Penrose limit and DLCQ of string theory

Assaf Shomer

Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel
and
Institute of Theoretical Physics, University of Amsterdam
Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands.

We argue that the Penrose limit of a general string background is a generalization of the Seiberg-Sen limit describing M(atrix) theory as the DLCQ of M theory in flat space. The BMN theory of type IIB strings on the maximally supersymmetric pp-wave background is understood as the exact analogue of the BFSS M(atrix) theory, namely, a DLCQ of IIB string theory on $AdS_5 \times S^5$ in the limit of infinite longitudinal momentum. This point of view is used to explain some features of the BMN duality.
1. Introduction.

One year ago Berenstein, Maldacena and Nastase (BMN) [1], building on recent work [2,3] showed that by taking a certain limit on both sides of the $AdS_5 \times S^5 / SYM_4$ duality [4] one can find a sector of the $N = 4$ SYM theory that describes, besides the supergravity modes, also the excited string states of type IIB string theory in a certain pp-wave background. String theory in that background is exactly solvable in lightcone gauge quantization [3,5] (see also [6]). BMN used this exact solution of type IIB string theory to predict the anomalous dimensions of operators in a certain sector of very large conformal dimensions and $R$ charge in the $N = 4$ theory [2]. The limit on the string side is called a “Penrose limit” [8]. The pp-wave background is the Penrose limit of $AdS_5 \times S^5$. It is a maximally supersymmetric solution to the type IIB supergravity equations of motion and an exact string background.

However, the exact nature of this “BMN duality” remains somewhat unclear. In the AdS/CFT correspondence [4,9] the duality had a very precise and operational meaning [10,11]. One identified the generating function of correlators in the $N = 4$ gauge theory with the partition function of string theory in $AdS_5 \times S^5$ with appropriate vertex operator insertions on the boundary. Type IIB strings on the maximally supersymmetric pp-wave are dual to a certain sector of the $N = 4$ theory. This is a little disturbing since the one side seems like a well defined and self contained (string) theory while the other side is just a part of a (gauge) theory. Described loosely, it is not clear what we are equating on both sides. For example, the pp-wave background can be roughly described as having a confining harmonic potential which effectively reduces the theory to become 1 + 1 dimensional [12]. This makes it hard to give an $S$ matrix interpretation along the lines of [10,11].

In this paper we point out a close analogy between the Penrose limit and the Seiberg-Sen limit [13,14] used in explaining M(atrix) theory. We interpret this in the following way. The Penrose limit of a string background should be viewed as a generalization of the Seiberg-Sen limit in flat space. String/M theory in the Penrose limit of a space should be understood as a generalized DLCQ description of string/M theory in that space. The BMN theory can be understood as a (generalized) DLCQ of type IIB strings on $AdS_5 \times S^5$, or equivalently, of $N = 4$ SYM. From this point of view type IIB strings on the pp-wave relate to type IIB strings on $AdS_5 \times S^5$ in a similar way to that which M(atrix) theory

\[2\] An alternative derivation of BMN’s prediction using a somewhat different perspective was given in [7].
relates to M theory. This enables us to explain many features of the BMN duality. In particular, we feel it gives a better understanding of its exact nature.

The structure of the paper is as follows. In section 2 we discuss briefly the “infinite momentum frame” (IMF) and “lightcone frame” (LCF). In section 3 and 4 we describe the Seiberg-Sen analysis of M(atrix) theory as DLCQ of M theory and of Matrix strings as DLCQ of type II string theories. We stress features of the Seiberg-Sen limit that prove useful later. In section 5 we discuss the possibility of generalizing the DLCQ of string/M theory in the Seiberg-Sen approach to other spacetimes. This leads us to the Penrose limit which is discussed in section 6. In section 7 we draw the analogy between the Seiberg-Sen limit and the Penrose limit and put forward our suggested interpretation, namely, that they are essentially the same process. In section 8 we employ this point of view to type IIB string theory on $\text{AdS}_5 \times S^5$. In section 9 we use this DLCQ perspective to understand some of the features of the BMN duality such as the renormalization of the ’t Hooft coupling. In section 10 we discuss in more detail some aspects of viewing BMN theory as DLCQ and point out an analogy between certain quiver theory operators discussed in [15] and the matrices of M(atrix) theory. In section 11 we discuss the relation between the Seiberg-Sen and Penrose limits and the Maldacena limit used in AdS/CFT. We end with a summary in section 12.

2. Infinite momentum frame and lightcone frame.

In this section we briefly review the issue of quantizing a system in the “Infinite momentum frame” (IMF) and in the “Lightcone frame” (LCF) in flat space.

2.1. IMF.

Consider a system (e.g. of particles) in a Lorentz invariant theory in $d+1$ dimensions. Pick out one space dimension, say the $d^{th}$, call it the longitudinal direction and denote it by $x_\parallel$. The other space directions will be called “transverse” and will be denoted by $x_\perp$. We thus use the basis \{t, $x_\parallel$, $x_1$, $x_{d-1}$\} = \{t, $x_\parallel$, $\vec{x}_\perp$\}.

The energy is given by the relativistic formula

$$E = \sum_a \sqrt{(p_a)_\parallel^2 + (p_a)_\perp^2 + m_a^2}, \quad \text{(2.1)}$$

where the index $a$ runs over particles. Boost now along the longitudinal direction in such a way that all particle states have a positive and “Infinite” (i.e. larger than any other
energy scale in the problem) longitudinal momentum

\[ |p_\perp|, m \ll |p_\parallel| \to \infty \]  

(2.2)

Such states will be called “proper” states, using the language of [17]. Denote the total longitudinal momentum by \( P_\parallel \equiv \sum_a (p_a)_\parallel \). Expanding (2.1) up to second order in \( |p_\perp| \) and \( |m|/|p_\parallel| \) the energy formula becomes non-relativistic

\[ E = |P_\parallel| + \sum_a \frac{(p_a)_\perp^2 + m_a^2}{2|(p_a)_\parallel|}, \]  

(2.3)

with \(|(p_a)_\parallel|\) playing the role of non-relativistic mass. All excitations not satisfying the condition above, i.e. whose momenta before the boost where comparable to \( P_\parallel \), such as modes with negative \((p_a)_\parallel\), are called “improper”. The improper modes decouple from the dynamics of the proper modes in the limit \( P_\parallel \to \infty \) since they become separated from them by an infinite energy gap. Compactifying the longitudinal direction on a circle of radius \( R_s \) the longitudinal momentum is quantized \( P_\parallel = N/R_s \), and the IMF limit (2.2) is \( N \to \infty \). Thus, analyzing the system in the IMF simplifies the problem.

1. The symmetry becomes Galilean.
2. In the limit \( P_\parallel = \infty \) the improper modes decouple.

2.2. LCF and DLCQ.

Here one changes from \( \{t, x_1, \ldots, x_d\} \) to “lightcone” coordinates \( \{x^+, x^-, x_1, \ldots, x_{d-1}\} \) via

\[ x^\pm \equiv \frac{t \pm x_d}{2}. \]  

(2.4)

Again denote \( \{x_1, \ldots, x_{d-1}\} \equiv \vec{x}_\perp \). The conjugate momenta become

\[ p_\pm = i\partial_\pm = i(\partial_t \pm \partial_d) = E \mp P_d. \]  

(2.5)

Choosing \( x^+ \) to play the role of time, the formula for the energy (the generator of \( x^+ \) translations) is (for the case \( (p_a)_- \neq 0 \))

\[ H_{lc} \equiv P_+ = \sum_a (p_a)_+ = \sum_a \frac{(p_a)_1^2 + m_a^2}{2(p_a)_-}. \]  

(2.6)

As in the IMF the longitudinal momentum \( p_- \) plays the role of non-relativistic mass. The symmetry group is not just the apparent \( SO(d-1) \). It is the full Galilean group in \( d \) space.
dimensions \([\mathbb{R}^4]\). Also here it is convenient to compactify the null direction \(x^−\) on a circle of radius \(R_l\). The momentum modes are now quantized as \(P_− = \frac{N}{R_l}\). Quantizing a system in LCF with a null circle is called “Discrete Light Cone Quantization” (DLCQ). Since \(p_−\) is conserved (it is like the mass in Galilean mechanics) the Hilbert space decomposes into superselection sectors labeled by the integer \(N\). Note that this analysis breaks down when \(p_− = 0\). The DLCQ does not obviously simplify the description of this “zero mode” sector.

