Inverse spectral problems for Sturm–Liouville operators with matrix-valued potentials

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Abstract
We give a complete description of the set of spectral data (eigenvalues and specially introduced norming constants) for Sturm–Liouville operators on the interval [0, 1] with matrix-valued potentials in the Sobolev space $W^{-1}_{2,1}$ and suggest an algorithm reconstructing the potential from the spectral data that is based on Krein’s accelerant method.

1. Introduction

The inverse spectral problem for a class $\mathcal{S}$ of differential operators consists in finding spectral characteristics of an operator $T$ in $\mathcal{S}$ that determine it uniquely and a method reconstructing $T$ from its spectral data. The main aim of this paper is to solve the inverse spectral problem for the class of Sturm–Liouville operators on the unit interval [0, 1] with matrix-valued distributional potentials in the Sobolev space $W^{-1}_{2,1}$. Before explaining the setting, we shall give a short overview of the known results; the reader is referred to the books [1–3] and the review paper [4] for further details.

1.1. Known results

The study of the inverse spectral problems for Sturm–Liouville and Dirac operators has a rather long history. The first result on unique determination of a Sturm–Liouville operator from its Neumann spectrum was given by Ambartsumyan already in 1929 [5], but the systematic development of the theory was initiated by the celebrated paper by Borg [6] of 1946 who proved that two spectra of a Sturm–Liouville equation determine uniquely the potential. Later Marchenko [7] showed that the potential is uniquely determined by the spectral function. In 1951, the paper by Gelfand and Levitan [8] appeared, giving an algorithm reconstructing the potential from the spectral measure, and the paper of Marchenko [9] with a detailed account of his method. Meanwhile Krein, in a series of papers [10–13], developed an alternative approach to inverse problems. As is well explained by Ramm [14], each of the three methods—that of...
Another important tool for the study of inverse spectral problems was developed by Trubowitz and his coauthors Isaacson [15], Isaacson and McKean [16], Dalhberg [17] and Pöschel [3]. It is based on the analytic dependence of an individual eigenvalue and norming constant on the potential. In some cases (e.g. in the inverse problem for the perturbed harmonic oscillator) the Trubowitz method is probably the only one yielding a satisfactory result [18, 19].

Recently, there has been an increasing interest in the Sturm–Liouville operators with matrix potentials, which are used to model the evolution of various vibrational systems, graphs, quantum networks, etc. Although nowadays the inverse spectral theory is well developed for scalar Sturm–Liouville operators, surprisingly, there are only particular results for the matrix case. One of the reasons for that is that the problem in the matrix case is much more complicated and requires more involved tools for its study.

We mention several important contributions to the area. A solution to the inverse scattering problem for matrix Sturm–Liouville operators on the half-line was given by Newton and Jost [20], Krein [13, 21], Agranovich and Marchenko [22]. Analogous results for Dirac operators on the half-line were obtained in the papers of Levitan and Gasymov [23] and Gasymov [24], and for more general first-order systems by Lesch and Malamud [25].

Inverse spectral problems for matrix-valued Sturm–Liouville and Dirac operators on a finite interval have been studied considerably less than those for the half-line. Uniqueness of reconstruction was studied in the papers by Carlson [26] and Yurko [27], and Malamud [28] investigated uniqueness for various settings of inverse problems for systems of differential equations of the first order. Yurko in [27, 29] suggested an algorithm reconstructing the potential from the spectral data, but did not give a complete description of the set of spectral data. Probably the closest to our work are the papers by Chelkak and Korotyaev [30, 31], which we next discuss in more detail.

In [30], the authors investigated uniqueness questions for operators

\[ T_q = -\frac{d^2}{dx^2} + q \]  

on (0, 1) subject to the Dirichlet boundary conditions and with Hermitian matrix-valued potentials \( q \) in \( L_1(0, 1) \) entrywise and studied isospectral transformations as well as some subtle properties of the local structure of the set of isospectral potentials. An important phenomenon was discovered there, namely the one of the so-called ‘forbidden’ subspaces. In [31], the inverse spectral problem for operators \( T_q \) with potentials \( q \) belonging to \( L_2(0, 1) \) entrywise was treated. The main theorem there gives a complete description of the set of spectral data. That result of [31] and theorem 1.2 of the present paper are similar but they treat different classes of potentials. Also the approaches are different: while the proofs in [31] are based on the Trubowitz method and some special isospectral transformations, we use the Krein accelerant method.

### 1.2. Setting of the problem

Let \( \mathcal{M}_r \) be the Banach algebra of the square \( r \times r \) matrices with complex entries, which we identify with the Banach algebra of linear operators \( \mathcal{C}^r \to \mathcal{C}^r \) endowed with the standard norm (see appendix A). We shall write \( I \) for the unit element of \( \mathcal{M}_r \) and \( \mathcal{M}_r^+ \) for the set of all \( A \in \mathcal{M}_r \) such that \( A \equiv A^* \geq 0 \). We shall use the abbreviations

\[ W^s_p := W^s_p((0, 1), \mathcal{C}^r), \quad W^s_p := W^s_p((0, 1), \mathcal{M}_r) \quad (s \in \mathbb{Z}, p \geq 1) \]
for the corresponding Sobolev spaces, as well as the notations

\[ L_p := L_p((0, 1), C'), \quad L_{p} := L_p((0, 1), M_p), \quad p \geq 1, \]

\[ \text{Re } L_{p} := \{u \in L_p | u = u^*\}, \quad \text{Re } W_2^{-1} := \{u \in W_2^{-1} | u = u^*\}. \]

Some further information on these spaces can be found in appendix A.

For an arbitrary \( \tau \in L_2 \), we consider the differential expression

\[ t_\tau (f) := - \left( \frac{d}{dx} + \tau \right) \left( \frac{d}{dx} - \tau \right) f \]

on the domain

\[ D(t_\tau) := \{ f \in W_2^1 | f^{[1]} := (f' - \tau f) \in W_2^1 \}. \]

The function \( f^{[1]} \) is usually called the quasi-derivative of \( f \).

Denote by \( t_{\tau,D} \) and \( t_{\tau,N} \) the restrictions of \( t_{\tau} \) onto the domains

\[ D(t_{\tau,D}) := \{ f \in D(t_\tau) | f \in W_{2,0} \}, \]
\[ D(t_{\tau,N}) := \{ f \in D(t_\tau) | f^{[1]} \in W_{1,0} \} \]

respectively. Here and hereafter, \( W_{2,0} := \{ f \in W_2^1 | f(0) = f(1) = 0 \} \).

The differential expression \( t_{\tau,D} \) (unlike \( t_{\tau,N} \)) can be written in the usual potential form. Namely, let us define the Miura map \( b \) via (cf [32])

\[ L_2 \ni u \mapsto u' + u^2 := b(u) \in W_2^{-1} \]

and the class of (Hermitian) Miura potentials

\[ \mathcal{M} := \{ q = b(u) | u \in L_2 \} \quad (\text{Re } \mathcal{M} := \{ q = b(u) | u \in \text{Re } L_2 \}). \]

It can be shown (see lemma 2.2) that for an arbitrary \( q \in \mathcal{M} \) and \( \tau \in b^{-1}(q) \) one has

\[ D(t_{\tau,D}) := \{ f \in W_{2,0} | (-f'' + qf) \in L_2 \}, \quad t_{\tau,D}(f) := -f'' + qf, \]

where the derivative \( f'' \) and the product \( qf \) should be understood in the distributional sense. In particular,

\[ t_{\tau_1,D} = t_{\tau_2,D} \quad \text{if} \quad b(\tau_1) = b(\tau_2). \quad (1.2) \]

For \( \tau \in L_2 \) and \( q := b(\tau) \), we consider the operators \( S_\tau \) and \( T_q \) acting in \( L_2 \) via

\[ S_\tau f := t_{-\tau,N} (f), \quad D(S_\tau) := D(t_{-\tau,N}), \]
\[ T_q f := -f'' + qf, \quad D(T_q) := D(t_{\tau,D}). \]

In this paper we shall concentrate ourselves on the study of the spectral properties of the operators \( S_\tau \) and \( T_q \) for \( \tau \in \text{Re } L_2 \) and \( q = b(\tau) \). In this case the operators \( S_\tau \) and \( T_q \) are self-adjoint; moreover, \( S_\tau \geq 0 \) and \( T_q > 0 \). Their spectra \( \sigma(S_\tau) \) and \( \sigma(T_q) \) consist of countably many isolated eigenvalues accumulating at \( +\infty \); moreover, \( \sigma(S_\tau) = \sigma(T_q) \cup \{0\} \).

Let \( \tau \in L_2 \) and \( \lambda \in \mathbb{C} \). We denote by \( \varphi(\cdot, \lambda, \tau) \) and \( \psi(\cdot, \lambda, \tau) \) the matrix-valued solutions of the Cauchy problems

\[ -\left( \frac{d}{dx} + \tau \right) \left( \frac{d}{dx} - \tau \right) \varphi = \lambda^2 \varphi, \quad \varphi(0) = 0, \quad \varphi^{[1]}(0) = \lambda I, \quad (1.3) \]
\[ -\left( \frac{d}{dx} - \tau \right) \left( \frac{d}{dx} + \tau \right) \psi = \lambda^2 \psi, \quad \psi(0) = I, \quad \psi^{[1]}(0) = 0. \quad (1.4) \]

They are related to each other via

\[ \left( \frac{d}{dx} - \tau \right) \varphi(\cdot, \lambda, \tau) = \lambda \psi(\cdot, \lambda, \tau), \quad \left( \frac{d}{dx} + \tau \right) \psi(\cdot, \lambda, \tau) = -\lambda \varphi(\cdot, \lambda, \tau) \]
and admit representations in the form (see theorem 2.1)

\[
\begin{align*}
\varphi(x, \lambda, \tau) &= \sin \lambda x I + \int_0^x (\sin \lambda t) K_{\tau,D}(x, t) \, dt, \\
\psi(x, \lambda, \tau) &= \cos \lambda x I + \int_0^x (\cos \lambda t) K_{\tau,N}(x, t) \, dt,
\end{align*}
\]

(1.5)

where the matrix-valued kernels \( K_{\tau,D} \) and \( K_{\tau,N} \) belong to the algebra \( G^2_+ \) (see appendix A). Equations (1.5) determine uniquely \( K_{\tau,D} \) and \( K_{\tau,N} \) within the class \( L_2(\Omega, M_r) \), \( \Omega := \{(x, t) \mid 0 \leq t \leq x \leq 1\} \).

The function \( \lambda \mapsto \varphi(1, \lambda, \tau)^{-1} \), as well as the Weyl–Titchmarsh function

\[ m_\tau(\lambda) := -\varphi(1, \lambda, \tau)^{-1} \psi(1, \lambda, -\tau), \]

is meromorphic in \( \mathbb{C} \); note that \( m_\tau(\lambda) = -\cot \lambda I \).

Let now \( \tau \in \mathbb{R} \) and \( q = b(\tau) \). We denote by \( \lambda_j(\tau) (j \in \mathbb{Z}_+) \) the square roots of the pairwise distinct eigenvalues of the operator \( S_\tau \) labelled in increasing order, i.e. \( 0 = \lambda_0(\tau) < \lambda_k(\tau) < \lambda_{k+1}(\tau), k \in \mathbb{N} \); then

\[ \sigma(S_\tau) = \{\lambda_j^2(\tau)\}_{j=0}^\infty, \quad \sigma(T_q) = \{\lambda_j^2(\tau)\}_{j=1}^\infty. \]

The function \( m_\tau \) is a matrix-valued Herglotz function (i.e. \( \text{Im } m_\tau(\lambda) \geq 0 \) for \( \text{Im } \lambda \geq 0 \), and the set \( \{\pm\lambda_j(\tau)\}_{j \in \mathbb{Z}_+} \) is the set of its poles. Put by definition

\[ \alpha_\tau(\lambda) := -\frac{1}{2} \underset{\lambda = \lambda_j}{\text{res}} m_\tau(\lambda), \quad \alpha_j(\tau) := -\underset{\lambda = \lambda_j}{\text{res}} m_\tau(\lambda), \quad j \in \mathbb{N}. \]

(1.6)

The matrix \( \alpha_j(\tau) \) for \( j \in \mathbb{Z}_+ \) (resp. for \( j \in \mathbb{N} \)) is called the norming constant of the operator \( S_\tau \) (resp. of the operator \( T_q \)) corresponding to the eigenvalue \( \lambda_j^2(\tau) \). We note that the multiplicity of the eigenvalue \( \lambda_j^2(\tau) \) of \( S_\tau \) or \( T_q \) equals rank \( \alpha_j(\tau) \).

It turns out that a sequence \( ((\lambda_j(\tau), \alpha_j(\tau)))_{j \in \mathbb{Z}_+} \) depends only on the function \( q = b(\tau) \). In view of this we call the collections \( a_\tau = ((\lambda_j(\tau), \alpha_j(\tau)))_{j \in \mathbb{Z}_+} \) and \( b_\tau = ((\lambda_j(\tau), \alpha_j(\tau)))_{j \in \mathbb{N}} \) a sequences of spectral data, and the matrix-valued measures

\[ \nu_\tau := \sum_{j=0}^\infty \alpha_j(\tau) \delta_{\lambda_j(\tau)}, \quad \mu_\tau := \sum_{j=1}^\infty \alpha_j(\tau) \delta_{\lambda_j(\tau)} \]

(1.7)

will be termed the spectral measures of the operators \( S_\tau \) and \( T_q \) respectively. Here \( \delta_\lambda \) is the Dirac delta-measure centred at the point \( \lambda \). In particular, if \( \tau = q = 0 \), then

\[ \nu_0 = \frac{1}{2} I \delta_0 + \sum_{n=1}^\infty I \delta_{\pi n}, \quad \mu_0 = \sum_{n=1}^\infty I \delta_{\pi n}. \]

(1.8)

Clearly, the spectral measures of the operators \( S_\tau \) and \( T_q \) are related by the simple formula

\[ \nu_\tau - \mu_\tau = \alpha_\tau(\tau) \delta_0, \quad \tau \in \mathbb{R} \}

A simple relation exists also between the function \( m_\tau \) and the measure \( \nu_\tau \). Namely,

\[ m_\tau(\lambda) = 2\lambda \int_0^\infty d\nu_\tau(\xi), \quad \lambda \in \mathbb{C}. \]

The main aim of the present paper is to give a complete description of the classes \( \mathfrak{A} := \{a_\tau \mid \tau \in \mathbb{R} \} \) and \( \mathfrak{B} := \{b_\tau \mid q \in \mathbb{R} \} \) of spectral data and to suggest an efficient method of reconstructing the functions \( \tau \) and \( q \) from the measures \( \nu_\tau \) and \( \mu_\tau \) respectively. We note that the description of the classes \( \mathfrak{A} \) and \( \mathfrak{B} \) is equivalent to the description of the families of measures \( \mathcal{V} := \{\nu_\tau \mid \tau \in \mathbb{R} \} \) and \( \mathcal{M} := \{\mu_\tau \mid q \in \mathbb{R} \} \).
1.3. Main results

We start with characterization of the spectral data for Sturm–Liouville operators under consideration. In what follows a (resp. b) will stand for an arbitrary sequence \((\lambda_j, \alpha_j))_{j \in \mathbb{Z}_+}\) (resp. \((\lambda_j, \alpha_j))_{j \in \mathbb{N}}\), in which \((\lambda_j)_{j \in \mathbb{Z}_+}\) is a strictly increasing sequence of non-negative numbers and \(\alpha_j\) are nonzero matrices in \(M_{\nu}^\tau\), and \(\nu^a\) and \(\mu^b\) will denote the measures given by

\[
\nu^a := \sum_{j=0}^{\infty} \alpha_j \delta_{\lambda_j}, \quad \mu^b := \sum_{j=1}^{\infty} \alpha_j \delta_{\lambda_j}.
\]

Next, we partition the semi-axis \([0, \infty)\) into pairwise disjoint intervals \(\Delta_n\) \((n \in \mathbb{Z}_+)\), namely

\[
\Delta_0 = \{0\}, \quad \Delta_1 = \left(0, \frac{3\pi}{2}\right], \quad \Delta_n = \left(\pi n - \frac{\pi}{2}, \pi n + \frac{\pi}{2}\right], \quad n > 1.
\]

A complete description of the classes \(\mathcal{A}\) and \(\mathcal{B}\) is given by the following two theorems.

**Theorem 1.1.** In order that a sequence \(a = ((\lambda_j, \alpha_j))_{j \in \mathbb{Z}_+}\) should belong to \(\mathcal{A}\) it is necessary and sufficient that the following conditions be satisfied:

\((A_1)\) \(\sum_{n=1}^{\infty} \sum_{j \in \Delta_n} |\lambda_j - \pi n|^2 < \infty\), \(\sup_{n \in \mathbb{N}} \sum_{j \in \Delta_n} 1 < \infty\), \(\sum_{n=1}^{\infty} \|I - \sum_{j \in \Delta_n} \alpha_j\|^2 < \infty\);

\((A_2)\) \(\exists N_0 \in \mathbb{N} \forall N \in \mathbb{N} (N \geq N_0) \implies \sum_{n=1}^{N} \sum_{j \in \Delta_n} \text{rank} \alpha_j = N\;

\((A_3)\) the system of functions \(\{d \cos \lambda_j x \mid j \in \mathbb{Z}_+, d \in \text{Ran} \alpha_j\}\) is complete in the space \(L_2\).

**Theorem 1.2.** In order that a sequence \(b = ((\lambda_j, \alpha_j))_{j \in \mathbb{N}}\) should belong to the class \(\mathcal{B}\) it is necessary and sufficient that conditions \((A_1)\), \((A_2)\), and the following condition \((A_4)\) be satisfied:

\((A_4)\) the system of functions \(\{d \sin \lambda_j x \mid j \in \mathbb{N}, d \in \text{Ran} \alpha_j\}\) is complete in the space \(L_2\).

There exists a simple relation between the classes \(\mathcal{A}\) and \(\mathcal{B}\).

