Control design for tracking problem of the bilinear control system using observation matrix and pole placement

Khozin Mu’tamar¹, Janson Naiborhu¹

¹Dept. of Mathematics, Faculty of Mathematics and Natural Sains, Institut Teknologi Bandung, Bandung, Indonesia
E-mail: mutamar.khozin@students.itb.ac.id

Abstract. Bilinear control system is widely used, especially in the chemistry and engineering to describe chemical reactions and biological populations. Many of them are discussing about how to stabilize the system. In this paper, control design for tracking problem of the bilinear control system is presented. It is assumed that bilinear control system has full relative degree and the system is observable. Control design is divided into two stages. First, using observation matrix, tracking problem of bilinear control system transformed into stabilization problem of linear control system. Then, the linear control system is stabilized using static feedback control with pole placement. Examples are given to show controller performance.

1. Introduction

Bilinear control systems are widely used in chemistry and engineering, such as chemical control reactions and biological populations, because these systems are more controlled and can produce better performance than linear control systems [1]. Bilinear control system is defined by

$$\dot{x} = Ax + \sum_{i}^{m} u_{i}B_{i}x$$

(1)

where $A, B \in \mathbb{R}^{n \times n}$. The word bilinear refers to the relationship $u, x$, each of which has a linear degree and is part of the nonlinear system [2]. The stability of bilinear control system can be found in [3, 4]. Some controller have been studied by other researchers. Linear feedback control technique is used in [5] for the bilinear control system, applied to nuclear reactor problems. Optimal control has been applied to the bilinear control system to stabilize the CSTR chemical reaction problem in [6] and chemotherapy in cancer involving multiple control in [7]. Ramenzapour et.al [8] study bilinear control system in chemical reaction problems using combination of optimal control and iterative procedures.

Nonminimum-phase makes some difficulties in control design [9]. Unstable internal dynamics makes state variables go to unbounded while the output is successfully stabilized. An example of nonminimum-phase bilinear control systems is DC/DC converter, explained in [10, 11]. In order to manipulate the output, Carrizosa et.al [11] uses the Galerkin method to approximate the output of the system. Meanwhile, different approach are used in [9, 12]. They design control...
by limiting state variables, so that the output is bounded in the defined kernel domain. It means that the system does not need to be linearized across the domain but only at several sampling points.

In this article, we present a control design for stabilization and tracking problem of bilinear control system. Differentiating from the previous articles, this article uses observation matrix and pole placement technique. Pole placement has an advantage because the desired system stability can be adjusted as needed. In stabilization problem, the nonminimum-phase bilinear control system is transformed into a linear control system using input-output feedback linearization. Virtual control of linear control system is designed using pole placement, then it’s transformed into initial control by matrix observation. In tracking problem, a new system is defined based on the assumptions that the system is observable dan exact linearizable. The control design is done using similar procedure in stabilization problem based on newly defined system.

This article is organized as follows. After this, input-output linearization (IOFL) is given. IOFL is used to transform a bilinear control system into a linear control system using the output of the system. Next, the control design for the stabilization problem and the tracking problem is discussed and is followed by numerical simulation to show control performance for a particular case. Conclusion about the problem and the results are given in the end of this article.

2. Input-Output Feedback Linearization

Given nonlinear control system

\[ \dot{x}(t) = f(x) + g(x)u(t) \]
\[ y(t) = h(x) \]

where \( x \) is state variable; \( x \in D \subseteq \mathbb{R}^n \), \( u \) is control variable; \( u \in \mathbb{R} \), \( f \) is state function; \( f: D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n \), and \( g \) is control function; \( g: D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n \). \( D \subseteq \mathbb{R}^n \) is the domain of state function containing the origin \( x_e \). Linearization problem using input-output is determining whether exist a transformation \( z(\eta, \psi) = T(x) \) and a control \( u = \alpha(x) + \beta(x)\nu \) which makes equation (2) into linear form

\[
\begin{align*}
\dot{\eta} &= f_0(\eta, \psi) \\
\dot{\psi} &= A\psi + B\nu \\
y &= C\psi
\end{align*}
\]

The transformation \( T(x) \) can be determined using lie derivative.