Note that the LCF is simpler than the IMF in several ways.

1. (2.6) is exact for any value of \((p_a)_− \neq 0\) while (2.3) was true only in the limit of infinite momentum. In the compact case this translates to the distinction between finite \(N\) and infinite \(N\).

2. Quantizing with respect to \(x^+\) as the time and demanding to have a non negative Hamiltonian we see that for \((p_a)_− \neq 0\) we must have

\[ P_+ = H_{lc} \geq 0, \quad (p_a)_− > 0, \]

while in IMF improper modes (such as negative longitudinal momentum modes) decouple only in the limit \(N \to \infty\).

To summarize, all the advantages of the IMF appear now in each DLCQ sector with finite longitudinal momentum \(N\). The two procedures are supposed to agree in the limit \(N = \infty\). In the next section we discuss the application of these old ideas \([16]\) to string/M theory.

3. M(atrix) Theory and The Seiberg-Sen limit

The Seiberg-Sen (SS) analysis \([13,14]\) explains why the DLCQ of M theory with \(N\) units of longitudinal momentum is (as conjectured by BFSS \([18]\) and refined by Susskind \([17]\)) the low energy limit of the 0+1 theory on the worldvolume of \(N\) D0 branes. The analysis consists of two steps. The first step is the realization that the seemingly suspicious procedure of compactification along a null direction in flat space is best thought of as the limit of a spacelike compactification. This makes contact with the IMF. It is demonstrated that this limit leads to a theory of tensionless strings and is thus difficult to analyze. It is then observed that the relevant energies in the IMF also vanish in the limit. In fact, they vanish faster than the string tension. This leads to the second step that involves rescaling the Planck unit. This rescaling achieves a “focusing” on the relevant energy scale. It is
then showed that with this rescaling one exactly ends up with the low energy limit of the 0+1 theory on the worldvolume of $N D0$ branes.

We now review the Seiberg-Sen procedure in some detail.

3.1. Step I - The null circle as a limit.

We first discuss how the null circle is viewed as the limit of a spacelike circle. The boost isometry of flat space with rapidity parameter $\alpha$ rescales the lightcone coordinates \((2.4)\) as follows

\[
x^\pm \rightarrow e^{\mp \alpha} x^\pm.
\]

(3.1)

Suppose $x_d$ is compact, i.e, $x_d \rightarrow x_d + 2 \pi R_s$ brings us back to the same point. In the boosted frame this translates into the following combined action which must be taken in order to return to the same point

\[
x^+ \rightarrow x^+ + e^{-\alpha} \pi R_s \quad \text{and} \quad x^- \rightarrow x^- + e^{\alpha} \pi R_s.
\]

(3.2)

Sending

\[
\alpha \rightarrow \infty, \quad R_s \rightarrow 0, \quad e^{\alpha} R_s \equiv 2 R_l \sim \text{fixed},
\]

(3.3)

the combined symmetry action reduces in the limit to $x^- \rightarrow x^- + 2 \pi R_l$ so we get a null circle\(^3\).

It is easy to see why this limit gives a “difficult” corner in the parameter space of string/M theory. Reducing M theory along $R_s$ we get a type IIA theory with the following coupling constant and string tension

\[
g_A = \left(\frac{R_s}{l_p}\right)^{3/2}, \quad T = 2 \pi R_s l_p^3,
\]

(3.4)

so when $R_s \rightarrow 0$ we get tensionless strings. The remedy also suggests itself, namely - a rescaling of $l_p$.

3.2. Step II - The rescaling of physical parameters.

Denoting the components of the momentum vector $P^\mu$ in the non-boosted frame (where one has the spacelike circle) by a subscript $s$ for “spacelike” and the (infinitely)

\(^3\) Note that at any finite boost parameter $x^-$ is not compact. Only in the exact limit does $e^{-\alpha} = 0$ and we do not need to supplement the translation in $x^-$ with a translation in $x^+$. 

5
boosted frame (the frame with the null circle) by a subscript \( l \) for “lightlike” we have at any intermediate finite boost parameter \( \alpha \)

\[
E_s = \frac{1}{2} e^\alpha \left( E_l + P_l + e^{-2\alpha} (E_l - P_l) \right)
\]

\[
P_s = \frac{1}{2} e^\alpha \left( E_l + P_l - e^{-2\alpha} (E_l - P_l) \right).
\]

Thus, in the “spacelike” frame all the momenta and energies are exponentially blue shifted with respect to the “lightlike” frame. This is so because we are considering momentum modes along a vanishing circle. However, the physics lies not in this diverging “zero point” energy due to the boost but in the fluctuations. Equivalently, since we have a Galilean symmetry the mass is a constant parameter that does not take part in the dynamics. Focusing on the fluctuations above the D0 brane mass we get using (2.5) (2.6)

\[
\Delta E_s = E_s - P_s = e^{-\alpha} (E_l - P_l) = \frac{R_s}{2R_l} H_{lc},
\]

which vanish in the limit (3.3). In order to focus on these as finite energy fluctuations we scale also the Planck mass \( \tilde{m}_p \to \infty \) and the transverse size of the manifold \( \tilde{R}_\perp \to 0 \) along with the boost parameter \( \alpha \to \infty \) as

\[
\tilde{m}_p \sim e^{\frac{\alpha}{2}} \to \infty
\]

\[
\tilde{m}_p^2 R_s = m_p^2 R_l \sim fixed
\]

\[
\tilde{m}_p \tilde{R}_\perp = m_p R_\perp \sim fixed.
\]

The most easily extendible rationale behind this rescaling is that according to (3.6) the relative energies scale like \( R_s \) under further boosting. On dimensional grounds they should thus be proportional to \( m_p^2 R_s \) (\( m_p \) being the only scale in M theory). Since the boost does not affect the transverse coordinates \( R_\perp \) it makes sense to scale those like \( \tilde{t}_p \) so that the transverse geometry in dimensionless units does not change during the limit. This rescaling gives us the desired corner of parameter space, since now

\[
\tilde{g}_A = (R_s \tilde{m}_p)^{3/2} = (R_s \tilde{m}_p^2)^{3/2} \tilde{m}_p^{-3/2} \to 0
\]

\[
\tilde{\alpha}' = (R_s \tilde{m}_p^3)^{-1} = (R_s \tilde{m}_p^2)^{-1} \tilde{m}_p^{-1} \to 0.
\]

Note also that if the transverse space contains a circle one can consider T-duality. Although the size of the transverse manifold vanishes like \( \tilde{t}_p \) the size of the T-dual circle remains fixed in the limit.

\[
\frac{\tilde{\alpha}'}{\tilde{R}_\perp} = \frac{1}{(\tilde{R}_\perp \tilde{m}_p)(R_s \tilde{m}_p^2)} \sim fixed.
\]

\[\text{Rescaled parameters will be denoted with tildes.}\]
This means that the natural physical description involves a T-duality. This is one way to see why the DLCQ of type IIA is given in terms of $D1$ strings and the DLCQ of IIB in terms of $D2$ branes.