**Proposition 1.3.** Let \(a = ((\lambda_j, \alpha_j))_{j \in \mathbb{Z}_+}\) and \(b = ((\lambda_j, \alpha_j))_{j \in \mathbb{N}}\). Then \((a \in \mathcal{A}) \iff (b \in \mathcal{B}) \land (\lambda_0 = 0, \alpha_0 > 0)\).

**Remark 1.4.** Proposition 1.3 implies that one always has \(\alpha_0 > 0\). Also, it is easy to see that conditions \((A_1)\) and \((A_2)\) yield uniform boundedness of \(\alpha_j\).

We also note that given \((A_2)\) condition \((A_1)\) can be written in the form

\[
\sum_{n=1}^{\infty} \left\| I - \sum_{j \in \Delta_n} \alpha_j \right\|^2 < \infty, \quad \sum_{n=1}^{\infty} \left\| \sum_{j \in \Delta_n} (\lambda_j - \pi n) \alpha_j \right\|^2 < \infty,
\]

which makes it easier to compare the results of this paper with the results of [31].

By definition, every \(a \in \mathcal{A}\) (resp., every \(b \in \mathcal{B}\)) forms spectral data for an operator \(S_\tau\) with some \(\tau \in \text{Re} L_2\) (resp., for an operator \(T_q\) with some \(q \in \text{Re} \mathcal{M}\)). It turns out that these spectral data determine the matrix-valued functions \(\tau\) and \(q\) uniquely, i.e. the following holds true.

**Theorem 1.5.** The mappings \(\text{Re} L_2 \ni \tau \mapsto a_\tau \in \mathcal{A}\) and \(\text{Re} \mathcal{M} \ni q \mapsto b_q \in \mathcal{B}\) are bijective.

As we mentioned earlier, we base our algorithm of reconstruction of the functions \(\tau\) and \(q\) on the Krein accelerant method developed in [33].
Definition 1.6. We say that a function \( H \in L^2((-1, 1), M_r) \) is the accelerant if it is even (i.e. \( H(-x) = H(x) \)) and if for every \( a \in [0, 1] \) the integral equation

\[
 f(x) + \int_0^a H(x - t) f(t) \, dt = 0, \quad x \in (0, 1),
\]

has no non-trivial solutions in the space \( L^2 \). The set of all accelerants is denoted by \( \mathcal{H} \) and is endowed with the metric of the space \( L^2((-1, 1), M_r) \). We shall write \( \text{Re} \mathcal{H} \) for the subset of \( \mathcal{H} \) of all Hermitian accelerants.

Spectral measure of the operator \( S_\tau \) naturally generates Krein accelerants, as explained in the following theorem.

Theorem 1.7. Take a sequence \( a = ((\lambda_j, \alpha_j))_{j \in \mathbb{Z}_+} \) satisfying condition \((A_1)\) and set \( \nu = \nu a \).

Then the limit

\[
 H_{\nu}(x) := \lim_{n \to \infty} \int_0^{\frac{\pi(n+1/2)}{2}} 2 \cos(2\lambda x) \, d(\nu - \nu_0)(\lambda), \quad x \in (-1, 1),
\]

exists in the topology of the space \( L^2((-1, 1), M_r) \). If, in addition, condition \((A_3)\) holds, then the function \( H_{\nu} \) belongs to \( \text{Re} \mathcal{H} \).

Conversely, every accelerant \( H \) determines a function in \( L^2 \) in the following way. It is known (see appendix A) that the associated Krein equation

\[
 R(x, t) + H(x - t) + \int_0^x R(x, \xi) H(\xi - t) \, d\xi = 0, \quad (x, t) \in \Omega, \quad (1.11)
\]

has a unique solution \( R_H \) in the class \( L^2(\Omega, M_r) \). We can now define a mapping \( \Theta \) from \( \mathcal{H} \) to \( L^2 \) given by the formula

\[
 [\Theta(H)](x) := H(x) + \int_0^x R_H(x, \xi) H(\xi) \, d\xi = -R_H(x, 0). \quad (1.12)
\]

The functions \( \Theta(H) \) and \( R_H \) are related to each other as follows:

Theorem 1.8. Assume that \( H \in \mathcal{H}_2 \), \( R = R_H \) and \( \tau = \Theta(H) \). Then

\[
 K_{\tau,D}(x, t) = \frac{1}{2} \left[ R\left(x, \frac{x+t}{2}\right) - R\left(x, \frac{x-t}{2}\right) \right],
\]

\[
 K_{\tau,N}(x, t) = \frac{1}{2} \left[ R\left(x, \frac{x+t}{2}\right) + R\left(x, \frac{x-t}{2}\right) \right],
\]

where \( (x, t) \in \Omega \) and \( K_{\tau,D} \) and \( K_{\tau,N} \) are the kernels of \((1.5)\).

Theorem 1.8 is an analogue of a theorem from the paper of Krein [33], where the accelerant theory was applied to the inverse scattering theory. Accelerants were used to solve the inverse spectral problems on finite intervals in the papers [34, 35].

The following theorem describes some additional properties of the mapping \( \Theta \).

Theorem 1.9. The mapping \( \Theta \) is a homeomorphism between the metric spaces \( \mathcal{H}_2 \) and \( L^2 \). Moreover, if \( H \in \mathcal{H}_2 \), then \( H^* \in \mathcal{H}_2 \) and \( \Theta(H^*) = (\Theta(H))^* \).

Finally, we show how the accelerants can be used to reconstruct \( \tau \) and \( q \) from the corresponding spectral data.
The paper is organized as follows. Section 2 deals with the direct spectral problem and consists of four parts. We establish the basic properties of the operators $T_q$ and $S_\tau$ (theorem 2.4) in subsection 2.1, find the asymptotics of eigenvalues and norming constants (theorem 2.5) in subsection 2.2, study the properties of the Weyl–Titchmarsh function in subsection 2.3, and, finally, prove necessity of conditions of theorems 1.1 and 1.2 in subsection 2.4.

In section 3 we study Krein accelerants. This section is divided into three parts. In the first two subsections we establish theorems 1.8 and 1.9, and in subsection 3.3 we prove propositions 3.7 and 3.8 on operators $\mathcal{H}_e$ and $\mathcal{H}_g$, which play an important role in section 4.

The inverse spectral problem is considered in section 4, which consists of three parts. In subsection 4.1 we study the properties of the function $H_e$ (theorem 1.7), in subsection 4.2 we finish the proof of theorems 1.1 and 1.2, and, finally, in subsection 4.3 we establish proposition 1.3 and theorems 1.5 and 1.10.

There are two short appendices. Some information on the spaces used in the paper and well-known facts from the theory of factorization of Fredholm operators are gathered in appendix A. Three auxiliary lemmata on orthogonal projectors are proved in appendix B.
2. The direct spectral problem

2.1. Basic properties of the operators \( T_q \) and \( S_t \)

In this subsection we prove self-adjointness of the operators \( T_q \) and \( S_t \) in the case where \( \tau \in \text{Re} \, L_2 \) and \( q = b(\tau) \) and also construct their resolvents and the resolutions of identity.

Recall that we have denoted by \( \psi(\cdot, \lambda, \tau) \) and \( \psi'\psi(\cdot, \lambda, \tau) \) the matrix-valued solutions of the Cauchy problems (1.3) and (1.4). The results of [36] imply the following statement.

Theorem 2.1. The problems (1.3) and (1.4) have unique solutions \( \psi(\cdot, \lambda, \tau) \) and \( \psi(\cdot, \lambda, \tau) \) in the classes \( \{ u \in W^1_2 | u^{(1)} \in W^1_2 \} \) and \( \{ u \in W^1_2 | u^{(1)} \in W^1_2 \} \), respectively. Moreover, the following statements hold:

(i) the functions \( \psi(\cdot, \lambda, \tau) \) and \( \psi(\cdot, \lambda, \tau) \) are related to each other via

\[
\left( \frac{d}{dx} - \tau \right) \psi(\cdot, \lambda, \tau) = \lambda \psi(\cdot, \lambda, \tau), \quad \left( \frac{d}{dx} + \tau \right) \psi(\cdot, \lambda, \tau) = -\lambda \psi(\cdot, \lambda, \tau); \tag{2.1}
\]

(ii) for every \( \tau \in L_2 \) there exist unique matrix-valued functions \( K_{\tau,D} \) and \( K_{\tau,N} \) belonging to the algebra \( G_2^+ \) (see appendix A) such that for any \( \lambda \in \mathbb{C} \) and \( x \in [0, 1] \)

\[
\psi(x, \lambda, \tau) = \sin \lambda x I + \int_0^x (\sin \lambda t) K_{\tau,D}(x, t) \, dt,
\]

\[
\psi(x, \lambda, \tau) = \cos \lambda x I + \int_0^x (\cos \lambda t) K_{\tau,N}(x, t) \, dt; \tag{2.2}
\]

(iii) the mappings \( L_2 \ni \tau \mapsto K_{\tau,D} \in G_2^+ \) and \( L_2 \ni \tau \mapsto K_{\tau,N} \in G_2^+ \) are continuous.

Lemma 2.2. Suppose that \( \tau \in L_2 \) and \( q = b(\tau) \). Then

\[
\text{D}(t_{\tau,D}) = \{ f \in W^1_{2,0} | (-f'' + qf) \in L_2 \}, \quad t_{\tau,D}(f) = -f'' + qf,
\]

where the derivative \( f'' \) and the product \( qf \) are interpreted in the distributional sense, i.e. as elements of the space \( W^{-1}_{2} \). Moreover, if \( \tau_1 \) is another element of \( L_2 \), then

(i) \( t_{\tau_1,D} = t_{\tau,D} \iff b(\tau_1) = b(\tau) \);

(ii) \( K_{\tau_1,D} = K_{\tau,D} \iff b(\tau_1) = b(\tau) \);

(iii) \( K_{\tau_1,N} = K_{\tau,N} \iff \tau_1 = \tau \).

Proof. If \( u \in W^{-1}_{2} \) and \( f \in W^1_{2,0} \), then we define the product \( uf \in W^{-1}_{2} \) as

\[
(u f)_j = \sum_{k=1}^{r} u_{jk} f_k, \quad f = (f_1, \ldots, f_r), \quad u = (u_{jk})_{1 \leq j, k \leq r}.
\]

Observe that for every \( f \in W^1_{2,0} \), the equality \( (\tau f)' = \tau' f + \tau f' \) holds in the distributional sense. Therefore, if \( f \in \text{D}(t_{\tau,D}) \), then

\[
t_{\tau,D}(f) = -f'' + (\tau f)' - \tau f' + \tau^2 f = -f'' + qf \in L_2.
\]

On the other hand, if \( f \in W^1_{2,0} \), and \((-f'' + qf) \in L_2 \), then taking into account the equality

\[
-(f' - \tau f)' = (-f'' + qf) + \tau (f' - \tau f), \tag{2.3}
\]

we obtain \( (f' - \tau f) \in W^1_{2} \). Moreover, the function \( f' - \tau f \) is bounded, and, using (2.3) again, we obtain that \( (f' - \tau f) \in W^1_{\tau} \), i.e. that \( f \in \text{D}(t_{\tau,D}) \). This establishes the claim about \( t_{\tau,D} \).

Part (i) is the straightforward consequence of the above considerations. Let us prove part (ii). Assume that \( K_{\tau_1,D} = K_{\tau,D} \). Then \( \psi(\cdot, 0, \tau_1) = \psi(\cdot, 0, \tau) \) in view of (2.2), and (2.1) yields the relations

...
\[(\tau_1 - \tau)\psi(\cdot, \lambda, \tau) = \lambda[\psi(\cdot, \lambda, \tau) - \psi(\cdot, \lambda, \tau_1)], \quad (2.4)\]

\[
\left( \frac{d}{dx} + \tau_1 \right) \left( \frac{d}{dx} + \tau \right) \psi(\cdot, \lambda, \tau) = \left( \frac{d}{dx} + \tau \right) \left( \frac{d}{dx} - \tau \right) \psi(\cdot, \lambda, \tau). \quad (2.5)\]

Note that according to the Riemann–Lebesgue lemma for every \(x_0 \in (0, 1]\) there exists \(\lambda_0 \in \mathbb{R}\) such that \(\det \psi(x_0, \lambda_0, \tau) \neq 0\). Therefore, we conclude from equation (2.4) that the function \(g = \tau_1 - \tau\) is absolutely continuous on the interval \((0, 1]\). Putting \(\tau_1 = \tau + g\) in (2.5), we easily obtain the equality

\[(g' + g\tau + \tau g + g^2)\psi(\cdot, \lambda, \tau) = 0, \quad \lambda \in \mathbb{C},\]

which implies that \(g' + g\tau + \tau g + g^2 = 0\) and, thus, that \(b(\tau_1) = b(\tau)\).

Now let \(b(\tau_1) = b(\tau)\). Then the function \(g = \tau_1 - \tau\) obeys \(g' = \tau^2 - \tau_1^2 \in L_1\) and thus is absolutely continuous on the interval \([0, 1]\). Next, the relation \(g' + g\tau + \tau g + g^2 = 0\) yields equality (2.5). It follows from (2.5) and the definition of the functions \(\psi(\cdot, \lambda, \tau)\) and \(\psi(\cdot, \lambda, \tau)\) that they are solutions of the Cauchy problem

\[
\left( \frac{d}{dx} + \tau_1 \right) y = -\lambda^2 y, \quad y(0) = 0, \quad y^{(1)}(0) = \lambda I.
\]

Thus, \(\psi(\cdot, \lambda, \tau) = \psi(\cdot, \lambda, \tau)\) for all \(\lambda \in \mathbb{C}\). Therefore, (see (2.2)),

\[
\int_0^x (\sin \lambda t)[K_{\tau_1, D}(x, t) - K_{\tau, D}(x, t)] \, dt = 0, \quad x \in [0, 1], \quad \lambda \in \mathbb{C}.
\]

Since \(\lambda\) and \(x\) are arbitrary, \(K_{\tau_1, D} = K_{\tau, D}\).

It remains to prove (iii). If \(\tau_1 = \tau\), then \(K_{\tau_1, N} = K_{\tau, N}\) by the uniqueness claim of theorem 2.1. Assume that \(K_{\tau_1, N} = K_{\tau, N}\); then, according to (2.1) and (2.2),

\[
\left( \frac{d}{dx} + \tau_1 \right) \psi(x, 0, \tau_1) = \left( \frac{d}{dx} + \tau \right) \psi(x, 0, \tau) = 0, \quad \psi(\cdot, 0, \tau_1) = \psi(\cdot, 0, \tau),
\]

and thus \((\tau_1 - \tau) \psi(\cdot, 0, \tau) = 0\). Therefore, to prove part (iii) it is enough to show that the matrix \(\psi(\cdot, 0, \tau)\) is invertible for every \(x \in [0, 1]\). Suppose the last statement is false. Then there exist \(x_0 \in (0, 1]\) and \(c \in C^b \setminus \{0\}\) such that \(\psi(x_0, 0, \tau)c = 0\). Therefore, the function \(f = \psi(\cdot, 0, \tau)c\) is a nonzero solution of the Cauchy problem \(f' + \tau f = 0\), \(f(x_0) = 0\), which is impossible. The proof is complete. \(\square\)

For every \(\tau \in L_2, \lambda \in \mathbb{C}\) and \(x \in [0, 1]\) we put

\[
\mathcal{W}(x, \lambda, \tau) = \begin{pmatrix} \psi(x, \lambda, \tau) & \psi(x, \lambda, -\tau) \\ -\psi(x, \lambda, \tau) & \psi(x, \lambda, -\tau) \end{pmatrix}, \quad Q(x, \lambda, \tau) = \begin{pmatrix} -\tau(x) & \lambda I \\ -\lambda I & \tau(x) \end{pmatrix}.
\]

It follows from (2.1) that \(\frac{d}{dx}\mathcal{W}(x, \lambda, \tau) = Q(x, \lambda, \tau)\mathcal{W}(x, \lambda, \tau)\), so that

\[
\frac{d}{dx}\mathcal{W}^*(x, \lambda, -\tau^*)\mathcal{W}(x, \lambda, \tau) = 0.
\]

Therefore, the matrices \(\mathcal{W}^*(x, \lambda, -\tau^*)\) and \(\mathcal{W}(x, \lambda, \tau)\) are inverse to each other, which yields the relations

\[
\psi(x, \lambda, \tau)\psi^*(x, \lambda, -\tau^*) + \psi(x, \lambda, -\tau)\psi^*(x, \lambda, \tau) = I, \quad \psi(x, \lambda, \tau)\psi^*(x, \lambda, -\tau^*) - \psi(x, \lambda, -\tau)\psi^*(x, \lambda, \tau) = 0. \quad (2.6)
\]

Let us denote by \(\Phi_\tau(\lambda)\) and \(\Psi_\tau(\lambda)\) (\(\lambda \in \mathbb{C}\)) the operators acting from \(C^b\) to \(L_2\) by the formulae

\[
[\Phi_\tau(\lambda)c](x) := \sqrt{2}\psi(x, \lambda, \tau)c, \quad [\Psi_\tau(\lambda)c](x) := \sqrt{2}\psi(x, \lambda, \tau)c. \quad (2.7)
\]
Taking into consideration (2.2), we obtain, for \( \lambda \in \mathbb{C} \),

\[
\Phi_\tau(\lambda) = (\mathcal{I} + K_{\tau,D})\Phi_0(\lambda), \quad \Psi_\tau(\lambda) = (\mathcal{I} + K_{\tau,N})\Psi_0(\lambda),
\]

where \( K_{\tau,D} \) and \( K_{\tau,N} \) are integral operators with kernels \( K_{\tau,D} \) and \( K_{\tau,N} \) respectively and \( \mathcal{I} \) is the identity operator in the algebra \( B(L_2) \), which is an algebra of bounded linear operators acting in \( L_2 \). Note that since \( K_{\tau,D} \) and \( K_{\tau,N} \) belong to \( G_2^+ \), the operators \( K_{\tau,D} \) and \( K_{\tau,N} \) belong to the algebra \( \mathcal{O}_1 \) (see appendix A), and hence they are Volterra operators [37, chapter IV].

