Effective Field Theory in Nuclear Many-Body Physics

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Abstract

Recent progress in Lorentz-covariant quantum field theories of the nuclear many-body problem (quantum hadrodynamics, or QHD) is discussed. The importance of modern perspectives in effective field theory and density functional theory for understanding the successes of QHD is emphasized.

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I. OVERVIEW

Reference [1] is a presentation entitled Relativistic Nuclear Many-Body Theory given at the Seventh International Conference on Recent Progress in Many-Body Theories held in Minneapolis, Minnesota, in August, 1991. This was a report on a long-term effort to understand the nuclear many-body system in terms of relativistic quantum field theories based on hadronic degrees of freedom [2,3], a topic we refer to as quantum hadrodynamics (or QHD). An extensive, more recent review of work in this area is contained in Ref. [4], and a text now exists [5] that provides background material. There has been significant recent progress in this area [6,7,8], and the goal of this contribution is to summarize briefly what has transpired since the presentation in Ref. [1].

The only consistent framework we have for discussing the relativistic many-body system is relativistic quantum field theory based on a local lagrangian density. In any lagrangian approach, one must first decide on the generalized coordinates, and hadronic

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1Extensive references to other work in this field are contained in Refs. [1] through [5].
degrees of freedom—baryons and mesons—are the most appropriate for ordinary nuclear systems (QHD). Early attempts involved simple renormalizable models, which reproduced some basic features of the nuclear interaction [2]. The advantage of such models is that in principle, one can consistently investigate and relate all aspects of nuclear structure to a small number of renormalized coupling constants and masses. The disadvantage, in addition to the strong coupling constants that make reliable approximation schemes difficult to come by, is that limiting the discussion to renormalizable lagrangians is too restrictive. Despite these drawbacks, the simple models led to interesting insights. In relativistic mean-field theory (MFT), nuclear densities, the level structure of the nuclear shell model, and the spin dependence of nucleon–nucleus scattering are reproduced [2]. The simplest model (QHD–I) consists of baryons and isoscalar scalar and vector mesons. A basic feature of all these models is that there are strong scalar and vector mean fields present in the nucleus, which cancel in the binding energy but which add to give the large spin-orbit interaction [7].

There is now overwhelming evidence that the underlying theory of the strong interaction is quantum chromodynamics (QCD), a Yang–Mills non-abelian gauge theory built on an internal color symmetry of a system of quarks and gluons. If mass terms for the $u$ and $d$ quarks are absent in the lagrangian, QCD possesses chiral symmetry in the nuclear domain; although spontaneously broken in manifestation, this symmetry should play an essential role in nuclear dynamics. The challenge [1] was to understand the theoretical basis of QHD, the successes that it had, and its limitations, in terms of QCD. Indeed, as we say in our summary in Ref. [1]:

More generally, it is probable that at low energies and large distances, QCD can be represented by an effective field theory formulated in terms of a few hadronic degrees of freedom. All possible couplings must be included in the low-energy effective lagrangian, which is then to be used at tree level. The underlying assumption of QHD is that of a local relativistic theory formulated in terms of baryons and the lightest mesons. The theory is assumed to be renormalizable, and one then attempts to extract predictions for long-range phenomena by computing both tree-level diagrams and renormalized quantum loop corrections. In the end, it may turn out that this assumption is untenable, and that the only meaningful interpretation of QHD is as an effective theory, to be used at the tree or one-loop level. The limitation to renormalizable couplings may then be too restrictive. Nevertheless, the phenomenological success of the MFT of QHD–I in the nuclear domain implies that whatever the effective field theory for low-energy, large-distance QCD, it must be dominated by linear, isoscalar, scalar and vector interactions.

The major progress since Ref. [1], in addition to the multitude of applications discussed in Ref. [4], has been the following [4][5][6]:

- The understanding of QHD as a low-energy, effective lagrangian for QCD, which can be used to improve MFT calculations systematically;

- The understanding of the way spontaneously broken chiral symmetry is realized in QHD;
• The development of a consistent, controlled expansion and approximation scheme that allows one to compute reliable results for bulk nuclear properties;

• The relation of relativistic MFT to density functional theory and Kohn–Sham potentials, placing it on a sounder theoretical basis;

• The understanding of the robustness of many of the QHD–I results.

