Line bundles of type \((1, \ldots, 1, 2, \ldots, 2, 4, \ldots, 4)\) on Abelian Varieties

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Abstract

We show birationality of the morphism associated to line bundles \(L\) of type 
\((1, \ldots, 1, 2, \ldots, 2, 4, \ldots, 4)\) on a generic \(g\)-dimensional abelian variety into its complete linear system such that \(h^0(L) = 2^g\). When \(g = 3\), we describe the image of the abelian threefold and from the geometry of the moduli space \(SU_C(2)\) in the linear system \(|2\theta_C|\), we obtain analogous results in \(IPH^0(L)\).

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1 Introduction

Let \(L\) be an ample line bundle of type \(\delta = (\delta_1, \delta_2, \ldots, \delta_g)\) on a \(g\)-dimensional abelian variety \(A\). Consider the associated rational map \(\phi_L : A \to IPH^0(A, L)\).

When \(g = 2\), Birkenhake, Lange and van Straten (see [3]) have studied line bundles of type \((1, 4)\) on abelian surfaces. Suppose \(L\) is an ample line bundle of type \((1, 4)\) on an abelian surface \(A\). Then there is a cyclic covering \(\pi : A \to B\) of degree 4 and a line bundle \(M\) on \(B\) such that \(\pi^*M = L\). Let \(X\) denote the unique divisor in \(|M|\) and put \(Y = \pi^{-1}(X)\). Their main theorem is
Theorem 1.1 1) $\phi_L : A \to A' \subset \mathbb{P}^3$ is birational onto a singular octic $A'$ in $\mathbb{P}^3$ if and only if $X$ and $Y$ do not admit elliptic involutions compatible with the action of the Galois group of $\pi$.

2) In the exceptional case $\phi_L : A \to A' \subset \mathbb{P}^3$ is a double covering of a singular quartic $A'$, which is birational to an elliptic scroll.

Here we generalise this situation to higher dimensions and show

Theorem 1.2 Suppose $L$ is an ample line bundle of type $\delta = (1, ..., 1, 2, ..., 2, 4, ..., 4)$ on a $g$-dimensional abelian variety $A$, $g \geq 3$, such that 1 and 4 occur equally often and at least once in $\delta$. Then, for a generic pair $(A, L)$, the following holds.

a) The associated morphism $\phi_L : A \to \mathbb{P}^H(0, L)$ is birational onto its image.

b) When $g = 3$, the image $\phi_L(A)$, can be described as follows, there are 4 curves $C_i$ on the image $\phi_L(A)$ such that the restricted morphism $\phi_L : \phi_L^{-1}(C_i) \to C_i \subset \phi_L(A)$ is of degree 2.

Birkenhake et al (see [3], Proposition 1.7, p.631) have shown the existence of the following commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\phi_L} & \phi_L(A) \subset \mathbb{P}^3 = \mathbb{P}^H(0, L) \\
\downarrow \pi & & \downarrow \pi \\
B & \xrightarrow{\phi_M^2} & K(B) \subset \mathbb{P}^3 = \mathbb{P}^H(0, M^2) \\
\end{array}
\]

where $p(z_0 : z_1 : z_2 : z_3) = (z_0^2 : z_1^2 : z_2^2 : z_3^2)$ and the pair $(B, M)$ is a principally polarized abelian surface. This diagram explains the geometry of the image $\phi_L(A)$ from the geometry of the Kummer surface $K(B)$ and it also gives the explicit equation of the surface $\phi_L(A)$ in $\mathbb{P}^3$.

Similarly, when $g \geq 3$ and the pair $(A, L)$ as in 1.2, we show that there is a commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\phi_L} & \phi_L(A) \subset \mathbb{P}^{2^g-1} = \mathbb{P}^H(0, L) \\
\downarrow \pi & & \downarrow \pi \\
B & \xrightarrow{\phi_M^2} & K(B) \subset \mathbb{P}^{2^g-1} = \mathbb{P}^H(0, M^2) \\
\end{array}
\]

where $p(z_0 : ... : z_{2^g-1}) = (z_0^2 : ... : z_{2^g-1}^2)$ and $\pi$ is an isogeny of degree $2^g$ and the pair $(B, M)$ is a principally polarized abelian variety. This will explain the birationality of the map $\phi_L$ and the geometry of the image $\phi_L(A)$, when $g = 3$, as asserted in 1.2.

Since $\text{deg}(\phi_M^2 \circ \pi) = 2^{g+1}$ and from the birationality of $\phi_L$, it follows that $\text{deg}(p|_{\phi_L(A)}) = 2^{g+1}$. But since $\text{deg}p = 2^{2g-1}$ the inverse image of the Kummer variety in $\mathbb{P}^H(0, L)$ has
components other than the image $\phi_L(A)$. Hence the image $\phi_L(A)$ will be defined by forms other than those coming from those forms which define the variety $K(B)$.

We study the situation when $g = 3$, in detail. Consider a pair $(A, L)$, with $L$ being an ample line bundle of type $(1, 2, 4)$ on an abelian threefold $A$. Consider an isogeny $A \rightarrow B = A/G$, where $G$ is a maximal isotropic subgroup of $K(L)$ of the type $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $B$ is a principally polarized abelian threefold. If $B$ is isomorphic to the Jacobian variety of $C$, $J(C)$, where $C$ is a smooth non-hyperelliptic curve of genus 3, then the situation becomes interesting because of the following results due to Narasimhan and Ramanan.

**Theorem 1.3** (See [12], Main Theorem, p.416) If $C$ is a non-hyperelliptic curve of genus 3, then the moduli space $SU_C(2)$ is isomorphic to a quartic hypersurface in $\mathbb{P}^7$.

(Here $\mathbb{P}^7 = |2\theta|$, where $\theta$ is the canonical principal polarization on the Jacobian $J(C)$ and $SU_C(2)$ is the moduli space of rank 2 semi-stable vector bundles with trivial determinant on the curve $C$).

**Theorem 1.4** (See [11]) The Kummer variety $K$ is precisely the singular locus of $SU_C(2)$, if $g(C) \geq 3$.

The quartic hypersurface, $F = 0$, is classically called the *Coble quartic* and is $G(2\theta)$-invariant in the linear system $|2\theta|$. We identify the group of projective transformations, $H$, of order 8, which acts on $\pi^{-1}K(C)$, (see 3.7). The $G(L)$-invariant octic hypersurface $R$, given as $F(z_0^2 : ... : z_7^2) = 0$ in $\mathbb{P}H^0(L)$, then contains the components $h(\phi_L(A)), h \in H$ in its singular locus.

Now we use the geometry of the moduli space $SU_C(2)$ in the linear system $|2\theta|$, which has been extensively studied (see [5], for instance), to get analogous results in $\mathbb{P}H^0(L)$.

We show

**Theorem 1.5** Consider a pair $(A, L)$, as above. Let $a \in K(L)$ be an element of order 2 such that $e^L(a, g) = -1$, for all $g \in G$, (here $e^L$ is the Weil form on the group $K(L)$). Let $\mathbb{P}W_a$ be an eigenspace in $\mathbb{P}H^0(L)$, for the action of $a$. Then there is a polarized abelian surface $(Z, N)$, $N$ is ample of type $(1, 4)$ and a commutative diagram

\[
\begin{array}{cccccc}
Z & \xrightarrow{\phi_N} & \phi_N(Z) & \subset & \mathbb{P}H^0(N) & \simeq \mathbb{P}W_a \\
\downarrow f & & \downarrow q & & \downarrow p \\
P_a & \xrightarrow{\phi_{2\theta_a}} & K(P_a) & \subset & \mathbb{P}H^0(2\theta_a) & \simeq \mathbb{P}V_a
\end{array}
\]
Here \((P_a, \theta_a)\) is the Prym variety associated to the 2-sheeted unramified cover of the curve \(C\), given by \(\pi(a)\) and \(IPV_a\) is the eigenspace in \(IPH^0(2\theta)\), for the action of \(\pi(a)\). The isomorphisms above are Heisenberg equivariant and the morphism \(q\) is given as \((r_0 : r_1 : r_2 : r_3) \mapsto (r_0^2 : r_1^2 : r_2^2 : r_3^2)\).

