Locally Lipschitz Vector Optimization Problems: Second-order Constraint Qualifications, Regularity Condition and KKT Necessary Optimality Conditions

Yi-Bin Xiao · Nguyen Van Tuyen · Jen-Chih Yao · Ching-Feng Wen

Abstract In the present paper, we are concerned with a class of constrained vector optimization problems, where the objective functions and active constraint functions are locally Lipschitz at the referee point. Some second-order constraint qualifications of Zangwill type, Abadie type and Mangasarian–Fromovitz type as well as a regularity condition of Abadie type are proposed in a nonsmooth setting. The connections between these proposed conditions are established. They are applied to develop second-order Karush–Kuhn–Tucker necessary optimality conditions for local (weak, Geoffrion properly) efficient solutions to the considered problems. Examples are also given to illustrate the obtained results.

Keywords Locally Lipschitz vector optimization · Second-order constraint qualification · Abadie second-order regularity condition · Second-order KKT necessary optimality conditions

Yi-Bin Xiao
School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, P.R. China.
E-mail: xiaoyb9999@hotmail.com

Nguyen Van Tuyen
School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, P.R. China; Department of Mathematics, Hanoi Pedagogical University 2, Xuan Hoa, Phuc Yen, Vinh Phuc, Vietnam
E-mail: tuyensp2@yahoo.com; nguyenvantuyen83@hpu2.edu.vn

Jen-Chih Yao
Center for General Education, China Medical University, Taichung, 40402, Taiwan
E-mail: yaojc@mail.cmu.edu.tw

Ching-Feng Wen
Center for Fundamental Science; and Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Kaohsiung, 80708, Taiwan; Department of Medical Research, Kaohsiung Medical University Hospital, Kaohsiung, 80708, Taiwan
E-mail: cfwen@kmu.edu.tw
1 Introduction

In this paper, we are interested in second-order optimality conditions for the following constrained vector optimization problem

\[
\min_{x} f(x) \quad \text{(VP)}
\]

subject to \( x \in Q \triangleq \{ x \in X : g(x) \leq 0 \} \),

where \( f = (f_i), i \in I \triangleq \{1, \ldots, p\} \), and \( g = (g_j), j \in J \triangleq \{1, \ldots, m\} \) are vector-valued functions defined on a Banach space \( X \).

As a mainstream in the study of vector optimization problems, optimality condition for vector optimization problems has attracted the attention of many researchers in the field of optimization due to their important applications in many disciplines, such as variational inequalities, equilibrium problems and fixed pointed problems; see, for example, [1–6]. It is well-known that if \( f_i, g_j \) are differentiable at \( \bar{x} \in Q \) and \( \bar{x} \) is a local weak efficient solution of (VP), then there exist Lagrange multipliers \((\lambda, \mu) \in \mathbb{R}^p \times \mathbb{R}^m\) satisfying

\[
\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) = 0, \quad (1)
\]

\[
\mu = (\mu_1, \ldots, \mu_m) \geq 0, \mu_j g_j(\bar{x}) = 0, \quad (2)
\]

\[
\lambda = (\lambda_1, \ldots, \lambda_p) \geq 0, (\lambda, \mu) \neq 0; \quad (3)
\]

see [7, Theorem 7.4]. Conditions (1)–(3) are called the first-order F.-John necessary optimality conditions. If \( \lambda \) is nonzero, then this conditions are called the first-order Karush–Kuhn–Tucker (\( KKT \)) optimality conditions. By Motzkin's theorem of the alternative [8, p.28], the existence of \( KKT \) multipliers is equivalent to the inconsistency of the following system

\[
\nabla f_i(\bar{x})(v) < 0, \quad i \in I, \quad (4)
\]

\[
\nabla g_j(\bar{x})(v) \leq 0, \quad j \in J(\bar{x}), \quad (5)
\]

with unknown \( v \in X \), where \( J(\bar{x}) \) is the active index set at \( \bar{x} \). Conditions (4)–(5) are called by the first-order \( KKT \) necessary conditions in primal form.

The first-order \( KKT \) optimality conditions play an important role in both the theory and practice of constrained optimization. In order to obtain these optimality conditions, constraint qualifications and regularity conditions are indispensable; see, for example, [9, 10]. We recall here that these assumptions are called constraint qualifications (\( CQ \)) when they have to be fulfilled by the constraints of the problem, and they are called regularity conditions (\( RC \)) when they have to be fulfilled by both the objectives and the constraints of the problem; see [11] for more details.
Second-order necessary optimality conditions complement first-order conditions in constructing numerical algorithms for finding optimal solutions and also in convergence analysis for numerical algorithms; see, for example, [12–14]. One of the first investigations to obtain second-order optimality conditions of KKT-type for smooth vector optimization problems was carried out by Wang [15]. Then, by introducing a new second-order constraint qualification in the sense of Abadie, Aghezzaf et al. [16] extended Wang’s results to the non-convex case. Maeda [17] was the first to propose an Abadie regularity condition and established second-order KKT necessary optimality conditions for C\(^1\) vector optimization problems. By using the second-order directional derivatives and introducing a new second-order constraint qualification of Zangwill-type, Ivanov [18] introduced some optimality conditions for C\(^1\) vector optimization problems with inequality constraints. Very recently, by proposing some types of the second-order Abadie regularity conditions, Huy et al. [19,20] have obtained some second-order KKT necessary optimality conditions for C\(^1\) vector optimization problems in terms of second-order symmetric subdifferentials. For other contributions to second-order KKT optimality conditions for vector optimization, the reader is invited to see the papers [21–26] with the references therein.

Our aim is to weaken the hypotheses of the optimality conditions in [15–18,20,21,26]. To obtain second-order KKT necessary conditions, by using second-order upper generalized directional derivatives and second-order tangent sets, we introduce some second-order constraint qualifications of Zangwill type, Abadie type and Mangasarian-Fromovitz type as well as a regularity condition of Abadie type. Our obtained results improve and generalize the corresponding results in [15–18,20,21,26], because the objective functions and the active constraint functions are only locally Lipschitz at the referee point and the required constraint qualifications are also weaker. Moreover, the connections between these proposed conditions are established.

The organization of the paper is as follows. In Section 2, we recall some notations, definitions and preliminary material. Section 3 is devoted to investigate second-order constraint qualifications and regularity conditions in a nonsmooth setting for vector optimization problems. In Section 4 and Section 5 we establish some second-order necessary optimality conditions of KKT-type for a local (weak, Geoffrion properly) efficient solution of \((\text{VP})\). Section 6 draws some conclusions.

2 Preliminaries

In this section, we recall some definitions and introduce basic results, which are useful in our study.

Let \(\mathbb{R}^p\) be the \(p\)-dimensional Euclidean space. For \(a, b \in \mathbb{R}^p\), by \(a \leq b\), we mean \(a_i \leq b_i\) for all \(i \in I\); by \(a \leq b\), we mean \(a \leq b\) and \(a \neq b\); and by \(a < b\), we mean \(a_i < b_i\) for all \(i \in I\).
We first recall the definition of local (weak, Geoffrion properly) efficient solutions for the considered problem (VP). Note that the concept of properly efficient solution has been introduced at first to eliminate the efficient solutions with unbounded trade-offs. This concept was introduced initially by Kuhn and Tucker [27] and was followed thereafter by Geoffrion [28]. Geoffrion’s concept enjoys economical interpretations, while Kuhn and Tucker’s one is useful for numerical and algorithmic purposes.

**Definition 2.1** Let $Q_0$ be the feasible set of (VP) and $\bar{x} \in Q_0$. We say that:

(i) $\bar{x}$ is an efficient solution (resp., a weak efficient solution) of (VP) iff there is no $x \in Q_0$ satisfying $f(x) \leq f(\bar{x})$ (resp., $f(x) < f(\bar{x})$).

(ii) $\bar{x}$ is a Geoffrion properly efficient solution of (VP) iff it is efficient and there exists $M > 0$ and such that, for each $i$,

$$\frac{f_i(x) - f_i(\bar{x})}{f_j(\bar{x}) - f_j(x)} \leq M,$$

for some $j$ such that $f_j(\bar{x}) < f_j(x)$ whenever $x \in Q_0$ and $f_i(\bar{x}) > f_i(x)$.

(iii) $\bar{x}$ is a local efficient solution (resp., local weak efficient solution, local Geoffrion properly efficient solution) of (VP) iff it is an efficient solution (resp., weak efficient solution, Geoffrion properly efficient solution) in $U \cap Q_0$, where $U$ is some neighborhood of $\bar{x}$.

Hereafter, we assume that $X$ is a Banach space equipped with the norm $\| \cdot \|$. Let $\Omega$ be a nonempty subset in $X$. The closure, convex hull and conic hull of $\Omega$ are denoted by $\text{cl} \Omega, \text{conv} \Omega$ and $\text{cone} \Omega$, respectively.

