KINETIC AND MACROSCOPIC MODELLING OF A RAREFIED POLYATOMIC GAS

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Abstract. In this paper, we consider the two kinetic models of continuous type for a polyatomic gas [12, 8], that introduce a single continuous variable supposed to capture all the phenomena related to the more complex structure of a molecule having more than one atom, such as internal degrees of freedom in a collision. In particular, we provide a direct comparison of these two models, and show their equivalence after the distribution function is rescaled and the cross section is reformulated, by elaborating the argument of [4]. We then focus on the kinetic model [8] for which the rigorous existence and uniqueness result in a space homogeneous setting is recently proven in [13]. Using the cross section proposed in this model together with the maximum entropy principle, we establish macroscopic models of six and fourteen fields. In the case of six moments, we calculate the exact, nonlinear, production term and prove its total agreement with extended thermodynamics [1], as it satisfies the entropy residual inequality on the whole range of model validity. Moreover, for the fourteen moments model, we provide new expressions for relaxation times and transport coefficients in a linearized setting, that yield both matching with the experimental data for dependence of the shear viscosity upon temperature and a satisfactory agreement with the theoretical value of the Prandtl number, on the room temperature range when only translational and rotational modes of molecules are taken into account, as much as on higher temperatures when vibrational modes appear as well.

1. Introduction

This manuscript is devoted to the both kinetic and macroscopic modelling of a polyatomic gas. In these models the core mechanism of particle interactions are molecular collisions, which are much more intricate in polyatomic gases than the ones in monatomic case, whether elastic or non-elastic. The main effect is presence of internal degree of freedom – except classical translation in the physical space (\(\mathbb{R}^3\)), rotation or vibration of a polyatomic molecule can occur during the collision process. At the kinetic level, this is reflected on microscopic energy conservation law during a molecular collision, where apart the usual kinetic energy of molecules it appears microscopic internal energy as well. On the other hand, at the macroscopic level, the trace of momentum flux is not related to the gas internal energy density anymore, that resulted into two types of moments equations describing polyatomic gas within extended thermodinamics, momentum and energy like hierarchies.

In the context of kinetic theory, the main difference in modelling comes with the parametrization of microscopic energy collision law.
In the semi-classical approach [29, 10, 17, 9, 14, 15], only the molecular velocity is parameterized, while the internal energy of molecules takes discrete values. Moreover, one distribution function is assigned to each energy level, leading to the system of kinetic equations. This model uses experimental data and is of great importance for computational tasks, but lacks mathematical background.

On the other hand, continuous kinetic models [8, 11, 12] introduce a single continuous variable which sees internal degrees of freedom as a communicable internal energy during collisions. Then both molecular velocities and molecular internal energies are parameterized: velocities are obtained by introducing the classical scattering direction which splits the pure kinetic energy of the colliding particles, while for microscopic internal energy an additional parameter is introduced in order to distribute the proportion of pure internal energy to each interacting molecule, following the ideas of Borgnakke-Larsen procedure [6]. This parametrization makes the model suitable for rigorous mathematical analysis. In particular, the existence and uniqueness theory for the Boltzmann equation from [8] in the space-homogeneous setting is recently established in [13], followed by the study of polynomial and exponential moments associated to this solution.

The continuous kinetic models incorporate microscopic internal energy to the list of arguments of the distribution function, which allows to write a single Boltzmann equation describing a polyatomic gas. The collision operator has the two key elements that are subject to the modelling: (i) the cross section which encodes microscopic interaction law, and (ii) the weight function that aims at recovering a proper energy law at the macroscopic level. Continuous models precisely differ in the use of functional space as an environment where physical intuition is achieved: for [8] physical quantities associated to the kinetic model (such as gas density, mean velocity, energy, etc.) are obtained by means of the plain $L^1$ space, and so we refer to this model as non-weighted, while in [11, 12] a weighted $L^1$ space arise and so we call this model weighted. Both models are accurate in the case of polytropic gases, when macroscopic internal energy of the gas is linear with respect to the gas temperature. For non-polytropic gases, when this dependence is nonlinear, there are first attempts to theoretically rewrite macroscopic models starting from the weighted model [5, 22], but still the weight function remains unknown.

In this manuscript we restrict to polytropic gases, and perform a direct comparison of these two continuous models. We show that they are equivalent, but only after the distribution function is appropriately rescaled with the weight function and the cross section is reformulated. Redefinition of the cross section, firstly pointed out in [4], Remark 1, removes the singularity of the strong form of collision operator from the weighted model, which opens the door to the mathematically rigorous theory. For instance, the first rigorous results are obtained in [13] for the non-weighted model in the case of space homogeneity.

The kinetic weighted model is extensively used as a basis for deriving macroscopic models of extended thermodynamics for a polyatomic gas starting from the kinetic theory [24].

This research path starts with the fourteen moments model firstly introduced in [20], and improved in many ways afterwards [21, 3, 25]. Kinetic theory provides an
insight by calculating the production terms that allows for explicit expressions of relaxation times, which are of phenomenological nature in the macroscopic theories.

On the other hand, when shear stresses and heat conduction are neglected, six fields model arise. In this model, the dominant non-equilibrium effect is the dynamic pressure, which is an excess normal pressure added to standard thermodynamic pressure. The physical motivation for such a study is the fact that the bulk viscosity, and consequently the relaxation time for dynamic pressure is several order of magnitudes greater than the shear viscosity and heat conductivity [23, 27, 26]. This model is of particular interest, since it is one of the rare systems that admits a non-linear closure of the governing equations using the entropy principle [1, 2, 28]. It also admits the exact solution of the variational problem of Maximum Entropy Principle, as shown in [23, 5, 19]. In particular, in [19] the dynamic pressure is not introduced a priori, but rather regarded as a measure of deviation of the system from an equilibrium state, through the analysis of pressure tensor trace. The source term is calculated in [5] for the discrete energy model, and in [19] for the weighted continuous model which can be related to the source term of extended thermodynamics. The price for this result was the choice of the cross section: it is not Galilean invariant, but yields the Galilean invariant production term.

In this manuscript we take another path and we start with the non-weighted model and build both six and fourteen moments models using the Maximum Entropy Principle. We are motivated by recent rigorous results from [13], where a new model for the cross section is proposed. Apart from the fact that it allows the mathematical theory, this cross section provides production terms that are in total agreement with the macroscopic theory of extended thermodynamics. More precisely, for six fields model we prove that the residual inequality from [1] is satisfied on the whole range of model validity. On the other hand, production terms for the fourteen moments model lead to new expressions for relaxation times and transport coefficients. Since the cross section contains one parameter, we will show that this parameter can be adjusted so that the shear viscosity dependence upon temperature matches with the experimental data given in [9] for room temperature range and in [18, 16] for high temperatures, and at the same time recovers a value of the Prandtl number that coincides at a satisfactory level with its theoretical estimate obtained by means of Eucken’s relation.

The paper is organized as follows. Section 2 studies collisions in polyatomic gases, and introduces the main notions of continuous kinetic models. Then in Section 3 we describe the non-weighted and weighed models, that are further compared in Section 4. For the non-weighted model we establish macroscopic models, namely six fields model in Section 6 and fourteen moments model in Section 7. The Appendix contains computations of Jacobian of collision transformation, and of production terms for both macroscopic models.

2. STUDY OF COLLISIONS IN A POLYATOMIC GAS

In this Section, we describe a collision process and introduce the main notions used in continuous kinetic models.

We assume that interactions between particles are binary collisions of polyatomic molecules. Due to the complex structure of a polyatomic molecule, we need to take
into account internal degrees of freedom, as apart from the usual translation, there is a possible rotation and vibration of molecules during the collision process. Idea of continuous kinetic models is to capture these phenomena with a unique continuous variable $I$, that we call microscopic internal energy of the molecule. In continuous models, a single Boltzmann equation governs evolution of the distribution function, that now has extended list of arguments – besides the usual molecular velocity $v \in \mathbb{R}^3$, it depends also on the microscopic internal energy $I \in [0, \infty)$.

In order to study a collision process, we attribute a velocity-internal energy pair $(v, I)$ to each molecule. Then we consider the two colliding molecules, with pre-collisional molecular velocities and microscopic internal energies $(v', I')$ and $(v'_*, I'_*)$. After the collision, these quantities transform to $(v, I)$ and $(v'_*, I'_*)$ respectively. Here we consider elastic collisions, meaning that the total (kinetic+microscopic internal) energy of the molecular pair is conserved, and thus conservation laws of momentum and energy hold during the collision process,

$$
v + v_* = v' + v'_* ,
\frac{m}{2} |v|^2 + I + \frac{m}{2} |v_*|^2 + I_* = \frac{m}{2} |v'|^2 + I' + \frac{m}{2} |v'_*|^2 + I'_* .
$$

These equations can be written in the reference frame of center-of-mass, by introducing velocity of the center of mass $V$ and relative velocity $u$,

$$V := \frac{v + v_*}{2} , \quad u := v - v_* .
$$

Then (2.1) can be rewritten,

$$V = V' ,
\frac{m}{4} |u|^2 + I + I_* = \frac{m}{4} |u'|^2 + I' + I'_* =: E .
$$

In order to describe the complete collision transformation, the aim is express all pre-collisional quantities in terms of post-collisional ones. To that end, we use Borgnakke-Larsen procedure that first introduces parameter $R \in [0, 1]$ in order to separate pre-collisional kinetic energy $\frac{m}{4} |u'|^2$ and total microscopic internal energy $I' + I'_*$,

$$\frac{m}{4} |u'|^2 = RE , \quad I' + I'_* = (1 - R)E .
$$

Then, parameter $r \in [0, 1]$ distributes the total microscopic internal energy among the two colliding molecules, which implies

$$I' = r(1 - R)E , \quad I'_* = (1 - r)(1 - R)E .
$$

Finally, we parametrize relative speed from (2.4) with a unit vector $\sigma \in S^2$, which yields expression for pre-collisional velocities using conservation of momentum (2.3),

$$v' = V + \sqrt{\frac{RE}{m}} \sigma , \quad v'_* = V - \sqrt{\frac{RE}{m}} \sigma .
$$

Relations (2.6)–(2.5) together with

$$r' = \frac{I}{I + I_*} , \quad R' = \frac{I}{E - \frac{m}{4} |u|^2} , \quad \sigma' = \frac{u}{|u|} .
$$

define the following collision transformation

$$T : (v, v_*, I, I_*, r, R, \sigma) \rightarrow (v', v'_*, I', I'_*, r', R', \sigma').
$$
The following Lemma gives the Jacobian of this transformation.

**Lemma 2.1.** The Jacobian of transformation $T$ given in (2.8) is given by

$$J_T = \frac{(1 - R)R^2}{(1 - R')R'^2} = \frac{(1 - R)|u'|}{(1 - R')|u|}. \quad (2.9)$$

The proof of this Lemma can be found in Appendix A.

Using relations (2.5) and (2.7) we are in position to prove the following Lemma which shows the invariance property of a function that involves the product $II_*$.

**Lemma 2.2.** The following invariance holds

$$II_*(r(1 - r)(1 - R)^2 = I'I'(1 - r')(1 - R')^2,$$

where the involved quantities are linked via the mapping (2.8).

**Proof.** We first write

$$r(1 - R) = \frac{I'}{E}, I = r'(1 - R')E, (1 - r)(1 - R) = \frac{I'}{E}, I_* = (1 - r')(1 - R')E,$$

so that

$$r(1 - R) I (1 - r)(1 - R) I_* = I'I'(1 - r')(1 - R')I'(1 - r)(1 - R),$$

which concludes the proof. \qed

### 3. Kinetic models for a polyatomic gas

In this Section we describe the two continuous kinetic models for a polyatomic gas available in the literature. The goal is to write the Boltzmann equation governing the distribution function that probabilistically describes the state of a polyatomic gas.

In both cases, distribution function depends on the usual macroscopic variables: time $t \geq 0$ and space position $x \in \mathbb{R}^3$, but also on extended list of microscopic variables: molecular velocity $v \in \mathbb{R}^3$ and microscopic internal energy $I \in [0, \infty)$, i.e.

$$f := f(t, x, v, I) \geq 0,$$

and is non-negative. Its measure of change, collision operator, acts only on microscopic variables $v$ and $I$. We have the two different definitions of collision operators, depending on the functional space we work in.

We define the plain $L^1$ space,

$$L^1 = \left\{ f \text{ measurable : } \int_{\mathbb{R}^3 \times [0, \infty)} |f(v, I)| \, dIdv < \infty \right\},$$

and the $L^1$ space weighted with the suitable function $\varphi(I)$,

$$L^1_\varphi = \left\{ f \text{ measurable : } \int_{\mathbb{R}^3 \times [0, \infty)} |f(v, I)| \varphi(I) \, dIdv < \infty \right\}.$$

We describe the two models below.
3.1. **Model 1: Non-weighted model.** The non-weighted model is introduced in [8]. For the distribution function

\[ f := f(t, x, v, I) \geq 0, \]

we write the Boltzmann equation

\[ \partial_t f + v \cdot \nabla_x f = Q^{nw}(f, f)(v, I), \] (3.1)

with the collision operator \( Q^{nw}(f, f) \) defined in the strong form below.

3.1.1. **Collision operator \( Q^{nw} \) in the strong form.** We first introduce functions

\[ \phi_\alpha(r) := (r(1 - r))^\alpha, \quad \psi_\alpha(R) := (1 - R)^2\alpha. \] (3.2)

The strong form of collision operator for the non-weighted model reads

\[ Q^{nw}(f, f)(v, I) = \int_{\mathbb{R}^3 \times [0, \infty)} \left( f' f'_* \left( \frac{I I'_*}{I'_* I} \right)^\alpha - f f_* \right) \]
\[ \times B^{nw}(1 - R)\frac{B_\alpha(r)}{\psi_\alpha(R)} \, d\sigma \, dr \, dI \, dv, \] (3.3)

with \( \alpha > -1 \), and where we have used the standard abbreviations

\[ f' := f(t, x, v', I'), \quad f'_* := f(t, x, v'_*, I'_*), \quad f_* := f(t, x, v_*, I_*), \] (3.4)

and quantities \( v', I', v'_*, I'_* \) are described in the collision transformation \( T \) from (2.8).

