A MIRROR CONSTRUCTION FOR THE BIG EQUIVARIANT QUANTUM COHOMOLOGY OF TORIC MANIFOLDS

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ABSTRACT. We identify a certain universal Landau-Ginzburg model as a mirror of the big equivariant quantum cohomology of a (not necessarily compact or semipositive) toric manifold. The mirror map and the primitive form are constructed via Seidel elements and shift operators for equivariant quantum cohomology. Primitive forms in non-equivariant theory are identified up to automorphisms of the mirror.

1. INTRODUCTION

Givental [24, 22] and Hori-Vafa [32] proposed that a mirror of a toric variety is given by a Laurent polynomial function (Landau-Ginzburg potential) on \((\mathbb{C}^\times)^D\) with \(D\) the dimension of the toric variety. The potential is of the form:

\[
F(x) = Q^{b_1}x^{b_1} + \cdots + Q^{b_m}x^{b_m}, \quad x \in (\mathbb{C}^\times)^D
\]

where \(b_1, \ldots, b_m \in \mathbb{N} \cong \mathbb{Z}^D\) are primitive generators of one-dimensional cones of the fan \(\Sigma\) of the toric variety and \(Q\) is the Novikov variable. Givental’s mirror theorem [22] implies, when the toric variety \(X_\Sigma\) is compact and \(c_1(X_\Sigma)\) is semipositive, that the Jacobi ring of \(F(x)\) is isomorphic to the small quantum cohomology and the twisted de Rham cohomology \(H^D(D, \Omega_{(\mathbb{C}^\times)^D}[z], zd + dF\wedge)\) is isomorphic to the small quantum connection. A mirror of equivariant quantum cohomology is given by adding a logarithmic term to the potential [22]:

\[
F_\lambda(x) = F(x) + \sum_{i=1}^D \lambda_i \log x_i, \quad \text{where } \lambda_1, \ldots, \lambda_D \text{ are the equivariant parameters for the torus } T \cong (\mathbb{C}^\times)^D \text{ acting on the toric variety.}
\]

A generalization to big quantum cohomology has been studied by Barannikov [3] and Douai-Sabbah [17]. They obtained big quantum cohomology mirrors of (weighted) projective spaces by adding to \(F(x)\) monomial terms which form a basis of the Jacobi ring. This leads to an isomorphism of Frobenius manifolds between the A-model (quantum cohomology) and the B-model (singularity theory).

In this paper we study mirror symmetry for both big and equivariant quantum cohomology. It turns out that this has a very simple description. Consider a universal Landau-Ginzburg potential \(F_\lambda(x; y)\) of the form:

\[
F_\lambda(x; y) = \sum_k y_k Q^{\beta(k)} x^k - \lambda \cdot \log x.
\]

Here the sum is taken over all lattice points \(k \in \mathbb{N}\) in the support \(|\Sigma|\) of the fan and \(y = \{y_k\}\) is an infinite set of parameters. We need infinitely many parameters \(y\) because the equivariant cohomology \(H^*_T(X_\Sigma)\) is infinite dimensional. Let \(GM(F_\lambda)\) denote the Gauss-Manin system.

\[\text{Small means that the parameter space is restricted to } H^2; \text{ big means that the parameter space is the whole cohomology group.}\]
There is a formal invertible change of variables (mirror map) between the A-model parameter quantum cohomology. In fact, the above theorem follows almost as a formal consequence of properties of these operators. A Seidel element is an invertible element of quantum cohomology associated to a Hamiltonian circle action on a symplectic manifold, introduced by Seidel \cite{Seidel}. This can be “lifted” to the equivariant setting and yields a shift operator for equivariant properties of these operators. A Seidel element is an invertible element of quantum cohomology associated to the \(F_\lambda\) of \(F_\lambda\) is isomorphic to the big equivariant quantum cohomology of \(X_\Sigma\). The key ingredients of the proof are Seidel elements and shift operators for equivariant quantum cohomology.

Theorem 1.1 (Theorem 3.20, Corollary 3.21). Let \(\mathbb{N} \cong \mathbb{Z}^D\) be a lattice and let \(\Sigma\) be a fan in \(\mathbb{N} \otimes \mathbb{R}\) which defines a smooth semi-projective toric variety \(X_\Sigma\) having a torus-fixed point. There is a formal invertible change of variables (mirror map) between the A-model parameter \(\tau \in H^*_T(X_\Sigma)\) and the B-model parameter \(y\) such that the Gauss-Manin system \(GM(F_\lambda)\) of \(F_\lambda\) is isomorphic to the big equivariant quantum connection of \(X_\Sigma\) and that the Jacobi ring of \(F_\lambda\) is isomorphic to the big equivariant quantum cohomology of \(X_\Sigma\).

The main theorem is characterized by the differential equation:

\[
\frac{\partial \tau(y)}{\partial y_k} = S_k(\tau(y))
\]

together with a certain asymptotic initial condition, where \(S_k(\tau)\) is the Seidel element associated to the \(\mathbb{C}^x\)-action \(k \in \mathbb{N}\).

Theorem 1.2 (a generalization of \cite{Iritani}; Proposition 5.6). The mirror map \(y \mapsto \tau(y)\) is characterized by the differential equation:

\[
\omega(y^+) = \exp \left( \sum_{k \in \mathbb{N} \cap |\Sigma|} \sum_{n=1}^{\infty} y_{k,n} z^{n-1} Q^{\beta(k)} x^k \right) \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_D}{x_D}.
\]

The image \(\Upsilon(y, y^+, z) \in H^*_T(X_\Sigma)[z][Q][y, y^+]\) of \(\omega(y^+)\) under the mirror isomorphism in Theorem 1.1 is characterized by the differential equation:

\[
\frac{\partial \Upsilon(y, y^+, z)}{\partial y_{k,n}} = [z^{n-1} S_k(\tau(y))] \Upsilon(y, y^+, z) \quad n = 0, 1, 2, \ldots,
\]

together with a certain asymptotic initial condition, where \(S_k(\tau)\) denotes the shift operator associated to the \(\mathbb{C}^x\)-action \(k \in \mathbb{N}\) and we set \(y_{k,0} := y_k\). In particular, a primitive form in the sense of K. Saito \cite{Saito} is given by \(\omega(y^+)\) with \(y^+\) satisfying \(\Upsilon(y, y^+, z) = 1\).

Mirror symmetry for non-equivariant big quantum cohomology follows immediately by taking a non-equivariant limit of Theorem 1.1. In order to obtain a Landau-Ginzburg potential and a (cochain-level) primitive form in the non-equivariant setting, we need to choose a formal map \((s, f) : H^*(X_\Sigma)[Q] \to H^*_T(X_\Sigma)[Q] \times H^*_T(X_\Sigma)[z][Q]\) such that the non-equivariant

\[\text{This is equivalent to } X_\Sigma \text{ being a GIT quotient of a vector space. We do not assume that } X_\Sigma \text{ is projective or } c_1(X_\Sigma) \text{ is semipositive.}\]
5.6). The hypergeometric series, called the other by reparametrizations of the mirror.

Theorem 1.4 (Theorems 4.4, 4.8 and Corollary 4.9). The Gauss-Manin system $\text{GM}(s^* F)$ of $s^* F$ is isomorphic to the non-equivariant big quantum connection of $X$. Moreover, oscillatory primitive forms $\exp(s^* F / z) \zeta_{(s, f)}$ associated with various data $(s, f)$ are related to each other by reparametrizations of the mirror.

We observe that reparametrizations of the mirror form an infinite-dimensional formal group $JG$. The group $JG$ reduces the equivariant theory to the non-equivariant one: in terms of Givental’s Lagrangian cone $[23]$, the non-equivariant Givental cone can be regarded as the orbit space of the equivariant Givental cone under a $JG$-action (see Theorem [4.8] and Remark [5.6]).

The mirror map and primitive forms can be calculated concretely in terms of the following hypergeometric series, called the extended $I$-function $[12]$:

$$I(y, z) = ze^{\sum_{i=1}^{m} u_i \log y_i / z} \sum_{\ell \in \mathbb{N}} y^\ell Q^{d(\ell)} \left( \prod_{i=1}^{m} \frac{\prod_{c=-\infty}^{0} (u_i + cz)}{\prod_{c=-\infty}^{0} (u_i + cz)} \right) \frac{1}{\prod_{k \in G} \ell_k ! z_k}.$$ 

We deduce the following theorem from basic properties of shift operators, without relying on the mirror theorem $[12]$ for the extended $I$-function.

Theorem 1.5 (Corollary 5.4). We set $y_k(z) = y_k + \sum_{n=1}^{\infty} y_{k,n} z^n$ and $y(z) = \{ y_k(z) : k \in \mathbb{N} \cap |\Sigma| \}$. The primitive form of the equivariant mirror is given by $\omega(y^+)$ for $y^+$ such that one has $I(y(z), z) = z(1 + O(z^{-1}))$. Moreover, for such $y^+$, the asymptotics $I(y(z), z) = z + \tau(y) + O(z^{-1})$ determines the mirror map $\tau(y)$.

The primitive form in this paper is given as a formal power series in the parameters $y$, and should be thought of as a “formal primitive form” in the sense of Li-Li-Saito $[39]$ (see also $[53]$). Note that our primitive form is defined over the Novikov ring.

Cho-Oh $[9]$ and Fukaya-Oh-Ohta-Ono $[19, 20, 18]$ constructed the Landau-Ginzburg potential as a generating function of open Gromov-Witten invariants. Their potentials were computed for compact semi-positive toric manifolds by Chan-Lau-Leung-Tseng $[6]$ via Seidel representation (see also $[29]$). It is natural to ask if our inverse mirror map $\tau \mapsto y_k(\tau)$ (and the function $\tau \mapsto y^+(\tau)$ giving the primitive form) is a generating function of certain open Gromov-Witten invariants. We also note the approach of Gross $[29]$ using tropical geometry and that of González-Woodward $[27]$ using quantum Kirwan maps.

Remark 1.6. When we talk about formal power series in $y$ and $y^+$, we always mean a function on the formal neighbourhood of the base point given by $y_b = \cdots = y_{bn} = 1$, $y_k = 0$ for $k \in G := (\mathbb{N} \cap |\Sigma|) \setminus \{ b_1, \ldots, b_m \}$ and $y_{k,n} = 0$ for all $k \in \mathbb{N} \cap |\Sigma|$ and $n \geq 1$. This base point corresponds to the original potential $[11]$.

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Notation.

- $T \cong (\mathbb{C}^*)^{\dim T}$ is an algebraic torus; we write $\hat{T} := T \times \mathbb{C}^*$;
- $X$ is a smooth $T$-variety (satisfying the assumption in §3);
- $X_{\Sigma}$ is a smooth toric variety associated to a fan $\Sigma$; in this case $T \cong (\mathbb{C}^*)^{\dim X_{\Sigma}}$;
- $(\lambda, z) \in \text{Lie}(T) \times \text{Lie}(\mathbb{C}^*) = \text{Lie}(\hat{T})$ denote variables on the Lie algebra;
- $H^{*}_{\hat{T}}(pt) \cong \mathbb{C}[\lambda, z]$ is the ring of polynomial functions on Lie($\hat{T}$);
- $H^{*}_{\hat{T}}(X) = H^{*}_{\hat{T}}(X)[z]$ where $\hat{T}$ acts on $X$ via the projection $\hat{T} \to T$;
- $H^{*}_{\hat{T}}(X)_{loc} := H^{*}_{\hat{T}}(X) \otimes_{H^{*}_{\hat{T}}(pt)} \text{Frac}(H^{*}_{\hat{T}}(pt)) = H^{*}_{\hat{T}}(X) \otimes H^{*}_{\hat{T}}(pt) \text{Frac}(H^{*}_{\hat{T}}(pt)[z])$;
- All (co)homology groups have $\mathbb{C}$ coefficients unless otherwise specified;
- $\Psi(k)$ is given in Notation §3.
- $y_k$ is a variable associated to a lattice point $k \in |\Sigma|$; $y_i = y_{b_i}$ for $1 \leq i \leq m$.

2. Shift operators in equivariant Gromov-Witten theory

In this section we recall basic definitions of equivariant Gromov-Witten invariants and shift operators. Shift operators first appeared in the work of Okounkov-Pandharipande [15] for quantum cohomology of Hilbert schemes of points on $\mathbb{C}^2$; they are also studied by Braverman-Maulik-Okounkov [5], Maulik-Okounkov [41] and the author [37]. Let $T \cong (\mathbb{C}^*)^{\dim T}$ be an algebraic torus. Let $X$ be a smooth variety over $\mathbb{C}$ equipped with an algebraic $T$-action. We assume the following conditions.

- $X$ is semi-projective, i.e. the natural map $X \to \text{Spec}(H^0(X, \mathcal{O}))$ is projective.
- The set of $T$-weights of $H^0(X, \mathcal{O})$ is contained in a strictly convex cone in $\text{Hom}(T, \mathbb{C}^*) \otimes \mathbb{R}$ and $H^0(X, \mathcal{O})^T = \mathbb{C}$.

In this paper we only need the case where $X$ is a toric variety, but the shift operator makes sense for general $X$ as above. The above conditions ensure that the $T$-fixed set $X^T$ is projective, and also that $X$ is equivariantly formal, i.e. $H^*_T(X)$ is a free $H^*_T(pt)$-module and one has a (non-canonical) isomorphism $H^*_T(X) \cong H^*(X) \otimes_{\mathbb{C}} H^*_{\hat{T}}(pt)$ as an $H^*_T(pt)$-module, see [37, Proposition 2.1]. These conditions make equivariant Gromov-Witten invariants well-defined and ensure the existence of a non-equivariant limit for quantum cohomology.

2.1. Formal power series ring. Let $\text{Eff}(X) \subset H_2(X, \mathbb{Z})$ denote the semigroup generated by effective curves. For a module (or a ring) $M$, we write $M[[Q]]$ for the space of formal power series of the form:

$$\sum_{d \in \text{Eff}(X)} a_d Q^d, \quad a_d \in M.$$  

Here $Q$ is a formal parameter called the Novikov variable. For a countable set $x = \{x_1, x_2, x_3, \ldots \}$ of variables, the space $M[x]$ of formal power series in $x$ with coefficients in $M$ consists of formal sums of the form:

$$\sum_{I} a_I x^I, \quad a_I \in M$$

where the index $I$ ranges over all sequences $(i_1, i_2, i_3, \ldots)$ of non-negative integers such that $\sum_{n=1}^{\infty} i_n < \infty$ and we set $x^I = \prod_{n=1}^{\infty} x^n_{i_n}$. The space $M[x]$ can be also described as the projective limit of the spaces $M[x_1, \ldots, x_n]$. Note that we shall abuse notation when we use the variable $y$ and write $M[y]$, see §3.2 and Remark [16].
Recall that a topology on a module (or ring) is said to be linear if the fundamental neighbourhood system of 0 is given by submodules (resp. ideals). Let $M$ be a linearly topologized module (or ring) and let $\{M_\nu \subset M\}$ denote the fundamental neighbourhood system of 0. The topology on $M[x]$ is defined by the following fundamental neighbourhood system of 0:

$$M[x]_{\nu,I} = \left\{ \sum_I a_I x^I : a_I \in M_\nu \text{ for all } I \in \mathcal{I} \right\}$$

where $\mathcal{I}$ ranges over all finite sets of exponents $I$. The topology on $M[Q]$ is defined similarly. The convergence in $M[x]$ (or in $M[Q]$) is the coefficient-wise convergence: a sequence in $M[x]$ converges if and only if the coefficient of $x^I$ converges in $M$ for each $I$. When $M$ is complete, the spaces $M[Q], M[x]$ are also complete.

### 2.2. Quantum cohomology and quantum connection.

For equivariant cohomology classes $\alpha_1, \ldots, \alpha_n \in H^*_T(X)$, $d \in H_2(X, \mathbb{Z})$ and non-negative integers $c_1, \ldots, c_n \in \mathbb{Z}_{\geq 0}$, we have equivariant Gromov-Witten invariants

$$\langle \alpha_1 \psi^{c_1}, \alpha_2 \psi^{c_2}, \ldots, \alpha_n \psi^{c_n} \rangle_{X,T}^{0,n,d} = \int_{[X_{0,n,d}]_{\text{vir}}} \prod_{i=1}^n \psi_i^{c_i} \text{ev}^*_i(\alpha_i)$$

taking values in the fraction ring Frac($H^*_T(pt)$) of $H^*_T(pt)$. Here $X_{0,n,d}$ denotes the moduli space of genus-zero stable maps to $X$ of degree $d$ and with $n$ markings, $[X_{0,n,d}]_{\text{vir}}$ denotes its virtual fundamental class, $\text{ev}_i : X_{0,n,d} \to X$ is the evaluation map at the $i$th marking and $\psi_i$ is the first Chern class of the universal cotangent line bundle at the $i$th marking. When $X$ is not proper, the right-hand side is defined by the Atiyah-Bott localization formula [2, 28] and thus belongs to Frac($H^*_T(pt)$).

The equivariant quantum product $*_{\tau}$ with $\tau \in H^*_T(X)$ is given by

$$(\alpha *_{\tau} \beta, \gamma) = \sum_{n \geq 0} \sum_{d \in \text{Eff}(X)} \frac{Q^d}{n!} \langle \alpha, \beta, \gamma, \tau, \ldots, \tau \rangle_{0,n+3,d}^{X,T}$$

for $\alpha, \beta, \gamma \in H^*_T(X)$, where $(\alpha, \beta) = \int_X \alpha \cup \beta$ is the equivariant Poincaré pairing taking values in Frac($H^*_T(pt)$). Let $T_0, \ldots, T_N$ be a basis of $H^*_T(X)$ over $H^*_T(pt)$ and write $\tau = \sum_{i=0}^N \tau^i T_i$. The product $*_{\tau}$ defines a commutative ring structure on

$$H^*_T(X)[Q][\tau] := H^*_T(X)[Q][\tau^0, \ldots, \tau^N].$$

Notice that the product $*_{\tau}$ is defined without localization. This follows from our assumption that $X$ is semi-projective (see [37, §2.3]).

