On the geometry of random polytopes

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February 6, 2019

Abstract

We present a simple proof to a fact recently established in [5]: let $\xi$ be a symmetric random variable that has variance 1, let $\Gamma = (\xi_{ij})$ be an $N \times n$ random matrix whose entries are independent copies of $\xi$, and set $X_1, \ldots, X_N$ to be the rows of $\Gamma$. Then under minimal assumptions on $\xi$ and as long as $N \geq c_1 n$,

$$c_2 (B_n^\infty \cap \sqrt{\log (e N / n)} B_2^n) \subset \text{absconv}(X_1, \ldots, X_N)$$

with high probability.

1 Introduction

Let $\xi$ be a symmetric random variable that has variance 1 and let $X = (\xi_1, \ldots, \xi_n)$ be the random vector whose coordinates are independent copies of $\xi$. Consider a random matrix $\Gamma$ whose rows $X_1, \ldots, X_N$ are independent copies of $X$. In this note we explore the geometry of the random polytope

$$K = \text{absconv}(X_1, \ldots, X_N) = \Gamma^* B_1^n;$$

specifically, we study whether $K$ is likely to contain a large canonical convex body.

One of the first results in this direction is from [4], where it is shown that if $\xi$ is the standard gaussian random variable, $0 < \alpha < 1$ and $N \geq c_0 (\alpha) n$, then

$$c_1 (\alpha) \sqrt{\log (e N / n)} B_2^n \subset \text{absconv}(X_1, \ldots, X_N)$$

(1.1)

with probability at least $1 - 2 \exp (-c_2 N^{1-\alpha} n^\alpha)$. It should be noted that this estimate cannot be improved—up to the dependence of the constants on $\alpha$ (see, for example, the discussion in Section 4 of [9]).

The proof of (1.1) relies heavily on the tail behaviour of the gaussian random variable. It is therefore natural to try and extend (1.1) beyond the gaussian case, to random polytopes generated by more general random variables that still have ‘well-behaved’ tails. The optimal subgaussian estimate was established in [9]:

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Theorem 1.1. Let $\xi$ be a mean-zero random variable that has variance 1 and is $L$-subgaussian. Let $0 < \alpha < 1$ and set $N \geq c_0(\alpha) n$. Then with probability at least $1 - 2 \exp(-c_1 N^{1-\alpha} n^\alpha)$
\begin{equation}
  c_2(\alpha) \left( B_\infty^n \cap \sqrt{\log(eN/n)} B_2^n \right) \subset \text{absconv}(X_1, ..., X_N),
\end{equation}
where $c_0$ and $c_2$ are constants that depend on $\alpha$ and $c_1$ is an absolute constant.

Remark 1.2. Note that the body $\text{absconv}(X_1, ..., X_N)$ contains in (1.2) is slightly smaller than in (1.1), as one has to intersect the Euclidean ball from (1.1) with the unit cube.

While Theorem 1.1 resolves the problem when $\xi$ is subgaussian, the situation is less clear when $\xi$ is heavy-tailed. That naturally leads to the following question:

Question 1.3. Under what conditions on $\xi$ one still has that for $N \geq c_1 n$, $c_2(\alpha) \left( B_\infty^n \cap \sqrt{\log(eN/n)} B_2^n \right) \subset \text{absconv}(X_1, ..., X_N)$ (1.3)
with high probability?

Following the progress in [7], where Question 1.3 had been studied under milder moment assumptions on $\xi$ than in Theorem 1.1, Question 1.3 was answered in [5] under a minimal small-ball condition on $\xi$.

Definition 1.4. A mean-zero random variable $\xi$ satisfies a small-ball condition with constants $\kappa$ and $\delta$ if
\begin{equation}
  \Pr(|\xi| \geq \kappa) \geq \delta.
\end{equation}

Theorem 1.5. [5] Let $\xi$ be a symmetric, variance 1 random variable that satisfies (1.4) with constants $\kappa$ and $\delta$. For $0 < \alpha < 1$ there are constants $c_1, c_2$ and $c_3$ that depend on $\kappa, \delta$ and $\alpha$ for which the following holds. If $N \geq c_1 n$ then with probability at least $1 - 2 \exp(-c_2 N^{1-\alpha} n^\alpha)$, $c_3(\alpha) \left( B_\infty^n \cap \sqrt{\log(eN/n)} B_2^n \right) \subset \text{absconv}(X_1, ..., X_N)$.

