We present a mathematical model to decompose a longitudinal deformation into normal and abnormal components. The goal is to detect and extract subtle quivers from periodic motions in a video sequence. It has important applications in medical image analysis. To achieve this goal, we consider a representation of the longitudinal deformation, called the Beltrami descriptor, based on quasiconformal theories. The Beltrami descriptor is a complex-valued matrix. Each longitudinal deformation is associated to a Beltrami descriptor and vice versa. To decompose the longitudinal deformation, we propose to carry out the low rank and sparse decomposition of the Beltrami descriptor. The low rank component corresponds to the periodic motions, whereas the sparse part corresponds to the abnormal motions of a longitudinal deformation. Experiments have been carried out on both synthetic and real video sequences. Results demonstrate the efficacy of our proposed model to decompose a longitudinal deformation into regular and irregular components.

Keywords: Longitudinal Deformation, Beltrami Descriptor; Low-rank, sparse, quasiconformal
descriptors is not sensitive to the error of the descriptors, which is crucial for the decomposition. To decompose the longitudinal deformation, we propose to extract the low rank part and sparse part of the Beltrami descriptor. The periodic motion of the deformation is characterized by the low rank component of the descriptor. On the other hand, subtle quivers are characterized by the sparse part of the Beltrami descriptor. This low rank and sparse pursuit problem can be relaxed to a complex-valued Robust Principal Component Analysis (RPCA) problem, which can be solved by alternating minimization method with multipliers (ADMM). We test our proposed model on both synthetic and real video sequences. Experimental results illustrate the efficacy of our proposed method for the decomposition of longitudinal deformations.

In short, our contributions of this paper are three-folded.

1. First, we propose to consider a special representation of longitudinal deformations, called the Beltrami descriptor, to decompose the deformation. The Beltrami descriptor captures the geometric information of the deformation, and hence manipulating the descriptor allows us to process and analyze the deformation according to its geometry.

2. Secondly, we consider the low rank and sparse decomposition of the Beltrami descriptor to decompose a longitudinal deformation into regular and irregular components. To the best of our knowledge, it is the first work to decompose a longitudinal deformation via low rank and sparse pursuit.

3. Thirdly, in practical applications, it is often desirable to extract bijective irregular longitudinal component, which detect and capture the abnormal subtle quivers from normal periodic motion. In this work, we theoretically show that the extracted irregular component is bijective under a suitable choice of parameters.

The paper is organized as follow: in section 2, we will briefly review some previous works related to this paper. In 3, some necessary mathematical tools will be described. The Beltrami descriptor and our proposed decomposition algorithm will be explained in details in section 4. Last but not least, experimental results will be shown in section 5 and we cap off with a conclusion in section 6.

## 2 Previous Work

Shape analysis of structures from images plays a fundamental role in various fields, such as computer visions and medical image analysis. One commonly used approach is done by analyzing the deformation fields between corresponding images. Deformation fields between images are often obtained through the image registration process. Registration aims to establish a meaningful one-to-one dense correspondence between images. Over years, various registration methods have been proposed, which can be categorized into feature-based [1][2], intensity-based[3][4], and combined-feature-intensity-based methods[5][6]. Amongst these methods, quasiconformal-based registration models have been widely used, with which our model in this paper is built upon. For instance, in [6], Lam et al. proposed an optimization model based on quasiconformal geometry to obtain landmark-based and intensity-based registration between images or surfaces.

Once the deformation fields are obtained, different shape analysis methods have been recently proposed. In [7][8], Lui et al. proposed to detect shape variation based on the Beltrami coefficients of the deformation field as well as the curvature mismatching. The method has been applied for Alzheimer’s disease analysis [9] and tooth morphometry [10].

A quasiconformal metric for deformation classification is also introduced to classify the left ventricle deformations of myopathic and control subjects [11]. The wavelet support vector machine (WSVM) has been proposed to study the deformation field [12]. Algorithms to analyze deformation field with different geometric scales and directions have also been recently developed. The basic idea is to decompose the vector field representing the deformation into various meaningful components. For instance, Tong et al. [13] proposed a variational model to decompose a vector field into the divergence-free part, the curl-free part, and the harmonic part using the idea of Helmholtz-Hodge decomposition. Recently, the morphlet transform has been proposed to obtain a multi-scale representation for diffeomorphisms [14].

Wavelet transform on the Beltrami coefficient of the deformation field has also been proposed to decompose a deformation into multiple components with various geometric scales [15]. However, to the best of our knowledge, an effective method to analyze time-dependent longitudinal deformation is still lacking.

In this work, our goal is to decompose a longitudinal deformation into normal and abnormal components. To do so, Robust Principal Component Analysis (RPCA) will be performed on the descriptor of the longitudinal deformation. RPCA has been widely studied in recent years and have been used for various applications. For example, Zhou et. al. [16] proposed “GoDec” that was adding one more noise term, so as to remove the noise captured by cameras. Also, Zhou et. al. [17] made an improvement by imposing one more constraint to ensure the moving objects are small and continuous pieces. Li et. al. [18] suggested another method, SSC-RPCA, that could work well when the background exhibits some minor motion, like flowing water of a lake or a river, or the moving object does not move fast enough,
with more terms into the original RPCA model to force the model to group different regions of the moving object in a roughly segmented video. Oreifej et. al. [19] presented another term to model turbulence to capture moving object in a badly turbulence-corrupted video. Sobral et. al. [20] proposed a way to improve detection of moving object by imposing shape constraints. Javed et. al. [21] put forward a superpixel-based matrix decomposition method with maximum norm regularizations and structured sparsity constraints to deal with the real-time challenge. The model designed by Ebdai et. al. [22] estimates the support of the foreground regions with a superpixel generation step, and then spatial coherence can be imposed. Cao et. al. [23] presented a novel method of RPCA, using tensor decomposition, as well as 3D total variation to enforce spatio-temporal continuity of the moving objects.

To compute the RPCA effectively, various numerical methods have been proposed. For example, Lin et. al. [24] compared two methods: accelerated proximal gradient algorithm applied to the primal and gradient algorithm applied to the dual problem. Another well-know optimization method, which is going to be used in this paper, is the Alternating Direction Method (ADM) proposed by Yuan et. al. [25], or similarly the Augmented Lagrange Multiplier Method proposed by Lin et. al. [26].

3 Mathematical Background

In this section, we will review some mathematical background related to this work.

3.1 Quasiconformal theories

In the following, some basic ideas of quasi-conformal geometry are discussed. For details, we refer readers to [27, 28].

