On the existence of solutions to stochastic quasi-variational inequality and complementarity problems

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Abstract

Variational inequality problems allow for capturing an expansive class of problems, including convex optimization problems, convex Nash games and economic equilibrium problems, amongst others. Yet in most practical settings, such problems are complicated by uncertainty, motivating the examination of a stochastic generalization of the variational inequality problem and its extensions in which the components of the mapping contain expectations. When the associated sets are unbounded, ascertaining existence requires having access to analytical forms of the expectations. Naturally, in practical settings, such expressions are often difficult to derive, severely limiting the applicability of such an approach. Consequently, our goal lies in developing techniques that obviate the need for integration and our emphasis lies in developing tractable and verifiable sufficiency conditions for claiming existence. We begin by recapping almost-sure sufficiency conditions for stochastic variational inequality problems with single-valued maps provided in our prior work [44] and provide extensions to multi-valued mappings. Next, we extend these statements to quasi-variational regimes where maps can be either single or set-valued. Finally, we refine the obtained results to accommodate stochastic complementarity problems where the maps are either general or co-coercive. The applicability of our results is demonstrated on practically occurring instances of stochastic quasi-variational inequality problems and stochastic complementarity problems, arising as nonsmooth generalized Nash-Cournot games and power markets, respectively.

1 Introduction

Motivation: In deterministic regimes, a wealth of conditions exist for characterizing the solution sets of variational inequality, quasi-variational inequality and complementarity problems (cf. [8, 9, 28]), including sufficiency statements of existence and uniqueness of solutions as well as more refined conditions regarding the compactness and connectedness of solution sets and a breadth of sensitivity and stability questions. Importantly, the analytical verifiability of such conditions from problem primitives (such as the underlying map and the set) is essential to ensure the applicability of such schemes, as evidenced by the use of such conditions in analyzing a range of problems arising in power markets [19, 22, 51], communication networks [10, 39, 49, 57], structural analysis [40, 41], amongst others. The first instance of a stochastic variational inequality problem was presented by King and Rockafellar [27] in 1993 and the resulting stochastic variational inequality problem requires an $x \in X$ such that

$$(y - x)^T E[F(x, \xi(\omega))] \geq 0, \quad \forall y \in X,$$

where $X \subseteq \mathbb{R}^n$, $\xi : \Omega \rightarrow \mathbb{R}^d$, $F : X \times \mathbb{R}^d \rightarrow \mathbb{R}^n$, $E[.]$ denotes the expectation, and $(\Omega, \mathcal{F}, \mathbb{P})$ represents the probability space. In the decade that followed, there was relatively little effort on addressing analytical and computational challenges arising from such problems. But in the last ten years, there has been an

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immense interest in the solution of such stochastic variational inequality problems via Monte-Carlo sampling methods [23, 24, 31, 52]. But verifiable conditions for characterization of solution sets have proved to be relatively elusive given the presence of an integral (arising from the expectation) in the map. Despite the rich literature in deterministic settings, direct application of deterministic results to stochastic regimes is relatively elusive given the presence of an integral (arising from the expectation) in the map. Despite the conditions for characterization of solution sets have proved to be very complicated by several challenges: First, a direct application of such techniques on stochastic problems requires the availability of closed-form expressions of the expectations. Analytical expressions for expectation are not easy to derive even for single-valued problems with relatively simple continuous distributions. Second, any statement is closely tied to the distribution. Together, these barriers severely limit the generalizability of such an approach. To illustrate the complexity of the problem class under consideration, we consider a simple stochastic linear complementarity problem.

**Example 1.** Consider the following stochastic linear complementarity problem:

\[ 0 \leq x \perp \mathbb{E} \left[ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x + \begin{pmatrix} -2 + \omega_1 \\ -4 + \omega_2 \end{pmatrix} \right] \geq 0. \]

Specifically, this can be cast as an affine stochastic variational inequality problem \( VI(K, F) \) where

\[ K \triangleq \mathbb{R}^2_+ \text{ and } F(x) \triangleq \mathbb{E}[F(x; \omega)], \text{ where } F(x; \omega) \triangleq \left[ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x + \begin{pmatrix} -2 + \omega_1 \\ -4 + \omega_2 \end{pmatrix} \right]. \]

Consider two cases that pertain to either when the expectation is available in closed-form (a); or not (b):

**a) Expectation \( \mathbb{E}[.] \) available:** Suppose in this instance, \( \omega \) is a random variable that takes values \( \omega^1 \) of \( \omega^2 \), given by the following:

\[ \omega^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ or } \omega^2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \text{ with probability } 0.5. \]

Consequently, the stochastic variational inequality problem can be expressed as

\[ 0 \leq x \perp \left[ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x + \begin{pmatrix} -2 \\ -4 \end{pmatrix} \right] \geq 0. \]

In fact, this problem is a strongly monotone linear complementarity problem and admits a unique solution given by \( x^* = (0, 2) \). If one cannot ascertain monotonicity, a common approach lies in examining coercivity properties of the map; specifically, \( VI(K, F) \) is solvable since there exists an \( x^{ref} \in K \) such that (cf. [9, Ch. 2])

\[ \lim_{\|x\| \to \infty, x \in K} \mathbb{E}[F(x)]^T (x - x^{ref}) > 0. \]

**b) Expectation \( \mathbb{E}[.] \) not available in closed-form:** However, in many practical settings, closed-form expressions of the expectation are unavailable. Two possible avenues are available:

(i) If \( K \) is compact, under continuity of the expected value map, \( VI(K, F) \) is solvable.

(ii) If there exists a single \( x \in K \) that solves \( VI(K, F(; \omega)) \) for almost every \( \omega \in \Omega \), \( VI(K, F) \) is solvable.

Clearly, \( K \) is a cone and (i) does not hold. Furthermore (ii) appears to be possible only for pathological examples and in this case, there does not exist a single \( x \) that solves the scenario-based \( VI(K, F(; \omega)) \) for every \( \omega \in \Omega \). Specifically, the unique solutions to \( VI(K, F(; \omega^1)) \) and \( VI(K, F(; \omega^2)) \) are \( x(\omega^1) = (0, 3/2) \) and \( x(\omega^2) = (1/3, 7/3) \), respectively and since \( x(\omega^1) \neq x(\omega^2) \), avenue (ii) cannot be traversed. Consequently, neither of the obvious approaches can be adopted yet \( VI(K, F) \) is indeed solvable with a solution given by \( x^* = (0, 2) \).

Consequently, unless the set is compact or the scenario-based VI is solvable by the same vector in an almost sure sense, ascertaining solvability of stochastic variational inequality problems for which the expectation is unavailable in closed form does not appear to be immediately possible through known techniques.

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1 Formal definitions for these problems are provided in Section 1.1.
In what could be seen as the first step in this direction, our prior work \cite{44} examined the solvability of convex stochastic Nash games by analyzing the equilibrium conditions, compactly stated as a stochastic variational inequality problem. Specifically, this work relies on utilizing Lebesgue convergence theorems to develop integration-free sufficiency conditions that could effectively address stochastic variational inequality problems, with both single-valued and a subclass of multi-valued maps arising from nonsmooth Nash games.

As a simple illustration of the avenue adopted, consider Example 1 again and assume that the expectation is unavailable in closed form, we examine whether the a.s. coercivity property holds (as presented in the next section). It can be easily seen that there exists an $x^{\text{ref}} \in K$, namely $x^{\text{ref}} \equiv 0$, such that

$$\lim_{\|x\| \to \infty, x \in K} F(x; \omega)^T (x - x^{\text{ref}}) > 0,$$

for $\omega = \omega^1$ and $\omega = \omega^2$.

It will be shown that satisfaction of this coercivity property in an almost-sure sense is sufficient for solvability. But such statements, as is natural with any first step, do not accommodate stochastic quasi-variational problems and can be refined significantly to accommodate complementarity problems. Moreover, they cannot accommodate multi-valued variational maps. The present work focuses on extending such sufficiency statements to quasi-variational inequality problems and complementarity problems and accommodate settings where the maps are multi-valued.

**Contributions:** This paper provides amongst the first attempts to examine and characterize solutions for the class of stochastic quasi-variational inequality and complementarity problems when expectations are unavailable in closed form. Our contributions can briefly be summarized as follows:

(i) **Stochastic quasi-variational inequality problems (SQVIs):** We begin by recapping our past integration-free statements for stochastic VIs that required the use of Lebesgue convergence theorems and variational analysis. Additionally, we provide extensions to regimes with multi-valued maps and specialize the conditions for settings with monotone maps and Cartesian sets. We then extend these conditions to stochastic quasi-variational inequality problems where in addition to a coercivity-like property, the set-valued mapping needs to satisfy continuity, apart from other “well-behavedness” properties to allow for concluding solvability. Finally, we extend the sufficiency conditions to accommodate multi-valued maps.

(ii) **Stochastic complementarity problems (SCPs):** Solvability of complementarity problems over cones requires a significantly different tack. We show that analogous verifiable integration-free statements can be provided for stochastic complementarity problems. Refinements of such statements are also provided in the context of co-coercive maps.

(iii) **Applications:** Naturally, the utility of any sufficiency conditions is based on its level of applicability. We describe two application problems. Of these, the first is a nonsmooth stochastic Nash-Cournot game which leads to an SQVI while the second is a stochastic equilibrium problem in power markets which can be recast as a stochastic complementarity problem. Importantly, both application settings are modeled with a relatively high level of fidelity.

**Remark:** Finally, we emphasize three points regarding the relevance and utility of the provided statements: (i) First, such techniques are of relevance when integration cannot be carried out easily and have less utility when sample spaces are finite; (ii) There are settings where alternate models for incorporating uncertainty have been developed \cite{7, 36, 38}. Such models assume relevance when the interest lies in robust solutions. Naturally, an expected-value formulation has less merit in such settings and correspondingly, such robust approaches cannot capture risk-neutral decision-making. Consequently, the challenge of analyzing existence of this problem cannot be done away with by merely changing the formulation, since an alternate formulation may be inappropriate. (iii) Third, we present sufficiency conditions for solvability. Still, there are simple examples which can be constructed in finite (and more general) sample spaces where such conditions will not hold and yet solvability does hold. We believe that this does not diminish the importance of our results; in fact, this is not unlike other sufficiency conditions for variational inequality problems. For instance, we may construct examples where the coercivity of a map may not hold over the given set \cite{9} but the variational inequality problem may be solvable. However, in our estimation, in some of the more practically occurring

\footnote{A similarly loose dichotomy exists between stochastic programming and robust optimization.}
engineering-economic systems, such conditions do appear to hold, reinforcing the relevance of such techniques. In particular, we show such conditions find applicability in a class of risk-neutral and risk-averse stochastic Nash games in [4]. In the present work, we show that such conditions can be employed for analyzing a class of stochastic generalized Nash-Cournot games with nonsmooth price functions as well as for a relatively more intricate networked power market in uncertain settings. Before proceeding to our results we provide a brief history of deterministic and stochastic variational inequalities.

Background and literature review: The variational inequality problem provides a broad and unifying framework for the study of a range of mathematical problems including convex optimization problems, Nash games, fixed point problems, economic equilibrium problems and traffic equilibrium problems [9]. More generally, the concept of an equilibrium is central to the study of economic phenomena and engineered systems, prompting the use of the variational inequality problem [18]. Harker and Pang [16] provide an excellent survey of the rich mathematical theory, solution algorithms, and important applications in engineering and economics while a more comprehensive review of the analytical and algorithmic tools is provided in the recent two volume monograph by Facchinei and Pang [9].

Increasingly, the deterministic model proves inadequate when contending with models complicated by risk and uncertainty. Uncertainty in variational inequality problems has been considered in a breadth of application regimes, ranging from traffic equilibrium problems [7, 15], cognitive radio networks [30, 50], Nash games [13, 35], amongst others. Compared to the field of optimization, where stochastic programming [4, 53] and robust optimization [3] have provided but two avenues for accommodating uncertainty in static optimization problems, far less is currently available either from a theoretical or an algorithmic standpoint in the context of stochastic variational inequality problems. Much of the efforts in this regime have been largely restricted to Monte-Carlo sampling schemes [13, 28, 27, 23, 50, 39, 60, 42], and a recent broader survey paper on stochastic variational inequality problems and stochastic complementarity problems [33].

1.1 Formulations

To help explain the two main formulations for stochastic variational inequality problems found in literature - the expected value formulation and almost-sure formulation; we now define variational inequality problems.

### 1.1.1 The expected-value formulation

We consider this formulation in the analysis in this paper.
Definition 1.1 (Stochastic variational inequality problem (SVI$(K,F)$)). Let $K \subseteq \mathbb{R}^n$ be a closed and convex set, $F : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ be a single-valued map and $F(x) \triangleq \mathbb{E}[F(x;\omega)]$. Then the stochastic variational inequality problem, denoted by SVI$(K,F)$, requires an $x \in K$ such that the following holds:

$$(y - x)^T \mathbb{E}[F(x;\omega)] \geq 0, \quad \forall y \in K.$$  \hspace{1cm} \text{(SVI$(K,F)$)} \tag{1.1}

Figure 1 provides a schematic of the stochastic variational inequality problem. Note that when $x$ solves SVI$(K,F)$, there may exist $\omega \in \Omega$ for which there exist $y \in K$ such that $(y - x)^TF(x;\omega) < 0$. Naturally, in instances where the expectation is simple to evaluate, as seen in Example 1 earlier, the resulting SVI$(K,F)$ is no harder than its deterministic counterpart.

For instance, if the sample space $\Omega$ is finite, then the expectation reduces to a finite summation of deterministic maps which is itself a deterministic map. Consequently, the analysis of this problem is as challenging as a deterministic variational inequality problem. Unfortunately, in most stochastic regimes, this evaluation relies on a multidimensional integration and is not a straightforward task. In fact, a more general risk functional can be introduced instead of the expectation leading to a risk-based variational inequality problem that requires an $x$ such that

$$(y - x)^T \rho[F(x;\omega)] \geq 0, \quad \forall y \in K,$$

where $\rho[F(x;\omega)] \triangleq \mathbb{E}[F(x;\omega)] + \mathbb{D}[F(x;\omega)]$ and $\mathbb{D}[\cdot]$ is a map incorporating dispersion measures such as standard deviation, upper semi-deviation, or the conditional value at risk (CVaR) (cf. [47, 48, 53] for recent advances in the optimization of these risk measures).

