\beta\) large enough,

\[
K^r(u_0) \geq \sup_{n \in S^{d-1}} \tau^q(n) \times H \left( \frac{1}{2} \inf_{n \in S^{d-1}} \tau^q(n) \right).
\]

The convexity and the lattice symmetries of the Wulff crystal \(\mathcal{W}^q\) imply that the ratio

\[
\frac{\inf_{n \in S^{d-1}} \tau^q(n)}{\sup_{n \in S^{d-1}} \tau^q(n)}
\]

is bounded from below by some constant \(c_d\) not depending on \(\beta\), hence

\[
K^r(u_0) \geq \sup_{n \in S^{d-1}} \tau^q(n) \times H(c_d) \geq C \beta
\]

for large \(\beta\). The claim (2.2) follows.

The proof of Theorem 3.21 requires three Lemmas, that are stated now together with their proof. We introduce the new surface energy

\[
\mathcal{F}^{r,-}(u) = \int_{\partial^* u \setminus \partial^* u_0} \tau^q(n_{x}^{u}) d\mathcal{H}^{d-1} - \int_{\partial^* u_0 \setminus \partial^* u_0} \tau^r d\mathcal{H}^{d-1}, \ \forall u \in BV.
\]

(3.37)
Lemma 3.22. Let \((u_0 = \chi_{U_0}, \tau^r) \in \text{IC}\) and \(u = \chi_U \in \text{BV}\). Then, the profiles \(v = \chi_{U_0 \setminus U}\) and \(w = \chi_{U \setminus U_0}\) satisfy

\[ \mathcal{F}^r(u) - \mathcal{F}^r(u_0) = \mathcal{F}^{r,-}(v) + \mathcal{F}^{r,-}(w). \]  

(3.38)

Proof. We remark first that

\[ \mathcal{F}^r(u) - \mathcal{F}^r(u_0) = \int_{\partial^ru \setminus \partial^r u_0} \tau^q(n_u) d\mathcal{H}^{d-1} - \int_{\partial^ru_0 \setminus \partial^r u} \tau^r d\mathcal{H}^{d-1}. \]  

(3.39)

As stated in Theorem 3.61 in [AFP00], given any \(u = \chi_U \in \text{BV}\) the local density of \(U\) at \(x\) is either 0, 1/2 or 1, for \(\mathcal{H}^{d-1}\)-almost all \(x \in \mathbb{R}^d\), and the set of points at which the local density is 1/2 coincides, up to a \(\mathcal{H}^{d-1}\)-negligible set, to the reduced boundary \(\partial^ru\) (denoted \(\mathcal{F}U\) in [AFP00]). This implies the two equalities

\[ (\partial^ru \setminus \partial^r u_0) = (\partial^r v \setminus \partial^r u_0) \cup (\partial^r w \setminus \partial^r u_0), \]  

(3.40)

\[ (\partial^r u_0 \setminus \partial^r u) = (\partial^r w \cap \partial^r u_0) \cup (\partial^r v \cap \partial^r u_0) \]  

(3.41)

up to \(\mathcal{H}^{d-1}\)-negligible sets, where \(\cup\) stands for the disjoint union (again, up to \(\mathcal{H}^{d-1}\)-negligible sets). Furthermore, the outer normal at \(x \in \partial^r u \setminus \partial^r w\) (resp. \(x \in \partial^r v \setminus \partial^r u_0\)) corresponds (\(\mathcal{H}^{d-1}\)-a.s.) to the outer normal at \(x \in \partial^r u \setminus \partial^r w\) (resp. to the opposite of the former). Thus, in conjunction with \(3.39\), equations \(3.40\) and \(3.41\) imply respectively the \(\tau^q\) and the \(\tau^r\) part of \(3.38\).

\[ \square \]

Lemma 3.23. Let \((u_0, \tau^r) \in \text{IC}\) and \(u = \chi_U \in \text{BV}\). For any \(h > 0\), there is a finite collection of droplets \((v_i = \chi_{V_i})_{i \in I}\) such that

i. The \(V_i\) are disjoint and their union is \(U\),

ii. Each \(V_i\) is included in a box of side-length at most \(h\),

iii. And

\[ \mathcal{F}^{r,-}(u) \geq \sum_{i \in I} \mathcal{F}^{r,-}(v_i) - \|1 - u\|_{L^1} \frac{dr^q(e_1)}{h}. \]  

(3.42)

