Pseudo-Finsleroid metric function of spatially anisotropic relativistic type

G.S. Asanov

Division of Theoretical Physics, Moscow State University
119992 Moscow, Russia
(e-mail: asanov@newmail.ru)
Abstract

The paper contributes to the important and urgent problem to extend the physical theory of space-time in a Finsler-type way under the assumption that the isotropy of space is violated by a single geometrically distinguished spatial direction which destroys the pseudo-Euclidean geometric nature of the relativistic metric and space. It proves possible to retain the fundamental geometrical property that the indicatrix should be of the constant curvature. Similar property appears to hold in the three-dimensional section space. The last property was the characteristic of three-dimensional positive-definite Finsleroid space proposed and developed in the previous work, so that the present paper lifts that space to the four-dimensional relativistic level. The respective pseudo-Finsleroid metric function is indicated. Numerous significant tensorial and geometrical consequences have been elucidated.

Keywords: Finsler metrics, relativistic spaces.
1. Introduction

It is well-known that the relativistic pseudo-Euclidean metric possesses the geometrical property that the involved indicatrix (mass-shell) is of the constant unit curvature (equal to $-1$), in short $R_{\text{pseudo-Euclidean indicatrix}} = -1$. This important property gives rise to numerous important relativistic implications. They can attractively be extended if the Finsler metric function entails the constant curvature of indicatrix, in which case we call the function the Finsleroid metric function and denote by $F$. Respectively, in the present paper we are aimed to have

$$R_{\text{pseudo-Finsleroid indicatrix}} = -H^2. \quad (1.1)$$

In each tangent space, the $H$ is a constant.

The entailed Finslerian metric tensor is implied to be of the time-space signature: $(+---\ldots)$. The approach is $N$-dimensional; $N=4$ in the proper space-time context.

We shall specify the space anisotropy by the attractive condition that the rotational symmetry of the space is really violated by presence of a single geometrically distinguished vector of spacelike nature, that is to say, the anisotropy bears the axial character.

Presence of such anisotropy is typical of many pictures appeared in physical applications. Can an occurred preferred spatial vector field, to be denoted as $i_{\{3\}i}(x)$, be geometrically distinguished, in the sense that the vector $i_{\{3\}i}(x)$ can make traces on dependence of the fundamental metric function $F(x,y)$ on local directions at the point $x$? If resolved in positive, the deviation of the $F$ from the pseudo-Euclidean metric will influence everywhere the relativistic kinematic and dynamic, as well as the equations for the relativistic physical fields.

The Finsler geometry [1,2] (against the Riemannian geometry [3]) presents various genius and systematic methods to incorporate in the theory any possible anisotropy of space-time. However, to apply the methods we must specify the Finsler-type metric function $F$ which can adequately reflect and underline the anisotropy occurred.

In the previous work [4-8], we introduced and developed the spatially anisotropic three-dimensional Finsler metric function which fulfils the properties of positive-definiteness and constancy of the indicatrix curvature. Numerous remarkable and outstanding properties of the arisen space $F_{3PD}$ has been studied. However, the possibility to extend the space to a four-dimensional pseudo-Finsleroid relativistic space $F_{4\text{relativistic}}$ was remained a hidden matter. The principle difficulty to proceed was rooted in the circumstance that, despite the Finsleroid metric function $F_{3}$ of the space $F_{3PD}$ is constructed from tangent vectors in a pure algebraic (and rather simple) way, no four-dimensional relativistic pseudo-Finsleroid metric function $F_{4}$ of the algebraic form can be found (and probably does not exist) to continue $F_{3}$ in a four-dimensional framework.

The four-dimensional pseudo-Riemannian space involves the two-fold indicatrix-constancy property. Namely, the constancy of curvature of the indicatrix (three-dimensional pseudo-sphere) as well as the spatial section of the indicatrix. The section is the two-dimensional sphere. Surprisingly, it proves possible to retain the two-fold indicatrix-constancy property under extending $F_{3}$ to a four-dimensional level. The respective pseudo-Finsleroid metric function is proposed and described in the present paper.

