Structural properties of LPV to LFR transformation: minimality, input-output behavior and identifiability.

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Abstract—In this paper, we introduce and study important properties of the transformation of Affine Linear Parameter-Varying (ALPV) state-space representations into Linear Fractional Representations (LFR). More precisely, we show that (i) state minimal ALPV representations yield minimal LFRs, and vice versa, (ii) the input-output behavior of the ALPV representation determines uniquely the input-output behavior of the resulting LFR, (iii) structurally identifiable ALPVs yield structurally identifiable LFRs, and vice versa. We then characterize LFR models which correspond to equivalent ALPV models based on their input-output maps. As illustrated all along the paper, these results have important consequences for identification and control of systems described by LFRs.

I. INTRODUCTION

Linear Fractional Transformation (LFT) is one of the main tools used in the past decades for studying uncertain systems (see, e.g., [22]). For instance, Linear Fractional Representations (LFRs) have been widely used in control synthesis of Liner Parameter Varying (LPV) systems or for $H_{\infty}$ optimal control (see [22], [16], [5] for overview). More recently, the LFRs have attracted a lot of attention as far as system identification is concerned, (see [8], [9], [10], [6], [20]). For LPV model-based controller design, several solutions first consist in transforming the LPV system into an LFR. This step indeed allows us to use control tools developed for LFRs to design a controller to guarantee the satisfactory closed loop operation of the LPV plant in many operating conditions. As far as system identification is considered, it is clear that this information is sufficient to determine uniquely the input-output behavior of the LPV model directly with LFRs. These observations mean that, in control design as well as in system identification, LPV models are often used as an intermediary representation, whose main purpose is to serve as a source for an LFT description. It is thus of prime interest in control in general to study the transformation of LPV to LFR closely.

In this paper, a specific attention is paid to important realization theory concepts like minimality and input-output equivalence of model representations. The reason why this last point (i.e., input-output equivalence of LPV models and LFRs) is a crucial and a challenging problem in system identification and controller design can be illustrated as follows. Input-output equivalence of two LPV models means that these two models yield the same outputs for the all inputs and scheduling signals. Input-output equivalence of the corresponding LFRs means that they both yield the same outputs for all input and all the choice of uncertainty block $\Delta$. The latter point (i.e. for all $\Delta$) leads us to the conclusion that the LFRs should behave the same way for any blocks which do not arise from scheduling variables. The uncertainty operator in $\Delta$ could be, for instance, any stable non-rational transfer function. In fact, we can not even conclude that two LFRs which arise from two input-output equivalent LPV models can be interconnected with the same uncertainty block $\Delta$. This observation is not an issue if the LPV model is known from first principles. However, if the LPV model is identified from data, leading to a black-box model, then different identification methods applied to the same measurements may yield different LPV models which are, at most, input-output equivalent. By keeping in mind that the existence of a controller for an LFR only depends on its input-output behavior (while the disturbances block $\Delta$ is assumed to have a bounded norm), the situation described above means that the outcome of controller synthesis may depend on the choice of the identification method, even under ideal conditions. This is clearly an undesirable situation.

This simple illustration of system identification for control clearly points out the fact that we need to understand the relationship between the input-output behavior of LPV models and the corresponding LFRs. Indeed, the measurements allow us to say something about the input-output behavior of the underlying LFR for those choices of the uncertainty block which correspond to scheduling variables. However, it is not a priori clear that this information is sufficient to determine the input-output behavior of the LFR for all other choices of the uncertainty structure.

In this paper, we first show that the transformation from ALPV models to LFRs preserves minimality. This enables us to show in a second step that the input-output behavior of an ALPV model uniquely determines the input-output behavior of the corresponding LFR. Indeed, from [1], it follows that minimal ALPV models with the same input-output behavior are related by a constant state isomor-

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$^4$Note that the interconnection of an LFR with a block $\Delta$ need not always be well-defined.
we will define trivial \([4], [21]\). In order to avoid to deal with this issue, there are several such conditions, and their relationship is not investigated before. Notice that, in deriving the results of the paper, we use realization theory of ALPV systems [15], [14], and realization theory of LFRs (viewed as multidimensional systems) [3], [2].

Outline of the paper In Section II we present formal definitions to setup the framework of this paper. Section III contains the main results dedicated to the connection between ALPV models and the corresponding LPV-LFR ones. Finally, Section IV concludes the paper.

II. THE FORMAL SETUP: LFR AND ALPV MODELS

In this section we present the formal setup of the problem considered in this paper. First, in subsections II-A and II-B we define LPV-LFR and ALPV models, respectively. In subsection II-C, the transformation from LPV-LFR to ALPV models and vice versa is presented. Motivating examples are presented in subsection II-D while in subsection II-E the studied problems are formulated.

A. General LFR models as multidimensional systems

A linear fractional representation (abbreviated as LFR) is presented in Figure I where \(M\) is a tuple of matrices \((A, B, C, D)\) representing an LTI system, and \(\Delta\) is a linear operator on suitable function spaces. Note the feedback loop in Figure I is not necessarily well-posed. Hence, in order to define the input-output behavior of an LFR formally, we have to impose additional conditions on \(M\) and \(\Delta\). There are several such conditions, and their relationship is not trivial [4], [21]. In order to avoid to deal with this issue, we will define formal input-output maps, by viewing LFRs as multidimensional systems [2]. We will see that the formal input-output map determines the input-output behavior of the LFR, for those cases which are of interest for this paper and for which the interconnection is well-posed.

