Accelerating universe: recent observations and implications for extended gravity theories

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Abstract. Recent high quality cosmological observations have confirmed and mapped in detail the accelerating expansion of the universe. These observations involve geometrical methods (use of standard candles and standard rulers) and dynamical methods (measured growth rate of cosmological perturbations). The Cosmological Constant (effective equation of state parameter of $w=-1$) remains consistent with all current data as a driving force of the acceleration and can be generated by quantum fluctuations of the vacuum with a proper cutoff. A signature in the Casimir effect would be expected in that case. An Evolving Dark Energy Density ($w=w(t)$) is also allowed by the data and a subset of the allowed evolving forms, which corresponds to crossing of the line $w=-1$, is inconsistent with most models based on General Relativity. On the other hand, scalar tensor extensions of General Relativity are consistent with the full range of allowed expansion histories. Demanding consistency of Scalar-Tensor theories with solar system tests and full range of allowed expansion histories implies constraints on Newton's constant evolution $G(t)$.

1. Introduction
The assumption of large scale homogeneity and isotropy of the universe combined with the assumption that general relativity is the correct theory on cosmological scales leads to the Friedman equation which in a flat universe takes the form

$$H^2(a) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(a),$$

where $a(t)$ is the scale factor of the universe and $\rho$ its average energy density. Both sides of this equation can be observationally probed directly: The left side using mainly geometrical methods (measuring the luminosity and angular diameter distances $d_L(z)$ [1, 2, 3] and $d_A(z)$ [4, 5, 6, 7] with standard candles and standard rulers) shows an accelerating expansion at recent redshifts and the matter - radiation density part of the right side using dynamical and other methods (cosmic microwave background [7], large scale structure observations [9], lensing [10] etc). These observations have indicated [11] that the two sides of the Friedman equation (44) can not be equal if $\rho(a) = \rho_m(a) \sim a^{-3}$ even if a non-zero curvature is assumed. There are two possible resolutions to this puzzle: Either modify the right side of the Friedman equation (44) introducing a new form of 'dark' energy ideal fluid component ($\rho(a) = \rho_m(a) + \rho_X(a)$) with suitable evolution in order to restore the equality or modify both sides by changing the way energy density affects geometry thus modifying the Einstein equations.
In the first class of approaches the required gravitational properties of dark energy (see [12, 13, 14, 15, 16, 17] for recent reviews) needed to induce the accelerating expansion are well described by its equation of state $w(z) = \frac{p_X(z)}{\rho_X(z)}$ which enters in the second Friedman equation as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_m + \rho_X(1 + 3w)),$$

implying that a negative pressure ($w < -1/3$) is necessary in order to induce accelerating expansion. The simplest viable example of dark energy is the cosmological constant [11, 18, 19] ($w = -1$). This example however even though consistent with present data lacks physical motivation. Questions like ‘What is the origin of the cosmological constant?’ or ‘Why is the cosmological constant $10^{120}$ times smaller than its natural scale so that it starts dominating at recent cosmological times (coincidence problem)?’ remain unanswered [20]. Attempts to replace the cosmological constant by a dynamical scalar field (quintessence [21, 22, 23]) have created a new problem regarding the initial conditions of quintessence which even though can be resolved in particular cases (tracker quintessence), can not answer the above questions in a satisfactory way.

The parameter $w(z)$ determines not only the gravitational properties of dark energy but also its evolution. This evolution is easily obtained from the energy momentum conservation

$$d(\rho_X a^3) = -p_X d(a^3),$$

which leads to

$$\rho_X = \rho_0 X e^{-3 \int_1^a \frac{dz}{a}(1+w(z'))} = \rho_0 X e^{3 \int_0^z \frac{dz'}{1+z'}(1+w(z'))}.$$  \hspace{1cm} (4)

Therefore the determination of $w(z)$ is equivalent to that of $\rho_X(z)$ which in turn is equivalent to the observed $H(z)$ from the Friedman equation (44) expressed as

$$H(z) = H_0[\Omega_{0m}(1+z)^3 + \Omega_0 X e^{3 \int_0^z \frac{dz'}{1+z'}(1+w(z'))}].$$ \hspace{1cm} (5)

Thus, knowledge of $\Omega_{0m}$ and $H(z)$ suffices to determine $w(z)$ which is obtained from equation (5) as [24]

$$w(z) = \frac{\frac{2}{3}(1+z)\frac{d\ln H}{dz} - 1}{1 - \frac{H_0^{2}}{H^2}\Omega_{0m}(1+z)^3}.$$  \hspace{1cm} (6)

In the second class of approaches the Einstein equations get modified and the new equations combined with the assumption of homogeneity and isotropy lead to a generalized Friedman equation of the form

$$f(H^2) = g(\rho_m),$$ \hspace{1cm} (7)

where $f$ and $g$ are appropriate functions determined by the modified gravity theory [25, 26, 27, 28, 29, 30]. In this class of models, the parameter $w(z)$ can also be defined from equation (6) but it can not be interpreted as $\frac{p_X}{\rho_X}$ of a perfect fluid.

The simplest (but quite general) examples of modified gravity theories are scalar tensor theories [25, 31, 32, 33, 34] where the Newton’s constant $G$ is promoted to a function of a field $\Phi$: $8\pi G \rightarrow \frac{1}{\Phi}$ whose dynamics at the Lagrangian level is determined by a potential $U(\Phi)$. Assuming homogeneity and isotropy, the modified Friedman equation in these theories takes the form

$$H^2 = \frac{1}{3F}(\rho_m + \frac{1}{2}\dot{\Phi}^2 + U - 3HF).$$ \hspace{1cm} (8)

This equation reduces to a regular minimally coupled scalar field dark energy (quintessence) in the general relativity limit of a constant $F = \frac{1}{8\pi G}$. The positive nature of the kinetic term

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however implies that certain types of behaviors of $H$ may not be reproducible without invoking a time-dependent $F$. These types of behavior of $H(z)$ which include a $w(z)$ crossing the Phantom Divide Line (PDL) $w = -1$ are potential signatures of extended gravity theories and will be discussed in the next section. To identify this type of signatures, a detailed form of the observed $H(z)$ is required which may be obtained by a combination of multiple dark energy probes. Observational probes may be divided in two classes [35] according to the methods used to obtain $H(z)$.

- **Geometric methods** probe the large scale geometry of space-time directly through the redshift dependence of cosmological distances ($d_L(z)$ or $d_A(z)$). They thus determine $H(z)$ independent of the validity of Einstein equations.

- **Dynamical methods** determine $H(z)$ by measuring the evolution of energy density (background or perturbations) and using a gravity theory to relate them with geometry i.e. with $H(z)$. These methods rely on knowledge of the dynamical equations that connect geometry with energy and may therefore be used in combination with geometric methods to test these dynamical equations.

### 2. Geometrical Probes of Expansion

Examples of geometric probes include

(i) The measured supernova distance redshift relation $d_L(z)$ [1, 2, 3] which for a flat universe, is connected to $H(z)$ as

$$d_L(z) = (1 + z) \int_0^z \frac{dz'}{H(z')}.$$  \quad (9)

(ii) The measured [7, 8] angular diameter distance $d_A(z_{rec})$ to the sound horizon $r_s(z_{rec})$ at recombination

$$d_A(z_{rec}) = \frac{1}{1 + z_{rec}} \int_0^{z_{rec}} \frac{dz'}{H(z')}.$$  \quad (10)

(iii) The scale of the sound horizon measured at more recent redshifts ($z_{BAO}$) through large scale structure redshift survey correlation functions [4]

$$D_V(z) = \left[ \left( \int_0^{z_{BAO}} \frac{dz}{H(z)} \right)^2 \frac{z_{BAO}}{H(z_{BAO})} \right]^{1/3}.$$  \quad (11)

Early SnIa data put together with more recent such data through the Gold dataset [2, 36] have been used to reconstruct $w(z)$ and have demonstrated a mild preference for a $w(z)$ that crossed the PDL [37, 38]. However, this dataset has been shown to suffer from systematics due to the inhomogeneous origin of the data [39]. Nevertheless, a cosmological constant remained consistent with the Gold dataset but only at the 2σ level. More recent SnIa data (SNLS, ESSENCE, HST) [40] have demonstrated a higher level of consistency with $\Lambda CDM$ and showed no trend for a redshift dependent equation of state cite.

On the other hand, the use of standard rulers (CMB+BAO) has rarely been studied independent of SnIa due to the small number of datapoints involved (see however [42]). It has been pointed out however that the latest BAO data [43] “require slightly stronger cosmological acceleration at low redshifts than $\Lambda CDM$”. This statement is equivalent to a trend towards a $w(z) < -1$ at low $z$ and therefore a possibility of crossing the PDL $w = -1$. In what follows we quantify this statement in some detail by comparing the best fit form of $w(z)$ obtained from the SnIa data to the corresponding form obtained from the CMB+BAO data. This comparison
is done quantitatively by identifying the quality of fit of $\Lambda CDM$ in the context of each dataset. In particular, we consider the Chevalier-Polarski-Linder (CPL) [44, 45] parametrization

$$w(z) = w_0 + w_1 \frac{z}{1+z},$$

and assuming flatness, we identify the ‘distance’ in units of $\sigma$ (distance) of the parameter space point $(w_0, w_1) = (-1, 0)$ corresponding to $\Lambda CDM$ from the best fit point $(\bar{w}_0, \bar{w}_1)$ for each dataset (SnIa standard candles or CMB+BAO standard rulers) and for several priors of $(\Omega_0, \Omega_b)$. We thus identify an interesting systematic difference in trends between the two datasets. We also discuss the implications of this difference in trends on the distance duality relation which measures quantitatively the agreement between luminosity and angular diameter distances. This relation has been shown to be respected when clusters of galaxies are used as standard rulers [46].

Thus, we assume a CPL parametrization for $w(z)$ and apply the maximum likelihood method separately for standard rulers (CMB+BAO) and standard candles (SnIa) assuming flatness. The corresponding late time form of $H(z)$ for the CPL parametrization used in what follows is

$$H^2(z) = H_0^2 [\Omega_0 (1+z)^3 + (1-\Omega_0)(1+z)^3(1+w_0+w_1)e^{-3w_1z}].$$

At earlier times this needs to be generalized taking into account radiation i.e.

$$E^2(a) = \frac{H(a)^2}{H_0^2} = \Omega_m (a + a_{eq}) a^{-4} + \Omega_{de} X(a),$$

where $a = 1/(1+z)$, $\Omega_{de} = 1 - \Omega_m - \Omega_{rad}$ and

$$X(a) = \exp \left[ -3 \int_1^a \frac{(1+w(a')) da'}{a'} \right] = a^{-3(1+w_0+w_1)} e^{-3w_1(1-a)},$$

with $w(a) = w_0 + w_1(1-a)$.

