Exactly solvable time-dependent models of two interacting two-level systems

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Two coupled two-level systems placed under external time-dependent magnetic fields are modelled by a general Hamiltonian endowed with a symmetry that enables us to reduce the total dynamics into two independent two-dimensional sub-dynamics. Each of the sub-dynamics is shown to be brought into an exactly solvable form by appropriately engineering the magnetic fields and thus we obtain an exact time evolution of the compound system. Several physically relevant and interesting quantities are evaluated exactly to disclose intriguing phenomena in such a system.

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I. INTRODUCTION

A rigid and localized dimeric structure (simply dimer) consists of a pair of independent distinguishable quantum subsystems living, by definition, in finite-dimensional Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) and, therefore, hereafter referred to as spins \( \hat{S}_1 = (S_{1x}, S_{1y}, S_{1z}) \) and \( \hat{S}_2 = (S_{2x}, S_{2y}, S_{2z}) \) respectively, \( \hat{S}_i^a (i = 1, 2; a = x, y, z) \) being the operator for the \( a \)-cartesian component of \( \hat{S}_i \) in the laboratory reference frame. The dimension of the Hilbert space \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) of the dimer is \((2S_i + 1)(2S_j + 1)\), indeed postulating the absence in the two subsystems as well as in the compound system of classical degrees of freedom (situation previously described using the adjectives ‘rigid’ and ‘localized’). The physical nature of \( \hat{S}_i \) depends on the particular scenario under scrutiny: it may be the spin of an electron or a nucleus, the angular momentum of an atom in its ground state or an effective representation of a few-level system dynamical variable. The Hamiltonian \( \mathcal{H} \) of the dimer is then a true or effective spin Hamiltonian where the terms linear in \( \hat{S}_i^a \) may (even fictitiously) be interpreted as Zeeman coupling of each of the two spins with classical, external, generally different and time-dependent effective magnetic fields \( \mathbf{B}_1(t) \) and \( \mathbf{B}_2(t) \) while the bilinear contributions may be thought of as stemming from the spin-spin interaction [1].

Over the last two decades a great deal of theoretical, experimental and applicative attention has been devoted to the field of Molecular Magnetic Materials, in particular after the discovery of the so-called Single Magnet Molecule (SSM), that is a single molecule behaving like a nanosized magnet associated to an unusual high value (even \( S = 10 \)) of the spin in the ground state of the molecule. It is a matter of fact that as a result of a successful, extraordinary and synergically interdisciplinary effort aimed at searching and producing SMM in laboratory, in the last few years we have witnessed a very fast growing of efficient protocols for synthesizing a variety of such molecular magnets with the added value of possessing a number of constituent paramagnetic ions embodied in the molecule running from 2 to 10 in different samples [3]. Such important technological advances on the one hand open very good applicative perspectives in many directions, from the realization of an experimental set up for testing theoretical prediction concerning qudits-based single purpose quantum computers to the availability of new materials with magnetic properties tailored on demand to meet specific tasks. On the other hand the production of crystalline or powder samples made up of molecular magnetic units, provides an ideal platform to investigate and reveal the emergence of nonclassical signatures in the quantum dynamics of two or few interacting spins.

The simplest coupled spin system we may conceive consists, of course, of two interacting spin 1/2’s only in a dimer, isolated from its environment (rest of the sample) degrees of freedom. Some binuclear copper(II) compounds, e.g. [4] [5], provide a possible scenario of this kind and in the previous references the values of the parameters characterizing the spin-spin interaction in such a molecule have been experimentally determined exploiting electron-paramagnetic resonance techniques. Motivations to investigate the emergence of quantum signatures in the behaviour of two coupled spins (\( \geq 1/2 \)) go beyond the area of magnetic materials. Two spin 1/2 Hamiltonians provide indeed experimentally implementable powerful effective models to capture quantum properties of such systems like two coupled semiconductor quantum dots [6] or a pair of two neutral cold atoms each nested into two adjacent sites of an optical lattice made up of an isolated double wells [7]. Spin models provide a successful language to investigate possible manipulations of the qubits aimed at quantum computing purposes and quantum information transfer between two spin-qubits [8], encompassing rather different physical contents like, for example, cavity QED [9] [10], superconductors [11] [12] and trapped ions [13] [14].

The most general Hamiltonian model of an isolated dimer hosting two spin 1/2’s may be written as a bilinear form involving the two sets of operators \( \{\hat{S}_{1x}, \hat{S}_{1y}, \hat{S}_{1z}, \hat{S}_{2x}, \hat{S}_{2y}, \hat{S}_{2z}\} \)
and \( \{ \hat{S}_x^1, \hat{S}_y^1, \hat{S}_z^1, \hat{S}_x^0, \hat{S}_y^0, \hat{S}_z^0 \} \), that is,

\[
H = \sum_{(i,j) \neq (0,0)} \gamma_{ij} \hat{S}_i^1 \otimes \hat{S}_j^1
\]  

(1)

where \( i \) and \( j \) run in the set \((x,y,z,0)\) and the operator \( \hat{S}_i^0 \) \((i = 1, 2)\) is the identity operator \( \mathbb{1}_i \) in \( \mathcal{H}_i \). The six real parameters \( \gamma_{i0} \) and \( \gamma_{0j} \) \((i \neq j)\) are assumed to be generally time dependent while all the other parameters characterizing the spin-spin coupling are real and time independent. Without further specific constraints on the 15 parameters \( \gamma_{ij} \) \((i, j = x, y, z, 0)\), the Hamiltonian possesses no symmetries and in particular it does not commute with the collective angular momentum operators \( \hat{S}^2 = (\hat{S}_1 + \hat{S}_2)^2 \) and/or \( \hat{S}^z = \hat{S}_1^z + \hat{S}_2^z \). In such a case, even if \( H \) is time independent, the four roots of the relative secular equation, albeit determinable, are rather involved functions of all the 15 parameters and then are practically not exploitable for extracting physical prediction on the physical system under scrutiny. Thus, either legitimated by investigations on specific physical situations or motivated by the interest in studying models possessing, by construction, constants of motion, some constraints on the parameters \( \gamma_{ij} \) have been introduced in the literature, making the Hamiltonian \( \mathbb{1} \) less general and at the same time non trivial and of physical interest. It is enough to quote the main declinations of the three-dimensional quantum Heisenberg models or the Dzyaloshinskii-Moriya (DM) models \([15, 16]\) in conformation or not with simplified contributions to terms describing anisotropy effects in the Hamiltonian.

In this paper we too investigate a Hamiltonian model included in (1), still general enough to remain not commuting with \( \hat{S}^2 \) and \( \hat{S}^z \), but such to possess a symmetry property at the origin of significant properties characterizing its quantum dynamics. A peculiar aspect of such a symmetry property is that it displays its usefulness even when we wish to study our physical system in a time-dependent scenario. Exploiting, indeed, the symmetry-induced reduction of the quantum dynamics generated by the time-dependent Hamiltonian we are going to propose to two dynamically invariant proper subspaces of \( \mathcal{H} \), we are able to successfully apply a recently reported \([17]\) systematic approach for generating exactly solvable quantum dynamics of a single spin \(1/2\) subjected to a time-dependent magnetic field. Thus the main result of this paper is twofold. First we report the exact explicit solution of the time-dependent Schrödinger equation of a system of two coupled spin \(1/2\)'s described by a time-dependent generalized Heisenberg model. Second, we demonstrate that the method reported in Ref. \([17]\), even as it stands, proves to be a useful tool to treat more complex time-dependent scenarios.

The paper is organized as follows. The Hamiltonian model and the decoupling procedure are discussed in Section II, where, in addition, the structure of the time-evolution operator is also constructed with the help reported in Ref. \([17]\). In the subsequent Section III, some exactly solvable time-dependent Hamiltonian models of the two coupled qubits are singled out and analysed. Sections IV and V are respectively dedicated to a systematic study of the time behaviour of exemplary collective spin operators and of the concurrence. Some conclusive remarks are finally reported in the last section VI.

## II. THE HAMILTONIAN MODEL

By construction, the Hamiltonian model (1) includes all possible contributions stemming from internal or external couplings of our two spin \(1/2\) system. It may be suggestively cast in the following form

\[
H' = \mu_B (\mathbf{B}_1 \cdot \mathbf{g}_1 \cdot \mathbf{S}_1 + \mathbf{B}_2 \cdot \mathbf{g}_2 \cdot \mathbf{S}_2) + \mathbf{S}_1 \cdot \mathbf{\Gamma}_{12} \cdot \mathbf{S}_2, 
\]  

(2)

where \( \mathbf{g}_1, \mathbf{g}_2 \) and \( \mathbf{\Gamma}_{12} \) are appropriate second-order cartesian tensors whose entries are related to the 15 parameters appearing in Eq. (1) and \( \mu_B \) denotes the Bohr magneton. Equation (2) mimics the usual way of representing the Hamiltonian used in a molecular or nuclear context to describe the coupling of two true spin \(1/2\)'s. In general we may claim that \( \mathbf{g}_1 \) and \( \mathbf{g}_2 \) include possible corrections to the coupling terms between each spin and its local time-dependent external magnetic field, while the other term includes contact term-like couplings as well as anisotropic-like spin-spin couplings.

