Theories of the Fractional Quantum Hall Effect

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Abstract. This is an introduction to the microscopic theories of the FQHE. After a brief description of experiments, trial wavefunctions and the physics they contain are discussed. This is followed by a description of the hamiltonian approach, wherein one goes from the electrons to the composite fermions by a series of transformations. The theory is then compared to other theoretical approaches and to experiment.

1 Introduction

In these lectures I will provide an introduction to some of the theories of the Fractional Quantum Hall Effect (FQHE), for a non-FQHE person who is familiar with graduate quantum mechanics and a little bit of second quantization. I will discuss only the simplest phenomena to illustrate the basic ideas and also focus on the parts of the subject I am most familiar with, since many excellent reviews exist [1], [2], [3]. A rule I will bear in mind is that things should be made as simple as possible but no simpler.

2 The problem

In the Hall effect, one takes a system of electrons in the $x-y$ plane, subject to a magnetic field in the $z$-direction. A current density $j_x$ is driven in the $x$-direction and the voltage $V_y$ is measured in the $y$-directions. At equilibrium, an electric field $E_y$ is set up in the $y$-direction which balances the Lorentz force $v_x B$, and the Hall conductance is

$$\sigma_{xy} = \frac{j_x}{E_y} = \frac{n ev_x}{v_x B} = \frac{n e}{B}$$

(1)

where $n$ is the number density. Thus we expect that $\sigma_{xy}$ will be linear in $ne/B$. Instead we find steps at some special values as in Figure 1.

$$\sigma_{xy} = \frac{e^2}{2\pi \hbar} \frac{p}{2ps + 1} \equiv \frac{e^2}{2\pi \hbar} \nu$$

(2)

$$p = 1, 2 \ldots \ s = 0, 1, 2 \ldots$$

(3)

The case $s = 0, \nu = p$ corresponds to the Integer Quantum Hall Effect (IQHE) discovered by von Klitzing et al [4]. The case $s > 0$ corresponding to
the FQHE was discovered by Tsui et al [5]. The case $p = 1$, $\nu = 1/(2s + 1)$ was explained by Laughlin [6] with his famous trial state. The more general case $s \geq 0$, $p \geq 1$, $\nu = \frac{p}{2ps+1}$ was explained by Jain using the idea of composite fermions (CF) [7]. (The case $\nu = \frac{p}{2ps-1}$ is a trivial extension.)

2.1 What is special about these values of $\sigma_{xy}$?

Note that where the straight line meets the plateaus,

$$\sigma_{xy} = \frac{e^2}{2\pi \hbar} \frac{p}{2ps+1} = \frac{ne}{B} \quad (4)$$

$$\frac{eBL^2}{2\pi \hbar nL^2} = 2s + \frac{1}{p} \quad (5)$$

$$\frac{[BL^2/(2\pi \hbar)]}{nL^2} = \frac{\Phi/\Phi_0}{N} = 2s + \frac{1}{p} \quad (6)$$

where $\Phi_0 = \frac{2\pi \hbar}{e}$ is a quantum of flux. At these points, there are $\nu^{-1} = 2s + \frac{1}{p}$ flux quanta per particle. Thus IQHE has $\frac{1}{p}$ flux quanta per electron. We will see that this makes life easy, in terms of coming up with a good wavefunction for the ground state. The FQHE has $2s + \frac{1}{p} > 1$ flux quanta per electron. This will be seen to frustrate the search for a ground state.

2.2 What makes FQHE so hard?

The usual approach of considering interactions perturbatively fails when the number of flux quanta per particle $> 1$. Consider one free electron.

$$H_0 = \frac{(p + eA)^2}{2m} \quad m \text{ is the bare mass} \quad (7)$$
Theories of the Fractional Quantum Hall Effect

\[ eA = eB \left( y, -x \right) = -\frac{1}{2l^2} \hat{z} \times \mathbf{r} \]  
\[ l^2 = \frac{\hbar}{eB} \]  
\[ \nabla \times eA = -eB \]

Note \( B \) points down the \( z \)-axis. If you do not like this complain to the person who defined the electron charge to be \(-e\) or \( z = x + iy \). Setting \( \hbar = c = 1 \),

\[ H_0 = \frac{\eta^2}{2ml^4} \]
\[ \eta = \frac{1}{2} \mathbf{r} + l^2 \hat{z} \times \mathbf{p} \]
\[ [\eta_x, \eta_y] = il^2 \]
\[ E = (n + \frac{1}{2}) \frac{eB}{m} \]

One calls \( \eta \) the cyclotron coordinate, even though its two components form a conjugate pair. This is why the Hamiltonian describes a harmonic oscillator whose energy levels are called Landau Levels (LL). Of special interest is the Lowest Landau Level (LLL) which has \( n = 0 \). In this state \( \langle \eta^2 \rangle_{n=0} = l^2 \).

Now a one-dimensional oscillator spectrum for a two dimensional problem implies the LL’s must be degenerate. The degeneracy is due to another canonical pair that does not enter the Hamiltonian, called the guiding center coordinates \( \mathbf{R} \) defined by

\[ \mathbf{R} = \frac{1}{2} \mathbf{r} - l^2 \hat{z} \times \mathbf{p} \]
\[ [R_x, R_y] = -il^2 \]

Note that the role of \( \hbar \) is played by \( l^2 \) and that

\[ \mathbf{r} = \mathbf{R} + \eta. \]

which plays the role of phase space and the degeneracy of the LLL is

\[ D = \frac{L^2}{\pi \hbar^2} = \frac{L^2}{2\pi l^2} = \frac{eBL^2}{2\pi \hbar} = \frac{\Phi}{\Phi_0} \]

where \( \Phi_0 \) is the flux quantum. The inverse filling factor is

\[ \nu^{-1} = \text{flux quanta or LLL states per electron} = \frac{eB}{2\pi \hbar} \]

When \( \nu^{-1} > 1 \) the noninteracting problem is macroscopically degenerate since there are more LLL states than electrons. However when \( \nu^{-1} > 1 \) has the special values where the steps cross the straight line, we know excellent approximations to ground states and low lying excitations. At these points

\[ \sigma_{xy} = \frac{ne}{B} = \frac{e^2}{2\pi \hbar} \frac{p}{2ps + 1} \]
which means that at these special points

$$\frac{eB}{2\pi \hbar} = \frac{\text{flux quanta density}}{\text{particle density}} = \frac{\text{LLL states}}{\text{particles}} = \frac{2ps + 1}{p} \quad (21)$$

**flux quanta per particle = states per particle = 2s + \frac{1}{p} \quad (22)**

Thus for example at $\nu = 2/5$ there are $5/2$ states per particle in the LLL and
the $V = 0$ problem is macroscopically degenerate. While this is also true when
there are 2.500001 states per electron, we will see that at these special values, a
single ground state built out of LLL states will emerge in accordance with the
expectation that as $m \to 0$ and $eB/m \to \infty$, a sensible low energy theory built
out of the LLL states must exist! Let us see how this happens and what happens
as we move along the plateaus.

**Getting to know the LLL** The LLL condition $a_\eta |LLL\rangle = 0$ gives

$$\psi = e^{-|z|^2/4l^2} f(z) \quad (23)$$

where $z = x + iy$. A basis for $\psi$ is

$$\psi_m(z) = z^m e^{-|z|^2/4l^2} \quad m = 0, 1, \ldots \quad (24)$$

The Gaussian is usually suppressed. The state has $L_z = m$.

If $\nu = 1$, (one electron per LLL state) there is a unique noninteracting
ground state (which may then be perturbed by standard means)

$$\chi_1 = \prod_{i<j} (z_i - z_j) \cdot \text{Gaussian} = \text{Det} \begin{vmatrix} z_0^0 & z_1^1 & z_2^2 & \cdots \\ z_0^0 & z_1^1 & z_2^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} \cdot \text{Gaussian} \quad (25)$$

For $\nu < 1$, we want to work entirely within the LLL. If in $H = T + V$ we set
$T$ equal to a constant ($eB/2m$ per particle), all the action is in $V$. Why is this
a problem if $V$ is a function of just coordinates? In terms of the density

$$\rho(r) = \sum_j \delta(r - r_j) \quad (26)$$

$$V = \frac{1}{2} \int d^2 r \int d^2 r' \rho(r) v(r - r') \rho(r') \quad (27)$$

$$= \frac{1}{2} \sum_q \rho(q) v(q) \rho(-q) \quad (28)$$

$$\rho(q) = \sum_j e^{-iqr_j} \quad (29)$$

The point is that if one sets $T = eB/2m$, the LLL value, one must project
the operator $r$ to the LLL. The coordinates $x$ and $y$ which commute in the full
Hilbert space, no longer commute in the LLL.
**Projection to the LLL** Let $\mathcal{P}$ denote projection to the LLL. Then

$$\mathcal{P} : \mathbf{r} = \mathbf{R} + \eta \Rightarrow \mathbf{R}. \quad (30)$$