**Lemma 2.3.** Let \( \tau \in \mathbb{L}_2 \). Then the following statements hold:

(i) the operator functions \( \lambda \mapsto \Phi_\tau(\lambda)/\lambda \) and \( \lambda \mapsto \Psi_\tau(\lambda) \) are analytic in \( \mathbb{C} \); moreover,

\[
\ker \Phi_\tau(\lambda) = \{0\}, \quad \text{Ran} \Phi_\tau(\lambda) = \mathbb{C}', \quad \lambda \in \mathbb{C}\setminus\{0\},
\]

\[
\ker \Psi_\tau(\lambda) = \{0\}, \quad \text{Ran} \Psi_\tau(\lambda) = \mathbb{C}', \quad \lambda \in \mathbb{C},
\]

where \( \mathcal{E}_\lambda := \ker \varphi(1, \lambda, \tau) \);

(ii) the operator functions \( \lambda \mapsto \varphi(1, \lambda, \tau)^{-1} \) and

\[
\lambda \mapsto m_\tau(\lambda) = -\varphi(1, \lambda, \tau)^{-1}\psi(1, \lambda, -\tau)
\]

are meromorphic in \( \mathbb{C} \). Moreover, \( m_0(\lambda) = -\cot \lambda I \) and

\[
\|m_\tau(\lambda) + \cot \lambda I\| = o(1)
\]

as \( \lambda \to \infty \) within the domain \( \mathcal{O} = \{z \in \mathbb{C} \mid \forall n \in \mathbb{Z} \mid z - \pi n > 1\} \).

**Proof.** Part (i) obviously follows from (2.7) and (2.8). From (2.2) we obtain that

\[
\varphi(1, \lambda, \tau) = \sin \lambda I + \int_0^1 (\sin \lambda t) K_{\tau,D}(1, t) \, dt,
\]

\[
\psi(1, \lambda, -\tau) = \cos \lambda I + \int_0^1 (\cos \lambda t) K_{-\tau,N}(1, t) \, dt.
\]

Since \( K_{\tau,D}, K_{-\tau,N} \in G_2^+ \), the functions \( K_{\tau,D}(1, \cdot) \) and \( K_{-\tau,N}(1, \cdot) \) belong to \( \mathbb{L}_2 \). The relations

\[
\lim_{|\lambda| \to \infty} e^{-|\text{Im} \lambda|} \int_0^1 f(x) \cos \lambda x \, dx = \lim_{|\lambda| \to \infty} e^{-|\text{Im} \lambda|} \int_0^1 f(x) \sin \lambda x \, dx = 0
\]

for \( f \in L_1(0, 1) \) that refine the classical Riemann–Lebesgue theorem [38, chapter 1] are proved in [1, chapter 1]; thus

\[
\lim_{|\lambda| \to \infty} e^{-|\text{Im} \lambda|} \|\varphi(1, \lambda, \tau) - \sin \lambda I\| = \lim_{|\lambda| \to \infty} e^{-|\text{Im} \lambda|} \|\psi(1, \lambda, -\tau) - \cos \lambda I\| = 0.
\]

In particular, \( \varphi(1, \lambda, \tau) \) is invertible for all \( \lambda \in \mathcal{O} \) large enough, so that \( m_\tau \) is meromorphic and relation (2.12) holds. \( \square \)

**Theorem 2.4.** Let \( \tau \in \text{Re} \mathbb{L}_2 \) and \( q = b(\tau) \). Then the following statements hold.

(i) The operators \( S_\tau \) and \( T_q \) are self-adjoint; moreover, \( S_\tau \geq 0 \) and \( T_q \geq 0 \).

(ii) The spectra \( \sigma(S_\tau) \) and \( \sigma(T_q) \) consist of isolated eigenvalues and

\[
\sigma(S_\tau) = \{\lambda^2 \mid \ker \varphi(1, \lambda, \tau) \neq \{0\}\} = \sigma(T_q) \cup \{0\}.
\]

(iii) Let \( \lambda_j = \lambda_j(\tau) \) and \( P_{j,\tau} \) (resp. \( Q_{k,q} \)) be the orthogonal projector on the eigensubspace \( \ker (S_\tau - \lambda_j^2 \mathcal{I}) \) (resp. \( \ker (T_q - \lambda_j^2 \mathcal{I}) \)); then

\[
\sum_{j=0}^{\infty} P_{j,\tau} = \mathcal{I}, \quad \sum_{k=1}^{\infty} Q_{k,q} = \mathcal{I}.
\]
(iv) The norming constants \( \alpha_j = \alpha_j(\tau) \) (see the introduction) satisfy the relations \( \alpha_0 > 0 \) and \( \alpha_j \geq 0, \ j \in \mathbb{N} \). Moreover, for \( j \in \mathbb{Z}_+ \) and \( k \in \mathbb{N} \) we have

\[
P_{j,\tau} = \Phi_\tau(\lambda_j)\alpha_j\Phi_\tau^*(\lambda_j), \quad Q_{k,\tau} = \Phi_\tau(\lambda_k)\alpha_k\Phi_\tau^*(\lambda_k),
\]

where \( \Phi_\tau^*(\lambda) = [\Phi_\tau(\lambda)]^* \), \( \Psi_\tau^*(\lambda) = [\Psi_\tau(\lambda)]^* \) (see (2.7)).

(v) If \( \tau_1 \in \mathbb{R} \) and \( b(\tau_1) = b(\tau) \), then \( \lambda_j(\tau_1) = \lambda_j(\tau) \) and \( \alpha_j(\tau_1) = \alpha_j(\tau) \) for every \( j \in \mathbb{N} \).

**Proof.** Writing \( S = S_\tau \) and \( T = T_q \) for short and integrating by parts, we get

\[
(Sf|g) = (f' - \tau f|g' - \tau g) = (f|Sg), \quad f, g \in D(S),
\]

\[
(Tf|g) = (f' - \tau f|g' - \tau g) = (f|Tg), \quad f, g \in D(T),
\]

where \((\cdot|\cdot)\) is the scalar product in the space \( L_2 \). Therefore, the operators \( S \) and \( T \) are symmetric and non-negative. Suppose that \( (Tf|f) = 0 \) for some \( f \in D(T) \). Then \( f \) is a solution of the Cauchy problem \( f' - \tau f = 0, \ f(0) = 0 \). The uniqueness theorem then gives \( f = 0 \), and therefore \( T > 0 \).

Take now an arbitrary \( f \in L_2 \) and a point \( \lambda \in \mathbb{C}\setminus\{0\} \) for which the matrix \( \varphi(1, \lambda, \tau) \) is nonsingular.

The function

\[
g(x) = [X(\lambda)f](x) := \frac{\varphi(x, \lambda, \tau)}{\lambda} \int_x^1 \psi^*(t, \lambda, -\tau) f(t) \, dt + \frac{\psi(x, \lambda, -\tau)}{\lambda} \int_0^x \psi^*(t, \lambda, \tau) f(t) \, dt
\]

vanishes at \( x = 0 \), belongs to the domain of the differential expression \( t_\tau \) and solves the equation \( \tau g = \lambda^2 g + f, \) as can be verified directly by using relations (2.1) and (2.6). A generic solution of the above differential equation that vanishes at \( x = 0 \) takes therefore the form \( \varphi(x, \lambda, \tau)c + g \) for some vector \( c \in \mathbb{C}^N \); the choice

\[
c := \frac{m_{\tau}(\lambda)}{\lambda} \int_0^1 \varphi(t, \lambda, \tau) f(t) \, dt
\]

makes this solution to vanish at the point \( x = 1 \) as well. This implies that the point \( \lambda^2 \) is a resolvent point of the operator \( T \) and that the resolvent of \( T \) is given by

\[
(T - \lambda^2 I)^{-1} = \frac{1}{2\lambda} \Phi_\tau(\lambda)m_{\tau}(\lambda)\Phi_\tau^*(\lambda) + X(\lambda).
\]

Similar arguments show that the resolvent of the operator \( S \) at the point \( \lambda^2 \) equals

\[
(S - \lambda^2 I)^{-1} = \frac{1}{2\lambda} \Psi_\tau(\lambda)m_{\tau}(\lambda)\Psi_\tau^*(\lambda) - Y(\lambda),
\]

where the operator \( Y \) is given via

\[
[Y(\lambda)f](x) := \frac{\varphi(x, \lambda, -\tau)}{\lambda} \int_x^1 \psi^*(t, \lambda, \tau) f(t) \, dt + \frac{\psi(x, \lambda, \tau)}{\lambda} \int_0^x \psi^*(t, \lambda, -\tau) f(t) \, dt.
\]

The above formulae show that \( S \) and \( T \) have compact resolvents and thus (i)–(iii) follow.

Recall that \( -\alpha_j(\tau) \) is the residue of the Weyl–Titchmarsh function \( m_{\tau} \) at the point \( \lambda_j(\tau) \), \( j \in \mathbb{N} \). Taking \( \varepsilon > 0 \) small enough, we get

\[
P_{j,\tau} = \frac{-1}{2\pi i} \oint_{|\zeta| = \varepsilon} (S - \zeta I)^{-1} \, d\zeta = \frac{-1}{2\pi i} \oint_{|\lambda_j| = \varepsilon} 2\lambda(S - \lambda^2 I)^{-1} \, d\lambda,
\]

\[
= \frac{-1}{2\pi i} \oint_{|\lambda_j| = \varepsilon} \Psi_{\tau}(\lambda)m_{\tau}(\lambda)\Psi_{\tau}^*(\lambda) d\lambda = \Psi_{\tau}(\lambda_j)\alpha_j\Psi_{\tau}^*(\lambda_j)
\]
for every $j \in \mathbb{N}$. Similarly, we obtain that
\[ R_{0, \tau} = \Psi_t(0) \alpha_0 \Psi_t^*(0), \quad Q_{j,q} = \Phi_t(\lambda_j) \alpha_j \Phi_t^*(\lambda_j), \quad j \in \mathbb{N}. \]
Recalling (2.9), we see that $\alpha_j \geq 0$ for all $j \in \mathbb{Z}_+$. Let us prove that $\alpha_0 > 0$. Assume the contrary; then
\[ \ker S = \text{Ran} R_{0, \tau} = \text{Ran} [\Psi_t(0) \alpha_0 \Psi_t^*(0)] \neq \text{Ran} \Psi_t(0). \]
On the other hand, (2.10) on account of the equality $\ker \varphi(1, 0, \tau) = \mathbb{C}^\tau$ yields $\ker S = \text{Ran} \Psi_t(0)$. The contradiction derived shows that $\alpha_0 > 0$.

It remains to prove (v). Let $\tau_1 \in \text{Re} \mathcal{L}_2$ be such that $b(\tau_1) = b(\tau)$. It follows from lemma 2.2 that $K_{\tau_1, D} = K_{\tau, D}$, and thus $\varphi(\cdot, \cdot, \tau_1) = \varphi(\cdot, \cdot, \tau)$. Therefore, in view of (2.14) we get $\lambda_j(\tau_1) = \lambda_j(\tau)$ for all $j \in \mathbb{N}$ and, moreover, $\Phi_{\tau_1}(\cdot) = \Phi_{\tau}(\cdot)$. It follows now from (2.16) that
\[ \Phi_t(\lambda_j) \alpha_j(\tau_1) \Phi_t^*(\lambda_j) = Q_{j,q} = \Phi_t(\lambda_j) \alpha_j(\tau) \Phi_t^*(\lambda_j), \quad j \in \mathbb{N}. \]
Hence, using (2.9), we obtain that $\alpha_j(\tau_1) = \alpha_j(\tau)$ for all $j \in \mathbb{N}$. The proof is complete. \hfill $\square$

2.2. The asymptotics of eigenvalues and norming constants

The main result of this section is the following theorem.

**Theorem 2.5.** Let $\tau \in \text{Re} \mathcal{L}_2$. Then for the sequence $a = a_\tau$ condition (A_1) holds.

First we prove two lemmas. In what follows, we shall use the following notation. If $(\lambda_j)_{j \in \mathbb{Z}_+}$ is a strictly increasing sequence of non-negative numbers and $(\alpha_j)_{j \in \mathbb{Z}_+}$ is a sequence in $M^+_\tau$, then
\begin{equation}
\beta_n := I - \sum_{\lambda_j \in \Delta_n} \alpha_j, \quad \tilde{\lambda}_j := \lambda_j - \pi n, \quad \lambda_j \in \Delta_n, \quad n \in \mathbb{N},
\end{equation}
with $\Delta_n$ defined in subsection 1.3.

**Lemma 2.6.** Let $\tau \in \text{Re} \mathcal{L}_2$ and $\lambda_j = \lambda_j(\tau), \ j \in \mathbb{Z}_+$. Then
\begin{equation}
\sum_{n=1}^{\infty} \sum_{\lambda_j \in \Delta_n} |\tilde{\lambda}_j|^2 < \infty, \quad \sup_{n \in \mathbb{N}} \sum_{\lambda_j \in \Delta_n} 1 < \infty.
\end{equation}

**Proof.** Let us fix $\tau \in \text{Re} \mathcal{L}_2$ and note that the numbers $\lambda_j(\tau), \ j \in \mathbb{Z}_+$, are non-negative zeros of an entire function $g(\lambda) := \det \varphi(1, \lambda, \tau), \ \lambda \in \mathbb{C}$. In view of (2.13) the function $g$ is odd. Taking into account (2.14), we conclude that zeros of the function $g$ are all real. The function $\lambda \mapsto \varphi(1, \lambda, \tau)$ belongs to the following class of function $C \Rightarrow M^\tau$: \[ F_f(\lambda) := \sin \lambda I + \int_{-1}^{1} f(t) e^{i\lambda t} \, dt, \quad \lambda \in \mathbb{C}, \]
where $f \in L_2((-1, 1), M_\tau)$. Indeed, it follows from (2.13) that $\varphi(1, \cdot, \tau) = F_f$ for $f(t) = (\text{sign} t) K_{\tau, D}(1, |t|/(2\tau))$. It is shown in the paper [39] that the set of zeros of a function $\det F_f$ with $F_f$ as above can be indexed (counting multiplicities) by the set $\mathbb{Z}$ so that the corresponding sequence $(\omega_n)_{n \in \mathbb{Z}}$ has the asymptotics
\[ \omega_{k+1} = \pi k + \tilde{\omega}_j k, \quad k \in \mathbb{Z}, \quad j = 0, \ldots, r - 1, \]
where the sequences $(\tilde{\omega}_j k)_{k \in \mathbb{Z}}$ belong to $\ell_2(\mathbb{Z})$. Therefore, (2.18) follows, and the proof is complete. \hfill $\square$
Lemma 2.7. Let the operators $A(z)$ and $B(z)$ ($z \in \mathbb{C}$) act from $L_2$ to $M_z$ by the formulae

$$A(z) f = \sqrt{2} \int_0^1 (\sin zt) f(t) \, dt, \quad B(z) g = \sqrt{2} \int_0^1 (\cos zt) g(t) \, dt. \quad (2.19)$$

Then for every $f_1, f_2 \in L_2$ and $\lambda \in \mathbb{D} := \{ \lambda \in \mathbb{C} \mid |\lambda| \leq 1 \}$ we have

$$\sum_{n=1}^{\infty} (\|A(\pi n + \lambda)f_1\|^2 + \|B(\pi n + \lambda)f_2\|^2) \leq 16r^2(\|f_1\|_{L_2}^2 + \|f_2\|_{L_2}^2). \quad (2.20)$$

Proof. Since $\{\sqrt{2}\sin n\pi t\}_{n \in \mathbb{N}}$ and $\{\sqrt{2}\cos n\pi t\}_{n \in \mathbb{N}}$ form orthonormal systems of $L_2(0, 1)$, it follows that

$$\sum_{n=1}^{\infty} (\|A(\pi n)f_1\|^2 + \|B(\pi n)f_2\|^2) \leq 2r^2(\|f_1\|_{L_2}^2 + \|f_2\|_{L_2}^2). \quad (2.21)$$

Relations (2.19) yield

$$A(\pi n + \lambda)f_1 = B(\pi n)f_{11} + A(\pi n)f_{12}, \quad B(\pi n + \lambda)f_2 = B(\pi n)f_{22} - A(\pi n)f_{21}, \quad (2.22)$$

where

$$f_{1j}(t) = (\sin \lambda t)f_j(t), \quad f_{2j}(t) = (\cos \lambda t)f_j(t), \quad t \in (0, 1), \quad j = 1, 2.$$

Since $|\sin \lambda|^2 + |\cos \lambda|^2 \leq (\sinh 1 + \cosh 1)^2 \leq 8$ for $\lambda \in \mathbb{D}$, we have

$$\|f_{1j}\|_{L_2}^2 + \|f_{2j}\|_{L_2}^2 \leq 8\|f_j\|_{L_2}^2, \quad j = 1, 2.$$ 

Hence, using (2.22) and (2.21), we arrive at (2.20). $\square$

Proof of theorem 2.5. Let $\tau \in \Re L_2$ and $\lambda_j = \lambda_j(\tau), \alpha_j = \alpha_j(\tau)$. According to lemma 2.6, it remains to show that $\sum_{n=1}^{\infty} \|\beta_n\|^2 < \infty$. Set

$$\mathbb{T}_\alpha := \{ \lambda \in \mathbb{C} \mid |\lambda - \pi n| = 1 \}, \quad \mathbb{D}_\alpha := \{ \lambda \in \mathbb{C} \mid |\lambda - \pi n| \leq 1 \}, \quad n \in \mathbb{Z},$$

$$f_1 := 2^{-1/2}K_{\tau,D}(1, \cdot), \quad f_2 := 2^{-1/2}K_{\tau,N}(1, \cdot).$$

Since $K_{\tau,D}, K_{\tau,N} \in G_2^1$, we have $f_1, f_2 \in L_2$. It follows from (2.13) and (2.19) that

$$\varphi(1, \lambda, \tau) = \sin \lambda I + A(\lambda)f_1, \quad \psi(1, \lambda, -\tau) = \cos \lambda I + B(\lambda)f_2.$$ 

Hence, using (2.11), we obtain the equality

$$m_{\tau}(\lambda) + \cot(\lambda) I = \varphi(1, \lambda, \tau)^{-1}[(\cot \lambda)A(\lambda)f_1 - B(\lambda)f_2]. \quad (2.23)$$

