Spaces with Torsion from Embedding, 
and the Special Role of Autoparallel Trajectories

Hagen KLEINERT and Sergei V. SHABANOV

Institute for Theoretical Physics, FU-Berlin, Arnimallee 14, D-14195, Berlin, Germany

Abstract

As a contribution to the ongoing discussion of trajectories of spinless par-
ticles in spaces with torsion we show that the geometry of such spaces can be
induced by embedding their curves in a euclidean space without torsion. Tech-
nically speaking, we define the tangent (velocity) space of the embedded space
imposing non-holonomic constraints upon the tangent space of the embedding
space. Parallel transport in the embedded space is determined as an induced
parallel transport on the surface of constraints. Gauss’ principle of least con-
straint is used to show that autoparallels realize a constrained motion that has a
minimal deviation from the free, unconstrained motion, this being a mathemat-
ical expression of the principle of inertia. In contrast, geodesics play no special
role in the constrained dynamics, making them less likely candidates for particle
trajectories.

1. On an affine manifold equipped with a metric, there exist two preferred connections
compatible with the metric \[ \Gamma_{\mu \nu \kappa} \]. One is the Riemann connection defined only by the
metric. In a coordinate basis in the tangent space of the manifold, the coefficients of
this connection are Christoffel symbols

\[
\bar{\Gamma}_{\mu \nu \kappa} = g_{\kappa \lambda} \bar{\Gamma}_{\nu \kappa \lambda} = \frac{1}{2} \left( g_{\mu \nu, \kappa} + g_{\mu \kappa, \nu} - g_{\nu \kappa, \mu} \right). \tag{1}
\]

Here \( g_{\mu \nu} \) are components of the metric tensor, and indices after a comma stand for
the corresponding derivatives, \( T_{\mu \nu \ldots \kappa \ldots} = \partial_{\lambda} \partial_{\kappa} \cdots T_{\mu \nu \ldots} \). By definition, the covariant
derivative formed with this Riemann connection satisfies the metricty condition

\[
\bar{D}_{\mu} g_{\nu \kappa} = \partial_{\mu} g_{\nu \kappa} - \bar{\Gamma}_{\nu \mu \lambda} g_{\lambda \kappa} - \bar{\Gamma}_{\kappa \mu \lambda} g_{\nu \lambda} = 0. \tag{2}
\]

Apart from \( \bar{\Gamma}_{\mu \nu \lambda} \), there exists also a Cartan connection \( \Gamma_{\mu \nu \kappa} \). It satisfies the same
compatibility condition with the metric:

\[
D_{\mu} g_{\nu \kappa} = \partial_{\mu} g_{\nu \kappa} - \Gamma_{\nu \mu \lambda} g_{\lambda \kappa} - \Gamma_{\kappa \mu \lambda} g_{\nu \lambda} = 0. \tag{3}
\]

\footnote{Email: kleinert@physik.fu-berlin.de; shabanov@physik.fu-berlin.de; URL: http://www.physik.fu-berlin.de/~kleinert Phone/Fax: 0049/30/8383034
\footnote{on leave from Laboratory of Theoretical Physics, JINR, Dubna, Russia; a DFG fellow.}
It can always be represented in the form [1]

$$\Gamma_{\mu\nu\kappa} = \bar{\Gamma}_{\mu\nu\kappa} + K_{\mu\nu\kappa},$$  \hspace{1cm} (4)

where $K_{\mu\nu\kappa}$ is any antisymmetric tensor in $\nu\kappa$, called the contorsion tensor [1]

$$K_{\mu\nu\kappa} = S_{\mu\nu\kappa} - S_{\nu\kappa\mu} + S_{\kappa\mu\nu},$$  \hspace{1cm} (5)

where $S_{\mu\nu\kappa} = g_{\kappa\lambda} S_{\mu\nu}^{\lambda}$ and $S_{\mu\nu}^{\lambda}$ is the torsion tensor

$$S_{\mu\nu}^{\lambda} = \frac{1}{2} \left( \Gamma_{\mu\nu}^{\lambda} - \Gamma_{\nu\mu}^{\lambda} \right).$$  \hspace{1cm} (6)

