ON THE Ext-COMPUTABILITY OF SERRE QUOTIENT CATEGORIES

MOHAMED BARAKAT AND MARKUS LANGE-HEGERMANN

ABSTRACT. To develop a constructive description of Ext in categories of coherent sheaves over certain schemes, we establish a binatural isomorphism between the Ext-groups in Serre quotient categories \( A/C \) and a direct limit of Ext-groups in the ambient Abelian category \( A \). For Ext\(^1\) the isomorphism follows if the thick subcategory \( C \subset A \) is localizing. For the higher extension groups we need further assumptions on \( C \). With these categories in mind we cannot assume \( A/C \) to have enough projectives or injectives and therefore use Yoneda’s description of Ext.

1. INTRODUCTION

Our original motivation is to develop a constructive and computer-friendly description of Abelian categories of coherent sheaves \( \mathcal{Coh}_X \) on various classes of Noetherian schemes \( X \). In this setup the functors Hom and Ext\(^c\) are ubiquitous, and any constructive approach needs to incorporate these functors. For example, the global section functor on \( \mathcal{Coh}_X \) can be defined as \( \Gamma := \text{Hom}(\mathcal{O}_X, -) \), i.e., in terms of the Hom functor and the structure sheaf \( \mathcal{O}_X \). The higher sheaf cohomology \( H^i \) is usually defined in the nonconstructive larger category of quasi-coherent sheaves on \( X \) as \( H^i = R^i\Gamma = \text{Ext}^i(\mathcal{O}_X, -) \).

In this paper we deal with computing the bivariate \( \text{Ext}^i(-, -) \), where for the special univariate case of sheaf cohomology \( H^i = \text{Ext}^i(\mathcal{O}_X, -) \) there often exist good algorithms. Our minimal assumption on \( X \) is that the category \( \mathcal{Coh}_X \) is equivalent to a Serre quotient category \( A/C \approx \mathcal{Coh}_X \) where \( A \) is a computable category (in the sense of Appendix A) of finitely presented graded modules and \( C \subset A \) is its thick subcategory of all modules with zero sheafification. The canonical functor \( Q : A \to A/C \) then plays the role of the exact sheafification functor \( \text{Sh} : A \to \mathcal{Coh}_X, M \mapsto \widetilde{M} \). Some classes of schemes for which this holds are listed in [BLH14b, Section 4], including projective and toric schemes.

The computability of Ext\(^c\) would usually follow from that of Hom in case the underlying category is computable and has constructively enough projectives or enough injectives. However, as categories of coherent sheaves do not in general admit enough injectives or projectives we cannot assume this for the computation of Ext\(^c\) in an abstract Serre quotient \( A/C \). Hence, Ext\(^c\) in such an \( A/C \) cannot even be defined constructively as a derived functor using projective or injective resolutions and we are left over with Yoneda’s description of Ext\(^c\) [Oor64]. Although Yoneda’s description does not a priori provide an algorithm to compute Ext\(^c\), it is sufficient to prove our main result: Under certain assumptions on \( C \) the computability of Ext\(^c\) in \( A/C \)

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can be reduced to the computability of $\Ext^c$ in $\mathcal{A}$, provided a certain (infinite) direct limit is constructive. More precisely:

**Theorem 1.1.** If $\mathcal{C}$ is an almost split localizing\(^1\) subcategory of an Abelian category $\mathcal{A}$ then the binatural transformation\(^2\)

$$\mathcal{Q}^{\Ext} : \lim_{\substack{M' \leq M, \\ M/M' \in \mathcal{C}}} \Ext^c_{\mathcal{A}}(M', N) \to \Ext^c_{\mathcal{A}/\mathcal{C}}(M, N)$$

is an isomorphism (of Abelian groups) for all $\mathcal{C}$-torsion-free $M \in \mathcal{A}$ and $\mathcal{C}$-saturated $N \in \mathcal{A}$.

For applications to coherent sheaves $\mathcal{A}/\mathcal{C} \simeq \mathsf{Coh} X$ (for $X$ as above) we need to prove that the thick subcategory $\mathcal{C}$ of modules with zero sheafification is almost split localizing. This is for example the case if $X$ is a projective space and $\mathcal{A}$ is suitably chosen (see Section 3 and Example 6.5). However, it is worth mentioning that Theorem 1.1 cannot cover\(^3\) the case of coherent sheaves on nonsmooth toric varieties. One can see this easily since the Cox ring and hence the category $\mathcal{A}$ of finitely presented graded modules over this ring is of finite global dimension while one can easily construct coherent sheaves on a nonsmooth toric variety with non-vanishing $\Ext^c$ for arbitrarily high $c$ (see Example 6.6).

The theorem suggests an algorithmic approach to the computability of $\Ext^c$ in $\mathcal{A}/\mathcal{C}$. To compute the left hand side $\lim_{\substack{M' \leq M, \\ M/M' \in \mathcal{C}}} \Ext^c_{\mathcal{A}}(M', N)$ we need to be able to compute $\Ext^c_{\mathcal{A}}$ and a direct limit of Abelian groups. For categories of graded modules $\mathcal{A}$ there are well-known algorithms to compute $\Ext^c_{\mathcal{A}}$. Proving that the (infinite) direct limit can be computed in finite terms depends on $\mathcal{A}$ and $\mathcal{C}$. For example, in the category $\mathcal{A}/\mathcal{C} \simeq \mathsf{Coh} X$ of coherent sheaves on a projective space the finiteness of this direct limit follows from the Castelnuovo-Mumford regularity. Thus, Theorem 1.1 is an abstract form of [Smi00, Theorem 1] and [Smi13], without the context-specific convergence analysis. If $\mathcal{A}$ is the category of graded modules and if the limit is reached for a certain $M' \leq M$ then one can use a graded free resolution of $M'$ in $\mathcal{A}$ to compute $\Ext^c_{\mathcal{A}/\mathcal{C}}(M, N)$. In this case this graded free resolution of $M'$ in $\mathcal{A}$ corresponds, under the canonical functor $\mathcal{Q} : \mathcal{A} \to \mathcal{A}/\mathcal{C}$, to a locally free resolution of $M$ in $\mathcal{A}/\mathcal{C} = \mathsf{Coh} X$ satisfying some regularity bounds. We believe that the multigraded Castelnuovo-Mumford [MS04, MS05, HSS06] can be used to prove the finiteness of the limit in the case of smooth projective toric varieties. We leave this for future work.