The model we are going to propose is based on the following assumptions: a) \( \mathbf{B}_1(t) \) and \( \mathbf{B}_2(t) \) are at any time directed along the \( z \)-axis of the laboratory frame, namely \( \mathbf{B}_i(t) = (0,0,B_i^z(t)) \) \((i = 1, 2)\); b) the cartesian tensor \( \mathbf{\Gamma}_{12} \) has the following form

\[
\mathbf{\Gamma}_{12} = \begin{pmatrix}
\gamma_{xx} & \gamma_{xy} & 0 \\
\gamma_{yx} & \gamma_{yy} & 0 \\
0 & 0 & \gamma_{zz}
\end{pmatrix};
\]  

(3)

c) the \( \mathbf{g} \)-tensors have the following form

\[
\mathbf{g}_i = \begin{pmatrix}
g_{i}^{xx} & g_{i}^{xy} & 0 \\
g_{i}^{yx} & g_{i}^{yy} & 0 \\
0 & 0 & g_{i}^{zz}
\end{pmatrix}
\]  

(4)

\((i = 1, 2)\). The structure of the previous tensors is, for example, appropriate when the dimer coincides with a binuclear unit characterized by a \( C_2 \)-symmetry with respect to the \( z \) axis as in Ref. \([5]\).

In accordance with our previous assumptions, in this paper we investigate the quantum dynamics of the following time-dependent two spin Hamiltonian model

\[
H = \hbar \omega_1 \hat{\sigma}_1^x + \hbar \omega_2 \hat{\sigma}_2^x + \gamma_{xx} \hat{\sigma}_1^x \hat{\sigma}_2^x + \gamma_{yy} \hat{\sigma}_1^y \hat{\sigma}_2^y + \gamma_{zz} \hat{\sigma}_1^z \hat{\sigma}_2^z + \gamma_{xy} \hat{\sigma}_1^x \hat{\sigma}_2^y + \gamma_{yx} \hat{\sigma}_1^y \hat{\sigma}_2^x + \gamma_{zx} \hat{\sigma}_1^z \hat{\sigma}_2^x + \gamma_{zy} \hat{\sigma}_1^y \hat{\sigma}_2^z + \gamma_{yz} \hat{\sigma}_1^y \hat{\sigma}_2^z + \gamma_{xz} \hat{\sigma}_1^x \hat{\sigma}_2^z,
\]  

(5)

where \( \hat{\sigma}_i^x, \hat{\sigma}_i^y \) and \( \hat{\sigma}_i^z \) \((i = 1, 2)\) are the Pauli matrices related to the respective components of the spin operator \( \mathbf{S}_i \) as

\[
\hat{S}_i = \frac{\hbar}{2} \hat{\sigma}_i.
\]  

(6)
with \( \sigma_i \equiv (\sigma_i^x, \sigma_i^y, \sigma_i^z) \), while

\[
\omega_i(t) = \frac{\mu_B g_i^z B_i^z(t)}{2}.
\]

Note that the identity operators \( \mathbb{1}_i \) are and will mostly be suppressed for notational simplicity.

A. Symmetry-based decoupling of the two spins

As anticipated, our Hamiltonian does not commute with \( S^2 \) and \( S^z \) but it has been constructed in such a way to exhibit the following canonical and symmetry transformation

\[
\tilde{\sigma}_i^z \to -\tilde{\sigma}_i^z, \quad \tilde{\sigma}_i^y \to -\tilde{\sigma}_i^y, \quad \tilde{\sigma}_i^z \to \tilde{\sigma}_i^z, \quad i = 1, 2.
\]

This fact implies the existence of a unitary time-independent operator accomplishing the transformation (8), which is by construction a constant of motion. It is easy to convince oneself that this unitary operator is given by \( \pm \tilde{\sigma}_1^z \tilde{\sigma}_2^z \), being the transformation (8) nothing but the rotations of \( \pi \) around the \( \hat{z} \) axis with respect to each spin. Indeed, we can write the unitary operator accomplishing this transformation as follows

\[
e^{i \tilde{S}_z^i / \hbar} \otimes e^{i \tilde{S}_z^j / \hbar} = -\tilde{\sigma}_1^z \tilde{\sigma}_2^z = \cos \left( \frac{T}{2} \tilde{S}_z \right),
\]

where we have defined \( \tilde{S}_z = \tilde{\sigma}_1^z + \tilde{\sigma}_2^z \). Equation (9) shows that the constant of motion \( \tilde{\sigma}_1^z \tilde{\sigma}_2^z \) is indeed a parity operator with respect to the collective spin-Pauli variable \( \tilde{S}_z \), since in correspondence to its integer eigenvalues \( M = 0, \pm 2 \), \( \tilde{\sigma}_1^z \tilde{\sigma}_2^z \) has eigenvalues +1 and -1 respectively.

It is important to underline that the existence of this constant of motion implies the existence of two sub-dynamics related to the two eigenvalues of \( \tilde{\sigma}_1^z \tilde{\sigma}_2^z \). We can extract these two sub-dynamics by considering that the operator \( \tilde{\sigma}_1^z \tilde{\sigma}_2^z \) has the same spectrum of \( \tilde{\sigma}_z \), i.e., the same eigenvalues \( \pm 1 \) with the same twofold degeneracy. Therefore there exists a unitary time-independent operator \( U \) transforming \( \tilde{\sigma}_1^z \tilde{\sigma}_2^z \) in \( \tilde{\sigma}_z \). It can be easily seen that the unitary and hermitian operator

\[
\Lambda = \frac{1}{2} \left[ \mathbb{1} + \tilde{\sigma}_1^z \tilde{\sigma}_2^z - \tilde{\sigma}_1^z \tilde{\sigma}_2^z \right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

in the standard ordered basis \( B = \{ |++\rangle, |--\rangle, |+\rangle, |-\rangle, |\rangle, \langle | \} \)

accomplishes the desired transformation:

\[
U \otimes \tilde{\sigma}_1^z \tilde{\sigma}_2^z U = \tilde{\sigma}_1^z \tilde{\sigma}_2^z.
\]

Transforming \( H \) into \( \tilde{H} = U \otimes H U \), we get

\[
\tilde{H} = \hbar \omega_1 \tilde{\sigma}_1^z + \hbar \omega_2 \tilde{\sigma}_2^z + \gamma_{xx} \tilde{\sigma}_1^z + \gamma_{xx} \tilde{\sigma}_2^z + \gamma_{yx} \tilde{\sigma}_1^z \tilde{\sigma}_2^z + \gamma_{yx} \tilde{\sigma}_1^z \tilde{\sigma}_2^z.
\]

It is easy to check that \( \tilde{\sigma}_z \) is a constant of motion of \( \tilde{H} \) and that, consequently, \( \tilde{H} \) may be represented as

\[
\tilde{H} = \sum_{\sigma} \tilde{H}_{\sigma} | \sigma_z \rangle \langle \sigma_z | = \tilde{H}_+ \otimes |+\rangle \langle + | + \tilde{H}_- \otimes |\rangle \langle - |.
\]

where

\[
\tilde{H}_{\sigma} = \gamma_{zz} \sigma_z^2 + \hbar (\omega_1 + \omega_2 \sigma_z^2) \tilde{\sigma}_1^z + \gamma_{yx} \sigma_z^2 \tilde{\sigma}_1^z \tilde{\sigma}_2^z + \gamma_{yx} \tilde{\sigma}_1^z \tilde{\sigma}_2^z.
\]

This implies the existence of two \( \sigma_z^2 = \pm 1 \) sub-dynamics relative to a fictitious spin \( 1/2 \) immersed in different magnetic fields, each one possessing three components with the \( z \) one only depending on time.

B. Evolution operator in the presence of inhomogeneous time-dependent magnetic field

If \( \omega_1 \) and \( \omega_2 \) were time independent, we would be able to solve exactly the time-evolution problem related to the Hamiltonian \( \tilde{H} \). Indeed it is straightforward to find the eigenstates of \( \tilde{H} \) as

\[
| \psi \rangle = | \phi_{1z} \rangle | \sigma_z \rangle \otimes | \sigma_z \rangle
\]

(\( i = 1, 2 \) where \( | \phi_{1z} \rangle \) are the two eigenvectors of \( \tilde{H}_\pm \), that is the two eigenvectors related to the sub-dynamics with \( \sigma_z^2 = \pm 1 \). Through the relation

\[
| \psi \rangle = U | \tilde{\psi} \rangle,
\]

we can in turn find the eigenvectors of \( H \) and the time evolution of an arbitrary state of the two spins.

In the time-dependent case (when \( \omega_1 \) and \( \omega_2 \) depend on time), thanks to the fact that the unitary and hermitian operator \( U \) is time independent, we succeed, in view of the structure possessed by \( \tilde{H} \) as given by Eq. (13), in decoupling the time-dependent Schrödinger equation into two time-dependent Schrödinger equations of single spin \( 1/2 \). Therefore, we can construct the time-evolution operator of the whole dynamics of the two interacting spin \( 1/2 \)’s, starting from the construction of the two time-evolution operators of the two sub-dynamics of single spin \( 1/2 \). Indeed, starting from the initial time-dependent Schrödinger equation for the evolution operator \( \tilde{U} \) generated by \( \tilde{H} \)

\[
i \hbar \tilde{U} = \tilde{H} \tilde{U}, \quad \tilde{U}(0) = \mathbb{1},
\]

we have, since \( \tilde{U} = 0 \),

\[
i \hbar \tilde{U} = \tilde{H} \tilde{U}, \quad \tilde{U}(0) = \mathbb{1},
\]

where \( \tilde{U} \equiv U \otimes U \). If we search the time-evolution operator of \( H \) in the form

\[
\tilde{U} = \tilde{U}_+ \otimes |+\rangle \langle + | + \tilde{U}_- \otimes |\rangle \langle - |
\]
the time-dependent Schrödinger equation for $\hat{U}$ in (19) is converted into the following two Cauchy problems related to the sub-dynamics associated to $\hat{H}_+$ and $\hat{H}_-$,

\[
\begin{align*}
    i\hbar \dot{\hat{U}}_+ &= \hat{H}_+ \hat{U}_+, & \hat{U}_+(0) &= \mathbb{I}, \\
    i\hbar \dot{\hat{U}}_- &= \hat{H}_- \hat{U}_-, & \hat{U}_-(0) &= \mathbb{I}.
\end{align*}
\]

If we are able to solve these two single spin-1/2 time-dependent Schrödinger equations we then can construct easily the unique time-evolution operator of the entire $\hat{H}$ as given in Eq. (20) and then the time-evolution operator of the initial $\hat{H}$ through the following relation

\[
\hat{U} = \hat{U}_+ \hat{U}_- = \hat{U}_- \hat{U}_+.
\]

The importance of this result consists in the possibility of applying the Messina-Nakazato approach [17] to each of the two sub-dynamics of single spin 1/2 to generate a class of time-dependent exactly solvable models whose Hamiltonian could be generally written as in (5). It is also important to stress that the procedure and the result are valid also if all the coupling constants were time-dependent, too, besides $\omega_1$ and $\omega_2$. To illustrate such a possibility we solve in detail the quantum dynamics of the two coupled spins taking advantage of some results reported in Ref. [17].

