1. Brascamp-Lieb forms and inequalities

The recently active area of Brascamp-Lieb inequalities focuses on invariant multi-linear forms in functions on Euclidean spaces. By the Schwartz kernel theorem, the multi-linear forms acting on n-tuples of Schwartz functions $F_j$ on $\mathbb{R}^{k_j}$ continuously in each argument are exactly the ones that can be written as

$$\Lambda(F_1 \otimes F_2 \otimes \cdots \otimes F_n)$$

with a unique tempered distribution $\Lambda$ on $\mathbb{R}^{k_1 + \cdots + k_n}$.

Brascamp-Lieb forms arise when the distribution $\Lambda$ specializes to integration over an affine subspace of $\mathbb{R}^{k_1 + \cdots + k_n}$ with respect to an invariant measure,

$$\int_{\mathbb{R}^{k_1 + \cdots + k_n}} \left( \prod_{j=1}^n F_j(x_j) \right) \delta(\Pi(x-z)) \, dx,$$

where $x$ denotes a vector with components $x_j$, $\Pi$ is a linear map whose ker, translated by the vector $z$, is the affine space of integration, and $\delta$ is the Dirac delta measure on the range of the map $\Pi$. Here we have called the zero set of a linear map the ker rather than the kernel of the map so as to distinguish it from an integral kernel such as for example in the Schwartz kernel theorem.

A change of variables equates this form with

$$\int_{\mathbb{R}^{k_1 + \cdots + k_n}} \left( \prod_{j=1}^n F_j(x_j + z_j) \right) \delta(\Pi x) \, dx,$$

which is a Brascamp-Lieb form with integration over a linear space, acting on translates of the functions $F_j$. Using such a reduction, we shall assume throughout this survey that the space of integration is linear, unless stated otherwise:

$$\int_{\mathbb{R}^{k_1 + \cdots + k_n}} \left( \prod_{j=1}^n F_j(x_j) \right) \delta(\Pi x) \, dx. \quad (1.1)$$

A further change of variables, replacing $x$ by $x - z$ with a vector $z$ in the ker of $\Pi$, shows an invariance of the Brascamp-Lieb integral under translation of the functions by amounts $z_j$. Similarly, one observes a homogeneity of the form under simultaneous dilations of the functions.

Using the Fourier transform, one may write for a Brascamp-Lieb form

$$\hat{\Lambda}(\hat{F}_1 \otimes \hat{F}_2 \otimes \cdots \otimes \hat{F}_n),$$

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where $\hat{\Lambda}$ is integration over the orthogonal complement of the subspace of integration of $\Lambda$. If $\Pi$ in (1.1) is an orthogonal projection, we may write for the Fourier transform integral
\[
\int_{\mathbb{R}^{k_1 + \cdots + k_n}} \left( \prod_{j=1}^{n} \hat{F}_j(\xi_j) \right) \delta((1 - \Pi)(\xi)) \, d\xi.
\] (1.2)

This allows to identify further invariances of the form under simultaneous translations of the Fourier transforms of the functions. A translation of the Fourier transform of a function is the same as a modulation of the function itself:
\[
M_\xi F(x) = F(x)e^{2\pi i x \cdot \xi}.
\]

Up to scalar multiples, the multi-linear forms of Brascamp-Lieb type are determined by their translation and modulation symmetries.

One may write the integral over the subspace also as a parameterized integral. Assume the subspace has dimension $m$ and let $I : \mathbb{R}^m \to \mathbb{R}^{k_1 + \cdots + k_n}$ be a parameterization. Denote by $I_j$ the composition of $I$ with the projection onto the $j$-th coordinate space $\mathbb{R}^{k_j}$. We may then write for (1.1), up to scalar multiple,
\[
\int_{\mathbb{R}^m} \left( \prod_{j=1}^{n} F_j(I_j x) \right) \, dx.
\] (1.3)

Writing each $F_j$ as Fourier integral, we obtain for (1.3)
\[
\int_{\mathbb{R}^{k_1 + \cdots + k_n}} \int_{\mathbb{R}^m} \left( \prod_{j=1}^{n} \hat{F}_j(\xi_j)e^{2\pi i \xi_j \cdot (I_j x)} \right) \, dx \, d\xi
\]
\[
= \int_{\mathbb{R}^{k_1 + \cdots + k_n}} \left( \prod_{j=1}^{n} \hat{F}_j(\xi_j) \right) \delta(\sum_{j=1}^{n} I_j^T(\xi_j)) \, d\xi,
\]
which is of the form (1.2) with $1 - \Pi = \sum_{j=1}^{n} I_j^T$.

It is natural to seek bounds for Brascamp-Lieb forms by products of norms of the functions, with a choice of norms respecting the symmetries of the form. Most common are Lebesgue norms $L^p$, which are invariant under translations and modulations and have a homogeneity under dilations. The corresponding bounds are called Brascamp-Lieb inequalities. With a choice of exponents $p_j$, these inequalities are written as
\[
\left| \int_{\mathbb{R}^m} \left( \prod_{j=1}^{n} F_j(I_j x) \right) \, dx \right| \leq C \prod_{j=1}^{n} \|F_j\|_{p_j}
\] (1.4)

with a constant $C$ depending on the $I_j$ and $p_j$ but not on the Schwartz functions $F_j$.

Given a tuple of exponents, if $p_j < \infty$ for some $j$, then a Brascamp-Lieb inequality can only hold if the map $I_j$ is surjective. To see this, assume $I_j$ is not surjective. Let $y$ and $z$ parameterize respectively the range of $I_j$ and the orthogonal complement of this range in $\mathbb{R}^{k_j}$. Then left-hand side of the Brascamp-Lieb inequality does not change under replacing $F_j$ by
\[
\widetilde{F}_j(y,z) := F_j(y,\lambda z),
\]
while the right-hand side scales with a power of $\lambda$ that is non-trivial if $p_j < \infty$.

If $p_j = \infty$, then the map $I_j$ need not be surjective. For example, if $m = 0$, then the projection $I_j$ is not surjective except in the pathological case $k_j = 0$. Nevertheless, as the Brascamp-Lieb integral becomes evaluation at a point, the Brascamp-Lieb inequality holds with all exponents equal to $\infty$. 

Well known cases of a Brascamp-Lieb inequality are Hölder’s inequality, where all maps $I_j$ are the identity map, Young’s convolution inequality, and the Loomis-Whitney inequality where $m = n$, $k_j = n - 1$ and the one dimensional kers of the maps $I_j$ span the full space $\mathbb{R}^m$.

Much research has been devoted to Brascamp-Lieb and related inequalities, we refer to [8], [3], [4], [5] and the references therein. In particular, [3] proves a necessary and sufficient dimensional condition for a Brascamp-Lieb inequality to hold, namely that

$$\dim(V) \leq \sum_{j=1}^{n} \frac{1}{p_j} \dim(I_j V)$$  \hspace{1cm} (1.5)$$

for every subspace $V$ of $\mathbb{R}^m$, with equality if $V = \mathbb{R}^m$. The easy direction of this equivalence is necessity of (1.5). It is seen by testing the Brascamp-Lieb inequality on suitable characteristic functions $F_j$, generating them as limits of Schwartz functions. The supports of these functions are such that the integrand on the left-hand side of (1.4) is nonzero on a disc, more precisely on a one-neighborhood in $\mathbb{R}^m$ of a large ball in $V$ of radius $R$. The left-hand side of the Brascamp-Lieb inequality grows in $R$ with the order $R^{\dim(V)}$. The suitable choice of the function $F_j$ is the characteristic function of the projection of the disc to $\mathbb{R}^{k_j}$. Its $L^{p_j}$ norms grow with the order $R^{\dim(I_j(V))/p_j}$. Letting $R$ tend to infinity, we obtain the lower bound of (1.5). The equality in case $V = \mathbb{R}^m$ is obtained by using in addition small balls in $\mathbb{R}^m$.