The cross section \( B^{nw} \) is supposed to satisfy the micro-reversibility conditions

\[ B^{nw} := B^{nw}(v, v, I, I, R, r, \sigma) = B^{nw}(v', v', I', I', R', r', \sigma') \]
\[ = B^{nw}(v_*, v, I, I, R, 1 - r, -\sigma). \] (3.5)

Let us explain the terms involved in the strong form (3.3). First, term \((1 - R)B_\alpha\) is coming from the Jacobian of collision transformation computed in the Lemma 2.1. Then, renormalization of a distribution function \( f \) by the factor \( I^\alpha \) will allow to obtain the proper caloric equation of state, which causes presence of functions \( \phi_\alpha(r) \) and \( \psi_\alpha(R) \) aiming to ensure the invariance property of the measure, as shows the upcoming Lemma 3.1. As we shall see later, \( \alpha \) will be strongly connected to the degrees of freedom of a molecule.

**Lemma 3.1.** The measure

\[ dA = B^{nw} \phi_\alpha(r) (1 - R)B_\alpha \psi_\alpha(R) I^\alpha I_*^\alpha \, d\sigma \, dr \, dI \, dv \] (3.6)

is invariant with respect to the changes

\[ (v, v_*, I, I_*, R, r, \sigma) \leftrightarrow (v', v'_*, I', I'_*, R', r', \sigma'), \] (3.7)
\[ (v, v_*, I, I_*, R, r, \sigma) \leftrightarrow (v_*, v, I, I_* I, R, 1 - r, -\sigma). \] (3.8)

**Proof.** The proof immediately follows from the property (3.5) of the cross section \( B^{nw} \), Lemma 2.2 and Jacobian of transformation (2.8) from Lemma 2.1.

Using the measure \( dA \), we can rewrite the strong form (3.3) in the following manner,

\[ Q^{nw}(f, f)(v, I) = \int_{\mathbb{R}^3 \times [0, \infty)} \left( \frac{f' f'_*}{I'_* I_*} - \frac{f f_*}{I I_*} \right) \, dA, \] (3.9)

obtained by pulling out the factor \((I I_*)^\alpha\).
3.1.2. Collision operator $Q^{nw}$ in the weak form. The choice of the functional space becomes evident in the definition of the weak form. We here work in the plain $L^1$ space.

Lemma 3.2 (The weak form of the collision operator $Q^{nw}$). For any test function $\chi(v, I)$ that makes the following left hand side meaningful, the collision operator (3.3) takes the following weak form

$$
\int_{\mathbb{R}^3 \times [0, \infty)} Q^{nw}(f, f)(v, I) \chi(v, I) dI dv
$$

$$
= \frac{1}{2} \int_{\mathbb{R}^6 \times [0, \infty)^2 \times [0, 1]^2 \times S^2} \frac{f f_*}{(II_*)^\alpha} (\chi(v', I') + \chi(v'_*, I'_*) - \chi(v, I) - \chi(v_*, I_*)) dA,
$$

(3.10)

with the measure $dA$ from (3.6).

Proof. We integrate the collision operator (3.3) against a test function $\chi(v, I)$ with respect to $v$ and $I$ variables and then perform changes of variables (3.7) and (3.8). Using invariance properties of the measure $dA$ (3.6) stated in Lemma 3.1, we obtain

$$
\int_{\mathbb{R}^3 \times [0, \infty)} Q^{nw}(f, f)(v, I) \chi(v, I) dI dv
$$

$$
= \int_{\mathbb{R}^6 \times [0, \infty)^2 \times [0, 1]^2 \times S^2} \frac{f f_*}{(II_*)^\alpha} (\chi(v', I') - \chi(v, I)) dA
$$

(3.11)

$$
= \int_{\mathbb{R}^6 \times [0, \infty)^2 \times [0, 1]^2 \times S^2} \frac{f f_*}{(II_*)^\alpha} (\chi(v'_*, I'_*) - \chi(v_*, I_*)) dA,
$$

which yields desired estimate (3.19). □

The conservation laws at the microscopic level (2.1) imply the annihilation of the weak form (3.10) for the conserved quantities. More precisely,

$$
\int_{\mathbb{R}^3 \times [0, \infty)} Q^{nw}(f, f)(v, I) \left( \frac{m}{mv^2 + I} \right) dI dv = 0.
$$

(3.12)

The functions $m$, $mv$ and $\frac{m}{mv^2 + I}$ are called the collision invariants.

Our next goal is to formulate the H-theorem for the collision operator $Q^{nw}$. To that end, we first define the entropy production,

$$
D^{nw}(f) = \int_{\mathbb{R}^3 \times [0, \infty)} Q^{nw}(f, f)(v, I) \log(f(v, I)I^{-\alpha}) dI dv,
$$

(3.13)

and then study its properties in the following theorem.

Theorem 3.3 (H-theorem). Let the cross section $B^{nw}$ be positive almost everywhere, and let $f \geq 0$ be such that the collision operator $Q^{nw}(f, f)$ and the entropy production $D^{nw}(f)$ are well defined. Then the following properties hold,

i. Entropy production is non-positive, that is

$$
D^{nw}(f) \leq 0.
$$

(3.14)

ii. The three following properties are equivalent

(1) $D^{nw}(f) = 0$ for all $v \in \mathbb{R}^3$, $I \in \mathbb{R}_+$,
$Q_{nw}(f,f) = 0$ for all $v \in \mathbb{R}^3$, $I \in \mathbb{R}_+$,

(3) There exists $n \geq 0, U \in \mathbb{R}^3$, and $T > 0$, such that

$$f(v, I) = \frac{n}{Z(T)} \left( \frac{m}{2\pi kT} \right)^{\frac{3}{2}} I^\alpha e^{-\frac{1}{kT} (\frac{m}{2} |v-U|^2 + I)},$$

where $Z(T)$ is a partition (normalization) function

$$Z(T) = \int_{[0,\infty)} I^\alpha e^{\frac{1}{kT} I} dI = (kT)^{\alpha+1} \Gamma(\alpha+1),$$

where $\Gamma$ represents the Gamma function.

The proof is given in [8].

3.2. Model 2: Weighted model. This model originates from [11, 12]. In this case distribution function

$$g := g(t, x, v, I) \geq 0,$$

satisfies the Boltzmann equation

$$\partial_t g + v \cdot \nabla_x g = Q_w(g, g)(v, I),$$

where $Q_w(g, g)$ is the collision operator that acts only on $(v, I)$ variables and is described below.

3.2.1. Collision operator $Q_w$ in the strong form. Weighted model is related to the weighted $L^1$ space, and in this case the collision operator is defined as

$$Q_w(g, g)(v, I) = \int_{\mathbb{R}^3 \times [0,\infty) \times [0,1]^2 \times S^2} \frac{(g' g'_s - gg_s)}{\varphi(I)}(g' g'_s - gg_s) \times B^w(1 - R) R^\frac{3}{2} \frac{1}{\varphi(I)} d\sigma dr dI dI' dv,$$

where we have used the standard conventions as in (3.4), with the primed quantities from (2.8), and the cross section

$$B^w := B^w(v, v_s, I, I_s, R, r, \sigma) = B^w(v', v'_s, I', I'_s, R', r', \sigma'),$$

The factor $(1 - R) R^\frac{3}{2}$ is Jacobian of the collision transformation (2.9). The measure $\varphi(I)$ aims at capturing the features of polyatomic gases at the macroscopic level, and notably to provide an agreement with the caloric equation of state. Contrary to the non-weighted model, this measure is not introduced a priori, which theoretically gives a room to obtain a general equation for polytropic or non-polytropic gases, corresponding to linear or non-linear dependence of the macroscopic internal energy with respect ot temperature, respectively.

3.2.2. Collision operator $Q_w$ in the weak form. For the weighted model, the weak form of collision operator is obtained by means of the integration against the weight function $\varphi(I)$, as described in the upcoming Lemma 3.4.
Lemma 3.4. For any test function \( \chi(v,I) \) that makes the following left hand side meaningful, the collision operator \( (3.17) \) has the following weak form

\[
\int_{\mathbb{R}^3 \times [0, \infty)} Q^w(g,g)(v,I) \chi(v,I) \varphi(I) \, dI \, dv \\
= \frac{1}{2} \int_{\mathbb{R}^6 \times [0,\infty)^2 \times [0,1]^2 \times S^2} g g_* \left( \chi(v',I') + \chi(v'_*,I'_*) - \chi(v,I) - \chi(v_*,I_*) \right) \\
\times B^w (1 - R) R^\frac{7}{2} \, d\sigma \, dr \, dR \, dv_* \, dI_* \, dv \, dI,
\]

where primed quantities as functions of non-primed ones are given in (2.8), and \( B^w \) is from (3.18).

Proof. We first integrate the strong form (3.17) against suitable test function \( \chi(v,I) \) in the velocity-internal energy space \( \mathbb{R}^3 \times [0, \infty) \) with the weight \( \varphi(I) \) in \( I \). Then we change the variables, first we interchange primes and non-primes (3.7) and then we replace particles by means of (3.8). This gives

\[
\int_{\mathbb{R}^3 \times [0, \infty)} Q^w(g,g)(v,I) \chi(v,I) \varphi(I) \, dI \, dv \\
= \int_{\mathbb{R}^6 \times [0,\infty)^2 \times [0,1]^2 \times S^2} g g_* \left( \chi(v',I') - \chi(v,I) \right) \\
\times B^w (1 - R) R^\frac{3}{2} \, d\sigma \, dr \, dR \, dv_* \, dI_* \, dv \, dI,
\]

where we have used invariance of the cross section \( B^w \) stated in (3.18) and Jacobian of the transformation (2.9), which concludes the estimate (3.19).

Microscopic conservation laws (2.1) imply that the weak form (3.19) vanishes when test functions are chosen as \( m, mv \) and \( \frac{m}{r} |v|^2 + I \),

\[
\int_{\mathbb{R}^3 \times [0, \infty)} Q^w(g,g)(v,I) \left( \frac{m}{r} \frac{mv}{|v|^2 + I} \right) \, dI \, dv = 0.
\]

We refer to those test functions as collision invariants.

We now formulate the H-theorem for the collision operator \( Q^w \). We first define the entropy production,

\[
D^w(g) = \int_{\mathbb{R}^3 \times [0, \infty)} Q^w(g,g)(v,I) \log(g(v,I)) \varphi(I) \, dI \, dv,
\]

Theorem 3.5 (H-theorem). Let the cross section \( B^w \) be positive almost everywhere, and let \( g \geq 0 \) such that the collision operator \( Q^w(g,g) \) and entropy production \( D^w(g) \) are well defined. Then the following properties hold

i. Entropy production is non-positive, that is

\[
D^w(g) \leq 0.
\]

ii. The three following properties are equivalent
(1) \(D^w(g) = 0\) for all \(v \in \mathbb{R}^3, \ I \in \mathbb{R}_+\),

(2) \(Q^w(g, g) = 0\) for all \(v \in \mathbb{R}^3, \ I \in \mathbb{R}_+\),

(3) There exists \(n \geq 0, U \in \mathbb{R}^3\), and \(T > 0\), such that
\[
g(v, I) = \frac{n}{Z(T)} \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-\frac{1}{kT} \left(\frac{m}{2} |v - U|^2 + I\right)},
\]
where \(Z(T)\) is a normalization function
\[
Z(T) = \int_{[0, \infty)} e^{-\frac{1}{kT} \varphi(I)} dI.
\]
The proof can be found in [12].

4. Comparison of the continuous kinetic models

In order to compare the two continuous kinetic models, we start with the weighted one presented in Section 3.2. We firstly change the distribution function \(g\) so that the weight \(\varphi(I)\) is detached from it. More precisely, we introduce the distribution function \(f\) by means of
\[
f(t, x, v, I) = g(t, x, v, I) \varphi(I).
\]
Then the Boltzmann equation (3.16) written in terms of \(g\) can be rewritten in terms of \(f\),
\[
\partial_t f + v \cdot \nabla_x f = \int_\Omega \left( f' f' \varphi(I) \left( \frac{I}{I'} \right) \varphi'(I') - ff' \right) \frac{B^w(I) \varphi(I)}{\varphi(I) \varphi(I')}(1 - R) R^2 \sigma dr dR dI_* d\sigma_* d\sigma,
\]
which coincides with the Boltzmann equation from [4], Remark 1. However, the effective collision cross section
\[
\frac{B^w(I) \varphi(I)}{\varphi(I) \varphi(I')},
\]
as it was named in [4], in general does not satisfy the micro-reversibility assumptions.

In order to overcome this drawback and to obtain the non-weighted model described in Section 3.1, we need to make a choice of the function \(\varphi(I)\), and so we take
\[
\varphi(I) = I^\alpha.
\]
With this choice of the weight function \(\varphi(I)\), the Boltzmann equation (4.2) becomes
\[
\partial_t f + v \cdot \nabla_x f = \int_\Omega \left( f' f' \left( \frac{I}{I'} \right) \varphi(I) \varphi'(I') - ff' \right) \frac{B^w(I) \varphi(I)}{I\alpha \varphi}(1 - R) R^2 \sigma dr dR dI_* d\sigma_* d\sigma_*.
\]
Now, it is clear that in order to recover non-weighted formula (3.1)-(3.3) we need to multiply and divide by a factor \((r(1 - r))^{\alpha} (1 - R)^{2\alpha} = \phi_\alpha(r) \psi_\alpha(R)\),
\[
\partial_t f + v \cdot \nabla_x f = \int_\Omega \left( f' f' \left( \frac{I}{I'} \right) \varphi(I) \varphi'(I') - ff' \right) \times \frac{B^w(I) \varphi(I)}{I\alpha \varphi \phi_\alpha(r) \psi_\alpha(R)} \phi_\alpha(r) \psi_\alpha(R)(1 - R)^{2\alpha} \sigma dr dR dI_* d\sigma_* d\sigma_*.
This factor makes the effective cross section (4.3) for the choice (4.4) micro-reversible, and so we are led to define the new cross section

\[ B^{nw} = \frac{B^w}{I^\alpha \phi_{\alpha}(r) \psi_{\alpha}(R)}, \tag{4.5} \]

that leads to the non-weighted model as in (3.1)-(3.3).

