Minimal Charts
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Abstract

In this paper, we give definitions of three kinds of minimal charts, and we investigate properties of minimal charts and establish fundamental theorems characterizing minimal charts. To classify charts with two or three crossings we use the fundamental theorems. In the future paper, we give an enumeration of the charts with two crossings.

1 Introduction

Charts are oriented labeled graphs in a disk with three kinds of vertices called black vertices, crossings, and white vertices (see page 3 for the precise definition of charts). From a chart, we can construct an oriented closed surface embedded in 4-space $\mathbb{R}^4$ (see [6, Chapter 14, Chapter 18 and Chapter 23]). A C-move is a local modification between two charts in a disk (see Section 2 for C-moves). A C-move between two charts induces an ambient isotopy between oriented closed surfaces corresponding to the two charts. Two charts are said to be C-move equivalent if there exists a finite sequence of C-moves which modifies one of the two charts to the other.

We will work in the PL or smooth category. All submanifolds are assumed to be locally flat. A surface link is a closed surface embedded in 4-space $\mathbb{R}^4$. A 2-link is a surface link each of whose connected component is a 2-sphere. A 2-knot is a surface link which is a 2-sphere. An orientable surface link is called a ribbon surface link if there exists an immersion of a 3-manifold $M$ into $\mathbb{R}^4$ sending the boundary of $M$ onto the surface link such that each connected component of $M$ is a handlebody and its singularity consists of ribbon singularities, here a ribbon singularity is a disk in the image of $M$ whose pre-image consists of two disks; one of the two disks is a proper disk of $M$ and the other is a disk in the interior of $M$. In the words of charts, a ribbon surface link is a surface link corresponding to a ribbon chart, a chart C-move equivalent to a chart without white vertices [4]. A chart is called a 2-link chart if a surface link corresponding to the chart is a 2-link.

In this paper, we denote the closure, the interior, the boundary, and the complement of (...) by $\text{Cl}(...), \text{Int}(...), \partial(...), (...)^c$ respectively. Also for a finite set $X$, the notation $|X|$ denotes the number of elements in $X$.

At the end of this paper, there is the index of new words and notations introduced in this paper.

Kamada showed that any 3-chart is a ribbon chart [4]. Kamada’s result was extended by Nagase and Hirota: Any 4-chart with at most one crossing

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Figure 1: The letter $m$ is a label and $\varepsilon = \pm 1$.

is a ribbon chart \[7\]. We showed that any $n$-chart with at most one crossing is a ribbon chart \[13\]. We also showed that any 2-link chart with at most two crossings is a ribbon chart \[14\], \[15\].

Charts have strong conditions on orientations of arcs around vertices. In a small neighborhood of each white vertex, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward (see Fig. 2(c)). Among six short arcs in a small neighborhood of a white vertex, a central arc of each three consecutive arcs oriented inward (resp. outward) is called a middle arc at the white vertex. Observing precisely middle arcs, orientations of edges, and a part of a chart cutting by a disk called a tangle, we shall prove the following theorem \[18\]:

Any 2-link chart with at most three crossings is C-move equivalent to either a ribbon chart, or the disjoint union of a ribbon chart and a chart as shown in Fig. 1 or its “reflection”.

In this paper we establish fundamental theorems characterizing $c$-minimal charts, $w$-minimal charts and $cw$-minimal charts. For the classification theorem above, we use the fundamental theorems obtained in this paper.

For a 4-chart as shown in Fig. 1 we obtain a 2-twist spun trefoil by setting $m = 2$ (see \[4\] p. 144], \[6\] p. 170]). It is well known that the 2-knot is not a ribbon 2-knot. On the other hand, Hasegawa showed that if a non-ribbon chart representing a 2-knot is minimal, then the chart must possess at least six white vertices \[2\] where a minimal chart $\Gamma$ means its complexity $(w(\Gamma), f(\Gamma))$ is minimal among the charts C-move equivalent to the chart $\Gamma$ with respect to the lexicographic order of pairs of integers, here $w(\Gamma)$ is the number of white vertices in $\Gamma$, $f(\Gamma)$ is the number of free edges in $\Gamma$. Here a free edge is an edge of $\Gamma$ containing two black vertices. Nagase, Ochiai, and Shima showed that there does not exist a minimal chart with exactly five white vertices \[19\]. Nagase and Shima show that there does not exist a minimal chart with exactly seven white vertices \[8\], \[9\], \[10\], \[11\], \[12\]. Ishida, Nagase, and Shima showed that any minimal chart with exactly four white vertices is C-move equivalent to a chart in two kinds of classes \[3\].
Let $n$ be a positive integer. An $n$-chart (a braid chart of degree $n$ \cite{1} or a surface braid chart of degree $n$ \cite{6}) is an oriented labeled graph in the interior of a disk, which may be empty or have closed edges without vertices satisfying the following four conditions (see Fig. 2):

(i) Every vertex has degree 1, 4, or 6.
(ii) The labels of edges are in $\{1, 2, \ldots, n-1\}$.
(iii) In a small neighborhood of each vertex of degree 6, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward, and these six are labeled $i$ and $i + 1$ alternately for some $i$, where the orientation and label of each arc are inherited from the edge containing the arc.
(iv) For each vertex of degree 4, diagonal edges have the same label and are oriented coherently, and the labels $i$ and $j$ of the diagonals satisfy $|i - j| > 1$.

We call a vertex of degree 1 a **black vertex**, a vertex of degree 4 a **crossing**, and a vertex of degree 6 a **white vertex** respectively. Among six short arcs in a small neighborhood of a white vertex, a central arc of each three consecutive arcs oriented inward (resp. outward) is called a **middle arc** at the white vertex (see Fig. 2(c)). For each white vertex $v$, there are two middle arcs at $v$ in a small neighborhood of the white vertex $v$.

Let $\Gamma$ be a chart. Let $c(\Gamma)$ and $b(\Gamma)$ be the number of crossings, and the number of bigons of $\Gamma$ respectively. We define complexities as follows (see \cite{4} for the original complexity of charts):

- The 4-tuple $(c(\Gamma), w(\Gamma), -f(\Gamma), -b(\Gamma))$ is called a **$c$-complexity** of $\Gamma$.
- The 4-tuple $(w(\Gamma), c(\Gamma), -f(\Gamma), -b(\Gamma))$ is called a **$w$-complexity** of $\Gamma$.
- The 3-tuple $(c(\Gamma) + w(\Gamma), -f(\Gamma), -b(\Gamma))$ is called a **cw-complexity** of $\Gamma$.

![Figure 2](image-url)
A chart $\Gamma$ is said to be $c$-minimal (resp. $w$-minimal or $cw$-minimal) if its $c$-complexity (resp. $w$-complexity or $cw$-complexity) is minimal among the charts which are C-move equivalent to the chart $\Gamma$ with respect to the lexicographical order of the 4-tuple (or 3-tuple) of the integers.

In this paper, if a chart is $c$-minimal, $w$-minimal or $cw$-minimal, then we say that the chart is minimal.

A hoop is a closed edge of a chart $\Gamma$ that contains neither crossings nor white vertices. Therefore a hoop decomposes $\Gamma$ into disjoint pieces: an inside, an outside, and itself. A hoop is said to be simple if one of the complementary domains of the hoop does not contain any white vertices.

Let $\Gamma$ be a chart. For each label $m$, we define

$$\Gamma_m = \text{the union of all the edges of label } m \text{ and their vertices in } \Gamma.$$ 

Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. Let $E$ be a disk with $\partial E \subset \Gamma_m$. Then the disk $E$ is called a 3-color disk provided that

(i) the disk $E$ does not contain any crossings, and

(ii) $\Gamma \cap E \subset \Gamma_{m-1} \cup \Gamma_m \cup \Gamma_{m+1}$.

Further a disk $E$ with $\partial E \subset \Gamma_m$ is called a 2-color disk provided that $\Gamma \cap E \subset \Gamma_m \cup \Gamma_{m-1}$ or $\Gamma \cap E \subset \Gamma_m \cup \Gamma_{m+1}$. A 2-color disk is a special kind of 3-color disk.

Now 3-color disks and 2-color disks often appear in charts. For example, let $m$ be any label of a chart $\Gamma$, and $E$ a disk with $\partial E \subset \Gamma_m$ but without crossings, free edges nor simple hoops. If $\Gamma$ is a minimal chart, then we can show that $E$ is a 3-color disk (see Lemma 11.2 in Section 11). Further, if $m$ is the minimal label or the maximal label of the chart, then $E$ is a 2-color disk. This indicates that it is important to investigate 3-color disks and 2-color disks.

An edge $e$ of $\Gamma$ is said to be middle at a white vertex $v$ if it contains a middle arc at the vertex $v$.

Let $m$ be a label of a chart $\Gamma$. A simple closed curve in $\Gamma_m$ is called a cycle of label $m$. Let $C$ be a cycle of label $m$ bounding a disk $E$. An edge $e$ of label $m$ is called an inside (resp. outside) edge for $C$ provided that

(i) $e \cap C$ consists of one white vertex or two white vertices, and

(ii) $e \subset E$ (resp. $e \subset Cl(E^c)$).
Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. For a cycle $C$ of label $m$, we define
\[ W(C) = \{ w \mid w \text{ is a white vertex in } C \}, \]
\[ W_{\text{Mid}}^I(C, m) = \{ w \in W(C) \mid \text{there exists an inside edge for } C \text{ middle at } w \}, \]
\[ W_{\text{Mid}}^O(C, m) = \{ w \in W(C) \mid \text{there exists an outside edge for } C \text{ middle at } w \}. \]

**Theorem 1.1** Let $\Gamma$ be a minimal chart, and $m$ a label of $\Gamma$. Let $E$ be a 3-color disk with $\partial E \subset \Gamma_m$ but without free edges nor simple hoops. If $\Gamma_m \cap E$ is connected, then we have
\[ |W_{\text{Mid}}^I(\partial E, m)| + 2 \leq |W_{\text{Mid}}^O(\partial E, m)|. \]

Let $\Gamma$ be a chart, and $D$ a disk. The pair $(\Gamma \cap D, D)$ is called a tangle provided that
(i) $\partial D$ does not contain any white vertices, black vertices nor crossings of $\Gamma$,

(ii) if an edge of $\Gamma$ intersects $\partial D$, then the edge intersects $\partial D$ transversely, and

(iii) $\Gamma \cap D \neq \emptyset$.

Let $\Gamma$ be a chart. A tangle $(\Gamma \cap D, D)$ is called an NS-tangle of label $m$ (new significant tangle) provided that
(i) if $i \neq m$, then $\Gamma_i \cap \partial D$ is at most one point,

(ii) $\Gamma \cap D$ contains at least one white vertex, and

(iii) for each label $i$, the intersection $\Gamma_i \cap D$ contains at most one crossing.

**Theorem 1.2** In a minimal chart, there does not exist an NS-tangle of any label.

The above theorem is an extended result of Theorem 3.5 in [14], and does a significant job for the classification of charts from the view point of the number of crossings. The above theorem and Consecutive Triplet Lemma (Lemma 3.1) are our main tools as we see how to use these two tools to prove Theorem 1.3 below.

Let $\Gamma$ be a chart, and $m$ a label of the chart. Let $W$ be the set of all the white vertices of $\Gamma$. The closure of a connected component of $\Gamma_m - W$ is called an internal edge of label $m$ if it contains a white vertex but does not contain any black vertex, here we consider $\Gamma_m$ as a topological set. Thus an internal edge begins at a white vertex passing through several crossings and ends at a white vertex.

Let $\Gamma$ be a chart. A tangle $(\Gamma \cap D, D)$ is said to be admissible provided that
(i) $D$ contains neither free edge nor simple hoop.
(ii) any edge intersecting $\partial D$ is contained in an internal edge,

(iii) if an internal edge $e$ intersects $\partial D$, then each connected component of $e \cap D$ contains a white vertex.

Let $\Gamma$ be a chart, and $D$ a disk. Suppose that an edge $e$ of $\Gamma$ transversely intersects $\partial D$. Let $p$ be a point in $e \cap \partial D$, and $N$ a regular neighbourhood of $p$. Then the orientation of $e$ induces the one of the arc $e \cap N$. The edge $e$ is said to be locally inward (resp. locally outward) at $p$ with respect to $D$ if the oriented arc $e \cap N$ is oriented from a point outside (resp. inside) $D$ to a point inside (resp. outside) $D$. We often say that $e$ is locally inward (resp. outward) at $p$ instead of saying that $e$ is locally inward (resp. outward) at $p$ with respect to $D$, if there is no confusion.

For a simple arc $X$, we set
\[ \partial X = \text{the set of its two endpoints}, \]
\[ \text{Int } X = X - \partial X. \]

Let $\Gamma$ be a chart. An edge of $\Gamma$ is called a terminal edge if it contains a white vertex and a black vertex.

Let $\Gamma$ be a chart, and $m$ a label of the chart. A tangle $(\Gamma \cap D, D)$ is called a IO-tangle of label $m$ provided that (see Fig. 4)

(i) $\partial D$ intersects neither terminal edge nor free edge,

(ii) there exists a label $k$ with $|m - k| = 1$ and $\Gamma \cap D \subset \Gamma_m \cup \Gamma_{k}$,

(iii) there exist two arcs $L_I, L_O$ on $\partial D$ with $L_I \cap L_O = \partial L_I = \partial L_O = \Gamma_m \cap \partial D$,

(iv) for any point $p \in \Gamma \cap \text{Int } L_I$, there is an edge of label $k$ locally inward at $p$, and

for any point $p \in \Gamma \cap \text{Int } L_O$, there is an edge of label $k$ locally outward at $p$.

An IO-tangle of label $m$ is said to be simple if all the terminal edge in $D$ is of label $m$.

**Theorem 1.3** If $(\Gamma \cap D, D)$ is an admissible tangle in a minimal chart $\Gamma$ such that

(a) $\Gamma \cap D \subset \Gamma_m \cup \Gamma_{m-1}$ or $\Gamma \cap D \subset \Gamma_m \cup \Gamma_{m+1}$ for some label $m$,

(b) $\Gamma_m \cap \partial D$ consists of exactly two points, and

(c) $\Gamma_m \cap D$ contains a cycle,

then the tangle $(\Gamma \cap D, D)$ is a simple IO-tangle of label $m$. 

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The theorem above determines the structure of minimal charts with two crossings ([16] and [17]), and give us an enumeration of the charts with two crossings. The enumeration is much complicated than the one of 2-bridge links in $\mathbb{R}^3$, of course. We enumerate charts with two crossings as follows (see [16], [17]): For any minimal $n$-chart $\Gamma$ with two crossings in a disk $D^2$, there exist two labels $1 \leq \alpha < \beta \leq n - 1$ such that $\Gamma_\alpha$ and $\Gamma_\beta$ contain cycles $C_\alpha$ and $C_\beta$ with $C_\alpha \cap C_\beta$ the two crossings and that for any label $k$ with $k < \alpha$ or $\beta < k$, the set $\Gamma_k$ does not contain a white vertex. If $\Gamma_\alpha$ or $\Gamma_\beta$ contains at least three white vertices, then after shifting all the free edges and simple hoops into a regular neighbourhood of $\partial D^2$ by applying C-I-M1 moves and C-I-M2 moves, we can find an annulus $A$ containing all the white vertices of $\Gamma$ but not intersecting any hoops nor free edge such that (see Fig. 5(a))

1. Each connected component of $Cl(D^2 - A)$ contains a crossing,
2. $\Gamma \cap \partial A = (C_\alpha \cup C_\beta) \cap \partial A$, and $\Gamma \cap \partial A$ consists of eight points.

We can show the annulus $A$ can be split into mutually disjoint four disks $D_1, D_2, D_3, D_4$ and mutually disjoint four disks $E_1, E_2, E_3, E_4$ such that

1. For each $i = 1, 3$ (resp. $i = 2, 4$) the tangle $(\Gamma \cap D_i, D_i)$ is an IO-tangle of label $\alpha$ (resp. label $\beta$),
2. For each $i = 1, 2, 3, 4$, $(\Gamma \cap E_i, E_i)$ is a tangle with $\Gamma \cap E_i \subset \cup_{j=\alpha+1}^{\beta-1} \Gamma_j$ as shown in Fig. 5(b).

We count the number of edges between terminal edges in Fig. 5(b) to enumerate charts with two crossings. As important results, from the enumeration we can calculate the fundamental group of the exterior of the surface link represented by $\Gamma$, and the braid monodromy of the surface braid represented by $\Gamma$. 

Figure 4: An IO-tangle of label $m$. The thick edges are of label $m$. The light gray arc is $L_1$, and the dark gray arc is $L_0$. 

[Diagram of an IO-tangle with labels and arcs]
Figure 5: (b) A tangle $(\Gamma \cap E_i, E_i)$ with $\Gamma \cap E_i \subset \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ for the case $\alpha = 1$ and $\beta = 5$, here all the free edges and simple hoops are in a regular neighbourhood of $\partial D^2$.

Our paper is organized as follows: In Section 2, we introduce the definition of C-moves and its related words. In Section 3, we introduce Consecutive Triplet Lemma (Lemma 3.1). For a label $m$ of a minimal chart $\Gamma$, we also investigate a complementary domain $U$ of $\Gamma_m$ with $\Gamma \cap U \subset \Gamma_{m-1} \cup \Gamma_{m+1}$. In Section 4, we give a proof of Theorem 1.1. In Section 5, for a chart $\Gamma$ and a label $m$, we investigate a neighbourhood $N$ of an arc in $\Gamma_m$ with $N \cap \Gamma \subset \Gamma_m \cup \Gamma_k$ for some label $k$. This situation occurs an arc in the boundary of a 2-color disk and an arc in an IO-tangle of label $m$. In Section 6, for a tangle $(\Gamma \cap D, D)$ and each label $m$ with $\Gamma_m \cap D \neq \emptyset$, we obtain an equation and search for conditions for the existence of a special cycle which never bounds a 2-color disk in a minimal chart. In Section 7, we give a proof of Theorem 1.2. In Section 8, we investigate the boundary of a 2-color disk. In Section 9, we investigate an arc in $\Gamma_m$ such that each white vertex in the arc is contained in a terminal edge. This situation occurs an arc in $\Gamma_m$ connecting vertices in the boundary of 2-color disks. In Section 10, we give a proof of Theorem 1.3. In Section 11, we give complementary lemmata to make our paper self-contained.

2 Preliminaries

In this section we give the definitions of C-moves and its related words. Now C-moves are local modifications of charts as shown in Fig. 6 (cf. [11, p. 117], [6, p. 142–143] and [20]). As one of C-moves, Kamada originally defined CI-moves as follows: A chart $\Gamma$ is obtained from a chart $\Gamma'$ in a disk $D^2$ by a CI-move, if there exists a disk $E$ in $D^2$ such that
Figure 6: For the C-III move, the edge containing the black vertex does not contain a middle arc at a white vertex in the left figure.

(i) the two charts $\Gamma$ and $\Gamma'$ intersect the boundary of $E$ transversely or do not intersect the boundary of $E$,

(ii) $\Gamma \cap E^c = \Gamma' \cap E^c$, and

(iii) neither $\Gamma \cap E$ nor $\Gamma' \cap E$ contains a black vertex,

where $E^c$ is the complement of $E$ in the disk $D^2$.

**Remark 2.1** Any CI-move is realized by a finite sequence of seven types: C-I-R2, C-I-R3, C-I-R4, C-I-M1, C-I-M2, C-I-M3, C-I-M4.

Let $D^2_1, D^2_2$ be disks, and $pr_2 : D^2_1 \times D^2_2 \to D^2_2$ the projection defined by $pr_2(x, y) = y$. Let $Q_n$ be a set of $n$ interior points of $D^2_1$. A *surface braid* $S$ is an oriented surface embedded properly in $D^2_1 \times D^2_2$ such that the map $pr_2|_S : S \to D^2_2$ is a branched covering of degree $n$ and $\partial S = Q_n \times \partial D^2_2$ [6, Chapter 14].

