Extended Kelvin theorem in relativistic magnetohydrodynamics

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We prove the existence of a generalization of Kelvin’s circulation theorem in general relativity which is applicable to perfect isentropic magnetohydrodynamic flow. The argument is based on a new version of the Lagrangian for perfect magnetohydrodynamics. We illustrate the new conserved circulation with the example of a relativistic magnetohydrodynamic flow possessing three symmetries.

I. INTRODUCTION

It is hard to overstate the important role that Kelvin’s theorem on the conservation of circulation of a simple perfect fluid has played in the development of hydrodynamics. Among other things it provided the basis for the discussion of potential flows, and showed that isolated vortices should exist, and that they obey the Helmholtz laws, etc. On the other hand, Kelvin’s theorem is fragile: as soon as dissipation comes in, it breaks down. And when the body force per unit mass of fluid is not a gradient, as happens for the Lorentz force in magnetohydrodynamics (MHD), Kelvin’s theorem ceases to apply.

Often fluids in the real world carry magnetic fields. For example, the fluid at the Earth’s core, the plasma in the sun and pulsars, the ionized gas in interstellar space and in supernova remnants, the plasma in intergalactic space, and many others, all carry magnetic fields. These are important systems for which we need insights of the sort Kelvin’s theorem bestowed on ordinary fluid dynamics. Can we extend Kelvin’s theorem to MHD?

The first such extension was found two decades ago by E. Oron working with the formalism of relativistic perfect MHD. This is a circulation theorem for a hybrid velocity-magnetic field combination. Oron’s derivation assumes both stationary symmetry and axisymmetry, while it is well known that Kelvin’s theorem requires neither of these. Yet it has not proved possible to rid Oron’s result of the symmetry assumptions.

In the present paper we follow, on the wake of our earlier paper, a different route. We use the least action principle to give a rather straightforward existence proof for a generically conserved hybrid velocity–magnetic field circulation in general relativistic MHD which does not depend on the presence of spacetime symmetries. Recently Elsässer has given a related result which he obtains by direct manipulation of the relativistic MHD equations.

A formal introduction to relativistic MHD is given by Lichnerowicz. As mentioned, we approach the whole problem not from equations of motion, but from the least action principle. In special relativity Penfield proposed a perfect fluid Lagrangian which admits vortical isentropic unmagnetized flow. The early general relativistic Lagrangian of Taub as well as the more recent one by Kodama et. al describe only irrotational perfect fluid flows.
The Lin device\(^4\) to include vortical flows is incorporated by Schutz\(^9\) in his Lagrangian. Carter\(^10\) has introduced a relativistic Lagrangian for particle-like motions from which the properties of fluid flows, including vortical ones, can be inferred. However, it does not correspond in detail to the MHD paradigm. Achterberg\(^11\) proposed a general relativistic MHD action, which, however, describes only “irrotational” flows. Thompson\(^12\) used this Lagrangian in the extreme relativistic limit. Heyl and Hernquist\(^13\) modified it to include QED effects. In this paper we follow Schutz’s\(^9\) approach while supplementing it by the introduction of magnetic fields.

In Sec. II.A we review the relativistic MHD equations. In Sec. II.B we describe our Lagrangian,\(^2\) and show that it gives the correct Maxwell and fluid equations, while in Sec. II.C we recover the relativistic Euler equation from it. In Sec. III.A we derive the general form of the conserved circulation, while in Sec. III.B we illustrate it with the special case of a MHD flow endowed with three symmetries.

II. RELATIVISTIC ACTION PRINCIPLE

In this section we construct a Lagrangian density for MHD flow in general relativity (GR). Greek indices run from 0 to 3. The coordinates are denoted \(x^\alpha = (x^0, x^1, x^2, x^3)\); \(x^0\) stands for time. A comma denotes the usual partial derivative; a semicolon covariant differentiation. Our signature is \(\{−,+,+,+\}\). We take \(c = 1\).

\[\text{A. Relativistic MHD Equations}\]

Our first step is enumerating all the correct general relativistic (GR) equations for MHD. These were developed by Lichnerowicz,\(^4\) Novikov and Thorne,\(^14\) Carter,\(^10\) Bekenstein and Oron\(^1\) and others.

