CUTOFF ON GRAPHS AND THE SARNAK-XUE DENSITY OF EIGENVALUES

Konstantin Golubev and Amitay Kamber

Abstract. It was recently shown in [23] and [28] that Ramanujan graphs, i.e., graphs with the optimal spectrum, exhibit cutoff of the simple random walk in an optimal time and have an optimal almost-diameter. We show that this spectral condition can be replaced by a weaker condition, the Sarnak-Xue density property, to deduce similar results. This allows us to prove that some natural families of Schreier graphs of the $\text{SL}_2(F_t)$-action on the projective line exhibit cutoff, thus proving a special case of a conjecture of Rivin and Sardari, [27].

1. Introduction

Various works (e.g. [5, 24]) proved that many families of $(q + 1)$-regular graphs are expanders, and in particular their diameter (i.e., the largest distance between a pair of vertices) is equal to $\log_q(n)$ up to a multiplicative constant, where $n$ is the number of vertices. In this paper, we show that under certain conditions, the distance between most of the vertices in a graph is approximately optimal, that is, for every $\epsilon > 0$ and $n$ large enough, the distance is bounded by $(1 + \epsilon) \log_q(n)$. We start with a special case of Schreier graphs of $\text{SL}_2(F_t)$, which follows from Theorem 1.8 below.

Theorem 1.1. Let $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3 > 0$ and $l \in \mathbb{N}$ be fixed. Then there exists $t_0$ such that the following holds.

Let $t > t_0$ be a prime, let $F_t$ be the finite field with $t$ elements, and let $P^1(F_t)$ be the projective line over $F_t$. Let $s_1, s_2, \ldots, s_l$ be elements in $\text{SL}_2(F_t)$ chosen uniformly at random. Construct a $2l$-regular Schreier graph by connecting each point $\begin{bmatrix} a \\ b \end{bmatrix}$ of $P^1(F_t)$ to $s_i^\pm_1 \begin{bmatrix} a \\ b \end{bmatrix}$, $i = 1, \ldots, l$. Then with probability at least $1 - \epsilon_0$ the following two statements hold:

- (Almost-Diameter) The distance between all but $\epsilon_1 t^2$ of the pairs $\begin{bmatrix} a \\ b \end{bmatrix}$, $\begin{bmatrix} a' \\ b' \end{bmatrix} \in P^1(F_t)$ satisfies

$$d\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a' \\ b' \end{bmatrix}\right) \leq (1 + \epsilon_2) \log_{2^{l-1}}(t).$$

- (Cutoff) Consider the distribution of the simple random walk $A^k \delta_{x_0}$, starting from some $x_0 = \begin{bmatrix} a \\ b \end{bmatrix} \in P^1(F_t)$. Here $A$ is the normalized adjacency operator on $L^2(P^1(F_t))$ defined by the graph structure, and $\delta_{x_0}$ is the probability $\delta$-function supported on $x_0$. Then for all but $\epsilon_1 t$ of $x_0 \in P^1(F_t)$, for every

Konstantin Golubev, k.golubev@gmail.com
D-MATH, ETH Zurich, Switzerland

Amitay Kamber, ak2356@dpmms.cam.ac.uk
Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, UK.
$k > (1 + \epsilon_2) \frac{2t}{2t-2} \log_{2t-1}(n)$, it holds that
\[ \|A^k \delta_{x_0} - \pi\|_1 \leq \epsilon_3, \]
where $\pi$ is the constant probability function on $P^1(F_t)$.

Bourgain and Gamburd proved in [5] that the Cayley graphs of $SL_2(F_t)$ with respect to random generators are expanders with probability tending to 1 as $t \to \infty$. Since the graphs of Theorem 1.1 are quotients of those graphs, they are expanders as well, which implies that the distance between every two elements is bounded by $C \log_{2t-1}(t)$, $C$ some constant. Theorem 1.1 further shows that for every $\epsilon > 0$ and $t$ large enough, this constant is $(1 + \epsilon)$ when we consider almost all the pairs.

Let us provide some context to this result. Let $\mathcal{F}$ be a family of finite $(q+1)$-regular connected graphs with the number of vertices tending to infinity. The graphs can have multiple edges and loops. Let $X \in \mathcal{F}$ be a graph from the family, and let $n$ denote its number of vertices. By $A: L^2(X) \to L^2(X)$ we denote the normalized adjacency operator of $X$

$$Af(x_0) = \frac{1}{q+1} \sum_{x_1 \sim x_0} f(x_1),$$

where $x_1 \sim x_0$ means that $x_1$ is adjacent to $x_0$, and the sum is with multiplicity according to the number of edges between $x_0$ and $x_1$.

There are various results relating the eigenvalues of $A$ with the geometry of $X$. In particular, if the largest eigenvalue of $A$ in absolute value, excluding $\pm 1$, is bounded by some $\lambda < 1$, i.e., $X$ is an expander, then it is well known that the diameter of the graph is logarithmic in its size ([19]). Explicitly, in [9] it is proven that the diameter is bounded by

$$\left\lfloor \frac{\cosh^{-1}(n-1)}{\cosh^{-1}(1/\lambda)} \right\rfloor + 1.$$

A special case is when the graph is a Ramanujan graph, which means that $\lambda \leq 2\sqrt{q}/(q+1)$. The Alon-Boppana theorem states that Ramanujan graphs are asymptotically the best spectral expanders. Then the upper bound on the diameter is $2\log_q(n) + O(1)$, as proved already by Lubotzky, Phillips and Sarnak ([24]). Recently, it was proven by Lubetzky and Peres ([23]) and independently by Sardari ([28]) that when one considers the almost-diameter of a Ramanujan graph $X$, i.e., the distance between most of the pairs of vertices in the graph, it is bounded by $(1 + \epsilon) \log_q(n)$, which is, up to the $\epsilon$ factor, an optimal result. In [23] it is also shown that the simple random walk on $X$ exhibits cutoff in the $L^1$-norm. The same results hold for a sequence of graphs that are almost-Ramanujan, i.e., for every $\epsilon > 0$ and $n$ large enough, they satisfy $\lambda \leq 2\sqrt{q}/(q+1) + \epsilon$.

In practice, there are very few families of $(q+1)$-regular graphs that are known to be Ramanujan or almost-Ramanujan. For example, the question is completely open for the graphs underlying Theorem 1.1, or other natural families of graphs we discuss below. It is widely believed that those graphs are almost-Ramanujan, but proving it seems to be currently out of reach. As a corollary, they should have an optimal almost-diameter and exhibit a cutoff of the simple random walk. We refer to the work of Rivin and Sardari ([27]) for those conjectures and some experimental verification of them.

In this work, we show that some of the corollaries can be proved by replacing the Ramanujan condition with a weaker spectral density condition. The density condition is based on the work of Sarnak and Xue.
([30]) on multiplicities of automorphic representations. More importantly, as Sarnak and Xue showed in the context of Lie groups of rank 1, the density condition is also equivalent to a natural combinatorial path counting property, see Definition 1.3 and Theorem 1.4, allowing us to prove that it holds in many situations.

We use the following notations. For a real function \( f(X, y, z) \) of the graph \( X \) and auxiliary parameters \( y, z \), we write \( f(X, y, z) \ll_{X, y} g(X, y, z) \) if there exist constants \( n_0, C \) depending on the family \( \mathcal{F} \) and \( y \), but not on \( X \) and \( z \), such that when the size \( n \) of \( X \) is greater than \( n_0 \), it holds that \( f(X, y, z) \leq C g(X, y, z) \). The constants may also depend on the regularity \( q \), but we do not mention it explicitly. The dependence on \( \mathcal{F} \) is used to allow the family of graphs to depend on some other parameters, and we stress that the constant \( C \) will be the same for all the graphs \( X \in \mathcal{F} \). We write \( f(\cdot) = O_{\mathcal{F}, y} (g(\cdot)) \) for \( f(\cdot) \ll_{X, y} g(\cdot) \), and \( f(\cdot) \asymp_{X, y} g(\cdot) \) if both \( f(\cdot) \ll_{X, y} g(\cdot) \) and \( g(\cdot) \ll_{X, y} f(\cdot) \) take place. We write \( f(X, y, z) = o_{\mathcal{F}, y} (g(X, y, z)) \), if for every \( c > 0 \) there exist \( n_0 \) depending on \( \mathcal{F}, y, c \) such that for \( n = |X| > n_0 \) it holds that \( f(X, y, z) \leq cg(X, y, z) \).

Let \( \lambda_0 = 1 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1} \geq -1 \) be the eigenvalues of \( A \). We associate to \( \lambda_i \) its \( p \)-value \( p_i \) as follows

\[
\begin{cases}
  p_i = 2 & \text{if } |\lambda_i| \leq \frac{q}{q+1} \sqrt{q}, \\
  2 < p_i \leq \infty & \text{else},
\end{cases}
\]

where we use the convention that \( 1/\infty = \lim_{p \to \infty} 1/p = 0 \). In particular, \( p_0 = \infty \), and if \( X \) is bipartite, then \( \lambda_{n-1} = -1 \) and \( p_{n-1} = \infty \) as well. The definition of \( p_i \) is based on the action of \( A \) on the \( L^p \)-functions on the \( (q + 1) \)-regular tree, namely, the \( L^p \)-norm of \( A \) on the \( (q + 1) \)-regular tree is \( \frac{1}{\sqrt{q+1}} (q^{1/p} + q^{1-1/p}) \), see [21].

**Definition 1.1.** We say that a family \( \mathcal{F} \) of graphs satisfies the Sarnak-Xue density property, if for every graph \( X \in \mathcal{F}, p > 2 \) and \( \epsilon > 0 \),

\[
\# \{ i : p_i \geq p \} \ll_{p, \epsilon} n^{2/p+\epsilon},
\]

where \( n \) is the number of vertices of \( X \).

We prove two theorems that follow from the density property and will show below how one may prove the property. Theorem 1.3 was recently proved independently in [4].

Since the number of vertices in a ball of radius \( r \) in the \( (q + 1) \)-regular tree is \( 1 + (q + 1) \frac{q^r - 1}{q-1} \leq 3q^r \), the distance from a certain vertex to all but \( o(n) \) of the other vertices is bounded from below by \( \log_q (n) - o(1) \). This shows that up to a \( (1 + o(1)) \) factor, the following theorem is optimal.

**Theorem 1.2.** If \( \mathcal{F} \) is a family of expander graphs which satisfies the Sarnak-Xue density property, then for every \( \epsilon_0, \epsilon_1, \epsilon_2 > 0 \), there is \( n_0 \) depending on \( \mathcal{F}, \epsilon_0, \epsilon_1, \epsilon_2 \), such that the following holds. For \( X \in \mathcal{F} \) of size \( |X| = n > n_0 \), for all but \( \epsilon_0 n \) of \( x_0 \in X \), all but \( \epsilon_0 n \) of the vertices of \( X \) are within \( (1 + \epsilon_1) \log_q (n) \) distance from \( x_0 \), i.e.,

\[
\# \{ y \in X : d(x_0, y) > (1 + \epsilon_1) \log_q (n) \} \leq \epsilon_0 n.
\]

If, in addition, \( X \) is vertex-transitive, then the statement is true for all \( x_0 \in X \). Moreover, in such a case, \( 2(1 + \epsilon_1) \log_q (n) \) is a bound on the diameter of \( X \).

The second theorem concerns the cutoff phenomena, as discussed by Lubetzky and Peres in [23]. To simplify the result, we assume that the graph \( X \) is a non-bipartite graph. We let \( \delta_{x_0} \in L^2 (X) \) be the delta probability function supported on \( x_0 \), defined as \( \delta_{x_0} (x_0) = 1 \) and \( \delta_{x_0} (x) = 0 \) if \( x \neq x_0 \). Then \( A^k \delta_0 \)
describes the probability distribution of the simple random walk that starts at $x_0$ after $k$ steps. Since $X$ is non-bipartite, this probability converges pointwise to the constant probability $\pi$, defined as $\pi(x) = \frac{1}{n}$ for all $x \in X$. The following theorem describes the speed of the convergence of this random walk in the $L^1$-norm.