3.3. Scaling of dimensionful quantities.

An essential part of the Seiberg-Sen analysis is the rescaling of the dimensionful quantity $m_p$. This point deserves an explanation. Why were we allowed to rescale the parameters of the theory? by doing so we naively change the theory. The point is that the scaling of dimensionful parameters is physically meaningless. The only physically invariant rescaling involve dimensionless parameters, so rescaling a dimensionful parameter is a way of focusing on some sector of the theory by sending a dimensionless parameter to zero. For instance we can use (2.3) and (2.5) to write the following relation for each particle in the IMF

$$p_0 - |p_{\parallel}| = p_{\perp}^{IMF} = \frac{p_{\perp}^2 + m^2}{2|p_{\parallel}|} = \frac{p_{\perp}^2 + m^2}{2N} R_s. \tag{3.10}$$

The Seiberg-Sen boost (3.1) takes $R_s \to 0$. So in order to focus on states of such energy one can introduce a mass scale $m_f$ and write

$$p_{\perp}^{IMF} = \left(\frac{p_{\perp}^2 + m^2}{m_f^2}\right) \left(\frac{R_s m_f^2}{2N}\right). \tag{3.11}$$

Keeping the first parenthesis and $N$ fixed in the limit dictates that we must scale $m_f$ exactly as in (3.7). Using (3.6) we can now derive (2.6). So states with energy of order $m_f$ survive the rescaling as finite energy fluctuations. Those are the “proper states”. States with energy of order $P_{\parallel}$ have infinite energy in $m_f$ units since $\frac{P_{\parallel}}{m_f} \sim \frac{1}{R_s m_f} \sim m_f \to \infty$. Those are the “improper modes” that decouple from the dynamics in DLCQ [17]. Thus, DLCQ achieves a simplification of the kinematics at the expense of focusing on the dynamics in a certain energy band.

To summarize, the Seiberg-Sen limit in the context of M(atrix) theory consists of viewing the null circle as the end point of a limiting procedure during which the radius $R_s$
and the size of the transverse manifold vanish\footnote{It is worth mentioning here that since the DLCQ of M theory involves a compactification on a vanishing circle supergravity is no longer a good description. This explains \cite{19} why various comparisons made in the literature between perturbative results in 11 dimensional DLCQ supergravity and calculations in the Matrix model turned out to agree only for quantities protected by supersymmetry. (for a recent review see \cite{20}).} in such a way so as to keep

\[ R_l \equiv \left( \frac{\tilde{m}_p}{m_p} \right)^2 R_s \sim fixed \]

\[ R_\perp \equiv \frac{\tilde{m}_p}{m_p} \tilde{R}_\perp \sim fixed. \]

4. Matrix strings and the Seiberg-Sen limit.

The original Seiberg-Sen analysis was done for the DLCQ of M theory. The DLCQ of string theory was treated later by \cite{21,22,23}. The essential new feature of this analysis is the 9–11 flip. In this section we briefly review Matrix strings.

4.1. Matrix strings.

In order to describe the DLCQ of type IIA string theory with parameters \( g_A, \alpha' \) having longitudinal momentum \( P_- = \frac{N}{R_l} \) along a null circle \( x^- \sim x^- + 2\pi R_l \) we need to perform a lift to M theory along another circle \( R_9 = g_A l_s \) and get M theory with \( l_p = g_{1/3} A l_s \). Reducing now along the null circle we get a different IIA theory with (hatted parameters)

\[ R_l = \hat{g}_A \hat{l}_s, \quad l_p = \hat{g}_A^{1/3} l_s = \hat{g}_{1/3} A \hat{l}_s \]

along with \( N D0 \) branes. The end result (after one T-duality) is the low energy theory on \( N D1s \) wrapping a circle with parameters

\[ R_{D1} = \frac{\alpha'}{R_l}, \quad g_{YM} = \frac{R_l}{g_A \alpha'}. \]

An analogous treatment is done for the DLCQ of type IIB on a circle by first T-dualizing to type IIA.

4.2. Seiberg-Sen analysis of Matrix strings.

Now we turn to the Seiberg-Sen point of view. The null circle with radius \( R_l \) is viewed as the limit of a vanishing spacelike circle of radii \( R_s \rightarrow 0 \) with \( 2R_l = e^\alpha R_s \sim fixed. \)
Repeating the above analysis but now the compactification done on the spacelike circle \( R_s \) we get essentially the same answer but with \( R_l \) replaced by \( R_s \), namely

\[
R_{D1} = \frac{\alpha'}{R_s}, \quad g_{YM} = \frac{R_s}{g_A \alpha'}.
\]

(4.3)

Now the limit seems to give a singular result. The D0 branes are transverse to a vanishing circle and the YM coupling vanishes as well. The solution is to rescale \( l_p \) as in (3.12), i.e.

\[
l_p \sim \sqrt{R_s} \quad \text{or} \quad R_s m_p^2 \sim \text{fixed}.
\]

(4.4)

The 9–11 flip gives us an interesting way of rescaling the 11 dimensional Planck length that has a clear physical interpretation in the original IIA string theory which we are describing in DLCQ. Looking at (4.1) we can achieve the appropriate rescaling of \( l_p \) by rescaling the string tension in the original IIA in following manner

\[
l_s \rightarrow \tilde{l}_s \sim \sqrt{R_s}, \quad g_s \sim \text{fixed}.
\]

(4.5)

In this scaling both quantities in (4.3) are being held fixed in the limit. If we have also transverse directions, keeping the geometry fixed in string units one needs to define \( \tilde{R}_\perp \sim \tilde{l}_s \) such that \( R_\perp / l_s = \tilde{R}_\perp / \tilde{l}_s \). These scaling in string theory exactly implement (3.12). Note that \( R_0 = g_A l_s \) which after the 9–11 flip is just a transverse circle indeed scales like \( l_s \) as it should. The same analysis with the rescaling (4.5) works also for the DLCQ of type IIB on a circle.

To summarize, in both the DLCQ of M theory and of type IIA/B string theory the Seiberg-Sen limit is being effectively implemented by correctly scaling the fundamental length scale of the theory which we will denote here by \( l_f = m_f^{-1} \). In all cases the scaling is a decoupling limit

\[
\tilde{m}_f \rightarrow \infty,
\]

(4.6)

along with the vanishing limit of the spacelike circle \( R_s \rightarrow 0 \) while keeping

\[
R_l \equiv \left( \frac{\tilde{m}_f}{m_f} \right)^2 R_s \sim \text{fixed}
\]

\[
R_\perp \equiv \frac{\tilde{m}_f}{m_f} \tilde{R}_\perp \sim \text{fixed}.
\]

(4.7)

In the next section we suggest how one can apply this procedure to more general string backgrounds.
5. An attempt to generalize DLCQ.

The analysis presented in the last section is not directly applicable to any background and tacitly assumes that the space is the product of flat space (where time and the null circle reside) and a general compact manifold of some generic size $R_{\perp}$

$$R^{1,p} \times M^{9-p}_{\perp}.$$  \hspace{1cm} (5.1)

In other words it assumes there is a boost isometry. In this section we will try to suggest a possible generalization of DLCQ to more general spacetimes.

In order to generalize the concept of DLCQ to backgrounds not of the form (5.1) we restate the basic idea behind DLCQ. We are describing a theory symmetric under a boost in the $\{t, x_d\}$ plane. Thus, we can use this symmetry to describe the physics from the LCF. In other words, since different inertial observers are related by the boost symmetry we are free to choose the observer that gives the simplest description. So what happens in more general spacetimes? In a theory of gravity (such as string/M theory) all observers are equally fit to describe the physics. Typically the simplest description appears in the frame of the appropriate generalization of “inertial observer” to curved spaces, namely, in the frame of a freely falling observer. Those observers move along geodesics. What made the LCF/IMF description simple was the introduction of a large quantity, namely, the longitudinal momentum which boosted the observer asymptotically close to the speed of light. In a general spacetime we should adopt a local version of this statement, namely, “moving asymptotically close to a null geodesic”. The same physical reasoning suggests that a simple theory might emerge if one also rescales the fundamental scale of the theory appropriately.

Our suggestion immediately raises several questions.

1. Does such a limit exist?
2. Why is this lightcone quantization discrete? where is the circle in a spacetime that does not have a null isometry like flat space?
3. Does such a procedure reduce to the usual DLCQ in flat space? for instance can we get M(atrix) theory that way?

The central point of this paper is to argue that indeed this is a well defined procedure known as “the Penrose limit of a spacetime”. In the next section we describe the Penrose limit and show that it generalizes in a precise way the Seiberg-Sen prescription. Later on we will address the other questions presented above.
6. The Penrose limit.

