Pulsar timing array observations of gravitational wave source timing parallax

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ABSTRACT

Pulsar timing arrays (PTAs) act to detect gravitational waves by observing the small, correlated effect the waves have on pulse arrival times at the Earth. This effect has conventionally been evaluated assuming the gravitational wave phase fronts are planar across the array, an assumption that is valid only for sources at distances \( R \gg 2\pi L^2/\lambda \), where \( L \) is physical extent of the array and \( \lambda \) is the radiation wavelength. In the case of PTAs, the array size is of the order of the pulsar–Earth distance (kpc) and \( \lambda \) is of the order of parsec. Correspondingly, for point gravitational wave source distances closer than \( \sim 100 \) Mpc, the PTA response is sensitive to the source parallax across the pulsar–Earth baseline. Here, we evaluate the PTA response to gravitational wave point sources including the important wavefront curvature effects. Taking the wavefront curvature into account, the relative amplitude and phase of the timing residuals associated with a collection of pulsars allow us to measure the distance to, and the sky position of, the source.

Key words: gravitational waves – methods: observational – distance scale.

1 INTRODUCTION

It has been just over 30 yr since Sazhin (1978) and Detweiler (1979) showed how passing gravitational waves cause disturbances in pulsar pulse arrival times that could be used to detect or limit gravitational wave signal strength, and 20 yr since Foster & Backer (1990) proposed using correlated timing residuals of a collection of pulsars – i.e. a pulsar timing array (PTA) – to achieve greater sensitivity. Thought of as a gravitational wave detector a PTA is large: its size \( L \) (of the order of kpc) is much greater than the gravitational radiation wavelength scale \( \lambda \) (of the order of pc) that we use it to probe. The conventional approximation of gravitational waves as plane-fronted applies only for sources at distances \( R \gg L^2/\lambda \); for less distant sources, the curvature of the gravitational radiation phase fronts contributes significantly to the detector response. Here, we evaluate the response of a PTA to discrete gravitational wave sources at distances close enough that gravitational wave phase front curvature is important and describe how the PTA observations of such sources may be used to measure the source distance and location on the sky.

Pulsar timing currently provides the best observational evidence for gravitational waves: as of this writing the orbital decay of the binary pulsar system PSR B1913+16, induced by gravitational wave emission and measured by timing measurements, is the most conclusive observational evidence for gravitational waves (Taylor, Fowler & McCulloch 1979; Weisberg & Taylor 2005). This evidence is often described, without any intent of disparagement, as indirect since it is the effect of gravitational wave emission – i.e. the effect of the emission on the emitter – i.e. observed.\textsuperscript{1} The effect of the gravitational radiation, independent of its source, on another system – a detector – itself remains to be observed.

As proposed by Sazhin (1978) and Detweiler (1979), precision pulsar timing can also be used to detect gravitational waves directly. Gravitational waves passing between the pulsar and the Earth – i.e. temporal variations in the space–time curvature along the pulsar–Earth null geodesics – change the time required by successive pulses to travel the path from pulsar to the Earth. Since successive pulses are emitted from the pulsar at regular intervals, the effect of passing gravitational waves is apparent as irregularities in the observed pulse arrival times. Through the use of precision timing, a collection of especially stable pulsars may thus be turned into a Galactic scale gravitational wave detector.

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\textsuperscript{1}Indirect observation is enough to inaugurate the field of gravitational wave astronomy, that is the study of the physical universe through the application of the laws and theories to astronomical observations involving gravitational waves. See, for example, Finn & Sutton (2002) and Sutton & Finn (2002) which use the effect of gravitational wave emission on several binary pulsar systems to bound the mass of the graviton.

