Precise Time Evolution of Superconductive Phase Qubits

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New procedure on precise analysis of superconducting phase qubits using the concept of Feynman path integral in quantum mechanics and quantum field theory has been introduced. This wave function and imaginary part of the energy of the pseudo-ground state of the Hamiltonian in Phase Qubits has been obtained from semi-classical approximation and we estimate decay rate, and thus the life time of meta-stable states using the approach of Instantons model. We devote the main efforts to study the evolution of spectrum of Hamiltonian in time after addition of interaction Hamiltonian, in order to obtain the high fidelity quantum gates.

I. INTRODUCTION

The potential to manipulate information efficiently with quantum mechanics and remarkable promise of quantum computation has led to a search and invention of a significant number of proposal for physical system that could implement a quantum computer in large size [1–4]. Superconducting circuits using Josephson junctions provide a promising approach towards the construction of a scalable solid-state quantum computer. These devices show the quantum effects in macroscopic scale and it’s the most advantage of these elements [5]. They can play the basic building blocks of quantum computers, which are qubits. In addion by manipulating the the qubits via external controlled current sources there is possibility to construct the specific quantum gates [6].

A viable quantum computer needs to has a the stable and long-live Qubits that make their coherency for long time before the manipulation and operation acts on them. Thus In order to building the quantum bits and quantum gates with high accuracy and high fidelity we need to have a deep recognition to the exact description physics of the system which in translation to quantum mechanics, it mean we should have a precise analysis on evolution of Propagator of the system which is completely time-dependent.

For the basics operation in quantum computation we fundamentally need to study the evolution of N two-level quantum systems which we can describe their states with the N-component vector $|\psi\rangle$. The evolution of this state can be given by propagator. In the general, the Hamiltonian is uncomutatable during the time, $[H(t), H(t_0)] \neq 0$ and we are bound to use the approximating method such as Dyson series for finding the genral form propagator. Unfortunately it’s not easy to calculate epecailly when we need analytical description of system. In the other hand There are various method to solve the time-independent quantum mechanics problem such as perturbation theory or WKB theory. But in fact in fact WKB is uncontrolled approximation in general and it is hard to say that the result of this methods is accurate or not. Therefore find the methods that help us to get the more accurate and reliable result is very important and essential.

In this paper we claim that functional formalism of quantum mechanics and Feynman path integral [8,9] give us the more accurate answer about estimating the ground states of energy and describing the meta stable states wave functions and decay rates of states in super-conducting phase qubits. Admittedly the formalism of path integral has been built completely time dependent and evolutionary proscess of the system will be tracked more convineint. This is the biggest treasure that lies down under this formalism.

At first section we review on structure of phase potential and we use from the Instanon model for finding the most properties of ground states of energy up to accuracy of $O(\hbar)$, after that we present the time-dependent propagator of the quansume system and this achieveent leads us to find the repersentation of the Hamiltonain across the time while the application of external manipulation, for the application of building the quantum gates.

II. SUPERCONDUTING PHASE QUBIT

Single Josephson junctions phase Qubits consists of one Josephson junction which use the quantum tunneling effect to produce the continuous current in the existence of external current source, $I_c$.

The Hamiltonian of system can be written as

$$H_{dc} = \frac{E_C}{2e} \frac{\partial^2}{\partial \delta^2} + E_J \cos \delta + \frac{\hbar}{2e} I_c \delta$$

(1)

Where $\delta$ present the phase of Josephson junction and $E_J = \hbar / 2e I_c$. In order to manipulate the system we need to evolve the system by time-dependent current, this manipulation introduce the Hamiltonian of interaction which can be given by

$$H_{\mu\nu} = \frac{\Phi_0}{2\pi} I_{\mu\nu} \delta = \frac{\Phi_0}{2\pi} I(t) \cos(\omega t + \phi) \delta$$

(2)

Here $\Phi_0 = \frac{\hbar}{2e}$ is quanat of flux, Thus the total Hamiltonian...
of system yields
\[ H(t) = H_0 + V(t) = H_0 + \frac{I_0 \Phi_0}{2\pi} I(t) \cos(\omega t + \phi) \] (3)