The quantum connection $\nabla$ is a pencil of flat connections on the tangent bundle $TH^*_T(X) = H^*_T(X) \times H^*_T(X)$ of $H^*_T(X)$ defined by

$$\nabla_\alpha = \partial_\alpha + z^{-1}(\alpha *_{\tau})$$

where $z$ is the pencil parameter, $\tau \in H^*_T(X)$ denotes a point on the base, $\alpha \in H^*_T(X)$ and $\partial_\alpha$ denotes the directional derivative. This is known to be flat, and admits a fundamental solution $M(\tau, z) \in \text{End}(H^*_T(X))[z^{-1}][Q][\tau]$ such that

$$\partial_\alpha \circ M(\tau, z) = M(\tau, z) \circ \nabla_\alpha.$$
In this paper, we use the following fundamental solution [21, §1], [47, Proposition 2]:

\[
(M(\tau, z)\alpha, \beta) = (\alpha, \beta) + \sum_{d \in \text{Eff}(X), n \geq 0 \atop (d,n) \neq (0,0)} \frac{Q_d}{n!} \left< \alpha, \tau, \ldots, \tau, \frac{\beta}{z - \psi} \right>_{0,n+2,d}^{X,T}
\]

where \(1/(z - \psi)\) should be expanded in power series \(\sum_{n=0}^{\infty} \psi^n z^{-n-1}\). We regard the pencil parameter \(z\) as an equivariant parameter for an additional \(\mathbb{C}^\times\). Set \(\hat{T} = T \times \mathbb{C}^\times\) and consider the \(\hat{T}\)-action on \(X\) induced by the natural projection \(\hat{T} \to T\). By the localization method, we find that \(M(\tau, z)\) defines an operator

\[
M(\tau, z) : H^*_T(X)[Q][\tau] \to H^*_T(X)_{\text{loc}}[Q][\tau].
\]

where \(H^*_T(X)_{\text{loc}} = H^*_T(X) \otimes H^*_T(\text{pt}) \text{Frac}(H^*_T(\text{pt}))\) is called the Givental space.

2.3. Shift operators. For a cocharacter \(k : \mathbb{C}^\times \to T\) of \(T\), we say that \(k\) is semi-negative if \(k\) pairs with every \(T\)-weight of \(H^0(X, \mathcal{O})\) non-positively. Here we adopt the convention that \(T\) acts on a function \(f \in H^0(X, \mathcal{O})\) as \((t \cdot f)(x) := f(t^{-1}x)\). We consider a shift operator associated to a semi-negative cocharacter.

For a cocharacter \(k\) of \(T\), we consider the space

\[
E_k := \left( X \times (\mathbb{C}^2 \setminus \{(0,0)\}) \right) / \mathbb{C}^\times
\]

where \(\mathbb{C}^\times\) acts on \(X \times \mathbb{C}^2\) by \(s \cdot (x, (v_1, v_2)) = (s^k x, (s^{-1}v_1, s^{-1}v_2))\) and \(s^k\) denotes the image of \(s \in \mathbb{C}^\times\) under \(k\). The space \(E_k\) is a fiber bundle over \(\mathbb{P}^1\) with fiber \(X\). The group \(\hat{T} = T \times \mathbb{C}^\times\) acts on \(E_k\) by \((t, u) \cdot (x, (v_1, v_2)) = [tx, (v_1, uv_2)]\). Let \(X_0\) denote the fiber of \(E_k\) at \([1,0] \in \mathbb{P}^1\) and \(X_{\infty}\) denote the fiber of \(E_k\) at \([0,1] \in \mathbb{P}^1\). Note that the induced \(\hat{T}\)-actions on \(X_0\) and \(X_{\infty}\) are given by

\[
(t, u) \cdot x = t \cdot x \quad \text{for } x \in X_0;
\]

\[
(t, u) \cdot x = tu^k \cdot x \quad \text{for } x \in X_{\infty}.
\]

We have an isomorphism \(\Phi_k : H^*_T(X_0) \cong H^*_\hat{T}(X_{\infty})\) induced by the identity map \(X_0 \cong X_{\infty}\) and the group automorphism \(\hat{T} \to \hat{T}, (t, u) \mapsto (tu^k, u)\). Notice that \(\Phi_k\) is not a homomorphism of \(H^*_\hat{T}(\text{pt})\)-modules but satisfies the property:

\[
\Phi_k(f(\lambda, z)\alpha) = f(\lambda + kz, z)\Phi_k(\alpha)
\]

for \(f(\lambda, z) \in H^*_\hat{T}(\text{pt})\). Here \(\lambda \in \text{Lie}(T)\) and \(z \in \text{Lie}(\mathbb{C}^\times)\) are equivariant parameters for \(T\) and \(\mathbb{C}^\times\) respectively and we identify \(H^*_T(\text{pt})\) with the ring of polynomial functions on \(\text{Lie}(T) \times \text{Lie}(\mathbb{C}^\times)\). We have an isomorphism [47, Lemma 3.7]

\[
H^*_\hat{T}(E_k) \cong \left\{ (\alpha, \beta) \in H^*_T(X_0) \oplus H^*_T(X_{\infty}) : \alpha - \Phi_k^{-1} \beta \equiv 0 \mod z \right\}
\]

which sends \(\gamma\) to \((\gamma|_{X_0}, \gamma|_{X_{\infty}})\). We write \(\hat{\tau} \in H^*_\hat{T}(E_k)\) for the lift of \(\tau \in H^*_T(X)\) such that \(\hat{\tau}|_{X_0} = \tau\) and \(\hat{\tau}|_{X_{\infty}} = \Phi_k(\tau)\). The assignment \(\tau \mapsto \hat{\tau}\) is not \(H^*_\hat{T}(\text{pt})\)-linear.

Let \(H^2_{\text{sec}}(E_k, \mathbb{Z})\) denote the subset of \(H^2(E_k, \mathbb{Z})\) consisting of section classes \(d \in H^2(E_k, \mathbb{Z})\) such that \(\pi_*(d) = [\mathbb{P}^1]\), where \(\pi : E_k \to \mathbb{P}^1\) is the natural projection. We set \(\text{Eff}(E_k)_{\text{sec}} := \text{Eff}(E_k) \cap H^2_{\text{sec}}(E_k, \mathbb{Z})\). Consider the \(\mathbb{C}^\times\)-action on \(X\) induced by \(k : \mathbb{C}^\times \to T\) and the \(T\)-action on \(X\). To each \(\mathbb{C}^\times\)-fixed point \(x \in X\), we can associate a section
\( \sigma_x = \{x\} \times \mathbb{P}^1 \) of \( \pi: E_k \to \mathbb{P}^1 \). When \( k \) is semi-negative, there exists a unique connected component \( F_{\min} \) of the \( \mathbb{C}^\times \)-fixed set \( X^\mathbb{C}^\times \) such that \( \mathbb{C}^\times \)-action has only positive weights on the normal bundle of \( F_{\min} \) in \( X \) (see [37, §3.2]). The section class associated to a fixed point in \( F_{\min} \) is called the minimal section class and is denoted by \( \sigma_{\min}(k) \).

**Lemma 2.1** ([37, Lemma 3.5, Lemma 3.6]). Let \( k \) be a semi-negative cocharacter of \( T \). Then \( E_k \) is semi-projective and \( \text{Eff}(E_k)^{\text{sec}} = \sigma_{\min}(k) + \text{Eff}(X) \).

**Definition 2.2** (shift operator). Let \( k: \mathbb{C}^\times \to T \) be a semi-negative cocharacter. For \( \tau\in H^*_T(X) \), define a map \( \tilde{S}_k(\tau) : H^*_T(X_0)[Q] \to H^*_T(X_\infty)[Q] \) by

\[
\left( \tilde{S}_k(\tau)(\alpha, \beta) \right) = \sum_{n=0}^{\infty} \sum_{d \in \text{Eff}(E_k)^{\text{sec}}} \frac{Q^{d-\sigma_{\min}(k)}}{n!} \left( t_0, \alpha, \ldots, \hat{t}_0, \ldots, t_\infty, \beta \right) \bigg|_{E_k, \tilde{T}}
\]

where \( \alpha \in H^*_T(X_0), \beta \in H^*_T(X_\infty) \) and \( t_0: X_0 \to E_k, t_\infty: X_\infty \to E_k \) are the natural inclusions. We define \( \tilde{S}_k(\tau) := \Phi_k^{-1} \circ \tilde{S}_k(\tau): H^*_T(X_0)[Q] \to H^*_T(X_0)[Q] \). Note that \( \tilde{S}_k(\tau), \tilde{S}_k(\tau) \) are defined without localization, which again follows from the fact that \( E_k \) is semi-projective, see [37, Remark 3.10]. Note also that \( \tilde{S}_k(\tau) \) is \( H^*_T(\text{pt}) \)-linear, but \( \tilde{S}_k(\tau) \) satisfies \( \tilde{S}_k(\tau)(f(\lambda, z)\alpha) = f(\lambda - kz, z)\tilde{S}_k(\tau)\alpha \) for \( f(\lambda, z) \in H^*_T(\text{pt}) \) and \( \alpha \in H^*_T(X) \).

We also introduce a (constant) shift operator acting on the Givental space \( H^*_T(X)^{\text{loc}} \).

**Definition 2.3** (shift operator on \( H^*_T(X)^{\text{loc}} \)). Let \( k: \mathbb{C}^\times \to T \) be a semi-negative cocharacter. Let \( X^T = \bigsqcup_i F_i \) denote the decomposition of the \( T \)-fixed set \( X^T \) into connected components. Let \( N_i \) be the normal bundle of \( F_i \) in \( X \) and let \( N_i = \bigoplus_\alpha N_i,\alpha \) be the decomposition into \( T \)-eigenbundles, where \( T \) acts on \( N_i,\alpha \) by the weight \( \alpha \in \text{Hom}(T, \mathbb{C}^\times) \). We write \( c(N_i,\alpha) = \prod_{\rho_{i,\alpha,j} \in \text{Hom}(T, \mathbb{C}^\times)} (1 + \rho_{i,\alpha,j}) \) with \( \rho_{i,\alpha,j} \) being the virtual Chern roots of \( N_i,\alpha \). Let \( \sigma_i(k) \in H^2(E_k, \mathbb{Z})^{\text{sec}} \) denote the section class of \( E_k \) given by a \( T \)-fixed point in \( F_i \). We set

\[
\Delta_i(k) := Q^{\sigma_i(k) - \sigma_{\min}(k)} \prod_\alpha \prod_{j=1}^{\text{rank}(N_i,\alpha)} \prod_{\rho_{i,\alpha,j} \in \text{Hom}(T, \mathbb{C}^\times)} \left( \rho_{i,\alpha,j} + \alpha + cz \right) \prod_{\rho_{i,\alpha,j} \in \text{Hom}(T, \mathbb{C}^\times)} \left( \rho_{i,\alpha,j} + \alpha + cz \right).
\]

Using the localization isomorphism [2]

\[
\iota^*: H^*_T(X)^{\text{loc}} \cong H^*_T(X^T)^{\text{loc}} = \bigoplus_i H^*(F_i) \otimes_{\mathbb{C}} \text{Frac}(H^*_T(\text{pt}))
\]

given by the restriction to the fixed set \( X^T \), we define the operator \( S_k : H^*_T(X)^{\text{loc}} \to H^*_T(X)^{\text{loc}} \) by the commutative diagram:

\[
\begin{array}{ccc}
H^*_T(X)^{\text{loc}} & \xrightarrow{S_k} & H^*_T(X)^{\text{loc}} \\
\iota^* \downarrow & & \downarrow \iota^* \\
H^*_T(X^T)^{\text{loc}} & \xrightarrow{\oplus_i \Delta_i(k)e^{-kz\partial_\lambda}} & H^*_T(X^T)^{\text{loc}}
\end{array}
\]

where \( e^{-kz\partial_\lambda} \) acts on \( \text{Frac}(H^*_T(\text{pt})) = \mathbb{C}(\lambda, z) \) by the shift of equivariant parameters \( f(\lambda, z) \mapsto f(\lambda - kz, z) \).
Proposition 2.4 ([5, 41], [57], Theorem 3.14, Corollaries 3.15, 3.16). Let \( k, l \) be semi-negative cocharacters of \( T \). We have the following properties.

1. \( M(\tau, z) \circ S_k(\tau) = S_k \circ M(\tau, z) \)
2. The shift operators commute with the quantum connection, i.e. \([\nabla_\alpha, S_k(\tau)] = 0\) for any \( \alpha \in H^*_T(X) \).
3. We have \( S_k(\tau) \circ S_l(\tau) = Q^{d(k,l)}S_{k+l}(\tau) \), \( S_k \circ S_l = Q^{d(k,l)}S_{k+l} \) for some \( d(k,l) \in H_2(X, \mathbb{Z}) \) which is symmetric in \( k \) and \( l \); in particular the shift operators commute each other: \([S_k, S_l] = 0\).

We give an explicit description for \( d(k, l) \) in the above proposition. Let \( ET \to BT \) denote a universal \( T \)-bundle with \( ET \cong (\mathbb{C}^\infty \setminus \{0\})^{\dim T} \) and \( BT \cong (\mathbb{P}^{\infty})^{\dim T} \). Consider the Borel construction \( X_T = X \times_T ET \) of \( X \). This is an \( X \)-bundle over \( BT \). A cocharacter \( k \) of \( T \) induces a map \( \varphi_k : \mathbb{P}^1 \subset \mathbb{P}^{\infty} \cong BC^\times \to BT \) and the \( X \)-bundle \( E_k \) can be naturally identified with the pull-back \( \varphi_k^*TX_T \) of the Borel construction. Therefore we have a natural map \( E_k \to X_T \). Using this, we can compare section classes in \( H_2^{sec}(E_k) \) for various \( k \) in the single space \( H_2^T(X) = H_2(X_T) \). Note that \( H_2^{sec}(E_k) \) becomes a subgroup of \( H_2^T(X) \) by the equivariant formality [57] Proposition 2.1] of \( X \). We have the following lemma:

Lemma 2.5. We have \( d(k, l) = \sigma_{\min}(k + l) - \sigma_{\min}(k) - \sigma_{\min}(l) \) in \( H_2^T(X) \).

Proof. A straightforward calculation shows that \( S_k \circ S_l = Q^{d(k,l)}S_{k+l} \) for some \( d(k,l) \in H_2(X) \) satisfying

\[
(\sigma_i(k) - \sigma_{\min}(k)) + (\sigma_i(l) - \sigma_{\min}(l)) = (\sigma_i(k + l) - \sigma_{\min}(k + l)) + d(k,l)
\]

for each fixed component \( F_i \subset X_T \), where \( \sigma_i(k) \in H_2^{sec}(E_k) \) denote the section class associated to \( F_i \) as in Definition 2.3. We may regard this as a relation in \( H_2(X_T) \) by pushing it forward along the inclusion \( X \hookrightarrow X_T \). Since a fixed point in \( F_i \) defines a section of the Borel construction \( X_T \), it follows that \( \sigma_i(k) + \sigma_i(l) = \sigma_i(k + l) \) in \( H_2(X_T) \). The conclusion follows immediately.

Definition 2.6 (Seidel elements). Let \( k : \mathbb{C}^\times \to T \) be a semi-negative cocharacter. The Seidel elements are defined as \( S_k(\tau) := \lim_{z \to 0} S_k(\tau) \).

By part (2) of Proposition 2.4, \( \lim_{z \to 0} S_k(\tau) \) commutes with the quantum multiplication and therefore we have

\[
\lim_{z \to 0} S_k(\tau) = (S_k(\tau) \ast_\tau).
\]

By part (3) of Proposition 2.4, we have \( S_k(\tau) \ast_\tau S_l(\tau) = Q^{d(k,l)}S_{k+l}(\tau) \). This gives the Seidel representation [34, 42, 57] of the monoid of semi-negative cocharacters on equivariant quantum cohomology.

Definition 2.7 (commuting vector fields [57, §4.3]). For a semi-negative cocharacter \( k \) of \( T \), we define a vector field \( V_k \) on \( H^*_T(X) \times H^*_T(X) = H^*_T(X) \times H^*_T(X)[z] \) by

\[
(V_k)_{\tau, \Upsilon} = (S_k(\tau), [z^{-1}S_k(\tau)]_+ \Upsilon)
\]

where \( (\tau, \Upsilon) \in H^*_T(X) \times H^*_T(X) \) and \([z^{-1}S_k(\tau)]_+ \Upsilon := z^{-1}S_k(\tau) \Upsilon - z^{-1}S_k(\tau) \ast_\tau \Upsilon \). The vector fields \( V_k \) commute each other: \([V_k, V_l] = 0\).
Remark 2.8. The fundamental solution $M(\tau, z)$ in (2.1) defines a map
\[ H^*_\tau(X) \times H^*_\tau(X) \rightarrow H^*_\tau(X)_{\text{loc}}, \quad (\tau, \chi) \mapsto z M(\tau, z) \chi. \]
The image of this map is known as the Givental cone [23]. Under this map, the vector field $V_k$ corresponds to the linear vector field on $H^*_\tau(X)_{\text{loc}}$ given by $f \mapsto z^{-1} S_k f$ (see [27, §4.3]). The commutativity of the vector fields $V_k$ follows by Proposition 2.4 (3).