Each connection compatible with the metric on the manifold defines a curve which parallel-transports its tangent vector along itself. Let $v^\mu$ stand for the tangent vector and $\dot{v}^\mu$ for its derivative with respect to the affine parameter, the proper time on the curve. Then the equation for the curve which parallel-transports its tangent vector along itself with respect to the Cartan connection reads

$$\frac{Dv^\mu}{dt} \equiv \dot{v}^\mu + \Gamma_{\lambda\nu}^\mu v^\lambda v^\nu = 0.$$  \hspace{1cm} (7)

It describes autoparallels, the straightest curves in Riemann-Cartan space. The reason for this name will be explained later in Section 5. The same equation with the Christoffel symbols describes geodesics, the shortest curves with respect to the metric $g_{\mu\nu}$. For the Cartan connection (4), the deviation from the geodesics is caused by a torsion force in (7) coming from the symmetrical part of the contorsion tensor $K_{(\mu\nu)\lambda} v^\mu v^\nu = -2S_{\lambda\nu} v^\mu v^\nu$.

Apart from extremizing a length between two fixed endpoints, geodesics in a Riemannian space can be obtained by embedding the Riemannian space in a euclidean space of a higher dimension. This is done by imposing certain constraints on the Cartesian coordinates spanning the euclidean space. The points on the constraint surface constitute the embedded Riemannian space. Straight lines in the euclidean space, which are geodesic and autoparallel and also determine a free motion in that space, become geodesics when the motion is restricted to the constraint surface. The restriction of the free motion to the constraint surface is done in a conventional way, i.e., by adding the equations of constraints to the equations of motion. When the constraint force is removed, geodesic trajectories turns into straight lines in the embedding space.

For dynamics in Riemann-Cartan spaces, no such embedding is known. The purpose of this letter is to fill this gap. The new embedding will have the property that autoparallels are realized as trajectories of a constrained free motion.

Key observation of our theory is the fact that in order to define a geometry (metric and the law of parallel-transport) by embedding it is not necessary to constrain the position in a euclidean space. One may impose constraints only on the tangent (or velocity) space, thereby defining a physical tangent space as a subspace embedded in a
bigger velocity space. This is sufficient to define all curves in the embedded space as a special subset of all curves in the embedding space because each curve is specified by its tangent vector. The metric in the embedded space is naturally induced by restricting the scalar product in the bigger (embedding) tangent space to the constraint surface. The induced connection is uniquely determined by the compatibility condition of the embedding of the tangent space with the parallel-transport law in the embedding space.

This means the following: Take a curve in the original space connecting points 1 and 2. This curve is then embedded into a bigger euclidean space by specifying the tangent vector of the image curve. A vector from the tangent space at point 1 is parallel-transported along the curve to point 2, and then it is embedded in the bigger space. We require that the resulting vector must be the same as the one obtained in the opposite way: The vector at point 1 is first embedded into a bigger euclidean space and then parallel-transported along the image of the curve connecting points 1 and 2. This compatibility condition ensures that the connection in the original space is uniquely determined by the embedding law.

Constraints imposed on the tangent space can be non-holonomic, and this is the source of torsion. The notion of ”holonomic” and ”non-holonomic” constraints is the same as in classical mechanics. For a mechanical system, generalized velocities are elements of the tangent space of its configuration space. Let the motion be subject to constraints linear in velocities. According to the Hertz classification, constraints are said to be holonomic if they are integrable (i.e., equivalent to some constraints on the configuration space only), and non-holonomic if they are non-integrable. Sometimes dynamical systems with non-holonomic constraints are simply called non-holonomic systems. It is important to realize that the motion of non-holonomic systems does not occur on any submanifold of the configuration space, nonetheless it is described by a less number of parameters than the corresponding unconstrained motion.