In section 2 we discuss the relation to density functional theory [9,10] and Kohn–Sham potentials [11]. Section 3 contains a brief presentation of the effective lagrangian, and section 4 summarizes some recent results.

II. DENSITY FUNCTIONAL THEORY

We begin with a discussion of nonrelativistic density functional theory (DFT) and generalize later to include relativity. The basic idea behind DFT is to compute the energy $E$ of the many-fermion system (or, at finite temperature, the grand potential $\Omega$) as a functional of the particle density. DFT is therefore a successor to Thomas–Fermi theory, which uses a crude energy functional, but eliminates the need to calculate the many-fermion wave function.

The strategy behind DFT can be seen most easily by working in analogy to thermodynamics [12]. For a uniform system in a box of volume $V$ at temperature $T$, one first computes the grand potential $\Omega(\mu, T, V)$, where $\mu$ is the chemical potential. It then follows that the number of particles $N$ is determined by

$$N = \langle \hat{N} \rangle = -\frac{\partial \Omega}{\partial \mu}.$$ (1)

The convexity of $\Omega$ implies that $N$ is a monotonically increasing function of $\mu$, so this relation can be inverted for $\mu(N)$. Finally, one makes a Legendre transformation to the Helmholtz free energy $F(N, T, V) = \Omega(\mu(N), T, V) + \mu(N)N$ to discuss systems with a fixed density $n = N/V$.

For a finite system, we replace the chemical potential with an external, single-particle potential $\sum_i v(r_i)$. The grand potential is now a functional: $\Omega([v(r)], T)$, and a functional derivative with respect to $v$ gives the particle density:

$$n(r) = \langle \hat{n}(r) \rangle = \frac{\delta \Omega}{\delta v(r)}.$$ (2)

The convexity of $\Omega$ allows us (in principle) to invert this relation and find $v(r)$ as a (complicated) functional of $n(r)$. Finally, we make a functional Legendre transformation to define the Hohenberg–Kohn free energy, which is a functional of $n(r)$:

$$F_{HK}[n(r)] = \Omega[v(r)] - \int dr \ n(r)v(r).$$ (3)

\[2\] In fact, one can absorb $\mu$ into the definition of $v$. We suppress all spin dependence at this point.

\[3\] Higher variational derivatives yield various correlation functions.
The variational derivative of this free energy functional with respect to \( n \) now gives

\[
\frac{\delta F_{\text{HK}}}{\delta n(r)} = -v(r). \tag{4}
\]

If we now restrict consideration to \( T = 0 \) and \( v(r) = 0 \), then the Hohenberg–Kohn theorem follows \([4,10]\): If the functional form of \( F_{\text{HK}}[n(r)] \) is known exactly, the ground-state expectation value of any observable is a unique functional of the exact ground-state density. Moreover, it follows immediately from Eq. (4) that the exact ground-state density can be found by minimizing the energy functional. Although we have assumed here that the ground state is non-degenerate, this assumption can be easily relaxed \([10]\).

The generalization of DFT to relativistic systems is straightforward \([13]\). The energy functional \( F_{\text{HK}} \) now becomes a functional of both scalar and vector densities (or more precisely, vector four-currents). Extremization of the functional gives rise to variational equations that determine the ground-state densities.

Significant progress in solving these equations was made by Kohn and Sham \([11]\), who introduced a complete set of single-particle wave functions. In our case, these wave functions allow us to recast the variational equations as Dirac equations for occupied orbitals. The single-particle hamiltonian contains local, density-dependent, scalar and vector potentials, even when the exact energy functional is used. Moreover, one can introduce auxiliary (scalar and vector) fields corresponding to the local potentials, so that the resulting equations resemble those in a relativistic MFT calculation \([4,6]\).

The strength of the approach rests on the following theorem:

The exact ground-state scalar and vector densities, energy, and chemical potential for the fully interacting many-fermion system can be reproduced by a collection of (quasi)fermions moving in appropriately defined, self-consistent, local, classical fields.