III. INSTANTON MODEL

In this section we study the metastable states in tilted-washboard potential of Josephson-junction phase qubit. We use from path integral approach for our study. If we consider the particle with unite mass which is under the influence of the one-dimentioanl potential \( V(x) \), then following the Euclidean form of the Feynman path integral we can describe the evolution of the particle with

\[ \langle x_f | e^{-\frac{\mathcal{H}T}{\hbar}} | x_i \rangle = N \int [dx] e^{-\frac{\mathcal{H}}{\hbar}} \] (4)

Here \( |x_f\rangle \) and \( |x_i\rangle \) are the eigenvalue of the space and the \( N \) refer to normalization factor, \( H \) represent the Hamiltonian of the system which can be depends on time and \( T \) shows the time interval of the evolution.

The symbol of \( [x] \) denotes the integration over all functions \( x(t) \) that obey from the boundary condition \( x(t) = x_i, x(t+\frac{T}{2}) = x_f \). If \( \bar{x} \) be any functions which obey the boundary condition then we can write \( x(t) = \bar{x}(t) + \sum_n c_n x_n(t) \) where the set \( x_n \) build the complete set, \( \int_{-\frac{T}{2}}^{\frac{T}{2}} x_n(t)x_m(t) = \delta_{mn} \) and \( x_n(\pm \frac{T}{2}) = 0 \). By these condition we can rewrite the mesure of the integral by

Calculation in order of \( \hbar \) and using the semi classical approximation lets us to write the evolution of the system by

\[ \langle x_f | e^{-\frac{\mathcal{H}t}{\hbar}} | x_i \rangle = N e^{-\frac{\mathcal{S}(\bar{x})}{\hbar}} \prod_n \lambda_n^{-\frac{1}{2}} [1 + O(\hbar)] \] (5)

If we define \( \omega^2 \) to be \( V''(0) \) then the standard calculation shows that for large \( T \)

\[ N[\det(-\partial^2 + \omega^2)]^{-\frac{1}{2}} = (\frac{\omega}{\pi \hbar})^{\frac{1}{2}} e^{-\frac{\mathcal{S}(x)}{2\hbar}} \] (6)

Now if we consider the desire potential shape as Fig III.

\[ B = \int_{-\infty}^{\infty} dt (\frac{dx}{dt})^2 = \int_0^\sigma dx [2V(x)]^{\frac{1}{2}} \] (7)

where \( x = \sigma \) is a place where potential is zero, \( V = 0 \). If we define the center of the bounce by the place where we have \( \frac{dx}{dt} = 0 \) there, we can see this point is invarinat under time translation. For large time, \( T \), we can put \( n \) separated points like that that have enough space from here and each of points plays the role of one signle Bounce motion. If we show the center of this points by \( \frac{T}{2} > t_1 > t_2 \cdots > t_n > -\frac{T}{2} \) in path integral we should notice that the value of action changes from \( S \) to \( nS \) and as this \( n \) points are located in the far distance from each other we can write the determinant as the product of determiniat of many single Bounce motion. in this way we obtain

\[ (\frac{\omega}{\pi \hbar})^{\frac{1}{2}} e^{-\omega \frac{T}{2}} K^n \] (8)

Where \( K \) is a factor that we will discuss on it later. Also the integration over time yeilds

\[ \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \int_{-\frac{T}{2}}^{t_1} dt_1 \cdots \int_{-\frac{T}{2}}^{t_{n-1}} dt_{n-1} \int_{-\frac{T}{2}}^{t_n} dt_n = \frac{T^n}{n!} \] (9)

finally we will have

\[ \sum_{n=0}^{\infty} \left( \frac{\omega}{\pi \hbar} \right)^{\frac{1}{2}} e^{-\omega \frac{T}{2}} (Ke^{-\frac{\mathcal{S}(x)}{2\hbar}})^n = \left( \frac{\omega}{\pi \hbar} \right)^{\frac{1}{2}} e^{[-\frac{\mathcal{S}(x)}{2\hbar} + K e^{-\frac{\mathcal{S}(x)}{2\hbar}}]} \] (10)