The projected components do not commute:

$$[R_x, R_y] = -il^2 \quad \text{or} \quad [z, \bar{z}] = -2l^2 \quad \text{in the LLL} \quad (31)$$

As for the densities

$$\mathcal{P} : \langle e^{-i\mathbf{q} \cdot \mathbf{r}} \rangle_{LLL} \Rightarrow \langle e^{-i\mathbf{q} \cdot \eta} \rangle_{LLL} \quad (32)$$

Thus the projected problem is defined by

$$\bar{\mathcal{H}} = \frac{1}{2} \sum_{\mathbf{q}} e^{-q^2 l^2 / 2} \bar{\rho}(\mathbf{q}) v(\mathbf{q}) \bar{\rho}(-\mathbf{q}) \quad (33)$$

$$\bar{\rho}(\mathbf{q}) = \sum_j e^{-i\mathbf{q} \cdot \mathbf{R}_j} \quad (34)$$

I shall refer to $\bar{\rho}$ as the projected density, although it is actually the Magnetic Translation Operator and differs from the density by a factor $e^{-q^2 l^2 / 4}$. (This explains the gaussian in Eqn. (33)). The $\bar{\rho}$ obey the Girvin-MacDonald-Platzman (GMP) algebra:

$$[\bar{\rho}(\mathbf{q}), \bar{\rho}(\mathbf{q}')] = 2i \sin \left( \frac{(\mathbf{q} \times \mathbf{q}') l^2}{2} \right) \bar{\rho}(\mathbf{q} + \mathbf{q}') \quad (35)$$

There is no small parameter in $\bar{\mathcal{H}}$ and the overall energy scale is set by $v(q)$. How is one to find the ground state?

### 2.3 Laughlin’s answer

For $\nu = 1/(2s + 1)$ Laughlin proposed a trial state:

$$\Psi_{\{z_i\}} = \prod_{i=2}^{N} \prod_{j<i} (z_i - z_j)^{2s+1} \exp(-\sum_i |z_i|^2 / 4l^2) \quad (36)$$

Let us now contemplate its many virtues.

- It lies in the LLL. (Analytic function times gaussian)
- It has definite angular momentum (homogeneous in $z$’s)
- It obeys the Pauli principle ($2s + 1$ is odd, spin is assumed fully polarized)
- It has the right filling.

Proof 1: Let us refer to the wavefunction as a function of $z_N$ with all the others fixed as $\Psi(z_N)$. Consider a finite sample and drag $z_N$ around it.

$$\text{# of zeros of } \Psi(z_N) = \frac{\text{Aharonov Bohm phase change}}{2\pi} = \frac{\Phi}{\Phi_0} \quad (37)$$
But we know
\[ \frac{\Phi}{\Phi_0} = N\nu^{-1} \]  
(38)

Thus \( \Psi_{1/2s+1}(z_N) \) must be a polynomial of degree \( N(2s+1) \) in \( z_N \) and it is.

Proof 2: Again we consider a finite droplet. Consider any single particle wavefunction in the LLL with angular momentum \( m \).

\[ |\psi_m|^2 = |z|^{2m}e^{-|z|^2/2l^2} \]
(39)

\[ \frac{\Phi}{\Phi_0} = 2\pi ml^2 \]
(40)

Thus the single particle state of \( L_z = m \) has a size (which is sharply defined at large \( m \)) such that it encloses \( m \) flux quanta. Now the largest value of \( m \) for any coordinate in

\[ \Psi_{\pi z_{\text{max}}} = \prod_{i=2}^{N}(z_i - z_j)^{2s+1} \exp(-\sum_{i} |z_i|^2/4l^2) \]

is \( N(2s+1) \). Thus the droplet encloses \( 2s+1 \) flux quanta per particle.

- Has great correlations and no wasted zeros. Halperin observed \[9\] that the Laughlin function has no wasted zeros in the following sense. Given that \( \psi(z_N) \) has to have \( N(2s+1) \) zeros per particle (given analyticity and the filling fraction) only \( N \) of these had to lie on other electrons by the Pauli principle. But they all lie on other electrons, thereby keeping them away from each other very effectively, producing a low potential energy.

- It is not enough to know what \( \Psi \) is, we need to know what kind of state it represents. Laughlin gave the following answer: it is an incompressible fluid, which means a fluid that abhors density changes. Whereas a Fermi gas would increase (decrease) its density globally when compressed (decompressed), an incompressible fluid is wedded to a certain density and would first show no response and then suddenly nucleate a localized region of different density. (Just the way a Type II superconductor, in which magnetic field is not welcome, will allow it to enter in quantized units in a region that turns normal.) Laughlin deduced this result based on the following plasma analogy.

Note that all averages in the ground state are found using the measure

\[ \langle \Omega \rangle = \int \prod_{i} d^2 z_i \Psi^\ast \Omega \Psi / \int \prod_{i} d^2 z_i \Psi^\ast \Psi \]  
(42)

\[ \Psi^\ast \Psi = e^{-E/kT^m} \]
(43)

\[ \frac{1}{kT^m} = 2(2s+1) \]
(44)

\[ ^*E^m = \sum_{i<j} -\ln|z_i - z_j| + \sum_{i} \frac{\pi n_i}{2} |z_i|^2 \]  
(45)
where I have used $1/l^2 = eB = 2\pi n/\nu$. This describes a plasma with $1/kT = 2(2s + 1)$, particles of unit charge interacting with a log potential and in a neutralizing background given by second term. The plasma is known to hate density changes away from that of the background because of the log potential.

Laughlin also provided the wavefunction for a state with a quasihole, a state with a charge deficit. If one imagines inserting a tiny flux tube at a point $z_0$ and slowly increasing the flux to one quantum, one must, by gauge invariance, return to an eigenstate of $H$, and each particle must undergo a $2\pi$ phase shift as it goes around $z_0$. This and analyticity imply the ansatz

$$\Psi_{qh} = \prod_i (z_i - z_0)\Psi_{2s+1} \tag{46}$$

This is a quasihole. (There is a more complicated state with a quasiparticle.) The prefactor is a vortex at $z_0$, which is an analytic zero at $z_0$ for every coordinate. It denotes a hole near $z_0$ with charge deficit $1/(2s + 1)$ (in electronic units) as can be seen using Laughlin’s wavefunction and the plasma analogy in which there is now an extra term in energy $\frac{1}{2s+1} \sum_i \ln |z_i - z_0|$. This positive charge embedded in the plasma is immediately screened by an equal negative quasihole charge.

A second proof of quasihole charge depends only on the state being gapped and incompressible. As the flux quantum $\Phi_0$, is inserted, the charge driven out to infinity is given by integrating the radial current density $j$ produced by the Hall response to the induced azimuthal $E$ field:

$$Q = -\int j(r,t)2\pi r dt = -\sigma_{xy}\int E\,2\pi r \,dt$$

$$= -\sigma_{xy}\int \frac{d\Phi}{dt} dt = -\Phi_0 \sigma_{xy} \tag{47}$$

$$= -\frac{2\pi \hbar}{e} \frac{e^2}{2\pi \hbar} \nu = -e\nu \tag{48}$$

Thus fractional charge is due to fractional $\sigma_{xy}$ and not vice versa.

Finally the plateaux are produced when you go off the magic fractions because the change in density appears in the form of quasiparticles (holes) which get localized, as per standard localization mythology in $d = 2$.

While a fairly comprehensive understanding of the fundamentals FQHE was provided by Laughlin, the subsequent years brought many phenomena that needed explanation. For example Girvin, MacDonald and Platzman [8] worked out the dispersion relation for a magnetoexciton in which has the quantum numbers of a quasiparticle and hole which are however not infinitely separated. Their work showed a roton minimum just as in superfluids.

We can ask how there is a Hall current at $\nu = 1/2$ if $e^* = 0$, i.e., the quasiparticles are neutral.

But our primary question next is this: what are the wavefunctions for fractions of the form $\frac{p}{2s+1}$? The corresponding Laughlin wavefunction with $2s+1 \rightarrow$
$2s + 1/p$ does not obey the Pauli principle. Jain gave us trial states for these fractions and also explained why they are natural, in terms of objects called composite fermions (CF).

3 Composite Fermions

It was pointed out by Leinaas and Myrheim that in $d = 2$ one could have particles (dubbed anyons by Wilczek) that suffered a phase change $e^{i\theta}$ upon exchange, with $\theta = 0, \pi$ corresponding to bosons and fermions respectively. To do this one takes a fermion and drives through its center a point flux tube. If this contains an even/odd number of flux quanta, the composite particle one gets is a fermion/boson. (One may ask how adding a flux quantum is going to make any difference since it will not show up as one goes fully around it. Consider two particles, $A$ at the origin and $B$ at $x = R$. To exchange them we rotate $B$ by $\pi$ around $A$ and shift both to the right by $R$.)

Jain exploited this idea as follows. Consider $\nu^{-1} = 2s + 1/p$, where each particle sees $2s + 1/p$ flux quanta of external $B$ per particle. Let us now convert our electrons into CF by attaching $2s$ point flux quanta pointing opposite to $B$.

