On Threshold Routing in a Service System with Highest-Bidder-First and FIFO Services

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Abstract—In this paper, we consider a two server system serving heterogeneous customers. One of the server has a FIFO scheduling policy and charges a fixed admission price to each customer. The second server follows the highest-bidder-first (HBF) policy where an arriving customer bids for its position in the queue. Customers make an individually optimal choice of the server and for such system, we characterize the equilibrium routing of customers. We specifically show that this routing is characterized by two thresholds.

I. INTRODUCTION

Consider two make-to-order firms manufacturing an identical product. Upon receiving an order, the firms must assemble the product and deliver it to the customer ordering it. Each firm can assemble only one quantity at a time and the time taken to assemble the product need not be deterministic. During an assembly of a product, if there are more orders being placed by other customers, then these orders have to be fulfilled by the firm by suitably scheduling the subsequent orders. The two firms differ in their pricing strategy and the scheduling policy for choosing subsequent orders. One of the firm charges a fixed admission price for the product and maintains a FIFO scheduling discipline. The second firm employs a bidding policy where subsequent customers place a bid and their queue position in the schedule in proportional to the bid placed. The customers ordering the product differ in their cost for unit delay and are thus sensitive to the delay in receiving the product. When placing an order, the customer does not know the number of pending orders but may be informed about the service rate and the arrival rate for the orders. When ordering the product, the customers have to decide which firm to choose and if they choose the bidding firm, then what is the optimal bid to be made such that the cost of obtaining the product (the sum of the monetary and the delay cost) is minimized. Motivated by this problem, our interest is to characterize the equilibrium choice of the firm made by the heterogeneous customers.

Applicable to more general setting, a formal description of the problem considered in this paper is as follows. Consider a two server service system with customers arriving according to a homogeneous Poisson process. The customers are heterogeneous in their cost for unit delay. The service system consists of two servers and the customers are required to obtain service at one of these two servers. There is no dispatcher available to route the customers and hence each arriving customer has to make an individually optimal queue join decision. Each server in the service system has an associated queue and the two queues differ from each other in their scheduling policy. One of the queue has the standard FIFO scheduling policy and to monetize the offered service, it charges a fixed admission price to its customers. The other queue has a non-preemptive priority scheduling discipline where after the current service completion, a customer with the highest priority level is next chosen for service from the pool of customers waiting in the queue. The priority of a customer in this queue is determined by the bid paid by each arriving customer. Naturally, a higher bid corresponds to a higher priority in the queue. Such a scheduling policy is also known as the highest-bidder-first (HBF) policy and was introduced by Kleinrock [1]. In this paper, our primary interest is to characterize the equilibrium routing satisfying the Wardrop conditions [2] and determine the bidding decision made by those customers choosing the HBF server.

Such a system with parallel HBF and FIFO services was first analyzed in [3]. To investigate the effect on the revenue from an HBF server, a free FIFO service was introduced in the system. Further a minimum bid was made mandatory for those choosing the HBF server. For such a system, the equilibrium routing and bidding strategy was first analyzed in [3]. It was shown that the Wardrop equilibrium routing is characterized by a single threshold and customers with delay sensitivity (cost per unit delay) above the threshold choose the HBF server while the rest choose the FIFO server. Two scenarios were considered for modeling the system; in the first scenario, a free FIFO server was added in parallel to an existing HBF server. In the second scenario, the total service capacity was shared between the HBF and the FIFO server. Assuming that the customers cannot balk, it was shown that an addition of a free FIFO server decreases the system revenue. On the contrary, with the help of numerical examples, it was conjectured that sharing capacity with a FIFO server improves the revenue from the HBF server.