Definition 1: An LFR is a tuple 

\[ \mathcal{M} = (p, m, d, \{n_i\}_{i=1}^d, A, B, C, D), \]

where \(p, m, d, n_i, i \in \{1, \ldots, d\}\) are positive integers, and \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}\) and \(D \in \mathbb{R}^{p \times m}\) are matrices, where \(n = \sum_{i=1}^d n_i\). In Definition II the choice of integers \(\{n_i\}_{i=1}^d\) expresses the tacit assumption that \(\Delta = \text{diag}[\delta_1 I_{n_1}, \ldots, \delta_d I_{n_d}]\) when defining the behavior of \(\mathcal{M}\), where \(\delta_i, \forall i = 1, \ldots, d\), are linear operators on scalar valued sequences, and \(I_{n_i}\) is the \(n_i \times n_i\) identity matrix. Here we used the standard notation of [4], [21]. That is, if \(\delta\) is a linear operator on scalar valued sequences, then \(\delta I_n\) stands for the linear operator on sequences with values from \(\mathbb{R}^n\), such that the result of applying \(\delta I_n\) to a sequence is obtained by applying \(\delta\) to each coordinate of the sequence \(x\), see [4], [21] for the formal definition. Next, we define what we mean by a formal input-output map of an LFR. To this end, we need the following notation.

Notation 1 (Free monoid \(\mathcal{X}\)): Let \(\mathcal{X}\) be the monoid generated by a nonempty finite set \(\mathcal{X}\). An element \(w \in \mathcal{X}\) of length \(|w| = n\), is a sequence of the form \(x_1 x_2 \ldots x_n\), where \(x_i \in \mathcal{X}, \forall i = 1, 2, \ldots, n\). Denote \(\epsilon\) the empty sequence where \(|\epsilon| = 0\).

Definition 2 (Canonical partitioning): Let \(\mathcal{M}\) be an LFR of the form (I). The collection \(\{H_i, F_{i,j}, G_j\}_{i,j=1}^d\) of matrices such that \(F_{i,j} \in \mathbb{R}^{n_i \times n_j}, G_j \in \mathbb{R}^{n_j \times m}, H_i \in \mathbb{R}^{p \times n_i}\), \(i, j = 1, \ldots, d\), and

\[ A = \begin{bmatrix} F_{1,1} & F_{1,2} & \cdots & F_{1,d} \\ F_{2,1} & F_{2,2} & \cdots & F_{2,d} \\ \vdots & \vdots & \ddots & \vdots \\ F_{d,1} & F_{d,2} & \cdots & F_{d,d} \end{bmatrix}, \quad B = \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_d \end{bmatrix}, \]

\[ C = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_d \end{bmatrix}, \]

is called the canonical partitioning of \(\mathcal{M}\).

Notice that \(D \in \mathbb{R}^{p \times m}\) is not subject to partitioning according to \(i\) and \(j\).

Definition 3: The formal input-output map of an LFR \(\mathcal{M}\) is a function \(Y_\mathcal{M} : \mathcal{X}^* \rightarrow \mathbb{R}^{p \times m}, \mathcal{X} = \{1, \ldots, d\}\), and it is defined as follows: \(Y_\mathcal{M}(\epsilon) = D\) and, for all \(s = i_1 \cdots i_k, i_1, \ldots, i_k \in \mathcal{X}, k > 0\),

\[ Y_\mathcal{M}(s) = \begin{cases} H_{i_1} F_{i_1,i_{k-1}} \cdots F_{i_{k-1},i_{k}} G_{i_{k}} & k = 1, \\ H_{i_1} F_{i_1,i_{k-1}} \cdots F_{i_{k-1},i_{k}} G_{i_{k}} & k > 1, \end{cases} \]

where \(\{H_i, F_{i,j}, G_j\}_{i,j=1}^d\) is the canonical partitioning of \(\mathcal{M}\).

Remark 1: (Formal input-output map and star product \(\mathcal{M} \ast \Delta\)) Below we explain the relationship between \(Y_\mathcal{M}\) and the usual star product \(\mathcal{M} \ast \Delta\). We use the notation and terminology of [4]. In particular, we denote by \(I_2(\mathbb{R}^n)\) the space of all \(I_2\) sequences taking values in \(\mathbb{R}^n\) and we denote...
by $L(l_2)$ the set of all bounded operators from $l_2(\mathbb{R})$ to $l_2(\mathbb{R})$, and we use $\|\cdot\|$ to denote the induced operator norm of an operator from $L(l_2)$. Following [4], for any $\gamma > 0$ we define the set

$$\Delta_\gamma = \{\Delta = \text{diag}[\delta_1 I_{n_1}, \ldots, \delta_d I_{n_d}] : \delta_i \in L(l_2), \|\delta_i\| \leq \gamma, i = 1, \ldots, d\}.$$ 

An LFR $M$ of the form (1) is called $\gamma$-stable, if $(I_n - A\Delta)$ is an invertible bounded linear operator on $l_2(\mathbb{R}^n)$ for all $\Delta \in \Delta_\gamma$. Recall from [4], [21] that if $M$ is $\gamma$-stable, then for any input $u \in l^2(\mathbb{R}^m)$ and uncertainty block $\Delta \in \Delta_\gamma$, the feedback interconnection on Fig. 1 is well defined, and the corresponding output $y \in l^2(\mathbb{R}^p)$ satisfies $y = (M + \Delta)u$, where the bounded linear operator $(M \ast \Delta) : l^2(\mathbb{R}^m) \to l^2(\mathbb{R}^p)$ is defined by $M \ast \Delta := D + C\Delta(I_n - A\Delta)^{-1}B$.

From [4], it follows that, if $M$ is a $\gamma$-stable LFR, then

$$(M \ast \Delta)u = \sum_{s \in X^n} Y_M \delta_s I_m u,$$

for all $\Delta \in \Delta_\gamma$, where $\delta_s$ is the identity operator and for $s = i_1 \cdots i_k, i_1, \ldots, i_k \in X, k > 0$, $\delta_s(z) = \delta_{i_k}(\delta_{i_{k-1}}(\cdots \delta_{i_1}(z) \cdots))$ for all $z \in l_2(\mathbb{R})$. That is, the formal input-output map $Y_M$ determines the star product $M \ast \Delta$ uniquely for all $\Delta \in \Delta_\gamma$, provided $M$ is stable.

