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The distribution of numbers with many ordered factorizations

par Noah LEBOWITZ-LOCKARD

RéSUMÉ. Soit $g(n)$ le nombre de factorisations de $n$ en produit ordonné de facteurs plus grands que 1. On trouve des bornes précises pour les moments positifs de $g$. On utilise ces résultats pour estimer le nombre de $n \leq x$ tels que $g(n) \geq x^\alpha$ pour tous les $\alpha$ positifs. En outre, soient $G(n)$ et $g_P(n)$ les nombres de factorisations de $n$ en produit ordonné de facteurs distincts plus grands que 1 et en produit ordonné de facteurs premiers respectivement. On donne des bornes inférieures pour les moments positifs de $G$ et $g_P$.

Abstract. Let $g(n)$ be the number of ordered factorizations of $n$ into numbers larger than 1. We find precise bounds on the positive moments of $g$. We use these results to estimate the number of $n \leq x$ satisfying $g(n) \geq x^\alpha$ for all positive $\alpha$. In addition, let $G(n)$ and $g_P(n)$ be the number of ordered factorizations of $n$ into distinct numbers larger than 1 and primes, respectively. We also bound the positive moments of $G$ and $g_P$ from below.

1. Introduction

Let $g(n)$ be the number of ordered factorizations of $n$ into numbers larger than 1. For example, $g(18) = 8$ because the ordered factorizations of 18 are

$$18, \quad 9 \cdot 2, \quad 2 \cdot 9, \quad 6 \cdot 3, \quad 3 \cdot 6, \quad 3 \cdot 3 \cdot 2, \quad 3 \cdot 2 \cdot 3, \quad 2 \cdot 3 \cdot 3.$$ 

In 1931, Kalmár [11] found an asymptotic estimate for the sum of $g(n)$ for $n \leq x$, namely

$$\sum_{n \leq x} g(n) \sim -\frac{1}{\rho \zeta'(\rho)} x^\rho,$$

where $\zeta$ is the Riemann zeta function and $s = \rho \approx 1.73$ is the unique solution to $\zeta(s) = 2$ in $(1, \infty)$. Kalmár found the first error term for this equation, which Ikehara [9] subsequently improved. Most recently, Hwang [8] proved that

$$\sum_{n \leq x} g(n) = -\frac{1}{\rho \zeta'(\rho)} x^\rho + O\left(x^\rho \exp\left(-c \log_2 x \left(\frac{3}{2}\right) - \epsilon\right)\right)$$

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Mots-clefs. Ordered factorizations.
for all positive $\epsilon$ where $c = c(\epsilon)$ is a positive constant. (Throughout this paper, $\log_k$ refers to the $k$th iterate of the logarithm. In addition, all error terms apply as $x \to \infty$.)

There have also been numerous results on the maximal order of $g(n)$. Clearly, $g(n) \ll n^\rho$ for all $n$. In 1936, Hille [7] proved that for any $\epsilon > 0$, there exist infinitely many $n$ for which $g(n) > n^{\rho-\epsilon}$. Multiple people [2, 3, 12] refined Hille’s bound. The best known result on the maximal order of $g(n)$ comes from Deléglise, Hernane, and Nicolas [2, Théorème 3], namely that there exist positive constants $C_1$ and $C_2$ such that

$$x^\rho \exp\left(-C_1 \frac{(\log x)^{1/\rho}}{\log_2 x}\right) \leq \max_{n \leq x} g(n) \leq x^\rho \exp\left(-C_2 \frac{(\log x)^{1/\rho}}{\log_2 x}\right)$$

for sufficiently large $x$. (The authors conjecture that there exists a positive constant $C$ for which

$$\max_{n \leq x} g(n) = x^\rho \exp\left(-(C + o(1)) \frac{(\log x)^{1/\rho}}{\log_2 x}\right).$$

For such a value of $C$, we would have $C_2 \leq C \leq C_1$.)

From here on, all instances of $C_1$ and $C_2$ refer to any pair of constants satisfying

$$x^\rho \exp\left(-C_1 \frac{(\log x)^{1/\rho}}{\log_2 x}\right) \leq \max_{n \leq x} g(n) \leq x^\rho \exp\left(-C_2 \frac{(\log x)^{1/\rho}}{\log_2 x}\right)$$

for sufficiently large $x$. In particular, they have the same values in Theorems 1.2 and 1.3. In Section 8, we introduce $C_3$ and $C_4$, which also have fixed values. If a result refers to a constant $C$, the value of $C$ is specific to that result. Beginning in the next section, we introduce a series of constants $c_1, c_2, \ldots$. The only constraint on a given $c_i$ is that it be large with respect to $c_{i-1}$ and $\beta$. Note that the $c_i$’s are only relevant when $\beta \leq 1/\rho$.

Throughout this paper, $o(1)$ means that a function goes to 0 as $x \to \infty$, at a rate depending on all other parameters. The rate at which this occurs is dependent upon $\beta$ unless otherwise stated. The constant multiple implied by the $\ll$ symbol also depends on $\beta$.

It is easy to bound the negative moments of $g$. If $\beta \geq 0$, then

$$\sum_{n \leq x} g(n)^{-\beta} = x^{1+o(1)}.$$

The sum is at most $x$ because $g(n)^{-\beta} \leq 1$ for all $n$ and $\gg x/\log x$ because $g(p)^{-\beta} = 1$ for all prime $p$. In fact, Just and the author [10] recently proved that

$$\sum_{n \leq x} g(n)^{-\beta}, \sum_{n \leq x} \tilde{g}(n)^{-\beta}, \sum_{n \leq x} G(n)^{-\beta}$$
are all
\[
\frac{x}{\log x} \exp \left((1 + o(1))(1 + \beta)(\log 2)^{\beta/(1+\beta)}(\log_2 x)^{1/(1+\beta)}\right),
\]
where \(\tilde{g}(n)\) (resp. \(G(n)\)) is the number of ordered factorizations of \(n\) into coprime (resp. distinct) parts larger than 1. In addition, we bounded the positive moments of \(\tilde{g}\) [10, Theorem 1.8]. If \(\beta \in (0, 1)\), then
\[
\sum_{n \leq x} \tilde{g}(n)^\beta = x \exp \left((1 + o(1)) - \frac{1 - \beta}{(\log 2)^{\beta/(1-\beta)}}(\log_2 x)^{1/(1-\beta)}\right).
\]
(For the corresponding sum with \(\beta \geq 1\), see [10, Theorems 1.2, 1.7].) Using a similar proof, we obtain a lower bound for the corresponding sum of \(g(n)^\beta\) which is larger than the bound we obtain from the sum of \(\tilde{g}(n)^\beta\). We also bound this quantity from above.