A lucky and attractive possibility occurs when we undertake the systematic analysis of the possibility to reveal a respective function $F_{4}$ in terms of the involved angles $\eta, \theta, \phi$. They play the role of the hyperbolic, azimuthal, and polar angle, respectively, and extend the angle triple used frequently in the pseudo-Euclidean relativistic space. We restrict the
anisotropy to the uni-directional spatial type, such that at each point of the underground manifold exists a preferred vector of spatial type. We naturally use the direction of the vector to be the polar axis.

In Section 2, the initial notions and underlying concepts are explained.

In Section 3, the required Finsleroid metric function $F = F(x, y)$ is constructed by the help of the angle set $\eta, \theta, \phi$. The polar angle $\phi$ is given by the function $\phi = \phi(x, y)$ of simple traditional form. The explicit function $\theta = \theta(x, y)$ can be found for the azimuthal angle $\theta$ from the function $\tan(x, \theta)$ written in Eq. (3.20). The functions $\eta = \eta(x, y)$ and $\theta = \theta(x, y)$ extend their pseudo-Euclidean precursors by involving the characteristic scalars $p$ and $H$. At the same time, the dependence $\eta = \eta(x, y)$ cannot be written in an explicit algebraic form. The reason is that the explicit inverse representations are admitted by all the structural functions entered the Finsleroid metric except for one function. The algebraic form of that function ($r = r(x, \eta)$ presented in Section 3) is not sufficiently simple to perform an explicit inversion.

This peculiarity refers also to the Finsleroid metric function $F = F(x, y)$, since in the basic relativistic representation $F = y^0 V$ the $V$, like to the case of the pseudo-Euclidean relativistic framework, is a function of the relativistic angle $\eta$. The very dependence $V = V(x, \eta)$ is found explicitly upon evaluation of a simple integral.

In the end of Section 3, all components of the angle representation

$$l^i = l^i(x; \eta, \theta, \phi)$$

of the unit vector $l^i = y^i / F$ will be presented in a simple algebraic form.

Nevertheless, the derivatives $\partial \eta / \partial y^i$, as well as $\partial \theta / \partial y^i$, and $\partial \phi / \partial y^i$, admit simple algebraic representations in terms of the angle triple $\eta, \theta, \phi$. The latter property opens up a direct way to evaluate the components of the unit vector and Finsleroid metric tensor in a concise form. With these key objects at hands, we prove able to evaluate the indicatrix metric tensor. The tensor, being written in the angle representation, just reveals the property that the indicatrix is a space of the constant curvature of the value $-H^2$ (versus the pseudo-Euclidean value -1). The property is the principle result of the present paper.

Another fundamental geometrical significance of the obtained Finsleroid metric function is that the three-dimensional section space reveals the property that the generalized unit sphere (obtained in the horizontal section of the indicatrix) is a space of the positive-definite constant curvature of the value $p^2$ (versus the Euclidean sphere curvature value 1). The property was the characteristic of three-dimensional Finsleroid space $F^P_3$. In fact, the present paper lifts that space to the four-dimensional relativistic level. The $H$ and $p$ play the role of the characteristic parameters of the Finsleroid extension of the ordinary pseudo-Euclidean relativistic space.

In Section 4, all components of the entailed unit vector and angular metric tensor have been evaluated.

In Section 5, we find out the angle derivatives of the unit vectors to evaluate the indicatrix metric tensor. The obtained result is surprisingly simple. Namely, the entailed line element is the $1/H$-factor of the conventional line element over the pseudo-Euclidean relativistic hyperboloid. Thus we demonstrate the remarkable phenomenon that $1/H$ plays the role of the radius of the pseudo-Finsleroid relativistic hyperboloid, whence the pseudo-Finsleroid indicatrix is indeed the space of the constant curvature value $-H^2$. Finally, we write down the result of evaluation of the determinant of the respective pseudo-Finsleroid metric tensor $g_{ij}$.

In Conclusions we emphasize several important aspects of the developed theory.
In the case \( p = 1 \), the proposed function \( F \) does not involve any spatial directions and, therefore, is spatially isotropic. In Appendix we present the evaluation chain which optimally leads to the conclusion that in this case the function \( F \) is precisely the pseudo-Finsleroid relativistic metric function introduced and applied in the previous work [5-7,9,10].