The Penrose limit \[8,24\] is defined by focusing on the geometry near a null geodesic. Locally, in the neighborhood 6 of a null geodesic one can introduce the following set of coordinates \( Y \equiv \{ y^\pm, y^i \} \) such that the line element is given by

\[
d s^2 = g_{\mu\nu} d y^\mu d y^\nu = -2 d y^- [d y^+ + A(Y) d y^+ + B_i(Y) d y^i] + C_{ij}(Y) d y^i d y^j, \tag{6.1}
\]

where \( i, j = 1, \ldots, 8 \) and \( C_{ij} \) is a positive definite symmetric matrix. The close-by null geodesics are parameterized by \( y^-, y^i = \text{const} \) and the affine parameter along them is \( y^+ \). The “original” null geodesic on which we focus is at \( y^- = 0 \).

The next step in taking the Penrose limit is to blow up this neighborhood to become the whole space 7. This is done by introducing an auxiliary dimensionless parameter \( \Omega \) and defining a rescaled set of coordinates \( X \equiv \{ x^\pm, x^i \} \)

\[
\begin{align*}
x^+ &\equiv y^+, \\
x^- &\equiv \Omega^2 y^-, \\
x^i &\equiv \Omega y^i.
\end{align*} \tag{6.2}
\]

Penrose then tells us that the rescaled metric

\[
G_{\mu\nu}(X) \equiv \Omega^2 g_{\mu\nu}(Y) \tag{6.3}
\]

has the following well defined limit 8 as \( \Omega \to \infty \)

\[
ds^2 = G_{\mu\nu} d x^\mu d x^\nu = -2 d x^+ d x^- + C_{ij}(x^+) d x^i d x^j, \tag{6.4}
\]

where \( C_{ij}(x^+) \equiv C_{ij}(x^+, 0, \vec{0}) \). This is the pp-wave in Rosen coordinates. One can change to Brinkman coordinates in which the line element is given in the more familiar form

\[
ds^2 = -4 d z^+ d z^- - H_{ij}(z^+) z^i z^j (d z^+)^2 + d z^i d z^i. \tag{6.5}
\]

We do not concern ourselves with other background fields such as gauge fields or p-form fields. All those can be scaled appropriately in the Penrose limit 24. In Rosen coordinates it is obvious that the pp-wave has many isometries. Note that there is always the null isometry along the coordinate in the “minus” direction. We next argue how one can naturally take the Penrose limit in such a way that effectively compactifies the null isometric direction so that in the end we get a null circle.

---

6 The neighborhood must not contain conjugate points.
7 This is reminiscent of the “near horizon limit” of Maldacena. We discuss this point later on.
8 This is easily seen by expanding \( C_{ij}(Y) d y^i d y^j = C_{ij}(x^+, \frac{x^-}{\Omega}, \frac{x^i}{\Omega}) \frac{d x^i}{\Omega} \frac{d x^j}{\Omega} \) in powers of \( \frac{1}{\Omega} \).
6.1. Penrose limit + identifications, or from null isometry to a null circle.

Near the null geodesic there is a coordinate system of the form (6.1). Let us introduce a local time and space coordinates

\[ t = y^+ + y^- , \quad s = y^+ - y^- . \]  

Assume that the metric has an isometric direction with some component along \( s \). Compactifying along that isometric direction involves

\[ s \rightarrow s + c_\parallel, \]  

with \( c_\parallel \) some constant. Of course, \( s \) need not be the isometric direction itself so the actually symmetry transformation may also involves an action on some other “transverse” spacelike coordinates \( y^i \) which we will symbolically denote here as

\[ \alpha_\perp \rightarrow \alpha_\perp + c_\perp, \]  

with \( c_\perp \) some other appropriate constants. The radius of the circle is

\[ R_s \sim c_\parallel \sim c_\perp. \]  

(6.7) induces the following combined action on the lightcone coordinates

\[ y^+ \rightarrow y^+ + \frac{1}{2} c_\parallel , \quad y^- \rightarrow y^- - \frac{1}{2} c_\parallel . \]  

In the Penrose limit the transverse coordinates get rescaled (6.2) so let us define \( \beta_\perp \equiv \Omega \alpha_\perp \) to be the rescaled spacelike coordinates involved in the circle identification. This gives

\[ x^+ \rightarrow x^+ + \frac{1}{2} c_\parallel , \]  

\[ x^- \rightarrow x^- - \frac{1}{2} c_\parallel \Omega^2 \]  

\[ \beta_\perp \rightarrow \beta_\perp + c_\perp \Omega. \]  

Note that this restricts the discussion to spacetimes that have locally at least one spacelike and one timelike killing vectors. Spacetimes that do not have those minimal requirements would probably be too difficult to analyze in any case.

If this is not the case, perhaps one can still approximate the space by another one that does have this isometry, in such a way that the difference between the metrics on the two spaces vanishes in the limit. We do not pursue such a generalization here.
The Penrose limit sends $\Omega \to \infty$. If we *supplement* the Penrose limit by the following *rescaled identification*, namely, (6.7) together with

$$c_\parallel \to 0, \quad c_\parallel \Omega^2 \sim \text{fixed},$$

we get in the limit that the combined action degenerated to an action only on the $x^-$ direction

$$x^- \sim x^- + 2\pi A_l,$$  \hspace{1cm} (6.13)

where we have defined $\frac{1}{2} c_\parallel \Omega^2 \equiv 2\pi A_l$. Namely, we get a null circle. The following general argument was demonstrated in specific cases when discussing the Penrose limit of orbifolds of the type $\text{AdS}_5 \times S^5/\mathbb{Z}_K$ [15,25]. These models will be addresses in detail later on and will make the above abstract procedure clearer.

7. Penrose limit, DLCQ and the Seiberg-Sen limit.

We saw that any spacetime has a Penrose limit. This limit is achieved by focusing on the neighborhood of a null geodesic and rescaling the coordinates in a way that blows up the neighborhood to become the whole space. The rescaling is universal (6.2)

A. The time coordinate does not get rescaled.

B. The null coordinate gets rescaled *quadratically* in an auxiliary parameter.

C. The spacelike transverse coordinates get rescaled *linearly* in that parameter.

We further argued how one can use a slight generalization of the Penrose limit so as to naturally end up with a null circle. This generalization involves a limit of discrete identifications that produce a vanishingly small radius (6.12) in the non-rescaled coordinates $Y$. The circle vanishes with the same rescaling as the null coordinate. These rescalings are easily recognized to be identical to those done in the Seiberg-Sen treatment of the DLCQ of string/M theory, namely (3.12) with the identification

$$\Omega \equiv \frac{\tilde{m}_p}{m_p}.$$  \hspace{1cm} (7.1)

We are thus led to the following statement. The analogue of M(atrix) theory for string/M theory on general curved backgrounds is given by string/M theory on the Penrose limit of that background. The Penrose limiting procedure supplemented by appropriately rescaled discrete identifications gives the DLCQ. The “usual” Penrose limit is the decompactification limit, or DLCQ in the infinite longitudinal momentum limit. The Penrose
limit automatically achieves both steps in the Seiberg-Sen procedure, namely, describing
the system from the point of view of an observer moving asymptotically close to the speed
of light and rescaling parameters so as to focus on finite energy fluctuations. We believe
that this “focusing” property of DLCQ is being mirrored in the geometry by the fact that
the Penrose limit blows up the neighborhood of a null geodesic to become the whole space
and “throws the rest” to infinity. This corresponds to the decoupling of heavy modes in
DLCQ according to the usual UV/IR relation in AdS/CFT. \[26\]

We feel this answers the first two questions posed above. It is also straightforward to
see that this procedure reduces to the Seiberg-Sen limiting procedure if one considers the
Penrose limit of 11 dimensional flat space compactified on a circle.