The proof is straightforward \([10]\). Start with a collection of noninteracting fermions moving in an externally specified, local, one-body potential. The exact ground state for this system is known: just calculate the lowest-energy orbitals and fill them up. Therefore, if one can find a suitable local, one-body potential based on an exact energy functional, the exact ground state of that system can be determined. But this potential is precisely what one obtains by differentiating the interaction parts of \( F_{\text{HK}} \) with respect to \( n(r) \) \([10]\). The resulting one-body potential will generally be density dependent and thus must be determined self-consistently.

Several points are noteworthy. As noted by Kohn \([10]\), the single-particle basis constructed as described above can be considered “density optimal”, in contrast to the Hartree (or Hartree–Fock) basis, which is “total-energy optimal”. Thus the exact scalar and vector densities are given by sums over the squares of the Dirac wave functions, with unit occupation probability. Moreover, since these densities are guaranteed to make the energy functional stationary [the external \( v(r) = 0 \)], the exact ground-state energy is also obtained.

\[\text{For simplicity, we assume that the least-bound orbital is completely filled, so the ground state is non-degenerate.}\]
The proof that the eigenvalue of the least-bound state is exactly the Fermi energy is given in Ref. [14]. Note, however, that aside from this association, the exact Kohn–Sham wave functions (and remaining eigenvalues) have no known, directly observable meaning.

If one knows the exact functional form of the energy on the density, one can describe the observables noted in the theorem exactly (and easily) in terms of the Kohn–Sham basis. Observables of this type are typically the ones calculated in relativistic MFT. Moreover, it has been known for many years [2] that the mean-field contributions dominate the single-particle potentials at ordinary densities. Thus, by parametrizing the energy functional in a mean-field (or “factorized”) form, and by fitting the parameters to empirical bulk and single-particle nuclear data, one should obtain an excellent approximation to the exact energy functional in the relevant density regime. This is the key to the success of relativistic MFT calculations, as we will verify below, using the effective lagrangian constructed in the next section.

III. EFFECTIVE LAGRANGIAN

We cannot give a detailed derivation and discussion of the effective lagrangian of QHD in this short article, but we can illustrate the basic principles. To exhibit how spontaneously broken chiral symmetry is incorporated quite generally into the hadronic theory, consider the linear $\sigma$-model with an additional linear coupling of an isoscalar $V^\mu$ to the baryon current, the so-called “chiral ($\sigma, \omega$) model” [5]. Define right- and left-handed nucleon fields by $\psi_{R,L} \equiv (1 \pm \gamma_5)\psi/2$ and the $SU(2)$ matrix $U \equiv \exp (i\vec{\tau} \cdot \vec{\pi}/s_0)$, where $\vec{\pi}$ is the isovector pion field. If $M$ is the nucleon mass, determined by the spontaneous breaking of chiral symmetry, and $s_0 = M/g_\pi$, then the lagrangian for the chiral ($\sigma, \omega$) model can be written as the $s_0 \to \infty$ limit of the following generalized lagrangian [4]

$$\mathcal{L} = i \left[ \bar{\psi}_R \gamma_\mu (\partial^\mu + ig_\pi V^\mu) \psi_R + \bar{\psi}_L \gamma_\mu (\partial^\mu + ig_\pi V^\mu) \psi_L \right] \right.
- g_\pi s_0 \left( 1 - \frac{\sigma}{s_0} \right) \left[ \bar{\psi}_R U^\dagger \psi_L + \bar{\psi}_L U \psi_R \right] + \frac{1}{2} (\partial_\mu \sigma \partial^\mu \sigma) + \frac{1}{4} s_0^2 \operatorname{tr} (\partial_\mu U \partial^\mu U^\dagger),$$

For $m_\pi^2 = 0$, this lagrangian is evidently invariant under chiral $SU(2)_L \times SU(2)_R$ transformations of the form ($\sigma$ and $V^\mu$ are unchanged)

$$\psi_L \to L \psi_L, \quad \psi_R \to R \psi_R, \quad U \to LUR^\dagger.$$  \hfill (6)

Here $L$ and $R$ are independent, global $SU(2)$ matrices, and the generalized potential $\mathcal{V}$ is chosen to be invariant, with the limit $\mathcal{V} \to m_\pi^2 \sigma^2/2 + O(1/s_0)$. Conventional notation is recovered with the identification

$$s_0 = M/g_\pi \equiv f_\pi.$$ \hfill (7)