2.1. Standard Rulers
2.1.1. CMB We use the datapoints $(R, l_a, \Omega_b h^2)$ of Ref. [42] where $R$, $l_a$ are two shift parameters:

- The scaled distance to recombination

$$R = \sqrt{\Omega_0 \frac{H_0^2}{c^2} r(z_{CMB})},$$

where $r(z_{CMB})$ is the comoving distance from the observer to redshift $z$ and is given by

$$r(z) = \frac{c}{H_0} \int_0^z \frac{dz}{E(z)},$$

with $E(z) = H(z)/H_0$.  




The angular scale of the sound horizon at recombination

\[ l_a = \pi \frac{r(a_{CMB})}{r_s(a_{CMB})}, \tag{18} \]

where \( r_s(a_{CMB}) \) is the comoving sound horizon at recombination given by

\[ r_s(a_{CMB}) = \frac{c}{H_0} \int_0^{a_{CMB}} \frac{c_s(a)}{a^2 E(a)} da, \tag{19} \]

with the sound speed being \( c_s(a) = \frac{1}{\sqrt{3(1 + \bar{R}_b a)}} \) and \( a_{CMB} = \frac{1}{1+z_{CMB}} \), where \( z_{CMB} = 1089 \). Actually, \( z_{CMB} \) has a weak dependence on \( \Omega_m \) and \( \Omega_b \), see Ref. [47] but we have checked that the sound horizon changes only to less than 0.1%. The quantity \( \bar{R}_b \) is actually the photon-baryon energy-density ratio, and its value can be calculated using

\[ \bar{R}_b = \frac{3}{4\pi^2} \left( T_{CMB}/2.7K \right)^{-4}. \]

For a flat prior, the 3-year WMAP data (WMAP3) [7] measured best fit values are [42]

\[ \bar{V}_{CMB} = \begin{pmatrix} \frac{\bar{R}}{l_a} \\ \bar{\Omega}_b h^2 \end{pmatrix} = \begin{pmatrix} 1.70 \pm 0.03 \\ 302.2 \pm 1.2 \\ 0.022 \pm 0.00082 \end{pmatrix}, \tag{20} \]

The corresponding normalized covariance matrix is [42]

\[ C_{CMB}^{norm} = \begin{pmatrix} 1 & -0.09047 & -0.01970 \\ -0.09047 & 1 & -0.6283 \\ -0.01970 & -0.6283 & 1 \end{pmatrix}, \tag{21} \]

from which the covariance matrix can be found to be:

\[ (C_{CMB})_{ij} = (C_{CMB}^{norm})_{ij} \sigma_{\bar{V}_{CMB}^i} \sigma_{\bar{V}_{CMB}^j}, \tag{22} \]

where \( \sigma_{\bar{V}_{CMB}} \) are the 1σ errors of the measured best fit values of eq. (20).

We thus use equations (20), (16) and (18) to define

\[ X_{CMB_s} = \begin{pmatrix} R - 1.70 \\ l_a - 302.2 \\ \bar{\Omega}_b h^2 - 0.022 \end{pmatrix}, \tag{23} \]

and construct the contribution of CMB to the \( \chi^2 \) as

\[ \chi^2_{CMB} = X_{CMB_s}^T C_{CMB}^{-1} X_{CMB_s}, \tag{24} \]

with

\[ C_{CMB}^{-1} = \begin{pmatrix} 1131.32 & 4.8061 & 5234.42 \\ 4.8061 & 1.1678 & 1077.22 \\ 5234.42 & 1077.22 & 2.48145 \times 10^6 \end{pmatrix}. \tag{25} \]

Notice that \( \chi^2_{CMB} \) depends on four parameters (\( \Omega, \Omega_b, w_0 \) and \( w_1 \)). Due to the large number of parameters involved, in what follows we will consider various different priors on the parameters \( \Omega, \Omega_b \).
2.1.2. BAO  As in the case of the CMB, we apply the maximum likelihood method using the datapoints \[43\]

\[
\vec{V}_{\text{BAO}} = \left( \begin{array}{c} \frac{r_s(z_{\text{CMB}})}{D_V(0.2)} = 0.1980 \pm 0.0058 \\ \frac{r_s(z_{\text{CMB}})}{D_V(0.35)} = 0.1094 \pm 0.0033 \end{array} \right),
\]

(26)

where the dilation scale (see also eq. (27))

\[
D_V(z_{\text{BAO}}) = \left[ \left( \int_0^{z_{\text{BAO}}} \frac{dz}{H(z)} \right)^2 \frac{z_{\text{BAO}}}{H(z_{\text{BAO}})} \right]^{1/3},
\]

(27)

encodes the visual distortion of a spherical object due to the non-euclidianity of a FRW spacetime, and is equivalent to the geometric mean of the distortion along the line of sight and two orthogonal directions. We thus construct

\[
X_{\text{BAO}} = \left( \begin{array}{c} \frac{r_s(z_{\text{dec}})}{D_V(0.2)} - 0.1980 \\ \frac{r_s(z_{\text{dec}})}{D_V(0.35)} - 0.1094 \end{array} \right),
\]

(28)

and using the inverse covariance matrix \[43\]

\[
C_{\text{BAO}}^{-1} = \left( \begin{array}{cc} 35059 & -24031 \\ -24031 & 108300 \end{array} \right),
\]

(29)

we find the contribution of BAO to \(\chi^2\) as

\[
\chi^2_{\text{BAO}} = X_{\text{BAO}}^T C_{\text{BAO}}^{-1} X_{\text{BAO}}.
\]

(30)

2.2. Standard Candles

2.2.1. SnIa  We use the SnIa dataset of Davis et al. \[40\] consisting of four subsets: ESSENCE \[41\] (60 points), SNLS \[3\] (57 points), nearby \[2\] (45 points) and HST \[36\] (30 points).

These observations provide the apparent magnitude \(m(z)\) of the supernovae at peak brightness after implementing correction for galactic extinction, K-correction and light curve width-luminosity correction. The resulting apparent magnitude \(m(z)\) is related to the luminosity distance \(D_L(z)\) through

\[
m_{\text{th}}(z) = \bar{M}(M, H_0) + 5 \log_{10}(D_L(z)),
\]

(31)

where in a flat cosmological model

\[
D_L(z) = (1 + z) \int_0^z \frac{dz'}{H(z', \Omega, w_0, w_1)},
\]

(32)

is the Hubble free luminosity distance \((H_0 d_L)\), and \(\bar{M}\) is the magnitude zero point offset and depends on the absolute magnitude \(M\) and on the present Hubble parameter \(H_0\) as

\[
\bar{M} = M + 5 \log_{10}(H_0^{-1} M_{\text{pc}}) + 25 = M - 5 \log_{10} h + 42.38.
\]

(33)

The parameter \(M\) is the absolute magnitude which is assumed to be constant after the above mentioned corrections have been implemented in \(m(z)\).
Figure 1. The 68.3% and 95.4% $\chi^2$ confidence contours in the $w_0 - w_1$ parameter space for each dataset category for $\Omega_{0m} = 0.24$, $\Omega_b = 0.042$. The blue dots correspond to the $w_0 - w_1$ best fit while the yellow dots to $\Lambda CDM (-1,0)$.

The SNIa datapoints are given after the corrections have been implemented, in terms of the distance modulus

$$\mu_{\text{obs}}(z_i) \equiv m_{\text{obs}}(z_i) - M, \quad (34)$$

The theoretical model parameters are determined by minimizing the quantity

$$\chi_{\text{SNIa}}^2(\Omega, w_0, w_1) = \sum_{i=1}^{N} \frac{(\mu_{\text{obs}}(z_i) - \mu_{\text{th}}(z_i))^2}{\sigma_{\mu_i}^2}, \quad (35)$$

where $N = 192$ and $\sigma_{\mu_i}^2$ are the errors due to flux uncertainties, intrinsic dispersion of SNIa absolute magnitude and peculiar velocity dispersion. These errors are assumed to be Gaussian and uncorrelated. The theoretical distance modulus is defined as

$$\mu_{\text{th}}(z_i) \equiv m_{\text{th}}(z_i) - M = 5\log_{10}(D_L(z)) + \mu_0, \quad (36)$$

where

$$\mu_0 = 42.38 - 5\log_{10}h, \quad (37)$$

and $\mu_{\text{obs}}$ is given by (34). The steps we followed for the minimization of (35) are described in detail in Refs [49, 37, 38].

2.3. Results

We consider separately the standard ruler data ($\chi_{\text{SR}}^2 \equiv \chi_{\text{CLM}}^2 + \chi_{\text{BAO}}^2$) and the standard candle data ($\chi_{\text{SNIa}}^2$) and perform minimization of the corresponding $\chi^2$ with respect to the parameters $w_0$ and $w_1$ for various priors of $\Omega$ and $\Omega_b$ in the 2$\sigma$ range of the the WMAP3 best fit i.e. $0.21 \leq \Omega \leq 0.27$, $0.034 \leq \Omega_b \leq 0.049$.

In Fig 1 we show the 68.3% and 95.4% $\chi^2$ confidence contours in the $w_0 - w_1$ parameter space for the two dataset categories (standard ruler and standard candle data) for $\Omega = 0.24$ (the best
fit of the WMAP3 CMB data [7]). Fig. 1a shows the \( w_0 - w_1 \) contours obtained using SnIa data [40] (standard candles) while Fig. 1b shows the corresponding contours assuming CMB+BAO data [42, 43] (standard rulers). The blue dots correspond to the \((w_0, w_1)\) best fit, the yellow dots to \( \Lambda CDM \) \((w_0, w_1) = (-1, 0)\). The distance in units of \( \sigma \) \((\sigma\text{-distance} \, d_\sigma)\) of the best fit to \( \Lambda CDM \) was found by converting \( \Delta \chi^2 = \chi^2_{\Lambda CDM} - \chi^2_{min} \) to \( d_\sigma \) i.e. solving \([50]\)

\[
1 - \Gamma(1, \Delta \chi^2 / 2) / \Gamma(1) = Erf(d_\sigma / \sqrt{2}),
\]

for \( d_\sigma \) \((\sigma\text{-distance})\), where \( \Delta \chi^2 \) is the \( \chi^2 \) difference between the best-fit and \( \Lambda CDM \) and \( Erf() \) is the error function. Notice that \( \Lambda CDM \) is consistent at less than 1\( \sigma \) level according to the SnIa data \( d_{\sigma \, SnIa} \simeq 0.5 \) in Fig. 1a while the corresponding consistency level reduces to \( d_{\sigma \, SNIa} 1.7\sigma \) for the standard ruler CMB+BAO data. This mild difference in trends between standard candles and standard rulers persists also for all values of \( \Omega \) in the 2\( \sigma \) range of WMAP3 best fit.

