Stable dark solitons in $\mathcal{PT}$-symmetric dual-core waveguides

Yu. V. Bludov$^1$, V. V. Konotop$^2$, and B. A. Malomed$^3$

1 Centro de Física, Universidade do Minho, Campus de Gualtar, Braga 4710-057, Portugal
2 Centro de Física Teórica e Computacional and Departamento de Física, Faculdade de Ciências, Universidade de Lisboa, Avenida Professor Gama Pinto 2, Lisboa 1649-003, Portugal
3 Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel

We construct dark solitons in the recently introduced model of the nonlinear dual-core coupler with the mutually balanced gain and loss applied to the two cores, which is a realization of parity-time symmetry in nonlinear optics. The main issue is stability of the dark solitons. The modulation stability of the CW (continuous-wave) background, which supports the dark solitons, is studied analytically, and the full stability is investigated in a numerical form, via computation of eigenvalues for modes of small perturbations. Stability regions are thus identified in the parameter space of the system, and verified in direct simulations. Collisions between stable dark solitons are briefly considered too.

PACS numbers: 42.65.Tg, 11.30.Er

I. INTRODUCTION

The concept of the parity-time $\mathcal{PT}$ symmetry was originally elaborated in the field theory [1], as a generalization of the canonical conservative systems, which are based on Hermitian Hamiltonians, for a special case of dissipative systems which include exactly balanced and spatially separated linear gain and loss. Such systems are described by non-Hermitian Hamiltonians, whose Hermitian and anti-Hermitian parts are spatially even and odd, respectively. A distinctive feature of the non-Hermitian Hamiltonians, which are subject to the condition of the $\mathcal{PT}$ symmetry, is the fact that, up to a certain critical value of the strength of their anti-Hermitian (dissipative) part, the spectrum of such Hamiltonians may remain purely real (physical). When this occurs a $\mathcal{PT}$-symmetric non-Hermitian Hamiltonian can be eventually transformed into Hermitian ones by means of similarity transformations [2].

In terms of the quantum theory, $\mathcal{PT}$-symmetric systems are the settings of theoretical interest. For the realization of the $\mathcal{PT}$ symmetry in real settings, one can make use of the fact that the linear propagation equation derived for optical beams in the paraxial approximation has essentially the same form as the Schrödinger equation in quantum mechanics, in one- and two-dimensional (1D and 2D) cases alike. In other words, the evolution of the wave function of a quantum particle may be emulated by the transmission of an optical beam, as in both cases the wave propagation follows the same principles. This fact makes it possible to simulate many quantum-mechanical phenomena by means of relatively simple settings which can be realized in classical optics [3]. In this vein, the realization of $\mathcal{PT}$-symmetric settings in optical systems, which combine spatially symmetric refractive-index landscapes and mutually balanced spatially separated gain and loss, was proposed in [4] (see also [5] for subsequent early development of optical applications) and experimentally demonstrated in [6].

Typically, the models amount to the 1D or 2D linear Schrödinger equations with a complex potential, whose real and imaginary parts are, respectively, spatially even and odd. Another possibility of the realization of the $\mathcal{PT}$-symmetric settings in optics, in the form a dual-core coupler, with the mutually balanced gain and loss applied to the two cores, was recently proposed in the works [6, 7] for stationary regime of light propagation and in [8, 10] for the bright optical solitons which exist when the arms of the coupler obey Kerr nonlinearity. In this last setting, the solitons are available in the exact analytical form, and their stability boundary can be found analytically too [8, 10].

A natural extension of the analysis of the nonlinear $\mathcal{PT}$-symmetric systems is to search for stable dark solitons in them, which is subject of the present work. We notice that the dark solitons in a parabolic potential with a $\mathcal{PT}$-symmetric non-Hermitian part, where they can be considered as the nonlinear modes, bifurcating from the first excited state of the linear $\mathcal{PT}$-symmetric parabolic potential, have recently been addressed in the literature [11].

