An infinite family of maximally superintegrable systems in a magnetic field with higher order integrals

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Abstract

We construct an additional independent integral of motion for a class of three dimensional minimally superintegrable systems with constant magnetic field. This class was introduced in \cite{1} and it is known to possess periodic closed orbits. In the present paper we demonstrate that it is maximally superintegrable. Depending on the values of the parameters of the system, the newly found integral can be of arbitrarily high polynomial order in momenta.

Keywords: integrability; superintegrability; higher order integrals; magnetic field.

1 Introduction

In our recent paper \cite{1} we found a class of minimally superintegrable systems in three spatial dimensions with constant magnetic field which possesses closed bounded periodic trajectories for a particular choice of parameters. Namely, a quantity $\kappa$ constructed out of them (cf. equation (8) below) must be rational, $\kappa = \frac{m}{n}$. For three particular choices of $\kappa$ this system is known to be maximally superintegrable with integrals of at most second order in momenta \cite{1}. Thus a natural question arises asking whether also for the remaining values of the parameters satisfying the rationality constraint a missing independent integral can be constructed.

In this paper we describe how the considered system can be reduced to the two dimensional anisotropic harmonic oscillator and how the known integrals of the anisotropic oscillator give rise to a new independent integral for the system with magnetic field. Assuming that we have $\kappa = \frac{m}{n}$ where $m$ and $n$ are incommensurable, the additional integral is of order $m + n - 1$ in momenta with leading order terms involving angular momenta linearly. I.e. we find that the system is indeed maximally superintegrable, as hinted by the shape of the trajectories, and the minimal polynomial order of the fifth integral can be arbitrarily high. Thus it provides one of the few known examples of maximally superintegrable systems with magnetic field whose integrals are not of second (or less) order in momenta.
For the sake of clearness, let us recall that in classical mechanics integrability means that there exist $N$ integrals of motion $X_j$, including the Hamiltonian, that Poisson commute pairwise, are well defined functions on the phase space and are functionally independent. The system is superintegrable if it allows $k$ further independent integrals $Y_a$ that Poisson commute with the Hamiltonian, but not necessarily with each other, nor with the integrals $X_j$. The integer $k$ satisfies $1 \leq k \leq N-1$ where $k = 1$ and $k = N-1$ correspond to minimal and maximal superintegrability, respectively. Similarly, in quantum mechanics we assume that the integrals are well-defined commuting self-adjoint operators, polynomial in the operators $\hat{x}_j$ and $\hat{p}_j$ representing the coordinates and the momenta, or more general objects, such as convergent series in these operators. Requirement of independence in this case means that no nontrivial fully symmetrized polynomial in the integrals of motion should vanish.

Maximally superintegrable systems are of special interest in classical physics because all their finite trajectories are closed in configuration space and the motion is periodic. In quantum mechanics the energy levels are degenerate and it has been conjectured that maximally superintegrable systems are exactly solvable [2].

Most of the recent research on superintegrability focused on “natural” Hamiltonians with scalar potentials. For early systematic study in three dimensions see [3, 4, 5, 6]. This paper belongs to a series of papers [7, 11, 8] studying superintegrability of three dimensional systems with magnetic fields. We refer the reader to the papers [7, 8] for discussion of a general motivation for our research and to [11] for details concerning the introduction of the system considered in this paper. For a general discussion of integrability and superintegrability see a recent review [9], for superintegrability in the presence of vector potentials e.g. [10, 12, 13, 14, 15, 16, 17].

The structure of the paper is as follows: In Section 2 we describe our system, its integrals and its trajectories as presented in [11]. In Section 3 we reduce its dynamics to the well-known case of two dimensional anisotropic harmonic oscillator. In Section 4 we use the known integrals of the anisotropic oscillator to construct previously unknown integrals for the three dimensional problem under investigation. In Section 5 we present an explicit example. In the last section we conclude with a summary of our results and comment on the superintegrability of the quantum analogue of our system.