A surface $S$ with a conformal structure is called a Riemann surface. Given two Riemann surfaces $M$ and $N$, a map $f : M \to N$ is conformal if it preserves the surface metric up to a multiplicative factor called the conformal factor. An immediate consequence is that every conformal map preserves angles. With the angle-preserving property, a conformal map effectively preserves the local geometry of the surface structure. A generalization of conformal maps is the quasi-conformal maps, which are orientation preserving homeomorphisms between Riemann surfaces with bounded conformality distortion, in the sense that their first order approximations take small circles to small ellipses of bounded eccentricity [27]. Mathematically, $f : \mathbb{C} \to \mathbb{C}$ is quasi-conformal provided that it satisfies the Beltrami equation:

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z},$$  \hspace{1cm} (1)

for some complex-valued function $\mu$ satisfying $||\mu||_\infty < 1$. $\mu$ is called the Beltrami coefficient, which is a measure of non-conformality. It measures how far the map at each point is deviated from a conformal map. In particular, the map $f$ is conformal around a small neighborhood of $p$ when $\mu(p) = 0$. Infinitesimally, around a point $p$, $f$ may be expressed with respect to its local parameter as follows:

$$f(z) = f(p) + f_x(p)z + f_z(p)\bar{z}$$
$$= f(p) + f_x(p)(z + \mu(p)\bar{z}).$$  \hspace{1cm} (2)

Obviously, $f$ is not conformal if and only if $\mu(p) \neq 0$. Inside the local parameter domain, $f$ may be considered as a map composed of a translation to $f(p)$ together with a stretch map $S(z) = z + \mu(p)\bar{z}$, which is composed by a multiplication of $f_x(p)$, which is conformal. All the conformal distortion of $S(z)$ is caused by $\mu(p)$. $S(z)$ is the map that causes $f$ to map a small circle to a small ellipse. From $\mu(p)$, we can determine the angles of the directions of maximal magnification and shrinking and the amount of them as well. Specifically, the angle of maximal magnification is $\arg(\mu(p))/2$ with magnifying factor $1 + |\mu(p)|$; The angle of maximal shrinking is the orthogonal angle $(\arg(\mu(p)) - \pi)/2$ with shrinking.
factor $1 - |\mu(p)|$. Thus, the Beltrami coefficient $\mu$ gives us lots of information about the properties of the map (See Figure[1]).

The maximal dilation of $f$ is given by:

$$K(f) = \frac{1 + ||\mu||_{\infty}}{1 - ||\mu||_{\infty}}.$$  

(3)

Given a Beltrami coefficient $\mu : \mathbb{C} \to \mathbb{C}$ with $||\mu||_{\infty} < 1$. There is always a quasiconformal mapping from $\mathbb{C}$ onto itself which satisfies the Beltrami equation in the distribution sense [27]. More precisely, we have the following theorem:

**Theorem 1 (Measurable Riemann Mapping Theorem).** Suppose $\mu : \mathbb{C} \to \mathbb{C}$ is Lebesgue measurable satisfying $||\mu||_{\infty} < 1$, then there is a quasiconformal homeomorphism $\phi$ from $\mathbb{C}$ onto itself, which is in the Sobolev space $W^{1,2}(\mathbb{C})$ and satisfied the Beltrami equation [7] in the distribution sense. Furthermore, by fixing 0, 1 and $\infty$, the associated quasiconformal homeomorphism $\phi$ is uniquely determined.

Theorem[1] suggests that under suitable normalization, a homeomorphism from $\mathbb{C}$ or $\mathbb{D}$ onto itself can be uniquely determined by its associated Beltrami coefficient.

### 3.2 Robust Principal Component Analysis (RPCA)

The RPCA problem is stated as follow: Suppose we are given a matrix $M \in \mathbb{R}^{m \times n}$. Then we would like to solve the following minimisation problem:

$$\min_{N,A} \text{rank}(N) + \alpha \|A\|_0, \text{ such that } M = N + A$$  

(4)

where $N$ and $A$ are supposed to be a low-rank and a sparse matrix respectively, and $\alpha$ a parameter describing the trade off between the rank of the low-rank matrix and the $L_0$ norm of the sparse matrix. Since the above problem is NP-hard, there is a common relaxation that is the following:

$$\min_{N,A} \|N\|_* + \alpha \|A\|_1, \text{ such that } M = N + A$$  

(5)

Given that equation[5] is a convex optimization problem, the ADM approach suggested by Yuan et. al. [25] is a suitable method. Namely, the Augmented Lagrangian function of equation[5] is:

$$\mathcal{L}(N, A, Z; M) = \|N\|_* + \alpha \|A\|_1 - \langle Z, N + A - M \rangle + \frac{\beta}{2} \|N + A - M\|_2^2$$  

(6)

where $Z$ is the multiplier of the linear constraint, $\beta$ the penalty parameter. Here, we use $\langle \cdot, \cdot \rangle$ to denote the trace inner product. A simple iterative scheme is as follow:

$$\begin{cases} N^{k+1} = \arg \min_{N \in \mathbb{R}^{m \times n}} \mathcal{L}(N, A^k, Z^k; M) \\ A^{k+1} = \arg \min_{A \in \mathbb{R}^{m \times n}} \mathcal{L}(N^{k+1}, A, Z^k; M) \\ Z^{k+1} = Z^k - \beta(N^{k+1} + A^{k+1} - M) \end{cases}$$  

(7)

[25][29][30][31] showed that there are closed formulas to update $N^{k+1}, A^{k+1}$ and, obviously, $Z^{k+1}$ at each step. To solve for $A^{k+1}$, we can use the explicit solution:

$$A^{k+1} = \frac{1}{\beta} Z^k - N^k + M - P_{\Omega_{\alpha/\beta}} \left( \frac{1}{\beta} Z^k - N^k + M \right)$$  

(8)

where $P_{\Omega_{\alpha/\beta}}$ denoted the Euclidean projection onto $\Omega_{\alpha/\beta}^{\alpha/\beta} := \{ X \in \mathbb{R}^{n \times n} \mid -\alpha/\beta \leq X_{ij} \leq \alpha/\beta \}$. For the subproblem $N^{k+1}$, the explicit solution is:

$$N^{k+1} = U^{k+1} \text{diag} \left( \max \{ \sigma_i^{k+1} - \frac{1}{\beta}, 0 \} \right) (V^{k+1})^T$$  

(9)

where $U^{k+1}, V^{k+1}, \sigma_i^{k+1}$ are obtained by SVD that is:

$$M - A^{k+1} + \frac{1}{\beta} Z^k = U^{k+1} \Sigma^{k+1} (V^{k+1})^T \text{ with } \Sigma^{k+1} = \text{diag} \left( \left\{ \sigma_i^{k+1} \right\}_{i=1}^r \right)$$  

(10)
4 Decomposition of Longitudinal Deformations

In this section, we explain our proposed main algorithm for the decomposition of longitudinal deformations. The goal is to separate abnormal deformations from normal deformations. To achieve this goal, it is necessary to have an effective representation of longitudinal deformations. The longitudinal deformation has to be easily restored from the corresponding representation. In addition, an effective algorithm to decompose the representation is also required.

