Global geometry of the 2+1 rotating black hole

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Abstract

The generic rotating BTZ black hole, obtained by identifications in $AdS_3$ space through a discrete subgroup of its isometry group, is investigated within a Lie theoretical context. This space is found to admit a foliation by two-dimensional leaves, orbits of a two-parameter subgroup of $\tilde{SL}(2, \mathbb{R})$ and invariant under the BTZ identification subgroup. A global expression for the metric is derived, allowing a better understanding of the causal structure of the black hole.

Introduction

Vacuum Einstein equations in 2+1 dimensions with negative cosmological constant admit black hole solutions [1], referred to as BTZ black holes [2]. These solutions arise from identifications of points of $AdS_3$ by a discrete subgroup of its isometry group [2, 3]. According to the type of subgroup, the vacuum, extremal or generic (rotating or non-rotating) massive black holes are obtained [3].

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The non-rotating massive black hole was investigated in [4] in the framework of Lie groups and symmetric spaces. This approach revealed a natural foliation (a trivial fibration) of $AdS_3$ by 2-dimensional leaves invariant under the action of the identification subgroup and admitting a regular Poisson structure. These surfaces were constructed as orbits of a bi-action of $SL(2, \mathbb{R})$ twisted by an external automorphism. As a result a global expression of the metric of the black hole region between the singularities was obtained, which yielded a dynamical picture of the non-rotating black hole evolution.

In this paper, we apply a similar group theoretical approach to generic rotating black holes. Here again, we obtain an intrinsic foliation adapted to the residual isometry of the black hole solution. However, the foliation is quite different than in the non-rotating case. Indeed, the leaves are orbits of a two-parameter subgroup of $\tilde{SL}(2, \mathbb{R})$. This foliation naturally leads to a globally defined metric (up to polar angle singularities), improving our understanding of the causal structure of such black holes.

**BTZ-adapted foliation of $AdS_3$**

Rotating BTZ black holes are obtained from identifications in $AdS_3$ space. This space is isometric to $\tilde{SL}(2, \mathbb{R})$ endowed with its invariant metric. This metric (the Killing metric), denoted $\beta$, is defined at the identity $e$ by $\beta_e(X, Y) = \frac{1}{2} \text{Tr}(XY)$, where $X$, $Y$ are elements of $sl(2, \mathbb{R})$. In the following we will take the elements $H$, $E$ and $F$ as generators of $sl(2, \mathbb{R})$, defined by the commutation relations:

$$\begin{align*}
[H, E] &= 2E, & [H, F] &= -2F, & [E, F] &= H.
\end{align*}$$  \hfill (1)

We will also use the generator $T = E - F$.

Rotating black holes are characterized by their mass $M$ and angular momentum $J$, with $|J| < M$. The identifications defining them are performed by means of the action of a discrete subgroup of its isometry group $[2, 3]$, whose component connected to the identity is isomorphic to $\tilde{SL}(2, \mathbb{R}) \times \tilde{SL}(2, \mathbb{R})/\mathbb{Z}_2$. This discrete subgroup, referred to as BTZ subgroup, is isomorphic to $\mathbb{Z}$ and generated by $\exp(2\pi m\Xi)$, where $\Xi$ is the Killing vector field $[4]$: $\Xi = L_+ \overline{H} - L_- H$; \hfill (2)

$\overline{H}$ and $H$ denote respectively the right- and left-invariant vector fields on $SL(2, \mathbb{R})$ associated to the element $H$ of its Lie algebra. The parameters $L_\pm$
are related to $M$ and $J$ by $L_\pm = \sqrt{M \pm J}$, hence clarifying the geometrical meaning of $M$ and $J$. The BTZ black holes are thus obtained from the identifications:

$$z \sim \exp(2\pi m L_+ H) \ z \ \exp(2\pi m L_- H) \quad , \quad (3)$$

with $m \in \mathbb{Z}$.