The two-dimensional matrix of each sub-dynamics we got before can be written as follows

\[
\hat{H}_\pm = \hat{H}_\pm' \pm \gamma_{zz} \mathbb{I} = \begin{pmatrix} \Omega_\pm(t) & \Gamma_\pm \\ \Gamma_\pm & -\Omega_\pm(t) \end{pmatrix} \pm \gamma_{zz} \mathbb{I},
\]

where we have put

\[
\begin{align*}
    \Omega_\pm(t) &= h(\omega_2(t) \pm \omega_1(t)), \\
    \Gamma_\pm &= (\gamma_{xx} \mp \gamma_{yy}) - i(\pm \gamma_{xy} + \gamma_{yx}).
\end{align*}
\]

Since, denoting by $\mathcal{E}_\pm$ the time-evolution operator generated by $\hat{H}_\pm'$, the evolution operator generated by $\hat{H}_\pm$ is simply given by

\[
\hat{U}_\pm = e^{\pm i\gamma_{zz}t/\hbar} \mathcal{E}_\pm,
\]

we will search directly the time-evolution operator $\mathcal{E}_\pm$ in accordance with the Ref. [17]. Following the example given in the section 3.3 of [17] and considering the case when the transverse component of the magnetic field is constant (because in our context all the internal coupling coefficients are time independent), the time-evolution operator for each sub-dynamics of single spin 1/2 may be cast in the form

\[
\mathcal{E}_\pm = \begin{pmatrix} |a_\pm| e^{i\phi_\pm} & |b_\pm| e^{i\phi_\pm} \\
-|b_\pm| e^{-i\phi_\pm} & |a_\pm| e^{-i\phi_\pm} \end{pmatrix}
\]

where

\[
|a_\pm| = \cos \left[ \frac{\Gamma_\pm}{\hbar} \int_0^t \cos \Theta_\pm(t') dt' \right]
\]

and, since $|a_\pm|^2 + |b_\pm|^2 = 1$,

\[
|b_\pm| = \sin \left[ \frac{\Gamma_\pm}{\hbar} \int_0^t \cos \Theta_\pm(t') dt' \right]
\]

and

\[
\phi_\pm = -\left( \frac{\Theta_\pm}{2} + R_\pm \right), \quad \phi_\pm = -\frac{\Theta_\pm}{2} + R_\pm - \frac{\pi}{2}
\]

with

\[
R_\pm = \frac{\Gamma_\pm}{\hbar} \int_0^t \sin \Theta_\pm dt' \int_0^t \cos \Theta_\pm dt'.
\]

Here $\Theta_\pm(t)$ are arbitrary well-behaved mathematical functions fulfilling the condition $\Theta_\pm(0) = 0$. Consistently, the longitudinal components of the effective magnetic fields of the two sub-dynamics vary over time such that

\[
\Omega_\pm = \frac{\hbar}{2} \frac{\partial \Theta_\pm}{\partial t} + |\Gamma_\pm| \sin \Theta_\pm \cot \left[ \frac{2|\Gamma_\pm|}{\hbar} \int_0^t \cos \Theta_\pm dt' \right]
\]

and we get easily the time dependence of $\omega_1$ and $\omega_2$ (and that of $B_1^z$ and $B_2^z$ through the relation (7)) resulting

\[
\omega_1 = \frac{\Omega_+ + \Omega_-}{2h}, \quad \omega_2 = \frac{\Omega_+ - \Omega_-}{2h}.
\]

These equations practically single out the class of time-dependent Hamiltonians exactly treatable when the realistic assumption that the tensors $\Gamma_{12}$, $g_1$ and $g_2$ are time independent is made. Deriving the explicit form of the time-evolution operators related to the two independent sub-dynamics (described in terms of a single fictitious spin 1/2), we can write the whole unitary evolution operator of $\hat{H}$ (\hat{U}) as prescribed in (20) and through relation (22) we get the unitary time-evolution operator of our initial problem, which reads

\[
\hat{U} = \begin{pmatrix} |a_+| e^{i\Phi_+} & 0 & 0 & |b_+| e^{i\Phi_+} \\
0 & |a_-| e^{i\Phi_-} & |b_-| e^{i\Phi_-} & 0 \\
0 & |a_-| e^{-i\Phi_-} & |b_-| e^{-i\Phi_-} & 0 \\
-|b_+| e^{-i\Phi_+} & 0 & 0 & |a_+| e^{-i\Phi_+} \end{pmatrix},
\]

where we have put

\[
\Phi_{a/b} = \phi_{a/b} \pm \frac{\gamma_{zz}}{\hbar} t
\]

\[
\Phi_{a/b} = \phi_{a/b} \mp \frac{\gamma_{zz}}{\hbar} t.
\]

It is important to point out that if $\omega_1(t) = \omega_2(t)$ we have $\Omega_\pm = 0$ and the sub-dynamic related to $\sigma_z^\pm = -1$ is generated by

\[
\hat{H}_\pm = \begin{pmatrix} 0 & \Gamma_\pm \\ \Gamma_\pm & 0 \end{pmatrix}
\]
and then becomes a time-independent problem. In this instance the relative evolution operator reads
\[
\hat{U}_- = e^{-i\frac{\Phi}{\hbar}t} \begin{pmatrix}
\cos \left( \frac{\Gamma_1}{\hbar} t \right) & e^{i\Phi} \sin \left( \frac{\Gamma_1}{\hbar} t \right) \\
-e^{-i\frac{\Phi}{\hbar}t} \sin \left( \frac{\Gamma_1}{\hbar} t \right) & e^{i\Phi} \cos \left( \frac{\Gamma_1}{\hbar} t \right)
\end{pmatrix},
\]
where \( \Phi = \arctan \left( \frac{2\gamma_x + \gamma_y}{\gamma_y - \gamma_x} \right) \), and the whole evolution operator of the initial dynamic becomes
\[
U = \begin{pmatrix}
|a_+| e^{i\Phi_{a_+}} & 0 & 0 & |b_+| e^{i\Phi_{b_+}} \\
0 & e^{-i\frac{\Phi}{\hbar}t} \cos \left( \frac{\Gamma_1}{\hbar} t \right) & e^{i(\Phi - i\frac{\Phi}{\hbar}t)} \sin \left( \frac{\Gamma_1}{\hbar} t \right) & 0 \\
0 & e^{-i(\Phi + i\frac{\Phi}{\hbar}t)} \sin \left( \frac{\Gamma_1}{\hbar} t \right) & e^{-i\frac{\Phi}{\hbar}t} \cos \left( \frac{\Gamma_1}{\hbar} t \right) & 0 \\
-|b_+| e^{-i\Phi_{b_+}} & 0 & 0 & |a_+| e^{-i\Phi_{a_+}}
\end{pmatrix}.
\]

Of course, the evolution operator has the same form as that given by Eq. (33), where the two-by-two internal block is now completely determined regardless of the way \( H \) depends on time. This means that when \( \omega_1(t) = \omega_2(t) = \omega(t) \) in the Hamiltonian model given in Eq. (5), the time evolutions of \( |+\rangle \) and \( |-\rangle \) (and so, of every linear combination of these states) are independent of \( \omega(t) \) and are characterized by Bohr frequencies related to the coupling constants appearing in \( H \).

It is useful to underline that the condition \( \omega_1(t) = \omega_2(t) \) is not implied simply by the condition \( B_1(t) = B_2(t) \) because in general we may have different \( g \)-tensors (or factors) for the two spins which “rule” the coupling with the magnetic field and are responsible for the different effective local magnetic fields in the two sites, even when \( B_1(t) = B_2(t) \). So, the more general condition implying \( \omega_1(t) = \omega_2(t) \) is
\[
B_1(t) \cdot g_1 = B_2(t) \cdot g_2.
\]

III. EXACTLY SOLVABLE TIME DEPENDENT SCENARIOS FOR THE TWO SPIN 1/2 MODEL

In this section we report and discuss some particular time-dependent physical scenarios leading to exact analytical solutions of the Schrödinger equation (18) by taking advantage of the approach described in the previous section. To this end we notice that on the basis of Eq. (20) the knowledge of \( \hat{U}_+ \) and \( \hat{U}_- \) is enough for determining the evolution operator \( \hat{U} \) generated by the Hamiltonian \( H \) governing the quantum dynamics of the two coupled spins. This means that in practice our task is the resolution of two dynamical problems each formally referred to a spin 1/2. It is right this point that makes of relevance the method in Ref. [17].

The following two subsections report two novel exact solutions of the quantum dynamics of a spin 1/2 based on the method developed in Ref. [17]. In practice to single out a treatable scenario amounts at engineering the time-dependent magnetic field acting upon the spin 1/2. Both scenarios are useful in our problem meaning that each of them allows the selection of appropriate time-dependent exactly solvable models for \( H_+ \) and \( H_- \). The last subsection is dedicated to the explicit construction of the two spin Hamiltonian models emerging from the intermediate steps leading to \( H_+ \) and \( H_- \).

