Taste non-Goldstone, flavor-charged pseudo-Goldstone boson decay constants in staggered chiral perturbation theory

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We calculate the axial current decay constants of taste non-Goldstone pions and kaons in staggered chiral perturbation theory through next-to-leading order. The results are a simple generalization of the results for the taste Goldstone case. New low-energy couplings are limited to analytic corrections that vanish in the continuum limit; certain coefficients of the chiral logarithms are modified, but they contain no new couplings. We report results for quenched, fully dynamical, and partially quenched cases of interest in the chiral SU(3) and SU(2) theories.

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I. INTRODUCTION

The decay constants $f_\pi$ and $f_K$ parametrize hadronic matrix elements entering the leptonic decays of $\pi$ and $K$ mesons. The values of the decay constants can be combined with the leptonic decay rates to extract the CKM matrix elements $|V_{ud}|$ and $|V_{us}|$ and test first-row CKM unitarity. Tighter constraints on new physics are obtained by taking the ratio $f_K/f_\pi$ and the form factor for the semileptonic decay $K \to \pi\ell\nu$ as theoretical inputs; doing so has led to impressive agreement between the Standard Model and experiment [1, 2].

Staggered quarks have 4 tastes per flavor by construction [3, 4]. The full taste symmetry group for a single flavor is $SU(4)_L \times SU(4)_R$ in the continuum limit ($a = 0$). At finite lattice spacing, lattice artifacts of $O(a^2)$ break the taste symmetry, and the remaining exact chiral symmetry is $U(1)_A$, which is enough to prevent the staggered quark mass from being additively renormalized. Hence, staggered fermions have an exact chiral symmetry at nonzero lattice spacing. In addition, lattice calculations with staggered fermions are comparatively fast. Staggered chiral perturbation theory (SChPT) was first developed to describe the lattice artifacts and light quark mass dependence of lattice data for the pseudo-Goldstone boson (PGB) masses [5–8]. Lattice data were extrapolated to the continuum limit and physical quark masses to determine the light quark masses, tree-level propagators, and low-energy couplings (LECs); these served as inputs to lattice calculations of the decay constants, semileptonic form factors, mixing parameters, and other quantities [9–27]. Lattice calculations of $f_\pi$ have become precise enough to use it to determine the lattice spacing [28].

While there have been a few attempts to calculate the decay constants for the taste non-Goldstone sectors [29, 30], most lattice calculations of the decay constants have been concentrated on the taste Goldstone sector associated with the exact chiral symmetry of the staggered action. In Ref. [31], Aubin and Bernard calculated the decay constants of the taste Goldstone pions and kaons through next-to-leading order (NLO) in SChPT. Here we extend their calculation to the taste non-Goldstone pions and kaons; we find that the general results are simply related to those in the taste Goldstone case. We use the notation of Ref. [32] throughout.

In Sec. II we recall the staggered chiral Lagrangian and the tree-level propagators. In Sec. III we consider the definition of the decay constants, recall the various contributions through NLO in SChPT, and write down the general results in the 4+4+4 theory. Sec. IV contains the results for specific cases of interest in the 1+1+1 theory, and we conclude in Sec. V. We use the notation of Ref. [32] throughout.

II. CHIRAL LAGRANGIAN FOR STAGGERED QUARKS

In this section, we write down the chiral Lagrangian for staggered quarks. The single-flavor Lagrangian was formulated by Lee and Sharpe [3] and generalized to multiple flavors by Aubin and Bernard [7]. Here, we consider the 4+4+4 theory, in which there are three flavors and four tastes per flavor. The exponential parameterization of the PGB fields is a $12 \times 12$ unitary matrix,

$$\Sigma = e^{i\phi/f} \in U(12),$$

where the PGB fields are

$$\phi = \sum_a \phi^a \otimes T^a,$$

$$\phi^a = \begin{pmatrix}
U_a & \pi_a^+ & K_0^a \\
p_a^- & D_a & K_0^0 \\
K_a^- & \bar{K}_a^0 & S_a
\end{pmatrix},$$

$$T^a \in \{\xi_5, i\xi_\mu\bar{\nu}, i\xi_{\mu\nu}(\mu < \nu), \xi_{\mu}, \xi_{\ell}\}.$$
where \( a \) runs over the 16 PGB tastes, and the \( T^a \) are \( 4 \times 4 \) generators of \( U(4) \); \( \xi_I \) is the identity matrix. Under a chiral transformation, \( \Sigma \) transforms as

\[
SU(12)_L \times SU(12)_R : \Sigma \to L \Sigma R^\dagger
\]

where \( L, R \in SU(12)_L R \). In the standard power counting,

\[
\mathcal{O}(p^2/\Lambda^2) \approx \mathcal{O}(m_\eta/\Lambda_x) \approx \mathcal{O}(a^2 \Lambda^2_x).
\]