We thus obtain the situation described by Birkenhake et.al in the case \(g = 2\), nested in the case \(g = 3\).

Moreover, the \(G(N)\)-invariant octic surface \(\phi_N(Z)\) is mapped isomorphically onto the \(a^\perp/a(\simeq \text{Heis}(4))\)-octic \(R \cap IPW_a\) and we identify the set \(\cap h \in H h(\phi_L(A))\) with the set of all pinch points and the coordinate points in \(\phi_N(Z)\), occurring in each of the eigenspace \(IPW_a\), (see 5.6). Finally, we make some remarks on the moduli space \(A^{(1,2,4)}\).

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**Notation**: Suppose \(L\) is a symmetric line bundle i.e. \(L \simeq i^*L\) for the involution \(i: A \rightarrow A, a \mapsto -a\).

- The fixed group of \(L\) is \(K(L) = \{a \in A : L \simeq t_a^*L\}\), \(t_a : A \rightarrow A, x \mapsto a + x\).
- The theta group of \(L\) is \(G(L) = \{(a, \phi) : L \simeq t_a^*L\}\).
- \(K_1(\delta) = \frac{\mathbb{Z}}{d_1\mathbb{Z}} \times \ldots \times \frac{\mathbb{Z}}{d_g\mathbb{Z}}\), and \(\hat{K}_1(\delta) = \text{Hom}(K_1(\delta), \mathbb{C}^*)\).
- The Heisenberg group of type \(\delta\), \(\text{Heis}(\delta) = \mathbb{C}^* \times K_1(\delta) \times \hat{K}_1(\delta)\) and \(V(\delta) = \{f : f : K_1(\delta) \rightarrow \mathbb{C}\}\).

The Weil form \(e^\nu : K(L) \times K(L) \rightarrow \mathbb{C}^*\), is the commutator map \((x, y) \mapsto x'y'x'^{-1}y'^{-1}\), for any lifts \(x', y' \in G(L)\) of \(x, y \in K(L)\).

For any \(a \in K(L)\), \(a^\perp = \{x \in K(L) : e^L(a, x) = 1\}\).

Consider the semi-direct product, \(G(L) \ltimes (i)\), of the theta group associated to \(L\) and the group generated by the involution \(i\). Let \(\gamma \in G(L) \ltimes (i)\) be an element of order 2.

- \(H^0(L)^\pm_\gamma = (\pm 1)\)-eigenspace of \(H^0(L)\) for the action of \(\gamma\).
- \(h^0(L)^\pm_\gamma = \dim H^0(L)^\pm_\gamma\).
- \(Q(V) = \text{function field of a variety } V\).
2 Birationality of the map $\phi_L$.

Let $L$ be an ample line bundle of type $\delta = (1,..2,..,4)$ on a $g$-dimensional abelian variety $A$. Here number of $2$'s= number of $4$'s in $\delta$. Let $K(L)=\{a \in A : t^*_aL \simeq L\}$, where $t_a$ denotes translation by $a$ on $A$. Choose a maximal isotropic subgroup $G$ of $K_L$ w.r.t. the Weil form $e^L$, containing $2K(L)$ and having only elements of order $2$. Then $G \simeq \mathbb{Z}/2\mathbb{Z} \times ... \times \mathbb{Z}/2\mathbb{Z}$, $g$-times. Consider the exact sequence

$$1 \longrightarrow \mathcal{O}^* \longrightarrow G(L) \longrightarrow K(L) \longrightarrow 0.$$ 

Let $G'$ be a lift of $G$ in $G(L)$. Consider the isogeny $A \xrightarrow{\pi} B = A/G$. Then $L$ descends to a principal polarization $M$ on $B$. By Projection formula and using the fact that $\pi_*\mathcal{O}_A = \bigoplus_{\chi \in \hat{G}} L_\chi$, where $L_\chi$ denotes the line bundle corresponding to the character $\chi$, we deduce that

$$H^0(L) = \bigoplus_{\chi \in \hat{G}} H^0(M \otimes L_\chi).$$

Hence $\{s_\chi \in H^0(M \otimes L_\chi) : \chi \in \hat{G}\}$ is a basis for the vector space $H^0(L)$ and since $M^2 \otimes L_\chi^2 \simeq M^2$, $s_\chi^2 = s_\chi \otimes s_\chi \in H^0(M^2)\forall \chi \in \hat{G}$.

Consider the homomorphism $\epsilon_2 : G(L) \longrightarrow G(L^2), (x, \phi) \mapsto (x, \phi \otimes 2)$ and the inclusion $K(L) \subset K(L^2)$.

Then the subgroup $G \subset K(L^2)$ is isotropic for the Weil form $e^{L^2}$. Moreover, if $x \in K(L)$ and $g \in G$, then

$$e^{L^2}(x, g) = e^L(x, g).e^L(x, g) = 1.$$ 

Hence $\epsilon_2(G(L)) \subset \mathcal{Z}(\epsilon_2(G'))$ and $\pi(K(L)) \subset K(M^2)$. (Here $\mathcal{Z}(\epsilon_2(G')) = \{a \in G(L^2) : a.g' = g'.a, \forall g' \in \epsilon_2(G')\}).$

Now $G(M^2) = \mathcal{Z}(\epsilon_2(G'))/\epsilon_2(G')$ and $H^0(M^2) = H^0(L^2)G'$, where $H^0(L^2)G'$ denotes the vector subspace of $\epsilon_2(G')$-fixed sections of $H^0(L^2)$. For $g' \in G'$ and $\chi \in \hat{G}$, $g'(s_\chi^2) = \chi^2(g).s_\chi^2 = s_\chi^2$. Hence $s_\chi^2 \in H^0(L^2)\epsilon_2^G$, for all $\chi \in \hat{G}$.

We now show that $\{s_\chi^2 : \chi \in \hat{G}\}$ is a basis for $H^0(M^2)$, for a generic pair $(A, L)$.

In fact, we show that the homomorphism

$$\sum_{\chi \in \hat{G}} H^0(M \otimes L_\chi).H^0(M \otimes L_\chi) \xrightarrow{\rho} H^0(M^2)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(*)$$

is an isomorphism, for a generic pair $(A, L)$.
Consider the pair \((A, L) = (E_1 \times \ldots \times E_r, A_1 \times \ldots A_s, p^*_1 L_1 \otimes \ldots \otimes p^*_{r+s} L_{r+s})\), where \(r\) is the number of 2’s occurring in \(\delta\), \(E_1, \ldots, E_r\) are elliptic curves with line bundles \(L_i\) on \(E_i\) of degree 2 and \(A_j\) are simple abelian surfaces with line bundles \(L_j\) on \(A_j\) of type \((1,4)\) (by 1.1, \(\phi_{L_j}(A_j) \subset |L_j|\) is an octic surface).

In this case, one can easily see that the homomorphism

\[
S = \text{Sym}^2 H^0(L_1) \otimes \ldots \otimes \text{Sym}^2 H^0(L_{r+s}) \to H^0(L_1^2) \otimes \ldots \otimes H^0(L_{r+s}^2) = H^0(L_1^2 \otimes \ldots \otimes L_{r+s}^2)
\]

is injective. Here, \((B, M) = (F_1, M_1) \times \ldots \times (F_r, M_r) \times (B_1, M_1') \times \ldots (B_s, M_s')\), where \((F_j, M_j)\) are polarised elliptic curves of degree 1 and \((B_j, M_j)\) are principally polarised abelian surfaces. Also, the group \(G_H\) is generated by elements of the type \((e_1, \ldots, e_r, a_{r+1}', \ldots, a_g')\), where each of \(e_j\) and \(a_j'\) are non-trivial 2 torsion elements of \(E_j\) and \(A_j\), respectively.

