TWO QUESTIONS ARISING
IN THE THEORY OF ATTRACTORS

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Abstract. In this note, we dwell on the notions of global and exponential attractors for strongly continuous semigroups acting on a complete metric space. Two natural questions arising in the theory are addressed.

1. Global and exponential attractors of semigroups. A strongly continuous semigroup is a family of selfmaps \( S(t) \), depending on a parameter \( t \geq 0 \), acting on a complete metric space \( (X, d) \) and satisfying the following axioms:

   (i) \( S(0) = I \) (the identity map).
   (ii) \( S(t + \tau) = S(t)S(\tau) \) for all \( t, \tau \geq 0 \).
   (iii) For every fixed \( t \geq 0 \), the function \( x \mapsto S(t)x \) belongs to \( C(X, X) \).
   (iv) For every fixed \( x \in X \), the function \( t \mapsto S(t)x \) belongs to \( C([0, \infty), X) \).

An important application comes from the study of autonomous differential equations in Banach spaces. Indeed, whenever a Cauchy problem on a Banach space \( X \) is well posed for all initial data \( x_0 \in X \) taken at \( t = 0 \) (for some suitable notion of weak solution), then the corresponding (forward) solutions \( x(t) \) can be written in the form

\[
x(t) = S(t)x_0,
\]

where the semigroup \( S(t) \) is uniquely determined by the equation.

A crucial issue in connections with semigroups is their asymptotic behavior. This is particularly relevant when dealing with semigroups generated by differential models describing concrete evolutionary phenomena, which are characterized by the presence of dissipation mechanisms. In mathematical terms, the dissipation translates into the existence of suitably small regions of the space \( X \) able to capture the longterm dynamics. These concepts fall within the theory of dynamical systems, a subject that, although relatively recent, is nowadays considered a well-established branch of mathematics. For a detailed presentation of the theory, we address the reader to the classical textbooks [1, 5, 6, 9] and the subsequent treatises [2, 7], among others.

In this context, a key notion is the one of global attractor, which, loosely speaking, is the set where the asymptotic dynamics is eventually confined. More precisely, the global attractor is a compact set \( \mathcal{A} \subset X \) which is at the same time

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fully invariant, i.e. \( S(t)A = A \) for all \( t \geq 0 \); and

- attracting for \( S(t) \) with respect to the Hausdorff semidistance, i.e. for every bounded set \( B \subset X \)

\[
\delta(S(t)B, A) := \sup_{x \in B} \inf_{\xi \in \mathcal{A}} d(S(t)x, \xi) \rightarrow 0, \quad \text{as } t \to \infty.
\]

In spite of the fact that the global attractor provides the best possible description of the “final” dynamics, it suffers from a drawback which turns out to be quite relevant for practical purposes: the attraction rate cannot be explicitly estimated and, in general, can be arbitrarily slow. Consequently, the global attractor is usually very sensitive to perturbations, even very small ones. An alternative (and more robust) object to describe the longterm dynamics is the exponential attractor, first introduced in [3] in the Hilbert space context, and further generalized in [4]. This is a compact set \( \mathcal{E} \subset X \) of finite fractal dimension with the following properties:

- \( \mathcal{E} \) is positively invariant for the semigroup, i.e. \( S(t)\mathcal{E} \subset \mathcal{E} \) for all \( t \geq 0 \);
- \( \mathcal{E} \) is exponentially attracting for \( S(t) \), i.e. there exists an exponential rate \( \omega > 0 \) and a nondecreasing positive function \( Q \) such that

\[
\delta(S(t)B, \mathcal{E}) \leq Q(||B||)e^{-\omega t}
\]

for every bounded subset \( B \subset X \).

Here \( ||B|| = \inf_{x \in B} d(x, x_0) \), where \( x_0 \) is a an arbitrarily fixed element of \( X \) (usually one takes \( x_0 = 0 \) when there is a linear structure). Contrary to the global attractor, the exponential attractor is not unique. Indeed, if there is one, then there exist infinitely many of them. More details on the concept of exponential attractor can be found in [7].

2. Two questions.

2.1. Global attractors with a single equilibrium. The global attractor, whenever exists, is the largest bounded fully invariant set. In particular, it contains all the stationary points, namely, those \( x \in X \) for which \( S(t)x = x \) for all \( t \geq 0 \). In this work, we are interested to the case when \( S(t) \) has a unique stationary point \( x_\star \in X \). Clearly, having a unique stationary point does not mean that \( S(t) \) has the global attractor, not even if all trajectories converge to that point in the longtime. Counterexamples can be given also for linear semigroups. Indeed, it is well-known that there are linear semigroups which are stable (i.e. all the trajectories go to zero, the unique stationary point), but they are not uniformly (which is the same as exponentially) stable. See e.g. [8] for more details. However, if a linear semigroup is stable but not uniformly stable, then it does not possess the attractor (or, in other words, \( \{0\} \) is not the global attractor). Instead, if we know from the very beginning that the semigroup has the global attractor \( \mathcal{A} \), then a natural question arises.