In fact, Remark 1 suggests the following intuitive interpretation: the formal input-output map determines the input-output behavior of an LFR for bounded uncertainty and internally stabilizing control inputs.

In order to compare formally the behaviors of LFRs and ALPVs, we need to recall from [2] some aspects of realization theory of LFRs. We say that two LFRs $M_1$ and $M_2$ are formally input-output equivalent, if their formal input-output maps are equals, i.e., $Y_{M_1} = Y_{M_2}$. If $M$ is an LFR of the form (1), then we call the number $n = n_1 + \cdots + n_d$ the dimension of $M$ and we denote it by dim $M$. We say that the LFR $M$ is minimal, if for any LFR $M'$ which is formally input-output equivalent to $M$, dim $M \leq$ dim $M'$.

Minimal LFRs can be characterized in terms of reachability and observability, and minimal LFRs which are also input-output equivalent are in fact isomorphic. In order to present this characterization formally, we need the following definitions. Let $M$ be an LFR of the form (1) and let $((H_i, F_{i,j}, G_j))_{i,j=1}^d$ the corresponding partitioning of $M$. Define the $k$-step observability $\{O_k^M\}_{k=1}^d$ and $k$-step reachability matrices $\{R_k^M\}_{k=1}^d$ of $M$ recursively as follows: for all $i = 1, \ldots, d$,

$$O^0_k(M) = H_i, \quad R^0_k(M) = G_i,$$

$$R_{k+1}^M = R_k^M F_{i,1} R_k^M(F_{i,1} \cdots F_{i,d} R_k^M),$$

$$O_{k+1}^M = [O_k^M \quad (O_k^M F_{i,1} \cdots F_{i,d} R_k^M) \cdots (O_k^M F_{i,1} \cdots F_{i,d} R_k^M)]^T.$$

We say that $M$ is reachable and observable, if rank $R_k^M = n_i$ and rank $O_k^M = n_i$ for some $k > 0$ respectively.

Hence, observability and reachability of LFRs can be verified numerically, and any LFR can be transformed to a reachable and observable LFR whose formal input-output map coincides with that of the original LFR. Let $M$ be an LFR of the form (1) and let $M = (p,m,d,A,B,C,D)$. A nonsingular matrix $T \in \mathbb{R}^{n \times n}$ is said to be an isomorphism from $M$ to $M$, if $D = D, T A T^{-1} = A, C = CT^{-1}, \bar{B} = TB$ and $T = \text{diag}[T_1, T_2, \ldots, T_d]$, where $T_i \in \mathbb{R}^{n_i \times n_i}, i = 1, \ldots, d$. Two LFRs are said to be isomorphic, if there exists an isomorphism from the one to the other.

**Theorem 1 (Minimality of LFRs, [2]):** An LFR is minimal if and only if it is reachable and observable. Two LFRs which are minimal and formally input-output equivalent are isomorphic. Any LFR can be transformed to a minimal LFR which is formally input-output equivalent to the original one. Note that stable LFRs in the sense of [4] are closed under minimization.

**Theorem 2 ([4]):** Any minimal LFR which is formally input-output equivalent to a $\gamma$-stable LFR is also $\gamma$-stable.

**Remark 2 (Significance of minimality for control):** If two LFRs are isomorphic, then they behave in the same manner when interconnected with a controller. Indeed, if the controller itself is an LFR, then its interconnection with two isomorphic LFRs yield two closed-loop systems which are also isomorphic LFRs. In particular, if one of the closed-loop systems is stable (in the sense of Remark 1) then so is the other, and vice versa, and the input-output behaviors defined by the star product of the two closed-loop systems are the same. Since all minimal and formally input-output equivalent LFRs are isomorphic, then any controller which stabilizes a minimal LFR and achieves certain input-output behavior will also stabilize and achieve the same input-output behavior for any other minimal and formally input-output equivalent LFR. To sum up, minimal and formally input-output equivalent LFRs yield the same closed-loop behavior when interconnected with any stabilizing LFR controller. This is why the preservation of minimality by ALPV to LFR transformation is so important.

**B. Affine LPV Systems**

Below, we recall some basic definitions for affine LPV models. We follow the terminology of [15]. A discrete-time Affine Linear Parameter-Varying (ALPV) model $(\Sigma)$ is defined as follows

$$\Sigma \begin{cases} x(k+1) = A(p(k))x(k) + B(p(k))u(k), \\ y(k) = C(p(k))x(k) + D(p(k))u(k) \end{cases} \tag{2}$$

where $x(k) \in X = \mathbb{R}^{n_x}$ is the state vector, $y(k) \in Y = \mathbb{R}^{n_y}$ is the (measured) output signals, $u(k) \in U = \mathbb{R}^{n_u}$ represents the input signals while $p(k) \in P = \mathbb{R}^{n_p}$ is the scheduling variables of the system represented by $\Sigma$, and for all $p \in P$,

$$A(p) = \sum_{i=1}^{n_p} A_i p_i, \quad B(p) = \sum_{i=1}^{n_p} B_i p_i, \tag{3}$$

$$C(p) = \sum_{i=1}^{n_p} C_i p_i, \quad D(p) = \sum_{i=1}^{n_p} D_i p_i,$$

for constant matrices $A_i \in \mathbb{R}^{n_x \times n_x}, B_i \in \mathbb{R}^{n_x \times n_u}, C_i \in \mathbb{R}^{n_y \times n_x}, D_i \in \mathbb{R}^{n_y \times n_u}, i \in \{0, \ldots, n_p\}$. In the sequel, we
will use the short notation
\[ \Sigma = (n_p, n_x, n_u, n_y, \{ A_i, B_i, C_i, D_i \}_{i=0}^{n_p}) \]
to define a model of the form (2). The dimension of \( \Sigma \) is the dimension \( n_x \) of its state-space. Note that the system dimension \( n_x \) does not depend on the number (dimension) of the scheduling parameters. By a solution of \( \Sigma \) we mean a tuple of trajectories \((x, y, u, p) \in (X, Y, U, P)\) satisfying (2) for all \( k \in \mathbb{N} \), where \( X = \mathbb{R}^{n_x}, Y = \mathbb{R}^{n_y}, U = \mathbb{R}^{n_u}, P = \mathbb{R}^{n_p} \), and we use the following notation: for a set \( \mathcal{A} \), we denote by \( \mathcal{A}^N \) the set of all functions of the form \( \phi : \mathbb{N} \to \mathcal{A} \). An element of \( \mathcal{A}^N \) can be thought of as a signal in discrete-time.