4.1 Representation of longitudinal deformations.

In this work, we consider to represent the longitudinal deformations based on quasiconformal theories. An effective representation of longitudinal deformations should satisfy the following criteria.

1. First, the representation should capture the geometric information about the deformations. More precisely, it should describe the local geometric distortions under the deformations, so that the decomposition of longitudinal deformations can be achieved based on the geometry.
2. The corresponding longitudinal deformations can be easily restored from the representation, so that the deformation fields can be obtained after the decomposition of the representation is carried out.
3. The bijectivity of the corresponding deformations can be easily controlled during the manipulation of the representation. In other words, the corresponding deformations will not be severely corrupted during the decomposition process of the representation.

To achieve these objectives, we will consider a longitudinal deformation matrix based on the Beltrami coefficients. Suppose \( \{I_i\}_{i=1}^t \) are the video frames, each of size \( m \times n \), capturing the longitudinal data. Let \( I_{ref} \) be a reference image. For each frame \( I_j \), we compute the image registration \( f_j : \Omega \rightarrow \Omega \) from \( I_{ref} \) to \( I_j \). Here, \( \Omega \) refers to the rectangular image domain. The image registration can be computed using existing registration algorithms. In this work, the quasiconformal image registration method is applied.

Note that the image domain \( \Omega \) is discretized into uniformly distributed pixels. As such, we can consider that \( \Omega \) is discretized by regular triangulation \( \{V,E,F\} \), where \( V \) is the collection of vertices given by pixels. \( E \) and \( F \) are the collections of edges and faces respectively. With these notations, we assume \( f_i := (u_i,v_i) \), where \( u_i : V \rightarrow \mathbb{R} \) and \( v_i : V \rightarrow \mathbb{R} \) are the coordinate functions defined on every vertices. \( f_i \) is regarded as piecewise linear on each face. The quasiconformality or local geometric distortion of \( f_i \) can then be measured by the Beltrami coefficient.

For the piecewise linear map \( f_i \), we compute its Beltrami Coefficient by the approximation of its partial derivatives on each face \( T \in F \). The restriction of \( f_i \) on each face \( T \) can be written as

\[
f_i|_T(x,y) = \begin{pmatrix} a_T x + b_T y + r_T \\ c_T x + d_T y + q_T \end{pmatrix}
\]

Hence, \( D_x f_i(T) = a_T + i c_T \) and \( D_y f_i(T) = b_T + i d_T \). Then the gradient \( \nabla_T f_i \) can be obtained by solving:

\[
\begin{pmatrix} v_1 - v_0 \\ v_2 - v_0 \end{pmatrix} \begin{pmatrix} a_T \\ b_T \\ c_T \\ d_T \end{pmatrix} = \begin{pmatrix} u(v_1) - u(v_0) \\ v(v_1) - v(v_0) \\ u(v_2) - u(v_0) \\ v(v_2) - v(v_0) \end{pmatrix}
\]

where \( v_0, v_1 \) and \( v_2 \) are the three vertices of the face \( T \). By solving the above linear system, \( a_T, b_T, c_T, d_T \) can be computed. And the Beltrami coefficient of \( f_i \) on \( T \) can be obtained by

\[
\mu_i(T) = \frac{(a_T - d_T) + i (c_T + b_T)}{(a_T + d_T) + i (c_T - b_T)}
\]

We thus have the following definition of longitudinal deformation descriptor to represent the longitudinal deformations.

**Definition 1** (Longitudinal deformation descriptor). With the notations above, the longitudinal deformation descriptor \( L^\mu \) for \( \{f_i\}_{i=1}^t \) is a \( mn \times t \) complex-valued matrix given by

\[
L^\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_t \end{pmatrix}
\]

\( L^\mu \) is formulated using Beltrami coefficients, which capture the local geometric distortions under the longitudinal deformations. As it will be explained in the next subsection, \( L^\mu \) has a one-one correspondence with the longitudinal deformations. In other words, given \( L^\mu \), the associated longitudinal deformations can be reconstructed. On the other hand, according to quasiconformal theories, the deformation \( f_j \) is bijective (or folding-free) if \( ||\mu_j||_\infty < 1 \).
4.2 Reconstruction of longitudinal deformations from descriptors.

In the last subsection, we introduce the descriptor \( L^\mu \) to represent the longitudinal deformations. In order to utilize the descriptor, a reconstruction algorithm from the descriptor to the corresponding longitudinal deformations is necessary.

Let’s discuss how the longitudinal deformations can be reconstructed from \( L^\mu \). Consider \( f_j |T \) restricted to a triangle \( T \in \mathcal{F} \). Suppose the three vertices of \( T \) is given by \( v_0, v_1 \) and \( v_2 \), whose coordinates are given by \( v_k = (g_k, h_k) \) for \( k = 0, 1 \) or 2. \( v_0, v_1 \) and \( v_2 \) are deformed by \( f_j |T \) to \( w_0, w_1 \) and \( w_2 \), whose coordinates are given by \( w_k = (s_k, t_k) \) for \( k = 0, 1, 2 \). Denote \( \mu_j(T) = \rho_j + i\tau_j \). Let \( \gamma_1(T) = \frac{(\rho_2 - 1)^2 + \tau_2^2}{1 - \rho_2^2 - \tau_2^2} \), \( \gamma_2(T) = \frac{-2\rho_2}{1 - \rho_2^2 - \tau_2^2} \) and \( \gamma_1(T) = \frac{(1 + \rho_2)^2 + \tau_2^2}{1 - \rho_2^2 - \tau_2^2} \).

By comparing the real and imaginary parts, Equation (13) can be formulated as follows:

\[
\begin{align*}
    a_T &= \alpha_T^0 s_0 + \alpha_T^1 s_1 + \alpha_T^2 s_2; \\
    b_T &= \beta_T^0 s_0 + \beta_T^1 s_1 + \beta_T^2 s_2; \\
    c_T &= \alpha_T^0 t_0 + \alpha_T^1 t_1 + \alpha_T^2 t_2; \\
    d_T &= \beta_T^0 t_0 + \beta_T^1 t_1 + \beta_T^2 t_2.
\end{align*}
\]

where

\[
\begin{align*}
    \alpha_T^0 &= (h_2 - h_3)/A_T; & \alpha_T^1 &= (h_2 - h_0)/A_T; & \alpha_T^2 &= (h_0 - h_1)/A_T; \\
    \beta_T^0 &= (g_2 - g_3)/A_T; & \beta_T^1 &= (g_2 - g_0)/A_T; & \beta_T^2 &= (g_0 - g_1)/A_T;
\end{align*}
\]

