ON THE NUMBER OF AFFINE EQUIVALENCE CLASSES OF
BOOLEAN FUNCTIONS AND $q$-ARY FUNCTIONS

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Abstract. Let $R_q(r,n)$ be the $r$th order $q$-ary Reed-Muller code of length $q^n$, which is the set of functions from $F_q^n$ to $F_q$ represented by polynomials of degree $\leq r$ in $F_q[X_1,\ldots,X_n]$. The affine linear group $AGL(n,F_q)$ acts naturally on $R_q(r,n)$. We derive two formulas concerning the number of orbits of this action: (i) an explicit formula for the number of AGL orbits of $R_q(n(q-1),n)$, and (ii) an asymptotic formula for the number of AGL orbits of $R_2(n,n)/R_2(1,n)$. The number of AGL orbits of $R_2(n,n)$ has been numerically computed by several authors for $n \leq 31$; the binary case of result (i) is a theoretic solution to the question. Result (ii) answers a question by MacWilliams and Sloane.

1. Introduction

Let $F_q$ be the finite field with $q$ elements and let $F_q^n = \{f : f:X_i \rightarrow F_q, 1 \leq i \leq n\}$ denote the set of all functions from $F_q^n$ to $F_q$. Every $g \in F_q^n$ is (uniquely) represented by a polynomial $f \in F_q[X_1,\ldots,X_n]$ with $\deg_X f < q$ for all $1 \leq i \leq n$; we define $\deg g = \deg f$. We shall not distinguish a function from $F_q^n$ to $F_q$ and a polynomial in $F_q[X_1,\ldots,X_n]$ that represents it. For $-1 \leq r \leq n(q-1)$, the $r$th order Reed-Muller code of length $q^n$ is

\begin{equation}
R_q(r,n) = \{f \in F_q^n : \deg f \leq r\}.
\end{equation}

Note that $F_q^n = R_q(n(q-1),n)$. Let

\begin{equation}
AGL(n,F_q) = \left\{ \begin{bmatrix} A & 0 \\ a & 1 \end{bmatrix} : A \in GL(n,F_q), a \in F_q^n \right\}
\end{equation}

be the affine linear group of degree $n$ over $F_q$. The set $F_q^n$ is an $F_q$-algebra on which $AGL(n,F_q)$ acts as automorphisms: For $\alpha = [A,a] \in AGL(n,F_q)$ and $f(X_1,\ldots,X_n) \in F_q^n$, $\alpha(f) = f((X_1,\ldots,X_n)a)$. Consequently, $AGL(n,F_q)$ acts on $R_q(r,n)$ and on $R_q(r,n)/R_q(s,n)$ for $-1 \leq s \leq r \leq n(q-1)$.

When $q = 2$, we write $R_2(r,n) = R(r,n)$. In this case, $F_2^n = R(n,n)$ is the set of all Boolean functions in $n$ variables. When two Boolean functions are said to be equivalent, it is meant, depending on different authors, that $f$ and $g$ are in the same AGL orbit of $F_2^n$, or $f + R(1,n)$ and $g + R(1,n)$ are in the same AGL orbit of $R(n,n)$ or $R(1,n)$ (The affine equivalence in the latter sense is referred to as extended affine equivalence in [2]). Most coding theoretic and cryptographic

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properties of Boolean functions are preserved under affine equivalence. Let \( N_{q,n} \) and \( M_n \) denote the number of AGL orbits of \( F(F_q^n, F_q^n) \) and the number of AGL orbits of \( R(n, n)/R(1, n) \), respectively. The number \( N_{q,n} \) has been computed by several authors for \( q = 2 \) and \( n \) up to 31 [3, 4, 9, 14, 15]. We will derive an explicit formula for \( N_{q,n} \), hence providing a theoretic solution to the question. Our approach differs from those in some previous works in that we do not use the cycle index, the generating function in Pólya’s counting; rather, we find direct application of Burnside’s lemma more suitable for this particular question. The number \( M_n \) has also been studied by a number of authors [1, 4, 7, 8, 11, 13]. (In fact, the number of AGL orbits of \( R(r, n)/R(s, n) \) has been computed recently for all \(-1 \leq s < r \leq n \leq 10 \) [13].) In [8], Maiorana not only computed \( M_6 \), but also classified \( R(6, 6)/R(1, 6) \). However, no explicit formula for \( M_n \) is known in general. An open question by MacWilliams and Sloane [7, Research Problem (14.2)] asks how fast the number \( M_n \) grows with \( n \). We will give an asymptotic formula for \( M_n \) as \( n \to \infty \). Both \( N_{2,n} \) and \( M_n \) are important sequences; in the On-line Encyclopedia of Integer sequences [10], they are listed as A000214 and A001289, respectively.

The paper is organized as follows: Section 2 is a review of some mathematical results to be used in the paper. In Section 3, we derive an explicit formula for \( N_{q,n} \). The asymptotic formula for \( M_n \) is proved in Section 4. We conclude the paper with a few brief remarks in Section 5 and a conclusion in Section 6. For readers’ convenience, a list of notations used in the paper is compiled in the appendix.

2. Mathematical Background

2.1. Burnside’s lemma.

Let \( G \) be a finite group acting on a finite set \( X \). For \( x \in X \), the subset \( Gx = \{ ax : a \in G \} \subset X \) is called the \( G \)-orbit of \( x \). The \( G \)-orbits form a partition of \( X \), and the number of \( G \)-orbits is given by the following formula referred to as Burnside’s lemma:

\[
\text{number of } G\text{-orbits} = \frac{1}{|G|} \sum_{a \in G} \text{Fix}(a),
\]

where \( \text{Fix}(a) = |\{ x \in X : ax = x \}| \) is the number of fixed points of \( a \) in \( X \). If \( a, b \in G \) are conjugate to each other, that is, \( b = gag^{-1} \) for some \( g \in G \), then \( \text{Fix}(a) = \text{Fix}(b) \). Let \( a_1, \ldots, a_k \) be the representatives of the conjugacy classes of \( G \) and let \( [a_i] \) denote the conjugacy class of \( a_i \). Then we have \( |[a_i]| = |G|/|c(a_i)| \), where

\[
c(a_i) = \{ g \in G : ga_i = a_ig \}
\]

is the centralizer of \( a_i \) in \( G \). Therefore (2.1) can be more effectively computed as follows:

\[
\text{number of } G\text{-orbits} = \frac{1}{|G|} \sum_{i=1}^k |[a_i]| \text{Fix}(a_i) = \sum_{i=1}^k \frac{\text{Fix}(a_i)}{|c(a_i)|}.
\]

2.2. Rational canonical form of a matrix.

Let \( F \) be any field and let \( f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in F[X] \) be a monic polynomial of degree \( n \). A companion matrix of \( f \) is an \( n \times n \) matrix \( A \) over \( F \)
whose minimal polynomial is \( f \); one can choose
\[
A = \begin{bmatrix}
0 & 1 & 0 & 1 & \cdots & 0 & 1 \\
& & & & & \ddots & \\
& & & & & & & -a_0 - a_1 - \cdots - a_{n-1}
\end{bmatrix},
\]

Every square matrix \( A \) over \( \mathbb{F} \) is similar (conjugate) to a \textit{rational canonical form}
\[
\begin{bmatrix}
A_1 \\
\vdots \\
A_m
\end{bmatrix},
\]
where each \( A_i \) is a companion matrix of some \( f_i \in \mathbb{F}[X] \) which is a power of an irreducible polynomial over \( \mathbb{F} \). The polynomials \( f_1, \ldots, f_m \) are the \textit{elementary divisors} of \( A \). Two square matrices are similar if and only if they have the same list (multiset) of elementary divisors.

\subsection*{2.3. Conjugacy classes of AGL\((n, \mathbb{F}_q)\).}

In this subsection, we describe the representatives of the conjugacy classes of \( \text{AGL}(n, \mathbb{F}_q) \) and recall the formulas for the sizes of the centralizers of these representatives. These results can be found in \cite[§6.4]{6}.

A \textit{partition} is a sequence of nonnegative integers \( \lambda = (\lambda_1, \lambda_2, \ldots) \) with only finitely many nonzero terms. We define \(|\lambda| = \sum_{i \geq 1} i \lambda_i \) and \( T(\lambda) = \{ i : \lambda_i > 0 \} \).