An alternative natural setting for the consideration of dark solitons in $\mathcal{PT}$-symmetric optical systems is provided by the above-mentioned dual-core system. As well as a broad class of other solutions, dark solitons in this system can be easily found in an exact form [8], the actual problem being the analysis of their stability and interactions. The model is introduced in Sec. II, and the modulational stability of the CW (continuous-wave) background, supporting the dark solitons, which is a necessary condition for their stability, is investigated in an analytical form in Sec. III. The mathematical framework for the full analysis of the dark-soliton stability is introduced in Sec. IV, and numerical results, which can be summarized in the form of stability diagrams for the $\mathcal{PT}$-symmetric dark solitons, are reported in Section V. Collisions between dark solitons are briefly considered in Sec. V too. The paper is concluded by Sec. VI.
II. THE MODEL

We start with the system of equations for scaled field variables \( q_{1,2} \):

\[
\begin{align*}
\frac{i}{\gamma} \frac{\partial q_1}{\partial z} &= -\frac{\partial^2 q_1}{\partial x^2} + (\chi_1 |q_1|^2 + \chi_2 |q_2|^2) q_1 + i \gamma_1 q_1 - q_2, \\
\frac{i}{\gamma} \frac{\partial q_2}{\partial z} &= -\frac{\partial^2 q_2}{\partial x^2} + (\chi_1 |q_1|^2 + \chi_2 |q_2|^2) q_2 - i \gamma_2 q_2 - q_1.
\end{align*}
\] (1a)

Here the linear-coupling constant is scaled to be one, positive coefficients \( \gamma_1 \) and \( \gamma_2 \) account for the gain and loss, respectively, in the two cores, while \( \chi \) and \( \chi_1, \chi_2 \) are real coefficients of cross-phase modulation (XPM) and self-phase modulation (SPM).

Since the subject of the work is the existence and dynamics of dark solitons, it is first necessary to address the existence and modulational stability of the carrier-wave (CW) background, i.e., solutions in the form of

\[ q_{1,2}(x, z) = u_{1,2}(z) e^{i \delta z}, \] (2)

with complex amplitudes \( u_{1,2} \) and real propagation constant \( b \). The substitution of this into Eqs. (1) yields

\[ |u_j|^2 = \frac{|\gamma_1 - \gamma_2| \sqrt{1 - \gamma_1 \gamma_2}}{|\gamma_2 (\chi_1 - \chi) + \gamma_1 (\chi - \chi_2)|} \sqrt{\frac{\gamma_1}{\gamma_j}}, \quad j = 1, 2, \] (3)

while the relative phase, \( \delta \equiv \arg u_2 - \arg u_1 \), is determined by relation

\[ \tan \delta = \frac{2 \Theta(\gamma_1 - \gamma_2) - 1}{\sqrt{\gamma_1 \gamma_2}}, \] (4)

where \( \Theta(x) \) is the Heaviside’s step function. The propagation constant of this solution is

\[ b = \frac{\cos \delta}{\sqrt{\gamma_1 \gamma_2}} \frac{\gamma_1 \chi_2 - \gamma_2 \chi_1}{\sqrt{1 - \gamma_1 \gamma_2}}. \] (5)

Note that, according to Eq. (3), the CW amplitudes in the two components are related by \( |u_2|^2/|u_1|^2 = \gamma_1/\gamma_2 \), which implies the balance between the gain and loss in the CW state. Further, it follows from Eq. (3) that the background amplitudes have a singularity at \( \gamma_2/\gamma_1 = (\chi - \chi_2)/(\chi - \chi_1) \neq 1 \), and this solution exists only at \( 0 < \gamma_1 \gamma_2 < 1 \). This last condition has simple physical explanation: it requires the gain (dissipation) in an arm to be small enough for being compensated by the energy flow from the other arm with dissipation (gain), the flow being limited by the strength of linear coupling (responsible for the power transfer between the arms) which in our case is normalized to one.

In what follows we concentrate on the case of the \( \mathcal{PT} \)-symmetry, with \( \gamma_1 = \gamma_2 = \gamma \). Then, it follows from Eq. (3) that the nonzero CW background may exist only with symmetric SPM coefficients, \( \chi_1 = \chi_2 \), and for \( \gamma < 1 \), hence it is convenient to define \( \gamma \equiv \sin \delta \), with \( 0 \leq \delta \leq \pi \), and rewrite Eqs. (1) as

\[
\begin{align*}
\frac{\partial q_1}{\partial z} &= -\frac{\partial^2 q_1}{\partial x^2} + (\chi_1 |q_1|^2 + \chi_2 |q_2|^2) q_1 + i \sin(\delta) q_1 - q_2, \\
\frac{\partial q_2}{\partial z} &= -\frac{\partial^2 q_2}{\partial x^2} + (\chi_1 |q_1|^2 + \chi_2 |q_2|^2) q_2 - i \sin(\delta) q_2 - q_1.
\end{align*}
\] (6a)

In comparison with the model of the \( \mathcal{PT} \)-symmetric dual-core fiber, which was introduced in [5, 8], Eqs. (6) include the XPM terms, which implies a non-negligible overlap between transverse modes supported by the two cores. Recently, it was demonstrated that, in comparison with the well-known results for the SPM-nonlinear dual-core system with the purely linear coupling [12], the addition of the XPM terms essentially affects the symmetry-breaking transformations of bright solitons [13] and patterns in the form of domain walls [14] in the conservative nonlinear coupler, whose model amounts to Eqs. (6) with \( \delta = 0 \).