From Eq.4 we can see for example that the correspond value of ground state of enrgy can be given by

\[ E_0 = (\hbar \omega - hKe^{-\frac{\mathcal{S}(x)}{2\hbar}}) [1 + O(\hbar)] \] (11)

We can see that one of the eigenvalue is zero and the corresponding eigenfunction that provide this eigenvalue can be easily given by \( x_1 = B^{-\frac{1}{2}} \frac{dx}{dt} \), therefore we need to omitte this trubling zero eigenvalue. we can use many ways to solve this problem and by the standardr way we
can calculate prime determinant which the zero eigenvalue has been omitted and we need to add coefficient to the $K$ like as \((\frac{B}{2\pi\hbar})\) [10, [11].

Now if we review on our solution way with more details we will find that as in one place the $\frac{B}{2\pi\hbar}$ become zero therfor $x$ has node and thus it cannot be the lowest eigenvalue of energy. It means that this system has the negatibe eigenvalue and the spectum of our systems is the special case and we have unstable state here and the unitarity of this location form the Hilbert sapce can be in doubt.

The key to point for solving this problem is that we need to add the coefficient one-half to our calculation and the relible result yields [10, [11], [12].

\[
Im[N\int dx e^{-\frac{B}{2\pi\hbar}\frac{1}{2}T}|\det[-\partial^2_t + V''(\tilde{x})]|-\frac{1}{2}] = 12
\]

and by comparing the result with the defination of $K$ we will find that

\[
Im K = \frac{1}{2} \frac{B}{2\pi\hbar}\frac{1}{2}T|\det[-\partial^2_t + V''(\tilde{x})]|-\frac{1}{2}
\]

(13)

In the stable situation, when the height of barrier penetration goes to infinity the solution of Schrodinger equation, corresponding to the ground state energy $E_0$ behaves as

\[
\psi_0(t) \sim e^{-i\frac{E_0}{\hbar}t}
\]

(14)

But for the case that we have not absolute minimum, $E_0$ becomes imaginary. Therefore for long times we have

\[
|\psi_0(t)| \sim e^{-\frac{Im E_0}{\hbar}t}
\]

(15)

It clearly shows that the amplitude and therefore the probability of state decays. The parameter $|\frac{\hbar}{Im E_0}|$ is the lifetime of a now metastable state with wave function $\psi(t)$. Let us to point out that the decay of state receives contributions from the continuation of all excited states. However, one expects, for intuitive reasons, that when the real part of the energy increases the corresponding contribution decreased faster with time, a property that can, indeed, be verified in examples. Thus, for large times, only the component corresponding to the pseudo-ground state survives. by considering the one-half calculation we have

\[
\Gamma = -2Im E_0/\hbar = \frac{B}{2\pi\hbar}\frac{1}{2}T|\det[-\partial^2_t + V''(\tilde{x})]|-\frac{1}{2}[1 + O(\hbar)]
\]

(16)

IV. ESTIMATION OF THE COEFINCEINT

Here we have similar situation to unstable states and bounces, therefore we follow the mentioned solving way that we discussed in previous sections.

In order to finding the classical path we should inverse the potential. If we call the turning point by $\sigma$, as is clear in fig 12, then $\sigma$ is the zero of $V(x) = \alpha x^2 + \beta x + \epsilon(x)$. The analytical solution of this equation obviously is not clear at first sight, specially the correction error, $\epsilon(x)$, has no simple formula. Thus it’s better to solve it with soft wares, depend on our parameter. By knowing the turning point then estimating the action, $S_0$ is easy. as usual

\[
S_0 = \int_{\delta_i}^{\delta_f} d\delta \sqrt{\frac{2V(\delta)}{m}}
\]

(17)