For example, if \( \lambda = (2, 0, 1, 3) \), then \(|\lambda| = 2 \cdot 1 + 0 \cdot 2 + 1 \cdot 3 + 3 \cdot 4 = 17 \) and \( T(\lambda) = \{1, 3, 4\} \). Let \( \mathcal{P} \) denote the set of all partitions. Let \( \mathcal{I} \) be the set of all monic irreducible polynomials in \( \mathbb{F}_q[X] \setminus \{X\} \). For \( f \in \mathcal{I} \) and \( \lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P} \), let \( f^\lambda \) denote the multiset
\[
\{f^1, f^1, f^2, f^2, \ldots\},
\]
i.e., the list with \( \lambda_1 \) copies of \( f \), \( \lambda_2 \) copies of \( f^2 \), and so on. Let \( \sigma f^\lambda \) be an element of \( \text{GL}(|\lambda|, \mathbb{F}_q) \) with elementary divisors \( f^\lambda \). For \( \alpha = [A \; 0] \in \text{AGL}(n_1, \mathbb{F}_q) \) and \( \beta = [B \; 0] \in \text{AGL}(n_2, \mathbb{F}_q) \), define
\[
\alpha \oplus \beta = \begin{bmatrix} A & 0 \\ a & b \\ 1 \end{bmatrix} \in \text{AGL}(n_1 + n_2, \mathbb{F}_q).
\]

Let
\[
N_n = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}_{n \times n}, \quad J_n = I + N_n,
\]
where \( I \) is the identity matrix. For \( \lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P} \), let
\[
\sigma f^\lambda = \bigoplus_{\lambda_1} \left[ \begin{bmatrix} J_1 & 0 \\ 0 & 1 \end{bmatrix} \right] \oplus \bigoplus_{\lambda_2} \left[ \begin{bmatrix} J_2 & 0 \\ 0 & 1 \end{bmatrix} \right] \oplus \cdots \in \text{AGL}(|\lambda|, \mathbb{F})
\]
and, for $t \in T(\lambda)$,

\[
(2.5) \quad \sigma_{\lambda,t} = \left[ \begin{array}{cc} J_1 & 1 \\ \end{array} \right] \oplus \cdots \oplus \left[ \begin{array}{cc} J_1 & 1 \\ \end{array} \right] \oplus \cdots \oplus \left[ \begin{array}{cc} J_{t-1} & 1 \\ \end{array} \right] \oplus \cdots \oplus \left[ \begin{array}{cc} J_{t-1} & 1 \\ \end{array} \right]
\]

where

\[
\epsilon_t = (1, 0, \ldots, 0) \in \mathbb{F}_q^d.
\]

By \cite{[6]} Theorem 6.23, a set of representatives of the conjugacy classes of $\text{AGL}(n, \mathbb{F}_q)$ is given by $C = C_1 \cup C_2$, where

\[
(2.6) \quad C_1 = \left\{ \sigma_{\lambda} \oplus \left( \bigoplus_{f \in \mathcal{I}} \sigma_{f,\lambda_f} \right) : \lambda, \lambda_f \in \mathcal{P}, |\lambda| + \sum_{f \in \mathcal{I}} |\lambda_f| \deg f = n \right\},
\]

\[
(2.7) \quad C_2 = \left\{ \sigma_{\lambda,t} \oplus \left( \bigoplus_{f \in \mathcal{I}} \sigma_{f,\lambda_f} \right) : \lambda, \lambda_f \in \mathcal{P}, |\lambda| > 0, t \in T(\lambda), |\lambda| + \sum_{f \in \mathcal{I}} |\lambda_f| \deg f = n \right\}.
\]

We shall refine the description of $C_1$ and $C_2$ to serve the purpose of the present paper. For $f \in \mathcal{I}$, the order of $f$, denoted by $\deg f$, is the multiplicative order of the roots of $f$. If $\deg f = d$, then $\deg f = \frac{\deg(q)}{d}$. For $d \geq 1$ with $p \nmid d$, the root of $f$ of $\mathbb{F}_q$ is denoted by $d$. Then $d = \frac{\deg(d)}{d}$, where $\deg$ is the Euler totient function. We order the partitions in the following manner: For $\lambda = (\lambda_1, \lambda_2, \ldots), \eta = (\eta_1, \eta_2, \ldots) \in \mathcal{P}$, "$\lambda < \eta$" means that $|\lambda| < |\eta|$, or $|\lambda| = |\eta|$ and for the largest $i$ such that $\lambda_i \neq \eta_i$ we have $\lambda_i < \eta_i$. (Partitions in this particular order can be easily generated by computer.) Let $D = \{d > 1 : d | q^i - 1 \text{ for some } 1 \leq i \leq n \}$. For $d \in D$, let

\[
\Lambda_d = \{(\lambda^{(1)}(d), \ldots, \lambda^{(\psi(d))}(d)) : \lambda^{(1)} \in \mathcal{P}, \lambda^{(1)} \leq \cdots \leq \lambda^{(\psi(d))}\},
\]

and for $\lambda = (\lambda^{(1)}(d), \ldots, \lambda^{(\psi(d))}) \in \Lambda_d$, let $|\lambda| = \sum_{i=1}^{\psi(d)} |\lambda^{(i)}|$. Let

\[
\Omega = \left\{ (\lambda, (\lambda_d)_{d \in D}) : \lambda \in \mathcal{P}, \lambda_d \in \Lambda_d, |\lambda| + \sum_{d \in D} \deg(q)|\lambda_d| = n \right\}.
\]

Then

\[
(2.8) \quad C_1 = \bigcup_{(\lambda, (\lambda_d)_{d \in D}) \in \Omega} \left\{ \sigma_{\lambda} \oplus \left( \bigoplus_{d \in D} \bigoplus_{f \in \mathcal{I}_d} \sigma_{f,\lambda_f} \right) : \lambda_f \in \mathcal{P}, (\lambda_f)_{f \in \mathcal{I}_d} \text{ is a permutation of } \lambda_d \right\}
\]

and

\[
(2.9) \quad C_2 = \bigcup_{(\lambda, (\lambda_d)_{d \in D}) \in \Omega} \left\{ \sigma_{\lambda,t} \oplus \left( \bigoplus_{d \in D} \bigoplus_{f \in \mathcal{I}_d} \sigma_{f,\lambda_f} \right) : t \in T(\lambda), \lambda_f \in \mathcal{P}, (\lambda_f)_{f \in \mathcal{I}_d} \text{ is a permutation of } \lambda_d \right\}.
\]
In $C_1$, let
\begin{equation}
\alpha = \sigma_{\lambda} \bigoplus_{d \in D} \bigg( \bigg( \bigoplus_{f \in I_d} \sigma_{f^{t_d}} \bigg) \bigg),
\end{equation}
and in $C_2$, let
\begin{equation}
\beta = \sigma_{\lambda,t} \bigoplus_{d \in D} \bigg( \bigg( \bigoplus_{f \in I_d} \sigma_{f^{t_d}} \bigg) \bigg).
\end{equation}
The sizes of the centralizers of $\alpha$ and $\beta$ in $\AGL(n, \mathbb{F}_q)$ are given by Theorem 6.24.

Write $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\lambda_d = (\lambda_d^{(1)}, \ldots, \lambda_d^{(\psi(d))})$, where $\lambda_d^{(i)} = (\lambda_{d,1}^{(i)}, \lambda_{d,2}^{(i)}, \ldots) \in \mathcal{P}$. We have
\begin{equation}
|c(\alpha)| := a_1(\lambda, (\lambda_d)_{d \in D}) = q \sum_{j=1}^{\lambda_1} \lambda_j \sum_{j,k \in \mathcal{N}} \min(j,k) \lambda_j \lambda_k + \sum_{d \in D} \min(d(q)) \sum_{i=1}^{\psi(d)} \sum_{j,k \in \mathcal{N}} \min(j,k) \lambda_{d,j}^{(i)} \lambda_{d,k}^{(i)}
\cdot \bigg( \prod_{j \in I_d} (1 - q^{-j}) \bigg) \bigg( \prod_{j \in I_d} \prod_{j \in I_d} (1 - q^{-\min(d(q))}) \bigg),
\end{equation}
\begin{equation}
|c(\beta)| := a_2(t, \lambda, (\lambda_d)_{d \in D}) = \frac{1}{q^{\lambda_t - 1}} q \sum_{j=1}^{\lambda_1} \lambda_j \sum_{j,k \in \mathcal{N}} \min(j,k) \lambda_j \lambda_k + \sum_{d \in D} \min(d(q)) \sum_{i=1}^{\psi(d)} \sum_{j,k \in \mathcal{N}} \min(j,k) \lambda_{d,j}^{(i)} \lambda_{d,k}^{(i)}
\cdot \bigg( \prod_{j \in I_d} (1 - q^{-j}) \bigg) \bigg( \prod_{j \in I_d} \prod_{j \in I_d} (1 - q^{-\min(d(q))}) \bigg).
\end{equation}

