ON THE GIT QUOTIENT SPACE OF QUINTIC SURFACES

PATRICIO GALLARDO

Abstract. We describe the GIT compactification for the moduli space of smooth quintic surfaces in $\mathbb{P}^3$. In particular, we show that a normal quintic surface with at worst isolated double points or minimal elliptic singularities is stable. We also describe the boundary of the GIT quotient, and we discuss the stability of the nonnormal surfaces.

1. Introduction

Horikawa showed that if $X$ is a minimal algebraic surface with invariants $p_g = 4$, $q = 0$, and $c_2^2 = 5$, and its canonical system $|K_X|$ has no base points, then there exists a birational holomorphic map from $X$ onto a quintic surface in $\mathbb{P}^3$ with at most ADE singularities (see [Hor75, Th. 1]). Therefore, our GIT quotient $\overline{M}_{5}^{GIT}$ is a weakly modular compactification of the locus parametrizing these surfaces (see [Laz16, sec. 1.1.5]). In a more general context, Gieseker [Gie77] proved the existence of a quasi-projective coarse moduli space for smooth projective surfaces of general type with fixed invariants $p_g$, $q$, and $c_2^2$. More recently, modular compactifications of these spaces were constructed by Kollár and Shepherd-Barron [KSBS88] and Alexeev [Ale94]. Currently, we have a limited understanding of the Kollár–Shepherd-Barron–Alexeev (KSBA) compactification $\overline{M}_{5}^{KSBA}$ of the locus parametrizing quintic surfaces with at most ADE singularities (see [Ran17], [Gal14]). Therefore, our results are relevant for shedding light on the degenerations of quintic surfaces.

1.1. Brief description of the GIT results. The GIT quotient space of quintic surfaces (see Figure 1) is a 40-dimensional projective variety. Next, we give a brief classification of the surfaces parametrized by it.

We characterize stability for normal quintic surfaces. We show that quintic surfaces with only isolated double point singularities and isolated triple point singularities with a reduced tangent cone are stable (see Corollary 2.5). Normal surfaces in which each singularity has either a Milnor number smaller than 22 or a modality smaller than 5 are stable (see Proposition 4.2). Quintic surfaces with minimal elliptic singularities are stable (Corollary 4.8); the minimal elliptic surface singularities are analogous to the curve singularities with a classical genus drop invariant equal to 1 (see Proposition 4.9). Surfaces with isolated triple point singularities with a nonreduced tangent cone can be both stable and unstable (see Proposition 4.1). Quintic surfaces with a quadruple point are unstable (see Proposition 2.7).
We give a partial description of stable nonnormal quintic surfaces. We show that quintic surfaces with an irreducible curve of singularities of genus greater than 1 are stable (see Corollary 2.9). A generic quintic surface with a curve of singularities of multiplicity 3 such that the support of that curve does not contain any line is stable (see Proposition 5.3). Surfaces with a triple line are unstable (see Proposition 2.6). Quintic surfaces that decompose in a union of a plane and a quartic surface are discussed in Proposition 5.1.

For strictly semistable surfaces, we focus only on describing the polystable, i.e., closed orbits. We show that the strictly semistable locus in the quotient, which we call the GIT boundary, is a union of four disjoint irreducible components $\Lambda_i$, $i = 1, 2, 3, 4$ of dimensions 6, 1, 0, and 1, respectively. The stabilizer of every GIT semistable quintic surface is either equal or contained in a $\text{SL}(2, \mathbb{C})$ (see Lemma 3.2). The generic quintic surface parametrized by the boundary component $\Lambda_1$ is normal, and it has two isolated singularities of multiplicity 3, geometric genus 3, modality 7, and Milnor number 24. The generic quintic surface parametrized by the $\Lambda_2$ component is singular along three lines supporting two nonisolated triple point singularities. The generic quintic surfaces parametrized by the components $\Lambda_3$ and $\Lambda_4$ are singular along double lines, and they also have isolated singularities of multiplicity 3, geometric genus 2, modality 5, and Milnor numbers 24 and 22, respectively (see section 3.1). Strictly semistable quintic surfaces that decompose as a union of a quartic surface and a hyperplane are described in Proposition 5.2. The only nonreduced semistable quintic surface is a union of a double smooth quadric surface and a hyperplane intersecting along a smooth conic (see Corollary 2.12).

1.2. Organization. In section 2 we present the combinatorial side of the GIT analysis. In particular, we list the critical one-parameter subgroups, and we give a description of nonstable surfaces. In section 3 we study the strictly semistable minimal orbits associated with the GIT compactification. A more geometric interpretation of the failure of stability in normal surfaces is described in section 4. The stability of quintic surfaces with nonisolated singularities is discussed in section 5. In some cases we encounter routine computations better done with the help of a computer. The online notebooks are available at the accompanying website [Web].
1.3. Related work. This work fits in a series of GIT constructions including Shah [Sha81], Laza [Laz09], Yokoyama [Yok02], Fedorchuck and Smyth [FS13], Lakhani [Lak10] and Swinarski [Swi12]. For analyzing the singularities, we benefited from the work of Laufer [Lau77], Prokhorov [Pro03], Arnold [Arn76], and others [EAF86], [Suz81], [YW78]. Quintic surfaces of a general type were also studied by Yang [Yan86]. We used the software Sage S+YY and Macaulay2 [GS]; in particular, we used the Macaulay2 package StatePolytope developed by D. Swinarski.

1.4. Notation. The homogeneous coordinates are denoted as $[x_0 : x_1 : x_2 : x_3]$. The homogenous polynomials of degree $d$ are denoted as $f_d(x_0, x_1, x_2, x_3)$. We work over the complex numbers. We denote $p_i$ as the point $(x_j = x_k = x_l = 0)$ with $i \neq j, k, l$; we denote $L_{ij}$ as the line $(x_k = x_l = 0)$ with $i, j \neq k, l$. Unless otherwise indicated, whenever a polynomial occurs, we suppose it has generic coefficients. However, it is written without nonzero coefficients. For example, $c_i x_i^2 + c_k x_k^2$ is written as $x_i^2 + x_k^2$. Furthermore, if we work at the completion of the local ring of a singularity, we do not write the coefficients whenever they are invertible elements. For example, $u(x, y, z)x^2 + v(x, y, z)y^2$ is written as $x^2 + y^2$ only if $u(x, y, z)$ and $v(x, y, z)$ are invertible power series. The equation of $X$ with respect to a given coordinate system is denoted as $F_X(x_0, x_1, x_2, x_3)$. We denote $\Xi_{F_X}$ as its set of nonzero monomials. Similarly, $V(I)$ denotes the zero set of an ideal $I$. Given a point $p \in X$, we refer to its projectivized tangent cone as tangent cone. Our computational framework follows that of Mukai [Muk03 sec. 7.2].

2. Geometric invariant theory analysis

Geometric invariant theory provides a standard way to compactify some moduli spaces. In particular, the moduli of smooth quintic surfaces is an open subset of the GIT compactification:

$$\overline{M}^\text{GIT}_5 = \mathbb{P}\left(\text{Sym}^5 \left(H^0\left(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)\right)\right)\right)^{ss} // \text{SL}(4, \mathbb{C}).$$

The stability of a given surface ($F_X = 0$) is determined by using the Hilbert–Mumford numerical criterion. Let $\lambda$ be a nontrivial one-parameter subgroup $\lambda(t) : \mathbb{G}_m \to \text{SL}(4, \mathbb{C})$. The Hilbert–Mumford numerical function can be defined as (for details see [Dol03 Ch. 9])

$$\mu(\lambda, X) = \min\{\lambda.m_k \mid m_k \in \Xi_{F_X}\}.$$ (2.1)

A nontrivial one-parameter subgroup (1-PS) $\lambda$ is called normalized if it has the form

$$\lambda = \text{diag}(t^{a_0}, t^{a_1}, t^{a_2}, t^{a_3}),$$

with $a_0 \geq a_1 \geq a_2 \geq a_3$ and $a_0 + a_1 + a_2 + a_3 = 0$.

We assume that our one-parameter subgroups are normalized. This is possible because any 1-PS is conjugated to a normalized one. The Hilbert–Mumford numerical criterion (see [Dol03 Th. 9.1]) implies that a quintic surface is stable (resp., semistable) if and only if for every normalized $\lambda$ it holds that $\mu(\lambda, X) < 0$ (resp., $\leq 0$).

The normalized one-parameter subgroups induce a partial order among the monomials. Indeed, given two monomials $m, m'$, then $m \geq m'$ if and only if $\lambda.m \geq \lambda.m'$ for every normalized 1-PS $\lambda$ (see [Muk03 eq. 7.11]). By the numerical criterion, the minimal monomials in a configuration $\Xi_{F_X}$ are the ones that determine
the sign of $\mu(\lambda, X)$, and $X$ is a nonstable surface if and only if there exist a coordinate system and at least one normalized parameter subgroup $\lambda = (a_0, a_1, a_2, a_3)$ such that its associated set of monomials $\Xi_{F_\lambda}$ is contained in

$$M^{\oplus}(\lambda) := \left\{ x_0^s x_1^{s_1} x_2^{s_2} x_3^{s_3} \mid a_0 i_0 + a_1 i_1 + a_2 i_2 + a_3 i_3 \geq 0, \sum_{s=0}^{s=3} i_s = 5, i_k \geq 0 \right\}.$$  

For the analysis of stability, it suffices to consider the maximal sets $M^{\oplus}(\lambda)$ with respect to the inclusion. We call them maximal nonstable configurations, and they are determined by a finite list of 1-PSs that we call critical one-parameter subgroups.

**Proposition 2.1.** A quintic surface $X$ is nonstable if and only if for a choice of a coordinate system its monomial configuration $\Xi_{F_\lambda}$ is contained in $M^{\oplus}(\lambda_i)$ for one of the following 1-PSs:

$$\lambda_1 = (1, 0, 0, -1), \quad \lambda_2 = (2, 1, -1, -2), \quad \lambda_3 = (4, 2, -1, -5),$$  
$$\lambda_4 = (2, 1, 0, -3), \quad \lambda_5 = (3, 0, -1, -2), \quad \lambda_6 = (5, 1, -2, -4),$$  
$$\lambda_7 = (2, 1, 1, -4), \quad \lambda_8 = (2, 2, -1, -3), \quad \lambda_9 = (7, 1, -4, -4), \quad \lambda_{10} = (8, -1, -2, -5),$$

Furthermore, if for a choice of coordinates $\Xi_{F_\lambda} \subseteq M^{\oplus}(\lambda_i)$ for $i \geq 7$, then $X$ is unstable.