The first equation states the conservation of the number of particles (we do not consider particle annihilation or creation processes),

\[
N^\alpha;\alpha = (nu^\alpha);\alpha = 0, \tag{2.1}
\]

where \(N^\alpha\) is the particle number 4–current density, \(n\) the particle proper number density and \(u^\alpha\) the fluid 4–velocity field normalized by \(u^\alpha u_\alpha = -1\). We consider flows which are inviscid and adiabatic, and therefore \(s\), the entropy per particle, is conserved along flow lines:

\[
(sN^\alpha);\alpha = 0 \quad \text{or} \quad u^\alpha s;\alpha = 0. \tag{2.2}
\]

The energy momentum tensor for the magnetized fluid is obtained by adding the electromagnetic energy–momentum tensor to that of an ideal fluid:

\[
T^{\alpha\beta} = pg^{\alpha\beta} + (p + \rho) u^\alpha u^\beta + (4\pi)^{-1}(F^{\alpha\gamma} F^{\beta\gamma} - \frac{1}{4} F^{\gamma\delta} F_{\gamma\delta} g^{\alpha\beta}). \tag{2.3}
\]

Here \(\rho\) denotes the fluid’s proper energy density (including rest mass) and \(p\) the scalar pressure (assumed isotropic), while \(F^{\alpha\beta}\) denotes the electromagnetic field tensor which satisfies the GR Maxwell’s equations.
\[ F^{\alpha_\beta}_{\gamma} = 4\pi j^{\alpha} \]  
\[ F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0, \]  
(2.4)  
(2.5)

where \( j^{\alpha} \) denotes the electric 4–current density.

The magnetic Euler equation for the fluid is derived from the vanishing covariant divergence law \( T^{\alpha\beta}_{\gamma} = 0 \):

\[ \frac{(\rho + p)u^{\beta}u^{\alpha}_{\beta}}{g} = -(g^{\alpha\beta} + u^{\alpha}u^{\beta}p_{\beta} + (4\pi)^{-1}F^{\alpha\beta}F_{\beta}\gamma\gamma. \]  
(2.6)

The term \( a^{\alpha} = u^{\beta}u^{\alpha}_{\beta} \) stands for the fluid’s acceleration 4–vector. The use of covariant derivatives and curved metric ensures that effects of gravitation are automatically included. In view of Eq. (2.4) the above yield the MHD Euler equation

\[ (\rho + p)u^{\alpha} = -h^{\alpha\beta}p_{\beta} + F^{\alpha\beta}j_{\beta}, \]  
(2.7)

where \( h^{\alpha\beta} \) is the projection tensor

\[ h^{\alpha\beta} \equiv g^{\alpha\beta} + u^{\alpha}u^{\beta}. \]  
(2.8)

The Euler equation describes a general electromagnetic field carrying flow. One needs an additional condition to distinguish MHD flow from all others. For any flow carrying an electromagnetic field, the (antisymmetric) Faraday tensor \( F_{\alpha\beta} \) may be split into electric and magnetic vectors with respect to the flow:

\[ E^{\alpha} = F_{\alpha\beta}u^{\beta} \]  
(2.9)

\[ B^{\alpha} = \ast F_{\beta\alpha}u^{\beta} \equiv \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F^{\gamma\delta}u^{\beta}. \]  
(2.10)

Here \( \epsilon^{\alpha\beta\gamma\delta} \) is the Levi-Civita totally antisymmetric tensor \( (\epsilon^{0123} = (-g)^{1/2} \) with \( g \) denoting the determinant of the metric \( g_{\alpha\beta} \) and \( \ast F^{\alpha\beta} \) is the dual of \( F_{\alpha\beta} \). In a frame comoving with the fluid, these 4–vectors have only spatial parts which correspond to the usual \( E \) and \( B \), respectively. One can use Eqs. (2.9,2.10) to express \( F_{\alpha\beta} \) using those 4–vectors

\[ F_{\alpha\beta} = u_{\alpha}E_{\beta} - u_{\beta}E_{\alpha} + \epsilon_{\alpha\beta\gamma\delta}u^{\gamma}B^{\delta} \]  
(2.11)

For an infinitely conducting (perfect MHD) fluid, the electric field in the fluid’s frame must vanish, i.e.,

\[ E^{\alpha} = F_{\alpha\beta}u^{\beta} = 0. \]  
(2.12)