Define the input-output function of \( \Sigma \) as the function \( Y_\Sigma : U \times P \to Y \) such that for any \((x, y, u, p) \in (X \times Y \times U \times P)\), \( y = Y_\Sigma(u, p) \) holds if and only if \((x, y, u, p)\) is a solution of \( \Sigma \) and \( x(0) = 0 \). Two ALPVs are said to be input-output equivalent, if their input-output maps coincide. An ALPV \( \Sigma \) is said to be a minimal, if for any ALPV \( \tilde{\Sigma} \) which is input-output equivalent to \( \Sigma \), \( \dim \Sigma \leq \dim \tilde{\Sigma} \).

From [14], [15], it follows that minimal ALPV systems can be characterized via observability and span-reachability, and input-output equivalent minimal ALPVs are isomorphic. In order to state this result precisely, define the \( n \)-step extended reachability matrix \( R_n(\Sigma) \) of \( \Sigma \), and \( n \)-step extended observability matrix \( O_n(\Sigma) \) of \( \Sigma \), \( n \in \mathbb{N} \), recursively as follows:

\[
R_0(\Sigma) = [B_0, B_1, \ldots, B_{n_p}], \\
R_{n+1}(\Sigma) = \left[ R_n(\Sigma), A_0 R_n(\Sigma), \ldots, A_{n_p} R_n(\Sigma) \right], \\
O_0(\Sigma) = [C_0^T, C_1^T, \ldots, C_{n_p}^T]^T, \\
O_{n+1}(\Sigma) = \left[ O_n(\Sigma), A_0^T O_n(\Sigma), \ldots, A_{n_p}^T O_n(\Sigma) \right]^T.
\]

Let us call \( \Sigma \) span-reachable, if \( \text{rank} R_{n_x-1} = n_x \), and let us call \( \Sigma \) observable, if \( \text{rank} O_{n_y-1} = n_y \). Finally, we consider an ALPV \( \Sigma' = (n_p, n_x, n_u, n_y, \{ A'_i, B'_i, C'_i, D'_i \}_{i=0}^{n_p}) \) with \( \dim \Sigma' = \dim \Sigma \). A nonsingular matrix \( T \in \mathbb{R}^{n_x \times n_x} \) is said to be an ALPV isomorphism from \( \Sigma \) to \( \Sigma' \), if for all \( i = 0, \ldots, n_p \),

\[
A'_i T = T A_i, \quad B'_i = T B_i, \quad C'_i T = C_i, \quad D'_i = D_i.
\]

Now we can recall the following result from [14].

**Theorem 3 ([14], [15]):** An ALPV is minimal if and only if it is span-reachable and observable. Any two minimal ALPVs which are input-output equivalent are isomorphic. Any ALPV can be transformed to an input-output equivalent minimal ALPV.

**C. Transforming ALPVs to LFRs**

One popular approach for control of ALPVs is to transform them to LFRs as follows [19].

**Definition 4:** Let \( \Sigma \) be an ALPV of the form (2). An LFR-ALPV \( \mathcal{M} \) of the form (1) is called an LFR calculated from \( \Sigma \), if \( d = n_p + 1, n_1 = n_y, p = n_y, m = n_x, D = D_0 \) and the canonical decomposition \( \{(H_i, F_{i,j}, G_j)\}_{i,j=1}^{d} \) of \( \mathcal{M} \) satisfies the following properties:

1. \( H_1 = C_0, G_1 = B_0, F_{1,1} = A_0, \)
2. for all \( i, j = 1, \ldots, d \), if \( i > 1 \) and \( j > 1 \), then \( F_{i,j} = 0 \),
3. for all \( i = 2, \ldots, d \),

\[
\begin{bmatrix}
A_{i-1} & B_{i-1} \\
C_{i-1} & D_{i-1}
\end{bmatrix} = \begin{bmatrix}
F_{1,i} & G_i \\
H_i & I
\end{bmatrix} \begin{bmatrix}
F_{1,1} & G_1
\end{bmatrix}.
\]

The intuition behind Definition 4 is as follows. A solution \((x, y, u, p)\), \( x(0) = 0 \) of the ALPV model \( \Sigma \) corresponds to a solution of the LFR \( \mathcal{M} \) for the following choice of \( \Delta \)

\[
\Delta = \Delta(p) = \text{diag}[\lambda_{i_1} I_{n_2}, \ldots, \lambda_{i_n} I_{n_n}]
\]

with \( \delta_i : \mathbb{R}^n \to \mathbb{R}^n \) defined by \( \delta_i(h)(t) = p_i(t)h(t) \), and \( \lambda(h)(t) = \begin{cases} h(t-1) & t > 0 \\ 0 & t = 0 \end{cases} \) for any sequence \( h \in \mathbb{R}^n \) and \( t \in \mathbb{N} \), i.e.,

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
A_0 & B_w & B_0 \\
C_0 & 0 & D_{zu}
\end{bmatrix} \begin{bmatrix}
x \\
w
\end{bmatrix} + \begin{bmatrix}
D_{yu}
\end{bmatrix} \begin{bmatrix}
I
\end{bmatrix} \begin{bmatrix}
h(t)
\end{bmatrix}
\]

\[
B_w = [F_{1,2}, \ldots, F_{1,d}], \quad C = [F_{T,1}, \ldots, F_{T,d}]^T, \quad D_{zu} = [G_{T,1}, \ldots, G_{T,d}]^T, \quad D_{yu} = [H_2, \ldots, H_d].
\]

The specific form of LFRs calculated from ALPVs serves as a motivation to define the following subset of LFRs.

