Structure behind Mechanics I: Foundation

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Abstract

This paper proposes a basic theory on physical reality and a new foundation for quantum mechanics and classical mechanics. It presents a scenario not only to solve the problem of the arbitrariness on the operator ordering for the quantization procedure, but also to clarify how the classical-limit occurs. This paper is the first of the three papers into which the previous paper quant-ph/9906130 has been separated for readability.

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1 INTRODUCTION

Seventeenth century saw Newtonian mechanics, published as "Principia: Mathematical principles of natural philosophy," the first attempt to understand this world under few principles rested on observation and experiment. It bases itself on the concept of the force acting on a body and on the laws relating it with the motion. In eighteenth century, Lagrange's analytical mechanics, originated by Mautertuis’ theological work, built the theory of motion on an analytic basis, and replaced forces by potentials; in the next century, Hamilton completed the foundation of analytical mechanics on the principle of least action in stead of Newton's laws. Besides, Maxwell’s theory of the electromagnetism has the Lorentz invariance inconsistent with the invariance under Galilean transformation, that Newtonian mechanics obeys. Twentieth century dawned with Einstein’s relativity changing the ordinary belief on the nature of time, to reveal the four-dimensional spacetime structure of the world. Relativity improved Newtonian mechanics based on the fact that the speed of light c is an invariant constant, and revised the self-consistency of the classical mechanics. Notwithstanding such a revolution, Hamiltonian mechanics was still effective not only for Newtonian mechanics but also for the Maxwell-Einstein theory, and the concept of energy and momentum played the most important role in the physics instead of force for Newtonian mechanics.

Experiments, however, indicated that microscopic systems seemed not to obey such classical mechanics so far. Almost one century has passed since Planck found his constant h; and almost three fourth since Heisenberg \(^\dagger\), Schrödinger \(^\dagger\) and their contemporaries constructed the basic formalism of quantum mechanics after the early days of Einstein and Bohr. The quantum mechanics based itself on the concept of wave functions instead of classical energy and momentum, or that of operators called as observables. This mechanics reconstructed the classical field theories except the general relativity. Nobody denies how quantum mechanics, especially quantum electrodynamics, succeeded in twentieth century and developed in the form of the standard model for the quantum field theories through the process to find new particles in the nature.

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Quantum mechanics, however, seems to have left some fundamental open problems on its formalism and its interpretation: the problem on the ambiguity of the operator ordering in quantum mechanics, which is crucial to quantize the Einstein gravity for instance, and that on the reality, which seems incompatible with the causality. These difficulties come from the problem how and why quantum mechanics relates itself with classical mechanics: the relationship between the quantization that constructs quantum mechanics based on classical mechanics and the classical-limit that induces classical mechanics from quantum mechanics as an approximation with Planck’s constant $h$ taken to be zero; the incompatibility between the ontological feature of classical mechanics and the epistemological feature of quantum mechanics in the Copenhagen interpretation.

Now, this paper proposes a basic theory on physical reality, and introduces a foundation for quantum mechanics and classical mechanics, named as protomechanics, that is motivated in the previous letter. It also attempts to revise the nonconstructive idea that the basic theory of motion is valid in a way independent of the describing scale, though the quantum mechanics has once destroyed such an idea that Newtonian mechanics held in eighteenth century. The present theory supposes that a field or a particle $X$ on the four-dimensional spacetime has its internal-time $\tilde{o}_A(X)$ relative to an domain $A$ of the spacetime, whose boundary and interior represent the present and the past, respectively. It further considers that object $X$ also has the external-time $\tilde{o}^*_A(X)$ relative to $A$ which is the internal-time of all the rest but $X$ in the universe. Object $X$ gains the actual existence on $A$ if and only if the internal-time coincides with the external-time:

$$\tilde{o}_A(X) = \tilde{o}^*_A(X).$$  \hspace{1cm} (1)

This condition discretizes or quantizes the ordinary time passing from the past to the future, and enables the deterministic structure of the basic theory to produce the nondeterministic characteristics of quantum mechanics. The both sides of relation further obey the variational principle as

$$\delta \tilde{o}_A(X) = 0, \quad \delta \tilde{o}^*_A(X) = 0.$$  \hspace{1cm} (2)

This relation reveals a geometric structure behind Hamiltonian mechanics based on the modified Einstein-de Broglie relation, and produces the conservation law of the emergence-frequency of a particle or a field based on the introduced quantization law of time. The obtained mechanics, protomechanics, rests on the concept of the synchronicity instead of energy-momentum or wave-functions, that synchronizes two intrinsic local clocks located at different points in the space of the objects on a present surface in the spacetime. It will finally solve the problem on the ambiguity of the operator ordering, and also give a self-consistent interpretation of quantum mechanics as an ontological theory.

The next section explains the basic laws on reality as discussed above, and leads to the protomechanics in Section 3, that produces the conservation laws of momentum and that of emergence-frequency. Section 4 presents the dynamical construction for the introduced protomechanics by utilizing the group-theoretic method called Lie-Poisson mechanics (consult APPENDIX). It provides the difference between classical mechanics and quantum mechanics as that of their function spaces: the function space of the observables for quantum mechanics includes that for classical mechanics; the dual space of the emergence-measures for classical mechanics includes that for quantum mechanics, viceversa. A brief statement of the conclusion immediately follows.

The present paper shall leave to the following paper the detail proof how the protomechanics deduces classical mechanics and quantum mechanics, since such proof needs an intricate mathematical technique strayed from the present context; and it will demonstrate there still valid for the description of a half-integer spin against the ordinary belief that the existence of such spin averts realistic approaches to the quantum mechanics from the completeness. It also has to leave to another paper the concluded implication how the present theory gives a self-consistent interpretation for quantum mechanics, since such discussion needs a

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1 The author of paper, "Tosch Ono," is the same person as that of the present paper, "Toshihiko Ono."

2 This naming of synchronicity is originated by Jung.
philosophical background beyond the scope of the present paper; and it will further prove there to provide the semantics of the regularization in a quantum field theory, the quantization of a phenomenological system, the causality in quantum mechanics and the origin of the thermodynamic irreversibility under the new insight.

The following diagram illustrates the construction of the present paper.

[Diagram]

larger scale ←

\[
\begin{array}{c}
\text{classical mechanics} [1] \\
\text{classical part: } \hbar \to 0 \\
\text{laws on reality (2)} \\
\text{protomechanics (3,4)}
\end{array}
\]

quantum mechanics [11]

* Numbers in bracket ( ) refer those of sections.

In this paper, I will use Einstein’s rule in the tensor calculus for Roman indices' \( i, j, k \in \mathbb{N} \) and Greek indices' \( \nu, \mu \in \mathbb{N} \), and not for Greek indices' \( \alpha, \beta, \gamma \in \mathbb{N} \). Consult the brief review on Lie-Poisson mechanics in APPENDIX. In addition, notice that the basic theory uses so-called \textbf{c-numbers}, while it will also utilize \textbf{q-numbers} to deduce the quantum mechanics in [11] for the help of calculations.

2 LAWS ON REALITY

Let \( M^{(4)} \) represent the spacetime, being a four-dimensional oriented \( C^\infty \) manifold, that has the topology or the family \( \mathcal{O} = \mathcal{O}_{M^{(4)}} \) of its open subsets, the topological \( \sigma \)-algebra \( \mathcal{B}(\mathcal{O}_{M^{(4)})} \), and the volume measure \( v^{(4)} \) induced from the metric \( g \) on \( M^{(4)} \). We shall certainly choose an arbitrary domain \( A \in \mathcal{O} \) in the discussion below, but we are interested in the case that domain \( A \) represents the past at a moment whose boundary \( \partial A \) is a three-dimensional present hypersurface in \( M^{(4)} \).

The space \( \tilde{M} \) represents that of the objects whose motion will be described, and has a projection operator \( \chi_A : \tilde{M} \to \tilde{M} \) for every domain \( A \in \mathcal{O} \) such that \( \chi_A^2 = \chi_A \). Every object \( X \in \tilde{M} \) has its own domain \( D(X) \) such that

\[
\chi_{D(X) \setminus A}(X) = X \iff D(X) \cap A = \emptyset.
\]

In particle theories, \( \tilde{M} \) is identified with the space of all the one-dimensional timelike manifolds or curves in \( M^{(4)} \), where \( \chi_A(l) = l \cap A \) for every domain \( A \) and \( D(l) = l \). In field theories, the space \( \Psi(M^{(4)}, V) \) of the complex valued or \( \mathbb{Z}_2 \)-graded fields over \( M^{(4)} \) such that \( \psi^{(4)} \in \Psi(M^{(4)}, V) \) is a mapping \( \psi^{(4)} : M^{(4)} \to V \)

\footnote{The paper of quant-ph/9906130 contains the information not only in the present paper but also in the following two papers [11, 12].}

\footnote{Such distinction between c-numbers and q-numbers does not play an important role in the present theory.}

\footnote{Spacetime \( M^{(4)} \) may be endowed with some additional structure.}
for a complex valued or \( \mathbb{Z}_2 \)-graded vector space \( V \). Mapping \( \chi_A \) satisfies that \( \chi_A(\psi^{(4)})(x) = \psi^{(4)}(x) \) if \( x \in A \) and that \( \chi_A(\psi^{(4)})(x) = 0 \) if \( x \notin A \), and \( D(\psi^{(4)}) \) gives the support of \( \psi^{(4)} \); \( D(\psi^{(4)}) = \text{supp}(\psi^{(4)}) \).

In addition, let us consider the set \( \mathcal{D}(\bar{M}) \) of all the differentiable mapping from \( \bar{M} \) to itself and the set \( \mathcal{D}(M^{(4)}) \) of all the diffeomorphisms of spacetime \( M^{(4)} \). In particle theories, set \( \mathcal{D}(\bar{M}) \) will be regarded as set \( \mathcal{D}(M^{(4)}) \); and, in field theories, it is the set of all the linear transformations of a field such that \( \Phi(\psi^{(4)}) = \psi^{(4)} + \phi^{(4)} \).