It is worthwhile to remark that in order to pass from the weighted to the non-weighted model or vice versa, it is not enough to only change distribution function by detaching or attaching the weight. In addition, we need to reformulate the cross section using (4.5). Therefore, one needs to be careful in choosing the cross section, since the collision operator (3.3) hides the micro-reversible part (3.2) in its cross section \( B^{nw} \). In another words, cross section \( B^{nw} \) that appears in the non-weighted model is not the same as \( B^w \) from the weighted one.

Reformulation of the cross section stated in (4.5) has some mathematical consequences as well. Namely, for the choice \( \varphi(I) = I^\alpha \) the strong form of collision operator in the weighted model (3.17) has singularity at zero for the variable \( I \), while the collision operator of the non-weighed model (3.3) does not have this drawback. The formula (4.5) reveals the reason for such a behavior: the cross section in (3.3) is multiplied by a factor that removes the singularity.

Therefore, we conclude that the equivalence of the models (3.16)-(3.17) and (3.1)-(3.3) holds after the distribution function is renormalized as in (4.1) and the cross section is reformulated by using the formula (4.5) for the choice (4.4) of the weight function.

5. Macroscopic models for a rarefied polyatomic gas

As it is very well known, the kinetic models can provide the macroscopic ones, using the so called moment method. The basic idea is to build infinite hierarchies of moment equations by integrating the Boltzmann equation over the microscopic molecular variables space – velocity space in the case of a monatomic gas leading to one hierarchy of moments, or velocity and microscopic internal energy space for a polyatomic gas causing the two hierarchies of moments, momentum and energy like. These infinite hierarchies are cut at some order of moments, that yield a non-closed system of moment equations. One of the possible ways to close the system is to formulate a variational problem, the maximum entropy principle (MEP), that seeks for a distribution function which maximizes the physical entropy subject to some constraints. These constrains are actually macroscopic moment densities related to the physical process at hand. Determination of the distribution function allows to obtain non-convective fluxes as functions of moment densities. Then it remains to calculate the production terms, when the choice of the cross section becomes a crucial aspect.

In Sections 6 and 7 we study six and fourteen fields models, respectively. The six moments model corresponds to the usual equilibrium macroscopic observables, mass \( \rho \), momentum \( \rho U \) and energy density \( \frac{1}{2} \rho |U|^2 + \rho e \), and the dynamic pressure \( \Pi \) as a dominant non-equilibrium variable. The fourteen moments model extends the list of non-equilibrium effects, by taking into account the stress tensor \(-p_{ij}, i, j \in \{1, 2, 3\}\)
and the heat flux \( q_i, i = 1, 2, 3 \). Our goal is to use MEP and to establish both six and fourteen moments models starting from the non-weighted Boltzmann equation described in the Section 3.1. The production terms are calculated for the following choice of the cross section

\[
B^{nw}(v, v^*, I, I^*, r, R, \sigma) = b(\hat{u} \cdot \sigma) \left( R \tilde{\sigma} |u|^{\gamma} + \left( r(1 - R) \frac{I}{m} \right) \tilde{\sigma} + \left( (1 - r)(1 - R) \frac{I^*}{m} \right) \tilde{\sigma} \right), \tag{5.1}
\]

where \( \gamma > 0 \), \( u := v - v^* \), \( \hat{u} = u / |u| \), that corresponds to the model 3 of the cross section in [13], shown to allow for mathematical rigorous theory in the space homogeneous setting. Function \( b \) will be assumed integrable over the unit sphere \( S^2 \) in the case of six fields equations, whereas will be taken constant for the fourteen moments model.

One of the main tools in the MEP procedure is the physical entropy density defined as follows.

**Definition 5.1 (Physical entropy).** For a distribution function \( f \geq 0 \) we define the physical entropy density

\[
h = -k \int_{\mathbb{R}^3 \times [0, \infty)} f \log(fI^{-\alpha}) \, dI \, dv, \tag{5.2}
\]

where \( k \) is the Boltzmann constant and \( \alpha > -1 \).

The entropy law is obtained by integration of the Boltzmann equation (3.1) against the test function \( \log(fI^{-\alpha}) \) multiplied by the factor \( -k \), i.e.

\[
\partial_t h + \sum_{j=1}^{3} \partial_{x_j} h_j = \Sigma,
\]

where \( h_j \) is the flux of entropy density in the direction \( x_j \) and \( \Sigma \) is the entropy production density, defined from the kinetic theory point of view via

\[
h_j = -k \int_{\mathbb{R}^3 \times [0, \infty)} v_j f \log(fI^{-\alpha}) \, dI \, dv, \quad \Sigma = -k D^{nw}(f), \tag{5.3}
\]

where \( D^{nw}(f) \) was already introduced in (3.13).

Both six and fourteen fields models are designated for non-equilibrium processes. We introduce the notion of local equilibrium using the idea of MEP. Namely, we seek for a distribution function such that the physical entropy (5.2) is maximized subject to the prescribed macroscopic densities obtained as its moments against the collision invariants (3.12). Then it can be shown that these densities satisfy Euler system of equations, that can be obtained from the integration of the Boltzmann equation against the collision invariants (3.12) over the velocity-microscopic internal energy space. The Euler system is conservative (i.e. production terms are all zero) which can be concluded from the kinetic theory point of view by the vanishing properties of the weak form (3.12).
Lemma 5.2 (Local equilibrium distribution function). The distribution function that solves the following problem

\[
\max_{f} h = -k \int_{\mathbb{R}^{3} \times [0, \infty)} f \log(f^\alpha) \, dI \, dv
\]

s.t.

\[
\begin{pmatrix}
\rho \\
0_i \\
(\alpha + \frac{5}{2}) n kT
\end{pmatrix} = \int_{\mathbb{R}^{3} \times [0, \infty)} \begin{pmatrix}
m_i \\
m c_i \\
m \frac{|c|^2 + I}{2}
\end{pmatrix} f \, dI \, dv,
\]

where we have used peculiar velocity \( c = v - U \), is the local equilibrium distribution function,

\[
f_M(t, x, v, I) := I^{\alpha} e^{-1 - \frac{m}{2} \lambda^{(0)}_m + \frac{3}{2} \sum_{i=1}^{3} \lambda^{(1)}_i m c_i + \mu^{(0)} \left( \frac{m}{2} |c|^2 + I \right)}.
\]

Proof. We follow the classical procedure of maximum entropy principle. Namely, we first introduce Lagrange multipliers \( \lambda^{(0)}, \lambda^{(1)}_i \) and \( \mu^{(0)} \) that correspond to the constraints (5.4). Then the extended functional reads

\[
\mathcal{L} = \int_{\mathbb{R}^{3} \times [0, \infty)} \left\{ -k f \log(f^\alpha) - f \left( \lambda^{(0)}_m + \sum_{i=1}^{3} \lambda^{(1)}_i m c_i + \mu^{(0)} \left( \frac{m}{2} |c|^2 + I \right) \right) \right\} dI \, dc.
\]

The solution of the Euler-Lagrange equation \( \delta \mathcal{L} / \delta f \) is given by

\[
f = I^{\alpha} e^{-1 - \frac{m}{2} \lambda^{(0)}_m - \frac{3}{2} \sum_{i=1}^{3} \lambda^{(1)}_i c_i - \frac{1}{2} \mu^{(0)} \left( \frac{m}{2} |c|^2 + I \right)}.
\]

Plugging this form into the constraints of the problem (5.4) we get a system of algebraic equations whose solution allows to express Lagrange multipliers in terms of macroscopic densities, which implies the solution (5.5).

Remark 1. We mention that Lagrange multipliers do not depend on the choice of the functional space, and thus they coincide with the ones obtained using the weighted model.

6. Six fields model

The six fields model corresponds to the moment equations obtained by integrating the Boltzmann equation (3.1) with respect to the microscopic set of variables – molecular velocity \( v \) and microscopic internal energy \( I \), against six test functions,

\[
m, \, mv, \, m|v|^2, \, \frac{m}{2} |v|^2 + I.
\]

Introducing the peculiar velocity \( c = v - U \) as a relative velocity of the molecules \( v \) with respect to the macroscopic velocity of the gas \( U \), we define densities of macroscopic observables that correspond to these test functions as follows

\[
\begin{pmatrix}
\rho \\
\rho U \\
3(p + \Pi) \\
\rho e
\end{pmatrix} = \int_{\mathbb{R}^{3} \times \mathbb{R}_+} \begin{pmatrix}
m \\
v \\
m |c|^2 \\
\frac{m}{2} |c|^2 + I
\end{pmatrix} f \, dI \, dv.
\]
Now integration of the Boltzmann equation against test functions (6.1) leads to the following set of six moments equations

\[
\begin{align*}
\partial_t \rho + \sum_{j=1}^{3} \partial_{x_j} (\rho U_j) &= 0, \\
\partial_t \rho U_i + \sum_{j=1}^{3} \partial_{x_j} (\rho U_j U_i + p_{ij}) &= 0, \\
\partial_t \left( \rho |U|^2 + 3(p + \Pi) \right) + \sum_{j=1}^{3} \partial_{x_j} \left\{ U_j \left( \rho |U|^2 + 3(p + \Pi) \right) + 2 \sum_{i=1}^{3} p_{ji} U_i + \sum_{i=1}^{3} p_{iij} \right\} &= \mathcal{P}, \\
\partial_t \left( \frac{1}{2} \rho |U|^2 + p e \right) + \sum_{j=1}^{3} \partial_{x_j} \left\{ U_j \left( \frac{1}{2} \rho |U|^2 + p e \right) + 3 \sum_{i=1}^{3} p_{ji} U_i + q_j \right\} &= 0,
\end{align*}
\]

(6.3)

for \( i = 1, 2, 3 \), and where we have assumed the following relations

\[
\sum_{i=1}^{3} p_{ii} = 3(p + \Pi), \quad pe = \left( \alpha + \frac{5}{2} \right) p.
\]

(6.4)

The undetermined non-convective fluxes are defined as

\[
\begin{pmatrix}
p_{ij} \\
\sum_{i=1}^{3} p_{iij}/q_j
\end{pmatrix}
= \int_{R^3 \times [0, \infty)} \begin{pmatrix}
m c_i c_j \\
\frac{m}{2} |c|^2 c_j \\
\left( \frac{m}{2} |c|^2 + I \right) c_j
\end{pmatrix}
f \, dI \, dc,
\]

(6.5)

while the production term reads

\[
\mathcal{P} = \int_{R^3 \times [0, \infty)} m |v|^2 Q^{nw}(f, f)(v, I) \, dI \, dv.
\]

(6.6)

The goal of this Section is to provide a closure to the system (6.3) via Maximum Entropy Principle, which is achieved by determining the six fields distribution function in the Lemma 6.1, and then calculating non-convective fluxes (6.5), as much as the production term (6.6) for a specific choice of the cross section (5.1). The obtained results are compared with the theory of extended thermodynamics in Section 6.1.

Lemma 6.1 (Six moments distribution function). Solution of the maximum entropy principle

\[
\begin{align*}
\max_f h &= -k \int_{R^3 \times [0, \infty)} f \log(f I^{-\alpha}) \, dI \, dv \\
\text{s.t. } &
\begin{pmatrix}
\rho \\
0 \\
3(p + \Pi) \\
\left( \alpha + \frac{5}{2} \right) p
\end{pmatrix}
= \int_{R^3 \times [0, \infty)} \begin{pmatrix}
m c_i \\
\frac{m}{2} |c|^2 \\
\frac{m}{2} |c|^2 + I
\end{pmatrix}
f \, dI \, dc.
\end{align*}
\]
is given with
\[
\hat{f}_6^6 = I^\alpha \frac{p}{m} \left( \frac{m}{2\pi kT} \right)^{\frac{\alpha}{2}} \frac{1}{(1 + \frac{\Pi}{p})^{\frac{\alpha}{2}}} \frac{1}{\Gamma(\alpha + 1)} \frac{1}{(kT)^{\alpha + 1}} \left( 1 - \frac{1}{2(\alpha + 1)nT} \right)^{\alpha + 1} \\
\times e^{-\frac{\hat{p}}{kT} \left( \frac{1}{(1 + \frac{\Pi}{p})^2} + \frac{1}{\Gamma(\alpha + 1)} \right)} I, \tag{6.7}
\]
that provides convergent moments if
\[
-1 < \frac{\Pi}{p} < \frac{2}{3}(\alpha + 1). \tag{6.8}
\]
The proof is very similar to the one in [23, 19], by virtue of Remark 1.

This distribution function allows to close the system of equations (6.3) for a particular choice of the cross section, as states the following Proposition.

