Noncommutative Common Cause Principles in Algebraic Quantum Field Theory

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Abstract
States in algebraic quantum field theory "typically" establish correlation between spacelike separated events. Reichenbach’s Common Cause Principle, generalized to the quantum field theoretical setting, offers an apt tool to causally account for these superluminal correlations. In the paper we motivate first why commutativity between the common cause and the correlating events should be abandoned in the definition of the common cause. Then we show that the Noncommutative Weak Common Cause Principle holds in algebraic quantum field theory with locally finite degrees of freedom. Namely, for any pair of projections $A, B$ supported in spacelike separated regions $V_A$ and $V_B$, respectively, there is a local projection $C$ not necessarily commuting with $A$ and $B$ such that $C$ is supported within the union of the backward light cones of $V_A$ and $V_B$ and the set $\{C, C^\perp\}$ screens off the correlation between $A$ and $B$.

Key words: algebraic quantum field theory, Reichenbach’s Common Cause Principle, Ising model

1 Introduction
An operationally well motivated and mathematically transparent approach towards quantum field theory is algebraic quantum field theory. In this theory observables (including quantum events) are represented by $C^*$-algebras associated to bounded regions of a given spacetime. The association of the algebras and the spacetime regions is established along the following lines.

1. Isotony. Let $S$ be a spacetime and let $\mathcal{K}$ be a collection of causally complete, bounded regions of $S$, such that $(\mathcal{K}, \subseteq)$ is a directed poset under inclusion $\subseteq$. The net of local observables is given by the isotope map $\mathcal{K} \ni V \mapsto \mathcal{A}(V)$ to unital $C^*$-algebras, that is $V_1 \subseteq V_2$ implies that $\mathcal{A}(V_1)$ is a unital $C^*$-subalgebra of $\mathcal{A}(V_2)$.

2. Quasilocal algebra. The quasilocal observable algebra $\mathcal{A}$ is defined to be the inductive limit $C^*$-algebra of the net $\{\mathcal{A}(V), V \in \mathcal{K}\}$ of local $C^*$-algebras.

3. Haag duality. The net $\{\mathcal{A}(V), V \in \mathcal{K}\}$ satisfies not only locality (Einstein causality) $\mathcal{A}(V')' \cap \mathcal{A} \supset \mathcal{A}(V), V \in \mathcal{K}$ but also algebraic Haag duality: $\mathcal{A}(V')' \cap \mathcal{A} = \mathcal{A}(V), V \in \mathcal{K}$, where primes denote spacelike complement and algebra commutant, respectively, and $\mathcal{A}(V')$ is the smallest $C^*$-algebra in $\mathcal{A}$ containing the local algebras $\mathcal{A}(V), V' \supset V \in \mathcal{K}$.

4. Covariance. Let $\mathcal{P}_\mathcal{K}$ be the subgroup of the group $\mathcal{P}$ of geometric symmetries of $S$ leaving the collection $\mathcal{K}$ invariant. A group homomorphism $\alpha: \mathcal{P}_\mathcal{K} \to \text{Aut} \mathcal{A}$ is given such that the automorphisms $\alpha_g, g \in \mathcal{P}_\mathcal{K}$ of $\mathcal{A}$ act covariantly on the observable net: $\alpha_g(\mathcal{A}(V)) = \mathcal{A}(g \cdot V), V \in \mathcal{K}$.

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To the net \{A(V), V \in K\} satisfying the above requirements we will refer as a \(\mathcal{P}_K\)-covariant local quantum theory. If \(\mathcal{S}\) is a Minkowski spacetime \(\mathcal{M}\) and \(K\) is the net of all double cones in \(\mathcal{M}\) then \(\mathcal{P}_K\) is the Poincaré group, and we obtain Poincaré covariant algebraic quantum field theories with locally infinite degrees of freedom. Restricting the collection \(K\), one can obtain \(\mathcal{P}_K\)-covariant local quantum theories with locally finite degrees of freedom (see Section 2).

A state \(\phi\) in a local quantum theory is defined as a state (normalized positive linear functional) on the quasi-local observable algebra \(A\). The corresponding GNS representation \(\pi_{\phi}: A \to B(\mathcal{H})\) converts the net of \(C^*\)-algebras into a net of subalgebras of \(B(\mathcal{H})\). Closing these subalgebras in the weak topology one arrives at a net of local von Neumann observable algebras: \(\mathcal{N}(V) := \pi_{\phi}(A(V))''\), \(V \in K\).

Here one can require further properties such as unitary implementability of \(\mathcal{P}_K\) on \(\mathcal{H}\), existence of a vacuum representation, weak additivity, etc. (See Haag 1992). Since von Neumann algebras are rich in projections, they offer a nice representation of quantum events: projections of a von Neumann algebra can be interpreted as 0-1-valued observables where the expectation value of a projection is the probability of the event that the observable takes on the value 1 in the given quantum state.