**Definition 1** (Lie derivative). Suppose \( h: \mathbb{R}^n \rightarrow \mathbb{R} \) is smooth scalar function and \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is smooth vector field in \( \mathbb{R}^n \). Lie derivative \( h \) over \( f \) is scalar function defined by

\[ L_f h = \nabla h f \]

Using lie derivative, the output of the system (3) is derived iteratively to form linear control system. The iteration process is carried out up to a number which is called the relative degree.

**Definition 2** (Relative degree). Given nonlinear control system (2) and output (3). Nonlinear control system (2) has relative degree \( \rho \), \( 1 \leq \rho \leq n \in \mathbb{N} \) if

\[ L_f h \neq 0, \quad k = 1, \ldots, \rho - 1 \]

Next, linearization process produces the normal form. Using the procedure given in [13], transformation \( T(x) \) which is transforming nonlinear control system (2) into normal form is given by next definition.
**Definition 3** (Isidori-Brynes Normal Form). Given nonlinear control system (2) with output (3). Suppose the system has relative degree \( \rho \leq n \). Then, transformation \( T \) which is normalized nonlinear control system using input-output given by

\[
z = T(x) = \begin{bmatrix} \phi(x) \\ - \psi(x) \end{bmatrix} \triangleq \begin{bmatrix} \eta \\ - \xi \end{bmatrix}
\]

(4)

where \( \psi(x) = \langle h(x), L_f h(x), \ldots, L_f^{\rho-1} h(x) \rangle \) and \( \phi(x) = (\phi_1, \phi_2, \ldots, \phi_{n-\rho}) \) is function that fulfills

\[
\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad 1 \leq i \leq n - \rho.
\]

Nonminimum-phase occur when zero dynamic \( \dot{\eta} = \Phi(\eta, \psi) \) from equation (4) is unstable.

**Definition 4** (Zero dynamic). Given internal dynamic of the normal form in (4)

\[
\dot{\eta} = \Phi(\eta, \psi)
\]

(5)

Zero dynamic is a system with the output being zero, or \( \psi = 0 \), so that the equation (5) become

\[
\dot{\eta} = \Phi(\eta, 0)
\]

3. Results and Discussion

Given single input and single output bilinear control system [2]

\[
x = Ax + uBx
\]

(6)

\[
y = h(x)
\]

(7)

where \( A, B \in \mathbb{R}^{n \times n}, x \in \mathbb{R}^n \) and \( u, y \in \mathbb{R} \). Suppose that output of the system (7) can be written as linear combination from state variable \( x \), that is \( y = cx \), with \( c \in \mathbb{R}^n \).

**Assumption 1.** The bilinear system (6) has a relative degree \( \rho = r \leq n \) so that \( cA^kBx = 0 \) for \( k = 0, 1, 2, \ldots, r - 2 \) and \( cA^{r-1}Bx \neq 0 \).

**Assumption 2.** The bilinear system (6) is observable, i.e. matrix defined as

\[
M = \begin{bmatrix} h(x) & L_f h(x) & \cdots & L_f^{n-1} h(x) \end{bmatrix}
\]

(8)

is full rank.

Define \( z = Mx \) as the transformation of state variable \( x \) and \( z \) where \( M \) is given in (8), we obtain

\[
\begin{align*}
z_1 &= cx \\
z_2 &= cAx \\
\vdots \\
z_n &= cA^{n-1}x
\end{align*}
\]

(9)

Time derivative of equation (9) yields

\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\vdots \\
\dot{z}_r &= z_{r+1} + cA^{r-1}Bxu \\
\dot{z}_{r+1} &= z_{r+2} + cA^rBxu \\
\vdots \\
\dot{z}_{n-1} &= z_n + cA^{n-2}Bxu \\
\dot{z}_n &= \nu
\end{align*}
\]

(10)
where \( \nu \) as new control variable in the system (10). The relation between control variable \( \nu \) and \( u \) is given by

\[
u = u = \nu - \text{cA}^n x \frac{\text{cA}^{n-1}Bx}{(\text{cA}^{n-1}Bx)^n} \quad \text{(11)}
\]

Substitute control variable \( u \) equation (11) to equation (10) and use the transformation \( z = Mx \), to obtain

\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\quad \vdots \\
\dot{z}_r &= z_{r+1} + \text{cA}^{r-1}BM^{-1}z \left( \nu - \text{cA}^n M^{-1}z \right) \\
\dot{z}_{r+1} &= z_{r+2} + \text{cA}^rBM^{-1}z \left( \nu - \text{cA}^n M^{-1}z \right) \\
&\quad \vdots \\
\dot{z}_{n-1} &= z_n + \text{cA}^{n-2}BM^{-1}z \left( \nu - \text{cA}^n M^{-1}z \right) \\
\dot{z}_n &= \nu \quad \text{(12)}
\end{align*}
\]