Now, let us assume that an object has its own internal-time relative to a domain of the spacetime.

**Law 1** For every domain \( A \in \mathcal{O} \), the mapping \( \tilde{o}_A : \bar{M} \to S^1 \) has an action \( S_A : \bar{M} \to \mathbb{R} \) and equips an object \( X \in M \) with the internal-time \( \tilde{o}_A(X) \):

\[
\tilde{o}_A(X) = e^{iS_A(X)}. \tag{4}
\]

For particle theories, a one-dimensional submanifold or a curve \( l \subset M^{(4)} \) represents the nonrelativistic motion for a particle such that \((t, x(t)) \in l \) for \( t \in T \), where \( M^{(4)} \) is the Newtonian spacetime \( M^{(4)} = T \times M^{(3)} \) for the Newtonian time \( T \subset \mathbb{R} \) and the three-dimensional Euclidean space \( M^{(3)} \); thereby, it has the following action for the ordinary Lagrangian \( L : TM \to \mathbb{R} \):

\[
S_A(l) = \frac{\hbar}{2} \int_{t \in A} dt \left( \frac{dx(t)}{dt} \right)^2,
\]

where \( \hbar = \hbar/4\pi \) or \( h = h/2\pi \) for Planck’s constant \( h \) (\( h = h/2\pi \)). The relativistic motion of a free particle whose mass is \( m \) has the following action for the proper-time \( \tau \in \mathbb{R} \):

\[
S_A(l) = \frac{\hbar}{m} \int_{\tau \in A} d\tau \left( \frac{dx(\tau)}{d\tau} \right)^2.
\]

For field theories, field variable \( X = \psi^{(4)} \) over spacetime \( M^{(4)} \) has the following action for the Lagrangian density \( \mathcal{L}_M \) of matters:

\[
S_A(\psi^{(4)}) = \frac{1}{\hbar c} \int_A dv^{(4)}(y) \mathcal{L}_M\left(\psi^{(4)}(y), d\psi^{(4)}(y)\right), \tag{7}
\]

where \( v^{(4)} \) is the volume measure of \( M^{(4)} \). In the standard field theory, \( \psi^{(4)} \) is a set of \( \mathbb{Z}_2 \)-graded fields over spacetime \( M^{(4)} \), the Dirac field for fermions, the Yang-Mills field for gauge bosons and other field under consideration. For the Einstein gravity, the Hilbert action includes the metric tensor \( g \) on \( M^{(4)} \) with a cosmological constant \( \Lambda \in \mathbb{R} \):

\[
S_A(\psi^{(4)}, g) = \frac{1}{\hbar c} \int_A dv^{(4)}(y) \mathcal{L}_M\left(\psi^{(4)}(y), d\psi^{(4)}(y)\right) - \frac{1}{\hbar c} \int_A dv^{(4)}(y) \left( \frac{c^4}{16\pi G} R_g + \Lambda \right) - \frac{2}{\hbar c} \int_{\partial A} dv^{(3)}_g \frac{c^4}{16\pi G} K_g,
\]

where \( R_g \) and \( K_g \) are the four-dimensional and the extrinsic three-dimensional scalar curvatures on domain \( A \) and on its boundary \( \partial A \); and \( G \) is the Newton’s constant of gravity. The last term of \( \mathcal{L}_M \) is necessary to produce the correct Einstein equation for gravity \([13]\).

Let us now consider the subset \( \mathcal{D}_A(M) \) of set \( \mathcal{D}(\bar{M}) \) such that every element \( \Phi \in \mathcal{D}_A(M) \) satisfies \( \chi_{\mathcal{D}(X) \setminus A}(\Phi(X)) = X \), and assume it as an infinite-dimensional Lie group. In particle theories, set \( \mathcal{D}_A(M) \) is the set \( \mathcal{D}_A(M) \) of all the diffeomorphisms of \( M \) such that \( \Phi(l) \setminus A = l \setminus A \); and, in field theories, it is the set of all the linear transformations of a field such that \( \Phi(\psi^{(4)}) = \psi^{(4)} + \phi^{(4)} \) for an element \( \phi^{(4)} \in \Psi(M^{(4)}, V) \) and that \( \phi^{(4)}(x) = 0 \) if \( x \notin A \). Mapping \( \tilde{o}_A \) may have the symmetry under a transformation \( \Phi \in \mathcal{D}(\bar{M}) \) such that it satisfies the following relation for every pair \((A, X)\):

\[
\tilde{o}_A(\Phi(X)) = \tilde{o}_A(X). \tag{9}
\]
Such symmetry verifies the existence of the conserved charge.  
Object $X$ and all the rest but $X$ composes the universe $U$. The internal-time $\Pi_A(U)$ of universe $U$ relative to domain $A$ would be separated into two parts:

$$\Pi_A(U) = \tilde{o}_A(X) \cdot \tilde{o}^*_A(X). \tag{10}$$

Let us call $\tilde{o}^*_A(X) \in S^1$ as the external-time of $X$ relative to $A$. Thus, the external-time of universe $U$ would always be unity: $\Pi_A^*(U) = 1$.

**Law 2** For every domain $A \in \tilde{O}$, the mapping $\tilde{o}_A^* : \tilde{M} \rightarrow S^1$ has an action $S_A^* : \tilde{M} \rightarrow \mathbb{R}$ and equips an object $X \in \tilde{M}$ with the external-time $\tilde{o}^*_A(X)$:

$$\tilde{o}^*_A(X) = e^{iS_A^*(X)}. \tag{11}$$

Let us also introduce the mapping $\tilde{s}_A(\tilde{o}) : \tilde{O} \rightarrow S^1$ that relates mappings $\tilde{o}^*_A$ and $\tilde{o}_A$:

$$\tilde{o}^*_A(X) = \tilde{o}_A(X) \cdot \tilde{s}_A(\tilde{o})(X). \tag{12}$$

It has a function $R_A(\tilde{o})$ such that

$$\tilde{s}_A(\tilde{o})(X) = e^{iR_A(\tilde{o})(X)}. \tag{13}$$

There is also the mapping $\tilde{s}_A^*(\tilde{o}^*) : \tilde{O} \rightarrow S^1$:

$$\tilde{o}^*_A(X) \cdot \tilde{s}_A^*(\tilde{o}^*)(X) = \tilde{o}_A(X). \tag{14}$$

Mapping $\tilde{n}_A$ may have the symmetry under a transformation $\Phi \in D(\tilde{M})$ such that it satisfies the following relation for every pair $(A,X)$:

$$\tilde{o}_A^*(\Phi(X)) = \tilde{o}_A(X). \tag{15}$$

If mapping $\tilde{n}_A$ also has symmetry (6) for the same transformation $\Phi$, they must satisfy the following invariance:

$$\tilde{s}_A(\tilde{o})(\Phi(X)) = \tilde{s}_A(\tilde{o})(X) \quad , \quad \tilde{s}_A^*(\tilde{o}^*)(\Phi(X)) = \tilde{s}_A^*(\tilde{o}^*)(X). \tag{16}$$

The following law further supplies the condition that an object has the actual existence on a domain of the spacetime.

**Law 3** Object $X \in \tilde{M}$ has actual existence on domain $A \in \tilde{O}$ when the internal-time coincides with the external-time:

$$\tilde{o}_A(X) = \tilde{o}_A^*(X). \tag{17}$$

Relation (17) requires the following quantization condition:

$$\tilde{s}_A(\tilde{o})(X) = 1, \tag{18}$$

or equivalently,

$$\tilde{s}_A(\tilde{o}^*)(X) = 1, \tag{19}$$

which quantizes spacetime $M^{(4)}$ for an object $X \in \tilde{M}$.

For the space $d_A(\tilde{M})$ of all the infinitesimal generators of $D_A(\tilde{M})$, let us consider an arbitrary element $\Phi_\epsilon \in D_A(\tilde{M})$, differentiable by parameter $\epsilon \in \mathbb{R}$:

$$\lim_{\epsilon \to 0} \frac{d\Phi_\epsilon}{dc} \circ \Phi_\epsilon^{-1} = \xi \in d_A(X). \tag{20}$$
Thus, we can introduce the variation $\delta$ as follows:

$$\left\langle i\dot{\sigma}_A (X)^{-1}\delta \sigma_A (X), \xi \right\rangle = i\dot{\sigma}_A (X)^{-1}\frac{d}{de}\bigg|_{e=0} \sigma_A (\Phi_e (X)),$$

(21)

$$\left\langle i\dot{\sigma}_A^* (X)^{-1}\delta \sigma^*_A (X), \xi \right\rangle = i\dot{\sigma}_A^* (X)^{-1}\frac{d}{de}\bigg|_{e=0} \sigma^*_A (\Phi_e (X))$$

(22)

where $\left\langle , \right\rangle : d_A^* (\tilde{M}) \times d_A (\tilde{M}) \rightarrow \mathbb{R}$ is the natural pairing for the dual space $d_A^* (\tilde{M})$ of $d_A (\tilde{M})$. This variation satisfies the variational principle of the following law.

**Law 4** *Object $X \in \tilde{M}$ must satisfy the variational principle for every domain $A \in \tilde{\Omega}$:*

$$\delta \sigma_A (X) = 0 \quad \delta \sigma_A^* (X) = 0.$$  

(23)

Thus, Law 4 keeps Law 3 under the above variation, and also has the following expression:

$$\delta \tilde{s}_A (\tilde{\partial}) (X) = 0 \quad (\delta \tilde{s}_A^* (\tilde{\partial}^*) (X) = 0.  

