ELECTRO-MAGNETIC NUCLEON FORM FACTORS AND THEIR SPECTRAL FUNCTIONS IN SOLITON MODELS

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Abstract

It is demonstrated that in simple soliton models essential features of the electro-magnetic nucleon form factors observed over three orders of magnitude in momentum transfer $t$ are naturally reproduced. The analysis shows that three basic ingredients are required: an extended object, partial coupling to vector mesons, and relativistic recoil corrections. We use for the extended object the standard skyrmion, one vector meson propagator for both isospin channels, and the relativistic boost to the Breit frame. Continuation to timelike $t$ leads to quite stable results for the spectral functions in the regime from the 2- or 3-pion threshold to about two rho masses. Especially the onset of the continuous part of the spectral functions at threshold can be reliably determined and there are strong analogies to the results imposed on dispersion theoretic approaches by the unitarity constraint.

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I. INTRODUCTION

Topological soliton models for structure and dynamics of baryons are based on effective nonlinear lagrangians for selected mesonic degrees of freedom. These usually comprise the pseudoscalar Goldstone boson octet of spontaneously broken chiral symmetry, but also the light vector and axial vector mesons have been included.

A decisive advantage of the soliton concept as compared to models where explicit point-like fermion fields are coupled to meson and gauge fields is the fact that already in leading classical approximation the spatial structure of the baryon as an extended object is obtained from the underlying effective action. Therefore all types of form factors can readily be extracted from the models, and comparison with the experimentally observed dependence on momentum transfer presents a stringent test for the resulting spatial profiles. Specifically, the electro-magnetic form factors of the nucleon for which we expect a wealth of precise data in the few (GeV/c) region from the new generation of electron accelerators pose a severe challenge for chiral soliton models.

Shortly after the initial work of [1] the e.m. nucleon form factors have been evaluated for various versions of effective meson models [2–4] and various sets of parameters, mainly for momentum transfers $Q < 1$ GeV/c. The conclusion was that the resulting $Q^2$-dependence follows roughly the standard dipole form, and that reasonable values for radii require the presence of explicit vector mesons. Unfortunately, in cases where these results have been included in more detailed comparisons with new data (e.g. in [9,10]), they have left the prejudice that soliton form factors are not satisfactory. However, for momentum transfers $Q > 1$ GeV/c it is important to incorporate relativistic kinematical corrections into the form factors. This is crucial for quark bag or cluster models [5–10] as well as for soliton models [11]. The implementation of these corrections is fairly easy for solitonic nucleons due to the Lorentz covariance of the underlying field equations.

The proton magnetic form factor is quite accurately known up to $30$ (GeV/c)$^2$ and it is not difficult to show [12] that with suitable choice of coupling constants the relativistically corrected soliton form factor calculated from a minimal vector meson model follows the data closely over more than three orders of magnitude in $Q^2$. This may be a bit disappointing because one might have hoped that in this region of high momentum transfer the characteristic shape of form factors could reflect dynamical features of QCD [13]. On the other hand, a similar conclusion had already been reached by Höhler [14] through a dispersion-theoretic fit of the spectral functions which included a continuous part to account for the Frazer-Fulco unitarity constraint [15].

In the following we would like to analyze which basic qualitative features of the e.m.
form factors and their spectral functions can be expected from the soliton concept for nucleons and to which extend these may correspond to observed facts. In order to expose the typical features of solitonic e.m. form factors in the most transparent way it is necessary to choose the essential ingredients as simple as possible. We therefore do not strive for optimal agreement with the data for some sophisticated model, but instead choose the most simple construction: $i$) a purely pionic soliton created through the standard Skyrme term; this represents the extended object with its spatial structure defined through the chiral profile $F(r)$; $ii$) its minimal coupling to the electro-magnetic field is partly mediated through vector mesons of mass $m_V$ represented by one Klein-Gordon propagator; $iii$) relativistic boost
factors to the Breit frame account for kinematic corrections.

In section II we show that these three ingredients are sufficient to produce for $Q^2 < 1$ (GeV/c)$^2$ perfect agreement with the standard dipole and for $Q^2 > 1$ (GeV/c)$^2$ the characteristic deviations from the dipole form. We also get indications at which point more precise data could lead to discrepancies.

Of course, it is most interesting to try to learn something about the spectral functions which underlie the resulting form factors. Especially intriguing is the question if and what the Skyrme soliton knows about the Frazer-Fulco constraint. A direct investigation of this problem by relating the pion-baryon phase shifts to the timelike form factor appears difficult, because the $\pi$-$N$ S-matrix in the soliton formulation depends only on one variable. Earlier analyses of the form factors for timelike momentum transfers [3] are not helpful in this respect because the method of analytic continuation used there [16], provides only averaged form factors with an averaging interval that conceals all detailed structures. Unfortunately, even within this simple model the continuation into the timelike regime remains ill-posed; but it seems that obtaining the spectral functions as Laplace inverse of the spatial densities provides rather stable results in the most interesting region from threshold to about 2 $m_\rho$. We discuss this procedure in detail in section III for the electric form factors; the magnetic spectral functions then are easily obtained by differentiation or integration. Once the spectral functions of the plain skyrmion are determined in its restframe, the modifications due to partial coupling to vector mesons, and the boost to the Breit frame pose no additional problem. They are discussed in sections IV and V.

II. RELATIVISTIC FORM FACTORS IN THE SOLITON MODEL

We create the nucleon as topological soliton through the standard Skyrme model [17] ($U \in SU(2)$)

$$\mathcal{L}(U) = \mathcal{L}^{(2)} + \mathcal{L}^{(4)}$$

$$\mathcal{L}^{(2)} = \frac{f_\pi^2}{4} \int \left( -tr L_\mu L^\mu + m_\pi^2 tr(U + U^\dagger - 2) \right) d^3x. \quad (2.2)$$

$$\mathcal{L}^{(4)} = \frac{1}{32e^2} \int tr[L_\mu, L_\nu]^2 d^3x \quad (2.3)$$

where $L_\mu$ denotes the chiral gradients

$$L_\mu = U^\dagger \partial_\mu U. \quad (2.4)$$

The pion decay constant $f_\pi$ and the pion mass $m_\pi$ have their physical values $f_\pi=93$ MeV and $m_\pi=138$ MeV, the Skyrme parameter $e$ is taken at $e=4.25$ which leads to the correct Delta-nucleon mass difference of $M_\Delta - M_N=295$ MeV.

The static soliton solution, the Skyrme hedgehog $U = \exp(i\vec{r} \cdot \vec{r} F(r))$ is characterized by the numerically determined chiral profile $F(r)$ which carries all the information about the spatial structure of the extended object.
In the Breit frame for spacelike $t = -Q^2$ the isoscalar ($T=0$) and isovector ($T=1$) electric and magnetic (Sachs) form factors are given as Fourier transforms of baryon density $B_0(r)$ (for $T=0$) and moment-of-inertia density $B_1(r)$ (for $T=1$):

\begin{align}
G_E^{T=0}(-Q^2) &= \frac{1}{2} \int d^3r \, j_0(Qr) B_0(r) \\
G_M^{T=0}(-Q^2) &= \frac{3}{2r_B^2} \int d^3r \, \frac{j_1(Qr)}{Qr} r^2 B_0(r) \\
G_E^{T=1}(-Q^2) &= \frac{1}{2} \int d^3r \, j_0(Qr) B_1(r) \\
G_M^{T=1}(-Q^2) &= \frac{3}{2} \int d^3r \, \frac{j_1(Qr)}{Qr} B_1(r)
\end{align}

Expressed in terms of the chiral profile $F(r)$ the densities $B_0(r)$ and $B_1(r)$ are

\begin{align}
B_0(r) &= -\frac{1}{2\pi^2} F' \sin^2 F \\
B_1(r) &= \frac{2}{3\Theta} \left[ f^2 \sin^2 F + \frac{1}{e^2} \sin^2 F \left( F'^2 + \frac{\sin^2 F}{r^2} \right) \right]
\end{align}

with normalization $\int d^3r \, B_0(r) = 1$, and $\int d^3r \, B_1(r) = 1$.