**Proposition 1** (Six fields system of equations). **Closed system of equations for six fields reads**

\[
\partial_t \rho + \sum_{j=1}^3 \partial_{x_j} (\rho U_j) = 0,
\]
\[
\partial_t \rho U_i + \sum_{j=1}^3 \partial_{x_j} (\rho U_i U_j + (\Pi + p) \delta_{ij}) = 0,
\]
\[
\partial_t (\rho |U|^2 + 3(p + \Pi)) + \sum_{j=1}^3 \partial_{x_j} \left\{ U_j \left( \rho |U|^2 + 5(p + \Pi) \right) \right\} = \mathcal{P},
\]
for \( i = 1, 2, 3 \), where the production term \( \mathcal{P} \) for the choice of the cross section (5.1) with the function \( b \in L^1(d\sigma) \), is given by
\[
\mathcal{P} = -\mathcal{C}_P \frac{p}{\rho} \frac{\alpha + \frac{5}{2}}{\rho (\alpha + 1)} \Pi, \tag{6.9}
\]
with the positive constant
\[
\mathcal{C}_P = \frac{\rho^{2-H}}{m} p^2 \sqrt{\frac{2}{\pi}} \Vert b \Vert_{L^1(d\sigma)} \Gamma \left( \frac{4\alpha + \gamma + 9}{2} \right) \left( k_1 \left( 2 \left( 1 + \frac{\Pi}{p} \right) \right)^{\frac{\alpha}{2}} + k_2 \left( 1 - \frac{3}{2(\alpha + 1) \Pi} \right)^{\frac{\alpha}{2}} \right), \tag{6.10}
\]
where constants \( k_1 \) and \( k_2 \) are
\[
k_1 = 2^{\frac{\alpha + 3}{2}} \Gamma(\alpha + 2) \Gamma(\alpha + 1) \Gamma \left( \frac{\gamma + 3}{2} \right) \Gamma \left( \frac{\gamma + 5}{2} \right), \tag{6.11}
\]
\[
k_2 = \frac{3\sqrt{\pi}}{4} \left( 2 \alpha + \gamma + 2 \right) \Gamma \left( \alpha + \frac{\gamma}{2} + 1 \right)^2,
\]
for any \( \alpha > -1 \), and \( \Pi \) is in the range (6.8).
Proof. The proof easily follows by plugging distribution function $\hat{f}^6$ into definitions of non-convective fluxes (6.5). Indeed, for $f = \hat{f}^6$ we obtain

$$p_{ij} = (p + \Pi)\delta_{ij}, \quad \sum_{i=1}^{3} p_{ii} = 0, \quad q_j = 0, \quad i, j = 1, 2, 3.$$  

Details of the calculation of production term (6.6) is given in Appendix Section B.1. □

**Proposition 2** (Entropy law for six fields model). The physical entropy density and its flux for the distribution function $\hat{f}^6$ from (6.7) are

$$h(\hat{f}^6) = -k \frac{\rho}{m} \left\{ -\left(\alpha + \frac{5}{2}\right) + \log \left( \frac{\frac{m}{2\pi kT}}{(1 + \frac{\Pi}{p})^{\frac{1}{\alpha + 1}}} \left( 1 + \frac{\Pi}{p} \right) \right) \right\},$$  

(6.12)

and

$$h_j(\hat{f}^6) = U_j h(\hat{f}^6), \quad j = 1, 2, 3.$$  

(6.13)

Moreover, the entropy density production term for the cross section (5.1) reads

$$\Sigma = -\frac{k \rho}{2mp} \left( 1 - \frac{3}{2(\alpha + 1)} \frac{\Pi}{p} \right)^{-1} \left( 1 + \frac{\Pi}{p} \right)^{-1} \frac{(\alpha + \frac{5}{2}) \Pi}{(\alpha + 1) p} \mathcal{P},$$  

(6.14)

where $\mathcal{P}$ is calculated in (6.9), and the non-negativity

$$\Sigma \geq 0$$

holds for every $\Pi/p$ in the range (6.8).

Proof. Plugging the distribution function (6.7) into the definition of the entropy density (5.2) and its flux (5.3) we obtain (6.12)–(6.13). The production term $\Sigma$ is by virtue of (5.3) proved in the Appendix B.2. Using the calculated $\mathcal{P}$ from (6.9), (6.14) becomes

$$\Sigma = \frac{k}{2m} \left( 1 - \frac{3}{2(\alpha + 1)} \frac{\Pi}{p} \right)^{-1} \left( 1 + \frac{\Pi}{p} \right)^{-1} \frac{(\alpha + \frac{5}{2})^2 \mathcal{P}}{(\alpha + 1)^2} \geq 0,$$

for every $\Pi/p$ in the range of the validity of the model (6.8). □

**6.1. Comparison with extended thermodynamics.** In extended thermodynamics for six moments [1], the non-equilibrium part of the entropy density was denoted with $\mathcal{K}$, for which the two conditions are prescribed: (i) it vanishes for $\Pi = 0$, and (ii) it satisfies partial differential equation (25) from [1]. Then applying the entropy principle it was shown that the entropy production term $\Sigma$ is related to the production term $\mathcal{P}$ with function $\mathcal{K}$ in the following way

$$\Sigma = \frac{1}{3} \frac{\partial \mathcal{K}}{\partial \Pi} \mathcal{P} > 0.$$  

(6.15)

In the present manuscript, starting from the kinetic theory we will determine function $\mathcal{K}$, show that it satisfies PDE (25) from [1], and then prove the above residual inequality (6.15) for the production term $\mathcal{P}$ from (6.9). Moreover, the entropy production term of the form (6.15) is equal to the one already calculated in
This ensures compatibility of our kinetic, as much as macroscopic six fields model, with the extended thermodynamics.

Starting from the kinetic theory, one solution of the PDE (25) from [1] can be found. As pointed out in [23, 19], function $\mathcal{K}$ can be a difference of the entropy density evaluated at the distribution function $\hat{f}_6$ corresponding to the six fields problem given in (6.7) and the local equilibrium distribution function $f_E$ from (5.5). In our notation,

$$\mathcal{K}(\rho, p, \Pi) := \frac{k_p}{m} \log \left( \left( 1 + \frac{\Pi}{p} \right)^{\frac{\gamma}{2}} \left( 1 - \frac{3\Pi}{2(\alpha + 1)p} \right)^{\alpha + 1} \right).$$

Using expressions (6.12), (5.5) and definition of the physical entropy (5.2), we obtain the following form of $\mathcal{K}$,

$$\mathcal{K}(\rho, p, \Pi) = \frac{3k_p}{2m p (\alpha + 1)} \left( 1 - \frac{3\Pi}{2(\alpha + 1)p} \right)^{-1} \left( 1 + \frac{\Pi}{p} \right)^{-1} \frac{\Pi}{p}. \quad (6.16)$$

With the expression above, it is easy to see that the entropy production (6.15) coincides with (6.14).

Therefore, we have proven that our kinetic model provides six fields model compatible with entropy principle from extended thermodynamics.

### 6.2. Relaxation time

Relaxation time $\tau_\Pi$ for the non-equilibrium variable $\Pi$ is obtained by linearizing the production term $\mathcal{P}$ around $\Pi = 0$, that yields

$$\mathcal{P} = -\frac{1}{\tau_\Pi}. \quad (7.1)$$

for

$$\tau_\Pi = \left\{ \left( \frac{\alpha + \frac{\gamma}{2}}{\alpha + 1} \right) \rho^\frac{1}{2} \sqrt{\frac{2}{\pi}} \|b\|_{L^1(\mathbb{R}^m)} \Gamma \left( \frac{4\alpha + \gamma + 9}{2} \right) \left( 2\tau_1 k_1 + k_2 \right) \right\}^{-1},$$

where constants $k_1$ and $k_2$ are from (6.11).

### 7. Fourteen moments model

The macroscopic model of fourteen moments is obtained by extending the list of test functions given in (6.1) that will allow to take into account evolution equations for momentum and heat fluxes. Namely, instead of (6.1) we consider the following test functions

$m, m v, m v_i v_j, \frac{m}{2} |v|^2 + I, \left( \frac{m}{2} |v|^2 + I \right) v_j. \quad (7.1)$

If addition to (6.2) and (6.5) we define the non-convective fluxes

$$\left( \begin{array}{c} p_{ijk} \\ q_{ij} \end{array} \right) = \int_{\mathbb{R}^3 \times \mathbb{R}^+} \left( \begin{array}{c} m c_i c_j c_k \\ m c_i c_j \end{array} \right) \left( \frac{m}{2} |c|^2 + I \right) c_i c_j \frac{dI}{dI} dv, \quad (7.2)$$
then integration of the Boltzmann equation (3.1) against the test functions (7.1) yields system of equations governing the 14 moments, namely,

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} \partial_{x_j} (\rho U_j) &= 0, \\
\frac{\partial (\rho U_i)}{\partial t} + \sum_{j=1}^{3} \partial_{x_j} (\rho U_i U_j + p_{ij}) &= 0, \\
\frac{\partial (\rho U_i U_j + p_{ij})}{\partial t} + \sum_{k=1}^{3} \partial_{x_k} \{\rho U_i U_j U_k + U_i p_{jk} + U_j p_{ki} + U_k p_{ij} + p_{ijk}\} &= P_{ij},
\end{align*}
\]

\[
\begin{align*}
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho |U|^2 + \rho e \right) + \sum_{i=1}^{3} \partial_{x_i} \left\{ \left( \frac{1}{2} \rho |U|^2 + \rho e \right) U_i + \sum_{j=1}^{3} p_{ij} U_j + q_i \right\} &= 0, \quad (7.3) \\
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho |U|^2 + \rho e \right) U_i + \sum_{j=1}^{3} p_{ij} U_j + q_i &= \left( \frac{1}{2} \rho |U|^2 + \rho e \right) U_i U_j \\
&+ \sum_{k=1}^{3} (U_i U_k p_{jk} + U_j U_k p_{ik}) + \frac{1}{2} \rho |U|^2 p_{ij} + \sum_{k=1}^{3} U_k p_{ijk} + q_i U_j + q_j U_i + q_{ij} \right\} &= Q_i,
\end{align*}
\]

with the production terms

\[
\begin{pmatrix}
P_{ij} \\
Q_i
\end{pmatrix} = \int_{\mathbb{R}^3 \times \mathbb{R}^+} \left( \frac{m}{2} \frac{v_i v_j}{|v|^2 + I} \right) Q_{vw}(f, f)(v, I) \, dI \, dv, \quad (7.4)
\]

for any \( i, j = 1, 2, 3 \). Our aim is to close the system above in an approximative setting by exploiting the maximum entropy principle and then proceeding with the appropriate linearization of the distribution function around local equilibrium. Production terms are linearized as well and calculated for the cross section (5.1). Linearization is unavoidable since the exact solution of the variational problem does not yield convergent moments.

**Lemma 7.1 (Fourteen moments distribution function).** The solution of the maximum entropy principle

\[
\operatorname{max} f \quad h = -k \int_{\mathbb{R}^3 \times [0, \infty)} f \log(f I^{-\alpha}) \, dI \, dv \\
\text{s.t.} \quad \begin{pmatrix}
\rho \\
0_i \\
p_{ij} \\
q_i
\end{pmatrix} = \int_{\mathbb{R}^3 \times \mathbb{R}^+} \begin{pmatrix}
m \\
m c_i \\
m c_i c_j \\
m c_i c_j
\end{pmatrix} \left( \frac{m}{2} \frac{|c|^2}{|c|^2 + I} c_i \right) f \, dI \, dc,
\]

where

\[
\begin{align*}
\rho &= \int_{\mathbb{R}^3 \times [0, \infty)} f \, dI \, dv, \\
0_i &= \int_{\mathbb{R}^3 \times [0, \infty)} x_i f \, dI \, dv, \\
p_{ij} &= \int_{\mathbb{R}^3 \times [0, \infty)} x_j f \, dI \, dv, \\
q_i &= \int_{\mathbb{R}^3 \times [0, \infty)} f \log(f) \, dI \, dv.
\end{align*}
\]

\[
\begin{align*}
d / d x_i &= \frac{\partial}{\partial x_i} (x_j) = \delta_{ij}, \quad \frac{d}{d x_j} (x_i) = 0, \\
|U|^2 &= \sum_{i=1}^{3} U_i^2, \\
|v|^2 &= \sum_{i=1}^{3} v_i^2, \\
|c|^2 &= \sum_{i=1}^{3} c_i^2.
\end{align*}
\]
where $\sum_{i=1}^{3} p_{ii} = 3(p + \Pi)$, linearized around a local equilibrium state $f_M$ defined in (5.5) is given with

\[
\hat{f}^{14} \approx f^{14} = f_M \left\{ 1 - \frac{\rho}{p^2} q \cdot c - \frac{3}{2(\alpha + 1)} \frac{\Pi \rho}{m p^2} \left( \frac{m}{2} |c|^2 + I \right) \right. \\
+ \frac{\rho}{2p^2} \sum_{i,j=1}^{3} \left( p_{ij} + \frac{\alpha + \frac{5}{2}}{\alpha + 1} \delta_{ij} \Pi \right) c_i c_j + \left( \alpha + \frac{7}{2} \right)^{-1} \frac{\rho^2}{m p^2} q \cdot c \left( \frac{m}{2} |c|^2 + I \right) \right\}. 
\]

(7.5)

The proof of this Lemma follows the one given in [20], by Remark 1.

This distribution function enables to close the system of equations corresponding to fourteen moments (7.3) in the linearized form, for the chosen cross section as in (5.1) under an additional assumption of the boundedness of the angular function $b(\hat{u} \cdot \sigma)$. Namely, the following Proposition holds.