Extensions to set-valued and conic regimes follow naturally. For instance, if $K$ is a point-to-set mapping defined as $K : \mathbb{R}^n \to \mathbb{R}^n$, then the resulting problem is a stochastic quasi-variational inequality, and is denoted by SQVI$(K,F)$. When $K$ is a cone, then VI$(K,F)$ is equivalent to a complementarity problem CP$(K,F)$ and its stochastic generalization is given next.

Definition 1.2 (Stochastic complementarity problem (SCP$(K,F)$)). Let $K$ be a closed and convex cone in $\mathbb{R}^n$, $F : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ be a single-valued mapping and $F(x) \triangleq \mathbb{E}[F(x;\omega)]$. Then the stochastic complementarity problem, denoted by SCP$(K,F)$, requires an $x \in K$ such that

$$K \ni x \perp \mathbb{E}[F(x;\omega)] \in K^*.$$  \hspace{1cm} \text{SCP$(K,F)$} \tag{1.2}

If the integrands of the expectation $(F(x;\omega))$ are multi-valued instead of single-valued, then we denote the mapping by $\Phi$. Accordingly, the associated variational problems are denoted by SVI$(K,\Phi)$, SQVI$(K,\Phi)$ and SCP$(K,\Phi)$.

The origin of the expected-value formulation can be traced to a paper by King and Rockafellar [27], where the authors considered a generalized equation [45, 46] with an expectation-valued mapping. Notably, the analysis and computation of the associated solutions are hindered significantly when the expectation is over a general measure space. Evaluating this integral is challenging, at best, and it is essential that specialized analytical and computational techniques be developed for this class of variational problems. From an analytical standpoint, Ravat and Shanbhag have developed existence statements for equilibria of stochastic Nash games that obviate the need to evaluate the expectation by combining Lebesgue convergence theorems with standard existence statements [43, 44]. Our earlier work focused on Nash games and examined such settings with nonsmooth payoff functions and stochastic constraints. It represents a starting point for our current study where we focus on more general variational inequality and complementarity problems.
and their generalizations and refinements. Accordingly, this paper is motivated by the need to provide sufficiency conditions for stochastic variational inequality problems, stochastic quasi-variational inequality problems, and stochastic complementarity problems. In addition, we consider variants when the variational maps are complicated by multi-valuedness. Stability statements for stochastic generalized equations have been provided by Liu, Römisch, and Xu [34].

From a computational standpoint, sampling approaches have addressed analogous stochastic optimization problems effectively [32, 52], but have focused relatively less on variational problems. In the latter context, there have been two distinct threads of research effort. Of these, the first employs sample-average approximation schemes [52]. In such an approach, the expectation is replaced by a sample-mean and the effort is on the asymptotic and rate analysis of the resulting estimators, which are obtainable by solving a deterministic variational inequality problem (cf. 13 [22, 53, 60]). A rather different track is adopted by Jiang and Xu [23] where a stochastic approximation scheme is developed for solving such stochastic variational inequality problems. Two regularized counterparts were presented by Koshal, Nedić, and Shaikhb [29, 31] where two distinct regularization schemes were overlaid on a standard stochastic approximation scheme (a Tikhonov regularization and a proximal-point scheme), both of which allow for almost-sure convergence. Importantly, this work also presents distributed schemes that can be implemented in networked regimes. A key shortcomings of standard stochastic approximation schemes is the relatively ad-hoc nature of the choice of steplength sequences. In [58], Yousefian, Nedić, and Shaikhb develop distributed stochastic approximation schemes where users can independently choose a steplength rule. Importantly, these rules collectively allow for minimizing a suitably defined error bound and are equipped with almost-sure convergence guarantees. Finally, Wang et al. [54] focus on developing a sample-average approximation method for expected-value formulations of the stochastic variational inequality problems while Lu and Budhiraja [35] examine the confidence statements associated with estimators associated with a sample-average approximation scheme for stochastic variational inequality problem, again with expectation-based maps.

1.1.2 The almost-sure formulation

While there are many problem settings where expected-value formulations are appropriate (such as when modeling risk-neutral decision-making in a competitive regime) but there are also instances where the expected-value formulation proves inappropriate. A case in point arises when attempting to obtain solutions to a variational inequality problem that are robust to parametric uncertainty; such problems might arise when faced with traffic equilibrium or structural design problems. In this setting, the almost-sure formulation of the stochastic variational inequality (see Figure 2) is more natural. Given a random mapping \( F \), the almost-sure formulation of the stochastic variational inequality problem requires a (deterministic) vector \( x \in K \) such that for almost every \( \omega \in \Omega \),

\[
(y - x)^T F(x; \omega) \geq 0, \quad \forall y \in K.
\] (1)

Naturally, if \( K \) is an \( n \)-dimensional cone, then (1) reduces to \( \text{CP}(K, F) \), a problem that requires an \( x \) such that for almost all \( \omega \in \Omega \),

\[
K \ni x \perp F(x; \omega) \in K^*.
\] (2)

A natural question is how one may relate solutions of the almost sure formulation of the SVI to that of the expected value formulation. The following result provided without a proof clarifies the relationship.

**Proposition 1.** Suppose there exists a single \( x \in K \) such that \( x \) solves \( \text{VI}(K, F(; \omega)) \) (\( \text{CP}(K, F(; \omega)) \)) for almost every \( \omega \in \Omega \). Then \( x \in K \) is a solution of \( \text{VI}(K, F) \) (\( \text{CP}(K, F) \)).
Yet, we believe that obtaining such an \( x \) is possible only in pathological settings and this condition is relatively useless in deriving solvability statements for \( \text{SVI}(K,F) \) and its variants. Furthermore, this is a **stronger condition** than the ones we develop as evidenced by Example 1. In this instance, there is no such \( x \) that solves \( \text{VI}(K,F(\cdot;\omega)) \) for every \( \omega \) but the problem is indeed solvable.

If \( K \triangleq \mathbb{R}^n_+ \), this problem is a nonlinear complementarity problem (NCP) and for a fixed but arbitrary realization \( \omega \in \Omega \), the residual of this system can be minimized as follows:

\[
\minimize_{x \geq 0} \| \Phi(x;\omega) \|
\]

where \( \Phi(x;\omega) \) denotes the equation reformulation of the NCP (See [9]). Consequently, a solution to the almost-sure formulation is obtainable by considering the following minimization problem:

\[
\minimize_{x \geq 0} \mathbb{E} [ \| \Phi(x;\omega) \| ] .
\]

(3)

More precisely, if \( x \) is a solution of the almost-sure formulation of NCP \( (K,F) \) if and only if \( x \) is a minimizer of \( (3) \) with \( \mathbb{E} [ \| \Phi(x;\omega) \| ] = 0 \). If the Fischer-Burmeister \( \phi_{FB} \) is chosen as the C-function, the expected residual minimization (ERM) problem in [6, 11] solves the following stochastic program to compute a solution of the stochastic NCP (2):

\[
\begin{align*}
\minimize_{x \geq 0} & \quad \mathbb{E} [ \| \Phi_{FB}(x;\omega) \| ] , \\
\text{where} & \quad \Phi_{FB}(x;\omega) \triangleq \left( \sqrt{x_i^2 + (\Phi_i(x;\omega))^2} - (x_i + \Phi_i(x;\omega)) \right)_{i=1}^n ;
\end{align*}
\]

(4)

see [33, equation (3.8)]. In [37], Luo and Lin consider the almost-sure formulation of a stochastic complementarity problem and minimize the expected residual. Convergence analysis of the expected residual minimization (ERM) technique has been carried out in the context of stochastic Nash games [38] and stochastic variational inequality problems [36]. In more recent work, Chen, Wets, and Zhang [7] revisit this problem and present an alternate ERM formulation, with the intent of developing a smoothed sample average approximation scheme. In contrast with the expected-value formulation and the almost-sure formulation, Gwinner and Raciti [14, 15] consider an infinite-dimensional formulation of the variational inequality for capturing randomness and provide discretization-based approximation procedures for such problems.

The remainder of the paper is organized as follows. In section 2, we outline our assumptions used, motivate our study by considering two application instances, and provide the relevant background on integrating set-valued maps. In section 3, we recap our sufficiency conditions for the solvability of stochastic variational inequality problems with single and multi-valued mappings and we provide results for the stochastic quasi-variational inequality problems with single and multi-valued mappings. Refinements of the statements for SVIs are provided for the complementarity regime in Section 4 under varying assumptions on the map. Finally, in section 5, we revisit the motivating examples of section 2 and verify that the results developed in this paper are indeed applicable.

### 2 Assumptions, examples, and background

In Section 2.1, we provide a summary of the main assumptions employed in the paper. The utility of such models is demonstrated by discussing some motivating examples in Section 2.2. Finally, some background is provided on the integrals of set-valued maps in Section 2.3.

#### 2.1 Assumptions

We now state the main assumptions used throughout the paper and refer to them when appropriate. The first of these pertains to the probability space.

**Assumption 1** (Nonatomicity of measure \( \mathbb{P} \)). The probability space \( \mathcal{P} = (\Omega, \mathcal{F}, \mathbb{P}) \) is nonatomic.
The next assumption focuses on the properties of the single-valued map, referred to as $F$.

**Assumption 2** (Continuity and integrability of $F$).

(i) $F : \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ is a single-valued map. Furthermore, $F(x; \omega)$ is continuous in $x$ for almost every $\omega \in \Omega$ and is integrable in $\omega$, for every $x$.

(ii) $F(x)$ is continuous in $x$.

Note that, the assumption of Lipschitz continuity of $F(x; \omega)$ with an integrable Lipschitz constant implies that $\mathbb{E}[F(x; \omega)]$ is also Lipschitz continuous. The next two assumptions pertain to the set-valued maps employed in this paper. When the map is multi-valued, to avoid confusion, we employ the notation $\Phi(x)$, defined as $\Phi(x) \triangleq \mathbb{E}[\Phi(x; \omega)]$, and impose the following assumptions.

**Assumption 3** (Lower semicontinuity and integrability of $\Phi$). $\Phi : \mathbb{R}^n \times \Omega \rightarrow 2^{\mathbb{R}^n}$ is a set-valued map satisfying the following:

(i) $\Phi(x; \omega)$ has nonempty and closed images for every $x$ and every $\omega \in \Omega$.

(ii) $\Phi(x; \omega)$ is lower semicontinuous in $x$ for almost all $\omega \in \Omega$ and integrably bounded for every $x$.

Finally, when considering quasi-variational inequality problems, $K$ is a set-valued map, rather than a constant map.

**Assumption 4.** The set-valued map $K : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ is deterministic, closed-valued, convex-valued.

We conclude this subsection with some notation. $\text{cl}(U)$ denotes the closure of a set $U \subset \mathbb{R}^n$, $\text{bd}(U)$ denotes the boundary of the set $U$ and $\text{dom}(K)$ denotes the domain of the mapping $K$.

### 2.2 Examples

We now provide two instances of where stochastic variational problems arise in practice.

**Nonsmooth stochastic Nash-Cournot equilibrium problems:** Cournot’s oligopolistic model is amongst the most widely used models for modeling strategic interactions between a set of noncooperative firms [12]. Under an assumption that firms produce a homogenous product, each firm attempts to maximize profits by making a quantity decision while taking as given, the quantity of its rivals. Under the Cournot assumption, the price of the good is assumed to be dependent on the aggregate output in the market. The resulting Nash equilibrium, qualified as the Nash-Cournot equilibrium, represents a set of quantity decisions at which no firm can increase profit through a unilateral change in quantity decisions. unilaterally changing the quantity of the product it produces. To accommodate uncertainty in costs and prices in Nash-Cournot models and loss of differentiability of price functions which can occur for example by introduction of price caps [22] we consider a stochastic generalization of the classical deterministic Nash-Cournot model and allow for piecewise smooth price functions, as captured by the following assumption on costs and prices.

**Assumption 5.** Suppose the cost function $c_i(x_i)$ is an increasing convex twice continuously differentiable function for all $i = 1, \ldots, N$. Let $X \triangleq \sum_{i=1}^{n} x_i$. Since $x_i$ denotes the quantity produced, $x_i \geq 0$. The price function $p(X; \omega)$ is a decreasing piecewise smooth convex function where $p(X; \omega)$ is given by

$$
p(X; \omega) = \begin{cases} 
p^1(X; \omega), & 0 \leq X \leq \beta^1 \\
p^j(X; \omega), & \beta^{j-1} \leq X \leq \beta^j, j = 2, \ldots, s \\
p^s(X; \omega), & \beta^s \leq X \end{cases}
$$

(5)

where $p^j(X; \omega) = a^j(\omega) - b^j(\omega)X$ is a strictly decreasing affine function of $X$ for $j = 1, \ldots, s$. Finally, $\beta^1, \ldots, \beta^s$ are a set of increasing positive scalars and $(a^j(\omega), b^j(\omega))$ are positive in an almost-sure sense and integrable for $j = 1, \ldots, s$. 


Consider an $N$-player generalized Nash-Cournot game. Given the tuple of rival strategies $x_{-i}$, the $i$th player’s strategy set is given by $K_i(x_{-i})$ while his objective function is given by $E[f_i(x; \omega)] \triangleq c_i(x_i) - \mathbb{E}[p(x; \omega)x_i]$. Then $\{x_i^*\}_{i=1}^N$ denotes a stochastic Nash-Cournot equilibrium if $x_i^*$ solves the convex optimization problem $G_i(x_{-i}^*)$, defined as

$$
\begin{align*}
\text{minimize} & \quad E[f_i(x; \omega)] \\
\text{subject to} & \quad x_i \in K_i(x_{-i}).
\end{align*}
$$

The equilibrium conditions of this problem are given a stochastic QVI with multi-valued mappings. In section 5, we revisit this problem with the intent of developing existence statements.

