Concurrence of assistance and Mermin inequality on three-qubit pure states

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We study a relation between the concurrence of assistance and the Mermin inequality on three-qubit pure states. We find that if a given three-qubit pure state has the minimal concurrence of assistance greater than 1/2 then the state violates some Mermin inequality.

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Bell-inequality violation in quantum mechanics tells us that quantum correlations are quite different from classical correlations. In the case of two-qubit states, the Clauser-Horne-Shimony-Holt (CHSH) inequality \[1\] is a well-known Bell inequality, and has an important property that any two-qubit pure state violates the CHSH inequality if and only if it is entangled. In particular, there exists an explicit relation between the degree of the CHSH-inequality violation and the amount of entanglement for two-qubit pure states \[2\]. This shows that entanglement of pure states in the two-qubit system can be certainly detected by employing the Bell inequality, and the Bell-inequality violation can be exactly determined according to the amount of entanglement for two-qubit pure states. On this account, there have been a lot of research works to attempt to generalize the explicit relation into the multiqubit pure states \[3, 4, 5, 6, 7\].

We here consider the Mermin inequality \[8\] for three-qubit pure states, which is a natural generalization of the CHSH inequality: Let \(B_M\) be the operator defined as

\[B_M = a_1 \cdot \sigma \otimes a_2 \cdot \sigma \otimes a_3 \cdot \sigma - a_1 \cdot \sigma \otimes b_2 \cdot \sigma \otimes b_3 \cdot \sigma - b_1 \cdot \sigma \otimes a_2 \cdot \sigma \otimes b_3 \cdot \sigma - b_1 \cdot \sigma \otimes b_2 \cdot \sigma \otimes a_3 \cdot \sigma - a_1 \cdot \sigma \otimes b_2 \cdot \sigma \otimes a_3 \cdot \sigma - b_1 \cdot \sigma \otimes a_2 \cdot \sigma \otimes a_3 \cdot \sigma,\]

where \(a_j\) and \(b_j\) are unit vectors in \(\mathbb{R}^3\), and \(\sigma = (\sigma_1, \sigma_2, \sigma_3)\) is the vector of the Pauli matrices. Then for a given three-qubit pure state \(|\psi\rangle\), the Mermin inequality is

\[|\langle \psi | B_M | \psi \rangle| \leq 2.\]

For the generalized Greenberger-Horne-Zeilinger (GHZ) states,

\[|\psi_{\text{GHZ}}\rangle = \cos \phi |000\rangle + \sin \phi |111\rangle,\]

it was numerically shown in Ref. \[3\] that the state \(|\psi_{\text{GHZ}}\rangle\) violates a Mermin inequality if and only if \(\sin 2\phi > 1/2\). This result implies that there exists a relation between the Mermin-inequality violation and the amount of entanglement for three-qubit pure states, since \(\sin 2\phi\) may represent the degree of entanglement in the state \(|\psi_{\text{GHZ}}\rangle\). Then one could naturally ask whether the same result can be obtained for any three-qubit pure state.

In order to answer this question, the proper quantity such as the value \(\sin 2\phi\) in the generalized GHZ states should be defined for general three-qubit pure states, and it should be investigated whether the Mermin inequality is violated, whenever the quantity is greater than some constant. In this paper, we consider the concurrence of assistance (CoA) \[9\] as such a quantity, and examine a relation between the CoA and the Mermin-inequality violation for several classes of three-qubit pure states including the generalized GHZ states, the states in the W class, and some coherent superpositions of well-known three-qubit pure states.

As a consequence, we analytically show that if a three-qubit pure state in those classes has the minimal CoA greater than 1/2 then the state violates a Mermin inequality, and furthermore find that our result can be generalized into all three-qubit pure states by exploiting the numerical work in Ref. \[5\].

We first take account of two simple but important measures of entanglement, the concurrence \[10\] and the CoA. The concurrence, \(C\), is defined as follows: For a pure state \(|\phi_{12}\rangle\) in \(2 \otimes d\) quantum systems \((d \geq 2)\), it is defined as

\[C(|\phi_{12}\rangle) = \sqrt{2(1 - \text{tr} \rho_1^2)} = 2\sqrt{\text{det} \rho_1},\]

where the minimum is taken over all possible decompositions, \(\rho_{12} = \sum_k p_k |\phi_k\rangle_{12} \langle \phi_k|\). The CoA, \(C^a\), is also defined in the similar way: For a pure state \(|\phi_{12}\rangle\), \(C^a(|\phi_{12}\rangle) \equiv C(|\phi_{12}\rangle)|\phi\rangle\). For a mixed state \(\rho_{12}\), it is defined as

\[C^a(\rho_{12}) = \max_k \sum p_k C(|\phi_k\rangle_{12} \langle \phi_k|),\]

where the maximum is taken over all possible decompositions of \(\rho_{12}\).