The order of a Lagrangian operator is defined as the sum of \( n_{p^2}, n_m \), and \( n_{a^2} \), which are the number of derivative pairs, powers of (light) quark masses, and powers of the squared lattice spacing, respectively, in the operator. At leading order, the Lagrangian operators fall into three classes: \((n_{p^2}, n_m, n_{a^2}) = (1,0,0), (1,1,0)\), and \((0,0,1)\), and we have

\[
\mathcal{L}_{LO} = \frac{f^2}{8} \text{Tr}(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger) - \frac{1}{4} \mu f^2 \text{Tr}(M \Sigma + M \Sigma^\dagger) \\
+ \frac{2m_0^2}{3}(U_I + D_I + S_I)^2 + \alpha^2(\mathcal{W} + \mathcal{W}')
\]

where \( f \) is the decay constant at leading order (LO), \( \mu \) is the condensate parameter, and \( M \) is the mass matrix,

\[
M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \otimes \xi_I.
\]

The term multiplied by \( m_0^2 \) is the anomaly contribution \cite{33}, and the potentials \( \mathcal{W} \) and \( \mathcal{W}' \) are the taste symmetry breaking potentials of Ref. \cite{7}.

At NLO, there are six classes of operators satisfying \( n_{p^2} + n_m + n_{a^2} = 2 \), but only two classes contribute to the decay constants: \((n_{p^2}, n_m, n_{a^2}) = (1,1,0)\) and \((1,0,1)\). Contributing operators in the former are Gasser-Leutwyler terms \cite{34},

\[
\mathcal{L}_{GL} = L_4 \text{Tr}(\partial_\mu \Sigma^\dagger \partial_\mu \Sigma) \text{Tr}(\chi \Sigma + \chi^\dagger \Sigma) \\
+ L_5 \text{Tr}(\partial_\mu \Sigma^\dagger \partial_\mu \Sigma (\chi \Sigma + \Sigma^\dagger \chi)),
\]

where \( \chi = 2 \mu M \), and contributing operators in the latter are \( \mathcal{O}(p^2 a^2) \) terms enumerated by Sharpe and Van de Water \cite{35}.

### III. DECAY CONSTANTS OF FLAVOR-CHARGED PSEUDO-GOLDSTONE BOSONS

For a flavor-charged PGB with taste \( t, P_t^+ \), the decay constant \( f_{P_t^+} \) is defined by the matrix element of the axial current, \( j_{\mu 5,t}^A \), between the single-particle state and the vacuum:

\[
\langle 0| j_{\mu 5,t}^A P_t^+(p) \rangle = -i f_{P_t^+} p_\mu.
\]

From the LO Lagrangian, the LO axial current is

\[
j_{\mu 5,t}^{P+} = -i f^2 \frac{8}{8} \text{Tr}\left[ T^a(3) \mathcal{B}^{P^+}(\partial_\mu \Sigma^\dagger + \Sigma^\dagger \partial_\mu \Sigma) \right],
\]

where \( T^a \equiv I_3 \otimes T^a \), \( I_3 \) is the identity matrix in flavor space, and \( \mathcal{B}^{P^+} \) is a projection operator that chooses the \( P^+ \) from the \( \Sigma \) field. For example, for \( \pi^+ \) it is \( \mathcal{B}_{\pi^+}^0 \equiv \delta_{x0} \delta_{y2} \), where \( x \) and \( y \) are the light quarks in \( P^+ \). For flavor-charged states, \( x \neq y \), by definition. Note that \( \mathcal{B}^{P^+} \) and \( T^{a}(3) \) commute with each other.

Expanding the exponentials \( \Sigma = e^{i\phi/\xi} \) in the LO current gives

\[
\partial_\mu \Sigma^\dagger + \Sigma^\dagger \partial_\mu \Sigma = \frac{2i}{f} \partial_\mu \phi \\
- \frac{i}{3f} (\partial_\mu \phi \phi^2 - 2\phi \partial_\mu \phi \phi + \phi^2 \partial_\mu \phi + \cdots).
\]

The \( \mathcal{O}(\phi) \) term of the axial current gives the LO term of the decay constants, \( f \), and NLO corrections from the wavefunction renormalization. The wavefunction renormalization consists of NLO analytic terms and one-loop chiral logarithms at NLO: we denote the former by \( \delta f^\text{analytic} \) and the latter by \( \delta f^\text{analytic} \). The \( \mathcal{O}(\phi^3) \) term of the axial current also gives one-loop chiral logarithms at NLO, \( \delta f^\text{current} \). Figs. 1(a) and 1(b) show the one-loop corrections to the decay constant. In addition, there is an analytic contribution to the decay constants from the NLO current. We denote the total of the NLO analytic terms by \( \delta f^\text{analytic} \), which consists of \( \delta f^\text{analytic} \) and analytic terms from the NLO current. Combining \( \delta f^\text{analytic} \) with the one-loop corrections, we write the decay constants up to NLO:

\[
f_{P_t^+} = f \left[ 1 + \frac{1}{16\pi^2 f^2} \left( \delta f^\text{analytic} + \delta f^\text{current} \right) + \delta f^\text{analytic} \right].
\]