**Strategic behavior in power markets:** Consider a power market model in which a collection of generation firms compete in a single-settlement market. Economic equilibria in power markets has been extensively studied using a complementarity framework; see [22, 21, 20]. Our model below is based on the model of Hobbs and Pang [22] which we modify to account for uncertainty in prices and costs.

Consider a set of nodes $\mathcal{N}$ of a network. The set of generation firms is indexed by $f$, where $f$ belongs to the finite set $\mathcal{F}$. At a node $i$ in the network, a firm $f$ may generate $g_{fi}$ units at node $i$ and sell $s_{fi}$ units to node $i$. The total amount of power sold to node $i$ by all generating firms is $S_i$. The generator firms’ profits are revenue less costs. If the nodal power price at node $i$ is a random variable $p_i(S_i; \omega)$ where $p_i(S_i; \omega)$ is a decreasing function of $S_i$, then the firms’ revenue is just the price times sales $s_{fi}$. The costs incurred by the firm $f$ at node $i$ are the costs of generating $g_{fi}$ and transmitting the excess ($s_{fi} - g_{fi}$). Let the cost of generation associated with firm $f$ at node $i$ be given by $c_{fi}(g_{fi}; \omega)$ and the cost of transmitting power from an arbitrary node (referred to as the hub) to node $i$ be given by $w_i$. The constraint set ensures a balance between sales and generation at all nodes, nonnegative sales and generation and generation subject to a capacity limit. The price and cost functions are assumed to satisfy the following requirement.

**Assumption 6.** For $i \in \mathcal{N}$, the price functions $p_i(S_i; \omega)$ is a decreasing function, bounded above by a nonnegative integrable function $\bar{p}_i(\omega)$. Furthermore, the cost functions $c_{fi}(g_{fi}; \omega)$ are nonnegative and increasing.

The resulting problem faced by the $f$th generating firm $f$ is to determine sales $s_{fi}$ and generation $g_{fi}$ at all nodes $i$ such that

$$
\begin{align*}
\text{maximize} & \quad \mathbb{E}\left[\sum_{i \in \mathcal{N}} (p_i(S_i; \omega)s_{fi} - c_{fi}(g_{fi}; \omega) - (s_{fi} - g_{fi})w_i)\right] \\
\text{subject to} & \quad \left\{\begin{array}{l}
0 \leq g_{fi} \leq \text{cap}_{fi} \\
0 \leq s_{fi}
\end{array}\right\}, \quad \forall i \in \mathcal{N},
\end{align*}
$$

and

$$
\sum_{i \in \mathcal{N}} (s_{fi} - g_{fi}) = 0.
$$

Note that, the generating firm sees the transmission fee $w_i$ and the rival firms’ sales $s_{-fi} \equiv \{s_{hi} : h \neq f\}$ as exogenous parameters to its optimization problem even though they are endogenous to the overall equilibrium model as we will see shortly.

The ISO sees the transmission fees $w = (w_i)_{i \in \mathcal{N}}$ as exogenous and prescribes flows $y = (y_i)_{i \in \mathcal{N}}$ as per the following linear program

$$
\begin{align*}
\text{maximize} & \quad \sum_{i \in \mathcal{N}} y_iw_i \\
\text{subject to} & \quad \sum_{i \in \mathcal{N}} \text{PDF}_{ij}y_i \leq T_j, \quad \forall j \in \mathcal{K},
\end{align*}
$$

where $\mathcal{K}$ is the set of all arcs or links in the network with node set $\mathcal{N}$, $T_j$ denotes the transmission capacity of link $j$, $y_i$ represents the transfer of power (in MW) by the system operator from a hub node to node node...
and PDF denotes the power transfer distribution factor, which specifies the MW flow through link $j$ as a consequence of unit MW injection at an arbitrary hub node and a unit withdrawal at node $i$.

Finally, to clear the market, the transmission flows $y_i$ must must balance the net sales at each node:

$$y_i = \sum_{f \in F} (s_{fi} - g_{fi}), \quad \forall i \in \mathcal{N}.$$

The above market equilibrium problem which comprises of each firm’s problem, the ISO’s problem and market-clearing condition, can be expressed as a stochastic complementarity problem by following the technique from [22]. This equivalent formulation of the above market equilibrium problem is described in the last section 5. We also illustrate the solvability of such problems using the framework developed in this paper.

### 2.3 Background on integrals of set-valued mappings

Recall that by Assumption 1, $(\Omega, \mathcal{F}, \mathbb{P})$ is a nonatomic continuous probability space. Consider a set-valued map $H$ that maps from $\Omega$ into nonempty, closed subsets of $\mathbb{R}^n$. We recall three definitions from [1, Ch. 8]

**Definition 2.1 (Measurable set-valued map).** A map $H$ is measurable if the inverse image of each open set in $\mathbb{R}^n$ is a measurable set: for all open sets $O \subseteq \mathbb{R}^n$, we have

$$H^{-1}(O) = \{ \omega \in \Omega \mid H(\omega) \cap O \neq \emptyset \} \in \mathcal{F}.$$

**Definition 2.2 (Integrably bounded set-valued map).** A map $H$ is integrably bounded if there exists a nonnegative integrable function $k \in L^1(\Omega, \mathbb{R}, \mathbb{P})$ such that

$$H(\omega) \subseteq k(\omega)B(0, 1) \text{ almost everywhere}.$$

**Definition 2.3 (Measurable selection).** Suppose a measurable map $h : \Omega \to \mathbb{R}^n$ satisfies $h(\omega) \in H(\omega)$ for almost all $\omega \in \Omega$. Then $h$ is called a measurable selection of $H$.

The existence of a measurable selection is proved in [1, Th. 8.1.3].

**Definition 2.4 (Integrable selection).** A measurable selection $h : \Omega \to \mathbb{R}^n$ is an integrable selection if $E[h(\omega)] < \infty$ where

$$E[h(\omega)] = \int_{\Omega} hd\mathbb{P} < \infty.$$

The set of all integrable selections of $H$ is denoted by $\mathcal{H}$ and is defined as follows:

$$\mathcal{H} \triangleq \{ h \in L^1(\Omega, \mathcal{F}, \mathbb{P}) : h(\omega) \in H(\omega) \text{ for almost all } \omega \in \Omega \}.$$

**Definition 2.5 (Expectation of a set-valued map).** The expectation of the set-valued map $H$, denoted by $E[H(\omega)]$, is the set of integrals of integrable selections of $H$:

$$E[H(\omega)] \triangleq \int_{\Omega} Hd\mathbb{P} \triangleq \left\{ \int_{\Omega} hd\mathbb{P} \mid h \in \mathcal{H} \right\}.$$

If the images of $H(\omega)$ are convex then this set-valued integral is convex [1, Definition 8.6.1]. If the assumption of convexity of images of $H$ does not hold, then the convexity of this integral follows from Th. 8.6.3 [1], provided that the probability measure is nonatomic. We make precisely such an assumption (See Assumption 1) and are therefore guaranteed that the integral of the set-valued map $H$ is a convex set [1] Th. 8.6.3.

Recall that, a point $\bar{z}$ of a convex set $K$ is said to be extremal if there are no two points $x, y \in K$ such that $\lambda x + (1 - \lambda)y = \bar{z}$ for $\lambda \in (0, 1)$ and is denoted by $\bar{z} \in \text{ext}(K)$. Similarly, as per Def. 8.6.5 [1], we define an extremal selection as follows:
**Definition 2.6** (Extremal selection). Given the convex set $\int_{\Omega} H dP$, an integrable selection $h \in \mathcal{H}$ is an extremal selection of $H$ if

$$\int_{\Omega} h dP$$

is an extremal point of the closure of the convex set $\int_{\Omega} H dP$.

The set of all extremal selections is denoted by $\mathcal{H}_e$ and is defined as follows:

$$\mathcal{H}_e \triangleq \left\{ h \in \mathcal{H} \mid \int_{\Omega} h dP \in \text{ext} \left( \text{cl} \left( \int_{\Omega} H dP \right) \right) \right\}.$$

By Theorem [1, Th. 8.6.3], we have the following Lemma for the representation of extremal points of closure of $\int_{\Omega} H dP$.

**Theorem 2** (Representation of extreme points of set-valued integral [1, Th. 8.6.3]). Suppose Assumption 1 holds and let $H$ be a measurable set-valued map from $\Omega$ to subsets of $\mathbb{R}^n$ with nonempty closed images. Then the following hold:

(a) $\int_{\Omega} H dP$ is convex and extremal points of $\text{cl}(\int_{\Omega} H dP)$ are contained in $\int_{\Omega} H dP$.

(b) If $x \in \text{ext} \left( \text{cl} \left( \int_{\Omega} H dP \right) \right)$, then there exists a unique $h \in \mathcal{H}_e$ with $x = \int_{\Omega} h dP$.

(c) If $H$ is integrably bounded, then the integral $\int_{\Omega} H dP$ is also compact.

As a corollary to the above theorem, we have a representation of points in a set-valued integral as

**Corollary 3** (Representation of points in a set-valued integral [1, Th. 8.6.6]). Let $H$ be a measurable integrably bounded set-valued map from $\Omega$ to subsets of $\mathbb{R}^n$ with nonempty closed images. If $P$ is nonatomic, then for every $x \in \int_{\Omega} H dP$, there exist $n + 1$ extremal selections $h_k \in \mathcal{H}_e$ and $n + 1$ measurable sets $A_k \in \mathcal{F}$, $k = 0, \cdots, n$, such that

$$x = \int_{\Omega} \left( \sum_{k=1}^{n} \chi_{A_k} h_k \right) dP$$

where $\chi_{A_k}$ is the characteristic function of $A_k$.

## 3 Stochastic quasi-variational inequality problems

In this section, we develop sufficiency conditions for the solvability of stochastic quasi-variational inequality problems under a diversity of assumptions on the map. More specifically, we begin by recapping sufficiency conditions for the solvability of stochastic variational inequality problems with single-valued and multi-valued maps in Section 3.1. In many settings, the variational inequality problems may prove incapable of capturing the problem in question. For instance, the equilibrium conditions of convex generalized Nash games are given by a quasi-variational inequality problem. As mentioned earlier, when the constant map $K$ is replaced by a set-valued map $K : \mathbb{R}^n \to \mathbb{R}^n$, the resulting problem is an SQVI. In this section, we extend the sufficiency conditions presented in the earlier section to accommodate the SQVI($K, F$) (Section 3.2) and SQVI($K, \Phi$) (Section 3.3), respectively. Throughout this section, Assumption 4 holds for the set-valued map $K$.

### 3.1 SVIs with single-valued and multi-valued mappings

In this section, we begin by assuming that the scenario-based mappings $F(\bullet; \omega)$ are single-valued for each $\omega \in \Omega$. With this assumption, we provide sufficient conditions that the scenario-based VI($K, F(\bullet; \omega)$) must satisfy in order to conclude the existence of solution to the stochastic SVI($K, F$) without requiring the evaluation of expectation operator. Recall that in SVI($K, F$), $F(x) = \mathbb{E}[F(x; \omega)]$. In particular, in the next proposition, we show that if a certain coercivity condition holds for the scenario-based map $F(\bullet; \omega)$ in an almost-sure sense then existence of the solution to the above SVI may be concluded without resorting to formal evaluation of the expectation.

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Proposition 4 (Solvability of SVI($K,F$)). Consider a stochastic variational inequality SVI($K,F$). Suppose Assumption 2 holds and $G(x;\omega) \triangleq F(x;\omega)^T(x-x^{ref})$. Suppose there exists an $x^{ref} \in K$ such that the following hold:

(i) $\liminf_{\|x\| \to \infty, x \in K} \left[ F(x;\omega)^T(x-x^{ref}) \right] > 0$ almost surely;

(ii) Suppose there exists a nonnegative integrable function $u(\omega)$ such that $G(x;\omega) \geq -u(\omega)$ holds almost surely for any $x$.

Then the stochastic variational inequality SVI($K,F$) has a solution.

Proof. See appendix.

In settings where $K$ is a Cartesian product, defined as $K \triangleq \prod_{\nu=1}^{N} K_{\nu}$, (6) VI($K,F$) is a partitioned variational inequality problem, as defined in [9, Ch. 3.5]. Accordingly, Proposition 4 can be weakened so that even if the coercivity property holds for just one index $\nu \in \{1,\ldots,N\}$, the stochastic variational inequality is solvable.

Proposition 5 (Solvability of SVI($K,F$) for Cartesian $K$). Consider a stochastic variational inequality SVI($K,F$) where $K$ is a Cartesian product of closed and convex sets as specified in (6). Suppose that Assumption 2 and the following hold:

(i) There exists an $x^{ref} \in K$ and an index $\nu \in \{1,\ldots,N\}$ such that for any $x \in K$,
$$\liminf_{\|x_{\nu}\| \to \infty, x_{\nu} \in K_{\nu}} \left[ F_{\nu}(x;\omega)^T(x_{\nu}-x_{\nu}^{ref}) \right] > 0,$$
holds in an almost sure sense; and

(ii) For the above $\nu$ and for any $x$, let $G(x;\omega) = F_{\nu}(x;\omega)^T(x_{\nu}-x_{\nu}^{ref})$. Suppose there exists a nonnegative integrable function $u(\omega)$ such that $G(x;\omega) \geq -u(\omega)$ holds almost surely for any $x$.

Then SVI($K,F$) admits a solution.

Proof. See appendix.

We now present a weaker set of sufficient conditions for existence under the assumption that the mapping $F(x;\omega)$ is a monotone mapping over $K$ for almost every $\omega \in \Omega$.

Corollary 6 (Solvability of SVI($K,F$) under monotonicity). Consider SVI($K,F$) and suppose that Assumption 2 holds. Further, let $F(x;\omega)$ be a monotone mapping on $K$ for almost every $\omega \in \Omega$. Suppose there exists an $x^{ref} \in K$ such that $G(x;\omega) = F(x^{ref};\omega)^T(x-x^{ref})$ and the following hold:

(i) $\liminf_{\|x\| \to \infty, x \in K} \left[ F(x^{ref};\omega)^T(x-x^{ref}) \right] > 0$ holds in almost sure sense;

(ii) Suppose there exists a nonnegative integrable function $u(\omega)$ such that $G(x;\omega) \geq -u(\omega)$ holds almost surely for any $x$.

