Embedding in MDS codes and Latin cubes*

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Abstract
An embedding of a code is a mapping that preserves distances between codewords. We prove that any code with code distance $\rho$ and length $d$ can be embedded into an MDS code with the same code distance and length but under a larger alphabet. As a corollary we obtain embeddings of systems of partial mutually orthogonal Latin cubes and $n$-ary quasigroups.

Keywords — Latin square, Latin cube, MOLS, MDS code, embedding

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1 Introduction

Let $Q_q = \{0, \ldots, q - 1\}$. Consider the Cartesian product $Q_q^d$ as the metric space with the Hamming distance $\rho$. A subset $C$ of $Q_q^d$ is called a code with distance $t + 1$ if $\min \rho(x, y) = t + 1$ for any $x, y \in C, x \neq y$. A subset $C$ of $Q_q^d$ is called an MDS$(t, d, q)$ code (of order $q$, code distance $t + 1$ and length $d$) if $|C \cap \Gamma| = 1$ for each $t$-dimensional axis-aligned plane $\Gamma$. Ethier and Mullen [6] proved that MDS$(t, t + s, q)$ codes are equivalent to a set of $t$ mutually orthogonal $s$-dimensional Latin cubes of order $q$. If $s = 2$ then an MDS$(t, t + 2, q)$ code is equivalent to a system of MOLS; if $t = 1$ then an MDS$(1, 1 + s, q)$ code is equivalent to an $s$-dimensional Latin cube or the Cayley table of an $s$-ary quasigroup of order $q$. The main result of the present paper is the following

**Theorem 1.** Let $C \subset Q_q^d$ be a code with distance $t + 1$. Then there exist $q', q' \geq q$ and an MDS$(t, d, q')$ code $M$ such that $C \subseteq M \subseteq Q_q^d$.

It means that $Q_q \subseteq Q_q'$ and $Q_q^d$ is a subset of $Q_q'^d$ where $q \leq q'$. Consequently, if $C \subset Q_q^d$ then $C \subset Q_q'^d$.

This theorem generalizes theorems on embeddings of partial Latin squares [7], $d$-dimensional Latin cubes [1] and systems of mutually orthogonal partial Latin squares [9], [3]. These embeddings preserves dimension $d$ and increases order $q$. In Theorem 1 the order (size of the alphabet) of a code increases exponentially. In a construction for embedding partial MOLS from [3] the order of code increases polynomially. In Theorem 3 we propose a construction with a polynomial grow of order for embeddings of partial $d$-dimensional Latin cubes. Note than in the special case of Latin squares our order $q' = q^2$ is worse than $q' = 2q$ in classical paper by Evans [7].

Another embedding that preserves the order is considered in [8] by Krotov and Sotnikova. More precisely, they proved that a code $C \subset Q_q^d$ with distance 3 can be embedded in a perfect 1-error-correcting code in $Q_q'^d$, $d' = \frac{q^d - 1}{q - 1}$, where $q$ is a prime power.

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In [4] and [5] Donovan, Grannell and Yazici consider a more complicated problem. In terms of the present paper, they construct MDS codes with partially predetermined projections.

2 Preliminaries

The proofs of propositions from this section can be found in [2] and [10]. Notation and propositions are also examined in detail in [11] and [12].

As defined above a subset \( C \) of \( Q^d_q \) is called an MDS \((t, d, q)\) code (of order \( q \), code distance \( t + 1 \) and length \( d \)) if \(|C \cap \Gamma| = 1\) for each \( t \)-dimensional axis-aligned plane \( \Gamma \). It means that \( Q^d_q \) is an MDS code with code distance 1. Note that the alphabetical list is not important in the definition of MDS codes. Moreover, we can use different alphabets in different coordinates if it is convenient.

**Proposition 1 (Singleton bound).** A subset \( M \subset Q^d_q \) with code distance \( t + 1 \) is an MDS code if and only if \(|M| = q^{d-t}\).

