Identity Testing for High-Dimensional Distributions via Entropy Tensorization

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Abstract

We present improved algorithms and matching statistical and computational lower bounds for the problem of identity testing $n$-dimensional distributions. In the identity testing problem, we are given as input an explicit distribution $\mu$, an $\varepsilon > 0$, and access to a sampling oracle for a hidden distribution $\pi$. The goal is to distinguish whether the two distributions $\mu$ and $\pi$ are identical or are at least $\varepsilon$-far apart. When there is only access to full samples from the hidden distribution $\pi$, it is known that exponentially many samples may be needed for identity testing, and hence previous works have studied identity testing with additional access to various conditional sampling oracles. We consider here a significantly weaker conditional sampling oracle, which we call the Coordinate Oracle, and provide a fairly complete computational and statistical characterization of the identity testing problem in this new model.

We prove that if an analytic property known as approximate tensorization of entropy holds for the visible distribution $\mu$, then there is an efficient identity testing algorithm for any hidden $\pi$ that uses $\tilde{O}(n/\varepsilon)$ queries to the Coordinate Oracle. Approximate tensorization of entropy is a classical tool for proving optimal mixing time bounds of Markov chains and concentration of Lipschitz functions for high-dimensional distributions. Recent work shows that spectral independence implies approximate tensorization and consequently establishes it for many families of $n$-dimensional distributions. We complement our algorithmic result for identity testing with a matching $\Omega(n/\varepsilon)$ statistical lower bound for the number of queries under the Coordinate Oracle. We also prove a computational phase transition: for a well-studied class of $n$-dimensional distributions, specifically sparse antiferromagnetic Ising models over $\{+1, -1\}^n$, we show that in the regime where approximate tensorization of entropy fails, there is no efficient identity testing algorithm unless $\text{RP}=\text{NP}$.

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1 Introduction

A fundamental problem in statistics and machine learning is the identity testing problem (also known as the goodness-of-fit problem). Roughly speaking, we are explicitly given a visible distribution $\mu$ and oracle access to samples from an unknown/hidden distribution $\pi$; the goal is to determine if these distributions are identical using as few samples from $\pi$ as possible.

The complexity of identity testing for the general distributions is now well-understood, including studies on what conditions on the visible and hidden distributions enable efficient identity testing. An intriguing line of work considers a different perspective: what additional assumptions on the sampling oracle for the hidden distribution are required to ensure efficient identity testing. We present tight results with more modest oracle assumptions than considered previously.

Let us begin with a formal definition of the classical identity testing framework. Let $\mathcal{X}$ be a finite state space of size $N = |\mathcal{X}|$, and let $d(\cdot, \cdot)$ denote some metric or divergence between distributions over $\mathcal{X}$; the standard choices for $d(\cdot, \cdot)$ are total variation distance or Kullback–Leibler divergence (KL divergence). For a distribution $\mu$ over $\mathcal{X}$ and a parameter $\varepsilon > 0$, denote by $\text{ID-TEST}(d, \varepsilon; \mu)$ the identity testing problem for $\mu$: having the full description of the visible distribution $\mu$ (as input), and provided with access to a sampling oracle for an unknown distribution $\pi$, the goal is to distinguish between $\pi = \mu$ vs. $d(\pi, \mu) \geq \varepsilon$ with probability at least $2/3$. The query or sample complexity is the number of calls to the sampling oracle required to solve this problem.

The complexity of the classical identity testing problem is widely studied. For the special case when $\mu$ is the uniform distribution over $\mathcal{X}$, also known as uniformity testing, there are efficient algorithms with sample complexity $O(\sqrt{N}/\varepsilon^2)$, which matches the information-theoretic lower bound [BFF+01, Pan08]. For general distributions $\mu$, there are also algorithms with optimal query complexity; that is, with optimal dependence on the state space size $N$, the distance parameter $\varepsilon$, and even the failure probability; see [CDVV14, ADK15, VV17a, DK16, Gol16, DGK+21, CS22] and the references therein.

In practice, data is typically high-dimensional, which naturally requires the study of identity testing for high-dimensional distributions, and this will be our focus. To be more precise, let $k \geq 2$ be an integer and $\mathcal{K} = \{1, \ldots, k\}$ be an alphabet (spin/color) set, and our state space is $\mathcal{X} = \mathcal{K}^n$, a product space of dimension $n$. We study the identity testing problem $\text{ID-TEST}(d, \varepsilon; \mu)$ for $n$-dimensional distributions $\mu$ over $\mathcal{X}$.

Note that for the problem to make sense, we need to be able to give a full description of the visible distribution $\mu$ (say in $\text{poly}(n)$ size) so that it can be provided as an input to the tester. Since the state space $|\mathcal{X}| = k^n$ is exponentially large in $n$, this is not true for all visible distribution $\mu$. Thus, we will focus our attention on special classes of $\mu$ that are polynomially representable; this includes product distributions (including the uniform distribution), Bayesian nets, and undirected graphical models (also known as spin systems) among others.

Identity testing for high-dimensional distributions has recently attracted a lot of attention, see, e.g., [DP17, DDK19, BBC+19, BCSV20, CDKS20, BGKV21, BCY22]. Despite the (almost) complete picture for identity testing in the classical setting, the picture is still less clear in the high-dimensional setting because of the exponential size of the state space. As mentioned earlier, even uniformity testing requires at least $\Omega(\sqrt{|\mathcal{X}|}) = \Omega(k^{n/2}/\varepsilon^2)$ samples which is exponentially large in the dimension; see also [BBC+19, BCSV20] for computational hardness results.

To overcome this difficulty, there are two types of further conditions that one may attach to the identity testing problem. The first approach is to restrict the unknown distribution $\pi$ to be in
some special class of distributions; a natural example is to require that \( \pi \) is from the same class as \( \mu \). For example, [BGKV21] studies the setting where both \( \mu \) and \( \pi \) are product distributions, [CDKS20, DP17] requires \( \mu \) and \( \pi \) to be Bayesian nets, [DDK19] studies the problem when \( \mu \) and \( \pi \) are Ising models. More recently [BCY22] assumes that \( \mu \) is a product distribution while \( \pi \) is a Bayesian net.

While such an approach leads to fruitful results for testing high-dimensional distributions, it is not ideal from a practical point of view, since \( \pi \) can be a “noisy” version of \( \mu \) and may no longer belong to a nice class of distributions; in fact, it may not be possible to describe \( \pi \) with polynomially many parameters.

An alternative approach to overcome the apparent intractability of identity testing in the high-dimensional setting is to assume access to stronger sampling oracles from the hidden distribution \( \pi \); specifically, access to different conditional sampling oracles for \( \pi \). This approach for high-dimensional distributions is the focus of this paper.

There are several types of conditional sampling oracles, and here we mention the most popular choices. The first is the general conditional sampling oracle (see \[\text{CRS15, CFGM16, FJO}^{+15}\]) which given any subset \( \mathcal{X}' \) of the space \( \mathcal{X} \) generates a sample from the projection of \( \pi \) to \( \mathcal{X}' \); that is, the oracle returns an element \( x \) from \( \mathcal{X}' \) with probability \( \pi(x) / \pi(\mathcal{X}') \). This oracle is not well-suited for the high-dimensional setting, as one can not hope to formulate the queries to the oracle efficiently (unless restricted to a special class of subsets \( \mathcal{X}' \)).

The second is the pairwise conditional sampling oracle (Pairwise Oracle) which takes a pair of configurations and generates a sample from the distribution restricted to these two choices: given \( x, y \in \mathcal{X} \) the oracle returns \( x \) with probability \( \pi(x) / (\pi(x) + \pi(y)) \) and \( y \) otherwise (see [CRS15]). The queries for Pairwise Oracle can be easily formulated for high-dimensional distributions, and identity testing has been studied in this setting. Recently, [Nar21] provided an identity testing algorithm with \( \tilde{O}(\sqrt{n}/\varepsilon^2) \) sample complexity and a matching statistical lower bound.

As far as we are aware, the other conditional sampling oracle previously studied in the high-dimensional setting is the subcube conditional sampling oracle (Subcube Oracle) introduced by [BC18] and also studied in [CCK+21, CJLW21]. This oracle is defined as follows. For an integer \( n \in \mathbb{N}^+ \), we let \([n] = \{1, \ldots, n\}\). For a vector \( x \in \mathcal{K}^n \) and \( S \subseteq [n] \), we let \( x_S = (x_i)_{i \in S} \in \mathcal{K}^S \) be the vector obtained from the coordinates of \( x \) in \( S \). A query to the Subcube Oracle consists of a subset \( \Lambda \subseteq [n] = \{1, \ldots, n\} \) and a configuration \( x \in \mathcal{K}^\Lambda \) on \( \Lambda \). If \( \pi(X_\Lambda = x) > 0 \), the oracle samples \( x' \in \mathcal{K}^{|n|\setminus\Lambda} \) from the conditional distribution \( \pi(x_{V\setminus\Lambda} = \cdot \mid X_\Lambda = x) \) (see Definition 3.1). For the Subcube Oracle, an identity testing algorithm using \( \tilde{O}(n^3/\varepsilon^3) \) queries was presented in [BC18]; further improved algorithms were presented for uniformity testing in [CCK+21] and for testing \( k \)-junta distributions in [CJLW21].

We propose a weaker conditional sampling oracle, which we call the Coordinate Oracle, and provide a fairly complete computational and statistical characterization of identity testing in this model. This oracle has already been implicitly used in [CCK+21] for uniformity testing on the binary hypercube \( \{0,1\}^n \) (which they call “edge tester”). We impose no conditions on the hidden distribution \( \pi \), other than access to the conditional sampling oracle, and explore which conditions on the visible distribution \( \mu \) are necessary and sufficient for identity testing. Since the visible distribution has to be explicitly provided, one immediate necessary condition is that \( \mu \) must have a polynomially sized (in \( n \)) description.

**Algorithmic results.** The new conditional sampling oracle we introduce, the Coordinate Oracle, corresponds to the Subcube Oracle restricted to query sets \( \Lambda \) where \( |\Lambda| = n - 1 \); thus we are fixing the configuration at all but one coordinate and looking at the conditional distribution at this
particular coordinate given a fixed configuration on the remaining coordinates. Hence, in many settings access to **Coordinate Oracle** is a very mild assumption since it corresponds to the conditional marginal distribution of a single coordinate.

In our algorithmic results we will assume access to the **Coordinate Oracle** for $\pi$ and also to the **General Oracle** that provides full (unconditional) samples from $\pi$. We note that the **General Oracle** corresponds to **Subcube Oracle** restricted to $\Lambda = \emptyset$, and previous work under the **Pairwise Oracle** also assumes access to the **General Oracle**. Moreover, since having access to both the **Coordinate Oracle** and the **General Oracle** is a weaker assumption than having access to a **Subcube Oracle**, an algorithm for **Coordinate Oracle + General Oracle** implies an algorithm for **Subcube Oracle**. For constant $k$, it also implies an algorithm for **Pairwise Oracle + General Oracle**, since one can simulate the **Coordinate Oracle** using just the **Pairwise Oracle**; see Remark 3.2.

For our algorithmic work we consider consider the identity testing problem with KL divergence, denoted $D_{KL}(\pi \parallel \mu)$ and formally defined in Section 3. The choice of KL divergence is natural since, by Pinsker’s inequality, an identity testing algorithm with KL divergence immediately gives one for total variation distance with similar query complexity and running time (there is only a quadratic loss in the distance parameter $\epsilon$).

We introduce next an important analytic property for the visible distribution, known as approximate tensorization of entropy [CMT15], which we will show is a sufficient (and essentially also necessary) condition for efficient identity testing. Approximate tensorization of entropy roughly states that the entropy of a distribution is bounded by the sum of the average conditional entropy at each coordinate. Approximate tensorization of entropy implies optimal mixing time bounds for the single-site update Markov chain, known as the Gibbs sampler or Glauber dynamics [Ces01, CLV21a]. It is also often used to establish modified log-Sobolev inequalities (MLSI) and thus the concentration of Lipschitz functions under the distribution [BG99, MT06].

**Definition 1.1** (Approximate Tensorization of Entropy). We say a distribution $\mu$ fully supported on $\mathcal{K}^n$ satisfies approximate tensorization of entropy with constant $C$ if for any distribution $\pi$ over $\mathcal{K}^n$ it holds

$$D_{KL}(\pi \parallel \mu) \leq C \sum_{i=1}^{n} E_{x \sim \pi_{n \setminus i}} \left[ D_{KL}(\pi_{i} (\cdot | x) \parallel \mu_{i} (\cdot | x)) \right],$$

where $\pi_{n \setminus i} (\cdot) = \pi(X_{n \setminus i} = \cdot)$ denotes the marginal distribution of $\pi$ on $[n \setminus \{i\}$, and $\pi_{i} (\cdot | x)$ (resp., $\mu_{i} (\cdot | x)$) denotes the marginal of $\pi$ (resp., of $\mu$) on the $i$-th coordinate conditional on $x$.

More details about approximate tensorization and equivalent formulations are given in Section 3.4. We note that the constant $C$ achieves the minimum $C = 1$ when $\mu$ is a product distribution.

There are a plethora of recent results establishing approximate tensorization in a wide variety of settings. In particular, [CLV20] showed that the spectral independence technique introduced by [ALG20] implies approximate tensorization for sparse undirected graphical models (i.e., spin systems on bounded degree graphs). Furthermore, recent works showed that spectral independence (and hence approximate tensorization) is implied by correlation decay arguments [CLV20, CGSV21, FGYZ21], path coupling for local Markov chains [BCC⁺22, Liu21], and stability of the partition function [CLV21b]. As such, approximate tensorization is now known to hold, for example, for a variety of spin systems on graphs of bounded degree in a tight range of parameters; this includes the classical Ising and hard-core models (weighted independent sets), monomer-dimer model (weighted matchings) [CLV21b], and uniform proper colorings [BCC⁺22, Liu21].

We show that approximate tensorization of the visible distribution $\mu$ yields an efficient identity testing algorithm, provide access to **Coordinate Oracle** and **General Oracle** for the hidden distribution $\pi$. Note we impose no restrictions on the hidden distribution beyond the oracle access.
We have three additional basic assumptions on the visible distribution. First, we assume that the Coordinate Oracle can be implemented efficiently for the visible distribution $\mu$; this is equivalent to requiring that a step of the Glauber dynamics for $\mu$ can be efficiently implemented. Formally, this means that for any coordinate $i$, any fixed feasible assignment $\sigma$ for the other $n-1$ coordinates, we can compute the conditional distribution at $i$ in polynomial time. The second assumption on $\mu$ is what we call $\eta$-balanced: namely, there is a lower bound $\eta$ so that the conditional probability of any spin $a \in \mathcal{K}$ at any coordinate $i$ is either 0 or at least $\eta$ (see Section 3.3). Finally, we also assume that $\mu$ is fully supported on $\mathcal{K}^n$. This last condition can actually be relaxed; we only require that the support of $\pi$ is a subset of the support of $\mu$ (see Section 4.5).

**Theorem 1.2.** Given a visible distribution $\mu$ over $\mathcal{X} = \mathcal{K}^n$ which satisfies Approximate Tensorization with constant $C$ and is $\eta$-balanced, there is an identity testing algorithm with Coordinate Oracle and General Oracle access with $\tilde{O}(n/\varepsilon)$ query complexity and polynomial running time.

We refer the reader to Theorem 4.1 for a more precise theorem statement indicating the explicit dependence on $C$ and $\eta$ in the query complexity.

We shall see that our algorithmic result in Theorem 1.2 is essentially tight, both statistically and computationally; i.e., we establish a $\Omega(n/\varepsilon^2)$ query lower bound in the same setting and show that there is a class of high-dimensional distributions where identity testing is computationally hard when approximate tensorization does not hold. A surprising feature of our algorithm is that it bypasses sampling from visible distribution $\mu$; it does not even require the concentration of any statistics under $\mu$. Our algorithm is also very simple and reminiscent of the one from [CCK+21] for uniformity testing. Finally, we point out that one may pursue an alternative algorithmic approach of simulating a Pairwise Oracle using a Coordinate Oracle and then utilizing the algorithm of Narayanan [Nar21]. However, there are several issues with this approach (e.g., one only obtains an approximate Pairwise Oracle sampler) and the resulting algorithm will have significantly worse sample complexity.

As mentioned, approximate tensorization is known to hold in a variety of settings, so the algorithmic result from Theorem 1.2 has a number of interesting applications, including product distributions [Ces01, CMT15], sparse undirected graphical models [CLV21a] (in the so-called tree uniqueness regime), and distributions satisfying a Dobrushin-type uniqueness condition [Mar19]. See Section 4.4 for more details and formal statements for these applications.

**Computational hardness results.** We demonstrate next that the above algorithmic result (Theorem 1.2) is tight from a computational perspective. We show, in the context of the Ising model, that approximate tensorization is a necessary condition for identity testing in polynomial time when given access to the Coordinate Oracle. In particular, we establish a computational phase transition for identity testing of the Ising model when approximate tensorization holds versus when approximate tensorization does not hold.

The Ising model is the simplest and most well-studied example of an undirected graphical model. Given a graph $G = (V, E)$, configurations of the model are denoted by $\Omega = \{+1, -1\}^V$. For a real-valued parameter $\beta$, the probability of a configuration $\sigma \in \Omega$ is

$$
\mu_{G,\beta}(\sigma) = \frac{1}{Z_{G,\beta}} \cdot \exp \left( \beta \sum_{\{v,w\} \in E} \sigma_v \sigma_w \right),
$$

where the normalizing constant $Z_{G,\beta}$ is known as the partition function. When $\beta > 0$ the model is ferromagnetic/attractive and when $\beta < 0$ then the model is antiferromagnetic/repulsive; see Section 4.4.2 for a more general definition of the model.
The antiferromagnetic Ising model undergoes an intriguing computational phase transition. For all constant \( d \geq 3 \), for all \( 0 > \beta > \beta_c(d) = -\frac{1}{2} \ln(\frac{d}{d-2}) \), on any graph \( G = (V,E) \) of maximum degree \( d \), the Glauber dynamics Markov chain has optimal \( O(|V| \log |V|) \) mixing time [CLV21a] and, consequently, the approximate sampling and counting (i.e., approximating the partition function \( Z_{G,\beta} \)) problems can be solved efficiently. Moreover, approximate tensorization holds in this regime, and hence Theorem 1.3 applies for identity testing in the Coordinate Oracle model. In contrast, when \( \beta < \beta_c(d) \) there are no polynomial-time approximate sampling or counting algorithms unless \( \text{NP} = \text{RP} \) [SS14, GSV16].

We establish here the computational hardness of identity testing with Coordinate Oracle + General Oracle access in the same parameter regime \( \beta < \beta_c(d) \), which thereby exhibits a sharp computational phase transition for identity testing with Coordinate Oracle + General Oracle access for the class of antiferromagnetic Ising models. We mention that the threshold \( \beta_c(d) = -\frac{1}{2} \ln(\frac{d}{d-2}) \) corresponds to the uniqueness/non-uniqueness phase transition on the infinite \( d \)-regular tree. Roughly speaking, in graphs of maximum degree at most \( d \), when \( \beta > \beta_c(d) \) long-range correlations die off, whereas in the non-uniqueness phase \( \beta < \beta_c(d) \) certain long-range correlations persist.

**Theorem 1.3.** For sufficiently large constant \( d \geq 3 \) and for constant \( \beta < 0 \), consider identity testing for the family of antiferromagnetic Ising models on graphs of maximum degree \( d \) with parameter \( \beta \).

i) **Easy Regime:** If \( \beta > \beta_c(d) \), then there exists a polynomial-time identity testing algorithm with Coordinate Oracle and General Oracle access with query complexity \( \tilde{O}(|V|/\varepsilon) \);

ii) **Hard Regime:** If \( \beta < \beta_c(d) \), then there is no polynomial-time identity testing algorithm with Coordinate Oracle and General Oracle access unless \( \text{RP} = \text{NP} \).

There are few analogous computational hardness results for identity testing; most hardness results in this setting are information theoretic. The few examples appeared in [BBC+19, BCSV20], which apply to the identity testing problem with only General Oracle access. These results establish the computational hardness of identity testing even when the hidden distribution is also assumed to an Ising model from the same class, but they require \(|\beta|d \gg \log |V| \). Our hardness result in Theorem 1.3 applies to the full non-uniqueness regime, where \(|\beta|d = \Omega(1) \) and allows access to both the Coordinate Oracle and the General Oracle.

We also note that our hardness result in Theorem 1.3 holds under any conditional sampling oracle that can be implemented in polynomial time for the antiferromagnetic Ising model, and thus applies to identity testing for the antiferromagnetic Ising model in the Pairwise Oracle model, which complements the algorithmic results from [CRS15, Nar21].

We prove the hardness result in Theorem 1.3 using a reduction from the maximum cut problem to identity testing. Our reduction is inspired by the one in [BBC+19], but we must use a different “degree reducing” gadget; namely, the one from [Sly10] used to establish the computational hardness of approximate counting antiferromagnetic spin systems. An interesting aspect of our proof is that we are also required to design sampling algorithms to simulate the hidden oracles. This is challenging because sampling from the antiferromagnetic Ising model is \( \text{NP} \)-hard in general, but we manage to do it for our instance using the recent algorithmic result of Koehler, Lee, and Risteski [KLR22], presenting an approximate sampling algorithm (using variational methods) for Ising models when the edge interaction matrix has low rank, and polymer models otherwise [JKP20].

**Statistical lower bounds.** We also present matching information-theoretic lower bounds for identity testing in the Coordinate Oracle model. We focus on uniformity testing when \( k = 2 \), i.e., the visible distribution \( \mu \) is the uniform distribution over the binary hypercube \( \{0,1\}^n \). We present
statistical lower bounds in two different settings: for the Subcube Oracle model with KL divergence and for the Coordinate Oracle + General Oracle model with total variation (TV) distance. In both cases, we show that identity testing requires a linear number of samples.

**Theorem 1.4** (Lower Bound for Subcube Oracle and KL Divergence). Any uniformity testing algorithm over \(\{0,1\}^n\) for KL divergence with access to Subcube Oracle requires \(\Omega(n/\varepsilon)\) queries.

**Theorem 1.5** (Lower Bound for Coordinate Oracle + General Oracle and TV Distance). Any uniformity testing algorithm over \(\{0,1\}^n\) for TV distance with access to both Coordinate Oracle and General Oracle requires at least \(\Omega(n/\varepsilon^2)\) queries.

Interestingly, if one considers uniformity testing with total variation distance and Subcube Oracle access, then the recent work [CCK+21] shows that \(\tilde{O}(\sqrt{n}/\varepsilon^2)\) queries suffice, which is sublinear.

As far as we know, it is unclear if the testing algorithm from [CCK+21] with sublinear sample complexity can be used for other distributions, e.g., general product distributions. We summarize our results along with the main results of [CCK+21] in Table 1.

|                | KL \(\geq \varepsilon\) | TV \(\geq \varepsilon\) |
|----------------|-------------------------|--------------------------|
| Coordinate Oracle | \(\tilde{\Theta}(n/\varepsilon)\) (this work) | \(\tilde{\Theta}(n/\varepsilon^2)\) (this work) |
| Subcube Oracle   | \(\tilde{\Theta}(n/\varepsilon)\) (this work) | \(\tilde{\Theta}(\sqrt{n}/\varepsilon^2)\) [CCK+21] |

Table 1: Query Complexity for Uniformity Testing over \(\{0,1\}^n\)

We complement the algorithmic result from Theorem 1.2 for the Coordinate Oracle with an improved identity testing algorithm for the Subcube Oracle. For this algorithmic result, we require the visible distribution to be \(b\)-marginally bounded, a slightly different (stronger but related) notion than \(\eta\)-balanced (see Section 3.3). See also Remark 7.3 for more discussions.

**Theorem 1.6** (Algorithmic Result for Subcube Oracle). For the visible distribution \(\mu\) that is \(b\)-marginally bounded, if we can compute the marginal of any coordinate conditioned on any partial configuration on a subset of coordinates, then there is an identity testing algorithm with access to the Subcube Oracle with \(\tilde{O}(n/\varepsilon)\) query complexity (the running time depends on the time to compute the conditional marginals).

Note that this theorem does not require approximate tensorization. We point out several interesting applications where Theorem 1.6 applies, but approximate tensorization fails (or we do not know if it holds) and hence Theorem 1.2 does not apply: undirected graphical models (e.g., Ising model) on trees, Bayesian networks, mixtures of product distributions, high-temperature Ising models, and monomer-dimer models on arbitrary graphs. Moreover, under the same assumptions as in Theorem 1.6 we show that one can estimate \(D_{KL}(\pi\parallel\mu)\) within additive error \(\varepsilon\) using \(\tilde{O}(n^4/\varepsilon^4)\) queries to the Subcube Oracle (see Theorem 7.9). This corresponds to the more challenging tolerant identity testing problem, where we want to distinguish between \(D_{KL}(\pi\parallel\mu) \leq s\) and \(D_{KL}(\pi\parallel\mu) \geq s + \varepsilon\) for \(s,\varepsilon > 0\). Note that identity testing corresponds to \(s = 0\). See Section 7 for more discussion.

### 2 Overview of Techniques

In this section, we present proof overviews for each of our main results: our algorithmic result (Theorem 1.2) and the complementary computational hardness (Theorem 1.3) and statistical lower bounds (Theorems 1.4 and 1.5) for the Coordinate Oracle model.
2.1 Algorithmic result for Coordinate Oracle model: Theorem 1.2

We begin with an overview of our identity testing algorithm in the Subcube Oracle model (Theorem 1.2). Suppose \( \mu \) is the visible distribution satisfying approximate tensorization of entropy, and \( \pi \) is an arbitrary distribution over \( \mathcal{K}^{n} \). If \( D_{KL}(\pi \parallel \mu) \geq \varepsilon \), then from the definition of approximate tensorization of entropy, we see that the following holds:

\[
D_{KL}(\pi \parallel \mu) \leq Cn \mathbb{E}_{(i,x)}[D_{KL}(p_{i}^{x} \parallel q_{i}^{x})],
\]

where \( i \in [n] \) is a uniformly random coordinate, \( x \in \mathcal{K}^{n \setminus i} \) is generated from the marginal distribution \( \pi_{n \setminus i}, p_{i}^{x} = \pi_{i}(\cdot \mid x) \), and \( q_{i}^{x} = \mu_{i}(\cdot \mid x) \). Therefore, the initial identity testing problem for \( \mu \) can be solved by distinguishing

\[
p_{i}^{x} = q_{i}^{x} \quad \text{for all pairs } (i,x) \quad \text{vs.} \quad \mathbb{E}_{(i,x)}[D_{KL}(p_{i}^{x} \parallel q_{i}^{x})] \geq \frac{\varepsilon}{Cn}.
\]

If the latter happens, then using standard facts for nonnegative random variables one can show that there exists an integer \( \ell \geq 1 \), such that \( 2^{\ell} = O(n) \) and

\[
\mathbb{P}_{(i,x)}\left( D_{KL}(p_{i}^{x} \parallel q_{i}^{x}) \geq 2^{\ell} \frac{\varepsilon}{Cn} \right) \geq \frac{1}{2^{\ell}}.
\]

Observe that we can use the General Oracle to generate the partial configuration \( x \) given \( i \). By exhaustively trying every \( \ell \) (note that \( \ell = O(\log n) \)), one can find a pair \((i,x)\) such that the event \( D_{KL}(p_{i}^{x} \parallel q_{i}^{x}) \geq 2^{\ell} \frac{\varepsilon}{Cn} \) occurs with probability \( \geq \frac{1}{2^{\ell}} \). Hence, we have reduced the initial to testing problem to solving identity testing for the two distributions \( p := p_{i}^{x} \) and \( q := q_{i}^{x} \) on a domain of size \( k \) with respect to the KL divergence. We assume \( q \) can be computed from the visible distribution \( \mu \), and we have access to a sampling oracle for \( p_{i}^{x} \) using the Coordinate Oracle for \( \pi \).

In the distribution testing literature, testing problems in terms of KL divergence seem to be largely overlooked. This is likely because one can have the KL divergence between the hidden distribution \( p \) and the visible distribution \( q \) to be infinity, but their TV distance could be arbitrary small, and so identity testing requires arbitrarily many samples. For example, as mentioned in [DKW18], this could happen when \( q \) is the distribution on a single point \( 0 \) and \( p \) is the Bernoulli distribution over \( \{0,1\} \) with mean \( \xi \) arbitrarily small. For this reason, in [DKW18] the identity testing problem for KL divergence is considered to be untestable, in the sense of worst-case (over \( q \)) sample complexity. However, the testing problem for KL divergence still makes sense for specific visible distributions \( q \); e.g., for the uniform distribution.

