Strong Analytic Controllability for Hydrogen Control Systems

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Abstract—The realization and representation of so(4, 2) associated with the hydrogen atom Hamiltonian are derived. By choosing operators from the realization of so(4, 2) as interacting Hamiltonians, a hydrogen atom control system is constructed, and it is proved that this control system is strongly analytically controllable based on a time-dependent strong analytic controllability theorem.

I. INTRODUCTION

In the microscopic world ruled by quantum mechanics, most interesting phenomena involve change of continuous quantum operators acting on infinite-dimensional state spaces. For instance, developments in quantum error correction [1], [2], [3] and quantum teleportation [4], [5] require addressing the question of continuous variables. It is noted that real world quantum systems are influenced by interaction with their changing environment to a greater or lesser extent. The structural Hamiltonians used to describe these open quantum systems are time-dependent ones, as is the case in [6], [7]. New research pays more and more attention in manipulation of time-dependent quantum systems acting on infinite-dimensional state spaces, which contrasts to the manipulation of discrete systems with finite-dimensional state spaces in traditional quantum computation.

From a theoretical point of view, the essential problem of the manipulation of quantum dynamical phenomena is the problem of control. For instance, the field of nuclear magnetic resonance is largely concerned with the geometric control of collections of interacting nuclear spins [8], [9], [10], [11]. The methods used to derive these results are those of geometric control applied to quantum systems: these group-theoretic methods allow easy mathematical treatment of Hamiltonian systems. Not only do certain systems possess obvious symmetries in their spectra which allows the application of group theory, but it has also been suggested that group theory might help to look for certain transformations which allow passing from one level state to another, and thus get new insight into the structure of the system [12].

In this paper, we use group-theoretic methods to study the control problem of a time-dependent hydrogen atom system acting on infinite-dimensional state spaces. As the simplest of all atoms, hydrogen atom plays a central role in quantum mechanical theory. It is the only atom for which the energy eigenvalue problem has been solved exactly. Wave functions of the hydrogen atom are the starting point for the description of all atoms and molecules. Understanding how to control the hydrogen atoms is the starting point for understanding how to control more complicated atoms and molecules. From the design of a control for atoms and molecules via perturbation/approximation method, the solution of the control problem of hydrogen atom furnishes the first step. From energy spectrum point of view, hydrogen atom possesses both discrete (bound state) and continuous (scattering states) spectra. Understanding the control problem of hydrogen atom will help us to understand the controlled transition from bound states to scattering states and vice versa.

It is well known that the bound state subspace of a hydrogen atom for a fixed energy \( E_n \) spans a representation space of dimension \( n^2 \) of the symmetry group \( SO(4) \) [13]. The symmetry group is \( SO(3, 1) \) for scattering states. When the atom interacts with an external electromagnetic field, it can make transitions to other energy subspaces, because energy is transferred to or from the atom. Thus we can connect a representation space of \( SO(4) \) or \( SO(3, 1) \) of some energy with representation spaces of other energies. It turns out that the totality of all states of the atom spans a representation space of \( SO(4, 2) \), which is the maximal kinematical space-time transformation that leave invariant the Schrodinger equation for Hydrogen Atom. Thus one can say that this dynamic group encompasses all the atomic states together with their transformations, including the transformations between states of different energy [14]. Although there are scattered results existing in the literature, to the authors’s knowledge, there is no result for both bound and scattering states with time-dependent Hamiltonians for hydrogen atom. Our study paves a way for understanding the control of decoherence problem, since the interaction Hamiltonians as a rule are time-dependent in the study of the decoherence problem.

In this paper, we begin in section 2 with summarizing the theorem of strong analytic controllability of time-dependent quantum systems [15]. As the tangent space to \( SO(4, 2) \) at the identity, \( so(4, 2) \) has a good correspondence geometric structure of the Lie group \( SO(4, 2) \). Moreover, \( so(4, 2) \) is
a vector space, a linear object, so it is easier to study than $SO(4, 2)$. In section 3, we give the realization of $so(4, 2)$ for negative energy and positive energy by solving differential equations, which is the representation states of $SO(4, 2)$ as well. Using operators from the realization of $so(4, 2)$ as interacting Hamiltonians, we construct a hydrogen atom control system in section 4. Based on the above and the strong analytic controllability theorem[15], we conclude that this hydrogen atom control system is strongly analytically controllable. The construction of this hydrogen atom control system provides theoretical direction on how to construct a new control system by adding controls and interacting Hamiltonians to the original Schrödinger equation so that this new control system is strongly analytically controllable. Finally we conclude this paper with section 5.