A surface braid $S$ can be represented by a motion picture method as follows: We identify the disk $D^2_2$ the product of the unit intervals $I_3, I_4$. For each $t \in I_4 = [0, 1]$, we define the subset $b_t$ in $D^2_1 \times I_3$ by $b_t \times \{t\} = S \cap (D^2_1 \times I_3 \times \{t\})$ such that $b_t$ is a geometric $n$-braid in $D^2_1 \times I_3$ except for a finite number of values $t_1, t_2, \ldots, t_m \in I_4$. Thus the surface braid $S$ is represented by the one-parameter family $\{b_t\}_{t \in [0, 1]}$ called a *motion picture* (see Fig. 7) (cf. [6]).
A chart can be constructed from a surface braid as follows: We identify the disk $D^2_1$ the product of the unit intervals $I_1, I_2$. Let $\pi : D^2_1 \times D^2_2 = I_1 \times I_2 \times D^2_2 \to I_1 \times D^2_2$ be the projection defined by $\pi(x_1, x_2, y) = (x_1, y)$. Then the image $\pi(S)$ of a surface braid $S$ has double points, triple points and branch points. The union of these points is a graph $G$ in $I_1 \times D^2_2$. We construct a chart from the surface braid $S$ by projecting the graph $G$ into $D^2_2$. The orientation of each edge is determined as follows: In a neighbourhood of each double point, there are two sheets; an upper sheet and a lower sheet. Let $v_1, v_2$ be the normal vectors at the double point in an upper sheet and a lower sheet determined from the orientation of $S$ respectively. Define the tangent vector $v_3$ of the graph $G$ at the double point such that the orientation of $I_1 \times D^2_2$ matches the triplet $(v_1, v_2, v_3)$. Then we have the orientated graph $G$; the chart each of whose edge is oriented. In a neighbourhood of each triple point, there are three sheets; a top sheet, a middle sheet and a bottom sheet. A white vertex is corresponding to a triple point, and middle arcs are recognizable as the intersection of a top sheet and a bottom sheet (see Fig. 8). The label of each edge is determined as follows: Let $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ be the standard generators of the classical braid group $B_n$. Any geometric braid $b$ can be represented by a product of $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and their inverse $\sigma^{-1}_1, \sigma^{-1}_2, \ldots, \sigma^{-1}_{n-1}$. A double point in an edge of the graph $G$ is corresponding to a crossing of a geometric braid $b_t$ for some $t \in I_4$, and the crossing is corresponding to some generator $\sigma_i$ or $\sigma^{-1}_i$. Then we define the label of the edge of $G$ by the label $i$. Thus we have the labeled graph $G$; the chart each of whose edge is labeled.

Now for any chart $\Gamma$ a disk $D^2$ is assigned so that the chart is contained in the disk $D^2$ by the definition of charts. If we want to emphasize that a domain $X$ is contained in $\text{Int} D^2$, then we say that $X$ is finite.
For any chart in a disk $D^2$ we can move free edges and simple hoops into a regular neighbourhood of $\partial D^2$ by C-I-M2 moves and ambient isotopies of $D^2$ as shown in Fig. 9. Even during argument, if free edges or simple hoops appear, we immediately move them into a regular neighbourhood of $\partial D^2$. Thus we assume the following.

**Assumption 1** For any chart in a disk $D^2$, all the free edges and simple hoops are in a regular neighbourhood of $\partial D^2$.

Let $\Gamma$ be a chart in a disk $D^2$, and $X$ the union of all the free edges and simple hoops. By Assumption 1 the set $X$ is in a regular neighbourhood $N$ of $\partial D^2$ in $D^2$. Define

$$\text{Main}(\Gamma) = \Gamma - X.$$ 

Let $\widehat{D} = Cl(D^2 - N)$. Then $\Gamma \cap \widehat{D} = \text{Main}(\Gamma)$. Hence $(\Gamma \cap \widehat{D}, \widehat{D})$ is a tangle without free edges and simple hoops.

**Assumption 2** In this paper, our arguments are done in the disk $\widehat{D}$, otherwise mentioned.

**Remark 2.2** Let $\Gamma$ be a chart. Let $\gamma_1, \gamma_2, \cdots, \gamma_6$ be six arcs around a white vertex $w$ lying clockwise in this order (see Fig. 10(a)). Then we have the following (see Fig. 10(b) and (c)).
(1) For each $j = 1, 2, \cdots, 6$, one of the two arcs $\gamma_j, \gamma_{j+1}$ is not a middle arc at $w$.

(2) For each $j = 1, 2$, one of the three arcs $\gamma_j, \gamma_{j+2}, \gamma_{j+4}$ of the same label is middle at $w$ but the others are not middle at $w$.

Here $\gamma_{j+6} = \gamma_j$ for each integer $j$.

Let $m$ be a label of a chart $\Gamma$. A simple closed curve in $\Gamma_m$ is called a ring, if it contains a crossing but does not contain a white vertex nor a black vertex. An arc is said to be internal if it is contained in an internal edge or a ring.

**Remark 2.3** Let $\Gamma$ be a minimal chart. Then we have the following:

1. *If an edge of $\Gamma$ contains a black vertex, then it is a terminal edge or a free edge.* For, if the edge contains a crossing, then we can eliminate the crossing on the edge by a C-II move. This contradicts that the chart is minimal.

2. *Any terminal edge of $\Gamma$ contains a middle arc at its white vertex.* For, if not, we can eliminate the white vertex by a C-III move.

3. *Each complementary domain of any ring must contain at least one white vertex (cf. Assumption 4).* For, suppose that there exists a ring $C$ such that a complementary domain of $C$ does not contain any white vertices. Let $F$ be the closure of the complementary domain. By (1), the ring $C$ does not intersect any terminal edge nor free edge. Thus any crossing on $C$ is contained in a proper internal arc of $F$. Since $F$ is a disk or an annulus, the domain $F$ contains a disk $D$ bounded by an arc $\ell_1$ on $C$ and a proper internal arc $\ell_2$ of $F$ such that any crossing on Int $\ell_1$ is contained in a proper internal arc $\ell$ of $D$ intersecting $\ell_2$ by a crossing (see Fig. 11(a)). Let $\ell'_2$ be an internal arc with $\ell'_2 \supset \ell_2$ such that $\ell'_2 - \ell_2$ does not contain a crossing. Let $\ell'_1$ be an arc outside $F$ parallel to $\ell_1$ with $\partial \ell'_1 = \partial \ell'_2$ (see Fig. 11(b)). Thanks to Assumption 1 and Assumption 2, there does not exist any black vertex in the disk bounded by $\ell'_1 \cup \ell'_2$. Thus we can shift the arc $\ell'_2$ to the arc $\ell'_1$ by a
CI-move so that the number of crossings decreases at least two (see Fig. 11(c)). This contradicts that the chart is minimal.

3 Admissible consecutive triplets

In this section we introduce one of our main tools, Consecutive Triplet Lemma, which minimal charts satisfy. Also an original idea for Theorem 1.1 is given in Lemma 3.3.

Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. An internal edge of label $m$ is called a loop if it contains exactly one white vertex.

Let $E$ be a disk, and $\ell_1, \ell_2, \ell_3$ three arcs on $\partial E$ such that each of $\ell_1 \cap \ell_2$ and $\ell_2 \cap \ell_3$ is one point and $\ell_1 \cap \ell_3 = \emptyset$ (see Fig. 12 a)), say $p = \ell_1 \cap \ell_2$, $q = \ell_2 \cap \ell_3$. Let $\Gamma$ be a chart in a disk $D^2$. Let $e_1$ be a terminal edge of $\Gamma$.

A triplet $(e_1, e_2, e_3)$ of mutually different edges of $\Gamma$ is called a consecutive triplet if there exists a continuous map $f$ from the disk $E$ to the disk $D^2$ such that (see Fig. 12 b) and (c))

(i) the map $f$ is injective on $E - \{p, q\}$,
(ii) $f(\ell_3)$ is an arc in $e_3$, and $f(\text{Int } E) \cap \Gamma = \emptyset$, $f(\ell_1) = e_1$, $f(\ell_2) = e_2$,
(iii) $f(p)$ and $f(q)$ are white vertices.

If the label of $e_3$ is different from the one of $e_1$ then the consecutive triplet is said to be admissible.

Lemma 3.1 (Consecutive Triplet Lemma) ([13, Lemma 1.1]) Any consecutive triplet in a minimal chart is admissible.

The above lemma was proven by the maximality of bigons in a minimal chart [13]. In Section 11 we shall give a proof of Consecutive Triplet Lemma to make our paper to be self-contained.

In the proof of Lemma 3.3 we must be careful of the next remark.
Remark 3.2 Let \( w \) be the white vertex of a loop. In a small neighborhood of the white vertex \( w \), the loop contains two short arcs \( \gamma, \gamma' \) with \( \gamma \cap \gamma' = w \). One of the two arcs \( \gamma, \gamma' \) is a middle arc at \( w \), but the other is not a middle arc at \( w \).

Let \( \Gamma \) be a chart. Let \( \gamma_1, \gamma_2, \cdots, \gamma_6 \) be six short arcs around a white vertex \( w \) lying clockwise in this order (see Fig. 10(a)). For each \( j = 1, 2, \cdots, 6 \) let \( e_j \) be the edge containing \( \gamma_j \) (possibly \( e_j = e_j+2 \)), here \( e_{j+6} = e_j \) for each integer \( j \). Let \( e = e_k \) for some \( k \in \{1, 2, \cdots, 6\} \). Then the two edges \( e_{k-1}, e_{k+1} \) are called the (\( e, w \))-edges.

Lemma 3.3 Let \( \Gamma \) be a minimal chart, and \( m \) a label of \( \Gamma \). Let \( U \) be a finite complementary domain of \( \Gamma_m \) with \( \Gamma \cap U \subset \Gamma_{m-1} \cup \Gamma_{m+1} \). Suppose that \( U \) does not contain any crossing. Then we have the following:

(a) The component \( U \) does not contain any white vertex.

(b) \( \text{Cl}(U) \) does not contain any terminal edge of label \( m \).

(c) If \( U \) is an open disk and if \( \text{Cl}(U) \) contains a white vertex in \( \Gamma_m \), then there exist at least two middle arcs of label \( m \pm 1 \) in \( \text{Cl}(U) \).

Proof. Since there is no white vertex contained in \( \Gamma_{m-1} \cap \Gamma_{m+1} \), Statement (a) holds.

We show Statement (b). Suppose that \( \text{Cl}(U) \) contains a terminal edge \( e_1 \) of label \( m \). Let \( v_1 \) be the white vertex of \( e_1 \). Then \( v_1 \in \partial U \) by Statement (a). Let \( e_2 \) be an \( (e_1, v_1) \)-edge. Then \( e_2 \) is of label \( m \pm 1 \) and \( e_2 \cap U \neq \emptyset \). The edge \( e_2 \) is not a terminal edge. For, if \( e_2 \) is a terminal edge, then by Remark 2.3(2) one of terminal edges \( e_1 \) and \( e_2 \) does not contain a middle arc at \( v_1 \). This contradicts Remark 2.3(2).

If \( e_2 \) is not a loop, then there exists an edge \( e_3 \) of label \( m \) in \( \partial U \) such that the consecutive triplet \((e_1, e_2, e_3)\) is not admissible (see Fig. 13(a)). This contradicts Consecutive Triplet Lemma (Lemma 3.1).

Suppose that \( e_2 \) is a loop. In a regular neighborhood \( N \) of \( v_1 \), the edge \( e_2 \) contains two short arcs \( \gamma_1, \gamma_2 \) with \( \gamma_1 \cap \gamma_2 = v_1 \). Now \( e_1 \) contains a short
arc $\gamma_3$ in $N$ with $\gamma_3 \ni v_1$. If $\gamma_1, \gamma_2, \gamma_3$ are not consecutive around $v_1$ (see Fig. 13(b)), then $v_1 \in \text{Int}(Cl(U))$. Hence again there exists an edge $e_3$ of label $m$ in $\partial U$ such that the consecutive triplet $(e_1, e_2, e_3)$ is admissible (see Fig. 13(b)). This contradicts Consecutive Triplet Lemma (Lemma 3.1).

Suppose that $\gamma_1, \gamma_2, \gamma_3$ are consecutive around $v_1$. Then $\gamma_1, \gamma_3, \gamma_2$ are consecutive arcs situated around $v_1$ in this order (see Fig. 13(c)). By Remark 2.3(2), the arc $\gamma_3$ is middle at $v_1$. Since $e_2$ is a loop, by Remark 3.2 one of $\gamma_1, \gamma_2$ is middle at $v_1$, say $\gamma_1$. Then the two consecutive arcs $\gamma_1, \gamma_3$ are middle at $v_1$. This contradicts Remark 2.2(1). Hence Statement (b) holds.

We show Statement (c). Let $W = \{v \mid v$ is a white vertex contained in a middle arc of label $m \pm 1$ in $Cl(U)\}$. We shall show $W \neq \emptyset$. Suppose $W = \emptyset$. By the assumption, $Cl(U)$ contains a white vertex $w$ in $\Gamma_m$. Then $w \in \partial U = Cl(U) - U$ by Statement (a). Let $e$ be an edge of label $m \pm 1$ intersecting $U$ and containing the white vertex $w$. Since $W = \emptyset$, the edge $e$ does not contain a middle arc at $w$. By Remark 2.3(2), the edge $e$ is not a terminal edge.

We claim that the edge $e$ is not a loop. For, suppose that the edge $e$ is a loop. Since there is no crossing in $\Gamma_{m+1} \cap \Gamma_m$, the edge $e$ must be contained in $Cl(U)$. By Remark 3.2 the edge $e$ contains a middle arc at $w$. Thus $w \in W$. This contradicts $W = \emptyset$. Hence the edge $e$ is not a loop.

Thus the edge $e$ contains two white vertices in $\partial U$, say $w_1 = w$ and $w_2$. Then for $i = 1, 2$, there exist two edges $e_{1i}$ and $e_{2i}$ of label $m$ in $\partial U$ containing $w_i$. Go around $\partial U$ starting from $e_{11}$ and next pass through $e_{12}$ and so on. Without loss of generality we can assume that $e_{11}$ is oriented inward at $w_1$ and that we pass $e_{11}, e_{12}, \cdots, e_{21}, e_{22}$ in this order. Since $Cl(U)$ does not contain a terminal edge of label $m$ by Statement (b), when we go around $\partial U$ (see Fig. 13(d)), we have that

(1) the orientation of the edge in $\partial U$ changes at a vertex in $W$.

Namely, on the way going around $\partial U$, if an edge and the next edge are inward (or outward) at a white vertex, then the white vertex is contained in $W$. Since $W = \emptyset$, Statement (1) implies that the edge $e_{12}$ is oriented outward at $w_1$, the edge $e_{21}$ is oriented inward at $w_2$, and the edge $e_{22}$ is oriented outward at $w_2$ (see Fig. 13(e)). Thus applying two C-I-M2 moves between $e_{11}$ and $e_{22}$ and between $e_{12}$ and $e_{21}$ (see Fig. 13(f)), we can eliminate the two white vertices $w_1$ and $w_2$ by a C-I-M3 move. This contradicts the fact that the chart $\Gamma$ is minimal. Thus $W \neq \emptyset$.

Again Statement (1) implies that the set $W$ consists of exactly even number of vertices. Thus there exist at least two middle arcs of label $m \pm 1$ in $Cl(U)$. □

Lemma 3.3(c) is our start point for Theorem 1.1.

**Corollary 3.4** Let $\Gamma$ be a minimal chart, and $m$ a label of $\Gamma$. Then we have the following:
Figure 13: In (a),(b),(c),(d),(e), each gray region is a finite complementary domain $U$ of $\Gamma_m$. In (c), the thick lines are the short arcs $\gamma_1, \gamma_2, \gamma_3$. In (d), each arc with three transversal short arcs is a middle arc in $Cl(U)$ of label $m \pm 1$.

(a) If $E$ is a 3-color disk with $\partial E \subset \Gamma_m$, then $E$ does not contain any terminal edge of label $m$.

(b) If $E$ is a 2-color disk with $\partial E \subset \Gamma_m$, then $E$ does not contain any terminal edge.

Proof. We show Statement (a). Since the 3-color disk $E$ does not contain any crossing, each component of $E - \Gamma_m$ does not contain any crossing. By Lemma 3.3(b), the closure of each component of $E - \Gamma_m$ does not contain any terminal edge of label $m$, and so does $E$. Thus Statement (a) holds.

We show Statement (b). Since a 2-color disk is a 3-color disk, the 2-color disk $E$ does not contain any terminal edge of label $m$ by Statement (a).

Let $k$ be the label with $|m - k| = 1$ and $\Gamma \cap E \subset \Gamma_m \cup \Gamma_k$. Suppose that there exists a terminal edge of label $k$ in $E$. Since any white vertex in $\Gamma \cap E$ is contained in $\Gamma_m \cap \Gamma_k$, we can find a non-admissible consecutive triplet
by a similar way to Lemma 3.3(b). This contradicts Consecutive Triplet Lemma (Lemma 3.1). Hence $E$ does not contain any terminal edge of label $k$. Therefore $E$ does not contain any terminal edge. \[\square\]

4 A proof of Theorem 1.1

In this section we shall prove Theorem 1.1 by using Lemma 3.3(c), Corollary 3.4(a), Lemma 4.2 and Lemma 4.3.

Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. A simple closed curve in $\Gamma_m$ is called a cycle of label $m$. We consider hoops, rings and loops as cycles.

Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. Let $C$ be a cycle of label $m$ bounding a disk $E$. An edge $e$ of label $m$ is called an inside (resp. outside) edge for $C$ provided that

(i) $e \cap C$ consists of one white vertex or two white vertices, and

(ii) $e \subset E$ (resp. $e \subset Cl(E^c)$).

For a cycle $C$ of label $m$ bounding a disk $E$, we define

$\mathcal{W}(E) = \{w \mid w$ is a white vertex in $E\}$,

$\mathcal{W}(Int \ E) = \{w \mid w$ is a white vertex in $Int \ E\}$,

$\mathcal{W}(C) = \{w \mid w$ is a white vertex in $C\}$,

$\mathcal{W}_I(C, m) = \{w \in \mathcal{W}(C) \mid w$ is contained in an inside edge for $C\}$,

$\mathcal{W}_{I\text{mid}}(C, m) = \{w \in \mathcal{W}(C) \mid$ there exists an inside edge for $C$ middle at $w\}$,

$\mathcal{W}_O(C, m) = \{w \in \mathcal{W}(C) \mid w$ is contained in an outside edge for $C\}$,

$\mathcal{W}_{O\text{mid}}(C, m) = \{w \in \mathcal{W}(C) \mid$ there exists an outside edge for $C$ middle at $w\}$.

Remark 4.1 Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. For a cycle $C$ of label $m$ bounding a disk $E$, we have the following:

1. Let $w$ be a white vertex in $\mathcal{W}_O(C, m)$. Then the vertex $w$ is in $\mathcal{W}_{O\text{mid}}(C, m)$ if and only if there exists an arc of label $m \pm 1$ in $E$ middle at the vertex $w$.

2. Let $w$ be a white vertex in $\mathcal{W}_I(C, m)$. Then the vertex $w$ is not in $\mathcal{W}_{I\text{mid}}(C, m)$ if and only if there exists an arc of label $m \pm 1$ in $E$ middle at the vertex $w$.

3. Let $w$ be a white vertex in $\mathcal{W}_I(C, m)$. Then the vertex $w$ is in $\mathcal{W}_{I\text{mid}}(C, m)$ if and only if there exists an arc of label $m \pm 1$ in $Cl(E^c)$ middle at the vertex $w$.

4. The set $\mathcal{W}(C)$ splits into disjoint subsets $\mathcal{W}_O(C, m)$ and $\mathcal{W}_I(C, m)$.

5. The set $\mathcal{W}(E)$ splits into three mutually disjoint subsets $\mathcal{W}(Int \ E)$, $\mathcal{W}_O(C, m)$ and $\mathcal{W}_I(C, m)$.
Lemma 4.2 Let $\Gamma$ be a minimal chart, and $m$ a label of $\Gamma$. Let $E$ be a 3-color disk with $\partial E \subset \Gamma_m$ but without free edges nor simple hoops. If $\Gamma_m \cap E$ is connected, then $E$ contains neither hoop nor ring.

Proof. Suppose that $E$ contains a hoop. By the assumption, the hoop is not simple. Thus there exists a white vertex $v$ in the interior of the disk bounded by the hoop. Any white vertex in $E$ is in $\Gamma_m \cap \Gamma_{m-1}$ or $\Gamma_m \cap \Gamma_{m+1}$. Thus $\Gamma_m \cap E$ contains at least two connected components; one containing the vertex $v$, and the other containing $\partial E$. Hence $\Gamma_m \cap E$ is not connected. This is a contradiction. Thus $E$ does not contain any hoop.