We are interested here in the instance-specific sample complexity instead of the worst-case one: for a given distribution \( q \), what is the number of samples required, potentially depending on \( q \), for identity testing against \( q \) for KL divergence? In the following lemma, we give a first attempt at solving this problem. The sample complexity of our testing algorithm depends on the minimum nonzero probability \( \eta = \min_{a} q(a) \). See Lemmas 4.3 and 4.9 for proofs and more discussions.

**Lemma 2.1.** Let \( k \in \mathbb{N}^{+} \) be an integer, and let \( \varepsilon > 0, \eta \in (0,1/2] \) be real numbers. Given a visible distribution \( q \) over domain \( \mathcal{K} \) of size \( k \) such that \( q(a) \geq \eta \) for any \( a \in \mathcal{K} \), and given sample access to an unknown distribution \( p \) over \( \mathcal{K} \), there exists a polynomial-time identity testing algorithm that distinguishes with probability at least 2/3 between the cases:

\[
p = q \quad \text{vs.} \quad D_{KL}(p \parallel q) \geq \varepsilon.
\]

(2)

For \( k \geq 3 \), the sample complexity of the identity testing algorithm is

\[
O\left( \min\left\{ \frac{1}{\sqrt{\eta}}, \frac{\sqrt{k\ln(1/\eta)}}{\varepsilon^{2}} \right\} \right).
\]

For \( k = 2 \), the sample complexity of the identity testing algorithm is

\[
O\left( \frac{\ln(1/\eta)}{\varepsilon} \right).
\]
We remark that the dependency on $\eta$ in the sample complexity is inevitable. This is because, for fixed $\varepsilon$, the TV distance $d_{TV}(p, q)$ could tend to zero if one lets $\eta$ go to zero, while the KL divergence stays the same value $D_{KL}(p \parallel q)$ for all $\eta$. For example, consider $k = 2$ and suppose both $p$ and $q$ are Bernoulli random variables with means $p, q \in (0, 1/2)$, respectively, with a slight abuse of notation. Then, one can have $D_{KL}(\text{Ber}(p) \parallel \text{Ber}(q)) = 0.1$ fixed for $p = p(q)$ as a function of $q$, and $d_{TV}(\text{Ber}(p), \text{Ber}(q)) = |p-q| \rightarrow 0$ as $q \rightarrow 0$. However, Lemma 2.1 shows that the sample complexity for identity testing with respect to KL divergence depends, in the worst case, logarithmically on $1/\eta$.

In particular, for uniformity testing under KL divergence, Lemma 2.1 gives an $O(\sqrt{k}/\varepsilon)$ sample complexity which matches what one expects from the $O(\sqrt{k}/\varepsilon^2)$ sample complexity for uniformity testing for TV distance and the Pinsker’s inequality relating KL divergence and TV distance.

### 2.2 Computational hardness of Coordinate Oracle model: Theorem 1.3

We establish hardness of the identity problem as stated in Theorem 1.3 for the antiferromagnetic Ising model on $\mathbb{F}$ with Coordinate Oracle and General Oracle access via a reduction from the maximum cut problem. Suppose $(G = (V_G, E_G), k)$ is an instance of the maximum cut problem. That is, we want to check whether $\max\text{-cut}(G) < k$ or $\max\text{-cut}(G) \geq k$. For our reduction, we construct an identity testing instance for the antiferromagnetic Ising model, feed it as input to a presumed testing algorithm, and claim that the output of the algorithm solves $(G = (V_G, E_G), k)$.

As the first step in our reduction, we construct the multi-graph $F = (V_F, E_F)$ by adding two special vertices $s$ and $t$ to $G$; i.e., $V_F = V_G \cup \{s, t\}$. These two vertices are connected with $N^2 - k$, where $N = |V_G|$. We also add $N$ edges between $s$ and each vertex of $V_G$ and do the same for $t$.

This way we ensure that:

1. When $\max\text{-cut}(G) < k$, the cut $(\{s, t\}, V_G)$ is the unique maximum cut of $F$ and has size $2N^2$;
2. When $\max\text{-cut}(G) \geq k$, there exists another cut in $F$—in addition to $(\{s, t\}, V_G)$—whose size is at least $2N^2$. This cut is obtained by taking the maximum cut $(S, V_G \setminus S)$ for $G$ and adding $s$ and $t$ to opposite sides of it; note that the size of this cut of $F$ is

$$\max\text{-cut}(G) + |S|N + |V_G \setminus S|N + N^2 - k = 2N^2 + \max\text{-cut}(G) - k.$$ 

We consider the antiferromagnetic Ising model on $F$. Observe that each cut $(S, V_F \setminus S)$ of $F$ corresponds to exactly two Ising configurations: $S$ is assigned $+1$ and $V_F \setminus S$ is assigned $-1$ and vice versa. Note also that the “ground states” of the Ising model on $F$, that is the configurations of maximum probability, correspond precisely to the maximum cuts of $F$.

Let $\Omega$ be the set of all cuts of $F$ and let $\Omega_0$ be the set of all cuts $(S, V_F \setminus S)$ of $F$ except those where $s \in S$, $t \in V_F \setminus S$, and the corresponding cut for $G$, i.e., $(S \setminus \{s\}, V_F \setminus \{S, t, s\})$, has size $\geq k$. Hence, if $\max\text{-cut}(G) < k$, then $\Omega_0 = \Omega$; also, if $\max\text{-cut}(G) \geq k$, then $\Omega \setminus \Omega_0$ contains (at least) a cut of $F$ corresponding to a cut of $G$ of size $\geq k$.

We set the visible distribution of our testing instance to be the Gibbs distribution $\mu_{F,\beta}$ of the antiferromagnetic Ising model on $F$ with $\beta < \beta_c(d) = -\frac{1}{2}\ln\left(\frac{d+2}{d}\right)$ in the non-uniqueness region (the requirement of $\beta < \beta_c(d)$ will be explained in what follows). The hidden distribution will be $\mu_{F,\beta}(\cdot \mid \Omega_0)$; that is, $\mu_{F,\beta}$ conditioned on configurations that correspond to cuts in $\Omega_0$. Our construction ensures that if $\max\text{-cut}(G) < k$, then $\mu_{F,\beta}(\cdot \mid \Omega_0) = \mu_{F,\beta}$. Moreover, it can be shown that when $\max\text{-cut}(G) \geq k$, the total variation distance between $\mu_{F,\beta}(\cdot \mid \Omega_0)$ and $\mu_{F,\beta}$ is $1 - o(1)$ since, intuitively, this is because $\Omega \setminus \Omega_0$ contains at least one other large cut of size at least $2N^2$, which is the size of $(\{s, t\}, V_G)$.

Our reduction is then completed by generating samples from $\mu_{F,\beta}(\cdot \mid \Omega_0)$ and giving these samples and $\mu_{F,\beta}$ to the identity testing algorithm as input. The testing algorithm is guaranteed
to succeed with probability at $2/3$. If the algorithm detects that the samples did not come from $\mu_{F,\beta}$, it means that $\max\text{-cut}(G) \geq k$; otherwise, it means that $\max\text{-cut}(G) < k$. Hence, we have a polynomial running time algorithm that solves the maximum cut problem with probability at least $2/3$, which is not possible unless $\text{RP} = \text{NP}$.

There are two important complications in this approach. First, note that $F$ is a multi-graph of unbounded degree, and our goal is to establish hardness for the class of antiferromagnetic Ising models graphs of maximum degree $d = O(1)$ when $\beta < \beta_c(d)$. Second, we do not know how to generate samples from $\mu_{F,\beta}(\cdot \mid \Omega_0)$ efficiently in polynomial time (approximately sampling from the antiferromagnetic Ising model is $\text{NP}$-hard in general).

Let us address first how we solve the issue of $F$ being a multi-graph with large maximum degree. For this, we use a “degree reducing” gadget introduced in [Sly10] to establish the computational hardness of approximate counting and sampling for antiferromagnetic spin systems in the non-uniqueness regime. In particular, each vertex in $F$ will be replaced by a gadget $H$ which consists of a (nearly) $d$-regular random bipartite graph with a relatively small number of trees attached to it; specifically, the leaves of each tree will be identified with unique vertices on the same side of the bipartite graph; see Section 5.1 for the precise construction. The root of these trees will be called ports and will be used to connect the gadgets as dictated by the edges of $F$. This results in a simple $d$-regular graph $\hat{F}$.

A key feature of the gadget $H$ is that in the tree non-uniqueness region $\beta < \beta_c(d)$, a sample from $\mu_{H,\beta}$ will have mostly $+1$’s on one side of $H$ and mostly $-1$’s on the other, or vice versa. Hence there are two possible “phases” ($+1$ and $-1$) for the gadget which simulate the spin assigned to the vertices of $F$; i.e., the phase of the gadget corresponds to the spin of the vertex of $F$. Therefore, in a configuration in $\hat{F}$, the phase of all the gadgets will determine a cut for $F$ and thus for $G$. Consequently, the same reduction described above for $F$ can be extended to $\hat{F}$.

As mentioned, we are also required to sample from $\mu_{\hat{F},\beta}(\cdot \mid \Omega_0)$. For this, we note that sampling a phase assignment from $\mu_{\hat{F},\beta}(\cdot \mid \Omega_0)$ is straightforward (see Lemma 5.3). We then sample the port configuration given the phase vector from $\Omega_0$. This is done via a rejection sampling procedure by noting that the marginal distribution on the ports is within $o(1)$ total variation distance of a suitably defined product distribution; this is the reason the trees were attached to the random bipartite graph in the gadget construction. Once the port configuration is sampled within the desired accuracy, we sample the configuration in each gadget given the configuration in the ports. For this we use a hybrid approach: we use the recent algorithm from [KLR22] for low-rank Ising models for one range of values of $\beta$ and polymer models (see [JKP20]) for the other. Both algorithms rely on the fact that the gadget is essentially a random bipartite graph and thus an expander.

Finally, we note that our reduction shows hardness for identity testing for the antiferromagnetic Ising model in the non-uniqueness $\beta < \beta_c(d)$ regime with access to General Oracle and to any (conditional) oracle for $\mu_{\hat{F},\beta}(\cdot \mid \Omega_0)$ that can be simulated in polynomial time. This includes the Coordinate Oracle and also the Pairwise Oracle.

### 2.3 Statistical lower bounds for Coordinate Oracle model: Theorems 1.4 and 1.5

We provide here an overview of our proof approach for Theorems 1.4 and 1.5 which establish information-theoretic lower bounds for uniformity testing over the binary hypercube $\{0,1\}^n$. Our proofs of the lower bounds use the same strategy as in several previous works in the distribution testing literature. We construct a family of “bad” distributions $B$, each of which has KL divergence or TV distance at least $\varepsilon$ from the uniform distribution over $\{0,1\}^n$. Then the lower bounds follow from, roughly speaking, the fact that the joint distribution of $L$ independent samples from the uniform distribution, and the joint distribution of $L$ independent samples from a bad $\pi \in B$ chosen
uniformly at random are close to each other.

Such an argument works nicely for non-adaptive identity testing algorithms, where the queries are predetermined before receiving any sample. To show the lower bounds for adaptive algorithms which is necessary with the presence of conditional sampling oracles, we need to consider the whole query history as was done in [CRS15, Nar21] for showing lower bounds with Pairwise Oracle. Informally speaking, a query history is a sequence of queries that the testing algorithm asks the oracle along with the outputs from the oracle. Each step, the tester determines, possibly randomly, a new query based on all previous queries that have been asked and all corresponding outputs from the oracle. The final output of the testing algorithm can be viewed as a (possibly randomized) function of the query history. Hence, we need to show that the following two processes generate close query histories in total variation distance.

The first is that in each step the algorithm computes a query and the oracle outputs a sample using the uniform distribution. In the second one, we first pick a bad distribution $\pi \in \mathcal{B}$ uniformly at random, and after that the oracle outputs samples using $\pi$. To show that the two query histories are close, we use ideas from [CRS15] and also the so-called hybrid argument in cryptography (see, e.g., [Gol04]). For each $\ell \leq L$ we consider a hybrid query history where the first $\ell$ queries are answered by the oracle using the uniform distribution, while the other $L - \ell$ queries are answered by a single $\pi \in \mathcal{B}$ chosen uniformly at random. Then it suffices to show that every pair of adjacent hybrid query histories are close to each other. Since two adjacent hybrid query histories differ only at one step, this can be done via careful calculations for a specific family $\mathcal{B}$ of bad distributions.

Our construction of bad distributions for Theorem 1.4 is inspired by studying approximate tensorization of entropy. In particular, our identity testing algorithm fails within $O(n/\varepsilon)$ steps if for most of pairs $(i, x)$ it holds $\pi_i(x \mid x) = \text{Ber}(1/2)$ but only for an $\varepsilon/n$ fraction of the pairs the KL divergence is large, which means we need $\Omega(n/\varepsilon)$ steps to be able to see it. For Subcube Oracle we would like to construct bad distributions with similar behavior. Namely, for most (random) conditioning on a (random) subset of coordinates the conditional distribution is the same as what one gets from the uniform distribution, and meanwhile with probability $O(\varepsilon/n)$ the KL divergence between the two conditional distributions is as large as $\Theta(n)$. We achieve this using the following type of construction. We pick a random subset $A$ of size $t$ such that $2^t = O(n/\varepsilon)$, and pick a vector $\sigma \in \mathcal{K}^n$. To generate a sample $x$ from the bad distribution $\pi = \pi_{A, \sigma}$, we first sample $x_A$ uniformly at random. If $x_A \neq \sigma_A$ then the other coordinates are sampled randomly, but if $x_A = \sigma_A$ then we take $x = \sigma$. One can check, with careful calculations, that such bad distributions satisfy our requirements. In particular, while the KL divergence for any such bad distribution to the uniform distribution is $\varepsilon$, the TV distance is $\varepsilon/n$ instead, and so a $\Omega(n/\varepsilon)$ lower bound is not a surprise for this construction of family of bad distributions.

Our bad distribution for Theorem 1.5 uses the constructions from the previous works [DDK19, CDKS20] for showing lower bounds when both visible and hidden distributions are from some structured family of high-dimensional distributions, in particular, Ising models and Bayesian networks. Each bad distribution is constructed by taking a perfect matching of all coordinates (we may assume $n$ is even) and consider the distribution such that coordinates from different pairs are independent of each other while in each pair the two coordinates are correlated with covariance $\Theta(\varepsilon/\sqrt{n})$. Then one can show that the joint distribution of $O(n/\varepsilon^2)$ samples from the uniform distribution and that from a bad distribution induced by a uniformly random perfect matching are close to each other. Furthermore, the Coordinate Oracle does not help much. For the uniform distribution, the outputs from Coordinate Oracle are just $\text{Ber}(1/2)$, the uniform Bernoulli random variables. Meanwhile, for a bad distribution the outputs from Coordinate Oracle are $\text{Ber}(1/2 \pm \xi)$ where $\xi = \Theta(\varepsilon/\sqrt{n})$. It is well-known that to distinguish between a fair coin $\text{Ber}(1/2)$ and a biased coin $\text{Ber}(1/2 \pm \xi)$ one needs at least $\Omega(1/\xi^2) = \Omega(n/\varepsilon^2)$ samples.
3 Preliminaries

In this section we gather a number of standard definitions and results that we will refer to in our proofs. Let \( k, n \in \mathbb{N}^+ \) be integers. Let \( \mathcal{K} = \{1, \ldots, k\} \) denote a finite alphabet set of size \( k \), and let \( \pi \) be an arbitrary distribution over \( \mathcal{K}^n \). Throughout the paper, we use \( n \) in the subscript and superscript to represent the set \( [n] = \{1, \ldots, n\} \) and use \( n \setminus i \) to represent the set \( [n] \setminus \{i\} \) to ease the notation.

### 3.1 Coordinate conditional sampling oracle

We recall next the formal definitions of the various sampling oracles discussed in the paper.

**Definition 3.1.** The sampling oracles for the hidden distribution \( \pi \) are defined as follows:

- **General Sampling Oracle** (General Oracle): Generate a sample \( x \) from \( \pi \).

- **Coordinate Conditional Sampling Oracle** (Coordinate Oracle): Given \( i \in [n] \) and \( x \in \mathcal{K}^{n \setminus i} \) as inputs to the oracle:
  - If \( \pi(X_{n \setminus i} = x) > 0 \), the oracle samples \( a \in \mathcal{K} \) from the conditional marginal distribution \( \pi(x_i = a \mid X_{n \setminus i} = x) \);
  - If \( \pi(X_{n \setminus i} = x) = 0 \), the oracle outputs \( a \in \mathcal{K} \) arbitrarily.

- **Subcube Conditional Sampling Oracle** (Subcube Oracle): Given \( \Lambda \subseteq [n] \) and \( x \in \mathcal{K}^\Lambda \) as inputs to the oracle:
  - If \( \pi(X_\Lambda = x) > 0 \), the oracle samples \( x' \in \mathcal{K}^{[n] \setminus \Lambda} \) from the conditional distribution \( \pi(x_{\Lambda \setminus \Lambda} = \cdot \mid X_\Lambda = x) \);
  - If \( \pi(X_\Lambda = x) = 0 \), the oracle outputs \( x' \in \mathcal{K}^{V \setminus \Lambda} \) arbitrarily.

- **Pairwise Conditional Sampling Oracle** (Pairwise Oracle): Given \( x, y \in \mathcal{K}^n \), the oracle returns \( x \) with probability \( \pi(x) / (\pi(x) + \pi(y)) \) and \( y \) otherwise.

We provide next two brief remarks noting that access to a **Coordinate Oracle** is a weaker assumption than access to a **Pairwise Oracle** or a **Subcube Oracle**.

**Remark 3.2.** **Pairwise Oracle** is generally a stronger oracle than **Coordinate Oracle**. When \( k = 2 \) and the state space is the binary hypercube \( \{0, 1\}^n \), this comparison is obvious, since **Coordinate Oracle** essentially generates samples conditioned in the set \( \{x, y\} \) where \( x \) and \( y \) differ in exactly one coordinate, while **Pairwise Oracle** can handle any pair vectors \( x, y \in \mathcal{K}^n \). If \( k \geq 3 \) is a constant (independent of \( n \)), then one can also simulate an \( \varepsilon \)-approximate **Coordinate Oracle** with **Pairwise Oracle** access in \( \text{poly}(k, \log(1/\varepsilon)) \) time (or a perfect one with \( \text{poly}(k) \) expected time) using a Markov chain. Given a query \((i, x)\) where \( i \in [n] \) and \( x \in \mathcal{K}^{n \setminus i} \), to generate a random value at the coordinate \( i \) conditional on \( x \), one can simulate the Markov chain that in each step picks an element \( a \in \mathcal{K} \) uniformly at random and lets \( a_{t+1} = a \) with probability \( \mu(x_{i,a}) / (\mu(x_{i,a}) + \mu(x_{i,a})) \) and \( a_{t+1} = a \) otherwise; \( x_{i,a} \) denotes the vector with the \( i \)-th coordinating being \( a \) and all other coordinates given by \( x \). Every step of the Markov chain can be perfectly implemented with the **Pairwise Oracle**, and a simple coupling argument shows that the \( \varepsilon \)-mixing time is \( \text{poly}(k, \log(1/\varepsilon)) \). For perfect sampling with \( \text{poly}(k) \) expected time, one can use the Coupling from the Past Method (see [PW96]).

**Remark 3.3.** As mentioned earlier, the power of **Subcube Oracle** subsumes the combination of **Coordinate Oracle** + **General Oracle**, implying that:
• Algorithms with both Coordinate Oracle + General Oracle access immediately give algorithms with Subcube Oracle access;
• Lower bounds for the Subcube Oracle model imply lower bounds with Coordinate Oracle + General Oracle access.

3.2 Identity testing

We provide next the formal definition of the identity testing problem for a distribution $\mu$ over $\mathcal{K}^n$. Let $d$ be any metric or divergence for distributions over $\mathcal{K}^n$.

**ID-TEST**$(d, \varepsilon; \mu)$.

**Input:** Description of a distribution $\mu$ over $\mathcal{K}^n$.

**Provided:** Access to Coordinate Oracle + General Oracle for an unknown distribution $\pi$ over $\mathcal{K}^n$.

**Goal:** Determine whether $\pi = \mu$, or $d(\pi, \mu) \geq \varepsilon$.

Let $F$ denote a family of distributions (with varying dimensions), each of which is supported on $\mathcal{K}^n$ for some integer $n \in \mathbb{N}^+$ and can be represented with $\text{poly}(n)$ parameters. We say an algorithm $A$ is an identity testing algorithm for the family $F$ if for every $\mu \in F$ it solves ID-TEST$(d, \varepsilon; \mu)$ with probability at least $2/3$. Note that the unknown distribution $\pi$ does not necessarily belong to the family $F$.

3.3 Coordinate balance and marginal boundedness

We say a distribution $\mu$ supported on $\mathcal{K}^n$ is $\eta$-balanced, if for every $i \in [n]$, every $x \in \mathcal{K}^n \setminus i$ with $\mu(X_{n \setminus i} = x) > 0$, and every $a \in \mathcal{K}$, one has

$$\text{either } \mu(X_i = a \mid X_{n \setminus i} = x) = 0, \text{ or } \mu(X_i = a \mid X_{n \setminus i} = x) \geq \eta.$$  

On the other hand, we say the distribution $\mu$ is $b$-marginally bounded if for every $\Lambda \subseteq [n]$, every $x \in \mathcal{K}^\Lambda$ with $\mu(X_{\Lambda} = x) > 0$, every $i \in [n] \setminus \Lambda$, and every $a \in \mathcal{K}$, one has

$$\text{either } \mu(X_i = a \mid X_{\Lambda} = x) = 0, \text{ or } \mu(X_i = a \mid X_{\Lambda} = x) \geq b.$$  

Note that marginal boundedness is a generalization of coordinate balance, and in particular, if in the definition of $b$-marginally bounded one restricts to $\Lambda$ where $|\Lambda| = n - 1$ then we obtain $b$-balanced. Hence, any $b$-marginally bounded distribution is also $b$-balanced. See also Remark 7.3 for a weaker version of marginal boundedness. Related notions to marginal boundedness appeared in [KM17, CLV21a].

3.4 Approximate tensorization of entropy

Let $\mu$ be a distribution over $\mathcal{K}^n$. For any non-negative function $f : \mathcal{K}^n \to \mathbb{R}_{\geq 0}$, the expectation of $f$ is defined to be $\mu(f) = \sum_{x \in \mathcal{K}^n} \mu(x)f(x)$, and the (relative) entropy of $f$ is defined as

$$\text{Ent}_\mu(f) = \mu(f \ln f) - \mu(f) \ln(\mu(f)),$$

with the convention that $0 \ln 0 = 0$.

Given a coordinate $i$ and a partial configuration $x \in \mathcal{K}^n \setminus i$ on all coordinates but $i$, one can define the entropy of the function $f$ with respect to the conditional distribution $\mu_i(\cdot \mid x)$, which we
denote by $\text{Ent}_i^x(f)$. Furthermore, we regard $\text{Ent}_i^x(f)$ as a function of $x$ and its expectation, when $x$ is generated from $\mu_{n\setminus i}$, is denoted as $\mu[\text{Ent}_i(f)]$. We are now ready to give the formal definition of approximate tensorization of entropy in the functional inequality form, as in [Ces01, CMT15, CP21].

**Definition 3.4** (Approximate Tensorization of Entropy: Functional Form). We say a distribution $\mu$ over $\mathcal{K}^n$ satisfies approximate tensorization of entropy with constant $C$ if for any nonnegative function $f : \mathcal{K}^n \to \mathbb{R}_{\geq 0}$ one has

$$\text{Ent}(f) \leq C \sum_{i=1}^{n} \mu[\text{Ent}_i(f)].$$

(3)

As mentioned in the introduction, approximate tensorization is an important tool for proving functional inequalities like the modified log-Sobolev inequality (MLSI); it is also useful for deriving optimal mixing time bounds for the Glauber dynamics. Although it is most often stated in this functional inequality form, mainly because of several useful analytic properties, in this paper we will consider its probabilistic version, as in [Mar19, GSS19].

For two distributions $\mu$ and $\pi$ over a discrete state space $\mathcal{K}^n$, we write $\pi \ll \mu$ if $\mu(x) = 0$ implies $\pi(x) = 0$ for any $x \in \mathcal{K}^n$, i.e., the support of $\pi$ is contained in the support of $\mu$. The Kullback–Leibler (KL) divergence is defined as

$$D_{\text{KL}}(\pi \| \mu) = \sum_{x \in \mathcal{K}^n} \pi(x) \ln \left( \frac{\pi(x)}{\mu(x)} \right).$$

The following definition of approximate tensorization is slightly more general than Definition 1.1 from the introduction.

**Definition 3.5** (Approximate Tensorization of Entropy: Probabilistic Form). We say a distribution $\mu$ over $\mathcal{K}^n$ satisfies approximate tensorization of entropy with constant $C$ if for any distribution $\pi$ over $\mathcal{K}^n$ such that $\pi \ll \mu$ one has

$$D_{\text{KL}}(\pi \| \mu) \leq C \sum_{i=1}^{n} \mathbb{E}_{x \sim \pi_{n\setminus i}} \left[ D_{\text{KL}}(\pi_i(\cdot \mid x) \| \mu_i(\cdot \mid x)) \right].$$

(4)

Note that in Definition 1.1 we required that $\mu$ has full support, instead of the more general assumption $\pi \ll \mu$. We remark that in Eq. (4) the partial configuration $x \in \mathcal{K}^n \setminus i$ is drawn from $\pi$ rather than $\mu$. It is easy to check that the two definitions Definitions 3.4 and 3.5 are equivalent to each other by letting $f = \pi/\mu$; see [Mar19].

### 4 Identity Testing via Approximate Tensorization

For integer $k \geq 2$ and real $C \geq 1, \eta > 0$, let $\mathcal{F}_k(C, \eta)$ denote the family of all distributions over $\mathcal{K}^n$ (for any $n \in \mathbb{N}^+$) with $\text{poly}(n)$ many parameters that are $\eta$-balanced and satisfy approximate tensorization of entropy with constant $C$. The goal of this section is to give an identity testing algorithm for the family $\mathcal{F}_k(C, \eta)$ in terms of the KL divergence. We observe that this also implies a tester for the TV distance by the Pinsker’s inequality.

For applications in Section 4.4 all the parameters $k, C, \eta$ are constants independent of $n$. In the theorem below, however, we consider a more general setting where these parameters are functions of the dimension $n$ with only mild assumptions on their growth rate. This allows us to have a clearer picture on the sample complexity and the dependency on all the parameters involved.
**Theorem 4.1.** Let \( k = k(n) \) be an integer and \( C = C(n) \geq 1, \eta = \eta(n) \in (0, 1/2] \) be reals. Suppose that \( \max\{\log C, \log \log(1/\eta)\} = O(\log n) \). There is an identity testing algorithm for the family \( \mathcal{F}_k(C, \eta) \) with query access to both **Coordinate Oracle** and **General Oracle** and for KL divergence with distance parameter \( \varepsilon > 0 \). The query complexity of the identity testing algorithm is

\[
O \left( \min \left\{ \frac{C}{\sqrt{\eta}} \cdot \frac{n}{\varepsilon} \log^3 \left( \frac{n}{\varepsilon} \right), \ C^2 \sqrt{k} \log \left( \frac{1}{\eta} \right) \cdot \frac{n^2}{\varepsilon^2} \log^2 \left( \frac{n}{\varepsilon} \right) \right\} \right).
\]

The running time of the algorithm is polynomial in all parameters (\( 1/\eta \) for the first bound, and \( \log(1/\eta) \) for the second) and also proportional to the time of computing the conditional marginal distributions \( \mu_i(\cdot | x) \) for any \( i \in [n] \) and any feasible \( x \in \mathcal{K}^{n \setminus i} \). Furthermore, if \( k = 2 \), i.e., we have a binary domain \( \mathcal{K} = \{0, 1\} \), the query complexity can be improved to

\[
O \left( C \log \left( \frac{1}{\eta} \right) \cdot \frac{n}{\varepsilon} \log^3 \left( \frac{n}{\varepsilon} \right) \right).
\]

### 4.1 Algorithm

Before presenting our algorithm, we first give a well-known fact, e.g., see [Nar21, Proposition 6.7].