**Theorem 1.3.** Assume that $X \in \mathcal{F}$ is a non-bipartite graph. Then for every $\epsilon_0, \epsilon_1, \epsilon_2 > 0$:

1. There exists $n_0$ depending on $\epsilon_1, \epsilon_2$ such that for $X \in \mathcal{F}$, $|X| = n > n_0$, for every $x_0 \in X$, for $k < (1 - \epsilon_1) \frac{\log_q(n)}{q - 1}$, it holds that

   $$\|A^k \delta_{x_0} - \pi\|_1 \geq 2 - \epsilon_2.$$ 

2. Assume moreover that $\mathcal{F}$ is a family of expander graphs which satisfies the Sarnak-Xue density property, then there exists $n_0$ depending on $\mathcal{F}, \epsilon_0, \epsilon_1, \epsilon_2$ such that for $X \in \mathcal{F}$, $|X| = n > n_0$, for all but $\epsilon_0 n$ of $x_0 \in X$, for every $k > (1 + \epsilon_1) \frac{\log_q(n)}{q - 1}$, it holds that

   $$\|A^k \delta_{x_0} - \pi\|_1 \leq \epsilon_2.$$ 

   If $X$ is moreover a vertex-transitive graph, the same result holds for every $x_0 \in X$.

Collectively, we may summarize Theorem 1.2 and Theorem 1.3 by saying that an expander graph that satisfies the Sarnak-Xue density property has an optimal almost-diameter and exhibits cutoff. We remark that Theorem 1.2 formally follows from Theorem 1.3, but its proof is simpler and we think that it is of independent interest.

The cutoff phenomenon has been studied extensively in different settings in recent years and in particular regarding its relationship with the Ramanujan property (e.g. [23, 22, 15, 8]). The relation between the optimality of the almost-diameter and the Ramanujan property is also studied in different contexts. We already mentioned the work of Sardari ([28]) and Lubetzky and Peres ([23]), but it is also closely related to the work of Parzanchevski and Sarnak about Golden Gates ([26]) and the general results of Ghosh, Gorodnik and Nevo on actions of semisimple groups ([14]).

The observation that the Ramanujan property can be replaced by the density condition was proved in the setting of hyperbolic surfaces in [29, 15]. Theorem 1.3, which proves the analogous statement in the graph setting (and is actually easier), was recently proved independently in [4].

The main novelty of our work is the verification of the density property in a number of interesting cases, such as Theorem 1.1. This is done by a path counting argument, which is explained in the rest of the introduction.

**The Weak Injectivity Radius Property.** We first give a parameterized version of the Sarnak-Xue density property defined above.

**Definition 1.2.** We say that a family $\mathcal{F}$ satisfies the Sarnak-Xue density property with parameter $0 < A \leq 1$, if for every $X \in \mathcal{F}$, $p > 2$ and $\epsilon > 0$,

$$\# \{ i : p_i \geq p \} \ll_{\epsilon, \mathcal{F}} n^{1 - A(1 - \frac{2}{p}) + \epsilon}.$$ 

Note that the trivial eigenvalue 1 (and also −1 if $X$ is bipartite) is always counted in the left hand side of the inequality, so the parameter $A$ is always bounded from above by 1. Ramanujan graphs automatically satisfy the density property with parameter $A = 1$, since for $p > 2$, $\# \{ i : p_i \geq p \} \in \{1, 2\}$ (depending on whether the
graph is bipartite or not). The density property does not necessarily imply uniform expansion of the family, since arbitrary large eigenvalues can appear as long as there are few of them. However, if $X$ is a Cayley graph of a quasirandom group $G$ in the sense of [16], meaning that the smallest non-trivial representation of $G$ is of dimension $\gg |G|^\beta$, then there is a lower bound on the multiplicity of every eigenvalue $\lambda_i$. Then if $A+\beta > 1$, it holds that for $p > \frac{2A}{\beta+A-1}$ and $n$ large enough, $\# \{i : p_i \geq p\} = 1$, implying expansion. This was indeed Sarnak and Xue’s approach to the proof of spectral gap for congruence subgroups of arithmetic cocompact subgroups of $\text{SL}_2(\mathbb{R})$ and $\text{SL}_2(\mathbb{C})$. We remark that the idea was actually used earlier in a similar context by Huxley ([20]). Notice that the group $\text{SL}_2(\mathbb{F}_1)$ is quasirandom with parameter $\beta = 1/3$, which implies that $A > 2/3$ guarantees expansion.

Another interesting result is the connection between Sarnak-Xue density and Benjamini-Schramm convergence. It follows from the results of Abert, Glasner and Virag ([2]) that the Sarnak-Xue density property with any parameter $A > 0$ implies Benjamini-Schramm convergence of the family to the $(q+1)$-regular tree (see Section 7).

We are finally ready to state the geometric definition underlying our work.

**Definition 1.3.** Let $X$ be a $(q+1)$-regular graph and $x_0 \in X$ a vertex. Let $P(X, k, x_0)$ be the number of non-backtracking paths of length $k$ starting and ending at $x_0$, and let $P(X, k) = \sum_{x_0 \in X} P(X, k, x_0)$.

We say that $X$ satisfies the *Sarnak-Xue weak injective radius property with parameter* $0 < A \leq 1$, if for $k = 2\lfloor A \log_q n \rfloor$, we have for every $\epsilon > 0$

$$P(X, k) \ll_{\epsilon, x} n^{1+\epsilon} q^{k/2} \asymp n^{1+A+\epsilon}. \quad (1.1)$$

For the parameter $A = 1$, the property is equivalent to the fact that

$$P(X, 2\lfloor \log_q n \rfloor) \ll_{\epsilon, x} n^{2+\epsilon}. \quad (1.2)$$

Sarnak and Xue ([30]) essentially proved the if part of the following theorem in the context of Lie groups of rank 1. They later showed that the weak injective radius property, and hence the density property, holds in all sequences of principal congruence subgroups of cocompact arithmetic subgroups of $\text{SL}_2(\mathbb{R})$ and $\text{SL}_2(\mathbb{C})$.

**Theorem 1.4.** The Sarnak-Xue density property with parameter $0 < A \leq 1$ is satisfied if and only if the Sarnak-Xue weak injective radius property with the same parameter $A$ is satisfied.

Therefore, if the weak injective radius property is satisfied with parameter $A = 1$ and the graphs of the family are expanders, then they have optimal almost-diameter and exhibit cutoff (as stated in Theorem 1.2 and Theorem 1.3).

The girth of a graph, i.e., the length of the shortest cycle, is equal to twice the injectivity radius of the graph (plus one, if the girth is odd). Therefore, if there are no cycles in $X$ of length $\leq 2A \log_q n$, or equivalently the graph has injective radius $\geq A \log_q n$, then the weak injective radius property with parameter $A$ is automatically satisfied. However, in many cases one can show that the weak injective radius property is satisfied for larger values of $A$. For example, Ramanujan graphs satisfy the weak injective radius property with parameter $A = 1$ (as follows from Theorem 1.4), while it is not known if Ramanujan graphs with girth close to $2\log_q(n)$ exist. The best known asymptotic result about girth is that the bipartite Ramanujan graphs constructed by Lubotzky, Phillips and Sarnak have girth at least $4/3 \log_q(n) - O(1)$. Another example is that by [13], random Cayley graphs in $\text{SL}_2(\mathbb{F}_1)$ have girth at least $(1/3 - o(1)) \log_q(n)$, while we show that
the weak injective radius property holds with parameter 1/3, that is essentially double of what follows from the girth.

**Graphs Satisfying Sarnak-Xue Density Property.** We focus on Sarnak-Xue density property with parameter $A = 1$. There are a number of simple examples of $(q + 1)$-regular graphs satisfying this property. Ramanujan graphs. This is straightforward- by definition, there are no eigenvalues with $p$-value greater than 2 except for the trivial ones. Let us remark that while the proof that the LPS graphs of [24] (or similar number theoretic graphs) are Ramanujan is far from being elementary and eventually relies heavily on the machinery of algebraic geometry, the fact that the Sarnak-Xue density property holds for them requires only elementary number theoretical tools, and is contained implicitly in [10, Theorem 4.4.4]. Therefore, our work provides an elementary proof for the cutoff phenomena for those graphs. See also Theorem 1.7 and its proof for this argument.

One-sided Ramanujan graphs. These are graphs with $\lambda_i \leq 2\sqrt{q}/(q + 1)$ for every $i > 0$. Such non-bipartite graphs are constructed by the interlacing polynomials method of [25] and [17]. See Proposition 4.3 for the proof of this case.

Random regular graphs. With high probability a random $(q + 1)$-regular graph of size $n$ and, more generally, with high probability a random $n$-cover (or $n$-lift) of a fixed $(q + 1)$-regular graph is almost-Ramanujan, and therefore satisfies the Sarnak-Xue density property. This follows from Alon’s conjecture, which was proved by Friedman ([11]), and its extension to random lifts proved by Bordenave ([3]). As a matter of fact, using Theorem 1.4 this actually follows from the much simpler work of Broder and Shamir [6].

**Schreier Graphs of $\text{SL}_2(\mathbb{F}_t)$**. Let $t$ be prime and $P^1(\mathbb{F}_t)$ be the projective line over $\mathbb{F}_t$, with an action of $\text{SL}_2(\mathbb{F}_t)$ by Mobius transformations. Since the action is transitive, we may consider $P^1(\mathbb{F}_t)$ as a quotient of $\text{SL}_2(\mathbb{F}_t)$. Let $S_t \subset \text{SL}_2(\mathbb{F}_t)$ be a symmetric generating set, and let $X_t = \text{Cayley}(\text{SL}_2(\mathbb{F}_t), S_t)$ be the corresponding Cayley graph. Let $Y_t$ be the Schreier quotient of $X_t$, whose vertex set is $P^1(\mathbb{F}_t)$.

Expansion and the Ramanujan condition on $X_t$ and $Y_t$ for various choices and generators was studied in various works. The two most important ones are [24] which found Ramanujan examples using deep results in number theory, and [5] which proved that such Cayley graphs with logarithmic girth are expanders, but the resulting bounds are quite far from the Ramanujan bound. In [27], Rivin and Sardari conjecture and verify experimentally that for many choices of generators both graphs should be almost-Ramanujan, and should therefore have an optimal almost-diameter and exhibit cutoff.

We note that, unlike expansion, the fact that density holds for a graph $X$ does not immediately imply that it holds for some quotient $Y$ of it. However, in this special case, we have the following *density amplification theorem*:

**Theorem 1.5.** If $X_t$ satisfies the Sarnak-Xue density property with parameter $A \geq 1/3$, then $Y_t$ satisfies this property with parameter $A = 1$. In particular, if the graph $Y_t$ is also an expander, then it has an optimal almost-diameter and exhibits cutoff (as stated in Theorem 1.2 and Theorem 1.3).

One can also replace $\text{SL}_2(\mathbb{F}_t)$ in Theorem 1.5 by either $\text{PGL}_2(\mathbb{F}_t)$ or $\text{PSL}_2(\mathbb{F}_t)$.

We remark that Theorem 1.5 is not based on quasirandomness, but on a simple counting argument. Indeed, quasirandomness implies that if we assume that the graph $X_t$ satisfies the Sarnak-Xue density property with
parameter $A$, then for the graph $Y_t$ it holds that
\[ \# \{ i : p_i \geq p \} \ll \epsilon, n^{2-3A(1-2/\rho)+\epsilon}, \]
which is a different estimate, and in particular has non-trivial implications only for $A \geq 2/3$.

Here are three interesting cases for which we can prove that $X_t$ satisfies the Sarnak-Xue density property with parameter $A \geq 1/3$. In all of these cases, one can also show expansion for $X_t$, and hence for $Y_t$, using the results of Bourgain and Gamburd ([5]). Recall that we assume that $t$ is prime.

**Theorem 1.6.** Let $S \subset SL_2(\mathbb{Z})$ be a symmetric set of size $|S| = (q + 1)$, which generates a free subgroup in $SL_2(\mathbb{Z})$, and assume that each element $s \in S$ has operator norm $\|s\| \leq q$. Then the graphs $X_t = Cayley(SL_2(\mathbb{F}_t), S \mod t)$ satisfy the Sarnak-Xue density property with parameter $A = 1/3$.

