CHOW CLASSES OF DIVISORS ON STACKS OF POINTED HYPERELLIPTIC CURVES

DAN EDIDIN AND ZHENNING HU

Abstract. We calculate the Chow classes of the universal hyperelliptic Weierstrass divisor $\mathcal{H}_{g,w}$ and the universal $g^2_1$ divisor $\mathcal{H}_{g,g^2_1}$. Our results are expressed in terms of a basis for $\text{Cl}(\mathcal{H}_{g,1})$ and $\text{Cl}(\mathcal{H}_{g,2})$ computed by Scavia [16].

1. Introduction

A problem inspired by Harris and Mumford’s paper [11] on the Kodaira dimension of the moduli space of curves is to compute the Picard classes of naturally occurring effective divisors in $\mathcal{M}_g$ and $\mathcal{M}_{g,n}$. Examples include divisors parametrizing $k$-gonal curves [11], divisors parametrizing curves with an exceptional Weierstrass point [6], the Weierstrass divisor on the universal stable curve $\mathcal{M}_{g,1}$ [5], and the divisor parametrizing curves with a linear series with Brill-Noether number $-1$ [8][9]. In this paper, we turn our attention to stacks of stable hyperelliptic curves. Since every hyperelliptic curve has a unique $g^2_1$, we cannot use exceptional linear series to impose conditions on hyperelliptic curves. However, the universal stable hyperelliptic curve $\mathcal{H}_{g,1}$ has a Weierstrass divisor $\mathcal{H}_{g,w}$, and the stack $\mathcal{H}_{g,2}$ contains the universal $g^2_1$, $\mathcal{H}_{g,g^2_1}$, parametrizing stable hyperelliptic curves $(C,p_1,p_2)$ where $p_1$ and $p_2$ sum to the $g^2_1$. The purpose of this paper is to compute the classes of these naturally occurring divisors in the divisor class groups of $\mathcal{H}_{g,1}$ and $\mathcal{H}_{g,2}$. We express our results in terms of a basis for $\text{Cl}(\mathcal{H}_{g,n})$ recently computed by Scavia [16].

Theorem 1.1.

$[\mathcal{H}_{g,n}] = \left( \frac{g+1}{g-1} \right) \psi - \frac{1}{2(2g+1)(g-1)} \eta_{\text{rr}} + \sum_{i=1}^{\lfloor (g-1)/2 \rfloor} \left[ -\frac{(i+1)(2i+1)}{(2g+1)(g-1)} \eta_{i,0} - \frac{(g-i)[2(g-i)-1]}{(2g+1)(g-1)} \eta_{i,1} \right] + \sum_{i=1}^{\lfloor g/2 \rfloor} \left[ -\frac{2i(2i+1)}{(2g+1)(g-1)} \delta_{i,0} - \frac{2(g-i)[2(g-i)+1]}{(2g+1)(g-1)} \delta_{i,1} \right].$

1991 Mathematics Subject Classification. 14H10, 14H51.
Theorem 1.2.

\[
[H_{g,g}] = \left( \frac{1}{g-1} \right) (\psi_1 + \psi_2) - \frac{1}{2(g-1)(2g+1)} \eta_{irr} - \left( \frac{g+1}{g-1} \right) \delta_{0,2} + \\
\sum_{i=1}^{\lfloor (g-1)/2 \rfloor} \left[ - \frac{(i+1)(2i+1)}{(g-1)(2g+1)} \eta_{i,0} + \frac{2i(g-i-1)-1}{(g-1)(2g+1)} \eta_{i,1} - \frac{(g-i)[2(g-i)-1]}{(g-1)(2g+1)} \eta_{i,2} \right] + \\
\sum_{i=1}^{\lfloor g/2 \rfloor} \left[ - \frac{2i(2i+1)}{(g-1)(2g+1)} \delta_{i,0} + \frac{(2i-1)[2(g-i)-1]-2}{(g-1)(2g+1)} \delta_{i,1} - \frac{2(g-i)[2(g-i)+1]}{(g-1)(2g+1)} \delta_{i,2} \right].
\]

Here \(\psi, \psi_1, \psi_2\) are the \(\psi\)-classes associated to the sections and the other classes are boundary divisors which we describe below. When \(g = 2\), our formula in Theorem 1.1 agrees with the formula proved by Eisenbud and Harris in [9] using Porteous’s formula. Unfortunately, we cannot apply their method in higher genus, because the corresponding degeneracy locus has expected codimension \(g-1\). Instead we use the method of test curves, and the main challenge is to find sufficiently interesting families of branched double covers of rational nodal curves. The key construction to do this is done in Section 4.3.

Conventions and notation. Throughout this paper, we fix a natural number \(g \geq 2\). Because Scavia uses results from topology to compute the divisor class group of \(\mathcal{H}_{g,n}\) we work over the field \(\mathbb{C}\) of complex numbers.

Acknowledgement

The first author was supported by Simons Collaboration Grants 315460 and 708560. The authors are very grateful to the referees for a careful reading and a number of helpful comments which improved the exposition.

2. Stacks of hyperelliptic curves

Let \(\mathcal{H}_g\) be the stack of smooth hyperelliptic curves of genus \(g \geq 2\). This is a closed smooth substack of the moduli stack \(\mathcal{M}_g\) and we denote by \(\overline{\mathcal{H}}_g\) its closure in \(\mathcal{M}_g\). This is a smooth substack of \(\overline{\mathcal{M}}_g\) by [2, Chapter XI, Lemma (6.15)]. As noted there, the stack \(\overline{\mathcal{H}}_g\) parametrizes curves which have a (unique) involution with isolated fixed points, whose quotient is a nodal rational curve.

Likewise, let \(\overline{\mathcal{H}}_{g,n}\) denote the stack of \(n\)-pointed hyperelliptic curves. It is defined as the stack-theoretic inverse image of \(\overline{\mathcal{H}}_g\) under the forgetful map \(\overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_g\); i.e. we set \(\overline{\mathcal{H}}_{g,n}\) to equal the fiber product \(\overline{\mathcal{H}}_g \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{M}}_{g,n}\). (Our definition of \(\overline{\mathcal{H}}_{g,n}\) differs slightly from Scavia’s, in that Scavia takes the reduced substack structure – an operation which does not change the divisor class group.)

In [3], the authors demonstrate that \(\overline{\mathcal{H}}_{g,n}\) is not smooth when \(g \geq 3\) and \(n \geq 2\), but Scavia [10] proves that \(\overline{\mathcal{H}}_{g,n}\) always contains a smooth open substack \(\mathcal{U}_{g,n}\) whose complement has codimension at least two. The stack \(\mathcal{U}_{g,n}\) is the union of two open substacks \(\mathcal{U}_{g,n}^1\) and \(\mathcal{U}_{g,n}^2\). The first open substack is the inverse image of \(\mathcal{H}_g\) under the projection \(\overline{\mathcal{H}}_{g,n} \rightarrow \overline{\mathcal{H}}_g\) and parametrizes curves with rational tails which contract to a smooth curve. The second open substack parametrizes stable pointed curves which remain stable after deleting the marked points.
2.1. Stable hyperelliptic curves as admissible covers. In order to study the boundary of $\overline{H}_g$ we use the fact that any stable hyperelliptic curve can be obtained by stabilizing an admissible cover. Once we do this we will interpret the divisors $\overline{H}_{g,w}$ and $\overline{H}_{g,2}$ in the context of admissible covers. This will allow us to determine their intersection with the boundary divisors of $\overline{H}_{g,1}$ and $\overline{H}_{g,2}$ respectively.

Let $\text{Adm}_{0,2g+2,2}$ be the stack whose sections over a base scheme $S$ parametrize the following data:

A stable pointed rational curve $(P \to S, \Sigma_{2g+2})$ and a double cover $C \to P$ which is étale over the complement of $\Sigma_{2g+2}$ and the nodes of $P$, and satisfies

1. $C \to S$ is a nodal curve.
2. Every node of $C$ maps to a node of $P$, and over the nodes of $P$ the map $C \to P$ can be described as in [1, Definition 4.1.1]. In particular they may be either branched or étale over the nodes.
3. $C \to P$ is ramified over the divisor $\Sigma_{2g+2}$.

Remark 2.1. Note that each fiber of $C \to S$ is a stable curve unless the corresponding fiber of $P \to S$ contains a tail with exactly two markings coming from $\Sigma_{2g+2}$. In this case it is the semi-stable curve obtained by replacing the node of an irreducible component with a rational bridge.

By [1], the stack $\text{Adm}_{0,2g+2,2}$ is a proper Deligne–Mumford stack. In general, stacks of admissible covers are not normal, but because double covers are always Galois, this stack is isomorphic to the stack of twisted stable maps to $B\mu_2$ which is smooth [1, Theorem 3.0.2].

The map $\text{Adm}_{0,2g+2,2} \to \overline{\mathcal{M}}_g$, which associates to an admissible double cover the corresponding stable curve of genus $g$, is a proper morphism with 0-dimensional fibers and the substack $\overline{\mathcal{H}}_g$ is the image of $\text{Adm}_{0,2g+2,2}$. Note that there is a natural free action of the symmetric group on $\text{Adm}_{0,2g+2,2}$ and the map $\text{Adm}_{0,2g+2,2} \to \overline{\mathcal{M}}_g$ factors through the quotient by $[\text{Adm}_{0,2g+2,2} / S_{2g+2}]$ in the sense of [15].

Thus we obtain a proper surjective morphism $[\text{Adm}_{0,2g+2,2} / S_{2g+2}] \to \overline{H}_g$ with 0-dimensional fibers which is a bijection on $\mathbb{C}$-valued points. However, the following example, which we learned from Andrea di Lorenzo, shows that the morphism $[\text{Adm}_{0,2g+2,2} / S_{2g+2}] \to \overline{H}_g$ is not representable, and therefore it is not an equivalence.