2. Preliminaries

Let \( M \) be an \( N \)-dimensional \( C^\infty \) differentiable manifold, \( T_x M \) denote the tangent space to \( M \) at a point \( x \in M \), and \( y \in T_x M \) mean tangent vectors. Suppose we are given on \( M \) a pseudo-Riemannian metric \( S = S(x,y) \). Denote by \( \mathcal{R}_N = (M,S) \) the obtained \( N \)-dimensional pseudo-Riemannian space. Let us also assume that the manifold \( M \) admits a non-vanishing time-like 1-form \( b = b(x,y) \) and a non-vanishing 1-form \( i_{\{3\}} = i_{\{3\}}(x,y) \) of the space-like type. The pseudo-Riemannian norms are taken to be unit: \(||b|| = 1, ||i_{\{3\}}|| = -1 \). Relative to natural local coordinates in the space \( \mathcal{R}_N \) we have the local representations

\[
 b_i = b_i(x), \quad i_{\{3\}} = i_{\{3\}}(x), \quad S_{ij} = a_{ij}(x),
\]

with a pseudo-Riemannian metric tensor \( a_{ij} \), so that the time-space signature \( \text{sign}(a_{ij}) = (+ - - - \ldots) \) takes place.

We also introduce the transversal tensor \( p_{ij} = p_{ij}(x) \) as follows:

\[
p_{ij} := -a_{ij} + b_i b_j - i_{\{3\}} i_{\{3\}},
\]

and construct the quadratic form \( p_{ij} y^i y^j \) in terms of which we introduce the variable

\[
y_\perp = \sqrt{p_{ij} y^i y^j},
\]

obtaining the decomposition

\[
S^2 = b^2 - (i_{\{3\}})^2 - (y_\perp)^2.
\]

Below, we confine our treatment to the four-dimensional case, \( N = 4 \). The transversal tensor can conveniently be spanned by a vector pair \((i_i, j_j)\):

\[
p_{ij} = i_i j_j.
\]

The pseudo-Riemannian metric tensor is decomposing to read

\[
a_{ij} = b_i b_j - i_i j_j - j_i j_j - i_{\{3\}} i_{\{3\}}.
\]

The contravariant reciprocal tensor

\[
a^{ij} = b^i b^j - i^i j^j - j^i j^j - i_{\{3\}} i_{\{3\}}
\]

obeys the reciprocity \( a^{ij} a_{jn} = \delta^i_n \), where \( \delta^i_n \) stands for the Kronecker symbol. The unit norm condition reads

\[
a^{ij} b_i b_j = 1, \quad a^{ij} i_i i_j = -1, \quad a^{ij} j_i j_j = -1, \quad a^{ij} i_{\{3\}} i_{\{3\}} j_j = -1.
\]

The covariant index of the vector \( b_i \) will be raised by means of the tensorial rule \( b^i = a^{ij} b_j \), which inverse reads \( b_i = a_{ij} b^j \).
It is convenient to use the 1-forms $i = i_i y^i$ and $j = j_i y^i$, in addition to $b$ and $i^{(3)}$, obtaining the variable set

$$z = w_3 = \frac{i^{(3)}}{b}, \quad w_1 = \frac{i}{b}, \quad w_2 = \frac{j}{b},$$

and

$$t = \frac{w_1}{w_2} \equiv \frac{c_1}{c_2} \equiv \frac{i}{j}, \quad c_1 = \frac{w_1}{z}, \quad c_2 = \frac{w_2}{z}. \quad (2.9)$$

Also, we shall apply the notation

$$w_\perp = \sqrt{(w_1)^2 + (w_2)^2}. \quad (2.10)$$

At each point $x \in M$, it proves convenient to use the triple $\eta, \theta, \phi$ with the meaning of the relativistic, azimuthal, and polar angle, respectively. It is the direction of the vector $i^{(3)}_i(x)$ that is regarded as the polar axis at point $x$.