(24)

Now, we will consider the mapping $\mathcal{P} : T \rightarrow \tilde{\Omega}$ for the time $T \subset \mathbb{R}$ of an observer’s clock $T$. Domain $\mathcal{P} (t)$ and its boundary $\partial \mathcal{P} (t) = \overline{\mathcal{P} (t)} \setminus \mathcal{P} (t)$ represent the past and the present at time $t \in T$, where $\overline{A}$ is the closure of $A \in \tilde{\Omega}$; and it satisfies the following conditions:

1. for every $X \in \tilde{M}$, $t_1 < t_2 \in T \Rightarrow \mathcal{P} (t_1) \cap D (X) \subset \mathcal{P} (t_2) \cap D (X)$ (ordering);
2. for every $X \in \tilde{M}$, the present $\partial \mathcal{P} (t) \cap D (X)$ is a spacelike hypersurface in $M^{(4)}$ for every time $t \in T$ (causality).

From Law 3, object $X$ emerges into the world at time $t \in T$ when it satisfies

$$\tilde{s}_{\mathcal{P} (t)} (\tilde{\partial}) (X) = 1.$$  

(25)

This condition of the emergence determines when object $X$ interacts with all the rest in the world, and discretizes time $T$ in Whitehead’s philosophy [14]. In other words, what a particle or a field $X$ gains actual existence or emerges into the world, here, means that it becomes exposed to or has the possibility to interact with the other elements or with the ambient world excluded from the description. Such occasional influences from the unknown factors can break the deterministic feature of the above description; and it would cause the irreversibility in general as considered in elsewhere [12]. The emergence further allows the observation of a particle or a field through an experiment even if the device or its environment is included in the description [12]. Besides, the variational principle of Law 4 produces the equation of motion and the conservation of the frequency of such emergence in the next section.

### 3 Foundation of Protomechanics

Let us consider the development of present $\partial \mathcal{P} (t)$ for short time $T = (t_i, t_f) \subset \mathbb{R}$, keeping the following description without the appearance of singularity; and suppose that the time interval extends long enough to keep the continuity of time beyond the discretization in the previous section, where such discretization would only affect the property of the emergence-measure, defined below, corresponding to the density matrices in quantum mechanics. Assume that present $\partial \mathcal{P} (t)$ is diffeomorphic to a three dimensional manifold $M^{(3)}$ by a diffeomorphism $\sigma_t : M^{(3)} \rightarrow \partial \mathcal{P} (t)$ for every $t \in T$. It induces a corresponding mapping $\tilde{\sigma}_t : \tilde{M} \rightarrow M$ for the space $M$ that is three-dimensional physical space $M^{(3)}$ for particle theories or the space $\tilde{M} = \Psi (M^{(3)}, V)$ of all the $C^\infty$-fields over $M^{(3)}$ for field theories. For particle theories, mapping $\tilde{\sigma}_t$ is defined as $\tilde{\sigma}_t (l) = \sigma_t^{-1} (l \cap \sigma_t (M^{(3)}))$ for a curve $l \subset M^{(4)}$; for field theories, it is defined as $\tilde{\sigma}_t (\psi^{(4)}) = \psi^{(4)} \circ \sigma_t$ for a field $\psi^{(4)}$.  


Let us assume that space $M$ is a $C^\infty$ manifold endowed with an appropriate topology and the induced topological $\sigma$-algebra.\footnote{$M$ is assumed as an ILH-manifold modeled by the Hilbert space endowed with an inverse-limit topology (consult [14]).} We will denote the tangent space as $TM$ and the cotangent space $T^*M$; and we shall consider the space of all the vector fields over $M$ as $X(M)$ and that of all the 1-forms over $M$ as $\Lambda^1(M)$. To add a one-dimensional cyclic freedom $S^1$ at each point of $M$ introduces the $S^1$-fiber bundle $E(M)$ over $M$.\footnote{The introduced freedom may not directly represent what is corresponding to the local clock in Weyl’s sense or the fifth-dimension in Kaluza’s sense for the four-dimensional spacetime $M^{(4)}$, if related.} Fiber $S^1$ represents an intrinsic clock of a particle or a field, which is located at every point on $M$. For the space $\Gamma[E(M)]$ of all the global sections of $E(M)$, every element $\eta \in \Gamma[E(M)]$ now represents the system that a particle or a field belongs to and carries with, and a synchronization of every two clocks located at different points in space $M$.

For past $\mathcal{P}(t)$ such that $\partial \mathcal{P}(t) = \sigma_t(M^{(3)})$, there is an mapping $o_t : TM \to \mathbb{R}$ such that every initial position $(x_0, \dot{x}_0) \in TM$ has an object $X \in \mathcal{M}_\mathcal{P}$ satisfying the following relation for $x_t = \hat{o}_t(X)$:

$$o_t (x_t, \dot{x}_t) = \hat{o}_{\mathcal{P}(t)} (X).$$

(26)

For the velocity field $v_t \in X(M)$ such that $v_t (x_t) = \frac{dx_t}{dt}$, we will introduce a section $\eta_t \in \Gamma [E(M)]$ and call it synchronicity over $M$:

$$\eta_t (x) = o_t (x, v_t (x)).$$

(27)

The Lagrangian $L^M_t : TM \to \mathbb{R}$ characterizes the speed of the internal-time:

$$L^M_t \left( x_t, \frac{dx_t}{dt} \right) = -i \hbar o_t \left( x_t, \frac{dx_t}{dt} \right)^{-1} \frac{d}{dt} o_t \left( x_t, \frac{dx_t}{dt} \right).$$

(28)

Since relation (28) is valid for every initial conditions of position $(x_t, \dot{x}_t) \in TM$, it determines the time-development of synchronicity $\eta_t$ in the following way for the Lie derivative $\mathcal{L}_v$ by velocity field $v_t \in X(M)$:

$$L^M_t (x, v_t (x)) = -i \hbar \eta_t (x)^{-1} \left( \frac{\partial}{\partial t} + \mathcal{L}_v \right) \eta_t (x).$$

(29)

Let us now consider the mapping $p : \Gamma [E(M)] \to \Lambda^1(M)$ satisfying the following relation:

$$p(\eta_t) = -i \hbar \eta_t^{-1} dv_t.$$

(30)

If the energy $E_t (\eta_t) : TM \to \mathbb{R}$ is defined as

$$E_t (\eta_t) (x) = i \hbar \eta_t (x)^{-1} \frac{\partial}{\partial t} \eta_t (x),$$

(31)

condition (29) satisfies the following relation:

$$E_t (\eta_t) (x) = v_t (x) \cdot p(\eta_t) (x) - L^M_t (x, v_t (x)).$$

(32)

Attention to the following calculation by definition (29):

$$- i \hbar \frac{\partial}{\partial v} \left\{ o_t (x, v_t (x))^{-1} \left( \frac{\partial}{\partial t} + \mathcal{L}_v \right) o_t (x, v_t (x)) \right\} = \frac{\partial L^M_t}{\partial v} (x, v_t (x)).$$

(33)

Since variational principle (23) in Law 1 implies that $o_t (x, \dot{x})$ is invariant under the variation of $\dot{x}$ at every point $(x, \dot{x})$, i.e.,

$$\frac{\partial}{\partial \dot{x}} o_t (x, \dot{x}) = 0 \iff \frac{\partial}{\partial v} o_t (x, v_t (x)) = 0$$

(34)
then formula (33) has the following different expression:

\[
-i\hbar \frac{\partial}{\partial v} \left\{ o_t(x, v_t(x))^{-1} \left( \frac{\partial}{\partial t} + L_{v_t} \right) o_t(x, v_t(x)) \right\} = \frac{\partial}{\partial v} \{ v_t(x) \cdot p(\eta_t(x)) \} = p(\eta_t(x)).
\] (35)

Equations (33) and (35) leads to the modified Einstein-de Broglie relation, that was \( p = h/\lambda \) for Planck’s constant \( h = 2\pi\hbar \) and wave number \( \lambda \) in quantum mechanics:

\[
p(\eta_t(x)) = \frac{\partial L^T_M}{\partial v}(x, v_t(x)).
\] (36)

Notice that this relation (36) produces the Euler-Lagrange equation resulting from the classical least action principle:

\[
dL^T_M(x, v_t(x)) - \left( \frac{\partial}{\partial t} + L_{v_t} \right) \frac{\partial L^T_M}{\partial v}(x, v_t(x)) = 0 \] (37)

\[
\iff \frac{\partial L^T_M}{\partial x^j}(x_t, \dot{x}_t) - \frac{d}{dt} \frac{\partial L^T_M}{\partial \dot{x}_j}(x_t, \dot{x}_t) = 0 ; \] (38)

thereby, relation (36) is stronger condition than the classical relation (38).