Here, \( A_T \) refers to the area of \( T \). According to computational quasiconformal theories [27], \( a_T, b_T, c_T \) and \( d_T \) also satisfy the following linear equations:

\[
\begin{align*}
    \sum_{T \in N_i} \alpha_T^j (\gamma_1(T)a_T + \gamma_2(T)b_T) + \beta_T^j (\gamma_2(T)a_T + \gamma_3(T)b_T) &= 0; \\
    \sum_{T \in N_i} \alpha_T^j (\gamma_1(T)c_T + \gamma_2(T)d_T) + \beta_T^j (\gamma_2(T)c_T + \gamma_3(T)d_T) &= 0;
\end{align*}
\]

where \( N_i \) denotes the set of faces attached to the vertex \( v_i \). Combining Equation (18) and (20), we obtain a linear system to solve for the coordinate functions \( u_j \) and \( v_j \) of \( f_j \), subject to a given boundary condition. In practice, we usually set \( f_j \) to be an identity map on the boundary as the boundary condition. Hence, we have \( D_j f_j = D_j (u_j, v_j) = (b_j^1, b_j^2) \), where \( D_j \) is an \( m \times m \) matrix \( D_j \) and \( (b_j^1, b_j^2) \) is a \( m \times 2 \) matrix given by the above non-singular linear system.

In summary, given \( L^\mu \), one can reconstruct the longitudinal deformations via solving a big linear system:

\[
\bar{D}f = \begin{pmatrix} D_1 & D_2 & \cdots & D_{n^2} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n^2} \end{pmatrix} = \begin{pmatrix} b^1 \\ b^2 \end{pmatrix}
\]

where \( \bar{D} \) is a \( m m^2 \times n^2 \) block diagonal matrix and hence the linear system can be solved in parallel, subject to the Dirichlet boundary condition that the map is an identity map on the boundary.

The above discussion gives rise to the following theorem about the relationship between the longitudinal deformation and its associated descriptors.

**Theorem 2.** Denote the longitudinal deformations by \( f \). \( f \) is associated with a unique descriptor \( L^\mu \), given by Equation (13), that satisfies \( ||L^\mu||_\infty < 1 \). Conversely, given a descriptor \( L^\mu \) of a longitudinal deformation, the corresponding longitudinal deformation \( f \) can be exactly reconstructed and is unique. In other words, if a longitudinal deformation \( g \) has a descriptor given by \( L^\mu \), then \( f = g \).

On the other hand, the bijectivity of the longitudinal deformation can be easily controlled by the norm of its descriptor. It can be explained by the following theorem.

**Theorem 3.** If \( ||L^\mu||_\infty < 1 \), then its associated longitudinal deformation is bijective.
Proof. Note that $||L^\mu||_\infty = \max_{i,j} \{|(L^\mu)_{ij}|\}$, where $(L^\mu)_{ij}$ denotes the $i$-th row and $j$-th column entry of $L^\mu$. Since $||L^\mu||_\infty < 1$, we have $|\mu_j| < 1$ for all $j = 1, 2, \ldots, l$. For every triangular face $T$, the restriction map $f_j|_T$ on $T$ is a linear map. The Jacobian $J_T$ of $f_j|_T$ is given by

$$J_T = \left| \frac{\partial (f_j|_T)}{\partial z} \right|^2 - \left| \frac{\partial (f_j|_T)}{\partial \bar{z}} \right|^2 = \left| \frac{\partial (f_j|_T)}{\partial z} \right|^2 (1 - |\mu_j(T)|^2) > 0$$

since $|\mu_j(T)| = \left| \frac{\partial f_j|_T^z}{\partial z} \right| / \left| \frac{\partial f_j|_T}{\partial \bar{z}} \right| < 1$ and $\left| \frac{\partial (f_j|_T)}{\partial \bar{z}} \right| > 0$ for a well-defined $\mu_j$. We conclude that $f_j|_T$ is orientation-preserving. Thus the piecewise linear deformation $f_j$ is locally injective on every one-ring neighborhood of a vertex. By Hadamard theorem, $f_j$ is globally bijective for all $j = 1, 2, \ldots, l$. We conclude that the longitudinal deformation associated to $L^\mu$ is bijective.

In addition, it is important to understand how the difference in two descriptors related to the difference in their corresponding longitudinal deformations.

Theorem 4. Let $L^\mu_1$ and $L^\mu_2$ be the descriptors of two longitudinal deformations $f$ and $g$ respectively. Suppose $||L^\mu_1 - L^\mu_2||_F < \epsilon$, where $|| \cdot ||_F$ denotes the Frobenius norm. Then:

$$||f - g||_F < C_1 \epsilon$$

$$||Df - Dg||_F < C_2 \epsilon$$

for some positive constants $C_1$ and $C_2$. Here,

$$Df = \begin{pmatrix} D_1 f_1 & D_2 f_1 & \cdots & D_l f_1 \\ D_1 f_2 & D_2 f_2 & \cdots & D_l f_2 \\ \vdots & \vdots & \ddots & \vdots \\ D_1 f_l & D_2 f_l & \cdots & D_l f_l \end{pmatrix} \in M_{l|F|\times 2l}$$

where $D_1 \varphi = \begin{pmatrix} \frac{\partial \varphi}{\partial z}(T_1) \\ \vdots \\ \frac{\partial \varphi}{\partial z}(T_{|F|}) \end{pmatrix} \in \mathbb{C}^{|F|}$ and $D_2 \varphi = \begin{pmatrix} |\frac{\partial \varphi}{\partial \bar{z}}(T_1)| \\ \vdots \\ |\frac{\partial \varphi}{\partial \bar{z}}(T_{|F|})| \end{pmatrix} \in \mathbb{C}^{|F|}$, where $\varphi$ is a piecewise linear map on $\Omega$ and $T_j \in F$ is a triangular face. $Dg$ is defined similarly.

Proof. Denote $f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_l \end{pmatrix}$, $g = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_l \end{pmatrix}$, $L^\mu_1 = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_l \end{pmatrix}$ and $L^\mu_2 = \begin{pmatrix} \nu_1 & \nu_2 & \cdots & \nu_l \end{pmatrix}$. For each $j$, $f_j$ and $g_j$ can be extended to $\mathbb{C}$, by letting $f_j$ and $g_j$ be the identity map outside the image domain $\Omega$. Without loss of generality, we can assume $f_j$ and $g_j$ are normalized quasiconformal maps associated to $\mu_j$ and $\nu_j$ respectively. If $\alpha > 1$ and $0 < p < 1$ satisfy $2 < 2\alpha < 1 + \frac{2}{p}$, then there exist a positive integer $C(k, \alpha)$ such that

$$||D_1 f_j - D_1 g_j||_2 \leq C(k, \alpha)||\mu_j - \nu_j||_q$$

and

$$||D_2 f_j - D_2 g_j||_2 \leq C(k, \alpha)||\mu_j - \nu_j||_q.$$

where $q = \frac{p}{2(2\alpha - 1)}$. Note that all matrix norms are equivalent. There exists a positive constant $A$ such that $|| \cdot ||_q \leq A || \cdot ||_2$. Hence,

$$||Df - Dg||_F = \left( \sum_{j=1}^l ||D_1 f_j - D_1 g_j||_2^2 + ||D_2 f_j - D_2 g_j||_2^2 \right)^{1/2}$$

$$\leq \left( \sum_{j=1}^l \frac{2C(k, \alpha)}{A^2} ||\mu_j - D_1 \nu_j||_2^2 \right)^{1/2}$$

$$= \sqrt{2} AC(k, \alpha)||L^\mu_1 - L^\mu_2||_F < \sqrt{2} AC(k, \alpha) \epsilon$$

The second inequality follows by letting $C_1 = \sqrt{2} AC(k, \alpha)$. 