Since $\dim(I_j \mathbb{R}^m) \leq \dim(\mathbb{R}^m)$, inequality (1.5) for $V = \mathbb{R}^m$ in case $m > 0$ implies that

$$1 \leq \sum_{j=1}^{n} \frac{1}{p_j}.$$  \hspace{1cm} (1.6)$$

When equality holds in (1.6), then each map $I_j$ is injective on $\mathbb{R}^m$ and we obtain $\dim(I_j V) = \dim(V)$ for all subspaces $V$ of $\mathbb{R}^m$. In this case, the condition (1.5) for $V = \mathbb{R}^m$ automatically implies the condition for all subspaces of $\mathbb{R}^m$. Assuming all $I_j$ are surjective as well, which is a mild assumption given the previous discussion, all $I_j$ are bijective. Reparameterizing the range of each $I_j$, we may assume that each $I_j$ is the identity map and thereby identify Hölder’s inequality.

While it may be tempting to study (1.4) with some $0 < p_j < 1$, such estimates are easily seen to fail. This is also reflected by (1.5). Assume for example a Brascamp-Lieb inequality with $p_1 < 1$ and denote the ker of $I_1$ by $W$. Then we obtain a contradiction by applying (1.5) twice:

$$m = k_1 + \dim(W) \leq k_1 + \sum_{j=2}^{n} \frac{1}{p_j} \dim(I_j W) < \frac{k_1}{p_1} + \sum_{j=2}^{n} \frac{k_j}{p_j} = m.$$ 

The endpoint case $p_j = 1$ reduces to Brascamp-Lieb inequalities of fewer functions. We show this in case $j = 1$. By a weak limiting process, the Brascamp-Lieb inequality extends to finite Borel measures in place of the first Schwartz function. In particular, one may insert translates of the Dirac delta measure. Conversely, bounds for the Brascamp-Lieb integral with translates of the Dirac delta measure as the first input imply by superposition the Brascamp-Lieb inequality for arbitrary Schwartz functions as first input. The Brascamp-Lieb inequality with a translate of the Dirac delta measure can be written as

$$\int_{\mathbb{R}^{k_1+\cdots+k_n}} \left( \prod_{j=2}^{n} F_j(x_j + y_j) \right) \delta(x_1 - y_1) \delta(\Pi x) \, dx,$$
which can be further written as
\[ \int_{\mathbb{R}^{k_1+\ldots+k_n}} \left( \prod_{j=1}^{n} F_j(x_j) \right) \delta(\Pi(x_1, \ldots, x_n)) \, dx_1, \ldots, dx_n. \]

Note that the range of the the restriction of \( \Pi \) to fixed \( y_1 \) is the same as the range of \( \Pi \) as a consequence of the assumption that \( I_1 \) is surjective. The last display is again a Brascamp-Lieb integral with an affine linear space of integration and one input function less. Thus we have shown the desired reduction.

This observation in reverse allows to interpret the Dirac delta measure in the general Brascamp-Lieb form (1.1) as coming from an \( L^1 \) function. Thus (1.4) is equivalent to the inequality
\[ \left| \int_{\mathbb{R}^{k_1+\ldots+k_n}} \left( \prod_{j=1}^{n} F_j(x_j) \right) \delta(x_{n+1} - \Pi(x_1, \ldots, x_n)) \, dx \right| \leq C \prod_{j=1}^{n} ||F_j||_{p_j} ||F_{n+1}||_1. \]

Here the subspace of integration is the graph of a function in the first \( n \) variables.

2. Singular Brascamp-Lieb inequalities

Coming to the main subject of this survey, one may ask whether a variant of the Brascamp-Lieb inequality continues to hold if one inserts singular integral kernels instead of finite measures into one or several input slots with \( p_j = 1 \). Singular integral kernels in general fail to be finite measures, but in many situations one retains inequalities thanks to cancellation between positive and negative parts of the kernel. Examples of singular integral kernels arise from integrating a mean zero Schwartz function over the group of dilations
\[ K(t) = \lim_{N \to \infty} \int_0^N \lambda^k \phi(\lambda t) \frac{d\lambda}{\lambda}, \quad \hat{K}(\tau) = \int_0^\infty \hat{\phi}\left(\frac{\tau}{\lambda}\right) \frac{d\lambda}{\lambda}. \]

Such kernels are homogeneous under dilations and smooth outside the origin. They are in general not locally integrable near the origin, yet they are tempered distributions in the sense that the limit in \( N \) has to be executed after the pairing with a Schwartz function. Tempered distributions with such limits are called principal value distributions. More generally, one may consider tempered distributions \( K \) on \( \mathbb{R}^k \) whose Fourier transform \( \hat{K} \), called the multiplier associated with \( K \), is a bounded measurable function satisfying the symbol estimates
\[ |\partial^\alpha \hat{K}(\tau)| \leq C|\tau|^{-|\alpha|} \]
for some constant \( C \), all \( \tau \neq 0 \) and all multi-indices \( \alpha \) up to suitably large order. This condition is satisfied for the above homogeneous kernels. For much of our survey it is sufficient to consider these homogeneous kernels. The Dirac delta measure is a singular integral kernel, it can be written in the form (2.1) with a Schwartz function of integral zero, and its Fourier transform is a constant function. A simple way to ensure that a Schwartz function has integral zero is to make it odd. Many of the interesting features of the theory can already be seen when restricting to odd kernels.

We write singular Brascamp-Lieb inequalities as
\[ \left| \int_{\mathbb{R}^m} \left( \prod_{j=1}^{h} F_j(I_j x) \right) \left( \prod_{j=h+1}^{n} K_j(\Pi_j x) \right) \, dx \right| \leq C \prod_{j=1}^{h} ||F_j||_{p_j} \]
with singular integral kernels $K_j$ on $\mathbb{R}^{k_j}$ and surjective maps

$$I_j, \Pi_j : \mathbb{R}^m \to \mathbb{R}^{k_j}.$$  

The constant $C$ is assumed to be independent of the functions $F_j$, and is assumed to depend on the kernels $K_j$ only through the constant in (2.2) and the bound on the order of derivatives in (2.2). For smooth homogeneous kernels, the constant $C$ is controlled by some Schwartz norm of the Schwartz function $\phi$ in (2.1).

As we ask a given singular Brascamp-Lieb inequality to hold for all choices of singular integral kernels, it needs to hold for the special choice of a Dirac delta measure. In particular, the bound (2.3) needs to hold when all kernels are the Dirac delta measure. Note that

$$\prod_{j=h+1}^n \delta(\Pi_j x) = \delta(\Pi_{h+1} x, \ldots, \Pi_n x),$$

where the Dirac delta measure on the right-hand side lives in dimension $k_{h+1} + \cdots + k_n$.

In order for the integral in (2.3) to be well defined, we need the map

$$x \mapsto (\Pi_{h+1} x, \ldots, \Pi_n x)$$

to be surjective. We assume this surjectivity and choose variables

$$t = (t_{h+1}, \ldots, t_n)$$

on the range of this map. Changing coordinates and choosing $y$ as vector of coordinates for the joint kernel

$$W = \bigcap_{j=h+1}^n \ker \Pi_j,$$  \hspace{1cm} (2.4)

we may rewrite the integral in (2.3) as

$$\int_{\mathbb{R}^m} \left( \prod_{j=1}^h F_j (I_j (y, t)) \right) \left( \prod_{j=h+1}^n K_j (t_j) \right) dy dt.$$  \hspace{1cm} (2.5)

Thanks to these conventions, it is particularly easy to reduce a singular integral by setting one kernel $K_j$ equal to the Dirac delta measure. One removes this kernel from (2.5), sets the coordinate $t_j$ equal to zero, and removes the integration over the variable $t_j$.