Since any 3-color disk does not contain any crossing, the disk $E$ does not contain any ring. \hfill \Box

Let $\Gamma$ be a chart $\Gamma$, and $m$ a label of $\Gamma$. For a 3-color disk $E$ with $\partial E \subset \Gamma_m$, we define

$$W^\text{Mid}(E, m \pm 1) = \{ v \in W(E) \mid \text{there is an arc of label } m \pm 1 \text{ in } E \text{ middle at } v \}.$$

Lemma 4.3 Let $\Gamma$ be a minimal chart, and $m$ a label of $\Gamma$. Let $E$ be a 3-color disk bounded by a cycle $C$ in $\Gamma_m$. Then we have

$$|W^\text{Mid}(E, m \pm 1)| = |W(\text{Int } E)| + |W^\text{Mid}_O(C, m)| + |W_I(C, m)| - |W^\text{Mid}_I(C, m)|.$$

Proof. By the definition of $W^\text{Mid}(E, m \pm 1)$, we have $W^\text{Mid}(E, m \pm 1) \subset W(E)$. By Remark 4.1(4) and (5), the set $W(E)$ is the disjoint union of $W(\text{Int } E)$ and $W(C)$. Thus

$$W^\text{Mid}(E, m \pm 1) - W(\text{Int } E) = W^\text{Mid}(E, m \pm 1) \cap W(C).$$

Since $W(C)$ is the disjoint union of $W_O(C, m)$ and $W_I(C, m)$ by Remark 4.1(4), we have

$$|W^\text{Mid}(E, m \pm 1) - W(\text{Int } E)| = |W^\text{Mid}(E, m \pm 1) \cap W(C)| = |(W^\text{Mid}(E, m \pm 1) \cap W_O(C, m) \cup W_I(C, m))| = |W^\text{Mid}(E, m \pm 1) \cap W_O(C, m) \cup (W^\text{Mid}(E, m \pm 1) \cap W_I(C, m))| = |W^\text{Mid}(E, m \pm 1) \cap W_O(C, m)| + |W^\text{Mid}(E, m \pm 1) \cap W_I(C, m)|.$$

Now by Remark 4.1(1), we have

$$W^\text{Mid}(E, m \pm 1) \cap W_O(C, m) = \{ v \in W_O(C, m) \mid \text{there exists an arc of } m \pm 1 \text{ in } E \text{ middle at } v \} = W^\text{Mid}_O(C, m).$$

Using Remark 4.1(2), we have

$$W^\text{Mid}(E, m \pm 1) \cap W_I(C, m) = \{ v \in W_I(C, m) \mid \text{there exists an arc of } m \pm 1 \text{ in } E \text{ middle at } v \} = \{ v \in W_I(C, m) \mid v \notin W^\text{Mid}_I(C, m) \} = W_I(C, m) - W^\text{Mid}_I(C, m).$$

Therefore

$$|W^\text{Mid}(E, m \pm 1)| = |W(\text{Int } E)| + |W^\text{Mid}(E, m \pm 1) - W(\text{Int } E)| = |W(\text{Int } E)| + |W^\text{Mid}(E, m \pm 1) \cap W_O(C, m)| + |W^\text{Mid}(E, m \pm 1) \cap W_I(C, m)| = |W(\text{Int } E)| + |W^\text{Mid}(E, m \pm 1) \cap W_O(C, m)| + |W_I(C, m) - W^\text{Mid}_I(C, m)| = |W(\text{Int } E)| + |W^\text{Mid}_O(C, m)| + |W_I(C, m)| - |W^\text{Mid}_I(C, m)|. \hfill \Box$$
Proof of Theorem 1.1. Let \( C = \partial E \). Since the disk \( E \) is a 3-color disk, (1) \( E \) does not contain any crossing by Condition (i) of a 3-color disk. Since \( \Gamma_m \cap E \) is connected by the assumption, (2) \( E \) does not contain a hoop nor a ring by Lemma 4.2.

Thus

(3) for each connected component of \( E - \Gamma_m \), its closure contains a white vertex.

Since \( E \) is a 3-color disk with \( \partial E \subset \Gamma_m \), by Corollary 3.4(a) the disk \( E \) does not contain any terminal edge of label \( m \). Hence all the vertices of \( \Gamma_m \) in \( E \) are white vertices. Let \( V \) be the number of white vertices of \( \Gamma_m \) in \( E \), and \( E \) the number of edges of \( \Gamma_m \) in \( E \), and \( F \) the number of connected components of \( E - \Gamma_m \). Since \( E \) is a 3-color disk with \( \partial E \subset \Gamma_m \), all the white vertices in \( E \) are white vertices of \( \Gamma_m \). Thus we have

(4) \( V = |W(E)| \).

By Remark 4.1(5), the set \( W(E) \) splits into three mutually disjoint subsets \( W(\text{Int } E) \), \( W_0(C, m) \) and \( W_I(C, m) \). Thus by (4), we have

(5) \( V = |W(\text{Int } E)| + |W_0(C, m)| + |W_I(C, m)| \).

Since the intersection \( \Gamma_m \cap E \) is connected by the assumption,

(6) each connected component of \( E - \Gamma_m \) is an open disk.

Claim 1. \( 2F = 2 + V - |W_0(C, m)| \).

Proof of Claim 1. In a small neighbourhood of a white vertex in \( \Gamma_m \), there are exactly three short arcs of label \( m \) intersecting each other at the white vertex. We fix the three short arcs of label \( m \) for each white vertex in \( \Gamma_m \). Each edge of \( \Gamma_m \) in \( E \) has two short arcs of label \( m \). Thus there are \( 2E \) short arcs in \( E \). On the other hand, any white vertex in \( W(\text{Int } E) \cup W_I(C, m) \) is incident with three short arcs of label \( m \) in \( E \), and any white vertex in \( W_0(C, m) \) is incident with two short arcs of label \( m \) in \( E \). Hence we have

\[
2E = 3(|W(\text{Int } E)| + |W_I(C, m)|) + 2|W_0(C, m)|.
\]

Thus by the equation (5), we have

\[
2E = 3(|W(\text{Int } E)| + |W_I(C, m)| + |W_0(C, m)|) - |W_0(C, m)|
= 3V - |W_0(C, m)|.
\]

Since \( E \) is a disk, we have the equation \( V - E + F = 1 \) by Euler formula. Hence

\[
2F = 2 - 2V + 2E
= 2 - 2V + (3V - |W_0(C, m)|)
= 2 + V - |W_0(C, m)|.
\]

This completes the proof of Claim 1.

Claim 2. \(|\mathcal{M}^\text{Med}(E, m \pm 1)| - 2F \geq 0\).

Proof of Claim 2. Let \( \mathcal{M}^*(E, m \pm 1) \) be the set of all the middle arcs of label \( m \pm 1 \) in \( E \). Now
(7) each white vertex is contained in exactly one middle arc of label $m \pm 1$. Hence we have

(8) $|M^*(E, m \pm 1)| = |W_{\text{Mid}}^E(m, m \pm 1)|$.

By (1), (3) and (6), Lemma 3.3(c) assures that for each connected component $U$ of $E - \Gamma_m$, the closure $\text{Cl}(U)$ contains at least two middle arcs of label $m \pm 1$ in $M^*(E, m \pm 1)$. Hence (7) assures us

$|M^*(E, m \pm 1)| - 2F \geq 0$.

Thus (8) implies $|W_{\text{Mid}}^E(m, m \pm 1)| - 2F \geq 0$. This completes the proof of Claim 2.

By Lemma 4.3 and (5), we have

$|W_{\text{Mid}}^E(m, m \pm 1)| = |W_{\text{Int}}^E| + |W_{O}^E(C, m)| + |W_{I}^E(C, m)| - |W_{\text{Mid}}^E(C, m)|$.

By Claim 1 and Claim 2, we have

$0 \leq |W_{\text{Mid}}^E(m, m \pm 1)| - 2F$

$= (V - |W_{O}^E(C, m)| + |W_{\text{Mid}}^E(C, m)| - |W_{I}^E(C, m)|) - (2 + V - |W_{O}^E(C, m)|)$

$= |W_{I}^E(C, m)| - |W_{\text{Mid}}^E(C, m)| - 2$.

Therefore $|W_{I}^E(C, m)| + 2 \leq |W_{\text{Mid}}^E(C, m)|$. This proves Theorem 1.1. □

Corollary 4.4 Let $\Gamma$ be a minimal chart, and $m$ a label of $\Gamma$. Let $E$ be a 3-color disk with $\partial E \subset \Gamma_m$ but without free edges nor simple hoops. If $\Gamma_m \cap E$ is connected, then we have

$|W_{\text{Mid}}^E(m, m \pm 1)| \geq 2$.

5 Dichromatic one-side pseudo paths and 2-color disks

In this section, we investigate an arc in the boundary of a 2-color disk $E$ with $\partial E \subset \Gamma_m$ for a minimal chart $\Gamma$. We shall show that if $\Gamma_m \cap E$ is connected and if $E$ contains neither free edge nor simple hoop, then $|W_{O}(\partial E, m) - W_{\text{Mid}}^E(\partial E, m)| \geq 2$.

Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. A simple arc $P$ in $\Gamma_m$ is called a path of label $m$ provided that the endpoints of $P$ are vertices of $\Gamma$. A simple arc $P^*$ in $\Gamma_m$ is called a pseudo path of label $m$ provided that

(i) $P^*$ contains at least one vertex of $\Gamma$, and

(ii) the endpoints of $P^*$ are not black vertices, crossings, nor white vertices.

Let $P^*$ be a pseudo path of label $m$ in a chart $\Gamma$. A disk $\Delta$ is called a side-disk of $P^*$ provided that $P^* \subset \partial \Delta$ (see Fig. 14). Let $N$ be a regular neighborhood of $P^*$ in the side-disk $\Delta$. Let $e$ be an edge of $\Gamma$, and $\gamma$ the
closure of a connected component of $e \cap \text{Int } N$. If $\gamma$ contains a white vertex in $P^*$, then $\gamma$ is called a side-arc of $P^*$ with respect to $\Delta$. A side-arc is said to be at a vertex $v$ if it contains the vertex $v$. Similarly, a side-arc is said to be middle at a vertex $v$ if it contains a middle arc at $v$.

Figure 14: The thick line is a pseudo path $P^*$, and the gray area is a side-disk $\Delta$.

Let $P^*$ be a pseudo path of label $m$ in a chart with a side-disk $\Delta$. The pseudo path $P^*$ is said to be inward (resp. outward) with respect to $\Delta$ provided that

(i) all the vertices in $P^*$ are white vertices, and

(ii) for each vertex $v$ in $P^*$, any side-arc at $v$ with respect to $\Delta$ is oriented inward (resp. outward) at $v$.

An inward pseudo path and an outward pseudo path are called $I/O$ pseudo paths.

Let $P^*$ be a pseudo path of label $m$ in a chart, and $v_1, v_2, \cdots, v_s$ all the vertices in $P^*$ which are situated in this order on $P^*$, here some of $v_1, v_2, \cdots, v_s$ may be crossings. For each $i = 1, 2, \cdots, s - 1$, let $e_i$ be the edge of label $m$ in $P^*$ with $\partial e_i = \{v_i, v_{i+1}\}$. Then the $s$-tuple $(v_1, v_2, \cdots, v_s)$ is called the associated vertex sequence for the pseudo path $P^*$, and the $(s - 1)$-tuple $(e_1, e_2, \cdots, e_{s-1})$ is called the associated edge sequence for the pseudo path $P^*$. The path $e_1 \cup e_2 \cup \cdots \cup e_{s-1}$ is denoted by $L(P^*)$. The path $L(P^*)$ is the maximal path contained in the pseudo path $P^*$. Let $\gamma_0$ and $\gamma_s$ be arcs in edges of label $m$ with $\gamma_0 \ni v_1, \gamma_s \ni v_s$ and $P^* = \gamma_0 \cup L(P^*) \cup \gamma_s$. Then $\gamma_0, \gamma_s$ are called the end-arcs of the pseudo path $P^*$.

Let $P^*$ be a pseudo path of label $m$ in a chart, and $(v_1, v_2, \cdots, v_s)$ the associated vertex sequence for $P^*$. The pseudo path $P^*$ is said to be admissible provided that

(i) the vertices $v_1, v_s$ are white vertices, possibly $v_1 = v_s$,

(ii) there exists a side-disk $\Delta$ so that for each $i = 1, s$, there does not exist a side-arc of label $m$ at $v_i$ with respect to the side-disk $\Delta$.

We also say that $P^*$ is admissible for the side-disk $\Delta$. 

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Important Notice. Let $P^*$ be a pseudo path of label $m$ in a chart, and $D$ a disk with $P^* \cap \partial D = \partial P^*$. Then $P^*$ splits $D$ into two disks $\Delta_1, \Delta_2$. Both of the two disks are side-disks of $P^*$. If $P^*$ is an admissible pseudo path, then $P^*$ is admissible for one of the two side-disks $\Delta_1, \Delta_2$, but NOT admissible for the other side-disk. Thus for admissible pseudo paths, we do not mention side-disks unless the side-disks are needed to be mentioned. Thus for an admissible pseudo path $P^*$, ’a side-arc’ means a side-arc with respect to a side-disk for which $P^*$ is admissible.

A pseudo path $P^*$ of label $m$ in a chart is called a dichromatic pseudo path if there exists a label $k$ with $|m - k| = 1$ such that any vertex in $P^*$ is contained in an edge of label $k$. The label $k$ is called the secondary label of the dichromatic pseudo path $P^*$.

A pseudo path $P^*$ of label $m$ in a chart is called a one-side pseudo path if there exists a side-disk $\Delta$ of $P^*$ such that (see Fig. 15)

(i) all the vertices in $P^*$ are white vertices, and

(ii) for each vertex in $P^*$ the label of any side-arc with respect to $\Delta$ is different from $m$.

Remark 5.1  (1) Any one-side pseudo path is admissible. Thus we do not mention side-disks for one-side pseudo paths.

(2) If $P^*$ is a one-side (resp. dichromatic) pseudo path, then any pseudo path in $P^*$ is a one-side (resp. dichromatic) pseudo path.

Let $P^*$ be a one-side pseudo path of label $m$ in a chart, and $(v_1, v_2, \cdots, v_s)$ the associated vertex sequence for $P^*$. Then for each vertex $v_i$ ($i = 1, 2, \cdots, s$), there exists a side-arc $\gamma'_i$ of label $m \pm 1$ at $v_i$. The $s$-tuple $(\gamma'_1, \gamma'_2, \cdots, \gamma'_s)$ is called an associated side-arc sequence for the one-side pseudo path $P^*$.

Lemma 5.2 Let $\Gamma$ be a minimal chart. Let $P^*$ be a dichromatic one-side pseudo path of label $m$ in $\Gamma$ with the associated vertex sequence $(v_1, v_2, \cdots, v_s)$, the associated edge sequence $(e_1, e_2, \cdots, e_{s-1})$ and an associated side-arc sequence $(\gamma'_1, \gamma'_2, \cdots, \gamma'_s)$. Let $\gamma_0$ and $\gamma_s$ be the end-arcs of $P^*$ with $\gamma_0 \ni v_1$ and $\gamma_s \ni v_s$. Suppose that each side-arc $\gamma'_i$ ($i = 1, 2, \cdots, s$) is not middle at $v_i$. Then we have the following:
(a) If the end-arc $\gamma_0$ is oriented inward at $v_1$, then each edge $e_i$ $(i = 1, 2, \cdots, s-1)$ is oriented inward at $v_{i+1}$, and the end-arc $\gamma_s$ is oriented outward at $v_s$.

(b) If the end-arc $\gamma_0$ is oriented outward at $v_1$, then each edge $e_i$ $(i = 1, 2, \cdots, s-1)$ is oriented outward at $v_{i+1}$, and the end-arc $\gamma_s$ is oriented inward at $v_s$.

(c) The pseudo path $P^*$ is an I/O pseudo path.

**Proof.** Set $e_0 = \gamma_0, e_s = \gamma_s$. Let $v_{s+1}$ be the endpoint of $\gamma_s$ different from $v_s$.

We show Statement (a). Now the end-arc $\gamma_0$ is oriented inward at $v_1$. Suppose that for some integer $i$ $(1 \leq i \leq s)$ $e_i$ is oriented outward at $v_{i+1}$. Let $t = \min \{ j \mid e_j$ is oriented outward at $v_{j+1} \}$. Then $t \geq 1$ and $e_t$ is oriented outward at $v_{t+1}$. Thus $e_t$ is oriented inward at $v_t$. But $e_{t-1}$ is oriented inward at $v_t$. Hence the side-arc $\gamma_t'$ is middle at $v_1$. This contradicts the assumption. Hence Statement (a) holds.

Similarly we can show Statement (b).

We show Statement (c). We only show the case that the end-arc $\gamma_0$ is oriented inward at $v_1$. Then by Statement (a), each edge $e_i$ $(i = 1, 2, \cdots, s-1)$ is oriented inward at $v_{i+1}$, and the end-arc $\gamma_s$ is oriented outward at $v_s$. Suppose that the pseudo path $P^*$ is not an I/O pseudo path. Then for some two integers $i, j$ with $1 \leq i < j \leq s$, one of the following occurs.

(1) The side-arc $\gamma_i'$ is oriented inward at $v_i$, and the side-arc $\gamma_j'$ is oriented outward at $v_j$.

(2) The side-arc $\gamma_i'$ is oriented outward at $v_i$, and the side-arc $\gamma_j'$ is oriented inward at $v_j$.

Without loss of generality we can assume $j = i + 1$. We show that we can eliminate the two white vertices $v_i$ and $v_{i+1}$ by C-moves. For, either Case (1) or Case (2), we can apply a C-I-M2 move between the two side-arcs $\gamma_i'$ and $\gamma_{i+1}'$ (see Fig. 16(a) and (b)). Since $e_{i-1}$ is oriented inward at $v_i$ and since $e_{i+1}$ is oriented inward at $v_{i+2}$, we can apply a C-I-M2 move between $e_{i-1}$ and $e_{i+1}$ (see Fig. 16(c)). Finally by applying a C-I-M3 move, we can eliminate the two white vertices $v_i$ and $v_{i+1}$. This contradicts the fact that the chart $\Gamma$ is minimal. Hence the pseudo path $P^*$ is an I/O pseudo path. Hence Statement (c) holds. \hfill $\Box$

**Lemma 5.3** Let $\Gamma$ be a minimal chart. Let $P^*$ be a dichromatic one-side pseudo path of label $m$ in $\Gamma$ with the associated vertex sequence $(v_1, v_2, \cdots, v_s)$ and an associated side-arc sequence $(\gamma'_1, \gamma'_2, \cdots, \gamma'_s)$. Let $\gamma_0$ and $\gamma_s$ be the end-arcs of $P^*$ with $\gamma_0 \ni v_1$ and $\gamma_s \ni v_s$. Suppose that

(a) the end-arc $\gamma_0$ is middle at $v_1$ or the end-arc $\gamma_s$ is middle at $v_s$, and

(b) the side-arc $\gamma_i'$ is middle at $v_i$ for all $1 \leq i \leq s$.

Then $\Gamma$ is not a minimal chart.

Proof. We show that we can eliminate two white vertices $v_i$ and $v_{i+1}$ by C-moves. For, either Case (1) or Case (2), we can apply a C-I-M2 move between the two side-arcs $\gamma_i'$ and $\gamma_{i+1}'$. Since $e_{i-1}$ is oriented inward at $v_i$ and since $e_{i+1}$ is oriented inward at $v_{i+2}$, we can apply a C-I-M2 move between $e_{i-1}$ and $e_{i+1}$. Finally by applying a C-I-M3 move, we can eliminate the two white vertices $v_i$ and $v_{i+1}$. This contradicts the fact that the chart $\Gamma$ is minimal. Hence $\Gamma$ is not a minimal chart. \hfill $\Box$
(b) for each $i = 2, 3, \ldots, s - 1$, the side-arc $\gamma'_i$ is not middle at $v_i$.

Then the pseudo path $P^*$ is an I/O pseudo path.

Proof. By (a), without loss of generality we can assume that

(1) the end-arc $\gamma_s$ is middle at $v_s$.

The side-arc $\gamma'_1$ is middle at $v_1$ or not middle at $v_1$.

First, suppose that the side-arc $\gamma'_1$ is not middle at $v_1$. Then by (b) and (1), we have that for each $i = 1, 2, \ldots, s$, the side-arc $\gamma'_i$ is not middle at $v_i$. Thus by Lemma 5.2(c), the pseudo path $P^*$ is an I/O pseudo path.

Next, suppose that the side-arc $\gamma'_1$ is middle at $v_1$. If $s = 1$ then our lemma is true. Thus we assume $s \geq 2$. Let $(e_1, e_2, \ldots, e_{s-1})$ be the associated edge sequence for $P^*$. Without loss of generality we can assume that

(2) the side-arc $\gamma'_1$ is oriented inward at $v_1$.