This corresponds to the usual MHD condition \( E + v \times B = 0. \)

**B. Relativistic Lagrangian density and equations of motion**

We now propose a Lagrangian density for GR MHD flow based on Schutz’s Lagrangian density for pure fluids in GR:

\[ \mathcal{L} = -\rho(n, s) - (16\pi)^{-1}F_{\alpha\beta}F^{\alpha\beta} + \phi N^{\alpha}_{,\alpha} + \eta (sN^{\alpha})_{,\alpha} + \lambda (\gamma N^{\alpha})_{,\alpha} + \tau^{\alpha}F_{\alpha\beta}N^{\beta}. \]  
(2.13)
As shown below, our Lagrangian reproduces Eqs. (2.1-2.2), (2.4-2.7) and (2.12). Here \( \phi \) is the Lagrange multiplier associated with the conservation of particle number Eq. (2.1) viewed as a constraint, \( \eta \) is that multiplier associated with the adiabatic flow constraint, Eq. (2.2), and \( \lambda \) is that associated with the conservation along the flow of Lin’s \(^{8}\) quantity \( \gamma \). \( \tau_\alpha \) is a quartet of Lagrange multipliers which enforce the field freezing condition Eq. (2.12). We should interject that \( \tau^\alpha \) is not determined uniquely, as we discuss in Sec. III.

We view \( \gamma, N^\alpha \) and \( s \) as the independent fluid variables, while \( n \) and \( u^\alpha \) are determined by the obvious relations

\[
-N^\alpha N_\alpha = n^2; \quad u^\alpha = n^{-1} N^\alpha. \quad (2.14)
\]

Some authors prefer to include in the Lagrangian the constraint \( N^\alpha N_\alpha + n^2 = 0 \), which stands for the normalization of the fluid’s 4–velocity. However we choose to impose this constraint later and thus remain with a simpler Lagrangian.

We can now vary the Lagrangian with respect to the independent variables. Variation of \( \phi \) recovers the conservation of particles \( N^\alpha \langle_\alpha = 0 \). Variation of \( \lambda \) with subsequent use of the previous result yields

\[
\gamma_{,\alpha} u^\alpha = 0. \quad (2.15)
\]

If we vary Lin’s \( \gamma \) we get

\[
\lambda_{,\alpha} u^\alpha = 0. \quad (2.16)
\]

These results inform us that \( \gamma \) and \( \lambda \) are both locally conserved along the flow. In view of the thermodynamic relation \( n^{-1} (\partial \rho/\partial s)_n = T \), with \( T \) the locally measured fluid temperature, variation of \( s \) gives

\[
u^\alpha \eta_{,\alpha} = -T. \quad (2.17)
\]

We now vary \( N^\alpha \) using the obvious consequence of Eq. (2.14),

\[
\delta n = -u_\alpha \delta N^\alpha, \quad (2.18)
\]

together with the thermodynamic relation \(^{14}\) involving the specific enthalpy \( \mu \),

\[
\mu \equiv (\partial \rho/\partial n)_s = n^{-1} (\rho + p). \quad (2.19)
\]

We thus get the most important equation herein:

\[
\mu u_\alpha = \phi_{,\alpha} + s \eta_{,\alpha} + \gamma \lambda_{,\alpha} + \tau^\beta F_{\alpha\beta}. \quad (2.20)
\]

By contracting Eq. (2.20) with \( u^\alpha \) and using \( u_\alpha u^\alpha = -1 \) as well as Eqs. (2.12) and (2.16-2.17), we get

\[
\phi_{,\alpha} u^\alpha = -\mu + Ts. \quad (2.21)
\]

Thus the proper time rate of change of \( \phi \) along the flow is just minus the specific Gibbs energy or chemical potential.
The importance of using Lin’s $\gamma$ is clear from Eq. (2.20). In the pure isentropic fluid case ($F^{\alpha\beta} = 0$ and $s = \text{const.}$), the Khalatnikov vorticity tensor given by

$$\omega_{\alpha\beta} = (\mu u_{\beta})_{,\alpha} - (\mu u_{\alpha})_{,\beta} = (\gamma \lambda_{\beta})_{,\alpha} - (\gamma \lambda_{\alpha})_{,\beta}$$

(2.22)

would vanish in the absence of $\gamma$, thus constraining us to discuss only irrotational flow. This problem is well known from non relativistic pure fluid Lagrangian theory. Lin [8] remarked that one can label each fluid element by its original Lagrangian coordinate. The requirement that this stay fixed adds an additional constraint (“label conservation”) to the Lagrangian function, and makes possible the description of vortical flow. While for isentropic flow Kelvin’s theorem forbids the creation of vorticity, the flow in any given region can be vortical due to conditions upstream.