For the magnetic form factors the isoscalar and isovector magnetic moments

\begin{align}
2\mu_0 = \mu_p + \mu_n &= \frac{M_N}{3\Theta} r_B^2, \\
2\mu_1 = \mu_p - \mu_n &= \frac{2}{3} M_N \Theta
\end{align}

(with baryonic square radius $r_B^2 = \int d^3r \, r^2 B_0(r)$) have, respectively, been divided out such that they both satisfy the same normalization condition as their electric counterparts

\begin{align}
G_M^0(0) = G_M^1(0) = \frac{1}{2}.
\end{align}

The Skyrme term $\mathcal{L}^{(4)}$ in (2.3) effectively represents the lowest local approximation to explicit inclusion of higher resonances, notably $\rho$ and $\omega$ mesons [18–20]. Propagating vector mesons introduce additional pole structures near the vector meson mass into the form factors, which cannot be accounted for by the local Skyrme term alone in a satisfactory way. However, instead of dealing with the ambiguities and numerous parameters of vector meson models it appears more appropriate to incorporate their effect into one common factor $\Lambda(t)$ to be multiplied with the pure Skyrme model form factors (2.5)-(2.8),

\begin{align}
\Lambda(t) = \lambda \left( \frac{m_V^2}{m_V^2 - t} \right) + (1 - \lambda).
\end{align}

With the physical vector meson mass $m_V = 770$ MeV this factor contains one additional parameter $\lambda$ which allows to interpolate between complete vector meson dominance ($\lambda = 1$) and the purely pionic Skyrme model ($\lambda = 0$). The resulting form factors
\[ G_{E,M}^{0,1}(t) = \Lambda(t) \, G_{E,M}^{0,1(\pi)}(t) \]  
\[ (2.14) \]

can be shown to represent good approximations to results obtained in models with explicit vector mesons included. (For the coupling to the \( \omega \)-meson in the minimal model it is exact.)

Although the definitions \( (2.5)-(2.8) \) hold in the Breit frame only, expressions \( (2.9) \) and \( (2.10) \) for the densities contain the chiral profile \( F(r) \) as obtained in the soliton restframe. However, both frames coincide only for infinite soliton mass \( M_S \). For values of \( Q^2 \) near and beyond the actual soliton mass \( M_S \) kinematic corrections have to be included by employing the hedgehog soliton boosted to the Breit frame, where it moves with velocity \( v \)

\[ \gamma^2 = (1 - v^2)^{-1} = 1 + \frac{Q^2}{4M^2} = 1 - \frac{t}{4M^2} \]  
\[ (2.15) \]

with spacelike momentum transfer \( t = -Q^2 \leq 0 \).

An approximate way to implement these recoil corrections has been suggested by Ji [11]. The electric form factors are Lorentz scalars, therefore one has

\[ G_E(t) = G_E^{(nr)}(t/\gamma^2) \]  
\[ (2.16) \]

where \( G_E^{(nr)} \) is the non-relativistic form factor evaluated in the soliton restframe.

For the magnetic form factors the boost to the Breit frame introduces an additional factor \( \gamma^{-2} \)

\[ G_M(t) = \frac{1}{\gamma^2} G_M^{(nr)}(t/\gamma^2). \]  
\[ (2.17) \]

These boost factors are well-known in models for relativistic quark clusters [5]. There, however, \( G_E \) carries the same factor \( \gamma^{-2} \) as \( G_M \) due to the normalization of the spectator quarks.

The kinematical transformations \( (2.16) \) and \( (2.17) \) determine the asymptotic behavior of the relativistic form factors for large spacelike \( t \to -\infty \):

\[ G_E(t) \to \left[ G_E^{(nr)}(-4M^2) + \frac{4M^2}{\gamma^2} \frac{\partial}{\partial t} G_E^{(nr)}(-4M^2) \right] + \mathcal{O}\left(\frac{1}{t^2}\right) \]  
\[ (2.18) \]

\[ G_M(t) \to \left[ \frac{1}{\gamma^2} G_M^{(nr)}(-4M^2) \right] + \mathcal{O}\left(\frac{1}{t^2}\right). \]  
\[ (2.19) \]

These asymptotic forms show a rather undesirable feature of the boost transformations \( (2.16), (2.17) \) because they violate the superconvergence rule expected for e.m. form factors [21][22].

\[ \lim_{t \to -\infty} t G_{E,M}(t) = 0. \]  
\[ (2.20) \]

 Practically, however, the absolute values of \( G_{E,M}^{(nr)} \) and its derivative at \( t = -4M^2 \) are so small that up to \( Q^2 = 10 \) (GeV/c)^2 the influence of the undesired terms is not important.
This is demonstrated in figs.1,2 where the dashed lines show the e.m. form factors after subtraction of the terms written out in square brackets in eqs.(2.18), (2.19).

In principle, within tree approximation, the kinematic mass \( M \) must be identified with the classical soliton mass \( M_S \). Ideally, of course, \( M \) should coincide with the physical nucleon mass \( M_N \). However, it is known, that this difference is related to quantum corrections \([23,24]\) which probably also affect the existence and positions of zeros in the non relativistic form factors. Therefore, \( M \) is not really well defined in the model and could be used as additional parameter to minimize the undesired terms in (2.18), (2.19) by choosing \( M \) such that 

\[
G_M(-4M^2) = 0.
\]

In this way superconvergence could be enforced at least for \( G_M \). However, to keep the number of parameters as small as possible we strictly adhere to the tree approximation and identify in the following the kinematical mass \( M \) in the boost transformation with the soliton mass \( M_S \) which is \( M_S=1648 \text{ MeV} \) for the Skyrme model with \( \epsilon=4.25 \).

Thus the only parameter remaining in the model is \( \lambda \). It enters sensitively into the radii and interpolates between the purely pionic radii (which are generally too small) and the completely vector meson dominated radii (which are too big):

\[
<r^2> = \frac{6}{G(0)} \frac{\partial}{\partial t} G(0) = \frac{\lambda}{m_V^2} + <r^2>^{\text{pionic}}. \tag{2.21}
\]

(For the electric neutron radius we take

\[
<r^2>^n_N = 6 \frac{\partial}{\partial t} G^n_N(0) = \frac{1}{2}(<r^2>^0_N - <r^2>^1_E); \tag{2.22}
\]

here the vector meson contributions cancel).

The form factors shown in figs.1,2 are calculated with \( \lambda=0.75 \), i.e. the e.m. coupling is strongly but not completely vector meson dominated. The resulting square radii are (in \( \text{fm}^2 \)):

\[
r_B^2 = 0.233; \quad <r^2>^E_B = 0.756; \quad <r^2>^p_M = 0.717; \quad <r^2>^n_E = -0.228; \quad <r^2>^n_M = 0.746;
\]

Up to \( Q^2 \sim 0.5 \text{ (GeV/c)}^2 \), \( G_M^p, G_E^p, G_M^n \) follow the standard dipole \( G_D(t) = (1-t/0.71)^{-2} \) very closely. The magnetic form factors rise above the dipole in the region between 0.5 and 5 \( \text{(GeV/c)}^2 \), and fall below the dipole above 5 \( \text{(GeV/c)}^2 \). The neutron form factor \( G_M^n \) rises slightly above the proton \( G_M^p \), which seems to be in conflict with the data of \([25]\) for \( G_M^n \). In fig.2 for \( G_M^n \) in addition to more recent data points \([25,28]\) we have also included a collection of older data \([29,34]\). They seem to indicate significant deviations from the standard dipole parametrization also for small \( Q^2 \). But we find that at least up to 1 \( \text{(GeV/c)}^2 \) the ratio \( G_M^p/G_M^n \) stays constant with good accuracy, and it appears difficult to accommodate such differences between proton and neutron data. With explicit dynamical inclusion of vector mesons it is not difficult to find a parameter choice such that the rise of \( G_M^p \) above the dipole coincides closely with that observed in the data (see e.g. \([12]\)). but it should be remembered that both, isoscalar and isovector magnetic moments \( \mu_0 \) and \( \mu_1 \) are known to be subject to sizable quantum corrections \([23]\), which may also affect the shape of the form factors.