Now it is easy to see that \(\sum_{\chi \in \mathcal{G}} H^0(M \otimes L_\chi) \cdot H^0(M \otimes L_\chi) \subset S\) and \(H^0(M^2) \subset H^0(L^2)\) and (\(\ast\)) is an isomorphism.

Hence, for a generic pair \((A, L)\) as above, (\(\ast\)) is an isomorphism.

As a consequence, we obtain the following

**Proposition 2.1** Consider a generic principally polarized abelian variety \((B', M')\) of dimension \(g\). Let \(H\) be a subgroup of 2-torsion points of \(B'\), of order \(g\). Then the image of \(H\) in \(\mathcal{K}(B')\) generates the linear system \(|2M'|\).

(This is well known if \(H\) consists of all the 2-torsion points of \(B'\), for any principally polarised pair \((B', M')\).)

**Proof:** Since the map \(B' \overset{\phi_{2M'}}{\to} |2M'|\) is given by \(a \mapsto t^*_a \theta + t^*_a \theta\), where \(\theta\) is the unique divisor in \(|M'|\), the assertion is equivalent to showing the surjectivity of the multiplication map

\[
\sum_{\chi \in \hat{H}} H^0(M' \otimes L_\chi) \otimes H^0(M' \otimes L_\chi) \to H^0(M'^2)\, !(\).
\]

Here \(\hat{H}\) is the dual image of \(H\) in \(\text{Pic}^0(B')\). But we showed above this isomorphism, if \(\hat{H}\) gives rise to a \(g\)-sheeted cover \((A', L')\) of \((B', M')\), where \(L'\) is of type \((1, \ldots, 2, \ldots, 4)\). Otherwise, \(\hat{H}\) gives a cover \((A', L')\) where \(L'\) is of type \((2, 2, \ldots, 2)\). By similar argument used in proving (\(\ast\)), (\(!\)) is still true when \(A' = E_1 \times \ldots \times E_g\) and \(L' = L_1 \times L_2 \ldots \times L_g\), where \(L_j\) are line bundles of degree 2 on the elliptic curves \(E_j\). Hence our assertion is true for a generic pair \((B', M')\). \(\square\)
So, for a generic pair \((A, L)\), the map \(\mathbb{P}H^0(L) \to \mathbb{P}H^0(M^2)\), given as \((..., s_x, ...) \mapsto \ldots, s_x^2, ...)\) is a morphism and we obtain a commutative diagram (I),

\[
\begin{array}{ccc}
A & \xrightarrow{\phi_L} & \phi_L(A) \\
\downarrow\pi & & \downarrow p \\
B = A/G & \xrightarrow{\phi_M^2} & K(B) \\
\end{array}
\]

where \(p(\ldots, s_x, ...) = (\ldots, s_x^2, ...)\).

**Remark 2.2** Since \(\phi_M^2 \circ \pi\) is a morphism, \(\phi_L\) is a morphism i.e. \(L\) is base point free.

**Lemma 2.3** Consider a pair \((A, L)\) as in 1.2. Let \(\gamma \in \mathcal{G}(L)\) be an element of order 2. Then \(H^0(L) \neq H^0(L)^\pm\).

**Proof:** Case 1: Suppose \(\gamma = g \in \mathcal{G}(L)\). Then the action of \(\gamma\) is fixed point free on \(A\). Hence by Atiyah- Bott fixed point theorem,

\[
h^0(L)^\pm = h^0(L)^{-} = h^0(L)/2.
\]

Case 2: Suppose \(\gamma = i\). Then

\[
h^0(L)^\pm_i = h^0(L)/2 \pm 2^{g-1}
\]

(see [1], 4.6.6), where \(s\) is the number of odd integers occurring in the type of \(L\).

Case 3: Suppose \(\gamma = i.g\) and \(H^0(L) = H^0(L)^+\), where \(g \in \mathcal{G}(L)\) is an element of order 2. Let \(s \in H^0(L)^-\). Then \(\gamma(s) = s\) gives \(i(s) = -s\), i.e. \(s \in H^0(L)^-\). Hence \(H^0(L)^- \subset H^0(L)^-\). But this contradicts the fact that \(h^0(L)^- = 2^{g-1}\) and \(h^0(L)^- = 2^{g-1} - 2^{g-s-1}\) (here \(s > 1\)). Similarly \(H^0(L) \neq H^0(L)^-\). \(\square\)

Suppose \(\phi_L\) is not birational and is a finite morphism of degree \(d\), \(d > 1\). Notice that \(A \xrightarrow{\phi_M \circ \pi} K(B)\) is a Galois covering with Galois group \((G, i) \simeq (\mathbb{Z}/2\mathbb{Z})^{g+1}\) and we have the extension of fields, \(Q(K(B)) \to Q(\phi_L(A)) \to Q(A)\). Hence the Galois group of \(Q(A)\) over \(Q(\phi_L(A))\) is a subgroup of \((G, i)\), say \(H\), of order \(d\). Let \(\gamma \in H\). Then \(\gamma\) is an involution on \(A\), given as \(a \mapsto \epsilon a + g\) where \(\epsilon = \pm 1\), \(g \in G\) and it induces an involution \(\gamma'\) on \(H^0(L)\).

Hence \(\phi_L\) factorizes as \(A \xrightarrow{\psi_1} A/\langle \gamma \rangle \xrightarrow{\psi_2} \phi_L(A) \subset \mathbb{P}^{2g-1}\). This means that the morphism \(\psi_2\) is given by the pair \((N, H^0(L)^+)\) or \((N', H^0(L)^-)\), where \(N\) and \(N'\) are line bundles on \(A/\langle \gamma \rangle\) whose pullback to \(A\) is \(L\). By 2.3, \(H^0(L) \neq H^0(L)^+\) and hence \(\phi_L(A)\) is a degenerate variety in \(\mathbb{P}^{2g-1}\). This contradicts the fact that the morphism \(\phi_L\) is given by a complete linear system. Hence \(\phi_L\) is a birational morphism.
3 Configuration when \( g = 3 \)

Assume \( g = 3 \). Choose a theta structure \( f : \mathcal{G}(L) \rightarrow \text{Heis}(2, 4) \), (i.e. \( f \) is an isomorphism which restricts to identity on \( \mathcal{A}^* \)). This induces an isomorphism \( H^0(L) \simeq V(2, 4) \) and a level structure \( K(L) \simeq \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \). Let \( \sigma_1, \tau_1, \sigma_2, \tau_2 \) be the generators of the summands such that \( o(\sigma_i) = 2 \) and \( o(\tau_i) = 4 \). The Weil form \( e^L \) is given as

\[
\begin{align*}
  e^L(\sigma_1, \sigma_2) &= -1 \\
  e^L(\tau_1, \tau_2) &= -i \\
  e^L(\sigma_i, \tau_j) &= 1
\end{align*}
\]

Then we see that the subgroup \( G = \langle \sigma_1, \tau_1^2, \tau_2^2 \rangle \) of \( K(L) \) is maximal isotropic for the form \( e^L \).

We may assume \( L \) is strongly symmetric (see [10], Remark 2.4., p.160), i.e., \( e^L_*(g) = 1 \) for all \( g \in K(L)_2 \), after choosing a normalized isomorphism \( \psi : L \simeq i^*(L) \), i.e. \( \psi(0) = +1 \). Here \( e^L_* : A_2 \rightarrow \{ \pm 1 \} \) is a quadratic form whose value at an element \( a \), of order 2 is the action of \( \psi \) at the fibre of \( L \) at \( a \).

Consider the exact sequence

\[
1 \rightarrow \mathcal{A}^* \rightarrow \mathcal{G}(L) \rightarrow K(L) \rightarrow 0
\]

and the homomorphism \( \delta_{-1} : \mathcal{G}(L) \rightarrow \mathcal{G}(L) \), \( z \mapsto izi \). Then \( \delta_{-1}(z) = az^{-1} \) for some \( a \in \mathcal{A}^* \).

