A GEOMETRIC INTERPRETATION OF THE HASSE-ARF THEOREM

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Abstract. We give an a geometric interpretation of the Hasse-Arf theorem for function fields using the recently proved Oort conjecture.

1. Introduction

The Hasse-Arf theorem (see [13, IV.3]) controls the jumps of the ramification filtration for abelian groups. Aim of this short note is to give an intuitive geometric argument, why the Hasse-Arf theorem is true. Unfortunately our approach does not provide a new proof of the Hasse-Arf theorem; It is based on the recently proved Oort conjecture for cyclic groups, and the known proof of the Oort conjecture is heavily based on the Hasse-Arf theorem. However, we have enjoyed this argument and we believe that it reveals the nature of the Hasse-Arf theorem. We would like to mention that the author has spent a lot of time trying to prove the Hasse-Arf theorem with the methods of the article [14] but without success.

After posting a first version on the arXiv, M. Mattignon informed the author that essentially the same observation was made in his article [3, section 6 p.16] jointly written by B. Green. He also explained that the proof of Sen’s [11] theorem by Lubin [8], implies the truth of the Hasse-Arf theorem and is really a geometric proof of the Hasse-Arf theorem. We would like to thank him for his response. We would like to also thank M. Romagny for his comments.

Let $O = k[[t]]$ be a complete local ring over an algebraically closed field $k$ of characteristic $p > 0$. This ring $O$ is equipped with a valuation $v$ and a local uniformiser $t$. Suppose that a finite $p$-group $G$ acts on $O$. Then a ramification filtration is defined by

$$G_0 := \{ \sigma \in G : v(\sigma(t) - t) \geq i + 1 \}.$$

The Hasse Arf theorem reduces [13, IV.3 exer. 3] to the following statement for the cyclic case:

**Theorem 1.** Let $G$ be a cyclic $p$-group of order $p^n$. The ramification filtration is given by

$$G_0 = G_1 = \cdots = G_{j_0} \supseteq G_{j_0+1} = \cdots = G_{j_1} \supseteq G_{j_1+1} = \cdots = G_{j_{n-1}} \supseteq \{1\},$$

i.e. the jumps of the ramification filtration appear at the integers $j_0, \ldots, j_{n-1}$. Then

$$j_k = i_0 + i_1 p + i_2 p^2 + \cdots i_k p^k.$$

Notice that since $G$ is assumed to be a $p$-group $G_1 = G$. The Harbater-Katz-Gabber compactification theorem asserts that there is a Galois cover $X \to \mathbb{P}^1$ ramified only at one point $P$ of $X$ with Galois group $G = \text{Gal}(X/\mathbb{P}^1) = G_1$ such that $G_1(P) = G_1$ and the action of $G_1(P)$ on the completed local ring $O_{X,P}$ coincides with the original action of $G_1$ on $\mathbb{P}^1$.

The Oort conjecture (now a theorem proved by A.Obus, S. Wewers [9] and F. Pop [10]) states:

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Theorem 2. Let $X$ be a projective nonsingular curve, defined over an algebraically closed field $k$ of characteristic $p > 0$, acted on by a cyclic group $G$. There is a proper smooth family of curves $\mathcal{X} \to \text{Spec} A$, where $A$ is a local ring with maximal ideal $m$ so that $A/m = k$ and the special fibre of $\mathcal{X}_m = \mathcal{X} \times_{\text{Spec} A} \text{Spec} k$ is the original curve $X$ and the generic fibre $\mathcal{X}_0$ is a curve defined over a field of characteristic 0.

We will apply the Oort conjecture on the Harbater-Katz-Gabber compactification $X$ coming from the local action of $G = \mathbb{Z}/p^n\mathbb{Z}$ on $\mathcal{O}$. We construct a relative notion of horizontal ramification divisor. This divisor intersects the special fibre on a single point $P$ and the generic fibre at a set with $p^k$ points. The Hasse-Arf has the following geometric interpretation:

Theorem 3. The numbers $i_k$ are numbers of different orbits of the action of the group $G$ on the ramified points in the generic fibres with size $p^k$.

The method of proof reflects the philosophy of lifting geometrical objects from positive characteristic to characteristic zero, and use the easier characteristic zero case. This was one of the main ideas of modular representation theory and is also one of the main ideas in defining cohomology theories over Witt rings, see for example [12].