Example 2.2 (Andrea Di Lorenzo). Consider the admissible cover of the rational curve $P$ which is the union of two copies of $\mathbb{P}^1$. Mark $2g$ general points on one component and two points on the other component. Let $C' \to P$ be the admissible double cover of $P$ branched at the marked points. The curve $C'$ is the union of a general hyperelliptic curve $D$ of genus $g - 1$ and a $\mathbb{P}^1$ which meets $D$ in two points. The automorphism group of the admissible cover $C' \to \mathbb{P}^1$ contains two non-trivial elements. The first is the automorphism which restricts to the hyperelliptic involution on $D$ and the automorphism of $\mathbb{P}^1$ which fixes the two ramification points of the double cover $\mathbb{P}^1 \to \mathbb{P}^1$, and exchanges the two points of attachment of the $\mathbb{P}^1$ to $D$.

The other automorphism restricts to the identity on $D$, and on the $\mathbb{P}^1$ fixes the points of attachment but exchanges the two ramification points. See Figure [1].

On the other hand, the stabilization of $C'$ is an irreducible nodal hyperelliptic curve. This curve has only one non-trivial automorphism which is the hyperelliptic involution.
Figure 1. Two non–trivial automorphisms

Remark 2.3. A section of the quotient \([\text{Adm}_{0,2g+2,2}/S_{2g+2}]\) over a base scheme \(S\) is given by the following data:

1. A rational curve \(P \to S\) and a divisor \(\Sigma_{2g+2} \subset P\) which is finite and étale over \(S\) and such that the geometric fibers of \(P \to S\) are stable when marked by the pullback of \(\Sigma_{2g+2}\) to the fiber.
2. An admissible double cover \(C \to P\) which is ramified over \(\Sigma_{2g+2}\).

2.2. The Weierstrass divisor in \(\mathcal{H}_{g,1}\). Let \(\mathcal{H}_{g,w}\) be the substack of \(\mathcal{H}_{g,1}\) parametrizing Weierstrass points in the fibers of the projection \(\mathcal{H}_{g,1} \to \mathcal{H}_g\). By [13, Corollary 6.8], \(\mathcal{H}_{g,w}\) is étale of degree \(2g+2\) over \(\mathcal{H}_g\) and hence \(\mathcal{H}_{g,w}\) is a smooth divisor in \(\mathcal{H}_{g,1}\). The following result is certainly well known but lacking a suitable reference, and we include a (sketch of a) proof.

Proposition 2.4. \(\mathcal{H}_{g,w}\) is irreducible.

Proof. We use a construction of \(\mathcal{H}_{g,1}\) as a quotient stack given by Pernice in [14]. Specifically he proves that \(\mathcal{H}_{g,1}\) is a quotient of the scheme \(\tilde{\mathcal{A}}_{\text{sm}}(2g+2) \subset \mathcal{A}^{2g+3}\times \mathcal{A}^1\) by a connected algebraic group \(B\). The scheme \(\tilde{\mathcal{A}}_{\text{sm}}(2g+2)\) parametrizes pairs \((f,s)\) where \(f\) is a binary form of degree \(2g+2\) with simple roots and \(f(0:1) = s^2\).

With this description the Weierstrass divisor is the quotient of the irreducible closed subscheme defined by setting \(s = 0\). Hence \(\mathcal{H}_{g,w}\) is irreducible since it has a smooth cover by an irreducible scheme. \(\square\)

We denote by \(\overline{\mathcal{H}}_{g,w}\) the closure (with its canonical closed substack structure) in \(\overline{\mathcal{H}}_{g,1}\) of the divisor \(\mathcal{H}_{g,w}\). Since \(\mathcal{H}_{g,w}\) is irreducible so is \(\overline{\mathcal{H}}_{g,w}\). The discussion after [4, Proposition 1] shows that \(\overline{\mathcal{H}}_{g,w}\) is a Cartier divisor in \(\overline{\mathcal{H}}_{g,1}\). Precisely if \(C \to S\) is a family of stable hyperelliptic curves with involution \(\tau\), then the restriction of \(\overline{\mathcal{H}}_{g,w}\) to \(C\) is the fixed locus of \(\tau\) minus the nodes of type \(\Delta_i\) for \(i > 0\). In particular if \((C \to S, \sigma)\) is a stable pointed family of hyperelliptic curves such that \(C \to S\) is still stable after forgetting the section (i.e. the fibers of \(C \to S\) do not contain rational bridges) then the pullback of \(\overline{\mathcal{H}}_{g,w}\) along the morphism \(S \to \overline{\mathcal{H}}_{g,1}\) is the divisor \(\sigma^*\tau^*\mathcal{O}(\sigma)\).

2.3. The \(g_2^1\) divisor in \(\overline{\mathcal{H}}_{g,2}\). Let \(\mathcal{H}_{g,g_2^1} \subset \mathcal{H}_{g,2}\) be the divisor parametrizing two–pointed hyperelliptic curves \((C,p_1,p_2)\) such that \(\mathcal{O}(p_1 + p_2)\) is the \(g_2^1\) on \(C\).

Note that the substack \(\mathcal{H}_{g,g_2^1}\) is representable since the hyperelliptic involution cannot fix the \(g_2^1\) divisor. It is equivalent to the representable open substack \(\mathcal{H}_{g,1} \setminus \mathcal{H}_{g,w}\) since for any non–Weierstrass point \(p\) on a hyperelliptic curve there is a unique...
point $q$ such that $p + q$ is a $g^1_2$. In particular, $\mathcal{H}_{g,g^1_2}$ is irreducible. Let $\mathcal{H}_{g,g^1_2}$ be its closure in $\mathcal{H}_{g,2}$ with its reduced induced substack structure.

Although we do not know if $\mathcal{H}_{g,g^1_2}$ is a Cartier divisor, we will use the following explicit description of $\mathcal{H}_{g,g^1_2}$ as a Cartier divisor on the smooth open substack $\mathcal{H}_{g,2}^{rt}$ parametrizing two-pointed curves with at most one rational tail and which stabilizes to a smooth curve.

The morphism $\mathcal{H}_{g,2}^{rt} \rightarrow \mathcal{H}_{g,1}$ which forgets the first section makes $\mathcal{H}_{g,1}^{rt}$ into the universal stable pointed hyperelliptic curve over $\mathcal{H}_{g,1}$. The fibers of $\mathcal{H}_{g,2}^{rt} \rightarrow \mathcal{H}_{g,1}$ are one-pointed smooth curves and there is a hyperelliptic involution $\tau: \mathcal{H}_{g,2}^{rt} \rightarrow \mathcal{H}_{g,1}^{rt}$ which commutes with the projection to $\mathcal{H}_{g,1}$. Identify the open set $\mathcal{H}_{g,2} \subset \mathcal{H}_{g,2}^{rt}$ as parametrizing pairs ($((C,p),q)$ where $(C,p)$ is a marked curve (i.e. an object of $\mathcal{H}_{g,1}$) and $q$ is a point on $C$ distinct from $p$. With our definition, $((C,p),q)$ is in $\mathcal{H}_{g,g^1_2}$ if and only if $q = \tau(p)$. In other words, we identify $\mathcal{H}_{g,g^1_2}$ with the intersection of the divisor $\tau(\sigma)$ with the open set $\mathcal{H}_{g,2} \subset \mathcal{H}_{g,2}^{rt}$. Thus the closure of $\mathcal{H}_{g,g^1_2}$ in $\mathcal{H}_{g,2}^{rt}$ is the Cartier divisor $\tau(\sigma)$ where $\sigma$ is the universal section.

This observation can be applied to compute the degree $\mathcal{H}_{g,g^1_2}$ on a family $(C \rightarrow S, \sigma_1, \sigma_2)$ where $S$ is a smooth projective which maps to $\mathcal{H}_{g,2}^{rt}$.

**Proposition 2.5.** Let $(C \rightarrow S, \sigma_1, \sigma_2)$ be a family as above and let $(C' \rightarrow S, \sigma)$ be the family obtained by deleting the section $\sigma_2$ and stabilizing. If $f: C \rightarrow C'$ is the stabilization map, then $\deg_S(\mathcal{H}_{g,1}^{rt}) = \deg \sigma^* f^*(\tau(\sigma))$ where $\tau$ is the hyperelliptic involution on $S$.

**Proof.** The map $S \rightarrow \mathcal{H}_{g,2}^{rt}$ factors as $S \xrightarrow{\sigma} C \xrightarrow{f} C' \rightarrow \mathcal{H}_{g,2}^{rt}$. By the discussion above, the pullback of $\mathcal{H}_{g,1}^{rt}$ to $C'$ is the Cartier divisor $\tau(\sigma)$. 

3. Divisor class groups of stacks of hyperelliptic curves

In this Section we recall from [2, Section 13.8] and [16] the description of the divisor class groups of $\mathcal{H}_g, \mathcal{H}_{g,1}, \mathcal{H}_{g,2}$.

### 3.1. The divisor class group of $\mathcal{H}_g$

The theory of admissible covers allows us to describe the boundary divisors of $\mathcal{H}_g$; i.e. the irreducible components of $\mathcal{H}_g \setminus \mathcal{H}_g$. See [2, Section 13.8] for a reference.