Then the edge $e_1$ is oriented inward at $v_1$. Thus the edge $e_1$ is oriented outward at $v_2$. Since the end-arc $\gamma_s$ is middle at $v_s$ by (1), the side-arc $\gamma'_s$ is not middle at $v_s$. Hence by (b), we have that for each $i = 2, 3, \ldots, s$, the side-arc $\gamma'_i$ is not middle at $v_i$. Let $\gamma$ be a short arc containing $v_2$ in the edge $e_1$. We can apply Lemma 5.2 to the dichromatic one-side pseudo path $P^*_1 = \gamma \cup e_2 \cup \cdots \cup e_{s-1} \cup \gamma_s$. Then the pseudo path $P^*_1$ is an I/O pseudo path, and the end-arc $\gamma_s$ is oriented inward at $v_s$ because $e_1$ is oriented outward at $v_2$ (i.e. the arc $\gamma$ is oriented outward at $v_2$). Thus $\gamma'_s$ is oriented inward at $v_s$ because the end-arc $\gamma_s$ is middle at $v_s$ by (1). Hence the pseudo path $P^*_1$ is an inward pseudo path. Thus for each $i = 2, 3, \ldots, s$, the side-arc $\gamma'_i$ is oriented inward at $v_i$. Considering (2), we have for each $i = 1, 2, \ldots, s$, the side-arc $\gamma'_i$ is oriented inward at $v_i$. Therefore the pseudo path $P^*$ is an inward pseudo path. This completes the proof of Lemma 5.3. □

Lemma 5.4 Let $\Gamma$ be a minimal chart. Let $P^*$ be a dichromatic one-side pseudo path of label $m$ in $\Gamma$ with the associated vertex sequence $(v_1, v_2, \ldots, v_s)$ and an associated side-arc sequence $(\gamma'_1, \gamma'_2, \ldots, \gamma'_s)$. Suppose that the end-arcs $\gamma_0$ and $\gamma_s$ of $P^*$ are middle at $v_1$ and $v_s$ respectively. Then we have the following:
Figure 17: The vertices $v_1, v_2$ are in $\mathcal{W}_O(C, m)$. (a) The thick line is a path $P$. (b) The thick line is the extended pseudo path $\hat{P}$.

(a) $s \geq 3$.

(b) For some integer $t$ with $2 \leq t \leq s - 1$, the side-arc $\gamma'_t$ is middle at $v_t$.

**Proof.** First, we shall prove Statement (b). Without loss of generality we can assume that the end-arc $\gamma_0$ is oriented inward at $v_1$. Since $\gamma_0$ is middle at $v_1$, we have

(1) the side-arc $\gamma'_1$ is oriented inward at $v_1$.

Since the end-arcs $\gamma_0, \gamma_s$ are middle at $v_1, v_s$ respectively, Remark 2.2(1) implies

(2) the side-arc $\gamma'_1$ (resp. $\gamma'_s$) is not middle at $v_1$ (resp. $v_s$).

Now suppose that for each $i = 2, \cdots, s - 1$ the side-arc $\gamma'_i$ is not middle at $v_i$. Then by (2), we have that for $i = 1, 2, \cdots, s$ the side-arc $\gamma'_i$ is not middle at $v_i$. Hence by Lemma 5.2(c), the dichromatic one-side pseudo path $P^*$ is an I/O pseudo path. Further since the end-arc $\gamma_0$ is oriented inward at $v_1$, by Lemma 5.4(a) the end-arc $\gamma_s$ is oriented outward at $v_s$. Since $\gamma_s$ is middle at $v_s$, we have

(3) the side-arc $\gamma'_s$ is oriented outward at $v_s$.

Hence the side-arc $\gamma'_1$ is oriented outward at $v_1$ because $P^*$ is an I/O pseudo path. This contradicts (1). Hence for some integer $t$ with $2 \leq t \leq s - 1$, the side-arc $\gamma'_t$ is middle at $v_t$. Thus Statement (b) holds.

We show Statement (a). By Lemma 5.4(b), we have $s - 1 \geq 2$. Thus $s \geq 3$. □

Let $C$ be a cycle of label $m$ in a chart $\Gamma$, and $P$ a path in $C$ with $\partial P = \{v_1, v_2\} \subset \mathcal{W}_O(C, m)$ (see Fig. 17(a)). For $i = 1, 2$, let $e_i$ be the outside edge of label $m$ for $C$ containing $v_i$, and $\gamma_i$ an arc in $e_i$ containing $v_i$ (see Fig. 17(b)). Set $\hat{P} = \gamma_1 \cup P \cup \gamma_2$. Then the union $\hat{P}$ is called an extended pseudo path of $P$.

**Remark 5.5** Any extended pseudo path is admissible.
Remark 5.6 By the condition of a 2-color disk, any 2-color disk does not contain a crossing. Hence any 2-color disk is also a 3-color disk.

Let $\Gamma$ be a chart. Let $C$ be a cycle or a path of label $m$ in $\Gamma$, and $S$ a set of white vertices in $C$, here we assume that $|S| \geq 2$ if $C$ is a cycle. By cutting $C$ at all the white vertices in $S$, the set $C$ splits into paths. Then the set of all the paths is called the path decomposition of $C$ by $S$, denoted by $P(C; S)$.

Let $m$ be a label of a chart $\Gamma$, and $E$ a 2-color disk with $\partial E \subset \Gamma_m$ and $\Gamma \cap E \subset \Gamma_m \cup \Gamma_k$ for some label $k$. The label $k$ is called the secondary label of the 2-color disk $E$.

Lemma 5.7 Let $\Gamma$ be a chart and $m$ a label of $\Gamma$. Let $C$ be a cycle of label $m$ in $\Gamma$ bounding a 2-color disk. Let $P$ be a path in $P(C; W_O(C, m))$. Then any extended pseudo path of $P$ is a dichromatic one-side pseudo path.

Proof. Let $E$ be the 2-color disk with $\partial E = C$, and $k$ the secondary label of the 2-color disk $E$. By Remark 5.6, the 2-color disk does not contain any crossing, neither does $P$. Let $v$ be a white vertex in $P - \partial P$. Then $v$ is not in $W_O(C, m)$ by the assumption. Hence the vertex $v$ is in $W_I(C, m)$. Thus there exists exactly one edge of label $m \pm 1$ containing the vertex $v$ in $Cl(E^c)$. Since the cycle $C$ bounds a 2-color disk, the label of the edge is $k$. Therefore the extended pseudo path of $P$ is a dichromatic one-side pseudo path. □

Lemma 5.8 Let $\Gamma$ be a minimal chart. Let $C$ be a cycle of label $m$ in $\Gamma$ bounding a 2-color disk $E$ without free edges nor simple hoops. If $\Gamma_m \cap E$ is connected, then

(a) in the path decomposition $P(C; W_O^{Mid}(C, m))$ there exist at least two paths each of which contains a white vertex in $W_O(C, m) - W_O^{Mid}(C, m)$,

(b) $|W_O(C, m) - W_O^{Mid}(C, m)| \geq 2$, and

(c) in the path decomposition $P(C; W_O(C, m) - W_O^{Mid}(C, m))$ there exist at least two paths each of which contains a white vertex in $W_O^{Mid}(C, m)$.

Proof. By Remark 5.6, the 2-color disk is also a 3-color disk. Let $s = |W_O^{Mid}(C, m)|$. Then $s \geq 2$ by Corollary 4.4. There are $s$ paths in the path decomposition $P(C; W_O^{Mid}(C, m))$.

We show Statement (a). Suppose that in $P(C; W_O^{Mid}(C, m))$ there exists at most one path containing a white vertex in $W_O(C, m) - W_O^{Mid}(C, m)$. Then in $P(C; W_O^{Mid}(C, m))$ there are $s - 1$ paths each of which does not contain any vertices in $W_O(C, m)$ except its end vertices. Hence the $s - 1$ paths are in $P(C; W_O(C, m))$. By Lemma 5.7, the extended pseudo paths of the $s - 1$ paths are dichromatic one-side pseudo paths. Let $k$ be the secondary label of the 2-color disk $E$. By Lemma 5.4(b), each of the pseudo paths
has a side-arc of label $k$ middle at a vertex in the pseudo path, namely each of the paths contains a vertex in $W^\text{Mid}_I(C, m)$ by Remark 4.1(3). Thus $|W^\text{Mid}_I(C, m)| \geq s - 1$.

By Remark 5.6, the 2-color disk is a 3-color disk. Thus by Theorem 1.1

\[ s + 1 = (s - 1) + 2 \leq |W^\text{Mid}_I(C, m)| + 2 \leq |W^\text{Mid}_O(C, m)| = s. \]

This is a contradiction. Therefore Statement (a) holds.

Now Statement (b) is a direct result of Statement (a).

We show Statement (c). By Statement (b), we have $|W_O(C, m) - W^\text{Mid}_O(C, m)| \geq 2$. Thus we can consider the path decomposition $P(C; W_O(C, m) - W^\text{Mid}_O(C, m))$

Suppose that in the path decomposition $P(C; W_O(C, m) - W^\text{Mid}_O(C, m))$ there exists a path $P$ containing all the white vertices in $W^\text{Mid}_O(C, m)$ (see Fig. 18). Then all the paths in $P(C; W^\text{Mid}_O(C, m))$ are contained in $P$ except one, say $Q$. Thus $P \cup Q = C$. Since $P \in P(C; W_O(C, m) - W^\text{Mid}_O(C, m))$ implies Int $P \cap (W_O(C, m) - W^\text{Mid}_O(C, m)) = \emptyset$, we have $W_O(C, m) - W^\text{Mid}_O(C, m) \subset Q$. This contradicts Statement (a). □

Figure 18: An example of a path $P$ in a cycle $C$ of label $m$ with $W^\text{Mid}_O(C, m) \subset P$. Each arc with three transversal short arcs is a middle arc. The thick lines are paths $P$ and $Q$ with $\partial P = \{v_1, v_3\}$ and $\partial Q = \{v_2, v_4\}$, $W_O(C, m) = \{v_1, v_2, \cdots, v_8\}$ and $W^\text{Mid}_O(C, m) = \{v_2, v_3, w_4\}$.

6 Suspicious cycles

In this section, for a tangle $(\Gamma \cap D, D)$ and each label $m$ with $\Gamma_m \cap D \neq \emptyset$, we obtain an equation and search for conditions for the existence of a special cycle which never bounds a 2-color disk in a minimal chart. This special cycle is crucial in the proof of Theorem 1.2.

Let $G$ be a graph. For each vertex $v$ of $G$, we denote by $\deg_G v$ the degree of the vertex $v$ in $G$.

Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. A tree $T$ in $\Gamma$ is called a reducible tree of label $m$ provided that

(i) each edge of $T$ is of label $m$,
(ii) the tree $T$ contains a white vertex,

(iii) if $v$ is a crossing in $T$, then $\deg_T v = 2$,

(iv) if $T$ contains exactly one white vertex $v$, then $\deg_T v = 3$,

(v) if $T$ contains at least two white vertices, then

(a) for each white vertex $v$ of $T$, we have $\deg_T v = 1$ or $3$,

(b) there exists at most one white vertex $v_0$ in $T$ with $\deg_T v_0 = 1$.

The white vertex $v_0$ with $\deg_T v_0 = 1$ is called the special vertex of the reducible tree.

**Lemma 6.1** Any minimal chart does not contain a reducible tree of any label.

*Proof.* Suppose that there exists a reducible tree $T$ of label $m$ in a minimal chart. If $T$ contains exactly one white vertex $w$, then $T$ contains three terminal edges by Remark 2.3(1). By Remark 2.2(2), two of the three terminal edges are not middle at $w$. This contradicts Remark 2.3(2). Hence $T$ contains at least two white vertices.

Let $T^*$ be the subtree obtained from the reducible tree $T$ by taking out all the terminal edges. Since $T^*$ is a tree containing at least two white vertices, there exist two white vertices each of whose degree in $T^*$ is 1. Let $w$ be one of them different from the special vertex of $T$. By Condition (v) for a reducible tree, we have $\deg_T w = 3$. Hence in the reducible tree $T$ there exist two terminal edges of label $m$ at $w$. By Remark 2.2(2), one of the two terminal edges is not middle at $w$. This contradicts Remark 2.3(2). □

Let $C$ be a cycle of label $m$ in a chart $\Gamma$. Then the cycle $C$ is said to be suspicious provided that

(i) the outside edges of label $m$ for $C$ are terminal edges except one,

(ii) the cycle contains a white vertex, and

(iii) the cycle bounds a disk $E$ with $\Gamma_m \cap E$ connected.

The following lemma is an easy consequence of the definition of a suspicious cycle.

**Lemma 6.2** In a minimal chart $\Gamma$, for any label $m$ of $\Gamma$ there does not exist a suspicious cycle of label $m$ bounding a 2-color disk.
Proof. Suppose that there exists a suspicious cycle $C$ of label $m$ bounding a 2-color disk $E$. By Assumption 1 and Assumption 2 the disk $E$ contains neither free edge nor simple hoop. By Condition (i) of a suspicious cycle, we have $|W_O(C,m) - W_O^{Mid}(C,m)| \leq 1$. On the other hand, by Condition (iii) of a suspicious cycle, Lemma 5.8(b) implies that $|W_O(C,m) - W_O^{Mid}(C,m)| \geq 2$. This is a contradiction. \hfill \Box

Let $X$ be a subset of a chart. A cycle in $X$ is said to be maximal in $X$ if it is not contained in the disk bounded by another cycle in $X$.

Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. Let $(\Gamma \cap D, D)$ be a tangle. Let $G$ be a connected component of $\Gamma_m \cap D$ containing a white vertex, and $E_1, E_2, \ldots, E_d$ all the disks bounded by the maximal cycles in $G$. For each $i = 1, 2, \ldots, d$, let

$$\hat{E}_i = E_i \cup (\cup \{ e \mid e \text{ is a terminal edge in } G \text{ intersecting } E_i \}).$$

Let $P_1, P_2, \ldots, P_p$ be all the closures of connected components of $G - \bigcup_{i=1}^d \hat{E}_i$ not intersecting $\partial D$, and $Q_1, Q_2, \ldots, Q_q$ all the closures of connected components of $G - \bigcup_{i=1}^d \hat{E}_i$ intersecting $\partial D$. Let

$$\mathcal{H} = (\bigcup_{i=1}^d \hat{E}_i) \cap (\bigcup_{j=1}^p P_j) \cup (\bigcup_{k=1}^q Q_k), \quad h = |\mathcal{H}|,$n

$$s = |(\bigcup_{i=1}^d \hat{E}_i) \cap (\bigcup_{j=1}^p P_j)|, \quad t = |(\bigcup_{i=1}^d \hat{E}_i) \cap (\bigcup_{k=1}^q Q_k)|.$$

For each $k = 0, 1, \ldots, h$, let

1. $x_k$ = the number of $E_i$’s containing exactly $k$ points in $\mathcal{H}$,
2. $y_k$ = the number of $P_j$’s containing exactly $k$ points in $\mathcal{H}$.

Then $(E_1, E_2, \ldots, E_d; P_1, P_2, \ldots, P_p; Q_1, Q_2, \ldots, Q_q)$ and $(\mathcal{H}, h, s, t; x_0, x_1, \ldots, x_h, y_0, y_1, \ldots, y_h)$ are called the primary fundamental information and the secondary fundamental information of $G$ respectively. These will be used in Lemma 6.3 and Lemma 6.4. For $G$ in Fig. 19 there are six disks $E_1, E_2, E_3, E_4, E_5, E_6$ and seven trees $P_1, P_2, P_3, P_4, P_5, Q_1, Q_2$. Thus we have

$$\mathcal{H} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\},$$

$h = 12, \quad s = 9, \quad t = 3,$

$x_0 = 0, x_1 = 3, x_2 = 1, x_3 = 1, x_4 = 1, x_5 = \cdots = x_{12} = 0,$

$y_0 = 0, y_1 = 2, y_2 = 2, y_3 = 1, y_4 = \cdots = y_{12} = 0.$

Let $X$ be a subset of a chart. A connected component $G$ of $X$ is called a small component of $X$ if any finite complementary domain of $G$ does not intersect $X$.

Lemma 6.3 Let $\Gamma$ be a minimal chart, and $m$ a label of $\Gamma$. Let $(\Gamma \cap D, D)$ be a tangle, and $G$ a connected component of $\Gamma_m \cap D$ containing a white vertex with the primary fundamental information $(E_1, E_2, \ldots, E_d; P_1, P_2, \ldots, P_p; Q_1, Q_2, \ldots, Q_q)$ and the secondary fundamental information $(\mathcal{H}, h, s, t; x_0, x_1, \ldots, x_h, y_0, y_1, \ldots, y_h)$. Then we have the following:
(a) $y_0 = y_1 = 0$.

(b) $2x_0 + x_1 = 2 - 2q + t + (x_3 + y_3) + 2(x_4 + y_4) + \cdots + (h - 2)(x_h + y_h)$.

(c) Suppose that $G$ is a small component of $\Gamma_m \cap D$. If $1 \leq x_0$ or $1 \leq x_1$, then $D$ contains a suspicious cycle of label $m$.

Proof. We show Statement (a). Suppose $1 \leq y_0$ or $1 \leq y_1$. Then there exists a tree $P_j$ ($1 \leq j \leq p$) containing at most one point of $H$. Hence $P_j$ is a reducible tree. This contradicts Lemma 6.1.

We show Statement (b). For each $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$ we have $P_i \cap Q_j = \emptyset$. Hence

$$h = s + t,$$

$$x_0 + x_1 + \cdots + x_h = d, y_0 + y_1 + \cdots + y_h = p,$$ and

$$1 \times x_1 + 2 \times x_2 + \cdots + h \times x_h = h, 1 \times y_1 + 2 \times y_2 + \cdots + h \times y_h = s = h - t.$$

By adding the two equations in (3) we have

$$1 \times x_1 + 2 \times x_2 + \cdots + h \times x_h + (1 \times y_1 + 2 \times y_2 + \cdots + h \times y_h) = 2 \times h - t.$$

For each $i = 1, 2, \cdots, d$, let

$$\hat{E}_i = E_i \cup (\cup \{e \mid e \text{ is a terminal edge in } G \text{ intersecting } E_i\}).$$

Let $E_* = \cup_{i=1}^d \hat{E}_i, P_* = \cup_{j=1}^p P_j, Q_* = \cup_{k=1}^q Q_k$ and $X = E_* \cup P_* \cup Q_*$. Then we have $X = G \cup (\cup_{i=1}^d E_i)$. Hence the set $X$ is connected and any cycle in $X$ bounds a disk in $X$, i.e. $X$ is simply connected. Hence by Euler formula, we have $\chi(X) = 1$. On the other hand, considering (1) we have

$$\chi(X) = \chi(E_* \cup P_* \cup Q_*)$$

$$= \chi(E_*) + \chi(P_*) + \chi(Q_*) - (\chi(E_* \cap P_*) + \chi(E_* \cap Q_*) + \chi(P_* \cap Q_*)) + \chi(E_* \cap P_* \cap Q_*)$$

$$= d + p + q - (s + t + 0) + 0 = d + p + q - h.$$
Hence

\(5\) \(d + p + q - h = 1\).

By doubling both sides of the equation (5), and eliminating \(d, p, h\) using (2) and (4), we have

\[
2(x_0 + x_1 + \cdots + x_h) + 2(y_0 + y_1 + \cdots + y_h) + 2q
- ((1 \times x_1 + 2 \times x_2 + \cdots + h \times x_h) + (1 \times y_1 + 2 \times y_2 + \cdots + h \times y_h) + t) = 2.
\]

Since \(y_0 = y_1 = 0\) by Statement (a), we have

\[
2x_0 + x_1 = 2 - 2q + t + (x_3 + y_3) + 2(x_4 + y_4) + \cdots + (h - 2)(x_h + y_h).
\]

We show Statement (c). Suppose \(1 \leq x_0\) or \(1 \leq x_1\). Then there exists an integer \(i\) with \(1 \leq i \leq d\) such that the disk \(E_i\) intersects \(H\) by at most one point. Thus the outside edges of label \(m\) for \(\partial E_i\) are terminal edges except one. Since \(G\) is a small component of \(\Gamma_m \cap D\), the intersection \(\Gamma_m \cap E_i\) is connected. Since \(G\) contains a white vertex, \(G\) is not a hoop nor a ring. Thus the cycle \(\partial E_i\) contains a white vertex. Therefore \(\partial E_i\) is a suspicious cycle of label \(m\).

\[\square\]

**Lemma 6.4** Let \(\Gamma\) be a minimal chart, and \(m\) a label of \(\Gamma\). Let \((\Gamma \cap D, D)\) be a tangle. Suppose that there exists a small component \(G\) of \(\Gamma_m \cap D\) containing a white vertex. If \(|G \cap \partial D| \leq 1\), then \(D\) contains a suspicious cycle of label \(m\).