### 3. The pseudo-Finsleroid metric function to geometrize space-time

Our guiding idea is to construct the desired Finsleroid metric function

$$F = bV(x, r) \quad (3.1)$$

by the help of the variable

$$r = r(x, w_3, w_\perp) \quad (3.2)$$

to be specified in the following axial way:

$$r = zU(x, f), \quad f = c_2Z(x, t), \quad (3.3)$$

subject to the angle representation

$$r = r(x, \eta), \quad f = f(x, \theta), \quad Z = Z(x, \phi), \quad (3.4)$$

specified by the following differential equations:

$$\frac{1}{V}V_\eta = -\frac{1}{H^2 R_1} \sinh \eta, \quad \frac{1}{U}U_\theta = \frac{1}{p^2 R_2} \sin \theta, \quad (3.5)$$

were

$$R_1 = \cosh \eta + \sqrt{1 - \frac{1}{p^2} + \left(1 - \frac{1}{H^2}\right) \sinh^2 \eta} \quad (3.6)$$

and

$$R_2 = \cos \theta + \sqrt{\frac{1}{p^2} - 1 \sin \theta}. \quad (3.7)$$

We assume

$$|p| \leq 1, \quad |H| \geq 1.$$

The equations can explicitly be integrated:

$$V = C_1(x) \frac{1}{R_1}J \quad (3.8)$$
with
\[ J = \exp \left( \frac{\sqrt{H^2 - 1}}{H} \arccosh \left( p \frac{\sqrt{H^2 - 1}}{\sqrt{H^2 - p^2}} \cosh \eta \right) \right), \quad (3.9) \]
or
\[ J = \left( \sqrt{1 - \frac{1}{H^2}} \cosh \eta + \sqrt{1 - \frac{1}{p^2} + \left( 1 - \frac{1}{H^2} \right) \sinh^2 \eta} \right)^{\sqrt{1 - \frac{1}{H^2}}}, \quad (3.10) \]
and also
\[ U = C_2(x) \frac{1}{R_2} I \quad (3.11) \]
with
\[ I = \exp \left( \sqrt{1 - \frac{1}{p^2} - 1} \theta \right). \quad (3.12) \]

Next, we set forth the representation
\[ \ln r = \frac{1}{p^2} \int \frac{d\eta}{R_1 \sinh \eta}, \quad (3.13) \]
which entails
\[ \frac{1}{r} \eta = \frac{1}{p^2 R_1} \frac{1}{\sinh \eta}, \quad \eta_r = \frac{p^2}{r} R_1 \sinh \eta. \quad (3.14) \]

By making the substitution
\[ r = \frac{C_2(x) \sinh \eta}{R_1} Y_1, \quad (3.15) \]
we can conclude that the function \( Y_1 \) must obey the equation
\[ (\ln Y_1) \eta = \left( \frac{1}{p^2} - 1 \right) \frac{1}{\sinh \eta \sqrt{1 - \frac{1}{p^2} + \left( 1 - \frac{1}{H^2} \right) \sinh^2 \eta}}. \quad (3.16) \]
The solution reads
\[ Y_1 = \exp \left( -\frac{1}{2} \sqrt{\frac{1}{p^2} - 1} \arctan \left( \sqrt{\frac{1}{p^2} - 1} - \frac{2 \cosh \eta}{\frac{1}{H^2} - \frac{1}{p^2} + \left( 1 - \frac{1}{H^2} \right) \cosh^2 \eta} \right) \right) \quad (3.17) \]
After that, we need to apply the integral
\[ \ln f = \int \frac{d\theta}{\sin \theta \left( \cos \theta + \sqrt{\frac{1}{p^2} - 1} \sin \theta \right)}. \quad (3.18) \]
so that

$$\frac{\partial \theta}{\partial f} = \frac{1}{f} \sin \theta \left( \cos \theta + \sqrt{\frac{1}{p^2} - 1} \sin \theta \right), \quad \theta_f = \frac{1}{f} \sin \theta \frac{1}{U} I.$$ 

It can readily be verified that

$$f = C_{17}(x) \frac{\sin \theta}{\cos \theta + \sqrt{\frac{1}{p^2} - 1} \sin \theta} \equiv C_{17}(x) \sin \theta \frac{1}{R_2}. \quad (3.19)$$

The last function can be inverted, yielding

$$\tan \theta = \frac{1}{C_{17} f} \frac{1}{1 - \sqrt{\frac{1}{p^2} - 1} \frac{1}{C_{17} f}}. \quad (3.20)$$

The property \( f(x, 0) = 0 \) holds.

