A FINITENESS THEOREM FOR WEIGHTED DISCRIMINANTS WITH MASS FORMULAS

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Abstract. We define the notion of a weighted discriminant and corresponding counting function for number fields, and what it means for these counting functions to have a mass formula for a set of primes. We prove that for $\Gamma$ an $\ell$-group, there are only finitely many weighted discriminants for $\Gamma$-extensions of $\mathbb{Q}$ that have a mass formula for all primes.

1. Introduction

A standard question in arithmetic statistics asks:

Question 1. Given a finite group $\Gamma$ and a transitive action of $\Gamma$ on a set of size $n$, how many number fields $K$ are there with $[K : \mathbb{Q}] = n$, $\text{Gal}(K/\mathbb{Q}) = \Gamma$, and $\text{Disc}(K/\mathbb{Q}) < X$?

The discriminant is not the only natural invariant of number fields by which we can count, and experience has shown us that the order in which we count number fields can make a qualitative difference in the answer to Question 1. Wood, in [8], proposes, in the specific case $\Gamma = D_4$ and $n = 4$, to replace the discriminant by another invariant, which is derived from the structure of $D_4$ as a wreath product $C_2 \wr C_2$. Wood proves that this invariant has a universal mass formula, which we will define in section 2. The discriminant does not have this when $\Gamma = D_4$, but it does when $\Gamma = S_n$ for any $n$ [4].

The notion of a counting function formalizes a set of invariants that are reasonable to substitute for the discriminant in Question 1. Most counting functions do not have universal mass formulas. By contrast, we show in Theorem 9 (which is a slight generalization of a result of Kedlaya [4]) that if any counting function is sufficiently “nice”, it must have a tame mass formula, defined in section 2, which falls just short of being universal due to bad behavior at primes dividing $|\Gamma|$.

We consider a particular type of counting function called a weighted discriminant counting function, defined in section 3, which generalize Wood’s invariant. In section 6 we prove our main theorem:

Theorem 2. Let $\Gamma$ be any finite $\ell$-group. Then there are only finitely many positive weighted discriminant counting functions $\Gamma$ which have a universal mass formula.
A natural question for future work, then, is:

**Question 3.** Does Theorem 2 hold for all finite groups?

We will discuss this and other potential generalizations of Theorem 2 in section 9.

2. **Counting Functions and Mass Formulas**

Let \( \Gamma \) be a finite group.

Let \( S_{\mathbb{Q}_p, \Gamma} \) be the set of continuous homomorphisms \( G_{\mathbb{Q}_p} \to \Gamma \), where \( G_{\mathbb{Q}_p} \) denotes the absolute Galois group of \( \mathbb{Q}_p \). We define a **counting function** for \( \Gamma \) to be any mapping

\[
c : \bigcup_p S_{\mathbb{Q}_p, \Gamma} \to \mathbb{Z}
\]

satisfying the following conditions:

- \( c(\phi) = c(\gamma \phi \gamma^{-1}) \) for any \( \gamma \in \Gamma \)
- \( c(\phi) = 0 \) if \( \phi \) is unramified

We use the notation of [8] here, except that we allow \( c \) to take negative values.

Also as in [8], we define the **total mass at** \( p \) of a counting function \( c \) to be

\[
M(\mathbb{Q}_p, \Gamma, c) = \frac{1}{|\Gamma|} \sum_{\phi \in S_{\mathbb{Q}_p, \Gamma}} \frac{1}{p^{e(\phi)}}
\]

Note that this sum is finite, so the right-hand side is well-defined. Kedlaya [4] and Wood [8] omit the factor of \( \frac{1}{|\Gamma|} \), but we will see in Theorem 9 that all the coefficients of the Laurent polynomial \( M(\mathbb{Q}_p, \Gamma, c) \) are still integers if we include it, so we divide it out for simplicity.

Define a **character Laurent polynomial** to be a sum

\[
f(x) = \sum_{i=k_1}^{k_2} \sigma_i(x) x^{-i}
\]

defined for integers \( x \), where each \( \sigma_i \) is a \( \mathbb{Z} \)-linear combination of Dirichlet characters modulo divisors of \( |\Gamma| \). Note that \( i \) may take negative values if \( k_1 < 0 \).

We use the convention that if \( \chi \) is a character with modulus \( n \) and \( (x, n) > 1 \), then \( \chi(x) = 0 \), and we assume that each character has its smallest possible modulus. That is, we exclude, for example, the character with \( \chi(x) = 1 \) when \( 5 \nmid x \), and \( \chi(x) = 0 \) when \( 5|x \); instead, we use \( \chi(x) = 1 \) for all \( x \). This is necessary for Theorem 9 to hold in the case where \( \Gamma \) is not an \( \ell \)-group, but it does not affect the value of any \( \chi(x) \) where \( (x, |\Gamma|) = 1 \).

If \( f \) is a character Laurent polynomial and \( S \) is a set of primes, we say that the pair \((c, \Gamma)\) has \( f \) as an \( S \)-mass formula (or a mass formula for \( S \)) if for all primes \( p \in S \),

\[
M(\mathbb{Q}_p, \Gamma, c) = f(p)
\]
We generally say that \( f \) is an \( S \)-mass formula for \( c \), or that \( c \) has an \( S \)-mass formula, since the reference to \( \Gamma \) is implicit in the counting function \( c \). If \( S \) is the set of all primes, then we call \( f \) a universal mass formula. If \( S \) is the set of all primes not dividing \( |\Gamma| \), then we call \( f \) a tame mass formula.

Masses and mass formulas can be defined over a base field other than \( \mathbb{Q} \) by replacing the fields \( \mathbb{Q}_p \) by all nonarchimedean completions of the base field, and replacing \( p \) elsewhere by the residue characteristic. We will currently consider only \( \mathbb{Q} \) as a base field, and consider more general base fields in future work.

**Remark.** If only the trivial character appears in the coefficients of \( f \) (i.e. \( f \) is a Laurent polynomial with integer coefficients), then we call \( f \) a pure mass formula. This corresponds to the definition of “mass formula” used by Kedlaya and Wood, except that we allow positive powers of \( p \) to appear in \( f \), accounting for counting functions that may take negative values.

Our definition of a mass formula, with characters allowed to appear in the coefficients, expands the definition used by Kedlaya and Wood, but it gives the more elegant result on tame mass formulas in Theorem 9.

A counting function is called proper if, given \( \phi : G_{\mathbb{Q}_p} \to \Gamma \) and \( \phi' : G_{\mathbb{Q}_{p'}} \to \Gamma \) with \( p, p' \not\mid |\Gamma| \) and \( \phi(I_{\mathbb{Q}_p}) = \phi'(I_{\mathbb{Q}_{p'}}) \), where \( I_K \) denotes the absolute inertia group of a local field \( K \), then \( c(\phi) = c(\phi') \) (even if \( p \neq p' \)). That is, for tame primes, \( c \) depends only on the image of the absolute inertia group.