We briefly recall the language of Serre quotient categories in Section 2 and deal with the $c = 0$ case of Theorem 1.1 in Section 3. In Section 4 we recall Yoneda’s description of $\Ext^c$ and in Section 5 we define the binatural transformation $\mathcal{Q}^{\Ext}$. In the main Section 6 we define the above mentioned condition which $\mathcal{C}$ needs to satisfy and prove Theorem 1.1. There it is also proved that the theorem is valid if $c = 1$ under the weaker condition that $\mathcal{C}$ is a localizing subcategory of $\mathcal{A}$ (cf. Theorem 6.2). Finally, in Appendix A we briefly sketch a constructive context for this paper.

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\(^1\)Cf. Definition 6.3.
\(^2\)We drop the canonical functor $\mathcal{Q}$ in $\Ext^c_{\mathcal{A}/\mathcal{C}}(\mathcal{Q}(M), \mathcal{Q}(N))$ since $\mathcal{Q}$ is the identity on objects.
\(^3\)Contrary to Theorem 6.2.
2. Preliminaries on Serre Quotients

In this section we recall some results about Serre quotients [Gab62]. From now on $\mathcal{A}$ is an Abelian category.

A non-empty full subcategory $\mathcal{C}$ of an Abelian category $\mathcal{A}$ is called thick if it is closed under passing to subobjects, factor objects, and extensions. In this case the (Serre) quotient category $\mathcal{A}/\mathcal{C}$ is a category with the same objects as $\mathcal{A}$ and Hom-groups

$$\text{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) := \lim_{M' \leq M, N' \leq N} \text{Hom}_\mathcal{A}(M', N/N').$$

The canonical functor $\mathcal{Q} : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{C}$ is defined to be the identity on objects and maps a morphism $\varphi \in \text{Hom}_\mathcal{A}(M, N)$ to its class in the direct limit $\text{Hom}_{\mathcal{A}/\mathcal{C}}(M, N)$. The category $\mathcal{A}/\mathcal{C}$ is Abelian and the canonical functor $\mathcal{Q} : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{C}$ is exact.

Let $\mathcal{C} \subset \mathcal{A}$ be thick. An object $M \in \mathcal{A}$ is called $\mathcal{C}$-torsion-free if $M$ has no nonzero subobjects in $\mathcal{C}$. We will need the following simple lemma.

**Lemma 2.1.** An extension of $\mathcal{C}$-torsion-free objects is again $\mathcal{C}$-torsion-free.

**Proof.** Let $E$ be an object in $\mathcal{A}$ with $\mathcal{C}$-torsion-free subobject $L$ and $\mathcal{C}$-torsion-free factor object $B = E/L$. Assume that $E$ has a nontrivial $\mathcal{C}$-subobject $T$. Since $L \cap T = 0$ we conclude that $T$ is isomorphic to the nontrivial $\mathcal{C}$-subobject $(T + L)/L$ of $B$, a contradiction. \square

If every object $M \in \mathcal{A}$ has a maximal $\mathcal{C}$-subobject $H(\mathcal{C})(M)$ then we call $\mathcal{C}$ a thick subcategory. An object $M \in \mathcal{A}$ is called $\mathcal{C}$-saturated if it is $\mathcal{C}$-torsion-free and every extension of $M$ by an object $C \in \mathcal{C}$ is trivial. Denote by $\text{Sat}_\mathcal{C}(\mathcal{A}) \subset \mathcal{A}$ the full subcategory of $\mathcal{C}$-saturated objects with embedding functor $\iota : \text{Sat}_\mathcal{C}(\mathcal{A}) \hookrightarrow \mathcal{A}$. The thick subcategory $\mathcal{C} \subset \mathcal{A}$ is called a localizing subcategory if the canonical functor $\mathcal{Q} : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{C}$ admits a right adjoint $\mathcal{F} : \mathcal{A}/\mathcal{C} \rightarrow \mathcal{A}$, called the section functor of $\mathcal{Q}$. The section functor $\mathcal{F} : \mathcal{A}/\mathcal{C} \rightarrow \mathcal{A}$ is left exact and preserves products, the counit of the adjunction $\delta : \mathcal{Q} \circ \mathcal{F} \cong \text{Id}_{\mathcal{A}/\mathcal{C}}$ is a natural isomorphism. Let $\eta : \text{Id}_{\mathcal{A}} \rightarrow \mathcal{F} \circ \mathcal{Q}$ denote the unit of the adjunction. The kernel $\ker(\eta_M : M \rightarrow (\mathcal{F} \circ \mathcal{Q})(M)))$ is then the maximal $\mathcal{C}$-subobject $H(\mathcal{C})(M)$ of $M$. The cokernel of $\eta_M$ lies in $\mathcal{C}$. We call $(\mathcal{F} \circ \mathcal{Q})(M)$ the $\mathcal{C}$-saturation of $M$. An object $M \in \mathcal{A}$ is $\mathcal{C}$-saturated if and only if $\eta_M$ is an isomorphism.

The image $\mathcal{F}(\mathcal{A}/\mathcal{C})$ of $\mathcal{F}$ is a subcategory of $\text{Sat}_\mathcal{C}(\mathcal{A})$ and the inclusion functor $\mathcal{F}(\mathcal{A}/\mathcal{C}) \hookrightarrow \text{Sat}_\mathcal{C}(\mathcal{A})$ is an equivalence of categories with the restricted-corestricted monad $\mathcal{F} \circ \mathcal{Q} : \text{Sat}_\mathcal{C}(\mathcal{A}) \rightarrow \mathcal{F}(\mathcal{A}/\mathcal{C})$ as a quasi-inverse. The restricted canonical functor $\mathcal{Q} : \text{Sat}_\mathcal{C}(\mathcal{A}) \rightarrow \mathcal{A}/\mathcal{C}$ and the corestricted section functor $\mathcal{F} : \mathcal{A}/\mathcal{C} \rightarrow \text{Sat}_\mathcal{C}(\mathcal{A})$ are quasi-inverse equivalences of categories. In particular, $\text{Sat}_\mathcal{C}(\mathcal{A}) \simeq \mathcal{F}(\mathcal{A}/\mathcal{C}) \simeq \mathcal{A}/\mathcal{C}$ is an Abelian category. Define the reflector $\mathcal{D} := \text{co-res}_{\text{Sat}_\mathcal{C}(\mathcal{A})}(\mathcal{F} \circ \mathcal{Q}) : \mathcal{A} \rightarrow \text{Sat}_\mathcal{C}(\mathcal{A})$. The adjunction $\mathcal{D} \dashv (\mathcal{F} : \mathcal{A}/\mathcal{C} \rightarrow \mathcal{A})$. They both share the same adjunction monad $\mathcal{F} \circ \mathcal{Q} = \iota \circ \mathcal{D} : \mathcal{A} \rightarrow \mathcal{A}$. In particular, the reflector $\mathcal{D}$ is exact and $\iota$ is left exact. $\mathcal{F}(\mathcal{A}/\mathcal{C}) \simeq \text{Sat}_\mathcal{C}(\mathcal{A})$ are not in general Abelian subcategories of $\mathcal{A}$, as short exact sequences in $\text{Sat}_\mathcal{C}(\mathcal{A})$ are not necessarily exact in $\mathcal{A}$. For more details and for a characterization of the monad $\mathcal{F} \circ \mathcal{Q}$ see [BLH13].

