Rotationally invariant noncommutative phase space of canonical type with recovered weak equivalence principle

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received 21 July 2018; accepted 3 September 2018
published online 27 September 2018

PACS 03.65.-w – Quantum mechanics
PACS 11.10.Nx – Noncommutative field theory

Abstract – We study the influence of noncommutativity of coordinates and noncommutativity of momenta on the motion of a particle (macroscopic body) in uniform and nonuniform gravitational fields in noncommutative phase space of canonical type with preserved rotational symmetry. It is shown that because of noncommutativity the motion of a particle in a gravitational field is determined by its mass. The trajectory of motion of a particle in a uniform gravitational field corresponds to the trajectory of a harmonic oscillator with frequency determined by the value of the parameter of momentum noncommutativity and mass of the particle. The equations of motion of a macroscopic body in a gravitational field depend on its mass and composition. From this follows a violation of the weak equivalence principle caused by noncommutativity. We conclude that the weak equivalence principle is recovered in rotationally invariant noncommutative phase space if we consider the tensors of noncommutativity to be dependent on mass. So, finally we construct noncommutative algebra which is rotationally invariant, equivalent to noncommutative algebra of canonical type, and does not lead to violation of the weak equivalence principle.

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Introduction. – Much attention has been devoted recently to studies of features of space structure at the Planck scale. To describe space quantization the idea of noncommutativity of coordinates was considered. In noncommutative phase space the following relations are satisfied:

\[ [X_i, X_j] = i\hbar \theta_{ij}, \]
\[ [X_i, P_j] = i\hbar \left( \delta_{ij} + \sum_k \frac{\theta_{jk} \eta_{ik}}{4} \right), \]
\[ [P_i, P_j] = i\hbar \eta_{ij}. \]

In the canonical version of noncommutative phase space \( \theta_{ij}, \eta_{ij} \) are elements of constant matrices.

Different problems were examined in noncommutative space, among them free particles [1–4], classical systems with various potentials [5–9], the Landau problem [10–15], many-particle systems [1,16–20], gravitational quantum wells [21,22], and many others.

Noncommutative algebra (1)–(3) with \( \theta_{ij}, \eta_{ij} \) being elements of constant matrices is not rotationally invariant. In noncommutative space of canonical type there is a problem of rotational symmetry breaking. Therefore, different types of noncommutative algebras were considered to preserve the rotational symmetry (see, for instance, [9,23–25]). Among them rotationally invariant noncommutative algebras with position-dependent noncommutativity (see, for instance [26–32], and reference therein), with involving spin degrees of freedom (see, for instance [33–36], and reference therein) have been widely studied.

To preserve the rotational symmetry and construct rotationally invariant noncommutative algebra equivalent to noncommutative algebra of canonical type the generalization of parameters of noncommutativity to the tensors was considered [37]. The tensors were supposed to be constructed with the help of additional coordinates and momenta

\[ \theta_{ij} = \frac{c_\theta l_P^2}{\hbar} \sum_k \varepsilon_{ijk} \tilde{a}_k, \quad \eta_{ij} = \frac{c_\eta l_P^2}{\hbar} \sum_k \varepsilon_{ijk} \tilde{p}_k. \]

Here \( l_P \) is the Planck length, \( c_\theta, c_\eta \) are dimensionless constants, \( \tilde{a}_i, \tilde{b}_i, \tilde{p}^a_i, \tilde{p}^b_i \) are additional dimensionless coordinates and momenta conjugate to them which are
governed by harmonic oscillators \( H_{osc}^{\pm} = \hbar \omega_{osc}((\vec{p}^2)^{\pm} + \vec{a}^2)/2 \), \( H_{osc}^0 = \hbar \omega_{osc}((\vec{p}^2)^0 + \vec{b}^2)/2 \), with \( \hbar/m_{osc}\omega_{osc} = \ell_P \) and very large frequency \( \omega_{osc} \) [37].

Additional coordinates and additional momenta satisfy 
\[ [\bar{a}_i, \bar{a}_j] = [\bar{b}_i, \bar{b}_j] = [\bar{a}_i, \bar{b}_j] = [\bar{b}_i, \bar{a}_j] = \bar{p}_i \bar{p}_j = \bar{p}_j \bar{p}_i = 0, \quad [\bar{a}_i, \bar{p}_j] = [\bar{b}_i, \bar{p}_j] = i \delta_{ij}, \quad [\bar{a}_i, \bar{p}_j] = [\bar{b}_i, \bar{p}_j] = 0. \]

Also, \( [\bar{a}_i, X_n] = [\bar{b}_i, P_j] = [\bar{p}_i, X_n] = [\bar{p}_i, P_j] = 0 \). Therefore, \( [\bar{a}_i, X_n] = [\theta_{ij}, P_k] = [\bar{a}_i, X_n] = [\eta_{ij}, P_k] = 0 \) as in the case of canonical noncommutativity \( (\theta_{ij}, \eta_{ij}) \) being constants). In this sense rotationally invariant algebra
\[
[X_i, X_j] = i\hbar \Gamma_{ij}^{mk} \sum_k \varepsilon_{ijk} \bar{a}_k, \tag{5}
\]
\[
[X_i, P_j] = i\hbar \left( \delta_{ij} + \frac{c^2 \varepsilon_{ij}}{4}(\bar{a} \cdot \bar{P}) \delta_{ij} - \frac{c^2 \varepsilon_{ij}}{4} \bar{a}_j \bar{P}_i \right), \tag{6}
\]
\[
[P_i, P_j] = \frac{c^2 \hbar^2}{4} \sum_k \varepsilon_{ijk} \bar{P}_k. \tag{7}
\]
is equivalent to noncommutative algebra of canonical type \( (1)–(3) \) [37].

