Impedance acoustic cloaking

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Abstract. The paper addresses a new approach to passive acoustic cloaking, which is an alternative to the well-known approach of transformation optics. It is shown first that the scattered acoustic field of a body is completely determined by special surface impedances defined at the body surface. The surface impedances of a non-scattering, i.e. acoustically transparent, body of any geometry are then derived explicitly, and a thin coating is proposed that can provide such impedances with controlled accuracy. The coating represents a doubly periodic surface structure made of elements of sub-wavelength size. The computer simulation shows that the coating being placed on any body can render it non-scattering and, hence, invisible to all acoustic observation systems in a range of low and medium frequencies. The width of the range and the coating efficiency depend on the number of internal couplings introduced between the coating elements.

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1. Introduction

The definition of invisibility directly depends on the observation method. Among the variety of physical methods of observation (magnetic, thermal, optical, etc), the present paper deals only with the so-called active acoustic methods based upon insonification of objects and analysis of reflected or scattered waves. Hence, an acoustically invisible body is understood here as a body that does not reflect or scatter sound. In this respect, the problem of this paper (how to render an arbitrary body acoustically invisible) is analogous to the well-known optical problem of ‘a cloak of invisibility’.

Let us begin with a brief review of the main results published on this problem. There are two main groups of active observation methods: pulse-echolocation or monostatic methods and tomographic or multistatic methods.

Echolocation is the simplest technique used in radar and sonar systems as well as in Nature by some animals (dolphins and bats). An echolocator is a device that combines a source and transmitter of probing signals, a receiver of reflected signals and a processing unit. From reflected signals, information on the distance to the body under observation, its size and some other parameters is extracted. Hence, to make a body invisible to an echolocator, it is necessary to suppress its backscattering. Because of the widespread practical use of radars and sonars, there are a lot of publications on the topic including works on protection from locators. As commonly accepted, absorbing coatings are one of the most efficient means of suppressing backscattering, and among the existing coatings, the best is believed to be the matched coating, which has the local impedance $\rho c$. At high frequencies (where the body dimensions are much larger than the wavelengths), bodies with matched coatings have reflecting properties very close to those of the black body of Kirchhoff and are therefore almost invisible to pulse–echo systems [1, 2]. In this paper, it will be shown below that the efficiency of the coatings with extended reaction proposed here is even higher than that of the matched coating, especially at low frequencies.

Other active observation methods, i.e. tomographic or multistatic methods, which are now being intensively developed in medical acoustics, ocean acoustics and other areas, are much more sophisticated [3]. In these methods, the sound field in a certain volume of the medium is generated by several sources and measured by many sensors. Processing of the received signals enables one, by solving the inverse scattering problem, to reconstruct the inhomogeneities of the medium and to trace all the foreign bodies in the volume under study. It is evident that even a perfect non-reflecting body, which is invisible to echolocation systems, is detectable by tomographic methods because, like the black body, it casts a shadow. A body is invisible with respect to tomographic methods only if it does not scatter waves in any direction and is fully transparent to incident fields. Note that invisibility in optics is equivalent to non-scattering.

Several theoretical solutions to the problem of how to make a body non-scattering have been obtained so far. The first mathematically exact method was proposed by G D Malyuzhinets in the 1960s [4]. It is an active method, ‘active’ meaning that it uses additional sound sources (actuators) to suppress the scattered field. In this method, a body to be protected is enveloped by two closed sound-transparent surfaces (Huygens surfaces). On the inner surface, pressure and velocity sensors are placed. They provide (with the help of the Helmholtz–Huygens integral operator [5]) decomposition of the total field into two components: the incident field and the scattered field. On the outer Huygens surface, actuators of the monopole and dipole types are
placed. They radiate into the exterior the extracted scattered component in counterphase, thus canceling the scattered field of the body. This and other feed-forward active control methods based on the Huygens principle have been validated in laboratory experiments and used by many authors, e.g. [6]–[10]. A similar study was also carried out in optics [11].

One more method of active control of scattered sound fields has been proposed in [12, 13]. In this method, sensors (accelerometers) and force actuators are placed flush on the surface of the elastic body to be protected. Theoretically, full cancellation of the scattered field component can be achieved by the proper action of the actuators on the elastic body surface, the actuators being controlled by the total vibration normal velocity signals measured by the sensors. Up to now, this feedback active control method has been verified only in computer simulations. In the literature, there are other modifications of the active methods of suppressing the radiated and scattered fields [7, 14, 15].

There is also one passive method of controlling scattered acoustic fields called the transformation method. It has come to acoustics from electromagnetics and optics—see e.g. [16]–[19] and the literature review in [20, 21]. Physically, the idea of this method is to guide incident waves around the body so that they return to their original trajectories without scattering and penetration into the body. Mathematically, the method consists in finding a coordinate transformation that maps the free space filled with a homogeneous fluid to the exterior of a certain cavity. As the transformation is of the one-to-one type everywhere (except at a single point, which is mapped into the cavity boundary), the sound field in the transformed space represents exactly the field in the original (untransformed) space deformed (stretched and bent) by the transformation. If, for example, the field in the untransformed free space is that of a traveling plane wave with the straight trajectories of the fluid particles, the field in the transformed space will represent the traveling wave with curved trajectories that run around the cavity. As a result, any body placed into the cavity will remain undisturbed by the field and, hence, will be acoustically invisible. The field in the transformed space is shown to satisfy the wave equation with coordinate-dependent tensorial coefficients. This means that to physically obtain such a field, the region around the cavity should be filled with inhomogeneous fluid having a tensorial density and compressibility. Such fluids are called metafluids. At present, the transformation cloaking method is intensively being studied theoretically in acoustics [22]–[29] and elastodynamics [20], [29]–[31], as well as in optics, electrodynamics and other areas, although some preliminary experimental results are also reported [28, 32].