**Proof.** Only finitely many configurations of monomials are relevant for the GIT analysis. To find them, with the aid of a computer program (see [Web]), we list all the configurations, and we identify the maximal ones. The computation complexity is greatly reduced by using two basic observations: First, it suffices to consider the configurations associated with $M^{\oplus}(\lambda)$ where $\lambda$ is such that there exist distinct monomials $m_1, m_2$ satisfying $\lambda.m_1 = \lambda.m_2$. Second, a configuration is characterized by its set of minimal monomials with respect to the previously defined partial order. We also ensure that our list of critical 1-PSs is complete. Indeed, by examining the equation $\lambda.m_1 = \lambda.m_2$, with $\lambda = (a_0, a_1, a_2, a_3)$, it is clear that $|a_i| < 3(5)^3$ with $a_i \in \mathbb{Z}$. Applying [Muk03, Prop. 7.19], we confirm that $M^{\oplus}(\lambda) \subset M^{\oplus}(\lambda_k)$ for every $\lambda$ of that form. Our implementation of the algorithm to find the maximal sets $M^{\oplus}(\lambda_k)$ and to ensure that our list of critical 1-PSs is complete follows similar cases in the literature (e.g., [Laz09], [Lak10]). Finally, the generic configuration associated with each $M^{\oplus}(\lambda_k)$ can be either semistable or unstable depending on the presence or absence of the point $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ in the convex hull spanned by the monomials in $M^{\oplus}(\lambda_k)$. We distinguish the semistable configurations from the unstable configurations by using the Macaulay2 package StatePolytope developed by D. Swinarski. \hfill \Box

For each $k = 1, \ldots, 10$ our goal is to describe the geometric properties of surfaces $X$ such that $\Xi_{F_\lambda} \subset M^{\oplus}(\lambda_k)$. Our first step is to recall that each normalized $\lambda$ acts on the vector space $W := H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(1))$, determining a weight decomposition $W = \oplus_s W_s$. This decomposition induces a (partial) flag of subspaces $(F_n)_m := \oplus_{s \leq m} W_s \subset W$ that determines a (partial) flag $(F_n)_\lambda := p_\lambda \subset L_\lambda \subset H_\lambda \subset \mathbb{P}^3$. For instance, in our coordinate system if the normalized $\lambda$ has different weights $a_i$, the flag $(F_n)_\lambda$ is

$$(p_\lambda := [0 : 0 : 0 : 1]) \in (L_\lambda := V(x_0, x_1)) \subset (H_\lambda := V(x_0)).$$
We say that \((F_n)_\lambda\) is a bad flag for the surface \(X\) with respect to \(\lambda\) if \(\mu(\lambda, X) \geq 0\). Next, we describe the singularities of \(X\) singled out by their bad flag with respect to \(\lambda_k\).

**Proposition 2.2.** Let \(X\) be a quintic surface, and let \(\Delta\) be its singular locus. If \(X\) is a strictly semistable quintic surface with isolated singularities, then the following applies:

1. \(\Delta\) contains a triple point singularity \(p \in X\) whose tangent cone is a union of a double plane \(H^2\) and another plane. The intersection multiplicity of the surface with any line in \(H\) containing the triple point is 5.

2. \(\Delta\) contains a triple point singularity \(p \in X\) whose tangent cone is a union of a double plane \(H^2\) and another plane intersecting \(H\) along a line \(L\) which is contained in \(X\). The intersection of the hyperplane \(H\) with the surface \(X\) is a union of a double line \(L^2\) and a nodal cubic plane curve such that the double line is tangent to the cubic curve at the node.

3. \(\Delta\) contains a triple point singularity \(p \in X\) whose tangent cone is a triple plane \(H^3\). The quintic plane curve obtained from the intersection of the surface \(X\) with \(H\) has a quadruple point whose tangent cone contains a triple line.

If \(X\) is an irreducible, strictly semistable quintic surface with nonisolated singularities, then

4. \(\Delta\) contains a double line \(L^2\) supporting a special double point whose tangent cone is \(H^2\). At the completion of the local ring, the equation associated with the double point has the form (see subsection 1.4)
   \[x^2 + y^2 z^2 f_2(y, z^2) + y^5.\]
   The intersection of \(X\) with \(H\) is a quintuple line supported on \(L\).

5. \(\Delta\) contains a double line \(L^2\) supporting a special triple point \(p \in X\). The tangent cone of the triple point is a union of three planes intersecting along \(L\). At the completion of the local ring, the equation associated with the triple point has the following form:
   \[xf_2(x, y) + y^3 z + y^4 + x^2 z^2 + xyz^3.\]
   The intersection of the surface with one of the above hyperplanes \(H\) is a union of a conic and a transversal triple line supported on \(L\).

6. \(\Delta\) contains a double line \(L^2\) supporting a special triple point whose tangent cone is a union of a double plane \(H^2\) and another plane. At the completion of the local ring, the equation associated with the triple point has the following form:
   \[x^2 y + x^4 + y^4 + x^3 z + x^2 z^2 + xy^3 + xy^2 z + xyz^3.\]
   The intersection of the surface with the hyperplane \(H\) is a union of a quadruple line supported on \(L\) and another line.

**Proof.** We suppose the quintic surface is strictly semistable. By Proposition 2.1 we need to find only the geometric characterization of the quintics defined by the equations \(F_{\lambda_i}\) for \(1 \leq i \leq 6\). The statement describes the intersection of these surfaces with the corresponding singular flag \((F_n)_{\lambda_i}\). We order the cases as in the statement. The equation associated with the first case is
\[F_{\lambda_1} = x^2_3 x^2_0 f_1(x_0, x_1, x_2) + x_3 x_0 f_3(x_0, x_1, x_2) + f_5(x_0, x_1, x_2).\]
The equation associated with the second case is
\[
F_{\lambda_3} = x_3^2 x_0^2 f_1(x_0, x_1) + x_3(x_0^2 x_2^2 + x_2 f_3(x_0, x_1)) + x_0^2 f_5(x_0, x_1, x_2).
\]

The equation associated with the third case is
\[
F_{\lambda_4} = x_3^2 x_0^3 + x_3 x_1^3 f_1(x_1, x_2) + x_3 x_0^2 h_1(x_0, x_1, x_2) + x_3 x_0^2 f_2(x_1, x_2) + x_3 x_0 x_1 g_2(x_1, x_2) + f_5(x_0, x_1, x_2).
\]

The equation associated with the fourth case is
\[
F_{\lambda_5} = x_3^3 x_0^2 + x_3^2 x_1^2 f_1(x_1, x_2) + x_3 x_0^2 f_2(x_0, x_1, x_2) + x_3 x_0 x_1^2 f_1(x_1, x_2) + x_0^2 f_3(x_0, x_1, x_2) + x_0 x_1 g_3(x_1, x_2) + a_1 x_1^5.
\]

The equation associated with the fifth case is
\[
F_{\lambda_2} = x_3^2 x_0^2 f_2(x_0, x_1) + x_3 x_1^2 f_1(x_1, x_2) + x_3 x_0 x_1^2 h_1(x_1, x_2) + x_3 x_0^2 g_2(x_0, x_1, x_2) + x_0^2 f_3(x_0, x_1, x_2) + x_0 x_1 g_3(x_1, x_2) + x_1^2 h_2(x_1, x_2).
\]

The equation associated with the last case is
\[
F_{\lambda_6} = x_3^2 x_0^2 f_1(x_0, x_1, x_2) + x_3 x_0^2 f_2(x_0, x_1, x_2) + x_3 x_0 x_1^2 h_1(x_1, x_2) + x_3 x_1^4 + x_1^2 g_1(x_1, x_2) + x_0 x_1 f_3(x_1, x_2) + x_0^2 g_3(x_0, x_1, x_2).
\]

To find the local equations of the singularities, we use an analytic change of coordinates as described by [Kol98 sec. 2.5].

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
1-PS & Associated Geometric Characteristics \\
\hline
\lambda_1, \lambda_3, \lambda_4 & Isolated triple point singularity with nonreduced tangent cone \\
\lambda_7 & Isolated ordinary quadruple point singularity \\
\lambda_2, \lambda_6, \lambda_8 & Double line of singularities supporting a nonisolated triple point \\
\lambda_5, \lambda_9 & Double line of singularities with a distinguished double point \\
\lambda_{10} & A union of a quartic surface and a hyperplane \\
\hline
\end{tabular}
\end{table}

Next, we describe the main geometric characteristics of the surfaces destabilized by the critical 1-PSs \lambda_7, \lambda_8, \lambda_9, and \lambda_{10}.

**Proposition 2.3.** Let \( X \) be a quintic surface, and let \( \Delta \) be its singular locus. Suppose that for some coordinate system \( \Xi_{F_X} \subseteq M^\oplus(\lambda_k) \), with \( k \geq 7 \). Then \( X \) is an unstable quintic surface, and one of the following cases holds:

1. \( \Delta \) contains an ordinary quadruple point.
2. \( \Delta \) contains a double line supporting a special triple point \( p \in X \) whose tangent cone is a union of three concurrent hyperplanes intersecting along a line \( L \). At the completion of the local ring, the equation associated with the triple point has the following form:
\[
f_3(x, y) + y^2 z^3 + xyz^3 + x^2 z^3.
\]
The intersection of the surface with one of the above hyperplanes is a union of a cubic curve and a tangent double line supported at $L$.

3. $\Delta$ contains a double line supporting a special double point whose tangent cone is $H^2$. At the completion of the local ring, the equation associated with the double point has the form $x^2 + y^3$. The intersection of the surface with $H$ is a union of a quadruple line supported on $L$ and another line.

4. $X$ is a union of a smooth quartic surface and a hyperplane such that
   * The intersection of the hyperplane with the quartic surface is a quartic plane curve with a triple line $L^3$.
   * The intersection of the quartic surface with this line $L$ is a quadruple point.

Proof. We write down the equations of surfaces destabilized by the critical 1-PS $\lambda_i$ with $i \geq 7$ and match them with the cases of the statement. The generic equation associated with the first case is given by

$$F_{\lambda_7} = x_3 f_4(x_0, x_1, x_2) + f_5(x_0, x_1, x_2, x_3).$$

The equation associated with the second case is given by

$$F_{\lambda_8} = x_3 f_3(x_0, x_1) + x_3(x_2 f_3(x_0, x_1) + f_4(x_0, x_1)) + x_2^2 f_2(x_0, x_1) + x_2 f_4(x_0, x_1) + f_5(x_0, x_1).$$

The equation associated with the third case is given by

$$F_{\lambda_9} = x_3^2 x_0^2 + x_3^2 x_0^2 f_1(x_0, x_1, x_2) + x_3 x_0 x_1 + x_3 x_2 x_0 f_2(x_0, x_1) + x_3 f_4(x_0, x_1) + x_2^2 x_0^2 + x_2 x_0 f_2(x_0, x_1) + x_2 f_4(x_0, x_1) + f_5(x_0, x_1).$$

The equation associated with the fourth case is given by

$$F_{\lambda_{10}} = x_3^2 x_0^2 + x_3^2 x_0^2 g_1(x_0, x_1, x_2) + x_3 x_2 x_0^2 f_1(x_0, x_1, x_2) + x_3 x_0 f_3(x_0, x_1) + x_0 f_4(x_1, x_2, x_0).$$

Next, we discuss some additional stability results.

