COMPANION FORMS FOR UNITARY AND SYMPLECTIC GROUPS

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ABSTRACT. We prove a companion forms theorem for ordinary \( n \)-dimensional automorphic Galois representations, by use of automorphy lifting theorems developed by the second author, and a technique for deducing companion forms theorems due to the first author. We deduce results about the possible Serre weights of mod \( l \) Galois representations corresponding to automorphic representations on unitary groups. We then use functoriality to prove similar results for automorphic representations of \( \text{GSp}_4 \) over totally real fields.

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1. Introduction.

1.1. The problem of companion forms was first introduced by Serre for modular forms in his seminal paper [Ser87]. Fix a prime \( l \), algebraic closures \( \mathbb{Q} \) and \( \overline{\mathbb{Q}}_l \) of \( \mathbb{Q} \) and \( \mathbb{Q}_l \) respectively, and an embedding of \( \mathbb{Q} \) into \( \overline{\mathbb{Q}}_l \). Suppose that \( f \) is a modular newform of weight \( k \geq 2 \) which is ordinary at \( l \), so that the corresponding \( l \)-adic Galois representation \( \rho_{f,l} \) becomes reducible when restricted to a decomposition group \( G_{\mathbb{Q}_l} \) at \( l \). Then the companion forms problem is essentially the question of determining for which other weights \( k' \) there is an ordinary newform \( g \) of weight \( k' \geq 2 \) such that the Galois representations \( \rho_{f,l} \) and \( \rho_{g,l} \) are congruent modulo \( l \). The problem is straightforward unless the restriction to \( G_{\mathbb{Q}_l} \) of \( \overline{\rho}_{f,l} \) (the reduction mod \( l \) of \( \rho_{f,l} \)) is split and non-scalar, in which case there are two possible Hida families whose corresponding Galois representations lift \( \overline{\rho}_{f,l} \); the restrictions of the corresponding Galois representations to a decomposition group at \( l \) are either “upper-triangular” or “lower-triangular”.

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This problem was essentially resolved by Gross and Coleman-Voloch (\cite{Gro90}, \cite{CV92}). In the paper \cite{Gee07}, the first author reproved these results, and generalised them to Hilbert modular forms, by a completely new technique. In essence, rather than working directly with modular forms, the method is to firstly obtain a Galois representation which should correspond to a modular form in the sought-after Hida family, and then to use a modularity lifting theorem to prove that this Galois representation is modular. In \cite{Gee07} the Galois representation is obtained by using a generalisation of a lifting technique of Ramakrishna, which is proved by purely deformation theory techniques. The modularity is then obtained from the $R = T$ theorem of Kisin for Hilbert modular forms of parallel weight 2 (\cite{Kis07b}).

These techniques seem amenable to generalisation (to other reductive groups over more general number fields), subject to some important caveats. In particular, it is necessary to have modularity lifting theorems available over fields in which $l$ is highly ramified. The current technology for modularity lifting theorems requires one to work with reductive groups which admit discrete series, and to work over totally real or CM fields; so it is impossible at present to work directly with $GL_n$ for $n > 2$. Instead, one works with closely related groups, such as unitary or symplectic groups, which do admit discrete series.

In the present paper we make use of $R = T$ theorems for unitary groups to deduce companion forms theorems for unitary groups (in arbitrary dimension), and thus for conjugate self-dual automorphic representations of $GL_n$ over CM fields. We then deduce similar theorems for $GSp_4$ by developing the relevant deformation theory and employing known instances of functoriality. The analogue for unitary groups of the $R = T$ theorems of \cite{Kis07b} seem to be out of reach at present, and we use the main theorems of \cite{Ger09} instead. As explained below, this in fact allows us to prove stronger theorems than the natural analogue of \cite{Kis07b} would permit. We replace the use of Ramakrishna’s techniques in \cite{Gee07} with a method of Khare and Wintenberger, which allows weaker hypotheses on local deformation problems.

To our knowledge the only results on companion forms for groups other than $GL_2$ are those announced for $GSp_4$ over $\mathbb{Q}$ in \cite{HT08} (see also \cite{Til09}). Our results are rather stronger than those of \cite{HT08} in several respects. We are able to work with arbitrary totally real fields (with no restriction on ramification at $l$), rather than just over $\mathbb{Q}$, and we do not need any assumption that the residual Galois representation occurs at minimal level (indeed, one may deduce results on level lowering for $GSp_4$ from our theorem). In addition, the results of \cite{HT08} apply only in one special case, effectively one of 8 cases (corresponding to the 8 elements of the Weyl group of $GSp_4$) where one could hope to prove a companion forms theorem; this is in part due to the fact that their techniques only apply to Galois representations in the Fontaine-Laffaille range. In contrast, we make no such restrictions. We hope that these results will prove useful for generalisations of the Buzzard-Taylor method to $GSp_4$, as part of a program of Tilouine.

In recent years there has been a good deal of interest in generalisations of Serre’s conjecture (cf. \cite{ADP02}) and in particular in the question of determining the set of weights of a given Galois representation (cf. \cite{Her09}). One of us (T.G.) has formulated a conjecture to the effect that the set of weights should be determined completely by the existence of (local) crystalline lifts (cf. \cite{Gee06}). In general this seems to be a very difficult conjecture to prove, but our methods give a substantial partial result; essentially we prove the conjecture (subject to mild technical
hypotheses) for ordinary weights for unitary groups which are compact at infinity. See section 6 for the precise statements.

We now outline the structure of the paper. In section 3 we develop the basic deformation theory that we need. We then recall in section 4 the necessary material on ordinary automorphic representations on unitary groups and modularity lifting theorems for the corresponding Galois representations; in particular we recall the main theorem of [Ger09].

Section 5 contains our main theorems for unitary groups; the corresponding Galois representations are conjugate self-dual representations of the absolute Galois group of an imaginary CM field. Using the results of section 3 we give a lower bound for the dimension of a universal deformation ring, and the results of section 4 then permit us to prove that this universal deformation ring is finite over $\mathbb{Z}_l$, which implies that it has $\overline{\mathbb{Q}_l}$-points, which correspond to the Galois representations we seek. The automorphy of these Galois representations follows at once from the modularity lifting theorems recalled in section 4. The particular universal deformation ring we consider is one for representations of the absolute Galois group of a totally real field, valued in a group $G_n$ defined in [CHT08]. Representations valued in this group correspond to representations which are self-dual with respect to some pairing; this permits us to prove results for both the conjugate self-dual representations considered in section 5, and the symplectic representations studied in later sections.

We remark that the $\overline{\mathbb{Q}_l}$-points of universal deformation rings that we study in section 5 always correspond to ordinary crystalline representations of a certain weight. This is in contrast to the approach of [Gee07], which used potentially crystalline representations corresponding to Hilbert modular forms of parallel weight 2 and non-trivial level at $l$. The required automorphic representations were then obtained by specialising Hida families through these points at the sought-for weight. The difficulty with following this approach in general is that if the weight is not sufficiently regular a specialisation of a Hida family at this weight may fail to be an unramified principal series at places dividing $l$ (for example, a specialisation of a Hida family of modular forms in weight 2 can correspond to a Steinberg representation at $l$). It is for this reason that we use modularity lifting theorems for crystalline lifts instead.

In section 6 we deduce results about the possible Serre weights of mod $l$ Galois representations corresponding to automorphic representations of compact at infinity unitary groups. In particular, we deduce that the possible ordinary weights are determined by the existence of local crystalline lifts. We remark that these are the first results in anything approaching this level of generality for any groups other than $GL_2$.

Finally in section 7 we study the analogous questions for automorphic representations of $GSp_4$ over totally real fields. We use the known functoriality between globally generic cuspidal representations of $GSp_4$ and $GL_4$ to apply the methods of the earlier sections. In particular, we prove results analogous to those of section 5 for Galois representations valued in $GSp_4$, and obtain a lower bound for the dimension of a universal deformation ring as in section 5. We then prove that this universal deformation ring is finite over the corresponding one for unitary representations, which allows us to deduce that our symplectic universal deformation ring
is also finite over \( \mathbb{Z}_l \). Our main results for symplectic representations follow from this.

We remark that in all our main theorems we actually obtain somewhat more precise results; we are also able to control the ramification of our Galois representations at places not dividing \( l \), and we are able to choose our Galois representations so as to correspond to points on any particular set of irreducible components of the local deformation rings. Thus as a direct corollary of our results one obtains strong results on level lowering and level raising for ordinary automorphic Galois representations. Similarly, our method yields modularity lifting theorems for ordinary representations of \( \text{GSp}_4 \) which are rather stronger than those of [GT05]; for example, we do not need to assume any form of level-lowering for \( \text{GSp}_4 \), we work over general totally real fields, and we are not restricted to weights in the Fontaine-Laffaille range.

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2. Notation

If \( M \) is a field, we let \( G_M \) denote its absolute Galois group. Let \( \epsilon \) denote the \( l \)-adic or mod \( l \) cyclotomic character of \( G_M \). If \( M \) is a finite extension of \( \mathbb{Q}_p \) for some \( p \), we write \( I_M \) for the inertia subgroup of \( G_M \). We write all matrix transposes on the left; so \( tA \) is the transpose of \( A \). If \( R \) is a local ring we write \( m_R \) for the maximal ideal of \( R \). We let \( \mathbb{Z}_n^+ \) denote the subset of elements \( \lambda \in \mathbb{Z}_n \) with \( \lambda_1 \geq \ldots \geq \lambda_n \).

3. Galois deformations

3.1. Local deformation rings. Let \( l \) be a prime number and \( K \) a finite extension of \( \mathbb{Q}_l \) with residue field \( k \) and ring of integers \( \mathcal{O} \). Let \( M \) be a finite extension of \( \mathbb{Q}_p \) (with \( p \) possibly equal to \( l \)). Let \( \overline{\rho} : G_M \to \text{GL}_n(k) \) be a continuous representation. Let \( \mathcal{C}_\mathcal{O} \) be the category of complete local Noetherian \( \mathcal{O} \)-algebras with residue field \( k \). Then the functor from \( \mathcal{C}_\mathcal{O} \) to \( \text{Sets} \) which takes \( A \in \mathcal{C}_\mathcal{O} \) to the set of liftings of \( \overline{\rho} \) to a continuous homomorphism \( \rho : G_M \to \text{GL}_n(A) \) is represented by a complete local Noetherian \( \mathcal{O} \)-algebra \( R_{\overline{\rho}} \). We call this ring the universal \( \mathcal{O} \)-lifting ring of \( \rho \).

We write \( \overline{\rho} : G_M \to \text{GL}_n(\mathbb{C}) \) for the universal lifting. We will need to consider certain quotients of \( R_{\overline{\rho}} \).

3.1.1. The case where \( p \neq l \). Firstly, we consider the case \( p \neq l \). In this case, the quotients we wish to consider will correspond to particular inertial types. Recall that \( \tau \) is an inertial type for \( G_M \) over \( K \) if \( \tau \) is a \( K \)-representation of \( I_M \) with open kernel which extends to a representation of \( G_M \), and that we say that an \( l \)-adic representation of \( G_M \) has type \( \tau \) if the restriction of the corresponding Weil-Deligne representation to \( I_M \) is equivalent to \( \tau \). For any such \( \tau \) there is a unique reduced, \( l \)-torsion free quotient \( R_{\overline{\rho}}^{\square, \tau} \) of \( R_{\overline{\rho}} \) with the property that if \( E/K \) is a finite extension, then a map of \( \mathcal{O} \)-algebras \( R_{\overline{\rho}} \to E \) factors through \( R_{\overline{\rho}}^{\square, \tau} \) if and only if the corresponding \( E \)-representation has type \( \tau \). Furthermore, we have:

**Lemma 3.1.1.** For any \( \tau \), if \( R_{\overline{\rho}}^{\square, \tau} \neq 0 \) then \( R_{\overline{\rho}}^{\square, \tau}[1/l] \) is equidimensional of dimension \( n^2 \) and is generically formally smooth.

**Proof.** This is Theorem 2.0.6 of [Gee06].
Of course, $R^B_{\tau} \neq 0 \text{ if and only if } \overline{\rho}$ has a lift of type $\tau$.

3.1.2. The case where $p = l$. Now assume that $p = l$. In this case, we wish to consider crystalline ordinary deformations of fixed weight. We assume from now on that $K$ is large enough that any embedding $M \hookrightarrow \overline{K}$ has image contained in $K$.

**Notation.** Recall that $\mathbb{Z}_l^n$ is the set of non-increasing $n$-tuples of integers. We say that $\lambda \in (\mathbb{Z}_l^n)^{\text{Hom}(M,K)}$ is regular if for each $j = 1, \ldots, n-1$ there exists $\tau : M \hookrightarrow K$ with $\lambda_{\tau,j} > \lambda_{\tau,j+1}$.

Let $\epsilon$ be the $l$-adic cyclotomic character and let $\text{Art}_M : M^\times \to W^af_M$ be the Artin map (normalized to take uniformizers to lifts of geometric Frobenius).

**Definition 3.1.2.** Let $\lambda$ be an element of $(\mathbb{Z}_l^n)^{\text{Hom}(M,K)}$. We associate to $\lambda$ an $n$-tuple of characters $I_M \to O^\times$ as follows. For $j = 1, \ldots, n$ define

$$\chi^\lambda_j : I_M \to O^\times \quad \sigma \mapsto \epsilon(\sigma)^{-(j-1)} \prod_{\tau : M \hookrightarrow K} \tau(\text{Art}_M^{-1}(\sigma))^{-\lambda_{\tau,j} - j + 1}.$$ 

Note that $\chi^\lambda_j$ can also be thought of as the restriction to $I_M$ of any crystalline character $G_M \to \overline{\mathbb{Q}}_l^\times$ whose Hodge-Tate weight with respect to $\tau : M \hookrightarrow \overline{\mathbb{Q}}_l$ is given by $(j-1) + \lambda_{\tau,n-j+1}$ for all $\tau$ (we use the convention that the Hodge-Tate weights of $\epsilon$ are all $-1$).

Let $\lambda$ be an element of $(\mathbb{Z}_l^n)^{\text{Hom}(M,K)}$. We associate to $\lambda$ an $l$-adic Hodge type $v_\lambda$ in the sense of section 2.6 of [Kis08] as follows. Let $D_K$ denote the vector space $K^n$. Let $D_{K,M} = D_K \otimes_{\mathbb{Q}_l} M$. For each embedding $\tau : M \hookrightarrow K$, we let $D_{K,\tau} = D_{K,M} \otimes_{K \otimes M, \tau} K$ so that $D_{K,M} = \oplus_{\tau} D_{K,\tau}$. For each $\tau$ choose a decreasing filtration $\text{Fil}_i D_{K,\tau}$ of $D_{K,\tau}$ so that $\dim_K \text{gr}_i D_{K,\tau} = 0$ unless $i = (j-1) + \lambda_{\tau,n-j+1}$ for some $j = 1, \ldots, n$ in which case $\dim_K \text{gr}_i D_{K,\tau} = 1$. We define a decreasing filtration of $D_{K,M}$ by $K \otimes_{\mathbb{Q}_l} M$-submodules by setting

$$\text{Fil}_i D_{K,M} = \oplus_{\tau} \text{Fil}_i D_{K,\tau}.$$ 

Let $v_\lambda = \{D_{K,f} \otimes \text{Fil}_i D_{K,M}\}$.

Let $B$ denote a finite, $K$-algebra and $\rho_B : G_M \to \text{GL}_n(B)$ a crystalline representation. Then $D_B := D_{\text{cris}}(\rho_B) = (\rho_B \otimes_{\mathbb{Q}_l} B_{\text{cris}})^{G_M}$ is a free $B \otimes_{\mathbb{Q}_l} M_0$-module of rank $n$ where $M_0$ is the maximal subfield of $M$ which is unramified over $\mathbb{Q}_l$. Moreover, $D_B$ is equipped with a $B$-linear and $\varphi_0$-semilinear morphism $\varphi_B$ where $\varphi_0$ denotes geometric Frobenius on $M_0$. For each embedding $\tau : M_0 \hookrightarrow K$, let $D_{B,\tau} = D_B \otimes_{B \otimes M_0, \tau} B$. Then $D_{B,\tau} = \oplus_{\tau} D_{B,\tau}$. Also, for each $\tau$, $\varphi_B$ defines an isomorphism of $B$-modules $\varphi_B : D_{B,\tau} \overset{\sim}{\longrightarrow} D_{B,\tau \circ \varphi_0^{-1}}$. Let $f = [M_0 : \mathbb{Q}_l]$. Then $\varphi_B^f$ is a $B$-linear endomorphism of $D_B$ which preserves each $D_{B,\tau}$. For each $\tau$, the isomorphism $\varphi_B : D_{B,\tau} \overset{\sim}{\longrightarrow} D_{B,\tau \circ \varphi_0^{-1}}$ takes $\phi_B^f$ to $\phi_B^f$. Let $P_B(M) \in B[M]$ denote the characteristic polynomial of $\phi_B^f$ on $D_{B,\tau}$ for any choice of $\tau$.

Let $\mathcal{F}$ denote the flag variety over $\text{Spec} O$ whose set of $A$-points, for any $O$-algebra $A$, corresponds to filtrations $0 = \text{Fil}_0 \subset \text{Fil}_1 \subset \ldots \subset \text{Fil}_{n} = A^n$ of $A^n$ by locally free submodules which, locally, are direct summands and are such that $\text{Fil}_j$ has rank $j$.

**Definition 3.1.3.** Let $E$ be an algebraic extension of $K$ let $B$ be a finite local $E$-algebra. Let $\rho : G_M \to \text{GL}_n(B)$ be a continuous homomorphism. We say that $\rho$
is ordinary of weight $\lambda \in (\mathbb{Z}_n)^{\text{Hom}(M,K)}$ if $\rho$ is conjugate to a representation of the form

$$
\begin{pmatrix}
\psi_1 & * & \ldots & * & *
0 & \psi_2 & \ldots & * & *
\vdots & \vdots & \ddots & \vdots & \vdots
0 & 0 & \ldots & \psi_{n-1} & *
0 & 0 & \ldots & 0 & \psi_n
\end{pmatrix}
$$

where for each $j = 1, \ldots, n$ the character $\psi_j$ agrees on an open subgroup of $I_M$ with the character $\chi_j^n$ introduced above.

Equivalently, $\rho$ is ordinary of weight $\lambda$ if there is a full flag $0 = \text{Fil}_0 \subset \text{Fil}_1 \subset \ldots \subset \text{Fil}_n = B^n$ of $B^n$ which preserved by $G_M$ and such that the representation of $G_M$ on $\text{gr}_j = \text{Fil}_j / \text{Fil}_{j-1}$ is potentially semistable and for each embedding $\tau : M \to K$, the Hodge-Tate weight of $\text{gr}_j$ with respect to $\tau$ is $(j-1) + \lambda_{\tau,n-j+1}$.

**Lemma 3.1.4.** Suppose that $E$ is an algebraic extension of $K$ and $\rho : G_M \to \text{GL}_n(E)$ is ordinary of weight $\lambda$. Let $\psi_1, \ldots, \psi_n : G_M \to E^\times$ be as above. Then

1. $\rho$ is potentially semistable.
2. If each $\psi_j$ is crystalline (which occurs if and only if $\psi_j$ agrees with $\chi_j^n$ on all of $I_M$), then $\rho$ is semistable.
3. If each $\psi_j$ is crystalline and $\lambda$ is regular, then $\rho$ is crystalline.

**Proof.** Part 2 follows from Proposition 1.28(2) of [Nek93] and part 1 follows from part 2. Part 3 follows from Proposition 1.26 of [Nek93] and the formulae in Proposition 1.24 of [Nek93]. \qed

**Lemma 3.1.5.** Let $\psi_i : G_M \to E^\times$ be as above (with respect to some $\lambda \in (\mathbb{Z}_n)^{\text{Hom}(M,K)}$), with each $\psi_i$ crystalline. Suppose that $\overline{\tau} : G_M \to \text{GL}_n(k)$ is of the form

$$
\begin{pmatrix}
\overline{\tau}_1 & * & \ldots & * & *
0 & \overline{\tau}_2 & \ldots & * & *
\vdots & \vdots & \ddots & \vdots & \vdots
0 & 0 & \ldots & \overline{\tau}_{n-1} & *
0 & 0 & \ldots & 0 & \overline{\tau}_n
\end{pmatrix}
$$

where $\overline{\psi}_i = \overline{\tau}_j$ for each $1 \leq i \leq n$. Suppose that for each $i < j$ we have $\overline{\tau}_i \overline{\tau}_j^{-1} \neq \overline{\tau}$. Then $\overline{\tau}$ has a lift to a crystalline representation $\rho : G_M \to \text{GL}_n(E)$ of the form

$$
\begin{pmatrix}
\psi_1 & * & \ldots & * & *
0 & \psi_2 & \ldots & * & *
\vdots & \vdots & \ddots & \vdots & \vdots
0 & 0 & \ldots & \psi_{n-1} & *
0 & 0 & \ldots & 0 & \psi_n
\end{pmatrix}
$$

**Proof.** The fact that any upper-triangular representation of this form is crystalline follows easily as in the proof of Lemma 3.1.4 because the assumption that $\overline{\tau}_i \overline{\tau}_j^{-1} \neq \overline{\tau}$ implies that $\psi_i \psi_j^{-1} \neq \overline{\tau}$. The fact that such an upper-triangular lift exists follows from the fact that $H^2(G_M,u) = 0$, where $u$ is the subspace of the Lie algebra $\text{ad} \overline{\tau}$ consisting of strictly upper-triangular matrices. The vanishing of this cohomology group follows from Tate local duality and the existence of a filtration on $u$ whose
graded pieces are one-dimensional with $G_M$ acting via the characters $\mathcal{P}_j^{-1} \neq \epsilon$, $i < j$ (cf. Lemma 3.2.3 of [Ger09]).

We now recall some results of Kisin. Let $\lambda$ be an element of $(\mathbb{Z}_p^\times)^{\text{Hom}(M,K)}$ and let $\psi_\lambda$ be the associated $l$-adic Hodge type.

**Definition 3.1.6.** If $B$ is a finite $K$-algebra and $V_B$ is a free $B$-module of rank $n$ with a continuous action of $G_M$ that makes $V_B$ into a de Rham representation, then we say that $V_B$ is of $l$-adic Hodge type $\psi_\lambda$ if for each $i$ there is an isomorphism of $B \otimes_{\mathbb{Q}_l} M$-modules

$$\text{gr}^i(V_B \otimes_{\mathbb{Q}_l} B_{dR})^{G_M} \cong B \otimes_K (\text{gr}^i D_{K,M}).$$

For example, if $E$ is a finite extension of $K$ and $\rho : G_M \to \text{GL}_n(E)$ is ordinary of weight $\lambda$, then $\rho$ is of $l$-adic Hodge type $\psi_\lambda$.

Corollary 2.7.7 of [Kis08] implies that there is a unique $l$-torsion-free quotient $R^{\psi_\lambda,crt}_{\mathcal{P}}$ of $R^{\psi_\lambda,crt}_{\mathcal{P}}$ with the property that for any finite $K$-algebra $B$, a homomorphism of $\mathcal{O}$-algebras $\zeta : R^{\psi_\lambda,crt}_{\mathcal{P}} \to B$ factors through $R^{\psi_\lambda,crt}_{\mathcal{P}}$ if and only if $\zeta \circ \rho^{\underline{\square}}$ is crystalline of $l$-adic Hodge type $\psi_\lambda$. Moreover, Theorem 3.3.8 of [Kis08] implies that $\text{Spec } R^{\psi_\lambda,crt}_{\mathcal{P}}[1/l]$ is formally smooth over $K$ and equidimensional of dimension $n^2 + \frac{1}{2}n(n - 1)[M : \mathbb{Q}_l]$.