**Question 1.** Assume that \( S(t) \) possesses the global attractor \( \mathcal{A} \). Assume also that the semigroup has a unique stationary point \( x_\star \) such that

\[
\lim_{t \to \infty} S(t)x = x_\star, \quad \forall x \in X.
\]

Does it follow that the attractor \( \mathcal{A} \) reduces to \( \{x_\star\} \)?

If \( S(t) \) is in addition a gradient system, namely, there exists a Lyapunov functional, then it is true that \( \mathcal{A} = \{x_\star\} \) (see e.g. [5, §3.8]). The same holds if, besides existing the attractor, the semigroup \( S(t) \) is Lipschitz on compact sets uniformly in
time (the proof is an easy exercise left to the reader). However, this is a very restrictive condition. It is then interesting to understand what happens in the general case.

2.2. On the intersection of exponential attractors. Another well-known property of the global attractor, which in fact is a characterizing one, is its minimality among compact attracting sets. Consequently, the intersection of any two compact attracting sets is still attracting (besides being obviously compact), for it contains the global attractor. Accordingly, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are two exponential attractors, then their intersection is certainly an attracting set. What is expected to change is the attraction rate. This leads to our second question.

**Question 2.** Assume that $S(t)$ possesses two exponential attractors $\mathcal{E}_1$ and $\mathcal{E}_2$. Is the intersection $\mathcal{E}_1 \cap \mathcal{E}_2$ still an exponential attractor?

3. **Answer to question 1.** Let $\mathbb{T}_1$ be the unit circle (viewed as a subset of $\mathbb{R}^2$ or $\mathbb{C}$), i.e.

$$\mathbb{T}_1 = \{e^{i\vartheta} : \vartheta \in [0, 2\pi)\}$$

Set then $X = \mathbb{T}_1$ and consider the semigroup $S(t)$ on $X$ defined as

$$S(t)e^{i\vartheta_0} = e^{i\vartheta(t)},$$

where $\vartheta$ solves the Cauchy problem

$$\begin{cases}
\frac{d}{dt}\vartheta = \vartheta(2\pi - \vartheta), \\
\vartheta(0) = \vartheta_0.
\end{cases}$$

Explicitly,

$$\vartheta(t) = \frac{2\pi \vartheta_0 e^{2\pi t}}{\vartheta_0(e^{2\pi t} - 1) + 2\pi}.$$  

Then, $x_* = 1$ is the unique stationary point, and

$$S(t)e^{i\vartheta_0} \to 1, \quad \forall \vartheta_0 \in [0, 2\pi).$$

However, it is easily seen that the global attractor $\mathcal{A}$ coincides with the whole space $X$, reflecting the fact that the dynamics is somehow “slow”. This shows that the answer to Question 1 is negative.

In fact, we can construct an example of this kind for a semigroup $S(t)$ acting on $X = \mathbb{R}^2$ (or $X = \mathbb{C}$). Let $\mathbb{D}$ be the unit disk, and write

$$\mathbb{D} = \bigcup_{\alpha \in [0, 1]} \mathbb{T}_\alpha$$

where

$$\mathbb{T}_\alpha = \{(1 - \alpha) + \alpha e^{i\vartheta} : \vartheta \in [0, 2\pi)\}.$$  

The action of $S(t)$ is as follows: on each $\mathbb{T}_\alpha$ the dynamics is the same as in $\mathbb{T}_1$. Outside $\mathbb{D}$, the trajectories form a vortex ending at the point $x_*$, as shown in fig. 1.

Again, $x_* = 1$ is the unique stationary point, which attracts all trajectories. But the attractor $\mathcal{A}$ in this case coincides with $\mathbb{D}$.
4. Answer to question 2. Let $X = \mathbb{R}^2$, and consider the semigroup $S(t)$ generated by the ODE on $X$

$$\frac{d}{dt} x = -x\|x\|,$$

which explicitly reads

$$S(t)x = \frac{x}{1 + \|x\|t}, \quad \forall x \in X.$$ 

It is then standard matter to see that there exists the global attractor $A = \{0\}$, which attracts bounded set of initial data polynomially fast with optimal rate $1/t$ (in particular $A$ is not an exponential attractor). We now construct two exponential attractors $E_1$ and $E_2$ whose intersection $E_1 \cap E_2$ is just $A$. This will provide a negative answer to Question 2.