3. Equivariant mirrors of toric manifolds

3.1. Toric manifolds. We collect basic definitions and facts about toric manifolds, for which we refer the reader to [14, 15]. Let $N$ be a free abelian group. Consider a rational simplicial fan $\Sigma$ in $N_R = N \otimes \mathbb{R}$ such that

- each cone of $\Sigma$ is generated by part of a $\mathbb{Z}$-basis of $N$;
- the support $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$ of the fan $\Sigma$ is full-dimensional and convex;
- there exists a strictly convex piecewise linear function $f : |\Sigma| \rightarrow \mathbb{R}$ which is linear on each cone of $\Sigma$.

These conditions ensure that the corresponding toric variety $X_\Sigma$ is smooth and satisfies the conditions in §2. Let $b_1, \ldots, b_m \in N$ denote primitive generators of the one-dimensional cones of $\Sigma$. These define the fan sequence
\begin{equation}
0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^m \xrightarrow{(b_1, \ldots, b_m)} N \longrightarrow 0
\end{equation}
where the third arrow sends the standard basis $e_i \in \mathbb{Z}^m$ to $b_i \in N$ and $\mathbb{L}$ is the kernel of $\mathbb{Z}^m \rightarrow N$. For a subset $I \subset \{1, \ldots, m\}$, we write $\sigma_I$ for the cone generated by $\{b_i : i \in I\}$. Define $K := \mathbb{L} \otimes \mathbb{C}^\times$. Since $\mathbb{L}$ is a subgroup of $\mathbb{Z}^m$, $K$ is a subgroup of $(\mathbb{C}^\times)^m$ and thus $K$ acts on $\mathbb{C}^m$. The toric variety $X_\Sigma$ is defined as the quotient
\[ X_\Sigma = (\mathbb{C}^m \setminus Z) / K \]
where $Z \subset \mathbb{C}^m$ is defined as the zero set of the ideal generated by monomials $\prod_{1 \leq i \leq m, i \notin I} z_i$ with $I \subset \{1, \ldots, m\}$ such that the cone $\sigma_I$ belongs to $\Sigma$. Here $z_1, \ldots, z_m$ are the standard co-ordinates on $\mathbb{C}^m$. The torus $T := (\mathbb{C}^\times)^m / K$ naturally acts on $X_\Sigma$. By tensoring the exact sequence (3.1) with $\mathbb{C}^\times$, we find $T \cong N \otimes \mathbb{C}^\times$. In particular the lattice $\text{Hom}(\mathbb{C}^\times, T)$ of cocharacters is identified with $N$. The toric variety $X_\Sigma$ contains the torus $T = (\mathbb{C}^\times)^m / K$, and a character $\chi \in \text{Hom}(T, \mathbb{C}^\times) = N^\times$ of $T$ extends to a regular function on $X_\Sigma$ if and only if $\chi \cdot v \geq 0$ for all $v \in |\Sigma| \subset N \otimes \mathbb{R}$. The space $H^0(X_\Sigma, \mathcal{O})$ of regular functions is generated by such characters, and therefore we find that a cocharacter $k \in N$ of $T$ is semi-negative if and only if $k \in |\Sigma|$.

Notation 3.1. For $k \in N \cap |\Sigma|$, take a cone $\sigma_I \in \Sigma$ containing $k$ and write $k = \sum_{i \in I} n_i b_i$. Define a vector $\Psi(k) = (\Psi_1(k), \ldots, \Psi_m(k)) \in (\mathbb{Z}_{\geq 0})^m$ as
\[ \Psi_i(k) = \begin{cases} n_i & \text{if } i \in I; \\ 0 & \text{otherwise}. \end{cases} \]
and set $|k| := \sum_{i \in I} n_i$.\footnote{Recall the convention on the $T$-action on $H^0(X_\Sigma, \mathcal{O})$ at the beginning of §2.3.
For $1 \leq i \leq m$, let $u_i \in H^2_T(X_\Sigma)$ denote the Poincaré dual of the $T$-invariant divisor \( \{ z_i = 0 \}/K \subset X_\Sigma \). The $T$-equivariant cohomology ring of $X_\Sigma$ is generated by $u_1, \ldots, u_m$ over $\mathbb{C}$ and has the following presentation:

$$H^*_T(X_\Sigma) = \mathbb{C}[u_1, \ldots, u_m]/\mathfrak{J}_{SR}$$

where $\mathfrak{J}_{SR}$ is the ideal generated by $\prod_{i \in I} u_i$ such that the cone $\sigma_I$ does not belong to $\Sigma$. An element $\chi \in H^2_T(pt, \mathbb{Z}) \cong \mathbb{N}^*$ can be expressed as a linear combination of $u_i$'s:

$$\chi = \sum_{i=1}^m (\chi \cdot b_i) u_i.$$  

For each $k \in \mathbb{N} \cap |\Sigma|$, define $\phi_k := \prod_{i=1}^m u_i^\Psi(k) \in H^*_T(X_\Sigma)$. The following lemma is obvious from the above presentation of $H^*_T(X_\Sigma)$.

**Lemma 3.2.** The set $\{ \phi_k : k \in \mathbb{N} \cap |\Sigma| \}$ is a basis of $H^*_T(X_\Sigma)$ over $\mathbb{C}$.

We have that $H^2_T(X_\Sigma, \mathbb{Z})$ is a free $\mathbb{Z}$-module with basis $u_1, \ldots, u_m$. In particular the equivariant homology $H^2_T(X_\Sigma, \mathbb{Z})$ is identified with $\mathbb{Z}^m$ via the dual basis of $u_1, \ldots, u_m$. Moreover the fan sequence (B.1) is identified with:

$$0 \longrightarrow H_2(X_\Sigma, \mathbb{Z}) \cong \mathbb{L} \longrightarrow H_T^2(X_\Sigma, \mathbb{Z}) \cong \mathbb{Z}^m \longrightarrow H_T^2(pt, \mathbb{Z}) \longrightarrow 0.$$

For a cone $\sigma_I \in \Sigma$, we set $C_I = \{ d \in H_2(X_\Sigma, \mathbb{R}) : d \cdot u_i \geq 0 \text{ for } i \notin I \}$. Then the cone of effective curves is generated by $C_I$ with $\sigma_I \in \Sigma$ and we have:

$$\text{Eff}(X_\Sigma) = \mathbb{L} \cap \sum_{\sigma_I \in \Sigma} C_I.$$

Let $k \in \mathbb{N} \cap |\Sigma|$ be a semi-negative cocharacter of $T$ and let $E_k$ be the associated $X_\Sigma$-bundle as in (2.2).

**Lemma 3.3.** The minimal section class $\sigma_{\min}(k)$ of $E_k$ is identified with the element $-\Psi(k) \in \mathbb{Z}^m \cong H^2_T(X_\Sigma, \mathbb{Z})$ under the natural inclusion $H^2_{sec}(E_k) \hookrightarrow H^2_T(X_\Sigma)$ described in the paragraph after Proposition 2.4.

**Proof.** Let $\sigma_I \in \Sigma$ be a cone containing $k$ and write $k = \sum_{i \in I} n_i b_i$. The minimal section $\sigma_{\min}(k)$ of $E_k$ is associated to a point in the toric subvariety $\bigcap_{i \in I} \{ z_i = 0 \}$ whose normal bundle has $\mathbb{C}^2$ weights $\{ n_i \}_{i \in I}$. The class $u_i \in H^2_T(X_\Sigma)$ corresponds, via the natural map $E_k \rightarrow (X_\Sigma)_T$, to the toric divisor $D_i = \{ z_i = 0 \} \times_{\mathbb{C}^2} (\mathbb{C}^2 \setminus \{ 0 \})$ in $E_k$. It suffices to compute the intersection number of $\sigma_{\min}(k)$ and $D_i$. It is easy to see that $D_i \cdot \sigma_{\min}(k)$ equals $-n_i$ if $i \in I$ and zero otherwise. The conclusion follows.

The above lemma and Lemma 2.5 imply:

**Corollary 3.4.** The class $d(k, l) \in H_2(X_\Sigma, \mathbb{Z})$ in Proposition 2.4 is given by $d(k, l) = \Psi(k) + \Psi(l) - \Psi(k+l)$ under the inclusion $H_2(X_\Sigma, \mathbb{Z}) \hookrightarrow H^2_T(X_\Sigma, \mathbb{Z}) \cong \mathbb{Z}^m$. In particular $d(k, l) \in \text{Eff}(X_\Sigma)$ by (3.3).

**Corollary 3.5.** We have $S_k(\tau) = \prod_{i=1}^m S_{b_i}(\tau)^{\Psi_i(k)}$.

**Proof.** By Corollary 2.4, $d(k, l) = 0$ whenever $k$ and $l$ belong to the same cone. The conclusion follows by the property $S_k(\tau) \circ S_l(\tau) = Q^{d(k,l)} S_{k+l}(\tau)$. 

\[ \square \]
3.2. Mirror map. We introduce an infinite set \( y = \{ y_k : k \in \mathbb{N} \cap |\Sigma| \} \) of variables which forms a natural co-ordinate system of the B-model. We set \( y_i := y_{b_i} \) for \( 1 \leq i \leq m \) and \( G := (\mathbb{N} \cap |\Sigma|) \setminus \{ b_1, \ldots, b_m \} \). By abuse of notation, we write \( M[y] \) for the space of formal power series in the variables \( \{ \log y_1, \ldots, \log y_m \} \cup \{ y_k : k \in G \} \) with coefficients in \( M \) (see \( \S 2.1 \)). We consider the lattice of infinite rank:

\[
\hat{L} = \left\{ \ell = (\ell_k)_{k \in \mathbb{N} \cap |\Sigma|} \in \mathbb{Z}^{\oplus \mathbb{N} \cap |\Sigma|} : \sum_{k \in \mathbb{N} \cap |\Sigma|} \ell_k k = 0 \right\}.
\]

By the inclusion \( \{ b_1, \ldots, b_m \} \subset \mathbb{N} \cap |\Sigma| \), we can regard \( L \) as a sublattice of \( \hat{L} \) and we have \( \hat{L}/L \cong \mathbb{Z}^{\oplus G} \). We define a splitting \( \hat{L} \to L \cong H_2(X_\Sigma, \mathbb{Z}) \), \( \ell \mapsto d(\ell) \) by

\[
u_i \cdot d(\ell) = \sum_{k \in \mathbb{N} \cap |\Sigma|} \ell_k \Psi_i(k).
\]

Note that we have the linear relation \( \sum_{i=1}^m (\nu_i \cdot d(\ell))b_i = 0 \). We set

\[
\hat{L}_{\text{eff}} := \{ \ell \in \hat{L} : d(\ell) \in \text{Eff}(X_\Sigma), \ell_k \geq 0 (\forall k \in G) \}
\]

\[
\cong \text{Eff}(X_\Sigma) \times (\mathbb{Z}_{\geq 0})^{\oplus G}
\]

where in the second line we used the splitting \( \hat{L} \cong L \oplus \mathbb{Z}^{\oplus G} \). We set, for \( \ell \in \mathbb{Z}^{\oplus (\mathbb{N} \cap |\Sigma|)} \),

\[
y^\ell := \prod_{\ell \in \mathbb{N} \cap |\Sigma|} y_k^\ell = \exp \left( \sum_{i=1}^m \ell_{b_i} \log y_i \right) \prod_{k \in G} y_k^{\ell_k}.
\]

Note that every neighbourhood of 0 in \( \mathbb{C}[Q][y] \) contains all but finite elements of \( \{ y^\ell Q^{d(\ell)} : \ell \in \hat{L}_{\text{eff}} \} \), thus a power series of the form \( \sum_{\ell \in \hat{L}_{\text{eff}}} a_\ell y^\ell Q^{d(\ell)} \) is well-defined. We introduce mirror maps as an integral submanifold of the commuting vector fields \( \{ V_k : k \in \mathbb{N} \cap |\Sigma| \} \) from Definition 2.7.

**Proposition 3.6.** There exist unique functions

\[
\tau(y) \in H^*_T(X_\Sigma)[Q][y], \quad \Upsilon(y, z) \in H^*_T(X_\Sigma)[Q][y]
\]

of the form

\[
\tau(y) = \sum_{i=1}^m u_i \log y_i + \sum_{\ell \in \hat{L}_{\text{eff}} \setminus \{0\}} \tau_\ell y^\ell Q^{d(\ell)}
\]

\[
(3.4)
\]

\[
\Upsilon(y, z) = 1 + \sum_{\ell \in \hat{L}_{\text{eff}} \setminus \{0\}} \Upsilon_\ell(z) y^\ell Q^{d(\ell)}
\]

with \( \tau_\ell \in H^*_T(X_\Sigma) \), \( \Upsilon_\ell(z) \in H^*_T(X_\Sigma) \) such that we have

\[
\frac{\partial \tau(y)}{\partial y_k} = S_k(\tau(y)), \quad \frac{\partial \Upsilon(y, z)}{\partial y_k} = [z^{-1}S_k(\tau(y))]_+ \Upsilon(y, z)
\]

for all \( k \in \mathbb{N} \cap |\Sigma| \). We call the function \( y \mapsto \tau(y) \) the mirror map.
Remark 3.7. By the definition of the variables \( y \) shows that proved in [37, Proposition 4.7]. Since the vector fields \( V_k \) commute each other, we have a unique solution \( (\tau(y), \Upsilon(y,z)) \) to (3.5) which takes the form (3.4) along \( \{ y_k = 0, \forall k \in G \} \). It suffices to show that \( \tau(y), \Upsilon(y,z) \) are expanded as in (3.4). Write \( \tau(y) = \sum_{i=1}^m u_i \log y_i + \tau'(y) \). By using the divisor equation [37, Remark 3.12], we find that

\[
\Sigma_k(\tau(y); Q) = y^{-\Phi(k)}\Sigma_k(\tau'(y); Qy)
\]

where \( y^{-\Phi(k)} = \prod_{i=1}^m y_i^{-\Phi_i(k)} \) and \( \Sigma_k(\sigma; Qy) \) is obtained from \( \Sigma_k(\sigma; Q) \) by replacing \( Q^d \) with \( Q^d y_1^{\alpha_1} \cdots y_m^{\alpha_m} \). The differential equation for \( \tau \) reads:

\[
y^{\Phi(k)} \frac{\partial \tau'(y)}{\partial y_k} = \Sigma_k(\tau'(y); Qy)
\]

Notice that \( e_k - \Phi(k) \) belongs to \( \hat{L}_{\text{eff}} \) and \( d(e_k - \Phi(k)) = 0 \), where \( e_k \in \mathbb{Z}^{\oplus (N^\vee | \Sigma)} \) denotes the standard basis vector whose \( k \)th component is \( \delta_{k,l} \). This shows, by induction on powers of the variables \( \{ y_k : k \in G \} \), that \( \tau(y) \) has an expansion of the form (3.4). A similar argument shows that \( \Upsilon(y,z) \) also has an expansion of the form (3.4).

\[
\square
\]

Remark 3.7. By the definition of \( \hat{L}, \tau(y) \) and \( \Upsilon(y,z) \) satisfy the following equations:

\[
\sum_{k \in N^\vee | \Sigma} k \otimes y_k \frac{\partial \tau(y)}{\partial y_k} = \sum_{i=1}^m b_i \otimes u_i \quad \text{in } N \otimes \mathbb{Z} H_\tau^*(X_{\Sigma}),
\]

\[
\sum_{k \in N^\vee | \Sigma} k \otimes y_k \frac{\partial \Upsilon(y,z)}{\partial y_k} = 0 \quad \text{in } N \otimes \mathbb{Z} H_\tau^*(X_{\Sigma}).
\]

Contracting the first equation with \( \chi \in N^* \otimes \mathbb{C} \cong H^2_\tau(pt) \) and using (3.2), we obtain

\[
\sum_{k \in N^\vee | \Sigma} (\chi \cdot k) y_k \frac{\partial \tau(y)}{\partial y_k} = \sum_{k \in N^\vee | \Sigma} (\chi \cdot k) y_k \Sigma_k(\tau(y)) = \chi.
\]

This is a generalization of the linear relation for Batyrev elements [25].

Lemma 3.8. The classical limit \( Q \to 0 \) of the shift operator \( \Sigma_k(\tau) \) is given by

\[
\lim_{Q \to 0} \Sigma_k(\tau)f(u) = e^{(\tau(u) - \Phi(k)z - \tau(u))/z} f(u - \Phi(k)z) \prod_{i=1}^m \prod_{c=0}^{\Psi_i(k)-1} (u_i - cz)
\]

where \( f(u), \tau = \tau(u) \in H_\tau^*(X_{\Sigma}) \) are equivariant cohomology classes expressed as polynomials in \( u_1, \ldots, u_m \) and \( u - \Phi(k)z = (u_1 - \Psi_1(k)z, \ldots, u_m - \Psi_m(k)z) \). In particular we have \( \lim_{Q \to 0} S_k(\tau) = \phi_k \exp(-\sum_{i=1}^m \Psi_i(k) \frac{\partial \tau(u)}{\partial u_i}) \).