8. DLCQ of strings in AdS space.

In this section we finally reach the model that motivated this line of research. We argue
that type IIB string theory on a Penrose limit of the orbifold background \( AdS_5 \times S^5 / Z_M \)
together with an appropriate scaling of the rank of the orbifold group \( M \) (see e.g. \[15,25\])
is a DLCQ of type IIB on \( AdS_5 \times S^5 \). The lightcone quantization here is discrete since the
resulting pp-wave space has a null circle. This theory is analogous to Susskind’s finite \( N \)
version of M(atrix) theory \[17\]. The limit of infinite longitudinal momentum corresponds
to the Penrose limit of \( AdS_5 \times S^5 \), namely, to the BMN theory. This theory is analogous
to the BFSS M(atrix) theory \[18\].

8.1. The DLCQ of \( AdS_5 \times S^5 \).

Just as in flat space, it is best to start from the DLCQ, namely with finite longitudinal
momentum. To that end we need to discuss strings on the Penrose limit of \( AdS_5 \times S^5 / Z_M \).
Those have been studied in e.g. \[14,25\]. We choose to focus on the case where the null
geodesic does not pass through singular points of the orbifold. This Penrose limit results
in the maximally supersymmetric type IIB pp-wave. Following \[15\] we write the metric on
\( AdS_5 \times S^5 / Z_M \) as

\[
\begin{align*}
ds^2 &= R^2 \left[ - \cosh^2 \rho \ dt^2 + d\rho^2 + \sinh^2 \rho \ d\Omega_3^2 + \right. \\
& \quad \left. d\alpha^2 + \sin^2 \alpha \ d\theta^2 + \cos^2 \alpha \ (d\gamma^2 + \cos^2 \gamma \ d\chi^2 + \sin^2 \gamma \ d\phi^2) \right],
\end{align*}
\]

This is reminiscent of what happens to the region outside the “throat” in Maldacena’s limit.
where the first line is the $AdS_5$ metric in global coordinates. The second line is the metric on $S^5$ embedded in $R^6 \simeq C^3$ with coordinates

$$ z_1 = R \sin \alpha e^{i\theta}, \quad z_2 = R \cos \alpha \cos \gamma e^{i\chi}, \quad z_3 = R \cos \alpha \sin \gamma e^{i\phi}. \quad (8.2) $$

The orbifold action identifies any point with the point resulting from the combined action

$$ \chi \rightarrow \chi + \frac{2\pi}{M}, \quad \phi \rightarrow \phi - \frac{2\pi}{M}. \quad (8.3) $$

This choice of the metric and of the orbifold action explicitly break the $SO(6)$ isometry group of the 5 sphere into $U(1) \times SO(4)$, where the $U(1)$ is parametrized by $\theta$.

In order to take the Penrose limit we choose to focus on the following null geodesic

$$ \chi = t, \quad \rho = \alpha = \gamma = 0, \quad (8.4) $$

and rescale the coordinates in its neighborhood as follows

$$ x^+ \equiv \frac{1}{\mu} \frac{t + \chi}{2}, \quad x^- \equiv \frac{\mu R^2 t - \chi}{2}, \quad (8.5) $$

$$ r \equiv R\rho, \quad \omega \equiv R\alpha, \quad y \equiv R\gamma, $$

with $\mu$ an arbitrary positive parameter of mass dimensions. Making the substitution (8.5) and taking the limit $R \rightarrow \infty$ one is left with the maximal supersymmetric pp-wave background

$$ ds^2 = -4dx^+dx^- - \mu^2 z^2 (dx^+)^2 + dz^2, \quad (8.6) $$

where

$$ dz^2 \equiv \sum_{i=1}^8 dz^i dz^i = dr^2 + r^2 d\Omega_3^2 + d\omega^2 + \omega^2 d\theta^2 + dy^2 + y^2 d\phi^2 \quad (8.7) $$

denote the 8 flat transverse coordinates (four originating from the $S^5$ and four from the $AdS_5$ factors). Again, we suppress the RR-form since it is of no importance to our discussion. However, as opposed to [4] here $x^-$ can be made compact by appropriately scaling $M$. This is so since the identification (8.3) sends any point to an identical point by the combined action

$$ x^+ \rightarrow x^+ + \frac{\pi}{\mu M}, \quad x^- \rightarrow x^- + \frac{\mu R^2 \pi}{M}, \quad \phi \rightarrow \phi - \frac{2\pi}{M}. \quad (8.8) $$
If we scale together

\[ R, M \to \infty, \quad \frac{\mu R^2}{M} \equiv 2R_l \sim \text{fixed}, \tag{8.9} \]

we see that in the limit any point is mapped to an identical point by only sending \( x^- \to x^- + 2\pi R_l \), i.e. \( x^- \) parameterizes a null circle of radius \( R_l \).

However \( R \) is a dimensionful parameter and thus there is no physics in the claim that \( R \to \infty \). The only physically meaningful scaling involve dimensionless quantities. Since we are talking about string theory in \( \text{AdS}_5 \times S^5/Z_M \) we can equivalently choose \( R/l_s \) or \( R/l_p \) as the dimensionless parameter, since the string coupling is constant. Choosing \( \Omega \equiv \frac{R}{l_s} \sim (g_s N)^{\frac{1}{4}} \) the Penrose limit can equally be taken by

\[ N \to \infty, \quad g_s \sim \text{fixed}, \tag{8.10} \]

which sends

\[ \alpha' \sim \frac{1}{\sqrt{N}} \to 0, \quad g_s \sim \text{fixed}. \tag{8.11} \]

So let us identify

\[ \frac{1}{M} \equiv \mu R_s, \quad \Omega \equiv \frac{R}{l_s} \sim m_s. \tag{8.12} \]

The relation (8.9) is equivalent to saying

\[ R_s \to 0, \quad m_s \to \infty, \quad R_s m_s^2 \sim \text{fixed}, \quad g_s \sim \text{fixed}. \tag{8.13} \]

This is the same limit as in the Matrix string (4.5). The transverse neighborhood, whose size (generally denoted by \( R_\perp \)) in this case is given by the three radii \( \rho, \alpha, \gamma \) also scale appropriately since \( R_\perp m_s \sim \text{fixed} \) in the limit. So we recover (4.7).

Now let us look back at the gauge theory side. Here \( \frac{1}{\pi T} \) is the periodicity of the difference angle \( \frac{\chi - \phi}{2} \) inside the \( S^5/Z_M \). The lightcone Hamiltonian and longitudinal momentum are

\[ H_{lc} = p_+ = \mu(\Delta - J), \quad p_- = \frac{\Delta + J}{\mu R^2}. \tag{8.14} \]

Notice that the orbifold identification (8.3) is not only within the great circle parameterized by \( \chi \) along which we boost by (8.3). So let us divide the current as follows [15]

\[ J = -i \partial \chi = J^+ + J^-, \tag{8.15} \]

16
where we define
\[ J^\pm \equiv -\frac{i}{2}(\partial \chi \pm \partial \phi) \equiv -i \frac{\partial}{\partial \varphi^\pm}, \] (8.16)
and
\[ \varphi^\pm = \chi \pm \phi. \] (8.17)
This choice of coordinates is the geometric manifestation of the group theory statement \( SO(4) = SU(2)_+ \times SU(2)_- \) \[23\], where the currents (8.16) generate the Cartan subalgebra. In particular the eigenvalues of \( J^\pm \) are half integral.

The orbifold identification (8.3) acts only on \( \varphi^- \) by
\[ \varphi^- \sim \varphi^- + \frac{\pi}{M}. \] (8.18)
Its effect is to project on the subspace of states periodic over \( 1/2M \) of the full period, namely, only states with
\[ J^- = M \times (2k), \] (8.19)
with \( k \) half integral, survive the orbifold projection.