The influence of noncommutativity on the implementation of the equivalence principle was studied in the case of coordinates noncommutativity [18,19,38–41], noncommutativity of coordinates and noncommutativity of momenta [42–44]. The authors of paper [43] concluded that the equivalence principle holds in the sense that an accelerated frame of reference is locally equivalent to a gravitational field, unless noncommutative parameters are anisotropic \( (\eta_{xy} \neq \eta_{xz}) \). In our previous papers we studied the possibility to recover the weak equivalence principle in a space with noncommutativity of coordinates [18,39,41], in four-dimensional noncommutative phase space of canonical type [44]. In the present paper we study the implementation of the weak equivalence principle in rotationally invariant noncommutative phase space \( (5)–(7) \). We examine features of motion of a particle (macroscopic body) in gravitational field in the space \( (5)–(7) \). The cases of uniform and nonuniform gravitational fields are studied and the dependence of the motion of a particle (macroscopic body) on its mass and composition is analyzed. We show that the assumption that particles with different masses feel the effect of noncommutativity with different parameters gives a possibility to preserve the weak equivalence principle.

The paper is organized as follows. In the next section features of the description of a composite system in rotationally invariant noncommutative phase space are presented. The motion of a particle (macroscopic body) in a uniform gravitational field in the noncommutative phase space is analyzed and the weak equivalence principle is considered in the third section. In the fourth section the results are generalized to the case of a nonuniform gravitational field. Conclusions are presented in the fifth section.

Description of many-particle system motion in noncommutative phase space with rotational symmetry. – The features of the description of the motion of a composite system in rotationally invariant noncommutative phase space were studied in our previous paper [45]. In the paper we considered the general case when coordinates and momenta of different particles may satisfy noncommutative algebra with different tensors of noncommutativity \( \theta_{ij}^{(n)} \) and \( \eta_{ij}^{(n)} \).

\[
[X_n^{(m)}, X_n^{(n)}] = i\hbar \delta_{mn}\theta_{ij}^{(m)}, \quad [X_n^{(m)}, p_j^{(n)}] = i\hbar \delta_{mn}\theta_{ij}^{(n)} \tag{8}
\]
\[
[X_n^{(m)}, p_j^{(n)}] = i\hbar \delta_{mn}\eta_{ij}^{(m)} \tag{9}
\]
\[
[p_i^{(m)}, p_j^{(n)}] = i\hbar \delta_{mn}\eta_{ij}^{(n)} \tag{10}
\]
\[
\theta_{ij}^{(n)} = \frac{c^2 \hbar^2}{\hbar} \sum_k \varepsilon_{ijk} \bar{a}_k, \quad \eta_{ij}^{(n)} = \frac{c^2 \hbar}{\ell_P^2} \sum_k \varepsilon_{ijk} \bar{P}_k, \tag{11}
\]

Here indices \( m, n = (1, \ldots, N) \) label the particles.

Coordinates and momenta which satisfy \( (8)–(10) \) can be represented as
\[
X_i^{(n)} = x_i^{(n)} - \frac{1}{2} \theta_{ij}^{(n)} p_j^{(n)}, \quad p_i^{(n)} = p_i^{(n)} + \frac{1}{2} \eta_{ij}^{(n)} x_j^{(n)}, \tag{12}
\]

The momenta and coordinates of the center-of-mass and the momenta and the coordinates of the relative motion defined in the traditional way read \( P^{(n)} = \sum_n P^{(n)} \), \( X^{(n)} = \sum_n X^{(n)} \), \( \Delta P^{(n)} = \mu_n P^{(n)} \), \( \Delta X^{(n)} = X^{(n)} - X^{(n)} \), here \( \mu_n = m_n/M \), \( M = \sum_{n=1}^{N} m_n \), coordinates \( X^{(n)} = \langle X_1^{(n)}, X_2^{(n)}, X_3^{(n)} \rangle \) and momenta \( P^{(n)} = \langle P_1^{(n)}, P_2^{(n)}, P_3^{(n)} \rangle \) satisfy \( (8)–(10) \). So, taking into account \( (8)–(10) \), we have the following commutation relations:
\[
[X_i^{(n)}, X_j^{(m)}] = i\hbar \sum_n \mu_n^2 \theta_{ij}^{(n)}, \quad [P_i^{(n)}, P_j^{(m)}] = i\hbar \sum_n \eta_{ij}^{(n)} \tag{13}
\]
\[
[X_i^{(n)}, p_j^{(m)}] = i\hbar \left( \delta_{ij} + \sum_n \mu_n \theta_{ik}^{(n)} \eta_{jk}^{(n)} \right), \tag{14}
\]
\[
[X_i^{(n)}, \Delta X_j^{(m)}] = i\hbar \left( \mu_n \theta_{ij}^{(n)} - \sum_m \mu_m \theta_{ij}^{(m)} \right), \tag{15}
\]
\[
[P_i^{(n)}, \Delta P_j^{(m)}] = i\hbar \left( \eta_{ij}^{(n)} - \mu_n \sum_m \eta_{ij}^{(m)} \right). \tag{16}
\]

