Conjunction of Conditional Events and T-norms

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Abstract. We study the relationship between a notion of conjunction among conditional events, introduced in recent papers, and the notion of Frank t-norm. By examining different cases, in the setting of coherence, we show each time that the conjunction coincides with a suitable Frank t-norm. In particular, the conjunction may coincide with the Product t-norm, the Minimum t-norm, and Lukasiewicz t-norm. We show by a counterexample, that the prevision assessments obtained by Lukasiewicz t-norm may be not coherent. Then, we give some conditions of coherence when using Lukasiewicz t-norm.

Keywords: Coherence, conditional event, conjunction, Frank t-norm.

1 Introduction

In this paper we use a notion of conjunction, which differently from other authors, is defined in the setting of coherence as a suitable conditional random quantity with values in the unit interval (see, e.g. \cite{2, 3, 4, 5, 9}). We study the relationship between our notion conjunction and the notion of Frank t-norm. We show that, under the hypothesis of logical independence, if the prevision assessments involved with the conjunction \((A | H) \wedge (B | K)\) of two conditional events are coherent, then the prevision of the conjunction coincides, for a suitable \(\lambda \in [0, +\infty]\), with the Frank t-norm \(T_\lambda(x, y)\), where \(x = P(A | H), y = P(B | K)\). Moreover, \((A | H) \wedge (B | K) = T_\lambda(A | H, B | K)\). Then, we consider the case \(A = B\), by determining the set of all coherent assessment \((x, y, z)\) on \(\{A | H, A | K, (A | H) \wedge (A | K)\}\). We show that, under coherence, it holds that \((A | H) \wedge (A | K) = T_\lambda(A | H, A | K)\), where \(\lambda \in [1, +\infty]\). We also study the particular case where \(A = B\) and \(H K = \emptyset\). Then, we consider conjunctions of three conditional events and we show that to make prevision assignments by means of the Product t-norm, or the Minimum t-norm, is coherent. Finally, we examine the Lukasiewicz t-norm and we show by a counterexample that coherence is in general not assured. We give some conditions for coherence when the prevision assessments are made by using the Lukasiewicz t-norm.

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2 Preliminary Notions and Results

In our approach, given two events $A$ and $H$, with $H \neq \emptyset$, the conditional event $A|H$ is looked at as a three-valued logical entity which is true, or false, or void, according to whether $AH$ is true, or $A\overline{H}$ is true, or $\overline{A}H$ is true. In numerical terms $A|H$ assumes one of the values 1, or 0, or $x = P(A|H)$. Then $A|H = AH + x\overline{H}$.

Given a family $F = \{X_1|H_1, \ldots, X_n|H_n\}$, for each $i \in \{1, \ldots, n\}$ we denote by $\{x_{i1}, \ldots, x_{ir_i}\}$ the set of possible values of $X_i$ when $H_i$ is true; then, for each $i$ and $j = 1, \ldots, r_i$, we set $A_{ij} = (X_i = x_{ij})$. We set $C_0 = H_1 \cdots H_n$ (it may be $C_0 = \emptyset$); moreover, we denote by $C_1, \ldots, C_m$ the constituents contained in $H_1 \vee \cdots \vee H_n$. Hence $\bigwedge_{i=1}^m (A_{1i} \vee \cdots \vee A_{ri} \vee H_i) = \bigvee_{h=0}^m C_h$. With each $C_h, h \in \{1, \ldots, m\}$, we associate a vector $Q_h = (q_{h1}, \ldots, q_{hn})$, where $q_{hi} = x_{ij}$ if $C_h \subseteq A_{ij}, j = 1, \ldots, r_i$, while $q_{hi} = \mu_i$ if $C_h \subseteq \overline{H_i}$; with $C_0$ it is associated $Q_0 = M = (\mu_1, \ldots, \mu_n)$. Denoting by $I$ the convex hull of $Q_1, \ldots, Q_m$, the condition $M \in I$ amounts to the existence of a vector $(\lambda_1, \ldots, \lambda_m)$ such that: $\sum_{h=1}^m \lambda_h Q_h = M$, $\sum_{h=1}^m \lambda_h = 1$, $\lambda_h \geq 0$, $h \in \{1, \ldots, m\}$; in other words, $M \in I$ is equivalent to the solvability of the system $(\Sigma)$, associated with $(F, M)$,

$$\begin{cases}
\sum_{h=1}^m \lambda_h q_{hi} = \mu_i, & i \in \{1, \ldots, n\}, \\
\sum_{h=1}^m \lambda_h = 1, & \lambda_h \geq 0, h \in \{1, \ldots, m\}.
\end{cases}$$

Given the assessment $M = (\mu_1, \ldots, \mu_n)$ on $F = \{X_1|H_1, \ldots, X_n|H_n\}$, let $S$ be the set of solutions $A = (\lambda_1, \ldots, \lambda_m)$ of system $(\Sigma)$. We point out that the solvability of system $(\Sigma)$ is a necessary (but not sufficient) condition for coherence of $M$ on $F$. By assuming the system $(\Sigma)$ solvable, that is $S \neq \emptyset$, we define:

$$I_0 = \{i : \max_{A \in S} \sum_{h \in C_h \subseteq H_i} \lambda_h = 0\},$$

$$F_0 = \{X_i|H_i, i \in I_0\}, \quad M_0 = (\mu_i, i \in I_0).$$

Then, the following theorem can be proved (Theorem 3)

**Theorem 1.** [Operative characterization of coherence] A conditional prevision assessment $M = (\mu_1, \ldots, \mu_n)$ on the family $F = \{X_1|H_1, \ldots, X_n|H_n\}$ is coherent if and only if the following conditions are satisfied:

(i) the system $(\Sigma)$ defined in (1) is solvable;

(ii) if $I_0 \neq \emptyset$, then $M_0$ is coherent.