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For the first inequality, note that \( f_j \) and \( g_j \) are both normalized quasiconformal map for \( j = 1, 2, ..., l \). Then,
\[
f_j = v + SD_2 f_j \quad \text{and} \quad g_j = v + SD_2 g_j
\]
where \( v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^n \) is the position vector of all vertices of \( \Omega \). \( S \in M_{n \times |F|}(\mathbb{C}) \) is defined in such a way that for any \( h \in \mathbb{C}^{|F|} \), \((Sh)_k = \sum_{m=1}^{|F|} w_{km}(h)_k \) where \( w_{km} = \frac{1}{\pi} \int_{T_m} \frac{1}{\rho^2} d\tau \) and \( T_m \) is the \( m \)-th triangular face of \( \Omega \). Thus, we have
\[
||f_j - g_j|| = ||S(D_2 f_j) - S(D_2 g_j)||_2 \\
\leq ||S||_2 ||D_2 f_j - D_2 g_j||_2 \\
\leq ||S||_2 C(k, \alpha) ||\mu_j - \nu_j||_q.
\]
We can conclude that \( ||f - g||_F \leq A||S||_2 C(k, \alpha) ||L_i^\mu - L_j^\mu||_F < A||S||_2 C(k, \alpha) \epsilon \). The first inequality follows by letting \( C_1 = A||S||_2 C(k, \alpha) \).

Theorem 4 tells us the fact that two longitudinal deformations are close if their Beltrami descriptors are close to each others. Furthermore, their smoothnesses are similar if their Beltrami descriptors are close. In other words, the longitudinal deformation is stable under the perturbation of descriptor. It is a crucial observation, so that the manipulation of longitudinal deformations through Beltrami descriptors is not sensitive to the error of the descriptors. On the other hand, to alleviate the issue of large storage requirement, \( L_i^\mu \) can be used to replace \( L_i^\mu \). Theorem 4 tells us the reconstruction error of the longitudinal deformation is small if \( L_i^\mu \) and \( L_i^F \) are close to each others.

### 4.3 Decomposition of normal and abnormal components.

In this subsection, we will explain how we can decompose a longitudinal deformation into normal and abnormal components. To achieve this goal, we propose to apply the low rank and sparse decomposition of the deformation descriptor.

Given a deformation descriptor \( L_i^\mu \), we assume \( L_i^\mu \) is composed of the normal deformation \( N \) and abnormal deformation \( A \). Normal deformation \( N \) is often characterized by repeating pattern. Mathematically, \( N \) can be regarded as periodic and hence it should be of low rank. On the other hand, the abnormal deformation often occurs at some particular region and time. Thus, \( A \) can be assumed to be sparse. As such, our problem can be formulated as finding \( N \) and \( A \) such that they minimize:
\[
\min_{N, A} ||N||_* + \alpha ||A||_1, \quad \text{subject to} \quad L_i^\mu = N + A \in \mathbb{C}^{m \times n \times t}
\]
(20)

The first term involves the nuclear norm, aiming to minimize the rank of \( N \). The second term aims to enhance the sparsity of \( A \). The optimization problem can be solved using the alternating minimization method with multiplier (ADMM) as in the real case with suitable modifications. We will describe it in details as follows.

The Augmented Lagrangian function can be written as
\[
E(N, A, Z; L_i^\mu) = ||N||_* + \alpha ||A||_1 - \langle Z, N + A - L_i^\mu \rangle + \frac{\beta}{2} \|N + A - L_i^\mu\|_2^2
\]
(21)

with \( \langle X, Y \rangle = \text{real}(\text{tr}(X^*Y)) = \text{real}(\text{tr}(XY^*)) \). ADMM to solve the optimizaton can be written as the following iterative scheme:
\[
\begin{align*}
\Lambda^{k+1} &= \arg\min_{N \in \mathbb{C}^{m \times n \times t}} E(N, A^k, Z^k; L_i^\mu) \quad (N\text{-subproblem}) \\
A^{k+1} &= \arg\min_{A \in \mathbb{C}^{m \times n \times t}} E(N^{k+1}, A, Z^k; L_i^\mu) \quad (A\text{-subproblem}) \\
Z^{k+1} &= Z^k - \beta (N^{k+1} + A^{k+1} - L_i^\mu)
\end{align*}
\]
(22)

We will now describe how each subproblems can be tackled. We begin by looking into the \( A\)-subproblem. Some definitions are needed to help our explanation.

**Definition 2.** For \( A \in \mathbb{C}^{M \times N} \), define the norm
\[
\|A\|_{1,2} = \sum_{i=1}^{M} \left( \sum_{j=1}^{N} |a_{ij}|^2 \right)^{\frac{1}{2}}
\]
(23)
It can be easily seen that equation 23 sums each row’s $L^2$ norm, and it clearly defines a matrix norm as well. Now, the $A$-subproblem can be solved via a modified Euclidean projection, as described in the following proposition.

**Lemma 1.** Define $f : \mathbb{R}^{N \times 2} \to \mathbb{R}$ by

$$f(X) = \alpha \|X\|_1 + \frac{1}{2} \|M - X\|_2^2$$

where $M$ is a matrix in $\mathbb{R}^{N \times 2}$. Then the minimiser $X^*$ of $f$ is given by

$$X^*_j = \left(1 - \frac{\alpha}{\|M_j\|_2}\right)_+ M_j$$

Note that for a matrix $A$, $A_j$ denotes the $j$-th row of $A$, $|A|_2$ is the usual $L^2$-norm of a vector and $(y)_+ = \max\{y, 0\}$ for $y \in \mathbb{R}$.