**Definition 2.2** Let $\bar{x} \in \Omega$ and $u \in X$.

(i) The tangent cone to $\Omega$ at $\bar{x} \in \Omega$ is defined by

$$T(\Omega; \bar{x}) := \{d \in X : \exists t_k \downarrow 0, \exists d_k \to d, \bar{x} + t_k d_k \in \Omega, \forall k \in \mathbb{N}\}.$$

(ii) The second-order tangent set to $\Omega$ at $\bar{x}$ with respect to the direction $u$ is defined by

$$T^2(\Omega; \bar{x}, u) := \left\{ v \in X : \exists t_k \downarrow 0, \exists v_k \to v, \bar{x} + t_k u + \frac{1}{2} t_k^2 v_k \in \Omega, \forall k \in \mathbb{N} \right\}.$$

Clearly, $T(\cdot; \bar{x})$ and $T^2(\cdot; \bar{x}, u)$ are isotone, i.e., if $\Omega^1 \subset \Omega^2$, then

$$T(\Omega^1; \bar{x}) \subset T(\Omega^2; \bar{x}),$$

$$T^2(\Omega^1; \bar{x}, u) \subset T^2(\Omega^2; \bar{x}, u).$$

It is well-known that $T(\Omega; \bar{x})$ is a nonempty closed cone. For each $u \in X$, the set $T^2(\Omega; \bar{x}, u)$ is closed, but may be empty. However, we see that the set $T^2(\Omega; \bar{x}, 0) = T(\Omega; \bar{x})$ is always nonempty.
Let $F: X \to \mathbb{R}$ be a real-valued function defined on $X$ and $\bar{x} \in X$. The function $F$ is said to be locally Lipschitz at $\bar{x}$ iff there exist a neighborhood $U$ of $\bar{x}$ and $L \geq 0$ such that

$$|F(x) - F(y)| \leq L\|x - y\|, \quad \forall x, y \in U.$$  

**Definition 2.3** Assume that $F: X \to \mathbb{R}$ is locally Lipschitz at $\bar{x} \in X$. Then:

(i) (See [29]) The Clarke’s generalized derivative of $F$ at $\bar{x}$ is defined by

$$F^c(\bar{x}, u) := \limsup_{x \to \bar{x}} \frac{F(x + tu) - F(x)}{t}, \quad u \in X.$$  

(ii) (See [30]) The second-order upper generalized directional derivative of $F$ at $\bar{x}$ is defined by

$$F^{cc}(\bar{x}, u) := \limsup_{t \downarrow 0} \frac{F(\bar{x} + tu) - F(\bar{x}) - tF^c(\bar{x}, u)}{\frac{1}{2}t^2}, \quad u \in X.$$  

It is easily seen that $F^c(\bar{x}, 0) = 0$ and $F^{cc}(\bar{x}, 0) = 0$. Furthermore, the function $u \mapsto F^c(\bar{x}, u)$ is finite, positively homogeneous, and subadditive on $X$; see, for example, [29, 31].

The following lemmas will be useful in our study.

**Lemma 2.1** Suppose that $F: X \to \mathbb{R}$ is locally Lipschitz at $\bar{x} \in X$. Let $u \in X$ and $\{(t_k, u^k)\}$ be a sequence converging to $(0^+, u)$. If

$$F(\bar{x} + t_k u^k) \geq F(\bar{x}) \quad \text{for all} \quad k \in \mathbb{N},$$

then $F^c(\bar{x}, u) \geq 0$.

**Proof** Since $F$ is locally Lipschitz at $\bar{x}$ and

$$\lim_{k \to \infty} (\bar{x} + t_k u^k) = \lim_{k \to \infty} (\bar{x} + t_k u) = \bar{x},$$

there exist $L \geq 0$ and $k_0 \in \mathbb{N}$ such that

$$|F(\bar{x} + t_k u^k) - F(\bar{x} + t_k u)| \leq Lt_k \|u^k - u\| \quad \text{for all} \quad k \geq k_0.$$  

Thus,

$$0 \leq F(\bar{x} + t_k u^k) - F(\bar{x}) = [F(\bar{x} + t_k u^k) - F(\bar{x} + t_k u)] + [F(\bar{x} + t_k u) - F(\bar{x})] \leq L t_k \|u^k - u\| + F(\bar{x} + t_k u) - F(\bar{x})$$

for all $k \geq k_0$. This implies that

$$0 \leq \lim_{k \to \infty} L \|u^k - u\| + \limsup_{k \to \infty} \frac{F(\bar{x} + t_k u) - F(\bar{x})}{t_k} \leq \limsup_{x \to \bar{x}} \frac{F(x + tu) - F(x)}{t}.$$  

Therefore, $F^c(\bar{x}, u) \geq 0$, as required.
Lemma 2.2 Suppose that $F: X \rightarrow \mathbb{R}$ is locally Lipschitz at $\bar{x} \in X$. Let $(u, v)$ be a vector in $X \times X$ and $\{(t_k, v_k)\}$ be a sequence converging to $(0^+, v)$ and satisfying

$$F(\bar{x} + t_k u + \frac{1}{2} t_k^2 v_k) \geq F(\bar{x}) \quad \text{for all } k \in \mathbb{N}.$$ 

If $F^o(\bar{x}, u) = 0$, then $F^o(\bar{x}, v) + F^{oo}(\bar{x}, u) \geq 0$.

Proof For each $k \in \mathbb{N}$, put $x^k := \bar{x} + t_k u + \frac{1}{2} t_k^2 v$ and $y^k := \bar{x} + t_k u + \frac{1}{2} t_k^2 v$. Since $F$ is locally Lipschitz at $\bar{x}$ and

$$\lim_{k \to \infty} x^k = \lim_{k \to \infty} y^k = \bar{x},$$

there exist $L \geq 0$ and $k_0 \in \mathbb{N}$ such that

$$|F(x^k) - F(\bar{x})| \leq \frac{1}{2} t_k^2 L \|v^k - v\| \quad \text{for all } k \geq k_0.$$ 

Thus,

$$0 \leq F(x^k) - F(\bar{x}) = [F(x^k) - F(y^k)] + [F(y^k) - F(\bar{x} + t_k u)] + [F(\bar{x} + t_k u) - F(\bar{x}) - t_k F^o(\bar{x}, u)] \leq \frac{1}{2} t_k^2 L \|v^k - v\| + [F(y^k) - F(\bar{x} + t_k u)] + [F(\bar{x} + t_k u) - F(\bar{x}) - t_k F^o(\bar{x}, u)]$$

for all $k \geq k_0$. This implies that

$$0 \leq \lim_{k \to \infty} L \|v^k - v\| + \limsup_{k \to \infty} \frac{F(\bar{x} + t_k u + \frac{1}{2} t_k^2 v) - F(\bar{x} + t_k u)}{\frac{1}{2} t_k^2} \leq \limsup_{x \to \bar{x}} \frac{F(x + tv) - F(x)}{t} + \limsup_{t \downarrow 0} \frac{F(\bar{x} + tu) - F(\bar{x}) - t F^o(\bar{x}, u)}{\frac{1}{2} t^2} = F^o(\bar{x}, v) + F^{oo}(\bar{x}, u).$$

Therefore, $F^o(\bar{x}, v) + F^{oo}(\bar{x}, u) \geq 0$. The proof is complete.

3 Second-order constraint qualification and regularity condition

From now on, we consider problem (VP) under the following assumptions:

- The functions $f_i, i \in I, g_j, j \in J(\bar{x})$, are locally Lipschitz at $\bar{x}$,
- The functions $g_j, j \in J \setminus J(\bar{x})$, are continuous at $\bar{x}$,
where $\bar{x}$ is a feasible point of (VP) and $J(\bar{x})$ is the \textit{active index set} at $\bar{x}$, that is,
\[ J(\bar{x}) := \{ j \in J : g_j(\bar{x}) = 0 \}. \]
For any vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathbb{R}^2$, we denote the lexicographic order by
\[ a \leq_{\text{lex}} b, \text{ iff } a_1 < b_1 \text{ or } (a_1 = b_1 \text{ and } a_2 \leq b_2), \]
\[ a <_{\text{lex}} b, \text{ iff } a_1 < b_1 \text{ or } (a_1 = b_1 \text{ and } a_2 < b_2). \]

Let us introduce some notations which are used in the sequel. For each $\bar{x} \in Q_0$ and $u \in X$, put
\[ Q := Q_0 \cap \{ x \in X : f_i(x) \leq f_i(\bar{x}), \ i \in I \}, \]
\[ J(\bar{x}; u) := \{ j \in J(\bar{x}) : g_j^2(\bar{x}, u) = 0 \}, \]
\[ I(\bar{x}; u) := \{ i \in I : f_i^2(\bar{x}, u) = 0 \}. \]
We say that $u$ is a \textit{critical direction} of (VP) at $\bar{x}$ iff
\[ f_i^2(\bar{x}, u) \leq 0, \ \forall i \in I, \]
\[ f_i^2(\bar{x}, u) = 0, \ \text{at least one } i \in I, \]
\[ g_j^2(\bar{x}, u) \leq 0, \ \forall j \in J(\bar{x}). \]

The set of all critical directions of (VP) at $\bar{x}$ is denoted by $\mathcal{C}(\bar{x})$. Obviously, $0 \in \mathcal{C}(\bar{x})$.