In view of (2.18) and (2.19), we can choose $n_0 \in \mathbb{N}$, such that

$$\sum_{n=n_0}^{\infty} \sum_{\lambda_j \in \Delta_n} |\hat{\lambda}_j|^2 < 1, \quad \sup_{n \geq n_0} \sup_{\lambda \in \Delta_n} \|A(\lambda)f_1\| \leq \frac{1}{4}. \quad (2.24)$$

Note that for every $\lambda \in \mathbb{T}_0$ we have

$$|\sin \lambda| = \prod_{k=1}^{\infty} \left| 1 - \frac{\lambda^2}{\pi^2 k^2} \right| \geq \prod_{k=1}^{\infty} \left( 1 - \frac{1}{\pi^2 k^2} \right) = \sin 1,$$

$$|\cot \lambda| = \frac{1}{\lambda} + \sum_{n=1}^{\infty} \frac{2\lambda}{\lambda^2 - (\pi k)^2} \leq 1 + \sum_{n=1}^{\infty} \frac{2}{(\pi k)^2 - 1} = 2 - \cot 1,$$
whence $|\sin \lambda| \geq \frac{1}{2}$ and $|\cot \lambda| \leq \sqrt{3}$ for $\lambda \in T_n$, $n \in \mathbb{Z}_+$. Therefore estimates (2.24) yield the following inequalities for $\lambda \in T_n$ with $n \geq n_0$:

$$\|\varphi(1, \lambda, \tau)^{-1}\| \leq |\sin \lambda|^{-1}(1 - |\sin \lambda|^{-1}\|A(\lambda)f_1\|^{-1}) \leq 4.$$  \hspace{1cm} (2.25)

For $n \geq n_0$ the function $m_\tau$ has no poles on the circle $T_n$ and $\{\lambda_j \mid \lambda_j \in \Delta_n\} \subset D_n$. Hence, (1.6) implies that

$$\beta_n = \frac{1}{2\pi i} \oint_{T_n} (m_\tau(\lambda) + \cot \lambda I) d\lambda, \quad n \geq n_0.$$  \hspace{1cm} (2.26)

Using (2.23) and (2.25), we get

$$\|\beta_n\|^2 \leq \frac{96}{2\pi} \int \{\|A(\pi n + e^{i\theta})f_1\|^2 + \|B(\pi n + e^{i\theta})f_2\|^2\} d\theta, \quad n \geq n_0,$$

and in view of (2.20) we get $\sum_{n=1}^{\infty} \|\beta_n\|^2 < \infty$. \hspace{1cm} \(\square\)

2.3. The Weyl–Titchmarsh function

**Proposition 2.8.** Let $\tau \in \text{Re} L_2$. Then $m_\tau$ is a Herglotz function and

$$m_\tau(\lambda) = 2\lambda \int_0^\infty \frac{dv_\tau(\xi)}{\xi^2 - \lambda^2}, \quad \lambda \in \mathbb{C}.$$  \hspace{1cm} (2.27)

**Proof.** That $m_\tau$ is Herglotz is clear, so we only need to prove (2.27). Set

$$h(\lambda) := m_\tau(\lambda) - 2\lambda \int_0^\infty \frac{dv_\tau(\xi)}{\xi^2 - \lambda^2}, \quad \lambda \in \mathbb{C}.$$  

It follows from (1.6) and (1.7) that $h$ is entire. To show that $h = 0$ it suffices to justify the estimate $\|h(\lambda)\| = o(1)$ as $\lambda$ goes to infinity within the domain

$$\mathcal{O} = \{z \in \mathbb{C} \mid \forall n \in \mathbb{Z} \mid |z - \pi n| > 1\}.$$  

We set

$$g(\lambda) := \cot \lambda I + 2\lambda \int_0^\infty \frac{dv_\tau(\xi)}{\xi^2 - \lambda^2}, \quad \lambda \in \mathbb{C}.$$  

In virtue of (2.12), it suffices to prove the estimate

$$\|g(\lambda)\| = o(1), \quad \mathcal{O} \ni \lambda \to \infty.$$  \hspace{1cm} (2.28)

Since

$$\cot \lambda = 1 + \sum_{n=1}^{\infty} \frac{2\lambda}{\lambda^2 - (\pi n)^2}, \quad \lambda \in \mathbb{C},$$

we find that

$$g(\lambda) = \frac{1}{\lambda}(I - 2\alpha_0) + \sum_{n=1}^{\infty} \frac{2\lambda I}{\lambda^2 - (\pi n)^2} - \sum_{n=1}^{\infty} \sum_{\lambda_j \in \Delta_n} \frac{2\lambda \alpha_j}{\lambda^2 - \lambda_j^2}.$$  

Let

$$g_n(\lambda) = \frac{2\lambda I}{\lambda^2 - (\pi n)^2} - \sum_{\lambda_j \in \Delta_n} \frac{2\lambda \alpha_j}{\lambda^2 - \lambda_j^2}, \quad n \in \mathbb{N}.
Taking into account the equality
\[
\frac{2\lambda}{\lambda^2 - (\pi n)^2} - \frac{2\lambda}{\lambda^2 - \lambda_j^2} = \frac{\lambda_j}{(\lambda + \pi n)(\lambda + \lambda_j)} - \frac{\lambda_j}{(\lambda - \pi n)(\lambda - \lambda_j)},
\]
for all \(\lambda \in \mathcal{O}\) and for all \(\lambda_j \in \Delta_n\) such that \(|\lambda_j| \leq 1/2\) we get
\[
\left| \frac{2\lambda}{\lambda^2 - (\pi n)^2} - \frac{2\lambda}{\lambda^2 - \lambda_j^2} \right| \leq \frac{2|\lambda_j|}{|\lambda - \pi n|} + \frac{2|\lambda_j|}{|\lambda + \pi n|}.
\]
Therefore, for big enough \(n\) it holds
\[
\|g_n(\lambda)\| \leq \left| \frac{2\lambda\beta_n}{\lambda^2 - (\pi n)^2} + \sum_{j \in \Delta_n} \|\alpha_j\| \right| \left| \frac{2\lambda}{\lambda^2 - (\pi n)^2} - \frac{2\lambda}{\lambda^2 - \lambda_j^2} \right| \leq \left( \|\beta_n\| + \sum_{j \in \Delta_n} 2|\lambda_j|\|\alpha_j\| \right) \left( \frac{1}{|\lambda - \pi n|} + \frac{1}{|\lambda + \pi n|} \right), \quad \lambda \in \mathcal{O}.
\]
By theorem 2.5 condition \((A_1)\) holds and (see remark 1.4) the sequence \((\alpha_j)_{j \in \mathbb{Z}_+}\) is bounded. Thus, we find that
\[
\|\beta_n\| + \sum_{j \in \Delta_n} 2|\lambda_j|\|\alpha_j\| \leq C \left( \|\beta_n\|^2 + \sum_{j \in \Delta_n} |\lambda_j|^2 \right)^{1/2}, \quad n \in \mathbb{Z}_+,
\]
where \(C\) is positive constant. Let \(\mathcal{O}_1 := \{\lambda \in \mathcal{O} \mid |\Re \lambda| \leq \pi/2\}\). Then
\[
\sup_{\lambda \in \mathcal{O}_1} \sum_{n \in \mathbb{Z}} \frac{1}{|\lambda - \pi n|^2} = \sup_{\lambda \in \mathcal{O}_1} \sum_{n \in \mathbb{Z}} \frac{1}{|\lambda - \pi n|^2} \leq 1 + \sum_{n=1}^{\infty} \frac{8}{(\pi n)^2} =: c.
\]
Using the last three inequalities and the Cauchy–Bunyakowski inequality, we get that for all big enough \(k\)
\[
\sum_{n=k}^{\infty} \|g_n(\lambda)\| \leq 2C\sqrt{c} \left( \sum_{n=k}^{\infty} \|\beta_n\|^2 + \sum_{n=k}^{\infty} \sum_{j \in \Delta_n} |\lambda_j|^2 \right)^{1/2}, \quad \lambda \in \mathcal{O}. \tag{2.29}
\]
Since \(\|g_n(\lambda)\| = O(1/\lambda)\) as \(\lambda \to \infty, (n \in \mathbb{N})\), (2.28) follows from (2.29).

\section{2.4. Necessity parts of theorems 1.1 and 1.2}

The theorem stated below implies necessity in theorems 1.1 and 1.2.

\textbf{Theorem 2.9.} Let \(\tau \in \Re L_2\). Then for a sequence \(\alpha_\tau\) conditions \((A_1)\–(A_4)\) hold.

First we prove four auxiliary but nonetheless important lemmas.

\textbf{Lemma 2.10.} Assume that \(\tau \in L_2\) and let \(a = (\lambda_j, \alpha_j)_{j \in \mathbb{Z}_+}\) satisfy condition \((A_1)\). Then
\[
\sum_{n=1}^{\infty} \sum_{\lambda_j \in \Delta_n} \|\Psi_n(\lambda_j) - \Psi_0(\pi n)\|^2 < \infty, \quad \sum_{n=1}^{\infty} \sum_{\lambda_j \in \Delta_n} \|\Phi_n(\lambda_j) - \Phi_0(\pi n)\|^2 < \infty. \tag{2.30}
\]

\textbf{Proof.} We prove the first inequality; the second one can be proved similarly. Recalling the definitions in (2.7)–(2.8), we conclude that for every \(\lambda, \xi \) in \(\mathbb{R}\) one has
\[
\|\Psi_n(\lambda)\| \leq C, \quad \|\Psi_n(\lambda) - \Psi_n(\xi)\| \leq C|\lambda - \xi|.
\]
with $C = \sqrt{2}(1 + \|K_{r,n}\|)$. By virtue of (2.17) we derive the estimate
\[
\sum_{n=1}^{\infty} \sum_{\lambda_j \in \Delta_n} \|\Psi_t(\lambda_j) - \Psi_t(\pi n)\|^2 \leq C^2 \sum_{n=1}^{\infty} \|\Delta_n\|^2 < \infty. \tag{2.31}
\]

We note that the operator $K_{r,n}$ belongs to the Hilbert–Schmidt class $B_2$ and that the sequence $(P_{n,0})_{n \in \mathbb{Z}}$ consists of pairwise orthogonal projectors. Therefore,
\[
\sum_{n=1}^{\infty} \|K_{r,n}P_{n,0}\|^2 \leq \sum_{n=1}^{\infty} \|K_{r,n}P_{n,0}\|_{B_2}^2 \leq \|K_{r,n}\|_{B_2}^2.
\]

It follows from (2.8) that $\Psi_t(\pi n) - \Psi_0(\pi n) = \mathcal{H}_{r,n}\Psi_0(\pi n)$. Since $\|\Psi_0(\pi n)\| = 1$ and $P_{n,0}\Psi_0(\pi n) = \Psi_0(\pi n)$,
\[
\sum_{n=1}^{\infty} \|\Psi_t(\pi n) - \Psi_0(\pi n)\|^2 \leq \sum_{n=1}^{\infty} \|K_{r,n}P_{n,0}\Psi_0(\pi n)\|^2 \leq \|K_{r,n}\|_{B_2}^2. \tag{2.32}
\]

Combining (2.31) and (2.32) with the uniform bound $\sup_{n \in \mathbb{N}} \sum_{\lambda_j \in \Delta_n} 1 < \infty$ satisfied due to (A1), we obtain inequality (2.30) for the operators $\Psi_t$.

**Lemma 2.11.** Assume that $\tau \in L_2$ and let $a = ((\lambda_j, \alpha_j))_{j \in \mathbb{Z}}$ satisfy condition (A1). Then for every $f \in L_2$ we get
\[
\sum_{j=0}^{\infty} \|\alpha_j \Psi_t^*(\lambda_j) f\|^2_{C'} < \infty, \quad \sum_{j=1}^{\infty} \|\alpha_j \Phi_t^*(\lambda_j) f\|^2_{C'} < \infty. \tag{2.33}
\]

Furthermore, for every sequence $(c_j) \in l_2(\mathbb{Z}_+, C')$ the series $\sum_{j=0}^{\infty} \Psi_t(\lambda_j)c_j$ and $\sum_{j=1}^{\infty} \Phi_t(\lambda_j)c_j$ converge in the space $L_2$.

**Proof.** We shall only prove the first inequality in (2.33) and convergence of the series $\sum_{j=0}^{\infty} \Psi_t(\lambda_j)c_j$; the other statements are justified analogously. Since the norms of the matrices $\alpha_j$ are uniformly bounded, it suffices to prove the inequality $\sum_{j=0}^{\infty} \|\Psi_t^*(\lambda_j) f\|^2_{C'} < \infty$. Observe that
\[
\sum_{j=0}^{\infty} \|\Psi_t^*(\lambda_j) f\|^2_{C'} \leq \sum_{n=0}^{\infty} \sum_{\lambda_j \in \Delta_n} \|\Psi_t^*(\lambda_j) f\|^2_{C'} \\
\leq 2 \sum_{n=0}^{\infty} \sum_{\lambda_j \in \Delta_n} \left\{ \|\Psi_t^*(\lambda_j) - \Psi_t^*(\pi n)\| f\|^2_{C'} + \|\Psi_t^*(\pi n)\| f\|^2_{C'} \right\}
\]
and that $\sum_{n=1}^{\infty} \|\Psi_t^*(\pi n)\|^2_{C'} \leq \| f\|^2$ by the Bessel inequality for the orthonormal system $\{\sqrt{2}\cos \pi nx\}_{n \in \mathbb{N}}$. The desired inequality follows now from lemma 2.10 and the fact that, by virtue of (A1), $\sup_{n \in \mathbb{N}} \sum_{\lambda_j \in \Delta_n} 1 < \infty$.

To prove convergence of the series $\sum_{j=0}^{\infty} \Psi_t(\lambda_j)c_j$, we similarly write
\[
\sum_{j=0}^{\infty} \Psi_t(\lambda_j)c_j = \sum_{n=0}^{\infty} \sum_{\lambda_j \in \Delta_n} [\Psi_t(\lambda_j) - \Psi_t(\pi n)]c_j + \sum_{n=0}^{\infty} \sum_{\lambda_j \in \Delta_n} \Psi_t(\pi n)c_j.
\]
Since $(c_j) \in l_2(\mathbb{Z}_+, C')$, the first series converges in $L_2$ due to (2.30) and the Cauchy–Bunyakowski inequality, while the second one due to the fact that the system $\{\sqrt{2}\cos \pi nx\}_{n \in \mathbb{N}}$ is orthonormal in $L_2(0, 1)$.
Lemma 2.12. Assume that \( \tau \in L_2 \), let \( a = ((\lambda_j, \alpha_j))_{j \in \mathbb{Z}_+} \) satisfy (A1) and set
\[
P_j = \Psi_\tau(\lambda_j) \alpha_j \Psi_\tau^*(\lambda_j), \quad \tilde{Q}_j = \Phi_\tau(\lambda_j) \alpha_j \Phi_\tau^*(\lambda_j), \quad j \in \mathbb{N}.
\]
Then the series \( \sum_{j=1}^{\infty} \tilde{Q}_j \) and \( \sum_{j=1}^{\infty} \tilde{P}_j \) converge in the strong operator topology and
\[
\sum_{n=1}^{\infty} \| P_{n,0} - \sum_{j \in \Delta_n} \tilde{P}_j \|^2 < \infty, \quad \sum_{n=1}^{\infty} \| Q_{n,0} - \sum_{j \in \Delta_n} \tilde{Q}_j \|^2 < \infty. \tag{2.34}
\]

Proof. Convergence of the series \( \sum_{j=1}^{\infty} \tilde{Q}_j \) and \( \sum_{j=1}^{\infty} \tilde{P}_j \) follows from lemma 2.11. We prove the first inequality in (2.34); the proof of the second one is similar. Recall that \( P_{n,0} = \Psi_0(\pi n)\Psi_0^*(\pi n) \) and that (see (2.17))
\[
\sum_{\lambda_j \in \Delta_n} \Psi_0(\pi n) \alpha_j \Psi_0^*(\pi n) = P_{n,0} - \Psi_0(\pi n) \beta_n \Psi_0^*(\pi n).
\]
Using this and the relation
\[
P_j - \Psi_0(\pi n) \alpha_j \Psi_0^*(\pi n) = \Psi_\tau(\lambda_j) \alpha_j \left[ \Psi_\tau^*(\lambda_j) - \Psi_0^*(\pi n) \right] + \left[ \Psi_\tau(\lambda_j) - \Psi_0(\pi n) \right] \alpha_j \Psi_0^*(\pi n),
\]
one concludes that
\[
\left\| P_{n,0} - \sum_{j \in \Delta_n} \tilde{P}_j \right\|^2 \leq C_1 \| \beta_n \|^2 + C_2 \sum_{j \in \Delta_n} \| \Psi_\tau(\lambda_j) - \Psi_0(\pi n) \|^2,
\]
where \( C_1 \) and \( C_2 \) are positive constants independent of \( n \in \mathbb{N} \). The result now follows from (2.30) and assumption (A1). \( \square \)

Let \( \tau \in L_2 \) and let \( a = ((\lambda_j, \alpha_j))_{j \in \mathbb{Z}_+} \) satisfy condition (A1). We denote by \( \mathcal{Y}_a, \tau \) and \( \mathcal{Y}_a, \tau \) the operators given by the formulæ
\[
\mathcal{Y}_a, \tau := \sum_{j=0}^{\infty} \Psi_\tau(\lambda_j) \alpha_j \Psi_\tau^*(\lambda_j), \quad \mathcal{Y}_a, \tau := \sum_{j=1}^{\infty} \Phi_\tau(\lambda_j) \alpha_j \Phi_\tau^*(\lambda_j). \tag{2.35}
\]

Remark 2.13. For every \( \tau \in \text{Re} L_2 \) the operators \( \mathcal{Y}_a, \tau \) and \( \mathcal{Y}_a, \tau \) are non-negative. In addition, in view of theorem 2.4 for every \( \tau \in \text{Re} L_2 \) we get the equality
\[
\mathcal{Y}_a, \tau = \mathcal{Y}_a, \tau . \tag{2.36}
\]

Lemma 2.14. Assume that \( \tau \in L_2 \) and that \( a = ((\lambda_j, \alpha_j))_{j \in \mathbb{Z}_+} \) satisfies condition (A1). Then the following equivalences hold:
\[
\left( \mathcal{Y}_a, \tau > 0 \right) \iff (A_3), \quad \left( \mathcal{Y}_a, \tau > 0 \right) \iff (A_4). \tag{2.37}
\]

Proof. Taking into account relations (2.8), we obtain that
\[
\mathcal{Y}_a, \tau = (I + \mathcal{X}_{\tau,N}) \mathcal{Y}_a, 0 (I + \mathcal{X}_{\tau,D}), \quad \mathcal{Y}_a, \tau = (I + \mathcal{X}_{\tau,N}) \mathcal{Y}_a, 0 (I + \mathcal{X}_{\tau,D}). \tag{2.38}
\]
Since the operators \( I + \mathcal{X}_{\tau,N} \) and \( I + \mathcal{X}_{\tau,D} \) are homeomorphisms of the space \( L_2 \), it is enough to prove equivalences (2.37) only for \( \tau = 0 \). Set
\[
\mathcal{X} = \{ d \cos \lambda_j x \mid j \in \mathbb{Z}_+, d \in \text{Ran} \alpha_j \}, \quad \mathcal{Y} = \{ d \sin \lambda_j x \mid j \in \mathbb{N}, d \in \text{Ran} \alpha_j \}
\]
and observe that conditions (A3) and (A4) are equivalent to completeness of the sets \( \mathcal{X} \) and \( \mathcal{Y} \) respectively. Since, in view of (2.7),
\[
\Psi_0^*(\lambda) f = \sqrt{2} \int_{0}^{1} (\cos \lambda x) f(x) \, dx, \quad \Phi_0^*(\lambda) f = \sqrt{2} \int_{0}^{1} (\sin \lambda x) f(x) \, dx,
\]
we conclude that
\[ \ker \mathcal{U}_{a,0} = \bigcap_{j \in \mathbb{Z}_+} \ker \alpha_j \Phi_j(\lambda_j) = \mathcal{Y}^\perp, \quad \ker \mathcal{U}_{a,0} = \bigcap_{j \in \mathbb{N}} \ker \alpha_j \Psi_j(\lambda_j) = \mathcal{X}^\perp. \]

This justifies the equivalences of (2.37).