Noting also (3.11), we obtain

$$\sin \theta = \frac{1}{U} f I, \quad \theta_f = \frac{1}{U^2} I^2. \quad (3.21)$$

The polar angle \( \phi \) is constructed in the traditional way:

$$\phi = \arctan t, \quad (3.22)$$

so that

$$\phi_t = \frac{1}{1 + t^2}.$$ 

The last equality entails \( Z = \sqrt{1 + t^2} C_{11}(x) \) and

$$f = \frac{1}{w_3} \sqrt{w_1 w_1 + w_2 w_2} C_{11}(x). \quad (3.23)$$

Henceforth, it is convenient to specify the integration constants according to \( C_2 = C_{17} = 1 \). Using (3.20) and (3.21), we can write

$$U^2 = \left( 1 - \sqrt{\frac{1}{p^2} - 1} f^2 + f^2 \right) I^2 = \left( 1 - 2 \sqrt{\frac{1}{p^2} - 1} f + \frac{1}{p^2} f^2 \right) I^2.$$ 

Making the choice \( C_{11} = p \), we come to

$$U^2 = \left( 1 - 2 \sqrt{1 - p^2 w} + w \right) I^2 \quad (3.24)$$

and

$$f = wp, \quad (3.25)$$

where

$$w = \frac{1}{w_3} \sqrt{w_1 w_1 + w_2 w_2}. \quad (3.26)$$

With the function \( U \) given by (3.24), the product \( r = zU \) is just the three-dimensional positive-definite Finsleroid metric function proposed and developed in the previous work.
(5-8] (where the notation $h$ was used instead of $p$). The axial structure and the positive-definiteness of the entailed metric tensor, taken in conjunction with the indicatrix curvature constancy of positive value, is the characteristic of the function.

Now, the angle representation of tangent vectors $\{y^i\}$ can be derived. Indeed, at any fixed background point $x \in M$, we have

$$w_3 = w_3(\eta, \theta), \quad w_\perp = w_\perp(\eta, \theta),$$

with

$$w_3 = \frac{r}{U}, \quad w_\perp = \frac{w_3 - f}{p} = \frac{w_3 - 1}{p} \frac{1}{R_2}.$$ 

so that

$$w_3 = r \frac{1}{I} R_2, \quad w_\perp = \frac{1}{p} \frac{1}{I} \sin \theta. \quad (3.27)$$

Also,

$$w_1 = w_\perp \cos \phi, \quad w_2 = w_\perp \sin \phi. \quad (3.28)$$

Thus, the sought result is given by the equalities

$$y^1 = w_1 b, \quad y^2 = w_2 b, \quad y^3 = w_3 b,$$

together with

$$y^0 = \frac{1}{V} F, \quad V = V(\eta) \quad (3.29)$$

(here $y^0 = b$).

4. Unit vectors and metric tensor

To study the geometrical aspects of the pseudo-Finsleroid space, we need to clarify the structure of the entailed metric tensor.

Choosing a fixed tangent space, we shall represent the involved vectors by the help of their components with respect to the base frame $\{b_i, i_i, j_i, i(3)i\}$ which enters the Riemannian metric tensor $a_{ij}$ according to (2.5).

It is convenient to use the notation

$$V_1 = \frac{\partial V}{\partial w_1}, \quad V_2 = \frac{\partial V}{\partial w_2}, \quad V_3 = \frac{\partial V}{\partial w_3}, \quad V_r = \frac{\partial V}{\partial r},$$

and

$$r_1 = \frac{\partial r}{\partial w_1}, \quad r_2 = \frac{\partial r}{\partial w_2}, \quad r_3 = \frac{\partial r}{\partial w_3}.$$ 

The identity

$$r = r_1 w_1 + r_2 w_2 + r_3 w_3 \quad (4.1)$$

holds fine due to the homogeneity involved.

We should derive the covariant unit vector components $l_i = \partial F/\partial y^i$ from the function

$$F = y^0 V.$$
We get
\[ l_1 = V_1, \quad l_2 = V_2, \quad l_3 = V_3, \tag{4.2} \]
and
\[ l_0 = V - (V_1 w_1 + V_2 w_2 + V_3 w_3) = V - V_r(r_1 w_1 + r_2 w_2 + r_3 w_3), \]
or
\[ l_0 = V - V_r r \]
(because of (4.1)).