The change of variables $U \equiv \xi \xi$, $N_L \equiv \xi \psi_L$, $N_R \equiv \xi \psi_R$ reduces the fermion terms in the preceding lagrangian to
\[ \mathcal{L}_{\text{fermion}} = \overline{N} \left[ i\gamma^\mu (\partial_\mu + iv_\mu + ig_\nu V_\nu) + \gamma^\mu \gamma_5 g_\mu - M + g_\pi \sigma \right] N , \]

\[ v_\mu \equiv - \frac{i}{2} (\xi^\dagger \partial_\mu \xi + \xi \partial_\mu \xi^\dagger) , \quad a_\mu \equiv - \frac{i}{2} (\xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger) . \] (8)

This lagrangian is invariant under the following nonlinear chiral transformation:

\[ \xi(x) \rightarrow L \xi(x) h^\dagger(x) \equiv h(x) \xi(x) B^\dagger , \quad N(x) \rightarrow h(x) N(x) , \] (9)

where \( h(x) \) is a local \( SU(2) \) matrix. It follows that \( U \) still transforms globally according to Eq. (8). Additional mesons and interactions can now be introduced requiring only invariance under the local isospin transformations of Eq. (9). While illustrated within the framework of a simple model, this nonlinear realization of \( SU(2)_L \times SU(2)_R \) is, in fact, quite general, and can be used as a basis for constructing the most general QHD lagrangian [4].

The effective lagrangian, which reflects the underlying spontaneously broken chiral symmetry of QCD, and from which the energy functional of the previous section is obtained, is constructed from the following series of steps [4,6]:

1. A baryon field and low-mass meson fields that concisely describe the important interaction channels, namely, \( \pi(0^-, 1) \), \( \phi(0^+, 0) \), \( V_\mu(1^-, 0) \), and \( \rho_\mu(1^-, 1) \), are the generalized coordinates of choice. The pion, a Goldstone boson, is treated as in the example above. Higher mass meson fields are assumed to be “integrated out” and their contributions contained in the effective coupling constants.

2. Dimensional analysis is first used to characterize the various terms in the effective lagrangian. Briefly, this is done as follows. The initial couplings of the meson fields to the baryon fields are linear, with a strong coupling constant \( g \). The dimensionless form of this combination is \( g \phi/M = \phi/f_\pi \) [see Eq. (7)]; non-Goldstone boson fields are assumed to enter in this dimensionless form. From the mass term of the meson fields \( \propto m^2 \phi, \) with \( m^2 \approx M^2 \), one then deduces an overall scale factor in the lagrangian density of \( f_\pi^2 M^2 \). From the baryon mass term \( M\overline{\psi}\psi \), one concludes that the appropriate dimensionless form of the baryon densities is \( \overline{\psi}\psi/Mf_\pi^2 \). This “naive” dimensionless analysis (NDA) then implies that, after appropriate combinatorial factors are included, the various terms in the effective lagrangian enter with dimensionless coefficients of order unity.

3. The various interaction terms allowed by the \( SU(2)_L \times SU(2)_R \) symmetry of QCD are then constructed using the nonlinear realization of chiral symmetry illustrated above. Simply writing down all possible terms does not get one very far unless there is an organizational principle, and the following provides the crucial insight:

4. Although the mean scalar and vector field energies are large compared to the nuclear binding energy, the dimensionless combinations \( g_\phi \phi_0/M \approx \phi_0/f_\pi \) and \( g_\nu V_0/M \approx V_0/f_\pi \) are roughly 1/3 and thus provide convenient expansion parameters. Furthermore, spatial variations of the meson fields and of the baryon densities in the nucleus are observed to occur over the scale of the nuclear surface region, and hence the dimensionless ratio \( \nabla/M \) also provides a useful expansion quantity (as does the characterization of chiral symmetry violation at the lagrangian level, \( m_\pi/M \)).
5. A combination of these observations allows one to construct a hierarchy of decreasing contributions to the effective lagrangian for the nuclear many-body system characterized by an integer $\nu$ defined by

$$\nu = d + \frac{n}{2} + b,$$  
(10)

where $d$ is the number of derivatives, $n$ is the number of nucleon fields, and $b$ is the number of non-Goldstone boson fields present in the interaction term. The effective lagrangian at various levels of $\nu$ is given in Refs. [4,6].