Now we invoke the idea used by Laughlin et al and argue that on average the CF’s see $1/p$ flux quanta per particle and fill up exactly $p$ LL’s. This gives the following trail state at mean field level:

$$\Psi_{p/2s+1} = \prod_{i<j} \left\{ \frac{(z_i - z_j)^{2s}}{|z_i - z_j|} \right\} \cdot \chi_p(z, \bar{z})$$

(49)

where $\chi_p$ is the CF wavefunction with $p$-filled LL’s and the prefactor takes you back to electrons. Jain however improved this ansatz in two ways and proposed:

$$\psi_{p/2s+1} = \mathcal{P} \prod_{i<j} (z_i - z_j)^{2s} \cdot \chi_p(z, \bar{z}))$$

(50)

• He replaces flux tubes by vortex attachment: $\prod_{i<j} (z_i - z_j)^{2s}$.
• He does a projection to LLL using $\mathcal{P}$:

$$\mathcal{P} : \bar{z} \rightarrow 2l^2 \frac{\partial}{\partial z}$$

(Recall $[z, \bar{z}] = -2l^2$.)

(51)

• At $p = 1$, $\chi(z, \bar{z}) = \prod (z_i - z_j)$, and we do not need $\mathcal{P}$ to get back Laughlin’s answer. At $p > 1$ we have concrete expression for $\Psi$ in terms of electron coordinates.

The Jastrow factor

$$J(2s) = \prod_{i<j} (z_i - z_j)^{2s}$$

(52)

describes $2s$-fold vortices on particles. This has two effects.

First, when one CF goes around a loop, it effectively sees $\nu^{-1} = 2s + 1/p - 2s = 1/p$ flux quanta per enclosed particle since the phase change due to encircling
vortices attached to CF’s neutralizes $2s$ of the flux quanta per particle of the Aharanov - Bohm phase of the external field.

Thus while degeneracy of the noninteracting problem is present for any $\nu < 1$, at the Jain fractions one can beat it by thinking in terms of composite fermions. As we move off the Jain fractions, the incremental CF (particles or holes) get localized, giving the plateaus.

Next, the vortices reduce $e$ down to

$$e^* = 1 - \frac{2ps}{2ps + 1} = \frac{1}{2ps + 1} \tag{53}$$

The idea that electrons bind to vortices, first pointed by Halperin$^9$, has been greatly emphasized by Read$^{12}$ who also give a physical picture to explain this binding: unlike the flux tube, the vortex is a physical LLL excitation, one which is charged and hence attracted to the electron.

It is however important to note that $2s$-fold vortices are associated with electrons only in the Laughlin states. In the Jain states this is only before projection by $\mathcal{P}$. The reason is that upon projection, zeros can move off the electrons (as will be explained shortly) and also get destroyed: for example at $\nu = 2/5$, after projection to the LLL (i.e., to analytic functions) there can only be $5/2$ zeros per electron; one must lie on other electron as per the Pauli’s principle and only $3/2$ are left to form vortices. Clearly they cannot all be on electrons or be associated with them uniquely.

This brings us to the following question. The quasiparticle charge, $e^* = 1 - \frac{2ps}{2ps + 1}$ is robust under $\mathcal{P}$ since $e^*$ is tied to $\sigma_{xy}$ which is presumed robust under projection. However the CF can no longer be viewed as an electron-vortex complex. What, if any, is the particle that binds to the electron to bring $e$ down to $e^*$? The hamiltonian theory will provide an answer.

$\nu = 1/2$ The wavefunction, also called the Rezayi-Read wavefunction$^{13}$ is

$$\Psi = \mathcal{P} \prod_{i<j} (z_i - z_j)^2 \text{Det} |e^{ik \cdot r}|. \tag{54}$$

At $\nu = 1/2$, $e^* = 0$ since the double vortex has charge = $-1$. Read$^{12}$ gave the following argument for the dipole moment of this neutral CF. Consider

$$e^{ik \cdot r} = \exp \frac{i}{2}(k\bar{z} + \bar{k}z) \quad k = k_x + ik_y \tag{55}$$

Since $e^{ik \cdot r}$ causes $z \rightarrow z + ikl^2$ in the Jastrow factor producing the dipole moment $d^* = kl^2$. The energy to separate the vortex from electron (coulomb attraction) becomes kinetic energy of the CF. The leading term is quadratic in separation i.e., $k$, and defines an $m^*$.

While this picture has many merits and will be regained in an operator version later on, there are some subtleties often neglected.

- Every $z_i$ gets translated so that $(z_i - z_j) \rightarrow (z_i - z_j + ik_i - ik_j)$.
• All this is for one assignment of $k$’s. Upon antisymmetrization, we can’t say where the zeros will be and there need not be a simple relation between the location of the electrons and the zeros in $\Psi_{LLL}$.

• Upon projection, vortices not only get moved around, they are reduced in number: at $\nu = 1/2$ there are 2 vortices per electron (before projection there were three zeros per particle: two in the Jastrow factor and one in the determinant). One is on electrons due to the Pauli principle and the other is a single vortex.

• The value of dipole moment $d^*$ is sensitive to wavefunction unlike $\sigma_{xy}$: it is zero before projection (since vortices sit on the electrons) and reaches some value $d^*$ in the LLL, presumably $d^* = kl^2$. Where is one to look for this moment and what does it really mean? The hamiltonian approach will show how the dipole is described in an operator treatment.

The enormous success of the trial wavefunctions notwithstanding, the following challenges exist and are addressed by the hamiltonian approach.

• Separate high and low energies at the cyclotron scale and LLL.

• For the latter, obtain a limit independent of $m$, and an energy scale set by $v(q)$, especially for $1/m^*$.

• Find the partner to the electron that turns it into a CF.

• Obtain the right $e^* = 1/(2ps + 1)$, $d^*$, and $\mu^* = e/2m^*$, the magnetic moment predicted by Simon, Stern and Halperin [14].

• Explain who carries Hall current if CF don’t. (At $\nu = 1/2$, $e^* = 0$.)

• Analyze unequal time correlations, $T > 0$ correlations and disorder.

4 Hamiltonian Theory I- Chern-Simons Approach

The aim here is to start with the electronic hamiltonian and try to reach, by a sequence of transformations, exact or approximate, fair or foul, a description of the final quasiparticles, the CF’s. So we begin with

$$H = \sum_i \frac{(p_i + eA(r_i))^2}{2m} + V$$

(56)

The next step is to attach flux tubes to electrons by the following Chern-Simons (CS) gauge transformation [15], introduced into the FQHE work by Lopez and Fradkin [16] in the functional approach. In the hamiltonian version one trades the electronic wavefunction $\Psi_e$ for $\Psi_{CS}$ defined as follows:

$$\Psi_e = \prod_{i<j} \frac{(z_i - z_j)^{2s}}{|z_i - z_j|^{2s}} \Psi_{CS} = \exp(2is \sum_{i<j} \phi_{ij}) \Psi_{CS}.$$  \hspace{1cm} (57)

$$H_{CS} = \sum_i \frac{(p_i + eA(r_i) + a_{cs}(r_i))^2}{2m} + V$$

(58)
In $H_{CS}$ there appears a CS gauge field, $a_{cs}$, that comes from the action of $p$ on the prefactor, which is the phase of the Jastrow factor:

$$a_{cs}(r_i) = 2s \nabla \sum_{j \neq i} \phi_{ij}$$

(59)

$$\oint a_{cs}(r_i) \cdot dr_i = 2s \oint \sum_{j \neq i} \nabla \phi_{ij} \cdot dr_i = 4\pi s \# \text{ enclosed}$$

(60)

$$\nabla \times a_{cs} = 4\pi s \rho$$

(61)

The above equations show that even though $a_{cs}$ is a gradient, it has a curl since $\phi_{ij}$ is multivalued. The curl is readily calculated by the use of Stokes' theorem. The idea of flux attachment was to cancel part of the applied field on average. To this end we separate $a_{cs}$ and $\rho$ into average and fluctuating parts:

$$\nabla \times \langle a_{cs} \rangle + \nabla \times : a_{cs} : = 4\pi sn + 4\pi s : \rho :$$

(63)

This gives

$$H_{CS} = \sum_i \left( \frac{p + eA + (a_{cs}) + : a_{cs} :}{2m} \right)^2 + V$$

$$= \sum_i \left( \frac{\Pi + : a_{cs} :}{2m} \right)^2 + V$$

(64)

$$\Pi = p + eA + (a_{cs})$$

(65)

$$\nabla \times (eA + (a_{cs})) = -eB + 4\pi ns$$

$$= -\frac{eB}{2ps + 1} = -eB^* \quad (A^* = \frac{A}{2ps + 1})$$

(66)

(67)

Here is the good news from above the picture derived by Fradkin and Lopez.

