TOEPLITZ OPERATORS AND CARLESON MEASURE BETWEEN WEIGHTED BERGMAN SPACES INDUCED BY REGULAR WEIGHTS

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ABSTRACT. In this paper, we give a universal description of the boundedness and compactness of Toeplitz operator $T_\mu^\omega$ between Bergman spaces $A_p^\eta$ and $A_q^\upsilon$ when $\mu$ is a positive Borel measure, $1 < p, q < \infty$ and $\omega, \eta, \upsilon$ are regular weights. By using Khinchin’s inequality and Kahane’s inequality, we get a new characterization of the Carleson measure for Bergman spaces induced by regular weights.

Keywords: Bergman space, Carleson measure, Toeplitz operator, regular weight.

1. INTRODUCTION

Suppose $\omega$ is a radial weight (i.e., $\omega$ is a positive, measurable and integrable function on $[0, 1)$ and $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$, the open unit disc in the complex plane). We say that $\omega$ is a doubling weight, denoted by $\omega \in \hat{\mathcal{D}}$, if

$$\hat{\omega}(r) \lesssim \hat{\omega}(r + \frac{1}{2})$$

for all $r \in (0, 1)$, where $\hat{\omega}(r) = \int_r^1 \omega(t)dt$. $\omega$ is a regular weight, denoted by $\omega \in \mathcal{R}$, if

$$\frac{\hat{\omega}(r)}{(1 - r)\omega(r)} \approx 1$$

for all $r \in (0, 1)$. Obviously, if $\omega \in \mathcal{R}, \omega \in \hat{\mathcal{D}}$ and $\omega(t) \approx \omega(s)$ whenever $s, t \in (0, 1)$ satisfying $1 - t \approx 1 - s$. See [10, 11, 15, 17] for more information about doubling weights and related topics.

For $z \in \mathbb{D}$, the Carleson square $S_z$ is defined by

$$S_z = \left\{ \xi = re^{i\theta} \in \mathbb{D} : |\xi| \leq 1; |\theta - \arg z| < \frac{1 - |z|}{2\pi} \right\}.$$ 

As usual, let $\beta(a, z) = \frac{1}{2} \log \frac{1 + |\varphi_a(z)|}{1 - |\varphi_a(z)|}$ denote the Bergman metric for $a, z \in \mathbb{D}$ and

$$D(z, r) = \{ a \in \mathbb{D} : \beta(z, a) < r \}$$

be the Bergman disk with radius $r$ and center $z$. Here $\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}$. It is easy to check that, for any given $\omega \in \mathcal{R}$ and $r \in (0, \infty)$,

$$\omega(D(z, r)) \approx \omega(S_z) \approx (1 - |z|)^2 \omega(z), \quad z \in \mathbb{D}.$$  

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For $0 < s < \infty$, a sequence $\{a_j\}_{j=1}^{\infty} \subset \mathbb{D}$ is called $s$-separated (or separated, for brief) if $\beta(a_i, a_j) \geq s$ for all $i \neq j$. A sequence $\{a_j\}_{j=1}^{\infty} \subset \mathbb{D}$ is called $r$-covering if 

$$
\mathbb{D} = \cup_{j=1}^{\infty} D(a_j, r)
$$

for some $r \in (0, \infty)$.

Let $0 < p < \infty$ and $\mu$ be a positive Borel measure on $\mathbb{D}$. The Lebesgue space $L_{\mu}^p$ consists of all measurable functions $f$ on $\mathbb{D}$ such that

$$
\|f\|_{L_{\mu}^p} = \int_{\mathbb{D}} |f(z)|^p d\mu(z) < \infty.
$$

In particular, when $d\mu(z) = \omega(z)dA(z)$, we denote $L_{\mu}^p$ by $L_{\omega}^p$. Here $dA$ is the normalized area measure on $\mathbb{D}$.

Let $0 < p < \infty$ and $\omega$ be a radial weight. Let $H(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$. The weighted Bergman space $A_{\omega}^p$ consists of all $f \in H(\mathbb{D})$ such that

$$
\|f\|_{A_{\omega}^p} = \int_{\mathbb{D}} |f(z)|^p \omega(z)dA(z) < \infty.
$$

Throughout this paper, we assume that $\omega(z) > 0$ for all $z \in \mathbb{D}$. For otherwise $A_{\omega}^p = H(\mathbb{D})$. As usual, we write $A_{\omega}^p$ for the standard weighted Bergman space induced by $\omega(z) = (\alpha + 1)(1 - |z|^2)^p$ with $-1 < \alpha < \infty$. For convenience, the weight $(\alpha + 1)(1 - |z|^2)^p(-1 < \alpha < \infty)$ will be called standard weight.

Let $\omega \in \hat{\mathbb{D}}$. It is easy to see that $A_{\omega}^2$ is a closed subspace of $L_{\omega}^2$ and the orthogonal Bergman projection $P_{\omega}$ from $L_{\omega}^2$ to $A_{\omega}^2$ is given by

$$
P_{\omega}f(z) = \int_{\mathbb{D}} f(\xi)\overline{B_{\omega}(\xi)}\omega(\xi)dA(\xi),
$$

where $B_{\omega}$ is the reproducing kernel of $A_{\omega}^2$. For more results about the Bergman projection $P_{\omega}$, see [13][15]. For a positive Borel measure $\mu$ on $\mathbb{D}$, the Toeplitz operator associated with $\mu$, a natural extension of $P_{\omega}$, is defined by

$$
T_{\mu}^\omega f(z) = \int_{\mathbb{D}} f(\xi)\overline{B_{\omega}(\xi)}d\mu(\xi).
$$

For the standard weight, we will write $P_{\omega}$ and $T_{\mu}^\omega$ as $P_\omega$ and $T_{\mu}^\omega$, respectively.

Toeplitz operators on Bergman spaces attract a lot of attentions in the past decades. See [24] Chapter 7 and the references therein for the theory of Toeplitz operators on standard weighted Bergman spaces $A_{\omega}^p$. In [13], Peláez and Rättyä characterized the Schatten class Toeplitz operators $T_{\mu}^\omega(\omega \in \hat{\mathbb{D}})$ on $A_{\omega}^2$. Peláez, Rättyä and Sierra [17] gave several characterizations of bounded and compact Toeplitz operators $T_{\mu}^\omega : A_{\omega}^p \rightarrow A_{\omega}^q$, when $\omega \in \mathcal{R}$ and $1 < p, q < \infty$. Recently, Duan, Guo, Wang and Wang [21] extended these results to $0 < p \leq 1$ and $q = 1$. See [3][6][8][21] for more discussions on this topic.

It is interesting to study the boundedness and compactness of $T_{\mu}^\omega : A_{\eta}^p \rightarrow A_{\nu}^q$ when $\omega, \eta, \nu$ are different weights. When $\omega, \eta, \nu$ are all standard weights, this problem was studied by Pau and Zhao in [9] in the case of the unit ball. Motivated by [9], we study the boundedness and compactness of $T_{\mu}^\omega : A_{\eta}^p \rightarrow A_{\nu}^q$ when $1 <
$p, q < \infty$ and $\omega, \eta, \upsilon$ are regular weights. Set

$$\sigma_{p,q}(r) = \left( \frac{\omega(r)}{\eta(r)} \right)^{p'} \quad \text{and} \quad A(p, \eta) = \sup_{0 \leq r < 1} \frac{\eta(r)^{\frac{1}{p}} \sigma_{p,q}(r)^{\frac{1}{p'}}}{\omega(r)}.$$  \hfill (2)

Here $p' = \frac{p}{p-1}$ is the conjugate number of $p$. The first result of this paper is stated as follows.