2 The system

We consider the Hamiltonian system on the phase space with coordinates $(\vec{x}, \vec{p})$ where $\vec{x} = (x, y, z)$ and $\vec{p} = (p_1, p_2, p_3)$. We assume that its magnetic
field and effective potential are given by

\[
\vec{B}(\vec{x}) = (-\Omega_1, \Omega_2, 0), \quad W(\vec{x}) = \frac{\Omega_1 \Omega_2}{2S}(Sx - y)^2
\]

(1)

where \(\Omega_1, \Omega_2, S\) are real constants such that \(S \neq 0\) and \(\Omega_1^2 + \Omega_2^2 \neq 0\). Its Hamiltonian can therefore be written as

\[
H = \frac{1}{2}(\vec{p} + \vec{A}(\vec{x}))^2 + W(\vec{x})
\]

(2)

and we shall work with the gauge chosen as

\[
\vec{A} = (0, 0, -\Omega_2 x - \Omega_1 y).
\]

(3)

This system is already known to be minimally superintegrable [1]. It has three Cartesian type integrals

\[
\begin{align*}
X_0 &= p_3, \\
X_1 &= p_1^2 - 2\Omega_2 x p_3 + \Omega_2 (S\Omega_1 + \Omega_2)x^2, \\
X_2 &= p_2^2 - 2\Omega_1 y p_3 + \frac{\Omega_1}{S}(S\Omega_1 + \Omega_2)y^2
\end{align*}
\]

(4)

on which the Hamiltonian is polynomially dependent, \(H = \frac{1}{2}(X_0^2 + X_1 + X_2)\), and an additional first order integral

\[
X_3 = p_1 + S p_2 - (S\Omega_1 + \Omega_2)z.
\]

(5)

The classical trajectories of the system [1] are known, cf. [1]

\[
\begin{align*}
x(t) &= \frac{1}{\omega_1} \left( (\omega_1^2 x_0 - \Omega_2 p_{30}) \cos(\omega_1 t) + \omega_1 p_{10} \sin(\omega_1 t) + \Omega_2 p_{30} \right), \\
y(t) &= \frac{1}{\omega_2} \left( (\omega_2^2 y_0 - \Omega_1 p_{30}) \cos(\omega_2 t) + \omega_2 p_{20} \sin(\omega_2 t) + \Omega_1 p_{30} \right), \\
z(t) &= \frac{1}{\Omega_1 S + \Omega_2} \left( p_{10} (\cos(\omega_1 t) - 1) + S p_{20} (\cos(\omega_2 t) - 1) + \frac{\Omega_2 p_{30} - \omega_2^2 y_0}{\omega_2} \sin(\omega_2 t) + \frac{\Omega_1 p_{30} - \omega_1^2 x_0}{\omega_1} \sin(\omega_1 t) + z_0, \right)
\end{align*}
\]

where we introduced the constants

\[
\omega_1 = \sqrt{\Omega_2(S\Omega_1 + \Omega_2)}, \quad \omega_2 = \sqrt{\frac{\Omega_2}{S}(\Omega_1 S + \Omega_2)} = \sqrt{\Omega_1 S \Omega_2 - \omega_1^2}
\]

in order to shorten the terms in (6).

In [1] it was also proven that in the special cases

\[
S = \frac{\Omega_1}{\Omega_2}, \quad S = 4\frac{\Omega_1}{\Omega_2} \quad \text{and} \quad S = \frac{\Omega_1}{4\Omega_2}
\]

(7)
the system (1) is maximally superintegrable, with the additional integral of order 1 and 2, respectively. In the following we prove that the system (1) is maximally superintegrable whenever the trajectories (6) are periodic (or, equivalently, closed), i.e. for

\[ S = \frac{\Omega_1}{\Omega_2} \kappa^2, \quad \text{where} \quad \kappa = \frac{m}{n}, \quad m, n \in \mathbb{N} \]  

are incommensurable, (8)

with the fifth integral of order \(m + n - 1\) in the momenta \(p_1, p_2, p_3\). We notice that systems with the parameters \(\Omega_1, \Omega_2, \kappa\) and \(\Omega_2, \Omega_1, \frac{1}{\kappa}\) are equivalent, they differ just by a choice of Cartesian coordinates, cf. [1].