Then SVI($K,F$) is solvable.

Proof. See appendix.

Next, we consider a stochastic variational inequality SVI($K,\Phi$) where $\Phi(x) \triangleq E[\Phi(x;\omega)]$ and $\Phi(x;\omega)$ is a multi-valued mapping. Before proceeding to prove existence of solutions to SVIs with multi-valued maps, we restate Corollary 3 for the case of the set-valued integral $\Phi(x) = E[\Phi(x;\omega)]$ of interest.
Lemma 7 (Representation of elements of set-valued integral as an integral of convex combination of extremal selections). Suppose Assumption 1 holds. Let \( \Phi \) be a measurable integrably bounded set-valued map from \( \mathbb{R}^n \times \Omega \) to subsets of \( \mathbb{R}^n \) with closed nonempty images. Then any \( w \in \mathbb{E}[\Phi(x; \omega)] \) can be expressed as

\[
w = \int_{\Omega} g(x; \omega) d\mathbb{P}
\]

where \( g(x; \omega) = \sum_{k=1}^{n} \lambda_k(x) f_k(x; \omega) \) and \( \lambda_k(x) \geq 0, \sum_{k=0}^{n} \lambda_k(x) = 1 \) and each \( f_k(x; \omega) \) is an extremal selection of \( \Phi(x; \omega) \).

Proof. Since \( w \in \mathbb{E}[\Phi(x; \omega)] \) and \( \mathbb{E}[\Phi(x; \omega)] \) is a convex set, thus by Carathéodory’s theorem for convex sets, there exists \( \lambda_k(x) \geq 0, w_k \in \text{ext}(\text{cl}(\mathbb{E}[\Phi(x; \omega)])) \) such that

\[
w = \sum_{k=0}^{n} \lambda_k(x) w_k, \quad \sum_{k=0}^{n} \lambda_k(x) = 1
\]

Now, since \( w_k \in \text{ext}(\text{cl}(\mathbb{E}[\Phi(x; \omega)])) \), by [1, Th. 8.6.3], for each index \( k \), there exists an extremal selection \( f_k(x; \omega) \) from \( \Phi(x; \omega) \) such that

\[
\int_{\Omega} f_k(x; \omega) d\mathbb{P} = w_k.
\]

Thus, we obtain

\[
w = \sum_{k=0}^{n} \lambda_k(x) \int_{\Omega} f_k(x; \omega) d\mathbb{P},
\]

which can be rewritten as

\[
w = \int_{\Omega} g(x; \omega) d\mathbb{P}
\]

where \( g(x; \omega) = \sum_{k=0}^{n} \lambda_k(x) f_k(x; \omega) \). The required representation result follows.

We begin by providing a coercivity-based sufficiency condition for deterministic multi-valued variational inequalities [26].

Theorem 8 (Solvability of VI \((K, \Phi)\)). Suppose \( K \) is a closed and convex set in \( \mathbb{R}^n \) and let \( \Phi : K \rightrightarrows \mathbb{R}^n \) be a lower semicontinuous multifunction with nonempty closed and convex images. Consider the following statements:

(a) Suppose there exists an \( x_{\text{ref}} \in K \) such that \( L_{<}(K, \Phi) \) is bounded (possibly empty) where

\[
L_{<}(K, \Phi) \triangleq \left\{ x \in K : \inf_{y \in \Phi(x)} (x - x_{\text{ref}})^T y < 0 \right\}.
\]

(b) The variational inequality VI \((K, \Phi)\) is solvable

Then, (a) implies (b). Furthermore, if \( \Phi(x) \) is a pseudomonotone mapping over \( K \), then (a) is equivalent to (b).

Using this condition, we proceed to develop sufficiency conditions for the existence of solutions to SVI \((K, \Phi)\).

Proposition 9 (Solvability of SVI \((K, \Phi)\)). Consider SVI \((K, \Phi)\) and suppose assumptions 1 and 3 hold. Further, suppose the following hold:

(i) Suppose there exists an \( x_{\text{ref}} \in K \) such that

\[
\liminf_{x \in K, \|x\| \to \infty} \left( \inf_{w \in \Phi(x; \omega)} w^T(x - x_{\text{ref}}) \right) > 0 \text{ almost surely.}
\]

(ii) For the above \( x_{\text{ref}} \), suppose there exists a nonnegative integrable function \( U(\omega) \) such that \( g(x; \omega)^T(x - x_{\text{ref}}) \geq -U(\omega) \) holds almost surely for any integrable selection \( g(x; \omega) \) of \( \Phi(x; \omega) \) and for any \( x \).

Then SVI \((K, \Phi)\) is solvable.
Proof. The proof proceeds in two parts.

(a) We first show that the following coercivity condition holds for the expected value map: there exists an \( x^{\text{ref}} \in K \) such that

\[
\liminf_{x \in K, \|x\| \to \infty} \left( \inf_{w \in \Phi(x)} w^T (x - x^{\text{ref}}) \right) > 0. \tag{7}
\]

(b) If (a) holds, then we show that for the given \( x^{\text{ref}} \in K \), the set \( L_<(K, \Phi) \) is bounded (possibly empty) where

\[
L_<(K, \Phi) \triangleq \left\{ x \in K : \inf_{y \in \Phi(x)} (x - x^{\text{ref}})^T y < 0 \right\}. \tag{8}
\]

Proof of (a): We proceed by a contradiction and assume that (7) does not hold for the expected value map. Thus, for any \( x \)

\[
\text{Since } \Phi(\cdot) \text{ for some } y.
\]

Now, \( \text{By hypothesis (ii), we may use Fatou’s Lemma to interchange the order of integration and limit infimum, as shown next:}

\[
\int_\Omega \liminf_{k \to \infty} \left[ \int_\Omega \Phi(x_k; \omega)\right]^T (x_k - x^{\text{ref}}) dP \leq 0.
\]

Consequently, there is a set of positive measure \( U \subseteq \Omega \), over which the integrand is nonpositive or

\[
\liminf_{k \to \infty} \left[ \int_\Omega \Phi(x_k; \omega)\right]^T (x_k - x^{\text{ref}}) dP \leq 0, \quad \forall \omega \in U.
\]

Substituting the expression for \( g_k \), we obtain the following inequality.

\[
\liminf_{k \to \infty} \left[ (x_k - x^{\text{ref}})^T \left( \sum_{l=1}^n \lambda_l(x_k) f_l(x_k; \omega) \right) \right] \leq 0, \quad \forall \omega \in U.
\]

As a result, for at least one index \( l \in \{1, ..., n\} \), we have that

\[
\liminf_{k \to \infty} \left[ \lambda_l(x_k) f_l(x_k; \omega)^T (x_k - x^{\text{ref}}) \right] \leq 0, \quad \forall \omega \in U.
\]

Since \( 0 \leq \lambda_l(x_k) \leq 1 \), the following must be true for the above \( l \):

\[
\liminf_{k \to \infty} \left[ f_l(x_k; \omega)^T (x_k - x^{\text{ref}}) \right] \leq 0, \quad \forall \omega \in U.
\]
Moreover, \( f_l(x_k; \omega) \in \Phi(x_k; \omega) \) since it is an extremal selection and we have that
\[
\liminf_{k \to \infty} \left[ \inf_{w \in \Phi(x_k; \omega)} w^T (x_k - x^\text{ref}) \right] \leq 0, \quad \forall \omega \in U.
\]

Since, this holds for any \( x^\text{ref} \), it holds for the vector \( x^\text{ref} \) in the hypothesis and for a set of positive measure \( U \), we have that
\[
\liminf_{x \in K, \|x\| \to \infty} \left[ \inf_{w \in \Phi(x; \omega)} w^T (x - x^\text{ref}) \right] \leq 0, \forall \omega \in U.
\]

This contradicts the hypothesis and condition (7) must hold for the expected value map.

**Proof of (b)** Next, we show that when the condition (7) holds for the expected value map, then for the given \( x^\text{ref} \in K \) the set \( L_<(K, \Phi) \) is bounded (possibly empty) where
\[
L_<(K, \Phi) \triangleq \left\{ x \in K : \inf_{y \in \Phi(x)} [(x - x^\text{ref})^T y] < 0 \right\}.
\]

If the set \( L_<(K, \Phi) \) is empty, then the result follows by Theorem 8. Suppose \( L_<(K, \Phi) \) is nonempty and unbounded. Then, there exists a sequence \( \{x_k\} \in L_<(K, \Phi) \) with \( \|x_k\| \to \infty \). Since \( x_k \in L_<(K, \Phi) \), we have for each \( k \),
\[
\inf_{y \in \Phi(x_k)} [(x_k - x^\text{ref})^T y] < 0.
\]

This implies that for the sequence \( \{x_k\} \), we have that
\[
\liminf_{x_k \in K, \|x_k\| \to \infty} \left[ \inf_{w \in \Phi(x_k)} w^T (x_k - x^\text{ref}) \right] \leq 0.
\]

But this contradicts the coercivity property of the expected value map proved earlier:
\[
\liminf_{x \in K, \|x\| \to \infty} \left[ \inf_{w \in \Phi(x)} w^T (x - x^\text{ref}) \right] > 0.
\]

This contradiction implies that \( L_<(K, \Phi) \) is bounded and by Theorem 8 SVI\((K, \Phi)\) is solvable.

### 3.2 SQVIs with single-valued mappings

Our first result is an extension of [9, Cor. 2.8.4] to the stochastic regime. In particular, we assume that the mapping \( \mathbb{E}[F(x; \omega)] \) cannot be directly obtained; instead, we provide an existence statement that relies on the scenario-based map \( F(x; \omega) \).

**Proposition 10** (Solvability of SQVI\((K, F)\)). Suppose Assumptions 2 and 3 hold. Furthermore, suppose there exists a bounded open set \( U \subset \mathbb{R}^n \) and a vector \( x^\text{ref} \in U \) such that the following hold:

(a) For every \( \bar{x} \in \text{cl}(U) \), the image \( K(\bar{x}) \) is nonempty and \( \lim_{\bar{x} \to \bar{x}} K(x) = K(\bar{x}) \);

(b) \( x^\text{ref} \in K(x) \) for every \( x \in \text{cl}(U) \);

(c) \( L_<(K, F; \omega) \cap \text{bd}(U) = \emptyset \) holds almost surely, where
\[
L_<(K, F; \omega) \triangleq \left\{ x \in K(x) : (x - x^\text{ref})^T F(x; \omega) < 0 \right\}.
\]

Then, SQVI\((K, F)\) has a solution.

**Proof.** Recall that by [9, Cor. 2.8.4], the stochastic SQVI\((K, F)\) is solvable if \( L_<(K, F) \cap \text{bd}(U) = \emptyset \), where
\[
L_<(K, F) \triangleq \left\{ x \in K(x) : (x - x^\text{ref})^T \mathbb{E}[F(x; \omega)] < 0 \right\}.
\]

We proceed by contradiction and assume that there exists an \( x \in L_<(K, F) \) and \( x \in \text{bd}(U) \). By assumption, \( x \notin L_<(K, F; \omega) \) for any \( \omega \) implying that \( (x - x^\text{ref})^T F(x; \omega) \geq 0 \) for all \( \omega \in \Omega \). It follows that \( (x - x^\text{ref})^T \mathbb{E}[F(x; \omega)] \geq 0 \), implying that \( x \notin L_<(K, F) \). This contradicts our assertion that \( x \in L_<(K, F) \). Therefore, we must have that \( L_<(K, F) \cap \text{bd}(U) = \emptyset \) and by [9, Cor. 2.8.4], the stochastic SQVI\((K, F)\) has a solution.

\[\square\]
Another avenue for ascertaining existence of equilibrium in stochastic regimes is an extension of Harker’s result [17, Th. 2] which we present next.

**Theorem 11** (Solvability of SQVI($K, F$) under compactness). Suppose Assumptions 2 and 4 hold and there exists a nonempty compact convex set $\Gamma$ such that the following hold:

(i) $K(x) \subseteq \Gamma, \forall x \in \Gamma$;

(ii) $K$ is a nonempty, continuous, closed and convex-valued mapping on $\Gamma$.

Then the SQVI($K, F$) has a solution.

**Proof.** Since $F$ is continuous by Assumption 2(ii), all conditions of [17, Th. 2] hold. Thus, the SQVI($K, F$) has a solution.

The above theorem relies on properties of the map $K$ and the continuity of the map $F$ to ascertain existence of solution. By Assumption 2, continuity of the map $F$ holds in the settings we consider and thus the solvability of SQVI($K, F$) follows readily. This theorem has a slightly different flavor compared to other results in this paper in the sense that we do not look at properties of the scenario-based map (other than continuity) that then guarantee existence of solution. We have listed this theorem here for completeness as it presents an alternate perspective of looking at the question of solvability of SQVI($K, F$).

### 3.3 SQVIs with multi-valued mappings

In this section, we relax the assumption of single-valuedness of the scenario-based mappings $F(\bullet; \omega)$ and instead allow for the map $\Phi(\bullet; \omega)$ to be multi-valued. In the spirit of the rest of this paper, our interest lies in deriving results that do not rely on evaluation of expectation. We use the concepts of set-valued integrals discussed in the previous section 2.3 and require that Assumption 4 holds throughout this subsection. Our first existence result relies on a sufficiency condition for generalized quasi-variational inequalities [5, Cor. 3.1]. We recall [5, Cor. 3.1] which can be applied to the multi-valued SQVI($K, \Phi$).

**Proposition 12.** Consider the SQVI($K, \Phi$). Suppose that there exists a nonempty compact convex set $C$ such that the following hold:

(a) $K(C) \subseteq C$;

(b) $E[\Phi(x; \omega)]$ is a nonempty contractible-valued and compact-valued upper semicontinuous mapping on $C$;

(c) $K$ is nonempty continuous convex-valued mapping on $C$.