In this section we outline the calculation of \( \delta f^\text{analytic} \), \( \delta f^\text{current} \), and \( \delta f^\text{analytic} \) and present results for the 4+4+4 theory.
A. Wavefunction renormalization correction

At $\mathcal{O}(\phi)$ the axial current, Eq. (11), is

$$j_{\mu5,t}^{P+,\phi} = f \left( \partial_\mu \phi^t_{gyz} \right),$$

(14)

where we used $\tau_a = 4 \delta_{ta}$, $\mathcal{O}_{ij}^{P+} = \delta_{ix} \delta_{jy}$, and performed the trace over taste indices. Here $\tau_{a_1a_2\cdots}$ is defined by

$$\tau_{a_1a_2\cdots} \equiv \text{Tr}(T_a T_b T_c \cdots).$$

(15)

The contributions of the $\mathcal{O}(\phi)$ current to the decay constants are defined by the matrix element

$$\langle 0 | j_{\mu5,t}^{P+,\phi} | P^+_t(p) \rangle = f(-ip_\mu) \langle 0 | \phi_{xy}^t | P^+_t(p) \rangle = f(-ip_\mu) \sqrt{Z_{P^+_t}},$$

(16)

where $Z_{P^+_t} = 1 + \delta Z_{P^+_t}$ is the wavefunction renormalization constant of the $\phi_{xy}^t$ field. At NLO the wavefunction renormalization corrections are

$$\frac{1}{16\pi^2f^2} \delta f_{P^+_t}^Z + \delta f_{P^+_t}^{\text{analyt.}} = \frac{1}{2} \frac{d\Sigma}{2 dp^2} |_{p^2=m_{P^+_t}^2},$$

(18)

where $\Sigma$ is the self-energy of $P^+_t$. Using the self-energy from Ref. [32], we find the one-loop corrections

$$\delta f_{P^+_t}^Z = \frac{1}{24f} \sum_a \left[ \sum_Q l(Q_a) + 16\pi^2 \frac{d^4q}{(2\pi)^4} \left( D_{xx}^a + D_{yy}^a - 2\theta^{at} D_{xy}^a \right) \right],$$

(19)

where $Q$ runs over the six flavor combinations $xi$ and $yi$ for $i \in \{u,d,s\}$, $a$ runs over the 16 PGB tastes in the SU(4)$_T$, $Q_a$ is the squared tree-level pseudoscalar meson with flavor $Q$ and taste $a$, and $\phi^t_{xy} = \frac{1}{2} \phi^{ab}_{xy}$. In Eq. (19), $l(Q_a)$ and $D_{ij}^a$ are chiral logarithms and the disconnected piece of the tree-level propagator, respectively [7, 32].

$$l(X) \equiv X \left( \ln X + \phi^2 + \phi \right),$$

(20)

where $\phi = (\sqrt{X}L)$ is the finite volume correction of Ref. [2], and

$$D_{ij}^a = \frac{\delta_{ij}}{(q^2 + I_a)(q^2 + J_a)} \times \frac{(q^2 + U_a)(q^2 + D_a)(q^2 + S_a)(q^2 + \eta_a)}{(q^2 + \eta_a)(q^2 + \eta_a)}. $$

(21)

Here the names of mesons denote the squares of their tree-level masses, and

$$\delta_I = 4m_0^2/3, \quad \delta_{I\mu} = 0, \quad \delta_5 = 0,$$

$$\delta_{\mu} = a^2 \Delta_{\mu}, \quad \delta_{\mu5} = a^2 \delta_A.$$

(22)

For $X \in \{I, J, U, D, S\}$,

$$X_a \equiv m^2_{X_a} = 2m_{x} + a^2 \Delta_a,$$

(24)

where $m_x$ is the mass of the quark of flavor $x \in \{i,j,u,d,s\}$, while for $X \in \{\pi^0, \eta, \eta'\}$, the squares of the tree-level meson masses are the eigenvalues of the matrix

$$\begin{pmatrix}
U_a + \delta_a & \delta_a & \delta_a \\
\delta_a & D_a + \delta_a & \delta_a \\
\delta_a & \delta_a & S_a + \delta_a
\end{pmatrix}.$$  

(25)

The squared tree-level mass of a flavor-charged meson $(\pi^\pm, \bar{K}^\pm, K^0, K^0)$ is

$$P^+_t = \frac{1}{2} (X_t + Y_t) = \mu(m_x + m_y) + a^2 \Delta_t,$$

(26)

where $X \neq Y \in \{U, D, S\}$ and $x \neq y \in \{u, d, s\}$. The hairpin coupling $\delta_{V,A}$ and taste splittings $\Delta_a$ are combinations of the couplings of the LO Lagrangian $\mathcal{L}$.

