SPRINGER CORRESPONDENCE FOR EXCEPTIONAL LIE ALGEBRAS AND THEIR DUALS IN SMALL CHARACTERISTIC

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Abstract. We describe the Springer correspondence explicitly for exceptional Lie algebras of type $G_2$ and $F_4$ and their duals in bad characteristics, i.e. in characteristics 2 and 3.

1. Introduction

In this paper we describe the Springer correspondence explicitly for exceptional Lie algebras of type $G_2$ and $F_4$ and their duals in bad characteristics. In what follows we describe the set-up of the paper in more detail.

Let $G$ be an almost-simple algebraic group defined over an algebraically closed field $k$ of prime characteristic and $\mathfrak{g}$ the Lie algebra of $G$. Let $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. Denote by $U_G$, $N_\mathfrak{g}$ and $N_{\mathfrak{g}^*}$ the varieties of unipotent elements in $G$, nilpotent elements in $\mathfrak{g}$, and nilpotent elements in $\mathfrak{g}^*$, respectively. We recall that an element in $\mathfrak{g}^*$ is called nilpotent if it annihilates a Borel subalgebra of $\mathfrak{g}$ (see [KW]). The group $G$ acts on $\mathfrak{g}$ and $\mathfrak{g}^*$ by adjoint and coadjoint action, respectively. It is known that the number of $G$-orbits in $U_G$, $N_\mathfrak{g}$ and $N_{\mathfrak{g}^*}$, respectively, is finite (for $N_{\mathfrak{g}^*}$ see [X4] and references there). We fix a prime $l \neq \text{char } k$. Let $\mathcal{A}_G$ (resp. $\mathcal{A}_\mathfrak{g}$, $\mathcal{A}_{\mathfrak{g}^*}$) denote the set of all pairs $(O, E)$, where $O \subset U_G$ (resp. $O \subset N_\mathfrak{g}$; $O \subset N_{\mathfrak{g}^*}$) is a $G$-orbit and $E$ is an irreducible $G$-equivariant $\overline{\mathbb{Q}}_l$-local system on $O$ (up to isomorphism). Let $W$ be the Weyl group of $G$ and let $\text{Irr}(W)$ denote the set of irreducible characters (over $\overline{\mathbb{Q}}_l$) of the Weyl group $W$. The Springer correspondence for $G$ (resp. $\mathfrak{g}$, $\mathfrak{g}^*$) maps the set $\text{Irr}(W)$ injectively into the set $\mathcal{A}_G$ (resp. $\mathcal{A}_\mathfrak{g}$, $\mathcal{A}_{\mathfrak{g}^*}$); we denote the map by $\gamma_G$ (resp. $\gamma_\mathfrak{g}$, $\gamma_{\mathfrak{g}^*}$). The construction was originally given for $\mathfrak{g}$ by Springer in large characteristic [Sp] and generalized to arbitrary characteristic for $G$ by Lusztig [L1]. The same construction as Lusztig’s works for $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) in any characteristic assuming that $G$ is adjoint (resp. simply connected), see for example [X1, X2]; this in turn gives rise to Springer correspondences for all $\mathfrak{g}$ and $\mathfrak{g}^*$ in any characteristic.

If the characteristic of $k$ is good for $G$, there exists a $G$-equivariant isomorphism between $U_G$ and $N_\mathfrak{g}$ by a theorem of Springer, moreover, there exists a $G$-invariant non-degenerate bilinear form on $\mathfrak{g}$ which we can use to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. Thus in such cases the sets $\mathcal{A}_G$, $\mathcal{A}_\mathfrak{g}$ and $\mathcal{A}_{\mathfrak{g}^*}$ can be naturally identified. The Springer correspondence for $G$, i.e. the map $\gamma_G : \text{Irr}(W) \rightarrow \mathcal{A}_G$ has been described explicitly in all characteristics [L1, LS2, Sp, Sh, AL, S2].

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Assume now that the characteristic of $k$ is bad for $G$. In general we can no longer identify the sets $\mathcal{A}_G, \mathcal{A}_g$ and $\mathcal{A}_{g^*}$. The Springer correspondence for $g$ and $g^*$ in bad characteristic has been determined explicitly when $G$ is a classical group [X3]. Assume that $G$ is an exceptional group of type $G_2$ or $F_4$ in bad characteristic. The Springer correspondence for $g$ has been described by Spaltenstein [S3] when $G$ is of type $F_4$ and char $k = 2$. In this paper we describe the Springer correspondence for $g$ and $g^*$ in the remaining cases. We will focus on the case of $g^*$ with details given and treat the case of $g$ briefly. The Springer correspondence for $g$ and $g^*$ turns out the same as in characteristic 0 when $G$ is of type $G_2$ and char $k = 2$, or when $G$ is of type $F_4$ and char $k = 3$. The same holds for $g^*$ when $G$ is of type $G_2$ and char $k = 3$.

The paper is organized as follows. In $\S 2$ we discuss some preliminaries. In particular, we determine the existence of $G$-invariant non-degenerate bilinear forms on $g$ (see Lemma 2.1). In $\S 3$ we describe the component groups of centralizers for nilpotent elements in $g^*$ and we compute the dimensions of the Springer fibers. In $\S 4$ we describe the Springer correspondence $\gamma_{g^*}: \text{Irr} W \to \mathcal{A}_{g^*}$ for $g^*$ explicitly. In particular, following [L3], we give an a priori description of the set $\gamma_{g^*}^{-1}(\bar{O}, \bar{Q}) | (\bar{O}, \bar{Q}) \in \mathcal{A}_{g^*}$; i.e. a set of Weyl group representations that parametrize nilpotent coadjoint orbits in $g^*$. Finally, in $\S 5$ we describe the maps $\gamma_g: \text{Irr} W \to \mathcal{A}_g$ for $g$ explicitly.

2. Preliminaries

In this section let $G$ be an exceptional group of type $G_2$ or $F_4$ defined over an algebraically closed field $k$.

2.1. Structural constants and Bruhat decomposition. As in [X3], we make use of the generators and relations in $G$, $g$, $g^*$, and the Bruhat decomposition for $G$. We recall some notations from [X3] §7.3-§7.5], for details, see loc.cit. For $g \in G$ and $\xi \in g^*$, we write $g.\xi$ for the coadjoint action, i.e., $g.\xi(x) = \xi(\text{Ad}(g^{-1})x)$ for all $x \in g$, where Ad denotes the adjoint action of $G$ on $g$.

Fix a maximal torus $T$ of $G$ and $B \supset T$ a Borel subgroup of $G$. Let $R, R^+$ and $\Pi$ be the set of roots, positive roots, and simple roots, determined by $(G, T, B)$, respectively. For each $\alpha \in R$, let $x_\alpha: \mathbb{G}_a \to U_\alpha \subset G$ be an isomorphism such that

$$sx_\alpha(t)s^{-1} = x_\alpha(\alpha(s)t)$$

for all $s \in T$ and $t \in \mathbb{G}_a$,

$$n_\alpha(t) := x_\alpha(t)x_{-\alpha}(t^{-1})x_\alpha(t) \in N_G(T)$$

and $n_\alpha(1) := n_\alpha$ has image $w_\alpha$ in $W$, where $w_\alpha$ denotes the reflection with respect to $\alpha$. Define

$$h_\alpha(t) = n_\alpha(t)n_\alpha(-1) \in T.$$ 

By Bruhat decomposition, each $g \in G$ can be written uniquely in the form

$$g = bn_wu_w,$$

for some $b \in B$, $u_w \in U_w := \{ \prod_{\alpha>0, w(\alpha)<0} x_\alpha(t_\alpha) | t_\alpha \in \mathbb{G}_a \},$

and $n_w \in N_G(T)$ a representative of $w \in W$.
Let \( \mathfrak{t}, \mathfrak{b}, \mathfrak{n} \) be the Lie algebra of \( T, B, \) and the unipotent radical \( U \) of \( B \) respectively. Define \( \mathfrak{n}^* = \{ \xi \in \mathfrak{g}^* \mid \xi(\mathfrak{b}) = 0 \} \). Then for any \( \xi \in \mathcal{N}_{\mathfrak{g}}^* \), there exists \( g \in G \) such that \( g.\xi \in \mathfrak{n}^* \). We make use of the Chevalley basis \( \{ h_\alpha, \alpha \in \Pi, e_\alpha, \alpha \in R \} \) of \( \mathfrak{g} \), in particular, the basis \( \{ e_\alpha, \alpha \in R^+ \} \) of \( \mathfrak{n} \), chosen in \[X3\], where the adjoint action of \( G \) is given as follows

\[
\text{Ad}(x_\alpha(t)) e_\beta = \sum_i t_i M_{\alpha, \beta, i} e_{i\alpha + \beta}, \quad \alpha, \beta \in R^+
\]

(2.2)

\[
\text{Ad}(h_\alpha(t)) e_\beta = t^{A_{\alpha \beta}} e_\beta, \quad \text{Ad}(n_\alpha) e_\beta = \eta_{\alpha, \beta} e_{w_\alpha(\beta)}, \quad \alpha \in \Pi,
\]

and the basis \( \{ e'_\alpha, \alpha \in R^+ \} \) of \( \mathfrak{n}^* \), chosen in \[X3\], where the coadjoint action of \( G \) is given as follows

\[
x_\alpha(t).e'_\beta = \sum_i (-1)^i t^i M_{\alpha, -i\alpha - \beta, i} e'_{i\alpha + \beta}, \quad \alpha, \beta \in R^+
\]

(2.3)

\[
h_\alpha(t).e'_\beta = t^{A_{\alpha \beta}} e'_\beta, \quad n_\alpha.e'_\beta = \eta_{\alpha, \beta} e_{w_\alpha(\beta)}, \quad \alpha \in \Pi.
\]

We will also make use of the following relations in \( G \) (see for example \[C\] §5.2, §7.2), for \( \alpha, \beta \in R \),

\[
x_\alpha(s)^{-1}x_\beta(t)^{-1}x_\alpha(s)x_\beta(t) = \prod_{i,j > 0} x_{i\beta + j\alpha}(c_{ij\beta\alpha} (-t)^i s^j), \quad \alpha \neq \pm \beta
\]

(2.4)

\[
n^2_{\alpha} = h_\alpha(-1), \quad n_\alpha x_\beta(t)n_\alpha^{-1} = x_{w_\alpha(\beta)}(\eta_{\alpha, \beta} t), \quad n_\alpha n_\beta n_\alpha^{-1} = h_{w_\alpha(\beta)}(\eta_{\alpha, \beta}) n_{w_\alpha(\beta)}.
\]

Here \( A_{\alpha, \beta}, M_{\alpha, \beta, i}, c_{ij\beta\alpha} \) and \( \eta_{\alpha, \beta}(= \pm 1) \) are structural constants (for their determination see \[X3\] and \[C\] pages 77 and 94), in particular, they are integers.