Therefore, by Theorem 1.5, the graphs $Y_t$ have an optimal almost-diameter and exhibit cutoff.

A particularly interesting case is the set
\[ S = \left\{ \begin{pmatrix} 1 & \pm 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 2 & 1 \end{pmatrix} \right\}. \]
If $\pm 2$ is replaced with $\pm 1$ then the subgroup generated in $SL_2$ is not free, so the corresponding graphs $X_t$ do not even Benjamini-Schramm converge to the 4-regular infinite tree and, in particular, do not satisfy the density property. On the other hand, for $S = \left\{ \begin{pmatrix} 1 & \pm 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 3 & 1 \end{pmatrix} \right\}$ the norm of every element is greater than $q = 3$, so it is not known whether the graphs satisfy the density property with parameter $A \geq 1/3$. We conjecture that they do, and moreover, in Theorem 1.6, the condition on the operator norm bound is redundant.

**Theorem 1.7.** For primes $q, t > 2$ let $X^{q,t}$ be the $(q + 1)$-regular Cayley graphs of $PSL_2(\mathbb{F}_t)$ or $PSL_2(\mathbb{F}_t)$ constructed by Davidoff, Sarnak and Valette in [10]. Let $S_t$ be a symmetric subset of the generators of $X^{q,t}$ of size $|S_t| = q' + 1 \geq \sqrt[q]{q} + 1$, and let $X_t$ be the $(q' + 1)$-regular Cayley graph generated by $S_t$. Then the graphs $X_t$ satisfy the Sarnak-Xue density property with parameter $A = 1/3$. Therefore, the graphs $Y_t$ have an optimal almost-diameter and exhibit cutoff.

The graphs $X^{q,t}$ above are usually denoted by $X^{p,q}$ but we denote differently to avoid confusion with the rest of the article. These graphs are a slight generalization of the LPS graphs of [24] since the congruence conditions for $p$ and $q$ is not assumed.

The following theorem, together with Theorem 1.5, implies Theorem 1.1.

**Theorem 1.8.** Let $S_i$ be a random set of size $(q + 1)/2$ in $SL_2(\mathbb{F}_t)$, and $X_t = Cayley(SL_2(\mathbb{F}_t), S_t \cup S_t^{-1})$. Then for every $\epsilon_0 > 0$, with probability $1 - \epsilon_0$, the graphs $X_t$ satisfy the Sarnak-Xue density property with parameter $A = 1/3$. Therefore, the graphs $Y_t$ have an optimal almost-diameter and exhibit cutoff.

As the probabilistic statement in the theorem is somewhat confusing, let us state it in another form. The theorem says that for every $\epsilon_0 > 0$, among all $M_t$ of the possible graphs $X_t$, we may choose $(1 - \epsilon_0)M_t$ of them, and the resulting family of graphs will satisfy the Sarnak-Xue density property with parameter $A = 1/3$. The underlying constants in Definition 1.2 will depend on $\epsilon_0$. 
The proof follows from the results of [13]. It is a natural question to ask whether density holds for $X_t$ with parameter $A = 1$. It is notable that the methods that are used to prove that random graphs and random covers of graphs are almost Ramanujan ([11, 4]) do not work in this case.

**Structure of the Paper.** We start with the application of the results to the Schreier graphs on the projective line in Section 2, where we prove Theorem 1.5, Theorem 1.6, Theorem 1.7 and Theorem 1.8.

In Section 3 we recall some basic results from the spectral theory of graphs. In Section 4 we prove Theorem 1.4 and discuss other conditions that are equivalent to the density property.

In Section 5 we prove Theorem 1.2 and Theorem 1.3 for Ramanujan graphs. These results also appear in [23, 28], but we provide the proof for them in order to simplify the proof of the next results. In Section 6 we prove Theorem 1.2 and Theorem 1.3. Similar results recently appeared independently in [4], and are included here for self-containment and since our proofs are slightly different.

Finally, in Section 7 we shortly discuss the connection between the density property and Benjamini-Schramm convergence.

**Acknowledgments.** The authors are thankful to Michael Chapman, Ori Parzanchevski and Peter Sarnak for fruitful discussions, and to the referees for their constructive remarks and suggestions. During the work on this project, the first author was supported by the SNF grant 200020-169106 at ETH Zurich. This work was part of the PhD thesis of the second author at the Hebrew University of Jerusalem, under the guidance of Prof. Alex Lubotzky, and was supported by the ERC grant 692854.

2. **Schreier Graphs on the Projective Line**

Let $t$ be a prime, $\mathbb{F}_t$ the finite field with $t$ elements, $G_t = \text{SL}_2(\mathbb{F}_t)$, and $P_t = P^1(\mathbb{F}_t)$ be the projective line over $\mathbb{F}_t$, i.e., the set of vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ such that $a, b \in \mathbb{F}_t$, not both zero, quotiented by the equivalence relation $\begin{pmatrix} a \\ b \end{pmatrix} \sim \begin{pmatrix} a' \\ b' \end{pmatrix}$ if and only if $ba' = ab'$.

The group $G_t$ acts transitively on $P^1(\mathbb{F}_t)$, the stabilizer of $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in P^1(\mathbb{F}_t)$ is

$$K_t = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{F}_t^*, b \in \mathbb{F}_t \right\},$$

and therefore $P_t$ can be identified with the space $P_t \simeq G_t/K_t$. Explicitly, $\begin{pmatrix} a \\ b \end{pmatrix} \in P^1(\mathbb{F}_t)$ corresponds to the set of matrices $A \in \text{SL}_2(\mathbb{F}_t)$, with first column equivalent to $\begin{pmatrix} a \\ b \end{pmatrix}$.

Let $S_t \subset G_t$ be a symmetric generating set of $G_t$, with $|S_t| = q + 1$. Let $X_t$ the Cayley graph of $G_t$ w.r.t. $S_t$, and let $Y_t$ be the Schreier graph of the quotient $G_t/K_t$ w.r.t. $S_t$.

**Theorem 2.1.** If $X_t$ has weak injective radius with parameter $A$, then $Y_t$ has weak injective radius with parameter $\min \{1, 3A\}$. 

Proof. For simplicity, we assume that $S_t = R_t \cup R_t^{-1}$ with $|R_t| = |S_t|/2$. We will also first assume that $G_t = \PSL_2(\mathbb{F}_t)$. Let $F_R$ be the free group generated by a set $R$ of size $|R| = |R_t| = (q + 1)/2$, and let $l: F_R \to \mathbb{N}_{\geq 0}$ be the standard length, i.e. $l(g)$ is the minimal length of a product of generators and their inverses which is equal to $g$. An identification $R \simeq R_t$ defines a homomorphism $\varphi_t: F_R \to G_t$.

Note that if $N \subset F_R$ is a finite index subgroup which defines a Schreier graph $X$ on the vertex set $F_R/N$, then the weak injective radius property with parameter $A$ holds for $X$ if and only if for $k \leq 2A \log_q(|X|)$ and every $\epsilon > 0$,

$$\# \{(g, xN) \in F_R \times F_R/N : l(g) = k, gxN = xN\} \ll \epsilon \cdot |X|^{1+\epsilon} q^{k/2},$$

Denote $N_t = \ker \varphi_t$ and $M_t = \left\{g \in F_R : \varphi_t(g) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$, where the equality is in $P^1(\mathbb{F}_t)$ (alternatively one can say that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{F}_t^2$ is an eigenvector of $\varphi_t(g)$). $M_t$ is a subgroup of $F_R$ and contains $N_t$, which is its normal core in $F_R$. $M_t$ defines the graph $Y_t$, while $N_t$ defines the graph $X_t$. Since $N_t$ is normal in $F_R$, and since $|X_t| = (t - 1)(t + 1)t/2 \asymp t^3$, the assumed Sarnak-Xue weak injective property with parameter $A$ for $X_t$ implies that for $k \leq 2A \log_q(|X_t|) = 6A \log_q(t) + O(1)$ and every $\epsilon > 0$, it holds that

$$\# \{(g \in N_t : l(g) = k) \ll \epsilon \cdot t^k q^{k/2}.$$

Let us bound the size of the set $M_{t,k} = \{(g, y) \in F_R \times P^1(\mathbb{F}_t) : l(g) = k, \varphi_t(g)y = y\}$. If $(g, y) \in M_{t,k}$, then $y \in P^1(\mathbb{F}_t)$ is a projective eigenvector of $\varphi_t(g)$, i.e., the lift of $y$ to $\mathbb{F}_t^2$ is an eigenvector of $\varphi_t(g)$. There are two options: either $\varphi_t(g) = I \in G_t$ (i.e., $g \in N_t$) and then $\varphi_t(g)$ has $(t + 1)$ projective eigenvectors, or $\varphi_t(g) \neq I \in G_t$, and $\varphi_t(g)$ has at most 2 projective eigenvectors (note that here the assumption that the group is actually $\PSL_2(\mathbb{F}_t)$ is used). Hence, for $k \leq 6A \log_q(t) - O(1) = 2(3A) \log_q(t) - O(1)$,

$$|M_{t,k}| \leq (t + 1) \# \{(g \in N_t : l(g) = k)\} + 2 \# \{g \in F_R : l(g) = k\} \ll \epsilon \cdot |Y_t|^{1+\epsilon} q^{k/2} + q^k.$$

Note that for $k \leq 2\log_q(|Y_t|)$ it holds that

$$q^k \leq |Y_t| q^{k/2},$$

and this implies that the weak injective property for $Y_t$ holds with parameter $\min \{1, 3A\}$.

The treatment of the case $\SL_2$ is similar to that of $\PSL_2$ except for the case of the elements $g \in F_R$ such that $\varphi_t(g) = -I$. Such elements also have $(t + 1)$ projective eigenvectors. We treat this case by showing that the number of such elements of length $k \leq 6A \log_q(t)$ is also bounded by $\ll \epsilon \cdot t^k q^{k/2}$. This follows from the general Lemma 6.5 in Section 6.

We now prove that the three examples from the introduction have Sarnak-Xue weak injective radius $A = 1/3$.

Proof of Theorem 1.6. Let $S \subset \SL_2(\mathbb{Z})$ be a symmetric set of size $|S| = (q + 1)$, which generates a free group in $\SL_2(\mathbb{Z})$, and assume that each element $s \in S$ has operator norm $\|s\| \leq q$. We need to show that $X_t = \text{Cayley}(\SL_2(\mathbb{F}_t), S \mod t)$ has weak injective radius with parameter $A = 1/3$. 

\[ \square \]
Denote $\Gamma = \text{SL}_2(\mathbb{Z})$, and let $\Gamma(t) = \{ \gamma \in \Gamma : \gamma \equiv I \mod{t} \}$ be its principal congruence subgroup of level $t$. Let $B_T = \{ A \in M_n(\mathbb{R}) : \|A\| \leq T \}$.

Then it is proved in ([12]) that

\begin{equation}
\label{eq:2.1}
|\Gamma(t) \cap B_T| \ll_q (T/t^2 + 1) (T/t + 1) T^\epsilon.
\end{equation}

For completeness, let us give a short proof of this fact. Write

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

It is sufficient to prove Equation (2.1) for the norm $\|\gamma\|_\infty = \max\{|a|, |b|, |c|, |d|\}$, since all the norms on $M_2(\mathbb{R})$ are equivalent. Then if $\gamma \in \Gamma(t)$, i.e. $\gamma = I \mod{t}$, it is a simple exercise to see that $a + d = 2 \mod{t^2}$.

Then if $\|\gamma\|_\infty \leq T$ there are $\ll_q (T/t^2 + 1)$ options for $a + d$, there are $\ll_q (T/t + 1)$ options for $a$, and therefore $\ll_q (T/t^2 + 1) (T/t + 1)$ options for $a, d$. Then if $ad \neq 1$ we have $bc = 1 - ad \neq 0$ and by standard bounds on the divisor function ([18, Theorem 315]) there are $\ll \epsilon_q T^\epsilon$ options for $b, c$. When $ad = 1$ then there are 2 options for $a, d$, and $bc = 0$, so there are $\ll_q (T/t + 1)$ options for $b, c$.