Upon an embedding via non-holonomic constraints on the tangent space, any curve that parallel-transports its tangent vector along itself has an image with same property. Therefore straight lines, which are autoparallels and geodesics with respect to a trivial euclidean connection, are natural images of autoparallels in the embedded space. Using Gauss’ principle of least constraint we show that autoparallels describe a constrained motion such that the acceleration (or the force) induced by the constraints has a least deviation from the acceleration of the corresponding unconstrained motion, while geodesics play no special role in the constrained dynamics. This comprises the main result of our paper.

We remark that torsion has been included into general relativity. The corresponding theory is known as the Cartan-Einstein theory. From the point of view of the general coordinate invariance, both geodesics and autoparallels are equally good candidates for trajectories of spinless test particles. The conventional way of obtaining the law of interaction between matter and spacetime connection is to apply the gauge principle and minimal coupling. In such an approach, scalar fields are decoupled from torsion, thus leading to geodesics as true trajectories of spinless point particles.
This, however, does not imply that the autoparallels are incompatible with the gauge principle. Equation (7) is just as gauge covariant as the geodesic equation $\frac{Dv^\mu}{dt} = 0$. But autoparallels do not seem to be compatible with the minimal gauge coupling principle, when all derivatives in a Lagrangian of a free theory are replaced by the covariant derivatives in order to obtain the interacting theory.

The embedding has been a powerful tool in studying geometrical forces in non-Euclidean configuration spaces. Our approach based on the constrained dynamics might be useful in constructing possible field models of interaction between matter and a generic spacetime connection where autoparallels are true trajectories of spinless particles. Up to now neither experiments nor theoretical principles forbid such theories.

2. Let $[x]$ be a Euclidean space and $x^i, i = 1, 2, ..., M = \dim[x]$, be a set of coordinates. Let $[q]$ be a space of a smaller dimension, $N = \dim[q]$, spanned by local coordinates $q^\mu, \mu = 1, 2, ..., N$. Tangent spaces of $[x]$ and $[q]$ are denoted as $T[x]$ and $T[q]$, respectively. Consider a curve $q^\mu(t)$ in $[q]$ and tangent spaces at each point of the curve, $T[q(t)]$. We define a curve $x^i(t)$ in the embedding space $[x]$ by specifying its tangent vector

$$v^i = \varepsilon^i_{\mu}(q)v^\mu, \quad v^\mu = \dot{q}^\mu(t), \quad v^i = \dot{x}^i(t),$$

where coefficients $\varepsilon^i_{\mu}(q)$ are some functions on $[q]$. From (8) follows

$$x^i(t) = x^i(0) + \int_0^t dq^\mu \varepsilon^i_{\mu}(q).$$

For any curve $q^\mu(t)$ in $[q]$, equation (9) determines a curve in the space $[x]$ up to a global translation on a vector $x^i(0)$. Thus it determines an embedding of the space of all paths in $[q]$ into the space of all paths in $[x]$. Assuming equation (8) to hold for all curves in the space $[q]$ passing through some point $q^\mu$, we specify the embedding of $T[q]$ at this point into the Euclidean space $T[x]$. Note that tangent spaces at all points of $[x]$ are the same and coincide with the space $[x]$ because the parallel transport is trivial in the space $[x]$ (the connection $\Gamma_{ij}^k$ vanishes identically). So, the shift of the image curve $x^i(t)$ on a constant vector in (8) is irrelevant for the embedding of the tangent space. The embedding of the path space or the tangent space includes the case of the pointwise embedding of the space $[q]$ itself into $[x]$, but it appears to be more general as we shall see.