From our Hamiltonian it is celer that $m = \frac{\hbar^2}{2V_0}$ and $V(\delta) = E_J \cos \delta + \frac{h}{2e} I_c \delta + \epsilon(\delta)$.

\[
S_0 = \int_{0}^{\sigma} d\delta \sqrt{\frac{4E_J}{\hbar^2}} \sqrt{E_J \cos \delta + \frac{h}{2e} I_c \delta + \epsilon(\delta) + c_0}
\]

(18)

Where $c_0$ is constant that appear form changing the coordinate in order to the hill point of potential locate at zero point of coordinate. Value of this integral can be calculate easily by soft wares. Now we try to find the classical path. For simplicity and consistency we previous formula in previous subsection we change the variable of motion $x \rightarrow q$ instead of $\delta$. From equation of motion we have

\[
\frac{1}{2}m \dot{q}_c^2 = V(x_c) + E_0
\]

(19)

Thus the classical path obey from this relation

\[
t = t_1 + \sqrt{\frac{2}{m}} \int_{0}^{x'} dx_c \sqrt{V(x_c) + E_0}
\]

(20)

\[
= \int_{0}^{x'} dx_c \sqrt{\frac{4E_J}{\hbar^2} \cos x_c + \frac{h}{2e} I_c x_c + \epsilon(x_c) + E_0}
\]

The $E_0$ is the constant of motion in must be selected which in $x \rightarrow 0$, $t \rightarrow -\infty$ and vice-versa. As we know the zero Eigenfunction of $[-\partial^2_t + V''(x_c)]$ is

\[
x_1 = S_0 - \frac{1}{2} \frac{dx_c}{dt}
\]

(21)

For the next estimation we strongly to know the behavior of $x_1$ respect to time. from previous equation we have function $t(x_c)$ and what we need is the inverse of this function $g = f^{-1} = x_c(t)$. Finding the analytical form for this function is complicated and it’s better to solve it numerically in exact case that we need and with desire parameters.

And for example, for the potentialional $V(x) = \frac{1}{2}x^2 + \frac{1}{3}g x^4$ the $x_c(t)$ has the form $x_c(t) = g(t) \sim \frac{1}{\cosh(t-t_0)}$, Fig 13. Hence we expect that $x_1$ behave exponentially when time goes to infinity.

\[
x_1 = S_0 - \frac{1}{2} \frac{dx_c}{dt} \rightarrow Ae^{-|t|}, t \rightarrow \pm \infty
\]

(22)
We consider estimating the quantities $S_0$ and $x_c(t)$ let us to estimate the $A$ factor which is constant and fundamentally is function of just $I_c$ and $L$ and capacitance and cross section area of Josephson junction.

It is easy to show that in general

$$\det\left[-\frac{\partial^2}{\partial x^2} + U''(x_c)\right] \det\left[-\frac{\partial^2}{\partial x^2} + \omega^2\right] = \frac{1}{2A^2}$$

And

$$K = \left(\frac{S_0}{2\pi\hbar}\right)^{\frac{1}{2}} \sqrt{\frac{1}{2A^2}}$$

Thus finally we will have

$$\Gamma = \hbar\left(\frac{S_0}{2\pi\hbar}\right)^{\frac{1}{2}} \sqrt{\frac{1}{2A^2}} e^{-\frac{S_0}{2}}$$

**V. PROPERTIES OF PROPAGATOR**

Propagator palys the fundemental rules for understanding the evolution of the system and the whole properties of the system basicaly can obtain from the Propagator and here we want find the element of the Hamiltonian of our sytem which are dependent on time. As we know

$$\langle x_f | U(t_f, t_i) | x_i \rangle = U(x_f, t_f; x_i, t_i) = \int_{x(t_f)}^{x(t_i)} D[x(t)] e^{iS[x(t)]}$$

It is clear that the Path Integral are invariant under the transformation of $x(t) \rightarrow x(t) + y(t), y(t_i) = y(t_f)$ Which yields to Equation which is so called Schwinger-Dyson equation:

$$\int_{x(t_i)}^{x(t_f)} [Dx(t)] \frac{\delta}{\delta x(t)} e^{iS[x(t)]} = 0$$

Which is equal to In the other hand as we know

$$S[x(t)] = \int_{t_i}^{t_f} L(x(t), \dot{x}(t); t) dt$$

Combining this with Schwinger-Dyson equation we found that the Identity

$$\int_{x(t_i)}^{x(t_f)} [Dx(t)] \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}(t)}\right) e^{iS[x(t)]} = 0$$

This Identity is some aspect of the Ehrenfest Theoerem.