Then the stochastic SQVI($K, \Phi$) admits a solution.

However, this result requires evaluating $E[\Phi(x; \omega)]$, an object that admits far less tractability; instead, we develop almost-sure sufficiency conditions that imply the requirements of Proposition 12.

**Proposition 13** (Solvability of SQVI($K, \Phi$))). Suppose Assumptions 4 and 3 hold. Furthermore, suppose there exists a nonempty compact convex set $C$ such that the following hold:

(a) $K(C) \subseteq C$;

(b) $\Phi(x; \omega)$ is a nonempty upper semicontinuous mapping for $x \in C$ in an almost-sure sense;

(c) $K$ is nonempty, continuous and convex-valued mapping on $C$.

Then the stochastic SQVI($K, \Phi$) admits a solution.

**Proof.** From Proposition 12 it suffices to show that under the above assumptions, $E[\Phi(x; \omega)]$ is a nonempty contractible-valued, compact-valued, upper semicontinuous mapping on $C$. 

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(i) $E[\Phi(x; \omega)]$ is nonempty: Since $\Phi(x; \omega)$ is lower semicontinuous, it is a measurable set-valued map. Since it is a measurable set-valued map with nonempty closed images, by Aumann’s measurable selection theorem [1, Th. 8.1.3], there exists a measurable selection $h$ from $\Phi(x; \omega)$. Since $\Phi(x; \omega)$ is integrably bounded, every measurable selection is integrable. Thus, $\int_{\Omega} h dP \in \int_{\Omega} \Phi(x; \omega)$, implying that $E[\Phi(x; \omega)]$ is nonempty.

(ii) $E[\Phi(x; \omega)]$ is contractible-valued: Since the probability space is nonatomic by definition, we have that $E[\Phi(x; \omega)]$ is a convex set. Since a convex set is contractible, we have that $E[\Phi(x; \omega)]$ is contractible.

(iii) $E[\Phi(x; \omega)]$ is compact-valued: Since $\Phi(x; \omega)$ is integrably bounded, by [1, Th. 8.6.3], we get that $E[\Phi(x; \omega)]$ is compact.

(iv) $E[\Phi(x; \omega)]$ is upper semicontinuous: By hypothesis, we have that $\Phi(x; \omega)$ is a measurable, integrably bounded and upper-semicontinuous $x \in C$. Thus, from [2, Cor. 5.2], it follows that $E[\Phi(x; \omega)]$ is upper semicontinuous.

The previous result relies on the compact-valuedness of $K$ with respect to a compact set $C$, a property that cannot be universally guaranteed. An alternate result for deterministic generalized QVI problems [5, Cor. 4.1] leverages coercivity properties of the map $\Phi(x)$ to claim existence of a solution. We state this result next.

**Proposition 14.** Let $K$ be a set-valued map from $\mathbb{R}^n$ to subsets of $\mathbb{R}^n$ and $\Phi$ from $\mathbb{R}^n$ to subsets of $\mathbb{R}^n$ be a measurable integrably bounded set-valued map with closed nonempty images. Suppose that there exists a vector $x^{ref}$ such that $x^{ref} \in \bigcap_{x \in \text{dom}(K)} K(x)$ and $\lim_{\|x\| \to \infty, x \in K(x)} \left[ \inf_{y \in \Phi(x)} \frac{(x - x^{ref})^T y}{\|x\|} \right] = \infty.$

Suppose the following hold:

(i) $\Phi(x)$ is a nonempty, contractible-valued, compact-valued, upper semicontinuous map on $\mathbb{R}^n$;

(ii) $K$ is convex-valued;

(iii) There exists a $\rho_0 > 0$ such that $K(x) \cap B_\rho$ is a continuous mapping for all $\rho > \rho_0$ where $B_\rho$ is a ball of radius $\rho$ centered at the origin.

Then for each vector $q$, SQVI($K, \Phi + q$) has a solution. Moreover, there exists an $r > 0$ such that $\|x^*\| < r$ for each solution $(x^*, y^*)$.

In the next proposition, by using the properties of $\Phi(x; \omega)$, we develop an integration-free analog of this result for multi-valued SQVI($K, \Phi$).

**Proposition 15 (Solvability of SQVI($K, \Phi$)).** Suppose Assumptions 3 and 4 hold. Suppose that there exists a vector $x^{ref}$ such that

(i) $x^{ref} \in \bigcap_{x \in \text{dom}(K)} K(x)$;

(ii) $\lim_{\|x\| \to \infty, x \in K(x)} \left[ \inf_{y \in \Phi(x; \omega)} \frac{(x - x^{ref})^T y}{\|x\|} \right] = \infty, \ a.s.$

(iii) For the above $x^{ref}$, suppose there exists a nonnegative integrable function $U(\omega)$ such that $g(x; \omega)^T (x - x^{ref}) \geq -U(\omega)$ holds almost surely for any integrable selection $g(x; \omega)$ of $\Phi(x; \omega)$ and for any $x$. 

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(iv) $\Phi(x; \omega)$ is an upper semicontinuous mapping on $\mathbb{R}^n$ in an almost-sure sense;

(v) There exists a $\rho_0 > 0$ such that $K(x) \cap B_{\rho}$ is a continuous mapping for all $\rho > \rho_0$.

Then for each vector $q$ the stochastic SQVI $K, \Phi + q$ has a solution. Moreover, there exists an $r > 0$ such that $\|x^*\| < r$ for each solution $(x^*, y^*)$.

**Proof.** First we show that (11) implies that the coercivity property (10) holds for the expected-value map $\|\cdot\|$. Suppose $\|x\| \rightarrow \infty$ for each $x 

consequently, for $j \in \mathbb{N}$, we obtain that

$$
\lim_{\|x\| \rightarrow \infty, x \in K(x)} \left[ \frac{(x - x^{ref})^T y}{\|x\|} \right] < \infty, \text{ a.s.}
$$

Substituting the expression for $g_k(x; \omega)$ in terms of extremal selections $f_j(x_k; \omega)$ from $\Phi(x; \omega)$, it follows that

$$
\lim_{\|x\| \rightarrow \infty, x \in K(x)} \left[ \frac{(x - x^{ref})^T \sum_{j=0}^n \lambda_j(x_k) f_j(x_k; \omega)}{\|x\|} \right] < \infty, \text{ a.s.}
$$

Consequently, for $j \in \{0, \cdots, n\}$, the following holds

$$
\lim_{\|x\| \rightarrow \infty, x \in K(x)} \left[ \frac{(x - x^{ref})^T \lambda_j(x_k) f_j(x_k; \omega)}{\|x\|} \right] < \infty, \text{ a.s.}
$$

(12)

We may now claim that for some index $j$,

$$
\lim_{\|x\| \rightarrow \infty, x \in K(x)} \left[ \frac{(x - x^{ref})^T f_j(x_k; \omega)}{\|x\|} \right] < \infty, \text{ a.s.}
$$

(13)

Suppose this claim is false, then for $j \in \{0, \cdots, n\}$, for $\omega \in U_j$, a set of positive measure, we have the following:

$$
\lim_{\|x\| \rightarrow \infty, x \in K(x)} \left[ \frac{(x - x^{ref})^T f_j(x_k; \omega)}{\|x\|} \right] = \infty.
$$

(14)
Since the denominator in (14) goes to $+\infty$ as $k \to \infty$, this implies that numerator goes to $+\infty$ at a faster rate than the denominator $\|x_k\|$; in effect, the numerator is $\Omega(\|x_k\|^n)$ where $n_j > 1$ and $v_k \in \Omega(u_k)$ implies that $v_k \geq u_k$ for sufficiently large $k$. Thus, for $j \in \{0, \ldots, n\}$ and $\omega \in U_j \subset \Omega$, where $U_j$ has positive measure, we have the following:

$$\begin{align*}
(x_k - x^{ref})^T f_j(x_k; \omega) &= \Omega(\|x_k\|^n), \quad \text{where } n_j > 1. 
\end{align*}$$

But from (12), for $j \in \{0, \ldots, n\}$ and for $\omega \in U_j$,

$$\lim_{\|x_k\| \to \infty, x_k \in K(x_k)} \left[ \frac{\lambda_j(x_k)(x_k - x^{ref})^T f_j(x_k; \omega)}{\|x_k\|} \right] < \infty,$$

whereby (15) implies that we must have that for each $j$, as $k \to \infty$, $\lambda_j(x_k) \to 0$ faster than $(x_k - x^{ref})^T f_j(x_k; \omega) \to +\infty$ to ensure that the limit given by (16) remains finite. But this leads to a contradiction since for each $k$, by construction, we have that for $j \in \{0, \ldots, n\}$, $\lambda_j(x_k) \geq 0$ and $\sum_{j=0}^n \lambda_j(x_k) = 1$. The resulting contradiction proves that for some $j$, we must have (13) holds, where $f_j(x_k; \omega)$ is an extremal selection of $\Phi(x_k; \omega)$. This implies that

$$\lim_{\|x_k\| \to \infty, x_k \in K(x_k)} \left[ \inf_{y \in \Phi(x_k; \omega)} (x_k - x^{ref})^T y \right] < \infty, \quad \text{a.s.}$$

which is in contradiction to hypothesis (11). It follows that the coercivity requirement (10) holds for $\Phi(x) = E[\Phi(x; \omega)]$.

Further, from the proof of the Proposition 13, we may claim that $E[\Phi(x; \omega)]$ is a nonempty contractible-valued, compact-valued, upper semicontinuous mapping on $\mathbb{R}^n$ and from Assumption 4, the map $K$ is convex-valued. Thus, all the hypotheses of Proposition 14 are satisfied and the multi-valued SQVI($K, \Phi$) admits a solution.

\section{Stochastic complementarity problems}

When the set $K$ in a VI($K, F$) is a cone in $\mathbb{R}^n$, then the VI($K, F$) is equivalent to a CP($K, F$) [25]. Our approach in the previous sections required us to assume that the map $K$ was deterministic. In practical settings, however, the map $K$ may take on a variety of forms. For instance, $K$ may be defined by a set of algebraic resource or budget constraints in financial applications, capacity constraints in network settings or supply and demand constraints in economic equilibrium settings. Naturally, these constraints may often be expectation or risk-based constraints. In such an instance, a complementarity approach assumes relevance; specifically, in such a case, this problem is defined in the joint space of primal variables and the Lagrange multipliers corresponding to the stochastic constraints. Such a transformation yields an SCP($K, H$) where the map $H$ may be expectation-valued while the set $K$ is a deterministic cone. However, such complementarity problems may also arise naturally, as is the case when modeling frictional contact problems [24]; and stochastic counterparts of such problems emerge from attempting to model risk and uncertainty. In the remainder of this section, we consider complementarity problems with single-valued maps.

Before proceeding, we provide a set of definitions.

\textbf{Definition 4.1} (CP($K, q, M$)). Given a cone $K$ in $\mathbb{R}^n$, an $n \times n$ matrix $M$ and a vector $q \in \mathbb{R}^n$, the complementarity problem CP($K, q, M$) requires an $x \in K, Mx + q \in K^*$ such that $x^T(Mx + q) = 0$.

Recall, from section 1.1 $K^* \triangleq \{ y : y^T d \geq 0, \forall d \in K \}$. The recession cone, denoted by $K_\infty$, is defined as follows.

\textbf{Definition 4.2} (Recession cone $K_\infty$). The recession cone associated with a set $K$ (not necessarily a cone) is defined as $K_\infty \triangleq \{ d : \text{for some } x \in K, \{ x + td : t \geq 0 \} \in K \}$.

Note that when $K$ is a closed and convex cone, $K_\infty = K$. Next, we define the CP kernel of a pair $(K, M)$ and define its $R_0$ variant.
Theorem 17. Let \( K \) be a closed and convex cone in \( \mathbb{R}^n \) and let \( F \) be a continuous map from \( K \) into \( \mathbb{R}^n \). If there exists a copositive matrix \( E \in \mathbb{R}^{n \times n} \) on \( K \) such that \((K,E)\) is an \( \mathbf{R}_0 \) pair and the union

\[
\bigcup_{\tau > 0} \text{SOL}(K,F + \tau E)
\]

is bounded, then the CP\((K,F)\) has a solution.

Recall that, in our notation for SCP\((K,H)\), \( H(x) = \mathbb{E}[H(x;\omega)] \). We now present an intermediate result required in deriving an integration-free sufficiency condition.

Lemma 18. Let \( K = \mathbb{R}^n_+ \) and let \( H(x;\omega) \) be a mapping that satisfies Assumption 2. Given an \( \omega \in \Omega \), suppose the following holds:

\[
\liminf_{\|x\| \to \infty, \omega \in K} H(x;\omega) > 0. \tag{19}
\]
Then, there exists a copositive matrix $M_\omega \in \mathbb{R}^{n \times n}$ on $K$ such that $(K, M_\omega)$ is an $R_0$ pair and $T_\omega$ is bounded where
\[
T_\omega \triangleq \bigcup_{\tau > 0} \text{SOL}(K, H(\bullet; \omega) + \tau M_\omega). \tag{20}
\]

Further, if $M_\omega \neq 0$, without loss of generality we may assume that $\|M_\omega\| = 1$.

**Proof.** We proceed by contradiction and assume that there is no copositive matrix $M_\omega$ where $(K, M_\omega)$ is an $R_0$ pair and the set $T_\omega$ is bounded where $T_\omega$ is given by (20). Therefore for any copositive matrix $M$ with $(K, M)$ an $R_0$ pair, the set $T$ is always unbounded where
\[
T \triangleq \bigcup_{\tau > 0} \text{SOL}(K, H(\bullet; \omega) + \tau M).
\]

Since $T$ is unbounded, by definition, there exists a sequence $\{x_k\} \in T$ and a sequence $\{\tau_k\} > 0$, with $\lim_{k \to \infty} \|x_k\| = \infty$, $x_k \geq 0$, $H(x_k; \omega) + \tau_k x_k^T M x_k \geq 0$ and $x_k^T H(x_k; \omega) + \tau_k x_k^T M x_k = 0$.