Example. In (2.10) and (2.11), let $q = 2$, $\lambda = (3, 0, 1)$, $t = 3$ and
\[ \bigg( \bigoplus_{d \in D} \bigg( \bigoplus_{f \in I_d} \sigma_{f^{t_d}} \bigg) \bigg) = \sigma_{f_1^{t_1}} \bigoplus \sigma_{f_2^{t_2}}, \]
where $f_1 = X^3 + X + 1$, $f_2 = X^3 + X^2 + 1 \in \mathbb{F}_2$, $\lambda_{f_1} = (1, 2)$ and $\lambda_{f_2} = (2, 0, 1)$. Let's see what $\alpha$ and $\beta$ look like. We have $ord f_1 = ord f_2 = 7$, i.e., $f_1, f_2 \in I_7$. Hence $\lambda_7 = (\lambda_7^{(1)}, \lambda_7^{(2)})$, where $\lambda_7^{(1)} = (1, 2)$ and $\lambda_7^{(2)} = (2, 0, 1)$, and $\lambda_d = ((0), \ldots, (0))$ for $7 \neq d \in D$. Let $A_i^{(j)}$ be a companion matrix of $f_i^j$. Then
\[ \alpha = \begin{bmatrix} J_1 \\ 0 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} J_1 \\ 0 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} J_1 \\ 0 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} J_3 \\ 0 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} A_1^{(1)} \\ 0 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} A_1^{(2)} \\ 0 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} A_1^{(2)} \\ 0 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} A_2^{(2)} \\ 0 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} A_2^{(1)} \\ 0 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} A_2^{(1)} \\ 0 & 1 \end{bmatrix}, \]
\[ \beta = \begin{bmatrix} J_1 \\ 0 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} J_1 \\ 0 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} J_1 \\ 0 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} J_3 \\ \epsilon_3 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} A_1^{(1)} \\ 0 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} A_1^{(2)} \\ 0 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} A_1^{(2)} \\ 0 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} A_2^{(2)} \\ 0 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} A_2^{(1)} \\ 0 & 1 \end{bmatrix} \bigoplus \begin{bmatrix} A_2^{(1)} \\ 0 & 1 \end{bmatrix}. \]
Note that $n = |\lambda| + |3| \lambda_7 = 6 + 3(5 + 5) = 36$, so $\alpha, \beta \in \AGL(36, \mathbb{F}_2)$. By (2.12) and (2.13) (with $\sigma_2(2) = 3$),
\[ |c(\alpha)| = 2^{a_2(1, (3, 0, 1))} \cdot (1 - 2^{-1})(1 - 2^{-2})(1 - 2^{-3})(1 - 2^{-1})(1 - 2^{-3})(1 - 2^{-3})(1 - 2^{-3})(1 - 2^{-3}) \]
\[ \cdot (1 - 2^{-3})(1 - 2^{-3})(1 - 2^{-3}) \cdot (1 - 2^{-3}) \]
\[ |c(\beta)| = \frac{1}{2^{21} - 1} 2^{1+3^2+2+2^2+2+2+2+2+2+3+1+3(1+2+2+2+2^2+2+2+2+3+1)} 
\cdot (1 - 2^{-1})(1 - 2^{-2})(1 - 2^{-3})(1 - 2^{-1})(1 - 2^{-3})(1 - 2^{-1})(1 - 2^{-3})(1 - 2^{-1}) 
\cdot (1 - 2^{-3})(1 - 2^{-2})(1 - 2^{-3})(1 - 2^{-3}
\cdot 2^{60} \cdot 3^5 \cdot 7^3. \]

3. A Formula for \( \mathfrak{R}_{q,n} \)

The objective of this section is to derive an explicit formula for \( \mathfrak{R}_{q,n} \), the number of AGL\((n, \mathbb{F}_q) \) orbits of \( \mathcal{F}(\mathbb{F}_q^n, \mathbb{F}_q) \). Although parts of the proof are rather technical, the general strategy is quite simple.

3.1. Strategy.

We shall identify GL\((n, \mathbb{F}_q) \) with the subgroup \( \{ [A, 0] : A \in \text{GL}(n, \mathbb{F}_q) \} \) of AGL\((n, \mathbb{F}_q) \). For \( \gamma = [A, 0] \in \text{AGL}(n, \mathbb{F}_q) \), we can identify \( \gamma \) with the affine map \( \mathbb{F}_q \rightarrow \mathbb{F}_q, x \mapsto xA + a \), whence \( \gamma(f) = f \circ \gamma \) for \( f \in \mathcal{F}(\mathbb{F}_q^n, \mathbb{F}_q) \). For \( \gamma \in \text{AGL}(n, \mathbb{F}_q) \), define

\[ \text{Fix}(\gamma) = \{ f \in \mathcal{F}(\mathbb{F}_q^n, \mathbb{F}_q) : \gamma(f) = f \} \]

and let \( c(\gamma) \) denote the centralizer of \( \gamma \) in AGL\((n, \mathbb{F}_q) \). Let \( \mathcal{C} \) be a set of representatives of the conjugacy classes of AGL\((n, \mathbb{F}_q) \). By Burnside’s lemma,

\[ \mathfrak{R}_{q,n} = \sum_{\gamma \in \mathcal{C}} \frac{\text{Fix}(\gamma)}{|c(\gamma)|}. \]

We choose \( \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \), where \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are given in (2.18) and (2.19), respectively. Then (3.2) becomes

\[ \mathfrak{R}_{q,n} = \sum_{\gamma \in \mathcal{C}_1} \frac{\text{Fix}(\gamma)}{|c(\gamma)|} + \sum_{\gamma \in \mathcal{C}_2} \frac{\text{Fix}(\gamma)}{|c(\gamma)|} \]

\[ = \sum_{(\lambda, (\lambda_d)_{d \in D}) \in \Omega} \sum_{(\lambda_f)_{f \in I_d}, \lambda_f \in \mathcal{P}, (\lambda_f)_{f \in I_d} \text{ is a permutation of } \lambda_d} \frac{\text{Fix}(\alpha)}{|c(\alpha)|} \]

\[ + \sum_{(\lambda, (\lambda_d)_{d \in D}) \in \Omega} \sum_{(\lambda_f)_{f \in I_d}, \lambda_f \in \mathcal{P}, (\lambda_f)_{f \in I_d} \text{ is a permutation of } \lambda_d} \frac{\text{Fix}(\beta)}{|c(\beta)|}, \]

where \( \alpha \) and \( \beta \) are given in (2.11) and (2.11), respectively, and \( |c(\alpha)| \) and \( |c(\beta)| \) are given in (2.12) and (2.13), respectively.

It remains to determine \( \text{Fix}(\alpha) \) and \( \text{Fix}(\beta) \). For entire Section 3, symbols \( \alpha \) and \( \beta \) are reserved for the elements of AGL\((n, \mathbb{F}_q) \) defined in (2.11) and (2.11).

3.2. Number of fixed points of \( \alpha \) and \( \beta \).

The objective of this subsection is to determine \( \text{Fix}(\alpha) \) and \( \text{Fix}(\beta) \). In general, for \( \gamma \in \text{AGL}(n, \mathbb{F}_q) \) and \( f \in \mathcal{F}(\mathbb{F}_q^n, \mathbb{F}_q) \), \( \gamma(f) = f \) if and only if \( f \circ \gamma = f \); this happens if and only if \( f \) is constant on every \( \langle \gamma \rangle \)-orbit in \( \mathbb{F}_q^n \), where \( \langle \gamma \rangle \) is the cyclic group generated by \( \gamma \). Hence

\[ \text{Fix}(\gamma) = q^{|c(\gamma)|}, \]
where $e(\gamma)$ is the number of $\langle \gamma \rangle$-orbits in $\mathbb{F}_q^n$. Let
\begin{equation}
(3.5) \quad \text{fix}(\gamma) = \left| \{x \in \mathbb{F}_q^n : \gamma(x) = x\} \right|.
\end{equation}
(We remind the reader that fix(\gamma) is the number fixed points of \gamma in \mathbb{F}_q^n, while Fix(\gamma) is the number fixed points of \gamma in \mathcal{F}(\mathbb{F}_q^n, \mathbb{F}_q).) By Burnside’s lemma (yes, another application of Burnside’s lemma),
\[ e(\gamma) = \frac{1}{o(\gamma)} \sum_{k=1}^{o(\gamma)} \text{fix}(\gamma^k). \]
Note that fix(\gamma^k) = fix(\gamma^{gcd(k,o(\gamma))}) and that for each $l \mid o(\gamma)$, the number of $k$ $(1 \leq k \leq o(\gamma))$ such that gcd($k, o(\gamma)) = l$ is $\phi(o(\gamma)/l)$. Hence
\begin{equation}
(3.6) \quad e(\gamma) = \frac{1}{o(\gamma)} \sum_{k \mid o(\gamma)} \phi(o(\gamma)/k) \text{fix}(\gamma^k).
\end{equation}
In the next four lemmas, we first compute $o(\gamma)$ and fix(\gamma^k) when \gamma is a component of \alpha or \beta, then we determine $o(\alpha)$ and $o(\beta)$, and finally we determine Fix(\alpha) and Fix(\beta). We will see that Fix(\alpha) depends only on $(\lambda_d, (\lambda_d)_{d \in D})$ and Fix(\beta) depends only on $(t, \lambda, (\lambda_d)_{d \in D})$.