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$^4$A functor is called a reflector if it has a fully faithful right adjoint.
3. The $c = 0$ Case

If the thick subcategory $C \subset A$ is torsion then the double direct limit in the definition of the Hom-groups in $A/C$ simplifies to a single direct limit

$$\text{Hom}_{A/C}(M, N) = \lim_{\overset{M' \leq M}{M'/M' \in C}} \text{Hom}_{A}(M', N).$$

If furthermore $C \subset A$ is localizing then the Hom-adjunction $^5$ between $Q$ and $S$ yields

$$(\text{Hom}) \quad \text{Hom}_{A}(M, (\mathcal{L} \circ \mathcal{Q})(N)) \cong \text{Hom}_{A/C}(M, N),$$

for all $M, N \in A$, avoiding the direct limit completely. Theorem 1.1 generalizes this last formula, being the $c = 0$ case.

The monad $\mathcal{L} \circ \mathcal{Q}$ together with its unit are constructive in the case $A/C \cong \text{Coh } \mathbb{P}^n_k$, i.e., of coherent sheaves on the projective space $X = \mathbb{P}^n_k$ over a field $k$. Hence, the above mentioned Hom-adjunction can be used to compute (global) Hom-groups. More precisely, let $A$ be the category of finitely presented $\mathbb{Z}$-graded $k[x_0, \ldots, x_n]$-modules generated in degree $\geq 0$ and $C$ be the thick subcategory of finite length modules. The $C$-saturation of an $N \in A$ is the truncated module of twisted global sections, i.e., $(\mathcal{L} \circ \mathcal{Q})(N) = \bigoplus_{i \geq 0} \Gamma(\tilde{N}(i))$, where $\tilde{N} \in \text{Coh } \mathbb{P}^n_k$ is the sheafifications of $N$. For $X = \mathbb{P}^n_k$, and hence for any projective scheme, there are already several algorithms to compute the monad $\mathcal{L} \circ \mathcal{Q}$; e.g., as an ideal transform $^{[BS98, \text{Theorem 20.3.15]}}$, or by the Beilinson monad $^{[Bei78, \text{EFS03, DE02}},$ or by the BGG-correspondence.

Recently, Perling $^{[Per14]}$ described the section functor $\mathcal{L}$ and hence the monad $\mathcal{L} \circ \mathcal{Q}$ for a larger class of schemes, but in a (yet) nonconstructive way. A constructive description is highly desirable as it would widen the applicability of Theorem 1.1 as an algorithm to compute Ext’s for further classes of smooth schemes.

4. Yoneda’s Description of Ext

Since in applications to coherent sheaves the quotient category $A/C$ does not have enough projectives or injectives we use Yoneda’s description of Ext$^c$ (cf. $^{[ML95, \text{Section III.5}]})$.

So let $B$ be an Abelian category. A $c$-cocycle in the $\text{Ext}_{B}^{c}(M, N)$ group ($c > 0$) is an equivalence class of $c$-extensions of $M$ by $N$, i.e., exact $B$-sequences

$$e : 0 \leftarrow M \leftarrow G_c \leftarrow \cdots \leftarrow G_1 \leftarrow N \leftarrow 0$$

of length $c + 2$. Two $c$-extensions $e, e'$ of $M$ by $N$ are in directed relation if there exists a chain morphism of the form

$$\begin{align*}
& e : 0 \leftarrow M \leftarrow G_c \leftarrow \cdots \leftarrow G_1 \leftarrow N \leftarrow 0 \\
& \downarrow \hspace{2cm} \downarrow \\
& e' : 0 \leftarrow M \leftarrow G'_c \leftarrow \cdots \leftarrow G'_1 \leftarrow N \leftarrow 0
\end{align*}$$

$^5$Again, we drop the canonical functor $\mathcal{Q}$ in $\text{Hom}_{A/C}(\mathcal{L}(M), \mathcal{L}(N))$ since $\mathcal{Q}$ is the identity on objects.
For $c > 1$ this directed relation is not symmetric. A $c$-cocycle is an equivalence class of the equivalence relation generated by this directed relation. Abusing the notation we will denote by $e$ the $c$-cocycle in $\text{Ext}_B^c(M, N)$ of a $c$-extension $e$ of $M$ by $N$.

We now recall the definition of the Yoneda composition $\text{Ext}_B^c(M, N) \times \text{Ext}_B^c(N, L) \to \text{Ext}_B^{c+c'}(M, L)$. We start with the case $c, c' > 0$. For $e^M_N \in \text{Ext}_B^c(M, N)$ and $e^N_L \in \text{Ext}_B^{c'}(N, L)$ represented by the extensions

\[ e^M_N : 0 \to M \to G_c \to \cdots \to G_1 \to N \to 0 \text{ and } e^N_L : 0 \to N \to G'_c \to \cdots \to G'_1 \to L \to 0, \]

respectively. The Yoneda composite $e^M_L = e^M_N e^N_L \in \text{Ext}_B^{c+c'}(M, L)$ is the $(c + c')$-cocycle represented by the $(c + c')$-extension

\[ e^M_L : 0 \to M \to G_c \to \cdots \to G'_1 \to L \to 0 \]

of $M$ by $L$, where the morphism $G_1 \to G'_1$ is the composition $G_1 \to N \to G'_c$.