Returning to the proof, by Equation (2.1), the number of paths of length $k$ in $X_t$ is bounded by

$$|\Gamma(t) \cap B_{q^k}| \ll_q (q^k/t^2 + 1) (q^k/t + 1) q^{k\epsilon}.$$ 

For $k \leq 2A \log_q (t^3) = 2 \log_q (t)$, i.e. $t \geq q^{k/2}$ it holds that

$$ (q^k/t^2 + 1) (q^k/t + 1) \ll q^{k/2},$$

which means that $X_t$ satisfies the Sarnak-Xue density property with parameter $A = 1/3$. \qed

Let us complete the analysis of $S = \left\{ \begin{pmatrix} 1 & \pm 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 2 & 1 \end{pmatrix} \right\}$.

We need to calculate the norm of the generators. Denote by $\lambda_{\text{max}}$ the maximal eigenvalue of a semi-definite matrix. Then

$$\left\| \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\|^2 = \lambda_{\text{max}} \left( \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right) = \lambda_{\text{max}} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= \frac{6 \pm \sqrt{36 - 1}}{2} \leq 6 \leq 3^2 = q^2,$$

and similarly for the other generators.

**Proof of Theorem 1.7.** For primes $q, t > 2$ let $X^{q,t}$ be the $(q+1)$-regular Cayley graphs of $\text{PSL}_2(\mathbb{F}_t)$ or $\text{PSL}_2(\mathbb{F}_t)$ constructed by Davidoff, Sarnak and Valette in [10]. Let $S_t$ be a symmetric subset of the generators of $X^{q,t}$ of size $|S_t| = q^t + 1 \geq \sqrt{q} + 1$, and let $X_t$ be the $(q^t + 1)$-regular Cayley graph generated by $S_t$. 


Then as shown in [10, Lemma 4.4.2], for \( k = 0 \mod 2 \), the number of cycles \( P(X^{p,t}, k, \text{id}) \) is bounded by the number \( s_Q(q^k) \) of ways to represent \( q^k \) by the quadratic form
\[
Q(x_0, x_1, x_2, x_3) = x_0^2 + 4t^2(x_1^2 + x_2^2 + x_3^2).
\]
Following the analysis in [10, Theorem 4.4.4] which we will not repeat but is quite elementary,
\[
s_Q(q^k) \ll_{c} q^{k\epsilon} \left( q^{k/3} + q^{k/2}/t \right)
\]
Therefore,
\[
P(X_t, k, \text{id}) \leq P(X^{p,t}, k, \text{id}) \ll_{c} q^{k\epsilon} \left( q^{k/3} + q^{k/2}/t \right)
\]
\[
\leq q^{2k\epsilon} \left( q^{k/3} + q^k/t \right)
\]
For \( k \leq 2 \log_q t \) it therefore holds that
\[
P(X_t, k, \text{id}) \ll_{c} t^c q^{k/2},
\]
as needed.

Proof of Theorem 1.8. Let \( S_t \) be a random set of size \((q + 1)/2\) in \( \text{SL}_2(F_t) \), and \( X_t = \text{Cayley} (\text{SL}_2(F_t), S_t \cup S_t^{-1}) \).
By [13, Lemma 10], the probability that a word \( w \) of length \( k \), when \( k = O(t/\log(t)) \), will evaluate to 1 is \( \leq \frac{k}{t} + o(t^{-2}) \). Let \( k \leq 2 \log_q(t) \). Then the expected number of cycles of length \( k \) is bounded by:
\[
E(P(X_t, k)) \ll q^{k/2} \leq kq^{k/2}.
\]
Let \( \epsilon_0 > 0, \epsilon > 0 \) and let \( t_0 = t_0(\epsilon) \) be large enough so that for \( t > t_0 \) it holds that \( \log_q(t) \leq t^c \). Then, by Markov’s inequality, the probability for \( t > t_0 \) that \( P(X_t, k) > C \epsilon q^{k/2} t^c \), is bounded by \( CC^{-1}t^{-c}k \leq 2C^{-1}\log_q(t) t^{-c} \ll C_\epsilon \). By choosing \( C_\epsilon = C_1 \epsilon_0 m^2 \) for \( \epsilon = \frac{1}{m} \), we can ensure that for every \( t \), with probability \((1 - \epsilon_0)\), for every \( \epsilon > 0 \) of the form \( \epsilon = \frac{1}{m} \), if \( t > t_0(\epsilon) \) it will hold that \( P(X_t, k) \leq C_\epsilon q^{k/2} t^c \).
Therefore, the same will hold for every \( \epsilon > 0 \), as needed.

3. Preliminaries on the Spectral Theory of Graphs

We start with some basics of the spectral theory of graphs. See [21] for some more details.

As before, let \( X \) be a finite \((q + 1)\)-regular graph with \( n \) vertices, possibly with multiple edges and self-loops. A non-backtracking path is a path in the graph such that no two consecutive steps take the same edge in the opposite direction. For \( x_0 \in X \), there are \((q + 1)q^{k-1}\) non-backtracking paths starting at \( x_0 \), and we write \( T_k(x_0, x_1) \) as the number of non-backtracking paths of length \( k \) that start at \( x_0 \) and finish at \( x_1 \).

For \( k \geq 0 \), define the distance \( k \) Hecke-operator \( A_k : L^2(X) \to L^2(X) \) by
\[
A_k f(x_0) = \frac{1}{(q + 1)q^{k-1}} \sum_{x_1 \in X} T_k(x_0, x_1) f(x_1).
\]
On the \((q + 1)\)-regular infinite tree \( A_k \) acts by averaging over the sphere of radius \( k \) around a vertex.

Note that the path-counting functions defined in Definition 1.3 can be expressed as
\[
P(X, k, x_0) = (q + 1)q^{k-1} \langle A_k \delta_{x_0}, \delta_{x_0} \rangle
\]
\[
P(X, k) = (q + 1)q^{k-1} \text{tr} A_k.
\]
It holds that \(A_0 = I_d, A_1 = A\), and the following recursive relation takes place for \(k \geq 1\)

\[
AA_k = \frac{q}{q + 1} A_{k+1} + \frac{1}{q + 1} A_{k-1}.
\]

Therefore, if \(v \in L^2(X)\) is an eigenfunction of \(A\), i.e., \(Av = \lambda v\), then \(A_k v = \lambda^{(k)} v\), where \(\lambda^{(k)}\) is a function of \(\lambda\) and \(k\). In order to calculate \(\lambda^{(k)}\), write

\[
\lambda = \frac{1}{q + 1} (\theta + q\theta^{-1}),
\]

for some \(0 \neq \theta \in \mathbb{C}\). This equation always has two (possibly, equal) solutions \(\theta_{\pm}\) satisfying \(\theta_{\pm} \theta = q\), and

\[
\theta_{\pm} = \frac{(q + 1) \lambda \pm \sqrt{(q + 1) \lambda^2 - 4q^2}}{2}.
\]

Solving this equation and recalling that \(A\) is a self-adjoint operator of norm 1, so its eigenvalues are real and satisfy \(-1 \leq \lambda \leq 1\), we have:

- If \(|\lambda| \leq \frac{2\sqrt{q}}{q + 1}\), then \(|\theta_{\pm}| = \sqrt{q}\). In this case, we let \(p(\lambda) = 2\) be the \(p\)-value of \(\lambda\), and we let \(\theta\) be one of \(\theta_{\pm}\).
- If \(\frac{2\sqrt{q}}{q + 1} < |\lambda| \leq 1\), then both \(\theta_{\pm}\) are real, and we let \(\theta\) be the larger one in absolute value. It holds that \(\sqrt{q} < |\theta| \leq q\) and \(\theta\) has the same sign as \(\lambda\). It also holds that \(|\theta| = q^{1 - 1/p}\), where \(2 < p = p(\lambda) \leq \infty\) is the \(p\)-value of \(\lambda\).

The relation between \(\theta\) and the eigenvalues \(\lambda^{(k)}\) of \(A_k\) is given by the following formula, which may be verified by induction

\[
\lambda^{(k)} = \frac{1}{(q + 1) q^{k-1}} \left( \theta^k + (q\theta^{-1})^k + (1 - q^{-1}) \sum_{i=1}^{k-1} q^i \theta^{k-2i} \right).
\]

The following proposition provides an upper bound and a lower bound for \(\lambda^{(k)}\).

**Corollary 3.1.** If \(p(\lambda) = 2\) then for every \(\epsilon > 0\),

\[
|\lambda^{(k)}| \leq (k + 1) q^{-k/2} \ll \epsilon q^{k(-1/2 + \epsilon)}.
\]

If \(p(\lambda) > 2\) then for every \(\epsilon > 0\),

\[
q^{-k/p(\lambda)} \leq |\lambda^{(k)}| \leq (k + 1) q^{-k/p(\lambda)} \ll \epsilon q^{k(-1/p(\lambda) + \epsilon)}.
\]

Moreover, if \(k \equiv 0 \mod 2\) then \(\lambda^{(k)}\) is positive, otherwise \(\lambda^{(k)}\) has the same sign as \(\lambda\).

**Proof.** The upper bounds follow directly from the explicit expression in Equation (3.1). Note that since \(|\theta| \geq |q\theta^{-1}|\), the value \(\theta^k\) is the largest one in absolute value in Equation (3.1).

It is left to prove the lower bound in the case of \(p = p(\lambda) > 2\). We can assume that \(\lambda > 0\), so \(\theta > 0\), and hence all the summands in Equation (3.1) are positive. Consider the function \(f: \mathbb{R} \to \mathbb{R}\) defined as \(f(x) = q^{-x/p}\). This function is decreasing, hence for any \(x_1, \ldots, x_N \in \mathbb{R}\)

\[
\frac{1}{N} \sum_{i=1}^{N} f(x_i) \geq f(\max \{x_i\}).
\]
Apply this to the following multiset (i.e., a set with repetitions) of numbers: $q^k$ times $k$, $(q - 1) q^{k-i-1}$ times $k - 2i$ for $0 < i < k$, and once the value $-k$. There are $N$ elements in this multiset, where

$$N = q^k + (q - 1) \sum_{i=1}^{k-1} q^{k-1-i} + 1 = q^k + q^{k-1} = (q + 1) q^{k-1}.$$ 

Therefore, we have

$$\frac{1}{(q + 1) q^{k-1}} \left( q^k q^{-k/p} + (q - 1) \sum_{i=1}^{k-1} q^{k-1-i} q^{-(k-2i)/p} + 1 \cdot q^{k/p} \right) \geq f(k) = q^{-k/p}.$$ 

The left hand side is equal to $\lambda^{(k)}$. \hfill \Box

4. Equivalence of the Density and the Path-Counting Properties

Recall that the Sarnak-Xue density property with parameter $A$ for a graph $X$ with $n$ vertices states that for every $p > 2$ and $\epsilon > 0$,

$$\# \left\{ i : p_i \geq p \right\} \ll_{\epsilon, X} n^{1-A(1-2/p)} + \epsilon.$$ 

Here $p_i$ is the p-value of the $i$-th eigenvalue.

Lemma 4.1. The Sarnak-Xue density property with parameter $A$ is equivalent to the fact that for every $\epsilon > 0$,

$$(4.1) \quad \sum_i n^{-1 + A(1-2/p_i)} \ll_{\epsilon, X} n^{\epsilon}.$$ 

Proof. Assume Inequality (4.1) holds. Then for every $\epsilon > 0$ and $p > 2$,

$$n^{\epsilon} \gg_{\epsilon, X} \sum_{i : p_i \geq p} n^{-1 + A(1-2/p_i)} \geq \sum_{i : p_i \geq p} n^{-1 + A(1-2/p)} = n^{-1 + A(1-2/p)} \# \left\{ i : p_i \geq p \right\},$$

which implies the Sarnak-Xue density property with parameter $A$.