Let $\mathcal{F}$ be the flag variety over $\text{Spec } \mathcal{O}$ as introduced above and let $\mathcal{G}^{\psi_\lambda}$ be the closed subscheme of $\mathcal{F} \times_{\text{Spec } \mathcal{O}} \text{Spec } R^{\psi_\lambda,crt}_{\mathcal{P}}$ corresponding to filtrations $\text{Fil}^{\underline{n}}$ which (i) are preserved by the induced action of $G_M$ and (ii) are such that $I_M$ acts on $\text{gr}_j = \text{Fil}_j / \text{Fil}_j^{-1}$ via the character $\chi_j^\lambda$ for each $j = 1, \ldots, n$. The fact that $\mathcal{G}^{\psi_\lambda}$ is a closed subscheme can proved in the same way as Lemma 3.1.2 of [Ger09]. Let $R^{\lambda,crt}_{\mathcal{P}}$ be the image of $R^{\psi_\lambda,crt}_{\mathcal{P}} \to \mathcal{O}_{\mathcal{G}^{\lambda}}(\mathcal{G}^{\lambda}[1/l])$.

In other words, $\text{Spec } R^{\lambda,crt}_{\mathcal{P}}$ is the scheme theoretic image of the morphism $\mathcal{G}^{\lambda}[1/l] \to \text{Spec } R^{\psi_\lambda,crt}_{\mathcal{P}}$. The next result follows from Lemma 3.3.3 of [Ger09].

**Lemma 3.1.7.** For any finite local $K$-algebra $B$, a homomorphism of $\mathcal{O}$-algebras $\zeta : R^{\psi_\lambda,crt}_{\mathcal{P}} \to B$ factors through $R^{\psi_\lambda,crt}_{\mathcal{P}}$ if and only if $\zeta \circ \rho^{\underline{\square}}$ is ordinary of weight $\lambda$. Moreover, Spec $R^{\lambda,crt}_{\mathcal{P}}$ is a union of irreducible components of $\text{Spec } R^{\psi_\lambda,crt}_{\mathcal{P}}$.

3.1.3. The $p = l$ case with a slight refinement. We continue to consider, as above, crystalline lifts of $\mathcal{P}$ which are ordinary of a given weight $\lambda$. A necessary condition for such lifts to exist is that $\mathcal{P}$ itself is conjugate to an upper triangular representation whose ordered $n$-tuple of diagonal characters, restricted to $I_M$, is given by $(\chi_1^\lambda, \ldots, \chi_n^\lambda)$. Let us assume that $\mathcal{P}$ has this property. In fact, let us fix characters $\mathcal{P}_1, \ldots, \mathcal{P}_n : G_M \to k^\times$ with $\mathcal{P}_j|I_M = \chi_j^\lambda$ and assume that $\mathcal{P}$ is conjugate to an upper triangular representation whose ordered $n$-tuple of diagonal characters is $\mathcal{P} := (\mathcal{P}_1, \ldots, \mathcal{P}_n)$. Note that if the characters $\chi_j^\lambda$ are not distinct, then we may have more than one choice for the ordered $n$-tuple $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$. We now would like to study crystalline lifts of $\mathcal{P}$ which are ordinary of weight $\lambda$ and are such that for each $j$, the character $\psi_j$ of Definition 3.1.3 lifts $\mathcal{P}_j$.

Let $R_{\mathcal{P}}$ denote the object of $\text{CO}_\mathcal{O}$ representing the functor which sends an object $A$ of $\text{CO}_\mathcal{O}$ to the set of lifts $(\psi_1, \ldots, \psi_n)$ of the ordered $n$-tuple $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ with $\psi_j|I_M = \chi_j^\lambda$ for each $j$. The ring $R_{\mathcal{P}}$ is non-canonically isomorphic to $\mathcal{O}[X_1, \ldots, X_n]$. Let $(\psi_1^{\text{univ}}, \ldots, \psi_n^{\text{univ}})$ be the universal lift of the tuple $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ to $R_{\mathcal{P}}$. Let $G_{\mathcal{P}}^{\lambda}$ denote
the closed subscheme of the flag variety $\mathcal{F} \times_{\text{Spec} \mathcal{O}} \text{Spec}(R_{\tilde{\mathfrak{p}}}^{\Delta,cr} \otimes_{\mathcal{O}} R_{\tilde{\mathfrak{p}}})$ corresponding to filtrations which are (i) preserved by the induced action of $G_M$ and (ii) such that $G_M$ acts on $\text{gr}_j$ via the pushforward of $\psi^\text{univ}_j$ for each $j = 1, \ldots, n$. Let $R_{\tilde{\mathfrak{p}},\mathfrak{p}}^{\Delta,cr}$ be the quotient of $R_{\tilde{\mathfrak{p}}}^{\Delta,cr} \otimes_{\mathcal{O}} R_{\tilde{\mathfrak{p}}}$ corresponding to the scheme theoretic image of $G_{\Delta,cr}[1/l]$. Note that we have a natural morphism $G_{\Delta,cr}[1/l] \to G^{\Delta,cr}[1/l]$ covering the morphism $\text{Spec} R_{\tilde{\mathfrak{p}},\mathfrak{p}}^{\Delta,cr} \to \text{Spec} R_{\tilde{\mathfrak{p}}}^{\Delta,cr}$.

**Lemma 3.1.8.** After inverting $l$, the morphism $\text{Spec} R_{\tilde{\mathfrak{p}},\mathfrak{p}}^{\Delta,cr} \to \text{Spec} R_{\tilde{\mathfrak{p}}}^{\Delta,cr}$ becomes a closed immersion and identifies $\text{Spec} R_{\tilde{\mathfrak{p}},\mathfrak{p}}^{\Delta,cr}[1/l]$ with a union of irreducible components of $\text{Spec} R_{\tilde{\mathfrak{p}}}^{\Delta,cr}[1/l]$. Moreover, every irreducible component of $\text{Spec} R_{\tilde{\mathfrak{p}}}^{\Delta,cr}$ arises in this way.

**Proof.** Let $X^{\text{ord,cr}} = \text{Spec} R_{\tilde{\mathfrak{p}}}^{\Delta,cr}$ and let $X_{\mathfrak{p}}^{\text{ord,cr}} = \text{Spec} R_{\mathfrak{p},\mathfrak{p}}^{\Delta,cr}$. Let $x$ be a closed point of $X_{\mathfrak{p}}^{\text{ord,cr}}[1/l]$ with residue field $E$. Let $z$ denote the image of $x$ in $X^{\text{ord,cr}}$. We claim that the natural map on completed local rings $\mathcal{O}_{X^{\text{ord,cr}},z} \to \mathcal{O}_{X_{\mathfrak{p}}^{\text{ord,cr}},x}$ is an isomorphism.

With this in mind, let $B$ denote a finite, local $E$-algebra and $\zeta: \mathcal{O}_{X_{\mathfrak{p}}^{\text{ord,cr}},x} \to B$ an $E$-algebra homomorphism. It follows from Lemma 3.1.4 that $\zeta$ corresponds to a crystalline representation $\rho_B: G_M \to \text{GL}_n(B)$ which preserves a full flag $0 \subset \text{Fil}_1 \subset \ldots \subset \text{Fil}_n = B^\times$ with $I_M$ acting on $\text{gr}_j$ via $\chi^\Delta_j$. Since the characters $\chi^\Delta_j$ are pairwise distinct, there is a unique such flag. Moreover, there exists a subring $A \subset B$ which is local and finite over $\mathcal{O}_E$ and such that the action of $G_M$ on $\text{gr}_j$ is given by a character $\psi_j: G_M \to A^\times$ which factors through $A^\times$ with $\psi_j \mod m_A = \tilde{\mathfrak{p}}_j \otimes_{\mathbb{F}_k} A/m_A$. We see that there is a unique lifting of $\zeta^\ast: \text{Spec} B \to \text{Spec} R_{\tilde{\mathfrak{p}}}^{\Delta,cr}$ to $\mathcal{O}_{X_{\mathfrak{p}}^{\text{ord,cr}},x}$ and that $\rho_B$ is formally smooth of relative dimension 0. It’s easy to see that both sides have the same residue field and hence the map is an isomorphism.

For each closed point in $X_{\mathfrak{p}}^{\text{ord,cr}}[1/l]$, there is at most 1 closed point of $X_{\mathfrak{p}}^{\text{ord,cr}}[1/l]$ lying over it. From this, and the claim just established, we deduce the first two statements. The final statement is clear. □

3.1.4. The $p = l$ case in non-fixed weight. In this section we assume that $p = l$, that $\tilde{\mathfrak{p}}: G_M \to \text{GL}_n(k)$ is the trivial homomorphism. Let $R_{\tilde{\mathfrak{p}}}^{\square}$ denote the universal $\mathcal{O}$-lifting ring of $\tilde{\mathfrak{p}}$ and let $\Lambda_\mathfrak{M} = \mathcal{O}[[I_M,\text{univ}]]$ where for a group $H$, $H(l)$ denotes its pro-$l$ completion. Then $\Lambda_\mathfrak{M}$ represents the functor $\mathcal{C}_\mathfrak{M} \to \text{Sets}$ sending an algebra $\mathcal{A}$ to the set of ordered $n$-tuples $(\chi_1, \ldots, \chi_n)$ of characters $\chi_j: I_{M,\text{univ}} \to A^\times$ lifting the trivial character modulo $m_\mathfrak{A}$. Let $\rho^{\square}$ denote the universal lift of $\tilde{\mathfrak{p}}$ to $R_{\tilde{\mathfrak{p}}}^{\square}$ and let $(\chi_1^{\text{univ}}, \ldots, \chi_n^{\text{univ}})$ denote the universal $n$-tuple of characters $I_{M,\text{univ}} \to \Lambda_\mathfrak{M}^\times$ lifting the trivial character modulo $m_\mathfrak{M}$.

Let $R_{\tilde{\mathfrak{p}},\Lambda_\mathfrak{M}}^{\square} = R_{\tilde{\mathfrak{p}}}^{\square} \otimes_{\mathcal{O}} \Lambda_\mathfrak{M}$. Let $\mathcal{G}$ denote the closed subscheme of the flag variety $\mathcal{F} \times_{\text{Spec} \mathcal{O}} \text{Spec} R_{\tilde{\mathfrak{p}},\Lambda_\mathfrak{M}}^{\square}$ corresponding to filtrations which are (i) preserved by the induced action of $G_M$ and (ii) such that $I_M$ acts on $\text{gr}_j$ via the pushforward of $\chi_j^{\text{univ}}$. Let $R_{\tilde{\mathfrak{p}},\Lambda_\mathfrak{M}}^{\Delta}$ be the quotient of $R_{\tilde{\mathfrak{p}},\Lambda_\mathfrak{M}}^{\square}$ corresponding to the scheme theoretic image of the morphism $\mathcal{G}[1/l] \to \text{Spec} R_{\tilde{\mathfrak{p}},\Lambda_\mathfrak{M}}^{\square}$.
If $E$ is a finite extension of $K$, a homomorphism of $O$-algebras $\zeta : R_{\mathfrak{m}^n,F}^\triangle \to E$ factors through $R_{\mathfrak{m}^n,F}^\triangle$ if and only if $\zeta \circ \rho^\square$ is conjugate to an upper triangular representation whose ordered $n$-tuple of diagonal characters, restricted to $I_M$, is the pushforward of $(\chi_1^{\text{univ}}, \ldots, \chi_n^{\text{univ}})$.

3.2. Global deformation rings.

3.2.1. The group $G_n$. Let $n$ be a positive integer, and let $G_n$ be the group scheme over $\mathbb{Z}$ which is the semidirect product of $\text{GL}_n \times \text{GL}_1$ by the group $\{1,j\}$, which acts on $\text{GL}_n \times \text{GL}_1$ by

$$j(g, \mu)j^{-1} = (\mu^tg^{-1}, \mu).$$

There is a homomorphism $\nu : G_n \to \text{GL}_1$ sending $(g, \mu)$ to $\mu$ and $j$ to $-1$. Write $\mathfrak{g}_n^0$ for the trace zero subspace of the Lie algebra of $\text{GL}_n$, regarded as a Lie subalgebra of the Lie algebra of $G_n$.

Definition 3.2.1. Let $F^+$ be a totally real field, and let $r : G_{F^+} \to G_n(L)$ be a continuous homomorphism, where $L$ is a topological field. Then we say that $r$ is odd if for all complex conjugations $c_v \in G_{F^+}$, $\nu \circ r(c_v) = -1$.

3.2.2. Bigness. Recall definition 2.5.1 of [CHT08].

Definition 3.2.2. Let $k$ be an algebraic extension of the finite field $F_l$. We say that a finite subgroup $H \subset \text{GL}_n(k)$ is big if the following conditions are satisfied.

- $H$ has no quotient of $l$-power order.
- $H^0(\mathfrak{g}_n^0(k)) = (0)$.
- $H^1(\mathfrak{g}_n^0(k)) = (0)$.
- For all irreducible $k[H]$-submodules $W$ of $\mathfrak{g}_n^0(k)$ we can find $h \in H$ and $\alpha \in k$ such that the $\alpha$-generalised eigenspace $V_{h,\alpha}$ of $h$ in $k^n$ is one-dimensional and furthermore $\pi_{h,\alpha} \circ W \circ i_{h,\alpha} \neq (0)$. Here $\pi_{h,\alpha} : k^n \to V_{h,\alpha}$ is the $h$-equivariant projection of $k^n$ to $V_{h,\alpha}$, and $i_{h,\alpha}$ is the $h$-equivariant injection of $V_{h,\alpha}$ into $k^n$.

We call a finite subgroup $H \subset G_n(k)$ big if $H$ surjects onto $G_n(k)/G_n^0(k)$ and $H \cap G_n^0(k)$ is big.

3.2.3. Deformation problems. Let $F/F^+$ be a totally imaginary quadratic extension of a totally real field $F^+$. Let $c$ denote the non-trivial element of $\text{Gal}(F/F^+)$. Let $k$ denote a finite field of characteristic $l$ and $K$ a finite extension of $\mathbb{Q}_l$, inside a fixed algebraic closure $\overline{\mathbb{Q}}_l$, with ring of integers $O$ and residue field $k$. Assume that $K$ contains the image of every embedding $F \to \overline{\mathbb{Q}}_l$ and that the prime $l$ is odd. Assume that every place in $F^+$ dividing $l$ splits in $F$. Let $S$ denote a finite set of finite places of $F^+$ which split in $F$, and assume that $S$ contains every place dividing $l$. Let $S_l$ denote the set of places of $F^+$ lying over $l$. Let $F(S)$ denote the maximal extension of $F$ unramified away from $S$. Let $G_{F^+} = \text{Gal}(F(S)/F^+)$ and $G_{F,S} = \text{Gal}(F(S)/F)$. For each $v \in S$ choose a place $\overline{v}$ of $F$ lying over $v$ and let $\overline{S}$ denote the set of $\overline{v}$ for $v \in S$. For each place $v|\infty$ of $F^+$ we let $c_v$ denote a choice of a complex conjugation at $v$ in $G_{F^+,S}$. For each place $w$ of $F$ we have a $G_{F,S}$-conjugacy class of homomorphisms $G_{F,w} \to G_{F,S}$. For $v \in S$ we fix a choice of homomorphism $G_{F,\overline{v}} \to G_{F,S}$.
If $R$ is a ring and $r : G_{F^+,S} \to \mathcal{G}_n(R)$ is a homomorphism with $r^{-1}(\text{GL}_n(R) \times \text{GL}_1(R)) = G_{F,S}$, we will make a slight abuse of notation and write $r|_{G_{F,S}}$ (respectively $r|_{G_{F,w}}$ for $w$ a place of $F$) to mean $r|_{G_{F}}$ (respectively $r|_{G_{F_w}}$) composed with the projection $\text{GL}_n(R) \times \text{GL}_1(R) \to \text{GL}_n(R)$.

Fix a continuous homomorphism

$$\bar{r} : G_{F^+} \to \mathcal{G}_n(k)$$

such that $G_{F,S} = \bar{r}^{-1}(\text{GL}_n(k) \times \text{GL}_1(k))$ and fix a continuous character $\chi : G_{F^+} \to \mathcal{O}_F^\times$ such that $\nu \circ \bar{r} = \chi$. Assume that $\bar{r}|_{G_{F,S}}$ is absolutely irreducible. As in Definition 1.2.1 of [CHT08], we define

- a lifting of $\bar{r}$ to an object $A$ of $\mathcal{C}_\mathcal{O}$ to be a continuous homomorphism $r : G_{F^+} \to \mathcal{G}_n(A)$ lifting $\bar{r}$ and with $\nu \circ r = \chi$;
- two liftings $r, r'$ of $\bar{r}$ to $A$ to be equivalent if they are conjugate by an element of $\ker(\text{GL}_n(A) \to \text{GL}_n(k))$;
- a deformation of $\bar{r}$ to an object $A$ of $\mathcal{C}_\mathcal{O}$ to be an equivalence class of liftings.

For each place $v \in S$, let $R^\square_{\bar{r}|_{G_{F_v}}}$ denote the universal $\mathcal{O}$-lifting ring of $\bar{r}|_{G_{F_v}}$ and let $R^\square_v$ denote a quotient of $R^\square_{\bar{r}|_{G_{F_v}}}$ which satisfies the following property:

(*) let $A$ be an object of $\mathcal{C}_\mathcal{O}$ and let $\zeta, \zeta' : R^\square_{\bar{r}|_{G_{F_v}}} \to A$ be homomorphisms corresponding to two lifts $r$ and $r'$ of $\bar{r}|_{G_{F_v}}$ which are conjugate by an element of $\ker(\text{GL}_n(A) \to \text{GL}_n(k))$. Then $\zeta$ factors through $R^\square_v$ if and only if $\zeta'$ does.

We consider the deformation problem

$$\mathcal{S} = (F/F^+, S, \tilde{S}, \mathcal{O}, \bar{r}, \chi, \{R^\square_v\}_{v \in S})$$

(see sections 2.2 and 2.3 of [CHT08] for this terminology). We say that a lifting $r : G_{F^+} \to \mathcal{G}_n(A)$ is of type $\mathcal{S}$ if for each place $v \in S$, the homomorphism $R^\square_{\bar{r}|_{G_{F_v}}} \to A$ corresponding to $r|_{G_{F_v}}$ factors through $R^\square_v$.

Let $\text{Def}_\mathcal{S}$ be the functor $\mathcal{C}_\mathcal{O} \to \text{Sets}$ which sends an algebra $A$ to the set of deformations of $\bar{r}$ to $A$ of type $\mathcal{S}$. By Proposition 2.2.9 of [CHT08] this functor is represented by an object $R^{\text{cr}}_\mathcal{S} \subseteq \mathcal{O}$.

**Lemma 3.2.3.** Let $M$ be a finite extension of $\mathbb{Q}_p$ for some prime $p$ and $\overline{\mathfrak{p}} : G_M \to \text{GL}_n(k)$ a continuous homomorphism. If $p \neq l$, let $r$ be an inertial type for $G_M$ over $K$ and let $R$ be a quotient of $R^{\text{cr}}_{\mathfrak{p}}$ corresponding to a union of irreducible components. If $p = l$, assume that $K$ contains the image of every embedding $M \hookrightarrow \overline{K}$, let $\lambda \in (\mathbb{Z}_p)^{\text{Hom}(M,K)}$ and let $R$ be a quotient of $R^{\lambda,\text{cr}}_{\mathfrak{p}}$ corresponding to a union of irreducible components. Then $R$ satisfies property (*) above.

**Proof.** We consider the case $p = l$; the other case is similar. Let $R^{\lambda,\text{cr}}_{\mathfrak{p}}[[X]] = R^{\lambda,\text{cr}}_{\mathfrak{p}}[[X_{ij} : 1 \leq i, j \leq n]]$ and consider the lift of $\overline{\mathfrak{p}}$ to $R^{\lambda,\text{cr}}_{\mathfrak{p}}[[X]]$ given by $(1_n + (X_{ij}))^\mathfrak{p}(1_n + (X_{ij}))^{-1}$. This lift gives rise to an $\mathcal{O}$-algebra homomorphism $R^{\lambda,\text{cr}}_{\mathfrak{p}} \to R^{\lambda,\text{cr}}_{\mathfrak{p}}[[X]]$. We claim that this homomorphism factors through $R^{\lambda,\text{cr}}_{\mathfrak{p}}$. This follows from the fact that $R^{\lambda,\text{cr}}_{\mathfrak{p}}[[X]]$ is reduced and $l$-torsion-free and every $\mathfrak{p}_{ij}$ point of this ring gives rise to a lift of $\mathfrak{p}$ which is crystalline of $l$-adic Hodge type $\mathfrak{v}_\lambda$. Let $\alpha$ denote the resulting $\mathcal{O}$-algebra homomorphism $R^{\lambda,\text{cr}}_{\mathfrak{p}} \to R^{\lambda,\text{cr}}_{\mathfrak{p}}[[X]]$. 


and let \( \iota : R^\lambda_{\mathcal{S},ct} \to R^\lambda_{\mathcal{S},ct}[X] \) denote the standard \( R^\lambda_{\mathcal{S},ct} \)-algebra structure on \( R^\lambda_{\mathcal{S},ct}[X] \).

The irreducible components of \( \text{Spec } R^\lambda_{\mathcal{S},ct}[X] \) and \( \text{Spec } R^\lambda_{\mathcal{S},ct} \) are in natural bijection (if \( \varphi \) is a minimal prime of \( R^\lambda_{\mathcal{S},ct} \), then \( \iota(\varphi) \) generates a minimal prime of \( R^\lambda_{\mathcal{S},ct}[X] \)). Let \( \varphi \) be a minimal prime of \( R^\lambda_{\mathcal{S},ct} \). We claim that the kernel of the map \( \beta : R^\lambda_{\mathcal{S},ct} \to R^\lambda_{\mathcal{S},ct}[X]/\iota(\varphi) = (R^\lambda_{\mathcal{S},ct}/\varphi)[X] \) induced by \( \alpha = \varphi \). To see this note that the map \( \gamma : R^\lambda_{\mathcal{S},ct}[X] \to R^\lambda_{\mathcal{S},ct}[X]/(X_{ij}) \equiv R^\lambda_{\mathcal{S},ct} \) satisfies \( \gamma \circ \alpha = \text{id} \). From this it follows that \( \ker \beta \subseteq \varphi \). Since \( \varphi \) is minimal, we must have \( \ker \beta = \varphi \). If \( \varphi_1, \ldots, \varphi_k \) are minimal primes of \( R^\lambda_{\mathcal{S},ct} \) and \( I = \varphi_1 \cap \cdots \cap \varphi_k \), we deduce that the kernel of the map \( R^\lambda_{\mathcal{S},ct} \to (R^\lambda_{\mathcal{S},ct}/I)[X] \) is \( I \). The lemma follows.

3.2.4. A lower bound. Let \( F, F^+, S, \tilde{S} \) and \( \tilde{r} \) be as in the previous section. In this section we will give a lower bound on the Krull dimension of the ring \( R^\mathbb{S}_{\text{inv}} \) for certain deformation problems \( S \).