By [6], Proposition 2.3, p.141, we further assume that \( f \) is a symmetric theta structure, i.e. \( f \circ \delta_{-1} = D_{-1} \circ f \), where \( D_{-1} : \text{Heis}(\delta) \rightarrow \text{Heis}(\delta) \) is the homomorphism \( (\alpha, x, l) \mapsto (\alpha, -x, -l) \).

**Lemma 3.1** If \( z \in \mathcal{G}(L) \) is an element of order 2 and \( z \neq \pm 1 \) then \( \delta_{-1}(z) = e^L_*(z)z \).

**Proof:** See [8], Proposition 3, p.309. \( \square \)

**Remark 3.2** Let \( \sigma'_1, \sigma'_2, \tau'_1, \tau'_2 \in \mathcal{G}(L) \) be lifts of \( \sigma_1, \sigma_2, \tau_1, \tau_2 \) such that \( o(\sigma'_i) = 2, o(\tau'_i) = 4 \). Since \( \tau'_i^2 \in G \), \( e^L_*(\tau'_i^2) = 1 \), hence by 3.1, \( \delta_{-1}((\tau'_i)^2) = (\tau'_i)^2 \). Hence \( \delta_{-1}(\tau'_i) = c \cdot \tau'_i^{-1}, c = \pm 1 \). We may assume \( c = +1 \), by suitably altering the lift \( \tau'_i \).
Let \( G' = < \sigma'_1, (\tau'_1)^2, (\tau'_2)^2 > \subset G(L) \).

Then \( L \) descends to a principal polarization \( M \) on \( B = A/G \).

As remarked in Section 2,

\[
H^0(L) = \bigoplus_{\chi \in \hat{G}} H^0(M \otimes L_\chi)
\]

and \( \{ s_\chi \in H^0(M \otimes L_\chi), \chi \in \hat{G} \} \) form a basis of \( H^0(L) \).

Consider the commutative diagram,

\[
\begin{array}{ccc}
A & \xrightarrow{\psi_L} & Pic^0(A) \\
\downarrow{\pi} & & \uparrow{\hat{\pi}} \\
B & \xrightarrow{\psi_M} & Pic^0(B)
\end{array}
\]

where \( \psi_L(a) = t_a^* L \otimes L^{-1} \) and \( \psi_M(b) = t_b^* M \otimes M^{-1} \). Then \( \psi_M \) is an isomorphism and since \( \hat{\pi}(L_\chi) = 0 \), we have \( \pi^{-1} \psi_M^{-1}(L_\chi) \in K(L) \forall \chi \in \hat{G} \). Hence \( M \otimes L_\chi \simeq t_b^* M \)

where \( b \in \pi(K(L)) \). The basis elements \( \{ s_\chi \}_{\chi \in \hat{G}} \) can be written as \( s_0, s_1 = \sigma'_2(s_0), s_2 = \tau'_1(s_0), s_3 = \tau'_2(s_0), s_4 = \sigma'_2 \tau'_1(s_0), s_5 = \sigma'_2 \tau'_2(s_0), s_6 = \tau'_1 \tau'_2(s_0), s_7 = \sigma'_2 \tau'_1 \tau'_2(s_0) \).

**Lemma 3.3** If \( a \in K(L)2 \), then \( a.i = i.a \).

**Proof:** By 3.1, \( \delta_{-1}(a) = e^L_+(a) a \). Since \( e^L_+(a) = 1, a.i = i.a \) \( \Box \)

In particular, \( g'i(s_0) = ig'(s_0) \), for all \( g' \in G' \). Since \( g's_0 = s_0, i(s_0) \in H^0(M) \). This implies that \( i(s_0) = \pm s_0 \). We may assume \( i(s_0) = s_0 \).

**Lemma 3.4** a) \( i\sigma'_{2}(s_0) = \sigma'_{2}(s_0) \).

b) \( i\tau'_{j}(s_0) = \tau'_{j}(s_0) \).

c) \( i\sigma'_{2} \tau'_{j}(s_0) = \sigma'_{2} \tau'_{j}(s_0) \).

d) \( i\tau'_{1} \tau'_{2}(s_0) = -\tau'_{1} \tau'_{2}(s_0) \).

e) \( i\sigma'_{2} \tau'_{1} \tau'_{2}(s_0) = -\sigma'_{2} \tau'_{1} \tau'_{2}(s_0) \)

**Proof:** We will use 3.3 and the fact that \( g'(s_0) = s_0 \), for all \( g' \in G' \).

a) \( i\sigma'_{2}(s_0) = \sigma'_{2}i(s_0) = \sigma'_{2}(s_0) \).

b) \( i\tau'_{j}(s_0) = \tau'^{-1}_{j}i(s_0) = \tau'^{2}_{j}(s_0) = \tau'_{j}(s_0) \), (since \( \tau'^{2}_{j} \in G' \)).

c) \( i\sigma'_{2} \tau'_{j}(s_0) = \sigma'_{2}i\tau'_{j}(s_0) = \sigma'_{2} \tau'_{j}(s_0) \).

d) \( i\tau'_{1} \tau'_{2}(s_0) = \tau'^{-1}_{1}i\tau'_{2}(s_0) = \tau'^{1}_{1} \tau'^{2}_{1}(s_0) = -\tau'_{1} \tau'_{2}(s_0) \) (since \( e^{L}(\tau'^{2}_{1}, \tau'^{2}_{2}) = -1, \tau'^{2}_{1} \in G' \)).

e) \( i\sigma'_{2} \tau'_{1} \tau'_{2}(s_0) = \sigma'_{2}i\tau'_{1} \tau'_{2}(s_0) - \sigma'_{2} \tau'_{1} \tau'_{2}(s_0) \) \( \Box \)

Hence we have shown the following.
Proposition 3.5 The vector subspace \( H^0(L)_{L}^+ \) of \( H^0(L) \) is generated by the sections \( s_0, s_1, s_2, s_3, s_4, s_5 \) and the subspace \( H^0(L)_{L}^- \) of \( H^0(L) \) is generated by the sections \( s_6 \) and \( s_7 \).

We then have the commutative diagram,

\[
\begin{array}{ccc}
A & \xrightarrow{\phi_L} & \phi_L(A) \\
\downarrow \pi & & \downarrow \pi \quad \cdots \quad (I).
\end{array}
\]

\[
B = A/G \xrightarrow{\phi_{M^2}^L} \mathcal{K}(B) \subset \mathbb{P}(H^0(M^2))
\]

Here \( \text{degree}(p) = 2^7 \) and \( \text{degree}(\pi) = 8 \). Since we have shown that \( \phi_L \) is a birational morphism, \( \text{degree}(\phi_L) = 1 \) and hence \( \text{degree}(p|_{\phi_L(A)}) = 2^4 \). The ramification locus of \( p|_{\phi_L(A)} \) is \( \bigcup_{i=0}^7 (H_i \cap \phi_L(A)) \), where \( H_i \) is the hyperplane \( \{ s_i = 0 \} \) in \( \mathbb{P}(H^0(L)) \), \( 0 \leq i \leq 7 \).

Consider the group \( J \) generated by the projective transformations \( \alpha_i \),

\[
(s_0, \ldots, s_i, \ldots, s_7) \mapsto (s_0, \ldots, -s_i, \ldots, s_7)
\]

for \( i = 1, \ldots, 7 \).

Then \( \text{order}(J) = 2^7 \) and the group \( J \) is the Galois group of the finite morphism \( p \).

Proposition 3.6 The group \( G' \times \langle i \rangle \) can be identified as a subgroup of \( J \).