The effective lagrangian with mean meson fields then determines the energy functional of the previous section, and a representation in terms of Dirac–Hartree orbitals leads to local, nonlinear Hartree equations, which can be solved numerically. The extent to which nuclei exhibit this hierarchy of interactions, and to which this effective lagrangian indeed describes the nucleus, is discussed in the next section.

IV. RESULTS AND SUMMARY

The QHD mean-field energy functional discussed above is given in several different forms in Refs. [4,6,8,15], as are the field equations that result from extremization. The equations are solved self-consistently for the closed-shell nuclei $^{16}$O, $^{40}$Ca, $^{48}$Ca, $^{88}$Sr, and $^{208}$Pb, and also in the nuclear matter limit. The parameters are then best-fit to empirical properties of the charge densities, the binding energies, and various splittings between energy levels near the Fermi surface using a figure of merit ($\chi^2$) defined by a weighted, squared deviation between the 29 calculated and empirical values. When working at the highest order of truncation (essentially $\nu = 4$), the calculated results are very accurate, as we illustrate shortly, but they are too numerous to reproduce here [6,15,8].

The critical question is whether the hierarchal organization of interaction terms is actually observed. This is illustrated in Fig. 1, where the nuclear matter energy/particle is shown as a function of the power of the mean fields, which is called $b$ in Eq. (10). (There are no gradient contributions in nuclear matter and $\langle \hat{\pi} \rangle = 0$.) The crosses and error bars are estimates based on NDA and naturalness, that is, overall coefficients are of order unity. It is clear that each successive term in the hierarchy is reduced by a roughly factor of five, and thus for any reasonable desired accuracy, the lagrangian can be truncated at a low value of $\nu$. Derivative terms and other coupling terms are analyzed in Ref. [8], with similar conclusions.

The quality of the fits to finite nuclei and the appropriate level of truncation is illustrated in Fig. 2, where the figure of merit is plotted as a function of truncation order and of various combinations of terms retained in $\mathcal{L}$. The full calculations (●) retain all allowed terms at a given level of $\nu$, while the other two choices keep only the indicated subset. There is clearly a great improvement in the fit (more than a factor of 35) in going from $\nu = 2$ to $\nu = 4$.

5The extension of the effective lagrangian to include electromagnetic interactions as an expansion in powers of derivatives is also discussed in these references.
FIG. 1. Nuclear matter energy/particle for two QHD parameter sets, one on the left and one on the right of the error bars. The power of fields is $b \equiv j + \ell$ for a term of the form $(g_s \phi_0)^j (g_v V_0)^\ell$ ($\ell$ is even). The arrow indicates the total binding energy, $\epsilon_0 = 16.1\text{MeV}$. Absolute values are shown.

FIG. 2. $\chi^2$ values for QHD parameter sets, as a function of the level of truncation.
but there is no further improvement in going to $\nu = 5$. Speaking chronologically, the $\nu = 2$ results show the level of accuracy obtained almost 20 years ago, while the $\nu = 4$ results were obtained four years ago [6]. Moreover, the $\phi^n$ only results at $\nu = 4$ show the state of the situation in the late 1980s, as discussed in Ref. [3]. Recent work [8] shows that the full complement of parameters at order $\nu = 4$ is underdetermined, and that only six or seven are determined by this data set, which explains the great success of these earlier models with a restricted set of parameters.

In summary, the hadronic theory of QHD is truly a manifestation of low-energy, strong-coupling QCD. The modern viewpoint of QHD based on effective field theory and density functional theory explains the accurate description of bulk and single-particle nuclear properties. Corrections to the mean-field parametrization of the energy functional can be calculated systematically using the effective hadronic lagrangian [16]. An important goal for the future is finding an efficient, tractable, nonperturbative way to match the low-energy, strong-coupling, effective field theory of QHD to the underlying QCD lagrangian.

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