### 2.2. Invariant non-degenerate bilinear forms on \( \mathfrak{g} \)

The following lemma was explained to the author by G. Lusztig. We include a proof here for completeness.

**Lemma 2.1.** Assume that \( G \) is of type \( G_2 \) (resp. \( F_4 \)). Then there exists a \( G \)-invariant non-degenerate bilinear form on \( \mathfrak{g} \) if and only if \( \text{char } k \neq 3 \) (resp. \( \text{char } k \neq 2 \)).

Thus in the above cases, we have a \( G \)-equivariant isomorphism \( \mathfrak{g} \cong \mathfrak{g}^* \). In turn the Springer correspondence maps \( \gamma_\beta \) and \( \gamma_{\beta^*} \) can be identified.

To prove Lemma 2.1 we first note that the Lie algebra \( \mathfrak{g} \) is simple in the above cases \[H\]. Assume that \( G \) is of type \( G_2 \) and \( \text{char } k \neq 3 \). Let \( \alpha \) be the short simple root and \( \beta \) the long simple root. Define a symmetric bilinear form \( (, ) \) on \( \mathfrak{g} \) as follows

\[
(h_\alpha, h_\alpha) = 6, \quad (h_\alpha, h_\beta) = -3, \quad (h_\beta, h_\beta) = 2
\]

(2.5)

\[
(e_\lambda, e_{\mu}) = 0 \text{ if } \lambda + \mu \neq 0, \quad (e_\lambda, e_{-\lambda}) = 3 \text{ (resp. 1)} \text{ if } \lambda \text{ is a short (resp. long) root}.
\]

Using \[K\] Theorem 2.2 one checks that (2.5) defines an invariant non-degenerate bilinear form on \( \mathfrak{g} \) over \( \mathbb{C} \), thus it is a scalar multiple of the Killing form, which is \( G \)-invariant. As all the structure constants are in \( \mathbb{Z} \), reducing mod \( \text{char } k \), we see that the above bilinear form remains non-degenerate and \( G \)-invariant.
Assume that $G$ is of type $F_4$ and $\text{char } k \neq 2$. Let $p, q$ be the long simple roots and $r, s$ the short simple roots such that $(q, r) \neq 0$. Define a symmetric bilinear form $(,)$ on $\mathfrak{g}$ as follows

$$
(h_p, h_p) = (h_q, h_q) = 2, \quad (h_p, h_q) = (h_q, h_r) = (h_p, h_s) = (h_q, h_s) = 0,
$$

$$(h_q, h_q) = 2, \quad (h_q, h_r) = (h_r, h_s) = -2, \quad (h_r, h_r) = (h_s, h_s) = 4,
$$

$$(e_\lambda, e_\mu) = 0 \text{ if } \lambda + \mu \neq 0, \quad (e_\lambda, e_{-\lambda}) = 2 \text{ (resp. 1) if } \lambda \text{ is a short (resp. long) root}.
$$

The same argument as in the case of $G_2$ shows that the bilinear form $(,)$ of (2.6) is non-degenerate and $G$-invariant.

Now assume that $\text{char } k = 3$ and $G$ is of type $G_2$, there are 6 nilpotent orbits in $\mathfrak{g}$ [St] and 5 nilpotent coadjoint orbits in $\mathfrak{g}^*$ [X4]. Similarly, when $\text{char } k = 2$ and $G$ is of type $F_4$, there are 22 nilpotent orbits in $\mathfrak{g}$ [St] and 18 nilpotent coadjoint orbits in $\mathfrak{g}^*$ [X4]. Thus in these cases there can not exist a $G$-invariant non-degenerate bilinear form on $\mathfrak{g}$. This completes the proof of Lemma 2.1.

2.3. Weyl group representations and truncated induction. Let $W$ be a Weyl group and let $\text{Irr } W$ denote the set of irreducible characters of $W$. For $\rho \in \text{Irr } W$,

$$
\text{let } b_\rho \text{ denote the minimal integer } d \text{ such that } \rho \text{ occurs in the } W\text{-module } \mathcal{S}_d(V) \text{ of all homogeneous polynomials of degree } d \text{ on the reflection space } V.
$$

For a subgroup $W' \subset W$, let $j'^W_W : \text{Irr } W' \to \text{Irr } W$ denote the truncated induction, i.e., for $\rho' \in \text{Irr } W'$, $j'^W_W \rho'$ is the unique $\rho \in \text{Irr } W$ such that $\rho \text{ occurs in } \text{Ind}_{W'}^W \rho'$, written as $\langle \rho, \text{Ind}_{W'}^W \rho' \rangle \neq 0$, and $b_\rho = b_{\rho'}$.

We label $\text{Irr } W$ using the notations of [A]. In particular, $\text{Irr } W(A_{n-1}) = \text{Irr } S_n$ is labelled by partitions of $n$, where $(n)$ denotes the trivial character and $(1^n)$ denotes the sign character, and $\text{Irr } W(B_n) = \text{Irr } W(C_n)$ is labelled by a pair of partition $[\lambda : \mu]$. For $W$ of type $G_2$ or $F_4$, $\chi_{i,j} \in \text{Irr } W$ has degree $i$. For the reader’s convenience, we list the values $b_\rho$ (see (2.7)) for $\rho \in \text{Irr } W(G_2)$ and $\rho \in \text{Irr } W(F_4)$ in the following tables (see for example [GP]).

| $\rho$ | $\chi_{1,1}$ | $\chi_{2,1}$ | $\chi_{2,2}$ | $\chi_{1,3}$ | $\chi_{1,4}$ | $\chi_{1,2}$ |
|--------|--------------|--------------|--------------|--------------|--------------|--------------|
| $b_\rho$ | 0 | 1 | 2 | 3 | 3 | 6 |

**Table 1. $b$-invariants for $W(G_2)$**

| $\rho$ | $\chi_{1,1}$ | $\chi_{4,2}$ | $\chi_{9,1}$ | $\chi_{8,1}$ | $\chi_{8,3}$ | $\chi_{2,1}$ | $\chi_{2,2}$ | $\chi_{12,1}$ | $\chi_{12,1}$ | $\chi_{16,1}$ | $\chi_{6,1}$ | $\chi_{6,2}$ | $\chi_{9,2}$ | $\chi_{9,3}$ |
|--------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| $b_\rho$ | 0 | 1 | 2 | 3 | 3 | 4 | 4 | 4 | 5 | 6 | 6 | 6 | 6 | 6 |

| $\rho$ | $\chi_{4,3}$ | $\chi_{4,4}$ | $\chi_{4,1}$ | $\chi_{8,2}$ | $\chi_{8,4}$ | $\chi_{9,4}$ | $\chi_{1,2}$ | $\chi_{1,3}$ | $\chi_{4,5}$ | $\chi_{2,2}$ | $\chi_{2,4}$ | $\chi_{1,1}$ |
|--------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| $b_\rho$ | 7 | 7 | 8 | 9 | 9 | 10 | 12 | 12 | 13 | 16 | 16 | 24 |

**Table 2. $b$-invariants for $W(F_4)$**
3. Nilpotent coadjoint orbits

In this section, we describe the component groups $A_G(\xi) := Z_G(\xi)/Z_G(\xi)^0$ for $\xi \in N_{g^*}$. Moreover, we compute the dimensions of Springer fibers. More precisely, let $B$ be a fixed Borel subgroup of $G$ with unipotent radical $U$. We write $b = \text{Lie } B$ and $n = \text{Lie } U$. Define

$$b^* = \{ \zeta \in g^* | \zeta(n) = 0 \}$$

and $B^G_\xi = \{ gB \in G/B | g^{-1}\xi \in b^* \}$, $\xi \in g^*$.

We show that

**Proposition 3.1.** For $\xi \in N_{g^*}$,

$$\dim B^G_\xi = \frac{\dim Z_G(\xi) - \text{rank } G}{2}. \quad (3.1)$$

The component groups $A_G(x)$ for $x \in N_0$ have been determined in all characteristics, as well as the dimensions of Springer fibers, see §5.1 for exposition. In view of Lemma 2.1, it suffices to consider the cases when $G$ is an exceptional group of type $G_2$ (resp. $F_4$) over an algebraically closed field $k$ of characteristic 3 (resp. 2), which we will assume in the remainder of this section.

The nilpotent coadjoint orbits in $g^*$ have been classified in [X4, §7]. We use the same notation and representatives for the orbits as in loc.cit. In particular, the orbits are labeled as in the Bala-Carter classification (while the two new orbits in bad characteristic are labeled as $(B_3)_2$ and $(A_2)_2$). The tilde in $\tilde{A}_1$ etc. indicates short roots.

The results are listed in Table 3 when $G$ is of type $G_2$ and char $k = 3$, and in Table 4 when $G$ is of type $F_4$ and char $k = 2$.

| Orbit       | representative $\xi$ | $\dim Z_G(\xi)$ | $A_G(\xi)$ | $\dim B^G_\xi$ |
|-------------|----------------------|-----------------|------------|-----------------|
| $G_2$       | $e'_\alpha + e'_\beta$ | 2               | 1          | 0               |
| $G_2(a_1)$  | $e'_\beta + e'_{2\alpha+\beta}$ | 4               | $S_3$      | 1               |
| $\tilde{A}_1$ | $e'_\alpha$          | 6               | 1          | 2               |
| $A_1$       | $e'_\beta$           | 8               | 1          | 3               |
| $\emptyset$ | 0                    | 14              | 1          | 6               |

**Table 3.** Type $G_2$, nilpotent coadjoint orbits, char $k = 3$
3.1. **Component groups of centralizers.** To describe the structure of the component groups $A_G(\xi)$, $\xi \in \mathcal{N}_g^*$, we make use of the following lemma.