Two commuting events \(A\) and \(B\) (represented as projections) are said to be correlating in a state \(\phi\) if \(\phi(AB) \neq \phi(A)\phi(B)\). If the events are supported in spatially separated spacetime regions \(V_A\) and \(V_B\), respectively, then the correlation between them is said to be superluminal. A remarkable characteristic of Poincaré covariant theories is that there exist "many" normal states establishing superluminal correlations. (See (Summers, Werner 1988) or (Halvorson, Clifton 2000) for the precise meaning of "many".) Since spacelike separation excludes direct causal influence, one looks for a causal explanation of these superluminal correlations in terms of common causes.

Reichenbach (1956) characterizes the notion of the common cause in the following probabilistic way. Let \((\Sigma, p)\) be a classical probability measure space and let \(A\) and \(B\) be two positively correlating events in \(\Sigma\) that is let

\[ p(A \land B) > p(A)p(B). \]  

**Definition 1.** An event \(C \in \Sigma\) is said to be the (Reichenbachian) common cause of the correlation between events \(A\) and \(B\) if the following conditions hold:

\[
\begin{align*}
p(A \land B | C) &= p(A | C)p(B | C) & (2) \\
p(A \land B | C^\perp) &= p(A | C^\perp)p(B | C^\perp) & (3) \\
p(A | C) &> p(A | C^\perp) & (4) \\
p(B | C) &> p(B | C^\perp) & (5)
\end{align*}
\]

where \(C^\perp\) denotes the orthocomplement of \(C\) and \(p(\cdot | \cdot)\) is the conditional probability.

The above definition, however, is too specific since (i) it allows only for causes with a positive impact on their effects and (ii) it excludes the possibility of a set of cooperating common causes. Hence we need to generalize the definition of the common cause in the following manner:

**Definition 2.** A partition \(\{C_k\}_{k \in K}\) in \(\Sigma\) is said to be the common cause system of the correlation between events \(A\) and \(B\) if the following screening-off condition holds for all \(k \in K\):

\[
p(A \land B | C_k) = p(A | C_k)p(B | C_k) \]  

where the cardinality \(|K|\) of \(K\) is said to be the size of the common cause system. A common cause system of size 2 is called a common cause (without the adjective 'Reichenbachian', indicating that the inequalities (4)-(5) are not required).

As a next step, the notion of the common cause system is generalized to the quantum case: First, one replaces the classical probability measure space \((\Sigma, p)\) by the non-classical probability measure space \((\mathcal{P}(\mathcal{N}), \phi)\) where \(\mathcal{P}(\mathcal{N})\) is the (non-distributive) lattice of projections (events) and \(\phi\) is a state.
of a von Neumann algebra $\mathcal{N}$. We note that in case of projection lattices we will use only algebra operations (products, linear combinations) instead of lattice operations ($\lor, \land$), because in case of commuting projections $A, B \in \mathcal{P}(\mathcal{N})$ we have $A \land B = AB$ and $A \lor B = A + B - AB$.

A set of mutually orthogonal projections $\{C_k\}_{k \in K} \subset \mathcal{P}(\mathcal{N})$ is called a partition of the unit $1 \in \mathcal{N}$ if $\sum_k C_k = 1$. Such a partition defines a conditional expectation

$$E : \mathcal{N} \to \mathcal{C}, \ A \mapsto E(A) := \sum_{k \in K} C_k AC_k,$$

that is $E$ is a unit preserving surjection onto the unital $C^*$-subalgebra $\mathcal{C} \subseteq \mathcal{N}$ obeying the bimodule property $E(B_1 AB_2) = B_1 E(A)B_2; A \in \mathcal{N}, B_1, B_2 \in \mathcal{C}$. We note that $\mathcal{C}$ contains exactly those elements of $\mathcal{N}$ that commute with $C_k, k \in K$. Since $\phi \circ E$ is also a state on $\mathcal{N}$ we can give the following

**Definition 3.** A partition of the unit $\{C_k\}_{k \in K} \subset \mathcal{P}(\mathcal{N})$ is said to be a (possibly) noncommuting common cause system of the commuting events $A, B \in \mathcal{P}(\mathcal{N})$, which correlate in the state $\phi: \mathcal{N} \to \mathcal{C}$, if

$$\frac{\phi(AB)C_k}{\phi(C_k)} = \frac{\phi(A)C_k}{\phi(C_k)} \frac{\phi(B)C_k}{\phi(C_k)},$$

for $k \in K$ with $\phi(C_k) \neq 0$. If $C_k$ commutes with both $A$ and $B$ for all $k \in K$ we call $\{C_k\}_{k \in K}$ a commuting common cause system. A common cause system of size $|K| = 2$ is called a common cause.