**Theorem 1.** Suppose $1 < p, q < \infty$, $\mu$ is a positive Borel measure on $\mathbb{D}$ and $\omega, \eta, \upsilon \in \mathcal{R}$ such that

$$A(p, \eta) < \infty, \quad A(q, \upsilon) < \infty.$$  

Let

$$W(z) = \eta(z)^{\frac{q}{\nu + q-p}} \sigma_{q,\omega}(z)^{\frac{q-p}{\nu + q-p}}.$$  

Then, $T_{\mu}^\omega : A_\eta^p \to A_\upsilon^q$ is bounded (compact) if and only if $\mu$ is a 1-Carleson measure (vanishing 1-Carleson measure) for $A_\omega^{\nu+a}$. Moreover,

$$\|T_{\mu}^\omega\|_{A_\eta^p \to A_\upsilon^q} \approx \|I_d\|_{A_\omega^{\nu+a} \to L_p^\upsilon}.$$  

For a function space $X$ and $0 < q < \infty$, a positive measure $\mu$ on $\mathbb{D}$ is called a $q$-Carleson measure (vanishing $q$-Carleson measure) for $X$ if the identity operator $I_d : X \to L_p^{\omega} \mu$ is bounded (compact). When $0 < p, q < \infty$, the $q$-Carleson measure for $A_\omega^q$ was characterized by many authors, we refer to \cite{21, 22} and the references therein. When $\omega \in \hat{\mathcal{D}}$, the problem was completely solved by Peláez and Rättyä in \cite{11, 12}. See \cite{11, 18, 16, 17} for more study of Carleson measure for Bergman spaces induced by various weights.

In \cite{9}, Pau and Zhao gave a new characterization of Carleson measures for standard weighted Bergman spaces in the unit ball by using the technique of sublinear operator and the characterizations of the boundedness and compactness of $T_{\mu}^\omega : A_\eta^p \to A_\upsilon^q$. Here, we extend their result (i.e., \cite{9} Theorem 1.1) to Bergman spaces induced by regular weights in a more direct way, without the using of Theorem 1. We state it as follows.

**Theorem 2.** Suppose $0 < p_i, q_i < \infty$, $\omega_i \in \mathcal{R}$, $i = 1, 2, \cdots, n$ and $\mu$ is a positive Borel measure on $\mathbb{D}$. Let

$$\lambda = \sum_{i=1}^{n} \frac{q_i}{p_i}, \quad \omega(z) = \prod_{i=1}^{n} \omega_i(z)^{\frac{q_i}{p_i}}.$$  

Then $\mu$ is a $\lambda$-Carleson measure for $A_\omega^1$ if and only if

$$M_n = \sup_{f_i \in A_{\omega_i}^{q_i}, i=1, \cdots, n} \left( \frac{\int_{\mathbb{D}} \prod_{i=1}^{n} |f_i(z)|^{q_i} d\mu(z)}{\prod_{i=1}^{n} \|f_i\|_{A_{\omega_i}^{q_i}}} \right) < \infty.$$  \hfill (3)

Moreover, $M_n \approx \|I_d\|_{A_\omega^{1/q} \to L_p^1}.$

The analogous characterizations of vanishing $\lambda$-Carleson measures for $A_{\omega}^1(\omega \in \mathcal{R})$ can be state as follows.
Theorem 3. Suppose \( 0 < p_i, q_i < \infty, \omega_i \in \mathcal{R}, i = 1, 2, \ldots, n \) and \( \mu \) is a positive Borel measure on \( \mathbb{D} \). Let
\[
\lambda = \sum_{i=1}^{n} \frac{q_i}{p_i}, \quad \omega(z) = \prod_{i=1}^{n} \omega_i(z)^{\frac{q_i}{p_i}}.
\]
Then the following statements are equivalent.

(i) \( \mu \) is a vanishing \( \lambda \)-Carleson measure for \( A^1_{\omega} \);
(ii) If \( \{f_{1,k}\}_{k=1}^{\infty} \) is bounded in \( A^{p_1}_{\omega_1} \) and converges to 0 uniformly on compact subsets of \( \mathbb{D} \),
\[
\lim_{k \to \infty} F(k) = 0,
\]
where
\[
F(k) = \sup \left\{ \int_{\mathbb{D}} |f_{1,k}(z)|^q \prod_{i=2}^{n} |f_i(z)|^q d\mu(z) : \|f_i\|_{A^{p_i}_{\omega_i}} \leq 1, i = 2, \ldots, n \right\};
\]
(iii) For any bounded sequences \( \{f_{1,k}\}_{k=1}^{\infty}, \ldots, \{f_{n,k}\}_{k=1}^{\infty} \) in \( A^{p_1}_{\omega_1}, \ldots, A^{p_n}_{\omega_n} \) respectively, all of which converge to 0 uniformly on compact subsets of \( \mathbb{D} \),
\[
\lim_{k \to \infty} \int_{\mathbb{D}} \prod_{i=1}^{n} |f_{i,k}(z)|^q d\mu(z) = 0.
\]

The paper is organized as follows. In Section 2, we state some preliminary results and lemmas, which will be used later. Section 3 is devoted to the proof of main results in this paper.

Throughout this paper, the letter \( C \) will denote constants and may differ from one occurrence to the other. For two real valued functions \( f \) and \( g \), we write \( f \lesssim g \) if there is a positive constant \( C \), independent of argument, such that \( f \leq Cg \). \( f \approx g \) means that \( f \lesssim g \) and \( g \lesssim f \).

2. Preliminaries

To prove our main results in this paper, we need some lemmas. First, we collect some characterizations of \( q \)-Carleson measure for \( A^{p}_{\omega} \) (\( \omega \in \mathcal{R} \)) from \([1,16,17]\) as follows.

**Theorem A.** Let \( 0 < p, q < \infty, \omega \in \mathcal{R} \) and \( \mu \) be a positive Borel measure on \( \mathbb{D} \). Then, the following statements hold.

(i) When \( p \leq q \), the following statements are equivalent:
   (ia) \( \mu \) is a \( q \)-Carleson measure for \( A^{p}_{\omega} \);
   (ib) \( \sup_{z \in \mathbb{D}} \frac{\mu(S_z)}{\omega(S_z)^{\frac{p}{q}}} < \infty \);
   (ic) \( \sup_{z \in \mathbb{D}} \frac{\mu(D(z,r))}{\omega(D(z,r))^{\frac{p}{q}}} < \infty \) for some (equivalently, for all) \( r \in (0, \infty) \).

Moreover,
\[
\|I_{\omega}\|_{A^{p}_{\omega} \to L^{q}_{\omega}} \approx \sup_{z \in \mathbb{D}} \frac{\mu(S_z)}{\omega(S_z)^{\frac{p}{q}}} \approx \sup_{z \in \mathbb{D}} \frac{\mu(D(z,r))}{\omega(D(z,r))^{\frac{p}{q}}}.
\]
(ii) When \( p \leq q \), the following statements are equivalent:
(iia) \( \mu \) is a vanishing \( q \)-Carleson measure for \( A^p_\omega \);
(iib) \( \lim_{|z| \to 1} \frac{\mu(S(z))}{\omega(S(z))} = 0 \);
(iic) \( \lim_{|z| \to 1} \frac{\mu(D(z,r))}{\omega(D(z,r))} = 0 \) for some (equivalently, for all) \( r \in (0, \infty) \).

(iii) When \( q < p \), the following statements are equivalent:

(iiiia) \( \mu \) is a \( q \)-Carleson measure for \( A^p_\omega \);
(iiiib) \( \mu \) is a vanishing \( q \)-Carleson measure for \( A^p_\omega \);
(iiiic) for any given \( r \in (0, \infty) \), \( \Phi(z) = \frac{\mu(D(z,r))}{\omega(D(z,r))} \in L^{p-q}_{\omega} \);
(iiiid) for each sufficiently large \( \gamma > 1 \), the function

\[
\Psi(z) = \int_D \left( \frac{1 - |\xi|}{1 - z \overline{\xi}} \right)^{\gamma} \frac{d\mu(\xi)}{\omega(S_\xi)}, \quad z \in \mathbb{D},
\]

belongs to \( L^{p-q}_{\omega} \).