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Timing of individual pulsars has been used to bound the gravitational wave power associated with stochastic gravitational waves of cosmological origin (Romani & Taylor 1983; Kaspi, Taylor & Ryba 1994; Lommen 2002); timing residual correlations among pulsar pairs have been used to bound the power in an isotropic stochastic gravitational wave background (Jenet et al. 2006); the absence of evidence for gravitational waves in the timing of a single pulsar has been used to rule out the presence of a proposed supermassive black hole binary in 3C 66B (Jenet et al. 2004); observations from multiple pulsars have been used to place limits on periodic gravitational waves from Sag A* (Lommen & Backer 2001); and timing analyses of individual pulsars have been combined to place a sky-averaged constraint on the merger rate of nearby black hole binaries in the early phases of coalescence (Yardley et al. 2010). Additionally, analyses of pulsar timing data have been proposed to search for the gravitational wave ‘memory’ effect (Seto 2009; Pshirkov, Baskaran & Postnov 2010; van Haasteren & Levin 2010) and more general gravitational wave bursts (Finn & Lommen 2010), which might be generated during, for example, the periastron passage of a highly eccentric binary system (Sesana 2010) or by cosmic string cusps or kinks (Damour & Vilenkin 2001; Siemens, Mandic & Creighton 2007; Leblond, Shlaer & Siemens 2009).

Analyses focused on discovering or bounding the intensity of a stochastic gravitational wave ‘background’ may rightly resolve the radiation into a superposition of plane waves with density \( \delta^2 \overline{h} / \partial f \partial \Omega \), where \( \overline{h}(t, x) \) is the gravitational wave strain and \( (f, \Omega) \) denotes a plane wave component at frequency \( f \) and propagating in direction \( \Omega \). Analyses focused on the discovery of gravitational wave point sources – for example, supermassive black hole binaries – must take proper account of the curvature of the radiation phase front, owing to the finite distance between the source and the detector, over the detector’s extent. When detector extent, measured in units of the radiation wavelength, is greater than the distance to the source, measured in units of detector extent, then curvature of the radiation wavefront over the detector contributes significantly to the detector response to the incident radiation.

In Section 2, we evaluate the pulse arrival-time disturbance associated with a spherically fronted gravitational wave traversing a pulsar–Earth line of sight and compare it with the response to the same wave in the plane-wave approximation. In Section 3, we describe how, owing to their sensitivity to gravitational radiation phase front curvature, the PTA observations of point sources can be used to measure or place lower bounds on the distance to the source distance and, more surprisingly, the distance to the array pulsars. Finally, we summarize our conclusions in Section 4.

2 PULSAR TIMING RESIDUALS FROM SPHERICALLY FRONTED GRAVITATIONAL WAVES

2.1 Timing residuals

Focus attention on the electromagnetic field associated with the pulsed emission of a pulsar and denote the field phase at the pulsar as \( \phi_0 \). We are interested in the time-dependent phase \( \phi(t) \) of the electromagnetic field associated with the pulsed emission measured at an Earth-based radio telescope, which we write as

\[
\phi(t) = \phi_0 [t - L(t) - \tau_0(t) - \tau_{GW}(t)],
\]

where

\[
\tau_0 = \begin{pmatrix}
\text{corrections owing exclusively to the spatial motion of the Earth} \\
\text{within the Solar system, the Solar system with respect to the pulsar,} \\
\text{and electromagnetic wave propagation through the interstellar medium (ISM)}
\end{pmatrix}
\]

\[
\tau_{GW} = \text{corrections owing exclusively to } \overline{h}(t, x)
\]

\[
L = \text{Earth–pulsar distance.}
\]

(Note that we work in units where \( c = G = 1 \).)

In the absence of gravitational waves \( \tau_{GW} \) vanishes and the phase front \( \phi_0(t) \) arrives at the Earth at time \( t_0(t) = t + L + \tau_0(t) \). In the presence of a gravitational wave signal, the phase front arrives at time \( t_0(t) + \tau_{GW}(t) \); thus, \( \tau_{GW} \) is the gravitational wave timing residual. Following Finn (2009) equations (3.26) and (3.12e), the arrival time correction \( \tau_{GW}(t) \) is

\[
\tau_{GW}(t) = -\frac{1}{2} \overline{h} \hat{n}^w \mathcal{H}_{\text{lin}}(t),
\]

where \( \mathcal{H}_{\text{lin}} \) is the integral of the transverse–traceless metric perturbation over the null geodesic ranging from the pulsar to the Earth:

\[
\mathcal{H}_{\text{lin}}(t) = L \int_{-1}^{0} \hat{h}_{\text{lin}} [t + L \xi, P - L(1 + \xi) \hat{n}] \, d\xi,
\]

\( P = \text{pulsar location,} \)

\( \hat{n} = \text{unit vector pointing from the Earth to the pulsar.} \)

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2.2 Gravitational waves from a compact source