- If we ignore $: a_{cs} :$ and $V$, the composite fermions see $1/p$ flux quanta each (since $2s + 1/p \rightarrow 1/p$) and have a unique ground state $\chi_p$ of $p$ filled LL's. One can go on to include the neglected terms perturbatively. Excitations are given by pushing fermions into higher CF LL's.
- At mean-field, the CF wavefunction $\chi_p$, transformed back to electrons is

$$\Psi_e = \prod_{i<j} \left( \frac{z_i - z_j}{|z_i - z_j|} \right)^{2s} \chi_p(z, \bar{z})$$

(68)

(See Rajaraman and Sondhi [17] for a way to get the entire Jastrow factor at the mean-field level, by introducing a complex gauge field.)

- Fluctuations at one loop give the square of the wavefunction and get rid of the $|z_i - z_j|$ but only for Laughlin like states.
The cyclotron mode appears with the right residue.

But there is also some bad news.

If we excite a fermion from level $p$ to $p + 1$, the energy cost (activation gap) of the particle-hole pair is $\Delta = eB^*/m$ plus corrections due to neglected terms. The dependence on $m$ is not good. We want $\Delta \approx e^2/\varepsilon_l$.

Jain does not have this problem: he does not use $H_{CS}$ or $\chi_p$ or its excitations directly. For him the CS picture is a step towards getting electronic wavefunctions for the ground and excited states by attaching the Jastrow factor and projecting. The energy gap is computed as the difference in $\langle \bar{V} \rangle$ between the ground and excited electronic wavefunctions.

Between cyclotron mode ($eB/m$) and the LLL excitations ($O(e^2/\varepsilon_l)$) there are many spurious modes once again due to the presence of $m$.

So far we have attached flux tubes and not vortices. Consequently the CF have electric charge unity and not $1/(2ps + 1)$.

The case of $\nu = 1/2$ Some remarkable predictions were made and verified at $\nu = 1/2$. Kalmeyer and Zhang \[18\] discussed it briefly and a very exhaustive study was made by Halperin, Lee and Read (HLR) \[19\].

At $\nu = 1/2$, $p = \infty$, $e^* = 0$ and $\Pi \rightarrow p$ and

$$H_{CS} = \sum_i \frac{(\mathbf{p} + \mathbf{a}_{cs})^2}{2m} + V$$  \hspace{1cm} (69)

HLR used the Random Phase Approximation (RPA) and made the following predictions that did not depend on $m$.

- A fermi surface should exist with $k_F$ determined by $n$.
- At $B^* = B - B_{1/2}$ CF should exhibit a bending with radius $R = \frac{k_F}{\kappa}$. This was verified by Kang et al [20], Goldman et al [21] and Smet et al [22].
- When an acoustic wave is coupled the electronic system, it must undergo a velocity shift and an attenuation described by

$$\frac{\delta v_s}{v_s} - \frac{i\kappa}{q} = \frac{\alpha^2/2}{1 + i\sigma_{xx}(q)/\sigma_m}$$  \hspace{1cm} (70)

where $\alpha$ is a piezoelectric constant, $v_s$ is the sound velocity, $\kappa$ describes the attenuation, $\sigma_{xx} = \frac{e^2}{\pi m} \frac{q}{v_F}$ and $\sigma_m = \frac{\omega_F}{v_s}$. Theory fits the experiments of Willett at al [23] with a $\sigma_m$ that is about five times larger than expected.
- They predicted divergences in $m^*$. These do not affect bosonic correlations (e.g., density-density) as shown by Kim et al [24].

Bosonic CS Theory There is an appealing CS theory of Zhang, Hansson and Kivelson [25] for Laughlin fractions, obtained by converting electrons to composite bosons in zero (mean) field upon attaching $2s + 1$ flux tubes. This has many
attractions: e.g., a Landau Ginsburg theory (see also Girvin, MacDonald [21], Read [27]) with FQHE ↔ superfluidity. However it too has problems due to the unwanted dependence on \( m \). Finally, unlike CF, which could be free in a first approximation, composite bosons had to be interacting for a stable ground state. Lack of time and space prevent further discussion of this alluring alternative.

5 Hamiltonian theory-II

I now turn to the extension of the CS theory that Murthy and I made in a series of papers [28], in order to cure it of some of its problems. We began by modifying the CS hamiltonian as follows.

\[
H = \sum_i \left( \frac{\Pi_i + :a_{cs} + :a}{2m} \right)^2 + V
\]

\[
0 = a|\text{physical}\rangle
\]

where \( a \) is a transverse vector field. Since \( d = 2 \), it has only one component at each \( q \) which we denote by \( a(q) \). We introduce a conjugate variable \( P(q) \):

\[
[a(q), P(-q')] = i\delta_{qq'}
\]

and define a longitudinal vector field \( P(q) \) in terms of it. What is going on? First note that we are now operating in a bigger Hilbert space. On vectors that are annihilated by \( a \), \( H \) is the same as \( H_{CS} \). Since \( a \) commutes with \( H \), such eigenstates of \( H \) and \( a \) will exist and we focus on only those with zero eigenvalue for \( a \). This is a trick adapted from the work of Bohm and Pines [29].

Why do we do this? We do this to fight fire with fire: \( a_{cs} \) is a transverse gauge field, but it is a complicated function of the particle coordinates. On the other hand \( a \) is an independent transverse field. Let us shift \( a \) by \(-a_{cs}\) so as to eliminate \( a_{cs} \) from \( H \). To shift \( a \) we use the exponential of \( P \):

\[
U = \exp \left\{ \sum_q P(q) \frac{4\pi is}{q} \rho(-q) \right\}. 
\]

The resulting hamiltonian and constraint on physical states are

\[
H = \sum_i \left( \frac{\Pi_i + a(r_i) + 4\pi s P(r_i)}{2m} \right)^2 + V
\]

\[
0 = (q a(q) - 4\pi s p(q))|\text{physical}\rangle
\]

The presence of \( P \) in \( H \) is due to the fact that \( U \) affects the particle sector as well. The nice thing is that \( H \) is written in terms of independent Fermi and Bose fields. Let us therefore focus on the quadratic part of \( H \):
\[ H = \sum_i \frac{\Pi_i^2}{2m} + \sum_q \left( a^2(q) + (4\pi s P(q))^2 \right) \]

\[ + V + : \rho : \text{term} + j \cdot A \text{ term} \]  \hspace{1cm} (77)

\[ 0 = (qa - 4\pi s \rho) \mid \text{physical} \rangle \]  \hspace{1cm} (78)

For those who want to know, here is how one gets the second term.

\[ \sum_i (a(r_i) + 4\pi s P(r_i))^2 = \]

\[ \int d^2 \rho(r)(a(r) + 4\pi s P(r))^2 = n \int d^2 r(a(r) + 4\pi s P(r))^2 + O : \rho : \]

\[ = n \sum_q (a^2(q) + (4\pi s P(q))^2 + O : \rho : \]

How does this help? Focus on first two terms in Eqn. (77). The ground state is a product: the fermions are in \( \chi_p \), the state with \( p \)-filled CF Landau levels, the oscillators, whose frequency is close to \( \omega_0 \), are in their ground state and

\[ \Psi_{CS} = e^{-C \sum q a^2 \chi_p} C \text{ is some constant} \]

\[ = \exp[-C' \sum q : \rho(q) : \frac{1}{q^2} : \rho(-q) :] \chi_p = |Jastrow| \chi_p \]

\[ \Psi_e = \text{Jastrow} \cdot \chi_p \]  \hspace{1cm} (80)

where \( C' \) is another constant and we have set \( a(q) \approx \rho(q)/q \) so that

\[ C' \sum q : \rho(q) : \frac{1}{q^2} : \rho(-q) : \approx \]

\[ \int d^2 r d^2 r' \sum_i s(\delta(r - r_i) - n) \ln|r - r'| (\sum_j \delta(r' - r_j) - n) \]

\[ = \sum_{i<j} 2s \ln|z_i - z_j| - \sum_i \frac{|z_i|^2}{4l_i^2} + \text{constant} \]

\[ \frac{1}{l_i^2} = \frac{1}{l^2} + \frac{2ps}{2ps + 1} \]

Going from the CS to the electron basis, we put in the phase of the Jastrow factor, and using \( 1/l_i^2 + 1/l^2 \) (from \( \chi_p \)) = \( 1/l^2 \), we even get the right gaussian.

Thus we get the Laughlin and unprojected Jain wavefunctions. However, if we had not known how good these wavefunctions were, we would not have...
bet much money on them since they came from an uncontrolled approximation. However, given what we know, it appears we are on the right track. But we still have complaints.

- The oscillators are at $\omega_0 = \frac{2}\sqrt{p+1}$ and violate Kohn’s Theorem.
- The kinetic energy scale for fermions is still set by $1/m$. The potential energy $V$ has played no role, instead of dominating the LLL physics.
- High and low energy degrees of freedom are still mixed up.
- No sign of $e^*,d^*$, or the moment $\mu^*=e/2m$ predicted by Simon et al [14].

