Multiplier Ideals, Vanishing Theorem
and Applications

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§0. Introduction.

The purpose of this note is to give a survey of the algebraic properties of multiplier ideals, and illustrate some of their applications to classical projective geometry.

Over the past ten or fifteen years, there have been two parallel developments of the classical Kodaira vanishing theorem. On the algebraic side, the Kawamata-Viehweg vanishing theorem for $\mathbb{Q}$-divisors has found many important applications, generally following the model of Kawamata’s arguments in [KMM]. From a more analytic viewpoint, the Nadel vanishing theorem [N] involving singular metrics has likewise proven of great utility. While a priori the analytic notion is more general, in actual applications these two approaches are essentially just different ways of packaging the same information. See Demailly’s CIME notes ([De3]) and Kollár’s lectures in this volume ([Kol4]) for proofs of this equivalence. While algebraic geometers are probably more comfortable with ideas occurring in Kawamata-Viehweg-based arguments, these arguments tend to be complicated by the necessity of passing to various resolutions of the ambient variety. By contrast, the Nadel-based approach essentially works directly on the variety of interest, which to some authors may seem more geometrically natural. One can suspect that the conceptual simplicity so achieved was one of the factors making possible the breakthrough of Anghern and Siu [AS]. It certainly was important to us in studying singularities of theta divisors [EL]. However already in the work of Esnault-Viehweg ([EV1]), it was realized that one could develop the multiplier-ideal viewpoint algebraically. The purpose of these notes is to outline this development, and indicate some applications. In term of actual content, much of the present material is covered in much greater generality from the Kawamata-Viehweg viewpoint in Kollár’s lectures ([Kol4]). Nonetheless, we hope that some readers may profit from seeing the material presented here in the language of multiplier ideals.

We start in §1 and §2 by recalling the basic constructions of multiplier ideals. Then we study the behaviour of these ideals under various standard geometric operations. In §3, we apply these results to study the singularities of theta divisors. In particular, we show that an irreducible theta divisor has only rational singularities. All the results in this section are joint work with R. Lazarsfeld. In §4, we study adjoint linear systems. These results generalize the classical theorems of Kodaira, Bombieri and Reider ([B]

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and [R]) on linear systems on surfaces. In the last section, we give a simple proof of a theorem of Levine on the invariance of plurigenra under deformations. I would like to mention that the idea of this proof is due to Siu. Finally, we discuss a result of Esnault and Viehweg on the zeros of polynomials ([EV2]). This paper was originally conceived as being a joint work with Lazarsfeld and Siu. We would like to thank them for sharing with us many of their ideas and many helpful discussions.

§1. Basic algebraic constructions of multiplier ideals and adjoint ideals.

To motivate the algebraic construction of multiplier ideals, it will be useful to recall the following well known theorems of Esnault and Viehweg ([EV1] 5.1 and 5.13). We consider the following setup. Let $X$ be a smooth complex projective variety and let $B$ be a Cartier divisor on $X$. Suppose that $B \equiv \Delta$, where $\Delta$ is a $\mathbb{Q}$-divisor of the form, $\Delta = \sum a_j F_j$, where the $F_j$’s are distinct smooth irreducible divisors. We also assume that the support of $\Delta$ is in normal crossing, and $0 < a_j < 1$ for all $j$. In other words, $\Delta$ is a boundary divisor.

Theorem 1.1. Let $D$ be any effective divisor on $X$ such that the support of $D$ is contained in the support of $\Delta$. Then the natural map,

$$H^i(\mathcal{O}_X(K_X + B)) \rightarrow H^i(\mathcal{O}_X(K_X + B + D))$$

is injective for every $i$.

The next proposition allows us to replace $X$ by another birational model.

Proposition 1.2. (a) Let $X$, $B$, and $\Delta$ be as in Theorem 1.1. Suppose that $g : Y \rightarrow X$ is a proper birational map such that the the support of $g^*(\Delta)$ is again a divisor in normal crossing. Then $g_*(\mathcal{O}_Y(K_Y + g^*B - [g^*\Delta])) = \mathcal{O}_X(K_X + B)$, and $R^i g_*(\mathcal{O}_Y(K_Y + B - [\Delta])) = 0$ for $i > 0$.

(b) Suppose that there is a nef and big divisor $H$ on $X$, such that $supp(H) \subset supp(\Delta)$. Then $H^i(\mathcal{O}_X(K_X + B)) = 0$ for $i > 0$.

These theorems are very useful. For instance, they imply the well known vanishing theorem of Kawamata and Viehweg and they also give Kollár’s theorem on higher direct images of dualizing sheaf.

Now we’ll recall the algebraic construction and the basic properties of multiplier ideals. We consider the following setup:

Let $X$ be a smooth variety and $G$ be an effective $\mathbb{Q}$-divisor on $X$. We construct an embedded resolution for $G$,

$$f : Y \rightarrow X.$$
We consider the following $\mathbb{Q}$-divisor on $Y$:

$$R = K_Y - f^*(K_X + G) = \sum a_j F_j,$$

where we assume that the $F_j$’s are distinct irreducible smooth divisors and their supports are in normal crossing. Then

$$[R] = \sum [a_j] F_j.$$

We write

$$[R] = P - N \quad \text{or} \quad R = P - N - \Delta$$

where $P$ and $N$ are effective integral divisors with no common components, and $\Delta$ is an effective $\mathbb{Q}$-divisor with all its coefficients between 0 and 1 and whose support is a divisor in normal crossing. We observe that

$$K_Y + \Delta \equiv f^*(K_X + G) + P - N.$$

We note that $P$ is $f$–exceptional. By a well known lemma of Fujita, we know that $f_*(\mathcal{O}_P(P)) = 0$ ([KMM]). Then

$$f_*(\mathcal{O}_Y(P - N)) = f_*(\mathcal{O}_Y(-N)) \subset f_*(\mathcal{O}_Y) = \mathcal{O}_X.$$

Hence, $f_*(\mathcal{O}_Y(P - N))$ is an ideal sheaf. We call this the multiplier ideal of $G$. Suppose that $Z(G)$ is the scheme defined by this ideal. We will denote the multiplier ideal by $I_{Z(G)}$.

**Remarks. 1.3.**

(a) By some fairly standard methods, one can check that the ideal $I_{Z(G)}$ is independent of the choice of the resolution. Let $p$ be a point in $X$. We say that the ideal $I_{Z(G)}$ is trivial at $p$, if $p$ does not belong to the scheme $Z(G)$. This is equivalent to saying that $f^{-1}(p)$ does not intersect the divisor $N$.

(b) Observe that $F_j \subset N$ if and only if $a_j \leq -1$. This leads to the interpretation that $Z(G)$ puts a scheme structure on the locus where the pair $(X, G)$ is not log-terminal.