At a point $X$ in the perturbative regime far from the source, the transverse–traceless gauge gravitational wave metric perturbation associated with radiation from a compact source may be written (Finn 1985) as

$$h_{lm}(t, X) = \int_{-\infty}^{\infty} \tilde{h}_{lm}(f, X) e^{-2\pi i f t} df$$

$$= \frac{1}{|X - S|} \int_{-\infty}^{\infty} \tilde{A}_{lm}(f, \hat{k}) e^{2\pi i (f |X - S|)} e^{-2\pi i f t} df \, d\hat{k},$$

where

$S = $ source location,

$\hat{k} = \frac{X - S}{|X - S|}$ = unit vector in direction of wave propagation at $X$.

The Fourier coefficient functions $\tilde{A}_{lm}(f, \hat{k})$ are everywhere symmetric, traceless and transverse with respect to $X - S$, that is $\tilde{A}_{lm}(f, \hat{k}) = \tilde{A}_{lm}(f, \hat{k}), e_{lm}^{(+)}/e_{lm}^{(-)}$, where $e_{lm}^{(+)}(\hat{k})$ and $e_{lm}^{(-)}(\hat{k})$ are the usual transverse–traceless gravitational wave polarization tensors for waves travelling in direction $\hat{k}$.

2.3 Response function

With expression (3) for $h_{lm}$, we can evaluate $\mathcal{H}_{lm}$ in the Fourier domain:

$$\tilde{\mathcal{H}}_{lm}(f) = L \int_{-1}^{0} \tilde{h}_{lm} \left[ f, \mathcal{P} - L(1 + \xi) \hat{n} \right] e^{-2\pi i f L \xi} \, d\xi$$

$$= \frac{L}{|\mathcal{P} - L(1 + \xi) \hat{n} - S|} \exp\left[ 2\pi i f \left( -L \xi + |\mathcal{P} - L(1 + \xi) \hat{n} - S| \right) \right] d\xi.$$ (5b)

Fig. 1 describes the Earth–pulsar–source geometry: note we define $R$ to be the Earth–source distance and $\theta$ is the angle between the Earth–pulsar and the Earth–source lines of sight. Along the path traversed by the pulsar’s electromagnetic pulse phase fronts ($\mathcal{P} - L \xi \hat{n}$ for $-1 \leq \xi \leq 0$), we may write as

$$\hat{k}(\xi) = \frac{\mathcal{P} - L(1 + \xi) \hat{n} - S}{|\mathcal{P} - L(1 + \xi) \hat{n} - S|},$$

$$r(\xi) = \text{distance between gravitational wave phase front and source}$$

$$= |\mathcal{P} - L(1 + \xi) \hat{n} - S| = R \left[ 1 + 2 \frac{L \xi}{R} \cos \theta + \left( \frac{L \xi}{R} \right)^2 \right]^{1/2}. \quad (6b)$$

From equations (5) and (6b), we note that the time of arrival disturbance owing to a passing gravitational wave depends on the dimensionless quantities $\pi f L$ and $L/R$. The PTA pulsar distances $L$ are all of the order of kpc. Gravitational waves detectable via pulsar timing observations have frequencies greater than the inverse duration of the observational data set and less than the typical sampling period, that is $10^{-5} \lesssim f \lesssim 10^{-3} \text{ Hz}$. Strong sources of gravitational waves (e.g. supermassive black hole binaries and triplets) in this band are all expected to be extragalactic, with significant numbers within on order 100 Mpc (Sesana & Vecchio 2010). The scales of interest are thus either very large or very small:

$$\epsilon = 10^{-5} \left( \frac{L}{1 \text{ kpc}} \right) \left( \frac{100 \text{ Mpc}}{R} \right), \quad (7a)$$

$$\pi f L = 10^4 \left( \frac{f}{1 \text{ yr}^{-1}} \right) \left( \frac{L}{1 \text{ kpc}} \right), \quad (7b)$$

$$\pi f L \epsilon = 10^{-1} \left( \frac{f}{1 \text{ yr}^{-1}} \right) \left( \frac{L}{1 \text{ kpc}} \right)^2 \left( \frac{100 \text{ Mpc}}{R} \right), \quad (7c)$$

where we have introduced $\epsilon$ to denote the dimensionless $L/R$. Taking advantage of these scales, the Fourier transform $\tilde{\mathcal{H}}_{lm}(f)$ may be expressed as

$$\tilde{\mathcal{H}}_{lm}(f) = L \int_{-1}^{0} \tilde{h}_{lm} \left[ f, \mathcal{P} - L(1 + \xi) \hat{n} \right] e^{-2\pi i f L \xi} \, d\xi$$

$$= \frac{L}{|\mathcal{P} - L(1 + \xi) \hat{n} - S|} \exp\left[ 2\pi i f (|\mathcal{P} - L(1 + \xi) \hat{n} - S|) \right] d\xi.$$ (8b)
Gravitational wave source timing parallax