Moreover,

\[
\|I_d\|_{A^p_{\sigma}} \to L^p_{\omega} \approx \|\Phi\|_{L^{p-q}_{\omega}} \approx \|\Psi\|_{L^{p-q}_{\omega}}.
\]

The duality relation between weighted Bergman space via \( A^2_\omega \)-pairing plays a very important role in the proof of our main results in this paper. See \([22]\) and \([23, \text{Theorem 2.11}]\) for related results for standard weighted Bergman spaces. For Bergman spaces induced by radial weights, we refer to \([15, 18, 19]\). The following theorem comes from \([15, \text{Theorem 13}]\) and \([19, \text{Theorem 3}]\).

**Theorem B.** Suppose \( 1 < p < \infty \) and \( \omega, \eta \in \mathcal{R} \). Then the following statements are equivalent:

(i) \( A(p, \eta) < \infty \);
(ii) \( P_\omega : L^p_\eta \to L^p_\eta \) is bounded;
(iii) \( (A^p_{\sigma_{p,\eta}})^* \simeq A^p_\eta \) via \( A^2_\omega \)-pairing

\[
\langle f, g \rangle_{A^p_\sigma} = \int_D f(z) \overline{g(z)} \omega(z) dA(z), \quad \forall f \in A^p_{\sigma_{p,\eta}}, \ g \in A^p_{\eta},
\]

with equivalent norms. Here \( \sigma_{p,\eta} \) and \( A(p, \eta) \) are defined as in \([2]\).

The following lemma can be proved by a standard technique. We omit the details of the proof.

**Lemma 1.** Suppose \( \eta, \nu \in \hat{D} \) and \( 0 < p, q < \infty \). If \( T : A^p_\eta \to A^q_\nu \) is bounded and linear, then \( T \) is compact if and only if \( \lim_{j \to \infty} \|T f_j\|_{A^q_\nu} = 0 \) whenever \( \{f_j\} \) is bounded in \( A^p_\eta \) and converges to 0 uniformly on compact subsets of \( \mathbb{D} \).

**Lemma 2.** Let \( 1 < p < \infty \) and \( \omega, \eta \in \mathcal{R} \). Then, \( A(p, \eta) < \infty \) if and only if \( \sigma_{p,\eta} \in \mathcal{R} \).

**Proof.** If \( \sigma_{p,\eta} \in \mathcal{R} \), from the fact that

\[
\overline{\sigma_{p,\eta}}(r) \approx (1 - r) \sigma_{p,\eta}(r),
\]

we have \( A(p, \eta) < \infty \).
Conversely, assume \( A(p, \eta) < \infty \). After a calculation,

\[
\sigma_{p,\eta}(r) \leq \sigma_{p,\eta}(r) \left( 1 - \frac{\eta(r)}{\eta(r)_{p,p'}} \right) \approx \sigma_{p,\eta}(r)(1 - r)\sigma_{p,\eta}(r)
\]

and

\[
\sigma_{p,\eta}(r) \geq \int_r^1 \sigma_{p,\eta}(t)dt \approx (1 - r)\sigma_{p,\eta}(r),
\]

which implies that \( \sigma_{p,\eta} \in \mathcal{R} \).

\[\□\]

**Lemma 3.** Let \( 1 < p < \infty \) and \( \omega, \eta \in \mathcal{R} \) such that \( A(p, \eta) < \infty \). If \( \{z_j\}_{j=1}^{\infty} \subseteq \mathbb{D} \) is separated, then, for any \( c = \{c_j\}_{j=1}^{\infty} \),

\[||F||_{A_p^\omega} \lesssim ||c||_{p'}.
\]

Here

\[F(z) = \sum_{j=1}^{\infty} \frac{c_j B_{z_j}^\omega}{||B_{z_j}^\omega||_{A_p^\omega}}.
\]

**Proof.** Since \( A(p, \eta) < \infty \), by *Theorem 3* \( (A_{p,\eta}^{p,p'})^* \approx A_{\eta}^p \) via \( A_{\omega}^{1,p} \)-pairing with equivalent norms. By Lemma 2 \( \sigma_{p,\eta} \in \mathcal{R} \). Then Lemma 2.4 in [11] implies that, if \( \gamma \) is large enough,

\[F_a(z) = \left( \frac{1 - |a|^2}{1 - \overline{a}z} \right)^\gamma \in A_{p,\eta}^{p,p'} \text{ and } ||F_a||_{A_{p,\eta}^{p,p'}} \approx \sigma_{p,\eta}(S_a).
\]

Then,

\[||B_z^\omega||_{A_p^\omega} \approx \sup_{||h||_{A_{p,\eta}^{p,p'}} \leq 1} |\langle h, B_z^\omega \rangle_{A_p^\omega}| \geq \frac{1}{\sigma_{p,\eta}(S_z)^{1/p'}} |\langle F_z, B_z^\omega \rangle_{A_p^\omega}| = \frac{1}{\sigma_{p,\eta}(S_z)^{1/p'}}
\]

and

\[||B_z^\omega||_{A_p^\omega} \approx \sup_{||h||_{A_{p,\eta}^{p,p'}} \leq 1} |\langle h, B_z^\omega \rangle_{A_p^\omega}| = \sup_{||h||_{A_{p,\eta}^{p,p'}} \leq 1} |h(z)| \leq \frac{1}{\sigma_{p,\eta}(S_z)^{1/p'}}.
\]

In the last estimate, we used the fact that \( \sigma_{p,\eta} \in \mathcal{R} \), and the subharmonicity of \( |h|_{p'} \), i.e., for any given \( 0 < r < \infty \),

\[|h(z)|_{p'} \leq \frac{1}{\sigma_{p,\eta}(D(z, r))} \int_{D(z,r)} |h(\xi)|_{p'} \sigma_{p,\eta}(\xi)dA(\xi)
\]

\[\leq \frac{1}{\sigma_{p,\eta}(S_z)} ||h||_{A_{p,\eta}^{p,p'}}., \quad z \in \mathbb{D}.
\]

Thus, by the fact that \( \sigma_{p,\eta} \in \mathcal{R} \) we have

\[||B_z^\omega||_{A_p^\omega} \approx \frac{1}{\sigma_{p,\eta}(S_z)^{1/p'}} \approx \frac{\eta(S_z)^{1/p}}{\omega(S_z)}. \quad (4)
\]
So, if \( \{z_j\} \) is 2r-separated, for any \( g \in A_{\sigma,r}^p \), by Hölder’s inequality, (1) and (4), we have

\[
\| g \|_{A_{\sigma,r}^p} \leq \left( \sum_{j=1}^{\infty} \frac{c_j \| g \|_{A_{\sigma,r}^p}}{\| B_{\eta_j} \|_{A_{\sigma,r}^p}} \right)^{\frac{1}{p'}} \leq \left( \sum_{j=1}^{\infty} \frac{\| g \|_{A_{\sigma,r}^p}}{\| B_{\eta_j} \|_{A_{\sigma,r}^p}} \right)^{\frac{1}{p'}} \leq \left( \sum_{j=1}^{\infty} \int_{D(z_j,r)} |g(z)|^{p'} \sigma(D(z_j,r)) dA(z) \right)^{\frac{1}{p'}} \leq \left( \sum_{j=1}^{\infty} \int_{D(z_j,r)} |g(z)|^{p'} \sigma_D(z) dA(z) \right)^{\frac{1}{p'}}.
\]

Therefore, \( \| F \|_{A_{\sigma,r}^p} \leq \| F \|_{A_{\sigma,r}^p} \). The proof is complete. \( \Box \)

**Remark.** It should be point out that, when \( \eta, \omega \in \hat{D} \) and \( p > 0 \), \( \| B_{\omega} \|_{A_{\sigma,r}^p} \) was estimated for the first time in [14, Theorem 1]. Here, for the benefit of readers, under some more assumptions, we estimate it in a simple way.