(c) If we do not assume that $G$ is an effective $\mathbb{Q}$-divisor in the construction of the multiplier ideal, then $P$ may not be $f$–exceptional. In this case, $f_*(\mathcal{O}_Y(P - N))$ will give a fractional ideal. In a similar manner, we say that multiplier fractional ideal is nontrivial at $p$, if $\mathcal{O}_{X, p}$ is not contained in the localization of this fractional ideal at $p$. Equivalently, this means that $f^{-1}(p)$ has an nonempty intersection with the divisor $N$. 
In this fashion, we construct fractional ideals, which enjoy many similar cohomological properties as the multiplier ideals.

(d) One can in fact carry out the same construction when $X$ is normal and $K_X + G$ is a $\mathbb{Q}$-Cartier divisor.

The next Proposition follows easily from the Kawamata-Viehweg-Nadel vanishing theorem or Theorem 1.1 and 1.2.

**Proposition 1.4.** (a) $R^i f_*(\mathcal{O}_Y(P - N)) = 0$ for $i > 0$.

(b) Assume that $X$ is complete and $A$ is a Cartier divisor on $X$ such that $A - (K_X + G)$ is nef and big. Then $H^i(\mathcal{O}_X(A) \otimes I_{Z(G)}) = 0$ for $i > 0$.

**Proof.** (a) Observe that

$$P - N = K_Y + f - (\text{nef divisor}) + \Delta,$$

where $\Delta$ is a boundary divisor with normal crossing support. Now (a) follows from the Kawamata-Viehweg vanishing theorem [KMM].

(b) It follows from (a) and the Leray spectral sequence that

$$H^i(f^*(\mathcal{O}_Y(A + P - N))) = H^i(\mathcal{O}_X(A) \otimes I_{Z(G)}).$$

Now (b) follows by applying the Kawamata-Viehweg vanishing theorem on $Y$. \qed

Next, we would like to review the notion of adjoint ideals to study hypersurface singularities [EL3]. Its construction is very similar to the construction of multiplier ideals. Let $X$ be a smooth variety. Let $H$ be a reduced effective Cartier divisor in $X$. Let $f : Y \to X$ be an embedded resolution and $F$ be the proper transform of $H$ in $Y$. Then we can write

$$K_Y + F - f^*(K_X + H) = P - N,$$

where $P$ and $N$ are effective divisors with no common components. Note that every component of $P$ is $f$-exceptional. Then the adjoint ideal $J = J_H$ is defined to be

$$J = f_*(\mathcal{O}_Y(P - N)) = f_*(\mathcal{O}_Y(-N)) \subset f_*\mathcal{O}_Y = \mathcal{O}_X.$$

We observe that

$$\mathcal{O}_Y(P - N)|_F = K_F - f^*(K_H).$$
This shows that $J$ is trivial if and only if $H$ has only canonical singularities. Since $H$ is Gorenstein, $H$ has only canonical singularities if and only if it has only rational singularities (cf. [Kol3, 11.10]).

**Proposition 1.5.** Let $J$ be the adjoint ideal of $H$. There is the following exact sequence:

\begin{equation}
0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + H) \otimes J \to f_*\mathcal{O}_F(K_F) \to 0.
\end{equation}

Moreover, $J = \mathcal{O}_X$ if and only if $H$ is normal and it has only rational singularities.

**Proof.** There is the following exact sequence on $Y$:

\begin{equation}
0 \to \mathcal{O}_Y(K_Y) \to \mathcal{O}_Y(K_Y + F) \to \mathcal{O}_F(K_F) \to 0.
\end{equation}

We note that $f_*\mathcal{O}_Y(K_Y) = \mathcal{O}_X(K_X)$ and $R^1f_*\mathcal{O}_Y(K_Y) = 0$. Also $f_*\mathcal{O}_Y(K_Y + F) = \mathcal{O}_X(K_X + H) \otimes J$ by 1.5.1. We see that 1.5.2 follows from pushing forward the above exact sequence under $f$. Since the map $f : F \to H$ factors through the normalization of $H$, one sees that $f_*\omega_F = \omega_H$ only if $H$ is normal. We have already observed the last statement. □

**Remark 1.6.** (a) One may view from (1.5.2) that the adjoint ideal measures the failure of adjunction.

(b) The above construction can also be carried out when $X$ has only normal Gorenstein canonical singularities.

§2. Geometric properties of multiplier ideals.

In this section, we will investigate the properties of the multiplier ideals under simple geometric operations such as taking hyperplane sections, specialization, and finite ramified coverings.

Keeping notations as in §1, let $H$ be a smooth irreducible hypersurface in $X$. Let $\alpha$ be the coefficient of $H$ in $G$. We will assume that $0 \leq \alpha < 1$. By Remark 1.3(a), the construction of the multiplier ideal is independent of the choice of the resolution. Then we may assume that $f^*H$, the exceptional divisors of $f$, and $f^*(G)$ are all in normal crossing in $Y$. Let $F$ be the proper transform of $H$ in $Y$. We set $G' = G - \alpha H$ and $G'|_H = G'|_H$. We observe that $K_H + G'|_H$ is a $\mathbb{Q}$-Cartier divisor on $H$. Then we can consider the multiplier ideal determined by $G'|_H$ in $\mathcal{O}_H$. The following comparison of $I_{Z(G)}$ and $I_{Z(G'|_H)}$ is essentially due to Esnault and Viehweg ([EV1], Proposition 7.5).

**Proposition 2.1.** $I_{Z(G'|_H)} \subset \text{Im}(I_{Z(G)} \to \mathcal{O}_H)$. 

Proof. Using the notations as before, we write \( f^*(H) - F = \sum h_j F_j \). Let \( R' = K_Y - f^*(K_X) - \sum h_j F_j - G' \). We note that \( R'|_F = K_F - f^*(K_H - G'_H) \). Then
\[
f_*(O_F([R'|_F])) = I_{Z(G'_H)}. \]
Next we consider the divisor,
\[
R'' = R' - F = K_Y - f^*(K_X - G' - H). \]
Since \( \alpha < 1 \), we see that
\[
(2.1.1) [R'] \subset [R]. \]
Consider the exact sequence,
\[
0 \rightarrow O_Y([R'']) \rightarrow O_Y([R']) \rightarrow O_F([R'|_F]) \rightarrow 0.
\]
By Proposition 1.2, \( R_1 f_*(O_Y([R''])) = 0 \). Set \( J = f_*([R']) \). Then \( J \) maps onto \( I_{Z(G'_H)} \). By (2.1.1), \( J \) is a subsheaf of \( I_{Z(G)} \). This completes the proof of Proposition 2.1. □

Remarks 2.2.