A. First exactly solvable time-dependent scenario for one spin 1/2

We may put
\[
|\Gamma| \int_0^t \cos \Theta dt' = \frac{1}{2} \arcsin \left[ \tanh(\gamma t) \right], \quad \gamma = 2|\Gamma|/\hbar.
\]

With this choice \( |a(t)|(|b(t)| \) goes from 1(0), at \( t = 0 \), to 1/\( \sqrt{2} \)(1/\( \sqrt{2} \)), as \( t \to \infty \). Indeed we have
\[
|a(t)| = \sqrt{\frac{\cosh(\gamma t) + 1}{2 \cosh(\gamma t)}}, \quad |b(t)| = \sqrt{\frac{\cosh(\gamma t) - 1}{2 \cosh(\gamma t)}}.
\]

Moreover, we have
\[
\cos \Theta(t) = \frac{1}{\cosh(\gamma t)} \quad (41a), \quad \sin \Theta(t) = \tanh(\gamma t) \quad (41b)
\]
and the integral \( R \) is trivially integrated to yield
\[
\mathcal{R} = \frac{\gamma t}{2}.
\]
From (41) we derive
\[ \dot{\Theta} = \frac{\gamma}{\cosh(\gamma t)} \]  
\[ \Theta = 2 \arctan \left[ \tanh \left( \frac{\gamma}{2} t \right) \right] \]
so that we get
\[ \phi_a = -\arctan \left[ \tanh \left( \frac{\gamma}{2} t \right) \right] - \frac{\gamma}{2} t \]  
\[ \phi_b = -\arctan \left[ \tanh \left( \frac{\gamma}{2} t \right) \right] + \frac{\gamma}{2} t - \frac{\pi}{2} \]
and the longitudinal component of the magnetic field varies as
\[ \Omega = 2|\Gamma| \frac{1}{\cosh(\gamma t)} \]

The plot of \( \frac{\Omega}{|\Gamma|} \) is shown in Fig. 1 as a function of \( \tau_1 = \frac{2|\Gamma|}{\hbar} t \).

**B. Second exactly solvable time-dependent scenario for one spin 1/2**

We have a monotonically decreasing trend of the function \( |a(t)| \) also by putting
\[ |\Gamma| \frac{\hbar}{\gamma} \int_0^t \cos \Theta dt' = \arcsin \left[ \tanh(\gamma t) \right], \quad \gamma = \frac{|\Gamma|}{\hbar} \]
which, in view of Eqs. (27) and (28), implies
\[ |a(t)| = \frac{1}{\cosh(\gamma t)} \]  
\[ |b(t)| = \tanh(\gamma t) \]
In this case, thus, \( |a(0)| \) varies from 1(0), at \( t = 0 \), to 0(1) when \( t \to \infty \), realizing a perfect inversion of the spin. The expressions of \( \cos \Theta(t) \) and \( \sin \Theta(t) \) are the same as those in (41) of the previous case and so also \( \dot{\Theta} \) and \( \Theta \) have the same expressions as those given in (43) (though the definition of \( \gamma \) is different in the two cases). What is different is the value of integral \( \mathcal{R} \) which, in this case, results in
\[ \mathcal{R} = \frac{1}{2} \sinh(\gamma t) \]
and for the phases of \( a \) and \( b \) we have
\[ \phi_a = -\arctan \left[ \tanh \left( \frac{\gamma}{2} t \right) \right] - \frac{1}{2} \sinh(\gamma t), \]  
\[ \phi_b = -\arctan \left[ \tanh \left( \frac{\gamma}{2} t \right) \right] + \frac{1}{2} \sinh(\gamma t) - \frac{\pi}{2} \]

With this choice the longitudinal component of magnetic field must be engineered as
\[ \Omega = |\Gamma| \frac{3}{2} \left[ \frac{1}{\cosh(\gamma t)} - \cos(\gamma t) \right] \]

Fig. 2 shows the behaviour of \( \frac{\Omega}{|\Gamma|} \) in this case against \( \tau_2 = \frac{|\Gamma|}{\hbar} t \).

It is important to point out that the value of the factor multiplying the function \( \arcsin[\tanh(\gamma t)] \) is crucial for the possibility to solve exactly the integral \( \mathcal{R} \). Furthermore, it has a remarkable role in determining deeply the time evolution of important physical quantities as \( |a| \), \( |b| \) and \( \Omega \). We saw, indeed, that the asymptotic \( (t \to \infty) \) values of \( |a| \) and \( |b| \) are very different in the two cases determining a completely different dynamical evolution in time. Finally, as we can see from Fig. 2, the multiplying factor significantly determines the time trend of the longitudinal component of magnetic field which must be engineered appropriately to have the exact dynamics we are studying.
C. Time-dependent scenarios for the two spin model

At closing this section, we emphasize the novelty of our results by explicitly giving all the time dependences (constructed on the basis of Eq. (32)) of \(\omega_1\) and \(\omega_2\) (and so of the two magnetic fields \(B_1^z\) and \(B_2^z\) in view of Eq. (7)) in the two spin Hamiltonian model (5) leading to exactly solvable and solved models. If we are interested in studying the time evolution of an initial state that belongs to one of the two dynamically invariant subspaces of \(H\), wherein the dynamics is described by \(\hat{H}_+\) or \(\hat{H}_-\), we get classes of time dependent scenarios which can be treated and solved exactly. Precisely, if we consider, e.g., the sub-dynamics characterized by \(\sigma_1^z\sigma_2^z = 1\) and described by \(\hat{H}_+\), the two classes of time-dependent exactly solvable problems of two spins interacting according to our model in (5) are given by

\[
h(\omega_1(t) + \omega_2(t)) = \frac{2|\Gamma_+|}{\cosh(2\Gamma_+/\hbar t)}; \quad (51a)
\]

\[
h(\omega_1(t) + \omega_2(t)) = \frac{|\Gamma_+|}{2} \left[ \frac{3}{\cosh(\Gamma_+/\hbar t)} - \cosh(\Gamma_+/\hbar t) \right]. \quad (51b)
\]

Equations (51) make clear the reason why we are talking of classes of time-dependent exactly solvable models. Indeed we see that we have different possible choices of the two magnetic fields \(B_1^z\) and \(B_2^z\) such that their combination, in accordance to Eq. (7), satisfies one of the previous conditions, getting different time-dependent scenarios in which we are able to know exactly the dynamics. Obviously we have the analogous situation also for the other sub-dynamics characterized by \(\sigma_1^z\sigma_2^z = -1\) and described by \(\hat{H}_-\). In this case the classes of exactly solvable models are due by the conditions

\[
h(\omega_1(t) - \omega_2(t)) = \frac{2|\Gamma_-|}{\cosh(2\Gamma_-/\hbar t)}; \quad (52a)
\]

\[
h(\omega_1(t) - \omega_2(t)) = \frac{|\Gamma_-|}{2} \left[ \frac{3}{\cosh(\Gamma_-/\hbar t)} - \cosh(\Gamma_-/\hbar t) \right]. \quad (52b)
\]

We stress that Eqs. (52) are not compatible with the situation corresponding to \(\omega_1(t) = \omega_2(t)\) for which, on the other hand, the quantum dynamics in the subspace under scrutiny has been completely solved as explicitly given by Eq. (37). In other words, Eqs. (52) display their usefulness, generating exactly solvable time-dependent Hamiltonian models of the two spins, only when \(\omega_1(t) \neq \omega_2(t)\). If we look, instead, at the entire dynamics of the two interacting spin 1/2’s, considering a general initial condition belonging to the total four dimensional Hilbert space \(\mathcal{H}\) we have the following four exactly solvable time dependent cases \([+ (-)\text{ in } \pm\text{ corresponds to } 1(2)]\)

\[
h\omega_{1/2}(t) = \frac{|\Gamma_+|}{\cosh(2\Gamma_+/\hbar t)} \pm \frac{|\Gamma_-|}{\cosh(2\Gamma_-/\hbar t)}; \quad (53a)
\]

\[
h\omega_{1/2}(t) = \frac{|\Gamma_+|}{\cosh(2\Gamma_+/\hbar t)} \pm \frac{|\Gamma_-|}{\cosh(2\Gamma_-/\hbar t)} \left[ \frac{3}{\cosh(\Gamma_+/\hbar t)} - \cosh(\Gamma_-/\hbar t) \right]; \quad (53b)
\]

\[
h\omega_{1/2}(t) = \frac{|\Gamma_+|}{4} \left[ \frac{3}{\cosh(\Gamma_+/\hbar t)} - \cosh(\Gamma_+/\hbar t) \right] \pm \frac{|\Gamma_-|}{4} \left[ \frac{3}{\cosh(\Gamma_-/\hbar t)} - \cosh(\Gamma_-/\hbar t) \right]. \quad (53c)
\]

\[
h\omega_{1/2}(t) = \frac{|\Gamma_+|}{4} \left[ \frac{3}{\cosh(\Gamma_+/\hbar t)} - \cosh(\Gamma_+/\hbar t) \right] \pm \frac{|\Gamma_-|}{4} \left[ \frac{3}{\cosh(\Gamma_-/\hbar t)} - \cosh(\Gamma_-/\hbar t) \right]. \quad (53d)
\]

For example, if we consider the time-dependent scenario given by Eq. (53a) with a particular choice

\[
\gamma_x = \gamma_y = 2\gamma_{xy} = 2\gamma_{yx} = c, \quad (54)
\]

we have, in view of Eqs. (24) and (39),

\[
|\Gamma_+| = c, \quad |\Gamma_-| = 2c \quad \gamma_+ = \frac{2c}{\hbar}, \quad \gamma_- = \frac{4c}{\hbar} \quad (55)
\]

and the time behaviour of \(\frac{\hbar \omega}{c}\) and \(\frac{\hbar \omega}{c}\) in terms of \(\tau_c = \frac{\text{ut}}{\hbar}\) is seen in Fig. 3 (\(\hbar = 1\)).