**Corollary 2.4.** Let $X$ be a normal quintic surface with a triple point whose tangent cone is nonreduced. Let $\tilde{X} \rightarrow X$ be the monomial transformation of $X$ with its center at the triple point. Then $\tilde{X}$ is nonnormal if and only if there is a coordinate system such that $\Xi_{F_{\lambda_1}} \subset \Xi_{F_{\lambda_i}}$.

Proof. We can select a coordinate system such that the triple point is supported at $p_3$ and its tangent cone is $(x_0^2 f_1(x_0, x_1, x_2) = 0)$. In that case, the equation of the quintic surface can be written as

$$x_3^2 x_0^2 f_1(x_0, x_1, x_2) + x_3 x_0 f_3(x_0, x_1, x_2) + f_5(x_0, x_1, x_2).$$

The singularities of $\tilde{X}$ are supported on the exceptional divisor. As found by Yang [Yan86, Prop. 4.2], this happens along the intersection of $(x_0 = 0)$ with $(f_4(x_0, x_1, x_2) = 0)$, and the failure of normality of $\tilde{X}$ is equivalent to $x_0 \mid f_4(x_0, x_1, x_2)$. Therefore, $\tilde{X}$ is nonnormal if and only if the equation of $X$ can be written as

$$x_3^2 x_0^2 f_1(x_0, x_1, x_2) + x_3 x_0 f_3(x_0, x_1, x_2) + f_5(x_0, x_1, x_2).$$

The statement follows by inspecting the equation $F_{\lambda_i}$ in the proof of Proposition 2.2.
Corollary 2.5. Let $X$ be a normal quintic surface such that each of its singularities is either an isolated double point or an isolated triple point whose tangent cone is reduced. Then $X$ is stable.

Proof. It follows from Propositions 2.2 and 2.3 because the singularities of nonstable normal quintic surfaces are necessarily worse than triple points with a nonreduced tangent cone.

Proposition 2.6. Let $X$ be a quintic surface containing a line $L$ of singularities such that $\text{mult}_p(X) = 3$ for all $p$’s $\in L$. Then $X$ is unstable.

Proof. We can suppose the triple line is supported at $(x_0 = x_1 = 0)$. Then the equation of $X$ can be written as

$$g_0(x_0, x_1, x_2, x_3)x_0^3 + f_2(x_2, x_3)x_0^2x_1 + h_2(x_0, x_1, x_2, x_3)x_0x_1^2 + p_2(x_1, x_2, x_3)x_1^3,$$

which is destabilized by $\lambda_8$ (see [Web]).

Proposition 2.7. Let $X$ be a quintic surface with a singularity of multiplicity greater than or equal to 4. Then $X$ is unstable.

Proof. Suppose that the quadruple point is supported at $p_3$. Then $X$ is destabilized by $\lambda_7$.

Lemma 2.8. Let $X$ be an irreducible quintic surface such that its singular locus $\Delta$ contains a nonplanar reduced curve $C$. Then $\text{deg}(C) \leq 6$. Furthermore, if $X$ has at least one triple point singularity, then $C$ is a twisted cubic, an elliptic quartic curve, or a degeneration of the two.

Proof. We apply the genus formula to the generic section of $X$ which is an irreducible plane quintic curve that cannot have more than six double points. Those double points are induced by $C$. Then if $X$ is irreducible, the degree of $C$ is less than 6. The same argument applies if $X$ has a triple point. We take a section of $X$ through it, and we assume that there is not a curve of degree 4 and genus 2 in $\mathbb{P}^3$.

Corollary 2.9. Let $X$ be an irreducible quintic surface with a curve of singularities supported on a reduced curve $C$. Suppose that the genus of $C$ is greater than 1, $C$ does not contain any line, and $X$ does not have an additional line of singularities. Then $X$ is stable.

Proof. Lemma 2.8 and our hypothesis about the genus of $C$ imply that $X$ has no triple point singularities. Then $X$ is either stable or $\Xi_{F_X} \subset M^\circ(\lambda_i)$ for $i \in \{5, 9\}$ (see Table 11). However, this last case implies that $X$ contains a line of singularities.

To decide the semistability of a quintic surface with a $\text{SL}(2, \mathbb{C})$-stabilizer, we make use of its symmetry to reduce the number of 1-PSs for which we have to check the Hilbert–Mumford numerical criterion (for a similar argument see [AFSL13, Prop. 2.4]).

Lemma 2.10. Let $X$ be a quintic surface that decomposes in a union of a quartic surface and a hyperplane. Suppose that there is an $\text{SL}(2, \mathbb{C}) \subset \text{Aut}(X)$ action that fixes a smooth conic, $C$, on $X$. Then there is a coordinate system $\{x_1\}$ such that the equation associated with $X$ has the form

$$x_1(f_2(x_0, x_2, x_3)^2 + x_1f_3(x_0, x_1, x_2, x_3)).$$
where \((x_1 = f_2(x_0, x_2, x_3) = 0)\) defines the invariant conic. Moreover, the quintic surface \(X\) is semistable if and only if it is semistable with respect to every 1-PS acting diagonally on \(\{x_1\}\) and of the form \(\lambda = \text{diag}(a_0, a_1, a_2, a_3)\), with \(a_0 \geq a_2 \geq a_3\).

**Proof.** A basis \(\{x_i\}\) of a vector space \(W\) is compatible with a reductive group if, given an equivariant decomposition of \(W\), the equivariant subspaces are spanned by a subset of the variables \(\{x_i\}\). For us, the group is \(\text{SL}(2, \mathbb{C})\), \(W := H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))\), and \(V \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))\) is the standard two-dimensional \(\text{SL}(2, \mathbb{C})\)-representation. We select a distinguished coordinate system \(\{x_i\}\) compatible with the \(\text{SL}(2, \mathbb{C})\)-decomposition

\[
W := H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \cong \text{Sym}^2(V) \oplus \text{Sym}^0(V)
\]

induced by the embedding \(C \hookrightarrow \mathbb{P}^3\). In particular, \(\mathbb{P}(\text{Sym}^2(V)) := (x_1 = 0)\) is the plane containing \(C\). We select a maximal torus \(T_{\max}\) compatible with \(\{x_i\}\). It follows that the plane \((x_1 = 0)\) is fixed by \(T_{\max}\), and the equation of \(X\) in this coordinate system is the one in the statement. We follow the notation of [MST11] Def. 4.6 by saying that \(T_{\max}\) determines stability for \(X\) because \(W\) has a multiplicity-free decomposition into irreducible \(\text{SL}(2, \mathbb{C})\)-representations and the basis \(\{x_i\}\) is compatible with the \(\text{SL}(2, \mathbb{C})\)-action. According to Morrison and Swinarski [MST11] Prop. 4.7, if \(X\) is \(T_{\max}\) semistable, then \(X\) is \(\text{SL}(4, \mathbb{C})\) semistable. Our result follows because we can take \(\lambda \subset T_{\max}\) to be \(\lambda = \text{diag}(a_0, a_1, a_2, a_3)\), with \(a_0 \geq a_2 \geq a_3\), where this last condition is achieved by relabelling. ∎

**Proposition 2.11.** Let \(X\) be a quintic surface that decomposes as a double quadric surface \(Q\) and a hyperplane \(H\). Then \(X\) is semistable if and only if \(Q\) is smooth and \(Q \cap H\) is smooth.

**Proof.** The quadric surface must be smooth; otherwise, \(X\) will contain a quadruple point. If the intersection \(Q \cap H\) is singular, then the equation associated with the quintic surface can be written as \(x_1(x_0^2 + x_0x_2 + x_1^2 + x_1x_3)^2\) because there is exactly one orbit of such quintics. This last quintic surface is destabilized by \(\lambda_8\). Next, we suppose that the conic \(Q \cap H\) is smooth. Then in some coordinate system \(F_X = x_1(x_1x_2 - x_0x_3 + x_3^2)\), where \(\alpha \neq 0\). The semistability follows from Lemma [2.10] and by noting that \(F_X\) is clearly semistable with respect to every 1-PS acting diagonally on the \(x_i\)’s. ∎

**Corollary 2.12.** A nonreduced quintic surface \(X\) is semistable if and only if \(X = 2Q + H\), where \(Q\) is a smooth quadric surface, and \(H\) is a hyperplane intersecting \(Q\) along a smooth conic.

**Proof.** If \(X\) decomposes as a union of a double plane and another cubic surface, then we can select a coordinate system so that \(F_X = x_0^2p_3(x_0, x_1, x_2, x_3)\), which is destabilized by \(\lambda_{10}\). By degree considerations, the other case is a union of a double quadric surface \(Q^2\) and a hyperplane \(H\). Then the statement follows from Proposition [2.11]. ∎

3. Minimal orbits of the GIT compactification

Recall that we refer to the image of the strictly semistable locus in the GIT quotient as the GIT boundary. Given a point \(q\) at the GIT boundary, there is a unique closed orbit associated with \(q\). If we say that \(X\) is parametrized by \(q\), then
we suppose that $X$ corresponds to that closed orbit. Next, we describe the generic surfaces parametrized by the GIT boundary and several aspects of its geometry.

**Theorem 3.1.** The strictly semistable locus in $\overline{M}_5^\text{GIT}$ has four disjoint irreducible components: $\Lambda_1$, $\Lambda_2$, $\Lambda_3$, and $\Lambda_4$, of dimensions 6, 1, 0, and 1, respectively (see Figure 1). Let $X_k$ be a generic surface parametrized by $\Lambda_k$. Then $X_k$ has the following geometric properties:

1. The surface $X_1$ is normal; it contains two isolated triple point singularities which are called $V_{24}$ in [EAF86, Table II]. This singularity has geometric genus 3, modality 7, and Milnor number 24.
2. The surface $X_2$ is singular along three lines that support two nonisolated triple point singularities.
3. The surface $X_3$ has a triple point isolated singularity of geometric genus 2, modality 5, and Milnor number 24, which is called $V_{24}^1$ in [Suz81, p. 244]. Additionally, the surface $X_3$ is singular along a line supporting a distinguished triple point singularity.
4. The surface $X_4$ has an isolated triple point singularity of geometric genus 2, modality 5, and Milnor number 22, which is called $V_{22}^1$ in [Suz81, p. 244]. Additionally, the surface $X_4$ is singular along a line supporting only singularities of multiplicity 2.

The boundary component $\Lambda_1$ is stabilized by the 1-PS $\lambda_1$, $\Lambda_2$ is stabilized by $\lambda_2$, $\Lambda_3$ is stabilized by $\lambda_3$ and $\lambda_6$, and $\Lambda_4$ is stabilized by $\lambda_4$ and $\lambda_5$.