(a) Proposition 2.1 implies that if \( I_{Z(G'_H)} \) is trivial at a point \( p \) in \( H \), then \( I_{Z(G)} \) is also trivial at \( p \).

(b) In Propostion 2.1, we can make the same conclusion when \( H \) is only assumed to be irreducible noraml and Cartier.

The following is a well known criterion to see whether the multiplier is nontrivial. See [Dem2] and [EV1].

Corollary 2.3. Let \( X \) be a smooth \( n \)-dimensional variety and \( G \) be an effective \( Q \)-divisor on \( X \). Let \( p \) be a point in \( X \).

(a) If \( Mult_p(G) \geq n \), then the multiplier ideal of \( G \) is nontrivial at \( p \).

(b) If \( Mult_p(G) < 1 \), then the multiplier ideal of \( G \) is trivial at \( p \).

Proof. (a) This follows from a simple calcuation by blowing up \( X \) at \( p \).

(b) This is clearly true when \( n = 1 \). For \( n > 1 \), if \( H \) is a general hyperplane section of \( X \) through \( p \), then the multiplier of ideal of \( G|_H \) is trivial by induction. Hence the multiplier ideal of \( G \) is also trivial at \( p \) by 2.2 (a).

Definition 2.4. Let \( p \) be a point in \( X \). We say that \( G \) is critical at \( p \), if
(a) $p$ is in $Z(G)$ and $p$ is not in $Z(\lambda G)$, for any $0 < \lambda < 1$.

(b) There is a unique irreducible component $F$ of $N$, such that $N = F + N_1$ where $N_1 \cap f^{-1}(p)$ is empty. Then $F$ is called the critical component of $G$ at $p$ and $f(F) = Z$ is called the critical variety of $G$ at $p$.

**Remark 2.5**

Suppose that $G$ is an effective $\mathbb{Q}$-divisor in satisfies 2.4 (a). In general, $G$ may not be critical at $p$. We would like to sketch a well known argument showing that we can perturb the divisor $G$ a little bit to obtain a new divisor $G'$ which is critical at $p$.

We'll assume that $G$ is ample and $f$ is a projective morphism. We can find an ample $\mathbb{Q}$-divisor in $Y$ of the following form:

$$A = f^*(\alpha G) - \sum \beta_j F_j,$$

where $0 < \beta_j << \alpha << 1$. By Bertini's theorem, we can find a sufficiently large and divisible integer $m$ with the following properties.

(a) There is a smooth irreducible divisor $H \equiv mA$ and $H + \sum F_j$ is a divisor of normal crossing.

(b) $m\beta_j$ are integers.

(c) $maG$ is an integral divisor.

Observe that $|H + m \sum \beta_j F_j| = |f^*(maG)|$. So $H + m \sum \beta_j F_j = f^*D$ for some effective divisor $D$ in $X$. Observe that $\frac{1}{m} D \equiv \alpha G$. Now consider the $\mathbb{Q}$-divisor $G_1 = G + \frac{1}{m} D$. By choosing $\beta_j$ sufficiently generic, we may assume there is a small positive rational number $\epsilon$, such that $G' = (1 - \epsilon)G_1$ is critical at $p$.

The following result has also been observed by Kawamata independently ([Ka2]).

**Proposition 2.6.** Suppose that $G$ is critical at $p$. Let $F_1$ be its critical component and $Z$ be its critical variety. Then the natural projection map

$$f : F_1 \longrightarrow Z,$$

has a trivial Stein factorization near $p$. Then $Z$ is normal at $p$. In particular, if $Z$ is a curve, then $Z$ is smooth at $p$. 
Proof. After replacing $X$ by an open neighborhood of $p$, we may assume that $[R] = P - F_1$. By a theorem of Fujita, we know that $f_*(\mathcal{O}_P(P)) = 0$ ([KMM]). This implies that $f_*(\mathcal{O}_Y) = f_*(\mathcal{O}_Y(P)) = \mathcal{O}_X$. By Proposition 1.2, $f_*(\mathcal{O}_Y(P))$ maps onto $f_*(\mathcal{O}_{F_1}(P))$. This implies that the natural map $\mathcal{O}_Y \rightarrow f_*(\mathcal{O}_{F_1})$ is onto. We conclude that $f_*(\mathcal{O}_{F_1}) = \mathcal{O}_Z$, and the Stein factorization of the map from $F_1$ to $Z$ is trivial. \(\square\)

It is an interesting problem to understand the singularity of $Z$ at $p$. It is well known that if $Z$ is of codimension 1, then $Z$ has only log-terminal singularities. More recently, Kawamata has shown that if $\text{codim}(Z) = 2$, then $Z$ again only has log-terminal singularity [Ka2]. One may ask that whether this is true in general.

The following result was first shown by Angehrn and Siu using analytic methods [AS]. See also [Kol4].

**Proposition 2.7.** Let $X$ be a smooth variety and let $T$ be a smooth curve. We consider the product variety $X \times T$. Let $G$ be an effective $\mathbb{Q}$-Cartier divisor on $X \times T$. We assume that the support of $G$ does not contain any fiber $X_t = X \times \{t\}$, for $t \in T$. Denote by $G_t$ the restriction of $G$ to $X_t$. Let

$$s : T \rightarrow X \times T$$

be a section. Assume that the multiplier ideal of $G_t$ is nontrivial at the point $s(t)$ for a general $t$. If $X_0 = X \times \{0\}$ is the special fiber, then the multiplier ideal of $G_0$ is also nontrivial at $s(0)$.

Proof. We construct an embedded resolution $f : Y \rightarrow X \times T$. By the theorem of generic smoothness, we see that for generic $t$ in $T$, $f^{-1}(X_t) \rightarrow X_t$ is also an embedded resolution. This implies that $Z(G) \cap X_t = Z(G_t)$ for generic $t$. If $s(0)$ is not in $Z(G_0)$, then $s(0)$ is not in $Z(G)$ by Proposition 2.1. This means that $s(t)$ is not in $Z(G)$ for generic $t$. This is a contradiction. \(\square\)

The following Proposition discusses the properties of a multiplier ideal, when the $\mathbb{Q}$-divisor is pulled back by a generic finite map. See also [Kol4].

**Proposition 2.8.** Let $X$ and $M$ be two smooth irreducible varieties. Let $G$ be an effective $\mathbb{Q}$-divisor on $X$. Let $\phi : M \rightarrow X$ be a proper generically finite map. Let $q$ be a point in $M$. Then the fractional multiplier ideal of the $\mathbb{Q}$-divisor $\phi^*G - K_{M/X}$ is nontrivial at $q$ if and only if the multiplier ideal of $G$ is nontrivial at $\phi(q)$.

