A NOTE ON A RELATION BETWEEN THE WEAK AND STRONG DOMINATION NUMBERS OF A GRAPH

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Abstract. In a graph $G = (V,E)$ a vertex is said to dominate itself and all its neighbors. A set $D \subseteq V$ is a weak (strong, respectively) dominating set of $G$ if every vertex $v \in V - S$ is adjacent to a vertex $u \in D$ such that $d_G(v) \geq d_G(u)$ (respectively). The weak (strong, respectively) domination number of $G$, denoted by $\gamma_w(G)$ ($\gamma_s(G)$, respectively), is the minimum cardinality of a weak (strong, respectively) dominating set of $G$. In this note we show that if $G$ is a connected graph of order $n \geq 3$, then $\gamma_w(G) + t\gamma_s(G) \leq n$, where $t = 3/(\Delta + 1)$ if $G$ is an arbitrary graph, $t = 3/5$ if $G$ is a block graph, and $t = 2/3$ if $G$ is a claw free graph.

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1. INTRODUCTION

We consider finite, undirected, simple graphs. Let $G$ be a graph, with vertex set $V$ and edge set $E$. The open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood is $N[v] = N(v) \cup \{v\}$. For a subset $S \subseteq V$, the open neighborhood is $N(S) = \cup_{v \in S} N(v)$ and the closed neighborhood is $N[S] = N(S) \cup S$. By $G[S]$ we denote the subgraph induced by the vertices of $S$. If $v$ is a vertex of $V$, then the degree of $v$ denoted by $d_G(v)$, is the size of its open neighborhood. A tree is a connected graph that contains no cycle. A star $K_{1,q}$ is a tree of order $q + 1$ with at least $q$ vertices of degree 1. A subdivided star $SS_q$ is obtained from a star $K_{1,q}$ by replacing each edge $uv$ of the star by a vertex $w$ and edges $uw$ and $vw$. The claw is the star $K_{1,3}$. Given any graph $H$, a graph $G$ is $H$-free if it does not have any induced subgraph isomorphic to $H$. A block graph is a graph in which every block (maximal 2-connected graph) is a clique. It is well-known that block graphs are exactly chordal graphs that do not contain $K_4 - \{e\}$ as induced subgraph.
In [5], Sampathkumar and Pushpa Latha have introduced the concept of weak and strong domination in graphs. A subset \( D \subseteq V \) is a weak dominating set (wd-set) if every vertex \( v \in V - S \) is adjacent to a vertex \( u \in D \), where \( d_G(v) \geq d_G(u) \). The subset \( D \) is a strong dominating set (sd-set) if every vertex \( v \in V - S \) is adjacent to a vertex \( u \in D \), where \( d_G(u) \geq d_G(v) \). The weak (strong, respectively) domination number \( \gamma_w(G) \) (\( \gamma_s(G) \), respectively) is the minimum cardinality of a wd-set (an sd-set, respectively) of \( G \). If \( D \) is an sd-set of \( G \) of size \( \gamma_s(G) \), then we call \( D \) a \( \gamma_s(G) \)-set.

Strong and weak domination have been studied for example in [1–4].

In their paper introducing weak and strong domination in graphs, Sampathkumar and Pushpa Latha showed that a graph \( G \) of order \( n \) satisfies \( \gamma_w(G) + \gamma_s(G) \leq n \) if \( G \) is a \( d \)-balanced graph (\( G \) has an sd-set \( D_1 \) and a wd-set \( D_2 \) such that \( D_1 \cap D_2 = \emptyset \)). However there exist graphs \( G \) for which \( \gamma_w(G) + \gamma_s(G) > n \). For example if \( G \) is a subdivided star \( SS_q \) with \( q \geq 3 \), then \( \gamma_w(SS_q) = \gamma_s(SS_q) = q + 1 = (n + 1)/2 \).

2. RESULTS

We begin by giving an observation and two useful lemmas.