II. A Strong Analytic Controllability Theorem for Time-dependent Quantum Systems

Quantum control systems as described by the Schrödinger equation are bilinear systems with respect to controls and states

$$\frac{\partial}{\partial t} \psi(t, x) = (H_0(t, x) + \sum_{i=1}^{r} u_i(t) H_i(t, x)) \psi(t, x),$$

where $H_0(t, x), H_i(t, x), i = 1, 2, \ldots, r$, are skew-Hermitian operators on unit sphere $S_H$, $\psi(t, x) \in S_H$, and $u_i(t), i = 1, \ldots, r$ are piecewise constant functions. In fact, the above skew-Hermitian operator $H(t, x)$ can be written as $iH'(t, x)$, where $H'$, being Hermitian, is an observable.

Operators involved in system (I) are generally unbounded and act on an infinite-dimensional Hilbert space which consists of quantum states $\psi$. In order to find a domain on which the exponentiations of these operators converge, we introduce a so-called analytic domain $D_\omega$ proposed by Nelson [16], a dense domain invariant under the action of operators in system (I). In addition, the solution of the Schrödinger equation can be expressed globally in exponential form on this domain, which is also invariant under the action of the exponentiations of these operators.

For system (I), a theorem [17], [18] tells us that the transitivity of states on $S_H$ requires infinite control manipulations based on the piecewise constant control set $\{u_i(t)\}$. In practice, switching control manipulations infinitely seems meaningless. So practical interest directs us to consider controllability on a finite-dimensional submanifold $M$ of the unit sphere Hilbert space $S_H$. Moreover, a finite-dimensional tangent space generated by $H_0(t, x)\psi(t, x), \ldots, H_r(t, x)\psi(t, x)$ is defined on $M \cap D_\omega$ and $\text{dim } M \cap D_\omega = m$ when assuming $\text{dim } M = m$. That is, the finite-dimensional tangent space generated by $H_0(t, x)\psi(t, x), \ldots, H_r(t, x)\psi(t, x)$ is densely defined on $M$. Further the manifold $M$ can be given by the closure of the set \( \{e^{t_0H_0}e^{t_1H_1}\ldots e^{t_rH_r}\psi_0\} \), with $(p0, p1, \ldots, pr)$ any permutation of $(0, 1, \ldots, r)$ and $t_i \in R$, $i = 0, 1, \ldots, r$.

On an analytic domain, the notion of controllability is modified by that of analytic controllability.

**Definition 2.1:** Let $\mathcal{L}$ be the Lie algebra generated by skew-Hermitian operators $H_0, \ldots, H_r$ on a unit sphere $S_H$. For system (I), if $D_\omega$ exists for $\mathcal{L}$, and for any $\psi_0, \psi_\delta \in D_\omega \cap M$, there exist $u_1(t), \ldots, u_r(t)$, and $T (\forall T)$ s.t. the solution of control system (I) satisfies $\psi(t_0, x) = \psi_\delta, \psi(T, x) = \psi_\delta$ and $\psi(t, x) \in D_\omega \cap M$, $t_0 \leq t \leq T$, then the system is called analytically controllable (strongly analytically controllable) on $M$, and we say the corresponding unitary Lie group is analytically transitive on $M$.

Let $R_t(\psi_0)$ denote the reachable set of system (I) from starting state $\psi_0$ at time $t$, and let $R(\psi_0) = \bigcup_{t \geq t_0} R_t(\psi_0)$ denote the reachable set starting from $\psi_0$ at time greater than $t_0$.