III. MODULATIONAL STABILITY OF THE CW BACKGROUND

CW solutions of Eqs. (6) with equal amplitudes follow from expressions [34]:

\[ q_j^{(0)} = \rho \exp \left[ i(-1)^j (\delta/2) - ibz \right], \quad b = \rho^2 (\chi_1 + \chi) - \cos \delta. \] (7)

Here \( j = 1, 2 \), and components have phase mismatch \( \delta \) imposed by the gain-loss coefficient.

To analyze the modulational stability of the CW (7), we use the standard ansatz with arbitrary real perturbation wavenumber \( k \), the corresponding eigenvalue, \( \beta \), and infinitesimal perturbation amplitudes, \( \eta_j, \nu_j \):

\[ q_j = \rho \left[ e^{i(-1)^j \delta/2} + \eta_j e^{-i(\beta z - kx)} + \nu_j e^{i(\beta z - kx)} \right] e^{-ibz}. \] (8)

Then, two branches \( \beta = \beta_{1,2}(k) \) of the linear excitations are readily found as

\[
\begin{align*}
\beta_1(k) &= \pm k \sqrt{2 \rho^2 (\chi_1 + \chi)}, \\
\beta_2(k) &= \pm \sqrt{k^2 + 2 \cos \delta \left[ k^2 + 2 \cos \delta + 2 \rho^2 (\chi_1 - \chi) \right]}.\end{align*}
\] (9)

From (9) it follows that, for the stability of the background, one has to require

\[ \chi_1 + \chi \geq 0, \] (11)

the constraint which is also necessary for the modulational stability of the CW background in the conservative system (\( \delta = 0 \)), and which is imposed in what follows. Equation (9) gives rise to two other conditions for the modulational stability,

\[ \cos \delta \geq 0, \quad \text{i.e.,} \quad 0 \leq \delta \leq \pi/2; \] (12)
\[(\chi_1 - \chi)\rho^2 + \cos \delta > 0.\]  

(13)

IV. STATIONARY DARK SOLITONS AND THEIR LINEAR STABILITY

Turning to the study of the dark-soliton solutions, we focus on the situation when both components have the same intensity profile, i.e.,

\[q_j(x, z) = u(x, z)e^{i(-1)^j\delta/2} \quad (j = 1, 2),\]

and thus reduce Eqs. (10) to the standard nonlinear Schrödinger equation,

\[i\frac{\partial u}{\partial z} = -\frac{\partial^2 u}{\partial x^2} + (\chi_1 + \chi)|u|^2 u - \cos(\delta)u,\]

(15)

whose dark soliton solution is commonly known\[15]\: 

\[u_s(x, z) = \frac{iv - w\tanh(w(x-vz)/2)}{\sqrt{2(\chi_1 + \chi)}} e^{-ibz}.\]

(16)

Here \(b\) is given by Eq. (14), and real parameters \(v\) and \(w\), which determine the “velocity” (in fact, the spatial tilt) and the depth of the soliton, are linked by relation \(w^2 + v^2 = 2(\chi_1 + \chi)\rho^2\).

Below we focus on the fundamental dark soliton with zero velocity \(v = 0\) (alias the black soliton), \(u_s(x, z) = u_0(x)e^{-ibz}\) where

\[u_0(x) = \rho\tanh\left(\rho\frac{\sqrt{\chi_1 + \chi}x}{2}\right).\]

(17)

To address its stability, we first notice that the CW background must be modulationally stable, hence the parameters to be considered are limited by constraints \([11 - 13]\). Further, to study the linear stability of the entire dark soliton \([14]\), we adopt the perturbation solution as

\[q_j(x, z) = [u_0(x) + u'_j(x, z) + ivu''_j(x, z)] e^{i(-1)^j\delta/2 - ibz},\]