To get finite quantities we send following \[1\] \( \Delta \sim J \sim \sqrt{N} \to \infty \). But we saw (8.9) that if we want a null circle also \( M \sim \sqrt{N} \) so \( J/M \sim fixed \). Therefore, we can keep finite quantum numbers for \( k \) and \( J^+ \) and still get the Penrose limit since \( M \to \infty \). Let us define the following integer number
\[ q \equiv 2k. \] (8.20)
From (8.14) we get using (8.19) and (8.9) that in the Penrose limit
\[ p_- \sim \frac{2J}{\mu R^2} \to \frac{q}{R_\ell}. \] (8.21)
So \( q \) is the quantum number denoting the longitudinal momentum along the null circle in the resulting pp-wave background. Remember that in M(atrix) theory, the number \( N \) of \( D0 \) branes was the number of longitudinal momentum quanta. We thus suggest to interpret \( q \) as the analogue of the number of \( D0 \) branes in Susskind’s finite \( N \) reformulation of M(atrix) theory \[17\]. The diverging quantum number \( J \) corresponds to the diverging mass of the \( D0 \) brane.

In the Penrose limit of \( AdS_5 \times S^5 \) studied by BMN no orbifold was taken so this is the special case \( M = 1 \). This means that \( q \sim J \to \infty \). In other words the original BMN paper analyzed the DLCQ of string theory on \( AdS_5 \times S^5 \) in the limit of infinite longitudinal momentum. This is analogous to the BFSS M(atrix) theory. To summarize, the study initiated in \[1\] of strings on the Penrose limit of \( AdS \) space is a concrete realization of the generalized DLCQ procedure proposed here.
9. Known facts about BMN from the DLCQ perspective.

In this section we employ the DLCQ perspective to understand some features of the BMN duality.

9.1. “Renormalization” of coupling constants.

The effective expansion parameter in the sector dual to type IIB on the pp-wave was shown\cite{12,27} to be not the ‘t Hooft coupling $\lambda = 4\pi g_B N$, which diverges in the Penrose limit, but rather

$$\lambda' \equiv \frac{\lambda}{J^2} \sim \text{fixed.} \quad (9.1)$$

Also the effective genus expansion parameter for the Feynman graphs is “renormalized”

$$g_2 = \frac{J^2}{N} \quad (9.2)$$

This phenomenon got a convincing combinatorial explanation by studying the relevant Feynman graphs (see e.g. \cite{12},\cite{27}). We now show that this renormalization is a direct manifestation of the Seiberg-Sen rescaling. Since we are really describing the DLCQ of strings in $AdS_5 \times S^5$ we can use the standard AdS/CFT relation \cite{4}

$$\left(\frac{R}{l_s}\right)^4 = \lambda = 4\pi g_B N. \quad (9.3)$$

In AdS/CFT we keep this quantity fixed (‘t Hooft limit) but here we are sending this dimensionless quantity to infinity. BMN tell us to send $R \to \infty$ and scale $J \sim R^2$. Physically this is the same as setting $R = 1$ and scaling $\alpha' \sim \frac{1}{J} \to 0$ while keeping $g_B$ fixed. It follows that

$$g_B \sim \frac{1}{N\alpha'^2} \sim \frac{J^2}{N} = g_2 \sim \text{fixed.} \quad (9.4)$$

We see that (9.2) is the original IIB string coupling on $AdS_5 \times S^5$. From this relation (9.1) follows accordingly.

9.2. Non-planar diagrams at infinite $N$ and second quantization.

A closely related fact is that even at the large $N$ limit the genus expansion is nontrivial, and one needs to take into account diagrams of all genera. In fact, the string coupling (9.4) is non vanishing (it is the original IIB string coupling of the $AdS_5 \times S^5$ background). This phenomenon seems reminiscent of a central property of the DLCQ of M theory, namely,
that M(atrix) theory is argued to be a second quantized theory. It is tempting to assume that also the DLCQ of string in AdS via the Penrose limit gives rise to a second quantized theory. Thus one should expects that even in the strict $N = \infty$ limit strings will interact, and the effect of higher genera will be indispensable.

9.3. \textit{Pp-wave algebra, Inonu-Wigner contraction and the Galilei group.}

One of the simplifying features of DLCQ and IMF in flat space is the appearance of a Galilean symmetry instead of the Lorentz symmetry. An analogous statement exist also for the DLCQ of strings on $AdS_5 \times S^5$. The symmetry algebra of the Penrose limit of $AdS_5 \times S^5$ is an Inonu-Wigner (IW) contraction \cite{28} of the symmetry algebra of the full space, namely, $SO(2, 4) \times SO(6)$. The Inonu-Wigner contraction is also the procedure by which one gets the Galilei group as a limit of small velocities (or infinite rest mass) from of the Lorentz group \cite{30}. This is exactly what happens in the Penrose limit. By moving close to a null geodesic we give a diverging lightcone “rest mass” to all the particles (2.6). Thus, the IW contraction of the symmetry algebra in the Penrose limit should be understood as the exact analogue of the appearance of the Galilean symmetry in flat space\footnote{13}.

9.4. \textit{Decoupling of negative modes and the BPS condition.}

Due to the BPS condition of the $N = 4$ supersymmetry algebra, $\Delta \geq |J|$, both $p^\pm$ in (8.14) are positive. It is clear that the lightcone Hamiltonian should be positive, but why is the longitudinal momentum positive? This is one of the features of DLCQ (see (2.7)). This is analogous to the reason that in M(atrix) theory one has only $D0$ and no anti-$D0$ branes. In fact, BMN argued that the insertion of a $\bar{Z}$ impurity decouples in the Penrose limit. This is the statement that in IMF and DLCQ the modes that have negative longitudinal momentum (in this case negative $J$) decouple.

\footnote{12} We discuss only the bosonic part of the symmetry algebra. This can be generalized to the supersymmetric case \cite{29}.

\footnote{13} Note that this gives a physically intuitive “explanation” of the fact (4.7) that the transverse coordinates scale linearly (momentum is linear in the velocity for small velocities) and the longitudinal quadratically (the energy is quadratic in small velocities).

\footnote{14} A related point noticed in \cite{17} is that performing a T-duality in the quiver theory space leads to a non-relativistic string theory (NRST) \cite{31}.
9.5. *The parameter* \( \mu \).

This (nonphysical) parameter appears due to a rescaling symmetry of the pp-wave. In the language of DLCQ this parameter reflects the fact that we have an infinite quantity in the problem, namely the infinite momentum. We are free to “rescale” infinity by a positive number. The extra freedom of choosing \( \mu \) is analogous to the freedom of performing additional boosts in the IMF.

9.6. *Supersymmetry.*

The BMN pp-wave background has 32 supercharges, however, only 16 of them are linearly realized \([2]\). This is reminiscent of the fact that M(atrix) theory gives a DLCQ description of a theory with 32 supercharges, in terms of \( D0 \) branes that are \( \frac{1}{2} \) BPS.

10. *A closer look at BMN as DLCQ.*

In this section we take a closer look at the BMN theory as a DLCQ of type IIB string theory on \( AdS_5 \times S^5 \). We also point out an analogy between the matrices of M(atrix) theory and a certain type of operators in the \( N = 2 \) quiver gauge theory dual to type IIB strings on \( AdS_5 \times S^5 / \mathbb{Z}_M \).

10.1. *Excited string states.*

Using AdS/CFT \([10]\) one can predict the following general behavior of conformal dimensions of single trace operators in \( N = 4 \) SYM in the supergravity approximation

\[
1 \ll \lambda = g_B N = \frac{R^4}{\alpha'^2} \ll N = \frac{R^4}{l_p^4}. \tag{10.1}
\]

A. dimensions of order \( \sim 1 \) correspond to KK modes of the reduction on the \( S^5 \). They are all BPS states in 10 dimensions, and thus all correspond to chiral primaries in the CFT. Those states exist up to \( \Delta = N \).