Figure 1. Gravitational wave source (depicted as a binary system), the Earth and the pulsar. The gravitational wave phase fronts intersect the pulsar–Earth path, travelled by the electromagnetic pulses, in a manner that depends on the distance to the source and the source–Earth–pulsar angle $\theta$.

\[
\tilde{A}_{lm}(f, \hat{k}_0) \exp \left[ \frac{\pi i f R (2 - \frac{1 - \cos \theta}{1 + \cos \theta})}{\sqrt{f R \sin \theta}} \right] \frac{e^{i\pi/4}}{2} \text{erf} \left( \frac{e^{i\pi/4}}{\sqrt{f R \sin \theta}} \right) \sqrt{\frac{\pi f R \sin \theta}{1 + \cos \theta}} e^{-i\pi/4} 2 \text{erf} \left( \frac{e^{i\pi/4}}{\sqrt{f R \sin \theta}} \right)
\]

\[
\times \left[ 1 + \mathcal{O}(\epsilon) + \mathcal{O} \left( \epsilon \frac{\partial \log \tilde{A}}{\partial k} \cdot \hat{n} \right) \right].
\]

(8c)

Except when $\theta \leq (\pi f R)^{-1/2} \ll 1$, the error function arguments are always large in magnitude. Taking advantage of the asymptotic expansion of erf$(z)$ about the point at infinity,

\[
\text{erf}(z) \sim 1 - \frac{\exp \left( -z^2 \right)}{z \sqrt{\pi}}.
\]

we may thus write

\[
\tilde{\tau}_{gw} = -\frac{1}{2} \tilde{A} \hat{n}^m \tilde{H}_{lm}(f),
\]

(10a)

\[
= \tilde{\tau}_{pw} + \tilde{\tau}_{cr},
\]

(10b)

where

\[
\tilde{\tau}_{pw} = \text{plane wave approximation timing residual}
\]

\[
= 2 \tilde{A}(f, \hat{k}_0) \exp \left( 2\tau i f L \sin^2 \frac{\theta}{2} \right) \text{sinc} \left( 2\pi f L \sin^2 \frac{\theta}{2} \right).
\]

(10c)
\[ \tilde{\tau}_\alpha = \text{phase front curvature correction to plane wave timing residual} \]
\[ = \epsilon (1 + \cos \theta) \tilde{A}(f, \hat{k}_0) \exp \left( i \pi f L \left( 4 \sin^2 \frac{\theta}{2} + \frac{\epsilon}{2} \sin^2 \theta \right) \right) \text{sinc} \left( \frac{\pi f L \epsilon}{2} \sin^2 \theta \right) \]  

(10d)

\[ \tilde{A}(f, \hat{k}_0) = \frac{i \epsilon}{4} \left[ \tilde{\hat{n}} \cdot \tilde{\hat{n}} \right] \tilde{A}_{\text{in}}(f, \hat{k}_0) \exp (2i \pi f R) \]  

(10e)

and

\[ \text{sinc}(x) = \left( \frac{\text{Unnormalized sinc function}}{x} \right) = \frac{\sin(x)}{x}. \]  

(10f)

These expressions are valid for all \( \pi f L \gg 1 \).

3 DISCUSSION

The Fourier amplitude and phase of the gravitational wave timing residual \( \tilde{\tau}_{GW} \) for any particular pulsar depend on the pulsar’s position and distance relative to the position and distance of the gravitational wave source, and the relative amplitude of the signal in the two gravitational wave polarizations. Correspondingly, observation in four or more pulsars of the gravitational wave timing residuals associated with a single source can, in principle, be used to infer the three-dimensional source position and location on the sky. In this section, we discuss the conditions under which sufficiently accurate measurements of the \( \tilde{\tau}_{GW} \) are possible and provide a proof-of-principle demonstration of how they may be combined to determine the sky location and distance to a gravitational wave source.