The primary difference between the system model considered in this paper and that of [3] is as follows. We assume that the FIFO server is not free but in fact comes with an admission price. This assumption makes the model more naturally applicable to a variety of revenue based service systems such as the above example for make-to-order firms. We relax the assumption of a minimum bid and analyze the equilibrium routing and bidding rule for this problem. As in [3], the equilibrium bids in the HBF server (by those choosing HBF under equilibrium routing) can be determined from the analysis in [3]. [5]. We begin by analyzing whether a single threshold routing function as in [3] satisfies the equilibrium
routing conditions. To our surprise, this is not the case. We then check for the threshold routing where customers with sensitivity above a threshold choose the FIFO server while the rest choose the HBF server. We show that such a candidate for equilibrium routing also does not satisfy the necessary conditions for Wardrop equilibrium. In our main result, we prove that the (Wardrop) equilibrium routing is of the following type: there exists two threshold and customers with sensitivity between the two thresholds choose the FIFO server while the rest choose the HBF server. To the best of our knowledge, the result is novel and has a useful insight. While the ‘middle class’ of the population (based on their sensitivities) choose the FIFO service, the remaining customers (specifically those with high and low delay sensitivity) choose the HBF server.

The rest of the paper is organized as follows. In the next section, we shall introduce the basic notation and recall some preliminary results from [3]. We prove our main results in Section III and this is followed by a discussion summarizing the results.

II. Notation and Preliminaries

In this section, we shall introduce the basic notation and recall some preliminary results from [3]. We assume as in [3] that the customers arrive to the service system according to a homogeneous Poisson process of rate \( \lambda \). Service times of customers are i.i.d. random variables with distribution \( G(\cdot) \) and unit mean. Each arriving customer has an associated parameter \( \beta \) which is a realization of the random variable \( \beta \), \( 0 \leq a \leq \beta \leq b < \infty \). \( \beta \) represents a customers cost per unit delay. Let \( F(\beta) \) denote the distribution of \( \beta \) which is assumed to be absolutely continuous in \((a, b)\). We also call \( \beta \) as the type of the customer and call \( F(\beta) \) as the type profile.

The first server uses the non preemptive HBF discipline and serves at rate \( \mu_1 \). The second server uses the FIFO discipline and serves at rate \( \mu_2 \). Customers choosing the HBF server will have to place a bid before joining its queue while those choosing the FIFO server have to pay a fixed admission price denoted by \( c \). We will assume that all arrivals will have to receive service from one of the two servers and they cannot balk. Thus an arriving customer now has to make the following decisions on arrival; which server to use, and, if it chooses the HBF server, then the value of its bid. As in [3], we assume oblivious decisions and let \( p(\beta) : [a, b] \to (0, 1] \) denote the probability that a customer of type \( \beta \) chooses the FIFO server. Further, let \( X(\beta) \) be the equilibrium bid if such a customer chooses the HBF server. For a preliminary analysis of the HBF queue, refer [1]. Lui [5] and Glazer and Hassin [4] were the first to consider the case with heterogeneous customers (characterized by \( \beta \)) and have determined the equilibrium bidding function \( X(\beta) \). The function \( X(\beta) \) determines the optimal value of that bid to be made by a customer of type \( \beta \) such that the sum of the bid and the expected waiting cost in the queue is minimized. Specifically, it was shown that \( X(\beta) \) is given by

\[
X(\beta) = \int_0^\beta \frac{2\rho W_0 y}{(1 - \rho + \rho F(y))^2} dF(y)
\]  

where \( \rho \) denotes the traffic intensity, \( F(\cdot) \) denotes the underlying distribution of \( \beta \) and \( W_0 \) denotes the expected waiting time in the HBF server added to that of an arriving customer due to the residual service time of an existing customer. This is given by,

\[
W_0 = \frac{\lambda}{2} \int_0^\infty \tau^2 dG(\mu\tau)
\]

where \( \lambda \) and \( \mu \) denote the arrival rate of customers and the service rate of the HBF server respectively. It was further shown that for a customer of type \( \beta \), its expected waiting time \( W(\beta) \) is given by

\[
W(\beta) = \frac{\mu^2 W_0}{(\mu - \lambda(1 - F(\beta)))^2}.
\]  

Now for a given \( p(\beta) \), it is easy to see that the arrival rate to the FIFO server is \( \lambda_2 := \lambda \int_0^\infty p(\beta)dF(\beta) \) while the arrival rate to the HBF server is \( \lambda_1 := \lambda - \lambda_2 \). Let \( \rho_1 := \lambda_1/\mu_1 \). A customer of type \( \beta \) that chooses the HBF server experiences a bid-dependent waiting time that will be denoted by \( W_1(\beta) \). The customers choosing the FIFO server experience an expected waiting time denoted by \( W_2(\lambda_2) \). Continuing with the notation of [3], let \( D_1(\beta) := W_1(\beta) + \frac{1}{\mu_1} \) and \( D_2 := W_2(\lambda_2) + \frac{1}{\mu_2} \) be the expected sojourn times in, respectively, the HBF and the FIFO servers. Refer Fig. 1 for an illustration of the system model.