For $c = 0$ and $c' > 0$ let $\varphi^M_N \in \text{Hom}_B(M, N)$ and $e^N_L \in \text{Ext}_B^{c'}(N, L)$ as above. The Yoneda composite $\varphi^M_N e^N_L \in \text{Ext}_B^c(M, L)$ is given by the pullback $c'$-extension

\[ e^N_L : 0 \to N \to G'_{c'} \to G'_{c'-1} \to \cdots \to G'_1 \to L \to 0 \]
\[ \varphi^M_N e^N_L : 0 \to M \to G'_{c'} \to G'_{c'-1} \to \cdots \to G'_1 \to L \to 0 \]

For $c > 0$ and $c' = 0$ let $e^M_N \in \text{Ext}_B^c(M, N)$ as above and $\psi^N_L \in \text{Hom}_B(N, L)$. The Yoneda composite $e^M_N \psi^N_L \in \text{Ext}_B^c(M, L)$ is given by the pushout $c$-extension

\[ e^M_N : 0 \to M \to G_c \to \cdots \to G_1 \to N \to 0 \]
\[ e^M_N \psi^N_L : 0 \to M \to G_c \to \cdots \to G'_1 \to L \to 0 \]

For more details see, e.g., [HS97, Section IV.9], [BB08, Appendix B].

5. THE BINATURAL TRANSFORMATION

Let $A$ is an Abelian category and $C \subset A$ a thick subcategory. Applying the exact canonical functor $\mathcal{Q} : A \to A/C$ to a cocycle $e \in \text{Ext}_A^c(M, N)$ we obtain a cocycle $\mathcal{Q}(e)$ in $\text{Ext}_A^c(M, N)$. In other words, the canonical functor $\mathcal{Q} : A \to A/C$ induces maps\(^6\)

\[ \text{Ext}_A^c(M', N'/N') \to \text{Ext}_{A/C}^c(M, N) \]

\(^6\)We write $\text{Ext}_{A/C}^c(M, N)$ for $\text{Ext}_{A/C}^c(\mathcal{Q}(M), \mathcal{Q}(N))$ since $\mathcal{Q}$ is the identity on objects.

\(^7\)By this we mean the composition $\text{Ext}_A^c(M', N/N') \to \text{Ext}_{A/C}^c(M', N/N') \cong \text{Ext}_{A/C}^c(M, N)$, where the last isomorphism is the inverse of the one induced by the $A$-mono $M' \to M$ and the $A$-epi $N \to N'$, as both become isomorphisms in $A/C$.\]
for all \( M, N \in A \), \( M' \leq M \), \( N' \leq N \) with \( M/M' \in \mathcal{C} \) and \( N' \in \mathcal{C} \). For \( M'' \leq M' \) with \( M'/M'' \in \mathcal{C} \) and \( N'' \leq N' \) with \( N''/N' \in \mathcal{C} \) the cocycle

\[
e'': 0 \leftarrow M'' \leftarrow G''_c \leftarrow G''_{c-1} \leftarrow \cdots \leftarrow G'_2 \leftarrow G'_1 \leftarrow N/N'' \leftarrow 0 \in \text{Ext}^e_A(M'', N/N''),
\]

induces a cocycle

\[
e'' = (M'' \hookrightarrow M')e'(N/N' \rightarrow N/N''),
\]
as the Yoneda composite \( e'' \). Sending the cocycle \( \varphi \) of the maps \( \varphi G''_c = G''_{c'} \) \( \leq c' \leq c-1 \).

Taking the pullback of a subobject \( \iota_{L'} : L' \hookrightarrow L \) with \( L/L' \in \mathcal{C} \) we obtain a subobject \( \iota'_M : M' \hookrightarrow M \) with \( M/M' \leq \mathcal{C} \). Sending the cocycle \( \iota_{L'}G''_c \in \text{Ext}^c_A(L', N) \) to \( \iota_MG''_c \in \text{Ext}^c_A(M', L) \) defines the first argument action of \( F \) on \( \varphi \). The proof of functoriality in the first argument follows from the identity \( \iota_MG''_c = \iota_M\varphi G''_c = \varphi |_{M'} \iota_{L'}G''_c = \varphi |_{M'}G''_c \).

For the functoriality in the second argument consider a morphism \( \psi : N \rightarrow L \) and take the colimit \( \varphi \) \( \varphi |_{M'}G''_c = \varphi |_{M'}G''_c \)

As \( \mathcal{D} \) is exact, the map \( \mathcal{D}^{\text{Ext}} \) respects Baer sums.
6. The Proof

Our goal is to give sufficient conditions for the binatural transformation $\mathcal{D}^{\text{Ext}}$ to be an isomorphism. For this we assume that $\mathcal{C} \subset \mathcal{A}$ is a localizing subcategory of the Abelian category $\mathcal{A}$. Then the restricted canonical functor $\mathcal{D} : \text{Sat}_C(\mathcal{A}) \to \mathcal{A}/\mathcal{C}$ and the corestricted section functor $\mathcal{S} : \mathcal{A}/\mathcal{C} \to \text{Sat}_C(\mathcal{A})$ are adjoint equivalences of categories.

Remark 6.1. We will use this equivalence to replace $\text{Ext}_{\mathcal{A}/\mathcal{C}}$ by the isomorphic $\text{Ext}_{\text{Sat}_C(\mathcal{A})}$, the functor $\mathcal{D} : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ by $\hat{\mathcal{D}} := \text{co-res}_{\text{Sat}_C(\mathcal{A})}(\mathcal{S} \circ \mathcal{D}) : \mathcal{A} \to \text{Sat}_C(\mathcal{A})$, and finally $\mathcal{D}^{\text{Ext}}$ by

$$\hat{\mathcal{D}}^{\text{Ext}} : \lim_{M' \leq M, \ M/M' \in \mathcal{C}} \text{Ext}^{1}_{\mathcal{A}}(M', N) \to \text{Ext}^{1}_{\text{Sat}_C(\mathcal{A})}(M, N).$$

For simplicity we write $\text{Ext}^{1}_{\text{Sat}_C(\mathcal{A})}(M, N)$ for $\text{Ext}^{1}_{\text{Sat}_C(\mathcal{A})}(\hat{\mathcal{D}}(M), \hat{\mathcal{D}}(N))$. Recall that in Theorem 1.1 we require $M$ to be $\mathcal{C}$-torsion-free and $N$ to be $\mathcal{C}$-saturated. Since the cokernel $(\mathcal{S} \circ \mathcal{D})(M)/M$ of $\eta_M$ lies in $\mathcal{C}$ we can, without loss of generality, as well assume $M$ to be $\mathcal{C}$-saturated as the limit does not distinguish between $M$ and its saturation $(\mathcal{S} \circ \mathcal{D})(M)$.