For the other direction, we apply discrete integration by parts (see [18, Theorem 421]). For a smooth function $f : [2, \infty) \to \mathbb{R}$ and a finite sequence of points $x_i$, let $M(x) = \# \left\{ i : x_i \geq x \right\}$. Then the following holds

$$\sum_i f(x_i) = M(2) f(2) + \int_2^\infty M(x) \frac{\partial}{\partial x} f(x) \, dx.$$
Let \( f(x) = n^{-1 + A(1 - 2/x)} \) and assume the Sarnak-Xue density property with parameter \( A \), then

\[
\sum_i n^{-1 + A(1 - 2/p_i)} = \sum_i f(p_i) = n^{-1} \# \{ i : p_i \geq 2 \} + \int_2^\infty \# \{ i : p_i \geq x \} \frac{\partial}{\partial x} n^{-1 + A(1 - 2/x)} \, dx \\
\ll \epsilon, x \cdot n \cdot \int_2^\infty n^{1 - A(1 + 2/x)} + \ln(n) A 2 \frac{1}{x^2} n^{-1 + A(1 - 2/x)} \, dx \\
\ll 1 + n^{2\epsilon} \int_2^\infty \frac{1}{x^2} \, dx \ll n^{2\epsilon}.
\]

We can now give a proof of Theorem 1.4.

**Proof of Theorem 1.4.** First assume that Sarnak-Xue density property with parameter \( A \) is satisfied for \( X \). We shall prove that \( P(X, k) \ll \epsilon, x \cdot n^{1 + \epsilon} q^{k/2} \) for \( k = 2 \lfloor A \log_q(n) \rfloor \). Indeed:

\[
P(X, k) = (q + 1) q^{k-1} \text{tr} A_k = \\
= (q + 1) q^{k-1} \sum_{i=1}^n \lambda_i^{(k)} \\
\leq (q + 1) q^{k-1} \sum_{i=1}^n |\lambda_i^{(k)}| \\
\leq (q + 1) q^{k-1} \sum_{i=1}^n (q + 1) q^{-k/p_i} \\
\ll \sum_{i=1}^n q^{k(1-1/p_i)} = \sum_{i=1}^n n^{2A(1-1/p_i)} \\
= n^A n \sum_{i=1}^n n^{-1 + A(1-1/p_i)} \\
\ll \epsilon, x \cdot q^{k/2} n n^{\epsilon} = n^{1 + \epsilon} q^{k/2}.
\]

Here we used the upper bounds of Corollary 3.1 in the fourth line, and Lemma 4.1 in the last line.
For the other direction, assume the injective radius property with parameter $A$. Let $k = 2 \lfloor A \log_q n \rfloor$. Then by the weak injective radius property we have for every $\epsilon > 0$,

$$n^{1+A+\epsilon} \propto n^{1+\epsilon}q^{k/2} \gg_{\epsilon,F} P(X,k) =$$

$$= (q + 1) q^{k-1} \sum_{i=1}^{n} \lambda_i^{(k)}$$

$$\gg q^k \sum_{i:p_i > 2} \lambda_i^{(k)} + q^k \sum_{i:p_i = 2} \lambda_i^{(k)}$$

$$\geq q^k \sum_{i:p_i > 2} q^{-k/p_i} + q^k \sum_{i:p_i = 2} \lambda_i^{(k)}$$

$$\geq q^k \sum_{i:p_i > 2} n^{-2A/p_i} + q^k \sum_{i:p_i = 2} \lambda_i^{(k)}.$$

Here we applied the lower bound of Corollary 3.1 in the last line and used the fact that $k$ is even. By re-arranging and applying the upper bounds of Corollary 3.1, we have

$$\sum_{i:p_i > 2} n^{-2A/p_i} \ll_{\epsilon,F} q^{-k} n^{1+A+\epsilon} - \sum_{i:p_i = 2} \lambda_i^{(k)}$$

$$\ll_{\epsilon} n^{1-A+\epsilon} + nq^{-k(1/2-\epsilon)}$$

$$\asymp n^{1-A+\epsilon}.$$

This proves that the Sarnak-Xue density property holds as in Lemma 4.1. \hfill \Box

**Remark 4.2.** The proof theorem implies that if the weak injective radius property holds for the parameter $A$, then it holds for any parameter $A' \leq A$, which is not obvious from the definition.

One may also weaken the Sarnak-Xue density property, so we have to consider only positive eigenvalues of $A$. We have:

**Proposition 4.3.** The Sarnak-Xue weak injective radius property with parameter $A$ ($0 < A \leq 1$) holds if for every $p > 2$ and $\epsilon > 0$,

$$\# \{i : \lambda_i > 0, p_i \geq p \} \ll_{\epsilon,F} n^{1-A(1-2/p)+\epsilon}.$$

Note that the difference between this property and the Sarnak-Xue density property (Definition 1.2) is that we restricted ourselves to positive eigenvalues. One cannot hope to prove the analogous statement for the negative eigenvalues only, as the smallest negative eigenvalue may stay within the Ramanujan range even without Benjamini-Schramm convergence to the $(q + 1)$-regular tree ([7]).

**Proof.** Assume the weaker condition. Let $k = 2 \lfloor A \log_q n \rfloor + 1$. From the geometric interpretation of $\text{tr}A_k$ as path counting, we know that $\text{tr}A_k \geq 0$. Therefore,

$$\sum_{i:p_i = 2} \lambda_i^{(k)} + \sum_{i:p_i > 2, \lambda_i > 0} \lambda_i^{(k)} + \sum_{i:p_i > 2, \lambda_i < 0} \lambda_i^{(k)} \geq 0.$$
Noticing that $k$ is odd, and using Corollary 3.1 and Lemma 4.1 we get for every $\epsilon > 0$,

$$\sum_{i:p_i > 2, \lambda_i < 0} n^{-2A/p_i} \preceq \sum_{i:p_i > 2, \lambda_i < 0} q^{-k/p_i} \leq - \sum_{i:p_i > 2, \lambda_i < 0} \lambda_i^{(k)} \leq \sum_{i:p_i > 2, \lambda_i > 0} \lambda_i^{(k)} + \sum_{i:p_i = 2} \lambda_i^{(k)} \ll_{\epsilon} \sum_{i:p_i > 2, \lambda_i > 0} q^{-k(1/p_i - \epsilon)} + \sum_{i:p_i = 2} q^{-k(1/2 - \epsilon)} \ll_{\epsilon} \sum_{i:p_i > 2, \lambda_i > 0} n^{-2A/p_i + \epsilon} + n \cdot n^{-A(1 + \epsilon)} \ll_{\epsilon} n^{1-A+\epsilon}.$$

Applying Lemma 4.1 again we have for every $p > 2$ and $\epsilon > 0$,

$$\# \{i : \lambda_i < 0, p_i \geq p\} \ll_{\epsilon, \mathcal{F}} n^{1-A(1-2/p) + \epsilon},$$

and together with the weaker condition we have the full Sarnak-Xue density property. \hfill \square

Let us discuss non-backtracking cycles, i.e. non-backtracking paths of length $k \geq 2$ such that the last edge is not the inverse of the first edge. We consider two cycles as equivalent if they are rotations of one another. We say that a cycle $C$ is primitive if there is no $l | k$ and a cycle $C'$ of length $l$, such that $C$ is equivalent to the concatenation of $C'$ to itself $k/l$ times.

The Sarnak-Xue density property with parameter $A$ is also equivalent to a bound on the number of primitive cycles of length $k \leq 2A \log_q (n)$. The theorem below shows a stronger result, that it is enough to consider only cycles of length $k = 2 \lfloor A \log_q (n) \rfloor$.

The theorem is based on the graph prime number theorem of [31].

**Theorem 4.1.** Denote by $\pi_X (k)$ the number of equivalence classes of primitive cycles of length $k$, and by $N_X (k)$ the number of cycles of length $k$ (i.e. proper cycles, not equivalence classes).

For a sequence of graphs $X \in \mathcal{F}$, the Sarnak-Xue density property with parameter $A$ is satisfied if and only if for $k = 2 \lfloor A \log_q (n) \rfloor$ and $\epsilon > 0$, either $\pi_X (k) \ll_{\epsilon, \mathcal{F}} n^{1+\epsilon} q^{k/2}$ or $N_X (k) \ll_{\epsilon, \mathcal{F}} n^{1+\epsilon} q^{k/2}$.

**Proof.** We first show that the last two conditions are equivalent. We have (see [31, Section 10])

$$N_X (k) = \sum_{m | k} m \pi_X (m),$$

which implies

$$\pi_X (k) = \frac{1}{k} \sum_{m | k} \mu \left( \frac{k}{m} \right) N_X (m),$$

where $\mu$ is the Mobius function. Then

$$\left| \pi_X (k) - \frac{1}{k} N_X (k) \right| = \frac{1}{k} \sum_{m | k, m \neq k} \mu \left( \frac{k}{m} \right) N_X (m).$$
Using the trivial bound

\[ N_X (m) \ll n q^m, \]
the fact that the largest divisor of \( k \) is \( k/2 \), and \( \sum_{m|k} 1 \leq k \), we have

\[ |\pi_X (k) - \frac{1}{k} N_X (k)| \ll \varepsilon n q^{k/2}. \]

Using \( k \ll n^{\varepsilon} \), we see that \( \pi_X (k) \ll n^{1+\varepsilon} q^{k/2} \) if and only if \( N_X (k) \ll n^{1+\varepsilon} q^{k/2} \).

We now show the claim for \( N_X (k) \). Let \( E_X \) be the set of directed edges of \( X \), for \( e \in E_X \) let \( o(e), t(e) \in X \) be the origin and terminus of \( e \), and let \( \bar{e} \in E_X \) be the opposite edge. Let \( H : L^2 (E_X) \to L^2 (E_X) \) be Hashimoto’s non-backtracking operator, defined by

\[ H f (e) = \sum_{e' : o(e') = t(e), e' \neq \bar{e}} f (e'). \]

The following is well-known, and follows from the theory of the graph Ihara zeta function, see [31, Section 10] :

1. \( N_X (k) = \text{tr} H^k \).
2. If the eigenvalues of \( A \) are \( \lambda_1, \ldots, \lambda_n \), then the eigenvalues of \( H \) are \( \theta_i \pm \), defined as in Section 3, by

\[ \theta_i \pm = \frac{(q + 1) \lambda_i \pm \sqrt{(q + 1) \lambda_i^2 - 4q^2}}{2}, \]

and also has the eigenvalues \( \pm 1 \), each with multiplicity \( |E_X|/2 - n \).

Therefore,

\[ N_X (k) = \left( 1 + (-1)^k \right) (|E_X|/2 - n) + \sum_{i=1}^{n} (\theta_{i,+}^k + \theta_{i,-}^k). \]

Note that for \( k = 0 \mod 2 \) and \( p_i > 2 \) we have

\[ \theta_{i,+}^k + \theta_{i,-}^k = q^{k(1-1/p_i)} + q^{k/p_i}. \]

Choose \( k = 2 \lfloor A \log_q (n) \rfloor \). We then have

\[ N_X (k) - \sum_{i:p_i>2} (\theta_{i,+}^k + \theta_{i,-}^k) = N_X (k) - \sum_{i:p_i>2} (q^{k(1-1/p_i)} + q^{k/p_i}) = \]

\[ = 2 (|E_X|/2 - n) + \sum_{i:p_i=2} (\theta_{i,+}^k + \theta_{i,-}^k). \]

Using the fact that for \( p = 2 \) it holds that \( |\theta| = \sqrt{q} \) we have

\[ \left| N_X (k) - \sum_{i:p_i>2} q^{k(1-1/p_i)} \right| \ll n q^{k/2}. \]

Therefore, \( N_X (k) \ll n^{1+\varepsilon} q^{k/2} \) if and only if \( \sum_{i:p_i>2} q^{k(1-1/p_i)} \ll n^{1+\varepsilon} q^{k/2} \). Using the fact that \( k = 2 \lfloor A \log_q (n) \rfloor \) and Lemma 4.1 we get the claim for \( N_X (k) \). \( \square \)
5. Ramanujan Graphs

In this section, we discuss the analogues of Theorems 1.2 and 1.3 for Ramanujan graphs. The results can essentially be found in the work of Lubetzky and Peres ([23]) or Sardari ([28]). We include them here in a slightly different form in order to make use of them later.