**Lemma 3.2 ([3], 3.4, p.176]).** Let $\sigma : G \to G$ be the Frobenius endomorphism induced by $k \mapsto a^k : a \mapsto a^q$. Let $\mathcal{O} \subset \mathfrak{g}^*$ be a $G$-orbit. Then the set of $G^\sigma$-orbits in $\mathcal{O}^\sigma$ is in bijection with $H^1(\sigma, A_G(\xi))$, where $\xi \in \mathcal{O}$.

Note that for $q$ large enough, $\sigma$ acts trivially on $A_G(\xi)$. Thus $|H^1(\sigma, A_G(\xi))|$ equals the number of conjugacy classes in $A_G(\xi)$. It then follows from [4, §7, Table 1 and Table 2] that we only need to consider the representative $\xi = e'_{\beta} + e'_{2 \alpha + \beta}$ for the orbit $G_2(a_1)$ and the representative $\xi_5 = e'_{pqr} + e'_{qrs} + e'_{pq2r} + e'_{q2r2s}$ for the orbit $F_4(a_3)$. Here we have used the fact that $S_2 \cong \mathbb{Z}/2\mathbb{Z}$ is the only finite group with exactly 2 conjugacy classes.

Assume that $G$ is of type $G_2$ and $\text{char } k = 3$. We determine $A_G(\xi)$ for $\xi = e'_{\beta} + e'_{2 \alpha + \beta}$. Using (2.1) and (2.3), one can check that $Z_G(\xi) \subset B$ and

$$A_G(\xi) \cong \langle \gamma_1, \gamma_2 \rangle \cong S_3,$$

where $\gamma_1 = e'_{2p3q2r}$ and $\gamma_2 = e'_{2p3q2r}$. The entries in Table 4 indicate that $\dim B_{\xi}$ is small enough, $\dim Z_G(\xi) = 1$, and $\dim B_{\xi} = 4$ for $\mathcal{O} = \mathcal{O}^{\sigma}$. It then follows from [4, §7, Table 1 and Table 2] that $S_2 \cong \mathbb{Z}/2\mathbb{Z}$ is the only finite group with exactly 2 conjugacy classes.

**Table 4.** Type $F_4$, nilpotent coadjoint orbits, char $k = 2$

| Orbit | representative $\xi$ | dim $Z_G(\xi)$ | $A_G(\xi)$ | dim $B_{\xi}$ |
|-------|----------------------|--------------|------------|--------------|
| $F_4$ | $e'_{p} + e'_{q} + e'_{r} + e'_{s}$ | 4 | 1 | 0 |
| $F_4(a_1)$ | $e'_{p} + e'_{qr} + e'_{q2r} + e'_{s}$ | 6 | $S_2$ | 1 |
| $F_4(a_2)$ | $e'_{p} + e'_{qr} + e'_{rs} + e'_{q2r2s}$ | 8 | 1 | 2 |
| $B_3$ | $e'_{p} + e'_{qrs} + e'_{q2r} + e'_{pq2rs}$ | 10 | 1 | 3 |
| $C_3$ | $e'_{s} + e'_{q2r} + e'_{pqr}$ | 10 | 1 | 3 |
| $F_4(a_3)$ | $e'_{pqr} + e'_{qrs} + e'_{pq2r} + e'_{q2r2s}$ | 12 | $S_4$ | 4 |
| $(B_3)_{2}$ | $e'_{p} + e'_{qr} + e'_{q2r2s}$ | 12 | 1 | 4 |
| $C_3(a_1)$ | $e'_{pq} + e'_{q2rs} + e'_{q2r2s}$ | 14 | $S_2$ | 5 |
| $B_2$ | $e'_{pq} + e'_{q2r2s}$ | 16 | $S_2$ | 6 |
| $A_2 + A_1$ | $e'_{pqr} + e'_{q2r2s}$ | 16 | 1 | 6 |
| $A_2 + A_1$ | $e'_{q2r2s} + e'_{pq2rs}$ | 18 | 1 | 7 |
| $A_2$ | $e'_{pqr} + e'_{q2r2s}$ | 22 | 1 | 9 |
| $A_2$ | $e'_{pq2r} + e'_{q2r2s} + e'_{pq2r2s}$ | 22 | 1 | 9 |
| $A_1 + A_1$ | $e'_{2q3r2s}$ | 24 | 1 | 10 |
| $(A_2)_{2}$ | $e'_{pq2r} + e'_{q2r2s}$ | 28 | 1 | 12 |
| $A_1$ | $e'_{2q3r2s}$ | 30 | $S_2$ | 13 |
| $A_1$ | $e'_{2p3q2r}$ | 36 | 1 | 16 |
| 0 | 0 | 52 | 1 | 24 |
where $\gamma_1 = h_\beta(-1)$ and $\gamma_2 = h_\beta(-1)x_\alpha(\eta)$ ($\eta^2 = -1$). In particular, we have used the following formula

$$u(t)\xi = \xi + (t_1^2 + t_1)e_{3a+\beta} + (t_5 + t_3)e_{3a+2\beta}.$$  

where $u(t) = x_\alpha(t_1)x_\beta(t_2)x_{a+\beta}(t_3)x_{2a+\beta}(t_4)x_{3a+\beta}(t_5)x_{3a+2\beta}(t_6)$, $t_i \in \mathbb{G}_a$.

Assume that $G$ is of type $F_4$ and char $k = 2$. We determine $A_G(\xi_5)$ for $\xi_5 = e'_{pq} + e'_{qs} + e'_{q2r} + e'_{q2r_2}$. Let $P$ be the standard parabolic group $P_J$, where $J = \{p, r, s\}$. Then using [CP, Theorem 7.3] and the description of nilpotent pieces in $g^*$ in [X4 §7.6] we see that $Z_G(\xi_5) \subset P$. Using (2.1) and (2.3), one checks that

$$Z^0_G(\xi_5) = Z_{U_P}(\xi_5), \quad A_G(\xi_5) \cong \langle \gamma_1, \gamma_2, \gamma_3 \rangle \cong S_4,$$

where

$$\gamma_1 = x_p(1)x_s(1), \quad \gamma_2 = n_p n_s, \quad \gamma_3 = x_p(1)x_s(1)x_r(1)n_r x_r(1)$$

with

$$\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = 1, \quad (\gamma_1 \gamma_2)^3 = (\gamma_2 \gamma_3)^3 = 1, \quad (\gamma_1 \gamma_3)^2 = 1.$$  

In particular we have used the following formula (here the assumption char $k = 2$ is used)

$$u(t)\xi_5 = e'_{pq} + e'_{qs} + (1 + t_2)e_{pq2r} + t_2 e_{q2r_2} + (t_1 + t_3)e'_{pqrs} + (1 + t_4)e'_{q2r_2}$$

$$+(t_1t_2 + t_2t_3 + t_4)e'_{pq2r_2} + (t_1 + t_3 - t_2^2 + t_4(t_1 + t_3))e'_{pq2r_2},$$

where $u(t) = x_p(t_1)x_r(t_2)x_s(t_3)x_r(t_4)$, $t_i \in \mathbb{G}_a$.

**Remark 3.3.** For the element $\xi_5$ above, we have another interesting way to deduce that $A_G(\xi_5) \cong S_4$ as follows. Taking $q$ big enough, we can choose a representative $\xi$ of the orbit of $\xi_5$ such that the Frobenius morphism $\sigma$ fixes $\xi$ and acts trivially on $A_G(\xi)$. This implies that $|Z_G(\xi)\sigma| = |A_G(\xi)||Z_G(\xi)^0\sigma|$. In view of [X4 Table 2], we conclude that $A_G(\xi)$ is a finite group of order 24 with exactly 5 conjugacy classes. It turns out that the finite groups with exactly 5 conjugacy classes have been classified by Miller [M] in 1919; the list is the following: the alternating group $A_5$, the non-cyclic group of order 21, the metacyclic group of order 20, the dihedral group of order 14, the dihedral group of order 8, the quaternion group, and $Z/5Z$. We can then easily identify our group from the list.

3.2. Springer fibers. Let $L$ denote a Levi subgroup of a proper parabolic subgroup $P$ of $G$. We write $l = \text{Lie } L$, $n_P = \text{Lie } U_P$, where $U_P$ is the unipotent radical of $P$. Let

$$l^* = \{ \xi \in g^* | \xi(n_P \oplus n_P^*) = 0 \} \quad \text{and} \quad n^*_P = \{ \xi \in g^* | \xi(l \oplus n_P) = 0 \},$$

where $g = l \oplus n_P \oplus n_P^*$. To prove Proposition 3.1 we first note that (see [X4 Proposition 3.1])

$$\text{the equality (3.1) holds for any } \xi' \in N_{l^*} (\text{replacing } G \text{ by } L).$$

Let $\xi \in N_{g^*}$. We prove Proposition 3.1 using the following statements.

Assume that $\xi$ lies in $l^*$ and $\dim Z_L(\xi) = 2 \dim B^L_\xi + \text{rank } L$.

Then $\dim Z_G(\xi) = 2 \dim B^G_\xi + \text{rank } G$. 

This is proved using the same argument as in \cite[II 3.14]{S1}. In particular, one uses that the equality in (3.1) holds if there exists \( w \in W \) such that \( \mathcal{O}_\xi \cap n^* \cap n_w^* \) is dense in \( n^* \cap n_w^* \), where \( n_w^* = \{ \zeta \in g^* \mid \zeta(ww^{-1}) = 0 \} \).

If \( \xi \) lies in the orbit obtained by inducing the orbit of \( \xi' \in \mathfrak{l}^* \) (see \cite[4.2]{X1}) and (3.6)
\[
\dim Z_L(\xi') = 2 \dim B^L_\xi + \text{rank } L, \quad \dim Z_G(\xi) = 2 \dim B^G_\xi + \text{rank } G
\]
This is proved using (3.3) and the same argument as in \cite[Theorem 1.3]{LS1}. In particular, the proof shows that if \( \mathcal{O}_\xi = \text{Ind}_{\xi'}^\xi \mathcal{O}_{\xi'}^L \), then
\[
\dim \mathcal{O}_\xi = \dim \mathcal{O}_{\xi'}^L + 2 \dim U_P.
\]

For \( G_2 \), one readily checks that Proposition 3.1 holds using the above (3.5) and (3.6). In fact, the orbit \( G_2 \) is induced from the zero orbit in a maximal torus, the orbit \( G_2(a_1) \) is induced from the zero orbit in a Levi subgroup of type \( A_1 \), and the remaining orbits satisfy the assumptions in (3.5).