**Proof.** The statements about $X_k$ follow from studying the equations associated with the monomial invariants with respect to $\lambda_k$ for $k \leq 6$ and comparing them to [Suz81, EAF86, YW78]. The main theoretical tool is the centralizer version of Luna’s criterion (see [Lun75, Remark, p. 237]). Luna’s result implies that, given $W = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5))$, the group $G = \text{SL}(4, \mathbb{C})$, and the stabilizer $G_x$ of $x$, the orbit $G \cdot x$ is closed in $W$ if and only if the orbit $C_G(G_x) \cdot x$ is closed in $W^{G_x}$, where $W^{G_x} \subset W$ denotes the invariant set under the $G_x$ action and $C_G(G_x)$ is the centralizer of $G_x$ in $G$. We also need some stability calculations to determine which monomials cannot vanish in the equations of $X_i$, $i = 1, 2, 3, 4$. Next, we complete the analysis of the boundary associated with each $\lambda_k$.

**Boundary stratum $\Lambda_1$.** The equation of $X_1$ can be written as

$$\overline{F}_{\lambda_1} = x_3^2 x_0^2 f_1(x_1, x_2) + x_3 x_0 f_3(x_1, x_2) + f_5(x_1, x_2).$$

The generic quintic surface defined by equation (3.1) has two triple point singularities at the points $[1 : 0 : 0 : 0]$ and $[0 : 0 : 0 : 1]$ of the type described in the statement. The centralizer $C_G(\lambda_1)$ is given by

$$C_G(\lambda_1) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \frac{1}{\text{det } A} \end{pmatrix} : A \in GL(2, \mathbb{C}), \ a \neq 0 \right\}.$$ 

The dimension of $\Lambda_1$ is computed by noting that $\dim(W^{\lambda_1}) = 12$, the centralizer has dimension 5, and $\overline{F}_{\lambda_1}$ can be defined up to a constant.

**Boundary stratum $\Lambda_2$.** The centralizer of $\lambda_2$ is the torus, and the stability analysis is the standard one. The equation of a semistable surface parametrized by
Let $\lambda_2$ be given by
\[ \mathcal{F}_{\lambda_2} = x_3^2x_0x_1^2 + x_3 x_2 x_1^2 + a_1 x_3 x_2 x_1^3 + a_2 x_0 x_1 x_2^3, \quad \text{where } [a_1 : a_2] \in \mathbb{P}^1. \]
This surface is singular along the lines
\[ (x_2 = x_3 = 0), \quad (x_1 = x_2 = 0), \quad (x_0 = x_1 = 0). \]
By analyzing equation (3.2) and its partial derivatives, it follows that if $a_1$ and $a_2$ are not equal to 0, then the surface defined by $\mathcal{F}_{\lambda_1}$ has only two nonisolated triple point singularities at $[1 : 0 : 0 : 0]$ and $[0 : 0 : 0 : 1]$. The points $[0 : 1]$ and $[1 : 0]$ in $\Lambda_2$ parametrize a union of a quartic surface and a hyperplane which is described in Proposition 5.2.

**Boundary stratum** $\Lambda_3$. The equation of a semistable surface stabilized by $\lambda_3$ is given by
\[ \mathcal{F}_{\lambda_3} = x_3^2 x_0^2 x_1 + x_3 x_1^2 x_2 + x_0 x_2^4. \]
This surface has an isolated triple point supported at $[0 : 0 : 0 : 1]$ and a line of singularities supported at $(x_2 = x_3 = 0)$. The line of singularities supports a triple point at $[1 : 0 : 0 : 0]$.

**Boundary stratum** $\Lambda_4$. The equation of a semistable surface stabilized by $\lambda_4$ is given by
\[ \mathcal{F}_{\lambda_4} = x_0^3 x_3^2 + x_3 x_2 x_1^3 + a_1 x_0 x_1 x_2 x_3 + a_2 x_2^5, \quad \text{where } [a_1 : a_2] \in \mathbb{P}^1. \]
If $a_2 \neq 0$, then the point $[a_1 : 1] \in \Lambda_4$ parametrizes a quintic surface with one isolated triple point at $[0 : 0 : 0 : 1]$ and a line of singularities of multiplicity 2 supported at $(x_2 = x_3 = 0)$. The point $[1 : 0] \in \Lambda_4$ parametrizes a union of a quartic surface and a hyperplane described in Proposition 5.2.

Finally, we observe that the equations associated with the monomial invariants with respect to $\lambda_5$ and $\lambda_6$ are, after a change of coordinates, equal to equations (3.4) and (3.3), respectively.

The following result is important for showing that the boundary components are mutually disjoint.

**Lemma 3.2.** Let $G^0_x$ be the connected component of the stabilizer associated with a closed orbit of a strictly semistable point. Then $\text{rank}(G^0_x) = 1$, and up to isogeny, the largest stabilizer for a semistable quintic surface is either $\text{SL}(2, \mathbb{C})$ or $\mathbb{G}_m$.

**Proof.** If follows from the fact that there is not a semistable surface with a $(\mathbb{C}^*)^2$ contained in its stabilizer. Indeed, it is enough to consider polystable, i.e., closed orbits. Let $X$ be a surface that is stabilized by two different 1-PSs that are not contained in the same $\mathbb{C}^*$. By Proposition 2.1, we can take one of those 1-PSs to be $\lambda_i$, $i = \{1, 2, 3, 4\}$. Since all maximal torus are conjugated, we can suppose that the other 1-PS is acting diagonally on the same coordinates. However, by equations (3.1), (3.2), (3.3), and (3.4) we can see that such a 1-PS does not exist. For example, if we consider a nontrivial normalized one-parameter subgroup $\lambda = (a_0, a_1, a_2, a_3)$ stabilizing the quintic
\[ (\mathcal{F}_{\lambda_1} = x_0^2 x_1^2 f_1(x_1, x_2) + x_3 x_0 f_3(x_1, x_2) + f_5(x_1, x_2) = 0). \]
Then we must have one of the following equations:
\[ 2a_3 + a_0 + a_1 = 0, \quad 2a_3 + a_0 + a_2 = 0. \]
because \( f_1(x_1, x_2) \neq 0 \). Otherwise, the quintic will acquire a quadruple point which makes it unstable. Similarly, the term \( f_5(x_1, x_2) \) cannot vanish completely, so we have an equation of the form \( ma_1 + (5 - m)a_2 \) for some case in which \( 0 \leq m \leq 5 \). These equations imply that \( \lambda \) is contained in the \( \mathbb{C}^* \) generated by \( \lambda_1 \). The same argument holds for the other cases.

Finally, Matsushima’s criterion (see [Mat92]) implies that \( G_x^0 \) is a reductive group. Therefore, our lemma follows from the classification of reductive groups over the complex numbers. \( \square \)

The above lemma concludes the proof of Theorem 3.1. Indeed, the components of the strictly semistable loci are disjoint because if \( X \) is parametrized by \( \Lambda_i \cap \Lambda_j \), with \( i = 1, 2, 3, 4 \), \( j = 1, 2, 3, 4 \) and \( i \neq j \), then the stabilizer of \( X \) contains a \( (\mathbb{C}^*)^2 \), which is impossible by Lemma 3.2. \( \square \)

3.1. Local analysis near the GIT boundary. Here, we discuss the local structure at selected points, in the étale topology, of our GIT quotient

\[
\overline{\mathcal{M}}^{GIT}_5 := (\mathbb{P}^N)^{ss} \sslash G,
\]

where \( G \cong SL(4, \mathbb{C}) \) and \( N = 55 \). The main technical tool is Luna’s slice theorem [Lun75 App. D]. Let \( x \in (\mathbb{P}^N)^{ss} \) be a strictly semistable point with stabilizer \( G_x \). There is a \( G_x \)-invariant slice \( V_x \) to the orbit \( G \cdot x \) which can be taken to be a smooth, affine, locally closed subvariety of \( (\mathbb{P}^N)^{ss} \) such that \( U = G \cdot V_x \) is open in \( (\mathbb{P}^N)^{ss} \). Given \( (G \times G_x V_x)/G_x \) where the action on the product is given by \( h \cdot (g, v) = (g \cdot h^{-1}, hv) \), and by considering the fiber of the normal bundle \( \mathcal{N}_x := (\mathcal{N}_{G \cdot x | \mathbb{P}^n}) |_x \), we have the following commutative diagram:

\[
\begin{array}{cccc}
G \times_{G_x} \mathcal{N}_x & \xrightarrow{\text{étale}} & G \times_{G_x} V_x & \xrightarrow{\text{étale}} & U \subset (\mathbb{P}^N)^{ss} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{N}_x/G_x & \xrightarrow{\text{étale}} & V_x/G_x & \xrightarrow{\text{étale}} & U/G \subset (\mathbb{P}^N)^{ss} \sslash G \cong \overline{\mathcal{M}}^{GIT}_5
\end{array}
\]

Recall that Kirwan constructed a partial desingularization of the GIT quotient by blowing up loci associated with positive dimensional stabilizers (see [Kir85]). The associated exceptional divisor \( \mathbb{P}(\mathcal{N}_x)^{ss}/G_x \) often carries, itself, a modular meaning. It is of special interest to understand the Kirwan blowup of \( \overline{\mathcal{M}}^{GIT}_5 \) at the point \( \omega \) that parametrizes a union of a double smooth quadric surface and a transversal hyperplane. Indeed, Rana [Ran17 Thms. 1.4 and 4.1] proves that on the KSBA compactification \( \overline{\mathcal{M}}^{KSB}_{5} \) there is a Cartier divisor \( D \) associated with the deformations of the \( \frac{1}{4}(1, 1) \) singularity. At least one component of this divisor is obtained from taking the stable replacement of the following family of quintic surfaces degenerating to \( \omega \):

\[
X_t = (f_2(x)^2f_1(x) + tf_2(x)f_3(x) + t^2f_5(x) = 0),
\]

where \( f(x) := f(x_0, x_1, x_2, x_3) \). Next, we give a local analysis of the GIT quotient near \( \omega \). (For a similar situation in degree 4, see Shah [Sha81]).

**Proposition 3.3.** Let \( \omega \in \overline{\mathcal{M}}^{GIT}_5 \) be the point parametrizing a union of a double smooth quadric surface \( Q \) and a transversal hyperplane \( H \). Let \( x \) be a semistable
point with closed orbit mapping to the point \( \omega \in \mathcal{M}_5^{GIT} \). Then the natural representation of its stabilizer \( G_x \cong \text{SL}(2, \mathbb{C}) \) on the normal bundle \( \mathcal{N}_x \) is isomorphic to

\[
\mathcal{N}_x = (\text{Sym}^5(V) \otimes \text{Sym}^5(V)) \oplus \text{Sym}^6(V),
\]

where \( V \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \) is the standard two-dimensional representation of \( \text{SL}(2, \mathbb{C}) \) coming from the action on the conic \( Q \cap H \).