**Definition 2.2:** A time-dependent control system is said to be completely controllable if $R(\psi_0) = M$ holds for all $\psi_0 \in M$. If $R_t(\psi_0) = M$ for all time $t > t_0$ and for any $\psi_0 \in M$, then this system is said to be strongly completely controllable on $M$. If $R_t(\psi_0) = M \cap D_\omega$ for all time $t > t_0$ and for any $\psi_0 \in M \cap D_\omega$, then this system is said to be strongly analytically controllable on $M$.

With the help of Kunita’s method [19], we derive the following strong analytic controllability theorem in [15], [20].

**Theorem 2.1:** Considering quantum control system (I), let

$$B(t, x) = \mathcal{L}\{H_1(t, x), \ldots, H_r(t, x)\}$$

$$B_1 = -[H_0, B] + \frac{\partial}{\partial t} B$$

$$\vdots$$

$$B_n = -[H_0, B_{n-1}] + \frac{\partial}{\partial t} B_{n-1}$$

$$\vdots$$

$$C = \mathcal{L}\{B, B_1, \ldots, B_n, \ldots\}.$$  

Suppose $\text{dim } \mathcal{L}(t, x)\psi(t, x) = m$ holds for all $\psi \in M \cap D_\omega$, and $[B, C](t, x) \subset B(t, x)$ holds for all $(t, x)$. Then the time-dependent quantum control system (I) is strongly analytically controllable.

III. The realization and the representation of $so(4, 2)$

Kleinert [21] pointed out that the internal structure of the quantum mechanical system of the hydrogen atom can be described completely in terms of simple group operations in the representation space of the non-compact group $SO(4, 2)$. $SO(4, 2)$ is the group of rotation in six-dimensional pseudo-Euclidean space with the metric

$$g = \text{diag}(1, 1, 1, 1, -1, -1).$$

It is a 15-parameter group and is generated by 15 generators $L_{\alpha \beta} = -L_{\beta \alpha}$, $\alpha, \beta = 1, \ldots, 6$ which form the Lie algebra $so(4, 2)$ with the commutation rules

$$[L_{\alpha \beta}, L_{\alpha \gamma}] = i\gamma_{\alpha \beta} L_{\alpha \gamma}.$$

All information contained in the Schrödinger theory of the hydrogen atom can be expressed in a completely algebraic language. That is, representation states of $so(4, 2)$ can be
brought in one-to-one correspondence with the states of the hydrogen atom.

It is known [22] that the Lie algebra so(4, 2) satisfies the following commutation relations:

\[
\begin{align*}
\{L, L\} &= iL, \\
\{L, A\} &= iA, \\
\{L, B\} &= iB, \\
\{L, S\} &= 0, \\
\{A, B\} &= IS, \\
\{A, C\} &= i\Gamma, \\
\{B, C\} &= iS, \\
\{C, A\} &= i\Gamma, \\
\{C, B\} &= iS, \\
\{C, L\} &= 0, \\
\{D, A\} &= 0, \\
\{D, B\} &= -i\Gamma, \\
\{D, C\} &= iS, \\
\{G, A\} &= -iC, \\
\{G, B\} &= -iD, \\
\{G, C\} &= -iA, \\
\{G, L\} &= -iL,
\end{align*}
\]

where \([A, B] = iS\) means \([A_i, B_j] = i\delta_{ij} S\).

A realization of the Lie algebra so(4, 2) is a homomorphism which associates a concrete set of operators with abstract basis vectors of the Lie algebra [22]. It is more of a physical concept than a mathematical one. In quantum mechanics, such operators are often differential operators expressed in terms of the position operator \(\mathbf{r}\) and momentum operator \(\mathbf{p}\), which act on a Hilbert space of quantum mechanical states. In the following we will give a realization of so(4, 2) for negative energy and positive energy respectively by solving differential equations based on Bacry’s method, with which Bacry derived the realization of so(4, 1) for negative energy [23]. Dothan [24] proposed that these generators constitute a finite-dimensional spectrum generating algebra. Adams [22] described a beautiful algebraic method to construct another type of 15 generators of so(4, 2). These two sets of 15 generators are equivalent according to the so(4, 2) commutating structure.