We defer discussing the analytic corrections $\delta f_{P^+_t}^{\text{analyt.}}$ to Sec. III C.

B. Current correction

The $\mathcal{O}(\phi^3)$ terms of the axial current are

$$j_{\mu5,t}^{P+,\phi^3} = -\frac{1}{24f} \tau_{abc} \left( \partial_\mu \phi^a_k \phi^b_k \phi^c_l \phi^t_{lx} \delta_{x}\right) - 2 \phi^a_k \partial_\mu \phi^b_k \phi^c_l \phi^t_{lx} + \phi^a_k \phi^b_k \phi^c_l \phi^t_{lx} \delta_{x},$$

(27)

In the calculation of the matrix element defined in Eq. (10), each term of Eq. (27) contributes only one contraction because the derivatively coupled fields in the current must contract with the external field to obtain a nonzero result. For example, the first term gives

$$\partial_\mu \phi^a_k \phi^b_k \phi^c_l \phi^t_{lx} \rightarrow -ip_\mu \phi^a_k \phi^b_k \phi^c_l \phi^t_{lx},$$

(28)

where $\phi^a_k$ [32, 36]

$$K^a_{ij,kl} \equiv \int \frac{d^4q}{(2\pi)^4} \langle \phi^a_k \phi^b_k \rangle \quad (\text{no sum}),$$

(29)

$$\langle \phi^a_k \phi^b_k \rangle = \delta^{ab} \left( \delta_{ij} \delta_{jk} \frac{1}{q^2 - 1/2 (I_a + J_a) + \delta_{ij} \delta_{kl} D_{ij}^a} \right).$$

(30)

Collecting the three contributions from Eq. (27), we find

$$\frac{1}{6f} \sum_a \left[ \sum_j \left( K^a_{x,ix} + K^a_{y,jy} \right) - 2 \theta^{at} K^a_{xx,yy} \right],$$

(31)
where $j$ runs over $\{ u, d, s \}$. Performing the integrals over the loop momenta gives the one-loop current corrections to the decay constants:

$$\delta f_{Pc}^{\text{current}} = -\frac{1}{6} \sum_a \left[ \sum_Q l(Q_a) \right] \left[ \sum_{Q} l(Q_a) \right] + 16\pi^2 \int \frac{d^4q}{(2\pi)^4} \left( D^{a}_{xx} + D^{a}_{yy} - 2\theta q D^{a}_{xy} \right).$$

(32)

Note that $\delta f_{Pc}^{\text{current}}$ is proportional to the one-loop wavefunction renormalization correction, $\delta f_{Pc}^{\text{ren}}$. This was shown in the taste Goldstone case in Ref. [31].

C. Next-to-leading order analytic contributions

Now we consider the NLO analytic contributions to the decay constants. They come from the $O(p^2 m)$ Gasser-Leutwyler Lagrangian in Eq. (9) and the $O(p^2 a^2)$ Sharpe-Van de Water Lagrangian of Ref. [35]. Both Lagrangians contribute to wavefunction renormalization and the current.

The analytic corrections to the self-energy give the wavefunction renormalization correction

$$\delta f_{Pc}^{\text{anal}, Z} = -\frac{64}{f^2} L_4 \mu (m_u + m_d + m_s) + \frac{8}{f^2} L_5 \mu (m_x + m_y) - \frac{8}{f^2} a^2 \mathcal{F}_t,$$

(33)

while the NLO current from the Gasser-Leutwyler terms gives the current correction

$$\delta f_{Pc}^{\text{current, GL}} = \frac{128}{f^2} L_4 \mu (m_u + m_d + m_s) + \frac{16}{f^2} L_5 \mu (m_x + m_y).$$

(34)

The contributions of the $O(p^2 a^2)$ operators coming from the Sharpe-Van de Water Lagrangian in Ref. [35] give the current correction

$$\delta f_{Pc}^{\text{current, SV}} = a^2 \mathcal{F}_t.$$  

(35)

The LECs $\mathcal{E}_t$ and $\mathcal{E}_t'$ are degenerate within the irreps of the lattice symmetry group. Sharpe and Van de Water observed that contributions from the $O(p^2 a^2)$ source operators destroy would-be relations between the $O(4)$-violations in the PGB masses and the (axial current) decay constants.

Collecting the analytic corrections, we have (in the 4+4+4 theory)

$$\delta f_{Pc}^{\text{anal}} = \frac{64}{f^2} L_4 \mu (m_u + m_d + m_s) + \frac{8}{f^2} L_5 \mu (m_x + m_y) + a^2 \mathcal{F}_t,$$

(36)

where the constants $\mathcal{F}_t$ subsume the constants $\mathcal{E}_t^{(i)}$. Examining Eqs. (19), (23), and (30), we see that the constants $\mathcal{F}_t$ (for $t \neq 4$) are the only new LECs entering the (NLO) expressions for the decay constants, in the sense that the others are present also in the taste Goldstone case.