It is well known that $\tilde{SL}(2,\mathbb{R})$ admits a globally defined decomposition, referred to as Iwasawa decomposition, which allows to represent each element $z \in \tilde{SL}(2,\mathbb{R})$ in a unique way as the product:

$$z = a \ n \ k \quad , \quad (4)$$

with:

$$a \in A = SO(1,1) = \{\exp(\phi \ H) \ | \ \phi \in \mathbb{R}\} \quad , \quad (5)$$

$$n \in N = \{\exp(u \ E) \ | \ u \in \mathbb{R}\} \quad , \quad (6)$$

$$k \in K = \tilde{SO}(2) = \{\exp(\tau \ T) \ | \ \tau \in \mathbb{R}\} \quad . \quad (7)$$

This decomposition is not adapted to the BTZ bi-action [3]. Instead we devise a modified, BTZ-adapted, Iwasawa decomposition defining a global diffeomorphism on $\tilde{SL}(2,\mathbb{R})$. It reads, for $L_+ > L_-$, i.e. $J > 0$:

$$z = a^{L_+} \ n \ k \ a^{L_-} \quad . \quad (8)$$

For $J < 0$, the decomposition to consider is $z = a^{L_-} \ n \ k \ a^{L_+}$; we hereafter assume $J > 0$.

The decomposition [3] naturally induces a foliation of $\tilde{SL}(2,\mathbb{R})$ whose 2-dimensional leaves $\mathcal{F}_\tau$, corresponding to constant $\tau$ sections, are stable with respect to the action of the BTZ subgroup. These leaves are obtained as follows. We introduce the action $\nu$ of the subgroup $R = AN$ on $\tilde{SL}(2,\mathbb{R})$ as:

$$\nu : R \times \tilde{SL}(2,\mathbb{R}) \to \tilde{SL}(2,\mathbb{R}) : (a \ n , z) \mapsto \nu_{an}(z) = a^{L_+} \ n \ z \ a^{L_-} \quad . \quad (9)$$

The orbits of this action are obtained by acting with $\nu$ on a given element of the subgroup $K$ transverse to the leaves:

$$\mathcal{F}_\tau = \{\nu_{an}(\exp(\tau \ T))\} \quad , \quad \forall \tau \in \mathbb{R} \quad . \quad (10)$$
In particular, the subgroup $R$ constitutes the leaf at $k = e$.

**Modified Iwasawa coordinate system (MICS)**

The BTZ-adapted map \( \Phi \) defines a global coordinate system \((\tau, u, \phi)\) by:

\[
\Phi : \mathbb{R}^3 \to \tilde{SL}(2, \mathbb{R}) : (\tau, u, \phi) \mapsto \nu_{\exp(\phi H)} \exp(u E) \exp(\tau T) ,
\]

referred hereafter as Modified Iwasawa Coordinate System (MICS). In these coordinates, the \( AdS_3 \) metric reads as:

\[
ds^2 = -d\tau^2 - du \, d\tau + L_+ d\phi \sin(2\tau) d\phi - L_- \sin(2\tau) d\phi^2 + 2 \left[ M + L_+ L_- (\cos(2\tau) - u \sin(2\tau)) \right] d\phi^2.
\]

The identification \( \Theta \), yielding BTZ black holes, is simply:

\[
(\tau, u, \phi) \mapsto (\tau, u, \phi + 2\pi m), \quad m \in \mathbb{Z} ;
\]

hence, when restricting the \( \phi \)-coordinate to \( 0 \leq \phi < 2\pi \), eq.\( [12] \) becomes a global expression for the metric for the rotating BTZ black hole. The relation between the MICS coordinates and more usual coordinate systems can easily be obtained at the level of \( SL(2, \mathbb{R}) \) by using an explicit \( SL(2, \mathbb{R}) \) matrix representation.