- $\eta_{rr}$: This parametrizes stable hyperelliptic curves $C$ with at least one non-separating node such that its partial normalization at such point does not have a separating node. A general curve in $\eta_{rr}$ has a single node. Its normalization is a smooth hyperelliptic curve of genus $g - 1$. If $\{p,q\}$ is the inverse image of the node, then $p + q$ equals $g^1_2$. (If $g = 2$ then the normalization has genus 1 and there is no condition on the points since any two points on an elliptic curve sum to a $g^1_2$.)

- $\delta_i$ for $1 \leq i \leq \lfloor \frac{g}{2} \rfloor$: For each $i$ this parametrizes curves $C$ with a disconnecting node such that the partial normalizations at the node is the disjoint union of a curve of genus $i$ and a curve of genus $g - i$. Moreover, such a node is fixed by the hyperelliptic involution of $C$, which means that each point in the inverse image of the node is a Weierstrass point on its corresponding component.
• \( \eta_i \) for \( 1 \leq i \leq \lfloor \frac{g-1}{2} \rfloor \): They parametrize curves \( C \) having two nodes that are conjugated by the hyperelliptic involution such that the partial normalization is the disjoint union of two curves of genera \( i \) and \( g-1-i \). The points of attachment on each component sum to a \( g^2_1 \).

**Theorem 3.1.** [2 Chapter XIII, Theorem (8.4)]. [4] For any \( g \geq 2 \), \( \text{Pic}(\overline{\mathcal{M}}_g) \mathbb{Q} \) is a vector space of dimension \( g \), freely generated by the classes \( \eta_{irr}, \{ \delta_i \}_{1 \leq i \leq \lfloor g/2 \rfloor} \), and \( \{ \eta_i \}_{1 \leq i \leq \lfloor (g-1)/2 \rfloor} \).

### 3.2. The divisor class groups of \( \mathcal{M}_{g,1} \) and \( \mathcal{M}_{g,2} \).

We begin by describing the inverse images of these boundary divisors under the projection \( \pi : \mathcal{M}_{g,1} \to \mathcal{M}_g \).

- \( \eta_{irr} \): The inverse image of \( \eta_{irr} \) is an irreducible component of \( \mathcal{M}_{g,1} \setminus \mathcal{M}_{g,1} \) and we again denote it by \( \eta_{irr} \).
- \( \delta_i \): If \( i < g/2 \) then the inverse image of \( \delta_i \) consists of two irreducible components \( \delta_{i,1} \) and \( \delta_{i,0} \) where the second index is one if the marked point is on the component of genus \( i \) and zero if the marked point is on the component of genus \( g-i \). If \( i = g/2 \) then the inverse image of \( \delta_i \) is irreducible.
- \( \eta_i \): If \( i < \frac{g-1}{2} \) then the inverse image of \( \eta_i \) again has two irreducible components \( \eta_{i,0} \) and \( \eta_{i,1} \) corresponding to whether the marked point is on the component of genus \( g-i-1 \) or the component of genus \( i \). When \( i = \frac{g-1}{2} \), then \( \eta_{i,0} = \eta_{i,1} \).

**Remark 3.2.** Note that the divisors \( \eta_{irr}, \delta_{i,0}, \delta_{i,1} \) are Cartier divisors because they can be identified with pullbacks of divisors on the smooth stacks \( \mathcal{M}_g \) and \( \mathcal{M}_{g,1} \) respectively. For \( 1 \leq i < g/2 \) the sum \( \eta_{i,0} + \eta_{i,1} \) is Cartier because it is the pullback of the Cartier divisor \( \eta_i \) on \( \mathcal{M}_g \). However, we do not know if each component is Cartier. A potential way to show that \( \mathcal{M}_{g,1} \) is singular is to show that the divisors \( \eta_{i,0} \) and \( \eta_{i,1} \) are not locally defined by a single equation along their intersection.

The boundary divisors of \( \mathcal{M}_{g,2} \) are defined in a similar manner. Again the inverse image of \( \eta_{irr} \) is irreducible but we need to keep track of the markings for the other components.

- \( \delta_i \): If \( i < g/2 \) then the inverse image of \( \delta_i \) consists of four irreducible components \( \delta_{i,0}, \delta_{i,2}, \delta_{i,(1)} \) and \( \delta_{i,(2)} \) defined as follows. The divisor \( \delta_{i,0} \) corresponds to both marked points being on the component of genus \( g-i \), \( \delta_{i,2} \) corresponds to both marked points being on the component of genus \( i \), and \( \delta_{i,(j)} \) for \( j = 1,2 \) means that the \( j \)th marked point is on the component of genus \( i \). We denote by \( \delta_{i,1} \) the sum \( \delta_{i,(1)} + \delta_{i,(2)} \). The divisor \( \delta_{i,1} \) is invariant under the involution of \( \mathcal{M}_{g,2} \) which exchanges the marked points. If \( i = \frac{g}{2} \) then the inverse image of \( \delta_i \) consists of two irreducible components \( \delta_{i,0} = \delta_{i,2} \) and \( \delta_{i,1} = \delta_{i,(1)} = \delta_{i,(2)} \).
- \( \eta_i \): If \( i < \frac{g-1}{2} \) then the inverse image of \( \eta_i \) again has four irreducible components \( \eta_{i,0}, \eta_{i,2}, \eta_{i,(1)}, \eta_{i,(2)} \) corresponding to the distribution of the marked points on the component of genus \( g-i-1 \) or the component of genus \( i \). Again we denote by \( \eta_{i,1} \) the sum \( \eta_{i,(1)} + \eta_{i,(2)} \) and the divisor \( \eta_{i,1} \) is invariant under the involution of \( \mathcal{M}_{g,2} \). When \( i = \frac{g-1}{2} \), then \( \eta_{i,0} = \eta_{i,2} \) and \( \eta_{i,1} = \eta_{i,(1)} = \eta_{i,(2)} \).
- \( \delta_{0,2} \): The general curve is a smooth hyperelliptic curve with a rational tail with the two marked points lying on it. Moreover, since a rational curve
with three distinct points does not move in moduli, in order to produce stratum $\delta_{0,2}$ with dimension $2g$, we impose no condition on the point on the genus $g$ component for attaching a rational tail with two marked points.

**Remark 3.3.** Again the divisors $\eta_{irr}, \delta_{i,j}$ are Cartier, but we do not know if the divisors $\eta_0, \eta_2, \eta_{i,(1)}$ and $\eta_{i,(2)}$ are Cartier along their intersections.

**Theorem 3.4.** [10] Theorem 1.1] $\text{Cl}(\Upsilon_{g,n})_\mathbb{Q} = \text{Pic}(\mathcal{U}_{g,n})_\mathbb{Q}$ is freely generated by the classes $\psi_1, \ldots, \psi_n$, and all boundary divisors, where the $\psi_i$ are the pullbacks to $\Upsilon_{g,n}$ of the corresponding $\psi$-classes$^1$ on $\mathcal{M}_{g,n}$.

By Theorem 3.4 we can write

\[
\Upsilon_{g,w} = \sum_{i=1}^{\lfloor g/2 \rfloor} [a_i,0 \delta_{i,0} + a_i,1 \delta_{i,1}] + \sum_{j=1}^{\lfloor (g-1)/2 \rfloor} [b_j,0 \eta_{j,0} + b_j,1 \eta_{j,1}] + c \eta_{irr} + d \psi
\]

for some rational numbers $a_i,0, a_i,1, b_j,0, b_j,1, c, d$ where $\psi = \psi_1 \in \text{Pic}(\Upsilon_{g,1})$. The divisor $\Upsilon_{g,g,1}$ is invariant under the involution that exchanges the marked points, so it can be expressed as a sum of invariant divisors

\[
\Upsilon_{g,g,1} = c \eta_{irr} + d(\psi_1 + \psi_2) + a_{0,2} \delta_{0,2} + \sum_{i=1}^{\lfloor 2/g \rfloor} [a_i,0 \delta_{i,0} + a_i,1 \delta_{i,1} + a_i,2 \delta_{i,2}] + \sum_{j=1}^{\lfloor (g-1)/2 \rfloor} [b_j,0 \eta_{j,0} + b_j,1 \eta_{j,1} + b_j,2 \eta_{j,2}]
\]

where the $a_{i,0}, a_{i,1}, a_{i,2}, b_{j,0}, b_{j,1}, b_{j,2}, c, d$ are rational numbers, and are unrelated to those of $\Upsilon_{g,1}$.

Our goal is to determine all the $\mathbb{Q}$-coefficients of these generators. We will use the method of test curves as has been used widely in the literature. To do this we will begin by determining how the divisors $\Upsilon_{g,w}$ and $\Upsilon_{g,g,1}$ intersect the boundary strata of $\Upsilon_{g,1}$ and $\Upsilon_{g,2}$ respectively. All of these assertions are easily checked using the the descriptions of the divisors $\Upsilon_{g,w}$ and $\Upsilon_{g,g,1}$ given in terms of admissible covers.

### 3.3. Intersection of $\Upsilon_{g,w}$ with the boundary of $\Upsilon_{g,1}$

We summarize our results as follows.

**Proposition 3.5.** $\Upsilon_{g,w}$ intersects each of the boundary divisors in codimension-one, and a general point of the intersection is described as follows.

1. The general point of the intersection $\Upsilon_{g,w}$ with $\eta_{irr}$ is an irreducible curve $C$ of arithmetic genus $g$ with a single node such that the marked point is a Weierstrass point of the normalization of $C$.
2. The general point of the intersection with $\delta_{1,1}$ is a general point of $\delta_1 \in \Upsilon_g$ with the marked point $p$ on the component $E$ of genus 1 having the property that $O(p-o)^2 = 0$ where $o \in E$ is the point of attachment of $E$.
3. The general point of the intersection with $\delta_{i,1}$ for $i > 1$ is a general point of $\delta_i$ with the marked point a Weierstrass point of the component of genus $i$.
4. The general point of the intersection with $\delta_{i,0}$ is a general point of $\delta_i$ with the marked point a Weierstrass point of the component of genus $g-i$.