Proof: : Since the action of \( g \in G \) on the abelian threefold is fixed point free, the \( \pm 1 \)-eigenspaces of \( H^0(L) \) under the transformation \( g \in G' \) are equidimensional. Also, \( g(s_\chi) = \chi(g).s_\chi \), for all \( \chi \in \hat{G} \), implies that \( g = \alpha_i \alpha_j \alpha_k \alpha_l \in J \), for some \( 0 \leq i < j < k < l \leq 7 \). Here \( \alpha_0 = \alpha_1 \alpha_2 \ldots \alpha_7 \). By 3.5, \( i(s_0 : \ldots : s_7) = (s_0 : \ldots : s_5 : -s_6 : -s_7) \). Hence the involution \( i = \alpha_0 \alpha_7 \). Hence we can identify \( G' \times \langle i \rangle \) as a subgroup of \( J \). □

Moreover, since the Galois group of the morphism \( p \), \( \text{Gal}(p) = J \) and the subgroup \( G' \times \langle i \rangle \subset J \), leaves the image \( \phi_L(A) \) invariant in \( \mathbb{P}H^0(L) \), we have the following

Proposition 3.7 Consider the commutative diagram \( (I) \). The inverse image of the variety, \( \mathcal{K}(B) \), has eight distinct components \( h(\phi_L(A)) \), where \( h \in J/(G' \times \langle i \rangle) \).

In Section 2, we have seen that \( \{ t_0 = s_0^2, t_1 = \sigma_2'(s_0^2), t_2 = \tau_1'(s_0^2), t_3 = \tau_2'(s_0^2), t_4 = \sigma_2'\tau_1'(s_0^2), t_5 = \sigma_2'\tau_1'(s_0^2), t_6 = \tau_1'(\tau_2'(s_0^2), t_7 = \sigma_2'\tau_1'(\tau_2'(s_0^2)) \} \)

form a basis of \( H^0(M^2) \).

Remark 3.8 (We use the same notations for the elements in \( K(L) \) and their images in \( K(M^2) \).) The elements \( \sigma_2', \tau_1', \tau_2' \) of \( G(M^2) \) act on these sections as follows.
Now let $H_i = \{s_i = 0\}$ denote the coordinate hyperplanes in $\mathbb{P}H^0(L)$, for $i = 0, 1, ..., 7$. Consider the curve $C = H_6 \cap H_7 \cap \phi_L(A)$. Then the involution $i$ acts trivially on the curve $C$ and hence the degree of the restricted morphism $\phi_L^{-1}(C) \to C$ is at least 2.

**Proposition 3.9** The restricted morphism $\phi_L' : \phi_L^{-1}(C) \to C$ is of degree 2.

**Proof:** Consider the commutative diagram

\[
\begin{array}{ccc}
\phi_L^{-1}(C) & \xrightarrow{\phi_L'} & C \\
\downarrow \pi' & & \downarrow p' \\
\phi_{M^2}^{-1}(p(C)) & \xrightarrow{\phi_M^2} & p(C)
\end{array}
\]

Suppose the degree of the restricted morphism $\phi_L'$ is greater than 2. Since the Galois group of the morphism $\phi_M^2 \circ \pi'$ is the group $G \times \langle i \rangle$, the Galois group of $\phi_L'$ contains an element $g \in G$. Hence the element $g$ acts trivially on the curve $C$. This means that $C$ is contained in one of the eigenspaces $\mathbb{P}W^\pm$ of $\mathbb{P}H^0(L)$, for the action of $g$. We claim that the intersection $\phi_L(A) \cap \mathbb{P}W^\pm$ is at most a finite set of points. This will give a contradiction.

If $g^+ = \{a \in K(L) : e^L(a, g) = 1\}$, then $\frac{g^+}{<g>}$ isomorphic to $Heis(1, 1, 4)$ or $Heis(1, 2, 2)$ and the group $\frac{g^+}{<g>}$ acts on the linear space $\mathbb{P}W^\pm$. Hence projecting from $\mathbb{P}W^\pm$ gives a map $\phi_g : \frac{A}{<g>} \to \mathbb{P}W^\pm$, which is base point free in the first case (by [2]) and has a finite base locus in the second case (by [10]). This proves our claim. $\Box$

Now, the group $G$ leaves the curve $C$ invariant and moreover since $\sigma_2(H_6) = H_7$, we get $\sigma_2(C) = C$. Hence the curves

$$\tau_1(C) = H_3 \cap H_5 \cap \phi_L(A)$$
\[ \tau_2(C) = H_2 \cap H_4 \cap \phi_L(A) \] 
\[ \tau_1.\tau_2(C) = H_0 \cap H_1 \cap \phi_L(A) \]
are also invariant for the action of \( \sigma_2 \) and since for \( x \in C, i(x) = x, i.\tau_j^2(\tau_j(x)) = \tau_j^2.\tau_j^{-1}i(x) = \tau_j(x) \). By \( K(L) \)-invariance of the image \( \phi_L(A) \), we get

**Corollary 3.10** The morphism \( \phi_L \) restricts to a morphism of degree 2 on the curves \( \phi_L^{-1}(C), \phi_L^{-1}(\tau_1(C)), \phi_L^{-1}(\tau_2(C)) \) and \( \phi_L^{-1}(\tau_1.\tau_2(C)) \), onto their respective images. Moreover, the Galois groups of these restricted morphisms are \(< i >, < i.\tau_1^2 >, < i.\tau_2^2 > \) and \(< i.\tau_1^2.\tau_2^2 > \), respectively.

Let \( A^+_2 \) denote the set of points of order 2 on \( A \) where the involution \( i \) acts on the fibre of \( L \) at those points as +1 and \( A^-_2 \) denote the set of points where \( i \) acts as −1. By [1], Remark 4.7.7, \( \text{cardinality}(A^+_2) = 48 \) and \( \text{cardinality}(A^-_2) = 16 \). Hence if \( a \in A^-_2 \) and \( s \in H^0(L)^+_2 \), then \( s(a) = 0 \). This implies that for \( a \in A^-_2, \phi_L(a) = (0 : 0 : \ldots : 0 : c_1 : c_2) \in IPH^0(L) \), for some \( c_1, c_2 \in \mathcal{C} \).

**Proposition 3.11** Let \( a \in A^+_2 \) (respectively \( A^-_2 \)) and \( g \in K(L)_2 \). Then \( a + g \in A^+_2 \) (respectively \( A^-_2 \)).

**Proof:** Let \( g \in K(L)_2 \) and \( (g, \phi) \in \mathcal{G}(L) \) be a lift of order 2 and \( \psi : L \to i^*(L) \) be the normalized isomorphism. By [7], Proposition 3, p.309,

\[ \delta_{-1}(g, \phi) = (g, (t^*_g\psi)^{-1} \circ i^*\phi \circ \psi) \]
\[ = e^L_+(g). (g, \phi) \]
\[ = (g, \phi) \text{ (since } L \text{ is strongly symmetric).} \]

Hence the following diagram commutes

\[
\begin{array}{ccc}
L & \overset{\psi}{\longrightarrow} & i^*(L) \\
\downarrow \phi & & \downarrow i^*(\phi) \\
t^*_gL & \overset{t^*_g(\psi)}{\sim} & i^*t^*_gL = t^*_g(i^*L)
\end{array}
\]

Evaluating at \( a \in A^+_2 \) (respectively \( A^-_2 \)), gives \( \psi(a) = t^*_g(\psi)(a) = \psi(a + g) \), i.e. \( a + g \in A^+_2 \) (respectively \( A^-_2 \)). \( \square \)

Now let \( a \in A^-_2 \) then \( \phi_L(a) = (0 : \ldots : c_1, c_2) \) for some \( c_1, c_2 \in \mathcal{C} \). Then \( \sigma_2\phi_L(a) = (0 : \ldots : c_2 : c_1) \). We may assume \( c_2 \neq 0 \). Let \( P_0 = \phi_L(a) = (0 : \ldots : c : 1) \) and \( Q_0 = p(P_0) = (0 : \ldots : c^2 : 1) \), for some \( c \in \mathcal{C} \).
Proposition 3.12  The points $h(P_0), h \in K(L)/ < \tau_1^2, \tau_2^2 >$ are of degree 4 on the image $\phi_L(A)$.