The class of singular integral kernels is invariant under dilation symmetries but not under translation or modulation symmetries. The translation symmetries of the Brascamp-Lieb integral discussed after (1.1) leave the singular Brascamp-Lieb form invariant only if the components $z_j$ in the notation after (1.1) are zero for $j > h$, that is those $j$ belonging to kernels. An analogous observation holds for the modulation symmetries.

The mean zero condition on the Schwartz function in (2.1) is an important theme in singular integral theory. To see necessity of the cancellation, consider a kernel $K_n$ of the form (2.1) generated by a non-negative Schwartz function that is not constant equal to zero, and assume there is only one kernel or reduce the complexity by replacing the other kernels by Dirac delta measures. Consider (2.3) with characteristic functions $F_j$ of standard unit balls in the respective dimensions similarly to the proof of necessity of (1.5). The right-hand side of (2.3) is finite. The integrand on the left-hand side is equal to $K_n(t_n)$ for $y$ in a small ball about the origin and $t_n$ in a small fixed interval around the origin. Uniformly in this ball in $y$, the integral in $t_n$ tends to $\infty$ with $N$, because the degree of homogeneity of the singular integral kernel is critical for integration. Thus the left-hand side of (2.3) is unbounded.
Singular Brascamp-Lieb inequalities have seen much development in recent years, but the level of understanding is far from establishing a general criterion mirroring the condition (1.5). We present some necessary and some sufficient conditions.

A necessary condition for (2.3) can be obtained by specifying all $K_j$ as Dirac delta measures, yielding a reduced Brascamp-Lieb inequality of lower order with integration over the joint ker defined in (2.4). We obtain that $I_j$ needs to map $W$ onto $\mathbb{R}^{k_j}$ if $p_j < \infty$, and (1.5) for the reduced inequality gives the necessary condition

$$\dim(V) \leq \sum_{j=1}^{h} \frac{1}{p_j} \dim(I_j V)$$

(2.6)

for all $V \subseteq W$, with equality if $V = W$.

Due to the importance of cancellation of the singular integral kernel, we may obtain further necessary conditions for (2.3), namely that

$$\ker I_1 + \ker \Pi_n = \mathbb{R}^m,$$

(2.7)

and similarly for other indices by permutation of the Schwartz functions and kernels. To see necessity, assume this condition is violated. By reduction we may assume $h = 1$. Then there is a non-zero linear functional $\lambda$ on $\mathbb{R}^m$ which vanishes on $\ker I_1$ and on $\ker \Pi_n$. This functional factors as

$$\lambda(x) = \rho_1(I_1 x) = \rho(\Pi_n x)$$

for some suitable maps $\rho_1$, $\rho$. Let $K_n$ be the kernel defined by (2.1) with the Schwartz function $e^{-|t|^2} \rho(t)$ and define for any tuple of Schwartz functions

$$\tilde{F}_1 = F_1 \times (\text{sgn} \circ \rho).$$

We obtain

$$\int_{\mathbb{R}^m} F_1(I_1 x) \left( \prod_{j=2}^{n-1} F_j(I_j x) \right) |K_n|(\Pi_n x) \, dx = \int_{\mathbb{R}^m} \tilde{F}_1(I_1 x) \left( \prod_{j=2}^{n-1} F_j(I_j x) \right) K_n(\Pi_n x) \, dx.$$

Approximating $\tilde{F}_1$ by Schwartz functions and applying a hypothetical singular Brascamp-Lieb inequality for the right-hand side, we obtain the same inequality for the left-hand side, contradicting the impossibility of the inequality for the non-negative kernel $|K_n|$.

If $p_1 = \infty$, we obtain another necessary condition for a singular Brascamp-Lieb inequality, which we adapt from [42], namely

$$\bigcap_{j=2}^{n-1} \ker I_j \subseteq \ker I_1 \cup \ker \Pi_n.$$

For assume this is not the case. Pick a vector $u$ which is in the space on the left-hand side but not in the space on the right-hand side. There is a linear functional $\lambda_1$ that factors as $\lambda_1(x) = \rho_1(I_1 x)$ and is positive on $u$. Let $F_1 = 1_+ \circ \rho_1$ with $1_+$ the characteristic function of the positive half line. Let $F_j$ for $2 \leq j \leq h$ be the characteristic function of the unit ball.

There is also a linear functional $\lambda$ that factors as $\lambda(x) = \rho(\Pi_n x)$ and is positive on $u$. Let $K_n$ be the homogeneous kernel (2.1) generated by $e^{-|t|^2} \rho(t)$. We split the singular Brascamp-Lieb integral (2.3) by first integrating along lines parallel to $u$:

$$\int_{\ker(\lambda)} \int_{\mathbb{R}} F_1(I_1(x + su)) \left( \prod_{j=2}^{n-1} F_j(I_j(x + su)) \right) K_n(\Pi_n(x + su)) \, ds \, dx$$
The middle factor in the integrand, the product over \(j\), is independent of \(s\) and equal to 1 for \(x\) in a small neighborhood of the origin. The first factor is bounded,

\[
F_1(I_1(x + su)) = 1 + (\lambda_1(x) + s\lambda_1(u)),
\]

and for some sufficiently large \(a\) it vanishes for \(s < -a\) and is constant 1 for \(s > a\). The third factor is positive for \(s > 0\). Hence the integral over \(s < -a\) vanishes, is a bounded number for \(-a < x < a\), and is plus infinity for \(s > a\) and \(x\) in a small neighborhood of the origin. Hence the singular Brascamp-Lieb integral is unbounded.

We come to some sufficient conditions for singular Brascamp-Lieb inequalities to hold. If one of the exponents \(p_j\) is equal to 1, we may reduce a singular Brascamp-Lieb inequality to one of lower complexity by the use of Dirac delta measures as discussed in the non-singular case. Validity of the reduced inequalities becomes a sufficient criterion for validity of the original inequality.

If \(1 \leq p_j \leq 2\) (2.8) for all \(1 \leq j \leq h\), then it is useful to pass to the integral on the Fourier transform side. If \(\Pi\) in (1.1) is an orthogonal projection, the Fourier transform integral reads as

\[
\int_{\mathbb{R}^k} (\prod_{j=1}^h \hat{F}_j(\xi_j)) \left( \prod_{j=h+1}^n \hat{K}_j(\xi_j) \right) \delta((1 - \Pi)\xi) d\xi. \tag{2.9}
\]

This is estimated by a non-singular Brascamp-Lieb inequality in the Fourier transforms of the functions, using that the multipliers \(\hat{K}_j\) are functions in \(L^\infty\). Aiming at the dual exponents \(p_j' = p_j / (p_j - 1)\), we need the condition (1.5):

\[
\dim(V) \leq \sum_{j=1}^h \frac{1}{p_j'} \dim(V_j),
\]

where \(V\) is a subspace of \(\ker(1 - \Pi)\), \(V_j\) is its projection onto the \(j\)-th coordinate space, and equality holds for \(V\) equal to \(\ker(1 - \Pi)\). We thus estimate (2.9) with the Brascamp-Lieb inequality by

\[
\leq C \prod_{j=1}^h \|F_j\|_{p_j'} \leq C \prod_{j=1}^h \|F_j\|_{p_j}.
\]

In the second inequality we have used the Hausdorff Young inequality, which is applicable by the assumption (2.8). An interesting variant of this theme is to estimate a singular Brascamp-Lieb integral by a mixed product of \(L^p\) norms of the functions and \(L^p\) norms of the Fourier transforms of the functions. An instance of this has been studied in [39].