Because of relations \( (15), (16) \) the motion of the center of mass is not independent of the relative motion in noncommutative phase space. In [45] we showed that in the case when tensors of noncommutativity corresponding to a particle are determined by its mass as
\[
\theta_{ij}^{(n)} = \frac{\tilde{\gamma} \ell_P^2}{\hbar m_n} \sum_k \varepsilon_{ijk} \bar{a}_k, \quad \eta_{ij}^{(n)} = \frac{\tilde{\alpha} \hbar m_n}{\ell_P^2} \sum_k \varepsilon_{ijk} \bar{P}_k, \tag{17}
\]

namely when the following conditions hold
\[
\tilde{c}_{\theta}^{(n)} m_n = \tilde{\gamma} = \text{const}, \tag{18}
\]
\[
\tilde{c}_{\eta}^{(n)} m_n = \tilde{\alpha} = \text{const}. \tag{19}
\]
where $\tilde{\gamma}$, $\tilde{\alpha}$ are constants which are the same for particles with different masses, one has

$$[X_i^c, \Delta X_j^{(n)}] = [P_i^c, \Delta P_j^{(n)}] = 0. \quad (20)$$

Also, if conditions (18), (19) hold the commutation relations for coordinates and momenta of the center of mass read

$$[X_i^c, X_j^c] = i\hbar \delta_{ij}, \quad [P_i^c, P_j^c] = i\hbar \delta_{ij}, \quad [X_i^c, P_j^c] = i\hbar \left( \delta_{ij} + \sum_k \frac{\theta_{ik} \eta_{kj}}{4} \right). \quad (22)$$

So, coordinates and momenta of the center of mass satisfy noncommutative algebra (1)–(3) with effective tensors of noncommutativity which depend on the total mass of the system and do not depend on its composition.

$$\theta_{ij}^c = \sum_n \mu_n^2 \theta_{ij}^{(n)} = \frac{\tilde{\gamma}_i^2 \rho_i}{\hbar M} \sum_k \varepsilon_{ijk} \tilde{a}_k, \quad (23)$$

$$\eta_{ij}^c = \sum_n \eta_{ij}^{(n)} = \frac{\tilde{\alpha}_i^2 \rho_i}{\hbar p} \sum_k \varepsilon_{ijk} \tilde{p}_k. \quad (24)$$

The noncommutative coordinates $X_i^c$ and noncommutative momenta $P_i^c$ can be represented as

$$X_i^c = \sum_n \mu_n x_i^{(n)} + \frac{1}{2} \theta_{ij}^{(n)} x_j^{(n)} = x_i - \frac{1}{2} \theta_{ij} x_j^c, \quad (25)$$

$$P_i^c = \sum_n \left( p_i^{(n)} + \frac{1}{2} \eta_{ij}^{(n)} x_j^{(n)} \right) = p_i^c + \frac{1}{2} \eta_{ij} x_j^c^c. \quad (26)$$

Here the $x_i^c = \sum_n n \mu_n x_i^{(n)}$, $p_i^c = \sum_n p_i^{(n)}$, coordinates and momenta $x_i^c$, $p_i^c$ satisfy the ordinary commutation relations. Also, on the basis of the definition of $\Delta X^{(n)}$, $\Delta P^{(n)}$ using (4), (12) we obtain

$$\Delta X_i^{(n)} = \Delta p_i^{(n)} + \frac{1}{2} \eta_{ij}^{(n)} \Delta x_j^{(n)} \quad (27),$$

$$\Delta P_i^{(n)} = \Delta p_i^{(n)} + \frac{1}{2} \eta_{ij}^{(n)} \Delta x_j^{(n)} \quad (28),$$

where $\Delta x_i^{(n)} = x_i^{(n)} - x_i$, $\Delta p_i^{(n)} = p_i^{(n)} - \mu_n \tilde{p}_i^c$.

In the next sections these results will be used for studies of the motion of the macroscopic body in a gravitational field in rotationally invariant noncommutative phase space.

**Influence of noncommutativity on the motion in a uniform gravitational field.** – First, let us study the motion of a particle of mass $m$ in noncommutative phase space with preserved rotational symmetry (5)–(7). For the particle in a uniform gravitational field one has the following Hamiltonian:

$$H_p = \frac{p_i^2}{2m} + mgX_1. \quad (29)$$

Here for convenience we choose the $X_1$-axis to be directed along the field direction. The coordinates and momenta $X_i$, $P_i$ satisfy (5)–(7). Because of the dependence of tensors of noncommutativity on additional coordinates and momenta $\tilde{a}_i$, $\tilde{p}_i^c$, $\tilde{p}_i^c$ one has to consider the total Hamiltonian as follows:

$$H = H_p + H_{osc}^a + H_{osc}^b \quad (30)$$

where $H_{osc}^a$, $H_{osc}^b$ are Hamiltonians of harmonic oscillators. Using the representation (12) one can write

$$H = \frac{p_i^2}{2m} + m g x_1 - \frac{(\eta_i \cdot L)}{2m} + \frac{m g}{2} [\theta \cdot p]_1 + \frac{[\eta \times x]^2}{8m} + H_{osc}^a + H_{osc}^b. \quad (31)$$

Here we introduce the following notations: $\theta = (\theta_1, \theta_2, \theta_3)$, $\eta = (\eta_1, \eta_2, \eta_3)$, $\theta_i = \sum_j \varepsilon_{ijk} \theta_{jk}/2$, $\eta_i = \sum_j \varepsilon_{ijk} \eta_{jk}/2$, and $L = [x \times p]$.