Proof: By 3.11, the action of $G$ on the set $A^2$ has two distinct orbits, namely $O_1 = \{a + g : g \in G\}$ and $O_2 = \{a + \sigma_2 + g : g \in G\}$. Then $\phi_{M^2} \circ \pi(O_1) = Q_0$ and $\phi_{M^2} \circ \pi(O_2) = \sigma_2(Q_0)$. Notice that $P_0 \in \tau_1(C) \cap \tau_2(C) \cap \tau_1. \tau_2(C)$. Hence, by 3.10, $\phi_L^{-1}(P_0) = \{a, a + 2\tau_1, a + 2\tau_2, a + 2\tau_1 + 2\tau_2\}$. The assertion now follows from the $K(L)$-invariance of the image $\phi_L(A)$. □

Corollary 3.13  The points $b(Q_0)$, where $b < \pi(\sigma_2), \pi(\tau_1), \pi(\tau_2) >$, lie on the Kummer $K(B)$.

4  Prym Varieties

We recall few facts on Prym varieties (see [5], [9], [12], for details).

Let $C$ be a smooth projective curve of genus $g$. We will assume $C$ has no vanishing theta-nulls. In particular, when $g = 3$, this means $C$ is a non-hyperelliptic curve. A point of order 2, in $X = \text{Jac}(C)$, say $x$, defines an unramified 2-sheeted cover $C_x$ of $C$, $q_x : C_x \rightarrow C$. Let $P_x = \text{Ker}(Nm(q_x) : \text{Jac}(C_x) \rightarrow X)^0$, where $'o'$ denotes the connected component containing 0 \in $\text{Jac}(C_x)$. Here $Nm(q_x)(\mathcal{O}(\sum r_i P_i)) = \mathcal{O}(\sum r_i q_x(P_i))$ is the norm map. This defines a principally polarized abelian variety $(P_x, \theta_{P_x})$, of dimension $g - 1$. Since the kernel of the dual map $q_x' : X \rightarrow \text{Jac}(C_x)$ is generated by the element $x$, $q_x'$ induces an isomorphism $x^\perp / x \rightarrow P_x[2]$. Since $q_x^* \mathcal{O}_{C_x} \simeq \mathcal{O}_C \oplus x$, we have $det(q_x^* \mathcal{O}_{C_x}) \simeq x$. Hence $det(q_x^*(p))$ is also $x$, for any $p \in ker(Nm(q_x))$.

Fix a $z \in X$ with $z^2 \simeq x$. This gives a map

$$\psi_x : \text{Ker}(Nm(q_x)) \simeq P_x \cup P_x \rightarrow SU_C(2).$$

where $\psi_x(p) = (q_x^*(p)) \otimes z$.

The image of $\psi_x$ is independent of the choice of $z$. Recall the map

$$SU_C(2) \xrightarrow{\phi} |2\theta_C| \simeq \mathbb{P}(H^0(SU_C(2), \mathcal{L}))$$

where $\mathcal{L}$ generates $\text{Pic}(SU_C(2)) \simeq \mathbb{Z}$.

Let $\mathcal{P}V_x^+$ and $\mathcal{P}V_x^-$ be the two eigenspaces for the action of $x$ on $|2\theta_C|$. Then there is one component of $\text{Ker}(Nm(q_x))$ in each eigenspace. So we get a map $\phi_x : P_x \rightarrow \mathcal{P}V_x$. 

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Proposition 4.1 The map $\phi_x : P_x \rightarrow \text{IPV}_x$ is the natural map

$$P_x \rightarrow \mathcal{K}(P_x) \subset \mathbb{P}(H^0(P_x, 2\theta P_x)) \cong \text{IPV}_x.$$ 

Proof: : See [5], Proposition 1, p.745.

Proposition 4.2 For any curve $C$ and any $x$ in $X[2] - \{0\}$, we have $\mathcal{K}(C) \cap \text{IPV}_x = \mathcal{K}(P_x[2])$, (the Schottky Jung relations).

Proof: : See [5], Proposition 2 (1), p.746.

5 Situation in $\mathbb{P}(H^0(L))$, when $g = 3$.

We now assume $B = J(C)$, where $J(C)$ is the Jacobian of a non-hyperelliptic curve $C$ of genus 3. (This is the generic situation, since the dimension of the moduli space of principally polarized abelian threefolds is 6 which equals the dimension of the moduli space of curves of genus 3.) Recall the results of Narasimhan and Ramanan (Theorem 1.3, Theorem 1.4), to obtain a morphism

$$J(C) \overset{\phi_{2\theta}}{\rightarrow} \mathcal{K}(C) \subset F \subset |2\theta|$$

where

1) $F$ is a quartic hypersurface and is the isomorphic image of the moduli space $SU_C(2)$ and

2) the Kummer variety $\mathcal{K}(C)$ is precisely the singular locus of $F$.

We will use the following

Proposition 5.1 Let $L$ be an ample line bundle of type $\delta = (d_1, d_2, ..., d_g)$ on an abelian variety $A$. Then the set of irreducible representations of the theta group $\mathcal{G}(L)$, where $\alpha \in \mathcal{G}^*$ acts as multiplication by $\alpha^n$ (called as of 'weight n'), is in bijection with the set of characters on the subgroup of $n$–torsion elements, $K(L)_n$, of $K(L)$. Moreover, the dimension of any such representation is $\frac{d_1.d_2...d_g}{(n,d_1)...(n,d_g)}$. ( $(n,d_i)$ denotes the greatest common divisor of $n$ and $d_i$.)

Proof: : When $n = 2$, the statement is proved in [6], Proposition 3.2, p.142. The same proof holds when $n > 2$, by choosing a section over the subgroup of $n$-torsion elements, $K(L)_n$, of $K(L)$ in the exact sequence

$$1 \rightarrow \mathcal{G}^* \rightarrow \mathcal{G}(L) \rightarrow K(L) \rightarrow 0$$
in the proof of [6], Proposition 3.2. □

**Corollary 5.2** The quartic $F$ in $|2\theta|$ is $\mathcal{G}(2\theta)$-invariant and the linear span of the eight cubics $\{ \frac{dF}{dt} \}$ for $i = 0, 1, \ldots, 7$ form an irreducible $\mathcal{G}(2\theta)$-module where $\alpha \in \mathcal{C}^*$ acts as multiplication by $\alpha^3$.

**Proof:** Consider the multiplication maps $\text{Sym}^n H^0(2\theta) \to H^0(2n\theta)$. Then $I_n = \text{Ker}(\rho_n) =$ vector space of degree $n$ forms containing the image $\mathcal{K}(B)$ in $\mathcal{I} H^0(2\theta)$. Since the vector spaces $\text{Sym}^n H^0(2\theta)$ and $H^0(2n\theta)$ (via the homomorphism $\mathcal{G}(2\theta) \to \mathcal{G}(2n\theta)$) are $\mathcal{G}(2\theta)$-modules, of weight $n$ and $\rho_n$ is equivariant for the $\mathcal{G}(2\theta)$-action, $I_n$ is also a $\mathcal{G}(2\theta)$-module of weight $n$. Now the homogenous polynomial $F \in I_4$ and the partial derivatives $\frac{dF}{dt} \in I_3$. By 5.1, it follows that $F$ is $\mathcal{G}(2\theta)$-invariant, upto scalars. If $z \in \mathcal{G}(2\theta)$, then $\frac{dF}{dt} = \frac{d(zF)}{dt} = \alpha \frac{dF}{dt}$, for some scalar $\alpha$. Hence $W$ is a $\mathcal{G}(2\theta)$-module of weight $3$. By 5.1, dimension of such an irreducible representation is $8$. This proves our assertion. □

Similarly, we see that $R = F(s_0^2, \ldots, s_7^2)$ is a $\mathcal{G}(L)$-invariant octic hypersurface in $\mathcal{I} H^0(L)$, by applying 5.1.