For \( F_4 \), we make use of the following table for induced orbits. The first row contains orbits \( \tilde{\mathcal{O}} \) in \( N_{\mathfrak{g}^*} \), the second row indicates the type of the Levi subgroup \( L \), and the third row contains orbits \( \mathcal{O} \) in \( N_{\mathfrak{r}} \) such that \( \tilde{\mathcal{O}} = \text{Ind}_{\xi'}^\xi \mathcal{O} \), where 0 denotes the zero orbit and we use the notation in \cite{X2} for orbits in a Levi of type \( B_3 \) or \( C_3 \). The last row of the table is included for later use and will be explained there. We remark that we did not attempt to identify all induced orbits here, rather only the cases that are needed for our purpose.

| \( \tilde{\mathcal{O}} \) | \( A_2 \) | \( A_2 \) | \( B_2 \) | \( F_4(a_3) \) | \( F_4(a_3) \) | \( B_3 \) | \( F_4(a_1) \) | \( F_4(a_2) \) |
|---|---|---|---|---|---|---|---|---|
| type of \( L \) | \( B_3 \) | \( C_3 \) | \( C_3 \) | \( A_2A_1 \) | \( B_3 \) | \( B_3 \) | \( B_3 \) | \( A_1A_1 \) |
| \( \mathcal{O} \) | 0 | 0 | 1^6 | 0 | (1; 1^4) | (0; 3^2) | (2; 1^2) | 0 |
| \( \rho_{\xi',1}^L \) | \([- : 1^3] \) | \([- : 1^3] \) | \([1^3 : -] \) | \([1^3 \times 1^2] \) | \([1 : 1^2] \) | \([- : 3] \) | \([2 : 1] \) | \([2 \times 2] \) |

**Table 5. Induced nilpotent coadjoint orbits, type \( F_4 \), \text{char } \mathbb{k} = 2**

As an example, we explain the case of the orbit \( F_4(a_3) \) being induced from the orbit \((1; 1^4)\) in \( l^* \) for \( L \) of type \( B_3 \); the other cases can be checked similarly. Let \( \xi = e'_{pqr} + e'_{qrs} + e'_{pq2r} + e_{q2r2s} \), which is in the orbit \( F_4(a_3) \), and let \( \xi' = e'_{pqr} + e'_{pq2r} \). Let \( P \) be the standard parabolic subgroup \( P_J, J = \{ p, q, r \} \) and \( L \) the standard Levi subgroup of \( P \). Then \( L \) is of type \( B_3 \). Moreover, \( \xi' \in \mathfrak{l}^* \) and \( \xi \in \xi' + n^*_p \). Using \cite{X2}, one checks that \( \xi' \) lies in the orbit \((1; 1^4)\) of \( \mathfrak{l}^* \), moreover, \( \dim \mathcal{O}_{\xi'}^L = 10 \). Note that \( \dim \mathcal{O}_\xi = 40 = \dim \mathcal{O}_{\xi'} + 2 \dim U_P = \dim \text{Ind}_{\xi'}^\xi \mathcal{O}_{\xi'}^L \) (in the last equality we use (3.7)). Since \( \text{Ind}_{\xi'}^\xi \mathcal{O}_{\xi'}^L \) is the unique dense orbit in \( G.(\mathcal{O}_{\xi'} + n^*_p) \), we obtain that
\[
\text{Ind}_{\xi'}^\xi \mathcal{O}_{\xi'}^L, \xi = e'_{pqr} + e'_{qrs} + e'_{pq2r} + e_{q2r2s}, \quad \xi' = e'_{pqr} + e'_{pq2r}.
\]

\footnote{I.e. \( \mathcal{O}_\xi \cap (\mathcal{O}_{\xi'}^L + n^*_p) \) is dense in \( \mathcal{O}_{\xi'}^L + n^*_p \), where \( \mathcal{O}_{\xi'}^L \) denotes the \( L \)-orbit of \( \xi' \); we write \( \mathcal{O}_\xi = \text{Ind}_{\xi'}^\xi \mathcal{O}_{\xi'}^L \).}
Now one checks readily that all $\xi \in \mathcal{N}_{g^*}$ satisfy either the assumptions in the above statement (3.5) or those of (3.6). This completes the proof of Proposition 3.1.

4. The Springer correspondence for $g^*$

Let $G$ be a connected simply connected algebraic group defined over an algebraically closed field $k$ of prime characteristic. Fix a prime $l \neq \text{char } k$. Let

$$\gamma_{g^*} : \text{Irr } W \rightarrow \mathcal{A}_{g^*}$$

be the injective Springer correspondence map constructed for $g^*$ as in [X2, §5], where $\text{Irr } W$ and $\mathcal{A}_{g^*}$ are the sets defined in the introduction. Namely, $\text{Irr } W$ denotes the set of irreducible characters of the Weyl group $W$ of $G$ and $\mathcal{A}_{g^*}$ is the set of all pairs $(\mathcal{O}, \mathcal{E})$ where $\mathcal{O} \subset \mathcal{N}_{g^*}$ is a $G$-orbit and $\mathcal{E}$ is an irreducible $G$-equivariant $\mathbb{Q}_l$-local system on $\mathcal{O}$ (up to isomorphism). The constructions and proofs in [X2, §5] apply for any simply connected $G$. In what follows we briefly recall the construction. Consider the following proper maps

$$\varphi_0 : G \times B^* \rightarrow \mathcal{N}_{g^*}, \quad \varphi : G \times B^* \rightarrow g^*.$$

The map $\varphi_0$ is semismall and the map $\varphi$ is small. Moreover,

$$\varphi_0, \mathcal{Q}_l[-] \cong \bigoplus_{(\mathcal{O}, \mathcal{E}) \in \mathcal{A}_{g^*}} \text{IC}(\mathcal{O}, \mathcal{E}) \otimes V_{\mathcal{O}, \mathcal{E}} \cong \bigoplus_{\rho \in \text{Irr } W} \rho \otimes (\varphi_0, \mathcal{Q}_l)_{\rho|\mathcal{N}_{g^*}[-]}$$

where $[-]$ denotes shift by $\text{dim } \mathcal{N}_{g^*}$, $\text{IC}(\mathcal{O}, \mathcal{E})$ is the perverse IC-extension of the local system $\mathcal{E}$ on $\mathcal{O}$, and $(\varphi_0, \mathcal{Q}_l)_\rho = \text{Hom}_{\mathcal{Q}_l[W]}(\rho, \mathcal{Q}_l)$. The map $\gamma_{g^*}$ maps $\rho \in \text{Irr } W$ to the unique pair $(\mathcal{O}, \mathcal{E}) \in \mathcal{A}_{g^*}$ such that $\text{IC}(\mathcal{O}, \mathcal{E}) \cong (\varphi_0, \mathcal{Q}_l)_{\rho|\mathcal{N}_{g^*}[-]}$. Note that this implies that $\rho \cong V_{\mathcal{O}, \mathcal{E}}$ under $W$-action.

In this section we describe the map $\gamma_{g^*}$ of (4.1) explicitly assuming that $G$ is of type $G_2$ (resp. $F_4$) and $\text{char } k = 3$ (resp. $\text{char } k = 2$). The results are given in Table 6 (resp. Table 7). For a pair $(\mathcal{O}, \mathcal{E}) \in \mathcal{A}_{g^*}$, we write $\rho_{\xi, \phi}$ (or simply $\rho_{\xi}$) for the inverse image $\gamma_{g^*}^{-1}(\mathcal{O}, \mathcal{E})$, where $\xi \in \mathcal{O}$ and $\phi \in \text{Irr } A_G(\xi)$. We correspond to the local system $\mathcal{E}$.

| Orbit of $\xi$ | $\phi$ | $\rho_{\xi, \phi}$ | Orbit of $\xi$ | $\phi$ | $\rho_{\xi, \phi}$ |
|----------------|-------|-----------------|----------------|-------|-----------------|
| $G_2$          | (1)   | $\chi_{1,1}$    | $\tilde{A}_1$  | (1)   | $\chi_{2,2}$    |
| $G_2(a_1)$     | (3)   | $\chi_{2,1}$    | $A_1$          | (1)   | $\chi_{1,4}$    |
| $G_2(a_1)$     | (2, 1)| $\chi_{1,3}$    | $\emptyset$    | (1)   | $\chi_{1,2}$    |
| $G_2(a_1)$     | (3)   | $-$             |                |       |                 |

Table 6. Springer correspondence for $g^*$, type $G_2$ and $\text{char } k = 3$
We remark that in view of the results in \textbf{[X3]}, Tables \textbf{[6]} and \textbf{[7]}, the pair \((G_2(a_1), \mathcal{E}_{(13)})\) for \(G_2\) and the pair \((F_4(a_3), \mathcal{E}_{(13)})\) for \(F_4\) are the cuspidal pairs in the sense of \textbf{[L1]}, where \(\mathcal{E}_{\phi}\) denotes the local system corresponding to \(\phi \in \text{Irr} A_G(\xi)\).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Orbit of \(\xi\) & \(\phi\) & \(\rho_{\xi,\phi}\) & Orbit of \(\xi\) & \(\phi\) & \(\rho_{\xi,\phi}\) \\
\hline
\(F_4\) & (1) & \(\chi_{1,1}\) & \(F_4(a_3)\) & (2, 1, 1) & \(\chi_{1,3}\) \\
\(F_4(a_1)\) & (2) & \(\chi_{4,2}\) & \(F_4(a_3)\) & (1) & \(-\) \\
\(F_4(a_1)\) & (1) & \(\chi_{2,3}\) & \((B_3)_{2}\) & (1) & \(\chi_{2,1}\) \\
\(F_4(a_2)\) & (1) & \(\chi_{9,1}\) & \(C_3(a_1)\) & (2) & \(\chi_{16,1}\) \\
\(B_3\) & (1) & \(\chi_{8,1}\) & \(C_3(a_1)\) & (1) & \(\chi_{4,4}\) \\
\(C_3\) & (1) & \(\chi_{8,3}\) & \(B_2\) & (2) & \(\chi_{9,2}\) \\
\(F_4(a_3)\) & (4) & \(\chi_{12,1}\) & \(B_2\) & (1) & \(\chi_{4,1}\) \\
\(F_4(a_3)\) & (3, 1) & \(\chi_{9,3}\) & \(\tilde{A}_2 + A_1\) & (1) & \(\chi_{6,1}\) \\
\(F_4(a_3)\) & (2, 2) & \(\chi_{6,2}\) & \(A_2 + \tilde{A}_1\) & (1) & \(\chi_{4,3}\) \\
\hline
\end{tabular}
\caption{Springer correspondence for \(g^*\), type \(F_4\) and \(\text{char } k = 2\)}
\end{table}