Choose \( \nu \) as control with constant feedback of state variables \( z \), that is \( \nu = -Kz \) where \( K = [k_i], k_i \in \mathbb{R} \). Equation (12) can be written

\[
\dot{z} = (A_c - \psi(k)(K + \text{cA}^n M^{-1}))z \quad \text{(13)}
\]

where \( \psi(k) \) is defined by

\[
\psi(k) = \begin{cases} 
\text{cA}^{k-1}BM^{-1}z & k = r, r+1, \ldots, n-1 \\
\frac{\text{cA}^n M^{-1}z}{(\text{cA}^{n-1}B)^n} & \text{lainnya}
\end{cases} \quad \text{(14)}
\]

and \( A_c \) is given by

\[
A_c = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -k_1 & -k_2 & -k_3 & -k_4 & \cdots & -k_n \end{bmatrix}
\]

Stabilization of (13) can be done by select \( k_i \) such that \( A_c - \psi(k)(K + \text{cA}^n M^{-1}) \) is Hurwitz.

3.1. Tracking Problem

Suppose the bilinear control system (6) has relative degree \( n \) and the output (7) is desired to follow the trajectory \( y_d(t) \). Define new variables

\[
\begin{align*}
e_1(t) &= h(x) - y_d(t) = cx - y_d(t) \\
e_2(t) &= L_1 h(x) - y_d(t) = \text{cA}x - \dot{y}_d(t) \\
e_3(t) &= L_2 h(x) - y_d(t) = \text{cA}^2x - \ddot{y}_d(t) \\
&\quad \vdots \\
e_n(t) &= L_n h(x) - y_d^{(n-1)}(t) = \text{cA}^{n-1}x - y_d^{(n-1)}(t)
\end{align*}
\]

(15)
Equation (15) can be written in matrix form,

\[ \mathbf{e} = \mathbf{Mx} - \mathbf{y}_d \]  

(16)

where \( \mathbf{e} = \langle e_1, e_2, \ldots, e_n \rangle \), \( \mathbf{y}_d = \langle y_d, \dot{y}_d, \ldots, y_d^{(n-1)} \rangle \), and \( \mathbf{M} \) is observation matrix in equation (8). Using assumption that the bilinear control system (6) has relative degree \( \rho = n \), time derivative of equation (15) yields

\[
\begin{aligned}
\dot{e}_1(t) &= \mathbf{L}_f h(x) - \dot{y}_d(t) \\
\dot{e}_2(t) &= \mathbf{L}_f^2 h(x) - \ddot{y}_d(t) \\
\dot{e}_3(t) &= \mathbf{L}_f^3 h(x) - \dddot{y}_d(t) \\
&\vdots \\
\dot{e}_{n-1}(t) &= \mathbf{L}_f^{n-1} h(x) - y_d^{(n-1)}(t) \\
\dot{e}_n(t) &= \mathbf{L}_f^n h(x) - y_d^{(n)}(t) + \mathbf{L}_g \mathbf{L}_f^{n-1} h(x) u(t)
\end{aligned}
\]

(17)

Using the transformation in equation (15), equation (17) become

\[
\begin{aligned}
\dot{e}_1(t) &= e_2(t) \\
\dot{e}_2(t) &= e_3(t) \\
&\vdots \\
\dot{e}_{r-1}(t) &= e_r(t) \\
\dot{e}_r(t) &= e_{r+1}(t) \\
&\vdots \\
\dot{e}_{n-1}(t) &= e_n(t) \\
\dot{e}_n(t) &= \nu - y_d^{(n)}(t)
\end{aligned}
\]

(18)

\( \nu \) is new control on linearized system in (18) whose relation to control \( u \) is given in (11). Stabilization of equation (18) can be determined by eliminating term \( y_d^{(n)}(t) \) and introducing constant feedback, so that control \( \nu \) is

\[
\nu = y_d^{(n)}(t) + \sum_{i=1}^{n} k_i e_i
\]

(19)

Substitute control \( \nu \) in (19) to equation (18) to produce linear form

\[
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \ddots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
k_1 & k_2 & k_3 & \cdots & k_n
\end{bmatrix}
\begin{bmatrix}
e \\
\dot{e}
\end{bmatrix}
\]

(20)

\( k_i \) can be selected specifically so that the system (20) is asymptotically stable, i.e. choosing \( k_i \) so that \( \text{Re}(\lambda) < 0 \) applies to all root characteristics of the system (20).