We remark that the CoA is an entanglement monotone on three-qubit pure states \[11\]. Thus, even though the definitions of the two entanglement measures are quite similar, the CoA can be thought of as a measure of entanglement on tripartite pure states, while the concurrence is a good measure of bipartite entanglement. Our aim in
this paper is to define an appropriate measure of entanglement for three-qubit pure states, and to investigate how the entanglement measure is related to the Mermin-inequality violation. Hence, the CoA may be one of good candidates for such a tripartite entanglement measure.

Furthermore, it was known in Ref. [12] that, for any three-qubit pure state $|\psi\rangle_{123}$, there exists the so-called monogamy equality in terms of the concurrence and the CoA as follows:

$$C_{1(23)}^2 = C_{12}^2 + (C_{13}^2)^2,$$

(6)

where $C_{1(23)} = C(|\psi\rangle_{1(23)} \langle \psi|)$, $C_{12} = C(\text{tr}_3|\psi\rangle_{123} \langle \psi|)$, and $C_{13} = C(\text{tr}_2|\psi\rangle_{123} \langle \psi|)$. Thus, for distinct $i$ and $j$ in $\{1,2,3\}$, we clearly have the equalities $C_i^2 = \sqrt{\tau + C_j^2}$, where $\tau$ is called the three-angle [13][14], defined as

$$\tau = C_{1(23)}^2 - C_{12}^2 - C_{13}^2,$$

(7)

and is known as an entanglement measure to distinguish the GHZ class from the W class [14]. Here, the GHZ class and the W class are the sets of all pure states with genuine three-qubit entanglement equivalent to the GHZ state [12].

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle),$$

(8)

under stochastic local operations and classical communication (SLOCC), and equivalent to the W state,

$$|\text{W}\rangle = \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle),$$

(9)

under SLOCC, respectively.

Now, we consider the Schmidt decomposition of three-qubit pure states as follows [14]:

$$|\psi\rangle_{123} = \lambda_0 |000\rangle_{123} + \lambda_1 e^{i\phi} |100\rangle_{123} + \lambda_2 |101\rangle_{123} + \lambda_3 |110\rangle_{123} + \lambda_4 |111\rangle_{123},$$

(10)

where $\phi = \sqrt{-1}$, $0 \leq \theta \leq \pi$, $\lambda_j \geq 0$, and $\sum_j \lambda_j^2 = 1$. Thus, in order to calculate the CoAs for three-qubit pure states, it suffices to consider the states in Eq. (10). By somewhat tedious but straightforward calculations, we obtain the following results on the CoAs $C_{ij}^a$ for $|\psi\rangle_{123}$:

$$C_{ij}^a = \begin{cases} 
2\lambda_0 \sqrt{\lambda_3^2 + \lambda_4^2}, & i = 1, j = 2, 3, 4; \\
2 \sqrt{\lambda_0^2 \lambda_3^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2 - 2 \lambda_1 \lambda_2 \lambda_3 \lambda_4 \cos \theta}, & i = 3, j = 2, 1; \\
2 \lambda_0 \sqrt{\lambda_2^2 + \lambda_3^2}, & i = 4, j = 3, 2.
\end{cases}$$

(11)

Let $C_{\text{min}}^a = \min \{C_{12}^a, C_{23}^a, C_{34}^a\}$. Then $C_{\text{min}}^a$ is called the minimal CoA, which is the very entanglement measure relevant to our purpose. Our claim is that, for a given three-qubit pure state, if its minimal CoA is greater than 1/2 then there exists a Mermin inequality which the state violates.

For the generalized GHZ states $|\psi_{\text{GHZ}}\rangle$ in Eq. (3), it is easy to calculate that $C_{\text{min}}^a = \sin 2\phi$ by Eqs. (11). Take $\tilde{a}_j = (1, 0, 0)$ and $\tilde{b}_j = (0, 1, 0)$ for all $j = 1, 2, 3$. Then the Mermin inequality in Eq. (2) becomes $4 \sin 2\phi \leq 2$. Hence, we can clearly obtain that if the generalized GHZ state has the minimal CoA greater than 1/2 then the state violates a Mermin inequality. In other words, our claim is true for the generalized GHZ states.

We now take the W class into account. It is known in [14][17] that any state $|\psi_W\rangle$ in the W class can be written as

$$|\psi_W\rangle = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle,$$

(12)

which has the simpler Schmidt decomposition than the general one in Eq. (10), since its three-tangle is zero. In this case, we take $\tilde{a}_j = (0, 0, 1)$ for all $j = 1, 2, 3$, $\tilde{b}_1 = (-1, 0, 0)$, and $\tilde{b}_2 = \tilde{b}_3 = (1, 0, 0)$. Then the Mermin inequality in Eq. (2) becomes

$$\lambda_0^2 - \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\lambda_0 \lambda_2 + 2\lambda_2 \lambda_3 + 2\lambda_3 \lambda_0 \leq 2.$$

(13)

By Eqs. (11), the minimal CoA, $C_{\text{min}}^a$, for the W class can be readily calculated as

$$C_{\text{min}}^a = 2 \min \{\lambda_0 \lambda_2, \lambda_0 \lambda_3, \lambda_2 \lambda_3\}.$$ 

(14)