3. Dynamical Tests of LCDM

Even though these tests are presently the most accurate probe of dark energy, the mere determination of the expansion rate \( H(z) \) is not able to provide significant insight into the properties of dark energy and distinguish it from models that attribute the accelerating expansion to modifications of general relativity. The additional observational input that is required is the growth function \( \delta(z) \equiv \delta\rho(z) \) of the linear matter density contrast as a function of redshift. The combination of the observed functions \( H(z) \) and \( \delta(z) \) can provide significant insight into the properties of dark energy (e.g. sound speed, existence of anisotropic stress etc) or even distinguish it from modified gravity theories.

The corresponding parametrization of the linear growth function \( \delta(z) \) can be made efficiently by introducing a growth index \( \gamma \) defined by

\[
\frac{d \ln \delta(a)}{d \ln a} = \Omega_m(a)^\gamma,
\]

where \( a = \frac{1}{1+z} \) is the scale factor and

\[
\Omega_m(a) \equiv \frac{H_0^2 \Omega_{0m} a^{-3}}{H^2(a)}.
\]

This parametrization was originally introduced by Wang and Steinhardt [51] (see also [52] for more recent discussions) and was shown to provide an excellent fit to \( \frac{d \ln \delta(a)}{d \ln a} \) corresponding to various general relativistic cosmological models for specific values of \( \gamma \). In particular, it was shown that for dark energy with slowly varying \( w(z) \simeq w_0 \) the parameter \( \gamma \) in a flat universe is

\[
\gamma = \frac{3(w_0 - 1)}{6w_0 - 5},
\]

which for the \( \Lambda CDM \) case \( (w = -1) \) reduces to \( \gamma = \frac{6}{11} \). It is therefore clear that the observational determination of the growth index \( \gamma \) can be used to test \( \Lambda CDM \).

The observational determination of the growth index \( \gamma \) requires knowledge not only of \( \delta(z) \) but also of \( H(z) \) and \( \Omega_{0m} \) (see equation (40)). It is possible however to construct more direct tests of \( \Lambda CDM \) which require knowledge only of \( \delta(z) \) and \( H(z) \). Such a null test of \( \Lambda CDM \) was recently proposed in Ref. [53] where it was suggested that the validity of \( \Lambda CDM \) requires the following equality of observables

\[
\frac{(H(z)^2)'(1+z)^2}{(1+z)^2\delta'(0)^2} \int_0^\infty \frac{\delta(z)\delta'(z)}{(1+z)} dz + 1 = 0,
\]

for
where $\equiv \frac{d}{dz}$. In fact, as discussed section III, there is an improved version of this null test that does not involve derivative of the observable $H(z)$ and therefore it is less prone to observational errors.

Both of the $\Lambda CDM$ tests discussed above (the growth index $\gamma$ and the null test) require observational determination of $\delta(z)$. There are several observational approaches that can lead to the determination of $\delta(z)$. For example, redshift distortions of galaxy power spectra [54], the rms mass fluctuation $\sigma_8(z)$ inferred from galaxy and $Ly - \alpha$ surveys at various redshifts [55]-[56], weak lensing statistics [57], baryon acoustic oscillations [58], Sachs-Wolfe effect etc. Unfortunately, the currently available data are limited in number and accuracy and come mainly from the first two categories. They involve significant error bars and non-trivial assumptions that hinder a reliable determination of $\delta(z)$. In addition, a large part of the available data are at high redshifts ($z > 1$) where $\Lambda CDM$ is degenerate with most other dark energy models since dark energy is subdominant compared to matter at high redshifts in most models.

Nevertheless, it is still instructive to consider the presently available data to investigate the possible weak constraints that can be imposed on $\Lambda CDM$. Such a task serves two purposes

(i) It can be used as a paradigm for the time when more accurate data will be available
(ii) It can provide constraints from a dynamical test which are orthogonal and completely independent from the usual geometrical tests.

Thus, in what follows we use a wide range of presently available data on both redshift distortions and $\sigma_8(z)$ to determine the observed growth index $\gamma$ and test $\Lambda CDM$ in two ways

(i) Comparing the measured value of $\gamma$ with the $\Lambda CDM$ prediction $\gamma = \frac{6}{11}$
(ii) Implementing a new null test that exploits the consistency between $H(z)$ and $\delta(z)$ in the context of $\Lambda CDM$.

3.1. Fitting the Growth Index

According to general relativity, the equations that determine the evolution of the density contrast $\delta$ in a flat background consisting of matter with density $\rho_m$ and dark energy with $\rho_{de} = \frac{\omega}{w}$ are of the form

\[
\ddot{\delta} + 2\dot{a}\frac{\dot{\delta}}{a} = 4\pi G \rho_m \delta, \tag{43}
\]

\[
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} (\rho_m + \rho_{de}), \tag{44}
\]

\[
2\ddot{a} + \left(\frac{\dot{a}}{a}\right)^2 = -8\pi G w \rho_{de}. \tag{45}
\]

It is straightforward to change variables in eq. (43) from $t$ to $\ln a \left(\dot{a} = H \frac{d}{da} \ln a\right)$ to obtain [51]

\[
(\ln \delta)^\prime + (\ln \delta)^\prime 2 + (\ln \delta)^\prime \left[\frac{1}{2} - \frac{3}{2} w(1 - \Omega_m(a))\right] = \frac{3}{2} \Omega_m(a), \tag{46}
\]

where we used (45) and

\[
\Omega_m(a) = \frac{\rho_m(a)}{\rho_m(a) + \rho_{de}(a)}, \tag{47}
\]

as in (40). A further change of variables from $\ln a$ to $\Omega_m(a)$ can be made by considering the differential of eq. (47) and using energy conservation ($d\rho = -3(\rho + p)d\ln a$) leads to

\[
d\Omega_m = 3w\Omega_m(1 + \Omega_m)d\ln a, \tag{48}
\]
to get [51]  
\[ 3w\Omega_m(1 - \Omega_m)\frac{df}{d\Omega_m} + f^2 + f\left [\frac{1}{2} \frac{3}{2}w(1 - \Omega_m)\right ] = \frac{3}{2} \Omega_m, \]  
(49)  
where we have set  
\[ f = \frac{d\ln \delta}{d\ln a}. \]  
(50)  
Using the ansatz  
\[ f = \Omega_m^{\gamma(\Omega_m)}, \]  
(51)  
in eq. (49) and expanding around \( \Omega_m = 1 \) (good approximation especially at \( z \gtrsim 1 \)) we find to lowest order  
\[ \gamma = \frac{3(w - 1)}{6w - 5}, \]  
(52)  
which reduces to \( \gamma = \frac{6}{11} \) for \( \Lambda CDM \) \((w = -1)\).

Equations (51) and (52) provide excellent approximations to the numerically obtained form of \( f(z) \). This is demonstrated in Fig. 2 where we plot the numerically obtained solution of eq. (46) for the normalized growth  
\[ g(z) \equiv \frac{\delta(z)}{\delta(0)}, \]  
(53)  
in the case of \( \Lambda CDM \) \((\Omega_{0m} = 0.3)\) along with the corresponding approximate result  
\[ g(z) = e^{\int_{0}^{z} \Omega_m(a)^{\gamma} \frac{da}{a}}, \]  
(54)  
with \( \gamma = \frac{6}{11} \) obtained from  
\[ f(\Omega_{0m}, \gamma, a) = a \frac{d\delta/da}{\delta} = \Omega_m(a)^{\gamma}. \]  
(55)  

Our goal in this section is to fit the parameter \( \gamma \) using observational data and compare it with the value \( \gamma = \frac{6}{11} \) of \( \Lambda CDM \). The most useful currently available data that can be used to constrain \( \delta(z) \) (and \( \gamma \)) involve the redshift distortion parameter \( \beta \) [59] observed through the

Figure 2. The numerically obtained solution of eq. (46) for the normalized growth of eq. (53) in the case of \( \Lambda CDM \) \((\Omega_{0m} = 0.3)\) along with the corresponding approximate result with \( \gamma = \frac{6}{11} \) obtained from eq. (55). The agreement between the two approaches is excellent.
Table 1. The currently available data for the parameters $\beta$ and $b$ at various redshifts along with the inferred growth rates and references.

| $z$   | $\beta$       | $b$       | $f_{\text{obs}}$ | Ref.   |
|-------|---------------|-----------|------------------|--------|
| 0.15  | 0.49 ± 0.09   | 1.04 ± 0.11 | 0.51 ± 0.11    | [54], [9] |
| 0.55  | 0.45 ± 0.05   | 1.66 ± 0.35 | 0.75 ± 0.18    | [62]   |
| 1.4   | 0.60$^{+0.14}_{-0.11}$ | 1.5 ± 0.20 | 0.90 ± 0.24    | [63]   |
| 3.0   | --            | --        | 1.46 ± 0.29    | [61]   |

Anisotropic pattern of galactic redshifts on cluster scales (for a pedagogical discussion see [38]). The parameter $\beta$ is related to the growth rate $f$ as

$$\beta = \frac{d \ln \delta / d \ln a}{b} = \frac{f}{b},$$

(56)

where $b$ is the bias factor connecting total matter perturbations $\delta$ and galaxy perturbations $\delta_g$ ($b = \frac{\delta_g}{\delta}$).

The currently available data for the parameters $\beta$ and $b$ at various redshifts are shown in Table I along with the inferred growth rates and references. This is an expanded version of the dataset used in Ref. [60] where a similar analysis was performed using a different parametrization suitable for modified gravity models. Notice however that these have assumed $\Lambda CDM$ (with $\Omega_0 m = 0.3$) when converting redshifts to distances for the power spectra and therefore their use to test models different from $\Lambda CDM$ may not be reliable. In the same Table we also show the growth rate obtained in Ref. [61] directly from the change of power spectrum $Ly - \alpha$ forest data in SDSS at various redshift slices.

Using the data of Table I we can perform a maximum likelihood analysis in order to find $\gamma$ and check its consistency with the $\Lambda CDM$ value $6_{-11}^{+11}$. We thus construct

$$\chi_f^2(\Omega_0 m, \gamma) = \sum_i \left[ \frac{f_{\text{obs}}(z_i) - f_{\text{th}}(z_i, \gamma)}{\sigma_{f_{\text{obs}}}} \right]^2,$$

(57)

where $f_{\text{obs}}$ and $\sigma_{f_{\text{obs}}}$ are obtained from Table I while $f_{\text{th}}(z_i, \gamma)$ is obtained from eq. (51).