**Proof.** First, we claim that

(1) \(G\) contains a cycle.

Suppose that \(G\) does not contain a cycle. If \(|G \cap \partial D| = 0\), then \(G\) is a reducible tree without the special vertex. This contradicts Lemma 6.1. If \(|G \cap \partial D| = 1\), let \(\tau\) be a terminal edge or an internal edge with \(\tau \supseteq G \cap \partial D\). Let \(T = G \cup \tau\). Now for any white vertex \(w\) of \(G\), we have \(\deg_T w = 3\). Hence \(T\) is a reducible tree. This contradicts Lemma 6.1. Thus Statement (1) holds.

Let \((E_1, E_2, \ldots, E_d; P_1, P_2, \ldots, P_i; Q_1, Q_2, \ldots, Q_q)\) and \((H, h, s, t; x_0, x_1, \ldots, x_h, y_0, y_1, \ldots, y_h)\) be the primary fundamental information and the secondary fundamental information of the small component \(G\). By Lemma 6.3(b), we have

(2) \(2x_0 + x_1 = 2 - 2q + t + (x_3 + y_3) + 2(x_4 + y_4) + \cdots + (h - 2)(x_h + y_h)\).

Now \(|G \cap \partial D| \leq 1\) implies that \(q = 0\) or \(1\). If \(q = 0\) then \(2 - 2q + t \geq 2\). Hence \(2x_0 + x_1 \geq 2\) by (2). Thus \(x_0 \geq 1\) or \(x_1 \geq 1\). Hence there exists a suspicious cycle of label \(m\) by Lemma 6.3(c).

Suppose \(q = 1\). By Statement (1), we have \(d \geq 1\). Then we have \(t \geq 1\), because \(G \cup \bigcup_{i=1}^{d} E_i\) is connected. Thus \(2 - 2q + t \geq 1\). Hence \(2x_0 + x_1 \geq 1\) by Statement (2). Thus \(x_0 \geq 1\) or \(x_1 \geq 1\). Therefore there exists a suspicious cycle of label \(m\) by Lemma 6.3(c). This proves Lemma 6.4. \[\square\]
7 Proof of Theorem 1.2

In this section we shall prove Theorem 1.2 by using Lemma 6.2, Lemma 6.4 and the three lemmata in this sections.

Let $\Gamma$ be a chart, and $D$ a disk. We define

- $\omega(D) =$ the number of white vertices in $\Gamma \cap D$,
- $x(D) =$ the number of crossings in $\Gamma \cap D$,
- $n(\partial D) =$ the number of points in $\Gamma \cap \partial D$.

For a tangle $(\Gamma \cap D, D)$ in a chart $\Gamma$, let $\tau(D) = (\omega(D), x(D), n(\partial D))$.

We call $\tau(D)$ the $\tau$-complexity of the tangle.

Let $\Gamma$ be a chart. An NS-tangle $(\Gamma \cap D, D)$ of label $m$ is said to be minimal if its $\tau$-complexity of the tangle is minimal amongst the NS-tangles of all the labels with respect to the lexicographical order of the triplet of integers.

**Lemma 7.1** Let $\Gamma$ be a minimal chart. Let $(\Gamma \cap D, D)$ be a minimal NS-tangle of label $m$. Then $D$ does not contain any ring.

**Proof.** Suppose that $D$ contains a ring $C$ of label $k$. Then $C$ bounds a disk $E$. Let $A$ be a regular neighbourhood of $\partial E$, and $D' = Cl(E - A)$. Then by Condition (iii) of an NS-tangle, the intersection $\Gamma_k \cap D$ contains at most one crossing. Thus the ring $C = \partial E$ contains at most one crossing. Hence

1. the intersection $\Gamma \cap \partial D'$ is at most one point.

By Remark 2.3(3), the disk $E$ contains a white vertex. Thus we have

2. the disk $D'$ contains a white vertex.

By Condition (iii) of an NS-tangle, for each label $i$ the intersection $\Gamma_i \cap D$ contains at most one crossing. Since $D' \subset D$, we have

3. for each label $i$, the intersection $\Gamma_i \cap D'$ contains at most one crossing.

Hence $(\Gamma \cap D', D')$ is an NS-tangle. Since $C$ is a ring with $C \cap D' = \emptyset$, we have $\omega(D') \leq \omega(D)$, and $x(D') < x(D)$. Hence we have $\tau(D') < \tau(D)$. This contradicts the fact that the NS-tangle $(\Gamma \cap D, D)$ is minimal. \(\square\)

**Lemma 7.2** Let $\Gamma$ be a minimal chart. If $(\Gamma \cap D, D)$ is a minimal NS-tangle of label $m$, then $\partial D$ does not intersect any terminal edge.

**Proof.** Suppose that $\partial D$ intersects a terminal edge $e$ with a black vertex $v$. Then there are two cases: $v \in D$ or $v \notin D$.

**Case 1.** $v \in D$.

Let $\ell$ be the connected component of $e \cap D$ with $v \in \ell$. And let $N$ be a regular neighbourhood of $\ell$ in $D$, and $D' = Cl(D - N)$. Then $D'$ is a disk. Since the terminal edge $e$ does not contain any crossings, we have $\Gamma \cap D' = (\Gamma \cap D) - \ell$ and $\Gamma \cap \partial D' = (\Gamma \cap \partial D) - \ell$. Thus $(\Gamma \cap D', D')$ is an NS-tangle with $\omega(D') = \omega(D)$, $x(D') = x(D)$, and $n(\partial D') = n(\partial D) - 1$. Hence $\tau(D') < \tau(D)$. This contradicts the fact that the NS-tangle $(\Gamma \cap D, D)$ is minimal.
Case 2. \( v \notin D \).

Let \( \ell^* \) be the arc in the terminal edge \( e \) connecting \( v \) and a point \( p \) in \( \partial D \) with \( \ell^* \cap D = \{p\} \). Let \( N^* \) be a regular neighbourhood of \( \ell^* \), \( D^* = D \cup N^* \). Then \( D^* \) is a disk. Since the terminal edge \( e \) does not contain any crossings, we have \( \Gamma \cap D^* = (\Gamma \cap D) \cup \ell^* \) and \( \Gamma \cap \partial D^* = (\Gamma \cap \partial D) - \{p\} \). Thus \( (\Gamma \cap D^*, D^*) \) is an NS-tangle.

Let \( \ell^* \) be the arc in the terminal edge \( e \) connecting \( v \) and a point \( p \) in \( \partial D \) with \( \ell^* \cap D = \{p\} \). Let \( N^* \) be a regular neighbourhood of \( \ell^* \), \( D^* = D \cup N^* \). Then \( D^* \) is a disk. Since the terminal edge \( e \) does not contain any crossings, we have \( \Gamma \cap D^* = (\Gamma \cap D) \cup \ell^* \) and \( \Gamma \cap \partial D^* = (\Gamma \cap \partial D) - \{p\} \). Thus \( (\Gamma \cap D^*, D^*) \) is an NS-tangle. This contradicts the fact that the NS-tangle \( (\Gamma \cap D, D) \) is minimal.

Therefore \( \partial D \) does not intersect any terminal edge. \( \square \)

Lemma 7.3 If there exists an NS-tangle in a minimal chart, then there exist a minimal chart \( \Gamma \) and a minimal NS-tangle \( (\Gamma \cap D, D) \) such that \( D \) does not contain any hoop.

Proof. Suppose that there exists an NS-tangle in a minimal chart \( \Gamma \). Let \( (\Gamma \cap D^*, D^*) \) be a minimal NS-tangle of \( \Gamma \). By Assumption 1 and Assumption 2, we can assume that

1. the disk \( D^* \) does not contain any simple hoop.

Suppose that \( D^* \) contains a hoop. Let \( C \) be an innermost hoop in \( D^* \). Let \( E \) be the disk bounded by the hoop \( C \), and \( A \) a regular neighbourhood of \( C \). Set \( D = Cl(\{E - A\}) \). Then \( \Gamma \cap \partial D = \emptyset \). Since \( C \) is an innermost hoop in \( D^* \), the disk \( D \) does not contain any hoop. Since \( C \) is not simple by (1), the disk \( E \) contains a white vertex and so does \( D \). Thus \( (\Gamma \cap D, D) \) is an NS-tangle. Since \( (\Gamma \cap D^*, D^*) \) is a minimal NS-tangle, so is \( (\Gamma \cap D, D) \). Hence the tangle \( (\Gamma \cap D, D) \) is a desired tangle. \( \square \)

Let \( C \) be a cycle of label \( m \) in a chart \( \Gamma \), and \( E \) the disk bounded by \( C \). Let

\[ E^* = E \cup (\cup \{e | e \text{ is a terminal edge intersecting } E\}) \]

Let \( D^* \) be a regular neighbourhood of \( E^* \). Since any terminal edge does not contain a crossing, the set \( D^* \) is a disk. Thus \( (\Gamma \cap D^*, D^*) \) is a tangle. The tangle \( (\Gamma \cap D^*, D^*) \) is called a tangle induced from the cycle \( C \).

Remark 7.4

1. Int \( D^* \) contains all terminal edges intersecting \( C \), and \( \partial D^* \) does not intersect any terminal edge.

2. If an edge intersects \( \partial D^* \), then it must intersect \( C \), because \( D^* \) is a regular neighborhood of \( E^* \).

For a subset \( X \) of a chart \( \Gamma \), we define

\[ \overline{\alpha}(X) = \min \{ i | \Gamma_i \cap X \neq \emptyset \}, \quad \overline{\beta}(X) = \max \{ i | \Gamma_i \cap X \neq \emptyset \} \]

Proof of Theorem 1.2. Suppose that there exists an NS-tangle in a minimal chart. By Lemma 7.1, Lemma 7.2 and Lemma 7.3 there exist a minimal chart \( \Gamma \) and a minimal NS-tangle \( (\Gamma \cap D, D) \) of label \( m \) such that
Further, since (i) of a suspicious cycle we have there exists at most one edge of label \( \cap \) of at most one crossing, and since (ii) of an NS-tangle, we have \(| \Gamma_\alpha \cap \partial D | \leq 1 \). Let \( G_\alpha \) be a small component of \( \Gamma_\alpha \cap D \). Then \(| G_\alpha \cap \partial D | \leq 1 \). Thus by (1), (2) and (3), the set \( G_\alpha \) contains a white vertex.

Hence by Lemma 6.4, there exists a suspicious cycle \( C^* \) of label \( \alpha \) in \( D \). Let \(( \Gamma \cap D^*, D^*) \) be a tangle induced from \( C^* \). By Remark 7.4(1), \( \partial D^* \) does not intersect any terminal edge.

**Claim 1.** The tangle \(( \Gamma \cap D^*, D^*) \) is an NS-tangle of label \( \alpha + 1 \).

**Proof of Claim 1.** Since \( C^* \) is a suspicious cycle, we have (5) the cycle \( C^* \) contains a white vertex.

Let \( i \) be a label with \( i \geq \alpha + 2 \). Since \( C^* \subset D \), the intersection \( \Gamma_i \cap C^* \) consists of at most one crossing by Condition (iii) of an NS-tangle. If an edge of label \( i \) intersects \( \partial D^* \), then it must intersect the cycle \( C^* \) by Remark 7.4(2). Hence there exists at most one edge of label \( i \) intersecting \( \partial D^* \). Thus (6) for any label \( i \geq \alpha + 2 \) the intersection \( \Gamma_i \cap \partial D^* \) is at most one point.

Further, since \( C^* \) is a suspicious cycle of label \( \alpha \), by Remark 7.4(1) and Condition (i) of a suspicious cycle we have

(7) the intersection \( \Gamma_\alpha \cap \partial D^* \) is at most one point.

Since \( \alpha \) is the lowest label in \( D \),

(8) for any label \( j < \alpha \), we have \( \Gamma_j \cap \partial D^* = \emptyset \).

Since \(( \Gamma \cap D, D) \) is an NS-tangle, for each label \( i \), \( \Gamma_i \cap D \) contains at most one crossing, and since \( D^* \subset D \), we have

(9) for each label \( i \), the intersection \( \Gamma_i \cap D^* \) contains at most one crossing.

The tangle \(( \Gamma \cap D^*, D^*) \) is an NS-tangle of label \( \alpha + 1 \) in \( D \) by (5), (6), (7), (8) and (9). Hence Claim 1 holds.

**Claim 2.** \( \Gamma \cap D^* \subset \Gamma_\alpha \cup \Gamma_{\alpha + 1} \).

**Proof of Claim 2.** Suppose that there exists an integer \( r \neq \alpha, \alpha + 1 \) with \( \Gamma_r \cap D^* \neq \emptyset \). Then \( r > \alpha + 1 \), because \( \alpha \) is the lowest label in \( D \). Let \( \beta^* = \overline{\beta}(\Gamma \cap D^*) \). Then \( \beta^* \geq r > \alpha + 1 \). Thus \( \Gamma_{\beta^*} \cap \partial D^* \) consists of at most one point by (6). Let \( G_{\beta^*} \) be a small component of \( \Gamma_{\beta^*} \cap D^* \). Then \(| G_{\beta^*} \cap \partial D^* | \leq 1 \). Thus by (1), (3) and (4), the set \( G_{\beta^*} \) contains a white vertex.

Hence Lemma 6.4 assures that there exists a suspicious cycle \( C^{**} \) of label \( \beta^* \) in \( D^* \). Let \(( \Gamma \cap D^{**}, D^{**}) \) be a tangle induced from \( C^{**} \). Since \( \beta^* \neq \alpha, \alpha + 1 \), the cycle \( C^{**} \) is contained in the interior of the disk bounded by \( C^* \). Hence (5) implies that

(10) \( \omega(D^{**}) < \omega(D^*) \leq \omega(D) \).

Now we can show that the tangle \(( \Gamma \cap D^{**}, D^{**}) \) is an NS-tangle of label \( \beta^* - 1 \) by a similar way as the one used to show Claim 1. Now by (10) we have \( \tau(D^{**}) < \tau(D) \). This contradicts the fact that the NS-tangle \(( \Gamma \cap D, D) \) is minimal. Hence Claim 2 holds.

Therefore the suspicious cycle \( C^* \) bounds a 2-color disk. This contradicts
Lemma 6.2. This proves Theorem 1.2. □

Let \( \Gamma \) be a chart, and \((\Gamma \cap D, D)\) a tangle. Let \( m \) be a label of \( \Gamma \) with \( \Gamma_m \cap D \neq \emptyset \), and \( G \) a connected closed subset of \( \Gamma_m \cap D \). Set \( X = \cup \{ E \mid E \text{ is a disk bounded by a cycle in } G \} \).

Let \( G^* = G \cup X \). Then \( G^* \) is simply connected. Let \( D^* \) be a regular neighbourhood of \( G^* \) in \( D \). Then \( D^* \) is a disk. Thus \((\Gamma \cap D^*, D^*)\) is a tangle called a tangle induced from \( G \) with respect to \( D \).

Remark 7.5

(1) If an edge intersects \( \partial D^* \), then the edge must intersect \( G \), because \( D^* \) is a regular neighborhood of \( G^* = G \cup X \) in \( D \).

(2) Suppose that \( G \) is a connected component of \( \Gamma_m \cap D \). Since \( D^* \) is a regular neighbourhood of \( G^* \) in \( D \), we have \( |\Gamma_m \cap \partial D^*| = |G \cap \partial D^*| = |G \cap \partial D| \).

Let \( \Gamma \) be a chart. A tangle \((\Gamma \cap D, D)\) is said to be 2-color if there exist two labels \( m, k \) with \( |m - k| = 1 \) and \( \Gamma \cap D \subset \Gamma_m \cup \Gamma_k \).

The following lemma will be used in the proof of Theorem 8.6 and Theorem 1.3.

Lemma 7.6

Let \( \Gamma \) be a minimal chart, and \( m \) a label of \( \Gamma \). Then we have the following:

(a) If \((\Gamma \cap D, D)\) is a 2-color admissible tangle with \( |\Gamma_m \cap \partial D| = 2 \), then \( \Gamma_m \cap D \) is connected.

(b) If \( E \) is a 2-color disk with \( \partial E \subset \Gamma_m \) but without free edges nor simple hoops, then \( \Gamma_m \cap E \) is connected.

Proof. We show Statement (a). Suppose that \( \Gamma_m \cap D \) is not connected. Then \( |\Gamma_m \cap \partial D| = 2 \) implies that

(1) there exists a connected component \( G \) of \( \Gamma_m \cap D \) with \( |G \cap \partial D| \leq 1 \).

Let \((\Gamma \cap D', D')\) be a tangle induced from \( G \) with respect to \( D \).

**Claim.** The disk \( D' \) contains at least one white vertex.

**Proof of Claim.** If \( |G \cap \partial D| = 1 \), then the point \( G \cap \partial D \) is contained in an internal edge \( \bar{e} \) by Condition (ii) of an admissible tangle. Hence \( G \) contains a white vertex by Condition (iii) of an admissible tangle. Thus \( D' \) contains a white vertex.

Suppose that \( |G \cap \partial D| = 0 \). If \( G \) contains a white vertex, then \( D' \) contains a white vertex. If \( G \) does not contain any white vertex, then \( G \) is a hoop or a ring or a free edge. By Condition (i) of an admissible tangle, the set \( G \) is neither free edge nor simple hoop. Hence \( G \) is a non-simple hoop or a ring. By the definition of a simple hoop and Remark 2.3(3), the curve \( G \) bounds a disk containing a white vertex. Hence \( D' \) contains a white vertex. Thus Claim holds.
By Remark 7.5(2) and Statement (1), we have $|\Gamma_m \cap \partial D'| = |G \cap \partial D| \leq 1$. Thus

(2) $\Gamma_m \cap \partial D'$ is at most one point.

Since $(\Gamma \cap D, D)$ is a 2-color tangle, there exists a label $k$ with $|m - k| = 1$ and $\Gamma \cap D \subset \Gamma_m \cup \Gamma_k$. Thus

(3) for any label $i$ with $i \neq m$ and $i \neq k$, we have $\Gamma_i \cap \partial D' = \emptyset$.

Since $(\Gamma \cap D, D)$ is a 2-color tangle, the disk $D$ does not contain any crossing. Hence

(4) for each label $i$, the intersection $\Gamma_i \cap D'$ does not contain any crossing. Thus by Claim, Statement (2), (3) and (4), we have that $(\Gamma \cap D', D')$ is an NS-tangle of label $k$. This contradicts Theorem 1.2. Hence Statement (a) holds.

We show Statement (b). Suppose that $\Gamma_m \cap E$ is not connected. Let $G$ be a connected component of $\Gamma_m \cap E$ with $G \cap \partial E = \emptyset$. Let $D$ be a regular neighbourhood of $E$. Then $(\Gamma \cap D, D)$ is a 2-color tangle with $G \cap \partial D = \emptyset$. Let $(\Gamma \cap D', D')$ be a tangle induced from $G$ with respect to $D$. In a similar way to (a), we can show that $(\Gamma \cap D', D')$ is an NS-tangle. This contradicts Theorem 1.2.

\[ \square \]

8 Inward pseudo paths and outward pseudo paths

In this section we investigate a 2-color disk $E$ with $\partial E \subset \Gamma_m$ and an arc on the boundary of $E$ for a minimal chart $\Gamma$.

Let $\Gamma$ be a chart, and $P_1^*, P_2^*$ pseudo paths of label $m$ in $\Gamma$. The pair $(P_1^*, P_2^*)$ is called an I/O pair of type I if there exist side-disks $\Delta_1, \Delta_2$ of $P_1^* \cup P_2^*$ such that

(i) $P_1^* \cup P_2^*$ is a pseudo path,

(ii) $\Delta_1 \cup \Delta_2$ is a side-disk of $P_1^* \cup P_2^*$, and

(iii) the intersection $P_1^* \cap P_2^*$ contains exactly one white vertex.

We denote the pseudo path $P_1^* \cup P_2^*$ by $P_1^* * P_2^*$. The union $\Delta_1 \cup \Delta_2$ is called an associated side-disk of $P_1^* * P_2^*$. The pair $(\Delta_1, \Delta_2)$ is called an associated side-disk pair of the I/O pair $(P_1^*, P_2^*)$.