**Proof of theorem 2.9.** Let \( \tau \in \text{Re } L_2 \) and \( a_\tau = ((\lambda_j(\tau), \alpha_j(\tau)))_{j \in \mathbb{Z}_+} \). Then (A1) holds by theorem 2.5, while (A3) and (A4) are satisfied in view of (2.36) and lemma 2.14. Next, in virtue of lemma 2.12,

\[ \sum_{n=0}^{\infty} \left\| P_{n,0} - \sum_{j \in \mathbb{Z}_+} P_{j,\tau} \right\|_2^2 < \infty. \]

Applying now lemma appendix B.1, we conclude that there exists \( N_0 \in \mathbb{N} \) such that

\[ \sum_{n=0}^{N_0} \sum_{j \in \mathbb{Z}_+} \text{rank } P_{j,\tau} = \sum_{n=0}^{N_0} \text{rank } P_{n,0}, \quad N \geq N_0. \]

(2.39)

It follows from (2.9) and (2.16) that \( \text{rank } P_{j,\tau} = \text{rank } \alpha_j \) and \( \text{rank } P_{n,0} = r \) for all \( j, n \in \mathbb{Z}_+ \). Also, \( \text{rank } \alpha_0 = r = \alpha_0 > 0 \). Together with (2.39) this justifies (A2). The proof is complete.

\[ \square \]

3. The Krein accelerant

3.1. **Proof of theorem 1.8**

Let \( H \) be an even function belonging to \( L_2((-1, 1), M_r) \). We denote by \( \mathcal{H} \) an operator in \( L^2 \) given by

\[ (\mathcal{H} f)(x) := \int_0^1 H(x - t) f(t) \, dt. \]

(3.1)

Set \( \mathcal{H}_a := \chi_a \mathcal{H} \chi_a, \quad a \in [0, 1] \), where \( \chi_a \) is an operator in \( L_2 \) of multiplication by the indicator of the interval \((0, a] \), i.e.

\[ (\chi_a f)(x) = \begin{cases} f(x), & \text{if } x \in (0, a], \\ 0, & \text{if } x \in (a, 1). \end{cases} \]

**Remark 3.1.** Taking into account definition 1.6, it is easy to see that the following equivalences take place:

\[ H \in \mathfrak{H} \iff \forall a \in [0, 1] \quad \ker (\mathcal{I} + \mathcal{H}_a) = \{0\}; \]

\[ H \in \mathfrak{H} \iff H^* \in \mathfrak{H}_2; \]

\[ H \in \text{Re } \mathfrak{H} \iff (\mathcal{I} + \mathcal{H}) > 0. \]

(3.2)

Definition 1.6 also implies that the set \( \mathfrak{H}_2 \) is open and, therefore, the set \( \mathfrak{H}_2 \cap C^1([-1, 1], M_r) \) is dense everywhere in \( \mathfrak{H}_2 \).

Suppose that \( H \in \mathfrak{H}_2 \). Then for every \( a \in [0, 1] \) the operator \( \mathcal{I} + \mathcal{H}_a \) is invertible in the algebra \( \mathcal{B}(L_2) \) of bounded linear operators acting in \( L_2 \). Since the operator \( \mathcal{H}_a \) depends continuously on \( a \in [0, 1] \), the mapping \([0, 1] \ni a \mapsto (\mathcal{I} + \mathcal{H}_a)^{-1} \in \mathcal{B}(L_2)\) is continuous.
Denote by $\Gamma_{a,H}$ the kernel of the integral operator $-\left((I+\mathcal{H}^a)^{-1}\mathcal{H}^a\right)$. Since $\mathcal{H}$ belongs to the class of the Hilbert–Schmidt operators, the mapping

$$[0, 1] \times \mathcal{S}_2 \ni (a, H) \mapsto \Gamma_{a,H} \in L_2((0, 1)^2, M_r)$$

is continuous.

It can be easily seen that, for all $(x, t) \in (0, 1)^2$ and $H \in \mathcal{S}_2$,

$$[\Gamma_{a,H}(x, t)]^* = \Gamma_{a,H^*}(t, x) = \Gamma_{a,H^*}(x, t).$$

(3.4)

For every $H \in \mathcal{S}_2$ and every $(x, t) \in \Omega$ we put

$$\tilde{R}_H(x, t) = \int_0^x H(x - y)H(y - t)\,dy + \int_0^x \int_0^x H(x - u)\Gamma_{x,H}(u, v)H(v - t)\,dv\,du. \quad (3.5)$$

Since functions (3.3) are continuous and the shift $t \mapsto H(\cdot - t)$ acts continuously in $L^2$-topology, one concludes that $\tilde{R}_H \in C(\Omega, M_r)$ and that the mapping

$$\mathcal{S}_2 \ni H \mapsto \tilde{R}_H \in C(\Omega, M_r) \quad (3.6)$$

is continuous.

**Lemma 3.2.** Let $H \in \mathcal{S}_2$. Then the function

$$R_H(x, t) := \begin{cases} \tilde{R}_H(x, t) - H(x - t), & 0 \leq t \leq x \leq 1, \\ 0, & 0 \leq x < t \leq 1, \end{cases} \quad (3.7)$$

belongs to $G_1^2$ and is a unique solution of equation (1.11) in the class $L_2(\Omega, M_r)$. Moreover, the mapping $\mathcal{S}_2 \ni H \mapsto R_H \in G_1^2$ is continuous.

**Proof.** That $R_H$ belongs to $G_1^2$ and the mapping $H \mapsto R_H$ is continuous easily follows from continuity of the mapping (3.6). Taking into account the definition of $\Gamma_{a,H}$, we obtain that

$$H(x - t) + \Gamma_{x,H}(x, t) + \int_0^x \Gamma_{x,H}(x, \xi)H(\xi - t)\,d\xi = 0, \quad (x, t) \in \Omega.$$

Straightforward calculations give

$$\Gamma_{x,H}(x, t) = \tilde{R}_H(x, t) - H(x - t) = R_H(x, t), \quad (x, t) \in \Omega. \quad (3.8)$$

Thus, $R_H$ is a solution of equation (1.11). Uniqueness follows from lemma A.3. □

The proposition below follows from the results of [13] (see also [37]).

**Proposition 3.3.** Suppose that $H \in \mathcal{S}_2 \cap C([-1, 1], M_r)$. Then, for every $x, t \in [0, a]$,

$$\frac{\partial \Gamma_{a,H}}{\partial a}(x, t) = \Gamma_{a,H}(x, a)\Gamma_{a,H}(a, t), \quad \Gamma_{a,H}(x, t) = \Gamma_{a,H}(a - x, a - t). \quad (3.9)$$

Now we study the properties of the mapping $\Theta$ defined by (1.12).

**Lemma 3.4.**

(i) The mapping $\Theta : \mathcal{S}_2 \rightarrow L_2$ is continuous and $\Theta(H^*) = [\Theta(H)]^*$ for every $H \in \mathcal{S}_2$;

(ii) if $H \in \mathcal{S}_2 \cap C^1([-1, 1], M_r)$, then $R_H \in C^1(\Omega, M_r)$ and

$$\frac{\partial}{\partial \xi}[R_H(x, x - t)] - R_H(x, 0)R_H(x, t) = 0, \quad (x, t) \in \Omega. \quad (3.10)$$
**Proof.** (i) Continuity of the mapping $\Theta$ easily follows from definition and the continuity of the mapping $H \mapsto RH$ (see lemma 3.2). By virtue of (3.8) and the relation

$$\Gamma_{\alpha,H}(x,\xi) = -H(x-\xi) - \int_0^x H(x-u)\Gamma_{\alpha,H}(u,\xi) \, du, \quad (x,\xi) \in \Omega,$$

we obtain that

$$\Theta(H)(x) = H(x) - \int_0^x H(x-\xi)H(\xi) \, d\xi - \int_0^x \int_0^x H(x-u)\Gamma_{\alpha,H}(u,v)H(v) \, du \, dv.$$  

Now assume that, in addition, $H \in C([-1,1], M_r)$; using then (3.4) and the second equality in (3.9), we get that

$$[\Theta(H)]^*(x) = H^*(x) - \int_0^x H^*(\xi)H^*(x-\xi) \, d\xi$$

$$- \int_0^x \int_0^x H^*(v)\Gamma_{\alpha,H}(v,u)H^*(x-u) \, du \, dv$$

$$= H^*(x) - \int_0^x H^*(x-y)H^*(y) \, dy$$

$$- \int_0^x \int_0^x H^*(x-y)\Gamma_{\alpha,H}(y,t)H^*(t) \, dy \, dt$$

$$= \Theta(H^*)(x)$$

as claimed. Since the set $\delta_2 \cap C([-1,1], M_r)$ is dense in $\delta_2$ and $\Theta$ is continuous, the equality $\Theta(H^*) = [\Theta(H)]^*$ holds for all $H \in \delta_2$.

(ii) Let $H \in \delta_2 \cap C^1([-1,1], M_r)$ and $R = RH$, so that

$$R(x,t) + H(x-t) + \int_0^x R(x,\xi)H(\xi-t) \, d\xi = 0, \quad (x, t) \in \Omega. \tag{3.11}$$

It follows from proposition 3.3 that the function $a \mapsto \Gamma_{\alpha,H}(u,v)$ is continuously differentiable for $a \geq \max[\mu,\nu]$. Therefore, taking into account (3.9), (3.5) and (3.7), we obtain $R \in C^1(\Omega, M_r)$. Since $R$ is a solution of equation (1.11) and the function $H$ is even, one derives the equality

$$R(x, x-t) + H(t) + \int_0^x R(x, x-\xi)H(\xi-t) \, d\xi = 0, \quad (x, t) \in \Omega. \tag{3.12}$$

Now differentiate (3.12) in the variable $x$ to get

$$\frac{\partial}{\partial x}[R(x, x-t)] + R(x,0)H(x-t) + \int_0^x \frac{\partial}{\partial x}[R(x, x-\xi)]H(\xi-t) \, d\xi = 0. \tag{3.13}$$

Multiplying both sides of equality (3.11) by $R(x,0)$ from the left, we obtain

$$R(x,0)R(x,t) + R(x,0)H(x-t) + \int_0^x R(x,0)R(x,\xi)H(\xi-t) \, d\xi = 0.$$  

Subtracting (3.13) from the above equality, we find that for the function

$$F(x,t) := \frac{\partial}{\partial x}[R(x, x-t)] - R(x,0)R(x,t)$$

the following relation holds for all $(x, t) \in \Omega$:

$$F(x,t) + \int_0^x F(x,\xi)H(\xi-t) \, d\xi = 0.$$  

Lemma A.3 now yields $F \equiv 0$, and the proof is complete. \qed
Proof of theorem 1.8. Assume that \( H \in \mathcal{H}_2 \) and set

\[
Q_{o,H}(x, t) := \frac{1}{2} \left[ R_H \left( x, \frac{x + t}{2} \right) - R_H \left( x, \frac{x - t}{2} \right) \right],
\]

\[
Q_{e,H}(x, t) := \frac{1}{2} \left[ R_H \left( x, \frac{x + t}{2} \right) + R_H \left( x, \frac{x - t}{2} \right) \right],
\]

where \((x, t) \in \Omega\). We extend the functions \( Q_{o,H} \) and \( Q_{e,H} \) onto the square \((0, 1)^2\) by setting \( Q_{o,H} \equiv 0 \) and \( Q_{e,H} \equiv 0 \) in \( \Omega^- = [0, 1]^2 \setminus \Omega \). It follows from (3.7) and the continuity of the mapping (3.6) that \( Q_{o,H} \) and \( Q_{e,H} \) belong to \( G^2 \) and that the mappings

\[
\mathcal{H}_2 \ni H \mapsto Q_{o,H} \in G^2, \quad \mathcal{H}_2 \ni H \mapsto Q_{e,H} \in G^2
\]

are continuous. For \( x \in [0, 1] \) and \( \lambda \in \mathbb{C} \), we consider the functions

\[
\varphi(x, \lambda) := \sin \lambda x + \int_0^x (\sin \lambda t) Q_{o,H}(x, t) \, dt, \\
\psi(x, \lambda) := \cos \lambda x + \int_0^x (\cos \lambda t) Q_{e,H}(x, t) \, dt.
\]

Straightforward transformations give

\[
\varphi(x, \lambda) = \sin \lambda x + \int_0^x (\sin \lambda(x - 2s)) R(x, x - s) \, ds, \\
\psi(x, \lambda) = \cos \lambda x + \int_0^x (\cos \lambda(x - 2s)) R(x, x - s) \, ds.
\]

Suppose that \( H \in \mathcal{H}_2 \cap C^1([-1, 1], M_r) \). According to lemma 3.4, the function \( R_H \) belongs to \( C^1(\Omega, M_r) \) and equality (3.10) holds. Using (3.15) and (3.10), we arrive at the relations

\[
\left( \frac{d}{dx} - \tau \right) \varphi(x, \lambda) = \lambda \psi(x, \lambda), \\
\left( \frac{d}{dx} + \tau \right) \psi(x, \lambda) = -\lambda \varphi(x, \lambda),
\]

where \( \tau = \Theta(H) \). Since \( \varphi(0, \lambda) = 0 \) and \( \psi(0, \lambda) = 1 \), the functions \( \varphi \) and \( \psi \) are solutions of problems (1.3) and (1.4). In view of theorem 2.1 this establishes the relations

\[
Q_{o,H} = K_{r,D}, \quad Q_{e,H} = K_{r,N}, \quad \tau = \Theta(H)
\]

for all \( H \in \mathcal{H}_2 \cap C^1([-1, 1], M_r) \). Since the set \( \mathcal{H}_2 \cap C^1([-1, 1], M_r) \) is dense everywhere in \( \mathcal{H}_2 \) and the mappings \( \Theta, (3.14), L_2 \ni \tau \mapsto K_{r,D} \in G^2_2 \) and \( L_2 \ni \tau \mapsto K_{r,N} \in G^2_2 \) are continuous (see theorem 2.1), it follows that equalities (3.16) hold for all \( H \in \mathcal{H}_2 \). The proof is complete. \( \square \)

3.2. Proof of theorem 1.9

Consider the integral equation

\[
P(x, x - s) + \tau(s) + \int_s^x \tau(u) P(u, s) \, du = 0, \quad (x, s) \in \Omega,
\]

where \( \tau \) is a known function from \( L_2 \), and \( P \) is an unknown function from the class \( G^2_2 \).

Lemma 3.5. Equation (3.17) has at most one solution. If \( H \in \mathcal{H}_2 \cap C^1([-1, 1], M_r) \) and \( \tau = \Theta(H) \), then the solution \( R_H \) of the Krein equation (1.11) is also a solution of equation (3.17).
Proof. To prove uniqueness, it is enough to show that the corresponding homogeneous equation
\[ F(x, x - s) = - \int_0^x \tau(u) F(u, s) \, du, \quad (x, s) \in \Omega, \tag{3.18} \]
has only zero solution in \( G_2^+ \). Let a function \( F \in G_2^+ \) be a solution of (3.18) and set
\[ g(x) := \int_0^x \|F(x, s)\| \, ds, \quad x \in [0, 1]. \]
Then \( g \) is continuous on \([0, 1]\) and, in view of (3.18),
\[ g(x) = \int_0^x \|F(x, x - s)\| \, ds \leq \int_0^x \|\tau(u)\| \|F(u, s)\| \, ds \, du \leq \int_0^x \|\tau(u)\| g(u) \, du. \]
Now Gronwall’s inequality implies that \( g \equiv 0 \), whence \( F = 0 \).