**Theorem 1.1.** If \(\beta \in (0, 1/\rho)\), then
\[
\sum_{n \leq x} g(n)^\beta \geq x \exp((C_g + o(1))(\log x)^{1/(1-\beta)}),
\]
with
\[
C_g = \frac{1 - \beta}{(\log 2)^{\beta/(1-\beta)}} \exp \left(\frac{\beta}{(\log 2)(1 - \beta)} \sum_p \frac{1}{e^{p^{1/\beta}} - 1}\right).
\]
In addition,
\[
\sum_{n \leq x} g(n)^\beta = x \exp((\log x)^{o(1)}).
\]

For the larger moments of \(g\), we obtain notably larger bounds. In particular, there is a significant increase at \(\beta = 1/\rho\). For all \(\beta < 1/\rho\), the exponent of \(\log x\) in the exponent is 0. However, at \(\beta = 1/\rho\), the exponent increases to \(1/\rho\).

**Theorem 1.2.** If \(\beta \in [1/\rho, 1)\), then
\[
x^{\rho \beta} \exp \left(C_2(1 - \beta)\frac{(\log x)^{1/\rho}}{\log_2 x}\right) \leq \sum_{n \leq x} g(n)^\beta
\leq x^{\rho \beta} \exp \left((1 + o(1))2\left(\frac{2}{\log 2}\right)^{1/\rho} (\log x)^{1/\rho}\log_2 x\right)
\]
for sufficiently large \(x\).

**Theorem 1.3.** If \(\beta > 1\), then
\[
x^{\rho \beta} \exp \left(-C_1\beta\frac{(\log x)^{1/\rho}}{\log_2 x}\right) \ll \sum_{n \leq x} g(n)^\beta \ll x^{\rho \beta} \exp \left(-C_2(\beta - 1)\frac{(\log x)^{1/\rho}}{\log_2 x}\right).
\]
Asymptotics for the moments and maximal order of the unordered factorization function are already known [1, 10, 16].

We also show that the $\beta = 1/\rho$ case of Theorem 1.2 implies the following result about the distribution of large values of $g(n)$.

**Theorem 1.4.** Fix $\epsilon > 0$. As $x \to \infty$, we have
\[
\# \{ n \leq x : g(n) \geq x^\alpha \} = x^{1-(\alpha/\rho)+o(1)}
\]
uniformly for all $\alpha \in [0, \rho - \epsilon]$.

Let $G(n)$ and $gp(n)$ be the number of ordered factorizations of $n$ into distinct parts greater than 1 and prime parts, respectively. As with $g(n)$, asymptotic formulas for the sum and negative moments for these functions are already known [5, 10, 14]. We find lower bounds for the positive moments of these functions using techniques similar to the ones we used for $g(n)$.

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**2. Preliminary results**

Let $c_1$ be a large constant. For a given number $n$, let $A$ and $B$ be the $(c_1(\log x)^\beta)$-smooth and $(c_1(\log x)^\beta)$-rough parts of $n$, respectively. In other words, $n = AB$, where every prime factor of $A$ is at most $c_1(\log x)^\beta$ and every prime factor of $B$ is greater than $c_1(\log x)^\beta$. We may write
\[
\sum_{n \leq x} g(n)^\beta = \sum_{A \leq x}^{A \text{ (c}_1(\log x)^\beta\text{-smooth)}} \sum_{B \leq x/A}^{B \text{ (c}_1(\log x)^\beta\text{-rough)}} g(AB)^\beta.
\]

Let $\Omega(n)$ be the number of (not necessarily distinct) prime factors of $n$. For a given $M$, let $\Omega_{> M}(n)$ be the number of prime factors of $n$ which are $> M$. Before proving our main theorems, we must write a few results.

**Lemma 2.1** ([12, Lemma 2.5]). For any two integers $n_1$ and $n_2$, we have
\[
g(n_1n_2) \leq g(n_1) \cdot (2\Omega(n_1n_2))^{\Omega(n_2)}.
\]

Because $A \leq x$, we have $\Omega(A) \leq (\log A)/(\log 2)$. Because $B \leq x$ is $(c_1(\log x)^\beta)$-rough, we have
\[
\Omega(B) \leq \frac{\log B}{\log(c_1(\log x)^\beta)} \leq \frac{1}{\beta \log_2 x} \log x.
\]

**Corollary 2.2.** For all $n \leq x$, we have
\[
g(n) \leq g(A) \cdot \left( \frac{2}{\log 2} \log x \right)^{\Omega(B)}.
\]
In the proof of [15, Lemma 8], Pollack proves the following result, but does not explicitly state it.

**Lemma 2.3.** For all \( y \leq x \), we have
\[
\sum_{n \leq T} y^{\Omega > 2y(n)} \leq T \exp(2y \log T),
\]
uniformly for \( T \in [1, x] \).

We close this section with a theorem about the distribution of smooth numbers. Let \( \Psi(x, y) \) be the number of \( y \)-smooth numbers up to \( x \).

**Theorem 2.4** ([18, Theorem III.5.2]). Fix \( x \geq y \geq 2 \). We have
\[
\Psi(x, y) = \exp\left(\left(1 + O\left(\frac{1}{\log x} + \frac{1}{\log y}\right)\right) Z\right),
\]
with
\[
Z = \frac{\log x}{\log y} \log \left(1 + \frac{y}{\log x}\right) + \frac{y}{\log y} \log \left(1 + \frac{\log x}{y}\right).
\]

3. Large values of \( \beta \)

We establish precise bounds on the \((1/\rho)\)-th moment of \( g(n) \), which we then use to obtain bounds on the \( \beta \)-th moment of \( g \) for all \( \beta > 1/\rho \).

**Theorem 3.1.** We have
\[
\sum_{n \leq x} g(n)^{1/\rho} \leq x \exp\left((1 + o(1))2 \left(\frac{2}{\log 2}\right)^{1/\rho} (\log x)^{1/\rho \log 2 x}\right).
\]

**Proof.** We rewrite \( g(n) \) as \( g(AB) \) and apply Corollary 2.2:
\[
\sum_{n \leq x} g(n)^{1/\rho} = \sum_{A \leq x} \sum_{(c_1(\log x)^{1/\rho})\text{-smooth}} g(AB)^{1/\rho}
\]
\[
\leq \sum_{A \leq x} g(A)^{1/\rho} \sum_{B \leq x/A} g(B^{(1/\rho)\Omega(B)})
\]
By definition, \( \Omega(B) = \Omega > c_1(\log x)^{1/\rho}(n) \). Lemma 2.3 gives us
\[
\sum_{B \leq x/A} g(B^{(1/\rho)\Omega > c_1(\log x)^{1/\rho}(n)}) 
\]
\[
\leq \frac{x}{A} \exp\left(2 \left(\frac{2}{\log 2}\right)^{1/\rho} (\log x)^{1/\rho \log 2 x}\right),
\]
which implies that
\[
\sum_{n \leq x} g(n)^{1/\rho} \leq x \exp \left( 2 \left( \frac{2}{\log 2} \right)^{1/\rho} \left( \log x \right)^{1/\rho} \log_2 x \right) \sum_{A \leq x} \frac{g(A)^{1/\rho}}{A}. 
\]

Because \( g(A) \ll A^{\rho} \), we have \( g(A)^{1/\rho}/A \ll 1 \). Hence,
\[
\sum_{n \leq x} g(n)^{1/\rho} \leq x \exp \left( 2 \left( \frac{2}{\log 2} \right)^{1/\rho} \left( \log x \right)^{1/\rho} \log_2 x \right) \Psi(x, c_1(\log x)^{1/\rho}).
\]
By Theorem 2.4,
\[
\Psi(x, c_1(\log x)^{1/\rho}) = \exp(O((\log x)^{1/\rho})),
\]
which implies that
\[
\sum_{n \leq x} g(n)^{1/\rho} \leq x \exp \left( (1 + o(1))2 \left( \frac{2}{\log 2} \right)^{1/\rho} \left( \log x \right)^{1/\rho} \log_2 x \right). \quad \square
\]

While the following corollary applies to all \( \beta > 1/\rho \), it is only useful when \( \beta \leq 1 \) as well. Theorem 3.4 supersedes this result when \( \beta > 1 \).