3 Reduction to the anisotropic oscillator

By using the known integrals, let us reduce the system (1) to the anisotropic harmonic oscillator in two degrees of freedom (in the following abbreviated as 2 d.o.f.). The Hamilton’s equations read

\[
\begin{align*}
\dot{x} &= p_1, \quad \dot{y} = p_2, \quad \dot{z} = -\Omega_1 y - \Omega_2 x + p_3, \quad \dot{p}_3 = 0, \\
\dot{p}_1 &= -(\kappa^2 \Omega_1^2 + \Omega_2^2) x + \Omega_2 p_3, \quad \dot{p}_2 = -\left(\Omega_1^2 + \frac{\Omega_2^2}{\kappa^2}\right) y + \Omega_1 p_3.
\end{align*}
\]

By substituting \(p_3 \equiv p_{30}\) and by the shift

\[
x = X + \frac{\Omega_2 p_{30}}{\Omega_1^2 \kappa^2 + \Omega_2^2}, \quad y = Y + \frac{\kappa^2 \Omega_1 p_{30}}{\Omega_1^2 \kappa^2 + \Omega_2^2}, \quad (10)
\]

the equations simplify to

\[
\begin{align*}
\dot{P}_1 &= -(\kappa^2 \Omega_1^2 + \Omega_2^2) X, \quad \dot{P}_2 = -\left(\Omega_1^2 + \frac{\Omega_2^2}{\kappa^2}\right) Y, \quad \dot{X} = P_1, \quad \dot{Y} = P_2, \\
\dot{z} &= -\Omega_1 Y - \Omega_2 X + p_{30}.
\end{align*}
\]

where \(P_1, P_2\) are the momenta conjugated to the new space coordinates \(X, Y\) (once evaluated they are equal to the original \(p_1, p_2\)). By solving the first two equations in (11) with respect to \(X\) and \(Y\) and substituting into (12) we find

\[
\Omega_2 \dot{P}_1 + \Omega_1 \kappa^2 \dot{P}_2 - (\Omega_2^2 + \Omega_1^2 \kappa^2) \dot{z} = 0
\]

corresponding to the integral (5). The dynamics are thus reduced to the dynamics of an anisotropic oscillator, whose frequency ratio is \(\kappa\) and canonical coordinates are \((X, Y, P_1, P_2)\). It is known [18] that if \(\kappa\) satisfies (8), such an oscillator is superintegrable. Let us henceforth restrict to this case and set

\[\omega^2 = \frac{\Omega_2^2}{n^2} + \frac{\Omega_1^2}{m^2}.\]  

(13)
The Hamiltonian of the 2 d.o.f. oscillator (11) is obtained by substituting (10) into the Hamiltonian (2) and reads

\[ H_2 = \frac{1}{2}(P_1^2 + P_2^2) + \frac{1}{2}\omega^2(m^2X^2 + n^2Y^2). \]  

(14)

By introducing complex coordinates

\[ z_1 = iP_1 + m\omega X, \quad z_2 = iP_2 + n\omega Y, \]

the ring of the invariants of the oscillator (14) is generated by

\[ I_1 = z_1\bar{z}_1, \quad I_2 = z_2\bar{z}_2, \quad I_3 = \text{Re} \left(z_1^n\bar{z}_2^m\right), \quad I_4 = \text{Im} \left(z_1^n\bar{z}_2^m\right). \]  

(15)

The invariants (15) are clearly not independent; they satisfy the relation

\[ I_2^3 + I_2^4 = I_1 I_2^m. \]  

(16)

Equivalently, the integrals \( I_3, I_4 \) can be expressed in terms of Chebyshev polynomials as

\[ I_3 = |z_1|^{n-1}|z_2|^{m-1} \left( |z_1||z_2| T_n \left( \frac{\text{Re} z_1}{|z_1|} \right) T_m \left( \frac{\text{Re} z_2}{|z_2|} \right) + \right. \]

\[ + \left. \text{Im} z_1\text{Im} z_2 U_{n-1} \left( \frac{\text{Re} z_1}{|z_1|} \right) U_{m-1} \left( \frac{\text{Re} z_2}{|z_2|} \right) \right), \]

\[ I_4 = |z_1|^{n-1}|z_2|^{m-1} \left( |z_2| \text{Im} z_1 U_{n-1} \left( \frac{\text{Re} z_1}{|z_1|} \right) T_m \left( \frac{\text{Re} z_2}{|z_2|} \right) - \right. \]

\[ - \left. |z_1|\text{Im} z_2 T_n \left( \frac{\text{Re} z_1}{|z_1|} \right) U_{m-1} \left( \frac{\text{Re} z_2}{|z_2|} \right) \right), \]

(17)

where \( T_n, U_n \) denote the Chebyshev polynomial of degree \( n \) of the first and second type, respectively. As we will show in the next section, the integrals \( I_1 \) and \( I_2 \) correspond to the Cartesian type integrals \( X_1 \) and \( X_2 \) of the original system while \( I_3 \) (or \( I_4 \)) gives a new independent integral for the system (1).