We now use the following second-order approximation sets for $Q$ and $Q_0$ to introduce second-order constraint qualifications and regularity condition. For each $\bar{x} \in Q_0$ and $u \in X$, set
\[ L^2(Q; \bar{x}, u) := \left\{ v \in X : F_i^2(\bar{x}; u, v) \leq_{\text{lex}} (0, 0), \ i \in I \right\}, \]
\[ L^2(Q_0; \bar{x}, u) := \left\{ v \in X : G_j^2(\bar{x}; u, v) \leq_{\text{lex}} (0, 0), \ j \in J(\bar{x}) \right\}, \]
\[ L^2_0(Q_0; \bar{x}, u) := \left\{ v \in X : G_j^2(\bar{x}; u, v) <_{\text{lex}} (0, 0), \ j \in J(\bar{x}) \right\}, \]
\[ A(\bar{x}; u) := \left\{ v \in X : \forall j \in J(\bar{x}; u) \exists \delta_j > 0 \text{ with } g_j \left( x + tu + \frac{1}{2}t^2v \right) \leq 0 \right\}, \]
\[ B(\bar{x}; u) := \left\{ v \in X : g_j^2(\bar{x}, v) + g_j^2(\bar{x}, u) \leq 0, \ \forall j \in J(\bar{x}; u) \right\}. \]
where
\[
F^2_i(\bar{x}; u, v) := (f_i^0(\bar{x}, u), f_i^0(\bar{x}, v) + f_i^0(\bar{x}, u)), \quad i \in I, \quad v \in X,
\]
\[
G^2_j(\bar{x}; u, v) := (g_j^0(\bar{x}, u), g_j^0(\bar{x}, v) + g_j^0(\bar{x}, u)), \quad j \in J(\bar{x}), \quad v \in X.
\]

For brevity, we denote \( L(Q; \bar{x}) := L^2(Q; \bar{x}, 0) \). It is easily seen that, for each \( u \in C(\bar{x}) \), we have
\[
L^2_0(Q_0; \bar{x}, u) = \{ v \in X : g_j^0(\bar{x}, v) + g_j^0(\bar{x}, u) < 0, \quad j \in J(\bar{x}, u) \}.
\]

**Definition 3.1** Let \( \bar{x} \in Q_0 \) and \( u \in X \). We say that:

(i) The Zangwill second-order constraint qualification holds at \( \bar{x} \) for the direction \( u \) iff
\[
B(\bar{x}; u) \subset \text{cl}A(\bar{x}; u). \quad (ZSCQ)
\]

(ii) The Abadie second-order constraint qualification holds at \( \bar{x} \) for the direction \( u \) iff
\[
L^2(Q_0; \bar{x}, u) \subset T^2(Q_0; \bar{x}, u). \quad (ASCQ)
\]

(iii) The Mangasarian–Fromovitz second-order constraint qualification holds at \( \bar{x} \) for the direction \( u \) iff
\[
L^2_0(Q_0; \bar{x}, u) \neq \emptyset. \quad (MFSCQ)
\]

(iv) The weak Abadie second-order regularity condition holds at \( \bar{x} \) for the direction \( u \) iff
\[
L^2(Q; \bar{x}, u) \subset T^2(Q_0; \bar{x}, u). \quad (WASRC)
\]

The \((ZSCQ)\) type was first introduced by Ivanov [18, Definition 3.2] for \( C^1 \) functions. The \((ASCQ)\) type was proposed by Aghezzaf and Hachimi for \((VP)\) with \( C^2 \) data; see [16, p.40]. The \((MFSCQ)\) type was first introduced in [32] for \( C^2 \) scalar optimization problems. The \((WASRC)\) type was used for \( C^{1,1} \) vector optimization problems in [20]. For problems with only locally Lipschitz active constraints and objective functions, these conditions are new.

**Definition 3.2** Let \( \bar{x} \in Q_0 \). We say that the Zangwill constraint qualification \((ZCQ)\) (resp., Abadie constraint qualification \((ACQ)\), Mangasarian–Fromovitz constraint qualification \((MFCQ)\), weak Abadie regularity condition \((WARC)\)) holds at \( \bar{x} \) iff the \((ZSCQ)\) (resp., \((ASCQ)\), \((MFSCQ)\), \((WASRC)\)) holds at \( \bar{x} \) for the direction \( 0 \).

The following result shows that the \((WASRC)\) is weaker than other constraint qualification conditions in Definition 3.1.

**Proposition 3.1** Let \( \bar{x} \in Q_0 \) and \( u \in X \). Then the following implications hold:
Let \( v \in v \). Indeed, if otherwise, there is a subsequence \( \{ h_j \} \) such that
\[
\lim_{j \to \infty} g_j(\bar{x}) = 0.
\]
We claim that there exists a subsequence \( \{ v_{h_j} \} \) converging to \( v \). Let \( \{ t_{h_j} \} \) be an arbitrary positive sequence converging to 0. We claim that there is a subsequence \( \{ t_{h_k} \} \subset \{ t_{h_j} \} \) such that
\[
\bar{x} + t_{h_k} u + \frac{1}{2} t_{h_k}^2 v_k^1 \in Q_0, \quad \forall k \in \mathbb{N}.
\]

We will prove this claim by induction on \( k \).

In case of \( k = 1 \), let \( \{ x_{h_k} \} \) be a sequence defined by
\[
x_{h_k} := \bar{x} + t_{h_k} u + \frac{1}{2} t_{h_k}^2 v^1 \quad \text{for all} \quad h \in \mathbb{N}.
\]

Let us consider the following possible cases for \( j \in J \).

**Case 1.** \( j \notin J(\bar{x}) \). This means that \( g_j(\bar{x}) < 0 \). Since \( g_j \) is continuous at \( \bar{x} \) and \( \lim_{h \to \infty} x^h = \bar{x} \), there is \( H_1 \in \mathbb{N} \) such that \( g_j(x^h) < 0 \) for all \( h \geq H_1 \).

**Case 2.** \( j \in J(\bar{x}) \). This means that \( g_j(\bar{x}) = 0 \) and \( g_j^2(\bar{x}, u) \leq 0 \).

We claim that there exists \( H_2 \in \mathbb{N} \) such that \( g_j(x^h) < 0 \) for all \( h \geq H_2 \). Indeed, if otherwise, there is a subsequence \( \{ t_{h_k} \} \subset \{ t_{h_j} \} \) satisfying
\[
g_j \left( \bar{x} + t_{h_k} u + \frac{1}{2} t_{h_k}^2 v^1 \right) \geq g_j(\bar{x}) = 0, \quad \forall l \in \mathbb{N},
\]
or, equivalently,
\[
g_j \left( \bar{x} + t_{h_k} \left( u + \frac{1}{2} t_{h_k} v^1 \right) \right) \geq g_j(\bar{x}), \quad \forall l \in \mathbb{N}.
\]
Clearly, \( \lim_{t \to \infty} (u + \frac{1}{2} t_h v^1) = u \). By Lemma 2.1, \( g^0_j(\bar{x}, u) \geq 0 \), and which contradicts with the fact that \( g^0_j(\bar{x}, u) < 0 \).

**Case 3.** \( j \in J(\bar{x}; u) \). Since \( v^1 \in A(\bar{x}; u) \) and \( j \in J(\bar{x}; u) \), there exists \( \delta_j > 0 \) such that
\[
g_j \left( \bar{x} + t u + \frac{1}{2} t^2 v^1 \right) \leq 0, \quad \forall t \in (0, \delta_j).
\]

From \( \lim_{h \to \infty} t_h = 0 \) it follows that there is \( H_3 \in \mathbb{N} \) such that \( t_h \in (0, \delta_j) \) for all \( h \geq H_3 \). Thus, \( g_j \left( x^h \right) \leq 0 \) for all \( h \geq H_3 \).