**Corollary 3.2.** If \( \beta \geq 1/\rho \), then
\[
\sum_{n \leq x} g(n)^{\beta} \leq x^{\rho \beta} \exp \left( 1 + o(1) \right) \left( \frac{2}{\log 2} \right)^{1/\rho} \left( \log x \right)^{1/\rho} \log_2 x.
\]

**Proof.** We have
\[
\begin{align*}
\sum_{n \leq x} g(n)^{\beta} &\leq \left( \max_{n \leq x} g(n) \right)^{\beta - (1/\rho)} \sum_{n \leq x} g(n)^{1/\rho} \\
&\leq \left( x^{\rho} \exp \left( -C_2 \frac{\left( \log x \right)^{1/\rho}}{\log_2 x} \right) \right)^{\beta - (1/\rho)} \\
&\quad \cdot x \exp \left( 1 + o(1) \right) \left( \frac{2}{\log 2} \right)^{1/\rho} \left( \log x \right)^{1/\rho} \log_2 x \\
&= x^{\rho \beta} \exp \left( 1 + o(1) \right) \left( \frac{2}{\log 2} \right)^{1/\rho} \left( \log x \right)^{1/\rho} \log_2 x. \quad \square
\end{align*}
\]

We close this section with a few short proofs of our remaining bounds.
Theorem 3.3. For $x$ sufficiently large, we have
\[ \sum_{n \leq x} g(n)^\beta \geq x^{\rho \beta} \exp \left( C_2 (1 - \beta) \frac{(\log x)^{1/\rho}}{\log_2 x} \right) \]
for all $\beta < 1$.

Proof. We have
\[ \sum_{n \leq x} g(n) \leq \left( \max_{n \leq x} g(n) \right)^{1-\beta} \sum_{n \leq x} g(n)^\beta. \]
Therefore,
\[ \sum_{n \leq x} g(n)^\beta \geq \left( \max_{n \leq x} g(n) \right)^{-(1-\beta)} \sum_{n \leq x} g(n) \geq x^{\rho \beta} \exp \left( C_2 (1 - \beta) \frac{(\log x)^{1/\rho}}{\log_2 x} \right). \]

Though this theorem applies to all $\beta \leq 1$, it is only useful when $\beta \geq 1/\rho$, as we already know that the sum is at least $\lceil x \rceil$.

Theorem 3.4. For $x$ sufficiently large, we have
\[ x^{\rho \beta} \exp \left( -C_1 \beta \frac{(\log x)^{1/\rho}}{\log_2 x} \right) \leq \sum_{n \leq x} g(n)^\beta \leq x^{\rho \beta} \exp \left( -C_2 (\beta - 1) \frac{(\log x)^{1/\rho}}{\log_2 x} \right) \]
for all $\beta > 1$.

Proof. For the lower bound, we have
\[ \sum_{n \leq x} g(n)^\beta \geq \left( \max_{n \leq x} g(n) \right)^{\beta} \geq x^{\rho \beta} \exp \left( -C_1 \beta \frac{(\log x)^{1/\rho}}{\log_2 x} \right). \]
In addition,
\[ \sum_{n \leq x} g(n)^\beta \leq \left( \max_{n \leq x} g(n) \right)^{\beta-1} \sum_{n \leq x} g(n) \leq x^{\rho \beta} \exp \left( -C_2 (\beta - 1) \frac{(\log x)^{1/\rho}}{\log_2 x} \right) \]
gives us the upper bound.

From Theorem 3.1, we obtain Theorem 1.4.

Theorem 3.5. Fix $\epsilon > 0$. As $x \to \infty$,
\[ \# \{ n \leq x : g(n) \geq x^\alpha \} = x^{1-(\alpha/\rho)+o(1)} \]
uniformly for all $\alpha \in [0, \rho - \epsilon]$. 
Proof. For a given $\alpha$, define

$$S_\alpha = \{n \leq x : g(n) \geq x^\alpha\}.$$ 

By definition,

$$\sum_{n \in S_\alpha} g(n)^{1/\rho} \geq \sum_{n \in S_\alpha} x^{\alpha/\rho} = x^{\alpha/\rho} \cdot \#S_\alpha.$$ 

From Theorem 3.1 we obtain

$$\sum_{n \in S_\alpha} g(n)^{1/\rho} \leq \sum_{n \leq x} g(n)^{1/\rho} = x \exp \left( (1 + o(1))2 \left( \frac{2}{\log 2} \right)^{1/\rho} (\log x)^{1/\rho} \log_2 x \right).$$ 

Putting these inequalities together gives us

$$\#S_\alpha \leq x^{1-(\alpha/\rho)} \exp \left( (1 + o(1))2 \left( \frac{2}{\log 2} \right)^{1/\rho} (\log x)^{1/\rho} \log_2 x \right) = x^{1-(\alpha/\rho) + o(1)}.$$ 

Fix $\delta > 0$. There exists some $m \leq x^{(1+\delta)\alpha/\rho}$ with the property that

$$g(m) > \left( x^{(1+\delta)\alpha/\rho} \right)^{\rho/(1+\delta)} = x^\alpha.$$ 

Therefore,

$$\#S_\alpha \geq \#\{n \leq x : m|n\} \sim x/m \geq x^{1-(1+\delta)(\alpha/\rho)}.$$ 

Taking the limit as $\delta \to 0$ shows that

$$\#S_\alpha \geq x^{1-(\alpha/\rho) + o(1)},$$

completing our proof. \qed

4. Small values of $\beta$

Using the $(1/\rho)$-th moment of $g(n)$ and the results from Section 2, we obtain the following upper bound for the small positive moments of $g(n)$. (For every result in the next two sections, we let $\beta \in (0, 1/\rho).$)

Theorem 4.1. For all $\beta$, we have

$$\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{\beta^2 + o(1)}).$$

In the next section, we prove the following theorem.

Theorem 4.2. If there exists a constant $C > 1$ such that

$$\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{\beta C + o(1)}).$$
uniformly for all $\beta$, then

$$\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{\beta C + 1 + o(1)})$$

uniformly for all $\beta$ as well.

Applying this result arbitrarily many times allows us to obtain the upper bound in Theorem 1.1, which we rewrite here.

**Theorem 4.3.** We have

$$\sum_{n \leq x} g(n)^\beta = x \exp((\log x)^o(1)).$$

Before doing any of this, we write a few lemmas. We first show that we may assume that $g(n)$ is small. Afterwards, we prove that we may assume that $A$ and $\Omega(B)$ are small as well.