In this paper, the primary interest is to obtain the equilibrium strategy henceforth denoted by \( (p^E(\beta), X^E(\beta)) \). Note that \( X^E(\beta) \) needs to be determined for only those customers that under equilibrium decide to join the HBF queue. Clearly, the system considered is non-atomic and all customers choose individually optimal strategies. The equilibrium attained is a Wardrop equilibrium that was first described in [2] and used extensively in transportation systems. The Wardrop equilibrium routing condition on \( p^E(\beta) \) for all \( \beta \) is that

\[
p^E(\beta) \geq 0 \quad \text{implies that} \quad c + \beta D_2 \leq X^E(\beta) + \beta D_1(\beta).
\]  

Fig. 1. System with a HBF server and a FIFO server.
Further $0 < p^E(\beta) < 1$ implies $c + \beta D_2 = X^E(\beta) + \beta D_1(\beta)$.

The following theorem recasts the equilibrium strategy $(p^E(\beta), X^E(\beta))$ for the system model considered in [3]. The key difference between the models is that the FIFO server in [3] charges no admission price and a customer joining the HBF server is required to pay a minimum bid $M$.

**Theorem I:** Using $\beta_1$ determined below, define $p^E(\beta)$, $F_1(\beta)$, and $W_0$ as follows.

$$
p^E(\beta) = \begin{cases} 
0 & \text{for } \beta > \beta_1, \\
t & \text{for } \beta = \beta_1, \\
1 & \text{for } \beta < \beta_1.
\end{cases}
$$

$$
F_1(\beta) := \int_0^\beta \frac{dF(x)}{dF(\beta)} \beta_1 \leq \beta \leq b, \\
1 & \beta > b.
\end{cases}
$$

$$
W_0 = \frac{\lambda_1}{2} \int_0^\infty \tau^2 dG(\mu_1),
$$

$$
X^E(\beta) = \int_0^\beta \frac{2 p_1 W_0 y}{(1 - p_1 + p_1 F_1(y))^3} dF_1(y).
$$

For the routing and bidding policy $(p^E(\beta), X^E(\beta))$, $\beta_1$ is determined as follows.

- If using $\beta_1 = a$ in (4)–(7) satisfies $M + a D_1(a) < a D_2$, then set $\beta_1 = a$.
- Else if using $\beta_1 = b$ in (4)–(7) satisfies $M + b D_1(b) > b D_2$ then set $\beta_1 = b$.
- Else find $\beta_1 \in (a, b)$ which when used in (4)–(7) satisfies $M + \beta_1 D_1(\beta_1) = \beta_1 D_2$. \hspace{1cm} (8)

$(p^E(\beta), X^E(\beta))$ is an equilibrium strategy with $\beta_1$ defined as above. Further, $\beta_1$ is unique. \hfill \square

It is important to point out that the Wardrop condition for the case with a minimum bid $M > 0$ and a free FIFO server ($c = 0$) that is used in the above theorem is that

$$
p^E(\beta) \geq 0, \text{ implies that } \beta D_2 \leq M + X(\beta) + \beta D_1(\beta).
$$

**Remark 1:** Note from the conditions for $\beta_1$ that when $\beta_1 \in (a, b)$, the expected cost experienced by a customer with $\beta = \beta_1$ at the two servers is same and hence the value of $t$ in Eq. (4) is arbitrary. Also, $F_1(\cdot)$ in the theorem corresponds to the type profile of those customers that choose to join the HBF server.