For the negative energy case, Greiner and Müller [25] describes a way to relate so(4) to the hydrogen atom Hamiltonian

\[
H = \frac{p^2}{2} + \frac{1}{r},
\]

expressed in the atomic units, where \(p\) is the momentum and \(r\) is the distance between the electron and the proton in the hydrogen atom. That is, they showed that the angular momentum \(L = \mathbf{r} \times \mathbf{p}\) and the Runge-Lenz vector \(A = \frac{1}{\sqrt{-2H}}(L \times p - p \times L) + \frac{r}{2}\) generate the Lie algebra so(4), which is part of the Lie algebra of so(4, 2). Moreover, L and A commute with Hamiltonian \(H\). Hence, these operators can only transform a quantum state to another with the same energy level. When quantum systems interact with the environment, they may absorb or release energy, thus changing their energy levels. so(4, 2) is more interesting because it permits to transform an energy level into another energy level [22], [24], [21], [26], [13]. Hence, so(4, 2) is represented fully on the energy eigenstates, be they discrete or continuous, and energy degenerate or nondegenerate.

In order to find the other operators \(B, \Gamma, S, C,\) and \(D\) in terms of \(r\) and \(p\), it is much simpler to use the Poisson bracket instead of commutators because equations \([4] - [12]\) lead to differential equations which can be integrated in a simple way under the Poisson bracket [23]. Let \(f(r, p)\) be a scalar function of the classical variables \(r\) and \(p\), satisfying the Poisson bracket relation \(\{L, f\} = 0\). \(f\) can be written as a function of three scalars \(r, H,\) and \(l, \) where \(l^2 = r^2 p^2 - (r \cdot p)^2[23]\). Then we solve the differential equations in the Poisson bracket formulism. The operators so calculated in terms of the Poisson bracket may need be symmetrized to be Hermitian to get the following negative-energy realization of so(4, 2) [20].

\[
\begin{align*}
L &= r \times p, \\
A &= \frac{1}{\sqrt{-2H}}[(L \times p - p \times L)/2 - \frac{r}{r}], \\
B &= \frac{p^r + r p}{2} \cos \zeta - \frac{1}{\sqrt{-2H}} \frac{p(r \cdot p) + (p \cdot r)p}{2} \sin \zeta \\
&\quad + \frac{1}{\sqrt{-2H}} \frac{r}{r} \sin \zeta, \\
\Gamma &= -\frac{pr + r p}{2} \sin \zeta - \frac{1}{\sqrt{-2H}} \frac{p(r \cdot p) + (p \cdot r)p}{2} \cos \zeta \\
&\quad + \frac{1}{\sqrt{-2H}} \frac{r}{r} \cos \zeta, \\
S &= -\frac{r \cdot p + p \cdot r}{2} \sin \zeta - \frac{1}{\sqrt{-2H}}(1 + 2Hr) \cos \zeta, \\
C &= -\frac{r \cdot p + p \cdot r}{2} \cos \zeta + \frac{1}{\sqrt{-2H}}(1 + 2Hr) \sin \zeta, \\
D &= \frac{1}{\sqrt{-2H}}.
\end{align*}
\]

where \(\zeta = \sqrt{-2H} (r \cdot p + p \cdot r)/2 + (-2H)^{3/2} t\).

Further, we compute the commutators between the Hamiltonian \(H\) and the generators of so(4, 2) as follows.

\[
\begin{align*}
[H, L] &= 0, \\
[H, A] &= 0, \\
[H, B] &= (-2H)^{3/2} i\Gamma, \\
[H, \Gamma] &= -(2H)^{3/2} iB, \\
[H, S] &= -(2H)^{3/2} iC, \\
[H, C] &= -(2H)^{3/2} iS, \\
[H, D] &= 0.
\end{align*}
\]

Obviously these so(4, 2) generators satisfy the condition

\[
i \frac{\partial G}{\partial t} - [H, G] = 0.
\]

In [24], the relation \([28]\) tells us that if \(\psi\) is an eigenstate of the Schrödinger equation

\[
i \frac{\partial \psi}{\partial t} = H \psi,
\]

then \(G \psi\) is an eigenstate of the above Schrödinger equation when \(G\) satisfies the condition \([29]\). So this realization of so(4, 2) forms a finite-dimensional spectrum-generating algebra [24]. All of the energy eigenfunctions of the physical problem form a basis for a single unitary irreducible representation.
Remark 3.1 We know that $D$, $L$, and $A$ commute with the Hamiltonian $H$, so the multiplets produced by the action of the transformations generated by $D$, $L$, and $A$ are degenerate eigenfunctions of $H$. The other generators of $so(4,2)$ do not commute with Hamiltonian $H$, so the multiplets produced by the transformation generated by these operators cannot be degenerate eigenfunctions of $H$. Rather, these transformations generate a spectrum of eigenfunctions of $H$.