(18)

with infinitesimal perturbation amplitudes \(u'_1(x, z)\) and \(u''_2(x, z)\). Then, substituting expressions \([15]\) into Eq. \((15)\), we end up with the eigenvalue problem:

\[\frac{\partial \mathbf{u}}{\partial z} = \mathbf{L}\mathbf{u}, \quad \mathbf{u} = \text{col}\{u'_1, u''_1, u'_2, u''_2\}\]

(19)

with operators

\[\mathbf{L} = \begin{pmatrix} \sin \delta & L_+ & -\sin \delta & -\cos \delta \\ -L_+ & \sin \delta & -L & -\sin \delta \\ \sin \delta & -\cos \delta & -\sin \delta & L_+ \\ -L & \sin \delta & -L & -\sin \delta \end{pmatrix},\]

(20)

\[L_{\pm} \equiv -\frac{\partial^2}{\partial x^2} - b + [(2 \pm 1)\chi_1 + \chi]\rho^2,\]

(21)

\[L \equiv 2\chi\rho^2 - \cos \delta.\]

(22)

Let us now prove that the stability analysis can be reduced to two separate problems,

\[L_j\psi = \lambda_j\psi \quad (j = 1, 2),\]

(23)
where the operators are

\begin{align}
L_1 &\equiv (L_+ - L)(L_+ + \cos \delta), \\
L_2 &\equiv (L_+ - \cos \delta)(L_+ + L)
\end{align}

(24a)

(24b)

such that \( \text{Im} \Lambda_{1,2} = 0 \) and \( \text{Re} \Lambda_{1,2} > 0 \) constitute necessary and sufficient conditions for the linear stability of the soliton.

To this end, we notice that SU(4) rotation

\[
P = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & -1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & -1 & 0 & -1 \\
-1 & 0 & 1 & 0
\end{pmatrix}
\]

provides for a unitary transformation, \( \mathcal{L}_0 = P \mathcal{L} P^{-1} \), with

\[
\mathcal{L}_0 = \begin{pmatrix}
0 & 0 & 0 & L - L_+ \\
2 \sin \delta & \cos \delta - \cos \delta - 2 \sin \delta \\
L_+ + L & 0 & 0 \\
L_+ + \cos \delta & 0 & 0 & 0
\end{pmatrix}
\]

(26)

Since the eigenvalues of \( \mathcal{L} \) and \( \mathcal{L}_0 \) coincide, we can consider the spectrum of the latter linear operator. Taking into account that both \( \mathcal{L} \) and \( \mathcal{L}_0 \) are built of real coefficients, solutions can be looked for in the form of \( u_1, u_2 \sim \exp(i \Lambda \lambda) \). Moreover, if \( \Lambda \) is an eigenvalue, then \( \Lambda \) is an eigenvalue as well (the overbar stands for the complex conjugate). In other words, the absence of an imaginary part of \( \Lambda \), which is equivalent to the condition that \( \Lambda = \lambda^2 \) is real and positive, is a necessary and sufficient condition for the absence of the instability.

As the next step, we consider the eigenvalue problem, \( -\mathcal{L}_0^2 \Psi = \Lambda \Psi \), where \( \Psi = Pu \), and we make use of the block structure of \( \mathcal{L}_0^2 \):

\[
-\mathcal{L}_0^2 = \begin{pmatrix}
L_1 & 0 & 0 & 0 \\
4 \sin \delta L_+ & L_2 & 0 & 0 \\
0 & 0 & L_2 & 4 \sin \delta L_+ \\
0 & 0 & 0 & L_1
\end{pmatrix}
\]

(27)

where \( L_{1,2} \) were introduced in Eqs. \((24)\). Now, a straightforward consideration demonstrates that \( \Lambda \) must coincide with either \( \Lambda_1 \) or \( \Lambda_2 \). Thus, the study of the stability of the dark solitons is reduced to eigenvalue problems \((23)\).

Now, we notice that

\[
L_+ \cos \delta = \frac{\partial^2}{\partial \chi^2} - (\chi_1 + \chi)(\rho^2 - u_0^2), \tag{28a}
\]

\[
L_+ + L = L_0 + 2(\chi_1 + \chi)u_0^2. \tag{28b}
\]

Therefore, taking into account Eq. \((11)\), we conclude that the eigenvalue problem for operator \( L_2 \) is nothing but the standard stability problem for the black soliton in the defocusing medium, with the effective nonlinearity \( \chi_1 + \chi \). This problem is very well studied \([16, 17]\). In particular, it is known that \( L_+ + L \) is positive definite and \( L_+ \cos \delta \) has only one negative eigenvalue and one zero eigenvalue \([16]\). Moreover, it is known too \([17]\) that the minimal eigenvalue of \( L_2 \) is positive. Thus, the eigenvalue problem for \( L_2 \) does not give instability, and our subsequent analysis is performed below for operator \( L_1 \), which may give rise to instability.