**Definition 5 (LPV-LFR):** An LPV-LFR is an LFR of the form (1), where \( d > 1 \), and the canonical decomposition \( \{(H_i, F_{i,j}, G_j)\}_{i,j=1}^{d} \) of \( \mathcal{M} \) has the property that \( F_{i,j} = 0 \), if \( i > 1 \) and \( j > 1 \).

It then follows that an LFR calculated from an ALPV model is an LPV-LFR. Note that not only ALPV models can be transformed to LPV-LFR models, but there is a transformation in the reverse direction.

**Definition 6 (From LPV-LFR to ALPV):** Let \( \mathcal{M} \) be an LPV-LFR of the form (1) and let \( \{(H_i, F_{i,j}, G_j)\}_{i,j=1}^{d} \) be its canonical decomposition. We can define the ALPV \( \Sigma_{\mathcal{M}} \) which corresponds to \( \mathcal{M} \) as the ALPV of the form (2), such that \( n_p = d - 1, n_x = n_1, n_y = p, m = n_x, D = D_0, A_0 = F_{1,1}, B_0 = G_1, C_0 = H_1, \) and for all \( i = 1, \ldots, n_p \),

\[
\begin{bmatrix}
A_i & B_i \\
C_i & D_i
\end{bmatrix} = \begin{bmatrix}
F_{i+1,1} & G_{i+1} \\
H_{i+1,1}
\end{bmatrix} \begin{bmatrix}
F_{1,i+1} & G_{i+1}
\end{bmatrix}.
\]

Note that, while there are many ways to transform an ALPV to and LPV-LFR, each LPV-LFR gives rise to a single ALPV. The operation of transforming an LPV-LFR to an ALPV is in a sense the inverse of the transformation of an ALPV to LPV-LFR: if \( \mathcal{M} \) is an LPV-LFR calculated from an ALPV \( \Sigma \) using Definition 4 then \( \Sigma_{\mathcal{M}} = \Sigma \). However, \( \mathcal{M} \) is any LPV-LFR, then the LPV-LFRs calculated from \( \mathcal{M}_{\Sigma} \) using Definition 4 are in general different from \( \mathcal{M} \) and they need not even be isomorphic to \( \mathcal{M} \).

That is, ALPVs yield LPV-LFRs and LPV-LFRs can be converted to ALPVs. Intuitively, the conversion is such that one could use control design techniques for LFRs to control ALPVs. If this path is taken, then the sole use of ALPV models is to serve as a source of LFR models, and hence instead of identifying ALPV models, one could identify LFR models directly. Unfortunately, as we will see in the next
D. Inconsistency of the ALPV to LFR transformation: motivating example

The transformations of Definition 4 and Definition 6 give rise to a number of fundamental questions, which have implications for control and system identification. To illustrate these problems, let us consider the following example.

Example 1 (Motivating example): Let us consider the ALPV model $\Sigma$ of the form (2), such that $n_p = 1$, $n_x = 2$, $n_u = n_y = 1$, with the following model matrices

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_{0}^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{1}^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and $D_0 = D_1 = 0$. Consider now the following two LFRs,

$$\mathcal{M} = (1, 1, 2, \{2, 3\}, (\hat{A}, \hat{B}, \hat{C}, 0),$$

$$\hat{\mathcal{M}} = (1, 1, 2, \{2, 3\}, (\hat{\hat{A}}, \hat{\hat{B}}, \hat{\hat{C}}, 0),$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0.2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0.5 & 0.5 \end{bmatrix},$$

$$\hat{A} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0.2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\hat{\hat{C}} = \begin{bmatrix} 1 & 0 & 0 & 0.5 & 0 \end{bmatrix}.$$

It is easy to see that both $\mathcal{M}$ and $\hat{\mathcal{M}}$ satisfy Definition 4 yet the matrices are completely different. In fact, it is easy to see that $\mathcal{M}$ and $\hat{\mathcal{M}}$ are not isomorphic. At a first glance, it is not clear that these two LFRs are in fact formally input-output equivalent. Concerning system identification, this example raises the question as to how to distinguish between these two LFRs, since clearly they originate from the same ALPV, and hence their behavior for $\Delta = \Delta(p)$ from (5) should be the same. From the point of view of control, these two LFRs behave quite differently. Using classical $H_\infty$ control, we computed LTI controllers for both LFRs which render the closed-loop $\gamma$-stable. If this controller is applied to the original ALPV, then it renders it stable for all scheduling sequences $p \in \mathcal{P}$ satisfying $|p(t)| < \gamma$ for all $t \in \mathbb{N}$. The largest $\gamma$ we could get with $\mathcal{M}$ is $\frac{1}{2}$, while the largest $\gamma$ we could get for $\hat{\mathcal{M}}$ is $\frac{1}{2\Delta}$, where $\Delta(p)$. That is, the guaranteed performance of the controller depends on the choice of the LFR!