**Proof.** Minimising Equation 24 is equivalent to minimising each row of $X$. In particular, we must have

$$0 \in \partial \left(\alpha |X^*_j|_2 + \frac{1}{2} \|M_j - X^*_j\|_2^2\right) \quad \text{for } j = 1, 2, ..., N.$$  

For $\|X^*_j\|_2 \neq 0$, we have

$$\frac{\alpha X^*_j}{\|X^*_j\|_2} + X^*_j - M_j = 0$$

From the above equation, it is observed that $X^*_j$ and $M_j$ are in the same direction. Thus

$$X^*_j = M_j - \frac{\alpha}{\|X^*_j\|_2} M_j$$

$$= M_j - \frac{\alpha}{\|M_j\|_2} M_j$$

$$= \left(1 - \frac{\alpha}{\|M_j\|_2}\right)_+ M_j$$

$$= M_j - \hat{P}_{D_{\alpha/\beta}}(M_j)$$

If $\|X^*_j\|_2 = 0$, then by calculating the subdifferential of equation 24, we get:

$$0 \in \alpha \{x + \frac{1}{\alpha} M_j : x \in \partial(|X^*_j|_2)\}$$

Hence, $0 \in \alpha \{g + \frac{1}{\alpha} M_j : \|g\|_2 \leq 1\}$. This implies $|M_j| \leq \alpha$. Putting everything together and iterating over each row, we arrive at equation 25.

**Theorem 5.** For $N, A, Z, L^\mu \in \mathbb{C}^{mn \times t}$, the solution to the $A$-subproblem is

$$A^{k+1} = \frac{1}{\beta} Z^k - N^k + L^\mu - \hat{P}_{D_{\alpha/\beta}} \left(\frac{1}{\beta} Z^k - N^k + L^\mu\right)$$

where $\hat{P}_{D_{\alpha/\beta}}$ denotes the Euclidean projection onto $D_{\alpha/\beta} := \{z \in \mathbb{C} : |z| \leq \alpha/\beta\}$.

**Proof.** To find the minimizer for the $A$-subproblem, it is equivalent to solving

$$A^{k+1} = \arg \min_A \alpha \|A\|_1 + \frac{\beta}{2} \|N^k + A - L^\mu - \frac{1}{\beta} Z^k\|_2^2$$

Let $\varphi : \mathbb{C}^{mn \times t} \to \mathbb{R}^{mnt \times 2}$ be the transformation defined by:

$$\varphi(X) = (\text{real(vec}(X)), \text{imag(vec}(X)))$$
According to [32], equation [31] is indeed equivalent to

\[
A^* = \arg \min_A \frac{\alpha}{\beta} \| \varphi(A) \|_{1,2} + \frac{1}{2} \| \varphi(A) + \varphi(L^\mu) - \frac{1}{\beta} \varphi(Z^k) \|_{F}^2
\]

where \( \| X \|_{1,2} = \sum_{j=1}^{n} \| X_{j} \|_{2} \) with \( X_{j} \) denotes the \( j \)-th row of \( X \).

Putting \( M = \frac{1}{\beta} Z^k - N^k + L^\mu \). Lemma [1] suggests that

\[
A_{j \rightarrow}^* = \left( 1 - \frac{\alpha}{\beta |M_{j \rightarrow}|} \right)_{+} M_{j \rightarrow}
\]

If \( |M_{j \rightarrow}| \leq \frac{\alpha}{\beta} \), \( A_{j \rightarrow}^* = 0 \). If \( |M_{j \rightarrow}| > \frac{\alpha}{\beta} \),

\[
A_{j \rightarrow}^* = \left( 1 - \frac{\alpha}{\beta |M_{j \rightarrow}|} \right)_{+} M_{j \rightarrow} = M_{j \rightarrow} - \hat{P}_{D_{\alpha/\beta}}(M_{j \rightarrow})
\]

Then formula [30] follows.

Theorem [31] is important as it gives us a closed form solution to solve the \( A \)-subproblem during the ADMM iteration. Next, we will look at the \( N \)-subproblem. Indeed, the \( N \)-subproblem can be treated exactly as in the real case, which is described as follows.

**Theorem 6.** For \( N, A, Z, L^\mu \in \mathbb{C}^{mn \times t} \), the solution to the low-rank subproblem in equation [9] is

\[ N^{k+1} = U^{k+1} \text{diag} \left( \max \{ \sigma_i^{k+1} - \frac{1}{\beta}, 0 \} \right) (V^{k+1})^T \]

where \( U^{k+1}, V^{k+1}, \sigma_i^{k+1} \) are obtained by SVD that is:

\[ L^\mu - A^{k+1} + \frac{1}{\beta} Z^k = U^{k+1} \Sigma^{k+1} (V^{k+1})^T \] with \( \Sigma^{k+1} = \text{diag} \left( \{ \sigma_i^{k+1} \}_{i=1}^{r} \right) \)

The proof of the above theorem follows similarly as in the case of real-valued matrices. We refer readers are referred to [29, 30] for the details of the proof.

It is worth mentioning that different literature has provided theoretical guarantee that the ADMM approach on this RPCA will converge. Readers can refer to [33, 25, 34, 35, 36, 37, 38, 39]. In particular, Hong et. al. [33] proved that the approach has linear convergence.

We summarize the algorithm for the decomposition of \( L^\mu \) into normal and abnormal deformations as follows.

**Algorithm 1:** Decomposition of \( L^\mu \)

**Input:** matrix \( L^\mu \in \mathbb{C}^{mn \times t} \), \( N \in \mathbb{N} \)

**Output:** Normal component \( \mathcal{N} \) and abnormal component \( \mathcal{A} \)

**Initialization:** \( \mathcal{N}_0 \) be a zero matrix, \( Z_0 = L^\mu / \| L^\mu \|_2, \beta_k(N) = \min \{ (1.5)^k \frac{1.25}{\| Z \|_2}, (1.5)^N \frac{1.25}{\| Z \|_2} \} \)

while not converge do

- Update \( \mathcal{N}^{k+1} \) using equation [36];
- Update \( A^{k+1} \) using equation [30];
- \( Z_{k+1} \leftarrow Z_k + \beta_k(n)(L^\mu - \mathcal{N}^{k+1} - A^{k+1}) \)

end

Here, \( N \) is a chosen integer parameter. Once \( L^\mu \) is decomposed into \( \mathcal{N} \) and \( \mathcal{A} \), the associated normal and abnormal longitudinal deformations can be reconstructed according to Equation [18].

The subtle quivers from a longitudinal deformation are naturally bijective without overlaps. A crucial question is whether our extracted abnormal deformation is indeed bijective. As a matter of fact, performing the low rank and sparse decomposition on the Beltrami descriptor is beneficial, since we can theoretically guarantee the bijectivity of the extracted abnormal deformation under suitable choice of the parameter. Hence, our algorithm can give a realistic and accurate extracted component for further deformation analysis. This fact is explained in details with the following theorem.
Theorem 7. Considering equation [39], there exists a constant $c(L^\mu)$ such that if

$$\alpha > c(L^\mu) = \frac{\|L^\mu\|_{\max}}{\|\mu\|_2^2} + \frac{1.25 p \ 1 - (1.5)^N q^N}{1 - 1.5 q} + \frac{\beta_N(N)pq^N}{1 - q}$$

where $p, q$ depend on $L^\mu$ and $\|L^\mu\|_{\max} < 1$, then our algorithm [7] would yield $\|A^k\|_{\max} < 1$ for all $k \in \mathbb{N}$.