Proof. We define \(I = \{0, \ldots, \lceil 1/h \rceil\}\) and call

\[ V_i = \{x \in U : 0 \leq x_k - z_k - i_k h \leq h, \ \forall k = 1, \ldots, d\}, \ \forall i \in I \]

where \(z\) is some point of \((0, h)^d\). The first two properties hold trivially. Now we chose \(z\) so that \(3.42\) holds. For every \(k \in \{1, \ldots, d\}\) we have

\[ \int_0^h \mathcal{H}^{d-1}(\{x \in U : (x_k - z_k)/h \in \mathbb{N}\}) dz_k = \mathcal{L}^d(U) \]

and therefore we can chose \(z_k \in (0, h)\) such that \(\mathcal{H}^{d-1}(\{x \in U : (x_k - z_k)/h \in \mathbb{N}\}) \leq \mathcal{L}^d(U)/h = \|1 - u\|_{L^1}/(2h)\). Equation \(3.42\) follows as the new portions of interface in the \(v_i\) that were not present in \(v\) are exactly the \(\{x \in U : (x_k - z_k)/h \in \mathbb{N}\}\) as \(k = 1, \ldots, d\).  

\[ \square \]
Lemma 3.24. Assume that \( u \in \text{BV}, \, n_0 \in S^{d-1} \) and \( \varepsilon > 0 \) satisfy
\[
\tau^r(x) + (\varepsilon + \| n_x^u - n_0 \|_2) \sup_{n \in S^{d-1}} \tau^q(n) \leq \tau^q(n_0) \tag{3.43}
\]
for all \( x \in \partial^* u \cap \partial^* u_0 \setminus \mathcal{N} \), where \( \mathcal{H}^{d-1}(\mathcal{N}) = 0 \). Then
\[
\mathcal{F}^{r,-}(u) \geq c_{d} \varepsilon \sup \tau^q(\| 1 - u \|_{L^1})^{\frac{d-1}{d}} \tag{3.44}
\]
where \( c_d > 0 \) is a constant that depends only on \( d \).

Proof. We let
\[
W^q = \{ z \in \mathbb{R}^d : z \cdot n \leq \tau^q(n), \forall n \in S^{d-1} \}
\]
and \( \tilde{W} = \{ z \in W^q : z \cdot n_x^u \leq -\tau^r(x), \forall x \in \partial^* u \cap \partial^* u_0 \setminus \mathcal{N} \} \).

The surface tension associated to \( \tilde{W} \) is
\[
\tilde{\tau}(n) = \sup_{z \in \tilde{W}} z \cdot n, \quad \forall n \in S^{d-1}
\]
and it satisfies obviously the relations
\[
\tilde{\tau}(n) \leq \tau^q(n), \forall n \in S^{d-1}
\]
\[
\tilde{\tau}(n_x^u) \leq -\tau^r(x), \forall x \in \partial^* u \cap \partial^* u_0 \setminus \mathcal{N}
\]
therefore
\[
\mathcal{F}^{r,-}(u) \geq \int \tilde{\tau}(n_x^u) d\mathcal{H}^{d-1}(x). \tag{3.45}
\]
The isoperimetric inequality for \( \tilde{\tau} \) (see [KP94]) implies in turn that
\[
\mathcal{F}^{r,-}(u) \geq d\mathcal{C}^{d}(\tilde{W})^{1/d} \left( \frac{\| 1 - u \|_{L^1}}{2} \right)^{\frac{d-1}{d}} \tag{3.46}
\]
so the claim will follows from an appropriate lower bound on the volume \( \tilde{W} \). There exists \( z_0 \in W^q \) such that \( z_0 \cdot n_0 = \tau^q(n_0) \). Now we take \( z \in W^q \), close to \(-z_0\), and prove that it lies in \( \tilde{W} \): if \( x \in \partial^* u \cap \partial^* u_0 \setminus \mathcal{N} \), then
\[
z \cdot n_x^u = (z + z_0) \cdot n_x^u - z_0 \cdot (n_x^u - n_0) - z_0 \cdot n_0 \leq \| z + z_0 \|_2 + \sup_{S^{d-1}} \tau^q \| n_x^u - n_0 \|_2 - \tau^q(n_0)
\]
as the Euclidean norm of \( z_0 \) is not larger than \( \tau^q(\| z_0 \|_2) \leq \sup_{S^{d-1}} \tau^q \). According to our assumption (3.43) it follows that
\[
\tilde{W} \supset W^q \cap B \left( -z_0, \varepsilon \sup_{S^{d-1}} \tau^q \right).
\]
The set \( W^q \) is convex and contains all the images of \( z_0 \) by the symmetries of \( \mathbb{Z}^d \), therefore it contains an hypercube which vertices are the images of \( z_0 \) by the above mentioned symmetries. Consequently the volume of \( \tilde{W} \) is at least \( 1/2^d \) of the volume of the ball \( B \left( -z_0, \varepsilon \sup_{S^{d-1}} \tau^q \right) \) and the proof is over. \( \square \)
Proof. (Theorem 3.21). Given $\delta > 0$, the continuity of $\tau^r$ and the smoothness of $\partial u_0$ imply that there exists $h(\delta) > 0$ such that