B. dimensions of order \( m_s \sim \lambda^{1/4} \) corresponds to excited string states.

C. dimensions of order \( 1/g_B \sim N \) correspond to D-branes.

D. dimensions of order \( 1/g_B^2 \sim N^2 \) correspond to NS5-branes.

Let us look at the energy formula in BMN

\[
H_{lc} = \mu \sum_{n=-\infty}^{\infty} N_n \sqrt{1 + \frac{n^2}{(\mu P_\alpha')^2}}, \tag{10.2}
\]
In the limit the contribution of each oscillator to the anomalous dimension is given by

\[(\Delta - J)_n = \sqrt{1 + \frac{4\pi g_B N n^2}{J^2}}.\]  

(10.3)

The \(n = 0\) sector describes the supergravity modes. We will discuss those momentarily. The excited string states correspond to \(n \neq 0\). Note that in the region where the second term in (10.2) dominates one gets back the usual string spectrum in flat space where we have the relation

\[E^2 \sim T n,\]  

(10.4)

with \(T\) the string tension. Looking at the same limit in (10.3) we identify

\[T \sim \sqrt{\lambda'},\]  

(10.5)

where \((9.1)\) \(\lambda' = \lambda/J^2\). Using the usual translation formula between energies and dimensions

\[\Delta \sim RE,\]  

(10.6)

and the rescaling used by BMN \(J \sim R^2\) we get

\[\Delta \sim \sqrt{R^2} \sim R(\lambda')^{1/4} \sim \sqrt{\lambda'}(\lambda')^{1/4} = (\lambda)^{1/4}.\]  

(10.7)

This is the correct region where one expects strings to appear in \(AdS_5 \times S^5\). We interpret this as evidence for the relation between the string excitations in the pp-wave and the string excitations in \(AdS_5 \times S^5\).

10.2. Supergravity modes.

The \(n = 0\) sector was identified in [1] as corresponding to the supergravity modes propagating in the plane wave geometry. From the point of view described here it should contain information also about supergravity modes in \(AdS_5 \times S^5\). At first sight this seems impossible for several reasons. The BMN states consist only of the part of the spectrum of the \(N = 4\) theory that has divergingly large dimensions. So how can we even expect to see the known states with small dimensions of order \(~ 1\). Also, the representations of the pp-wave algebra are very different from the representations of \(SO(2, 4) \times SO(6)\). We believe the answer lies in the way DLCQ “blows up” bands in the spectrum. This operation “distorts” the spectrum but does not change the Hilbert space. Indeed, the action of the
Seiberg-Sen/Penrose limit on the Hilbert space can be understood in two equivalent ways (the usual passive/active descriptions)

A. Keeping the $AdS_5 \times S^5$ symmetry algebra, namely we are still classifying states according to scaling dimensions and Lorentz quantum numbers, but we are looking at states with diverging dimensions and $R$ charge. This is the point of view presented in BMN.

B. We keep the full Hilbert space of supergravity modes on $AdS_5 \times S^5$ but we contract the $AdS_5 \times S^5$ symmetry algebra to the pp-wave algebra. The Hilbert space is always there, but the symmetry operators get rescaled so that many states end up with either zero or $\infty$ eigenvalues.

Let us illustrate this point using an example. Consider the simplest group contraction, namely, that of $SU(2)$

$$[J^3, J^\pm] = \pm J^\pm, \quad [J^+, J^-] = 2J^3,$$  \tag{10.8}

with Casimir

$$J^2 = J^+J^- + (J^3)^2 - J^3.$$  \tag{10.9}

The group contraction amounts to the rescaling

$$J^3 = \Omega \tilde{J}^3, \quad \Omega \to \infty.$$  \tag{10.10}

In the limit $\Omega \to \infty$ $\tilde{J}^3$ becomes a central element since

$$[\tilde{J}^3, J^\pm] = \pm \frac{J^\pm}{\Omega} \to 0.$$  \tag{10.11}

So the Hilbert space decomposes into sectors labeled by the $\tilde{J}^3$ quantum number. Let us denote this quantum number by $\tilde{m}$. The parent $J^3$ quantum number will be denoted by $m$ and the relation is

$$\tilde{m} = \frac{m}{\Omega}.$$  \tag{10.12}

Now focus on a sector labeled by a given $\tilde{m}$. In this sector the $J^\pm$ satisfy the following relation

$$[J^+, J^-] = 2\tilde{m}\Omega.$$  \tag{10.13}

We recognize this to be the algebra of a harmonic oscillator by defining

$$J^\pm = \sqrt{2\tilde{m}\Omega} \tilde{J}^\pm.$$  \tag{10.14}
which satisfy in the limit $\Omega \to \infty$

\[ [\tilde{J}^3, \tilde{J}^\pm] = 0, \quad [\tilde{J}^+, \tilde{J}^-] = 1. \quad (10.15) \]

So now we have the algebra of a harmonic oscillator and another central element. Clearly the representations of the harmonic oscillator are very different from representations of $SU(2)$. For one thing, they are infinite dimensional. But what really happened here is just a relabeling of the Hilbert space. Let us look at the Casimir (10.9) in the tilded variables. In the limit we are considering the $\tilde{m}$ is just a number in each superselection sector, so we can just as well label the representations by

\[ \hat{N} = \tilde{J}^+ \tilde{J}^-, \quad (10.16) \]

which is the number operator, or the Hamiltonian of the harmonic oscillator. Thus, $\tilde{J}^\pm$ are now ladder operators of (what is left of) the $SU(2)$ Casimir. This means that they take us from one $SU(2)$ representation to another along an infinite “equal $\tilde{J}^3$ line.” So now instead of a “symmetry algebra” we have a “spectrum generating algebra” since the operators do not commute with the Hamiltonian. But we do not lose states. All the states are there but under different “names” (quantum numbers). Note that all the states with finite $J^3$ quantum numbers end up in the zero mode sector of the contracted algebra due to (10.12).

We believe a similar phenomenon occurs in the contraction of the $SO(2, 4) \times SO(6)$ to the pp-wave. The pp-wave algebra has the same number of generators and the same number of supersymmetries as the original $AdS_5 \times S^5$. Some of the symmetries do not commute with the lightcone Hamiltonian but seen as a spectrum generating algebra it has to generate the full Hilbert space. It simply organizes it differently. The “missing” states in the pp-wave with respect to $AdS_5 \times S^5$ should map to the zero mode sector $p_- = 0$ since they did not scale fast enough to “keep up” with the diverging denominator in (8.14).

10.3. Matrices and quivers.

We end this section by pointing out an analogy between a class of operators discussed in [15] and the matrices of M(atrix) theory. The size of the matrices in M(atrix) theory with $N$ units of longitudinal momentum [17] is $N \times N$. In our case the $q$ units of longitudinal momentum (8.21) relate to winding around a quiver diagram [13]. The $|q = 1, m = 0)$
states introduced in [15] wind once around the quiver diagram (see equation (27) and figure 2. in [15])

\[ |q = 1, m = 0\rangle \sim \text{Tr}(A_1 A_2 \ldots A_M), \quad (10.17) \]

with \( A_i \) the bi-fundamental fields in the \( (N, \bar{N}) \) of \( SU(N)_i \times SU(N)_{i+1} \). The general \( |q, m = 0\rangle \) state winds \( q \) times around the quiver.

\[ |q, m = 0\rangle \sim \text{Tr}(A_1 A_2 \ldots A_M \ldots \ldots A_1 A_2 \ldots A_M), \quad (10.18) \]

It is suggestive to relate (10.17) with \( 1 \times 1 \) matrices in M(atrix) theory and (10.18) with \( q \times q \) matrices. Perhaps the analogue of a “block diagonal” matrix is a “multi-trace” operator of the general form (e.g. in the case of minimal blocks)

\[ |q, m = 0\rangle \sim \text{Tr}(A_1 A_2 \ldots A_M)\text{Tr}(A_1 A_2 \ldots A_M) \ldots \ldots \text{Tr}(A_1 A_2 \ldots A_M), \quad (10.19) \]

It would be very interesting to follow this suggestion further by interpreting the mixing between single and multi trace operators of this form as interactions between partons. It is also interesting to understand the role of the winding modes \( m \neq 0 \) from this point of view.