**Proposition 3 (Fourteen fields system of equations).** Closed system of equations for fourteen moments reads

\[
\partial_t \rho + \sum_{j=1}^{3} \partial_{x_j} (\rho U_j) = 0, \\
\partial_t (\rho U_i) + \sum_{j=1}^{3} \partial_{x_j} (\rho U_i U_j + p_{ij}) = 0, \\
\partial_t (\rho U_i U_j + p_{ij}) + \sum_{k=1}^{3} \partial_{x_k} \left\{ \rho U_i U_j U_k + U_i p_{jk} + U_j p_{ki} + U_k p_{ij} \right\} \\
+ \left( \alpha + \frac{7}{2} \right)^{-1} \left( q_i \delta_{jk} + q_j \delta_{ki} + q_k \delta_{ij} \right) = P_{ij},
\]

\[
\partial_t \left( \frac{1}{2} \rho |U|^2 + \rho e \right) + \sum_{i=1}^{3} \partial_{x_i} \left\{ \left( \frac{1}{2} \rho |U|^2 + \rho e \right) U_i + \sum_{j=1}^{3} p_{ij} U_j + q_i \right\} = 0,
\]

\[
\partial_t \left\{ \left( \frac{1}{2} \rho |U|^2 + \rho e \right) U_i + \sum_{j=1}^{3} p_{ij} U_j + q_i \right\} + \sum_{j=1}^{3} \partial_{x_j} \left\{ \left( \frac{1}{2} \rho |U|^2 + \rho e \right) U_i U_j \right. \\
+ \sum_{k=1}^{3} (U_i U_k p_{jk} + U_j U_k p_{ik}) + \frac{1}{2} \rho |U|^2 p_{ij} + \left( \alpha + \frac{9}{2} \right) \left( \alpha + \frac{7}{2} \right)^{-1} (q_i U_j + q_j U_i) \\
+ \left. \left( \alpha + \frac{7}{2} \right)^{-1} q U \delta_{ij} + \left( \alpha + \frac{9}{2} \right) P_{ij} - \frac{p^2}{\rho} \delta_{ij} \right\} = Q_i,
\]

for $i, j = 1, 2, 3$, where we have assumed the relations (6.4). The production terms for the cross section (5.1) with $b(\hat{u} \cdot \sigma) = K$, $K$ is a constant, linearized around the
global equilibrium state (3.15) read

\[ P_{ij}^{14} = -K \frac{\rho}{m} \frac{\sqrt{\pi}}{2\Gamma \left( \frac{4\alpha + \gamma + 7}{2} \right)} \left( \frac{kT}{m} \right)^{\frac{7}{2}} \left\{ \frac{1}{10} (4\alpha + \gamma + 7) (2^{7+2}(\gamma + 5)n_1 + 15n_2) p_{ij} \right. \\
+ \left. \left( \frac{\alpha + 5}{2} \right) \left( \frac{2^{7+4}}{3} n_1 + \frac{(4\alpha + \gamma + 4)}{(\alpha + 1)} n_2 \right) \Pi \delta_{ij} \right\} \] (7.6)

\[ Q_{i}^{14} = \sum_{k=1}^{3} U_k P_{ki}^{14} = \left( \frac{\alpha + 7}{2} \right)^{-1} K \frac{\rho}{m} \frac{\sqrt{\pi}}{144\Gamma \left( \frac{4\alpha + \gamma + 9}{2} \right)} \left( \frac{kT}{m} \right)^{\frac{7}{2}} q_i \times \left( 2^{7+5}((4\alpha + \gamma)(3\alpha + \gamma) + 57\alpha + 15\gamma + 60)n_1 \right. \\
+ \left. 9((4\alpha + \gamma)(2(4\alpha + \gamma) + \gamma^2 + 38) + 7\gamma^2 + 160)n_2 \right) \] (7.7)

with the constants

\[ n_1 = \Gamma \left( \frac{\alpha + 1}{2} \right)^2 \Gamma \left( \frac{\gamma + 3}{2} \right), \quad n_2 = \pi \Gamma \left( \frac{\alpha + \gamma + 1}{2} \right)^2. \] (7.8)

**Proof.** The non-convective fluxes are obtained by plugging distribution function (7.5) into their definition (7.2),

\[ p_{ij} = \left( \frac{\alpha + 7}{2} \right)^{-1} \left( q_i \delta_{jk} + q_j \delta_{ki} + q_k \delta_{ij} \right), \]
\[ q_{ij} = \left( \frac{\alpha + 9}{2} \right) \frac{p}{\rho} p_{ij} - \frac{p^2}{\rho} \delta_{ij}, \]

while the production terms are calculated in the Appendix C. \[ \square \]

**7.1. Relaxation times and transport coefficients.** In sense of extended thermodynamics, and the theory of hyperbolic systems of balance laws, production terms can be represented in following form:

\[ P_{ij}^{14} = -\frac{1}{\tau_s} p_{ij} - \frac{1}{\tau_{\Pi}} \Pi \delta_{ij}, \quad Q_{i}^{14} = \sum_{k=1}^{3} U_k P_{ki}^{14} - \frac{1}{\tau_{q}} q_i, \]

where \( \tau_s, \tau_{\Pi}, \tau_{q} \) are appropriate relaxation times. It is also known that relaxation times can be related to the transport coefficients - shear viscosity \( \mu \), bulk viscosity \( \nu \), and heat conductivity \( \kappa \) in the following manner,

\[ \mu = p \tau_s, \quad \nu = \frac{4(\alpha + 1)}{3(2\alpha + 5)} p \tau_{\Pi}, \quad \kappa = \left( \alpha + \frac{7}{2} \right) \frac{p^2}{\rho T} \tau_{q}. \]

In extended thermodynamics those parameters are of phenomenological nature. Starting from the Boltzmann equation, calculation of the production terms allows to obtain their explicit expressions, that will depend on \( \alpha > -1 \) related to the number of internal degrees of freedom and potential \( \gamma > 0 \) from the cross section (5.1). More precisely, from (7.6) and (7.7) it can be easily recognized

\[ \mu = \frac{m}{K} \frac{\Gamma \left( \frac{4\alpha + \gamma + 9}{2} \right)}{\sqrt{\pi}} \left( kT \right)^{1 - \frac{7}{2}} 30 \left( (4\alpha + \gamma + 7) (2^{7+2}(\gamma + 5)n_1 + 15n_2) \right)^{-1}. \] (7.9)
\[ \nu = \frac{2(\alpha + 1)^2 m^{\gamma/2}}{3(\alpha + \frac{7}{2})^2} \frac{\Gamma\left(\frac{4\alpha + \gamma + 9}{2}\right)}{K \sqrt{\pi}} (kT)^{1-\gamma/2} \]
\[ \times \left(\frac{2^{\gamma+4}}{3}(\alpha + 1)n_1 + (4\alpha + \gamma + 4)n_2\right)^{-1}, \]
\[ \kappa = \left(\alpha + \frac{7}{2}\right)^2 \frac{km^{\frac{\gamma}{2}}}{K} \frac{144\Gamma\left(\frac{4\alpha + \gamma + 9}{2}\right)}{\sqrt{\pi}} (kT)^{1-\gamma/2} \]
\[ \times \left(2^{\gamma+5}((4\alpha + \gamma)(3\alpha + \gamma) + 57\alpha + 15\gamma + 60)n_1 \right. \]
\[ + 9((4\alpha + \gamma)(2(4\alpha + \gamma) + \gamma^2 + 38) + 7\gamma^2 + 160)n_2\left.^{-1}, \right. \]

where \(n_1\) and \(n_2\) are from (7.8).

### 7.2. Prandtl number.

One of the tests for the validity of the transport coefficients given above is to verify the value of the Prandtl number, defined in our notation as

\[ \text{Pr} = \left(\alpha + \frac{7}{2}\right) \frac{k \mu}{m \kappa}. \] (7.10)

From the other side, theoretical value of the Prandtl number for polyatomic gases can be obtained by Eucken’s relation that in our notation reads

\[ \text{Pr} = \frac{4\alpha + 14}{4\alpha + 19}. \] (7.11)

The goal is to find a \(\gamma > 0\) such that for certain values of \(\alpha > -1\) the two expressions (7.10) and (7.11) are equal.

The value of \(\alpha\) is related to modes of a polyatomic molecule, as shows Table 1.

**Table 1.** Degrees of freedom corresponding to different modes (combinations of translation/rotation/vibration) with the number of atoms in a polyatomic molecule \(N \geq 2\), value of \(\alpha\), theoretical value of the Prandtl number from (7.11) and the value of \(\gamma\) enabling that the two Prandtl numbers (7.10) and (7.11) are equal.

|                | Translation and rotation | Translation, rotation and vibration |
|----------------|---------------------------|-------------------------------------|
|                | Linear molecule           | Non-linear molecule                  |                               |
| Degrees of freedom | 5                         | 6                                   | \(3N\)                         |
| \(\alpha\)      | 0                         | \(\frac{1}{2}\)                     | \(\frac{1}{2}(3N - 5)\)       |
| \(\text{Pr from (7.11)}\) | \(\frac{14}{17}\)       | \(\frac{16}{17}\)                  | \(\frac{6N+4}{6N+9}\)         |
| \(\gamma\)      | 2.153                     | 2.368                               | Table 2                       |

As it can be seen in Table 1, for \(\alpha = 0\) the theoretical value of the Prandtl number \(\frac{14}{17}\) is obtained from (7.10) by taking \(\gamma = 2.153\). Considering \(\alpha = \frac{1}{2}\), the value \(\gamma = 2.368\) in (7.10) recovers \(\text{Pr} = \frac{16}{17}\). When vibrational modes are also taken into account, for any number of atoms \(N \geq 2\) we can find the value \(\gamma > 0\) such that the correct value of the Prandtl number (7.11) is obtained, as shows Table 2.
Table 2. The number $N$ of atoms in a polyatomic molecule and the corresponding value of potential $\gamma$ such that the theoretical value of the Prandtl number from (7.11) is equal to the one in (7.10).

| $N$ | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\gamma$ | 4.063 | 9.469 | 17.262 | 25.801 | 34.705 | 43.835 | 53.123 | 62.526 |

7.3. Dependence of the shear viscosity on temperature. Another physical validity of the proposed model can be provided by studying temperature dependence of the shear viscosity (7.9). Our goal is to compare the shear viscosity (7.9) with experimental data given in [9] for room temperature range 293-373K, and in [18, 16] for high temperatures in the range 600-2000K.

The shear viscosity issuing from the kinetic theory (7.9), provides the following dependence on temperature,

$$
\mu \sim T^{1-\frac{2}{s}},
$$

(7.12)

where $\gamma$ is related to the choice of the cross-section (5.1) with constant angular part, as stated in Proposition 3. We point out that such a relation makes sense only if $\gamma < 2$, since it is observed that shear viscosity of gases increases as temperature grows.

In [9] the following relation is assumed,

$$
\mu \sim T^s,
$$

(7.13)

and in Table 14, page 232, provides experimental values for $s$ on the temperature range 293-373 K. Direct comparison of (7.12) and (7.13) gives the relation between $s$ and $\gamma$,

$$
\gamma = -2s + 2.
$$

(7.14)

Note that the comparison is only possible for $s < 1$, in order to ensure positivity of $\gamma$.

For different polyatomic molecules, the goal is to adjust the value of $\gamma$ in order to match the experimentally measured $s$ related to $\gamma$ by virtue of (7.14). Combining this $\gamma$ with $\alpha$ coming from the structure of a molecule gives the value of the Prandtl number using (7.10). That value can be compared to the theoretical one obtained in (7.11). Table 3 shows the results.

We mention that similar analysis was performed in [21] only for CO, where the exponent of temperature depends on $\alpha$ which is not the case here as it can be seen in (7.12), because of the additional term in the collision operator weak form involving $I^{\alpha}I^{\alpha}$ that subtracts dependence on $\alpha$.

On the other side, for higher temperatures, we have to consider vibrational modes as well. In [18, 16] experimental data for pointwise values of shear viscosity at certain high temperatures in the range 100-2000 K can be found. Those data can be fit in the manner of (7.12) leading to the value of $\gamma$. The results are shown in the Table 4.
Table 3. Value of $\gamma$ matching the experimental data for dependence of shear viscosity $\mu$ upon temperature for different molecules and the Prandtl number from (7.10). This value of the Prandtl number is further compared to the theoretical one (7.11) and the relative error is provided.

| Gas  | Value of $\gamma$ | Pr from (7.10) | Pr from (7.11) | Relative error |
|------|-------------------|----------------|----------------|----------------|
| H$_2$ | 0.664             | 0.816          | 0.737          | 10.7%          |
| CO   | 0.532             | 0.819          | 0.737          | 11.1%          |
| N$_2$ | 0.524             | 0.819          | 0.737          | 11.1%          |
| NO   | 0.424             | 0.82           | 0.737          | 11.3%          |
| O$_2$ | 0.454             | 0.82           | 0.737          | 11.3%          |
| CO$_2$ | 0.134             | 0.819          | 0.737          | 11.1%          |
| N$_2$O | 0.114             | 0.819          | 0.737          | 11.1%          |
| CH$_4$ | 0.328             | 0.849          | 0.762          | 10.3%          |

Table 4. Value of $\gamma$ matching the experimental data for dependence of shear viscosity $\mu$ of the shape (7.12) upon high temperature for different molecules and the Prandtl number from (7.10). This value of the Prandtl number is further compared to the theoretical one (7.11) and the relative error is provided.

| Gas  | Value of $\gamma$ | Pr from (7.10) | Pr from (7.11) | Relative error |
|------|-------------------|----------------|----------------|----------------|
| H$_2$ | 0.624             | 0.847          | 0.762          | 11.2%          |
| N$_2$ | 0.704             | 0.846          | 0.762          | 11.0%          |
| CO$_2$ | 0.599             | 0.894          | 0.815          | 9.7%           |
| CH$_4$ | 0.419             | 0.930          | 0.872          | 6.8%           |

We conclude that for a fixed polyatomic molecule the same $\gamma$ can provide agreement with the experimental data concerning dependence of shear viscosity on temperature and a satisfactory value of the Prandtl number, which coincides with the theoretical one at a relative error ranging from 6.8 - 11.3%. These results are valid in any temperature range as long as the power in (7.13) is less than one.

Finally, we put in evidence the key of success of this analysis. We first introduce the difference of the two Prandtl numbers (7.10) and (7.11),

$$\Delta(\gamma, \alpha) = \left(\alpha + \frac{7}{2}\right) \frac{k \mu m}{\kappa} - \frac{4\alpha + 14}{4\alpha + 19}.$$  

Then Tables 1 and 2 show the values $\gamma^*$ such that $\Delta(\gamma^*, \alpha) = 0$ for a fixed $\alpha$. Although $\gamma^*$ grows with the increase of $\alpha$, the analysis of viscosity dependence on temperature yields $\gamma < 2$ and at the same time provides satisfactory agreement with the Prandtl number. The reason is that $\Delta(\gamma, \alpha)$ is close to zero for $\gamma < 2$, for any $\alpha$, as shows the Figure 1.

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APPENDIX A. PROOF OF THE LEMMA 2.1 (JACOBIAN OF THE COLLISION TRANSFORMATION)

Proof. Using ideas from [12], we decompose the mapping $T$ from (2.8) into a sequence of mappings and calculate Jacobian of each of them. Then the Jacobian of $T$ will be a product of those Jacobians. More precisely, $T$ can be understood as a composition of the following transformations

$$T = T_9 \circ T_8 \circ T_7 \circ T_6 \circ T_5 \circ T_4 \circ T_2 \circ T_1,$$

where composition is understood as $(f \circ g)(x) = f(g(x))$ and each $T_i$ is described below.