Now, $M$ is copositive on $K$ and $\|x_k\| \to \infty$ implies that the sequence $\{x_k^T M x_k\}$ goes to $\infty$. Since $\tau_k > 0$ and $\{x_k^T M x_k\}$ goes to $\infty$, the sequence $\{\tau_k x_k^T M x_k\}$ either goes to $\infty$ or 0. This together with $x_k^T H(x_k; \omega) + \tau_k x_k^T M x_k = 0$ implies that $x_k^T H(x_k; \omega)$ goes to $-\infty$ or 0. From (19) and since $\|x_k\| \to \infty$, we get that $x_k^T H(x_k; \omega) > 0$ for large $k$, which contradicts the assertion that $x_k^T H(x_k; \omega)$ goes to $-\infty$ or 0. The boundedness of $T$ follows.

Further, if $M_\omega \neq 0$, taking $\beta = \frac{1}{\|M_\omega\|}$, by Lemma 16, we may use $\beta M_\omega$ instead of $M_\omega$ in (20). Therefore, without loss of generality, we may assume that $\|M_\omega\| = 1$. \hfill \square

We are now prepared to prove our main result.

**Theorem 19 (Solvability of SCP($K, H$)).** Consider the stochastic complementarity problem SCP($K, H$) where $K$ is the nonnegative orthant. Suppose Assumption 2 holds for the mapping $H$ and $G(x; \omega) \triangleq x^T H(x; \omega)$. Further suppose the following hold:

(i) \[
\liminf_{\|x\| \to \infty, x \in K} H(x; \omega) > 0, \text{ a.s.} \tag{21}
\]

(ii) Suppose there exists a nonnegative integrable function $u(\omega)$ such that $G(x; \omega) \geq -u(\omega)$ holds almost surely for any $x$.

Then the stochastic complementarity problem SCP($K, H$) admits a solution.

**Proof.** Note that if (21) holds, since $K$ is the nonnegative orthant and $\|x_k\| \to \infty$ as $k \to \infty$, we must have $\|x_k\| > 0$ for sufficiently large $k$. This allows us to conclude that \[
\liminf_{\|x\| \to \infty, x \in K} \left[ x^T H(x; \omega) \right] > 0, \text{ a.s.} \tag{22}
\]

In other words, if hypothesis (21) in Lemma 18 holds, then (22) holds.

From hypothesis (21) in Lemma 18, we may conclude that for almost every $\omega \in \Omega$, there exists a copositive matrix $M_\omega \in \mathbb{R}^{n \times n}$ on $K$ such that $(K, M_\omega)$ is an $R_0$ pair and the union $T_\omega$ is bounded where \[
T_\omega \triangleq \bigcup_{\tau > 0} \text{SOL}(K, H(\bullet; \omega) + \tau M_\omega). \tag{23}
\]

Observe that, by Lemma 18 for each $\omega$ for which $M_\omega \neq 0$, we may assume that $\|M_\omega\| = 1$. Consequently, $E[\|M_\omega\|] < +\infty$ and the integrability of $M_\omega$ follows.

We will prove our main result by using Theorem 17. In particular, we show that there exists a copositive matrix $M$, defined as $M \triangleq E[M_\omega] \in \mathbb{R}^{n \times n}$, where $M$ is copositive on $K$, $(K, M)$ is an $R_0$ pair, and the set $T$ is bounded, where
\[
T \triangleq \bigcup_{\tau > 0} \text{SOL}(K, H + \tau M).
\]

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(i) \( M = \mathbb{E}[M_\omega] \) is copositive on \( K \): Consider the matrix \( \mathbb{E}[M_\omega] \). Since \( M_\omega \) is copositive on \( K \) in an almost-sure sense, it follows that

\[
x^T \mathbb{E}[M_\omega] x = \mathbb{E} \left[ x^T M_\omega x \right] \geq 0, \quad \forall \, x \in K,
\]

implying that \( M = \mathbb{E}[M_\omega] \) is a copositive matrix on \( K \).

(ii) \((K, M)\) is an \( R_0 \) pair: We need to show that \( \text{SOL}(K_\infty, 0, \mathbb{E}[M_\omega]) = \{0\} \). Since \( K \) is a closed and convex cone, \( K_\infty = K \) and it suffices to show that \( \text{SOL}(K, 0, \mathbb{E}[M_\omega]) = \{0\} \). Clearly, \( 0 \in \text{SOL}(K, 0, \mathbb{E}[M_\omega]) \). It remains to show that \( \text{SOL}(K, 0, \mathbb{E}[M_\omega]) \neq 0 \). Suppose this claim is false and there exists a \( d \in K \) such that \( d \neq 0 \) and \( d \in \text{SOL}(K, 0, \mathbb{E}[M_\omega]) \). It follows that \( d^T \mathbb{E}[M_\omega] d = 0 \). This can be written as \( \mathbb{E} \left[ d^T M_\omega d \right] = 0 \) and there are sets \( U_>, U_< \) and \( U_= \) such that

\[
d^T M_\omega d = \begin{cases} > 0, & \omega \in U_> \\ = 0, & \omega \in U_= \\ < 0, & \omega \in U_< 
\end{cases}
\]

We consider each of these possibilities next.

(a) Suppose \( \mathbb{P}(U_=) > 0 \). We have \( d^T M_\omega d = 0 \) for \( \omega \in U_= \). Since each \((K, M_\omega)\) is an \( R_0 \) pair, we obtain that \( d = 0 \). But this contradicts our assumption that \( d \neq 0 \), implying that \( \mathbb{P}(U_=) = 0 \).

(b) Suppose \( \mathbb{P}(U_<) > 0 \). For \( \omega \in U_< \), we have \( d^T M_\omega d < 0 \). Since \( M_\omega \) is copositive, for \( d \in K \) we must have that \( d^T M_\omega d \geq 0 \). This contradiction implies that \( \mathbb{P}(U_<) = 0 \).

(a) and (b) above imply that \( \mathbb{P}(U_>\setminus\Omega) = 1 \) or in other words, we have \( d^T M_\omega d > 0 \) for \( \omega \in \Omega \). This implies that \( \mathbb{E} \left[ d^T M_\omega d \right] > 0 \). This contradicts our assertion that \( \mathbb{E} \left[ d^T M_\omega d \right] = 0 \). Thus, we must have \( d = 0 \) and \( \text{SOL}(K, 0, \mathbb{E}[M_\omega]) = \{0\} \). Therefore, \((K, M)\) is indeed an \( R_0 \) pair.

(iii) The set \( T \) is bounded: We proceed to show that the set \( T \) is bounded where

\[
T \triangleq \bigcup_{\tau > 0} \text{SOL}(K, H + \tau M).
\]

It suffices to show that there exists an \( m > 0 \) such that for all \( x \in K, ||x|| > m \) implies \( x \notin T \). Suppose there is no such finite \( m \), implying that

\[
\text{for any } m > 0, \, \exists \, x \in K, ||x|| > m \text{ and } x \in T.
\]

For each \( k > 0 \), choose \( x_k \in K \) such that \( ||x_k|| > k \) and \( x_k \in T \). For this sequence \( ||x_k|| \to \infty \). Since \( ||x_k|| > k \) observe that for any \( k \), \( x_k \neq 0 \). Now, for each \( k \), since \( x_k \in T \), it follows that \( x_k \in \text{SOL}(K, H + \tau_k M) \) for some \( \tau_k > 0 \). Thus, for each \( k \) we have \( x_k^T H(x_k) + \tau_k x_k^T M x_k = 0 \). Since \( x_k \neq 0 \) this means that for each \( k \),

\[
x_k^T H(x_k) = \mathbb{E}[x_k^T H(x_k; \omega)] = -\tau_k x_k^T M x_k.
\]

Observe that, since \( x_k \in K \) and \( ||x_k|| \to \infty \), we have \( x_k \neq 0 \). Further, since \( M \) is copositive we have \( x_k^T M x_k \geq 0 \). Since \( ||x_k|| \to \infty \), we have that \( x_k^T M x_k \geq 0 \). Since \( \tau_k > 0 \), there are two possibilities for the sequence \( \tau_k x_k^T M x_k \): either it \( \tau_k x_k^T M x_k \to +\infty \) or \( \tau_k x_k^T M x_k \to 0 \) as \( k \to \infty \). In either case, as \( k \to \infty \) from \( \{26\} \) we can conclude

\[
\liminf_{k \to \infty} x_k^T H(x_k) = \liminf_{k \to \infty} \left[ \mathbb{E}[x_k^T H(x_k; \omega)] \right] = \liminf_{k \to \infty} \left[ -\tau_k x_k^T M x_k \right] \leq 0.
\]

On the other hand, by \( \{22\} \) we have that

\[
\liminf_{k \to \infty} x_k^T H(x_k; \omega) > 0 \quad \text{a.s.}
\]
By hypothesis (ii), Fatou’s lemma can be applied, giving us

$$ \liminf_{k \to \infty} x_k^T H(x_k) = \liminf_{k \to \infty} \left[ \mathbb{E} \left[ x_k^T H(x_k; \omega) \right] \right] \geq \mathbb{E} \left[ \liminf_{k \to \infty} x_k^T H(x_k; \omega) \right] > 0, $$

(29)

where the last inequality follows from (28). But this contradicts (27) and implies that there is a scalar $m$ such that $x \in K, \|x\| > m$ implies $x \notin T$. In other words, $T$ is bounded. We have shown that all the conditions of Theorem 17 are satisfied and we may conclude that the stochastic complementarity problem $SCP(K, H)$ has a solution.

In the above proposition, hypothesis (21) guaranteed the existence of a copositive matrix $M$, so that all of the conditions of Theorem 17 are satisfied for the SCP. In certain applications, it may be possible to show the existence of a copositive matrix $M$ such that $(K, M)$ is an $R_0$ pair, for example, choosing $M$ as the identity matrix may suffice. In fact, if we assume existence of a copositive matrix $M$ such that $(K, M)$ is an $R_0$ pair, then we do not require hypothesis (21) above but merely equation (22) suffices. This is demonstrated in the proposition below.

**Proposition 20.** Consider the stochastic complementarity problem $SCP(K, H)$ where $K$ is the nonnegative orthant. Suppose Assumption 2 holds for the mapping $H, G(x; \omega) = x^T H(x; \omega)$, and there exists a copositive matrix $M$ on $K$ such that $(K, M)$ is an $R_0$ pair and the following hold:

(i)

$$ \liminf_{x \in K, \|x\| \to \infty} [x^T H(x; \omega)] > 0, \text{ almost surely.} \quad (30) $$

(ii) Suppose there exists a nonnegative integrable function $u(\omega)$ such that $G(x; \omega) \geq -u(\omega)$ holds almost surely for any $x$.

Then $SCP(K, H)$ has a solution.

**Proof.** We proceed to show that the set $T$ is bounded where

$$ T \triangleq \bigcup_{\tau > 0} \text{SOL}(K, H + \tau M). \quad (31) $$

As earlier, it suffices to show that there exists an $m > 0$ such that for all $x \in K, \|x\| > m$ implies $x \notin T$. Suppose there is no such $m$ implying that

$$ \text{for any } m > 0 \exists x \in K, \|x\| > m \text{ and } x \in T. \quad (32) $$

Construct a sequence $\{x_k\}$ as follows: For each $m = k > 0$, choose $x_k \in K$ such that $x_k \in K, \|x_k\| > k$ and $x_k \in T$. For this sequence $\|x_k\| \to \infty$. Since $\|x_k\| > k$ observe that for any $k$, $x_k \notin T$. Now, for each $k$ since $x_k \in T$, it follows that $x_k \in \text{SOL}(K, H + \tau_k M)$ for some $\tau_k > 0$. Thus, for each $k$, we have $x_k \in T$, $H(x_k) + \tau_k M x_k \geq 0, \tau_k > 0$ and $x_k^T H(x_k) + \tau_k x_k^T M x_k = 0$. Since $x_k \notin T$ and $M$ is copositive and $(K, M)$ is an $R_0$ pair we have that $x_k^T M x_K \geq 0$. This together with $\tau_k > 0$ means that for each $k$,

$$ \mathbb{E}[x_k^T H(x_k; \omega)] = x_k^T H(x_k) = -\tau_k x_k^T M x_k \leq 0. $$

This gives us that

$$ \liminf_{k \to \infty} \mathbb{E}[x_k^T H(x_k; \omega)] = \liminf_{k \to \infty} \left[ -\tau_k x_k^T M x_k \right] \leq 0. \quad (33) $$

On the other hand, since $x_k \in K$ and $\|x_k\| \to \infty$, by hypothesis (30) we have that

$$ \liminf_{k \to \infty} x_k^T H(x_k; \omega) > 0 \text{ a.s.} $$

This means that

$$ \mathbb{E} \left[ \liminf_{k \to \infty} x_k^T H(x_k; \omega) \right] > 0. \quad (34) $$

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Now, by hypothesis (ii), Fatou’s lemma is applicable, implying that
\[
\liminf_{k \to \infty} \mathbb{E} [x_k^T H(x_k; \omega)] \geq \mathbb{E} \left[ \liminf_{k \to \infty} x_k^T H(x_k; \omega) \right] > 0,
\]
where the last inequality follows from (34). This contradicts (33). This contradiction implies that there exists an \( m \) such that \( x \in K, \|x\| > m \) implies \( x \notin T \). In other words, \( T \) is bounded. By hypothesis, we have that there exists a copositive matrix \( M \) on \( K \) such that \((K, M)\) is an \( R_0 \) pair and we have shown that \( T \) is bounded. Thus, all conditions of Theorem 17 are satisfied and we may conclude that the stochastic complementarity problem \( SCP(K, H) \) has a solution.

We now consider several corollaries, the first of which requires defining a co-coercive mapping.