Now if looking for the variation of Action with conditions which $\delta x(t_i) = 0$, $\delta x(t_f) \neq 0$, then we have

$$\delta S[x(t)] = \int_{t_i}^{t_f} dt \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}(t)}\right) \delta x(t) + \frac{d}{dt} \delta x(t_f)$$

$$= \int_{t_i}^{t_f} dt \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}(t)}\right) \delta x(t) + p(t_f) \delta x(t_f)$$

If we define $p(t_f) = p_f$ then

$$p_f = \frac{\partial S[x_c]}{\partial x_f}$$

Now the variation of the propagator means

$$\delta U(x_f, t_f; x_i, t_i) = \delta x(t_f) \frac{\partial}{\partial x_f} U(x_f, t_f; x_i, t_i)$$

Therefore

$$\frac{\partial}{\partial x_f} U(x_f, t_f; x_i, t_i) = \frac{i}{\hbar} \int_{x(t_i)=x_i}^{x(t_f)=x_f} D[x(t)] e^{iS[x(t)]}$$

The other property that we need to know is the variation of the action with respect to time. As we know

$$L(x_f, \dot{x}_f) = \frac{d}{dt} S[x(t)] = \frac{\partial S}{\partial t_f} + \frac{\partial S[x(t)]}{\partial x_f} \frac{dx_f}{dt} = \frac{\partial S}{\partial t_f} + p_f \dot{x}_f$$

Thus

$$\frac{\partial S[x_c]}{\partial t_f} = L(x_f, \dot{x}_f) - p_f \dot{x}_f = -H(x_f, p_f)$$

Now we can see that

$$i\hbar \frac{\partial}{\partial t_f} \int_{x(t_i)=x_i}^{x(t_f)=x_f} [Dx(t)] e^{iS[x(t)]} = \int_{x(t_i)=x_i}^{x(t_f)=x_f} [Dx(t)] H(x_f, p_f) e^{iS[x(t)]}$$

Therefore the Propagator is the kernel or The green function of the Schroodinger equation

$$[i\hbar \frac{\partial}{\partial t_f} - H(x_f, p_f)] U(x_f, t_f; x_i, t_i) = 0$$

**VI. EVOLUTION OF TIME-DEPENDENT HAMILTONIAN**

Discution when the system is open ....
As we works in the Semi-classica regime we develop our result end evolution with this approximation. If we consider the Action and change it’s variable to

$$x(\tau) := y(\tau) + \bar{x}(\tau)$$

$$\int L dt \quad = S(x(\tau)) = S(y(\tau) + \bar{x}(\tau))$$

$$= S(\bar{x}(\tau)) + \frac{\delta S}{\delta \bar{x}} |\bar{x}(\tau)|^2 + \frac{1}{2} \frac{\delta^2 S}{\delta \bar{x}^2} |\bar{x}(\tau)|^2$$

Then by semi-classical approximation we have
\[ S(\vec{x} + y) \simeq S(\vec{x}) + \frac{1}{2} \delta^2 S y^2 \]  

(41)

the most advantage of the Path Integral for our works lie under this property that the path integral is completely time-dependent formalism and it’s not important does the Hamiltonina is time dependent or not and it’s the vital property which we need, in the other hand if we works with canonical formalism there is no clear connection between the time-dependent perturbation and time-independent one.

\[
U(x_f, t; x_i, t_i) = \int_{x(t_f)}^{x(t_i)} D[x(t)] e^{iS[x(t)]} \]