Since
\[ \frac{1}{V} V_r = - \frac{1}{H^2} \frac{p^2}{r} \sinh^2 \eta \]
(see (3.5) and (3.14)), we obtain the clear and useful representation
\[ l_0 = V \left( 1 + \frac{p^2}{H^2} \sinh^2 \eta \right). \tag{4.3} \]

With this preparation, all the derivatives
\[ l_{ij} = \frac{\partial l_i}{\partial y^j} \]
can readily be found. At first, we obtain
\[ l_{00} = \frac{1}{y^0} V_{rr} r^2, \quad l_{01} = -\frac{1}{y^0} V_{rr} r r_1, \quad l_{02} = -\frac{1}{y^0} V_{rr} r r_2, \quad l_{03} = -\frac{1}{y^0} V_{rr} r r_3. \]
After that, we can use \( V_1 = V_r r_1 \equiv l_1 \) and conclude that
\[ y^0 l_{11} = V_r r_1 r_1 + V_r r_{11}. \]

By following this method, we can find all the components of the angular metric tensor
\[ h_{ij} = F l_{ij}. \]
The result reads
\[ h_{00} = V V_{rr} r^2, \quad h_{01} = -V V_{rr} r r_1, \quad h_{02} = -V V_{rr} r r_2, \quad h_{03} = -V V_{rr} r r_3, \tag{4.4} \]
\[ h_{11} = V V_{rr} r_1 r_1 + V V_r r_{11}, \quad h_{12} = V V_{rr} r_1 r_2 + V V_r r_{12}, \quad h_{22} = V V_{rr} r_2 r_2 + V V_r r_{22}, \tag{4.5} \]
\[ h_{13} = V V_{rr} r_1 r_3 + V V_r r_{13}, \quad h_{23} = V V_{rr} r_2 r_3 + V V_r r_{23}, \quad h_{33} = V V_{rr} r_3 r_3 + V V_r r_{33}. \tag{4.6} \]

Using the derivative values
\[ \frac{1}{V} V_r = - \frac{1}{H^2} \frac{p^2}{r} \sinh^2 \eta, \quad \eta_r = \frac{p^2}{r} \sinh \eta R_1, \quad \frac{1}{V} V_{rr} = - \frac{1}{V} V_{rr}, \quad \frac{1}{V} V_{q} = - \frac{1}{H^2} \frac{1}{R_1} \sinh \eta,
we straightforwardly obtain the angle representation
\[ h_{ij} = -\frac{1}{H^2} \left( \eta_i \eta_j + \sinh^2 \eta (\theta_i \theta_j + \sin^2 \theta \phi_i \phi_j) \right) F^2. \]  
\hfill (4.7)

5. The indicatrix curvature

With the help of the formulas written in the end of Section 3, the differentiation of
the unit vector components
\[ l^0 = \frac{1}{V}, \quad l^a = \frac{1}{V} w_a \]
with respect to the used angles yields the following list:
\[ l^0_\eta = -\frac{1}{V} V l^0, \quad l^0_\theta = l^0_\phi = 0, \]
\[ l^1_\eta = -\frac{1}{V} V l^1 + \frac{1}{r} r l^1, \quad l^1_\theta = \left( -\frac{1}{I} I_\theta + \frac{\cos \theta}{\sin \theta} \right) l^1, \quad l^1_\phi = -\frac{\sin \phi}{\cos \phi} l^1, \]
\[ l^2_\eta = -\frac{1}{V} V l^2 + \frac{1}{r} r l^2, \quad l^2_\theta = \left( -\frac{1}{I} I_\theta + \frac{\cos \theta}{\sin \theta} \right) l^2, \quad l^2_\phi = \frac{\cos \phi}{\sin \phi} l^2, \]
\[ l^3_\eta = -\frac{1}{V} V l^3 + \frac{1}{r} r l^3, \quad l^3_\theta = \left( \frac{1}{R_2} R_2 \theta - \frac{1}{I} I_\theta \right) l^3, \quad l^3_\phi = 0, \]

together with
\[ l^3_\theta = -\frac{1}{p^2 R_2} \sin \theta l^3 = -\frac{1}{p^2 \sin \theta} \frac{1}{V} \frac{1}{r}. \]

From (3.7) and (3.12) we can conclude that
\[ \frac{1}{I} I_\theta = \sqrt{\frac{1}{p^2} - 1}, \quad R_2 \theta = -\sin \theta + \sqrt{\frac{1}{p^2} - 1} \cos \theta. \]