For each place \( v \in S \) away from \( l \), fix an inertial type \( \tau_v \) for \( I_{F_v} \) and assume that \( \tilde{r}|_{G_{F_v}} \) has a lift of type \( \tau_v \) (in other words, \( R^\mathbb{S}_{\tilde{r}|_{G_{F_v}}} \) is non-zero). Let \( R_v \) be a quotient of \( R^\mathbb{S}_{\tilde{r}|_{G_{F_v}}} \) corresponding to a union of irreducible components.

For each place \( v \in S \) lying above \( l \), let \( \lambda_v \) be an element of \( (\mathbb{Z}^*_v)^{\text{Hom}(F_v, K)} \), and assume that \( \tilde{r}|_{G_{F_v}} \) has a crystalline lift which is ordinary of weight \( \lambda_v \) and let \( R_v \) be a quotient of the ring \( R^\mathbb{S}_{\tilde{r}|_{G_{F_v}}} \) corresponding to a union of irreducible components. Let

\[
S = (F/F^+, S, \tilde{S}, O, \tilde{r}, \chi, \{ R_v \}_{v \in S}).
\]

Lemma 3.2.4. Assume that \( \tilde{r} \) is odd, and that \( H^0(G_{F^+, S}, \text{ad } \tilde{r}(1)) = \{0\} \). For \( S \) as above, the Krull dimension of \( R^\mathbb{S}_{\text{inv}} \) is at least 1.

Proof. By Corollary 2.3.5 of [CHT08] (noting that \( \chi(c_v) = -1 \) for all \( v|\infty \)) we see that this dimension is at least

\[
1 + \sum_{v \in S} (\dim R_v - n^2 - 1) - \dim_k H^0(G_{F^+, S}, \text{ad } \tilde{r}(1)) - \sum_{v|\infty} n(n-1)/2.
\]

For \( v \in S \) away from \( l \), we have \( \dim R_v = n^2 + 1 \) by Lemma 3.1.1. For \( v \in S \) lying over \( l \) we have \( \dim R_v = n^2 + 1 + \frac{1}{2}n(n-1)|F_v^+ : \mathbb{Q}_l| \) by Lemma 3.1.7 and the remark preceding it. We therefore have

\[
\sum_{v \in S} (\dim R_v - n^2 - 1) = \sum_{v|l} \frac{1}{2}n(n-1)|F_v^+ : \mathbb{Q}_l| = \frac{1}{2}n(n-1)|F^+ : \mathbb{Q}| = \sum_{v|\infty} n(n-1)/2,
\]

giving the required bound.

3.2.5. A finiteness result. Let \( F, F^+, S, \tilde{S} \) and \( \tilde{r} \) be as in the previous two sections. Suppose that \( L^+/F^+ \) is a finite totally real extension. Let \( L = L^+F \). Let \( S' \) (resp. \( \tilde{S}' \)) denote a set of places of \( L^+ \) (resp. \( L \)) all of which split in \( L \), containing all places lying over a place in \( S \) (resp. containing exactly one place above each
place in \( S' \), and containing every place lying above a place in \( \tilde{S} \). Let \( G_{L+,S'} = \text{Gal}(L(S')/L^+) \), where \( L(S') \) is the maximal extension of \( L \) unramified outside \( S' \). Let \( G_{L,S'} = \text{Gal}(L(S')/L) \). We assume that \( \overline{r}|_{G_{L,S'}} \) is absolutely irreducible.

Let

\[
\mathcal{S}_0 = (F/F^+, S, \tilde{S}, \mathcal{O}, \overline{r}, \chi, \{ R_{\overline{r}|_{G_{F,v}}} \}_{v \in S})
\]

and

\[
\mathcal{S}'_0 = (L/L^+, S', \tilde{S}', \mathcal{O}, \overline{r}|_{G_{L^+,S'}}, \chi|_{G_{L^+,S'}}, \{ R_{\overline{r}|_{G_{L,v'}}} \}_{v' \in S'})
\]

and let \( R^{\text{univ}}_{\mathcal{S}_0} \) and \( R^{\text{univ}}_{\mathcal{S}'_0} \) denote the rings representing the functors \( \text{Def}_{\mathcal{S}_0} \) and \( \text{Def}_{\mathcal{S}'_0} \).

Restricting the universal deformation valued in \( R^{\text{univ}}_{\mathcal{S}_0} \) to \( G_{L^+,S'} \) gives \( R^{\text{univ}}_{\mathcal{S}'_0} \) the structure of a \( R^{\text{univ}} \)-algebra.

**Lemma 3.2.5.** \( R^{\text{univ}}_{\mathcal{S}'_0} \) is a finite \( R^{\text{univ}}_{\mathcal{S}_0} \)-algebra.

**Proof.** The argument is extremely similar to that of Lemma 3.6 of [KW08]. Write \( m_{L^+} \) for the maximal ideal of \( R^{\text{univ}}_{\mathcal{S}_0} \), and let \( r_{F^+,L^+} \) denote the \( R^{\text{univ}}_{\mathcal{S}_0}/m_{L^+} + R^{\text{univ}}_{\mathcal{S}_0} \)-representation of \( G_{F^+,S} \) obtained from the universal representation over \( R^{\text{univ}}_{\mathcal{S}_0} \). By definition, \( r_{F^+,L^+}|_{G_{L^+,S'}} \) is equivalent to \( \overline{r}|_{G_{L^+,S'}} \). As a consequence, if \( M \) denotes the normal closure of the composite of \( L^+ \) and the fixed field of \( \ker \overline{r} \), then \( r_{F^+,L^+} \) factors through \( \text{Gal}(M/F^+) \), and the image of \( r_{F^+,L^+} \) is necessarily finite.

Now, by Lemma 2.1.12 of [CHT08], we see that \( R^{\text{univ}}_{\mathcal{S}'_0}/m_{L^+} + R^{\text{univ}}_{\mathcal{S}_0} \) is generated by the traces of the \( r_{L^+,F^+}(g) \) for \( g \in G_{F,S} \). Consider a prime ideal \( p \) of \( \mathcal{T}^{\text{univ}} := R^{\text{univ}}_{\mathcal{S}_0}/m_{L^+} + R^{\text{univ}}_{\mathcal{S}_0} \). Because the image of \( r_{F^+,L^+} \) is finite, we see that the images of these traces in \( \mathcal{T}^{\text{univ}}/p \) are sums of roots of unity of bounded degree, so that \( \mathcal{T}^{\text{univ}}/p \) is finite. Thus \( \mathcal{T}^{\text{univ}} \) is a 0-dimensional Noetherian ring, so it is Artinian, and thus a direct product of Artin local rings with finite residue fields. Thus \( \mathcal{T}^{\text{univ}} \) is finite. It follows that \( R^{\text{univ}}_{\mathcal{S}'_0} \) is a finitely generated module over the complete local ring \( R^{\text{univ}}_{\mathcal{S}_0} \).

\( \square \)

### 4. Ordinary automorphic representations

#### 4.1. **Ordinary automorphic representations of \( GL_n \).** Let \( L \) be either a totally real number field or a quadratic totally imaginary extension of a totally real number field. Let \( \lambda \in (\mathbb{Z}_+^n)^{\text{Hom}(L,\mathbb{C})} \). Let \( \pi \) be an automorphic representation of \( GL_n(\mathbb{A}_L) \) which is

- RAESDC (regular, algebraic, essentially-self-dual, cuspidal) of weight \( \lambda \) if \( L \) is totally real, or
- RACSDC (regular, algebraic, conjugate-self-dual, cuspidal) of weight \( \lambda \) if \( L \) is totally imaginary.

See section 5 of [Tay08] or section 4 of [CHT08] for definitions of these terms. Let \( l \) be a prime number and \( i: \mathbb{Q}_l \rightarrow \mathbb{C} \) an isomorphism of fields. Let \( v \) be a place of \( L \) dividing \( l \) and \( \varpi_v \) a uniformizer in \( \mathcal{O}_{L_v} \). For each \( b > 0 \), let \( Iw(i^{b}, \varpi_v^b) \) denote the open compact subgroup of \( GL_n(\mathcal{O}_{L_v}) \) consisting of matrices which reduce modulo \( \varpi_v^b \) to a unipotent upper triangular matrix. The space \( (i^{-1} \pi_v^{b})^{Iw(i^{b}, \varpi_v^b)} \) carries commuting
actions of the scaled Hecke operators

\[ U_{\tau,\lambda,\varpi}^{(j)} := \left( \prod_{i=1}^{j} \prod_{\tau:L_i \to Q} \tau(\varpi)^{-\lambda_{i+1}-1} \right) \left[ \text{Iw}(v^{h,b}) \begin{pmatrix} \varpi v 1_j & 0 \\ 0 & 1_{n-j} \end{pmatrix} \text{Iw}(v^{h,b}) \right] \]

for \( j = 1, \ldots, n \). We define the ordinary part \((\iota^{-1}\pi_v)^{\text{Iw}(v^{h,b})}\text{ord}\) of \((\iota^{-1}\pi_v)^{\text{Iw}(v^{h,b})}\) to be the maximal subspace which is invariant under each \( U_{\tau,\lambda,\varpi}^{(j)} \) and such that every eigenvalue of each \( U_{\tau,\lambda,\varpi}^{(j)} \) is an \( l \)-adic unit. We define

\[ (\iota^{-1}\pi_v)^{\text{ord}} := \lim_{b>0}(\iota^{-1}\pi_v)^{\text{Iw}(v^{h,b})}\text{ord}. \]

We say that \( \pi \) is \( \iota \)-ordinary at \( v \) if the space \((\iota^{-1}\pi_v)^{\text{ord}}\) is non-zero.

### 4.2. \( l \)-adic automorphic forms on definite unitary groups

Let \( F^+ \) denote a totally real number field and \( n \) a positive integer. Let \( F/F^+ \) be a totally imaginary quadratic extension of \( F^+ \) and let \( c \) denote the non-trivial element of \( \text{Gal}(F/F^+) \). Suppose that the extension \( F/F^+ \) is unramified at all finite places. Assume that \( \nu[F^+:\mathbb{Q}] \) is divisible by 4. Under this assumption, we can find a reductive algebraic group \( G \) over \( F^+ \) with the following properties:

- \( G \) is an outer form of \( \text{GL}_n \) with \( G/F \cong \text{GL}_n/F \);
- for every finite place \( v \) of \( F^+ \), \( G \) is quasi-split at \( v \);
- for every infinite place \( v \) of \( F^+ \), \( G(F_v^+) \cong \text{Un}_n(\mathbb{R}) \).

We can and do fix a model for \( G \) over the ring of integers \( \mathcal{O}_{F^+} \) of \( F^+ \) as in section 2.1 of [Ger09]. For each place \( v \) of \( F^+ \) which splits as \( \mathbb{Q}w^c \) in \( F \) there is a natural isomorphism

\[ \iota_w : G(F_v^+) \cong \text{GL}_n(F_w) \]

which restricts to an isomorphism between \( G(\mathcal{O}_{F^+}) \) and \( \text{GL}_n(\mathcal{O}_{F_w}) \). If \( v \) is a place of \( F^+ \) split over \( F \) and \( \overline{v} \) is a place of \( F \) dividing \( v \), then we let

- \( \text{Iw}(\overline{v}) \) denote the subgroup of \( \text{GL}_n(\mathcal{O}_{F_\overline{v}}) \) consisting of matrices which reduce to an upper triangular matrix modulo \( \overline{v} \);
- \( \text{Iw}(\overline{v}^{b,c}) \), for \( 0 \leq b \leq c \), denote the subgroup of \( \text{GL}_n(\mathcal{O}_{F_\overline{v}}) \) consisting of matrices which reduce to an upper triangular matrix modulo \( \overline{v}^c \) and to a unipotent matrix modulo \( \overline{v}^b \). In particular \( \text{Iw}(\overline{v}^{0,0}) = \text{GL}_n(\mathcal{O}_{F_\overline{v}}) \).

Let \( l > n \) be prime number with the property that every place of \( F^+ \) dividing \( l \) splits in \( F \). Fix an algebraic closure \( \overline{\mathbb{Q}_l} \) of \( \mathbb{Q}_l \). Let \( K \) be an algebraic extension of \( \mathbb{Q}_l \) in \( \overline{\mathbb{Q}_l} \) such that every embedding \( F \rightarrow \overline{\mathbb{Q}_l} \) has image contained in \( K \) and such that \( K \) contains a primitive \( l \)-th root of unity. Let \( \mathcal{O} \) denote the ring of integers in \( K \) and \( k \) the residue field. Let \( S_l \) denote the set of places of \( F^+ \) dividing \( l \) and for each \( v \in S_l \), let \( \overline{v} \) be a place of \( F \) over \( v \). Let \( \overline{S}_l \) be the set of \( \overline{v} \) for \( v \in S_l \).

Let \( W \) be an \( \mathcal{O} \)-module with an action of \( G(\mathcal{O}_{F^+,l}) \). Let \( V \subset G(\mathbb{A}_{F^+,e}) \) be a compact open subgroup with \( v_l \in G(\mathcal{O}_{F^+,l}) \) for all \( v \in V \), where \( v_l \) denotes the projection of \( v \) to \( G(F_l^+) \). We let \( S(V,W) \) denote the space of \( l \)-adic automorphic forms on \( G \) of weight \( W \) and level \( V \), that is, the space of functions

\[ f : G(F^+)^{\backslash}G(\mathbb{A}_{F^+,e}) \rightarrow W \]

with \( f(gv) = v_l^{-1}f(g) \) for all \( v \in V \).
Let \( \tilde{I}_l \) denote the set of embeddings \( F \hookrightarrow K \) giving rise to a place in \( \tilde{S}_l \). To each \( \lambda \in (\mathbb{Z}_p^\times)^{\tilde{I}_l} \) we associate a finite free \( \mathcal{O} \)-module \( M_\lambda \) with a continuous action of \( G(\mathcal{O}_{F^+,l}) \) as in Definition 2.2.3 of [Ger09]. The representation \( M_\lambda \) is the tensor product over \( \tau \in \tilde{I}_l \) of the irreducible algebraic representations of \( GL_n \) of highest weights given by the \( \lambda_\tau \). We write \( S_\lambda(V, \mathcal{O}) \) instead of \( S(V, M_\lambda) \) and similarly for any \( \mathcal{O} \)-module \( A \), we write \( S_\lambda(V, A) \) for \( S(V, M_\lambda \otimes \mathcal{O} A) \).

Assume from now on that \( K \) is a finite extension of \( \mathbb{Q}_l \). Let \( I \) denote the product of all places in \( S_l \). Let \( R \) and \( S_\alpha \) denote finite sets of finite places of \( F^+ \) disjoint from each other and from \( S_l \) and consisting only of places which split in \( F \). Assume that each \( v \in S_\alpha \) is unramified over a rational prime \( p \) with \( |F(\zeta_p) : F| > n \).

Let \( T = S_l \prod R \prod S_\alpha \). For each \( v \in T \) fix a place \( \tilde{v} \) of \( F \) dividing \( v \), extending the choice of \( \tilde{v} \) for \( v \in S_l \). We henceforth identify \( G(F_v^+) \) with \( GL_n(F_\tilde{v}) \) via \( \iota_v \) for \( v \in T \) without comment. Let \( U = \prod_v U_v \) be the compact open subgroup of \( G(\mathbb{A}_{F^+}^\infty) \) with

- \( U_v = G(\mathcal{O}_{F_v^+}) \) if \( v \not\in R \cup S_\alpha \) splits in \( F \);
- \( U_v = Iw(\tilde{v}) \) if \( v \in R \);
- \( U_v = \ker(GL_n(\mathcal{O}_{F_v}) \rightarrow GL_n(k(\tilde{v}))) \) if \( v \in S_\alpha \);
- \( U_v \) is a hyperspecial maximal compact subgroup of \( G(F_v^+) \) if \( v \in S_l \).

If \( 0 \leq b \leq c \), we let \( U^{(b,c)} = U^t \times \prod_v Iw(\tilde{v}^{b,c}) \). We note that if \( S_\alpha \) is non-empty then \( U \) is sufficiently small (which means that its projection to \( G(F_v^+) \) for some \( v \in F^+ \) contains no finite order elements other than the identity).

For each \( v \in S_l \) fix a uniformizer \( \varpi_v \) in \( \mathcal{O}_{F_\tilde{v}} \). For \( 0 \leq b \leq c \) with \( c > 0 \) and \( j = 1, \ldots, n \), consider the scaled Hecke operator

\[
U_{\lambda, \varpi_v}^{(j)} := \left( \prod_{i=1}^{j} \prod_{\tau : \mathbb{F}_p \rightarrow \mathbb{Q}_l} \tau(\varpi_v)^{-\lambda_{\tau_i} p^{-n-1}} \right) \left[ Iw(\tilde{v}^{b,c}) \begin{pmatrix} \varpi_v^{1,j} & 0 \\ 0 & 1_{n-j} \end{pmatrix} Iw(\tilde{v}^{b,c}) \right]
\]

acting on the space \( S_\lambda(U^{(b,c)}, \mathcal{O}) \). We let \( S_\lambda^{ord}(U^{(b,c)}, \mathcal{O}) \) denote the ordinary part of \( S_\lambda(U^{(b,c)}, \mathcal{O}) \) as defined in section 2.4 of [Ger09] (noting that the space \( S_\lambda(U^{(b,c)}, \mathcal{O}) \) is denoted \( S_\lambda^{(1)}(U^{(b,c)}, \mathcal{O}) \) in [Ger09]). This is the maximal sub-module on which each of the operators \( U_{\lambda, \varpi_v}^{(j)} \) acts invertibly. This space is preserved by the Hecke operators

- \( T_w^{(j)} := \iota_w^{-1} \left( GL_n(\mathcal{O}_{F_w}) \begin{pmatrix} \varpi_w^{1,j} & 0 \\ 0 & 1_{n-j} \end{pmatrix} GL_n(\mathcal{O}_{F_w}) \right) \)

for \( w \) a place of \( F \), split over \( F^+ \) and not lying over \( T \), \( j = 1, \ldots, n \) and \( \varpi_w \) a uniformizer in \( \mathcal{O}_{F_w} \), and

- \( \langle u \rangle := \left( \prod_{v \in S_l} Iw(\tilde{v}^{b,c}) \text{diag}(u_{\tilde{v}}) Iw(\tilde{v}^{b,c}) \right) \)

for \( u = (u_v)_{v \in S_l} \in \prod_{v \in S_l} (\mathcal{O}_{F_v}^\times)^n \).

We let \( T_\lambda^{ord}(U^{(b,c)}, \mathcal{O}) \) denote the \( \mathcal{O} \)-subalgebra of \( \text{End}_{\mathcal{O}}(S_\lambda^{ord}(U^{(b,c)}, \mathcal{O})) \) generated by the operators \( T_w^{(j)} \), \( (T_w^{(n)})^{-1} \) and \( \langle u \rangle \). We let

\[
T_\lambda^{ord}(U^{(\infty)}, \mathcal{O}) := \lim_{c \rightarrow \infty} T_\lambda^{ord}(U^{(c)}, \mathcal{O})
\]
Let \( T_n \) denote the diagonal torus in \( GL_n \). We define \( T_n(1) \) as the pro-\( t \) part of \( T_n(\mathcal{O}_{F^+,1}) = \prod_{v \in S_F} T_n(\mathcal{O}_{F^+,v}) \). In other words, we have an exact sequence

\[
0 \to T_n(1) \to T_n(\mathcal{O}_{F^+,1}) \to T_n(\mathcal{O}_{F^+}/1) \to 0.
\]

Define the completed group algebras

\[
\Lambda^+ := \mathcal{O}[T_n(\mathcal{O}_{F^+,1})] \\
\Lambda := \mathcal{O}[T_n(1)].
\]

Identifying \( T_n(\mathcal{O}_{F^+,1}) \) with \( \prod_{v \in S_F} T_n(\mathcal{O}_{F^+_v}) \) in the natural way gives \( T_{\Lambda}^{t,\mathrm{ord}}(U(\infty),\mathcal{O}) \) the structure of a \( \Lambda^+ \)-algebra (via the operators \( \langle \rangle \)).

It is shown in section 2.6 of [Ger09] that the algebra \( T_{\Lambda}^{t,\mathrm{ord}}(U(\infty),\mathcal{O}) \) is independent of the weight \( \lambda \) in the sense that for each \( \lambda \) there is a natural isomorphism \( T_{\lambda}^{t,\mathrm{ord}}(U(\infty),\mathcal{O}) \cong T_0^{t,\mathrm{ord}}(U(\infty),\mathcal{O}) \). We let \( T_{\Lambda}^{t,\mathrm{ord}}(U(\infty),\mathcal{O}) \) denote the universal ordinary Hecke algebra as in Definition 2.6.2 of [Ger09]. By definition, this is just \( T_0^{t,\mathrm{ord}}(U(\infty),\mathcal{O}) \) with a modified \( \Lambda^+ \)-structure which is more convenient from the point of view of Galois representations.

4.3. An \( R^{\mathrm{red}} = T \) Theorem. Let \( T_{\Lambda}^{t,\mathrm{ord}}(U(\infty),\mathcal{O}) \) be the algebra introduced above. Let \( \mathfrak{m} \) be a maximal ideal of \( T_{\Lambda}^{t,\mathrm{ord}}(U(\infty),\mathcal{O}) \) with residue field \( k \) which is non-Eisenstein in the sense of section 2.7 of [Ger09]. According to propositions 2.7.3 and 2.7.4 of [Ger00] one can choose a continuous homomorphism

\[
\tilde{r}_\mathfrak{m} : G_{F^+} \to \mathcal{G}_n(T_{\Lambda}^{t,\mathrm{ord}}(U(\infty),\mathcal{O})/\mathfrak{m})
\]

with \( \tilde{r}_\mathfrak{m}|_{G_F} \) absolutely irreducible and a continuous lifting

\[
r_\mathfrak{m} : G_{F^+} \to \mathcal{G}_n(T_{\Lambda}^{t,\mathrm{ord}}(U(\infty),\mathcal{O})_{\mathfrak{m}})
\]

with the following properties:

1. \( r_\mathfrak{m} \) is unramified outside \( T \).
2. If \( v \not\in T \) is a place of \( F^+ \), which splits as \( w u \) in \( F \) and \( \text{Frob}_u \) is the geometric Frobenius element of \( G_{F_u}/I_{F_u} \), then \( r_\mathfrak{m}(\text{Frob}_w) \) has characteristic polynomial

\[
X^n - T_1^{(1)} X^{n-1} + \ldots + (-1)^j T_1^{(j-1)} T_1^{(j)} X^{n-j} + \ldots + (-1)^n (Nw)^{n(n-1)/2} T_1^{(n)}.
\]

3. \( \nu \circ r_\mathfrak{m} = \epsilon^{1-n} \delta_{F/P^+}^{\mu_\mathfrak{m}} \) where \( \delta_{F/P^+} \) is the non-trivial character of \( \text{Gal}(F/F^+) \) and \( \mu_\mathfrak{m} \in \mathbb{Z}/2\mathbb{Z} \).
4. If \( v \in R \) and \( \sigma \in I_{F_v} \), then \( r_\mathfrak{m}(\sigma) \) has characteristic polynomial \( (X - 1)^n \).