**Definition 1.** Given any pair of conditional events $A|H$ and $B|K$, with $P(A|H) = x$ and $P(B|K) = y$, we define their conjunction as the conditional random quantity $(A|H) \land (B|K) = Z \|(H \lor K)$, where $Z = \min \{A|H, B|K\}$. In betting terms, $z = P[\{(A|H) \land (B|K)\}]$ represents the amount you agree to pay, with the proviso that you will receive the quantity:

$$(A|H) \land (B|K) = \begin{cases}
1, & \text{if } AH BK \text{ is true,} \\
0, & \text{if } A\overline{H} \lor B \overline{K} \text{ is true,} \\
x, & \text{if } \overline{A}H BK \text{ is true,} \\
y, & \text{if } A\overline{H} \overline{K} \text{ is true,} \\
z, & \text{if } \overline{A}H \overline{K} \text{ is true.}
\end{cases}$$
Different approaches to compounded conditionals, not based on coherence, have been developed by other authors (see, e.g., [10]).

We recall a result which shows that Fréchet-Hoeffding bounds still hold for the conjunction of conditional events ([3, Theorem 7]).

**Theorem 2.** Given any coherent assessment \((x, y)\) on \(\{A|H, B|K\}\), with \(A, H, B, K\) logically independent, \(H \neq \emptyset, K \neq \emptyset\), the extension \(z = \mathbb{P}[A|H) \land (B|K)]\) is coherent if and only if the following Fréchet-Hoeffding bounds are satisfied:

\[
\max\{x + y - 1, 0\} = z' \leq z \leq z'' = \min\{x, y\}. \tag{4}
\]

**Remark 1.** From Theorem 2 as the assessment \((x, y)\) on \(\{A|H, B|K\}\) is coherent for every \((x, y) \in [0, 1]^2\), the set \(\Pi\) of coherent assessments \((x, y, z)\) on \(\{A|H, B|K, (A|H) \land (B|K)\}\) is

\[
\Pi = \{(x, y, z) : (x, y) \in [0, 1]^2, \max\{x + y - 1, 0\} \leq z \leq \min\{x, y\}\}. \tag{5}
\]

The set \(\Pi\) is the tetrahedron with vertices the points \((1, 1, 1), (1, 0, 0), (0, 1, 0), (0, 0, 0)\).

For other definition of conjunctions, where the conjunction is a conditional event, some results on lower and upper bounds have been given in [10].

**Definition 2.** Let be given \(n\) conditional events \(E_1|H_1, \ldots, E_n|H_n\). For each non-empty subset \(S\) of \(\{1, \ldots, n\}\), let \(x_S\) be a prevision assessment on \(\bigwedge_{i \in S}(E_i|H_i)\). Then, the conjunction \(C_{1\cdots n} = (E_1|H_1) \land \cdots \land (E_n|H_n)\) is defined as

\[
C_{1\cdots n} = \begin{cases} 
1, & \text{if } \bigwedge_{i=1}^{n} E_i|H_i, \text{ is true} \\
0, & \text{if } \bigvee_{i=1}^{n} \overline{E_i}|H_i, \text{ is true,} \\
x_S, & \text{if } \bigwedge_{i \in S} E_i|H_i \text{ is true.}
\end{cases} \tag{6}
\]

In particular \(C_i = E_i|H_i, i = 1, \ldots, n\); \(C_{ij} = (E_i|H_i) \land (E_j|H_j); \{i, j\} \subseteq \{1, 2, \ldots, n\}\), and so on. Then, for instance, differently from other papers, here the symbol \(C_n\) represents the conditional event \(E_n|H_n\) (and not the conjunction \(\bigwedge_{i=1}^{n}(E_i|H_i)\)). Moreover, if \(S = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}\), the conjunction \(\bigwedge_{i \in S}(E_i|H_i)\) is denoted by \(C_{i_1 \cdots i_k}\) and \(x_S\) is also denoted by \(x_{i_1 \cdots i_k}\).

In the betting framework, you agree to pay \(\mu = \mathbb{P}(C_{1\cdots n})\) with the proviso that you will receive:

- 1, if all conditional events are true;
- 0, if at least one of the conditional events is false;
- the prevision of the conjunction of that conditional events which are void, otherwise.

The operation of conjunction is associative and commutative.

**Theorem 3.** Assume that the events \(E_1, E_2, E_3, H_1, H_2, H_3\) are logically independent, with \(H_1 \neq \emptyset, H_2 \neq \emptyset, H_3 \neq \emptyset\). Then, the set \(\Pi\) of all coherent assessments \(\mathcal{M} = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})\) on \(\mathcal{F} = \{C_1, C_2, C_3, C_{12}, C_{13}, C_{23}, C_{123}\}\).
is the set of points \((x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})\) which satisfy the following conditions

\[
\begin{align*}
(x_1, x_2, x_3) & \in [0, 1]^3, \\
\max\{x_1 + x_2 - 1, x_{13} + x_{23} - x_3, 0\} & \leq x_{12} \leq \min\{x_1, x_2\}, \\
\max\{x_1 + x_3 - 1, x_{12} + x_{23} - x_2, 0\} & \leq x_{13} \leq \min\{x_1, x_3\}, \\
\max\{x_2 + x_3 - 1, x_{13} + x_{23} - x_1, 0\} & \leq x_{23} \leq \min\{x_2, x_3\}, \\
1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23} & \geq 0, \\
x_{123} & \geq \max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13} + x_{23} - x_3\}, \\
x_{123} & \leq \min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\}.
\end{align*}
\] (7)

Remark 2. As shown in (7), the coherence of \((x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})\) amounts to the condition

\[
\begin{align*}
\max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13} + x_{23} - x_3\} & \leq x_{123} \leq \\
\min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\}. \\
\end{align*}
\] (8)

Then, in particular, the extension \(x_{123}\) on \((E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)\) is coherent if and only if \(x_{123} \in [x'_{123}, x''_{123}]\), where

\[
\begin{align*}
x'_{123} & = \max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13} + x_{23} - x_3\}, \\
x''_{123} & = \min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\}.
\end{align*}
\]

Then, by Theorem 3 it follows Corollary 1.