Some remarks are in place here. First, in case of a commuting common cause system $\phi \circ E$ can be replaced by $\phi$ in $\S$ since $\phi(AB)C_k = \phi(ABC_k), k \in K$. Second, using the decompositions of the unit, $1 = A + A^\perp = B + B^\perp$, $\S$ can be rewritten in an equivalent form:

$$(\phi \circ E)(AB)C_k(\phi \circ E)(A B^\perp C_k) = (\phi \circ E)(A B C_k)(\phi \circ E)(A B^\perp C_k), \quad k \in K.$$  (9)

One can even allow here the case $\phi(C_k) = 0$ since then both sides of $\S$ are zero. Finally, it is obvious from $\S$ that if $C_k \leq X$ with $X = A, A^\perp, B$ or $B^\perp$ for all $k \in K$ then $\{C_k\}_{k \in K}$ serves as a (commuting) common cause system of the given correlation independently of the chosen state $\phi$. These solutions are called trivial common cause systems. In case of common cause, $|K| = 2$, triviality means that $\{C_k\} = \{A, A^\perp\}$ or $\{C_k\} = \{B, B^\perp\}$.

Attached to the definition of the common cause (system), Reichenbach’s Common Cause Principle (CCP) is the following hypothesis: *if there is a correlation between two events and there is no direct causal (or logical) connection between the correlating events then there exists a common cause of the correlation*. The CCP in its present form, however, does not refer to the spatiotemporal localization of the common cause (system). Since in a local quantum theory all local events are supported in a well-defined spacetime region, the CCP needs some tailoring to make it fit well to the local quantum field theoretical setting. To address this point, one has to specify the localization of the possible causes of the correlations. One can define three different pasts of the regions $V_1$ and $V_2$ in a spacetime $\mathcal{S}$ as (Rédei, Summers 2007):

- weak common past: $wpast(V_1, V_2) := I_-(V_1) \cup I_-(V_2)$
- common past: $cpast(V_1, V_2) := I_-(V_1) \cap I_-(V_2)$
- strong common past: $spast(V_1, V_2) := \cap_{x \in V_1 \cup V_2} I_-(x)$

where $I_-(V)$ denotes the union of the backward light cones $I_-(x)$ of every point $x$ in $V$ (Rédei, Summers 2007). With these different localizations of the common cause in hand now we can define various CCPs according to (i) whether commutativity is required and (ii) where the common cause system is localized.
Definition 4. A $\mathcal{P}_K$-covariant local quantum theory $\{\mathcal{A}(V), V \in \mathcal{K}\}$ is said to satisfy the Commutative/Noncommutative (Weak/Strong) Common Cause Principle if for any pair $A \in \mathcal{A}(V_1)$ and $B \in \mathcal{A}(V_2)$ of projections supported in spacelike separated regions $V_1, V_2 \in \mathcal{K}$ and for every locally faithful state $\phi : \mathcal{A} \to \mathbb{C}$ such that

$$\phi(AB) \neq \phi(A) \phi(B)$$

there exists a nontrivial commuting/noncommuting common cause system $\{C_k\}_{k \in \mathcal{K}} \subset \mathcal{A}(V)$, $V \in \mathcal{K}$ of the correlation in the sense of Definition 3 such that the localization region $V$ is in the (weak/strong) common past of $V_1$ and $V_2$.

In the next Section we summarize the results concerning the Commutative CCPs. In Section 3 we first motivate why commutativity is a physically unjustifiable requirement for common cause systems in local quantum theories and why it should be given up. Then our main theorem will be proven stating that the Noncommutative Weak CCP holds in algebraic quantum field theory with locally finite degrees of freedom. Section 4 concludes the paper.

2 Commutative Common Cause Principles

The question whether the Commutative Common Cause Principles are valid in a Poincaré covariant local quantum theory in the von Neumann algebraic setting was first raised by Rédei (1997, 1998). As an answer to this question, Rédei and Summers (2002, 2007) have shown that the Commutative Weak CCP is valid in algebraic quantum field theory with locally infinite degrees of freedom. Namely, in the von Neumann setting they proved that for every locally normal and faithful state and for every superluminally correlating pair of projections there exists a weak common cause, that is a common cause system of size 2, in the weak past of the correlating projections. They have also shown (Rédei and Summers, 2002, p 352) that the localization of the common cause cannot be restricted to $wpast(V_1, V_2) \setminus I_-(V_1)$ or $wpast(V_1, V_2) \setminus I_-(V_2)$.

Concerning the Commutative (Strong) CCP less is known. If one also admits projections localized only in unbounded regions then the Strong CCP is known to be false: von Neumann algebras pertaining to complementary wedges contain correlated projections but the strong past of such wedges is empty (see Summers and Werner, 1988 and Summers, 1990). However, restricting ourselves to local algebras the situation is not clear. We are of the opinion that one cannot decide on the validity of the (Strong) CCP without an explicit reference to the dynamics since there is no bounded region $V$ in $cpast(V_1, V_2)$ (hence neither in $spast(V_1, V_2)$) for which isotony would ensure that $\mathcal{A}(V_1 \cup V_2) \subset \mathcal{A}(V''')$. But dynamics relates the local algebras since $\mathcal{A}(V_1 \cup V_2) \subset \mathcal{A}(V''' + t) = \alpha_t(\mathcal{A}(V'''))$ can be fulfilled for certain $V \in cpast(V_1, V_2)$ and for certain time translation by $t$.