Let \( H \in \mathcal{S}_2 \) and \( \tau = \Theta(H) \). Since \( \tau = \Theta(H) = - R_H (\cdot, 0) \), we get by virtue of equality (3.10) that
\[ R_H (x, x - t) + \tau(t) + \int_t^x \tau(\xi) R_H (\xi, t) \, d\xi = 0, \quad (x, t) \in \Omega. \tag{3.19} \]
Thus, \( R_H \) is a solution of equation (3.17). \( \square \)

Lemma 3.6. For every \( \tau \in L_2 \) equation (3.17) has a unique solution \( P_\tau \) in \( G_2^+ \) and the mapping \( L_2 \ni \tau \mapsto P_\tau \in G_2^+ \) is continuous. Moreover, if \( \tau \in C([0, 1], M_c) \), then \( P_\tau \) is continuous in \( \Omega \), the function \( (x, t) \mapsto P_\tau (x, x - t) \) has a continuous partial derivative in the variable \( x \) and
\[ \frac{\partial}{\partial x} [P_\tau (x, x - t)] = - \tau(x) P_\tau (x, t), \quad (x, t) \in \Omega. \tag{3.20} \]
Proof. For every \( n \in \mathbb{N} \) and \((x, s) \in \Omega\), we set
\[ P_{\tau,n+1}(x, x - s) := \tau(s), \quad P_{\tau,n+1}(x, x - s) := \int_s^x \tau(u) P_{\tau,n}(u, s) \, du. \tag{3.21} \]
It follows by induction that the recurrence relation (3.20) yields the equality
\[ P_{\tau,n+1}(x, s) = \int_{\Pi_{n+1}^n} \tau \left( x - s + \xi_n(y) \right) \tau(n_1) \cdots \tau(n_2) \, dy_1 \cdots dy_n, \tag{3.21} \]
where, for \( n \in \mathbb{N} \) and \((x, s) \in \Omega\), we set \( \xi_n(y) := \sum_{j=1}^n (-1)^{j+1} y_j \) and
\[ \Pi_n^2 := \{ y = (y_1, \ldots, y_n) \in \mathbb{R}^n \mid 0 \leq y_n \leq y_{n-1} \leq \cdots \leq y_1 \leq x - s + \xi_n(y) \leq x \}. \]
We extend the functions \( P_{\tau,n}, n \in \mathbb{N} \), onto the square \([0, 1] \times [0, 1]\) by setting \( P_{\tau,n} \equiv 0 \) in \( \Omega^- \). Since the function \( \tau \) belongs to \( L_2 \), for any \( n \in \mathbb{N} \) the functions \( P_{\tau,n+1} \) are continuous in the square \([0, 1]^2\). Taking into account (3.20) and using the Cauchy–Bunyakowski inequality and Fubini’s theorem, we find that for every \( n \in \mathbb{N} \) and \( x \in [0, 1] \) it holds
\[ \| P_{\tau,n+1}(x, \cdot) \|_{L_2}^2 \leq \int_0^1 \| P_{\tau,n+1}(x, s) \|^2 \, ds \]
\[ \leq \frac{1}{n!} \int_0^1 \int_{\Pi_n^2} \| \tau \left( x - s + \xi_n(y) \right) \|^2 \| \tau(n_1) \|^2 \cdots \| \tau(n_2) \|^2 \, dy_1 \cdots dy_n \, ds \]
\[ \leq \frac{1}{n!} \int_{\Pi_{n+1}^n} \| \tau(n_1) \|^2 \cdots \| \tau(n_{n+1}) \|^2 \, dn_1 \cdots dn_{n+1} \]
\[ = \frac{1}{(n!)(n+1)!} \left( \int_0^1 \| \tau(\xi) \|^2 \, d\xi \right)^{n+1}, \]

where $\Pi_n = \{ t := (t_1, \ldots, t_n) \in \mathbb{R}^n \mid 0 \leq t_n \leq \cdots \leq t_1 \leq 1 \}$. Therefore,
\[
\| P_{r,n+1}(x, \cdot) \|_{L_2} \leq (n!)^{-1} \| r \|_{L_2}^{n+1}.
\]
Similarly we obtain that
\[
\| P_{r,n+1}(\cdot, x) \|_{L_2} \leq (n!)^{-1} \| r \|_{L_2}^{n+1}.
\]
Now, taking into account continuity of the functions $P_{r,n+1}$, we conclude that $P_{r,n+1}$ belongs to $G^{n+1}_2$, and, moreover,
\[
\| P_{r,n+1} \|_{G^n_2} \leq (n!)^{-1} \| r \|_{L_2}^{n+1}, \quad n \in \mathbb{N}.
\]
Taking into account (3.20) and (3.22), we see that the function
\[
P_r := \sum_{n=1}^{\infty} (-1)^n P_{r,n}
\]
is a solution of equation (3.17). Uniqueness of solution of equation (3.17) follows from lemma 3.5.

Next we prove continuous dependence of $P_r$ on $r$. We shall use the inequality
\[
\left( \prod_{k=1}^{n} a_k - \prod_{k=1}^{n} b_k \right)^2 \leq n \sum_{k=1}^{n} |a_k - b_k|^2 \prod_{j \neq k} (|a_j| + |b_j|)^2
\]
for arbitrary $(a_k)_{k=1}^{n}, (b_k)_{k=1}^{n} \in \mathbb{C}^n$, which easily follows from the relation
\[
\prod_{k=1}^{n} a_k - \prod_{k=1}^{n} b_k = \sum_{k=1}^{n} a_1 \cdots a_{k-1}(a_k - b_k) b_{k+1} \cdots b_n
\]
after applying the Cauchy–Bunyakowski inequality. It follows from (3.22) and (3.23) that it is sufficient to prove continuity of the mappings
\[
L_2 \ni r \mapsto P_{r,n} \in G^{n+1}_2, \quad n \in \mathbb{N}.
\]
Let $\tau_1, \tau_2 \in L_2$. Reasoning similarly as when deriving the estimate (3.22) and using (3.24), we obtain the estimate
\[
\| P_{\tau_1,n+1} - P_{\tau_2,n+1} \|_{G^n_2} \leq \left( \frac{n+1}{n!} \right)^2 \| \tau_1 - \tau_2 \|_{L_2}^2 \| \tau_1 \|_{L_2} \| \tau_2 \|_{L_2} 2n,
\]
which yields continuity of the mappings (3.25).

Finally, assume that $\tau \in C([-1, 1], M_r)$. Since the function $P_r$ verifies (3.17), $P_r$ is continuous, and the function $(x, s) \mapsto P_r(x, x - s)$ is absolutely continuous in $x$. Differentiation of (3.17) in the variable $x$ leads to (3.19). The proof is complete. □

**Proof of theorem 1.9.** For any $r \in L_2$ we denote by $\mathcal{P}_r$ the integral operator with kernel $P_r$. It follows from lemma 3.6 that $\mathcal{P}_r$ belongs to $\Theta^2_2$ and that the mapping $L_2 \ni r \mapsto \mathcal{P}_r \in \Theta^2_2$ is continuous. Hence in view of lemma A.4 we obtain that the mapping $\mathcal{Y}$ that acts from $L_2$ to $L_2((-1, 1), M_r)$ via
\[
\mathcal{Y}(\tau)(x) = \left[ (I + \mathcal{P}_r)^{-1} \tau \right]|x|, \quad x \in (-1, 1),
\]
is also continuous. Next we show that the mapping $\mathcal{Y}$ is the left inverse of $\Theta$, i.e. that the relation
\[
\mathcal{Y}(\Theta(H)) = H
\]
holds for all $H \in \mathcal{H}_2$. Since the set $\mathcal{H}_2 \cap C([-1, 1], M_r)$ is dense everywhere in $\mathcal{H}_2$ and the mappings $\Theta$ and $\mathcal{Y}$ are continuous, it is enough to prove equality (3.26) for
\( H \in \mathcal{F}_2 \cap C^1([-1, 1], M_r) \). Assume \( H \) is such and set \( \tau = \Theta(H) \). Then it follows from lemmas 3.5 and 3.6 that \( P_\tau = R_H \). Therefore, recalling (1.12), we obtain that

\[
\tau(x) = [\Theta(H)](x) = H(x) + \int_0^x P_\tau(x, \xi) H(\xi) \, d\xi, \quad x \in (0, 1),
\]

and, therefore, \( \Upsilon(\Theta(H)) = \Upsilon(\tau) = H \).

It remains to show that \( \Upsilon \) is also the right inverse of \( \Theta \), i.e. that, for every \( \tau \in L_2 \),

\[
\Upsilon(\tau) \in H^2 \quad \text{and} \quad \Theta(\Upsilon(\tau)) = \tau.
\]

Continuity of the mappings \( \Theta \) and \( \Upsilon \) implies that it is enough to prove (3.27) for continuous \( \tau \). In this case for \( H := \Upsilon(\tau) \) we get the relation

\[
\tau(x) = H(x) + \int_0^x P_\tau(x, \xi) H(\xi) \, d\xi, \quad x \in [0, 1].
\]

According to lemma 3.6, the function \( P_\tau \) is continuous in \( \Omega \), and, therefore, the function \( H \) is continuous as well. Consider a function \( F \) defined for \( (x, t) \in \Omega \) by

\[
F(x, t) := P_\tau(x, x - t) + H(t) + \int_0^t P_\tau(x, x - \xi) H(\xi - t) \, d\xi.
\]

Then the function \( F \) is continuous in \( \Omega \), has a continuous partial derivative in the variable \( x \) by virtue of lemma 3.6 and

\[
\frac{\partial}{\partial x} [F(x, x - t)] = -\tau(x)F(x, t), \quad (x, t) \in \Omega.
\]

Since

\[
F(t, 0) = -\tau(t) + H(t) + \int_0^t P_\tau(t, \xi) H(\xi) \, d\xi = 0, \quad t \in [0, 1],
\]

we obtain that

\[
F(x, x - t) = -\int_t^x \tau(\xi)F(\xi, t) \, d\xi, \quad t \leq x \leq 1.
\]

However, as was shown in the proof of lemma 3.5, the last equality is only possible for \( F \equiv 0 \). Therefore, \( P_\tau \) is a solution of the Krein equation (1.11) with \( H = \Upsilon(\tau) \). Now proposition A.2 implies that \( H \) is an accelerant; moreover, \( \Theta(\Upsilon(\tau)) = \tau \) in view of (3.28). Finally, the equality \( \Theta(H^*) = [\Theta(H)]^* \) follows from lemma 3.4 (i). The proof is complete. \( \Box \)

3.3. The operators \( \mathcal{H}_o \) and \( \mathcal{H}_e \)

For any \( H \in L_2((-1, 1), M_r) \) we denote by \( \mathcal{H}_o \) and \( \mathcal{H}_e \) integral operators that act in \( L_2 \) by the formulae

\[
(\mathcal{H}_o f)(x) := \int_0^1 H_o(x, t) f(t) \, dt, \quad (\mathcal{H}_e f)(x) := \int_0^1 H_e(x, t) f(t) \, dt,
\]

where

\[
H_o(x, t) := \frac{1}{2} \left[ H \left( \frac{x - t}{2} \right) - H \left( \frac{x + t}{2} \right) \right],
\]

\[
H_e(x, t) := \frac{1}{2} \left[ H \left( \frac{x - t}{2} \right) + H \left( \frac{x + t}{2} \right) \right].
\]
Proposition 3.7. Let $H \in S_2$ and $\tau = \Theta(H)$. Then
\begin{equation}
(I + \mathcal{K}_{\nu, N})(I + \mathcal{H}_c)(I + \mathcal{K}_{\nu, D}) = I = (I + \mathcal{K}_{\nu, D})(I + \mathcal{H}_c)(I + \mathcal{K}_{\nu, D}).
\end{equation}

Proof. We shall prove the first equality in (3.30); the second one is proved similarly. It follows from the Krein equation (1.11) that
\begin{align}
R_H(x, \frac{x-t}{2}) + H\left(\frac{x+t}{2}\right) + \int_0^x R_H(x, \xi)H\left(\xi - \frac{x-t}{2}\right) d\xi = 0,
\end{align}
\begin{align}
R_H(x, \frac{x+t}{2}) + H\left(\frac{x-t}{2}\right) + \int_0^x R_H(x, \xi)H\left(\xi + \frac{x+t}{2}\right) d\xi = 0
\end{align}
for $(x, t) \in \Omega$. Combining these relations and taking into account theorem 1.8 proved above, we arrive at the equality
\begin{equation}
K_{\tau, N}(x, t) + H_e(x, t) + \int_0^x K_{\tau, N}(x, \xi)H_e(\xi, t) d\xi = 0
\end{equation}
for all $(x, t) \in \Omega$. By virtue of lemma 3.4, $\Theta(H^*) = [\Theta(H)]^* = \tau^*$. Thus, according to (3.31), we find that for $(x, t) \in \Omega$ it holds
\begin{equation}
K_{\tau, N}(x, t) + H_e^*(x, t) + \int_0^x K_{\tau, N}(x, \xi)H_e^*(\xi, t) d\xi = 0.
\end{equation}
Equalities (3.31), (3.32) and theorem A.1 now lead to the relation
\begin{equation}
I + \mathcal{H}_c = (I + \mathcal{K}_{\tau, N})^{-1}(I + \mathcal{K}_{\tau, N})^{-1},
\end{equation}
which yields the first equality in (3.30). The lemma is proved.

Proposition 3.8. If ker$(I + \mathcal{H}_c) = \{0\}$, then $I + \mathcal{H}_c$ is a bijection of $W^1_2$.

Proof. It is enough to prove that $\mathcal{H}_c$ maps $W^1_2$ to itself and that the operator $\mathcal{H}_c : W^1_2 \rightarrow W^1_2$ is compact. If $H \in C^1([-1, 1], M_\nu)$, then integration by parts gives
\begin{equation}
(\mathcal{H}_c f)' = \mathcal{H}_c f' - H_e(-1, 1) f(1), \quad f \in W^1_2.
\end{equation}
Using the fact that the operator of differentiation considered on the domain $W^1_2$ is closed in $L_2$ and that the space $C^1([-1, 1], M_\nu)$ is dense in $L_2([-1, 1], M_\nu)$, we conclude that equality (3.33) holds also for an arbitrary function $H \in L_2([-1, 1], M_\nu)$. Equality (3.33) implies that the operator $\mathcal{H}_c$ maps the space $W^1_2$ to itself; moreover,
\begin{equation}
\|\mathcal{H}_c\|_{W^1_2 \rightarrow W^1_2} \leq C\|H\|_{L_2((-1, 1), M_\nu)},
\end{equation}
where $C$ is a constant independent of $H$. If the function $H$ is a trigonometric polynomial, then $\mathcal{H}_c$ is an operator of finite rank. Approximating an arbitrary function $H \in L_2((-1, 1), M_\nu)$ by a sequence of trigonometric polynomials and taking into account (3.34), we conclude that the operator $\mathcal{H}_c : W^1_2 \rightarrow W^1_2$ is compact.

4. The inverse spectral problem

4.1. Proof of theorem 1.7

We shall base the proof of theorem 1.7 on the following two lemmas.

Lemma 4.1. Let for a sequence $a = (\lambda_j, \alpha_j)_{j \in \mathbb{Z}}$, condition $(A_1)$ hold and $v = v^a$. Then the limit (1.10) exists in the topology of the space $L_2((-1, 1), M_\nu)$. Moreover, the function $H = H_e$ is even and hermitian, and for the operators $\mathcal{H}_c$ and $\mathcal{H}_o$ the following relations hold:
\begin{equation}
I + \mathcal{H}_c = \mathcal{U}_{a, 0}, \quad I + \mathcal{H}_o = \mathcal{U}_{a, 0}.
\end{equation}
Proof. In view of (1.7) and (1.8) we have to show that the series
\[ \sum_{n=1}^{\infty} \left[ \cos(2\pi nx) I - \sum_{\lambda_j \in \Delta_n} \cos(2\lambda_j x) \alpha_j \right] \]
converges in \( L_2((-1, 1), \mathcal{M}_r) \). Let \( \tilde{\lambda}_j \) and \( \beta_n \) be given by (2.17). Then
\[ \cos 2\lambda_j x = \cos 2\pi n x - 2x\tilde{\lambda}_j \sin 2\pi n x - \vartheta_{n,j} (x), \quad n \in \mathbb{N}, \quad \lambda_j \in \Delta_n, \]
where
\[ \vartheta_{n,j} (x) := (1 - \cos 2x\tilde{\lambda}_j) \cos 2\pi n x - (2\tilde{\lambda}_j x - \sin 2\tilde{\lambda}_j x) \sin 2\pi n x. \]
This yields the equality
\[ \cos(2\pi nx) I - \sum_{\lambda_j \in \Delta_n} \cos(2\lambda_j x) \alpha_j = \cos(2\pi nx) \beta_n + 2x \sin(2\pi nx) \gamma_n + \sum_{\lambda_j \in \Delta_n} \vartheta_{n,j} (x) \alpha_j, \]
in which \( \gamma_n := \sum_{\lambda_j \in \Delta_n} \tilde{\lambda}_j \alpha_j \). According to condition (A_1)
\[ \sum_{n=1}^{\infty} \| \beta_n \|^2 < \infty, \quad \sup_{j \in \mathbb{Z}_n} \| \alpha_j \| < \infty, \quad \sum_{j=1}^{\infty} |\tilde{\lambda}_j|^2 < \infty, \quad \sup_{n \in \mathbb{N}} \sum_{\lambda_j \in \Delta_n} 1 < \infty, \quad (4.3) \]
which, in particular, implies that \( \sum_{n=1}^{\infty} \| \gamma_n \|^2 < \infty \). From the above we conclude that the series \( \sum_{n=1}^{\infty} \cos(2\pi nx) \beta_n \) and \( \sum_{n=1}^{\infty} \sin(2\pi nx) \gamma_n \) converge in the topology of the space \( L_2((-1, 1), \mathcal{M}_r) \). It is easy to see that
\[ \max_{|s| \leq 1} |\vartheta_{n,j} (s)| \leq 6 |\tilde{\lambda}_j|^2, \quad n \in \mathbb{N}, \quad \lambda_j \in \Delta_n. \]
Therefore,
\[ \sum_{n=1}^{\infty} \sum_{\lambda_j \in \Delta_n} \| \vartheta_{n,j} \| \| \alpha_j \| \| L_2((-1, 1), \mathcal{M}_r) \| \leq 6 \left( \sup_{j \in \mathbb{Z}_n} \| \alpha_j \| \right) \sum_{j=1}^{\infty} |\tilde{\lambda}_j|^2 < \infty. \]
Combining the above, we conclude that the series (4.2) converges. The fact that the function \( H_r \) is even and hermitian is obvious.