Proof. We can construct embedded resolutions, $f : Y \rightarrow X$ and $g : W \rightarrow M$. We may suppose that $\phi$ extends to a proper generically finite map $\phi' : W \rightarrow Y$. Let $R = K_Y - f^*(K_X + G)$ and $R_1 = K_W - \phi'^*f^*(K_X + G)$. Then the multiplier ideal of
G is given by $f_* \mathcal{O}_Y([R])$. Similarly the fractional multiplier ideal for $\phi^* G - K_{M/X}$ is given by $g_* \mathcal{O}_W([R_1])$. Let $F$ be an irreducible component of $R$ with coefficient $a$ and let $E$ be an irreducible divisor in $W$ which maps onto $F$. Denote by $m$ the coefficient of $E$ in $K_W/Y$ and $b$ the coefficient of $E$ in $R_1$. Then

\[(2.7.1) \quad b = a(m + 1) - m.\]

We note that $a \leq -1$ if and only if $b \leq -1$. Conversely if $E$ is an irreducible component of $R_1$, after further blowing up of $Y$ and $W$, we may assume that $\phi'(E)$ is a divisor in $Y$. It follows that the multiplier ideal of $G$ is nontrivial at $\phi(q)$ if and only if the fractional multiplier ideal of $\phi^* G - K_{M/X}$ is nontrivial at $q$. □

**Example 2.9** (Demailly) Let $D$ be the divisor in $\mathbb{C}^n$ defined by the equation $x_1^{d_1} + x_2^{d_2} + \ldots + x_n^{d_n}$. Assume that $\frac{1}{d_1} + \frac{1}{d_2} + \ldots + \frac{1}{d_n} = \lambda < 1$. Then $\lambda D$ is critical at the origin. One can check this by considering the finite map $\phi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ given by $(x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n})$, where $m_i = \frac{d_1 d_2 \ldots d_n}{d_i}$. Then $\phi^* D$ is defined by the Fermat equation $x_1^d + x_2^d + \ldots + x_n^d$ where $d = d_1 \ldots d_n$. The relative canonical divisor of $\phi$ is defined by $x_1^{m_1-1} \ldots x_n^{m_n-1}$. After blowing up the origin, we obtain an embedded resolution of $\phi^* D$ and $K_\phi$. Now 2.7 follows from 2.6 by a simple calculation. See [Dem3] for an argument using the analytic techniques and see also [Kol4] for another argument using weighted blowing up.

Observe that if $d_i >> d_1$ for $i > 1$, then the multiplicity of $\lambda D$ at the origin is of the form of $1 + \epsilon$ and it is an isolated singularity. This shows that Proposition 1.5(b) is essentially the best possible result.

§3 Singularities of theta divisors

In this section, we'll apply the techniques from the first two sections to study singularities of theta divisors. The results in this section are my joint work with R. Lazarsfeld. For more details see [EL3]. In the following, we let $(A, \Theta)$ be a principally polarized abelian variety. First we note the following well known result.

**Proposition 3.1.** Let $Z$ be a nonempty proper closed subscheme of $A$. Then

$$H^0(\mathcal{O}_A(\Theta) \otimes P \otimes I_Z) = 0 \quad \text{for a general } P \in \text{Pic}^0 A.$$

**Proof.** We first note that $\mathcal{O}_A(\Theta) \otimes P$ is linearly equivalent to a translate of $\Theta$. Now the Proposition follows from $h^0(\mathcal{O}_A(\Theta) \otimes P) = 1$ and the fact that a general translate of the divisor $\Theta$ does not contain $Z$. □
The following is an extension of a theorem of Kollár to the pluritheta divisors, as proposed in [Kol2, Problem 17.15].

**Proposition 3.2.** Let \((A, \Theta)\) be a p.p.a.v., and for a positive integer \(m\) fix any divisor \(D \in |m\Theta|\). Then the pair \((A, \frac{1}{m}D)\) is log-canonical. In particular for each positive integer \(k\) the set,

\[
\Sigma_{mk}(D) = \{x \in D | \text{mult}_x(D) \geq mk\}
\]
is of codimension greater or equal to \(k\).

Proof. Suppose for contradiction that \((A, \frac{1}{m}D)\) is not log-canonical. Then for sufficiently small positive \(\epsilon\), the multiplier ideal \(I_Z\) for the divisor \(G = (1 - \epsilon)\frac{1}{m}D\) is nontrivial. By Proposition 1.4,

\[
H^i(O_A(\Theta) \otimes P \otimes I_Z) = 0 \quad \text{for all } i > 0 \quad \text{and } P \in \text{Pic}^0(A).
\]

It follows from Proposition 3.1 that \(\chi(O_A(\Theta) \otimes P \otimes I_Z) = 0\) for general \(P\). This implies that \(\chi(O_A(\Theta) \otimes P \otimes I_Z) = 0\) for all \(P\). We conclude that

\[
H^i(O_A(\Theta) \otimes P \otimes I_Z) = 0 \quad \text{for all } i \geq 0 \quad \text{and for all } P \in \text{Pic}^0(A).
\]

It follows from Mukai’s theory [Muk] of the Fourier functor that \(O_A(\Theta) \otimes I_Z = 0\). Thus we have a contradiction. \(\square\)

**Theorem 3.3.** Assume that \(\Theta\) is irreducible. Then the adjoint ideal of \(\Theta\) is trivial. In particular, \(\Theta\) is normal and has only rational singularities.

**Proof.** Let \(f : Y \rightarrow A\) be an embedded resolution. Let \(W\) be the proper transform of \(\Theta\) in \(Y\). Let \(J\) be the adjoint ideal of \(\Theta\). It follows from 1.5.2 that we have the following exact sequence:

\[
0 \rightarrow P \rightarrow O_A(\Theta) \otimes P \otimes J \rightarrow f_*\omega_W \otimes P \rightarrow 0
\]

for all \(P \in \text{Pic}^0(A)\). By the generic vanishing theorem of Green and Lazarsfeld [GL],

\[
H^i(f_*(\omega_W \otimes P)) = 0 \quad \text{for } i > 0 \quad \text{and for generic } P \in \text{Pic}^0(A).
\]

If \(J\) is nontrivial, then by Proposition 3.1, \(h^0(O_A(\Theta) \otimes P \otimes J) = 0\) for generic \(P\). We conclude that \(H^0(f_*(\omega_W \otimes P)) = 0\) for generic \(P\). This implies that \(\chi(f_*(\omega_W \otimes P)) = 0\) for general \(P\). This shows that \(\chi(f_*(\omega_W)) = 0\). Since \(\Theta\) is an irreducible ample divisor, it is well known that \(\Theta\) is of general type. It follows from a theorem of Kawamata and Viehweg that \(\chi(f_*(\omega_W)) > 0\) [KV]. This gives a contradiction. \(\square\)
Remark 3.4. More generally, Lazarsefeld and I ([EL3]) have shown the following. If \( X \) is an irreducible closed subvariety of an abelian variety. Let \( Y \) be a resolution of singularities of \( X \). Then \( X \) is of general type if and only if \( \chi(\omega_Y) > 0 \). This result was conjectured by Kollár.