As customary, we write $F_{\alpha\beta} = A_{\beta;\alpha} - A_{\alpha;\beta} = A_{\beta} - A_{\alpha}$, which ensures that the Maxwell Eqs. (2.5) are automatically satisfied. The other half, Eqs. (2.4), are obtained by varying with respect to the components of the vector potential $A_{\alpha}$. Because of the antisymmetry of $F_{\alpha\beta}$, the last term of the Lagrangian, Eq. (2.13), can be written as $(\tau^\beta N^\alpha - \tau^\alpha N^\beta) A_{\alpha;\beta}$. The variation of $A_{\alpha}$ in the corresponding term in the action produces, after integration by parts, the term $[(1/2)(\tau^\alpha N^\beta - \tau^\beta N^\alpha)]_{;\beta} \delta A_{\alpha}$. Because for any antisymmetric tensor $t_{\alpha\beta}$, $(-g)^{1/2}t_{\alpha\beta;\beta} = [(-g)^{1/2}t_{\alpha\beta}]_{;\beta}$, variation of $A_{\alpha}$ leads to the equation

$$F_{\alpha;\beta} = 4\pi \left( \tau^\alpha N^\beta - \tau^\beta N^\alpha \right)_{;\beta}. \quad (2.23)$$

We see that this is just Eq. (2.4) provided we identify the electric current density $j^\alpha$ as

$$j^\alpha = \left( \tau^\alpha N^\beta - \tau^\beta N^\alpha \right)_{;\beta} \quad (2.24)$$

Since the divergence of the divergence of any antisymmetric tensor vanishes, the charge conservation equation ($j^{\alpha;\alpha} = 0$) is satisfied automatically. Formally Eq. (2.23) determines the Lagrange multiplier 4–vector $\tau^\alpha$, modulo the freedom inherent in it, as we discuss in Sec. III.

C. MHD Euler equation in General Relativity

We now go on to tie the equations of motion together to yield the MHD Euler equation (2.7). We begin by writing the Khalatnikov vorticity $\omega_{\beta\alpha}$ in two forms,

$$\omega_{\beta\alpha} = \mu_{,\beta} u_{\alpha} - \mu_{,\alpha} u_{\beta} + \mu u_{\alpha;\beta} - \mu u_{\beta;\alpha}, \quad (2.25)$$

as well as by means of Eq. (2.20)

$$\omega_{\beta\alpha} = s_{,\beta} \eta_{,\alpha} - s_{,\alpha} \eta_{,\beta} + \gamma_{,\beta} \lambda_{,\alpha} - \gamma_{,\alpha} \lambda_{,\beta} + \tau^\delta_{;\beta} F_{\alpha\delta} - \tau^\delta_{;\alpha} F_{\beta\delta} + \tau^\delta F_{\alpha\delta;\beta} - \tau^\delta F_{\beta\delta;\alpha}. \quad (2.26)$$

Contracting the left hand side of the first with $N^{\alpha}$, recalling Eq. (2.14) and that by normalization $u^{\alpha} u_{\alpha;\beta} = 0$ whereas $u^{\alpha} u_{\alpha;\beta} = a_{\alpha}$ (recall that $a^{\alpha}$ is the fluid’s 4–acceleration), we get
\[ \omega_{\beta\alpha} N^\alpha = -n \mu_\beta - n \mu_\alpha u^\alpha u_\beta - n \mu a_\beta = -n h_{\beta\alpha} - n \mu a_\beta. \tag{2.27} \]