Remark 1.6. The assumption made in [5] is slightly stronger than in Theorem 1.5, namely, that for every $x \in \mathbb{R}$, $\Pr(|\xi - x| \geq \kappa) \geq \delta$. However, (1.4) suffices for the proof. At the same time, in [5] the random variables $(\xi_{ij})$, each one of the $\xi_{ij}$’s satisfying (1.4) with the same constants $\kappa$ and $\delta$. In what follows we consider only the case in which $\xi_{ij}$ are independent copies of a single random variable $\xi$—though extending the presentation to the independent case is straightforward.

The original proof of Theorem 1.5 is based on the construction of a well-chosen net, and that construction is rather involved. Here we present a much simpler argument that is based on the small-ball method (see, e.g., [10, 11, 12]). As an added value, the method presented here gives more information than the assertion of Theorem 1.5, as is explained in what follows.

The starting point of the proof of Theorem 1.5 is straightforward: let
\begin{equation}
  K = \text{absconv}(X_1, ..., X_n) = \Gamma^* B_1^n
\end{equation}
and set
\begin{equation}
  L = \left( B_\infty^n \cap \sqrt{\log(eN/n)} B_2^n \right).
\end{equation}

\[^1\text{A centred random variable is } L\text{-subgaussian if for every } p \geq 2, \|\xi\|_{L_p} \leq L \sqrt{p} \|\xi\|_{L_2}.\]
By comparing the support functions of $L$ and of $K$, one has to show that with the wanted probability, for every $z \in \mathbb{R}^n$, $h_L(z) \leq h_{cK}(z)$. And, since $h_{cK}(z) = c\|z\|_\infty$, Theorem 1.5 can be established by showing that for suitable constants $c_0$ and $c_1$,

$$Pr(\exists z \in \partial L^o \ | \ |\Gamma z\|_\infty \leq c_0) \leq 2 \exp(-c_1 N^{1-\alpha} n^\alpha).$$  \hfill (1.5)

What we actually show is a stronger statement than (1.5): not only is there a high probability event on which

$$\inf_{z \in \partial L^o \parallel \Gamma z \parallel_\infty \geq c_0},$$

but in fact, on that “good event”, for each $z \in \partial L^o$, $\Gamma z$ has $\sim N^{1-\alpha} n^\alpha$ large coordinates, with each one of these coordinates satisfying that $|\langle z, X_i \rangle| \geq c_0$. Thus, the fact that $\|\Gamma z\|_\infty \geq c_0$ is exhibited by many coordinates and not just by a single one.

Proving that indeed, with high probability the smallest cardinality

$$\inf_{z \in \partial L^o \{i : |\langle z, X_i \rangle| \geq c_0}\}$$

is large is carried out in two steps:

**Controlling a single point.** For $0 < \alpha < 1$ and a well chosen $c_0 = c_0(\alpha)$ one establishes an *individual estimate*: that for every fixed $z \in \partial L^o$,

$$Pr(|\langle z, X \rangle| \geq 2c_0) \geq 4 \left(\frac{n}{N}\right)^\alpha.$$

In particular, if $X_1, \ldots, X_N$ are independent copies of $X$ then with probability at least $1 - 2 \exp(-c_2 N^{1-\alpha} n^\alpha)$,

$$|\{i : |\langle z, X_i \rangle| \geq 2c_0\}| \geq 2N^{1-\alpha} n^\alpha.$$ \hfill (1.6)

**From a single function to uniform control.** Thanks to the high probability estimate with which (1.6) holds, it is possible to control uniformly any subset of $\partial L^o$ whose cardinality is at most $\exp(c_2 N^{1-\alpha} n^\alpha/2)$. Let $T$ be a minimal $\rho$-cover of $\partial L^o$ with respect to the $\ell_2$ norm of the allowed cardinality. For every $z \in \partial L^o$, let $\pi z \in T$ that satisfies $\|z - \pi z\|_2 \leq \rho$. The wanted uniform control is achieved by showing that

$$\sup_{z \in \partial L^o \{i : |\langle z - \pi z, X_i \rangle| \geq c_0\}\} \leq N^{1-\alpha} n^\alpha$$

with probability at least $1 - 2 \exp(-c_3(\alpha) N^{1-\alpha} n^\alpha)$.