The approach to acoustic cloaking in the present paper is also of the passive type and, from the physical point of view, is rather close to the transformation-based approach: incident waves propagate through the coating and the body (if it is resilient) without producing a scattered field. But the mathematics behind the method and the structure of the proposed coatings are quite different. The basis of the approach is a new impedance theory of sound scattering [13, 33]. One of the results of this theory states that the field scattered by a body is completely determined by special impedances defined at its surface. Therefore, to make an arbitrary body non-scattering and acoustically invisible, there is no need to surround it by a shell of a metafluid that mimics all the field details of the untransformed region, as in the case of the transformation method. Instead, it is sufficient to cover the body with a coating that provides the needed impedances on its outer surface. The internal structure of the coating does not matter: it may be fabricated of a composite elastic material or engineered from discrete mechanical parts or have other structure. The only requirement is that its surface impedances should be equal to those of the non-scattering body. That is the main idea of the proposed approach.
Below, analytical expressions for the surface impedances of the ideally transparent body having any geometry are derived. They show that this body has a surface of global reaction, and a coating that possesses such a property should have a very complicated structure in which every pair of coating elements is coupled, which can hardly be realized in practice. For this reason, a so-called coating with extended reaction (CER) has been proposed [34] whose surface reaction lies between the global and local ones and which is rather feasible for practical realization because only neighboring elements are coupled. As shown below in computer simulation, even the simplest modifications of the CERs have an efficiency that is higher than that of the existing means. It is also shown that the more coating elements that are coupled, the smaller the scattered sound power. In the limit when all the elements are coupled, the coating becomes globally reactive and renders the body it covers fully transparent and acoustically invisible in a wide frequency range.

The layout of the paper is the following. In section 2, analytical equations for the surface impedances of a non-scattering body are derived. The proposed coatings with extended reaction are described in section 3. The results of computer simulation of plane and cylindrical CERs are presented in sections 4 and 5. Finally, the main findings of the paper are summarized in section 6.

2. Surface impedances of ideally transparent bodies

In this section, the basic equations of the impedance theory of sound scattering [13, 33] are briefly documented, from which the surface impedance matrix of an ideally transparent body is explicitly derived.

Consider a finite-size linearly elastic body (scatterer) of arbitrary geometry, with volume $V$ and outer surface $A$, immersed in an acoustic fluid medium. The medium is assumed linear and lossless but may be inhomogeneous and bounded. In the medium outside the body, some acoustic sources are present that, in the absence of the body, produce an incident pressure field with the complex amplitude $p_i(x)$, with $x$ being the coordinates of a point in the medium. In the presence of the body, the total pressure field is represented by the sum of two components, $p(x) = p_i(x) + p_s(x)$, where $p_s(x)$ is the scattered component. The scattering problem is stated as follows: to determine the operator that relates the scattered field component to the incident field component, which is assumed prescribed.

There are a lot of methods for solving the scattering problem [5, 35, 36]. Most of them are based on directly solving the acousto-elastic problem, i.e. on obtaining solutions to the set of differential equations that describe vibrations of the elastic body and medium together with the boundary conditions including those at the interface $A$ and infinity and using various simplifying assumptions. The essence of the impedance method proposed in [13, 33] consists of two steps. In the first step, vibrations of the body and medium are independently described by the impedance characteristics defined with respect to the contact surface $A$. The differential equations for the body and medium are solved separately together with all the boundary conditions but one on $A$. In the second step, these impedances are used for satisfying the boundary condition on $A$ to give the solution of the scattering problem. To obtain the needed impedance characteristics, surface $A$ is represented as a set of $N$ small-size surface elements $\Delta A_n$. Considering the pressure and normal velocity as constant valued within each of the elements, one can replace continuous functions by $N$-vectors. For example, the total pressure amplitude $p(y)$ and the total normal velocity $v(y)$, $y \in A$, are replaced by the vectors $p = [p(y_1)\Delta A_1, \ldots, p(y_N)\Delta A_N]^T$, $v = [v(y_1)\Delta A_1, \ldots, v(y_N)\Delta A_N]^T$. 

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\( v = [v(y_1), \ldots, v(y_N)]^T \), where \( y_n \) are the coordinates of a point within the area \( \Delta A_n \). Vectors \( p_s, v_s \) and \( p_i, v_i \) for the scattered and incident components are determined in a similar manner.