In what follows in this section, we assume the graphs to be non-bipartite. However, the bipartite case can be treated similarly with the main difference that the eigenvalue \((-1)\) has to be taken into account just as 1, as we explain now. If \(X\) is non-bipartite, define

\[
L^2_0(X) = \left\{ f \in L^2(X) : \sum_{x \in X} f(x) = 0 \right\}.
\]

If \(X\) is an expander then the norm of \(A\) on \(L^2_0(X)\) is bounded by \(\lambda_0 < 1\). Let \(\pi(x) = \frac{1}{n}\) be the constant probability function. Then we may write every delta function as

\[(5.1) \quad \delta_{x_0} = \pi + (\delta_{x_0} - \pi),\]

with \(\delta_{x_0} - \pi \in L^2_0(X)\). It holds that \(A_k \pi = \pi\) for every \(k \geq 0\).

If \(X\) is bipartite, the vertex set can be decomposed into two equal parts \(X = X_L \cup X_R\), where \(|X_L| = |X_R| = n/2\), and all the edges are between a vertex in \(X_L\) and a vertex in \(X_R\). Then define

\[
L^2_{00}(X) = \left\{ f \in L^2(X) : \sum_{x \in X_L} f(x) = \sum_{x \in X_R} f(x) = 0 \right\}.
\]

In this case, if \(X\) is an expander then the norm of \(A\) on \(L^2_{00}(X)\) is bounded by \(\lambda_0 < 1\). Let \(\pi_- \in L^2(X)\) be the function

\[
\pi_-(x) = \begin{cases} 
\frac{1}{n} & x \in X_L \\
-\frac{1}{n} & x \in X_R.
\end{cases}
\]

Then the delta function for \(x_0 \in X_L\) can be written as

\[(5.2) \quad \delta_{x_0} = \pi + \pi_- + (\delta_{x_0} - \pi - \pi_-)\]

with \(\delta_{x_0} - \pi - \pi_- \in L^2_{00}(X)\), and analogously for \(x_0 \in X_R\) it holds that

\[(5.3) \quad \delta_{x_0} = \pi - \pi_- + (\delta_{x_0} - \pi + \pi_-),\]

with \(\delta_{x_0} - \pi + \pi_- \in L^2_{00}(X)\). It holds that \(A_k \pi = \pi\) for every \(k \geq 0\) and \(A^k \pi_- = (-1)^k \pi_-\).

Using Equations (5.1), (5.2) and (5.3) we can understand the action of \(A_k\), for \(k \geq 0\), on \(\delta_{x_0}\).

For simplicity, we consider from now on the non-bipartite case only. The statements and proofs can be extended to the bipartite case with minor adjustments.

Lemma 5.1. Let \(X\) be a \((q+1)\)-regular non-bipartite graph of size \(n\). Assume that for \(b \in \mathbb{R}, c > 0,\)

\[
\|A_k (\delta_{x_0} - \pi)\|_2^2 \leq cn^{-b}.
\]

Then

\[
\# \{ y : d(x_0, y) > k \} \leq cn^{-b+2}.
\]
Proof. The result follows from the fact that if $d(x_0, y) > k$, then $A_k(\delta_{x_0} - \pi)(y) = -\frac{1}{n}$ and therefore,
$$\|A_k(\delta_{x_0} - \pi)\|_2^2 \geq \# \{ y : d(x_0, y) > k \} n^{-2}.$$ 

□

**Theorem 5.1.** If $X$ is a Ramanujan graph, then for every $\tau > 0$, 
$$\# \{ y \in X : d(x_0, y) > (1 + \tau) \log_q(n) \} \leq ((1 + \tau) \log_q(n) + 1)^2 n^{1-\tau} = o_\tau(n),$$
and for every $\epsilon > 0$, for $n$ large enough, $(2 + \epsilon) \log_q(n)$ is an upper bound on the diameter of $X$.

Proof. For simplicity assume that $X$ is non-bipartite, the bipartite case is similar.

Write $\delta_{x_0}$ as 
$$\delta_{x_0} = \pi + (\delta_{x_0} - \pi),$$
with $\delta_{x_0} - \pi \in L^2_0(X)$. Then, by Corollary 3.1 
$$\|A_k(\delta_{x_0} - \pi)\|_2 \leq (k + 1) q^{-k/2} \|\delta_{x_0}\|_2 = (k + 1) q^{-k/2}.$$ 
And for $k = (1 + \tau) \log_q(n)$,
$$\|A_k(\delta_{x_0} - \pi)\|_2^2 \leq ((1 + \tau) \log_q(n) + 1)^2 n^{1-\tau}$$
which implies, by Lemma 5.1, that 
$$\# \{ y : d(x_0, y) > (1 + \epsilon) \log_q(n) \} \leq ((1 + \tau) \log_q(n) + 1)^2 n^{1-\tau}.$$ 

The diameter bound can be deduced from the almost-diameter bound, since for $n$ large enough every two vertices have a third vertex of distance at most $(1 + \epsilon/2) \log_q(n)$ from the both of them. However, it can also be deduced directly by taking $\tau = 1 + \epsilon$. □

Let us give some remarks on the proof and some generalizations.

1. The same proof shows that if every non-trivial eigenvalue $\lambda$ of $X$ is bounded by $|\lambda| \leq \lambda_0 = \frac{1}{q^{1/p} + q^{-1/p}}$, then the almost-diameter is bounded by $(1 + \epsilon) (p/2) \log_q(n)$ and the diameter is bounded by $(1 + \epsilon) \rho \log_q(n)$.
2. By taking $\tau = (2 + \epsilon) \log_q(\log_q(n))$ or $\tau = 1 + (2 + \epsilon) \log_q(\log_q(n))$, we get that for every $\epsilon > 0$ and $n$ large enough the almost-diameter is actually bounded by $\log_q(n) + (2 + \epsilon) \log_q(\log_q(n))$ and the diameter is bounded by $2 \log_q(n) + (2 + \epsilon) \log_q(\log_q(n))$.
3. Both previous remarks can be improved, following the method in [9]. Assume $P \in \mathbb{R}[X]$ is a polynomial of degree $k$, such that $P(1) = 1$ and $P(\lambda) = o\left(\frac{1}{\sqrt{q}}\right)$ for $|\lambda| \leq \lambda_0$. Then using the operator $P(A)$ instead of the operator $A_k$, the same proof implies that the almost-diameter is bounded by $k$. In [9] it is shown that the optimal choice of $P$ (depending on $\lambda_0$) is some twist of the Chebyshev polynomial of the first kind, which satisfies for $|\lambda| \leq \lambda_0$ that $|P(\lambda)| \leq \cosh\left(k \frac{1}{\lambda_0}\right)^{-1}$. For Ramanujan graphs, the bound is $|P(\lambda)| \leq 2 q^{-k/2}$. It has the effect of replacing $((1 + \tau) \log_q(n) + 1)^2$ in Theorem 5.1 by 4, and it thus reduces the almost-diameter of Ramanujan graphs to $\log_q(n) + O(g(n))$, where $g(n) \to \infty$ arbitrarily slowly. In the non-Ramanujan case this analysis is even better, and improves the coefficient of $\log_q(n)$ to $\frac{\ln(g)}{2} \left(\cosh\left(\frac{1}{\lambda_0}\right)\right)^{-1}$. Similar improvements can be made to
the diameter. The main results of our work cannot be improved similarly, since all the gain from the better analysis is lost due to the weaker assumptions.

**Theorem 5.2.** For every \( \epsilon_0 > 0, \epsilon_1 > 0 \) there is \( n_0 \) such that the following holds. Let \( X \) be a non-bipartite graph of size \( n > n_0 \). Then for every \( x_0 \in X \) the probability distributions \( A^k \delta_{x_0} \) of the simple random walk on \( X \) satisfy the following:

1. For \( k < (1 - \epsilon_0) \frac{q + 1}{q - 1} \log_q (n) \), it holds that
   \[
   \| A^k \delta_{x_0} - \pi \|_1 \geq 2 - \epsilon_1.
   \]

2. If \( X \) is Ramanujan, for \( k > (1 + \epsilon_0) \frac{q + 1}{q - 1} \log_q (n) \), it holds that
   \[
   \| A^k \delta_{x_0} - \pi \|_1 \leq \epsilon_1.
   \]

The proof of this theorem is based on the following lemma, whose proof can be found in [23, Section 2].

**Lemma 5.2.** It holds that \( A^k = \sum_{i=0}^{k} \alpha_i^{(k)} A_i \) for some constants \( \alpha_i^{(k)} \), satisfying \( 0 \leq \alpha_i^{(k)} \) and \( \sum_{i=0}^{k} \alpha_i^{(k)} = 1 \). Moreover, for every \( \epsilon_2, \epsilon_3 > 0 \), for \( k \) large enough,
\[
\sum_{i:|i-\frac{k}{q+1}k|>\epsilon_2k} \alpha_i^{(k)} \leq \epsilon_3.
\]

The constants \( \alpha_i^{(k)} \) in the lemma are the probability that the simple random walk on the \((q + 1)\)-regular tree starting from some vertex \( x_0 \) is at distance \( i \) from \( x_0 \) after \( k \) steps. Therefore, the lemma is a crude estimate on the rate of escape of the simple random walk, and it follows from the fact that the random walk is transient almost-surely, and once we leave for the last time the root \( x_0 \) we move away from \( x_0 \) with probability \( \frac{q}{q+1} \) and move towards \( x_0 \) with probability \( \frac{1}{q+1} \). For more precise statements, including a Central Limit Theorem for this deviation, see [23, Section 2].

**Proof of Theorem 5.2.** For (1), assume that \( k < (1 - \epsilon_0) \frac{q + 1}{q - 1} \log_q (n) \). Choose \( \epsilon_2 > 0 \) small enough relatively to \( \epsilon_0 \). It holds, by Lemma 5.2, that
\[
\| A^k \delta_{x_0} - \pi \|_1 = \left\| \sum_{i=0}^{k} \alpha_i^{(k)} (A_i \delta_{x_0} - \pi) \right\|_1 \geq - \sum_{i>(\frac{q-1}{q+1}+\epsilon_2)k}^{k} \alpha_i^{(k)} \| A_i \delta_{x_0} - \pi \|_1 + \left\| \sum_{i\leq(\frac{q-1}{q+1}+\epsilon_2)k} \alpha_i^{(k)} (A_i \delta_{x_0} - \pi) \right\|_1.
\]

For \( n \) large enough, the first term is bounded by \( \epsilon_1/3 \) by Lemma 5.2. In the second term, for \( i \leq \left( \frac{q-1}{q+1} + \epsilon_2 \right) k \), \( A_i \delta_{x_0} \) is supported on a ball of radius at most
\[
\left( \frac{q-1}{q+1} + \epsilon_2 \right) \left( 1 - \epsilon_0 \right) \frac{q+1}{q-1} \log_q (n) \leq (1 - \epsilon_0/2) \log_q (n)
\]
around \( x_0 \), where we choose \( \epsilon_2 \) to be small enough relative to \( \epsilon_0 \) so that the above will hold. Therefore, \( \sum_{i\leq(\frac{q-1}{q+1}+\epsilon_2)k} \alpha_i^{(k)} A_i \delta_{x_0} \) is non-zero on at most
\[
O \left( q^{(1-\epsilon_0/2) \log_q (n)} \right) \ll n^{1-\epsilon_0/2}
\]
of the vertices of the graph, which implies, using Lemma 5.2,
\[
\left\| \sum_{i \leq (\frac{n}{\ell} + \epsilon_2)k} \alpha_i^{(k)} (A_i \delta_{x_0} - \pi) \right\|_1 \geq \left( \sum_{i \leq (\frac{n}{\ell} + \epsilon_2)k} \alpha_i^{(k)} \right) \left( 2 - O(n^{-\epsilon_0/2}) \right) \\
\geq (1 - \epsilon_1/3) \left( 2 - O(n^{-\epsilon_0/2}) \right).
\]