The above theorem determines the equilibrium routing and bidding pair $(p^E(\beta), X^E(\beta))$ where $p^E(\beta)$ is characterized by a unique threshold $\beta_1$. It is natural therefore to ask whether a similar single threshold routing equilibrium holds when $M = 0$ and the FIFO server charges an admission price. This question is analyzed in the next section.

**III. Characterizing the Wardrop Equilibrium**

In this section, we return to our main model and drop the assumption of a free FIFO service and minimum bid $M$ made in [3]. To simplify the analysis and restrict the number of cases arising due to the heterogeneity of servers, we shall henceforth assume that $\mu_1 = \mu_2 = \mu$, i.e., the servers are identical with service rate $\mu$. We shall further assume that $c > 0$ and that the minimum bid $M = 0$.

A possible candidate for the Wardrop equilibrium routing and bidding policy for this case is the one that was identified in Theorem I. Recall that such a policy was characterized by a threshold $\beta_1 \in [a, b]$ and customers with $\beta > \beta_1$ choose the HBF server while those with $\beta < \beta_1$ choose the FIFO server at equilibrium. We begin the analysis of this section by first investigating whether the routing and bidding policy $p^E(\beta)$ and $X^E(\beta)$ as described in Eq. (4)–(7) of Theorem I holds for some $\beta_1 \in [a, b]$. In the following lemma, we will show that when $c > 0$ and $M = 0$, $p^E(\beta)$ and $X^E(\beta)$ as described in Eq. (4)–(7) with $\beta_1 \in [a, b]$ is not possible. Further if $p^E(\beta)$ and $X^E(\beta)$ satisfy Eq. (4)–(7), then the underlying threshold $\beta_1$ must satisfy $\beta_1 = a$, and the corresponding equilibrium strategy for all customers is to choose the HBF server.

**Lemma I:** Consider a routing and bidding policy $(p^E(\beta), X^E(\beta))$ as described in Eq. (4)–(7) of Theorem I and assume that $c > 0$ and $M = 0$. Then such a $(p^E(\beta), X^E(\beta))$ with $\beta_1 \in (a, b)$ and satisfying the Wardrop equilibrium conditions of Eq. (3) does not exist.

**Proof:** We first prove that $\beta_1 \notin (a, b)$ using contradiction. Assume that when $c > 0$ and $M = 0$, the corresponding $(p^E(\beta), X^E(\beta))$ satisfy Eq. (4)–(7) with $\beta_1 \in (a, b)$. Since $\beta_1 \in (a, b)$, the equilibrium routing function $p^E(\beta)$ requires that $c + \beta_1 D_2 = \beta_1 D_1(\beta_1)$. This implies that $D_2 < D_1(\beta_1)$ and hence for any $\beta < \beta_1$, we have

$$
c = \beta_1(D_1(\beta_1) - D_2) > \beta_1 D_1(\beta_1) - D_2 = \beta_1 D_1(\beta_1) - D_2 = \beta_1 D_1(\beta_1) - D_2. \hspace{1cm} (9)
$$

To see how the last equality is true, consider a customer with $\beta < \beta_1$ (that deviates from the prescribed $p^E(\beta)$) and chooses the HBF server instead of the FIFO server. Note from the definition of $F_1(\cdot)$ and $X^E(\cdot)$ in Eq. (4) Eq. (7) respectively that $X^E(\beta_1) = 0$. Using the property from [3], that $X^E(\beta)$ is increasing in $\beta$ and the fact that the $\beta$ customer is infinitesimal, we can conclude that $X^E(\beta) = 0$. The deviating $\beta$ customer therefore does not pay a bid and will occupy the end of the queue. The infinitesimal nature of this customer does not affect the delay at any of the two servers and hence for such a deviating customer with $\beta < \beta_1$ we have $D_1(\beta) = D_1(\beta_1)$. From Eq. (9), we have $c + \beta_2 D_2 < \beta_1 D_1(\beta)$. Thus for this customer, the cost at the FIFO queue is more than the cost it would experience at the HBF server. This customer has an incentive to deviate from $p^E(\beta)$ and hence $p^E(\beta)$ is not an equilibrium for any $\beta_1 \in (a, b)$.