As shown in \([13, 33]\), three basic impedance \( N \times N \) matrices, defined at the interface \( A \), are necessary for solving the scattering problem: \( Z, Z_i \) and \( Z_r \). The input surface impedance matrix \( Z \) of the body in vacuum characterizes the body as a scatterer, while matrices \( Z_i \) and \( Z_r \) describe vibrations of the medium. Matrix \( Z_i \) is the surface input impedance matrix of the volume \( V \) filled with the medium and \( Z_r \) is the matrix of the radiation impedances, i.e. the impedances of the medium in the exterior of \( A \). These matrices linearly relate the pressure vectors to the normal velocity vectors

\[
\begin{align*}
p_i + Z v_i &= 0, \\
p_i + p_s + Z(v_i + v_s) &= 0, \\
p_s - Z r v_s &= 0.
\end{align*}
\]  

(1)

In the impedance theory, two scattering matrices, \( Q \) and \( S \), are introduced. Matrix \( Q \) relates vector \( v_s \) of the normal velocity of the scattered component to vector \( v_i \) of the normal velocity of the incident component, while matrix \( S \) relates the corresponding pressure vectors:

\[
\begin{align*}
v_s &= Q v_i, \\
p_s &= S p_i.
\end{align*}
\]  

(2)

The main result of the theory is the ‘theorem of three impedances’: the scattering matrices \( Q \) are uniquely expressed through the three impedance matrices introduced above, or their inverses: three mobility matrices \( Y_k, k = \emptyset, i, r \),

\[
Q = (Z_r + Z)^{-1}(Z_i - Z), \quad S = (Y_r + Y)^{-1}(Y_i - Y).
\]  

(3)

The proof of the theorem readily follows from equations (1). The two scattering matrices are not independent; they satisfy the matrix equation \( Z_i Q + S Z_r = 0 \). Equations (3) generalize the classical Fresnel’s equation for the plane wave coefficient of reflection from a plane interface \([5]\). Equations (2) and (3) give the solution to the pressure and normal velocity only on interface \( A \). The scattered field at other points of the fluid can, if necessary, be obtained using the Helmholtz–Huygens integral operator \([5]\).

It is evident from equation (3) that the scattering matrices, \( Q \) and \( S \), are zero matrices if the \textit{in vacuo} surface impedance matrix of the body \( Z \) is equal to that of the fluid in the volume \( V \):

\[
Z = Z_i \quad \text{or} \quad Y = Y_i.
\]  

(4)

A body that satisfies condition (4) is ideally non-scattering, i.e. absolutely transparent or indiscernible from the medium and, hence, acoustically invisible to any mono- or multistatic methods of detection. Equation (4) means, in particular, that the internal structure of the scatterer is unimportant. What really makes it invisible is its surface reaction, which is responsible for the values of the surface impedances. This also means that an arbitrary body can, in principle, be made ideally invisible with the help of a thin coating if, when placed on the body, it has the surface impedance matrix \( Z_i \).

The most important physical property of the ideally non-scattering body (or coating) is the global reaction of its surface. The property follows from the fact that a finite volume filled with a lossless fluid represents a high-quality vibratory system. The response of such a system to a surface point excitation is noticeable not only at the driven point (as in the case of locally reacting surfaces) but at all points of the volume and at its surface. For that reason, the matrix \( Z_i \) in equation (4) is fully populated. And this means that in a discrete coating that realizes the invisibility condition (4), each coating element should be coupled to all other elements, leading to great complexity of the coating structure and making it impractical. The proposed coatings having only neighboring elements coupled are much easier for practical implementation, and
their efficiency, although not ideal, is higher than that of commonly used coatings, which mostly are locally reacting.

It should be noted that the above results do not explicitly depend on the surface discretization number \( N \). They are also valid in the limit \( \Delta A_n \to 0 \) \( (N \to \infty) \). In this case, the velocity and pressure vectors on \( A \) are continuous functions, while the matrices of impedance and mobility turn into integral or differential operators. For example, the mobility matrix \( Y_i \) changes into the integral operator with the surface Green’s function \( g_i \) of the fluid volume \( V \) as a kernel so that the equation \( v_i + Y_i p_i = 0 \) that is equivalent to the first equation (1) is replaced by the integral equation

\[
v_i(y_1, y_2) + \int_A g_i(y_1, y_2/\xi_1, \xi_2) p_i(\xi_1, \xi_2) \, d\xi_1 \, d\xi_2 = 0, \quad y, \xi \in A, \quad \text{etc.}
\]

Generally, computation of the ideal impedance and mobility matrices in equations (4) for bodies of complex configurations can be performed using one of the well-developed numerical methods—finite element method, boundary element method or the finite difference method. For a few geometries (plane, cylinder or sphere), they can be obtained analytically. Furthermore, if the fluid and scattering body possess the same translational or rotational symmetry as the interface \( A \), then all three fundamental matrices, \( Z_i, Z_r \) and \( Z \), have identical eigenvectors, and all the equations (1)–(4) become valid for each eigenmode separately. Matrix equation (4) in this case is split into a set of independent equations for the corresponding modal impedances or modal mobilities that have a very simple analytical form — see the examples below in sections 4 and 5.

3. Coatings with extended reaction

A schematic diagram of one of the simplest modifications of the proposed coating with extended reaction is shown in figure 1. It is a two-dimensional doubly periodic structure (with the period \( l \) along the surface coordinate axes \( \xi_1 \) and \( \xi_2 \)) composed of identical elements (cells) of small wave dimensions, \( l \ll \lambda \). The cell is characterized by an impedance \( Z_1 \): if a time harmonic force \( f \ast \exp(-i\omega t) \) acts on the face of an isolated cell in the normal direction \( \xi_3 \), its normal velocity response amplitude \( v \) relates to the force amplitude via this impedance, \( f = Z_1 v \).