**Proof.** The lemma follows from calculating an appropriate \( G_x \cong \text{SL}(2, \mathbb{C}) \) equivariant decomposition of the summands in the normal exact sequence

\[
0 \to \mathcal{T}_{G,x} \to \mathcal{T}_{\mathbb{P}^N} \to \mathcal{N}_{G,x|\mathbb{P}^N} \to 0,
\]

which we localize at \( x \). To find the equivariant decomposition of \( T_{\mathbb{P}^N} \), we use [FH91, Exercise 11.14] together with the decomposition on expression (2.2) to calculate the following decomposition of \( H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \):

\[
\text{Sym}^{10}(V) \oplus \text{Sym}^8(V) \oplus (\text{Sym}^6(V))^{\oplus 2} \oplus (\text{Sym}^4(V))^{\oplus 2} \oplus (\text{Sym}^2(V))^{\oplus 3} \oplus (\text{Sym}^0(V))^{\oplus 3}.
\]

Next, we use the Euler sequence to obtain

\[
0 \to \mathcal{O}_{\mathbb{P}^N}|_x \to \mathcal{O}_{\mathbb{P}^N}(1)|_x \otimes H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \to \mathcal{T}_{\mathbb{P}^N}|_x \to 0,
\]

so the decomposition of \( H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \) induces a decomposition at the tangent space \( \mathcal{T}_{\mathbb{P}^N}|_x \). To calculate the decomposition of the tangent space \( \mathcal{T}_{G,x}|_x \), we use the exact sequence

\[
0 \to \mathcal{T}_{G,x} \to \mathcal{T}_G \to \mathcal{T}_{G,x} \to 0.
\]

The tangent space \( \mathcal{T}_{G,x}|_x \) is identified with the adjoint representation of \( \mathfrak{sl}(2, \mathbb{C}) \), which is isomorphic to \( \text{Sym}^3(V) \). The tangent space of \( \mathcal{T}_{G}|_x \) corresponds to the Lie algebra \( \mathfrak{sl}(4, \mathbb{C}) \), which has a 15-dimensional adjoint representation. The embedding \( C \hookrightarrow \mathbb{P}^3 \) induces a decomposition as

\[
\mathcal{T}_G|_x \cong \text{Sym}^4(V) \oplus (\text{Sym}^2(V))^{\oplus 3} \oplus \text{Sym}^0(V),
\]

from which we obtain \( \mathcal{T}_{G,x}|_x \). Therefore, by comparing irreducible summands in the exact sequence (3.8), we obtain the following decomposition for \( \mathcal{N}_{G,x|\mathbb{P}^N}|_x \):

\[
\text{Sym}^{10}(V) \oplus \text{Sym}^8(V) \oplus (\text{Sym}^6(V))^{\oplus 2} \oplus \text{Sym}^4(V) \oplus \text{Sym}^2(V) \oplus \text{Sym}^0(V),
\]

from which we obtain our statement by [FH91, Exer. 11.11]. \( \square \)

Finally, we describe a similar analysis for the other boundary components. In the following statement, an exponent \( n \) means that the corresponding entry is repeated \( n \) times.

**Proposition 3.4.** The fiber of the Kirwan blowup over \( x \in \Lambda_2 \) is

\[
\mathbb{P}(10, 9, 8, 7^2, 6^3, 5^2, 4^2, 3^2, 2^3, 1^2) \times \mathbb{P}(10, 9, 8, 7^2, 6^3, 5^3, 4^2, 3^2, 2^3, 1^2).
\]

The exceptional divisor associated with the Kirwan blowup of \( x \in \Lambda_3 \) is \( W_{\mathbb{P}_{18}} \times W_{\mathbb{P}_{21}} \), where

\[
W_{\mathbb{P}_{18}} \cong \mathbb{P}(25, 21, 18, 17, 16, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1)
\]

and

\[
W_{\mathbb{P}_{21}} \cong \mathbb{P}(1^2, 2, 3, 4, 5, 6^2, 7, 8, 9, 10^2, 11^2, 12, 13, 14, 15, 16, 18, 20).
\]
The fiber of the Kirwan blowup over \( x \in \Lambda_4 \) is
\[ \mathbb{P}(15, 12, 11, 10, 9, 8, 7^2, 6^2, 5, 4, 3^2, 2^2, 1) \times \mathbb{P}(1^2, 2^2, 3^3, 4^3, 5^3, 6^3, 7^2, 8^2, 9, 10). \]

**Proof.** Let \( x \) be a semistable point with closed orbit mapping to the GIT boundary \( \Lambda_2, \Lambda_3, \) or \( \Lambda_4. \) Our statement follows after finding the eigenvalues associated with the action of the stabilizer \( G^x_0 \cong \mathbb{C}^* \) on the normal bundle \( N_x. \) Given a one-parameter subgroup \( \lambda_k \) with \( k = \{2, 3, 4\}, \) the \( \lambda_k \) equivariant decomposition \( W \cong \bigoplus a_i V_{a_i} \) induces a decomposition of the space of monomials \( \text{Sym}^5(W) = \bigoplus_{\alpha} V_{\alpha}^{\otimes n_{\alpha}}. \) We can choose the point \( x, \) so it parametrizes quintic surfaces given by equations (3.2), (3.3), and (3.4). To calculate \( T_{P^N}, \) we use the Euler sequence (3.9). The line bundle \( O_{P^N} \) and \( O_{P^N}(1) = V_0 \) has weight 0. At \( x, \) from the Euler sequence we obtain
\[ 0 \to V_0 \to V_0 \otimes \bigoplus_{\alpha} V_{\alpha}^{\otimes n_{\alpha}} \to T_{P^N}|_x \to 0, \]
from which we obtain the decomposition of \( T_{P^N}|_x. \) To obtain the decomposition of \( T_{G.x}|_x, \) we use the exact sequence (3.10). The tangent space to \( G \) is the Lie algebra \( \mathfrak{sl}(4, \mathbb{C}), \) and \( T_{G.x} \) is the adjoint representation of \( \lambda_k \cong \mathbb{G}_m. \) The one-parameter subgroup \( \lambda_k \) acts by conjugation on \( \mathfrak{sl}(4, \mathbb{C}) \) with eigenvalues of the form \( a_i - a_j \) for all \( i \)'s, \( j \)'s. Therefore, the exact sequence (3.10) becomes
\[ 0 \to V_0 \to \bigoplus_{i,j} V_{(a_i - a_j)} \to T_{G.x}|_x \to 0. \]
The expression of the normal bundle for each \( \lambda_k \) follows from the exact sequence (3.8). \( \square \)

4. Stable isolated singularities

In this section we complete the characterization of stable normal quintic surfaces. We also discuss the role of invariants from singularity theory in determining the stability of these surfaces.

4.1. Stability of triple point singularities. By Corollary 2.5 we know that a quintic surface whose singularities are at worst isolated double points or isolated triple points with reduced tangent cone is stable. Quadruple points are unstable by Proposition 2.7. Next, we consider triple points singularities with nonreduced tangent cone.

Recall that a surface singularity is of type \( \tilde{E}_8 \) if, at the completion of the local ring, its equation is equivalent to \( z^2 + x^3 + y^6 + tx^2y^2 \) with \( 4t^3 + 27 \neq 0. \) Similarly, a singularity is of type \( Z_{13} \) if its equation can be written as \( z^2 + x^2y + y^6 + txy^5. \) A singularity is of type \( W_{1.0} \) if its equation can be written as \( x^4 + (t + y)x^2y^3 + y^6, \) with \( t^2 \neq 4 \) (for details see [Arn76]).

**Proposition 4.1.** Let \( X \) be a normal quintic surface with a triple point singularity with nonreduced tangent cone at \( p \in X. \) We suppose that any other singularity of \( X \) is either an isolated double point or a triple point with a reduced tangent cone. Let \( Bl_pX \) be the monomial transformation of \( X \) with a center at \( p. \)

1. If the tangent cone of \( p \in X \) is a union of a double plane and another plane, then \( X \) is nonstable if and only if \( Bl_pX \) has a line of singularities, a singularity of type \( Z_{13}, \) or a degeneration of it.
2. If the tangent cone of \( p \in X \) is a triple plane, then \( X \) is nonstable if and only if \( Bl_pX \) has a line of singularities, a singularity of type \( \tilde{E}_8 \), a singularity of type \( W_{1,0} \), or a degeneration of them.

**Proof.** We first describe representations of quintic surfaces with a triple point as double covers of \( \mathbb{P}^2 \). Let \( p \in X \) be a triple point on a reduced quintic surface which contains only a finite number of lines through \( p \). If \( Bl_pX \to X \) is the monomial transformation of \( X \) centered at the triple point, then we have a natural morphism \( Bl_pX \to \mathbb{P}^2 \). Its Stein factorization is \( Bl_pX \to X^* \to \mathbb{P}^2 \), where \( X^* \) is a double cover of \( \mathbb{P}^2 \) branched over an octic plane curve \( B(X) \). If the equation of the quintic surface is

\[
(4.1) \quad F_X(x_0, x_1, x_2, x_3) := x_3^2 f_3(x_0, x_1, x_2) + x_3 f_4(x_0, x_1, x_2) + f_5(x_0, x_1, x_2),
\]

then the equation of \( B(X) \) is

\[
(4.2) \quad F_{B(X)} = f_3(x_0, x_1, x_2)f_5(x_0, x_1, x_2) - f_4(x_0, x_1, x_2)^2.
\]

The map \( Bl_pX \to X^* \) contracts the proper transform of the lines \( L \subset X \) through the triple point, and it is an isomorphism everywhere else. Thus, if no line in \( X \) passes through \( p \), then \( Bl_pX \cong X^* \). If the singularity of \( X \) is supported at \( p \), then the singularities on \( Bl_pX \) are supported in the exceptional divisor, \( E \), of the monomial transformation. The reduced image of \( E \) in \( \mathbb{P}^2 \) is the curve defined by \( f_3(x_0, x_1, x_2) = 0 \). By using partial derivatives, we can see that the singularities of \( Bl_pX \) are supported at

\[\text{Sing} \left( f_3(x_0, x_1, x_2) = 0 \right) \cap \left( f_4(x_0, x_1, x_2) = 0 \right).\]

Notice that under our hypotheses, \( B(X) \) completely determines both \( Bl_pX \) and \( X \) because either \( Bl_pX \) is the double cover of \( \mathbb{P}^2 \) branched over \( B(X) \) or it resolves the rational map from \( X^* \) to \( X \). Clearly, equation \((4.1)\) determines such a surface \( X \). This equation is enough for us because we will prove the lack of stability of \( X \). Then we should exhibit an \( F_X \) and a subgroup of \( \lambda \) such that \( \Xi_{F_X} \subset M^\oplus(\lambda) \). We now proceed to prove Proposition 4.2.

First, we suppose that \( X \) is nonstable. By our hypotheses and the results of section 2, there is a change of coordinates such that \( \Xi_{F_X} \subset M^\oplus(\lambda_i) \) for \( i \in \{1, 3, 4\} \) (see Table 1). From the equations in the proof of Proposition 2.2, we obtain the following:

1. If \( \Xi_{F_X} \subset M^\oplus(\lambda_1) \), then the tangent cone of \( T_pX \) is a union of a double plane and another plane, and \( Bl_pX \) has a line of singularities.