To prove our main results, we will use the classical Khinchin’s inequality and Kahane’s inequality, which can be found in [2, Chapters 1 and 11] for example. For \( k \in \mathbb{N} \) and \( t \in (0, 1) \), let \( r_k(t) = \text{sign}(\sin(2^k \pi t)) \) be a sequence of Rademacher functions.

**Khinchin’s inequality:** Let \( 0 < p < \infty \). Then for any sequence \( \{c_k\} \) of complex numbers,

\[
\left( \sum_{k=1}^{\infty} |c_k|^2 \right)^{\frac{1}{p}} \approx \left( \int_0^1 \left( \sum_{k=1}^{\infty} |c_k r_k(t)|^p \right)^{\frac{1}{p}} dt \right)^{\frac{1}{p}}.
\]

**Kahane’s inequality:** Let \( X \) be a quasi-Banach space, and \( 0 < p, q < \infty \). For any sequence \( \{x_k\} \subset X \),

\[
\left( \int_0^1 \left( \sum_{k=1}^{\infty} r_k(t) x_k \right)^p X dt \right)^{\frac{1}{p}} \approx \left( \int_0^1 \left( \sum_{k=1}^{\infty} r_k(t) x_k \right)^q X dt \right)^{\frac{1}{q}}.
\]

Moreover, the implicit constants can be chosen to depend only on \( p \) and \( q \), and independent of the space \( X \).

### 3. Proof of main results

By Theorem 1, Theorem 2 can be stated in a more direct way as follows. Hence, to prove Theorem 2, we only need to prove the following theorem.

**Theorem 1.** Suppose \( 1 < p, q < \infty \), \( \mu \) is a positive Borel measure on \( \mathbb{D} \) and \( \omega, \eta, \nu \in \mathcal{R} \) such that

\[
A(p, \eta) < \infty, \quad A(q, \nu) < \infty.
\]

Then, the following statements hold.
(i) When $1 < p \leq q < \infty$, $\mathcal{T}_\mu^\omega : A^p_\eta \to A^q_\nu$ is bounded if and only if for some (equivalently, for all) $r \in (0, \infty)$,  

$$M_0 = \sup_{z \in \mathbb{D}} \frac{\mu(D(z, r))}{\omega((D(z, r)))} \frac{\nu(D(z, r))^{\frac{1}{q}}}{\eta(D(z, r))^{\frac{p}{q}}} < \infty.$$  

Moreover, $\|\mathcal{T}_\mu^\omega\|_{A^p_\eta \to A^q_\nu} \approx M_0$.  

(ii) When $1 < p \leq q < \infty$, $\mathcal{T}_\mu^\omega : A^p_\eta \to A^q_\nu$ is compact if and only if for some (equivalently, for all) $r \in (0, \infty)$,  

$$\lim_{|z| \to 1} \frac{\mu(D(z, r))}{\omega((D(z, r)))} \frac{\nu(D(z, r))^{\frac{1}{q}}}{\eta(D(z, r))^{\frac{p}{q}}} = 0.$$  

(iii) When $1 < q < p < \infty$, the following statements are equivalent:  

(iiiia) $\mathcal{T}_\mu^\omega : A^p_\eta \to A^q_\nu$ is compact;  

(iiib) $\mathcal{T}_\mu^\omega : A^p_\eta \to A^q_\nu$ is bounded;  

(iiiic) for some (equivalently, for all) separated and $r$-covering $(0 < r < \infty)$ sequence $\{z_j\}_{j=1}^\infty$,  

$$\lambda_j = \frac{\mu(D(z_j, r))}{\omega(D(z_j, r))} \frac{\nu(D(z_j, r))^{\frac{1}{q}}}{\eta(D(z_j, r))^{\frac{p}{q}}}, \quad j = 1, 2, \ldots ,$$  

is a sequence in $l^\frac{p}{q}$;  

(iiiid) for some (equivalently, for all) $r \in (0, \infty)$, $\widehat{\mu}_r \in L^{\frac{p}{q}}_{\omega^{\frac{1}{q}}}$, where  

$$\widehat{\mu}_r(z) = \frac{\mu(D(z, r))}{\omega(D(z, r))} \frac{\nu(D(z, r))^{\frac{1}{q}}}{\eta(D(z, r))^{\frac{p}{q}}}.$$  

Moreover,  

$$\|\mathcal{T}_\mu^\omega\|_{A^p_\eta \to A^q_\nu} \approx \|\{\lambda_j\}_{j=1}^\infty\|_{l^\frac{p}{q}} \approx \|\widehat{\mu}_r\|_{L^{\frac{p}{q}}_{\omega^{\frac{1}{q}}}}.$$  

Proof. For convenience, write $\|\mathcal{T}_\mu^\omega\| = \|\mathcal{T}_\mu^\omega\|_{A^p_\eta \to A^q_\nu}$. Since (5) holds, by Lemma 2 and Theorem 3 we have that $\sigma_{p, \eta} \in \mathcal{R}$, $\sigma_{q, \nu} \in \mathcal{R}$ and  

$$(A^p_{\sigma_{p, \eta}})^* \approx A^p_\eta, \quad (A^q_{\sigma_{q, \nu}})^* \approx A^q_\nu$$  

(8) via $A^2_\omega$-pairing with equivalent norms.  

(i). Suppose $1 < p \leq q < \infty$ and $\mathcal{T}_\mu^\omega : A^p_\eta \to A^q_\nu$ is bounded. By (4), we have  

$$|\mathcal{T}_\mu^\omega B^\omega_{\epsilon}(z)| = \left|\langle \mathcal{T}_\mu^\omega B^\omega_{\epsilon}, B^\omega_{\epsilon}\rangle_{A^q_\nu} \right| \leq \|\mathcal{T}_\mu^\omega B^\omega_{\epsilon}\|_{A^q_\nu}\|B^\omega_{\epsilon}\|_{A^q_\nu} \leq \|\mathcal{T}_\mu^\omega\结构调整 \cdot \|B^\omega_{\epsilon}\|_{A^q_\nu}\|B^\omega_{\epsilon}\|_{A^q_\nu} \approx \|\mathcal{T}_\mu^\omega\|\eta(S)^{\frac{1}{p}} \frac{1}{\omega(S)^{\frac{1}{q}}}. $$  

Meanwhile, by [17, Lemma 8], there exists a real number $r > 0$ such that  

$$|B^\omega_{\epsilon}(\xi)| \approx B^\omega_{\epsilon}(z) \quad \text{for all} \quad \xi \in D(z, r) \quad \text{and} \quad z \in \mathbb{D}. $$  

(9)
Then, (4) implies
\[ |B^ω_z(z)| = (B^ω_z, B^ω_z)_{A^ω_0} = \|B^ω_z\|_{A^ω_0}^2 \approx \frac{1}{\omega(S_z)}. \]
Thus,
\[ (T^ω_\mu B^ω_z)(z) = \int_D |B^ω_z(\xi)|^2 d\mu(\xi) \geq \frac{\mu(D(z,r))}{\omega(S_z)^2}. \] (10)
Therefore, using (1),
\[ D \approx \frac{\mu(D(z,r))}{\omega(S_z)^2}. \]