We make the following assumptions:

a) The subgroup \( \mathcal{T}_\mathfrak{m}(G_{F^+}(\mathbb{Q})) \) of \( \mathcal{G}_n(k) \) is big;
(b) For \( v \in S_t \cup R, \mathcal{T}_\mathfrak{m}(G_{F_v}) = \{ 1_v \} \);
(c) The set \( S_\mathcal{A} \) is non-empty and for \( v \in S_\mathcal{A}, \mathcal{T}_\mathfrak{m}|_{G_{F_v}} \) is unramified and \( H^0(G_{F_v}, \text{ad}\mathcal{T}_\mathfrak{m}(1)) = \{ 0 \} \).

For \( v \in S_t \), let

\[
\Lambda_{F_v} := \mathcal{O}[I_{F_v}^{\text{red}}(l)^n].
\]
where for a group $H$, $H(l)$ denotes the pro-$l$ completion. The inverses of the Artin maps $\text{Art}_{F_v}$ for $v \in S_l$ give rise to an isomorphism

$$\prod_{v \in S_l} (I_{F_v}(l))^{\mu} \rightarrow \prod_{v \in S_l} (1 + \omega_v \mathcal{O}_{F_v})^{\mu} \cong T_n(l)$$

and hence an isomorphism

$$\tilde{\otimes}_{v \in S_l} \Lambda_{F_v} \rightarrow \Lambda.$$ 

Corollary 3.1.4 of [Ger09] shows that $r_m$ satisfies the following property, in addition to (0)-(4) above:

(5) For $v \in S_l$, the homomorphism $R_{\tilde{\otimes} G_{F_v}}^\Lambda \otimes \Lambda_{F_v} \rightarrow T^{\sigma,\text{ord}}(U(l^{\infty}), \mathcal{O}_m)$ coming from $r_m|_{G_{F_v}}$ and the $\Lambda_{F_v}$-algebra structure on $T^{\sigma,\text{ord}}(U(l^{\infty}), \mathcal{O}_m)$ factors through the quotient $R_{\tilde{\otimes} G_{F_v}}^\Lambda \otimes \Lambda_{F_v}$ of $R_{\tilde{\otimes} G_{F_v}}^\Lambda$ constructed in section 3.1.4.

We now turn to deformation rings. For each $v \in S_l$, let $R_{\tilde{\otimes} G_{F_v}}^\Lambda$ be the quotient of $R_{\tilde{\otimes} G_{F_v}}^\Lambda \otimes \Lambda_{F_v}$ constructed in section 3.1.4. For $v$ in $R$, let $R_{\tilde{\otimes} G_{F_v}}^\Lambda$ denote the quotient of $R_{\tilde{\otimes} G_{F_v}}^\Lambda$ corresponding to lifts $r$ for which $r(\sigma)$ has characteristic polynomial $(X - 1)^n$ for each $\sigma \in I_{F_v}$. This ring is studied in section 3 of [Tay08]. Let

$$\mathcal{S}_\Lambda = \left( \frac{F}{F^+}, T, \tilde{T}, \Lambda, \tilde{\Lambda}, \hat{\epsilon}, \hat{\epsilon}, \eta \right) \cup \left\{ R_{\tilde{\otimes} G_{F_v}}^\Lambda \mid v \in S_l, \{ R_{\tilde{\otimes} G_{F_v}}^\Lambda \mid v \in S_l \} \cup \{ R_{\tilde{\otimes} G_{F_v}}^\Lambda \mid v \in R, \{ R_{\tilde{\otimes} G_{F_v}}^\Lambda \mid v \in S_l \} \right\}$$

Let $\mathcal{C}_\Lambda$ denote the category of complete local Noetherian $\Lambda$-algebras with residue field $k$. We say that a lift $r$ of $\tilde{r}$ to an object $A$ of $\mathcal{C}_\Lambda$ is of type $\mathcal{S}_\Lambda$ if for each $v \in S_l$, the homomorphism $R_{\tilde{\otimes} G_{F_v}}^\Lambda \otimes \Lambda_{F_v} \rightarrow A$ coming from $r|_{G_{F_v}}$ and the $\Lambda$-structure on $A$ factors through $R_{\tilde{\otimes} G_{F_v}}^\Lambda$ and if for each $v \in R$ the homomorphism $R_{\tilde{\otimes} G_{F_v}}^\Lambda \rightarrow A$ coming from $r|_{G_{F_v}}$ factors through $R_{\tilde{\otimes} G_{F_v}}^\Lambda$. We define deformations of type $\mathcal{S}_\Lambda$ in the same way. Let $\text{Def}_{\mathcal{S}_\Lambda} : \mathcal{C}_\Lambda \rightarrow \text{Sets}$ be the functor which sends an object $A$ to the set of deformations of $\tilde{r}$ to $A$ of type $\mathcal{S}_\Lambda$. This functor is represented by an object $R_{\mathcal{S}_\Lambda}^\text{univ}$ of $\mathcal{C}_\Lambda$.

Properties (0)-(5) above imply that the lift $r_m$ of $\tilde{r}$ to $T^{\sigma,\text{ord}}(U(l^{\infty}), \mathcal{O}_m)$ is of type $\mathcal{S}_\Lambda$ and hence gives rise to a homomorphism of $\Lambda$-algebras

$$R_{\mathcal{S}_\Lambda}^\text{univ} \rightarrow T^{\sigma,\text{ord}}(U(l^{\infty}), \mathcal{O}_m).$$

The following result is contained in Theorem 4.3.1 of [Ger09].

**Theorem 4.3.1.** The map $R_{\mathcal{S}_\Lambda}^\text{univ} \rightarrow T^{\sigma,\text{ord}}(U(l^{\infty}), \mathcal{O}_m)$ induces an isomorphism

$$R_{\mathcal{S}_\Lambda}^\text{univ,red} \rightarrow T^{\sigma,\text{ord}}(U(l^{\infty}), \mathcal{O}_m)$$

and $\mu_m \equiv n \mod 2$ so that $r_m$ is odd.

Let $\lambda \in (\mathbb{Z}/n)^\hat{I}$ and for each $v \in S_l$, let $\lambda_v$ denote the element of $(\mathbb{Z}/n)^{\text{Hom}(F_v, K)}$ given by the $\lambda_v|_{F_v}$ for $\tau : F_v \rightarrow K$. In section 3.1.2, we associated to $\lambda_v$ an $n$-tuple of characters $(\chi_1^\lambda_v, \ldots, \chi_n^\lambda_v)$ from $I_{F_v}$ to $\mathcal{O}^\times$. These characters induce an $\mathcal{O}$-algebra homomorphism

$$\chi^\lambda_v : \Lambda_{F_v} \rightarrow \mathcal{O}$$
and taking the tensor product over the places \( v \in S_l \), we get a homomorphism
\[
\chi^\lambda : \Lambda \to \mathcal{O}.
\]

We denote the kernels of these homomorphisms by \( \wp_\lambda \) and \( \varphi_\lambda \). The next result follows from Corollary 2.5.4 and Lemma 2.6.4 of [Ger09] (noting that \( U \) is sufficiently small since \( S_\alpha \) is non-empty).

**Proposition 4.3.2.** The algebra \( \mathbb{T}^{T,\text{ord}}(U(\infty), \mathcal{O}) \) is finite and faithal as a \( \Lambda \)-module and for every \( \lambda \in \left( \mathbb{Z}_l^n \right)^{\hat{f}} \) there is a natural surjection
\[
\mathbb{T}^{T,\text{ord}}(U(\infty), \mathcal{O}) \otimes_\Lambda \Lambda_{\wp_\lambda}/\wp_\lambda \to \mathbb{T}^{T,\text{ord}}(U(1,1), \mathcal{O}) \otimes_\mathcal{O} K
\]
whose kernel is nilpotent.

Let \( \lambda \) and \( \lambda_v \) for \( v \in S_l \) be as above. Consider the deformation problem
\[
S_\lambda = \{ F/F^+, T, \tilde{T}, \mathcal{O}, \tilde{r}, c^1-n \delta_{\text{reg}} / F^+, \{ R_{\mathcal{O}_v} \}_{v \in S_{\mathcal{O}}}, \{ R_{\mathcal{O}_{1/2}} \}_{v \in S_{\mathcal{O}}}, \{ R_{\mathcal{O}_{1/2}} \}_{v \in S_{\mathcal{O}}} \}
\]
and the corresponding deformation ring \( R_{S_\lambda}^{\text{univ}} \).

**Corollary 4.3.3.** The ring \( R_{S_\lambda}^{\text{univ}} \) is a finite \( \mathcal{O} \)-algebra.

**Proof.** First of all, observe that if \( R \) is an object of \( \mathcal{C}_\mathcal{O} \), then \( R \) is a finite \( \mathcal{O} \)-algebra if \( R_{\text{red}} \) is. Indeed, if \( R_{\text{red}} \) is finite over \( \mathcal{O} \) then \( R/m_{\mathcal{O}} R \) is Noetherian and 0-dimensional and hence Artinian. It follows from the topological Nakayama lemma that \( R \) is finite over \( \mathcal{O} \).

The ring \( (R_{S_\lambda}^{\text{univ}})_{\text{red}} \) is a quotient of \( (R_{S_\lambda}^{\text{univ}})_{\wp_\lambda} \). By Theorem 4.3.1 and Proposition 4.3.2 \( (R_{S_\lambda}^{\text{univ}})_{\text{red}}/\wp_\lambda \) is a finite \( \mathcal{O} \)-algebra. The result follows. \( \square \)

5. **Existence of Lifts**

5.1. Let \( F \) be an imaginary CM field, \( F^+ \) its maximal totally real subfield and \( c \) the non-trivial element of \( \text{Gal}(F/F^+) \). Let \( \pi \) be a RACSDC automorphic representation of \( \text{GL}_n(A_F) \) and \( \iota \) an isomorphism \( \mathbb{Q}_l \cong \mathbb{C} \). In [CH09] it is shown that there is a semisimple representation
\[
r_{l,\tau}(\pi) : G_F \to \text{GL}_n(\overline{\mathbb{Q}_l})
\]
uniquely determined by the following properties:

1. \( r_{l,\tau}(\pi)^c = r_{l,\tau}(\pi)^c \iota^{-1-n} \);
2. for \( w \) a place of \( F \) not dividing \( l \) we have
\[
| r_{l,\tau}(\pi)|_{G_{F_w}}^{\text{ss}} \cong r_l(\iota^{-1} \pi_w)^{\chi}(1-n)^{ss}
\]
where \( r_l(\iota^{-1} \pi_w) \) is the \( l \)-adic representation associated to the Weil-Deligne representation \( \text{rec}_l(\pi_w) \otimes |^{\chi}(1-n)/2 \) (and \( \text{rec}_l \) is the local Langlands correspondence of [HT01]).

If \( F \) and \( c \) are as above, we let \( (\mathbb{Z}_l^n)_{c}^{\text{Hom}(F, \mathbb{Q}_l)} \) denote the subset of \( (\mathbb{Z}_l^n)_{c}^{\text{Hom}(F, \mathbb{Q}_l)} \) consisting of elements \( \lambda \) with \( \lambda_{\tau,c,j} = -\lambda_{\tau,n-j+1} \) for all \( \tau : F \hookrightarrow \mathbb{Q}_l \) and \( j = 1, \ldots, n \).

If \( \lambda \in \left( \mathbb{Z}_l^n \right)_{c}^{\text{Hom}(F, \mathbb{Q}_l)} \) and \( w \) is a place of \( F \) dividing \( l \), we let \( \lambda_w = (\lambda_{w,\sigma})_{\sigma} \) be the element of \( (\mathbb{Z}_l^n)_{c}^{\text{Hom}(F, \mathbb{Q}_l)} \) determined by \( \lambda_{w,\sigma} = \lambda_{w,F} \) for all \( \sigma : F_w \to \mathbb{Q}_l \).

One can find a finite extension \( K \) of \( \mathbb{Q}_l \) with ring of integers \( \mathcal{O} \) so that \( r_{l,\tau}(\pi) \) can be conjugated to take values in \( \text{GL}_n(\mathcal{O}) \). Reducing modulo the maximal ideal
of $\mathcal{O}$, extending scalars to $\mathbb{F}_l$ and semisimplifying, one obtains a representation $\tilde{r}_{l,\iota}(\pi) : G_F \to \text{GL}_n(\mathbb{F}_l)$ which is independent of any choices made.

If $K$ (resp. $k$) is an algebraic extension of $\mathbb{Q}_l$ (resp. $\mathbb{F}_l$) and $\rho : G_F \to \text{GL}_n(K)$ (resp. $\overline{\rho} : G_F \to \text{GL}_n(k)$) is a continuous representation, we say that $\rho$ (resp. $\overline{\rho}$) is automorphic if there exists a $\pi$ and $\iota$ as above with $r_{l,\iota}(\pi)$ (resp. $\tilde{r}_{l,\iota}(\pi)$) isomorphic to $\rho \otimes_K \overline{\mathbb{Q}_l}$ (resp. $\overline{\rho} \otimes_k \mathbb{F}_l$). We say that $\rho$ (or $\overline{\rho}$) is ordinarily automorphic if in addition $\pi$ and $\iota$ can be chosen so that $\pi$ is $\iota$-ordinary at every place dividing $l$. We say that $\rho$ is ordinary automorphic of weight $\lambda \in (\mathbb{Z}_l^n)_{c}^{\text{Hom}(F, \mathbb{F}_l)}$ if $\rho$ is automorphic and $\rho|_{G_{F_w}}$ is ordinary of weight $\lambda_w \in (\mathbb{Z}_l^n)_{c}^{\text{Hom}(F_w, \mathbb{F}_l)}$ for each place $w | l$ of $F$. We say that $\rho$ is ordinary automorphic if it is ordinary automorphic of some weight. If $\rho$ is ordinarily automorphic and its reduction $\overline{\rho}$ is absolutely irreducible, then $\rho$ is ordinary automorphic by Proposition 5.3.1 of [Ger09].

We are now ready to prove our main theorem. For the convenience of the reader, we recall all our assumptions in the statement of the theorem.

**Theorem 5.1.1.** Let $F$ be an imaginary CM field with maximal totally real subfield $F^+$. Let $n \geq 2$ be an integer and $l > n$ a prime number. Suppose that $\zeta_l \notin F$. Assume that the extension $F/F^+$ is split at all places dividing $l$. Suppose that

$$\overline{\rho} : G_F \to \text{GL}_n(\mathbb{F}_l)$$

is an irreducible representation satisfying the following assumptions.

1. The representation $\overline{\rho}$ is ordinarily automorphic (so in particular $\overline{\rho} \cong \overline{\rho}^{\iota - 1 - n}$).
2. Any place of $F$ at which $\overline{\rho}$ is ramified splits over $F^+$.
3. The image $\overline{\rho}(G_{F(\zeta_l)})$ is big.
4. $\text{ker ad} \overline{\rho}$ does not contain $F(\zeta_l)$.
5. There is an element $\lambda \in (\mathbb{Z}_l^n)_c^{\text{Hom}(F, \mathbb{F}_l)}$ such that for every place $w | l$ of $F$,

$$\overline{\rho}|_{G_{F_w}}$$

has an ordinary crystalline lift of weight $\lambda_w$.

Then $\overline{\rho}$ has an automorphic lift $\rho$ which is crystalline and ordinary of weight $\lambda_w$ at each place $w$ of $F$ dividing $l$, and which is ordinarily automorphic of level prime to $l$.

In fact, suppose we are given a set of places $S$ of $F^+$ which split in $F$, a choice of a place $v$ of $F$ above each place $v$ of $F^+$, and an inertial type $\tau_v$ for $I_{F_v}$ for each $v \in S$ not dividing $l$ such that $\overline{\rho}|_{G_{F_v}}$ has a lift of type $\tau_v$. Then $\rho$ can be chosen to be of type $\tau_v$ at $v$ for all places $v \in S$, $v \nmid l$. More precisely, given a choice of irreducible component of each ring $R_{\overline{\rho}|_{G_{F_v}}}$ with $v \in S$, $v \nmid l$ and of each ring $R_{\overline{\rho}|_{G_{F_v}}}$ with $v | l$, $\rho$ may be chosen so as to give a point on each of these components.

**Proof.** It suffices to prove the final statement. Choose an isomorphism $\iota : \mathbb{C}_l \cong \mathbb{C}$ and a RACSDC automorphic representation $\pi$ of $\text{GL}_n(k_F)$ which is $\iota$-ordinary at all places dividing $l$ such that $\tilde{r}_{l,\iota}(\pi) \cong \overline{\rho}$. By Lemma 2.1.4 of [CHT08], $\overline{\rho}$ extends to a representation $\tilde{r} : G_{F^+} \to G_n(\mathbb{F}_l)$ with $\tilde{r}^{-1}(\text{GL}_n(\mathbb{F}_l) \times \text{GL}_1(\mathbb{F}_l)) = G_F$. Extending $S$ if necessary, we may assume that $S$ contains all places above $l$ and that $\overline{\rho}$ is unramified away from $S$. Indeed, for the places $v \nmid l$ just added to $S$, the lift $r_{l,\iota}(\pi)|_{G_{F_v}}$ determines an inertial type $\tau_v$ for $I_{F_v}$ and at least one irreducible component of $R_{\overline{\rho}|_{G_{F_v}}}$.

Choose a finite extension $K$ of $\mathbb{Q}_l$ inside $\mathbb{C}_l$ with residue field $k$ and ring of integers $\mathcal{O}$ containing a primitive $l$-th root of unity so that $r_{l,\iota}(\pi)$ can be conjugated
to take values in $G_\alpha(\mathcal{O})$, so that $K$ contains the image of every embedding $F \hookrightarrow \mathcal{O}_l$ and so that each type $\tau_v$ for $v \in S, v \nmid l$ is defined over $K$. Assume from now on that $\nu_{l,\epsilon}(\pi)$ takes values in $G_\alpha(\mathcal{O})$.

By Lemma 2.1.4 of [10], $\nu \circ \varphi = \tau^{1-n}\delta_{F/F^+}$ for some $\mu \in \mathbb{Z}/2\mathbb{Z}$, where $\delta_{F/F^+}$ is the quadratic character of $G_{F^+}$ corresponding to $F$. By Theorem 4.3.3, $\varphi$ is odd, so in fact $\mu \equiv n \mod 2$. For each $v \in S$, let $\mathcal{O}_v$ be the chosen irreducible component of $\mathcal{O}_{\varphi/F_v}$ when $v \nmid l$ or $\mathcal{O}_{\varphi/F_v}$ when $v|l$. Let $\mathcal{S}$ denote the set of $\varphi$ for $v \in S$ and let

$$\mathcal{S} = (F/F^+, S, \mathcal{O} \varphi, \tau^{1-n}\delta_{F/F^+}, \rho_{l,\epsilon}, \{\mathcal{O}_v\}_{v \in \mathcal{S}}).$$

To prove the theorem it suffices to show that we can find a closed point of $R^G\text{univ}[1/l]$ so that the corresponding representation restricted to $G_F$ is automorphic.

Choose a finite place $v_1$ of $F$ not lying over $S$ so that

1. $v_1$ is unramified over a rational prime $p$ with $[F(\zeta_p) : F] > n$;
2. $v_1$ does not split completely in $F(\zeta_l)$;
3. $\text{ad} \varphi(F_{v_1}) = 1$.

The last two conditions imply that $H^0(G_{F_{v_1}}, \text{ad} \varphi(1)) = \{0\}$. Choose a CM extension $L$ of $F$ with the following properties:

- $L/F$ is Galois and soluble;
- $L$ is linearly disjoint from $\varphi(\ker(\text{ad} \varphi))(\zeta_l)$ over $F$;
- all primes of $L$ lying above $S$ or $\{v_1\}$ are split over $L^+$ where $L^+$ is the maximal totally real subfield of $L$;
- the extension $L/L^+$ is unramified at all finite places;
- $4|\{L^+/ F^+\}$;
- for each place $\varphi \in \mathcal{S}$ away from $l$ and each place $w$ of $L$ lying over $\varphi$, we have $\mathcal{N}w \equiv 1 \mod l, \mathcal{P}(G_{L_w}) = \{1_n\}$, the type $\tau_v$ becomes trivial upon restriction to $I_{L_w}$ and if $\pi_{L_w}$ denote the base change of $\pi$ to $L$, then $(\pi_{L_w})_{w} \neq 0$.
- the places $\{v_1, c_{v_1}\}$ split completely in $L$;
- for each place $\varphi \in \mathcal{S}$ dividing $l$ and each place $w$ of $L$ lying over $\varphi$ we have $\hat{\varphi}(G_{L_w}) = \{1_n\}$.
- if $w$ is a place of $L$ not lying over $l$ such that $(\pi_{L_w})$ is ramified, then $w$ lies over a place of $L^+$ which splits in $L$, and $(\pi_{L_w})_{w} \neq 0$.

Let $T$ denote the set of places of $L^+$ comprised of those lying above $S \cup \{v_1|F^+\}$, together with any places of $L^+$ over which there is a place of $L$ with $(\pi_{L_w})$ ramified. Let $\tilde{T}$ denote a set of places of $L$, containing all places lying above $S \cup \{v_1\}$, such that $\tilde{T}$ consists of one place $\tilde{w}$ for each place $w \in T$. For each $\tilde{w} \in \tilde{T}$ lying above $v_1$, let $\mathcal{R}_{\tilde{w}} = \mathcal{R}_{\hat{\varphi}[\mathcal{O} \varphi]_{L_\tilde{w}}}$. For $\tilde{w} \in \tilde{T}$ not dividing $l$ or $v_1$, let $\mathcal{R}_{\tilde{w}}$ denote the quotient $\mathcal{R}_{\hat{\varphi}[\mathcal{O} \varphi]_{L_\tilde{w}}}$ of $\mathcal{R}_{\mathcal{O} \varphi} \mathcal{L}_\tilde{w}$ corresponding to lifts for which each element of inertia has characteristic polynomial $(X - 1)^n$. Let $\lambda_{\tilde{w}}$ be the element of $(\mathbb{Z}/l)^\text{Hom}(L_{\tilde{w}}, \mathcal{O})$ determined by $(\lambda_{\tilde{w}})_{\tau} = \lambda_{\tau|\tilde{w}}$ for all $\tau : L \hookrightarrow \mathcal{O}_l$. Extend $K$ if necessary so that it contains the image of every embedding $L \hookrightarrow \mathcal{O}_l$. For $\tilde{w} \in \tilde{T}$ lying above $l$, let $\mathcal{R}_{\tilde{w}} = \mathcal{R}_{\hat{\varphi}[\mathcal{O} \varphi]_{L_\tilde{w}}}$.

$$S' = (L/L^+, T, \tilde{T}, \mathcal{O}, \varphi, \lambda_{L^+, T}, \epsilon^{1-n}\delta_{L/L^+}, \{\mathcal{O}_w\}_{w \in T}).$$
Restricting the universal deformation over $R_{\text{univ}}^S$ to $G_{L^+,T}$ gives rise to a map $R_{\text{univ}}^S \to R_{\text{univ}}^S$ and by Lemma 3.2.4, this map is finite.