The solution is to decouple oscillators and fermions. They are coupled by the $j \cdot A$ term. Since we have an $A^\dagger A$ term, we want to shift the oscillators to complete the squares using (schematically)

$$U = \exp\left[ C \sum_\mathbf{q} j(q)A^\dagger(q) + h.c. \right]$$ (81)

The transformation due to $U$ is implemented with two approximations:

- Infrared limit: $ql << 1$
- RPA: $\sum_j e^{-i(q-q') \cdot r_j} \simeq n\delta(q - q')$

Upon decoupling, the theory takes the following form (for $s = 1$) [28]:

$$H = \sum_\mathbf{q} \omega_0 A^\dagger(q)A(q) + \sum_i \frac{e\delta B(r_i)}{2m}$$
$$+ \sum_j \frac{|\Pi_j|^2}{2m} - \frac{1}{2mn} \sum_i \sum_j \Pi_i^-e^{-i\mathbf{q} \cdot (r_i-r_j)}\Pi_j^+$$
$$+ V(\rho)$$ (82)

$$J^+(\mathbf{q}) = \frac{e(q_x+iq_y)}{q\sqrt{2\pi}}\omega_0 c A(\mathbf{q}) \quad J^\pm = J_x \pm iJ_y$$ (83)

$$c^2 = \frac{2p}{2p+1}$$ (84)

$$\rho(\mathbf{q}) = \sum_j e^{-i\mathbf{q} \cdot r_j} \left( 1 - il^2 \frac{\mathbf{q} \times \Pi_j}{1+c} + .. \right)$$
$$+ \frac{eq}{\sqrt{8\pi}}(A(\mathbf{q}) + A^\dagger(\mathbf{q}))$$ (85)

$$\chi(\mathbf{q}) = \sum_j e^{-i\mathbf{q} \cdot r_j} \left( 1 + il^2 \frac{\mathbf{q} \times \Pi_j}{c(1+c)} + .. \right)$$ (87)

$$0 = \chi(\mathbf{q})|\text{physical}$$ (88)
Let us digest the various terms. First consider $H_{os}$, the oscillator part, to which we will add a coupling between the external potential $\Phi_{ext}$ and the oscillator part of the charge density:

$$\rho(q) = \frac{cq}{\sqrt{8\pi}}(A(q) + A^\dagger(q))$$  \hspace{1cm} (89)

$$H_{os} = \sum_q \omega_0 A^\dagger(q)A(q) + e\Phi_{ext}(q)\frac{cq}{\sqrt{8\pi}}(A(q) + A^\dagger(q))$$ \hspace{1cm} (90)

Note that the current depends only on $A$:

$$J^+(q) = \frac{eq}{q\sqrt{2\pi}}\omega_0 cA(q)$$ \hspace{1cm} (91)

- Kohn’s theorem is obeyed upon decoupling.
- The constraint $\chi$ makes no reference to $A$, which is thus truly decoupled.
- We have separated the high and low energy physics in the infrared limit.
- Both $J$ and $\sigma_{xy}$ are due to $A$ alone. We find

$$\langle A \rangle = \frac{ecq}{\omega_0 \sqrt{8\pi}}\Phi_{ext}$$ \hspace{1cm} (92)

Plugging it into $J(A)$ and dividing by $E$ we get

$$\sigma_{xy} = \frac{e^2}{4\pi}c^2 = \frac{e^2}{2\pi} \frac{p}{2p + 1}$$ \hspace{1cm} (93)

**Thus the oscillators carry the Hall current, known to be non LLL effect:**

$$\sigma_{xy} \simeq \frac{1}{m} \cdot \frac{1}{(eB/m)} \simeq m^0$$ \hspace{1cm} (94)

- The matrix elements of the oscillator part of $\rho$ saturates the sum rule

$$\int_0^\infty S(q,\omega)\omega d\omega = \frac{q^2 n}{2m}$$ \hspace{1cm} (95)

$$S(q,\omega) = \sum_N \langle \langle N | \rho(q) | 0 \rangle \rangle^2 \delta(\omega - E_N)$$ \hspace{1cm} (96)

where

$$\rho(q) = \bar{\rho}(q) + ^nA + A^\dagger^n \text{part}$$ \hspace{1cm} (97)

The LLL (CF) part (due to $\bar{\rho}(q)$) must therefore go as $q^4$. Now we turn to the particle sector.

- The particles have the magnetic moment $\frac{e^2}{2m}$ deduced by Simon et al. [1].
The kinetic term becomes
\[
\sum_j |\Pi_j|^2 - \frac{1}{2m} \sum_i \sum_j \Pi_i e^{-i\mathbf{q} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \Pi_j^+ \tag{98}
\]

The \(i = j\) term when combined with the first, renormalizes the mass
\[
\frac{1}{m^*} = \frac{1}{m} (1 - \sum_q \frac{q}{n}) \tag{99}
\]

As oscillators are added and decoupled, \(1/m^*\) decreases. We can make it go away if we choose \(\sum_q = \text{number of oscillators} = n\), the number of particles. Let us then make this choice.

The \(i \neq j\) terms in Eqn. (98) can be shown to be convertible to a short range piece \[28\]. The infrared low energy hamiltonian and constraint become
\[
H = \bar{V} = \frac{1}{2} \sum_q \rho(q)v(q)\rho(-q) \tag{100}
\]

\[
\rho = \sum_j e^{-i\mathbf{q} \cdot \mathbf{r}_j} \left[ 1 - \frac{i l^2}{(1 + c)} \mathbf{q} \times \Pi_j \right] + (\nu A + A^\dagger \nu) \nonumber
\]

\[
\tilde{\chi} = \sum_j e^{-i\mathbf{q} \cdot \mathbf{r}_j} \left[ 1 + \frac{i l^2}{c(1 + c)} \mathbf{q} \times \Pi_j + \ldots \right] \nonumber
\]

\[
e_2^2 = \frac{2ps}{2ps + 1} \nonumber
\]

Hereafter we will set oscillators in their ground state and \(A = A^\dagger = 0\).

Murthy and I used this low \(q\) theory to study gaps \[30\] for potentials that vanished quickly at large \(q\). There were however some conceptual problems that bothered me. For example the constraint algebra does not close: \([\chi_\alpha, \chi_\beta] \neq f_{\alpha\beta\gamma} \chi_\gamma\). The error was of course due to higher order terms in \(q\). It did not make sense that two constraints each annihilated the physical states but their commutator did not. (The charge algebra did not close either, but this at least did not pose the same problem.) Secondly, it was not clear what the constraints stood for. I proposed the following remedy \[31\]. Let us assume that the power series for charge and constraint mark the beginning of exponential series. Thus we assume that they extend to:

\[
\tilde{\rho} = \sum_j \exp(-i\mathbf{q} \cdot (\mathbf{r}_j - \frac{l^2}{1 + c} \hat{z} \times \Pi_j)) \equiv \sum_j e^{-i\mathbf{q} \cdot \mathbf{R}_ej} \nonumber
\]

\[
\tilde{\chi} = \sum_j \exp(-i\mathbf{q} \cdot (\mathbf{r}_j + \frac{l^2}{c + e^2} \hat{z} \times \Pi_j)) \equiv \sum_j e^{-i\mathbf{q} \cdot \mathbf{R}_ej} \nonumber
\]
Note that \( \mathbf{R}_e \) and \( \mathbf{R}_v \) were fully determined by the small \( qL \) theory. Suddenly a lot of things fall into place.

- Consider \( \mathbf{R}_e \) and its commutation relations

\[
\mathbf{R}_e = \mathbf{r} - \frac{l^2}{(1 + c)} \mathbf{z} \times \mathbf{\Pi},
\]  

\[
[R_{ex}, R_{ey}] = -il^2,
\]

which correspond to the electron guiding center, but represented in the CF basis. In particular, it is written in terms of the velocity operator \( \mathbf{\Pi} \), which sees a weaker field \( A^* \) and will lead to a nondegenerate ground state with \( p \) filled CF Landau levels in Hartree-Fock calculations.

- The commutation relations of

\[
\mathbf{R}_v = \mathbf{r} + \frac{l^2}{c(1 + c)} \mathbf{z} \times \mathbf{\Pi}
\]

\[
[R_{vx}, R_{vy}] = il^2/c^2
\]

correspond to the guiding center coordinates of a particle of charge \(-c^2 = -2ps/(2ps + 1)\). This is the right charge for an object that should bind with the electron to form the CF. We shall refer to it as the vortex even though, as we have seen, in the Jain case there aren’t enough zeros to associate a \( 2s \)-fold vortex with each electron.

- The two sets of coordinates commute:

\[
[R_e, R_v] = 0.
\]

- Will the vortex pair with the electron? Figure 2 shows that the separation between the two is of order \( \mathbf{\Pi} \). Since there will be terms in \( H \) that associate an energy with \( \mathbf{\Pi} \), the two will bind. Finally since \( H \) is just \( V \) in the new basis, the binding has its origin in the electrostatic interaction between electrons, as envisaged by Read.

- The constraint \( \bar{\chi} = 0 \) states that the vortices will have no density fluctuations. In other words, one member of every CF (the vortex) obeys a collective constraint on density.