We straightforwardly obtain the angle representation
\[ h_{ij} = -\frac{1}{H^2} \left( \eta_i \eta_j + \sinh^2 \eta (\theta_i \theta_j + \sin^2 \theta \phi_i \phi_j) \right) F^2. \]  
\hfill (5.1)

Now we are prepared to evaluate the indicatrix metric tensor
\[ i_{ab} = -h_{ij} l^i_a l^j_b, \quad (a, b = \eta, \theta, \phi) \]  
\hfill (5.2)
in the angle representation. The following simple result is obtained:
\[ i_{\eta\eta} = -\frac{1}{H^2}, \quad i_{\theta\theta} = -\frac{1}{H^2} \sinh^2 \eta, \quad i_{\phi\phi} = -\frac{1}{H^2} \sinh^2 \eta \sin^2 \theta, \]
\[ i_{\eta\theta} = i_{\theta\eta} = 0, \quad i_{\eta\phi} = i_{\phi\eta} = 0, \quad i_{\phi\phi} = 0, \]
The entailed line element square reads

\[(ds)^2 = (dF)^2 - F^2(dl)^2\]

with

\[(dl)^2 = \frac{1}{H^2}((d\eta)^2 + \sinh^2 \eta((d\theta)^2 + (d\phi)^2)).\]

This is just the $1/H^2$-factor of the metric on the pseudo-Euclidean relativistic hyperboloid. We observe the remarkable phenomenon that $1/H$ plays the role of the radius of the pseudo-Finsleroid relativistic hyperboloid, whence the pseudo-Finsleroid indicatrix is indeed the space of the constant curvature value $-H^2$.

The attentive evaluation of the determinant of the respective pseudo-Finsleroid metric tensor

\[g_{ij} = h_{ij} + l_il_j\]

leads to the following result:

\[\det(g_{ij}) = -\frac{1}{H^6} \left(p^4 I^4 V^4 R_1\right)^2 \frac{1}{r^6} \sinh^6 \eta. \quad (5.3)\]

We observe here the independence of the polar angle $\phi$. The dependence on the angle $\theta$ enters through the function $I$ written in (3.12). The right-hand part of the determinant (5.3) tends to $-1$ when $H \to 1, p \to 1$.

Conclusions: Important aspects

The pseudo-Finsleroid approach developed in the present paper involves two characteristic parameters of extension, $H$ and $p$. The metric function $F$ reduces to the pseudo-Riemannian metric function when $H \to 0, p \to 0$. The geometrical significance of the parameters is remarkable, namely they represent the indicatrix curvature values. The entailed Finslerian metric tensor is of the time-space signature $(+−−−)$. The respective pseudo-Finsleroid metric function is implicitly proposed by a simple algebraic equation. However, the components of the Finslerian unit vector and metric tensor, as well as the partial derivatives of the angles $\eta, \theta, \phi$ are evidenced to show sufficiently transparent and handy structure.

Unlike to the pseudo-Riemannian geometry, the curvature tensor of the tangent space does not vanish identically. The metric tensor does depend on tangent vectors: $g_{ij} = g_{ij}(x,y)$ in contrast to the Riemannian preassumption $a_{ij} = a_{ij}(x)$.

Above, we have found the function $F$ by starting from the attractive idea to preserve the key pseudo-Euclidean geometrical property that the curvature of the associated relativistic indicatrix is everywhere constant.

In the preceding papers [6-10], the pseudo-Finsleroid relativistic metric function revealing such a property was constructed upon assuming that the involved geometrically distinguished vector (that was denoted by $b_i$) is time-like and the spatial isotropy holds fine. In the preceding sections of the present paper it is shown that the metric admits a due extension which also involves the preferred vector of space-like type.