6.1. The proof for $\text{Ext}^{1}$. For $c = 1$ it turns out that assuming $\mathcal{C} \subset \mathcal{A}$ to be localizing is already sufficient for $\mathcal{D}^{\text{Ext}}$ to be an isomorphism.

Theorem 6.2. If $\mathcal{C}$ is a localizing subcategory of the Abelian category $\mathcal{A}$ then

$$\hat{\mathcal{D}}^{\text{Ext}} : \lim_{M' \leq M, \ M/M' \in \mathcal{C}} \text{Ext}^{1}_{\mathcal{A}}(M', N) \to \text{Ext}^{1}_{\text{Sat}_C(\mathcal{A})}(M, N) \cong \text{Ext}^{1}_{\mathcal{A}/\mathcal{C}}(M, N)$$

is an isomorphism (of Abelian groups) for all $\mathcal{C}$-saturated $M, N \in \mathcal{A}$.

Proof. Recall that a short exact sequence $e : 0 \to M \xleftarrow{\pi} E \xleftarrow{\iota} N \to 0$ in $\text{Sat}_C(\mathcal{A})$ is in general only left exact in $\mathcal{A}$, since the embedding functor $\iota : \text{Sat}_C(\mathcal{A}) \to \mathcal{A}$ is in general only left exact. The $\mathcal{A}$-cokernel of $\pi$ lies in $\mathcal{C}$, i.e., for $M' := \text{im} \pi$ the sequence $0 \to M' \xleftarrow{\pi} E \xleftarrow{\iota} N \to 0$ is exact in $\mathcal{A}$ and $M/M' = \text{coker} \pi \in \mathcal{C}$. This yields the preimage of $e$ under $\mathcal{D}^{\text{Ext}}$ and shows surjectivity.

For the injectivity take an exact $\mathcal{A}$-sequence $e : 0 \to M' \xleftarrow{E} E \xleftarrow{\iota} N \to 0$ such that the corresponding exact $\text{Sat}_C(\mathcal{A})$-sequence

$$\hat{\mathcal{D}}^{\text{Ext}}(e) : 0 \to \hat{\mathcal{D}}(M') \xleftarrow{\hat{\mathcal{D}}(E)} \hat{\mathcal{D}}(E) \xleftarrow{\hat{\mathcal{D}}(\iota)} \hat{\mathcal{D}}(N) \to 0$$

is split, i.e., $e$ is in the kernel of $\mathcal{D}^{\text{Ext}}$. By definition of split short exact sequences, there is a $\hat{\psi} : \hat{\mathcal{D}}(E) \to \hat{\mathcal{D}}(N)$ such that $\hat{\psi} \circ \hat{\mathcal{D}}(\iota) = \text{Id}_{\hat{\mathcal{D}}(N)}$. Since $N$ is $\mathcal{C}$-saturated the unit $\eta_N : N \to \iota(\hat{\mathcal{D}}(N))$ is an isomorphism and we can define $\psi := \eta^{-1}_N \circ \iota(\hat{\psi}) \circ \eta_E$. Note that $\eta_E \circ \varphi = \iota(\hat{\mathcal{D}}(\varphi)) \circ \eta_N$, by the naturality of $\eta$. Then $\psi \circ \varphi = \eta^{-1}_N \circ \iota(\hat{\psi}) \circ \eta_E \circ \varphi = \eta^{-1}_N \circ \iota(\hat{\psi}) \circ \iota(\hat{\mathcal{D}}(\varphi)) \circ \eta_N = \eta^{-1}_N \circ \iota(\text{Id}_{\hat{\mathcal{D}}(N)}) \circ \eta_N = \eta^{-1}_N \circ \text{Id}_{\hat{\mathcal{D}}(N)} \circ \eta_N = \text{Id}_N$ implies that $e$ is split, i.e., zero in $\text{Ext}^{1}_{\mathcal{A}}(M', N)$. 

\[\square\]
6.2. The proof of surjectivity for higher Ext’s. For $c \geq 2$ we need further conditions on the categories $\mathcal{A}$ and $\mathcal{C}$.

**Definition 6.3.** Let $\mathcal{A}$ be an Abelian category and $\mathcal{C} \subset \mathcal{A}$ a thick subcategory. For an object $a \in \mathcal{A}$ we call a subobject $a^\perp \leq a$ an **almost $\mathcal{C}$-complement** if $a^\perp$ is $\mathcal{C}$-torsion-free and $a/a^\perp \in \mathcal{C}$. We call $\mathcal{C}$ an **almost split** (thick) subcategory if for each object $a \in \mathcal{A}$ there exists an almost $\mathcal{C}$-complement $a^\perp$.

**Remark 6.4.** Let $\mathcal{A}$ be an Abelian Noetherian category and $\mathcal{C} \subset \mathcal{A}$ a thick subcategory. The following two properties are equivalent:

(a) $\mathcal{C}$ is almost split.

(b) For each object $a \in \mathcal{A}$ which does not lie in $\mathcal{C}$ there exists a nontrivial $\mathcal{C}$-torsion-free subobject of $a$.

**Proof.** We only discuss the nontrivial direction. Start with a nontrivial $\mathcal{C}$-torsion-free subobject $a_1 \leq a$. If $a/a_1$ lies in $\mathcal{C}$ we are done. Otherwise define $a_2$ to be the preimage in $a$ of a nontrivial $\mathcal{C}$-torsion-free subobject in $a/a_1$. By Lemma 6.1, $a_2$ is $\mathcal{C}$-torsion-free. Iterating the process yields a strictly ascending chain of $\mathcal{C}$-torsion-free subobjects of $a$. Due to Noetherianity this iteration has to stop, say at $a_n$, and it can only stop at $a_n$ if $a/a_n$ lies in $\mathcal{C}$. $\square$

**Example 6.5.** Let $S$ be the polynomial ring $k[x_0, \ldots, x_n]$ graded by total degree and $\mathcal{A}$ the category of f.g. graded $S$-module. Consider the thick subcategory $\mathcal{C}$ of 0-dimensional graded modules. These are the modules living in a finite degree interval. For any $M \in \mathcal{A}$ there exists a maximal submodule $N \in \mathcal{C}$, and let $d$ be the smallest integer with $N_{\geq d} = 0$. Then $M_{\geq d}$ is a nontrivial $\mathcal{C}$-torsion-free submodule of $M$. Recall that $\mathcal{A}/\mathcal{C} \simeq \mathcal{Coh}\mathbb{P}^n_k$.