\section{Numerical Results}

The results of the numerical analysis of the linear stability are depicted in Fig. 2. For the defocusing XPM, in the subdomain \( -\chi_1 \leq \chi \leq \chi_1^* \) (where \( \chi_1^* \) is the critical value, denoted in Fig. 2(a) by the blue vertical line), the dark-soliton stability region coincides with that for the CW background, which is \( -\cos \delta \leq b \leq 2 \chi_1 (\cos \delta) / (\chi - \chi_1) \) for, equivalently, \( \rho^2 \leq (\cos \delta) (\chi - \chi_1) \) for \( \chi_1 < \chi \) and \( b \geq -\cos \delta \) for \( \chi \geq \chi_1 \) \( \leq \chi_1^* \), see Fig. 1(a). At the same time, in subdomain \( \chi_1 \geq \chi_1^* \) a dark-soliton’s instability “wedge” is present: as seen from Fig. 2(a), the dark soliton is stable when \( -\cos \delta \leq b \leq b_1 \) or \( b > b_2 \). The value of the propagation constant, \( b_1 \), at the lower edge of the “wedge” \([16]\) is almost independent of SPM coefficient \( \chi_1 \) (except for a small region in a vicinity of the critical value \( \chi_1^* \)) for the CW background, while the upper edge, \( b = b_2 \), has a quasi-linear function of \( \chi_1 \). With the increase of \( \gamma = \sin \delta \) the dark-soliton instability “wedge” gradually shrinks \([16]\), disappearing at \( \gamma = 1 \). For the focusing or zero XPM, with \( \chi = -1 \) or \( \chi = 0 \), the dark soliton is stable at \( -\cos \delta \leq b \leq b_1 \), and unstable at \( b > b_1 \), see Fig. 2(c). It is relevant to note that \( b_1 \) does not depend on \( \chi_1 \), and almost coincides with \( b_1 \) corresponding to the defocusing XPM \([16]\), while \( b_2 \) coincides with the vertical line, \( \chi_1 = -\chi_1^* \).

The linear stability analysis was completed by the direct simulations of Eqs. \((4)\). Typical examples of the perturbed evolution of stable and unstable dark solitons are presented in Fig. 3. The predicted stability of the dark
soliton below the lower edge of the instability “wedge”, i.e., at $b < b_1$, is confirmed by the simulations. For the dark-soliton parameters corresponding to point A in Figs. 2(a) and 2(b), the evolution of fields component $q_1(x,z)$ is shown in Fig. 3(a). Similarly, the evolution of the dark soliton with parameters corresponding to point B, as well as the stability of the dark soliton with the parameters corresponding to point C in Figs. 2(a) and 2(b), are demonstrated in Figs. 3(b) and 3(c), respectively.

For the focusing XPM, the stability and instability of the dark solitons [points $A'$ and $B'$ in Fig. 2(c)] is also confirmed by direct simulations, see Figs. 4(a) and 4(b), respectively.

For the symmetric system the two dark soliton always [for both the focusing – Fig. 4(a) and defocusing – Figs. 4(b), 4(c) signs of the XPM] repel each other and start motion in opposite directions without self-destruction. The repulsion from the boundaries of the $x$-domain in Fig. 5 happens due to the implied periodic boundary conditions, and is equivalent to the repulsion between the dark soliton. As can be seen from the comparison of Figs. 5(b) and 5(c), the increase of $b$ (while separation $\ell$ between the dark solitons is kept unchanged) results in reduction of the repulsion between the solitons, and, consequently, decrease of the solitons’ “velocities”. The reason for this
where use separate components in the initial condition, i.e., take to the soliton’s width. \[ \ell \]

phenomenon is that larger \( b \) corresponds to a smaller soliton width, and, as a result, a larger ratio of separation \( \ell \) to the soliton’s width.