**Definition 4.6** (Co-coercive function). A mapping \( F : K \subseteq \mathbb{R}^n \to \mathbb{R}^n \) is said to be co-coercive on \( K \) if there exists a constant \( c > 0 \) such that
\[
(F(x) - F(y))^T (x - y) \geq c\|F(x) - F(y)\|^2, \quad \forall x, y \in K.
\]

We state Cor. [9, Cor. 2.6.3], which is used in the proof of the next proposition.

**Corollary 21.** Let \( K \) be a pointed, closed, convex cone in \( \mathbb{R}^n \) and let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a continuous map. If \( F \) is co-coercive on \( \mathbb{R}^n \), then \( CP(K, F) \) has a nonempty compact solution set if and only if there exists a vector \( u \in \mathbb{R}^n \) satisfying \( F(u) \in \text{int}(K^*) \).

Our next result provides sufficiency conditions for the existence of a solution to an SCP when an additional co-coercivity assumption is imposed on the mapping. In particular, we assume that the mapping \( H(z; \omega) \) is co-coercive for almost every \( \omega \in \Omega \).

**Proposition 22** (Solvability under co-coercivity). Let \( K \) be a pointed, closed, convex cone in \( \mathbb{R}^n \). Suppose Assumption 2 holds for the mapping \( H(x; \omega) \) and \( H(x; \omega) \) is co-coercive on \( K \) with constant \( \eta(\omega) > 0 \). Suppose \( \eta(\omega) \geq \bar{\eta} > 0 \) for all \( \omega \in \Omega \) where \( \mathbb{P}(\bar{\Omega}) = 1 \) and there exists a deterministic vector \( u \in \mathbb{R}^n \) satisfying \( H(u; \omega) \in \text{int}(K^*) \) in an almost sure sense. Then the solution set of the SCP\((K, H)\) is a nonempty and compact set.

**Proof.** First we show that, \( \mathbb{E} [H(x; \omega)] \) is co-coercive in \( x \). We begin by noting that
\[
(x - y)^T (\mathbb{E} [H(x; \omega)] - \mathbb{E} [H(y; \omega)]) = \int_{\Omega} (x - y)^T (H(x; \omega) - H(y; \omega)) d\mathbb{P}
\]
\[
= \int_{\Omega} (x - y)^T (H(x; \omega) - H(y; \omega)) d\mathbb{P},
\]
where the second equality follows by noting that \( \mathbb{P}(\bar{\Omega}) = 1 \). This allows us to conclude that
\[
\int_{\Omega} (x - y)^T (H(x; \omega) - H(y; \omega)) d\mathbb{P} \geq \int_{\Omega} \eta(\omega)\|H(x; \omega) - H(y; \omega)\|^2 d\mathbb{P}
\]
\[
\geq \bar{\eta} \int_{\Omega} \|H(x; \omega) - H(y; \omega)\|^2 d\mathbb{P},
\]
where the first inequality follows from the co-coercivity of \( H(x; \omega) \), the second inequality follows from noting that \( \eta(\omega) \geq \bar{\eta} \) for \( \omega \in \bar{\Omega} \), a set of unitary measure. Finally, by again recalling that \( \bar{\Omega} \) has measure one and by leveraging Jensen’s inequality since \( \|\cdot\|^2 \) is a convex function, the required result follows:
\[
\bar{\eta} \int_{\Omega} \|H(x; \omega) - H(y; \omega)\|^2 d\mathbb{P} = \bar{\eta} \int_{\Omega} \|H(x; \omega) - H(y; \omega)\|^2 d\mathbb{P} \geq \bar{\eta} \|\mathbb{E} [H(x; \omega)] - \mathbb{E} [H(y; \omega)]\|^2.
\]
Further, since \( H(u; \omega) \in \text{int}(K^*) \) holds almost surely for a deterministic vector \( u \), we have that for all \( x \in K \), \( H(u; \omega)^T x \geq 0 \) holds almost surely. This implies that for all \( x \in K \), \( \mathbb{E} [H(u; \omega)^T x] \geq 0 \) holds. Thus, there exists a \( u \in \mathbb{R}^n \) such that \( \mathbb{E} [H(u; \omega)] \in K^* \).
It remains to show that $\mathbb{E}[H(u; \omega)]$ lies in $\text{int}(K^*)$. If $\mathbb{E}[H(u; \omega)] \notin \text{int}(K^*)$, then there exists an $x \in K$ such that $\mathbb{E}[H(u; \omega)]^T x = 0$. Since $x \in K$ and by assumption, $H(u; \omega) \in \text{int}(K^*)$ almost surely, it follows that $H(u; \omega)^T x > 0$ almost surely, implying that $\mathbb{E}[H(u; \omega)]^T x > 0$. Thus, we arrive at a contradiction, proving that $\mathbb{E}[H(u; \omega)] \in \text{int}(K^*)$. Thus, by Corollary 21, since $\mathbb{E}[H(z; \omega)]$ is co-coercive and there is a vector $u \in \mathbb{R}^n$ such that $\mathbb{E}[H(u; \omega)] \in \text{int}(K^*)$, it follows that $\text{SCP}(K, H)$ has a nonempty compact solution set.

The next corollary is a direct application of Theorem 17 to $\text{SCP}(K, H)$ when $E$ is the identity matrix and can be viewed as a theorem of the alternative for CPs.

**Corollary 23** (Cor. 2.6.2 [9]). Let $K$ be a closed convex cone in $\mathbb{R}^n$ and let $H(x; \omega)$ satisfy Assumption 2. Either $\text{SCP}(K, H)$ has a solution or there exists an unbounded sequence of vectors $\{x_k\}$ and a sequence of positive scalars $\{\tau_k\}$ such that for every $k$, the following complementarity condition holds:

$$K \ni x_k \perp \mathbb{E}[H(x_k; \omega)] + \tau_k x_k \in K^*.$$  

We may leverage this result in deriving a stochastic generalization.

**Proposition 24** (Theorem of the alternative). Let $K$ be a closed convex cone in $\mathbb{R}^n$ and let $H(x; \omega)$ be a mapping that satisfies Assumption 2. Either $\text{SCP}(K, H)$ has a solution or there exists an unbounded sequence of vectors $\{x_k\}$ and a sequence of positive scalars $\{\tau_k\}$ such that for every $k$, the following complementarity condition holds almost-surely:

$$K \ni x_k \perp H(x_k; \omega) + \tau_k x_k \in K^*. \quad (35)$$

**Proof.** Suppose (35) holds almost surely. Consequently, it also holds in expectation or

$$K \ni x_k \perp \mathbb{E}[H(x_k; \omega)] + \tau_k x_k \in K^*. \quad (36)$$

Therefore by Cor. 23 $\text{SCP}(K, H)$ does not admit a solution. \qed

## 5 Examples revisited

We now revisit the motivating examples presented in Section 2 and show the applicability of the developed sufficiency conditions in the context of such problems.

### 5.1 Stochastic Nash-Cournot games with nonsmooth price functions

In Section 2.1 we described a stochastic Nash-Cournot game in which the price functions were nonsmooth. We revisit this example in showing the associated stochastic quasi-variational inequality problem is solvable.

Before proceeding, we recall that $f_i(x_i; \omega)$ is a convex function of $x_i$, given $x_{-i}$ (see [22] Lemma 1).

**Lemma 25.** Consider the function $f_i(x; \omega) = c_i(x_i) - x_i p(X; \omega)$ where $p(X; \omega)$ is given by (5). Then $f_i(x_i; x_{-i})$ is a convex function in $x_i$ for all $x_{-i}$.

The convexity of $f_i$ and $K_i(x_{-i})$ allows us to claim that the first-order optimality conditions are sufficient; these conditions are given by a multi-valued quasi-variational inequality SQVI($K, \Phi$) where $\Phi$, the Clarke generalized gradient, is defined as

$$\Phi(x) \triangleq \mathbb{E} \left[ \prod_{i=1}^N \partial x_i f_i(x; \omega) \right],$$

and $\Phi(x; \omega)$ is defined as $\prod_{i=1}^N \partial x_i f_i(x; \omega)$. The subdifferential set of $f_i(x; \omega)$ is defined as

$$\partial x_i f_i(x; \omega) = c_i'(x_i) - \partial x_i (x_i p(X; \omega)) = c_i'(x_i) - p(X; \omega) - x_i \partial x_i p(X; \omega).$$

**Proof.**
Thus, if \( w \in \Phi(x; \omega) \), then \( w = \prod_{i=1}^{n} w_i \) where \( w_i \in \partial_{x_i} f_i(x; \omega) \). Based on the piecewise smooth nature of \( p(X; \omega) \), the Clarke generalized gradient of \( p \) is defined as follows:

\[
\partial_x p(X; \omega) \in \begin{cases} 
\{ -b_1(\omega) \}, & 0 \leq X < \beta^1 \\
- [b_j^{-1}(\omega), b_j(\omega)], & \beta^{j-1} = X, \ j = 2, \ldots, s \\
\{ -b^s(\omega) \}, & \beta^s < X \end{cases}
\]  \hspace{1cm} (37)

Since our interest lies in showing the applicability of our sufficiency conditions when the map \( \Phi \) is expectation valued, we impose the required assumptions on the map \( K \) as captured by Prop. 15 (i) and (v). Existence of a nonsmooth stochastic Nash-Cournot equilibrium follows from showing that hypotheses (ii) – (iv) of Prop. 15 do indeed hold.

**Theorem 26** (Existence of stochastic Nash-Cournot equilibrium). Consider the stochastic generalized Nash-Cournot game and suppose Assumptions 3 and 5 hold. Further, assume that conditions (i) and (v) of Prop. 15 hold. Then, this game admits an equilibrium.

**Proof.** Since \( \partial_x f_i(x; \omega) \) is a Clarke generalized gradient, it is a nonempty upper semicontinuous mapping at \( x_i \), given \( x_{-i} \). Furthermore, the integrability of \((a^j(\omega), b^j(\omega))\) for \( j = 1, \ldots, s \) allows us to claim that \( \partial_x f_i(x; \omega) \) is integrably bounded. Consequently, hypothesis (iv) in Proposition 15 holds.

By Assumption 5 since \( a_i(\omega) \) and \( b_i(\omega) \) are positive, we have that they are bounded below by the nonnegative constant (integrable) function 0. From, this and the description of \( \Phi \) derived above, we see that hypothesis (iii) in Prop. 15 holds. Thus Fatou’s lemma can be applied.

We now proceed to show that hypothesis (ii) in Proposition 15 holds. It suffices to show that there exists an \( x^{ref} \in K(x) \) such that

\[
\lim_{\|x\| \to \infty, x \in K(x)} \left( \inf_{w \in \Phi(x; \omega)} \frac{(x - x^{ref})^T w}{\|x\|} \right) = \infty.
\]

Consider a vector \( x^{ref} \) such that

\[
x^{ref} \in \bigcap_{x \in \text{dom}(K)} K(x).
\]

Then \( w^T(x - x^{ref}) \) can be expressed as the sum of several terms:

\[
w^T(x - x^{ref}) = \sum_{i=1}^{N} c'_i(x_i)(x_i - x_i^{ref}) - p(X; \omega)(x - x^{ref}) - \sum_{i=1}^{N} x_i(x_i - x_i^{ref})\partial_x p(X; \omega).
\]

When \( \|x\| \to \infty \), from the nonnegativity of \( x \), it follows that \( X \to \infty \) and for sufficiently large \( X \), we have that \( \partial_x p(X; \omega) = -b^s(\omega) \). Consequently, for almost every \( \omega \in \Omega \), we have that

\[
\lim_{\|x\| \to \infty, x \in K(x)} \inf_{w \in \Phi(x; \omega)} \frac{(x - x^{ref})^T w}{\|x\|}
\]

\[
\lim_{\|x\| \to \infty, x \in K(x)} \left( \sum_{i=1}^{N} c'_i(x_i)(x_i - x_i^{ref}) \right) - \lim_{\|x\| \to \infty, x \in K(x)} \left( p(X; \omega)(x - x^{ref}) \right)
\]

\[
= \lim_{\|x\| \to \infty, x \in K(x)} \left( \frac{\sum_{i=1}^{N} (c'_i(x_i) + b^s(\omega)(x + x_i))(x_i - x_i^{ref})}{\|x\|} \right) - \lim_{\|x\| \to \infty, x \in K(x)} \left( \frac{a^s(\omega)(x - x^{ref})}{\|x\|} \right) = \infty,
\]

where the last equality is a consequence of noting that the numerator of Term (a) tends to \( +\infty \) at a quadratic rate while the numerator of Term (b) tends to \( +\infty \) at a linear rate. The existence of an equilibrium follows from the application of Prop. 15.
5.2 Strategic behavior in power markets

In Section 2.1, we have presented a model for strategic behavior in imperfectly competitive electricity markets. We will now develop a stochastic complementarity-based formulation of such a problem. The developed sufficiency conditions will then be applied to this problem.