§4 Adjoint linear systems

Let \( X \) be a smooth \( n \)-dimensional complex projective variety and \( L \) be an ample line bundle on \( X \). In classical projective geometry, the adjoint linear system \( |K_X + L| \) and the pluricanonical systems \( |mK_X| \) play very important roles in studying the properties of curves and surfaces. One would expect that they would be important in studying higher dimensional varieties as well. The general philosophy is that if \( L \) is "sufficiently" ample, then the adjoint linear system \( |K_X + L| \) should be free or very ample. More precisely, we expect that if \( |K_X + L| \) is not free or very ample, then \( X \) has a subvariety of low degree with respect to \( L \). For instance, we have the following conjectures of Fujita.

(1) \( |K_X + (n + 1)L| \) is free.

(2) \( |K_X + (n + 2)L| \) is very ample.

(3) If \( X \) is a smooth minimal \( n \)-fold of general type, then \( |(n + 2)K_X| \) is free and \( |(n + 3)K_X| \) is very ample on the canonical model of \( X \).

One may conjecture the following refinement of (1).

(4) Suppose that \( L^n > n^2 \). Also assume that for every irreducible subvariety \( Z \) of \( X \),

\[
L^{\dim Z} \cdot Z \geq (n)^{\dim Z}.
\]

Then the linear system \( |K_X + L| \) is free.

In the last few years, these questions have generated a great deal of work. In this section, we'll describe some of the recent results on these problems. When \( X \) is a surface, these conjectures hold by the results of Bombieri and Reider ([B] and [R]). In higher dimension, the first breakthrough is due to Demailly [Dem1]. More specifically, he shows that \( |2K_X + 12n^nL| \) is very ample using some very powerful analytic techniques. Using cohomological methods developed by Kawamata and Shokurov, the author and Lazarsfeld [EL1] proved that the freeness part of Fujita conjecture is true for threefolds. In a very recent work, Kawamata [Ka1] has proved a similar result for fourfolds. Recently, Lee has shown that if \( X \) is minimal Gorenstein threefold of general type, then \( |5K_X| \) gives a birational morphism [Lee]. Ein and Lazarsfeld and independently Helmke have shown that (4) holds when \( X \) is a threefold [H]. See [Lee] for more
precise results for threefolds. In general for a smooth $n-$dimensional projective variety $X$, the work of Angehrn and Siu shows that if $L^{\dim Z} \cdot Z > \binom{n+1}{2}^{\dim Z}$ for every subvariety $Z$ in $X$, then $|K_X + L|$ is free. Furthermore, Kollár has extended the theorem to singular varieties with only log-terminal singularities. See [Kol4] for more details.

In this section, we would describe how we can use multiplier ideals to solve some of these problems on adjoint linear systems. This approach is inspired by the work of Anhern and Siu. The strategy is fairly simple. Suppose $p$ is a given point in $X$. We would like to construct an effective $\mathbb{Q}$-divisor $G$ which is linearly equivalent to $\lambda L$ where $\lambda < 1$, such that the multiplier scheme $Z(G)$ defined by $G$ is zero dimensional at $p$. Then the vanishing theorem for multiplier ideals will imply that the restriction map,

$$H^0(O_X(K_X + L)) \to H^0(O_X(K_X + L)|Z)$$

is surjective. This in turn implies that we can find a section of $O_X(K_X + L)$ that does not vanish at $p$. In order to start, by the Riemann-Roch theorem, if $L^n > n^n$, then we can construct an effective $\mathbb{Q}$-divisor $G$ such that $G$ is equivalent to $\lambda_1 L$ with $\lambda_1 < 1$ and $\text{Mult}_p(G) \geq n$. Then the multiplier ideal of $G$ is nontrivial at $p$. After replacing $G$ by a smaller multiple of $G$ and adding a small perturbation term as in 2.5, we may assume that $G$ is critical at $p$. The difficulty is that in general we do not know that the critical variety of $G$ at $p$ is zero-dimensional. The crucial new idea of Angehrn and Siu is to construct a new $\mathbb{Q}$-divisor $G'$ of the following form,

$$G' = (1 - \epsilon)G + D',$$

such that $G'$ is equivalent to $\lambda' L$, where $\lambda' < 1$. Furthermore, $G'$ is critical at $p$ and the critical variety of $G'$ is a proper subset of the critical variety of $G$. After repeating this process a finite number of times, we would then be able to construct an effective $\mathbb{Q}$-divisor with the property that its multiplier scheme is of zero-dimensional. In the rest of this section, we’ll describe a bit more of the technical details in constructing the $\mathbb{Q}$-divisor $G'$ as above.

First we’ll introduce an invariant $def_p(G)$, the deficit of $G$. This number roughly speaking is a measure on the difficulty in constructing the new $\mathbb{Q}$-divisor $G'$ as above with a 0-dimensional multiplier scheme at $p$. We consider the following setup. Let $X$ be a smooth variety and $p$ be a point in $X$. Let $G$ be an effective $\mathbb{Q}$-divisor in $X$. First we will assume that the multiplier ideal of $G$ is trivial at $p$. Let $\pi : X' \to X$ be the blowup of $X$ at $p$ and $E \subset X'$ be the exceptional divisor. We construct an embedded resolution for $G$, $f : Y \to X$, which factors through $\pi$. We write the factorization as $f = \pi \circ g$. Suppose that $f^*(G) = \sum g_j F_j$, $K_{Y/X} = \sum b_j F_j$, $K_Y - f^*(K_X + G) = \sum a_j F_j$ and $g^*(E) = \sum e_j F_j$. Now we define the deficit of $G$ as

$$def_p(G) = \inf_{f(F_j) = p} \{ \frac{a_j + 1}{e_j} \}.$$
One checks easily that $\text{def}_p(G) \leq c$ if and only for every effective $\mathbb{Q}$-divisor $D$ where $\text{Mult}_p D \geq c$, we have $p \in Z(G + D)$. In particular, the definition for the deficit is independent of the choice of the embedded resolution. Next we consider the case that when $G$ has a nontrivial multiplier ideal at $p$, but the multiplier ideal of $(1 - t)G$ is trivial at $p$ for any $t > 0$. In this case, we define the deficit of $G$ at $p$ as:

$$\text{def}_p(G) = \lim_{t \to 0^+} \text{def}_p((1 - t)G).$$

For $t > 0$, one notes that $\text{def}_p((1 - t)G) - t \text{Max}(\frac{g_j}{e_j}) \leq \text{def}_p(G) \leq \text{def}_p((1 - t)G)$.