Under the modified Einstein-de Broglie relation (36), relation (32) gives the Legendre transformation and introduces Hamiltonian \( H^T_M \) as a real function on cotangent space \( T^*M \) such that

\[
E_t(\eta_t(x)) = H^T_M(x, p(\eta_t(x))).
\] (39)

This satisfies the first equation of Hamilton’s canonical equations of motion:

\[
v_t(x) = \frac{\partial H^T_M}{\partial p}(x, p(\eta_t(x))).
\] (40)

Solvability \( [\frac{\partial}{\partial t}, d] = 0 \) further leads to the second equation of Hamilton’s canonical equations of motion:

\[
\frac{\partial}{\partial t} p(\eta_t(x)) = -dH^T_M(x, p(\eta_t(x))),
\] (41)

which is equivalent to equation (37) of motion under condition (36). If Lagrangian \( L^T_M \) satisfies

\[
\frac{\partial L^T_M}{\partial t} = 0 ,
\] (42)

then equations (40) and (41) of motion prove the conservation of energy:

\[
\left( \frac{\partial}{\partial t} + L_{v_t} \right) H^T_M(x, p(\eta_t(x))) = 0.
\] (43)

On the other hand, the mapping \( \tilde{s}_{P(t)}(\partial) \) induces a mapping \( s_t(o_t) : T M \rightarrow S^1 \) such that every initial position \( (x_0, \dot{x}_0) \in T M \) has an object \( X \in M_P \) satisfying the following relation:

\[
s_t(o_t) \left( x_t, \frac{dx_t}{dt} \right) = \tilde{s}_{P(t)}(\partial)(X).
\] (44)

For velocity field \( v_t \), we can define the following section \( \varsigma_t(\eta_t) \in \Gamma [E(M)] \) and call it shadow over \( M \):

\[
\varsigma_t(\eta_t)(x) = s_t(o_t)(x, v_t(x)).
\] (45)
Condition (22) of emergence now has the following form:

\[ s_t (\eta_t) \left( x_t, \frac{dx_t}{dt} \right) = 1 \iff \varsigma_t (\eta_t) (x) = 1, \]

(46)

when synchronicity \( \eta_t \) comes across the section \( \eta_t^* = \eta_t \cdot \varsigma_t (\eta_t) \) at position \( x \in M \). Let us introduce the function \( T_t(\eta_t)^{TM} : TM \to \mathbb{R} \) such that

\[ T_t(\eta_t)^{TM} \left( x_t, \frac{dx_t}{dt} \right) = -i\hbar s_t(\eta_t) \left( x_t, \frac{dx_t}{dt} \right)^{-1} \frac{d}{dt} s_t(\eta_t) \left( x_t, \frac{dx_t}{dt} \right). \]

(47)

Since relation (47) is valid for every initial conditions of position \( x_t \in M \), it determines the time-development of shadow \( \varsigma_t (\eta_t) \) in the following way for the Lie derivative \( \mathcal{L}_{v_t} \) by the velocity field \( v_t \in X(M) \) such that \( v_t (x_t) = \frac{dx_t}{dt} \):

\[ T_t(\eta_t)^{TM} (x, v_t(x)) = -i\hbar \varsigma_t (\eta_t)^{-1} \left\{ \frac{\partial}{\partial t} + \mathcal{L}_{v_t} \right\} \varsigma_t (\eta_t). \]

(48)

In stead of Hamiltonian for a synchronicity, we will consider the emergence-frequency \( f_t (\eta_t) : M \to \mathbb{R} \) for a shadow such that

\[ 2\pi \hbar f_t (\eta_t) (x) = i\hbar \varsigma_t (\eta_t)^{-1} \frac{\partial}{\partial t} \varsigma_t (\eta_t) (x), \]

(49)

which represents the frequency that a particle or a field emerges into the world. Condition (18) satisfies the following relation:

\[ 2\pi \hbar f_t (\eta_t) (x) = v_t(x) \cdot p (s_t (\eta_t) (x)) - T_t(\eta_t)^{TM} (x, v_t(x)). \]

(50)

Variational principle (24) from Law 4 implies that \( s_t (\eta_t) (x, \dot{x}) \) is invariant under the variation of \( \dot{x} \) at every point \( (x, \dot{x}) \), i.e.,

\[ \frac{\partial}{\partial \dot{x}} s_t (\eta_t) (x, \dot{x}) = 0 \iff \frac{\partial}{\partial v_t} s_t (\eta_t) (x, v_t(x)) = 0, \]

(51)

which leads to the following relation corresponding to the modified Einstein-de Broglie relation for synchronicity \( \eta_t \):

\[ p \left( s_t (\eta_t) \right) (x) = \frac{\partial T_t^{TM}(\eta_t)}{\partial v_t} (x, v_t(x)). \]

(52)

Relation (22) proves the conservation of emergence-frequency in the same way as relation (31) proved that of energy (13):

\[ \left( \frac{\partial}{\partial t} + \mathcal{L}_{v_t} \right) f_t (\eta_t) (x) = 0. \]

(53)

Notice that emergence-frequency \( f_t (\eta_t) \) can be negative as well as positive, and that it produces a similar property of the Wigner function for a wave function in quantum mechanics (1).

In addition, the probability measure \( \check{\nu} \) on \( M \) induces the probability measure \( \nu_t \) on \( M \) at time \( t \in T \) such that

\[ d\nu_t \left( x_t, \frac{dx_t}{dt} \right) = d\check{\nu} (X), \]

(54)

that represents the ignorance of the initial position in \( M \); thereby it satisfies the conservation law:

\[ \frac{d}{dt} d\nu_t \left( x_t, \frac{dx_t}{dt} \right) = 0. \]

(55)

This relation can be described by using the Lie derivative \( \mathcal{L}_{v_t} \) as

\[ \left( \frac{\partial}{\partial t} + \mathcal{L}_{v_t} \right) d\nu_t (x, v_t(x)) = 0. \]

(56)
Since the velocity field $v_t$ has relation (40) with synchronicity $\eta_t$, we can define the emergence-measure $\mu_t(\eta_t)$ as the product of the probability measure with the emergence-frequency:

$$d\mu_t(\eta_t)(x) = d\nu_t(x, v_t(x)) \cdot f_t(\eta_t)(x).$$

(57)

Thus, we will obtain the following equation of motion for emergence-measure $d\mu_t(\eta_t)$:

$$\left(\frac{\partial}{\partial t} + L_{v_t}\right) d\mu_t(\eta_t) = 0.$$ 

(58)

Let me summarize the obtained mechanics or protomechanics based on equations (29) and (58) of motion with relation (40) in the following theorem that this section proved.

**Theorem 1 (Protomechanics)** Hamiltonian $H_t^{TM} : T^*M \to \mathbb{R}$ defines the velocity field $v_t \in \mathcal{X}(M)$ and Lagrangian $L_t^{TM} : TM \to \mathbb{R}$ as follows:

$$v_t(x) = \frac{\partial H_t^{TM}}{\partial p}(x, p(\eta_t)(x)),$$

(59)

$$L_t^{TM}(x, v(x)) = v(x) \cdot p(\eta_t)(x) - H_t^{TM}(x, p(\eta_t)(x)),$$

(60)

where mapping $p : \Gamma[E(M)] \to \Lambda^1(M)$ satisfies the modified Einstein-de Broglie relation:

$$p(\eta_t) = -\bar{\hbar}^{-1} d\eta_t.$$ 

(61)

The equation of motion is the set of the following equations:

$$\left(\frac{\partial}{\partial t} + L_{v_t}\right) \eta_t(x) = -\bar{\hbar}^{-1} L_t^{TM}(x, v_t(x)) \eta_t(x),$$

(62)

$$\left(\frac{\partial}{\partial t} + L_{v_t}\right) d\mu_t(\eta_t) = 0.$$ 

(63)

## 4 DYNAMICAL CONSTRUCTION OF PROTOMECHANICS

Let us express the introduced protomechanics in the statistical way for the ensemble of all the synchronicities on $M$, and construct the dynamical description for the collective motion of the sections of $E(M)$. Such statistical description realizes the description within a long-time interval through the introduced relabeling process so as to change the labeling time, that is the time for the initial condition before analytical problems occur. In addition, it clarifies the relationship between classical mechanics and quantum mechanics under the assumption that the present theory safely induces them, and that will be proved in the following paper [11]. For mathematical simplicity, the discussion below suppose that $M$ is a $N$-dimensional manifold for a finite natural number $N \in \mathbb{N}$.

The derivative operator $D = \bar{\hbar} dx^j \partial_j : T^n_0(M) \to T^{n+1}_0(M)$ ($m \in \mathbb{N}$) for the space $T^n_0(M)$ of all the $(0,n)$-tensors on $M$ can be described as

$$D^n p(x) = \bar{\hbar}^n \left( \prod_{k=1}^n \partial_j p_j(x) \right) dx^j \otimes \left( \otimes_{k=1}^n dx^{jk} \right).$$

(64)

By utilizing this derivative operator $D$, the following Banach norm endows the space $\Gamma[E(M)]$ of all the $C^\infty$ sections of $E(M)$ with a norm topology for the family $\mathcal{O}_{\Gamma[E(M)]}$ of the induced open balls:

$$\|p(\eta)\| = \sup_M \sum_{\kappa \in \mathbb{Z}_{\geq 0}} \bar{\hbar}^\kappa |D^\kappa p(\eta)(x)|_x,$$

(65)

\[\text{In another way, consult quant-ph/9906130.}\]
where \( |_x \) is a norm of covectors at \( x \in M \).

In terms of the corresponding norm topology on \( \Lambda^1(M) \) we can consider the space \( C^\infty \left( \Lambda^1(M), C^\infty(M) \right) \) of all the \( C^\infty \)-differentiable mapping from \( \Lambda^1(M) \) to \( C^\infty(M) = C^\infty(M, \mathbb{R}) \) and the subspaces of the space \( C(\Gamma[E(M)]) \) such that

\[
C \left( \Gamma \left[ E(M) \right] \right) = \left\{ p^*F : \Gamma \left[ E(M) \right] \to C^\infty(M) \mid F \in C^\infty \left( \Lambda^1(M), C^\infty(M) \right) \right\}. 
\]

(66)

Classical mechanics requires the local dependence on the momentum for functionals, while quantum mechanics needs the wider class of functions that depends on their derivatives. The space of the classical functionals and that of the quantum functionals are defined as

\[
C_{cl} \left( \Gamma \left[ E(M) \right] \right) = \left\{ p^*F \in C \left( \Gamma \left[ E(M) \right] \right) \mid p^*F(\eta)(x) = F^{\kappa M}(x, p(\eta)(x)) \right\} 
\]

(67)

\[
C_q \left( \Gamma \left[ E(M) \right] \right) = \left\{ p^*F \in C \left( \Gamma \left[ E(M) \right] \right) \mid p^*F(\eta)(x) = F^{q_1}(x, p(\eta)(x), ..., D^np(\eta)(x), ...) \right\},
\]

(68)

(69)

and related with each other as

\[
C_{cl} \left( \Gamma \left[ E(M) \right] \right) \subset C_q \left( \Gamma \left[ E(M) \right] \right) \subset C \left( \Gamma \left[ E(M) \right] \right).
\]

(70)