Now, let $G$ be a reductive group over $\mathcal{O}_{L^+}$ as in section 3.2.4 (with $L^+$ replacing $F^+$). By Théorème 5.4 and Corollaire 5.3 of [Lab09], the assumption that $L/L^+$ is unramified at all finite places, $\pi_L$ is the strong base change of an automorphic representation $\Pi$ of $G(\mathbb{A}_{L^+})$. By Lemma 5.1.6 of [Ger09], $\pi_L$ is $l$-ordinary at each place of $L$ dividing $l$. Let $U \subset G(\mathbb{A}_{L^+})$ be the compact open subgroup defined as in section 4.2 with $S_0$ the set of places of $T$ above $v_1|_{F^+}$ and $R$ the set of places of $T$ not dividing $l$ and not in $S_0$. Then extending $\mathcal{O}$ if necessary, the Hecke eigenvalues on $(l^{-1}\Pi_{\infty})^U \otimes \mathcal{O}_{\mathcal{V}}(l^{-1}\Pi_{v})^{\text{ord}}$ give rise to an $\mathcal{O}$-algebra homomorphism $T_{\text{ord}}\left(U(\mathbb{A}_{\infty}),\mathcal{O}\right) \to \mathcal{O}$. Reducing this modulo the maximal ideal of $\mathcal{O}$ gives a maximal ideal $m$ of $T_{\text{ord}}\left(U(\mathbb{A}_{\infty}),\mathcal{O}\right)$ which is non-Eisenstein by the second of our conditions on $L$ above. All of the hypotheses of section 4.3 are satisfied and we deduce from Corollary 4.3.3 that $R_{\text{univ}}^S$ is finite over $\mathcal{O}$. Theorem 4.3.1 and Proposition 4.3.2 imply that every closed point of $R_{\text{univ}}^S[1/l]$ gives rise to a representation of $G_L$ which is ordinarily automorphic.

Since $R_{\text{univ}}^S$ is finite over $\mathcal{O}$ and has Krull dimension at least one by Lemma 3.2.4, the ring $R_{\text{univ}}^S[1/l]$ is non-zero. Any closed point on this ring gives rise to a crystalline ordinary representation $\rho$ of $G_F$ which is ordinarily automorphic upon restriction to $G_L$. By Lemma 1.4 of [BLGHT09], any such $\rho$ is automorphic and hence, by Lemma 5.1.6 of [Ger09], is in fact ordinarily automorphic. Finally, it follows from Theorem 5.3.2 of [Ger09] that such a $\rho$ is ordinarily automorphic of level prime $l$.

We can frequently make this rather more explicit.

**Corollary 5.1.2.** Let $F$ be an imaginary CM field with maximal totally real subfield $F^+$. Let $n \geq 2$ be an integer and $l > n$ a prime number. Suppose that $\zeta_l \notin F$. Assume that the extension $F/F^+$ is split at all places dividing $l$. Let $\tilde{S}_l$ be a set of places of $F$ lying over $l$, containing exactly one place above each place of $F^+$ dividing $l$. Suppose that

$$\overline{\rho} : G_F \to \text{GL}_n(\mathbb{F}_l)$$

is an irreducible representation satisfying the following assumptions.

1. The representation $\overline{\rho}$ is ordinary automorphic (so in particular $\overline{\rho} = \overline{\rho}^\vee \epsilon^{-n}$).
2. Any place of $F^+$ at which $\overline{\rho}$ is ramified splits in $F$.
3. The image $\overline{\rho}(G_{F(G)})$ is big.
4. $\text{ker ad} \overline{\rho}$ does not contain $F(\zeta_l)$.
5. There is an element $\lambda \in (\mathbb{Z}_l^n)_{\text{c}}^{\text{Hom}(F,\mathbb{F}_l)}$ such that for every place $v \in \tilde{S}_l$, $\overline{\rho}|_{G_{F_v}}$ is isomorphic to a representation

$$\left( \begin{array}{cccc} \overline{\rho}_{v,1} & * & \ldots & * \\ 0 & \overline{\rho}_{v,2} & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \overline{\rho}_{v,n-1} \\ 0 & 0 & \ldots & \overline{\rho}_{v,n} \end{array} \right)$$

where $\overline{\rho}_{v,1}|_{F_v} = \chi_i^{\lambda_v}|_{F_v}$ (where $\chi_i^{\lambda_v}$ is the crystalline character of Definition 3.1.2), and for each $i < j$ we have $\overline{\rho}_{v,i} \overline{\rho}_{v,j}^{-1} \notin \epsilon$. 

Then $\overline{\rho}$ has an ordinarily automorphic lift (of level prime to $l$) $\rho$ of weight $\lambda$ which is crystalline at all places dividing $l$; furthermore for each place $v \in \tilde{S}_l$, $\rho|_{G_{F_v}}$ is isomorphic to a representation

$$
\begin{pmatrix}
\psi_{v,1} & * & \ldots & * & * \\
0 & \psi_{v,2} & \ldots & * & * \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \psi_{v,n-1} & * \\
0 & 0 & \ldots & 0 & \psi_{v,n}
\end{pmatrix}
$$

with $\psi_{v,i}$ a lift of $\overline{\pi}_{v,i}$ agreeing with $\chi_\lambda^{\psi_v}$ on $I_{F_v}$.

**Proof.** This is an immediate consequence of Theorem 5.1.1, Lemma 3.1.5 and Lemma 3.1.8. \hfill \Box

We now state a corollary to the proof of Theorem 5.1.1 which we will use in section 7.

**Corollary 5.1.3.** Let $l, n, F, F^+, \overline{\rho}, \pi, \bar{r}, S, \tilde{S}, \{\tau_v\}_{v \in S, v|l}, \{\lambda_{v}^\psi\}_{v \in S_l}$ and $\{R_v\}_{v \in S}$ be as in Theorem 5.1.1 and its proof. Let

$$S = (F/F^+, S, \tilde{S}, O, \bar{r}, \epsilon^{1-n} \delta_{F/F^+}, \{R_v\}_{v \in S}).$$

Then $R_S^{\text{inv}}$ is a finite $O$-module of rank at least 1.

6. Serre weights

6.1. We now put ourselves in the setting of section 4.2. In particular, we let $F^+$ denote a totally real number field and $n$ a positive integer. Let $F/F^+$ be a totally imaginary quadratic extension of $F^+$ and let $c$ denote the non-trivial element of $\text{Gal}(F/F^+)$. Suppose that the extension $F/F^+$ is unramified at all finite places. Assume that $n[F^+: \mathbb{Q}]$ is divisible by 4. We note that we could make somewhat weaker assumptions, but the necessity of considering definite unitary groups which fail to be quasi-split at some finite places would complicate the notation unnecessarily.

Let $G$ be the reductive algebraic group over $F^+$ defined in section 4.2 together with a fixed model over $O_{F^+}$ as before. Again, we take a prime number $l > n$ so that every place in the set $S_l$ of places of $F^+$ dividing $l$ splits in $F$. Fix a set $\tilde{S}_{l}$ of places of $F$ consisting of exactly one place above each place in $S_l$. Let $O$ be the ring of integers of $\overline{\mathbb{Q}}$, with residue field $\overline{\mathbb{Q}}_l$. Let $I_{\bar{r}}$ denote the set of embeddings $F \hookrightarrow \overline{\mathbb{Q}}_l$ giving rise to one of the places $\tilde{v} \in \tilde{S}_l$. Let $I_{\bar{r}}$ denote the subset of $I_{\bar{r}}$ giving rise to $\tilde{v}$. Let the residue field of $F_{\tilde{v}}$ be $k(\tilde{v})$. Then any element $\sigma \in I_{\bar{r}}$ induces an embedding $\overline{\sigma} : \bar{k}(\tilde{v}) \hookrightarrow \overline{\mathbb{Q}}_l$. For an embedding $\tau : k(\tilde{v}) \hookrightarrow \overline{\mathbb{Q}}_l$, we let $I_{\tilde{r}}$ denote the subset of $I_{\bar{r}}$ consisting of the $\sigma$ with $\overline{\sigma} = \tau$. We let $I_{\tilde{r}}$ be the set of embeddings $\tau : k(\tilde{v}) \hookrightarrow \overline{\mathbb{Q}}_l$ (running over all $v$).

Define $(\mathbb{Z}_l^+)^{I_{\tilde{r}}}$ as in section 4.2. Let $(\mathbb{Z}_l^n)^{I_{\tilde{r}}}$ be the subset of $(\mathbb{Z}_l^n)^{I_{\tilde{r}}}$ consisting of $\lambda$ with $l - 1 \geq \lambda_{\tau,i} - \lambda_{\tau,i+1} \geq 0$ for all $\tau$ and all $i = 1, \ldots, n - 1$. Let $(\mathbb{Z}_l^n)^{I_{\tilde{r}}}$ denote the subset of $(\mathbb{Z}_l^n)^{I_{\tilde{r}}}$ consisting of weights $\lambda$ with the property that for each $\tilde{v}$ and $\tau : k(\tilde{v}) \hookrightarrow \overline{\mathbb{Q}}_l$, it is possible to write $\tilde{I}_\tau = \{\sigma_{\tau,1}, \ldots, \sigma_{\tau,e}\}$ with $\lambda_{\sigma_{\tau,i},j} = 0$ if $i > 1$ and $l - 1 \geq \lambda_{\sigma_{\tau,i-1},j} - \lambda_{\sigma_{\tau,i-1,j+1}} \geq 0$ for all $j = 1, \ldots, n - 1$. This being the case,
we define $\lambda_{r,j} := \lambda_{r-1,j}$. In this way, we define a surjective map $\pi$ from $(\mathbb{Z}_p^n, r_{\text{ex}})^{\bar{t}}$ to $(\mathbb{Z}_p^n, \bar{t})$.

Fix $\lambda \in (\mathbb{Z}_p^n, r_{\text{ex}})^{\bar{t}}$. We now consider the finite free $\mathcal{O}$-module $M_\lambda$ of Definition 2.2.3 of [Ger09]. This has a continuous action of $G(\mathcal{O}_{F^+}) = \prod v \in S_l \text{GL}_n(\mathcal{O}_{F_v})$. The action on $M_\lambda \otimes \overline{\mathbb{F}}_l$ factors through $\prod_{v \in S_l} \text{GL}_n(k(v))$.

Let $S_a$ be a nonempty set of finite places of $F^+$, disjoint from $S_l$, such that any place $v$ of $S_a$ splits in $F$ and is unramified over a rational prime $p$ with $[F(\zeta_p) : F] > n$. Choose a place $\tilde{v}$ of $F$ lying over each $v \in S_a$. Let $U = \prod_{v} U_v$ be a compact open subgroup of $G(F_v^+)$ such that

- $U_v \subset G(\mathcal{O}_{F_v^+})$ for all $v$;
- $U_v = \iota_v^{-1} \ker(\text{GL}_n(\mathcal{O}_{F_v}) \to \text{GL}_n(k(\tilde{v})))$ if $v \in S_a$;
- $U_v = G(\mathcal{O}_{F_v^+})$ if $v \mid \mathfrak{p}$;
- $U_v$ is a hyperspecial maximal compact subgroup of $G(F_v^+)$ if $v$ is inert in $F$.

Then (because $S_a$ is nonempty) $U$ is sufficiently small, and

$$S_\lambda(U, \mathcal{O}) \otimes \overline{\mathbb{F}}_l = S(U, M_\lambda) \otimes \overline{\mathbb{F}}_l = S(U, M_\lambda \otimes \overline{\mathbb{F}}_l).$$

Let $T$ be a finite set of finite places of $F^+$ which split in $F$, containing $S_l$ and all the places $v$ which split in $F$ for which $U_v \neq G(\mathcal{O}_{F_v^+})$. We let $\mathcal{T}_\lambda^{T, \text{univ}}$ be the commutative $\mathcal{O}$-polynomial algebra generated by formal variables $T_{\lambda, v}^{(j)}$ for all $1 \leq j \leq n$, $w$ a place of $F$ lying over a place $v$ of $F^+$ which splits in $F$ and is not contained in $T$, together with variables $T_{\lambda, v}^{(j)}$ for all $v \in S_l$ and $j = 1, \ldots, n$. We now fix a uniformiser $\varpi_v$ of $\mathcal{O}_{F_v}$ for each $v \in S_l$. The algebra $\mathcal{T}_\lambda^{T, \text{univ}}$ acts on $S(U, M_\lambda)$ via the following Hecke operators:

- $T_{\lambda, v}^{(j)} := \iota_v^{-1} \left[ \text{GL}_n(\mathcal{O}_{F_v}) \left( \begin{array}{cc} \varpi_v & 1_j \\ 0 & 1_{n-j} \end{array} \right) \right] \text{GL}_n(\mathcal{O}_{F_v})$,

for $w \not\in T$ and $\varpi_v$ a uniformiser in $\mathcal{O}_{F_w}$, and

- $T_{\lambda, v}^{(j)} := \prod_{i=1}^{j} \prod_{\gamma : F_v \to F_{\gamma}} \tau(\varpi_v^{-\lambda_{i,F_v,n-i-1}}) \iota_v^{-1} \left[ \text{GL}_n(\mathcal{O}_{F_v}) \left( \begin{array}{cc} \varpi_v & 1_j \\ 0 & 1_{n-j} \end{array} \right) \right] \text{GL}_n(\mathcal{O}_{F_v})$,

for $v \in S_l$.

Suppose that $\mathfrak{m}$ is a maximal ideal of $\mathcal{T}_\lambda^{T, \text{univ}}$ with residue field $\overline{\mathbb{F}}_l$ such that $S(U, M_\lambda)_\mathfrak{m} \neq 0$. Then as in section 4.3 there is a continuous semisimple representation

$$\overline{\rho}_\mathfrak{m} : G_{F^+} \to G_n(\overline{\mathbb{F}}_l)$$

naturally associated to $\mathfrak{m}$. We have the following definition.

**Definition 6.1.1.** Suppose that $\overline{\rho} : G_F \to \text{GL}_n(\overline{\mathbb{F}}_l)$ is a continuous irreducible representation. Then we say that $\overline{\rho}$ is modular and ordinary of weight $\lambda \in (\mathbb{Z}_p^n, r_{\text{ex}})^{\bar{t}}$ if there is a $T$, $U$ as above and a maximal ideal $\mathfrak{m}$ of $\mathcal{T}_\lambda^{T, \text{univ}}$ with residue field $\overline{\mathbb{F}}_l$ such that

- $S(U, M_\lambda)_\mathfrak{m} \neq 0$,
- $\mathfrak{m}$ does not contain any of the operators $T_{\lambda, v}^{(j)}$, and
\( \bullet \mathcal{P} \cong \tilde{r}_m|_{G_F}. \)

We say that \( \mathcal{P} \) is modular and ordinary if it is modular and ordinary of some weight \( \lambda \in (\mathbb{Z}_+^{n, \text{res}})^{\tilde{l}}. \)

Note in particular that if \( \mathcal{P} \) is ordinary and modular then it is unramified at any place of \( F \) which doesn’t split over \( F^+ \). We have the following theorem.

**Theorem 6.1.2.** Suppose that

\[ \mathcal{P}: G_F \to \text{GL}_n(\bar{F}_l) \]

is an irreducible representation satisfying the following assumptions.

1. The representation \( \mathcal{P} \) is modular and ordinary (so in particular \( \mathcal{P}^c = \mathcal{P}^c \varepsilon^{1-n} \)).
2. The image \( \mathcal{P}(G_F(\zeta)) \) is big.
3. \( \mathcal{P}^{\text{ker ad}} \) does not contain \( F(\zeta_l) \).

Then \( \mathcal{P} \) is modular and ordinary of weight \( \lambda \in (\mathbb{Z}_+^{n, \text{res}})^{\tilde{l}} \) if and only if

- For every place \( v|l \) of \( F^+ \), \( \mathcal{P}|_{G_{F_v}} \) has an ordinary crystalline lift of weight \( \lambda_{\tilde{v}} \).

**Proof.** Suppose firstly that \( \mathcal{P} \) is modular and ordinary of weight \( \lambda \in (\mathbb{Z}_+^{n, \text{res}})^{\tilde{l}} \). Then by definition we see that there is a \( U, T, m \) as above such that \( S_\lambda(U, \mathcal{O})_m \neq 0 \) and \( \mathcal{P}_m \cong \mathcal{P} \). Then \( S_\lambda(U, \mathcal{O})_m \neq 0 \). Choose an isomorphism \( \iota : \bar{\mathcal{O}} \to \mathbb{C} \). We see by Corollaire 5.3 of [Lab09] and Lemma 2.2.5 of [Ger09] that there is a RACSDC representation \( \pi \) of \( \text{GL}_n(\bar{A}_F) \) which is unramified at \( l \), \( \nu \)-ordinary at all \( v|l \) (by Lemma 2.7.6 of [Ger09]) and which satisfies \( r_{l,\nu}(\pi) \cong \mathcal{P} \). Thus \( r_{l,\nu}(\pi) \) is ordinary and crystalline of weight \( \lambda \). The representations \( r_{l,\nu}(\pi)|_{G_{F_v}} \) then provide the required lifts.

For the converse, if the condition holds then by Theorem 6.1.1 \( \mathcal{P} \) has a lift to a representation \( \rho \) which is crystalline and ordinary of weight \( \lambda \) and ordinarily automorphic of level prime to \( l \). Say \( \rho = r_{l,\nu}(\pi) \). The result now follows from Corollaire 5.3 and Théorème 5.4 of [Lab09], the strong multiplicity one theorem for \( \text{GL}_n \) and Lemma 2.2.5 of [Ger09]. \( \square \)

We now show that if \( \mathcal{P} \) is modular and ordinary of weight \( \lambda \), then it is modular of weight \( \pi(\lambda) \) in the sense of generalisations of Serre’s conjecture (cf. [Her09]). This is a straightforward consequence of the elementary calculations underlying Hida theory, as we now explain.

Let \( v_\lambda \) be the rank one \( \mathcal{O} \)-submodule of \( M_\lambda \) on which the usual maximal torus of \( \text{GL}_n \) acts via the highest weight \( \lambda \). Let \( v_{w_0,\lambda} \) be the rank one \( \mathcal{O} \)-submodule of \( M_\lambda \) on which the usual maximal torus of \( \text{GL}_n \) acts via the lowest weight \( w_0,\lambda \).

The irreducible \( \bar{F}_l \)-representations of \( \prod_{v \in S_l} \text{GL}_n(k(\tilde{v})) \) are tensor products of irreducible representations of the \( \text{GL}_n(k(\tilde{v})) \). From the standard classification of the irreducible \( \bar{F}_l \)-representations of \( \text{GL}_n(k(\tilde{v})) \) (see for example the appendix to [Her09]), one sees that:

1. There is an irreducible \( \bar{F}_l \)-representation \( F_\lambda \) of \( \prod_{v \in S_l} \text{GL}_n(k(\tilde{v})) \) for each \( \lambda \in (\mathbb{Z}_+^{n, \text{res}})^{\tilde{l}} \), and every irreducible \( \bar{F}_l \)-representation of \( \prod_{v \in S_l} \text{GL}_n(k(\tilde{v})) \) is equivalent to some \( F_\lambda \).
2. Take \( \lambda \in (\mathbb{Z}_+^{n, \text{res}})^{\tilde{l}} \). Let \( P_\lambda \) be the sub-\( \prod_{v \in S_l} \text{GL}_n(k(\tilde{v})) \)-representation of \( M_\lambda \otimes \bar{F}_l \) generated by \( v_\lambda \otimes \bar{F}_l \). Then \( P_\lambda \cong F_{\pi(\lambda)} \) (see II.8.8(1) of [Jan03]).
Let $T_l$ be the Hecke operator $\prod_{v \mid l} T_{v, l}^{(j)}$. Then $T_l$ acts by 0 on $S(U, (M_\lambda \otimes \overline{F}_l)/P_\lambda)$.

**Proof.** We employ the notation of section 2 of [Ger09]; specifically, let $\alpha$ be the element of $G(\mathbb{A}_\mathbb{F}_l)$ defined in the proof of Lemma 2.5.2 of [Ger09]. Then $(w_0 \lambda)(\alpha)^{-1} \alpha$ acts by an $l$-adic unit on $v_{w_0 \lambda}$, but by an eigenvalue with positive $l$-adic valuation on every other weight space in $M_\lambda$. The result follows immediately from the fact that $P_\lambda$ contains $v_{w_0 \lambda} \otimes \overline{F}_l$.

Note that if $U, T, \lambda$ are as above and $m$ is a maximal ideal of $\mathbb{F}_l$ with residue field $\overline{F}_l$ such that $S(U, P_\lambda)_m \neq 0$, then $S(U, M_\lambda)_m \neq 0$, and we have a Galois representation $\overline{r}_m$ as before.

**Corollary 6.1.4.** $\overline{r}$ is modular and ordinary of weight $\lambda$ if and only if there is a $U, T$ as above and a maximal ideal $m$ of $\mathbb{F}_l$ with residue field $\overline{F}_l$ such that

- $S(U, P_\lambda)_m \neq 0$,
- $m$ does not contain any of the operators $T_{\lambda, v}^{(j)}$, and
- $\overline{r} \cong \overline{r}|_{G_F}$.

**Proof.** This is an immediate consequence of the definitions and of Lemma 6.1.3.

Fix now an element $\mu \in (\mathbb{Z}_l^n)^\mathbb{F}_l$. Fix $\lambda \in (\mathbb{Z}_l^n)^{\mathbb{F}_l}$ with $\pi(\lambda) = \mu$. Then there is an equivalence $P_\lambda \cong F_\mu$, so that $T_{\lambda, \text{univ}}$ acts on $S(U, F_\mu)$. Suppose that $\lambda' \in (\mathbb{Z}_l^n)^{\mathbb{F}_l}$ with $\pi(\lambda') = \mu$. Then we also have an action of $T_{\lambda', \text{univ}}$ on $S(U, F_\mu)$, and it is easy to check that under the natural isomorphism between $T_{\lambda, \text{univ}}$ and $T_{\lambda', \text{univ}}$ the Hecke operators at places not dividing $l$ act identically on $S(U, F_\mu)$, while those at places dividing $l$ differ only by units (this is just a statement about the rescaling in the definition of the Hecke operators at places dividing $l$). We thus have the following result/definition.

**Lemma 6.1.5.** Suppose that $\overline{r} : G_F \to \text{GL}_n(\overline{F}_l)$ is a continuous irreducible representation. Then we say that $\overline{r}$ is modular and ordinary of weight $\mu \in (\mathbb{Z}_l^n)^{\mathbb{F}_l}$ if there is a $U, T$ as above, and for some (equivalently, any) $\lambda \in (\mathbb{Z}_l^n)^{\mathbb{F}_l}$ with $\pi(\lambda) = \mu$ there is a maximal ideal $m$ of $\mathbb{F}_l$ with residue field $\overline{F}_l$ such that

- $S(U, F_\mu)_m \neq 0$,
- $m$ does not contain any of the operators $T_{\lambda, v}^{(j)}$, and
- $\overline{r} \cong \overline{r}|_{G_F}$.

We can then reinterpret Theorem 6.1.2.

**Theorem 6.1.6.** Suppose that $\overline{r} : G_F \to \text{GL}_n(\overline{F}_l)$ is an irreducible representation satisfying the following assumptions.

1. The representation $\overline{r}$ is ordinary and modular (so in particular $\overline{r} = \overline{r}^\vee \cdot 1^n$).
2. The image $\overline{r}(G_{F(\zeta)})$ is big.
3. $\overline{r}_{\text{ker ad}} \overline{r}$ does not contain $F(\zeta)$. 