Corollary 1. For any coherent assessment \((x_1, x_2, x_3, x_{12}, x_{13}, x_{23})\) on

\[
\{E_1|H_1, E_2|H_2, E_3|H_3, (E_1|H_1) \wedge (E_2|H_2), (E_1|H_1) \wedge (E_3|H_3), (E_2|H_2) \wedge (E_3|H_3)\}
\]

the extension \(x_{123}\) on \((E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)\) is coherent if and only if \(x_{123} \in [x'_{123}, x''_{123}]\), where

\[
\begin{align*}
x'_{123} & = \max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13} + x_{23} - x_3\}, \\
x''_{123} & = \min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\}.
\end{align*}
\] (9)

We recall that in case of logical dependencies, the set of all coherent assessments may be smaller than that one associated with the case of logical independence. However, as shown in the next result, the set of coherent assessments is the same when the conditioning events \(H_1 = H_2 = H_3 = H\) (where possibly \(H = \emptyset\)).

Theorem 4. Let be given any logically independent events \(E_1, E_2, E_3, H\), with \(H \neq \emptyset\). Then, the set \(\Pi\) of all coherent assessments \(\mathcal{M} = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})\) on \(\mathcal{F} = \{C_1, C_2, C_3, C_{12}, C_{13}, C_{23}, C_{123}\}\) is the set of points \((x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})\) which satisfy the conditions in formula (7). A corollary similar to Corollary 1 could be associated to Theorem 4.
3 Representation by Frank t-norms for \((A|H) \wedge (B|K)\)

In the next result we study the relation between our notion of conjunction and t-norms.

**Theorem 5.** Let us consider the conjunction \((A|H) \wedge (B|K)\), with \(A, B, H, K\) logically independent and with \(P(A|H) = x, P(B|K) = y\). Moreover, given any \(\lambda \in [0, +\infty]\), let \(T_\lambda\) be the Frank t-norm with parameter \(\lambda\). Then, the assessment \(z = T_\lambda(x, y)\) on \((A|H) \wedge (B|K)\) is a coherent extension of \((x, y)\) on \(\{A|H, B|K\}\); moreover \((A|H) \wedge (B|K) = T_\lambda(A|H, B|K)\). Conversely, given any coherent extension \(z = P[(A|H) \wedge (B|K)]\) of \((x, y)\), there exists \(\lambda \in [0, +\infty]\) such that \(z = T_\lambda(x, y)\).

**Proof.** We recall that for every \(\lambda \in [0, +\infty]\) the Frank t-norm \(T_\lambda: [0, 1]^2 \to [0, 1]\) with parameter \(\lambda\) is defined as

\[
T_\lambda(u, v) = \begin{cases} 
T_M(u, v) = \min\{u, v\}, & \text{if } \lambda = 0, \\
T_P(u, v) = uv, & \text{if } \lambda = 1, \\
T_L(u, v) = \max\{u + v - 1, 0\}, & \text{if } \lambda = +\infty, \\
\log_\lambda(1 + (\frac{\lambda - 1}{\lambda - 1})^n), & \text{otherwise.}
\end{cases} \tag{10}
\]

For every \(\lambda \in [0, +\infty]\), it holds that \(T_L(u, v) \leq T_\lambda(u, v) \leq T_M(u, v)\), for every \((u, v) \in [0, 1]^2\) (see, e.g., [7]). Then, from Theorem 2 for any given \(\lambda\) the assessment \(z = T_\lambda(x, y)\) is a coherent extension of \((x, y)\) on \(\{A|H, B|K\}\). Moreover, from (10) it holds that \(T_\lambda(1, 1) = 1, T_\lambda(0, 0) = T_\lambda(0, v) = 0, T_\lambda(u, 1) = u, T_\lambda(1, v) = v\). Hence,

\[
T_\lambda(A|H, B|K) = \begin{cases} 
1, & \text{if } AHBK \text{ is true,} \\
0, & \text{if } \overline{AH} \text{ is true or } \overline{BK} \text{ is true,} \\
x, & \text{if } \overline{HBK} \text{ is true,} \\
y, & \text{if } \overline{KAH} \text{ is true,} \\
T_\lambda(x, y), & \text{if } \overline{KBK} \text{ is true,}
\end{cases} \tag{11}
\]

and, if we choose \(z = T_\lambda(x, y)\), from (10) and (11) it follows that \((A|H) \wedge (B|K) = T_\lambda(A|H, B|K)\).