Conversely, suppose (6) holds. Let
\[ W(z) = \eta(z) \frac{\omega(S_z)^{-\frac{1}{2}}}{\xi} \sigma_{q,v}(z)^{\frac{1}{2}}. \] (11)
By Hölder’s inequality and \( \sigma_{q,v} \in \mathcal{R} \),
\[ \hat{W}(t) \leq \hat{\eta}(t) \frac{\omega(S_z)^{-\frac{1}{2}}}{\xi} \sigma_{q,v}(t)^{\frac{1}{2}} \approx (1-t)W(t), \]
and
\[ \hat{W}(t) \geq \int_t^{t+1} W(r)dr \approx (1-t)W(t). \]
Therefore, \( W \in \mathcal{R} \) and
\[ W(S_z) \approx \omega(S_z) \eta(S_z)^{\frac{1}{2}}. \]
So, for any \( f \in A_\eta^p \) and \( h \in A_{q,v}^q \), by (6), (11), Theorem A and H"older’s inequality,
\[ \int_D |h(z)f(z)|d\mu(z) \leq M_0 \left( \int_D |f(z)|^{\frac{p^*}{q^*}} |h(z)|^{\frac{q^*}{q}} W(z)dA(z) \right)^{\frac{q}{q^*}} \leq M_0 \|f\|_{A^p_\eta} \|h\|_{A^q_{q,v}}. \]
By Fubini’s theorem and \( (A_{q,v}^q)^* \approx A^q_0 \) via \( A^q_0 \)-pairing, we have
\[ \|T^\omega_\mu f\|_{A^q_0} \approx \sup_{h \in A_{q,v}^q} \frac{|(h, T^\omega_\mu f)_{A^q_0}|}{\|h\|_{A_{q,v}^q}} = \sup_{h \in A_{q,v}^q} \frac{|\int_D h(z)f(z)d\mu(z)|}{\|h\|_{A_{q,v}^q}} \leq M_0 \|f\|_{A^q_0}, \]
i.e., \( \|T^\omega_\mu\| \leq M_0. \)

(ii) Suppose \( 1 < p \leq q < \infty \) and \( T^\omega_\mu : A_\eta^p \rightarrow A^q_0 \) is compact. Let
\[ b_\eta(w) = \frac{B^\omega_z(w)}{\|B^\omega_z\|_{A^q_0}}. \]
By (4) and \( \sigma_{p,q} \in \mathcal{R} \), \( \{b_\eta\} \) is bounded in \( A^q_0 \) and converges to 0 uniformly on compact subsets of \( D \) as \( |z| \rightarrow 1 \). By Lemma \[ \lim_{|z| \rightarrow 1} \|T^\omega_\mu b_\eta\|_{A^q_0} = 0. \] Then, (7) is obtained from (11), (4),
\[ |T^\omega_\mu b_\eta(z)| = \left| (T^\omega_\mu b_\eta, B^\omega_z)_{A^q_0} \right| \leq \|T^\omega_\mu b_\eta\|_{A^q_0} \|B^\omega_z\|_{A^q_0} \approx \|T^\omega_\mu b_\eta\|_{A^q_0} \frac{1}{\omega(S_z)^{\frac{1}{2}}}. \]
By Khinchin’s inequality, we have
\[ |T_\mu^\omega b_k(z)| = \frac{1}{\|B_z\|_{A^\mu_0}} \int_{\mathbb{D}} |B_z^\omega(\xi)|^2 d\mu(\xi) \gtrsim \frac{\omega(D(z, r)) \mu(D(z, r))}{\eta(D(z, r))^{\frac{1}{p}}} \omega(D(z, r))^2. \]

Here, \( r \) is that in (9).

Conversely, suppose (7) holds. For any \( s \in (0, 1) \), let
\[ d\mu_s = \chi_s(z) d\mu(z), \quad d\mu_s = d\mu - d\mu_s \]
and
\[ M_s = \sup_{z \in \mathbb{D}} \frac{\mu_s(D(z, r))}{\omega(D(z, r))^{\frac{1}{p}}} \nu(D(z, r))^{\frac{1}{q}}. \]

Here, \( \chi_s(z) \) is the characteristic function of \( \{ z \in \mathbb{D} : |z| \geq s \} \). Then, \( \lim_{s \to 1} M_s = 0 \).

Let \( \{ f_n \} \) be bounded in \( A^\mu_q \) and converge to 0 uniformly on compact subsets of \( \mathbb{D} \) as \( n \to \infty \). Then, by statement (i), we have
\[ \lim_{n \to \infty} \| T_\mu^\omega f_n \|_{A^\mu_q} \leq \lim_{n \to \infty} \| T_{\mu_s}^\omega f_n \|_{A^\mu_q} + \lim_{n \to \infty} \| T_{\mu_{s_0}}^\omega f_n \|_{A^\mu_q} \leq C(s) \limsup_{n \to \infty} |f_n(z)| + M_s \| f_n \|_{A^\mu_q} = M_s \| f_n \|_{A^\mu_q}. \]

Letting \( s \to 1 \), \( \lim_{n \to \infty} \| T_\mu^\omega f_n \|_{A^\mu_q} = 0 \). By Lemma 1, \( T_\mu^\omega : A^\mu_q \to A^\mu_q \) is compact.

(iii), (iiia) \( \Rightarrow \) (iiib). It is obvious.

(iiib) \( \Rightarrow \) (iiic). Suppose \( T_\mu^\omega : A^\mu_q \to A^\mu_q \) is bounded. Let \( \{ z_j \}_{j=1}^{\infty} \subset \mathbb{D} \) be separated and \( r \)-covering. Firstly, we choose \( r \in (0, \infty) \) as that in (9). For any \( c = \{ c_j \}_{j=1}^{\infty} \in l^p \) and \( r_j(t) = \text{sign} \left( \sin \left( \frac{2}{t} \right) \right) \), let
\[ F_i(z) = \sum_{j=1}^{\infty} c_j r_j(t) \frac{B_z^\omega}{\| B_z^\omega \|_{A^\mu_q}}. \]

Then, Lemma 5 implies
\[ \| T_\mu^\omega F_i \|^q_{A^\mu_q} \leq \| T_\mu^\omega \|^q \| c \|^q_{l^p}. \]

By Khinchin’s inequality, we have
\[
\int_0^1 \| T_\mu^\omega F_i \|^q_{A^\mu_q} dt = \int_{\mathbb{D}} \left( \int_0^1 \left| \sum_{j=1}^{\infty} c_j r_j(t) \frac{\| T_\mu^\omega B_z^\omega(z) \|^q}{\| B_z^\omega \|_{A^\mu_q}} \right| dt \right) \nu(z) dA(z)
\geq \int_{\mathbb{D}} \left( \sum_{j=1}^{\infty} \frac{|c_j|^q \| T_\mu^\omega B_z^\omega(z) \|^q}{\| B_z^\omega \|_{A^\mu_q}^q} \right)^{\frac{q}{2}} \nu(z) dA(z)
\geq \sum_{j=1}^{\infty} \frac{|c_j|^q}{\| B_z^\omega \|_{A^\mu_q}^q} \int_{D(z, r)} |T_\mu^\omega B_z^\omega(z)|^q \nu(z) dA(z).
\]
Then, by (1), (4), (10) and subharmonicity of $|T_\mu B_c(z_j)|$, we have

$$\int_0^1 ||T_\mu F_i||_{A_0^q}^q \, dt \geq \sum_{j=1}^{\infty} |c_j|^q \left( \frac{\mu(D(z_j, r)) \nu(D(z_j, r))^{\frac{1}{p}}}{\eta(D(z_j, r))^{\frac{1}{p}}} |T_\mu B_c(z_j)| \right)^q$$

$$\geq \sum_{j=1}^{\infty} |c_j|^q \left( \frac{\mu(D(z_j, r)) \nu(D(z_j, r))^{\frac{1}{p}}}{\eta(D(z_j, r))^{\frac{1}{p}}} \right)^q.$$