Consider two sets of tangent spaces $T[q(t)]$ and $T[x(t)]$ at points of the curve $q^\mu(t)$ and of its image $x^i(t)$ defined by (9). We embed the space $T[q(t)]$ in $T[x(t)]$ so that for any element $T^\mu$ of $T[q(t)]$, its image in $T[x(t)]$ is

$$T^i = \varepsilon^i_{\mu}(q(t))T^\mu.$$
the total derivative with respect to the affine parameter
\[ \frac{DT_i}{dt} = v^j D_j T^i = \frac{dT_i}{dt}. \] (11)

Similarly, an infinitesimal change of the vector \( T^\mu \) under the parallel transport along the curve \( q^\mu(t) \) is specified by the covariant derivative
\[ \frac{DT^\mu}{dt} = v^\nu D_\nu T^\mu = v^\nu (\partial_\nu T^\mu + \Gamma^\mu_{\kappa\nu} T^\kappa) \] (12)
with \( \Gamma^\mu_{\nu\kappa} \) being the connection on the space \([q]\). Now we come to the crucial condition for our embedding procedure: We require that the vector \( DT_i/dt \) must be the image of the vector \( DT^\mu/dt \), that is,
\[ \frac{DT_i}{dt} = \varepsilon^i_{\mu} \frac{DT^\mu}{dt}. \] (13)

Equation (13) has a transparent geometrical meaning. The parallel transport of the vector \( T^\mu \) along any curve \( q^\mu(t) \) and the subsequent embedding of the resulting vector into the bigger space \( T[x] \) via the relation (10) gives an element of \( T[x] \). This element must coincide with the one obtained in the opposite way in which the vector \( T^\mu \) is first embedded and then parallel-transported along the image \( x^i(t) \) of the curve \( q^\mu(t) \). This implies that the embedding of the tangent space \( T[q] \) at any point of \([q]\) is compatible with the parallel transport on \([q]\). The compatibility condition (13) uniquely determines the connection coefficients \( \Gamma^\mu_{\nu\kappa} \) via the embedding coefficients \( \varepsilon^i_{\mu} \).

Before we proceed to prove this statement, let us introduce some useful notations. For any two vectors from \( T[x] \), one can introduce a scalar product associated with the Cartesian metric on \([x]\)
\[ (\tilde{T}, T) = \delta_{ij} \tilde{T}^i T^j. \] (14)

If the vectors \( T^i \) and \( \tilde{T}^i \) are the images of \( T^\mu \) and \( \tilde{T}^\mu \), respectively, then the embedding coefficients \( \varepsilon^i_{\mu} \) determine an induced metric on \([q]\)
\[ (\tilde{T}, T) = g_{\mu\nu} \tilde{T}^\mu T^\nu, \quad g_{\mu\nu} = (\varepsilon_{\mu}, \varepsilon_{\nu}). \] (15)

It is useful to introduce the quantity
\[ \varepsilon^{ij} = \varepsilon^i_{\mu} g^{\mu\nu}, \] (16)
where \( g^{\mu\lambda} g_{\lambda\nu} = \delta_{\mu\nu} \). From (17) follows that
\[ (\varepsilon^\mu, \varepsilon^\nu) = g^{\mu\nu}, \quad (\varepsilon^\mu, \varepsilon_{\nu}) = \delta^\mu_{\nu}. \] (17)

Assuming that the metrics \( g_{\mu\nu} \) and \( \delta_{ij} \) are used to lower or raise indices of tensors on \([q]\) and \([x]\), respectively, the embedding condition (13) can be written in a more general form
\[ \frac{dT^i_{\kappa\lambda\cdots}}{dt} = \frac{d}{dt} \left( \varepsilon^i_{\mu} \varepsilon^j_{\nu} \cdots \varepsilon^k_{\lambda} \varepsilon^\beta_\cdots T^\mu_{\nu\cdots} \right) = \varepsilon^i_{\mu} \varepsilon^j_{\nu} \cdots \varepsilon^k_{\lambda} \varepsilon^\beta_\cdots \frac{DT^\mu_{\nu\cdots}}{dt}. \] (18)
where the covariant derivative reads
\[
\frac{DT_{\lambda \beta \cdots}^{\mu \nu \cdots}}{dt} = v^\lambda \left( \partial_\lambda T_{\alpha \beta \cdots}^{\mu \nu \cdots} + \Gamma_\mu \lambda T_{\alpha \beta \cdots}^{\mu \nu \cdots} + \cdots - \Gamma_\alpha \lambda T_{\lambda \beta \cdots}^{\mu \nu \cdots} - \cdots \right) .
\] (19)