**Lemma 4.2.** Let \( \varepsilon, M > 0 \) be reals and let \( L = \lceil \log_2(M/\varepsilon) \rceil \). If \( Y \) is a non-negative random variable such that \( Y \leq M \) always and \( \mathbb{E}Y \geq \varepsilon \), then there exists a non-negative integer \( \ell \leq L \) such that

\[
\Pr(Y \geq 2^{\ell-1} \varepsilon) \geq \frac{1}{2^\ell(L+1)}.
\]

**Proof.** Suppose for sake of contradiction that for all \( 0 \leq \ell \leq L \) it holds

\[
\Pr(Y \geq 2^{\ell-1} \varepsilon) < \frac{1}{2^\ell(L+1)}.
\]

Notice that \( 2^L \varepsilon \geq M \). Then we have

\[
\mathbb{E}Y = \int_0^M \Pr(Y \geq y) dy = \int_0^{\varepsilon/2} \Pr(Y \geq y) dy + \sum_{\ell=0}^L \int_{2^{\ell-1} \varepsilon}^{2^\ell \varepsilon} \Pr(Y \geq y) dy
\]

\[
\leq \frac{\varepsilon}{2} + \sum_{\ell=0}^L \frac{(2^\ell \varepsilon - 2^{\ell-1} \varepsilon) \Pr(Y \geq 2^{\ell-1} \varepsilon)}{2^{\ell-1} \varepsilon}
\]

\[
< \frac{\varepsilon}{2} + \sum_{\ell=0}^L 2^{\ell-1} \varepsilon \cdot \frac{1}{2^\ell(L+1)} = \varepsilon,
\]

which is a contradiction. \( \square \)

For \( i \in [n] \) and \( x \in \mathcal{K}^{n \setminus i} \), we define \( q_i^x = \mu_i(\cdot | x) \) to be a distribution over \( \mathcal{K} \) induced by the pair \( (i, x) \) from \( \mu \), where we think of \( i \) and \( x \) as the parameters. Similarly, we define \( p_i^x = \pi_i(\cdot | x) \) with respect to \( \pi \).

Recall that the approximate tensorization of entropy for \( \mu \) can be written as

\[
D_{\text{KL}}(\pi \| \mu) \leq C \sum_{i=1}^n \mathbb{E}_{x \sim \pi_{n \setminus i}} \left[ D_{\text{KL}}(\pi_i(\cdot | x) \| \mu_i(\cdot | x)) \right] = Cn \mathbb{E}_{(i, x)} \left[ D_{\text{KL}}(q_i^x \| q_i^x) \right],
\]

14
where \( i \in [n] \) is a uniformly random coordinate and \( x \) is generated from the marginal distribution \( \pi_{n \setminus i} \). Therefore, the original identity testing problem boils down to the following testing problem:

\[
\Pr_{(i,x)}(p_i^x = q_i^x) = 1 \quad \text{v.s.} \quad \mathbb{E}_{(i,x)}[D_{\text{KL}}(p_i^x \| q_i^x)] \geq \varepsilon'
\]

where \( \varepsilon' = \varepsilon/(Cn) \). Notice that \( D_{\text{KL}}(p_i^x \| q_i^x) \leq \ln(1/\eta) \) for all \((i,x)\) assuming \( \eta \)-balance and \( p \ll q \). By Lemma 4.2 it further boils down to the following sequence of testing problems: let \( L = \lceil \log_2(\ln(1/\eta)/\varepsilon') \rceil \) and for each \( \ell \leq L \), for a random pair \((i,x)\), distinguish between \( p_i^x = q_i^x \) surely versus

\[
\Pr_{(i,x)}(D_{\text{KL}}(p_i^x \| q_i^x) \geq 2^{\ell-1}\varepsilon') \geq \frac{1}{2^\ell(L+1)}.
\]

For this testing problem, we sample \((i,x)\) randomly for \( O(2^\ell(L+1)) \) times to see the event \( D_{\text{KL}}(p_i^x \| q_i^x) \geq 2^{\ell-1}\varepsilon' \) happens if in the latter case, and when it happens the problem is reduced to a classical identity testing setting on a finite state space where we can apply previously known identity testing algorithm. To accomplish this we also give an identity testing algorithm for the KL divergence, which is missing in the literature; see Lemmas 4.3 and 4.9.

We give a few more definitions before presenting our algorithm formally. For a distribution \( \pi \) over \( \mathcal{X} = \mathcal{K}^n \), we define the set \( \mathcal{X}' \) by

\[
\mathcal{X}' = \{(i,x) : i \in [n], x \in \mathcal{K}^{n \setminus i}\}
\]

to be the set of all pairs \((i,x)\) where \( i \) is one coordinate and \( x \) contains the values of all coordinates other than \( i \). We then define a distribution \( \pi' \) over \( \mathcal{X}' \) by

\[
\pi'(i,x) = \frac{1}{n} \pi_{n \setminus i}(x) = \frac{1}{n} \pi(X_{n \setminus i} = x),
\]

so that a sample from \( \pi' \) can be obtained in the following way: first pick \( i \in [n] \) uniformly at random, and then sample \( x \) from the marginal distribution \( \pi_{n \setminus i} \).

Our algorithm is given in Algorithm 1, which also appeared in the previous work [CCK+21] for uniformity testing over the binary hypercube \([0,1]^n\). We now give our proof of Theorem 4.1.

**Proof of Theorem 4.1.** Suppose first that \( \pi = \mu \). Then each time we call the KL tester in Line 9, it returns Yes with probability at least \( 1 - \delta \) since \( p_i^x = q_i^x \) for any \((i,x)\). If every time the result is Yes then Algorithm 1 will return Yes (i.e., \( \pi = \mu \)). By a simple union bound, the probability that Algorithm 1 mistakenly outputs No is at most

\[
\sum_{\ell=0}^L T_\ell \cdot \delta \leq 2^{L+3}(L+1) \cdot 2^{-2L-6} \leq \frac{1}{8},
\]

where the last inequality is due to \( L + 1 \leq 2L \).

Next assume that \( D_{\text{KL}}(\pi \| \mu) \geq \varepsilon \). Then by approximate tensorization of entropy we have \( \mathbb{E}_{(i,x)}[D_{\text{KL}}(p_i^x \| q_i^x)] \geq \varepsilon' \) where \( \varepsilon' = \varepsilon/(Cn) \). By Lemma 4.2, there exists a non-negative integer \( \ell \leq L \) such that

\[
\Pr(D_{\text{KL}}(p_i^x \| q_i^x) \geq 2^{\ell-1}\varepsilon') \geq \frac{1}{2^\ell(L+1)}.
\]

For this \( \ell \), the algorithm repeats for \( T_\ell = 2^{\ell+2}(L+1) \) times to find such a pair \((i,x)\) via the general sampling oracle; the probability that the algorithm fails to find such \((i,x)\) is upper bounded by

\[
\left(1 - \frac{1}{2^\ell(L+1)}\right)^{T_\ell} \leq \exp\left(-\frac{T_\ell}{2^\ell(L+1)}\right) = e^{-4} \leq \frac{1}{50}.
\]
Algorithm 1: Identity Testing for $F_k(C, \eta)$ for KL divergence

**Input:** Description (parametrization) of a given distribution $\mu \in F_k(C, \eta)$, query access to both Coordinate Oracle and General Oracle for an unknown distribution $\pi$, and distance parameter $\epsilon > 0$.

1. $\epsilon' \leftarrow \epsilon/(Cn)$;
2. $L \leftarrow \lceil \log_2(\ln(1/\eta)/\epsilon') \rceil$;
3. for $0 \leq \ell \leq L$ do
   4. $\epsilon_\ell \leftarrow 2^{\ell-1}\epsilon'$; /* Distance parameter */
   5. $\delta \leftarrow 2^{-2L-6}$; /* Failure probability */
   6. $T_\ell \leftarrow 2^{\ell+2}(L + 1)$; /* Need $T_\ell$ samples of $(i, x)$ to see $D_{KL}(p^i_x \parallel q^i_x) \geq \epsilon_\ell$ */
   7. for $t = 1, 2, \ldots, T_\ell$ do
      8. Sample $(i, x)$ from $\pi'$ via General Oracle for $\pi$;
      9. Call $A_{KL-ID}$ from Lemmas 4.3 and 4.9 to distinguish between $p^i_x$ and $q^i_x$ with distance parameter $\epsilon_\ell$ and failure probability $1/n^3$ (samples from $p^i_x$ are obtained via Coordinate Oracle for $\pi$); /* Check whether $D_{KL}(p^i_x \parallel q^i_x) \geq \epsilon_\ell$ */
     10. if $A_{KL-ID}$ returns No (i.e., $D_{KL}(p^i_x \parallel q^i_x) \geq \epsilon_\ell$) then
         11. Output: No (i.e., $D_{KL}(\pi \parallel \mu) \geq \epsilon$), and the algorithm ends;
     12. end
   end
   end
13. end

**Output:** Yes (i.e., $\pi = \mu$)

In the case that such a pair is successfully found, the KL tester in Line 9 will return No with probability at least $1 - \delta$, and hence then Algorithm 1 returns No. Therefore, if Algorithm 1 wrongly outputs Yes then either a good pair $(i, x)$ is not found, or the KL tester in Line 9 makes a mistake on a good pair $(i, x)$. The probability of outputting Yes is then upper bounded by $1/50 + \delta \leq 1/8$.

Finally, Lemma 4.3, combined with the amplification technique for failure probability (e.g., see [Can22, Lemma 1.1.1]), implies that the number of samples required by Algorithm 1 is at most

$$
\sum_{\ell=0}^{L} T_\ell \cdot O\left(\min\left\{\frac{\ln(1/\delta)}{\epsilon_\ell \sqrt{\eta}}, \frac{\sqrt{k} \ln(1/\eta) \ln(1/\delta)}{\epsilon_\ell^2}\right\}\right)
= \sum_{\ell=0}^{L} O\left(\min\left\{\frac{CL^2n}{\epsilon \sqrt{\eta}}, \frac{C^2 \sqrt{k} L^2 n^2 \ln(1/\eta)}{2^\ell \epsilon^2}\right\}\right)
= O\left(\min\left\{\frac{CL^3 n}{\epsilon \sqrt{\eta}}, \frac{C^2 \sqrt{k} L^2 n^2 \ln(1/\eta)}{\epsilon^2}\right\}\right).
$$

Since $L = O(\log(n/\epsilon))$ under our assumptions $\log(C) = O(\log n)$ and $\log \log(1/\eta) = O(\log n)$, we obtain the sample complexity upper bound from the theorem. For $k = 2$ the sample complexity is obtained in the same way, using Lemma 4.9 instead.

4.2 Identity testing for KL divergence on general domain

In this and next subsection, we prove Lemma 2.1 from Section 2.1, which is also a key subroutine of Algorithm 1. We first consider general $k \geq 2$ in this subsection, and then give an improved sample
complexity bound for $k = 2$ in the next subsection.

**Lemma 4.3.** Let $k \in \mathbb{N}^+$ be an integer, and let $\varepsilon > 0$, $\eta \in (0, 1/2]$ be reals. Given a target distribution $q$ over domain $\mathcal{K}$ of size $k$ such that either $q(a) = 0$ or $q(a) \geq \eta$ for any $a \in \mathcal{K}$, and given sample access to an unknown distribution $p \ll q$ over $\mathcal{K}$, there exists a polynomial-time identity testing algorithm that distinguishes with probability at least $2/3$ between the two cases

$$p = q \quad \text{and} \quad D_{\text{KL}}(p \| q) \geq \varepsilon$$

with sample complexity

$$O \left( \min \left\{ \frac{1}{\varepsilon \sqrt{\eta}}, \frac{\sqrt{k} \ln(1/\eta)}{\varepsilon^2} \right\} \right).$$

We remark that the running time is polynomial in $1/\eta$ if we apply the first bound, and $\log(1/\eta)$ for the second.

We need the following standard comparisons among difference statistical divergence between two distributions. See [Can22] for more.

**Lemma 4.4.** Let $q$ be a distribution fully supported on a finite set $\mathcal{K}$, and let $\eta = \min_{a \in \mathcal{K}} q(a)$. Then for any distribution $p$ over $\mathcal{K}$ with $p \ll q$ it holds

$$D_{\text{KL}}(p \| q) \leq \chi^2(p \| q) \leq \frac{1}{\eta} \|p - q\|_2^2 \leq \frac{2}{\eta} d_{\text{TV}}(p, q)^2.$$

Our KL tester uses the following identity testing algorithm from [DK16] for $\ell_2$ distance, and also the flattening technique proposed there to reduce the $\ell_2$ norm. See also [Can22, Theorem 2.2.2] for an exposition of the algorithms and techniques.

**Lemma 4.5 ([DK16]).** Given the distance parameter $\varepsilon > 0$, full description of the target distribution $q$ with domain $\mathcal{K}$ of size $k$, and general sample access to an unknown distribution $p$ over $\mathcal{K}$, there exists a polynomial-time identity testing algorithm $A_{\ell_2\text{-id}}$ that distinguishes with probability at least $2/3$ between the two cases

$$\|p - q\|_2 \leq \frac{\varepsilon}{2} \quad \text{and} \quad \|p - q\|_2 \geq \varepsilon$$

with sample complexity

$$O \left( \max \left\{ \frac{\|q\|_2}{\varepsilon^2}, \frac{1}{\varepsilon} \right\} \right).$$

We are now ready to give our proof of Lemma 4.3.

**Proof of Lemma 4.3.** Without loss of generality we may assume that $q$ is fully supported over $\mathcal{K}$; i.e., $q(a) \geq \eta$ for all $a \in \mathcal{K}$. We establish the two bounds in the lemma separately using two testing algorithms depending on the range of parameters, both based on the $\ell_2$ tester in [DK16]. To clarify, our testing algorithm will check the two bounds $1/(\varepsilon \sqrt{\eta})$ and $\sqrt{k/\varepsilon^2} \ln(1/\eta)$, find the smaller one, and run the algorithm for that bound.

**Algorithm 1.** Our plan is to use Lemma 4.4 to transform the original problem for KL into an identity testing problem for $\ell_2$ distance, and then apply the $\ell_2$ tester from Lemma 4.5 given by [DK16]. To get a nice bound on the number of samples needed, we would like to have the $\ell_2$ norm $\|q\|_2$ as small as possible, while still keeping $q_{\min} := \min_{a \in \mathcal{K}} q(a) \geq \eta$; the reason for the latter is that we will lose a factor of $1/q_{\min}$ when applying Lemma 4.4. Hence, ideally we would like to have
\(q(a) = \Theta(\eta)\) for all \(a \in \mathcal{K}\), and then one has \(k = \Theta(1/\eta)\) and \(\|q\|_2 = O(\sqrt{\eta^2 k}) = O(\sqrt{\eta})\). This is achievable via the flattening technique from [DK16], which we explain below.

We construct a new instance \(p', q'\) (including the oracle for \(p'\)) of the identity testing problem from \(p, q\) such that \(\eta/2 \leq q'(a) \leq \eta\) for all \(a \in \mathcal{K}'\), where \(\mathcal{K}'\) is a new domain of size \(k' = \Theta(1/\eta)\). We show that the new identity testing problem with \(p', q'\) is equivalent to the original one with \(p, q\), requiring the same number of samples. And we can apply Lemma 4.5 to the new instance \(p', q'\) with nicer properties to obtain a better bound on the sample complexity instead of doing it directly to \(p, q\). For each \(a \in \mathcal{K}\), define

\[
k_a = \left\lfloor \frac{q(a)}{\eta} \right\rfloor + 1.
\]

We split each \(a \in \mathcal{K}\) into \(k_a\) distinct copies, denoted by \(a_1, \ldots, a_{k_a}\), which constitute the new domain \(\mathcal{K}'\); namely,

\[
\mathcal{K}' = \{a_i : a \in \mathcal{K}, 1 \leq i \leq k_a\}.
\]

Notice that the size \(k' = |\mathcal{K}'|\) of the new domain is bounded by

\[
k' = \sum_{a \in \mathcal{K}} k_a \leq \sum_{a \in \mathcal{K}} \left( \frac{q(a)}{\eta} + 1 \right) = \frac{1}{\eta} + k \leq \frac{2}{\eta},
\]

where the last inequality follows from \(\eta \leq 1/k\) since \(q\) is fully supported on \(\mathcal{K}\). The new target distribution \(q'\) is given by for every \(a \in \mathcal{K}\) and \(i \in [k_a]\),

\[
q'(a_i) = \frac{q(a)}{k_a},
\]

and similarly \(p'\) is given by \(p'(a_i) = p(a)/k_a\). We can easily transform a sample from \(p\) into a sample from \(p'\): if we receive \(a\) as a sample from \(p\), then we can compute \(k_a\) (since the target distribution \(q\) is given in full description) and generate \(i \in [k_a]\) uniformly at random, so that \(a_i\) is a sample from the distribution \(p'\). The crucial fact here is that the KL divergence (more generally, any \(f\)-divergence) is preserved under flattening. Indeed, observe that

\[
D_{KL}(p' \| q') = \sum_{a \in \mathcal{K}} \sum_{i \in [k_a]} p'(a_i) \ln \frac{p'(a_i)}{q'(a_i)} = \sum_{a \in \mathcal{K}} \sum_{i \in [k_a]} \frac{p(a)}{k_a} \ln \frac{p(a)}{q(a)} = D_{KL}(p \| q).
\]

Thus, we only need to solve the identity testing problem for the flattened distribution \(p'\) and \(q'\). Moreover, we observe that for all \(a \in \mathcal{K}\) and \(i \in [k_a]\) it holds

\[
\frac{\eta}{2} \leq q'(a_i) = \frac{q(a)}{k_a} \leq \eta,
\]

since we have

\[
\frac{2q(a)}{\eta} \geq k_a = \left\lfloor \frac{q(a)}{\eta} \right\rfloor + 1 \geq \frac{q(a)}{\eta},
\]

where the first inequality follows from \(q(a) \geq \eta\). Therefore, we observe that

\[
\|q'\|_2^2 = \sum_{a_i \in \mathcal{K}'} q'(a_i)^2 \leq \eta \sum_{a_i \in \mathcal{K}'} q'(a_i) = \eta.
\]
Note that $D_{KL}(p' \parallel q') \leq (2/\eta) \|p' - q'\|_2^2$ by Lemma 4.4. Applying Lemma 4.5, we are able to distinguish between $p' = q'$ versus $\|p' - q'\|_2^2 \geq \varepsilon \eta /2$, and hence between $p' = q'$ versus $D_{KL}(p' \parallel q') \geq \varepsilon$, using
\[
O\left( \max \left\{ \frac{\|q'\|_2}{\varepsilon \eta}, \frac{1}{\sqrt{\varepsilon \eta}} \right\} \right) = O\left( \frac{1}{\varepsilon \sqrt{\eta}} \right)
\]
samples from the unknown distribution $p'$. This then gives an identity testing algorithm for $p$ and $q$ for KL divergence using the same number of samples from $p$.

**Algorithm 2.** The previous algorithm works well when $1/\eta$ is not too large. To get a better dependency on $1/\eta$ as in the second bound, more work is required. Our first step is still flattening the distributions, but up to the scale $1/k$ instead of $\eta$. This is done exactly in [DK16] and [Can22, Theorem 2.2.2]. Let $k_a = \lfloor kq(a) \rfloor + 1$ for each $a \in \mathcal{K}$ and let $q'(a_i) = q(a)/k_a$ for each $a \in \mathcal{K}$ and $i \in [k_a]$. The flattened distributions $p'$ and $q'$ satisfies the following properties:

(a) Given an explicit description of $q$, one can efficiently give an explicit description of the flattened distribution $q$;

(b) Given access to the sampling oracle for $p$, one can efficiently generate samples from the flattened distribution $p'$;

(c) The KL divergence is preserved, i.e., $D_{KL}(p' \parallel q') = D_{KL}(p \parallel q)$;

(d) The size of the new domain is $k' \leq 2k$;

(e) For every $a_i \in \mathcal{K}'$, we have $\eta /2 \leq q'(a_i) \leq 2/k'$;

(f) We have $\|q'\|_2 \leq \sqrt{2/k'}$.

The proofs of these properties are the same as before or as in [Can22, Theorem 2.2.2] so we omit here. We only mention the lower bound on $q'(a_i)$: since $k_a = \lfloor kq(a) \rfloor + 1 \leq kq(a) + q(a)/\eta$ we have that
\[
q'(a_i) = \frac{q(a)}{k_a} \geq \frac{kq(a)}{kq(a) + q(a)/\eta} = \frac{\eta}{k\eta + 1} \geq \frac{\eta}{2}.
\]

Therefore, it suffices to consider the identity testing problem with respect to distributions $p'$ and $q'$ satisfying properties (e) and (f). For ease of notation, in the rest of the proof we assume that our $p, q$ are already flattened to satisfy (e) and (f), instead of writing $p', q'$ and $k'$.

Our second step is to divide elements in $\mathcal{K}$ into two classes, those with larger probability mass and those with smaller one, and to upper bound the KL divergence by dealing with the two classes separately. Let
\[
\zeta = \frac{\varepsilon}{10k \ln(2/\eta)},
\]
and let $\mathcal{K}_1 = \{a \in \mathcal{K} : q(a) \geq \zeta \}$ and $\mathcal{K}_2 = \{a \in \mathcal{K} : \eta /2 \leq q(a) < \zeta \}$. Hence, $\mathcal{K}_1, \mathcal{K}_2$ forms a partition of $\mathcal{K}$. We upper bound the KL divergence of $p$ and $q$ as follows. Observe that
\[
D_{KL}(p \parallel q) = \sum_{a \in \mathcal{K}_1} p(a) \ln \frac{p(a)}{q(a)} + \sum_{a \in \mathcal{K}_2} p(a) \ln \frac{p(a)}{q(a)}.
\]  (7)

For the second term, we have
\[
\sum_{a \in \mathcal{K}_2} p(a) \ln \left( \frac{p(a)}{q(a)} \right) \leq \sum_{a \in \mathcal{K}_2} p(a) \ln (2 / \eta) = (\ln (2 / \eta)) p(\mathcal{K}_2).
\]  (8)
For the first term, we have
\[
\sum_{a \in \mathcal{K}_1} p(a) \ln \left( \frac{p(a)}{q(a)} \right) \leq \sum_{a \in \mathcal{K}_1} p(a) \left( \frac{p(a)}{q(a)} - 1 \right)
\]
\[= \sum_{a \in \mathcal{K}_1} (p(a) - q(a)) \left( \frac{p(a)}{q(a)} - 1 \right) + \sum_{a \in \mathcal{K}_1} q(a) \left( \frac{p(a)}{q(a)} - 1 \right)
\]
\[= \sum_{a \in \mathcal{K}_1} \frac{(p(a) - q(a))^2}{q(a)} + p(\mathcal{K}_1) - q(\mathcal{K}_1)
\]
\[\leq \frac{1}{\zeta} \|p - q\|^2 + q(\mathcal{K}_2) - p(\mathcal{K}_2),
\]
where the last inequality is because \( q(a) \geq \zeta \) for \( a \in \mathcal{K}_1 \). Therefore, combining Eqs. (7) to (9) we obtain
\[D_{KL}(p \parallel q) \leq \frac{1}{\zeta} \|p - q\|^2 + (\ln(2/\eta) - 1) p(\mathcal{K}_2) + q(\mathcal{K}_2).
\]
In particular, Eq. (10) directly implies the following fact.

**Fact 4.6.** If \( D_{KL}(p \parallel q) \geq \varepsilon \), then
\[\text{either } p(\mathcal{K}_2) \geq \frac{\varepsilon}{5 \ln(2/\eta)}, \text{ or } \|p - q\|^2 \geq \frac{4}{5} \varepsilon \zeta.
\]
To see this, suppose on contrary that \( p(\mathcal{K}_2) < \varepsilon/(5 \ln(2/\eta)) \) and \( \|p - q\|^2 < \frac{4}{5} \varepsilon \zeta \). Since we know
\[q(\mathcal{K}_2) \leq \zeta k = \frac{\varepsilon}{10 \ln(2/\eta)} < \frac{\varepsilon}{5 \ln(2/\eta)},
\]
we deduce from Eq. (10) that
\[\varepsilon \leq D_{KL}(p \parallel q) < \frac{4}{5} \varepsilon + \frac{1}{5} \varepsilon = \varepsilon,
\]
which is a contradiction.

Our identity testing algorithm proceeds by conducting two tests independently. In the first test, we try to distinguish between \( p(\mathcal{K}_2) = q(\mathcal{K}_2) \) and \( p(\mathcal{K}_2) \geq \varepsilon/(5 \ln(2/\eta)) \) with failure probability 1/6 and sample complexity \( m_1 = O(\ln(1/\eta)/\varepsilon) \). (If \( \mathcal{K}_2 = \emptyset \) then we do nothing in this first stage.) To be more precise, let \( X \) (respectively \( Y \)) be the indicator of the event that a sample drawn from \( p \) (respectively \( q \)) is contained in \( \mathcal{K}_2 \). So both \( X \) and \( Y \) are Bernoulli random variables, where the expectation \( q(\mathcal{K}_2) > 0 \) of \( Y \) is known to the algorithm, while the expectation \( p(\mathcal{K}_2) \) of \( X \) is unknown but we have sample access to \( X \) via samples from \( p \). We would like to distinguish between the two cases \( p(\mathcal{K}_2) = q(\mathcal{K}_2) \), i.e. \( X \) and \( Y \) are the same, and \( p(\mathcal{K}_2) \geq \varepsilon/(5 \ln(2/\eta)) \), i.e. \( X \) and \( Y \) are far from each other since \( q(\mathcal{K}_2) \leq \varepsilon/(10 \ln(2/\eta)) \). This is a standard property testing problem for Bernoulli random variables. We use the testing algorithm from Lemma 4.8 for
\[\gamma = \frac{\varepsilon}{5q(\mathcal{K}_2) \ln(2/\eta)} - 1 \geq 1
\]
with failure probability 1/6, using \( m_1 = O(\ln(1/\eta)/\varepsilon) \) samples from \( X \).
In the second stage, we run the tester from Lemma 4.5 to distinguish between \( p = q \) and \( \|p - q\|_2^2 \geq \frac{4}{5}\varepsilon \zeta \) with failure probability 1/6. Let \( m_2 \) be the number of samples that the \( \ell_2 \) tester uses, and we obtain from Lemma 4.5 that

\[
m_2 = O \left( \max \left\{ \frac{\|q\|_2}{\varepsilon \zeta}, \frac{1}{\sqrt{\varepsilon \zeta}} \right\} \right) = O \left( \frac{\sqrt{k} \ln(1/\eta)}{\varepsilon^2} \right),
\]

where we use the property (f) from flattening.

Suppose in both tests the outputs are Yes (i.e., \( p(K_2) = q(K_2) \) in the first and \( p = q \) in the second), then our identity testing algorithm will output Yes (i.e., \( p = q \)). If in at least one test the output is No, then our identity testing algorithm will output No (i.e., \( D_{\text{KL}}(p \parallel q) \geq \varepsilon \)). To finish up the proof, we still need to bound the failure probability and the number of samples needed for our testing algorithm. Suppose first that \( p = q \), and hence \( p(K_2) = q(K_2) \). Our testing algorithm wrongly outputs No if at least one of the two tests makes a mistake and outputs No. By a simple union bound, the probability of this is at most \( 1/6 + 1/6 = 1/3 \). On the other hand, if \( D_{\text{KL}}(p \parallel q) \geq \varepsilon \), then either \( p(K_2) \geq \varepsilon/(5\ln(2/\eta)) \) or \( \|p - q\|_2^2 \geq \frac{1}{5}\varepsilon \zeta \) by Fact 4.6, and so at least one of the two tests should output No if it does not make a mistake. Hence, the failure probability is at most 1/6. Finally, the number of samples we need is

\[
m_1 + m_2 = O \left( \frac{\ln(1/\eta)}{\varepsilon} \right) + O \left( \frac{\sqrt{k} \ln(1/\eta)}{\varepsilon^2} \right) = O \left( \frac{\sqrt{k} \ln(1/\eta)}{\varepsilon^2} \right).
\]

This establishes the second bound of the lemma.

\[\square\]

### 4.3 Identity testing for KL divergence on binary domain

If \( k = 2 \), i.e., we have a binary domain \( K = \{0, 1\} \), then the sample complexity for the KL tester is better.