In \( G_E^p \) the corresponding rise above the dipole is suppressed and the form factor is dominated by a zero near 10 \( \text{(GeV/c)}^2 \). From our discussion of the boost transformation
it is clear that the high-$Q^2$ behavior of the ratios $G/G_D$ is sensitive to the position of the first zeros in the form factors (relative to $4M^2$). From (2.3) we expect that the first zero in $G_E$ occurs at lower $Q^2$ than the first zero in $G_M$. (It may be noted that in the plain Skyrme model $G_M$ is monotonously decreasing and has no zeros; however, inclusion of sixth-order terms or explicit vector mesons leads to small fluctuations and corresponding zeros also in $G_M$). As long as these conditions are not inverted by quantum corrections we would therefore conclude that the rapid fall-off in $G_E$ relative to $G_M$ is a typical and rather stable feature of soliton models. A similar behavior is, however, also observed in relativistic quark bag or cluster models (see e.g. [7,9,10]).

Data points for the electric neutron form factor extracted from $e-d$ scattering depend sensitively on the choice of the deuteron wave functions. This is illustrated in fig.2 for $G_E^n$ by the dotted lines which represent fits of [35] of the form

$$G_E^n(-Q^2) = -a\mu_n\frac{Q^2}{4M^2_N}G_D(-Q^2)(1 + b\frac{Q^2}{4M^2_N})^{-1},$$

(2.23)

where $a$ and $b$ depend on the choice of the n-p potential used in the deuteron wave function. Included in fig.2 is the data set of [35] for the Paris potential and an older data collection of [36] based on the Feshbach-Lomon wave functions. The dash-dotted line is the Galster fit [36] with $a=1$, $b=5.6$. The shape of the calculated $G_E^n$ (full line) follows closely the Galster parametrization. (The absolute values of $G_E^n$ and thus the value of the slope near $Q^2 = 0$ can easily be reduced by allowing $\lambda$ to be slightly different for isoscalar and isovector mesons. Again, however, without quantum corrections it is not very meaningful to adjust parameters to the very sensitive electric neutron square radius, which experimentally is near $<r^2>_E^{2n} = -0.12\text{ fm}^2$).

So, apart from choosing $\lambda=0.75$ we have made no attempt to further improve agreement with the experimental data. The results sufficiently demonstrate that this extremely simple model is capable to cover the essential features of the observed form factors. Significant quantum corrections are expected to the magnetic moments, to the radii, and to $M$. By the choice of $\lambda$ we have compensated their possible influence on the radii. The high-$Q^2$ behavior of the form factors is sensitive to the choice of $M$ and expected differences in loop corrections could possibly be absorbed into slightly different values of $M$ for different form factors. But here all form factors in figs.1 and 2 have been calculated with the same value of $M$ (the soliton mass). As long as rotational and loop corrections to the currents are not included, the accuracy shown in figs.1,2 is satisfactory and we now turn to the typical features of the spectral functions which underlie these form factors.

III. SPECTRAL FUNCTIONS FOR NUCLEON FORM FACTORS IN THE NONRELATIVISTIC SKYRME MODEL

It is generally assumed that e.m. form factors $G(t)$ fulfill unsubtracted dispersion relations

$$G(t) = \frac{1}{\pi} \int_{t_0}^{\infty} \frac{\Gamma(t')}{{t'}^2-t}dt'$$

(3.1)
where the spectral function $\Gamma(t) = \text{Im}\ G(t)$ is given by the boundary value of $G(t+i0)$ along the upper edge of the cut which extends from $t_0$ along the positive real axis in the $t$-plane. The lower limit is $t_0 = 4m^2_\pi$ for isovector and $t_0 = 9m^2_\pi$ for isoscalar form factors. The determination of spectral functions from experimental form factor data in the spacelike region $t < 0$ requires an analytical continuation to the cut which is known to be mathematically an ill-posed problem. Therefore such extrapolations have to be performed under certain stabilizing assumptions and the resulting spectral functions reflect these assumptions.

It is therefore of interest to find out what successful nucleon models may tell us about the structure of the spectral functions. But it should be noted that (unless the models provide analytical expressions for the form factors) in practice we face the same ill-posed problem of inverting (3.1) with $G(t)$ given with finite numerical accuracy for all values $t < t_0$. However, apart from the fact that the numerical errors of the ‘model data’ for $G(t)$ can be made (arbitrarily) small, the models in general will provide at least some analytical constraints on form factors (or on the underlying spatial densities) which will help to stabilize the analytic continuation.

In the following we analyze this procedure for the Sachs form factors of the simple Skyrme model. Inclusion of additional explicit vector meson poles and relativistic corrections then will be straightforward.

**A. Isoscalar electric $G_{E}^{0}(t)$**

We repeat that in the Breit frame for spacelike $t = -Q^2$ the isoscalar electric form factor is given by the Fourier transform of the baryon density $B_{0}(r)$

$$G_{E}^{0}(-Q^2) = \frac{1}{2} \int d^3r\ j_0(Qr)B_{0}(r) . \quad (3.2)$$

If we assume the existence of a function $\Gamma(t')$ such that the integral in (3.1) is convergent then spatial density $B_{0}(r)$ and spectral function $\Gamma(t')$ are related through

$$rB_{0}(r) = \frac{1}{\pi^2} \int_{\mu_0}^{\infty} e^{-\mu r}\mu\Gamma(\mu^2)d\mu \quad (3.3)$$

(with notation $t' = \mu^2$), i.e. the function $\mu\Gamma(\mu^2)$ is the Laplace inverse of $\pi^2rB_{0}(r)$. Although mathematically the problem (3.3) is no less ill-posed than (3.1), models generally supply analytical information for the behavior of $B_{0}(r)$ for $r \to 0$ and $r \to \infty$, in which case (3.3) appears better suited for obtaining the spectral functions.

Specifically, the baryon density $B_{0}(r)$ for the Skyrme hedgehog is given by

$$B_{0}(r) = -\frac{F'\sin^2 F}{2\pi^2r^2} \quad (3.4)$$

and the asymptotic form of the chiral angle $F(r)$ is

$$F(r) \xrightarrow{r \to \infty} \frac{A}{r^2}(1 + m_\pi r)e^{-m_\pi r} \quad (3.5)$$
where the constants $A$, $B$ are fixed by solving the nonlinear equation for the soliton profile. The strategy for constructing $\Gamma(\mu^2)$ then is to approximate the numerically determined density $rB_0(r)$ by a set of functions with known Laplace inverse such that the constraints imposed by (3.5), (3.6) are fulfilled and (3.3) is satisfied for all values of $r$.

Commonly, spectral functions are set up as sums of several discrete ‘monopoles’ at positions $\mu = \nu_0$ with strengths $a_0$

$$\Gamma^{(0)} (\mu^2) = \sum a_0 \delta(\mu^2 - \nu_0^2).$$

Similarly, one may consider sums of ‘dipoles’ at positions $\mu = \nu_1$ with strengths $a_1$

$$\Gamma^{(1)} (\mu^2) = \sum a_1 \nu_1^2 \delta'(\mu^2 - \nu_1^2).$$

Their contributions to $rB_0(r)$ according to (3.3) are of the form

$$2\pi^2 rB_0(r) = \left\{ \sum a_0 e^{-\nu_0 r} + \sum a_1 \left( \frac{\nu_1 r}{2} \right) e^{-\nu_1 r} \right\}.$$

In addition to (3.7), (3.8) we define higher-order pole structures in the spectral function

$$\Gamma^{(2)} (\mu^2) = a_2 \left( \nu_2^4 \delta''(\mu^2 - \nu_2^2) - \frac{1}{2} \nu_2^2 \delta'(\mu^2 - \nu_2^2) \right),$$

$$\Gamma^{(3)} (\mu^2) = a_3 \left( \nu_3^6 \delta''(\mu^2 - \nu_3^2) - \frac{3}{2} \nu_3^4 \delta'(\mu^2 - \nu_3^2) \right),$$

$$\Gamma^{(4)} (\mu^2) = a_4 \left( \nu_4^8 \delta''(\mu^2 - \nu_4^2) - 3 \nu_4^6 \delta'(\mu^2 - \nu_4^2) + \frac{3}{4} \nu_4^2 \delta''(\mu^2 - \nu_4^2) \right),$$

such that their contributions to $rB_0(r)$ are

$$2\pi^2 rB_0(r) = \left\{ \sum_{i=0,1,...} a_i \left( \frac{\nu_i r}{2} \right)^i e^{-\nu_i r} \right\}.$$

This convenient form proves very efficient for an accurate representation of numerically obtained functions $rB_0(r)$; we therefore prefer to use the pole combinations $\Gamma^{(i)}$ instead of the ‘pure multipoles’ $\delta^{(i)}$.