### 3.2. Numerical Simulation

**Example 1.** Consider bilinear control system

\[
\begin{aligned}
\dot{x}_1 &= 2x_1 + x_2 + x_3 - x_4 \\
\dot{x}_2 &= x_1 + x_2 - x_3 + x_4 + (x_1 + x_2 + 2x_3 + x_4) u \\
\dot{x}_3 &= x_1 + x_3 - 2x_4 \\
\dot{x}_4 &= x_1 + 3x_3 + 4x_4
\end{aligned}
\]

(21)
with output \( y = x_1 \). A control \( u \) will be determined which stabilizes the desired output. Bilinear control system (21) is written in matrix form \( \dot{x} = Ax + uBx \) where \( A \) and \( B \) are given by

\[
A = \begin{bmatrix}
2 & 1 & 1 & -1 \\
1 & 1 & -1 & 1 \\
1 & 0 & 1 & -2 \\
1 & 0 & 3 & 4
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

and output \( y = cx \) with \( c = [1, 0, 0, 0] \). Bilinear control system (21) has relative degree \( \rho = 2 \) and is non-minimum phase. Observation matrix \( M \)

\[
M = \begin{bmatrix}
1 & 0 & 0 & 0 \\
2 & 1 & 1 & -1 \\
5 & 3 & -1 & -7 \\
5 & 8 & -20 & -28
\end{bmatrix}
\]

has \( \text{rank}(M) = 4 \) so that the bilinear control system (21) is observable.

From equation (14), we have \( \psi = \langle 0, \theta, 3\theta, 0 \rangle \) where \( \theta = \left( \frac{k_1}{8} + \frac{3}{4} \right) z_1 + \left( \frac{k_2}{8} + \frac{27}{8} \right) z_2 + \left( \frac{k_3}{8} - \frac{13}{4} \right) z_3 + \left( \frac{k_4}{8} + 1 \right) z_4 \). Suppose \( A_c = A - \psi I \), equation (13) becomes \( \dot{z} = A_c z \) with

\[
A_c = \begin{bmatrix}
0 & \frac{k_1}{8} + \frac{3}{4} & 0 & 0 \\
- \left( \frac{k_2}{8} + \frac{27}{8} \right) & -1 & 0 & 0 \\
-3 \left( \frac{k_1}{8} + \frac{3}{4} \right) & -3 \left( \frac{k_2}{8} + \frac{27}{8} \right) & -3 \left( \frac{k_3}{8} - \frac{13}{4} \right) & - \left( \frac{k_4}{8} + 1 \right) \\
-k_1 & -k_2 & -k_3 & -k_4
\end{bmatrix}
\]

(23)

Next, \( k_i \) is manually selected for matrix \( A_c \) in equation (23) to be Hurwitz. The characteristic polynomial of the \( A_c \) is given by

\[
C_{A_c} = \lambda^4 + \left( -\frac{5k_1}{8} + \frac{k_2}{8} + \frac{3k_3}{8} + k_4 \right) \lambda^3 + \left( \frac{k_1}{8} - \frac{5k_2}{8} - 2k_3 - \frac{51k_4}{8} + \frac{87}{8} \right) \lambda^2 + \left( -\frac{5k_1}{8} + \frac{10k_2}{8} + \frac{27k_3}{8} + \frac{87k_4}{8} + \frac{9}{4} \right) \lambda + \left( \frac{5k_1}{4} + \frac{3k_3}{4} + \frac{9k_4}{4} \right)
\]

(24)

Assuming our system is stable in the poles \( \lambda = -1 \pm 2I, -2 \pm I \). The characteristic polynomial for these poles are

\[
C_p = \lambda^4 + 6\lambda^3 + 18\lambda^2 + 30\lambda + 25
\]

(25)

It’s obvious that both characteristic polynomial (24) and (25) must have equal coefficients. Thus, we need to solve linear equation to find feedback control.