In the coating shown in figure 1, each cell is coupled with four first neighbors (two in each direction) the coupling being characterized by an impedance \( Z_2 \). This implies that the force of interaction between two neighboring cells is proportional to the difference of their velocities with the coefficient of proportionality \( Z_2 \). For more complex modifications of CER, coupling of impedance type may be introduced between the second, third, etc neighbors. CER may also be anisotropic having different periods and different coupling impedances along the coordinate axes. For brevity, a coating with the first neighbors coupled will be called CER1. If the second, third, etc neighbors are also coupled, the coating will be called CER2, CER3, etc. In particular, if there is no coupling between cells and a coating is locally reacting, it will be called CER0.

Forced vibrations of a CER are described by a set of finite difference equations. For the modified CER1 shown in figure 1, these equations are

\[
f_{m,n} = Z_1 v_{m,n} + Z_2 (2v_{m,n} - v_{m+1,n} - v_{m-1,n}) + Z_2 (2v_{m,n} - v_{m,n+1} - v_{m,n-1}). \quad (5)
\]
Figure 1. Schematic diagram of a coating with extended reaction in which the first neighbors are coupled (CER1).

Here \( v_{m,n} \) is the complex amplitude of the normal velocity of the cell with the coordinates \( \xi_1 = ml, \xi_2 = nl \); \( f_{m,n} \) is the amplitude of an external force produced, e.g., by an incident wave; \( m, n \) are integers.

Since the wave dimensions of a coating cell are assumed small, it is appropriate for further analysis to pass from finite difference equations (5) to the corresponding continuous equation with partial derivatives. Dividing equation (5) by the cell area \( l^2 \) and replacing the finite differences by derivatives, one obtains the following equation for CER1 in figure 1:

\[
z_1 v(\xi_1, \xi_2) - Z_2 \Delta v(\xi_1, \xi_2) = q(\xi_1, \xi_2),
\]

where \( q = f/l^2 \) is the surface density of the external force or pressure \( z_1 = Z_1/l^2 \); \( \Delta = \partial^2 / \partial \xi_1^2 + \partial^2 / \partial \xi_2^2 \) is the Laplace operator. This is a second-order equation with two structural impedance parameters \( Z_1 \) and \( Z_2 \) that should be chosen from a condition of closeness of the coating surface impedance (obtained from equation (5) or (6)) to the impedance (4) of the ideally transparent body—see sections 4 and 5. For more complex CER2, CER3, etc, the continuous equations of type (6) are of the fourth, sixth, etc order with respect to the surface coordinates. They contain more structural impedance parameters (6 for CER2, 10 for CER3, etc), which allows one to better approximate the ideal characteristics.
4. Non-reflective plane

As the first example consider the well-known problem of wave reflection from an impedance plane. It will be shown that even simple modifications of CERs are more efficient non-reflectors than the commonly used local coatings including the best of them—the matched coating.

Consider a half-space $\xi_3 \geq 0$ filled with a homogeneous fluid $\rho c$ and bounded by a plane $(\xi_1, \xi_2, 0)$, which is the interface between the fluid and a coating with extended reaction. Let a plane pressure wave with complex amplitude $p_i$ be incident upon the plane at an angle $\theta$ with the axis $\xi_3$. Reflected from the plane is also a plane wave of amplitude $p_r$. According to Fresnel’s equation \[ R(\theta) = \frac{p_r}{p_i} = \frac{z - z_\theta}{z + z_\theta}, \] (7) where $z$ is a surface impedance of the coating and
\[ z_\theta = \frac{\rho c}{\cos \theta} \] (8)
is the so-called acoustic impedance of the fluid equal to the surface impedance of the fluid half-space. Both surface impedances, $z$ and $z_\theta$, are, by definition, the ratios of the external exponentially distributed force exerted by the incident wave on the interface $\xi_3 = 0$ to the normal velocity response of the surface, which is also exponentially dependent on the surface coordinates. Hence, the surface impedance of CER1 of figure 1 is obtained from equation (6) after substitution of the force (9) into its right-hand side
\[ q(\xi_1, \xi_2) = q \exp[i(k_1 \xi_1 + k_2 \xi_2 - \omega t)], \] (9)
to the normal velocity response of the surface, which is also exponentially dependent on the surface coordinates. Hence, the surface impedance of CER1 of figure 1 is obtained from equation (6) after substitution of the force (9) into its right-hand side
\[ z = \frac{q}{v} = z_1 + z_2 (k_i l)^2 \sin^2 \theta, \] (10)
where $z_{1,2} = Z_{1,2}/l^2$. Acoustic impedance (8) is obtained in a similar way.

As shown in [13], Fresnel’s solution (7) and (8) follows from the general impedance solution (3) as a particular case. Since the system configuration in this case possesses translational symmetry (uniformity) in the $\xi_1$- and $\xi_2$-directions, the fundamental impedance matrices, which are here of the infinite order, have exponential eigenfunctions of the type (9) and continuous spectra of simple eigenvalues $z$ and $z_\theta$ that are continuous functions of the angle of incidence. Besides, matrices $Z_i$ and $Z_r$, as well as their eigenvalues, coincide since they characterize the same fluid half-space. Thus, equation (7) represents equation (3) written for one mode that corresponds to incident angle $\theta$ with $R(\theta) = S = -Q$.