(42)

\[
= \int_{x(t_f)}^{x(t_i)} D[y(t)] e^{i(S(\vec{x}) + \frac{1}{2} \delta^2 S y^2)}
\]

Therfore we found the most important lemma that help us to develope the calculation :

\[
U(x_f, t_f; x_i, t_i) = e^{iS(x_f, t_f; x_i, t_i)} U(0, 0, t_f; 0, t_i) \]

(43)

which the \( U(0, 0, t_f; 0, t_i) \) is the propagator of the system that hase the hamiltonian \( H = H(t) \) for the especial case that the \( x_f = x_i = 0 \).

by this relation we can obtain the general propgator that thar the initial and the final positions are arbitrary.

\section{VII. ELEMENT OF THE HAMILTONIAN}

In principle we can obtain the path integral formalism by accepting the two fundemental property for propagator, the first one, we consider that the propagator has time-independent one.

We consider a bounded operator in Hilbert space, \( U(t, t') \), \( t \geq t' \), which describes the evolution from time \( t' \) to time \( t \) and satisfies a Markov property in time \[\text{[10]}\]

\[
U(t, t') U(t'', t') = U(t, t'') \quad \text{for} \quad t \geq t'' \geq t
\]

(44)

Also we consider \( U(t', t') = 1 \). moreover, we assume that \( U(t, t') \) is differentiable with a continues derivative. We set

\[
\frac{\partial U(t, t')}{\partial t} \bigg|_{t=t'} = \frac{H(t)}{i\hbar}
\]

(45)

here \( \hbar \) is real parameter and as we know later it becomes Planck’s constant. With this two fundamental properties we can obtain interesting result. By differentiating the Eq.\[\text{[13]}\]with respect to \( t \) and take the \( t'' = t \) we find

\[
\frac{i\hbar}{\partial t} U(t, t') = H(t) U(t, t')
\]

(46)

Now we use from this Identity to estimate the Hamiltonian

\[
\langle y| \frac{\partial U(t, t')}{\partial t} \big|_{t=t'} |x \rangle = \frac{1}{i\hbar} \langle y|H(t)|x \rangle
\]

(47)

as

\[
\langle y| \frac{\partial U(t, t')}{\partial t} \big|_{t=t'} |x \rangle = \frac{\partial}{\partial t} \langle y|U(t, t')|x \rangle \big|_{t=t'}
\]

(48)

Thus

\[
\langle y|H(t)|x \rangle = i\hbar \left( \frac{\partial}{\partial t} \langle y|U(t, t')|x \rangle \big|_{t=t'} \right)
\]

(49)

\[
+ \left( \frac{\partial}{\partial t} \langle y|U(t, t')|x \rangle \big|_{t=t'} \right)
\]

Hence we find that

\[
\langle y|H(t)|x \rangle = i\hbar \left( \frac{\partial}{\partial t} \langle y|U(t, t')|x \rangle \big|_{t=t'} \right)
\]

(50)

here for obtaining the result we need to know the \( \frac{\partial}{\partial t} \langle y|U(0, 0, t_f; 0, t_i)\rangle \big|_{t_i=t_f} \) that by estimating for our system we can easily derivve from it. But as we estimated, the propagator for \( x_f = x_i = 0 \)

\[
U(0, 0, t_f; 0, t_i) = \left( \frac{\omega}{\pi \hbar} \right)^\frac{1}{2} e^{-i\omega(t_f-t_i)} e^{-\Gamma(t_f-t_i)}
\]

(51)

Where \( \Gamma \) can be given by

\[
\Gamma = \hbar |K| e^{-\frac{\delta_0}{\pi}}
\]

(52)

Thus

\[
\frac{\partial}{\partial t} \langle y|U(0, 0, t_f; 0, t_i)\rangle \big|_{t_i=t_f} = \left( \frac{\omega}{\pi \hbar} \right)^\frac{1}{2} \left[ -\frac{i\omega}{2} - \Gamma \right]
\]

(53)