**Lemma 4.4.** For all $\beta$, we have

$$\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^o(1))$$

for some positive constant $c_2$.

**Proof.** Fix a large number $M$. We consider

$$\sum_{k > M} \sum_{\substack{n \leq x \\ g(n) > \exp(c_2 \log x)^{1/\rho \log_2 x}}} g(n)^\beta.$$

We then show that for any $k$, the inner sum is sufficiently small. Note that the number of $k$ for which $g(n) < e^{k+1}$ for some $n \leq x$ is on the order of $\log x$. We have

$$\sum_{\substack{n \leq x \\ e^k \leq g(n) < e^{k+1}}} g(n)^\beta \ll e^{\beta k} \# \{n \leq x : g(n) \geq e^k\}.$$

From the proof of Theorem 3.5, we see that

$$\# \{n \leq x : g(n) \geq e^k\}$$

$$\leq xe^{-k/\rho} \exp \left( (1 + o(1))2 \left( \frac{2}{\log 2} \right)^{1/\rho} (\log x)^{1/\rho \log_2 x} \right),$$
which implies that
\[
\sum_{n \leq x} g(n)^\beta 
\leq x \exp\left(-k\left(\frac{1}{\rho} - \beta\right) + (1 + o(1))2\left(\frac{2}{\log 2}\right)^{1/\rho} (\log x)^{1/\rho} \log_2 x\right).
\]
If
\[
k > \left(\frac{2\rho}{1 - \rho \beta}\left(\frac{2}{\log 2}\right)^{1/\rho} + \epsilon\right) (\log x)^{1/\rho} \log_2 x
\]
for some \(\epsilon > 0\), then the upper bound is \(O(x)\).

From here on, we assume that
\[
g(n) \leq \exp(c_2(\log x)^{1/\rho} \log_2 x).
\]
From [12, Lemma 2.6], we have
\[
g(n) \gg 2^{\Omega(n)},
\]
allowing us to assume that
\[
\Omega(n) \leq (\log x)^{(1/\rho) + o(1)}.
\]

**Lemma 4.5.** We have
\[
\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{(\beta/\rho) + o(1)}) \sum_{A \leq x} \frac{g(A)^\beta}{A}.\]

**Proof.** Recall that
\[
\sum_{n \leq x} g(n)^\beta = \sum_{A \leq x} \sum_{B \leq x/A} g(AB)^\beta.
\]
By Lemma 2.1,
\[
g(AB) \leq g(A) \cdot (2\Omega(n))^{\Omega(B)} \leq g(A) \cdot ((\log x)^{(1/\rho) + o(1)})^{\Omega_{>c_1(\log x)^\beta}(n)}.
\]
Therefore,
\[
\sum_{n \leq x} g(n)^\beta 
\leq \sum_{A \leq x} g(A)^\beta \sum_{B \leq x/A} ((\log x)^{(\beta/\rho) + o(1)})^{\Omega_{>c_1(\log x)^\beta}(n)}.
\]
By Lemma 2.3, the final sum in this expression is at most
\[
\frac{x}{A} \exp((\log x)^{(\beta/\rho) + o(1)}).
\]
We now have
\[ \sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{(\beta/\rho)+o(1)}) \sum_{A \leq x} \frac{g(A)^\beta}{A} \text{ (} A (c_1 (\log x)^\beta)\text{-smooth)} \]
\[ \sum_{A \leq x} \frac{g(A)^\beta}{A} \leq \frac{\exp((\log x)^{(\beta/\rho)+o(1)})}{\exp(c_1 (\log x)^\beta)} \text{ (} A (c_1 (\log x)^\beta)\text{-smooth)} \]
\[ \sum_{n \leq x} \frac{g(n)^\beta}{A} \leq \frac{\exp((\log x)^{(\beta/\rho)+o(1)})}{\exp(c_1 (\log x)^\beta)} \text{ (} A (c_1 (\log x)^\beta)\text{-smooth)} \]

This result allows us to bound \( A \) and \( \Omega(B) \).

**Lemma 4.6.** For a sufficiently large constant \( c_3 \),
\[ \sum_{n \leq x} g(n)^\beta = o(x). \]

*Proof.* Fix a large number \( M \). By the previous result, we have
\[ \sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{(\beta/\rho)+o(1)}) \sum_{A \leq x} \frac{g(A)^\beta}{A}. \]
Note that \( g(A)^\beta \ll A^{\rho \beta} \), which implies that
\[ \sum_{M < A \leq x} \frac{g(A)^\beta}{A} \ll M^{-(1-\rho \beta)} \Psi(x, c_1 (\log x)^\beta). \]
By Theorem 2.4,
\[ \Psi(x, c_1 (\log x)^\beta) = \exp\left( (1 + o(1)) \frac{c_1(1 - \beta)}{\beta}(\log x)^\beta \right). \]
If
\[ M > \exp\left( \left( \frac{c_1(1 - \beta)}{\beta(1 - \rho \beta)} + \epsilon \right)(\log x)^\beta \right) \]
for some \( \epsilon > 0 \), then
\[ M^{-(1-\rho \beta)} \Psi(x, c_1 (\log x)^\beta) \leq \exp(-(\epsilon + o(1))(\log x)^\beta), \]
which implies that
\[ \sum_{n \leq x} g(n)^\beta = o(x). \square \]

**Lemma 4.7.** For all \( \epsilon > 0 \), we have
\[ \sum_{n \leq x} g(n)^\beta = o(x). \]

*Proof.* Once again, let \( M \) be a large number. By the previous theorem, we may assume that \( A \leq \exp(c_3 (\log x)^\beta) \). We have
\[ \sum_{n \leq x} g(n)^\beta \leq \sum_{A \leq \exp(c_3 (\log x)^\beta)} g(A)^\beta \sum_{B \leq x/A} \frac{(2\Omega(n))^{\beta \Omega(B)}}{\Omega(B) > M}. \]
By assumption, \( \Omega(n) \leq (\log x)^{(1/\rho)+o(1)} \).

By definition, \( \Omega(B) = \Omega_{c_1(\log x)^{\beta}}(B) \). In addition, multiplying each term by 
\[ \exp(\beta \Omega_{c_1(\log x)^{\beta}}(B) - \beta M) \]

increases the sum. Hence,
\[
\sum_{B \leq x/A \atop \Omega(B) > M} (2\Omega(n))^\beta \Omega(B) \leq \sum_{B \leq x/A} ((\log x)^{(1/\rho)+o(1)})^\beta \Omega_{c_1(\log x)^{\beta}}(n) \cdot \exp(\beta \Omega_{c_1(\log x)^{\beta}}(B) - \beta M) = \exp(-\beta M) \sum_{B \leq x/A} ((\log x)^{(1/\rho)+o(1)})^\beta \Omega_{c_1(\log x)^{\beta}}(B) \leq \frac{x}{A} \exp((\log x)^{(\beta/\rho)+o(1)} - \beta M).
\]

If \( M > (\log x)^{\beta+\epsilon} \), then this sum is at most
\[
\frac{x \exp(- (\log x)^{\beta+\epsilon+o(1)})}{A}.
\]
Plugging this back into our original formula gives us
\[
\sum_{n \leq x \atop A \leq \exp(c_3(\log x)^{\beta})} g(n)^\beta \leq x \exp(- (\log x)^{\beta+\epsilon+o(1)}) \sum_{A \leq \exp(c_3(\log x)^{\beta})} \frac{g(A)^\beta}{A}.
\]

Note that the rightmost sum is \( O(\exp(c_3(\log x)^{\beta})) \) because \( g(A)^\beta / A = O(1) \). We now have
\[
\sum_{n \leq x \atop A \leq \exp(c_3(\log x)^{\beta})} g(n)^\beta \leq x \exp(- (\log x)^{\beta+\epsilon+o(1)}). \quad \square
\]

To summarize, we may now assume that \( \log A, \Omega(B) \leq (\log x)^{\beta+o(1)} \).