So, for all $c = \{c_j\}_{j=1}^{\infty} \in l^p$, we have

$$\sum_{j=1}^{\infty} |c_j|^q \leq ||T_\mu||^q ||c||_p^q.$$

The classical duality relation $(l^p)^* \simeq l^{\frac{p}{p-q}}$ implies that $||\lambda_j|| \in l^{\frac{p}{p-q}}$ and

$$||\lambda_j||_{l^{\frac{p}{p-q}}} \leq ||T_\mu||.$$

Suppose $r_0 > r$ and $\{z_j\}_{j=1}^{\infty}$ is separated and $r_0$-covering. Let $\chi(j, i) = 1$ when $D(z_j, r_0) \cap D(z_i, r) \neq \emptyset$ and $\chi(i, j) = 0$ otherwise. It is easy to check that there exists a natural number $K$ such that

$$1 \leq \sum_{j=1}^{\infty} \chi(j, i) \leq K, \quad 1 \leq \sum_{i=1}^{\infty} \chi(j, i) \leq K.$$

Then

$$\sum_{j=1}^{\infty} \left| \frac{\mu(D(z_j, r_0)) \nu(D(z_j, r_0))^{\frac{1}{p}}}{\omega(D(z_j, r_0)) \eta(D(z_j, r_0))^{\frac{1}{p}}} \chi(j, i) \right| \leq \sum_{j=1}^{\infty} \left| \frac{\mu(D(z_i, r)) \nu(D(z_i, r))^{\frac{1}{p}}}{\omega(D(z_i, r)) \eta(D(z_i, r))^{\frac{1}{p}}} \chi(j, i) \right|$$

$$\geq \sum_{i=1}^{\infty} \left| \frac{\mu(D(z_i, r)) \nu(D(z_i, r))^{\frac{1}{p}}}{\omega(D(z_i, r)) \eta(D(z_i, r))^{\frac{1}{p}}} \chi(j, i) \right|$$

$$\geq \sum_{i=1}^{\infty} \left| \frac{\mu(D(z_i, r)) \nu(D(z_i, r))^{\frac{1}{p}}}{\omega(D(z_i, r)) \eta(D(z_i, r))^{\frac{1}{p}}} \chi(j, i) \right|.$$

Therefore, (iiiic) holds and $||\lambda_j||_{l^{\frac{p}{p-q}}} \leq ||T_\mu||$ for all $0 < r < \infty$.

(iiiic) $\Rightarrow$ (iid). Suppose (iiiic) holds. Let $\{z_j\}_{j=1}^{\infty}$ be separated and $s$-covering. For convenience, let

$$W_1(z) = \frac{\omega(D(z, r))^{1+\frac{1}{q} - \frac{s}{p}} \eta(D(z, r))^{\frac{1}{q}}}{\nu(D(z, r))^{\frac{1}{q}}}.$$
Then
\[
\|\hat{\mu}_r\|_{L_{1,\mu}^{\frac{\mu}{p}}} \leq \sum_{j=1}^{\infty} \int_{D(z_j, r)} \left( \frac{\mu(D(z, r))}{W_1(z)} \right)^{\frac{\mu}{p-q}} \omega(z) dA(z)
\]
\[
\leq \sum_{j=1}^{\infty} \left( \frac{\mu(D(z_j, r+s))}{W_1(z_j)} \right)^{\frac{\mu}{p-q}} \omega(D(z_j, r+s))
\]
\[
\approx \sum_{j=1}^{\infty} \left( \frac{\mu(D(z_j, r+s))}{\omega(D(z_j, r+s))} \right)^{\frac{\mu}{p-q}} \mu(D(z_j, r+s))^{\frac{\mu}{p-q}} \cdot \omega(D(z_j, r+s))^{\frac{\mu}{p-q}}
\].

Therefore, \(\hat{\mu}_r \in L_{1,\mu}^{\frac{\mu}{p}}\) and
\[
\|\hat{\mu}_r\|_{L_{1,\mu}^{\frac{\mu}{p}}} \leq \|\lambda_j\|_{L_{\mu}^{\frac{\mu}{p}}}. 
\]

(iii) \(\Rightarrow\) (iiib). Suppose (iiic) holds. Let \(W\) be defined as that in (11). Then we have
\[
\int_D \left( \frac{\mu(D(z, r))}{W(D(z, r))} \right)^{\frac{\mu}{p-q}} W(z) dA(z) \approx \|\hat{\mu}_r\|_{L_{1,\mu}^{\frac{\mu}{p}}} < \infty.
\]

So, by Theorem [A] \(Id : A_{W}^{\frac{\mu}{p-q}, q} \rightarrow L_{\mu}^{1}\) is bounded and
\[
\|Id\|_{A_{W}^{\frac{\mu}{p-q}, q} \rightarrow L_{\mu}^{1}} \approx \|\hat{\mu}_r\|_{L_{1,\mu}^{\frac{\mu}{p}}}.
\]

Therefore, for any \(f \in A_q^p\) and \(h \in A_{p-q, \mu}^q\), by Fubini’s theorem and Hölder’s inequality, we have
\[
|h, T_{\mu}^\omega f|_{A_q^p} \leq \int_D |h(z)f(z)| d\mu(z)
\]
\[
\leq \|\hat{\mu}_r\|_{L_{1,\mu}^{\frac{\mu}{p-q}}} \left( \int_D |f(z)h(z)|^{\frac{\mu}{p-q}} W(z) dA(z) \right)^{\frac{p-q}{\mu}}
\]
\[
\leq \|\hat{\mu}_r\|_{L_{1,\mu}^{\frac{\mu}{p-q}}} \|f\|_{A_q^p} \|h\|_{A_{p-q, \mu}^q}.
\]

Then, (13) implies \(\|T_{\mu}^\omega\| \leq \|\hat{\mu}_r\|_{L_{1,\mu}^{\frac{\mu}{p-q}}}\).

(iiib) \(\Rightarrow\) (iiia). By [20] Theorem 3.2, \(A_q^p\) and \(A_{p-q, \mu}^q\) are isomorphic to \(l^p\) and \(l^q\), respectively. Theorem 1.2.7 in [7] shows that every bounded operator from \(l^p\) to \(l^q\) is compact when \(0 \leq q < p < \infty\). Therefore, (iiib) \(\Rightarrow\) (iiia). The proof is complete.