Evidently, however, the representation (3.12) does not allow to satisfy the asymptotic constraint (3.5) which requires that $rB_0(r)$ decreases for large $r$ like

$$2\pi^2 rB_0(r) \to A^3 \frac{N_4(r)}{r^8} e^{-3\pi r}$$

with degree-four polynomial

$$N_4(r) = (1 + m_\pi r)^2 \left( 2 + 2m_\pi r + (m_\pi r)^2 \right).$$

In order to correctly reproduce the asymptotic behavior of the spatial density a term like (3.13) has to be included on the right-hand side of (3.12). However, to be able to satisfy the
constraint (3.14) for \( r \to 0 \), the denominator in (3.13) has to be replaced by a degree-eight polynomial

\[
D_8(r) = \sum_{i=0}^{8} c_i r^i \quad \text{with} \quad c_8 = 1. \tag{3.15}
\]

Then, the analytical knowledge about \( rB_0(r) \) and its derivatives at \( r = 0 \) fixes the amplitudes of monopoles, dipoles,... in terms of the coefficients \( c_i \). In contrast to the discrete structures (3.7)-(3.11) of the spectral function which underlie the right-hand side of (3.12) the rational function \( N_i(r)/D_8(r) \) corresponds to a continuous part \( \Gamma_{\text{cont}}(\mu^2) \) of the spectral distribution \( \Gamma(\mu^2) \) which (due to the exponential factor \( \exp(-3m_\pi r) \)) is nonzero only for \( \mu \geq 3m_\pi \). Its form near threshold is determined by the asymptotics (3.5).

Through an extended analysis of the nonlinear differential equation for \( F(r) \) additional terms in (3.3) for \( r \to \infty \) can be determined which lead to improved numerator polynomials \( N_k(r) \) in (3.13) with \( k > 4 \). Further terms in the expansion of \( \sin^2 F \) in (3.4) carry the exponential factor \( \exp(-5m_\pi r) \) and create additional continuous contributions along the cut for \( \mu \geq 5m_\pi \). The evaluation of further terms of \( F(r) \) in (3.3) for \( r \to 0 \) provide additional equations to fix more amplitudes in (3.12). In this way increasingly accurate approximations to \( rB_0(r) \) could be constructed.

There is, however, a severe constraint which has to be imposed on acceptable denominator polynomials \( D_k(r) \): the Laplace inverse of \( N_i/D_k \) (where \( k > i \)) contains the exponentials \( \exp(\alpha_i \mu) \) where \( \alpha_i \) are the roots of \( D_k \). Therefore the integral in (3.3) converges only for \( r > \Re \alpha_{\text{max}} \), where \( \alpha_{\text{max}} \) is the root with the largest positive real part. Consequently all roots of acceptable denominator polynomials must lie in the left complex half plane. (For simple roots the imaginary axis is still acceptable).

The fact that the ”data”, the numerically obtained function \( B_0(r) \), is known with great precision allows to create very satisfactory fits by including a sufficient number of terms at different positions \( \nu_i \) in the sum on the r.h.s. of

\[
2\pi^2 rB_0(r) = A^3 \frac{N_4(r)}{D_8(r)} e^{-3m_\pi r} + \sum_{i=0,1,...} a_i \left( \frac{\nu_i r}{2} \right)^i e^{-\nu_i r} \tag{3.16}
\]

and constrain their amplitudes \( a_i \) through boundary conditions at \( r = 0 \)

\[
\begin{align*}
0 &= 2A^3 \frac{1}{c_0} + a_0 \\
2\pi^2 B_0(0) &= -2A^3 \frac{c_1}{c_0} - a_0 \nu_0 + \frac{1}{2} a_1 \nu_1 \\
0 &= -4A^3 \frac{m_\pi^2 c_2^2}{c_0} + a_0 \nu_0^2 - a_1 \nu_1^2 + \frac{1}{2} a_2 \nu_2^2 \\
6\pi^2 B_0''(0) &= 2A^3 \left[ 3m_\pi^2 c_0^3 + 6m_\pi^2 c_0^2 c_1 + 12c_0 c_1 c_2 - 6c_0^2 c_3 - 6c_1^2 \right] - a_0 \nu_0^3 + \frac{3}{2} a_1 \nu_1^3 - \frac{3}{2} a_2 \nu_2^3 + \frac{3}{4} a_3 \nu_3^3.
\end{align*}
\tag{3.17}
\]

or, explicitly,

\[
0 = 2A^3 \frac{1}{c_0} + a_0 \tag{3.18}
\]

\[
2\pi^2 B_0(0) = -2A^3 \frac{c_1}{c_0} - a_0 \nu_0 + \frac{1}{2} a_1 \nu_1 
\]

\[
0 = -4A^3 \frac{m_\pi^2 c_2^2}{c_0} + a_0 \nu_0^2 - a_1 \nu_1^2 + \frac{1}{2} a_2 \nu_2^2 
\]

\[
6\pi^2 B_0''(0) = 2A^3 \left[ 3m_\pi^2 c_0^3 + 6m_\pi^2 c_0^2 c_1 + 12c_0 c_1 c_2 - 6c_0^2 c_3 - 6c_1^2 \right] - a_0 \nu_0^3 + \frac{3}{2} a_1 \nu_1^3 - \frac{3}{2} a_2 \nu_2^3 + \frac{3}{4} a_3 \nu_3^3.
\]
The first of these equations shows that at least one discrete monopole is necessary to allow for $rB_0(r)|_a = 0$, and that its discrete monopole strength $a_0$, however, is completely compensated by the continuous part of the spectral function, because we have from (3.16) and (3.3) that

$$\int \Gamma_{\text{cont}}(\mu^2)d\mu^2 = A^3 \frac{N_4}{D_8}|_{r=0} = \frac{2A^3}{c_0}. \quad (3.19)$$

So, evidently, the origin of the dipole nature of the form factor lies in the fact that the density $B_0(r)$ has a finite limit at $r=0$.

Due to the fact that the set of functions which appear on the r.h.s. of (3.16) is neither orthogonal nor complete, there remain severe ambiguities in their selection. From the physical point of view we might prefer to include only a set of monopoles at different positions. This, however, proves very inefficient for an accurate fit. Including more terms in (3.16) leads to improved fits; however, as usual, it does not make sense to include many more terms, because different fits of comparable (improved) accuracy may differ appreciably in their resulting distribution of multipole strengths. We find the best results (with respect to stability and accuracy) by including the four terms with $i = 0, \ldots, 3$ in (3.16), i.e. just one term for each value of $i$ at positions $\nu_i$ with the four amplitudes $a_i$ fixed through the four relations (3.18). The resulting pole structure can only be interpreted as an effective discrete representation of some undetermined underlying strength distribution. As is evident from (3.18), location and strength of these effective pole structures are in close connection to the additional continuous part which is severely constrained through the asymptotic behavior of the spatial density. Thus (3.16) with $i = 0, \ldots, 3$ appears as a simple and natural form which satisfies the asymptotics for $r \to 0$ and $r \to \infty$ and allows for a very accurate fit of $rB_0(r)$.