Then $\overline{\rho}$ is modular and ordinary of weight $\mu \in (\mathbb{Z}_p^n)^{T_i}$ if and only if for some (equivalently, any) $\lambda \in (\mathbb{Z}_{p,r,c}^{\gamma})^{T_i}$ with $\pi(\lambda) = \mu$, the following condition holds.

- For every place $v|l$ of $F^+$, $\overline{\rho}_{G_{E_v}}$ has an ordinary crystalline lift of weight $\lambda_{\overline{v}}$.

Proof. This follows at once from Theorem 6.1.2 Lemma 6.1.3 and Lemma 6.1.5 □

7. $GSp_4$

7.1. Definitions. We define $GSp_4$ to be the reductive group over $\mathbb{Z}$ defined as a subgroup of $GL_4$ by

$$GSp_4(R) = \{g \in GL_4(R) : gJ^4g = \mu(g)J\}$$

where $\mu(g)$ is the similitude factor (which is uniquely determined by $g$), and $J$ is the antisymmetric matrix

$$\begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}$$

where $X$ is the $2 \times 2$ antisymmetric matrix with all entries on the antidiagonal equal to 1. Note that the map $\mu : g \mapsto \mu(g)$ gives a homomorphism $GSp_4 \to G_m$.

Lemma 7.1.1. Let $\Gamma$ be a profinite group, and $S \subset R$ be complete local Noetherian rings with $m_R \cap S = m_S$ and common residue field $k$ of characteristic $> 2$. Let $\rho : \Gamma \to GSp_4(R)$ be a continuous representation. Suppose that $\rho$ mod $m_R$ is absolutely irreducible and that $tr\rho(\Gamma) \subset S$. Then there is a $ker(GSp_4(R) \to GSp_4(k))$-conjugate of $\rho$ whose image is contained in $GSp_4(S)$.

Proof. By Lemma 2.1.10 of [CHT08], we see that $\rho$ is $ker(GL_4(R) \to GL_4(k))$-conjugate to a representation $\rho'$ valued in $GL_4(S)$. Now, $(\mu \circ \rho)^2 = det \rho = det \rho'$ is valued in $S$, which by Hensel’s lemma means that $\mu \circ \rho$ is valued in $S$. Thus $t_1(\rho')^{-1}(\mu \circ \rho)$ is also valued in $GL_4(S)$. Because $\rho'$ and $t_1(\rho')^{-1}(\mu \circ \rho)$ are conjugate in $GL_4(R)$ they are also conjugate in $GL_4(S)$, by Théorème 1 of [Car94]. Suppose that $\rho' = Bt_1(\rho')^{-1}(\mu \circ \rho)B^{-1}$. The matrix $B$ is antisymmetric (because $\rho$ is symplectic). By choosing a symplectic basis for the symplectic form determined by $B$, we see that $\rho$ is $GL_4(R)$-conjugate to a representation valued in $GSp_4(S)$, and it is easy to check that one may choose the symplectic basis so that the conjugating matrix is in $ker(GL_4(R) \to GL_4(k))$. It remains to check that the conjugating matrix is also in $GSp_4(R)$; but this is an immediate consequence of Schur’s lemma. □

7.2. Symplectic lifting rings (local case). Fix as before a finite field $k$ of characteristic $l > 2$, and a finite totally ramified extension $K$ of $W(k)[1/l]$ with ring of integers $O$. Let the maximal ideal of $O$ be $m_K = (\pi_K)$. Let $M$ be a finite extension of $\mathbb{Q}_p$ for some prime $p$, possibly equal to $l$. In the case where $p = l$, we assume that $K$ contains the image of every embedding of $M$ into $\overline{K}$. Let

$$\overline{\rho} : G_M \to GSp_4(k)$$

be a continuous representation. Since $GSp_4$ is an algebraic subgroup of $GL_4$, we can also view it as a representation to $GL_4(k)$. Then there is a universal $O$-lifting

$$\rho : G_M \to GL_4(R_\square)$$,
and it is immediate that there is a quotient $R_{\square, sympl}^{\square}$ of $R_{\square}$ and a universal symplectic lifting

$$\rho_{\square, sympl} : G_M \to \mathrm{GSp}_4(R_{\square, sympl}^{\square}).$$

We will need to study certain refined lifting problems. Suppose that $p = l$. Let $\lambda$ be an element of $(\mathbb{Z}_4^\times)^{\text{Hom}(M,K)}$ and let $v_{\lambda}$ be the associated $l$-adic Hodge type (see section 3.1.2). Corollary 2.7.7 of [Kis08] shows that there is a unique $\text{GL}_H$-Hodge type if and only if the same is true of the composite homomorphism to the matrix representation of $\beta$. Let $\rho$ be an element of $(\mathbb{Z}_4^\times)$, and let $\alpha$ be the associated $l$-adic Hodge type $v_{\lambda}$ (where as usual we define a homomorphism $G_M \to \text{GSp}_4(B)$ to be crystalline of a particular Hodge type if and only if the same is true of the composite homomorphism to $\text{GL}_4(B)$).

The following discussion will be useful below. Let $E$ be a finite extension of $K$ and let $C_E$ be the category of finite, local $E$-algebras with residue field $E$. If $B$ is an object of $C_E$, a symplectic $B$-module is a pair $(V_B, \alpha_B)$ where $V_B$ is a free $B$-module of rank 4 with a continuous action of $G_M$ and $\alpha_B$ is a symplectic pairing $V_B \times V_B \to B$ satisfying

$$\alpha_B(gx, gy) = \psi_B(g)\alpha_B(x, y)$$

for all $x, y \in V_B$ and $g \in G_M$, for some continuous character $\psi_B : G_M \to B^\times$. A symplectic basis of such a pair $(V_B, \alpha_B)$ is a basis $\beta_B = \{e_1, e_2, e_3, e_4\}$ of $V_B$ where the matrix $(\alpha_B(e_i, e_j))$ equals $\lambda J$ for some $\lambda \in B^\times$. Two symplectic $B$-modules $(V_B, \alpha_B)$ and $(V_B', \alpha_B')$ are isomorphic if $\psi_B = \psi_B'$ and there is an isomorphism of $B[G_M]$-modules $V_B \cong V_B'$ under which $\alpha_B'$ pulls back to $\alpha_B$.

Fix a symplectic $E$-module $(V_E, \alpha_E)$ together with a symplectic basis $\beta_E$. A deformation of $(V_E, \alpha_E)$ to an object $B$ of $C_E$ is a triple $(V_B, \alpha_B, \iota_B)$ where $(V_B, \alpha_B)$ is a symplectic $B$-module and $\iota_B$ is an isomorphism $(V_B \otimes_B B/\mathfrak{m}_B, \alpha_B \otimes_B B/\mathfrak{m}_B) \cong (V_E, \alpha_E)$ of symplectic $E$-modules. A framed deformation of $(V_E, \alpha_E, \beta_E)$ is a deformation $(V_B, \alpha_B, \iota_B)$ together with a symplectic basis $\beta_B$ of $(V_B, \alpha_B)$ reducing to $\beta_E$ under $\iota_B$. Let $\rho_E : G_M \to \text{GSp}_4(E)$ be the matrix of $V_E$ with respect to $\beta_E$. For an object $B$ of $C_E$ there is a natural bijection between the set of isomorphism classes of framed deformations of $(V_E, \alpha_E, \beta_E)$ to $B$ and the set of lifts $\rho_B : G_M \to \text{GSp}_4(B)$: the class of a framed deformation $(V_B, \alpha_B, \beta_B)$ corresponds to the matrix representation of $\rho_B$ with respect to the basis $\beta_B$. Similarly, there is a natural bijection between the set of isomorphism classes of deformations of $(V_B, \alpha_B)$ to $B$ and the set of deformations of $\rho_E$ to $B$, that is, $\ker(\text{GSp}_4(B) \to \text{GSp}_4(E))$-conjugacy classes of lifts $\rho_B : G_M \to \text{GSp}_4(B)$ of $\rho_E$: the class of a deformation $(V_B, \alpha_B)$ corresponds to the conjugacy class of the matrix representation of $\rho_B$ with respect to any symplectic basis $\beta_B$ lifting $\beta_E$.

Suppose that $(V_B, \alpha_B)$ is a crystalline symplectic $B$-module and let $D_B := D_{\text{cris}}(V_B) = (V_B \otimes_{\mathbb{Q}_l} B_{\text{cris}})_{G_M}$ be the associated weakly admissible filtered $\varphi$-module. Let $D_{\psi_B} = D_{\text{cris}}(\psi_B)$. There is an associated alternating pairing

$$D(\alpha_B) : D_B \times D_B \to D(\varphi_B)$$

which is a map of filtered $\varphi$-modules and is non-degenerate in the sense that it induces an isomorphism $D_B \to \text{Hom}(D_B, D(\varphi_B))$. This pairing is defined by taking the $B_{\text{cris}}$-linear extension of $\alpha_B$ to $V_B \otimes_{\mathbb{Q}_l} B_{\text{cris}}$ and then taking $G_M$-invariants. Suppose in addition that $V_B$ has $l$-adic Hodge type $v_{\lambda}$. Let $\tau : M \hookrightarrow K$ be
an embedding and let $D_{B,\tau} = (D_B \otimes_{M_0} M) \otimes_{B \otimes_{M,1} \otimes T} B$ and $D_{\psi_B,\tau} = (D_{\psi_B} \otimes_{M_0} M) \otimes_{B \otimes_{M,1} \otimes T} B$. Then $D(\alpha_B)$ induces a symplectic pairing $D_{B,\tau} \times D_{B,\tau} \to D_{\psi_B,\tau}$. For $j = 1, \ldots, 4$, let $i_j = \lambda_{\tau,j} + (4 - j)$ be the Hodge-Tate weights of $B$ with respect to $\tau$. Let $i_\psi$ be the Hodge-Tate weight of $\psi_B$ with respect to $\tau$. Then $i_\psi = i_1 + i_4 = i_2 + i_3$ since $V_B \cong \text{Hom}_B(V_B, \psi_B)$. Let $\text{Fil}^i$ be the filtration on $D_{B,\tau}$. In order for $D(\alpha_B)$ to respect filtrations and to be non-degenerate we must have $D(\alpha_B)(\text{Fil}^i, \text{Fil}^j) = \{0\}$, $D(\alpha_B)(\text{Fil}^1, \text{Fil}^2) = D_{\psi_B,\tau}$ and $D(\alpha_B)(\text{Fil}^1, \text{Fil}^2) = D_{\psi_B,\tau}$. In other words, we can find a symplectic basis $e_1, e_2, e_3, e_4$ for $D_{B,\tau}$ such that $\text{Fil}^v = B e_1 + \ldots + B e_4$ for $j = 1, \ldots, 4$.

We define a symplectic filtered $\varphi$-module over an object $B$ in $C_E$ to be a pair $(D_B, D(\alpha_B))$ consisting of a weakly admissible rank 4 filtered $\varphi$-module $D_B$ over $B \otimes_{Q} M_0$ and an alternating, non-degenerate morphism of filtered $\varphi$-modules

$$D(\alpha_B) : D_B \times D_B \to D_{\psi_B}$$

where $D_{\psi_B}$ is a weakly admissible rank 1 filtered $\varphi$-module over $B \otimes_{Q} M_0$. There is an obvious notion of isomorphism between symplectic filtered $\varphi$-modules and also an obvious notion of a deformation of a symplectic filtered $\varphi$-module over $E$ to an object $B$ of $C_E$. The functors $D_{\text{cris}}$ and $V_{\text{cris}}$ are quasi-inverse equivalences of categories between the category of crystalline symplectic $B$-modules and the category of symplectic filtered $\varphi$-modules over $B$ (all morphisms in these categories are isomorphisms).

Suppose now that $M$ is a finite extension of $Q_p$, $p \neq l$. Then it is easy to check (for example by considering the Weil-Designe representation corresponding to the universal lifting) that the inertial type at a closed point of the generic fibre is an invariant of the irreducible components of $R_{\varphi}^{\square, \text{sympl}}[1/l]$. Thus for any 4-dimensional inertial type $\tau$ of $I_M$ which is defined over $K$, there is a unique reduced $l$-torsion-free quotient $R_{\varphi}^{\square, \text{sympl}} \cdot \tau$ of $R_{\varphi}^{\square, \text{sympl}}$, corresponding to a union of irreducible components of $R_{\varphi}^{\square, \text{sympl}}[1/l]$, with the property that for any finite extension $L$ of $K$, a homomorphism of $O$-algebras $R_{\varphi}^{\square, \text{sympl}} \to L$ factors through $R_{\varphi}^{\square, \text{sympl}} \cdot \tau$ if and only if the corresponding lifting of $\overline{\varphi}$ (considered as a representation to $\text{GL}_4(L)$) has type $\tau$.

In applications we will fix the similitude character of our deformations. To this end, fix a character $\psi : G_M \to O_{\psi}$ lifting the character $\mu \circ \overline{\varphi}$ which is crystalline if $p = l$. Let $R_{\varphi}^{\square, \text{sympl}, \psi}$ denote the universal lifting ring for lifts with similitude factor $\psi$. Similarly, we let $R_{\varphi}^{\square, \text{sympl}, \tau, \psi}$, $R_{\varphi}^{\square, \text{sympl}, \lambda, \text{cr}, \psi}$ denote the corresponding quotients of $R_{\varphi}^{\square, \text{sympl}, \psi}$.

Let $\text{ad} \overline{\varphi}$ denote the Lie algebra of $GSp_4$ over $k$, and $\text{ad}^0 \overline{\varphi}$ the Lie algebra of $Sp_4$. These have a natural action of $G_M$ via $\overline{\varphi}$ and the adjoint action of $GSp_4(k)$, and are respectively 11-dimensional and 10-dimensional $k$-vector spaces.

We have the following result on the dimensions of these local lifting rings.

**Proposition 7.2.1.** Let $M$ be a finite extension of $Q_p$. If $p \neq l$, and $\tau$ is such that the ring $R_{\varphi}^{\square, \text{sympl}, \tau, \psi}$ is non-zero, then any irreducible component of $R_{\varphi}^{\square, \text{sympl}, \tau, \psi}$ has dimension at least 11. If $p = l$ and $\psi_\lambda$ is such that $R_{\varphi}^{\square, \text{sympl}, \lambda, \text{cr}, \psi}$ is non-zero, then this ring is equidimensional of dimension $11 + 4[M : Q_l]$.

**Proof.** Firstly, suppose $p = l$ and let $X = \text{Spec} R_{\varphi}^{\square, \text{sympl}, \lambda, \text{cr}, \psi}$. Let $x$ be a closed point of $X[1/l]$ with residue field $E$. It suffices to show that the completed local
ring $O_{X,x}^\wedge$ is formally smooth over $E$ of dimension $10 + 4[M : \mathbb{Q}_l]$. We first establish formal smoothness. Let $\rho_E : G_M \to \text{GSp}_4(E)$ be the representation associated to $x$. Let $B$ denote a finite local $E$-algebra with residue field $E$ and let $I$ be an ideal of $B$ with $m_B I = \{0\}$. Let $\zeta : R_{\text{sympl},v,cr}^\wedge \to B/I$ be an $O$-algebra homomorphism corresponding to a crystalline lift $\rho_{B/I} : G_M \to \text{GSp}_4(B/I)$ of $\rho_E$. We need to show that we can lift $\zeta$ to $B$, or equivalently, that we can find a crystalline lift $G_M \to \text{GSp}_4(B)$ of $\rho_{B/I}$ with similitude character $\psi$.

Let $V_{B/I} = (B/I)^4$ regarded as $G_M$-module via $\rho_{B/I}$ and let $\alpha_{B/I} : V_{B/I} \times V_{B/I} \to (B/I)(\psi)$ be the symplectic pairing associated to the matrix $J$ (that is, $\alpha_{B/I}(x, y) = \langle x, Jy \rangle$ where $x$ and $y$ are regarded as column vectors). Let $(D_{B/I}, D(\alpha_{B/I}))$ be the symplectic, filtered $\varphi$-module over $B/I$ associated to $(V_{B/I}, \alpha_{B/I})$. To construct the required lift of $\rho_{B/I}$, it suffices (by applying $V_{\text{cris}}$) to construct a symplectic filtered $\varphi$-module $(D_B, D(\alpha_B))$ over $B$ (with $D(\alpha_B)$ valued in $B \otimes E D_{\text{cris}}(\psi)$) lifting $(D_{B/I}, D(\alpha_{B/I}))$.

Let $\bar{b}$ be an $E \otimes_{\mathbb{Q}_l} M_0$-generator of $D_\psi := D_{\text{cris}}(\psi)$. Choose a $(B/I) \otimes_{\mathbb{Q}_l} M_0$-basis $e_1, e_2, e_3, e_4$ for $D_{B/I}$ so that the matrix $(D(\alpha_{B/I})(e_i, e_j))$ is $(1 \otimes b) J \in M_{4 \times 4}(B/I) \otimes_{\mathbb{Q}_l} D_\psi$. The matrix $M_\varphi$ of $\varphi$ with respect to this basis is an element of $\text{GSp}_4((B/I) \otimes_{\mathbb{Q}_l} M_0)$ with similitude factor $(\varphi(b)/b) \in (E \otimes_{\mathbb{Q}_l} M_0)^\times \subset ((B/I) \otimes_{\mathbb{Q}_l} M_0)^\times$. Let $M_\varphi$ be a lifting of this matrix to an element of $\text{GSp}_4(B \otimes_{\mathbb{Q}_l} M_0)$ with the same similitude factor. Let $D_B$ be the free $B \otimes_{\mathbb{Q}_l} M_0$-module on generators $\bar{e}_1, e_2, e_3, e_4$. Endow it with the symplectic form $D(\alpha_B) : D_B \times D_B \to B \otimes E D_\psi$ defined by $(D(\alpha_B)(\bar{e}_i, \bar{e}_j)) = (1 \otimes b) J$. Let $\tilde{\varphi}$ be the $\varphi_0$-semilinear automorphism of $D_B$ whose matrix with respect to the basis $\bar{e}_i$ is $\tilde{M}_\varphi$. Now choose a filtration on $D_B \otimes_{M_0} M$ lifting the filtration on $D_{B/I} \otimes_{M_0} M$ and such that $D(\alpha_B)$ becomes a map of filtered $\varphi$-modules. Then $D_B$ becomes a weakly admissible filtered $\varphi$-module and we have shown that $O_{X,x}^\wedge$ is formally smooth over $E$.

We now determine the relative dimension $d$ of $O_{X,x}^\wedge$ over $E$. Let $\mathfrak{g}$ denote the Lie algebra of $\text{GSp}_4(E)$ and $\mathfrak{g}^\circ$ the Lie algebra of $\text{Sp}_4(E)$. Let $D_{\rho_E}(E[\varepsilon])$ (resp. $D_{\rho_E}(E[\varepsilon])$) denote the set of crystalline lifts (resp. deformations) $G_M \to \text{GSp}_4(E[\varepsilon])$ of $\rho_E$ with similitude character $\psi$. These sets are naturally $E$-vector spaces. Since the natural map $D_{\rho_E}(E[\varepsilon]) \to D_{\rho_E}(E[\varepsilon])$ is a $\mathfrak{g}^\circ/(\mathfrak{g}^\circ)^{G_M}$-torsor, we have

$$d = \dim_E D_{\rho_E}(E[\varepsilon]) = \dim_E \left( \mathfrak{g}^\circ/(\mathfrak{g}^\circ)^{G_M} \right) + \dim_E D_{\rho_E}(E[\varepsilon]).$$

Let $D_{\rho_E}(E[\varepsilon])$ denote the set of equivalence classes of deformations $(D, D(\alpha))$ to $E[\varepsilon]$ of the symplectic filtered $\varphi$-module $(D_E, D(\alpha_E))$ where the pairing $D(\alpha)$ takes values in $E[\varepsilon] \otimes E D_\psi$. By the discussion preceding the proposition, we see that there is a natural bijection between $D_{\rho_E}(E[\varepsilon])$ and $D_{\rho_E}(E[\varepsilon])$.

Choose any deformation $(D', D(\alpha'))$ in $D_{\rho_E}(E[\varepsilon])$. Given any other such deformation $(D', D(\alpha'))$, we can choose an isomorphism of $E[\varepsilon] \otimes_{\mathbb{Q}_l} M_0$-modules $j : D' \to D$ taking $D(\alpha)$ to $D(\alpha)'$. Let $\varphi$ denote the $\varphi$-operator on $D$ and Fil the filtration on $D \otimes_{M_0} M$. Let $\varphi'$ denote the operator on $D$ corresponding under $j$ to the $\varphi$-operator on $D'$. Similarly, let $\text{Fil}'$ denote the filtration on $D \otimes_{M_0} M$ corresponding under $j$ to the filtration on $D' \otimes_{M_0} M$. Choose an isomorphism of $E[\varepsilon] \otimes_{\mathbb{Q}_l} M_0$ modules between $D$ and $D_0 \otimes E E[\varepsilon]$ which identifies $D(\alpha)$ with $D(\alpha_E) \otimes 1$. Let $\mathfrak{g}_{D,\alpha}$ denote the Lie algebra of $\text{Sp}(D_E, D(\alpha_E))$ and $\text{Sp}(D_E, D(\alpha_E))$ respectively. Similarly, let $\mathfrak{g}_{D,\alpha,E,M}$ denote the Lie algebra of $\text{GSp}(D_E \otimes_{M_0} M, \alpha_E \otimes 1)$. Then there exists $X \in \mathfrak{g}_{D,E}$ and $Y \in \mathfrak{g}_{D,E}$
such that $\varphi' = (1 + \varepsilon X)\varphi$ and $\text{Fil}' = (1 + \varepsilon Y)\text{Fil}$. Moreover, any such pair $X, Y$ gives rise to a deformation of $(D_E, D(\alpha_E))$ and we get a surjective linear map

$$g D_E \oplus g D_{E,M}/b D_{E,M} \twoheadrightarrow D_E(E[\varepsilon]).$$

The kernel of this map is the image of the map

$$g D_E \twoheadrightarrow g D_E \oplus g D_{E,M}/b D_{E,M}$$

sending $Z$ to the pair $(Z - \varphi \circ Z \circ \varphi^{-1}, Z)$. Denote the kernel of this last map by $(g D_E)^{\varphi = 1, \text{Fil}}$. We have shown that

$$d = \dim_E \left( \frac{(g^o)^{G_M}}{(g^o)^{G_M}} \right) + \dim_E g D_{E,M}/b D_{E,M} + \dim_E (g D_E)^{\varphi = 1, \text{Fil}}.$$

The result now follows from the fact that $\dim_E g^o = 10$, $\dim_E g D_{E,M}/b D_{E,M} = 4[M : \mathbb{Q}]$ and $(g^o)^{G_M} \cong (g D_E)^{\varphi = 1, \text{Fil}}$ via $D_{\text{cris}}$.

Now suppose that $p \neq l$. In this case we only need to establish a lower bound on the dimension, and we do this by means of a slight variant of Mazur’s lower bound for the dimension of an unrestricted deformation ring (see Proposition 2 of [Marz99]). Note that by the construction of the ring $R^{\text{sympl,}\tau,\psi}$, we need only show that each irreducible component of $R^{\square,\text{sympl,}\psi}$ has dimension at least 11.