Let $o( )$ denote the order of a group element and $\nu( )$ denote the $p$-adic order of integers, where $p = \text{char} \mathbb{F}_q$.

**Lemma 3.1.** Let $f \in I_d$, where $d \in D \cup \{1\}$, and let $A \in \text{GL}(o_d(q), \mathbb{F}_q)$ be a companion matrix of $f^l$, treated as an element of AGL($o_d(q), \mathbb{F}_q$). Then $o(A) = dp^{\lceil \log_p l \rceil}$ and
\begin{equation}
(3.7) \quad \text{fix}(A^k) = q^{\nu(p^\nu(k), l) \nu(d, k)}, \quad k \geq 0,
\end{equation}
where
\begin{equation}
(3.8) \quad \nu(d, k) = \begin{cases} o_d(q) & \text{if } d \mid k, \\ 0 & \text{if } d \nmid k. \end{cases}
\end{equation}

**Proof.** Write $k = p^\nu(k) k_1$, where $p \nmid k_1$. Then by [6] Lemma 6.11, the nullity of $A^k - I$ is
\[ \text{null}(A^k - I) = \text{deg gcd}(X^k - 1, f^l) = \text{deg gcd}((X^{k_1} - 1)^{p^\nu(k)}, f^l) \]
\[ = \min(p^\nu(k), l) \text{deg gcd}(X^{k_1} - 1, f) \]
\[ = \min(p^\nu(k), l) \nu(d, k), \]
which gives (3.7). Note that
\[ A^k = I \iff \min(p^\nu(k), l) \nu(d, k) = o_d(q)l \]
\[ \iff d \mid k \text{ and } p^\nu(k) \geq l \]
\[ \iff d \mid k \text{ and } \nu(k) \geq \log_p l. \]
Hence $o(A) = dp^{\lceil \log_p l \rceil}$. \hfill $\square$

**Lemma 3.2.** Let $\sigma = \begin{bmatrix} J_m & 0 \\ \epsilon_m & 1 \end{bmatrix} \in \text{AGL}(m, \mathbb{F}_q)$. Then $o(\sigma) = p^{1 + \lceil \log_p m \rceil}$ and for $k \geq 0$,
\begin{equation}
(3.9) \quad \text{fix}(\sigma^k) = \begin{cases} q^m & \text{if } \nu(k) \geq 1 + \lceil \log_p m \rceil, \\ 0 & \text{if } \nu(k) < 1 + \lceil \log_p m \rceil. \end{cases}
\end{equation}
Proof. We have
\[ \sigma^k = \begin{bmatrix} \epsilon_m(I + J_m^k + \cdots + J_m^{k-1}) & 0 \\ 1 & 1 \end{bmatrix}, \]
where
\[ J_m^k = (I + N_m)^k = \sum_{i=0}^{k} \binom{k}{i} N_m^i \]
and
\[ I + J_m + \cdots + J_m^{k-1} = \sum_{i=0}^{k-1} \sum_{j=0}^{i} \binom{i}{j} N_m^j = \sum_{j=0}^{k-1} \left( \sum_{i=j}^{k-1} \binom{i}{j} \right) N_m^j = \sum_{j=0}^{k-1} \left( \binom{k}{j+1} \right) N_m^j. \]
In the above, \( J_m^k = I \) if and only if \( (k) = \cdots = \binom{k}{m-1} = 0 \) and \( \epsilon_m(I + J_m + \cdots + J_m^{k-1}) = 0 \) if and only if \( (k) = \cdots = \binom{k}{m} = 0 \). Let \( \text{id} \) denote the identity of AGL(m, \( F_2 \)). Then
\[ \sigma^k = \text{id} \iff (k) = \cdots = \binom{k}{m} = 0 \iff p^{\nu(k)} > m \]
\[ \iff \nu(k) > \log_p m \iff \nu(k) \geq 1 + \lfloor \log_p m \rfloor, \]
so \( o(\sigma) = p^{1 + \lfloor \log_p m \rfloor} \).
For \( x \in \mathbb{F}_q^m \), the equation \( \sigma^k(x) = x \) is equivalent to
\[ \epsilon_m(I + J_m + \cdots + J_m^{k-1}) = -x(J_m^k - I), \]
i.e.,
\[ ((1), (2), \ldots, (m)) = -x \begin{bmatrix} 0 & \binom{k}{1} & \cdots & \binom{k}{m-1} \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \cdots & \binom{k}{2} & 0 \end{bmatrix}. \]
This holds if and only if \( (1) = \cdots = (m) = 0 \), i.e., \( \nu(k) \geq 1 + \lfloor \log_p m \rfloor \). Hence we have (3.9). \qed

For \( \lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P} \), define \( m(\lambda) = \max\{i : \lambda_i > 0\} \). For \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(\psi(d))}) \in \Lambda_d \), where \( d \in D, \lambda^{(i)} \in \mathcal{P} \), define \( m(\lambda) = \max_{1 \leq i \leq \psi(d)} m(\lambda^{(i)}) \).

Lemma 3.3. (i) We have
\[ o(\alpha) := \begin{aligned} b_1(\lambda, (\lambda_d)_{d \in D}) &= \text{lcm}\{d : |\lambda_d| > 0\} p^{[\log_p \max(m(\lambda)), m(\lambda_d) : d \in D]} \end{aligned}, \]
(We define \( \text{lcm}(\emptyset) = 1. \))

(ii) We have
\[ o(\beta) := \begin{aligned} b_2(t, \lambda, (\lambda_d)_{d \in D}) &= \begin{cases} p & \text{if } t = m(\lambda) \geq \max\{m(\lambda_d) : d \in D\} \text{ and } t \text{ is a power of } p, \\
1 & \text{otherwise}. \end{cases} \end{aligned} \]
Proof: (i) By (2.10), $\alpha$ is a direct sum of affine transformations $\sigma_j$ and $\sigma_{f^j}$ ($f \in I_d, d \in D$). Therefore, the order of $\alpha$ equals the lcm of the orders of its components, that is,

\begin{equation}
(3.12) \quad o(\alpha) = \text{lcm}\left(\{o(\sigma_j)\} \cup \left(\bigcup_{d \in D} \{o(\sigma_{f^j}) : f \in I_d\}\right)\right).
\end{equation}

By (2.14), $o(\sigma_j) = \text{lcm}\{o(J_i) : \lambda_i > 0\}$, where $\lambda = (\lambda_1, \lambda_2, \ldots)$, and by Lemma 3.1 (with $f^j = (x-1)^i$), $o(J_i) = p^{\lceil \log_p i \rceil}$. Thus

\begin{equation}
(3.13) \quad o(\sigma_j) = p^{\max\{\lceil \log_p i \rceil : \lambda_i > 0\}} = p^{\lceil \log_p m(\lambda) \rceil}.
\end{equation}

For $d \in D$ and $f \in I_d$, write $\lambda_f = (\lambda_{f,1}, \lambda_{f,2}, \ldots)$. Since the elementary divisors of $\sigma_{f^j}$ are

\begin{equation}
\sigma_{f^j} = \text{lcm}\{o(A_i) : \lambda_{f,i} > 0\}, \quad \text{where } A_i \text{ is a companion matrix of } f^i \text{ and } o(A_i) = dp^{\lceil \log_p i \rceil}\end{equation}

\begin{equation}
(3.14) \quad \text{lcm}\{o(\sigma_{f^j}) : f \in I_d\} = dp^{\lceil \log_p m(\lambda_d) \rceil}.
\end{equation}

Combining (3.12) = (3.14) gives

\begin{equation}
(3.15) \quad o(\alpha) = \text{lcm}\{d : |\lambda_d| > 0\} p^{\lceil \log_p \max\{m(\lambda) \cup \{m(\lambda_d) : d \in D\}\} \rceil}.
\end{equation}