It is convenient to rewrite the Hamiltonian (31) in the following form: $H = H_0 + \Delta H$ with $H_0$, $\Delta H$ being defined as $H_0 = (H_p)_{ab} + H_{osc}^a + H_{osc}^b$, $\Delta H = H - H_0 = (H_p)_{ab}$. Notation $(\ldots)_{ab} = \{\psi_{0,0,0}^{a,b}, \psi_{0,0,0}^{a,b}, \ldots, \psi_{0,0,0}^{a,b}\}$ is used for averaging over the degrees of freedom of harmonic oscillators $H_{osc}^a$, $H_{osc}^b$ in the ground states. The oscillators are considered to be in the ground states because the frequency $\omega_{osc}$ is supposed to be very large, which leads to the large distance between their energy levels. So, harmonic oscillators in the ground states remain in them. Functions $\psi_{0,0,0}^{a,b}, \psi_{0,0,0}$ are well-known eigenstates of harmonic oscillators $H_{osc}^a$, $H_{osc}^b$. We have

$$H_0 = \frac{p_i^2}{2m} + m g x_1 - \frac{(\eta_i \cdot L)}{2m} + \frac{m g}{2} [\theta \cdot p]_1 + \frac{[\eta \times x]^2}{8m} - \frac{\langle \eta^2 \rangle x^2}{12m} \quad (32)$$

$$\Delta H = -\frac{\langle \eta \cdot L \rangle}{2m} + \frac{m g}{2} [\theta \cdot p]_1 + \frac{m g}{2} \frac{[\eta \times x]^2}{8m} - \frac{\langle \eta^2 \rangle x^2}{12m} \quad (33)$$

where we take into account that $\langle \psi_{0,0,0}^{a,b} | \theta_i | \psi_{0,0,0}^{a,b} \rangle = 0$, $\langle \psi_{0,0,0}^{a,b} | \eta | \psi_{0,0,0}^{a,b} \rangle = 0$,

$$\langle \theta_i \theta_j \rangle = \frac{c_{214}^2}{h^2} \delta_{ij} \langle \psi_{0,0,0}^{a,b} | \theta_i \theta_j | \psi_{0,0,0}^{a,b} \rangle = \frac{c_{214}^2 p}{h^2} \delta_{ij}, \quad (34)$$

$$\langle \eta_i \eta_j \rangle = \frac{h^2 c_{214}^2}{h^2} \delta_{ij} \langle \psi_{0,0,0}^{a,b} | \eta_i \eta_j | \psi_{0,0,0}^{a,b} \rangle = \frac{h^2 c_{214}^2}{h^2} \delta_{ij}. \quad (35)$$

and $\langle \theta_i^2 \rangle = \langle \eta^2 \rangle \delta_{ij}/3$, $\langle \eta_i \eta_j \rangle = \langle \eta^2 \rangle \delta_{ij}/3$. In our paper [45] we showed that up to the second order in $\Delta H$ one can consider the Hamiltonian $H_0$ because the corrections to the spectrum of $H_0$ up to the second order in the perturbation theory vanish. Namely, in the first order of the perturbation theory the corrections to the spectrum of $H_0$ caused by $\Delta H$ read

$$\Delta E^{(1)} = \langle \psi_{(n_p)}^{f,b} | \Delta H | \psi_{(n_p)}^{f,a} \rangle = \langle \psi_{(n_p)}^{f,b} | (H_p)_{ab} - (H_p)_{ab} | \psi_{(n_p)}^{f,a} \rangle = 0. \quad (36)$$

Here $\psi_{(n_p)}^{f,a}$ are eigenfunctions of $(H_p)_{ab}$ which corresponds to the well-known eigenfunctions of a harmonic oscillator.
of mass \( m \) and frequency \( \sqrt{\eta}/\sqrt{6m} \) in a uniform field \( mg \) (\( \{n_p\} \) being quantum numbers). In the second order of the perturbation theory we can write

\[
\Delta E^{(2)} = \lim_{\omega_{osc} \to \infty} \sum_{\{n_p\}, \{n^*\}} \left| \langle \psi_{\{n_p\}, \{n^*\}, \{n^*\}} | \Delta H | \psi_{\{n_p\}, \{0\}}, \{0\} \rangle \right|^2
\]

\[
\times \left( E^p_{\{n_p\}} - E^p_{\{0\}} - \sum_{i=1}^{3} h \omega_{osc} (n_i^2 + n_i^1) \right)^{-1} = 0.
\]

The sets of numbers \( \{n_p^r\}, \{n^s\}, \{n^b\} \) and \( \{n_p\}, \{0\}, \{0\} \) do not coincide. The notation \( E^p_{\{n_p\}} \) corresponds to the energy levels of \( \langle H_p \rangle_{ab} \).