$$x, y \in \partial^r u_0, \|x - y\|_2 \leq h(\delta) \Rightarrow \left\{ \begin{array}{c} |\tau^r(x) - \tau^r(y)| \leq \delta \\ \|n^u_{x_0} - n^u_{y_0}\|_2 \leq \delta. \end{array} \right.$$ 

As we claimed, $h(\delta)$ depends only on $(u_0, \tau^r)$, and of course on $\delta$. Also, it is clear that one can chose the function $h$ non-decreasing. Furthermore, to a given direction $n \in S^{d-1}$ corresponds $z \in \mathcal{W}$ such that $z \cdot n = \tau^q(n)$ and therefore

$$\tau^q(n') \geq z \cdot n' \geq \tau^q(n) - \|n - n'\|_2 \sup \tau^q.$$ 

It follows that any $u \in \text{BV}$ with diameter not greater than $h(\delta)$ satisfies assumption (3.43) of Lemma 3.24 when one takes

$$\varepsilon = \delta = \frac{\inf_{x \in \partial^r u_0} (\tau^q(n^u_{x_0}) - \tau^r(x))}{6 \max(\sup_{n \in S^{d-1}} \tau^q(n), 1)}.$$ 

Now we consider some phase profile $u \in \text{BV}$, call $v$ and $w$ the profiles given by Lemma 3.22, and then $(v_i)_{i \in I}$ the union of the droplets associated to both $v$ and $w$ by Lemma 3.23. In conjunction with Lemma 3.24, we obtain

$$\mathcal{F}^r(u) - \mathcal{F}^r(u_0) \geq c_d \varepsilon \sup \tau^q \sum_{i \in I} \left( \|1 - v_i\|_{L^1} \right)^{\frac{d-1}{d}}$$

$$- \left( \|1 - v\|_{L^1} + \|1 - w\|_{L^1} \right) \frac{d r^q(e_1)}{h(\delta)}.$$ 

According to the definition of $v, w$ and of the $v_i$ we have

$$\|u - u_0\|_{L^1} = \|1 - v\|_{L^1} + \|1 - w\|_{L^1}$$

$$= \sum_{i \in I} \|1 - v_i\|_{L^1}$$

therefore, the trivial inequality $x^\alpha + y^\alpha \geq (x + y)^\alpha$ for $\alpha \in (0, 1)$ and $x, y > 0$ implies that, for every $u \in \text{BV},$

$$\mathcal{F}^r(u) - \mathcal{F}^r(u_0) \geq c_d \delta \sup \tau^q (\|u - u_0\|_{L^1})^{\frac{d-1}{d}} - \|u - u_0\|_{L^1} \frac{d r^q(e_1)}{h(\delta)}.$$ 

But $x^{1-1/d} - bx$ is maximized for $x = (\frac{d-1}{bd})^d$ and equals $c_d(1/b)^{d-1}$. So, taking $b = \frac{d r^q(e_1)}{c_d \sup \tau^q \delta h(\delta)}$, for any $u \in \text{BV}$ at $L^1$-distance $(\frac{d-1}{bd})^d$ from $u$ (this distance is smaller than the $L^1$ distance between $u_0$ and 1 if we require that $\sup_{\delta > 0} h(\delta)$ be small), we get

$$\mathcal{F}^r(u) - \mathcal{F}^r(u_0) \geq c'_d \delta \sup \tau^q \left( \delta h(\delta) \right)^{d-1}$$

and we have proved (3.34) with $H(6\delta) = c'_d \delta^{d-1}(\delta)$, as $\varepsilon = \delta$. \qed
3.4.2 Continuous evolution and continuous separation