11. Relation to Maldacena’s limit.

The Penrose limit blows up the vicinity of a null geodesic to become the whole space. This is very reminiscent of the “near horizon limit” (NHL) of Maldacena. Note that the horizon is also a null hypersurface. Indeed the NHL is exactly analogous to the Penrose limit where instead of the null geodesic there is the null hypersurface constituting the horizon. The scaling introduced by Maldacena, namely

\[ \alpha' \to 0, \quad \frac{r}{\alpha'} = m^2 s \sim \text{fixed} \quad (11.1) \]

is very reminiscent of the one investigated in this paper (5.2)-(3.12)-(4.7). In Maldacena’s limit the radial coordinate transverse to the D3 branes plays the role of the longitudinal \( x^- \) coordinate. This radial coordinate is transverse to the null surface just as in the Penrose limit the \( x^- \) coordinate is transverse to the \( x^+ \) coordinate parameterizing the null geodesic. In both cases this direction is rescaled like the second power of the fundamental mass scale.
However, there are also differences, e.g. the clean “decoupling” property of Maldacena’s limit may not occur in this case. Also, in Maldacena’s case there is no compactification involved. This is analogous to BFSS and to BMN. Maldacena’s limit seems to be related to a further generalization of DLCQ in the infinite longitudinal momentum limit. We suspect that both procedures are special cases of a general rule (see also [32,33]).

12. Summary.

In this paper we argued that string/M theory on the background of the Penrose limit of a spacetime is a generalization of the DLCQ procedure introduced by Seiberg and Sen. We analyzed the case of type IIB strings on the maximally supersymmetric pp-wave which is the Penrose limit of $AdS_5 \times S^5$ and argued that it is analogous to the BFSS M(atrix) theory. The Penrose limit of an appropriate orbifold space $AdS_5 \times S^5 / Z_M$ was understood as the analogue of Susskind’s finite $N$ M(atrix) theory. We used this perspective to explain some of the features of the BMN theory. We feel this gives better understanding of the nature of the BMN duality.

Acknowledgments

I would like to thank Emiliano Imeroni, Barak Kol, Kostas Skenderis and Jelper Striet for discussions and suggestions. I am very pleased to thank Riccardo Argurio for many helpful comments on the manuscript. Special thanks go to Jan de Boer for many discussions, suggestions and sharp questions as well as for comments on the manuscript. This work is supported by a Clore fellowship.
References

[1] D. Berenstein, J. M. Maldacena and H. Nastase, “Strings in flat space and pp waves from N = 4 super Yang Mills,” JHEP 0204 (2002) 013 ; [hep-th/0202021].
[2] M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, “Penrose limits and maximal supersymmetry,” Class. Quant. Grav. 19 (2002) L87 ; [hep-th/0201081]. M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, “A new maximally supersymmetric background of IIB superstring theory,” JHEP 0201 (2002) 047 ; [hep-th/0110242]. J. Figueroa-O’Farrill and G. Papadopoulos, “Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory,” JHEP 0108 (2001) 036 ; [hep-th/0105308].
[3] R. R. Metsaev and A. A. Tseytlin, Phys. Rev. D65 (2002) 126004 ; [hep-th/0202109]. R. R. Metsaev, Nucl. Phys. B625 (2002) 70–96 ; [hep-th/0112044].
[4] J. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231 ; Int. J. Theor. Phys. 38 (1998) 1113 ; [hep-th/9711200].
[5] J. G. Russo and A. A. Tseytlin, JHEP 0209 (2002) 035 ; [hep-th/0208114]. J. G. Russo and A. A. Tseytlin, JHEP 0204 (2002) 021 ; [hep-th/0202179]. R. R. Metsaev and A. A. Tseytlin, Phys. Rev. D 65 (2002) 126004 ; [hep-th/0202109].
[6] K. Sfetsos, Phys. Lett. B 324 (1994) 335 ; [hep-th/9311010]. D. I. Olive, E. Rabinovici and A. Schwimmer, Phys. Lett. B 321 (1994) 361 ; [hep-th/9311081].
[7] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Nucl. Phys. B 636 (2002) 99 ; [hep-th/0204051].
[8] R. Penrose, “Any space-time has a plane wave as a limit,” in Differential geometry and relativity, Reidel, Dordrecht (1976) 271–275.
[9] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323 (2000) 183 ; [hep-th/9905111]. E. D’Hoker and D. Z. Freedman, “Supersymmetric gauge theories and the AdS/CFT correspondence,” ; [hep-th/0201253].
[10] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2 (1998) 253 ; [hep-th/9802150].
[11] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428 (1998) 105 ; [hep-th/9802109].
[12] N. R. Constable et al., “Pp-wave string interactions from perturbative Yang–Mills theory,” JHEP 07 (2002) 017 ; [hep-th/0205089].
[13] N. Seiberg, “Why is the matrix model correct?,” Phys. Rev. Lett. 79 (1997) 3577 ; [hep-th/9710009].
[14] A. Sen, “D0 branes on T(n) and matrix theory,” Adv. Theor. Math. Phys. 2 (1998) 51 ; [hep-th/9709220].
[15] S. Mukhi, M. Rangamani and E. Verlinde, “Strings from quivers, membranes from moose,” JHEP 0205 (2002) 023; hep-th/0204147.

[16] S. Weinberg, Phys. Rev. 150 (1966) 1313. J. B. Kogut and L. Susskind, Phys. Rept. 8 (1973) 75.

[17] L. Susskind, “Another conjecture about M(atrix) theory,”; hep-th/9704080.

[18] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “M theory as a matrix model: A conjecture,” Phys. Rev. D 55 (1997) 5112; hep-th/9610043.

[19] R. Helling, J. Plefka, M. Serone and A. Waldron, “Three graviton scattering in M-theory,” Nucl. Phys. B 559 (1999) 184; hep-th/9905183.

[20] W. Taylor, “M(atrix) theory: Matrix quantum mechanics as a fundamental theory,” Rev. Mod. Phys. 73 (2001) 419; hep-th/0101120.

[21] R. Dijkgraaf, E. Verlinde and H. Verlinde, “Matrix string theory,” Nucl. Phys. B 500 (1997) 43; hep-th/9703030.

[22] T. Banks and N. Seiberg, “Strings from matrices,” Nucl. Phys. B 497 (1997) 41; hep-th/9702187.

[23] L. Motl, “Proposals on nonperturbative superstring interactions”; hep-th/9701025.

[24] R. Gueven, “Plane wave limits and T-duality,” Phys. Lett. B 482 (2000) 255; hep-th/0005061.

[25] M. Bertolini, J. de Boer, T. Harmark, E. Imeroni and N. A. Obers, “Gauge theory description of compactified pp-waves,” JHEP 0301 (2003) 016; hep-th/0209201.

[26] L. Susskind and E. Witten, ; hep-th/9805114.

[27] C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, “A new double-scaling limit of N = 4 super Yang-Mills theory and PP-wave strings,” Nucl. Phys. B 643, 3 (2002); hep-th/0205033.

[28] E. Inonu and E. P. Wigner, Proc. Nat. Acad. Sci. 39 (1953) 510.

[29] M. Hatsuda, K. Kamimura and M. Sakaguchi, Nucl. Phys. B 632 (2002) 114; hep-th/0202190. M. Hatsuda, K. Kamimura and M. Sakaguchi, Nucl. Phys. B 637 (2002) 168; hep-th/0204002.

[30] S. Weinberg, “The Quantum Theory Of Fields. Vol. 1: Foundations”

[31] J. Gomis and H. Ooguri, “Non-relativistic closed string theory” J. Math. Phys. 42 (2001) 3127; hep-th/0009187. U. H. Danielsson, A. Guijosa and M. Kruczenski, “IIA/B, wound and wrapped,” JHEP 0010, 020 (2000); hep-th/0009182.

[32] F. Antonuccio, A. Hashimoto, O. Lunin and S. Pinsky, JHEP 9907 (1999) 029; hep-th/9906087. I. Chepelev, Phys. Lett. B 453 (1999) 245; hep-th/9901033.

[33] J. Polchinski, “M-theory and the light cone,” Prog. Theor. Phys. Suppl. 134 (1999) 158; hep-th/9903163.