1. Recall that the class $\psi_1$ is the line bundle on the stack $\mathcal{M}_{g,n}$ whose pullback to a pointed family of stable curves $(\mathcal{C} \twoheadrightarrow \mathcal{B}, \sigma_1, \ldots, \sigma_n)$ is the line bundle $\sigma_i^*\omega_{\mathcal{C}/\mathcal{B}}$. 

(5) The general point of the intersection with $\eta_{1,1}$ is a general point of $\eta_1$ with the marked point $p$ on the component of genus 1 satisfying the condition $\mathcal{O}(2p) = \mathcal{O}(o_1 + o_2)$ where $o_1, o_2$ are the points of attachment on the component of genus 1.

(6) The general point of the intersection with $\eta_{i,1}$ with $i > 1$ is a general point of $\eta_i$ with the marked point a Weierstrass point of the component of genus $i$.

(7) The general point of the intersection with $\eta_{i,0}$ is a general point of $\eta_i$ with the marked point being a Weierstrass point of the component of genus $g - i - 1$.

Proof. The proof can be illustrated by admissible covers. We consider degenerations of a $2g + 2$-pointed rational curve as a stable $2g + 2$-pointed rational curve with a single node, with various distributions of the $2g + 2$ points. When taking double cover, the single node may or may not be branched, depending on the parity of the number of markings on each component. For instance, if there are two points on one twig and $2g$ on the other, then the marked Weierstrass point must lie over one of the $2g$ points which proves (1). For any $i > 1$, if there are $2i + 1$ points on one twig and the rest $2g - 2i + 1$ on the other, then the single node is branched and thus there are two choices for the marked Weierstrass point, corresponding to $\delta_{i,0}$ and $\delta_{i,1}$. It proves (3) and (4). The rest follows similarly. □

3.4. Intersection of $\overline{\mathcal{H}}_{g, \frac{g_2}{2}}$ with the boundary of $\overline{\mathcal{H}}_{g,2}$.

Proposition 3.6. $\overline{\mathcal{H}}_{g, \frac{g_2}{2}}$ intersects each of the boundary divisors in codimension one, and a general point of the intersection is described as follows.

(1) The general point of the intersection $\overline{\mathcal{H}}_{g, \frac{g_2}{2}}$ with $\eta_{irr}$ is an irreducible curve $C$ of arithmetic genus $g$ with a single node such that the two marked points sum to a $g_2$ on the normalization of $C$.

(2) The general point of the intersection with $\delta_{0,2}$ is a curve where the point of attachment on the curve of genus $g$ is a Weierstrass point.

(3) The general point of the intersection with $\delta_{1,2}$ is a general point of $\delta_1 \in \overline{\mathcal{H}}_g$ with the marked points $p, q$ on the component of genus 1 having the property that $\mathcal{O}(p + q) = \mathcal{O}(2\sigma)$, where $\sigma$ is the point of attachment of the component of genus 1.

(4) The general point of the intersection with $\delta_{i,2}$ for $i > 1$ is a general point of $\delta_i$ with the marked points $p, q$ summing to a $g_2$ on the component of genus $i$.

(5) The general point of the intersection with $\delta_{i,0}$ is a general point of $\delta_i$ with the marked points summing to a $g_2$ on the component of genus $g - i$.

(6) The general point of the intersection with $\eta_{1,2}$ is a general point of $\eta_1$ with the marked points $p, q$ on the component of genus 1 satisfying the condition $\mathcal{O}(p + q) = \mathcal{O}(o_1 + o_2)$ where $o_1, o_2$ are the points of attachment on the component of genus 1.

(7) The general point of the intersection with $\eta_{i,2}$ for $i > 1$ is a general point of $\eta_i$ with the marked points summing to a $g_2$ on the component of genus $i$.

(8) The general point of the intersection with $\eta_{i,0}$ is a general point of $\eta_i$ with the marked points summing to a $g_2$ on the component of genus $g - i - 1$. 
Proof. The proof is similar to that of Proposition 3.3 except while taking the double cover, the pair of $g_2^1$ points lies over a point on a twig away from any of the $2g + 2$ markings.

4. Methods of constructing test curves

We describe four basic methods to construct complete one-parameter families of curves in $\mathcal{H}_{g,1}$, $\mathcal{H}_{g,2}$. All of our test curves will be based on these constructions and will lie in the smooth locus of $\mathcal{H}_{g,1}$ and $\mathcal{H}_{g,2}$ respectively.

4.1. Test curves with trivial moduli. The simplest examples of one-parameter families of pointed hyperelliptic curves are families $(X \to B, \sigma)$ where the section moves but the moduli of the curves does not vary. In this case the composite map $B \to \mathcal{H}_{g,1} \to \mathcal{H}_g$ is constant. The basic construction is as follows. Let $C$ be a fixed smooth hyperelliptic curve of genus $h$ and consider the family $C \times C \to C$ with the section $\Delta_C: C \to C \times C$ being the diagonal.

Choosing $k$-points $p_1, \ldots, p_k$ on $C$ we can obtain a family of hyperelliptic curves with $k + 1$ sections $X \to C$ where $X$ is the surface obtained by blowing up $C \times C$ at the points $(p_j, p_j)$ for $j = 1, \ldots, k$. The sections $\sigma_0, \sigma_1, \ldots, \sigma_k$ are the strict transforms of the sections $\Delta_C$, $\{p_1\} \times C, \ldots, \{p_k\} \times C$.

Using test curves of this form we can easily compute the coefficient $d$ of $\psi$ (respectively $\psi_1 + \psi_2$) in the expressions for $[\mathcal{H}_{g,w}]$ and $[\mathcal{H}_{g,g}]$ respectively.

4.2. A special pencil of pointed hyperelliptic curves with non-trivial moduli. Similar to the idea of varying an elliptic curve in an elliptic surface, we can also vary a hyperelliptic curve in some special pencil of hyperelliptic curves to obtain a family of pointed hyperelliptic curves with at worst a single node.

If $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth curve of bidegree $(2, g + 1)$, then by adjunction $g(C) = g$ and the projection $\pi_1: C \to \mathbb{P}^1$ expresses $C$ as a double cover; i.e. $C$ is hyperelliptic. Let $C \to \mathbb{P}^1$ be a general pencil of curves of bidegree $(2, g + 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$; i.e.

$$C = Z(\lambda G(x, y, u, v) + \mu H(x, y, u, v)) \subset \mathbb{P}^1_{\lambda,\mu} \times \mathbb{P}^1_{x,y} \times \mathbb{P}^1_{u,v}$$

where both $G$ and $H$ are some fixed general bihomogeneous polynomials of bidegree $(2, g + 1)$.

There are $4(g + 1)$ base points in this pencil corresponding to the intersection of the two curves defined by $G$ and $H$ respectively which define sections of the map $C \to \mathbb{P}^1_{\lambda,\mu}$. The projection to the second and third factors $C \to \mathbb{P}^1 \times \mathbb{P}^1$ realizes $C$ as the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at these $4(g + 1)$ points with the sections being the exceptional divisors. Let $\sigma: \mathbb{P}^1 \to C$ be the section determined by one of the base points.

Lemma 4.1.

$$\deg_{\mathbb{P}^1}(\mathcal{H}_{g,w}) = 1.$$  

Proof. The degree of $\mathcal{H}_{g,w}$ on this family can be calculated as $\deg \sigma^*(\tau^*O(\sigma))$ where $\tau$ is the hyperelliptic involution. The divisor $\sigma + \tau^*\sigma$ is the pullback of a divisor under the projection to $C \to \mathbb{P}^1_{x,y}$. Hence $(\sigma + \tau^*\sigma)^2 = 0$. On the other hand, $\sigma^2 = -1$ because $\sigma$ is an exceptional divisor of the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at the base point. Thus $(\tau^*\sigma)^2 = -1$ as well, so we conclude that $\sigma \cdot \tau^*\sigma = 1$; i.e. $\deg_{\mathbb{P}^1}(\mathcal{H}_{g,w}) = 1$. □
Lemma 4.2. The degree of $\eta_{\text{irr}}$ on this pencil is $4(2g + 1)$ and the degrees of all other boundary divisors are zero.

Proof. Since $\mathbb{P}^1 \times \mathbb{P}^1$ is a hypersurface in $\mathbb{P}^3$ the argument outlined in [10] Section 7.4.2 shows that a general pencil of curves of type $(a, b)$ has a finite number of nodal fibers and each fiber has a single node which is not one of the base points. It follows that our pencil hyperelliptic curves embedded as curves of degree $(2, g + 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ intersects the boundary of $\overline{\mathcal{H}}_{g,1}$ only in $\eta_{\text{irr}}$. The divisor $\eta_{\text{irr}}$ on this family is the pullback from $\mathcal{M}_{g,1}$ of the divisor $\delta_0$ because there are no fibers of type $\eta_i$ for $i > 0$. Given that only nodes appear in the singular fibers, the degree of this divisor is the degree of the discriminant of the linear systems $|O(2, g + 1)|$ on $\mathbb{P}^1 \times \mathbb{P}^1$. By [10] Proposition 7.9 this degree is $4(2g + 1)$. □

Using this special test curve with non-trivial moduli, we can easily get a $\mathbb{Q}$-linear relation between coefficients $d$ and $c$. We also get the similar statement for the 2-pointed case by choosing two base points in the pencil. The detailed calculations will be given in Section 5.1 and Section 5.5.