Conversely, given any coherent extension \(z\) of \((x, y)\), there exists \(\lambda\) such that \(z = T_\lambda(x, y)\). Indeed, if \(z = \min\{x, y\}\), then \(\lambda = 0\); if \(z = \max\{x + y - 1, 0\}\), then \(\lambda = +\infty\); if \(\max\{x + y - 1, 0\} < z < \min\{x, y\}\), then by continuity of \(T_\lambda\) with respect to \(\lambda\) it holds that \(z = T_\lambda(x, y)\) for some \(\lambda \in [0, +\infty]\) (for instance, if \(z = xy\), then \(z = T_1(x, y)\)) and hence \((A|H) \wedge (B|K) = T_\lambda(A|H, B|K)\).

**Remark 3.** As we can see from [8] and Theorem 5 in case of logically independent events, if the assessed values \(x, y, z\) are such that \(z = T_\lambda(x, y)\) for a given \(\lambda\), then the conjunction \((A|H) \wedge (B|K) = T_\lambda(A|H, B|K)\). For instance, if \(z = T_1(x, y) = xy\), then \((A|H) \wedge (B|K) = (A|H, B|K) = (A|H) \cdot (B|K)\). Conversely, if \((A|H) \wedge (B|K) = T_\lambda(A|H, B|K)\) for a given \(\lambda\), then \(z = T_\lambda(x, y)\). Then, the set \(\Pi\) given in (5) can be written as

\[
\Pi = \{(x, y, z) : (x, y) \in [0, 1]^2, z = T_\lambda(x, y), \lambda \in [0, +\infty]\}. \tag{12}
\]
4 Conjunction of \((A|H)\) and \((A|K)\)

In this section we examine the conjunction of two conditional events in the particular case when \(A = B\), that is \((A|H) \land (A|K)\). By setting \(P(A|H) = x\), \(P(A|K) = y\) and \(\mathbb{P}[(A|H) \land (A|K)] = z\), it holds that

\[
(A|H) \land (A|K) = \begin{cases} 
1, & \text{if } AHK \text{ is true,} \\
0, & \text{if } \overline{A}HK \text{ is true,} \\
0, & \text{if } \overline{A}\overline{H}K \text{ is true,} \\
x, & \text{if } \overline{A}AK \text{ is true,} \\
y, & \text{if } \overline{A}H\overline{K} \text{ is true,} \\
z, & \text{if } \overline{H}K \text{ is true.}
\end{cases}
\]

**Theorem 6.** Let \(A, H, K\) be three logically independent events, with \(H \neq \emptyset\), \(K \neq \emptyset\). The set \(\Pi\) of all coherent assessments \((x, y, z)\) on the family \(\mathcal{F} = \{A|H, A|K, (A|H) \land (A|K)\}\) is given by

\[
\Pi = \{(x, y, z) : (x, y) \in [0, 1]^2, T_P(x, y) = xy \leq z \leq \min\{x, y\} = T_M(x, y)\}.
\]

**(13)**

**Proof.** Let \(\mathcal{M} = (x, y, z)\) be a prevision assessment on \(\mathcal{F}\). The constituents associated with the pair \((\mathcal{F}, \mathcal{M})\) and contained in \(H \lor K\) are: \(C_1 = AHK\), \(C_2 = \overline{A}HK\), \(C_3 = \overline{A}\overline{H}K\), \(C_4 = \overline{A}K\), \(C_5 = \overline{A}\overline{H}K\), \(C_6 = AH\overline{K}\). The associated points \(Q_i\)'s are \(Q_1 = (1, 1, 1), Q_2 = (0, 0, 0), Q_3 = (x, 0, 0), Q_4 = (0, y, 0), Q_5 = (x, 1, x), Q_6 = (1, y, y)\). With the further constituent \(C_0 = \overline{H}K\) it is associated the point \(Q_0 = \mathcal{M} = (x, y, z)\).

Considering the convex hull \(\mathcal{I}\) (see Figure 11) of \(Q_1, \ldots, Q_6\), a necessary condition for the coherence of the prevision assessment \(\mathcal{M} = (x, y, z)\) on \(\mathcal{F}\) is that \(\mathcal{M} \in \mathcal{I}\), that is the following system must be solvable

\[
(\Sigma) \left\{ \begin{align*}
\lambda_1 + x\lambda_3 + x\lambda_5 + \lambda_6 &= x, \\
\lambda_1 + y\lambda_4 + \lambda_5 + y\lambda_6 &= y, \\
\sum_{h=1}^{6} \lambda_h &= 1, & \lambda_h &\geq 0, & h = 1, \ldots, 6.
\end{align*} \right.
\]

First of all, we observe that solvability of \((\Sigma)\) requires that \(z \leq x\) and \(z \leq y\), that is \(z \leq \min\{x, y\}\). We now verify that \((x, y, z)\), with \((x, y) \in [0, 1]^2\) and \(z = \min\{x, y\}\), is coherent. We distinguish two cases: (i) \(x \leq y\) and (ii) \(x > y\).

**Case (i).** In this case \(z = \min\{x, y\} = x\). If \(y = 0\) the system \((\Sigma)\) becomes

\[
\lambda_1 + \lambda_6 = 0, \quad \lambda_1 + \lambda_5 = 0, \quad \lambda_1 = 0, \quad \lambda_2 + \lambda_3 + \lambda_4 = 1, \quad \lambda_h \geq 0, \quad h = 1, \ldots, 6.
\]

which is clearly solvable. In particular there exist solutions with \(\lambda_2 > 0, \lambda_3 > 0, \lambda_4 > 0\), by Theorem 14 as the set \(I_0\) is empty the solvability of \((\Sigma)\) is sufficient for coherence of the assessment \((0, 0, 0)\). If \(y > 0\) the system \((\Sigma)\) is solvable and a solution is

\[
A = (\lambda_1, \ldots, \lambda_6) = (x, \frac{x(1-y)}{y}, 0, \frac{y-x}{y}, 0, 0).
\]
Fig. 1. Convex hull $\mathcal{I}$ of the points $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$. $M' = (x, y, z')$, $M'' = (x, y, z'')$, where $(x, y) \in [0, 1]^2$, $z' = xy$, $z'' = \min\{x, y\}$. In the figure the numerical values are: $x = 0.35$, $y = 0.45$, $z' = 0.1575$, and $z'' = 0.35$.