For \( p \in [0, 1] \), the Bernoulli distribution denoted by \( \text{Ber}(p) \) is the distribution over \( \{0, 1\} \) such that \( \Pr(X = 1) = p \). We record below the standard Chernoff bounds.

**Lemma 4.7** (Chernoff bounds). Suppose \( X_1, \ldots, X_m \) are independent Bernoulli random variables from \( \text{Ber}(p) \) where \( p \in [0, 1] \). Let \( \hat{p} = \frac{1}{m} \sum_{i=1}^{m} X_i \) denote the sample mean. Then for all \( \delta \geq 0 \),

\[
\Pr \left( \hat{p} \leq (1 - \delta)p \right) \leq e^{-\delta^2 pm/2};
\]

\[
\Pr \left( \hat{p} \geq (1 + \delta)p \right) \leq e^{-\delta^2 pm/(2+\delta)} \leq \begin{cases} e^{-\delta^2 pm/3}, & 0 \leq \delta \leq 1; \\ e^{-\delta pm/3}, & \delta \geq 1. \end{cases}
\]

The following is a folklore fact.

**Lemma 4.8.** Let \( \gamma > 0 \) be a real number. Given \( q \in (0, 1/(1 + \gamma)] \) and sample access to \( \text{Ber}(p) \) with unknown \( p \in [0, 1] \), there exists a polynomial-time identity testing algorithm that distinguishes with probability at least 2/3 between the two cases

\[
p = q \quad \text{and} \quad p \geq (1 + \gamma)q
\]

with sample complexity

\[
O \left( \frac{1 + \gamma}{\gamma^2 q} \right) = \begin{cases} O \left( \frac{1}{\gamma^2 q} \right), & 0 < \gamma \leq 1; \\ O \left( \frac{1}{(1 + \gamma)q} \right), & \gamma \geq 1. \end{cases}
\]
Proof. Let \( \hat{p} \) denote the sample mean of \( m \) independent samples from \( \text{Ber}(p) \) where
\[
m = \left\lceil \frac{10(1 + \gamma)}{\gamma^2 q} \right\rceil.
\]
If \( \hat{p} \leq (1 + \gamma/2)q \), then the tester concludes \( p = q \); otherwise, it concludes \( p \geq (1 + \gamma)q \).

Suppose first \( p = q \). Then by the Chernoff bound Lemma 4.7 we have
\[
\Pr \left( \hat{p} \geq \left( 1 + \frac{\gamma}{2} \right) q \right) \leq \exp \left( -\frac{\gamma^2 q m}{2(4 + \gamma)} \right) \leq \frac{1}{3}.
\]
If \( p \geq (1 + \gamma)q \), then again by the Chernoff bound Lemma 4.7 we have
\[
\Pr \left( \hat{p} \leq \left( 1 + \frac{\gamma}{2} \right) q \right) \leq \Pr \left( \hat{p} \leq \left( 1 - \frac{\gamma}{2(1 + \gamma)} \right) q \right)
\leq \exp \left( -\frac{\gamma^2 q m}{8(1 + \gamma)^2} \right) \leq \exp \left( -\frac{\gamma^2 q m}{8(1 + \gamma)} \right) \leq \frac{1}{3}.
\]
Finally, for \( 0 < \gamma \leq 1 \) one has \( (1 + \gamma)/\gamma^2 \leq 2/\gamma^2 \), and for \( \gamma \geq 1 \) one has \( (1 + \gamma)/\gamma^2 \leq 4/(1 + \gamma) \), which completes the proof of the lemma.

We now give our testing algorithm for Bernoulli random variables.

**Lemma 4.9.** Let \( \epsilon > 0 \) be a real number. Given \( q \in (0, 1) \) and sample access to \( \text{Ber}(p) \) with unknown \( p \in [0, 1] \), there exists a polynomial-time identity testing algorithm that distinguishes with probability at least \( 2/3 \) between the two cases
\[
p = q \quad \text{and} \quad D_{\text{KL}}(\text{Ber}(p) \parallel \text{Ber}(q)) \geq \epsilon
\]
with sample complexity
\[
O \left( \frac{\ln(1/\eta)}{\epsilon} \right)
\]
where \( \eta = \min\{q, 1 - q\} \).

Proof. We may assume without loss of generality that \( q \leq 1/2 \) and \( \eta = q \), as otherwise we can flip the Bernoulli. For \( q \in (0, 1) \) and \( p \in [0, 1] \), we define
\[
\varphi_{\text{KL}}(p, q) = D_{\text{KL}}(\text{Ber}(p) \parallel \text{Ber}(q)) = p \ln \frac{p}{q} + (1 - p) \ln \frac{1 - p}{1 - q}.
\]
The testing algorithm is as follows. Let \( S < T \) be parameters which, as will be clear soon, depend on the distance parameter \( \epsilon \) and the mean \( q \) of the given Bernoulli (note that both \( \epsilon \) and \( q \) are known to the algorithm). Compute the sample mean \( \hat{p} \) for \( p \) using \( m \) independent samples from \( \text{Ber}(p) \). The testing algorithm determines \( p = q \) or \( \varphi_{\text{KL}}(p, q) \geq \epsilon \) by checking whether \( \hat{p} \) belongs to the interval \([S, T]\) or not. More specifically, if \( \hat{p} \in [S, T] \), then it outputs \( p = q \). If \( \hat{p} \notin [S, T] \), then it outputs \( \varphi_{\text{KL}}(p, q) \geq \epsilon \). We need to choose suitable \( S \) and \( T \) so that the algorithm is accurate with high probability and the number of samples required is minimized. Given \( q \in (0, 1/2] \) and \( \epsilon > 0 \), we will consider three separate cases.

**Case 1:** \( \epsilon \leq 2q \). We choose \( S = q - \sqrt{\epsilon q}/8 \), \( T = q + \sqrt{\epsilon q}/8 \), and let
\[
m = \left\lceil \frac{64}{\epsilon} \right\rceil.
\]
be the number of samples. If \( p = q \), then by the Chernoff bound Lemma 4.7 we have
\[
\Pr (\hat{p} \notin [S, T]) \leq \Pr \left( |\hat{p} - q| \geq \sqrt{\frac{\varepsilon q}{8}} \right) \leq 2 \exp \left( -\frac{\varepsilon m}{24} \right) \leq \frac{1}{3},
\]
where we use \( \sqrt{\varepsilon q/8} \leq q/2 \) by the assumption \( \varepsilon \leq 2q \).

Now suppose \( \varphi_{kl}(p, q) \geq \varepsilon \). By Lemma 4.4, we have
\[
\varepsilon \leq \varphi_{kl}(p, q) \leq \frac{2}{q}(p - q)^2,
\]
and hence either \( p \leq q - \sqrt{\varepsilon q/2} \) or \( p \geq q + \sqrt{\varepsilon q/2} \). Suppose \( p \leq 0 \) then trivially \( \Pr (\hat{p} \in [S, T]) = 0 \) since \( S = q - \sqrt{\varepsilon q/8} \geq q/2 > 0 \). If \( 0 < p \leq q - \sqrt{\varepsilon q/2} \), then again by the Chernoff bound Lemma 4.7 we have
\[
\Pr (\hat{p} \in [S, T]) \leq \Pr (\hat{p} \geq S) \leq \exp \left( -\frac{(S - p)^2 m}{2p + (S - p)} \right) \leq \exp \left( -\frac{\varepsilon m}{16} \right) \leq \frac{1}{3},
\]
where the second to last inequality follows from \( S + p \leq 2q \) and \( (S - p)^2 \geq \varepsilon q/8 \). If \( p \geq q + \sqrt{\varepsilon q/2} \), then Lemma 4.7 gives
\[
\Pr (\hat{p} \in [S, T]) \leq \Pr (\hat{p} \leq T) \leq \exp \left( -\frac{(p - T)^2 m}{2p} \right) \leq \exp \left( -\frac{\varepsilon m}{32} \right) \leq \frac{1}{3},
\]
where the second to last inequality follows from the fact that \( (p - T)^2/(2p) \) is minimized at \( p = q + \sqrt{\varepsilon q/2} \leq 2q \) and hence
\[
\frac{(p - T)^2}{2p} \geq \frac{q + \sqrt{\varepsilon q/2} - T}{2q + \sqrt{\varepsilon q/2}} \geq \frac{\varepsilon}{32}.
\]

**Case 2:** \( 2q < \varepsilon \leq 2q \ln(1/q) \). (This case is possible only for \( q < 1/e \).) Again if \( \varphi_{kl}(p, q) \geq \varepsilon \) then either \( p \leq q - \sqrt{\varepsilon q/2} \) or \( p \geq q + \sqrt{\varepsilon q/2} \). But since \( \varepsilon > 2q \), we have \( q - \sqrt{\varepsilon q/2} < 0 \) and hence it must be \( p \geq q + \sqrt{\varepsilon q/2} \). This means that, we need to distinguish between \( p = q \) versus
\[
p \geq q + \sqrt{\varepsilon q/2} \geq 2q
\]
as \( \varepsilon > 2q \). Therefore, we can apply the identity tester from Lemma 4.8 for \( \gamma = 1 \) with sample complexity
\[
m = O \left( \frac{1}{q} \right) = O \left( \frac{\ln(1/q)}{\varepsilon} \right)
\]
since \( \varepsilon \leq 2q \ln(1/q) \).

**Case 3:** \( \varepsilon > \max\{2q, 2q \ln(1/q)\} \). Just as in Case 2, if \( \varphi_{kl}(p, q) \geq \varepsilon \) then one must have \( p \geq q \), since \( p < q \) implies
\[
\varphi_{kl}(p, q) \leq \varphi_{kl}(0, q) = \ln \left( \frac{1}{1 - q} \right) \leq \frac{q}{1 - q} \leq 2q.
\]
Since \( p \geq q \), we have \( 1 - p \leq 1 - q \) and thus
\[
\varepsilon \leq \varphi_{kl}(p, q) = p \ln \frac{p}{q} + (1 - p) \ln \frac{1 - p}{1 - q} \leq p \ln \frac{p}{q} \leq p \ln \frac{1}{q}.
\]
Therefore, it suffices to distinguish between \( p = q \) and \( p \geq \varepsilon / (q \ln(1/q)) > 2q \). The identity tester from Lemma 4.8 for \( \gamma = \varepsilon / (q \ln(1/q)) - 1 \geq 1 \) can achieve \( 2/3 \) success probability with sample complexity
\[
m = O \left( \frac{\ln(1/q)}{\varepsilon} \right).
\]
This completes the proof of the lemma. \( \square \)
4.4 Applications

Here we give several applications of Theorem 4.1.

4.4.1 Product distributions

For each \( i \in [n] \) let \( \mu_i \) be an arbitrary distribution over \( \mathcal{K} \), and define a product distribution \( \mu = \mu_1 \otimes \cdots \otimes \mu_n \) over \( \mathcal{K}^n \). It is well-known that every product distribution satisfies approximate tensorization of entropy with an optimal constant \( C = 1 \).

**Lemma 4.10** ([Ces01, MSW03, CMT15]). Let \( \mu \) be an arbitrary product distribution over \( \mathcal{K}^n \). For any distribution \( \pi \) over \( \mathcal{K}^n \) such that \( \pi \ll \mu \), we have

\[
D_{KL}(\pi \parallel \mu) \leq n \sum_{i=1}^{n} \mathbb{E}_{x \sim \pi_{\setminus i}} \left[ D_{KL}(\pi_i(\cdot \mid x) \parallel \mu_i(\cdot \mid x)) \right].
\]

Namely, every product distribution satisfies approximate tensorization of entropy with constant 1.

For a product distribution \( \mu \), define \( \eta(\mu) = \min_{i \in [n]} \min_{a \in \mathcal{K}}: \mu_i(a) > 0 \). Observe that \( \mu \) is \( \eta(\mu) \)-balanced. Let \( \mathcal{P}(\eta) \) denote the collection of all product distributions \( \mu \) such that \( \eta(\mu) \geq \eta \).

The following corollary follows immediately from Theorem 4.1 and Lemma 4.10.

**Corollary 4.11.** Let \( \eta \in (0, 1/2] \) be real. There is a polynomial-time identity testing algorithm for the family \( \mathcal{P}(\eta) \) of \( \eta \)-balanced product distributions with query access to both Coordinate Oracle and General Oracle and for KL divergence with distance parameter \( \varepsilon > 0 \). The query complexity of the identity testing algorithm is \( O((n/\varepsilon) \log^3(n/\varepsilon)) \).

4.4.2 Sparse Ising models in the uniqueness region

An Ising model is a tuple \((G, \beta, h)\) where

- \( G = (V, E) \) is a finite simple graph;
- \( \beta : E \to \mathbb{R} \) is a function of edge couplings;
- \( h : V \to \mathbb{R} \) is a function of vertex external fields.

We may also view \( \beta \) and \( h \) as vectors; in particular, we write \( \beta_{uv} \) to represent the edge coupling of an edge \( \{u, v\} \in E \), and write \( h_v \) to represent the external field of a vertex \( v \in V \).

The Gibbs distribution of an Ising model \((G, \beta, h)\) is given by

\[
\mu_{(G, \beta, h)}(\sigma) = \frac{1}{Z_{(G, \beta, h)}} \exp \left( \sum_{\{u, v\} \in E} \beta_{uv} \sigma_u \sigma_v + \sum_{v \in V} h_v \sigma_v \right), \quad \forall \sigma \in \{+, -\}^V,
\]

where

\[
Z_{(G, \beta, h)} = \sum_{\sigma \in \{+, -\}^V} \exp \left( \sum_{\{u, v\} \in E} \beta_{uv} \sigma_u \sigma_v + \sum_{v \in V} h_v \sigma_v \right)
\]
is the partition function.

**Definition 4.12** (The family \( \mathcal{I}_S(\Delta, \delta, h^*) \) of Ising models in tree-uniqueness). For an integer \( \Delta \geq 3 \) and reals \( \delta \in (0, 1), h^* > 0 \), let \( \mathcal{I}_S(\Delta, \delta, h^*) \) be the family of Gibbs distributions of Ising models \((G, \beta, h)\) satisfying:
(1) The maximum degree of $G$ is at most $\Delta$;

(2) We have $(\Delta - 1) \tanh(\beta^*) \leq 1 - \delta$, where $\beta^* = \max_{(u,v)\in E} |\beta_{uv}|$ denotes the maximum edge coupling in absolute value;

(3) For each $v \in V(G)$, we have $|h_v| \leq h^*$.

Recent works towards establishing optimal mixing of Glauber dynamics have shown approximate tensorization of entropy for the family $\mathcal{IS}(\Delta, \delta, h^*)$.

**Lemma 4.13** ([CLV20, CLV21a]). For any integer $\Delta \geq 3$ and reals $\delta \in (0,1), h^* > 0$, there exists a constant $C = C(\Delta, \delta, h^*) \geq 1$, such that every Ising distribution $\mu$ from the family $\mathcal{IS}(\Delta, \delta, h^*)$ satisfies approximate tensorization of entropy with constant $C$.

We then deduce the following corollary from **Theorem 4.1** and **Lemma 4.13**.

**Corollary 4.14.** Suppose $\Delta \geq 3$ is an integer and $\delta \in (0,1), h^* > 0$ are reals. There is a polynomial-time identity testing algorithm for the family $\mathcal{IS}(\Delta, \delta, h^*)$ of Ising models with query access to both Coordinate Oracle and General Oracle and for KL divergence with distance parameter $\varepsilon > 0$. The query complexity of the identity testing algorithm is $O((n/\varepsilon) \log^3(n/\varepsilon))$.

### 4.4.3 Distributions satisfying Dobrushin uniqueness condition

Let $\mu$ be a distribution over $\mathcal{K}^n$. For $i, j \in [n]$, the Dobrushin influence of $i$ on $j$ is given by

$$a_{u,v} = \max_{(x,x') \in \mathcal{C}_{i,j}} d_{TV}(\mu_j(\cdot | X_n \setminus j = x), \mu_j(\cdot | X_n \setminus j = x')),$$

where $\mathcal{C}_{i,j}$ denotes the collection of all pairs $(x,x')$ of vectors in $\mathcal{K}^{n\setminus i}$ such that $\mu(X_n \setminus j = x) > 0$, $\mu(X_n \setminus j = x') > 0$, and $x,x'$ either are the same or differ exactly at the coordinate $i$. The Dobrushin influence matrix $A$ is an $n \times n$ matrix with entries given as above. Note that $A$ is not symmetric in general.

For $b \in (0,1/2]$, we say the distribution $\mu$ is $b$-marginally bounded if for every $\Lambda \subseteq [n]$, every $x \in \mathcal{K}^\Lambda$ with $\mu(X_\Lambda = x) > 0$, every $i \in [n] \setminus \Lambda$, and every $a \in \mathcal{K}$, one has

$$\text{either } \mu(X_i = a | X_\Lambda = x) \geq b \text{ or } \mu(X_i = a | X_\Lambda = x) = 0.$$

Note that though seemingly similar, the notion of marginal boundedness is not the same as the coordinate balance defined in **Section 3.3**. We observe that any $b$-marginally bounded distribution is also $b$-balanced.

For $\delta \in (0,1)$ and $b \in (0,1/2]$, let $\mathcal{D}(\delta, b)$ be the family of all distributions over $\mathcal{K}^n$ satisfying the following conditions:

(1) The Dobrushin influence matrix $A$ of $\mu$ satisfies $\|A\|_2 \leq 1 - \delta$;

(2) $\mu$ is $b$-marginally bounded.

Marton proved that every distribution from the family $\mathcal{D}(\delta, b)$ satisfies approximate tensorization of entropy.

**Lemma 4.15** ([Mar19]). Suppose $\delta \in (0,1)$ and $b \in (0,1/2]$ are reals. Every distribution $\mu$ from the family $\mathcal{D}(\delta, b)$ satisfies approximate tensorization of entropy with constant $C = 1/(b\delta^2)$.

The following corollary follows from **Theorem 4.1** and **Lemma 4.15**.
Corollary 4.16. Suppose \( \delta \in (0,1) \) and \( b \in (0,1/2) \) are reals. There is a polynomial-time identity testing algorithm for the family \( \mathcal{D}(\delta,b) \) with query access to both Coordinate Oracle and General Oracle and for KL divergence with distance parameter \( \varepsilon > 0 \). The query complexity of the identity testing algorithm is \( O((n/\varepsilon)\log^3(n/\varepsilon)) \).

For Ising models, there is also a stronger version of Dobrushin uniqueness in literature.

Definition 4.17 (The family \( \mathcal{I}_D(\delta,h^*) \) of Ising models in Dobrushin-uniqueness). For \( \delta \in (0,1) \) and \( h^* > 0 \), let \( \mathcal{I}_D(\delta,h^*) \) be the family of Gibbs distributions of Ising models \((G,\beta,h)\) satisfying:

1. For each \( v \in V(G) \), we have \( \sum_{u \in N(v)} |\beta_{uv}| \leq 1 - \delta \);
2. For each \( v \in V(G) \), we have \( |h_v| \leq h^* \).

Notice that in the Ising model we have \( a_{u,v} \leq \tanh(|\beta_{uv}|) \leq |\beta_{uv}| \) for \( \{u,v\} \in E \) and \( a_{u,v} = 0 \) for non-edges. So we have \( \mathcal{I}_D(\delta,h^*) \subseteq D(\delta,b) \) for \( b \geq 1/(e^{2(h^*+1)} + 1) \). Hence, the following corollary follows immediately from Corollary 4.16.

Corollary 4.18. Suppose \( \delta \in (0,1) \) and \( h^* > 0 \) are reals. There is a polynomial-time identity testing algorithm for the family \( \mathcal{I}_D(\delta,h^*) \) of Ising models with query access to both Coordinate Oracle and General Oracle and for KL divergence with distance parameter \( \varepsilon > 0 \). The query complexity of the identity testing algorithm is \( O((n/\varepsilon)\log^3(n/\varepsilon)) \).

4.5 Identity testing for TV distance

One of the main goals of this paper is to give efficient identity testing algorithms without any restriction on the noisy, unknown distribution \( \pi \). However, since we work with KL divergence in most parts of our algorithmic results, one assumption we have to make is that the support of the hidden distribution \( \pi \) is contained in that of the visible \( \mu \), denoted by \( \pi \ll \mu \). This is necessary for the KL divergence \( D_{KL}(\pi \| \mu) \) to be finite. However, we emphasize that this assumption is fairly mild and does not introduce any restriction in many settings for the following two reasons. (1) In many cases the visible distribution \( \mu \) is already fully supported on \( \mathcal{X}^n \) and hence the hidden one \( \pi \) can be arbitrary, e.g., \( \mu \) is the uniform distribution or from an Ising model. (2) Testing algorithms for KL divergence can be easily applied as a black box to obtain identity testing algorithms for TV distance, where in the latter we do not require \( \pi \ll \mu \). Here we show how our identity testing algorithm Algorithm 1 can be used to test for TV distance.

Lemma 4.19. Suppose \( A_{KL-ID} \) is an identity testing algorithm for a family \( F \) of distributions with query access to both Coordinate Oracle and General Oracle and for KL divergence with distance parameter \( \varepsilon > 0 \). The query complexity of \( A_{KL-ID} \) is \( m(n,1/\varepsilon) \) and the running time of \( A_{KL-ID} \) is polynomial in \( n \) and \( 1/\varepsilon \). Then there exists a polynomial-time identity testing algorithm \( A_{TV-ID} \) for \( F \) with the same query access and for TV distance with distance parameter \( \varepsilon > 0 \). The query complexity of \( A_{TV-ID} \) is \( O(m(n,2/\varepsilon^2) + 1/\varepsilon) \).

Proof. Let \( X^\mu \subseteq X \) denote the support of \( \mu \). By the law of total probability we have \( \pi(\cdot) = \pi(X^\mu)\pi(\cdot | X^\mu) + \pi(X^\mu)\pi(\cdot | X^\mu) \) where \( X^\mu = X \setminus X^\mu \) is the complement. Therefore, we obtain from the triangle inequality that

\[ d_{TV}(\pi,\mu) \leq \pi(X^\mu) + d_{TV}(\pi(\cdot | X^\mu),\mu). \]

In particular, if \( d_{TV}(\pi,\mu) \geq \varepsilon \), then either \( \pi(X^\mu) \geq \varepsilon/2 \) or \( d_{TV}(\pi(\cdot | X^\mu),\mu) \geq \varepsilon/2 \), where the latter implies \( D_{KL}(\pi(\cdot | X^\mu) \| \mu) \geq \varepsilon/2 \) via the Pinsker’s inequality.
Our testing algorithm $A_{TV-ID}$ runs in two stages. In the first stage, we distinguish between $\pi(X^c_\mu) = 0$ versus $\pi(X^c_\mu) \geq \varepsilon/2$ using $O(1/\varepsilon)$ samples from $\pi$, and we say $\pi$ passes this stage if none of these samples is in $X^c_\mu$. In particular, by choosing suitable constants we can make the failure probability at most $1/3$, i.e., if $\pi(X^c_\mu) \geq \varepsilon/2$ then the probability that $\pi$ passes is at most $1/3$. Observe that if $\pi(X^c_\mu) = 0$ then it always passes the first stage.

In the second stage, we test between $\pi(\cdot | X_\mu) = \mu$ versus $D_{KL}(\pi(\cdot | X_\mu) \| \mu) \geq \varepsilon^2/2$, using $A_{KL-ID}$ with failure probability $1/3$. Note that if we saw samples that belong to $X^c_\mu$ when running $A_{KL-ID}$, either from calls of Coordinate Oracle or from calls of General Oracle, we can safely conclude that $\pi \neq \mu$ and hence $d_{TV}(\pi, \mu) \geq \varepsilon$. Otherwise, these samples can be viewed as generated perfectly from the conditional distribution $\pi(\cdot | X_\mu)$. We say $\pi$ passes the second stage if $A_{KL-ID}$ outputs $\text{Yes}$ (i.e., $\pi(\cdot | X_\mu) = \mu$).

If $\pi$ passes both stages then $A_{TV-ID}$ outputs $\text{Yes}$ (i.e., $\pi = \mu$); otherwise it outputs $\text{No}$ (i.e., $d_{TV}(\pi, \mu) \geq \varepsilon$). Observe that, if $\pi = \mu$ then it passes the first stage always and passes the second stage with probability at least $2/3$. Meanwhile, if $d_{TV}(\pi, \mu) \geq \varepsilon$ then either $\pi(X^c_\mu) \geq \varepsilon/2$ or $D_{KL}(\pi(\cdot | X_\mu) \| \mu) \geq \varepsilon^2/2$. If $\pi(X^c_\mu) \geq \varepsilon/2$ then it passes the first stage with probability at most $1/3$. And if $D_{KL}(\pi(\cdot | X_\mu) \| \mu) \geq \varepsilon^2/2$ it passes the second stage with probability at most $1/3$. Hence, the probability that $\pi$ passes both stages is at most $1/3$. Therefore, $A_{TV-ID}$ is a polynomial-time identity testing algorithm with sample complexity $O(m(n, 2/\varepsilon^2) + 1/\varepsilon)$. □

5 Hardness of Identity Testing When Approximate Tensorization Fails

In this section we show that approximate tensorization is essentially a necessary condition for efficient identity testing, in the sense that there are high-dimensional distributions, specifically the antiferromagnetic Ising model, for which either approximate tensorization holds with constant $C = O(1)$ (and thus there is an efficient identity algorithm from Theorem 1.2), or there is no polynomial-time identity testing algorithm with General Oracle and Coordinate Oracle access unless RP = NP.

We prove the hardness result in Theorem 1.3 from the introduction in the following sections. We use a reduction from the maximum cut problem to identity testing. In particular, given a hard maximum cut instance, we construct an identity testing instance whose outputs provides the maximum cut. Our reduction is inspired by the one in [BBC+19], but we use a different “degree reducing” gadget (namely, the one from [Sly10]), and we are also required to design an algorithm to sample from the hidden model we construct. This is challenging because sampling from the antiferromagnetic Ising model is NP-hard in general, but for our instance we manage to do it using a hybrid approach. Specifically, we use the recent algorithm from [KLR22] for low-rank Ising model for one range of parameters and polymer models [JKP20] for the other. Both algorithms rely on the fact that the graph in our testing is a random bipartite graph with trees attached to it that happens to be a good expander.

Our proof is organized as follows. First, we introduce our degree reducing gadget in Section 5.1. The testing instance construction and the reduction is then provided in Section 5.2. Finally, Sections 5.3 and 5.4 contain our sampling algorithm.

5.1 The degree reducing gadget

The gadget construction has as parameters integers $n \geq 1$, $d \geq 3$ and real numbers $0 < \theta, \psi < 1/8$. Let $\ell = 2^\psi \log_{d-1} n$, $t = (d-1)^{\theta \log_{d-1} n}$ and $m = t(d-1)^{\ell}$. The gadget is constructed as follows:
1. Let \( \hat{G} = (V_{\hat{G}}, E_{\hat{G}}) \) be a random bipartite graph with \( n + m \) vertices on each side.

2. For \( s \in \{+,-\} \), let the vertices on the \( s \)-side of \( \hat{G} \) be \( W_s \cup U_s \), where \( |W_s| = n \) and \( |U_s| = m \).

3. Let \( M_1, \ldots, M_{d-1} \) be \( d-1 \) random perfect matchings between \( W_+ \cup U_+ \) and \( W_- \cup U_- \); that is, each \( M_i \) is drawn uniformly at random from the set of all perfect matching between \( W_+ \cup U_+ \) and \( W_- \cup U_- \);

4. Let \( M' \) be a random perfect matching between \( W_+ \) and \( W_- \);

5. Set \( E_{\hat{G}} = M' \cup \left( \bigcup_{i=1}^{d-1} M_i \right) \);

6. Construct collections \( T_+ \) and \( T_- \) each of \( t \) disjoint \( (d-1) \)-ary trees of height \( \ell \).

7. Adjoin \( T_+ \) (resp., \( T_- \)) to \( \hat{G} \) by identifying each vertex of \( U_+ \) (resp., of \( U_- \)) with one of the leaves of the trees in \( T_+ \) (resp., \( T_- \)). We denote the set of roots of the trees in \( T_+ \) (resp., \( T_- \)) by \( R_+ \) (resp., \( R_- \)).

Let \( G = (V_G, E_G) \) be the random multi-graph resulting from this construction.

### 5.2 The reduction

Let \( (K = (V_K, E_K), k) \) be an instance of the maximum cut problem. Namely, we want to distinguish between the cases \( \text{max-cut}(K) < k \) and \( \text{max-cut}(K) \geq k \), where \( \text{max-cut}(K) \) denotes the size of the maximum cut of the graph \( K \).