IV. RESULTS

To formulate the full QCD and (partially) quenched results in rooted SchPT, we employ the replica method [37, 38]. Rooting introduces a factor of 1/4 in front of the explicit chiral logarithms $l(Q_a)$ in Eqs. (40) and (42) and in the $L_4$ term in Eq. (46). We must also replace the eigenvalues of the mass matrix with those of the matrix obtained by sending $\delta_a \to \delta_a/4$ there. We have

$$\delta f_{Pc}^{\text{con}} = \frac{1}{32} \sum_{Q, B} g_B l(Q_B),$$

(39)

$$\delta f_{Pc}^{\text{disc}} = -\frac{2}{f^2} \int \frac{d^4q}{(2\pi)^4} \left( D^I_{xx} + D^I_{yy} - 2D^I_{xy} \right) + 4D^V_{xx} + 4D^V_{yy} - 2\Theta^{VF} D^V_{xy} + 4D^A_{xx} + 4D^A_{yy} - 2\Theta^{AF} D^A_{xy}.$$  

(40)

In Eq. (41), the flavor-neutral, tree-level masses $(\pi^0, \eta, \eta')$ appearing in $P_{c\bar{c}}$ have been replaced with the masses obtained by sending $\delta_a \to \delta_a/4$ in the flavor-neutral meson mass matrix. In Eqs. (39) and (41), we summed over $a$ within each $SO(4)$ irrep in Eqs. (39) and (42). $B$ and $F$ represent (taste) $SO(4)$ irreps,

$$B, F \in \{ I, V, T, A, P \},$$

(41)

$$t \in F,$$  

(42)

and

$$\Theta^{BF} = \sum_{a \in B} \theta^{at}, \quad g_B = \sum_{a \in B} 1.$$  

The coefficients $\Theta^{BF}$ are given in Table I. The loop corrections differ from those in the taste Goldstone case only in the values of the coefficients $\Theta^{BF}$.

Eq. (48) subsumes the NLO analytic corrections in fully dynamical and partially quenched SU(3) ChPT; in the former case, $m_x, m_y$ are chosen from $m_u, m_d,$ and $m_s$. In the quenched case, the $L_4$ term is dropped. To obtain the NLO analytic corrections in SU(2) ChPT, we drop terms with the heavy quark mass(es), and the LECs become heavy quark mass dependent [40].
Below we give the one-loop contributions to the decay constants for each of these cases. In Sec. IV A 1 and Sec. IV A 2 fully dynamical and partially quenched results for the 1+1+1 and 2+1 flavor cases in SU(3) chiral perturbation theory are given. The analogous results in SU(2) chiral perturbation theory are given. The analogous results for the 2+1 flavor case are easily obtained from Sec. IV A 2.

### A. SU(3) chiral perturbation theory

#### 1. Fully dynamical case

In Eq. (39) \( Q \) runs over six flavor combinations, \( x_i \) and \( g_i \) for \( i \in \{u, d, s\} \). Setting \( xy = ud, us, ds \) gives the results for the \( \pi^+, K^+, \) and \( K^0 \) in the fully dynamical 1+1+1 flavor case:

\[
\delta f_{\pi_F}^{\text{con}} = -\frac{1}{32} \sum_B g_B \left( l(U_B) + 2l(\pi_B^+) \right) + l(K_B^+) + l(D_B) + l(K_B^0),
\]

\[
\delta f_{K_F}^{\text{con}} = -\frac{1}{32} \sum_B g_B \left( l(\pi_B^+) + l(D_B) \right) + 2l(K_B^+) + l(K_B^0) + l(S_B).
\]

\[
\delta f_{K^0_F}^{\text{con}} = -\frac{1}{32} \sum_B g_B \left( l(\pi_B^+) + l(D_B) \right) + 2l(K_B^+) + l(K_B^0) + l(S_B).
\]

In the disconnected parts, Eq. (40), the integrals can be performed as explained in Ref. 24. After performing the integrals and decoupling the \( \eta^\pi \) by taking \( m_0^2 \to \infty \), we find

\[
\delta f_{\pi_F}^{\text{disc}} = \sum_X \left[ \frac{1}{6} \left\{ R_{D\pi^+}^{US}(X_I)l(X_I) \right. \right.
\]

\[
+ R_{S\pi^+}^{US}(X_I)l(X_I) - 2R_{\pi^+}^{US}(X_I)l(X_I) \right] + \frac{1}{4} a^2 \delta'_V \left\{ 2R_{D\pi^+}^{US}(X_V)l(X_V) \right.
\]

\[
+ 2R_{S\pi^+}^{US}(X_V)l(X_V) - \Theta^{VF} R_{\pi^+}^{US}(X_V)l(X_V) \right] + (V \to A) ,
\]

\[
\delta f_{K_F}^{\text{disc}} = \sum_X \left[ \frac{1}{6} \left\{ R_{D\pi^+}^{US}(X_I)l(X_I) \right. \right.
\]

\[
+ R_{S\pi^+}^{US}(X_I)l(X_I) - 2R_{\pi^+}^{US}(X_I)l(X_I) \right] + \frac{1}{4} a^2 \delta'_V \left\{ 2R_{D\pi^+}^{US}(X_V)l(X_V) \right.
\]

\[
+ 2R_{S\pi^+}^{US}(X_V)l(X_V) - \Theta^{VF} R_{\pi^+}^{US}(X_V)l(X_V) \right] + (V \to A) .
\]