We observe that, if $x > 0$, then $\lambda_1 > 0$ and $I_0 = \emptyset$ because $C_1 = HK \subseteq H \lor K$, so that $M = (x, y, x)$ is coherent. If $x = 0$ (and hence $z = 0$), then $\lambda_4 = 1$ and $I_0 \subseteq \{2\}$. Then, as the sub-assessment $P(A|K) = y$ is coherent, it follows that the assessment $M = (0, y, 0)$ is coherent too. Case (ii). The system is solvable and a solution is

$$A = (\lambda_1, \ldots, \lambda_6) = (y, \frac{y(1-x)}{x}, \frac{x-y}{x}, 0, 0, 0).$$

We observe that, if $y > 0$, then $\lambda_1 > 0$ and $I_0 = \emptyset$ because $C_1 = HK \subseteq H \lor K$, so that $M = (x, y, y)$ is coherent. If $y = 0$ (and hence $z = 0$), then $\lambda_3 = 1$ and $I_0 \subseteq \{1\}$. Then, as the sub-assessment $P(A|H) = x$ is coherent, it follows that the assessment $M = (x, 0, 0)$ is coherent too.

Thus, for every $(x, y) \in [0, 1]^2$, the assessment $(x, y, \min\{x, y\})$ is coherent and, by recalling that $z \leq \min\{x, y\}$, the upper bound on $z$ is $z = T_M(x, y)$.

We now verify that $(x, y, xy)$, with $(x, y) \in [0, 1]^2$ is coherent; moreover we will show that $(x, y, z)$, with $z < xy$, is not coherent, in other words the lower bound for $z$ is $xy$. First of all, we observe that $M = (1-x)Q_4 + xQ_6$, so that a solution of $(\Sigma)$ is $A_1 = (0, 0, 0, 1-x, 0, x)$. Moreover, $M = (1-y)Q_3 + yQ_5$, so that another solution is $A_2 = (0, 0, 1-y, 0, y, 0)$. Then

$$A = \frac{A_1 + A_2}{2} = (0, 0, \frac{1-y}{2}, \frac{1-x}{2}, \frac{y}{2}, \frac{x}{2})$$

is a solution of $(\Sigma)$ such that $I_0 = \emptyset$. Thus the assessment $(x, y, xy)$ is coherent for every $(x, y) \in [0, 1]^2$. 

Conjunction of Conditional Events and T-norms 7
In order to verify that \( xy \) is the lower bound on \( z \) we observe that the points \( Q_3, Q_4, Q_5, Q_6 \) belong to a plane \( \pi \) of equation: \( yX + xY - Z = xy \), where \( X, Y, Z \) are the axis' coordinates. Now, by considering the function \( f(X, Y, Z) = yX + xY - Z \), we observe that for each constant \( k \) the equation \( f(X, Y, Z) = k \) represents a plane which is parallel to \( \pi \) and coincides with \( \pi \) when \( k = xy \). We also observe that

\[
\begin{align*}
    f(Q_1) &= f(1,1,1) = x + y - 1 = T_L(x, y) \leq xy = T_P(x, y); \\
    f(Q_2) &= f(0,0,0) = 0 \leq xy = T_P(x, y); \\
    f(Q_3) &= f(Q_4) = f(Q_5) = f(Q_6) = xy = T_P(x, y).
\end{align*}
\]

Then, for every \( \mathcal{P} = \sum_{h=1}^{6} \lambda_h Q_h \), with \( \lambda_h \geq 0 \) and \( \sum_{h=1}^{6} \lambda_h = 1 \), that is \( \mathcal{P} \in \mathcal{I} \), it holds that

\[
f(\mathcal{P}) = f \left( \sum_{h=1}^{6} \lambda_h Q_h \right) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leq xy.
\]

On the other hand, given any \( a > 0 \), by considering \( \mathcal{P} = (x, y, xy - a) \) it holds that

\[
f(\mathcal{P}) = f(x, y, xy - a) = xy + xy - xy + a = xy + a > xy.
\]

Therefore, for any given \( a > 0 \) the assessment \((x, y, xy - a)\) is not coherent because \((x, y, xy - a) \notin \mathcal{I} \). Then the lower bound on \( z \) is \( xy = T_P(x, y) \). Finally, the set of all coherent assessments \((x, y, z)\) on \( \mathcal{F} \) is the set \( \Pi \) in (14).

Based on Theorem 6 we can give an analogous version for the Theorem 5 (when \( A = B \)).

**Theorem 7.** Let us consider the conjunction \((A|H) \land (A|K)\), with \( A, H, K \) logically independent and with \( P(A|H) = x, P(A|K) = y \). Moreover, given any \( \lambda \in [1, +\infty] \), let \( T_\lambda \) be the Frank t-norm with parameter \( \lambda \). Then, the assessment \( z = T_\lambda(x, y) \) on \((A|H) \land (A|K)\) is a coherent extension of \((x, y)\) on \((A|H, A|K)\); moreover \((A|H) \land (A|K) = T_\lambda(A|H, A|K) \). Conversely, given any coherent extension \( z = \mathbb{P}[(A|H) \land (A|K)] \) of \((x, y)\), there exists \( \lambda \in [1, +\infty] \) such that \( z = T_\lambda(x, y) \).