3. Inequalities with one singular kernel and Hölder scaling

As seen in the previous section, when all exponents \(p_j\) are at most 2, then one has a good sufficient criterion for a singular Brascamp-Lieb inequality. At the other end of the spectrum, when the \(p_j\) are large, one finds the special case of Hölder scaling

\[
\sum_{j=1}^h \frac{1}{p_j} = 1,
\]

where in an average sense the \(p_j\) are as large as they can be. This is a heavily studied case and we shall assume it throughout the rest of the survey.

Recall that in the Hölder case the condition (2.6) needs only to be checked for \(V = W\). Each map \(I_j\) restricted to \(W\) needs to be injective. Neglecting some trivial extensions
for $p_j = \infty$, we may also assume that this map is surjective for each $j$. As a consequence, all $k_j$, $1 \leq j \leq n - 1$ are equal and in particular $k_j = k_1$ and

$$m = k_1 + k_n.$$  

The singular Brascamp-Lieb integral may then be written as

$$\int_{\mathbb{R}^{k_1}} \int_{\mathbb{R}^{k_n}} \left( \prod_{j=1}^{n-1} F_j(A_jy + B_j t) \right) K_n(t) dtdy,$$

with matrices $A_j$ and $B_j$. Each of the matrices $A_j$ has to be regular. Changing $F_j$ by precomposing with the matrix $A_j$, we may assume that all $A_j$ are equal to the identity matrix,

$$\int_{\mathbb{R}^{k_1}} \int_{\mathbb{R}^{k_n}} \left( \prod_{j=1}^{n-1} F_j(y + B_j t) \right) K_n(t) dtdy. \quad (3.1)$$

Interchanging the order of integration so that $y$ becomes the inner variable and replacing it by $y - B_1 t$, we may in addition assume that $B_1 = 0$.

Writing each $F_j$ as Fourier integral we obtain for (3.1)

$$\int_{\mathbb{R}^{(n-1)k_1}} \int_{\mathbb{R}^{k_1}} \int_{\mathbb{R}^{k_n}} \left( \prod_{j=1}^{n-1} \hat{F}_j(\eta_j) e^{2\pi i \eta_j \cdot (y + B_j t)} \right) K_n(t) dtdy d\eta_1 \ldots d\eta_{n-1}$$

$$= \int_{\mathbb{R}^{(n-1)k_1}} \int_{\mathbb{R}^{n-1} \eta_1 + \ldots + \eta_{n-1} = 0} \int_{\mathbb{R}^{k_n}} \left( \prod_{j=1}^{n-1} \hat{F}_j(\eta_j) \right) \hat{K}_n(-\sum_j B_j^T \eta_j) dtd\gamma,$$

where $d\gamma$ is the Lebesgue measure on the subspace $\eta_1 + \ldots + \eta_{n-1} = 0$ in $\mathbb{R}^{(n-1)k_1}$.

We look at small values of $n$. For $n = 2$, the singular Brascamp-Lieb integral in the discussed variables becomes

$$\int_{\mathbb{R}^{k_1}} \int_{\mathbb{R}^{k_2}} F_1(y) K_2(t) dtdy.$$

Taking formally the Fourier transform, one obtains

$$\hat{F}_1(0) \hat{K}_2(0),$$

which is undetermined by (2.2) and does not lead to an interesting theory.

The case $n = 3$ describes bilinear forms which dualize to linear operators. In the above coordinates, the singular Brascamp-Lieb integral can be written as

$$\int_{\mathbb{R}^{k_1}} \int_{\mathbb{R}^{k_3}} F_1(y) F_2(y + Bt) K_3(t) dtdy.$$

If $B$ is not injective, we may integrate the ker of $B$ first. This integrates the singular integral kernel towards a lower dimensional kernel, reducing the problem to a similar problem where $B$ is injective. If $B$ is not surjective, we may split the integration over $y$ into integration over the range of $B$ and the complement of the range. The integral over the range is a similar singular Brascamp-Lieb with smaller dimension, which can be estimated first. Subsequently, one can estimate the complementary integral by Hölder’s inequality. Hence we may assume without loss of generality that $B$ is regular. By changing variables and replacing the kernel $K_3$ by its composition with the inverse of $B$, we obtain the form

$$\int_{\mathbb{R}^{k_1}} \int_{\mathbb{R}^{k_3}} F_1(y) F_2(y + t) K_3(t) dtdy.$$
The dual linear operator is the classical convolution with a singular integral kernel, which is well understood. As a consequence, we have the desired singular Brascamp-Lieb inequality with Hölder scaling and $1 < p_1, p_2 < \infty$. The restriction $1 < p_j$ can be understood as a condition of the type (2.7) after a reduction by a Dirac delta function as in the discussion after (2.7).

We turn to the genuinely multi-linear case $n \geq 4$. Fixing $n$ and $k_1$, singular Brascamp-Lieb inequalities become easier with growing $k_n$. In case of odd kernels this can be made rigorous by the method of rotations, which we will discuss more thoroughly later.

The largest and thus easiest interesting case is $k_n = (n-2)k_1$. Beyond that, one would necessarily violate condition (2.7) or be able to integrate out some of the $t$ variables of $K$ to reduce to a kernel of smaller dimension. The case $k_n = (n-2)k_1$ is the classical theory of multi-linear operators of Coifman-Meyer type [12]. Note that the map $(B_2 \otimes \ldots \otimes B_n)$ has to be surjective or else one could again reduce the problem by integrating a trivial kernel.

As a consequence, there are no translations of this subspace which leave the multiplier invariant. Hence the Coifman-Meyer case does not exhibit modulation symmetries. As one lowers $k$ from the maximal interesting $(n-2)k_1$, one may no longer uniquely determine the embedding map $I$ up to change of coordinates. The discussion bifurcates depending on the geometry of $I$, and the classification of cases leads to quite elaborate linear algebraic questions. One case in every dimension is distinguished as the generic position of these projections. It can be obtained almost surely by picking $I$ randomly with respect to suitable Gaussian probability measures. The study of this generic situation has begun in the work on the bilinear Hilbert transform [47] and [30].
$k_1 = 1$, the best sufficient dimensional condition in the generic situation is \([60]\). In the notation

$$
\int_{\mathbb{R}} \int_{\mathbb{R}^{kn}} \left( \prod_{j=1}^{n-1} F_j(y + B_j t) \right) K_n(t) dt dx,
$$

the generic case is when each tuple of the linear functionals $B_j$ spans the maximal possible space. One obtains the singular Brascamp-Lieb inequality with Hölder scaling for all $1 < p_j \leq \infty$ provided one has the dimensional condition

$$
n - 3 < 2 k_n
$$

for any $n \geq 3$. Unlike the Coifman-Meyer case, the generic singular Brascamp-Lieb integral for $k < (n - 2)k_1$ exhibits modulation symmetries. The proof of the above result employs a modulation invariant counterpart of Calderón-Zygmund techniques called time-frequency analysis. This technique originates in the works of \([19], [29]\) and was first applied to singular Brascamp-Lieb forms in the work \([17]\) on the bilinear Hilbert transform. An approach to time-frequency analysis through outer measures was described in \([20]\). The principal value limit in \((2.1)\) in the context of time-frequency analysis and in particular the bilinear Hilbert transform is studied in \([10], [18], [19]\).