Let \( N = |V_K| \); we may assume that \( N = n^{\theta/12} \), where \( n \) and \( \theta \) are the parameters for the degree reducing gadget construction in the previous section. Form the multi-graph \( F = (V_F, E_F) \) by adding two special vertices \( s \) and \( t \) to \( K \) (i.e., \( V_F = V_K \cup \{s,t\} \)), connecting \( s \) and \( t \) with \( N^2 - k \) edges, and adding \( N \) edges between each \( s \) and \( t \) and each vertex in \( V_K \); note that \( F \) has \( |E_K| + 3N^2 - k \). This construction ensures that:

1. When \( \text{max-cut}(K) < k \), then \( \{\{s,t\}, V_K\} \) is the unique maximum cut of \( F \) and has size \( 2N^2 \);

2. When \( \text{max-cut}(K) \geq k \), then there exists another cut in \( F \) whose size is at least \( 2N^2 \); this cut is obtained by taking the maximum cut for \( K \) and adding \( s \) and \( t \) to opposite sides of it.

Next, we generate an instance \( G = (V_G, E_G) \) of the degree reducing gadget from Section 5.1. We then obtain the multi-graph \( \hat{F} = (V_{\hat{F}}, E_{\hat{F}}) \) by replacing every vertex \( v \in V_F \) with a copy \( G_v \); we label each copy of \( G \) by \( G^v \) and let \( R^v_+ \) and \( R^v_- \) denote \( R_+ \) and \( R_- \) for \( G_v \). Moreover, for each edge \( \{u,v\} \in E_F \), we add a matching of size \( n^{30/4} \) between \( R^v_+ \) and \( R^u_+ \), and another matching of the same size between \( R^v_- \) and \( R^u_- \). Note that \( \hat{F} \) is a \( d \)-regular multi-graph.

We will consider the antiferromagnetic Ising model on the multi-graph \( \hat{F} \). (See Section 4.4.2 for the definition of the Ising model on a simple graph. The definition extends to the multi-graph setting by simply considering multi-edges in the summation.) For a configuration \( \sigma \in \{+1, -1\}^{V_G} \), we define its phase \( Y_v(\sigma) \) as \(+1\) if the number of vertices assigned \(-1\) in \( W_+ \) is greater than the number of vertices assigned \(-1\) in \( W_- \); otherwise we set \( Y(\sigma) = -1 \). For a configuration \( \sigma \in \{+1, -1\}^{V_F} \), we let \( Y(\sigma) \) denote the phase vector of \( \sigma \), which contains as coordinates the phase of \( \sigma \) in each gadget \( G_v \).

Let \( \Omega = \{+1, -1\}^{V_F} \) be the set of all phase vectors. Let \( \xi^+_s \in \Omega \) (resp., \( \xi^-_s \in \Omega \)) be the phase vector that assigns \(+1\) (resp., \(-1\)) to \( s \) and \(-1\) (resp., \(+1\)) to every other gadget in \( \hat{F} \). Let
\( \Omega_{st} = \{ \xi^+_s, \xi^-_t \} \). Observe that each phase vector \( Y(\sigma) \) corresponds to a cut in the graph \( F \), with the phase determining the side of the cut for each vertex.

Let \( \Omega'_0 \subseteq \Omega \) be the collection of all phase vectors corresponding to cuts (\( \{ s \} \cup U, \{ t \} \cup V_F \setminus U \)) of \( F \), which in turn correspond to cuts (\( U, V_K \setminus U \)) of \( K \) of size \( < k \). Let \( \Omega_0 \) be \( \Omega'_0 \) together with the phase vectors for cuts (\( \{ s, t \} \cup U, V_F \setminus U \)) of \( F \). Then:

1. if \( \text{max-cut}(K) < k \), then \( \Omega_0 = \Omega \);
2. if \( \text{max-cut}(K) \geq k \), then \( \Omega_0 \subset \Omega \) and \( \Omega \setminus \Omega_0 \) contains at least phase vector corresponding to a cut (\( \{ s \} \cup U, \{ t \} \cup V_F \setminus U \)) of \( F \), where \( (U, V_F \setminus U) \) is a maximum cut for \( K \).

We are now ready to describe our instance for the identity testing problem. Let \( \beta < \beta_c(d) := -\frac{1}{2} \ln \left( \frac{d}{d-2} \right) \); this parameter regime corresponds to the so-called tree uniqueness region for \((d-1)-ary infinite trees. The visible distribution of our testing instance will be the Gibbs distribution \( \mu_{\hat{F},\beta} \) for the antiferromagnetic Ising model on \( \hat{F} \). The hidden distribution will be \( \mu_{\hat{F},\beta}(\cdot \mid Y(\sigma) \in \Omega_0) \); that is, \( \mu_{\hat{F},\beta} \) conditioned on the phase vector being in \( \Omega_0 \). Our construction ensures that if \( \text{max-cut}(K) < k \), then \( \mu_{\hat{F},\beta}(\cdot \mid Y(\sigma) \in \Omega_0) = \mu_{\hat{F},\beta} \). In addition, we have the following fact.

**Lemma 5.1.** If \( \text{max-cut}(K) \geq k \) and \( \beta < \beta_c(d) \), then \( d_{TV}(\mu_{\hat{F},\beta}(\cdot \mid Y(\sigma) \in \Omega_0), \mu_{\hat{F},\beta}) = 1 - o(1) \).

**Proof.** Observe that

\[
d_{TV}(\mu_{\hat{F},\beta}(\cdot \mid Y(\sigma) \in \Omega_0), \mu_{\hat{F},\beta}) = \sum_{\sigma : Y(\sigma) \in \Omega \setminus \Omega_0} \mu_{\hat{F},\beta}(\sigma).
\]

Since \( \text{max-cut}(K) \geq k \), the set \( \Omega \setminus \Omega_0 \) contains (at least) the phase vector corresponding to a maximum cut of \( F \). Hence, \( \sum_{\sigma : Y(\sigma) \in \Omega \setminus \Omega_0} \mu_{\hat{F},\beta} \) is at least the probability that a sample from \( \mu_{\hat{F},\beta} \) reveals a maximum cut for \( F \). The results in [Sly10, GSV16, CGG+16] imply that this probability is indeed \( 1 - 1/2^{n/4} \), as desired. Specifically, the argument in the proof of Theorems 1 and 2 in [Sly10] shows that this holds (under certain conditions) for the hard-core model; [GSV16] extends the argument for any antiferromagnetic spin system (including the Ising model); and Lemma 22 from [CGG+16] shows that the required condition holds for all \( \beta < -\frac{1}{2} \ln \left( \frac{d}{d-2} \right) \) in the tree uniqueness region.

The idea of our reduction is to provide this testing instance to an identity testing algorithm and use its output to determine whether \( \mu_{\hat{F},\beta}(\cdot \mid Y(\sigma) \in \Omega_0) = \mu_{\hat{F},\beta} \) or

\[
d_{TV}(\mu_{\hat{F},\beta}(\cdot \mid Y(\sigma) \in \Omega_0), \mu_{\hat{F},\beta}(\cdot)) = 1 - o(1).
\]

This gives whether \( \Omega_0 = \Omega \) or not, and thus whether the \( \text{max-cut}(K) < k \) or not. All that remains to complete the reduction is that we show how to sample (in polynomial time) from the hidden distribution \( \mu_{\hat{F},\beta}(\cdot \mid Y(\sigma) \in \Omega_0) \) and how to simulate the Coordinate Oracle for it.

Simulating the conditional marginal oracle for \( \mu_{\hat{F},\beta}(\cdot \mid Y(\sigma) \in \Omega_0) \) is straightforward. Given a vertex \( v \in V_{\hat{F}} \) and a configuration \( \sigma \in \{+1, -1\}^{V_{\hat{F}} \setminus \{v\}} \), we can first check if \( Y(\sigma) \not\in \Omega_0 \); if this the case, we output \( \{+1, -1\} \) arbitrarily. Otherwise, we sample from the vertex marginal \( \mu_{\hat{F},\beta}(\cdot \mid \sigma) \), which can be done in \( O(d) \) time. Sampling from \( \mu_{\hat{F},\beta}(\cdot \mid Y(\sigma) \in \Omega_0) \) is much trickier, but it can be done relying heavily on the structure of the graph \( \hat{F} \); note that the problem of approximately sampling antiferromagnetic is computationally hard, even in the bounded degree case. We prove the following.
Lemma 5.2. For any \( \varepsilon \in (0,1) \) and any phase vector \( \mathcal{Y} \in \Omega_0 \) there is an algorithm that generates a sample from a distribution \( \mu_{\text{ALG}} \) such that \( d_{TV} \left( \mu_{\text{ALG}}, \mu_{\hat{F},\beta}(\cdot \mid \mathcal{Y}) \right) \leq \varepsilon + \log(1/\varepsilon)e^{-\Omega(n^{\theta/4})} \) with running time \( \text{poly}(|V_F|, 1/\varepsilon) \).

The proof of this lemma is provided in Section 5.3. We are now ready to prove Theorem 1.3 from the introduction.

Proof of Theorem 1.3. The first part of the theorem was proved in Section 4.4.2. For the second part, suppose there is an identity testing algorithm with polynomial running time and sample complexity.

Let \( (K = (V_K, E_K), k) \) be an instance of the maximum cut problem with \( |V_K| = n^{\theta/12} \). Set \( \mu_{\hat{F},\beta} \) to be the visible distribution and \( \mu_{\hat{F},\beta}(\cdot \mid Y(\sigma) \in \Omega_0) \) to be the hidden one. Suppose \( L = \text{poly}(n) \) is the sample complexity of the testing algorithm in this instance. Generate a set \( S \) of \( L \) samples from the distribution \( \mu_{\text{ALG}} \) from Lemma 5.2 setting \( \varepsilon = 1/(100L) \), so that

\[
d_{TV} \left( \mu_{\text{ALG}}^{\otimes L}, \mu_{\hat{F},\beta}^{\otimes L}(\cdot \mid Y(\sigma) \in \Omega_0) \right) \leq L \cdot d_{TV} \left( \mu_{\text{ALG}}, \mu_{\hat{F},\beta}(\cdot \mid Y(\sigma) \in \Omega_0) \right) \leq \frac{1}{50},
\]

where \( \mu_{\text{ALG}}^{\otimes L} \) and \( \mu_{\hat{F},\beta}^{\otimes L}(\cdot \mid Y(\sigma) \in \Omega_0) \) denote the product distributions corresponding to \( L \) independent samples from \( \mu_{\text{ALG}} \) and \( \mu_{\hat{F},\beta}(\cdot \mid Y(\sigma) \in \Omega_0) \) respectively.

Our algorithm for solving \( (K = (V_K, E_K), k) \) gives \( S \) to the testing algorithm. Recall that our construction ensures that if \( \text{max-cut}(K) < k \), then \( \mu_{\hat{F},\beta}(\cdot \mid Y(\sigma) \in \Omega_0) = \mu_{\hat{F},\beta} \) and that if \( \text{max-cut}(K) \geq k \) then

\[
d_{TV} \left( \mu_{\hat{F},\beta}(\cdot \mid Y(\sigma) \in \Omega_0), \mu_{\hat{F},\beta} \right) = 1 - o(1); \tag{13}
\]

see Lemma 5.1.

If \( \pi^{\otimes L} \) is the optimal coupling of the distributions \( \mu_{\text{ALG}}^{\otimes L} \) and \( \mu_{\hat{F},\beta}^{\otimes L}(\cdot \mid Y(\sigma) \in \Omega_0) \), and \( (S, S') \) is sampled from \( \pi^{\otimes L} \), then \( S' = S \) with probability at least \( 49/50 \), \( S \sim \mu_{\text{ALG}}^{\otimes L} \) and \( S' \sim \mu_{\hat{F},\beta}^{\otimes L}(\cdot \mid Y(\sigma) \in \Omega_0) \). Therefore, if Eq. (13) holds (i.e., \( \text{max-cut}(K) \geq k \)), then

\[
\Pr[\text{TESTER outputs Yes when given samples } S \text{ where } S \sim \mu_{\text{ALG}}^{\otimes L}] = \Pr[\text{TESTER outputs No when given samples } S \text{ where } (S, S') \sim \pi^{\otimes L}] \leq \Pr[\text{TESTER outputs No when given samples } S' \text{ where } (S, S') \sim \pi^{\otimes L}] + \pi^{\otimes L}(S \neq S') = \Pr[\text{TESTER outputs No when given samples } S' \text{ where } S' \sim \mu_{\text{M}}^{\otimes L}] + \pi^{\otimes L}(S \neq S') \leq \frac{1}{3} + \frac{1}{50} = \frac{53}{150}. \tag{14}
\]

Hence, the TESTER returns NO with probability at least \( 3/5 \) in this case.

Now, when \( \text{max-cut}(K) < k \) and \( \mu_{\hat{F},\beta}(\cdot \mid Y(\sigma) \in \Omega_0) = \mu_{\hat{F},\beta} \), we can analogously deduce that the TESTER returns YES with probability at least \( 2/3 \). Therefore, our algorithm can solve any maximum cut instance \( (K = (V_K, E_K), k) \) in polynomial time with probability at least \( 3/5 \), and the result follows.

5.3 Sampling conditional on the phase vector: proof of Lemma 5.2

We start with a number of definitions and facts required to describe and analyze our algorithm to establish Lemma 5.2. The proofs of these facts are provided in Section 5.4. The first lemma states that it essentially suffices to sample from the simpler conditional distribution \( \mu_{\hat{F},\beta}(\cdot \mid Y(\sigma) \in \Omega_0) \).
Lemma 5.3. \(d_{TV} \left( \mu_{\hat{G}, \beta}(\cdot \mid Y(\sigma) \in \Omega_{st}), \mu_{\hat{G}, \beta}(\cdot \mid Y(\sigma) \in \Omega_0) \right) \leq \frac{1}{2n^{1/4}}. \)

We call the roots in \( \bigcup_{v \in V_P} (R^v_+ \cup R^v_-) \) used to connect the degree reducing gadgets *ports*. Let \( P \) denote the set of all ports of \( \hat{G} \); we also use \( P^v \subset P \) to denote the set of ports of the gadget \( G_v \).

For a configuration \( \{+1, -1\}^P_v \), let \( Z_{G_v, \beta}(\sigma_{P_v}) \) denote the sum of the weights of all the configurations on \( G_v \) that agree with \( \sigma_{P_v} \). We will need an approximation algorithm for this quantity and an approximate sampling algorithm for \( \mu_{G_v, \beta}(\cdot \mid \sigma_{P_v}) \). A fully polynomial-time randomized approximation scheme (FPRAS) for \( Z_{G_v, \beta}(\sigma_{P_v}) \) is an algorithm that for every \( \varepsilon > 0 \) and \( \delta \in (0, 1) \) outputs \( \hat{Z} \) so that, with probability at least \( 1 - \delta \), \( e^{-\varepsilon \hat{Z}} \leq Z_{G_v, \beta}(\sigma_{P_v}) \leq e^{\varepsilon \hat{Z}} \) and runs in time polynomial in \( |V_{G_v}|, 1/\varepsilon \) and \( \log(1/\delta) \). A polynomial-time sampling algorithm for \( \mu_{G_v, \beta}(\cdot \mid \sigma_{P_v}) \) is a randomized algorithm that for every \( \varepsilon > 0 \) runs in time polynomial in \( |V_{G_v}| \) and \( 1/\varepsilon \) and outputs a sample from a distribution \( \varepsilon \)-close in total variation distance to \( \mu_{G_v, \beta}(\cdot \mid \sigma_{P_v}) \).

Lemma 5.4. Let \( \sigma_{P_v} \in \{+1, -1\}^P_v \) be an arbitrary spin configuration on \( P_v \). For all sufficiently large \( d = O(1) \), with probability \( 1 - o(1) \) over the choice of the random multi-graph \( G_v \), for all \( \beta < 0 \) there is an FPRAS for \( Z_{G_v, \beta}(\sigma_{P_v}) \) and a polynomial-time sampling algorithm for \( \mu_{G_v, \beta}(\cdot \mid \sigma_{P_v}) \).

For \( \mathcal{Y} \in \Omega \), let \( \mu_{P, \beta}(\cdot \mid \mathcal{Y}) \) denote the marginal distribution of \( \mu_{\hat{G}, \beta}(\cdot \mid \mathcal{Y}) \) on \( P \). When \( \beta = \beta_c(d) \), in the non-uniqueness regime for the infinite \((d-1)\)-ary tree, there are two semi-translation invariant measures, denoted \( \mu^+ \) and \( \mu^- \). These measures can be obtained by conditioning on the leaves at level \( 2h \) (resp., \( 2h+1 \)) to have spin \(-1\), and then taking the weak limits as \( h \to \infty \). Let \( p^+ \) (resp., \( p^- \)) be the probability that the root of the tree is assigned \(-1\) under \( \mu^+ \) (resp., \( p^- \)).

Let \( P^+_v = R^v_+ \cap P_v \) and \( P^-_v = R^v_- \cap P_v \). For \( i \in \{+1, -1\} \), we define the following product distribution over configurations \( \sigma \in \{+1, -1\}^P_v \) on \( P_v \):

\[
Q^i_v(\sigma) = (p^i)|\sigma^-(\cdot)\cap P^+_v|(1 - (p^i))|\sigma^-\cap P^-_v|, \quad \text{where } \sigma^-(i) \text{ denotes the set of vertices from } P_v \text{ assigned spin } \iota \text{ in } \sigma.
\]

The product distribution \( Q^+_v \) (resp., \( Q^-_v \)) is known to be a good approximation for \( \mu_{G_v, \beta}(\cdot \mid \mathcal{Y}_v = +1) \) (resp., \( \mu_{G_v, \beta}(\cdot \mid \mathcal{Y}_v = -1) \)), as formalized in the following lemma. Here \( \mathcal{Y}_v \) denote the phase of the gadget \( G_v \).

Lemma 5.5 (Lemma 22 [CGG+16] & Lemma 19 [GŠV16]). Let \( \beta < \beta_c(d) \). Then, there exists \( \theta \) and \( \psi \) such that for \( s \in \{+1, -1\} \), with probability \( 1 - o(1) \) over the choice of the random \( n \)-vertex multi-graph \( G_v \), for any \( \sigma_{P_v} \in \{+1, -1\}^P_v \) we have

\[
1 - n^{-2\theta} \leq \frac{\mu_{G_v, \beta}(\sigma_{P_v} \mid \mathcal{Y}_v = s)}{Q^s_v(\sigma_{P_v})} \leq 1 + n^{-2\theta}. \tag{15}
\]

Moreover, for \( s \in \{+1, -1\} \) we have \( \mu_{G_v, \beta}(Y_v = s) \geq \frac{1}{n} \).

Next, for a phase vector \( \mathcal{Y} \in \Omega \) we define another product measure this time over configurations \( \sigma \in \{+1, -1\}^P \) on \( P \). Let

\[
w^\mathcal{Y}_P(\sigma) = \prod_{v \in V_P} Q^v_{\mathcal{Y}_v}(\sigma_{P_v}) \prod_{(u,v) \in E(P)} e^{\beta_{u,v} \sigma_u \sigma_v},
\]

where \( E(P) \) is the set of edges with both endpoints in \( P \). Let \( Z^\mathcal{Y}_P = \sum_{\sigma \in \{+1, -1\}^P} w^\mathcal{Y}_P(\sigma) \) and define

\[
Q^\mathcal{Y}_P(\sigma) = \frac{w^\mathcal{Y}_P(\sigma)}{Z^\mathcal{Y}_P}.
\]

We have the following approximation for \( \mu_{P, \beta}(\cdot \mid \mathcal{Y}) \) in terms of \( Q^\mathcal{Y}_P \).
Lemma 5.6. Let $\beta < \beta_c(d)$. For every $\mathcal{Y} \in \Omega$ and $\sigma \in \{+1,-1\}^P$, we have

$$\left| \frac{\mu_{P,\beta}(\sigma | \mathcal{Y})}{Q_P^\mathcal{Y}(\sigma)} - 1 \right| = o(1).$$

Finally, we will also use the following fact.

Lemma 5.7. Let $\mathcal{Y} \in \Omega$. Suppose $\sigma_P \in \{+1,-1\}^P$ is sampled from $Q_P^\mathcal{Y}$ and that $\sigma \in \{+1,-1\}^{\bar{V}_P}$ is then sampled from $\mu_{\hat{F},\beta}(\cdot | \sigma_P)$. Then, the phase vector of $\sigma$ is $\mathcal{Y}$ with probability at least $1 - e^{-\Omega(n^{3d/4})}$.

We can now provide the proof of Lemma 5.2.

**Proof of Lemma 5.2.** Let $\sigma^+$ denote the all-plus configuration on $P$. For $\sigma \in \{+1,-1\}^P$, recall that we use $Z_{\hat{F},\beta}(\sigma,\mathcal{Y})$ to denote the total weight of the configurations of $\hat{F}$ that agree with $\sigma$ on $P$ and have phase vector $\mathcal{Y}$. The algorithm is as follows:

1. Sample $\mathcal{Y} \in \{\xi_{st}^+, \xi_{st}^-, \xi_{lr}^+, \xi_{lr}^-, \xi_{lr}^+\} \cup \{\xi_{lr}^-, \xi_{lr}^+\}$ uniformly at random. Note that by ignoring all other phase vectors in $\Omega_0$, the error is at most $1/2n^{3d/4}$ by Lemma 5.3.

2. Sample $\sigma_P \in \{+1,-1\}^P$ from a distribution $\varepsilon/3$-close in total variation distance to $\mu_{P,\beta}(\cdot | \mathcal{Y})$ with the following rejection sampling algorithm:

   2.1 Generate $\sigma_P \in \{+1,-1\}^P$ from the product distribution $Q_P^\mathcal{Y}$;

   2.2 Compute the approximation $\hat{Z}(\sigma_P)$ for $Z_{\hat{F},\beta}(\sigma_P,\mathcal{Y})$ such that

   $$\left(1 - \frac{\varepsilon}{10}\right) Z_{\hat{F},\beta}(\sigma_P,\mathcal{Y}) \leq \hat{Z}(\sigma_P) \leq \left(1 + \frac{\varepsilon}{10}\right) Z_{\hat{F},\beta}(\sigma_P,\mathcal{Y});$$

   this can be done in time $\text{poly}(|V_{\bar{F}}|,1/\varepsilon)$ with success probability at least $1 - \varepsilon/3$ by Lemma 5.4.

2.3 Accept $\sigma_P$ with probability:

   $$r(\sigma_P) = \frac{1}{10} \cdot \frac{Q_P^\mathcal{Y}(\sigma^+)}{Q_P^\mathcal{Y}(\sigma_P)} \cdot \frac{\hat{Z}(\sigma_P)}{Z(\sigma^+)}.$$

2.4 Repeat until accept or exceed $T = c \log(1/\varepsilon)$ rounds, for a suitable constant $c > 0$, in which case we let $\sigma_P = \sigma^+$.

3. Sample the configuration of each gadget $G_v$ conditional on the port configuration $\sigma_{P_v}$ on $P_v$ from a distribution $\varepsilon/|V_{\bar{F}}|$-close to $\mu_{G_v,\beta}(\cdot | \sigma_{P_v})$ with the algorithm from Lemma 5.4.

4. Output the resulting configuration $\sigma$.

For the analysis of this algorithm, let us focus first on the rejection sampling process in Step 2. First, note that the process is well-defined since:

$$r(\sigma_P) = \frac{1}{10} \cdot \frac{Q_P^\mathcal{Y}(\sigma^+)}{Q_P^\mathcal{Y}(\sigma_P)} \cdot \frac{\hat{Z}(\sigma_P)}{Z(\sigma^+)} \leq 1,$$

where the last inequality follows from Lemma 5.6. Second, each iteration of the rejection sampling algorithm can be implemented in polynomial time; in particular, we can compute compute
Lemma 5.4 and union bound, the phase vector of Lemma 5.6. Finally, we claim that the output of this algorithm is at least $\epsilon/3$-close to $\mu_{P,\beta}(|Y|$ in total variation distance. To see this, note that for each $\sigma P \in \{+1, -1\}^P$, the probability that process outputs $\sigma P$ in one round is:

$$Q_P^y(\sigma P) r(\sigma P) = \frac{1}{10} \cdot Q_P^y(\sigma^+) \cdot \frac{\hat{Z}(\sigma P)}{\hat{Z}(\sigma^+)} \approx \hat{Z}(\sigma P).$$

Therefore, conditioned on the algorithm accepting on the first $T = c\log(1/\epsilon)$ rounds, the probability that $\sigma P$ is the output is

$$\frac{\hat{Z}(\sigma P)}{\sum_{\sigma'} \hat{Z}(\sigma')} \leq \left(1 + \frac{\epsilon}{5}\right) \cdot \frac{Z_{\hat{F},\beta}(\sigma P, \mathcal{Y})}{\sum_{\sigma'} Z_{\hat{F},\beta}(\sigma', \mathcal{Y})} = \left(1 + \frac{\epsilon}{5}\right) \mu_{P,\beta}(\sigma P \mid \mathcal{Y}),$$

and similarly for the lower bound. Moreover, since by Lemma 5.6

$$r(\sigma P) = \frac{1}{10} \cdot Q_P^y(\sigma^+) \cdot \frac{\hat{Z}(\sigma P)}{\hat{Z}(\sigma^+)} \geq \frac{1}{20} \cdot Q_P^y(\sigma^+) \cdot \frac{Z_{\hat{F},\beta}(\sigma P, \mathcal{Y})}{Z_{\hat{F},\beta}(\sigma^+, \mathcal{Y})}$$

$$= \frac{1}{20} \cdot \frac{\mu_{P,\beta}(\sigma P \mid \mathcal{Y})}{\mu_{P,\beta}(\sigma^+ \mid \mathcal{Y})} \cdot \frac{Q_P^y(\sigma^+)}{Q_P^y(\sigma P)} \geq \frac{1}{100},$$

the probability that the algorithm accepts in the first $T = c\log(1/\epsilon)$ rounds is at least

$$1 - \left(1 - \frac{1}{100}\right)^{c\log(1/\epsilon)} \geq 1 - \frac{\epsilon}{10},$$

for a suitable constant $c > 0$.

Now, note that by Lemma 5.7 and union bound, the phase vector of $\sigma$ agrees with $\mathcal{Y}$ with probability at least $1 - |T|e^{-\Omega(n^{\theta/4})}$; hence, the output distribution of the algorithm satisfies:

$$d_{TV} \left(\mu_{\text{alg}}, \mu_{\hat{F},\beta} \mid \mathcal{Y}\right) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + |V_F| \cdot \frac{\epsilon}{3|V_F|} + |T|e^{-\Omega(n^{\theta/4})} + 2^{-n^{\theta/4}} \leq \epsilon + \log(1/\epsilon)e^{-\Omega(n^{\theta/4})},$$

as claimed. \hfill \Box

### 5.4 Sampling conditional on the phase vector: proof of auxiliary facts

We provide in this section the proofs of Lemmas 5.3, 5.4, 5.6 and 5.7.