The following Proposition follows fairly immediately from the definition.

**Proposition 4.1.** (a) $\text{def}_p(G) \leq \text{dim}(X) - \text{mult}_p(G)$.

(b) If $G$ is critical at $p$ and if the critical variety is zero-dimensional at $p$, then $\text{def}_p(G) = 0$.

In the following, we will assume that $G$ is critical at $p$. We may achieve this by perturbing the divisor $G$ a bit as in 2.5. Alternatively, one may work with reducible critical varieties as in [Ka1]. We will investigate the behaviour of the deficit function when we restrict $G$ to a general hyperplane section $H$ through the given point $p$. Let $\Lambda$ be the proper transform of $H$ in $Y$. Then $f^*(H) = \Lambda + g^*(E)$. By adjunction formula,

$$K_{\Lambda/H} = (K_{Y/X} - g^*(E))|_\Lambda.$$

Let $\phi = f|_\Lambda$. By Bertini’s theorem, we may assume that $f^*(G)$ intersects $\Lambda$ transversely. If we write

$$[K_{Y/X} - g^*(E) - f^*(G)] = P_1 - N_1,$$

where $P_1$ and $N_1$ are effective divisors with no common components, then $\phi_* (\mathcal{O}_\Lambda((P_1 - N_1)|_\Lambda)$ is the multiplier ideal of $G|_H$. By Proposition 2.1, we know that the multiplier ideal of of $G|_H$ is nontrivial.

**Proposition 4.2** (a) If $\text{def}_p(G) \geq 1$, then $\text{def}_p(G|_H) = \text{def}_p(G) - 1$ and $G|_H$ is critical at $p$.

(b) Suppose that $Z$ is the critical variety of $G$ at $p$. Then $\text{def}_p(G) \leq \text{dim}(Z)$

**Proof.** (a) One can check this using the formula 4.2.1.

(b) If $\text{dim}(Z) = 0$, then $\text{def}_p(G) = 0$ by 4.1. In general, we induct on the dimension of $Z$. Now (b) follows from (a) by considering $G|_H$ and induction. □
The following result of Helmke shows that we can use the deficit to control the multiplicity of the critical variety \([H]\). Suppose that \(X\) is a smooth \(n\)-dimensional variety and let \(p\) be a given point on \(X\). Assume that \(G\) is an \(\mathbb{Q}\)-effective divisor on \(X\) and \(G\) is critical at \(p\). Let \(F \subset Y\) be the critical component of \(G\) and let \(Z\) be its critical variety. Suppose that the embedded dimension of \(Z\) at \(p\) is \(n_1\) and \(\dim(Z) = r > 0\). Let \(d = def_p(G)\). Set \(k = [r - d]\).

**Theorem 4.3.** (Helmke) \(\text{Mult}_p(Z) \leq \binom{n_1 - r + k}{k}\).

**Corollary 4.4** Suppose that the critical variety \(Z\) is a surface and the embedded dimension of \(Z\) at \(p\) is \(n_1\). Then \(\text{Mult}_p(Z) \leq n_1 - 1\).

**Proof.** If \(def_p(G) > 1\), then \(Z\) is smooth by 2.6 and 4.2. If we assume that \(def_p(G) \leq 1\), then \(\text{Mult}_p(Z) \leq n_1 - 1\) by 4.3. \(\square\)

Now we continue our discussion of constructing the divisor \(G\) with a zero dimensional multiplier scheme. We consider the following definition motivated by the work of Angehrn and Siu [AS].

**Definition 4.5.** Let \(B\) be an effective \(\mathbb{Q}\)-Cartier divisor on the critical variety \(Z\). An effective \(\mathbb{Q}\)-Cartier divisor \(D\) on \(X\) is said to be a nice lifting of \(B\), if \(D\) satisfies the following two properties:

(a) \(D|_Z = B\).

(b) \(Z((G + D)|_{X - Z}) = Z(G|_{X - Z})\).

Let \(B\) be an effective \(\mathbb{Q}\)-Cartier divisor on the critical variety \(Z\). Suppose that \(D'\) and \(D''\) are two nice liftings of \(B\). Let \(f : Y \to X\) be an embedded resolution for \(D'\) and \(D''\). Let \(f^*D' = \sum d'_j F_j\), \(f^*D'' = \sum d''_j F_j\), \(f^*G = \sum g_j F_j\), and \(R = K_Y - f^*(K_X + G) = \sum a_j F_j\). After replacing \(X\) by an open neighborhood of \(p\), we may assume that

\[ [R] = P - F_1 \]

where \(P\) consists of \(f\)-exceptional divisors. Then \(f(F_1) = Z\) is a critical variety of \(G\). In applying this result, \(G\) usually would be an ample divisor. As in 2.5, we can perturb \(G\) a little bit. So we can assume the following addition simplifying assumption that

\[ \{a_j\} > 0 \quad \text{for all} \quad j \neq 1. \]

The following basic lemma will allow us to prove that certain multiplier ideals are
nontrivial. The idea is that the picture is controlled by the restriction of the divisor to the critical variety.

**Lemma 4.6.** Keep assumptions as above.

(a) Suppose that \(p\) is in the multiplier scheme \(Z((1 - s)D' + (1 - t)G)\) for all sufficiently small positive \(s\) and \(t\). Then \(p\) is also in \(Z((1 - s')D'' + (1 - t')G)\) for all sufficiently small positive \(s'\) and \(t'\). Furthermore the multiplier scheme \(Z((1 - s)D'' + (1 - t')G)\) is a proper closed subset of the critical variety \(Z\) of \(G\).

(b) Suppose that \(p\) is a smooth point of \(Z\) and

\[
\text{Mult}_p(B) > \text{def}_p(G).
\]

Then for any nice lifting \(D''\) of \(B\), the multiplier scheme for \(D'' + (1 - t)G\) is nottrivial at the point \(p\) for all sufficiently small positive \(t\).

(c) Let \(G'\) be a \(Q\)-effective divisor of the form \(G' = (1 - t)G + D''\) where \(t\) is a sufficiently small positive number. Assume that the multiplier ideal of \(G'\) is nontrivial at \(p\), but the multiplier ideal \((1 - s)G'\) is trivial for all positive \(s\). Then \(\text{def}_p(G') \leq \text{def}_p((1 - t)G) - \text{Mult}_p(B)\).