Various interesting physical applications of the proposed pseudo-Finsleroid metric function can be elaborated. Respective extensions of the relativistic kinematic and dynamic are the nearest ways.
Appendix: Spatially isotropic limit

Let us put $p = 1$ in the representations given in Section 3. Among them, we choose the following:

$$J = \exp \left( \sqrt{1 - \frac{1}{H^2}} \eta \right), \quad R_1 = \cosh \eta + \sqrt{1 - \frac{1}{H^2}} \sinh \eta, \quad r = C_2 \frac{\sinh \eta}{R_1}.$$ 

It follows that

$$\frac{1}{C_2} \cosh \eta + \sqrt{1 - \frac{1}{H^2}} \sinh \eta = \sinh \eta, \quad \frac{1}{C_2} r \cosh \eta = \left( 1 - \frac{1}{C_2} \sqrt{1 - \frac{1}{H^2}} \right) \sinh \eta.$$ 

This equation can be solved, yielding

$$\frac{1}{\tanh \eta} = C_2 \frac{1}{r} - \sqrt{1 - \frac{1}{H^2}}, \quad \frac{1}{\sinh^2 \eta} = \left( C_2 \frac{1}{r} - \sqrt{1 - \frac{1}{H^2}} \right)^2 - 1.$$ 

In this way, we obtain the angle functions

$$\sinh \eta = \frac{1}{\sqrt{\left( C_2 \frac{1}{r} - \sqrt{1 - \frac{1}{H^2}} \right)^2 - 1}}, \quad \cosh \eta = \frac{C_2 \frac{1}{r} - \sqrt{1 - \frac{1}{H^2}}}{\sqrt{\left( C_2 \frac{1}{r} - \sqrt{1 - \frac{1}{H^2}} \right)^2 - 1}},$$

which entail the representations

$$R_1 = \frac{C_2 \frac{1}{r}}{\sqrt{\left( C_2 \frac{1}{r} - \sqrt{1 - \frac{1}{H^2}} \right)^2 - 1}}, \quad e^n = \frac{C_2 \frac{1}{r} - \sqrt{1 - \frac{1}{H^2}} + 1}{\sqrt{\left( C_2 \frac{1}{r} - \sqrt{1 - \frac{1}{H^2}} \right)^2 - 1}},$$

so that

$$J = \left( \frac{C_2 \frac{1}{r} - \sqrt{1 - \frac{1}{H^2}} + 1}{\sqrt{\left( C_2 \frac{1}{r} - \sqrt{1 - \frac{1}{H^2}} \right)^2 - 1}} \right)^{1 - \frac{1}{H^2}}.$$

The function $V$ written in Section 3 reduces to

$$V = \frac{1}{C_2} \sqrt{\left( C_2 \frac{1}{r} - \sqrt{1 - \frac{1}{H^2}} \right)^2 - 1} \left( \frac{C_2 \frac{1}{r} - \sqrt{1 - \frac{1}{H^2}} + 1}{\sqrt{\left( C_2 \frac{1}{r} - \sqrt{1 - \frac{1}{H^2}} \right)^2 - 1}} \right)^{1 - \frac{1}{H^2}}.$$
(we have taken the integration constant $C_1$ to be 1).

Let us square this function:

$$V^2 = \frac{1}{(C_2)^2} r^2 \left[ \left( C_2 \frac{1}{r} - \sqrt{1 - \frac{1}{H^2}} \right)^2 - 1 \right] \left( C_2 \frac{1}{r} - \frac{1 - \frac{1}{H^2} + 1}{\sqrt{1 - \frac{1}{H^2} - 1}} \right)^{1 - \frac{1}{H^2}}.$$

Convenient cancelation is possible here. We obtain

$$V^2 = \frac{1}{(C_2)^2} \left( C_2 - \left( 1 + \sqrt{1 - \frac{1}{H^2}} \right) r \right)^{1 - \sqrt{1 - \frac{1}{H^2}}} \left( C_2 + \left( 1 - \sqrt{1 - \frac{1}{H^2}} \right) r \right)^{1 + \sqrt{1 - \frac{1}{H^2}}}.$$

Since $w_3 = r \cos \theta, w_\perp = r \sin \theta$, we have $r = \sqrt{(w_1)^2 + (w_2)^2 + (w_3)^2} \equiv |w|$.

Using the notation

$$g_+ = -\sqrt{H^2 - 1} + H, \quad g_- = -\sqrt{H^2 - 1} - H,$$

and $\tilde{C}_2 = HC_2$, we can write

$$V^2 = \frac{1}{(\tilde{C}_2)^2} \left( \tilde{C}_2 + g_- |w| \right)^{g_+ / H} \left( \tilde{C}_2 + g_+ |w| \right)^{-g_- / H}.$$  (A.2)

This $V^2$ does coincide with the respective function used in [5-10].

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