**Remark 8.1** Any vertex in an I/O pseudo path is a white vertex by the definition of I/O pseudo paths.
Lemma 8.2  Let $\Gamma$ be a chart, and $P_1^*, P_2^*$ pseudo paths of label $m$ in $\Gamma$. If $(P_1^*, P_2^*)$ is an I/O pair of type I, then $P_1^* * P_2^*$ is an I/O pseudo path with respect to its associated side-disk.

Proof. We use the notations in the definition of an I/O pair of type I. Without loss of generality we can assume that the pseudo path $P_1^*$ is inward with respect to the side-disk $\Delta_1$. Then for each vertex $v$ in $P_1^*$, any side-arc at $v$ of $P_1^*$ with respect to $\Delta_1$ is oriented inward at $v$. Thus we have

(1) for each vertex $v$ in $P_1^* \subset P_1^* * P_2^*$, any side-arc at $v$ of $P_1^* * P_2^*$ with respect to $\Delta_1 \cup \Delta_2$ is oriented inward at $v$.

By Condition (iv) for the definition of an I/O pair of type I, there exists exactly one vertex $v_0$ in $P_1^* \cap P_2^*$. Since the white vertex $v_0$ is a common vertex of $P_1^*$ and $P_2^*$, any side-arc at $v_0$ of $P_2^*$ with respect to $\Delta_2$ is oriented inward at $v_0$, too. Hence the I/O pseudo path $P_2^*$ is inward with respect to $\Delta_2$. Thus for each vertex $v$ in $P_2^*$, any side-arc at $v$ of $P_2^*$ with respect to $\Delta_2$ is oriented inward at $v$. Hence

(2) for each vertex $v$ in $P_2^* \subset P_1^* * P_2^*$, any side-arc at $v$ of $P_1^* * P_2^*$ with respect to $\Delta_1 \cup \Delta_2$ is oriented inward at $v$.

Thus by (1) and (2), the pseudo path $P_1^* * P_2^*$ is inward with respect to $\Delta_1 \cup \Delta_2$. □

Let $\Gamma$ be a chart, and $P_1^*, P_2^*$ pseudo paths of label $m$ in $\Gamma$. The pair $(P_1^*, P_2^*)$ is called an I/O pair of type II if there exist side-disks $\Delta_1, \Delta_2$ of $P_1^*, P_2^*$ respectively (see Fig. 21) such that

(i) the pseudo paths $P_1^*$ and $P_2^*$ are I/O pseudo paths with respect to the side-disks $\Delta_1, \Delta_2$ respectively,

(ii) the intersection $\gamma = \Delta_1 \cap \Delta_2$ is an end-arc of $P_1^*$ and $P_2^*$ respectively,

(iii) $\gamma$ is middle at the white vertex $v_0$ in $\gamma$. 

Figure 20: The thick lines are pseudo paths $P_1^*, P_2^*$. 

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Figure 21: The thick line is a pseudo path $P^*$. The end-arc $\gamma$ is a middle arc at $v_0$. And $\Delta_1, \Delta_2$ are side-disks of $P_1^*, P_2^*$ respectively.

(iv) $P^* = \text{Cl}((P_1^* \cup P_2^*) - \gamma)$ is a pseudo path, and $P^* \cap \gamma = v_0$ and

(v) the union $\Delta_1 \cup \Delta_2$ is a side-disk of $P^*$.

We denote $P_1^* \ast P_2^*$ by $P_1^* \ast P_2^* \ast P_3^* \ast \cdots \ast P_s^*$. The union $\Delta_1 \cup \Delta_2$ is called an associated side-disk of $P_1^* \ast P_2^*$. The pair $(\Delta_1, \Delta_2)$ is called an associated side-disk pair of the I/O pair $(P_1^*, P_2^*)$.

Lemma 8.3 Let $\Gamma$ be a chart. Let $P_1^*, P_2^*$ be pseudo paths of label $m$ in $\Gamma$. If $(P_1^*, P_2^*)$ is an I/O pair of type II, then $P_1^* \ast P_2^*$ is an I/O pseudo path with respect to its associated side-disk.

Proof. We use the notations in the definition of an I/O pair of type II. The end-arc $\gamma$ is a side-arc of $P_1^* \ast P_2^*$ middle at the vertex $v_0$. Without loss of generality we can assume that the side-arc $\gamma$ is oriented inward at $v_0$. Let $\gamma'$ be a side-arc at $v_0$ of $P_1^*$ with respect to $\Delta_1$, and $\gamma''$ a side-arc at $v_0$ of $P_2^*$ with respect to $\Delta_2$ (see Fig. 21). Since $\gamma$ is middle at $v_0$, the side-arcs $\gamma', \gamma''$ are oriented inward at $v_0$. Thus the pseudo paths $P_1^*$ and $P_2^*$ are inward with respect to $\Delta_1$ and $\Delta_2$ respectively. Hence

(1) for each vertex $v$ in $P_1^*$, if $v \neq v_0$, then any side-arc at $v$ of $P_1^* \ast P_2^*$ with respect to $\Delta_1 \cup \Delta_2$ is oriented inward at $v$, and

(2) for each vertex $v$ in $P_2^*$, if $v \neq v_0$, then any side-arc at $v$ of $P_1^* \ast P_2^*$ with respect to $\Delta_1 \cup \Delta_2$ is oriented inward at $v$.

The three side-arcs $\gamma', \gamma, \gamma''$ are oriented inward at $v_0$. Therefore by (1) and (2), the pseudo path $P_1^* \ast P_2^*$ is inward with respect to $\Delta_1 \cup \Delta_2$. □

Let $\Gamma$ be a chart, $m$ a label of $\Gamma$, and $s \in \mathbb{N}$ with $s \geq 2$. Let $P^*$ be a pseudo path of label $m$ in $\Gamma$ with a side-disk $\Delta$, and $P_1^*, P_2^*, \ldots, P_s^*$ pseudo paths of label $m$ in $\Gamma$ such that

(i) for $i, j \in \{1, 2, \ldots, s\}$, if $|i - j| > 1$, then $P_i^* \cap P_j^* = \emptyset$;

(ii) for each $k = 1, 2, \ldots, s$, $L_k = \cup_{i=1}^k L(P_i^*)$ is a path in $P^*$, here $L(P_i^*)$ is the maximal path contained in the pseudo path $P_i^*$ (see Section 5) and

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(iii) \( L_s = L(P^*) \), here \( L(P^*) \) is the maximal path contained in \( P^* \).

The \( s \)-tuple \( (P_1^*, P_2^*, \ldots, P_s^*) \) is called an I/O sequence for \((P^*, \Delta)\) if there exist side-disks \( \Delta_1, \Delta_2, \ldots, \Delta_s \) of \( P_1^*, P_2^*, \ldots, P_s^* \) respectively such that

(i) for each \( k = 1, 2, \ldots, s \), \( \Delta_k' = \cup_{i=1}^k \Delta_i \) is a disk in \( \Delta \),

(ii) for each \( i = 1, 2, \ldots, s-1 \), the pair \((P_i^*, P_{i+1}^*)\) is an I/O pair of type I or type II with an associated side-disk pair \((\Delta_i, \Delta_{i+1})\), and

(iii) \( \Delta'_s \) is a side-disk of \( P^* \).

The \( s \)-tuple \((\Delta_1, \Delta_2, \ldots, \Delta_s)\) is called an associated side-disk sequence of the I/O sequence \((P_1^*, P_2^*, \ldots, P_s^*)\).

**Remark 8.4** Let \((P_1^*, P_2^*, \ldots, P_s^*)\) be an I/O sequence for \((P^*, \Delta)\) with an associated side-disk sequence \((\Delta_1, \Delta_2, \ldots, \Delta_s)\). Let \( Q_1^* = P_1^* \). Since the side-disk \( \Delta \) of the pseudo path \( P^* \) contains all the side-disks \( \Delta_1, \Delta_2, \ldots, \Delta_s \) of \( P_1^*, P_2^*, \ldots, P_s^* \), we can inductively show that for each \( i = 2, 3, \ldots, s \), the pair \((Q_{i-1}^*, P_i^*)\) is an I/O pair of type I (resp. type II) with an associated side-disk pair \( (\Delta_{i-1}', \Delta_i) \) if the pair \((P_{i-1}^*, P_i^*)\) is an I/O pair of type I (resp. type II) and that \( Q_i^* = Q_{i-1}^* \ast P_i^* \) is an I/O pseudo path with respect to \( \Delta_i \) by Lemma 8.2 (resp. Lemma 8.3). Hence \( Q_s^* \) is an I/O pseudo path. Therefore \( P^* \) is an I/O pseudo path. We denote the I/O pseudo path \( Q_s^* \) by \( P_1^* \ast P_2^* \ast \cdots \ast P_s^* \).

**Lemma 8.5** Let \( \Gamma \) be a minimal chart. Let \( P^* \) be a dichromatic one-side pseudo path of label \( m \) in \( \Gamma \) with the associated vertex sequence \((v_1, v_2, \ldots, v_s)\) and an associated side-arc sequence \((\gamma'_1, \gamma'_2, \ldots, \gamma'_s)\). Suppose that the end-arcs \( \gamma_0 \) and \( \gamma_s \) of \( P^* \) are middle at \( v_1 \) and \( v_s \) respectively. Suppose that there exists an integer \( t \) with \( 2 \leq t \leq s-1 \) such that

(i) the side-arc \( \gamma'_t \) is middle at \( v_t \), and

(ii) for each \( i = 1, 2, \ldots, s \) with \( i \neq t \), the side-arc \( \gamma'_t \) is not middle at \( v_i \).

Then the pseudo path \( P^* \) is an I/O pseudo path.

**Proof.** Let \((e_1, e_2, \ldots, e_{s-1})\) be the associated edge sequence for \( P^* \). Let \( \gamma_{t-1} \) and \( \gamma_t \) be short arcs containing \( v_t \) in the edges \( e_{t-1} \) and \( e_t \) respectively. Set \( P_1^* = \gamma_0 \cup e_1 \cup e_2 \cup \cdots \cup e_{t-1} \cup \gamma_t \), and \( P_2^* = \gamma_{t-1} \cup e_2 \cup \cdots \cup e_{s-1} \cup \gamma_s \). Then \( P_1^* \) and \( P_2^* \) are dichromatic one-side pseudo paths by Remark 5.1(2). Thus \( P_1^* \) and \( P_2^* \) are I/O pseudo paths by Lemma 5.3. Thus \((P_1^*, P_2^*)\) is an I/O pair of type I. Therefore the pseudo path \( P^* \) is an I/O pseudo path by Lemma 8.2. \( \square \)
Theorem 8.6 Let $\Gamma$ be a minimal chart. Let $C$ be a cycle of label $m$ in $\Gamma$ bounding a 2-color disk $E$ such that the disk $E$ contains neither free edge nor simple hoop. If there exist two paths $S, T$ in the path decomposition $\mathcal{P}(C; \mathcal{W}_O(C, m) - \mathcal{W}_O^{\text{Mid}}(C, m))$ with $\mathcal{W}_O^{\text{Mid}}(C, m) \subset S \cup T$, then

(a) each of the paths $S$ and $T$ contains at least one white vertex in $\mathcal{W}_O^{\text{Mid}}(C, m)$,

(b) the extended pseudo paths $\hat{S}, \hat{T}$ of $S, T$ are I/O pseudo paths.

Proof. By Lemma 7.6(b), the intersection $\Gamma_m \cap E$ is connected. Let $s = |S \cap \mathcal{W}_O^{\text{Mid}}(C, m)|$, and $t = |T \cap \mathcal{W}_O^{\text{Mid}}(C, m)|$. Suppose that $s = 0$ or $t = 0$. Then $\mathcal{W}_O^{\text{Mid}}(C, m) \subset S \cup T$ implies that $\mathcal{W}_O^{\text{Mid}}(C, m)$ is contained in one of the two paths $S, T$ in $\mathcal{P}(C; \mathcal{W}_O(C, m) - \mathcal{W}_O^{\text{Mid}}(C, m))$. This contradicts Lemma 5.8(c). Thus we have $s \geq 1, t \geq 1$. Namely, Statement (a) holds.

We show Statement (b). Let

$S = \mathcal{P}(S; \mathcal{W}_O^{\text{Mid}}(C, m) \cap S) = \{S_1, S_2, \ldots, S_{s+1}\},$ and

$T = \mathcal{P}(T; \mathcal{W}_O^{\text{Mid}}(C, m) \cap T) = \{T_1, T_2, \ldots, T_{t+1}\}.$

Since $S, T \in \mathcal{P}(C; \mathcal{W}_O(C, m) - \mathcal{W}_O^{\text{Mid}}(C, m))$, we have

(1) $\{S_1, S_2, \ldots, S_{s+1}, T_1, T_2, \ldots, T_{t+1}\} \subset \mathcal{P}(C; \mathcal{W}_O(C, m)).$

Further, without loss of generality we can assume that

(2) each of $\partial S_1, \partial S_{s+1}, \partial T_1, \partial T_{t+1}$ consists of a vertex in $\mathcal{W}_O(C, m) - \mathcal{W}_O^{\text{Mid}}(C, m)$ and a vertex in $\mathcal{W}_O^{\text{Mid}}(C, m)$,

(3) for $i = 2, 3, \ldots, s$ and $j = 2, 3, \ldots, t$, we have $\partial S_i \subset \mathcal{W}_O^{\text{Mid}}(C, m)$ and $\partial T_j \subset \mathcal{W}_O^{\text{Mid}}(C, m)$.

Let

$X = \{P \mid P \in S \cup T, \partial P \subset \mathcal{W}_O^{\text{Mid}}(C, m)\} = \{S_2, S_3, \ldots, S_s, T_2, T_3, \ldots, T_t\}.$

Since $\mathcal{W}_O^{\text{Mid}}(C, m) \subset S \cup T$ by the assumption, we have

(4) $s + t = |\mathcal{W}_O^{\text{Mid}}(C, m)|$.

Let $k$ be the secondary label of the 2-color disk $E$. By Remark 4.1(3), we have

(5) for each path $P$ in $S \cup T$, a side-arc of label $k$ of the extended pseudo path $\hat{P}$ of $P$ is middle at a vertex $v$ in $\text{Int} P$ if and only if $v \in \mathcal{W}_O^{\text{Mid}}(C, m)$.

By Statement (1), Lemma 5.7 implies that

(6) for each path $P \in S \cup T$, the extended pseudo path $\hat{P}$ of $P$ is a dichromatic one-side pseudo path.
Further, Statement (3) assures that for each path $P \in X$ we can apply Lemma 5.4(b) to $\hat{P}$ so that the extended pseudo path $\hat{P}$ has a side-arc of label $k$ middle at a vertex in $\text{Int} \ P$. Namely by Statement (5)

(7) for each $P \in X$, the set $\text{Int} \ P$ contains at least one vertex in $\mathcal{W}_I^{\text{Mid}}(C, m)$.

For each $i = 1, 2, \cdots , s + 1$ and $j = 1, 2, \cdots , t + 1$, let

$$\sigma_i = |S_i \cap \mathcal{W}_I^{\text{Mid}}(C, m)|, \quad \tau_j = |T_j \cap \mathcal{W}_I^{\text{Mid}}(C, m)|.$$

Then Statement (7) implies

(8) for each $i = 2, 3, \cdots , s$ and $j = 2, 3, \cdots , t$, we have $1 \leq \sigma_i$ and $1 \leq \tau_j$.

**Claim 1.** $\sigma_2 = \sigma_3 = \cdots = \sigma_s = \tau_2 = \tau_3 = \cdots = \tau_t = 1$.

**Proof of Claim 1.** If not, then by Statement (8), we have

$$\sigma_i > 0, \text{ or } \sigma_{s+1} > 0, \text{ or } \tau_1 > 0, \text{ or } \tau_{t+1} > 0, \text{ or }$$

$$\sigma_i > 1 \text{ for some } 2 \leq i \leq s, \text{ or } \tau_j > 1 \text{ for some } 2 \leq j \leq t.$$

Thus we have

$$|\mathcal{W}_I^{\text{Mid}}(C, m)| \geq \sigma_1 + (\sigma_2 + \cdots + \sigma_s) + \sigma_{s+1} + \tau_1 + (\tau_2 + \cdots + \tau_t) + \tau_{t+1}$$

$$> 0 + (1 + \cdots + 1) + 0 + 0 + (1 + \cdots + 1) + 0$$

$$= (s - 1) + (t - 1) = s + t - 2.$$

Using the equation (4) and the inequality (9), Theorem 1.1 implies

$$s + t = (s + t - 2) + 2 < |\mathcal{W}_I^{\text{Mid}}(C, m)| + 2 \leq |\mathcal{W}_I^{\text{Mid}}(C, m)| = s + t.$$

This is a contradiction. Therefore Claim 1 holds.

Now Claim 1 implies that

(10) each path in $X$ contains exactly one white vertex in $\mathcal{W}_I^{\text{Mid}}(C, m)$.

**Claim 2.** $C - (\cup_{P \in X} P)$ does not contain any white vertex in $\mathcal{W}_I^{\text{Mid}}(C, m)$.

**Proof of Claim 2.** If $C - (\cup_{P \in X} P)$ contains a white vertex in $\mathcal{W}_I^{\text{Mid}}(C, m)$, then by Statement (10) we have the inequality $|\mathcal{W}_I^{\text{Mid}}(C, m)| > |X| = s + t - 2$. Thus we have the same contradiction as the one of Claim 1. Thus Claim 2 holds.

Without loss of generality we can assume that $S_1, S_2, \cdots , S_{s+1}$ are situated on $S$ in this order. By Statement (3), (6) and (10), Lemma 8.5 implies that the extended pseudo paths $\hat{S}_2, \hat{S}_3, \cdots , \hat{S}_s$ are I/O pseudo paths (see $S_2$ and $S_3$ in Fig. 22). Further, Claim 2 and Statement (5) imply that for any vertex $v \in \text{Int} \ S_1$ (resp. $v \in \text{Int} \ S_{s+1}$) any side-arc at $v$ of $\hat{S}_1$ (resp. $\hat{S}_{s+1}$) is not middle at $v$. Hence by Statement (2) and Statement (6), Lemma 5.3 implies that the extended pseudo paths $\hat{S}_1, \hat{S}_{s+1}$ are I/O pseudo paths (see $S_1$ and $S_4$ in Fig. 22). Now for each $i = 1, 2, \cdots , s$, we have that $(\hat{S}_i, \hat{S}_{i+1})$ is an I/O pair of type II. Thus $(\hat{S}_1, \hat{S}_2, \cdots , \hat{S}_s, \hat{S}_{s+1})$ is an I/O sequence for $(\hat{S}, \Delta)$ here $\Delta$ is a side-disk of $\hat{S}$. Hence by Remark 8.4 the extended pseudo path $\hat{S}_1 \hat{S}_2 \cdots \hat{S}_s \hat{S}_{s+1}$ is an I/O pseudo path and so is $\hat{S}$. Similarly we can show that $\hat{T}$ is an I/O pseudo path. This completes the proof of Theorem 8.6. \qed
Figure 22: The thick lines are of label $m$. Each arc with three transversal short arcs is a middle arc at the white vertex. For paths $S_1, S_2, S_3, S_4$ of label $m$, $\partial S_1 = \{v_0, v_1\}$, $\partial S_2 = \{v_1, v_2\}$, $\partial S_3 = \{v_2, v_3\}$, $\partial S_4 = \{v_3, v_4\}$.

9 Bridges

In this section, we investigate a path of label $m$ such that each white vertex in the path is contained in a terminal edge.

Let $\Gamma$ be a chart. Let $B$ be an admissible pseudo path of label $m$ in $\Gamma$ with the associated vertex sequence $(v_1, v_2, \cdots, v_s)$ and the associated edge sequence $(e_1, e_2, \cdots, e_{s-1})$. The pseudo path $B$ is called a bridge provided that

(i) $s \geq 2$ and all the vertices in $B$ are white vertices,

(ii) the edge $e_1$ is not middle at $v_1$ and the edge $e_{s-1}$ is not middle at $v_s$, and

(iii) for each $i = 2, 3, \cdots, s-1$, there exists a terminal edge of label $m$ at $v_i$.

Let $\gamma_0$ and $\gamma_s$ be the end-arcs of $B$ with $\gamma_0 \ni v_1$ and $\gamma_s \ni v_s$. Let $\Delta$ be a side-disk for which $B$ is admissible. Let $\gamma^*_0$ and $\gamma^*_s$ be short arcs in edges of label $m$ in $\Gamma$ with $\gamma^*_0 \cap \Delta = v_1$ and $\gamma^*_s \cap \Delta = v_s$. Then $B^* = \gamma^*_0 \cup e_1 \cup e_2 \cup \cdots \cup e_{s-1} \cup \gamma^*_s$ is a bridge called the co-bridge of $B$ (see Fig. 23). It is clear that the bridge $B$ is the co-bridge of $B^*$. If there exists a label $k$ with $|m - k| = 1$ and $v_1, \cdots, v_s \in \Gamma_m \cap \Gamma_k$, then the bridge $B$ is called a dichromatic bridge.