Recall the Weil form $\epsilon_L$ on $K(L)$ and the isotropic subgroup $G = <\sigma_1, \tau_1^2, \tau_2^2 > \subset K(L)$. Then $\epsilon_L(\sigma_2 + g, \sigma_1) = -1$, for all $g \in G$. Let $a = \sigma_2 + g$, for $g \in G$ and $a' = \sigma_2' + g' \in \mathcal{G}(L)$.

Recall the basis $\{ s_0, s_1, \ldots, s_7 \}$ of $H^0(L)$ and $\{ s_0^2, \ldots, s_7^2 \}$ of $H^0(M^2)$, (see Section 3). Let $W_a^+$ and $W_a^-$ denote the eigen spaces in $H^0(L)$, for the action of $a'$. Now $\mathcal{I} P W_{a}^{\pm} = \{ s = 0 : s \in W_a^{\pm} \}$ and $\mathcal{I} P V_{a}^{+} = \{ t = 0 : t \in H^0(M^2)^{\pm}_a \}$. Now $W_{a}^{\pm} = \{ s_0 \pm s_1, s_2 \pm s_4, s_3 \pm s_5, s_6 \pm s_7 \}$ and $H^0(M^2)^{\pm}_a = \{ s_0^2 - s_1^2, s_2^2 - s_4^2, s_3^2 - s_5^2, s_6^2 - s_7^2 \}$.

Then $p$ restricts on $\mathcal{I} P W_{\sigma_2}^{\pm} \to \mathcal{I} P V_{\sigma_2}^{+}$ as $(s_0 ; s_2 ; s_3, s_6) \mapsto (s_0^2 : s_2^2 : s_3^2 : s_6^2)$, of degree $2^3$. Similarly, one checks that if $a = \sigma_2 + g, g \in G$ then $p$ restricts to $\mathcal{I} P W_{a}^{\pm} \to \mathcal{I} P V_{\sigma_2}^{+}$ as $(z_0 ; \ldots ; z_3) \mapsto (z_0^2 ; \ldots ; z_3^2)$ of degree $2^3$.

**Proposition 5.3** Consider a principally polarized abelian surface $(Y, P)$, which is not a product of elliptic curves. Let $y_1, y_2 \in Y$ be elements of order $2$, such that $e^{Pz}(y_1, y_2) = -1$. Then we have the following.

1) There is a polarized abelian surface $(Z, N)$, such that $N$ is strongly symmetric of type $(1, 4)$ and there is a covering map $f : Z \to Y$ with the Galois group of the map $f$ being isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
2) The vector space $H^0(N)$ can be written as

$$H^0(N) = H^0(P) \oplus H^0(t_{y_1}^*P) \oplus H^0(t_{y_2}^*P) \oplus H^0(t_{y_1+y_2}^*P).$$

and there is a commutative diagram

\[
\begin{array}{cccc}
Z & \xrightarrow{\phi_N} & \phi_N(Z) & \subset \mathbb{P}^3 = \mathbb{P}H^0(N) \\
\downarrow f & & \downarrow q & \\
Y & \xrightarrow{\phi_{Y}} & K(Y) & \subset \mathbb{P}^3 = \mathbb{P}H^0(M^2)
\end{array}
\]

where $q(r_0 : r_1 : r_2 : r_3) = (r_0^2 : r_1^2 : r_2^2 : r_3^2)$. Here \(\{r_0, r_1, r_2, r_3\}\) is a basis obtained from above decomposition of $H^0(N)$, such that $r_0, r_1, r_2 \in H^0(N)_i^+$ and $r_3 \in H^0(N)_i^-$.

**Proof:** 1) Consider the isomorphism $\phi_P : Y \rightarrow \text{Pic}^0(Y)$, $b \mapsto t_b^*P \otimes P^{-1}$. Let $L_{y_1}$ and $L_{y_2}$ denote the images of $y_1$ and $y_2$ under this map. These two line bundles define an unramified cover, $f : Z \rightarrow Y$, whose Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, as asserted.

Then $N = f^*P$ is an ample line bundle and $\text{dim} H^0(N) = 4$. So to see that $N$ is of type $(1,4)$, it is enough to show that $K(N)$ has an element of order 4. Consider the commutative diagram

\[
\begin{array}{cccc}
Z & \xrightarrow{\psi_N} & \text{Pic}^0(Z) & \\
\downarrow f & & \uparrow \hat{f} & \\
Y & \xrightarrow{\psi_M} & \text{Pic}^0(Y)
\end{array}
\]

Then $\hat{f} \circ \psi_M(y_i) = 0$. This implies that if $z_1$ and $z_2$ are in $Z$ such that $f(z_i) = y_i$, then $z_1, z_2 \in K(N)$. Moreover, since $e^{P^2}(y_1, y_2) = -1$ and $N^2 \simeq f^*(P^2)$, we have $e^{N^2}(z_1, z_2) = -1$. This gives $e^N(z_1, z_2) = \pm i$. Hence the elements $z_1, z_2 \in K(N)$ are of order 4.

2) Clearly, $f_*N = P \oplus (P \otimes L_{y_1}) \oplus (P \otimes L_{y_2}) \oplus (P \otimes L_{y_1+y_2})$. Now, in the algebraic equivalence class of $N$, there are strongly symmetric line bundles. Hence, by tensoring $P$ with a suitable line bundle of order 2, we may assume that $N = f^*P$ is strongly symmetric and $r_0 \in H^0(P)$ is such that $i(r_0) = r_0$.

Since $N$ is strongly symmetric, by 3.1, $\delta_{-1}(z_j^2) = (z_j')^2$, for some lifts $z_j' \in \mathcal{G}(N)$ of $z_j \in K(N)$. We may further choose the lifts such that $\delta_{-1}(z_j') = (z_j')^{-1}$, (as in 3.2). In particular, the descent data of $N$ to $P$ is $K' = \langle (z_1')^2, (z_2')^2 \rangle \supset \mathcal{G}(N)$, which is a splitting over $K = \langle z_1^2, z_2^2 \rangle \supset K(N)$ in the exact sequence

$$1 \rightarrow \mathcal{C}' \rightarrow \mathcal{G}(N) \rightarrow K(N) \rightarrow 0.$$
This means \((z_j')^2r_0 = r_0\). Also this gives

As in 3.5, we see that

\[ i.z_j'(r_0) = z_j'(r_0) \]

and

\[ i.z_1',z_2'(r_0) = -z_1',z_2'(r_0). \]

Thus \(r_0, r_1 = z_1'(r_0), r_2 = z_2'(r_0) \in H^0(N)^+\) and \(r_3 = z_1',z_2'(r_0) \in H^0(N)^-\).

Hence one sees as earlier that \(Gal(q) = \langle z_1^2, z_2^2, i \rangle\), with a commutative diagram as in 5.3. □

**Proposition 5.4** Let \(a = \sigma_2 + g, g \in G\) and \(IPW_a\) denote an eigenspace of \(a\) in \(PH^0(L)\). Then there is an abelian surface \(Z\) and a symmetric line bundle \(N\) on \(Z\) of type \((1,4)\) such that \(Z \xrightarrow{\phi_N} PH^0(N) \cong IPW_a \subset PH^0(L)\). Moreover, under this isomorphism, the image \(\phi_N(Z)\) is mapped onto the Heis\((4)\)-invariant surface \(S = R \cap IPW_a\), where \(R\) is the Heis\((2,4)\)-invariant hypersurface of degree 8 in \(PH^0(L)\), defined by \(F(s_0^2 : s_1^2 : \ldots : s_7^2) = 0\). (\(F\) being the Coble quartic).