An alternative observational probe of the growth function $\delta(z)$ is the redshift dependence of the rms mass fluctuation $\sigma_S(z)$ defined by

$$\sigma^2(R, z) = \int_0^\infty W^2(kR) \Delta^2(k, z) \frac{dk}{k},$$

(58)

with

$$W(kR) = 3 \left( \frac{\sin(kR)}{(kR)^3} - \frac{\cos(kR)}{(kR)^2} \right)r,$$

(59)

$$\Delta^2(k, z) = 4\pi k^3 P_S(k, z),$$

(60)

with $R = 8h^{-1} \text{Mpc}$ and $P_S(k, z)$ the mass power spectrum at redshift $z$. The function $\sigma_S(z)$ is connected with $\delta(z)$ as

$$\sigma_S(z) = \frac{\delta(z)}{\delta(0)} \sigma_S(z = 0),$$

(61)
Table 2. The currently available data for the rms fluctuation $\sigma_8(z)$ at various redshifts and references.

| $z$  | $\sigma_8$ | $\sigma_{\sigma_8}$ | Ref. |
|------|------------|---------------------|------|
| 2.125| 0.95       | 0.17                | [55] |
| 2.72 | 0.92       | 0.17                |      |
| 2.2  | 0.92       | 0.16                | [56] |
| 2.4  | 0.89       | 0.11                |      |
| 2.6  | 0.98       | 0.13                |      |
| 2.8  | 1.02       | 0.09                |      |
| 3.0  | 0.94       | 0.08                |      |
| 3.2  | 0.88       | 0.09                |      |
| 3.4  | 0.87       | 0.12                |      |
| 3.6  | 0.95       | 0.16                |      |
| 3.8  | 0.90       | 0.17                |      |
| 0.35 | 0.55       | 0.10                | [64] |
| 0.6  | 0.62       | 0.12                |      |
| 0.8  | 0.71       | 0.11                |      |
| 1.0  | 0.69       | 0.14                |      |
| 1.2  | 0.75       | 0.14                |      |
| 1.65 | 0.92       | 0.20                |      |

which implies

$$s_{th}(z_1, z_2) \equiv \frac{\sigma_8(z_1)}{\sigma_8(z_2)} = \frac{\delta(z_1)}{\delta(z_2)} = \frac{\int_{1}^{1+z_1} \Omega_m(a) \frac{da}{a}}{\int_{1}^{1+z_2} \Omega_m(a) \frac{da}{a}},$$

(62)

where we made use of eq. (54). Most of the currently available datapoints $\sigma_8(z_i)$ originate from the observed redshift evolution of the flux power spectrum of Ly $\alpha$ forest [56]. These datapoints are shown in Table II along with the corresponding reference sources. These data are not as useful as the redshift distortion factors for the determination of $\gamma$ for two reasons

(i) The rms fluctuation $\sigma_8(z)$ is not connected directly with the growth rate $f(z)$. Instead, it is related with $f(z)$ through the integral of eq. (54).

(ii) Most of the Ly $\alpha \sigma_8$ data appear at high redshifts where $\Lambda CDM$ is degenerate with most other dark energy models.

Using the data of Table II we construct the corresponding $\chi^2_s$ defined as

$$\chi^2_s(\Omega_0 m, \gamma) = \sum_i \left[ \frac{s_{\text{obs}}(z_i, z_i+1) - s_{\text{th}}(z_i, z_i+1)}{\sigma_{\text{obs}}} \right]^2,$$

(63)

where $s_{\text{obs}}$ is derived by error propagation from the corresponding $1\sigma$ errors of $\sigma_8(z_i)$ and $\sigma_8(z_{i+1})$ while $s_{\text{th}}(z_i, z_{i+1})$ is defined in eq. (62). We can thus construct the combined $\chi^2_{\text{tot}}(\Omega_0 m, \gamma)$ as

$$\chi^2_{\text{tot}}(\Omega_0 m, \gamma) \equiv \chi^2_f(\Omega_0 m, \gamma) + \chi^2_s(\Omega_0 m, \gamma).$$

(64)
Figure 3. The cosmological data for the growth rate $f(z)$ along with the best theoretical fit $f = \Omega_m(z)^\gamma$ with $\Omega_{0m} = 0.3$ and the corresponding $1\sigma$ errors (shaded region). The errorboxes on f are obtained using the ratios at the specific redshifts. Clearly, the best fit shows a minor difference from $\Lambda CDM$ only at low redshifts.

Setting $\Omega_{0m} = 0.3$ and minimizing $\chi^2_{tot}$ with respect to $\gamma$ we find

$$\gamma = 0.707^{+0.226}_{-0.189}, \quad (65)$$

which differs somewhat from the corresponding result of Ref. [60] because we have used a broader dataset, we have used a different parametrization for $f$ and we have assumed $\Lambda CDM$ as our fiducial model thus avoiding the marginalization of the parameter $w_0$. The result (65) indicates that the $\Lambda CDM$ value of $\gamma = \frac{6}{11} = 0.545$ is well within $1\sigma$ from the best fit and is clearly consistent with data. The imposed constraints however are rather weak and even a flat model with matter only (SCDM) predicting $\gamma = 0.6$ (set $w = 0$ in eq. (52)) is consistent with the data.

Ignoring the $\sigma_8(z)$ data of Table II leads to a negligible change in the best fit to $\gamma = 0.693^{\pm}0.2$. This is consistent with the above discussion on the usefulness of these data. Alternatively, assuming the $\Lambda CDM$ value for $\gamma$ ($\gamma = \frac{6}{11}$) and minimizing with respect to $\Omega_{0m}$ we find $\Omega_{0m} = 0.226^{+0.09}_{-0.07}$.

The cosmological data for the growth rate $f(z)$ are shown in Fig.3 along with the best theoretical fit $f = \Omega_m(z)^\gamma$ with $\Omega_{0m} = 0.3$ and the corresponding $1\sigma$ errors (shaded region). In the same plot we show the corresponding $f_{\Lambda CDM}(z)$ obtained by solving numerically eq. (46) for $w = -1$ and $\Omega_{0m} = 0.3$. Clearly, the best fit shows a minor difference from $\Lambda CDM$ only at low redshifts. We therefore conclude that $\Lambda CDM$ is consistent with current data for the growth factor $\delta(z)$ and the consistency is maximized for $\Omega_{0m} = 0.23 \pm 0.08$.

4. A Possible Origin of the Cosmological Constant

The observed accelerating expansion of the universe indicates that either the universe is dominated by an energy form with repulsive gravitational properties (dark energy) or general relativity needs to be modified on cosmological scales (modified gravity) (see [12] and references therein). Dark energy may appear in the form of a cosmological constant (constant energy density [65]), quintessence (variable energy density due to an evolving scalar field [66]), a perfect fluid filling throughout space [67] etc. As discussed in the previous sections, the simplest among the above possibilities is the cosmological constant which is also favored by most cosmological data [68] compared to alternative more complicated models. It is therefore important to identify the possible origins of such a cosmological constant.
There have been several proposals concerning the origin of the cosmological constant even though none of them is completely satisfactory. Some of them include the zero point energy of the vacuum [69], anthropic considerations [70], brane cosmology [71], degenerate vacua [72] etc. The simplest physical mechanism that could lead to a cosmological constant is the zero-point energy of the vacuum made finite by an ultraviolet cutoff. A natural value of such a cutoff is the Planck scale leading to an energy density of the vacuum ρ_V = (2.44 × 10^{27}) eV^4. However, such a value of the energy density leads to a cosmological constant which is 123 orders of magnitude larger than the observed one.

If the accelerating expansion of the universe is due to a cosmological constant emerging due to zero point fluctuations of the vacuum, the required energy density of the vacuum would be ρ_Λ ≃ 10^{-11} eV^4 corresponding to a cutoff scale (see eq. (66) below) of ω_c ≃ 10^{-3} eV (l_c ≃ 0.1 mm). Even though there is no apparent physical motivation for such a cutoff scale, it can not be a priori excluded unless it is shown to be in conflict with specific experimental data.

There are various types of laboratory experiments which are able to probe directly quantities related to the zero point energy of the vacuum. Such experiments include measurements of the Casimir force [73] and measurements of the current noise in Josephson junctions (see [74, 75] for a debate on the effectiveness of Josephson junction experiments on probing the energy of the vacuum). Even though these non-gravitational experiments can only probe changes of the zero-point energy under variations of system parameters or of external couplings, these changes can be modified in the presence of a cutoff of the absolute value of the zero-point energy. Therefore, these experiments can become indirect probes of the absolute value of the zero-point energy under the assumption of the presence of a cutoff.

According to the Casimir effect [76, 77, 78], the presence of macroscopic bodies (like a pair of conducting plates) leads to a modification of the zero point vacuum fluctuations due to the introduction of non-trivial boundary conditions. This modification manifests itself as a force between the macroscopic bodies that distort the vacuum. When the role of the macroscopic bodies is played by a pair of conducting plates, the discreteness imposed on the fluctuation field modes, lowers the vacuum energy and leads to an attractive force between the plates (the \textit{Casimir force}) which has been measured by several experiments [73]. In the presence of a finite cutoff length scale [79] l_c, electromagnetic field modes with wavelengths λ < l_c are suppressed and the vacuum energy lowering due to the presence of the plates gets modified. This in turn leads to modifications of the Casimir force between the plates compared to the case where no cutoff has been imposed. In fact it may be shown [80, 75, 81] that when the plate separation d becomes significantly less than the cutoff scale l_c then the force between the plates changes sign and becomes repulsive! However, there is no experimental indication for change of sign of the Casimir force down to separations d ≃ 100 nm. This imposes a constraint on the vacuum energy cutoff as l_c < 100 nm corresponding to a vacuum energy ρ_V > 1 eV^4. Such a value of ρ_V is clearly inconsistent with the vacuum energy corresponding to the cosmological constant (ρ_Λ ≃ 10^{-11} eV^4). This inconsistency increases further if the contribution of interactions other that electromagnetism is included since in that case the predicted ρ_V from the Casimir effect would get even larger to include the contribution of other fields. It is therefore clear that either vacuum energy is not responsible for the accelerating expansion of the universe or there is a missing ingredient in the calculation of the vacuum energy as a function of the cutoff scale.