On the other hand, contracting Eq. (2.26) with \( N^\alpha \) and using Eqs. (2.13-2.17) and (2.12) to drop a number of terms, we get

\[ \omega_{\beta\alpha} N^\alpha = -n T s_{\beta\alpha} - \tau^\delta\gamma_{\gamma\alpha} F_{\beta\delta} N^\alpha + \tau^\delta F_{\alpha\delta\beta} N^\alpha - \tau^\delta F_{\beta\delta\alpha} N^\alpha. \tag{2.28} \]

By virtue of Eq. (2.2), \( -n T s_{\beta\alpha} \) is the same as \( -n T h_{\beta\alpha} \). It is convenient to use the thermodynamic identity \( d\mu = n - 1 dp + T ds \), which follows from Eq. (2.19) and the first law \( d(\rho/n) = T ds - pd(1/n) \), to replace \( -n T s_{\beta\alpha} \) in Eq. (2.28) by \( h_{\beta\alpha} (n \mu_\alpha + p_\alpha) \). Equating our two expressions for \( \omega_{\beta\alpha} N^\alpha \) gives, after a cancellation,

\[ - (n \mu a_\beta + h_{\beta\alpha} p_\alpha) = -\tau^\delta\gamma_{\gamma\alpha} F_{\beta\delta} N^\alpha + \tau^\delta F_{\alpha\delta\beta} N^\alpha - \tau^\delta F_{\beta\delta\alpha} N^\alpha. \tag{2.29} \]

The last two terms in this equation can be combined into a single one by virtue of Eq. (2.5), which can be written with covariant as well as ordinary derivatives. Further, by Eq. (2.19) we may replace \( n \mu \) by \( \rho + p \). In this manner we get

\[ (\rho + p) a_\beta = -h_{\beta\alpha} p_\alpha + F_{\beta\alpha} \tau^\delta N^\alpha + F_{\beta\delta} \tau^\delta N^\alpha. \tag{2.30} \]

The term \( \tau^\delta\gamma_{\gamma\alpha} N^\alpha \) here can be replaced by two others with help of Eq. (2.23) if we take into account that \( N^\delta_{\gamma\beta} = 0 \):

\[ (\rho + p) a_\beta = -h_{\beta\alpha} p_\alpha + (4\pi)^{-1} F_{\beta\delta} F_{\delta\alpha} + F_{\beta\delta} \tau^\delta N^\alpha + F_{\beta\delta} \left( \tau^\alpha N^\delta \right)_{\alpha}. \tag{2.31} \]

We note that the last two terms on the right hand side combine into \( \left(F_{\beta\alpha} N^\alpha \right)^\delta_{\beta} \), which vanishes by Eq. (2.12). Now substituting from the Maxwell equations (2.4) we arrive at the final equation

\[ (\rho + p) a_\beta = -h_{\beta\alpha} p_\alpha + F_{\beta\delta} j^\delta, \tag{2.32} \]

which is the correct GR MHD Euler equation (2.7).

Note that we have not used any information about \( \tau^\alpha \) beyond Eq. (2.23); hence Euler’s equation is valid for all choices of \( \tau^\alpha \), of which there are many as we shall explain. Since we are able to obtain all equations of motion for GR MHD from our Lagrangian density, we may regard it as correct, and go on to look at some consequences.

### III. NEW CIRCULATION CONSERVATION LAW

#### A. General Remarks

Eqs. (2.20) and (2.15-2.16) lead immediately to a law of circulation conservation for relativistic perfect isentropic MHD flow. Define the vector field

\[ z_\alpha \equiv \mu u_\alpha - \tau^\beta F_{\alpha\beta}. \tag{3.1} \]
and its associated circulation $\Gamma$

$$\Gamma = \oint_{C} z_{\alpha} dx^{\alpha}, \quad (3.2)$$

where $C$ is a closed simply connected curve drifting with the fluid. According to Eq. (2.20), $z_{\alpha} = \phi_{,\alpha} + s\eta_{,\alpha} + \gamma \lambda_{,\alpha}$. Since $\phi_{,\alpha}$ is a gradient, its contribution to $\Gamma$ vanishes. Likewise, for isentropic flow ($s = \text{const}$.) the term involving $s\eta_{,\alpha}$ makes no contribution to $\Gamma$. Thus

$$\Gamma = \oint_{C} \gamma \lambda_{,\alpha} dx^{\alpha} = \oint_{C} \gamma d\lambda. \quad (3.3)$$

By Eqs. (2.15-2.16) both $\gamma$ and $\lambda$ are conserved with the flow. Thus $\Gamma$ is a circulation which is conserved along the flow.