![Fig. 3. Plots of \(\frac{\hbar \omega}{c}\) (red solid line) and \(\frac{\hbar \omega}{c}\) (blue dashed line), in terms of \(\tau_c = \frac{\text{ut}}{\hbar}\), according to the time-dependent model satisfying Eqs. (53a) and (55).](image)

The above four cases, so, provide \(\omega_1(t)\) and \(\omega_2(t)\) (and consequently the magnetic fields \(B_1^z(t)\) and \(B_2^z(t)\)) such that our corresponding time-dependent Hamiltonian model given by Eq. (5) turns out to be exactly solvable and the related global time-evolution operator \(\mathcal{U}\),
given by Eq. (33), can be derived by plugging Eqs. (40) and (44) or (47) and (49) in place of \(|a_+|, |b_+|, \phi^+_\alpha, \phi^+_\beta\)
or of \(|a_-|, |b_-|, \phi^-_\alpha, \phi^-_\beta\) at will, depending on what of the four time-dependent scenarios, given in Eqs. (53), we choose.

We notice that when \(g_1^zz \neq g_2^zz\), Eqs. (53), by specializing Eq. (5), generate time-dependent Hamiltonian models of the two spins which cannot exactly be solved when a same external homogeneous magnetic field \(B_1^z(t) = B_2^z(t)\) is applied on the two spins. In such a case, indeed, the time dependence of the magnetic field determined from Eq. (51) is incompatible with that of the same magnetic field derivable from Eq. (52). However, it is of relevance to point out that, in this case, if we choose the time dependence of the unique magnetic field derived from either Eqs. (51a) or (51b) (52a) or (52b), we get particular time-dependent Hamiltonian models for which we are able to solve exactly the sub-dynamics in the subspace singled out by the condition \(\sigma^z_1 \sigma^z_2 = 1\) (\(\sigma^z_1 \sigma^z_2 = -1\)). It is finally useful to underline that when the physical system may be described assuming \(g_1^zz = g_2^zz\), a homogeneous time-dependent magnetic field as derivable from either Eq. (51a) or (51b), leads to \(\omega_1(t) = \omega_2(t)\) whose implications on the two spin quantum dynamics, have already been discussed after Eq. (34).

IV. DYNAMICAL PROPERTIES OF THE TWO SPIN 1/2 MODEL

In the previous sections we have built exactly solvable models for two coupled spin 1/2’s and solved them as well. This result is important for two reasons. The first one is that it shows that the systematic route reported in Ref. [17] may be successfully applied to physical systems living in an \(f\)-dimensional Hilbert space with \(f > 2\). The second one is related to the construction of new time-dependent exactly solvable Hamiltonian models in its own, since solutions of such problems, generally speaking, are very rare. Thus we are going to exploit the knowledge of the solutions we have found, in the subsections IIIA, the first time-dependent scenario, and in IIIB, the second time-dependent scenario, to investigate physical properties exhibited by our two spin system under the corresponding engineered magnetic fields.

A. Quantum evolution in the dynamically invariant subspace with parity +

In the subspace where the constant of motion \(\hat{S}_1^z \hat{S}_2^z\) assumes the value \(\frac{N^2}{4}\) with certainty, in view of Eq. (33), the initial states

\[ |\psi(0)\rangle = |\psi^+_\alpha(0)\rangle \equiv |++\rangle \]  

and

\[ |\psi(0)\rangle = |\psi^-_\beta(0)\rangle \equiv |--\rangle \]  

at time \(t\), respectively, become

\[ |\psi^+_\alpha(t)\rangle = |a_+|e^{i\phi^+_\alpha(t)}|++\rangle - |b_+|e^{-i\phi^+_\alpha(t)}|--\rangle, \]

\[ |\psi^-_\beta(t)\rangle = |b_+|e^{i\phi^-_\beta(t)}|++\rangle + |a_+|e^{-i\phi^-_\beta(t)}|--\rangle. \]

Since both \(|++\rangle\) and \|--\rangle\) are eigenvectors of \(\hat{S}_2^z\), the subspace they span pertains to the quantum number \(S = 1\) of such collective observable. It is worthwhile to observe that the magnetization \(\langle \hat{S}_2^z(t)\rangle_{\alpha/\beta}\) of the system in this subspace is not known with certainty. Indeed, the mean value of \(\hat{S}_2^z = \hat{S}_1^z + \hat{S}_2^z\) on the states \(|\psi^+_\alpha(t)\rangle\) and \(|\psi^-_\beta(t)\rangle\), may be respectively cast as follows

\[ \langle \hat{S}_2^z(t)\rangle_{\alpha} \equiv \langle \psi^+_\alpha(t)\rangle|\hat{S}_2^z\rangle|\psi^+_\alpha(t)\rangle = \hbar(|a_+|^2 - |b_+|^2), \]

\[ \langle \hat{S}_2^z(t)\rangle_{\beta} \equiv \langle \psi^-_\beta(t)\rangle|\hat{S}_2^z\rangle|\psi^-_\beta(t)\rangle = -\hbar(|a_+|^2 - |b_+|^2). \]

where \(|a_+(t)|\) and \(|b_+(t)|\) appear as entries of the matrix for \(\mathcal{U}\).

1. First time-dependent scenario

As soon as \(\omega_1(t)\) and \(\omega_2(t)\) satisfy Eq. (51a) we get the following magnetization time dependences

\[ \langle \hat{S}_2^z(t)\rangle_{\alpha/\beta} = \pm \frac{\hbar}{\cosh(2\Gamma t)} \]  

As a rule, the upper (lower) sign corresponds to \(\alpha (\beta)\) here and in what follows.

It should be appreciated that the recipe provided by Eq. (51a) enables in principle the construction of infinitely many time-dependent Hamiltonian models, all of them exactly predicting such an evolution of the magnetization of the coupled two spin system. In Fig. 4 we plot such time dependences (for \(\alpha\) and \(\beta\)) against the dimensionless time \(\frac{2t}{\hbar}\).

The common asymptotic value of \(\langle \hat{S}_2^z(t)\rangle_{\alpha}\) and \(\langle \hat{S}_2^z(t)\rangle_{\beta}\) can be understood by noticing that \(|\psi^+_\alpha(t)\rangle\) and \(|\psi^-_\beta(t)\rangle\), up to inessential phase factors, for large \(t\) evolve into the following entangled Bell states of the two spins

\[ |\psi^+_\alpha(t)\rangle \rightarrow e^{-i\left(\frac{\gamma + i\hbar}{\hbar}\right)t} \frac{1}{\sqrt{2}} |++\rangle + |--\rangle, \]

\[ |\psi^-_\beta(t)\rangle \rightarrow e^{i\left(\frac{\gamma + i\hbar}{\hbar}\right)t} \frac{1}{\sqrt{2}} |++\rangle - |--\rangle, \]

both having vanishing average magnetizations for \(t \rightarrow \infty\). Equations (61) clearly evidence that engineering the magnetic fields on the two spins, in accordance with the constraint imposed by Eq. (51a), might be, in principle, at the heart of many possible experimental schemes successfully exploitable for generating such fully entangled states whatever the internal coupling coefficients appearing in the Hamiltonian model, given by Eq. (5), are.
Eq. (62), which is plotted in Fig. 5 as functions of $\frac{|\Gamma_+|}{\hbar} t$.

In this case a gradual inversion of the magnetization of the system occurs, due to the fact that as $t \to \infty$ we have a perfect inversion of the probability of finding the two spins in the state $|++\rangle (|--\rangle)$ when its initial state is $|--\rangle (|++\rangle)$. The asymptotic states for $t \to \infty$, in this scenario, are, indeed

$$|\psi_\alpha^+(t)\rangle \to e^{-i(\Phi^L_+ + \pi)}|--\rangle,$$

$$|\psi_\beta^+(t)\rangle \to e^{i\Phi^L_+} |--\rangle,$$

implying immediately that

$$\left|\langle \psi_\alpha^+_{\alpha/\beta}(\infty) | \psi_\beta^+_{\alpha/\beta}(0) \rangle \right|^2 = 0.$$  \hspace{1cm} (64)

B. Quantum evolution in the dynamically invariant subspace with parity $-$

When $\hat{S}_1^z \hat{S}_2^z$ assumes the value $-\frac{\Gamma_+^2}{\hbar}$ with certainty, the initial states

$$|\psi(0)\rangle = |\psi_\alpha^-(0)\rangle \equiv |+-\rangle$$  \hspace{1cm} (65)

and

$$|\psi(0)\rangle = |\psi_\beta^-(0)\rangle \equiv |--\rangle,$$  \hspace{1cm} (66)

evolve, respectively, as follows

$$|\psi_\alpha^-(t)\rangle = |a_-|e^{i\Phi^-_+} |--\rangle - |b_-|e^{-i\Phi^-_+} |+-\rangle,$$  \hspace{1cm} (67a)

$$|\psi_\beta^-(t)\rangle = |b_-|e^{i\Phi^-_+} |--\rangle + |a_-|e^{-i\Phi^-_+} |+-\rangle.$$  \hspace{1cm} (67b)

The states $|+-\rangle$ and $|--\rangle$ generate the eigenspace of $\hat{S}_z$ of eigenvalue $M = 0$, which, in turn, is not invariant with respect to the observable $\hat{S}_z$, whose mean value runs from 0 to $2\hbar^2$ in accordance with the following time evolutions:

$$\langle \hat{S}^2(t) \rangle_\alpha \equiv \langle \hat{S}_\alpha^-(t) | \hat{S}^2 | \hat{S}_\alpha^-(t) \rangle =$$

$$= \hbar^2 \left[ 1 - 2 |a_-| |b_-| \cos(\phi_a + \phi_b) \right],$$  \hspace{1cm} (68)

$$\langle \hat{S}^2(t) \rangle_\beta \equiv \langle \hat{S}_\beta^-(t) | \hat{S}^2 | \hat{S}_\beta^-(t) \rangle =$$

$$= \hbar^2 \left[ 1 + 2 |a_-| |b_-| \cos(\phi_a + \phi_b) \right].$$

Equations (52) prescribe constraints on $\omega_1(t)$ and $\omega_2(t)$, with $\omega_1(t) \neq \omega_2(t)$, to guarantee the existence of exact dynamical behaviour of the corresponding models of the two spins, when the system is initially prepared in the subspace of parity $-$. When $\omega_1(t) = \omega_2(t)$, exploiting Eq. (37), we may easily get

$$\langle \hat{S}^2(t) \rangle_{\alpha/\beta} = \hbar^2 \left[ 1 \pm \sin \left( \frac{2|\Gamma_+|}{\hbar} t \right) \cos(\Phi) \right],$$  \hspace{1cm} (69)

exhibiting oscillations at the Bohr frequency $\frac{2|\Gamma_+|}{\hbar}$ as expected on the basis of Eq. (35).