Let \( C \subset Q^d_q \). The set
\[
\widehat{C}^i_a = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) : (x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_d) \in C\}
\]
is called a retract of \( C \). Note that \( \widehat{C}^i_a \subset Q^{d-1}_q \).

An embedded retract of \( C \) is the set
\[
C^i_a = \{(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_d) \in C : x_i = a\}
\]
Note that \( C^i_a \subset Q^d_q \).

A projection of \( C \) along the \( i \)th direction is the set
\[
C_i = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) : \exists a \in Q_q \ (x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_d) \in C\}
\]
Note that \( C_i \subset Q^{d-1}_q \).

**Proposition 2.**

- Every projection of an MDS code with distance \( t + 1 \), \( t \geq 1 \), is an MDS code with code distance \( t \).
- Every retract of an MDS code with distance \( t < d \) is an MDS code with the same code distance.

**Proposition 3.** Let \( M \subset Q^d_q \) be an MDS\((1, d, q)\) code. Then the set
\[
\{(x_1, \ldots, x_{d-1}, x_d + a(\mod q), a) : a \in Q_q, (x_1, \ldots, x_d) \in M\}
\]
is an MDS\((1, d, q)\) code.

Therefore, any MDS\((1, d, q)\) code can be included as a retract in an MDS\((1, d', q)\), where \( d' > d \). An analogous property for MDS codes with code distance greater than 2 is not true.

A subset \( T \) of an MDS code \( M \subset Q^d_q \) is called an MDS subcode of the code if \( T \) is an MDS code in \( A_1 \times \cdots \times A_d \) with the same code distance as \( M \) and \( T = M \cap (A_1 \times \cdots \times A_d) \) where \( A_i \subset Q_q, i \in \{1, \ldots, d\} \). Obviously \(|A_1| = \cdots = |A_d| = q^t\) and \( q' \) is the order of the subcode \( T \). Note that the definition of a Latin subsquare is analogous.
Proposition 4. Suppose that $C$ is an MDS code, $C_1$ is an MDS subcode of $C$ in a subcube $A_1 \times \cdots \times A_d$, $C_2$ is an MDS code in a subcube $A_1 \times \cdots \times A_d$ with the same distance as $C_1$. Then if we switch $C_1$ by $C_2$ then we obtain the MDS code $C'$ with the same parameters as $C$.

This exchanging of subcodes is called a switching of $C$.

Corollary 1. Suppose that $C$ is an MDS code, $C_1$ is an MDS subcode of $C$ in a subcube $A_1 \times \cdots \times A_d$ and $u \in A_1 \times \cdots \times A_d$. Then there exists an MDS code $C' = (C \setminus C_1) \cup C_2$ such that $u \in C'$.

Proposition 5. If $C_1$ and $B_1$ are disjoint MDS subcodes of $C$ and $C_2$ is an MDS code in the same subcube as $C_1$ then $B_1$ is an MDS subcode of $(C \setminus C_1) \cup C_2$.

The set $Q_{q_1,q_2}$ can be considered as the Cartesian product $Q_{q_1} \times Q_{q_2}$. Consequently, we can identify $Q_{q_1}^d \times Q_{q_2}^d$ and the hypercube $Q_{q_1,q_2}^d$. Thus, if $C_1 \subset Q_{q_1}^d$ and $C_2 \subset Q_{q_2}^d$ then

$$C_1 \times C_2 = \{(x_1, y_1), (x_2, y_2), \ldots, (x_d, y_d) : (x_1, \ldots, x_d) \in C_1, (y_1, \ldots, y_d) \in C_2\} \subset Q_{q_1,q_2}^d.$$

Theorem 2 (McNeish). Suppose $M_1$ is an $MDS(t,d,q_1)$ code and $M_2$ is an $MDS(t,d,q_2)$ code. Then $M_1 \times M_2$ is an $MDS(t,d,q_1q_2)$ code.

It is well known the following generalization of McNeish’s theorem.