**Example.** Let \( \Gamma = C_2 \). Then each surjective \( \phi \in \bigcup_p S_{\mathbb{Q}_p, \Gamma} \) corresponds to a distinct quadratic extension of \( \mathbb{Q}_p \). Define a counting function \( c \) so that \( c(\phi) \) is the discriminant exponent (the power of \( p \) appearing in the discriminant) of this extension.

This counting function is proper, and it has a universal pure mass formula, as we can verify by computing masses explicitly using \([3]\). If \( p \neq 2 \), there are two ramified quadratic extensions of \( \mathbb{Q}_p \), each with discriminant exponent 1. In addition, there is one unramified quadratic extension, and one non-surjective map \( G_{\mathbb{Q}_p} \to C_2 \) (the trivial map), so the mass at \( p \) is \( 1 + p^{-1} \).

For \( p = 2 \), there are still two unramified maps \( G_{\mathbb{Q}_p} \to C_2 \), but now there are two quadratic extensions of \( \mathbb{Q}_2 \) with discriminant exponent 2, and four quadratic extensions with discriminant exponent 3. The mass at 2 is thus \( 1 + 2^{-2} + 2 \cdot 2^{-3} = 1 + 2^{-1} \). Since this agrees numerically with the mass at all other primes, the mass formula \( f(p) = 1 + p^{-1} \) is universal.

The following two results are due to Kedlaya \([4]\,\text{Corollaries 5.4-5.5]}\):

**Proposition 4.** Let \( a \) be an integer not divisible by \( |\Gamma| \). Then for any proper counting function \( c \), \((\Gamma, c)\) has a pure \( S \)-mass formula, where \( S \) is the set of all primes congruent to a modulo \(|\Gamma|\).

**Proposition 5.** Let \( c \) be any proper counting function. Then \((\Gamma, c)\) has a pure tame mass formula if and only if \( \Gamma \) has a rational character table. This also holds for any group \( \Gamma \).
Kedlaya only considers a subset of the counting functions we allow here, so we will show that Proposition 4 extends to the set we are considering. We will omit this for Proposition 5, since we will later generalize this theorem for non-pure mass formulas.

Proof. Consider the quotient $G_{Q_p}/G_{1,Q_p}$, for $p \mid |\Gamma|$, where the latter group is the absolute wild inertia group. This quotient is a semidirect product of the absolute tame inertia group $G_{0,Q_p}/G_{1,Q_p}$ with $\hat{\mathbb{Z}}$. Let the topological generators of $G_{0,Q_p}/G_{1,Q_p}$ and $\hat{\mathbb{Z}}$ be $s$ and $t$, respectively. Then a continuous homomorphism $\phi : G_{Q_p}/G_{1,Q_p} \to \Gamma$ is described entirely by $\phi(s)$ and $\phi(t)$, where these choices must be compatible with the relation $tsts^{-1} = sp$.

Furthermore, if $c$ is a proper counting function, then $c(\phi)$ is determined only by the choice of $\phi(s)$.

Now suppose $q$ is another prime with $q = p + a \cdot |\Gamma|$, where $a \in \mathbb{Z}$. Then for any $\sigma \in \Gamma$, $\sigma^q = \sigma^p \cdot \sigma^{a \cdot |\Gamma|} = \sigma^p$. This shows that the number of pairs $(\sigma, \tau)$ with $\sigma, \tau \in \Gamma$ and $\tau \sigma \tau^{-1} = \sigma^p$ is the same as the number with $\tau \sigma \tau^{-1} = \sigma^q$, and thus there is a one-to-one correspondence between $S_{Q_p,\Gamma}$ and $S_{Q_q,\Gamma}$, which preserves the value of any proper counting function $c$.

From this, it follows that the total masses of $c$ at $p$ and $q$ are the same Laurent polynomial in $p$ and $q$, and thus $c$ has a pure mass formula for all primes congruent to $p$ modulo $|\Gamma|$. □

Remark. In the following sections, we will discuss global maps $\phi : G_Q \to \Gamma$. We call such a map a $\Gamma$-extension of $\mathbb{Q}$.

A $\Gamma$-extension of $\mathbb{Q}$ is also equivalent to the data of a Galois extension $K/\mathbb{Q}$ together with a choice of isomorphism $\text{Gal}(K/\mathbb{Q}) \cong \Gamma$. We will sometimes refer to these extensions in terms of the map $\phi$, and sometimes in terms of the field $K$, taking the isomorphism $\text{Gal}(K/\mathbb{Q}) \to \Gamma$ to be implicit.

By way of notation, if $K$ is such a field and $H$ is a subgroup of $\Gamma$, then $K_H$ will denote the fixed field of $H$ in $K$.

3. Weighted discriminants

We use the term alternate discriminant to refer to any reasonable rational-valued function on the set of $\Gamma$-extensions of $\mathbb{Q}$. A “reasonable” function, broadly speaking, should be one determined locally at each rational prime $p$ by the restriction of $\phi$ to $G_{Q_p}$.

If we require alternate discriminants to be determined locally in this way, then an alternate discriminant is equivalent to a counting function. Given a counting function $c$ for $\Gamma$, we can build an alternate discriminant corresponding to $c$: Let $\phi : G_Q \to \Gamma$ be a $\Gamma$-extension. Then if $\phi_p$ is the restriction of $\phi$ to $G_{Q_p}$, and

$$D_c(\phi) = \prod_p p^{c(\phi_p)}$$

Conversely, if an alternate discriminant $D$ is defined locally, then we can construct a counting function corresponding to it. If $\phi_p$ is the restriction of
some $\Gamma$-extension to $G_{\mathbb{Q}_p}$, then we define $c(\phi_p)$ to be the power of $p$ appearing in $D(\phi)$.

However, from the perspective of searching for universal mass formulas, this broad class of invariants is not very interesting, even if we require the counting functions to be proper. As we will see in Theorem 9 any proper counting function $c$ is guaranteed to have a tame mass formula. Then, since the condition of properness imposes no restrictions on how the counting function can behave at primes dividing $|\Gamma|$, we can assign values to $c$ in such a way that it forces the tame mass formula to be universal.

Thus, we seek a natural way to define counting functions (or alternate discriminants) globally, and prohibit entirely contrived behavior at the wild prime. To that end, in this paper we consider weighted discriminants, a class of alternate discriminants defined as follows:

**Definition 6.** We say that $w$ is a weight function for $\Gamma$ if $w : \{(H, H')\} \to \mathbb{Z}$, where the domain of $w$ consists of ordered pairs $(H, H')$ where $H \subset \Gamma$ and $H'$ is a maximal subgroup of $H$.