(ii) By (2.3),

\begin{equation}
(3.16) \quad o(\sigma_{\lambda,i}) = \text{lcm}\left(\{o(J_i) : i \neq t, \lambda_i > 0\} \cup \left\{o\left(\begin{bmatrix} J_i \\ i & 1 \end{bmatrix}\right)\right\}\right),
\end{equation}

where $o(J_i) = p^{\lceil \log_p i \rceil}$ (Lemma 3.1) and $o\left(\begin{bmatrix} J_i \\ i & 1 \end{bmatrix}\right) = p^{1 + \lceil \log_p i \rceil}$ (Lemma 3.2). Hence

\begin{equation}
(3.17) \quad o(\sigma_{\lambda}) = p^{\max\{1 + \lceil \log_p i \rceil \cup \{1 + \lceil \log_p i \rceil : i \neq t, \lambda_i > 0\}\}}.
\end{equation}

Now by (2.11), (3.14) and (3.15),

\begin{equation}
(3.18) \quad o(\beta) = \text{lcm}\left(\{o(\sigma_{\lambda,i})\} \cup \left(\bigcup_{d \in D} \{o(\sigma_{f^j}) : f \in I_d\}\right)\right) = \text{lcm}\{d : |\lambda_d| > 0\} p^e,
\end{equation}

where

\begin{equation}
e = \max\{1 + \lceil \log_p t \rceil \cup \{1 + \lceil \log_p i \rceil : i \neq t, \lambda_i > 0\} \cup \{1 + \lceil \log_p m(\lambda_d) \rceil : d \in D\}\}.
\end{equation}

When $t < \max\{m(\lambda) \cup \{m(\lambda_d) : d \in D\})$,

\begin{equation}
1 + \lceil \log_p t \rceil \leq \lceil \log_p \max\{m(\lambda) \cup \{m(\lambda_d) : d \in D\}\} \rceil,
\end{equation}

whence

\begin{equation}
(3.19) \quad e = \lceil \log_p \max\{m(\lambda) \cup \{m(\lambda_d) : d \in D\}\} \rceil.
\end{equation}

When $t = \max\{m(\lambda) \cup \{m(\lambda_d) : d \in D\})$, that is, $t = m(\lambda) \geq \max\{m(\lambda_d) : d \in D\}$,

\begin{equation}
1 + \lceil \log_p t \rceil \geq \max\{1 + \lceil \log_p i \rceil : i \neq t, \lambda_i > 0\} \cup \{1 + \lceil \log_p m(\lambda_d) \rceil : d \in D\},
\end{equation}

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whence

\[(3.18) \quad e = 1 + \lfloor \log_p t \rfloor = \begin{cases} 1 + \lfloor \log_p t \rfloor & \text{if } t \text{ is a power of } p, \\ \lfloor \log_p t \rfloor & \text{otherwise.} \end{cases}\]

Combining (3.10) – (3.13) gives (3.11). \(_\square\)

We remind the reader that for \(\lambda \in \mathcal{P} \), we write \(\lambda = (\lambda_1, \lambda_2, \ldots)\), and for \(\lambda_d \in \Lambda_d\), we write \(\lambda_d = (\lambda_d^{(1)}, \ldots, \lambda_d^{(v(d))})\), where \(\lambda_d^{(i)} = (\lambda_d^{(i)}; \lambda_d^{(i)}; \ldots) \in \mathcal{P}\).

**Lemma 3.4.** (i) We have \(\text{Fix}(\alpha) = q^{e_1(1, (\lambda_d)_{d \in D})}\), where

\[(3.19) \quad e_1(1, (\lambda_d)_{d \in D}) =\]

\[\frac{1}{b_1(1, (\lambda_d)_{d \in D})} \sum_{k|\beta(1, (\lambda_d)_{d \in D})} \phi(b_1(1, (\lambda_d)_{d \in D}))/k \]

\[\cdot q \sum_{j \geq 1} \min(p, d, j) \sum_{d \in D} a_d(q) \sum_{d \leq \psi(d)} \sum_{j \geq 1} \min(p, d, j) \lambda_d^{(j)} .\]

(ii) We have \(\text{Fix}(\beta) = q^{e_2(t, (\lambda_d)_{d \in D})}\), where

\[(3.20) \quad e_2(t, (\lambda_d)_{d \in D}) =\]

\[\frac{1}{b_2(t, (\lambda_d)_{d \in D})} \sum_{k|\beta(1, (\lambda_d)_{d \in D})} \phi(b_2(t, (\lambda_d)_{d \in D}))/k \]

\[\cdot q \sum_{j \geq 1} \min(p, d, j) \sum_{d \in D} a_d(q) \sum_{d \leq \psi(d)} \sum_{j \geq 1} \min(p, d, j) \lambda_d^{(j)} .\]

**Proof.** (i) By (3.4), it suffices to show that \(e(\alpha) = e_1(1, (\lambda_d)_{d \in D})\), where \(e(\alpha)\) is given in (3.6). Recall from (3.6) that

\[(3.21) \quad e(\alpha) = \frac{1}{o(\alpha)} \sum_{k|o(\alpha)} \phi(o(\alpha)/k) \text{fix}(\alpha^k),\]

where \(o(\alpha) = b_1(1, (\lambda_d)_{d \in D})\). In the above, by (2.10),

\[\text{fix}(\alpha^k) = \text{fix}(\sigma^k) \prod_{d \in D} \prod_{f \in I_d} \text{fix}(\sigma^k_{f, d}),\]

where \(\text{fix}(\sigma^k)\) and \(\text{fix}(\sigma^k_{f, d})\) are computed as follows: By (2.4) and (3.7),

\[\text{fix}(\sigma^k) = \prod_{j \geq 1} \text{fix}(J^k) \lambda_j = \prod_{j \geq 1} q^{\min(p, d, j) \lambda_j}.\]

Let \(A_f\) be a companion matrix of \(f^j\) and \(\lambda_f = (\lambda_{f, 1}, \lambda_{f, 2}, \ldots)\). By (3.7),

\[\text{fix}(\sigma^k_{f, d}) = \prod_{j \geq 1} \text{fix}(A^k_{f, d}) \lambda_{f, j} = \prod_{j \geq 1} q^{\min(p, d, j) \epsilon(d, k) \lambda_{f, j}}.\]

Hence

\[(3.22) \quad \text{fix}(\alpha^k) = q^{\sum_{j \geq 1} \min(p, d, j) \lambda_j} \prod_{d \in D} \prod_{f \in I_d} q^{\sum_{j \geq 1} \min(p, d, j) \epsilon(d, k) \lambda_{f, j}} \]

\[= q^{\sum_{j \geq 1} \min(p, d, j) \lambda_j} + \sum_{d \in D} \sum_{d \leq \psi(d)} \sum_{j \geq 1} \min(p, d, j) \epsilon(d, k) \lambda_d^{(j)} \]

\[(\text{since (} \lambda_f \text{) is a permutation of } \lambda_d).\]
Therefore, (3.3) and (3.20) gives

\[ e \]

In the above, \( o \) are given in (3.24).

Thus, by comparing \( \sigma \) and \( \sigma_{\lambda,t} \) (2.4) and (2.5), we have

\[ \text{fix}(\sigma_{\lambda,t}^k) = \begin{cases} \text{fix}(\sigma_k^\lambda) & \text{if } \nu(k) \geq 1 + \lfloor \log_t t \rfloor, \\ 0 & \text{otherwise}. \end{cases} \]

Hence

\[ \text{fix}(\beta^k) = \text{fix}(\sigma_{\lambda,t}^k) \prod_{d \in D} \prod_{f \in I_d} \text{fix}(\sigma_{f,t}^k) = \begin{cases} \text{fix}(\alpha^k) & \text{if } \nu(k) \geq 1 + \lfloor \log_t t \rfloor, \\ 0 & \text{otherwise}. \end{cases} \]

Therefore,

\[ (3.23) \quad e(\beta) = \frac{1}{\alpha(\beta)} \sum_{k \mid \alpha(\beta)} \phi(\alpha(\beta))/k \text{fix}(\beta^k) = \frac{1}{\alpha(\beta)} \sum_{k \mid \alpha(\beta)} \phi(\alpha(\beta))/k \text{fix}(\alpha^k). \]

In the above, \( o(\beta) = b_2(t, \lambda, (\lambda_d)_{d \in D}) \) and \( \text{fix}(\alpha^k) \) is given in (3.22). Now comparing (3.23) and (3.20) gives \( e(\beta) = e_2(t, \lambda, (\lambda_d)_{d \in D}) \).

3.3. The formula for \( \mathfrak{N}_{q,n} \).

We now assemble the formula for \( \mathfrak{N}_{q,n} \).