**Example 6.6.** Consider the cone $\sigma = \text{Cone}((1,0), (1,2)) \subset \mathbb{R}^2$ and the nonsmooth affine toric variety $U_\sigma$ with Cox ring $S = k[x,y], \deg x = \deg y = 1 \in \text{Cl}(U_\sigma) = \mathbb{Z}/2\mathbb{Z}$, and affine coordinate ring $S_0 = k[x^2, xy, y^2]$. The kernel of the sheafification from the category $\mathcal{A}$ of f.g. graded $S$-modules to $\mathcal{Coh} U_\sigma$ is the thick subcategory $\mathcal{C}$ of graded modules with $M_0 = 0$ (cf. [CLS11, Proposition 5.3.3]). The graded $S$-module $M := S/\langle x^2, xy, y^2 \rangle$ violates condition (b) of the previous remark. Hence, $\mathcal{C} \subset \mathcal{A}$ is not almost split.

Now we show that $\mathcal{D}_{\mathcal{Ext}}$ is not surjective. Note that all $\text{Ext}^c_{\mathcal{A}}$ vanish for $c > 2$, where 2 is the global dimension of $S$. However, for the $S_0$-module $k = S_0/\langle x^2, xy, y^2 \rangle$ the group $\text{Ext}^c_{\mathcal{A}/\mathcal{C}}(k, k) \neq 0$ for all $c \in \mathbb{Z}_{\geq 0}$ (in fact, $\text{Ext}^c_{\mathcal{A}/\mathcal{C}}(k, k) \cong k^4$ for all $c > 1$). The sheafification of $k$ is the skyscraper sheaf in $\mathcal{Coh} U_\sigma \simeq \mathcal{A}/\mathcal{C}$ on the singular point of $\text{Spec}(S_0)$.

One can replace $\mathcal{C}$-torsion-free $\mathcal{A}$-complexes having defects in $\mathcal{C}$ with exact $\mathcal{A}$-complexes, which are equivalent in the following sense:

**Definition 6.7.** Let $\mathcal{C}$ be a thick subcategory of the Abelian category $\mathcal{A}$ and $e$ an $\mathcal{A}$-complex. We say a subcomplex $e'$ **equals $e$ up to $\mathcal{C}$-factors** if $e/e'$ is a complex in $\mathcal{C}$.

**Lemma 6.8.** Let $\mathcal{C}$ be an almost split thick subcategory of the Abelian category $\mathcal{A}$ and

$e : 0 \leftarrow M \leftarrow G_c \leftarrow \cdots \leftarrow G_1 \leftarrow N \leftarrow 0.$
a \( C \)-torsion-free \( A \)-complex which is exact up to \( C \)-defects. Then there exists an exact (\( C \)-torsion-free) \( A \)-subcomplex \( e^\perp \) of \( e \)

\[
e^\perp : 0 \leftarrow M^\perp \leftarrow G_c^\perp \leftarrow \cdots \leftarrow G_1^\perp \leftarrow N \leftarrow 0
\]

which equals \( e \) up to \( C \)-factors.

\begin{proof}
Define \( \overline{G}_1 := G_1/\text{im}_A(G_1 \leftrightarrow N) \). Construct the subobject \( G_1^\perp \leq G_1 \) as the preimage in \( G_1 \) of an almost \( C \)-complement in \( G_1 \). Since \( G_1^\perp \) is the preimage of an almost \( C \)-complement it follows that \( G_1/G_1^\perp \in \mathcal{C} \).

For \( i > 1 \) we assume to have constructed \( G_i^\perp \leq G_i \) with \( G_i/G_i^\perp \in \mathcal{C} \). We proceed inductively and consider \( \overline{G}_i := G_i/\text{im}_A(G_i \leftrightarrow G_{i-1}^\perp \leftrightarrow \cdots \leftrightarrow G_1^\perp \leftrightarrow G_{c-1}^\perp \leftrightarrow N) \). Again construct the subobject \( G_i^\perp \leq G_i \) as the preimage in \( G_i \) of an almost \( C \)-complement in \( G_i \). As above \( G_i/G_i^\perp \in \mathcal{C} \).

Finally define the subobject \( M^\perp \leq M \) as the \( A \)-image \( \text{im}_A(M \leftrightarrow G_{c-1} \leftrightarrow G_{c-1}^\perp) \). Let \( \widehat{M} := \text{im}_A(M \leftrightarrow G_{c-1}) \). Then \( \widehat{M}/M^\perp \in \mathcal{C} \) as an epimorphic image of \( G_{c-1}/G_{c-1}^\perp \) under \( M \leftrightarrow G_{c-1} \). Since also \( M/\widehat{M} \in \mathcal{C} \) it follows that \( M/M^\perp \in \mathcal{C} \) as an extension of two objects in \( \mathcal{C} \). The whole argument is visualized in the diagram\(^8\) below, where the dotted lines stand for (factor) objects in \( \mathcal{C} \).
\end{proof}

The above lemma yields the preimages needed to prove the surjectivity of \( \check{\mathfrak{D}}^{\text{Ext}} \).

**Proposition 6.9.** Let \( \mathcal{C} \) be an almost split localizing subcategory of the Abelian category \( A \). Then

\[
\check{\mathfrak{D}}^{\text{Ext}} : \varinjlim_{\substack{M \leq M', \ M/M' \in \mathcal{C} \}} \text{Ext}^c_A(M', N) \to \text{Ext}^c_{\text{Sat}_C(A)}(M, N)
\]

is an epimorphism (of Abelian groups) for all \( \mathcal{C} \)-saturated \( M, N \in \mathcal{A} \).

\begin{proof}
For the surjectivity consider a \( c \)-extension \( \widehat{e} \in \text{Ext}^c_{\text{Sat}_C(A)}(M, N) \) for \( c > 0 \), represented by an exact \( \text{Sat}_C(A) \)-complex

\[
\widehat{e} : 0 \leftrightarrow M \leftrightarrow G_c \leftrightarrow \cdots \leftrightarrow G_1 \leftrightarrow N \leftrightarrow 0.
\]

\(^8\)Cf. [Bar09] for the use of Hasse diagrams to prove statements in Abelian categories.
Lemma 6.8 applied to the $\mathcal{A}$-complex $e = \iota(\hat{e})$ which is exact up to $\mathcal{C}$-defects yields a preimage $e^\perp$ of $\hat{e}$. 

Due to the left exactness of $\iota$ we can even choose $G_1^\perp := G_1$ when applying Lemma 6.8 in the proof of Proposition 6.9. This is illustrated by the diagram below.