From these assumptions, we may improve Lemma 4.5.

**Theorem 4.8.** We have
\[
\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{\beta^2+o(1)}) \sum_{A \leq \exp((\log x)^{\beta^2+o(1)})} \frac{g(A)^\beta}{A}.
\]

**Proof.** Once again, we have
\[ g(AB) \leq g(A) \cdot (2\Omega(n))^{\Omega_{c_1(\log x)^{\beta}}(n)}. \]
In this case, we have a more precise bound for \( \Omega(n) \). Note that
\[
\Omega(A) \ll \log A \leq (\log x)^{\beta+o(1)}, \quad \Omega(B) \leq (\log x)^{\beta+o(1)}.
\]
Therefore,
\[ g(AB) \leq g(A) \cdot ((\log x)^{\beta+o(1)} \Omega_{>c_1(\log x)^{\beta}(n)}. \]

We now have
\[ \sum_{n \leq x} g(AB)^{\beta} \leq \sum_{A \leq \exp(c_2(\log x)^{\beta})} g(A)^{\beta} \sum_{B \leq x/A} ((\log x)^{\beta+o(1)} \Omega_{>c_1(\log x)^{\beta}(n)}. \]

By Lemma 2.3, the rightmost sum is at most
\[ \frac{x}{A} \exp((\log x)^{\beta^2+o(1)}). \]

Using these results, we obtain a new upper bound on the sum of \( g(n)^{\beta} \).

**Theorem 4.9.** We have
\[ \sum_{n \leq x} g(n)^{\beta} \leq x \exp((\log x)^{\beta^2+o(1)}). \]

**Proof.** Because of the previous result, it is sufficient to show that
\[ \sum_{A \leq c_3 \exp((\log x)^{\beta})} g(A)^{\beta} \frac{A}{A} = \exp((\log x)^{o(1)}). \]

We break the sum into “e-adic” intervals, based on the sizes of \( A \) and \( g(A) \):
\[ \sum_{A \leq \exp(c_3(\log x)^{\beta})} g(A)^{\beta} \frac{A}{A} \]
\[ = \sum_{k \leq c_2(\log x)^{\beta} \log_2 x} \sum_{m \leq \rho c_2(\log x)^{\beta} \log_2 x} \sum_{e^k \leq A < e^{k+1}} \sum_{e^m \leq g(A) < e^{m+1}} g(A)^{\beta} \frac{A}{A}. \]

Because the number of possible \( k \) and \( m \) is sufficiently small, we only need to show that
\[ \sum_{e^k \leq A < e^{k+1}} g(A)^{\beta} \frac{A}{A} = \exp((\log x)^{o(1)}) \]
for all possible \( k \) and \( m \).

For any \( k, m \), we have
\[ \sum_{e^k \leq A < e^{k+1}} g(A)^{\beta} \frac{A}{A} \ll \sum_{e^k \leq A < e^{k+1}} e^{m\beta-k} \]
\[ \leq e^{m\beta-k} \# \{ A < e^{k+1} : g(A) \geq e^m \}. \]

By Theorem 3.5,
\[ \# \{ A < e^{k+1} : g(A) \geq e^m \} \leq e^{k-(m/\rho)+o(k)}, \]
which implies that
\[
\sum_{e^k \leq A < e^{k+1}} \frac{g(A) \beta}{A} \leq e^{m \beta - (m/\rho) + o(k)}.
\]

Fix a constant \( \epsilon \). If \( k < m/\epsilon \), then the exponent is negative for \( x \) sufficiently large because \( \beta < 1/\rho \).

Suppose \( m \leq \epsilon k \). We have
\[
\sum_{e^k \leq A < e^{k+1}} \frac{g(A) \beta}{A} \leq \sum_{e^k \leq A < e^{k+1}} A^{-(1-\epsilon+o(1))}
\]
\[\leq e^{-(1-\epsilon+o(1))k \Psi(e^{k+1}, c_1 (\log x)^\beta)}.
\]

By assumption, \( k \leq c_3 (\log x)^\beta \). If \( k = o((\log x)^\beta) \), then
\[
\Psi(e^{k+1}, c_1 (\log x)^\beta) = \exp\left((1 + o(1)) \left(k - \frac{1}{\beta} \frac{k \log k}{\log_2 x}\right)\right)
\]
by Theorem 2.4. We now have
\[
\sum_{e^k \leq A < e^{k+1}} \frac{g(A) \beta}{A} \leq \exp\left(-(1 + o(1)) \left(\frac{1}{\beta} \frac{k \log k}{\log_2 x} - \epsilon k\right) + o(k)\right).
\]

If \( k > (\log x)^C \) for some constant \( C > \beta \epsilon \), then this quantity is \( o(1) \). Otherwise, the sum is still at most \( \exp((\log x)^{\beta + o(1)}) \leq \exp((\log x)^{(\epsilon/\rho)+o(1)}) \).

Letting \( \epsilon \) go to 0 gives us our desired result. \( \square \)

5. Improving our bound

In the previous section, we obtained an upper bound for the sum of \( g(n)^\beta \) for all \( \beta < 1/\rho \). Using this bound, we can obtain a substantially better result using the following theorem.

**Theorem 5.1.** Let \( C > 1 \) be a constant. Suppose
\[
\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{\beta C + o(1)})
\]
for all \( \beta \in (0, 1/\rho) \). Then,
\[
\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{\beta C + 1 + o(1)})
\]
for all such \( \beta \) as well.
Proof. Let $k$ be a large number. Let $S$ be the set of all $n \leq x$ satisfying $e^k \leq g(n) < e^{k+1}$. We consider the sum of $g(n)^\beta$ over all $n \in S$. Note that
\[
\sum_{n \in S} g(n)^\beta \leq e^{\beta k} \#S \leq e^{1/\rho} e^{\beta k} \#S.
\]
We bound the righthand side of the inequality above. Let $\beta_0 \in (\beta, 1/\rho)$. We have
\[
\sum_{n \in S} g(n)^{\beta_0} \geq e^{\beta_0 k} \#S.
\]
By assumption,
\[
\sum_{n \leq x} g(n)^{\beta_0} \leq x \exp((\log x)^{\beta_0^C + o(1)}).
\]
In particular, for any $\epsilon$, there exists a number $N$ such that if $x > N$, then
\[
\sum_{n \leq x} g(n)^{\beta_0} \leq x \exp((\log x)^{\beta_0^C + \epsilon}),
\]
which implies that
\[
\#S \leq xe^{-\beta_0 k} \exp((\log x)^{\beta_0^C + \epsilon})
\]
for all sufficiently large $x$. (Note that $N$ is independent of $k$.) Plugging this into our $g(n)^\beta$ sum gives us
\[
\sum_{n \in S} g(n)^\beta \leq xe^{-(\beta_0 - \beta)k} \exp((\log x)^{\beta_0^C + \epsilon}).
\]
If $k > (\log x)^{\beta_0^C + \epsilon}$, then this quantity is $o(x)$. Because the number of such $k$ is sufficiently small, we may assume that $k \leq (\log x)^{\beta_0^C + \epsilon}$. We may therefore assume that $g(n) \leq \exp((\log x)^{\beta_0^C + \epsilon + o(1)})$ and $\Omega(n) \leq (\log x)^{\beta_0^C + \epsilon + o(1)}$. If $x$ is sufficiently large, we have $\Omega(n) \leq (\log x)^{\beta_0^C + 2\epsilon}$.

