ENHANCED INTERFACE REPULSION
FROM QUENCHED HARD–WALL RANDOMNESS

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Abstract. We consider the harmonic crystal, or massless free field, \( \varphi = \{ \varphi_x \}_{x \in \mathbb{Z}^d}, d \geq 3 \), that is the centered Gaussian field with covariance given by the Green function of the simple random walk on \( \mathbb{Z}^d \). Our main aim is to obtain quantitative information on the repulsion phenomenon that arises when we condition \( \varphi_x \) to be larger than \( \sigma_x \), \( \sigma = \{ \sigma_x \}_{x \in \mathbb{Z}^d} \) is an IID field (which is also independent of \( \varphi \)), for every \( x \) in a large region \( D_N = ND \cap \mathbb{Z}^d \), with \( N \) a positive integer and \( D \subset \mathbb{R}^d \) a rather general bounded subset of \( \mathbb{R}^d \). We are mostly motivated by results for given typical realizations of \( \sigma \) (quenched set–up), since the conditioned harmonic crystal may be seen as a model for an equilibrium interface, living in a \((d+1)\)-dimensional space, constrained not to go below a inhomogeneous substrate that acts as a hard wall. This substrate is mostly flat, but presents some rare anomalous spikes. We consider various types of substrate and we observe that the interface is pushed away from the wall much more than in the case of a flat wall as soon as the upward tail of \( \sigma_0 \) is heavier than Gaussian, while essentially no effect is observed if the tail is sub–Gaussian. In the critical case, that is the one of approximately Gaussian tail, the interplay of the two sources of randomness, \( \varphi \) and \( \sigma \), leads to an enhanced repulsion effect of additive type. This generalizes work done in the case of a flat wall and also in our case the crucial estimates are optimal Large Deviation type asymptotics as \( N \to \infty \) of the probability that \( \varphi \) lies above \( \sigma \) in \( D_N \). We will consider the annealed case too. It turns out that quenched and annealed asymptotics coincide and this fact plays a role in the proofs and concurs to building an understanding of the phenomenon.

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1. Introduction and main results

1.1. The harmonic crystal. An harmonic crystal is (for us) the Gaussian random field \( \varphi = \{ \varphi_x \}_{x \in \mathbb{Z}^d}, d \geq 3 \), such that \( \mathbb{E} (\varphi_x) = 0 \) and \( \mathbb{E} (\varphi_x \varphi_y) = G(x, y) \) for every \( x \) and \( y \) in \( \mathbb{Z}^d \), where \( G : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}^+ \) is the Green function of the simple random walk on \( \mathbb{Z}^d \):

\[
G(x, y) = \sum_{n=0}^{\infty} p_n(x, y),
\]

where \( p_n(x, y) \) is the probability that the simple random walk \( \{ X_k \}_{k=0,1,\ldots} \) with \( X_0 = x \) and hopping to nearest neighbor sites with probability \( 1/2d \), is at site \( y \) after \( n \) time steps. We remark that \( G(x, y) = G(x - y, 0) = G(y - x, 0) \) for every \( x, y \in \mathbb{Z}^d \). In short we will write \( \varphi \sim \mathcal{N}(0, G(\cdot, \cdot)) \): the same notation will be used for (finite dimensional) Gaussian vectors. We observe that \( \varphi \) is a Gibbsian field (cf. [3]) and can be characterized by its one point conditional probability: for every
and every measurable bounded function $h : \mathbb{R} \to \mathbb{R}$

$$E \left[ h(\varphi_x), F^c_x \right] (\psi) = E(h(Z_\psi)), \quad \mathbb{P}(d\psi) \text{ a.s.},$$

(1.2)

where $Z_\psi \sim \mathcal{N}((1/2d) \sum_{y : |y-x| = 1} \psi_y, 1)$ and, for $A \subset \mathbb{Z}^d$, $F_A^\varphi = \sigma(\varphi_x : x \in A)$. In equation (1.2) we also introduced the notation $E(P)$ for expectation (probability) of the random variable involved: if we need to insist on the measure (say $\mu$) on the probability space we write $E_\mu(P_\mu)$. We will reserve the use of $E$ (and $P$) for the random field $\varphi$. Notice that from (1.2) one easily extracts the fact that $\varphi$ is also a Markov field.

In more informal way we may simply say that $\varphi$ is a Gibbsian field with respect to the formal Hamiltonian

$$H(\varphi) = \frac{1}{8d} \sum_{x,y : |x-y| = 1} (\varphi_x - \varphi_y)^2. \quad (1.3)$$

This imprecise statement helps getting an intuitive grasp on the special features of the harmonic crystal. We stress in particular two facts (see [13], particularly Ch. 13, for a detailed treatment):

- **Existence of a Gibbsian field associated to a certain $H$ is not guaranteed in $\mathbb{R}^\mathbb{Z}^d$, due to the lack of compactness and general results to tackle this problem do not apply to the case of (1.3).**
  As a matter of fact there exists no Gibbs measure associated to (1.3) if $d = 1, 2$. Of course the fact that $H$ is a quadratic form allows for a full solution of the existence problem and the characterization of the space of all the Gibbs measures (associated to $H$): in particular one easily arrives to formula (1.1) for the covariances and one understands the necessity of being on a lattice in which a simple symmetric random walk is transient in order to have existence of a (infinite volume) Gibbs measure.

- **The space of Gibbs measures associated to $H$ is extremely large (as soon as it is non empty, of course).** One can show that $\mu$ is an extremal element of such (convex) space of Gibbs measures if and only if $\mu \sim \mathcal{N}(u, G(\cdot, \cdot))$, with $(\Delta u)_x = (1/2d) \sum_{e \in \mathbb{Z}^d, |e| = 1} (u_{x+e} - u_x) = 0$ for every $x$, that is $u$ is harmonic. In particular we may choose $u_x = a + v \cdot x$ for every choice of $a \in \mathbb{R}$ and $v \in \mathbb{R}^d$. We can interpret $\varphi_x$ as the height of the interface above a reference plane: to a certain extent the richness of the Gibbs space is intimately connected with the interest of the model as a very simplified caricature of a physical interface (this issue is developed at length in [1] and [12]).

As the reader may have noticed, we have made the choice not to distinguish between random and numerical variables when talking about $\varphi$.

1.2. A model for entropic repulsion: the case of a flat wall. In [6] (see however [3] for a review of the various improvements obtained since then) the authors considered the problem of
identifying the asymptotics of the probability of the event
\[
\Omega^+_N = \{ \varphi : \varphi_x \geq 0 \text{ for every } x \in D_N \}
\]
where \( N \) is a positive integer, \( D_N = ND \cap \mathbb{Z}^d, \) \( D = (-1/2, 1/2)^d, \) and the asymptotics is with respect to \( N \uparrow \infty. \) Their main results are essentially two, we restate them here informally:

- In the sense of exponential asymptotics, \( \mathbb{P}(\Omega^+_N) \) behaves like \( \exp(-\alpha N^{d-2} \log N). \) The constant \( \alpha \) has been determined:
  \[
  \alpha = 2G \text{Cap}(D), \quad G(0,0) = G,
  \]
where \( \text{Cap}(D), D \) an open subset of \( \mathbb{R}^d \) is the Newtonian capacity of \( D: \)
\[
\text{Cap}(D) = \inf \left\{ \frac{1}{2d} \| \partial f \|_2^2 : f \in C_0^\infty(\mathbb{R}^d; [0, \infty)), \; f(r) = 1 \text{ for all } r \in D \right\},
\]

\[
R_d = \lim_{x \to \infty} |x|^{d-2}G(0, x) \in (0, \infty).
\]

The equivalence between the two definitions of capacity in (1.6) can be found for example in [4, Lemma A.8] and the existence of the non–degenerate limit in (1.7) we refer to [13].

- The trajectories of the field \( \varphi \) that are typical with respect to \( \mathbb{P}(d\varphi|\Omega^+_N) \) are pushed to infinity as \( N \uparrow \infty \) in the sense that (\[6, \text{Prop. 1.3 and Lemma 4.7} \) and \([9, \text{Lemma 3.3} \])
\[
\lim_{N \to \infty} \sup_{x \in D_N} \frac{\mathbb{E}(\varphi_x|\Omega^+_N)}{\sqrt{4G \log N}} - 1 = 0.
\]

Essentially what happens is that the field stays flat and does not change its structure (see \([6, \text{Th. 3.3} \]), but it flees the wall: and it does this to regain its freedom of fluctuating (this effect is indeed called entropic repulsion).

The two issues are intimately connected. In fact having a good guess for the behavior of the trajectories of \( \mathbb{P}(d\varphi|\Omega^+_N) \) leads to a good lower bound on the asymptotics of \( \mathbb{P}(\Omega^+_N) \) (and may suggest a strategy for the upper bound). On the other side the same probability asymptotics enter in a crucial way in proving that the good guess on \( \mathbb{P}(d\varphi|\Omega^+_N) \) is really close to \( \mathbb{P}(d\varphi|\Omega^+_N) \) itself. The asymptotic behavior of \( \mathbb{P}(\Omega^+_N) \) is however not the only ingredient and this two way argument (from probability estimates to path properties and viceversa) is by no means general. For further discussions on physical aspects of entropic repulsion we refer for example to \([8, \text{[16]} \) and the several references therein.

To understand the results in \([6] \) one may start with the most naive guess for the behavior of \( \varphi_x \) under \( \mathbb{P}(d\varphi|\Omega^+_N) \) in the repulsion region: if in the region \( D_N \) the field just translates up of \( \min_{x \in D_N} \varphi_x, \) whose typical behavior under \( \mathbb{P}(d\varphi) \) is approximately \( \sqrt{2dG \log N}, \) then the field
would not be bothered by the presence of the wall. This (sloppy) argument is easily translated into a rigorous lower bound on the probability of $\Omega_N^+$ (see [3, Lemma 2.3]): one would then verify that it is not the optimal lower bound. That translating to height $\sqrt{2dG\log N}$ is not a very good guess is also clear from the result on the typical height of the field that we just mentioned above: the field moves up to (and not beyond!) $\sqrt{4G\log N}$. Therefore the guess that the field simply translates globally isn’t really correct and something more complex is happening. What happens can be synthetised in the following way: it does not cost too much (in a Large Deviation sense) to modify extrema of the $\varphi$ field (they happen only on one site). But we can go beyond: preventing the field $\varphi_x$ from fluctuating freely on $o(N^{d-2})$ sites (say: sparsely chosen in $D_N$) is not a substantial modification. This is a non obvious fact, and it is essentially a consequence of the fact that a random walk that leaves from a site $x \in D_N$ reaches with probability (almost) one $D_N^C$ even if we have put $o(N^{d-2})$ traps in $D_N$. It turns out that the typical cardinality of $\{x : \varphi_x \leq -\sqrt{4G\log N}\}$ is about $N^{d-2}$. One may therefore believe that translating by slightly more than $\sqrt{4G\log N}$ should suffice: on (and around) the rebel sites something different happens, but apart from these sites, that are few, we should still believe that translating is a good guess. In the present paper we present a proof of the lower bound that implements in a direct way this heuristics and that we believe is more direct than the original proof (and the modified version proposed in [9]).

1.3. The model with a random substrate. In the physical literature much effort is devoted to investigating a variety of random surface phenomena, including entropic repulsion effects, in the presence of a rough or disordered substrate: the analysis covers a variety of interface–substrate models (do for example a general search on the physics archive [http://xxx.lanl.gov] for the key–phrase 'disordered substrate'), most of which seem at the moment out of the reach of mathematical treatment. Here we look for rigorous results on purely entropic repulsion effects in the presence of a disordered substrate in the simplified framework of the high dimensional harmonic crystal.

This substrate, or wall, will be modelled via a random field $\sigma = \{\sigma\}_{x \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$: the law of $\sigma$ will be denoted by $P$ ($E$). The hypotheses on $\sigma$ are:

H.1 Independence: $\sigma$ is an IID field.

H.2 Almost Gaussian behavior of upward (or $\sigma^+$) tails: there exists $Q > 0$ such that

$$\lim_{r \to \infty} \frac{1}{r^2} \log P(\sigma_0 > r) = -\frac{1}{2Q}. \quad (1.9)$$

H.3 Weak control on downward (or $\sigma^-$) tails: $E(\sigma_0^-) < \infty$.