4.1. \textbf{The methods.} We describe the map \(\gamma_{g^*}\) of \textbf{(4.1)} following the methods in \textbf{[AL S2 S4]}. In particular, we use extensively the tables of induce/restrict matrix for Weyl group representations given by Alvis in \textbf{[A]}. More specifically, we make use of the following statements, for which the proofs follow the same arguments as in \textbf{[AL S2 S4]} and the references there. Thus we only briefly comment on the proofs. Throughout this subsection \(\xi \in \mathcal{N}_{g^*}\), \(L\) is a Levi subgroup of a proper parabolic subgroup \(P\) of \(G\), \(\mathfrak{l} = \text{Lie } L\), \(U_P\) is the unipotent radical of \(P\), and \(W_L\) is the Weyl group of \(L\), regarded naturally as a subgroup of \(W\).

To begin with, let \(b_{\rho}\) be defined as in \textbf{(2.7)} for \(\rho \in \text{Irr } W\). Taking stalks at 0 in \textbf{(4.2)} and using that \(H^{2i}(\mathcal{B}, \mathbb{Q}_l) \cong \mathcal{G}_i(V)\), we see that (here we use Proposition \textbf{3.1})

\begin{equation}
\tag{4.3}
b_{\rho_{\xi,\phi}} = \dim B_{\xi}^G.
\end{equation}

Next let \(\xi' \in \mathcal{N}_{g^*}\). Consider the permutation representation \(\varepsilon_{\xi,\xi'}\) of \(A_G(\xi) \times A_L(\xi')\) afforded by the finite set \(S_{\xi,\xi'}\) of irreducible components of dimension \(d_{\xi,\xi'}\) of \(Y_{\xi,\xi'} := \{ g \in G \mid g^{-1} \xi \in \xi' + n_{P}\}\). The same proof as in \textbf{[L1]} (see also \textbf{[X3]}) shows that for \(\phi \in \text{Irr } A_G(\xi)\) and \(\phi' \in \text{Irr } A_L(\xi')\), we have

\begin{equation}
\tag{4.4}
\langle \phi \otimes \phi', \varepsilon_{\xi,\xi'} \rangle = \langle \text{Res}^W_{W_L} \rho_{\xi,\phi,\xi'}^G, \rho_{\xi',\phi'}^L \rangle_{W_L} := n_{\xi,\xi',\phi,\phi'}.
\end{equation}

In particular, we make use of the following special cases of \textbf{(4.4)}.

\begin{equation}
\tag{4.5}
\text{If } n_{\xi,\xi',1,1} = 0, \text{ then } S_{\xi,\xi'} = \emptyset \text{ and all } n_{\xi,\xi',\phi,\phi'} = 0.
\end{equation}

\footnote{Here \(d_{\xi,\xi'} = (\dim Z_G(\xi) + \dim Z_L(\xi'))/2 + \dim U_P\).}
If the orbit of $\xi$ is induced from that of $\xi'$ in $\mathfrak{l}^*$, then we can assume that $\xi \in \xi' + n_{\mathfrak{l}'}$. Using the same argument as in [LS1, 1.3, 1.5] and [S4], we see that in this case 
(4.6) $S_{\xi, \xi'}$ is isomorphic to $A_G(\xi)/N$ as sets with $A_G(\xi) \times A_L(\xi')$-actions, where

(4.7) $N = Z_{Z^0(\xi)\mathfrak{l}'}(\xi)/Z^0(\xi)$ is a normal subgroup of $H = Z_{\mathfrak{l}'}(\xi)/Z^0(\xi)$, $H/N \cong A_L(\xi')$, and the group $A_G(\xi) \times H/N \cong A_G(\xi) \times A_L(\xi')$ acts on $A_G(\xi)/N$ by $(a, hN)(xN) = axh^{-1}N$. Thus it follows from (4.4) and (4.6) that for $\phi \in \text{Irr } A_G(\xi)$ and $\phi' \in \text{Irr } A_L(\xi')$,

(4.8) $n_{\xi, \xi', \phi, \phi'} = \langle \tilde{\phi}', \text{Res}_{H}^{A_G(\xi)}(\phi) \rangle$, where $\tilde{\phi}'$ is the lift of $\phi'$ to $H$.

Moreover, the same argument as in [LS1, 1.3] shows that $\dim B_{\xi}^G = \dim B_{\xi'}^L$. It then follows from (4.3), the definition of the truncated induction operator $j$ (see §2.3) and the above discussion that

(4.9) If the orbit of $\xi$ is induced from that of $\xi' \in \mathcal{N}_{\mathfrak{t}^*}$, then $\rho_{\xi, 1}^G = j_{W_{\mathfrak{l}'}(\rho_{\xi', 1}^L)}$.

Finally, the same argument as in [AL] shows that

(4.10) If $\xi \in \mathfrak{l}^*$, then $\langle \rho_{\xi, 1}^G, \text{Ind}_{W_{\mathfrak{l}'}(\rho_{\xi', 1}^L)} \rangle \neq 0$, where $\rho_{\xi, 1}^L = \sum (-1)^i H^i(B_{\xi}^L, \mathfrak{q}_l)$.

In addition, we remark that the decomposition (4.2) implies that we have a decomposition

$$H^{2d_\xi}(B_{\xi}^G, \mathfrak{q}_l) \cong \bigoplus_{\phi \in \text{Irr } A_G(\xi)} \phi \otimes \rho_{\xi, \phi}^G$$

as $A_G(\xi) \times W$ representations, where $d_\xi = \dim B_{\xi}^G$ and $A_G(\xi)$ acts via the permutation representation $\epsilon_\xi$ afforded by the set of irreducible components of $B_{\xi}^G$. In particular, we see that

(4.11) $\rho_{\xi, \phi}^G \neq 0$ if and only if $\phi$ occurs in the above permutation representation $\epsilon_\xi$.

4.2. Type $G_2$ in characteristic 3. We verify in this subsection the description of the map $\gamma_{\mathfrak{g}^*}$ given in Table 6. First, using (4.3), Table 1 and Table 3 one determines easily the images of $\chi_{1, 1}$, $\chi_{1, 2}$, $\chi_{2, 1}$ and $\chi_{2, 2}$ under $\gamma_{\mathfrak{g}^*}$.

For $\xi_4 = e'_\beta$ in the orbit $A_1$, note that $\xi_4$ is a regular nilpotent element in $\mathfrak{l}^*$, where $L$ is a Levi subgroup of type $A_1$. Using (4.11), we see that $\langle \rho_{\xi_4, 1}, \text{Ind}_{W(A_1)}^{W(e_1)}[2] \rangle \neq 0$. Thus it follows from [A, Table 64] that $\rho_{\xi_4, 1} = \chi_{1, 4}$.

It remains to determine $\gamma_{\mathfrak{g}^*}(\chi_{1, 3})$. To this end, it is enough to determine the pair $(\xi_2, \phi)$, $\xi_2 = e'_\alpha + e'_{2\alpha + \beta}$ in the orbit $G_2(a_1)$ and $\phi \neq 1$, that appears in the image of $\gamma_{\mathfrak{g}^*}$. In view of (4.11) we study the permutation representation $\epsilon_{\xi_2}$ of $A_G(\xi_2) \cong S_3$ afforded by the set of irreducible components of $B_{\xi_2}$. Using the Bruhat decomposition (2.1) and (2.3), one can check that $g^{-1}\xi_2 \in \mathfrak{b}^*$ if and only if $g^{-1} \in B, g^{-1} \in BS_{a}x_{a}(t_1)$, or $g^{-1} \in BS_{\beta}S_{a}x_{a}(\varpi)x_{3\alpha + \beta}(t_5)$, where $\varpi^3 + \varpi = 0$. Thus $B_{\xi_2}$ has 4 irreducible components. Using the description of
$A_G(\xi_2)$ in (3.2), one readily checks that the permutation representation $\epsilon_{\xi_2}$ decomposes as $(3) \oplus (3) \oplus (2,1)$. Thus $\gamma_{0^*}(\chi_{1,3}) = (\xi_2, (2,1))$. This completes the verification of Table 4

4.3. Type $F_4$ in characteristic 2. We verify in this subsection the description of the map $\gamma_{0^*}$ given in Table 7. First, using (1.3), Table 2, and Table 4, one determines easily the image of $\chi_{1,1}, \chi_{4,2}, \chi_{9,1}, \chi_{16,1}, \chi_{9,4}, \chi_{4,5}$, and $\chi_{1,4}$, under $\gamma_{0^*}$.

Using (1.3), (4.9), [A, Tables 60-61], and Table 5 (where the last row indicates $\rho_{\xi,1}$ for $\xi' \in \mathcal{O}$), we see that $\gamma_{0^*}(\chi_{8,2}) = (\tilde{A}_2, \tilde{Q}_l)$, $\gamma_{0^*}(\chi_{8,4}) = (A_2, \tilde{Q}_l)$, $\gamma_{0^*}(\chi_{9,2}) = (B_2, \tilde{Q}_l)$, $\gamma_{0^*}(\chi_{12,1}) = (F_4(a_3), \tilde{Q}_l)$ and $\gamma_{0^*}(\chi_{8,1}) = (B_3, \tilde{Q}_l)$.