- The density and constraint obey the following algebras:

\[
[\bar{\rho}(q), \bar{\rho}(q')] = 2i \sin \left[ \frac{l^2(q \times q')}{2} \right] \bar{\rho}(q + q')
\]

\[
[\bar{\chi}(q), \bar{\chi}(q')] = -2i \sin \left[ \frac{l^2(q \times q')}{2c^2} \right] \bar{\chi}(q + q')
\]

\[
[\bar{\chi}(q), \bar{\rho}(q')] = 0
\]  

(106)
Pasquier and Haldane [32], working in the LLL, studied bosons at $\nu = 1$ and obtained an algebra that coincides with the above for the case $c = 1$.

Here is where we stand:

$$H = \frac{1}{2} \sum \bar{\rho}(q) \ v(q) e^{-\left(ql\right)^2/2} \ \bar{\rho}(q)$$  \hspace{1cm} (107)$$
$$0 = [H, \tilde{\chi}]$$  \hspace{1cm} (108)$$
$$\tilde{\chi} = 0$$  \hspace{1cm} (109)$$

As in Yang-Mills, the constraints form an algebra that commutes with $H$.

Could we perhaps ignore the constraint $\tilde{\chi} = 0$? Consider $\nu = 1/2$ where

$$\bar{\rho}(q) = \sum_j e^{-i q \cdot r_j} \left[ 1 - \frac{itq^2}{2} + \cdots \right]$$  \hspace{1cm} (110)$$

The fermion has unit charge and half the expected dipole moment. Presumably $\tilde{\chi}$ should send $e \to e^*$ and $d \to d^*$. Read [33] showed in a conserving approximation that $\tilde{\chi}$ generates a gauge field whose longitudinal part makes $e^* = 0$ and the surviving dipoles interact via the transverse field whose propagator is peaked at $\omega \simeq q^3$ and thus expected to matter only at $T = 0$ and at the Fermi surface.

Murthy [34] computed magnetoexciton dispersion in a conserving approximation. The results obeyed Kohn’s theorem ($S(q) \simeq q^4$).

A shortcut to $\chi$ in the safe region ($T > 0$, and/or $\omega$ not $<< q^3$) was provided by Murthy and myself [28,30]. We used the preferred combination

$$\bar{\rho}' = \bar{\rho} - c^2 \tilde{\chi}$$  \hspace{1cm} (111)$$

equivalent to $\bar{\rho}$ and weakly gauge invariant:

$$[\tilde{\chi}, \bar{\rho}'] \simeq \tilde{\chi}.$$  \hspace{1cm} (112)
Clearly $H(\bar{\rho}^p)$ is also weakly gauge invariant. Consider the series

$$\bar{\rho}^p = \sum_j e^{-i\mathbf{q} \cdot \mathbf{r}_j} \left( \frac{1}{2ps+1} - i\mathbf{q} \times \mathbf{\Pi}_j + 0 \cdot (\mathbf{q} \times \mathbf{\Pi}_j)^2 + \cdots \right)$$  \hspace{1cm} (113)

Amazingly this single combination yields the correct $e^*$ and $d^*$ and can be verified to have $q^2$ matrix elements between Hartree-Fock states, in compliance with Kohn’s theorem. It also makes physical sense: $\bar{\rho}^p = \bar{\rho} - c^2 \bar{\chi}$ adds the charge of electrons to that of correlation hole, namely $-c^2 = -\frac{2ps}{2ps+1}$ and describes the correlated entity, the CF.

Henceforth we will work with $H(\bar{\rho}^p)$. Its significance is the following. In the constrained space $\bar{\chi} = 0$, there are many equivalent hamiltonians. In the HF approximation, these are not equivalent and $H(\bar{\rho}^p)$ best approximates (within HF states and in the infrared) the true hamiltonian between true eig enstates. In contrast to a variational calculation where one searches among states for an optimal one, here the HF states are the same for a class of hamiltonians (where $\bar{\chi}$ is introduced into $H$ in any fashion as long as rotational invariance holds) and we seek the best hamiltonian: $\bar{\rho}^p$ encodes the fact that every electron is accompanied by a correlation hole of some sort which leads to a certain $e^*$, $d^*$ and obeys Kohn’s theorem ($q^2$ matrix element in the LLL).

However when gauge invariance (constraints) are crucial, one must not use $H(\bar{\rho}^p)$ but revert to the conserving approximation. Here is an example.

**Compressibility at $\nu = 1/2$** A question that came up when the dipolar form of the charge operator was derived was this. If $\bar{\rho}^p$ is dipolar, is the system compressible? Early approximate calculations suggested that compressibility vanishes as $q^2$ due to the extra $q$ in the dipole moment.

Halperin and Stern [35] showed through an example that dipolar fermions can be compressible if $H$ has the right symmetries. These are part of the symmetries generated by $\bar{\chi}$. In particular as $q \to 0$, $\bar{\chi}(\mathbf{q}) \simeq \sum_j e^{-i\mathbf{q} \cdot \mathbf{r}_j}$. Its action is

$$\delta \mathbf{r}_j = [\bar{\chi}, \mathbf{r}_j] = 0$$  \hspace{1cm} (114)

$$\delta \mathbf{p}_j = [\bar{\chi}(\mathbf{q}_0), \mathbf{p}_j] \simeq \mathbf{q}_0 \quad \text{as} \quad \mathbf{q}_0 \to 0$$  \hspace{1cm} (115)

This symmetry of $H$ is called K-invariance and had been noticed earlier by Haldane. It makes the Fermi surface very squishy and produces a response whose singularity cancels the $q^2$. The detailed work of Stern et al [35], showed that finite compressibility is ensured if the CF has a Landau parameter $F_1 = -1$.

Compressibility at $\nu = 1/2$ was also confirmed by Read [33] within the conserving approximation and by D.H.Lee [36].

That the correct compressibility obtains only if the constraint $\bar{\chi} = 0$ is correctly imposed makes physical sense: if one end of every dipolar fermion (the vortex) is actually part of an inert system with no density fluctuations (for this is what $\bar{\chi} = 0$ means) the other, electronic, end (with charge unity) must respond with a nonvanishing compressibility.
5.1 My Final Answer

For the rest of these lectures I will use

\[ H^p = \frac{1}{2} \sum_q \tilde{\rho}^p(q) \tilde{v}(q) \tilde{\rho}^p(-q) \]  

(116)

\[ \tilde{\rho}(q) = \sum_j e^{-i q \cdot r_j} \left[ \frac{1}{2 ps + 1} - i t^2 q \times \Pi_j + \text{known series} \right] \]

\[ \tilde{v}(q) = v(q) e^{-q^2 l^2 / 2} = \frac{2 \pi e^2 e^{-q\lambda}}{q} e^{-q^2 l^2 / 2} \text{ ZDS} \]  

(117)

Note that \( v(q) \) is the Zhang-Das Sarma potential [37]. I use it just to illustrate the formalism: it has a free parameter \( \lambda \) which allows one to suppress large \( q \). Roughly speaking, \( \lambda \) is a measure of sample thickness in units of \( l \).

I illustrate the rather unusual form of \( H \) we have been led to by considering the simplest case of \( \nu = 1/2 \). When we square \( \tilde{\rho} \) we get a double sum over particles whose diagonal part is the one particle (free field) term:

\[ H^0_{\nu=1/2} = 2 \sum_j \int \frac{d^2 q}{4\pi^2} \sin^2 \left[ \frac{q \times k_j l^2}{2} \right] \tilde{v}(q) \]  

(118)

This is not a hamiltonian of the form \( k^2 / 2m^* \). However if the potential is peaked at very small \( q \), we can expand the sin and read off an approximate \( 1/m^* \)

\[ \frac{1}{m^*} = \int \frac{q dq d\theta}{4\pi^2} \left[ (\sin^2 \theta) (q l)^2 \right] \tilde{v}(q) \]  

(119)

which has its origin in electron-electron interactions. However we can do more: we have full \( H_0 \) as well as the interactions. The point I want to emphasize is that \( H \) is not of the traditional form and that there is no reason it had to be.

6 Computation of Gaps

The formalism will be illustrated with some examples, starting with activation gaps for a fully polarized sample. The expression for the gap is

\[ \Delta_a = \langle p + PH | H | p + PH \rangle - \langle p | H | p \rangle \]  

(120)

where \( |p \rangle \) stands for a HF ground state with \( p \)-filled LL’s and \( |p + PH \rangle \) for the state with a widely separated particle-hole pair. The hamiltonian is \( H = H^p \).

It turns out all the matrix elements can be analytically evaluated. Figure 3 compares the numbers so obtained with those obtained by Park, Meskini, and Jain [38] using Jain wavefunctions. The gap formula looks the same for them, except that the states are not the simple ones (like \( \chi_p \)) mentioned above, but these multiplied by the Jastrow factor and projected down to the LLL. On
the other hand \( H = V(\rho) \) is very simple in the electronic basis with \( \rho(q) = \sum_j e^{-i\mathbf{q}\cdot\mathbf{r}_j} \).