Examples of $\sigma$ fields of course include the case of $\sigma_0 \sim \mathcal{N}(0,Q)$, $Q > 0$, or the absolute value of such a variable. We discuss in Subsection 1.6 each one of this hypotheses. The model corresponding to H.2 turns out to be the most interesting, but for completness we consider also the cases:
H.2–1 **Sub-Gaussian behavior of upward tails:**

\[
\lim_{r \to \infty} \frac{1}{r^2} \log P(\sigma_0 > r) = -\infty.
\]  

(1.10)

H.2–2 **Super-Gaussian behavior of upward tails:** there exists \( \beta \in (0, 1) \) and \( Q > 0 \) such that

\[
\lim_{r \to \infty} \frac{1}{r^{2\beta}} \log P(\sigma_0 > r) = -\frac{1}{2Q}.
\]  

(1.11)

Also for \( \sigma \) our notation does not distinguish between random and numerical variables.

The random fields \( \varphi \) and \( \sigma \) are assumed to be independent of each other. We may therefore think the configuration space to be \( \mathbb{R}^{Z^d} \times \mathbb{R}^{Z^d} \), endowed with the (local) product topology and equipped with the Borel \( \sigma \)-algebra: on this space the measure is \( \mathbf{P} \otimes \mathbf{P} \). Therefore \((\sigma, \varphi) \in \mathbb{R}^{Z^d} \times \mathbb{R}^{Z^d}\) is a wall–interface configuration. We introduce an interaction between \( \varphi \) and \( \sigma \) by conditioning with respect to a suitable event: given \( \sigma \in \mathbb{R}^{Z^d} \) and \( A \subset Z^d \), the \( \sigma \)-entropic repulsion event on \( A \) is defined by

\[
\Omega_{A,\sigma}^+ = \{ \varphi : \varphi_x \geq \sigma_x \text{ for every } x \in A \}.
\]  

(1.12)

We mostly impose the repulsion on a rather general domain \( D_N = ND \cap Z^d \), \( D \) a bounded connected domain with piecewise smooth boundary and containing the origin: we use the shortcut notation \( \Omega_{N,\sigma}^+ = \Omega_{D_N,\sigma}^+ \).

We talk about **quenched results** in the cases in which a \( \mathbf{P} \)-typical configuration \( \sigma \) is chosen and kept fixed (while \( \varphi \) is considered random): in this case we prefer to work on the measure space \((\mathbb{R}^{Z^d}, \mathcal{B}(\mathbb{R}^{Z^d}), \mathbf{P})\) rather than introducing complicated conditioning notations. Of course it is in this **quenched set-up** that \( \Omega_{A,\sigma}^+ \) is an event.

We talk instead of **annealed results** when both \( \sigma \) and \( \varphi \) are averaged at the same time. In the annealed set–up, with abuse of notation, \( \Omega_{A,\sigma}^+ \) is rather the event \( \{(\sigma, \varphi) : \varphi_x \geq \sigma_x \text{ for every } x \in A \} \).

1.4. **Main results: the case of almost Gaussian \( \sigma^+ \) tails.** One of the main results that we are going to prove is that the quenched probability of the entropic repulsion event is vanishing exponentially and we identify its asymptotic behavior. Moreover quenched and annealed asymptotics coincide.

**Theorem 1.1.** Under hypotheses \( H[1], H[2] \) and \( H[3] \) we have that

\[
\lim_{N \to \infty} \frac{1}{N^{d-2} \log N} \log \mathbf{P} \left( \Omega_{N,\sigma}^+ \right) = \lim_{N \to \infty} \frac{1}{N^{d-2} \log N} \log \mathbf{P} \otimes \mathbf{P} \left( \Omega_{N,\sigma}^+ \right) = -2(G + Q) \text{Cap}(D).
\]  

(1.13)

\( \mathbf{P}(d\sigma) \)-a.s..
The proof of Theorem 1.1 requires the sharpest probability estimates that we obtain in this work and sheds light on the behavior of the conditional measure $\mathbb{P}(\cdot | \Omega^+_{N,\sigma})$: this is the measure that contains the information directly related to the physical situation we are modelling. The next result concerns the asymptotics of this measure. For $\epsilon > 0$ and a configuration $\varphi$ call $N_\epsilon(\varphi)$ the cardinality of the set

$$\left\{ x \in D_N : \left| \frac{\varphi_x}{\sqrt{4(G+Q)} \log N} - 1 \right| \geq \epsilon \right\}$$

(1.14)

**Theorem 1.2.** Under hypotheses $H.1$, $H.2$ and $H.3$, $\mathbb{P}(d\sigma)$–a.s. for every $\epsilon > 0$, $N_\epsilon(\varphi)/|D_N|$ tends to 0 in probability with respect to $\mathbb{P}(d\varphi|\Omega^+_{N,\sigma})$.

We refer to Section 4 for further results on $\mathbb{P}(d\varphi|\Omega^+_{N,\sigma})$.

1.5. **Super/Sub–Gaussian $\sigma^+$ tails.** Call $N_\epsilon^p(\varphi)$, $\kappa = 0, 1$, the cardinality of the set

$$\left\{ x \in D_N : \left| \frac{\varphi_x}{4((1-\kappa)G + \kappa Q) \log N)^{(\kappa \beta - 1 + (1-\kappa))/2}} - 1 \right| \geq \epsilon \right\}$$

(1.15)

**Theorem 1.3.** Assume $H.1$ and $H.3$.

1. Under Hypothesis $H.2–1$, $\mathbb{P}(d\sigma)$–a.s. we have that

$$\lim_{N \to \infty} \frac{1}{N d - 2 \log N} \log \mathbb{P} \left( \Omega^+_{N,\sigma} \right) = \lim_{N \to \infty} \frac{1}{N d - 2 \log N} \log \mathbb{P} \otimes \mathbb{P} \left( \Omega^+_{N,\sigma} \right) = -2G\text{Cap}(D),$$

and for every $\epsilon > 0$, $N_\epsilon^0(\varphi)/|D_N|$ tends to 0 in probability with respect to $\mathbb{P}(d\varphi|\Omega^+_{N,\sigma})$.

2. Under Hypothesis $H.2–2$, $\mathbb{P}(d\sigma)$–a.s. we have that

$$\lim_{N \to \infty} \frac{1}{N d - 2 \log N)^{1/\beta}} \log \mathbb{P} \left( \Omega^+_{N,\sigma} \right) = \lim_{N \to \infty} \frac{1}{N d - 2 \log N^{1/\beta}} \log \mathbb{P} \otimes \mathbb{P} \left( \Omega^+_{N,\sigma} \right) = -(4Q)^{1/\beta} \text{Cap}(D) \frac{2}{2},$$

(1.17)

and for every $\epsilon > 0$, $N_\epsilon^1(\varphi)/|D_N|$ tends to 0 in probability with respect to $\mathbb{P}(d\varphi|\Omega^+_{N,\sigma})$.

We see therefore that in the sub–Gaussian regime the behavior is not far from the one found in the case of a flat wall (and in fact, under stronger conditions on the law of $\sigma^-$, this part of the result is a direct consequence of the results in [3], see Section 5). But in the super–Gaussian regime the fluctuations of the substrate are dominating: we can say that in this regime the entropic contribution to the phenomenon is, to leading order, coming from $\sigma$, while of course the energy contribution is still coming from the $\varphi$–field and it appears in the capacity term.
1.6. **On the results, on the strategies of proof and possible generalizations.** First of all we try to extend the heuristic ideas that we sketched at the end of Subsection 1.2. We assume H.2, and we start with an observation that seems to suggest that the effect of a random quenched hard wall should be the same of that of a perfectly flat wall: it is an immediate consequence of the results in [1] that if \(|\sigma_x| = o(\sqrt{\log N})\) for every \(x \in D_N\), then one obtains Theorem 1.1 and Theorem 1.2 with \(Q = 0\): that is the phenomenology of the flat wall. The argument is not totally convincing, because under such conditions on \(\sigma\) we are on a set which is \(P\)-negligible (notice that this is not true under H.2 [1]). However one can show that a typical \(\sigma\) is such that for sufficiently large \(N\) the cardinality of \(\{x \in D_N : |\sigma_x| = o(\sqrt{\log N})\}\) is larger than \(|D_N|/(1 - \delta_N)\), for any choice of \(\{\delta_N\}_N\), such that \(\delta_N \searrow 0\) and \(\delta_N N^\epsilon \to \infty\) for every \(\epsilon > 0\). So the game is clearly to understand if large excursions of \(\sigma\), that happen on thin sets, affect the \(\varphi\)-field: quantitatively we observe that, by Hypotheses H.1 and H.2, there are about \(N^{d-(\alpha/2Q)}\) (\(\alpha > 0\)) sites \(x\) on which \(\sigma_x\) is approximately \(\sqrt{\alpha \log N}\). Let us accept that the \(\sigma\)-levels with \(\alpha > 4Q\) do not have any effect (recall the discussion at the end of Subsection 1.2): we remain with all the levels with \(\alpha \in (0,4Q)\). Higher levels in principle affect the \(\varphi\) field more seriously, but they are substantially less than lower levels: and on the other side one can repeat a similar discussion for the \(\varphi\) field. It turns out that the relevant \(\alpha\) is \(2G/\sqrt{G+Q}\) and these levels mostly interact with \(\varphi\)-downward spikes of height \(\approx 2Q/\sqrt{G+Q}\), and to accommodate both \(\sigma\) and \(\varphi\), the \(\varphi\) field translates up to \(\approx \sqrt{4(G+Q) \log N}\). Reasoning this way, the appearance of a final result that depends only on \((G+Q)\) looks quite miraculous.

The quantity \((G + Q)\) appears naturally if we restrict to the case \(\sigma_0 \sim \mathcal{N}(0,0,\mathcal{N}(G + Q\mathcal{I})(.,.))\), with \(\mathcal{I}(x,x) = 1\) and \(\mathcal{I}(x,y) = 0\) if \(x \neq y\). Observe that the long range part of the covariance is still given by the Green function (and this is the part responsible for the appearance of the capacity): the large excursions essentially depend only on the diagonal and this justifies the appearance of \((G + Q)\). This is of course not a proof, but it can be turned into a proof: note that \(\varphi - \sigma\) is an FKG field, see the next subsection, and apply for example the argument in [3, §4] for the lower bound; a proof of the upper bound is given in Section 3. But of course in this case we have solved the annealed model and quenched probabilities may be smaller (Corollary 2.3). We have therefore transferred the problem to the slippery issue of quenched = annealed. We take this occasion to stress that probability estimates can be really viewed as free energy estimates: one can insert the conditioning with respect to \(\Omega_{N,\sigma}^+\) directly in the Hamiltonian, just by adding the site dependent 1–body potential \(V_x(\varphi_x) = \infty \mathbb{1}_{(-\infty,\sigma_x)}(\varphi_x)\).

Let us now address the issue of the necessity of the hypotheses on \(\sigma\) and \(\varphi\):

1. Hypothesis H.1 can be relaxed and the result extended to a large class of mildly correlated fields. However, even in easy cases (like \(\sigma\) a Markov field with exponentially decaying correlations), the extension turns out to be heavy. Moreover one should also observe that the first part of the proof of the probability upper bound (Proposition 3.1) fails for strongly correlated...
fields. As a matter of fact, it is in the class of strongly correlated $\sigma$–fields that we can exhibit examples in which quenched≠annealed (work in progress).

2. Hypothesis H.2 is not necessary to carry out a full analysis, but as we argued, it captures models in which the randomness of wall and interface act on the same scale. Heavier tails (H.2–3) lead to a predominance of the wall randomness. Lighter tails (H.2–1) lead to the phenomenology of the flat wall (to leading order, of course).

3. Hypothesis H.3 should not be needed at all. We believe that imposing, as an extreme case, the hard wall condition only with positive probability should not change the phenomena. However having some a priori lower bound at every site for $\varphi$ under the conditioned field comes really handy.

4. We have chosen the most elementary harmonic crystal to simplify the exposition: essentially nothing changes if we choose $G(\cdot,\cdot)$ to be the Green function of a more general symmetric translation invariant irreducible random walk which performs jumps of finite range $k$. However $\varphi$ in this case is $k$–steps Markov and the conditioning arguments become more cumbersome. Even cases of infinite range jumps can in principle be tackled: but then one has to resort (as in [3]) to hypercontractive estimates, while here we simply play on conditioning (in a way similar to the case treated in [3]).