Lemma 9.1 Let $\Gamma$ be a minimal chart. Let $B$ be a dichromatic bridge of label $m$ in $\Gamma$, and $B^*$ a co-bridge of $B$. Then one of $B$ and $B^*$ is an inward pseudo path, and the other is an outward pseudo path.

Proof. We use the notations in the definition of a bridge and a co-bridge. Since the edge $e_1$ is not middle at $v_1$ by Condition (ii) of a bridge, one of the two end-arcs $\gamma_0$ and $\gamma^*_0$ is middle at $v_1$ by Remark 2.2(2). Since the co-bridge of $B^*$ is $B$, without loss of generality we can assume that

(1) the end-arc $\gamma_0$ is middle at $v_1$.

Further, we can assume that
First, we prove the case $s = 2$. Now $B = \gamma_0 \cup e_1 \cup \gamma_2$ and $B^* = \gamma_0^* \cup e_1 \cup \gamma_2^*$ are dichromatic one-side pseudo paths (see Fig. 23(a)). Let $(\gamma_1', \gamma_2')$ and $(\gamma_1'', \gamma_2'')$ be side-arc sequences of $B$ and $B^*$ respectively. Then by (1) and (2), we have

(3) the side-arc $\gamma_1'$ is oriented inward at $v_1$, and the edge $e_1$ and the side-arc $\gamma_2''$ are oriented outward at $v_1$.

Since the end-arc $\gamma_0$ is middle at $v_1$ by (1), and since $s = 2$, the bridge $B$ is an I/O pseudo path by Lemma 5.3. Since the side-arc $\gamma_1'$ is oriented inward at $v_1$ by (3), the bridge $B$ is an inward pseudo path. Thus the side-arc $\gamma_2$ is oriented inward at $v_2$. Now $e_1$ is oriented inward at $v_2$ by (3). Since $e_1$ is not middle at $v_2$ by Condition (ii) of a bridge, the side-arc $\gamma_2''$ is oriented outward at $v_2$. Since the side-arc $\gamma_1''$ is oriented outward at $v_1$ by (3), the co-bridge $B^*$ is an outward pseudo path.

Suppose that $s \geq 3$. Now for each $i = 2, 3, \ldots, s - 1$ there exists a terminal edge $\tilde{e}_i$ of label $m$ at $v_i$ by Condition (iii) of a bridge. Since $\tilde{e}_2$ is a terminal edge at $v_2$, by Remark 2.3(2) we have

(4) the edge $\tilde{e}_2$ is middle at $v_2$.

If $\tilde{e}_2$ contains a side-arc of $B$, say $\tilde{\gamma}_2$, then $P = \gamma_0 \cup e_1 \cup \tilde{\gamma}_2$ is a dichromatic one-side pseudo path (see Fig. 23(b)). But the end-arc $\gamma_0$ of $P$ is middle at $v_1$ by (1), and the end-arc $\tilde{\gamma}_2$ of $P$ is middle at $v_2$ by (4). Further, the dichromatic one-side pseudo path $P$ contains only two white vertices $v_1, v_2$. This contradicts Lemma 5.4(a). Hence $\tilde{e}_2$ contains a side-arc of $B^*$, say $\tilde{\gamma}_2^*$ (see Fig. 23(c)). By (4),

(5) the side-arc $\tilde{\gamma}_2^*$ is middle at $v_2$. 

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Figure 23: The thickened lines are bridges $B$. 

(2) the end-arc $\gamma_0$ is oriented inward at $v_1$. 

(3) the side-arc $\gamma_1'$ is oriented inward at $v_1$, and the edge $e_1$ and the side-arc $\gamma_2''$ are oriented outward at $v_1$. 

(4) the edge $\tilde{e}_2$ is middle at $v_2$. 

(5) the side-arc $\tilde{\gamma}_2^*$ is middle at $v_2$. 

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Let $\gamma_1$ (resp. $\gamma_2$) be a short arc in the edge $e_1$ (resp. $e_2$) containing $v_2$. Let $B_1 = \gamma_0 \cup e_1 \cup \gamma_2$, $B_2 = \gamma_1 \cup e_2 \cup \ldots \cup e_{s-1} \cup \gamma_s$, $B_1^* = \gamma_0^* \cup e_1 \cup \gamma_2^*$, $B_2^* = \gamma_1^* \cup e_2 \cup \ldots \cup e_{s-1} \cup \gamma_s^*$. Since $\gamma_2^*$ is middle at $v_2$ by (5), the edges $e_1$ and $\gamma_2$ are not middle at $v_2$ by Remark 2.2(2). Thus $B_1, B_2, B_1^*, B_2^*$ are bridges.

Now $B_1^*, B_2^*$ are co-bridges of $B_1, B_2$ respectively. Thus $B_1$ is inward and $B_1^*$ is outward by the case $s = 2$. By induction on the number of edges of a bridge, we can show that $B_2$ and the co-bridge of $B_2$ are I/O pseudo paths.

Since $(B_1, B_2)$ is an I/O pair of type I, the bridge $B$ is an I/O pseudo path by Lemma 8.2. Since $B_1$ is an inward pseudo path, so is $B$. Since $(B_1^*, B_2^*)$ is an I/O pair of type II, the bridge $B^*$ is an I/O pseudo path by Lemma 8.3. Since $B_1^*$ is an outward pseudo path, so is $B^*$. This proves Lemma 9.1.

Let $\Gamma$ be a chart. Let $B$ be a pseudo path of label $m$ in $\Gamma$ with the associated vertex sequence $(v_1, v_2, \ldots, v_s)$ and the associated edge sequence $(e_1, e_2, \ldots, e_{s-1})$. Let $\gamma_0$ and $\gamma_s$ be the end-arcs of $B$ with $\gamma_0 \ni v_1$ and $\gamma_s \ni v_s$. The pseudo path $B$ is called a pier provided that (see Fig. 24)

(i) all the vertices in $B$ are white vertices,

(ii) the edge $e_1$ is not middle at $v_1$, and

(iii) for each $i = 2, 3, \ldots, s$, there exists a terminal edge of label $m$ at $v_i$ which does not contain the end-arc $\gamma_s$.

Let $\gamma_0^*$ be a short arc in an edge of label $m$ in $\Gamma$ with $v_1 = \gamma_0^* \cap e_1 = \gamma_0^* \cap \gamma_0$. Then $B^* = \gamma_0^* \cup e_1 \cup e_2 \cup \ldots \cup e_{s-1} \cup \gamma_s$ is a pier called the co-pier of $B$. It is clear that the pier $B$ is the co-pier of $B^*$. There exist side-disks $\Delta$ and $\Delta^*$ of $B$ and $B^*$ respectively with $\Delta \cap \Delta^* = B \cap B^*$. The side-disk $\Delta$ (resp. $\Delta^*$) is called a nice side-disk for $B$ (resp. $B^*$). If there exists a label $k$ with $|m - k| = 1$ and $v_1, \ldots, v_s \in \Gamma_m \cap \Gamma_k$, then the pier $B$ is called a dichromatic pier.

**Corollary 9.2** Let $\Gamma$ be a minimal chart. Let $B$ be a dichromatic pier of label $m$ in $\Gamma$, and $B^*$ the co-pier of $B$. Then one of $B$ and $B^*$ is an inward pseudo path with respect to a nice side-disk, and the other is an outward pseudo path with respect to a nice side-disk.

**Proof.** We use the notations in the definition of a pier. Let $\Delta$ and $\Delta^*$ be nice side-disks of $B$ and $B^*$ respectively. Our result is true for the case $s = 1$, because $e_1$ is not middle at $v_1$. Suppose that $s \geq 2$. Let $\tilde{e}_s$ be the terminal edge of label $m$ at $v_s$ not containing the end-arc $\gamma_s$. If $\tilde{e}_s \cap \text{Int} \Delta^* \neq \emptyset$, then $B$ is an admissible pseudo path. If $\tilde{e}_s \cap \text{Int} \Delta \neq \emptyset$, then $B^*$ is an admissible pseudo path. Since $B$ and $B^*$ are co-piers of each other, without loss of generality we can assume that $B$ is an admissible pseudo path (see Fig. 24). Now the terminal edge $\tilde{e}_s$ is middle at $v_s$ by Remark 2.3(2). By Remark 2.2(2), the edge $e_{s-1}$ is not middle at $v_s$. Thus $B$ is a bridge. Let $\tilde{\gamma}_s$ be a short arc in the terminal edge $\tilde{e}_s$ of label $m$ with $v_s \in \tilde{\gamma}_s$. Then
$B^\dagger = \gamma_s^* \cup e_1 \cup \cdots \cup e_{s-1} \cup \tilde{\gamma}_s$ is a co-bridge of $B$. Thus $B^\dagger$ and $B$ are I/O pseudo paths by Lemma 9.1. Without loss of generality we can assume that $B$ is inward and $B^\dagger$ is outward. Then a side-arc at $v_s$ of $B$ with respect to $\Delta$ is oriented inward at $v_s$. Hence $\tilde{e}_s$ is oriented outward at $v_s$. Since the terminal edge $\tilde{\gamma}_s$ is middle at $v_s$, the side-arcs at $v_s$ of $B^*$ with respect to $\Delta^*$ is oriented outward at $v_s$. Thus the pier $B^*$ is an outward pseudo path with respect to $\Delta^*$. This proves Corollary 9.2. □

10 Proof of Theorem 1.3

In this section we shall prove Theorem 1.3 by using Theorem 8.6(b), Lemma 9.1, Corollary 9.2, and Lemma 10.1 below.

Lemma 10.1 Let $\Gamma$ be a minimal chart, and $m$ a label of $\Gamma$. Let $(\Gamma \cap D, D)$ be a 2-color admissible tangle with $|\Gamma_m \cap \partial D| = 2$. If $\Gamma_m \cap D$ contains a cycle, then there exist disks $E_1, E_2, \cdots, E_d$ in $\text{Int} \, D$ and simple arcs $L_0, L_1, \cdots, L_d$ in $\text{Cl} \,(D - \bigcup_{i=1}^d E_i)$ such that

(a) $\partial E_i \subset \Gamma_m$ for each $i = 1, 2, \cdots, d$ and $L_j \subset \Gamma_m$ for each $j = 0, 1, \cdots, d$,

(b) for each $j = 1, \cdots, d-1$, $L_j$ connects a vertex in $\partial E_j$ and a vertex in $\partial E_{j+1}$,

the arc $L_0$ connects a point in $\partial D$ and a vertex in $\partial E_1$, and

the arc $L_d$ connects a vertex in $\partial E_d$ and a point in $\partial D$.

(c) if an edge of label $m$ intersects $D - ((\cup_{i=1}^d E_i) \cup (\cup_{j=0}^d L_j))$, then it is a terminal edge.

Proof. Since $(\Gamma \cap D, D)$ is a 2-color tangle,

(1) there exists a label $k$ with $|m - k| = 1$ and $\Gamma \cap D \subset \Gamma_m \cup \Gamma_k$,

(2) the disk $D$ does not contain any crossing.

By Lemma 7.6(a),

(3) $\Gamma_m \cap D$ is connected.
Let $E_1, E_2, \ldots, E_d$ be all the disks bounded by the maximal cycles in $\Gamma_m \cap D$. Since $\Gamma_m \cap D$ contains a cycle, we have $d \geq 1$. For each $i = 1, 2, \ldots, d$, let
\[ \hat{E}_i = E_i \cup (\cup \{ e \mid e \text{ is a terminal edge in } \Gamma_m \cap D \text{ intersecting } E_i \}). \]

Let $P_1, P_2, \ldots, P_p$ be all the closures of connected components of $(\Gamma_m \cap D) - \bigcup_{i=1}^d \hat{E}_i$ not intersecting $\partial D$, and $Q_1, Q_2, \ldots, Q_q$ all the closures of connected components of $(\Gamma_m \cap D) - \bigcup_{i=1}^d \hat{E}_i$ intersecting $\partial D$. Then

(4) for each $i = 1, 2, \ldots, p$, $P_i \cap \partial D = \emptyset$, and the tree $P_i$ is not a terminal edge.

By the condition $|\Gamma_m \cap \partial D| = 2$, we have $q \leq 2$. Set
\[ \mathcal{E} = \{ E_1, E_2, \cdots, E_d \}, \]
\[ \mathcal{P} = \{ P_1, P_2, \cdots, P_p \}, \]
and
\[ \mathcal{Q} = \{ Q_1, Q_2, \cdots, Q_q \}. \]

Since $\Gamma_m \cap D$ is connected by (3) and since $|\Gamma_m \cap \partial D| = 2$,

(5) for each $i = 1, 2, \cdots, d$, the cycle $\partial E_i$ contains a white vertex.

Further, by (3) and (4), we have

(6) for each $i = 1, 2, \cdots, p$, the tree $P_i$ contains at least two white vertices.

Claim 1. For each $i = 1, 2, \cdots, d$, the disk $E_i$ intersects exactly two of $\mathcal{P} \cup \mathcal{Q}$.

**Proof of Claim 1.** If there exists a disk $E_i$ intersecting at most one of $\mathcal{P} \cup \mathcal{Q}$, let $D'$ be a regular neighbourhood of $\hat{E}_i$. Then $\Gamma_m \cap \partial D'$ consists of at most one point. Thus by (5), the tangle $(\Gamma \cap D', D')$ is an NS-tangle of label $k$. This contradicts Theorem 1.2.

Suppose that there exists a disk $E_i$ intersecting at least three of $\mathcal{P} \cup \mathcal{Q}$. Let $D'$ be a regular neighbourhood of $\hat{E}_i$. Then $\text{Cl}((\Gamma_m \cap D) - D')$ consists of at least three connected components. Thus the condition $|\Gamma_m \cap \partial D| = 2$ implies that there exists a connected component $X$ of $\text{Cl}((\Gamma_m \cap D) - D')$ with $X \cap \partial D = \emptyset$. Let $(\Gamma \cap D'', D'')$ be a tangle induced from $X$ with respect to $D$. Then $\Gamma_m \cap \partial D''$ consists of one point. Further, by (6), the component $X$ contains at least one white vertex. Thus $(\Gamma \cap D'', D'')$ is an NS-tangle of label $k$. This contradicts Theorem 1.2. Therefore Claim 1 holds.

Claim 2. For each $i = 1, 2, \cdots, p$, the tree $P_i$ intersects exactly two of $\mathcal{P} \cup \mathcal{E}$.

**Proof of Claim 2.** If there exists an element $P_i$ intersecting at most one of $\mathcal{E}$, then $P_i$ is a reducible tree. This contradicts Lemma 6.1.

Suppose that there exists an element $P_i$ intersecting at least three of $\mathcal{E}$. Then $\text{Cl}((\Gamma_m \cap D) - P_i)$ consists of at least three connected components. Again, the condition $|\Gamma_m \cap \partial D| = 2$ implies that there exists a connected
component $X$ of $Cl((\Gamma_m \cap D) - P_i)$ with $X \cap \partial D = \emptyset$. Let $(\Gamma \cap D', D')$ be a tangle induced from $X$ with respect to $D$. Then $\Gamma_m \cap \partial D'$ consists of one point. Thus by (5), the tangle $(\Gamma \cap D', D')$ is an NS-tangle of label $k$. This contradicts Theorem 1.2. Therefore Claim 2 holds.

Claim 3. The set $Q$ consists of exactly two elements each of which intersects exactly one of $E$.

Proof of Claim 3. If $Q$ consists of only one element $Q_1$, then $Cl((\Gamma_m \cap D) - Q_1)$ does not intersect $\partial D$. Let $(\Gamma \cap D', D')$ be a tangle induced from a connected component of $Cl((\Gamma_m \cap D) - Q_1)$ with respect to $D$. Then $\Gamma_m \cap \partial D'$ consists of one point. Thus by (5), the tangle $(\Gamma \cap D', D')$ is an NS-tangle of label $k$. This contradicts Theorem 1.2. Hence the condition $|\Gamma_m \cap \partial D| = 2$ implies that $Q$ consists of exactly two elements.

Now by (3), each element of $Q$ intersects at least one of $E$. Suppose that there exists an element $Q_i$ intersecting at least two of $E$. Thus $Cl((\Gamma_m \cap D) - Q_i)$ consists of at least two connected components. Since $Q_i \cap \partial D \neq \emptyset$, the condition $|\Gamma_m \cap \partial D| = 2$ implies that $Cl((\Gamma_m \cap D) - Q_i) \cap \partial D$ consists of at most one point. Hence there exists a connected component $X$ of $Cl((\Gamma_m \cap D) - Q_i)$ with $X \cap \partial D = \emptyset$. Let $(\Gamma \cap D'', D'')$ be a tangle induced from $X$ with respect to $D$. Then $\Gamma_m \cap \partial D''$ consists of one point. Hence by (5), the tangle $(\Gamma \cap D'', D'')$ is an NS-tangle of label $k$. This contradicts Theorem 1.2. Therefore Claim 3 holds.

By Claim 1, Claim 2, and Claim 3, we have $d = p + 1$. Thus by renumbering $E_1, E_2, \ldots, E_d$ and $P_1, P_2, \ldots, P_{d-1}$, by setting $P_0 = Q_1, P_d = Q_2$ we can assume that

(7) for each $i = 1, 2, \ldots, d$ and $j = 0, 1, \ldots, d$,

$$E_i \cap P_j = \begin{cases} \text{one point} & \text{if } j = i - 1 \text{ or } i, \\ \emptyset & \text{otherwise.} \end{cases}$$

Set $w_0 = P_0 \cap \partial D$, and $v_{d+1} = P_d \cap \partial D$. For each $i = 1, 2, \ldots, d$ and $j = 0, 1, \ldots, d$, let $v_i = E_i \cap P_{i-1}$, $w_i = E_i \cap P_i$, and

$$L_j = \text{the simple arc in } P_j \text{ connecting } w_j \text{ and } v_{j+1}.$$ 

Then $E_1, E_2, \ldots, E_d$ and $L_0, L_1, \ldots, L_d$ satisfy Condition (a) and Condition (b) in Lemma 10.1.

Let $Y = (\bigcup_{i=1}^d E_i) \cup (\bigcup_{j=0}^d L_j)$. Then $Y$ is simply connected and $Y \cap \partial D = \{w_0, v_{d+1}\}$.

Claim 4. If an edge of label $m$ intersects $D - Y$, then it is a terminal edge.

Proof of Claim 4. If not, then there exists the closure $T$ of a connected component of $\Gamma_m \cap (D - Y)$ such that $T$ is a tree containing at least two white vertices. Now Statement (3) implies that $|T \cap Y| \geq 1$. Suppose that $|T \cap Y| > 1$. Since $Y$ is connected, we can find a new cycle of label $m$ not contained in $\bigcup_{i=1}^d E_i$. This contradicts that $E_1, E_2, \ldots, E_d$ are all the disks bounded by the maximal cycles in $D$. Hence $|T \cap Y| = 1$. Thus $T$ is a
reducible tree with the special vertex \( T \cap Y \). This contradicts Lemma \ref{6}. Hence Claim 4 holds.

Therefore Lemma \ref{10} holds. \( \square \)

**Proof of Theorem 1.3.** Neither terminal edge nor free edge is an internal edge. By Condition (ii) of an admissible tangle, the boundary \( \partial D \) intersects neither terminal edge nor free edge. Thus the tangle \((\Gamma \cap D, D)\) satisfies Condition (i) of an IO-tangle.

By Lemma \ref{10}, there exist disks \( E_1, E_2, \cdots, E_d \) in \( \text{Int} D \) and simple arcs \( L_0, L_1, \cdots, L_d \) in \( \text{Cl}(D - \cup_{i=1}^d E_i) \) satisfying Conditions (a), (b), (c) in Lemma \ref{10}.

The condition \(|\Gamma_m \cap \partial D| = 2\) implies that \( \partial D - (\Gamma_m \cap \partial D) \) consists of two connected components. Let \( L_I, L_O \) be the closures of the connected components. Then \( L_I, L_O \) are arcs on \( \partial D \) with

\[
(1) \quad L_I \cap L_O = \partial L_I = \partial L_O = \Gamma_m \cap \partial D.
\]

Thus the two arcs \( L_I, L_O \) satisfy Condition (iii) of an IO-tangle.