### TABLE I. The coefficient \( \Theta^{BF} \) defined in Eq. (42) is in row \( B \) and column \( F \).
Using the tree-level masses of the taste singlet channel, $\delta f^{\text{disc}}$, and masses, $m_u$, $m_d$ and $m_s$. The connected contributions to the decay constants in the partially quenched 1+1+1 flavor case are

$$
\delta f_{P^+}^{\text{con}} = -\frac{1}{32} \sum_{Q,B} g_B \, l(Q_B).
$$

Performing the integrals in Eq. (59), keeping all quark masses distinct gives the disconnected contributions for the partially quenched 1+1+1 flavor case:

$$
\delta f_{P^+}^{\text{disc}} = \sum_z \left[ \frac{1}{6} \left\{ D_{X\pi^0\eta}^{UDS}(Z_I) l(Z_I) + D_{Y\pi^0\eta}^{UDS}(Z_I) l(Z_I) \right\} + \frac{1}{4} a^2 \delta' \left\{ 2 D_{X\pi^0\eta}^{UDS}(Z_V) l(Z_V) + 2 D_{Y\pi^0\eta}^{UDS}(Z_V) l(Z_V) \right\} - \Theta^{\text{VF}} R^{\text{UDS}}_{Y\pi^0\eta}(Z_V) l(Z_V) \right\} + (V \to A)
$$

They simplify the results, Eqs. (52) and (53):

$$
\delta f_{\pi^+}^{\text{disc}} = \frac{1}{4} a^2 \delta' (4 - \Theta^{\text{VF}}) \left[ \frac{S_{\eta^0} - \eta}{(\eta - \eta')(\eta' - \eta)} l(\eta) \right] + (V \to A),
$$

$$
\delta f_{K^+}^{\text{disc}} = \frac{1}{12} \left[ 3l(\pi^0) - 5l(\eta^0) + 6l(S_I) - 4l(\eta) \right] + \frac{1}{4} a^2 \delta' \left[ \frac{S_{\eta^0} - \eta}{(\eta - \eta')(\eta' - \eta)} l(\eta) \right] + \frac{1}{2} \Theta^{\text{VF}} \frac{1}{\eta - \eta'} \left\{ l(\eta') - l(\eta) \right\} + (V \to A).
$$

2. Partially quenched case

In the partially quenched case, the valence quark masses, $m_x$ and $m_y$ are not degenerate with the sea quark masses, $m_u$, $m_d$ and $m_s$. The connected contributions to the decay constants in the partially quenched 1+1+1 flavor case are

$$
\delta f_{P^+}^{\text{con}} = -\frac{1}{32} \sum_{Q,B} g_B \, l(Q_B).
$$

Here $\delta_3(\sqrt{X}L)$ is the finite volume correction defined in Ref. [4], and $X$ and $Y$ represent the squared tree-level masses of $x\bar{x}$ and $y\bar{y}$ PGBs, respectively.

For $m_x = m_y$, we find

$$
\delta f_{P^+}^{\text{disc}} = \frac{1}{4} a^2 \delta' (4 - \Theta^{\text{VF}}) \left[ R^{UDS}_{X\pi^0\eta}(X_V) l(X_V) \right] + \sum_z D^{UDS}_{X\pi^0\eta}(Z_V) l(Z_V) + (V \to A).
$$

The connected contributions in the 2+1 flavor case are obtained by setting $m_u = m_d$ in Eq. (63). To obtain the disconnected contributions, we perform the integrals in
Eq. (10) setting $m_u = m_d$. For $m_x \neq m_y$, we find

\[
\delta f_{P_x}^{\text{disc}, m_x \neq m_y} = \sum_Z \left\{ \frac{1}{6} \left( D_{X_{\eta},X}(Z_I)l(Z_I) + D_{Y_{\eta},Y}(Z_I)l(Z_I) \right) + \frac{1}{4} a^2 \delta' (4 - \Theta V^F) \frac{l(X_V)}{Y_V - X_V} \right\} + (V \to A). 
\]

Here, note that $I$ represents the taste-singlet irrep.

For $m_x = m_y$, we find

\[
\delta f_{P_x}^{\text{disc}, m_x = m_y} = \frac{1}{4} a^2 \delta' (4 - \Theta V^F) \frac{l(X_V)}{Y_V - X_V} + (V \to A). 
\]

and for $m_x = m_y$,

\[
\delta f_{P_x}^{\text{disc}, m_x = m_y} = \frac{1}{4} a^2 \delta' (4 - \Theta V^F) \frac{l(X_V)}{Y_V - X_V} + (V \to A). 
\]

Quenching the sea quarks also affects the analytic terms. In the quenched version of Eq. (60), there is no $L_4$ term of Eq. (60), which is coming from the sea quarks.