Allowing the roots of $D_8$ to vary freely in the left complex half plane leads to two distinct double roots and their complex conjugates, i.e.

$$D_8 = |(r - \alpha_1)^2(r - \alpha_2)^2|^2. \quad (3.20)$$

Imposing this form for $D_8$ leaves two complex roots $\alpha_1$, $\alpha_2$ and four pole positions $\nu_i$, $(i = 0, \ldots, 3)$ as parameters for the fit (3.16). We find that the position of the roots $\alpha_1$, $\alpha_2$ is quite stable with respect to different pole combinations, and the smallest $\chi^2$ is obtained for the combination given in (3.16).

With (3.20) the decomposition of $N_4/D_8$ into partial fractions is

$$\frac{N_4(r)}{D_8(r)} = \frac{\gamma_1}{r - \alpha_1} + \frac{\gamma_{11}}{(r - \alpha_1)^2} + \frac{\gamma_2}{r - \alpha_2} + \frac{\gamma_{22}}{(r - \alpha_2)^2} + \text{c.c.} \quad (3.21)$$

where the coefficients $\gamma_i$ satisfy relations like

\[
\begin{align*}
\text{Re} \sum_{i=1,2} \gamma_i &= 0 \\
\text{Re} \sum_{i=1,2} (\gamma_{ii} + \alpha_i \gamma_{i}) &= 0 \\
\text{Re} \sum_{i=1,2} (2\alpha_i \gamma_{ii} + \alpha_i^2 \gamma_{i}) &= 0
\end{align*}
\quad (3.22)
\]
and so forth, which imply that the Laplace inverse \( \gamma(\mu) \) of \( N_4/D_8 \)

\[
\gamma(\mu) = 2 \Re \sum_{i=1,2} (\gamma_i + \gamma_{ii} \mu) e^{\alpha_i \mu}
\]  

(3.23)

satisfies at threshold \( \mu = 0 \)

\[
\gamma(0) = 0; \quad \gamma'(0) = 0; \quad \gamma''(0) = 0; \quad \gamma'''(0) = m_\pi^4; \quad \text{etc.}
\]  

(3.24)

(These relations, of course, follow also quite generally by comparing the coefficients for positive powers of \( r \) for \( r \to \infty \) in the definition of \( \gamma(\mu) \)

\[
N_i(r) = D_k(r) \int_0^\infty e^{-\mu r} \gamma(\mu) d\mu
\]  

(3.25)

for polynomials \( N_i(r) \) and \( D_k(r) \).) From (3.3) and (3.16) the continuous part of \( \Gamma(\mu^2) \) then is given by

\[
\Gamma_{\text{cont}}(\mu^2) = \frac{A^3}{2 \mu} \gamma(\mu - 3m_\pi)
\]  

(3.26)

for \( \mu \geq 3m_\pi \). At threshold \( \mu^2 \to (3m_\pi)^2 \) its behavior is fixed by (3.22) as

\[
\Gamma_{\text{cont}}(\mu^2) \to \frac{A^3}{6^5} (\mu^2 - (3m_\pi)^2)^3.
\]  

(3.27)

It should be noted that this onset of the spectral density at threshold is completely determined by the form of the baryon density (3.4) and the asymptotics of the chiral angle (3.5) and is independent of the discrete structures added in (3.16). It cannot be obtained in parametrizations of spectral functions in terms of discrete pole structures only, but the existence of this continuous part affects positions and amplitudes of poles through equations like (3.18).

Altogether, the isoscalar electric spectral function is obtained in the form

\[
\Gamma^{I=0}(t) = \Gamma_{\text{cont}}(t) + \sum_{i=0}^3 \Gamma^{(i)}(t)
\]  

(3.28)

where the continuous part is given by (3.26) with (3.23) and the amplitudes of the discrete pole structures (3.7)-(3.10) are fixed through (3.18).

For \( \mu^2 \gg (3m_\pi)^2 \) the continuous part \( \Gamma_{\text{cont}}(\mu^2) \) is dominated by the exponential \( \exp \alpha_i \mu \) corresponding to the root \( \alpha_i = -\varepsilon_i \pm i \varphi_i \) of \( D_8 \) with the smallest (absolute) value of its real part \( \varepsilon_i \), i.e. \( \Gamma_{\text{cont}} \) oscillates with slowly decreasing amplitude. It turns out that the imaginary part \( \varphi_i \) (i.e. the period of the oscillations) is sharply fixed by the fit, while variations of the real part \( -\varepsilon_i \) within a limited range affect the quality of the fit only very little. (By putting a constraint on \( \varepsilon_i \) the oscillations thus can be damped without severe consequences for the fit).

As an example we present the result of such an analysis for the standard Skyrme model (with \( f_\pi = 93 \text{ MeV} \) and \( e = 4.25 \)). A typical fit leads to the (two-fold) roots (in units of (inverse) \( m_\pi = 770 \text{ MeV} \))
Here the real part of \( \alpha_2 \) has been kept fixed at \(-0.5\). If \( \varepsilon_2 \) is allowed to move freely it will slowly approach \( \varepsilon_2 = 0 \) with only marginal improvement of the fit and little change in pole positions. The choice (3.29) efficiently cuts down the oscillations in the continuous part of the spectral function for large \( \mu \) (see fig.3). The pole locations and strengths for the same fit are listed in table I.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\Gamma^{(i)} & i = 0 & i = 1 & i = 2 & i = 3 \\
\nu_i [m_\phi] & 1.61 & 5.14 & 1.93 & 2.09 \\
\alpha_i [m_\phi^2] & -8.80 & 0.47 & 15.99 & -4.89 \\
\hline
\end{array}
\]

**TABLE I.** Positions and strengths of the discrete structures for the isoscalar electric spectral function (in the soliton restframe)

A very stable feature of this result is the discrete monopole strength \( \Gamma^{(0)}(t) \) near 1.6 \( m_\phi \). Of course, this discrete monopole strength is completely compensated by the continuous part \( \Gamma_{\text{cont}}(t) \) which is very small near \( t \approx m_\phi^2 \) but has a first pronounced maximum near \( (2.5m_\phi)^2 \) (see fig.3). Also this feature of \( \Gamma_{\text{cont}}(t) \) is very stable. There is no discrete low-lying dipole strength. But \( \Gamma^{(2)} \) contains dipole strength and strong quadrupole strength near 1.5 GeV. Together with some octupole strength this may indicate broader underlying structures, although the details are certainly peculiar to our specific choice of allowed pole structures in the ansatz (3.16). Altogether it appears that in this way we obtain a quite reliable picture of the spectral function which underlies the isoscalar electric form factor of the Skyrme model up to about 2 - 3 \( m_\phi \).

**B. Isovector electric \( G_1^E(t) \)**

For the isovector electric form factor

\[
G_1^E(-Q^2) = \frac{1}{2} \int d^3r \, j_0(Qr)B_1(r)
\]  

(3.30)

the baryon density \( B_0(r) \) in (3.2) and (3.3) is replaced by the (isorotation) inertia density

\[
B_1(r) = \frac{2}{3\Theta} \sin^2 F \left( f_\pi^2 + \frac{1}{e^2} (F^2 + \frac{\sin^2 F}{r^2}) \right),
\]

(3.31)

normalized by the moment of inertia \( \Theta \) such that \( \int B_1(r) d^3r = 1 \). The asymptotic constraint (3.5) then suggests a rational function \( N_2(D_4) \) on the right-hand side in the representation of \( B_1(r) \) (corresponding to (3.19))

\[
2\pi^2 B_1(r) = f_1 \frac{N_2(r)}{D_4(r)} e^{-2m_\pi r} + \frac{1}{r} \left\{ \sum a_0 e^{-\nu_0 r} + \sum a_1 \frac{\nu_1}{2} r e^{-\nu_1 r} + \cdots \right\}
\]

(3.32)

with
\[ f_1 = \frac{4 \pi^2}{3 \Theta} f_\pi^2 A^2 \]

\[ N_2(r) = (1 + m_\pi r)^2 \]

\[ D_4 = \sum_{i=0}^{4} c_i r^i \quad \text{with} \quad c_4 = 1. \] (3.33)