Let \( Y = (\cup_{i=1}^d E_i) \cup (\cup_{j=0}^d L_j) \). Then \( Y \cap \partial D \) consists of a point in \( \partial L_0 \) and a point in \( \partial L_d \). Now \( Y \) is simply connected. Hence \(|Y \cap \partial D| = |\Gamma_m \cap \partial D| = 2\) implies that the set \( D \setminus Y \) consists of two connected components. Let \( \Delta_I, \Delta_O \) be the closures of connected components with \( \Delta_I \supset L_I \) and \( \Delta_O \supset L_O \). Then \( \Delta_I \) and \( \Delta_O \) are disks. Let \( T_I = \text{Cl}(\partial \Delta_I - L_I) \) and \( T_O = \text{Cl}(\partial \Delta_O - L_O) \).

Then \( T_I \) and \( T_O \) are pseudo paths of label \( m \) with side-disks \( \Delta_I \) and \( \Delta_O \) respectively.

For each \( i = 1, 2, \cdots, d \), let (see Fig. 25)

\[
v_i = E_i \cap L_{i-1}, \quad w_i = E_i \cap L_i, \\
\gamma_i = \text{an arc in the edge in } E_i \cap \Delta_I \text{ with } v_i \in \gamma_i, \\
\delta_i = \text{an arc in the edge in } E_i \cap \Delta_I \text{ with } w_i \in \delta_i, \\
\phi_i = \text{an arc in the edge in } L_{i-1} \text{ with } v_i \in \phi_i, \\
\psi_i = \text{an arc in the edge in } L_i \text{ with } w_i \in \psi_i, \\
T_i = E_i \cap \Delta_I, \quad \hat{T}_i = \phi_i \cup T_i \cup \psi_i.
\]

For each \( i = 0, 1, \cdots, d \), let \( \hat{L}_i = \delta_i \cup L_i \cup \gamma_{i+1} \), here \( \delta_0 = \emptyset, \gamma_{d+1} = \emptyset \).

By Lemma \ref{10}(c), for each \( i = 1, 2, \cdots, d \), the outside edges of label \( m \) for \( \partial E_i \) are terminal edges except two edges containing one of \( v_i, w_i \). Thus by Remark 2.3(2) we have

\[
(2) \quad W_O(\partial E_i, m) - W_O^{\text{Mid}}(\partial E_i, m) \subset \{v_i, w_i\}.
\]

Since \((\Gamma \cap D, D)\) is an admissible tangle, the disk \( D \) contains neither free edge nor simple hoop. Since \((\Gamma \cap D, D)\) is a 2-color tangle, for each \( i = 1, 2, \cdots, d \) the disk \( E_i \) is a 2-color disk with \( \partial E_i \subset \Gamma_m \). Thus by Lemma \ref{7}(b),

\[
(3) \quad \text{for each } i = 1, 2, \cdots, d, \text{ the intersection } \Gamma_m \cap E_i \text{ is connected.}
\]
Figure 25: (a) The thick lines are arcs $\gamma_i, \delta_i, \phi_i$ and $\psi_i$. (b) The thick line is a pseudo path $\hat{T}_i$.

By (2), (3) and Lemma 5.8(b), we have $|W_0(\partial E_i, m) - W_0^\text{Mid}(\partial E_i, m)| = 2$. Hence

$$\{v_i, w_i\} = W_0(\partial E_i, m) - W_0^\text{Mid}(\partial E_i, m).$$

**Claim 1.** The pseudo paths $\hat{L}_0, \hat{L}_1, \cdots, \hat{L}_d$ are I/O pseudo paths with respect to $\Delta_I$.

**Proof of Claim 1.** By (4), the arc $\phi_i$ is not middle at $v_i$, nor the arc $\psi_i$ is not middle at $w_i$. Hence by Lemma 10.1(c), for each $i = 1, 2, \cdots, d - 1$ the pseudo path $\hat{L}_i$ is a dichromatic bridge. Further $\hat{L}_0$ and $\hat{L}_d$ are dichromatic piers because $\phi_1$ and $\psi_d$ are not middle at $v_1$ and $w_d$ respectively. Thus Claim 1 follows from Lemma 9.1 and Corollary 9.2. Hence Claim 1 holds.

By (4), Theorem 8.6(b) implies that

$$\text{(5) for each } i = 1, 2, \cdots, d, \text{ the extended pseudo path } \hat{T}_i \text{ of the path } T_i \text{ is an I/O pseudo path with respect to } \Delta_I.$$  

**Claim 2.** $T_I, T_O$ are I/O pseudo paths with respect to $\Delta_I, \Delta_O$ respectively.

**Proof of Claim 2.** Let $i \in \{1, 2, \cdots, d\}$. Since $\hat{L}_{i-1} \cap \hat{T}_i$ contains the white vertex $v_i$, and since $\hat{T}_i \cap \hat{L}_i$ contains the white vertex $w_i$, the pairs $(\hat{L}_{i-1}, \hat{T}_i)$ and $(\hat{T}_i, \hat{L}_i)$ are I/O pairs of type I by Claim 1 and (5). Thus $(\hat{L}_0, \hat{T}_1, \hat{L}_1, \hat{T}_2, \cdots, \hat{L}_{d-1}, \hat{T}_d, \hat{L}_d)$ is an I/O sequence for $(T_I, \Delta_I)$. Hence by Remark 8.4 $T_I = \hat{L}_0 * \hat{T}_1 * \hat{L}_1 * \hat{T}_2 * \cdots * \hat{L}_{d-1} * \hat{T}_d * \hat{L}_d$ is an I/O pseudo path with respect to $\Delta_I$.

Similarly $T_O$ is an I/O pseudo path with respect to $\Delta_O$. Thus Claim 2 holds.

Let $\gamma'_I$ be a side-arc of $T_I$ at $v_1$ with respect to $\Delta_I$, and $\gamma'_O$ a side-arc of $T_O$ at $v_1$ with respect to $\Delta_O$. Without loss of generality we can assume that the arc $\gamma'_I$ is oriented inward at $v_1$. Suppose that $\phi_1$ is oriented inward at $v_1$. Since $\phi_1$ is not middle at $v_1$ by (4), the arc $\gamma'_O$ is oriented outward at $v_1$ (see
Fig. 26(a)). Suppose that $\phi_1$ is oriented outward at $v_1$. Since $\gamma'_I$ is oriented inward at $v_1$, the arc $\gamma'_O$ is oriented outward at $v_1$ (see Fig. 26(b)). Namely, if $\gamma'_I$ is oriented inward at $v_1$, then the arc $\gamma'_O$ is oriented outward at $v_1$. Since $\gamma'_I$ is a side-arc of $T_I$ with respect to $\Delta_I$ and since $\gamma'_O$ is a side-arc of $T_O$ with respect to $\Delta_O$, we have that the I/O pseudo path $T_I$ is inward with respect to $\Delta_I$ and the I/O pseudo path $T_O$ is outward with respect to $\Delta_O$.

Let $p$ be a point in $\Gamma \cap \text{Int } L_I$, and $e$ an edge of $\Gamma$ containing the point $p$. Then by (1), the label of $e$ is $k$. Since the tangle $(\Gamma \cap D, D)$ is admissible, there exists an internal edge $\bar{e}$ of label $k$ containing $e$ such that each connected component of $\bar{\gamma} \cap D$ contains a white vertex. Let $\ell$ be the arc in $\bar{\gamma} \cap D$ connecting the point $p$ and a white vertex $v$ of $\bar{\gamma}$. Now $p \in \Delta_I$ implies $v \in \Delta_I$. Since an edge of label $m$ intersecting $D - Y$ is a terminal edge by Lemma 10.1a), (6) the set $D - Y$ does not contain any white vertex. Hence $v \in T_I$. Thus $\ell$ is oriented inward at $v$. Hence the edge $e$ is locally inward at $p$. Similarly for any point $p \in \Gamma \cap \text{Int } L_O$ there exists an edge of label $k$ locally outward at $p$. Hence $(\Gamma \cap D, D)$ is an IO-tangle.

**Claim 3.** $D$ does not contain any terminal edge of label $k$.

**Proof of Claim 3.** Suppose that $D$ contains a terminal edge $e$ of label $k$. Let $w$ be the white vertex in $e$. Since for each $i = 1, 2, \ldots, d$ the 2-color disk $E_i$ does not contain any terminal edge by Corollary 3.1 b), the terminal edge $e$ is not contained in $\bigcup_{i=1}^d E_i$. Thus $e$ intersects $D - \bigcup_{i=1}^d E_i$. Hence by (6), the white vertex $w$ is in $(\bigcup_{i=1}^d \partial E_i) \cup (\bigcup_{j=0}^d L_j)$. Thus $w \in \partial E_i$ for some $i \in \{1, 2, \ldots, d\}$ or $w \in \text{Int } L_i$ for some $i \in \{0, 1, 2, \ldots, d\}$.

Suppose that $w \in \partial E_i$ for some $i \in \{1, 2, \ldots, d\}$. There are exactly three edges of label $m$ at $w$. Since $\partial E_i$ contains at least two white vertices by (4), and since $w \in \partial E_i$, two of the three edges are contained in $\partial E_i$. Thus one of $(e, w)$-edges is in $\partial E_i$. Let $e'$ be an $(e, w)$-edge in $\partial E_i$. Then $e'$ is an edge of label $m$ in $D$. Since $\partial E_i$ contains at least two white vertices by (4), the edge $e'$ contains two white vertices. Let $w'$ be the white vertex of $e'$ different from $w$. Since $(\Gamma \cap D, D)$ is a 2-color tangle, there exists an edge $e''$ of label $k$ at $w'$ such that $(e, e', e'')$ is a non-admissible consecutive triple. This contradicts Consecutive Triplet Lemma (Lemma 3.1).

Suppose that $w \in \text{Int } L_i$ for some $i \in \{0, 1, 2, \ldots, d\}$. There are exactly three edges of label $m$ at $w$. Since $L_i$ is an arc, there exists exactly one edge $e^*$ of label $m$ with $L_i \cap e^* = w$. Thus $e^* \cap (D - Y) \neq \emptyset$. Hence by Lemma 10.1c), the edge $e^*$ is a terminal edge. Since $e$ is a terminal edge, neither $(e, w)$-edges are terminal edges of label $m$. Thus $e^*$ is not an $(e, w)$-edge. Hence both of the two $(e, w)$-edges contain short arcs contained in $L_i$. Since $|L_i \cap \partial D| \leq 1$, one of the two $(e, w)$-edges is in $L_i$. Let $e'$ be an $(e, w)$-edge in $L_i$, and $w'$ the white vertex of $e'$ different from $w$. Since $(\Gamma \cap D, D)$ is a 2-color tangle, there exists an edge $e''$ of label $k$ at $w'$ such that $(e, e', e'')$ is a non-admissible consecutive triple. This contradicts Consecutive Triplet Lemma (Lemma 3.1). Thus $D$ does not contain any terminal edge of label $k$. Hence Claim 3 holds.
Since \((\Gamma \cap D, D)\) is a 2-color tangle, by Claim 3 all the terminal edge in \(D\) is of label \(m\). Thus \((\Gamma \cap D, D)\) is a simple IO-tangle. Therefore this proves Theorem 1.3. □

11 Appendix

In this section, we give complementary lemmata to make our paper self-contained.

**Proof of Consecutive Triplet Lemma (Lemma 3.1).** Suppose that there exists a non-admissible consecutive triplet \((e_1, e_2, e_3)\) in a minimal chart. Then \(e_1\) is a terminal edge and the labels of \(e_1\) and \(e_3\) are same. Let \(\partial e_2 = \{w_1, w_2\}, w_1 \in e_1, \text{ and } w_2 \in e_3\) (possibly \(w_1 = w_2\)). Suppose that \(\partial e_3 = \{w_2, w_3\}\) (possibly \(w_2 = w_3\)).

To make argument simple we assume that the three points \(w_1, w_2, \text{ and } w_3\) are mutually different. The vertex \(w_3\) may not be a white vertex. Without loss of generality, we can assume that the edge \(e_1\) is oriented outward at the white vertex \(w_1\). In a minimal chart, by Remark 2.3(2) any terminal edge must contain a middle arc at its white vertex. Hence \(e_1\) contains a middle arc at \(w_1\). Thus the edge \(e_2\) is oriented outward at the white vertex \(w_1\), too.

The edge \(e_3\) is oriented inward at the white vertex \(w_2\). For, if the edge \(e_3\) is oriented outward at \(w_2\), then the edge \(e_3\) contains a non-middle arc at \(w_2\). By applying a C-I-M2 move between \(e_1\) and \(e_3\) we can get a new terminal edge which contains a non-middle arc at the white vertex \(w_2\). Hence we can eliminate the white vertex \(w_2\) by a C-III move. This contradicts that the chart is minimal. Therefore the edge \(e_3\) is oriented inward at the white vertex \(w_2\) (see Fig. 27(a)).

Let \(e_4\) be the \((e_2, w_1)\)-edge different from the edge \(e_1\). Since \(e_1\) contains a middle arc and oriented outward at \(w_1\), the edge \(e_4\) is oriented inward at the white vertex \(w_1\). Let \(e_5\) be the \((e_2, w_2)\)-edge different from the edge \(e_3\). The edge \(e_5\) is oriented inward at the white vertex \(w_2\). For, if the edge \(e_5\) is oriented outward at the white vertex \(w_2\), then by applying a C-I-M2 move

Figure 26: The thick lines are of label \(m\).
between $e_1$ and $e_3$ and further applying a C-I-M2 move between $e_4$ and $e_5$, we get three consecutive edges connecting the two white vertices $w_1$ and $w_2$. Hence by applying a C-I-M3 move we can eliminate the two white vertices. This contradicts that the chart is minimal. Therefore the edge $e_5$ is oriented inward at the white vertex $w_2$ (see Fig. 27(b)).

Let $e_6$ be the $(e_3, w_2)$-edge different from $e_2$. Since the three edges $e_2, e_3, e_5$ are oriented inward at the white vertex $w_2$, the edge $e_6$ is oriented outward at the white vertex $w_2$ (see Fig. 27(b)).

The vertex $w_3$ is a white vertex. For, if $w_3$ is not a white vertex, then $w_3$ is a crossing or a black vertex. Hence the edge $e_3$ is not contained in a bigon. By applying a C-I-M2 move between $e_1$ and $e_3$, we can get a new bigon without destroying old bigons. Thus the number of bigons increases. This contradicts that the chart is minimal. Hence the vertex $w_3$ must be a white vertex.

Since we can apply a C-I-M2 move between $e_1$ and $e_3$, the edge $e_3$ must contain a middle arc at the white vertex $w_3$. Hence the $(e_3, w_3)$-edges are oriented outward at $w_3$. Since the edge $e_6$ is oriented outward at $w_2$ (see Fig. 27(c)), neither of the $(e_3, w_3)$-edges is equal to $e_6$. Hence the edge $e_3$ is not contained in a bigon.

Now by applying a C-I-M2 move between $e_1$ and $e_3$ we can get a new bigon without destroying old bigons. Thus the number of bigons increases. This contradicts that the chart is minimal. □

![Figure 27](image-url)
As a consequence of Theorem 1.2, we can show the following lemma.

**Lemma 11.1** (Boundary Condition Lemma) ([14, Lemma 4.1]) Let \((\Gamma \cap D, D)\) be a tangle in a minimal chart \(\Gamma\) such that \(D\) does not contain any crossing, free edge nor simple hoop. Let \(a = \overline{\pi}(\Gamma \cap \partial D)\) and \(b = \overline{\beta}(\Gamma \cap \partial D)\). Then \(\Gamma \cap D = \emptyset\) except for \(a \leq i \leq b\).

**Proof.** We claim that \(D\) does not contain any hoop. If \(D\) contains a hoop, say \(C'\), then \(C'\) is not a simple hoop by the assumption. Hence \(C'\) bounds a disk \(E'\) containing a white vertex. Let \(D'\) be a regular neighborhood of \(E'\). Then \(\Gamma \cap \partial D' = \emptyset\). Since \(D\) does not contain any crossing, neither does \(D'\). Thus \((\Gamma \cap D', D')\) is an NS-tangle. This contradicts Theorem 1.2. Hence \((1)\) \(D\) does not contain any hoop.

Let \(i\) be a label with \(\Gamma_i \cap D \neq \emptyset\). Suppose that \(i < a\) or \(b < i\). Without loss of generality we can assume \(i < a\). Let \(\alpha = \overline{\pi}(\Gamma \cap D)\). Since \(\alpha \leq i < a = \overline{\pi}(\Gamma \cap \partial D)\), we have \(\Gamma_\alpha \cap \partial D = \emptyset\). Let \(G\) be a connected component of \(\Gamma_\alpha \cap D\). Then \(G \cap \partial D = \emptyset\). Let \((\Gamma \cap D^*, D^*)\) be a tangle induced from \(G\) with respect to \(D\). Then \(D^* \subset D\). Since \(G \cap \partial D = \emptyset\), by Remark 7.5(2) we have \(|\Gamma_\alpha \cap \partial D^*| = |G \cap \partial D| = 0\). Hence \((2)\) \(\Gamma_\alpha \cap \partial D^* = \emptyset\).

Since \(\alpha\) is the lowest label in \(D\),

(3) for each label \(j < \alpha\), we have \(\Gamma_j \cap \partial D^* = \emptyset\).

Since \(D\) does not contain any crossing by assumption, for each label \(j > \alpha + 1\) we have \(G \cap \Gamma_j = \emptyset\). Hence by Remark 7.5(1)

(4) for each label \(j > \alpha + 1\), we have \(\Gamma_j \cap \partial D^* = \emptyset\).

Since \(D^* \subset D\), by assumption

(5) \(D^*\) does not contain any crossing.

By (1), the disk \(D^*\) does not contain any hoop. By the assumption, the disk \(D^*\) does not contain any free edge. By (5), the disk \(D^*\) does not contain any ring. Hence \(D^*\) contains at least one white vertex. Thus by (2), (3), (4) and (5), the tangle \((\Gamma \cap D^*, D^*)\) is an NS-tangle of label \(\alpha + 1\). This contradicts Theorem 1.2. Therefore \(a \leq i \leq b\). □

In Section 1 we have mentioned the following: Let \(m\) be any label of a chart \(\Gamma\), and \(E\) a disk with \(\partial E \subset \Gamma_m\) but without crossings, free edges nor simple hoops. If \(\Gamma\) is a minimal chart, then we can show that \(E\) is a 3-color disk. Further, if \(m\) is the minimal label or the maximal label of the chart, then \(E\) is a 2-color disk.

This can be shown by the following lemma.

**Lemma 11.2** Let \(\Gamma\) be a minimal chart. Let \(C\) be a cycle of label \(m\) bounding a disk \(E\) without crossings, free edges nor simple hoops. Then we have the following:

(a) \(E\) is a 3-color disk.
(b) If there exists a label $k$ with $|m-k| = 1$ such that all the white vertices in $C$ are in $\Gamma_m \cap \Gamma_k$, then $E$ is a 2-color disk.

Proof. Let $(\Gamma \cap D, D)$ be a tangle induced from the cycle $C$. Then $E \subseteq D$. Since $\partial E$ is a cycle of label $m$, we have $m - 1 \leq \overline{\pi}(\Gamma \cap \partial D)$ and $\overline{\beta}(\Gamma \cap \partial D) \leq m + 1$. By Boundary Condition Lemma (Lemma 11.1), we have $m - 1 \leq \overline{\pi}(\Gamma \cap D)$ and $\overline{\beta}(\Gamma \cap D) \leq m + 1$. Thus $\Gamma \cap D \subseteq \Gamma_{m-1} \cup \Gamma_{m} \cup \Gamma_{m+1}$. Since $E \subseteq D$, the disk $E$ is a 3-color disk.

Similarly we can show Statement (b). \hfill \Box

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| $w(\Gamma)$ | $p_2$ | $f(\Gamma)$ | $p_2$ | $c(\Gamma)$ | $p_3$ | $b(\Gamma)$ | $p_3$ | $\Gamma_m$ | $p_4$ | $\mathcal{W}^{\text{Mid}}_j(C, m)$ | $p_5, p_{17}$ | $\mathcal{W}^{\text{Mid}}_O(C, m)$ | $p_5, p_{17}$ | $\mathcal{P}(C; S)$ | $p_{26}$ | $\deg_G v$ | $p_{27}$ | $\tau(D)$ | $p_{32}$ | $\pi(X)$ | $p_{33}$ | $\beta(X)$ | $p_{33}$ | $P_1^* \ast P_2^* \ast \cdots \ast P_s^*$ | $p_{36}, p_{38}$ | $P_1^* \ast P_2^* \ast \cdots \ast P_s^*$ | $p_{39}$ |
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