In (37) we take into account that

\[
\left| \langle \psi_{\{n_p\}, \{n^*\}, \{n^*\}} | \Delta H | \psi_{\{n_p\}, \{0\}}, \{0\} \rangle \right|^2
\]

does not depend on \( \omega_{osc} \) because \( \sqrt{\hbar/m \omega_{osc}} = l_P \).

So, taking into account the expression for \( \Delta H \) (33), we can state that up to the second order in the parameters of noncommutativity for a particle in a uniform gravitational field one can consider the Hamiltonian (32). In this approximation the equations of motion of the particle read

\[
\dot{x}_i = \frac{\dot{p}_i}{m}, \quad \dot{p}_i = -mg \delta_{i,1} - \frac{\langle \eta^2 \rangle x_i}{6m^2}.
\]

From (38) the trajectory of a particle in a uniform field in rotationally invariant noncommutative phase space is the following:

\[
x_i(t) = \left( x_{0i} + 6g \frac{m^2}{\langle \eta^2 \rangle} \frac{\delta_{1,i}}{6m^2} \right) \cos \left( \sqrt{\frac{\langle \eta^2 \rangle}{6m^2}} t \right)
\]

\[
+ \frac{6m^2}{\langle \eta^2 \rangle} \sin \left( \sqrt{\frac{\langle \eta^2 \rangle}{6m^2}} t \right) - 6g \frac{m^2}{\langle \eta^2 \rangle} \delta_{1,i}.
\]

Here \( x_{0i}, v_{0i} \) are initial coordinates and velocities of the particle. Note that the motion of the particle is affected only by the momentum noncommutativity. In the limit \( \langle \eta^2 \rangle \to 0 \) one obtains \( x_i(t) = \delta_{1,i} g t^2/2 + x_{0i} \) as it should be. From (39) we have that particles with different masses move on different trajectories in a uniform gravitational field in rotationally invariant noncommutative phase space. From this we can conclude that the weak equivalence principle also known as the uniqueness of the free-fall principle is violated. The principle states that velocity and position of a point mass in a gravitational field are independent of mass, composition and structure and depend only on its initial position and velocity.

We would like to stress that if condition (19) holds from (35), (39), one has

\[
\frac{\langle \eta^2 \rangle}{m^2} = \frac{3h^2 \bar{a}^2}{2l_P^2} = B = \text{const},
\]

\[
x_i(t) = \left( x_{0i} + 6g \frac{m^2}{B} \frac{\delta_{1,i}}{6} \right) \cos \left( \sqrt{\frac{B}{6}} t \right)
\]

\[
+ v_{0i} \sqrt{\frac{6}{B}} \sin \left( \sqrt{\frac{B}{6}} t \right) - 6g \frac{m^2}{B} \delta_{1,i}.
\]

Constant \( B \) does not depend on mass. So, if condition (19) is satisfied, the trajectory of a particle in a uniform gravitational field does not depend on its mass as it has to be and the weak equivalence principle is recovered in noncommutative phase space with rotational symmetry.

The same conclusion can be done in the case of the motion of a composite system in the uniform gravitational field. For a composite system (macroscopic body) of mass \( M \) in a uniform field we can write

\[
H_k = \frac{(p^c)^2}{2M} + Mg X_c^1 + H_{rel}.
\]

Here \( p^c, X_c \) are momenta and coordinates of the center of mass of the composite system (macroscopic body). The Hamiltonian \( H_{rel} \) corresponds to the relative motion. As was shown in the previous section, if conditions (18), (19) hold, the coordinates and the momenta of the center of mass and the coordinates and the momenta of the relative motion satisfy (20)–(22), and can be represented as (25)–(28). So, we can write

\[
H_0 = \frac{(p^c)^2}{2M} + Mg x_1^c + \frac{\langle \eta^2 \rangle^2 (x^c)^2}{12M} + \langle H_{rel} \rangle_{ab}
\]

\[
+ H_{osc}^a + H_{osc}^b.
\]

Note that \( H_0, \langle H_{rel} \rangle_{ab} = 0 \), because \( \langle H_{rel} \rangle_{ab} \) depends on \( \Delta x_i^{(n)}, \Delta p_i^{(n)} \) which commute with \( x_i^c, p_i^c, \bar{a}_i, \bar{b}_i, \bar{p}_i^a, \bar{p}_i^b \). So, the trajectory of the center of mass of a composite system is as follows:

\[
x_i^c(t) = \left( x_{0i}^c + \frac{6g B^2}{M^2} \right) \cos \left( \sqrt{\frac{B}{6M^2}} t \right)
\]

\[
+ v_{0i} \sqrt{\frac{6}{B}} \sin \left( \sqrt{\frac{B}{6M^2}} t \right) - \frac{6g B^2}{M^2} \delta_{1,i}.
\]

If relation (19) does not hold, we have that the trajectory of the center of mass of a body depends on its mass. When relation (19) is satisfied the effective tensor of momentum noncommutativity is defined as (24) and we can write

\[
\frac{\langle \eta^2 \rangle^2}{M^2} = \frac{3h^2 \bar{a}^2}{2l_P^2} = B = \text{const},
\]

\[
x_i^c(t) = \left( x_{0i}^c + \frac{6g B^2}{B} \frac{\delta_{1,i}}{6} \right) \cos \left( \sqrt{\frac{B}{6}} t \right)
\]

\[
+ v_{0i} \sqrt{\frac{6}{B}} \sin \left( \sqrt{\frac{B}{6}} t \right) - 6g \frac{B^2}{B} \delta_{1,i}.
\]