Proof. This is proved when \( k = b_i \) in [37, Lemma 4.5]. Note that we considered a redundant \((\mathbb{C}^\times)^m\)-action on \( X_{\Sigma} \) in [37] and there is some difference in notation. The conclusion follows from Corollary 3.5 and the case where \( k = b_i \).

\[
\square
\]
We consider the co-ordinates \( t = \{ t_k : k \in \mathbb{N} \cap |\Sigma| \} \) on the equivariant cohomology \( H^*_T(X_\Sigma) \) given by

\[
3.6 \quad t \mapsto \tau = \sum_{k \in \mathbb{N} \cap |\Sigma|} t_k \phi_k \in H^*_T(X_\Sigma)
\]

where \( \{ \phi_k \} \) is the basis in Lemma 3.2. We define the formal neighbourhood of the origin of \( H^*_T(X_\Sigma)[Q] \) to be \( \text{Spf}(\mathbb{C}[Q][[t]]) \). The mirror map identifies \( \text{Spf}(\mathbb{C}[Q][[t]]) \) with the formal neighbourhood \( \text{Spf}(\mathbb{C}[Q][[y]]) \) of the base point \( y^* \) in the \( y \)-parameter space.

\[
3.7 \quad y^* := \{ y_1 = \cdots = y_m = 1, \ y_k = 0 \ (\forall k \in G) \}.
\]

**Lemma 3.9.** The mirror map \( y \mapsto \tau(y) \) in Proposition 3.4 defines an isomorphism between \( \text{Spf}(\mathbb{C}[Q][[y]]) \) and \( \text{Spf}(\mathbb{C}[Q][[t]]) \).

**Proof.** Expand the mirror map as \( \tau(y) = \sum_{k \in \mathbb{N} \cap |\Sigma|} t_k(y) \phi_k \). One can check using the expansion (3.4) that \( \lim_{|k| \to \infty} t_k(y) = 0 \) in the topology of \( \mathbb{C}[Q][y] \) (see §2.1). We also have \( t_k(y) \big|_{y=y^*,Q=0} = 0 \). Therefore the map \( y \mapsto \tau(y) \) defines a well-defined morphism \( \text{Spf} \mathbb{C}[Q][[y]] \to \text{Spf} \mathbb{C}[Q][[t]] \) of formal schemes. By Lemma 3.8, we have

\[
3.8 \quad \frac{\partial \tau(y)}{\partial y_k} \bigg|_{y=y^*,Q=0} = \lim_{Q \to 0} S_k(0) = \phi_k \quad \text{for } k \in \mathbb{N} \cap |\Sigma|.
\]

The conclusion follows by the formal inverse function theorem (Theorem A.1). \( \square \)

**Lemma 3.10.** The cohomology classes \( \tau_\ell, \gamma_\ell(z) \) appearing in equation (3.4) are homogeneous. We have \( \deg \tau_\ell = 2(1 - \sum_{k \in \mathbb{N} \cap |\Sigma|} \ell_k) \) and \( \deg \gamma_\ell(z) = -2 \sum_{k \in \mathbb{N} \cap |\Sigma|} \ell_k \).

**Proof.** Note that the lemma implies the homogeneity of \( \tau(y) \) and \( \gamma(y, z) \) with respect to the degree \( \deg y_k = 2 \) of variables (except for the leading term \( \sum_{i=1}^m u_i \log y_i \) of \( \tau(y) \)). We start with the homogeneity of \( S_k(\tau) \). Let \( \gamma_1, \ldots, \gamma_s \) be classes in \( H^*_T(X_\Sigma) \). We claim that for \( \tau = \sum_{i=1}^m u_i \log y_i + \sum_{j=1}^s x_j \gamma_j \), \( S_k(\tau) \) is a homogeneous endomorphism of degree 0 if we define the degree of \( x_j \) to be \( 2 - \deg \gamma_j \) and the degree of \( y_i \) to be 2. Using the divisor equation [37, Remark 3.12], we can write \( \tilde{S}_k(\tau) \alpha, \beta \) as the sum of the following terms:

\[
\langle \iota_{0,\alpha}^* \tau, \hat{\gamma}_{j_1}, \ldots, \hat{\gamma}_{j_n}, \epsilon_{\infty, x^* \beta} \rangle_{0, n+2, d, \sigma_{\min}} \frac{Q^d}{n!} x_{j_1}, \ldots, x_{j_n} \prod_{i=1}^m y_i^{u_i d - \Psi_i(k)}
\]

with \( d \in \text{Eff}(X_\Sigma), 1 \leq j_1, \ldots, j_n \leq s \) and \( n \geq 0 \). By the virtual dimension formula of the moduli space of stable maps, the degree of this term is \( \deg \alpha + \deg \beta - 2 \dim X_\Sigma \), where we used \( c_1(E_k) \cdot \sigma_{\min} = 2 - |k| \). Therefore the claim follows. The lemma follows from this claim and the recursive construction of \( \tau_\ell, \gamma_\ell(z) \) in [37, Proposition 4.6] and in Proposition 3.6. \( \square \)

Define the Euler vector field on the \( y \)-space and on the \( t \)-space (i.e. \( H^*_T(X_\Sigma) \)) by the formula:

\[
3.9 \quad \mathcal{E}_y = \sum_{k \in \mathbb{N} \cap |\Sigma|} y_k \frac{\partial}{\partial y_k}, \quad \mathcal{E}_t = \sum_{i=1}^m \frac{\partial}{\partial t_i} + \sum_{k \in \mathbb{N} \cap |\Sigma|} \left( 1 - \frac{1}{2} \deg \phi_k \right) t_k \frac{\partial}{\partial t_k}.
\]
Note that $\frac{1}{2} \deg \phi_k = |k| := \sum_{i=1}^m \Psi_i(k)$. Let $\text{Gr}_0 \colon H^*_T(X_\Sigma) \to H^*_T(X_\Sigma)$ be the grading operator defined by $\text{Gr}_0(\alpha) = p\alpha$ for $\alpha \in H^*_{2p}(X_\Sigma)$. Note that $\text{Gr}_0(z\alpha) = z\alpha + z\text{Gr}_0(\alpha)$. Lemma 3.10 together with its proof implies the following:

**Lemma 3.11.** The mirror map $\tau(y)$ preserves the Euler vector field: $\tau_*E_y = E_t$. Moreover, we have

$$[E_t + \text{Gr}_0, S_k(\tau)] = 0, \quad (E_y + \text{Gr}_0)\Upsilon(y, z) = 0.$$ 

In particular we have $[E_y + \text{Gr}_0, S_k(\tau(y))] = 0$.

**Remark 3.12.** The relation $\tau_*E_y = E_t$ implies the following equality:

$$\sum_{k \in N\cap|\Sigma|} y_k S_k(\tau) = \sum_{k \in N\cap|\Sigma|} y_k \frac{\partial \tau(y)}{\partial y_k} = c_1^T(X_\Sigma) + \sum_{k \in N\cap|\Sigma|} (1 - |k|) t_k \phi_k.$$ 

Since $\{S_k(\tau) : k \in N \cap |\Sigma|\}$ forms a $\mathbb{C}[Q][t]$-basis of $H^*_T(X_\Sigma)[Q][t]$, the inverse mirror map $y_k(t)$ is obtained as the expansion coefficients of the right-hand side in $\{S_k(\tau)\}$.

**Remark 3.13** (divisor equation). Let $Q_i \frac{\partial}{\partial Q_i}$ be the differential operator in the Novikov variable such that $(Q_i \frac{\partial}{\partial Q_i})Q^d = (u_i \cdot d)Q^d$. By (3.4), $\tau(y)$ and $\Upsilon(y, z)$ satisfy the following analogue of the divisor equation:

$$Q_i \frac{\partial}{\partial Q_i} \tau(y) = D_i \tau(y) - u_i, \quad Q_i \frac{\partial}{\partial Q_i} \Upsilon(y, z) = D_i \Upsilon(y, z),$$

where we set $D_i = \sum_{k \in N\cap|\Sigma|} \Psi_i(k) y_k \frac{\partial}{\partial y_k}$. The divisor equation for $S_k(\tau)$ can be written in the following form:

$$Q_i \frac{\partial}{\partial Q_i} S_k(\tau) = \left( \frac{\partial}{\partial b_i} + \Psi_i(k) \right) S_k(\tau),$$

$$Q_i \frac{\partial}{\partial Q_i} S_k(\tau(y)) = (D_i + \Psi_i(k)) S_k(\tau(y)).$$

**Remark 3.14.** In this paper we did not consider the degree of the Novikov variable; it is simply set to be zero. It is also conventional to set $\deg Q^d = c_1(X_\Sigma) \cdot d$. The divisor equation above replaces this degree of $Q^d$ with the part $\sum_{i=1}^m \frac{\partial}{\partial b_i}$ of the Euler vector field $E_t$.

3.3. **Gauss-Manin system.** Regarding $\text{Eff}(X_\Sigma) \subset H_2(X, \mathbb{Z}) \cong \mathbb{L}$ as a subset of $\mathbb{Z}^m$, we consider the semigroup

$$\mathbb{M} = \text{Eff}(X_\Sigma) + (\mathbb{L}_{\geq 0})^m.$$ 

For a ring $R$, we introduce a certain completion $R[\mathbb{M}]$ of the semigroup ring $R[\mathbb{M}]$. We write $w_i$ for the element of $R[\mathbb{M}]$ corresponding to the $i$th basis vector $e_i \in (\mathbb{L}_{\geq 0})^m$ and write $Q^d$ for the element of $R[\mathbb{M}]$ corresponding to $d \in \text{Eff}(X_\Sigma)$. The uncompleted Novikov ring $R[Q] := R[\text{Eff}(X_\Sigma)]$ is naturally contained in $R[\mathbb{M}]$. Consider the ideal of $R[\mathbb{M}]$ generated by $Q^d$ with $d \in \text{Eff}(X_\Sigma) \setminus \{\emptyset\}$ and write $R\{\mathbb{M}\}$ for the completion with respect to this ideal. Then $R\{\mathbb{M}\}$ is an $R[Q]$-algebra.

The mirror Landau-Ginzburg model is defined on the space $\text{Spf}(\mathbb{C}\{\mathbb{M}\})$. We introduce a convenient co-ordinate system $(x, Q)$ on it. We consider the semigroup ring $\mathbb{C}[N \cap |\Sigma|]$ of $N \cap |\Sigma|$ and denote by $x^k \in \mathbb{C}[N \cap |\Sigma|]$ for the element corresponding to $k \in N \cap |\Sigma|$. 

Choose a maximal cone $\sigma_{b_i}$ of $\Sigma$. Since $\{b_i : i \in I_0\}$ is a $\mathbb{Z}$-basis of $N$, we can define a splitting $\varsigma: N \to \mathbb{Z}^m$ of the fan sequence (3.1) by sending $b_i \in N$ with $i \in I_0$ to $e_i \in \mathbb{Z}^m$. This splitting defines an embedding $C[M] \hookrightarrow C[Q] \otimes C[N \cap |\Sigma|]$ of $C[Q]$-algebras by the assignment:

$$w_j = Q^{e_j - \varsigma(b_j)}x^b.$$ 

Note that $e_j - \varsigma(b_j) \in \mathbb{Z}^m$ lies in $\text{Eff}(X_\Sigma)$ (see (3.3)). This exhibits $C[M]$ as a subalgebra of $C[N \cap |\Sigma|][Q]$. For $k \in N \cap |\Sigma|$, we write

$$w^{\Psi(k)} := \prod_{i=1}^m w_i^{\psi_i(k)} = Q^{\beta(k)}x^k$$

where $\beta(k) := \Psi(k) - \varsigma(k) \in \text{Eff}(X_\Sigma)$. Then we have $C[M] = (\bigoplus_{k \in N \cap |\Sigma|} Cw^{\Psi(k)})[Q]$, i.e. $\{w^{\Psi(k)} : k \in N \cap |\Sigma|\}$ is a topological $C[Q]$-basis of $C[M]$. We define the universal Landau-Ginzburg potential by

$$F(x;y) := \sum_{k \in N \cap |\Sigma|} y_k w^{\Psi(k)} = \sum_{k \in N \cap |\Sigma|} y_k Q^{\beta(k)}x^k.$$ 

This parametrizes all elements of $C[M]$ and belongs to $C[M][y]$. We also consider the equivariant version:

$$F_\lambda(x;y) = F(x;y) - \lambda \cdot \log x$$

where $\lambda \in N \otimes \mathbb{C} = \text{Lie}(T)$ is an equivariant parameter and $\log x$ is regarded as a point in $N^* \otimes \mathbb{C}$. Choosing an auxiliary basis of $N$, we write $x = (x_1, \ldots, x_D)$ and $\lambda = (\lambda_1, \ldots, \lambda_D)$ so that $\lambda \cdot \log x = \sum_{i=1}^D \lambda_i \log x_i$ where $D = \text{rank } N$.

**Definition 3.15.** Set $\omega = \frac{dx_1}{x_1} \cdots \frac{dx_D}{x_D}$ and $\omega_i = (x_i \frac{\partial}{\partial x_i}) \omega$. The (logarithmic) Gauss-Manin system $\text{GM}(F_\lambda)$ of $F_\lambda(x;y)$ is defined to be the cokernel of the map

$$zd + dF_\lambda \wedge: \bigoplus_{i=1}^D C[z]{\{M\}}[y][\lambda] \omega_i \to C[z]{\{M\}}[y][\lambda]\omega.$$ 

where $d$ is the derivation with respect to the $x$-variables which is linear over the ground ring $C[z][Q][y][\lambda]$ and defined on generators by $dw^{\Psi(k)} = \sum_{i=1}^D k_i Q^{\beta(k)}x^k dx_i/x_i = \sum_{i=1}^D k_i w^{\Psi(k)} dx_i/x_i$. The grading operator on the Gauss-Manin system is given by:

$$\text{Gr} = \mathcal{E}_y + z \frac{\partial}{\partial z} + \sum_{i=1}^D \lambda_i \frac{\partial}{\partial \lambda_i},$$

where $\mathcal{E}_y$ is the Euler vector field in (3.9).

**Remark 3.16.** The term $\lambda \cdot \log x$ in $F_\lambda(x;y)$ depends on the choice of a splitting $\varsigma: N \to \mathbb{Z}^m$, but the Gauss-Manin system $\text{GM}(F_\lambda)$ itself does not.

**Remark 3.17.** Each element $f(z,x,y)\omega \in \text{GM}(F_\lambda)$ associates the following oscillatory integral:

$$\int e^{F_\lambda(x;y)/z} f(z,x,y)\omega.$$ 

The image of $zd + dF_\lambda \wedge$ corresponds to exact oscillatory forms. This cohomology has been used in singularity theory [51, 50] and also in the context of the GKZ system [1, 4, 48, 49].
The image of $zd + dF_{\lambda} \wedge$ is topologically generated by

$$\left( zd + dF_{\lambda} \wedge \right) w^{\Psi(l)} \omega_i = \left( zl_i w^{\Psi(l)} + \sum_{k \in \mathbb{N} \cap |\Sigma|} k_i y_k w^{\Psi(k) + \Psi(l)} - \lambda_i w^{\Psi(l)} \right) \omega$$

as a $\mathbb{C}[z][Q][y][\lambda]$-module where $k_i, l_i$ denote the $i$th components of $k, l \in \mathbb{N} \cap |\Sigma|$ (with respect to the auxiliary basis of $\mathbb{N}$). This gives a relation in $\text{GM}(F_{\lambda})$ and can be thought of as defining the action of $\lambda_i$. Therefore, we have:

$$\text{GM}(F_{\lambda}) \cong \mathbb{C}[z]\{M\}[y] \omega.$$

**Proposition 3.18.** There exists a unique connection $\nabla_{\partial y_k} : \text{GM}(F_{\lambda}) \to z^{-1} \text{GM}(F_{\lambda})$ in the $y$-direction which satisfies

$$\nabla_{\partial y_k} (fs) = \frac{\partial f}{\partial y_k} s + f \nabla_{\partial y_k} s,$$

$$\left[ \nabla_{\partial y_k}, \nabla_{\partial y_l} \right] = 0, \quad \left[ \nabla_{\partial y_k}, \text{Gr} \right] = \nabla_{\partial y_k},$$

$$\nabla_{\partial y_k} w^{\Psi(l)} \omega = z^{-1} \frac{\partial F_{\lambda}}{\partial y_k} w^{\Psi(l)} \omega = z^{-1} w^{\Psi(k) + \Psi(l)} \omega$$

for $f \in \mathbb{C}[z][Q][y][\lambda]$, $s \in \text{GM}(F_{\lambda})$, $k, l \in \mathbb{N} \cap |\Sigma|$. We call $\nabla$ the Gauss-Manin connection.

**Proof.** Since $\{w^{\Psi(k)} : k \in \mathbb{N} \cap |\Sigma|\}$ is a topological $\mathbb{C}[z][Q][y][\lambda]$-basis of $\mathbb{C}[z]\{M\}[y][\lambda]$, we have a unique connection $\nabla_{\partial y_k}$ on $\mathbb{C}[z]\{M\}[y][\lambda]$ satisfying the above properties. It suffices to check that $z \nabla_{\partial y_k}$ preserves the image of $(zd + dF_{\lambda} \wedge)$. Using Corollary 3.4, we find that

$$z \nabla_{\partial y_k} \left[ (zd + dF_{\lambda} \wedge) w^{\Psi(l)} \omega_i \right] = Q^{d(k,l)} (zd + dF_{\lambda} \wedge) w^{\Psi(k+l)} \omega_i.$$

The conclusion follows. $\square$

We can introduce shift operators for the Gauss-Manin system.