Doing the differentiation in the left-hand side of (18) and applying relations (17) we find
\[
v^\lambda \Gamma_{\nu \lambda}^\mu = \left( \varepsilon^\mu, \frac{d}{dt} \varepsilon_\nu \right) = - \left( \varepsilon_\nu, \frac{d}{dt} \varepsilon^\mu \right) ,
\] (20)

which should hold for any curve in \([q]\) (for any \(v^\mu\)). Thus, we conclude that
\[
\Gamma_{\mu \nu \lambda} = g^{\lambda \alpha} (\varepsilon_\alpha, \varepsilon_{\mu \nu}) .
\] (21)

Equation (20) ensures that along any curve \(q^\mu(t)\), the fields \(\varepsilon_i^\mu(q(t))\) and \(\varepsilon^i\mu(q(t))\) are transported parallel, as expressed by the relations \(D\varepsilon_i^\mu(q(t))/dt = 0\), \(D\varepsilon^i\mu(q(t))/dt = 0\).

Applying the covariant derivative \(D/dt\) to the metric (15) we obtain from the chain rule of differentiation \(Dg_{\mu \nu}(q(t))/dt = 0\) for any curve in \([q]\), which ensures the compatibility of the induced connection with the induced metric.

Thus we have succeeded in determining metric and parallel transport in the space \([q]\) by an embedding of all paths in \([q]\) into the space of all paths in the bigger euclidean space. The embedding of the path space implies the embedding of the tangent space, thus determining the induced metric on \([q]\). By imposing the condition that the parallel transport law is compatible with the embedding of the tangent space, the connection in the space \([q]\) is uniquely determined, too.

3. Let us now turn to the analysis of the connection (21). First of all, we observe that the torsion tensor is, in general, nonzero
\[
S_{\nu \kappa}^\mu = \frac{1}{2} g^{\mu \lambda} \left[ (\varepsilon_\lambda, \varepsilon_{\nu \kappa}) - (\varepsilon_\lambda, \varepsilon_{\kappa \nu}) \right] .
\] (22)

The torsion induced by the embedding is zero iff
\[
\varepsilon^i_{\nu, \mu} = \varepsilon^i_{\mu, \nu} .
\] (23)

If this condition is satisfied, the matrix elements in relations (8) are the derivatives of \(M\) functions \(\varepsilon^i(q)\)
\[
\varepsilon^i_{\mu}(q) = \partial_\mu \varepsilon^i(q) .
\] (24)

In this case, the path embedding (8) can be achieved by a pointwise embedding of the space \([q]\) in \([x]\). Relation (8) can then be written in the form
\[
dx^i = \varepsilon^i_{\mu}(q) dq^\mu = d\varepsilon^i(q) ,
\] (25)

i.e., we get the pointwise embedding
\[
x^i = \varepsilon^i(q) .
\] (26)
The condition (8) may be thought as constraints on the velocity $v^i$. The torsion tensor (22) vanishes when the constraints are integrable as can be seen from (23). In this case, the path embedding and the tangent space embedding can be obtained right-away from the space embedding (26). When the constraints are non-integrable, there is no pointwise embedding of $[q]$ into $[x]$, while the path space or the tangent space can still be embedded in the corresponding larger space. The latter is sufficient to specify the metric and connection induced by the embedding. In fact, in this approach the connection appears to be the most general connection compatible with the metric.