So, the trajectory of a body in a uniform field does not depend on its mass and composition because of the relation (19).
Note also that taking into account (39), the motion of the center of mass of a system of \(N\) noninteracting particles with masses \(m_a\) in a uniform gravitational field is described by the following trajectory:

\[
\begin{align*}
  x_a^i(t) &= \sum_a \mu_ax_{0a}^i(t) = -\sum_a 6g\mu_a \frac{m_a^2}{\langle \eta^a \rangle^2} \delta_{i,a} \\
  &+ \sum_a \mu_a \left( x_{0a}^i + 6g \frac{m_a^2}{\langle \eta^a \rangle^2} \delta_{i,a} \right) \cos \left( \sqrt{\frac{\langle \eta^a \rangle^2}{6m_a^2}} t \right) \\
  &+ \sum_a \mu_a v_{0a}^i \sqrt{\frac{6m_a^2}{\langle \eta^a \rangle^2}} \sin \left( \sqrt{\frac{\langle \eta^a \rangle^2}{6m_a^2}} t \right),
\end{align*}
\]

(47)

where the index \(a\) labels the particles, \(x_{0a}^i, v_{0a}^i\) are initial coordinates and initial velocities of the particle with mass \(m_a\). It is important to mention that if relation (19) holds, using \(x_{0a}^i = \sum_a \mu_a x_{0a}^i\) and \(v_{0a}^i = \sum_a \mu_a v_{0a}^i\), the expression (47) reduces to (44).

**Motion in the nonuniform gravitational field and the weak equivalence principle.** Let us study the Hamiltonian which corresponds to a particle of mass \(m\) in a nonuniform gravitational field

\[
\text{H}_p = \frac{p^2}{2m} - G\bar{M}m \frac{x}{X}
\]

(48)

Here \(X = |x| = \sqrt{\sum x_i^2}\). Coordinates \(x_i\) and momenta \(p_i\) satisfy the relations (5)–(7). Using the representation (12) we can write \(X = \sqrt{x^2 - (\theta \cdot L)} + \theta \cdot p/4\).

Up to the second order in the parameters of noncommutativity the Hamiltonian \(H_p\) reads

\[
\begin{align*}
  H_p &= \frac{p^2}{2m} - \frac{G\bar{M}m}{x} - \frac{\langle \theta \cdot L \rangle}{8m} + \frac{\langle \eta \times x \rangle^2}{8m} \\
  &- \frac{G\bar{M}m}{2x^3} (\theta \cdot L) - \frac{3G\bar{M}m}{8x^5} (\theta \cdot L)^2 + \frac{G\bar{M}m}{16} \\
  &\times \left( \frac{1}{x^2} \frac{\theta \times p}{x} \bigg|^1 x + \frac{1}{x} \frac{\theta \times p}{x} \bigg| x^2 + \frac{h^2}{x^3} (\theta \times x)^2 \right).
\end{align*}
\]

(49)

\(x = |x|\). Note that the last term in (49) appears because of noncommutativity of operators \(x^2\) and \(\theta \times p\) under the square root in \(X\). The details of the calculation of the expansion of \(1/X\) over the parameters of noncommutativity can be found in [9]. So, one can write the expression for \(\Delta H\),

\[
\Delta H = -\frac{\langle \theta \cdot L \rangle}{8m} + \frac{\langle \eta \times x \rangle^2}{8m} - \frac{\langle \eta^2 \rangle x^2}{8m} - \frac{G\bar{M}mL^2(\theta^2)}{2x^3} + \frac{G\bar{M}mL^2(\theta^2)}{12m} \\
+ \frac{G\bar{M}mL^2(\theta^2)}{8x^5} + \frac{G\bar{M}m}{16} \left( \frac{1}{x^2} \frac{\theta \times p}{x} \bigg| x^2 \right) \\
+ \frac{1}{x^2} \frac{\theta \times p}{x} \bigg| x^2 + \frac{h^2}{x^3} (\theta \times x)^2 \right) - 3G\bar{M}m (\theta \cdot L)^2 \\
- \frac{G\bar{M}m(\theta^2)}{24} \left( \frac{1}{x^2} p \bigg| x^2 + \frac{1}{x^2} p \bigg| x^2 + \frac{h^2}{x^3} \right).
\]

(50)

Similarly as in the case of uniform gravitational field up to the second order in parameters of noncommutativity one can study the following Hamiltonian:

\[
\begin{align*}
  H_0 &= \frac{p^2}{2m} - \frac{G\bar{M}m}{x} + \frac{\langle \eta^2 \rangle x^2}{12m} - \frac{G\bar{M}mL^2(\theta^2)}{8x^5} \\
  &+ \frac{G\bar{M}m(\theta^2) (2)}{24} \left( \frac{2}{x^3} p^2 + \frac{6\hbar h}{x^5} (x \cdot p) - \frac{\hbar^2}{x^3} \right) \\
  &+ \mu_a \left( \frac{x_{0a}}{a} + \frac{H_{osc}}{H_{osc}} \right).
\end{align*}
\]

(51)

