The Oriented Graph Complexes

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Abstract: The oriented graph complexes $G_{or}^n$ are complexes of directed graphs without directed cycles. They govern, for example, the quantization of Lie bialgebras and infinite dimensional deformation quantization. Similar to the ordinary graph complexes $G_C^n$ introduced by Kontsevich they come in two essentially different versions, depending on the parity of $n$. It is shown that, surprisingly, the oriented graph complex $G_{or}^n$ is quasi-isomorphic to the ordinary commutative graph complex of opposite parity $G_C^{n-1}$, up to some known classes. This yields in particular a combinatorial description of the action of $grt_1 \cong H^0(GC_2)$ on Lie bialgebras, and shows that a cycle-free formality morphism in the sense of Shoikhet can be constructed rationally without reference to configuration space integrals. Curiously, the obstruction class in the oriented graph complex found by Shoikhet corresponds to the well known theta graph in the ordinary graph complex.

1. Introduction

Graph complexes are differential graded vector spaces whose elements are linear combinations or series of isomorphism classes of graphs. Various flavors of graph complexes exist, depending on the type of graphs that are allowed in the series. The most commonly encountered complexes are the ribbon graph complex, which computes the cohomology of moduli spaces of curves [4], the “Lie” graph complex which computes the cohomologies of the automorphism groups of free groups [1], and the “commutative” graph complexes which govern the deformations of the $E_n$ operads [4, 18]. The common feature of all these graph complexes is that their cohomology is very hard to compute, and usually only very few facts are known beyond the computer accessible regime.

In more detail, the elements of Kontsevich’s “commutative” graph complexes $fcG_C^n$ (for $n$ a fixed integer) are series of certain isomorphism classes of ordinary undirected connected graphs, e.g.,
The differential acts by splitting vertices, pictorially:

The signs and cohomological degrees are determined such that a vertex carries degree \( n \), while an edge carries degree \( 1 - n \). For more details, see the precise definition in Sect. 3 below. Physically the above graph complex may be understood as the complex of vacuum Feynman diagrams for an \( n \)-dimensional topological field theory. Mathematically these complexes are important since they control deformations of the \( E_n \) operads, see [18] for details. One can check that the graph complexes carry a differential graded Lie algebra structure.

In this paper we will compare the above graph complexes to another version of graph complexes first introduced (to the knowledge of the author) by Merkulov, the oriented graph complexes \( G^{or}_n \). The elements of \( G^{or}_n \) are (essentially) series of isomorphism classes of directed graphs, which do not contain directed loops. In other words, the directions of the edges naturally endow the set of vertices with the structure of a partially ordered set. For example, the graph on the left hand side is admissible, that on the right hand side is not:

The differential again splits vertices as before. For more precise definitions see Sect. 3. The complexes \( G^{or}_n \) also carry a natural dg Lie algebra structure.

These oriented graph complexes appear for example in the quantization of Lie bialgebras (for \( n = 3 \)), where they act on Lie bialgebra structures, and in infinite dimensional deformation quantization (for \( n = 2 \)), see also [16].

Slightly different versions of oriented graph complexes with external legs have been considered in [11]. These complexes are highly related, for an indication of the link see Sect. 3.3.

The main result of this paper is a computation of the cohomology of the oriented graph complexes.

**Theorem 1.** The cohomology of the oriented graph complex \( G^{or}_n \) is isomorphic to the cohomology of the ordinary commutative graph complex \( fcG_{n-1} \) as dg Lie algebra. In particular,

\[
H(G^{or}_n) \cong H(fcG_{n-1}) = H(GC_{n-1}) \bigoplus \bigoplus_{j \geq 1, \mod 4} \mathbb{K}[n - j].
\]

Furthermore, the identification preserves the additional grading by the first Betti numbers of graphs on both sides.