Figures 2 and 3 present some results of computer simulation of this problem. Both figures show the absolute value of the reflection coefficient (7) as a function of the incident angle. Curves 1 in these figures relate to locally reacting coatings that are widely used as sound absorbing and non-reflecting means [37]. The reason why most practical non-reflective coatings and devices are just locally reacting is that they all are made of highly absorptive materials in which disturbances propagate with a large spatial attenuation. The surface impedance of a local coating does not depend on the angle of incidence of the plane wave. As is also evident from equation (7), each local coating has one angle of full absorption, say $\theta_1$. When the surface impedance of the coating is equal to the acoustic impedance (8) corresponding to $\theta_1$,
\[ z = z_{\theta_1} = \frac{\rho c}{\cos \theta_1}, \]
Figure 2. Angular dependence of the reflection coefficient of: (1) the matched local coating: \( \theta_1 = 0^\circ, \ z = \rho c, \ R = 0.48 \); (2) one of the matched CER1’s: \( \theta_1 = 0^\circ, \ \theta_2 = 50^\circ, \ R = 0.35 \); and (3) the best matched CER1: \( \theta_1 = 0^\circ, \ \theta_2 = 71.1^\circ, \ R = 0.29 \).

the plane wave that is incident at this angle does not get reflected and is completely absorbed by the impedance surface.

Among the variety of locally reacting coatings, two of them, the matched coating and optimal coating, are accepted as the best non-reflective (or absorptive) coatings. The matched coating has the surface impedance \( \rho c \). It does not reflect normally incident waves (\( \theta_1 = 0^\circ \)). The angle dependence of the absolute value of its reflection coefficient (7) is shown in figure 2 by curve 1. The corresponding averaged reflection coefficient, which is defined as

\[
R = \left[ \int_0^{\pi/2} |R(\theta)|^2 \sin \theta \, d\theta \right]^{1/2},
\]

is equal to \( R = 0.48 \).

The mean coefficient \( R \) reaches its minimum at the angle of full absorption \( \theta_1 = 62.46^\circ \), where it is equal to \( R = 0.37 \). This is the optimal local coating, which is, on average, the best among all local coatings. Its surface impedance is \( z = 2.18 \rho c \). The modulus of its reflection coefficient as a function of the incident angle is presented in figure 3 by curve 1. Note that the parameters of the optimal coating (\( \theta_1 \) and \( z \)) depend on criterion (11). If the so-called diffuse absorption coefficient \( \alpha \) was used as a goal function instead of the mean coefficient \( R \), these parameters would be \( \theta_1 = 50^\circ \) and \( z = 1.56 \rho c \) [37].

Consider now the reflection properties of coatings with extended reaction restricting the analysis by CER1. Contrary to local coatings, which may have only one incident angle of full absorption, this coating is characterized by two angles of full absorption—see equations (7) and (10). If \( \theta_1 \) and \( \theta_2 \) are these angles, the two impedance parameters, \( z_1 \) and \( z_2 \), can be
Figure 3. Angular dependence of the reflection coefficient for: (1) optimal local coating: $\theta_1 = 62.46^\circ$, $z = 2.18\rho c$, $R = 0.37$; and (2) optimal CER1: $\theta_1 = 25.4^\circ$, $\theta_2 = 72.1^\circ$, $R = 0.283$.

determined from the equality of the surface impedance (10) to the acoustic impedance (8) at these angles:

\[
\begin{align*}
z_1 + z_2 (k_j l)^2 \sin^2 \theta_1 &= \frac{\rho c}{\cos \theta_1}, \\
z_1 + z_2 (k_j l)^2 \sin^2 \theta_2 &= \frac{\rho c}{\cos \theta_2}.
\end{align*}
\]

Solution of this set of linear equations gives the coating impedance parameters as

\[
\begin{align*}
z_1 &= \frac{\sin^2 \theta_2 / \cos \theta_1 - \sin^2 \theta_1 / \cos \theta_2}{\rho c}, \\
\frac{(k_j l)^2 z_2}{\rho c} &= \frac{1 / \cos \theta_2 - 1 / \cos \theta_1}{\sin^2 \theta_2 - \sin^2 \theta_1}.
\end{align*}
\]

To be physically realizable, these impedances should be real valued and positive. From this condition it follows that the smaller angle should not exceed $\arccos(1/\sqrt{3})$.

The computations performed for CER1 gave the following results. The best pair of the full absorption angles that minimizes the mean reflection coefficient (11) are $\theta_1 = 25.4^\circ$ and $\theta_2 = 72.1^\circ$. This is the optimal CER1. Its impedances (12) are equal to $z_1/\rho c = 0.51$ and $z_2 (k_j l)^2/\rho c = 3.03$, and the mean coefficient (11) is $R = 0.283$. The angular dependence of the plane wave reflection coefficient is shown in figure 3 by curve 2. The optimal CER1 is obviously more efficient, on average, than any local coating. Practically, it can be used as a protection against multistatic observation systems for scatterers with plane boundaries that are large compared to the wavelengths.
As to invisibility to monostatic systems, the most efficient are those coatings that do not reflect waves at small angles of incidence, since the backscattered fields are composed of plane waves reflected in directions close to the surface normal. That is why the matched local coating (curve 1 in figure 2) is often considered the best. However, CER1 works even better if one of its two angles of full absorption is set as zero, \( \theta_1 = 0 \) (as for the local coating), and the other angle is chosen appropriately to enhance non-reflection in the needed range of small angles. For example, curve 2 in figure 2 corresponds to CER1 with \( \theta_2 = 50^\circ \), the reflection coefficient of which is less than 0.032 (the reflected energy is less than \( 10^{-3} \)) for angles \( \theta = 0–55^\circ \), while for the local matched coating, this range is only \( \theta = 0–20^\circ \). Curve 3 in figure 2 corresponds to the best matched CER1 (\( \theta_2 = 72.1^\circ \)), which, among all the matched coatings CER1, has the minimum value of the mean reflection coefficient (\( R = 0.29 \)).