4 The fifth integral

Let us first invert the shift (11). The integrals \( I_1 \) and \( I_2 \) give, after neglecting terms proportional to the integral \( p_{30}^2 \),

\[ \tilde{I}_1 = p_1^2 - 2\Omega_2xp_{30} + \frac{m^2\Omega_1^2 + n^2\Omega_2^2}{n^2} x^2, \]  

(18)

\[ \tilde{I}_2 = p_2^2 - 2\Omega_1yp_{30} + \frac{m^2\Omega_1^2 + n^2\Omega_2^2}{m^2} y^2. \]  

(19)

By substituting back \( p_3 = p_{30} \) into (18)–(19) we see that they correspond to the Cartesian type integrals \( X_1 \) and \( X_2 \) of (1).
Similarly, we can find the expressions of the integrals \( I_3 \) and \( I_4 \) in the original phase space coordinates. We find it convenient to work with the following series expressions for the Chebyshev polynomials

\[
T_n(a) = \sum_{k=0}^{[n/2]} \binom{n}{2k} a^{n-2k} (a^2 - 1)^k, \quad (20)
\]

\[
U_n(a) = \sum_{k=0}^{[n/2]} \binom{n+1}{2k+1} a^{n-2k} (a^2 - 1)^k, \quad (21)
\]

so that we can explicitly write the two integrals as polynomials in the momenta. Namely, after inverting the shift (10) and substituting \( p_{30} = p_3 \) we have, in gauge covariant form,

\[
X_4 = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} (m \omega \tilde{X}^A)^{n-2k} (p_1^A)^{2k} \sum_{k=0}^{[m/2]} (-1)^k \binom{m}{2k} (n \omega \tilde{Y}^A)^{m-2k} (p_2^A)^{2k}
+ \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k+1} (m \omega \tilde{X}^A)^{n-2k-1} (p_1^A)^{2k+1}
\cdot \sum_{k=0}^{[m/2]} (-1)^k \binom{m}{2k+1} (n \omega \tilde{Y}^A)^{m-2k-1} (p_2^A)^{2k+1}, \quad (22)
\]

and

\[
X_5 = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k+1} (m \omega \tilde{X}^A)^{n-2k-1} (p_1^A)^{2k+1} \sum_{k=0}^{[m/2]} (-1)^k \binom{m}{2k} (n \omega \tilde{Y}^A)^{m-2k} (p_2^A)^{2k}
- \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} (m \omega \tilde{X}^A)^{n-2k} (p_2^A)^{2k} \sum_{k=0}^{[m/2]} (-1)^k \binom{m}{2k+1} (n \omega \tilde{Y}^A)^{m-2k-1} (p_1^A)^{2k+1}, \quad (23)
\]

where

\[
\tilde{X}^A = x - \frac{n^2 \Omega_2 (p_3^d + \Omega_2 x + \Omega_1 y)}{m^2 \Omega_1^2 + n^2 \Omega_2^2}, \quad \tilde{Y}^A = y - \frac{m^2 \Omega_1 (p_3^d + \Omega_2 x + \Omega_1 y)}{m^2 \Omega_1^2 + n^2 \Omega_2^2}
\]

and

\[
p_j^A = p_j + A_j(\tilde{x}) \quad (25)
\]

are the components of the velocities, i.e. are gauge invariant.

Since \( I_1, I_2, I_3 \) are independent integrals for the oscillator system, by applying the chain rule we see that the integrals \( X_0, X_1, X_2, X_4 \) form a set of independent integrals for the original system, where \( X_0 = p_3 \). Moreover, if the gauge is chosen as in (3), none of them depends on the \( z \) variable, while
Therefore the five integrals $X_0, X_1, X_2, X_3, X_4$ are independent. This implies the maximal superintegrability of the system (1). Notice that the same argument also applies to $X_0, X_1, X_2, X_3, X_5$.