Note that for \( p = 3, 4 \), the HF answer is not necessarily above the PMJ results, perhaps because the PMJ results may not be ideal benchmarks. Indeed when I compare my results to the exact diagonalization work of Morf et al.\cite{morf}, I find that the HF numbers are systematically above, as shown in Fig.\cite{fig3}. (The \( b \) parameter is like \( \lambda \) and defines a very similar potential.)

I also compared the numbers to PMJ for a gaussian potential and found almost perfect agreement except for \( 1/3 \). In general it appears the theory works best for potentials that are soft at large \( q \), say for \( \lambda > 1 \).

Are CF weakly interacting? Given that two different mass scales control activation and polarization processes, one expects the answer to be negative, though only the hamiltonian formalism, with a concrete \( H \), allows us to ask this in a meaningful way. In Fig\cite{fig5} I compare the effect of turning off the interaction on gaps. Note that interactions seem less important for \( \nu = 1/4 \) and systematically get less important as \( p \) increases. I do not understand this.

I next compare the theory to the experiments of Du et al.\cite{du} in Table\cite{table1}. This is very tricky since we do not know the exact form of interactions and have not included disorder. The idea is to see what sort of \( \lambda \) fit the data.

The value of \( \lambda \approx 2 \) is roughly double what one expects for a pure system. Additionally, the gaps, when extrapolated to \( \nu = 1/2 \) have a negative intercept of a few Kelvin. These suggest that disorder is very strong and that it is not possible to describe a disordered system by an effective \( \lambda \).

![Figure 3](image-url) 

**Fig. 3.** Dimensionless activation gaps \( \delta = \Delta_a/(e^2/\varepsilon l) \) compared the work of PMJ.
Consider now the results of Pan et al. [41] who found that
\[
m_{n}^{\text{nor}} = \frac{m_{a}}{m_{e} \sqrt{B(T)}} \simeq .25 - .35. \tag{121}
\]

near \( \nu = 1/2 \) and \( 1/4 \). How does this rough equality of normalized masses of fermions with two and four vortices fit in the present theory? I find
\[
m_{n}^{\text{nor}} = .163 \lambda \quad (s = 1) \tag{122}
\]
\[
= .175 \lambda^{5/4} \quad (s = 2) \tag{123}
\]

where the powers of \( \lambda \) are approximate. The theory does explain the near equality of these masses, but clearly does not attribute any fundamental significance to it since the answer is sensitive to the potential \( (\lambda) \).

\[\text{Fig. 4. Dimensionless gaps compared to exact diagonalization by Morf et al.}\]

\[\text{Fig. 5. Effect of turning off the interaction on gaps.}\]
6.1 Magnetic phenomena

So far we have assumed the spin to be fully polarized along the applied field. Thus in the fraction \( \frac{p}{2ps + 1} \), the CF fill \( p \)-LL with spins up. This however costs a lot of kinetic energy, which would favor filling spin-up and down LL’s equally. If one could vary the Zeeman coupling (by placing the sample in a tilted field whose normal part remains fixed) one could drive the system through various transitions. If \( E(p-r, r) \) is the energy of the state with \( p-r \) up and \( r \) down LL’s (not including Zeeman energy) then the transition \( r \to r + 1 \) will take place at a field \( B^c \) given by

\[
E(p-r, r) - E(p-r-1, r+1) = \frac{eB^c}{2m_e} \frac{n}{p} \tag{124}
\]

When the energies are computed, a strange fact, previously seen by Park and Jain, emerges: even though the CF are not free, the energy differences behave as if CF were free and occupied LL with a (polarization) gap \( \Delta_p \). That is, we find

\[
E(p-r, r) - E(p-r-1, r+1) = \frac{n(p-2r-1)}{p} \Delta_p \tag{125}
\]

(This would be the relation in a free theory since \( \frac{n}{p} \) spin-up fermions of energy \( (p-r-1 + \frac{1}{2})\Delta_p \) drop to the spin-down level with energy \( (r+\frac{1}{2})\Delta_p \)).

In the gapless case, the polarization is determined by

\[
\mathcal{E}_+(k_{+F}) - \mathcal{E}_-(k_{-F}) = g \frac{e}{2m_e \cos \theta} B^c \tag{126}
\]

where \( k_{\pm F} \) are the Fermi momenta of up and down fermions and \( \mathcal{E}_{\pm F} \) the corresponding Fermi energies. Calculations show once again that the Fermi energy differences may be fit by

\[
\mathcal{E}_+(k_{+F}) - \mathcal{E}_-(k_{-F}) = \frac{k_{+F}^2 - k_{-F}^2}{2m_p} \tag{127}
\]

where \( m_p \) is the polarization mass.

But we know CF cannot be not free because the activation gap \( \Delta_a = eB^*/m_a \) and polarization gap \( \Delta_p = eB^*/m_p \) are very different. Here is my argument [42] that this free field behaviour for magnetic phenomena is accidental and reflects \( d = 2 \) and rotational invariance. Let us assume that the energy as a function of total spin \( S \) has the form

\[
E(S) = E(0) + \frac{\alpha}{2} S^2 \tag{128}
\]

where \( \alpha \) is the inverse linear static susceptibility. Consider the gapless case for simplicity. When \( dn \) particles go from spin-down to spin-up,

\[
dE = \alpha S \ dS = \alpha S (2 \ dn) \tag{129}
\]

\[
= \alpha \frac{k_{+F}^2 - k_{-F}^2}{2m_p} (2 \ dn) \tag{130}
\]
using the areas of the circular Fermi seas. We see that $dE$ has precisely the form of the kinetic energy difference of particles of mass $m_p$ given by

$$\frac{1}{m_p} = \frac{\alpha}{\pi}. \quad (131)$$

Thus $m_p$ is essentially the static susceptibility, which happens to have dimensions of mass in $d = 2$. With this understanding I give the calculated values

$$\frac{1}{m_p^{(2)}} = \frac{e^2 l}{\varepsilon} C_p^{(2)}(\lambda) = \frac{0.087}{\lambda^{7/4}} \quad (132)$$

$$\frac{1}{m_p^{(4)}} = \frac{e^2 l}{\varepsilon} C_p^{(4)}(\lambda) = \frac{0.120}{\lambda^{7/4}} \quad (133)$$

where the superscripts on $C$ refer to the number of vortices attached and the exponent $7/4$ is approximate.

**Comparison to $T = 0$ experiments** Since magnetic transitions are controlled by ground state energies, perhaps disorder can be incorporated via an effective potential. *I have shown [43] that under certain restrictive conditions this is possible and will try to fit theory to experiment by using one data point to extract an effective $\lambda$ and use it to make predictions for other measurements made on that sample (using simple scaling laws for $\lambda$ if needed).

Kukushkin et al [44] vary both $n$ and a perpendicular $B$ to drive the system through various $T = 0$ (by extrapolation) transitions. I will compare the hamiltonian theory to these experiments by computing the $B_c$’s at which the systems at $1/4, 2/5, 3/7, 4/9$, and $1/2$ lose full polarization ($r = 0$ for gapped cases, saturation for the gapless cases) and, for $4/9$, also the $r = 1$ transition, $|3, 1\rangle \rightarrow |2, 2\rangle$. I fit $\lambda$ to the $n = 3/7$ transition $|3, 0\rangle \rightarrow |2, 1\rangle$ at $B^c = 4.5T$. The results are in Table 2.

**7 Physics at $T > 0$**

The hamiltonian formalism is particularly suited to study physics at $T > 0$ in the infinite volume limit. Given a concrete hamiltonian, one can work out the HF energies at $T > 0$

$$E^Z_\pm(k) =$$

$$\pm \frac{1}{2} \left[ \frac{eB}{2m} \right] + 2 \int \frac{d^2q}{4\pi^2} \hat{v}(q) \sin^2 \left[ \frac{k \times q l^2}{2} \right]$$

$$- 4 \int \frac{d^2k'}{4\pi^2} \hat{v}_{\pm}^{F}(\lambda') \hat{v}(\lambda) \sin^2 \left[ \frac{k' \times k l^2}{2} \right]$$
Table 2. Comparison to Kukushkin et al. Critical fields based on a fit at 3/7. The rows are ordered by the last column which measures density.

| $\nu$ | comment | $B^c$ (exp) | $B^c$ (theo) | $\nu B^c$ (exp) |
|-------|---------|-------------|-------------|-----------------|
| 4/9   | (3, 1) → (2, 2) | 2.7 T | 1.6 T | 1.2 |
| 2/5   | (2, 0) → (1, 1) | 3 T | 2.65 T | 1.2 |
| 1/4   | saturation | 5.2 T | 4.4 T | 1.3 |
| 3/7   | (3, 0) → (2, 1) | 4.5 T | 4.5 T | 1.93 |
| 4/9   | (4, 0) → (3, 1) | 5.9 T | 5.9 T | 2.62 |
| 1/2   | saturation | 9.3 T | 11.8 T | 4.65 |

where the superscript on $E^Z_\pm$ reminds us it is the total energy including the Zeeman part, the Fermi functions

$$n^F_\pm(|k|) = \frac{1}{\exp \left[ (E^Z_\pm(k) - \mu) / kT \right] + 1}$$

depend on the energies $E^Z_\pm(k)$ and the chemical potential $\mu$. At each $T$, one must choose a $\mu$, solve for $E^Z_\pm(k)$ till a self-consistent answer with the right total particle density $n$ is obtained. From this one may obtain the polarization by taking the difference of up and down densities.