We conclude this discussion by addressing the question about the optimality of Theorem 1.2. This theorem should be compared with the result (1.8) obtained in the case of a flat wall. Since the models coincide for $Q = 0$ (at least if one chooses $\sigma_0 \sim N(0,Q)$) one naturally suspects that Theorem 1.2 could be improved. While in principle Theorem 1.2 should be improvable, for general $G$ and $Q$ one certainly cannot prove a result like the one we just mentioned for the flat wall. Observe in fact that if $dQ > (G + Q)$ extrema of $\sigma$ field pierce the interface and therefore no uniformity with respect to $x$ is to be expected, at least as long as we consider upper bounds: local (or almost local) deformation of the interface over a sparse lattice of points, the sites of the large excursions of $\sigma$, are necessarily present.

1.7. Overview of the sections and some further notation and preliminaries. In Section 2 the main result is the quenched lower bound on the probability of $\Omega_{N,\sigma}^+$. The annealed bound follows then by a standard argument (that we detail in Corollary 2.5). In Section 3 we take the opposite route: the main result is an annealed upper bound, from which the quenched upper bound follows. Therefore the proof of Theorem 1.1 follows from Proposition 2.1, Corollary 2.3, Proposition 3.1 and Corollary 3.2.

In Section 4 we present the proof of Theorem 1.2 (which is the combination of Proposition 4.6 and Proposition 4.9, along with some other results, see in particular Remark 4.8).

In Section 5 we sketch the proof of Theorem 1.3.
An important role is played by the FKG (Fortuin–Kasteleyn–Ginibre) inequality (or positive association property): if \( E, F \subset \mathbb{R}_Z^d \) are two increasing events \( (E \text{ is increasing if } \phi \in E \text{ implies that } \phi + \psi \in E \text{ for every } \psi \in [0, \infty)^Z) \) then \( \mathbb{P}(E \cap F) \geq \mathbb{P}(E)\mathbb{P}(F) \). Positively correlated Gaussian fields satisfy the FKG inequality: of this fact there exist several proofs (see for example [14]).

We conclude with some notations: \( \lfloor \cdot \rfloor \) denotes the integer part of the positive real number \( \cdot \). Unless otherwise stated, \( o(1) \) is always considered with respect to \( N \to \infty \) (and no uniformity should be assumed with respect to other parameters which may be present). With standard notation we set \( \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)dz \). We keep \( | \cdot | \) to denote the Euclidean norm, or the absolute value in the one–dimensional case: if we write \( \| r \| \) we mean \( \max_{i=1, \ldots, d} | r_i | \), \( r \in \mathbb{R}^d \). For \( A \subset \mathbb{Z}^d \) we denote by \( \mathcal{F}^\sigma_{\sigma, \phi} \) the \( \sigma \)–algebra generated by \( \sigma_x \) and \( \phi_x, x \in A \), and \( \mathcal{F}^{\sigma}_{\sigma} \) is the \( \sigma \)–algebra generated by the \( \sigma \)–variables indexed by \( A \), and analogous for \( \phi \). Moreover if \( A \) is missing in this notation, it means \( A = \mathbb{Z}^d \).

2. Probability lower bounds: quenched (and annealed) estimates

In this section we work under the hypotheses H.1 and H.2. The main result that we prove is the following:

**Proposition 2.1.** \( \mathbb{P}(d\sigma) \)-a.s.

\[
\liminf_{N \to \infty} \frac{1}{Nd^2 \log N} \log \mathbb{P} \left( \Omega_{N, \sigma}^+ \right) \geq -2(G + Q)\text{Cap}(D). \tag{2.1}
\]

**Proof.** For every choice of a large integer parameter \( k \) we define the auxiliary field \( \tilde{\sigma} \) by setting \( \theta = \sqrt{4Q(1 + (1/2k))} / k \), \( \tilde{k} = \lfloor (\sqrt{2(d + 2)Q}) / \theta \rfloor \) and

\[
\tilde{\sigma}_x = \begin{cases} 
\theta \sqrt{\log N}, & \text{if } \sigma_x \leq \theta \sqrt{\log N}, \\
 k\theta \sqrt{\log N}, & \text{if } \sigma_x \in (\theta \sqrt{\log N}, k\theta \sqrt{\log N}) \text{ for } k = 2, 3, \ldots, \tilde{k}, \\
 \tilde{k}\theta \sqrt{\log N}, & \text{if } \sigma_x \in (k\theta \sqrt{\log N}, \tilde{k}\theta \sqrt{\log N}] \\
 \infty & \text{if } \sigma_x > \tilde{k}\theta \sqrt{\log N},
\end{cases} \tag{2.2}
\]

and set \( L_N(k) = \{ x \in D_N : \tilde{\sigma}_x = k\theta \sqrt{\log N} \} \) for \( k \in \{1, 2, \ldots, \tilde{k}, \infty\} \).

Let us now select a good \( \sigma \)–set. Call \( N_k \) the cardinality of the random set \( L_N(k) \). First we define \( G_N \in \sigma(\sigma_x : x \in D_N) \) as the event specified by

\[
|N_k - \mathbb{E}[N_k]| \leq \frac{\mathbb{E}[N_k]}{2}, \quad \text{for } k = 2, 3, \ldots, \tilde{k}, \tilde{k} \tag{2.3}
\]

and by

\[
N_\infty = 0. \tag{2.4}
\]

The good \( \sigma \)–set is \( (G_N, \text{ev}) = \bigcup_{N} \bigcap_{k \geq N} G_k \).
Lemma 2.2. $P((\mathcal{G}_N, ev)) = 1.$

Proof of Lemma 2.2. Set $p_N^k = P(\sigma_0 \in ((k - 1)\theta \sqrt{\log N}, k\theta \sqrt{\log N}))$ and $f_N(k) = N^{d-((k-1)\theta^2/(2Q))}$ for $k = 2, 3, \ldots, \tilde{k};$ $p_N^k = P(\sigma_0 \in (\tilde{k}\theta \sqrt{\log N}, k\theta \sqrt{\log N}))$ and $f_N(k) = N^{d-(\tilde{k}\theta^2/(2Q))}$. Then $E[N_k] = |D_N|p_N^k$ and $\var_P(N_k) = |D_N|p_N^k(1 - p_N^k)$, therefore by assumption H.2 for every $\epsilon > 0$ we have that

$$\lim_{N \to \infty} N^{-\epsilon} \left( \frac{E[N_k]}{f_N(k)} \right) = 0 \quad \text{and} \quad \lim_{N \to \infty} \frac{\var_P(N_k)}{E[N_k]} = 1,$$

for $k = 2, 3, \ldots, \tilde{k}, \tilde{k}$.

We use the following inequality due to G. Bennett (cf. [2]) that says that if $\{X_j\}_{j=1,2,\ldots}$ is a collection of centered IID variables such that $|X_1| \leq 1$, then for every $t \geq 0$

$$P \left( \left| \sum_{j=1}^{n} X_j \right| > t \right) \leq 2 \exp \left( -\frac{t^2}{2\var(X_1) + 2t/3} \right).$$

Therefore for $k = 2, 3, \ldots, \tilde{k}, \tilde{k}$

$$P \left( \left| N_k - E[N_k] \right| > \frac{E[N_k]}{2} \right) \leq 2 \exp \left( -\frac{(E[N_k])^2}{8\var_P(N_k) + 4E[N_k]/3} \right),$$

and applying (2.5) we obtain that for every sufficiently large $\tilde{k}$ there exists $c > 0$ such that for $k = 2, 3, \ldots, \tilde{k}, \tilde{k}$

$$P \left( \left| N_k - E[N_k] \right| > \frac{E[N_k]}{2} \right) \leq 2 \exp(-cE[N_k]) \leq 2 \exp \left( -cN^{(d-2)/2} \right).$$

Moreover by direct computation $P(N_{\infty} > 0) \leq N^{-3/2}$ for sufficiently large $N$. The first Lemma of Borel–Cantelli completes the proof. □

(Lemma 2.2)

From now on we simply assume that $\sigma \in (\mathcal{G}_N, ev)$. So, in particular, $L_N(\infty) = \emptyset$ and for every $\epsilon > 0$ there exists $\overline{N}$ such that for $N \geq \overline{N}$

$$N_k = |L_N(k)| \leq N^{d-((k-1)\theta^2/(2Q)) + \epsilon}, \quad k = 1, 2, \ldots, \overline{k},$$

$$N_{\overline{k}} = |L_N(\overline{k})| \leq N^{d-(\overline{k}\theta^2/(2Q)) + \epsilon},$$

(notice that the result is trivial for $k = 1$).

Let us go back to the analysis of the $\varphi$ field: we have that

$$P(\varphi_x \geq \sigma_x, \ x \in D_N) \geq P(\varphi_x \geq \tilde{\sigma}_x, \ x \in D_N) \geq P(\varphi_x \geq \tilde{\sigma}_x, \ x \in D_{\overline{N}}) \cdot P(\overline{\Omega}_N^+(\overline{k})),$$

in which the first step is immediate consequence of $\tilde{\sigma} \geq \sigma$ and in the second one we used the FKG inequality with the notations $\overline{\Omega}_N^+(\overline{k}) := \{\varphi : \varphi_x \geq \overline{k}\theta \sqrt{\log N}, x \in L_N(\overline{k})\}$ and $D_{\overline{N}} = D_N \setminus L_N(\overline{k})$. 

""
Therefore
\[
\liminf_{N \to \infty} \frac{1}{N^{d-2} \log N} \log \mathbb{P}\left( \Omega_{N, \sigma}^{+} \right) \geq \liminf_{N \to \infty} \frac{1}{N^{d-2} \log N} \log \mathbb{P}\left( \tilde{\Omega}_{N}^{+}(\tilde{k}) \right) + \liminf_{N \to \infty} \frac{1}{N^{d-2} \log N} \log \mathbb{P}\left( \varphi_{x} \geq \tilde{\sigma}_{x}, \ x \in D_{N}^{-} \right). \tag{2.11}
\]

The following straightforward entropy estimate deals with the first term in the right-hand side of the above expression: let us introduce the map \(T_{\psi}: (T_{\psi} \varphi)_{x} = \varphi_{x} + \psi_{x}, \) for \(\psi \in \Omega_{N} \) and \(x \in \mathbb{Z}^{d}.\)

If \(\mu\) and \(\nu\) are two probability measures defined on the same measurable space and if \(\mu\) is absolutely continuous with respect to \(\nu\) we denote by \(H(\mu|\nu)\) the relative entropy \(E_{\mu}[\log(d\mu/d\nu)].\) If we choose \(\psi_{x} = \sqrt{2G(d+2)} \log N + \kappa \theta \sqrt{\log N}\) for \(x \in L_{N}(\tilde{k})\) and \(\psi_{x} = 0\) otherwise, by direct computation
\[
H \left( \mathbb{P}T_{\psi}^{-1}[\mathbb{P}] \right) = \frac{1}{4d} \sum_{x,y:|x-y|=1} (\psi_{x} - \psi_{y})^{2} \leq \left( \sqrt{2G(d+2)} \log N + \kappa \theta \sqrt{\log N} \right)^{2} |L_{N}(\tilde{k})|. \tag{2.12}
\]

By (2.3) and (2.5) we therefore have that for \(N\) and \(\mathcal{F}\) sufficiently large
\[
H \left( \mathbb{P}T_{\psi}^{-1}[\mathbb{P}] \right) \leq N^{d-2(1+(1/\mathcal{F}))^{2}}. \tag{2.13}
\]

Moreover by using the FKG inequality we obtain that for sufficiently large \(N\)
\[
\mathbb{P}T_{\psi}^{-1}\left( \tilde{\Omega}_{N}^{+}(\tilde{k}) \right) \leq \mathbb{P}\left( \varphi_{x} \geq -\sqrt{2G(d+2)} \log N \right. \text{ for every } x \in L_{N}(\tilde{k})
\begin{equation}
\geq \prod_{x \in L_{N}(\tilde{k})} \mathbb{P}\left( \varphi_{x} \geq -\sqrt{2G(d+2)} \log N \right) \geq \left(1 - (1/N^{d+1})\right)^{N^{d}} \geq 1/2, \tag{2.14}
\end{equation}

and therefore by applying the standard entropy inequality
\[
\log \left( \frac{P_{\nu}(E)}{P_{\mu}(E)} \right) \geq -\frac{1}{P_{\mu}(E)} \left( H(\mu|\nu) + e^{-1} \right), \tag{2.15}
\]
in which \(\mu\) and \(\nu\) are two probabilities and \(E\) is an event of positive \(\mu\) measure, we obtain that
\[
\mathbb{P} \left( \tilde{\Omega}_{N}^{+}(\tilde{k}) \right) \geq \exp \left( -N^{d-2(1+(1/\mathcal{F}))^{2}} \right), \tag{2.16}
\]
for sufficiently large \(N\), which shows that the first term in the right-hand side of (2.11) vanishes.