Note furthermore that the rotating BTZ black hole is canonically endowed with a regular Poisson structure, whose characteristic foliation coincides with the foliation induced by the action \( \nu \) defined in eq. \( [9] \). In \((\tau, u, \phi)\) coordinates, the normalized Poisson structure reads as:

\[
\{ , \} = \frac{2}{L_+ + L_- \cos(2\tau)} \partial u \wedge \partial \phi.
\]

**Causal structure**

The identifications \( \Theta \) are known to induce acausal regions containing closed time-like curves passing through every point. The BTZ subgroup generator \( \Xi \) given by eq.\( [2] \) is time-like in these regions, whereas it is space-like in the physical regions. The boundaries between physical and non-physical regions are the BTZ singularities defined by:

\[
\mathcal{S} = \{ z \in AdS_3 \text{ such that } \beta_{\Xi}(z, \Xi) = 0 \}.
\]
In terms of MICS, these singularities are given by the $\phi$-invariant surfaces:

$$q(\tau, u) \equiv M + L_+ L_- [\cos(2\tau) - u \sin(2\tau)] = 0 \quad ; \quad (16)$$

$q(\tau, u) > 0$ corresponds to causally safe regions and $q(\tau, u) \leq 0$ to regions with closed causal curves.

Let us now express the BTZ horizons $\mathcal{H}$ in terms of MICS. For that purpose, we first note that a lightray passing through the point $z_0$ can always be written as:

$$z(s) = z_0 \exp(sL), \quad s \in \mathbb{R} \quad , \quad (17)$$

with $L$ a future pointing null vector obtained by rotation of $E$ around the $T$-axis, expressed as

$$L = \text{Ad}(\exp(\kappa T)) E \quad , \quad 0 \leq \kappa < \pi \quad . \quad (18)$$

This null vector can also be seen as resulting from another null vector boosted by a Lorentz rotation with axis along $H$. These two remarks allow us to parametrize the light-rays passing through a point $z = a^{L+} n k a^{L-}$ of $AdS_3$, with direction specified by $\kappa$, as:

$$\ell_\kappa^z(s) = a^{L+} n k \exp[s \text{Ad}(\exp(\kappa T)) E] a^{L-} \quad . \quad (19)$$

The future and past light-cones at point $z$ are defined as:

$$C^\pm_z = \{ \ell_\kappa^z(s) | 0 \leq \kappa < \pi, \quad s \in \mathbb{R}^\pm \} \quad . \quad (20)$$

The position of $z$ with respect to the BTZ horizons can be determined according to the number of directions $\kappa$ allowing the light-rays to escape a given singularity, either in the past or in the future. If this number is zero, $z$ is situated behind the horizon, whereas if this number is infinite, $z$ is before the horizon. By continuity, the horizons are defined as the set of points $z$ for which a finite number of $\kappa$’s permit to escape the singularity for infinite values of $s$, i.e. for which the equation of the intersections $C^\pm_z \cap S$:

$$\beta_{\ell_\kappa^z(s)}(\Xi, \Xi) = 0 \quad , \quad (21)$$

has no solution for finite $s$. In terms of MICS, this equation reads as:

$$2L_+ L_- \beta_{\ell_\kappa^z(s)}(\mathbf{H}, \mathbf{H}) + M = 0 \quad . \quad (22)$$
It is a second order equation in $s$, invariant for the substitution $z \mapsto a z$, whose coefficients depend on the coordinates $u$ and $\tau$ of the starting point $z(u, \tau, \phi)$ of the light-ray and on its direction $\kappa$. Requiring that eq. (22) has no solution for finite $s$, neither in the past nor in the future, comes to impose the coefficients of $s$ and $s^2$ to vanish. In this way we obtain the equations of the inner and outer horizons $\mathcal{H}^-$ and $\mathcal{H}^+$:

$$
\mathcal{H}_1^+ : \tau = m\pi , \quad \mathcal{H}_2^+ : u = -\tan(\tau) , \quad (23)
$$

$$
\mathcal{H}_1^- : \tau = \frac{\pi}{2} + m\pi , \quad \mathcal{H}_2^- : u = \cot(\tau) , \quad
$$

where $m \in \mathbb{Z}$.

Strikingly, the horizons of the rotating BTZ black hole correspond to the union of left classes of the subgroups $R = AN$ and $\overline{R} = A\overline{N}$ $(\overline{N} = \{\exp(v F) \mid v \in \mathbb{R}\})$ of the form $z R$ and $z \overline{R}$, with $z = \exp(m\frac{\pi}{2} T), m \in \mathbb{Z}$.