Now consider the case when $\beta_1 = b$. This implies that none of the customers choose the HBF server and we have $c + \beta_2 D_2 < \beta_1 D_1(\beta)$ for all $\beta$. Note that since $\beta_1 = b$, $\lambda_1 = 0$ and hence if a customer of type $\beta$ were to choose HBF, the delay would have been $D_1(\beta) = \frac{1}{\mu}$. At the FIFO server, we have $\lambda_2 = \lambda$ and hence $D_2 = \frac{1}{\mu - \lambda_1}$. Now $\beta_1 = b$ and hence $c + \beta_2 D_2 < \beta_1 D_1(\beta)$ implies that $c + \frac{\beta_2}{\mu - \lambda_2} < \frac{\beta_1}{\mu}$ for all $\beta$. However this is not possible for any $\beta$ as $c > 0$ and $\frac{1}{\mu - \lambda_2} > \frac{1}{\mu}$.

Next, we outline the conditions on the parameters for $\beta_1 = (a, b)$.
a. From the definition of $p^E(\beta)$, this corresponds to the case when none of the customers choose the FIFO server i.e., when $c + \beta D_2 > X(\beta) + \beta D_1(\beta)$ for all $\beta$. It is straightforward to see that this case is possible when $c$ is set to any arbitrarily large value such that for all $\beta$ we have $c + \beta D_2 > X(\beta) + \beta D_1(\beta)$. This completes the proof.

An alternative candidate for the equilibrium routing and bidding policy is as follows.

$$
\hat{p}(\beta) = \begin{cases} 
1 & \text{for } \beta > \beta_1, \\
t & \text{for } \beta = \beta_1, \\
0 & \text{for } \beta < \beta_1.
\end{cases}
\tag{10}
$$

with the threshold $\beta_1 \in (a, b)$ and the corresponding distribution $F_1(\beta)$ satisfying

$$
F_1(\beta) := \begin{cases} 
0 & \beta < a \\
\int_a^\beta dF(x) & a \leq \beta \leq \beta_1, \\
\int_a^{\beta_1} dF(x) & \beta > \beta_1.
\end{cases}
\tag{11}
$$

It is easy to see that $\hat{p}(\beta)$ for $\beta_1 = a$ (resp. $b$) is equal to $p^E(\beta)$ (Eq. (4)) with $\beta_1 = b$ (resp. $a$) and the conditions for such equilibria with $\beta_1 = a$ (or $b$) is already outlined in the previous lemma. In the following lemma, we shall now show that when $c > 0$ and $M = 0$, a routing equilibrium satisfying Eq. (10) with $\beta_1 \in (a, b)$ is not possible.

**Lemma 2:** Let $c > 0$, $M = 0$ and consider a routing policy that satisfies Eq. (10) with $\beta \in (a, b)$ and has the type profile of customers to the HBF server given by Eq. (11). Such a routing policy does not satisfy the Wardrop equilibrium conditions of Eq. (3).

**Proof:** The proof is by contradiction. Suppose $\hat{p}(\beta)$ as in Eq. (10) satisfies the conditions of Eq. (3). Since $\beta \in (a, b)$, we have

$$
c + \beta_1 D_2 = X(\beta_1) + \beta_1 D_1(\beta_1).
$$

Now since $X(\beta)$ is increasing in $\beta$ and all customers with $\beta < \beta_1$ choose the HBF server, we have $X(\beta_1) > 0$ and $D_1(\beta_1) = \frac{1}{\mu}$. Further, $\lambda_2 > 0$ implies $D_2 > \frac{1}{\mu} = D_1(\beta_1)$ and hence $X(\beta_1) > c$. For $\beta > \beta_1$, we therefore have the following.

$$
X(\beta_1) - c = \beta_1 \left( D_2 - \frac{1}{\mu} \right) < \beta \left( D_2 - \frac{1}{\mu} \right).
$$

Thus for any customer with $\beta > \beta_1$ we have

$$
X(\beta_1) + \frac{\beta}{\mu} < c + \beta D_2.
\tag{12}
$$

Now consider a customer with $\beta > \beta_1$ that deviates from $\hat{p}(\beta)$ and chooses the HBF server instead of the FIFO server. The bid paid by this marginal customer is $X(\beta) = X(\beta_1)$ resulting in $D_1(\beta_1) = \frac{1}{\mu}$. From Eq. (12), the cost at the HBF server for such a customer $(X(\beta_1) + \frac{\beta}{\mu})$ is lower than the corresponding cost $(c + \beta D_2)$ at the FIFO server. This customer now has an incentive to deviate and the therefore the routing policy $\hat{p}(\beta)$ of Eq. (10) is not an equilibrium policy.