**Proposition 3.19.** Consider the operator $w^{\Psi(k)} : \text{GM}(F_{\lambda}) \to \text{GM}(F_{\lambda})$ with $k \in \mathbb{N} \cap |\Sigma|$ given by multiplication by $w^{\Psi(k)}$ on $\mathbb{C}[z]\{M\}[y][\lambda] \cong \text{GM}(F_{\lambda})$. This satisfies the following properties:

$$w^{\Psi(k)} \circ w^{\Psi(l)} = w^{\Psi(l)} \circ w^{\Psi(k)} = Q^{d(k,l)} w^{\Psi(k+l)}$$

$$w^{\Psi(k)} \circ f(Q,y,\lambda) = f(Q,y,\lambda - kz) \circ w^{\Psi(k)}$$

$$w^{\Psi(k)} \circ \nabla_{\partial y_k} = \nabla_{\partial y_k} \circ w^{\Psi(k)}, \quad w^{\Psi(k)} \circ \text{Gr} = \text{Gr} \circ w^{\Psi(k)},$$

where $k, l \in \mathbb{N} \cap |\Sigma|$ and $f(Q,y,\lambda) \in \mathbb{C}[Q][y][\lambda]$.

**Proof.** The first equation follows from Corollary 3.4. The second equation follows from the relation (3.11) defining the action of $\lambda_i$. The third and the fourth are obvious. $\square$

We now state our main result. Recall that $t = \{t_k : k \in \mathbb{N} \cap |\Sigma|\}$ is a co-ordinate system on $H^*_c(X_{\Sigma})$ given by (3.6) and the mirror map gives a formal invertible change of variables between $y$ and $t$ (Lemma 3.9).
\textbf{Theorem 3.20.} We identify the parameters $y = \{y_k\}$ and $t = \{t_k\}$ via the mirror map in Proposition 3.13. Then we have a unique isomorphism
\[
\Theta : \text{GM}(F_\lambda) \cong H^*_T(X_\Sigma)[Q][t]
\]
of $\mathbb{C}[z][Q][y]$-modules such that

(1) $\Theta(\omega) = \Upsilon(y, z)$;
(2) $\Theta$ intertwines $w_{\Psi(k)}$ with $S_k(\tau(y))$ for $k \in \mathbb{N} \cap |\Sigma|$;
(3) $\Theta$ intertwines the Gauss-Manin connection with the quantum connection;
(4) $\Theta$ intertwines the action of equivariant parameters $\lambda^i, i = 1, \ldots, D$;
(5) $\Theta$ preserves the grading, i.e. $\Theta \circ \text{Gr} = \text{(E}_y + \text{Gr}_0) \circ \Theta$,

where $\Upsilon(y, z)$ is as in Proposition 3.6.

\textbf{Proof.} Since $\{w_{\Psi(k)} \omega : k \in \mathbb{N} \cap |\Sigma|\}$ gives a topological $\mathbb{C}[z][Q][y]$-basis of $\text{GM}(F_\lambda)$, we can define a $\mathbb{C}[z][Q][y]$-module homomorphism $\Theta$ by setting $\Theta(w_{\Psi(k)} \omega) = S_k(\tau(y)) \Upsilon(y, z)$. By Lemma 3.8 and the form (3.4) of $\tau(y)$, we find that
\[
S_k(\tau(y)) \Upsilon(y, z)|_{y^* = Q = 0} = \prod_{i=1}^m \prod_{c=0}^{\Psi(k)-1} (u_i - cz) = \phi_k + O(z)
\]
where $y^*$ is given in (3.7). Thus $\Theta$ is an isomorphism. This preserves the grading by Lemma 3.11. Proposition 2.4 (3) and Proposition 3.19 show that $\Theta$ intertwines $w_{\Psi(k)}$ with $S_k(\tau(y))$.

The differential equation (3.5) for $\Upsilon$ gives:
\[
\frac{\partial}{\partial y_k} \Upsilon(y, z) + z^{-1} \frac{\partial \tau(y)}{\partial y_k} \Upsilon(y, z) = z^{-1} S_k(\tau(y)) \Upsilon(y, z)
\]
where we used $\frac{\partial \tau(y)}{\partial y_k} = S_k(\tau(y))$. This implies that
\[
\nabla^\text{QC} \Theta(\omega) = \Theta \left( \nabla^\text{GM} \frac{\partial \omega}{\partial y_k} \right)
\]
where QC stands for quantum connection and GM stands for Gauss-Manin. Since $\nabla^\text{QC}$ commutes with $S_k(\tau(y))$, $\Theta$ intertwines $S_k(\tau(y))$ with $w_{\Psi(k)}$, and $w_{\Psi(k)}$ commutes with $\nabla^\text{GM}$, it follows that we can replace $\omega$ with $w_{\Psi(k)}\omega$ in this formula. Part (3) follows. In view of the relation (3.11), part (4) is equivalent to the relation:
\[
\lambda_i S_l(\tau(y)) \Upsilon(y, z) = z l_S_l(\tau(y)) \Upsilon(y, z) + \sum_{k \in \mathbb{N} \cap |\Sigma|} k_i y_k S_k(\tau(y)) S_l(\tau(y)) \Upsilon(y, z).
\]
It suffices to show the equality for $l = 0$. Using the differential equation (3.5) again, we find that this is equivalent to:
\[
\lambda_i \Upsilon(y, z) = \sum_{k \in \mathbb{N} \cap |\Sigma|} z k_i y_k \frac{\partial \Upsilon(y, z)}{\partial y_k} + \sum_{k \in \mathbb{N} \cap |\Sigma|} k_i y_k \frac{\partial \tau(y)}{\partial y_k} \Upsilon(y, z).
\]
This follows from Remark 3.7. \hfill $\square$

We define the Jacobi ring of $F_\lambda$ to be
\[
J(F_\lambda) := \mathbb{C}[\mathbb{M}][y][\lambda]/(x_1 \partial_{x_1} F_\lambda(x; y), \ldots, x_D \partial_{x_D} F_\lambda(x; y)) \cong \mathbb{C}[\mathbb{M}][y].
\]
Theorem 3.23. The following theorem follows easily from Proposition 3.6 and Theorem 3.20.

\[ \text{Corollary 3.21. We have a } C[Q][y][\lambda]-\text{algebra isomorphism } J(F_\lambda) \cong C\{M\}[y] \xrightarrow{\sim} (H^*_T(X_\Sigma)[Q][t], *) \text{ that sends } w^{\psi(k)} = \partial_{y_k} F(x; y) \text{ to } \frac{\partial r(y)}{\partial y_k} \text{ for } k \in \mathbb{N} \cap |\Sigma|. \]

Proof. Note that we have \( \text{GM}(F_\lambda)/z \text{GM}(F_\lambda) \cong J(F_\lambda) \cdot \omega. \) Since \( \Theta \) intertwines the Gauss-Manin connection with the quantum connection, it induces an isomorphism \( J(F_\lambda) \cdot \omega \cong H^*_T(X_\Sigma)[Q][t] \) intertwining the action of \( w^{\psi(k)} \) with the quantum product \( \frac{\partial r(y)}{\partial y_k} \ast r(y). \)

3.4. Primitive form. We define the primitive form \( \zeta \in \text{GM}(F_\lambda) \) as the inverse image of the identity class \( 1 \)

\[ \zeta := \Theta^{-1}(1) = \sum_{k \in \mathbb{N} \cap |\Sigma|} c_k(z, y) w^{\psi(k)} \omega \]

under the isomorphism \( \Theta \) in Theorem 3.20, where \( c_k(z, y) \in C[z][Q][y] \). Since \( \Theta \) intertwines the Gauss-Manin connection with the quantum connection, we obtain:

**Proposition 3.22.** When we identify the parameters \( y \) and \( t \) via the mirror map, the primitive form satisfies the differential equation:

\[ z \nabla \frac{\partial \phi}{\partial x_k} \nabla \frac{\partial \phi}{\partial x_l} \zeta = \sum_{j \in \mathbb{N} \cap |\Sigma|} c^j_{k,l}(t) \nabla \frac{\partial \phi}{\partial x_j} \zeta \]

where \( c^j_{k,l}(t) \in C[Q][t] \) are the structure constants of the equivariant quantum product, i.e. \( \phi_k \ast \phi_l = \sum_j c^j_{k,l}(t) \phi_j \).

We give an alternative description for the mirror isomorphism \( \Theta \) and the primitive form. Introduce an infinite set \( y^+ = \{y_{k,n} : k \in \mathbb{N} \cap |\Sigma|, n = 1, 2, 3, \ldots \} \) of parameters and consider the formal deformation of \( \omega \):

\[ \omega(y^+) = \exp \left( \sum_{k \in \mathbb{N} \cap |\Sigma|} \sum_{n=1}^{\infty} y_{k,n} z^{n-1} w^{\psi(k)} \right) \omega. \]

In view of the oscillatory integral in Remark 3.17, this formal deformation corresponds to adding to the potential \( F_\lambda \) the \( z \)-dependent term \( \sum_k \sum_{n \geq 1} y_{k,n} z^n w^{\psi(k)} \). The original parameters \( y = \{y_k\} \) correspond to \( \{y_{k,0}\} \) in this sense. Note that the primitive form \( \zeta \) can be written in this form (3.12) since \( c_k(z, y)|_{y=y^*,Q=0} = \delta_{k,0} \). We will work with formal power series in all these variables \( \{\log y_1, \ldots, \log y_m\} \cup \{y_k : k \in G\} \cup \{y_{k,n} : k \in \mathbb{N} \cap |\Sigma|, n \geq 1\} \).

The following theorem follows easily from Proposition 3.6 and Theorem 3.20.

**Theorem 3.23.** The image \( \Upsilon(y, y^+, z) = \Theta(\omega(y^+)) \) is characterized by the following differential equation:

\[ \frac{\partial \Upsilon(y, y^+, z)}{\partial y_{k,n}} = [z^{n-1} \delta_k(\tau(y))] \Upsilon(y, y^+, z) \quad n = 0, 1, 2, \ldots \]
where we set \( y_{k,0} := y_k, \left[ z^{n-1}S_k(\tau(y)) \right]_0, \) \( \Upsilon := z^{n-1}S_k(\tau(y))\Upsilon - \delta_{n,0} z^{-1} S_k(\tau(y))*_{\tau(y)} \Upsilon, \)
together with the expansion
\[
(3.14) \quad \Upsilon(y,y^+,z) = 1 + \sum_{(\ell, \ell^+)} \Upsilon_{\ell,\ell^+}(z) y^\ell(y^+)^{\ell^+} Q^d(\ell,\ell^+)
\]
with \( \Upsilon_{\ell,\ell^+}(z) \in H^*_T(X_\Sigma), \) where the sum is taken over \((\ell, \ell^+) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}, \) and such that

- \( \sum_{k\in\mathbb{N}_{|\Sigma|}} (\ell_k + \sum_{n=1}^{\infty} \ell_{k,n}) = 0; \)
- \( \ell_k \geq 0 \text{ for all } k \in G, \ell_{k,n} \geq 0 \text{ for all } (k,n) \in (\mathbb{N} \cap |\Sigma|) \times \mathbb{N} \); and
- \( d(\ell, \ell^+) \in \text{Eff}(X_\Sigma) \)

where \( \mathbb{N} = \{1, 2, 3, \ldots\} \) is the set of natural numbers and \( d(\ell, \ell^+) \in H_2(X_\Sigma, \mathbb{Z}) \) is determined by \( u_i \cdot d(\ell, \ell^+) = \sum_{k\in\mathbb{N}_{|\Sigma|}} (\ell_k + \sum_{n=1}^{\infty} \ell_{k,n}) \Psi_i(k). \)

We consider the formal neighbourhood of \((0,1)\) in \( H^*_T(X_\Sigma)[[Q]] \times H^*_T(X_\Sigma)[[Q]]. \) This is defined similarly to the formal neighbourhood of \((0,1)\) in \( H^*_T(X_\Sigma)[[Q]] \) (see the discussion around (3.8)) by choosing a \( \mathbb{C} \)-linear basis of \( H^*_T(X_\Sigma) \times H^*_T(X_\Sigma). \)

**Lemma 3.24.** The map \((y, \omega(y^+)) \mapsto (\tau(y), \Upsilon(y,y^+,z))\) defines an isomorphism between the formal neighbourhood \( \text{Spf}(\mathbb{C}[Q[[y,y^+]]) \) of \( \omega \) in the Gauss-Manin system and the formal neighbourhood of \((0,1)\) in \( H^*_T(X_\Sigma)[[Q]] \times H^*_T(X_\Sigma)[[Q]]. \)

**Proof.** The map \((y, \omega(y^+)) \mapsto (\tau(y), \Upsilon(y,y^+,z))\) defines a morphism of formal schemes for a reason similar to the proof of Lemma 3.9. By the formal inverse function theorem (Theorem A.1), it suffices to check that the differential of the map at \( y = y^*, y^+ = 0, Q = 0 \) is an isomorphism (see (8.7) for \( y^* \)). We checked that \( y \mapsto \tau(y) \) is an isomorphism in Lemma 3.9. By Lemma 3.8, we have for \( n \geq 1 \)

\[
(3.15) \quad \left. \frac{\partial \Upsilon(y,y^+,z)}{\partial y_{k,n}} \right|_{y=y^*,y^+=0,Q=0} = z^{n-1} \prod_{i=1}^{m} \prod_{c=0}^{\Psi_i(k)-1} (u_i - cz).
\]

These form a \( \mathbb{C} \)-linear basis of \( H^*_T(X_\Sigma), \) and the conclusion follows. \( \square \)

**Corollary 3.25.** The primitive form \( \zeta \) is given by \( \omega(y^+) \) with \( \Upsilon(y,y^+,z) = 1. \)

**Remark 3.26** (cf. Remark 3.12). Extending the argument in Lemma 3.11, we can easily show the homogeneity \( (\mathcal{E}_{y,y^*} + \text{Gr}_0) \Upsilon(y,y^+,z) = 0 \) with respect to the extended Euler vector field \( \mathcal{E}_{y,y^*} = \sum_{k\in\mathbb{N}_{|\Sigma|}} \sum_{n=0}^{\infty} (1-n) y_{k,n} \frac{\partial}{\partial y_{k,n}}, \) where we set \( y_{k,0} = y_k. \) From this we obtain

\[
\sum_{k,n} (1-n) y_{k,n} \left[ z^{n-1}S_k(\tau(y)) \right]_0 \Upsilon(y,y^+,z) + \text{Gr}_0 \Upsilon(y,y^+,z) = 0.
\]

Suppose that \( y^+ \) is chosen so that \( \Upsilon(y,y^+,z) = 1 \) and \( \omega(y^+) \) is the primitive form. Then one obtains by (3.10) that

\[
\sum_{k\in\mathbb{N}_{|\Sigma|}} \sum_{n=0}^{\infty} (1-n) y_{k,n} z^n S_k(\tau(y)) 1 = c^*_T(X_\Sigma) + \sum_{k\in\mathbb{N}_{|\Sigma|}} (1-|k|) t_k \phi_k.
\]

This determines \( y_{k,n} \) with \( n \neq 1 \) as the expansion coefficients of the right-hand side, as \( z^n S_k(\tau) \) form a basis of \( H^*_T(X_\Sigma). \)
**Remark 3.27.** In this paper, we do not study the higher residue pairing \([51]\) for the Gauss-Manin system. It would be interesting to define such a structure in our setting. Since \(X_\Sigma\) is not necessarily compact, the higher residue pairing should take values in \(\mathbb{C}(\lambda_1, \ldots, \lambda_D)[z][Q][y]\) in general and coincide with the Poincaré pairing on quantum cohomology.

### 4. Non-equivariant mirrors

#### 4.1. Non-equivariant mirror isomorphism.

We obtain a non-equivariant mirror isomorphism by taking the non-equivariant limit \(\lambda \to 0\) of Theorem 3.20. The non-equivariant Gauss-Manin system \(\text{GM}(F)\) is defined to be the cokernel of the map:

\[
zd + dF \wedge: \bigoplus_{i=1}^{D} \mathbb{C}[z]\{M_i\}[y] \omega_i \to \mathbb{C}[z]\{M_i\}[y]\omega
\]

which is just \(\text{GM}(F_\lambda)/\sum_{i=1}^{D} \lambda_i \text{GM}(F_\lambda)\). The mirror isomorphism \(\Theta\) in Theorem 3.20 induces an isomorphism

\[
\Theta_{\text{noneq}}: \text{GM}(F) \cong H^*(X_\Sigma)[z][Q][t].
\]

In the non-equivariant limit, we can extend the flat connection in the \(z\)-direction using the homogeneity. For the non-equivariant Gauss-Manin system \(\text{GM}(F)\), we define:

\[
\nabla_{z \frac{\partial}{\partial z}} := \text{Gr}_t - \nabla_{\varepsilon_t}.
\]

Explicitly this is given by

\[
\nabla_{z \frac{\partial}{\partial z}}(f(x, z, y)\omega) = \left( z \frac{\partial f(x, z, y)}{\partial z} - z^{-1}F(x; y)f(x, z, y) \right) \omega
\]

for \(f(x, z, y) \in \mathbb{C}[z]\{M_i\}[y]\). For the non-equivariant quantum cohomology module \(H^*(X_\Sigma)[z][Q][t]\), we define:

\[
\nabla_{z \frac{\partial}{\partial z}} := (\text{Gr}_t + \mathcal{E}_t) - \nabla_{\varepsilon_t}
\]

which is given explicitly as:

\[
\nabla_{z \frac{\partial}{\partial z}} f(z, t) = \text{Gr}_0(f(z, t)) - \frac{1}{z} (c_1(X_\Sigma) + \sigma - \text{Gr}_0(\sigma)) *_{\sigma} f(z, t)
\]

for \(f(z, t) \in H^*(X_\Sigma)[z][Q][t]\), where \(\sigma\) is the non-equivariant limit of \(\tau = \sum_{k \in \mathbb{N}^{|\Sigma|}} t_k \phi_k\)

and \(*_{\sigma}\) denotes the non-equivariant quantum product. Note that \(\text{Gr}_0\) contains the derivation \(z \frac{\partial}{\partial z}\). These operators \(\nabla_{z \frac{\partial}{\partial z}}\) have a pole of order one along \(z = 0\). By Theorem 3.20, it is clear that the isomorphism \(\Theta_{\text{noneq}}\) intertwines the quantum connection and the Gauss-Manin connection including in the \(z\)-direction.