Put \( h_1 := \max \{ H_1, H_2, H_3 \} \). Then, we have \( g_j \left( x^h \right) \leq 0 \) for all \( h \geq h_1 \) and \( j \in J \). This implies that
\[
\bar{x} + t_h u + \frac{1}{2} t_h^2 v^1 \in Q_0, \quad \forall h \geq h_1.
\]

Thus, by induction on \( k \), there exists a subsequence \( \{ t_{h_k} \} \subset \{ t_h \} \) such that
\[
\bar{x} + t_{h_k} u + \frac{1}{2} t_{h_k}^2 v^k \in Q_0, \quad \forall k \in \mathbb{N}.
\]

From this, \( \lim_{k \to \infty} t_{h_k} = 0 \), and \( \lim_{k \to \infty} v^k = v \), it follows that \( v \in T^2(Q_0; \bar{x}, u) \).

Since \( v \) is arbitrary in \( L^2(Q_0; \bar{x}, u) \), we have
\[
L^2(Q_0; \bar{x}, u) \subset T^2(Q_0; \bar{x}, u).
\]

Thus the (ASCQ) holds at \( \bar{x} \) for the direction \( u \).

(ii) We now assume that the (MFSCQ) holds at \( \bar{x} \) for the direction \( u \in X \) and \( v^0 \in L^0(Q_0; \bar{x}, u) \). Fix \( v \in L^2(Q_0; \bar{x}, u) \). Then,
\[
g^0_j(\bar{x}, u) \leq 0, \quad \forall j \in J(\bar{x}),
\]
\[
g^0_j(\bar{x}, v) + g^\infty_j(\bar{x}, u) \leq 0, \quad \forall j \in J(\bar{x}; u).
\]

Let \( \{ s_k \} \) and \( \{ t_h \} \) be any positive sequences converging to zero. For each \( k \in \mathbb{N} \), put \( v^k := s_k v^0 + (1 - s_k) v \). Then, \( \lim_{k \to \infty} v^k = v \). We claim that there exists a subsequence \( \{ t_{h_{k_i}} \} \) of \( \{ t_h \} \) such that
\[
\bar{x} + t_{h_{k_i}} u + \frac{1}{2} t_{h_{k_i}}^2 v^k \in Q_0, \quad \forall k \in \mathbb{N}.
\]

Consequently, \( v \in T^2(Q_0; \bar{x}, u) \) and we therefore get the (ASCQ).

Indeed, for \( k = 1 \), we have that \( v^1 = s_1 v^0 + (1 - s_1) v \). Fix \( j \in J \). If \( j \in J(\bar{x}; u) \), then, we prove as in Case 1 and Case 2 of the proof of assertion
(i) that there exists \( H_1 \in \mathbb{N} \) such that
\[
g_j \left( x^h \right) < 0, \quad \forall h \geq H_1,
\]
where \( x^h := \bar{x} + t_h u + \frac{1}{2} t_h^2 v^1 \). If \( j \in J(\bar{x}; u) \), then
\[
g^0_j(\bar{x}, v^0) + g^\infty_j(\bar{x}, u) < 0.
\]
Thus, 

\[ g_j^o(\bar{x}, v^1) + g_j^{\infty}(\bar{x}, u) \leq s_1 g_j^o(\bar{x}, v^0) + (1 - s_1) g_j^o(\bar{x}, v) + g_j^{\infty}(\bar{x}, u) \]

\[ = s_1 [g_j^o(\bar{x}, v^0) + g_j^{\infty}(\bar{x}, u)] + (1 - s_1) [g_j^o(\bar{x}, v) + g_j^{\infty}(\bar{x}, u)] \]

\[ < 0. \]

Thus, 

\[ \limsup_{h \to \infty} \frac{g_j(x^h)}{\frac{1}{2} t_h^2} = \limsup_{h \to \infty} \frac{g_j(x^h) - g_j(\bar{x}) - t_h g_j^o(\bar{x}; u)}{\frac{1}{2} t_h^2} \]

\[ \leq \limsup_{h \to \infty} \frac{g_j(\bar{x} + t_h u) + \frac{1}{2} t_h^2 v^1 - g_j(\bar{x} + t_h u)}{\frac{1}{2} t_h^2} \]

\[ + \limsup_{h \to \infty} \frac{g_j(\bar{x} + t_h u) - g_j(\bar{x}) - t_h g_j^o(\bar{x}; u)}{\frac{1}{2} t_h^2} \]

\[ \leq g_j^o(\bar{x}; v^1) + g_j^{\infty}(\bar{x}; u) \]

\[ < 0. \]

This implies that there exists \( H_2 \in \mathbb{N} \) such that \( g_j(x^h) < 0 \) for all \( h \geq H_2 \).

Put \( h_1 := \max \{ H_1, H_2 \} \). Then we have \( g_j(x^h) < 0 \) for all \( h \geq h_1 \) and \( j \in J \).

Thus, 

\[ \bar{x} + t_h u + \frac{1}{2} t_h^2 v^1 \in Q_0 \ \forall h \geq h_1, \]

and the assertion follows by induction on \( k \).

(iii) Assume that there exists \( v^0 \in L_0^2(Q_0; \bar{x}, 0) \). Then \( g_j^o(\bar{x}, v^0) < 0 \) for all \( j \in J(\bar{x}) \). Let \( u \in C(\bar{x}) \). For each \( t > 0 \), put \( v(t) := u + t v^0 \). We claim that there exists \( t > 0 \) such that \( v(t) \in L_0^2(Q_0; \bar{x}, u) \). Indeed, for each \( j \in J(\bar{x}; u) \), one has

\[ g_j^o(\bar{x}, v(t)) + g_j^{\infty}(\bar{x}, u) \leq g_j^o(\bar{x}, u) + t g_j^o(\bar{x}, v^0) + g_j^{\infty}(\bar{x}, u) \]

\[ = t g_j^o(\bar{x}, v^0) + g_j^{\infty}(\bar{x}, u) \]

\[ < 0 \]

for \( t \) large enough. This implies that \( v(t) \in L_0^2(Q_0; \bar{x}, u) \) for \( t \) large enough, as required.

The relations between second-order constraint qualifications are summarized in Figure 1.

Remark 3.1 The forthcoming Examples 4.1 and 4.2 show that \( (WASRC) \not\Rightarrow (ZSCQ) \) and \( (WASRC) \not\Rightarrow (MFSCQ) \).

For the remainder of this paper, we apply the \( (WASRC) \) to establish some second-order \( KKT \) necessary optimality conditions for efficient solutions of \( (VP) \). We point out that, by proposition 3.1, all the results concerning \( KKT \) necessary optimality conditions remain valid when the \( (WASRC) \) are replaced by one of \( (ZSCQ), (ASCQ) \) and \( (MFSCQ) \).
4 Second-order optimality conditions for efficiencies

In this section, we apply the (WASRC) to establish some second-order KKT necessary optimality conditions in primal form for local (weak) efficient solutions of (VP).

The following theorem gives a first-order necessary optimality condition for (VP) under the regularity condition (WARC).

**Theorem 4.1** If $\bar{x} \in Q_0$ is a local (weak) efficient solution of (VP) and (WARC) holds at $\bar{x}$, then the system

\begin{align}
  f^0_i(\bar{x}, u) &< 0, \quad i \in I, \tag{6} \\
  g^y_j(\bar{x}, u) &\leq 0, \quad j \in J(\bar{x}), \tag{7}
\end{align}

has no solution $u \in X$.

**Proof** Arguing by contradiction, assume that there exists $u \in X$ satisfying conditions (6) and (7). This implies that $u \in L(Q_0; \bar{x})$. Since the (WARC) holds at $\bar{x}$, one has $L(Q_0; \bar{x}) \subset T(Q_0; \bar{x})$.

Consequently, $u \in T(Q_0; \bar{x})$. Thus there exist $t_k \to 0^+$ and $u_k \to u$ such that

$\bar{x} + t_k u_k \in Q_0$

for all $k \in \mathbb{N}$. We claim that, for each $i \in I$, there exists $K_i \in \mathbb{N}$ satisfying

$f_i(\bar{x} + t_k u_k) < f_i(\bar{x}), \quad \forall k \geq K_i.$

Indeed, if otherwise, there exist $i \in I$ and a sequence $\{k_l\} \subset \mathbb{N}$ such that

$f_i(\bar{x} + t_{k_l} u_{k_l}) \geq f_i(\bar{x}), \quad \forall l \in \mathbb{N}.$

By Lemma 2.1, we have $f_i^0(\bar{x}, u) \geq 0$, contrary to (6).