Let us therefore concentrate on the second term and on the event \(\tilde{\Omega}_{N}^{+} = \{ \varphi_{x} \geq \tilde{\sigma}_{x}, \ x \in D_{N}^{-} \} \): the proof of Proposition 2.1 is complete once we have shown that
\[
\liminf_{N \to \infty} \frac{1}{N^{d-2} \log N} \log \mathbb{P}\left( \tilde{\Omega}_{N}^{+} \right) \geq -2(G + Q)\text{Cap}(D). \tag{2.17}
\]

In order to prove (2.17) we set \(\alpha_{N} = \alpha \sqrt{\log N}, \alpha > 0.\) For \(\psi_{N} \in \Omega_{N} \) such that \((\psi_{N})_{x} = \alpha_{N} f(x/N), \ f \in C_{0}^{\infty}(\mathbb{R}^{d}; [0, \infty)) \) and \(f(r) = 1\) if \(r \in D,\) we set \(\mathbb{P}_{N} = \mathbb{P}T_{\psi_{N}}^{-1}\) and \(\tilde{\mathbb{P}}_{N}(\cdot) = \mathbb{P}_{N}(\cdot; \Omega_{N}^{+}).\) Therefore \(d\mathbb{P}_{N}/d\mathbb{P} = (d\mathbb{P}_{N}/d\mathbb{P}_{N})(d\mathbb{P}_{N}/d\mathbb{P})\) and by the entropy inequality (2.15)
\[
\log \mathbb{P}\left( \tilde{\Omega}_{N}^{+} \right) \geq -H \left( \tilde{\mathbb{P}}_{N}[\mathbb{P}] \right) - e^{-1}
= -H \left( \tilde{\mathbb{P}}_{N}[\mathbb{P}] \right) - \mathbb{E}_{N} \left( \log \left( \frac{d\mathbb{P}_{N}}{d\mathbb{P}} \right) \right) - e^{-1} \equiv -H_{1} - H_{2} - e^{-1}. \tag{2.18}
\]
First of all by direct evaluation and FKG we have

\[
H_1 = -\log \mathbb{P}_{T_{\psi_N}}(\widehat{\Omega}_N^+) \leq - \sum_{x \in D_N^+} \log \mathbb{P}(\varphi_x \geq \tilde{\sigma}_x - \alpha_N) \\
= - \sum_{k=1}^{\overline{r}} |L_N(k)| \log \left(1 - \mathbb{P}\left(\varphi_0 < k\theta \sqrt{\log N} - \alpha_N\right)\right). \tag{2.19}
\]

One checks directly that if

\[
\alpha > \overline{r}\theta, \tag{2.20}
\]

and

\[
\frac{(k - 1)^2 \theta^2}{2Q} + \frac{(k\theta - \alpha)^2}{2G} > 2, \tag{2.21}
\]

for all \( k \leq \overline{r} \), then each of the \( \overline{r} \) summands in (2.19) is \( o(N^{d-2}) \), and therefore negligible:

\[
\lim_{N \to \infty} \frac{1}{N^{d-2}} H_1 = 0. \tag{2.22}
\]

Observe that, by (2.20) and (2.21), a more explicit assumption that implies (2.22) is

\[
\alpha > k\theta + \sqrt{4G - (k - 1)^2 \theta^2 \left(\frac{G}{Q}\right)}, \text{ for every } k \leq \overline{r}. \tag{2.23}
\]

Note that \( 4G - (k - 1)^2 \theta^2 (G/Q) \geq 0 \) holds for every \( k \leq \overline{r} \) using the definition of \( \theta \). If we observe that

\[
\max_{1 \leq k \leq \overline{r}} k\theta + \sqrt{4G - (k - 1)^2 \theta^2 \left(\frac{G}{Q}\right)} \leq \theta + \max_{x \in [0, 2\sqrt{G}]} \left\{ x + \sqrt{4G - x^2 \left(\frac{G}{Q}\right)} \right\} \leq \theta + 2\sqrt{G + Q}, \tag{2.24}
\]

we conclude that (2.23) is satisfied if

\[
\alpha > 2\sqrt{G + Q} + \theta. \tag{2.25}
\]

Therefore under this hypothesis on \( \alpha \) the estimate (2.22) holds.

Let us consider \( H_2 \): observe that

\[
\log \left(\frac{d\mathbb{P}_N}{d\mathbb{P}}(T_{\psi_N} \varphi)\right) = \frac{1}{4d} \sum_{x,y:|x-y|=1} \left(2\alpha_N(f(y/N) - f(x/N))(\varphi_x - \varphi_y) + (\alpha_N(f(x/N) - f(y/N)))^2\right), \tag{2.26}
\]
and therefore
\[ \frac{1}{N^{d-2} \log N} H_2 = \frac{\alpha_N^2}{4dN^d \log N} \sum_{x,y:|x-y|=1} (N(f(x/N) - f(y/N)))^2 \]
\[ + \frac{\alpha_N}{2dN^{d-2} \log N} \mathbb{E} \left[ \sum_{x,y:|x-y|=1} (f(y/N) - f(x/N))(\varphi_x - \varphi_y) \right] \left[ T_{\psi_N}^{-1} \Omega^+_N \right] \]
\[ \equiv C_N + R_N. \]

It is easy to see that $C_N$ converges for $N \to \infty$ to $\alpha^2 \| \partial f \|_2^2 / (4d)$. We show now that $\lim_{N \to \infty} R_N = 0$ if (2.25) holds. Observe in fact that
\[ \frac{1}{2d} \sum_{x,y:|x-y|=1} (f(x/N) - f(y/N))(\varphi_x - \varphi_y) = - \sum_x (\Delta f(x/N))(\varphi_x) \sim N(0, \sigma_N^2), \]
where $\sigma_N^2 = (\| \partial f \|_2^2 / 2d)N^{d-2}(1+o(1))$. We use now the following consequence of Jensen inequality ($Y$ a random variable, $E$ a positive probability event, $t > 0$)
\[ \mathbb{E} |Y| \leq \frac{1}{t} \log \mathbb{E} [\exp(tY)] - \frac{1}{t} \log \mathbb{P}(E), \]
to obtain with $t = N^{d-2}$ (recall (2.19)) that
\[ |R_N| \leq \frac{1}{t} \log \mathbb{E} \left[ \exp \left( \frac{t\alpha_N}{2dN^d \log N} \sum_{x,y} (f(y/N) - f(x/N))(\varphi_x - \varphi_y) \right) \right] - \frac{1}{t} \log \mathbb{P} \left( T_{\psi_N}^{-1} \Omega^+_N \right) \]
\[ \leq \frac{\alpha^2 \sigma_N^2}{2(2d)^2 N^{d-2} \log N} + \frac{H_1}{N^{d-2}} = o(1). \]

This shows that under the hypothesis (2.25) on $\alpha$
\[ \lim_{N \to \infty} \frac{1}{N^{d-2} \log N} H_2 = \frac{\alpha^2 \| \partial f \|_2^2}{2d}. \]

Since $\mathbb{P}_N(\Omega_N^+)^p = 1$ we may apply the entropy inequality (2.15), and (2.27) is obtained by optimising the choices of $f$ and $\alpha$, by the definition of the capacity (cf. (1.4), first line) and using the fact that $\theta$ can be taken arbitrarily small (that is, $\mathbb{E}$ arbitrarily large).

**Remark 2.3.** One may wonder if a more general estimate like the one proven in [2, §2] holds in this case too. The answer is positive: Proposition 2.1 can be extended in the sense that if $\{b_N\}_{N \in \mathbb{N}}$ is a sequence of real numbers such that $\lim_{N \to \infty} b_N / \sqrt{N} = b \geq -2\sqrt{G+Q}$, then $\mathbb{P}(d\sigma)$–a.s.
\[ \lim_{N \to \infty} \frac{1}{N^{d-2} \log N} \log \mathbb{P}(\varphi_x \geq \sigma_x + b_N \text{ for every } x \in D_N) \geq - \left(2\sqrt{G+Q+b}\right)^2 \text{Cap}(D)/2. \]

We do not give a proof of this result, except for the extreme case $b = -2\sqrt{G+Q}$, which follows immediately from Lemma 2.4 below.
In Section 4, the proof of an upper bound on the height of the conditioned field requires the following technical estimate.

**Lemma 2.4.** For every $\epsilon > 0$, if we choose $\alpha_N = (2\sqrt{G + Q} + \epsilon)\sqrt{\log N}$ we have that
\[
\lim_{N \to \infty} \frac{1}{N^{d-2}} \log P \left( \Omega_{N,\sigma}^{+} - \alpha_N \right) = 0.
\] (2.33)

**Proof.** It is a simplified version of the preceding proof: let us keep the same notations. As for (2.11) we have
\[
\lim_{N \to \infty} \frac{1}{N^{d-2}} \log P \left( \Omega_{N,\sigma}^{+} - \alpha_N \right) \geq \lim_{N \to \infty} \frac{1}{N^{d-2}} \log P \left( \varphi_x \geq \bar{\sigma}_x - \alpha_N, \ x \in D_N^{-} \right) + \lim_{N \to \infty} \frac{1}{N^{d-2}} \log P \left( \left\{ \varphi_x \geq k\theta \sqrt{\log N} - \alpha_N, x \in L_N(\tilde{k}) \right\} \right). \tag{2.34}
\]

The second term on the right-hand side of (2.34) is not smaller than the first term in the right-hand side of (2.11) and it is therefore equal to zero. The first term is dealt by applying the FKG inequality very much in the spirit of (2.19):
\[
\frac{1}{N^{d-2}} \log P \left( \varphi_x \geq \bar{\sigma}_x - \alpha_N, \ x \in D_N^{-} \right) \geq \sum_{k=1}^{\tilde{k}} |L_N(k)|N^{2-d} \log \left( 1 - P \left( \varphi_0 < k\theta \sqrt{\log N} - \alpha_N \right) \right). \tag{2.35}
\]

As for (2.19), the term on the right-hand side of (2.33) vanishes as $N \to \infty$ if
\[
\frac{(k - 1)^2 \theta^2}{2Q} + \frac{(k\theta - 2\sqrt{G + Q} - \epsilon)^2}{2G} > 2, \tag{2.36}
\]
for every $k \leq \tilde{k}$. But this is true as long as $\epsilon > \theta$: since $\theta$ can be chosen arbitrarily small, we are done. \qed

We conclude this section by observing that the quenched lower bound provides also an annealed lower bound.

**Corollary 2.5.**
\[
\liminf_{N \to \infty} \frac{1}{N^{d-2} \log N} \log P \otimes P \left( \Omega_{N,\sigma}^{+} \right) \geq -2(G + Q)\text{Cap}(D). \tag{2.37}
\]

**Proof.** Since by Proposition 2.1 for every $\epsilon > 0$ we have that the $P$ probability that $P(\Omega_{N,\sigma}^{+}) \geq \exp(-2(G + Q + \epsilon)\text{Cap}(D)N^{d-2} \log N)$ is larger than $1/2$ for sufficiently large $N$, the result is immediate. \qed
3. Probability upper bounds: annealed (and quenched) estimates

In this section we need also some control on the downward tails of the \( \sigma \) field. We recall that in this section we commit abuse of notation for \( \Omega_{\Lambda,\sigma}^+ \).