From (51), the equations of motion of a particle in a nonuniform gravitational filed read

\[
\begin{align*}
  \dot{x} &= \frac{p}{m} - \frac{G\bar{M}m^2(\theta^2)}{12} \left( \frac{1}{x^3} p - 3x \frac{3x}{x^5} (x \cdot p) \right), \\
  \dot{p} &= -\frac{G\bar{M}m}{x^3} - 6m \frac{6m}{6m} \frac{G\bar{M}m(\theta^2)}{4 \frac{1}{x^3} (x \cdot p)p} \\
  &- \frac{2x}{x^3} p^2 + \frac{5x}{2x^2} L^2 + \frac{5h^2}{x} \frac{5h^2}{x} - \frac{5h^2}{x} (x \cdot p) \right).
\end{align*}
\]

(52)

In the limit \(h \to 0\) from (52), (53) we obtain

\[
\begin{align*}
  \dot{x} &= \frac{p}{m} - \frac{G\bar{M}m^2(\theta^2)}{12} \left( \frac{1}{x^3} p - 3x \frac{3x}{x^5} (x \cdot p) \right), \\
  \dot{p} &= -\frac{G\bar{M}m}{x^3} - 6m \frac{6m}{6m} \frac{G\bar{M}m(\theta^2)}{4 \frac{1}{x^3} (x \cdot p)p} \\
  &- \frac{2x}{x^3} p^2 + \frac{5x}{2x^2} L^2 + \frac{5h^2}{x} \frac{5h^2}{x} - \frac{5h^2}{x} (x \cdot p) \right).
\end{align*}
\]

(53)

where the vector \(\mathbf{v} = \mathbf{p}/m\) is introduced. The obtained equations of motion depend on the products \(m^2(\theta^2)\) and \(\langle \eta^2 \rangle / m^2\). Supposing that conditions (18), (19) hold we have

\[
\begin{align*}
  \dot{x} &= \frac{p}{m} - \frac{G\bar{M}m^2(\theta^2)}{12} \left( \frac{1}{x^3} p - 3x \frac{3x}{x^5} (x \cdot p) \right), \\
  \dot{p} &= -\frac{G\bar{M}m}{x^3} - 6m \frac{6m}{6m} \frac{G\bar{M}m(\theta^2)}{4 \frac{1}{x^3} (x \cdot p)p} \\
  &- \frac{2x}{x^3} p^2 + \frac{5x}{2x^2} L^2 + \frac{5h^2}{x} \frac{5h^2}{x} - \frac{5h^2}{x} (x \cdot p) \right).
\end{align*}
\]

(54)

Here we use (40) and take into account that on condition (18) the following relation is satisfied:

\[
\langle \theta^2 \rangle m^2 = \frac{3\bar{a}^2 L^2 m^2}{2h^2} = A = \text{const},
\]

(58)

where \(A\) is a constant which does not depend on mass. The obtained equations of motion (56), (57) depend on the constants \(A, B\) which are the same for different particles.

So, the weak equivalence principle is recovered in noncommutative phase space with preserved rotational symmetry due to relations (18), (19).

In the quantum case, if (18), (19) are satisfied the equations (52), (53) read

\[
\begin{align*}
  \dot{x} &= \frac{p}{m} - \frac{G\bar{M}m^2(\theta^2)}{12} \left( \frac{1}{x^3} p - 3x \frac{3x}{x^5} (x \cdot p) \right), \\
  \dot{p} &= -\frac{G\bar{M}m}{x^3} - 6m \frac{6m}{6m} \frac{G\bar{M}m(\theta^2)}{4 \frac{1}{x^3} (x \cdot p)p} \\
  &- \frac{2x}{x^3} p^2 + \frac{5x}{2x^2} L^2 + \frac{5h^2}{x} \frac{5h^2}{x} - \frac{5h^2}{x} (x \cdot p) \right).
\end{align*}
\]

(59)
\[ \dot{\mathbf{v}} = -\frac{GM\mathbf{x}}{x^3} - \frac{B\mathbf{x}}{6} - \frac{GM\mathbf{A}}{4} \left( \frac{1}{x^5} (\mathbf{x} \cdot \mathbf{v}) \mathbf{v} - \frac{2\mathbf{x}}{x^5} \mathbf{v} \right) + \frac{5\mathbf{x}}{2x^6}[\mathbf{x} \times \mathbf{v}]^2 + \frac{5\hbar x^2}{6m^2x^4} - \frac{5i\hbar}{mx^2}\mathbf{x}(\mathbf{x} \cdot \mathbf{v}) \, . \] (60)

The equations depend on \( h/m \), as it has to be. This is because of the commutation relation \([x, v] = i\hbar \dot{I}/m \) [46].

So, in the case of the preserving of conditions (18), (19), the motion of a particle in a nonuniform gravitational field does not depend on its mass as it is in the ordinary space.