Coming back to the proof of Rédei and Summers, the proof had a crucial premise, namely that the algebras in question are von Neumann algebras of type III. Although these algebras arise in a natural way in the context of Poincaré covariant theories other local quantum theories apply von Neumann algebras of other type. For example, theories with locally finite degrees of freedom are based on finite dimensional (type I) local von Neumann algebras. This raised the question whether the Commutative Weak CCP is valid in other local quantum theories. To address the problem Hofer-Szabó and Vecsernyés (2012) have chosen the local quantum Ising model (see Müller, Vecsernyés) having locally finite degrees of freedom. It turned out that the Commutative Weak CCP is not valid in the local quantum Ising model and it cannot be valid either in theories with locally finite degrees of freedom in general.

Since this model, which is a prototype of local UHF-type quantum theories, will play a role in our theorem proven in the next Section, we introduce it here. First, consider the net of ‘intervals’ $(i, j) := \{i, i + \frac{1}{2}, \ldots, j - \frac{1}{2}, j\} \subset \frac{1}{2}\mathbb{Z}$ of half-integers. The set of half-integers can be interpreted as the space coordinates of the center $(0, x), x \in \mathbb{Z}$ and $(-1/2, x), x \in \mathbb{Z} + 1/2$ of minimal double
cones $O^m_i$ of unit diameter on a ‘thickened’ Cauchy surface in two dimensional Minkowski space $M^2$ (See Fig. 1.) An interval $(i, j) \subset \frac{1}{2}\mathbb{Z}$ can be interpreted as the smallest double cone $O^m_{i,j} \subset M^2$ containing both $O^m_i$ and $O^m_j$. They determine a directed subset $K^m_{CS}$ of double cones in $M^2$, which is left invariant by the group of space-translations with integer values.

![Figure 1: A thickened Cauchy surface in the two dimensional Minkowski space $M^2$](image)

The net of local algebras is defined as follows. Consider the ‘one-point’ observable algebras $A(O^m_i), i \in \frac{1}{2}\mathbb{Z}$ associated to the minimal double cone $O^m_i$. Let $U_i$ denote the selfadjoint unitary generator of the algebra $A(O^m_i) \simeq M_1(\mathbb{C}) \oplus M_1(\mathbb{C})$ for any $i \in \frac{1}{2}\mathbb{Z}$. Between the generators one demands the following commutation relations:

$$U_i U_j = \begin{cases} -U_j U_i, & \text{if } |i - j| = \frac{1}{2}, \\ U_j U_i, & \text{otherwise}. \end{cases} \quad (11)$$

Now, the local algebra $A(O_{i,j})$ associated to the double cone $O_{i,j} \in K^m_{CS}$ are linearly spanned by the monomials

$$U_i^{k_i} U_{i+\frac{1}{2}}^{k_{i+\frac{1}{2}}} \ldots U_j^{k_j} U_{j-\frac{1}{2}}^{k_{j-\frac{1}{2}}} \quad (12)$$

where $k_i, k_{i+\frac{1}{2}} \ldots k_{j-\frac{1}{2}}, k_j \in \{0, 1\}$. Since the local algebras $A(O_{i,i-\frac{1}{2}+n}), i \in \frac{1}{2}\mathbb{Z}$ for $n \in \mathbb{N}$ are isomorphic to the full matrix algebra $M_2^n(\mathbb{C})$ the quasilocal observable algebra $A$ is a uniformly hyperfinite (UHF) $C^*$-algebra.

In case of the Ising model the causal (integer valued) time evolutions are classified (see Müller, Vecsernyé). That is the possible causal and (discrete) time translation covariant extensions of the "Cauchy surface net" \( \{A(O), O \in K^m_{CS}\} \) to \( \{A(O), O \in K^m\} \) are given, where $K^m$ is the subset of double cones in $M^2$ that are spanned by minimal double cones being integer time translates of those in $K^m_{CS}$, i.e. those in the original Cauchy surface. The set $K^m$ is left invariant by integer space and time translations, and the extended net also satisfies isotony, Einstein causality and algebraic Haag duality: $A(O') \cap A = A(O), O \in K^m$. Moreover, the commuting (unit) time and (unit) space translation automorphisms $\beta$ and $\alpha$ of the quasilocal algebra $A$ act covariantly on the local algebras. The causal time translation automorphisms $\beta$ of $A$ can be parametrized by $\theta_1, \theta_2; \eta_1, \eta_2$ with $-\pi/2 < \theta_1, \theta_2 \leq \pi/2$ and $\eta_1, \eta_2 \in \{-1, 1\}$ and they are given on the algebraic generator set \( \{U_i \in A(O^m_i), i \in \frac{1}{2}\mathbb{Z}\} \) of $A$. The automorphisms $\beta = \beta(\theta_1, \theta_2, \eta_1, \eta_2)$ of $A$ corresponding to causal