While the time-frequency analysis in \([60]\) breaks down if the condition \((3.2)\) is violated, it remains an open problem whether \((3.2)\) is necessary for singular Brascamp-Lieb inequalities to hold. Even under condition \((3.2)\), interesting open questions remain concerning the extension of singular Brascamp-Lieb inequalities to restricted type inequalities beyond the threshold at $p_j = \infty$. This is discussed in \([60]\), see also \([16]\) for a discussion near the boundary of the range of exponents with known bounds.

The extension of the above result of \([60]\) to $k_1 > 1$ is addressed in \([14]\), proving singular Brascamp-Lieb inequalities on the form

$$
\int_{\mathbb{R}^{k_1}} \int_{\mathbb{R}^{kn}} \left( \prod_{j=1}^{n-1} F_j(y + B_j t) \right) K_n(t) dt dy
$$

assuming $B_j : \mathbb{R}^{kn} \to \mathbb{R}^{kj}$ are in generic position and

$$
k_1(n - 3) < 2 k_n.
$$

If $k_n$ is an integer multiple of $k_1$, this follows rather quickly from the methods of \([60]\).

For the fractional multiple case, \([14]\) uses some additional arguments from additive combinatorics. The authors restrict attention to the range $2 < p_j \leq \infty$. It is not known whether the restriction $2 < p_j$ is necessary.

A partial explanation for the break down of modulation invariant time-frequency analysis beyond \((3.2), (3.3)\) is the occurrence of more general symmetries. For example, consider the case of the trilinear Hilbert transform

$$
\int_{\mathbb{R}} \int_{\mathbb{R}} \left( \prod_{j=1}^{4} F_j(y + B_j t) \right) \frac{1}{t} dt dy
$$

with generic, that is pairwise different, numbers $B_j$. This form exhibits a symmetry under quadratic modulation

$$
Q_{\alpha_j} F_j(x) = F_j(x) e^{i \alpha_j x^2}
$$

where the four numbers $\alpha_j$ are all non-zero and satisfy

$$
\sum_j \alpha_j(y + B_j t)^2 = 0.
$$
It would be interesting to find extensions of time-frequency analysis that are invariant under more general symmetries and address boundedness of the trilinear Hilbert transform. This starts with a solid understanding of the type of symmetries, we refer to related work on inverse theorems for Gowers norm \[34\] involving generalized quadratic phase functions possibly relevant for the trilinear Hilbert transform and the more general symmetries in \[35\]. Additional symmetries may not be the only obstruction to go beyond \(3.2\), because it is not clear that all cases beyond \(3.2\) exhibit additional symmetries.

Shrinking \(k_n\) further, the minimal non-trivial case is \(k_n = 1\). The distance to \(k_1\) is maximized if \(k_1 = n - 1 = h\). If \(k_1\) is greater than or equal to \(h\), then the vectors \(B_j\), \(2 \leq j \leq h\) span a space of dimension less than \(k_1\) and one may reduce to a singular Brascamp-Lieb integral of lower order as discussed in the case \(n = 3\). By the same token, if \(k_1 = h\), then these vectors have to be linearly independent and thus a basis of \(\mathbb{R}^{k_1}\). Since all bases are equivalent up to change of variables, one can write the singular Brascamp-Lieb integral without loss of generality in symmetric form as

\[
\int_{\mathbb{R}^h} \left( \prod_{j=1}^{h} F_j(x_1, \ldots, x_j-1, x_{j+1}, \ldots, x_h) \right) \frac{1}{x_1 + \ldots + x_h} \, dx. \tag{3.4}
\]

This form is called the simplex Hilbert form. Maybe the biggest challenge in the area is to understand whether this form satisfies any singular Brascamp-Lieb inequalities. By symmetry and interpolation techniques, the easiest bound to prove should be the one with all exponents equal. We formulate this as a conjecture.

**Conjecture 1.** There exists a constant \(C\) such that for all tuples of Schwartz functions \((F_j)_{j=1}^h\) the form \((3.4)\) is bounded by

\[
C \prod_{j=1}^{h} \|F_j\|_h.
\]

By the method of rotations, bounds for the simplex Hilbert form imply bounds for many singular Brascamp-Lieb integrals, including for the multi-linear Hilbert transform, another major open problem. Moreover, bounds for the simplex Hilbert form imply bounds for the Carleson and polynomial Carleson operator

\[
\int_{\mathbb{R}} f(x - t)e^{i(N_1(x)t + N_2(x)t^2 + \ldots + N_d t^d)} \frac{dt}{t},
\]

which was for general \(d\) studied in \[49\], \[50\] and \[71\]. Partial progress on the simplex Hilbert form in the case \(h = 3\) can be found in \[45\], which in particular establishes the above conjectured bound in a dyadic model when one of the functions takes a special form. Further results concerning truncations of the simplex Hilbert form and effective bounds in the parameter of truncation are discussed in \[70\] based on the approach in \[60\] and in \[26\].

Having discussed generic choices of \(B_j\) in the spectrum from large \(k_n\) to small \(k_n\), we turn attention to some of the phenomena arising when we do not ask the \(B_j\) to be in generic positions. We begin with the simplest case which displays some of the phenomena,

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \left( \prod_{j=1}^{3} F_j(y + B_j t) \right) K_4(t) \, dt \, dy.
\]
The generic case has three different real numbers $B_j$, this is the classical bilinear Hilbert transform. All generic cases have the same proof of Brascamp-Lieb bounds using time-frequency analysis. If two values of $B_j$ are equal, the form changes its nature. One identifies the pointwise product of two functions, and replacing the product by a new function we obtain a singular Brascamp-Lieb integral with $n = 3$. Applying the classical theory without time-frequency analysis and then applying Hölder’s inequality to resolve the product proves $L^p$ bounds in this degenerate situation. The case that all three values of $B_j$ are equal is even further degenerate but of no interest, it leads to the pointwise product of three functions together with the indeterminate integral in case $n = 2$. If two of the values of $B_j$ approach each other, the first proof of the bilinear Hilbert transform produced a growing constant in the singular Brascamp-Lieb inequality. It was natural to seek uniform bounds, which was achieved in a series of papers \[67\], \[48\], \[32\], \[63\], \[68\] in the full Hölder range of exponents with $1 < p_j \leq \infty$. Some of these results were generalized to uniform bounds on other families of singular Brascamp-Lieb integrals in \[61\].

A more complicated classification of cases occurs for the two dimensional bilinear Hilbert transform
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( \prod_{j=1}^{3} F_j(y + B_j t) \right) K_4(t) dtdy,
\]
a situation first considered by \[15\] and then thoroughly discussed in the PhD thesis \[69\]. The thesis classifies the possibilities for the parameters $B_1$, $B_2$, $B_3$ into nine cases. Most cases can be normalized such that $B_1 = 0$ and $B_2 = I$, leaving only $B = B_3$ as indetermined matrix, which may be assumed to be in Jordan canonical form. A trivial pointwise product occurs if $B = 0$ or $B = I$, this results in a reduction of the complexity of the integral as in the one dimensional case. The case that all eigenvalues of $B$ are different from 0 and 1 is the generic case covered by previous results. The case that one eigenvalue of $B$ is equal to 0 or 1 and the other eigenvalue is different from 0 and 1 is an interesting hybrid case discussed in \[15\], likewise the case of a non-trivial Jordan block with eigenvalue 0 or 1. The case when $B$ has both 0 and 1 as eigenvalue is called the twisted paraproduct and is an instance of the forms in Theorem \[2\] below with $m = 2$, albeit with the fourth function set constant equal to 1.