For the positive energy, it is known [22] that the angular momentum $L = r \times p$ and the Runge-Lenz vector $B = \frac{1}{\sqrt{2H}}(L \times p - p \times L) + \frac{1}{2} \frac{L \cdot p}{H} \frac{p}{r}$ generate the Lie algebra $so(3, 1)$, which is part of the Lie algebra $so(4,2)$. Similar to the negative energy case, we give the positive-energy realization of $so(4,2)$ as follows.

$$L = r \times p$$

$$A = \frac{pr + rP}{2} \sinh \zeta - \frac{1}{\sqrt{2H}} \frac{p(r \cdot p) + (p \cdot r)p}{2} \cosh \zeta$$

$$+ \frac{1}{\sqrt{2H}} \frac{r}{r} \cosh \zeta$$

$$B = \frac{1}{\sqrt{2H}} [(L \times p - p \times L)/2 - \frac{1}{r}]$$

$$\Gamma = \frac{pr + rP}{2} \cosh \zeta - \frac{1}{\sqrt{2H}} \frac{p(r \cdot p) + (p \cdot r)p}{2} \sinh \zeta$$

$$+ \frac{1}{\sqrt{2H}} \frac{r}{r} \sinh \zeta$$

$$S = \frac{1}{\sqrt{2H}} (2Hr + 1) \sinh \zeta - \frac{r \cdot p + p \cdot r}{2} \cosh \zeta$$

$$D = \frac{r \cdot p + p \cdot r}{2} \sinh \zeta - \frac{1}{\sqrt{2H}} (2Hr + 1) \cosh \zeta$$

$$C = \frac{1}{\sqrt{2H}}$$

where $\zeta = \sqrt{2H} (r \cdot p + p \cdot r)/2 - (2H)t$.

Similar to the negative energy case, we compute the commutation relations between the Hamiltonian $H$ and these fifteen positive-energy generators of $so(4,2)$ as follows.

$$[H, L] = 0$$

$$[H, A] = -(2H)^{\frac{3}{2}} i \Gamma$$

$$[H, B] = 0$$

$$[H, \Gamma] = -(2H)^{\frac{3}{2}} i A$$

$$[H, S] = (2H)^{\frac{3}{2}} i D$$

$$[H, D] = (2H)^{\frac{3}{2}} i S$$

$$[H, C] = 0.$$  

It is easily verified that the realization of $so(4,2)$ for positive energy satisfies the relationship (25), based on the relationships (37) — (43).

Remark 3.2 In fact, relation (28) is a special case of the following relation. Let $Q = i \partial_t - H$, where $H$ is the Hamiltonian. Then the sufficient condition for operator $Z$ to belong to a symmetry algebra is [27]

$$[Z, Q] = R_L(t, x)Q,$$

where $R_L(t, x)$ is a operator having consistent orders of $\partial_t$ and $\partial x$, and $x \in R^n$. When $R_L(t, x) = 0$, relation (44) is reduced to relation (28). In [27], Miller uses relation (44) to find the basis of the symmetry algebra, then diagonalize different operators in this symmetry algebra to separate variables in different coordinate systems, thereby finding solutions for differential equations.