One such possible missing ingredient could be the presence of a universal compact extra dimension [83] with compactification scale R. Even though the current experimental constraints on R are quite stringent (R < (300 GeV)^{-1} ≃ 10^{-9} mm [84]) it is still instructive to consider this possibility in the context of the Casimir effect with a cutoff in the vacuum energy. The propagation of vacuum energy modes along this extra dimension modifies the field spectrum and affects accordingly the Casimir force.

In this section we address the following question: ‘What is the effect of an extra compact
Figure 4. The normalized vacuum energy \( \delta u \equiv \frac{\Delta u (d, R)}{\frac{1}{3} \bar{h} \omega_c} \) as a function \( \alpha \) (dimensionless form of \( d \)) for \( \beta = 0 \) (dashed line) and \( \beta = 1 \) (continuous line). The curve corresponding to \( \beta = 1 \) has been shifted to lower values so that the locations of the minima can be compared more easily. Notice that the minimum shifts slightly to larger values as we increase the extra dimension size.

We start by deriving the vacuum energy in the region between two parallel plates. Consider a 3 dimensional Euclidean space with an extra compact dimension of scale \( R \). In this space consider a box of square parallel conducting plates of large surface \( L^2 \) parallel to the \( xy \) plane placed at a small distance \( d \) apart. The vacuum energy of the quantized electromagnetic field in the region between the plates \([76, 82, 83]\) is

\[
\mathcal{E} = \frac{1}{2} \sum_{\vec{k}, \lambda} \hbar \omega_{\vec{k}} g(\frac{\omega_{\vec{k}}}{\omega_c}),
\]

where \( g(x) \) is a UV cutoff regulator (\( \lim_{x \to 0} g(x) = 1, \lim_{x \to \infty} g(x) = 0 \)) and \( \lambda \) counts the polarization modes.

Also

\[
\omega_{\vec{k}} = c \sqrt{k_x^2 + k_y^2 + k_d^2 + k_R^2},
\]

where

\[
k_x = \frac{\pi m_x}{L}, \quad k_y = \frac{\pi m_y}{L}, \quad k_d = \frac{\pi n}{d}, \quad k_R = \frac{N}{R},
\]

with \( m_x, m_y, n, N = 0, 1, 2, \ldots \). In the large \( L \) limit eq. (66) takes the form

\[
\frac{\mathcal{E}}{L^2} = \frac{3\hbar c}{4\pi} \sum_{n, N=0}^{\infty} \int_{0}^{\infty} dq q \omega_q g(\frac{\omega_q}{\omega_c}),
\]

where \( \omega_q^2 = c^2 \sqrt{q^2 + (\frac{\pi n}{L})^2 + (\frac{N}{R})^2} \) and the factor 3 is due to the three polarization degrees of freedom in the presence of the extra dimension.

The prime (’) on the sum implies that when \( N = n = 0 \) we should put a factor \( \frac{1}{3} \) (only one degree of freedom from polarization) while if only one of the \( N, n \) is 0 we should put an extra factor \( \frac{2}{3} \) (only two degrees of freedom from polarization). It is straightforward to show that the modification of the vacuum energy due to the presence of the plates is

\[
\Delta u(d, R) \equiv u_{\text{vac}}(d, R) - u_{\text{vac}}^{\text{vac}}(d, R) =
\]
Figure 5. The plate distance $d_0(\beta)$ in mm for which the Casimir force changes sign, as a function of the dimensionless scale of compactification $\beta$. In converting from the dimensionless distance $\alpha_0(\beta)$ to the dimensionful distance $d_0(\beta)$ in mm we have assumed that the cutoff is equal to $\omega_0(\beta) \ (\text{eq. (77)})$ required to reproduce the observed cosmological constant with vacuum energy.

$$\delta u = \frac{3\omega_c^3 h}{4\pi c^2} \sum_{N=0}^{\infty} \left( \sum_{n=0}^{\infty} F(n,N) - \int_0^{\infty} dq F(q,N) \right) - h(n,N),$$

where

$$h(n,N) = \frac{3\omega_c^3 h}{4\pi c^2} \frac{2}{3} F(0,0) + \frac{1}{3} \sum_{n=1}^{\infty} F(n,0) +$$

$$- \frac{1}{3} \sum_{N=1}^{\infty} F(0,N) - \frac{1}{3} \int_0^{\infty} dq F(q,0),$$

and

$$F(n,N) \equiv \int_0^{\infty} \frac{dv}{\sqrt{(\frac{n}{\alpha})^2 + (\frac{N}{\beta})^2}} g(v),$$

with

$$\alpha \equiv \frac{\omega_c d}{c\pi} \quad \beta \equiv \frac{\omega_c R}{c}.$$ 

In the limit $\beta \to 0$ corresponding to $R \to 0$ (no extra dimension), only the $N = 0$ mode contributes and the above expressions reduce to the well known \cite{82,81} forms of the regularized vacuum energy.

We now specify an exponential form for the UV cutoff function $g(v)$. Other smooth forms of $g(v)$ also lead to similar results. For $g(v) = e^{-v}$ we have $F(n,N) = e^{-s(2+s(s+2))}$ with $s = \sqrt{(\frac{n}{\alpha})^2 + (\frac{N}{\beta})^2}$. In Fig. 4 we show the normalized vacuum energy $\delta u \equiv \frac{\Delta u}{\omega_c^2}$ as a function of $\alpha$ for $\beta = 0$ and $\beta = 1$. Notice that for a small value of $R$ ($R\omega_c \to 0$) we recover the result of ref. \cite{81} where the repulsive nature of the Casimir force was demonstrated for plate separations $d$ much smaller than the cutoff scale $\omega_c^{-1}$ without the presence of extra dimensions. For $\beta = 0$ (no extra dimension), the Casimir force becomes repulsive for $\alpha < \alpha_0 \approx 0.36$. For a cutoff leading to the observed value of cosmological constant ($\omega_c \approx 10^{-3}eV$) we find the critical separation $d_0 \approx 0.6mm$ such that for separations $d < d_0$ the Casimir force becomes
repulsive. Since the Casimir force has been experimentally shown to be attractive down to plate separations \( d \simeq 100 \text{nm} \) \[73\] it becomes clear that the observed cosmological constant can not be due to vacuum energy with appropriate cutoff and no extra dimensions.

The introduction of a compact extra dimension with finite size has two effects:

- The cutoff scale \( \omega_c(\beta) \) required to match the observed value of the cosmological constant slowly decreases
- The critical dimensionless separation \( \alpha_0(\beta) \) for which the Casimir force changes sign increases (see Fig. 4).

In order to demonstrate the first effect we may evaluate the predicted vacuum energy density as a function of the dimensionless size \( \beta \) of the extra dimension. We have

\[
\rho_V(\omega_c, \beta) = \frac{u_V^{\omega_c}(d)}{d} = \frac{3\hbar \omega_c^4}{4\pi^2 c^3} Q(\beta),
\]

where

\[
Q(\beta) = \sum_{N=0}^{\infty} \int_0^\infty dq \int_0^{\infty} \frac{v^2 e^{-v}}{\sqrt{q^2 + \left(\frac{N}{\beta}\right)^2}} dv.
\]

Demanding

\[
\rho_V(\omega_c, \beta) = \rho_\Lambda = 10^{-11} eV^4,
\]

we find

\[
\omega_c^\Lambda(\beta) = \left(\frac{4\pi^2}{3}Q(\beta)^{-1}10^{-11}\right)^{1/4} eV.
\]

In Fig. 5 we show a plot of the dimensionful plate distance \( d_0(\beta) \) corresponding to a sign change of the Casimir force. This is found by converting the dimensionless minima \( \alpha_0 \) of Fig. 4 to the corresponding dimensionful plate distances \( d_0 \) assuming a cutoff equal to \( \omega_c^\Lambda(\beta) \) in eq. (73) for \( \alpha \). As shown in Fig. 5 \( d_0(\beta) \) slowly increases with \( \beta \) as expected by inspection of Fig. 4 which shows the minimum \( \alpha_0 \) of \( \Delta u(\alpha, \beta) \) shifts to larger values as we increase the dimensionless size \( \beta \) of the extra dimension.

It is easy to convert the dimensionless parameter \( \beta \) to the compactification scale \( R \) for a particular value of the cutoff \( \omega_c \). For example for the cosmological constant cutoff \( \omega_c = \omega_c^\Lambda(\beta) \) we obtain using eqs. (73), (77)

\[
R = \frac{3\beta}{(8.2 \times Q(\beta))^{-1/4}}.
\]
Finally, it is straightforward to use eqs (70), (73) to evaluate the Casimir force for \( \omega_c = \omega_{c,\Lambda}(\beta) \). The resulting force is shown in Fig. 6 for \( \beta = 0 \) \((R = 0)\) and \( \beta = 1 \) \((R \simeq 0.7\,\text{mm})\). Clearly, the plate separation where the force changes sign depends weakly on the size of the extra dimension and is always larger than 0.6mm. We conclude that even in the presence of compact extra dimensions a cosmological constant induced by zero point vacuum fluctuations with appropriate cutoff is in conflict with experimental measurements of the Casimir force which indicate an attractive force down to separations of \( d \simeq 100\,\text{nm} \). However, if future Casimir experiments detect a change of sign of the Casimir force at small plate separations, this could be interpreted as an indication of a finite cutoff of the vacuum energy. Even though such a vacuum energy would be too large to be consistent with the observed cosmological constant, the inclusion of other effects (eg fermions which contribute negative vacuum energy) could restore the consistency with cosmology.

5. Observational Signature of Extended Gravity Theories

Two central current questions in cosmology research are the following:

- What theoretical models are consistent with the currently detected form of \( H(z) \)?
- What are the generic predictions of these models with respect to \( H(z) \) so that they can be ruled out or confirmed by more detailed observations of \( H(z) \)?

In a class of approaches the required gravitational properties of dark energy (see [12, 13, 14, 17] for recent reviews) needed to induce the accelerating expansion are well described by its equation of state \( w(z) = \frac{p_X(z)}{\rho_X(z)} \). In this case the simplest model consistent with the currently detected form of \( H(z) \) is the flat \( \Lambda\text{CDM} \) model. As discussed in section 1, according to this model, the universe is flat and its evolution is determined by general relativity with a cosmological constant through the Friedman equation

\[
H(z)^2 = H_0^2(\Omega_{0m}(1+z)^3 + \Omega_\Lambda),
\]

where \( \Omega_{0m} = \frac{\rho_m}{\rho_c} \) is the current matter density normalized on the critical density for flatness \( \rho_c \) and \( \Omega_\Lambda = 1 - \Omega_{0m} \) is a constant density due to the cosmological constant. The main advantages of this model are simplicity and predictability. It has a single free parameter and it can be definitively ruled out by future observations. Its disadvantages are lack of theoretical motivation and fine tuning: The observed value of the cosmological constant is 120 orders of magnitude smaller than its theoretically expected value [19, 11].