The weighted discriminant given by a weight function $w$ is

$$D_w(K) = \prod_{(H, H')} N_{K_H/K_H}(\text{Disc}(K_{H'}/K_H))^{w(H, H')}$$

where Disc is the standard relative discriminant and $N$ is the norm.

Since $N_{K_H/K_H}(\text{Disc}(K_{H'}/K_H))^{w(H, H')}$ can be determined locally from the ramification groups of $K/\mathbb{Q}$, $D_w$ is an alternate discriminant, and can also be defined in terms of a counting function $c_w$. We call a counting function of this form a weighted discriminant counting function. If $w(H, H') \geq 0$ for each $(H, H')$, we call $c_w$ positive.

**Remark.** Changing the isomorphism $\text{Gal}(K/\mathbb{Q}) \to \Gamma$ by an outer automorphism of $\Gamma$ may change the value of $D_w(K)$, but an inner automorphism will not.

**Remark.** If the weight function $w$ takes only nonnegative values, then $D_w$ integer-valued; however, this restriction is not needed for any of our results.

**Remark.** It is possible for two different weight functions to give the same counting function. For example, let $\Gamma = C_2 \times C_2$, and let $H_1$, $H_2$, and $H_3$ be its order-2 subgroups, with 1 denoting the trivial subgroup. If we let $w$ be the weight function with $w(\Gamma, H_1) = 1$ and all other weights equal to 0, and $w'$ be the weight function with $w(H_2, 1) = 2$ and all other weights zero, then $c_w = c'_w$.

4. **An Explicit Formula for** $c_w$

In this section, we give an explicit formula for $c_w(\phi)$ in terms of the weight function $w$ and the ramification groups of the map $\phi$, which we will use in the proof of Theorem 2.
Let \( \phi : G_{\mathbb{Q}} \to \Gamma \) be a map, and let \( K/\mathbb{Q} \) be the corresponding \( \Gamma \)-extension. If \( p \) is a prime of \( K \) above \( p \), we denote by \( I_{p,i} \) the \( i \)th ramification group in lower numbering at \( p \), for the extension \( K/\mathbb{Q} \). As in [7], \( i = 0 \) and \( i = -1 \) correspond to the inertia and decomposition groups, respectively. Throughout this section, \( \text{Disc} \) denotes the standard discriminant ideal, and \( \mathcal{D} \) the different ideal.

Let \( H' \) be a maximal subgroup of \( H \subseteq \Gamma \). Recall from before that \( K_H \) and \( K_{H'} \) are the fixed fields of \( H \) and \( H' \) in \( K \).

Using the fact that the discriminant of a field extension is the norm of the different ideal, and that

\[
\text{Disc} K/K_H = N_{K_{H'}/K_H} (\text{Disc} K/H') \cdot (\text{Disc} K_{H'}/K_H)|^{H'}/H)
\]

we first obtain

\[
\text{Disc} K_{H'}/K_H = \left( \frac{N_{K/K_H} \mathcal{D}(K/K_H)}{N_{K/K_H} \mathcal{D}(K/K_{H'})} \right)^{\frac{1}{|H'|}}
\]

Now norming down to \( \mathbb{Q} \) gives:

\[
N_{K_{H'}/\mathbb{Q}}(\text{Disc}(K_{H'}/K_H)) = N_{K/\mathbb{Q}} \left( \frac{\mathcal{D}(K/K_H)}{\mathcal{D}(K/K_{H'})} \right)^{\frac{1}{|H'|}}
\]

Now we take the valuation at \( p \) of both sides, and use the fact that if \( p \) is a prime above \( p \) and \( K/\mathbb{Q} \) is Galois, then \( N_{K/\mathbb{Q}}(p) = p^{f_{K/\mathbb{Q}}(p)} \), where \( f \) denotes the degree of the residue field extension.

\[
v_p(N_{K_{H'}/\mathbb{Q}}(\text{Disc}(K_{H'}/K_H))) = \frac{f_{K/\mathbb{Q}}(p)}{|H'|} \sum_{p \mid p} (v_p(\mathcal{D}(K/H)) - v_p(\mathcal{D}(K/H')))
\]

Using the formula in [7] for the different in terms of the ramification groups of an extension, and that \( f_{K/\mathbb{Q}}(p) = \frac{|I_{p,-1}|}{|I_{p,0}|} \), the right side becomes

\[
\frac{|I_{p,-1}|}{|I_{p,0}| \cdot |H'|} \sum_{p \mid p} \left[ \sum_{i \geq 0} (|I_{p,i} \cap H| - |I_{p,i} \cap H'|) \right]
\]

Now choose any prime \( p \) above \( p \). The ramification groups of the other primes above \( p \) are conjugates of \( I_{p,i} \). There are \( |\Gamma|/|I_{p,-1}| \) of these, so we can rewrite the previous line as

\[
\frac{|I_{p,-1}|}{|I_{p,0}| \cdot |H'|} \cdot \frac{1}{|I_{p,-1}|} \sum_{\gamma \in \Gamma} \left[ \sum_{i \geq 0} (|\gamma I_{p,i} \gamma^{-1} \cap H| - |\gamma I_{p,i} \gamma^{-1} \cap H'|) \right]
\]

If \( \phi \) is a map \( G_{\mathbb{Q}} \to \Gamma \) with inertia groups \( I_{p,i} \), and \( \phi_p \) is its restriction to \( G_{\mathbb{Q}_p} \), then we set

\[
e_{H,H'}(\phi_p) := \frac{1}{|I_{p,0}| \cdot |H'|} \cdot \sum_{\gamma \in \Gamma} \left[ \sum_{i \geq 0} (|\gamma I_{p,i} \gamma^{-1} \cap H| - |\gamma I_{p,i} \gamma^{-1} \cap H'|) \right]
\]
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Note that this expression does not depend on the choice of \( p \), since we sum over all conjugates of \( I_p, i \).

Now if \( w \) is any weight function with corresponding weighted discriminant \( D_w \), define the counting function

\[
c_w(\phi_p) = \sum_{(H, H')} c_{H, H'}(\phi_p) \cdot w(H, H')
\]

Let \( \phi : G_\mathbb{Q} \to \Gamma \), with \( \phi_p \) the restriction of \( \phi \) to \( G_{\mathbb{Q}_p} \). Since

\[
c_w(\phi_p) = \sum_{(H, H')} v_p(N_{K_H/\mathbb{Q}}(\text{Disc}(K_{H'}/K_H))) \cdot w(H, H')
\]

we have

\[
D_{c_w}(K) = \prod_p p^{c_w(\phi_p)}
= \prod_p \prod_{(H, H')} p^{v_p(N_{K_H/\mathbb{Q}}(\text{Disc}(K_{H'}/K_H))) \cdot w(H, H')}
= \prod_{(H, H')} N_{K_H/\mathbb{Q}}(\text{Disc}(K_{H'}/K_H))^{w(H, H')}
= D_w(K)
\]

Thus if \( c_{H, H'} \) and \( c_w \) are defined as above, then \( c_w \) is the counting function corresponding to the weighted discriminant \( D_w \).