3.1 Plane wave approximation timing residuals

Focus attention first on \( \tilde{\tau}_{pw} \): the plane-wave approximation to \( \tilde{\tau}_{gw} \). In order to relate the angular location of a gravitational wave source to the relative amplitude and phase of the \( \tau_{pw} \) for two or more PTA pulsars, the uncertainties \( \sigma_L \) in the Earth–pulsar distances, and \( \sigma_\theta \) in pulsar sky locations, must satisfy

\[ \sigma_L \lesssim 2 \text{ pc} \left( \frac{0.1 \text{ yr}^{-1}}{f} \right) \left( \frac{1/2}{\sin^2(\theta/2)} \right), \]  

(11a)

\[ \sigma_\theta \lesssim 2.5 \left( \frac{0.1 \text{ yr}^{-1}}{f} \right) \left( \frac{\text{kpc}}{L} \right) \left( \frac{\pi/4}{\sin \theta} \right). \]  

(11b)

The constraint on pulsar angular position accuracy is not a severe one; however, the constraint on pulsar distance accuracy is quite significant. For this reason, previous pulsar timing searches for periodic gravitational waves (Lommen & Backer 2001; Lommen 2001; Jenet et al. 2004; Yardley et al. 2010) have sought only to identify an anomalous periodic contribution to pulse arrival times, avoiding explicit use of the amplitude of \( \tilde{\tau}_{gw} \).

Improvements in instrumentation and timing techniques are making possible pulsar distance measurements of the accuracy and precision necessary to relate the \( \tilde{\tau}_{pw} \) to the gravitational wave source angular location. For example, in recent years, a succession of very long baseline interferometry (VLBI) observations have established the distance to J0437–4715 with an accuracy of \( \pm 1.3 \) pc (Deller et al. 2008; Verbiest et al. 2008; Deller 2009); correspondingly, for J0437–4715 predictions of the amplitude and phase of \( \tau_{pw} \) can be made and take part in the analysis of timing data to search for gravitational waves at frequencies \( f \lesssim 0.1 \text{ yr} \). Accurate pulsar distance measurements have been identified as a crucial element of a continuing program to use precision pulsar timing and interferometry to test gravity and measure neutron star properties (Cordes et al. 2009); correspondingly, we can reasonably expect that the number of pulsars with high-precision distances will increase rapidly over the next several years.

Looking towards the future, observations with the Square Kilometre Array (SKA) (Smits et al. 2009) using existing timing techniques are expected to be capable of measuring timing parallax distances to millisecond pulsars at 20 kpc with an accuracy of 20 per cent (Tingay et al. 2010). When measured by VLBI or timing parallax, the fractional uncertainty in the pulsar distance is equal to the fractional uncertainty in the semi-annual pulse arrival time variation owing to the curvature of the pulsar’s electromagnetic phase fronts, that is

\[ \frac{\sigma_L}{L} = \frac{\sigma_\tau}{\Delta \tau}, \]  

(12a)

where \( \sigma_\tau \) is the timing noise power in the bandwidth \( T^{-1} \) (for \( T \) the observation duration) about a frequency of 2 yr and

\[ \Delta \tau = \left( \frac{\text{au}}{2c} \right) \left( \frac{\text{au}}{L} \right) = 1.2 \text{ \mu s} \left( \frac{\text{kpc}}{L} \right). \]  

(12b)

is the peak-to-peak variation in the pulse arrival time residual owing to the curvature of the pulsar’s electromagnetic phase fronts. Correspondingly, a 20 per cent accuracy measurement of the distance to a pulsar at 20 kpc corresponds to an 0.5 per cent accuracy measurement of the distance to a pulsar of the \( \sigma_\tau \), at 500 pc, that is 2.5 pc.