In our case as the second derivstion of time dependent potential, \( V'' \), is independent for time, this estimation is likes to time independent mode.

\[
\langle y|H(t)|x \rangle = i\hbar \left( -\frac{\hbar}{\pi} \right) \left[ -\frac{i\omega}{2} - \Gamma \right]
\]

(54)

now if the \( |n \rangle \) and \( |m \rangle \) be the the two metastable of the system that are time-dependent foundemantal, then

\[
\langle m|H(t)|n \rangle = \int_x \int_y dx dy \langle m|x \rangle \langle x|H(t)|y \rangle \langle y|n \rangle
\]

(55)

Or

\[
\langle m|H(t)|n \rangle = \int_x \int_y m^*(x) \langle x|H(t)|y \rangle n(y)
\]
\( \langle m | H(t) | n \rangle = \int_x \int_y dxdy \langle m | x \rangle \langle x | H(t) | y \rangle \langle y | n \rangle \) \hspace{1em} (56) \\
\hspace{1em} = \int_x \int_y dxdy \psi_m^*(x) i\hbar \{ -\frac{i}{\hbar} H(x,y) \} \psi_n(y) + \{ \frac{\omega}{\pi \hbar} \}^{\frac{1}{2}} \{ -\frac{i\omega}{2} - \Gamma \} \psi_n(y) \)

**Appendix A  WAVE FUNCTIONS**

As we saw in previous section, estimating the element of the Hamiltonian requires the eigenfunction of the energy and we need the corresponding wave functions in the representation of space. Here we want to obtain the semi-classical approximation or in the other word we use the WKB approximation and expand wave function by order of \( \hbar \).

\[
\psi_p = a \text{Ai}(ax) + b \text{Bi}(ax) \quad (57)
\]

By defining
\[
\theta := \frac{1}{\hbar} \int_{x_1}^{x_2} p(x') dx', \quad \gamma := \int_{x_2}^{x_3} |p(x)| dx \quad (58)
\]

By comparit the coefficient and by using the patch function near each point we find that

\[
\psi(x) \simeq \begin{cases} 
-\frac{D}{\sqrt{|p(x)|}} \sin \left[ \int_{x_1}^{x_2} p(x') dx' - \theta - \frac{\pi}{2} \right] & x < x_1 \\
-\frac{D}{\sqrt{|p(x)|}} \sin \left[ \int_{x_2}^{x_3} p(x') dx' \right] + \sin \theta e^{-\frac{1}{\hbar} \int_{x_3}^{x_3} |p(x')| dx'} & x_1 < x < x_2 \\
\frac{1}{\sqrt{|p(x)|}} e^{ie^{-\frac{1}{\hbar} \int_{x_3}^{x_3} |p(x')| dx'}} & x > x_3
\end{cases}
\]

As the amount of energy for metastable states has imaginary part, therefore the amplitude of wavefunction decay gradually and the state disappear after long time.

**FIG. 2.** Tilted Washboard potential.

In the WKB approximation regime we use the the Patch function as a auxiliary function to connecting the coefficient if the wavefunction in the two side of the returning point. Therefore near the turning point the wavefunction is near to solution of the differential equation which their answer given by airy function.

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\psi(x) \simeq \begin{cases} 
-\frac{D}{\sqrt{|p(x)|}} \sin \left[ \int_{x_1}^{x_2} p(x') dx' - \theta - \frac{\pi}{2} \right] & x < x_1 \\
-\frac{D}{\sqrt{|p(x)|}} \sin \left[ \int_{x_2}^{x_3} p(x') dx' \right] + \sin \theta e^{-\frac{1}{\hbar} \int_{x_3}^{x_3} |p(x')| dx'} & x_1 < x < x_2 \\
\frac{1}{\sqrt{|p(x)|}} e^{ie^{-\frac{1}{\hbar} \int_{x_3}^{x_3} |p(x')| dx'}} & x > x_3
\end{cases}
\]

As the amount of energy for metastable states has imaginary part, therefore the amplitude of wavefunction decay gradually and the state disappear after long time.

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