Attempts to replace the cosmological constant by a minimally coupled dynamical scalar field (minimally coupled quintessence (MCQ) [23, 21, 22]) have led to models with a vastly larger number of parameters fueled by the arbitrariness of the scalar field potential. Despite this arbitrariness and vast parameter space, quintessence models are generically constrained to predicting a limited range of functional forms for \( H(z) \) [25, 34]. This is a welcome feature which provides ways to either rule out or confirm this class of theories.

The allowed functional space of \( H(z) \) can be further increased by considering models based on extensions of general relativity such as braneworlds [27, 85, 86, 87], \( f(R) \) theories [88] or scalar-tensor theories [31, 25, 34] (extended quintessence (EXQ) [89]) where the accelerated expansion of the universe is provided by a non-minimally coupled scalar field. This class of theories is strongly motivated theoretically as it is predicted by all theories that attempt to quantize gravity and unify it with the other interactions. On the other hand, its parameter space is even larger than the corresponding space of MCQ since the later is a special case of EXQ. Local (eg solar system) gravitational experiments and cosmological observations constrain the allowed parameter space to be close to general relativity. Despite of these constraints however, the allowed by EXQ functional forms of \( H(z) \) are significantly more than those allowed by MCQ.
The detailed identification of the forbidden $H(z)$ functional forms for both MCQ and EXQ is particularly important since it may allow future observations determining $H(z)$ to rule out one or both of these theories.

Previous studies [90] have mainly focused on the $H(z)$ limits of MCQ using a combination of plausibility arguments and numerical simulations of several classes of potentials. The low redshift $H(z)$ parameter space was divided in three sectors: a forbidden sector which could not correspond to any plausible quintessence model, a sector corresponding to the freezing quintessence scenario and a sector corresponding to the thawing quintessence scenario. In the freezing quintessence models, a field $\Phi$ which was already rolling towards its potential minimum prior to the onset of acceleration slows down ($\ddot{\Phi} < 0$) and creeps to a halt (freezes) mimicking a cosmological constant as it comes to dominate the universe. In the thawing quintessence models, the field has been initially halted by Hubble damping at a value displaced from its minimum until recently when it ‘thaws’ and starts to roll down to the minimum ($\dot{\Phi} > 0$).

In this section, we extend these studies to the case of EXQ. Instead of using numerical simulations however, applied to specific potential classes, we use generic arguments demanding only the internal consistency of the theory. Thus our ‘forbidden’ sector when reduced to MCQ is smaller but more generic (applicable to a more general class of models) than that of Ref. [90] (see also Refs. [91, 92]).

The size and location of the sectors of the low $z$, $H(z)$ parameter space, depends sensitively on the assumed current time derivatives of the Newton’s constant $G(t)$ and reduce to well known results in the MCQ limit of $G(t) = G_0 = \text{const}$. Therefore an interesting interplay develops between local gravitational experimental constraints of the current time derivatives of $G(t)$ (eg $\frac{\dot{G}_0}{G_0}$ or $\frac{\ddot{G}_0}{G_0}$) and cosmological observations of $H(z)$ at low redshifts. For example, a constraint on $\frac{\dot{G}_0}{G_0}$ from local gravitational experiments defines the forbidden sector in the low $z$ expansion coefficients of $H(z)^2$ in the context of EXQ. If such coefficients are measured to be in the forbidden sector by cosmological observations then EXQ could be ruled out. Alternatively, if such coefficients are measured to be in the forbidden sector for MCQ but in the allowed sector of EXQ (either ‘freezing’ or ‘thawing’) then this would rule out MCQ in favor of EXQ. As shown in what follows, current observational constraints on $H(z)$ imply significant overlap with the allowed sectors of both MCQ and EXQ. This however may well change in the near future with more accurate determinations of $H(z)$ and the time derivatives of $G_0$.

6. The Boundaries of Extended Quintessence

Extended quintessence is based on the simplest but very general (given its simplicity) extension of general relativity: Scalar-Tensor theories. In these theories Newton’s constant obtains dynamical properties expressed through the potential $F(\Phi)$. The dynamics are determined by the Lagrangian density [25, 31]

$$\mathcal{L} = \frac{F(\Phi)}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - U(\Phi) + \mathcal{L}_m[\psi_m; g_{\mu\nu}] ,$$

(80)

where $\mathcal{L}_m[\psi_m; g_{\mu\nu}]$ represents matter fields approximated by a pressureless perfect fluid. The function $F(\Phi)$ is observationally constrained as follows:

- $F(\Phi) > 0$ so that gravitons carry positive energy [31].
- $\frac{1}{F} \left( \frac{\partial F}{\partial \Phi} \right)^2 |_{z=0} < \frac{1}{4} 10^{-4}$ from solar system observations [93].
In such a model the effective Newton’s constant for the attraction between two test masses is given by

$$G_{\text{eff}}(t) = \frac{1}{F(t)} \left( \frac{dF}{dt}(t) \right)^2 + \frac{2}{F(t)} \left( \frac{d^2F}{dt^2}(t) \right) \approx \frac{1}{F(t)} = G(t), \quad (*)$$

where the approximation of equation (81) is valid at low redshifts. Assuming a homogeneous $\Phi$ and varying the action corresponding to (80) in a background of a flat FRW metric, we find the coupled system of generalized Friedman equations

$$3FH^2 = \rho + \frac{1}{2} \Phi^2 - 3H \dot{F} + U, \quad (82)$$

$$-2FH = \rho + p + \dot{\Phi}^2 + \ddot{F} - H \dot{F} \quad (83)$$

where we have assumed the presence of a perfect fluid ($\rho = \rho_m, p \approx 0$) playing the role of matter fields. Expressing in terms of redshift and eliminating the potential $U$ from equations (82), (83) we find [31, 34]

$$\Phi'^2 = -F'' - (\ln H)' + \frac{2}{1 + z} \left[ \Phi' - \frac{6}{(1 + z)^2} - \frac{4}{1 + z} (\ln H)' \right] F - 3(1 + z) \Omega_{0m} \left( \frac{H_0}{H} \right)^2 F_0, \quad (84)$$

where $'$ denotes derivative with respect to redshift and $F_0$ is set to 1 in units of $\frac{1}{8\pi G_0}$ and corresponds to the present value of $F$. Alternatively, expressing in terms of redshift and eliminating the kinetic term $\Phi'^2$ from equations (82), (83) we find

$$U = \frac{(1 + z)^2 H^2}{2} [F'' + (\ln H)' - \frac{4}{1 + z} F' +$$

$$+ \left[ \frac{6}{(1 + z)^2} - \frac{2}{1 + z} (\ln H)' \right] F -$$

$$- 3(1 + z) \left( \frac{H_0}{H} \right)^2 F_0 \Omega_{m,0}]. \quad (85)$$

We now wish to explore the observational consequences that emerge from the following generic inequalities anticipated on a purely theoretical level

$$\Phi'(z)^2 > 0, \quad (86)$$

$$U'(z) > 0, \quad (87)$$

$$(\Phi'(z)^2)' > 0, \quad (\text{freezing}), \quad (88)$$

$$(\Phi'(z)^2)' < 0, \quad (\text{thawing}). \quad (89)$$

The inequality (86) is generic and merely states that the scalar field in scalar-tensor theories is real as it is directly connected to an observable quantity (Newton’s constant). The inequality (87) is also very general as it merely states that the scalar field rolls down (not up) its potential. This inequality is not as generic as (86) since it implicitly assumes a monotonic potential. Finally, the inequality (88) ((89)) denotes a scalar field which decelerates (accelerates) as it rolls down its potential thus corresponding to a freezing (thawing) quintessence model.
Since we are interested in the observational consequences of equations (86)-(89) at low redshifts, we consider expansions of equations (84) and (85) around \( z = 0 \) expanding \( F(z) \), \( H(z)^2 \) \( U(z) \), \( \Phi(z) \) as follows [33]:

\[
F(z) = 1 + F_1 z + F_2 z^2 + ..., \tag{90}
\]
\[
H(z)^2 = 1 + h_1 z + h_2 z^2 + ..., \tag{91}
\]
\[
U(z) = 1 + U_1 z + U_2 z^2 + ..., \tag{92}
\]
\[
\Phi(z) = 1 + \Phi_1 z + \Phi_2 z^2 + ..., \tag{93}
\]

where we have implicitly normalized over \( F_0, H_0, U_0 \) and \( \Phi_0 \).

It is straightforward to connect the expansion coefficients \( F_i \) of equation (90) with the current time derivatives of \( G(t) \) using equation (81) and the time-redshift relation

\[
\frac{dt}{dz} = -\frac{1}{H(z)(1 + z)} \tag{94}
\]

For example for \( F_1 \) we have

\[
F_1 = \frac{1}{F_0} \left. \frac{dF}{dz} \right|_{z=0} = \frac{\dot{G}_0}{G_0 H_0} = g_1, \tag{95}
\]

where the subscript 0 denotes the present time and \( H_0 \simeq 10^{-10} \) \( h \) \( yrs^{-1} \). Similarly, for \( F_2 \) we find

\[
F_2 = g_1 (g_1 - \frac{h_1 + 2}{4}) - \frac{g_2}{2}, \tag{96}
\]

where we have defined

\[
g_n \equiv \frac{G_0^{(n)}}{G_0 H_0^n}. \tag{97}
\]

with the superscript \(^{(n)}\) denoting the \( n^{th} \) time derivative. We may now substitute the expansions (90), (91), (93) in equation (84) replacing the coefficients \( F_i \) by the appropriate combination of \( g_n \). Equating terms order by order in \( z \) and ignoring terms proportional to \( g_1 \) due to solar system constraints which imply [94, 96]

\[
|g_1| < 10^{-13} \text{yr}^{-1} \text{s}^{-1} H_0^{-1} \simeq 10^{-3} h^{-1} \ll 1, \tag{98}
\]

we find for the zeroth and first order in \( z \)

\[
h_1 - 3 \Omega_{0m} + g_2 = \Phi_1^2 > 0, \tag{99}
\]
\[
-h_1(1 + h_1) + 2h_2 - 3 \Omega_{0m}(1 - h_1) - g_3 = 4\Phi_1 \Phi_2 = (\Phi^2)'(z = 0) \tag{100}
\]

The inequality (99) defines a forbidden sector for extended quintessence for each value of \( g_2 \). For \( g_2 = 0 \) this reduces to the well known result that MCQ can not cross the phantom divide line (see equation (116) below). Equation (100) can be used along with (88), (89) to divide the allowed \( (h_1, h_2) \) parameter sector into a freezing quintessence sector \( (\Phi'(z)^2)' > 0 \) and a thawing quintessence sector \( (\Phi'(z)^2)' < 0 \) for each set of \( (g_2, g_3) \).