The metric tensor $g_{\mu\nu}$ has $N(N+1)/2$ independent components. The torsion tensor $S^\nu{}_{\kappa\mu}$ has $N^2(N-1)/2$ independent components. To embed a general metric space with torsion, the number $NM$, being the number of independent embedding coefficients $\varepsilon^i{}_{\mu}$, should be greater or equal to $N(N^2 + 1)/2$. This leads to the relation between the dimensions of the spaces $[q]$ and $[x]$

$$(\dim[q])^2 + 1 \leq 2 \dim[x].$$

(27)

4. We are now ready to show that the autoparallel curves are specially favored geometric curves in the space $[q]$ since our embedding procedure maps them into the straight lines in the embedding space $[x]$. A straight line parallel-transports its tangent vector along itself with respect to a trivial connection $\Gamma_{ij}{}^k = 0$:

$$\frac{Dv^i}{dt} = \dot{v}^i = 0. \tag{28}$$

Applying the compatibility condition (13) to the velocity vector $T^\mu = v^\mu$, we conclude that the straight line is the image of a curve satisfying the equation $Dv^\mu/dt = 0$, which is the autoparallel, as announced above. Indeed, upon substituting (8) into (28) and multiplying the result by $\epsilon^i{}_{\mu}$ we get

$$(\varepsilon^\mu, \dot{v}) = \dot{v}^\mu + \left( \epsilon^\mu, \frac{d}{dt} \epsilon^i{}_{\nu} \right) v^\nu = \frac{Dv^\mu}{dt} = 0. \tag{29}$$

For every path in $[q]$, the compatibility of the parallel transport law with the embedding yields the condition (21). Therefore equation (29) turns into the autoparallel equation (4). In particular, when the constraint (8) is integrable, the torsion vanishes, and equation (29) describes geodesics in the embedded manifold (26).

5. It is useful to give a mechanical interpretation of why autoparallels should be favored as particle trajectories. They describe a constrained motion with an acceleration that deviates minimally from the acceleration of the corresponding unconstrained motion. This property can be formulated mathematically by means of Gauss’ principle of least constraint [2, 3].

Consider a Lagrangian system in the space $[x]$ with a Lagrangian $L = L(x, v)$. At each moment of time, a state of the system can labeled by the pair $\psi = (x^i(t), v^i(t))$. 
At the state $\psi$, construct a matrix $H_{ij} = \partial^2 L/\partial v^i \partial v^j$ called a Hessian of the system. Consider two paths $x_1^i(t)$ and $x_2^i(t)$ going through the state $\psi$. Gauss’ deviation function (sometimes also called Gauss’ constraint) for two paths at the state $\psi$ reads

$$G_\psi = \frac{1}{2} \left( \dot{v}_1^i - \dot{v}_2^i \right) H_{ij} \left( \dot{v}_1^j - \dot{v}_2^j \right).$$

(30)

It measures the deviation of two motions from one another at the same system state $\psi$. Let the motion in $[x]$ be subject to constraints. All paths $x^i(t)$ allowed by the constraints and going through a state $\psi$ are called conceivable motions. A path $\bar{x}^i(t)$ is called released motion if it satisfies the Euler-Lagrange equations for the Lagrangian $L$. Gauss’ principle of least constraint says that the deviation of conceivable motions from a released motion takes a stationary value on the actual motion.

In our case, the released motion is a free motion with zero acceleration $\ddot{x}^i = 0$ and $H_{ij} = \delta_{ij}$. Accelerations of the conceivable motions are

$$\dot{v}^i = \varepsilon_{\mu}^i \dot{v}^\mu + \varepsilon_{\mu,\nu}^i v^\mu v^\nu.$$

(31)

Gauss’ deviation function (30) assumes the form

$$G_\psi = \frac{1}{2} \left[ \dot{v}^\mu + \varepsilon_{\mu,\nu} \frac{d}{dt} \varepsilon_{\nu}^i \right] v^\nu.$$

(32)

It is non-negative, $G_\psi \geq 0$, for any state $\psi$ and achieves its absolute minima, $G_\psi = 0$, for the acceleration determined by equation (29). Thus, the actual motion is realized by autoparallels.