The situation becomes even more interesting, if we notice that the following LFR $\hat{\hat{\mathcal{M}}} = (1, 1, 2, \{2, 2\}, \hat{\hat{A}}, \hat{\hat{B}}, \hat{\hat{C}}, 0)$ with

$$\hat{\hat{A}} = \begin{bmatrix} 1 & 0 & -1.196 & 0.5429 \\ 0 & 0.2 & -0.8668 & -0.9364 \\ -0.3413 & -1.519 & 0 & 0 \\ -0.752 & 0.338 & 0 & 0 \end{bmatrix},$$

$$\hat{\hat{B}} = \begin{bmatrix} 1 & 0 & -0.3413 & -0.752 \end{bmatrix}^T,$$

$$\hat{\hat{C}} = \begin{bmatrix} 1 & 0 & -0.598 & 0.2714 \end{bmatrix},$$

also satisfies Definition 4. The dimension of $\hat{\hat{\mathcal{M}}}$ is smaller than the dimension of $\mathcal{M}$ and $\hat{\mathcal{M}}$. This means $\mathcal{M}$ and $\hat{\mathcal{M}}$ are not minimal dimensional LFR representations of the ALPV $\Sigma$, and hence we might be tempted to think that our problems are caused by parasitic dynamics which are present in $\mathcal{M}$ and $\hat{\mathcal{M}}$, but which are absent from the ALPV $\Sigma$. But how can we be sure $\mathcal{M}$ is itself minimal? How to modify Definition 4 so that we cannot get LFRs of higher dimension than it is strictly necessary? What if we can find another minimal dimensional LFR which satisfies Definition 4 and which is not isomorphic to $\mathcal{M}$?

E. Problem formulation

The questions raised above can be addressed by answering the following questions:

1) Is it true that two ALPVs which are input-output equivalent yield LFRs which are formally input-output equivalent?

2) Can we modify Definition 4 so that minimal ALPVs get transformed to minimal LFRs in the sense of Theorem 1 and that this transformation preserves isomorphism?

3) Is the ALPV calculated from a minimal LPV-LFR according to Definition 6 minimal?

4) Can we transform an LPV-LFR to a minimal LFR which is also an LPV-LFR?

If the answers to these questions are positive, then the situation in Example 1 can be handled easily, by using Theorem 1 and Theorem 3. Namely, it is enough to restrict attention to minimal ALPVs and LFRs. Then, the modified transformation will guarantee that minimal and input-output equivalent ALPVs are transformed to minimal and formally input-output equivalent, and thus isomorphic LFRs. That is, the result of transforming ALPVs to LFRs is essentially unique, when minimal ALPVs are concerned. From Remark 2 it then follows that the result of control synthesis will not depend on which particular minimal ALPV or LFR was chosen, as long as the chosen ALPV is input-output equivalent to the original one. If one would like to bypass ALPVs altogether, then one should work with minimal LPV-LFRs, where minimality is understood in the sense of Theorem 1. Then, if two minimal LPV-LFRs describe two input-output equivalent ALPVs, then they will be isomorphic, and hence equivalent for control synthesis. Moreover, with some more work we can show that identifiability of LPV-LFRs is equivalent to that of ALPVs, so for identification it will not matter which modelling framework is used.
In the next section we state the answers to the questions above formally.

III. EQUIVALENCE BETWEEN ALPV AND LFR-LPV: PRESERVATION OF INPUT-OUTPUT BEHAVIOR, MINIMALITY AND IDENTIFIABILITY

We start with presenting a special case of Definition 4 which will have some useful properties.

Definition 7 (MR factorization): Let $\Sigma$ be an ALPV of the form (2). We say that an LFR $\mathcal{M}$ of the form (1) is calculated by Matrix Full Rank (MR) factorization from $\Sigma$, if $\mathcal{M}$ satisfies Definition 4 and in addition, for any $i = 2, \ldots, d$, $[F_{i,1} \ G_i]$ and $[F_{i,i} \ H_i^T]$ are full row and column rank respectively, where $(H_i, F_{i,j}, G_j)_{i,j=1}^d$ is the canonical partitioning of $\mathcal{M}$.

That is, the only distinguishing feature of MR factorization is that it explicitly requires the factorization (4) to be full rank. Note that LPV-LFRs calculated by MR factorization are unique up to isomorphism.

The next theorem, which is the main result of the paper, tells us that MR transformation preserves minimality and isomorphism. Finally, Part (iii) of Theorem 5 allows us to characterize the formal input-output maps of ALPV models.

Theorem 4 (Transforming ALPV to LPV-LFRs): Let $\Sigma, \Sigma_1, \Sigma_2$ be ALPVs and let $\mathcal{M}_1, \mathcal{M}_2$ be an LPV-LFR calculated from $\Sigma, \Sigma_1, \Sigma_2$ respectively by MR factorization.

1. $\Sigma$ is a minimal ALPV $\iff$ $\mathcal{M}$ is a minimal LFR.
2. The ALPVs $\Sigma_1$ and $\Sigma_2$ are input-output equivalent $\iff \mathcal{M}_1$ and $\mathcal{M}_2$ are formally input-output equivalent.
3. $\Sigma_1$ and $\Sigma_2$ are isomorphic $\iff$ the corresponding LFRs $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic.

Note that Theorem 4 allows us to derive the following useful properties for the transformation from LPV-LFRs to ALPVs.

Theorem 5 (Transformation from LPV-LFR to ALPV): Let $\mathcal{M}, \hat{\mathcal{M}}, \tilde{\mathcal{M}}$ be LPV-LFRs, and let $\Sigma = \Sigma_{\mathcal{M}}, \tilde{\Sigma} = \Sigma_{\tilde{\mathcal{M}}}$ be the ALPVs associated with $\mathcal{M}$ and $\tilde{\mathcal{M}}$ respectively.

(i) If $\mathcal{M}$ is minimal, then $\Sigma$ is a minimal ALPV.
(ii) $\mathcal{M}$ and $\hat{\mathcal{M}}$ are formally input-output equivalent, if and only if $\Sigma$ and $\hat{\Sigma}$ are input-output equivalent.
(iii) $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are isomorphic, then $\Sigma$ and $\tilde{\Sigma}$ are isomorphic.
(iv) Let $\mathcal{M}$ be the LPV-LFR computed from $\Sigma$ by MR factorization. Then $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are formally input-output equivalent. If $\mathcal{M}$ is minimal, then so is $\tilde{\mathcal{M}}$ and it is isomorphic to $\mathcal{M}$.