Unfortunately, solar system gravitational experiments have so far provided constraints for \( g_1 \) [94] (equation (98)) but not for \( g_i \) with \( i \geq 2 \). This lack of constraints is not due to lack of observational data quality but simply due to the fact that existing codes have parameterized \( G(t) \) in the simplest possible way i.e. as a linear function of \( t \). It is therefore straightforward
Figure 7. The $h_1 - h_2$ sectors of EXQ: Sector I is the forbidden sector where the scalar field becomes imaginary. Sector II is also forbidden and corresponds to a scalar field that rolls up its potential. Sector III corresponds to ‘freezing’ EXQ where the field is decelerating down its potential. Sector IV corresponds to ‘thawing’ EXQ where the scalar field is accelerating down its potential. The forbidden sector I shrinks for a Newton’s constant $G$ that decreases with time ($g_2 > 0$) while the ‘freezing sector’ expands (Fig. 7a). The boundaries of the sectors are provided by (99), (100) and (110). The $1\sigma$, $2\sigma \chi^2$ contours obtained from the SNLS SNIa dataset for $\Omega_{0m} = 0.24$ using the CPL parameterization, are also shown.

to extend this parameterizations to include more parameters thus obtaining constraints of $g_i$ with $i \geq 2$. Such an analysis is currently in progress [95] but even before the results become available we can use some heuristic arguments to estimate the order of magnitude of the expected constraints on $g_i$ ($i \geq 2$) given the current solar system data.

Current solar system gravity experiments are utilizing lunar laser ranging [96] and high precision planet ephemerides data [94] to compare the trajectories of celestial bodies with those predicted by general relativity. The possible deviation from the general relativity predictions is parameterized [96] using the PPN parameters $\beta$ and $\gamma$, the first current derivative of Newton’s constant $\dot{G}/G_0$ and a Yukawa coupling correction to Newton’s inverse square law. These experiments have been collecting data for a time $\Delta t$ of several decades [96, 94] i.e. $\Delta t = O(100\text{yrs})$. The current $1\sigma$ constraint [94] of

$$|\frac{\dot{G}}{G_0}| = | - 0.2 \pm 0.5| \times 10^{-13} \text{yrs}^{-1} < 10^{-13} \text{yrs}^{-1},$$

(101)

implies an upper bound on the total variation $\frac{\Delta G}{G_0}$ over the time-scale $\Delta t$ of approximately

$$|\frac{\Delta G}{G_0}| \approx \frac{G_0}{G_0} |\Delta t| < 10^{-11}. $$

(102)

The same bound is obtained by considering the relative error in the orbital periods $T$ of the Earth and other planets which at $1\sigma$ is [94]

$$\frac{\Delta T}{T} < 10^{-12.}$$

(103)

Given that the Keplerian orbital period is

$$T_K = \sqrt{\frac{4\pi^2 r^3}{Gm}} \sim G^{-\frac{1}{2}},$$

(104)
we find
\[ \left| \frac{\Delta T}{T} \right| = \frac{1}{2} \frac{\Delta G}{G_0} < 10^{-12}, \tag{105} \]
in rough agreement with (102).

Using the upper bound (102) and attributing any variation of \( G \) to a term quadratic in \( \Delta t \) we get
\[ \frac{\Delta G}{G_0} \simeq \frac{\ddot{G}_0}{G_0} (\Delta t)^2 < 10^{-11}, \tag{106} \]
which implies that
\[ \left| \frac{\ddot{G}_0}{G_0} \right| < 10^{-15} \text{yr}^{-2} \rightarrow |g_2| < 10^5 h^{-2}, \tag{107} \]
giving a rough order of magnitude estimate of the upcoming constraints on \( g_2 \). Preliminary results from the analysis of solar system data indicate that \[ |g_2| < 10^{-15} \text{yr}^{-2} \rightarrow |g_2| < 10^5 h^{-2}, \tag{108} \]
which is not far off the rough estimate of equation (107). Generalizing the above arguments to arbitrary order in \( \Delta t \) we find
\[ |g_n| < 10^{8n-11} h^{-n}, \tag{109} \]
where \( n \geq 2 \).

We now return to the constraint equations (99) and (100) and supplement them by the constraint obtained from the inequality (87). Using the expansions (90), (91) and (92) in equation (85) and equating terms of first order in \( z \) we find
\[ U_1 = U'(z = 0) = \frac{1}{2}(5h_1 - 2h_2 - 9\Omega_{0m} + 5g_2 + g_3) > 0, \tag{110} \]
where as usual we have ignored terms proportional to \( g_1 \) due to (98). We now may use (99), (100) and (110) to define the following sectors in the \( h_1 - h_2 \) parameter space for fixed \( g_1, g_2 \):

- A forbidden sector I where the inequality (99) is violated.
- A forbidden sector II where the inequality (110) is violated but not the inequality (99).
- An allowed sector III of freezing EXQ where the inequalities (86), (87) and (88) are respected while (89) is violated.
- An allowed sector IV of thawing EXQ where the inequalities (86), (87) and (88) are respected while (88) is violated.

These sectors are shown in Fig. 7 for \( \Omega_{0m} = 0.24 \) \[7\], \( g_3 = 0 \) and three values of \( g_2 \): \( g_2 = 1.97, g_2 = 0 \) and \( g_2 = -1.91 \).

It would be interesting to superpose on these sectors of Fig. 7 the parameter \( \chi^2 \) contours obtained by fitting to the SnIa data. It is not legitimate however to extrapolate the expansions in powers of \( z \) out to \( z = \mathcal{O}(1) \) where the SnIa data extend. We thus use the CPL ansatz (12), (13). We superpose on the sectors of Fig. 7, the \( 1\sigma \) and \( 2\sigma \) \( h_1 - h_2 \) \( \chi^2 \) contours obtained by fitting the Chevalier-Polarski-Linder (CPL) [45, 98] parametrization (12), (13) to the SNLS dataset [3] with the same \( \Omega_{0m} \) prior (\( \Omega_{0m} = 0.24 \)). Using equation (13) along with the expansion (91) it is easy to show that
\[ h_1 = 3(1 + w_0 - \Omega_{0m} w_0), \tag{111} \]
\[ h_2 = \frac{3}{2}(2 + 5w_0(1 - \Omega_{0m}) + (1 - \Omega_{0m})(3w_0^2 + w_1)), \tag{112} \]
Figure 8. The sectors I-IV of Fig. 7 mapped on the $w_0 - w_1$ parameter space. The allowed sectors extend to $w_0 < -1$ for $g_2 > 0$.

so that the standard $\chi^2$ contours in the $w_0 - w_1$ space (see eg [37]) can be easily translated to the $h_1 - h_2$ space of Fig. 7. The parameter $w(z)$ of eq. (6) is particularly useful and physically relevant in the MCQ limit of $g_i \rightarrow 0$. In that limit $w(z)$ becomes the MCQ equation of state parameter i.e.

$$w(z) = \frac{p_{\text{MCQ}}}{\rho_{\text{MCQ}}} = \frac{\frac{1}{2} \dot{\Phi}^2 - U(\Phi)}{\frac{1}{2} \dot{\Phi}^2 + U(\Phi)},$$

(113)

as may be shown from the MCQ Friedman equations. The values of $g_2$ used in Figs 7a and 7c were motivated by demanding minimum and maximum overlap of the forbidden sector I with the $2\sigma$ $h_1 - h_2$ contour.

The following comments can be made with respect to Fig. 7:

- For values of $g_2 < -1.91$, the $2\sigma$ parameter contour obtained from SNLS lie entirely in the forbidden sector. This bound is independent of $g_3$ which does not enter in the inequality (99). Thus in the context of EXQ the constraint on $g_2$ obtained by the SNLS data at $2\sigma$ level is

$$g_2 = \frac{G_0}{C_0 H_0^2} > -1.91.$$  (114)

Notice the dramatic improvement of this constraint (with respect to the lower bound) compared to the anticipated constraint of $-10^5 < g_2 < 10^5$ anticipated from solar system data (equation (107))!

- The parameter sector III of freezing quintessence is significantly smaller than sector IV of thawing quintessence and the difference is more prominent for smaller $g_2$.

- For $g_2 > 0$ the allowed parameter space increases significantly compared to MCQ ($g_2 = 0$). Therefore, if future cosmological observations show preference to the forbidden sectors I or II of Fig. 7b (MCQ) this could be interpreted as evidence for EXQ with $g_2 > 0$ (see also comments in Refs. [34, 38, 33])

Even though the plots of Fig. 7 capture the full physical content of our results it is useful to express the sectors I-IV in terms of parameter pairs other than $h_1 - h_2$ which are more common in the literature. Such parameters are the expansion coefficients $w_i$ of $w(z)$

$$w(z) = w_0 + w_1 z + w_2 z^2 + ..., $$

(115)
which is connected to $H(z)$ via equation (6). By expanding both sides of equation (6) using equations (91) and (115) we may express $h_i$ in terms of $w_i$ thus rederiving equations (111) and (112) for $i = 0$ and $i = 1$. The result is identical since the $w_0$, $w_1$ expansion coefficients of equation (115) coincide with the $w_0$, $w_1$ coefficients of the CPL parametrization for a $w(z)$ given by equation (12). The advantage of using the parameters $w_0 - w_1$ instead of $h_1 - h_2$ is that they provide better contact with previous studies and can illustrate clearly the fact that a $g_2 > 0$ can provide a phantom behavior $w_0 < -1$ and crossing of the phantom divide line $w = -1$ in EXQ models. Using equations (111) and (112) we may express the constraint equations (99), (100) and (110) that define the sectors I-IV in Fig. 7 in terms of $w_0 - w_1$. The resulting equations are

\begin{align}
3(1 - \Omega_{0m})(1 + w_0) + g_2 &= \Phi_1^2 > 0, \\
3(1 - \Omega_{0m})(1 + w_0)(3\Omega_{0m}w_0 - 2) + w_1 - g_2(1 + 3(1 - \Omega_{0m})w_0) - g_3 &= (\Phi_2^2)'(z = 0), \\
\frac{2}{3}(1 - \Omega_{0m})(3(1 - w_0^2) - w_1) + \frac{5}{2}g_2 + \frac{3}{2}g_3 &> 0.
\end{align}

In the MCQ limit ($g_2 \to 0$, $g_3 \to 0$) equation (118) has also been obtained in Ref. [91] as a generic limit of MCQ. Using now (116)-(118) along with the $w_0 - w_1 \chi^2$ contours obtained from the SNLS dataset [3, 37] we construct Fig. 8 which is a mapping of Fig. 7 on the $w_0 - w_1$ parameter space. An interesting point of Fig. 8 is that for $g_2 > 0$ (Fig. 8a) the forbidden sector I shrinks significantly compared to MCQ (Fig. 8b) and allows for a $w_0 < -1$.