Using this bound on $\Omega(n)$, we may bound the sum of $g(n)^\beta$. We have
\[
\sum_{\Omega(n) \leq (\log x)^{\beta_0^C + 2\epsilon}} g(n)^\beta \leq \sum_{\Omega(A) \leq (\log x)^{\beta_0^C + 2\epsilon}} g(A)^\beta \sum_{\Omega(B) \leq (\log x)^{\beta_0^C + 2\epsilon}} (2\Omega(n))^{\beta \Omega(B)}.
\]
Suppose $\beta_0^C + \epsilon < \beta$. Plugging in our bound on $\Omega(n)$ allows us to bound the rightmost sum:
\[
\sum_{\Omega(B) \leq (\log x)^{\beta_0^C + 2\epsilon}} (2\Omega(n))^{\beta \Omega(B)} \leq \sum_{\Omega(B) \leq (\log x)^{\beta_0^C + 2\epsilon}} ((\log x)^{\beta_0^C + 2\epsilon})^{\beta \Omega_{>e_1(\log x)^\beta}(n)} \leq \frac{x}{A} \exp((\log x)^{3\beta_0^C + 2\beta \epsilon + o(1)}).
For $x$ sufficiently large, we may bound the sum by
\[
\frac{x}{A} \exp((\log x)^{\beta_0^C + 3\epsilon}).
\]
We now have
\[
\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{\beta_0^C + 3\epsilon})
\]
\[
\sum_{A \leq x} \frac{g(A)^\beta}{A}.
\]
Our bound on $\Omega(A)$ gives us $A \leq x \exp((\log x)^{\beta_0 + \epsilon + o(1)})$. In the proof of Theorem 4.9, we showed that the sum of $g(A)^\beta / A$ over all such $A$ is $x \exp((\log x)^{o(1)})$. For $x$ sufficiently large, the sum is at most $x \exp((\log x)^{\epsilon})$. Putting everything together gives us
\[
\sum_{n \leq x} g(n)^\beta \leq x \exp((\log x)^{\beta_0^C + 4\epsilon}).
\]
Our desired result comes from the fact that we may assume that $\beta_0 - \beta$ and $\epsilon$ are both arbitrarily small.

Applying this result arbitrarily many times proves that
\[
\sum_{n \leq x} g(n)^\beta = x \exp((\log x)^{o(1)}).
\]

6. A lower bound for the small moments

Let $\tilde{g}(n)$ be the number of factorizations of $n$ into coprime parts greater than 1. Just and the author [10] recently proved that
\[
\sum_{n \leq x} \tilde{g}(n)^\beta = x \exp\left((1 + o(1)) \frac{1 - \beta}{(\log 2)^{\beta/(1 - \beta)}} (\log_2 x)^{1/(1 - \beta)} \right)
\]
for all $\beta \in (0, 1)$. Because $g(n) \geq \tilde{g}(n)$ for all $n$, this quantity is a lower bound for the sum of $g(n)^\beta$. We provide a slightly larger lower bound for this sum. Before doing so, we write a theorem that will prove useful [17, Theorem 3.1] (see [17, Theorem 4.1] for a corresponding upper bound and [6, Corollary 2] for a more precise version of this result on a smaller interval). Let $\pi(x, k)$ be the number of $n \leq x$ with exactly $k$ distinct prime factors.

**Theorem 6.1.** For
\[
\log_2 x (\log x)^2 \leq k \leq \frac{\log x}{3 \log_2 x},
\]
we have
\[
\pi(x, k) \geq \frac{x}{k! \log x} \exp\left(\left(\log L_0 + \frac{\log L_0}{L_0} + O\left(\frac{1}{L_0}\right)\right) k\right),
\]
with
\[
L_0 = \log_2 x - \log k - \log_2 k.
\]
We also impose a lower bound on the smallest prime factors of our values of \( n \). For a given number \( R \), we let \( \pi(x, k, R) \) be the number of \( R \)-rough \( n \leq x \) with exactly \( k \) distinct prime factors.

**Corollary 6.2.** Let \( x \) and \( k \) satisfy the conditions of the previous theorem and let \( R \) be a fixed positive number. As \( x \to \infty \), we have

\[
\pi(x, k, R) \geq \frac{x}{k! \log x} \exp \left( \left( \log L_0 + \frac{\log L_0}{L_0} + O \left( \frac{1}{L_0} \right) \right) k \right).
\]

**Proof.** Let \( n \leq x \) be an \( R \)-rough number with exactly \( k \) distinct prime factors. In the proof of the previous theorem, Pomerance already assumes that every prime factor of \( n \) is greater than or equal to \( \sqrt{k^2} \), which we may assume is greater than \( R \). In addition, this proof is entirely self-contained except for a reference to [17, Proposition 2.1]. However, it is straightforward to modify the proof of this result to assume that \( n \) is \( R \)-rough. \( \Box \)

Using the results of [2], we bound \( g(n)^\beta \) on a suitable set of \( n \leq x \) and multiply this bound by the size of the set.

**Definition 6.3 ([2, Définition 2.1] (see also [4, Theorem 1])).** For a tuple \( (a_1, \ldots, a_r) \), let \( c = c(a_1, \ldots, a_r) \) be the unique solution to the equation

\[
\prod_{i=1}^{r} \left( 1 + \frac{a_i}{c} \right) = 2.
\]

**Definition 6.4 ([2, Définition 3.1]).** With \( c \) defined above, we have

\[
F := F(a_1, \ldots, a_r) = \sum_{i=1}^{r} a_i \log \left( 1 + \frac{c}{a_i} \right).
\]

**Lemma 6.5 ([2, Théorème 2]).** Let \( n = p_1^{a_1} \cdots p_r^{a_r} \). We have

\[
g(n) \gg \frac{\exp(F - r)}{\sqrt{a_1 \cdots a_r}}.
\]

**Theorem 6.6.** If \( \beta \in (0, 1/\rho) \), then

\[
\sum_{n \leq x} g(n)^\beta \geq x \exp((C_g + o(1))(\log x)^{1/(1-\beta)}),
\]

with \( C_g = \frac{1 - \beta}{(\log 2)^{\beta/(1-\beta)}} \exp \left( \frac{\beta}{(\log 2)(1-\beta)} \sum_p \frac{1}{e_p^{1/\beta} - 1} \right) \).