If \( p \nmid |\Gamma| \), then \( c_w \) depends only on the inertia groups \( I_{p, 0} \), and in particular not on the decomposition group. This implies:

**Corollary 7.** Given any weight function \( w \), the corresponding counting function \( c_w \) is proper.

In addition, the following property of \( c_{H, H'} \) will be useful in the proof of Theorem 2:

**Corollary 8.** If all of the following hold:

- \( \Gamma \) is an \( \ell \)-group
- \( c \) is positive (i.e each \( w(H, H') \geq 0 \))
- \( \phi_p : G_{\mathbb{Q}_p} \to \Gamma \) and \( \phi'_\ell : G_{\mathbb{Q}_\ell} \to \Gamma \), with \( p \neq \ell \)
- \( \phi_p(I_{\mathbb{Q}_p}) = \phi'_\ell(I_{\mathbb{Q}_\ell}) \)

then \( c_{H, H'}(\phi'_\ell) \geq 2c_{H, H'}(\phi_p) \).

**Proof.** If the image of inertia under \( \phi \) and \( \phi' \) is the same, but \( \phi' \) is wildly ramified, then \( I_{\ell, 0}(\phi') = I_{\ell, 1}(\phi') \) because \( \Gamma \) is an \( \ell \)-group. Thus for any \((H, H')\),

\[
\sum_{i \geq 0} \left( |I_{\ell, i} \cap H| - |I_{\ell, i} \cap H'| \right) \geq \sum_{0 \leq i \leq 1} \left( |I_{\ell, i} \cap H| - |I_{\ell, i} \cap H'| \right) = 2 \left( |I_{\ell, 0} \cap H| - |I_{\ell, 0} \cap H'| \right)
\]

\[\square\]
5. Tame Mass Formulas and Their Coefficients

In this section, we prove a more general form of Proposition 5 for non-pure mass formulas.

**Theorem 9.** Any proper counting function $c$ has exactly one tame mass formula. The tame mass formula is of the form

$$f(x) = \sum_{C} \sigma_C(x) x^{-i_C}$$

where the sum ranges over conjugacy classes of cyclic subgroups $C \subseteq \Gamma$. Each “coefficient” $\sigma_C$ is a sum of distinct Dirichlet characters modulo divisors of $|\Gamma|$, containing the trivial character.

**Remark.** This is an extension of Kedlaya’s result (Proposition 5) to include non-pure mass formulas. Proposition 5 implies that as long as $c$ only takes nonnegative values, then the mass formula given by Theorem 9 is pure if and only if $\Gamma$ has a rational character table.

We will need the following fact from representation theory:

**Proposition 10.** Let $A$ be an abelian group, and $B$ a subgroup of $A$. Let $\sigma$ be the sum of all irreducible characters of $A$ that are trivial on $B$. Then

$$\sigma(a) = \begin{cases} 0 & \text{if } a \notin B \\ [A : B] & \text{if } a \in B \end{cases}$$

Also, we use the notation $g_1 \sim g_2$ to mean that $g_1$ and $g_2$ are conjugate as elements of $\Gamma$, and $[g]$ to denote the conjugacy class of $g$.

We now prove Theorem 9.

**Proof.** Let $a$ be an integer relatively prime to $|\Gamma|$. Since $c$ is proper, there exists a pure mass formula $f_a$ for all primes congruent to $a$ modulo $|\Gamma|$, by Proposition 4. This is unique, since if there were another such pure mass formula $f'_a$, then $f_a$ and $f'_a$ would be two different Laurent polynomials which agree at infinitely many values, which is impossible.

Now let $f$ be a character Laurent polynomial, and assume $f$ is a tame mass formula for $c$. If $p \equiv a \pmod{|\Gamma|}$, then we must have $f(p) = f_a(p)$. Suppose

$$f(x) = \sum_i \left( \sum_{j} b_{i,j}(x) \chi_j(x) \right) x^{-i}$$

and

$$f_a(x) = \sum_i b'_i x^{-i}$$

where the inner sum in the first line runs over all Dirichlet characters modulo divisors of $|\Gamma|$. Since $f(p) = f_a(p)$, we must have If $p$ is sufficiently large
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compared to all the $b_{i,j}$ and $b'_i$, then for every $i$, we must have

$$\sum_{\chi_j} b_{i,j} \chi_j(p) = b'_i$$

for all $i$ and all $p \equiv a \bmod |\Gamma|$ sufficiently large compared to the $b_{i,j}$ and $b'_i$, which implies it holds for all $p \equiv a \bmod |\Gamma|$, since the $\chi_j$ are periodic.

Each coefficient of $f$ is a function on the conjugacy classes (i.e. the elements) of $(\mathbb{Z}/|\Gamma|\mathbb{Z})^*$, and its value on each $a \in (\mathbb{Z}/|\Gamma|\mathbb{Z})^*$ is determined by the corresponding coefficient of $f_a$. There is thus a unique $\mathbb{C}$-linear combination of irreducible characters of $(\mathbb{Z}/|\Gamma|\mathbb{Z})^*$ for each coefficient of $f$ that makes that coefficient agree with the corresponding coefficient of each of the $f_a$.

Finally, each irreducible character of $(\mathbb{Z}/|\Gamma|\mathbb{Z})^*$ is equal to a unique Dirichlet character with modulus a divisor of $|\Gamma|$ that is as small as possible.

This shows that there is a unique function $f$ of the form

$$f(x) = \sum_C \sigma_C(x)x^{-ic}$$

such that $f(p)$ agrees with the mass of $c$ at $p$ for all $p \nmid |\Gamma|$, where each $\sigma_C$ is a complex linear combination of Dirichlet characters modulo divisors of $|\Gamma|$.

To prove Theorem 9, it remains to show that each $\sigma_C$ is a sum of distinct characters, with the trivial character appearing in the sum.

Let $p$ be a prime not dividing $|\Gamma|$, and let $f_p$ be the pure mass formula for the set of primes congruent to $p$ modulo $|\Gamma|$, as discussed above. Like $f$, $f_p$ also has a term corresponding to each conjugacy class of cyclic subgroups of $\Gamma$.