The computation of $1/T_1$ is more involved. The theory predicts

$$\frac{1}{T_1} = -16\pi^3 k_B T \left( \frac{K_{\mu}^{\text{max}}}{n} \right)^2$$

$$\times \int_{E_0}^{\infty} dE \left( \frac{d n^F(E)}{dE} \right) \rho_+(E) \rho_-(E) F(k_+, k_-)$$

$$F = e^{-(k_+^2 + k_-^2)/2} I_0(k_+ k_- t^2)$$

$$\rho_\pm(E) = \int \frac{kd\mathbf{k}}{2\pi} \delta(E - E_\pm^Z(k))$$

where $E_0$ is the lowest possible energy for up spin fermions, $k_\pm$ are defined by $E^Z_\pm(k_\pm) = E$ and $K_{\mu}^{\text{max}}$ is the measured maximum Knight shift.

**Comparison to experiment** We now compare to some experiments at $\nu = 1/2$ and $T > 0$. Consider first Dementyev et al [44]. From their data point $P = 0.75$ for $B = B_\perp = 5.52T$ at 300 mK I deduce $\lambda = 1.75$. I have once again chosen to match my HF results with the above data point, and see to what extent a sole parameter $\lambda$, can describe $P$ and $1/T_1$ for the given sample at a given $B_\perp$, but various temperatures and tilts. The results are shown in Fig. 6. Dementyev et al had pointed out that a two parameter fit (mass $m$ and interaction $J$), led to
Fig. 6. Comparison to the work of Dementiev et al. The value of $\lambda$ is fit to $P$ at $300 mK$, $B_\perp = 5.52 T$ and the rest follows from the theory.

four disjoint sets for these four curves. Given that $H$ is neither free nor of the standard form ($p^2/2m + V(x)$) this is to be expected. By contrast, a single $\lambda$ is able to describe the data since $H$ has the right functional form.

Next I compare the results to that of Melinte et al.\cite{16} for polarization and Freytag et al.\cite{18} for $1/T_1$. A value of $\lambda = 1.6$ was extracted from one polarization point for sample M280. Note that a factor of two difference in $1/T_1$. I do not

Fig. 7. Comparison to the work of Freytag et al. The value of $\lambda$ was fit to 1.6 and the rest follows from the theory.
have an explanation for this. If the theory is correct, the figure implies that there are other means of relaxation besides the one introduced into the calculation.

8 Summary

Electrons in a magnetic field form degenerate Landau Levels and for the cases we were interested in ($\nu < 1$), all the electrons could be fit into the Lowest Landau Level (LLL) with room to spare. The macroscopic degeneracy of the noninteracting ground state ruled out perturbative treatments. Interactions were expected to lift the degeneracy and produce a unique ground state and tower of excited states with a scale set by the interactions. How was one to find them?

One way out was to write down inspired trial wavefunctions in the LLL that had all the right properties, a trail blazed by Laughlin for $\nu = 1/(2s + 1)$. From his work we learnt also that the system was an incompressible fluid that allowed only localized density deviations. For example a vortex at $z_0$ described a quasihole with a charge deficit of $1/(2s + 1)$.

Jain extended the wavefunctions to fractions $\nu = p/(2ps + 1)$. By trading an electron for CF which carried $2s$ units of flux that, on average, canceled $2s$ of the $2s + 1/p$ flux quanta per particle, Jain obtained a particle that saw $1/p$ flux quanta per particle, which was just right to fill $p$ LL’s in the noninteracting case, a state we called $\chi_p$. The electronic wavefunction was obtained by undoing the flux attachment. In practice, the flux tubes were replaced by vortices and a projection was done to the LLL.

From these wavefunctions we learnt [7],[12] that in the Laughlin case electrons were bound to $2s$-fold vortices to form CF with a charge $e^*$. This was true in the Jain case only before projection, which kills many zeros and also moves them off the particles. It was however true in both cases that the CF had a charge $e^*$.

We then asked how all these could be derived directly from the Hamiltonian. We saw how flux could be attached by the Linneas and Myrheim or Chern-Simons transformation following Fradkin and Lopez. Then the work of Murthy and myself was described. Here one introduces additional (oscillator) degrees of freedom at the cyclotron scale and a corresponding number of constraints $\chi$. Placing the oscillators in the ground states gave us the modulus of the Jastrow factor which combined with the phase in the CS transformation to give the Laughlin and unprojected Jain wavefunctions. These oscillators were then decoupled from the fermions in the infrared-RPA approximation. The oscillators, now at exactly the cyclotron frequency, were seen to carry all the Hall current. The particles’ kinetic energy could be quenched if the number of oscillators was chosen to equal the number of particles. In the low energy sector we were then left with just the potential energy $V$ written in terms of the projected density $\bar{\rho}$ (in the new basis) and a set of constraints $\bar{\chi}$.

We then turned to my extension of these expressions for all $ql$ (denoted by $\bar{\rho}$ and $\bar{\chi}$) which not only made the problem mathematically more alluring, but captured much of the known CF physics in operator form. In particular, one could see a CF made of the electron and another object which had the
charge of the 2s-fold vortex (called vortex for simplicity). The potential energy of electrons was seen to bind the two and to give the CF its charge and internal structure, as well as kinetic and interaction energies exactly as desired. The Hilbert space of the CF was seen to spawn two guiding center coordinates, one for the electron and one for the vortex. The constraints prevented the latter from having density fluctuations. Hence the proper implementation of constraints was needed to recover the HLR result of finite compressibility at \( \nu = 1/2 \): with one end of every CF frozen, only the electronic end responded to any applied potential as a unit charge object.

There were two approaches to solving the theory defined by \( H = V(\vec{p}) \) and subject to the constraints \( \bar{\chi} = 0 \). Since the \([\bar{\chi}, H] = 0\) one could find an approximation that respected the constraint algebra, as was done by Read for the gapless case or Murthy [34] for magnetoplasmons. This is ideal when gauge invariance is important, typically at \( T = 0 \), in the gapless cases, at the Fermi surface. For the other cases, (emphasized here), we use the one proposed by Murthy and myself: make the replacement \( \vec{p} \rightarrow \vec{p} - e^2 \bar{\chi} \equiv \vec{p}' \) which incorporates many of the effects of the constraints in a HF calculation and respects Kohn’s theorem.

We focused on a few illustrative examples of this formalism. We saw that gaps could be computed for the polarized case in HF in closed form and that they were within about 10% of the wavefunction based results or exact diagonalization results, for potentials that were soft for large \( q \), which meant roughly that \( \lambda > 1 \), \( \lambda \) being the parameter in the Zhang-Das Sarma potential.

We could see that CF were not free by turning off the interaction term and observing sizeable changes in gaps. (Without a hamiltonian such a question did not even have a meaning.) We could understand the rough equality of the renormalized masses observed by Pan et al.

Turning to magnetic phenomena, we understood how polarization phenomena could be mimicked by a free theory with a LL gap \( \Delta_p \) due a conspiracy involving \( d = 2 \) and rotational invariance. Magnetic transitions at \( T = 0 \) observed by Kukushkin et al were described by the theory given one data point from which an effective \( \lambda \) was deduced. (Heuristic arguments suggesting that magnetic phenomena in a disordered system could be described by an effective \( \lambda \) was demonstrated elsewhere [33].)

We saw how it was possible to compute polarization and relaxation at \( \nu = 1/2 \) as a function of \( T \) using just one data point to extract \( \lambda \). The power of this approach was evident in the comparison to the data of Dementyev et al: whereas a single \( \lambda = 1.75 \) gave a very good description of two polarization and two relaxation rate graphs, a fit to the data with a canonical mass plus interaction term required four disjoint sets of values. The theory was also in fair agreement with the Melinte et al and Freytag et al data.

In summary, we understand the FQHE in two complimentary fronts: trial wavefunctions and hamiltonian approaches. The former give excellent numbers where applicable and the latter provide many interpolating steps, insight, and facilitate otherwise impossible computations such as unequal-time correlations, coupling to disorder or relaxation rates at \( T > 0 \), all in the infinite volume limit.
After this brief introduction, inevitably limited and idiosyncratic, you should be ready to pursue variants of the simple FQHE system studied here. Some are reviewed in Ref. 2 by Eisenstein (experiments on double layer systems), Girvin and MacDonald (theory of the same), and Kane and Fisher (edge physics). Other possible areas that might interest you are areas are drag [47] and skyrmions [48].

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