\[\text{For detailed Hopf algebraic description of the local quantum spin models see (Szlachányi, Vecsernyé, 1993), (Nill, Szlachányi, 1997), (Müller, Vecsernyé)}\]
Proposition 1. proved the following Proposition: in simple local algebras with spacelike separated supports contained in the weak future/past wpast I. Commutative Weak CCP, in general. That is there exist non-zero projections \( B \in A \) establishing a correlation between the other two Commutative CCPs as well. Obviously, the Weak CCP is weaker than the (Strong) CCP, hence the Proposition above falsified the other two Commutative CCPs as well.

where \( x \in \mathbb{Z} \). The causal evolutions clearly show how local primitive causality holds in the discrete case: If \( V \) consists of three neighbouring minimal double cones on a thickened Cauchy surface then \( \mathcal{A}(V) = \mathcal{A}(V') \) due to (13) and (14). Moreover, the following algebra isomorphisms hold (see Müller, Vecsényes): If \( O \in \mathcal{K}^m \) is a double cone containing \( n_+ \) and \( n_- \) minimal double cones in the right forward and left forward lightlike directions, respectively, then \( |\mathcal{A}(O)| \), the linear dimension of the corresponding local algebra is \( 2^{n(O)} \), \( n(O) := n_+ + n_- - 1 \) and

\[
\mathcal{A}(O) \simeq \begin{cases} 
M_{2^{n(O)/2}}(\mathbb{C}), & \text{if } n(O) \text{ is even}, \\
M_{2^{(n(O)-1)/2}}(\mathbb{C}) \oplus M_{2^{((n(O)-1)/2)}}(\mathbb{C}), & \text{if } n(O) \text{ is odd}. 
\end{cases}
\]

These properties lead us to the following definition:

**Definition 5.** Let \( W^{L/R}(O) \) and \( I_{\pm}(O) \) be the smallest left/right wedge region and the smallest forward/backward light cone, respectively, in the 2-dimensional Minkowski space \( \mathcal{M}^2 \) that contain the double cone \( O \in \mathcal{K}^m \). The local quantum theory \( \{ \mathcal{A}(O), O \in \mathcal{K}^m \} \) in \( \mathcal{M}^2 \) is called local UHF-type quantum theory if

(i) \( \mathcal{A}(O), O \in \mathcal{K}^m \) are finite dimensional C*-algebras,

(ii) for any \( O \in \mathcal{K}^m \) there exist double cones \( O_{L/R}^{\pm} \in \mathcal{K}^m \) with \( O \subset O_{L/R}^{\pm} \subset W^{L/R}(O) \cap I_{\pm}(O) \) such that \( \mathcal{A}(O_{L/R}^{\pm}) \) are simple C*-algebras,

(iii) local primitive causality holds: if \( V \) consists of three neighbouring minimal double cones on a thickened Cauchy surface then \( \mathcal{A}(V) = \mathcal{A}(V') \).

Clearly, the quasilocal algebra \( \mathcal{A} \) of a local UHF-type quantum theory is a UHF C*-algebra. What is more, any two local algebras with spacelike separated supports \( O_1, O_2 \in \mathcal{K}^m \) are embedded in simple local algebras with spacelike separated supports contained in the weak future/past \( I_{\pm}(O_1) \cup I_{\pm}(O_2) \) of \( O_1 \) and \( O_2 \). Local primitive causality implies that for any triple \( O_1, O_2, O_{\pm} \in \mathcal{K}^m \) with \( O_1 \cup O_2 \subset O_{\pm} \subset I_{\pm}(O_1 \cup O_2) \) the algebras obey \( \mathcal{A}(O_{\pm}) = \mathcal{A}(O_{\pm} \cap I_{\pm}(O_1)) \vee \mathcal{A}(O_{\pm} \cap I_{\pm}(O_2)) \).

Based on this \( \mathbb{Z} \times \mathbb{Z} \)-covariant local quantum Ising model Hofer-Szabó and Vecsényes (2012) proved the following Proposition:

**Proposition 1.** A local UHF-type quantum theory \( \{ \mathcal{A}(O), O \in \mathcal{K}^m \} \) in \( \mathcal{M}^2 \) does not satisfy the Commutative Weak CCP, in general. That is there exist nonzero projections \( A \in \mathcal{A}(O_a) \) and \( B \in \mathcal{A}(O_b) \) localized in two spacelike separated double cones \( O_a, O_b \in \mathcal{K}^m \) and a faithful state on \( A \) establishing a correlation between \( A \) and \( B \) such that there exists no nontrivial common cause system of the correlation localized in \( wpast(O_a, O_b) \).