Let $x$ be an element of $\Gamma$. The coefficient $\sigma_{\langle x \rangle}$ is $\frac{1}{|\Gamma|}$ times the number of maps $G_{Q_p} \to \Gamma$ with inertia group conjugate to $\langle x \rangle$. Each such map is specified by an ordered pair $(s,t) \in \Gamma^2$, where $\langle t \rangle$ is conjugate to $\langle x \rangle$, and $sts^{-1} = t^p$. ($t$ is the generator of inertia, and $s$ is the Frobenius element.)

If $x^p \not\in \langle x \rangle$, then there are no such pairs. Otherwise, the number of choices for $t$ is the number of elements of $\Gamma$ generating a subgroup conjugate to $\langle x \rangle$, and the number of choices for $s$ is equal to the number of elements of $E(x)$, the centralizer of $x$ in $\Gamma$.

In the latter case, let $n$ be the order of $x$ in $\Gamma$. If $a$ and $b$ are coprime to $n$, and $x \sim x^a$ and $x \sim x^b$, then we have $g_1xg_1^{-1} = x^a$, and $g_2xg_2^{-1} = x^b$, and

$$g_2g_1xg_1^{-1}g_2^{-1} = g_2x^ag_2^{-1} = x^{ab}$$

Thus $\langle x \rangle \cap \langle x \rangle$ is naturally in bijection with a subgroup $S \subseteq (\mathbb{Z}/n\mathbb{Z})^*$, via $x^k \mapsto k$.

We can now calculate $\sigma_{\langle x \rangle}(p)$. For each element of $\langle x \rangle$, we have one choice for $t$, but we also need to count elements of $\Gamma$ not in $\langle x \rangle$ but generating a subgroup conjugate to $\langle x \rangle$. Overall, then, a choice of $t$ is described by a choice of an element of $\langle x \rangle$ and a coset of $S$ in $(\mathbb{Z}/n\mathbb{Z})^*$. The number of
choices for $s$, as above, is $|C(x)|$. The coefficient is then
$$\frac{1}{|\Gamma|} \cdot |[x]| \cdot \frac{\phi(n)}{|S|} \cdot |\mathcal{C}(x)| = \frac{\phi(n)}{|S|} = [\mathbb{Z}/n\mathbb{Z}^* : S]$$

since $|[x]| \cdot |\mathcal{C}(x)| = |\Gamma|$.

Now, if $x \sim x^p$, then $p \in S$ when $p$ is taken as an element of $(\mathbb{Z}/n\mathbb{Z})^*$. Thus $\sigma_{(x)}(p)$ should be 0 if $p \notin S$ and $|(\mathbb{Z}/n\mathbb{Z})^* : S|$ if $p \in S$.

Let $\sigma_{n,S}$ be the sum of all irreducible characters of $(\mathbb{Z}/n\mathbb{Z})^*$ that are trivial on $S$. By Proposition 10, $\sigma_{(x)}(p) = \sigma_{n,S}(p)$. Thus $\sigma_{n,S} = \sigma_{(x)}$, the “coefficient” of $f$ corresponding to the conjugacy class of $(x)$. Finally, $\sigma_{n,S}$ is a sum of distinct Dirichlet characters including the trivial character, as desired.

Remark. When $\Gamma$ is an $\ell$-group, as in the proof of Theorem 2, $\ell$ divides the modulus of every nontrivial Dirichlet character in the coefficients of $f$, so Theorem 9 implies that $f(\ell)$ is a polynomial with one term for each conjugacy class of cyclic subgroups of $\Gamma$ and all coefficients equal to 1.

6. Proof of Theorem 2

We now are equipped to prove our main theorem, Theorem 2. Let $\Gamma$ be an $\ell$-group, and $c$ a positive weighted discriminant counting function for $\Gamma$, with weight function $w$, and corresponding weighted discriminant $D_w$. Assume that $c$ has a universal mass formula $f$. Our method of proof will be to show that $c$ is completely determined by the values of certain linear combinations of the weights $w(H,H')$, and then to put an upper bound on the value of each such linear combination.

6.1. Preliminaries. If $f$ is universal, it must be exactly the unique tame mass formula described in Theorem 9. Let $[C_1], \ldots, [C_s]$ be the conjugacy classes of cyclic subgroups of $\Gamma$. By Theorem 9 $f$ is of the form
$$f(p) = \sum_{[C_j]} \sigma_{C_j}(p)p^{-n_j}$$

where each $\sigma_{C_j}$ is a sum of Dirichlet characters containing the trivial character exactly once, and the integers $n_j$ are determined by the counting function $c_w$. Since each nontrivial character vanishes at $\ell$, we have
$$f(\ell) = \sum_{[C_j]} \ell^{-n_j}$$

$f$ is universal if and only if this quantity is equal to the total mass of $c$ at $\ell$.

Note that $f(\ell)$ need only be numerically equal to the total mass; the two quantities will never be abstractly the same polynomial in $\ell$. For example, if we take $\Gamma = C_2$, and $D_w$ to be the standard discriminant, then the tame mass formula is
$$f(p) = 2 + 2p^{-1}$$
At $\ell = 2$, there are two quadratic extensions of $\mathbb{Q}_2$ of discriminant 4 and four extensions of discriminant 8 [3], so the total mass is

$$2 + 2\ell^{-2} + 4\ell^{-3}$$

However, since

$$2 + 2 \cdot 2^{-1} = 2 + 2 \cdot 2^{-2} + 4 \cdot 2^{-3} = 3$$

the mass formula $2 + 2p^{-1}$ is universal.

6.2. Linear forms in the weights. Now we consider the weights $w(H, H')$ as variables.

If $p$ is any prime, and $\phi : G_{\mathbb{Q}_p} \to \Gamma$, then the value of $c(\phi)$ is an integer linear combination of different $w(H, H')$, as described in section [1] If $p \neq \ell$, then $c(\phi)$ depends only on the image of inertia, $\phi(I_{\mathbb{Q}_p})$, and there are only finitely many possible inertia groups since $\Gamma$ is finite. If $p = \ell$, then $c(\phi)$ also depends on higher ramification groups, but in this case there are only finitely many possible maps $\phi : G_{\mathbb{Q}_\ell} \to \Gamma$.

This shows:

**Proposition 11.** There is a finite set of linear forms $\{L_1, \ldots, L_r\}$ in the variables $w(H, H')$ with the following property: for any prime $p$ and any map $\phi : G_{\mathbb{Q}_p} \to \Gamma$, there is an $i$ such that for any weighted discriminant counting function $c$ with weight function $w$, we have $c(\phi) = L_i$.

This implies that the counting function $c$ is not determined by the weights $w(H, H')$, but only by the values of the linear forms $L_1, \ldots, L_r$. Thus to prove that there are only finitely many equivalence classes of weighted discriminants with universal mass formulas, it suffices to put upper and lower bounds on the values of the $L_i$.