**Proposition 3.1.** Under hypotheses \( H[4] \), \( H[5] \) and \( H[6] \) we have that

\[
\limsup_{N \to \infty} \frac{1}{N^{d-2}} \log N \, \mathbb{P} \otimes \mathbb{P} \left( \Omega_{N,\sigma}^+ \right) \leq -2 (G + Q) \operatorname{Cap}(D). \tag{3.1}
\]

**Proof.** Let us choose an even natural number \( L \) and for \( y \in 2L \mathbb{Z}^d \) let us set

\[
B(y) = B_L(y) = \left\{ x : \max_{i=1,\ldots,d} |x_i - y_i| = L/2 \right\},
\]

and \( \Lambda_c \) is the set of \( y \in 2L \mathbb{Z}^d \) such that \( B(y) \subset D_N \). Set also \( \Lambda = \bigcup_{y \in \Lambda_c} B(y) \). We have that

\[
\mathbb{P} \otimes \mathbb{P} \left( \Omega_{N,\sigma}^+ \right) \leq \mathbb{P} \otimes \mathbb{P} \left( \Omega_{\Lambda_c,\sigma}^+ \right) = \mathbb{E} \otimes \mathbb{E} \left[ \prod_{y \in \Lambda_c} \mathbb{P} \otimes \mathbb{P} \left( \varphi_y \geq \sigma_y \mid \mathcal{F}_B(y) \right) ; \Omega_{\Lambda,\sigma}^+ \right], \tag{3.3}
\]

in which we used the Markov property of the \( \varphi \)-field and the independence of the \( \sigma \)-field. Observe now that, under \( \mathbb{P} \otimes \mathbb{P} (\cdot | \mathcal{F}_{B(y)}^\varphi)(\psi) \), \( \varphi_y \sim \mathcal{N}(\sum_{z \in B(y)} q(z) \psi_z, G_L) \) where \( q(z) = q_L(z) \) is the probability that a simple random walk leaving at \( y \) hits \( B(y) \) at \( z \) and \( G_L \) is a positive number with the property that \( G_L^\varphi \sim G \) as \( L \to \infty \). We set \( M_y^\varphi(\psi) = \sum_{z \in B(y)} q(z) \psi_z \).

We now take a positive number \( \kappa \) and we consider the inner \( \kappa \)-discretization of \( D \): that is for \( r \in \kappa \mathbb{Z}^d \), set \( A_r = r + [0, \kappa)^d \) and define \( I = \{ r \in \kappa \mathbb{Z}^d : A_r \subset D \} \) (assume \( I \neq \emptyset \)). We are interested in this decomposition at the lattice level or, more precisely, on the \( 2L \)-rarified lattice level (the sublattice \( \Lambda_c \) of centers): so define \( C_r = NA_r \cap \Lambda_c \) and remark that \( |C_r| = c(N\kappa/2L)^d (1 + o(1)) \).

For \( \eta \in (0, 1/4) \) and \( \alpha \in (0, 4(G_L + Q)) \) let us now consider the event

\[
E_{\eta,\alpha} = \left\{ (\sigma, \varphi) : \text{there exists } r \in I \text{ such that } |\{ y \in C_r : M_y^\varphi(\varphi) \leq \sqrt{\alpha \log N} \}| \geq \eta |C_r| \right\}. \tag{3.4}
\]

Of course \( E_{\eta,\alpha} \) is \( \mathcal{F}_\varphi \)-measurable. Observe that on \( E_{\eta,\alpha} \)

\[
\prod_{y \in \Lambda_c} \mathbb{P} \otimes \mathbb{P} \left( \varphi_y \geq \sigma_y \mid \mathcal{F}_{B(y)}^\varphi \right)
\leq \prod_{y \in \Lambda_c} \left( 1 - \mathbb{P} \left( \varphi_y \leq - \frac{G_L}{G_L + Q} \sqrt{\alpha \log N} \right) \mathbb{P} \left( \sigma_y \geq \frac{Q}{G_L + Q} \sqrt{\alpha \log N} \right) \right)
\leq \left( 1 - N^{-\frac{\alpha + \epsilon}{2(G_L + Q)}} \right)^{|C_r|}, \tag{3.5}
\]

where \( r \) is any element of \( I \) and \( \epsilon \in (0, 4(G_L + Q) - \alpha) \). Then for sufficiently large \( N \) and a suitable constant \( \epsilon' \) we have that

\[
\mathbb{E} \otimes \mathbb{E} \left[ \prod_{y \in \Lambda_c} \mathbb{P} \otimes \mathbb{P} \left( \varphi_y \geq \sigma_y \mid \mathcal{F}_{B(y)}^\varphi \right) ; E_{\eta,\alpha} \right] \leq \left( 1 - N^{-\frac{\alpha + \epsilon'}{2(G_L + Q)}} \right)^{c\eta (N\kappa/L)^d}
\leq \exp \left( - \epsilon' N^{d-\frac{\alpha + \epsilon'}{2(G_L + Q)}} \right), \tag{3.6}
\]

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which is negligible (recall (3.1) and the choice of $\epsilon$ and $\alpha$).

Let us consider now the event

$$ E_N = \left\{ (\sigma, \varphi) : \left\{ y \in \Lambda_c : M_y^\varphi(\sigma) \leq -(\log N)^{1/4} \right\} \geq \delta_N |\Lambda_c| \right\}, $$

with $\delta_N = \max(\sqrt{\mathbb{P}(M_y^\varphi(\sigma) \leq -(\log N)^{1/4})}, N^{-1})$. By H.3 and the Markov inequality, $\delta_N$ vanishes as $N \to \infty$. A direct application of (2.6) leads to the existence of a constant $c > 0$ such that for every $N$

$$ \mathbb{P} \otimes \mathbb{P}(E_N) \leq c \exp(-\delta_N |\Lambda_c|). $$

which again is negligible, in view of the result we are after (cf. (3.1)). We observe that on $E_N^c$ we may select a set (depending on $\sigma$ and $N$) $\Lambda_c^G \subset \Lambda_c$ with the property that $|\Lambda_c^G|/|\Lambda_c| \geq (1 - \delta_N)$, on which $M_y^\varphi(\sigma) > -(\log N)^{1/4}$: this implies that if we define $C_r^G = C_r \cap \Lambda_c^G$ we can find a positive constant $c = c(\kappa, D)$ such that $|C_r^G|/|C_r| \geq (1 - \delta_N) \lor 0$ for every $r$. We choose $N$ such that $c\delta_N < \eta$.

By the last two observations ((3.6) and (3.8)), we are allowed to replace the event $\Omega_{\Lambda, \sigma}^+$ with $\Omega_{\Lambda, \sigma}^+ \cap E_{\eta, \alpha}^c \cap E_N^c$ in the rightmost expression in (3.3). If $(\sigma, \varphi) \in \Omega_{\Lambda, \sigma}^+ \cap E_{\eta, \alpha}^c \cap E_N^c$ then for every $r \in I$ there are at least $(1 - 2\eta)|C_r^G|$ sites $y \in C_r^G$ such that $M_y^\varphi(\varphi - \sigma) \geq 0$ and $M_y^\varphi(\sigma) > -(\log N)^{1/4}$. Therefore for every choice of $f_r \geq 0$, $r \in I$,

$$ \sum_{r \in I} f_r \frac{1}{|C_r^G|} \sum_{y \in C_r^G} M_y^\varphi(\varphi) \geq (1 - 3\eta)\sqrt{\alpha \log N} \sum_{r \in I} f_r. $$

Therefore if we call $F_N$ the event specified by $(\sigma, \varphi)$ such that (3.9) holds, we have shown that

$$ \limsup_{N \to \infty} \frac{1}{N^{d-2} \log N} \mathbb{P} \otimes \mathbb{P}(\Omega_{\Lambda, \sigma}^+) \leq \limsup_{N \to \infty} \frac{1}{N^{d-2} \log N} \log \mathbb{P} \otimes \mathbb{P}(F_N \cap E_N^c). $$

In order to deal with Gaussian computations we condition with respect to $\sigma$: notice that $E_N$ is measurable with respect to $\mathcal{F}^\sigma$. We have that on $E_N^c$

$$ \mathbb{P} \otimes \mathbb{P}(F_N | \mathcal{F}^\sigma) \leq \exp \left( - \frac{(1 - 3\eta)^2 \alpha \log N \left( \sum_{r \in I} f_r \right)^2}{2 \var_{\mathbb{P}} \left( \sum_{r \in I} f_r \frac{1}{|C_r^G|} \sum_{y \in C_r^G} M_y^\varphi(\varphi) \right) \sum_{r \in I} f_r \frac{1}{|C_r^G|} \sum_{y \in C_r^G} \var_{\mathbb{P}}(\varphi_y) \right). $$

We estimate the variance with respect to the Gaussian measure $\mathbb{P} \otimes \mathbb{P}(d\varphi | \mathcal{F}^\sigma)$: by Jensen’s inequality

$$ \var \left( \sum_{r \in I} f_r \frac{1}{|C_r^G|} \sum_{y \in C_r^G} M_y^\varphi(\varphi) \right) \leq \var_{\mathbb{P}} \left( \sum_{r \in I} f_r \frac{1}{|C_r^G|} \sum_{y \in C_r^G} \varphi_y \right), $$

and observe that, if we define the function $f_\kappa : \mathbb{R}^d \to \mathbb{R}$ as $f_\kappa(x) = \sum_{r \in I} f_r \mathbb{I}_{A_r}(x)$, we can write

$$ \sum_{r \in I} f_r \frac{1}{|C_r^G|} \sum_{y \in C_r^G} \varphi_y = \frac{1}{|C_r^G|} \sum_{y \in \Lambda_c^G} \gamma_N(y) f_\kappa(y/N) \varphi_y, $$

where $\gamma_N(y) = \sum_{r \in I} f_r \mathbb{I}_{A_r}(y/N)$. For $y \in \Lambda_c^G$, we have $\gamma_N(y) \leq \gamma_N(x)$ for any $x \in A_r$, and therefore $\gamma_N(y)$ is bounded on compact sets. By (3.12), (3.13) and the proposed function $f_\kappa$, one has the claimed result.
where $\gamma_N(y) = |C_r|/|C^G_{r(y)}|$, with $r(y)$ the index of the $C$–box that contains $y$. Note that $\gamma_N(y)$ is a $\sigma$–dependent function, but, uniformly in $\sigma$, $1 \leq \gamma_N(y) \leq 1/(1 - c\delta_N)$ for every $r$ and every sufficiently large $N$ (chosen before). Observe also that

$$\sum_{r \in I} f_r = \frac{1}{|C_r|} \sum_{y \in \Lambda_c} f_\kappa(y/N).$$

(3.14)

By direct computation we obtain that

$$\lim_{N \to \infty} \frac{1}{N^{d-2}\log N} \mathbb{E} \left( \sum_{y \in \Lambda_c} f_\kappa(y/N) \varphi_y \right)^2 = \frac{(\int_D f_\kappa(x) dx)^2}{\int_D \int_D f_\kappa(x) f_\kappa(x') R_d |x - x'|^{-d+2} dx dx'} \equiv C(f_\kappa),$$

(3.15)

where $R_d$ is defined in (1.7). We stress once again that the variance appearing on the left–most term of (3.15) depends on $\sigma$: but for $\sigma \in E^G_N$ and fixed $f$ this convergence is uniform. This tells us that for every $\epsilon > 0$ and for every sufficiently large $N$

$$\sup_{\sigma \in E^G_N} \mathbb{P} \otimes \mathbb{P} (F_\sigma) (\sigma) \leq \exp \left( -N^{d-2}\log N \left( (1 - 4\eta)^2 \alpha C(f_\kappa) + \epsilon \right) /2 \right),$$

(3.16)

and recalling (3.10) we obtain

$$\limsup_{N \to \infty} \frac{1}{N^{d-2}\log N} \log \mathbb{P} \otimes \mathbb{P} (\Omega^+_{N,\sigma}) \leq -(1 - 4\eta)^2 \alpha C(f_\kappa)/2.$$  

(3.17)

We can then let $\alpha \not\to 4(G_L + Q)$, $L \not\to \infty$, $\eta \not\to 0$ and $\kappa \not\to 0$ and optimise over the choice of $f_\kappa$, which is now any function which is piecewise constant over an arbitrarily thin regular grid and equal to zero outside $D$. By the second line in (3.10), the capacity of $D$ appears and we are done.

$$\square$$

We complete this section by observing that one can extract from an annealed upper bound on the probability of $\Omega^+_{N,\sigma}$ a quenched upper bound. The annealed upper bound is provided in Proposition 3.1.