We now come to the main result of this section. Having discarded two candidate policies, we shall provide the routing and bidding policy satisfying the Wardrop equilibrium condition of Eq. (3). As we shall show in the following theorem, this policy is characterized by two thresholds $\beta_1$ and $\beta_2$ that satisfy $a < \beta_1 < \beta_2 < b$. In this policy, customers with $\beta$ satisfying $\beta_1 < \beta < \beta_2$ choose the FIFO server while the rest choose the HBF server. We will also characterize the conditions for such an equilibrium policy in the following theorem.

**Theorem 2:** Let $\beta_1$ and $\beta_2$ be two thresholds satisfying the conditions outlined below. Define $p^E(\beta)$, $F_1(\beta)$, and $W_0$ as follows.

$$
p^E(\beta) = \begin{cases} 
1 & \text{for } \beta_1 < \beta < \beta_2, \\
t & \text{for } \beta \in \{\beta_1, \beta_2\}, \\
0 & \text{elsewhere}.
\end{cases}
\tag{13}
$$

$$
F_1(\beta) := \begin{cases} 
\int_a^\beta dF(x) & \beta < \beta_1 \\
\int_a^{\beta_1} dF(x) & \beta_1 \leq \beta \leq \beta_2, \\
\int_a^{\beta_2} dF(x) & \beta_2 < \beta \leq b, \\
1 & \beta > b.
\end{cases}
\tag{14}
$$

$$
W_0 = \frac{\lambda_1}{2} \int_0^\infty \tau^2 dG(\mu \tau),
\tag{15}
$$

$$
X^E(\beta) = \int_0^\beta \frac{2p_1 W_0 y}{(1 - p_1 + p_1 F_1(y))} dF_1(y) \tag{16}
$$

The conditions that need to be satisfied by $\beta_1$ and $\beta_2$ are as follows.

1) $a < \beta_1 < \beta_2 < b$. 
2) $X^E(\beta_1) = c$.
3) $X^E(\beta_1) + \beta_1 D_1(\beta_1) = c + \beta_1 D_2(\lambda_2)$ where $\lambda_2 = \int_{\beta_1}^{\beta_2} \lambda dF(x)$.

Then, $(p^E(\beta), X^E(\beta))$ is an equilibrium strategy satisfying the Wardrop conditions of Eq. (3).

**Proof:** As in the case of Theorem 1, we will prove that $p^E(\beta)$ and $X^E(\beta)$ as described in (13)-(16) satisfy the Wardrop condition. Note that with $p^E(\beta)$ as in (13), the arrival rate to the FIFO server is $\lambda_2 = \lambda \int_{\beta_1}^{\beta_2} dF(\tau)$. The arrival rate to the HBF server is $\lambda - \lambda_2$ and the type profile of customers choosing this server will be as in Eq. (14). As in Theorem 1, customers that join the HBF server will use the equilibrium bidding policy of Eq. (16). We will now verify that $p^E(\beta)$ of (13) satisfies the corresponding Wardrop condition.