Finally, we have for \( n \) large enough:
\[
\left\| A^k \delta_{x_0} - \pi \right\|_1 \geq -\epsilon_1/3 + 2 - \epsilon_1/2 \\
\geq 2 - \epsilon_1.
\]

For (2), assume that \( k > (1 + \epsilon_0) \frac{q + 1}{q - 1} \log_q (n) \). Choose \( \epsilon > 0 \) small enough relatively to \( \epsilon_0 \). Then
\[
\left\| A^k \delta_{x_0} - \pi \right\|_1 = \left\| \sum_{i=0}^{k} \alpha_i^{(k)} (A_i \delta_{x_0} - \pi) \right\|_1 \\
\leq \sum_{i \leq (\frac{n}{\ell} + \epsilon_2)k} \alpha_i^{(k)} \left\| A_i \delta_{x_0} - \pi \right\|_1 + \sum_{i > (\frac{n}{\ell} + \epsilon_2)k} \alpha_i^{(k)} \left\| A_i \delta_{x_0} - \pi \right\|_1.
\]

The first term is bounded for \( n \) large enough by \( \epsilon_1/2 \) by Lemma 5.2. The second term can be bounded by Cauchy-Schwartz
\[
\sum_{i > (\frac{n}{\ell} + \epsilon_2)k} \alpha_i^{(k)} \left\| (A_i \delta_{x_0} - \pi) \right\|_1 \leq \sup_{i > (\frac{n}{\ell} + \epsilon_2)k} \left\| A_i \delta_{x_0} - \pi \right\|_1 \\
\leq \sup_{i > (\frac{n}{\ell} + \epsilon_2)k} \sqrt{n} \left\| A_i \delta_{x_0} - \pi \right\|_2.
\]

But since \( k > (1 + \epsilon_0) \frac{q + 1}{q - 1} \log_q (n) \) and \( \epsilon_2 \) is small enough relatively to \( \epsilon_0 \), it holds that \( i > \left( \frac{q - 1}{q + 1} - \epsilon_2 \right) k \) satisfies \( i > (1 + \epsilon_0/2) \log_q (n) \). Then
\[
\sqrt{n} \left\| A_i \delta_{x_0} - \pi \right\|_2 \leq \sqrt{n} \left\| A_i \right\|_{L^2_q(X)} \left\| \delta_{x_0} - \pi \right\|_2 \\
\ll \epsilon_3 \sqrt{n} q^{(1/2 + \epsilon_3)} \\
\ll n^{1/2 + (1 + \epsilon_0/2) - (1/2 + \epsilon_3)}
\]

By choosing \( \epsilon_3 > 0 \) small enough relatively to \( \epsilon_0 \) and \( n \) large enough, we get that the last value is also bounded by \( \epsilon_1/2 \). \( \square \)

Remark 5.3. As in Theorem 5.1, the results can be improved in the Ramanujan case by a more careful analysis. In particular, the condition \( k < (1 - \epsilon_0) \frac{q + 1}{q - 1} \log_q (n) \) can be replaced by
\[
k < \frac{q + 1}{q - 1} \log_q (n) - \left( \log_q (n) \right)^{1/2 + \epsilon_0}
\]
and similarly for the upper bound (see [23] for more details).

6. Proof of Theorem 1.2 and Theorem 1.3

Before proving the theorems we give some other useful definition.
Definition 6.1. We say that \( x_0 \in X \) has local Sarnak-Xue property with parameter \( A \) if for every \( k \leq 2A \log_q (n) \) and \( \epsilon > 0 \) it holds that
\[
P(X, k, x_0) \ll_{\epsilon, F} n^\epsilon q^{k/2}.
\]

Let \( u_0, \ldots, u_{n-1} \) be an orthogonal basis of \( L^2(X) \) composed of eigenvectors of \( A \), with eigenvalue \( \lambda_i \) and \( p \)-value \( p_i \).

Lemma 6.2. The vertex \( x_0 \in X \) has local Sarnak-Xue property with parameter \( A \) if and only if for every \( \epsilon > 0 \),
\[
\sum_{i=0}^{n-1} |\langle \delta_{x_0}, u_i \rangle|^2 n^A(1-2/p_i) \ll_{\epsilon, F} n^\epsilon.
\]

Proof. It holds that
\[
P(X, k, x_0) = (q + 1) q^{k-1} \langle \delta_{x_0}, A_k \delta_{x_0} \rangle.
\]
By the spectral decomposition
\[
\delta_{x_0} = \sum_{i=0}^{n-1} \langle \delta_{x_0}, u_i \rangle u_i
\]
we have
\[
P(X, k, x_0) \asymp q^k \sum_{i=0}^{n-1} |\langle \delta_{x_0}, u_i \rangle|^2 \lambda_i^{(k)}
\]
\[
= q^k \sum_{i:p_i=2} |\langle \delta_{x_0}, u_i \rangle|^2 \lambda_i^{(k)} + q^k \sum_{i:p_i>2} |\langle \delta_{x_0}, u_i \rangle|^2 \lambda_i^{(k)}.
\]
the first term always satisfies for every \( \epsilon > 0 \),
\[
q^k \left| \sum_{i:p_i=2} |\langle \delta_{x_0}, u_i \rangle|^2 \lambda_i^{(k)} \right| \ll_{\epsilon, F} q^k q^{-k/2} q^k \sum_{i:p_i=2} |\langle \delta_{x_0}, u_i \rangle|^2 \ll_{\epsilon, F} q^{k/2}.
\]
Therefore, the local Sarnak-Xue density is satisfied if and only if for every \( k \leq 2A \log_q (n) \) and \( \epsilon > 0 \),
\[
q^k \sum_{i:p_i>2} |\langle \delta_{x_0}, u_i \rangle|^2 \lambda_i^{(k)} \ll_{\epsilon, F} n^\epsilon q^{k/2}.
\]
Since for \( k = 0 \mod 2 \) it holds that \( q^{k/p_i} \leq \lambda_i^{(k)} \ll_{\epsilon} q^{k/(p_i+\epsilon)} \) and for \( k = 1 \mod 2 \) we know the upper bound in absolute value, the last condition holds if and only if for every \( k \leq 2A \log_q (n) \) and \( \epsilon > 0 \),
\[
q^k \sum_{i:p_i>2} |\langle \delta_{x_0}, u_i \rangle|^2 q^{-k/(p_i+\epsilon)} \ll_{\epsilon, F} n^\epsilon q^{k/2}
\]
i.e., the local Sarnak-Xue density with parameter \( A \) is satisfied if and only if for every \( k \leq 2A \log_q (n) \) and \( \epsilon > 0 \),
\[
\sum_{i:p_i>2} |\langle \delta_{x_0}, u_i \rangle|^2 q^{k(1/2-1/p_i)} \ll_{\epsilon, F} n^\epsilon.
\]
This condition for \( k = 2A \left\lfloor \log_q (n) \right\rfloor \) shows that local Sarnak-Xue density with parameter \( A \) implies Equation (6.1). But if this condition holds for \( k \) then it obviously holds for \( k' \leq k \). Therefore, Equation (6.1) also implies the local Sarnak-Xue density with parameter \( A \).

Lemma 6.3. If a vertex \( x_0 \in X \) has local Sarnak-Xue density with parameter \( A = 1 \) and \( X \) is an expander then:

- For every \( \epsilon_0, \epsilon_1 > 0 \), for \( n \) large enough depending on \( \mathcal{F}, \epsilon_0, \epsilon_1 \), for all but \( \epsilon_0 n \) of \( y \in X \) it holds that \( d(x, y) \leq (1 + \epsilon_1) \log_q (n) \), i.e. \( R(n) = (1 + \epsilon_1) \log_q (n) \) is an almost-radius of \( X \) at \( x_0 \).
- For every \( \epsilon_0, \epsilon_1 > 0 \), for \( n \) large enough depending on \( \mathcal{F}, \epsilon_0, \epsilon_1 \), for \( k > (1 + \epsilon_1) \frac{q + 1}{q - 1} \log_q (n) \), it holds that

\[
\| A^k \delta_{x_0} - \pi \|_1 \leq \epsilon_0.
\]

Proof. As before, we assume that \( X \) is non-bipartite, and the bipartite case can be treated similarly. We let \( u_0, \ldots, u_{n-1} \) be the orthogonal basis of \( L^2(X) \) as above. We assume that \( u_0 = \frac{1}{\sqrt{n}} \) the \( L^2 \)-normalized constant function.

The fact that \( X \) is an expander means that there exists \( p' < \infty \) (depending only on \( \mathcal{F} \)) such that for all \( i > 0 \) it holds that \( p_i \leq p' \).

Let \( k \geq \left\lfloor (1 + \epsilon_1) \log_q (n) \right\rfloor \). By Lemma 5.1 and the proof of Theorem 5.2, to prove both claims in the theorem it suffices to prove that

\[
\| A_k (\delta_{x_0} - \pi) \|_2^2 \leq \epsilon_0 n^{-1}.
\]

Decompose \( \delta_{x_0} - \pi = \sum_{i>1} \left\langle \delta_{x_0}, u_i \right\rangle u_i \). Then by Corollary 3.1, for every \( \epsilon_2 > 0 \).

\[
\| A_k (\delta_{x_0} - \pi) \|_2^2 = \sum_{i=1}^{n-1} |\left\langle \delta_{x_0}, u_i \right\rangle|^2 \lambda_i^{(k)} \leq \epsilon_2 \sum_{i=1}^{n-1} |\left\langle \delta_{x_0}, u_i \right\rangle|^2 q^{2k(-1/p_i + \epsilon_2)}
\]

\[
\leq \epsilon_2 \sum_{i=1}^{n-1} |\left\langle \delta_{x_0}, u_i \right\rangle|^2 n^{2(-1/p_i + \epsilon_2)(1 + \epsilon_1)}
\]

\[
= \sum_{i=1}^{n-1} |\left\langle \delta_{x_0}, u_i \right\rangle|^2 n^{-2/p_i} n^{2((-1/p_i + \epsilon_2)\epsilon_1 + \epsilon_2)}
\]

\[
\leq n^{2((-1/p' + \epsilon_2)\epsilon_1 + \epsilon_2)} \sum_{i=1}^{n-1} |\left\langle \delta_{x_0}, u_i \right\rangle|^2 n^{-2/p_i}.
\]

If we choose \( \epsilon_2 > 0 \) small enough relative to \( \epsilon_1, p' \) it holds that for every \( \epsilon_3 > 0 \),

\[
\ll n^{-\epsilon_1/p'} \sum_{i=2}^{n} |\left\langle \delta_{x_0}, u_i \right\rangle|^2 n^{-2/p_i} \ll \epsilon_3, \mathcal{F} n^{-\epsilon_1/p'} n^{-1+\epsilon_3},
\]

where we used Lemma 6.2 and the local Sarnak-Xue condition. Choose a small enough constant \( c_\mathcal{F} \), and then choose \( \epsilon_3 > 0 \) small enough and \( n \) large enough relative to \( \epsilon_1, p' \), so that the last value is \( \leq c_\mathcal{F} \epsilon_0 n^{-1} \).
By adjusting $c_F$, we get eventually that
\[
\|A_k(\delta_{x_0} - \pi)\|_2^2 \leq \epsilon_0 n^{-1},
\]
as needed. \qed

**Lemma 6.4.** Assume that $\mathcal{F}$ is a family of graphs satisfying the Sarnak-Xue density with parameter $A = 1$. Let $\epsilon_0 > 0$. Then we may choose for each graph $X \in \mathcal{F}$ a subset $Y \subset X$ of the vertices with $|Y| \geq |X| (1 - \epsilon_0)$ such that for every $x_0 \in Y$ the local Sarnak-Xue property with parameter $A = 1$ is satisfied at $x_0$.

**Proof.** Since $P(X, k) = \sum P(X, k, x_0)$, if the Sarnak-Xue density property (or equivalently, the weak injective radius property) with parameter $A = 1$ holds then the local Sarnak-Xue density holds on average over all the vertices $x_0$. More precisely, for every $\epsilon_1 > 0$, the number of $x_0 \in X$ satisfying $P(X, 2 \lfloor \log_q(n) \rfloor, x_0) > n^{1+2\epsilon_1}$ is at most $C_\epsilon n^{1-\epsilon_1}$, $C_\epsilon$ some constant.