We further restrict the base space to the non-equivariant cohomology \(H^*(X_\Sigma)\). Let \(T_0, \ldots, T_N\) denote a homogeneous basis of \(H^*(X_\Sigma)\) and let \(s_0, \ldots, s_N\) be the co-ordinates on \(H^*(X_\Sigma)\) dual to \(T_0, \ldots, T_N\). We denote by \(\sigma = \sum_{i=0}^{N} s_i T_i\) a general point on \(H^*(X_\Sigma)\). We choose a formal section \(s: H^*(X_\Sigma)[Q] \to H_T^*(X_\Sigma)[Q]\) of the natural map \(H_T^*(X_\Sigma)[Q] \to H^*(X_\Sigma)[Q]\) of the form:

\[
s(\sigma) = \sum_{k \in \mathbb{N}^{|\Sigma|}} s_k(\sigma) \phi_k \quad \text{with} \quad s_k(\sigma) \in \mathbb{C}[Q][s_0, \ldots, s_N]
\]
such that \( s_k(0)|_{Q=0} = 0, \lim_{|k| \to \infty} s_k(\sigma) = 0 \) (in the adic topology of \( \mathbb{C}[Q][[s_0, \ldots, s_N]] \)) and that the non-equivariant limit of \( s(\sigma) \) equals \( \sigma \). The section \( s \) is a morphism \( \text{Spf} \mathbb{C}[Q][s_0, \ldots, s_N] \to \text{Spf} \mathbb{C}[Q][t] \) of formal schemes. By pulling back \( F \) by \( s \), we obtain a Landau-Ginzburg potential

\[
(s^*F)(x; \sigma) = \sum_{k \in \mathbb{N} \cap |\Sigma|} y_k(s(\sigma))w^{\psi(k)}
\]

parametrized by \( \sigma \in H^*(X_\Sigma) \), where \( y_k = y_k(\tau) \) denotes the inverse mirror map. Define the Gauss-Manin system \( \text{GM}(s^*F) \) of \( s^*F \) to be the cokernel of the map

\[
z d + d(s^*F)\wedge: \bigoplus_{i=1}^D \mathbb{C}[z]\{M_i\}[s_0, \ldots, s_N]\omega_i \to \mathbb{C}[z]\{M_i\}[s_0, \ldots, s_N]\omega.
\]

**Lemma 4.1.** The map \( \Theta_{\text{noneq}} \) in (4.1) induces an isomorphism:

\[
s^*\Theta_{\text{noneq}}: \text{GM}(s^*F) \xrightarrow{\simeq} H^*(X_\Sigma)[z][Q][s_0, \ldots, s_N].
\]

**Proof.** This is slightly subtle as the completed tensor product is not right exact in general. For any module \( M \), write \( M[s] = M[s_0, \ldots, s_N] \) for simplicity. The section \( s \) defines a continuous homomorphism \( s^*: \mathbb{C}[z][Q][y] \to \mathbb{C}[z][Q][s] \) by \( y_k \mapsto y_k(s(\sigma)) \); this is surjective as \( s \) is a section. We have the commutative diagram:

\[
\begin{array}{ccc}
\bigoplus_{i=1}^D \mathbb{C}[z]\{M_i\}[s]\omega_i & \xrightarrow{z d + d(s^*F)\wedge} & \mathbb{C}[z]\{M_i\}[s]\omega \\
\downarrow{s^*} & & \downarrow{s^*} \\
\bigoplus_{i=1}^D \mathbb{C}[z]\{M_i\}[s]\omega_i & \xrightarrow{z d + d(s^*F)\wedge} & \mathbb{C}[z]\{M_i\}[s]\omega
\end{array}
\]

where the top row is exact and the vertical arrows (induced by \( s \)) are all surjective. We need to show that the bottom row is exact. The surjectivity of \( s^*\Theta_{\text{noneq}} \) is obvious. Let \( \varphi \in \mathbb{C}[z]\{M_i\}[s]\omega \) be in the kernel of \( s^*\Theta_{\text{noneq}} \). Choose a lift \( \tilde{\varphi} \in \mathbb{C}[z]\{M_i\}[y]\omega \) of \( \varphi \) such that \( s^*\tilde{\varphi} = \varphi \). Then \( s^*(\Theta_{\text{noneq}}(\tilde{\varphi})) = 0 \). When we write \( \Theta_{\text{noneq}}(\tilde{\varphi}) = \sum_{i=0}^N f_i(z, Q, y)T_i \) with \( f_i \in \mathbb{C}[z][Q][y] \), this means \( s^*f_i = 0 \) for all \( 0 \leq i \leq N \). Choosing a preimage \( \tilde{T}_i \in \mathbb{C}[z]\{M_i\}[y]\omega \) of \( T_i \) under \( \Theta_{\text{noneq}} \), we have that \( \tilde{\varphi} - \sum_{i=0}^N f_i(z, Q, y)\tilde{T}_i \) is in the kernel of \( \Theta_{\text{noneq}} \) and this maps to \( \varphi \) under \( s^* \). Now a diagram chasing shows \( \varphi \) is in the image of \( z d + d(s^*F)\wedge \).

The pulled-back quantum connection \( s^*\nabla \) on \( H^*(X_\Sigma)[z][Q][s_0, \ldots, s_N] \) is given by:

\[
s^*\nabla \tilde{\varphi} = \frac{\partial}{\partial s_i} + \frac{1}{z} T_i \ast \sigma
\]

\[
s^*\nabla z \tilde{\varphi} = \text{Gr}_0 - \frac{1}{z} \left( c_1(X_\Sigma) + \sum_{i=0}^N \left( 1 - \frac{1}{2} \deg T_i \right) s_i T_i \right) \ast \sigma
\]

where \( \text{Gr}_0: H^*(X_\Sigma)[z] \to H^*(X_\Sigma)[z] \) is the linear operator given by \( \text{Gr}(\alpha z^n) = (n + p)\alpha \) for \( \alpha \in H^{2p}(X_\Sigma) \) and \( \ast \sigma \) is the non-equivariant quantum product. This is the usual quantum connection in non-equivariant theory. Thus we obtain:
**Theorem 4.2.** For any formal section \( s : H^*(X_\Sigma) [Q] \to H^*_T(X_\Sigma) [Q] \), the non-equivariant mirror isomorphism \( s^* \Theta_{\text{noneq}} \) intertwines the quantum connection and the Gauss-Manin connection including in the \( z \)-direction.

Define the Jacobi ring of \( s^* F \) to be

\[
J(s^* F) = \mathbb{C}[M]\{s_0, \ldots, s_N\}/(x_1 \partial_{x_1}(s^* F)(x; \sigma), \ldots, x_D \partial_{x_D}(s^* F)(x; \sigma)).
\]

Exactly in the same way as we deduced Corollary 3.2 from Theorem 3.2, we deduce the following corollary from Theorem 4.2:

**Corollary 4.3.** We have a \( \mathbb{C}[Q]\{s_0, \ldots, s_N\} \)-algebra isomorphism \( J(s^* F) \cong (H^*(X_\Sigma))[Q]\{s_0, \ldots, s_N\}, \star_\sigma \) that sends \( \partial_{s_i}(s^* F) \) to \( T_i \).

We now point out that the potential \( s^* F \) gives a universal unfolding of the original potential \( F(x; y^*) \) (see (3.7) for \( y^* \)), and conversely, that any universal unfolding of \( F(x; y^*) \) is equivalent to \( s^* F \) for some formal section \( s \). We say that \( G(x; r) \in \mathbb{C}[M]\{r_0, \ldots, r_N\} \) is a universal unfolding of \( F(x; y^*) \) if

1. \( G(x; 0) = F(x; y^*) \) and
2. \( \partial_{r_i}G(x; r)|_{r_0=0, \ldots, r_N} \) form a \( \mathbb{C}[Q] \)-basis of the Jacobi ring

\[
J(F(x; y^*)) = \mathbb{C}[M]/\langle x_i \partial_{x_i} F(x; y^*), i = 1, \ldots, D \rangle.
\]

Notice that the above Corollary 4.3 at \( \sigma = 0 \) implies that the Jacobi ring \( J(F(x; y^*)) \) is a free \( \mathbb{C}[Q] \)-module of rank \( \dim H^*(X_\Sigma) \).

**Proposition 4.4.** A function \( G(x; r) \in \mathbb{C}[M]\{r_0, \ldots, r_N\} \) is a universal unfolding of \( F(x; y^*) \) if and only if there exists a formal section \( s : H^*(X_\Sigma)[Q] \to H^*_T(X_\Sigma)[Q] \) as above such that \( G(x; r) \) equals \( s^* F \) under a formal invertible change of variables between \( \{r_0, \ldots, r_N\} \) and \( \{s_0, \ldots, s_N\} \) over \( \mathbb{C}[Q] \).

**Proof.** We write \( M[r] = M\{r_0, \ldots, r_N\} \) for simplicity. Corollary 4.3 implies the ‘if’ part of the statement. Conversely, suppose that a universal unfolding \( G(x; r) \) is given. We can write \( G(x; r) = F(x; y(r)) \) for some formal morphism \( r \mapsto y(r) \), \( \text{Spf}(\mathbb{C}[Q][r]) \to \text{Spf}(\mathbb{C}[Q][y]) \). Composing this with the mirror map \( \tau(y) \) and the non-equivariant limit, we obtain a map from \( \text{Spf}(\mathbb{C}[Q][r]) \) to the formal neighbourhood of zero in \( H^*(X_\Sigma)[Q] \). It suffices to show that this is an isomorphism. By the formal inverse function theorem, it suffices to show that the derivative at \( r = 0 \) is an isomorphism, i.e. the non-equivariant limits of \( \frac{\partial r_i(y(r))}{\partial r_i} \)|_{r=0}, \( i = 0, \ldots, N \) form a basis of \( H^*(X_\Sigma)[Q] \) over \( \mathbb{C}[Q] \). On the other hand, the non-equivariant limit of Corollary 3.2 gives an isomorphism of \( \mathbb{C}[Q][y] \)-algebras:

\[
J(F) \cong (H^*(X_\Sigma))[Q][t], \star_{\tau}
\]

which sends \( [\partial_{y_k} F(x; y)] \) to the non-equivariant limit of \( \frac{\partial r_i(y)}{\partial r_i} \). This isomorphism restricted to the base point \( y^* \) sends \( [\partial_{t_i} G(x; r)]_{r=0} \) to the non-equivariant limit of \( \frac{\partial r_i(y(r))^*}{\partial r_i} |_{r=0} \). The conclusion follows. □

**Remark 4.5.** Mirror symmetry for toric varieties has been studied by many people. As noted in the Introduction, Givental [24, 22] and Hori-Vafa [32] proposed Landau-Ginzburg mirrors for toric varieties. There are studies on non-compact case (local mirror symmetry) [40, 21, 8, 38, 43], Frobenius manifold [8, 17, 35, 48], semi-simplicity [34, 46], toric orbifolds
[13, 56, 50, 16, 27, 14, 4, 56], an approach using Lagrangian Floer theory [19, 24, 18, 6, tropical geometry [29], quantum Kirwan maps [53, 27] and quasimap spaces [10, 7], etc. In non-semipositive case, we need to take a certain “$Q$-adic” completion of the Gauss-Manin system; this has been pointed out by the author [55], [13, Theorem 1.2]. An isomorphism between a “completed” Jacobi ring and quantum cohomology was proved by Fukaya-Oh-Ohta-Ono [19, Theorem 1.9], [13, Theorem 1.2.34] using Lagrangian Floer theory and by González-Woodward [27, Theorems 1.16, 4.23] using quantum Kirwan map. (There are differences on Novikov rings and the choice of completions among the literature.) In the analytic setting (in semipositive case), results analogous to Theorem 4.2 are given, e.g. in [56, Proposition 4.8], [13, Theorem 4.11], [16, Theorem 5.1.1], [43, Theorem 7.43].

4.2. Reparametrization group. In the equivariant setting, the primitive form was given by an actual differential form (see [3, 4]). In the non-equivariant setting, however, there are many choices for cochain-level primitive forms. Consider the commutative diagram:

$$
\begin{array}{ccc}
\text{GM}(F_\lambda) & \cong & \mathbb{C}[z]\{M\}[y] \omega \\
| & & | \\
\text{GM}(F) & \cong & \Theta_{\text{noneq}} \quad H^*(X_\Sigma)[z][Q][y].
\end{array}
$$

A primitive form can be chosen to be any element in $\mathbb{C}[z]\{M\}[y]\omega$ which maps to 1 in the bottom right corner. We also have the freedom to choose a formal section $s \colon H^*(X_\Sigma)[Q] \rightarrow H^*_c(X_\Sigma)[Q]$ as in §4.1 to pull it back to the base $H^*(X_\Sigma)[Q]$. We show that all cochain-level primitive forms which are obtained in this way and coincide with $\omega$ at the origin $\sigma = Q = 0$ are related by reparametrizations of the $x$-variables.

**Definition 4.6** (reparametrization group). Consider a formal change of variables of the form:

$$
 x_i \mapsto \tilde{x}_i = x_i \exp \left( \sum_{k \in N \cap |\Sigma|} \epsilon_{k,i} w^{\Psi(k)} \right) \quad 1 \leq i \leq D,
$$

where $\epsilon = \{\epsilon_{k,i} : k \in N \cap |\Sigma|, 1 \leq i \leq D\}$ is a set of formal parameters. These transformations form a (non-commutative) formal group $G$ over $\mathbb{C}[Q]$ by composition. As a formal scheme, $G$ is isomorphic to $\text{Spf}(\mathbb{C}[Q][\epsilon])$. We also consider the jet group $JG$ of $G$, which consists of formal transformations:

$$
 x_i \mapsto \tilde{x}_i = x_i \exp \left( \sum_{k \in N \cap |\Sigma|} \sum_{n=0}^{\infty} \epsilon_{k,i,n} w^{\Psi(k)} z^n \right) \quad 1 \leq i \leq D
$$

containing the parameter $z$. The group $JG$ is isomorphic to $\text{Spf}(\mathbb{C}[Q][\tilde{\epsilon}])$ with $\tilde{\epsilon} = \{\epsilon_{k,i,n} : k \in N \cap |\Sigma|, 1 \leq i \leq D, n = 0, 1, 2, \ldots \}$. Note that $G$ acts on the module $\mathbb{C}\{M\}$ and $JG$ acts on the module $\mathbb{C}[z]\{M\}$. The generators $[\partial/\partial \epsilon_{k,i,n}]_e$ of the Lie algebra $T_e(JG)$ correspond to the vector fields:

$$
 W_{k,i,n} := z^n w^{\Psi(k)} x_i \frac{\partial}{\partial x_i}
$$

satisfying the commutation relation $[W_{k,i,n}, W_{l,j,p}] = Q^{d(k,l)}(l_i W_{k+l,j,n+p} - k_j W_{k+l,i,n+p})$. 


The formal group $JG$ acts, by change of variables, on oscillatory $D$-forms of the form:

$$e^{F(x,y)/z} \omega(y^+) = \exp \left( \frac{1}{z} \sum_{k \in \mathbb{N} \cap |\Sigma|} \sum_{n=0}^{\infty} y_{k,n} z^n \omega^{(k)} \right) \omega$$

where we set $y_{k,0} = y_k$. This defines the $JG$-action on $\text{Spf}(\mathbb{C}[Q][y, y^+, z])$, and the generator $W_{k,i,n}$ corresponds to the following vector field:

$$\tilde{W}_{k,i,n} := k_i \frac{\partial}{\partial y_{k,n+1}} + \sum_{l \in \mathbb{N} \cap |\Sigma|} \sum_{m=0}^{\infty} l_i Q^{d(k,l)} y_{l,m} \frac{\partial}{\partial y_{l+k,n+m}}$$

where the first term in the right-hand side comes from the Lie derivative of $\omega = \frac{dx_1}{x_1} \cdots \frac{dx_D}{x_D}$.