It is worth noting that the results presented above for infinite plane reflectors are approximately valid for finite-size bodies only at very high frequencies where the sound wavelength is negligibly small compared to the dimensions of the interface and its radii of curvature. In low and medium frequency ranges, the invisibility conditions and the parameters of the ‘invisible’ coatings are quite different, as is well demonstrated by the example of the next section.

5. Invisibility of a cylinder

Consider the scattering and reflection problem for an infinite cylinder of radius \( a \) in an unbounded space filled with a homogeneous fluid \( \rho_c \). The cylinder (with or without a coating) is assumed to be uniform along its axis \( x \) and symmetric with respect to rotations about this axis. The objective is to study the efficiency of coatings with extended reaction.

This problem is tractable analytically. Its solution can be found in one of the existing textbooks, e.g. in [36]. Below it is rewritten in terms of the surface impedances of the cylinder and fluid defined at the interface \( r = a \) in accordance with the impedance theory—see section 2. The three basic impedance matrices of the theory are, in this case, of the order of infinity. However, from the symmetry, it follows that the matrices have continuous spectra of simple eigenvalues and identical eigenfunctions in a form that is exponential in the \( x \)-direction and trigonometric of angle \( \phi \) of rotation about the axis:

\[
\exp(ikx) \cos(m\phi),
\]

where \( m \) is an integer — the number of a circular harmonic. It can be shown with the help of the wave equation that the eigenvalues of the fluid impedance matrices, \( z_i \) and \( z_r \), which correspond to the \( m \)th mode (13), are

\[
\frac{z_{im}}{\rho c} = -\frac{iJ_m(k_r a) k_i}{J_m'(k_r a) k_i}, \quad \frac{z_{rm}}{\rho c} = \frac{iH_m(k_r a) k_i}{H_m'(k_r a) k_i},
\]

where \( J_m \) and \( H_m \) are the Bessel and the Hankel functions of the first kind, \( k_r = (k_f^2 - k_0^2)^{1/2} \) is the radial component of the fluid wave number \( k_f \). The surface impedance matrix \( Z \) of the cylinder in vacuum also has similar modal impedances. They are denoted here as \( z^{(m)} \). Since the circular harmonics (13) are independent, i.e. orthogonal over the cylinder surface, the scattering (reflection) problem may be solved for each harmonic separately.

Let a plane wave of the complex amplitude \( p_i \) be incident on the cylinder at angle \( (90^\circ - \theta) \) to the \( x \)-axis. In terms of cylindrical coordinates \( (x, r, \phi) \), this wave can be represented as a
Figure 4. Specific impedance of the optimal local coating rendering minimum backscattering: 1, real part (resistance); 2, imaginary part (reactance).

series in the incident cylindrical waves [36]:

\[ p_i(x, r, \varphi) = p_i e^{ikx} \sum_{m=0}^{\infty} P_{im} \frac{J_m(k_r r)}{J_m(k_r a)} \cos(m\varphi), \]

\[ v_i(x, r, \varphi) = \frac{p_i}{\rho c} e^{ikx} \sum_{m=0}^{\infty} V_{im} \frac{J'_m(k_r r)}{J'_m(k_r a)} \cos(m\varphi), \]

\[ P_{im} = \varepsilon_m i^m J_m(k_r a), \quad P_{im} + (z_{im} / \rho c) V_{im} = 0. \]

Here \( k = k_f \sin \theta \) is the \( x \)-component of the fluid wave number, \( v_i \) is the radial component of the particle velocity of the medium; \( \varepsilon_0 = 1, \varepsilon_m = 2 \) for \( m > 0 \); \( z_{im} \) is the modal internal fluid impedance given in equation (14). Similarly, the field scattered by the cylinder can be expanded in the outgoing cylindrical waves as [36]

\[ p_s(x, r, \varphi) = p_s e^{ikx} \sum_{m=0}^{\infty} P_{sm} \frac{H_m(k_r r)}{H_m(k_r a)} \cos(m\varphi), \]

\[ v_s(x, r, \varphi) = \frac{p_s}{\rho c} e^{ikx} \sum_{m=0}^{\infty} V_{sm} \frac{H'_m(k_r r)}{H'_m(k_r a)} \cos(m\varphi), \]

\[ P_{sm} - (z_{sm} / \rho c) V_{sm} = 0, \]

where \( z_{sm} \) is the radiation impedance (14) of the \( m \)th circular mode. Each term of expansions (15) and (16) represents, on the cylinder surface \( r = a \), the circular harmonic (13). According to the impedance theory, the modal scattering coefficients, \( Q_m \) and \( S_m \), for velocity and pressure...
Figure 5. Backscattered pressure amplitude (18) as a function of frequency for: 1, a rigid cylinder; 2, the matched local coating; 3 and 4, the optimal CER0.

defined as $v_{sm} = Q_m v_{im}$, $p_{sm} = S_m p_{im}$, are obtained in the form

$$Q_m = \frac{z_{im} - z_{r0}}{z_{rm} + z_{r0}}, \quad S_m = \frac{y_{im} - y_{r0}}{y_{rm} + y_{r0}}, \quad z_{r0} Q_m + z_{im} S_m = 0. \quad (17)$$

Equations (14)–(17) give the general solution of the sound scattering and reflection problem for an infinite cylinder in terms of impedances and mobilities.