### B. SU(2) chiral perturbation theory

We obtain the SU(2) SChPT results from the SU(3) SChPT results using the prescription of Ref. [40]. SU(2) chiral perturbation theory was developed in Ref. [42] and applied to simulation data for the taste Goldstone decay constants in Refs. [13–15]. The results of this section extend the results of Refs. [13, 14] to the taste non-Goldstone case.

1. Fully dynamical case

From Eqs. (13), (14), and (15), we obtain the connected contributions for the fully dynamical $1+1+1$ flavor case ($m_u \neq m_d \ll m_s$):

\[
\delta f_{\pi^+}^{\text{con}} = -\frac{1}{32} \sum_B g_B \left( l(U_B) + 2l(\pi^+_B) + l(D_B) \right), 
\]

\[
\delta f_{K^+}^{\text{con}} = -\frac{1}{64} \sum_B g_B \left( l(U_B) + 2l(\pi^+_B) + l(D_B) \right), 
\]

For the disconnected contributions, we find from Eqs. (10), (17) and (19):

\[
\delta f_{f_{\pi^+}}^{\text{disc}} = \frac{1}{2} \left( l(U_I) + l(D_I) \right) - l(\pi^0_I) + \frac{1}{4} a^2 \delta' \sum_X \left\{ 2R_{U\pi_0}(X_V)l(X_V) + 2R_{D\pi_0}(X_V)l(X_V) \right\} + \frac{1}{4} a^2 \delta' \frac{\Theta V^F (l(\pi^0_V) - l(\pi^0_V))}{Y_V - \pi^0_V} 
\]

\[
+ (V \to A), 
\]

\[
\delta f_{f_{K^+}}^{\text{disc}} = \frac{1}{2} l(U_I) - l(\pi^0_I) + \frac{1}{4} a^2 \delta' \sum_X R_{U\pi_0}(X_V)l(X_V) + (V \to A), 
\]
and
\[
\delta f_{K^+}^{\text{disc}} = \frac{1}{2} l(D_l) - \frac{1}{4} l(\pi_l^0) + \frac{1}{2} a^2 \delta l V \sum_X R^{U}_{D_v} \eta(XV) l(XV) + (V \to A). \tag{75}
\]

The connected contributions in the fully dynamical 2+1 flavor case (\(m_u = m_d \ll m_s\)) are
\[
\delta f^{\text{con}}_{\pi^+} = -\frac{1}{8} \sum_B g_{Bl}(\pi_B), \tag{76}
\]
\[
\delta f^{\text{con}}_{K^+} = -\frac{1}{16} \sum_B g_{Bl}(\pi_B). \tag{77}
\]

For the disconnected contributions in the fully dynamical 2+1 flavor case, we find
\[
\delta f^{\text{disc}}_{\pi^+} = \left\{ 4 - \Theta^{VF} \right\} \left( l(\pi_V) - l(\eta_V) \right) + (V \to A), \tag{78}
\]
\[
\delta f^{\text{disc}}_{K^+} = \frac{1}{4} l(\pi_I) + l(\pi_V) - l(\eta_V) + (V \to A). \tag{79}
\]

2. Partially quenched case

Considering \(x\) and \(y\) to be light quarks (\(m_s \gg m_u, m_d, m_x, m_y\)), the connected contributions to the decay constants in the partially quenched 1+1+1 flavor case can be obtained by dropping terms corresponding to strange sea quark loops from Eq. (69). Eqs. (60), (63) and (64) give the disconnected contributions:
\[
\delta f^{\text{disc}}_{\pi^+, m_x \neq m_y} = \sum_Z \left\{ \frac{1}{4} \left[ D^{UD}_{\pi^0,X} (Z_l) l(Z_l) + D^{UD}_{\pi^0,Y} (Z_l) l(Z_l) - 2 R^{UD}_{XY \pi^0} (Z_l) l(Z_l) \right] \right. \\
+ \frac{1}{4} a^2 \delta l V \left\{ 2 D^{UD}_{\pi^0,Y} (Z_l) l(Z_l) + 2 D^{UD}_{\pi^0,Y} (Z_l) l(Z_l) - \Theta^{VF} R^{UD}_{XY \pi^0} (Z_l) l(Z_l) \right\} + (V \to A) \right. \\
+ \frac{1}{4} \left\{ R^{UD}_{\pi^0,Y} (X_l) l(X_l) + R^{UD}_{\pi^0,Y} (Y_l) l(Y_l) \right\} + (V \to A), \tag{80}
\]
and
\[
\delta f^{\text{disc}}_{\pi^+, m_x = m_y} = \frac{1}{4} a^2 \delta l V \left[ R^{UD}_{\pi^0,Y} (X_l) l(X_l) + \sum_Z D^{UD}_{\pi^0,Y} (Z_l) l(Z_l) \right] + (V \to A). \tag{81}
\]