Gauss’ principle of least constraint implies that the geometrical force caused by the constraints, must be minimal for the actual constrained motion. Thus, in the framework of constrained dynamics, autoparallels have the least deviation from the straight lines describing free, unconstrained motions. That is why they can rightfully be called the straightest lines on a manifold. The fact that particles must follow straightest lines is a consequence of the physical phenomenon inertia. It is hard to understand how a particle should know where to go to make its trajectory the shortest path to a distant point.

Finally, we remark that in the case of integrable constraints, Gauss’ principle of least constraint leads to geodesics, while for non-integrable constraints, geodesics do not have the least deviation from the free, unconstrained motion and do not play any special role in the dynamics.

Another interpretation of the autoparallel equation (29) rests on the d’Alembert-Lagrange principle [2, 3]. In theoretical mechanics, elements of the tangent space are also called virtual velocities. The embedding condition (10) determines virtual velocities of the constrained motion. Let us denote the Lagrange derivative as $[L]_i = d/dt \partial L/\partial v^i - \partial L/\partial x^i$. The d’Alembert-Lagrange principle asserts that the conceivable
motion of a system with the Lagrangian $L$ is an actual motion if for every moment of time

$$(T, [L]) = 0,$$  \hspace{1cm} (33)

for all virtual velocities of the constrained motion. Taking the free motion Lagrangian $L = (v, v)/2$ with the constraint (33) and substituting them into (33) we find that the autoparallel equation (29) follows from (33) for an arbitrary virtual velocity $T^\mu$.

In addition we remark that autoparallels can be obtained from Hölder’s variational principle [2, 3] applied to the free action. Let $\delta x^i \in T[x]$ be a variation vector field. Amongst all variation vector fields we single out those that are virtual velocities of the constrained motion,

$$\delta x^i = \varepsilon^i_{\mu} \delta q^\mu, \hspace{0.5cm} \delta q^\mu \in T[q].$$  \hspace{1cm} (34)

A conceivable path is called a critical point of the action functional if its variation vanishes when restricted on the subspace of virtual velocities of the constrained motion. Hölder’s variational principle suggests that the actual constrained motion is a critical point of the action. Making a variation of the action we find the equation of motion

$$(\delta x^i, [L]) = 0.$$  \hspace{1cm} (35)

Substituting the expression (34) for conceivable velocities and admissible variations (34) in (35), we obtain the autoparallel equation (29) for the free Lagrangian, $[L]_i = \delta_{ij} \dot{v}^j$.

Another remark concerns relativistic motion. The autoparallel motion may also be embedded into a Minkovski space in the same fashion as for a euclidean space. In all formulas given above, the euclidean metric $\delta_{ij}$ has to be replaced by a corresponding indefinite metric of the Minkovski space. Similarly, in the three variational principles for autoparallels considered in this section, the free motion in the embedding space should be described by the corresponding Lagrangian of a free relativistic particle, while all time derivatives in the equations of motion must be replaced by the derivatives with respect to the proper time defined on the constraint surface.

Finally we point out that the motion of a holonomic system is completely determined by the restriction of the Lagrangian to the constraint surface [4]. Thus, holonomic constrained systems are indistinguishable from ordinary unconstrained Lagrangian systems. This is not true for non-holonomic systems, meaning that the Euler-Lagrange equations for the Lagrangian restricted on the constraint surface do not coincide with the original equations for the constrained motion. This difficulty prevents us from applying a conventional Hamiltonian formalism to the autoparallel motion, and subjecting it to a canonical quantization. In other words, Dirac’s method of quantizing constrained systems [7] does not apply to non-holonomic systems because their motion is not described by the conventional Lagrange formalism [2]. The problem requires a further study.

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