2. Horizontal Ramification Divisors

Let $P$ be the unique ramification point on the special fibre. Let $\sigma \in G_1(P)$, $\sigma \neq 1$, and let $\tilde{\sigma}$ be a lift of $\sigma$ in $\mathcal{X}'$. The scheme $\mathcal{X}'$ is regular at $P$, and the completion of $\mathcal{O}_{\mathcal{X}', P}$ is isomorphic to the ring $R[[T]]$. Weierstrass preparation theorem [3] prop. VII.6 implies that:

$$\tilde{\sigma}(T) - T = g_\tilde{\sigma}(T)u_\tilde{\sigma}(T),$$

where $g_\tilde{\sigma}(T)$ is a distinguished Weierstrass polynomial of degree $m + 1$ and $u_\tilde{\sigma}(T)$ is a unit in $R[[T]]$.

The polynomial $g_\tilde{\sigma}(T)$ gives rise to a horizontal divisor that corresponds to the fixed points of $\tilde{\sigma}$. This horizontal divisor might not be reducible. The branch divisor corresponds to the union of the fixed points of any $\sigma \in G_1(P)$. Next lemma gives an alternative definition of a horizontal branch divisor for the relative curves $\mathcal{X} \to \mathcal{Y}$, that works even when $G$ is not a cyclic group.

Lemma 4. Let $\mathcal{X} \to \text{Spec} A$ be an $A$-curve, admitting a fibrewise action of the finite group $G$, where $A$ is a Noetherian local ring. Let $S = \text{Spec} A$, and $\Omega_{\mathcal{X}/S}$, $\Omega_{\mathcal{Y}/S}$ be the sheaves of relative differentials of $\mathcal{X}$ over $S$ and $\mathcal{Y}$ over $S$, respectively. Let $\pi : \mathcal{X} \to \mathcal{Y}$ be the quotient map. The sheaf

$$\mathcal{L}(\mathcal{L}^{-1}_{\mathcal{X}/\mathcal{Y}}) = \Omega_{\mathcal{X}/S}^{-1} \otimes_S \pi^* \Omega_{\mathcal{Y}/S},$$

is the ideal sheaf the horizontal Cartier divisor $D_{\mathcal{X}/\mathcal{Y}}$. The intersection of $D_{\mathcal{X}/\mathcal{Y}}$ with the special and generic fibre of $\mathcal{X}$ gives the ordinary branch divisors for curves.

Proof. We will first prove that the above defined divisor $D_{\mathcal{X}/\mathcal{Y}}$ is indeed an effective Cartier divisor. According to [5 Cor. 1.1.5.2] it is enough to prove that

- $D_{\mathcal{X}/\mathcal{Y}}$ is a closed subscheme which is flat over $S$.
- for all geometric points $\text{Spec} k$ of $S$, the closed subscheme $D_{\mathcal{X}/\mathcal{Y}} \otimes_S k$ of $\mathcal{X} \otimes_S k$ is a Cartier divisor in $\mathcal{X} \otimes_S k/k$.

In our case the special fibre is a nonsingular curve. Since the base is a local ring and the special fibre is nonsingular, the deformation $\mathcal{X} \to \text{Spec} A$ is smooth. (See the remark after the definition 3.35 p.142 in [7]). The smoothness of the curves $\mathcal{X} \to S$, and $\mathcal{Y} \to S$, implies that the sheaves $\Omega_{\mathcal{X}/S}$ and $\Omega_{\mathcal{Y}/S}$ are $S$-flat, [7 cor. 2.6 p.222].
On the other hand the sheaf $\Omega_{\mathcal{Y}, \text{Spec}A}$ is by \cite{4} Prop. 1.I.5.1 $\mathcal{O}_{\mathcal{X}}$-flat. Therefore, $\pi^*(\Omega_{\mathcal{Y}, \text{Spec}A})$ is $\mathcal{O}_{\mathcal{X}}$-flat and Spec$A$-flat \cite{4} Prop. 9.2. Finally, observe that the intersection with the special and generic fibre is the ordinary branch divisor for curves according to \cite{4} IV p.301].