Now let us study the matrix representation of $L$. If $\{ |n\rangle : n = 1, \ldots, N\}$, where $N$ can be infinity, is a basis for vector space $\mathcal{V}$, and if $P_i = T(h_i), i = 1, \ldots, l$, where $h_i \in L$, $h_i$ is a basis for $L$, and $T$ is a homomorphism, then $P_i |n\rangle = \sum_m \langle m | P_i |n\rangle |m\rangle$, $m, n = 1, \ldots, N$, where $\langle m | P_i |n\rangle$ denotes matrix element $(m, n)$ of $P_i$. These matrices are a basis for a matrix representation of the Lie algebra $L$. The above realizations of $so(4,2)$ can now be employed for us to look for the representation states $\{ |n\rangle : n = 1, \ldots, N\}$. First, we decompose $so(4,2)$ as follows

$$so(4,2) \supset so(4) \otimes so(2,1)$$

Following the method in [22] of expanding the representation states of $so(4)$ and the representation states of $so(2,1)$, we obtain the representation states of $so(4,2)$ for negative energy case as follows:

$$L_3 |nlm\rangle = m |nlm\rangle$$

$$L_+ |nlm\rangle = \omega_n |nl, m + 1\rangle$$

$$L_- |nlm\rangle = \omega_n |nl, m - 1\rangle$$

$$A_3 |nlm\rangle = \alpha_m c_n^l |n, l - 1,m\rangle + \alpha_n^{l+1} c_{l+1}^n |n, l + 1,m\rangle$$

$$A_+ |nlm\rangle = \beta_m^l c_n^l |n, l - 1,m + 1\rangle$$

$$- \gamma_m^{l+1} c_{l+1}^n |n, l, m + 1\rangle$$

$$A_- |nlm\rangle = - \beta_m^{l-1} c_n^l |n, l - 1,m - 1\rangle$$

$$+ \gamma_m^{l-1} c_{l-1}^n |n, l, m - 1\rangle$$

$$B_3 |nlm\rangle = \alpha_m u_n^l |n - 1, l, m\rangle + \alpha_n^{l} v_n^l$$

$$\cdot |n + 1, l, m\rangle + \alpha_m^{l+1} u_{n-1}^l |n - 1, l + 1, m\rangle$$

$$+ \alpha_n^{l+1} u_{n-l+1}^l |n + 1, l, m + 1\rangle$$

$$B_+ |nlm\rangle = \beta_m^{l+1} u_n^l |n, l - 1, m + 1\rangle + \beta_m^{l-1} v_n^l$$

$$\cdot |n - 1, l + 1, m\rangle - \gamma_m^{l+1} v_{n+1}^l |n - 1, l + 1, m + 1\rangle$$

$$+ \gamma_m^{l-1} v_{n-1}^l |n + 1, l + 1, m - 1\rangle$$

$$B_- |nlm\rangle = - \beta_m^{l-1} u_n^l |n - 1, l - 1, m\rangle - \beta_m^{l+1} v_n^l$$

$$\cdot |n + 1, l - 1, m\rangle - i \alpha_m^{l} v_n^l$$

$$+ i \alpha_n^{l+1} v_{n+1}^l |n - 1, l + 1, m\rangle$$

$$\cdot |n + 1, l + 1, m + 1\rangle + i \alpha_n^{l-1} v_{n-1}^l |n - 1, l, m - 1\rangle$$

$$\cdot |n + 1, l, m - 1\rangle - i \alpha_m^{l} v_n^l$$

$$- i \alpha_n^{l-1} v_{n-1}^l |n + 1, l - 1, m\rangle - i \alpha_m^{l+1} v_{n+1}^l |n - 1, l - 1, m + 1\rangle$$

$$- i \alpha_n^{l} v_n^l |n + 1, l, m + 1\rangle.$$
\[ \Gamma_{-|nlm} = i\beta^{-1}_{-m}u_{l}^{n}|n - 1, l - 1, m - 1\rangle - i\beta^{-1}_{-m}v_{l}^{n} \]
\[ \cdot |n + 1, l - 1, m - 1\rangle - i\beta^{+1}_{-m}v_{l+1}^{n+1}|n + 1, l + 1, m - 1\rangle \]
\[ + i\beta^{+1}_{-m}u_{l+1}^{n+1}|n + 1, l + 1, m - 1\rangle \]
\[ \cdot \langle n + 1, l + 1, m - 1 | \langle n - 1, l - 1, m - 1 | \]
\[ D|nlm\rangle = n|nlm\rangle \]
\[ T_{+}|nlm\rangle = \omega_{l}^{n}|n + 1, lm\rangle \]
\[ T_{-}|nlm\rangle = \omega_{l}^{-n}|n - 1, lm\rangle, \]