Hence, in contrast with local quantum theories having locally infinite degrees of freedom, the Commutative Weak CCP is not valid in general in models having locally finite degrees of freedom. Obviously, the Weak CCP is weaker than the (Strong) CCP, hence the Proposition above falsified the other two Commutative CCPs as well.
As it was argued at the end of (Hofer-Szabó and Vecsernyés 2012), one can react to this fact in two different ways. On the one hand, one may regard a discrete model as an approximation of a "more detailed" (discrete or continuous) model. Then the failure of Commutative CCPs is explained by the fact that the approximate model does not contain all the observables; the common cause remains buried beyond the coarse description of the physical situation in question. Only a refined extended model could reveal the hidden common causes.

The other strategy, however, is to accept the discrete model as a self-contained physical model describing a specific physical phenomenon and to endorse the consequence of Proposition 1 as to the Commutative CCP cannot be a universally valid principle in local quantum theories. But this immediately raises the following question: What is the situation if we abandon commutativity between the common cause (system) and the correlating events? Are the Noncommutative CCPs valid in AQFT?

In (Hofer-Szabó and Vecsernyés 2012) there have been some indications that replacing commuting common causes by noncommuting ones can help to regain the CCP: for the same state falsifying the Commutative Weak CCP a noncommuting common cause could be given (located in the common past of the correlating events). In the next Section we will show that this specific example is indeed the sign of a more general fact, namely the validity of the Noncommutative Weak CCP.

3 Noncommutative Common Cause Principles

Commutativity has a well-specified role in quantum theories. Observables should commute to be simultaneously measurable in quantum mechanics; commutativity of observables with spacelike separated supports is one of the axioms of local quantum theories. However, as the possible causal time evolutions in the local quantum Ising model show, the local observable $U_\beta$ does not commute even with its own unit time translates $\beta(U_x)$ (unless $\theta_1 = 0$ or $\pi/2$). This happens already in ordinary quantum mechanics.

Let us consider the quantum harmonic oscillator given by the Hamiltonian

$$H = \frac{\hbar}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2 \quad (16)$$

acting on the Hilbert space $\mathcal{H}$ of square integrable functions on the real line $\mathbb{R}$. The eigenfunctions $\psi_n \in \mathcal{H}, n = 0, 1, 2, \ldots$ of $H$ are

$$\psi_n(x) = \sqrt{\frac{1}{2^n n!} \frac{m\omega}{\pi \hbar}} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \quad (17)$$

where $H_n$ is the Hermite polynomial

$$H_n(x) = (-1)^n \exp\left(x^2 \frac{d^n}{dx^n}\right) \exp(-x^2). \quad (18)$$

The corresponding energy eigenvalues are $E_n = \hbar\omega(n + 1/2)$. The unitary time evolution operator $U(t) := \exp(iHt)$ is diagonal in the basis $\{\psi_n\} \subset \mathcal{H}$ with entries $\exp(iE_nt)$. Using the recursion relation

$$xH_n(x) = \frac{1}{2}H_{n+1}(x) + nH_{n-1}(x) \quad (19)$$

for the Hermite polynomials, it is easy to see that the time evolved position operator $x(t) := U(-t)xU(t)$ does not commute with $x \equiv x(0)$ for generic $t$, i.e for $\hbar\omega t \not\in 2\pi\mathbb{Z}$. For example, in

See e.g. (Borthwick, Garibaldi, 2010) as a recent experimental development concerning the Ising model.
the ground state
\[
\frac{m \omega}{\hbar} \left[ x, x(t) \right] \psi_0 = \left( e^{i(E_0 - E_1)t} - e^{i(E_1 - E_2)t} \right) \frac{1}{\sqrt{2}} \psi_2 + \left( e^{i(E_0 - E_1)t} - e^{i(E_1 - E_0)t} \right) \frac{1}{2} \psi_0
\]
\[
= -i \sin(\hbar \omega t) \psi_0 \neq 0.
\]
(20)

Thus, if an observable $A$ is not a conserved quantity, that is $A(t) \neq A$, then the commutator $[A, A(t)] \neq 0$ in general. So why should the commutators $[A, C]$ and $[B, C]$ vanish for the events $A, B$ and for their common cause $C$ supported in their (weak/common/strong) past? We think that commuting common causes are only unnecessary reminiscence of their classical formulation. Due to their relative spacetime localization, that is due to the time delay between the correlating events and the common cause, it is also an unreasonable assumption. The benefit of allowing noncommuting common causes is that they help to maintain the validity of the Weak CCP also in local UHF-type quantum theories.