First note that second and third conditions of the theorem imply that $D_1(\beta_1) = D_2(\lambda_2)$. Further, since $F_1(\beta)$ is a constant for $\beta \in (\beta_1, \beta_2)$, we have $X(\beta_1) = X(\beta_2)$ and $D_1(\beta_1) = D_1(\beta_2)$. Now consider a customer with $\beta < \beta_1$ that chooses the HBF server for the routing policy of Eq. (13). For this customer, we know from (3) in Property 1 of (13) that $X(\beta)$ is the optimal bid minimizing its individual cost i.e.,

$$
X(\beta) + \beta D_1(\beta) < X(\beta_1) + \beta D_1(\beta_1).
$$

This implies that

$$
X(\beta) + \beta D_1(\beta) < c + \beta D_2(\lambda_2)
$$
and hence any customer with $\beta < \beta_1$ has no incentive to choose the FIFO server. Similarly, for a customer with $\beta > \beta_2$ we have

$$X(\beta) + \beta D_1(\beta) < X(\beta_2) + \beta D_1(\beta_2) = c + \beta D_2(\lambda_2)$$

and clearly customers with $\beta > \beta_2$ have no incentive to choose the FIFO server.

Now consider any customer with $\beta \in (\beta_1, \beta_2)$. If such a customer were to choose the HBF server instead of the prescribed FIFO server, its cost at the HBF server will be $X(\beta_1) + \beta D_1(\beta_1)$. This follows from (1) and (3) of Property 1 in [3] and the fact that a deviation by a marginal customer does not change Eq. (13) and (14) due to the absolute continuity of $X$. Now $X(\beta_1) + \beta D_1(\beta_1) = c + \beta D_2(\lambda_2)$, i.e., the cost at the two servers is the same and there is no incentive for this customer to deviate from $p^F(\beta)$ given by Eq. (13). This completes the proof.

**Remark 2:** In the above discussion, we have assumed $c > 0$ and $M = 0$. W.l.o.g the observations are true even for the case when $c > M$ and $M \neq 0$. For this case, let $\hat{c} = c - M$. The above theorem and the previous lemmas will hold with $c$ being replaced by $\hat{c}$. It is easy to see that the case $c < M$ and $M \neq 0$ corresponds to the Wardrop equilibrium as described in Theorem 1.

We now provide an example to verify the claim in the above theorem.

**Example:** Consider an HBF and FIFO server with identical service rates $\mu = 5$. The arrival rate for the customers is $\lambda = 4$. $F(\cdot)$ is a uniform distribution with support $[a, b]$ where $a = 0$ and $b = 10$. Now suppose that the admission price at the FIFO server is $c = 0.2017$. Assume that the service requirement of all customers is identical with an exponential distribution of unit mean. For a system with the specified parameters, the interest is to characterize the Wardrop equilibrium routing function. It can be numerically verified that corresponding $p^E(\beta)$ is characterized by the two thresholds $\beta_1 = 1.67$ and $\beta_2 = 5.66$. With these thresholds, the corresponding arrival rates to the two servers are $\lambda_1 = 1.596$ and $\lambda_2 = 3.404$. Due to the exponential service requirement, it is easy to see that $D_2 = \frac{1}{\mu - \lambda_2} = 0.2938$. From the definition of $W_0$ and Eq. (2), the expected waiting time of customers in HBF queue expressed as a function of $\beta$ is

$$W_1(\beta) = \frac{1}{(\mu - \lambda(1 - F_1(\beta)))^2}.$$ 

It can be verified that the value of $D_2(\beta_1) = W_1(\beta_1) + \frac{1}{\mu} = 0.2938$. Further, evaluating $X^E(\beta_1)$ using Eq. (16) gives us $X^E(\beta_1) = c = 0.2017$. Clearly, all the three conditions are satisfied by the choice of $\beta_1$ and $\beta_2$ validating the theorem.

**IV. Discussion**

The present work characterizes a two threshold equilibrium when the service system has parallel FIFO and HBF servers. It should be noted that our analysis holds not only in the case of the FIFO scheduling policy but also for policies such as processor sharing or last-in, first-out (LCFS) or any other policy that does not differentiate between its customers on the basis of their $\beta$. As in [3], the application for this model can be in the form of a new pricing and auction mechanism in on-demand resource provisioning, e.g., cloud computing systems.

Some interesting extensions present themselves. If the two servers belong to the same system operator, it would be interesting to know the revenue maximizing admission price $c$ at the FIFO server. On the other hand, if the FIFO server is perceived as a competing system, then the revenue optimal $c$ obtained in this case would certainly be different.

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