**Corollary 3.2.** Assume $H_1$, $H_2$ and $H_3$, we have that

$$\limsup_{N \to \infty} \frac{1}{N^{d-2}\log N} \log \mathbb{P} \left( \Omega^+_{N,\sigma} \right) \leq -2(G + Q)\text{Cap}(D), \quad \mathbb{P}(d\sigma) – a.s.$$

(3.18)

**Proof.** Let $X_N(\sigma)$ be the random variable $\log \mathbb{P}(\Omega^+_{N,\sigma})/N^{d-2}\log N$ and choose $\ell = 2(G + Q)\text{Cap}(D) - \epsilon, \epsilon > 0$. By Markov inequality we have that

$$\frac{1}{N^{d-2}\log N} \log \mathbb{P} (X_N \geq -\ell) \leq \ell + \frac{1}{N^{d-2}\log N} \log \mathbb{P} \otimes \mathbb{P} (\Omega^+_{N,\sigma}).$$

(3.19)

Taking now the lim sup on both sides, we get that for sufficiently large $N$

$$\mathbb{P}(X_N \geq -\ell) \leq \exp(-\epsilon N^{d-2}\log N).$$

(3.20)
Thus by Borel-Cantelli I, for all $\epsilon > 0$,
\[ P(X_N \geq -2(G + Q)\text{Cap}(D) + \epsilon \text{ i.o.}) = 0, \]  
whence the thesis. \hfill \Box

4. Entropic repulsion

This section is devoted to the proof of Theorem 1.2. It is roughly split into two parts (lower and upper bounds) even if some of the arguments require both upper and lower estimates at the same time.

4.1. Lower bounds. We need the following preliminary result on the hitting probabilities of the simple random walk on $\mathbb{Z}^d$. We denote by $\{X^x_j\}_{j \geq 0}$ the simple random walk for which $X^x_0 = x$, and its law by $P^x$.

**Lemma 4.1.** For any positive integer $n$ let $S_n = \{y \in \mathbb{Z}^d : |y| \geq n\}$ and there exists $x \in \mathbb{Z}^d$ such that $|x| < n$ and $|y - x| = 1\}$, $\tau_x = \inf\{j \geq 0 : X^x_j \in S_n\}$ and
\[ H(x, y) = P^x(\tau_x = y), \]  
for $|x| < n$ and $y \in S_n$. Then there exist $c_1, c_2, c_3 > 0$ such that for every $\epsilon \in (0, 1/4)$
\[ c_1 n^{1-d} \leq H(x, y) \leq c_2 n^{1-d}, \]  
\[ |H(x, y) - H(x', y)| \leq c_3 \epsilon n^{1-d}, \]  
for every $x, x'$ such that $\|x\| \vee \|x'\| \leq \epsilon n$ and every $y \in S_n$.

**Proof.** In [15, Lemma 1.7.4] it is shown that
\[ c_1 n^{1-d} \leq H(0, y) \leq c_2 n^{1-d}, \]  
and from the proof of Theorem 1.7.1 in [15], where the author proves that for every fixed $u$ one can find a positive constant $c$ such that $|H(u, y) - H(0, y)| \leq c u O(n^{-1}) n^{1-d}$, it is not difficult to see that one can choose $c_u = c_3 |u|$, for some fixed constant $c_3$, so that if $\|x\| \leq \epsilon n$
\[ |H(x, y) - H(0, y)| \leq c_3 \epsilon n^{1-d}. \]  
By combining (4.3) and (4.4), possibly redefining $c_1$, $c_2$ and $c_3$, we get (4.2). \hfill \Box

For what follows it turns out to be convenient to introduce the notion of empirical measure: given $A$ finite subset of $\mathbb{Z}^d$ and $I \subset \mathbb{R}$ we define that function $L_A(I) : \mathbb{R}^{\mathbb{Z}^d} \to [0, 1]$ as
\[ (L_A(I))(\varphi) = \frac{1}{|A|} \sum_{x \in A} \mathbb{1}_I(\varphi_x). \]  
(4.5)
If $I$ is an interval, say $I = (a, b]$, then we drop the extra parentheses: $L_A(I) = L_A(a, b]$. The main result of this subsection is the following:
Proposition 4.2. For any \( a < 4(G + Q) \) and every \( \delta > 0 \)
\[
\lim_{N \to \infty} \mathbb{P} \left( L_{DN}(-\infty, \sqrt{a \log N}) \geq \delta \left| \Omega_{N,\sigma}^{+} \right| \right) = 0, \quad \mathbb{P}(d\sigma)\text{-a.s.} \tag{4.6}
\]

Proof. We adopt the notation of Section 3. The essential difference here is that \( L \) is not a fixed (large) number: rather we choose \( L = L(N) \not\to \infty \) as \( N \not\to \infty \). In what follows \( \epsilon \) is a small positive number, that we will choose in the last steps of the proof, and we use the short-cut notation \( B_{\epsilon L}(y) = B_{[\epsilon L]}(y) \) and \( D_{N}^{\epsilon} = \bigcup_{y \in \Lambda_{\epsilon}} B_{\epsilon L}(y) \). Of course \( \lim_{N \to \infty} |D_{N}^{\epsilon}|/|D_{N}| = \epsilon^{d} \).

We start with the following remark: it suffices to prove that for every \( \delta > 0 \) there exists \( \epsilon > 0 \) such that
\[
\lim_{N \to \infty} \mathbb{P} \left( L_{DN}(-\infty, \sqrt{a \log N}) \geq \delta \left| \Omega_{N,\sigma}^{+} \right| \right) = 0, \quad \mathbb{P}(d\sigma)\text{-a.s.} \tag{4.7}
\]
In fact the full result, i.e. (4.6), is a direct consequence of a finite number (approximately \( \epsilon^{-d} \)) of repetitions of the same argument applied to shifted copies of \( \Lambda_{\epsilon} \).

We prove two lemmas with which we select \( L = L(N) \) and a good subset of \( \Lambda_{\epsilon} \): note that these two lemmas concern \( \mathbb{P} \) and not \( \mathbb{P} \).

Lemma 4.3. For every \( \varrho \in (0, 2) \) and \( \zeta < 2Q \varrho \) choose \( L = 2\lfloor N^{\varrho/d} \rfloor \). Then \( \mathbb{P}(d\sigma)\text{-a.s.} \) there exists \( N_{0}(\sigma) < \infty \) such that for every \( N > N_{0}(\sigma) \) the following holds: for every \( y \in \Lambda_{\epsilon} \) there exists \( \bar{z}(y) \in B_{\epsilon L}(y) \) such that \( \sigma_{\bar{z}(y)} \geq \sqrt{\zeta \log N} \).

Proof. Set \( E_{N} = \{ \sigma : \text{there exists } y \in \Lambda_{\epsilon} \text{ such that } \sigma_{x} < \sqrt{\zeta \log N} \text{ for every } x \in B_{\epsilon L}(y) \} \). We need to show that \( \mathbb{P}(E_{N} \text{ i.o.}) = 0 \). Since the \( \sigma \)-field is IID we have that for sufficiently large \( N \)
\[
\mathbb{P}(E_{N}) = 1 - \left( 1 - \mathbb{P} \left( \sigma_{x} < \sqrt{\zeta \log N}, \text{ for every } x \in B_{\epsilon L}(0) \right) \right)^{|\Lambda_{\epsilon}|} \tag{4.8}
\]
\[
\leq 1 - \left( 1 - p_{N}^{c_{1}N^{\varrho}} \right)^{c_{2}N^{d-\epsilon}},
\]
where \( c_{1} \) and \( c_{2} \) are positive constants and \( p_{N} = 1 - N^{-\varrho/2Q} - \epsilon' \); \( \epsilon' \) is any strictly positive real number (we have applied \( \mathbb{H}[\mathbb{P}] \) that we choose smaller than \( (\varrho/2) - \zeta/(4Q) \). We conclude that \( \mathbb{P}(E_{N}) \leq \exp(-N^{\varrho/2} - \zeta/(4Q)) \) for sufficiently large \( N \) and therefore, by Borel–Cantelli I, the proof is complete. ☐ (Lemma 4.3).

For the second lemma we need some notation: set \( S_{L}(y) = \{ z : |z - y| \geq L \text{ and there exists } x \text{ such that } |x - y| < L \text{ and } |x - z| = 1 \} \). For any \( \varphi \in \mathbb{R}^{2d} \) and every \( x \in B_{\epsilon L}(y) \) define \( M_{x}^{\varphi}(\varphi) = \sum_{z \in S_{L}(y)} H(x, z)\varphi_{z} \) (note that this \( M_{x}^{\varphi}(\varphi) \) is different from \( M_{x}^{N}(\varphi) \) as defined in the previous section).
Lemma 4.4. Let $a > 0$ and choose $L = L(N)$ such that $\lim_{N \to \infty} L(N)/N^q$ is positive and finite for a given $q \in (0,1)$. Then for every $\delta > 0$ there exists $\epsilon_0 > 0$ such that $P(d\sigma)$–a.s. there exist $\Lambda^G \subset \Lambda$ and a finite number $N_0$ satisfying the following properties:

- for every choice of $y \in \Lambda^G$ and every $\epsilon \in (0,\epsilon_0)$, if $\varphi \in \Omega^+_{N,\sigma}$ and if there exists $x \in B_{L}(y)$ such that $M^y_x(\varphi) \leq \sqrt{a \log N}$ then
  \[
  \max_{x \in B_{L}(y)} |M^y_x(\varphi) - M^0_x(\varphi)| \leq \delta \sqrt{\log N},
  \]
  \[
  \text{for every } N \geq N_0,
  \]

\[
\lim_{N \to \infty} \frac{\Lambda^G_N}{|\Lambda|} = 1.
\]

Proof. Let $y \in \Lambda$. By applying repeatedly Lemma 4.1 we obtain

\[
|M^y_x(\varphi) - M^0_x(\varphi)| \leq \sum_{z \in S_L(y)} |H(x', z) - H(x, z)|\varphi_z| \leq c_3 c_1^{-1} \epsilon \sum_{z \in S_L(y)} H(x, z)|\varphi_z| = c_3 c_1^{-1} \epsilon M^y_x(\varphi) + 2 c_3 c_1^{-1} \epsilon \sum_{z \in S_L(y): \varphi_z < 0} H(x, z)|\varphi_z| \leq c_3 c_1^{-1} \epsilon \sqrt{a \log N} + \frac{2 c_3 c_1^{-1} \epsilon c}{|S_L(y)|} \sum_{z \in S_L(y)} |\sigma_z|.
\]

The Lemma is therefore proven once we show for example that there exists a sequence $\delta_N$ of positive numbers, $\delta_N \nearrow 0$ as $N \nearrow \infty$, such that

\[
\sum_N P \left( \left\{ y \in \Lambda : \frac{1}{|S_L(y)|} \sum_{z \in S_L(y)} |\sigma_z| > \frac{\sqrt{a \log N}}{2c_3} \right\} \right) > \delta_N |\Lambda| < \infty.
\]

Choose

\[
\delta^2_N = \max \left( P \left( \frac{1}{|S_L(y)|} \sum_{z \in S_L(y)} |\sigma_z| > \frac{\sqrt{a \log N}}{2c_3} \right), |\Lambda|^{-1/2} \right).
\]

By H.3 $\delta_N$ vanishes as $N$ tends to infinity. The rest of the proof of (4.12) follows from a direct application of (2.6).

We now choose $\sigma$ in the good set specified by Lemma 4.3 and Lemma 4.3. Let us fix the choice of the parameters with the help of an extra parameter $\tilde{\epsilon} > 0$:

\[
g = (2Q/(G + Q)) - \tilde{\epsilon} > 0, \quad \zeta = \left( \sqrt{\frac{2Q}{\sqrt{G + Q}}} - \tilde{\epsilon} \right)^2 > 0, \quad L = 2\lfloor N^{\theta/d} \rfloor.
\]
The claim follows since (4.7) is proven if we show that there exists $\tilde{\delta} > 0$ such that for every $x \in \mathcal{B}_L(0)$
\[
\lim_{N \to \infty} \frac{1}{N^{d-\epsilon}} \log \mathbb{P} \left( L_{\Lambda_c^G}(-\infty, \sqrt{a \log N}) \geq \tilde{\delta} | \mathbb{Q}_N^+ \right) < 0, \quad \mathbb{P}(d\sigma)-\text{a.s.} \tag{4.15}
\]
The claim follows since
\[
\left\{ \varphi : L_{\mathcal{D}_N}(-\infty, \sqrt{a \log N}) \geq \delta \right\} \subset \bigcup_{x \in \mathcal{B}_L} \left\{ \varphi : L_{\Lambda_c^G+x}(-\infty, \sqrt{a \log N}) \geq \delta \right\}, \tag{4.16}
\]
and $|\mathcal{B}_L| \exp(-cN^{d/2})$ vanishes as $N \to \infty$. Lemma 4.3 guarantees that we may substitute $\Lambda_c$ with $\Lambda_{c'}^G$.

We think now of $x$ as fixed and observe that $M_y^\rho(\varphi)$ and $\varphi_y$, for $y \in \Lambda_{c'}^G + x$ are close in the sense specified by the following lemma.