Note that the representative $\xi$ in Table 4 for the orbit $A_2 + \tilde{A}_1$ (resp. $(B_3)_2, C_3, (A_2)_2, A_1$) is a regular nilpotent element in $\mathfrak{t}^*$ for a Levi $L$ of type $A_2 A_1$ (resp. $B_3, C_3, A_2, A_1$). Thus using (1.3), (4.10) and again [A, Tables 60-61], we see that $\gamma_{0^*}^{-1}(\mathcal{O}_{\xi}, \tilde{Q}_l) = \chi_{4,3}$ (resp. $\chi_{2,1}, \chi_{8,3}, \chi_{1,2}, \chi_{2,4}$). For the orbit $(A_2)_2$, we have used that $\text{Ind}_{W(W_{A_2})}^W((3 : -) \oplus [2 : 1] \oplus [1 : 2] \oplus [- : 3])$, and for the orbit $(A_1)$, we have used that $\text{Ind}_{W(W_{A_1})}^W(2) = \text{Ind}_{W(W_{B_3})}^W([3 : -] \oplus [2 : 1] \oplus [1 : 2] \oplus [- : 3] \oplus [21 : -] \oplus [2 : 1] \oplus [1 : 2] \oplus [- : 21] \oplus [1^2 : 1] \oplus [1 : 1^2])$.

For the orbit $\tilde{A}_2 + A_1$, again the representative $\xi$ is a regular nilpotent element in $\mathfrak{t}^*$ for a Levi $L$ of type $\tilde{A}_2 A_1$. Thus $\gamma_{0^*}^{-1}(\mathcal{O}_{\xi}, \tilde{Q}_l) \in \{\chi_{6,1}, \chi_{9,3}\}$. Note that $\langle [21 : -], \text{Res}_{W(W_{B_3})}^W(\chi_{9,3}) \rangle \neq 0$ while $\langle [2 : 1], \text{Res}_{W(W_{B_3})}^W(\chi_{9,3}) \rangle = 0$. This contradicts with (4.5) as $[2 : 1] = (\gamma_{0^*}^{-1})(\mathcal{O}_{\xi}, \tilde{Q}_l)$ and $[21 : -] = (\gamma_{0^*}^{-1})(\mathcal{O}_{\xi}, \tilde{Q}_l)$.

Let $\xi$ be in the orbit $F_4(a_3)$. Taking $\xi'$ varying in $\mathfrak{t}^*$ for a Levi subgroup $L$ of type $B_3$ and using [A, Table 60], we see that $n_{\xi, \xi', 1, 1} \neq 0$ if and only if $\xi'$ satisfies $\rho_{\xi,1}^L = [3 : -]$ or $\rho_{\xi,1}^L = [2 : 1]$. In the former case $A_L(\xi') = 1$ and in the latter case we have $A_L(\xi') = S_2$ and $\rho_{\xi,1}^L = [21 : -]$ (see [X3]). By (4.9), $(\text{Res}_{W(W_{B_3})}^W(\rho_{\xi,1}^L) \rho_{\xi,1}^L)^W(W_{B_3}) = 0$ for any $\rho' \in \text{Irr}(W(B_3)) - \{[3 : -], [2 : 1], [21 : -]\}$. Thus $\rho_{\xi,1}^L = \chi_{2,3}$. The same argument shows that for $\xi$ in the orbit $B_2$, $(\text{Res}_{W(W_{B_3})}^W(\rho_{\xi,1}^L) \rho_{\xi,1}^L)^W(W_{B_3}) = 0$ for any $\rho' \in \text{Irr}(W(B_3)) - \{[2 : 1], [21 : -], [1 : 2], [- : 3], [- : 21]\}$, thus $\rho_{\xi,1}^L = \chi_{4,1}$.

Consider now $\xi_6$ in the orbit $F_4(a_3)$ and $1 \neq \phi \in \text{Irr}(A_G(\xi_6))$. The same argument as above shows that $\rho_{\xi_6,^\phi} \neq \chi_{2,2}$. Similarly, for $\xi_8$ in the orbit $C_3(a_1)$, $\rho_{\xi_8,^\phi} \neq \chi_{2,2}$. This forces that $\chi_{2,2} = \gamma_{0^*}(\tilde{A}_1, \mathcal{E}_{12})$.

It remains to determine the images of $\chi_{9,3}, \chi_{6,2}, \chi_{4,4}$ and $\chi_{1,3}$ under $\gamma_{0^*}$. We make use of (4.8). Let $\xi = e'_{pq} + e'_{qr} + e'_{pq_{2r}} + e'_{q_{2r}2s}$ in the orbit $F_4(a_3)$ and $\xi' = e'_{pq} + e'_{pq_{2r}}$. Let $P$ be the standard parabolic subgroup $P_J, J = \{p, q, r\}$ and $L$ the standard Levi subgroup of $P$. Then $L$ is of type $B_3$. Moreover, $\xi' \in \mathfrak{t}^*$ and $\xi \in \xi' + n_P$. It has been shown that the orbit of $\xi$ is induced from that of $\xi'$ (see (3.8)). Using the description of the component group $A_G(\xi)$ in (3.3), one can check that the groups defined in (4.7) are as follows

$H = Z_P(\xi)/Z_G^0(\xi) \cong \langle \gamma_1, \gamma_3 \rangle \cong S_2 \times S_2$ and $N = Z_{Z_G^0(\xi)}(J_P)(\xi)/Z_G^0(\xi) \cong \langle \gamma_1 \rangle \cong S_2$. 


One checks that
\[(4.12) \quad \text{Ind}_{H}^{A_{G}(\xi)} 1 = \text{Ind}_{S_{2} \times S_{2}}^{S_{4}} 1 = (4) \oplus (3, 1) \oplus (2, 2), \quad \text{Ind}_{H}^{A_{G}(\xi)} \tilde{\phi}' = (3, 1) \oplus (2, 1, 1),\]
where 1 is the trivial representation and \(\tilde{\phi}'\) is the lift of the sign character \(\phi'\) of \(H/N \cong S_{2}\) to \(H \cong S_{2} \times S_{2}\). It is shown in [X3] that
\[
\rho_{\xi,1}^{L} = [1 : 1^2] \quad \text{and} \quad \rho_{\xi,\phi'}^{L} = [1^3 : -].
\]
Let \(\phi \in \text{Irr} A_{G}(\xi)\). Using (4.4), (4.8) and (4.12), we conclude that
\[
\langle \rho_{\xi,\phi}^{G}, \text{Ind}_{W(B_{0})}^{W}[1; 1^2]\rangle_{W} = \langle \rho_{\xi,\phi}^{G}, \chi_{12,1} + \chi_{6,2} + \chi_{9,3}\rangle_{W} = \langle \phi, (4) \oplus (3, 1) \oplus (2, 2)\rangle_{s_{4}}
\]
\[
\langle \rho_{\xi,\phi}^{G}, \text{Ind}_{W(B_{0})}^{W}[1^3; 0]\rangle_{W} = \langle \rho_{\xi,\phi}^{G}, \chi_{4,4} + \chi_{9,3} + \chi_{1,3}\rangle_{W} = \langle \phi, (3, 1) \oplus (2, 1, 1)\rangle_{s_{4}}
\]
where in the first identity we have used [A] Table 60. It thus follows that
\[(4.13) \quad \rho_{\xi,(3,1)} = \chi_{9,3}, \quad \rho_{\xi,(2,2)} = \chi_{6,2}, \quad \rho_{\xi,(2,1,1)} \in \{\chi_{4,4}, \chi_{1,3}\}.
\]
Now let \(P\) be the standard parabolic subgroup \(P_{J}\), where \(J = \{p, q, s\}\), and let \(L\) be the standard Levi subgroup of \(P\), which is of type \(A_{2}A_{1}\). Then \(\xi \in \mathfrak{n}_{P}^{*}\) and \(\xi\) is induced from the zero orbit in \(P^{*}\). Again using [3,3] we see that in this case
\[
H = Z_{P}(\xi)/Z_{G}(\xi) \cong \langle \gamma_{1}, \gamma_{2}\rangle = N \cong S_{3}.
\]
One checks that
\[(4.14) \quad \text{Ind}_{H}^{A_{G}(\xi)} 1 = \text{Ind}_{S_{3}}^{S_{4}} 1 = (4) \oplus (3, 1).\]
The same argument as above shows that
\[
\langle \rho_{\xi,\phi}^{G}, \chi_{4,4} + \chi_{12,1} + \chi_{9,3}\rangle_{W} = \langle \phi, (4) \oplus (3, 1)\rangle_{s_{4}}.
\]
In view of (4.13), it follows from the above equation that
\[
\rho_{\xi,(2,1,1)} = \chi_{1,3} \quad \text{and} \quad \chi_{4,4} \neq \rho_{\xi,(1^4)}.
\]
The last inequality forces that \(\chi_{4,4} = \rho_{\xi_{8},(1^2)}\), where \(\xi_{8}\) is in the orbit \(C_{3}(a_{1})\). This completes the verification of Table [X].

4.4. A set of Weyl group representations. In this subsection we define a subset of \(\text{Irr}(W)\) following [L3] which gives an a priori definition of the set \(\{\gamma_{p}^{-1}(O, Q_{l}) \mid (O, Q_{l}) \in \mathcal{A}_{p}^{*}\}\).

Let \(R^{\vee}\) be the set of coroots. For \(\alpha \in R\), let \(\alpha^{\vee}\) denote the corresponding coroot. Define
\[
\tilde{\Theta} = \{\beta \in R \mid \beta^{\vee} - \alpha^{\vee} \notin R^{\vee}, \forall \alpha \in \Pi\};
\]
\[
\tilde{\Theta}_{r} = \{J \subset \tilde{\Theta} \mid J \text{ linearly independent, } |\sum_{\alpha \in \Pi} \mathbb{Z}\alpha^{\vee} / \sum_{\beta \in J} \mathbb{Z}\beta^{\vee}| = r^{k} \text{ for some } k \in \mathbb{Z}_{>0}\}.
\]
For \(J \in \tilde{\Theta}_{r}\), let \(W_{J}\) be the subgroup of \(W\) generated by the reflections \(s_{\alpha}, \alpha \in J\).