Indeed, combining the two estimates it follows that with probability at least

$$1 - 2 \exp(-c(\alpha) N^{1-\alpha} n^\alpha),$$

for every $z \in \partial L^o$, one has that

$$|\{i : |\langle \pi z, X_i \rangle| \geq 2c_0\}| \geq 2N^{1-\alpha} n^\alpha$$

and

$$|\{i : |\langle z - \pi z, X_i \rangle| \geq c_0\}| \leq N^{1-\alpha} n^\alpha.$$
Hence, on that event, for every $z \in \partial L^\circ$ there is $J_z \subset \{1, ..., n\}$ of cardinality at least $N^{1-\alpha}n^\alpha$, and for every $j \in J_z$,

$$| \langle z, X_i \rangle | \geq | \langle \pi z, X_i \rangle | - | \langle z - \pi z, X_i \rangle | \geq c_0,$$

implying that

$$\inf_{z \in \partial L^\circ} \{i : | \langle z, X_i \rangle | \geq c_0\} \geq N^{1-\alpha}n^\alpha;$$

in particular, $\inf_{z \in \partial L^\circ} \| \Gamma z \|_\infty \geq c_0$ as required.

In the next section this line of reasoning is used to prove Theorem 1.5.

2 Proof of Theorem 1.5

Before we begin the proof, let us introduce some notation. Throughout, absolute constant are denoted by $c, c_1, c'$ etc. Unless specified otherwise, the value of these constants may change from line to line. Constants that depend on some parameter $\alpha$ are denoted by $c(\alpha)$. We write $a \lessapprox b$ if there is an absolute constant $c$ such that $a \leq cb$;

$$a \lessapprox \alpha b \implies a \leq c(\alpha)b; \text{ and } a \sim b \text{ if both } a \lessapprox b \text{ and } b \lessapprox a.$$ 

The required estimate for a single point follows very closely ideas from [13], which had been developed for obtaining lower estimates on the tails of marginals of the Rademacher vector $(\varepsilon_i)_{i=1}^n$, that is, on

$$\Pr(|\sum_{i=1}^n \varepsilon_i z_i| > t)$$

as a function of the ‘location’ in $\mathbb{R}^n$ of $(z_i)_{i=1}^n$.

Fix $1 \leq r \leq n$ and consider the interpolation body $L_r = B^n_\infty \cap \sqrt{r}B^2_2$ and its dual $L_r^\circ = \text{conv}(B^1_1 \cup (1/\sqrt{r})B^2_2)$. The key estimate one needs to establish the wanted individual control is:

**Theorem 2.1.** There exist constants $c'$ and $c''$ that depend only on the small-ball constants of $\xi$ ($\kappa$ and $\delta$) such that if $z \in \partial L_r^\circ$ then

$$\Pr(|\langle z, X \rangle | \geq c') \geq 2\exp(-c''r).$$

The proof of Theorem 2.1 is based on some well-known facts on the interpolation norm $\| \cdot \|_{L_r^\circ}$.

**Lemma 2.2.** There exists an absolute constant $c_0$ such that for every $z \in \mathbb{R}^n$,

$$\|z\|_{L_r^\circ} \leq \sum_{i=1}^r z_i^* + \sqrt{r}(\sum_{i>r} (z_i^*)^2)^{1/2} \leq c_0\|z\|_{L_2^\circ},$$

where $(z_i^*)_{i=1}^n$ is the nonincreasing rearrangement of $(|z_i|)_{i=1}^n$.