**Proof of Lemma 5.3.** We have

$$d_{TV} \left(\mu_{\hat{F},\beta} \mid Y(\sigma) \in \Omega_0), \mu_{\hat{F},\beta} \mid Y(\sigma) \in \Omega_{st}\right) = \sum_{\sigma:Y(\sigma) \in \Omega_0 \setminus \Omega_{st}} \mu_{\hat{F},\beta}(\sigma \mid Y(\sigma) \in \Omega_0).$$

This is the probability of obtaining a phase vector in $\Omega_0 \setminus \Omega_{st}$ under $\mu_{\hat{F},\beta} \mid Y(\sigma) \in \Omega_0$. Observe that among all the cuts of $F$ corresponding to phase vectors in $\Omega_0$, the largest ones are those corresponding to the phase vectors in $\Omega_{st}$. Hence, following the argument in the proof of Lemma 5.1, we get from the results in [Sly10, GSV16, CGC+16] that this probability is at least $1 - 1/2^{n^{\theta/4}}$ as desired. \hfill \Box
Proof of Lemma 5.6. Let $\sigma \in \{+1, -1\}^P$. Let $Z_{F, \beta}(\sigma, Y)$ be the sum of the weights of all the configurations of $\hat{F}$ with phase vector $Y$ that agree with $\sigma$ in $P$ and, similarly, define $Z_{\hat{F}, \beta}(Y)$ as the sum of the weights of all configurations with phase vector $Y$, so that

$$
\mu_{P, \beta}(\sigma \mid Y) = \frac{Z_{F, \beta}(\sigma, Y)}{Z_{F, \beta}(Y)} = \frac{1}{Z_{\hat{F}, \beta}(Y)} \prod_{v \in V_F} Z_{G_v, \beta}(\sigma_{P_v}, Y_v) \prod_{\{u, v\} \in E(P)} e^{\beta u_v}. 
$$

By Lemma 5.5, we have

$$
\mu_{P, \beta}(\sigma \mid Y) \leq \frac{(1 + n^{-2\theta})^{|V_F|}}{Z_{F, \beta}(Y)} \prod_{v \in V_F} Z_{G_v, \beta}(Y_v) Q^\beta_{P_v}(\sigma_{P_v}) \prod_{\{u, v\} \in E(P)} e^{\beta u_v} = \frac{(1 + n^{-2\theta})^{|V_F|}}{Z_{\hat{F}, \beta}} \cdot w^\beta_{P}(\sigma) \cdot \prod_{v \in V_F} Z_{G_v, \beta}(Y_v).
$$

Then,

$$
\frac{\mu_{P, \beta}(\sigma \mid Y)}{Q^\beta_{P}(\sigma)} \leq \frac{Z^\beta_{\hat{F}}(\sigma)}{Z_{\hat{F}, \beta}} \cdot \prod_{v \in V_F} Z_{G_v, \beta}(Y_v).
$$

Now,

$$
Z^\beta_{\hat{F}} \cdot \prod_{v \in V_F} Z_{G_v, \beta}(Y_v) = \sum_{\sigma \in \{+1, -1\}^P} \prod_{\{u, v\} \in E(P)} e^{\beta u_v} \cdot \prod_{v \in V_F} Q^\beta_{P_v}(\sigma_{P_v}) Z_{G_v, \beta}(Y_v).
$$

From Eq. (15), we have

$$
\frac{1}{1 + n^{-2\theta}} \mu_{P, \beta}(\sigma_{P_v} \mid Y_v) \leq Q^\beta_{P_v}(\sigma_{P_v}) \leq \frac{1}{1 - n^{-2\theta}} \mu_{P, \beta}(\sigma_{P_v} \mid Y_v),
$$

and so

$$
Z^\beta_{\hat{F}} \cdot \prod_{v \in V_F} Z_{G_v, \beta}(Y_v) \leq \frac{1}{(1 - n^{-2\theta})^{|V_F|}} \sum_{\sigma \in \{+1, -1\}^P} \prod_{\{u, v\} \in E(P)} e^{\beta u_v} \cdot \prod_{v \in V_F} \mu_{P_v, \beta}(\sigma_{P_v} \mid Y_v) Z_{G_v, \beta}(Y_v) = \frac{1}{(1 - n^{-2\theta})^{|V_F|}} Z_{\hat{F}, \beta}.
$$

Thus, we have obtained the upper bound

$$
\frac{\mu_{P, \beta}(\sigma \mid Y)}{Q^\beta_{P}(\sigma)} \leq \left( \frac{1 + n^{-2\theta}}{1 - n^{-2\theta}} \right)^{|V_F|},
$$

and in we can deduce analogously that

$$
\frac{\mu_{P, \beta}(\sigma \mid Y)}{Q^\beta_{P}(\sigma)} \geq \left( \frac{1 - n^{-2\theta}}{1 + n^{-2\theta}} \right)^{|V_F|}.
$$

Recall that $|V_F| = n^\theta/12 + 2$, so that

$$
1 - o(1) \leq \frac{\mu_{P, \beta}(\sigma \mid Y)}{Q^\beta_{P}(\sigma)} \leq 1 + o(1)
$$

and the result follows. $\Box$

34
Proof of Lemma 5.7. Consider a gadget $G_v$ of $\hat{F}$ such that $Y_v = +1$. Let $u \in P^+_v$. We claim that since $p^+ > 1/2$, then $Q^+_P(u = 1) > 1/2$. To see this, let $w$ be the neighbor of $u$ in $P$ and suppose that the phase of the gadget containing $w$ is $+1$. Then,

$$
\frac{Q^+_P(u = -1)}{Q^+_P(u = +1)} = \frac{(p^+)^2 e^\beta + p^+(1 - p^+)e^{-\beta}}{(1 - p^+)^2 e^\beta + p^+(1 - p^+)e^{-\beta}} > 1,
$$

which implies that $Q^+_P(u = -1) > 1/2$. An analogous calculation shows that the same holds when the gadget containing $w$ is in the $-1$ phase. With the same reasoning, we can similarly deduce that when $u \in P^-_v$, then $Q^-_P(u = +1) > 1/2$. This implies by a Chernoff bound that if $\sigma_P \sim Q^+_P$ and $Y_v = +1$, then there exists a constant $\delta > 0$ such that:

$$
|\sigma^-_{P_v}(1) \cap P^+_v| - |\sigma^-_{P_v}(1) \cap P^-_v| \geq \delta|P_v|,
$$

and

$$
|\sigma^+_{P_v}(1) \cap P^+_v| - |\sigma^+_{P_v}(1) \cap P^-_v| \geq \delta|P_v|,
$$

with probability at least $1 - \exp(-\Omega(|P_v|))$.

Now,

$$
\frac{\mu_{G_v,\beta}(Y_v = +1 | \sigma_{P_v})}{\mu_{G_v,\beta}(Y_v = -1 | \sigma_{P_v})} = \frac{\mu_{G_v,\beta}(\sigma_{P_v} | Y_v = +1)}{\mu_{G_v,\beta}(\sigma_{P_v} | Y_v = -1)} \geq 1 - \frac{\epsilon}{n^{2\theta}} \cdot \frac{Q^+_P(\sigma_{P_v})}{Q^-_P(\sigma_{P_v})} \cdot \frac{\mu_{\hat{F},\beta}(Y_v = +1)}{\mu_{\hat{F},\beta}(Y_v = -1)},
$$

by Lemma 5.5. Lemma 5.5 also implies that $\frac{\mu_{G_v,\beta}(Y_v = +1)}{\mu_{G_v,\beta}(Y_v = -1)} \geq \frac{1}{n}$. Moreover, from the definition of $Q^+_P$ and $Q^-_P$ we have

$$
\frac{Q^+_P(\sigma_{P_v})}{Q^-_P(\sigma_{P_v})} = \frac{(p^+) |\sigma^-_{P_v}(1) \cap P^+_v| (1 - p^+) |\sigma^-_{P_v}(1) \cap P^-_v| (p^-) |\sigma^+_{P_v}(1) \cap P^-_v| (1 - p^-) |\sigma^+_{P_v}(1) \cap P^+_v|}{(p^-) |\sigma^-_{P_v}(1) \cap P^+_v| (1 - p^-) |\sigma^-_{P_v}(1) \cap P^-_v| (p^+) |\sigma^+_{P_v}(1) \cap P^-_v| (1 - p^+) |\sigma^+_{P_v}(1) \cap P^+_v|} \geq a^\delta|P_v|,
$$

for a suitable constant $a > 1$ since $p^+ > 1/2$ and $p^- < 1/2$. This implies that for a suitable constant $c_0 > 0$, we have

$$
\mu_{G_v,\beta}(Y_v = +1 | \sigma_{P_v}) \geq 1 - \frac{c_0 n}{a^{\delta}|P_v|} \geq 1 - \frac{c_0 n}{a^{5n^{3\theta/4}}},
$$

since $|P_v| \geq n^{3\theta/4}$. Finally, we note that

$$
\mu_{\hat{F},\beta}(Y_v | \sigma_P) = \prod_{v \in V\setminus P} \mu_{G_v,\beta}(Y_v | \sigma_{P_v}) \geq \left(1 - \frac{c_0 n}{a^{5n^{3\theta/4}}}\right)^{|V\setminus P|} \geq 1 - \frac{c_0 n^{1+\theta/12}}{a^{5n^{3\theta/4}}}
$$

since $|V\setminus P| = n^{3\theta/4} + 2$, and the result follows. \qed
5.4.1 Sampling from the degree reducing gadget

We focus now in proving Lemma 5.4. Let $\mu_{G,\beta}$ and $Z_{G,\beta}$ be the antiferromagnetic Ising distribution and its corresponding partition function on a degree reducing gadget $G$. We need to show to prove Lemma 5.4 how approximately sample from $\mu_{G,\beta}$ and how to compute $Z_{G,\beta}$ when conditioning on an arbitrary configuration on the ports $P$ of $G$. (Note that with a slight abuse of notation we are using $P$ for the set of ports of a single gadget $G$ throughout this section.) Let $\tau\{+1,-1\}^P$ be a configuration on the ports. Let $Z_{G,\beta}^\tau$ and $\mu_{G,\beta}^\tau$ denote the conditional Ising distribution distribution and the corresponding partition function.

To establish Lemma 5.4, we provide two different algorithms: one based on the recent results from [KLR22] that works when $\beta \geq -1/\sqrt{10d}$, and another based on polymer models that works when $\beta \leq -\frac{c\log d}{d}$ (for a sufficiently large constant $c > 0$), so that each value of the regime $\beta < 0$ is covered by one of these algorithms provided $d$ is a large enough.

Both algorithms use facts about the spectrum of the multi-graph induced by $V_G \setminus P$. Hence, let

$$H = (V_H, E_H)$$

be the multi-graph that results from removing $P$ from $V_G$. Let $A_H$ be the adjacency matrix for the multi-graph $H$; that is, $A_H(u, v)$ is the multiplicity of the edge $\{u, v\}$ in $H$. For $S \subseteq V_H$, let $\partial_v(S)$ be the set of edges from $E_H$ with one endpoint in $S$ and one $V_H \setminus S$. For any real symmetric matrix $Q$, let $\lambda_i(Q)$ denote its $i$-th largest eigenvalue.

**Fact 5.8.** Suppose $d = O(1)$. Then, with probability $1 - o(1)$:

1. $d - 2\sqrt{d} - 2 \leq \lambda_1(A_H) \leq d + 2\sqrt{d}$;
2. $-d - 2\sqrt{d} \leq \lambda_{|V_H|}(A_H) \leq -d + 2\sqrt{d} + 2$;
3. For $i \geq 2$, $-4\sqrt{d} - 2 \leq \lambda_2(A_H) \leq 4\sqrt{d} + 2$;
4. For every $S \subseteq V_H$ such that $|S| \leq |V_H|/2$, we have $|\partial_v(S)| \geq \frac{d-4\sqrt{d}-2}{2}|S|$.

**Proof.** Consider the symmetric matrices $A$, $B$, and $T$, of dimension $|V_H| \times |V_H|$ defined by:

- $A(u, v) = \kappa$ if $u \in W_+ \cup U_+$ and $v \in W_- \cup U_-$ (or vice versa) and the edge $\{u, v\}$ appears $\kappa$ times in $\bigcup_{i=1}^{d-1} M_i$; all other entries of $A$ are 0.
- $B(u, v) = 1$ if $u \in W_+$ and $v \in W_-$ (or vice versa) and $\{u, v\} \in M$; all other entries of $B$ are 0.
- $T(u, v) = 1$ if $\{u, v\} \in E_H$ and either $u$ or $v$ (or both) are vertices in $V_H \setminus (W_+ \cup U_+ \cup W_- \cup U_-)$; all other entries of $T$ are 0.

Note that $A_H = A + B + T$, so it follows from Weyl’s inequality (see [Fra12]) that

$$\lambda_i(A) + \lambda_{|V_H|}(B) + \lambda_{|V_H|}(T) \leq \lambda_i(A_H) \leq \lambda_i(A) + \lambda_1(B) + \lambda_1(T).$$

From Theorem 4 in [BDH22] and contiguity (see Theorem 4 and Corollary 1 in [MRRW97]), we know that $A$ has real eigenvalues $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_{|V_H|}(A)$, where $\lambda_1(A) = d - 1$, $\lambda_1(A) = -\lambda_{|V_H|-i+1}(A)$, and $2\sqrt{d} - 1 \leq \lambda_2(A) \leq 2\sqrt{d} + 1$ with probability $1 - o(1)$. The matrix $B$ has eigenvalues 1 and $-1$. Also, all the eigenvalues of the matrix $T$ are real and belong to the interval $[-2\sqrt{d}, 2\sqrt{d}]$ (see [HLW06]). Combining these facts, we obtain parts 1, 2 and 3; part 4 follows from Cheeger’s inequality (for multi-graphs). \qed
Proof of Lemma 5.4. Let $J$ be a $|V_H| \times |V_H|$ matrix indexed by the vertices of $H$ with entries $J(u,v) = \beta \cdot A_H(u,v)$ for $u \neq v$ and $J(u,u) = \alpha$ where $\alpha$ is a real number we choose later. Let $\partial P \subset V_H$ be the set of vertices of $H$ that were incident to $P$ in $G$. Define a magnetic field $h$ by letting $h_v = \beta$ (resp., $h_v = -\beta$) if $v \in \partial P$ and the vertex adjacent to $v$ in $P$ has $+1$ (resp., $-1$) spin in $\tau$; we set $h_v = 0$ otherwise. The Ising model on $H$ with edge interaction $\beta$ and external field $h$ assigns to each configuration $\sigma$ on $H$ probability:

$$
\mu_{H,\beta}(\sigma) = \frac{1}{Z_{H,\beta}} \exp \left( \beta \sum_{\{u,v\} \in E_H} \sigma_u \sigma_v + \sum_{v \in V_H} h_v \sigma_v \right) = \frac{1}{Z_{H,\beta}} \exp \left( \frac{1}{2} \langle \sigma, J \sigma \rangle + \langle h, \sigma \rangle \right),
$$

where $\hat{Z}_{H,\beta} = e^{\alpha |V_H|} Z_{H,\beta}$, and we interpret $\sigma$ and $h$ as vectors indexed by the vertices of $H$. By construction, $Z_{G,\beta}^\tau = Z_{H,\beta}$ and $\mu_{G,\beta}^\tau(\sigma) = \mu_{H,\beta}(\sigma)$ for every $\sigma \in \{+, -\}^{|V_H|}$.

The matrix $J$ has real eigenvalues $\lambda_1(J) \geq \lambda_2(A) \geq \cdots \geq \lambda_{|V_H|}(J)$. To bound the spectrum of $J$, we note that $J = \beta A_H + \alpha I$ and so $\lambda_i(J) = \beta \lambda_i(A_H) + \alpha$. Hence, setting $\alpha = -\beta(4\sqrt{d} + 2)$ and assuming that $0 > \beta \geq -1/(10\sqrt{d})$ and that $d$ is sufficiently large, we obtain from Fact 5.8 that with probability $1 - o(1)$: $\lambda_1(J) = \Theta(\sqrt{d})$, $\lambda_{|V(H)|}(J) = -\Theta(\sqrt{d})$ and that every other eigenvalue of $J$ is in the interval $[0, 1]$. Then, Theorem 1.1 from [KLR22] implies that:

1. There is an algorithm that with probability $1 - e^{-|V_H|}$ produces an $\varepsilon$-multiplicative approximation for $\hat{Z}_{H,\beta} = Z_{G,\beta}^\tau$ with running time poly$(|V_H|, 1/\varepsilon)$; and

2. There is an algorithm to sample from a distribution within $\varepsilon$ total variation distance from $\mu_{H,\beta} = \mu_{G,\beta}^\tau$ with running time poly$(|V_H|, \log(1/\varepsilon))$.

Hence, we have established the result for the case when $\beta \geq -\frac{1}{10\sqrt{d}}$. We consider next the case when $\beta \leq -\frac{1}{10\sqrt{d}}$, for a suitably large constant $c > 0$. For this, we introduce the notion of polymer models.

For a fixed configuration $\tau$ in $P$, let $P^+ \subset \partial P$ be the set of vertices of $\partial P$ adjacent to a vertex assigned “+” in $\tau$; define $P^- \subset \partial P$ similarly. For a configuration $\sigma$ on $H$, let $p^+(\sigma)$ (resp., $p^-(\sigma)$) denote the number of vertices from $P^+$ (resp., $P^-$) that are assigned spin $-1$ (resp., $+1$) in $\sigma$. Let also $D(H, \sigma)$ denote the number of edges incident two vertices with different spins in $\sigma$. Then, we can renormalize the Ising distribution Eq. (16) as:

$$
\mu_{H,\beta}(\sigma) = \frac{1}{\hat{Z}_{H,\beta}} e^{-2\beta(D(H, \sigma) + p^+(\sigma) + p^-(\sigma))} = \frac{w(\sigma)}{\hat{Z}_{H,\beta}},
$$

where $\hat{Z}_{H,\beta} = e^{-\beta|E_H| + |\partial P|} Z_{H,\beta}$.

Let $\Omega = \{+1, -1\}^{|V_H|}$. Observe the graph $H$ is bipartite with partition $(L, R)$ where $L \cup R = V_H$ and $|L| = |R|$. Let $\Omega^\pm \subset \Omega$ be the subset of configurations where the number vertices that are assigned $+1$ in $L$ and $-1$ in $R$ is more than $|V_H|/2$. Define $\Omega^\mp$ analogously. Let $Z_H^\pm = \sum_{\sigma \in \Omega^\pm} w(\sigma)$ and $Z_H^\mp = \sum_{\sigma \in \Omega^\mp} w(\sigma)$ so that $\hat{Z}_{H,\beta} = Z_H^+ + Z_H^-$. We define a polymer model whose partition function will serve as a good approximation for $Z_H^+$ and $Z_H^-$. We say $\gamma \subset V_H$ is a polymer if the subgraph induced by $\gamma$ is connected and $|\gamma| < |V_H|/2$. Two polymers are compatible if the graph distance between them is at least 2. Let $\mathcal{G}$ be the family of all sets of mutually compatible polymers. To each polymer $\gamma$ we assign the weight:

$$
w_\gamma = e^{-2\beta(|\partial_\gamma(\gamma)| + |P^- \cap L \cap \gamma| + |P^+ \cap R \cap \gamma| - |P^+ \cap L \cap \gamma| - |P^- \cap R \cap \gamma|)}.
$$

Define the polymer partition function:

$$
\Phi = \sum_{\Gamma \in \mathcal{G}} \prod_{\gamma \in \Gamma} w_\gamma.
$$
We say $S \subset V_H$ is sparse if every connected component of $S$ has size less than $|V_H|/2$. Note that there is a one-to-one correspondence between the sparse subsets of $V_H$ and polymer configurations from $G$. Then:

$$
\hat{\Phi} := e^{-2\beta(|E_H|+|P^-\cap R|+|P^+\cap L|)} \cdot \Phi = e^{-2\beta(|E_H|+|P^-\cap R|+|P^+\cap L|)} \sum_{\Gamma \in \mathcal{G}} \prod_{\gamma \in \Gamma} w_\gamma
$$

$$
= \sum_{S \text{ sparse}} e^{-2\beta(|E_H|\text{−}|\partial_v(S)|+|P^-\cap R||P^+\cap L|+|P^-\cap R\cap S|+|P^+\cap L\cap S|+|P^-\cap R\cap S|)}
$$

$$
= \sum_{S \text{ sparse}} e^{-2\beta(|E_H|\text{−}|\partial_v(S)|+|{(R\setminus S)\cup (S\cap L)}|\cap P^-|+|{(L\setminus S)\cup (S\cap R)}|\cap P^+|)}
$$

Now, we say $S \subset V_H$ is small if $|S| < |V_H|/2$ (otherwise we say it is large), so that

$$
Z_H^\mp = \sum_{S \text{ small}} e^{-2\beta(|E_H|\text{−}|\partial_v(S)|+|{(R\setminus S)\cup (S\cap L)}|\cap P^-|+|{(L\setminus S)\cup (S\cap R)}|\cap P^+|)}.
$$

Hence,

$$
0 \leq \hat{\Phi} - Z_H^\mp \leq \sum_{S \text{ sparse, large}} e^{-2\beta(|E_H|\text{−}|\partial_v(S)|+|{(R\setminus S)\cup (S\cap L)}|\cap P^-|+|{(L\setminus S)\cup (S\cap R)}|\cap P^+|)}.
$$

If $S$ is sparse, by part 4 of Fact 5.8, each connected component $S_i$ of $S$ satisfies $\partial_v(S_i) \geq \theta|S_i|$ with $\theta = \frac{d-\sqrt{d^2-4}}{2}$. Summing over the components of $S$ we get $\partial_v(S) \geq \theta|S| \geq \theta|V_H|/2$ when $S$ is large. Then,

$$
|\hat{\Phi} - Z_H^\mp| \leq \sum_{S \text{ sparse, large}} e^{-2\beta(|E_H|\text{−}\theta|V_H|/2+|{(R\setminus S)\cup (S\cap L)}|\cap P^-|+|{(L\setminus S)\cup (S\cap R)}|\cap P^+|)}
$$

and since $Z_H^\mp \geq e^{-2\beta(|E_H|+|L\cap P^+|+|R\cap P^-|)}$ and $|S| < |V_H|/2$, we have

$$
\left| 1 - \frac{\hat{\Phi}}{Z_H^\mp} \right| \leq 2|V_H|e^{-2\beta(-\theta|V_H|/2+|V_H|/2)} \leq e^{-|V_H|},
$$

(17)

provided $\theta > 1$ and $-\beta > \frac{1+\ln 2}{\theta-1}$. An analogous argument yields the same bound for $Z_H^\pm$.

Our goal now is to use Theorem 8 from [JKP20] to obtain an approximation for $\Phi$ and consequently for $\hat{\Phi}$, $Z_H^\mp$, and ultimately for $Z_H^\pm$. For this, it suffices to check that our polymer model satisfies the so-called Kotecký-Preiss condition (see, e.g., equation (3) from [JKP20]). This condition requires that for every polymer $\gamma$:

$$
\sum_{\gamma':d(\gamma,\gamma') \leq 1} w_{\gamma'} e^{2|\gamma'|} \leq |\gamma|,
$$

(18)

where $d(\cdot, \cdot)$ denotes graph distance. First note that

$$
w_{\gamma'} = e^{-2\beta(-|\partial_v(\gamma')|+|P^-\cap L \cap \gamma'|+|P^+\cap R \cap \gamma'|+|P^-\cap R \cap \gamma'|+|P^+\cap L \cap \gamma'|-|P^-\cap R \cap \gamma'|)} \leq e^{-2\beta(-\theta/2+1)|\gamma'|}.
$$

Hence,

$$
\sum_{\gamma':d(\gamma,\gamma') \leq 1} w_{\gamma'} e^{2|\gamma'|} \leq \sum_{\gamma':d(\gamma,\gamma') \leq 1} e^{2|\gamma'| (2-2\beta(-\theta/2+1))} \leq \sum_{v \in \gamma' \setminus \partial_v(\gamma)} \sum_{\gamma':v \in \gamma'} e^{2|\gamma'| (2-2\beta(-\theta/2+1))}.
$$
The number of polymers of size $k$ that contain a given vertex is at most $(ed)^k$ (see Lemma 2.1 in [GK04]), so
\[
\sum_{\gamma' \in \gamma'} e^{\gamma'((2-2\beta(-\theta/2+1))} \leq \sum_{t \geq 1} \left( (de(3-2\beta(-\theta/2+1)))^t \right) \leq \frac{1}{d+1}
\]
when $-\beta \geq \frac{3+\ln(d+2)}{d}$ and $\theta > 2$. (Note that the latter is true when $d$ is large enough.) Since $|\gamma \cup \partial_v(\gamma)| \leq (d+1)|\gamma|$, Eq. (18) follows. Hence, for sufficiently large $d = O(1)$, for a suitable constant $c > 0$, Theorem 8 from [JKP20] gives an FPTAS for $\Phi$ when $-\beta \geq \frac{\ln d}{d}$. This yields the desired FPTAS for $Z_{H,\beta}$. Note that if the desired approximation factor is smaller than $e^{-|V_H|}$, which is the best approximation for $Z_{H,\beta}$ we could using the polymer function $\Phi$ (see Eq. (17)), then we could use instead brute force for counting and sampling, since the running time would be allowed to be exponential in $|V_H|$. Finally, Theorem 9 from [JKP20] gives the a polynomial-time approximate sampling algorithm for the distribution:
\[
\nu(\Gamma) = \frac{\prod_{\gamma \in \Gamma} w_{\gamma}}{\Phi}.
\]
Once a polymer configuration $\Gamma$ is sampled from $\nu$, it can be easily transformed into an Ising configuration by setting the vertices in $L \setminus \Gamma$ and $R \cap \Gamma$ to +1 with probability $\frac{Z_{H}^+}{Z_{H}^+ + Z_{H}^-}$ and all other vertices to −1, and doing the opposite with the remaining probability. \hfill \square

6 Statistical Lower Bounds

In this section we establish lower bounds on the number of samples required to perform uniformity testing over the hypercube $\{0,1\}^n$ (i.e., $k = 2$), with a focus on comparisons between testing for KL divergence and TV distance, and between testing with Coordinate Oracle + General Oracle and Subcube Oracle. Throughout this section, we assume $k = 2$ and $\mathcal{K} = \{0,1\}$. Let $u_n$ denote the uniform distribution over $\mathcal{X}_n = \{0,1\}^n$ for an integer $n \in \mathbb{N}$. We omit the subscript $n$ when it is clear from context.

Our proofs of theorems use the recipe given in [CRS15, Theorem 8] for showing sample lower bounds with the pairwise conditional sampling oracle. We also note that the arguments here and in [CRS15] are similar to the hybrid argument in cryptography.

6.1 Uniformity testing with Subcube Oracle for KL divergence

In this subsection we consider uniformity testing over $\{0,1\}^n$ with access to Subcube Oracle for KL divergence, and give an information-theoretic lower bound of $\Omega(n/\varepsilon)$ on the number of samples needed.

Let $\text{Alg}$ denote an arbitrary uniformity testing algorithm (possibly randomized and adaptive), and for simplicity let $\text{Ora}[\pi]$ denote the Subcube Oracle with respect to a distribution $\pi$ over $\{0,1\}^n$. A pinning $\tau$ is a partial configuration on a subset of coordinates, namely $\tau \in \{0,1\}^\Lambda$ where $\Lambda \subseteq [n]$. Observe that pinnings are exactly the inputs to the subcube oracle $\text{Ora}[\pi]$. Let $\mathcal{T}$ be the collection of all pinnings on all subsets of coordinates. For $L \in \mathbb{N}$, define the query history with respect to $\text{Alg}$ and $\text{Ora}[\pi]$ of length $L$ to be the random vector in $(\mathcal{T} \times \{0,1\}^n)^L$ generated as follows:

- For $i = 1, \ldots, L$:
  - $\text{Alg}$ receives $((\tau_1, x_1), \ldots, (\tau_{i-1}, x_{i-1}))$ as input and generates $\tau_i \in \mathcal{T}$ (randomly) as output;
Theorem 1.4

- Ora[π] receives τi as input and generates xi ∈ {0,1}n as output.

- The query history is H = ((τ1,x1),...,τL,xL)).

The output of Alg with sample complexity L is a (randomized) function of the query history H of length L.

Our main theorem is stated as below in terms of the query history, from which Theorem 1.4 follows immediately.

**Theorem 6.1.** Let n ∈ N+ be a sufficiently large integer and ε > 0 be a real. Let u = un denote the uniform distribution over {0,1}n. There is no algorithm which can achieve the following properties using only L ≤ n/(64ε) samples:

- PrH(output = Yes) ≥ 2/3 for a random query history H of length L with respect to Alg and Ora[u];
- PrH’(output = No) ≥ 2/3 for a random query history H’ of length L with respect to Alg and Ora[π] where π is any distribution such that DKL(π∥u) ≥ ε.

Our plan, as in many previous works, is to construct a family B of bad distributions that are all ε far away from u in KL divergence, such that when picking a bad distribution from B uniformly at random and drawing limited number of samples, the joint distributions of these samples are close to that of samples drawn from u. We present now our construction of the bad family B. Let t = ⌈log2(n/ε)⌉ − 3 for sufficiently large n. For any A ⊆ [n] with |A| = t and any σ ∈ {0,1}n, define the distribution πA,σ in the following way. A sample from πA,σ is generated by:

- For each i ∈ A independently sample xi ∈ {0,1} uniformly at random;
- If xA ≠ σA, then for each j ∈ [n] \ A independently sample xj ∈ {0,1} uniformly at random and output x;
- If XA = σA then output x = σ.