We define a set \(\mathcal{T}_{W}^{*r} \subset \text{Irr}(W)\) by induction on \(|W|\) as follows. If \(W = \{1\}\), \(\mathcal{T}_{W}^{*r} = \text{Irr}(W)\). If \(W \neq \{1\}\), then
\[
\mathcal{T}_{W}^{*r} = S_{W}^{1} \cup \{\rho \in \text{Irr}(W) \mid \rho = J_{W,r}^{W}\rho' \text{ for some } J \in \tilde{\Theta}_{r} \text{ and some } \rho' \in \mathcal{T}_{W,J}^{*r}\},
\]
where $S^1_W$ is defined as in [L2, 1.3], i.e. it is the set of irreducible characters of $W$ that correspond to the pairs $(O, \bar{Q}_l)$ in characteristic 0.

One can verify that the set
\[
\{\gamma_{g'}^{-1}(O, \bar{Q}_l) \mid (O, \bar{Q}_l) \in \mathcal{A}_{g^*}\}
\]
coinsides with the set $T_{W^*}^3$ (resp. $T_{W^*}^2$) for $G_2$ (resp. $F_4$) in characteristic 3 (resp. 2).

In fact for $G_2$, we have $\tilde{\Theta}_3 = \{J\}$, where $J = \{\alpha, -(2\alpha + \beta)\}$ and $W_J = W(A_2)$, and in characteristic 3
\[
\{\gamma_{g'}^{-1}(O, \bar{Q}_l) \mid (O, \bar{Q}_l) \in \mathcal{A}_{g^*}\} = S^1_W = T_{W^*}^3;
\]
for $F_4$, we have $\tilde{\Theta}_2 = \{J_1, J_2, J_3\}$, where $J_1 = \{p, q, r, -p2q3r2s\}$, $W_{J_1} = W(B_3A_1)$, $J_2 = \{p, r, s, -p2q3r2s\}$, $W_{J_2} = W(A_3A_1)$, $J_3 = \{q, r, s, -p2q3r2s\}$ and $W(J_4) = W(C_4)$, and in characteristic 2
\[
\{\gamma_{g'}^{-1}(O, \bar{Q}_l) \mid (O, \bar{Q}_l) \in \mathcal{A}_{g^*}\} = S^1_W \cup \{\chi_{2,1}, \chi_{1,2}\} = T_{W^*}^2.
\]

Here we have used again the tables in [A] and the description of $T_{W^*}^2$ in [X3] when $W$ is of type $B$ or $C$.

5. Springer correspondence for $\mathfrak{g}$

Let $G$ be of type $G_2$ (resp. $F_4$) defined over an algebraically closed field of characteristic 2 or 3 (resp. characteristic 3). In this section, we describe the Springer correspondence maps $\gamma_{g'} : \text{Irr } W \to \mathcal{A}_{g^*}$ explicitly, see Tables 8, 9 and 10. In view of Lemma 2.1 and 4.1, this completes the description of the Springer correspondence for $\mathfrak{g}$ and $\mathfrak{g}^*$ when $G$ is of type $G_2$ and $F_4$ in all characteristics, as explained in the introduction.

As in [H] the orbits are labelled as in the Bala-Carter classification, and $\rho_{x,\phi}$, for $x \in \mathcal{N}_\mathfrak{g}$ and $\phi \in \text{Irr } A_G(x)$, denotes the inverse image of $(O_x, \mathcal{E})$ under $\gamma_{g}$, where $\mathcal{E}$ is the local system on the orbit $O_x$ of $x$ corresponding to $\phi$.

| Orbit of $x$ | $\phi$ | $\rho_{x,\phi}$ | Orbit of $x$ | $\phi$ | $\rho_{x,\phi}$ |
|--------------|--------|----------------|--------------|--------|----------------|
| $G_2$        | (1)    | $\chi_{1,1}$   | $\tilde{A}_1$| (1)    | $\chi_{2,2}$   |
| $G_2(a_1)$   | (3)    | $\chi_{2,1}$   | $A_1$        | (1)    | $\chi_{1,4}$   |
| $G_2(a_1)$   | (2, 1) | $\chi_{1,3}$   | $\emptyset$  | (1)    | $\chi_{1,2}$   |
| $G_2(a_1)$   | (1)    | $\chi_{1,3}$   |              |        |                |

Table 8. Springer correspondence for $\mathfrak{g}$, type $G_2$ and char $k = 2$
| Orbit of $x$ | $\phi$ | $\rho_{x,\phi}$ | Orbit of $x$ | $\phi$ | $\rho_{x,\phi}$ |
|-------------|--------|----------------|-------------|--------|----------------|
| $G_2$       | (1) $\chi_{1,1}$ |               | ($\tilde{A}_1)_2$ | (1) $\chi_{1,3}$ |           |
| $G_2(a_1)$  | (2) $\chi_{2,1}$ |              |          | $A_1$  | (1) $\chi_{1,4}$ |           |
| $G_2(a_1)$  | (1, 1) $-$ |                |          | $\emptyset$  | (1) $\chi_{1,2}$ |           |
| $\tilde{A}_1$ | (1) $\chi_{2,2}$ |           |          |          |               |           |

Table 9. Springer correspondence for $g$, type $G_2$ and char $k = 3$

| Orbit of $x$ | $\phi$ | $\rho_{x,\phi}$ | Orbit of $x$ | $\phi$ | $\rho_{x,\phi}$ | Orbit of $x$ | $\phi$ | $\rho_{x,\phi}$ |
|-------------|--------|----------------|-------------|--------|----------------|-------------|--------|----------------|
| $F_4$       | (1) $\chi_{1,1}$ |               | $F_4(a_3)$  | (2) $\chi_{6,2}$ |          | $A_2$  | (1) $\chi_{8,2}$ |           |
| $F_4(a_1)$  | (2) $\chi_{4,2}$ |              | $F_4(a_3)$  | (2, 1, 1) $\chi_{1,3}$ |   | $A_2$  | (2) $\chi_{8,4}$ |           |
| $F_4(a_2)$  | (1) $\chi_{2,3}$ |              | $F_4(a_3)$  | (1) $-$ |          | $A_2$  | (1$^2$) $\chi_{1,2}$ |           |
| $F_4(a_2)$  | (2) $\chi_{9,1}$ |              | $C_3(a_1)$  | (2) $\chi_{6,1}$ |          | $A_1 + \tilde{A}_1$ | (1) $\chi_{9,4}$ |           |
| $B_3$       | (1) $\chi_{8,1}$ |              | $B_2$       | (2) $\chi_{9,2}$ |          | $\tilde{A}_1$  | (2) $\chi_{4,5}$ |           |
| $C_3$       | (1) $\chi_{8,3}$ |              | $B_2$       | (1) $\chi_{6,1}$ |          | $A_1$  | (1$^2$) $\chi_{2,2}$ |           |
| $F_4(a_3)$  | (4) $\chi_{12,1}$ |             | $\tilde{A}_2 + A_1$ | (1) $\chi_{4,1}$ |          | $\emptyset$  | (1) $\chi_{1,4}$ |           |
| $F_4(a_3)$  | (3, 1) $\chi_{9,3}$ |             | $A_2 + \tilde{A}_1$ | (1) $\chi_{4,3}$ |          |          |               |           |

Table 10. Springer correspondence for $g$, type $F_4$ and char $k = 3$

5.1. **Nilpotent orbits in $g$.** The nilpotent orbits in $g$ for $G_2$ have been classified by Stuhler [St] and those for $F_4$ by Holt and Spaltenstein [HS]. It has been shown (see [HS]) that for $x \in \mathcal{N}_g$,

$$\dim B_x = \frac{\dim Z_G(x) - \text{rank} G}{2},$$

where $B_x = \{gB \mid \text{Ad}(g^{-1})x \in \mathfrak{b}\}$. We list the results here for the reader’s convenience, see Tables 11, 12 and 13.

Note that the nilpotent orbits in $g$ for $F_4$ in char $k = 3$ and component groups of centralizers coincide with those in characteristic 0. In particular, there are 4 distinguished nilpotent orbits (where $x \in \mathcal{N}_g$ is distinguished if $Z(G)^0$ is a maximal torus in $Z_G(x)$) and every orbit contains an element that is distinguished in a Levi subgroup. As char $k = 3$, all distinguished nilpotent elements in a proper Levi subgroup is regular.
| Orbit | representative \( x \) | \( \dim Z_G(x) \) | \( A_G(x) \) | \( \dim B_x \) |
|-------|-----------------|----------------|--------|--------|
| \( G_2 \) | \( e_\alpha + e_\beta \) | 2 | 1 | 0 |
| \( G_2(a_1) \) | \( e_\beta + e_{2\alpha + \beta} \) | 4 | \( S_3 \) | 1 |
| \( \tilde{A}_1 \) | \( e_\alpha \) | 6 | 1 | 2 |
| \( A_1 \) | \( e_\beta \) | 8 | 1 | 3 |
| \( \emptyset \) | 0 | 14 | 1 | 6 |

Table 11. Type \( G_2 \), nilpotent orbits, \( \text{char} \ k = 2 \)

| Orbit | representative \( x \) | \( \dim Z_G(x) \) | \( A_G(x) \) | \( \dim B_x \) |
|-------|-----------------|----------------|--------|--------|
| \( G_2 \) | \( e_\alpha + e_\beta \) | 2 | 1 | 0 |
| \( G_2(a_1) \) | \( e_\beta + e_{2\alpha + \beta} \) | 4 | \( S_2 \) | 1 |
| \( \tilde{A}_1 \) | \( e_\alpha + e_\beta \) | 6 | 1 | 2 |
| \( (\tilde{A}_1)_2 \) | \( e_\alpha \) | 8 | 1 | 3 |
| \( A_1 \) | \( e_\beta \) | 8 | 1 | 3 |
| \( \emptyset \) | 0 | 14 | 1 | 6 |