One should think of this lemma as saying that we can use the divisor \(D'\) to study the multiplier ideal constructed from \(D''\) provided only that \(D'\) and \(D''\) are nice liftings of the same divisor. For local questions, this allows us to pretend we are working with a particularly nice lifting. This Lemma is an algebraic substitute for the computation appearing in [AS] §4.

**Proof of 4.6.** Replacing \(D'\) by \((1 - s)D'\) for sufficiently small and generic \(s\), by 4.3.1 we may assume that

\[
\{a_j - d_j\} > 0 \quad \text{for} \quad j \neq 1.
\]

This means that for sufficiently small \(t\) and \(j \neq 1\),

\[
[a_j - d_j + tg_j] = [a_j - d_j].
\]

This implies that

\[
[K_Y - f^*(K_X + G + D')] = P' - N' - F_1,
\]

and

\[
[K_Y - f^*(K_X + D' + (1 - t)G')] = P' - N'.
\]
Consider the following exact sequence on $Y$:

$$0 \rightarrow \mathcal{O}_Y(P' - N' - F_1) \rightarrow \mathcal{O}_Y(P' - N') \rightarrow \mathcal{O}_{F_1}(P' - N') \rightarrow 0.$$ 

By the vanishing theorem, $R^1 f_* (\mathcal{O}_Y(P' - N' - F_1)) = 0$. Let $I_{Z'} = f_* (\mathcal{O}_Y(P' - N'))$ be the multiplier ideal of $D' + (1 - t)G$. By our assumption this is a nontrivial ideal at $p$. It follows that its image in $\mathcal{O}_Z$, which is equal to $f_* (\mathcal{O}_{F_1}(P' - N'))$, is a proper ideal subsheaf at $p$. In the similar fashion, we write

$$[K_Y - f^*(K_X + G + D'')] = P'' - N'' - F_1,$$

and

$$[K_Y - f^*(K_X + D'' + (1 - t')G)] = P'' - N''.$$ 

Let $I_{Z''} = f_* (\mathcal{O}_Y(P'' - N''))$ be the multiplier ideal of $D'' + (1 - t')G$. Since $D'$ and $D''$ both are liftings of $B$, we observe that $(P' - N')|_{F_1} = (P'' - N'')|_{F_1}$. We conclude that the image of the restriction map from $I_{Z''}$ to $\mathcal{O}_Z$ is equal to $f_* (\mathcal{O}_{F_1}(P' - N'))$ which is a proper ideal subsheaf of $\mathcal{O}_Z$ at $p$. We conclude that $I_{Z''}$ is nontrivial at $p$. The fact that $Z''$ is a proper closed subscheme of $Z$ follows from the fact $D'$ is a nice lifting.

(b) This is a local question. We will assume that $X$ is affine. Let $I_{p/X}$ and $I_{p/Z}$ be the maximal ideals corresponding to the point $p$ in $X$ and $Z$ respectively. Note the restriction map $I^k_{p/X} \rightarrow I^k_{p/Z}$ is surjective for each nonnegative integer $k$. By Bertini’s theorem, there is a nice lifting $D'$ of $B$ such that $\text{Mult}_p(D') = \text{Mult}_p(B)$. Then

$$\text{Mult}_p((1 - t)D') > \text{def}_p(G)$$

for all sufficiently small $t$. From the definition of deficit, we see that $p \in Z((1 - t)D' + (1 - s)G)$ for all sufficiently small $s$ and $t$. Now (b) follows from (a). (c) follows from the proof of (a) and (b). □

We’ll illustrate how we can apply these techniques to study adjoint linear systems by giving a quick sketch for the proof of the theorem of Angehrn and Siu [AS]. See also [Kol4].

**Theorem 4.7.** Let $X$ be a smooth $n$-dimensional complex projective variety and $L$ be an ample divisor on $X$. Suppose that for every subvariety $Z$ in $X$,

$$L^{\dim Z} : Z > \left( \frac{\dim X + 1}{2} \right)^{\dim Z}.$$ 

Then $|K_X + L|$ is free.
First we note the following lemma.

**Lemma 4.8** Let $X$ be a smooth complex projective variety and $p$ be a given point in $X$. Suppose that $G$ is an effective $\mathbb{Q}$-divisor on $X$ such that $G$ is critical at $p$. Let $Z$ be the critical variety of $G$ at $p$ and $\text{dim}(Z) = r$. Suppose that $A$ is an ample $\mathbb{Q}$-divisor in $X$. We assume that

$$A^r : Z > r^r.$$ 

Then there is an effective $\mathbb{Q}$-divisor $D$ on $X$ such that $D$ is equivalent to $A$ and $p \in Z((1 - t)G + D)$ for all sufficiently small positive $t$. Furthermore, locally near $p$, $Z((1 - t)G + D)$ is a proper subset $Z$.

**Proof.** Let $q$ be a general point of $Z$. Then by the Riemann-Roch theorem, we can find a $\mathbb{Q}$-Cartier divisor $B_q$ on $Z$ such that $B_q$ is equivalent to $A|_Z$ and

$$\text{Mult}_q(B_q) > r \geq \text{def}_q(G).$$

If $D_q$ is nice lifting of $B_q$ which is equivalent to $A$, then $q \in Z((1 - t)G + D_q)$ by 4.6. As we specialize $q$ to $p$, we obtain the desired $\mathbb{Q}$-divisor by Proposition 2.7. □

**Proof of 4.7.** The idea of the proof is the following. We will construct a sequence of $\mathbb{Q}$-Cartier divisors $G_i \equiv \lambda_i L$ where $\lambda_1 < \lambda_2 < ... < 1$ satisfying the following properties.

(a) $G_i$ is critical at $p$ with critical variety $Z_i$.

(b) If $\text{dim}(Z_i) > 0$, then $Z_{i+1}$ is a proper subvariety of $Z_i$.

(c) $G_i \equiv \lambda_i L$ and $\lambda_i < \frac{(n) + (n-1) + ... + (n-i+1)}{m}$, where $m = \binom{n+1}{2}$.

In $k$ steps ($k \leq n$), we will reach the case where $Z_k$ is zero dimensional at $p$.