Moreover, for every $z \in \mathbb{R}^n$ there is a partition of $\{1, ..., n\}$ to $r$ disjoint blocks $I_1, ..., I_r$ such that

$$\frac{\|z\|_{L_r^\circ}}{\sqrt{2}} \leq \sum_{j=1}^r (\sum_{i \in I_j} z_i^2)^{1/2} \leq \|z\|_{L_2^\circ}.$$
The first part of Lemma 2.2 is due to Holmstedt (see Theorem 4.1 in [6]) and it gives useful intuition on the nature of the norm $\| \cdot \|_{L^r}$. The second part is Lemma 2 from [13] and it plays an essential role in what follows.

Before proving Theorem 2.1, we require an additional observation that is based on the small-ball condition satisfied by $\xi$.

**Lemma 2.3.** Let $J \subset \{1, \ldots, n\}$ and set $Y = \sum_{j \in J} z_j \xi_j$. Then

$$\mathbb{E}|Y| \geq c(\kappa, \delta)(\sum_{j \in J} z_j^2)^{1/2},$$

where $c(\kappa, \delta) < 1$ is a constant that depends only on $\xi$’s small-ball constants $\kappa$ and $\delta$.

**Proof.** Let $(\varepsilon_j)_{j \in J}$ be independent, symmetric, $\{-1, 1\}$-valued random variables that are also independent of $(\xi_j)_{j \in J}$. Recall that $\xi$ is symmetric and therefore $(\xi_j)_{j \in J}$ has the same distribution as $(\varepsilon_j \xi_j)_{j \in J}$. By Khintchine’s inequality it is straightforward to verify that

$$\mathbb{E}|Y| = \mathbb{E}_\xi \mathbb{E}_\varepsilon \sum_{j \in J} \varepsilon_j z_j \xi_j \geq \mathbb{E}_\xi \left(\sum_{j \in J} z_j^2 \xi_j^2\right)^{1/2}.$$

Let $(\eta_j)_{j \in J} = 1_{\{|\xi_j| \geq \kappa\}}$; thus, the $\eta_j$’s are iid $\{0, 1\}$-valued random variables whose mean is at least $\delta$, and point-wise

$$\left(\sum_{j \in J} z_j^2 \xi_j^2\right)^{1/2} \geq \kappa \left(\sum_{j \in J} \eta_j z_j^2\right)^{1/2}.$$

Hence, and all that is left to complete the proof is to show that

$$\mathbb{E}\left(\sum_{j \in J} \eta_j z_j^2\right)^{1/2} \geq c(\delta) \left(\sum_{j \in J} z_j^2\right)^{1/2}.$$

Let $a_j = z_j^2 / (\sum_{j \in J} z_j^2)$ and in particular, $\|(a_j)_{j \in J}\|_1 = 1$. Assume without loss of generality that $J = \{1, \ldots, \ell\}$ and that the $a_j$’s are non-increasing, let $\gamma > 0$ be a parameter to be specified in what follows, and set $p = \mathbb{E}\eta_1 \geq \delta$.

Consider two cases:

- If $a_1 \geq \gamma p$ then with probability at least $p$, $\sum_{j=1}^\ell \eta_j a_j \geq a_1 \geq \gamma p$. In that case

  $$\mathbb{E}\left(\sum_{j=1}^\ell \eta_j a_j\right)^{1/2} \geq \sqrt{\gamma} p^{3/2} \geq \sqrt{\gamma} \delta^{3/2}.$$

- Alternatively, $a_1 \leq \gamma p$, implying that

  $$A = \sum_{j=1}^\ell a_j^2 \leq a_1 \sum_{j=1}^\ell a_j \leq \gamma p$$

  because $\|(a_j)_{j=1}^\ell\|_1 = 1$.