Table 12. Type \( G_2 \), nilpotent orbits, \( \text{char} \ k = 3 \)

| Orbit | representative \( x \) | \( \dim Z_G(x) \) | \( A_G(x) \) | \( \dim B_x \) |
|-------|-----------------|----------------|--------|--------|
| \( F_4 \) | \( e_p + e_q + e_r + e_s \) | 4 | 1 | 0 |
| \( F_4(a_1) \) | \( e_p + e_{qr} + e_{q2r} + e_s \) | 6 | \( S_2 \) | 1 |
| \( F_4(a_2) \) | \( e_p + e_{qr} + e_{rs} + e_{q2r2s} \) | 8 | \( S_2 \) | 2 |
| \( B_3 \) | \( e_p + e_{qr} + e_{q2r2s} \) | 10 | 1 | 3 |
| \( C_3 \) | \( e_s + e_{q2r} + e_{pqr} \) | 10 | 1 | 3 |
| \( F_4(a_3) \) | \( e_{pqr} + e_{qrs} + e_{pq2r} + e_{q2r2s} \) | 12 | \( S_4 \) | 4 |
| \( C_3(a_1) \) | \( e_{pqr} + e_{q2rs} + e_{q2r2s} \) | 14 | \( S_2 \) | 5 |
| \( B_2 \) | \( e_{pqr} + e_{q2r2s} \) | 16 | \( S_2 \) | 6 |
| \( \tilde{A}_2 + A_1 \) | \( e_{pqr} + e_{q2rs} + e_{pq2r2s} \) | 16 | 1 | 6 |
| \( A_2 + \tilde{A}_1 \) | \( e_{pq2r} + e_{q2r2s} + e_{pq2rs} \) | 18 | 1 | 7 |
| \( \tilde{A}_2 \) | \( e_{pqrs} + e_{q2rs} \) | 22 | 1 | 9 |
| \( A_2 \) | \( e_{pq2r} + e_{pq2rs} \) | 22 | \( S_2 \) | 9 |
| \( A_1 + \tilde{A}_1 \) | \( e_{pq2r2s} + e_{pq3rs} \) | 24 | 1 | 10 |
| \( \tilde{A}_1 \) | \( e_{pq2q3r2s} \) | 30 | \( S_2 \) | 13 |
| \( A_1 \) | \( e_{p3q4r2s} \) | 36 | 1 | 16 |
| \( \emptyset \) | 0 | 52 | 1 | 24 |

Table 13. Type \( F_4 \), nilpotent orbits, \( \text{char} \ k = 3 \)
5.2. **Springer correspondence.** We use the same strategy as in [41] to obtain the explicit description of the maps $\gamma_6$, following the methods in [AL, S2, S3]. As the arguments are entirely similar, we omit the details and only explain one most complicated case.

For $F_4$, we make use of Table 14 on induced nilpotent orbits. Here $\tilde{O} = \text{Ind}^L_0 O$, $0$ denotes the zero orbit in the third row, and the nilpotent orbits in $I$ correspond to a partition.

| $\tilde{O}$ | $\tilde{A}_2$ | $A_2$ | $B_2$ | $F_4(a_3)$ | $F_4(a_3)$ | $F_4(a_1)$ | $F_4(a_2)$ |
|------------|---------------|-------|-------|------------|------------|------------|------------|
| type of $L$ | $B_3$ | $C_3$ | $C_3$ | $\tilde{A}_2 A_1$ | $B_3$ | $B_3$ | $A_1 \tilde{A}_1$ |
| $O$ | 0 | 0 | $2^{114}$ | 0 | $3^{114}$ | $5^{112}$ | 0 |
| $\rho^L_{x^*,1}$ | $[\vdash : 1^3]$ | $[\vdash : 1^3]$ | $[1^3 : -]$ | $[1^3 \times [1^2]$ | $[1 : 1^2]$ | $[2 : 1]$ | $[2 \times 2]$ |

Table 14. Induced nilpotent orbits, type $F_4$, char $k = 3$

The most complicated case is the orbit $F_4(a_3)$. Let $x = e_{pqr} + e_{qrs} + e_{pq2r} + e_{q2r2s}$ be an element in the orbit $F_4(a_3)$. By direct computations using (2.2) and (2.4), one verifies that

$$A_G(x) \cong \langle \gamma_1, \gamma_2, \gamma_3 \rangle \cong S_4,$$

where

$$\gamma_1 = h_p(-1)h_q(-1)h_r(-1)h_s(-1)x_p(-1)x_s(-1)x_r(1),$$

$$\gamma_2 = h_p(-1)h_q(-1)h_r(-1)h_s(-1) n_p n_s,$$

$$\gamma_3 = h_p(-1)h_q(-1)x_p(-1)x_s(1)x_r(1)x_r(-1) n_r x_r(-1)$$

with

$$\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = 1, \text{ } (\gamma_1 \gamma_2)^3 = (\gamma_2 \gamma_3)^3 = 1, \text{ } (\gamma_1 \gamma_3)^2 = 1.$$

In particular we have the following formula

$$\text{Ad}(u(t))x = e_{pqr} + e_{qrs} + (1 - 2t_2)e_{pq2r} + t_2 e_{q2r2s} + (t_1 - t_3) e_{pqrs} + (1 - 2t_4)e_{q2r2s} + (t_1 + t_2 - t_3 + t_4) e_{pq2rs} + (t_1 + t_2 - 2t_4(t_1 - t_3)) e_{pq2r2s},$$

where $u(t) = x_p(t_1) x_r(t_2) x_s(t_3) x_r(t_4), t_i \in \mathbb{G}_a$. One can then proceed the same way as in the case of orbit $F_4(a_3)$ in $g^*$ in characteristic 2.

**References**

[A] Alvis, D. Induce/restrict matrices for exceptional Weyl groups. [arXiv:math/0506377](http://arxiv.org/abs/math/0506377)

[AL] Alvis, D. and Lusztig, G. On Springer’s correspondence for simple groups of type $E_n$ ($n = 6, 7, 8$). Math. Proc. Cambridge Philos. Soc. 92 (1982), no. 1, 65–72.

[B] Borel. A et al. Seminar on Algebraic Groups and Related Finite Groups. (Held at The Institute for Advanced Study, Princeton, N. J., 1968/69). Lecture Notes in Mathematics, Vol. 131 Springer-Verlag, Berlin-New York 1970.

[C] Carter, Roger W. Simple groups of Lie type. Reprint of the 1972 original. Wiley Classics Library. New York, 1989.

[CP] Clarke, Matthew C.; Premet, Alexander. The Hesselink stratification of nullcones and base change. Invent. Math. 191 (2013), no. 3, 631–669.
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[GP] Geck, Meinolf; Pfeiffer, Götz. Characters of finite Coxeter groups and Iwahori-Hecke algebras. London Mathematical Society Monographs. New Series, 21. The Clarendon Press, Oxford University Press, New York, 2000.

[H] Hogeweij, G. M. D. Almost-classical Lie algebras. I. Nederl. Akad. Wetensch. Indag. Math. 44 (1982), no. 4, 441–452.

[HS] Holt, D. F.; Spaltenstein, N. Nilpotent orbits of exceptional Lie algebras over algebraically closed fields of bad characteristic. J. Austral. Math. Soc. Ser. A 38 (1985), no. 3, 330–350.

[K] Kac, Victor G. Infinite-dimensional Lie algebras. Third edition. Cambridge University Press, Cambridge, 1990.

[KW] Kac, V.; Weisfeiler, B. Coadjoint action of a semi-simple algebraic group and the center of the enveloping algebra in characteristic p. Nederl. Akad. Wetensch. Proc. Ser. A 79=Indag. Math. 38 (1976), no. 2, 136–151.

[L1] Lusztig, G. Intersection cohomology complexes on a reductive group. Invent. Math. 75 (1984), no. 2, 205–272.

[L2] Lusztig, G. Unipotent elements in small characteristic. Transform. Groups 10 (2005), no. 3-4, 449–487.

[L3] Lusztig, G. Remarks on Springer’s representations. Represent. Theory 13 (2009), 391–400.

[LS1] Lusztig, G.; Spaltenstein, N. Induced unipotent classes. J. London Math. Soc. (2) 19 (1979), no. 1, 41–52.

[LS2] Lusztig, G.; Spaltenstein, N. On the generalized Springer correspondence for classical groups. Algebraic groups and related topics (Kyoto/Nagoya, 1983), 289–316, Adv. Stud. Pure Math., 6, North-Holland, Amsterdam, 1985.

[M] Miller, G. A. Groups possessing a small number of sets of conjugate operators. Trans. Amer. Math. Soc. 20 (1919), no. 3, 260–270.

[Sh] Shoji, T. On the Springer representations of Chevalley groups of type $F_4$. Comm. Algebra, 8 (1980), 409–440.

[S1] Spaltenstein, N. Classes unipotentes et sous-groupes de Borel. Lecture Notes in Mathematics, 946. Springer-Verlag, Berlin-New York, 1982.

[S2] Spaltenstein, N. Appendix. Math. Proc. Cambridge Philos. Soc. 92 (1982), no. 1, 73–78.

[S3] Spaltenstein, N. Nilpotent classes in Lie algebras of type $F_4$ over fields of characteristic 2. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 30 (1984), no. 3, 517–524.

[S4] Spaltenstein, N. On the generalized Springer correspondence for exceptional groups. Algebraic groups and related topics (Kyoto/Nagoya, 1983), 317–338, Adv. Stud. Pure Math., 6, North-Holland, Amsterdam, 1985.

[Sp] Springer, T. A., Trigonometric sums, Green functions of finite groups and representations of Weyl groups, Invent. Math., 36 (1976), 173–207.

[St] Stuhler, U. Unipotente und nilpotente Klassen in einfachen Gruppen und Liealgebren vom Typ G2. Nederl. Akad. Wetensch. Proc. Ser. A 74=Indag. Math. 33 (1971), 365–378.

[X1] Xue, T. Nilpotent orbits in classical Lie algebras over finite fields of characteristic 2 and the Springer correspondence. Represent. Theory 13 (2009), 371–390. (electronic).

[X2] Xue, T. Nilpotent orbits in the dual of classical Lie algebras in characteristic 2 and the Springer correspondence. Represent. Theory 13 (2009), 609–635. (electronic).

[X3] Xue, T. Combinatorics of the Springer correspondence for classical Lie algebras and their duals in characteristic 2. Adv. Math. 230 (2012), no. 1, 229–262.

[X4] Xue, T. Nilpotent coadjoint orbits in small characteristic. J. Algebra 397 (2014), 111–140.

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