For a curve $X$ and a branch point $P$ of $X$ we will denote by $i_G, P$ the order function of the filtration of $G$ at $P$. The Artin representation of the group $G$ is defined by $ar_P(\sigma) = -f_P i_G, P(\sigma)$ for $\sigma \neq 1$ and $ar_P(1) = f_P \sum_{\sigma \neq 1} i_G, P(\sigma)$ \cite{13} VI.2]. We are going to use the Artin representation at both the special and generic fibre. In the special fibre we always have $f_P = 1$ since the field $k$ is algebraically closed. Therefore a fixed point there might have not be algebraically closed therefore a fixed point there might have $f_P \geq 1$. The integer $i_G, P(\sigma)$ is equal to the multiplicity of $P \times P$ in the intersection of $\Delta, \Gamma_\sigma$ in the relative $A$-surface $\mathcal{X} \times_{\text{Spec}A} \mathcal{X}$, where $\Delta$ is the diagonal and $\Gamma_\sigma$ is the graph of $\sigma$ \cite{13} p. 105]

Since the diagonals $\Delta_0, \Delta_n$ and the graphs of $\sigma$ in the special and generic fibres respectively of $\mathcal{X} \times_{\text{Spec}A} \mathcal{X}$ are algebraically equivalent divisors we have:

**Proposition 5.** Assume that $A$ is an integral domain, and let $\mathcal{X} \to \text{Spec}A$ be a deformation of $X$. Let $\tilde{P}_i, i = 1, \ldots, s$ be the horizontal branch divisors that intersect at the special fibre, at point $P$, and let $P_i$ be the corresponding points on the generic fibre. For the Artin representations attached to the points $P, P_i$ we have:

\[
ar_P(\sigma) = \sum_{i=1}^s ar_{P_i}(\sigma).
\]

This generalizes a result of J. Bertin \cite{1}. Moreover if we set $\sigma = 1$ to the above formula we obtain a relation for the valuations of the differents in the special and the generic fibre, since the value of the Artin’s representation at 1 is the valuation of the different \cite{13} prop. 4.IV,prop. 4.VI]. This observation is equivalent to claim 3.2 in \cite{3} and is one direction of a local criterion for good reduction theorem proved in \cite{3} 3.4], \cite{5} sec. 5].

2.1. **The Artin representation on the generic fibre.** We can assume that after a base change of the family $\mathcal{X} \to \text{Spec}(A)$ the points $P_i$ at the generic fibre have degree 1. Observe also that at the generic fibre the Artin representation can be computed as follows:

\[
ar_Q(\sigma) = \begin{cases} 1 & \text{if } \sigma(Q) = Q, \\ 0 & \text{if } \sigma(Q) \neq Q. \end{cases}
\]

The set of points $S := \{P_1, \ldots, P_s\}$ that are the intersections of the ramification divisor and the generic fibre are acted on by the group $G$. Let $S_k$ be the subset of $S$ fixed by $\mathbb{Z}/p^{n-k}\mathbb{Z}$, i.e.

\[
P \in S_k \text{ if and only if } G(P) = \mathbb{Z}/p^{n-k}\mathbb{Z}.
\]

Let $s_k$ be the order of $S_k$. Observe that since for a point $Q$ in the generic fibre $\sigma(Q)$ and $Q$ have the same stabilizers (they are conjugate but $G$ is abelian) the sets $S_k$ are acted on by $G$. Therefore orders of $S_k$ are $s_k = p^{k}i_k$ where $i_k$ is the number of orbits of the action of $G$ on $S_k$.

Observe that

\[
G_{j_k} = \begin{cases} \mathbb{Z}/p^{n-k}\mathbb{Z} & \text{for } 0 \leq k \leq n-1 \\ \{1\} & \text{for } k \geq n. \end{cases}
\]

An element in $G_{j_k}$ fixes only elements with stabilizers which contain $G_{j_k}$. So $G_{j_k}$ fixes only $S_0$, $G_{j_k}$ fixes both $S_0$ and $S_1$ and $G_{j_k}$ fixes all elements in $S_0, S_1, \ldots, S_{s_k}$. So eq. (1) implies that an element $\sigma$ in $G_{j_k} - G_{j_{k+1}}$ satisfies $ar_P(\sigma) = j_k$ and by using equation (1) we arrive at

\[
j_k = i_0 + p i_1 + \cdots p^k i_k.
\]

This completes the proof of the Hasse-Arf theorem. The argument is illustrated in figure 3
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