The next result follows from Theorem 6 when \( H, K \) are incompatible.

**Theorem 8.** Let \( A, H, K \) be three events, with \( A \) logically independent from both \( H \) and \( K \), with \( H \neq \emptyset \), \( K \neq \emptyset \), \( HK = \emptyset \). The set \( \Pi \) of all coherent assessments \((x, y, z)\) on the family \( \mathcal{F} = \{A|H, A|K, (A|H) \land (A|K)\} \) is given by

\[
\Pi = \{(x, y, z) : (x, y) \in [0, 1]^2, z = xy = T_P(x, y)\}.
\]  

**Proof.** We observe that

\[
(A|H) \land (A|K) = \begin{cases} 
0, & \text{if } \overline{A|H} K \text{ is true;} \\
0, & \text{if } \overline{A|H} K \text{ is true;} \\
x, & \text{if } H \land AK \text{ is true;} \\
y, & \text{if } A|H K \text{ is true;} \\
z, & \text{if } H \text{ is true.}
\end{cases}
\]
Moreover, as $HK = \emptyset$, the points $Q_h$'s are $(x, 0, 0), (0, y, 0), (x, 1, x), (1, y, y)$, which coincide with the points $Q_3, \ldots, Q_6$ of the case $HK \neq \emptyset$. Then, as shown in the proof of Theorem 8, the condition $\mathcal{M} = (x, y, z)$ belongs to the convex hull of $(x, 0, 0), (0, y, 0), (x, 1, x), (1, y, y)$ amounts to the condition $z = xy$. □

Remark 4. From Theorem 3 when $HK = \emptyset$ it holds that $(A|H) \wedge (A|K) = (A|H) \cdot (A|K) = T_P(A|H, A|K)$, where $x = P(A|H)$ and $y = P(A|K)$.

5 Further Results on Frank t-norms

In this section we give some results which concern Frank t-norms and the family $\mathcal{F} = \{C_1, C_2, C_3, C_{12}, C_{13}, C_{23}, C_{123}\}$. We recall that, given any t-norm $T(x_1, x_2)$ it holds that $T(x_1, x_2, x_3) = T(T(x_1, x_2), x_3)$. We observe that, based on Definition 2 when $n = 3$ we obtain

$$C_{123} = \begin{cases} 1, & \text{if } E_1H_1E_2H_2E_3H_3 \text{ is true,} \\ 0, & \text{if } \overline{E}_1H_1 \vee \overline{E}_2H_2 \vee \overline{E}_3H_3 \text{ is true,} \\ x_1, & \text{if } \overline{T}_1E_2H_2E_3H_3 \text{ is true,} \\ x_2, & \text{if } \overline{T}_2E_1H_1E_3H_3 \text{ is true,} \\ x_3, & \text{if } \overline{T}_3E_1H_1E_2H_2 \text{ is true,} \\ x_{12}, & \text{if } \overline{T}_1\overline{T}_2E_3H_3 \text{ is true,} \\ x_{13}, & \text{if } \overline{T}_1\overline{T}_3E_2H_2 \text{ is true,} \\ x_{23}, & \text{if } \overline{T}_2\overline{T}_3E_1H_1 \text{ is true,} \\ x_{123}, & \text{if } \overline{T}_1\overline{T}_2\overline{T}_3H_3 \text{ is true.} \end{cases} \tag{15}$$

5.1 On the Product t-norm

Theorem 9. Assume that the events $E_1, E_2, E_3, H_1, H_2, H_3$ are logically independent, with $H_1 \neq \emptyset, H_2 \neq \emptyset, H_3 \neq \emptyset$. If the assessment $\mathcal{M} = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ on $\mathcal{F} = \{C_1, C_2, C_3, C_{12}, C_{13}, C_{23}, C_{123}\}$ is such that $(x_1, x_2, x_3) \in [0, 1]^3$, $x_{ij} = T_1(x_i, x_j) = x_i x_j$, $i \neq j$, and $x_{123} = T_1(x_1, x_2, x_3) = x_1 x_2 x_3$, then $\mathcal{M}$ is coherent. Moreover, $C_{ij} = T_1(C_i, C_j) = C_i C_j$, $i \neq j$, and $C_{123} = T_1(C_1, C_2, C_3) = C_1 C_2 C_3$.

Proof. From Remark 2 the coherence of $\mathcal{M}$ amounts to the inequalities in (8). As $x_{ij} = T_1(x_i, x_j) = x_i x_j$, $i \neq j$, and $x_{123} = T_1(x_1, x_2, x_3) = x_1 x_2 x_3$, the inequalities (8) become

$$\max\{0, x_1(x_2 + x_3 - 1), x_2(x_1 + x_3 - 1), x_3(x_1 + x_2 - 1)\} \leq x_1 x_2 x_3 \leq \min\{x_1 x_2, x_1 x_3, x_2 x_3, (1 - x_1)(1 - x_2)(1 - x_3) + x_1 x_2 x_3\}. \tag{16}$$

Thus, by recalling that $x_i + x_j - 1 \leq x_i x_j$, the inequalities are satisfied and hence $\mathcal{M}$ is coherent. Moreover, from (8) and (16) it follows that

$$C_{ij} = T_1(C_i, C_j) = C_i C_j, \quad i \neq j, \quad C_{123} = T_1(C_1, C_2, C_3) = C_1 C_2 C_3.$$