Let us now discuss the order of the new integrals $X_4, X_5$. From the expressions (22) and (23), it is clear that their order as polynomials in momenta is at most $m + n$. However from (22) we see that the terms of order $m + n$ in $X_4$ are only of the form

$$\alpha_k \gamma_j p_1^{2k} p_2^{2j} p_3^{n+m-2(k+j)}, \quad k = 0, \ldots, \left[ \frac{n}{2} \right], \quad j = 0, \ldots, \left[ \frac{m}{2} \right]$$

or

$$\beta_k \delta_j p_1^{2k+1} p_2^{2j+1} p_3^{n+m-2(k+j+1)}, \quad k = 0, \ldots, \left[ \frac{n-1}{2} \right], \quad j = 0, \ldots, \left[ \frac{m-1}{2} \right]$$

where $\alpha_j, \beta_j, \gamma_j, \delta_j$ are coefficients whose explicit expression can be found through (22). We can eliminate all the terms of the form (26) by subtracting the integrals

$$\alpha_k \gamma_j X_0^{n+m-2(k+j)} X_1^k X_2^j, \quad k = 0, \ldots, \left[ \frac{n}{2} \right], \quad j = 0, \ldots, \left[ \frac{m}{2} \right].$$

Similarly, we can subtract

$$\frac{\beta_k \delta_j}{2} X_0^{n+m-2(k+j+1)} X_1^k X_2^j \left( \frac{\Omega_2}{\kappa^2 \Omega_1} (X_3^2 - X_1) - \frac{\kappa^2 \Omega_1}{\Omega_2} X_2 \right),$$

where $k = 0, \ldots, \left[ \frac{n-1}{2} \right], \quad j = 0, \ldots, \left[ \frac{m-1}{2} \right]$, and eliminate all the terms (27). Therefore the order of the integral $X_4$ can be reduced to $m + n - 1$. By construction the terms of order $m + n - 1$ of the reduced integral $\tilde{X}_4$ take the form of products of $m + n - 1$ linear momenta and one coordinate. Since the highest order terms of any integral must belong to the enveloping algebra of the Euclidean algebra [9, 7], we deduce that each of the highest order terms of $\tilde{X}_4$ is a product of $m + n - 2$ linear momenta and one angular momentum. Since the leading order terms of all the other independent integrals $X_0, X_1, X_2, X_3$ contain only linear momenta, it is not possible to further reduce the order of the integral $\tilde{X}_4$ by polynomial combinations of the other integrals.

Concerning the order of $X_5$, we notice that this integral contains the highest order terms of the type

$$\alpha_k \gamma_j p_1^{2k} p_2^{2j+1} p_3^{m+n-2(k+j)-1}, \quad k = 0, \ldots, \left[ \frac{m}{2} \right], \quad j = 0, \ldots, \left[ \frac{n-1}{2} \right]$$

or

$$\beta_k \delta_j p_1^{2k+1} p_2^{2j+1} p_3^{m+n-2(k+j)-1}, \quad k = 0, \ldots, \left[ \frac{m-1}{2} \right], \quad j = 0, \ldots, \left[ \frac{n}{2} \right]$$

whose order as polynomials in $p_1$ and $p_2$ is odd. Therefore, it is not possible to eliminate them by polynomial combinations of the other integrals. As far as we can see, the order of $X_5$ cannot be reduced and it is $m + n$. 

7
5 Example: $n = 2, m = 3$

In order to illustrate the concepts and general results introduced above let us consider a particular nontrivial example. We choose the constants $n = 2$ and $m = 3$, i.e. $κ = \frac{3}{2}$. Thus the Hamiltonian of the 2 d.o.f. oscillator reads

$$H_2 = \frac{1}{2}(P_1^2 + P_2^2) + \frac{1}{2} ω^2(9X^2 + 4Y^2), \quad ω^2 = \frac{1}{4}Ω_1^2 + \frac{1}{9}Ω_2^2.$$