A final set of parameters we consider is the set of the expansion coefficients of the luminosity distance $d_L(z)$ which in a flat universe is connected to the Hubble expansion history $H(z)$ as

\begin{equation}
H(z)^{-1} = \frac{d}{dz} \left( \frac{d_L(z)}{1 + z} \right).
\end{equation}

Expanding $d_L(z)$ as

\begin{equation}
d_L(z) = z + d_{L2}z^2 + d_{L3}z^3 + ..., \tag{120}
\end{equation}

and using the expansion (91) in equation (119) we may express the coefficients $h_1 - h_2$ in terms of $d_{L2} - d_{L3}$ as follows

\begin{align}
h_1 &= 4(1 - d_{L2}), \\
h_2 &= 6(1 - 3d_{L2} + 2d_{L2}^2 - d_{L3}).
\end{align}

Figure 9. The sectors I-IV of Fig. 7 mapped on the $d_{L2} - d_{L3}$ parameter space. Notice that in this case the forbidden sector I is on the right (large $d_{L2}$).
Substituting now equations (121), (122) in equations (99), (100) and (110) we obtain the sector equations in $dL_2 - dL_3$ space as

\begin{equation}
4(1-dL_2) - 3\Omega_{0m} + g_2 = \Phi_2 > 0,
\end{equation}

\begin{equation}
4dL_2(2dL_2 + g_2 - 3\Omega_{0m}) + 9\Omega_{0m} - 8 - 12dL_3 - 5g_2 - g_3 = (\Phi^2)'(z = 0),
\end{equation}

\begin{equation}
4 + 8dL_2 - 12dL_2 + 6dL_3 - \frac{9}{2}\Omega_{0m} + \frac{2}{g_2} + \frac{g_2}{2} > 0.
\end{equation}

Using now equations (121), (122) to translate the $h_1 - h_2 \chi^2$ contours to the $dL_2 - dL_3$ parameter space and equations (123)-(125) to construct the sectors I-IV in the $dL_2 - dL_3$ space we obtain Fig. 9. The advantage of Fig. 9 compared to Figs. 7 and 8 is that it refers to the parameters $dL_2 - dL_3$ which are directly observable through the luminosity distances of SNIa without the need of any differentiation.

We have used generic theoretically motivated inequalities to investigate the space of observable cosmological expansion parameters that admits viable MCQ and EXQ theoretical models. Our inequalities are generic in the sense that they are independent of the specific features of any scalar field potential (eg scale, tracking behavior etc) and they only require that the models are internally consistent. The derived forbidden sectors which violate the above inequalities already have significant overlap with the parameter space which is consistent with observations at the 2σ level. This overlap which depends on the time derivatives of the Newton’s constant $G$ has lead to a useful constraint to the second derivative of $G$ (equation (114)) which is significantly more stringent compared to the corresponding constraint (107) anticipated from solar system gravity experiments.

An important reason that limits the observable parameter space consistent with MCQ and EXQ is the fact that the scalar field potential energy can induce accelerating expansion but not beyond the limit corresponding to the cosmological constant ($w(z) = -1$) obtained when the field’s evolution is frozen. Additional acceleration (superacceleration) can only be provided in the context of EXQ through the time variation of Newton’s constant $G$. A decreasing $G(t)$ with time favors accelerating expansion and allows for superacceleration. This physical argument is reflected in our results. We found that given $\dot{G}_0 \simeq 0$ (neglecting $g_1$ since $|g_1| < 10^{-4}$ from solar system tests) a $\dot{G}_0 > 0$ ($g_2 > 0$) decreases the forbidden sector and allows superacceleration ($w_0 < -1$). But $\dot{G}_0 = 0$ with $\ddot{G}_0 > 0$ implies that we are currently close to a minimum of $G(t)$ with $G(t)$ being larger in the past i.e.

$$\frac{G(t)}{G_0} \simeq 1 + \frac{1}{2}g_2(H_0(t - t_0))^2.$$  

(126)

Therefore a decreasing $G(t)$ corresponds to $\ddot{G}_0 > 0$ ($g_2 > 0$) (see Fig. 10) which in turn implies smaller forbidden sectors and allows for superacceleration in agreement with the above physical argument.

An additional interesting point is related to the construction of the $\chi^2$ contours of Figs. 7-9. The SNLS data analysis involved in the construction of these contours did not take into account the possible evolution of SNIa due to the evolving $G(t)$. It is straightforward to take into account the evolution of $G$ in the SNIa data analysis along the lines of Ref. [99]. In order to test the sensitivity of these contours with respect to the evolution of $G$ we have repeated the $\chi^2$ contour construction assuming a varying $G$ according to the ansatz

$$G(z) = G_0(1 + \alpha \frac{z^2}{(1 + z)^2}).$$

(127)
Figure 10. A $g_2 > 0$ implies a decreasing Newton’s constant $(\frac{G(t)}{G_0} \simeq 1 + \frac{1}{2} g_2 (H_0(t - t_0))^2$ around $t = t_0$).

which smoothly interpolates between the present value of $G = G_0$ and the high redshift value $G = G_0(1 + \alpha)$ implying that

$$\alpha = \frac{\Delta G}{G_0},$$

(128)

The parametrization (127) is consistent with both the solar system tests [94] ($\dot{G}/G_0 \simeq 0$) and the nucleosynthesis constraints [100]

$$|\frac{G_{nuc} - G_0}{G_0}| < 0.2.$$  

(129)

at 1σ, for $|\alpha| < 0.2$.

It is straightforward to evaluate $g_1, g_2, g_3$ in terms of $\alpha$ using the parametrization (127) and equation (94). The result is

$$g_1 = 0,$$

(130)

$$g_2 = 2\alpha,$$

(131)

$$g_3 = 3\alpha (-1 + 3w_0 (-1 + \Omega_{0m})).$$

(132)

We have considered the value of

$$\alpha = 0.2,$$

(133)

and repeated the SNLS data analysis taking into account the evolution of the SnIa absolute magnitude $M$ due to the evolving $G$ [101, 102, 103, 99] as

$$M = M_0 + \frac{15}{4} \log \frac{G}{G_0}.$$  

(134)

The steps involved in this analysis may be summarized as follows:

- Use the following magnitude redshift relation to fit to the SnIa data

$$m_{th}(z) = M_0 + 5 \log d_L(z) + \frac{15}{4} \log \frac{G(z)}{G_0},$$

(135)

where $G(z)$ is given by (127) and $d_L(z)$ is connected to $H(z)$ in the usual geometrically defined way (119) i.e.

$$d_L(z) = (1 + z) \int_0^z \frac{dz'}{H(z')}.$$  

(136)
Figure 11. Taking into account a $G$ variation consistent with the nucleosynthesis and solar system bounds introduces modifications to the $w_0 - w_1$ $\chi^2$ contours which are not significant for the currently available SnIa data (for the dashed line $\alpha = 0.2$ and the continuous $\alpha = 0$).

- Use an $H(z)$ parametrization that incorporates the evolution of $G$ (given by (127)) in the context of the CPL parametrization (13) i.e.

$$H^2(z) = \frac{G(z)}{G_0} H_0^2 [\Omega_{0m}(1+z)^3 + (1-\Omega_{0m})(1+z)^3(1+w_0+w_1)e^{-\frac{3w_1z}{1+w_1}}], \quad (137)$$

- Minimize the $\chi^2$ expression

$$\chi^2(w_0, w_1) = \sum_i \frac{(m_{\text{obs}}(z_i) - m_{\text{th}}(z_i; w_0, w_1))^2}{\sigma_{m_{\text{obs}}(z_i)}^2}, \quad (138)$$

and find the corresponding $1\sigma$ and $2\sigma$ $\chi^2$ contours along the lines of Refs. [37, 99]

The resulting $\chi^2$ contours in the $w_0 - w_1$ space are shown in Fig. 11 (dashed lines) superposed with the corresponding contours constructed by neglecting the evolution of $G$ (continuous lines). The change of the best fit $w_0 - w_1$ values and of the corresponding errorbars is minor (especially in the $w_0$ direction) and such that our main conclusion regarding the limiting values of $g_2$ remain practically unchanged even after the evolution of $G$ is taken into account in the analysis. This justifies neglecting the evolution of $G$ in the construction of the $\chi^2$ contours for our purposes. However, as the quality of SnIa data improves, it becomes clear from Fig. 11 that the effects of an evolving $G$ consistent with nucleosynthesis and solar system constraints on the data analysis can be significant! This possibility is further amplified if there are additional effects of an evolving $G$ on the SnIa data analysis. For example [104, 105] it is possible that the time scale stretch factor $s$ involved in the SnIa data analysis [3, 37] and arising from opacity effects in the stellar atmosphere may have a dependence on the Chandrasekhar mass and therefore on Newton’s constant $G$. We have shown however that even if such effects are included in the SnIa data analysis our results of Fig. 11 (dashed line) do not change more than 10%.

7. Summary-Conclusions
The main points of this review may be summarized as follows:
• Current geometrical and dynamical cosmological observations constraining the Hubble expansion history are consistent with the existence of a cosmological constant corresponding to an equation of state parameter \( w = -1 \) but they are also consistent with dark energy having an effective evolving \( w(z) \) that crosses the \( w = -1 \) line.

• The cosmological constant is currently the simplest and most probable cause of the accelerating expansion of the universe and may be generated by quantum fluctuations of the vacuum with a cutoff. A change of sign of the Casimir force is predicted in that case.

• An evolving equation of state parameter \( w(z) \) that is crossing the phantom divide line \( w = -1 \) is inconsistent with minimally coupled quintessence and also with scalar tensor quintessence if the Newton’s constant \( G(t) \) is increasing with time.

• Demanding consistency of scalar-tensor quintessence with local gravity experiments and with accelerating expansion crossing the \( w = -1 \) line implies a small present time derivative of the Newton’s constant \( G_0 \) i.e.

\[
\left| \frac{\dot{G}}{G_0 H_0} \right| < 10^{-4},
\]  

(139)

and a positive second derivative of \( G_0 \) i.e.

\[
\frac{\ddot{G}}{G_0 H_0^2} > O(1).
\]  

(140)

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