Now construct operators \( \mathcal{H} \) and \( \mathcal{H}_0 \) via formulae (3.29) taking there \( H = H_r \). We shall prove the first equality in (4.1), since the second one is proved similarly. Let
\[ H_{r,n}(x) := \int_0^{\pi(n+1)/2} 2 \cos(2\lambda x) d(v - v_0)(\lambda), \quad x \in [-1, 1], \quad n \in \mathbb{N}. \]
We denote by \( \mathcal{H} \) the operator in \( L_2 \) that acts via the formula
\[ (\mathcal{H} f)(x) := \frac{1}{2} \int_0^{\pi(n+1)/2} \left[ H_{r,n} \left( x - t \right) + H_{r,n} \left( x + t \right) \right] f(t) \, dt, \quad x \in [-1, 1]. \]
Since \( H_{r,n} \to H_r \) as \( n \to \infty \), the sequence \( (\mathcal{H} \mathcal{H}_n)_{n \in \mathbb{N}} \) converges in the operator norm to the operator \( \mathcal{H} \). It is easy to see that
\[ \mathcal{H} \mathcal{H}_n = \sum_{\lambda_j \leq \pi(n+1)/2} \Psi_0(\lambda_j) \alpha_j \Psi_0(\lambda_j) - \sum_{k=0}^{n} P_{k,0}. \]
The required equality follows now from (2.15) and (2.35).

Lemma 4.2. Let a sequence \( \alpha = ((\lambda_j, \alpha_j))_{j \in \mathbb{Z}_n} \) verifies condition (A_1). Then the following implications are true: (A_3) \( \implies \) (A_4) and [(A_4) \& (\lambda_0 = 0, \alpha_0 > 0)] \( \implies \) (A_3).
Proof. Let conditions (A1) and (A3) hold. Then it follows from (4.1) and (2.37) that $\ker(\mathcal{I} + \mathcal{H}_0) = \{0\}$. Hence, taking into account proposition 3.8, we obtain that $(\mathcal{I} + \mathcal{H}_0)W_1^1 = W_1^2$. Therefore, for an arbitrary $g \in L_2$ there exists $f \in W_1^2$ such that $g = (\mathcal{I} + \mathcal{H}_0)f'$. Put $c_j := \alpha_j \Psi_0^*(\lambda_j) f$, $j \in \mathbb{Z}_+$. In view of (4.1) and (2.35)

$$(\mathcal{I} + \mathcal{H}_0)f = \sum_{j=0}^{\infty} \Psi_0(\lambda_j)c_j.$$  

(4.4)

Taking into account the relation

$$\lambda_j \Psi_0^*(\lambda_j) f = \sqrt{2} \int_{0}^{1} (\lambda_j \cos \lambda_j t) f(t) \, dt = (-1)^{\eta} \sqrt{2} (\sin \lambda_j) f(1) - \Phi_0^*(\lambda_j) f', \quad \lambda_j \in \Delta_0,$$

the inequalities in (4.3) and lemma 2.11, we conclude that $\sum_{j=0}^{\infty} \|\lambda_j c_j\|_{L^2} < \infty$. In view of lemma 2.11 the series in (4.4) is termwise differentiable and therefore $g = - \sum_{j=1}^{\infty} \lambda_j \Phi_0(\lambda_j)c_j$. Since $g$ is arbitrary, the system $\{d \sin \lambda_j x \mid j \in \mathbb{N}, d \in \text{Ran} \alpha_j\}$ is complete in the space $L_2$ and the implication $(A_3) \implies (A_4)$ follows.

Now let conditions (A1), (A4) hold and let $\lambda_0 = 0$ and $\alpha_0 > 0$. Then $\ker(\mathcal{I} + \mathcal{H}_0) = \{0\}$ by virtue of (4.1) and (2.37). Since the operator $\mathcal{H}_0 : L_2 \to L_2$ is compact, $\text{Ran}(\mathcal{I} + \mathcal{H}_0) = L_2$. Let $J : L_2 \to L_2$ be an integration operator, i.e.

$$(Jf)(x) = \int_{0}^{1} f(t) \, dt.$$  

The range of the operator $J$ is everywhere dense in $L_2$. Therefore, the range of the operator $J(\mathcal{I} + \mathcal{H}_0)$ is also everywhere dense in $L_2$. In view of (4.1) for an arbitrary $f \in L_2$ we get

$$J(\mathcal{I} + \mathcal{H}_0)f = \sum_{j=1}^{\infty} \Phi_0(\lambda_j) \alpha_j \Psi_0^*(\lambda_j) f = \sum_{j=1}^{\infty} \frac{\sqrt{2}}{\lambda_j} d_j - \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \Psi_0(\lambda_j) d_j,$$

where $d_j := \alpha_j \Phi_0^*(\lambda_j) f$. Since the matrix $\alpha_0$ is nonsingular, the vector $\sum_{j=1}^{\infty} \sqrt{2} d_j / \lambda_j$ is in the range of $\alpha_0$. Therefore, the closed linear hull of the system

$$\{\{\cos \lambda_j x \} d \mid j \in \mathbb{Z}_+, d \in \text{Ran} \alpha_j\}$$

contains the range of $J(\mathcal{I} + \mathcal{H}_0)$ and thus (A3) holds. \qed

Proof of theorem 1.7. In view of lemma 4.1 only the second part of the theorem needs to be proved. In fact, it suffices to prove that conditions (A1) and (A3) imply that $H \in \text{Re} \mathcal{F}_2$.

Let therefore conditions (A1) and (A3) hold. It follows then from (4.1), (2.37), and lemma 4.2 that

$$\mathcal{I} + \mathcal{H}_0 > 0, \quad \mathcal{I} + \mathcal{H}_0^* > 0.$$  

(4.5)

Let us show that (4.5) implies positivity of the operator $\mathcal{I} + \mathcal{H}$. Let $L_{2,e}$ and $L_{2,o}$ be subspaces of $L_2$ consisting of functions that are respectively even and odd with respect to $\frac{1}{2}$, i.e.

$L_{2,e} := \{f \in L_2 \mid f(1-x) = f(x)\}, \quad L_{2,o} := \{f \in L_2 \mid f(1-x) = -f(x)\}.$

$L_{2,e}$ and $L_{2,o}$ are invariant subspaces of the operator $\mathcal{H}$. Consider unitary operators $\mathcal{H}_e : L_2 \to L_{2,e}$ and $\mathcal{H}_o : L_2 \to L_{2,o}$ acting via the formulae

$(\mathcal{H}_e g)(x) = \text{sgn}(2x-1)g(2x-1), \quad (\mathcal{H}_o g)(x) = g(2x-1), \quad x \in (0, 1).$

Simple calculations show that for an arbitrary $g \in L_2$ we have

$$(\mathcal{I} + \mathcal{H})\mathcal{H}_e g |\mathcal{H}_e g) = ((\mathcal{I} + \mathcal{H}_e)g) |g), \quad ((\mathcal{I} + \mathcal{H})\mathcal{H}_o g |\mathcal{H}_o g) = ((\mathcal{I} + \mathcal{H}_o)g) |g).$$

Taking into account (4.5) and the equality $L_{2,o} \oplus L_{2,e} = L_2$, we obtain that $\mathcal{I} + \mathcal{H} > 0$. Therefore, due to (3.2) the function $\hat{H}$ belongs to $\text{Re} \mathcal{F}_2$. \qed
4.2. Proof of sufficiency in theorems 1.1 and 1.2

In section 2 the necessity parts of theorems 1.1 and 1.2 were justified, and we still have to establish the sufficiency parts. By virtue of lemma 4.2 it suffices to prove the following theorem.

**Theorem 4.3.** Let for a sequence \( a = ((\lambda_j, \alpha_j))_{j \in \mathbb{Z}} \) conditions \((A_1)-(A_3)\) hold and let \( \nu = \nu^a \) and \( \tau = \Theta(H_\nu) \). Then \( \tau \in \mathbb{R} \mathbf{L}_2 \) and \( a = a_\tau \).

First we prove the following lemma.

**Lemma 4.4.** Let the assumptions of theorem 4.3 hold. Then the formulae

\[
\hat{P}_j = \Psi_\tau(\lambda_j) \alpha_j \Psi_\tau^*(\lambda_j), \quad \hat{Q}_k = \Phi_\tau(\lambda_k) \alpha_k \Phi_\tau^*(\lambda_k)
\]

(4.6)

determine complete systems \( \{\hat{P}_j\}_{j \in \mathbb{Z}} \) and \( \{\hat{Q}_k\}_{k \in \mathbb{N}} \) of pairwise orthogonal projectors.

**Proof.** Taking into account proposition 3.7 and equalities (4.1) and (2.38), we arrive at the equality

\[
\sum_{n=0}^{\infty} \sum_{\lambda_j \in \Delta_n} \hat{P}_j = I = \sum_{n=1}^{\infty} \sum_{\lambda_j \in \Delta_n} \hat{Q}_j,
\]

(4.7)

and that both series converge in the strong operator topology. We show that \( \{\hat{P}_j\}_{j \in \mathbb{Z}} \) is the sequence of pairwise orthogonal projectors; the proof for \( \{\hat{Q}_k\}_{k \in \mathbb{N}} \) is analogous.

In view of lemma 2.12,

\[
\sum_{n=1}^{\infty} \sum_{\lambda_j \in \Delta_n} \|P_{n,0} - \sum_{\lambda_j \in \Delta_n} \hat{P}_j\| < \infty.
\]

Thus, there exists a natural \( N_0 \) such that

\[
\sum_{n=N_0}^{\infty} \sum_{\lambda_j \in \Delta_n} \|P_{n,0} - \sum_{\lambda_j \in \Delta_n} \hat{P}_j\| < 1.
\]

(4.8)

Moreover, \( N_0 \) can be taken so large that \((A_2)\) holds, i.e. that for every \( N \geq N_0 \) it holds

\[
\sum_{n=0}^{N} \sum_{\lambda_j \in \Delta_n} \text{rank} \alpha_j = (N+1)r.
\]

(4.9)

Let us fix \( N > N_0 \) and set

\[
A = \sum_{n=N+1}^{\infty} \sum_{\lambda_j \in \Delta_n} \hat{P}_j, \quad P = \sum_{n=0}^{N} P_{n,0}, \quad A_k = \sum_{\lambda_j \in \Delta_k} \hat{P}_j.
\]

Since \( \text{rank} \hat{P}_j = \text{rank} \alpha_j \) for all \( j \in \mathbb{Z} \) in view of (2.9) and (4.6), we conclude by virtue of (4.9) that, for all \( k > N_0 \),

\[
\text{rank} A_k \leq \sum_{\lambda_j \in \Delta_k} \text{rank} \hat{P}_j = \sum_{\lambda_j \in \Delta_k} \text{rank} \alpha_j = r = \text{rank} P_{k,0}.
\]

(4.10)

Since \( (P_{n,0})_{n \in \mathbb{Z}_a} \) is a complete sequence of pairwise orthogonal projectors, by virtue of (4.8), (4.10) and lemma B.2 we obtain that

\[
\text{codim Ran } A \geq \text{rank } P = (N+1)r.
\]

(4.11)
It follows from (4.7), (4.9) and (4.11) that

\[ A + \sum_{n=0}^{N} \sum_{\lambda_j \in \Delta_n} \hat{P}_j = I, \]

\[ \sum_{n=0}^{N} \sum_{\lambda_j \in \Delta_n} \text{rank} \hat{P}_j = \sum_{n=0}^{N} \sum_{\lambda_j \in \Delta_n} \text{rank} \alpha_j = (N + 1) r \leq \text{codim Ran} \ A. \]

The above relations imply by virtue of lemma B.3 that the set \( \{ \hat{P}_j : \lambda_j \leq \pi/2 \} \) is a set of pairwise orthogonal projectors. Since \( N \) is arbitrary, we conclude that \( (\hat{P}_j)_{j \in \mathbb{Z}_+} \) is a sequence of pairwise orthogonal projectors. The proof is complete.

**Proof of theorem 4.3.** Let the assumptions of theorem 4.3 hold and let \( \hat{P}_j \) and \( \hat{Q}_j \) be operators of lemma 4.2. It follows from theorems 1.7 and 1.9 that \( \tau \in \text{Re L}_2 \). Theorem 4.3 will be proved if we show that

\[ \text{Ran} \ \hat{P}_j \subset \ker (S_\tau - \lambda_j^2 I), \quad j \in \mathbb{Z}_+. \tag{4.12} \]

Indeed, (4.12) together with (4.7) implies that \( \lambda_j(\tau) = \lambda_j \) for all \( j \in \mathbb{Z}_+ \). The last equality and (4.12) yield the inequality \( P_{j,\tau} - \hat{P}_j \geq 0, \ j \in \mathbb{Z}_+ \). However, in view of (4.7) and (2.15) we have

\[ \sum_{j=0}^{\infty} (P_{j,\tau} - \hat{P}_j) = 0. \]

Thus, \( P_{j,\tau} - \hat{P}_j = 0, \ j \in \mathbb{Z}_+ \), and, therefore, (see (2.16) and (4.6))

\[ \Psi_\tau(\lambda_j)(\alpha_j(\tau) - \alpha_j)\Psi_\tau^*(\lambda_j) = 0, \quad j \in \mathbb{Z}_+. \]

Now by virtue of (2.9) we obtain the equality \( \alpha_j(\tau) = \alpha_j, \ j \in \mathbb{Z}_+ \), i.e. the equality \( a = \alpha \).

It thus indeed only remains to prove (4.12). In view of (2.7) and (2.1) it is enough to show that \( \psi(1, \lambda_j, \tau)\alpha_j = 0, \ j \in \mathbb{Z}_+ \). Let \( j, k \in \mathbb{Z}_+ \) and \( c, d \in \mathbb{C}^r \). Since \( \tau = r^* \), taking into account (2.7) and (2.1) and integrating by parts, we find that

\[ \lambda_j(\Psi(\lambda_j)c|\Psi(\lambda_j)d)_{C_r^c} = \left( \frac{d}{dx} + \tau \right) \Phi(\lambda_j)c|\Psi(\lambda_j)d)_{C_r^c} = 2\psi(1, \lambda_j, \tau)c|\psi(1, \lambda_k, \tau)d)_{C_r^c} + \lambda_k(\Phi(\lambda_j)c|\Phi(\lambda_k)d)_{C_r^c} \]

and, therefore,

\[ \lambda_j(\Psi^*(\lambda_k)\Psi(\lambda_j) - \lambda_k(\Phi^*(\lambda_k)\Phi(\lambda_j)) = 2\psi^*(1, \lambda_k, \tau)\psi(1, \lambda_j, \tau). \]

It follows from the orthogonality of \( (\hat{P}_j)_{j \in \mathbb{Z}_+} \) and \( \hat{Q}_j \) \( j \in \mathbb{N} \) and relations (2.9) that

\[ a_k(\Psi^*(\lambda_k)|\Psi(\lambda_j)) = 0 = a_k(\Phi^*(\lambda_k)|\Phi(\lambda_j)) \alpha_j, \quad k \neq j. \]

Using this, we obtain the equality

\[ \left( \sum_{\lambda_k \in \Delta_n} (-1)^n a_k(\Psi^*(1, \lambda_k, \tau)\psi(1, \lambda_j, \tau)\alpha_j = 0, \quad n \in \mathbb{N}, \ \lambda_j \notin \Delta_n. \tag{4.13} \]

Since (see (2.13))

\[ \psi(1, \lambda_k, \tau) = \cos \lambda_k I + \int_0^1 (\cos \lambda_k t) K_{r,N}(1, t) \, dt, \quad k \in \mathbb{N}, \]

we can use the Riemann–Lebesgue lemma and the asymptotic behaviour of the sequences \( (\lambda_k) \) and \( (a_k) \) to prove that

\[ \lim_{n \to \infty} \sum_{\lambda_k \in \Delta_n} (-1)^n \psi(1, \lambda_k, \tau)\alpha_k = \lim_{n \to \infty} \sum_{\lambda_k \in \Delta_n} \alpha_k = I. \]

Passing to the limit in (4.13) as \( n \to \infty \), we derive the equality \( \psi(1, \lambda_j, \tau)\alpha_j = 0. \)

Theorem 4.3 is proved.
4.3. Proof of proposition 1.3 and theorems 1.5 and 1.10

Proof of proposition 1.3. It follows from theorems 1.1 and 1.2 and lemma 4.2 proved above that it is sufficient to prove the implication \((a \in \mathfrak{M}) \implies (\lambda_0 = 0, \omega_0 > 0)\). This fact, however, is an immediate corollary of theorem 2.4.

Proof of theorem 1.5. (i) Let \(\tau_1, \tau \in \text{Re} \mathfrak{M}_2\) and \(a_{\tau_1} = a = a_\tau\). In view of (2.36) \(\mathcal{U}_{a,\tau_1} = \mathcal{I}_c = \mathcal{U}_{a,\tau}\). We next prove the following implication:

\[
\mathcal{U}_{a,\tau_1} = \mathcal{U}_{a,\tau} \implies \mathcal{K}_{\tau_1,N} = \mathcal{K}_{\tau,N}.
\]

(4.14)

According to (4.1), \(\mathcal{U}_{a,0} = \mathcal{I} + \mathcal{K}_c\), and thus we conclude that the operator \((\mathcal{U}_{a,0} - \mathcal{I})\) belongs to \(\mathcal{B}_2\). Since \(\mathcal{U}_{a,0} > 0\) (see (4.5)), the operator \(\mathcal{U}_{a,0}\) admits a factorization:

\[
\mathcal{U}_{a,0} = (\mathcal{I} + \mathcal{K}_c)^{-1}(\mathcal{I} + \mathcal{K}_c^*)^{-1}, \quad \mathcal{K} \in \mathcal{B}_2^+.\]

It follows from the equality \(\mathcal{U}_{a,\tau} = \mathcal{U}_{a,\tau_1}\) and (2.38) that

\[
(\mathcal{I} + \mathcal{K}_{\tau_1,N})(\mathcal{I} + \mathcal{K}_c)^{-1}(\mathcal{I} + \mathcal{K}_c^*)^{-1}(\mathcal{I} + \mathcal{K}_{\tau_1,N}^*)
\]

\[
= (\mathcal{I} + \mathcal{K}_{\tau,N})(\mathcal{I} + \mathcal{K}_c)^{-1}(\mathcal{I} + \mathcal{K}_c^*)^{-1}(\mathcal{I} + \mathcal{K}_{\tau,N}^*).\]

Therefore, in view of theorem A.1 we have

\[
(\mathcal{I} + \mathcal{K}_{\tau_1,N})(\mathcal{I} + \mathcal{K}_c)^{-1} = (\mathcal{I} + \mathcal{K}_{\tau,N})(\mathcal{I} + \mathcal{K}_c)^{-1},
\]

which implies that \(\mathcal{K}_{\tau_1,N} = \mathcal{K}_{\tau,N}\).