This estimate of \( \sigma_L = 2.5 \) pc for a pulsar at 500 pc assumes constant signal-to-noise ratio and, correspondingly, constant timing precision \( \sigma_\tau \). Closer pulsars will have greater fluxes and, correspondingly, better timing precision; consequently, we may regard this as an upper bound.
on the distance measurement error $\sigma_L$. If we assume that $\sigma_\tau^{-2}$ is proportional to the pulsar energy flux, then

$$\sigma_\tau = 0.6 \text{ ns} \left( \frac{L}{\text{kpc}} \right),$$

(13a)

$$\sigma_L = 0.5 \text{ pc} \left( \frac{L}{\text{kpc}} \right)^3;$$

(13b)

that is, if the only barrier to improved timing precision is signal-to-noise ratio then subparsec precision distance measurements should be possible in the SKA era for pulsars within a kpc.\(^2\)

In practice, other contributions to the noise budget will eventually limit the timing precision. Among these the most important are intrinsic pulsar timing noise, pulse broadening owing to scattering in the ISM and tropospheric scintillation. Recent improvements in pulsar timing techniques (Lyne et al. 2010) suggest that intrinsic pulsar timing noise need not limit timing precision measurements in either present or future instrumentation. Unresolved pulse broadening owing to scattering in the ISM will limit the attainable $\sigma_\tau$ and, thus, $\sigma_L$. For nearby ($L < 1 \text{ kpc}$) pulsars, the measured pulse broadening at $\nu = 1 \text{ GHz}$ is typically in the 1–10 ns range (Manchester et al. 2005). Since pulse broadening is scaled with observation frequency as $\nu^{-3/2}$ (though observations suggest the index may be closer to $-3.9$ (Bhat et al. 2004)), scattering in the ISM also need not limit our ability to measure the distance to nearby pulsars. Finally, recent estimates by Jenet, Armstrong & Tinto (2011) indicate that, for gravitational wave frequencies $\lesssim 1 \text{ yr}^{-1}$, tropospheric scintillation will contribute to the noise budget at not more than the 0.03 ns rms level. Consequently, over the next decade, $\sigma_L \lesssim 1 \text{ pc}$ is, thus, a reasonable expectation for millisecond pulsars at distances up to a kpc.

### 3.2 Correction to plane wave approximation

Turn now to the ratio of Fourier amplitudes,

$$\rho(f, \hat{k}_0) = \frac{\tau_\alpha}{\tau_{gw}}$$

$$= \exp \left[ \pi f L \left( 2 \sin^2 \frac{\theta}{2} + \frac{\epsilon}{2} \sin^2 \theta \right) \right]$$

$$\times \frac{\sin \left( \frac{\pi}{4} f L \sin^2 \theta \right)}{\sin \left( 2\pi f L \sin^2 \frac{\theta}{2} \right)}. \quad (14)$$

When $\pi f L \sin^2 \theta \gg 1$, each of the sin functions whose ratio determines $|\rho|$ has order unity magnitude and $|\rho|$ oscillates rapidly from 0 to $\infty$. For a typical PTA pulsar (distance $L \approx \text{kpc}$), this will be the case for source frequencies greater than or of the order of $\text{yr}^{-1}$ and distances $R$ less than or of the order of 100 Mpc (cf. equation 7). In this regime, the gravitational wave phase front curvature plays an essential role in determining the response of a PTA.

As an example consider the residuals in the pulse arrival times for PSR J1939+2134 owing to gravitational waves from a supermassive black hole binary system like that proposed by Sudou et al. (2003) in the radio galaxy 3C 66B at RA 2\(^{h}\)23\(^{m}\)11\(^{s}\).4 Dec. 42\(^{\circ}\)59\('\)31\(\).4 and luminosity distance $R = 85.8 \text{ Mpc}$ (NASA/JPL 2010). The binary, ruled out by the subsequent analysis of Jenet et al. (2004), was supposed to have a total mass of $5.4 \times 10^{10} \text{M}_\odot$, a mass ratio of 0.1 and a period of 1.05 yr, corresponding to a gravitational wave frequency of 1.9 $\text{yr}^{-1}$. Assuming these binary parameters to be exact and taking the distance and location of J1939+2134 as provided by the ATNF Pulsar Catalogue (Manchester et al. 2005) yields $\rho = 1.1 \exp(-0.86\pi i)$, that is the magnitude of the ‘correction’ $\tau_\alpha$ is greater than the ‘leading-order’ plane-wave approximation contribution, the magnitude of $\tau_{gw}$ is 46 per cent of that predicted by the plane wave approximation and the phase of $\tau_{gw}$ is retarded by $\pi/2$ radians relative to $\tau_{pw}$.\(^3\)

### 3.3 Gravitational wave source distance and location on sky

The magnitude and phase of the plane-wave approximation Fourier coefficient function $\tilde{\tau}_{gw}$ depend on the distance to the associated pulsar and the pulsar–Earth–source angle. The correction depends on these and, in addition, the source distance. Measurements of $\tau_{gw}$ in a collection of pulsars, whose angular location and distance are known sufficiently accurate, can thus be used to measure the curvature of the gravitational radiation phase fronts and, thus, the luminosity distance and direction to the gravitational wave source.