In other words, the classical-limit indicates the limit of \( \hbar \to 0 \) with fixing \( |p(\eta)(x)| \) finite at every \( x \in M \), or what the characteristic length \( |x| \) and momentum \( |p| \) such that \( x/|x| \approx 1 \) and \( p/|p| \approx 1 \) satisfies

\[
|p|^{-n-1}D^n p(\eta)(x) \ll 1.
\]

(71)

In addition, the \( n \)-th semi-classical system can have the following functional space:

\[
C_{n+1} \left( \Gamma \left[ E(M) \right] \right) = \left\{ p^*F \in C \left( \Gamma \left[ E(M) \right] \right) \mid p^*F(\eta)(x) = F_{<n, >}(x, p(\eta)(x), ..., D^np(\eta)(x)) \right\}.
\]

(72)

Thus, there is the increasing series of subsets as

\[
C_1 \left( \Gamma \left[ E(M) \right] \right) \subset \cdots \subset C_n \left( \Gamma \left[ E(M) \right] \right) \subset C_\infty \left( \Gamma \left[ E(M) \right] \right) \subset C \left( \Gamma \left[ E(M) \right] \right),
\]

(73)

where \( F_{<1>} = F^{cl} \) and \( F_{<\infty>} = F^q \):

\[
C_1 \left( \Gamma \left[ E(M) \right] \right) = C_{cl} \left( \Gamma \left[ E(M) \right] \right)
\]

(74)

\[
C_\infty \left( \Gamma \left[ E(M) \right] \right) = C_q \left( \Gamma \left[ E(M) \right] \right).
\]

(75)

On the other hand, the emergence-measure \( \mu(\eta) \) has the Radon measure \( \tilde{\mu}(\eta) \) for section \( \eta \in \Gamma[E(M)] \) such that

\[
\tilde{\mu}(\eta) \left( F(p(\eta)) \right) = \int_M d\mu(\eta)(x) F(p(\eta))(x).
\]

(76)

The introduced norm topology on \( \Gamma \left( E(M) \right) \) induces the topological \( \sigma \)-algebra \( \mathcal{B} \left( \mathcal{O}_{\Gamma(E(M))} \right) \); thereby manifold \( \Gamma \left( E(M) \right) \) becomes a measure space having the probability measure \( \mathcal{M} \) such that

\[
\mathcal{M} \left( \Gamma \left( E(M) \right) \right) = 1.
\]

(77)

For a subset \( C_n \left( \Gamma \left( E(M) \right) \right) \subset C \left( \Gamma \left( E(M) \right) \right) \), an element \( \tilde{\mu} \in C_n \left( \Gamma \left( E(M) \right) \right)^* \) is a linear functional \( \tilde{\mu} : C_n \left( \Gamma \left[ E(M) \right] \right) \to \mathbb{R} \) such that

\[
\tilde{\mu} \left( p^*F \right) = \int_{\Gamma \left[ E(M) \right]} d\mathcal{M}(\eta) \tilde{\mu}(\eta) \left( F(p(\eta)) \right)
\]

(78)

\[
= \int_{\Gamma \left[ E(M) \right]} d\mathcal{M}(\eta) \int_M dv(x) \rho(\eta)(x) F(p(\eta))(x),
\]

(79)

\footnote{Assume here that \( \Lambda^1(M) \) has the Banach norm such that \( \|p\| = \sup_M \sum_{n \in \mathbb{Z}_{\geq 0}} |D^np(x)|_x \), for \( p \in \Lambda^1(M) \).}
where \( d\mu(\eta) = dv \rho(\eta) \). Let us call mapping \( \rho : \Gamma[E(M)] \to C^\infty(M) \) as the emergence-density. The dual spaces make an decreasing series of subsets (consult [17] in the definition of the Gelfand triplet):

\[
C_1 (\Gamma [E(M)])^* \supset ... C_n (\Gamma [E(M)])^* \supset ... C^\infty (\Gamma [E(M)])^* \supset C (\Gamma [E(M)])^* .
\]  

(80)

Thus, relation (80) requires the opposite sequence for the dual spaces:

\[
C^\infty (\Gamma [E(M)])^* \supset C_1 (\Gamma [E(M)])^* \supset ... C_n (\Gamma [E(M)])^* \supset ... C (\Gamma [E(M)])^* .
\]  

(81)

Let us summarize how the relation between quantum mechanics and classical mechanics in the following diagram.

\[
\begin{array}{ccc}
C(q)(\Gamma) & \rightarrow & C(cl)(\Gamma) \\
\uparrow \text{classical-limit} & | & \downarrow \text{quantization} \\
\uparrow & | & \\
C(cl)(\Gamma) & \rightarrow & C(q)(\Gamma)
\end{array}
\]

To investigate the time-development of the statistical state discussed so far, we will introduce the related group. The group \( D(M) \) of all the \( C^\infty \)-diffeomorphisms of \( M \) and the abelian group \( C^\infty (M) \) of all the \( C^\infty \)-functions on \( M \) construct the semidirect product \( S(M) = D(M) \times_{\text{semi}} C^\infty (M) \) of \( D(M) \) with \( C^\infty (M) \), and define the multiplication \( \cdot \) between \( \Phi_1 = (\varphi_1, s_1) \) and \( \Phi_2 = (\varphi_2, s_2) \in S(M) \) as

\[
\Phi_1 \cdot \Phi_2 = (\varphi_1 \circ \varphi_2, (\varphi_2^* s_1) \cdot s_2),
\]  

(82)

for the pullback \( \varphi^* \) by \( \varphi \in D(M) \). The Lie algebra \( s(M) \) of \( S(M) \) has the Lie bracket such that, for \( V_1 = (v_1, U_1) \) and \( V_2 = (v_2, U_2) \in s(M) \),

\[
[V_1, V_2] = ([v_1, v_2], v_1 U_2 - v_2 U_1 + [U_1, U_2]);
\]  

(83)

and its dual space \( s(M)^* \) is defined by natural pairing \( \langle , \rangle \). Lie group \( S(M) \) now acts on every \( C^\infty \) section of \( E(M) \) (consult APPENDIX). We shall further introduce the group \( Q(M) = Map (\Gamma [E(M)], S(M)) \) of all the mapping from \( \Gamma [E(M)] \) into \( S(M) \), that has the Lie algebra \( q(M) = Map (\Gamma [E(M)], s(M))^* \) and its dual space \( q(M)^* = Map (\Gamma [E(M)], s(M))^* \).

Let us further define the emergence-momentum \( J \in q(M)^* \) as follows:

\[
J (\eta) = dM(\eta) \ (\hat{\mu}(\eta) \otimes p(\eta), \hat{\mu}(\eta)).
\]  

(84)

Thus, the functional \( \mathcal{F} : q(M)^* \to \mathbb{R} \) can always be defined as

\[
\mathcal{F}(J) = \hat{\mu}(p^* \mathcal{F}).
\]  

(85)
On the other hand, the derivative $\mathcal{D}_\rho F (p)$ can be introduced as follows excepting the point where the distribution $\rho$ becomes zero:

$$
\mathcal{D}_\rho F (p) (x) = \sum_{(n_1, \ldots, n_N) \in \mathbb{N}^N} \frac{1}{\rho(x)} \left\{ \prod_{i} (-\partial_i)^{n_i} \left( \frac{\partial F}{\partial \left( \prod_{i} \partial_i^{n_i} \rho_{i}} \right) \right) \right\} \partial_j,
$$

(86)

Then, operator $\hat{F} (\eta) = \frac{d}{d\mathcal{F}} (\mathcal{J} (\eta))$ is defined as

$$
\frac{d}{d\epsilon} \bigg|_{\epsilon = 0} \mathcal{F} (\mathcal{J} + \epsilon \mathcal{K}) = \left< \mathcal{K}, \hat{F} \right>,
$$

(87)

i.e.,

$$
\hat{F} (\eta) = (\mathcal{D}_{\rho(\eta)} F (p(\eta))), -p(\eta) \cdot \mathcal{D}_{\rho(\eta)} F (p(\eta)) + F (p(\eta)) ;
$$

(88)

thereby, the following null-lagrangian relation can be obtained:

$$
\mathcal{F} (\mathcal{J}) = \langle \mathcal{J}, \hat{F} \rangle.
$$

(89)

Let us consider the time-development of the section $\eta^\tau_\tau (\eta) \in \Gamma [E(M)]$ such that the labeling time $\tau$ satisfies $\eta^\tau_\tau (\eta) = \eta$. It has the momentum $\tilde{p}^\tau (\eta) = -i \hbar \eta^\tau (\eta)^{-1} d\eta^\tau (\eta)$ and the emergence-measure $\mu^\tau (\eta)$ such that

$$
d\mathcal{M} (\eta) \tilde{\mu}^\tau (\eta) = d\mathcal{M} (\eta \eta^\tau (\eta)) \tilde{\mu} (\eta \eta^\tau (\eta)) ;
$$

(90)

$$
\tilde{\mu} (p^* F_i) = \int_{\Gamma [E(M)]} d\mathcal{M}(\eta \eta^\tau (\eta)) \tilde{\mu} (\eta \eta^\tau (\eta)) (p^* F_i (\eta))
$$

(91)

$$
= \int_{\Gamma [E(M)]} d\mathcal{M}(\eta \eta^\tau (\eta)) \tilde{\mu} (\eta \eta^\tau (\eta)) (p^* F (\eta \eta^\tau (\eta)))
$$

(92)

$$
= \int_{\Gamma [E(M)]} d\mathcal{M}(\eta \eta^\tau (\eta) \int_{M} d\nu (x) \rho^\tau (\eta)(x) F_i (p^\tau (\eta))(x).
$$

(93)

The introduced labeling time $\tau$ can always be chosen such that $\eta^\tau_\tau (\eta)$ does not have any singularity within a short time for every $\eta \in \Gamma [E(M)]$. The emergence-momentum $\mathcal{J}^\tau_\tau \in q(M)^* \mathcal{J}$ such that

$$
\mathcal{J}^\tau_\tau (\eta) = \mathcal{J}^\tau_\tau (\eta \eta^\tau (\eta)) = d\mathcal{M} (\eta \eta^\tau (\eta)) (\tilde{\mu} (\eta \eta^\tau (\eta)) \otimes p^\tau (\eta), \tilde{\mu} (\eta \eta^\tau (\eta)))
$$