It follows now from lemma 2.2 (iii) and (4.14) that \(\tau_1 = \tau\).

(ii) Let \(q_1, q \in \text{Re} \mathfrak{M}\) and \(b_{q_1} = b_q\). Let us fix \(\tau_1 = b^{-1}(q_1)\) and \(\tau = b^{-1}(q)\). Since \(b_{q_1} = b_q\), the sequences \(a = a_\tau\) and \(a_\tau = a_{\tau_1}\) differ at most in the first element. Thus, taking into account (2.36), we obtain that

\[
\mathcal{U}_{a,\tau_1} = \mathcal{U}_{a,\tau} = \mathcal{I} = \mathcal{U}_{a_{\tau_1},\tau_1} = \mathcal{U}_{a_{\tau_1},\tau}.
\]

(4.15)

Reasoning as in the proof of (4.14), we establish the implication

\[
\mathcal{U}_{a,\tau_1} = \mathcal{U}_{a,\tau} \implies (\mathcal{K}_{\tau_1,D} = \mathcal{K}_{\tau,D}).
\]

Therefore, taking into account (4.15) and the statement (ii) of lemma 2.2, we get that \(b(\tau_1) = b(\tau)\), i.e. \(q_1 = q\). The proof is complete.

Proof of theorem 1.10. Part (i) is a straightforward consequence of theorems 4.3 and 1.5. Let us prove part (ii). Assume that \(q \in \text{Re} \mathfrak{M}\) and that \(b = b_q\) belongs to \(\mathfrak{M}\). Augmenting the sequence \(b\) to a sequence \(a\) with \(\lambda_0 = 0\) and \(\omega_0 = 1\), we get by proposition 1.3 that \(a \in \mathfrak{M}\). Set

\[
\nu := \nu a = I b_0 + \mu b, \quad \tau := \Theta(H_\nu).
\]

From the already proved statement (i) we obtain that \(a = a_\tau\). Then \(b = b_{\bar{q}}\) for \(\bar{q} = b(\tau)\). It follows from theorem 1.5 that \(q = \bar{q} = b(\tau)\). The proof is complete.

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Appendix A. The spaces

A.1. Sobolev spaces

Suppose that $X$ is a Banach space. Denote by $L_p((a, b), X)$, $1 \leq p < \infty$, the Banach space of all strongly measurable functions $f : (a, b) \to X$, for which the norm

$$\|f\|_{L_p((a,b),X)} := \left(\int_a^b \|f(t)\|_X^p \, dt\right)^{1/p}$$

is finite. Denote by $C^k([a, b], X)$ the Banach space of $k$ times continuously differentiable functions $[a, b] \to X$ with the standard norm.

We also denote by $W^p_k((0, 1), X)$, $1 \leq p < \infty$, the Sobolev space that is the completion of the linear space $C^1([0, 1], X)$ by the norm

$$\|f\|_{W^p_k} := \left(\int_0^1 \|f(t)\|_X^p \, dt\right)^{1/p} + \left(\int_0^1 \|f'(t)\|_X^p \, dt\right)^{1/p}.$$ 

Every function $f \in W^p_k((0, 1), X)$ has the derivative $f'$ belonging to $L_p((0, 1), X)$.

We denote by $M$, the Banach algebra of $r \times r$ matrices with complex entries. In the standard way the algebra $M_r$ is identified with the Banach algebra of linear operators $A : C' \to C'$ and inherits the operator norm

$$\|A\| = \sup_{\|y\|_{C'} = 1} \|Ay\|_{C'},$$

where $\|\cdot\|_{C'}$ is the Euclidean norm of $C'$ generated by the standard scalar product $(\cdot, \cdot)_{C'}$. We shall write $L_2$ instead of $L_2((0, 1), C')$ and denote by $(\cdot, \cdot)$ the scalar product in $L_2$, i.e.

$$(f, g) = \int_0^1 (f(x), g(x))_{C'} \, dx, \quad f, g \in L_2.$$

We also use the following notations:

$$L_p := L_p((0, 1), M_r), \quad \text{Re} \, L_p := \{u \in L_p : u = u^*\}, \quad p \geq 1,$$

$$W^p_s := W^p_s((0, 1), C'), \quad W^p_s := W^p_s((0, 1), M_r) \quad (s \in \mathbb{Z}, p \geq 1).$$

Set $W^{-1}_2(0, 1) = \{f \in W^1_2((0, 1)) : f(0) = f(1) = 0\}$. Recall (see [40]) that $W^{-1}_2(0, 1)$ is the dual space of $W^1_2(0, 1)$. Denote by $W^1_2$ the Banach space of matrices $f = (f_{ij})_{1 \leq i, j \leq r}$ with entries $f_{ij} \in W^1_2(0, 1)$ and with the inherited norm

$$\|f\|_{W^1_2} = \left(\sum_{1 \leq i, j \leq r} \|f_{ij}\|_{W^1_2((0,1))}^2\right)^{1/2}.$$ 

Let $\text{Re} \, W^{-1}_2 := \{f \in W^{-1}_2 : f = f^*\}$. Here $f^* := (f_{ij})_{1 \leq i, j \leq r}$ for $f = (f_{ij})_{1 \leq i, j \leq r}$.

A.2. Factorization of operators

We state below some necessary facts related to the factorization of Fredholm operators [37, chapter 4]. Denote by $B_2$ the Hilbert space of all Hilbert–Schmidt operators in $L_2$. Every operator $X \in B_2$ is an integral operator with kernel $K \in L_2((0, 1)^2, M_r)$. Set $\Omega := \{(x, t) : 0 \leq t \leq x \leq 1\}$, $\Omega^- := [0, 1] \setminus \Omega$ and denote by $B_2^\Gamma$ and $B_2^\Lambda$ the subalgebras of $B_2$ consisting of all operators with kernels $K$ satisfying the condition $K(x, t) = 0$ a.e. in $\Omega^-$ and $K(x, t) = 0$ a.e. in $\Omega$, respectively. It is obvious that $B_2 = B_2^\Gamma \oplus B_2^\Lambda$; moreover, the
operators in $B^2_+$ are Volterra ones [37, chapter IV]. Denote by $P^\pm$ an orthogonal projector in $B^2_+$.

We say that an operator $I + \mathcal{K}$ with $\mathcal{K} \in B^2_-$ admits a factorization in $B^2_-$ if

$$I + \mathcal{K} = (I + \mathcal{K}_+)^{-1}(I + \mathcal{K}_-)^{-1}$$

(A.1) with some $\mathcal{K}_+ \in B^2_+$ and $\mathcal{K}_- \in B^2_-$.

**Theorem A.1.** Assume that $\mathcal{K} \in B^2_-$ with kernel $K$ is such that $I + \mathcal{K}$ admits a factorization in $B^2_-$. Then the operators $\mathcal{K}_+$ and $\mathcal{K}_-$ in (A.1) are determined uniquely by $\mathcal{K}$. Moreover, the operators $\mathcal{K}_+$ and $\mathcal{K}_-^*$ are solutions for $\mathcal{X} \in B^2_-$ of the equations

$$\mathcal{X} + P^+ \mathcal{K} + \mathcal{K}^* P^+ = 0, \quad \mathcal{X} + P^+ \mathcal{K}^* + \mathcal{K}^* P^+ = 0$$

(A.2) respectively. For the operator $I + \mathcal{K}$ to admit a factorization in $B^2_-$, it is necessary and sufficient that one of the following conditions should hold:

(i) for every $a \in [0, 1]$ the integral equation

$$f(x) + \int_0^a K(x, t) f(t) \, dt = 0, \quad x \in (0, 1),$$

has only the trivial solution in $L^2$;

(ii) at least one of the equations in (A.2) has a solution.

In terms of the kernels the first equation in (A.2) takes the form

$$X(s, t) + K(s, t) + \int_0^s X(s, \xi) K(\xi, t) \, d\xi = 0, \quad (s, t) \in \Omega.$$  

(A.3)

Theorem A.1 yields the following connection between the Krein accelerants and the Krein equation, namely

**Proposition A.2.** Assume that $H \in L^2((−1, 1), M_r)$ and that $\mathcal{K}$ is an integral operator of (3.1). Then the following statements are equivalent:

(i) the function $H$ belongs to the class $B^2_-$;

(ii) the operator $I + \mathcal{K}$ admits a factorization;

(iii) the Krein equation (1.11) has a solution in $L^2(\Omega, M_r)$.

Theorem A.1 also implies the following lemma.

**Lemma A.3.** If $K$ belongs to $L^2((0, 1)^2, M_r)$, then equation (A.3) has at most one solution, and the equation

$$X(s, t) + \int_0^s X(s, \xi) K(\xi, t) \, d\xi = 0, \quad (s, t) \in \Omega,$$

has only zero solution in $L^2(\Omega, M_r)$.

We denote by $G_2$ the set of all functions $K : [0, 1]^2 \to M_r$ having the property that the mappings

$$[0, 1] \ni x \mapsto K(x, \cdot) \in L^2, \quad [0, 1] \ni t \mapsto K(\cdot, t) \in L^2$$

are continuous on the interval $[0, 1]$. It is obvious that $G_2 \subset L^2((0, 1)^2, M_r)$. The set $G_2$ becomes a Banach space upon introducing the norm

$$\|K\|_{G_2} := \max \left\{ \max_{x \in [0, 1]} \|K(x, \cdot)\|_{L^2}, \max_{t \in [0, 1]} \|K(\cdot, t)\|_{L^2} \right\}.$$
Denote by $G^+_2$ and $G^-_2$ the subspaces of $G_2$ consisting of all functions $K$ that satisfy the condition $K(x, t) = 0$ a.e. in $\Omega^-$ and $K(x, t) = 0$ a.e. in $\Omega$, respectively.

Also denote by $\mathfrak{G}_2$ a Banach algebra consisting of integral operators $\mathcal{X}$ with kernels $K$ belonging to $G_2$. The norm in $\mathfrak{G}_2$ is defined via
\[
\|\mathcal{X}\|_{\mathfrak{G}_2} = \|K\|_{G_2}, \quad \mathcal{X} \in \mathfrak{G}_2.
\]

Denote by $\mathfrak{G}_2^+$ and $\mathfrak{G}_2^-$ the set of those $\mathcal{X} \in \mathfrak{G}_2$ whose kernels belong respectively to $G_2^+$ and $G_2^-$. Observe that $\mathfrak{G}_2^+ \subset \mathfrak{G}_2$ and $\mathfrak{G}_2^- \subset \mathfrak{G}_2$.

**Lemma A.4.** The mapping $A \mapsto (I + A)^{-1} - I =: \eta(A)$ acts continuously in $G_2^+$.

**Proof.** Using the Cauchy–Bunyakowski inequality, we find that for every $A \in G^+_2$
\[
\|A^n\|_{G^+_2} \leqslant (n!)^{-1/2}\|A\|_{G^+_2}^{n+1}, \quad n \in \mathbb{Z}_+.
\]  
(A.4)

Let $A, B \in G^+_2$ and $\|A\|_{G^+_2}, \|B\|_{G^+_2} \leqslant r$ for some fixed $r > 0$. Since
\[
\eta(A) - \eta(B) = \sum_{n=1}^\infty [(A)^n - (B)^n],
\]
in view of (A.4) and the identity $A^{n+1} - B^{n+1} = \sum_{k=0}^n A^n B^k (A - B)$, straightforward calculations show that the mapping $\eta$ acts continuously in $G_2^+$ and that
\[
\|\eta(A) - \eta(B)\|_{G^+_2} \leqslant C(r)\|A - B\|_{G^+_2}
\]
with
\[
C(r) := 1 + \sum_{k=0}^\infty (k!)^{-1/2} r^{k+1}.
\]
The proof is complete. \(\square\)

**Appendix B. Lemmas about orthogonal projectors**

Suppose that $H$ is a Hilbert space and denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators in $H$.

**Lemma B.1.** Let $(P_n)_{n=1}^\infty$ and $(G_n)_{n=1}^\infty$ be sequences of pairwise orthogonal projectors in $H$ of finite rank such that $\sum_{n=1}^\infty P_n = \sum_{n=1}^\infty G_n = I_H$ and $\sum_{n=1}^\infty \|P_n - G_n\|^2 < \infty$. Then there exists $N_0 \in \mathbb{N}$ such that, for all $N \geqslant N_0$,
\[
\sum_{n=1}^N \text{rank } P_n = \sum_{n=1}^N \text{rank } G_n.
\]

**Proof.** Let us choose a natural $N_0$ such that $\sum_{n=N_0}^\infty \|P_n - G_n\|^2 < \frac{1}{4}$. Fix an arbitrary natural $N \geqslant N_0$ and consider the orthogonal projectors $P = \sum_{n=1}^N P_n$ and $G = \sum_{n=1}^N G_n$. We shall now show that $\|P - G\| < 1$. Indeed, the assumptions of the lemma imply that
\[
G - P = \sum_{n=N+1}^\infty (P_n - G_n) = \sum_{n=N+1}^\infty P_n (P_n - G_n) + \sum_{n=N+1}^\infty (P_n - G_n) G_n.
\]

Thus, for every $f \in H$ of norm 1 we get the inequality
\[
\|(P - G)f\|_H \leqslant \sum_{n=N+1}^\infty (\|P_n f\| + \|G_n f\|) \|(P_n - G_n)f\|.
\]
Using the Cauchy–Bunyakowski inequality, we obtain that
\[ \| P - G \|^2 \leq 4 \sum_{n=N+1}^{\infty} \| P_n - G_n \|^2 < 1. \]

The inequality \( \| P - G \| < 1 \) implies (see [41, chapter 1]) that \( \text{rank } P = \text{rank } G \), and the proof is complete. \( \square \)

**Lemma B.2.** Assume that \( (A_j)_{j=1}^{\infty} \) is a sequence in \( \mathcal{B}(H) \) and that \( (G_j)_{j=1}^{\infty} \) is a sequence of pairwise orthogonal projectors such that the following holds:

(i) the series \( \sum_{j=1}^{\infty} A_j \) converges in the strong operator topology to an operator \( A \);
(ii) the orthogonal projector \( G := I_H - \sum_{j=1}^{\infty} G_j \) is of finite rank;
(iii) \( \sum_{j=1}^{\infty} \| A_j - G_j \|^2 < 1 \) and \( \text{rank } A_j \leq \text{rank } G_j \leq \infty \) for every \( j \in \mathbb{N} \).

Then \( \text{codim Ran } A \geq \text{rank } G \).

**Proof.** It follows from the assumptions of the lemma that, for every \( f \in L_2 \),
\[ \sum_{j=1}^{\infty} \| (A_j G_j - G_j) f \| \leq \sum_{j=1}^{\infty} \| A_j - G_j \| \| G_j f \| \leq \left( \sum_{j=1}^{\infty} \| A_j - G_j \|^2 \right)^{1/2} \| f \| \]
and, therefore, the series \( \sum_{j=1}^{\infty} (A_j G_j - G_j) \) strongly converges to some operator \( B \in \mathcal{B}(H) \) with \( \| B \| < 1 \). Hence, the series \( \sum_{j=1}^{\infty} A_j G_j \) strongly converges to the operator \( \hat{A} := (I_H + B)(I_H - G) \). Since the operator \( I_H + B \) is invertible, \( \hat{A} \) has a closed range and \( \text{codim Ran } \hat{A} = \text{rank } G \). Since \( \text{rank } A_j \leq \text{rank } G_j < \infty \), we have
\[ \text{Ran } A_j G_j = \text{Ran } A_j \quad (B.1) \]
for all \( j \in \mathbb{N} \). Indeed, if (B.1) does not hold for some \( k \in \mathbb{N} \), then there exists nonzero \( c \in \text{Ran } G_k \) such that \( A_k c = 0 \). Therefore, \( \| (A_k - G_k) c \| = \| c \| \), which implies that \( \| A_k - G_k \| \geq 1 \) thus contradicting assumption (iii).

We denote by \( X \) the closed linear hull of the set \( \bigcup_{j \in \mathbb{N}} \text{Ran } A_j \). It is obvious that \( \text{Ran } A \subset X \). It follows from (B.1) and the definition of the operator \( \hat{A} \) that \( \text{Ran } \hat{A} \) contains the set \( \bigcup_{j \in \mathbb{N}} \text{Ran } A_j \). But, as already noted, \( \text{Ran } \hat{A} \) is a closed set. Thus, \( \text{Ran } A \subset X \subset \text{Ran } \hat{A} \) and, therefore, \( \text{codim Ran } A \geq \text{codim Ran } \hat{A} = \text{rank } G \). The proof is complete. \( \square \)

**Lemma B.3.** Let \( \{A_j\}_{j=0}^{n} \) be a set of self-adjoint operators from the algebra \( \mathcal{B}(H) \) that are of finite rank for \( j \neq 0 \). If
\[ \sum_{j=0}^{n} A_j = I_H, \quad \sum_{j=1}^{n} \text{rank } A_j \leq \text{codim Ran } A_0, \]
then \( \{A_j\}_{j=0}^{n} \) is the set of pairwise orthogonal projectors.

**Proof.** It follows from the assumptions of the lemma that the space \( H \) is the direct sum of the subspaces \( \text{Ran } A_j \) for \( j = 0, \ldots, n \) and that \( A_k (A_k - I_H) + \sum_{j \neq k} A_j A_k = 0 \) for every \( k \leq n \).
It is obvious that the equality is possible only if all the summands on the left-hand side are equal to zero. This immediately yields the statement of the lemma. \( \square \)
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