In the absence of electromagnetic fields and in the nonrelativistic limit ($\mu \to m$ where $m$ is a fluid particle’s rest mass), $\Gamma$ for a curve $C$ taken at constant time reduces to Kelvin’s circulation. On this ground our result can be considered a proof that a generalization of Kelvin’s circulation theorem to GR MHD exists. This conclusion goes beyond Oron’s original conserved circulation in MHD \cite{1} in that no symmetry is necessary here for the circulation to be conserved.

To make full use of the new conservation law to solve or simplify problems in MHD, one must evidently know $\tau_{\alpha}$ explicitly. There is a certain amount of freedom in $\tau_{\alpha}$ which we have already discussed.\footnote{\textcircled{3}} Here it is important that the Lagrangian density (2.13) is invariant under the addition of $fu_{\alpha}$ to $\tau_{\alpha}$, where $f$ is an arbitrary scalar, because $fF_{\alpha\beta}u^{\alpha}u^{\beta}$ vanishes identically. We use this freedom to demand that $\tau_{\alpha}u_{\alpha} = 0$. We now recast the circulation law in a form eminently suitable for use in problems with symmetry.

First from Eq. (2.23) we infer that

$$\tau_{\alpha} N^{\beta} - \tau_{\beta} N^{\alpha} = (4\pi)^{-1}(F^{\alpha\beta} - W_{\alpha\beta}) ; \quad W_{\alpha\beta} :_{,\beta} = 0 \quad (3.4)$$

where $W_{\alpha\beta}$ is an antisymmetric tensor. Since $W_{\alpha\beta}$ must be divergenceless, it can generically be written as the dual of a curl,

$$W_{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = *F^{\alpha\beta} \quad (3.5)$$

where $F_{\gamma\delta} = A_{\delta,\gamma} - A_{\gamma,\delta}$. $A_{\alpha}$ here is to be distinguished from the ordinary vector potential $A_{\alpha}$ of $F_{\alpha\beta}$. Parenthetically we mention that one can independently make gauge transformations of $A_{\alpha}$ and of $A_{\alpha}$. This underscores the little known fact that MHD is a theory with $U(1) \times U(1)$ gauge symmetry.

By taking the dual of Eq. (3.4) and then contracting it with $u^{\gamma}$ (remembering that twice dual is equivalent to changing the sign) we get, with the help of Eq. (2.10),

$$B_{\delta} = F_{\delta,\gamma} u^{\gamma}. \quad (3.6)$$

Contracting Eq. (3.4) with $u_{\beta}$ and recalling that $F^{\alpha\beta}u_{\beta} = 0$ and $\tau_{\alpha}u^{\alpha} = 0$ gives

$$\tau^{\alpha} = (8\pi n)^{-1} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} u_{\beta}. \quad (3.7)$$

Now using Eq. (2.11) with $E_{\alpha} = 0$ and Eq. (3.7) we have
\[
F_{\alpha\beta\tau} = (8\pi n)^{-1} \epsilon_{\alpha\beta\gamma\delta} \epsilon^{\beta\xi\mu\nu} B^\gamma u^\delta u_\xi F_{\mu\nu}
\]  \hfill (3.8)

With the easily checked identity
\[
\epsilon_{\alpha\beta\gamma\delta} \epsilon^{\beta\xi\mu\nu} = \delta^\xi_{\alpha} \delta^\mu_{\gamma} \delta^\nu_{\delta} - \delta^\xi_{\alpha} \delta^\nu_{\gamma} \delta^\mu_{\delta} + \delta^\nu_{\alpha} \delta^\xi_{\gamma} \delta^\mu_{\delta} - \delta^\mu_{\alpha} \delta^\xi_{\gamma} \delta^\nu_{\delta} - \delta^\nu_{\alpha} \delta^\mu_{\gamma} \delta^\xi_{\delta} + \delta^\mu_{\alpha} \delta^\nu_{\gamma} \delta^\xi_{\delta}
\]  \hfill (3.9)