In the next two subsections, $\omega_1(t) \neq \omega_2(t)$ is assumed.

1. First time-dependent scenario

The sub-dynamics with parity $-$ is exactly solvable for all the possible time-dependent models deducible from
Eq. (52a). Accordingly, the time dependences of $\langle \hat{S}^2(t) \rangle_{\alpha/\beta}$ turn out to be
\[
\langle \hat{S}^2(t) \rangle_{\alpha/\beta} = \hbar^2 \left[ 1 \pm \tanh^2 \left( \frac{2|\Gamma|}{\hbar} t \right) \right],
\]
which is graphically represented in Fig. 6 against the dimensionless time $\frac{2|\Gamma|}{\hbar} t$.

\[
\begin{array}{c}
\text{FIG. 6. Time dependences of } \langle \hat{S}^2(t) \rangle_{\alpha} \text{ (red solid line) and } \langle \hat{S}^2(t) \rangle_{\beta} \text{ (blue dashed line) in the sub-dynamics with parity } - \text{ for the class of time-dependent models characterized by Eq. (52a).}
\end{array}
\]

The limiting values of $\langle \hat{S}^2(t) \rangle_{\alpha/\beta}$ for $t \to \infty$ suggest that the two spins asymptotically tend toward states identifiable as eigenstates of $\hat{S}^2$ of eigenvalue $S = 1$, $|S = 1, M = 0\rangle$, and $S = 0$, $|S = 0, M = 0\rangle$, in correspondence to $\alpha$ and $\beta$, respectively. Thus, whatever $\omega_1(t) \neq \omega_2(t)$, fulfilling Eq. (52a), are, the envisioned time-dependent scenario under scrutiny leads to the generation of the following Bell states
\[
\begin{align}
|\psi_\alpha^+(t)\rangle &\to e^{-i\left(\frac{\gamma+|\Gamma|}{\hbar} t + \frac{\pi}{4}\right)}|++\rangle + |+-\rangle, \quad (71a) \\
|\psi_\beta^-(t)\rangle &\to e^{i\left(\frac{-\gamma+|\Gamma|}{\hbar} t + \frac{\pi}{4}\right)}|+\rangle - |-\rangle. \quad (71b)
\end{align}
\]
Even in this case, then, our results, as expressed by Eq. (71), might provide implementable experimental strategies for the generation of two maximally entangled states $|S = 1, M = 0\rangle$ and $|S = 0, M = 0\rangle$.

2. Second time-dependent scenario

All the time-dependent models satisfying Eq. (52b) are characterized by time dependences of $\langle \hat{S}^2(t) \rangle_{\alpha/\beta}$ in the sub-dynamics with parity $-$ of the following form
\[
\langle \hat{S}^2(t) \rangle_{\alpha/\beta} = \hbar^2 \left[ 1 \pm \frac{2 \tanh^2 \left( \frac{2|\Gamma|}{\hbar} t \right)}{\cosh \left( \frac{2|\Gamma|}{\hbar} t \right)} \right],
\]
whose behaviour is illustrated in Fig. 7 against $\frac{|\Gamma|}{\hbar} t$.

\[
\begin{array}{c}
\text{FIG. 7. Time dependences of } \langle \hat{S}^2(t) \rangle_{\alpha} \text{ (red solid line) and } \langle \hat{S}^2(t) \rangle_{\beta} \text{ (blue dashed line) in the sub-dynamics with parity } - \text{ for the class of time-dependent models characterized by Eq. (52b).}
\end{array}
\]

Once again, the asymptotic trend of the curves reflects the $t \to \infty$ asymptotic states
\[
|\psi_\alpha^- (t)\rangle \to e^{-i\left(\frac{\gamma+|\Gamma|}{\hbar} t + \frac{\pi}{4}\right)}|+-\rangle, \quad |\psi_\beta^- (t)\rangle \to e^{i\phi} |+-\rangle, \quad (73)
\]
which means that under the strategy dictated by Eq. (52b) the initial state $|++\rangle$ is converted into the state $|+-\rangle$ while $|+-\rangle$ undergoes the analogous complete inversion.

C. Quantum dynamics from an arbitrary initial condition

In this last subsection we report the time evolution of a generic initial state in $\mathcal{H}$
\[
|\psi(0)\rangle = c_{++}|++\rangle + c_{+-}|+-\rangle + c_{-+}|-+\rangle + c_{--}|--\rangle \quad (74)
\]
generated by one of the Hamiltonian models Eqs. (53). Taking advantage of (58a), (58b), (67a) and (67b), we get formally
\[
|\psi(t)\rangle = c_{++}(t)|++\rangle + c_{+-}(t)|+-\rangle + c_{-+}(t)|-+\rangle + c_{--}(t)|--\rangle \quad (75)
\]
with

\[ c_{++}(t) = e^{i 2 \Delta t} \left( |a_+(t)| e^{i \phi_a^+(t)} c_{++} + |b_+(t)| e^{i \phi_b^+(t)} c_{++} \right) \] (76a)

\[ c_{+-}(t) = e^{-i 2 \Delta t} \left( |a_-(t)| e^{i \phi_a^-(t)} c_{+-} + |b_-(t)| e^{i \phi_b^-(t)} c_{+-} \right) \] (76b)

\[ c_{-+}(t) = e^{-i 2 \Delta t} \left( |a_-(t)| e^{i \phi_a^-(t)} c_{-+} - |b_+(t)| e^{i \phi_b^-(t)} c_{-+} \right) \] (76c)

\[ c_{-\pm}(t) = e^{i 2 \Delta t} \left( |a_+(t)| e^{i \phi_a^+(t)} c_{-\pm} - |b_-(t)| e^{-i \phi_a^-(t)} c_{-\pm} \right) \] (76d)

where \( |a_\pm(t)|, \phi_a^\pm(t), |b_\pm(t)|, \phi_b^\pm(t) \) appear as entries in the matrix representation of the evolution operator \( \mathcal{U} \) (33) generated by the specific two spin Hamiltonian model under scrutiny (that is determined by one of (53)).

The time evolutions of \( \hat{S}^z(t) \) as well as of \( \hat{S}^x \) exhibit no interference terms stemming from the presence of states of different parity in (75). For example, in view of Eq. (59), we have

\[ \langle \psi(t) | \hat{S}^z | \psi(t) \rangle = 2 |c_+ + (t)|^2 - |c_- (t)|^2 \] (77)

because the mean value of \( \hat{S}^z \) in any state of negative parity identically vanishes. It is thus interesting to evaluate the time evolution of the mean value of an observable which has nonvanishing matrix elements between states of different parities, for example \( \hat{S}^x = \hat{S}^x_1 + \hat{S}^x_2 \). We limit ourselves to an exemplary case, namely the one obtained by choosing \( H \) with \( \omega_1(t) \) and \( \omega_2(t) \) as prescribed in Eq. (53a) and the amplitudes of \( |\psi(0)\rangle \) real and such to make the initial state a common eigenstate of \( \hat{S}^2 \) and \( \hat{S}^x \) with maximum eigenvalues, that is,

\[ c_{++} = c_{+-} = c_{-+} = c_{-\pm} = \frac{1}{2}. \] (78)