The integral $X_4$ is of order $n + m - 1 = 4$. Thus its leading order terms are fourth order terms in the enveloping algebra of the Euclidean algebra, linear in the angular momenta $l_j$ and cubic in the linear momenta $p_j$. Explicitly, they are as follows

$$X_4^{(h.o.)} = \frac{1}{\sqrt{9Ω_1^2 + 4Ω_2^2}} \left( \frac{16Ω_3^2}{9Ω_1} + 4Ω_1Ω_2 \right) l_2p_2^2p_3 - 4Ω_1Ω_2 (3l_2p_3 + 8l_3p_2) p_3^2 - (4Ω_2^2 + 9Ω_1^2) (l_1p_3 + l_3p_1) p_2^2 + 27Ω_1^2 (l_1p_3 + l_3p_1) p_3^2.$$

The highest order terms read the same also when $X_4$ is expressed in a gauge covariant way, using $p_j^A = p_j + A_j(x)$ and $l_j^A = \sum_k e_{ijkl} x_k p^l$ instead of $p_j, l_j$.

For the sake of completeness let us also write down the rather unwieldy expression for the lower order terms (for our fixed gauge $\mathfrak{g}$, since the gauge covariant expression is even more cumbersome)

$$X_4 - X_4^{(h.o.)} = 2Ω_1τy^2p_1^2p_3 - 2τ \left( 3Ω_1x + \frac{8}{9}Ω_2y \right) yp_1p_2p_3 - \frac{8Ω_1τyzp_3}{9} - \frac{1}{2τ} \left( \frac{16Ω_3^2}{9Ω_1} + 4Ω_1Ω_2 \right) l_2p_2^2p_3 - \frac{1}{2τ} \left( 27 \left( x^2 - \frac{1}{3}y^2 - z^2 \right) Ω_3^2 - 36Ω_2^2Ω_2xy \right) p_3^2 + 4Ω_2Ω_1 (3x^2 + 4y^2 - 3z^2) - \frac{64Ω_3^2}{9} xy \right) p_3^2 - 2Ω_1τyzp_2p_3^2 - \frac{Ω_1^2}{2} \left( \frac{x^2}{4} + 2y^2 - z^2 \right) + \frac{4Ω_2^2}{9} \left( x^2 - \frac{1}{3}y^2 - z^2 \right) + \frac{16Ω_3^2}{81Ω_1} xy \right) yp_3^2 + \frac{1}{18Ω_1} r^3 \left( Ω_1 y - \frac{2}{3}Ω_2 x \right)^2 - \left( Ω_1^2 + \frac{4}{9}Ω_2^2 \right) z^2 \right) y^2p_3^2 + \frac{r^5}{108} y^2 x^2,$$

where $τ = \sqrt{9Ω_1^2 + 4Ω_2^2} = 6Ω$. Sample trajectories for two different choices of the frequencies $Ω_{1,2}$ are shown in Figure 14.
Figure 1: Sample trajectories for $n = 2$, $m = 3$, $\vec{x}_0 = (1, 0, 0)$, $\vec{p}_0 = (0, 1, \frac{1}{2})$ and $\Omega_1 = 1, \Omega_2 = \frac{3}{2}$ (solid line) versus $\Omega_1 = 1, \Omega_2 = \frac{1}{2}$ (dashed line).

6 Conclusions

We have demonstrated in a constructive way that the classical system defined by (1) is maximally superintegrable whenever the parameters satisfy the rationality constraint (8). The constructed fifth independent integral is polynomial in the momenta and coordinates and it is of order $m + n - 1$ where $m$ and $n$ are incommensurable integers such that $S = \frac{m^2\Omega_1}{n\Omega_2}$. Its leading order terms contain angular momenta, in contrast with all the other, previously known integrals for the system (1).

The explicit form of the integral $X_4$ given as the expression (22) minus terms of the form (28) and (29) is unfortunately rather complicated, cf. (31). We were not yet able to obtain any better insight into the structure of the monomials in (31). In particular it would be of interest to be able to predict the monomials appearing in the highest order terms for arbitrary $m, n$, together with relations between their coefficients, without performing the full tedious calculation.
Up to this point our analysis was purely classical. Thus a natural question arises whether its results can be taken over into the quantum case. We notice that the quantum analogues of the expressions $z_1$ and $z_2$

$$\hat{z}_1 = i\hat{P}_1 + m\omega\hat{X}, \quad \hat{z}_2 = i\hat{P}_2 + n\omega\hat{Y}$$