□
5.2 On the Minimum t-norm

Theorem 10. Assume that the events $E_1, E_2, E_3, H_1, H_2, H_3$ are logically independent, with $H_1 \neq \emptyset, H_2 \neq \emptyset, H_3 \neq \emptyset$. If the assessment $M = \{x_1, x_2, x_3, x_12, x_13, x_23, x_{123}\}$ on $F = \{C_1, C_2, C_3, C_{12}, C_{13}, C_{23}, C_{123}\}$ is such that $(x_1, x_2, x_3) \in [0, 1]^3$, $x_{ij} = T_M(x_i, x_j) = \min\{x_i, x_j\}$, $i \neq j$, and $x_{123} = T_M(x_1, x_2, x_3) = \min\{x_1, x_2, x_3\}$, then $M$ is coherent. Moreover, $C_{ij} = T_M(C_i, C_j) = \min\{C_i, C_j\}$, $i \neq j$, and $C_{123} = T_M(C_1, C_2, C_3) = \min\{C_1, C_2, C_3\}$.

Proof. From Remark 2 the coherence of $M$ amounts to the inequalities in (8). Without loss of generality, we assume that $x_1 \leq x_2 \leq x_3$. Then $x_{12} = T_M(x_1, x_2) = x_1$, $x_{13} = T_M(x_1, x_3) = x_1$, $x_{23} = T_M(x_2, x_3) = x_2$, and $x_{123} = T_M(x_1, x_2, x_3) = x_1$. The inequalities (8) become

$$\max\{0, x_1, x_1 + x_2 - x_3\} \leq x_1 \leq x_1 \leq x_1 = \min\{x_1, x_2, 1 - x_3 + x_1\}. \quad (17)$$

Thus, the inequalities are satisfied and hence $M$ is coherent. Moreover, from (8) and (15) it follows that

$$C_{ij} = T_M(C_i, C_j) = \min\{C_i, C_j\}, \quad i \neq j, \quad C_{123} = T_M(C_1, C_2, C_3) = \min\{C_1, C_2, C_3\}. \quad \Box$$

Remark 5. As we can see from (17) and Corollary 1 the assessment $x_{123} = \min\{x_1, x_2, x_3\}$ is the unique coherent extension on $C_{123}$ of the assessment

$$(x_1, x_2, x_3, \min\{x_1, x_2\}, \min\{x_1, x_3\}, \min\{x_2, x_3\})$$

on $\{C_1, C_2, C_3, C_{12}, C_{13}, C_{23}\}$.

We also notice that, if $C_1 \leq C_2 \leq C_3$, then $C_{12} = C_1, C_{13} = C_1, C_{23} = C_2$, and $C_{123} = C_1$. Moreover, $x_{12} = x_1, x_{13} = x_1, x_{23} = x_2$, and $x_{123} = x_1$.

5.3 On Lukasiewicz t-norm

We observe that in general the results of Theorems 9 and 10 do not hold for the Lukasiewicz t-norm (and hence for any given Frank t-norm), as shown in the example below. We recall that $T_L(x_1, x_2, x_3) = \max\{x_1 + x_2 + x_3 - 2, 0\}$.

Example 1. The assessment $(x_1, x_2, x_3, T_L(x_1, x_2), T_L(x_1, x_3), T_L(x_2, x_3), T_L(x_1, x_2, x_3))$ on the family $F = \{C_1, C_2, C_3, C_{12}, C_{13}, C_{23}, C_{123}\}$, with $(x_1, x_2, x_3) = (0.5, 0.6, 0.7)$ is not coherent. Indeed, by observing that $T_L(x_1, x_2) = 0.1, T_L(x_1, x_3) = 0.2, T_L(x_2, x_3) = 0.3$, and $T_L(x_1, x_2, x_3) = 0$, formula (8) becomes

$$\max\{0, 0.1 + 0.2 - 0.5, 0.1 + 0.3 - 0.6, 0.2 + 0.3 - 0.7\} \leq 0 \leq \min\{0.1, 0.2, 0.3, 1 - 0.5 - 0.6 - 0.7 + 0.1 + 0.2 + 0.3\},$$

that is: $\max\{0, -0.2\} \leq 0 \leq \min\{0.1, 0.2, 0.3, -0.2\}$; thus the inequalities are not satisfied and the assessment is not coherent.
More in general we have

**Theorem 11.** The assessment \((x_1, x_2, x_3, T_L(x_1, x_2), T_L(x_1, x_3), T_L(x_2, x_3))\) on the family \(\mathcal{F} = \{C_1, C_2, C_3, C_{12}, C_{13}, C_{23}\}\), with \(T_L(x_1, x_2) > 0\), \(T_L(x_1, x_3) > 0\), \(T_L(x_2, x_3) > 0\) is coherent if and only if \(x_1 + x_2 + x_3 - 2 \geq 0\). Moreover, when \(x_1 + x_2 + x_3 - 2 \geq 0\) the unique coherent extension \(x_{123}\) on \(C_{123}\) is \(x_{123} = T_L(x_1, x_2, x_3)\).

**Proof.** We distinguish two cases: (i) \(x_1 + x_2 + x_3 - 2 < 0\); (ii) \(x_1 + x_2 + x_3 - 2 \geq 0\).

Case (i). From (7) the inequality \(1 - x_1 - x_2 - x_3 + x_12 + x_13 + x_{23} \geq 0\) is not satisfied because \(1 - x_1 - x_2 - x_3 + x_12 + x_13 + x_{23} = x_1 + x_2 + x_3 - 2 < 0\). Therefore the assessment is not coherent.