**Lemma 1.** Let $\mathcal{N}$ be a tensor product matrix algebra $\mathcal{N} = M_{n_1}(\mathbb{C}) \otimes M_{n_2}(\mathbb{C})$. Let $\mathcal{N}_1$ and $\mathcal{N}_2 = \mathcal{N}_1^I \cap \mathcal{N}$ denote the unital subalgebras $M_{n_1}(\mathbb{C}) \otimes 1$ and $1 \otimes M_{n_2}(\mathbb{C})$, respectively. Let $\phi$ be a faithful state on $\mathcal{N}$ that leads to a correlation between two nontrivial (commuting) projections $A \in \mathcal{N}_1$ and $B \in \mathcal{N}_2$. Then there exists a (noncommuting) common cause $\{C_1 \equiv C, C_2 \equiv C^\perp\} \subset \mathcal{P}(\mathcal{N})$ such that
\[
\phi(C_k A B C_k) = \phi(C_k A B^\perp C_k) \phi(C_k A^\perp B C_k), \quad k = 1, 2.
\]
(21)

**Proof.** Let $Tr$ be the (unique) normalized trace on $\mathcal{N}$. We can suppose that $Tr B \geq Tr B^\perp$. (If it fails then change the pair $A, B$ to $A^\perp, B^\perp$, (21) is invariant with respect to this change.) Let us choose the projection $C := AB'$ such that $B' \in \mathcal{N}_2$ is an equivalent projection to $B$, $B' \sim B$, that is $Tr B' = Tr B$. Clearly $C$ is a subprojection of $A, C < A$, therefore (21) fulfills for $k = 1$ because both sides are zero. For $k = 2$ one obtains
\[
\phi(C^\perp A B C^\perp) \phi(A^\perp B^\perp) = \phi(C^\perp A B^\perp C^\perp) \phi(A^\perp B)
\]
(22)
since $C^\perp > A^\perp$. Substituting $B = 1 - B^\perp$ into (22) one arrives at
\[
\frac{\phi(A^\perp B^\perp)}{\phi(A^\perp)} = \frac{\phi(C^\perp A B^\perp C^\perp)}{\phi(C^\perp A)}.
\]
(23)

Due to faithfulness of $\phi$ the left hand side is in the interval $(0, 1)$. Let us choose a continuous map $U : [0, 1] \to U(\mathcal{N}_2)$ from the unit interval into the unitary elements of $\mathcal{N}_2$ such that $U(0) = 1$ and $U(1) := U(1)BU(1)^* \geq B^\perp$. Since projections having the same dimensions (the same trace) in a simple algebra $\mathcal{N}_2$ are equivalent and the group of unitaries in $\mathcal{N}_2$ is connected, such a path exists. Then
\[
[0, 1] \ni t \mapsto C(t) := AU(t)BU(t)^* = U(t)ABU(t)^* \in \mathcal{P}(\mathcal{N})
\]
(24)
defines a continuous map with $C(0) = AB$ and $C(1) = AB(1) \geq AB^\perp$. Hence,
\[
[0, 1] \ni t \mapsto F(t) := \frac{\phi(C(t) \perp AB C(t) \perp)}{\phi(C(t) \perp A)} \in [0, 1]
\]
(25)
defines a continuous function with $F(0) = 1$ and $F(1) = 0$ since $C(0) \perp A = AB^\perp$ (hence, in the nomin-ator $C(0) \perp AB C(0) \perp = AB^\perp$) and $C(1) \perp A \leq AB$ (hence, in the nominator $C(1) \perp AB C(1) \perp = 0$), respectively. Therefore there exists $t' \in (0, 1)$ such that $F(t')$ is equal to the left hand side of (23) and the corresponding pair of projections $\{AB' := C(t'), C(t')^\perp\}$ serves as a common cause.

The lemma above is enough to prove the validity of the Noncommutative Weak CCP in the local quantum Ising model for all possible causal time evolutions. We use only that they are local UHF-type quantum theories for all causal time evolutions.
Proof. We can suppose $O_a$ to be on the left of $O_b$, that is $O_a \in W_L(O_b)$. Since we are dealing with a local UHF-type quantum theory there exist spacelike separated double cones $O_{a-}^L, O_{b-}^R \in K^m$ with $O_a \subset O_{a-}^L \subset L_-(O_a) \cap W_L(O_a)$ and $O_b \subset O_{b-}^R \subset L_-(O_b) \cap W_R(O_b)$ with simple local algebras $A(O_{a-}^L)$ and $A(O_{b-}^R)$, respectively. Moreover, if $\bar{O} := O_{a-}^L \vee O_{b-}^R \in K^m$ then there exists $\bar{O}_R \in K^m$ with $\bar{O} \subset \bar{O}_R \subset L_-(O_a \vee O_b)$ with simple local algebra $A(\bar{O}_R)$, which is equal to $A(\bar{O}_R \cap L_-(O_a)) \cup A(\bar{O}_R \cap L_-(O_b))$ due to local primitive causality. Since the restriction of the state $\phi$ to $\mathcal{N}$ has been required to be faithful one can apply Lemma 1 for the quantities $\mathcal{N}, A \in \mathcal{N}_1, B \in \mathcal{N}_2$ to prove the existence of a common cause $\{C, C^\perp\}$ in $A(\bar{O}_R)$, that is in the weak past of $O_a$ and $O_b$. 