We remark that all steps are independent. Finally, we define

\[ B = \left\{ \pi_{A,\sigma} : A \in \binom{[n]}{t}, \sigma \in \{0,1\}^n \right\}. \]

We first show that the distributions in B are all bad in the sense that their KL divergence to the uniform distribution is at least ε. A key intuition in our construction of πA,σ here is that while the KL divergence DKL(πA,σ∥u) = Θ(ε), the TV distance is much smaller than ε and is dTV(πA,σ,u) = Θ(2−t) = Θ(ε/n). Hence, intuitively, it will take Θ(1/dTV(π,u)) = Θ(n/ε) samples to test between the family B and the uniform distribution u.

**Claim 6.2.** For all π ∈ B one has

\[ DKL(\pi∥u) \geq \varepsilon. \]

**Proof.** Suppose π = πA,σ ∈ B is a bad distribution. By definition we have π(x) = u(x) = 2−n if xA ≠ σA, and π(σ) = 2−t. Hence, we get

\[ DKL(\pi∥u) = π(σ) \ln \left( \frac{π(σ)}{u(σ)} \right) = \ln \frac{2^t}{2^t} (n-t) \geq \frac{2\varepsilon}{n} (n-t) \geq \varepsilon, \]

for n sufficiently large. □
Define $H$ to be the random query history of length $L$ with respect to $\text{Alg}$ and $\text{Ora}[u]$, and let $\text{output}$ denote the random output with respect to $H$ and $\text{Alg}$. Define $H'$ to be the random query history of length $L$ generated by

- Pick $\pi \in \mathcal{B}$ uniformly at random;
- Let $H'$ be the random query history of length $L$ with respect to $\text{Alg}$ and $\text{Ora}[\pi]$.

Further, let $\text{output}'$ denote the random output with respect to $H'$ and $\text{Alg}$. Our goal is to show that the TV distance between the two query histories $H$ and $H'$ is small and therefore by the data processing inequality the TV distance between $\text{output}$ and $\text{output}'$ is also small so the two properties in Theorem 6.1 cannot simultaneously hold.

**Lemma 6.3.** For the family $\mathcal{B}$ of bad distributions, query histories $H, H'$ of length $L \leq n/(64\varepsilon)$, and $\text{output}, \text{output}'$ defined as above, we have

$$d_{TV}(\text{output}, \text{output}') \leq d_{TV}(H, H') \leq \frac{1}{4}.$$

We present next the proof of Theorem 6.1 provided Lemma 6.3. The proof of the latter is postponed to Section 6.1.1.

**Proof of Theorem 6.1.** Suppose for sake of contradiction that $\text{Alg}$ satisfies both properties as in Theorem 6.1. Then for the family $\mathcal{B}$ of bad distributions, query histories $H, H'$ of length $L \leq n/(64\varepsilon)$, and $\text{output}, \text{output}'$ defined as above, we know from these two properties that

$$\Pr(\text{output} = \text{Yes}) \geq \frac{2}{3} \quad \text{and} \quad \Pr(\text{output}' = \text{Yes}) \leq \frac{1}{3}.$$

This implies $d_{TV}(\text{output}, \text{output}') \geq 1/3$ which contradicts Lemma 6.3. \hfill $\Box$

### 6.1.1 Proof of Lemma 6.3

To prove the lemma, we use the hybrid argument in cryptography which was already applied in previous works [CRS15]. For $0 \leq \ell \leq L$, define the *hybrid query history* $H^{(\ell)}$ with respect to $\text{Alg}$, $\text{Ora}[u]$, and $\text{Ora}[\pi]$ to be the random vector in $(T \times \{0, 1\}^n)^L$ generated as follows:

- For $i = 1, \ldots, \ell$:
  - $\text{Alg}$ receives $((\tau_1, x_1), \ldots, (\tau_{i-1}, x_{i-1}))$ as input and generates $\tau_i \in T$ (randomly) as output;
  - $\text{Ora}[u]$ receives $\tau_i$ as input and generates $x_i \in \{0, 1\}^n$ as output.
- Pick $\pi \in \mathcal{B}$ uniformly at random;
- For $i = \ell + 1, \ldots, L$:
  - $\text{Alg}$ receives $((\tau_1, x_1), \ldots, (\tau_{i-1}, x_{i-1}))$ as input and generates $\tau_i \in T$ (randomly) as output;
  - $\text{Ora}[\pi]$ receives $\tau_i$ as input and generates $x_i \in \{0, 1\}^n$ as output.
- The hybrid query history is $H^{(i)} = ((\tau_1, x_1), \ldots, (\tau_L, x_L))$.

Observe that $H^{(0)} = H'$ and $H^{(L)} = H$ in distribution. We will prove the following lemma regarding the distance between two adjacent hybrid query histories.
Lemma 6.4. For every $1 \leq \ell \leq L$, we have

$$d_{TV}(H^{(\ell-1)}, H^{(\ell)}) \leq \frac{16\varepsilon}{n}.$$ 

Note that Lemma 6.3 is an immediate consequence of Lemma 6.4.

Proof of Lemma 6.3. By the triangle inequality and Lemma 6.4, we have that

$$d_{TV}(H, H') \leq \sum_{\ell=1}^{L} d_{TV}(H^{(\ell-1)}, H^{(\ell)}) \leq L \cdot \frac{16\varepsilon}{n} \leq \frac{1}{4},$$

as claimed. 

It remains to prove Lemma 6.4. Inspecting the definitions of $H^{(\ell-1)}$ and $H^{(\ell)}$, we see that they only differ locally at one place, which we describe as follows. For $0 \leq i \leq L$ let $H_i = ((\tau_1, x_1), \ldots, (\tau_i, x_i))$ denote the first $i$ entries of a random hybrid query history (notice that $H_0 = \emptyset$). We write $H^{(\ell-1)}$ and $H^{(\ell)}$ in the following form.

Generation of $H^{(\ell-1)}$: 

1. $H_0 \xrightarrow{\text{Alg}, \text{Ora}[u]} H_{\ell-1};$
2. $\pi \sim \text{unif}(B);$ 
3. $H_{\ell-1} \xrightarrow{\text{Alg}, \text{Ora}[\pi]} x_{\ell};$
4. $H_{\ell} \leftarrow H_{\ell-1} \text{ append } (\tau_{\ell}, x_{\ell});$
5. $H_{\ell} \xrightarrow{\text{Alg}, \text{Ora}[\pi]} H_L = H^{(\ell-1)}.$

Generation of $H^{(\ell)}$: 

1. $H_0 \xrightarrow{\text{Alg}, \text{Ora}[u]} H_{\ell-1};$
2. $H_{\ell-1} \xrightarrow{\tau_{\ell}} \text{Ora}[\pi] \xrightarrow{\text{Alg}} x_{\ell};$
3. $H_{\ell} \leftarrow H_{\ell-1} \text{ append } (\tau_{\ell}, x_{\ell});$
4. $\pi \sim \text{unif}(B);$ 
5. $H_{\ell} \xrightarrow{\text{Alg}, \text{Ora}[\pi]} H_L = H^{(\ell)}.$

In fact the ordering of the steps (2)-(4) can be changed appropriately without having any influence on the final distribution of both $H^{(\ell-1)}$ and $H^{(\ell)}$, which will be helpful for a coupling argument. We rewrite the generating processes of $H^{(\ell-1)}$ and $H^{(\ell)}$ equivalently as follows:

Generation of $H^{(\ell-1)}$: 

1. $H_0 \xrightarrow{\text{Alg}, \text{Ora}[u]} H_{\ell-1};$
2. $H_{\ell-1} \xrightarrow{\text{Alg}} \tau_{\ell};$
3. $\pi \sim \text{unif}(B), \tau_{\ell} \xrightarrow{\text{Ora}[\pi]} x_{\ell};$
4. $H_{\ell} \leftarrow H_{\ell-1} \text{ append } (\tau_{\ell}, x_{\ell});$
5. $H_{\ell} \xrightarrow{\text{Alg}, \text{Ora}[\pi]} H_L = H^{(\ell-1)}.$

Generation of $H^{(\ell)}$: 

1. $H_0 \xrightarrow{\text{Alg}, \text{Ora}[u]} H_{\ell-1};$
2. $H_{\ell-1} \xrightarrow{\text{Alg}} \tau_{\ell};$
3. $\pi \sim \text{unif}(B), \tau_{\ell} \xrightarrow{\text{Ora}[\pi]} x_{\ell};$
4. $H_{\ell} \leftarrow H_{\ell-1} \text{ append } (\tau_{\ell}, x_{\ell});$
5. $H_{\ell} \xrightarrow{\text{Alg}, \text{Ora}[\pi]} H_L = H^{(\ell)}.$

Note that before and after the third step, the two processes have exactly the same steps. In the third step for $H^{(\ell-1)}$, we pick a bad distribution $\pi \in B$ uniformly at random, and $\text{Ora}[\pi]$ receives the pinning $\tau_{\ell}$ as input and generates $x_{\ell} \in \{0,1\}^n$ according to $\pi$ conditioned on $\tau_{\ell}$. Meanwhile, in the third step for $H^{(\ell)}$, we still pick a bad distribution $\pi \in B$ but do not use it (for now), and $\text{Ora}[u]$ receives $\tau_{\ell}$ as input and generates $x_{\ell} \in \{0,1\}^n$ according to $u$ instead of $\pi$. It is enough to
show that, in this step, conditional on that \(H_{\ell-1}\) and \(\tau_\ell\) are the same, the \(x_\ell\) generated in the two processes are the same with high probability. Since before and after this step the two processes are doing the same thing, we can then couple these two processes to produce the same hybrid query history with high probability, i.e., \(H_{\ell-1} = H_\ell\).

The following technical lemma bounds the probability that \(x_\ell\)'s are the same in both processes in the third step, which is crucial to us as explained earlier. The proof of it can be found in Section 6.1.2.

**Lemma 6.5.** Let \(\tau \in \mathcal{T}\) be an arbitrary pinning on some subset \(\Lambda \subseteq V\) of size \(m\). Then for a random distribution \(\pi\) chosen uniformly at random from \(\mathcal{B}\), we have

\[
\mathbb{E}_{\pi \sim \text{unif}(\mathcal{B})}[d_{TV}(u(\cdot \mid \tau), \pi(\cdot \mid \tau))] \leq \frac{16\varepsilon}{n}.
\]

We give below the proof of Lemma 6.4.

**Proof of Lemma 6.4.** We construct a coupling of \(H_{\ell-1}\) and \(H_\ell\) via coupling step-by-step the two processes generating \(H_{\ell-1}\) and \(H_\ell\). Initially \(H_0 = \emptyset\) for both processes. Then we can couple \(H_{\ell-1}\) and \(\tau_\ell\) since they are generated in the same way in both processes. For the third step, the bad distribution \(\pi\) can be chosen to be the same and we deduce from Lemma 6.5 that \(x_\ell\)'s can be coupled with probability at least \(1 - \varepsilon/n\). After that, suppose we couple \(x_\ell, H_\ell\) and then the final outputs are coupled. Hence, for this coupling \(\mathbb{P}\) we have

\[
d_{TV}(H_{\ell-1}, H_\ell) \leq \mathbb{P}(H_{\ell-1} \neq H_\ell) \leq \frac{16\varepsilon}{n},
\]

as wanted. \(\square\)

### 6.1.2 Proof of Lemma 6.5

Here we give the proof of the technical lemma, Lemma 6.5.

**Proof of Lemma 6.5.** The distribution \(\pi \in \mathcal{B}\) depends on \(A\) and \(\sigma\). We will show that for any choice of \(A \in \binom{[n]}{t}\) one has

\[
\mathbb{E}_\sigma[d_{TV}(u(\cdot \mid \tau), \pi_{A,\sigma}(\cdot \mid \tau))] \leq \frac{16\varepsilon}{n},
\]

where \(\sigma\) is a uniformly random configuration in \(\{0, 1\}^n\).

Suppose \(|\Lambda| = \ell\). Suppose \(|A \cap \Lambda| = j\) and hence \(|A \setminus \Lambda| = t - j\). Notice that \(j \leq \min\{t, \ell\}\). We partition \(X = \{0, 1\}^n\) into three disjoint subsets.

**Case 1.** \(X_1 = \{\sigma \in \{0, 1\}^n : \sigma_{A \cap \Lambda} \neq \tau_{A \cap \Lambda}\}\). We have

\[
\Pr(\sigma \in X_1) = \frac{|X_1|}{2^n} = 1 - \frac{1}{2^t},
\]

and also

\[
d_{TV}(u(\cdot \mid \tau), \pi_{A,\sigma}(\cdot \mid \tau)) = 0, \quad \forall \sigma \in X_1.
\]

**Case 2.** \(X_2 = \{\sigma \in \{0, 1\}^n : \sigma_{A \cap \Lambda} = \tau_{A \cap \Lambda}, \sigma_{A \setminus \Lambda} \neq \tau_{A \setminus \Lambda}\}\). We have

\[
\Pr(\sigma \in X_2) = \frac{|X_2|}{2^n} = \frac{1}{2^j} - \frac{1}{2^t}.
\]

43
By definition we have
\[ \pi_{A, \sigma} (x \mid \tau) = \begin{cases} 0, & \text{if } x_{A \setminus \Lambda} = \sigma_{A \setminus \Lambda}; \\ \frac{1}{2^{n-\ell} - 2^{n-\ell-1} - j}, & \text{if } x_{A \setminus \Lambda} \neq \sigma_{A \setminus \Lambda}. \end{cases} \]

It follows that
\[ d_{TV} (u (\cdot \mid \tau), \pi_{A, \sigma} (\cdot \mid \tau)) = \frac{1}{2^{n-\ell}}, \quad \forall \sigma \in X_2. \]

**Case 3.** \( X_3 = \{ \sigma \in \{0, 1\}^n : \sigma_{\Lambda} = \tau_{\Lambda} \} \). We have
\[ \Pr_{\sigma} (\sigma \in X_3) = \frac{|X_3|}{2^n} = \frac{1}{2^\ell}. \]

By definition we have
\[ \pi_{A, \sigma} (x \mid \tau) = \begin{cases} \frac{1}{2^\ell} + \frac{1}{2^{n-\ell} - 2^{n-\ell-1} - j}, & \text{if } x_{A \setminus \Lambda} \neq \sigma_{A \setminus \Lambda}; \\ 0, & \text{if } x_{A \setminus \Lambda} = \sigma_{A \setminus \Lambda} \text{ and } x_{[n] \setminus \Lambda \setminus A} \neq \sigma_{[n] \setminus \Lambda \setminus A}; \\ \frac{1}{2^t} + \frac{1}{2^{n-\ell} - 2^{n-\ell-1} - j}, & \text{if } x_{[n] \setminus \Lambda} = \sigma_{[n] \setminus \Lambda}. \end{cases} \]

It follows that
\[ d_{TV} (u (\cdot \mid \tau), \pi_{A, \sigma} (\cdot \mid \tau)) = \frac{2^t}{2^t + 2^{\ell} - 2^j} \leq \frac{2^\ell}{2^t + 2^{\ell} - 2^j - 1} \quad \forall \sigma \in X_3. \]

Therefore, combining all three cases we get from the law of total expectation that
\[ \mathbb{E}_\sigma [d_{TV} (u (\cdot \mid \tau), \pi_{A, \sigma} (\cdot \mid \tau))] \leq \frac{1}{2^t} + \frac{1}{2^t + 2^{\ell} - 2^j}. \]

Note that the second term is monotone increasing in \( j \) and by definition \( j \leq \min \{ t, \ell \} \). Hence, we deduce that
\[ \frac{1}{2^t + 2^{\ell} - 2^j} \leq \frac{1}{2^t + 2^{\ell} - 2^{\max \{ t, \ell \}}} = \frac{1}{2^{\max \{ t, \ell \}}} \leq \frac{1}{2^\ell}. \]

We conclude that for any \( A \),
\[ \mathbb{E}_\sigma [d_{TV} (u (\cdot \mid \tau), \pi_{A, \sigma} (\cdot \mid \tau))] \leq \frac{1}{2^t - 1} \leq \frac{16 \varepsilon}{n}, \]
where in the last inequality we recall that \( t = \lceil \log_2 (n / \varepsilon) \rceil - 3 \geq \log_2 (n / \varepsilon) - 3. \)

**6.2 Uniformity testing with Coordinate Oracle and General Oracle for TV distance**

In this section we consider uniformity testing over the binary hypercube for TV distance when we have access to Coordinate Oracle and General Oracle. We assume the binary hypercube is denoted by \( X_n = \{ +1, -1 \}^n \) instead of \( \{0, 1\}^n \), since our bad distributions will be Ising models where \( +1, -1 \) are more often used.
Let $\text{Alg}$ denote an arbitrary uniformity testing algorithm (possibly randomized and adaptive) with Coordinate Oracle and General Oracle access. We assume that $\text{Alg}$ receives $L$ independent full samples from the General Oracle, and is allowed to make $L$ queries to the Coordinate Oracle. Let $\mathcal{T}$ denote the set of all pinnings on $n-1$ coordinates, which is exactly all possible inputs to the Coordinate Oracle. For ease of notation we denote by $Ora[\pi]$ the Subcube Oracle with respect to a distribution $\pi$ over $\{+1,-1\}^n$. For integer $L \in \mathbb{N}^+$, we define the query history with respect to $\text{Alg}$ and $Ora[\pi]$ of length $2L$ to be the random vector in $\{+1\}$ generated as follows:

- Let $x_1,\ldots,x_L$ be $L$ independent samples from $\pi$;
- For $i = 1,\ldots,L$:
  - $\text{Alg}$ receives $((\tau_1,a_1),\ldots,(\tau_{i-1},a_{i-1}))$ as input and generates $\tau_i \in \mathcal{T}$ (randomly) as output;
  - $Ora[\pi]$ receives $\tau_i$ as input and generates $a_i \in \{0,1\}$ as output.
- The query history is $H = (x_1,\ldots,x_L; (\tau_1,a_1),\ldots,(\tau_L,a_L))$.

The output of $\text{Alg}$ with sample complexity $2L$ is a (randomized) function of the query history $H$ of length $2L$. Our main theorem is then stated as follows.

**Theorem 6.6.** There exists a universal constant $c > 0$ such that the following holds. Let $n \in \mathbb{N}^+$ be a sufficiently large integer and $\varepsilon > 0$ be a real. Let $u = u_n$ denote the uniform distribution over $\{+1,-1\}^n$. Then there is no algorithm which can achieve the following properties using $L$ samples from General Oracle and $L$ queries from Coordinate Oracle where $L \leq cn/\varepsilon^2$:

- $\Pr_{H}(\text{output} = \text{Yes}) \geq 2/3$ for a random query history $H$ of length $2L$ with respect to $\text{Alg}$ and $Ora[u]$;
- $\Pr_{H'}(\text{output} = \text{No}) \geq 2/3$ for a random query history $H'$ of length $2L$ with respect to $\text{Alg}$ and $Ora[\pi]$ where $\pi$ is any distribution such that $d_{TV}(\pi,u) \geq \varepsilon$.

We observe that Theorem 1.5 follows immediately from Theorem 6.6.

In [DDK19, Theorem 14] it was shown that $\Omega(n/\varepsilon^2)$ samples are necessary for uniformity testing with only General Oracle access but assuming the hidden distribution $\pi$ is an Ising model. See also [CDKS20, Theorem 14] for very similar lower bounds in the setting of Bayesian networks. Note that though Theorem 14 from [DDK19] is stated for symmetric KL divergence, it actually works for TV distance as well, see [DDK19, Remark 4]. We use the same constructions from [DDK19, CDKS20] for the family of bad distributions for our purpose. Assume that $n$ is even; the case of odd $n$ can be easily reduced to even $n$ by adding an extra uniform, independent coordinate. Suppose $M$ is a perfect matching of $n$ coordinates, i.e., $M$ is a collection of $n/2$ pairs of coordinates such that each coordinate appears in exactly one pair. Let $\mathcal{M}$ be the set of all perfect matchings on $[n]$. Each bad distribution $\pi_M$ where $M \in \mathcal{M}$ corresponds to an Ising model on the graph $G = ([n],M)$ of $n/2$ edges, with the edge coupling set to be $\beta = \rho \varepsilon / \sqrt{n}$ where $\rho$ is a universal constant sufficiently large. The following are established in [CDKS20, DDK19].

**Claim 6.7 ([CDKS20, DDK19]).** (1) For $\rho > 0$ sufficiently large, for all $M \in \mathcal{M}$, it holds

$$d_{TV}(\pi_M,u) \geq \varepsilon.$$
(2) For any $\rho > 0$ there exists $c_1 = c_1(\rho) > 0$ such that the following holds. Suppose $L \leq c_1 n/\varepsilon$. Let $X = (x_1, \ldots, x_L)$ be $L$ independent samples from $u$. Independently, let $M \in \mathcal{M}$ be chosen uniformly at random, and let $X' = (x'_1, \ldots, x'_L)$ be $L$ independent samples from $\pi_M$. Then $d_{\text{TV}}(X, X') \leq 0.98$.

Proof. (1) follows from Lemma 8 in [CDKS20]. (2) is proved in Section 8.3.2 in [DDK19]. See also in Section 8.1 from [CDKS20] the same result for a slightly different construction of $\pi_M$, where every edge is set to be ferromagnetic with probability 1/2 and antiferromagnetic otherwise. 

Define $H$ to be the random query history of length $2L$ with respect to $\text{Alg}$ and $\text{Ora}[u]$, and let output denote the random output with respect to $H$ and $\text{Alg}$. Define $H'$ to be the random query history of length $2L$ generated by

- Pick $M \in \mathcal{M}$ uniformly at random and let $\pi = \pi_M$;
- Let $H'$ be the random query history of length $2L$ with respect to $\text{Alg}$ and $\text{Ora}[\pi]$.

Further, let output' denote the random output with respect to $H'$ and $\text{Alg}$. Then we can show the following key lemma.

**Lemma 6.8.** For query histories $H, H'$ of length $2L$ where $L \leq cn/\varepsilon$ and output, output' defined as above, we have

$$d_{\text{TV}}(\text{output}, \text{output}') \leq d_{\text{TV}}(H, H') \leq 0.99.$$ 

Proof. The first inequality follows from the data processing inequality. We focus on the second one. For $M \in \mathcal{M}$ and $t \in \{0, 1\}$, let $\pi_{M,t}$ denote the Ising model on $G = ([n], M)$ with edge coupling $t\beta = t\rho\varepsilon/\sqrt{n}$. Observe that $\pi_{M,0} = u$ and $\pi_{M,1} = \pi_M$. We rewrite the process for generating the query histories $H$ and $H'$ of length $2L$ in the following equivalent form:

- Let $M \in \mathcal{M}$ be chosen uniformly at random from $\mathcal{M}$;
- Let $X_t = (x_1, \ldots, x_L) \in \mathcal{X}_n^L$ be $L$ independent samples from $\pi_{M,t}$;
- Let $R_t = (r_1, \ldots, r_L) \in \{0, 1\}^L$ be $L$ independent Bernoulli random variables with mean $(1 + \tanh(t\rho\varepsilon/\sqrt{n}))/2$;
- For $i = 1, \ldots, L$:
  - $\text{Alg}$ receives $((\tau_1, a_1), \ldots, (\tau_{i-1}, a_{i-1}))$ as input and generates $\tau_i \in \mathcal{T}$ (randomly) as output;
  - $\text{Ora}[\pi]$ receives $\tau_i$ as input, which fixes all coordinates but one say $j$, and suppose $j'$ is matched to $j$ in $M$; then $\text{Ora}[\pi]$ outputs $a_i = (\tau_i)_{j'}$ (the $j'$-th coordinate of $\tau_i$) as the sampled value at the $j$-th coordinate if $r_i = 1$, and outputs $a_i = 1 - (\tau_i)_{j'}$ otherwise;
- The query history is $H_t = (x_1, \ldots, x_L; (\tau_1, a_1), \ldots, (\tau_L, a_L))$.

Observe that if $t = 0$, then the final query history $H_0$ is distributed as $H$; meanwhile, if $t = 1$, then it is distributed as $H'$. Moreover, the process above can be viewed as a random mapping from the vector $(M, X_t, R_t)$ to the query history $H_t$ where, for fixed $(M, X_t, R_t)$, the randomness purely comes from the decision-making of $\text{Alg}$. Therefore, we can apply the data processing inequality and obtain

$$d_{\text{TV}}(H, H') \leq d_{\text{TV}}((M, X_0, R_0), (M, X_1, R_1)) \leq d_{\text{TV}}(X_0, X_1) + d_{\text{TV}}(R_0, R_1).$$

46
Note that $d_{TV}(X_0, X_1) \leq 0.98$ by Claim 6.7. For the second term, we have

$$d_{TV}(R_0, R_1) = d_{TV} \left( \text{Bin} \left( L, \frac{1}{2} \right), \text{Bin} \left( L, \frac{1}{2} \left( 1 + \tanh \frac{\rho \varepsilon}{\sqrt{n}} \right) \right) \right) \leq c' \cdot \sqrt{L} \cdot \frac{\rho \varepsilon}{\sqrt{n}} \leq 0.01,$$

where $c' > 0$ is a universal large constant, and $L \leq cn/\varepsilon^2$ for $c$ sufficiently small. Therefore, we deduce that $d_{TV}(H, H') \leq 0.98 + 0.01 = 0.99$ as claimed.

We end this section with the proof of Theorem 6.6.

**Proof of Theorem 6.6.** Suppose for sake of contradiction that Alg satisfies both properties as in Theorem 6.6. Then by a standard amplification technique for failure probability, one can decrease the failure probability from $1/3$ to $0.001$ with the number of samples needed increases only by a constant factor (see [Can22, Lemma 1.1.1]). In particular, for query histories $H, H'$ of length $2L$ where $L \leq cn/\varepsilon$ and output, output' defined as above, we have

$$\Pr(\text{output} = \text{Yes}) \geq 0.999 \quad \text{and} \quad \Pr(\text{output}' = \text{Yes}) \leq 0.001.$$

This implies $d_{TV}(\text{output}, \text{output}') \geq 0.998$ which contradicts Lemma 6.8.

## 7 Identity Testing with Subcube Oracle

In this section we give our algorithmic results for identity testing with access to the Subcube Oracle.

### 7.1 Identity testing with exact conditional marginal distributions

Recall that $[i] = \{1, \ldots, i\}$ for an integer $i \in \mathbb{N}^+$. The following factorization of entropy is well-known, see e.g. [Ces01, MSW03, CP21].

**Lemma 7.1.** For any nonnegative function $f: \mathcal{K}^n \to \mathbb{R}_{\geq 0}$ we have

$$\operatorname{Ent} f = \sum_{i=1}^n \mu_i \left[ \operatorname{Ent}_{[n] \setminus [i-1]} \left( \mu_{[n] \setminus [i]} f \right) \right]. \quad (19)$$

Equivalently, for any distribution $\pi$ over $\mathcal{K}^n$ such that $\pi \ll \mu$ we have

$$D_{KL}(\pi \| \mu) = \sum_{i=1}^n \mathbb{E}_{x \sim \pi_{[i-1]}} \left[ D_{KL}(\pi_i(\cdot \mid x) \| \mu_i(\cdot \mid x)) \right]. \quad (20)$$

The functional form, Eq. (19), is equivalent to the probabilistic form, Eq. (20), by taking $f = \pi/\mu$ to be the relative density.

We now give our testing algorithm with Subcube Oracle.

**Theorem 7.2.** Let $k = k(n)$ be an integer and let $b = b(n) \in (0, 1/2]$ be a real. Suppose that $\log \log (1/b) = O(\log n)$. There is an identity testing algorithm for all $b$-marginally bounded distributions with query access to Subcube Oracle and for KL divergence with distance parameter $\varepsilon > 0$. The query complexity of the identity testing algorithm is

$$O \left( \min \left\{ \frac{1}{\sqrt{b}} \cdot \frac{n \log^3 \left( \frac{n}{\varepsilon} \right)}{\varepsilon}, \sqrt{k} \log \left( \frac{1}{b} \right) \cdot \frac{n^2}{\varepsilon^2 \log^2 \left( \frac{n}{\varepsilon} \right)} \right\} \right).$$
The running time of the algorithm is polynomial in all parameters and also proportional to the time of computing the conditional marginal distributions \( \mu_i(\cdot \mid x) \) for any \( i \in [n] \) and any feasible \( x \in \mathcal{K}^{[i-1]} \). Furthermore, if \( k = 2 \), i.e., we have a binary domain \( \mathcal{K} = \{0, 1\} \), the query complexity can be improved to

\[
O \left( \log \left( \frac{1}{b} \right) \cdot \frac{n}{\varepsilon} \log^3 \left( \frac{n}{\varepsilon} \right) \right).
\]

Proof. We observe that Eq. (20) can be equivalently written as

\[
D_{\text{KL}}(\pi \parallel \mu) = n \mathbb{E}_{(i,x)} [D_{\text{KL}}(\pi_i(\cdot \mid x) \parallel \mu_i(\cdot \mid x))],
\]

where \( i \in [n] \) is a uniformly random coordinate and \( x \) is generated from \( \pi_{[i-1]} \). Therefore, Algorithm 1 still works once we generate the pair \((i,x)\) in Line 8 from the correct distribution as just described, and define \( p^x_i = \pi_i(\cdot \mid x) \), \( q^x_i = \mu_i(\cdot \mid x) \) correspondingly. The analysis is exactly the same with the constant \( C \) for approximate tensorization replaced by 1. We omit the proofs here and only highlight the differences: the coordinate balance \( \eta \) is now replaced by the marginal boundedness \( b \), and the running time depends on the time to compute the conditional marginal distributions \( \mu_i(\cdot \mid x) \) for any \( i \in [n] \) and any \( x \in \mathcal{K}^{[i-1]} \) such that \( \mu_{[i-1]}(x) > 0 \).