Proof of Theorem 2. Suppose \(\mu\) is a \(\lambda\)-Carleson measure for \(A_{\omega_1}^1\) and \(n \geq 2\). Let \(h_i \in A_{\omega_i}^{p_i/q_i}(i = 1, 2, \cdots, n)\). By Hölder’s inequality,
\[
\left\| \prod_{i=1}^{n} h_i \right\|_{A_{\omega_1}^{1/\mu}} \leq \left( \int_D \prod_{i=1}^{n} \left( |h_i(z)|^{q_i} \omega_i(z)^{\frac{q_i}{p_i}} \right) dA(z) \right)^{\frac{1}{q}} \leq \prod_{i=1}^{n} \|h_i\|_{A_{\omega_i}^{p_i/q_i}}.
\]
Then, letting $C_{\mu,\omega,\lambda} = \|I_d\|_{A^{1/\gamma}_{\omega} \to L^{1}_{\mu}}$, we have
\[
\left\|\sum_{i=1}^{n} h_{i}\right\|_{p} \leq C_{\mu,\omega,\lambda} \left\|\sum_{i=1}^{n} h_{i}\right\|_{A^{1/\gamma}_{\omega}} \leq C_{\mu,\omega,\lambda} \prod_{i=1}^{n} \|h_{i}\|_{A^{p_{i}/q_{i}}_{\omega}}. \tag{12}
\]
Let
\[
d\mu_{1}(z) = \prod_{i=2}^{n} |h_{i}(z)| \ d\mu(z).
\]
Then, Theorem A and (12) imply
\[
\|I_d\|_{A^{p_{1}/q_{1}}_{\omega_{1}} \to L^{p_{1}/q_{1}}_{\mu}} \approx \|I_d\|_{A^{p_{1}/q_{1}}_{\omega_{1}} \to L^{1}_{\mu}} \leq C_{\mu,\omega,\lambda} \prod_{i=2}^{n} \|h_{i}\|_{A^{p_{i}/q_{i}}_{\omega}}. \tag{13}
\]
Therefore, for all $f_{1} \in A^{p_{1}}_{\omega}$,
\[
\int_{\mathbb{D}} |f_{1}(z)|^{q_{1}} d\mu_{1}(z) \leq \|I_d\|_{A^{p_{1}/q_{1}}_{\omega_{1}} \to L^{1}_{\mu}} \|f_{1}\|_{A^{p_{1}}_{\omega_{1}}} \leq C_{\mu,\omega,\lambda} \prod_{i=2}^{n} \|h_{i}\|_{A^{p_{i}/q_{i}}_{\omega}} \|f_{1}\|_{A^{p_{1}}_{\omega_{1}}}. \tag{13}
\]
Similarly, letting
\[
d\mu_{2}(z) = |f_{1}(z)|^{q_{1}} |h_{3}(z)h_{4}(z) \cdots h_{n}(z)| \ d\mu(z),
\]
by Theorem A and (13), we have
\[
\|I_d\|_{A^{p_{2}/q_{2}}_{\omega_{2}} \to L^{p_{2}/q_{2}}_{\mu}} \approx \|I_d\|_{A^{p_{2}/q_{2}}_{\omega_{2}} \to L^{1}_{\mu}} \leq C_{\mu,\omega,\lambda} \prod_{i=3}^{n} \|h_{i}\|_{A^{p_{i}/q_{i}}_{\omega}}.
\]
and, for all $f_{2} \in A^{p_{2}}_{\omega_{2}}$,
\[
\int_{\mathbb{D}} |f_{2}(z)|^{q_{2}} d\mu_{2}(z) \leq \|I_d\|_{A^{p_{2}/q_{2}}_{\omega_{2}} \to L^{1}_{\mu}} \|f_{2}\|_{A^{p_{2}}_{\omega_{2}}} \leq C_{\mu,\omega,\lambda} \prod_{i=3}^{n} \|h_{i}\|_{A^{p_{i}/q_{i}}_{\omega}}.
\]
Continuing this process, we get (13) and $M_{n} \approx \|I_d\|_{A^{1/\gamma}_{\omega} \to L^{1}_{\mu}}$.

Conversely, suppose (13) holds. When $\lambda \geq 1$, by Lemma 2.4 in [11], we can choose $\gamma$ large enough such that
\[
\|F_{a}\|_{A^{p}_{\omega}} \approx \omega_{\lambda}(S_{a}), \quad \text{for all} \ i = 1, 2, \cdots, n \quad \text{and} \ a \in \mathbb{D},
\]
where $F_{a}(z) = \left(1 - \frac{|a|^{2}}{1 - |a|^{2}}\right)^{\gamma}$. For any $z \in S_{a}$ and $a \in \mathbb{D}$, we have $|1 - az| \approx 1 - |a|$. So, (13) implies
\[
\mu(S_{a}) \lesssim \int_{\mathbb{D}} \left(1 - \frac{|a|^{2}}{1 - |a|^{2}}\right)^{\gamma(q_{1} + q_{2} + \cdots + q_{n})} d\mu(z) \lesssim M_{n} \omega_{\lambda}(S_{a})^{\lambda}.
\]
By Theorem A, $\mu$ is a $\lambda$-Carleson measure for $A^{1}_{\omega}$ and $M_{n} \gtrsim \|I_d\|_{A^{1/\gamma}_{\omega} \to L^{1}_{\mu}}$.  

So, we only need to prove the case of $0 < \lambda < 1$. We use induction on $n$. When $n = 1$, it is just the definition of Carleson measure for Bergman spaces. Now, let $n \geq 2$ and the result holds for $n - 1$ functions. Set

\[ \lambda_{n-1} = \sum_{i=1}^{n-1} \frac{q_i}{p_i}, \quad \eta(z) = \prod_{i=1}^{n-1} \omega_i(z)^{\frac{q_i}{n-1}}, \quad d\mu_{f_n,q_n}(z) = |f_n(z)|^{q_n} d\mu(z), \]

and

\[ M_{n-1,f_n,q_n} = \sup_{f \in A^q_{\lambda_n}} \frac{\int_{D_n} \prod_{i=1}^{n-1} |f(z)|^{q_i} d\mu_{f_n,q_n}(z)}{\prod_{i=1}^{n-1} \|f\|_{A^{q_{p_i}}}}. \]

Then, (3) and the assumption imply

\[ \|I_d\|_{A^{\lambda_n}_{\lambda_n-1} \rightarrow L^{q_n}_{f_n,q_n}} \approx M_{n-1,f_n,q_n} \leq M_n \|f\|_{A^{q_n}_{\lambda_n}}. \]

Since $\lambda_{n-1} < 1$, by Theorem [A] if $\gamma$ is large enough and fixed, we have

\[ S(f_n) := \int_{D_n} \left( \int_{D_n} \left( \frac{1 - |\xi|^\gamma}{|1 - z\xi|^\gamma} \right)^{\frac{1}{\gamma}} |f_n(\xi)|^{q_n} d\mu(\xi) \right)^{\frac{1}{q_n}} \eta(z) dA(z) \]

\[ \leq \lambda_{n-1}^{1-n} \|f\|_{A^{q_n}_{\lambda_n}}. \]  

(14)

Let $\delta$ be large enough, $(a_k)_{k=1}^\infty$ be separated and $r$-covering,

\[ dV(\xi) = \frac{(1 - |\xi|^\gamma)}{|1 - z\xi|^\gamma} \frac{d\mu(\xi)}{(1 - |\xi|^2)\eta(\xi)}, \quad b_{ak} = \frac{1}{\omega_n(S_{ak})} \left( 1 - \frac{|a_k|}{1 - \overline{a}_k z} \right)^\delta. \]

Then, for any $c = \{c_k\} \in L^{p_0}_n$, from Theorem 3.2 in [20], we have

\[ \|f\|_{A^{q_n}_{\lambda_n}} \leq \|c\|_{L^{p_0}_n}, \]

where $f_n(\xi) = \sum_{k=1}^\infty c_k r_k(t) b_{ak}(\xi)$. Then, by Fubini’s theorem, Kahane’s inequality and Khinchin’s inequality, we have

\[ \int_0^1 S(f(t)) dt = \int_D \int_0^1 \left( \sum_{k=1}^\infty c_k r_k(t) b_{ak}(\xi) \right)^{\frac{q_n}{1 - \lambda_{n-1}}} dt \eta(z) dA(z) \]

\[ \geq \int_D \left( \int_0^1 \left( \sum_{k=1}^\infty c_k r_k(t) b_{ak}(\xi) \right)^{\frac{q_n}{1 - \lambda_{n-1}}} dt \right)^{\frac{1}{1 - \lambda_{n-1}}} \eta(z) dA(z) \]

\[ \geq \int_D \left( \int_0^1 \left( \sum_{k=1}^\infty |c_k|^2 b_{ak}(\xi)^2 \right)^{\frac{q_n}{2}} dV(\xi) \right)^{\frac{1}{1 - \lambda_{n-1}}} \eta(z) dA(z) \]