For \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(\psi(d))}) \in \Lambda_d \), let \( s(\lambda) \) be the number of permutations of \( (\lambda^{(1)}, \ldots, \lambda^{(\psi(d))}) \); that is, if \( (\lambda^{(1)}, \ldots, \lambda^{(\psi(d))}) \) has \( t \) distinct components with respective multiplicities \( k_1, \ldots, k_t \), then

\[ (3.24) \quad s(\lambda) = \frac{\psi(d)}{k_1! \cdots k_t!} \cdot \frac{\psi(d)}{k_1! \cdots k_t!}. \]

**Theorem 3.5.** We have

\[ \mathfrak{N}_{q,n} = \sum_{(\lambda, (\lambda_d)_{d \in D}) \in \Omega} s(\lambda_d) \frac{\psi_1(\lambda, (\lambda_d)_{d \in D})}{a_1(\lambda, (\lambda_d)_{d \in D})} + \sum_{(\lambda, (\lambda_d)_{d \in D}) \in \Omega} s(\lambda_d) \frac{\psi_2(t, \lambda, (\lambda_d)_{d \in D})}{a_2(t, \lambda, (\lambda_d)_{d \in D})}, \]

where \( s(\lambda_d), a_1(\lambda, (\lambda_d)_{d \in D}), e_1(\lambda, (\lambda_d)_{d \in D}), a_2(t, \lambda, (\lambda_d)_{d \in D}) \) and \( e_2(t, \lambda, (\lambda_d)_{d \in D}) \) are given in (3.24), (2.12), (3.19), (2.13) and (3.20), respectively.

**Proof.** Recall from (3.3) that

\[ \mathfrak{N}_{q,n} = \sum_{(\lambda, (\lambda_d)_{d \in D}) \in \Omega} s(\lambda_d) \frac{\text{Fix}(\alpha)}{|e(\alpha)|} \]

\[ + \sum_{(\lambda, (\lambda_d)_{d \in D}) \in \Omega} \sum_{(\lambda_f)_{f \in I_d} \text{ is a permutation of } \lambda_d} \frac{\text{Fix}(\beta)}{|e(\beta)|}. \]
In the above, by Lemma 3.4, 2.12 and 2.13,
\[
\frac{\text{Fix}(\alpha)}{|c(\alpha)|} = \frac{q^{e_1(\lambda; (\lambda_d)_{d \in D})}}{a_1(\lambda; (\lambda_d)_{d \in D})}
\]
and
\[
\frac{\text{Fix}(\beta)}{|c(\beta)|} = \frac{q^{e_2(t, \lambda; (\lambda_d)_{d \in D})}}{a_2(t, \lambda; (\lambda_d)_{d \in D})}.
\]
Both expressions depend on $\lambda_d$ rather than $(\lambda_f)_{f \in I_d}$. Hence
\[
\mathfrak{M}_{q,n} = \sum_{(\lambda, (\lambda_d)_{d \in D}) \in \Omega} s(\lambda_d) \frac{q^{e_1(\lambda, (\lambda_d)_{d \in D})}}{a_1(\lambda, (\lambda_d)_{d \in D})} + \sum_{(\lambda, (\lambda_d)_{d \in D}) \in \Omega} s(\lambda_d) \frac{q^{e_2(t, \lambda, (\lambda_d)_{d \in D})}}{a_2(t, \lambda, (\lambda_d)_{d \in D})}.
\]

\[\Box\]

4. An Asymptotic Formula for $\mathfrak{M}_n$

Recall that $\mathfrak{M}_n$ is the number of $\text{AGL}(n, \mathbb{F}_2)$ orbits of $R(n, n)/R(1, n)$. By [4] Theorem 5.1, $\mathfrak{M}_n$ is also the number of $\text{AGL}$ orbits of $R(n - 2, n)$. The main result of this section is the following asymptotic formula for $\mathfrak{M}_n$ as $n \to \infty$.

**Theorem 4.1.** We have
\[
\lim_{n \to \infty} \mathfrak{M}_n \cdot \frac{\prod_{i=1}^{\infty} (1 - 2^{-i})}{2^{2n - n/2 - 1}} = 1.
\]

To prove this theorem, we need some preparatory results, mainly about compound matrices.

4.1. Compound matrices and preparatory results.

For $0 \leq r \leq n$, let $C_r^n$ denote the set of all subsets of $\{1, \ldots, n\}$ of size $r$. Let $A$ be an $n \times n$ matrix (over any field). The $r$th compound matrix of $A$, denoted by $C_r(A)$, is the $\binom{n}{r} \times \binom{n}{r}$ matrix whose rows and columns are indexed by $C_r^n$ and whose $(S, T)$-entry ($S, T \in C_r^n$) is $\det(A(S, T))$, where $A(S, T)$ is the submatrix of $A$ with row indices from $S$ and column indices from $T$. For general properties of compound matrices, see [12] Chapter V. If the eigenvalues of $A$ are $\lambda_1, \ldots, \lambda_n$ (counting multiplicity), then the eigenvalues of $C_r(A)$ are $\prod_{i \in S} \lambda_i$, $S \in C_r^n$.

The quotient space $\text{R}(r, n)/\text{R}(r - 1, n)$ has a basis $\{X_S : S \in C_r^n\}$, where $X_S = \prod_{i \in S} X_i$. When $A \in \text{GL}(n, \mathbb{F}_2)$ acts on $\text{R}(r, n)/\text{R}(r - 1, n)$, its matrix with respect to the basis $(X_S)_{S \in C_r^n}$ of $\text{R}(r, n)/\text{R}(r - 1, n)$, displayed in a row, is $C_r(A)$, i.e.,
\[
A((X_S)_{S \in C_r^n}) = (X_S)_{S \in C_r^n} C_r(A);
\]
see [5] §4. More generally, when $\sigma = [A_{0 \times 1}] \in \text{AGL}(n, \mathbb{F}_q)$ acts on $\text{R}(r, n)/\text{R}(s, n)$, $-1 \leq s < r \leq n$, its matrix with respect to the basis $\{X_S : S \subseteq \{1, \ldots, n\}, s < |S| \leq r\}$ is
\[
\begin{bmatrix}
C_r(A) & 0 & \cdots & 0 \\
* & C_{r-1}(A) & \cdots & 0 \\
& \vdots & \ddots & \vdots \\
& * & \cdots & C_{s+1}(A)
\end{bmatrix}.
\]

**Lemma 4.2.** Let $\mathbb{F}$ be a field and $\lambda_1, \ldots, \lambda_n \in \mathbb{F}^*$ ($n \geq 1$) with $\lambda_1 \neq 1$. Then for at least half of the subsets $S$ of $\{1, \ldots, n\}$, $\prod_{i \in S} \lambda_i \neq 1$. 


For every $S \subseteq \{2, \ldots, n\}$, at most one of $\prod_{i \in S} \lambda_i$ and $\prod_{i \in (1) \cup S} \lambda_i$ is 1. \hfill $\square$

**Lemma 4.3.** Let $A$ be an $m \times m$ matrix, $B$ be an $n \times n$ matrix, and $0 \leq k \leq m$, $0 \leq l \leq n$. Then $C_k(A) \otimes C_l(B)$ is a principal submatrix of $C_{k+l}(A \oplus B)$, where $A \oplus B = [A_{ij} \oplus B_{ij}]$. ($C_0(A)$ is defined to be the $1 \times 1$ identity matrix [1].)

**Proof.** For $T \subseteq \{1, \ldots, n\}$, let $T' = \{m + j : j \in T\}$. Let $C$ be the principal submatrix of $C_{k+l}(A \oplus B)$ labeled by all $S \cup T'$ with $S \in C_k^m$ and $T \in C_l^n$. For $S_1, S_2 \in C_k^m$ and $T_1, T_2 \in C_l^n$, the $(S_1 \cup T_1', S_2 \cup T_2')$-entry of $C_{k+l}(A \oplus B)$ is

$$
\det[(A \oplus B)(S_1 \cup T_1', S_2 \cup T_2')] = \det(A(S_1, S_2)) \det(B(T_1, T_2));
$$

see Figure 1. Hence $C = C_k(A) \otimes C_l(B)$. \hfill $\square$

**Lemma 4.4.** Let $J_n$ be given by (2.3). Then

$$
C_r(J_n) = \begin{bmatrix}
C_r(J_{n-1}) & \ast \\
0 & C_{r-1}(J_{n-1})
\end{bmatrix},
$$

where the rows and columns of $C_r(J_{n-1})$ are labeled by $C_r^{n-1}$ and the rows and columns of $C_{r-1}(J_{n-1})$ are labeled by $\{S \cup \{n\} : S \in C_{r-1}^{n-1}\}$.