Recall that, the resulting problem faced by firm $f$ can be stated as follows:

$$\text{maximize } \mathbb{E} \left[ \sum_{i \in \mathcal{N}} (p_i(S_i; \omega)s_{fi} - c_{fi}(g_{fi}; \omega) - (s_{fi} - g_{fi})w_i) \right]$$

subject to

$$g_{fi} \leq \text{cap}_{fi}(\mu_{fi}), \quad \forall i \in \mathcal{N}$$

$$0 \leq g_{fi}, \quad \forall i \in \mathcal{N}$$

$$0 \leq s_{fi}, \quad \forall i \in \mathcal{N}$$

and

$$\sum_{i \in \mathcal{N}} (s_{fi} - g_{fi}) = 0. \quad (\lambda_f)$$

The equilibrium conditions of this problem are given by the following complementarity problem.

$$0 \leq s_{fi} \perp \mathbb{E} [-p_i'(S_i; \omega)s_{fi} - p_i(S_i; \omega) + w_i] - \lambda_f \geq 0, \quad \forall i \in \mathcal{N}$$

$$0 \leq g_{fi} \perp \mathbb{E} [c_{fi}'(g_{fi}; \omega) - w_i] + \mu_{fi} + \lambda_f \geq 0, \quad \forall i \in \mathcal{N}$$

$$0 \leq \mu_{fi} \perp \text{cap}_{fi} - g_{fi} \geq 0, \quad \forall i \in \mathcal{N}$$

$$\lambda_f \perp \sum_{i \in \mathcal{N}} (s_{fi} - g_{fi}) = 0. \quad (\forall f \in \mathcal{F})$$

The ISO’s optimization problem is given by

$$\text{maximize } \sum_{i \in \mathcal{N}} y_i w_i$$

subject to

$$\sum_{i \in \mathcal{N}} \text{PDF}_{ij} y_i \leq T_j, \quad (\eta_j) \quad \forall j \in \mathcal{K}$$

and its optimality conditions are as follows:

$$w_i = \sum_{j \in \mathcal{K}} \eta_j \text{PDF}_{ij} \quad \forall i \in \mathcal{N},$$

$$0 \leq \eta_j \perp T_j - \sum_{i \in \mathcal{N}} \text{PDF}_{ij} y_i \geq 0 \quad \forall j \in \mathcal{K}. \quad (38)$$

The market clearing conditions are given by the following.

$$y_i = \sum_{h \in \mathcal{F}} \left( s_{hi} - g_{hi} \right), \quad \forall i \in \mathcal{N}.$$  

Next, we define $\ell_i(\omega)$ and $h_i(\omega)$ as follows:

$$\ell_i(\omega) = -p_i'(S_i; \omega)s_{fi} - p_i(S_i; \omega) + w_i = -p_i'(S_i; \omega)s_{fi} - p_i(S_i; \omega) + \sum_{j \in \mathcal{K}} \eta_j \text{PDF}_{ij} \quad (39)$$

$$h_i(\omega) = c_{fi}'(g_{fi}; \omega) - w_i = c_{fi}'(g_{fi}; \omega) - \sum_{j \in \mathcal{K}} \eta_j \text{PDF}_{ij}. \quad (40)$$

Then, by aggregating all the equilibrium conditions together and eliminating $w_i$ and $y_i$ based on the equality
It follows that the inner product in the coercivity condition (30) reduces to

\[ 0 \leq s_{fi} \perp \mathbb{E}[f_i(\omega)] - \lambda_f \geq 0, \quad \forall i \in \mathcal{N} \]
\[ 0 \leq g_{fi} \perp \mathbb{E}[h_i(\omega)] + \mu_{fi} + \lambda_f \geq 0, \quad \forall i \in \mathcal{N} \]
\[ 0 \leq \mu_{fi} \perp \text{cap}_{fi} - g_{fi} \geq 0, \quad \forall i \in \mathcal{N} \]
\[ \lambda_f \perp \sum_{i \in \mathcal{N}} (s_{fi} - g_{fi}) = 0, \]

and \( 0 \leq \eta_j \perp T_j - \sum_{i \in \mathcal{N}} \text{PDF}_{ij} \sum_{h \in \mathcal{F}} (s_{hi} - g_{hi}) \geq 0. \quad \forall j \in \mathcal{K} \)

This can be viewed as the following stochastic (mixed)-complementarity problem where \( x, B, \) and \( H(x; \omega) \) are appropriately defined:

\[
0 \leq x \perp \mathbb{E}[H(x; \omega)] - B^T \lambda \geq 0 \\
\lambda \perp Bx = 0.
\]

It follows that the inner product in the coercivity condition (30) reduces to \( x^T \mathbb{E}[H(x; \omega)] \) as observed by this simplification:

\[
\begin{pmatrix} x^T \\ \lambda \end{pmatrix} \begin{pmatrix} \mathbb{E}[H(x; \omega)] - B^T \lambda \\ Bx \end{pmatrix} = x^T \mathbb{E}[H(x; \omega)].
\]

Next, we show that this inner product is bounded from below by \(-u(\omega)\) where \(u(\omega)\) is a nonnegative integrable function.

**Lemma 27.** For the stochastic complementarity problem SCP(\( K, H \)) above that represents the strategic behavior in power markets, there exists a nonnegative integrable function \(u(\omega)\) such that have that the following holds:

\[
G(x; \omega) = x^T H(x; \omega) \geq -u(\omega) \text{ almost surely for all } x \in K.
\]

**Proof.** The product \( x^T H(x; \omega) \) can be expressed as follows:

\[
\sum_{f,i} \left( -p'_i(S_i; \omega)s_{f_i}^2 - p_i(S_i; \omega)s_{f_i} + \left( \sum_{j \in \mathcal{K}} \eta_j \text{PDF}_{ij} \right) s_{f_i} \right) \\
+ \sum_{f,i} \left( c'_{f_i}(g_{fi}; \omega)g_{fi} - \left( \sum_{j \in \mathcal{K}} \eta_j \text{PDF}_{ij} \right) g_{fi} + \mu_{fi}g_{fi} \right) \\
+ \sum_{f,i} (\mu_{fi}\text{cap}_{fi} - \mu_{fi}g_{fi}) + \sum_j \eta_j \left( T_j - \sum_{i \in \mathcal{N}} \text{PDF}_{ij} \sum_{h \in \mathcal{F}} (s_{hi} - g_{hi}) \right).
\]

After appropriate cancellations, this reduces to

\[
\sum_{f,i} (-p'_i(S_i; \omega)s_{f_i}^2 - p_i(S_i; \omega)s_{f_i}) + \sum_{f,i} (c'_{f_i}(g_{fi}; \omega)g_{fi}) + \sum_{f,i} (\mu_{fi}\text{cap}_{fi}) + \sum_j \eta_j T_j.
\]

By Assumption 6 the price functions are decreasing functions bounded above by an integrable function and the cost functions are non-decreasing. Furthermore, \( K \) is the nonnegative orthant, \( \mu_{fi}, \eta_j \) are nonnegative, and \( \text{cap}_{fi}, T_j \) denote nonnegative capacities. Consequently, we have the following sequence of inequalities.

\[
\sum_{f,i} (-p'_i(S_i; \omega)s_{f_i}^2 - p_i(S_i; \omega)s_{f_i}) + \sum_{f,i} (c'_{f_i}(g_{fi}; \omega)g_{fi}) + \sum_{f,i} (\mu_{fi}\text{cap}_{fi}) + \sum_j \eta_j T_j \\
\geq \sum_{f,i} (-p_i(S_i; \omega)s_{f_i}) \geq -\left( \max_i \bar{p}_i(\omega) \right) \sum_{f,i} \text{cap}_{fi} \triangleq -u(\omega),
\]

where \( p_i(S_i; \omega) \leq \bar{p}_i(\omega) \) for all nonnegative \( S_i \) and \( \sum_{f,i} s_{f_i} \leq \sum_{f,i} \text{cap}_{fi}. \) Integrability of \( u(\omega) \) follows immediately by its definition. \[ \square \]
Having presented the supporting results, we now prove the existence of an equilibrium.

**Proposition 28** (Existence of an imperfectly competitive equilibrium). Consider the imperfectly competitive model in power markets. Under Assumption 6 this problem admits a solution.

**Proof.** The result follows by showing that Proposition 20 can be applied. Lemma 27 shows that hypothesis (ii) of Proposition 20 holds. We proceed to show that hypothesis (i) of Proposition 20 also holds. We show that the following property holds almost surely:

\[ \lim_{\|x\| \to \infty} x^T H(x; \omega) > 0. \]  
(41)

Consider the expression for \( x^T H(x; \omega) \) derived in Lemma 27:

\[ x^T H(x; \omega) = \sum_{f,i} \left( -p_i'(S_i; \omega)s_i^2 - p_i(S_i; \omega)s_{fi} \right) + \sum_{f,i} \left( c'_{fi}(g_{fi}; \omega)g_{fi} \right) + \sum_{f,i} \left( \mu_{fi} \text{cap}_{fi} \right) + \sum_j \eta_j T_j. \]

For large \( \|x\| \), the first summation is dominated by its first term and by Assumption 6 as \( \|x\| \) goes to \( \infty \), this term goes to \( \infty \). The other terms are all nonnegative by Assumption 6. Thus, the entire expression can only increase to \( \infty \) as \( \|x\| \) goes to \( \infty \). This proves that (41) holds and the required result follows.

\[ \square \]

### 6 Concluding Remarks

Finite-dimensional variational inequality and complementarity problems have proved to be extraordinarily useful tools for modeling a range of equilibrium problems in engineering, economics, and finance. This avenue of study is facilitated by the presence of a comprehensive theory for the solvability of variational inequality problems and their variants. When such problems are complicated by uncertainty, a subclass of models lead to variational problems whose maps contain expectations. A direct application of available theory requires access to analytical forms of such integrals and their derivatives, severely limiting the utility of existing sufficiency conditions for solvability.

To resolve this gap, we provide a set of integration-free sufficiency conditions for the existence of solutions to variational inequality problems, quasi-variational generalizations, and complementarity problems in settings where the maps are either single-valued or multi-valued. These conditions find utility in the existence of equilibria in the context of generalized nonsmooth stochastic Nash-Cournot games and strategic problems in power markets. We believe that these statements are but a first step in examining a range of problems in stochastic regimes. These include the development of stability and sensitivity statements as well as the consideration of broader mathematical objects such as stochastic differential variational inequality problems.

### 7 Appendix

**Proof of Prop. 4.**

**Proof.** Recall from [9, Ch. 2] that the solvability of \( \text{SVI}(K, F) \) requires showing that there exists an \( x^{\text{ref}} \) such that

\[ \lim_{\|x\| \to \infty, x \in K} \frac{F(x)^T (x - x^{\text{ref}})}{} > 0. \]  
(42)

But we have that

\[ \lim_{\|x\| \to \infty, x \in K} \frac{F(x)^T (x - x^{\text{ref}})}{} = \lim_{\|x\| \to \infty, x \in K} \left[ \int_{\Omega} F(x; \omega)^T (x - x^{\text{ref}}) d\Omega \right]. \]
By hypothesis (ii), we may apply Fatou’s lemma to obtain the following inequality:

\[
\liminf_{\|x\| \to \infty, x \in K} \mathbb{E} \left[ F(x; \omega)^T (x - x^{\text{ref}}) \right] \geq \int_{\Omega} \liminf_{\|x\| \to \infty, x \in K} \mathbb{E} \left[ F(x; \omega)^T (x - x^{\text{ref}}) \right] d\mathbb{P} > 0,
\]

where the last inequality follows from the given hypothesis. Thus (42) holds and therefore SVI(\(K, F\)) has a solution.

**Proof of proposition 5.**

*Proof.* For the given \(x^{\text{ref}} \in K\) and for any \(x \in K\), there exists a \(\nu \in \{1, \ldots, N\}\), such that

\[
\liminf_{\|x_\nu\| \to \infty, x_\nu \in K_\nu} \mathbb{E} \left[ F_\nu(x; \omega)^T (x_\nu - x^{\text{ref}}_\nu) \right] > 0
\]

holds almost surely. Thus we obtain

\[
\mathbb{E} \left[ \liminf_{\|x_\nu\| \to \infty, x_\nu \in K_\nu} F_\nu(x; \omega)^T (x_\nu - x^{\text{ref}}_\nu) \right] > 0.
\]

By hypothesis (ii) above we may apply Fatou’s Lemma to get

\[
\liminf_{\|x_\nu\| \to \infty, x_\nu \in K_\nu} \mathbb{E} \left[ F_\nu(x; \omega)^T (x_\nu - x^{\text{ref}}_\nu) \right] > 0.
\]

This implies that \(C_{\leq}\) is bounded where

\[
C_{\leq} := \left\{ x \in K : \max_{1 \leq \nu \leq N} \mathbb{E} \left[ F_\nu(x; \omega)^T (x_\nu - x^{\text{ref}}_\nu) \right] \leq 0 \right\}.
\]

From [9, Prop. 3.5.1], boundedness of \(C_{\leq}\) allows us to conclude that SVI(\(K, F\)) is solvable.

**Proof of Corollary 6.**

*Proof.* We begin with the observation that the monotonicity of \(F(x; \omega)\) allows us to bound \(F(x; \omega)^T (x - x^{\text{ref}})\) from below as follows:

\[
F(x; \omega)^T (x - x^{\text{ref}}) = \left( F(x; \omega) - F(x^{\text{ref}}; \omega) \right)^T (x - x^{\text{ref}}) + F(x^{\text{ref}}; \omega)^T (x - x^{\text{ref}})
\]

\[
\geq F(x^{\text{ref}}; \omega)^T (x - x^{\text{ref}}).
\]

Taking expectations on both sides gives us

\[
\mathbb{E} \left[ F(x; \omega)^T (x - x^{\text{ref}}) \right] \geq \mathbb{E} \left[ F(x^{\text{ref}}; \omega)^T (x - x^{\text{ref}}) \right].
\]

This implies that

\[
\liminf_{\|x\| \to \infty, x \in K} \mathbb{E} \left[ F(x; \omega)^T (x - x^{\text{ref}}) \right] \geq \liminf_{\|x\| \to \infty, x \in K} \mathbb{E} \left[ F(x^{\text{ref}}; \omega)^T (x - x^{\text{ref}}) \right]. \tag{43}
\]

By hypothesis (ii) above, Fatou’s Lemma can be employed in the last inequality to interchange limits and expectations leading to

\[
\liminf_{\|x\| \to \infty, x \in K} \mathbb{E} \left[ F(x^{\text{ref}}; \omega)^T (x - x^{\text{ref}}) \right] \geq \mathbb{E} \left[ \liminf_{\|x\| \to \infty, x \in K} F(x^{\text{ref}}; \omega)^T (x - x^{\text{ref}}) \right].
\]

But by assumption, we have that

\[
\liminf_{\|x\| \to \infty, x \in K} \mathbb{E} \left[ F(x^{\text{ref}}; \omega)^T (x - x^{\text{ref}}) \right] > 0
\]

holds in almost sure sense, implying that from (43), we have that

\[
\liminf_{\|x\| \to \infty, x \in K} \mathbb{E} \left[ F(x; \omega)^T (x - x^{\text{ref}}) \right] > 0.
\]

Now, Proposition 5 allows us to conclude the solvability of the stochastic variational inequality SVI(\(K, F\)).
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