**Proof.** Let \( k \) be a number on the order of \( (\log x)^C \) for some \( C > 1 \) and let \( (\alpha_1, \ldots, \alpha_r) \) be a tuple of positive real numbers which is independent of \( x \). Let \( S \) be the set of numbers \( \leq x \) of the form \( p_1^{\alpha_1} \cdots p_r^{\alpha_r} m \), where \( p_i \) is the \( i \)th prime and \( m \) is a \( p_r \)-rough number with exactly \( k \) distinct prime
factors. We bound \#S from below, in addition to providing a lower bound for \( g(n) \) for all \( n \in S \).

Because the \( p_i \) and \( \alpha_i \) are fixed, the number of elements of \( S \) is equal to the number of possible values of \( m \). By assumption,

\[
m \leq \frac{x}{p_1^{\alpha_1 k} \cdots p_r^{\alpha_r k}}.
\]

Therefore,

\[
\#S = \pi \left( \frac{x}{p_1^{\alpha_1 k} \cdots p_r^{\alpha_r k}, k, p_r} \right) \\
\geq x \exp \left( k \log_3 x - k \log k + \left( 1 - \sum_{i=1}^{r} (\log p_i) \alpha_i \right) + o(1) \right) k.
\]

At this point, we bound \( g(n) \). Because we only need a lower bound, we assume that \( m \) is squarefree. In our case, we have

\[
\left( 1 + \frac{1}{c} \right)^k \prod_{i=1}^{r} \left( 1 + \frac{\alpha_i k}{c} \right) = 2.
\]

Though we cannot determine \( c \) exactly, we can still obtain a suitable lower bound. Because

\[
\left( 1 + \frac{1}{c} \right)^k \leq 2,
\]

we have

\[
c \geq \frac{1}{2^{1/k} - 1} \sim \frac{k}{\log 2},
\]

giving us

\[
F = k \log(1 + c) + \sum_{i=1}^{r} \alpha_i k \log \left( 1 + \frac{c}{\alpha_i k} \right) \\
\geq k \log k + \left( \sum_{i=1}^{r} \alpha_i \log \left( 1 + \frac{1}{(\log 2) \alpha_i} \right) \right) - \log_2 2 + o(1) k.
\]

Note that \( (\alpha_1 k) \cdots (\alpha_r k) = \exp(o(k)) \). Hence,

\[
g(n) \geq \exp \left( k \log k + \left( \sum_{i=1}^{r} \alpha_i \log \left( 1 + \frac{1}{(\log 2) \alpha_i} \right) \right) - 1 - \log_2 2 + o(1) \right) k
\]

for all \( n \in S \).
We combine our estimates in order to bound the sum:

$$\sum_{n \leq x} g(n)^\beta \geq \sum_{n \in S} g(n)^\beta \geq \left( \min_{n \in S} g(n) \right)^\beta \#S \geq x \exp(k \log_3 x - (1 - \beta)k \log k + (M + o(1))k),$$

with

$$M = 1 - (1 + \log_2 2)\beta + \sum_{i=1}^r \alpha_i \left( \beta \log \left( 1 + \frac{1}{(\log 2)\alpha_i} \right) - \log p_i \right).$$

At this point, we select the $\alpha_i$’s in order to maximize $M$. For all $i$, we have

$$\frac{\partial M}{\partial \alpha_i} = \beta \log \left( 1 + \frac{1}{(\log 2)\alpha_i} \right) - \log p_i - \frac{\beta}{1 + (\log 2)\alpha_i} = 0.$$

Because we cannot write $\alpha_i$ in terms of $p_i$ nicely, we instead solve a similar equation and plug our result into our formula for $M$. While this result is not optimal, it still provides a lower bound. As $i \to \infty$, $\alpha_i \to 0$. Setting $\alpha_i$ to 0 in the final term gives us

$$\beta \log \left( 1 + \frac{1}{(\log 2)\alpha_i} \right) - \log p_i - \beta = 0,$$

which implies that

$$\alpha_i = \frac{1}{(\log 2)(e^{p_i^{1/\beta}} - 1)}.$$

(Technically, $\alpha_i{k}$ must be an integer, but rounding $\alpha_i{k}$ down does not change the final result.) Hence,

$$M = 1 - (1 + \log_2 2)\beta + \frac{\beta}{\log 2} \sum_{i=1}^r \frac{1}{e^{p_i^{1/\beta}} - 1}.$$

Letting $i$ go to $\infty$ gives us

$$M = 1 - (1 + \log_2 2)\beta + \frac{\beta}{\log 2} \sum_p \frac{1}{e^{p^{1/\beta}} - 1}.$$

In order to finish the proof, we choose $k$ to maximize the sum of $g(n)^\beta$. Recall that

$$\sum_{n \leq x} g(n)^\beta \geq x \exp(k \log_3 x - (1 - \beta)k \log k + (M + o(1))k).$$

If $k > (\log_2 x)^{(1/(1-\beta)) + \epsilon}$ for some $\epsilon > 0$, then our bound is $o(x)$. If $k < (\log_2 x)^{(1/(1-\beta)) - \epsilon}$, then the bound is $x \exp((\log_2 x)^{(1/(1-\beta)) - \epsilon + o(1)})$. Let $k =$
\[ R(\log_2 x)^{1/(1-\beta)} \] for some \( R = (\log_2 x)^{o(1)} \). We have

\[
\sum_{n \leq x} g(n)^\beta \geq x \exp(R(M - (1 - \beta) \log R + o(1))(\log_2 x)^{1/(1-\beta)}).
\]

The optimal value of \( R \) is the solution to the equation

\[
\frac{d}{dR}(R(M - (1 - \beta) \log R)) = M - (1 - \beta) \log R - (1 - \beta) = 0,
\]

namely

\[
R = \exp\left(\frac{M}{1 - \beta} - 1\right),
\]

which implies that

\[
\sum_{n \leq x} g(n)^\beta \geq x \exp((1 + o(1))(1 - \beta)R(\log_2 x)^{1/(1-\beta)}).
\]

We have

\[
R = \frac{1}{(\log 2)^{\beta/(1-\beta)}} \exp\left(\frac{\beta}{(\log 2)(1 - \beta)} \sum_p \frac{1}{ep^{1/\beta} - 1}\right),
\]

completing the proof.

The lower bound for the sum of \( \tilde{g}(n)^\beta \) is the result one obtains by letting \((\alpha_1, \ldots, \alpha_r)\) be the empty tuple.