Equation (75) immediately yields

\[ \langle \hat{S}^x(t) \rangle \equiv \langle \psi(t) | \hat{S}^x | \psi(t) \rangle = \hbar \left[ \cos \left( \frac{2 \gamma_+ t}{\hbar} \right) \right] \left[ |a_+||a_-| \cos(\phi_a^+) \cos(\phi_a^-) + |b_+||b_-| \sin(\phi_b^+) \sin(\phi_b^-) \right] + \sin \left( \frac{2 \gamma_+ t}{\hbar} \right) \left[ |a_+||b_-| \cos(\phi_a^+) \sin(\phi_b^+) - |a_-||b_+| \cos(\phi_a^-) \sin(\phi_b^-) \right] \] (79)

with

\[ |a_\pm(\tau_\pm)| = \sqrt{\frac{\cosh(\tau_\pm) + 1}{2 \cosh(\tau_\pm)}}, \]

\[ |b_\pm(\tau_\pm)| = \sqrt{\frac{\cosh(\tau_\pm) - 1}{2 \cosh(\tau_\pm)}}, \] (80)

\[ \phi_a^\pm(\tau_\pm) = - \arctan \left[ \frac{\tanh \left( \frac{\tau_\pm}{2} \right)}{ \frac{\tau_\pm}{2} + \frac{\tau_\pm}{2} - \pi / 2 } \right], \]

\[ \phi_b^\pm(\tau_\pm) = - \arctan \left[ \frac{\tanh \left( \frac{\tau_\pm}{2} \right)}{ \frac{\tau_\pm}{2} - \pi / 2 } \right], \]

where \( \tau_\pm = \gamma_\pm t \) and \( \gamma_\pm = \frac{2 |\Gamma_\pm|}{\hbar} \), according to the time-dependent scenario (53a) (that is, the “first time-dependent scenario” for each sub-dynamics). The plot of \( \langle \hat{S}^x(t) \rangle \), in this instance and considering the special case characterized by Eqs. (54) and (55), is given in Fig. 8 against \( \tau_+ \). It is possible to understand the peculiar behaviour for large \( t \), characterized evidently by one frequency, by deriving analytically the asymptotic expression of \( \langle \hat{S}^x(t) \rangle \), which, indeed, acquires the following clear form

\[ \langle \hat{S}^x(t) \rangle = \frac{\hbar}{2} \cos \left[ \left( \frac{\gamma_+}{|\Gamma_+|} + \frac{|\Gamma_-|}{2 |\Gamma_+|} - \frac{1}{2} \right) \gamma_+ \right] \] (81)

FIG. 8. Plot of \( \langle \hat{S}^x(t) \rangle \) (blue solid line) starting from \( c_{++} = c_{+-} = c_{-+} = c_{-\pm} = \frac{1}{2} \) according to the time-dependent scenario (53a) and the special choice in Eqs. (54) and (55); the red upper (green lower) dashed straight line represents \( \langle \hat{S}^x(t) \rangle = \frac{3}{2} \langle \hat{S}^x(t) \rangle = - \frac{1}{2} \).

V. CONCURRENCE IN THE TWO SUB-DYNAMICS

Spurred by the results of the previous section, which, in particular cases allow direct and first glance compar-
isomer between the initial level of entanglement with that
get stored in the asymptotic states, in this section we are
going to derive and analyse the exact time-evolution law
of the entanglement established in the two spin system
when it is initially prepared in the generic state given by
Eq. (74). For a pair of qubits, a good measure of entan-
glement is the concurrence \( C \) introduced by Wooters [18]
as well as the negativity, introduced by G. Vidal and R.
F. Werner [19], which in a generic state coincides with \( C \)[20]. At a generic time instant \( t \), it may be expressed as

\[
C(t) = 2|c_{++}(t)c_{--}(t) - c_{-+}(t)c_{+-}(t)|, \tag{82}
\]

where the four time-dependent coefficients are the com-
plex amplitudes of the normalized state \( |\psi(t)\rangle \), into
which \( |\psi(0)\rangle \) evolves, and are given in Eqs. (76). As
expected, when the system starts in a state of de-
finite parity, Eq. (82) yields \( C(t) = 2|c_{++}(t)c_{--}(t)| = 2|c_{-+}(t)c_{+-}(t)| \) for parity \(+(-)\). When \( c_{++}(0) = 1 \) or
\( c_{--}(0) = 1 \) and the first time-dependent scenario for
this sub-dynamics is assumed, that is (51a), the concurrence
results

\[
C(t) = 2|a(t)||b(t)| = \tanh\left(\frac{2|\Gamma_+|}{h}t\right), \tag{83}
\]

whose plot is reported in Fig. 9 against \( \tau_+ = \frac{2|\Gamma_+|}{h}t \).
The asymptotic behaviour of \( C(t) \) in this case is easily
understood in view of Eqs. (61).

Considering, instead, \( c_{++}(0) = c_{--}(0) = \frac{1}{\sqrt{2}} \), still to-
gogether with (51a), we obtain

\[
C(t) = \sqrt{1 - \tanh^2\left(\frac{2|\Gamma_+|}{h}t\right) \sin^2\left(\frac{2|\Gamma_+|}{h}t\right)}, \tag{84}
\]

which is plotted in Fig. 10 against \( \tau_+ = \frac{2|\Gamma_+|}{h}t \).

![FIG. 9. Plot of C(t) starting from ++ when the time-
dependent scenario (51a) is adopted.](image)

It shows that, after a transient regime \( (\tau_+ < 2\pi) \),
the concurrence oscillates between 0 and 1 as \( |\cos(\tau_+)| \),
which is immediately deduced from Eq. (84) for large \( t \).
The meaning of this behaviour is that the system peri-
odically evolves alternating factorized states and the Bell
states. To understand and better appreciate quantita-
tively this statement, we exploit Eqs. (61) to recover the
asymptotic expression of \( |\psi(t)\rangle \) \( (\gamma_+ = \frac{2|\Gamma_+|}{h}) \)

\[
|\psi(t)\rangle = e^{i\frac{2\pi}{\sqrt{2}} t} \left[ e^{-i\frac{\gamma_+}{2} t} \sin\left(\frac{\gamma_+}{2} t + \frac{\pi}{4}\right) |++\rangle + \cos\left(\frac{\gamma_+}{2} t + \frac{\pi}{4}\right) |--\rangle \right], \tag{85}
\]

which clearly exhibits oscillations of period \( T_+ = \frac{4\pi}{\gamma_+} = 2\pi \frac{h}{|\Gamma_+|} \) in accordance with the asymptotic expression
of \( C(t) \). Equation (85) easily explains the oscillations exhib-
ted by \( C(t) \) since it predicts that the two spin system,
up to a global phase factor, comes back to its initial condi-
tion and after a time \( T_+ \) \( (T_+ \approx \frac{2\pi}{\gamma_+}) \) it reaches the factorized
(Bell-like) state \( |++\rangle \) \( (\frac{1}{\sqrt{2}} |++\rangle + \frac{1}{\sqrt{2}} |--\rangle) \) whereas in the last
semi-period it reaches first the factorized state \( |--\rangle \) and
eventually its initial state. The reason why the concur-
rence does not vanish in this case in the transient region
stems from the fact that \( C(t) = 0 \) necessarily implies
\( c_{++}(t) = 0 \) or \( c_{--}(t) = 0 \). In view of the structure of the
two amplitudes \( c_{++}(t) \) and \( c_{--}(t) \), see Eqs. (76a) and
(76d), we deduce that \( |a_+(t)| = |b_-(t)| \) is a necessary
condition in order for \( c_{++}(t) \) or \( c_{--}(t) \) to vanish. Since
such a condition is only asymptotically reached by the
system, the concurrence cannot vanish during the trans-
ient regime.

We now study \( C(t) \) when the system is initially pre-
pared in one of the same three states considered above,
adopting this time as Hamiltonian model the one stem-
ning from Eq. (51b) (called “second time-dependent sce-
nario” in the previous section). When \( c_{++}(0) = 1 \) or
\( c_{--}(0) = 1 \) and the second time dependent scenario (51b)
is assumed, the concurrence becomes

$$ C(t) = 2 \frac{\tanh\left(\frac{\Gamma_+}{\hbar} t\right)}{\cosh\left(\frac{\Gamma_+}{\hbar} t\right)} $$

(86)

and this is plotted in Fig. 11 as a function of $\tau^t_+ = \frac{\Gamma_+}{\hbar} t$. We note that at time instant $(\tau_+)_0 = \arcsinh(1) \approx 0.88$

---

FIG. 11. Plot of $C(t)$ starting from $|++\rangle$ according to the time-dependent scenario (51b).

the system of the two spins reaches maximally entangled state from which it asymptotically evolves toward a factorized state. Even in this case it is useful to exploit Eqs. (58a) and (58b) together with Eqs. (47) and (49) predicting that at the particular time instant $(\tau_+)_0$ the evolved states respectively become

$$|\psi^+_{\tau_+} (\tau_+)_0\rangle = e^{i\left(\frac{\tau_+}{\hbar} + \phi_0\right)(\tau_+)_0}|++\rangle - e^{i\left(\frac{\tau_+}{\hbar} + \phi_0\right)(\tau_+)_0}|--\rangle, $$

(87a)

$$|\psi^-_{\tau_+} (\tau_+)_0\rangle = e^{i\left(\frac{\tau_+}{\hbar} + \phi_0\right)(\tau_+)_0}|++\rangle + e^{-i\left(\frac{\tau_+}{\hbar} + \phi_0\right)(\tau_+)_0}|--\rangle, $$

(87b)

which is in accordance with the result on the concurrence. Equations (63) provide the asymptotic form of the two evolutions under scrutiny, confirming the expectation of vanishing concurrence for large $t$.

If the two spin system is initially prepared in the Bell state $|\psi(0)\rangle = \frac{|++\rangle + |--\rangle}{\sqrt{2}}$, the concurrence may be expressed as

$$ C(t) = \sqrt{1 - 4 \frac{\tanh^2\left(\frac{\Gamma_+}{\hbar} t\right)}{\cosh^2\left(\frac{\Gamma_+}{\hbar} t\right)} \sin^2[\sinh\left(\frac{\Gamma_+}{\hbar} t\right)]}, $$

(88)

which is graphically represented in Fig. 12 as a function of $\tau^t_+ = \frac{\Gamma_+}{\hbar} t$. Since on the basis of Eqs. (63) the state $|++\rangle (|--\rangle)$ asymptotically evolves into $|--\rangle (|++\rangle)$, up to an initial state-dependent global phase factor, the concurrence in the case under scrutiny must asymptotically come back to its maximum value. The peculiar oscillatory behaviour as time goes on, is due to the fact that the time evolution of $|c_{++}(t)|$ and $|c_{--}(t)|$ is dominated by progressive oscillations of decreasing amplitudes around $1/\sqrt{2}$ until they asymptotically stabilize at such values as we can transparently appreciate in Fig. 13. The normalisation of $|\psi(t)\rangle$ justifies the coincidence of the time instants where $|c_{++}(t)|$ and $|c_{--}(t)|$ assume the value $1/\sqrt{2}$. On the other hand the independence of $C(t)$ on the both phases of $c_{++}(t)$ and $c_{--}(t)$ explains why this time instants are exactly those at which $C(t) = 1$. Moreover, Fig. 12 makes evident that all the infinitely-many minima of the concurrence occur at those time instants where $|c_{++}(t) - c_{--}(t)|$ reaches local maxima in time (maximally unbalanced condition).