Consider a periodic source of gravitational waves observed in an array of pulsars whose distances and relative locations are known to high precision. The gravitational wave contribution to the pulse arrival times for array pulsar $\ell$ may be represented by an amplitude and a

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\(^2\) These estimates are confirmed by recent work of Lee et al. (in preparation), who consider the role played by pulsar timing parallax in identifying gravitational waves from supermassive black hole binaries. Corbin & Cornish (2010) have also looked at how well pulsar distances may be determined from gravitational wave observations of ‘chirping’ black hole binary systems, where the combination of period evolution, source location and signal amplitude in the gravitational wave polarizations is sufficient to determine the source distance. They find modest improvements in array pulsar distance estimates over a priori estimates may be possible, but are by no means certain.

\(^3\) This comparison is meant only to illustrate the importance of $\tau_\alpha$ in estimating $\tau_{gw}$. In this case, the distance to J1939+2134 is not known accurately enough allow a prediction for an accurate estimate of either $\tau_{pw}$ or $\tau_{gw}$. © 2011 The Authors, MNRAS 414, 50–58
phase, which will depend on the different $\epsilon_\ell = L_\ell/R$, $\pi f L_\ell$ and $\theta$ for each pulsar $\ell$ in the array. Requiring that these phases and amplitudes all be consistent is a powerful constraint on the source distance and location on the celestial sphere.

As a crude but effective proof-of-principle demonstration of distance measurement by gravitational wave timing parallax consider observations made with a selection of International Pulsar Timing Array (IPTA) pulsars with distances whose precision has been projected, as described above, into the SKA era. Focus attention on the example source in 3C 66B. For this source, $(\pi f)^{-1} \approx 0.051$ pc. Following equation (13), four pulsars currently monitored as part of the IPTA are close enough that we may expect their distance to be measured with a 1σ uncertainty of less than 0.025 pc. Table 1 lists these pulsars and their currently measured distances. The following steps are taken for these four pulsars.

(i) Calculate the four $\tilde{r}_{gw}$ for the example source in 3C 66B described above. These complex amplitudes, corresponding to periodic timing residual amplitude and phase, are our ‘observations’. Denote the $\tilde{r}_{gw}$ for pulsar $k$ by $\tilde{r}_k$.

(ii) Adopt approximate distances to each of these four pulsars consistent with normally distributed measurement error with standard deviation as given in equation (13).

(iii) Using these approximate distances evaluate

$$\psi^2(r) = \sum_k \left[ \log \left( \frac{\tilde{r}_k(r)}{\tilde{r}_k} \right) \right]^2,$$

where $\tilde{r}_k(r)$ is the expected $\tilde{r}_{gw}$ for pulsar $k$ assuming the source at distance $r$ and pulsar $k$ at the approximate distance found in step (ii).

The quantity $\psi^2(r)$ is a measure of the misfit between the observations $\tilde{r}_k$ and the prediction $\tilde{r}_k$ assuming pulsars at the approximate distances and source at distance $r$. The $r$ that minimizes $\psi^2(r)$ is thus an estimator for the distance to the source. Fig. 2 shows a set of three scatter plots of the estimated $r$, declination $\theta$ and right ascension $\phi$ relative to their actual values over $10^4$ realizations of pulsar distance errors, for this example. Table 2 provides quantitative descriptive statistics for the distribution of errors. Clearly, if the pulsar distances can be sufficiently accurately determined, the distance and location of a detected periodic gravitational wave source can be accurately measured.

We emphasize that this is no more than a proof-of-principle demonstration. It is naive in its treatment of uncertainties in the $L_\ell$, ignores pulsar timing noise in the measured $\tilde{r}_k$, presumes a sufficiently accurate knowledge of pulsar locations $\theta_\ell$ and makes use of an ad hoc misfit statistic $\psi^2(r)$. Nevertheless, taken as a proof-of-principle demonstration, it shows that the potential exists for the PTA observations to determine the distance to periodic gravitational wave sources if the pulse arrival time dependence on the passing waves is properly modelled.