  By Bernstein’s inequality,

  $$\Pr\left(\sum_{j=1}^\ell (\eta_j - p)a_j \geq \frac{p}{2}\right) \leq 2 \exp\left(-c_0 \min\left\{\frac{(p/2)^2}{pA}, \frac{p/2}{a_1}\right\}\right) \leq 2 \exp(-c_1 / \gamma) \leq \frac{1}{2}.$$
provided that $\gamma$ is a small-enough absolute constant. Using, once again, that $\|a_j\|_1 = 1$ it is evident that with probability $1/2$, $\sum_{j=1}^\ell \eta_j a_j \geq (1/2)p$ and therefore
\[
\mathbb{E}(\sum_{j=1}^\ell \eta_j a_j)^{1/2} \geq \frac{\sqrt{p}}{4} \geq \frac{\sqrt{\delta}}{4}.
\]
Thus, setting $c(\kappa, \delta) \sim \kappa \delta^{3/2}$ one has that
\[
\left(\sum_{j=1}^\ell z_j^2 \xi_j^2\right)^{1/2} \geq c(\kappa, \delta) \left(\sum_{j=1}^\ell z_j^2\right)^{1/2},
\]
as claimed.

**Proof of Theorem 2.1.** Fix $z \in \partial L_r^\circ$ and recall that by Lemma 2.2 there is a decomposition of $\{1, \ldots, n\}$ to disjoint blocks $(I_j)_{j=1}^r$ such that
\[
\sum_{j=1}^r \left(\sum_{i \in I_j} z_i^2\right)^{1/2} \geq \frac{1}{\sqrt{2}}.
\]
(2.1)

Let $Y_j = \sum_{i \in I_j} z_i \xi_i$; observe that $Y_1, \ldots, Y_r$ are independent random variables and that by Lemma 2.3
\[
\mathbb{E}|Y_j| \geq c(\kappa, \delta) \left(\sum_{i \in I_j} z_i^2\right)^{1/2}
\]
for a constant $0 < c(\kappa, \delta) < 1$.

At the same time,
\[
\mathbb{E}|Y_j|^2 = \sum_{i \in I_j} z_i^2 \mathbb{E}\xi_i^2 = \sum_{i \in I_j} z_i^2.
\]
Therefore, by the Paley-Zygmund inequality (see, e.g., [2]), for any $0 < \theta < 1$,
\[
Pr(|Y_j| \geq \theta \mathbb{E}|Y_j|) \geq (1 - \theta^2) \frac{(\mathbb{E}|Y_j|)^2}{\mathbb{E}|Y_j|^2}.
\]
Setting $\theta = 1/2$,
\[
Pr\left(|Y_j| \geq \frac{1}{2} c(\kappa, \delta) \left(\sum_{i \in I_j} z_i^2\right)^{1/2}\right) \geq \frac{3}{4} c^2(\kappa, \delta),
\]
and since $Y_j$ is a symmetric random variable (because the $\xi_i$’s are symmetric), it follows that
\[
Pr\left(Y_j \geq \frac{1}{2} c(\kappa, \delta) \left(\sum_{i \in I_j} z_i^2\right)^{1/2}\right) \geq \frac{3}{8} c^2(\kappa, \delta) \equiv c_1(\kappa, \delta).
\]
For $1 \leq j \leq r$ let
\[
B_j = \left\{Y_j \geq \frac{1}{2} c(\kappa, \delta) \left(\sum_{i \in I_j} z_i^2\right)^{1/2}\right\}
\]
which are independent events. Hence,

\[
Pr \left( \sum_{i=1}^{n} \xi_i z_i \geq \frac{1}{2} c(\kappa, \delta) \sum_{j=1}^{r} \left( \sum_{i \in I_j} z_i^2 \right)^{1/2} \right) = Pr \left( \sum_{j=1}^{r} Y_j \geq \frac{1}{2} c(\kappa, \delta) \sum_{i \in I_j} \left( z_i^2 \right)^{1/2} \right) \\
\geq \prod_{j=1}^{r} Pr(B_j) \geq c_1^r(\kappa, \delta).
\]

Thus, by (2.1), if \( c' = \frac{1}{4} c(\kappa, \delta) \) and \( c'' = \log(1/c_1(\kappa, \delta)) > 0 \), one has

\[
Pr \left( \sum_{i=1}^{n} \xi_i z_i \geq c' \right) \geq \exp(-c'' r).
\]

\[
\square
\]

From here on, the constants \( c' \) and \( c'' \) denote the constants from Theorem 2.1.