Only in one of the nine cases it is not known whether the singular Brascamp-Lieb inequality holds at a nontrivial set of exponents. This is the case where the first columns of all three matrices $B_1$, $B_2$, $B_3$ vanish, while the second columns respectively are $(0,0)$, $(0,1)$, $(1,0)$. This case is a simplex Hilbert form discussed in the above conjecture. All remaining cases reduce to easier objects and are of lesser interest. An abundance of questions concerning uniform bounds arise between these various cases. While the method of rotations would prove uniform bounds for odd kernels from Conjecture \[1\] lacking a proof of the latter it may be of interest to study these uniform questions.

We turn to a class of Brascamp-Lieb integrals where the modulation symmetry group is spanned by rich modulations symmetries. A rich modulation symmetry is a modulation symmetry which generalizes to arbitrary phase functions. For example the Hölder form
\[
\int_{\mathbb{R}} F_1(x) F_2(x) dx
\]
is invariant not only under replacing $F_1$ and $F_2$ by $M_\xi F_1$ and $M_{-\xi} F_2$ respectively, but also under replacing them by
\[
F_1(x) e^{i\phi(x)}, \quad F_2(x) e^{-i\phi(x)}
\]
for arbitrary real phase functions $\phi$. If we consider each input function as a function in $k_1$ arguments, then one way that rich modulations symmetries occur is when slots of different functions share the same argument.

We consider an example where each of the $k = k_1$ slots carries two possible variables, making it $2^k$ integration variables, which we denote as

$$(x_1^0, \ldots, x_k^0, x_1^1, \ldots, x_k^1) = x.$$ 

Each possible combination of the variable occurs in one of the functions. This requires $2^k$ input functions parameterized by the cube $Q$, the set of all

$$j : \{1, 2, \ldots, k\} \to \{0, 1\}.$$ 

Consequently, for $j \in Q$, we have

$$I_j x = (x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(k)}).$$

We further consider a singular integral kernel $K$ in $\mathbb{R}^k$ and an arbitrary surjective $\Pi : \mathbb{R}^{2^k} \to \mathbb{R}^k$. The Brascamp-Lieb integral in question then writes as

$$\int_{\mathbb{R}^m} \left( \prod_{j \in Q} F_j(I_j x) \right) K(\Pi x) \, dx.$$ (3.5)

**Theorem 2** (from [28]). Given $k \geq 1$, the form (3.5) satisfies a singular Brascamp-Lieb inequality with $p_j = 2^k$ for all $j \in Q$ if and only if for all $j$

$$k = \dim(I_j(\ker \Pi)).$$ (3.6)

The condition (3.6) is the specialization of (2.6) in this situation.

While rich symmetries are very large symmetry groups and restrict techniques to those that are invariant under these symmetries, at least they have a very generic structure and one does not need to delve into the theory of polynomial or other structured symmetries. The main technique in the context of rich symmetries was pioneered in the context of the so-called twisted paraproduct in [42] and is sometimes called twisted technology. Brascamp-Lieb integrals involving rich symmetries were also studied in [41], [7], [44], [21], [22] and also in [25], [65] with applications to quantitative convergence of ergodic averages, and in [24], [23] with applications to some problems in Euclidean Ramsey theory. An application to stochastic integrals was studied in [43]. Further higher dimensional generalizations are discussed in [64].

It would be desirable to study some natural extensions of Theorem 2. One obvious generalization would be a more general range of exponents than the symmetric exponent point. Somewhat related to that is the question what happens if the corners of the cube are not fully occupied, that is the number of functions is strictly less than $2^k$. In case one has $L^\infty$ bounds, it is trivial to omit the corresponding function by estimating the constant function in $L^\infty$, but it is not clear that all inequalities with constant functions arise from more general $L^\infty$ bounds.

One further extension is to allow more than two variables in one slot, that is for $k_1 \geq 1$ and $l \geq 2$ we may consider $\mathbb{R}^m$ with coordinates

$$x = ( (x_1^0, \ldots, x_k^0), (x_1^1, \ldots, x_k^1), \ldots, (x_1^{l-1}, \ldots, x_k^{l-1}) ) \in \mathbb{R}^{kl}.$$ 

Then for all $j : \{1, 2, \ldots, k\} \to \{0, \ldots, l-1\}$ we may define

$$I_j x = (x_1^{j(1)}, x_2^{j(2)}, \ldots, x_k^{j(k)}).$$

One may then ask the analogous result as Theorem 2.

Note that also the simplex Hilbert forms of Conjecture [11] have many rich modulation symmetries. Indeed, the group of modulation symmetries of the simplex Hilbert form is spanned by rich symmetries. The space of integration in Fourier space has dimension
Since the singular integral kernel is one dimensional, this gives an \( n(n-2) \) dimensional group of modulation symmetries of the simplex Hilbert form. However, for each of the \( n \) variables one can find \( n-2 \) pairs of functions so that independent rich symmetries akin to the above shown apply between this pair of functions. The forms in Theorem 2 and the suggested generalization above have the structure that each variable has a fixed slot number in which it may occur. Note that this is not the case in the simplex Hilbert form. For example, the variable \( x_2 \) typically appears in the second slot, unless in the function \( F_1 \), where the variable \( x_1 \) is omitted and the variable \( x_2 \) appears in the first slot. This mismatch is the main obstacle to apply twisted technology to the simplex Hilbert form.

4. Method of rotations and more general kernels

The method of rotation allows to write a singular Brascamp-Lieb form with one singular integral kernel as a superposition of a family of forms with lower dimensional kernels. The family of forms is generated by rotations or more general linear transformations of the space of integration.

Turning to details, a singular Brascamp-Lieb form with a homogeneous smooth kernel can be written as

\[
\int_0^\infty \int_{\mathbb{R}^{k_1+\cdots+k_n}} \left( \prod_{j=1}^{n-1} F_j(x_j) \right) t^{k_n} \psi(tx_n) \phi(\Pi x) \, dx \, dt
\]

with a smooth and compactly supported function \( \psi \) with integral zero. Assume there is a vector \( v \) such that the inner product \( v \cdot x_n \) is bounded away from zero on the support of \( \psi \). The following display is a superposition by a weight function \( \phi \) of a family of forms generated by rank one perturbations of \( \Pi \) using a further fixed vector \( w \) and a varying scalar parameter \( a \):

\[
\int_0^\infty \int_{\mathbb{R}^{k_1+\cdots+k_n}} \left( \prod_{j=1}^{n-1} F_j(x_j) \right) t^{k_n} \psi(tx_n) \phi(a) \delta(\Pi x + wa(v \cdot x_n)) \, dx \, dt \, da.
\]

We assume \( \phi \) is smooth and compactly supported. Rescaling the variable \( a \) and combining it with the vector \( x_n \) to a vector of dimension \( k_n + 1 \), we recognize a new singular Brascamp-Lieb form

\[
\int_0^\infty \int_{\mathbb{R}^{k_1+\cdots+(k_n+1)}} \left( \prod_{j=1}^{n-1} F_j(x_j) \right) t^{k_n+1} \tilde{\psi}(tx_n, ta) \delta(\Pi x + wa) \, dx \, dt \, da
\]

with the compactly supported smooth function

\[
\tilde{\psi}(x_n, a) := \psi(x_n) \left\| \frac{a}{v \cdot x_n} \right\| \phi \left( \frac{a}{v \cdot x_n} \right).
\]

One verifies that \( \tilde{\psi} \) has integral zero by first integrating in \( a \) and then in \( x_n \). If we can prove bounds for the singular Brascamp-Lieb forms uniformly for all maps \( \Pi \) in the perturbed family, then by superposition we obtain a bound with the same exponents for (4.2).