(94)

$$
= d\mathcal{M} (\eta (\eta \eta^\tau (\eta)) (\tilde{\mu} (\eta (\eta \eta^\tau (\eta)) \otimes p^\tau (\eta), \tilde{\mu} (\eta (\eta \eta^\tau (\eta)))))
$$

(95)

$$
= d\mathcal{M} (\eta) (\tilde{\mu} (\eta \eta^\tau (\eta))) \otimes p^\tau (\eta), \tilde{\mu} (\eta \eta^\tau (\eta)))
$$

(96)

satisfies the following relation for the functional $\mathcal{F}^\tau_\tau : q(M)^* \rightarrow \mathbb{R}$:

$$
\mathcal{F}^\tau_\tau (\mathcal{J}^\tau_\tau) = \mu^\tau (p^* F_i) ,
$$

(97)

whose value is independent of labeling time $\tau$. The operator $\hat{F}^\tau_\tau = \frac{d}{d\mathcal{F}^\tau_\tau} (\mathcal{J}^\tau_\tau)$ is defined as

$$
\frac{d}{d\epsilon} \bigg|_{\epsilon = 0} \mathcal{F}^\tau_\tau (\mathcal{J}^\tau_\tau + \epsilon \mathcal{K}) = \left< \mathcal{K}, \hat{F}^\tau_\tau \right>,
$$

(98)

i.e.,

$$
\hat{F}^\tau_\tau = (\mathcal{D}_{\rho^\tau_\tau(p(\eta))} F_i (p^\tau_\tau (\eta))), -p^\tau_\tau (\eta) \cdot \mathcal{D}_{\rho^\tau_\tau (\eta)} F_i (p^\tau_\tau (\eta)) + F_i (p^\tau_\tau (\eta)) .
$$

(99)

Thus, the following null-lagrangian relation can be obtained:

$$
\mathcal{F}^\tau_\tau (\mathcal{J}^\tau_\tau) = \langle \mathcal{J}^\tau_\tau, \hat{F}^\tau_\tau \rangle,
$$

(100)
while the normalization condition has the following expression:
\[
\mathcal{I}(\mathcal{J}_t^\tau) = 1 \quad \text{for} \quad \mathcal{I}(\mathcal{J}_t^\tau) = \int_{\Gamma[E(M)]} d\mathcal{M}(\eta) \mu_t(\eta)(M).
\] (101)

**Theorem 2** For Hamiltonian operator \(\hat{H}_t^\tau = \frac{\partial H}{\partial p} (\mathcal{J}_t^\tau) \in \mathfrak{g}(\mathcal{M})\) corresponding to Hamiltonian \(p^* H_t^\tau(\eta)(x) = H_t^{T^*M}(x, p(\eta))\), equations (102) and (103) of motion becomes Lie-Poisson equation
\[
\frac{\partial \mathcal{J}_t^\tau}{\partial t} = \text{ad}^*_{\hat{H}_t^\tau} \mathcal{J}_t^\tau,
\] (102)
which can be expressed as
\[
\frac{\partial}{\partial t} \rho_t^\tau(\eta)(x) = -\sqrt{-1} \partial_j \left( \frac{\partial H_t^{T^*M}}{\partial p_j} (x, p_t^\tau(\eta)(x)) \rho_t^\tau(\eta)(x) \sqrt{\rho_t^\tau(\eta)}(x) \right),
\] (103)
\[
\frac{\partial}{\partial t} (\rho_t^\tau(\eta)(x)p_t^\tau(\eta)(x)) = -\sqrt{-1} \partial_j \left( \frac{\partial H_t^{T^*M}}{\partial p_j} (x, p_t^\tau(\eta)(x)) \rho_t^\tau(\eta)(x)p_t^\tau(\eta)(x) \sqrt{\rho_t^\tau(\eta)}(x) \right)
- \rho_t^\tau(\eta)(x)p_t^\tau(\eta)(x) \partial_k \frac{\partial H_t^{T^*M}}{\partial p_j} (x, p_t^\tau(\eta)(x))
+ \rho_t^\tau(\eta)(x) \partial_k \left( p_t^\tau(\eta)(x) \cdot \frac{\partial H_t^{T^*M}}{\partial p_j} (x, p_t^\tau(\eta)(x)) \right)
- H_t^{T^*M}(x, p_t^\tau(\eta)(x)).
\] (104)

**Proof.** Lie-Poisson equation (103) is calculated for \(\mathcal{D} H_t^\tau(\eta) = \mathcal{D}^\tau \rho_t^\tau(\eta) H_t^\tau(p_t^\tau(\eta))\) as follows:
\[
\frac{\partial}{\partial t} \rho_t^\tau(\eta)(x) = -\sqrt{-1} \partial_j \left( \mathcal{D}^\tau H_t^\tau(\eta)(x) \rho_t^\tau(\eta)(x) \sqrt{\rho_t^\tau(\eta)}(x) \right),
\] (106)
\[
\frac{\partial}{\partial t} (\rho_t^\tau(\eta)(x)p_t^\tau(\eta)(x)) = -\sqrt{-1} \partial_j \left( \mathcal{D}^\tau H_t^\tau(\eta)(x) \rho_t^\tau(\eta)(x)p_t^\tau(\eta)(x) \sqrt{\rho_t^\tau(\eta)}(x) \right)
- \rho_t^\tau(\eta)(x)p_t^\tau(\eta)(x) \partial_k \mathcal{D}^\tau H_t^\tau(\eta)(x)
+ \rho_t^\tau(\eta)(x) \partial_k \left( p_t^\tau(\eta)(x) \cdot \mathcal{D} H_t^\tau(\eta)(x) - H_t(\rho_t^\tau(\eta)) (x) \right),
\] (107)
where \(dv = dx^1 \wedge ... dx^N \sqrt{\mathcal{J}_t^\tau}\) and \(\sqrt{\mathcal{J}_t^\tau} = \sqrt{\det |g^{ij}|}\) for the local coordinate \(x = (x^1, x^2, ..., x^N)\). Second equation (107) can be rewritten in conjunction with the conservation (106) of the emergence-density as
\[
\frac{\partial}{\partial t} p_t^\tau(\eta)(x) + \mathcal{D}^\tau H_t^\tau(\eta)(x) \partial_j p_t^\tau(\eta)(x) + p_t^\tau(\eta)(x) \partial_k \mathcal{D}^\tau H_t^\tau(\eta)(x) = \partial_k L_t^\tau(\eta)(x),
\] (108)
where
\[
L_t^\tau(\eta)(x) = p_t^\tau(\eta)(x) \cdot \mathcal{D} H_t^\tau(\eta)(x) - H_t(p_t^\tau(\eta))(x),
\] (109)
or, by using Lie derivatives,
\[
\mathcal{L}_{\mathcal{D} H_t^\tau(\eta)} p_t^\tau(\eta) = dL_t^\tau(\eta).
\] (110)
Thus, we can obtain the equation of motion in the following simpler form by using Lie derivatives:
\[
\mathcal{L}_{\mathcal{D} H_t^\tau(\eta)} \eta_t^\tau = -ih\mathcal{L}_{\eta_t^\tau} \eta_t^\tau
\] (111)
\[
\mathcal{L}_{\mathcal{D} H_t^\tau(\eta)} \rho_t^\tau(\eta) \ dv = 0,
\] (112)
which is equivalent to the equations (29) and (58) when $p^* H_t (\eta) (x) = H^T_t M (x, p(\eta))$

Equation (102) will prove in the following two sections to include the Schrödinger equation in canonical quantum mechanics and the classical Liouville equations in classical mechanics.

For $U^\tau_t \in Q(M)$ such that $\frac{\partial U^\tau_t}{\partial t} \circ (U^\tau_t)^{-1} = \dot{H}^\tau_t (\eta) \in q(M)$, let us introduce the following operators:

$$\tilde{H}^\tau_t (\eta) = Ad_{U^\tau_t}^{-1} \dot{H}^\tau_t (\eta) \quad \text{and} \quad \tilde{F}^\tau_t (\eta) = Ad_{U^\tau_t}^{-1} \dot{F}^\tau_t (\eta).$$ (113)

It satisfies the following theorem.

**Theorem 3** Lie-Poisson equation (102) is equivalent to the following equation:

$$\frac{\partial}{\partial t} \tilde{F}^\tau_t = [\tilde{H}^\tau_t, \tilde{F}^\tau_t] + \left( \frac{\partial \tilde{F}^\tau_t}{\partial t} \right).$$ (114)

**Proof.** Equation (102) of motion concludes the following equation:

$$\left\langle \frac{\partial}{\partial t} J^\tau_t, \tilde{F}^\tau_t \right\rangle = \left\langle ad^*_{\tilde{H}^\tau_t} J^\tau_t, \tilde{F}^\tau_t \right\rangle,$$ (115)

The left hand side can be calculated as

$$\text{L.H.S.} = \frac{d}{dt} F_t (J^\tau_t) - \frac{\partial F_t}{\partial t} (J^\tau_t)$$

$$= \left\langle \left( \frac{\partial}{\partial t} Ad_{U^\tau_t} J^\tau_t \right), \tilde{F}^\tau_t \right\rangle - \left\langle Ad_{U^\tau_t} J^\tau_t, \frac{\partial \tilde{F}^\tau_t}{\partial t} \right\rangle$$

$$= \left\langle J^\tau_t, \frac{\partial}{\partial t} \tilde{F}^\tau_t \right\rangle - \left\langle J^\tau_t, \frac{\partial \tilde{F}^\tau_t}{\partial t} \right\rangle;$$ (118)

and the right hand side becomes

$$\text{R.H.S.} = \left\langle ad^*_{\tilde{H}^\tau_t} J^\tau_t, \tilde{F}^\tau_t \right\rangle$$

$$= \left\langle ad_{U^\tau_t} J^\tau_t, \tilde{F}^\tau_t \right\rangle$$

$$= \left\langle J^\tau_t, \tilde{F}^\tau_t \right\rangle.$$ (119)

Thus, we can obtain this theorem.