where
\[ \alpha_{m}^{l} = \sqrt{(l - m)(l + m)} \]
\[ u_{l}^{n} = \frac{1}{2} \sqrt{(n + l - 1)(n + l)} \]
\[ c_{m}^{l} = \sqrt{(l - m)(l + m)} \]
\[ v_{l}^{n} = \frac{1}{2} \sqrt{(n - l + 1)(n - l)} \]

and \( m = -l, \ldots, -1, 1, \ldots, l, l = 0, 1, \ldots, n - 1, n = 1, 2, 3, \ldots \) 

Remark 3.3 Vector operators \( L \) and \( A \) act on the angular portion of the basis kets \( |nlm\rangle \), affecting only the magnetic quantum number \( m \) and orbital angular momentum quantum number \( l \), while operators \( S, C, D \) act only on the remaining part. Further, vector operators \( B \) and \( \Gamma \) act on both parts.

Construction of the representation of \( so(4, 2) \) for positive energy follows a procedure similar to that used above for the negative energy case, based on the following alternative decomposition
\[ \text{so}(4, 2) \supset \text{so}(3, 1) \oplus \text{so}(2, 1). \]

For the radial part \( S, C, D \) of this realization we need to diagonalize \( C \), which has a continuous spectrum, instead of diagonalizing \( D \) in the negative energy case.

\[ C|vlm\rangle = v|vlm\rangle, \quad v \in (-\infty, \infty), \]

where \( |vlm\rangle \) denote the continuous eigenstates. Thus, using \( |vlm\rangle \) to denote the continuous eigenstates, the representation of \( so(4, 2) \) for positive energy is similar to that for negative energy with the following exceptions — replacing quantum number \( n \) in the negative energy case with \( iv \), \( l = 0, 1, \ldots, \infty \), and \( v \in (-\infty, \infty) \) being continuous \([13], [21]\).

IV. A STRONGLY AND ANALYTICALLY CONTROLLABLE HYDROGEN ATOM SYSTEM

Now let us consider the hydrogen atom control system
\[ i \frac{\partial}{\partial t} \psi(t, x) = [H + u_{1}(t)L_{1} + u_{2}(t)L_{2} + u_{3}(t)L_{3} + \ldots + u_{15}(t)D_{15}] \psi(t, x), \]

where \( H, L_{1}, A_{1}, B_{1}, \Gamma_{1}, S, C, D, i = 1, 2, 3 \) are Hermitian operators defined as before, and \( u_{j}(t), j = 1, 2, \ldots, 15 \) are piecewise constant control functions. Divide the operators in the right hand side of equation \((60)\) by \( i \), converting the control system \((60)\) to the standard form
\[ \frac{\partial}{\partial t} \psi(t, x) = [H' + u_{1}(t)L_{1}' + u_{2}(t)L_{2}' + u_{3}(t)L_{3}'] \psi(t, x), \]

where \( H', L_{1}', A_{1}', B_{1}', \Gamma_{1}', S', C', D', i = 1, 2, 3 \) are skew-Hermitian operators, and the \( u_{j}(t), j = 1, 2, \ldots, 15 \) are piecewise-constant functions. Corresponding to the commutators \( [B, \Gamma] \), we have
\[ [L', L'] = L' \quad [L', A'] = A' \quad [L', B'] = B' \]
\[ [L', S] = 0 \quad [A', A'] = L' \quad [S, A'] = B' \]
\[ [C', A'] = 0 \quad [C', B'] = 0 \quad [S', B'] = A' \]
\[ [D', B'] = -\Gamma' \quad [C', L'] = 0 \quad [D', L'] = 0 \]
\[ [L', \Gamma'] = 0 \quad [S', C'] = D' \]
\[ [D', C'] = 0 \quad [C', B'] = -D' \]

From the commutators \((56) - (59)\), we can see that \( L', A', B', \Gamma', S', C', D' \) form a real Lie algebra, which we denote by \( B \). In addition, it has already been shown that both of the realizations of \( so(4, 2) \) for positive and negative energy satisfy relation \((28)\). So we can rewrite relation \((28)\) as
\[ \frac{\partial G'}{\partial t} + [H', G'] = 0, \quad G' \in B. \]