**Lemma 4.7.** Let $\tau(y)$, $\Upsilon(y, y^+, z)$ be the functions in Proposition 3.6 and Theorem 3.23. We have

$$\tilde{W}_{k,i,n} \tau(y) = \lambda_i S_k(\tau(y)) \delta_{n,0}$$

$$\tilde{W}_{k,i,n} \Upsilon(y, y^+, z) = \lambda_i [z^{n-1} S_k(\tau(y))]_+ \Upsilon(y, y^+, z).$$

**Proof.** This is just a calculation. It is obvious that $\tilde{W}_{k,i,n} \tau(y) = 0$ for $n > 0$. For $n = 0$, using the differential equation (3.14), we have

$$\tilde{W}_{k,i,0} \tau(y) = \sum_{l \in \mathbb{N} \cap |\Sigma|} l_i Q^{d(k,l)} y_{l,0} \frac{\partial \tau(y)}{\partial y_{l+k}} = \sum_{l \in \mathbb{N} \cap |\Sigma|} l_i Q^{d(k,l)} y_{l,0} S_{l+k}(\tau(y))$$

$$= \sum_{l \in \mathbb{N} \cap |\Sigma|} l_i y_{l,0} S_k(\tau(y)) \ast_{\tau(y)} S_l(\tau(y)) \quad \text{by Proposition 2.4 (3)}$$

$$= \lambda_i S_k(\tau(y)) \quad \text{by Remark 3.7}$$

On the other hand, using the differential equation (3.13), we have

$$\tilde{W}_{k,i,n} \Upsilon = k_i z^n S_k(\tau(y)) \Upsilon + \sum_{l \in \mathbb{N} \cap |\Sigma|} \sum_{m=0}^{\infty} l_i Q^{d(k,l)} y_{l,m} [z^{n+m-1} S_{l+k}(\tau(y))]_+ \Upsilon$$

$$= k_i z^n S_k(\tau(y)) \Upsilon + \sum_{l \in \mathbb{N} \cap |\Sigma|} \sum_{m=0}^{\infty} l_i y_{l,m} z^{n+m-1} S_k(\tau(y)) S_l(\tau(y)) \Upsilon$$

$$- \delta_{n,0} \sum_{l \in \mathbb{N} \cap |\Sigma|} l_i y_{l,m} z^{-1} S_k(\tau(y)) \ast_{\tau(y)} S_l(\tau(y)) \ast_{\tau(y)} \Upsilon \quad \text{by Proposition 2.4 (3)}$$

$$= z^n S_k(\tau(y)) \left[ k_i \Upsilon + \sum_{l \in \mathbb{N} \cap |\Sigma|} \sum_{m=0}^{\infty} l_i y_{l,m} \left( \frac{\partial \Upsilon}{\partial y_{l,m}} + \delta_{m,0} z^{-1} S_l(\tau(y)) \ast_{\tau(y)} \Upsilon \right) \right]$$

$$- \delta_{n,0} z^{-1} \lambda_i S_k(\tau(y)) \ast_{\tau(y)} \Upsilon \quad \text{by Remark 3.7}$$

$$= z^n S_k(\tau(y)) (k_i + z^{-1} \lambda_i) \Upsilon - \delta_{n,0} \lambda_i S_k(\tau(y)) \ast_{\tau(y)} \Upsilon = \lambda_i [z^{n-1} S_k(\tau(y))]_+ \Upsilon.$$

where in the last line we used the equation $\sum_{l \in \mathbb{N} \cap |\Sigma|} \sum_{m=0}^{\infty} l_i y_{l,m} \partial_{y_{l,m}} \Upsilon = 0$, which follows from the expansion (3.14). \qed
Let $R$ be a linearly topologized $\mathbb{C}[Q]$-algebra. Let $R_{\text{nilp}}$ denote the ideal of $R$ consisting of topologically nilpotent elements, i.e. elements $x \in R$ such that $\lim_{n \to \infty} x^n = 0$. We assume that $R$ is complete and Hausdorff and that, for any neighbourhood $U$ of zero in $R$, there exists $n \in \mathbb{N}$ such that $(R_{\text{nilp}})^n \subset U$. An $R$-valued point on $\text{Spf}(\mathbb{C}[Q][y,y^+])$ is given by a collection $\{\log y_1, \ldots, \log y_m\} \cup \{y_k : k \in G\} \cup \{y_{n,k} : k \in \mathbb{N} \cap |\Sigma|, n = 1, 2, \ldots\}$ of elements of $R_{\text{nilp}}$ such that every neighbourhood $U \subset R$ of 0 contains all but finitely many elements of this collection. An $R$-valued point on the formal group $JG$ is described similarly. For an $R$-valued point $(y, y^+)$, we can make sense of $\tau(y) \in H^*_T(X_\Sigma) \otimes_R R$ and $\Upsilon(y, y^+, z) \in H^*_T(X_\Sigma)[z] \otimes_R R$ where $\otimes$ denotes the completed tensor product. Consider the map

$$ (y, y^+) \mapsto (\sigma(y), \Xi(y, y^+, z)) \in H^*(X_\Sigma)[Q] \times H^*(X_\Sigma)[z][Q] $$

defined as the non-equivariant limit of $(\tau(y), \Upsilon(y, y^+, z))$. We will see that this map classifies the $JG$-orbit on the space $\text{Spf}(\mathbb{C}[Q][y,y^+])$ of oscillatory forms.

**Theorem 4.8.** Let $R$ be as above, and let $(y_1, y^+_1)$, $(y_2, y^+_2)$ be two $R$-valued points of $\text{Spf} \mathbb{C}[Q][y,y^+]$. Then the following are equivalent:

1. there exists an $R$-valued point $g \in JG(R)$ of $JG$ such that $g : [e^{F(x,y_1)}/z \omega(y^+_1)] = [e^{F(x,y_2)}/z \omega(y^+_2)];$
2. one has $\sigma(y_1) = \sigma(y_2)$ and $\Xi(y_1, y^+_1, z) = \Xi(y_2, y^+_2, z)$.

**Proof.** (1) $\Rightarrow$ (2): Since the exponential map identifies the formal neighbourhood of the origin of $T_e(JG)$ with $JG$, we can work at the Lie algebra level. Lemma 4.7 implies that the generators $\tilde{W}_{k,i,n}$ of $T_e(JG)$ act on $\sigma(y), \Xi(y, y^+, z)$ trivially. Thus $\sigma$ and $\Xi$ are constant along a $JG$-orbit.

(2) $\Rightarrow$ (1): It follows from (3.8), (3.13) that

$$ (\delta_{n,0}S_k(\tau(y)), [z^{n-1}S_k(\tau(y))] + \Upsilon(y, y^+, z))|_{y=y^+, z=0, Q=0} $$

form a $\mathbb{C}$-basis of $H^*_T(X_\Sigma) \times H^*_T(X_\Sigma)[z]$. Define a $\mathbb{C}$-basis $\{v_{l,p} : l \in \mathbb{N} \cap |\Sigma|, p = 0, 1, 2, \ldots\}$ of $H^*_T(X_\Sigma) \times H^*_T(X_\Sigma)[z]$ by

$$ v_{l,p} = \begin{cases} (\phi_l, 0) & p = 0; \\ (0, z^{p-1} \phi_l) & p > 0. \end{cases} $$

Then we can write

$$ v_{l,p} = \sum_{k \in \mathbb{N} \cap |\Sigma|} \sum_{n=0}^{\infty} c_{k,n,l,p} (\delta_{n,0}S_k(\tau(y)), [z^{n-1}S_k(\tau(y))]) + \Upsilon(y, y^+, z) $$

for some (unique) coefficients $c_{k,n,l,p} \in \mathbb{C}[Q][y,y^+]$ such that $\lim_{|k| + n \to \infty} c_{k,n,l,p} = 0$. Define the vector fields $\xi_{l,p,i} = \sum_{k \in \mathbb{N} \cap |\Sigma|} \sum_{n=0}^{\infty} c_{k,n,l,p} \tilde{W}_{k,i,n}$. Then Lemma 4.7 implies:

$$ \xi_{l,p,i} (\tau(y), \Upsilon(y, y^+, z)) = \lambda_i v_{l,p}. $$

When we assume (2), we can find $r_{l,p,i} \in R_{\text{nilp}}$ such that $\lim_{|l| + p \to \infty} r_{l,p,i} = 0$ and

$$ (\tau(y_2), \Upsilon(y_2, y^+_2, z)) - (\tau(y_1), \Upsilon(y_1, y^+_1, z)) = \sum_{l \in \mathbb{N} \cap |\Sigma|} \sum_{p=0}^{D} \sum_{i=1}^{D} r_{l,p,i} \lambda_i v_{l,p}. $$
Such $r_{l,p,i}$ are not unique. They define an “$R$-dependent” vector field $\sum_{l,p,i} r_{l,p,i} X_{l,p,i}$ on the formal scheme $\mathcal{M}_R := \text{Spf}(\mathbb{C}[Q][y, y^+]) \times_{\text{Spf}(\mathbb{C}[Q])} \text{Spf}(R) = \text{Spf}(R[y, y^+])$ over $R$. Since $r_{l,p,i}$ are topologically nilpotent, we can integrate this vector field to obtain an automorphism of $\mathcal{M}_R$ over $R$ (see Theorem [A.2]). By construction, this automorphism sends the $R$-valued point $(y_1, y_2^+)$ to $(y_2, y_2^+)$. By integrating the corresponding Lie algebra element in the formal group $(J\mathcal{G})_R = J\mathcal{G} \times_{\text{Spf}(\mathbb{C}[Q])} \text{Spf}(R)$ over $R$, we obtain an element $g \in J\mathcal{G}(R)$ which sends $e^{F(x; y_1)/z} \omega(y_1)$ to $e^{F(x; y_2)/z} \omega(y_2)$.

To obtain a primitive form in the non-equivariant setting, we need to choose a formal section $s: H^*(X_\Sigma)[Q] \to H^*_T(X_\Sigma)[Q]$ as in [41] and a formal function $f: H^*(X_\Sigma)[Q] \to H^*_T(X_\Sigma)[Q]$ that is expanded as

$$f(\sigma) = \sum_{k \in \mathbb{N} \cap |\Sigma|} \sum_{n=0}^{\infty} f_{k,n}(\sigma) z^n \phi_k$$

with $f_{k,n}(\sigma) \in \mathbb{C}[Q][s_0, \ldots, s_N]$ such that $f_{k,n}(0)|_{Q=0} = \delta_{k,0} \delta_{n,0}$, $\lim_{|\sigma| \to \infty} f_{k,n}(\sigma) = 0$ in the topology of $\mathbb{C}[Q][s_0, \ldots, s_N]$ and that the image of $f(\sigma)$ under the natural map $H^*_T(X_\Sigma)[Q] \to H^*(X_\Sigma)[z][Q]$ is 1. By the isomorphism in Lemma [3.24], we obtain a Landau-Ginzburg potential $s^* F = F(x; y(\sigma))$ and a primitive form $\zeta(s, f) = \omega(y^+(\sigma))$ such that $s(\sigma) = \tau(y(\sigma)), f(\sigma) = \Upsilon(y(\sigma), y^+(\sigma), z)$.

The cohomology class $[\zeta(s, f)]$ maps to 1 under the isomorphism $s^* \Theta_{\text{noneq}}: \text{GM}(s^* F) \cong H^*(X_\Sigma)[Q][s_0, \ldots, s_N]$.

**Corollary 4.9.** Any oscillatory primitive forms $\exp(s^* F/z) \zeta(s, f)$ associated to various data $(s, f)$ as above are related to each other by a co-ordinate change in the $x$-variables, i.e. they are contained in a single $J\mathcal{G}(\mathbb{C}[Q][s_0, \ldots, s_N])$-orbit.

## 5. Extended I-function

In this section we relate the mirror map $\tau(y)$, the function $\Upsilon(y, z)$ (or $\Upsilon(y, y^+, z)$) and the primitive form $\zeta$ with certain hypergeometric series called the (extended) $I$-function. This gives us a concrete algorithm to calculate these quantities, although actual computations could be very complicated.

**Definition 5.1 ([12]).** Define a cohomology-valued hypergeometric series in the variables $y = \{ y_k : k \in \mathbb{N} \cap |\Sigma| \}$ as follows:

$$I(y, z) = z e^{\sum_{i=1}^n u_i \log y_i/z} \sum_{\ell \in \mathbb{Z}_{\text{eff}}} y^\ell Q^{d(\ell)} \left( \prod_{i=1}^m \frac{\prod_{c=-\infty}^{0}(u_i + cz)}{\prod_{c=-\infty}^{0}(u_i + cz)} \right) \frac{1}{\prod_{k \in G} \ell_k! z^{\ell_k}}$$

where we used the notation from [33.2]. This belongs to $H^*_T(X_\Sigma)_{\text{loc}}[Q][y]$ and is called the extended $I$-function.

Recall from Remark [22.8] that the image of the fundamental solution $M(\tau, z)$ sweeps the Givental cone in $H^*_T(X_\Sigma)_{\text{loc}}$. We show that the extended $I$-function is on the Givental cone ([22, 12]).
Proposition 5.2. Let $\tau(y)$, $\Upsilon(y, z)$ denote the functions from Proposition 3.3. We have $I(y, z) = zM(\tau(y), z)\Upsilon(y, z)$.

Proof. This proposition was proved in [37, §4.3] along the locus $\{y_k = 0 : k \in G\}$. It suffices to show that both $I(y, z)$ and $zM(\tau(y), z)\Upsilon(y, z)$ satisfy the same differential equation in $y_k$ for $k \in G$. We claim that both functions satisfy:

$$\frac{\partial f(y, z)}{\partial y_k} = z^{-1}S_k f(y, z).$$

Since $V_k$ corresponds to the linear vector field $f \mapsto z^{-1}S_k f$ on the Givental space $\hat{H}^*_T(X_\Sigma)_\text{loc}$ (see Remark 2.8, [37, §4.3]), the differential equation holds for $f = zM(\tau(y), z)\Upsilon(y, z)$. We show that the differential equation holds for $f = I(y, z)$. Let $x \in X_\Sigma$ be a $T$-fixed point. Let $I_x(y, z)$ denote the restriction of $I(y, z)$ to $x$. By Definition 2.3, we need to show that:

$$(5.1) \quad z \frac{\partial}{\partial y_k} I_x(y, z) = \Delta_x(k) e^{-ck\partial_y} I_x(y, z)$$

with

$$\Delta_x(k) = Q^{\Psi(k) - \sigma_{\min(k)}} \prod_{i=1}^m \frac{\prod_{c=-\infty}^{0} (u_i(x) + cz)}{\prod_{c=-\infty}^{-u_i(x)k} (u_i(x) + cz)} z^{\sum_{i=1}^m (u_i(x) \log y_i / z - (u_i(x) - k) \log y_i)}$$

where $u_i(x) \in H^2_T(pt)$ denotes the restriction of $u_i$ to $x$. Recall that $\sigma_{\min(k)} \in H^2_T(E_k)$ corresponds to $-\Psi(k) \in \mathbb{Z}^m \cong H^2_T(X_\Sigma, \mathbb{Z})$ by Lemma 3.3. A similar argument shows that the section class $\sigma_x$ associated to the fixed point $x$ corresponds to $(-u_i(x) \cdot k)_{i=1}^m$. Therefore the right-hand side of (5.1) equals:

$$Q^{\Psi(k) - \sum_{i=1}^m (u_i(x)k) e_i} \prod_{i=1}^m \frac{\prod_{c=-\infty}^{0} (u_i(x) + cz)}{\prod_{c=-\infty}^{-u_i(x)k} (u_i(x) + cz)} z^{\sum_{i=1}^m (u_i(x) \log y_i / z - (u_i(x) - k) \log y_i)}$$

$$\times \sum_{\ell \in \mathbb{L}_{\text{eff}}} Q^{d(\ell)} y^\ell \left(\prod_{i=1}^m \frac{\prod_{c=-\infty}^{-u_i(x)k} (u_i(x) + cz)}{\prod_{c=-\infty}^{-u_i(x)k} (u_i(x) + cz)}\right) \frac{1}{\prod_{l \in G \setminus l_i! z^l_i}}$$

Note that we can write the extended $I$-function as a sum over $\ell \in \hat{\mathbb{L}}$ (by replacing $l_i!$ with $\Gamma(1 + \ell_i)$) since the summand corresponding to $\ell \notin \mathbb{L}_{\text{eff}}$ automatically vanishes. Shifting the index $\ell$ as $\ell \rightarrow \ell + \sum_{i=1}^m (u_i(x) \cdot k)e_{b_i} - e_k$, we find that this equals the left-hand side of (5.1).

We explain that the functions $(\tau(y), \Upsilon(y, z))$ are obtained from $I(y, z)$ via the Birkhoff factorization [14]. Consider the $\mathbb{C}[z][Q][y]$-linear map $dI: H^*_T(X_\Sigma)[z][Q][y] \to H^*_T(X_\Sigma)[(z^{-1})][Q][y]$ sending $\phi_k$ to $\frac{\partial I(y, z)}{\partial y_k}$ for $k \in \mathbb{N} \cap |\Sigma|$. Here we used the embedding $H^*_T(X_\Sigma)_\text{loc} \hookrightarrow H^*_T(X_\Sigma)[(z^{-1})]$ given by the Laurent expansion at $z = \infty$. We have

$$\frac{\partial I(y, z)}{\partial y_k} = M(\tau(y), z) \left( z \frac{\partial \Upsilon(y, z)}{\partial y_k} + \frac{\partial \tau(y)}{\partial y_k} \Upsilon(y, z) \right)$$

$$= M(\tau(y), z) S_k(\tau(y)) \Upsilon(y, z).$$
Let $\mathcal{S}\Upsilon : H^*_p(X_\Sigma)[z][Q][y] \to H^*_p(X_\Sigma)[z][Q][y]$ denote the $\mathbb{C}[z][Q][y]$-linear map sending $\phi_k$ to $\mathcal{S}_k(\tau(y))\Upsilon(y,z)$. Then we have:

$$dI = M(\tau(y), z) \circ \mathcal{S}\Upsilon.$$ 

This can be viewed as the Birkhoff factorization of $dI$ when we regard $z$ as a loop parameter; notice that $M(\tau(y), z)$ belongs to $\text{End}_\mathbb{C}(H^*_p(X_\Sigma))[z^{-1}][Q][y]$ and that $M(\tau(y), z = \infty) = \text{id}$. The Birkhoff factorization can be performed recursively in powers in $Q$ and $y$, and this gives a concrete algorithm to compute $M(\tau(y), z)$ and $\Upsilon(y)$. The mirror map $\tau(y)$ is then obtained from the expansion:

$$M(\tau(y), z)1 = 1 + \frac{\tau(y)}{z} + o(z^{-1}).$$

Once we obtain $\tau(y)$ and $\mathcal{S}\Upsilon$, we can calculate the inverse mirror map $y = y(t)$ and the primitive form $\zeta = \sum_{k \in \mathbb{N} \cap |\Sigma|} c_k(y, z)w^{\Phi(k)}\omega$ by the requirement (see §3.4)

$$\sum_{k \in \mathbb{N} \cap |\Sigma|} c_k(y, z)S_k(\tau(y))\Upsilon(y, z) = 1.$$ 

Finally we extend Proposition 5.2 to the function $\Upsilon(y, y^+, z)$ in Theorem 3.23 and describe an alternative method to calculate the primitive form. Let $y^+ = \{y_{k,n} : k \in \mathbb{N} \cap |\Sigma|, n = 1, 2, 3, \ldots \}$ be the variables in §3.4 and consider

$$y_k(z) = y_k + \sum_{n=1}^{\infty} y_{k,n} z^n.$$ 

We write $y(z) = \{y_k(z) : k \in \mathbb{N} \cap |\Sigma|\}$.