Put $K_0 := \max \{K_1, \ldots, K_p\}$. Then,

$f_i((\bar{x} + t_k u_k) < f_i(\bar{x})$

for all $k \geq K_0$ and $i \in I$, which contradicts the hypothesis of the theorem.
Remark 4.1  (i) Recently, Gupta et al. [33, Theorems 3.1] shows that “If \( \bar{x} \) is an efficient solution of (VP), \( X = \mathbb{R}^n \), for each \( i \in I \), \( f_i \) is \( \partial^c \)-quasiconcave at \( \bar{x} \), and there exists \( i \in I \) such that
\[
L(M_i; \bar{x}) \subset T(M_i; \bar{x}),
\]
where
\[
M_i := \{ x \in Q_0 : f_i(x) \leq f_i(\bar{x}) \},
\]
\[
L(M_i; \bar{x}) := \{ u \in X : f_i^*(x; u) \leq 0, g_j^*(\bar{x}; u) \leq 0, j \in J(\bar{x}) \},
\]
then the system (8)–(7) has no solution”.
Clearly,
\[
T(M_i; \bar{x}) \subset T(Q_0; \bar{x}),
\]
\[
L(Q_0; \bar{x}) \subset L(M_i; \bar{x}).
\]
This implies that if condition (8) holds at \( \bar{x} \), then so does the (WARC). Thus, Theorem 4.1 improves [33, Theorems 3.1]. We note here that the assumption that \( f_i \) is \( \partial^c \)-quasiconcave at \( \bar{x} \) is not necessary in our result.

(ii) Theorem 4.1 also improves [33, Theorems 3.3]. Theorem 3.3 in [33] is as follows: “If \( \bar{x} \) is a weak efficient solution of (VP), \( X = \mathbb{R}^n \), \( Q_0 \) is convex, for each \( i \in I \), \( f_i \) is \( \partial^c \)-quasiconcave at \( \bar{x} \), and there exists \( i \in I \) such that
\[
L(M_i; \bar{x}) \subset \text{cl conv } T(M_i; \bar{x}),
\]
then the system (6)–(7) has no solution”.
Since \( T(M_i; \bar{x}) \subset T(Q_0; \bar{x}) \) and \( Q_0 \) is a closed convex set, we have
\[
\text{cl conv } T(M_i; \bar{x}) \subset T(Q_0; \bar{x}).
\]
This implies the (WARC) is weaker than condition (9) and so Theorem 4.1 sharpens Gupta et al.’s result. We would like to remark that our result does not require any convexity assumptions.

Now we are ready to present our result of second-order KKT optimality conditions for local (weak) efficient solutions of (VP) under the (WASRC).

Theorem 4.2  Let \( \bar{x} \) be a local (weak) efficient solution of (VP). Suppose that the (WASRC) holds at \( \bar{x} \) for any critical direction. Then, the system
\[
F_i^2(\bar{x}; u, v) \leq_{\text{lex}} (0, 0), \quad i \in I,
\]
\[
G_j^2(\bar{x}; u, v) \leq_{\text{lex}} (0, 0), \quad j \in J(\bar{x}).
\]
has no solution \((u, v) \in X \times X\).
Proof Arguing by contradiction, assume that there exists \((u, v) \in X \times X\) satisfying conditions (10) and (11). It follows that \(v \in L^2(Q; \bar{x}, u)\) and
\[
\begin{align*}
  f_i^o(\bar{x}, u) &\leq 0, \quad i \in I, \\
  g_j^o(\bar{x}, u) &\leq 0, \quad j \in J(\bar{x}).
\end{align*}
\]
Since the \((WASRC)\) holds at \(\bar{x}\), so does the \((WARC)\). By Theorem 4.1, there exists \(i \in I\) such that \(f_i^o(\bar{x}, u) = 0\). This means that \(u\) is a critical direction of \((VP)\) at \(\bar{x}\). Since the \((WASRC)\) holds at \(\bar{x}\) for the critical direction \(u\), we have
\[v \in T^2(Q_0; \bar{x}, u).\]
Thus there exist a sequence \(\{v^k\}\) converging to \(v\) and a positive sequence \(\{t_k\}\) converging to 0 such that
\[
x^k := \bar{x} + t_k u + \frac{1}{2} t_k^2 v^k \in Q_0, \quad \forall k \in \mathbb{N}.
\]
We claim that, for each \(i \in I\), there exists \(K_i \in \mathbb{N}\) such that
\[f_i(x^k) < f_i(\bar{x})\]
for all \(k \geq K_i\). Indeed, if otherwise, there exist \(i_0 \in I\) and a sequence \(\{k_l\} \subset \mathbb{N}\) satisfying
\[
f_{i_0} \left( \bar{x} + t_{k_l} u + \frac{1}{2} t_{k_l}^2 v^{k_l} \right) \geq f_{i_0}(\bar{x}), \quad \forall l \in \mathbb{N}.
\] (12)
We consider the following possible cases for \(i_0\).

Case 1. \(i_0 \in I(\bar{x}; u)\). This means that \(f_{i_0}^o(\bar{x}, u) = 0\). From (10) it follows that
\[
f_{i_0}^o(\bar{x}, v) + f_{i_0}^{oo}(\bar{x}, u) < 0.
\] (13)
From (12), \(\lim_{l \to \infty} t_{k_l} = 0\), \(\lim_{l \to \infty} v^{k_l} = v\), and Lemma 2.2 it follows that
\[f_i^o(\bar{x}, v) + f_i^{oo}(\bar{x}, u) \geq 0,
\] contrary to (13).

Case 2. \(i_0 \notin I(\bar{x}; u)\). This means that \(f_{i_0}^o(\bar{x}, u) < 0\). In this case we now rewrite (12) as
\[
f_{i_0} \left( \bar{x} + t_{k_l} \left( u + \frac{1}{2} t_{k_l} v^{k_l} \right) \right) \geq f_{i_0}(\bar{x}), \quad \forall l \in \mathbb{N}.
\]
From \(\lim_{l \to \infty} t_{k_l} = 0\), \(\lim_{l \to \infty} (u + \frac{1}{2} t_{k_l} v^{k_l}) = u\), and Lemma 2.1 it follows that \(f_{i_0}^o(\bar{x}, u) \geq 0\). This contradicts the fact that \(f_{i_0}^o(\bar{x}, u) < 0\).

Put \(K_0 := \max\{K_i : i \in I\}\). Then, we have
\[f_i(x^k) < f_i(\bar{x})\]
for all \(k \geq K_0\) and \(i \in I\), which contradicts the hypothesis of the theorem.
An immediate consequence of the above theorem is the following corollary.

**Corollary 4.1** Let \( \bar{x} \) be a local (weak) efficient solution of \((VP)\) and \( u \in \mathcal{C}(\bar{x}) \). Suppose that the \WASRC holds at \( \bar{x} \) for the direction \( u \). Then the following system

\[
\begin{align*}
    f_i^o(\bar{x}, v) + f_i^o(\bar{x}, u) &< 0, \; \; \; i \in I(\bar{x}; u), \\
    g_j^o(\bar{x}, v) + g_j^o(\bar{x}, u) &\leq 0, \; \; \; j \in J(\bar{x}, u),
\end{align*}
\]

has no solution \( v \in X \).

**Remark 4.2** Suppose that \( F: X \to \mathbb{R} \) is of class \( C^1(X) \), i.e., \( F \) is Fréchet differentiable and its gradient mapping is continuous on \( X \). If \( F \) is second-order directionally differentiable at \( \bar{x} \), i.e., there exists

\[
F''(\bar{x}, u) := \lim_{t \to 0} \frac{F(\bar{x} + tu) - F(\bar{x}) - t\langle \nabla F(\bar{x}), u \rangle}{t^2}, \; \; \; u \in X,
\]

then \( F''(\bar{x}, u) = F^{\infty}(\bar{x}, u) \) for all \( u \in X \). In \cite{ivanov}, Ivanov considered problem \((VP)\) under the following conditions:

\[
\begin{align*}
    &\text{The functions } g_j, \; j \notin J(\bar{x}) \text{ are continuous at } \bar{x}; \\
    &\text{The functions } f_i, \; i \in I, \; g_j, \; j \in J(\bar{x}) \text{ are of class } C^1(X); \\
    &\text{If } \langle \nabla f_i(\bar{x}), u \rangle = 0, \text{ then there exists } f_i''(\bar{x}, u); \\
    &\text{If } \langle \nabla g_j(\bar{x}), u \rangle = 0, \; j \in J(\bar{x}), \text{ then there exists } g_j''(\bar{x}, u).
\end{align*}
\]

If condition \( \mathfrak{C} \) holds at \( \bar{x} \) for the direction \( u \), then the system \((13)-(15)\) becomes

\[
\begin{align*}
    &\langle \nabla f_i(\bar{x}), v \rangle + f_i''(\bar{x}, u) < 0, \; \; \; i \in I(\bar{x}, u), \\
    &\langle \nabla g_j(\bar{x}), v \rangle + g_j''(\bar{x}, u) \leq 0, \; \; \; j \in J(\bar{x}, u).
\end{align*}
\]

Since the \WASRC is weaker than the \ZSCQ, Corollary 4.1 improves and extends result of Ivanov \cite{ivanov} Theorem 4.1 and of Huy et al. \cite{huy} Theorem 3.2. To illustrate, we consider the following examples.