(1) We first pass to the center-of-mass reference frame

$$T_1 : (v, e^*, I, I^*, r, R, \sigma) \mapsto (u, V, I, I^*, r, R, \sigma),$$

where $u$ and $V$ are relative velocity and velocity of center of mass from (2.2). It is clear that Jacobian of this transformation is 1,

$$J_{T_1} = 1.$$
(2) For the relative velocity \( u \) we pass to its spherical coordinates \( (|u|, \frac{u}{|u|}) \), where \( u/|u| \in S^2 \) is the angular variable, with the transformation \( T_2 \),

\[
(u, V, I, I_*, r, R, \sigma) \mapsto \left( |u|, \frac{u}{|u|}, V, I, I_*, r, R, \sigma \right),
\]

whose Jacobian is

\[ J_{T_2} = |u|^{-2}. \]

(3) In order to facilitate further calculation, we consider square of relative speed instead of relative speed itself, \( T_3 \),

\[
T_3 : \left( |u|, \frac{u}{|u|}, V, I, I_*, r, R, \sigma \right) \mapsto \left( |u|^2, \frac{u}{|u|}, V, I, I_*, r, R, \sigma \right)
\]

with the Jacobian

\[ J_{T_3} = 2 |u|. \]

(4) Instead of \( I_* \) we will use total energy \( E \), linked with the equation (2.3), \( T_4 \),

\[
T_4 : \left( |u|^2, \frac{u}{|u|}, V, I, r, R, \sigma \right) \mapsto \left( |u|^2, \frac{u}{|u|}, V, I, E, r, R, \sigma \right)
\]

whose Jacobian is 1,

\[ J_{T_4} = 1. \]

(5) Moreover, instead of \( R \) we want to have \( ER \), \( T_5 \),

\[
T_5 : \left( |u|^2, \frac{u}{|u|}, V, I, E, r, R, \sigma \right) \mapsto \left( |u|^2, \frac{u}{|u|}, V, I, E, r, ER, \sigma \right)
\]

with Jacobian

\[ J_{T_5} = E. \]

(6) Finally, we pass to pre-collisional quantities with the following mapping

\[
T_6 : \left( |u|^2, \frac{u}{|u|}, V, I, E, r, ER, \sigma \right) \mapsto \left( |u'|^2, \frac{u'}{|u'|}, V', I', I'_*, r', R', \sigma' \right).
\]

Let us compute Jacobian of this central transformation. First, for \( V \) we are using conservation law (2.3). Change of the unit vectors \( \frac{u}{|u|} \) and \( \sigma \) can be considered as a rotation. Thus we eliminate these variables and for the rest of variables, we use the following relations

\[
|u'|^2 = \frac{4RE}{m}, \quad I' = r(1 - R)E, \quad I'_* = (1 - r)(1 - R)E,
\]

\[
r' = \frac{I}{E - \frac{m}{4} |u|^2}, \quad R' = \frac{m |u|^2}{4E},
\]
and compute the corresponding Jacobian

$$J(|u|^2, I, E, r, E, R) \rightarrow (|u'|^2, I', I'_*, r', R')$$



$$= \begin{bmatrix}
0 & 0 & \frac{4R}{m} & 0 & \frac{1}{m} \\
0 & 0 & r(1-R) & (1-R)E & -r \\
0 & 0 & (1-r)(1-R) & -(1-R)E & 1-r \\
\frac{ml}{4E} & \frac{1}{E-\frac{m}{4}|u|^2} & -\frac{l}{m|u|^2-E^2} & 0 & 0 \\
\frac{m}{4E} & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

$$= \frac{(-1)^{i+2}}{E-\frac{m}{4}|u|^2} \frac{(-1)^{i+1}m}{4E} \begin{bmatrix}
\frac{4R}{m} & 0 & \frac{4}{m} \\
r(1-R) & (1-R)E & -r \\
(1-r)(1-R) & -(1-R)E & 1-r \\
\end{bmatrix}$$

$$= \frac{-m(1-R)E}{(E-\frac{m}{4}|u|^2)4E} \begin{bmatrix}
\frac{4R}{m} & 0 & \frac{1}{m} \\
r(1-R) & 1 & -r \\
(1-r)(1-R) & -1 & 1-r \\
\end{bmatrix}$$

$$= \frac{-m(1-R)}{4(E-\frac{m}{4}|u|^2)} \begin{bmatrix}
\frac{4R}{m} & 0 & \frac{4}{m} \\
1-R & 0 & -1 \\
(1-r)(1-R) & -1 & 1-r \\
\end{bmatrix}$$

$$= \frac{-m(1-R)}{4(E-\frac{m}{4}|u|^2)} \left( \frac{-4R}{m} + \frac{4(R-1)}{m} \right)$$

Finally,

$$J_{T_6} = \frac{1-R}{(E-\frac{m}{4}|u|^2)} = \frac{1-R}{I + I_*} = \frac{1-R}{(1-R')E}.$$  

(7) Now we go back, first from squares to squares of relative speed to relative speed itself,

$$T_7 : \left(|u'|^2, \frac{u'}{|u'|}, V', I', I'_*, r', R', \sigma' \right) \rightarrow \left(|u'|, \frac{u'}{|u'|}, V', I', I'_*, r', R', \sigma' \right).$$

with Jacobian

$$J_{T_7} = \frac{1}{2|u'|}.$$  

(8) For \(u'|\) we pass from spherical coordinates to Cartesian ones,

$$T_8 : \left(|u'|, \frac{u'}{|u'|}, V', I', I'_*, r', R', \sigma' \right) \rightarrow \left(u', V', I', I'_*, r', R', \sigma' \right).$$

with Jacobian

$$J_{T_8} = |u'|^2.$$  

(9) We go back from center-of-mass reference frame,

$$T_9 : (u', V', I', I'_*, r', R', \sigma') \rightarrow (v', v'_*, I', I'_*, r', R', \sigma').$$
with unit Jacobian

\[ J_{T_0} = 1. \]

Finally, we get the Jacobian of transformation \( T \),

\[ J_T = \prod_{i=1}^{9} J_{T_i} = \frac{(1 - R)|u'|}{(1 - R')|u|} = \frac{(1 - R)R^\frac{1}{4}}{(1 - R')R^\frac{1}{4}}, \]

where for the last equality we have used \(|u'| = \sqrt{\frac{4RE}{m}}\) and \(|u| = \sqrt{\frac{4RE}{m}}\).

\[ \square \]

**APPENDIX B. COMPUTATION OF THE PRODUCTION TERMS FOR THE SIX FIELDS MODEL**

For the sake of simplicity, we introduce the following notation

\[ M = \frac{m}{2kT} \left( \frac{1}{1 + \frac{R}{kT}} \right), \quad N = \frac{1}{kT} \left( \frac{1}{1 - \frac{3}{2}(\alpha + 1) \frac{I}{p}} \right), \quad L = \frac{\rho}{m} \left( \frac{M}{\pi} \right)^{\frac{3}{2}} N^{\alpha + 1} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)}. \]

(B.1)

We remind that the range for \( \Pi/p \) imposed in (6.8) implies positivity of these coefficients,

\[ M > 0, \quad N > 0. \]

With these coefficients, six moments distribution function (6.7) reads

\[ \hat{f}^6 = L I^\alpha e^{-M|c|^2 - N I}. \]

(B.2)

The aim here is to calculate the production term \( \mathcal{P} \) defined in (6.6) and the entropy production term \( D^{uw}(\hat{f}^6) \) from (3.13) for the cross section (5.1), namely

\[ \mathcal{B}^{uw}(v, v_s, I, I_s, r, R, \sigma) = b(\hat{u} \cdot \sigma) \left( R^\frac{1}{2} |v|^\gamma + \left( r(1 - R) \frac{I}{m} \right)^{\frac{3}{2}} + \left( (1 - r)(1 - R) \frac{I_s}{m} \right)^{\frac{3}{2}} \right), \quad \gamma > 0 \]

where we have denoted \( u := v - v_s, \hat{u} = u/|u| \).

**B.1. Computation of the production term \( \mathcal{P} \).** For the production term (6.6) we first note that taking the square of \( v = c + U \), it reduces to

\[ \mathcal{P} = \int_{\mathbb{R}^3 \times [0, \infty)} m |c|^2 Q^{uw}(\hat{f}_6, \hat{f}_6)(c, I) \, dc \, dI. \]

where we remind that the collision operator \( Q^{uw} \) is defined in (3.9). The weak form (3.11) yields

\[ \mathcal{P} = \frac{m}{2} \int_{\mathbb{R}^6 \times [0, \infty)^2 \times [0, 1]^2 \times S^2} \left( |c'|^2 + |c_s'|^2 - |c|^2 - |c_s|^2 \right) \hat{f}_6 \hat{f}_6 \]

\[ \times \mathcal{B}^{uw} \phi_\alpha(r) (1 - R)R^\frac{1}{2} \psi_\alpha(R) \, d\sigma \, dr \, dR \, dI_s \, dc_s \, dI \, dc \]

\[ = \frac{m}{2} L^2 \int_{\mathbb{R}^6 \times [0, \infty)^2 \times [0, 1]^2 \times S^2} \left( |c'|^2 + |c_s'|^2 - |c|^2 - |c_s|^2 \right) e^{-M(|c|^2 + |c_s|^2)} e^{-N(I + I_s)} \]

\[ \times \mathcal{B}^{uw} \phi_\alpha(r) (1 - R)R^\frac{1}{2} \psi_\alpha(R) I^\alpha I_s^\alpha \, d\sigma \, dr \, dR \, dI_s \, dc_s \, dI \, dc, \quad \text{(B.3)} \]
where $\phi_\alpha(r)$ and $\psi_\alpha(R)$ are defined in (3.2) and $B^{nw}$ is cross section (5.1). Now we pass to the relative $u$ and center of mass peculiar velocity $V_c$ by means of the following change of variables

$$(c, c_*) \rightarrow \left( u := c - c_*, V_c = \frac{c + c_*}{2} \right) \Rightarrow c = V_c + \frac{u}{2}, \ c_* = V_c - \frac{u}{2}. \quad (B.4)$$

with unit Jacobian. Therefore, the terms under integral in new variables become

$$|c'|^2 + |c_*|^2 - |c|^2 - |c_*|^2 = \frac{1}{2} (R - 1) |u|^2 + \frac{2R}{m} (I + I_*)$$

and

$$|c|^2 + |c_*|^2 = 2|V_c|^2 + \frac{1}{2} |u|^2. \quad (B.5)$$

Therefore, we can express the primed quantities from (B.3) in center-of-mass framework,

$$\mathcal{P} = \frac{m}{2} L^2 \int_{\mathbb{R}^6 \times [0,\infty)^2 \times [0,1]^2 \times S^2} \left( \frac{1}{2} (R - 1) |u|^2 + \frac{2R}{m} (I + I_*) \right) e^{-M(2|V_c|^2 + |u|^2)} e^{-N(I+I_*)} B^{nw} \phi_\alpha(r) \left( 1 - R \right) R^{\frac{3}{2}} \psi_\alpha(R) I^\alpha I^\beta \ d\sigma \ dr \ dI_* \ dc_\theta \ dI \ du,$$

Using that the cross section is of the form (5.1), i.e.

$$B^{nw}(v, v_*, I, I_*, r, R, \sigma) = b(v \cdot \sigma) \tilde{B}(|v|, I, I_*, r, R),$$

we can perform the integration with respect to $V_c$ and $\sigma$,

$$\mathcal{P} = m L^2 \left( \frac{\pi}{2M} \right)^{\frac{3}{2}} ||b||_{L^1(d\sigma)} \int_{\mathbb{R}^6 \times [0,\infty)^2 \times [0,1]^2} e^{-\frac{1}{2} |u|^2} e^{-N(I+I_*)}$$

$$\times \left( \frac{1}{4} (R - 1) |u|^2 + \frac{R}{m} (I + I_*) \right) \tilde{B}(|u|, I, I_*, r, R) \times \phi_\alpha(r) \left( 1 - R \right) R^{\frac{3}{2}} \psi_\alpha(R) I^\alpha I^\beta \ dr \ dI_* \ dI \ du.$$

We now pass to the spherical coordinates for the relative velocity $u$. Denoting $y = |u|$, and performing integration with respect to the angular part, we obtain

$$\mathcal{P} = m L^2 \left( \frac{\pi}{2M} \right)^{\frac{3}{2}} ||b||_{L^1(d\sigma)} \ 4 \pi \int_{[0,\infty)^3 \times [0,1]^2} e^{-\frac{1}{2} y^2} e^{-N(I+I_*)} y^2$$

$$\times \left( \frac{1}{4} (R - 1) y^2 + \frac{R}{m} (I + I_*) \right) \times \left( R^{\gamma/2} y^\gamma + \left( r(1 - R) \frac{I_\gamma}{m} \right)^\gamma + \left( (1 - r)(1 - R) \frac{I_\gamma}{m} \right)^\gamma \right),$$

$$\times \phi_\alpha(r) \left( 1 - R \right) R^{\frac{3}{2}} \psi_\alpha(R) I^\alpha I^\beta \ dr \ dI_* \ dI \ dy.$$
To that end, we introduce the following constant

\[ C_{(a,b,c)} = \int_{[0,1]^2} (r(1 - r))^{2\alpha} (1 - R)^{2\alpha + 1} R^2 (1 - R)^a R^{b} c \, dr \, dR \]

\[ = \frac{\Gamma(2\alpha + a + 2) \Gamma(b + \frac{3}{2}) \Gamma(\alpha + c + 1) \Gamma(a + 1)}{\Gamma(2\alpha + a + b + \frac{3}{2}) \Gamma(2\alpha + a + c + 2)}, \tag{B.7} \]

where \( \Gamma \) stands for the Gamma function. Then the production term becomes

\[ \mathcal{P} = m L^2 \left( \frac{\pi}{2 M} \right)^{\frac{3}{2}} \| b \|_{L^1(d\sigma)} \int_{[0,\infty)^3 \times [0,1]^2} e^{-\frac{\| b \|^2}{2}} e^{-N(I+I_*)} I^a I_*^a \]

\[ \times \left\{ -\frac{1}{4} C(\frac{3}{2},0) y^2 + C(0,\frac{5}{2}+1) y^2 \left( \frac{I}{m} + \frac{I_*}{m} \right)^\frac{2}{5} \right\} \]

\[ + C(\frac{5}{2},1,1) \left( \frac{I}{m} + \frac{I_*}{m} \right) \left( \frac{I}{m} + \frac{I_*}{m} \right)^\frac{2}{5} \}

\[ dI_* dI dy. \]

It remains to integrate with respect to \( I, I_* \) and \( y \). Introducing the positive constant

\[ C_\mathcal{P} = m L^2 \left( \frac{\pi}{2 M} \right)^{\frac{3}{2}} \| b \|_{L^1(d\sigma)} \int_{[0,\infty)^3 \times [0,1]^2} e^{-\frac{\| b \|^2}{2}} e^{-N(I+I_*)} I^a I_*^a \]

\[ \times \left( k_1 M^{-\frac{2}{5}} + k_2 (MN)^{-\frac{2}{5}} \right), \tag{B.8} \]

where \( k_1 \) and \( k_2 \) are from (6.11), we finally obtain the expression for production term

\[ \mathcal{P} = C_\mathcal{P} \left( -\frac{1}{2} M^{-1} + (MN)^{-1} \right). \tag{B.9} \]

It remains to come back to the original variables using (B.1), that yields

\[ \left( -\frac{1}{2} M^{-1} + (MN)^{-1} \right) = -\frac{kT \left( \frac{\alpha + \frac{I}{2}}{2} \right)}{m (\alpha + 1)} \frac{\Pi}{p} = -\frac{\rho (\alpha + \frac{I}{2})}{p (\alpha + 1)} \frac{\Pi}{p}, \]

and for \( C_\mathcal{P} \) we get (6.10).