The connected contributions to the decay constants in the partially quenched 2+1 flavor case can be obtained by setting \(m_u = m_d\) and decoupling the strange quark in the 1+1+1 flavor case, Eq. (69). From Eqs. (60) and (63), we find the disconnected contributions in the 2+1 flavor case:
\[
\delta f^{\text{disc}}_{\pi^+, m_x \neq m_y} = \sum_Z \left\{ -\frac{1}{2} R^{UD}_{XY \pi^0} (Z_l) l(Z_l) + \frac{1}{4} a^2 \delta l V \left\{ 2 D^{UD}_{\pi^0,X} (Z_l) l(Z_l) + 2 D^{UD}_{\pi^0,Y} (Z_l) l(Z_l) - \Theta^{VF} R^{UD}_{XY \pi^0} (Z_l) l(Z_l) \right\} + (V \to A) \right. \\
+ \frac{1}{4} \left\{ l(X_l) + (\pi_I - X_l) l(X_l) + l(Y_l) + (\pi_I - Y_l) l(Y_l) \right\} \\
+ \frac{1}{2} a^2 \delta l V \left\{ R^{UD}_{\pi^0,Y} (X_l) l(X_l) + R^{UD}_{\pi^0,Y} (Y_l) l(Y_l) \right\} + (V \to A), \right. \tag{82}
\]
and
\[
\delta f^{\text{disc}}_{\pi^+, m_x = m_y} = \frac{1}{4} a^2 \delta l V \left[ R^{UD}_{\pi^0,Y} (X_l) l(X_l) + \sum_Z D^{UD}_{\pi^0,Y} (Z_l) l(Z_l) \right] + (V \to A). \tag{83}
\]

Considering \(x\) to be a light quark and \(y\) to be a heavy quark (\(m_s \gg m_u, m_d, m_x, m_y\)), the connected contributions to the decay constants can be obtained by dropping terms from Eq. (59) corresponding to strange sea quarks and \(y\) valence quarks circulating in loops; \(i.e.,\) only the \(xU\) and \(xD\) terms survive in the sum over \(Q\). From Eqs. (60), (62), and (64) (or alternatively, Eqs. (60) and (64)), we find the disconnected contribution for the partially quenched 1+1+1 flavor case,
\[
\delta f^{\text{disc}}_{\pi^+} = \frac{1}{4} \sum_Z \left[ D^{UD}_{\pi^0,X} (Z_l) l(Z_l) + \sum_Z D^{UD}_{\pi^0,Y} (Z_l) l(Z_l) \right] + (V \to A). \tag{84}
\]
For the 2+1 flavor case, we find

\[
\delta f_{P^V}^{\text{disc}} = \sum \theta \left[ \frac{1}{2} a^2 \theta_v D_{\pi,\pi}(ZV)\tilde{l}(ZV) + (V \to A) \right] \\
+ \frac{1}{4} \left\{ \tilde{l}(X_I) + (\pi_I - X_I) \tilde{l}(X_I) \right\} \\
+ \frac{1}{2} a^2 \theta_v P_{\pi,\pi}(XV)\tilde{l}(XV) + (V \to A). \tag{85}
\]

V. CONCLUSION

Our results for the decay constants are given compactly by Eq. (13) with Eqs. (37) through (58): they reduce to those of Ref. [31] in the taste Goldstone sector. The only new LECs are those parametrizing the analytic corrections proportional to \(a^2\); the SO(4)-violating contributions are independent of those in the masses. As shown in Table I, the factors \(B_{\pi}^{BF}\) multiplying the disconnected pieces of the propagators \(D_{\pi,\pi}^{V}\) differ from the coefficients in the taste Goldstone case, but no new LECs arise in the loop diagrams. In SU(2) chiral perturbation theory with a heavy valence quark, the chiral logarithms are the same in all taste channels; only the analytic \(\theta(a^2)\) corrections differ.

Results for special cases of interest can be obtained by expanding the disconnected pieces of the propagators in Eq. (10). For the fully dynamical case with three non-degenerate quarks, the loop corrections in the SU(3) chiral theory are in Eqs. (43)-(48). Results in the isospin limit are in Eqs. (49)-(53). For the partially quenched case with three non-degenerate sea quarks, loop corrections in the SU(3) chiral theory are in Eqs. (55)-(59). Results in the isospin limit are in Eqs. (60)-(62). For the quenched case the results are in Eqs. (63)-(65). Results in SU(2) chiral perturbation theory are in Eqs. (70)-(74) and Eqs. (80)-(85). These results can be used to improve determinations of the decay constants, quark masses, and the Gasser-Leutwyler LECs by analyzing lattice data from taste non-Goldstone channels.

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