2. If \( \Xi_{F_X} \subset M^\oplus(\lambda_3) \), then we have two options: First, the tangent cone of \( T_pX \) is a union of a double plane and a different plane, and \( Bl_pX \) has either a singularity of type \( Z_{13} \) or a degeneration of it. Second, the tangent cone of \( T_pX \) is a triple plane and \( Bl_pX \) has either a singularity of type \( W_{1,0} \) or a degeneration of it.

3. If \( \Xi_{F_X} \subset M^\oplus(\lambda_4) \), then the tangent cone of \( T_pX \) is a triple plane and \( Bl_pX \) has either a \( \tilde{E}_8 \) singularity or a degeneration of it.

Next, we show that if \( Bl_pX \) has singularities as in the statement of the proposition, then \( X \) is nonstable. The case when \( Bl_pX \) is nonnormal follows from Corollary 2.3. Now we suppose that \( Bl_pX \) has a singularity of type \( Z_{13} \) and the tangent cone of \( X \) at \( p \) is the union of a plane and another one. We may assume that the triple point is supported at \( p_3 \) and that its tangent cone is given by
singularity of degree 6 with respect to the weights
Proof.
Invariants of singularities and GIT stability.
Proposition 4.2. If
et al. [EAF86].
singularities due to Arnold [Arn76], Suzuki, Yoshinaga [YW78, Suz81], and Estrada
ity of a normal quintic surface to the numerical invariants of its singularities. We
is supported at [0 : 0 : 0 : 1], and the tangent cone is supported at (x0 = 0). Then the equation of B(X) can be
written as
\[
F_{B(X)} = x_0^2 f_1(x_0, x_1) \left( x_2^4 x_0 + x_2^3 f_2(x_0, x_1) + x_2^2 f_3(x_0, x_1) + x_2 f_4(x_0, x_1) \right) \\
+ f_5(x_0, x_1) + \left( x_2^2 x_0^2 + x_2 f_3(x_0, x_1) + f_4(x_0, x_1) \right)^2.
\]
By comparing \(F_X\), as constructed in equation (4.1), with \(F_{\lambda_3}\) we obtain that \(\Xi_{F_X} = M^{\Sigma}(\lambda_3)\).

Now we suppose that the tangent cone of \(p \in X\) is a triple plane and \(B_{lp}X\) has an
\(\tilde{E}_8\) singularity. We may select a coordinate system such that the triple point of \(X\)
is supported at [0 : 0 : 0 : 1], and the tangent cone is supported at (x0 = 0). By our
hypothesis, \(B_{lp}X\) has a \(\tilde{E}_8\) singularity. Then \(B(X)\) has a semi–quasi-homogeneous
singularity of degree 6 with respect to the weights \(w(x) = 3\) and \(w(y) = 2\) at [0 : 0 : 1]. These weights determine the \(\tilde{E}_8\) singularity in the double cover. The
most general equation for such an octic plane curve can be written as
\[
F_{B_X} = x_0^3 f_5(x_0, x_1, x_2) - (x_2^2 x_0 f_1(x_0, x_1) + x_2 f_3(x_0, x_1) + f_4(x_0, x_1))^2.
\]
By comparing \(F_X\), as constructed in equation (4.1), to \(F_{\lambda_4}\), we obtain that \(\Xi_{F_X} = M^{\Sigma}(\lambda_4)\).

Now we suppose that the tangent cone of \(p \in X\) is a triple plane, and \(B_{lp}X\) has an
\(W_{1,0}\) singularity. By the same argument as above, we find that
\[
F_{B(X)} = x_0^3 \left( x_2^4 x_0 + x_2^3 f_2(x_0, x_1) + x_2^2 f_3(x_0, x_1) + x_2 f_4(x_0, x_1) + f_5(x_0, x_1) \right) \\
+ \left( x_2^2 x_0^2 + x_2 f_3(x_0, x_1) + f_4(x_0, x_1) \right)^2.
\]
By comparing \(F_X\), as constructed in equation (4.1), to \(F_{\lambda_3}\), we obtain that \(\Xi_{F_X} \subset M^{\Sigma}(\lambda_3)\).

4.2. Invariants of singularities and GIT stability. Next, we relate the stability
of a normal quintic surface to the numerical invariants of its singularities. We
start with a Milnor number and modality which are used in the classification of
singularities due to Arnold [Arn76], Suzuki, Yoshinaga [YW78, Suz81], and Estrada
et al. [EAF86].

Proposition 4.2. If \(X\) is a normal quintic surface where each of its singularities
has either a Milnor number smaller than 22 or a modality smaller than 5, then \(X\)
is stable.

Proof. We argue by contradiction. If the surface \(X\) is not stable, then Proposition
2.1 implies that there is a coordinate system such that \(\Xi_{F_X}\) is contained in one
of the \(M^{\Sigma}(\lambda_i)\)’s for \(i = 1, 3, 4, 7\). In particular, we can assume that a destabilizing
isolated singularity of \(X\) is supported at \(p_3\). We consider a deformation of \(X\) to the
general quintic (\(F_{\lambda_i} = 0\)). Then the destabilizing singularity of \(X\) deforms to the
singularity of (\(F_{\lambda_i} = 0\)). By Theorem 3.1 the singularities at (\(F_{\lambda_i} = 0\)) are either
\(V^*_{24}\) (notation as [EAF86]), \(V^{*1}_{24}\) (notation as [Suz81]), \(V^*_{22}\) (notation as [Suz81]), or
an ordinary quadruple point.
Now we use the fact that the Milnor number of the $V_{24}^*$ singularity is 24 and that its modality is 7. The Milnor number of the $V_{24}^1$ singularity is 24 and its modality is 5. The Milnor number of the $V_{22}^1$ singularity is 22 and its modality is 5. The Milnor number of a quadruple point is at least 27 and its modality is at least 6 (see [Suz81], [EAF86]). Therefore, the statement follows from the upper semicontinuity of both the Milnor number and the modality.

**Example 4.3.** The previous bound in the Milnor number is not a necessary condition for stability. Indeed, at $(t \neq 0)$ the zero set of the equation

$$F_t(x_0, x_1, x_2, x_3) = x_0^2x_3^2 + 2x_2x_1^3x_3 + x_2^5 + t(x_3x_2^3 - 3x_0^2x_1^3 + 3x_0x_1^4 + x_2^2x_0^2)$$

has a weakly elliptic singularity at $p_3$, which is formally equivalent to the singularity induced by the equation $x^2 + y^3 + z^{13}$ (see [Yan86] p. 452). The Milnor number of this singularity is 24; however, the surface $(F_t(x_0, x_1, x_2, x_3) = 0)$ is stable by Corollary 2.5. The zero set $(F_0(x_0, x_1, x_2, x_3) = 0)$ is a nonnormal surface parametrized by $\Lambda_4$.

For a non–log-canonical singularity $p \in X$, the log-canonical threshold $c_p(X)$ is a number valued between 0 and 1 such that the smaller its value, the worse the singularity (see [Ko97] p. 45) for definitions and details). The relationship between the log-canonical threshold and the GIT stability given in Lemma 4.4 below was first noticed by Hacking [Hac01] Prop. 10.4 and Kim and Lee [KL04] Rmk. 2.4.

**Lemma 4.4.** Suppose $X$ is a normal quintic surface such that every singularity $p \in X$ satisfies $c_p(X) < 4/5$ ($\leq 4/5$). Then $X$ is stable (semistable).

**Remark 4.5.** The converse of [KL04] Rmk. 2.4 does not hold in general. For example, there are semistable quartic plane curves with an $A_5$ singularity.

Next, we describe a natural family of singularities, called minimal elliptic, with a log-canonical threshold greater than 4/5.

**Definition 4.6.** Let $X$ be a normal surface singular at $p$. The geometric genus of the singularity $p \in X$ is $\dim(R^1\pi_*O_Y)$, where $\pi : Y \to X$ is a resolution of $X$ at $p$.

This invariant induces a well-known classification of singularities: **Rational singularities** are those for which the geometric genus is 0. For surfaces, the rational Gorenstein surface singularities are the ADE ones. After rational surface singularities we find the family of minimal elliptic singularities classified by Laufer [Lau77]. Next, we provide not the original definition of minimal elliptic singularities but, rather, a convenient one. Recall that we work with isolated hypersurface singularities which are always Gorenstein.

**Definition 4.7 (Lau77 Th. 3.10).** A surface singularity is minimal elliptic if and only if it is Gorenstein and $\dim R^1\pi_* (O_Y) = 1$.

An important application of Lemma 4.4 is the GIT stability of the minimal elliptic singularities.

**Proposition 4.8.** Let $X \subset \mathbb{P}^3$ be a normal quintic surface with either ADE or minimal elliptic singularities. Then $X$ is stable.

**Proof.** Log-canonical thresholds for minimal elliptic singularities for surfaces in $\mathbb{P}^3$ were computed by Prokhorov in [Pro03] Tables 1–3. In particular, they are always greater than or equal to $(\frac{4}{5} + \frac{1}{180})$. Therefore, the claim follows by Lemma 4.4 and the fact that ADE singularities are canonical.
The genus of a singularity \( p \in X \) can be interpreted by its effect on the geometric genus, \( p_g(X) \), of the variety \( X \). We include a proof for completeness.

**Proposition 4.9.** Given the minimal resolution \( \pi: Y \to X \) of a normal hypersurface of degree \( d \geq 4 \), with a unique non-ADE singularity of genus \( R^1(\pi_*O_Y) \), we have

\[
\frac{(d - 1)(d - 2)(d - 3)}{6} - p_g(Y) + q(Y) = R^1(\pi_*O_Y).
\]

Furthermore, if \( X \) is a quintic surface and \( Y \) is of general type, then \( q(Y) = 0 \), and we have

\[
4 - p_g(Y) = R^1(\pi_*O_Y).
\]

**Proof.** On a normal hypersurface \( X \) of degree \( d \geq 4 \), we have \( H^1(X, O_X) = q(X) = 0 \) and

\[
H^2(X, O_X) = p_g(X) = (d - 1)(d - 2)(d - 3)/6.
\]

From those values and the exact sequence (see [Yan86], p. 433)

\[
0 \to H^1(X, O_X) \to H^1(Y, O_Y) \to R^1(\pi_*O_Y) \to H^2(X, O_X) \to H^2(Y, O_Y) \to 0,
\]

we obtain \( p_g(X) - \dim R^1(\pi_*O_Y) = p_g(Y) - q(Y) = p_a(Y) \). If \( X \) is a quintic surface and \( Y \) is of general type, then the irregularity \( q(Y) \) vanishes by [Ume94]. \( \Box \)

The following result illustrates the complexity of the singularities in a semistable surface.