Proposition 6. Let $M$ be an $MDS(t,d,q_1)$ code and let $U[x]$ be an $MDS(t,d,q_2)$ code for each $x \in M$. Then the set $\bigcup_{x \in M} x \times U[x]$ is an $MDS(t,d,q_1q_2)$ code.

Let $q$ be a prime power and let $Q_q = GF(q)$. A linear $k$-dimensional subspace $C \subset Q_q^d$ with distance $t$ is called $[d,k,t]_q$ code over $GF(q)$. By Proposition 1 we see that any $[d,d-t,t+1]_q$ code over $GF(q)$ is an $MDS(t,d,q)$ code.

For each linear code $C$ there exists a check matrix $A_C$ such that

$$C = \{x \in (GF(q))^d : A_Cx = 0\},$$

where $A_C$ is a matrix of size $(d-k) \times d$.

Proposition 7. A linear code $C \subset Q_q^d$ is an MDS code if and only if all minors of $A_C$ of order $d-k$ is nonzero.

In this case $C$ is $MDS(d-k+1,d,q)$ code. It is easy to see that we can choose check matrix of special type $A_C = (I|A')$ where $I$ is identity matrix of size $t \times t$, $t = d-k$.

Corollary 2. Let $A_C$ be the check matrix of an MDS code with size $(d-k) \times d$. Then every nontrivial linear combination of rows of $A_C$ contains greater than $k+1$ nonzero elements.

Proposition 8. Let $q$ be a prime power. Then for each integer $d \leq q + 1$ and $\rho$, $3 \leq \rho < d$, there exists a linear (over $GF(q)$) MDS code $C \subset Q_q^d$ with the code distance $\rho$.

We will consider the Cartesian product of linear codes. Let $V = (GF(q))^m$. We consider elements of $V^d$ as matrices of size $d \times m$ over $GF(q)$. Let $A$ be the check matrix of an MDS code $C$ over $GF(q)$. Then

$$M = \{Y \in V^d : AY = 0\}$$

is an MDS code by the McNeish’s theorem. Moreover, for a linear subspace $W \subset V^d$ the set $M|_W = \{Y \in W^d : AY = 0\}$ is an MDS code and $M|_W = M \cap W^d$ is an MDS subcode of $M$.

Proposition 9. Let $W$ be a linear subspace of $V$ and $u \in M$, where $M$ is defined by (1). Then the set $M_{u,W} = u + M|_W$ is an MDS code.
Proof. By the definitions $M$ is a linear code and $M_{|W}$ is a linear subspace of $M$. Then $M$ contains an affine subspace $M_{u,W} = u + M_{|W}$. The code distances of $M_{|W}$ and $M_{u,W}$ are the same as for $M$. Firstly, we estimate the cardinality of $M_{|W}$. Given an $s$-dimensional subspace $W$, there exists a non-degenerate matrix $B_W$ such that $W = \{xB_W : x \in (GF(q))^s\}$. The rows of $B_W$ are base of $W$. Let $Z$ be a $d \times s$ matrix over $GF(q)$ such that $A_M Z = 0$. Then the matrix $Z B_W$ belongs to $M \cap W = M_{|W}$. By the McNeish’s theorem, the number of such matrices $Z$ is equal to $(q^s)^{d-t}$. The cardinality of $M_{u,W}$ is the same as one of $M_{|W}$. Moreover, alphabets of any coordinate of elements of $M_{u,W}$ have the same cardinalities as ones of $M_{|W}$. Then $M_{u,W}$ is an MDS subcode of $M$ by definition. ■

3 Proof of Theorem 1

By Proposition\[\text{5}\] we can find a prime power $p \geq q$ such that there exists a linear MDS code $M$ over $GF(p)$ of length $d$ and code distance $t + 1$. Suppose that $A = (I|A')$ is the check matrix of this code over $GF(p)$. Consider elements of $C$. Let $s_i$ be a number of different symbols in position $i$, $i = 1, \ldots, d$. We use alphabet $V = (GF(p))^n$ where $n = \sum_i s_i$. Without loss of generality we suppose that $C \subset (GF(p))^n \times \cdots \times (GF(p))^n = V^d$, moreover, we claim that all coordinates of $d$-tuples from $C$ are unit vectors $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in V$ and the sets of unit vectors in different positions are disjoint.