(32)
satisfy $[\hat{z}_1, \hat{z}_2] = 0$. Thus the hermitian expressions

$$\hat{I}_3 = \frac{1}{2} \left( \hat{z}_1^n (\hat{z}_2^\dagger)^m + \hat{z}_2^m (\hat{z}_1^\dagger)^n \right), \quad \hat{I}_4 = \frac{1}{2i} \left( \hat{z}_1^n (\hat{z}_2^\dagger)^m - \hat{z}_2^m (\hat{z}_1^\dagger)^n \right),$$

(33)

are again integrals of motion,

$$[\hat{H}, \hat{I}_3] = 0, \quad [\hat{H}, \hat{I}_4] = 0$$

(34)

(this claim can be also verified directly through a simple commutator evaluation, see e.g. [19]). Thus the integrals of the 2 d.o.f. anisotropic oscillator are preserved by the quantization although their explicit expression as polynomials in $\hat{X}, \hat{Y}$ and $\hat{P}_1, \hat{P}_2$ needs to be symmetrized due to presence of terms involving the same component of both the coordinate and the momentum, e.g. $\hat{X}$ and $\hat{P}_1$ in $\hat{z}_1$.

In order to return to the system (1) we assume that the gauge is fixed as in (3). We notice that the Hamiltonian as well as the integrals (4) and (5) contain only commuting terms in each of their monomials, thus they can be taken into quantum mechanics without any need for symmetrization. In the substitution

$$\hat{X} = \hat{x} - \frac{n^2\Omega_2\hat{p}_3}{m^2\Omega_1^2 + n^2\Omega_2^2}, \quad \hat{Y} = \hat{y} - \frac{m^2\Omega_4\hat{p}_3}{m^2\Omega_1^2 + n^2\Omega_2^2}$$

(35)

we have only commuting variables $\hat{x}, \hat{y}$ and $\hat{p}_3$. The momentum $\hat{p}_3$ also commutes with $\hat{p}_1$ and $\hat{p}_2$. Thus substituting (35) into the expressions (33) for $\hat{I}_3$ and $\hat{I}_4$ one can directly obtain quantum integrals

$$\hat{X}_4 = \frac{1}{2} \left( \left( i\hat{p}_1 - \frac{\Omega_2\hat{p}_3}{m\omega} + m\omega\hat{x} \right)^n \left( -i\hat{p}_2 - \frac{\Omega_4\hat{p}_3}{n\omega} + n\omega\hat{y} \right)^m + \text{h.c.} \right)$$

(36)

and

$$\hat{X}_5 = \frac{1}{2i} \left( \left( i\hat{p}_1 - \frac{\Omega_2\hat{p}_3}{m\omega} + m\omega\hat{x} \right)^n \left( -i\hat{p}_2 - \frac{\Omega_4\hat{p}_3}{n\omega} + n\omega\hat{y} \right)^m - \text{h.c.} \right),$$

(37)

where “h.c.” stands for hermitian conjugate. Expanding the powers in (36) and (37) one obtains quantum analogues of equations (22) and (23) as their properly symmetrized versions.

Also the argument concerning the lowering of the order of the integral $X_4$ remains the same in the quantum case, thus the integral $\hat{X}_4$ makes the quantum system maximally superintegrable of order $(m + n - 1)$. 

10
Let us notice that in accordance with [21] and [22] both the Hamilton–Jacobi and the Schrödinger equations separate in Cartesian coordinates. E.g. the Hamilton’s principal function $S(\vec{x}, t)$ can be written as

$$S(\vec{x}, t) = -Et + K_{3}z + S_{1}(x) + S_{2}(y)$$ (38)

where the functions $S_{1,2}$ are solutions of the quadratures

$$S_{1}'(x) = \pm \sqrt{-\left(\kappa^2 \Omega_{1}^2 + \Omega_{2}^2\right)x^2 + 2K_{3}x\Omega_{2} + 2K_{1}},$$

$$S_{2}'(y) = \pm \sqrt{-\left(\Omega_{1}^2 + \frac{\Omega_{2}^2}{\kappa^2}\right)y^2 + 2K_{3}y\Omega_{1} - K_{2}^2 - 2K_{1} - 2E},$$ (39)

expressible in terms of square roots and inverse trigonometric functions. Whether they separate also in some other coordinate system, i.e. whether the maximally superintegrable system [1] is multiseparable, remains to our knowledge an open question.

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