The general theory for Lie-Poisson systems certifies that, if a group action of Lie group $Q(M)$ keeps the Hamiltonian $H_t : q(M)^* \to R$ invariant, there exists an invariant charge functional $Q : \Gamma [E(M)] \to C(M)$ and the induced function $\hat{Q} : q(M)^* \to R$ such that

$$\left[ \hat{H}_t, \hat{Q} \right] = 0,$$ (122)

where $\hat{Q}$ is expressed as

$$\hat{Q} = \langle \mathcal{D}_{\rho(\eta)} Q (p(\eta)), -p(\eta) \cdot \mathcal{D}_{\rho(\eta)} Q (p(\eta)) + Q (p(\eta)) \rangle.$$ (123)
5 CONCLUSION

The present paper attempted to reveal the structure behind mechanics, and proposed a basic theory of physical reality realizing Whitehead’s philosophy. It induced protomechanics that deepened Hamiltonian mechanics under the modified Einstein-de Broglie relation. In the following papers [11, 12], the present theory will prove to induce both classical mechanics and quantum mechanics, to solve the problem of the operator ordering in quantum mechanics and to give its realistic, self-consistent interpretation.

APPENDIX: LIE-POISSON MECHANICS

Over a century ago, in an effort to elucidate the relationship between Lie group theory and classical mechanics, Lie [19] introduced the Lie-Poisson system, being a Hamiltonian system on the dual space of an arbitrary finite-dimensional Lie algebra. Several years later, as a generalization of the Euler equation of a rigid body, Poincaré [24] applied the standard variational principle on the tangent space of an arbitrary finite-dimensional Lie group and independently obtained the Euler-Poincaré equation on the Lie algebra, being equivalent to the Lie-Poisson equation on its dual space if considering no analytical difficulties. These mechanics structures for Lie groups were reconsidered in the 1960’s (see [21] for the historical information). Marsden and Weinstein [22], in 1974, proposed the Marsden-Weinstein reduction method that allows a Hamiltonian system to be reduced due to the symmetry determined by an appropriate Lie group, while Guillemin and Sternberg [23] introduced the collective-Hamiltonian method that describes the equation of motion for a Hamiltonian system as the Lie-Poisson equation of a reduced Lie-Poisson system.

Let $G$ be taken to be a finite- or infinite-dimensional Lie group and $g$ the Lie algebra of $G$; i.e., the multiplications $\cdot : G \times G \to G : (\phi_1, \phi_2) \to \phi_1 \cdot \phi_2$ with a unit $e \in G$ satisfy $\phi_1^{-1} \cdot \phi_2 \in G$ and induce the commutation relation $[\cdot , ] : g \times g \to g : (v_1, v_2) \to [v_1, v_2]$. For a function $F \in C^\infty(G, R)$, two types of derivatives respectively define the left- and the right-invariant vector field $v^+$ and $v^- \in \mathcal{X}(G)$ in the space $\mathcal{X}(G)$ of all smooth vector fields on $G$:

\begin{align}
   v^+ F(\phi) &= \frac{d}{d\tau}|_{\tau=0} F(\phi \cdot e^{\tau v}) \quad \text{(A1)} \\
   v^- F(\phi) &= \frac{d}{d\tau}|_{\tau=0} F(e^{\tau v} \cdot \phi). \quad \text{(A2)}
\end{align}

Accordingly, the left- and the right-invariant element of the space $\mathcal{X}(G)$ satisfy

\begin{align}
   [v_1^+, v_2^+] = [v_1, v_2]^+, \quad [v_1^-, v_2^-] = -[v_1, v_2]^-, \quad \text{and} \quad [v_1^+, v_2^-] = 0. \quad \text{(A3)}
\end{align}

In the subsequent formulation, $+$ and $-$ denote left- and right-invariance, respectively. In addition, $\langle \cdot , \cdot \rangle : g^* \times g \to R : (\mu, v) \to \langle \mu, v \rangle$ denotes the nondegenerate natural pairing (that is weak in general [24]) for the dual space $g^*$ of the Lie algebra $g$, defining the left- or right-invariant 1-form $\mu^\pm \in \Lambda^1(G)$ corresponding to $\mu \in g^*$ by introducing the natural pairing $\langle \cdot , \cdot \rangle : T^*_gG \times T_gG \to R$ for $\phi \in G$ as

\begin{align}
   \langle \mu^\pm(\phi), v^\pm(\phi) \rangle = \langle \mu, v \rangle. \quad \text{(A4)}
\end{align}

Let us now consider how the motion on a Poisson manifold $P$ can be represented by the Lie-Poisson equation for $G$ (or its central extension [24]), where $P$ is a finite or infinite Poisson manifold modeled on $C^\infty$ Banach spaces with Poisson bracket $\{ \cdot , \cdot \} : C^\infty(P, R) \times C^\infty(P, R) \to C^\infty(P, R)$. Also, $\Psi : G \times P \to P$ is an action of $G$ on $P$ such that the mapping $\Psi_\phi : P \to P$ is a Poisson mapping for each $\phi \in G$ in which $\Psi_\phi(y) = \Psi(\phi, y)$ for $y \in P$. It is assumed that the Hamiltonian mapping $\hat{J} : g \to C^\infty(P, R)$ is obtained for this action s.t. $X_{\hat{J}(v)} = v_P$ for $v \in g$, where $X_{\hat{J}(v)}$ and $v_P \in \mathcal{X}(P)$ denote the Hamiltonian vector field for $\hat{J}(v) \in C^\infty(P, R)$ and the infinitesimal generator of the action on $P$ corresponding to $v \in g$, respectively. As
such, the momentum (moment) mapping \( J: P \to g^* \) is defined by \( \hat{J}(v)(y) = \langle J(y), v \rangle \). For the special case in which \((P, \omega)\) is a symplectic manifold with a symplectic 2-form \( \omega \in \Lambda^2(G) \) (i.e., \( \delta \omega = 0 \) and \( \omega \) is weak nondegenerate), this momentum mapping is equivalent to that defined by \( d\hat{J}(v) = v_P [\omega] \).

\[
\begin{align*}
v \in g \\
\downarrow \\
d\hat{J}(v) = v_P [\omega] \text{ or } X_{\hat{J}(v)} = v_P \\
\omega \text{ or } \{ , \} \\
\hat{J}(v) \in C^\infty(P)
\end{align*}
\]

In twenty-first century, lots of mathematicians would have based their study especially on the Poisson structure or the symplectic structure in the above diagram, while the physicists would usually have made importance the functions as the Hamiltonian and the other invariance of motions as some physical matter. In Lie-Poisson mechanics, the Lie group plays the most important role as “motion” itself, while the present theory inherits such an idea.

For the trivial topology of \( G \) (consult [24] in the nontrivial cases), the Poisson bracket satisfies

\[
\{ \hat{J}(v_1), \hat{J}(v_2) \} = \pm \hat{J}([v_1, v_2]). \tag{A5}
\]

The Collective Hamiltonian Theorem [21] concludes the Poisson bracket for \( A \circ J \) and \( B \circ J \in C^\infty(P, R) \) can be expressed for \( \mu = J(y) \in g^* \) as

\[
\{ A \circ J, B \circ J \}(y) = \pm \langle J(y), \frac{\partial A}{\partial \mu}(\mu), \frac{\partial B}{\partial \mu}(\mu) \rangle, \tag{A6}
\]

where \( \frac{\partial F}{\partial \mu} : g^* \to g \) is the Fréchet derivative of \( F \in C^\infty(g^*, R) \) that every \( \mu \in g^* \) and \( \xi \in g \) satisfies

\[
\frac{d}{d\tau} \Big|_{\tau=0} F(\mu + \tau \xi) = \left\langle \xi, \frac{\partial F}{\partial \mu}(\mu) \right\rangle. \tag{A7}
\]

Thus, the collective Hamiltonian \( H \in C^\infty(g, R) \) such that \( H_P = H \circ J \) collects or reduces the Poisson equation of motion into the following Lie-Poisson equation of motion:

\[
\frac{d}{dt} \mu_t = \pm ad \frac{\partial H}{\partial \mu}(\mu_t) \mu_t, \tag{A8}
\]

where \( \mu_t = J(x_t) \) for \( x_t \in P \). We can further obtain the formal solution of Lie-Poisson equation of motion [A8] as

\[
\mu_t = Ad_{\phi_t(\mu_0)} \mu_0, \tag{A9}
\]

where generator \( \phi_t \in \tilde{G} \) satisfies \( \{ \frac{\partial H}{\partial \mu}(\mu_t) \}^+ = \phi_t^{-1} \cdot \frac{d\phi_t}{dt} \) or \( \{ \frac{\partial H}{\partial \mu}(\mu_t) \}^- = \frac{d\phi_t}{dt} \cdot \phi_t^{-1} \). The existence of this solution, however, should independently verified (see [23] for example).

In particular, Arnold [26] applies such group-theoretic method not only to the equations of motion of a rigid body but also to that of an ideal incompressible fluid, and constructs them as the motion of a particle on the three-dimensional special orthogonal group \( SO(3) \) and as that on the infinite-dimensional Lie group.
\[ \mathcal{D}_\nu(M) \text{ of all } \mathcal{C}_\infty \text{ volume-preserving diffeomorphisms on a compact oriented manifold } M. \] By introducing semidirect products of Lie algebras, Holm and Kupershmidt \cite{27} and Marsden et al. \cite{28} went on to complete the method such that various Hamiltonian systems can be treated as Lie-Poisson systems, e.g., the motion of a top under gravity and that of an ideal magnetohydrodynamics (MHD) fluid.