4.3. A general construction of pencils of pointed hyperelliptic curves with at worst nodes as singularities. To construct pencils of pointed hyperelliptic curves which have both non-trivial moduli and lie entirely in the boundary of $\overline{\mathcal{H}}_{g,1}$ or $\overline{\mathcal{H}}_{g,2}$, we want to attach a moving pointed family of hyperelliptic curves of genus $h$ to a fixed pointed hyperelliptic curve of genus $g - h$ or $g - h - 1$ depending on whether we want the pencil to lie in $\delta_h$ or $\eta_h$. For pencils lying in $\delta_h$ we need the attachment point to be a Weierstrass point, and for pencils lying in $\eta_h$ we need the two attachment points to sum to a $g_1^2$.

Unfortunately, the construction of Section 4.2 is not sufficiently flexible to allow us to make the section a Weierstrass section, or to mark two points which sum to a $g_1^2$. To rectify this problem we will consider a variant on the construction of the pencil shown in Section 4.2 which allows more flexibility for creating families with sections which have certain properties. Similar constructions have been previously considered in [2] Section 13.8 and [12] Section 3.2.

Any smooth hyperelliptic curve $C$ of genus $h$ can be embedded in the weighted projective space $\mathbb{P}(1, 1, h + 1)$ and expressed by an equation

$$z^2 = f(x, y) = f_0x^{2h+2} + f_1x^{2h+1}y + \cdots + f_{2h+2}y^{2h+2},$$

where the binary form $f(x, y)$ has distinct roots. Note that the weighted projective space $\mathbb{P}(1, 1, h + 1)$ is singular at the point $(0 : 0 : 1)$ but the image of $C$ misses this point.

Letting the coefficients $f_0, \ldots, f_{2h+2}$ vary in a one-parameter family produces a pencil of hyperelliptic curves embedded in $\mathbb{P}(1, 1, h + 1) \times \mathbb{P}^{2h+2}$. Since the discriminant locus of forms with multiple roots is an ample divisor, any complete curve necessarily contains singular curves, but we can arrange that the singular curves have a single node, corresponding to forms that have a single double root and all other roots simple. The basic pencil of hyperelliptic curves we consider is the pencil

$$X \subset \mathbb{P}(1, 1, h + 1) \times \mathbb{P}^1_{\lambda,\mu},$$

defined as

$$z^2 = f_{\lambda,\mu}(x, y) = \prod_{i=1}^{2}(a_i\lambda + b_i\mu)x + (c_i\lambda + d_i\mu)y \prod_{j=1}^{2h}(x + \alpha_j y),$$

where $a_i, b_i, c_i, d_i, \alpha_j$ are fixed constants in $k$. 
Lemma 4.3. The pencil $X \rightarrow \mathbb{P}^1$ intersects the divisor $\eta_{irr}$ at $4h + 2$ points, each of multiplicity 2.

Proof. It is clear that for general $a_i, b_i, c_i, d_i, \alpha_j$ there are $4h + 2$ values of $(\lambda : \mu)$ for which the form $f_{\lambda, \mu}$ has a single double root. On the other hand the pencil we constructed induces morphism $\mathbb{P}^1 \rightarrow \mathcal{M}_{X, 1}$ factors through an embedding of $\mathbb{P}^1 \rightarrow \mathbb{P}^{2h + 2}$ of degree 2. of degree two. The pullback of $\eta_{irr}$ to $\mathbb{P}^{2h + 2}$ is the discriminant which has degree $4h + 2$. Thus, the intersection of our pencil with the discriminant has degree $2(4h + 2)$. Moreover, the local structure of the intersection does not depend on the double point, so each intersection point necessarily has the same multiplicity 2.

Thus the family of hyperelliptic curves $X \rightarrow \mathbb{P}^1_{\lambda, \mu}$ has $4h + 2$ fibers each with a single node and the intersection with $\eta_{irr}$ has multiplicity two at each point of $\mathbb{P}^1_{\lambda, \mu}$ with a singular fiber. \qed

The binary form $f_{\lambda, \mu}(x, y)$ defines a divisor of degree $(2, 2h + 2)$ in $\mathbb{P}^1_{\lambda, \mu} \times \mathbb{P}^1_{x, y}$. Since this divisor is a square in Pic($\mathbb{P}^1 \times \mathbb{P}^1$) = $\mathbb{Z} \times \mathbb{Z}$, the equation $z^2 = f_{\lambda, \mu}(x, y)$ expresses the family $X \rightarrow \mathbb{P}^1$ as a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along the divisor $B_0 = Z(f_{\lambda, \mu}(x, y))$.

The divisor $B_0$ decomposes as a sum of $2h + 2$ irreducible components $\Delta_1 + \Delta_2 + \sigma_1 + \ldots + \sigma_{2h}$ which gives sections of the projection $\mathbb{P}^1_{\lambda, \mu} \times \mathbb{P}^1_{x, y} \rightarrow \mathbb{P}^1_{\lambda, \mu}$. The sections $\Delta_1, \Delta_2$ are given by the formula

\[(\lambda : \mu) \mapsto (- (c_i \lambda + d_i \mu) : (a_i \lambda + b_i \mu))\]

for $i = 1, 2$ and have degree $(1, 1)$. The sections $\sigma_i$ are the constant sections defined by $(\lambda : \mu) \mapsto (- \alpha_j : 1)$.

Since the sections intersect transversally in $4h + 2$ points, the total space $X$ has $(4h + 2)$ $A_1$-singularities at each of the nodes in the fibers of the map $X \rightarrow \mathbb{P}^1_{\lambda, \mu}$.

Let $P$ be the blow up of the surface $\mathbb{P}^1_{\lambda, \mu} \times \mathbb{P}^1_{x, y}$ at the $4h + 2$ intersection points of the irreducible components of $B_0$. The strict transforms $\tilde{\Delta}_1, \tilde{\Delta}_2, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_{2h}$ of the sections $\Delta_1, \Delta_2, \sigma_1, \ldots, \sigma_{2h}$ make $P \rightarrow \mathbb{P}^1_{\lambda, \mu}$ into a family of $(2h + 2)$-pointed rational curves. Let $\pi : \tilde{X} \rightarrow P$ be the double cover of $P$ branched along the divisor $\tilde{B}_0 = \tilde{\Delta}_1 + \tilde{\Delta}_2 + \tilde{\sigma}_1 + \ldots + \tilde{\sigma}_{2h}$. Then the data $(\pi : \tilde{X} \rightarrow P, \tilde{B}_0)$ is an admissible covering whose stabilization is our original family of hyperelliptic curves $X \rightarrow \mathbb{P}^1_{\lambda, \mu}$.

Here are some intersection multiplicities that we will use quite often in later computations:

\[(\sigma_i \cdot \sigma_i)_{\mathbb{P}^1 \times \mathbb{P}^1} = 0, \quad (\Delta_j \cdot \Delta_j)_{\mathbb{P}^1 \times \mathbb{P}^1} = 2.\]

Since each $\sigma_i$ gets blown up twice; each $\Delta_j$ gets blown up $(2h + 2)$ times and we have

\[(\tilde{\sigma}_i \cdot \tilde{\sigma}_i)_{\tilde{X}} = -2, \quad (\tilde{\Delta}_j \cdot \tilde{\Delta}_j)_{\tilde{X}} = -2h.\]

After taking the double cover $\tilde{X}$, since all $\tilde{\sigma}_i$’s and $\tilde{\Delta}_j$’s are contained in the branch locus $\tilde{B}_0$, we have

\[(s_i \cdot s_i)_{\tilde{X}} = -1, \quad (t_j \cdot t_j)_{\tilde{X}} = -h,\]

where $s_i = \pi^{-1}(\tilde{\sigma}_i)$ and $t_j = \pi^{-1}(\tilde{\Delta}_j)$, with $\pi^*(\tilde{\sigma}_i) = 2s_i$ and $\pi^*(\tilde{\Delta}_j) = 2t_j$.

Our main purpose for using this construction (see Figure 2) is to obtain Weierstrass sections of the family of hyperelliptic curves $f : \tilde{X} \rightarrow \mathbb{P}^1$ which can be used to attach another component or for marking a Weierstrass point. We will use the following four ways to get sections.
In order to get a Weierstrass section either for marking, or for attaching to another component we can do the following:

1. Choose the section $\tau: \mathbb{P}^1 \rightarrow \tilde{X}$ to be the inverse image of any of the sections $\tilde{\sigma}_i$ under the double cover map $\tilde{X}$. By the above calculation $\tau^2 = -1$ in $\tilde{X}$.

2. Choose the section $\theta: \mathbb{P}^1 \rightarrow \tilde{X}$ to be the inverse image of either section $\tilde{\Delta}_i$, which lies in the branch locus $\tilde{B}_0$. By the above calculation $\theta^2 = -h$.

In order to get two sections adding up to $g_{1/2}$ we use two constructions. These families can be viewed as either being entirely contained in $\mathcal{H}_{g, g_{1/2}}$ or they can be used for gluing to produce pencils contained in $\eta_i$ for some $i$.

1. Start with a general horizontal ruling of $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Since it intersects the branch locus at two distinct points, after taking the double cover, it becomes a double cover of $\mathbb{P}^1$ branched at two points. The family obtained by pullback along this double cover map is a family of hyperelliptic curves with two sections entirely contained in the divisor $\mathcal{H}_{g, g_{1/2}}$, and each section is the inverse image of a smooth point of the fiber of $X \rightarrow \mathbb{P}^1$ under the morphism which forgets the section.

2. Start with a horizontal ruling which intersects $\Delta_1$ and $\Delta_2$ at a common point. The inverse image of this ruling is a singular divisor with two irreducible components. After blowing up the point of intersection, we obtain a family with two sections, again entirely contained in $\mathcal{H}_{g, g_{1/2}}$.