B.2. Computation of the entropy production term \( D_{nw}(\tilde{I}^\alpha) \). With the notation (B.2) the test function corresponding to the entropy law becomes

\[ \log \left( \tilde{I}^\alpha I^{-\alpha} \right) = \log L - M |c|^2 - NI. \]

The weak form (3.10) allows to write

\[ D_{nw}(\tilde{I}^\alpha) = \frac{1}{2} L^2 \int_{R^6 \times [0,\infty)^2 \times [0,1]^2 \times S^2} \left( -M \left( |c'|^2 + |c_*'|^2 - |c|^2 - |c_*|^2 \right) \right. \]

\[ -N \left( I' + I_*' - I - I_* \right) \times e^{-M(|c|^2 + |c_*|^2)} e^{-N(I+I_*)} \]

\[ B_{nw} \phi_\alpha(r) (1 - R) R^2 \psi_\alpha(R) I^a I_*^a \, d\sigma \, dR \, dI_* \, dc \, dc_* \, dI \, dc, \tag{B.10} \]

with \( \phi_\alpha(r) \) and \( \psi_\alpha(R) \) from (3.2) and \( B_{nw} \) is cross section (5.1). The next step is to use coordinates of the center of mass by means of (B.4). Indeed, in addition to
(2.3) and (B.5) we also have

\[ I' + I_* I - I_* = -\frac{m}{2} \left( \frac{1}{2} (R - 1) |u|^2 + \frac{2R}{m} (I + I_*) \right). \]

These considerations allow to write (B.10) in terms of the production term \( P \) by virtue of (B.6),

\[ D^{nw}(\hat{f}^6) = \left( \frac{m}{2} N - M \right) \frac{P}{m}. \]  

Therefore, using the results of the previous Section B.1 and notably its final result (B.9) we obtain

\[ D^{nw}(\hat{f}^6) = -\left( \frac{m}{2} N - M \right)^2 \frac{C_P}{m N M}, \]

with the positive constant \( C_P \) from (B.8). Now is clear that \( D^{nw}(f) \) is non-positive, as claimed in the H-theorem (3.14).

The final result follows from (B.11) by exploiting

\[ \left( \frac{m}{2} N - M \right) = \frac{m}{2kT} \left( 1 - \frac{\Pi}{p} \right)^{-1} \left( 1 + \frac{\Pi}{p} \right)^{-1} \left( \frac{\alpha + \frac{5}{2}}{(\alpha + 1)^2} \right) \frac{1}{p}. \]

We note that the shorter notation in terms of \( M, N \), allow to rewrite the derivative (6.16),

\[ K_{\Pi} = -\frac{3k}{m} \left( \frac{m}{2} N - M \right). \]

Combining the last equation with (B.9) we get

\[ \frac{1}{3} \frac{\partial K}{\partial \Pi} = -\frac{k}{m} \left( \frac{m}{2} N - M \right) P = -kD^{nw}(\hat{f}^6) \geq 0. \]

**APPENDIX C. COMPUTATION OF THE PRODUCTION TERMS FOR THE FOURTEEN FIELDS MODEL**

For the sake of simplicity, equilibrium distribution function (5.5) will be written as
\[ f_M = I^a L_0 e^{-\frac{1}{kT} \left( \frac{1}{2} |v|^2 + I \right)} \quad \text{where} \quad L_0 = \frac{\rho}{m} \left( \frac{m}{2\pi kT} \right)^{\frac{3}{2}} \frac{1}{\Gamma(\alpha + 1)} \frac{1}{(kT)^{\alpha + 1}}. \]

Our aim is to compute the production terms in the 14 moments approximation, which amounts to plug the approximative distribution function \( f^{14} \) into the definition of the product terms (7.4),

\[ P_{ij}^{14} = \int_{\mathbb{R}^3 \times \mathbb{R}_+} mv_i v_j Q^{nw}(\hat{f}^{14}, \hat{f}^{14})(v, I) dI dv, \]

\[ Q_{ij}^{14} = \int_{\mathbb{R}^3 \times \mathbb{R}_+} \left( \frac{m}{2} |v|^2 + I \right) v_i Q^{nw}(\hat{f}^{14}, \hat{f}^{14})(v, I) dI dv. \]

Introducing the peculiar velocity \( c = v - u \) and using annihilations of the collision operator weak form (3.12), and after the change of variables \( v \mapsto c \) the expressions (7.4) simplify to

\[ P_{ij}^{14} = \int_{\mathbb{R}^3 \times \mathbb{R}_+} mc_i c_j Q^{nw}(\hat{f}^{14}, \hat{f}^{14})(c + u, I) dI dc, \]  
\[ Q_{ij}^{14} = \sum_{k=1}^{3} u_k P_{ki} + \int_{\mathbb{R}^3 \times \mathbb{R}_+} \left( \frac{m}{2} |c|^2 + I \right) Q^{nw}(\hat{f}^{14}, \hat{f}^{14})(c + u, I) dI dc. \]
As non-equilibrium effects are supposed small, products of the distribution functions appearing in the collision integral can be linearized with respect to the non-equilibrium quantities, \( p_{(ij)}, \Pi, q_i \). Using the microscopic conservation laws (2.1), it follows

\[
\dot{f}^{k'\ell'}_{\ell i} - \dot{f}^{kl}_{li} = f_M f_M^* \left\{ \sum_{k,l=1}^3 \frac{\rho}{2p^2} \left( p_{(kl)} + \left( \alpha + \frac{5}{2} \right) (\alpha + 1)^{-1} \Pi \delta_{kl} \right) 
\times (c'_k c'_l + c'_k c'_l - c_k c_l - c_{k*} c_{l*}) + \sum_{n=1}^3 \left( \alpha + \frac{7}{2} \right)^{-1} \frac{\rho^2}{mp^2} q_n
\times \left( \left( \frac{m}{2} |c'|^2 + I' \right) c_n + \left( \frac{m}{2} |c'_l|^2 + I'_l \right) c_{n*} - \left( \frac{m}{2} |c|^2 + I \right) c_n - \left( \frac{m}{2} |c|^2 + I_{\ast} \right) c_{n*} \right) \right\}. \tag{C.3}
\]

Placing (C.3) into (C.1) and (C.2) yields a suitable approximation for the source terms \( P^{14}_{ij} \) and \( Q^{14}_{ij} \), denoted by \( \overline{P}^{14}_{ij} \) and \( \overline{Q}^{14}_{ij} \), respectively.

We now introduce the following notation,

\[
P_{ijkl} = \frac{1}{2} \int m c_i c_j (c'_k c'_l + c'_k c'_l - c_k c_l - c_{k*} c_{l*}) f_M f_M^* 
\times B^{nw} \phi_n (r) (1 - R) R^2 \psi_\alpha (R) d\sigma dr dR dI_1 dc_1 dI ddc_1 ddc_2 ddc_3
\]

\[
Q_{in} = \frac{1}{2} \int \left( \frac{m}{2} |c|^2 + I \right) c_i \left( \left( \frac{m}{2} |c'|^2 + I' \right) c_n + \left( \frac{m}{2} |c'_l|^2 + I'_l \right) c_{n*} - \left( \frac{m}{2} |c|^2 + I_{\ast} \right) c_n - \left( \frac{m}{2} |c|^2 + I_{\ast} \right) c_{n*} \right) f_M f_M^* 
\times B^{nw} \phi_n (r) (1 - R) R^2 \psi_\alpha (R) d\sigma dr dR dI_1 dc_1 dI ddc_1 ddc_2 ddc_3.
\]

Now the parity arguments imply

\[
\overline{P}^{14}_{ij} = \frac{\rho}{2p^2} \sum_{k,l=1}^3 \left( p_{(kl)} + \left( \alpha + \frac{5}{2} \right) (\alpha + 1)^{-1} \Pi \delta_{kl} \right) P_{ijkl}, \tag{C.4}
\]

\[
\overline{Q}^{14}_{ij} = \sum_{k=1}^3 u_k \overline{P}^{14}_{ik} + \sum_{n=1}^3 \left( \alpha + \frac{7}{2} \right)^{-1} \frac{\rho^2}{mp^2} q_n Q_{in}. \tag{C.5}
\]

We calculate the production terms \( \overline{P}^{14}_{ij} \) and \( \overline{Q}^{14}_{ij} \) in separate Sections for the cross section

\[
B^{nw}(v, v_*, I, I_*, r, R, \sigma)
= K \left( R^2 |u|^2 + (r(1 - R) I_1 \frac{I}{m})^\frac{2}{2} + (1 - r)(1 - R) I_2 \frac{I}{m} \right) ^{\frac{2}{2}}, \tag{C.6}
\]

where \( K \) is a constant, \( u := v - v_*, \gamma > 0 \).

C.1. Computation of \( \overline{P}^{14}_{ij} \). Firstly, we exploit the parity arguments for the term \( P_{ijkl} \). Note that it vanishes unless indices are equal by pairs - the integral is non zero when \( i = j \) and \( k = l \) or \( i = k \) and \( j = l \) or \( i = l \) and \( j = k \). By symmetry, the last two terms lead to the same result and thus \( P_{ijkl} \) can be represented in the following form:

\[
P_{ijkl} = P_1 \delta_{ij} \delta_{kl} + P_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \tag{C.7}
\]
Combining (C.4) and (C.7) with the fact that pressure tensor is symmetric, we get

\[ P_{ij}^{14} = \frac{\rho}{2p_0^2} \left( 2p_{(ij)}P_2 + \frac{1}{3} \delta_{ij} \left( \alpha + \frac{5}{2} \right) (\alpha + 1)^{-1} \Pi \sum_{r,t=1}^3 P_{rtrt} \right). \]

The term \( P_2 \) can be determined from the system of equations obtained from the representation (C.7)

\[ \sum_{r,t=1}^3 P_{rtrt} = 9P_1 + 6P_2, \quad \sum_{r,t=1}^3 P_{rtrt} = 3P_1 + 12P_2, \]

whose solution is

\[ P_1 = \frac{1}{15} \sum_{r,t=1}^3 (2P_{rtrt} - P_{rtrt}), \quad P_2 = \frac{1}{30} \sum_{r,t=1}^3 (3P_{rtrt} - P_{rtrt}). \]

### C.1.1. Computation of \( \sum_{r,t=1}^3 P_{rtrt} \)

We first concentrate on the term

\[ \sum_{r,t=1}^3 P_{rtrt} = \frac{mL_0^2}{2} \int_{\mathbb{R}^6 \times [0,\infty)^2 \times [0,1]^2 \times S^2} e^{-\frac{1}{\hbar} \left( \frac{1}{2} \left( |c'|^2 + |c_s|^2 \right) + I + I_s \right)} \times \left( |c|^2 \left( |c'|^2 + |c_s|^2 - |c|^2 - |c_s|^2 \right) \right) \times \mathcal{B}^{tw} \phi_\alpha(r) (1 - R) \hat{R}^2 \psi_\alpha(R) I^a I_a^* \, d\sigma \, dR \, dI_s \, dc_s \, dI \, dc. \]

Passing to the center-of-mass reference frame, by changing variables to (B.4) yields

\[ \sum_{r,t=1}^3 P_{rtrt} = \frac{mL_0^2}{2} \int_{\mathbb{R}^6 \times [0,\infty)^2 \times [0,1]^2 \times S^2} e^{-\frac{1}{\hbar} \left( |V_c|^2 + \hat{V}_c \cdot u + \frac{1}{4} |u|^2 \right) \left( \frac{1}{2} (R - 1) |u|^2 + \frac{2R}{m} (I + I_s) \right)} \times \mathcal{B}^{tw} \phi_\alpha(r) (1 - R) \hat{R}^2 \psi_\alpha(R) I^a I_a^* \, d\sigma \, dR \, dI_s \, dc_s \, dI \, du \, dV_c. \]

The form of the cross-section (C.6)

\[ \mathcal{B}^{tw}(v, u, I, I_s, r, R, \sigma) = K \hat{B}(|u|, I, I_s, r, R), \]

allows to immediately integrate with respect to \( V_c \) and \( \sigma \),

\[ \sum_{r,t=1}^3 P_{rtrt} = mL_0^2 \left( \frac{\pi kT}{m} \right)^{3/2} \pi \int_{\mathbb{R}^3 \times [0,\infty)^2 \times [0,1]^2} e^{-\frac{1}{\hbar} \left( |u|^2 + I + I_s \right)} \times \left( 3 \frac{kT}{m} + \frac{1}{2} |u|^2 \right) \left( \frac{1}{2} (R - 1) |u|^2 + \frac{2R}{m} (I + I_s) \right) \times \hat{B}_\phi_\alpha(r) (1 - R) \hat{R}^2 \psi_\alpha(R) I^a I_a^* \, dR \, dI_s \, dI \, du. \]
Next, we pass to the spherical coordinates for the relative velocity \( u \). Denoting \( y = |u| \), and performing integration with respect to the angular part, we obtain

\[
\sum_{r,t=1}^{3} \mathcal{P}_{rtrt} = m K L_0 \left( \frac{\pi k T}{m} \right)^{\frac{2}{3}} 4\pi r^2 \int_{[0,\infty)^3} e^{-\frac{1}{2}r(\frac{2}{3}y^2 + l + I_s)} \\
\times \left( \frac{3kT}{m} + \frac{1}{2}y^2 \right) \left( \frac{1}{2}(R - 1)y + \frac{2R}{m}(I + I_s) \right) \\
\times \left( R^\frac{2}{3}|u|^\gamma + \left( r(1-R)\frac{l}{m} \right)^\frac{2}{3} + \left( (1-r)(1-R)\frac{l}{m} \right)^\frac{2}{3} \right) \\
\times \frac{\pi}{3} \cdot \frac{y^2}{\phi_\alpha(r)} (1 - R) R^\frac{2}{3}\psi_\alpha(R) I^\alpha I^\alpha \, dr \, dI_s \, dI \, dy.
\]