**Proof.** We have

$$
J_n = \begin{bmatrix}
J_{n-1} & \ast \\
0 & 1
\end{bmatrix}.
$$

Let $S, T \in C^n$. We show that $\det J_n(S, T)$, the $(S, T)$-entry of $C_r(J_n)$, equals the corresponding entry in the right side of (4.3).

If $n \notin S$ and $n \notin T$, then $J_n(S, T) = J_{n-1}(S, T)$, whence $\det J_n(S, T) = \det J_{n-1}(S, T)$.

---

**Figure 1.** $A(S_1, S_2)$ and $B(T_1, T_2)$ in $A \oplus B$
If \( n \in S \) and \( n \in T \), write \( S = S' \cup \{n\} \) and \( T = T' \cup \{n\} \), where \( S', T' \in \mathcal{C}_{n-1}^n \). Then
\[
J_n(S, T) = \begin{bmatrix}
J_{n-1}(S', T') & *\\
0 & 1
\end{bmatrix},
\]
whence \( \det J_n(S, T) = \det J_{n-1}(S', T') \).

If \( n \in S \) and \( n \notin T \), then the last row of \( J_n(S, T) \) is 0, whence \( \det J_n(S, T) = 0 \). \( \square \)

**Lemma 4.5.** We have
\[
\text{rank}(C_r(J_n) - I) \geq \binom{n-1}{r}, \quad r \geq 1.
\]

**Proof.** Let \( \rho(n, r) = \text{rank}(C_r(J_n) - I) \). By Lemma 4.4 we have
\[
\begin{cases}
\rho(n, r) \geq \rho(n-1, r-1) + \rho(n-1, r), \\
\rho(n, r) = 0 \text{ unless } 1 \leq r \leq n-1, \\
\rho(n, 1) = n-1 \text{ for } n \geq 1.
\end{cases}
\]

Using these conditions, it follows by induction on \( r \) that \( \rho(n, r) \geq \binom{n-1}{r} \) for \( r \geq 1 \). \( \square \)

### 4.2. Proof of Theorem 4.1

We are now ready to prove Theorem 4.1. First, recall that

\[
|\text{AGL}(n, \mathbb{F}_2)| = (2^n - 2^0)(2^n - 2^1) \cdots (2^n - 2^{n-1}) \cdot 2^n = 2^{n^2 + n} \prod_{i=1}^{n} (1 - 2^{-i})
\]

and
\[
|R(n-2, n)| = 2^{2^n-n-1}.
\]

For \( \sigma \in \text{AGL}(n, \mathbb{F}_2) \), let \( \text{Fix}(\sigma) \) be the number of fixed points of \( \sigma \) in \( R(n-2, n) \).

By Burnside’s lemma,
\[
\mathfrak{M}_n = \frac{1}{|\text{AGL}(n, \mathbb{F}_2)|} \sum_{\sigma \in \text{AGL}(n, \mathbb{F}_2)} \text{Fix}(\sigma)
\]
\[
= \frac{1}{|\text{AGL}(n, \mathbb{F}_2)|} \left( |R(n-2, n)| + \sum_{\text{id} \neq \sigma \in \text{AGL}(n, \mathbb{F}_2)} \text{Fix}(\sigma) \right),
\]

where
\[
|R(n-2, n)| = \frac{2^{2^n-n^2-2n-1}}{\prod_{i=1}^{n} (1 - 2^{-i})}.
\]

Therefore, to prove 4.1.1, it suffices to show that
\[
\sum_{\text{id} \neq \sigma \in \text{AGL}(n, \mathbb{F}_2)} \text{Fix}(\sigma) = o(|R(n-2, n)|) = o(2^{2^n-n-1}).
\]

Let \( \text{id} \neq \sigma = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \in \text{AGL}(n, \mathbb{F}_2) \). By 4.2, the matrix of \( \sigma \) with respect to the basis \( \{X_S : S \subset \{1, \ldots, n\}, |S| \leq n-2\} \) of \( R(n-2, n) \) is
\[
A = \begin{bmatrix}
C_{n-2}(A) & 0 & \cdots & 0 \\
* & C_{n-3}(A) & \cdots & 0 \\
& \vdots & \ddots & \vdots \\
& \ast & \cdots & C_0(A)
\end{bmatrix}.
\]
Note that
\begin{equation}
\text{Fix}(\sigma) = 2^{\text{null}(A-I)}.
\end{equation}

We estimate Fix(\sigma) in several cases.

**Case 1.** Assume that \(A\) has an eigenvalue \(\neq 1\).

By Lemma 4.2, the algebraic multiplicity of the eigenvalue 1 of \(A\) is \(\leq \frac{1}{2} \sum_{r=0}^{n-2} \binom{n}{r} < 2^{n-1}\). Hence
\begin{equation}
\text{Fix}(\sigma) < 2^{2^n-1}.
\end{equation}

**Case 2.** Assume that 1 is the only eigenvalue of \(A\) and \(A\) has an elementary divisor \((X-1)^m\) with \(m \geq \lfloor n/2 \rfloor + 1\).

We may assume that \(A = J_m \oplus A_1\) for some \(A_1 \in \text{GL}(n-m, \mathbb{F}_2)\). By Lemma 4.3, \(C_r(J_m)\) is a principal matrix of \(C_r(A)\) for all \(0 \leq r \leq m\). Thus by (4.7) and Lemma 4.5,
\begin{align*}
\text{rank}(A-I) &\geq \sum_{r=0}^{n-2} \text{rank}(C_r(A-I)) \\
&\geq \text{rank}(C_3(J_m) - I) \geq \binom{m-1}{3} \geq \binom{\lfloor n/2 \rfloor}{3}.
\end{align*}

Hence
\begin{equation}
\text{Fix}(\sigma) \leq 2^{\binom{n}{0} + \cdots + \binom{n-m}{m-2} - \binom{n/2}{3}} < 2^{2^n - \binom{n}{3}^{-1}}.
\end{equation}

**Case 3.** Assume that 1 is the only eigenvalue of \(A\) and \(A\) has an elementary divisor \((X-1)^m\) with \(2 \leq m \leq \lfloor n/2 \rfloor\).

Again, we may assume that \(A = J_m \oplus A_1\) for some \(A_1 \in \text{GL}(n-m, \mathbb{F}_2)\). By Lemma 4.3, \(C_1(J_m) \otimes C_3(A_1)\) is a principal submatrix of \(C_4(A)\). Since
\begin{align*}
C_1(J_m) \otimes C_3(A_1) &= \begin{bmatrix}
C_3(A_1) & C_3(A_1) \\
& \ddots & \ddots \\
& & \ddots & C_3(A_1) \\
& & & C_3(A_1)
\end{bmatrix}_{m \times m \text{ blocks}}
\end{align*}
and \(m \geq 2\), we have
\begin{equation}
\text{rank}(C_1(J_m) \otimes C_3(A_1) - I) \geq \text{rank} C_3(A_1) = \binom{n-m}{3} \geq \binom{\lfloor n/2 \rfloor}{3}.
\end{equation}

Therefore,
\begin{equation}
\text{rank}(A-I) \geq \text{rank}(C_4(A) - I) \geq \text{rank}(C_1(J_m) \otimes C_3(A_1) - I) \geq \binom{\lfloor n/2 \rfloor}{3},
\end{equation}
and hence
\begin{equation}
\text{Fix}(\sigma) < 2^{2^n - \binom{n}{3}^{-1}}.
\end{equation}

**Case 4.** Assume that \(A = I\) but \(a \neq (0, \ldots, 0)\).

We may assume that \(a = (0, \ldots, 0, 1)\), i.e.,
\[\sigma(x) = x + (0, \ldots, 0, 1) \text{ for all } x \in \mathbb{F}_2^n.\]
In this case, for $f \in R(n-2,n)$,
$$\sigma(f) = f \iff f = f(X_1, \ldots, X_{n-1}) \in R(n-2,n-1).$$
Thus
$$\text{Fix}(\sigma) = |R(n-2,n-1)| = 2^{2^{n-1}-1}. \tag{4.12}$$

In all four cases, we always have
$$\text{Fix}(\sigma) < 2^{2^n-\left(\binom{n}{2}\right)}$$
for $n$ sufficiently large.

Therefore
$$\sum_{id \neq \sigma \in AGL(n,\mathbb{F}_2)} \text{Fix}(\sigma) < |AGL(n,\mathbb{F}_2)|2^{2^n-\left(\binom{n}{2}\right)} \leq 2^{2^n + n + 2^n - \left(\binom{n}{3}\right)} = o(2^{2^n-n-1}).$$

This completes the proof of Theorem 4.1.