In this paragraph we give an alternative formulation of the surface energy gap \( K_r(u_0) \). We introduce a set of continuous evolution of the phase profile, for which the boundary splits continuously from its original location:

\[
\mathcal{C}(u_0) = \left\{ v : t \in [0, 1] \mapsto v_t \in BV: v_0 = u_0, v_1 \equiv 1, t \mapsto v_t \text{ is continuous for the } L^1 \text{-norm and } t \mapsto 1_{\partial^* v_t \cap \partial^* u_0} \text{ is continuous for the } L^1 \text{-norm associated to the measure } \mathcal{H}^{d-1} \right\}
\]

and then define \( K_{cs}^r(u_0) \), the gap in the free energy associated to the optimal droplet removal in this class of evolutions:

\[
K_{cs}^r(u_0) = \inf_{v \in \mathcal{C}(u_0)} \sup_{t \in [0, 1]} F^r(v_t) - F^r(u_0).
\]

**Theorem 3.25.** For all \((u_0, \tau^r) \in IC\),

\[
K_{cs}^r(u_0) = K^r(u_0).
\]

The inequality \( K_{cs}^r(u_0) \geq K^r(u_0) \) is clear since the set \( \mathcal{C}(u_0) \) is more restrictive that those evolutions considered in the definition \((1.24)\) of \( K^r(u_0) \). We prove the reverse inequality with the help of an interpolation: given an evolution \( v = (v_i)_{i=0 \ldots k} \in \mathcal{C}_\varepsilon(u_0) \) (see \((1.23)\)) we interpolate from \( v \) a continuous evolution \( v' \) with continuous separation from \( \partial^* u_0 \), at the price of a small increase in the maximal cost, negligible as \( \varepsilon \to 0 \) (and uniform over \( v \)):

\[
v' \in \mathcal{C}(u_0); \quad v'_{i/k} = v_i \quad \text{and} \quad \sup_{t \in [0, 1]} F^r(v'_t) \leq \max_{i=0 \ldots k} F^r(v_i) + o(1).
\]

The next lemma is one of the keys to the proof of Theorem 3.25.

**Lemma 3.26.** Let \((u_0, \tau^r) \in IC\) and \( \delta > 0 \). For any Borel set \( \Delta \subset [0, 1]^d \) with volume \( \mathcal{L}^d(\Delta) \leq \delta \), there exists a collection of measurable sets \( U = (U_t)_{t \in \mathbb{R}} \) such that:

i. \( t \mapsto U_t \) is a non-decreasing measurable function with

\[
\lim_{t \to -\infty} U_t = \emptyset \quad \text{and} \quad \lim_{t \to +\infty} U_t = \mathcal{T}
\]

where \( \mathcal{T} \) is the tube \( \mathcal{T} = [0, 1]^d \times \mathbb{R} \).

ii. The function \( t \mapsto U_t - t e_d \) is 1-periodic.

iii. The volume \( t \mapsto \mathcal{L}^d(U_t \cap [0, 1]^d) \) is a continuous function of \( t \)

iv. The area \( t \mapsto \mathcal{H}^{d-1}(U_t \cap \partial^* u_0) \) is a continuous function of \( t \)

v. The portion of the boundary of \( U_t \) that intersects \( \Delta + \mathbb{Z} e_d \) in \( \mathcal{T} \) has a small area:

\[
\sup_{t \in [0, 1]} \mathcal{H}^{d-1}\left(\partial U_t \cap (\Delta + \mathbb{Z} e_d) \cap \mathcal{T}\right) \leq 7\sqrt{\delta}
\]

for \( \delta > 0 \) small enough.
Proof. See Figure 2 for an illustration of the proof. To begin with, we partition $[0, 1]^d$ in horizontal slabs: let $n = \lfloor 1/\sqrt{\delta} \rfloor$ and call

$$A_i = \left\{ x \in [0, 1]^d : \frac{i-1}{n} \leq x \cdot e_d \leq \frac{i}{n} \right\}$$

for $i \in \{1, \ldots, n\}$. Because each slab has a volume at least $\sqrt{\delta}$, for each $i \in \{0, \ldots n-1\}$ there exists $z_i \in (i/n, (i+1)/n)$ such that the density $L^{d-1} (\Delta \cap \{z : z \cdot e_d = z_i\})$ of $\Delta$ at height $z_i$ is not larger than $\sqrt{\delta}$. We extend then the definition of $z_i$ by periodicity, letting

$$z_{i+n} = 1 + z_i, \forall i \in \mathbb{Z}.$$ 

Then, we let $\mathbf{n} = \cos \alpha \mathbf{e}_1 + \sin \alpha \mathbf{e}_d$ for some $\alpha \in [0, \pi/3]$ such that $\partial^* u_0$ has no face orthogonal to $\mathbf{n}$ and define