**Lemma 4.5.** For every $\eta > 0$, $\delta' > 0$,
\[
\limsup_{N \to \infty} \frac{1}{Nd-\epsilon} \log \mathbb{P} \left( \left| \left\{ y \in \Lambda_{c'}^G + x : |\varphi_y - M_y^\rho(\varphi)| \geq \eta \sqrt{\log N} \right\} \right| \geq \delta' |\Lambda_{c'}^G| \right) < -c, \tag{4.17}
\]
for some $c > 0$.

**Proof.** We observe that $\{\varphi_y - M_y^\rho(\varphi)\}_{y \in \Lambda_{c'}^G + x}$ forms an IID collection of centered Gaussian random variables of variance that is not larger than $G$. Therefore for every $\eta > 0$
\[
\limsup_{N \to \infty} \frac{1}{Nd-\epsilon} \log \mathbb{P} \left( \frac{1}{|\Lambda_{c'}^G|} \sum_{y \in \Lambda_{c'}^G + x} \left| \varphi_y - M_y^\rho(\varphi) \right| > \eta \sqrt{\log N} \right) < 0. \tag{4.18}
\]
Now note that the probability in (4.17) is not larger than
\[
\mathbb{P} \left( \frac{1}{|\Lambda_{c'}^G|} \sum_{y \in \Lambda_{c'}^G + x} \left| \varphi_y - M_y^\rho(\varphi) \right| \geq \eta \delta' \sqrt{\log N} \right), \tag{4.19}
\]
whence the thesis. \(\square\) (Lemma 4.3)

We define
\[
E_N^c = \left\{ \left\{ y \in \Lambda_{c'}^G + x : |\varphi_y - M_y^\rho(\varphi)| \leq c \sqrt{\log N} \right\} \right\} \left( 1 - \frac{\delta}{2} |\Lambda_{c'}^G| \right). \tag{4.20}
\]
By Lemma 4.3 we know that $\mathbb{P}(E_N^c) < \exp(-cN^{d-\epsilon})$ for sufficiently large $N$ and for some positive $c$. In order to prove (4.15), we analyze
\[
\mathbb{P} \left( \left\{ y \in \Lambda_{c'}^G + x : \varphi_y < \sqrt{a \log N} \right\} \geq \delta |\Lambda_{c'}^G| ; \mathbb{Q}_N^+ \right)
= \mathbb{P} \left( \left\{ y \in \Lambda_{c'}^G + x : \varphi_y < \sqrt{a \log N} \right\} \geq \delta |\Lambda_{c'}^G| ; E_N^c \right)
+ \mathbb{P} \left( \left\{ y \in \Lambda_{c'}^G + x : \varphi_y < \sqrt{a \log N} \right\} \geq \delta |\Lambda_{c'}^G| ; E_N^c \right) \tag{4.21}
\]
The first term in the right-hand side of (4.21) is not larger than \(\exp(-cN^{d-\varrho})\). We focus on the second term:

\[
P\left( \left\{ y \in \tilde{\Lambda}^G_c + x : \varphi_y < \sqrt{a \log N} \right\} \geq \delta |\tilde{\Lambda}^G_c| / E_N ; \Omega^+_{N,\varrho} \right) 
\leq P\left( \left\{ y \in \tilde{\Lambda}^G_c + x : M_y^G(\varphi) < \sqrt{a \log N} + \varepsilon \sqrt{\log N} \right\} \geq \delta |\tilde{\Lambda}^G_c| / 2 ; \Omega^+_{N,\varrho} \right).  
\]

(4.22)

Now we use the fact that, by Lemma 4.4, when \(M_y^G(\varphi) \leq \sqrt{b \log N}\) for some \(b > 0\), \(|M_y^G(\varphi) - M_y^G(\varphi)| \leq \varepsilon \sqrt{\log N}\) if we choose \(\varepsilon\) sufficiently small and \(N\) sufficiently large. We recall that \(\varepsilon\) was introduced at the beginning of the proof. Hence the last term in (4.22) is not larger than

\[
P\left( \left\{ y \in \tilde{\Lambda}^G_c + x : M_y^G(\varphi) < (\sqrt{a} + 2\varepsilon) \sqrt{\log N} \right\} \geq \delta |\tilde{\Lambda}^G_c| / 2 ; \Omega^+_{N,\varrho} \right).  
\]

(4.23)

Set

\[
\tilde{E}_N = \left\{ \varphi : \left\{ y \in \tilde{\Lambda}^G_c + x : M_y^G(\varphi) < (\sqrt{a} + 2\varepsilon) \sqrt{\log N} \right\} \geq \delta |\tilde{\Lambda}^G_c| / 2 \right\},
\]

with this notation the last term in (4.21) is dominated by

\[
P(\tilde{E}_N \cap \Omega^+_{N,\varrho}) \leq E \left[ \prod_{y \in \tilde{\Lambda}^G_c} P\left( \varphi_{\tilde{x}(y)} \geq \sigma_{\tilde{x}(y)} | F^G_{S_L(y)} \right) ; \tilde{E}_N \right] 
\leq E \left[ \prod_{y \in \tilde{\Lambda}^G_c} P\left( \varphi_{\tilde{x}(y)} \geq \sqrt{\log N} | F^G_{S_L(y)} \right) ; \tilde{E}_N \right].
\]

(4.25)

But on \(\tilde{E}_N\) we have that (we may think \(a \geq 2Q/\sqrt{G + Q}\))

\[
\prod_{y \in \tilde{\Lambda}^G_c} P\left( \varphi_{\tilde{x}(y)} \geq \sqrt{\log N} | F^G_{S_L(y)} \right) \leq \left(1 - \Phi \left(-\frac{\sqrt{a} + 2\varepsilon - \sqrt{\zeta} \sqrt{\log N}}{\sqrt{G_L}}\right)\right)^{|\tilde{\Lambda}^G_c|/2}
\leq \exp \left(-N^{d-\varrho - \frac{(\sqrt{a} + 2\varepsilon - \sqrt{\zeta})^2}{2G}}\right),
\]

(4.26)

where \(G_L = \min \left\{ \text{var} \left( \varphi_x | F^G_{S_L(y)} \right) : x \in B_{\text{d}_L}(y) \right\}\) (note that \(G_L\) does not depend on \(y \in \Lambda_c\) and \(G_L \neq G\) as \(L \neq \infty\)). The last step holds for sufficiently large \(N\).

Recalling the estimate of \(P(\Omega^+_{N,\varrho})\) (Theorem 1.1), equation 4.13 follows if

\[
d - \varrho - \frac{(\sqrt{a} + 3\varepsilon - \sqrt{\zeta})^2}{2G} > d - 2.
\]

(4.27)

A straightforward computation shows that if \(\varepsilon < (\sqrt{4(G + Q)} - a)/4\) then the left-hand side of (4.27) is bounded below by \(d - 2 + \bar{\varepsilon}\) and we are done.
4.2. Upper bounds. For \( \Lambda \subset \mathbb{R}^d \), \( N \in \mathbb{N} \) and \( \varphi \in \mathbb{R}^{Z_d} \) we set \( M^N_\Lambda (\varphi) = \sum_{x \in A_N} \varphi_x / |A_N| \): we always consider \( \Lambda \) a bounded open set with piecewise smooth boundary (even if this condition could be very much relaxed). We observe that \( M^N_N (\varphi) \sim \mathcal{N}(0, N^{2-d}(c(\Lambda) + o(1))) \), where \( c(\Lambda) > 0 \).

We give the following upper bound on the path of the interface above the rough wall:

**Proposition 4.6.** For every \( \Lambda \subset D \) we have that

\[
\limsup_{N \to \infty} \frac{\mathbb{E} \left[ M^N_\Lambda (\varphi) | \Omega^+_{N, \sigma} \right]}{\sqrt{\log N}} \leq \sqrt{4(G + Q)},
\]

for every \( \sigma \), \( \Psi \)-a.s.,

\[\mathbb{P}(d\sigma)\text{-a.s.} \]

**Proof.** Set \( \mathbb{P}_N = \mathbb{P}_{T^{-1}_\Psi^N} \), \( \psi_{N,x} \in [0, \infty) \) independent of \( x \). We observe that \( \mathbb{P}(\cdot | \Omega^+_{N, \sigma}) \) is dominated by \( \mathbb{P}_N(\cdot | \Omega^+_{N, \sigma}) \). This follows for example by writing a finite volume approximation of \( \mathbb{P} \), with \( 0 \)-boundary conditions, namely \( \mu_\alpha(\cdot) = \mathbb{P}(\cdot | F_{D_n}) , \psi^0 \), \( \psi^0 \equiv 0 \): we view this measure as a measure on \( \mathbb{R}^{D_n} \). One verifies directly that if \( T : \mathbb{R}^{D_n} \to \mathbb{R}^{D_n} \) is defined by \( (T \varphi)_x = \varphi_x + a \), \( a \geq 0 \), then \( \mu_n T^{-1} \) dominates \( \mu_n \) in the strong FKG sense (that is the two measures satisfy Holley’s inequality, cf. [17]). Therefore, if \( n \geq N \), we can define \( \mu_n T^{-1}(d\varphi) \exp(- \sum U_x(\varphi_x)) / Z \) and \( \mu_n (d\varphi) \exp(- \sum U_x(\varphi_x)) / Z' \), with \( U_x(\cdot) \) a potential that for definitiveness we choose equal to \( \beta r^4 \mathbb{1}_{(0,1)}(r) \mathbb{1}_{D_N}(x) \), \( \beta > 0 \) and \( Z, Z' \) are the normalization constants, and this two new measures are still ordered in the strong FKG sense. The limit for \( n \to \infty \) and then \( \beta \to \infty \) recovers the desired inequality.

We choose \( \psi_{N,x} = \alpha_N \), with \( \alpha_N = \sqrt{(4(G + Q) + \epsilon) \log N} \) and \( \epsilon > 0 \). Therefore

\[
\mathbb{E} \left[ M^N_\Lambda (\varphi) | \Omega^+_{N, \sigma} \right] \leq \mathbb{E}_N \left[ M^N_\Lambda (\varphi) | \Omega^+_{N, \sigma} \right] \leq \alpha_N + \mathbb{E} \left[ M^N_\Lambda (\varphi) | \Omega^+_{N, \sigma - \psi_N} \right].
\]

By applying (2.23) with \( Y = \pm M^N_\Lambda (\varphi) \), \( t = \delta N^{d-2} \) (\( \delta > 0 \)) and \( E = \Omega^+_{N, \sigma - \psi_N} \) we have that

\[
\mathbb{E} \left[ M^N_\Lambda (\varphi) | \Omega^+_{N, \sigma - \psi_N} \right] \leq \delta(c(\Lambda) + o(1)) - \frac{1}{\delta N^{d-2}} \log \mathbb{P} \left( \Omega^+_{N, \sigma - \psi_N} \right).
\]

In Section 2, Lemma 2.4, we have shown that for every \( \epsilon > 0 \)

\[
\lim_{N \to \infty} \frac{1}{N^{d-2}} \log \mathbb{P} \left( \Omega^+_{N, \sigma - \psi_N} \right) = 0.
\]

\[\mathbb{P}(d\sigma)\text{-a.s.} \]

This is more than we need: apply it in (4.28) to get (4.28). \( \square \)
Proposition 4.7. For every \( \Lambda \subset D \) we have that
\[
\liminf_{N \to \infty} \frac{\mathbb{E} \left[ M_N^\Lambda(\varphi) | \Omega_{N,\sigma}^+ \right]}{\sqrt{\log N}} \geq \sqrt{4(G + Q)},
\]
\( \mathbb{P}(d\sigma) \)-a.s.

Proof. Since Proposition 4.2 may be proved with \( \Lambda \) in place of \( D \), for all positive \( \epsilon \) and sufficiently large \( N \), there exists \( \Omega_{\epsilon} \subset \mathbb{R}^d \) such that \( \mathbb{P} \left( \Omega_{\epsilon} | \Omega_{N,\sigma}^+ \right) \geq (1 - \epsilon) \), for all \( \varphi \in \Omega_{\epsilon} \) there exists \( A_{\epsilon} \subset \Lambda_N \), \( |A_{\epsilon}| > (1 - \epsilon)|\Lambda_N| \) and \( \varphi_x \geq \sqrt{(4(G + Q) - \epsilon)} \log N \) for every \( x \in A_{\epsilon} \).