7. Factorizations into distinct parts

Let \( G(n) \) be the number of ordered factorizations of \( n \) into distinct parts greater than 1. Warlimont [19] showed that

\[
\sum_{n \leq x} G(n) = x \cdot L(x)^{O(1)},
\]

where

\[
L(x) = \exp\left(\frac{\log x \log_3 x}{\log_2 x}\right).
\]

The author and Pollack [14] recently improved this result, showing that

\[
\sum_{n \leq x} G(n) = x \cdot L(x)^{1+o(1)}.
\]

In addition, we proved that for any \( \epsilon > 0 \), there exist infinitely many \( n \) for which

\[
G(n) > n \cdot L(n)^{1-\epsilon}.
\]

A slight modification of the proof shows that

\[
\max_{n \leq x} G(n) = x \cdot L(x)^{1+o(1)}.
\]
From these bounds, we can obtain a formula for the $\beta$-th moments of $G$ for all $\beta > 1$. We have

$$
\left( \max_{n \leq x} G(n) \right)^\beta \leq \sum_{n \leq x} G(n)^\beta \leq \left( \sum_{n \leq x} G(n) \right)^\beta,
$$

which implies that

$$
\sum_{n \leq x} G(n)^\beta = x^\beta \cdot L(x)^{\beta+o(1)}.
$$

Just and the author [10] also showed that the negative moments of $G$ have the same formula as the negative moments of $g$, up to a negligible error. If $\beta \geq 0$, then

$$
\sum_{n \leq x} G(n)^{-\beta} = \frac{x}{\log x} \exp((1 + o(1))(1 + \beta)(\log 2)^{\beta/(1+\beta)}(\log x)^{1/(1+\beta)}).
$$

All that remains is to estimate the small positive moments of $G$. We do not provide an upper bound, but we can prove a lower bound using an argument similar to the proof of Theorem 6.6. Because we do not have an asymptotic formula for $G(n)$, we use a combinatorial argument.

Once again, let $S$ be the set of $n \leq x$ of the form $p_1^{\alpha_1} \cdots p_r^{\alpha_r} m$, where $p_i$ is the $i$th prime, $m$ is a $p_r$-rough number with exactly $k$ distinct prime factors, and $k$ is on the order of $(\log x)^{1/(1-\beta)}$. In the previous section, we established that

$$
\#S \geq x \exp \left( k \log_3 x - k \log k + \left( 1 - \sum_{i=1}^r (\log p_i) \alpha_i \right) + o(1) \right) k.
$$

We now bound $G(n)$ for all $n \in S$. First, we write $m$ as a product of exactly $k$ coprime numbers greater than 1, which we can do in $k!$ ways. Then, for each $i$, we write $p_i^{\alpha_i} m$ as a product of exactly $k$ numbers (not necessarily greater than 1). For each $i$, we can do this in

$$
\binom{(1 + \alpha_i)k - 1}{k}
$$

ways. We then combine our factorizations into one $k$-term product. The terms are distinct because they have distinct $p_r$-rough parts. Hence,

$$
G(n) \geq k! \prod_{i=1}^r \binom{(1 + \alpha_i)k - 1}{k} \\
= \exp(k \log k \\
+ \left( \left( \sum_{i=1}^r (1 + \alpha_i) \log(1 + \alpha_i) - \alpha_i \log \alpha_i \right) - 1 + o(1) \right) k).
$$
Repeating the argument from the previous section gives us
\[
\sum_{n \leq x} G(n)^\beta
\geq x \exp\left( (1 + o(1))(1 - \beta) \left( \prod_p \left( 1 + \frac{1}{p^{1/\beta} - 1} \right)^{\beta/(1 - \beta)} \right) (\log_2 x)^{1/(1 - \beta)} \right)
\]
for all $\beta \in (0, 1)$.

For all $n$, $G(n) \geq \tilde{g}(n)$. Our result is an improvement over the sum of $\tilde{g}(n)^\beta$ when
\[
\prod_p \left( 1 + \frac{1}{p^{1/\beta} - 1} \right) > \frac{1}{\log 2},
\]
which occurs when $\beta > 0.438$.

8. Factorizations into prime parts

Let $g_P(n)$ be the number of factorizations of $n$ into prime parts. Hernane and Nicolas [5] note that a result from [13] implies that
\[
\sum_{n \leq x} g_P(n) \sim -\frac{1}{\lambda \zeta_P'(\lambda)} x^\lambda,
\]
where $\zeta_P$ is the Riemann zeta function restricted to prime terms and $\lambda \approx 1.40$ is the unique solution in $(1, \infty)$ to $\zeta_P(\lambda) = 2$. They also showed that there exist positive constants $C_3$ and $C_4$ such that
\[
x^\lambda \exp\left( -C_3 \frac{(\log x)^\lambda}{\log_2 x} \right) \leq \max_{n \leq x} g_P(n) \leq x^\lambda \exp\left( -C_4 \frac{(\log x)^\lambda}{\log_2 x} \right),
\]
for all sufficiently large $x$. An argument similar to the proof of Theorem 3.4 shows that if $\beta \geq 1$, then
\[
x^{\lambda \beta} \exp\left( -C_3 \beta \frac{(\log x)^{1/\beta}}{\log_2 x} \right) \leq \sum_{n \leq x} g_P(n)^\beta
\leq x^{\lambda \beta} \exp\left( -C_4 (\beta - 1) \frac{(\log x)^{1/\beta}}{\log_2 x} \right)
\]
for all sufficiently large $x$ as well.

Recall that Lemma 2.1 states that for any $n_1, n_2 \in \mathbb{Z}_+$, we have
\[
g(n_1n_2) \leq g(n_1) \cdot (2\Omega(n_1n_2))^{\Omega(n_2)}.
\]
It is straightforward to modify Klazar and Luca’s proof of this result to apply to $g_P$.

Lemma 8.1. For any two integers $n_1$ and $n_2$, we have
\[
g_P(n_1n_2) \leq g_P(n_1) \cdot (2\Omega(n_1n_2))^{\Omega(n_2)}.
\]
Ordered factorizations

From this result, we obtain variants of Corollary 3.2 and Theorem 3.3. If $\beta \in [1/\lambda, 1)$, then

$$x^{\lambda \beta} \exp \left(C_4(1 - \beta) \left(\frac{\log x}{\log_2 x}\right)^{1/\lambda}\right) \leq \sum_{n \leq x} g_P(n)^\beta$$

$$\leq x^{\lambda \beta} \exp \left((1 + o(1)) \left(\frac{2}{\log 2}\right)^{1/\lambda} \left(\log x\right)^{1/\lambda \log_2 x}\right)$$

for all sufficiently large $x$. Applying the lemma and repeating the techniques of Sections 4 and 5 shows that if $\beta \in (0, 1/\lambda)$, then

$$\sum_{n \leq x} g_P(n)^\beta = x \exp((\log x)^o(1)),$$

Finally, we note that for any tuple $(a_1, \ldots, a_r)$, we have

$$g_P(p_1^{a_1} \cdots p_r^{a_r}) = \left(a_1 + \cdots + a_r\right)_{a_1, \ldots, a_r}.$$  

Using this result, we obtain a lower bound for the small moments of $g_P$. If $\beta < 1/\lambda$, then

$$\sum_{n \leq x} g_P(n)^\beta$$

$$\geq x \exp \left((1 + o(1))(1 - \beta) \left(1 - \sum_p \frac{1}{p^{1/\beta}}\right)^{-\beta/(1-\beta)} \left(\log_2 x\right)^{1/(1-\beta)}\right).$$

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