Based on the strong analytic controllability theorem given in section 2, the following holds:
\[ B_{1} = L\{ -[H', G'] + \frac{\partial G'}{\partial t}, G' \in B \} = 0, \]

so \( C = L\{ B, B_{1}, \ldots \} = B \) and \( \overline{B, C} \subset B \). In addition, the representation space of \( so(4, 2) \) spanned by \( |nlm\rangle \) and \( |vlm\rangle \) for negative and positive energies respectively, is indeed an analytic domain, because this space is invariant both under the Lie algebra \( so(4, 2) \) and under the Lie group \( SO(4, 2) \). That is, exponentiation of any operator from \( so(4, 2) \) acting on this representation space converges. Many examples can be found in \([28]\). Let \( M \) denote the closure of a manifold spanned by a finite set of eigenfunctions of \( so(4, 2) \) as is the case in \([17], [18], [29], [30]\). It is not hard to see that \( \dim\overline{C} \psi(t, x) = \dim M \) \([17], [18], [29], [30]\) for any \( x, t \) according to the representation of \( so(4, 2) \) in the last section. Hence, by Theorem \(2.1\) the control system \((60)\) is strongly analytically controllable on \( M \). In fact, the effect of the drift term \( H \psi \) on the control system can be compensated by the effect of \( B \psi \).

Remark 4.1 In control system \((61)\), we do not need 15 generators to appear in the control system. We can selectively
pick up at least five generators, for example, \( L_1', L_2', A_1', S' \), and \( C' \), and then by the commutation relations \((62) - (60)\) we can generate the other generators so as to make the system \((61)\) strongly analytically controllable. Because system \((61)\) corresponds to system \((60)\), we immediately conclude that the system
\[
\frac{\partial}{\partial t} \psi(t, x) = [H + u_1(t)L_1 + u_2(t)L_2 + u_3(t)A_3 + u_4(t)S + u_5(t)C] \psi(t, x)
\] (72)
is strongly analytically controllable.

The detailed analysis presented in this section provides theoretical guidance for choosing and imposing controls and interacting Hamiltonians on an original unperturbed system described by the Schrödinger equation for the hydrogen atom, such that the constructed control system is strongly analytically controllable. In addition, we can see that the unperturbed Hamiltonian \((15)\) and the interacting Hamiltonians from the realization of \(so(4, 2)\) act on the same state space. That is, addition of interacting Hamiltonians does not affect the state space of the original hydrogen atom system. The controllability problem can still be analyzed on the original state space, but with imposing controls and the interacting Hamiltonians.

Note that the angular momentum and the Runge-Lenz vector commute with the hydrogen atom Hamiltonian, so they cannot cause a transition from one energy level quantum state to another. However, the rest of operators of the realization of \(so(4, 2)\) can produce such transitions. Hence, at least one operator besides the angular momentum and the Runge-Lenz vector must be chosen as interacting Hamiltonians (as we did in control systems \((60)\) and \((72)\)) in order that the system is strongly analytically controllable and different quantum states of different energies.

V. Conclustion

Studying the manipulation of hydrogen atom — the simplest atom — can help us understand the manipulation of other atom systems. It is known that representation states of \(so(4, 2)\) can be brought in one-to-one correspondence with the discrete, continuous, degenerate and non-degenerate states of the hydrogen atom \([22], [24]\). In this paper, we first have given the realization and representation of the Lie algebra of \(so(4, 2)\). Then picking up operators from the realization of \(so(4, 2)\) as interacting Hamiltonians, and choosing piecewise constant functions as controls, we have constructed an ideal hydrogen atom control system which is strongly analytically controllable. When the system is interacting with environment, one has to consider adding disturbing noises. The results of this paper are the strong analytic controllability for hydrogen atom control systems without considering design schemes for control inputs. Based on results of this paper, we are investigating optimal transitions from bound states to bound states, scattering states to scattering states, bound states to scattering states and vice versa. This will facilitate the selection of desired control inputs to the system.

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