The boundary conditions at \( r = 0 \) are

\[ B_1(0) = 0; \quad B_1'(0) = 0; \quad B_1''(0) = 4 \pi^2 \Theta B_2^2(\pi^2 + 2 \frac{\pi^2}{e^2} B^2); \quad B_1'''(0) = 0; \] (3.34)

These conditions exclude discrete monopole strength. It turns out that at least four discrete structures are needed for satisfactory fits. Again we find that by far the best result is obtained if we allow for just one term of each power of \( r \) in (3.32), i.e. we use

\[ 2\pi^2 B_1(r) = f_1 \frac{N_2}{D_4} e^{-2m_\pi r} + \frac{1}{r} \left( \sum_{i=1}^{4} a_i \left( \frac{\nu_i}{2} r \right) e^{-\nu_i r} \right), \] (3.35)

where the four amplitudes \( a_i \) are fixed through relations (3.34), which read explicitly

\[
\begin{align*}
0 &= f_1 \frac{1}{c_0} + a_1 \frac{\nu_1}{2} \\
0 &= -f_1 \frac{c_1}{c_0} - a_1 \frac{\nu_1^2}{2} + a_2 \frac{\nu_2^2}{4} \\
2\pi^2 B_1''(0) &= -2f_1 \frac{(m_\pi^2 c_0 + c_0 c_2 - c_1^2)}{c_0^3} + a_1 \frac{\nu_1^3}{2} - a_2 \frac{\nu_2^3}{2} + a_3 \frac{\nu_3^3}{4} \\
0 &= f_1 \frac{4m_\pi^3 c_3 - 6m_\pi^2 c_1 c_0 - 6c_0^2 c_3 + 12c_0 c_1 c_2 - 6c_1^2}{c_0^4} - a_1 \frac{\nu_1^4}{2} + a_2 \frac{3\nu_2^4}{4} - a_3 \frac{3\nu_3^4}{4} + a_4 \frac{3\nu_4^4}{8}
\end{align*}
\]

The two complex roots \( \alpha_1, \alpha_2 \) of the denominator polynomial \( D_4(r) \) of degree 4

\[ D_4(r) = |(r - \alpha_1)(r - \alpha_2)|^2 \] (3.36)

together with the pole positions \( \nu_i \ (i=1,...,4) \) provide 8 real parameters for a very accurate fit. Again the roots \( \alpha_i = -\varepsilon_i \pm i\varrho_i \) are restricted to the left complex half plane.

It is instructive to notice that (in contrast to the isoscalar case) the different asymptotic behavior of \( B_1(r) \) for \( r \to \infty \) causes a discontinuity of the spectral function at threshold \( \mu^2 \to (2m_\pi)^2 \): the decomposition of \( N_2/D_4 \) into partial fractions

\[ \frac{N_2(r)}{D_4(r)} = \frac{\gamma_1}{r - \alpha_1} + \frac{\gamma_2}{r - \alpha_2} + c.c. \] (3.37)

with relations like

\[
\begin{align*}
\text{Re} \sum_{i=1,2} \gamma_i &= 0, \\
2\text{Re} \sum_{i=1,2} \alpha_i \gamma_i &= m_\pi^2, \\
\text{Re} \sum_{i=1,2} \alpha_i (\alpha_i \gamma_i - m_\pi^2) &= m_\pi
\end{align*}
\] (3.38)
shows that the Laplace inverse $\gamma(\mu)$ of $rN_2/D_4$

$$\gamma(\mu) = 2\text{Re} \sum_{i=1,2} \alpha_i \gamma_i e^{\alpha_i \mu} \quad (\mu \geq 0) \quad (3.39)$$

as $\mu \to 0$ takes on the value

$$\gamma(0) = m_\pi^2 \quad (3.40)$$

with a slope of

$$\gamma'(0) = 2m_\pi (1 + m_\pi \text{Re} \sum_{i=1,2} \alpha_i) \quad . (3.41)$$

This slope is positive and, due to the small value of $m_\pi$, it is not very sensitive to small variations in $\text{Re}(\alpha_1 + \alpha_2)$.

Altogether, we obtain for the isovector electric spectral function

$$\Gamma^{I=1}(t') = \Gamma_{\text{cont}}(t') + \sum_{i=1}^{4} \Gamma^{(i)}(t') \quad (3.42)$$

where the continuous part is given by

$$\Gamma_{\text{cont}}(\mu^2) = \frac{f_1}{2\mu} \gamma(\mu - 2m_\pi) \quad (3.43)$$

with (3.39) and the amplitudes $a_i$ of the discrete pole structures (3.8)-(3.11) are fixed through (3.34).

As before it turns out that the quality of the fit is not very sensitive to variations of $\varepsilon_2$ within $(0.2 < \varepsilon_2 < 0.4)$. Keeping $\varepsilon_2$ fixed at 0.3 the positions of the roots of $D_4$ are determined as

$$\alpha_1 = -0.550 \pm i \, 0.215 \quad ; \quad \alpha_2 = -0.3 \pm i \, 0.626 \quad ; \quad [m_\rho^{-1}] \quad (3.44)$$

with pole locations and amplitudes given in table II.

| $\nu_i \ [m_\rho]$ | $a_i \ [m_\pi^2]$ |
|-------------------|------------------|
| 4.39              | -7.59            |
| 0.72              | 0.009            |
| 2.60              | -5.59            |
| 2.92              | 3.49             |

| $i$ |
|-----|
| 1   |
| 2   |
| 3   |
| 4   |

TABLE II. Positions and strengths of the discrete structures for the isovector electric spectral function (in the soliton restframe)

The stable feature of the resulting isovector spectral function is that there is no significant low-lying discrete strength, while the continuous part rises steeply from its finite limit

$$\Gamma_{\text{cont}}(\mu^2 \to (2m_\pi)^2) \to \frac{f_1 m_\pi}{4} \quad (3.45)$$

at threshold to a pronounced maximum near $1.5m_\rho$ (see fig.4). Again the discrete structures above $2m_\rho$ mainly reflect our specific choice (3.35).
C. Magnetic form factors $G^0_M(t)$ and $G^1_M(t)$

In the purely pionic Skyrme model, the normalized magnetic isoscalar and isovector form factors are given by ($t = -Q^2$)

$$G^0_M(t) = \frac{3}{2 r_B^2} \int d^3r \frac{j_1(Qr)}{Qr} r^2 B_0(r),$$
$$G^1_M(t) = \frac{3}{2} \int d^3r \frac{j_1(Qr)}{Qr} B_1(r).$$

(3.46)

Thus, with the convenient supplementary definition

$$\tilde{G}^1_E(t) \equiv \frac{1}{2} \int d^3r \frac{j_0(Qr) B_1(r)}{r^2},$$

(3.47)

we have

$$G^0_M(t) = \frac{6}{r_B^2} \frac{\partial}{\partial t} G^0_E(t),$$
$$G^1_M(t) = 6 \frac{\partial}{\partial t} \tilde{G}^1_E(t).$$

(3.48)

and, correspondingly, for the spectral functions $\Gamma(t')$

$$\Gamma^0_M(t') = \frac{6}{r_B^2} \frac{\partial}{\partial t'} \Gamma^0_E(t'),$$
$$\Gamma^1_M(t') = 6 \frac{\partial}{\partial t'} \tilde{\Gamma}^1_E(t')$$

(3.49)

for timelike $t' = \mu^2$.

The Laplace transformation (3.3) relates $\Gamma^1_E(\mu^2)$ and $\tilde{\Gamma}^1_E(\mu^2)$ by

$$\mu \tilde{\Gamma}^1_E(\mu^2) = \int_0^\mu \int_0^{\nu} \nu \Gamma^1_E(\nu^2) d\nu d\nu'$$

(3.50)

(with $\mu_0 = 2m_\pi$).