Case (ii). We set \(x_{123} = T_L(x_1, x_2, x_3) = x_1 + x_2 + x_3 - 2\). Then, by observing that \(0 < x_i + x_j - 1 \leq x_1 + x_2 + x_3 - 2, i \neq j\), formula (8) becomes

\[
\max \{0, x_1 + x_2 + x_3 - 2\} \leq x_1 + x_2 + x_3 - 2 \leq \min \{x_1 + x_2 - 1, x_1 + x_3 - 1, x_2 + x_3 - 1, x_1 + x_2 + x_3 - 2\},
\]

that is: \(x_1 + x_2 + x_3 - 2 \leq x_1 + x_2 + x_3 - 2 \leq x_1 + x_2 + x_3 - 2\). Thus, the inequalities are satisfied and the assessment

\[(x_1, x_2, x_3, T_L(x_1, x_2), T_L(x_1, x_3), T_L(x_2, x_3), T_L(x_1, x_2, x_3))\]

on \(\{C_1, C_2, C_3, C_{12}, C_{13}, C_{23}, C_{123}\}\) is coherent and the sub-assessment

\[(x_1, x_2, x_3, T_L(x_1, x_2), T_L(x_1, x_3), T_L(x_2, x_3))\]

on \(\mathcal{F}\) is coherent too. \(\square\)

A result related with Theorem 11 is given below.

**Theorem 12.** If the assessment \((x_1, x_2, x_3, T_L(x_1, x_2), T_L(x_1, x_3), T_L(x_2, x_3), T_L(x_1, x_2, x_3))\) on the family \(\mathcal{F} = \{C_1, C_2, C_3, C_{12}, C_{13}, C_{23}, C_{123}\}\), is such that \(T_L(x_1, x_2, x_3) > 0\), then the assessment is coherent.

**Proof.** We observe that \(T_L(x_1, x_2, x_3) = x_1 + x_2 + x_3 - 2 > 0\); then \(x_i > 0\), \(i = 1, 2, 3\), and \(0 < x_i + x_j - 1 \leq x_1 + x_2 + x_3 - 2, i \neq j\). Then formula (8) becomes: \(\max \{0, x_1 + x_2 + x_3 - 2\} \leq x_1 + x_2 + x_3 - 2 \leq \min \{x_1 + x_2 - 1, x_1 + x_3 - 1, x_2 + x_3 - 1, x_1 + x_2 + x_3 - 2\}\), that is: \(x_1 + x_2 + x_3 - 2 \leq x_1 + x_2 + x_3 - 2 \leq x_1 + x_2 + x_3 - 2\). Thus, the inequalities are satisfied and the assessment is coherent. \(\square\)

6 Conclusions

We have studied the relationship between the notions of conjunction and of Frank t-norms. We have shown that, under logical independence of events and coherency of prevision assessments, for a suitable \(\lambda \in [0, +\infty]\) it holds that \(P((A|H) \wedge (B|K)) = T_\lambda(x, y)\) and \((A|H) \wedge (B|K) = T_\lambda(A|H, B|K)\). Then, we
have considered the case $A = B$, by determining the set of all coherent assessment $(x, y, z)$ on $(A|H, B|K, (A|H) \land (A|K))$. We have shown that, under coherence, for a suitable $\lambda \in [1, +\infty]$ it holds that $(A|H) \land (A|K) = T_\lambda(A|H, A|K)$. We have also studied the particular case where $A = B$ and $HK = \emptyset$. Then, we have considered the conjunction of three conditional events and we have shown that the prevision assessments produced by the Product t-norm, or the Minimum t-norm, are coherent. Finally, we have examined the Łukasiewicz t-norm and we have shown, by a counterexample, that coherence in general is not assured. We have given some conditions for coherence when the prevision assessments are based on the Łukasiewicz t-norm. Future work should concern the deepening and generalization of the results of this paper.

References

1. V. Biazzo, A. Gilio, and G. Sanfilippo. Generalized coherence and connection property of imprecise conditional previsions. In Proc. IPMU 2008, Malaga, Spain, June 22 - 27, pages 907–914, 2008.
2. A. Gilio and G. Sanfilippo. Conditional random quantities and iterated conditioning in the setting of coherence. In L. C. van der Gaag, editor, ECSQARU 2013, volume 7958 of LNCS, pages 218–229. Springer, Berlin, Heidelberg, 2013.
3. A. Gilio and G. Sanfilippo. Conjunction, disjunction and iterated conditioning of conditional events. In Synergies of Soft Computing and Statistics for Intelligent Data Analysis, volume 190 of AISC, pages 399–407. Springer, Berlin, 2013.
4. A. Gilio and G. Sanfilippo. Conditional random quantities and compounds of conditionals. Studia Logica, 102(4):709–729, 2014.
5. Angelo Gilio and Giuseppe Sanfilippo. Generalized logical operations among conditional events. Applied Intelligence, 49(1):79–102, Jan 2019.
6. S. Kaufmann. Conditionals right and left: Probabilities for the whole family. Journal of Philosophical Logic, 38:1–53, 2009.
7. Erich Peter Klement, Radko Mesiar, and Endre Pap. Triangular Norms. Springer, 2000.
8. V. McGee. Conditional probabilities and compounds of conditionals. Philosophical Review, 98:485–541, 1989.
9. G. Sanfilippo, N. Pfeifer, D.E. Over, and A. Gilio. Probabilistic inferences from conjoined to iterated conditionals. International Journal of Approximate Reasoning, 93(Supplement C):103 – 118, 2018.
10. Giuseppe Sanfilippo. Lower and upper probability bounds for some conjunctions of two conditional events. In SUM 2018, volume 11142 of LNCS, pages 260–275. Springer International Publishing, Cham, 2018.