Now we expand all the involved expressions and perform integration with respect to \( r \) and \( R \). Using the notation (B.7), we obtain

\[
\sum_{r,t=1}^{3} \mathcal{P}_{rtrt} = m K L_0 \left( \frac{\pi k T}{m} \right)^{\frac{2}{3}} 4\pi r^2 \int_{[0,\infty)^3} e^{-\frac{1}{2}r(\frac{2}{3}y^2 + l + I_s)} y^2 I^\alpha I^\alpha \\
\times \left( \frac{3kT}{m} + \frac{1}{2}y^2 \right) \left\{ \frac{1}{2}C(1,\frac{2}{3},0) y^{\gamma+2} + 2C(0,\frac{2}{3}+1,0) y^{\gamma} \left( \frac{l}{m} + \frac{l_s}{m} \right) \right\} \\
+ C(\frac{2}{3},0,\frac{2}{3}) \left( \left( \frac{l}{m} \right)^{\frac{2}{3}+1} + \left( \frac{l}{m} \right)^{\frac{2}{3}} \left( \frac{l_s}{m} \right)^{\frac{2}{3}} + \left( \frac{l_s}{m} \right)^{\frac{2}{3}} \right) \right\} I_s \, dI_s \, dI \, dy.
\]

Finally, after performing integration with respect to \( I, I_s \) and \( y \), we get

\[
\sum_{r,t=1}^{3} \mathcal{P}_{rtrt} = -K \rho^2 \sqrt{\frac{\pi}{\Gamma(4\alpha + 2\gamma + 1)}} (kT)^{\frac{2}{3}+2} \\
x \left\{ 3(4\alpha + \gamma + 4)\pi \Gamma \left( \alpha + \gamma + 1 \right)^2 + 2^{\gamma+1}(\alpha + 1)\Gamma \left( \alpha + 2 \right)^2 \right\}.
\]

C.1.2. Computation of \( \sum_{r,t=1}^{3} \mathcal{P}_{rtrt} \). For the term

\[
\sum_{r,t=1}^{3} \mathcal{P}_{rtrt} = \frac{m}{2} I_0 \int_{[0,\infty)^2} e^{-\frac{1}{2}r(\frac{2}{3}(c^2 + c_s^2) + l + I_s)} \\
\times \left( (c \cdot c')^2 + (c \cdot c')^2 - (c \cdot c)^2 - (c \cdot c_s)^2 \right) \\
\times B^{\mu \nu} \phi_\alpha(r) (1 - R) R^\frac{2}{3}\psi_\alpha(R) I^\alpha I^\alpha \, d\sigma \, dr \, dI_s \, d\sigma \, dc. 
\]

After change of variable (B.4) term under integral yields

\[
(c \cdot c')^2 + (c \cdot c')^2 - (c \cdot c)^2 - (c \cdot c_s)^2 = \\
\frac{1}{2}(V_c \cdot u')^2 + \frac{1}{2}(V_c \cdot u')(V_c \cdot u) + \frac{1}{8}(u \cdot u')^2 - \frac{1}{2}(V_c \cdot u)^2 - \frac{1}{2}(V_c \cdot u)|u|^2 - \frac{1}{8}|u|^4.
\]
Finally, performing the integration with respect to $V_r$, 

$$\sum_{r,t=1}^{3} P_{rtrt} = mKL_0^2 \left( \frac{\pi kT}{m} \right)^{\frac{3}{2}} \frac{1}{8} \int_{R^3 \times [0, \infty) \times [0, 1]^2 \times S^2} e^{-\frac{1}{8\pi} (\bar{\gamma} u^2 + I + I_*)} \times\left( \frac{kT}{m} |u'|^2 - |u| - \frac{1}{2} (|u'|^2 - |u|^4) \right) \times \tilde{B}_\phi(u) (1 - R) R^2 \psi_\alpha(R) I^\alpha I_*^\alpha \, d\sigma \, dR \, dI_* \, dI \, du.$$

Using relations (2.4) and (2.5), we can express

$$|u'|^2 = \frac{4RE}{m} = R|u|^2 + \frac{4R}{m} (I + I_*),$$

and integrate with respect to $R$ and $I$ using the notation (B.7) for the constants coming up from this integration,

$$\sum_{r,t=1}^{3} P_{rtrt} = mKL_0^2 \left( \frac{\pi kT}{m} \right)^{\frac{3}{2}} \frac{\pi}{2} \int_{[0, \infty)^3} e^{-\frac{1}{8\pi} (\bar{\gamma} y^2 + I + I_*)} \times\left( \frac{kT}{m} (R - 1) |y|^2 + \frac{1}{2} \left( \frac{R}{3} \frac{I + I_*}{m} \right) |y|^4 + \frac{4R}{m} (I + I_*) \left( \frac{kT}{m} + \frac{1}{6} |y|^2 \right) \right) \times \tilde{B}_\phi(u) (1 - R) R^2 \psi_\alpha(R) I^\alpha I_*^\alpha \, d\sigma \, dR \, dI_* \, dI \, dy.$$

Now we switch to spherical coordinates for the relative velocity $u$, and integrate with respect to $r$ and $R$ using the notation (B.7) for the constants coming up from this integration,

$$\sum_{r,t=1}^{3} P_{rtrt} = mKL_0^2 \left( \frac{\pi kT}{m} \right)^{\frac{3}{2}} \frac{\pi}{2} \int_{[0, \infty)^3} e^{-\frac{1}{8\pi} (\bar{\gamma} y^2 + I + I_*)} \times\left( \frac{-kT}{m} C_{(1, 0, 0)} y^2 + \frac{1}{2} \left( \frac{1}{3} C_{(0, 1, 0)} - C_{(0, 0, 0)} \right) y^4 \right) \times\left( \frac{kT}{m} y^2 + \frac{1}{6} |y|^2 \right) (I + I_*) y^\gamma \right) \times \left\{ \frac{-kT}{m} C_{\left( \frac{1}{2} + 1, 0, \frac{1}{2} \right)} y^2 + \frac{1}{2} \left( \frac{1}{3} C_{\left( \frac{1}{2}, 1, \frac{1}{2} \right)} - C_{\left( \frac{1}{2}, 0, \frac{1}{2} \right)} \right) y^4 \right\} \left( \frac{I}{m} \right)^{\frac{\gamma}{2}} + \left( \frac{I_*}{m} \right)^{\frac{\gamma}{2}} \right\} dI_* \, dI \, dy.$$

Finally, performing the integration with respect to $I$, $I_*$, and $y$ yields

$$\sum_{r,t=1}^{3} P_{rtrt} = -K \rho^2 \frac{\sqrt{\pi}}{\Gamma \left( \frac{\alpha + \gamma + 9}{2} \right)} \left( \frac{kT}{m} \right)^{\frac{\gamma + 2}{2}} \left\{ 9(8\alpha + 2\gamma + 13)\pi \Gamma \left( \alpha + \frac{\gamma}{2} + 1 \right)^2 + 2^{\gamma + 2}(4\alpha(\gamma + 6) + (\gamma + 12) + 39)\Gamma \left( \alpha + 1 \right)^2 \Gamma \left( \frac{\gamma + 3}{2} \right) \Gamma \left( \frac{\gamma + 5}{2} \right) \right\}.$$
C.2. Computation of $Q_i^{14}$. The parity arguments imply that $Q_m$ vanishes unless $i = n$, which for the production term (C.5) implies

$$Q_i^{14} = \sum_{k=1}^{3} u_k P_{ki}^{14} + \left(\alpha + \frac{7}{2}\right)^{-1} \frac{\rho^2}{mp^2} \frac{1}{3} \sum_{r=1}^{3} Q_{rr}.$$

C.2.1. Computation of $\sum_{r=1}^{3} Q_{rr}$. We now compute the term

$$\sum_{r=1}^{3} Q_{rr} = \frac{1}{2} L_0^2 \int_{R^6 \times [0, \infty)^2 \times [0, 1]^2 \times S^2} e^{-\pi \sum_{i=1}^{2} \left(\frac{1}{2} |\alpha|^2 + |c_i|^2\right) + I_{1*}} \left(\left(\frac{m}{2} |c'|^2 + I'\right) c' \cdot c + \left(\frac{m}{2} |c'_*|^2 + I'_*\right) c'_* \cdot c\right)$$

$$- \left(\frac{m}{2} |c|^2 + I\right) |c| - \left(\frac{m}{2} |c_*|^2 + I_*\right) c \cdot c \right)$$

$$\times \hat{B}^{\mu \nu} \phi_\alpha(r) \left(1 - R\right) R \psi_\alpha(R) I^n I^n_\alpha d\sigma dr dR dI_* d\epsilon c dI.$$

Switching to the center-of-mass framework by means of the change of variables (B.4), the term under integral becomes

$$\left(\frac{m}{2} |c'|^2 + I'\right) c' \cdot c + \left(\frac{m}{2} |c'_*|^2 + I'_*\right) c'_* \cdot c$$

$$- \left(\frac{m}{2} |c|^2 + I\right) |c| - \left(\frac{m}{2} |c_*|^2 + I_*\right) c \cdot c \right)$$

$$= \frac{m}{2} (u' \cdot V_c)^2 - \frac{m}{2} (u \cdot V_c)^2 + \frac{m}{4} (u - u')(u' - V_c) - \frac{m}{4} (u \cdot V_c)|u|^2$$

$$+ \frac{1}{2} (I' - I'_*) \left(u' \cdot V_c + \frac{1}{2} u \cdot u'\right) - \frac{1}{2} (I' - I'_*) \left(u \cdot V_c + \frac{1}{2} |u|^2\right).$$

The form of the cross section (C.8) allows for the first integration with respect to $V_c$ and $\sigma$,

$$\sum_{r=1}^{3} Q_{rr} = KL_0^2 \left(\frac{\pi kT}{m}\right)^{3} 2\pi \int_{R^3 \times [0, \infty)^2 \times [0, 1]^2} e^{-\pi \sum_{i=1}^{2} \left(\frac{1}{2} |\alpha|^2 + |c_i|^2\right) + I_{1*}}$$

$$\left\{ - \frac{1}{4} \left(\frac{5}{4} kT + \frac{m}{8} |u|^2 + I\right) (I - I_*) |u|^2$$

$$+ \frac{mkT}{32} \left(\left(\frac{5}{3} R - 3\right) |u|^2 + \frac{20R}{3m} (I + I_*)\right)$$

$$+ \left(\frac{1}{4} I + \frac{5}{16} kT\right) kT \left((R - 1) |u|^2 + \frac{4R}{m} (I + I_*)\right)\right\}$$

$$\times \hat{B} \phi_\alpha(r) (1 - R) R \psi_\alpha(R) I^n I^n_\alpha d\sigma dr dR dI_* dI \ du.$$
Next, passing to the spherical coordinates for the relative velocity \( u \), denoting \( |u| = y \), and integrating with respect to \( R \) and \( r \) we obtain

\[
\sum_{r=1}^{3} Q_{rr} = K \rho^2 \left( \frac{\pi kT}{m} \right)^{\frac{3}{2}} \int_{(0, \infty)^3} e^{-\frac{1}{kT} C(0, \Phi, 0)} \sum_{r=1}^{3} m k T \left( \frac{5}{3} C_{(0,1+\bar{\Phi}, 0)} - 3 C_{(0, \bar{\Phi}, 0)} \right) + \frac{4}{m} (I + I_{*}) C_{(0, \bar{\Phi} + 1, \bar{\Phi} + 1, \bar{\Phi})} \left( \frac{5}{36} m k T y^2 + \left( \frac{1}{4} I + \frac{5}{16} k T \right) k^2 C_{(1, 1, 1, \bar{\Phi})} \right) + \left( \frac{1}{m} \right)^{\frac{3}{2}} \left( 1 + I_{*} \right) C_{(0, \bar{\Phi} + 1, \bar{\Phi} + 1, \bar{\Phi})} \left( \frac{5}{90} m k T y^2 + \left( \frac{1}{4} I + \frac{5}{16} k T \right) k^2 C_{(1, 1, 1, \bar{\Phi})} \right) \right) I^{0} I_{*}^{0} \, dI_{*} \, dI \, du,
\]

where the constants are defined in (B.7). Finally, preforming integration with respect to \( I, I_{*}, y \) yields

\[
\sum_{r=1}^{3} Q_{rr} = -K \rho^2 \sqrt{\pi} \frac{\sqrt{3}}{48 \Gamma\left(\frac{4\alpha + \gamma + 3}{2}\right)} \left( \frac{k T}{m} \right)^{\frac{3}{2} + 3} \times \left\{ 9 \left( (4\alpha + \gamma) (2 (4\alpha + \gamma) + \gamma^2 + 38) + 7 \gamma^2 + 160 \right) \pi \Gamma \left( \alpha + \frac{\gamma + 1}{2} \right)^2 \right. \\
+ 2^{\gamma+5} \left( (4\alpha + \gamma) (3\alpha + \gamma) + 57 \alpha + 15 \gamma + 60 \right) \Gamma \left( \alpha + 1 \right)^2 \Gamma \left( \frac{\gamma + 3}{2} \right) \Gamma \left( \frac{\gamma + 5}{2} \right) \right\}.
\]

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