**Proposition 5.3.** Let $\Upsilon(y, y^+, z)$ be as in Theorem 3.23. We have

$$I(y(z), z) = zM(\tau(y), z)\Upsilon(y, y^+, z)$$

where $I(y(z), z)$ is obtained from the extended $I$-function $I(y, z)$ by replacing $y_k$ with $y_k(z)$.

**Proof.** It suffices to show that both sides satisfy the same differential equation:

$$\frac{\partial f(y, y^+)}{\partial y_{k,n}} = z^{n-1}S_k f(y, y^+).$$

In the proof of Proposition 5.2, we showed that $z \frac{\partial I(y,z)}{\partial y_k} = S_k I(y, z)$. Thus $f = I(y(z), z)$ satisfies the above differential equation. The differential equation for $f = zM(\tau(y), z)\Upsilon(y, y^+, z)$ follows easily from Proposition 2.4 (1) and Theorem 3.23. □

Recall from §3.4 that the primitive form $\zeta$ is given by $\omega(y^+)$ in (3.12) such that $\Upsilon(y, y^+, z) = 1$. Thus the asymptotics (5.2) implies:

**Corollary 5.4.** The primitive form $\zeta$ is given by $\omega(y^+)$ for $y^+$ such that the asymptotics $I(y(z), z) = z(1 + O(z^{-1}))$ holds. Moreover, for such $y^+$, the asymptotics $I(y(z), z) = z + \tau(y) + O(z^{-1})$ determines the mirror map $\tau(y)$.

**Remark 5.5.** Proposition 5.3 implies that $I(y(z), z)$ lies on the Givental cone. In fact, the family of vectors $(y, y^+) \mapsto I(y(z), z)$ covers the whole Givental cone and $(y, y^+)$ may be viewed as a B-model co-ordinate system on the cone.
Remark 5.6. When we identify the space \( \text{Spf}(\mathbb{C}[Q][y, y^+]) \) with the Givental cone as in the above remark, we can interpret Theorem 4.8 as follows: the non-equivariant Givental cone is the orbit space of the equivariant Givental cone under the action of the group \( JG \) of reparametrizations of the mirror.

Remark 5.7. The formal geometry appearing in this paper is very similar to the treatment of the Givental cone as a formal scheme in [11, Appendix B].

Remark 5.8. It should be possible to generalize the results in this paper to toric orbifolds (or toric Deligne-Mumford stacks). This is interesting since toric orbifolds correspond to arbitrary Laurent polynomials. See [13, 30, 16, 27, 12, 7, 56] for related works.

APPENDIX A. FORMAL GEOMETRY IN INFINITE DIMENSIONS

For the sake of completeness, we prove a formal inverse function theorem and the existence of a flow of a vector field in infinite dimensions. The results here are straightforward generalizations of well-known results in finite dimensions, but we could not find a reference. Throughout the section, we assume that \( R \) is a linearly topologized ring containing \( \mathbb{Q} \) and that \( R \) is complete and Hausdorff. We denote by \( \{R_\nu\} \) a fundamental neighbourhood system of zero consisting of ideals of \( R \).

Let \( x = \{x_1, x_2, x_3, \ldots\} \) be a countably infinite set of variables. A morphism \( f : \text{Spf}(R[x]) \rightarrow \text{Spf}(R[x]) \) of formal schemes over \( R \) (see §2.1 for \( R[x] \)) is given by a tuple \( \{f^*(x_1), f^*(x_2), f^*(x_3), \ldots\} \) of elements in \( R[x] \) such that \( f^*(x_i)|_{x=0} \in R_{\text{nilp}} \) and \( \lim_{n \to \infty} f^*(x_n) = 0 \), where \( R_{\text{nilp}} = \{x \in R : \lim_{n \to \infty} x^n = 0\} \). Consider the \( R \)-module

\[
T := \left\{(r_n)_{n=1}^\infty \in \mathbb{R}^N : \lim_{n \to \infty} r_n = 0\right\} \cong \left(\mathbb{Q}^\mathbb{N}\right) \widehat{\otimes} R.
\]

The topology on \( T \) is defined by submodules \( \left(\mathbb{Q}^\mathbb{N}\right) \widehat{\otimes} R_\nu \). A morphism \( f \) associates the (continuous) tangent map \( df : T \rightarrow T \) defined by \( df(e_i) = \sum_{j=1}^\infty \left. \frac{\partial f^*(x_j)}{\partial x_i} \right|_{x=0} e_j \). The following gives two important classes of morphisms.

- for a continuous \( R \)-module homomorphism \( A : T \rightarrow T \) with \( A(e_i) = \sum_{j=1}^\infty a_{ij}e_j \), we have a linear map \( f \) given by \( f^*(x_j) = \sum_{j=1}^\infty a_{ij}x_i \);
- for an element \( (r_j)_{j=1}^\infty \in T \) with \( r_j \in R_{\text{nilp}} \), we have a translation map \( f \) given by \( f^*(x_j) = x_j + r_j \).

Theorem A.1 (formal inverse function theorem). Let \( f : \text{Spf}(R[x]) \rightarrow \text{Spf}(R[x]) \) be a morphism of formal schemes over \( R \). If the tangent map \( df : T \rightarrow T \) at \( x = 0 \) is an isomorphism, \( f \) is an isomorphism.

**Proof.** By composing with a linear map and a translation, we may assume that \( f(0) = 0 \) and the tangent map \( df \) is the identity. Then the truncation of \( f^* \) given by \( R[x_1, \ldots, x_n] \subset R[x] \xrightarrow{f^*} R[x_1, \ldots, x_{n-1}] \) is an isomorphism, by the inverse function theorem in finite dimensions (see [31, Appendix A]; the proof over a discrete ring works verbatim over \( R \)). It follows easily that \( f^* \) is an isomorphism. \( \square \)

Next we discuss the integrability of a formal vector field. A formal vector field on \( \text{Spf}(R[x]) \) over \( R \) is a formal sum \( V = \sum_{n=1}^\infty V_n(x) \frac{\partial}{\partial x_n} \) with \( V_n(x) \in R[x] \) such that
\[ \lim_{n \to \infty} V_n(x) = 0. \] We consider the flow \( t \mapsto x(t) = (x_n(t))_{n=1}^{\infty} \) satisfying

\[ \frac{dx_n(t)}{dt} = V_n(x(t)) \quad \text{with } x_n(0) = x_n. \tag{A.1} \]

**Theorem A.2.** There exists a unique solution \( x(t) = (x_n(t))_{n=1}^{\infty} \) to the equation \((A.1)\) which defines a morphism \( \text{Spf}(R[[x]] \langle t \rangle) \to \text{Spf}(R[[x]]) \) of formal schemes. Let \( I \subset R \) be an ideal such that, for any \( \nu \), there exists \( n \in \mathbb{N} \) such that \( I^n \subset R_\nu \). If \( V_n(x) \in I[x] \) for all \( n \), then the substitution \( t = 1 \) in the solution \( x(t) \) is well-defined and we obtain a time-one flow map \( \text{Spf}(R[[x]]) \to \text{Spf}(R[[x]]) \).

**Proof.** Note that \( V \) defines a well-defined continuous mapping \( V : R[x] \to R[x] \). The flow is given by a continuous ring homomorphism \( R[[x]] \to R[[x]] \langle t \rangle \) defined by \( \varphi \mapsto \exp(tV)\varphi = \sum_{k=0}^{\infty} \frac{1}{k!} V^k(\varphi) \), where \( V^k \) is the \( k \)-fold composition of \( V \), see \([33, 3C]\). The former statement follows. To see the latter, it suffices to notice that \( \lim_{k \to \infty} V^k(\varphi) = 0 \) uniformly for all \( \varphi \in R[[x]] \) under the assumption. \( \qed \)

**References**

[1] Alan Adolphson. Hypergeometric functions and rings generated by monomials. *Duke Math. J.*, 73(2):269–290, 1994.
[2] M. F. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984.
[3] Sergeui Barannikov. Semi-infinite Hodge structure and mirror symmetry for projective spaces. arXiv:math.AG/0010157, 2001.
[4] Lev A. Borisov and R. Paul Horja. On the better behaved version of the GKZ hypergeometric system. *Math. Ann.*, 357(2):585–603, 2013.
[5] Alexander Braverman, Davesh Maulik, and Andrei Okounkov. Quantum cohomology of the Springer resolution. *Adv. Math.*, 227(1):421–458, 2011.
[6] Kwokwai Chan, Siu-Cheong Lau, Naichung-Conan Leung, and Hsian-Hua Tseng. Open Gromov-Witten invariants, mirror maps, and Seidel representations for toric manifolds. arXiv:1209.6119, 2012.
[7] Daewoong Cheong, Ionut Ciocan-Fontanine, and Bumsig Kim. Orbifold quasimap theory. arXiv:1405.7160 [math.AG], 2014.
[8] T.-M. Chiang, A. Klemm, S.-T. Yau, and E. Zaslow. Local mirror symmetry: calculations and interpretations. *Adv. Theor. Math. Phys.*, 3(3):495–565, 1999.
[9] Cheol-Hyun Cho and Yong-Geun Oh. Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds. *Asian J. Math.*, 10(4):773–814, 2006.
[10] Ionut Ciocan-Fontanine and Bumsig Kim. Wall-crossing in genus-zero quasimap theory and mirror maps. 2013. arXiv:1304.7506.
[11] Tom Coates, Alessio Corti, Hiroshi Iritani, and Hsian-Hua Tseng. Computing genus-zero twisted Gromov-Witten invariants. *Duke Math. J.*, 147(3):377–438, 2009.
[12] Tom Coates, Alessio Corti, Hiroshi Iritani, and Hsian-Hua Tseng. A mirror theorem for toric stacks. arXiv:1310.4163 [math.AG], 2013.
[13] Tom Coates, Alessio Corti, Yuan-Pin Lee, and Hsian-Hua Tseng. The quantum orbifold cohomology of weighted projective spaces. *Acta Math.*, 202(2):139–193, 2009.
[14] Tom Coates and Alexander Givental. Quantum Riemann-Roch, Lefschetz and Serre. *Ann. of Math.* (2), 165(1):15–53, 2007.
[15] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.
[16] Antoine Douai and Etienne Mann. The small quantum cohomology of a weighted projective space, a mirror \( D \)-module and their classical limits. *Geom. Dedicata*, 164:187–226, 2013.
[17] Antoine Douai and Claude Sabbah. Gauss-Manin systems, Brieskorn lattices and Frobenius structures. II. In *Frobenius manifolds*, Aspects Math., E36, pages 1–18. Vieweg, Wiesbaden, 2004.
[18] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. Lagrangian floer theory and mirror symmetry on compact toric manifolds. arXiv:1009.1648, 2010.

[19] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. Lagrangian Floer theory on compact toric manifolds. I. Duke Math. J., 151(1):23–174, 2010.

[20] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. Lagrangian Floer theory on compact toric manifolds II: bulk deformations. Selecta Math. (N.S.), 17(3):609–711, 2011.

[21] Alexander Givental. Elliptic Gromov-Witten invariants and the generalized mirror conjecture. In Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), pages 107–155. World Sci. Publ., River Edge, NJ, 1998.

[22] Alexander Givental. A mirror theorem for toric complete intersections. In Topological field theory, primitive forms and related topics (Kyoto, 1996), volume 160 of Progr. Math., pages 141–175. Birkhäuser Boston, Boston, MA, 1998.

[23] Alexander Givental. Symplectic geometry of Frobenius structures. In Frobenius manifolds, Aspects Math., E36, pages 91–112. Friedr. Vieweg, Wiesbaden, 2004.

[24] Alexander B. Givental. Homological geometry and mirror symmetry. In Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), pages 472–480. Birkhäuser, Basel, 1995.

[25] Eduardo González and Hiroshi Iritani. Seidel elements and mirror transformations. Selecta Math. (N.S.), 18(3):557–590, 2012.

[26] Eduardo González and Hiroshi Iritani. Seidel elements and potential functions for holomorphic disc counting. arXiv:1301.5454, 2013.

[27] Eduardo González and Chris Woodward. Quantum cohomology and toric minimal model programs. 2012. arXiv:1010.2118.

[28] Tom Graber and Rahul Pandharipande. Localization of virtual classes. Invent. Math., 135(2):487–518, 1999.

[29] Mark Gross. Mirror symmetry for $\mathbb{P}^2$ and tropical geometry. Adv. Math., 224(1):169–245, 2010.

[30] Martin Guest and Hironori Sakai. Orbifold quantum D-modules associated to weighted projective spaces. Comment. Math. Helv., 89(2):273–297, 2014.

[31] Michiel Hazewinkel. Formal groups and applications, volume 78 of Pure and Applied Mathematics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[32] Kentaro Hori and Cumrum Vafa. Mirror symmetry. arXiv:hep-th/0002222, 2000.

[33] Yulij Ilyashenko and Sergei Yakovenko. Lectures on analytic differential equations, volume 86 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2008.

[34] Hiroshi Iritani. Convergence of quantum cohomology by quantum Lefschetz. J. Reine Angew. Math., 610:29–69, 2007.

[35] Hiroshi Iritani. Quantum D-modules and generalized mirror transformations. Topology, 47(4):225–276, 2008.

[36] Hiroshi Iritani. An integral structure in quantum cohomology and mirror symmetry for toric orbifolds. Adv. Math., 222(3):1016–1079, 2009.

[37] Hiroshi Iritani. Shift operators and toric mirror theorem. arXiv:1411.6840 [math.AG], 2014.

[38] Yukiko Konishi and Satoshi Minabe. Local B-model and mixed Hodge structure. Adv. Theor. Math. Phys., 14(4):1089–1145, 2010.

[39] Changzheng Li, Si Li, and Kyoji Saito. Primitive forms via polyvector fields. arXiv:1311.1655, 2013.

[40] Bong H. Lian, Kefeng Liu, and Shing-Tung Yau. Mirror principle. I. Asian J. Math., 1(4):729–763, 1997.

[41] Davesh Maulik and Andrei Okounkov. Quantum groups and quantum cohomology. arXiv:1211.1287 [math.AG], 2012.

[42] Dusa McDuff and Susan Tolman. Topological properties of Hamiltonian circle actions. IMRP Int. Math. Res. Pap., pages 72826, 1–77, 2006.

[43] Takao Mochizuki. Twistor property of GKZ-hypergeometric systems. 2015. arXiv:1501.04146.

[44] Tadao Oda. Convex bodies and algebraic geometry, volume 15 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1988. An introduction to the theory of toric varieties, Translated from the Japanese.
[45] Andrei Okounkov and Rahul Pandharipande. The quantum differential equation of the Hilbert scheme of points in the plane. *Transform. Groups*, 15(4):965–982, 2010.

[46] Yaron Ostrover and Ilya Tyomkin. On the quantum homology algebra of toric Fano manifolds. *Selecta Math. (N.S.)*, 15(1):121–149, 2009.

[47] Rahul Pandharipande. Rational curves on hypersurfaces (after A. Givental). *Asterisque*, (252):Exp. No. 848, 5, 307–340, 1998. Séminaire Bourbaki. Vol. 1997/98.

[48] Thomas Reichelt and Christian Sevenheck. Logarithmic Frobenius manifolds, hypergeometric systems and quantum d-modules. 2010. [arXiv:1010.2118](http://arxiv.org/abs/1010.2118).

[49] Thomas Reichelt and Christian Sevenheck. Non-affine Landau-Ginzburg models and intersection cohomology. [arXiv:1210.6527 [math.AG]](http://arxiv.org/abs/1210.6527).

[50] Claude Sabbah. Hypergeometric period for a tame polynomial. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(7):603–608, 1999. A longer version published in: Port. Math. (N.S.) 63 (2006), no.2, 173–226.

[51] Kyoji Saito. The higher residue pairings $K_p^{(k)}$ for a family of hypersurface singular points. 40:441–463, 1983.

[52] Kyoji Saito. Period mapping associated to a primitive form. *Publ. Res. Inst. Math. Sci.*, 19(3):1231–1264, 1983.

[53] Morihiko Saito. On the structure of Brieskorn lattices, II. [arXiv:1312.6629](http://arxiv.org/abs/1312.6629), 2013.

[54] Paul Seidel. $\pi_1$ of symplectic automorphism groups and invertibles in quantum homology rings. *Geom. Funct. Anal.*, 7(6):1046–1095, 1997. [arXiv:dg-ga/9511011](http://arxiv.org/abs/dg-ga/9511011).

[55] Chris Woodward. Quantum Kirwan morphism and Gromov-Witten invariants of quotients. 2012. [arXiv:1204.1765](http://arxiv.org/abs/1204.1765).

[56] Fenglong You. Seidel elements and mirror transformations for toric stacks. [arXiv:1411.7732](http://arxiv.org/abs/1411.7732), 2014.

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