Let $v \in (GF(p))^d$ and $\overline{w} \in V^d$. Define $v \cdot \overline{w} = \sum_i v_i w_i \in V$. Consider $\overline{w} = (e_1, \ldots, e_d)^T \in C$. For each row $a'$, $i = 1, \ldots, t$, of the matrix $A = (I|A')$ we define vector $g'_i = a' \cdot \overline{w}$. Let $W = W(w) = L(g^1, \ldots, g^t) \subset V$ be a linear hull of vectors $g^1, \ldots, g^t$. By Proposition\[\text{7}\] the set $M = \{x \in V^d : Ax = 0\}$ is an MDS code. It is easy to see that $\overline{w} = (-g^1 + e_1, \ldots, -g^t + e_t, e_{t+1}, \ldots, e_d)^T$ belongs to $M$ because

$$A \overline{w} = -A(g^1, \ldots, g^t, 0, \ldots, 0)^T + A\overline{w} = -(g^1, \ldots, g^t)^T + (a^1 \overline{w}, \ldots, a^d \overline{w})^T = \overline{0}.$$ 

By Proposition\[\text{[\[\text{5}\]\]}\] the affine subspace $\overline{w} + M_{|W}$ is an MDS subcode of $M$. Moreover, for each $i = 1, \ldots, d$ an alphabet of $i$th coordinate of $\overline{w} + M_{|W}$ contains $e_i$. By Corollary\[\text{[\[\text{4}\]\]}\] there is a switching of $M$ such that the resulting MDS code $M'$ contains $\overline{w}$.

At last, let us prove that we can independently make such switchings for all $\overline{w} \in C$. By Proposition\[\text{[\[\text{5}\]\]}\] it is sufficient to prove that subcodes $\overline{w} + M_{|W}$ are pairwise disjoint. Let us show that we can reconstruct initial $\overline{w}$ from $\overline{w} + M_{|W}$. By definition, every nonzero linear combination of $g'_i$ is a linear combination of rows of $A$ multiplying by $\overline{w}$. By Corollary\[\text{[\[\text{2}\]\]}\] and definition of $e_i$, this linear combination contains $d - t + 1$ different vectors $e_i$. If $\overline{x} \in W^d$ contains a nonzero element in one of the last $d - t$ coordinates then $\overline{w} + \overline{x}$ contains at least $d - t$ different vectors $e_i$ in this coordinate. Since the code distance of $C$ is equal to $t + 1$, we conclude that $\overline{w} + \overline{x}$ belongs to only one subcode. If $\overline{x} \in W^d$ has zeros in all last $d - t$ coordinates then $\overline{w} + \overline{x} = (f^1, \ldots, f^t, e_{t+1}, \ldots, e_d)^T$. Since the code distance of $C$ is equal to $t + 1$ and the Hamming distance between $\overline{w} + \overline{x}$ and $\overline{w}$ is less than $t + 1$, we conclude that $\overline{w}$ is the initial vector for this subcode.

4 Embedding into Latin hypercube

**Theorem 3.** Let $C \subset Q_d^d$ be a code with code distance 2. Then there exist $q', q \leq q' \leq q^{d-1}$, and an MDS$(1, d, q')$ code $M$ such that $C \subset M \subset Q_d^{q'}$.

In other words we can embed a partial Latin $(d - 1)$-dimensional hypercube of order $q$ into $(d - 1)$-dimensional Latin hypercube of order $q^{d-1}$.
Lemma 1. Let $C \subset Q^d_q$ be a code with distance 2. Suppose that for each $a \in Q_q$ the retract $C^d_a$ is a subset of an MDS code $M_a \subset Q^d_q$. Then there exists an MDS code $M \subset Q^d_q$ such that $C \subseteq M$.