**Proposition 4.10.** There is at least one semistable hypersurface \( X \subset \mathbb{P}^3 \) of degree \( d \geq 4 \) with an isolated quasi-homogeneous singularity of genus \( g(d) \) where

\[
g(d) = \begin{cases} 
\frac{d(d-2)(4d-10)}{48} & \text{if } d \text{ is even}, \\
\frac{(d-1)(d-3)(4d-2)}{48} & \text{if } d \text{ is odd}.
\end{cases}
\]

Note that \( g(4) = 1 \), \( g(5) = 3 \), and \( g(6) = 7 \).

**Proof.** From the combinatorics of the GIT setting, it is clear that for any degree \( d \geq 4 \) the one-parameter subgroup \( \lambda_1 = (1, 0, 0, -1) \) is always critical and \( M^{\mathbb{G}}(\lambda_1) \) is a maximal semistable set. The generic associated semistable surface is the zero set of the polynomial \( F_{\lambda_1}(x_0, x_1, x_2, x_3) \) stabilized by \( \lambda_1 \). If \( d = 2m + 1 \), then

\[
F_{\lambda_1}(x_0, x_1, x_2, x_3) = x_3^m x_0^m f_1(x_1, x_2) + x_3^{m-1} x_0^{m-1} f_3(x_1, x_2) + \cdots + f_{2m+1}(x_1, x_2),
\]

and a similar equation is associated with the case \( d = 2m \). After localizing, we have a quasi-homogeneous polynomial with weights \( w = (2, 1, 1) \) and weighted multiplicity \( d \), and whose weighted leading term induces an isolated singularity. The geometric genus of a quasi-homogeneous isolated singularity hypersurface is determined by its weights. We apply a lemma [YWW78], p. 48] using the expressions \( n_i/d = (a_i - a_3)/w(f) \), where \( (a_0, a_1, a_2, a_3) = (1, 0, 0, -1) \) and \( w(f_a) - d = 0 \). Therefore, the geometric genus of the singularity at \( F_{\lambda_1} \) is given by the number of nonnegative integer solutions of the following equations:

\[
d = i_0 + i_1 + i_2 + i_3,
\]

\[
|a_3| (d - 4) \geq (a_0 - a_3) i_0 + (a_1 - a_3) i_1 + (a_2 - a_3) i_2.
\]
This is calculated by a standard method which was shown to us by E. Rosu. From it, we find that the geometric genus is equal to
\[
\left\lfloor \frac{d-4}{2} \right\rfloor \sum_{k=0}^{d-2} \left( \frac{d-2 - 2k}{2} \right).
\]
This formula becomes the expression of the statement after some algebraic manipulations.

5. Stability and nonisolated singularities

In this section we give a partial description of the stability for reducible quintic surfaces, surfaces with a certain curve of singularities of multiplicity 3, and quintic surfaces with a line of singularities of multiplicity 2. These cases will complement the ones discussed in previous sections (see Proposition 2.6 and Corollaries 2.9 and 2.12).

5.1. Reducible quintic surfaces. A generic quintic surface that decomposes as a union of a quartic surface and a hyperplane is GIT stable. The locus, called \(\overline{M}(4,1)\), in the GIT quotient that parametrizes those surfaces is 22 dimensional: 19 dimensions are associated with the moduli of \(K3\) surfaces, and three dimensions are associated with the choice of a hyperplane in \(\mathbb{P}^3\).

**Proposition 5.1.** Let \(X\) be a quintic surface that decomposes as a union of a hyperplane \(H\) and a quartic normal surface \(Y\) with isolated singularities such that the following apply:

1. The isolated singularities in our quartic surface do not destabilize the quintic surface.
2. The singular locus of the quartic surface \(Y\) is disjoint from the hyperplane, and the quartic plane curve \(Y \cap H\) has at worst a triple point whose tangent cone has a double line.

Then \(X\) is stable.

**Proof.** If the quintic surface is nonstable, then there is a coordinate system and a normalized 1-PS \(\lambda\) such that \(\mu(X, \lambda) \geq 0\) and the equation of \(X\) is \(F_X\). By condition (1) in the statement, \(p_\lambda\) must be supported in the intersection of the hyperplane with the quartic surface, and \(p_\lambda\) must have multiplicity 2 because \(\text{Sing}(Y)\) is disjoint from \(H\). By our results in section 2, \(\Xi_{F_X}\) is contained in \(M^\oplus(\lambda_k)\) for \(k = 5, 9, 10\) (see Table 1). However, this is not possible according to our hypothesis about \(Y \cap H\), the fourth case of Proposition 2.2 and the third and the fourth cases of Proposition 2.3.

Next, we describe the intersection between \(\overline{M}(4,1)\) and our GIT boundary. We can say, somewhat informally, that these are the worst unions of a quartic surface and a hyperplane parametrized by our GIT quotient.

**Proposition 5.2.** Let \(X\) be a quintic surface parametrized by a point in the intersection between the locus \(\overline{M}(4,1)\) and \(\Lambda_1\). Then one of the following conditions holds.

1. The surface \(X\) is parametrized by \(\Lambda_1\) and satisfies the following:
   * The quartic surface has two \(\tilde{E}_7\) singularities.
The intersection of the hyperplane and the quartic surface is a union of two conics of the form
\[(xy - a_1 z^2)(xy - a_2 z^2) = 0\].

The hyperplane does not intersect the singularities along their tangent cones.

2. The surface $X$ is parametrized by $\Lambda_2$ and satisfies the following:
* The singular locus of the quartic surface decomposes as a union of two coplanar double lines $L_1$ and $L_2$ intersecting at a nonisolated triple point with an associated equation of the form $x^2y + x^3z + y^2z^2$.
* The intersection of the hyperplane and the quartic surface decomposes as a union of a cuspidal plane curve and a line. That line is contained in the quartic surface. The singularity of the cuspidal curve is away from the triple point.

3. The surface $X$ is parametrized by $\Lambda_4$ and satisfies the following:
* The singular locus of the quartic surface has a double line $L$ and a distinguished triple point given by the equation $x^3 - xyz^2 + zy^3$ which is away from the hyperplane.
* The intersection of the hyperplane and the quartic surface is a union of two lines and a conic tangent to one of them.

We represent those geometric characteristics in Figure 2.

![Figure 2](image-url)

**Figure 2.** $Y_i$’s are our quartic surfaces, the dotted lines are the intersection $Y_i \cap H$, bold lines are the singular loci of the $Y_i$’s, and the numbers are the multiplicity of the singularities at those points.

**Proof.** Let $X$ be such a quintic surface. Then there is a one-parameter subgroup $\lambda$ such that $X$ is invariant under the action of it. By construction, $X = Y \cup H$ and it is easily seen that the hyperplane is also invariant under the action of $\lambda$. In particular, this implies that in our coordinate system the equation associated with $H$ must be $(x_i = 0)$. From our results in section 3 and up to a change of coordinates, we have the equations of these surfaces. So the statement reduces to describing their geometric characteristics, which follow from their equations:

\[
\tilde{F}_{\lambda_2} = x_0(x_3^2x_1^2 + x_3x_0x_2^2 + x_1x_2^3), \\
\tilde{F}_{\lambda_4} = x_3(x_0^3x_3 + x_2x_3^3 + x_0x_1x_2^2), \\
\tilde{F}_{\lambda_1} = x_1\left(x_3^2x_0^2 + x_0x_3f_2(x_1, x_2) + f_4(x_1, x_2)\right).
\]

Next, we show that a quintic surface with a nonlinear curve of singularities of multiplicity 3 decomposes as a union of a quartic surface and a hyperplane, and it is generically stable.
Proposition 5.3. Let $X$ be a quintic surface with a curve of singularities $C$ such that $C$ does not contain a line and $\text{mult}_p(X) = 3$ for every $p \in C$. Then $X$ decomposes as a union of a hyperplane and a quartic surface, and there is a coordinate system such that its associated equation can be written as

$$x_i \left( f_2(x_j, x_k, x_l)^2 + x_i^2 g_2(x_0, x_1, x_2, x_3) + x_i f_2(x_j, x_k, x_l) f_1(x_0, x_1, x_2, x_3) \right).$$

Moreover, this surface is generically stable (compare this with Proposition 2.6).

Proof. Let $C$ be such a curve. Consider two generic distinct points $p$ and $q$ on it, and let $L_{p,q}$ be the line that joins them. Since $p$ and $q$ are triple points, $L_{p,q}$ intersects $X$ with multiplicity greater than or equal to 6. However, since $X$ is a quintic surface, this implies that the surface contains the line $L_{p,q}$ for every $p$ and $q$ on $C$. Then $X$ contains the secant variety $\text{Sec}(C)$ of $C$. For a curve $C$ in $\mathbb{P}^3$, the secant variety of $C$ is either the whole $\mathbb{P}^3$ or a hyperplane, with the latter option happening only if $C$ is a plane curve itself (see [Har92, p. 144]). Then $C$ is a plane curve, and $X$ decomposes as a hyperplane $H$ and a quartic surface $Y$. Moreover, from the hypotheses and by degree considerations, $C$ is a smooth conic. Let our coordinate system be such that the critical one-parameter subgroups are the ones in Proposition 2.1 and the hyperplane is given by some $(x_i = 0)$. Then the equation associated with the quintic surface can be written as

$$x_i \left( f_4(x_j, x_k, x_l) + x_i g_3(x_0, x_1, x_2, x_3) \right).$$

By our hypotheses, $m_p(X) = 3$ for every point $p \in C \cap H$ and $C$ does not contain a line. Then it holds that $f_1(x_j, x_k, x_l) = (f_2(x_j, x_k, x_l))^2$ and either $x_i$ or $f_2(x_j, x_k, x_l)$ divides $g_3(x_0, x_1, x_2, x_3)$. In our coordinate system the most general equation satisfying these properties is the one of the statement.

Given a normalized one-parameter subgroup $\lambda = (a_0, a_1, a_2, a_3)$, we have

$$\mu(\lambda, X) \leq \min\{a_i + 2\mu(\lambda, f_2), 3a_i + \mu(\lambda, g_2), 2a_i + \mu(\lambda, f_2) + \mu(\lambda, l)\}.$$

In our coordinate system the curve $C$ cannot be supported at $(x_3 = 0)$ because a triple point is supported at both $p_3$ and $H$. By the construction and smoothness of $C$, we have

$$f_2(x_j, x_l, x_3) = x_3 f_2(x_j, x_l) + p_2(x_j, x_l),$$

with the set of monomials $\Xi_{f_2}$ containing at least $\{x_3 x_j x_l^2\}$ with $j \neq l$ and $j, l \neq i$. Additionally, it holds generically that $\mu(\lambda, g_2) \leq 2a_i$. Therefore,

$$\mu(\lambda, X) \leq \min\{a_i + 2(a_3 + a_j), a_i + 2a_2, a_i + (a_3 + a_j) + a_0, 2a_i + 2a_i + a_1\}.$$

A direct calculation shows that $\mu(\lambda_k, X) \leq 0$. Thus, $X$ is semistable. \qed

Next, we consider a quintic surface that decomposes as a union of a cubic and a quadric surface. On the moduli space, the locus that parametrizes these surfaces is 13 dimensional: Nine dimensions arise from the genus 4 curve defined by the intersection of the cubic and the quadric surface. The other four dimensions arise from the fact that we can add a multiple of the quadratic equation to the cubic surface equation without changing the genus 4 curve.