Proof. Using mathematical induction, we first check the base case

$$\|A^1\|_{\max} = \left\| \frac{1}{\beta_0} Z^0 + L^\mu - \hat{P}_{\beta_0/\beta_0} \left( \frac{1}{\beta_0} Z^0 + L^\mu \right) \right\|_{\max} < 1$$

Clearly, $\|L\|_{\max} < 1$ and since $Z^0$ is defined to be $\frac{L^\mu}{\|L^\mu\|_2}$, we proved the base case.

Assume it is true that $\|A^k\|_{\max} < 1$. From [33] by Hong et. al., we have, for some constant $p > 0, q \in (0, 1)$

$$\|L^\mu - N^\tau - A^\tau\| \leq pq^\tau$$

which is known as the R-linearity of convergence of ADMM. Note that using similar logic in equation [39]

$$\|A^{k+1}\|_{\max} < 1 \iff \left\| \frac{1}{\beta_k(N)} Z^k - N^k + L^\mu \right\|_{\max} < 1$$

Considering this specific term, we deduce that

$$\left\| \frac{1}{\beta_k(N)} Z^k - N^k + L^\mu \right\|_{\max} = \left\| \frac{1}{\beta_0} Z^0 + L^\mu - \frac{1}{\beta_0} \sum_{i=1}^{k-1} \beta_i(N)(L^\mu - N^i - A^i) \right\|_{\max}$$

$$\leq \frac{1}{\beta_k(N)} \left\| L^\mu \right\|_2^2 + \sum_{i=1}^{k} \beta_i(N)(L^\mu - N^i - A^i)\right\|_{\max} + \|A^k\|_{\max}$$

Notice that using equation [40] we have

$$\left\| \frac{L^\mu}{\|L^\mu\|_2^2} + \sum_{i=1}^{k} \beta_i(N)(L^\mu - N^i - A^i)\right\|_{\max}$$

$$= \frac{\|L^\mu\|_{\max}}{\|L^\mu\|_2^2} + \sum_{i=1}^{k} \beta_i(N)\|L^\mu - N^i - A^i\|_{\max} + \sum_{i=n}^{k} \beta_i(N)\|L^\mu - N^i - A^i\|_{\max}$$

$$\leq \frac{\|L^\mu\|_{\max}}{\|L^\mu\|_2^2} + \sum_{i=1}^{k} \beta_i(N)pq^i + \sum_{i=n}^{k} \beta_i(N)pq^i$$

$$= \frac{\|L^\mu\|_{\max}}{\|L^\mu\|_2^2} + \frac{1.25 p \ 1 - (1.5)^N q^N}{1 - 1.5 q} + \frac{\beta_N(N)pq^N}{1 - q}$$

$$\leq \frac{\alpha}{\beta_k(N)} + 1$$

Thus, putting the last inequality [43] into equation [42] we have

$$\left\| \frac{1}{\beta_k(N)} Z^k - N^k + L^\mu \right\|_{\max} \leq \frac{\alpha}{\beta_k(N)} + 1$$

which in turns imply that $\|A^{k+1}\|_{\max} < 1$. The induction is completed. □
With all tools introduced, we can now describe our whole algorithm as follows.

Algorithm 2: Abnormal Deformation Extraction and Recovery Algorithm

Input: Reference frame \( I_{ref} \), and video frame \( \{I_i\}_{i=1}^t \)

Output: Low-rank frames \( \{l_i\}_{i=1}^t \) and sparse frames \( \{s_i\}_{i=1}^t \)

for each frame \( I_i \) (parallel-computation compatible) do
  Register \( I_{ref} \) to \( I_i \) get the deformation field;
  Compute the Beltrami descriptor \( \mathcal{L}^\mu \);
end

Using algorithm \([\text{I}]\) decompose \( \mathcal{L}^\mu = \mathcal{N} + \mathcal{A} \);

for each column \( l_i \) of \( \mathcal{N} \) (parallel-computation compatible) do
  Using LBS, recover \( l_i \) to a map \( f_i^{\mathcal{N}} \);
  Deform \( I_{ref} \) with the map \( f_i^{\mathcal{N}} \) and we obtain \( l_i \);
end

for each column \( s_i \) of \( \mathcal{A} \) (parallel-computation compatible) do
  Using LBS, recover \( s_i \) to a map \( f_i^{\mathcal{A}} \);
  Deform \( I_{ref} \) with the map \( f_i^{\mathcal{A}} \) and we obtain \( s_i \);
end

5 Experimental Result

In this section, we present our experimental results on synthetic images, as well as on real medical images.

Example 1: We first test our proposed method on a synthetic image sequence. The input data is a sequence of binary images that shows a circle shrinks and expands, and repeats this process for a few cycles. Readers can refer to Figure 2 to visualise this process. The total frames of this process is 48, which means that the ground-truth rank of the Fourier Transformed BC matrix is 24. The whole expansion and contraction process is repeated 9 times, and 3 of which are perturbed by adding some deformations around the boundary of the circle. After adding perturbation on the cycles, the rank of the Beltrami descriptor matrix raises to 47, while our algorithm successfully reduces the rank of the low-rank matrix to 27. We remark that since we took the smallest circle as the reference image, one can observe that the recovered sparse image has a circle that is far smaller than perturbed frames. Table 1 shows the result of our algorithm on one of the three perturbations.

A straightforward method to decompose the longitudinal deformation is done by applying the RPCA on the vector fields of the deformation. As mentioned, vector fields cannot effectively capture the geometric information of the deformation. As such, RPCA on vector fields cannot yield satisfactory results. The last two columns of Table 1 show the results of pursuing the low-rank and sparse part on the deformation vector fields obtained from registering the reference frame to each of the video frames. We view each vector in the vector fields as an element in \( \mathbb{C} \), and we stacked them horizontally and obtain a giant matrix. Then we run the complex matrix decomposition algorithm on this matrix. Although the decomposed low-rank matrix is of rank 24, the last two columns of Table 1 clearly shows that the recovered results are far from the ground-truth and to be useful. The circles are distorted to ellipses. Compared to the results obtained from our original longitudinal deformation descriptor, this decomposition is not meaningful.