$$U_t = \left\{ x \in \bar{T} : x \cdot e_d \leq z_{\lfloor nt \rfloor} \text{ or } x \cdot e_d \leq z_{\lfloor nt \rfloor} \text{ and } x \cdot n \leq l_t \right\}, \quad \forall t \in \mathbb{R}$$

where $l$ is the piecewise linear function defined by: $\forall i \in \mathbb{Z}$,

$$l(i/n)^+ = z_i e_d \cdot n$$
$$l((i+1)/n) = (z_{i+1} e_d + e_1) \cdot n$$

and $l$ linear on each interval $(i/n, (i+1)/n]$. The set $U_t$ evolves as follows: between times $i/n$ and $(i+1)/n$, $U_t$ invades the region $\{x \in [0, 1]^d : z_i \leq x \cdot e_d \leq z_{i+1}\}$ by the mean of a front line normal to $\mathbf{n}$, that moves at a constant speed.

It is immediate from the definition that $U_t - te_d$ is 1-periodic and that $U_t$ increases continuously in volume. The $\mathcal{H}^{d-1}$ measure of $U_t \cap \partial^* u_0$ is non-decreasing and the assumption on $\mathbf{n}$ ensures that it increases continuously. We consider at last the portion of the surface of $U_t$ in $\bar{T}$ that might intersect $\Delta + \mathbb{Z} e_d$. We just have to take into account the upper portion of $\partial U$, made of the two planes at height $z_i, z_{i+1}$, and of a portion of plane normal to $\mathbf{n}$. Recall that the $z_i$ have been chosen so that the density of $\Delta + \mathbb{Z} e_d$ at height $z_i$ does not exceed $\sqrt{\delta}$. Similarly, because $\alpha \leq \pi/3$, the piece of plane orthogonal to $\mathbf{n}$ has a surface at most $4/n \leq 5/\sqrt{\delta}$ for $\delta > 0$ small enough. The claim follows.

The second key argument is periodicity: as seen on Figure 2 the interpolation between $v_0$ and $v_1$ has to choose first the region where the cost of $v_1$ is smaller than that of $v_0$ – which means that we have to fix $t_0$ carefully.

Proof. (Theorem 3.25). Let $\delta > 0$. There exists $\varepsilon \in (0, 2\delta]$ and $k \in \mathbb{N}$, together with $v \in C_c(u_0)$, such that

$$\max_{i=0, \ldots, k} F^r(v) \leq K^r(u_0) + \delta.$$ 

Starting from $v$, we construct a continuous evolution $u \in C(u_0)$ that has a maximal cost not much larger than that of $v$. It is enough to do the interpolation between two successive $v_i$, as one can paste together the successive interpolations to deduce the continuous evolution $u$.

Hence we consider $v_0, v_1 \in BV$ and assume that $\|v_0 - v_1\|_{L^1} \leq 2\delta$. We let

$$\Delta = \{ x : v_0(x) \neq v_1(x) \}$$
where

\[ u = \text{cost of the surface} \]

for any \( t \in [0,1] \) it could be that \( G_{t_0,t} \) selects first the region where \( v_1 \) has a larger cost than \( v_0 \), leading to a maximal cost larger than expected. We rule out this possibility with an appropriate choice for \( t_0 \) – see Figure 2 for an illustration of the discussion below.

For \( t_0 \in \mathbb{R} \) and \( t \in [0,1] \) we consider

\[ f(t_0,t) = \mathcal{F}^r_{G_{t_0,t}}(v_1) - \mathcal{F}^r_{G_{t_0,t}}(v_0). \]

Figure 2: The construction of \( U_t \), and the interpolation \( u_t \) between \( v_0 \) and \( v_1 \).
Our aim is to extend $f$ to arbitrary values of $t \in \mathbb{R}^+$. For $k \in \mathbb{Z}$, we denote by $v_i^k$ the translated of $v_i$ by $k e_d$, then for any $t_0 \in \mathbb{R}$ and $t \in [0,1)$ we have

$$f(t_0, t) = \sum_{k \in \mathbb{Z}} \left( F_{U_{t_0+t} \setminus U_{t_0}}(v_1^k) - F_{U_{t_0+t} \setminus U_{t_0}}(v_0^k) \right)$$

from the definition of $G_{t_0,t}$. The latter formula permits to extend $f$ to $\mathbb{R} \times \mathbb{R}^+$ and puts in evidence the existence of a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(t_0, t) = g(t_0 + t) - g(t_0), \quad \forall (t_0, t) \in \mathbb{R} \times \mathbb{R}^+.$$ 