Conversely, given a Brascamp-Lieb integral as in (4.2), one may seek to write it as superposition of Brascamp-Lieb forms with lower dimensional kernels. A general procedure exists, when the function \( \tilde{\psi} \) is odd. In addition, we assume \( \tilde{\psi} \) is compactly supported away from the origin. After a decomposition by a finite smooth partition of unity, and a suitable rotation of the coordinate system for each piece, we can assume that there is a vector \( v \) of dimension \( k_n \) such that \( \tilde{\psi} \) is supported in the union of two small neighborhoods respectively of \((v,0)\) and \((-v,0)\).
With suitable compactly supported functions \( \varphi \) and \( \rho \) we may write

\[
\tilde{\psi}(x_n, a) = \frac{1}{v \cdot x_n} \varphi \left( x_n, \frac{a}{v \cdot x_n} \right) = \frac{1}{v \cdot x_n} \varphi \left( x_n, \frac{a}{v \cdot x_n} \right) \rho \left( \frac{a}{v \cdot x_n} \right)
\]

and note that \( \varphi \) is odd in the first variable for fixed second variable. Taking a Fourier integral of \( \varphi \) in the second variable and denoting that by \( \hat{\varphi} \), we obtain

\[
\tilde{\psi}(x_n, a) = \int_{\mathbb{R}} \hat{\varphi}(x_n, \xi) \frac{1}{v \cdot x_n} e^{2\pi i \xi \frac{a}{v \cdot x_n}} \rho \left( \frac{a}{v \cdot x_n} \right) d\xi.
\]

For fixed \( \xi \), the integrand is a function of the form \([4.3]\) with an odd function \( \psi \). If we can prove bounds for the family of Brascamp-Lieb integrals of lower dimensional kernels uniformly for fixed Schwartz norm of \( \psi \) of some order, then we may integrate these bounds in \( \xi \) as the Schwarz norm of \( \hat{\varphi} \) in the first variable is rapidly decreasing as a function in the second variable.

One can iterate rank one perturbations to obtain the more general superposition

\[
\int_{\mathbb{R}^l} \int_0^\infty \int_{\mathbb{R}^{k_1+\cdots+k_n}} \left( \prod_{j=1}^{n-1} F_j(x_j) \right) t^{k_n} \psi(t x_n) \phi(a) \delta \left( \Pi x + \sum_{i=1}^l w_i a_i (v_i \cdot x_n) \right) dx dt \frac{dt}{t} da.
\]

If the function \( \phi \) in the above calculation is replaced by a finite Borel measure, in particular a Dirac delta measure, estimates for the form \([4.2]\) are equivalent to estimates for the form \([4.1]\) with lower dimensional kernel uniformly over the perturbation parameters in the support of \( \phi \). Choosing \( \phi \) with any intermediate regularity between smooth function and Borel measure, one can view the difficulty of estimates for the superposed operator as intermediate between the two endpoint cases. Estimates for such forms with rough singular integral kernel can be of their own interest, if estimates for the lower dimensional kernels are not known or maybe known to be false in general.

An early example of this principle is provided by the Calderón commutator \([9]\), which later appeared in the investigation of the Cauchy integral along Lipschitz curves, see \([11]\) and the references therein. The commutator can be viewed as a rough superposition of bilinear Hilbert transforms. Calderón proposed the study of the bilinear Hilbert transform and uniform bounds for it as a stepping stone towards the commutator. However, the bilinear Hilbert transform remained an open problem for many years after bounds for the Calderón commutator were obtained using different techniques. A recent account and approach to the Calderón commutator and higher order commutators was given in \([51]\) and in \([52]\). These higher order commutators can be seen as a suitable superposition of multi-linear Hilbert transforms which by themselves are not known to be bounded.

If \( \Pi \) as in \([4.1]\) is perturbed by a rank one map, then the embedding map \( I \) as in \([4.3]\) can also be identified as perturbed by a rank one matrix. To be more precise, we assume that the perturbation is \( \Pi + \Pi(u) \otimes v \) where \( v \) is a vector in \( \ker \Pi \) and \( u \) is orthogonal to \( \ker \Pi \). This representation can be found if the perturbation is small and the dimension of the ker of the perturbed map is equal to that of the original map, namely \( m \), but the kers are different. As we have a rank one perturbation, the two kers intersect in a space of dimension \( m - 1 \), and we may choose a unit vector \( v \) in \( \ker \Pi \) perpendicular to this subspace. Using that the perturbation is small, we may chose \( u \) perpendicular to \( \ker \Pi \) so that \( v - u \) is in the ker of the perturbation. Then \( \Pi + \Pi(u) \otimes v \) has the same ker as the perturbation and we may assume it is the perturbation. The perturbation of the embedding map \( I \) can then be written as \( I - u \otimes T^* v \). To verify this, one checks separately that the vectors that embed under \( I \) into the intersection of the kers of \( \Pi \) have the same image under the perturbed map, and that the vector that maps to \( v \) under \( I \) maps to \( v - u \) under the perturbation.
If the perturbations are such that only one component \(u_j\) of \(u\) and only the component \((I^T v)_n\) of \(I^T v\) is non-zero, we may view the averaging of the form as an averaging of the function \(F_j\). If we iterate several perturbations like that, then the averaged function takes the form

\[
F(I_j x, x) = \int_{\mathbb{R}^i} \phi(a) F_j \left( I_j x - \left( \sum_l a_l u_l (v_l \cdot I x_n) \right)_j \right) da.
\]

If there are enough averages so that the rank one matrices add to a regular matrix, and if \(F\) is in \(L^\infty\), then the averaged function \(F(y, z)\) becomes a \(y\) dependent symbol in the variable \(z\) in the sense

\[
|\partial^\alpha_y \partial^\beta_z F(y, z)| \leq C |z|^{-|\alpha| - |\beta|}
\]

for all multi-indices up to some degree depending on the regularity of the averaging function \(\phi\). Multiplying this symbol with the singular integral kernel gives a "space dependent" singular integral form which is nowadays seen within in the theory of \(T(1)\) theorems originating in [13]. Therefore, bounds for the averaged operator can be viewed as a Brascamp-Lieb version of a \(T(1)\) theorem.

In this spirit, a multi-linear \(T(1)\) theorem with a variant of the bilinear Hilbert transform with space dependent singular integral kernel was proven in [6] and applied in [62] in a singular variant of a higher Calderón commutator. \(T(1)\) theorems with rich modulation symmetries were proven in [44], [64] in dyadic models, it would be interesting to extend these results to the continuous setting and extend to further averaged singular Brascamp-Lieb forms.

The paper [27] discusses averages of the simplex Hilbert forms which yield singular Brascamp-Lieb forms with rich modulation symmetries. The averaged forms are such that they can be treated by twisted technology. More precisely, [27] proves bounds in cases \(n = 4\) and \(n = 5\) on

\[
\int_{(0,1)^{n-3}} \int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}} \left( \prod_{j=1}^{n-3} F_j(y + \alpha_j B_j t) \right) F_{n-2}(y + B_{n-1} t) F_{n-1}(y) K_n(t) dt dy d\alpha
\]

for linearly independent vectors \(B_j\).

5. **Inequalities with two singular kernels and Hölder scaling**

Singular Brascamp-Lieb integrals in the case of several singular integral kernels fall into the scope of multi-parameter theory. We display some of the features of multi-parameter theory using the example of two kernels. We continue to assume Hölder scaling.