**Observation 2.1.** 1) For a cycle \( C_n \) we have \( \gamma_w(C_n) = \gamma_s(C_n) = \lceil n/3 \rceil \).

2) For a nontrivial path \( P_n \) we have

\[
\gamma_s(P_n) = \lceil n/3 \rceil \quad \text{and} \quad \gamma_w(P_n) = \begin{cases} 
\lceil n/3 \rceil , & \text{if } n \equiv 1 \pmod{3}, \\
\lceil n/3 \rceil + 1, & \text{otherwise}.
\end{cases}
\]

**Lemma 2.2.** Let \( G = (V, E) \) be a nontrivial connected graph. Then \( G \) has a \( \gamma_s(G) \)-set \( D \) such that for every vertex \( x \in D \) having at least one neighbor in \( V - D \), there is a vertex \( y \in V - D \) adjacent to \( x \) such that \( d_G(y) \leq d_G(x) \).

**Proof.** Among all \( \gamma_s(G) \)-sets let \( D \) be a one such vertex such that \( \sum_{u \in D} d_G(u) \) is maximum. Obviously the result is valid if \( |V| = 2 \). Hence let \( |V| \geq 3 \) and assume that \( D \) contains a vertex \( x \) such that \( N(x) \cap (V - D) \neq \emptyset \) and \( d_G(y) > d_G(x) \) for every \( y \in N(x) \cap (V - D) \). Then \( \{y\} \cup D - \{x\} = D' \) is a \( \gamma_s(G) \)-set such that \( \sum_{u \in D'} d_G(u) > \sum_{u \in D} d_G(u) \), contradicting our choice of \( D \).

**Lemma 2.3.** Let \( X \) be an independent set of a connected graph \( G \) such that every vertex of \( X \) has degree at least three. Then:

(i) if \( G \) is a claw free graph, then \( 3|X| \leq 2|N(X)| \).

(ii) if \( G \) is a block graph, then \( 2|X| + 1 \leq |N(X)| \).

**Proof.** (i) Let \( E' \) be the set of edges between \( X \) and \( N(X) \). Then \( 3|X| \leq |E'| \). Also since \( G \) is claw free and \( X \) is independent, every vertex of \( G \) has at most two neighbors in \( X \), implying that \( |E'| \leq 2|N(X)| \). Therefore, \( 3|X| \leq |E'| \leq 2|N(X)| \).

(ii) Assume now that \( G \) is a block graph and let \( A = N(X) \). Consider the graph \( G'[X,A] \) induced by the vertices of \( X \) and \( A \). We can suppose that \( G'[X,A] \) is connected, for otherwise we can repeat the procedure below for each component. Let
Therefore, $x_3$ we have $\cup$. A note on a relation between the weak and strong domination numbers of a graph $\gamma$ or a block graph. Hence we obtain (i) and (ii), respectively. We omit the details.

is claw free or a block graph. Note that $\Delta G$ contains two adjacent vertices $u v$. Let $G$ be a connected graph of order $n$. If $G$ is claw free, then $\gamma_w(G)$ is a block graph, then $\gamma_w(G)$ is a connected block graph, each vertex $x_k$ for $k \geq 2$ has exactly one neighbor in $\cup_{j=1}^{k-1} A_j$. Using this fact and the fact that every vertex of $X$ has degree at least three, it follows that $|A_k| \geq 2$ for $2 \leq k \leq t$. Therefore, $|N(X)| = |A| = |A_1| + |A_2| + \ldots + |A_t| \geq 3 + 2(t - 1) = 2|X| + 1$. □

Now we are ready to state our main result.

**Theorem 2.4.** Let $G$ be a connected graph of order $n \geq 3$ and maximum degree $\Delta$. Then $\gamma_w(G) + 3\gamma_s(G)/(\Delta + 1) \leq n$. Moreover,

(i) if $G$ is a claw free graph, then $\gamma_w(G) + 3\gamma_s(G)/5 \leq n$, and

(ii) if $G$ is a block graph, then $\gamma_w(G) + 2\gamma_s(G)/3 \leq (3n - 1)/3$.