**Corollary 2.4.** For \( 0 < \alpha < 1 \), \( \kappa \) and \( \delta \) there are constants \( c_0 \) and \( c_1 \) that depend on \( \alpha, \kappa \) and \( \delta \), and an absolute constant \( c_2 \) for which the following holds. If \( N \geq c_0 n, r \leq c_1 \sqrt{\log(eN/n)} \) and \( z \in \partial L_\rho^0 \) then with probability at least \( 1 - 2 \exp(-c_2 N^{1-\alpha} n^\alpha) \),

\[
\left| \{ i : | \langle z, X_i \rangle | \geq c' \} \right| \geq 2N^{1-\alpha} n^\alpha.
\]

**Proof.** Let \( z \in \partial L_\rho^0 \), and invoking Theorem 2.1

\[
Pr \left( | \langle z, X \rangle | \geq c' \right) \geq \exp(-c'' r)
\]

where \( c' \) and \( c'' \) depend only on \( \kappa \) and \( \delta \).

Let \( r_0 = c_1 \log(eN/n) \) such that \( \exp(-c'' r_0) \geq 4(n/N)^\alpha \); thus, \( c_1 = c_1(\alpha, \kappa, \delta) \). If \( r \leq r_0 \), \( X_1, ..., X_N \) are independent copies of \( X \) and \( \eta_i = 1_{\{ | \langle z, X_i \rangle | \geq c' \}} \), then \( E \eta_i \geq 4(n/N)^\alpha \). Hence, by a standard concentration argument, with probability at least \( 1 - 2 \exp(-c_2 N^{1-\alpha} n^\alpha) \),

\[
\left| \{ i : | \langle z, X_i \rangle | \geq c' \} \right| \geq 2N^{1-\alpha} n^\alpha,
\]

where \( c_2 \) is an absolute constant.

Thanks to the high probability estimate with which Corollary 2.4 holds, one can control uniformly all the elements of a set \( T \subset \partial L_\rho^0 \) as long as \( |T| \leq \exp(c_0 N^{1-\alpha} n^\alpha) \) for a suitable absolute constant \( c_0 \), and as long as \( r \leq c(\alpha, \kappa, \delta) \log(eN/n) \). In that case, there is an event of probability at least \( 1 - 2 \exp(-c_1 N^{1-\alpha} n^\alpha) \) such that for every \( z \in T \),

\[
\left| \{ i : | \langle z, X_i \rangle | \geq c' \} \right| \geq 2N^{1-\alpha} n^\alpha.
\] (2.2)

The natural choice of a set \( T \) is a minimal \( \rho \)-cover of \( \partial L_\rho^0 \) with respect to the \( \ell_2 \) norm. Note that \( L_\rho^0 = \text{absconv}(B_0^0 \cup r^{-1/2} B_2^0) \subset B_2^0 \), and so there is a \( \rho \)-cover of the allowed cardinality for

\[
\rho \leq 5 \exp(-c_2 (N/n)^{1-\alpha}),
\]

where \( c_2 \) is an absolute constant.

Clearly, \( \{ z - \pi z : z \in \partial L_\rho^0 \} \subset \rho B_2^0 \), and to complete the proof of Theorem 1.3 it suffices to show that with probability at least \( 1 - 2 \exp(-c_3 N^{1-\alpha} n^\alpha) \)

\[
Q = \sup_{u \in \rho B_2^0} \left| \{ i : | \langle u, X_i \rangle | \geq c' / 2 \} \right| \leq N^{1-\alpha} n^\alpha.
\] (2.3)
To prove (2.3), observe that $Q$ is the supremum of an empirical process indexed by a class of binary valued functions

$$F = \{ f_z = 1_{\{ \langle z, \cdot \rangle \geq c' / 2 \}} : z \in \rho B_2^n \};$$

in particular, for every $f_z \in F$,

$$\| f_z \|_{L_2} = Pr^{1/2}( \| \langle z, X \rangle \| \geq c' / 2 ) \leq \frac{2\| \langle z, X \rangle \|_{L_2}}{c'} \leq \frac{2\rho}{c'} = c_4(\kappa, \delta) \exp(-c_2(N/n)^{1-\alpha}).$$