A direct consequence of Proposition 2 is that the Noncommutative Weak CCP holds in local UHF-type quantum theories. It is obvious from the proof that the given common cause $\{C, C^\perp\}$ in $wpast(O_a, O_b)$ is not sensitive to the explicit dynamics: we have used only local primitive causality of the dynamics and isotony of the local algebras. Namely, using the notations of the previous proof, both the correlating events and the common cause are elements of the algebra $A(\bar{O}_R) \equiv A(\bar{O}_R \cap L_-(O_a)) \cup A(\bar{O}_R \cap L_-(O_b))$. This property is missing in the case of the CCP, thus we discuss the consequences shortly in the rest of this Section.

The reason why the other CCPs are a more subtle problem than the weak CCP is that the algebra $A(O_a) \cup A(O_b)$ is not contained in any local algebra supported in $cpast(O_a, O_b)$. Isotony does not help, because $O_a \cup O_b \not\subset cpast(O_a, O_b)$. Hence, we need the explicit dynamics to relate the correlating events in $A$ to a common cause supported in $cpast(O_a, O_b)$. Although it is not clear how far one has to go back in the common past $cpast(O_a, O_b)$ for finding a common cause (if it exists at all) located in $O_c$ as a first guess one can use $O_c := O_a \cup O_b - (t, 0) \in K^m$, i.e. the time shifted double cone generated by $O_a$ and $O_b$, for the smallest time $t$ for which $O_c \subset cpast(O_a, O_b)$. (See Fig. 2) Since a proof of the existence or non-existence of a common causes in $cpast(O_a, O_b)$ seems

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{A plausible localization of a common cause in the common past of the correlating events $A$ and $B$.}
\end{figure}
to be difficult some examples may give a hint.

In our previous paper (Hofer-Szabó, Vecsernyés 2012) we have found common causes for events $A, B$ localized in two adjacent but spacelike separated minimal double cones $O_a = O_n^m + (1,0)$ and $O_b = O_n^m + (1,0)$ for one of the simplest dynamics of (13)-(14), namely for $\theta_1 = 0, \eta_1 = 1$. (See Fig. 3) The faithful states $\phi_\lambda(\cdot) := Tr(\rho_\lambda \cdot)$ on $A$ parametrized by $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, which lead to

$$\rho_\lambda = \lambda_1 A B + \lambda_2 A \perp B \perp + \lambda_3 A \perp B + \lambda_4 A B \perp, \quad \lambda_i > 0, \sum_{i=1}^4 \lambda_i = 4, \lambda_1 \lambda_2 \in \mathbb{Q}, \lambda_3 \lambda_4 \notin \mathbb{Q}$$

(26)

with the restriction $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4$. The common cause was supported in $O_a \cup O_b - (1,0) \subset cpast(O_a, O_b)$ fulfilling the first guess mentioned above. However, for generic time evolutions (13)-(14) we have not found a common cause localized in this region for all the states given in (26).

Requiring also $\lambda_1 = \lambda_2$ the set $\{C, C^\perp\} = \{\frac{1}{2}(1 \pm U_0)\}$ localized in $O^m(0,0) \subset O_a \cup O_b - (1,0)$ serves already as a common cause for all dynamics. (See Fig. 4) But the shortcoming of this

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solution is that the common cause and the correlating events lie within a common Cauchy surface, i.e. the solution is not sensitive to the dynamics.
4 Conclusions

In the paper we have shown that the Weak Common Cause Principle can be maintained also in local quantum theories with locally finite degrees of freedom if one allows noncommuting common causes as well. Since observables do not commute even with its own time translates in general and since the common causes are required to supported in the (weak) past of the correlating events we think that noncommuting generalization of common causes is a natural assumption in quantum theories. The investigation of the (Strong) Common Cause Principles needs some knowledge about the dynamics, hence their validity is a more difficult problem. Maybe in the simplest local quantum theory, in the local quantum Ising model where the possible dynamics are known one can tackle this problem as well.

Finally, we mention a further possible direction of research. As we saw in this paper, abandoning commutativity gives us extra freedom in the search of common causes for correlations. But how big is this freedom? Is it big enough to find a common cause for a set of correlations? Or in other words, if \( \{(A_m,B_n) \mid m \in M, n \in N\} \) is a set of correlating events supported in spacelike separated regions \( V_A \) and \( V_B \), respectively, then does there exist a (weak/strong) common common cause system of these correlations, that is a common cause system screening off all the correlations?

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