**Example 4.1** Let \( f: \mathbb{R}^2 \to \mathbb{R}^3 \) and \( g: \mathbb{R}^2 \to \mathbb{R} \) be two maps defined by

\[
\begin{align*}
    f(x) := (f_1(x), f_2(x), f_3(x)) &= (x_2, x_1 + x_2^2, -x_1 - x_1|x_1| + x_2^2) \\
    g(x) := |x_1| + x_2^3 - x_1^2, \; \forall x = (x_1, x_2) \in \mathbb{R}^2.
\end{align*}
\]

Then the feasible set of \((VP)\) is

\[
Q_0 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| + x_2^3 - x_1^2 \leq 0\}.
\]

Let \( \bar{x} = (0, 0) \in Q_0 \). It is easy to check that \( \bar{x} \) is an efficient solution of \((VP)\). For each \( u = (u_1, u_2) \in \mathbb{R}^2 \), we have

\[
\begin{align*}
    f_1''(\bar{x}, u) &= \langle \nabla f_1(\bar{x}), u \rangle = u_2, \\
    f_2''(\bar{x}, u) &= \langle \nabla f_2(\bar{x}), u \rangle = u_1, \\
    f_3''(\bar{x}, u) &= \langle \nabla f_3(\bar{x}), u \rangle = -u_1, \\
    g''(\bar{x}, u) &= |u_1|.
\end{align*}
\]
Thus,
\[ C(\bar{x}) = \{(u_1, u_2) \in \mathbb{R}^2 : u_1 = 0, u_2 \leq 0 \} \]

Clearly, \(0_{\mathbb{R}^2} := (0, 0)\) is a critical direction at \(\bar{x}\). We claim that the \([WASRC]\) holds at \(\bar{x}\) for the direction \(0_{\mathbb{R}^2}\). Indeed, we have
\[ L^2(Q; \bar{x}, 0_{\mathbb{R}^2}) = \{(v_1, v_2) \in \mathbb{R}^2 : v_1 = 0, v_2 \leq 0 \} \]

An easy computation shows that
\[ T^2(Q_0; \bar{x}, 0_{\mathbb{R}^2}) = T(Q_0; \bar{x}) = \{(v_1, v_2) \in \mathbb{R}^2 : v_1 = 0, v_2 \leq 0 \} \]

This implies that the \([WASRC]\) holds at \(\bar{x}\) for the direction \(0_{\mathbb{R}^2}\). By Corollary 4.1, the system
\[ f_i^c(\bar{x}, v) + f_i^{\infty}(\bar{x}, 0_{\mathbb{R}^2}) < 0, \quad i \in I(\bar{x}; 0_{\mathbb{R}^2}), \]
\[ g^c(\bar{x}, v) + g^{\infty}(\bar{x}, 0_{\mathbb{R}^2}) \leq 0, \]

has no solution \(v \in \mathbb{R}^2\). The second-order necessary conditions of Huy et al. [20, Theorem 3.2] and of Ivanov [18, Theorem 4.1] are not applicable to this example as the constraint function \(g\) is not Fréchet differentiable at \(\bar{x}\). Furthermore, the \([ZSCQ]\) does not hold at \(\bar{x}\) for the direction \(0_{\mathbb{R}^2}\). Indeed, we have
\[ B(\bar{x}; 0_{\mathbb{R}^2}) = \{(v_1, v_2) \in \mathbb{R}^2 : v_1 = 0, v_2 \in \mathbb{R} \} \]

Let \(v = (v_1, v_2) \in \mathbb{R}^2\). We have \(v \in A(\bar{x}; 0_{\mathbb{R}^2})\) if and only if there exists \(\delta > 0\) such that
\[ g\left(\bar{x} + t0_{\mathbb{R}^2} + \frac{1}{2}t^2v\right) \leq 0, \quad \forall t \in (0, \delta) \]

or, equivalently,
\[ |v_1| - \frac{1}{2}t^2v_1^2 + \frac{1}{4}t^4v_2^3 \leq 0, \quad \forall t \in (0, \delta). \quad (16) \]

It is easy to check that (16) is true if and only if \(v_1 = 0\) and \(v_2 \leq 0\). Thus,
\[ A(\bar{x}; 0_{\mathbb{R}^2}) = \{(v_1, v_2) \in \mathbb{R}^2 : v_1 = 0, v_2 \leq 0 \} \]

Clearly, \(B(\bar{x}; 0_{\mathbb{R}^2}) \not\subseteq \text{cl} \ A(\bar{x}; 0_{\mathbb{R}^2})\). This means that the \([ZSCQ]\) does not hold at \(\bar{x}\) for the direction \(0_{\mathbb{R}^2}\).

Remark 4.3 Recently, by using the \([MFSCQ]\), Luu [26, Corollary 5.2] derived some second-order KKT necessary conditions for weak efficient solutions of differentiable vector problems in terms of the second-order upper generalized directional derivatives. By Proposition 3.1, the \([WASRC]\) is weaker than the \([MFSCQ]\). Thus, Corollary 4.1 improves [26, Corollary 5.2]. To see this, let us consider the next example.
Example 4.2 Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) and \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) be two maps defined by
\[
\begin{align*}
f(x) &:= (f_1(x), f_2(x)) = (x_1 + x_2^2, -x_1 - x_1|x_1| + x_2^2) \\
g(x) &:= (g_1(x), g_2(x)) = (x_1 - x_2^2, -x_1 - x_2^2), \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.
\end{align*}
\]
Then the feasible set of \( \text{(VP)} \) is \( Q_0 = \{(x_1, x_2) \in \mathbb{R}^2 : -x_2^2 \leq x_1 \leq x_2^2\} \).

Let \( \bar{x} = (0, 0) \in Q_0 \). Clearly, \( \bar{x} \) is an efficient solution of \( \text{(VP)} \). It is easy to check that the \( WASRC \) holds at \( \bar{x} \) for the critical direction \( 0_{\mathbb{R}^2} \) but not the \( MFSCQ \). Thus Corollary 4.1 can be applied for this example, but not \[26, \text{Corollary 5.2}\].

5 Strong second-order optimality condition for local Geoffrion properly efficiencies

In this section, we apply the \( WASRC \) to establish a strong second-order KKT necessary optimality condition for a local Geoffrion properly efficient solution of \( \text{(VP)} \).

**Theorem 5.1** Let \( \bar{x} \in Q_0 \) be a local Geoffrion properly efficient solution of \( \text{(VP)} \). Suppose that the \( WASRC \) holds at \( \bar{x} \) for any critical direction. Then the system
\[
\begin{align*}
F_i^2(\bar{x}; u, v) &\leq_{lex} (0, 0), \quad i \in I, \quad (17) \\
F_i^2(\bar{x}; u, v) &<_{lex} (0, 0), \quad \text{at least one } i \in I(\bar{x}; u), \quad (18) \\
G_j^2(\bar{x}; u, v) &\leq_{lex} (0, 0), \quad j \in J(\bar{x}) \quad (19)
\end{align*}
\]
has no solution \((u, v) \in X \times X\).

**Proof** Arguing by contradiction, assume that the system (17)–(19) admits a solution \((u, v) \in X \times X\). Without any loss of generality we may assume that
\[
F_i^2(\bar{x}; u, v) <_{lex} (0, 0),
\]
where \( 1 \in I(\bar{x}; u) \). This implies that
\[
f_i^2(\bar{x}, v) + f_i^2(\bar{x}, u) < 0.
\]
From (17) and (19) it follows that \( v \in L^2(Q; \bar{x}, u) \) and
\[
f_i^2(\bar{x}, u) \leq 0, \quad i \in I, \\
g_j^2(\bar{x}, u) \leq 0, \quad j \in J(\bar{x}).
\]
This and \( 1 \in I(\bar{x}; u) \) imply that \( u \) is a critical direction at \( \bar{x} \). Since the \( WASRC \) holds at \( \bar{x} \) for the critical direction \( u \), we have \( v \in T^2(Q_0; \bar{x}, u) \).
Thus there exist a sequence \( \{ v^k \} \) converging to \( v \) and a positive sequence \( \{ t_k \} \) converging to 0 such that

\[
x^k := x + t_k u + \frac{1}{2} t_k^2 v^k \in Q_0, \quad \forall k \in \mathbb{N}.
\]

Since \( 1 \in I(\bar{x}; u) \) and (20), as in the proof of Case 1 of Theorem 4.2, there exists \( K_1 \in \mathbb{N} \) such that

\[
f_1(x^k) < f_1(\bar{x})
\]

for all \( k \geq K_1 \).