Let

$$\mathcal{K} = \{0\} \cup \bigcup_{n=2}^{\infty} \mathcal{K}_n \quad \text{where} \quad \mathcal{K}_n = \{x \in X : \|x\| = \frac{1}{\log n}\}.$$ 

Then $\mathcal{K}$ is a compact exponentially attracting set of rate $\omega = 1$. Indeed, it is enough to show that $\mathcal{K}$ (exponentially) attracts the unit ball of $X$, which is an absorbing set. Note first that $S(t)0 = 0$ for all $t$. Let then $x \neq 0$ be in the unit ball $B_1$ of $X$, and denote $a = \|x\| \leq 1$. Then

$$\delta(S(t)x, \mathcal{K}) = \inf_{n \geq 2} \frac{a}{1 + at} - \frac{1}{\log n}.$$ 

Writing $t = \log(n + \tau) - \frac{1}{a}$ with $\tau \in [0,1)$ and $n \geq 2$, we get

$$\delta(S(t)x, \mathcal{K}) \leq \frac{1}{\log(n + \tau)} - \frac{1}{\log n} = \frac{\log (1 + \frac{\tau}{n})}{\log(n + \tau) \log n} < \frac{3}{n} < 3e^{-t},$$

the last inequality being an immediate consequence of the relation

$$n = e^{t}e^{1/a} - \tau \geq e^t.$$ 

We conclude that

$$\delta(S(t)B_1, \mathcal{K}) \leq 3e^{-t}.$$ 

Unfortunately, such a $\mathcal{K}$ is not invariant. To recover invariance, we replace $\mathcal{K}$ with the set $\mathcal{E}$ constructed as follows: let $r_q$ be the straight line passing through the
origin of angular coefficient equal to \( \tan \left( \frac{\pi}{2} \right) \) (when \( q = 1 \) it is the vertical line). Then, calling \( D_n = \{ x \in X : \| x \| \leq \frac{1}{\log n} \} \), define
\[
\mathcal{E} = \bigcup_{n=2}^{\infty} I_n \quad \text{where} \quad I_n = \bigcup_{k=0}^{2^{n-1}-1} [r_{k2^{-n+2}} \cap D_n].
\]
Let now \( x \in B_1 \setminus \{ 0 \} \) and \( t > 0 \) be given. Then there is \( x_n \in \mathcal{K}_n \) such that
\[
\| S(t)x - x_n \| \leq \frac{3}{n}.
\]
On the other hand, since \( I_n \cap \mathcal{K}_n \) is made by \( 2^n \) equispaced points, there exists \( y_n \in I_n \) such that
\[
\| x_n - y_n \| \leq \frac{2\pi}{2^n \log n}.
\]
Thus, for \( t \) large enough (hence \( n \) large enough) independent of \( x \in B_1 \),
\[
\delta(S(t)x, \mathcal{E}) \leq \frac{4}{n} \leq 4e^{-t}.
\]
Since \( \| S(t)x \| \leq 1 \), it is then clear that the relation
\[
\delta(S(t)B_1, \mathcal{E}) \leq Ce^{-t}
\]
holds for every \( t \geq 0 \) and some \( C > 0 \). We conclude that \( \mathcal{E} \) is an exponential attractor for \( S(t) \).

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fig. 2 Portrait of the exponential attractor \( \mathcal{E} \).

At this point, it is easy to construct the two required exponential attractors \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \), both attracting at the same rate \( \omega = 1 \). Just take \( \mathcal{E}_1 = \mathcal{E} \) and \( \mathcal{E}_2 \) to be a rigid rotation of \( \mathcal{E} \) of angle \( \gamma \pi \) with \( \gamma \) irrational.

Actually, it is even possible to construct a continuous family \( \{ \mathcal{E}_\alpha \}_{\alpha \in U} \) of exponential attractors, each attracting at the same rate \( \omega = 1 \), with the property that
\[
\mathcal{E}_\alpha \cap \mathcal{E}_\beta = \mathcal{A} = \{ 0 \},
\]
for all $\alpha, \beta \in U$ with $\alpha \neq \beta$. To this end, just note that $[0, 1]$ contains an uncountable set $U$ of pairwise rationally independent numbers. Then, for any $\alpha \in U$, just set $\mathcal{E}_\alpha$ to be a rigid rotation of $\mathcal{E}$ of angle $\alpha \pi$.

As a final comment, it is worth noting that both the examples in §3 and in §4 are finite dimensional, although the pathologies in the theory are generally expected from infinite dimensional dynamics.

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