For the motion of an isentropic fluid, the governing Lie group is a semidirect product of the Lie group \( \mathcal{D}(M) \) of all \( \mathcal{C}_\infty \)-diffeomorphisms on \( M \) with \( \mathcal{C}_\infty(M) \times \mathcal{C}_\infty(M) \), i.e.,

\[
G(M) = \mathcal{D}(M) \rtimes_{\text{semi.}} \{ \mathcal{C}_\infty(M) \times \mathcal{C}_\infty(M) \}.
\]

For \( \tilde{\phi}_1 = (\phi_1, f_1, g_1), \tilde{\phi}_2 = (\phi_2, f_2, g_2) \in I(M) \), the product of two elements of \( I(M) \) is defined as follows:

\[
\tilde{\phi}_1 \cdot \tilde{\phi}_2 = (\phi_1, f_1, g_1) \cdot (\phi_2, f_2, g_2) = (\phi_1 \circ \phi_2, \phi_2^* f_1 + f_2, \phi_2^* g_1 + g_2),
\]

where \( \phi^* \) denotes the pullback by \( \phi \in \mathcal{D}(M) \) and the unit element of \( G(M) \) can be denoted as \( (\text{id.}, 0, 0) \in G(M) \), where \( \text{id.} \in \mathcal{D}(M) \) is the identity mapping from \( M \) to itself.

The Lie bracket for \( \tilde{v}_1 = (v^i_1 \partial_i, U_1, W_1) \) and \( \tilde{v}_2 = (v^i_2 \partial_i, U_2, W_2) \in g(M) \) becomes

\[
[\tilde{v}_1, \tilde{v}_2] = \left( v^i_1 \partial_i, v^i_2 \partial_i, \right) , v^i_1 \partial_j U_2 - v^i_2 \partial_j U_1 , v^i_1 \partial_j W_2 - v^i_2 \partial_j W_1 \right).
\]

For the volume measure \( v \) of \( M \), the element of the dual space \( g(M)^* \) of the Lie algebra \( g(M) \) can be described as

\[
\mathcal{J}_t = (dv \rho_t \otimes p_t, dv \rho_t, dv \sigma_t),
\]

in that \( p_t \in \Lambda^1(M) \), \( dv \rho_t \in \Lambda^3(M) \) and \( dv \sigma_t \in \Lambda^3(M) \) physically means the momentum, the mass density, and the entropy density.

For the thermodynamic internal energy \( U(\rho(x), \sigma(x)) \), the Hamiltonian for the motion of an isentropic fluid is introduced as

\[
\mathcal{H}(\mathcal{J}) = \frac{1}{2} \int_M dv(x) \rho_t(x) g^{ij}(x) p_t p_{ij} + \int_M dv(x) \rho_t(x) U(\rho_t(x), \sigma_t(x)).
\]

Define the operator \( \tilde{F}_t = \frac{\partial}{\partial \mathcal{J}_t} \mathcal{J}_t \) \( \in g(M) \) for every functional \( F : g(M)^* \to \mathbb{R} \) as

\[
\frac{d}{dt} \bigg|_{t=0} \mathcal{F}(\mathcal{J}_t + \epsilon \mathcal{K}) = \langle \mathcal{K}, \tilde{F}_t \rangle,
\]

then, the Hamiltonian operator \( \mathcal{H}_t = \frac{\partial \mathcal{H}}{\partial \mathcal{J}_t} \mathcal{J}_t \) \( \in g(M) \) is calculated for the velocity field \( v_t = g^{ij} p_t \partial_j \in X^1(M) \) as

\[
\mathcal{H}_t = \left( v^i \partial_i , - \frac{1}{2} g^{ij} p_t p_{ij} + U(\rho_t(x), \sigma_t(x)) + \rho_t(x) \frac{\partial U}{\partial \rho} (\rho_t(x), \sigma_t(x)) , \rho_t(x) \frac{\partial U}{\partial \sigma} (\rho_t(x), \sigma_t(x)) \right).
\]

The equation of motion becomes the following Lie-Poisson equation:

\[
\frac{d}{dt} \mathcal{J}_t = ad^*_H \mathcal{J}_t,
\]

which is calculated as follows:

1. the conservation laws of mass and entropy:

\[
\frac{\partial \rho_t}{\partial t} + \nabla^1 \partial_j \left( \rho_t v^i \sqrt{\ } \right) = 0,
\]

\[
\frac{\partial \sigma_t}{\partial t} + \nabla^1 \partial_j \left( \sigma_t v^i \sqrt{\ } \right) = 0,
\]

where \( \sqrt{\ } = \sqrt{|\det g^{ij}|} \);
2. the conservation law of momentum:
\[
\frac{\partial}{\partial t} (\rho v_i p_{ik}) + \sqrt{-1} \partial_j \left( v^j \rho v_i p_{ik} \right) + \partial_k p_t = 0,
\]
(A20)

where the pressure \( p_t \) satisfies the following condition:
\[
P_t(x) = \rho_t(x) \left\{ \rho_t(x) \frac{\partial U}{\partial \rho} + \sigma_t(x) \frac{\partial U}{\partial \sigma} \right\} (\rho_t(x), \sigma_t(x)),
\]
(A21)

which is consistent with the first law of thermodynamics.

Next, we consider \( D_v(M) \), being the Lie group of volume-preserving diffeomorphisms of \( M \), where every element \( \phi \in D_v(M) \) satisfies \( dv(\phi(x)) = dv(x) \). Lie group \( D_v(M) \) is a subgroup of \( G(M) \), and inherits its Lie-algebraic structure of. A right-invariant vector at \( T_xD_v(M) \) is identified with the corresponding divergence-free vector field on \( M \), i.e.,
\[
u^- (e) = u^i \partial_i \quad \nabla \cdot u = 0 \quad \text{for all} \quad x \in M.
\]
(A22)

We can define an operator \( P_\phi (\cdot) \) that orthogonally projects the elements of \( T_\phi G(M) \) onto \( T_\phi D_v(M) \) for \( \phi \in D_v(M) \subset G(M) \) such that
\[
P_\phi [v^- (\phi)] = P[v^- (\phi)]
\]
(A23)

and
\[
P[v^- (e)] = (v^i - \partial^i \theta) \partial_i,
\]
(A24)

where \( \theta : M \to \mathbb{R} \) satisfies \( \partial_i (v^i(x) - \partial^i \theta(x)) = 0 \) for every \( x \in M \). This projection changes Lie Poisson equation (A17) into the new Lie-Poisson equation representing the Euler equation for the motion of an incompressible fluid:
\[
\frac{\partial u_t}{\partial t} + u_t \cdot \nabla u_t + \nabla p = 0,
\]
(A25)

where the pressure \( p : M \to \mathbb{R} \) is determined by the condition \( \nabla \cdot u_t = 0 \).

References

[1] W. Heisenberg, Zeit. Phys. 33, 879 (1925).
[2] E. Schrödinger, Annalen der Physik 79, 361 (1926).
[3] H.J. Groenwald, Physica 12, 405 (1946).
[4] van Hove, Mem. Acad. Roy. Belg. 26, 61 (1951).
[5] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[6] J.S. Bell, Physics 1 (1964) 195; Speakable and unspeakable in quantum mechanics (Cambridge University Press, 1987).
[7] A. Aspect, P. Grangier, and G. Roger, Phys. Rev. Lett. 49, 91 (1982).
[8] N. Bohr, Dialectica 1, 312 (1948).
[9] T. Ono, Phys. Lett. A 230, 253 (1997)./ the doctoral dissertation in University of Tokyo (1997).
[10] C.G. Jung and W. Pauli, The interpretation of Nature and the Psyche, (Bollingen Foundation Inc., New York, 1955).
[11] T. Ono, Found. Phys., to be submitted.
[12] T. Ono, Found. Phys., to be submitted.
[13] R.M. Wald, General Relativity, (Univ. Chicago Press, USA, 1984).
[14] H. Omori, Infinite-Dimensional Lie Groups, (AMS, Providence, Rhode Island, 1997), Trans. Math. Mono. Vol.158./ (Kinokuniya Co., Ltd. Japan, 1979), in Japanese.
[15] Von TH. Kaluza, in: Modern Klauza-Klein Theories, ed. T. Appelquist, A. Chodos, and P.G.O. Freund (Addison-Wesley, USA/Canada, 1987).
[16] A.N. Whitehead, Process and Reality, edited by D.R. Griffin and D.W. Sherburne (The Free Press, New York and London, 1979).
[17] N.N. Bogolubov, A.A. Logunov, A.I. Oksak, and I.T. Todorov, General Principles of Quantum Field Theory, (Kluwer, 1990).
[18] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry (John Wiley, New York, 1963).
[19] S. Lie, Theorie der Transformationsgruppen (Zweiter Abschnitt, Teubner, 1890).
[20] H. Poincaré, C.R. Acad. Sci. 132, 369 (1901).
[21] J.E. Marsden and T.S. Ratiu, Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems (Springer-Verlag, New York, 1994).
[22] J.E. Marsden and A. Weinstein, Rep. on Math. Phys. 5, 121 (1974).
[23] V. Guillemin and S. Sternberg, Ann. Phys. 127, 220 (1980)./ Symplectic Techniques in Physics (Cambridge University Press, New York, 1991).
[24] R. Abraham and J. Marsden, Foundation of Mechanics (Addison-Wesley, Reading, MA, 1978), second edition,
[25] D. Ebin and J. Marsden, Ann. Math. 90, 102 (1970).
[26] V.I. Arnold, Ann. Inst. Fourier 16, 319 (1966); Mathematical Methods of Classical Mechanics (Springer, New York, 1978), Appendix 2.
[27] D. Holm and B. Kupershmidt, Physica D 6, 347 (1983).
[28] J. Marsden, T. Ratiu, and A. Weinstein, Trans. Am. Math. Soc. 281, 147 (1984).