6.3. The proof of injectivity for higher Ext’s. To prove the injectivity we show that almost $\mathcal{C}$-complements exist on the level of exact $\mathcal{A}$-complexes.

**Definition 6.10.** Let $\mathcal{C}$ be a thick subcategory of the Abelian category $\mathcal{A}$ and

$$e : 0 \leftarrow M \leftarrow G_c \leftarrow \cdots \leftarrow G_1 \leftarrow N \leftarrow 0$$

an $\mathcal{A}$-complex where $M, N$ are $\mathcal{C}$-torsion-free. We call a $\mathcal{C}$-torsion-free $\mathcal{A}$-subcomplex $\bar{e} \leq e$

$$\bar{e} : 0 \leftarrow \bar{M} \leftarrow \bar{G}_c \leftarrow \cdots \leftarrow \bar{G}_1 \leftarrow N \leftarrow 0,$$

which equals $e$ up to $\mathcal{C}$-factors an **almost $\mathcal{C}$-complement** in $e$.

**Proposition 6.11.** Let $\mathcal{C}$ be an almost split thick subcategory of the Abelian category $\mathcal{A}$ and

$$e : 0 \leftarrow M \leftarrow G_c \leftarrow \cdots \leftarrow G_1 \leftarrow N \leftarrow 0$$

an exact $\mathcal{A}$-complex where $M, N$ are $\mathcal{C}$-torsion-free. Then there exists an exact almost $\mathcal{C}$-complement $\tilde{e} \in \text{Ext}^c_{\mathcal{A}}(M, N)$ in $e$.

The value of this proposition lies in the following fact:

**Lemma 6.12.** The $\mathcal{A}$-subcomplex $\tilde{e} \leq e$ in the previous proposition represents in the colimit

$$\lim_{\substack{M' \leq M, M/M' \in \mathcal{C}}} \text{Ext}^c_{\mathcal{A}}(M', N)$$

the same $c$-cocycle as $e$.

**Proof.** The cocycle $e \in \text{Ext}^c_{\mathcal{A}}(M, N)$ is identical to the Yoneda product

$$e' := (\bar{M} \hookrightarrow M)e : 0 \leftarrow \bar{M} \leftarrow G_c' \leftarrow G_{c-1} \leftarrow \cdots \leftarrow G_1 \leftarrow N \leftarrow 0 \in \text{Ext}^c_{\mathcal{A}}(\bar{M}, N),$$

in the colimit $\lim_{\substack{M' \leq M, M/M' \in \mathcal{C}}} \text{Ext}^c_{\mathcal{A}}(M', N)$. 


By definition \( \widetilde{M} \leftarrow G_c' \leftarrow G_c \) is the pullback of \( \widetilde{M} \leftarrow M \leftarrow G_c \). The two morphisms \( \widetilde{M} \leftarrow G_c' \leftarrow G_c \) form a commutative square with \( M \leftarrow M \leftarrow G_c \). The universal property of the pullback yields a mono \( \widetilde{G}_c \leftarrow G_c' \) and we get an embedding of \( \epsilon \) in \( \epsilon' \) yielding the same \( c \)-cocycle in \( \text{Ext}_A^2(\widetilde{M}, N) \).

For the induction proof of Proposition 6.11 we need the next lemma, which shows how to replace short exact sequences in \( \mathcal{A} \) by short exact sequences of \( \mathcal{C} \)-torsion-free objects.

**Lemma 6.13.** Let \( \mathcal{C} \) be an almost split thick subcategory of the Abelian category \( \mathcal{A} \) and \( e : 0 \leftarrow M \leftarrow G \leftarrow L \leftarrow 0 \) a short exact \( \mathcal{A} \)-sequence with \( \mathcal{C} \)-torsion-free \( M \). Then there exists a short exact \( \mathcal{A} \)-subsequence \( \epsilon' : 0 \leftarrow M' \leftarrow G' \leftarrow L' \leftarrow 0 \) with \( \mathcal{C} \)-torsion-free objects which equals \( e \) up to \( \mathcal{C} \)-factors.

**Proof.** We interpret \( L \) as a subobject of \( G \) with factor object \( M \). Let \( G' \) be an almost \( \mathcal{C} \)-complement in \( G \). Hence \( G/G' \) lies in \( \mathcal{C} \). Now define \( L' = L \cap G' \) and \( M' = G'/L' \). The object \( L' \) is \( \mathcal{C} \)-torsion-free as a subobject of \( G' \) and \( M' \) is \( \mathcal{C} \)-torson-free since it is isomorphic to the subobject \( M'' = (G' + L)/L \) of \( M = G/L \). Finally \( M/M'' \) lies in \( \mathcal{C} \) as a factor of \( G/G' \) and \( L/L' \) lies in \( \mathcal{C} \) since it is isomorphic to the subobject \( (G' + L)/G' \) of \( G/G' \).

**Proof of Proposition 6.11.** We will construct \( \tilde{e} \) by induction on \( c \). For \( c = 1 \) take \( \tilde{e} = e \) by Lemma 2.1. Now assume the statement is true for \( c - 1 \) (i.e., for complexes of length \( c + 1 \)). Define \( L := \text{im}(G_c \leftarrow G_{c-1}) \). Write \( e \) as the Yoneda product (i.e., concatenation) \( e_1e_2 \) of the short exact \( \mathcal{A} \)-sequence \( e_1 : 0 \leftarrow M \leftarrow G_c \leftarrow L \leftarrow 0 \) and the exact \( \mathcal{A} \)-complex \( e_2 : 0 \leftarrow L \leftarrow G_{c-1} \leftarrow \cdots \leftarrow G_1 \leftarrow N \leftarrow 0 \). First apply Lemma 6.13 to \( e_1 \) and obtain the short exact \( \mathcal{C} \)-torsion-free \( \mathcal{A} \)-sequence \( e_1' : 0 \leftarrow M' \leftarrow G'_c \leftarrow L' \leftarrow 0 \). Now define the exact \( \mathcal{A} \)-subcomplex

\[
e_2' := (L' \leftarrow L)e_2 : 0 \leftarrow L' \leftarrow G'_{c-1} \leftarrow G'_{c-2} \leftarrow \cdots \leftarrow G_1 \leftarrow N \leftarrow 0
\]

of \( e_2 \). By the induction hypothesis there exists an exact \( \mathcal{C} \)-torsion-free \( \mathcal{A} \)-subcomplex \( \tilde{e}_2' : 0 \leftarrow L' \leftarrow G'_{c-1} \leftarrow G'_{c-2} \leftarrow \cdots \leftarrow G_1 \leftarrow N \leftarrow 0 \) of \( e_2' \). Replacing \( L' \) by its subobject \( \tilde{L} \) in \( e' \), we obtain the \( \mathcal{C} \)-torsion-free \( \mathcal{A} \)-complex \( \tilde{e}_1' : 0 \leftarrow M' \leftarrow G'_c \leftarrow \tilde{L} \leftarrow 0 \) which is exact up to a \( \mathcal{C} \)-defect. Now apply Lemma 6.8 to \( \tilde{e}_1' \) and obtain the short exact \( \mathcal{C} \)-torsion-free \( \mathcal{A} \)-sequence \( \tilde{e}_1 : 0 \leftarrow \tilde{M} \leftarrow \tilde{G}_c \leftarrow \tilde{L} \leftarrow 0 \). Finally define \( \tilde{e} := \tilde{e}_1\tilde{e}_2 \).