Proposition 5.4. Let $X$ be a union of a smooth quadric surface $Q$ and a cubic surface $Y$ with a triple point at $p \notin Q$. This triple point destabilizes the quintic surface if and only if the tangent cone of $Y$ at $p$ is either a union of a conic and tangent line or a degeneration of it.
Proof. Suppose $X$ is not stable, and our coordinate system is such that the critical 1-PSs are the ones in Proposition 2.4 and the triple point is supported at $p_3$. The cubic surface is a cone over a plane cubic curve $C$, and the equation of the quintic surface is $f_3(x_0, x_1, x_2)g_2(x_0, x_1, x_2, x_3)$. By our hypothesis the quadratic surface is away from the triple point. Therefore, the monomial $x_3^2$ is always present in $\mathfrak{E}_{g_2}$, which implies that $\mu(\lambda, X) = 2a_3 + \mu(\lambda, f_3)$. The following analysis is divided by the singularities of the cubic curve.

(1) If $C$ has a triple point, then $X$ is unstable because it has either a triple line or a double plane (see Proposition 2.6 or 2.12).
(2) If $C$ is a union of a conic with a transversal line, then the equation of the quintic surface can be written as $F_X = x_0(x_2x_0 - x_1^2)f_2(x_0, x_1, x_2, x_3)$, which is destabilized by $\lambda_9$.
(3) If $C$ is a union of three nonconcurrent lines, then an equation of the quintic surface is

$$f_1(x_0, x_1, x_2)g_1(x_0, x_1, x_2)h_1(x_0, x_1, x_2)f_2(x_0, x_1, x_2, x_3),$$

and the monomial $x_0x_1x_2x_3^2$ must have a coefficient other than 0 because the lines are not concurrent. The presence of this monomial implies that $\mu(\lambda_k, X) < 0$ for all $\lambda_k$’s.
(4) If $C$ is a union of a conic and a tangent line, then it deforms to three nonconcurrent lines and the stability of $X$ follows by the previous case.
(5) By considering the partial order among monomials (see section 2), if $C$ has a cuspidal singularity, then in our coordinate system any surface $X$, as in the statement, satisfies $\mu(\lambda, X) \leq \mu(\lambda, X_0)$, where

$$F_{X_0} = (x_i^2x_j + x_i^3 + p_3(x_0, x_i))g_2(x_0, x_1, x_2, x_3).$$

The statement follows from the inequality $\mu(\lambda_k, X_0) \leq \min\{2a_i + a_j + 2a_3, 3a_0 + 2a_3 | i, j \neq 0\} < 0$.
(6) Finally, if $C$ is a node, then $C$ deforms to a curve with cuspidal singularity and the statement follows from the previous case. $\square$

5.2. Quintic surfaces with a curve of singularities of multiplicity 2. Given a quintic surface with a curve of multiplicity 2, we use a sequence of blowups for constructing a triple cover of $\mathbb{P}^2$ branched over a curve of degree 12. Our purpose is to describe their general form and illustrate the diversity of singular surfaces parametrized by our quotient.

Lemma 5.5. Let $X$ be a quintic surface defined by the equation

$$(5.1) \quad F_X := x_3^2x_0^2 + x_3^2x_0g_2(x_0, x_1, x_2) + x_3f_4(x_0, x_1, x_2) + f_5(x_0, x_1, x_2),$$

and consider the surface in $\mathbb{P}(2, 1, 1, 1)$ defined by the equation

$$G_{F_X} = \psi^3 + \left(f_4 - \frac{g_2^2}{3}\right)\psi + \left(x_0f_5 + \frac{2}{27}g_2^2 - \frac{g_2f_4}{3}\right)$$
$$:= \psi^3 + h_4(x_0, x_1, x_2)\psi + h_6(x_0, x_1, x_3).$$

The following statements are equivalent:

(i) The set $\Xi_{F_X}$ is contained in either $M^\oplus(\lambda_5)$ or $M^\oplus(\lambda_9)$. 
(ii) The polynomials \( h_4(x_0, x_1, x_2) \) and \( h_6(x_0, x_1, x_2) \) are obtained from a linear combination of the monomials in the sets

\[
\Xi_{h_4} = \left\{ x_0^j x_1^i x_2^j \mid w_0 j_0 + w_1 j_1 + w_2 j_2 \geq c_1(k) \ ; \ j_0 + j_1 + j_2 = 4 \right\},
\]
\[
\Xi_{h_6} = \left\{ x_0^j x_1^i x_2^j \mid w_0 j_0 + w_1 j_1 + w_2 j_2 \geq c_2(k) \ ; \ j_0 + j_1 + j_2 = 6 \right\}.
\]

For the one-parameter subgroup \( \lambda_5 \), we have \( w_0 = 5, w_1 = 2, w_2 = 1, c_1(5) = 10, \) and \( c_2(5) = 15 \). For the one-parameter subgroup \( \lambda_9 \) we have \( w_0 = 11, w_1 = 5, w_2 = 0, c_1(9) = 20, \) and \( c_2(9) = 30 \).

**Proof.** We recall the representation of quintic surfaces with a double point as a finite cover of the plane (see [Yan86, p. 471]). Let \( \tilde{X} \to X \) be the monomial transformation of \( X \) with a center at \( p = [0 : 0 : 0 : 1] \). There is a morphism \( \tilde{X} \to \mathbb{P}^2 \) induced by the projection from the point \( p \in X \) that is generically finite of degree 3. The surface \( \tilde{X} \) is given by the equation

\[
t^3 x_0^2 + t^2 s x_0 g_2(x_0, x_1, x_2) + ts^2 f_4(x_0, x_1, x_2) + s^3 f_5(x_0, x_1, x_2),
\]

with \([t : s] \in \mathbb{P}^1\). From the equation we see \( \tilde{X} \) is singular along the line \((s = x_0 = 0)\). Blowing up \( \tilde{X} \) along this line in one of the charts, the total transform \( X' \) is given by

\[
x_0^2 (t^3 + t^2 s g_2(x_0, x_1, x_2) + ts^2 f_4(x_0, x_1, x_2) + x_0 s^3 f_5(x_0, x_1, x_2)).
\]

In this chart, we take \( \psi = t/s \) and substitute \( \psi - g_2(x_0, x_1, x_2) \) to obtain the equation \( G_{F_X} \).

**Claim:** (i) implies (ii). We suppose that \( \Xi_{F_X} \subset M^\oplus(\lambda_k) \) for \( k \in \{5, 9\} \). Given a polynomial \( h_d \) of degree \( d \), we denote its set of nonzero monomials as \( \Xi_{h_d} \). For \( \lambda_k \)

with \( k \in \{5, 9\} \) and the monomial \( x_0^{i_0} x_1^{i_1} x_2^{i_2} x_3^{i_3} \) (which we denote as \([i_0, i_1, i_2, i_3]\)), with \( i_0 + i_1 + i_2 + i_3 = 5 \), it holds that

\[
\lambda_k [i_0, i_1, i_2, i_3] = w(i_0, i_1, i_2) + 5a_3,
\]

where \( w(i_0, i_1, i_2) \) is equal to the weighted degree of the monomial \([i_0, i_1, i_2]\) with respect to \( w_0, w_1, \) and \( w_2 \), as in the statement. By equation (5.2) and simple arithmetic, we find that the following statements are equivalent:

(*) The set \( M^\oplus(\lambda_k) \) contains \( \Xi_{x_3 f_4}, \Xi_{x_2 x_0 g_2}, \) and \( \Xi_{f_5} \).

(*) The weighted degrees of \( h_4 \) and \( h_6 \) satisfy \( w(h_4) \geq c_1(k) \) and \( w(h_6) \geq c_2(k) \) for the corresponding set of weights \( w \).

Our claim follows.

**Claim:** (ii) implies (i). We suppose that the monomials \([i_0, i_1, i_2]\) in \( \Xi_{h_4} \) and \( \Xi_{h_6} \) satisfy condition (ii), and there is a quintic surface such that the equation of \( G_{F_X} \) is induced by \( F_X \). Let \( m = [i_0, i_1, i_2] \) be a monomial in \( \Xi_{h_4} \), and by construction either \( m \in \Xi_{f_4} \) or there are two monomials \( m_1, m_2 \) in \( \Xi_{g_2} \) such that \( m = m_1 m_2 \). In the first case, by equation (5.3) we obtain that \( \lambda_k [i_0, i_1, i_2, 1] \geq 0 \).

The second case follows because, if \( w(m_1 m_2) \geq c_1(k) \), then \( \lambda_k \cdot x_3^2 x_0 m_i \geq 0 \). The same argument applies for monomials in \( h_6 \). Then conditions (5.2) imply \( \lambda_k \cdot m \geq 0 \). □

**Proposition 5.6.** An irreducible quintic surface \( X \) with a curve of singularities of multiplicity 2 is nonstable if and only if there is a coordinate system such that
$F_X$ is given as equation $[5,1]$ and the branch locus associated with the morphism 
$(G_{FX}(v,x_0,x_1,x_2) = 0) \rightarrow \mathbb{P}^2$ can be written as one of the following equations:

$$D_{\lambda_9}(x_0,x_1,x_2) = x_0^2 \left( x_2^7 x_0^3 + \sum_{k=1}^{2} x_k^2 \sum_{i=0}^{2} x_2^{3k-i} f_{10-4k+i}(x_0,x_1) + f_0(x_0,x_1) \right),$$

$$D_{\lambda_9}(x_0,x_1,x_2) = x_2^6 x_0^5 x_1 + \sum_{i=0}^{5} x_2^{i} x_0 f_{12-2i} f(x_0,x_1).$$

Proof. The morphism $(G_{X,p} = 0) \rightarrow \mathbb{P}^2$ is generically finite of degree 3. Its associated
branch locus is given by the equation $4h_4(x_0,x_1,x_2)^3 + 27h_6(x_0,x_1,x_2)^2$. By the results of section $2$, a quintic surface, as in the statement, is nonstable if and only if there is a coordinate system where $\Xi_{FX} \subset M_{\Theta}(\lambda_k)$, with $k \in \{5,9\}$. Then the polynomials $h_4$ and $h_6$ in the equation of the branch locus satisfy inequalities as described in Lemma $5.5$. Our statement describes the most general branch loci that satisfy those inequalities. \hfill $\square$

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Department of Mathematics, Washington University at St. Louis, 1 Brookings Drive, St. Louis, Missouri 63130

Email address: pgallardocandela@wustl.edu