\[ \geq \sum_{j=1}^\infty \int_{D(a_j,r)} \left( \int_{D(a_j,r)} (|c_j|^2 b_{aj}(\xi)^2)^{\frac{q_n}{2}} dV(\xi) \right)^{\frac{1}{1 - \lambda_{n-1}}} \eta(z) dA(z) \]
So, for any given \( \varepsilon > 0 \), by Theorem 2, we have
\[
\limsup_{n \to \infty} \left( \frac{\mu(D(a_j, r))}{\omega_n(S_{a_j})^{1/\alpha_n}} \right)^{1/\alpha_n} \leq \frac{1}{\varepsilon}.
\]
Set
\[
C_j = \left\{ \frac{\mu(D(a_j, r))}{\omega(D(a_j, r))^{1/\alpha}} \right\}^{1/\alpha}.
\]
From (14) and the classical duality relation
\[
\left( I^{1/\alpha_n} \right)^* \approx I^{1/\alpha_n}
\]
the duality relation of the classical form
\[
\left\{ \left( \frac{\mu(D(a_k, r))}{\omega(D(a_k, r))^{1/\alpha}} \right)^{1/\alpha} \right\}^{1/\alpha_n} \leq M_n \left( \frac{1}{\varepsilon} \right)^{1/\alpha_n}.
\]
Then, by Theorem [A] we have
\[
\lim_{n \to \infty} \left( \frac{\mu(D(a_n, r))}{\omega(D(a_n, r))^{1/\alpha}} \right)^{1/\alpha_n} \approx \left( \int_{D_n} \frac{\mu(D(z, r))}{\omega(D(z, r))^{1/\alpha}} \omega(z) dA(z) \right)^{1/\alpha_n} \approx \left( \sum_{k=1}^{\infty} \frac{\mu(D(a_k, r))}{\omega(D(a_k, r))^{1/\alpha}} \right)^{1/\alpha_n} \leq M_n.
\]
The proof is complete. \( \square \)

The proof of Theorem 3 can be deduced in a standard way, see the proof of Theorem 4.1 in [9] for example. For the benefits of readers, we prove it here.

**Proof of Theorem 3.** (i) \( \Rightarrow \) (ii). Suppose (i) holds, i.e., \( \mu \) is a vanishing \( \lambda \)-Carleson measure for \( A_{\lambda} \). If \( s \in (0, 1) \), let \( \chi_s(z) \) be the characteristic function of \( \{ z \in D : |z| \geq s \} \) and \( d\mu_s(z) = \chi_s(z) d\mu(z) \). By Theorem A, we have
\[
\lim_{s \to 1} \|I_d\|_{A_{\lambda}^{1/1} \to L_p^{1/1}} = \lim_{s \to 1} \|I_d\|_{A_{\lambda}^{1/1} \to L_p^{1/1}} = 0.
\]
By Theorem [2] we have
\[
\lim_{s \to 1} \sup_{f \in A_{\lambda}^{q_1}, \ldots, t, n} \frac{\|f(z)|d\mu(z)\|_{A_{\lambda}^{q_1}}}{\prod_{j=1}^{\alpha} \|f(z)|d\mu(z)\|_{A_{\lambda}^{q_1}}} = 0.
\]
So, for any given \( \varepsilon > 0 \), there exists a real number \( s_0 \in (0, 1) \) such that
\[
\sup_{f \in A_{\lambda}^{q_1}, \ldots, t, n} \frac{\|f(z)|d\mu(z)\|_{A_{\lambda}^{q_1}}}{\prod_{j=1}^{\alpha} \|f(z)|d\mu(z)\|_{A_{\lambda}^{q_1}}} < \varepsilon.
\]
Meanwhile, when \( i = 2, \ldots, n \), for any \( \|f_i\|_{A_{\lambda}^{q_1}} \leq 1 \) and \( |z| \leq s_0 \), it is easy to check that
\[
|f_i(z)| \leq \frac{\|f_i\|_{A_{\lambda}^{q_1}}}{(1 - |z|)^{\frac{1}{\alpha}} \omega_i(z)} \leq \frac{1}{(1 - s_0)^{\frac{1}{\alpha}} \omega_i(s_0)}.
\]
Since \( \{ f_{1,k} \}_{k=1}^{\infty} \) converges to 0 uniformly on \( s_0 \mathbb{D} \), there exists a natural number \( K \) such that, for all \( k > K \),
\[
\int_{s_0 \mathbb{D}} |f_{1,k}(z)|^{\nu_1} \prod_{i=2}^{n} |f_i(z)|^{\nu_i} \, d\mu(z) < \varepsilon. \tag{16}
\]
Then, (15) and (16) imply \( \lim_{k \to \infty} F(k) = 0 \), i.e., (ii) holds.

(ii) \( \Rightarrow \) (iii). It is obvious.

(iii) \( \Rightarrow \) (i). Suppose (iii) holds. Then, \( \mu \) is a \( \lambda \)-Carleson measure for \( A_{\omega}^{1} \). Otherwise, by Theorem [2] there exist \( \{ f_{1,k} \}_{k=1}^{\infty}, \cdots, \{ f_{n,k} \}_{k=1}^{\infty} \) in the unit ball of \( A_{\nu_1}^{p_1}, \cdots, A_{\nu_2}^{p_2} \), respectively, such that
\[
\int_{\mathbb{D}} \prod_{i=1}^{n} \left| \frac{f_{i,k}(z)}{k} \right|^{\nu_i} \, d\mu(z) > 1.
\]
Meanwhile, since \( \lim_{k \to \infty} \| \frac{f_{1,k}}{k} \|_{A_{\nu_1}^{p_1}} = 0 \), by statement (iii), we have
\[
\int_{\mathbb{D}} \prod_{i=1}^{n} \left| \frac{f_{i,k}(z)}{k} \right|^{\nu_i} \, d\mu(z) = 0.
\]
This is a contradiction. So, \( \mu \) is a \( \lambda \)-Carleson measure for \( A_{\omega}^{1} \). Moreover, when \( 0 < \lambda < 1 \), by Theorem [3], \( I_{\lambda} : A_{\omega}^{1} \to L_{\mu}^{\lambda} \) is compact.

Suppose \( \lambda > 1 \). By Lemma 2.4 in [11], we can choose \( \gamma \) large enough such that
\[
\| F_a \|_{A_{\nu_1}^{p_1}} \approx \omega(S_a), \quad \text{for all } i = 1, 2, \cdots, n \text{ and } a \in \mathbb{D},
\]
where \( F_a(z) = \left( \frac{1-|a|^2}{|1-\overline{a}z|} \right)^{\gamma} \). Therefore, for \( i = 1, 2, \cdots, n \), \( f_{i,a}(z) = \frac{F_a(z)}{\| F_a \|_{A_{\nu_1}^{p_1}}} \) is bounded in \( A_{\nu_i}^{p_i} \) and converges to 0 uniformly on compact subsets of \( \mathbb{D} \) as \( |a| \to 1 \). From (iii), we have
\[
\lim_{|a| \to 1} \int_{\mathbb{D}} \left( \frac{1-|a|}{|1-\overline{a}z|} \right)^{(q_1+q_2+\cdots+q_n)\gamma} \frac{d\mu(z)}{\omega(S_a)^{\lambda}} = 0. \tag{17}
\]
For brief, let \( \gamma_n = (q_1 + q_2 + \cdots + q_n)\gamma \). Then, for any fixed \( r > 0 \), by (1) we have
\[
\frac{\mu(D(a, r))}{\omega(D(a, r))^{\lambda}} \approx \int_{D(a, r)} \left( \frac{1-|a|}{|1-\overline{a}z|} \right)^{\gamma_n} \frac{d\mu(z)}{\omega(S_a)^{\lambda}} \leq \int_{\mathbb{D}} \left( \frac{1-|a|}{|1-\overline{a}z|} \right)^{\gamma_n} \frac{d\mu(z)}{\omega(S_a)^{\lambda}}.
\]
Therefore, (17) and Theorem [3] imply that \( \mu \) is a vanishing \( \lambda \)-Carleson measure for \( A_{\omega}^{1} \). The proof is complete.

\[\square\]

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