For the continuous part of the isoscalar magnetic spectral function we obtain from (3.26) and (3.49)

$$\Gamma^0_{M \text{ cont}}(\mu^2) \to 3 \frac{1}{r_B^2} \frac{\partial}{\partial \mu} \frac{A^3}{2\mu} \gamma(\mu - 3m_\pi)$$

(3.51)

with $\gamma(\mu)$ given by (3.23). At threshold $\mu \to 3m_\pi$ $\Gamma^0_M$ rises like

$$\Gamma^0_{M \text{ cont}}(\mu^2) \to 3 \frac{A^3}{r_B^2} \frac{1}{64} (\mu^2 - (3m_\pi)^2)^2.$$
IV. COUPLING TO VECTOR MESONS

We have accounted for the influence of vector mesons on the form factors by multiplying the purely pionic parts $G^\pi(t)$ with

$$\Lambda(t) = \lambda D_V(t) + 1 - \lambda$$

(4.1)

where $D_V(t)$ is just the vector meson propagator. To illustrate its influence on the spectral functions we consider the isovector electric case where the pionic part of the spectral function is given by (3.42)

$$\Gamma^\pi(t') = \Gamma_{\text{cont}}(t') + \sum_{i=1}^{4} \Gamma^{(i)}(t')$$

(4.2)

with the continuous part $\Gamma_{\text{cont}}$ shown in fig.4 and positions and amplitudes of the discrete part $\sum \Gamma^{(i)}$ listed in table II.

If we denote the imaginary part of the $\rho$-propagator $\Im D_V = \Gamma^\rho(t')$ we have for the combined form factor (for $\lambda=1$)

$$G(t) = D_V(t)G^\pi(t) = \frac{1}{\pi^2} \int \frac{\Gamma^\rho(t') \Gamma^\pi(t'')}{t' - t''} dt' dt''.$$  

(4.3)

For the spectral function $\Gamma(t')$ of $G(t)$ this implies

$$\Gamma(t') = \frac{1}{\pi} \left( \Gamma^\rho(t') \mathcal{P} \int \frac{\Gamma^\pi(t'')}{t'' - t'} dt'' + \Gamma^\pi(t') \mathcal{P} \int \frac{\Gamma^\rho(t'')}{t'' - m_V^2} dt'' \right)$$

(4.4)

where $\mathcal{P}$ denotes the principal value. The first integral gives the resulting strength distribution of the vector-meson pole. It receives contributions from the continuous and the discrete part of $\Gamma^\pi$ which are slowly varying functions of $t'$ within the resonance region. At $t'=m_V^2$ we obtain the values

$$\mathcal{P} \int \frac{\Gamma_{\text{cont}}(t'')}{t'' - m_V^2} dt'' = 0.947, \quad \int \sum \Gamma^{(i)}(t'') dt'' = 1.192$$

(4.5)

so that the total strength of the vector-meson pole is increased by a factor of 2.14.

The second integral in (4.4) modifies the pionic spectral function. The discrete multipole structures $\Gamma^{(i)}$ in $\Gamma^\pi$ of course remain discrete, but receive additional lower multipoles at the same position. (Only monopoles remain monopoles, with modified strength). Of particular interest is the modification in the continuous part of the spectral function: The $t'$-dependence of the second integral in (4.4) cuts down the tails in $\Gamma_{\text{cont}}(t')$ for larger values of $t'$ and creates a sign change near the vector-meson pole. The result is shown in fig.5: For the demonstrative purpose we use for the $\rho$-pole the imaginary part (dotted line in fig.5)

$$\Gamma^\rho(t') = \frac{m_V^2 \varepsilon}{(m_V^2 - t')^4 + m_V^2 \varepsilon^2}$$

(4.6)

with the experimental full width $\varepsilon = 150$ MeV (and a smooth cutoff near the two-pion threshold). The dashed line in fig.5 is the continuous pionic part $\Gamma_{\text{cont}}$ from (3.43) and fig.4. The
The dash-dotted line shows the continuous spectral function \( \Gamma_E^{(T=1)}(t') \) as resulting from (4.4). The increased strength of the resonance due to the integrals (4.5) is evident. Furthermore, the figure clearly shows that the interplay between the vector meson and the solitonic pion cloud leads to appreciable enhancement of the resonance structure on the low-energy side and a corresponding depletion on the high-energy side. This is strongly reminiscent of the Frazer-Fulco unitarity constraint which enforces a similar behavior for the isovector spectral functions above the two-pion threshold [14,40]. Notable from the figure is also the finite limit (3.45) of \( \Gamma_E^{(T=1)}(t'=4m_\pi^2) = f_1 m_\pi / 4 \) at threshold.

The actual value of \( \lambda \) used in sect.1 for the form factors was \( \lambda = 0.75 \). Therefore the total spectral functions are obtained as the corresponding superpositions of \( \lambda \Gamma(t') \) from (4.4) and the purely pionic \( (1-\lambda) \Gamma^\pi(t') \) of (3.42). This final result for \( \Gamma_E^{(T=1)}(t') \) is shown by the full line in fig.5.

V. RELATIVISTIC CORRECTIONS

The Lorentz boosts (2.16) and (2.17) which relate the form factors as evaluated in the soliton restframe and in the Breit frame can be directly transferred to their respective spectral functions. If we denote by \( \Gamma^{(nr)}(t') \) the spectral function as evaluated in the soliton restframe, we have from (2.16)

\[
G_E(t) = \frac{1}{\pi} \int_{t_0}^{\infty} \frac{\Gamma^{(nr)}(t')}{t'-t/t'\gamma^2} dt' = \frac{1}{\pi} \int_{t_0}^{4M^2} \frac{\Gamma^{(nr)}(\tilde{t}/\tilde{\gamma}^2)}{\tilde{t}-t} d\tilde{t} + G_E^{(nr)}(-4M^2)
\]

with

\[
\tilde{\gamma}^2 = 1 - \frac{\tilde{t}}{4M^2} \quad \text{for timelike \( \tilde{t} > 0 \).}
\]

Thus, with

\[
G_E(t) = \frac{1}{\pi} \int_{t_0}^{4M^2} \frac{\Gamma_E(\tilde{t})}{\tilde{t}-t} d\tilde{t} + \text{const.}
\]

we have

\[
\Gamma_E(\tilde{t}) = \Gamma_E^{(nr)}(\tilde{t}/\tilde{\gamma}^2).
\]

The additional constant which appears in (5.1) and (5.3) is the undesirable consequence of (2.16) for \( t \to \infty \)

\[
G_E(t \to \pm\infty) \to G_E^{(nr)}(-4M^2)
\]

which we have met already in (2.18).

The integration interval in (5.3) is determined by the threshold of \( \Gamma_E^{(nr)} \), i.e. \( \tilde{t}/\tilde{\gamma}^2 = t_0 \), or
\[
\tilde{t}_0 = \frac{t_0}{1 + t_0/(4M^2)}. \tag{5.6}
\]

The upper limit is the physical threshold for soliton-antisoliton creation which follows from (5.4) for \(\tilde{\gamma}^2 \to 0\).

For the magnetic form factors the boost to the Breit frame (2.17) leads to

\[
G_M(t) = \frac{1}{\pi} \int_{\tilde{t}_0}^{4M^2} \frac{\Gamma_M(\tilde{t})}{\tilde{t} - t} d\tilde{t} \tag{5.7}
\]

with

\[
\Gamma_M(\tilde{t}) = \frac{1}{\gamma^2} \Gamma_M^{(nr)}(\tilde{t}/\gamma^2). \tag{5.8}
\]

In this case there remains no additional constant term in (5.4) such that \(G_M(t)\) satisfies \(G_M(t \to \pm \infty) \to 0\). The factor \(1/\tilde{\gamma}^2\) in the relativistically corrected magnetic spectral functions diverges as \(\tilde{t} \to 4M^2\). This amplifies the oscillations of the non relativistic \(\Gamma_M^{(nr)}(t')\) for \(t' \gg m_\pi^2\) which are not well determined by the fit (through \(\varepsilon_\eta\)) and thus should not be taken too seriously.