Example 2: The next example is on a sequence of real medical images of a beating heart. The original video contains 341 frames with repeated periodic beating of 31 times. In this example, artificial abnormal deformations are introduced.
Table 1: Results of Example 1

| Original Frame | Perturbed Frame | Recovered Low-Rank Frame | Recovered Sparse Frame | Recovered Low-Rank Frame on Vector Field | Recovered Sparse Frame on Vector Field |
|----------------|-----------------|--------------------------|------------------------|----------------------------------------|---------------------------------------|
| ![Image]        | ![Image]        | ![Image]                 | ![Image]               | ![Image]                                | ![Image]                              |
| ![Image]        | ![Image]        | ![Image]                 | ![Image]               | ![Image]                                | ![Image]                              |
| ![Image]        | ![Image]        | ![Image]                 | ![Image]               | ![Image]                                | ![Image]                              |
| ![Image]        | ![Image]        | ![Image]                 | ![Image]               | ![Image]                                | ![Image]                              |

To one of the 31 cycles, and so ground-truth images are available to study the accuracy of our proposed model. Table 2 shows the result. The second column shows the frames with manual deformation added on images in the first column, and the red box area is where deformation is added. We can see that our algorithm can almost recover the low-rank frames to the original frames and the sparse frames to the perturbed frames. The size of the input Beltrami descriptor is $19602 \times 341$, and the rank of the original video and perturbed video are 11 and 15 respectively. After running our algorithm on the matrix, the rank of the recovered low-rank matrix is reduced to 11.

Beside the recovered rank, from Table 2, we can see that our algorithm can capture and recover both the normal and abnormal deformation on the beating heart to great details. It can be seen that the recovered low-rank frames looks very much alike to the original frames and recovered sparse frames can effective capture the subtle quivers.

To better observe the result of our experiment, Figure 3 shows the second row of Table 2. The area bounded by the red box is the periphery of the beating heart and also where we added deformation, which can be clearly observed on Figure 3b. After running our algorithm, Figure 3c and Figure 3d respectively show the recovered regular and irregular motion obtained from the decomposed low-rank and sparse matrix of the Beltrami Descriptor. We can see that in Figure 3c, the heart gains back almost the complete shape as in Figure 3a, and the deformation of the periphery of it in Figure 3d is very similar to that in Figure 3b.
Example 3: In this example, we test our algorithm on another medical video of a beating heart with abnormal perturbations. The original rank of the video is 36. After performing our proposed method on the Beltrami descriptor, the rank of the low-rank matrix is reduced to 20. Table 2 displays one of the perturbation and its recovery by our method. As shown in the table, the results show that our algorithm can recover the normal and abnormal deformation. Readers can compare the first column with the third, and the second with the fourth.

Example 4: In this example, we test our algorithm on another medical video of a lung under respiration. The original video captures 31 cycles with some perturbation at some frames. The rank of the input longitudinal Beltrami descriptor is 23, which was reduced to 10 after performing our algorithm on it. Table 4 showed the pictures of one of the perturbation. In addition to running this experiment on our algorithm, we again test decomposing the vector field matrix as in Example 1. We stacked the registration deformation vector fields from the reference image to all other frames in the video into one giant matrix over complex field. Then, running the complex low-rank and sparse component pursuit on the matrix gave the last two columns in Table 4. It is clear that the decomposed sparse matrix can barely capture any abnormal deformation as the Beltrami descriptor does. This once again shows that applying the algorithm to decompose on vector field matrices is not a viable option.

Example 5: In this example, we test our proposed method on another medical video of a breathing lung with abnormal perturbation. The original video captures 36 cycles. The rank of the Beltrami descriptor matrix is 26. Our proposed method recovers the low rank matrix with rank 12. Table 5 displays the results of one of the perturbation using our algorithm. Again, our proposed method effectively decompose the longitudinal deformation into the normal periodic component and abnormal component.
Finally, we summarize the rank of the decomposed sparse component for each example in Table 6. The ranks of the original input Beltrami descriptors are also recorded. Note that our proposed algorithm can effectively obtain the sparse component that reduces the rank. The rank of the sparse component closely resemble to the rank of the Beltrami descriptor of the video without abnormal perturbations.

6 Conclusion

We address the problem of decomposing a longitudinal deformation into the normal periodic component and the abnormal irregular component. Our strategy is to represent the longitudinal deformation by the proposed Beltrami descriptor and apply RPCA on it. The low rank part effectively extracts the normal component, while the sparse part effectively captures the irregular quivers. The Beltrami descriptor describes the geometric information about the deformation, and hence performing the decomposition on the Beltrami descriptor yields meaningful results. In particular, we can prove that the extracted abnormal motion is guaranteed to be bijective under suitable choice of parameters. Extensive experiments on both synthetic and real data give encouraging results.

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| Frame without quiver | Frame with quiver | Recovered Low-Rank Frame | Recovered Sparse Frame |
|----------------------|-------------------|--------------------------|-----------------------|
| ![Frame without quiver](image1) | ![Frame with quiver](image2) | ![Recovered Low-Rank Frame](image3) | ![Recovered Sparse Frame](image4) |
| ![Frame without quiver](image5) | ![Frame with quiver](image6) | ![Recovered Low-Rank Frame](image7) | ![Recovered Sparse Frame](image8) |
| ![Frame without quiver](image9) | ![Frame with quiver](image10) | ![Recovered Low-Rank Frame](image11) | ![Recovered Sparse Frame](image12) |
| ![Frame without quiver](image13) | ![Frame with quiver](image14) | ![Recovered Low-Rank Frame](image15) | ![Recovered Sparse Frame](image16) |

Table 3: Result of Example 3

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| Original Frame | Perturbed Frame | Recovered Low-Rank Frame | Recovered Sparse Frame | Recovered Sparse Frame on Vector Field with FFT | Recovered Sparse Frame on Vector Field without FFT |
|---------------|----------------|--------------------------|------------------------|-----------------------------------------------|--------------------------------------------------|
| [Image]       | [Image]        | [Image]                   | [Image]                | [Image]                                       | [Image]                                          |
| [Image]       | [Image]        | [Image]                   | [Image]                | [Image]                                       | [Image]                                          |
| [Image]       | [Image]        | [Image]                   | [Image]                | [Image]                                       | [Image]                                          |
| [Image]       | [Image]        | [Image]                   | [Image]                | [Image]                                       | [Image]                                          |

Table 4: Result of Example 4

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Table 5: Result of Example 5

| Example Type          | Size of Input Matrix | Rank of Input Matrix Without Perturbation | Rank of Input Matrix With Perturbation | Rank of Recovered Low-Rank Matrix |
|-----------------------|----------------------|-----------------------------------------|---------------------------------------|-----------------------------------|
| Synthetic Circle      | 19602 × 431          | 24                                      | 47                                    | 27                                |
| First Heart Example   | 19602 × 341          | 11                                      | 15                                    | 11                                |
| Second Heart Example  | 19602 × 378          | 18                                      | 36                                    | 20                                |
| First Lung Example    | 19602 × 310          | 10                                      | 23                                    | 10                                |
| Second Lung Example   | 19602 × 360          | 10                                      | 26                                    | 12                                |

Table 6: Summary of Recovered Rank by Our Proposed Algorithm

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