Another possibility to set dark soliton in motion is to separate components in the initial condition, i.e., take

\[ q_j(x, 0) = u(x + (-1)^j D/2)e^{i(-1)^j \delta/2}, \tag{30} \]

where \( u(x) \) is borrowed from Eq. (17), and \( D \) is the initially imposed separation between the components. The results of the corresponding simulations are represented in Fig. 6 Here, for the defocusing XPM the “velocity” of the dark soliton does not strongly depend upon propagation constant \( b \) [cf. Figs. 30(b) and 6(c)]. The situation is completely different from the previous case [cf. Figs. 5(b) and 5(c)]. It should be noted that, in both Figs. 5 and 6 the simulations were run for dark solitons with the propagation constants far enough from the stability margins \( b_{1,2} \) [points A’, A and C in Fig. 2]. In the opposite situation, for the dark solitons with propagation constants close to stability margins \( b_{1,2} \), their motion may result in destruction, under certain conditions.

VI. CONCLUSIONS

To conclude we have reported the existence of stable vector solitons in the \( PT \)-symmetric coupled nonlinear Schrödinger equation one of which has gain and another dissipation, whose strengths are equal. The found solitons have identical amplitude profiles but the phase difference imposed by the gain-loss coefficients what ensures the balance between gain and losses. The stability of either backgrounds against which solitons propagate or of the solitons themselves are modified by dissipation and gain, what was confirmed by direct numerical simulations of the soliton propagation and interactions as well by the linear stability analysis.

Acknowledgements

VVK acknowledges support of Fundação para a Ciência e a Tecnologia (Portugal) under grants PTDC/FIS/112624/2009 and PEst-OE/FIS/UI0618/2011.

[1] C. M. Bender, Rep. Prog. Phys. 70, 947 (2007).
[2] A. Mostafazadeh, J. Phys. A: Math. Gen. 36, 7081 (2003); H. F. Jones, J. Phys. A: Math. Gen. 38, 1741 (2005).
[3] S. Longhi, Appl. Phys. B 104, 453 (2011).
[4] A. Ruschhaupt, F. Delgado and J. G. Muga, J. Phys. A: Math. Gen. 38, L171 (2005).
[5] R. El-Ganainy, K. G. Makris, D. N. Christodoulides, and Z. H. Musslimani, Opt. Lett. 32, 2632 (2007); Z. H. Musslimani, K. G. Makris, R. El-Ganainy, and D. N. Christodoulides, Phys. Rev. Lett. 100, 030402 (2008); M. V. Berry, J. Phys. A: Math. Theor. 41, 244007 (2008); S. Klaiman, U. Guenther, and N. Moiseyev, Phys. Rev. Lett. 101, 080402 (2008).
[6] C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Nature Phys. 6, 192 (2010).
[7] H. Ramezani, T. Kottos, R. El-Ganainy, and D. N. Christodoulides, Phys. Rev. A 82, 043803 (2010); A. A. Sukhorukov, Z. Xu, and Yu. S. Kivshar, Phys. Rev. A 82, 043818 (2010).
[8] R. Driben and B.A. Malomed, Opt. Lett. 36, 4323 (2011); R. Driben and B. A. Malomed, EPL 96, 51001 (2011).
[9] F. K. Abdullaev, V. V. Konotop, M. Ögren, and M. P. Sørensen, Opt. Lett. 36, 4566 (2011).
[10] N. V. Alexeeva, I. V. Barashenkov, A. A. Sukhorukov, and Y. S. Kivshar, Phys. Rev. A 85, 063837 (2012).
[11] D. A. Zezyulin and V. V. Konotop, Phys. Rev. A 85, 043840 (2012); V. Achilleos, P. G. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-González, Phys. Rev. A 86, 013808 (2012).
[12] E. M. Wright, G. I. Stegeman, and S. Wabnitz, Phys. Rev. A 40, 4455 (1989); C. Paré and M. Florjanczyk, Phys. Rev. A 41, 6287 (1990); A. I. Maimistov, Sov. J. Quant. Electr. 21, 687 (1991); P. L. Chu, B. A. Malomed, and G. D. Peng, J. Opt. Soc. Am. B 10, 1379 (1993); J. M. Soto-Crespo and N. Akhmediev, Phys. Rev. E 48, 4710 (1993).
[13] H. Sakaguchi and B. A. Malomed, Phys. Rev. E 83, 036608 (2011).
[14] N. Dror, B. A. Malomed, and J. Zeng, Phys. Rev. A 84, 046602 (2011).
[15] T. Tsuzuki, J. Low Temp. Phys. 4, 441 (1971)
[16] I. V. Barashenkov, Phys. Rev. Lett. 77, 1193 (1996)
[17] Y. Chen, Opt. Lett. 462 (1996)