Eq. (3.8) reduces to
\[
F_{\alpha\beta\tau} = (4\pi n)^{-1}(F_{\alpha\nu} B^\nu - B^\nu B^\nu u_\alpha)
\]  \hfill (3.10)

where use has been made of Eq. (3.6), \(B^\mu u_\nu = 0\) and \(u_\mu u^\mu = -1\). The conservation law (3.2) now takes the form
\[
\Gamma = \oint_C \left[ \chi u_\alpha - (4\pi n)^{-1} F_{\alpha\beta} B^\beta \right] dx^\alpha,
\]  \hfill (3.11)

with \(\chi \equiv \mu + (4\pi n)^{-1} B_\beta B^\beta\). The quantity \(\chi\) plays an important role in Oron’s generalization of Kelvin’s theorem to MHD.

**B. Example: MHD Flow with Three Symmetries**

Since \(F_{\alpha\beta}\) is not known explicitly, we cannot work out the conserved circulation without further work. Here we shall make some progress in this direction in the case of flow which is both stationary, and possesses two additional spatial symmetries (these last we assume not to be of the angular type). This means the physical quantities such as \(u^\alpha\), \(N^\alpha\), \(j^\alpha\) or \(B^\alpha\) are unchanged upon being Lie dragged along either of the three Killing vectors. We cannot automatically require the same of \(F_{\alpha\beta}\) because it is not a directly measurable quantity. However, as mentioned in our earlier work, we can make a transformation \(F_{\alpha\beta} \rightarrow F_{\alpha\beta} + f_{\alpha\beta}\), where \(f_{\alpha\beta}\) is a curl and orthogonal to \(u^\beta\), without changing the values of \(j^\alpha\) or \(B^\alpha\); this transformation at most adds to \(\Gamma\) a conserved quantity leaving it conserved. We shall assume here that by means of such a transformation we can make \(F_{\alpha\beta}\) share the symmetries of \(j^\alpha\) or \(B^\alpha\).

The following remarks apply when we choose the coordinates \(x^2\) and \(x^3\) to extend along the integral curves of the two spatial Killing vectors; by assumption these curves are non-compact, and so are \(x^2\) and \(x^3\). We also assume \(x^1\) is noncompact. The most general form for the “vector potential” \(A_\alpha\) for which \(F_{\alpha\beta}\) is independent of \(x^0\), \(x^2\) and \(x^3\), is (here we sacrifice manifest covariance in order to make the symmetries manifest)
\[
A_\alpha = x^0 \hat{\Phi}_\alpha + x^2 \hat{\Psi}_\alpha + x^3 \hat{\Xi}_\alpha + V_\alpha
\]  \hfill (3.12)

where \(\hat{\Phi}, \hat{\Psi}\) and \(\hat{\Xi}\) are each a linear combination of \(x^0\), \(x^2\) and \(x^3\) with constant coefficients plus a function of the nontrivial coordinate \(x^1\) only, while the components of the “vector” \(V_\alpha\) also depend only on \(x^1\). Thus
\[
F_{01} = (\hat{\Phi} - V_0)_1 \equiv \Phi_1
\]  \hfill (3.13)
\[
F_{21} = (\hat{\Psi} - V_2)_1 \equiv \Psi_1
\]  \hfill (3.14)
\[
F_{31} = (\hat{\Xi} - V_3)_1 \equiv \Xi_1
\]  \hfill (3.15)
while $F_{02}, F_{03}$ and $F_{23}$ are all strictly constant, and thus stand for global parameters of the flow.

By means of Eq. (3.4) we may now compute the components of $B_\alpha$:

$$B_0 = \Phi_1 u^1 + F_{02} u^2 + F_{03} u^3 \quad (3.16)$$

$$B_1 = -\Phi_1 u^0 - \Psi_2 u^2 - \Xi_1 u^3 \quad (3.17)$$

$$B_2 = \Psi_1 u^1 + F_{20} u^0 + F_{23} u^3 \quad (3.18)$$

$$B_3 = \Xi_1 u^1 + F_{30} u^0 + F_{32} u^2 \quad (3.19)$$

Solving these last for the derivatives of $\Phi, \Psi$ and $\Xi$ we get from (3.13)-(3.15)