Parts (i) – (iii) of Theorem 5 say that the transformation from LPV-LFR to ALPVs preserves input-output equivalence, minimality and isomorphism. Finally, Part (iv) says that, if attention is restricted to minimal LFRs, then the transformations from Definition 6 and Definition 4 are each others’ inverses, if isomorphic models are viewed as equals.

Theorem 5 and Theorem 4 have several consequences.

Corollary 1: Any minimal LFR which is formally input-output equivalent to an LPV-LFR is also an LPV-LFR.

In other words, Corollary 1 states that the class of LPV-LFRs are closed under minimization, i.e., when working with LPV-LFRs we can always assume they are minimal.

Let us say that two LPV-LFRs are input-output equivalent, if their associated ALPVs, as defined in Definition 6 are input-output equivalent. Intuitively, if two LPV-LFRs are input-output equivalent, then their input-output behavior is the same for any $\Delta$ of the form (5). From Theorem 5 and Theorem 4 we can deduce the following.

Corollary 2: Two LPV-LFRs are input-output equivalent, if and only if they are formally input-output equivalent.

That is, for LPV-LFRs, their formal input-output map uniquely determines the input-output function of the associated ALPV! In fact, the proof of Theorem 5 and Theorem 4 allows us to characterize the formal input-output maps of LPV-LFRs.

Theorem 6: An LFR $\mathcal{M}$ is formally input-output equivalent to an LPV-LFR, if and only if $Y_{\mathcal{M}}(s) = 0$ for all $s \in \mathbb{C}^n$, such that $s$ not of the form in $\{i_1, \ldots, i_k\}$ for some $i_1, \ldots, i_k \in \{1, \ldots, d\}$, $k > 0$.

This allows us to determine if an arbitrary LFR can be represented as a LPV-LFR.

The results of Theorem 4 allow us to conclude that identifiability of ALPV and LFRs parameterizations is in fact equivalent. In order to present the results formally, we define the concepts of LPV-LFR parameterizations and their structural identifiability. To this end, let us fix integers $d, m$ and $p$, and denote by $\mathcal{LPF}(d, m, p, \{n_i\}_{i=1}^d)$ the set of all LPV-LFRs of the form (1).

Definition 8 (Parameterizations): Let $\Theta \subseteq \mathbb{R}^{n_x}$ be the space of parameters. An LPV-LFR parametrization is a function $\mathbf{M} : \Theta \rightarrow \mathcal{LPF}(d, m, p, \{n_i\}_{i=1}^d)$. We will say that an LPV-LFR parametrization $\mathbf{M}$ is structurally identifiable, if for any two distinct parameter values $\theta_1, \theta_2 \in \Theta$, $\theta_1 \neq \theta_2$, $\mathbf{M}(\theta_1)$ and $\mathbf{M}(\theta_2)$ are not input-output equivalent. We say that the LPV-LFR parametrization $\mathbf{M}$ is formally structurally identifiable, if for any two distinct parameter values $\theta_1, \theta_2 \in \Theta$, $\theta_1 \neq \theta_2$, $\mathbf{M}(\theta_1)$ and $\mathbf{M}(\theta_2)$ are not formally input-output equivalent.

Structural identifiability means that the parameter values can be uniquely determined by observing the output of the underlying system for some input and for some choice of the uncertainty block $\Delta$ of the form (5), which corresponds to a choice of the scheduling variables. In contrast, formal structural identifiability means that it is possible to determine the parameter value by observing the output for some input and some choice of $\Delta$, but the chosen $\Delta$ need not arise from a scheduling variable. It is not a-priori clear that these two identifiability notions are equivalent.

In fact, we can show that structural identifiability of ALPV models, and (formal) structural identifiability of LPV-LFRs are equivalent. To this end, recall from [1] the notions of parametrization, structural identifiability and minimality for ALPVs. Denote by $\mathcal{LPV}(n_{p_1}, n_{c_p}, n_{u_p}, n_{y_p})$ the set of all ALPV models of the form (4). An ALPV parametrization is a function $\Sigma : \Theta \rightarrow \mathcal{LPV}(n_{p_1}, n_{c_p}, n_{u_p}, n_{y_p})$. An ALPV parametrization $\mathbf{L}$ is structurally identifiable, if for any two
distinct parameter values \( \theta_1, \theta_2 \in \Theta, \theta_1 \neq \theta_2 \), the input-output maps of \( L(\theta_1) \) and \( L(\theta_2) \) are not equal. We say that an LPV-LFR parametrization \( M \) originates from an ALPV parametrization \( L \) by MR factorization, if for every \( \theta \in \Theta, M(\theta) \) is an LPV-LFR which is calculated from the ALPV \( L(\theta) \) by using an MR factorization. Likewise, we say that an ALPV parametrization \( L \) arises from the LPV-LFR parametrization \( M \), if for every \( \theta \in \Theta, L(\theta) \) is the ALPV associated with \( M(\theta) \), defined in Definition 6.

**Theorem 7 (Identifiability of LPV-LFR and ALPVs):** Consider an ALPV parametrization \( L \) and a LPV-LFR parametrization \( M \).

1) \( M \) is structurally identifiable \( \iff \) \( M \) is formally structurally identifiable.
2) If \( M \) originates from \( L \) by an MR factorization, then, \( L \) is structurally identifiable \( \iff \) \( M \) is structurally identifiable.
3) If \( L \) arises from \( M \), then, \( L \) is structurally identifiable \( \iff \) \( M \) is structurally identifiable.

Theorem 7 implies that in order to identify LPV-LFRs, it is sufficient to identify the corresponding ALPVs, and vice versa. In particular, in order to identify LPV-LFR models, it is enough to test them for uncertainty blocks of the form (5) which come from scheduling variables. Theorem 7 allows us to use the recent results of [1] to investigate identifiability of LPV-LFRs.