As a measure of backscattering or a measure of invisibility to echolocation systems, the normalized reflected pressure amplitude at $r = r_0 \gg a$, $\psi = \pi$, is used:

$$BS = \frac{|p_0|}{|p_{ref}|} = \sqrt{\frac{2r_0}{a}} \sum_{m=0}^{\infty} (-1)^m v_{im} z_{r0} Q_m H_m(k r_0) H_m(k_0 a), \quad (18)$$

the reference amplitude being equal to $p_{ref} = p_r(a/2r_0)^{1/2}[5]$.

As a measure of transparency or a measure of invisibility to multistatic detection systems, the normalized scattering cross section $SCS$ is used. It is defined as the ratio of the power of sound scattered by a unit-length part of the cylinder to the power of the sound wave incident on the unit length: the smaller the $SCS$ the more invisible the cylinder. The equations for $SCS$ are derived from (14)–(17):

$$SCS = \frac{4}{k_r a} \left[ \frac{|Q_0|^2 J_1^2}{2|H_1|^2} + \sum_{m=1}^{\infty} \frac{|Q_m|^2 |J_m'|^2}{|H_m'|^2} \right] = \frac{4}{k_r a} \left[ \frac{|S_0|^2 J_2^2}{2|H_0|^2} + \sum_{m=1}^{\infty} \frac{|S_m|^2 |J_m|^2}{|H_m|^2} \right], \quad (19)$$

where the argument of all the Bessel and Hankel functions is $k_r a$.

To make the analysis of efficiency of the proposed coatings more evident, characteristics (18) and (19) were also computed for two cases well studied in the literature: for a rigid cylinder and a cylinder with matched local coating (see curves 1 and 2 in figures 5...
Figure 6. The total field pressure amplitude levels for a cylinder with optimal non-reflecting coating CER0.

Figure 7. Directivity pattern for a cylinder with optimal non-reflecting coating CER0 (solid line) and for a rigid cylinder (dotted line).
Figure 8. Scattering cross section as a function of frequency for: 1, a rigid cylinder; 2, the matched local coating; 3, the optimal CER0, 4, the optimal CER1.

and 8). The plane wave is assumed here to be incident normally on the cylinder. It is seen from the figures that the rigid cylinder is a good non-scatterer at very low frequencies \((k_0a < 0.2)\), while the matched local coating \((z^{(m)} = \rho c \text{ for all } m)\) is a rather efficient non-reflector at high frequencies \((k_0a > 5)\).

Coatings with extended reaction for a cylinder have the same structure as for plane CERs (figure 1) although bent to cover the cylindrical surface \(r = a\). Therefore, the equations of motion of cylindrical CERs coincide with those of plane CERs. In particular, for cylindrical CER1, equations (5) and (6) are valid. The \(m\)th modal surface impedance of the CER1 is obtained from equation (6) with the force excitation in the form of circular harmonic (13):

\[
z^{(m)} = z_1 + (m^2 + k^2a^2)z_2,
\]

where \(z_1 = Z_1/l^2\) and \(z_2 = Z_2/a^2\).

Values of the coating parameters can, in principle, be chosen using the same modal approach as in the case of plane geometry described in section 4. More exactly, the coating impedances, \(z_1\) and \(z_2\), can be determined from equality of impedance (20) to the first two modal fluid internal impedances (14): \(z_1 = z_{i0}(1 + k^2a^2) - z_{i1}k^2a^2; z_2 = z_{i1} - z_{i0}\). The CER1 with such parameters is fully transparent to the zero-order and the first-order incoming cylindrical waves (corresponding to \(m = 0\) and \(1\) in equation (15)), so that the scattered field (16) does not contain the first two outgoing waves, \(p_{s0} = p_{s1} = 0\)—see equations (17). More complex modifications CER2, CER3, etc with the coating parameters chosen according to this modal strategy provide full transparency to three, four and more incoming cylindrical waves. If these waves make the main contribution to the incident field, it is possible, with the use of such CERs, to achieve a considerable degree of invisibility for the protected cylindrical body as a whole.
Figure 9. Directivity pattern for a cylinder with optimal non-scattering coating CER1 (solid line) and for a rigid cylinder (dotted line) at frequency $k_f a = 1$.

The modal approach for designing CERs, due to its simplicity and physical evidence, is very attractive and useful. However, as is shown in [38], it has some disadvantages, the main one of which is the so-called spillover. This phenomenon consists in a redistribution of the scattered energy among the cylindrical waves, caused by the use of a coating, and in a possible increase in the energy of uncontrolled scattered waves. Manifestation of the spillover depends on the peculiarities of the problem under study, e.g. on its geometry. The results of the present author’s computations [34, 38] show that this phenomenon is thoroughly absent in plane CERs. In cylindrical CERs, it is pronounced in CER0 in rather wide frequency ranges, whereas in CER2, CER3 and other high-order CERs, it is noticeable in some narrow bands of medium and high frequencies.

In light of these results, a more preferable designing strategy for CERs would be the optimal strategy based on minimization of the invisibility goal function (18) or (19). The optimal strategy is not simple to realize and is less physically evident, but it is free from the spillover and other shortcomings of the modal strategy. Figures 4–10 present some results of the investigation of optimal CERs.