\[
\begin{cases}
\frac{k_2}{8} + \frac{3k_3}{8} + k_4 = 99 \\
\frac{k_1}{8} - \frac{5k_2}{8} - 2k_3 - \frac{51k_4}{8} = \frac{8}{5} \\
\frac{5k_1}{8} + \frac{10k_2}{8} + \frac{27k_3}{8} + \frac{87k_4}{8} = \frac{8}{11} \\
\frac{5k_1}{4} + \frac{3k_3}{4} + \frac{9k_4}{4} = 25
\end{cases}
\]

(26)
The solution of linear system (26) are feedback gain $K$ for control $\nu$, given by

$$K = \begin{bmatrix} 377 & 857 & 3 & -167 \\ 4 & 2 & 2 & 4 \end{bmatrix}$$

The simulation results of the bilinear control system (21) and the control $u(t)$ equation (11) are shown in Figures 1 and 2.

![Figure 1](image1.png)

**Figure 1.** Output bilinear control system (21), $y(t) = x_1(t)$ in $t \in [0, 15]$

![Figure 2](image2.png)

**Figure 2.** Control profile $u(t)$ stabilize bilinear control system (21).

**Example 2.** Consider bilinear system

$$\begin{cases} 
\dot{x}_1 = x_2 \\
\dot{x}_2 = x_3 + x_3u \\
\dot{x}_3 = x_1 + x_3 
\end{cases}$$

with output $y = x_3$. The output of the bilinear control system (27) is desired to follow the trajectory $y_d(t) = \sin t$. Unforced system (27) is unstable because there is positive eigen value $\lambda = 1.466$. Observation matrix $M$ of this system is

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

with $\text{rank}(M) = 3$ so that the bilinear control system (27) is observable. The linear dynamic system (20) which will be stabilized becomes

$$\dot{e} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k_1 & k_2 & k_3 \end{bmatrix} e$$

The characteristic polynomial from (29) is

$$C_\lambda = \lambda^3 - k_3\lambda^2 - k_2\lambda - k_1$$

If the system (29) is asymptotic stable in poles $\lambda = -2 \pm 2i, -1$ then the characteristic polynomial from these poles is

$$\hat{C}_\lambda = \lambda^3 + 5\lambda^2 + 12\lambda + 8$$
The similarity of both characteristic polynomials result $k_1 = -8, k_2 = -12, k_3 = -5$. Using equation (19), we obtain control $\nu$ that is

$$\nu = -\cos(t) - (8e_1 + 12e_2 + 5e_3)$$

(30)

Using relation between control $u$ dan $\nu$ in equation (11) and (30), observation matrix in (28) and transformation in (16), the control $u$ is

$$u(t) = \frac{11\cos(t) + 3\sin(t) - 18x_1 - 6x_2 - 27x_3}{x_3}$$

(31)

Control $u(t)$ is not well-defined while $x_3 \approx 0$. In order to handle control $u(t)$ while $x_3 \approx 0$, then control $u(t)$ can be written

$$u^*(t) = \begin{cases} 
10 \cdot \text{sign}(x_3), & |x_3| < 1e - 2 \\
\nu(t), & \text{other}
\end{cases}$$

(32)

The simulation result showing the output of bilinear control system (27) and given trajectory $y_d(t) = \sin(t)$ is shown in figure 3 and control profile is shown in figure 4.

Figure 3. Comparison between the output $y(t) = x_3(t)$ of the system (21) with a given path $y_d(t) = \sin(t)$.

Figure 4. Control profile $u$ which drive the output $y(t) = x_3(t)$ to follow the trajectory $y_d(t) = \sin(t)$.

Figure 3 shows $y(t) = x_3$ follow trajectory $y_d(t) = \sin(t)$ in $t \approx 5$. There is a strange behavior in $t \approx 2$ when $x_3$ goes in the wrong direction and then returns in the desired path. This behavior is consistent with that shown in the control profile in figure 4. The control profile in $t \in [0,5]$ has a different pattern. There is a repeating pattern after $t > 5$ when the system output successfully follows the desired path.

4. Summary

Stabilization and tracking problem in bilinear control systems have been discussed. The controls are designed using the assumption that the system is observable and using poles placement technique. From the stabilization problem, it’s shown that nonminimum-phase bilinear system can be controlled using this technique. The tracking problem simulation results show that the system output succeeds in following the given path in a short time. In addition, the output moves in the wrong direction before following the desired path.
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