**Proof:** Consider the restricted morphism \(p : IPW_a \longrightarrow IPV_a\), given as \((z_0 : \ldots : z_3) \mapsto (z_0^2 : \ldots : z_3^2)\). Then \(a\) acts trivially on \(IPW_a\) and \(a^\perp\) on \(IPW_a\), (here \(a^\perp = \{ y \in K(L) : e^t(a,y) = 1 \}\)). Hence there is a Heis\((4)\)-action on \(IPW_a\) and similarly a Heis\((2,2)\)-action on \(IPV_a\). By 4.1, there is a principally polarized abelian surface \((P_a, \theta_{C_a})\), \((P_a\) being the Prym variety associated to the element \(\pi(a) \in K(M^2))\), such that

\[ P_a \longrightarrow K(P_a) \subset |2\theta_{C_a}| \cong IPV_a\]

Consider the images of \(\tau_1, \tau_2\), which are elements of order 2 in \(J(C)\). Since \(e^{L^2}(\tau_1, a) = 1\), for the Weil form \(e^{2g}\) on \(J(C)[2]\), \(\tau_1, \tau_2 \in \pi(a)^\perp/\pi(a)\). Moreover, \(e^{2g}(\tau_1, \tau_2) = -1\). By 4.2, the points \(\phi_{M^2} \circ \pi(\tau_i)\), are nodes in the Kummer of the Prym variety \(P_a\). These nodes correspond to elements of order 2 in \(P_a\), say \(\beta_1\) and \(\beta_2\). Since the Weil form \(e^{2gC_a}\) on \(P_a[2]\) is induced from the Weil form \(e^{2g}\), we have \(e^{2gC_a}(\beta_1, \beta_2) = -1\). By 5.3, there is a polarized abelian surface \((Z, N)\) of type \((1,4)\), such that the following diagram commutes

\[
\begin{array}{ccc}
Z & \xrightarrow{\phi_N} & PH^0(N) \\
\downarrow f & & \downarrow q \\
P_a & \xrightarrow{\phi_{2gC_a}} & K(P_a) \subset |2\theta_{C_a}|
\end{array}
\]

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and for the choice of basis \( \{ r_0, r_1, r_2, r_3 \} \), in 5.3 2), the morphism \( q \) is defined as 
\[
(r_0 : r_1 : r_2 : r_3) \mapsto (r_0^2 : r_1^2 : r_2^2 : r_3^2),
\]
with \( \text{Gal}(q) = \langle z_1^2, z_2^2, i \rangle \), \( (z_j \text{ as in 5.3}).
\]

Now, \( R \) is the \( \text{Heis}(2,4) \)-invariant octic \( F(s_0^2 : \ldots : s_7^2) = 0 \), where \( F \) is the Coble quartic. Note that \( S = R \cap \mathcal{P} \mathcal{W}_a \) is \( a^\perp/a \)-invariant and is mapped onto the Kummer, \( K(P_a) \), under the restriction morphism. Moreover, the Galois group of \( p|_S \) is \( \langle \tau_1^2, \tau_2^2, i \rangle \) which is isomorphic to the Galois group of \( q \). Hence there is a \( \text{Heis}(4) \)-isomorphism \( \mathcal{P} \mathcal{H}^0(N) \longrightarrow \mathcal{P} \mathcal{W}_a \), such that the Heisenberg invariant octic surface \( \phi_N(Z) \) is mapped onto the \( \text{Heis}(4) \)-invariant octic surface \( S = R \cap \mathcal{P} \mathcal{W}_a \). This proves the assertion. \( \square \)

It is known that the Kummer \( \mathcal{K}(P_a) \), has 6 of its nodes in each of the coordinate hyperplane, namely the coordinate points and 3 other distinct points. The preimages of the coordinate points are the coordinate points in \( \mathcal{P} \mathcal{H}^0(N) \) and \( q \) is etale over the other 3 points which are the pinch points of \( \phi_N(Z) \) in the respective coordinate hyperplane.

**Proposition 5.5** \( \phi_N(Z) \) has exactly 48 pinch points, 12 in each coordinate hyperplane.

**Proof:** : See [3], Proposition 2.2, p.633.

Let \( T_a \) denote the set of pinch points and the coordinate points in \( \phi_N(Z) \).

**Proposition 5.6** The components \( h(\phi_L(A)) \), \( h \in H \) (here \( H = J/(G' \times i) \)) and \( \mathcal{P} \mathcal{W}_a \) intersect at the subset \( T_a \) of \( \phi_N(Z) \). In particular \( \cap_{h \in H} h(\phi_L(A)) = \cup_{a=\sigma_2+g, g \in G} T_a \).

**Proof:** : Since \( \pi^{-1}\mathcal{K}(C) = \cup_{h \in H} h(\phi_L(A)) \), by 4.2 and 5.5, we conclude that \( h(\phi_L(A)) \cap \mathcal{P} \mathcal{W}_a = T_a \), for all \( h \in H \). This gives the assertion. \( \square \)

### 6 Some remarks

a) Consider the moduli space \( \mathcal{A}^{l}_{(1,2,4)} \) of triples \( (A, c_1(L), f) \), where \( f : K(L) \longrightarrow Z/DZ \times Z/DZ \) is a level structure, ( here \( D = (1,2,4) \)). Consider the subset of \( \mathcal{A}^{l}_{(1,2,4)}, a_{\phi_L(A)}, \) parametrizing triples which admit a \( (Z/2Z)^3 \)-isogeny to the Jacobian of a non-hyperelliptic curve.

Since \( \text{dim} \mathcal{A}^{l}_{(1,2,4)} = \text{dim} \mathcal{A}^{l}_{(1,2,4)} = 6 \) and \( c_1(L) \) gives a birational morphism, \( \mathcal{A}^{l}_{(1,2,4)} \) is an open subset of \( \mathcal{A}^{l}_{(1,2,4)} \).

Consider a triple \( (A, c_1(L), f) \in \mathcal{A}^{l}_{(1,2,4)} \). We have seen that there is a \( \text{Heis}(2,4) \)-invariant octic hypersurface \( R \), defined by \( F(s_0^2 : s_1^2 : \ldots : s_7^2) = 0 \), ( \( F \) being the Coble quartic), such that \( \phi_L(A) \subset R \subset \mathcal{P} \mathcal{V}(2,4) \). In fact \( h(\phi_L(A)) \subset \text{Sing}(R) \), for all \( h \in H \), ( \( H \) as in 5.6).
Now $F$ is a $\text{Heis}(2,2,2)$-invariant quartic polynomial in $\mathbb{P}V(2,2,2)$. Since the space of $\text{Heis}(2,2,2)$-invariant quartics is 14-dimensional, (see [4], p.186), the space of $\text{Heis}(2,4)$-invariant octics in $\mathbb{P}^7$ which are of the form $R = F(s_0^2 : ... : s_7^2)$ where $F$ is a $\text{Heis}(2,2,2)$-invariant quartic, is also 14-dimensional. Call this space as $P(\text{Sym}^8V(2,4)^{\text{Heis}(2,4)}) = \mathbb{P}^{14}$.

So there is a morphism

$$\mathcal{A}_{(1,2,4)}^{lo} \xrightarrow{T} \mathbb{P}^{14}$$

where $T$ is defined as $(A, c_1(L), f) \mapsto R$.

One may try to study this morphism, from a moduli point of view.

b) Consider the special basis $\{s_0^2, ..., s_7^2\}$ (which is different from the usual Heisenberg basis) of $H^0(2\theta)$ and the action of the elements of the subgroup $<\sigma_2, \tau_1^2, \tau_2^2> \subset K(2\theta)$ on this basis (see 3.8).

Also, by 3.12, the points $b(P_0) \in \phi_L(A)$, where $b \in <\sigma_2, \tau_1, \tau_2> \subset K(L)$, $P_0 = (0 : ... : 0 : c : 1)$ and the point $Q_0 = (0 : ... : 0 : c^2 : 1) \in K(C)$, for some non-zero $c \in \mathfrak{C}$. With these data, in addition to knowing the geometry of $SU_C(2)$ in $|2\theta|$-linear system one may try to know the equation of the Coble quartic, in terms of this basis $\{s_0^2, ..., s_7^2\}$.

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