5. Final Remarks

In general, let $\theta(n; s, t)$ denote the number of $AGL$ orbits of $R(r,n)/R(s-1,n)$, $0 \leq s \leq r \leq n$. (Thus $M_{2,n} = \theta(n; 0, n)$ and $M_n = \theta(n; 0, n-2)$.) Since $\theta(n; s, r) = \theta(n; n-r, n-s)$ [4, Theorem 5.1], we may assume that $s + r \leq n$, that is, $0 \leq s \leq r \leq n-s$. In this range, $\theta(n; s, r)$ is numerically computed for $n \leq 10 [13]$ and is theoretically determined for $r \leq 2$ (linear and quadratic functions) and for $(s, r) = (0, n)$ (this paper). It appears that with due effort, $\theta(n; 0, n-1)$ can also be determined theoretically. For other values of $(s, r)$, explicit formulas for $\theta(n; s, r)$ appears to be out of immediate reach. For example, to determine $\theta(n; 3, 3)$, one needs to know $\text{null}(C_3(A) - I)$ for every $A \in GL(n,\mathbb{F}_2)$ in a canonical form under conjugation, or one needs to know the classification of cubic forms over $\mathbb{F}_2$; the former is difficult and the latter is probably impossible.

As for the asymptotics, we have only solved the question for $\theta(n; 0, n-2)$. However, it seems that the method should work for all $(s, r)$.

6. Conclusion

We derived an explicit formula for the number of equivalence classes of functions from $\mathbb{F}_q^n$ to $\mathbb{F}_q$ under the action of the affine linear group $AGL(n,\mathbb{F}_q)$. These numbers are enormous unless both $q$ and $n$ are small, hence complete classification of functions from $\mathbb{F}_q^n$ to $\mathbb{F}_q$ is not practical. However, the group theoretic approach in the paper may lead to solutions of similar problems. We also proved an asymptotic formula for the number of equivalence classes of cosets of the first order Reed-Muller code under the action of $AGL(n,\mathbb{F}_2)$. The asymptotic formula indicates that for most cosets of the first order Reed-Muller code, the subgroup of $AGL(n,\mathbb{F}_2)$ that stabilizes them is trivial.

APPENDIX

List of Notation

$\oplus \quad A \oplus B = \begin{bmatrix} A & B \\ \end{bmatrix}$
$\boxplus \quad \begin{bmatrix} A \\ a & 1 \end{bmatrix} \boxplus \begin{bmatrix} B \\ b & 1 \end{bmatrix} = \begin{bmatrix} A & B \\ a & b & 1 \end{bmatrix}$

$A \quad \text{formula in (4.7)}$

$a_1(\lambda, (\lambda_d)_{d \in D}) \quad |c(\alpha)|$, formula in (2.12)
\[ a_2(t, \lambda, (\lambda_d)_{d \in D}) \mid c(\beta) \mid, \text{ formula in (2.13)} \]  
\[ A(S, T) \]  
submatrix of \( A \) with row (column) indices in \( S (T) \)  
\[ AGL(n, F_q) \]  
\( \{ [a_0] : A \in \text{GL}(n, F_q), \ a \in F_q^n \} \), affine linear group  
\[ b_1(\lambda, (\lambda_d)_{d \in D}) \]  
\( \alpha(\alpha) \), formula in (3.10)  
\[ b_2(t, \lambda, (\lambda_d)_{d \in D}) \]  
\( \alpha(\beta) \), formula in (3.11)  
\[ C \]  
set of representatives of conjugacy classes of \( AGL(n, F_q) \)  
\[ C_1, C_2 \]  
defined in (2.6) and (2.7)  
\[ C^r \]  
set of subsets of \( \{1, \ldots, n\} \) of size \( r \)  
\[ C_r(A) \]  
\( r \)th compound matrix of \( A \)  
\[ c(\alpha) \]  
centralizer of \( \alpha \) in \( AGL(n, F_q) \)  
\[ D \]  
\( \{ d > 1 : d \mid q^i - 1 \text{ for some } 1 \leq i \leq n \} \)  
\[ e_1(\lambda, (\lambda_d)_{d \in D}) \]  
derived by \( \text{Fix}(\alpha) = q^{e_1(\lambda, (\lambda_d)_{d \in D})} \), formula in (3.19)  
\[ e_2(t, \lambda, (\lambda_d)_{d \in D}) \]  
derived by \( \text{Fix}(\beta) = q^{e_2(t, \lambda, (\lambda_d)_{d \in D})} \), formula in (3.20)  
\[ f^\lambda \]  
\( \{ f_1^1, \ldots, f_1^d, f_2^1, \ldots, f_2^d, \ldots \} \), where \( \lambda = (\lambda_1, \lambda_2, \ldots) \)  
\[ F(F_q^n, F_q) \]  
set of functions from \( F_q^n \) to \( F_q \)  
\[ \text{Fix}(\alpha) \]  
(§3) number of fixed points of \( \alpha \) in \( F(F_q^n, F_q) \)  
\[ \text{Fix}(\sigma) \]  
(§4) number of fixed points of \( \sigma \) in \( R(n, n - 2) \)  
\[ \text{fix}(\alpha) \]  
number of fixed points of \( \alpha \) in \( F_q^n \)  
\[ I \]  
identity matrix  
\[ I_d \]  
set of monic irreducible polynomials in \( F_q[X] \) \( \setminus \{X\} \)  
\[ \text{id} \]  
identity of the affine linear group  
\[ \mathcal{M}_r \]  
number of \( AGL \) orbits of \( R(n, n)/R(1, n) \)  
\[ m(\lambda) \]  
\( \max\{i : \lambda_i > 0\} \), where \( \lambda = (\lambda_1, \lambda_2, \ldots) \in P \)  
\[ m(\lambda) \]  
maximum of \( \psi(d) \) over \( \psi(\lambda) \), where \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(\psi(d))}) \in \Lambda_d \)  
\[ \mathcal{M}_{r, n} \]  
number of \( AGL \) orbits of \( F(F_q^n, F_q) \)  
\[ \text{null}(A) \]  
nullity of \( A \)  
\[ o(\ ) \]  
(§3) order of a group element  
\[ o(\ ) \]  
(§4) little-\( o \)-asymptotic  
\[ \omega_d(q) \]  
multiplicative order of \( q \) in \( \mathbb{Z}/d\mathbb{Z} \)  
\[ \text{ord} \]  
order of \( f \) in \( I \)  
\[ P \]  
set of all partitions  
\[ R_q(r, n) \]  
\( \{ f \in F(F_q^n, F_q) : \deg f \leq r \} \), \( q \)-ary Reed-Muller code  
\[ R(r, n) \]  
\( R_2(r, n) \), binary Reed-Muller code  
\[ s(\lambda) \]  
number of permutations of \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(\psi(d))}) \in \Lambda_d \)  
\[ T(\lambda) \]  
\( \{ i : \lambda_i > 0 \} \), where \( \lambda = (\lambda_1, \lambda_2, \ldots) \in P \)  
\[ X_S \]  
\( \prod_{i \in S} X_i \)  
\[ \alpha, \beta \]  
defined in (2.10) and (2.11) and protected in §3  
\[ \epsilon \]  
(1, 0, \ldots, 0) \in \mathbb{F}_q^d  
\[ \epsilon(d, k) \]  
(3.5)  
\[ \theta(n; s, r) \]  
number of \( AGL \) orbits of \( R(r, n)/R(s - 1, n) \)  
\[ |\lambda| \]  
\( \sum_{i \geq 1} \lambda_i \), where \( \lambda = (\lambda_1, \lambda_2, \ldots) \in P \)  
\[ |\lambda| \]  
\( \sum_{i = 1}^{\psi(d)} |\lambda^{(i)}| \), where \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(\psi(d))}) \in \Lambda_d \)  
\[ \Lambda_d \]  
\( \{ (\lambda^{(1)}, \ldots, \lambda^{(\psi(d))}) : \lambda^{(i)} \in P, \lambda^{(i)} \leq \cdots \leq \lambda^{(\psi(d))} \}, d \in D \)  
\[ \nu(\ ) \]  
p-adic order
\( \sigma_{f^\lambda} \) matrix with elementary divisors \( f^\lambda \)

\( \sigma_{\lambda} \)

\( \psi(d) \)

\( \phi \) Euler totient function

\[ \phi(d)/\phi(d) \]

\[ \psi(d) \]

\[ \Omega \]

\[ \{(\lambda, (\lambda_d)_{d \in D}) : \lambda \in \mathcal{P}, \lambda_d \in \Lambda_d, |\lambda| + \sum_{d \in D} \phi(d) |\lambda| = n \} \]

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