The requirement that the threshold \(\tilde{t}_0\) in (5.3) should appear at multiples of the physical pion mass \(\tilde{t}_0 = (n m_\pi)^2\) (\(n = 2\) or \(3\) for isovector or isoscalar spectral functions) can only be met if in the soliton restframe an effective pion mass \(\mu_\pi\) is employed:

\[
t_0 = (n \mu_\pi)^2 = \frac{(n m_\pi)^2}{4M^2 - (n m_\pi)^2}. \tag{5.9}
\]

This is an extension to timelike \(t_0\) and \(\tilde{t}_0\) of the usual 'reduced mass' formula

\[
(n \mu_\pi)^2 = \frac{(n m_\pi)^2}{4M^2 + (n m_\pi)^2} \tag{5.10}
\]

for spacelike momentum transfer. This effect, however, is very small.

The main effect of the transformations (5.4) and (5.8) is that the point \(t' = \infty\) is mapped onto the threshold \(\tilde{t} = 4M^2\), i.e. the nonrelativistic spectral functions are squeezed into the finite interval \(\tilde{t}_0 < \tilde{t} < 4M^2\). For the continuous parts of the pionic spectral functions this is shown in the lower sections of figs.3,4. Of course, the low-energy region is almost unaffected, especially the region around the vector meson resonance (after inclusion of the \(\varrho\)-pole) remains essentially as shown in fig.5. But the positions \(\nu_i\) of the higher discrete structures listed in tables I,II are shifted to \(\tilde{\nu}_i = \nu_i/\sqrt{1 + \nu_i^2/4M^2}\).

If we here use for \(M\) the physical nucleon mass \(M_N = 940\) MeV in order to relate these shifted positions to the actual physical \(N-\bar{N}\)–threshold we obtain the values listed in table III. Although we cannot really expect any close correspondence to the actually observed isoscalar (\(\Phi(1020), \omega(1420), \omega(1600), \Phi(1680)\)) and isovector (\(\varrho(1450), \varrho(1700)\)) resonances it is satisfactory to find the calculated structures in this energy range. (The isovector structure at 0.72 \(m_\varrho\) appears with almost vanishing strength 0.009 in table II and may indicate that the continuous part expects small modifications near 5 \(m_\pi\).)
|      | isoscalar |      | isovector |      |      |      |      |
|------|-----------|------|-----------|------|------|------|------|
|      | $i = 0$   | $i = 1$ | $i = 2$   | $i = 3$ | $i = 1$ | $i = 2$ | $i = 3$ | $i = 4$ |
| $\nu_i$ $[m_0]$ | 1.61 | 5.14 | 1.93 | 2.09 | 4.39 | (0.72) | 2.60 | 2.92 |
| $\tilde{\nu}_i$ $[MeV]$ | 1034 | 1695 | 1165 | 1221 | 1640 | 1369 | 1440 |      |

TABLE III. Positions of the calculated discrete structures: $\nu_i$, in the soliton restframe (taken from tables I,II), and $\tilde{\nu}_i$, in the Breit frame (as obtained from $\tilde{\nu}_i = \nu_i / \sqrt{1 + \nu_i^2 / 4M^2}$ with $M=M_N$).

VI. SUMMARY

In view of the widespread prejudice that baryons considered as solitons in effective mesonic theories provide only poor approximations for the e.m. nucleon form factors within a very limited range of $Q^2$ we demonstrate here that even in the most simple soliton model the essential features of form factors observed over three orders of magnitude in momentum transfer are naturally reproduced. The analysis shows that in order to achieve such a result it is necessary to incorporate three basic ingredients: an extended object, partial coupling to vector mesons, and relativistic recoil corrections. We have used here the standard skyrmion as the extended object, and (for simplicity) the same vector meson propagator in both isospin channels. Apart from the Skyrme constant $e$ which is adjusted to the Delta-nucleon mass split the model contains only one parameter of direct relevance to the form factors: the amount $\lambda$ which measures the admixing of photon-vector meson coupling. We use it to bring the e.m. proton radii close to their experimental values. This is sufficient to keep the form factors $G_{E,M}^p$ and $G_{M}^n$ very close to the standard dipole up to almost 1 GeV/c.

Naturally, the typical shape above $Q^2 \sim M_N^2$ is sensitive to the recoil corrections. We employ the boost transformation to the Breit frame suggested by Ji [11] which pulls the point $Q^2 = 4M^2$ (in the restframe) to $Q^2 = \infty$ (in the Breit frame). Thus the behavior of the form factors for $Q^2 \to \infty$ is, of course, very sensitive to the value of $M$, the kinematical mass of the particle represented by the soliton. In tree approximation $M$ is the classical soliton mass, but it is known to be subject to large quantum corrections. The boost to the Breit frame has the undesired feature that it violates superconvergence unless $4M^2$ coincides with a zero in $G_M$ (or a double zero in $G_E$). It should be noted, however, that the boost transformation is not an exact result, but its derivation relies on neglecting all commutators between center-of-mass and total momentum [11]. Thus it appears as a reasonable procedure to subtract the very small undesired terms from the boosted form factors. We have shown here that up to 10 (GeV/c)$^2$ their effect is qualitatively unimportant and the typical features observed in $G_M^p$ (which is the only form factor known sufficiently well up to high $Q^2$) clearly emerge from the model. Due to these difficulties connected with the treatment of recoil corrections the predictive power at tree level is poor for the high-$Q^2$ behavior. This flexibility is unfortunate because it prevents definite conclusions and conceals possible evidence for importance of QCD effects. From this point of view it would be highly desirable to learn more about loop and rotational corrections to the form factors.

The continuation to timelike $t$ leads to quite stable results for the spectral functions in the regime from the 2- or 3-pion threshold to about two rho masses. Beyond this region well-known ambiguities prevent definite results. Especially the onset of the continuous part of
the spectral functions at threshold can be reliably determined and there are strong analogies to the results imposed on dispersion theoretic approaches by the unitarity constraint. On the other hand, the region near the physical nucleon-antinucleon threshold \( t = 4M^2 \) is most poorly defined, because through the boost transformation it is an image of the nonrelativistic form factors near \( Q^2 \to \infty \). Both, \( G_M^{(nr)} \) and \( G_E^{(nr)} \), vanish in this limit, so the possibility of finite results depends on the power of (diverging) factors \((1 - t/4M^2)^{-1}\) introduced through the boost transformation. This does not seem a reliable way for investigating this regime; it might prove more promising to try to extract the timelike form factors above the \( N-\bar{N} \) threshold directly from soliton-antisoliton configurations.

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FIG. 1. Proton magnetic and electric form factors $G_M^p$ and $G_E^p$ (divided by the standard dipole $G_D$) as obtained from the Skyrme model with $e=4.25$, (soliton mass $M=1648$ MeV), and vector meson coupling $\lambda=0.75$ in (2.13). The dashed lines show the same form factors after subtraction of the square brackets in (2.18), (2.19). The data are from the compilations of [14] (open circles), [39] (open squares), [37] (full circles), and [38] (full triangles).

FIG. 2. Neutron magnetic form factors $G_M^n$ (divided by the standard dipole $G_D$) and neutron electric form factor $G_E^n$ (full lines) calculated as in fig.1. The data for $G_M^n$ are from [25] (full circles), [27] (diamonds), [26] (triangles), and [28] (square). The open circles represent older data from [23–34]. The data points for $G_E^n$ are from [35] for the Paris potential (full circles) and from [36] for Feshbach-Lomon wave functions (open circles). The dash-dotted line for $G_E^n$ is the Galster fit [36], the dotted lines are fits from [35] deduced for three different potentials.

FIG. 3. The continuous parts $\Gamma_{\text{cont}}$ of the isoscalar electric (full line) and magnetic (dashed line) spectral functions for the skyrmion with $e=4.25$. In the upper part they are shown in the soliton rest frame, in the lower part they are boosted to the Breit frame ($M=1648$ MeV).

FIG. 4. Same as fig.3 for the isovector spectral functions.

FIG. 5. The continuous part of the isovector electric spectral function after inclusion of the $\varphi$-resonance. The dotted line is the imaginary part of the $\varphi$-propagator (with width 150 MeV); the dashed line shows the continuous pionic part $\Gamma_{\text{cont}}$ from (3.43) and fig.4; the dash-dotted line shows $\Gamma_{E}^{(T=1)}(t')$ as resulting from (4.4) (i.e. for $\lambda=1$). The full line is the total result for $\lambda=0.75$. 

24