6.3. The “Mass Formula Formula”. For any prime $p$ and any of the linear forms $L_i$, let $a_{p,i}$ be the number of maps $\phi : G_{\mathbb{Q}_p} \to \Gamma$ with $c(\phi) = L_i$. The total mass of $c$ at $p$ is then

$$\sum_{L_i} a_{p,i} p^{-L_i}$$

On the other hand, for each weight function $w$, we have a tame mass formula $f_w$, which is universal if and only if $f_w(\ell)$, which we computed in [6.1], is equal to the total mass at $\ell$. Since $w$ is completely determined by the values of the $L_i$, a weighted discriminant counting function with a universal mass formula is equivalent to a choice of $L_1, \ldots, L_r$ such that

$$\sum_{[C_j]} \ell^{-n_j} = \sum_{L_i} a_{\ell,i} \ell^{-L_i}$$

using the notation of [6.1], and allowing the $n_j$ to depend on $w$.

By the argument in [6.2], each exponent on the left side is equal to one of the linear forms $L_i$. Namely, the exponent of the $[C_j]$ term is such that
if \( \phi : G_{Q_p} \to \Gamma \) with \( p \neq \ell \), and the image of inertia under \( \phi \) is \( C_j \), then \( c(\phi) = n_j \). Call this exponent \( L_{C_j} \).

Using cyclotomic extensions of \( Q_\ell \), we can also construct a totally ramified map \( \phi' : G_{Q_\ell} \to \Gamma \) with image \( C_j \). \( c(\phi') \) is then equal to another one of the linear forms \( L_i \); call this one \( L'_{C_j} \). By Corollary 3,

\[ L'_{C_j} \geq 2L_{C_j} \]

We can now rewrite (1) as follows:

\[ (2) \quad \sum_{[C_j]} \ell^{-L_{C_j}} = \sum_{[C_j]} \ell^{-b_j L_{C_j}} + \sum \ell^{-M_i} \]

where each \( b_j \) is at least 2. On the right side, we have taken one copy of each term with exponent \( L'_{C_j} \) and included it in the first sum, and rolled all other terms into the second sum. Thus each \( M_i \) is equal to one of the original \( L_i \), but the \( M_i \) may not be distinct, and some of them may also appear in the first sum.

We will now put upper bounds on the \( L_{C_j} \) and \( M_i \) separately by studying the \( \ell \)-adic valuation of equation (2).

6.4. Upper bounds. Let \( B_L \) be the largest of the \( L_{C_j} \), and assume \( B_L > 0 \).

Let \( t \) be the number of terms on the right side of equation (2). Each term is a power of \( \ell \), and one of these terms has \( \ell \)-adic valuation less than or equal to \(-2B_L \). The largest possible valuation of the right side is then \(-2B_L + t - 1 \).

Meanwhile, the valuation of the left side is greater than or equal to \(-B_L \), because no term has a valuation smaller than this. Thus for \( f \) to be universal, we must have

\[ -B_L \leq -2B_L + t - 1 \]

This implies that \( B_L \leq t - 1 \).

Now let \( B_M \) be the largest of the \( M_i \).

As before, the valuation of the right side of (2) is less than or equal to \(-B_M + t - 1 \), and the valuation of the left side is greater than or equal to \(-B_L \), which is greater than or equal to \(-t + 1 \). Thus for \( f \) to be universal, we must have

\[ -t + 1 \leq -B_M + t - 1 \]

and so \( B_M \leq 2t - 2 \).

This establishes upper bounds on all the linear combinations \( L_i \), and hence completes the proof of Theorem 2.

7. LATTICES OF WEIGHTS AND COUNTING FUNCTIONS

Let \( \Lambda_w \) be the space of weight functions for \( \Gamma \), and \( \Lambda_c \) be the space of proper counting functions for \( \Gamma \) (we are no longer assuming the counting functions are positive). For simplicity, assume \( \Gamma \) is an \( \ell \)-group in this section.

\( \Lambda_w \) and \( \Lambda_c \) are both integer lattices of finite rank. The rank of \( \Lambda_w \) corresponds to the number of pairs \((H, H')\) where \( H, H' \subset \Gamma \), and \( H' \) is a
maximal subgroup of $H$. For $\Lambda_c$, there is one generator for each nonidentity conjugacy class of cyclic subgroups of $\Gamma$, corresponding to the possible inertia groups at tame primes in $\Gamma$-extensions of $\mathbb{Q}$, and one generator for each nontrivial map $\phi : G_{\mathbb{Q}_p} \to \Gamma$, corresponding to the choices of $c(\phi)$ for these maps.

Corollary 7 gives a map $\Lambda_w \to \Lambda_c$, and the image $\Lambda_{cw} \subseteq \Lambda_c$ of this map is the space of weighted discriminant counting functions for $\Gamma$. The following example, which expands on a briefer discussion in Section 3, shows that the rank of $\Lambda_{cw}$ may be smaller than the rank of $\Lambda_w$.

**Example.** Let $\Gamma = C_2 \times C_2$, and let $H_1, H_2, H_3$ be its order-2 subgroups, with 1 denoting the trivial subgroup. For this $\Gamma$, $\Lambda_w$ has rank 6, and $\Lambda_c$ has rank 63, since $\Gamma$ has 3 cyclic subgroups, and there are 60 maps $\phi : G_{\mathbb{Q}_2} \to \Gamma$. (The large rank of $\Lambda_c$ is typical and reflects the absence of any restrictions on how proper counting functions may behave on wildly ramified maps.)

If $w$ is a weight function with counting function $c_w$, set

$$L_1 = 2w(\Gamma, H_1) + w(H_2, 1) + w(H_3, 1)$$
$$L_2 = 2w(\Gamma, H_2) + w(H_1, 1) + w(H_3, 1)$$
$$L_3 = 2w(\Gamma, H_3) + w(H_1, 1) + w(H_2, 1)$$

Then for any prime $p \neq 2$ and any $\phi_p : G_{\mathbb{Q}_p} \to \Gamma$, we have

$$c_w(\phi_p) \in \{L_1, L_2, L_3\}$$

With $p = 2$, if $\phi$ is any of the 60 maps $G_{\mathbb{Q}_2} \to \Gamma$,

$$c_w(\phi) = \{2L_i, 3L_i, L_1 + L_2 + L_3 + L_i\}$$

for $i \in \{1, 2, 3\}$.

This shows that $\Lambda_{cw}$ is a rank-3 lattice, with generators corresponding to $L_1$, $L_2$, and $L_3$.