This geometrical discussion is summarized in the Penrose diagram depicted in Fig. 1. The diagram is obtained by considering a constant $\phi$ section and making the coordinate transformation $u = \tan(p)$, with $-\frac{\pi}{2} \leq p < \frac{\pi}{2}$. On this diagram, each point corresponds to a circle (the orbit of the point under the action of the one parameter subgroup generated by $\phi$ translation). The horizons $\mathcal{H}^\pm$ are depicted by straight lines inclined at 45 degrees, crossing the $\tau$-axis at $\tau = m\frac{\pi}{2}$. The singularities, defined in [16], are situated beyond the inner horizons $\mathcal{H}^-$.

The causal structure of the BTZ spaces is obtained by considering the two fields of directions

$$
d\tau \overline{du} = \frac{-(L_+L_-u \sin(2\tau) + 2q(\tau, u)) \pm (L_+ + L_- \cos(2\tau))\sqrt{2q(\tau, u)}}{2(2q(\tau, u) + L_+^2 u^2)} , \quad (24)
$$

with $q(\tau, u)$, defined in eq. (16), positive in the causally safe region. These direction fields result from a projection on the tangent subspace, at each point of the $(\tau, p)$ coordinate plane, of the light-cone generators, parallelly to the $\partial_\phi$ direction. Note that as the vector field $\partial_\phi$ is not orthogonal to the $(\tau, p)$ coordinate surface, they are not simply given by the intersections of the light-cones with this plane.

We would like to emphasize that causal curves in the BTZ geometry are projected on the $(\tau, p)$ coordinate plane onto curves that never leave the projected cones. Hence, this Penrose diagram provides the causal structure of rotating BTZ black holes.
Finally, let us note that the leaves of the foliation, the surfaces of constant \( \tau \), are flat. To see this, notice that the induced metric on the surface \( \tau = \tau_0 \) reads as:

\[
 ds^2 = \frac{d\zeta d\phi}{2L_+} + \zeta d\phi^2,
\]

with \( \zeta = 2q(\tau_0, u) \) \( (\tau_0 \neq m\pi) \). We can further perform the change of coordinates \( U = e^{-2L_+\phi} \) and \( V = \frac{\zeta e^{2L_+\phi}}{2L_+} \), in terms of which the metric becomes:

\[
 ds^2 = -dUdV, \quad 0 < U < \infty, \quad -\infty < V < \infty,
\]

which corresponds to half a Minkowski space. The identifications according to the bi-action read in these coordinates as:

\[
 (U, V) \mapsto (U e^{-4\pi mL_+}, V e^{4\pi mL_+})
\]

They yield closed timelike curves for \( V < 0 \). Thus, when performing the identifications, the \( \tau = \) constant surfaces exhibit a Misner-space causal structure due to the fact that they intersect the singularity.

**Perspectives**

We showed in this paper that \( \text{AdS}_3 \) and rotating BTZ black holes possess a privileged foliation, the surfaces of constant \( \tau \), on which the group \( \mathbb{R} = AN \) acts (see \( \text{(9)} \)). A remarkable property of this group is the existence of a star product which differs from the usual Rieffel-Moyal product. The latter product is indeed adapted only to spaces admitting an action of \( \mathbb{R}^d \). The physical implications, namely in the context of string theory, of the existence of such a noncommutative structure in BTZ spaces will be addressed in the near future.

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Figure 1: The causal structure of the black hole is summarized in this Penrose diagram. The axis of the time-like coordinate $\tau$ is vertical, while that corresponding to the light-like coordinate $p$ is inclined at 45 degrees. Each point is a circle, orbit of the isometry subgroup generated by $\partial_\phi = L_+ \overline{H} - L_- H$. The projection parallel to $\partial_\phi$ of the lightcone generators are depicted at several points. Curves always lying inside the projected light-cones can be lifted to causal curves joining a point $(\tau_0, p_0, \phi_0)$ to a point $(\tau_1, p_1, \phi_1)$, with $\phi_0$ not necessarily equal to $\phi_1$. 