By the Riemann-Roch theorem, we can find an effective $\mathbb{Q}$-Cartier divisor $G_1$ such that $G_1$ is critical at $p$ with critical variety $Z_1$, and $G_1$ is equivalent to $\lambda_1 L$, where

$$\lambda_1 < \frac{n}{m}.$$ 

If $\text{dim}(Z_1) = 0$, then we are done. If $\text{dim}(Z_1) = d_1 > 0$, we continue our construction. Suppose inductively we have constructed $\mathbb{Q}$-Cartier divisor $G_i$ with $G_i$ critical at $p$ with critical subvariety $Z_i$ of dimension $d_i$, where

$$d_i \leq n - i.$$ 

The divisor $G_i$ is equivalent to $\lambda_i L$, where

$$\lambda_i < \frac{(n) + (n-1) + ... + (n-i+1)}{m}.$$
If $d_i = 0$, then we are done. Now assume that $d_i > 0$. Observe that $(\frac{d_i}{m}L)^{d_i} Z_i > d_i^{d_i}$. Using Lemma 4.8 and adding a small perturbation term, we find $G_{i+1}$ which is critical at $p$, with critical subvariety $Z_{i+1}$, which is a proper subvariety of $Z_i$. Furthermore we can choose $G_{i+1}$ which is equivalent to $\lambda_{i+1} L$ where $\lambda_{i+1} = \lambda_i + \beta_i$ with $\beta_i < \frac{d_i}{m} \leq \frac{n-i}{m}$.

It follows that

$$\dim(Z_{i+1}) \leq n - i - 1.$$ 

Also,

$$\lambda_{i+1} < \frac{(n) + (n-1) + \ldots + (n-i)}{m}.$$ 

Now we can complete the proof by induction. \(\square\)

**Remark 4.9.** Using Corollary 4.4 and an argument as above, one can show fairly easily that the freeness part of the Fujita conjecture is true for threefolds and fourfolds [EL1] and [Ka1].

§5. **Additional applications**

In this section, we’ll give two further applications. First, we’ll give a simple proof of a result of M. Levine on the invariance of plurigenra under deformations. The idea of this proof is due to Siu. The second application is to give a simple proof for a result of Esnault and Viehweg on the relation between singular hypersurfaces and postulation of a finite set in a complex projective space ([EV2]). We’ll need the following Lemma.

**Lemma 5.1** Let $L$ be a big line bundle on a smooth complex projective variety $X$. Suppose that there is an effective $\mathbb{Q}$-divisor $G$ on $X$ which is linearly equivalent to $L$, such that the multiplier ideal of $G$ is trivial. Then $H^i(X, \mathcal{O}_X(K_X + L)) = 0$ for $i > 0$.

**Proof.** Let $m$ be a sufficiently large and divisible positive integer, such that $mG$ is an integral Cartier-divisor. Let $f : Y \to X$ be an embedded resolution for the linear system $|mL|$ and $G$. We can write $f^*(mL) \sim A + B$, such that $A$ is free and big and $B$ is an effective divisor in normal crossing which is the fixed component of $|f^*(mL)|$. Since $mf^*G \in |f^*(mL)|$, $mf^*G - B$ is effective. By our assumption, we can write

$$K_{Y/X} - f^*(G) = \sum a_j F_j,$$

where $F_j$’s are distinct irreducible divisor in normal crossing and $a_j > -1$. Then $f^*L \sim \frac{A}{m} + \frac{B}{m}$ and

$$K_{Y/X} - \frac{A}{m} - \frac{B}{m} = \sum c_j F_j,$$

where $c_j > -1$. Then there is an effective $f$-exceptional divisor $P_1$, such that

$$f^*(K_X + L) + P_1 \sim K_Y + \Delta_1,$$
where $\Delta_1$ is a boundary divisor with normal crossing support. By Bertini’s theorem, we may assume that $\text{supp}A \subset \text{supp}\Delta_1$. By Proposition 1.2, this implies that

$$H^i(O_Y(f^*(K_X + L) + P_1)) = 0 \text{ for } i > 0.$$ 

Now since $f_*(f^*(K_X + L) + P_1) = K_X + L$ and $R^i f_*(f^*(K_X + L) + P) = 0$ for $i > 0$. This implies that $H^i(O_X(K_X + L)) = 0$ for $i > 0$. □

**Proposition 5.2** (Levine) Let $f : X \rightarrow T$ be a smooth projective morphism with connected fibers and $X_0 = f^{-1}(t_0)$ be a closed fiber of $f$. Suppose that $X_0$ is of general type. Let $m \geq 2$ be a positive integer. We suppose that there is a divisor $D \in |mK_{X_0}|$ and that the multiplier ideal for the $\mathbb{Q}$-divisor $G = \frac{m-1}{m}D \sim (m-1)K_{X_0}$ is trivial. Then $H^i(O_{X_0}(aK_{X_0})) = 0$ for $i > 0$ and $2 \leq a \leq m$. In particular, the plurigenera $h^0(O_{X_0}(aK_{X_0}))$ are locally constant for all $t$ in a neighborhood of $t_0$.

**Proof.** Observe that the multiplier ideal of $\lambda G$ is trivial for $\lambda < 1$. We note that $(a - 1)K_{X_0} \sim \frac{a-1}{m-1}G$. Now the Proposition follows from 5.1 and the semicontinuity theorem. □

**Remark 5.3** (a) Levine actually showed that the plurigenera are constant without the additional assumption that $X_0$ is of general type. The observation that one can use multiplier ideals to give a simple proof of this result, when $X_0$ is of general type, is due to Siu.

(b) The assumption in 5.2 is equivalent to saying that $G$ is log-terminal.

The final application is a theorem of Esnault and Viehweg [EV2] on the zeros of the polynomials. Let $S \subset \mathbb{P}^n$ be a finite subset. Suppose that there is a hypersurface $D$ of degree $d$ in $\mathbb{P}^n$, such that $\text{Mult}_p D \geq k$ for each $p \in S$.

**Proposition 5.4.** There is a hypersurface of degree $\lceil \frac{nd}{k} \rceil$ that contains $S$.

Proof. Let $G = \frac{d}{k}D$. Then $\text{Mult}_p G \geq n$ for each point $p \in S$. Let $I_Z$ be the multiplier ideal of $G$. Then $S \subset Z$. Let $H$ be the hyperplane class of $\mathbb{P}^n$. Let $m = \lceil \frac{nd}{k} \rceil$. Then $(m+1)H - G$ is ample. Since $K_{\mathbb{P}^n} \sim (-n-1)H$, we conclude by the vanishing theorem that $H^i(I_Z \otimes O_{\mathbb{P}^n}(t)) = 0$ for $t \geq m-n$ and $i > 0$. Since the Hilbert polynomial of $I_Z$ is a polynomial in $t$ of degree less than or equal to $n$, we can find an integer $t_0$ where $m-n \leq t_0 \leq m$, such that $h^0(I_Z \otimes O_{\mathbb{P}^n}(t_0))$ is nonzero. Since $t_0 \leq m$ and $S \subset Z$, this implies the proposition. □

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