It is possible to find exactly the infinite sequence of
states at which the concurrence assumes its maximum value. To this end we firstly calculate the time instants at which $C(t) = 1$ and $|c_{++}(t)| = |c_{--}(t)| = 1/\sqrt{2}$ simultaneously. They are given by

$$\tau_+^n = \text{arcsinh}(n\pi) \quad (89)$$

with $n = 0, 1, 2, \ldots$. Plugging $(\tau_+^n)_n$ into the state given in Eq. (75) after making explicit its time dependence with the help of Eqs. (76a), (76d) and (80) yields the following sequence of maximally entangled states progressively emerging in the time evolution of the initial Bell state:

$$|\psi[(\tau_+^n)_n]\rangle = e^{i\left(\frac{\tau_+^n}{k+n}+\phi_n+\theta_n\right)}|++\rangle + e^{-i\phi_n}|--\rangle \quad (90)$$

where

$$\phi_n = \phi_n^+[(\tau_+^n)_n] = \sqrt{\frac{1+\frac{(n\pi)^2}{2} - 1}{1+\frac{(n\pi)^2}{2} + 1}} \quad (91)$$

and

$$\theta_n = \arctan\left((-1)^{n+1}n\pi\right), \quad (92)$$

$n$ being an arbitrary nonnegative integer.

We emphasize that if we start from the initial condition $|+-\rangle$ adopting the time-dependent scenario $(52a)$ $(52b)$ we might again go through the arguments previously used to discuss the case $|++\rangle$ in conjunction with $(51a)$ $(51b)$, getting results that coincide with those expressed by Eqs. (83) $(86)$ provided that $\gamma_+$ is substituted by $\gamma_-$ and $\gamma_{zz}$ with $-\gamma_{zz}$. Analogously, had we started from the Bell state $(|+-\rangle + |--\rangle)/\sqrt{2}$, Eq. (84) $(88)$ represents a valid result in this case too, providing the same substitution of $\gamma_+$ and $\gamma_{zz}$ are made and Eq. $(52a)$ $(52b)$ is adopted.

It is worth noticing that in the parity-constrained dynamical evolution under scrutiny, the concurrence $C(t)$ at a generic time instant may be expressed as [21]

$$C(t) = \sqrt{C_{xx}^2 + C_{yy}^2} \quad (93)$$

where

$$C_{xx}(t) = \langle \psi(t)|\hat{\sigma}_1^x\hat{\sigma}_2^x|\psi(t)\rangle - \langle \psi(t)|\hat{\sigma}_1^x|\psi(t)\rangle\langle \psi(t)|\hat{\sigma}_2^x|\psi(t)\rangle \quad (94)$$

is the covariance of $\hat{\sigma}_1^x$ and $\hat{\sigma}_2^x$ and analogously $C_{yy}(t)$ is the covariance of $\hat{\sigma}_1^y$ and $\hat{\sigma}_2^y$. It is simple to show that, since [21] $C_{xx}(t) = 2Re[c_{++}(t)c_{--}(t)]$ and $C_{xx}(t) = 2Im[c_{++}(t)c_{--}(t)]$,

$$C_{xx}[(\tau_+^n)_n] = \cos(2\phi_n) \quad (95a)$$

and

$$C_{yy}[(\tau_+^n)_n] = \sin(2\phi_n) \quad (95b)$$

in accordance with the property $C[(\tau_+^n)_n] = 1$ for any $n = 0, 1, 2, \ldots$. Since, in view of Eq. (91), $2\phi_n$ spans an infinite countable number set between 0 and 2 made up of irrationally related elements, $C_{xx}[(\tau_+^n)_n]$ is a decreasing function of $n$, changing its sign as soon as $2(\tau_+^n)_n > \pi$ and asymptotically tending to $\cos(2)$, whereas $C_{yy}[(\tau_+^n)_n]$ is an increasing (decreasing) function of $n$ for $2(\tau_+^n)_n < \pi$ ($2(\tau_+^n)_n > \pi$), asymptotically tending to $\sin(2)$. It is remarkable that the quantitative link expressed by Eq. (93) enables a direct measurement of the level of the entanglement established in the system at any time instant.

VI. CONCLUSIVE REMARKS

The Hamiltonian model given by Eq. (5) adopted in this paper contains seven parameters and then it is potentially useful to describe a huge variety of physical systems and/or physical situations in its parameter space. The key guidance leading us to extract this model from the general one given in Eq. (1) is the idea of assuring to our model the existence of a constant of motion with two eigenvalues only, holding at the same time the non-commutativity with $S^2$ and/or $S^z$. Such a constant of motion, by construction, subdivides $\mathcal{H}$ into two dynamically invariant and orthogonal subspaces sharing the same dimension 2. The merit of such a decomposition is that it paves the way for extending our Hamiltonian model to a time-dependent scenario, namely that wherein the two spins are subjected to an appropriate inhomogeneous time-dependent magnetic field.

In this paper we report the exact time evolutions generated by such a time-dependent Hamiltonian. This result is first of all important in its own since exact solvable problems involving two coupled bodies driven by time-dependent external fields are rare. In connection with the last consideration, we point out that our exact treatment holds its validity even when the spin-spin coupling constants are time-dependent as for example happens when two neutral atoms located in the left and right sites of a double well are induced to merge in a single well by carefully adjusting (that is, time controlling) the trapping potential [7]. Our treatment possesses an additional merit of providing not a lucky trick confined to the problem under scrutiny only, but indeed an exportable route. This claim on the one hand stems from the circumstance that the symmetry condition imposed to our Hamiltonian may be easily attributed to other Hamiltonian models representing dimers hosting two spins higher than $\sqrt{2}$, and/or of different values. On the other hand the consequent emergence of invariant sub-dynamics is traceable back to such a symmetry leading indeed to the possibility of taking advantage of the method reported in Ref. [17]. Our treatment is illustrated finding the time behaviour of the two spins in correspondence to different choices of the inhomogeneous time-dependent magnetic field. In particular the time evolution of the mean value of some physically transparent observables as well as of the entanglement exhibited by the two qubit system during its time evolution is carefully reported and discussed. Summing up, we wish to remark
that providing exact solutions of a class of rather general Hamiltonian models describing two coupled qubits, although of relevance, is not the only result reported in this paper. We emphasize indeed that the strategic double exploitation of the decoupling treatment and of the systematic approach of Ref. [17] demonstrates in a very transparent way the usefulness of such an approach beyond the original application to the quantum dynamics of a spin 1/2 subjected to a time-dependent magnetic field.

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[1] John A. Weil, James R. Bolton, Electron Paramagnetic Resonance - Elementary Theory and Practical Applications (Second Edition), John Wiley & Sons (2007), Hoboken, New Jersey.
[2] L. Thomas, F. Lionti, R. Ballou, D. Gatteschi, R. Sessoni and B. Barbara, Nature (London) 383, 145 (1996).
[3] V. Calbucci, Ph.D. dissertation: Metodi matematici nelle scienze fisiche. Praxis: software per la simulazione del comportamento delle molecole magnetiche (2012).
[4] L. M. B. Napolitano, O. R. Nascimento, S. Cabaleiro, J. Castro and R. Calvo, Phys. Rev. B 77, 214423 (2008).
[5] R. Calvo, J. E. Abud, R. P. Sartoris and R. C. Santana, Phys. Rev. B 84, 104433 (2011).
[6] M. C. Baldiotti, V. G. Bagrov and D. M. Gitman, Physics of Particles and Nuclei Letters, 2009, Vol. 6, No. 7.
[7] M. Anderlini, P. J. Lee, B. L. Brown, J. Sebby-Strabley,2, W. D. Phillips and J. V. Porto, Nature 448, 452-456 (2007).
[8] Van Hieu Nguyen, J. Phys.: Condens. Matter 21 (2009) 273201.
[9] A. Imamoglu, D. D. Awschalom, G. Burkard, D. P. DiVincenzo, D. Loss, M. Sherwin, and A. Small, Phys. Rev. Lett. 83, 4204 (1999).
[10] Shi-Biao Zheng and Guang-Can Guo, Phys. Rev. Lett. 85, 2392 (2000).
[11] Xiaoqiang Wang, Phys. Rev. A 64, 012313 (2001).
[12] M. C. Arnesen, S. Bose, and V. Vedral, Phys. Rev. Lett. 87, 017901 (2001).
[13] X. X. Yi, H. T. Cui, and L. C. Wang, Phys. Rev. A 74, 054102 (2006).
[14] D. Porras and J. I. Cirac, Phys. Rev. Lett. 92, 207901 (2004).
[15] E. Albayrak, Eur. Phys. J. B 72, 491496 (2009).
[16] R. J. Guerrero and M. F. Rojas, Quantum Inf Process 14: 1973-1996 (2015).
[17] A. Messina and H. Nakazato, J. Phys. A: Math. Theor. 47, 445302 (2014).
[18] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[19] G. Vidal, R.F. Werner, Phys. Rev. A 65, 032314 (2002).
[20] A. Miranowicz and A. Grudka, Phys. Rev. A 70, 032326 (2004).
[21] F. Palumbo, A. Napoli and A. Messina, Open Syst. Inf. Dyn. 13, 309 (2006).