Let $m^{\text{sympl}}$ denote the maximal ideal of $R^{\square,\text{sympl,}\psi}$. Then $R^{\square,\text{sympl,}\psi}$ is the quotient of a power series ring over $\mathcal{O}$ in $\dim_k m^{\text{sympl}}/((m^{\text{sympl}})^2, \pi K)$ variables. The argument of the proof of Lemma 4.1.1 of [Kis07a] shows that it is necessary to quotient out by at most $\dim_k H^2(G_M, \text{ad}^0 \overline{\rho})$ relations. Thus every component of $R^{\square,\text{sympl,}\psi}$ has dimension at least

$$1 + \dim_k m^{\text{sympl}}/((m^{\text{sympl}})^2, \pi K) - \dim_k H^2(G_M, \text{ad}^0 \overline{\rho}).$$

Now, $m^{\text{sympl}}/((m^{\text{sympl}})^2, \pi K)$ is dual to the tangent space

$$D^{\square,\text{sympl}}(k[\varepsilon]/(\varepsilon^2)),$$

where $D^{\square,\text{sympl}}$ is the functor represented by $R^{\square,\text{sympl,}\psi}$. The elements of this space are 1-cocycles in $Z^1(G_M, \text{ad}^0 \overline{\rho})$, so we see that

$$\dim_k m^{\text{sympl}}/((m^{\text{sympl}})^2, \pi K) = \dim_k Z^1(G_M, \text{ad}^0 \overline{\rho}) = \dim_k H^1(G_M, \text{ad}^0 \overline{\rho}) + \dim_k \text{ad}^0 \overline{\rho} - \dim_k H^0(G_M, \text{ad}^0 \overline{\rho}).$$

Thus every component of $R^{\square,\text{sympl,}\psi}$ has dimension at least

$$1 + \dim_k H^1(G_M, \text{ad}^0 \overline{\rho}) + \dim_k \text{ad}^0 \overline{\rho} - \dim_k H^0(G_M, \text{ad}^0 \overline{\rho}) - \dim_k H^2(G_M, \text{ad}^0 \overline{\rho}) = 11$$

by the local Euler characteristic formula, as required. □

Remark 7.2.2. It is presumably possible to use the techniques of [Kis08] to prove that if $p \neq l$, and $\tau$ is such that the ring $R^{\text{sympl,}\tau,\psi}$ is non-zero, then it is equidimensional of dimension 11. As we do not need this result we have not attempted to verify this.

The following lemma can be proved in exactly the same way as Lemma 3.3.3 of [Ger09].
Lemma 7.2.3. Let $M$ be a finite extension of $\mathbb{Q}_l$. There is a quotient $R^\text{sympl,cr,ψ}_\overline{\mathcal{T}}$ of $R^\text{sympl,χ,cr,ψ}_\overline{\mathcal{T}}$ corresponding to a union of irreducible components such that for any finite local $K$-algebra $B$, a homomorphism of $\mathcal{O}$-algebras $\zeta : R^\text{sympl,χ,cr,ψ}_\overline{\mathcal{T}} \to B$ factors through $R^\text{sympl,Δ,cr,ψ}_\overline{\mathcal{T}}$ if and only if $\zeta \circ \overline{\rho}$ is ordinary of weight $\lambda$.

7.3. A lower bound on the dimension of a symplectic deformation ring.
In this section we outline a proof of a lower bound on the dimension of a global deformation ring. This material is by now rather standard (see for example section 4 of [Kis07a] or Corollary 2.3.5 of [CHT08]), and we content ourselves with a sketch of the proofs.

Suppose that $F^+$ is a totally real field. Let $\overline{\mathcal{T}} : G_{F^+} \to \text{GSp}_4(k)$ be absolutely irreducible. Let $S$ be a finite set of finite places of $F^+$, including all places at which $\overline{\mathcal{T}}$ is ramified, and all places dividing $l$. Let $F^+(S)$ be the maximal extension of $F^+$ unramified outside of $S$, and write $G_{F^+(S)}/F^+$ for the Galois group $\text{Gal}(F^+(S)/F^+)$ (note that we do not use the more usual notation $G_{F^+,v}$ for this group, as to do so would be inconsistent with [CHT08] and the earlier sections of this paper). Fix a character $\psi : G_{F^+(S)}/F^+ \to \mathcal{O}^\times$ lifting the character $\mu \circ \overline{\rho}$. If $R$ is a complete local Noetherian $\mathcal{O}$-algebra with residue field $k$, then an $R$-valued deformation of $\overline{\mathcal{T}}$ is a $\ker(G_{\text{Sp}_4}(R) \to \text{GSp}_4(k))$-conjugacy class of liftings of $\overline{\mathcal{T}}$ to $\text{GSp}_4(R)$. Since $\overline{\mathcal{T}}$ is absolutely irreducible, it is an easy consequence of Schur’s lemma that $\overline{\mathcal{T}}$ has a universal symplectic deformation with fixed similitude factor $\psi$ to a complete local Noetherian $\mathcal{O}$-algebra $R^\square_{F^+,S}$ (see for example Theorem 3.3 of [Til96]).

Let $R^\square_{F^+,S}$ denote the complete local Noetherian $\mathcal{O}$-algebra representing the functor $\widehat{\mathcal{T}}_{F^+,S}^\square$ which assigns to a complete local Noetherian $\mathcal{O}$-algebra $R$ with residue field $k$ the set of equivalence classes of tuples $(\rho, \{\alpha_v\}_{v \in S})$ where $\rho$ is a lifting of $\overline{\mathcal{T}}$ to $R$ with similitude character $\psi$ and for each $v \in S$, $\alpha_v \in \ker(G_{\text{Sp}_4}(R) \to \text{GSp}_4(k))$. Two such tuples $(\rho, \{\alpha_v\}_{v \in S})$ and $(\rho', \{\alpha'_v\}_{v \in S})$ are said to be equivalent if there exists an element $\beta \in \ker(G_{\text{Sp}_4}(R) \to \text{GSp}_4(k))$ with $\rho' = \beta \rho \beta^{-1}$ and $\alpha'_v = \beta \alpha_v$ for all $v \in S$. Note that $R^\square_{F^+,S}$ is formally smooth over $R^\square_{F^+,S}$ of relative dimension $11|S| - 1$. For each $v \in S$ let $R^\square_{v,\text{sympl,ψ}}$ denote the universal $\mathcal{O}$-lifting ring for symplectic liftings of $\overline{\mathcal{T}}_{\mathcal{O}_{F^v}}$ with similitude character $\psi$. Let $R^\square_S = \bigotimes_{v \in S} R^\square_{v,\text{sympl,ψ}}$. There is a natural map $R^\square_S \to R^\square_{F^+,S}$ given on $R$-points by sending a tuple $(\rho, \{\alpha_v\}_{v \in S})$ to the tuple $(\alpha_v \circ \rho|_{G_{\text{Sp}_4}}, \alpha_v)_{v \in S}$ (note that this map is well-defined by the definition of equivalence for these tuples).

For $i = 1, 2$ we let $h^i_S$ denote the $k$-dimension of the kernel of the natural map

$$H^i(G_{F^+(S)}/F^+, \text{ad}^0 \overline{\rho}) \to \prod_{v \in S} H^i(G_{F^+,v}, \text{ad}^0 \overline{\rho}).$$

Let $m_{F^+,S}$ denote the maximal ideal of $R^\square_{F^+,S}$, and $m_S$ the maximal ideal of $R^\square_S$.

Proposition 7.3.1. Let

$$\eta : m_S/(m^2_S, \pi_K) \to m_{F^+,S}/(m^2_{F^+,S}, \pi_K)$$

be the natural map. Then $R^\square_{F^+,S}$ is a quotient of a power series ring over $R^\square_S$ in $\dim_k \text{coker} \eta$ variables by at most $\dim_k \ker \eta + h_2^S$ relations.
Proof: This may be proved in exactly the same fashion as Proposition 4.1.4 of [Kis07a]. □

Corollary 7.3.2. Suppose that $H^0(G_{F^+}(S)/F^+, (\text{ad}^0 \overline{\rho})^*(1)) = 0$. Let $s = \sum_{v \mid \infty} \dim_k H^0(G_{F^+, v}, \text{ad}^0 \overline{\rho})$. Then for some non-negative integer $r$ and some $f_1, \ldots, f_{r+s}$, there is an isomorphism

$$R^\square_{F^+, S} \cong R^\square_S[[x_1, \ldots, x_{r+s}]]/(f_1, \ldots, f_{r+s}).$$

Proof. This is very similar to the proof of Proposition 4.1.5 of [Kis07a]. By Proposition 7.3.1 we see that the result will hold with $s$ chosen such that

$$|S| - s - 1 = \dim_k m_{F^+, S}/(m_{F^+, S}^2, \pi_K) - \dim_k m_S/(m_S^2, \pi_K) - h_S^2,$$

so it suffices to show that this agrees with the value of $s$ in the statement of the corollary. Note firstly that $\text{Hom}_k(m_{F^+, S}/(m_{F^+, S}^2, \pi_K), k)$ is naturally isomorphic to $D^{\square, \text{sympl}, \psi}(k[\epsilon]/(\epsilon^2))$. Consideration of the equivalence relation defining $D^{\square, \text{sympl}, \psi}_{F^+, S}$ shows that this space has $k$-dimension

$$11|S| + \dim_k H^1(G_{F^+}(S)/F^+, \text{ad}^0 \overline{\rho}) - \dim_k H^0(G_{F^+}(S)/F^+, \text{ad}^0 \overline{\rho}) - 1.$$

Similarly,

$$\dim_k m_S/(m_S^2, \pi_K) = \sum_{v \in S} \dim \text{ad}^0 \overline{\rho} + \dim_k H^1(G_{F^+}, \text{ad}^0 \overline{\rho}) - \dim_k H^0(G_{F^+}, \text{ad}^0 \overline{\rho})$$

$$= \sum_{v \in S} (10 + \dim_k H^1(G_{F^+}, \text{ad}^0 \overline{\rho}) - \dim_k H^0(G_{F^+}, \text{ad}^0 \overline{\rho})).$$

The condition that $H^0(G_{F^+}(S)/F^+, (\text{ad}^0 \overline{\rho})^*(1)) = 0$, together with the last 3 terms of the Poitou-Tate sequence, shows that the map $\theta^2$ is surjective, so that

$$h_S^2 = \dim_k H^2(G_{F^+}(S)/F^+, \text{ad}^0 \overline{\rho}) - \sum_{v \in S} \dim_k H^2(G_{F^+}, \text{ad}^0 \overline{\rho}).$$

Thus

$$\dim_k m_{F^+, S}/(m_{F^+, S}^2, \pi_K) - \dim_k m_S/(m_S^2, \pi_K) - h_S^2 = |S| + \sum_{v \in S} \chi(G_{F^+}, \text{ad}^0 \overline{\rho}) - \chi(G_{F^+}(S)/F^+, \text{ad}^0 \overline{\rho}) - 1,$$

where $\chi$ denotes the Euler characteristic as a $k$-vector space, and it suffices to show that

$$\sum_{v \in S} \chi(G_{F^+}, \text{ad}^0 \overline{\rho}) - \chi(G_{F^+}(S)/F^+, \text{ad}^0 \overline{\rho}) = \sum_{v \mid \infty} \dim_k H^0(G_{F^+}, \text{ad}^0 \overline{\rho}).$$

This follows at once from the local and global Euler characteristic formulae. □

For each place $v \in S$ not dividing $l$ we fix a type $\tau_v$ such that $\overline{\rho}|_{G_{F_v^+}}$ has a symplectic lifting of type $\tau_v$ and similitude character $\psi|_{G_{F_v^+}}$, and we fix a quotient $R_v$ of $R^{\text{sympl}, \tau_v, \psi}_{v}$ corresponding to a union of irreducible components. For each $v|l$ we fix a weight $\lambda_v$ such that $\overline{\rho}|_{G_{F_v^+}}$ has a crystalline symplectic lift of weight $\lambda_v$ and similitude character $\psi|_{G_{F_v^+}}$, and we fix a quotient $R_v$ of $R^{\text{sympl}, \Delta \lambda_v, cr, \psi}_{v}$ corresponding to a union of irreducible components. Let $R^\psi_S := \otimes_{v \in S} R_v$, and
let $R_{F^+, S}^{\square, \tau, \psi} = R_{F^+, S}^{\square, \text{sympl}, \psi} \otimes_{R_S} R_S^{\psi, \tau}$. Let $R_{F^+}^{\text{sympl}, \tau, \psi}$ be the universal deformation $\mathcal{O}$-algebra representing the functor which assigns to $R$ the $\ker(\text{GSp}_4(R) \to \text{GSp}_4(k))$-conjugacy classes of liftings of $\overline{\rho}$ with the property that for each $v \in S$ the corresponding lifting of $\overline{\rho}_{|G_{F_v^+}}$ gives an $R$-point of $R_v$ (that this functor is well defined follows from the symplectic analogue of Lemma 3.2.3 which can be proved in the same way). Thus $R_{F^+}^{\text{sympl}, \psi, \tau}$ is formally smooth over $R_{F^+}^{\text{sympl}, \psi, \tau}$ of relative dimension $11|S| - 1$.

**Definition 7.3.3.** We say that $\overline{\rho}$ is odd if for all complex conjugations $c \in G_{F^+}$, $(\mu \circ \overline{\rho})(c) = -1$.

**Proposition 7.3.4.** Assume that $\overline{\rho}$ is odd and that $H^0(G_{F^+}, (\text{ad}^0 \overline{\rho})^* (1)) = 0$. Then the Krull dimension of $R_{F^+}^{\text{sympl}, \psi, \tau}$ is at least one.

**Proof.** It suffices to check that the dimension of $R_{F^+}^{\square, \psi, \tau}$ is at least $11|S|$. By Corollary 7.3.2 it would be enough to check that

$$\dim R_{S}^{\psi, \tau} + |S| - 1 - \sum_{v \mid \infty} \dim_k H^0(G_{F_v^+}, \text{ad}^0 \overline{\rho}) \geq 11|S|.$$ 

By Propositions 7.2.1 and 7.2.3

$$\dim R_{S}^{\psi, \tau} \geq 1 + 10|S| + 4[F^+ : \mathbb{Q}].$$

An easy calculation using the fact that $\overline{\rho}$ is odd shows that for each $v \mid \infty$, $\dim_k H^0(G_{F_v^+}, \text{ad}^0 \overline{\rho}) = 4$ (for example, one easily checks that if $c_v$ is a corresponding complex conjugation then $\overline{\rho}(c_v)$ is conjugate to the diagonal matrix $\text{diag}(1, 1, -1, -1)$, and one may then compute explicitly). Thus

$$\dim R_{S}^{\psi, \tau} + |S| - 1 - \sum_{v \mid \infty} \dim_k H^0(G_{F_v^+}, \text{ad}^0 \overline{\rho}) \geq 10|S| + 4[F^+ : \mathbb{Q}] + |S| - 4[F^+ : \mathbb{Q}] = 11|S|,$$

as required. □

7.4. **Relationship to unitary representations.** Let $F$ be a totally imaginary CM field with maximal totally real field $F^+$, with the property that all primes in $S$ split in $F$. Let $\mathcal{S}$ denote a set of places of $F$ consisting of one place dividing each place in $S$. Recall that we let $G_{F^+, S} = \text{Gal}(F(S)/F^+)$. Let $\rho : G_{F^+} \to \text{GSp}_4(R)$ be a continuous representation, with $R$ a complete local Noetherian ring. Then, as in Lemma 2.1.2 of [CH108], there is a continuous homomorphism $r : G_{F^+} \to \mathcal{G}_4(R)$ determined by

$$r(g) = (\rho(g), (\mu \circ \rho)(g))$$

if $g \in G_F$, and

$$r(g) = (\rho(g)J^{-1}, -(\mu \circ \rho)(g))j$$

if $g \notin G_F$. We have

$$\nu \circ r = \mu \circ \rho.$$

Furthermore, this construction is obviously compatible with deformations, in the sense that if $B \in \ker(\text{GSp}_4(R) \to \text{GSp}_4(k))$ and $\rho$ is replaced by $\rho_B$ with

$$\rho_B(g) := B\rho(g)B^{-1},$$

then $r$ is replaced by $r_B$ with

$$r_B(g) := (aB, 1)r(g)(aB, 1)^{-1},$$
where \(a^2 = \mu(B)^{-1}\) (such an \(a\) exists because \(\mu(B) \in 1 + m_B\) and \(l > 2\)). Applying this construction to the universal symplectic deformation of the previous section

\[ \rho^{univ}: G_{F+}(S)/F^+ \to \text{GSp}_4(R_{F+}^{sympl,\psi,\tau}), \]

we obtain a deformation

\[ r^{sympl}: G_{F+}(S)/F^+ \to \mathcal{G}_4(R_{F+}^{sympl,\psi,\tau}). \]

We may also consider the corresponding residual representation

\[ \tilde{r}: G_{F+,S} \to \mathcal{G}_4(k), \]

and (in the notation of sections 2.2 and 2.3 of [CHT08]) the deformation problem

\[ S = (F/F^+, S, \bar{S}, \mathcal{O}, \tilde{r}, \psi, R_u) \]

with corresponding universal deformation

\[ r_S: G_{F+,S} \to \mathcal{G}_n(R_{S}^{univ}). \]

Since \(G_{F+}(S)/F^+\) is a quotient of \(G_{F+,S}\), there is a homomorphism \(\theta: R_{S}^{univ} \to R_{F+}^{sympl,\psi,\tau}\) such that there is an equality of deformations

\[ r^{sympl} = \theta \circ r_S. \]

**Lemma 7.4.1.** \(R_{F+}^{sympl,\psi,\tau}\) is finite over \(R_{S}^{univ}\).

**Proof.** Let \(\rho_{F,F^+}\) denote the \(\text{GSp}_4(R_{F+}^{sympl,\psi,\tau}/\theta(m_{R_{S}^{univ}}))\)-valued representation obtained from \(\rho^{univ}\), and let \(r_{F,F^+}\) denote the corresponding representation to \(\mathcal{G}_4(R_{F+}^{sympl,\psi,\tau}/\theta(m_{R_{S}^{univ}}))\).

Then \(r_{F,F^+}\) is equivalent to \(\tilde{r}\), so it has finite image, and thus the image of \(\rho_{F,F^+}\) is also finite. An argument exactly as in the proof of Lemma 3.2.5 (using Lemma 7.1.1 to see that the universal deformation ring is generated by traces) shows that \(R_{F+}^{sympl,\psi,\tau}/\theta(m_{R_{S}^{univ}})\) is finite, as required. \(\square\)

### 7.5. Companion forms for symplectic Galois representations and automorphic representations for \(\text{GL}_4\)

We now prove our first companion forms theorem for symplectic representations. This theorem applies to automorphic representations of \(\text{GL}_4\); in the next section we will use functoriality to deduce a result for automorphic representations of \(\text{GSp}_4\).

Suppose that \(\pi\) is a RAESDC representation of \(\text{GL}_4(\mathbb{A}_{F^+})\), with \(\pi^\vee \cong \chi_\pi\). Let \(\iota: \mathbb{Q}_l \xrightarrow{\sim} \mathbb{C}\). Then there is a continuous semisimple representation

\[ \rho_{\iota,\lambda}(\pi): G_{F^+} \to \text{GL}_4(\mathbb{Q}_l) \]

associated to \(\pi\) (see theorem 1.1 of [BLGHT09]). We say that a representation \(\rho: G_{F^+} \to \text{GL}_4(\mathbb{Q}_l)\) is automorphic if \(\rho \cong \rho_{\iota,\lambda}(\pi)\) for some \(\iota, \lambda, \pi\).

The representation \(\overline{p}_{\iota,\lambda}(\pi)\) may be conjugated to be valued in the ring of integers of a finite extension of \(\mathbb{Q}_l\), and we may reduce it modulo the maximal ideal of this ring of integers and semisimplify to obtain a well-defined continuous representation

\[ \overline{p}_{\iota,\lambda}(\pi): G_{F^+} \to \text{GL}_4(\mathbb{F}_l). \]

We say that a representation \(\overline{p}: G_{F^+} \to \text{GL}_4(\mathbb{F}_l)\) is automorphic if \(\overline{p} \cong \overline{p}_{\iota,\lambda}(\pi)\) for some \(\iota, \lambda, \pi\). We say that \(\overline{p}\) is symplectic ordinarily automorphic if \(\overline{p} \cong \overline{p}_{\iota,\lambda}(\pi)\), where \(\pi\) is \(\iota\)-ordinary and \(\rho_{\iota,\lambda}(\pi)\) is symplectic. We say that \(\overline{p}\) is symplectic ordinarily automorphic of level prime to \(l\) if furthermore \(\pi\) may be taken to be unramified at all places dividing \(l\).
Corollary 7.5.1. Assume that \( \overline{\rho} \) is symplectic ordinarily automorphic, that \( \overline{\rho}(G_{F^+(\zeta)}) \) is big, and that \((\overline{F^+})^{\ker \text{ad}}\overline{\rho}\) does not contain \( F^+(\zeta) \). Then \( R^{\text{sympl},\psi,\tau}_{F^+,S} \) is a finite \( \mathcal{O} \)-module of rank at least one.

Proof. This follows from Lemma 7.2.4, Corollary 5.1.3 and Proposition 7.3.4 (note we are free to choose \( F \) linearly disjoint from \((\overline{F^+})^{\ker \text{ad}}\overline{\rho}(\zeta)\) over \( F^+ \)). \( \square \)

Suppose that \( \rho : G_{F^+} \to \text{GSp}_4(\mathbb{Q}_l) \) is crystalline. Then the similitude factor \( \psi \) of \( \rho \) is a crystalline character of \( G_{F^+} \), so there is an integer \( n \) such that for all places \( v \mid l \), \( \psi|_{F_v^+} = e^n \). Suppose now that \( \rho' : G_{F^+} \to \text{GSp}_4(\mathbb{Q}_l) \) is another crystalline representation with similitude factor \( \psi' \), and that \( \overline{\rho} = \overline{\rho}' \). Then \( \overline{\psi} = \overline{\psi}' \), and there is an integer \( n' \) such that for all places \( v \mid l \), \( \psi|_{F_v^+} = e^{n'} \). Thus \( e^{n'-n} \) is a crystalline character of \( G_{F^+} \) whose reduction mod \( l \) is everywhere unramified. This motivates the choice of similitude factor in the following theorem (in particular, it shows that our choice of similitude factor does not exclude any possibilities for the Hodge-Tate weights of the Galois-representations we construct).

Theorem 7.5.2. Let \( F^+ \) be a totally real field. Let \( l \geq 5 \) be a prime number such that \( [F^+(\zeta) : F^+] > 2 \). Suppose that \( \overline{\rho} : G_{F^+} \to \text{GSp}_4(\mathbb{F}_l) \) is an irreducible representation, and let \( n \) be an integer such that \( \overline{\rho}^\sigma \) is an unramified character of \( G_{F^+} \). Suppose that \( \overline{\rho} \) satisfies the following assumptions.