Proof. Consider an arbitrary MDS code $B \subset Q^d_q$ with code distance 2. By Proposition 6 there exists an MDS code

$$M' = \{((x_1, y_1), \ldots, (x_d, y_d)) : x \in B, y \in M_a \text{ if } x_d = a\}$$

constructed as the generalized Cartesian product. For $a \in Q_q$ and $\overline{\tau} = (z_1, \ldots, z_{d-1}, a) \in M_a$ we consider the subcube $E_\overline{\tau} = (Q_q, z_1) \times \cdots \times (Q_q, z_{d-1}) \times H$, where $H = \{(a, a) : a \in Q_q\}$. The set $E_\overline{\tau} \cap M'$ is an MDS subcode because all retracts

$$(E_\overline{\tau} \cap M')^d_{(a,a)} = \widehat{B}_a^d \times (z_1, \ldots, z_{d-1})$$

are disjoint MDS codes under alphabet $Q_q \times z_i$ on $i$th coordinate, where $i = 1, \ldots, d - 1$. It is easy to see that $\overline{\tau} = ((0, z_1), \ldots, (0, z_{d-1}), (a, a)) \in E_\overline{\tau}$. By Proposition 4 and Corollary 1 there exists an MDS subcode $D_\overline{\tau}$ such that $M'' = (M' \setminus (E_\overline{\tau} \cap M')) \cup D_\overline{\tau}$ is an MDS code and $\overline{\tau} \in M''$. Since for different $\overline{\tau} \in C$ subcubes $E_\overline{\tau}$ are disjoint, all such switchings are independent. By switchings all $\overline{\tau} \in C$, we obtain an MDS code $M$ containing $((0, z_1), \ldots, (0, z_{d-1}), (a, a))$ for every $(z_1, \ldots, z_{d-1}, a) \in C$.

Proof of Theorem 3. We use induction on $d$. For $d = 2$ a code $C$ is a partial permutation. It is easy to see that any partial permutation is embedded into permutation. Suppose that the theorem is true for $d$. Consider a code $C \subset Q^{d+1}_q$ with distance 2. A retract $\widehat{C}^{d+1}_a \subset Q^d_q$ is a code with distance 2. By the induction hypothesis, there exists an MDS code $\widehat{M}_a \subset Q^d_q$ that includes $\widehat{C}^{d+1}_a$. By Proposition 3 for every $d$-dimensional MDS code $B$ with distance 2 there exists a $(d + 1)$-dimensional MDS code with distance 2 which includes $B$ as a retract. Then for each $a \in Q_q$ we obtain a $(d + 1)$-dimensional MDS code $M_a$ containing $\widehat{M}_a \subset Q^d_q$ as a retract and $\widehat{C}^{d+1}_a$ as the subset. It remains to apply Lemma 1 to complete the induction step.

For example, consider a partial Latin square of order 3 $C_3 = \begin{array}{ccc} 0 & 3 & 6 \\ 3 & 0 & 6 \\ 6 & 6 & 6 \end{array}$. By the construction from Theorem 3 we can embed $C_3$ into a Latin square of order 9. Consider generalized Cartesian product of $A = \begin{array}{ccc} a & b & c \\ b & c & a \\ c & a & b \end{array}$ and Latin squares $U_a = \begin{array}{ccc} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{array}$, $U_b = \begin{array}{ccc} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{array}$, $U_c = \begin{array}{ccc} 8 & 7 & 6 \\ 7 & 8 & 6 \\ 6 & 7 & 8 \end{array}$, $U_b = \begin{array}{ccc} 4 & 3 & 5 \\ 3 & 5 & 4 \\ 5 & 4 & 3 \end{array}$.

Elements of the target partial Latin square are bold. Elements of the MDS subcode which we choose for switching are italic. We need to perform 4 independent switchings.
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