To get the final expressions in Theorem 1.1 and 1.2, only a few of the above variants are needed.

4.4. The test family $F_{2i+1}$. Here we briefly recall the construction of $F_{2i+1}$ in \cite{2} Section 13.8 that produces a one-parameter family of hyperelliptic curves whose general fiber is smooth and whose singular fibers either have a node of type $\eta_{irr}$ or are in the divisor $\eta_i$.

Start with $2i+2$ divisors in $|\mathcal{O}(1, 1)|$ passing through a common point $p \in \mathbb{P}^1 \times \mathbb{P}^1$, and $2g-2i$ divisors in $|\mathcal{O}(1, 0)|$. After resolving the singularities in the branch locus we are allowed to take a double cover of the blown up surface $\tilde{P}$ since the branch divisor is a square. After stabilizing the double cover we will obtain a family with $(i+1)(4g-2i+1)$ fibers each with a single non-disconnecting node and one fiber of type $\eta_i$. As was the case for our previous family, each fiber with an ordinary node
contributes to $\eta_{irr}$ with multiplicity two. In particular, we have
\[
\deg_{F_{2i+1}}(\eta_i) = 2,
\deg_{F_{2i+1}}(\eta_{irr}) = 2(i + 1)(4g - 2i + 1).
\]
Taking the inverse images of the strict transforms to $\tilde{P}$ of other sections in $\mathbb{P}^1 \times \mathbb{P}^1$ gives a method of producing families with sections and specified fiber types. If the section is a component of the branch divisor, then we obtain a family with Weierstrass section; if the section is not contained in the branch divisor, we obtain a multi-section. After a degree 2 base change to the section, we obtain a family with two sections which sum to a $g_2^1$.

5. Computations

In this section, we use the methods introduced in Section 4 to prove Theorem 1.1 and Theorem 1.2.

5.1. Proof of Theorem 1.1 We will use the method of test curves to get express $[\mathcal{H}_{g,w}]$ in terms of a basis for $\text{Cl}(\mathcal{H}_{g,1}) \otimes \mathbb{Q}$ given in Theorem 3.4.

5.1.1. The first test curve – calculating the coefficient of $\psi$ in (3.1).

Proposition 5.1. The coefficient $d$ of $\psi$ in equation (3.1) equals \( \frac{g + 1}{g - 1} \).

Proof. Let $C$ be a fixed smooth hyperelliptic curve and consider the family of pointed curves over $C$, defined by $(C \times C \to C, \Delta)$ where $\Delta$ denotes the diagonal section. Since all fibers of this family are smooth we know that all boundary divisors vanish on $C$. Also $\deg_C([\mathcal{H}_{g,w}]) = 2g + 2$ since the curve $C$ has $2g + 2$ Weierstrass points. Since the section is the diagonal, $\deg_C(\psi) = -(\Delta)^2 = 2g - 2$. Hence $2g + 2 = (2g - 2)d$ or equivalently, $d = \frac{g + 1}{g - 1}$. \(\square\)

5.1.2. Calculation of the coefficient of $\eta_{irr}$ in (3.1).

Proposition 5.2. The coefficient $c$ of $\eta_{irr}$ in equation (3.1) equals \( \frac{1}{2(2g+1)(g-1)} \).

Proof. Let $(X \to \mathbb{P}^1, \sigma)$ be the pencil of hyperelliptic curves constructed in Section 4.2. By Lemma 4.2, we know the degree of the boundary class $\eta_{irr}$ on this family is
\[
\deg_{\mathbb{P}^1}(\eta_{irr}) = 4(2g + 1).
\]
Furthermore, since the marked point gets blown up once,
\[
\deg_{\mathbb{P}^1}(\psi) = -(\sigma)^2 = 1.
\]
Together with the fact that this family intersects other boundary divisors trivially, we obtain the following relation:
\[
1 = 4(2g + 1)c + d, \quad \text{where } d = \frac{g + 1}{g - 1},
\]
which implies that $c = \frac{-1}{2(2g+1)(g-1)}$ in (3.1). \(\square\)
5.1.3. Calculation the coefficients $a_{i,0}$, $a_{i,1}$, $b_{i,0}$ and $b_{i,1}$.

Proposition 5.3. For $1 \leq i \leq \lfloor g/2 \rfloor$, the coefficients $a_{i,0}$ and $a_{i,1}$ of $\delta_{i,0}$ and $\delta_{i,1}$ in (5.1) respectively, are

\begin{equation}
  a_{i,0} = -\frac{2i(2i + 1)}{(2g + 1)(g - 1)^2},
\end{equation}

\begin{equation}
  a_{i,1} = -\frac{2(g - i)(2g - i + 1)}{(2g + 1)(g - 1)}.\end{equation}

For $1 \leq i \leq \lfloor (g - 1)/2 \rfloor$, the coefficients $b_{i,0}$ and $b_{i,1}$ of $\eta_{i,0}$ and $\eta_{i,1}$ respectively, are

\begin{equation}
  b_{i,0} = -\frac{(2i + 1)(i + 1)}{(2g + 1)(g - 1)},
\end{equation}

\begin{equation}
  b_{i,1} = -\frac{2(g - i)(g - i - 1)}{(2g + 1)(g - 1)}.\end{equation}

Proof. We will compute these coefficients using the construction introduced in Section 4.3 to produce test curves contained in the boundary divisor $\delta_{i,0}$ and $\delta_{i,1}$ for $1 \leq i \leq \lfloor g/2 \rfloor$ which miss the Weierstrass divisor $\eta_{g,w}$.

(i) Let $\bar{X} \to \mathbb{P}^1$ be a family of admissible covers of genus $i$ with Weierstrass section $\tau$ as in family (I). We form a family of pointed hyperelliptic curves contained in the divisor $\delta_{i,0}$ by attaching a fixed hyperelliptic curve of genus $g - i$ at a Weierstrass point and choosing a general point for the marking. In this family we have three types of fibers as illustrated in Figure 3.

Note that the fibers of the middle type in Figure 3 are contained in the boundary divisors $\delta_{i,0}$ and $\eta_{i-1,0}$, but not in $\eta_{i\text{err}}$, since if they were contained in $\eta_{i\text{err}}$, they should appear when the Weierstrass point collides with a pair of points summing to the $g^1_2$. The only possible way this occurs is when the pair of points summing to the $g^1_2$ collide at a Weierstrass point. If this were the case the stable curve would necessarily have an elliptic component. The image of our family $\mathbb{P}^1$ in $\overline{\mathcal{M}}_{g,1}$ misses the intersection of the divisor $\eta_{i-1,0} \cap \eta_{i-1,1}$ so we may view $\eta_{i-1,0}$ as a Cartier divisor and define its degree on our family. Moreover we claim that the divisor $\eta_{i-1,0}$ intersects this family $\mathbb{P}^1$ transversally. To see this we...
argue as follows. The morphism $\mathbb{P}^1 \to \mathcal{H}_{g,1}$ factors through the clutching morphism $\mathcal{H}_{i,w} \times \mathcal{H}_{g-i,w} \to \delta_{i,1} \subset \mathcal{H}_{g,1}$. Let $D \subset \mathcal{H}_{i,w}$ be divisor whose general point parametrizes a Weierstrass pointed nodal curve where the marking is on a rational bridge as illustrated in Figure 4.

The first factor of the map $\mathbb{P}^1 \to \mathcal{H}_{g-i,w} \times \mathcal{H}_{i,w}$ is constant and the fibers of type $\eta_{i-r} \cdot 0$ are obtained by attaching a fixed curve in $\mathcal{H}_{g-i,w}$ to a moving curve in $\mathcal{H}_{i,w}$. It follows that the pullback of $\eta_{i-r} \cdot 0$ to $\mathbb{P}^1$ equals the pullback of the divisor $D$ along the morphism $\mathbb{P}^1 \to \mathcal{H}_{i,w}$ induced by the Weierstrass pointed family $\tilde{X} \to \mathbb{P}^1 (1)$.

To show that the intersection of our family with $D$ is transverse we need to understand the local structure of $\mathcal{H}_{i,w}$ in a neighborhood of a point of $D$. As noted in the proof of Proposition 2.4 the stack $\mathcal{H}_{i,w}$ can be identified with the quotient stack $[\mathbb{H}_{sm}(2i+2,w)/B]$ (see also [7]) where $\mathbb{H}_{sm}(2i+2,w)$ is the space of binary forms of degree $2i+2$ with distinct roots and which vanish at $(0 : 1)$. The map $f \mapsto x_0 f$ identifies $\mathbb{H}_{sm}(2i+2,w)$ with $\mathbb{A}_{sm}(2i+1) \setminus L$ where $\mathbb{A}_{sm}(2i+1)$ is the space of binary forms of degree $2i+1$ with distinct roots and $L$ is the hyperplane parametrizing forms which vanish at $(0 : 1)$. The quotient $[\mathbb{A}_{sm}(2i+1)/B]$ is a partial compactification of $\mathcal{H}_{i,w}$ which includes the Weierstrass pointed curves of type $D$ and the divisor $D$ is the quotient $[L/B]$.