The conclusion can be generalized to the case of motion of a composite system (macroscopic body) in a nonuniform gravitational field in noncommutative phase space with rotational symmetry. Similarly, as was shown in the previous section, for the motion of a composite system in a nonuniform gravitational field up to the second order in the parameters of noncommutativity we can write the following Hamiltonian:

\[ H_s = \frac{(p^c)^2}{2M} - \frac{GM\mathbf{M}}{x^3} + H_{rel}, \] (61)
\[ H_0 = \frac{(p^c)^2}{2M} - \frac{GM\mathbf{M}}{x^3} + \frac{\langle \eta^2 \rangle^2}{24}\left( \frac{2}{(x^3)^3} (p^c)^2 - \frac{6i\hbar}{(x^3)^5} (\mathbf{x} \cdot \mathbf{p})^2 - \frac{\hbar^2}{(x^3)^5} \right) + \langle H_{rel} \rangle_{ab} + H_{osc} + H_{osc}^b. \] (62)

Here we use (25)–(28). Taking into account (18), (19), the equations of motion do not depend on the composition of a system and on its mass and they read

\[ \dot{\mathbf{x}}^c = \mathbf{v}^c - \frac{GM\mathbf{M}}{12} \left( \frac{1}{(x^3)^3} \mathbf{v}^c - \frac{3\mathbf{x}^c}{(x^3)^3} (\mathbf{x} \cdot \mathbf{v}) \right), \] (63)
\[ \dot{\mathbf{v}}^c = \frac{GM\mathbf{x}}{x^3} - \frac{B\mathbf{x}}{6} - \frac{GM\mathbf{A}}{4} \left( \frac{1}{(x^3)^3} (\mathbf{x} \cdot \mathbf{v}) \mathbf{v}^c \right) - \frac{2\mathbf{c}}{(x^3)^3} \mathbf{v}^c + \frac{5\mathbf{x}}{2(x^3)^2} [\mathbf{v} \times \mathbf{v}^c] \, . \] (64)

We would like to stress that if conditions (18), (19) are not satisfied the equations of motion of a body (composite system) depends on its mass and \( \langle \theta^2 \rangle^2, \langle \eta^2 \rangle^2 \) which according to definitions \( \theta_{ij} = \sum_n h_{n\theta_{ij}^{(n)}}, \eta_{ij} = \sum_n h_{n\eta_{ij}^{(n)}} \) (see (23), (24)) depend on the composition. This dependence is an additional fact which causes the violation of the weak equivalence principle.

At the end of this section we would like to note that conditions (18), (19) are in agreement with the conditions which were proposed to solve the list of problems in four-dimensional (2D configurational space and 2D momentum space) noncommutative phase space of canonical type in [44,47]. Namely in these papers we proposed the parameter of coordinate noncommutativity to be inversely proportional to mass, and the parameter of momentum noncommutativity to be proportional to mass.

The same dependence we have in the case of rotationally invariant noncommutative space for the tensors of noncommutativity (17).

**Conclusions.** In this paper we studied the influence of noncommutativity on the motion in a gravitational field in rotationally invariant space with noncommutativity of coordinates and noncommutativity of momenta (5)–(7). The rotationally invariant noncommutative algebra is constructed with the help of the generalization of parameters of noncommutativity to tensors (4).

The motion of a particle in a uniform gravitational field was studied. We showed that in rotationally invariant noncommutative phase space up to the second order in the parameters of noncommutativity this motion is affected only by the noncommutativity of momenta. The trajectory of a particle in a uniform field corresponds to the trajectory of the motion of a harmonic oscillator and depends on the particle mass and the value of the parameter of momentum noncommutativity (39). It was concluded that in the case when particles with different masses feel noncommutativity with different parameters, namely when condition (19) is satisfied the trajectory of a particle does not depend on mass (41) and the weak equivalence principle is recovered in rotationally invariant noncommutative phase space. The same conclusion was done for the case of the motion of a particle in a nonuniform gravitational field. We showed that due to relations (18), (19) quantum equations of motion of the particle depend on \( h/m \) (59), (60), as it has to be, and in the classical limit they do not depend on mass (56), (57).

Besides the conditions (18), (19) are important in the consideration of the motion of a composite system (macroscopic body) in rotationally invariant noncommutative phase space. Namely, preserving of relations (18), (19) leads to commutativity of the coordinates of the center of mass and the relative coordinates, momenta the center of mass and the relative momenta (20) and to recovering the independence of motion of the center of mass of a composite system in a gravitational field from its mass and composition (63), (64). So, the noncommutative algebra (8)–(10) with (17) is rotationally invariant and does not lead to the violation of the weak equivalence principle.

We would like also to mention here another important conclusion which can be drawn due to relations (18), (19). In our paper [45] we showed that when conditions (18), (19) are satisfied, the noncommutative coordinates can be considered as kinematic variables and noncommutative momenta are proportional to mass; the noncommutative algebra for coordinates and momenta of the center of mass corresponds to noncommutative algebra for coordinates and momenta of individual particles with effective parameters of noncommutativity (21), (22).

So, the idea of dependence of parameters of noncommutativity on the mass (18), (19) gives the possibility to solve the list of problems in rotationally invariant noncommutative phase space, among them the violation of the weak
equivalence principle. In addition the idea is important not only in noncommutative space. In deformed space with minimal length $[X, P] = i\hbar(1 + \beta P^2)$ the special dependence of the parameter of deformation $\beta$ on mass leads to recovering the weak equivalence principle, preserving of the properties of the kinetic energy, independence of Galilean and Lorentz transformations of mass [48–50].

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The author thanks Prof. V. M. Tkachuk and Dr. Yu. S. Krynytskyi for their advice and great support during research studies. This work was partly supported by the grant of the President of Ukraine for support of scientific researches of young scientists (F-75/148-2018).

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