Considerations analogous to those leading to (5.1) from (2.5) turn the singular Brascamp-Lieb integral with two kernels into the form

\[
\int_{\mathbb{R}^{k_1}} \int_{\mathbb{R}^{k_{n-1}}} \int_{\mathbb{R}^{k_n}} \left( \prod_{j=1}^{n-2} F_j(y + B_j s + C_j t) \right) K_{n-1}(s) K_n(t) dt ds dy. \tag{5.1}
\]

Applying the Fourier transform as after (5.1) we obtain the alternative expression

\[
\int_{\Gamma} \left( \prod_{j=1}^{n-2} \hat{F}_j(\xi_j) \right) \hat{K}_{n-1}(-\sum_{j=1}^{n-2} B_j^T \xi_j) \hat{K}_n(-\sum_{j=1}^{n-2} C_j^T \xi_j) d\gamma, \tag{5.2}
\]

where \(\Gamma\) is the subspace of \(\mathbb{R}^{(n-2)k_1}\) determined by \(\xi_1 + \cdots + \xi_{n-2} = 0\) and \(d\gamma\) is the Lebesgue measure on this subspace.

Simplifying degenerations may occur. The arguments of the two multipliers in (5.2) can be identical, that is each \(C_j\) is equal to \(B_j\). As the product of two multipliers is
again a multiplier with analogous symbol bounds, this reduces to a singular Brascamp-Lieb with one kernel. Another simplifying degeneration of (5.1) may be separation. If for every \( j \) one of the matrices \( B_j \) or \( C_j \) is zero, then we may write the integral in \( s \) and \( t \) as a product of two integrals, one in \( s \) and one in \( t \). Then we may apply Hölder’s inequality in the variable \( x \) on this product. Resolving the resulting \( L^p \) norms by pairing with a dual function, we obtain two singular Brascamp-Lieb integrals with one kernel each. Separation in (5.1) may occur after replacing the variable \( y \) by \( y + B_s + C_t \) for suitable matrices \( B \) and \( C \).

A family of cases occurs with counterexamples to a singular Brascamp-Lieb inequality that show a phenomenon not possible for one kernel. Assume we have a family of quadratic forms \( Q_j \) on \( \mathbb{R}^{k_1} \) such that

\[
\sum_{j=1}^{n-2} Q_j(y + B_j s + C_j t) = s_1 t_1
\]

where \( s_1 \) and \( t_1 \) are the first components of \( s \) and \( t \), there being no loss in generality choosing these particular components. For \( n \) large enough compared to \( k_1, k_{n-1}, k_n \), such quadratic forms will exist in the case of generic matrices \( B_j \) and \( C_j \). Choose functions of the form

\[
F_j(x) = \phi(x) e^{-2\pi i Q_j(x)}
\]

where \( \phi \) is a non-negative smooth approximation of the characteristic function of a very large ball about the origin. Choose the kernel

\[
K_n(t) = \lim_{N \to \infty} \int_0^N \lambda^{k_1} \psi'(\lambda t_1) \phi(\lambda t_2, \ldots, \lambda t_n) \frac{d\lambda}{\lambda}
\]

with odd \( \psi \) which is non-negative on the positive half axis and with non-negative \( \phi \), and similarly for \( K_{n-1} \) with odd \( \bar{\psi} \) such that \( \bar{\psi} = \psi \). Zooming into the critical integrals in \( s_1 \) and \( t_1 \) in the expression (5.1), we see

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \hat{e}^{-2\pi i s_1 t_1} \hat{\psi}(\mu s_1) \psi(\lambda t_1) \, ds_1 \, dt_1 = \mu^{-1} \psi(\mu^{-1} t_1) \psi(\lambda t_1).
\]

The right-hand side is an even function in \( t_1 \) and non-negative on the positive half axis, hence it is non-negative, and it is not identically zero as one can see considering \( \mu^{-1} \) near \( \lambda \). The effect is that the cancellation of the kernel \( K_n \) is destroyed, resulting in unboundedness as \( N \) tends to \( \infty \). More details of this calculation can be found in [55] for the two examples

\[
\int_{\mathbb{R}^4} F_1(x_1, x_2) F_2(x_1 - t, x_2 - s) F_3(x_1 + t, x_2 + s) \frac{ds \, dt}{s \, t} dx
\]

and

\[
\int_{\mathbb{R}^3} F_1(x) F_2(x + t) F_3(x + s) F_4(x + t + s) \frac{ds \, dt}{s \, t} dx. \tag{5.3}
\]

Multi-parameter theory is named after the various scaling parameters occurring in a product of singular integral kernels. We call the product of the multipliers in (5.2) the joint multiplier and write it with scaling parameters \( \mu \) and \( \lambda \) as

\[
m(\sigma, \tau) = \hat{K}_{n-1}(\sigma) \hat{K}_n(\tau) = \lim_{N \to \infty} \int_0^N \int_0^M \hat{\phi}_{n-1}(\sigma, \mu) \hat{\phi}_n(\tau, \lambda) \frac{d\mu \, d\lambda}{\mu \, \lambda}.
\]

A typical step in multi-parameter theory is the cone decomposition, which is a sorting of an integral in several scaling parameters by the size of the scaling parameters as follows:

\[
m_1(\sigma, \tau) + m_2(\sigma, \tau) = \lim_{N \to \infty} \int_{0 < \mu < \lambda < N} \frac{d\mu \, d\lambda}{\mu \, \lambda} + \lim_{M \to \infty} \int_{0 < \lambda < \mu < M} \frac{d\mu \, d\lambda}{\mu \, \lambda}.
\]
Note that the joint multiplier $m$ in (5.2) satisfies the multi-parameter symbol estimate

$$|\partial_\sigma^\alpha \partial_\tau^\beta m(\sigma, \tau)| \leq C |\sigma|^{-|\alpha|} |\tau|^{-|\beta|},$$

(5.4)

where $\partial_\sigma$ and $\partial_\tau$ are any partial derivatives in the $\sigma$ and $\tau$ variables respectively. The cone multipliers $m_1$ and $m_2$ satisfy

$$|\partial_\sigma^\alpha \partial_\tau^\beta m_1(\sigma, \tau)| \leq C |\sigma|^{-|\alpha|} |\tau|^{-|\beta|},$$

(5.5)

$$|\partial_\sigma^\alpha \partial_\tau^\beta m_2(\sigma, \tau)| \leq C |\tau|^{-|\alpha|} |\sigma|^{-|\beta|}.$$  

(5.6)

In some instances, bounds for the variants of (5.2) with the joint multiplier replaced by the cone multipliers can be established, based on the symbol estimates (5.5), (5.6). Note that these symbol estimates, say (5.5), are generalizations of the single kernel case by the cone multipliers can be established, based on the symbol estimates (5.5), (5.6). However, already a simple modification of the above such as interchanging $s_2$ and $t_2$ in the argument of $F_3$ is not addressed by the discussion in [55].
A hybrid between the generic case and the flag paraproduct case is called the biest and studied in [57], [58].

\[ \int_{\mathbb{R}^3} F_1(x)F_2(x + t)F_3(x + s)F_4(x - t - s) \frac{ds}{s} \frac{dt}{t} dx. \]

It arises in the theory of iterated Fourier integrals, which occur in multi-linear expansions of certain ordinary differential equations. Singular Brascamp-Lieb inequalities for this form are known and require time frequency analysis because the bilinear Hilbert transform is embedded into this object. Compare with the similar form [53]. For a study of objects related to the biest see [59], [38], [39], [40], [36], [17].

A more recent development is the theory of vector valued inequalities in the context of singular Brascamp-Lieb inequalities. The helicoidal method was introduced in [1] to study forms similar to the biest through mixed norm spaces and vector-valued inequalities. A survey of the helicoidal method can be found in [2].

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