For a suitable choice of coordinates on $\mathbb{P}^1$ the stabilization of the family of admissible covers $\tilde{X} \to \mathbb{P}^1 (1)$ is given by the equation

$$z^2 = x(\lambda x + \mu y)(\mu x + \lambda y) \prod_{i=4}^{2i+2} (a_i x + b_i y)$$

where $a_i, b_i$ are general constants and $(\lambda : \mu)$ are coordinates on $\mathbb{P}^1$. With this choice of coordinates the map $\mathbb{P}^1 \to \mathcal{H}_{i,w}$ factors through a map $\mathbb{P}^1 \to \mathbb{A}_{sm}(2i+1)$ given by $(\lambda : \mu) \mapsto (\lambda x + \mu y)(\mu x + \lambda y) \prod_{i=4}^{2i+2} (a_i x + b_i y)$. Direct inspection shows that this $\mathbb{P}^1$ intersects the divisor $L$ transversally at the image of the points $(\lambda : \mu) = (0 : 1)$ and $(\lambda : \mu) = (1 : 0)$ thereby demonstrating our transversality assertion.

Therefore

\begin{align}
\text{deg}_{\mathbb{P}^1}(\eta_{i-r}) &= 2(4i), \\
\text{deg}_{\mathbb{P}^1}(\delta_{i,0}) &= r^2 = -1,
\end{align}

(5.5) \hspace{1cm} (5.6)

\begin{align}
\text{deg}_{\mathbb{P}^1}(\eta_{i-1,0}) &= 2,
\end{align}
where the multiple 2 in equation (5.5) is due to Lemma 4.3.

Since the marked point is never a Weierstrass point we have \( \deg_{P_1} [H_{g,w}] = 0 \). Also, the section is constant so \( \deg_B \psi = 0 \). Thus we obtain the following relation

\[
0 = -a_{i,0} + 2(4i)c + 2b_{i-1,0}.
\]

Switching the genera of these two components gives the following relation

\[
0 = -a_{i,1} + 2[4(g - i)]c + 2b_{i,1}.
\]

(ii) Now let \( \tilde{X} \to \mathbb{P}^1 \) be a family of admissible covers of genus \( i \) with Weierstrass section \( \theta \) as in family (2). Again, we form a family of pointed hyperelliptic curves contained in the divisor \( \delta_{i,0} \) by attaching a fixed hyperelliptic curve of genus \( g - i \) at a Weierstrass point and choosing an arbitrary point for the marking. Again \( \deg_{P_1} [H_{g,w}] = 0 \) and \( \deg_{P_1} \psi = 0 \) and we have the following:

\[
\begin{align*}
\deg_{P_1}(\eta_{i,r}) &= 2(2i), \\
\deg_{P_1}(\delta_{i,0}) &= \theta^2 = -i, \\
\deg_{P_1}(\eta_{i-1,0}) &= 2i + 2.
\end{align*}
\]

These give the relation:

\[
0 = 2(2i)c - ia_{i,0} + (2i + 2)b_{i-1,0}.
\]

Similarly, by symmetry,

\[
0 = 2[2(g - i)]c - (g - i)a_{i,1} + [2(g - i) + 2]b_{i,1}.
\]

Using these relations (5.7), (5.8), (5.9) and (5.10), we get the formula for the coefficients of \( \delta_{i,0}, \delta_{i,1} \) and \( \eta_{i,0}, \eta_{i,1} \):

\[
\begin{align*}
a_{i,0} &= \frac{2i(2i + 1)}{(2g + 1)(g - 1)}, \\
a_{i,1} &= \frac{2(g - i)(2g - i + 1)}{(2g + 1)(g - 1)}, \\
b_{i,0} &= \frac{(i + 1)(2i + 1)}{(2g + 1)(g - 1)}, \\
b_{i,1} &= \frac{(g - i)[2(g - i) - 1]}{(2g + 1)(g - 1)}.
\end{align*}
\]

which prove the proposition. \( \square \)

Therefore, together with the results from the previous sections, we have proved Theorem 1.1.

5.2. **Proof of Theorem 1.2.** The expression of \( \mathcal{F}_{g,2} \) will be computed in terms of a basis for \( \text{Cl}(\mathcal{F}_{g,2}) \otimes \mathbb{Q} \) given in Theorem 3.4. Recall that there is an extra boundary divisor \( \delta_{0,2} \) in \( \mathcal{F}_{g,2} \).
5.2.1. The first and second test curves – calculating the coefficients \( a_{0,2} \) and \( d \) in (3.2).

Proposition 5.4. The coefficient \( d \) of \( \psi_1 + \psi_2 \) equals \( \frac{1}{g-1} \) and the coefficient \( a_{0,2} \) of \( \delta_{0,2} \) equals \( -\frac{2+1}{g-1} \) in equation (5.2).

Proof. We will consider two families which intersect the boundary only in \( \delta_{0,2} \) and use it to obtain two independent relations between \( a_{0,2} \) and \( d \).

Let \( C \) be a fixed smooth hyperelliptic curve of genus \( g \), and let \( p \in C \) be a non-Weierstrass point. Let \( \widetilde{C} \times C \) be the blowup of \( C \) at the point \((p,p)\) and let \( \tilde{\sigma} \) and \( \tilde{\Delta} \) be the strict transforms of \( C \times \{p\} \) and the diagonal respectively. The family \( \widetilde{C} \times C \to C \) with the sections \( \sigma_1 = \tilde{\sigma} \), and \( \sigma_2 = \tilde{\Delta} \) is a family of two-pointed hyperelliptic curves.

All fibers are non-singular except for the fiber over the point \( p \in C \) which is in \( \delta_{0,2} \). Thus, \( \deg_C(\delta_{0,2}) = 1 \) and all other boundary divisors have degree 0. Since \( p \) is not a Weierstrass point, there is a unique point \( q \in C \) such that \( p + q \) is a \( g_1^1 \) hence \( \deg_C(\mathcal{H}_{g_1^1}) = 1 \) by Proposition 2.3. Also note that \( \tilde{\sigma}^2 = \sigma^2 - 1 = -1 \) so \( \deg_C(\psi_1) = 1 \), and \( \tilde{\Delta}^2 = \Delta^2 - 1 = 1 - 2g \) so \( \deg_C(\psi_2) = 2g - 1 \). Putting these together we obtain the relation

\[
(5.11) \quad 1 = a_{0,2} + d + (2g - 1)d = a_{0,2} + d(2g).
\]

Next let \( \rho: \widetilde{C} \times C \to C \times C \) be the blowup of the \( 2g + 2 \) points \((p,p)\) where \( p \) is a Weierstrass point of \( C \). Let \( \sigma_1 \) be the strict transform of the diagonal and \( \sigma_2 \) be the strict transform of the section of \( C \times C \to C \) given by \( x \mapsto (x, \tau(x)) \) where \( \tau \) is the hyperelliptic involution.

Since the blowup map \( \rho \) has \( (2g + 2) \) exceptional curves \( E_1, \ldots, E_{2g+2} \) each with self-intersection \(-1\), we have that \( \sigma_1^2 = \sigma_2^2 = 2 - 2g - (2g + 2) = -4g \). Hence, \( \deg_C(\psi_1) = \deg_C(\psi_2) = 4g \).

The family \( \widetilde{C} \times C \to C \) is contained entirely in \( \mathcal{H}_{g_1^1} \). Since \( \sigma_2 \) is obtained from \( \sigma_1 \) by the hyperelliptic involution, Proposition 2.3 implies that \( \deg_C(\mathcal{H}_{g_1^1}) = \sigma_1 \cdot (p^* \Delta) = \sigma_1 \cdot (\sigma_1 + E_1 + \ldots E_{2g+2}) = 2 - 2g \).

Finally our family has \( 2g + 2 \) fibers of type \( \delta_{0,2} \) so \( \deg_C(\delta_{0,2}) = 2g + 2 \). This yields the relation

\[
(5.12) \quad 2 - 2g = (2g + 2)a_{0,2} + 8gd.
\]

Combining (5.11) and (5.12) proves the proposition. \( \square \)

5.2.2. Calculating the coefficient of \( \eta_{\text{irr}} \) in (3.2).

Proposition 5.5. The coefficient \( c \) for \( \eta_{\text{irr}} \) in (3.2) equals \( \frac{1}{2(g-1)(2g+1)} \).

Proof. We use the construction in Section 4.2 where we take two of the \( 4(g + 1) \) base points which we denote by \( p_1, p_2 \) to produce two sections \( \sigma_1, \sigma_2 \) of the pencil. Since the pencil \( C \to \mathbb{P}_1 \) is general, the two points \( p_1, p_2 \) are not contained in any ruling of the quadric \( Q = \mathbb{P}^1 \times \mathbb{P}^1 \). It means that no section of a \( g_1^1 \) contains both
and $p_2$. Thus $\deg_{p_1}([\mathcal{H}_{g,g_1^2}]) = 0$. We also have:
\[
\begin{align*}
\deg_{\eta}(\eta_{irr}) &= 4(2g + 1), \\
\deg_{p_1}(\psi_1) &= -\sigma_1^2 = 1, \\
\deg_{p_1}(\psi_2) &= -\sigma_2^2 = 1,
\end{align*}
\]
which gives us the relation
\[0 = 4(2g + 1)c + 2d. \tag{5.13}\]
Since $d = \frac{1}{g-1}$ we obtain the desired formula for $c$. \hfill \square

5.2.3. Calculation of the remaining coefficients in (3.2). We use the same construction as in the proof of Proposition 5.3 of families of hyperelliptic curves of genus $i$ (resp. genus $g-i$) with a Weierstrass section except that we attach a fixed two-pointed curve of genus $g-i$ (resp. genus $i$) along the Weierstrass section. Since we can ensure that the marked points on the fixed curve do not sum to a $g_1^2$ we can assume that our family misses $\mathcal{F}_{g,g_1^2}$ and we obtain the following relations.