This function is, apart from a linear correction, 1-periodic: for all $t \in \mathbb{R}$,

$$g(t + 1) = g(t) + F^r(v_1) - F^r(v_0),$$

in other words,

$$g(t) = h(t) + t (F^r(v_1) - F^r(v_0))$$

where $h$ is a 1-periodic function. Now we fix $t_0$ such that $h(t_0) \geq \sup_t h(t) - \sqrt{\delta}$, it is immediate that

$$f(t_0, t) = g(t_0 + t) - g(t_0) = h(t_0 + t) - h(t_0) + t (F^r(v_1) - F^r(v_0)) \leq t (F^r(v_1) - F^r(v_0)) + \sqrt{\delta}.$$ 

Reporting into (3.49) we conclude that $u_t$ is a satisfactory interpolation between $v_0$ and $v_1$: provided that $\delta > 0$ is small enough,

$$F^r(u_t) \leq (1-t) F^r(v_0) + t F^r(v_1) + 15 \sqrt{\delta}, \quad \forall t \in [0,1).$$

3.4.3 The gap in surface energy in the isotropic case

In this paragraph we use Theorem 3.25 to compute the gap in surface energy in the two dimensional, isotropic case.

**Lemma 3.27.** Assume $d = 2$, $\tau^q(n) = 1$ for all $n \in S^{d-1}$ and consider

$$u_0 = \chi_B \quad \text{and} \quad \tau^r(x) = \lambda, \forall x \in \partial^*u_0$$

where $B$ is the disk of radius $r < 1/2$ centered at $(1/2, 1/2)$, and $\lambda \in (0,1)$. Then

$$\mathcal{K}^r(u_0) = 2r \left[ \sqrt{1 - \lambda^2} - \lambda \cos \lambda \right].$$ 

An optimal continuous evolution in this setting is illustrated on Figure 3.
Proof. In order to simplify the notation we will consider \( r = 1/2 \). The upper bound
\[
K^r(u_0) \leq \sup_{\theta \in [0, \pi]} (\sin \theta - \lambda \theta)
\]
is immediate if one considers the continuous evolution \((u_t)_{t \in [0, 1]}\) defined by
\[
u_t(x) = \begin{cases} 
-1 & \text{if } x \in B \text{ and } x \cdot e_2 \leq 1 - t \\
1 & \text{else}
\end{cases}
\]
and \( \theta \) satisfying \( 1 - t = 1/2 + \sin \theta \), as illustrated on Figure 3. The lower bound is scarcely more difficult to establish: given \((u_t)_{t \in [0, 1]} \in \mathcal{E}(u_0)\) a continuous evolution with continuous detachment, there is \( t \in (0, 1) \) such that
\[
\mathcal{H}^1(\partial^* u_0 \setminus \partial^* u_t) = \acos \lambda.
\]

Figure 4: Reduction of \( u \) to a portion of disk in three steps. The surface energy decreases, the length of contact is preserved.

Optimizing the droplets of \( u_t \) as in Figure 4 – we replace each portion of the interface not in \( \partial^* u_0 \) with a segment (step (1)) – we obtain a profile \( u'_t \) with a lower cost, yet it still has the
same contact length $\cos \lambda$ with $\partial^* u_0$. By isotropy of surface tension it is possible to aggregate the droplets together (step (2)) and obtain $u''$ with a unique droplet and a lower cost, preserving again the length of contact. At last, inverting the order of the segments and arcs and optimizing again we see that the profile of lower cost that satisfies $\mathcal{H}^1(\partial^* u_0 \setminus \partial^* u) = \cos \lambda$ coincides, apart from a rotation, with the profiles considered in the upper bound. The claim follows.

### 3.4.4 The gap in surface energy when the Wulff crystal is a square

Here we compute the gap in surface energy for another simple case, when the Wulff crystal $W^q$ is a square. We also show that the gap in surface energy is strictly bigger than the cost of the less likely overall magnetization.