For each \( i \in I \setminus I(\bar{x}; u) \), we have \( f_i^o(\bar{x}, u) < 0 \). As in the proof of Case 2 of Theorem 4.2, there exists \( K_i \in \mathbb{N} \) such that

\[
f_i(x^k) < f_i(\bar{x})
\]

for all \( k \geq K_i \). Without any loss of generality we may assume that

\[
f_i(x^k) < f_i(\bar{x})
\]

for all \( k \in \mathbb{N} \) and \( i \in \{ 1 \} \cup [I \setminus I(\bar{x}; u)] \). For each \( k \in \mathbb{N} \), put

\[
I_k := \{ i \in I(\bar{x}; u) \setminus \{ 1 \} : f_i(x^k) > f_i(\bar{x}) \}.
\]

We claim that \( I_k \) is nonempty for all \( k \in \mathbb{N} \). Indeed, if \( I_k = \emptyset \) for some \( k \in \mathbb{N} \), then we have

\[
f_i(x^k) \leq f_i(\bar{x}) \quad \forall i \in I(\bar{x}; u) \setminus \{ 1 \}.
\]

Using also the fact that \( f_i(x^k) < f_i(\bar{x}) \) for all \( i \in \{ 1 \} \cup [I \setminus I(\bar{x}; u)] \), we arrive at a contradiction with the efficiency of \( \bar{x} \).

Since \( I_k \subset I(\bar{x}; u) \setminus \{ 1 \} \) for all \( k \in \mathbb{N} \), without any loss of generality, we may assume that \( I_k = \bar{I} \) is constant for all \( k \in \mathbb{N} \). Thus, for each \( i \in \bar{I} \), we have

\[
f_i(x^k) > f_i(\bar{x}), \quad \forall k \in \mathbb{N}.
\]

By Lemma 2.2, we have

\[
f_i^o(\bar{x}, v) + f_i^{oo}(\bar{x}, u) \geq 0, \quad i \in \bar{I}.
\]

Since (17), for each \( i \in \bar{I} \subset I(\bar{x}; u) \setminus \{ 1 \} \), we have

\[
f_i^o(\bar{x}, v) + f_i^{oo}(\bar{x}, u) \leq 0.
\]

Thus,

\[
f_i^o(\bar{x}, v) + f_i^{oo}(\bar{x}, u) = 0, \quad i \in \bar{I}.
\]

(21)

Let \( \delta \) be a real number satisfying

\[
f_i^o(\bar{x}, v) + f_i^{oo}(\bar{x}, u) < \delta < 0,
\]

or, equivalently,

\[-[f_i^o(\bar{x}, v) + f_i^{oo}(\bar{x}, u)] > -\delta > 0.
\]
It is easily seen that
\[
\limsup_{k \to \infty} \frac{f_1(x^k) - f_1(\bar{x})}{t_k^2} \leq f_1^o(\bar{x}, v) + f_1^{\circ\circ}(\bar{x}, u).
\]

Thus there exists \(k_0 \in \mathbb{N}\) such that
\[
f_1(\bar{x}) - f_1(x^k) > -\frac{1}{2} \delta t_k^2 > 0
\]
for all \(k \geq k_0\). Then, for any \(i \in \bar{I}\) and \(k \geq k_0\), we have
\[
0 < \frac{f_1(x^k) - f_1(\bar{x})}{f_1(\bar{x}) - f_1(x^k)} \leq \frac{f_1(x^k) - f_1(\bar{x})}{-\frac{1}{2} \delta t_k^2}.
\]

From this and (21), we have
\[
0 \leq \limsup_{k \to \infty} \frac{f_i(x^k) - f_i(\bar{x})}{f_i(\bar{x}) - f_i(x^k)} \leq \limsup_{k \to \infty} \frac{f_i(x^k) - f_i(\bar{x} + t_k u)}{-\frac{1}{2} \delta t_k^2}
\]
\[
+ \limsup_{k \to \infty} \frac{f_i(\bar{x} + t_k u) - f_i(\bar{x}) - t_k f_i^o(\bar{x}; u)}{-\frac{1}{2} \delta t_k^2}
\]
\[
\leq -\frac{1}{\delta} [f_i^o(\bar{x}, v) + f_i^{\circ\circ}(\bar{x}, u)]
\]
\[
= 0.
\]

Thus,
\[
\lim_{k \to \infty} \frac{f_i(x^k) - f_i(\bar{x})}{f_i(\bar{x}) - f_i(x^k)} = +\infty,
\]
contrary to the fact that \(\bar{x}\) is a local Geoffrion properly efficient solution of (VP). The proof is complete.

The following corollary is immediate from Theorem 5.1.

**Corollary 5.1** Let \(\bar{x} \in Q_0\) be a local Geoffrion properly efficient solution of (VP) and \(u \in C(\bar{x})\). Suppose that the (WARC) holds at \(\bar{x}\) for the direction \(u\). Then the system
\[
f_i^o(\bar{x}, v) + f_i^{\circ\circ}(\bar{x}, u) \leq 0, \quad i \in I(\bar{x}; u),
\]
\[
f_i^o(\bar{x}, v) + f_i^{\circ\circ}(\bar{x}, u) < 0, \quad \text{at least one } i \in I(\bar{x}; u),
\]
\[
g_j^o(\bar{x}, v) + g_j^{\circ\circ}(\bar{x}, u) \leq 0, \quad j \in J(\bar{x}, u),
\]
has no solution \(v \in X\).

The next corollary shows that if the (WARC) holds at \(\bar{x}\), then every Geoffrion properly efficient solution of (VP) is also proper in the sense of Kuhn and Tucker [27].
Corollary 5.2 Let \( \bar{x} \in Q_0 \) be a local Geoffrion properly efficient solution of (VP). Suppose that the \((\text{WARC})\) holds at \( \bar{x} \). Then the system

\[
\begin{align*}
    f_i^\circ(\bar{x}, u) &\leq 0, \quad i \in I, \\
    f_i^\circ(\bar{x}, u) &< 0, \quad \text{at least one } i \in I, \\
    g_j^\circ(\bar{x}, u) &\leq 0, \quad j \in J(\bar{x}),
\end{align*}
\]

has no solution \( u \in X \).

Proof Since the \((\text{WARC})\) holds at \( \bar{x} \), the \((\text{WASRC})\) holds at \( \bar{x} \) for the critical direction 0. Clearly, \( I(\bar{x}; 0) = I \) and \( J(\bar{x}; 0) = J(\bar{x}) \). Thus, applying Corollary 5.1, the system (22)–(24) has no solution \( u \in X \).

Remark 5.1 Conditions (22)–(24) are also known as strong first-order KKT \((\text{SFKKT})\) necessary conditions in primal form. In [11], Burachik et al. introduced a generalized Abadie regularity condition \((\text{GARC})\) and established \(\text{SFKKT}\) necessary conditions for Geoffrion properly efficient solutions of differentiable vector optimization problems. Later on, Zhao [34] proposed an extended generalized Abadie regularity condition \((\text{EGARC})\) and then obtained \(\text{SFKKT}\) necessary conditions for problems with locally Lipschitz data in terms of Clarke’s directional derivatives. Recall that the \((\text{EGARC})\) holds at \( \bar{x} \in Q_0 \) if

\[ L(Q; \bar{x}) \subset \bigcap_{i=1}^I T(M^i; \bar{x}), \tag{25} \]

for all \( i \in I \); see [34, Definition 3.1]. If \( f_i \) and \( g_i \) are of class \( C^1(X) \), then condition (25) is called by the generalized Abadie regularity condition \((\text{GARC})\); see [11, p.483]. By the isotony of \( T(\cdot; \bar{x}) \) and \( M^i \subset Q_0 \), we have

\[ T(M^i; \bar{x}) \subset T(Q_0; \bar{x}) \quad \text{for all } i \in I. \]

Thus the \((\text{WARC})\) is weaker than the \((\text{EGARC})\) \((\text{GARC})\). The following example illustrates our results in which the condition \((\text{WARC})\) is satisfied, but the condition \((\text{EGARC})\) \((\text{GARC})\) is not fulfilled. It turns out that Corollary 5.2 improves and extends results of Zhao [34] Theorem 4.1 and Burachik et al. [11] Theorem 4.3.