When the Weierstrass section is obtained from a horizontal ruling we have:
\[
\begin{align*}
0 &= -a_{i,0} + 2(4i)c + 2b_{i-1,0}, \\
0 &= -a_{i,2} + 2[4(g-i)c + 2b_{i}]. \tag{5.14}
\end{align*}
\]
When the Weierstrass section is obtained from a ruling of degree $(1,1)$ we get the following relations:
\[
\begin{align*}
0 &= -ia_{i,0} + 2(2i)c + 2b_{i-1,0}, \\
0 &= -(g-i)a_{i,2} + 2 \cdot 2(g-i)c + 2(g-i+1)b_{i}. \tag{5.15}
\end{align*}
\]
By relations (5.14) and (5.16), together with Proposition 5.5, we obtain
\[
\begin{align*}
a_{i,0} &= \frac{2i(2i+1)}{(g-1)(2g+1)}, \\
b_{i,0} &= \frac{(i+1)(2i+1)}{(g-1)(2g+1)}. \tag{5.18}
\end{align*}
\]
Likewise, by relations (5.15) and (5.17), together with Proposition 5.5, we obtain
\[
\begin{align*}
a_{i,2} &= \frac{2(g-i)[2(g-i)+1]}{(g-1)(2g+1)}, \\
b_{i,2} &= \frac{(g-i)[2(g-i)-1]}{(g-1)(2g+1)}. \tag{5.20}
\end{align*}
\]
Now let’s complete the calculation by finding the coefficients $a_{i,1}$ and $b_{i,1}$ of the classes $\delta_{i,1}$ and $\eta_{i,1}$ respectively.

**Proposition 5.6.** For $1 \leq i \leq \lfloor g/2 \rfloor$, the coefficient $a_{i,1}$ of $\delta_{i,1}$ in (3.2) is
\[a_{i,1} = \frac{(2i-1)[2(g-i)-1] - 2}{(g-1)(2g+2)}. \tag{5.22}\]
For $1 \leq i \leq \lfloor (g-1)/2 \rfloor$, the coefficient $b_{i,1}$ of $\eta_{i,1}$ in (3.2) is
\[b_{i,1} = \frac{2i(g-i-1) - 1}{(g-1)(2g+1)}. \tag{5.23}\]
Proof. We start with the basic construction introduced in Section 1.3. We will take an extra general horizontal ruling $\sigma_0 \in |\mathcal{O}(1,0)|$ for the marked point on the genus $i$ component, and again take $\sigma_1$ which is contained in the branch locus to be the Weierstrass section for gluing. After taking the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$, the inverse image $s_0$ of $\sigma_0$ becomes a double cover of $\mathbb{P}^1$ branched at 2 points. Thus, after a necessary degree 2 base change to $s_0$ we obtain a stable one-parameter family of hyperelliptic curves of genus $i$ over the base $\mathbb{P}^1$ which is the double cover of $\mathbb{P}^1$ branched at two points. This family is equipped with 2 sections, one of which is a Weierstrass section. Then we attach each curve in this family to a fixed smooth genus $g - i$ at a Weierstrass point and mark an additional arbitrary point on the component of genus $g - i$. The result of this construction is a one-parameter family of stable genus $g$ hyperelliptic curves with two marked points – one on the component of genus $i$ and the other on the component of genus $g - i$. This family has empty intersection with the divisor $\mathcal{P}_{g,g_i}$ and we deduce the following relation among $c, d$ and $a_i, b_{i-1,1}$. The computation of the degrees is similar to that of Proposition 5.3.

\begin{equation}
-2a_i + 4b_{i-1,1} + 2 \cdot 2(4i)c + 2d = 0.
\end{equation}

By switching the genera, we get the symmetric relation

\begin{equation}
-2a_i + 4b_{i+1} + 2 \cdot 2 \cdot 4(g - i)c + 2d = 0.
\end{equation}

There is a second family of this form where we use the diagonal section of the Weierstrass divisor for gluing. Unfortunately, this section will hit the node of one special fiber producing after blowup a family with a rational bridge. Since we cannot ensure that this family is contained in the smooth locus of $\mathcal{P}_{g,2}$, we cannot compute the degrees of various divisors on the family.

To remedy this problem we use the family $F_{2i+1}$ of [2, Section 13.8] which was described in Section 1.3. To obtain a family with two sections we consider the inverse image of a general section $\sigma_0 \in |\mathcal{O}(1,0)|$ and a special section $\sigma_0' \in |\mathcal{O}(1,0)|$ passing the common point where $2i + 2$ divisors in the linear system $|\mathcal{O}(1,1)|$ meet. After resolving the singularities in the branch locus and then taking the double cover, the inverse image of $\sigma_0$, denoted by $s_0$, is now a double cover of $\mathbb{P}^1$ branched at $2i + 2$ points and the inverse image of $\sigma_0'$ now gets separated into two disjoint lines passing through the exceptional divisor, either one of which is denoted by $s_0'$. It can be checked that $s_0', s_0 = -1$.

After a degree 2 base change to $s_0$, we obtain a one-parameter family with base $B := s_0$ of pointed genus $g$ hyperelliptic curves with sections. Notice that $s_0$ is a smooth genus $i$ curve, and its inverse image under the base change consists of two components, which we denote by $s_0^{(1)}$ and $s_0^{(2)}$, satisfying $s_0^{(1)} \cdot s_0^{(2)} = 2i + 2$. By easy computation on $(s_0^{(1)} + s_0^{(2)})^2 = 2s_0^2 = 0$, we get $s_0^{(j)} \cdot s_0^{(j)} = -(2i + 2)$ for $j = 1, 2$. Choosing one section to be either $s_0^{(1)}$ or $s_0^{(2)}$, and one section to be the inverse image of $s_0'$, we obtain a two-pointed family of stable hyperelliptic curves contained in the smooth locus of $\mathcal{P}_{g,2}$ which has empty intersection with $\mathcal{P}_{g,g_i}$. On this family, together with the results from [2, Section 13.8] stated in Section 4.3.
we have
\[
\deg_B(\psi_1) = 2(i + 1), \\
\deg_B(\psi_2) = 2, \\
\deg_B(\eta_{i,1}) = 2, \\
\deg_B(\eta_{irr}) = 2 \cdot 2(i + 1)(4g - 2i + 1),
\]
which yields the relation
\[(5.26) \quad b_{i,1} + 2(i + 1)(4g - 2i + 1)c + (i + 2)d = 0.\]
Together with the facts from Proposition 5.5 and 5.4, we get
\[
b_{i,1} = \frac{2i(g - i - 1) - 1}{(g - 1)(2g + 1)}.\]
Using either one of relations mentioned above involving \(a_{i,1}\), we get
\[
a_{i,1} = \frac{(2i - 1)[2(g - i) - 1] - 2}{(g - 1)(2g + 2)},
\]
which finishes the proof.

Combining the results from Proposition 5.5, 5.4 and 5.6, with (5.18), (5.20), (5.19), and (5.21), we complete the proof of Theorem 1.2.

References

[1] Dan Abramovich, Alessio Corti, and Angelo Vistoli, Twisted bundles and admissible covers, 2003, pp. 3547–3618. Special issue in honor of Steven L. Kleiman. MR2007376
[2] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths, Geometry of algebraic curves. Volume II, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 268, Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris. MR2807457
[3] Ignacio Barros and Scott Mullane, The Kodaira classification of the moduli of hyperelliptic curves, arXiv preprint arXiv:2106.13774 (2021).
[4] Maurizio Cornalba, The Picard group of the moduli stack of stable hyperelliptic curves, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 18 (2007), no. 1, 109–115. MR2314467
[5] Fernando Cukierman, Families of Weierstrass points, Duke Math. J. 58 (1989), no. 2, 317–346. MR1016424
[6] Steven Diaz, Exceptional Weierstrass points and the divisor on moduli space that they define, Mem. Amer. Math. Soc. 56 (1985), no. 327, iv+69. MR791679
[7] Dan Edidin and Zhengning Hu, The integral Chow rings of the stacks of hyperelliptic Weierstrass points, preprint (2022).
[8] David Eisenbud and Joe Harris, Irreducibility and monodromy of some families of linear series, Ann. Sci. École Norm. Sup. (4) 20 (1987), no. 1, 65–87. MR892142
[9] ________, The Kodaira dimension of the moduli space of curves of genus \(\geq 23\), Invent. Math. 90 (1987), no. 2, 359–387. MR910206
[10] ________, 3264 and all that—a second course in algebraic geometry, Cambridge University Press, Cambridge, 2016. MR3617981
[11] Joe Harris and David Mumford, On the Kodaira dimension of the moduli space of curves, Invent. Math. 67 (1982), no. 1, 23–88. With an appendix by William Fulton. MR664324
[12] Eric Larson, The integral Chow ring of \(\overline{M}_2\), Algebr. Geom. 8 (2021), no. 3, 286–318. MR4206438
[13] Knud Lønsted and Steven L. Kleiman, Basics on families of hyperelliptic curves, Compositio Math. 38 (1979), no. 1, 83–111. MR523266
[14] Michele Pernice, *The Integral Chow Ring of the Stack of 1-Pointed Hyperelliptic Curves*, International Mathematics Research Notices (2021), available at https://academic.oup.com/imrn/advance-article-pdf/doi/10.1093/imrn/rnab072/37038844/rnab072.pdf.

[15] Matthieu Romagny, *Group actions on stacks and applications*, Michigan Math. J. 53 (2005), no. 1, 209–236. MR2125542

[16] Federico Scavia, *Rational Picard group of moduli of pointed hyperelliptic curves*, Int. Math. Res. Not. IMRN 21 (2020), 8027–8056. MR4184614

Department of Mathematics, University of Missouri, Columbia, MO 65211

Email address: edidind@missouri.edu

Email address: zhengning.hu@mail.missouri.edu