$$F_{01} = (B_0 - F_{02} u^2 - F_{03} u^3)/u^1 \quad (3.20)$$

$$F_{21} = (B_2 + F_{02} u^0 - F_{23} u^3)/u^1 \quad (3.21)$$

$$F_{31} = (B_3 + F_{03} u^0 + F_{23} u^2)/u^1 \quad (3.22)$$

We now calculate the quantity appearing last in Eq. (3.11) with the help of Eqs. (3.20) - (3.22). If temporarily we take $F_{02} = F_{03} = 0$, we get

$$F_{\alpha\beta} B^\beta dx^\alpha = (B^1/u^1)(B_0 dx^0 + B_2 dx^2 + B_3 dx^3) - (B_0 B^0 + B_2 B^2 + B_3 B^3)(dx^1/u^1)$$

$$+ (F_{23}/u^1)[(u^3 B^2 - u^2 B^3)dx^1 + (u^1 B^3 - u^3 B^1)dx^2 + (u^2 B^1 - u^1 B^2)dx^3] \quad (3.23)$$

This is much simplified by adding $B_1 B^1(dx^1/u^1)$ to the first term and subtracting it from the second term. In addition, one can unify the terms in square brackets by employing the Levi-Civita tensor. Putting all this together and using the expression (2.11) we have

$$\Gamma = \oint_C \left[ \chi u_\alpha + (4\pi nu^1)^{-1} \left( B_\beta B^\beta \delta_\alpha^1 - B^1 B_\alpha + (F_{23}/\sqrt{-g}) F_{0\alpha} \right) \right] dx^\alpha \quad (3.24)$$

We now show that the last term in the integrand does not contribute. First of all we can take the constant $F_{23}$ out of the integral. Next we realize that the law of particle number conservation (2.1) and the assumed symmetries tell us that $-C^{-1} \equiv \sqrt{-g} nu^1$ does not depend on $x^1$ either, and so can also be taken out of the integral. We are left with a term proportional to $C \oint F_{0\alpha} dx^\alpha$. Now, just like $F_{0\beta}, F_{\alpha\beta}$ derives from a vector potential, and must have all the symmetries we have assumed. We can thus take its vector potential $A_\alpha$ to have the form (3.12), but with new functions $\Phi', \Psi', \Xi'$ (sans circumflex), each of which is, again, a linear combination of $x^0, x^2$ and $x^3$ with constant coefficients plus a function of the nontrivial coordinate $x^1$ only. It follows that $F_{0\alpha} dx^\alpha = \Phi'_1 dx^1 + \text{const.} dx^2 + \text{const.} dx^3$. But this is a perfect differential; hence the term proportional to $F_{23}$ in Eq. (3.24) vanishes. If we now reinstate $F_{02}$ and $F_{03}$ we find that they contribute to $\Gamma$ terms proportional to $C F_{02} \oint F_{3\alpha} dx^\alpha$ and $C F_{03} \oint F_{2\alpha} dx^\alpha$, both of which are found to vanish by reasoning analogous to the above.

By the symmetries the term $B_\beta B^\beta (nu^1)^{-1}$ in the integrand of Eq. (3.24) can only depend on $x^1$. It is integrated over $x^1$ only, once forward and once backward because $C$ is a closed curve. Hence this term makes no contribution to $\Gamma$. Further, it is a consequence of Euler’s equation (2.7) that $(\mu B^\alpha)_\alpha = 0$. Only the radial derivative survives here, so that we have $\sqrt{-g} B^1 = D/\mu$ with $D$ a constant, a further global parameter of the flow. We thus obtain the final form of the conserved circulation:
\[ \Gamma = \oint_C \left[ \chi u_\alpha + DC(4\pi\mu)^{-1}B_\alpha \right] dx^\alpha \] (3.25)

This is exactly Oron’s original conserved circulation for two symmetries (we have here defined \( C \) and \( D \) to correspond to the two quantities with the same names which are conserved along streamlines for the case of two symmetries). Thus the imposition of a third symmetry does not cause any changes in \( \Gamma \). In future work we shall endeavor to recover Oron’s conserved quantity in the presence of two symmetries from the present approach, as well as to explore the conserved circulation in stationary flow with no spatial symmetry. The general case of no symmetries is a more distant goal.

ACKNOWLEDGMENTS

This work is supported by a grant from the Israel Science Foundation, which was established by the Israel National Academy of Sciences.

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