All constructions in this proof yield subcomplexes equal to their super-complexes up to \( \mathcal{C} \)-factors. Thus, we conclude the \( \tilde{e} \) equals \( e \) up to \( \mathcal{C} \)-factors.
By Remark 6.1 the following theorem is the equivalent “saturated form” of Theorem 1.1.

**Theorem 6.14.** If $C$ is an almost split localizing subcategory of the Abelian category $A$ then

$$\widehat{\mathcal{Q}}^\text{Ext}: \lim_{\mathcal{C}} \mathcal{Q}(\mathcal{C}) \to \mathcal{Q}(\mathcal{C})$$

is an isomorphism (of Abelian groups) for all $\mathcal{C}$-saturated $M, N \in A$.

**Proof.** Let $e' \in \mathcal{Q}(\mathcal{C})$ and $e'' \in \mathcal{Q}(\mathcal{C})$ be two cocycles which map to the same element $\hat{e} := \mathcal{Q}(e') = \mathcal{Q}(e'')$ in $\mathcal{Q}(\mathcal{C})$. By Proposition 6.11 we can pass in the colimit to $\mathcal{C}$-torsion-free representatives $\bar{e}' \in \mathcal{Q}(\mathcal{C})$ and $\bar{e}'' \in \mathcal{Q}(\mathcal{C})$, which are exact $\mathcal{A}$-subcomplexes of $e'$ and $e''$, respectively. Furthermore, $\bar{e}'$ is an $\mathcal{A}$-subcomplex of $\iota(e)$, as it is $\mathcal{C}$-torsion-free and the kernel of $e' \to \iota(e)$ is $H_C(e')$; the same holds for $\bar{e}''$. Taking the intersection of $\bar{e}'$ and $\bar{e}''$ as subcomplexes of $\iota(e)$ we obtain an $\mathcal{A}$-subcomplex $\bar{e}$ of $\iota(e)$, which is not necessarily exact. Lemma 6.8 yields an exact $\mathcal{A}$-subcomplex $e' \leq \bar{e}$ which still represents the same cocycle as $\bar{e}'$ and $\bar{e}''$ and hence $e'$ and $e''$ in the colimit, and thus all these cocycles are equal in the colimit. \qed

**APPENDIX A. SKETCH OF THE PROPER CONSTRUCTIVE SETUP**

We now roughly describe the constructive context of this paper. A detailed description would require a more elaborate preparation and would distract from the main result of this paper, which in this form should already be self-contained. The standard way to express mathematical notions constructively is to provide algorithms for all disjunctions and all existential quantifiers appearing in the defining axioms of a mathematical structure. In the case of Abelian categories this led us to the notion of a computable Abelian or constructively Abelian category [BLH11]. Given that, all constructions which only depend on a category being Abelian become computable\(^9\). The computability of $\mathcal{A}$ implies, in particular, that we can compute in its Hom-groups only locally, i.e., we can decide element membership in the Hom-sets, whether morphisms are zero, add and subtract morphisms, and hence decide the equality of two morphisms. This does not imply that we can “oversee” a Hom-group in any way, not even being able to decide its triviality (see Hom-computability below). For an Abelian category $\mathcal{A}$ with thick subcategory $\mathcal{C} \subset \mathcal{A}$ we prove in [BLH14c] that $\mathcal{A}/\mathcal{C}$ is computable once the Abelian category $\mathcal{A}$ is computable and the membership in $\mathcal{C} \subset \mathcal{A}$ is constructively decidable.

\(^9\)A constructive treatment of spectral sequences along these lines can be found in [Bar09] with a computer implementation in [BLH14a].
We call $C \subset A$ **constructively localizing** if there exists algorithms to compute the Gabriel monad $\mathcal{S} \circ \mathcal{Q}$ together with its unit. Formula (Hom) in Section 3 proves that if $A$ is Hom-computable and $C \subset A$ is constructively localizing then $A/C$ is Hom-computable, where Hom-computability means the computability as an enriched category over a computable monoidal category.

Theorem 1.1 implies that $A/C$ is Ext-computable if $A$ is Ext-computable and $C \subset A$ is almost split localizing and constructively localizing and the direct limit is constructive. We would define **Ext-computability** to be the Hom-computability of the derived category of $A$. This would lead too far away.

Finally, we note that the entire proof of Theorem 1.1 is constructive and suited for computer implementation. So if we assume that $A$ is computable and $C \subset A$ is constructively almost split localizing then the proof of Theorem 1.1 provides an algorithm to compute images and preimages of elements represented as Yoneda cocycles under $\hat{\mathcal{Q}}^{\text{Ext}} : \lim_{\rightarrow} M' \leq M, \text{Ext}_A^c(M', N) \to \text{Ext}_{\text{Sat}_C(A)}^c(M, N)$. Furthermore, if $A$ is Hom-computable and has constructively enough projectives or injectives then we can decide equality of (Yoneda) cocycles (cf. [BB08, Appendix B]).

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10 as we call it in [BLH13].

11 Enriched categories are usually required to be small. In an algorithmic setting any category is small, as the possible states of the computer memory is a set.

12 ... of Abelian groups, $k$-vector spaces, etc., depending on the context.

13 I.e., constructively localizing and that we can algorithmically construct the maximal almost $C$-complement of objects in $A$. 

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**Department of Mathematics, University of Kaiserslautern, 67653 Kaiserslautern, Germany**

*E-mail address: barakat@mathematik.uni-kl.de*

**Lehrstuhl B für Mathematik, RWTH Aachen University, 52062 Aachen, Germany**

*E-mail address: markus.lange.hegermann@rwth-aachen.de*