### IV. Conclusion

Structural properties of the transformation between ALPV and LFR models are studied. More precisely, minimal, input-output equivalent and identifiable ALPV models are shown to yield minimal, input-output equivalent and identifiable LPV-LFRs respectively, under the condition that the transformation is performed via a minimal rank factorization. LFR models that can be obtained from ALPV models are characterized using their input-output equivalent input-output maps. In a close future, these equivalence results will allow us to extend system identification solutions for ALPV to LFRs.

### Appendix

**Proof:** [Sketch of the proof of Theorem 4] (i) It is sufficient to prove that if \( M \) is reachable (resp. observable), then \( \Sigma \) is reachable (resp. observable). This can be shown in the same way as the corresponding implication in (1) of Theorem 4. More precisely, we can show that (6) holds. Hence, if \( M \) is reachable, then \( \Sigma \) is span-reachable by the same argument as in the proof of Theorem 4. Observability can be handled in a similar manner.

(ii) The proof of (ii) is similar to the proof of (2) of Theorem 4: it can be shown that the values of \( Y_M \) are either zeros or they coincide with the Markov-parameters of \( Y_\Sigma \). Hence, \( M \) and \( \Sigma \) are formally input-output equivalent, if and only if the Markov parameters of \( \Sigma \) and \( \Sigma \) coincide, and by [14], [15], the latter is equivalent to \( \Sigma \) and \( \Sigma \) being input-output equivalent.

(iii) Easy exercise.

(iv) Note that \( \Sigma \) is the ALPV associated with both \( M \) and \( \Sigma \). Hence, by part (ii) \( M \) and \( \Sigma \) have to be formally input-output equivalent. If \( M \) is minimal, then by part (i) so is \( \Sigma \), and by Theorem 4 \( M \) is minimal. Since \( M \) and \( \Sigma \) are both formally input-output equivalent and minimal, then, by Theorem 1 they are isomorphic.

**Proof:** [Proof of Corollary 1] Let \( M \) be an LPV-LFR, and compute its associated ALPV \( \Sigma \). From [14], [15] it follows that \( \Sigma \) can be converted to a minimal ALPV \( \Sigma_m \) which is input-output equivalent to \( \Sigma \). Let us calculate...
an LPV-LFR $\mathcal{M}_m$ from $\Sigma_m$ using MR factorization. By Theorem 4, $\mathcal{M}_m$ is minimal, and by part (iv), $\mathcal{M}_m$ and $\mathcal{M}$ are formally input-output equivalent. Note that $\mathcal{M}_m$ is an LPV-LFR. If $\mathcal{M}$ is any other minimal LFR which is formally input-output equivalent to $\mathcal{M}$, then $\mathcal{M}'$ is isomorphic to $\mathcal{M}_m$. It is easy to see that if an LFR satisfies the definition of LPV-LFRs, then any LFR which is isomorphic to it will also satisfy the definition of LPV-LFRs.

Proof: [Proof of Corollary 2] The corollary is just a reformulation of part (ii) of Theorem 5.

Proof: [Proof of Theorem 6] It is clear from the definition of an LPV-LFR that if $\mathcal{M}$ is an LPV-LFR, then $Y_\mathcal{M}$ satisfies the conditions of the theorem. It is left to show that if $Y_\mathcal{M}$ satisfies the conditions of the theorem, then $\mathcal{M}$ is input-output equivalent to an LPV-LFR. To this end, we can always assume that $\mathcal{M}$ is minimal, without loss of generality. Otherwise, we can always transform it to an equivalent minimal one. We also know that for a minimal LFR, the reachability and observability matrices are full row and column rank respectively. We will prove now that if the series has zero terms for the sequences which are not of the form $i_1 i_2 \cdots i_k 1$, then the matrix blocks $F_{i,j}$ will be zero when both $i, j$ are bigger than 1, and hence $\mathcal{M}$ is an LPV-LFR.

For this purpose, we will show that the block matrix $F_{r,q}=0$, whenever $r > 1$ and $q > 1$. We claim that if $O_rF_{r,q}G_q^e = 0$, where $O_r = O_r(\mathcal{M})$ and $G_q = G_q(\mathcal{M})$ are such that $\text{rank} O_r = n_r$ and $\text{rank} G_q = n_q$. Then, $F_{r,q} = 0$ due to the fact that $R_r^q(\mathcal{M})$ and $O_r(\mathcal{M})$ are full row and column rank respectively.

Let us take the row $H_{i_{k}}F_{i_{k},i_{k-1}}\cdots F_{i_{1},r}$ of $O_r(\mathcal{M})$, and the column $F_{q,j_{k-1}}\cdots F_{q,j_{0}}G_{j_{0}}$ of $R_{q}^r(\mathcal{M})$, then,

$$H_{i_{k}}F_{i_{k},i_{k-1}}\cdots F_{i_{1},r} F_{r,q} F_{q,j_{k-1}}\cdots F_{j_{1},j_{0}} G_{j_{0}} = S_{i_{k}i_{k-1}\cdots i_{1}, q, j_{k-1}\cdots j_{0}} = 0. \ (7)$$

This means that $O_{r}(\mathcal{M})F_{r,q} R_{q}^r(\mathcal{M}) = 0$, i.e., $F_{r,q} = 0$.

Proof: [Sketch of the proof of Theorem 7 (i)] The statement follows by noticing that for any $\theta_1, \theta_2 \in \Theta$, due to Corollary 2, $M(\theta_1)$ and $M(\theta_2)$ are formally input-output equivalent if and only if they are input-output equivalent.

(ii)-(iii) Assume that $M$ originates from $L$ by MR factorization, or $L$ originates from $M$. In both case, for any $\theta_1, \theta_2 \in \Theta$, $L(\theta_1)$ and $L(\theta_2)$ are input-output equivalent, if and only if $M(\theta_1)$ and $M(\theta_2)$ are input-output equivalent. Both statements of the theorem then follow from the definition of structural identifiability for $M$ and $L$.

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