**Lemma 3.28.** Assume $d = 2$, $\tau^q(n) = \|n\|_1$ for all $n \in S^{d-1}$ and $u_0 = \chi_C$ and $\tau^r(x) = \lambda, \forall x \in \partial^* u_0$ where $C = [1/2 - r, 1/2 + r]^2$, $r < 1/2$ and $\lambda \in [0, 1]$. Then, we have

$$ K^r(u_0) = 2r [1 - \lambda]. $$

**Proof.** Again we consider $r = 1/2$ in order to simplify the notations. The upper bound on the additional cost is immediate considering $u_t = \chi_{\{x \cdot e_2 \leq 1-t\}}$. For the lower bound we need a finer analysis. First, it is a consequence of the assumption on $\tau^q$ that for any open, connected $U \subset \mathbb{R}^2$ with extension $h^1, h^2$ in the canonical directions, that is:

$$ h(k) = \sup_{x \in U} x \cdot e_k - \inf_{x \in U} x \cdot e_k, $$

we have

$$ F^q(\chi_U) \geq 2h^1 + 2h^2. $$

Then, we decompose a profile configuration $u$ into its droplets $(U_i)_{i \geq 0}$. We call $h^1_i, h^2_i$ the extension of $U_i$ in the canonical directions and let $l_i$ the length of contact between $\partial^* \chi_{U_i}$ and $\partial^* u_0$, so that, for all $i$:

$$ F^r(\chi_{U_i}) \geq 2h^1_i + 2h^2_i - (1 - \lambda)l_i $$

If a droplet $U_i$ touches two opposite faces of $\partial^* u_0$, say $h^1 = 1$, then its extension in the orthogonal direction is at least $h^2 \geq (l_i - 1)/2$ and the inequality

$$ F^r(\chi_{U_i}) \geq 1 + \lambda l_i $$

follows. If on the opposite the droplet is in contact with at most two adjacent sides of $\partial^* u_0$, we have $h^1 + h^2 \geq l_i$ and hence

$$ F^r(\chi_{U_i}) \geq (1 + \lambda)l_i. $$

Assume now that the total length of contact is 3, i.e. that $\sum_i l_i = 3$. A consequence of the former lower bounds is that, whether or not some droplet touches two opposite faces, the cost of $u$ is at least $F^r(u) \geq 1 + 3\lambda$. The claim follows. \qed
We conclude this work with a comparison between the bottleneck due to the positivity of $\mathcal{K}(u_0)$ and the one due to the continuous evolution of the magnetization:

**Lemma 3.29.** In the settings of the former lemma, with furthermore $\lambda \in (0,1)$, we have

$$\mathcal{K}(u_0) > \sup_{m \in [-1,1]} \inf_{u \in BV : \int_{[0,1]^d} u = m} \mathcal{F}(u) - \mathcal{F}(u_0).$$

![Diagram](image.png)

Figure 5: The two profiles $u^1_m$ and $u^2_m$: for $m \simeq 1$ the first one is better, for $m \simeq -1$ the second one has a smaller cost.

**Proof.** We provide an upper bound for the right hand term, considering for a given $m$ the two profiles (see Figure 5)

$$u^1_m = \begin{cases} -1 & \text{if } x \in [0,1]^d \text{ and } x \cdot e_2 \leq \frac{1-m}{2} \\ 1 & \text{else} \end{cases}$$

and

$$u^2_m = \begin{cases} -1 & \text{if } x \in [0,1]^d \text{ and } \min(x \cdot e_1, x \cdot e_2) \leq 1 - \sqrt{\frac{1+m}{2}} \\ 1 & \text{else} \end{cases}$$

that both satisfy the volume constraint $\int_{[0,1]^d} u = m$. It is immediate that

$$\mathcal{F}(u^1_m) = 1 + (2 - m)\lambda \quad \text{and} \quad \mathcal{F}(u^2_m) = 4\lambda + 2(1 - \lambda)\sqrt{\frac{1+m}{2}},$$

hence

$$\sup_{m \in [-1,1]} \inf_{u \in BV : \int_{[0,1]^d} u = m} \mathcal{F}(u) \leq \sup_{m \in [-1,1]} \min(\mathcal{F}(u^1_m), \mathcal{F}(u^2_m)).$$

Note that $\mathcal{F}(u^1_m)$ decreases with $m$ while $\mathcal{F}(u^2_m)$ increases with $m$. Because of their extremal values there exists some $m_0 \in (0,1)$ at which $\mathcal{F}(u^1_{m_0}) = \mathcal{F}(u^2_{m_0}) = \sup_{m \in [-1,1]} \min(\mathcal{F}(u^1_m), \mathcal{F}(u^2_m))$, and since $m_0 < 1$ we have in particular $\mathcal{F}(u^1_{m_0}) < \mathcal{F}(u^2_{1}) = 1 + 3\lambda$ which is the maximal cost of an optimal continuous detachment evolution.