Remark 7.3. We remark that the assumption of marginal boundedness can be relaxed to the following slightly weaker version: for a fixed ordering of the coordinates, for every \( i \in [n] \), every \( x \in \mathcal{K}^{[i-1]} \) with \( \mu_{[i-1]}(x) > 0 \), and every \( a \in \mathcal{K} \), one has

\[
\text{either } \mu_i(a \mid x) = 0, \quad \text{or } \mu_i(a \mid x) \geq b.
\]

In some circumstances, this weaker notion of marginal boundedness can give a better bound on the sample complexity.

Theorem 7.2 indicates that identity testing can be done efficiently for a wide variety of families of distributions with the power of Subcube Oracle, assuming that one can efficiently compute the exact marginal probabilities under any conditioning. Below we give a few examples where Theorem 7.2 applies. Instead of going into detailed definitions and technical lemmas, we only describe the models without specifications to illustrate the usage of Theorem 7.2.

- Consider any undirected graphical model (e.g., Ising model, Potts model) defined on trees of constant degrees. Then the distributions are \( \Omega(1) \)-marginally bounded, and one can efficiently compute the marginal probabilities under any pinning via, e.g., Belief Propagation. Hence, there is a polynomial-time identity testing algorithm for undirected graphical models on bounded-degree trees with Subcube Oracle access. The sample complexity is \( O((n/\varepsilon) \log^3(n/\varepsilon)) \) where \( n \) is the number of vertices. If the degree is unbounded, then the marginal bound \( b \) can be as small as \( e^{-\Theta(n)} \). Still, by the second bound in Theorem 7.2 the number of samples needed is at most \( O((n^3/\varepsilon^2) \log^2(n/\varepsilon)) \).

- Consider the Bayesian network on DAGs, and assume without loss of generality that \([n] = \{1, \ldots, n\}\) is the topological ordering of the DAG. In particular, all conditional marginal probabilities at any coordinate \( i \in [n] \) and conditioned on any feasible pinning \( x \in \mathcal{K}^{[i-1]} \) are given by the Bayesian network. If these conditional marginal probabilities are lower bounded by \( b = \Omega(1) \), then there is a polynomial-time identity testing algorithm for such Bayesian networks with Subcube Oracle access, and the sample complexity is \( O((n/\varepsilon) \log^3(n/\varepsilon)) \). If \( b \) is exponentially small, then similarly as before the sample complexity is \( O((n^3/\varepsilon^2) \log^2(n/\varepsilon)) \). See also Remark 7.3 above on relaxing the marginal boundedness condition to specifically the topological ordering.
Consider mixtures of polynomially many product distributions, each of which has \( \eta(\mu) = \Omega(1) \) as defined in Section 4.4.1. One can efficiently compute the conditional marginal probabilities by the simple nature of mixtures of product distributions. Then by Theorem 7.2, we have an efficient identity testing algorithm with Subcube Oracle access and the sample complexity is \( O((n/\varepsilon)\log^3(n/\varepsilon)) \). Similarly as before, the sample complexity becomes \( O((n^3/\varepsilon^2)\log^2(n/\varepsilon)) \) when the minimum \( \eta(\mu) \) is exponentially small.

### 7.2 Identity testing with approximate conditional marginal distributions

In Theorem 7.2 we assume that one can compute exactly any conditional marginal distribution in polynomial time. In some applications the exact computation is not possible and one can get, at the best, an estimator of the conditional marginal probabilities. As we will show in this subsection, identity testing can still be done efficiently in this setting.

We first need a more robust version of Lemmas 4.3 and 4.9. We say there is an FPRAS for a distribution \( q \) over \( \mathcal{K} \) if for any \( \varepsilon > 0 \) and \( \delta \in (0,1) \), one can compute a distribution \( \hat{q} \) over \( \mathcal{K} \) as an approximation of \( q \) such that, with probability \( 1 - \delta \), we have that for every \( a \in \mathcal{K} \),

\[
\frac{e^{-\varepsilon}}{q(a)} \leq \frac{\hat{q}(a)}{\hat{q}(a)} \leq e^{\varepsilon},
\]

and \( \hat{q} \) can be computed with running time polynomial in \( k, 1/\varepsilon, \log(1/\delta) \), and the input size of \( q \) (e.g., the number of parameters representing \( q \)). We remark that if \( q(a) = 0 \) then \( \hat{q}(a) = 0 \).

**Lemma 7.4.** Let \( k \in \mathbb{N}^+ \) be an integer, and let \( \varepsilon > 0, b \in (0,1/2) \) be reals. Given an FPRAS for a target distribution \( q \) over domain \( \mathcal{K} \) of size \( k \) such that either \( q(a) = 0 \) or \( q(a) \geq b \) for any \( a \in \mathcal{K} \), and given sample access to an unknown distribution \( p \ll q \) over \( \mathcal{K} \), there exists a polynomial-time identity testing algorithm that distinguishes with probability at least \( 2/3 \) between the two cases

\[
p = q \quad \text{and} \quad D_{KL}(p\|q) \geq \varepsilon.
\]

For \( k \geq 3 \), the sample complexity of the identity testing algorithm is

\[
O \left( \min \left\{ \frac{1}{\varepsilon\sqrt{b}}, \frac{\sqrt{k} \ln(1/b)}{\varepsilon^2} \right\} \right).
\]

For \( k = 2 \), the sample complexity of the identity testing algorithm is

\[
O \left( \frac{\ln(1/b)}{\varepsilon} \right).
\]

**Proof.** Let \( m \) be an upper bound for the number of samples required in Lemmas 4.3 and 4.9, with the assumption being either \( q(a) = 0 \) or \( q(a) \geq b/2 \) for any \( a \in \mathcal{K} \), distance parameter \( \varepsilon/2 \), and failure probability \( 1/10 \). Let \( \xi = O(\min\{\varepsilon, 1/m\}) \) be a small constant, and let \( \hat{q} \) be an approximation of \( q \) such that with probability \( 9/10 \) we have \( e^{-\xi} \leq \hat{q}(a)/q(a) \leq e^\xi \) for every \( a \in \mathcal{K} \). Notice that if this holds then

\[
|D_{KL}(p\|\hat{q}) - D_{KL}(p\|q)| \leq \sum_{a\in\mathcal{K}} p(a) \left| \ln \left( \frac{q}{\hat{q}} \right) \right| \leq \xi.
\]

We then apply the identity testing algorithm \( A_{KL-id} \) from Lemmas 4.3 and 4.9 to the distributions \( p, \hat{q} \) with distance parameter \( \varepsilon/2 \) and failure probability \( 1/10 \), and returns the output of \( A_{KL-id} \) as our output. Note that \( \hat{q}(a) = 0 \) if \( q(a) = 0 \) and \( \hat{q}(a) \geq e^{-\xi}q(a) \geq b/2 \) if \( q(a) \geq b \), assuming
\( \hat{q} \) is a \( \xi \)-approximation of \( q \). Thus, the number of samples required by \( \mathcal{A}_{\text{KL-ID}} \) is at most \( m \). If \( D_{\text{KL}}(p \parallel q) \geq \varepsilon \), then

\[
D_{\text{KL}}(p \parallel \hat{q}) \geq D_{\text{KL}}(p \parallel q) - |D_{\text{KL}}(p \parallel q) - D_{\text{KL}}(p \parallel q)| \geq \varepsilon - \xi \geq \frac{\varepsilon}{2}.
\]

Hence, the testing algorithm wrongly outputs Yes only if at least one of the following happens

1. \( \hat{q} \) is not a \( \xi \)-approximation of \( q \), which happens with probability at most 1/10;
2. \( \mathcal{A}_{\text{KL-ID}} \) makes a mistake, which happens with probability at most 1/10.

This shows that the failure probability is at most 1/5. If \( p = q \), then notice that

\[
d_{\text{TV}}(p, \hat{q}) = d_{\text{TV}}(q, \hat{q}) = O(\xi) \leq \frac{1}{10m}.
\]

We consider an optimal coupling between \( m \) independent samples from \( p \) and \( m \) independent samples from \( \hat{q} \), so the probability that these two sets of \( m \) samples are not exactly the same is at most 1/10. One can think of the testing process as follows: we try to send \( m \) samples from \( \hat{q} \) to \( \mathcal{A}_{\text{KL-ID}} \), and it succeeds only when the samples are coupled with those from \( p \). Therefore, the failure probability, in addition to (1) and (2) above, also includes this uncoupled probability, and hence is at most 3/10. Finally, the number of samples needed, \( m \), is bounded in Lemmas 4.3 and 4.9. \( \square \)

Lemma 7.4, combined with the proof of Theorem 7.2, immediately implies the following theorem. See also Remark 7.3 for the discussion on relaxing marginal boundedness.

**Theorem 7.5.** Let \( k = k(n) \) be an integer and let \( b = b(n) \in (0, 1/2] \) be a real. Suppose that \( \log \log(1/b) = O(\log n) \). There is an identity testing algorithm for all \( b \)-marginally bounded distributions with query access to \( \text{Subcube Oracle} \) and for KL divergence with distance parameter \( \varepsilon > 0 \). The query complexity of the identity testing algorithm is

\[
O \left( \min \left\{ \frac{1}{\sqrt{b}} \cdot \frac{n}{\varepsilon} \log^3 \left( \frac{n}{\varepsilon} \right), \sqrt{k} \log \left( \frac{1}{b} \right) \cdot \frac{n^2}{\varepsilon^2} \log^2 \left( \frac{n}{\varepsilon} \right) \right\} \right).
\]

The running time of the algorithm is polynomial in all parameters assuming that there is an FPRAS for the conditional marginal distributions \( \mu_i(\cdot \mid x) \) for any \( i \in [n] \) and any feasible \( x \in K^{i-1} \). Furthermore, if \( k = 2 \), i.e., we have a binary domain \( K = \{0, 1\} \), the query complexity can be improved to

\[
O \left( \log \left( \frac{1}{b} \right) \cdot \frac{n}{\varepsilon} \log^3 \left( \frac{n}{\varepsilon} \right) \right).
\]

Again, we give a few examples as applications of Theorem 7.5, omitting all the technical details.

- Consider the Ising model with the interaction matrix \( J \) (with entries being \( \beta_{uv} \)’s and assumed to be positive semidefinite). We know from recent works [EKZ22, AJK+22, KLR22] that one can efficiently estimate all conditional marginal probabilities when \( \| J \|_2 < 1 \) under any external fields. There are two special features for this application. The first is that the marginal bounds could potentially be as small as \( e^{-\Theta(\sqrt{n})} \). The second is that we can only approximate the conditional marginal probabilities rather than get the exact values, and hence we should apply Theorem 7.5 instead of Theorem 7.2. With access to the \( \text{Subcube Oracle} \), one can obtain a polynomial-time identity testing algorithm for this family of Ising models with sample complexity \( O((n^{3/2}/\varepsilon) \log^3(n/\varepsilon)) \) (note that \( k = 2 \)).
Consider the monomer-dimer model (weighted matchings) on arbitrary (unbounded-degree) graphs. We know from the classical work [JS89] that one can approximate the conditional marginal distributions for all pinnings. Similar to the previous example, the marginal probabilities can be exponentially small (in the number of vertices) and one can at the best approximate them efficiently rather than computing them exactly. Still, we can apply Theorem 7.5 to obtain an efficient identity testing algorithm with access to the Subcube Oracle with sample complexity \( O((mn/\varepsilon)\log^3(n/\varepsilon)) \) where \( m \) is the number of edges of the graph (which is the dimension) and \( n \) is the number of vertices (note that \( k = 2 \)).

7.3 Estimating KL divergence with additive error

With access to the Subcube Oracle, we can also estimate the KL divergence from an unknown distribution \( \pi \) to a given distribution \( \mu \) within an arbitrary additive error in polynomial time. This corresponds to the tolerant identity testing problem for KL divergence; that is, given \( s, \varepsilon > 0 \), we want to distinguish between \( D_{KL}(\pi \parallel \mu) \leq s \) and \( D_{KL}(\pi \parallel \mu) \geq s + \varepsilon \).

We first consider estimating KL divergence for distribution \( p \) on a finite domain of size \( k \).

**Lemma 7.6.** Let \( k \in \mathbb{N}^+ \) be an integer, and let \( \varepsilon > 0 \), \( b \in (0, 1/2) \) be reals. Given an FPRAS for a target distribution \( q \) over domain \( \mathcal{K} \) of size \( k \) such that either \( q(a) = 0 \) or \( q(a) \geq b \) for any \( a \in \mathcal{K} \), and given sample access to an unknown distribution \( p \ll q \) over \( \mathcal{K} \), there exists a polynomial-time algorithm that computes \( \hat{H} \) such that with probability at least \( 2/3 \) it holds

\[
\left| \hat{R} - D_{KL}(p \parallel q) \right| \leq \varepsilon, \tag{22}
\]

with sample complexity

\[
O\left( \frac{k}{\varepsilon \log(k/\varepsilon)} + \frac{\log^2(1/b)}{\varepsilon^2} \right). \tag{23}
\]

For a distribution \( p \) over a finite domain \( \mathcal{K} \), the (Shannon) entropy of \( p \) is defined as

\[
H(p) = \sum_{a \in \mathcal{K}} p(a) \ln \left( \frac{1}{p(a)} \right). \tag{24}
\]

Observe that if \( p \ll q \) are two distributions over \( \mathcal{K} \), then

\[
D_{KL}(p \parallel q) = \sum_{a \in \mathcal{K}} p(a) \ln \left( \frac{1}{q(a)} \right) - \sum_{a \in \mathcal{K}} p(a) \ln \left( \frac{1}{p(a)} \right) = E_{a \sim p} \left[ \ln \left( \frac{1}{q(a)} \right) \right] - H(p). \tag{25}
\]

It suffices to estimate the two terms on the right-hand side of \( Eq. (23) \) respectively with good enough accuracy.

We need the following well-known result from [VV17b] for estimating the entropy of an unknown distribution from samples, see also [VV11, JVHW15, WY16].

**Lemma 7.7 ([VV17b]).** Let \( k \in \mathbb{N}^+ \) be an integer, and let \( \varepsilon > 0 \) be a real. Given sample access to an unknown distribution \( p \) over domain \( \mathcal{K} \) of size \( k \), there exists a polynomial-time algorithm that computes \( \hat{H} \) such that with probability at least \( 9/10 \) it holds

\[
\left| \hat{H} - H(p) \right| \leq \varepsilon, \tag{26}
\]

with sample complexity

\[
O\left( \frac{k}{\varepsilon \log(k/\varepsilon)} + \frac{\log^2 k}{\varepsilon^2} \right). \tag{27}
\]
For the first term in Eq. (23), we show the following estimator.

**Lemma 7.8.** Let $k \in \mathbb{N}^+$ be an integer, and let $\varepsilon > 0$, $b \in (0, 1/2]$ be reals. Given an FPRAS for a target distribution $q$ over domain $\mathcal{K}$ of size $k$ such that either $q(a) = 0$ or $q(a) \geq b$ for any $a \in \mathcal{K}$, and given sample access to an unknown distribution $p \ll q$ over $\mathcal{K}$, there exists a polynomial-time algorithm that computes $\hat{G}$ such that with probability at least $4/5$ it holds

$$
\left| \hat{G} - E_{a \sim p} \left[ \ln \left( \frac{1}{q(a)} \right) \right] \right| \leq \varepsilon,
$$

with sample complexity

$$
O \left( \frac{\ln^2(1/b)}{\varepsilon^2} \right).
$$

**Proof.** Compute an approximation $\hat{q}$ of $q$ such that, with probability $9/10$, we have that $e^{-\varepsilon/2} \leq \hat{q}(a)/q(a) \leq e^{\varepsilon/2}$ for every $a \in \mathcal{K}$ with $q(a) > 0$, and $\hat{q}(a) = 0$ for $q(a) = 0$. Generate $m$ independent samples from $p$, denoted by $a_1, \ldots, a_m$. Then our estimator is defined as

$$
\hat{G} = \frac{1}{m} \sum_{j=1}^{m} \ln \left( \frac{1}{\hat{q}(a_j)} \right).
$$

We will show that $\hat{G}$ satisfies Eq. (25) with probability at least $4/5$ for

$$
m = \left\lfloor \frac{8 \ln^2(1/b)}{\varepsilon^2} \right\rfloor.
$$

Observe that

$$
\left| \hat{G} - E_{a \sim p} \left[ \ln \left( \frac{1}{q(a)} \right) \right] \right|
\leq \left| \frac{1}{m} \sum_{j=1}^{m} \ln \left( \frac{1}{\hat{q}(a_j)} \right) - \frac{1}{m} \sum_{j=1}^{m} \ln \left( \frac{1}{q(a_j)} \right) \right| + \left| \frac{1}{m} \sum_{j=1}^{m} \ln \left( \frac{1}{q(a_j)} \right) - E_{a \sim p} \left[ \ln \left( \frac{1}{q(a)} \right) \right] \right|.
$$

Assuming $\hat{q}$ is an $(\varepsilon/2)$-approximation of $q$, we can upper bound the first term in Eq. (26) by

$$
\frac{1}{m} \sum_{j=1}^{m} \left| \ln \left( \frac{\hat{q}(a_j)}{q(a_j)} \right) \right| \leq \frac{1}{m} \sum_{j=1}^{m} \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.
$$

Meanwhile, for the second term in Eq. (26), since $0 \leq \ln(1/q(a)) \leq \ln(1/b)$ for all $a \in \mathcal{K}$ with $q(a) > 0$ and since $p \ll q$, we deduce from Hoeffding’s inequality that

$$
\Pr \left( \left| \frac{1}{m} \sum_{j=1}^{m} \ln \left( \frac{1}{q(a_j)} \right) - E_{a \sim p} \left[ \ln \left( \frac{1}{q(a)} \right) \right] \right| \geq \frac{\varepsilon}{2} \right) \leq 2 \exp \left( -\frac{\varepsilon^2 m}{2 \ln^2(1/b)} \right) \leq \frac{1}{10}, \quad \text{provided } m \geq (8/\varepsilon^2) \ln^2(1/b).
$$

Therefore, we deduce from Eq. (26) that

$$
\left| \hat{G} - E_{a \sim p} \left[ \ln \left( \frac{1}{q(a)} \right) \right] \right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
$$

with failure probability at most $1/10 + 1/10 = 1/5$ by the union bound, as wanted. □
Lemma 7.6 then follows easily from Lemmas 7.7 and 7.8.

**Proof Lemma 7.6.** Since the sample complexity upper bound we want to show is monotone increasing in $k$, we can safely assume without loss of generality that $q$ is fully supported on $\mathcal{K}$, i.e., $q(a) \geq b$ for each $a \in \mathcal{K}$. In particular, this implies that $b \leq 1/k$. Take $\hat{G}$ from Lemma 7.8 and $\hat{H}$ from Lemma 7.7, and let $\tilde{R} = \hat{G} - \hat{H}$. We then deduce from Eq. (23) that

$$\left| \tilde{R} - D_{\text{KL}} (p \parallel q) \right| \leq \left| \hat{G} - \mathbb{E}_{a \sim p} \left[ \ln \left( \frac{1}{q(a)} \right) \right] \right| + \left| \hat{H} - H(p) \right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which fails with probability at most $1/5 + 1/10 = 3/10$ by the union bound. The running time is polynomial in all parameters and depends on the given FPRAS for $q$. The sample complexity is given by

$$O \left( \frac{k}{\varepsilon \log(k/\varepsilon)} + \frac{\log^2 k}{\varepsilon^2} \right) + O \left( \frac{\log^2(1/b)}{\varepsilon^2} \right) = O \left( \frac{k}{\varepsilon \log(k/\varepsilon)} + \frac{\log^2(1/b)}{\varepsilon^2} \right),$$

since we have $k \leq 1/b$. \hfill \Box

We now give our main theorem for estimating KL divergence with Subcube Oracle access.

**Theorem 7.9.** Let $k = k(n)$ be an integer and let $b = b(n) \in (0, 1/2]$ be a real. Suppose that $k = O(n)$ and $\log \log(1/b) = O(\log n)$. Given a visible distribution $\mu$ over $\mathcal{K}^n$ that is $b$-marginally bounded, and given access to Subcube Oracle for a hidden distribution $\pi \ll \mu$ over $\mathcal{K}^n$, there is an algorithm that for any $\varepsilon > 0$ computes $\hat{S}$ such that with probability at least $2/3$ it holds

$$\left| \hat{S} - D_{\text{KL}} (\pi \parallel \mu) \right| \leq \varepsilon. \quad (27)$$

The query complexity of the algorithm is

$$O \left( \log^4 \left( \frac{1}{b} \right) \cdot \frac{n^4}{\varepsilon^2 \log \left( \frac{n}{\varepsilon} \right)} \right).$$

The running time of the algorithm is polynomial in all parameters assuming that there is an FPRAS for the conditional marginal distributions $\mu_i(\cdot \mid x)$ for any $i \in [n]$ and any feasible $x \in \mathcal{K}^{[i-1]}$.

**Proof.** From Eq. (20) we observe that

$$D_{\text{KL}} (\pi \parallel \mu) = n \mathbb{E}_{(i,x)} \left[ D_{\text{KL}} (\pi_i(\cdot \mid x) \parallel \mu_i(\cdot \mid x)) \right],$$

where $(i, x)$ is a random pair generated by taking a uniformly random coordinate $i \in [n]$ and sampling $x \in \mathcal{K}^{[i-1]}$ from the marginal of $\pi$ on the first $i - 1$ coordinates. Hence, it suffices to estimate $\mathbb{E}_{(i,x)} \left[ D_{\text{KL}} (\pi_i(\cdot \mid x) \parallel \mu_i(\cdot \mid x)) \right]$ with additive error $\varepsilon/n$.

Let $(i_1, x_1), \ldots, (i_L, x_L)$ be $L$ independent random pairs generated via the General Oracle (which is contained in the power of Subcube Oracle), where we define

$$L = \left\lceil \frac{8n^2 \ln^2(1/b)}{\varepsilon^2} \right\rceil.$$

For $1 \leq \ell \leq L$, we let

$$R_\ell = D_{\text{KL}} (\pi_{i_\ell}(\cdot \mid x_\ell) \parallel \mu_{i_\ell}(\cdot \mid x_\ell)).$$
Furthermore, for each $\ell$ let $\hat{R}_\ell$ be an estimate of $R_\ell$ which is obtained from Lemma 7.6 using the Subcube Oracle, such that
\[
\Pr \left( \left| \hat{R}_\ell - R_\ell \right| \geq \frac{\varepsilon}{2n} \right) \leq \frac{1}{10L}.
\]
Then, by the union bound we have
\[
\Pr \left( \left| \frac{1}{L} \sum_{\ell=1}^{L} \hat{R}_\ell - \frac{1}{L} \sum_{\ell=1}^{L} R_\ell \right| \geq \frac{\varepsilon}{2n} \right) \leq \sum_{\ell=1}^{L} \Pr \left( \left| \hat{R}_\ell - R_\ell \right| \geq \frac{\varepsilon}{2n} \right) \leq L \cdot \frac{1}{10L} = \frac{1}{10}.
\]
Note that using the standard amplification technique for the failure probability, the number of samples we need for each $\ell$ is
\[
O \left( \frac{kn}{\varepsilon \log(kn/\varepsilon)} + \frac{n^2 \log^2(1/b)}{\varepsilon^2} \right) \cdot O \left( \log L \right) = O \left( \log^2 \left( \frac{1}{b} \right) \cdot \frac{n^2}{\varepsilon^2} \log \left( \frac{n}{\varepsilon} \right) \right),
\]
where we use the assumptions $k = O(n)$ and $\log \log(1/b) = O(\log n)$.

Meanwhile, we observe $0 \leq D_{\text{KL}}(\pi_i(. \mid x) \parallel \mu_i(. \mid x)) \leq \ln(1/b)$ for any feasible pair $(i, x)$ since $\mu$ is $b$-marginally bounded and $\pi \ll \mu$. Hence, Hoeffding’s inequality implies that
\[
\Pr \left( \left| \frac{1}{L} \sum_{\ell=1}^{L} R_\ell - \mathbb{E}_{(i,x)} \left[ D_{\text{KL}}(\pi_i(. \mid x) \parallel \mu_i(. \mid x)) \right] \right| \geq \frac{\varepsilon}{2n} \right) \leq 2 \exp \left( -\frac{\varepsilon^2 L}{2n^2 \ln^2(1/b)} \right) \leq \frac{1}{10},
\]
provided $L \geq (8n^2/\varepsilon^2) \ln^2(1/b)$.

Therefore, by letting our estimator to be
\[
\hat{S} = \frac{n}{L} \sum_{\ell=1}^{L} \hat{R}_\ell,
\]
we deduce that
\[
\left| \hat{S} - D_{\text{KL}}(\pi \parallel \mu) \right| \leq n \left| \frac{1}{L} \sum_{\ell=1}^{L} \hat{R}_\ell - \frac{1}{L} \sum_{\ell=1}^{L} R_\ell \right| + n \left| \frac{1}{L} \sum_{\ell=1}^{L} R_\ell - \mathbb{E}_{(i,x)} \left[ D_{\text{KL}}(\pi_i(. \mid x) \parallel \mu_i(. \mid x)) \right] \right|
\leq n \cdot \frac{\varepsilon}{2n} + n \cdot \frac{\varepsilon}{2n} = \varepsilon,
\]
with failure probability at most $1/10 + 1/10 = 1/5$ by the union bound. Finally, the query complexity is given by
\[
O \left( \log^2 \left( \frac{1}{b} \right) \cdot \frac{n^2}{\varepsilon^2} \log \left( \frac{n}{\varepsilon} \right) \right) \cdot L = O \left( \log^4 \left( \frac{1}{b} \right) \cdot \frac{n^4}{\varepsilon^4} \log \left( \frac{n}{\varepsilon} \right) \right),
\]
as claimed.

We remark that Theorem 7.9 is applicable to all the examples mentioned in Sections 7.1 and 7.2. See also Remark 7.3 on relaxing the marginal boundedness condition.
8 Conclusion and Open Problems

In this paper we give efficient algorithms for identity testing with access to Coordinate Oracle and General Oracle, and also establish matching computational hardness and information-theoretical lower bounds. Our algorithmic result builds on the fact that the visible distribution satisfies approximate tensorization of entropy. While we show that for the antiferromagnetic Ising model, there is no polynomial-time identity testing algorithm when approximate tensorization fails, it is in general unclear if one can get a testing algorithm running in polynomial time without approximate tensorization, using either Coordinate Oracle or Subcube Oracle in addition to General Oracle. One important example is the ferromagnetic Ising model at all temperatures. We know that approximate tensorization fails at low temperature (large $\beta$) since the Glauber dynamics has exponential mixing time. We do not know whether an efficient identity testing algorithm exists or not even with access to the more powerful Subcube Oracle. Note that our Theorem 7.2 does not apply to ferromagnetic Ising models since we cannot estimate conditional marginal probabilities under an arbitrary pinning (corresponding to ferromagnetic Ising models with inconsistent local fields). Another important example is mixtures of product distributions. It is easy to show that approximate tensorization could fail even for a mixture of two product distributions with equal weights. We know from Theorem 7.2 that there is an efficient identity testing algorithm for the family of mixtures of polynomially many balanced product distributions given access to the Subcube Oracle. It is unclear to us, however, that if there is a polynomial-time testing algorithm using only the weaker Coordinate Oracle.

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