By Talagrand’s concentration inequality for bounded empirical processes (14, see also [1]), with probability at least $1 - 2 \exp(-t)$,

$$Q \lesssim \mathbb{E}Q + \sqrt{t} \sqrt{N} \sup_{f_z \in F} \| f_z \|_{L_2} + t \sup_{f_z \in F} \| f_z \|_{L_\infty}$$

$$\lesssim \mathbb{E}Q + \sqrt{t} \sqrt{N} c_4(\kappa, \delta) \exp(-c_2(N/n)^{1-\alpha}) + t$$

$$= (1) + (2) + (3);$$

Let us show that for the right choice of $t$ and $N$ large enough, $Q \leq N^{1-\alpha} n^\alpha$.

The required estimate on (2) and (3) clearly holds as long as

$$t \lesssim_{\kappa, \delta} N^{1-\alpha} n^\alpha \text{ and } N \gtrsim \alpha n.$$ 

As for $\mathbb{E}Q$, note that point-wise

$$\sup_{u \in \rho B_2^n} \left| \left\{ i : \langle u, X_i \rangle \geq c' / 2 \right\} \right| \leq \frac{2}{c'} \sup_{u \in \rho B_2^n} \sum_{i=1}^N | \langle u, X_i \rangle |.$$ 

Let $(\varepsilon_i)_{i=1}^N$ be independent, symmetric, $\{-1, 1\}$-valued random variables that are independent of $(X_i)_{i=1}^N$. By the Giné-Zinn symmetrization theorem [3] and the contraction inequality for Bernoulli processes [8],

$$\mathbb{E}Q \leq \frac{2}{c'} \mathbb{E} \sup_{u \in \rho B_2^n} \sum_{i=1}^N | \langle u, X_i \rangle |$$

$$\leq \frac{4}{c'} \mathbb{E} \sup_{u \in \rho B_2^n} \sum_{i=1}^N \varepsilon_i | \langle u, X_i \rangle | + \frac{2N}{c'} \sup_{u \in \rho B_2^n} \mathbb{E} | \langle u, X_i \rangle |$$

$$\leq \frac{4}{c'} \mathbb{E} \sup_{u \in \rho B_2^n} \left( \sum_{i=1}^N \varepsilon_i X_i, u \right) + \frac{2N}{c'} \rho$$

$$\leq \frac{4\rho}{c'} (\sqrt{Nn} + N) \lesssim_{\kappa, \delta} N \exp(-c_2(N/n)^{1-\alpha}),$$

which is sufficiently small as long as $N \gtrsim_{\alpha, \kappa, \delta} n$. 

3 Concluding Remarks

This proof of Theorem [3] is based on the small-ball method and follows an almost identical path to previous results that use the method: first, one obtains an individual estimate that
implies that for each $v$ in a fine-enough net, many of the values $(|\langle X_i, v \rangle|)_{i=1}^N$ are in the ‘right range’; and then, that the ‘oscillation vector’ $(|\langle X_i, z - v \rangle|)_{i=1}^N$ does not spoil too many coordinates when $v$ is ‘close enough’ to $z$. Thus, with high probability and uniformly in $z$, many of the values $(|\langle X_i, z \rangle|)_{i=1}^N$ are in the right range.

Having said that, there is one substantial difference between this proof and other instances in which the small-ball method had been used. Perviously, individual estimates had been obtained in the small-ball regime; here the necessary regime is different: one requires a lower estimate on the tails of marginals of $X = (\xi_i)_{i=1}^n$. And indeed, the core of the proof is the individual estimate from Theorem 2.1 where one shows that if $\xi$ satisfies a small-ball condition and $X$ has iid coordinates distributed as $\xi$ then its marginals exhibit a ‘supergaussian’ behaviour at the right level.

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