In this example, the generators of $\Lambda_{cw}$ are exactly the generators of $\Lambda_c$ corresponding to cyclic subgroups of $\Gamma$ (as opposed to the generators corresponding to maps $G_{\mathbb{Q}_2} \to \Gamma$). This leads to the following question:

**Question 12.** Is it the case that for any $\Gamma$, two weighted discriminant counting functions that agree at all tame primes must also agree at wild primes? That is, if $c_1(\phi) = c_2(\phi)$ for any $\phi : G_{\mathbb{Q}_p} \to \Gamma$ with $p \nmid |\Gamma|$, then $c_1(\phi) = c_2(\phi)$ for any $\phi : G_{\mathbb{Q}_p} \to \Gamma$ for any $p$?

A closely related (but not necessarily quite equivalent) question is:

**Question 13.** Is the rank of $\Lambda_{cw}$ always equal to the number of conjugacy classes of cyclic subgroups of $\Gamma$?

In all examples we have worked out for small groups, the answer to each of these questions is “yes”. If this always holds, it would suggest that weighted
discriminant counting functions are a natural subset of proper counting functions to consider, since they would completely dictate the behavior at wild primes, while maintaining as much freedom as possible at tame primes.

8. Example: Weighted discriminants for $D_4$

Let $Γ = D_4$, the dihedral group with 8 elements, and let $K$ be a Galois $Γ$-extension of $Q$. Let $H'$ be the subgroup of $Γ$ generated by a non-central element of order 2, and let $H$ be the subgroup of order 4 generated by $H'$ and the center of $Γ$. Note that $Γ$ is the Galois closure of the quartic field $K_{H'}$.

Any proper counting function is guaranteed by Theorem 9 to have exactly one tame mass formula. Since $D_4$ has a rational character table, this mass formula is pure by Proposition 5 at least if the counting function takes only nonnegative values. As in (6.3), for any given counting function, determining whether or not the tame mass formula is universal reduces to determining if it gives the correct value for the total mass at 2.

The standard discriminant $\text{Disc} K_{H'/Q}$ of the quartic subfield corresponds to a weighted discriminant for $K$ with the weights $w(H,H') = 1$ and $w(Γ,H) = 2$, and all other weights equal to zero. This does not have a universal mass formula; its total mass at each odd prime $p$ is given in [3] by $1 + p^{-1} + 2p^{-2} + p^{-3}$, and this must be the unique tame mass formula by Theorem 7. The total mass at 2 is $\frac{121}{64}$ (this can be computed exhaustively using [3]), whereas the tame mass formula evaluates to $\frac{17}{8}$ at $p = 2$, so this mass formula is not universal.

However, Wood [8] has found another weighted discriminant which does have a universal mass formula. This invariant has $w(H,H') = 1$, $w(Γ,H, ) = 1$ (instead of 2) and all other weights equal to zero. For this invariant, the total mass at each prime $p$ is $1 + 2p^{-1} + 2p^{-2}$.

Using the Database of Local Fields [3], we can obtain equation (2) for this $Γ$ and a weight function $w$ with counting function $c_w$. $D_4$ has 4 conjugacy classes of cyclic subgroups. Using the language of (6.2), there are linear forms $L_1, L_2, L_3, \text{ and } L_4$ in the weights corresponding to each of these. Let $L_1$ and $L_2$ correspond to the two non-central conjugacy classes of $C_2$'s, $L_3$ correspond to the center of $Γ$, and $L_4$ correspond to the $C_4$ subgroup of $Γ$. Then equation (2) for $D_4$ is

$$1 + 2^{-L_1} + 2^{-L_2} + 2^{-L_3} + 2^{-L_4}$$

$$= 1 + 2^{-2L_1} + 2^{-2L_2} + 2^{-2L_3} + 2 \cdot 2^{-3L_1} + 2 \cdot 2^{-3L_2} + 2 \cdot 2^{-3L_3}$$

$$+ 2^{-2(L_1+L_3)} + 2^{-2(L_2+L_3)} + 2 \cdot 2^{-2(L_1+2L_3)} + 2 \cdot 2^{-2(L_2+2L_3)}$$

$$+ 2 \cdot 2^{-3(L_1+L_3)} + 2 \cdot 2^{-3(L_2+L_3)} + 4 \cdot 2^{-3(L_3+L_4)} + 4 \cdot 2^{-2(L_1+L_2+L_3+L_4)}$$

$$+ 4 \cdot 2^{-2(L_1+L_2+L_3+L_4)} + 4 \cdot 2^{-2(L_1+L_2+L_3+2L_4)}$$

If $H$ is a subgroup of $Γ$ isomorphic to $C_2 \times C_2$, and $H'$ is a subgroup of $H$ of order 2 not equal to the center of $Γ$, then Wood’s invariant is given by
the weight function \( w(\Gamma, H) = 1, w(H, H') = 1, \) and \( w = 0 \) otherwise. From these weights, one can compute that \( L_1 = L_2 = 1 \) and \( L_3 = L_4 = 2 \). Since these values satisfy the equation above, this invariant has a universal mass formula. In fact:

**Proposition 14.** The invariant found by Wood is the only positive weighted discriminant counting function for \( D_4 \) with a universal mass formula.

**Proof.** Using slight refinements of the techniques in (6.4), tailored to give the best possible bounds for the equation above, we obtain \( L_1 \leq 10 \) for \( 1 \leq i \leq 4 \), if this equation is to hold. We then search by computer for solutions within these bounds. The only such solution is \((L_1, L_2, L_3, L_4) = (1, 1, 2, 2)\), as desired. \( \square \)

8.1. **Weighted Discriminants for \( Q_8 \).** Let \( \Gamma = Q_8 \), the quaternion group. As with \( D_4 \), any weighted discriminant counting function \( c_w \) is determined by the values of four linear forms \( L_1, ..., L_4 \) in the weights, corresponding to the four conjugacy classes of cyclic subgroups of \( Q_8 \). A similar calculation to the above shows that \( c_w \) has a universal mass formula if and only if \( L_1 = L_2 = L_3 = L_4 = 1 \).

There is no integer-valued weight function for \( Q_8 \) that produces these values of \( L_1, ..., L_4 \); some of the weights must be set to \( \frac{1}{4} \) to construct the correct counting function. However, with these weights, the corresponding weighted discriminant still happens to be integer-valued, which indicates that it may in some cases be desirable to allow fractional weights, as long as the weighted discriminant remains integer-valued. This does not have any effect on the validity of Theorem 2 since the \( L_i \) will need to have bounded denominator in order to keep the weighted discriminant integer-valued.

8.2. **Weighted Discriminants for \( S_3 \).** Let \( \Gamma = S_3 \) provides an instructive example in how Theorem 2 extends, in some cases, to non-\( \ell \)-groups.