Now, for each $k = 1, 2, \ldots$ let $\epsilon_k = 1/k$. Let $N_k$ be large enough so that $C_\epsilon k N_k^{1-\epsilon_k/2} \leq \epsilon_0$. Now for each $X$ choose $k$ maximal such that $n = |X| > N_k$. Let $Y \subset X$ be the set of vertices $x_0 \in X$ such that $P(X, 2 \lfloor \log_q(n) \rfloor, x_0) \leq n^{1+\epsilon_k}$. By construction $|Y| \geq |X|(1 - \epsilon_0)$.

We show that the local Sarnak-Xue density holds: for every $\epsilon > 0$ let $k$ be such that $\epsilon_k < 2\epsilon$. Then for $|X| > N_k$, for $x_0 \in Y$, $P(X, 2 \lfloor \log_q(n) \rfloor, x_0) \leq n^{1+\epsilon_k/2} \leq n^{1+\epsilon}$. Since there is a finite number of graphs $X \in \mathcal{F}$ with $|X| < N_k$ we are done. \qed

We can now prove Theorem 1.2 and Theorem 1.3.

**Proof of Theorem 1.2 and Theorem 1.3.** The proof follows from the combination of Lemma 6.4 and Lemma 6.3. \qed

As a final claim, we prove the following lemma, needed in the proof of Theorem 1.5.

**Lemma 6.5.** If $X$ is a Cayley graph which satisfies the Sarnak-Xue density property with parameter $A$, then for every $x_0, y_0 \in X$ and $k < 2A \log_q n$, we have for every $\epsilon > 0$
\[
P(X, k, x_0, y_0) \ll_{\epsilon, \mathcal{F}} n^{\epsilon} q^{k/2},
\]
where $P(X, k, x_0, y_0)$ is the number of non-backtracking paths from $x_0$ to $y_0$.

**Proof.** It holds that
\[
P(X, k, x_0, y_0) \leq (q + 1)^2 q^{k-2} \langle A_{[k/2]} \delta_{x_0}, A_{[k/2]} \delta_{x_0} \rangle,
\]
since the right hand side counts more paths than just non-backtracking paths between $x_0$ and $y_0$. Therefore, by the Cauchy-Schwartz inequality,
\[
P(X, k, x_0, y_0) \ll q^k \|A_{[k/2]} \delta_{x_0}\|_2 \|A_{[k/2]} \delta_{x_0}\|_2.
\]
Since $X$ is a Cayley graph, both $x_0$ and $y_0$ have local Sarnak-Xue property with parameter $A$, which means that if for every $k \leq 2A \log_q(n)$ and $\epsilon > 0$ it holds that
\[
P(X, k, x_0) = P(X, k, y_0) \ll_{\epsilon, \mathcal{F}} n^{\epsilon} q^{k/2}.
\]
As in Lemma 6.2, this is equivalent to the fact that for every $\epsilon > 0$
\[
\sum_{i=0}^{n-1} |\langle \delta x_0, u_i \rangle|^2 n^{A(1-2/p_i)} \ll_{F,\epsilon} n^\epsilon.
\]
Therefore, for $k \leq 2A \log_q(n)$ and $\epsilon > 0$,
\[
\|A_{\lfloor k/2 \rfloor} \delta x_0\|_2^2 = \sum_{i=0}^{n-1} |\chi_{\lfloor (k/2) \rfloor}(i)|^2 |\langle \delta x_0, u_i \rangle|^2
\]
\[
\ll \epsilon \sum_{i=0}^{n-1} q^{-k/p_i} |\langle \delta x_0, u_i \rangle|^2
\]
\[
= q^{-k/2} \sum_{i=0}^{n-1} q^{k/2-k/p_i} |\langle \delta x_0, u_i \rangle|^2
\]
\[
\ll q^{-k/2} \sum_{i=0}^{n-1} q^{A(1-2/p_i)} |\langle \delta x_0, u_i \rangle|^2
\]
\[
\ll_{\epsilon,F} q^{-k/2} n^\epsilon.
\]
Inserting this and the similar claim for $\lceil k/2 \rceil$ into Equation (6.2) we have for every $\epsilon > 0$,
\[
P(X, k, \delta x_0, y_0) \ll_{\epsilon,F} n^\epsilon q^{k/2},
\]
as needed. □

7. Benjamini-Schramm Convergence

Our goal in this section is to state the following theorem, which follows from [1] and [2].

**Theorem 7.1.** Let $F$ be a family of $(q+1)$-regular graphs with the number of vertices growing to infinity. The following are equivalent, and if they hold we say that the sequence of graphs Benjamini-Schramm converges to the $(q+1)$-regular tree:

1. For every $k > 0$, as $n \to \infty$
\[
\lim_{n \to \infty} P(X, k)/n \to 0.
\]
2. For every $k > 0$
\[
P(X, k)/n \ll_{\epsilon,k} n^\epsilon q^{k/2}.
\]
3. For every $\epsilon > 0$,
\[
\# \{\lambda \in \text{spec } A : |\lambda| > (1 + \epsilon) 2\sqrt{q}\}/n \to 0.
\]
4. The spectral measure of $A$ converges to the spectral measure of the tree.

**Proof.** The equivalence of (1), (3) and (4) is a consequence of [2, Theorem 5] (the result is slightly more general than [2, Theorem 4], but follows in the same way).

The fact that (2) and (3) are equivalent is proven in the same way as Theorem 1.4, and is a slightly stronger version of [2, Corollary 7]. □
One interesting corollary of the proof of Theorem 1.5 for the context of Benjamini-Schramm convergence is the following:

**Corollary 7.1.** If we have a family Cayley$(\text{SL}_2(\mathbb{F}_t), S_t)$, where $t$ is prime, which Benjamini-Schramm converges to the $(q+1)$-regular tree, then the corresponding family of Schreier graphs on $P^1_1(\mathbb{F}_t)$ also Benjamini-Schramm converges to the $(q+1)$-regular tree.

**References**

1. Miklós Abért, Yair Glasner, and Bálint Virág, *Kesten’s theorem for invariant random subgroups*, Duke Mathematical Journal **163** (2014), no. 3, 465–488. 7
2. , *The measurable Kesten theorem*, The Annals of Probability **44** (2016), no. 3, 1601–1646. 1, 7, 7
3. Charles Bordenave, *A new proof of Friedman’s second eigenvalue theorem and its extension to random lifts*, Annales Scientifiques de l’École Normale Supérieure. Quatrième Série **53** (2020), no. 6, 1393–1439. 1
4. Charles Bordenave and Hubert Lacoin, *Cutoff at the entropic time for random walks on covered expander graphs*, Journal of the Institute of Mathematics of Jussieu (2021), 1–46. 1, 1, 1, 1
5. Jean Bourgain and Alex Gamburd, *Uniform expansion bounds for Cayley graphs of $\text{SL}_2(\mathbb{F}_p)$*, Annals of Mathematics **167** (2008), no. 2, 625–642. 1, 1, 1, 1
6. Andrei Broder and Eli Shamir, *On the second eigenvalue of random regular graphs*, 28th Annual Symposium on Foundations of Computer Science (sfcs 1987), IEEE, 1987, pp. 286–294. 1
7. PJ Cameron, JM Goethals, JJ Seidel, and EE Shult, *Line graphs, root systems, and elliptic geometry*, Journal of Algebra **43** (1976), no. 1, 305–327. 4
8. Michael Chapman and Ori Parzanchevski, *Cutoff on Ramanujan complexes and classical groups*, arXiv preprint arXiv:1901.09383 (2019). 1
9. Fan RK Chung, Vance Faber, and Thomas A Manteuffel, *An upper bound on the diameter of a graph from eigenvalues associated with its Laplacian*, SIAM Journal on Discrete Mathematics **7** (1994), no. 3, 443–457. 1, 3
10. Giuliana Davidoff, Peter Sarnak, and Alain Valette, *Elementary number theory, group theory and Ramanujan graphs*, vol. 55, Cambridge University Press, 2003. 1, 1, 1, 1
11. Joel Friedman, *A proof of Alon’s second eigenvalue conjecture*, Proceedings of the thirty-fifth annual ACM symposium on Theory of computing, ACM, 2003, pp. 720–724. 1, 1
12. Alex Gamburd, *On the spectral gap for infinite index "congruence" subgroups of $\text{SL}_2(\mathbb{Z})$*, Israel Journal of Mathematics **127** (2002), no. 1, 157–200. 2
13. Alexander Gamburd, Shlomo Hoory, Mehrdad Shahshahani, Aner Shalev, and Balint Virág, *On the girth of random Cayley graphs*, Random Structures & Algorithms **35** (2009), no. 1, 100–117. 1, 1, 2
14. Anish Ghosh, Alexander Gorodnik, and Amos Nevo, *Best possible rates of distribution of dense lattice orbits in homogeneous spaces*, Journal für die reine und angewandte Mathematik (Crelles Journal) (2014). 1
15. Konstantin Golubev and Amitay Kamber, *Cutoff on hyperbolic surfaces*, Geometriae Dedicata (2019). 1–31. 1
16. William T Gowers, *Quasirandom groups*, Combinatorics, Probability and Computing **17** (2008), no. 3, 363–387. 1
17. Chris Hall, Doron Puder, and William F Sawin, *Ramanujan coverings of graphs*, Advances in Mathematics **323** (2018), 367–410. 1
18. Godfrey Harold Hardy and Edward Maitland Wright, *An introduction to the theory of numbers*, Oxford university press, 1979. 2, 4
19. Shlomo Hoory, Nathan Linial, and Avi Wigderson, *Expander graphs and their applications*, Bulletin of the American Mathematical Society **43** (2006), no. 4, 439–561. 1
20. Matin N Huxley, *Exceptional eigenvalues and congruence subgroups*, The Selberg Trace Formula and Related Topics, Contemp. Math **53** (1986), 341–349. 1
21. Amitay Kamber, *$L^p$-expander graphs*, Israel Journal of Mathematics **234** (2019), no. 2, 863–905. 1, 3
22. Eyal Lubetzky, Alexander Lubotsky, and Ori Parzanchevski, *Random walks on Ramanujan complexes and digraphs*, Journal of the European Mathematical Society **22** (2020), no. 11, 3441–3466. 1
23. Eyal Lubetzky and Yuval Peres, *Cutoff on all Ramanujan graphs*, Geometric and Functional Analysis **26** (2016), no. 4, 1190–1216. (document), 1, 1, 1, 5, 5, 5, 5.

24. Alexander Lubotzky, Ralph Phillips, and Peter Sarnak, *Ramanujan graphs*, Combinatorica **8** (1988), no. 3, 261–277. 1, 1, 1, 1, 5.

25. Adam Marcus, Daniel A Spielman, and Nikhil Srivastava, *Interlacing families I: bipartite Ramanujan graphs of all degrees*, Foundations of Computer Science (FOCS), 2013 IEEE 54th Annual Symposium on, IEEE, 2013, pp. 529–537.

26. Ori Parzanchevski and Peter Sarnak, *Super-golden-gates for PU(2)*, Advances in Mathematics **327** (2018), 869–901.

27. Igor Rivin and Naser T Sardari, *Quantum chaos on random cayley graphs of SL₂[Z/pZ]*, Experimental Mathematics **28** (2019), no. 3, 328–341. (document), 1, 1

28. Naser T Sardari, *Diameter of Ramanujan graphs and random Cayley graphs*, Combinatorica **39** (2019), no. 2, 427–446. (document), 1, 1, 1, 5.

29. Peter Sarnak, *Letter to Stephen D. Miller and Naser Talebizadeh Sardari on optimal strong approximation by integral points on quadrics*, 2015. 1

30. Peter Sarnak and Xiao Xi Xue, *Bounds for multiplicities of automorphic representations*, Duke Math. J **64** (1991), no. 1, 207–227. 1, 1

31. Audrey Terras, *Zeta functions of graphs: a stroll through the garden*, vol. 128, Cambridge University Press, 2010. 4, 4