1. There are finite fields \( \mathbb{F}_l \subset k \subset k' \) such that \( \text{Sp}_4(k) \subset \overline{\rho}(G_{F^+}) \subset (k')^\times \text{GSp}_4(k) \).
2. The representation \( \overline{\rho} \) is symplectic ordinarily automorphic of level prime to \( l \); say \( \overline{\rho} \cong \overline{\rho}_{l,1}(\pi) \), and write \( \psi \) for the similitude factor of \( \rho_{l,1}(\pi) \).
3. Define \( \psi_n := \psi e^n \overline{\omega}^{-n} \), where \( \overline{\omega} \) is the Teichmüller lift of the mod \( l \) cyclotomic character (so \( \psi_n = \overline{\psi} \), and \( \psi_n \) is crystalline). There is an element \( \lambda \in (\mathbb{Z}^4)^{\text{Hom}(F^+,\mathbb{Q}_l)} \) such that

- for all \( \tau \in \text{Hom}(F^+,\mathbb{Q}_l) \), we have \( \lambda_{\tau,1} \geq \cdots \geq \lambda_{\tau,4} \),
- for every place \( v \mid l \) of \( F^+ \), \( \overline{\rho}|_{G_{F_v^+}} \) has an ordinary crystalline symplectic lift of weight \( (\lambda_{\tau})_\tau \) (where the indexing set runs over the embeddings \( \tau \in \text{Hom}(F^+,\mathbb{Q}_l) \) inducing \( v \)) and similitude factor \( \psi_n \).

Then \( \overline{\rho} \) has an ordinary crystalline symplectic lift \( \rho \) of weight \( \lambda \) which is ordinarily automorphic of level prime to \( l \).

Given any finite set of places \( S \) of \( F^+ \), and an inertial type \( \tau_v \) for each \( v \in S \) not dividing \( l \), there is a finite set of places \( S' \) of \( F^+ \) dividing \( l \) such that \( \overline{\rho}|_{G_{F_v^+}} \) has a symplectic lift of type \( \tau_v \) and similitude factor \( \psi_n \), \( \rho \) can be chosen to be of similitude factor \( \psi_n \) and of type \( \tau_v \) at \( v \) for all places \( v \in S, v \nmid l \). More precisely, given a choice of a component of each ring \( R^{\text{sympl},\Delta,cr,\psi_n}_{F_v^+} (\mathbb{F}_l) \), \( \rho \) may be chosen so as to give a point on each of these components.

Proof. It suffices to prove the last statement. Enlarge \( S \) if necessary so that \( S \) contains all places of \( F^+ \) dividing \( l \) and all places at which \( \overline{\rho} \) is ramified. Choose a totally imaginary quadratic extension \( F/F^+ \) such that all places in \( S \) split in \( F \), and such that \( F \) is linearly disjoint from \((\overline{F^+})^{\ker \overline{\rho}}\). Note that we can choose a type \( \tau_v \) at any place not dividing \( l \) such that \( \overline{\rho}|_{G_{F_v^+}} \) has a symplectic lift of type \( \tau_v \) and similitude factor \( \psi_n \); \( \rho_{l,1}(\pi)|_{G_{F_v^+}} \) provides such a lift if \( n = 0 \), and we may twist
this lift in the general case (note that $\psi_v\psi_v^{-1}$ is unramified at $v$, and there is no
obstruction to taking a square root of an unramified character). We then consider
deformation problems as in the previous section. By Lemma 2.5.5 of [CHT08],
and the fact that $\text{PSp}_4(k)$ is simple, we see that $\overline{\rho}(G_{F^+}(\zeta_i))$ is big. Again, because
$\text{PSp}_4(k)$ is simple, the abelianisation of $\overline{\rho}(G_{F^+})$ is a subgroup of
$$\text{PGSp}_4(k)/\text{PSp}_4(k) \xrightarrow{\sim} k^\times/(k^\times)^2.$$ 

As this latter group has cardinality 2 and $|F^+(\zeta_i) : F^+| > 2$, we see that $\overline{\rho}^{\ker \text{ad}}\overline{\rho}$
does not contain $F^+(\zeta_i)$. Then Corollary 6.3.1 gives the existence of a Galois rep-
resentation satisfying every property except possibly automorphicity, which follows
from Theorem 4.3.1 and Lemmas 1.4 and 1.5 of [BLGHT09].

7.6. Companion forms for $\text{GSp}_4$. We now prove results for automorphic rep-
resentations for $\text{GSp}_4$ over totally real fields by making use of known cases of
funcioriality between $\text{GSp}_4$ and $\text{GL}_4$. The main result we need is the following.

**Theorem 7.6.1.** Let $M$ be a number field. There is an injective map $\pi \mapsto \Pi \boxtimes \theta$
from the set of globally generic cuspidal representations $\pi$ of $\text{GSp}_4$ over $M$ to the
set of globally generic representations $\Pi \boxtimes \theta$ of $\text{GL}_4 \times \text{GL}_1$ over $M$. This map has
the following properties:

1. $\theta = \omega_\pi$ (the central character of $\pi$), and the central character of $\Pi$ is $\omega_\Pi^2$.
2. $\Pi \cong \Pi^\vee \otimes \omega_\pi$.
3. For each place $v$ of $M$ there is an equality of Weil-Deligne representations
   $\text{rec}(\pi_v) = \text{rec}(\Pi_v)$, where we also denote the local Langlands correspondence
   of $\text{GSp}_4$ by $\text{rec}$, and consider $\text{GSp}_4$ as a subgroup of $\text{GL}_4$.
4. If $\Pi \boxtimes \theta$ is such that $\Pi$ is cuspidal, then $\Pi \boxtimes \theta$ is in the image of the map
   if and only if the partial L-function $L^S(s, \Pi, \Pi^\vee \otimes \theta^{-1})$ has a pole at $s = 1$
   (where $S$ is any finite set of places of $M$).
5. If $\Pi \boxtimes \theta$ is in the image of the map and $\Pi$ is not cuspidal, then $\Pi$ is an
   isobaric direct sum of two cuspidal representations of $\text{GL}_2$.

**Proof.** This is a special case of Theorem 13.1 of [GT07].

**Definition 7.6.2.** Let $F^+$ be a totally real field, and let $\pi$ be a cuspidal auto-
morphic representation of $\text{GSp}_4$ over $F^+$. Assume further that $\pi$ is automorphic of
weight $\mu = (\mu_{v,1}, \mu_{v,2}; \alpha_v)_{v | \infty} \in (\mathbb{Z}^3)_{\text{Hom}(F^+, \mathbb{R})}$, in the sense that for each $v | \infty$, $\pi_v$ is
a discrete series representation with the same central and infinitesimal characters as
the finite-dimensional irreducible algebraic representation of highest weight given by
$$t = \text{diag}(t_1, t_2, t_3, t_4) \mapsto t_1^{\mu_{v,1}} t_2^{\mu_{v,2}} \mu(t)^{-\mu_{v,1} + \mu_{v,2} + \alpha_v}/2.$$ 

Here $\mu_{v,1} \geq \mu_{v,2} \geq 0$ and $\mu_{v,1} + \mu_{v,2}$ has the same parity as $\alpha_v$. Fix an isomorphism
$\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Then we say that there is a Galois representation associated to $\pi$ if
there is a continuous semisimple representation
$$\rho_{\pi,\iota} : G_{F^+} \to \text{GSp}_4(\overline{\mathbb{Q}}_l)$$
such that:

- for each finite place $v \nmid l$,
$$\iota WD(\rho_{\pi,\iota}|W_{v^+})^{ss} \cong \text{rec}(\pi_v \otimes | \cdot |^{-3/2})^{ss},$$
where \(\text{rec}\) is the local Langlands correspondence of \([GT07]\) and \(|·|\) is the composition of the similitude character and the norm character.

- If \(\pi_v\) is unramified at a place \(v|l\) then \(\rho_{\pi_v}\) is crystalline at \(v\), and in any case it is de Rham.
- Define \(\lambda_{\nu,\mu} \in (\mathbb{Z}_l^4)^{\text{Hom}(F^+,\mathbb{Q}_l)}\) by letting
  \[
  \lambda_{\nu,\mu,\tau} = (\delta_{10\tau} + \mu_{10\tau,1} + \mu_{10\tau,2}, \delta_{20\tau} + \mu_{20\tau,1}, \delta_{30\tau} + \mu_{30\tau,2}, \delta_{40\tau})
  \]
  for each embedding \(\tau : F^+ \hookrightarrow \mathbb{Q}_l\), where
  \[
  \delta_v := \frac{1}{2}(\mu_{v,1} + \mu_{v,2} + \alpha_v)
  \]
  for each \(v|\infty\). Then for each \(\tau : F^+ \hookrightarrow \mathbb{Q}_l\) lying over a place \(v\) of \(F^+\), the Hodge-Tate weights of \(\rho_{\pi_v}|G_{F_v}\) with respect to \(\tau\) are the \(\lambda_{\nu,\mu,\tau,j} + 4 - j\).

We now define what it means for a cuspidal automorphic representation of \(\text{GSp}_4\) to be ordinary. We could do this directly in terms of Hecke operators on \(\text{GSp}_4\), but for the sake of brevity we use the local Langlands correspondences for \(\text{GL}_4\) and \(\text{GSp}_4\) and the definition of ordinarity for \(\text{GL}_4\).

**Definition 7.6.3.** Let \(\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}\) be an isomorphism, and let \(\pi\) be a cuspidal automorphic representation of \(\text{GSp}_4(F^+)\) which is of weight \(\mu = (\mu_{v,1}, \mu_{v,2}; \alpha_v)_{v|\infty}\) in the above sense. Let \(\lambda \in (\mathbb{Z}_l^4)^{\text{Hom}(F^+,\mathbb{R})}\) be defined by \(\lambda_v = (\delta_v + \mu_{v,1} + \mu_{v,2}, \delta_v + \mu_{v,1}, \delta_v + \mu_{v,2}, \delta_v)\). We say that \(\pi\) is \(\iota\)-ordinary if for each \(v|l\), the irreducible admissible representation \(\Pi_v\) of \(\text{GL}_4(F_v^+)\) with
  \[
  \text{rec}(\pi_v) = \text{rec}(\Pi_v)
  \]
  satisfies \((\iota^{-1}\Pi_v)^{\text{ord}} \neq 0\), where the space \((\iota^{-1}\Pi_v)^{\text{ord}}\) is defined as in section 1.1

**Definition 7.6.4.** We say that a continuous irreducible representation
  \[
  \rho : G_{F^+} \rightarrow \text{GSp}_4(\overline{\mathbb{Q}}_l)
  \]
  is \(\text{GSp}_4\)-automorphic (of weight \(\lambda \in (\mathbb{Z}_l^4)^{\text{Hom}(F^+,\mathbb{Q}_l)}\)) if there is a \(\pi\) and an \(\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}\) with \(\rho \cong \rho_{\pi,\iota}\), and for each \(\tau : F^+ \hookrightarrow \mathbb{Q}_l\) lying over a place \(v\) of \(F^+\), the Hodge-Tate weights of \(\rho_{\pi_v}\) with respect to \(\tau\) are the \(\lambda_{\nu,\mu,\tau,j} + 4 - j\). By the above definitions, we see that this is equivalent to \(\pi\) being automorphic of weight \(\mu\) with \(\lambda_{\nu,\mu} = \lambda\). We say that \(\rho\) is \(\text{GSp}_4\)-automorphic and holomorphic if \(\pi\) can be chosen to be a holomorphic discrete series at all infinite places, and that \(\rho\) is \(\text{GSp}_4\)-automorphic and generic if \(\pi\) can be chosen to be globally generic (note that it is possible for \(\rho\) to be both holomorphic and generic, corresponding to different choices of \(\pi\) in the same global \(L\)-packet). We say that \(\rho\) is \(\text{GSp}_4\)-ordinarily automorphic if \(\pi\) can be chosen to be \(\iota\)-ordinary. We say that \(\rho\) is \(\text{GSp}_4\)-ordinarily automorphic and holomorphic (respectively generic) if \(\pi\) may be chosen to be simultaneously \(\iota\)-ordinary and holomorphic discrete series at all infinite places (respectively globally generic). Finally, we say in addition that \(\rho\) is automorphic of level prime to \(l\) if \(\pi_l\) is unramified.

In recent work ([Sor08]) Sorensen has used Theorem 7.6.1 and the constructions of [HT01] to associate Galois representations to certain globally generic cuspidal representations of \(\text{GSp}_4\) over totally real fields. In particular, he obtains the following theorem, which gives a ready supply of Galois representations associated to automorphic representations of \(\text{GSp}_4\).
Theorem 7.6.5. Let $F^+$ be a totally real field, and let $\pi$ be a globally generic cuspidal automorphic representation of $\text{GSp}_4$ over $F^+$ of weight $\mu$ for some $\mu$. Assume that for some finite place $v$ the local component $\pi_v$ is an unramified twist of the Steinberg representation. Then there is a Galois representation associated to $\pi$.

It is now straightforward to use the results of the previous sections to deduce a theorem about companion forms for automorphic representations of $\text{GSp}_4$ over $F^+$.

Theorem 7.6.6. Let $F^+$ be a totally real field. Let $l \geq 5$ be a prime number such that $[F^+(\zeta_l) : F^+] > 2$. Fix $i : \overline{\mathbb{Q}}_l \to \mathbb{C}$. Suppose that

$$\overline{\rho} : G_{F^+} \to \text{GSp}_4(\overline{\mathbb{Q}}_l)$$

is an irreducible representation, and let $n$ be an integer such that $\overline{\mathbb{Q}}^n$ is an unramified character of $G_{F^+}$. Suppose that $\overline{\rho}$ satisfies the following assumptions.

1. There are finite fields $\mathbb{F}_l \subset k \subset k'$ such that $\text{Sp}_4(k) \subset \overline{\rho}(G_{F^+}) \subset (k')^\times \text{GSp}_4(k)$.
2. The representation $\overline{\rho}$ has a lift which is $\text{GSp}_4$-ordinarily automorphic and generic of level prime to $l$, with similitude factor $\psi$, say.
3. Define $\psi_n := \psi^\sigma \tilde{\omega}^{-n}$, where $\tilde{\omega}$ is the Teichmüller lift of the mod $l$ cyclotomic character (so $\psi_n = \psi$, and $\psi_n$ is crystalline). There is a $\lambda \in (\mathbb{Z}_l^\times)^{\text{Hom}(F^+, \overline{\mathbb{Q}}_l)}$ such that

- for every place $v | l$ of $F^+$, $\overline{\rho}|_{\text{GSp}_4^+_{F^+}}$ has an ordinary crystalline symplectic lift of weight $(\lambda_v)_\tau$ (where the indexing set runs over the embeddings $\tau \in \text{Hom}(F^+, \overline{\mathbb{Q}}_l)$ inducing $v$) and similitude factor $\psi_n$.

Then $\overline{\rho}$ has an ordinary crystalline symplectic lift $\rho$ of weight $\lambda$ and similitude factor $\psi_n$, which is $\text{GSp}_4$-ordinarily automorphic of level prime to $l$ and generic. If $F^+ = \mathbb{Q}$ then $\rho$ is also $\text{GSp}_4$-ordinarily automorphic of level prime to $l$ and holomorphic.

Given any set of places $S$ of $F^+$, and an inertial type $\tau_v$ for each $v \in S$ not dividing $l$ such that $\overline{\rho}|_{\text{GSp}_4^+_{F^+}}$ has a symplectic lift of type $\tau_v$ and similitude factor $\psi_n$, $\rho$ can be chosen to have type $\tau_v$ at $v$ for all places $v \in S, v \nmid l$. More precisely, given a choice of a component of each ring $R^{\text{sympl}, \tau_v} \psi_n$ ($v \in S, v \nmid l$) and $R^{\text{sympl}, \Delta, cr} \psi_n$ ($v | l$), $\rho$ may be chosen so as to give a point on each of these components.

Proof. This follows from Theorems (16.1) and (25.2). Note that if $\pi$ is a globally generic automorphic representation of $\text{GSp}_4$ with $\overline{\pi}_{\pi_e} \cong \overline{\rho}$, then the transfer of $\pi$ to $\text{GL}_4$ is cuspidal (because $\overline{\rho}$ is irreducible). Conversely, if $\Pi$ is a RAEDC automorphic representation of $\text{GL}_4(F^+)$ with $\Pi^\vee \cong \chi \Pi$, and $\rho_{\Pi_e}(\Pi)$ is symplectic, it follows that $L^S(s, \Pi, \Delta^2 \otimes \chi^{-1})$ has a pole at $s = 1$ (because the corresponding statement is true for $\rho_{\Pi_e}(\Pi)$).

In the case $F^+ = \mathbb{Q}$, the fact that $\rho$ is also $\text{GSp}_4$-automorphic and holomorphic follows from Proposition 1.5 of [Wet03] (because our assumptions on $\overline{\rho}$ obviously imply that if $\overline{\rho} \cong \overline{\pi}_{\pi_e}$ then $\pi$ is neither CAP nor weak endoscopic).

In many cases we can make this rather more explicit, just as in the unitary case.

Lemma 7.6.7. Let $M$ be a finite extension of $\mathbb{Q}_l$. Take $\lambda \in (\mathbb{Z}_l^\times)^{\text{Hom}(M, \overline{\mathbb{Q}}_l)}$. Let $E$ be a finite extension of $\mathbb{Q}_l$ with residue field $k$. Let $\psi_i, 1 \leq i \leq 4$, be crystalline
characters \( G_M \to E^\times \), with \( \psi_i|_{I_M} = \chi^\lambda_i|_{I_M} \) in the notation of Definition 3.1.3. Assume that \( \psi_1\psi_4 = \psi_2\psi_3 \). Suppose that \( \overline{\rho} : G_M \to \text{GSp}_4(k) \) is of the form

\[
\begin{pmatrix}
\overline{\rho}_1 & * & * & * \\
0 & \overline{\rho}_2 & * & * \\
0 & 0 & \overline{\rho}_3 & * \\
0 & 0 & 0 & \overline{\rho}_4
\end{pmatrix}
\]

where \( \overline{\psi}_i = \overline{\rho}_i \) for \( 1 \leq i \leq 4 \). Suppose that none of the characters \( \overline{\rho}_i \overline{\rho}_j^{-1}, i < j \), are equal to \( \tau \). Then \( \overline{\rho} \) has a lift to a crystalline representation \( \rho : G_M \to \text{GSp}_4(E) \) of the form

\[
\begin{pmatrix}
\psi_1 & * & * & * \\
0 & \psi_2 & * & * \\
0 & 0 & \psi_3 & * \\
0 & 0 & 0 & \psi_4
\end{pmatrix}
\]

Proof. This is proved in exactly the same way as Lemma 3.1.6. \( \square \)

Just as in section 3.1.3, we can consider ordinary crystalline lifts of a particular form. Given \( \overline{\rho}_i \), \( \lambda \) as in the previous lemma (but no longer requiring that the characters \( \overline{\rho}_i \overline{\rho}_j^{-1} \neq \epsilon \)), we can consider ordinary lifts where we demand that \( \psi_i|_{I_M} = \chi^\lambda_i|_{I_M} \) and \( \overline{\psi}_i = \overline{\rho}_i \), \( 1 \leq i \leq 4 \). This gives a deformation ring \( R^{\text{sympl},\lambda,\text{cr},\psi}_{\overline{\rho}} \) and the following lemma may be proved in exactly the same way as Lemma 3.1.8.

Lemma 7.6.8. After inverting \( l \), the morphism \( \text{Spec } R^{\text{sympl},\lambda,\text{cr},\psi}_{\overline{\rho}} \to \text{Spec } R^{\text{sympl},\lambda,\text{cr},\psi}_{\overline{\rho}} \) becomes a closed immersion identifying \( \text{Spec } R^{\text{sympl},\lambda,\text{cr},\psi}_{\overline{\rho}}[1/l] \) with a union of irreducible components of \( \text{Spec } R^{\text{sympl},\lambda,\text{cr},\psi}_{\overline{\rho}}[1/l] \).

Theorem 7.6.9. Let \( F^+ \) be a totally real field. Let \( l \geq 5 \) be a prime number such that \([F^+(\xi_1) : F^+] > 2\). Fix \( i : \overline{O}_l \to \mathbb{C} \). Suppose that

\( \overline{\rho} : G_{F^+} \to \text{GSp}_4(\mathbb{F}_l) \)

is an irreducible representation. Suppose that the following conditions hold.

1. There are finite fields \( F_l \subset k \subset k' \) such that \( \text{Sp}_4(k) \subset \overline{\rho}(G_{F^+}) \subset (k')^\times \text{GSp}_4(k) \).
2. The representation \( \overline{\rho} \) has a lift which is \( \text{GSp}_4 \)-ordinarily automorphic and generic of level prime to \( l \).
3. There is a \( \lambda \in (\mathbb{Z}/l\mathbb{Z})^{\text{Hom}(F^+,\overline{O}_l)} \) such that
   - \( m := \lambda_{v,1} + \lambda_{v,4} = \lambda_{v,2} + \lambda_{v,3} \) is independent of \( \tau \), and
   - for every place \( v | l \), \( \overline{\rho}|_{G_{F_v^+}} \) is isomorphic to a representation

\[
\begin{pmatrix}
\overline{\rho}_{v,1} & * & * & * \\
0 & \overline{\rho}_{v,2} & * & * \\
0 & 0 & \overline{\rho}_{v,3} & * \\
0 & 0 & 0 & \overline{\rho}_{v,4}
\end{pmatrix}
\]

where none of \( \overline{\rho}_{v,i} \overline{\rho}_{v,j}^{-1}, i < j \), are equal to \( \tau \). Furthermore, \( \overline{\rho}_{v,i}|_{E_{F_v^+}} = \chi^\lambda_i|_{E_{F_v^+}} \) for each \( i \) (in the notation of Definition 3.1.3).

Then \( \overline{\rho} \) has an ordinary crystalline symplectic lift \( \rho_0 \) of weight \( \lambda \), which is \( \text{GSp}_4 \)-ordinarily automorphic of level prime to \( l \) and generic, with similitude factor \( \psi \),
say. Furthermore $\psi^{m+3}$ is a finite order character, and for every place $v|l$, $\rho|G_{F_v}$ is isomorphic to a representation of the form

$$
\begin{pmatrix}
\psi_{v,1} & * & * & * \\
0 & \psi_{v,2} & * & * \\
0 & 0 & \psi_{v,3} & * \\
0 & 0 & 0 & \psi_{v,4}
\end{pmatrix}
$$

where the $\psi_{v,i}$ are crystalline characters such that $\psi_{v,i} = \overline{\mu}_{v,i}$ and $\psi_{v,i}|_{I_{F_v}} = \chi^\lambda_{F_v}$. Finally, if $F^+ = \mathbb{Q}$ then $\rho$ is also GSp$_4$-ordinarily automorphic and holomorphic (of level prime to $l$).

Proof. This follows from Theorem 7.6.6, together with Lemma 7.6.7 and Lemma 7.6.8.

Remark 7.6.10. It is expected that whenever $\pi$ is a cuspidal automorphic representation of GSp$_4(\mathbb{A}_M)$, $M$ a number field, and $\pi$ is neither CAP nor weak endoscopic, then $\pi$ is stable. In the special case that $\pi$ is a discrete series representation at each infinite place, this means that if $\pi = \pi_f \otimes \pi_\infty$ (with $\pi_f$, $\pi_\infty$ respectively denoting the finite and infinite factors of $\pi$) then $\pi_f \otimes \pi'_\infty$ is also automorphic for any $\pi'_\infty$ in the same $L$-packet as $\pi_\infty$, i.e. we are free to change between holomorphic and generic discrete series at any infinite place. Assuming this result, which is expected to follow from Arthur’s work on the trace formula (cf. [Art04]), one could conclude that the representation $\rho$ in the above theorems is also GSp$_4$-automorphic and holomorphic, even if $F^+ \neq \mathbb{Q}$.

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