Example 5.1 Consider the following problem:

\[
\begin{align*}
    \min f(x) := (f_1(x), f_2(x)) \\
    \text{subject to } x &\in Q_0 := \{x \in \mathbb{R}^2 \mid g(x) \leq 0\},
\end{align*}
\]

where

\[ f_1(x) := |x_1| + x_2^2, \quad f_2(x) := -f_1(x), \quad g(x) := x_2 \quad \text{for all } x = (x_1, x_2) \in \mathbb{R}^2. \]

Clearly, \( \bar{x} = (0, 0) \) is a Geoffrion properly efficient solution. The optimality conditions of Burachik et al. [11] Theorem 4.3 cannot be used for this problem as the functions \( f_1 \) and \( f_2 \) are not differentiable at \( \bar{x} \).
For each \( u = (u_1, u_2) \in \mathbb{R}^2 \), we have
\[
f_1^u(\bar{x}, u) = |u_1|, f_2^u(\bar{x}, u) = -|u_1|, g^u(\bar{x}, u) = \langle \nabla g(\bar{x}), u \rangle = u_2.
\]
It is easy to check that
\[
C(\bar{x}) = L(Q; \bar{x}) = \{(u_1, u_2) \in \mathbb{R}^2 : u_1 = 0, u_2 \leq 0\}.
\]
We claim that the \((EGARC)\) does not hold at \( \bar{x} \). Indeed, since
\[
M_1 = \{(x_1, x_2) \in \mathbb{R}^2 : f_1(x_1, x_2) \leq 0, g(x_1, x_2) \leq 0\} = \{\bar{x}\},
M_2 = \{(x_1, x_2) \in \mathbb{R}^2 : f_2(x_1, x_2) \leq 0, g(x_1, x_2) \leq 0\} = Q_0,
\]
we have \( T(M_1; \bar{x}) = \{\bar{x}\} \) and \( T(M_2; \bar{x}) = Q_0 \). Thus, \( T(Q_0; \bar{x}) = Q_0 \) and
\[
\bigcap_{i=1}^2 T(M_i; \bar{x}) = \{\bar{x}\}.
\]
Consequently,
\[
L(Q; \bar{x}) \not\subseteq \bigcap_{i=1}^2 T(M_i; \bar{x}),
\]
as required. This shows that the result of Zhao [34, Theorem 4.1] cannot be applied for this example.

Next we check the first-order necessary optimality conditions of our Corollary 5.2. Since \( T(Q_0; \bar{x}) = Q_0 \), we have
\[
L(Q; \bar{x}) \subseteq T(Q_0; \bar{x}).
\]
This means that the \((WARC)\) holds at \( \bar{x} \). By Corollary 5.2 the system (22)–(24) has no solution \( u \in \mathbb{R}^2 \).

6 Concluding remarks

In this paper we obtain primal second-order \( KKT \) necessary conditions for vector optimization problems with inequality constraints in a nonsmooth setting using second-order upper generalized directional derivatives. We suppose that the objective functions and active constraints are only locally Lipschitz. Some second-order constraint qualifications of Zangwill type, Abadie type and Mangasarian-Fromovitz type as well as a regularity condition of Abadie type are proposed. They are applied in the optimality conditions. Our results improve and generalize the corresponding results of Aghezza et al. [16, Theorem 3.3], Gupta et al. [33, Theorems 3.1 and 3.3], Huy et al. [20, Theorem 3.2], Ivanov [18, Theorem 4.1], Constantin [21, Theorem 2], Luu [26, Corollary 5.2], Zhao [34, Theorem 4.1], and Burachik et al. [11, Theorem 4.3].

To obtain second-order \( KKT \) necessary conditions in dual form, we need assume that the objective functions and constraint functions are of class \( C^1(X) \). Then one can follow the scheme of the proof of [16, Theorem 3.4] and we leave the details to the reader.
Acknowledgments

Y.-B. Xiao and N. V. Tuyen are supported by the National Natural Science Foundation of China (11771067). C.-F. Wen and J.-C. Yao are supported by the Taiwan MOST [grant number 106-2115-M-037-001], [grant number 106-2923-E-039-001-MY3], respectively, as well as the grant from Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Taiwan.

References

1. Lee GM, Kim DS, Lee BS, Yen ND. Vector variational inequality as a tool for studying vector optimization problems. Nonlinear Anal. 1998;34:745–765.
2. Lu J, Xiao YB, Huang NJ. A Stackelberg quasi-equilibrium problem via quasi-variational inequalities. Carpathian J Math. (to appear)
3. Mordukhovich BS. Variational analysis and generalized differentiation II. Applications. Berlin: Springer; 2006.
4. Petrușel A, Petrușel G, Xiao YB, Yao JC. Fixed point theorems for generalized contractions with applications to coupled fixed point theory. J. Nonlinear Convex Anal. 2018;19:71–87.
5. Wang YM, Xiao YB, Wang X, Cho YJ. Equivalence of well-posedness between systems of hemivariational inequalities and inclusion problems. J. Nonlinear Sci. Appl. 2016;9:1178–1192.
6. Xiao YB, Huang NJ, Cho YJ. A class of generalized evolution variational inequalities in Banach space. Appl. Math. Lett. 2012;25:914–920.
7. Jahn J. Vector optimization. Berlin: Springer-Verlag; 2011.
8. Mangasarian OL. Nonlinear programming. New York: McGraw Hill; 1969.
9. Andreani R, Haeser G, Martínez JM. On sequential optimality conditions for smooth constrained optimization. Optimization. 2011;60:627–641.
10. Tuyen NV, Yao JC, Wen CF. A note on approximate Karush–Kuhn–Tucker conditions in locally Lipschitz multiobjective optimization. Optim. Lett. 2018. DOI 10.1007/s11590-018-1261-y
11. Burachik RS, Rizvi MM. On weak and strong Kuhn–Tucker conditions for smooth multiobjective optimization. J. Optim. Theory Appl. 2012;155:477–491.
12. Bertsekas DP. Nonlinear programming. Belmont: Athena Scientific; 1999.
13. Izmailov AF, Solodov MV. An active-set newton method for mathematical programs with complementarity constraints. SIAM J. Optim. 2008;19:1003–1027.
14. Nocedal J, Wright SJ. Numerical optimization. New York: Springer; 1999.
15. Wang S. Second order necessary and sufficient conditions in multiobjective programming. Numer. Funct. Anal. Optim. 1991;12:237–252.
16. Aghezzaf B, Hachimi M. Second-order optimality conditions in multiobjective optimization problems. J. Optim. Theory Appl. 1999;102:37–50.
17. Maeda T. Second-order conditions for efficiency in nonsmooth multiobjective optimization. J. Optim. Theory Appl. 2004;122:521–538.
18. Ivanov VI. Second-order optimality conditions for vector problems with continuously Fréchet differentiable data and second-order constraint qualifications. J. Optim. Theory Appl. 2015;166:777–790.
19. Huy NQ, Kim DS, Tuyen NV. Strong second-Order Karush–Kuhn–Tucker optimality conditions for vector optimization. Appl. Anal. (to appear)
20. Huy NQ, Kim DS, Tuyen NV. New second-order Karush–Kuhn–Tucker optimality conditions for vector optimization. Appl. Math. Optim. 2017. DOI 10.1007/s00245-017-9432-2
21. Constantin E. Second-order necessary conditions in locally Lipschitz optimization with inequality constraints. Optim. Lett. 2015;9:245–261.
22. Ginchev I, Ivanov VI. Second-order optimality conditions for problems with $C^1$ data. J. Math. Anal. Appl. 2008;340:646–657.
23. Giorgi G, Jiménez B, Novo V. Strong Kuhn–Tucker conditions and constraint qualifications in locally Lipschitz multiobjective optimization problems. TOP 2009;17:288–304.
24. Ivanov VI. Second-order optimality conditions with arbitrary nondifferentiable function in scalar and vector optimization. Nonlinear Anal. 2015;125:270–289.
25. Ivanov VI. Second-order optimality conditions for inequality constrained problems with locally Lipschitz data. Optim. Lett. 2010;4:597–608.
26. Luu DV. Second-order necessary efficiency conditions for nonsmooth vector equilibrium problems. J. Global Optim. 2018;70:437–453.
27. Kuhn HW, Tucker AW. Nonlinear programming. In: Neyman J, editor. Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability; 1950 Jul 31-Aug 12; Berkeley (CA): University of California press; 1951. p. 481–492.
28. Geoffrion AM. Proper efficiency and the theory of vector maximization. J. Math. Anal. Appl. 1968;22:618–630.
29. Clarke FH. Optimization and nonsmooth analysis. New York: Wiley Interscience; 1983.
30. Páles Z, Zeidan VM. Nonsmooth optimum problems with constraints. SIAM J. Control Optim. 1994;32:1476–1502.
31. Xiao YB, Huang NJ, Cho YJ. Some equivalence results for well-posedness of hemivariational inequalities. J. Global Optim. 2015;61:789–802.
32. Ben-Tal A. Second-order and related extremality conditions in nonlinear programming. J. Optim. Theory Appl. 1980;31:143–165.
33. Gupta R, Srivastava M. Constraint qualifications in nonsmooth multiobjective optimization problem. Filomat. 2017;31:781–797.
34. Zhao KQ. Strong Kuhn–Tucker optimality in nonsmooth multiobjective optimization problems. Pac. J. Optim. 2015;11:483–494.