One of the most interesting results is that practically full invisibility with respect to echolocation systems can be achieved with the help of the simplest optimal coating. Consider a local cylindrical CER0 that is characterized by a structural parameter: the cell impedance $Z_1$. According to the optimal strategy, the problem lies in finding those realizable values of $Z_1$ that render the minimum backscattered function (18). This variational problem, as well as other similar problems for more complex CERs, is not tractable analytically. Therefore, all the results given below are obtained numerically.
Figure 10. The total field pressure amplitude levels for a cylinder with optimal non-scattering coating CER1 at frequency $k_\lambda a = 1$.

Figure 4 presents the real and imaginary parts of the optimal values of the surface impedance $Z_1/\rho c l^2$ that minimizes backscattering. At high frequencies, the impedance is rather close to $\rho c$. At medium and low frequencies, it exhibits low-quality resonances. Curves 3 and 4 in figure 5 show the efficiency of CER0, which has this optimal impedance. In the wide frequency range, the efficiency is evidently higher than even the efficiency of the matched coating. Figure 6 presents the total field pressure levels at one frequency of the range: a plane wave that is traveling from left to right remains practically unchanged on the left of the cylinder, although on the right of the cylinder the field is completely misshaped. Figure 7 explains the result: the directivity pattern of the cylinder with the optimal CER0 (solid line) demonstrates the absence of backscattering and large scattered amplitudes in other directions, compared with the amplitude scattered by the rigid cylinder, especially in the forward direction. The optimal CER0 thus completely solves the problem of suppressing backscattering. There is, however, one theoretical question that deserves mentioning.

Curves 3 and 4 in figure 5 have been obtained by numerical minimization of $BS$ (18) on a 2D grid with a certain mesh size $d$ in the plane $(\text{Real}(z_1/\rho c), \text{Imag}(z_1/\rho c))$. Curve 3 corresponds to the size $d = 0.02$, whereas for curve 4 the size is $d = 0.001$. The computations showed that further decrease in $d$ led to a proportional decrease in $BS$. This might mean that if the optimal value of the impedance $z_1$ were known mathematically exactly, the backscattered field would be exactly zero. In other words, $BS$ as a function of the cell impedance $z_1$ seems to have a very narrow zero-valued minimum. Interested readers are invited to try to prove mathematically (or reject) the following statement: for a finite-size body, there exists a realizable (passive, causal) locally reacting coating, which provides zero backscattering in a wide frequency range. Up to the moment of submission of this paper, the present author failed to do it by himself or to find the solution in the literature.

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While the problem of invisibility to acoustic echolocation systems has a rather simple solution, invisibility to multistatic systems is a much more difficult problem. To achieve a high degree of transparency, it is necessary to not only suppress backscattering but also suppress side scattering and forward scattering. In other words, it is necessary to eliminate the total power of the scattered field. Note that forward scattering is the most important angular component of the scattered field, since it is responsible for forming the shadow. Besides, it is directly related to the total scattered power, according to the well-known optical theorem [39]. To suppress the forward scattering, a coating, besides absorbing the incident field on the illuminated side of the body, should transmit the disturbance to the opposite side and reradiate it in the incident direction. This is possible only if the coating is globally reacting or at least with extended reaction.

Curve 3 in figure 8 corresponds to the optimal local coating the impedance of which minimizes the scattered power (19). This coating cannot transmit energy and, hence, does not reradiate it. Therefore its efficiency, although higher than that of the matched coating, is good only at very low frequencies \( k_0a < 0.2 \). In this frequency range, the coating behaves like a rigid body, while at high frequencies it is close to the matched coating.

When couplings between the coating elements are introduced and the coating becomes a structure with extended reaction, its efficiency increases. Curve 4 in figure 8 shows the scattering cross section (19) of the optimal CER1. Couplings between the adjacent coating cells widen the low frequency range of transparency: the body becomes practically invisible at low frequencies \( k_0a < 1 \) and partly transparent at medium frequencies \( k_0a < 4 \). Figure 9 presents the directivity pattern of the scattered field at frequency \( k_0a = 1 \) (solid line): the scattered amplitude decreases equally in all directions. As a result, the total pressure field represents in this case the field of the incident plane wave slightly distorted in the whole space (figure 10). The author verified that introduction of coupling between other neighbors further widens the low frequency range of invisibility. In the limit, when each pair of the coating elements is coupled and the coating is globally reactive, the body is expected to be fully transparent (non-scattering) in a wide frequency range.

6. Summary

The problem of how to render an arbitrary body undetectable by acoustic methods of observation based on insonification and reception of the reflected and scattered signals is considered. As follows from the literature review, great progress on the problem has been achieved by developing active acoustic methods of suppressing the scattered field by additional sound sources. One passive method is also under intensive study now—the method of transformation optics, which can lead to a practical solution provided that corresponding metafluids (new materials unknown in nature) are realized.

In this paper, an alternative passive method that, in the author’s opinion, is more feasible for practical implementation is proposed. It is demonstrated that the invisibility problem can be solved with the help of a thin passive coating whose surface impedances are equal to or close to the surface impedances of the ideally transparent body for which explicit equations are derived. The results of the computer simulation presented show that almost perfect invisibility to echolocation systems can be achieved using coatings of very simple structure, whereas for invisibility to tomographic (multistatic) detection systems, rather complicated coatings are needed.
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