In this case, \( c_w \) depends only on the values of two linear forms, again corresponding to the two conjugacy classes of cyclic subgroups of \( S_3 \): \( L_2 \) corresponding to \( C_2 \), and \( L_3 \) corresponding to \( C_3 \). The tame mass formula for \( S_3 \) is \( f(p) = 1 + p^{-L_2} + p^{-L_3} \). Instead of a single equation determining whether or not a counting function has a universal mass formula, there are two: one given by the mass at 2, and one given by the mass at 3. These equations are:

\[
\begin{align*}
1 + 2^{-L_2} + 2^{-L_3} &= 1 + 2^{-2L_2} + 2 \cdot 2^{-3L_2} + 2^{-L_3} \\
1 + 3^{-L_2} + 3^{-L_3} &= 1 + 3^{-L_2} + 2 \cdot 3^{-2L_3} + 2 \cdot 3^{-3(L_2+L_3)} + 3 \cdot 3^{-2(L_2+2L_3)}
\end{align*}
\]

Solving these equations directly gives \( L_2 = 1 \) and \( L_3 = 2 \). The resulting invariant is the standard discriminant of a cubic \( S_3 \)-extension of \( \mathbb{Q} \), so this is the only weighted discriminant for \( S_3 \) with a universal mass formula.

**Remark.** There are at least two weighted discriminants for \( S_3 \) for which the Malle-Bhargava field counting heuristics give the correct count. Davenport
and Heilbronn [2] show that for $S_3$ cubic fields counted by discriminant, the asymptotic is $\frac{1}{\zeta(3)} X$, as expected. More recently, Bhargava and Wood [1] show that when $S_3$ sextic fields (i.e. the Galois closure) are counted by discriminant, the asymptotic is $\left(\frac{1}{3} \prod_p c_p\right) X^{1/3}$, where the Euler product $\prod_p c_p$ is exactly as predicted. The former invariant has a universal mass formula, but the latter does not.

9. Further Work

9.1. Extending Theorem 2. There are three natural ways in which Theorem 2 could be generalized. Allowing ground fields other than $\mathbb{Q}$ should be straightforward; all of the definitions given extend naturally to other ground fields, and the results should transfer essentially unmodified, with the possible exception of the fact about cyclotomic extensions of $\mathbb{Q}_p$ used in (6.3).

Extending the theorem to non-positive counting functions poses slightly more difficulty, as Corollary 8 would need to be reworked or avoided.

The case where $\Gamma$ is not an $\ell$-group is considerably harder. On the surface, the techniques used to prove Theorem 2 seem as if they should adapt to this case, but in practice, the methods for bounding the $L_i$ are not as directly applicable.

Most importantly, equation (2) is no longer a single equation; instead, there is one such equation for each prime dividing $|\Gamma|$. Additionally, Corollary 8 fails when $\Gamma$ is not an $\ell$-group. As a result, it is no longer the case that each term on the left side of equation (2) has a corresponding term on the right side with exponent at least twice as large, which was the key ingredient in establishing bounds on these exponents.

The smallest $\Gamma$ for which this is a problem is $\Gamma = C_{15}$. For some non-$\ell$-groups, including $C_6$, $S_3$, $C_{10}$, and $D_5$, the techniques of Section 6 can be adapted in a somewhat ad hoc manner to bound the weights. In fact, it appears this may be possible whenever all the elements of $\Gamma$ have prime-power order. For $C_{15}$, however, there is no obvious way to adapt this method to bound the weights.

9.2. Artin Conductors. We have thus far considered two types of alternate discriminants: the generic type arising from a proper counting functions, and weighted discriminants. We proved Theorem 2 only for the latter, since properness is not a strong enough restriction on the behavior of a counting function. However, there is another class of alternate discriminants we could consider for any finite group $\Gamma$.

Fix a character $\chi$ of $\Gamma$. Then, for any map $\phi : G_{\mathbb{Q}_p} \to \Gamma$, let

$$c_\chi(\phi) = f(\chi \circ \phi)$$

where $f$ is the Artin conductor. In this way, we obtain a proper counting function for $\Gamma$. 
Based on our Theorem 2 and the rarity of universal mass formulas for small groups \( \Gamma \) that we have examined in detail, we conjecture the following:

**Conjecture 15.** For any finite group \( \Gamma \) (or at least any \( p \)-group), there are only finitely many characters \( \chi \) for which the counting function \( c_\chi \) has a universal mass formula.

In addition, the set of counting functions arising from Artin conductors of characters can be made into a lattice, if we allow virtual characters (arbitrary integer linear combinations of irreducible characters). Since the Artin conductor of a sum of characters is the sum of the Artin conductors, these “characters” still make sense in the definition of \( c_\chi \) even if they involve negative coefficients.

It would be interesting to study the relationship between this lattice and the lattice of weight functions, as mentioned before. As an example, if \( \Gamma = C_2 \times C_2 \) as in Example 7 there are three irreducible characters, and thus the lattice of counting functions from these has rank 3. This lattice turns out to be identical to the image of the lattice of weight functions. Even better, the three natural generators (given by the irreducible characters) exactly correspond to the three natural generators of the lattice of weight functions, with \((L_1, L_2, L_3)\) in Example 7 equal to \((1, 0, 0)\), \((0, 1, 0)\), and \((0, 0, 1)\).

**9.3. Infinite Weights.** Also of interest is the case of weighted discriminant counting functions, where \( \infty \) is allowed as a value for the weights. Setting \( w(H, H') = \infty \) has the effect of excluding from all calculations any \( \phi : \mathbb{Q}_p \to \Gamma \) in which

\[
\phi(I_p) \cap \gamma H' \gamma^{-1} \neq \phi(I_p) \cap \gamma H \gamma^{-1}
\]

where \( I_p \) denotes the inertia subgroup of \( G_{\mathbb{Q}_p} \). In terms of global number field invariants, this means that any field in which any prime above \( p \) is ramified in the extension \( K_{H'}/K_H \) is assigned a value of \( \infty \) for this alternate discriminant.

If we count number fields by some alternate discriminant instead of the standard discriminant, this allows us to exclude fields with certain types of ramification. In particular, if \( G \) and \( A \) are finite groups with \( A \) abelian, then we can in some cases use this technique to rephrase questions about unramified \( A \)-extensions of \( G \)-extensions of \( \mathbb{Q} \) as questions about counting number fields by some alternate discriminant.

For example, if \( G = C_2 \) and \( A \) is any finite abelian group, then any unramified \( A \)-extension of a quadratic field has Galois group \( \Gamma = A \rtimes C_2 \). By choosing appropriate weights, we can turn the study of the \( A \)-moment of class groups of quadratic fields into the study of \( \Gamma \)-extensions of \( \mathbb{Q} \), counted by an alternate discriminant. Unfortunately, it is not always this simple to pin down what \( \Gamma \) must be, so this technique becomes much harder to use for more complicated Cohen-Lenstra-type questions.
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