If \( \varphi \in \Omega_{\epsilon} \), we decompose
\[
M_N^\Lambda(\varphi) = \frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} \varphi_x + \frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N \setminus A_{\epsilon}} \varphi_x.
\]
Under \( \mathbb{P}(\cdot | \Omega_{N,\sigma}^+) \), the first term in equation (4.34) is larger than \((1 - \epsilon) \sqrt{(4(G + Q) - \epsilon)} \log N \), while the second term is larger than \(- \sum_{x \in \Lambda_N} |\sigma_x|/|\Lambda_N| \). Observe that the last quantity is also a minorant for \( M_N^\Lambda(\varphi) \) when \( \varphi \in \Omega_{\epsilon}^6 \). But \(- \sum_{x \in \Lambda_N} |\sigma_x|/|\Lambda_N| \) converges \( \mathbb{P}(d\sigma) \)-a.s. to \( \mathbb{E}[|\sigma_0|] \), thus
\[
\liminf_{N \to \infty} \frac{\mathbb{E} \left[ M_N^\Lambda(\varphi) | \Omega_{N,\sigma}^+ \right]}{\sqrt{\log N}} \geq (1 - \epsilon)^2 \sqrt{(4(G + Q) - \epsilon)}.
\]
The thesis follows taking \( \epsilon \to 0 \). \( \square \)

Remark 4.8. We have therefore that, for every choice of \( \Lambda \), \( \mathbb{P}(d\sigma) \)-a.s.
\[
\lim_{N \to \infty} \frac{\mathbb{E} \left[ M_N^\Lambda(\varphi) | \Omega_{N,\sigma}^+ \right]}{\sqrt{\log N}} = \sqrt{4(G + Q)}.
\]
By the Brascamp–Lieb inequality \( \mathbb{H} \) the random variable \( M_N^\Lambda(\varphi) \), (even) under the conditioned measure \( \mathbb{P}(d\varphi| \Omega_{N,\sigma}^+) \) \( \mathbb{H} \), has a sub–Gaussian behavior: the exponential centered moment of \( M_N^\Lambda(\varphi) \) is bounded by \( \exp(cN^{1-d}) \). This immediately yields the hydrostatic limit of the field: if for \( r \in D \) we define \( u_N(r) = \varphi_{r_N}/\sqrt{\log N} \) for \( rN \in \mathbb{Z}^d \) and if we extend \( u_N \) to a function from \( D \) to \( \mathbb{R} \) (for example) by a polynomial interpolation, then \( u_N \) converges weakly to \( u \equiv \sqrt{4(G + Q)} \) (that is \( \int_D u_N f \to \int f \) for every \( f \in C_0^0(D; \mathbb{R}) \)), in probability, with respect to \( \mathbb{P}(d\varphi| \Omega_{N,\sigma}^+) \), and \( \mathbb{P}(d\sigma) \)-a.s.

Proposition 4.9. For any \( b > 4(G + Q) \) and every \( \delta > 0 \),
\[
\lim_{N \to \infty} \mathbb{P} \left( L_{D_N}(\sqrt{b \log N}, +\infty) \geq \delta | \Omega_{N,\sigma}^+ \right) \to 0,
\]
\( \mathbb{P}(d\sigma) \)-a.s.
Proof. Fix $b$ and define
\[
\overline{N}_b(\varphi) = \left| \{ x \in D_N : \varphi_x > \sqrt{b \log N} \} \right|. \tag{4.38}
\]
By Proposition 4.2, for all positive $\epsilon$ and sufficiently large $N$, there exists $\Omega_\epsilon$ such that $\mathbb{P} \left( \Omega_\epsilon | \Omega_N^{+} \right) > (1 - \epsilon)$ and on $\Omega_\epsilon$, $\varphi_x \geq \sqrt{(4(G + Q) - \epsilon) \log N}$ on at least $(1 - \epsilon) |D_N|$ sites $x$. Thus on $\Omega_\epsilon$, $\varphi_x / \sqrt{\log N}$ is larger than $\sqrt{b}$ on at least $\overline{N}_b(\varphi)$ sites, larger than $\sqrt{4(G + Q) - \epsilon}$ on at least $(1 - \epsilon) |D_N| - \overline{N}_b(\varphi)$ sites and on the remaining (at most $\epsilon |D_N|$) sites it is larger than $-|\sigma_x|/\sqrt{\log N}$ (thanks to the conditioning on $\Omega_N^{+}$).

Thus on $\Omega_\epsilon$, $\varphi_x / \sqrt{\log N}$ is larger than $\sqrt{b}$ on at least $\overline{N}_b(\varphi)$ sites, larger than $\sqrt{4(G + Q) - \epsilon}$ on at least $(1 - \epsilon) |D_N| - \overline{N}_b(\varphi)$ sites and on the remaining (at most $\epsilon |D_N|$) sites it is larger than $-|\sigma_x|/\sqrt{\log N}$ (thanks to the conditioning on $\Omega_N^{+}$).

Thus
\[
\mathbb{E} \left[ \frac{M_N^0(\varphi)}{\sqrt{\log N}} \right] > (1 - \epsilon) f(b, \epsilon) - (1 + \epsilon) \sum_{x \in D_N} |\sigma_x|/\sqrt{\log N} |D_N|, \tag{4.39}
\]
where
\[
f(b, \epsilon) = \sqrt{4(G + Q) - \epsilon} + \mathbb{E} \left[ \frac{\overline{N}_b(\varphi)}{|D_N|} \right] \Omega_N^{+} (\sqrt{b} - \sqrt{4(G + Q) - \epsilon}). \tag{4.40}
\]

Now we let $N$ grow to infinity: by Remark 4.8 we obtain
\[
\sqrt{4(G + Q)} \geq \sqrt{4(G + Q) - \epsilon} + \limsup_{N \to \infty} \mathbb{E} \left[ \frac{\overline{N}_b(\varphi)}{|D_N|} \right] \Omega_N^{+} (\sqrt{b} - \sqrt{4(G + Q) - \epsilon}). \tag{4.41}
\]
Now let $\epsilon \searrow 0$: since $b$ is chosen strictly larger than $\sqrt{4(G + Q)}$ we get that
\[
\limsup_{N \to \infty} \mathbb{E} \left[ \frac{\overline{N}_b(\varphi)}{|D_N|} \right] \Omega_N^{+} = 0. \tag{4.42}
\]
This leads to the conclusion, once we observe that $L_{D_N} (\sqrt{b \log N}, +\infty) = \overline{N}_b(\varphi)/|D_N|$. □

5. Super–Gaussian and sub–Gaussian regimes

The proof of Theorem 1.3 can be obtained following and modifying step by step the proof of Theorem 1.1 and Theorem 1.2. However many of the steps in such an approach would be superfluous: we therefore sketch the proof pointing out the most substantial simplifications. On the way we also give some results that sharpen Theorem 1.3. We assume H.1 and H.3.

5.1. The sub–Gaussian regime. Under H.2 one immediately sees that for every $\theta > 0$ and every $k > 0$
\[
\lim_{N \to \infty} N^k \mathbb{P} \left( \max_{x \in D_N} \sigma_x > \theta \sqrt{\log N} \right) = 0. \tag{5.1}
\]
Therefore in proving the lower bound corresponding to (1.16) we may substitute the auxiliary field $\tilde{\sigma}$, previously defined in (2.2), with $\tilde{\sigma}_x = \theta \sqrt{\log N}$, with $\theta$ arbitrarily small. At this point we may directly apply [3, Prop. 2.1], that is the lower bound in the case of a flat wall: by sending $\theta$ to zero we obtain the result.

For what concerns a proof of the upper bound corresponding to (1.16), due to the weakness of the assumption H.3 the results in [6] are no longer applicable and one need some argument in the
spirit of the proof of (1.1). The guideline is the following: leave Definition (3.4) unchanged and replace (3.5) with
\[
\prod_{y \in \Lambda_c} \mathbb{P} \otimes \mathbb{P} \left( \varphi_y \geq \sigma_y | F_{B(y)}^{\sigma, \varphi} \right) \leq \left( 1 - N^{-\frac{\alpha + \epsilon}{4Q + \epsilon}} \right)^{|C_r|}. \tag{5.2}
\]
The rest of the steps are identical (set \( Q = 0 \)).

The very same arguments apply in extending the proof of Theorem 1.2 to cover the second part of Theorem 1.3(1). \( \square \)

We stress that observation (5.1), which allows a natural comparison argument, together with the result (1.8) immediately yields the following sharpening of Theorem 1.3(1):

**Proposition 5.1.** Under Hypothesis H.2 we have that
\[
\limsup_{N \to \infty} \sup_{x \in D_N} \frac{E \left[ \varphi_x \Omega_{W, \sigma}^+ \right]}{\sqrt{4G \log N}} \leq 1. \tag{5.3}
\]
Moreover if also \(-\sigma\) satisfies H.2 then
\[
\limsup_{N \to \infty} \sup_{x \in D_N} \left| \frac{E \left[ \varphi_x \Omega_{W, \sigma}^+ \right]}{\sqrt{4G \log N}} - 1 \right| = 0. \tag{5.4}
\]

5.2. The super–Gaussian regime. Once again a look to the proof of (2.1) is sufficient to understand that the multiscale decomposition in (2.2) is superfluous. In proving the lower bound of (1.17) we may substitute (2.2) with the much rougher discretization
\[
\sigma_x = \begin{cases} 
(4Q + \epsilon \log N)^{1/(2\beta)} & \text{if } \sigma_x \leq (4Q + \epsilon \log N)^{1/(2\beta)}, \\
((2d + 2)Q \log N)^{1/(2\beta)} & \text{if } \sigma_x \in ((4Q + \epsilon \log N)^{1/(2\beta)}, ((2d + 2)Q \log N)^{1/(2\beta)}], \\
\infty & \text{otherwise},
\end{cases} \tag{5.5}
\]
with \( \epsilon > 0 \). The rest of the proof of the lower bound for (1.17) follows in an absolutely analogous, but simpler, way as the proof of (2.1): the optimization over the levels of the \( \sigma \)-field is trivial.

For what concerns the upper bound for (1.17) it suffices to redefine \( E_{\eta, \alpha} \), cf. (3.4) in the proof of Proposition 3.1, in the following way:
\[
E_{\eta, \alpha} = \left\{ (\sigma, \varphi) : \text{there exists } r \in I \text{ such that } |\{ y \in C_r : M_{y}^{\square} (\varphi) \leq (\alpha \log N)^{1/2\beta}) | \geq \eta |C_r| \right\}. \tag{5.6}
\]
and one obtains (compare with (3.13))
\[
\prod_{y \in \Lambda_c} \mathbb{P} \otimes \mathbb{P} \left( \varphi_y \geq \sigma_y | F_{B(y)}^{\sigma, \varphi} \right) \leq \left( 1 - N^{-\frac{\alpha + \epsilon}{4Q}} \right)^{|C_r|}. \tag{5.7}
\]
The rest of the proof is essentially identical: just substitute \( \sqrt{\alpha \log N} \) with \((\alpha \log N)^{1/2\beta}\). Analogous modifications to the proof of Theorem 1.2 completes the proof of Theorem 1.3(2). \( \square \)
Remark 5.2. Since there are several spikes of the $\sigma$–field going beyond the level of the interface, in fact $\max_{x \in D_N} \sigma_x \approx (2dQ \log N)^{1/2\beta}$, one cannot hope to have a bound of the type $\mathbb{E}[\varphi_x | \Omega_{N,\sigma}^+] \leq \left((4Q + \epsilon) \log N \right)^{1/2\beta}$ uniformly in $x$ for $\epsilon$ arbitrarily small.

Remark 5.3. A word on heavier $\sigma$–tails is due: a new phenomenon is expected to arise if $\sigma$ has power law upward tails. In this case $\max_{x \in D_N} \sigma_x$, suitably normalized, converges to a nondegenerate random variable and this is sharply different of what happens in all the cases that we considered. Moreover excursion of the $\sigma$–field beyond the level $N^\delta$, some $\delta > 0$, would now be possible, even on more than $N^\epsilon$ sites, for some $\epsilon > 0$ depending on $\delta$ and on the tail behavior. It is quite clear from an entropy argument that, even if $\epsilon < d - 2$, these spikes may have a very strong effect on the field: almost local deformatons of the $\varphi$–field are not the optimal strategy to accomodate the presence of the wall. This is in stark contrast with the situation we dealt with, since (roughly) excursions of the $\sigma$–field beyond level $(\log N)^k$ on $o(N^{d-2})$ sites produce only almost local modifications of the $\varphi$–field.

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