### 4 CONCLUSION

Gravitational waves crossing the pulsar–Earth line of sight lead to a disturbance in the pulsar pulse arrival time. All previous derivations of this disturbance have assumed planar gravitational wave phase fronts. The gravitational wave phase fronts from point gravitational wave sources – for example, coalescing supermassive black hole binary systems – are curved, with curvature radius equal to the source luminosity distance. The approximation of planar wavefronts is thus valid only at distances $R \gg 2\pi f L^2/c$, where $R$ is the Earth–source distance, $L$ is the Earth–pulsar distance and $f$ is the gravitational radiation frequency. For typical pulsars distances (kpc) and relevant gravitational wave frequencies ($f < \approx 10^{-3}$ yr$^{-1}$), the correction to the disturbance magnitude and phase owing to phase front curvature is thus significant for sources within $\sim 100$ Mpc. Here, we have derived the curved wavefront corrections to the pulsar timing response for such ‘nearby’ sources, described when they are important, and shown that future gravitational wave observations using PTAs may be capable of measuring luminosity distances to supermassive black hole binary systems, and other periodic gravitational wave sources, approaching or exceeding 100 Mpc.

The gravitational wave source distance measurement described here is properly considered a parallax distance measurement. The baselines over which the parallax is measured are, in this case, the timing array pulsar–Earth baselines. Crucial to the ability to observe the effects of gravitational wave phase front curvature is a knowledge of the pulsar–Earth distance $L$ with an uncertainty $\sigma_L \lesssim (\pi f)^{-1} \approx 2$ pc (0.1 yr$^{-1}$/f). We argue that pulsar distance measurements of this accuracy, while beyond present capabilities (except in the case of J0437–4715, which is exceptionally bright and close), are within the capability of SKA-era observations for pulsars at distances $L \lesssim$ kpc.

When considered against the cosmic distance ladder, the distance measurement described here involves three rungs to reach distances greater than tens of Mpc: first, the determination of the astronomical unit; secondly, the distance to the array pulsars; finally, the distance to
Gravitational wave source timing parallax

Figure 2. Scatter plots of the timing parallax estimates of the distance, declination and right ascension of a simulated gravitational wave source in 3C 66B (i.e. $r = 87.8$ Mpc, Dec. = $2^\circ59'31''$ and RA $2^h23^m11^s$). See Section 3.3 for details.

Table 2. Mean, median and standard deviation of the timing parallax estimated distance and location on the sky of a simulated gravitational wave source in 3C 66B. In this example, the source was actually located at $r = 87.8$ Mpc, Dec. = $2^\circ59'31''$ and RA $2^h23^m11^s$. Only the four pulsars described in Table 1 were used to make these estimates. See Section 3.3 for details.

|        | Mean  | Median | Standard deviation |
|--------|-------|--------|--------------------|
| $r$    | 85.72 Mpc | 85.79 Mpc | 0.99 Mpc          |
| $\theta$ | $42^\circ59'31''$ | $42^\circ59'31''$ | $0''0'55'3''$    |
| $\phi$ | $2^h23^m11^s00^s$ | $2^h23^m11^s00^s$ | $0^h0^m0^s3''$   |

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the gravitational wave source. Since the method described here also provides precise source angular position, the likelihood of identifying an electromagnetic counterpart (e.g. host galaxy) to the gravitational wave source is great, raising the possibility of a precise measurement of both the redshift and luminosity distance to a single object at cosmological distances, with the obvious consequences for independent verification of the parameters describing our expanding Universe.

Even in the absence of an electromagnetic counterpart, measurement of the luminosity distance and angular location of a supermassive black hole binary system – the most likely source of periodic gravitational waves – enables the determination of the system’s so-called ‘chirp mass’, or $(m_1m_2)^{3/5}/(m_1 + m_2)^{1/5}$, without the need to observe the binary’s evolution.

We have only begun to plumb the potential of gravitational wave observations as a tool of astronomical discovery. While this potential will not be realized until gravitational waves are detected and observations become, if not routine, more than occasional, explorations like these will position us to more readily exploit the opportunities that future observations present.

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