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References

[ACCN87] M. Aizenman, J. T. Chayes, L. Chayes, and C. M. Newman. The phase boundary in dilute and random Ising and Potts ferromagnets. *J. Phys. A*, 20(5):L313–L318, 1987.

[ACCN88] M. Aizenman, J. T. Chayes, L. Chayes, and C. M. Newman. Discontinuity of the magnetization in one-dimensional $1/|x-y|^2$ Ising and Potts models. *J. Statist. Phys.*, 50(1-2):1–40, 1988.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.

[BGW12] T. Bodineau, B. Graham, and M. Wouts. Metastability in the dilute Ising model. *Probab. Theory Related Fields*, to appear.

[BI04] T. Bodineau and D. Ioffe. Stability of interfaces and stochastic dynamics in the regime of partial wetting. *Ann. Henri Poincaré*, 5:871–914, 2004.

[BM02] T. Bodineau and F. Martinelli. Some new results on the kinetic Ising model in a pure phase. *J. Statist. Phys.*, 109(1-2):207–235, 2002.

[Cer06] R. Cerf. *The Wulff crystal in Ising and percolation models*, volume 1878 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2006.

[CMST10] P. Caputo, F. Martinelli, F. Simenhaus, and F. L. Toninelli. “Zero” temperature stochastic 3D Ising model and dimer covering fluctuations: a first step towards interface mean curvature motion. [http://arxiv.org/abs/1007.3599](http://arxiv.org/abs/1007.3599) 2010.

[ES88] R. G. Edwards and A. D. Sokal. Generalization of the Fortuin-Kasteleyn-Swendsen-Wang representation and Monte Carlo algorithm. *Phys. Rev. D (3)*, 38(6):2009–2012, 1988.

[Fed69] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag, New York, 1969.

[HF87] D. A. Huse and D. S. Fisher. Dynamics of droplet fluctuations in pure and random Ising systems. *Phys. Rev. B*, 35(13):6841–6846, 1987.

[KP94] R. Kotecký and C.-E. Pfister. Equilibrium shapes of crystals attached to walls. *J. Statist. Phys.*, 76(1-2):419–445, 1994.

[LMST10] E. Lubetzky, F. Martinelli, Allan Sly, and Fabio Lucio Toninelli. Quasi-polynomial mixing of the 2D stochastic Ising model with “plus” boundary up to criticality. [http://arxiv.org/abs/1012.1271](http://arxiv.org/abs/1012.1271) 2010.

[LS10] E. Lubetzky and A. Sly. Critical ising on the square lattice mixes in polynomial time. [http://arxiv.org/abs/1001.1613](http://arxiv.org/abs/1001.1613) 2010.
[Mar99] F. Martinelli. Lectures on Glauber dynamics for discrete spin models. In Lectures on probability theory and statistics (Saint-Flour, 1997), volume 1717 of Lecture Notes in Math., pages 93–191. Springer, Berlin, 1999.

[MT10] F. Martinelli and F. L. Toninelli. On the mixing time of the 2D stochastic Ising model with “plus” boundary conditions at low temperature. Comm. Math. Phys., 296(1):175–213, 2010.

[Sch94] R. H. Schonmann. Slow droplet-driven relaxation of stochastic Ising models in the vicinity of the phase coexistence region. Comm. Math. Phys., 161(1):1–49, 1994.

[SS98] R. H. Schonmann and S. B. Shlosman. Wulff droplets and the metastable relaxation of kinetic Ising models. Comm. Math. Phys., 194(2):389–462, 1998.

[Tho89] L. Thomas. Bounds on the mass gap for the finite volume stochastic Ising models at low temperatures. Comm. Math. Phys., 126(1):1–11, 1989.

[Wou07] M. Wouts. The dilute Ising model: phase coexistence at equilibrium & dynamics in the region of phase transition. PhD thesis, Université Paris 7 - Paris Diderot, http://tel.archives-ouvertes.fr/tel-00272899, 2007.

[Wou08] M. Wouts. A coarse graining for the Fortuin-Kasteleyn measure in random media. Stochastic Process. Appl., 118(11):1929–1972, 2008.

[Wou09] M. Wouts. Surface tension in the dilute Ising model. The Wulff construction. Comm. Math. Phys., 289(1):157–204, 2009.