**Proof.** Clearly since $n \geq 3$, we have $\Delta \geq 2$. If $\Delta = 2$, then $G$ is either a cycle $C_n$ or a path $P_n$, and by Observation 2.1 the result holds. Thus we may assume that $\Delta \geq 3$. Let $D$ be a $\gamma_s(G)$-set satisfying the conditions of Lemma 2.2. Let $A = \{x \in D : N(x) \cap (V - D) \neq \emptyset\}$ and $X = D - A$. Observe that by our choice of $D$, the set $V - D$ weakly dominates $A$. If $X = \emptyset$, then $A = D$, and consequently, $\gamma_w(G) \leq |V - D| = n - \gamma_s(G)$. Hence the result is valid even for (i) and (ii) when $G$ is claw free or a block graph, respectively. From now on we will assume that $X \neq \emptyset$. If $X$ contains two adjacent vertices $u$ and $v$, then one of $D - \{u\}$ or $D - \{v\}$ is a strong dominating set of $G$, a contradiction. Hence $X$ is an independent set. Note that every vertex of $D$ has degree at least two, otherwise $n = 2$ or $G$ is not connected. Also since $N(X) \subseteq A$ we have $d_G(u) \geq 3$ for every $u \in X$; otherwise $D - \{u\}$ is an sd-set of $G$, a contradiction. Now since $V - D$ weakly dominates $A$, the set $(V - D) \cup X$ weakly dominates $G$, and therefore

$$\gamma_w(G) \leq |(V - D) \cup X| = n - |D| + |X|.$$ 

Now let us show how to bound $|X|$ by $|D|$ when $G$ is an arbitrary graph, claw free, or a block graph. Note that $|D| = |X| + |A| \geq |X| + |N(X)|$. Let $E(X, N(X))$ be the set of edges between $X$ and $N(X)$. Since $d_G(u) \geq 3$ for every $u \in X$ and $N(X) \subset D$ we have $3|X| \leq |E(X, N(X))|$. Also each vertex $y$ of $N(X)$ has degree at most $\Delta - 1$, otherwise $D - N(y) \cap X$ would be an sd-set of $G$, a contradiction. It follows that every vertex of $N(X)$ has at most $\Delta - 2$ neighbors in $X$, thus $|E(X, N(X))| \leq (\Delta - 2)|N(X)|$. This implies that $3|X| \leq |E(X, N(X))| \leq (\Delta - 2)|N(X)|$, and consequently, $|N(X)| \geq 3|X|/(\Delta - 2)$. Since $|D| \geq |X| + |N(X)|$, we obtain $|X| \leq (\Delta - 2)|D|/(\Delta + 1)$. Now we get $\gamma_w(G) \leq n - |D| + |X| = n - 3|D|/(\Delta + 1)$.

Using Lemma 2.3, one can improve the previous result when $G$ is a claw free graph or a block graph. Hence we obtain (i) and (ii), respectively. We omit the details. □
Since the class of trees is contained in the class of block graphs we obtain the following corollary.

**Corollary 2.5.** If $T$ is a tree of order $n \geq 3$, then $\gamma_w(T) + 2\gamma_s(T)/3 \leq (3n - 1)/3$.

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**REFERENCES**

[1] J.H. Hattingh, M.A. Henning, *On strong domination in graphs*, J. Combin. Math. Combin. Comput. 26 (1998) 73–92.

[2] J.H. Hattingh, R.C. Laskar, *On weak domination in graphs*, Ars Combinatoria 49 (1998).

[3] D. Rautenbach, *Bounds on the weak domination number*, Austral. J. Combin. 18 (1998), 245–251.

[4] D. Rautenbach, *Bounds on the strong domination number*, Discrete Math. 215 (2000), 201–212.

[5] E. Sampathkumar, L. Pushpa Latha, *Strong, weak domination and domination balance in graphs*, Discrete Math. 161 (1996), 235–242.

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