Since $\lambda_0^2 + \lambda_2^2 \geq 2\lambda_0 \lambda_2$, and $\lambda_0^2 + \lambda_2^2 + \lambda_3^2 = 1$, we can obtain the following inequalities:

$$2(1 - \lambda_0^2) = 2 (\lambda_2^2 + \lambda_3^2) \geq 2 \lambda_0 \lambda_2 + 2 \lambda_2 \lambda_3 + 2 \lambda_3 \lambda_0 \geq 3C_{\text{min}}^a.$$ 

(15)

Thus, if $C_{\text{min}}^a > 1/2$ then $\lambda_0^2 < 1/4$, and hence the left-hand side of the inequality in (13) is greater than 2, that is, its Mermin inequality is violated. Therefore, our claim is also true for the W class.

We now consider three coherent superpositions of well-known states. First, let us see a coherent superposition of the generalized GHZ state and a separable state $|101\rangle$,

$$|\psi_{\text{GHZ}} : S\rangle = \sqrt{1 - p} |\psi_{\text{GHZ}}\rangle + \sqrt{p} |101\rangle,$$

(16)

where $0 < p < 1$. Taking account of the Mermin inequality used in the case of the generalized GHZ states, we can show that the Mermin inequality is violated if and only if $4(1 - p) \sin 2\phi > 2$, and that the minimal CoA for $|\psi_{\text{GHZ}} : S\rangle$ equals $(1 - p) \sin 2\phi$ by Eqs. (11). Hence, it is clear that if the minimal CoA for the state $|\psi_{\text{GHZ}} : S\rangle$ is more than 1/2 then it violates the same Mermin inequality as the inequality for the generalized GHZ states, and vice versa.

The second coherent superposition which we deal with is a superposition of the W state and a separable state $|000\rangle$,

$$|\text{W} : S\rangle = \sqrt{1 - p} |\text{W}\rangle + \sqrt{p} |000\rangle,$$

(17)
where $0 < p < 1$. We now take $\mathbf{a}_j = (0,0,-1)$ and $\mathbf{b}_j = (1,0,0)$ for all $j = 1,2,3$. Then the Mermin inequality becomes $3 - 4p \leq 2$, that is, $p \geq 1/4$. It can be obtained from simple calculations \[18\] that its minimal CoA is $2(1 - p)/3$. Thus, the minimal CoA for the state $|W : S\rangle$ is greater than $1/2$ if and only if the Mermin inequality with respect to $\mathbf{a}_j = (0,0,-1)$ and $\mathbf{b}_j = (1,0,0)$ is violated.

Let us now deal with the coherent superposition of the GHZ state and the W state,

$$|\text{GHZ : W}\rangle = \sqrt{1 - p}|\text{GHZ}\rangle + \sqrt{p}|\text{W}\rangle,$$

where $0 < p < 1$. Then it follows from direct computations that for all $0 < p < 1$ the states $|\text{GHZ : W}\rangle$ have the minimal CoA more than $1/2$. Thus, it suffices to show that the state violates some Mermin inequality for each $0 < p < 1$. We here use three Mermin inequalities to show the violation, according to the value of the parameter $p$. We first consider the Mermin inequality with respect to $\mathbf{a}_j = (1/2,0,-\sqrt{3}/2)$ and $\mathbf{b}_j = (0,1,0)$ for all $j = 1,2,3$. Then the Mermin inequality for the states $|\text{GHZ : W}\rangle$ becomes

$$\frac{1}{8} \left( 3\sqrt{2}(5 + \sqrt{3})\sqrt{p(1 - p)} + 13(1 - p) + 9\sqrt{3}p \right) \leq 2.$$  \([19]\)

Let $M(p)$ be the left-hand side of the inequality in \([19]\). Then it can be shown that $M(p)$ is greater than two if $0.011 \leq p \leq 0.999$, and hence the Mermin inequality is violated in this case, which is depicted in FIG. 1.

Furthermore, it can be readily shown that the states $|\text{GHZ : W}\rangle$ for $0 < p < 1/2$ and $1 > p > (9 + \sqrt{21})/15 \approx 0.9055$ violate the Mermin inequalities taken in the cases of the states $|\psi_{\text{GHZ}}\rangle$ and the states $|\text{W : S}\rangle$, respectively. Therefore, our claim also holds for the states $|\text{GHZ : W}\rangle$.

In addition to our analytical results, our claim can be generalized into any three-qubit pure states, by exploiting the numerical work of Emary and Beenakker in Ref. \[5\]. In their work, an entanglement measure $\sigma$ was defined as

$$\sigma \equiv \min \left( \frac{C_X^2 + C_Y^2 - C_{XY}^2}{2} \right),$$  \((20)\)

where the minimization is over the permutations $X,Y,Z$ in $\{1,2,3\}$. Then we can obtain the simple relation among the three-tangle, the minimal CoA, and this measure of entanglement $\sigma$ as follows:

$$0 \leq \tau \leq (C_{\text{min}}^a)^2 \leq \sigma \leq 1.$$  \((21)\)

Since their numerical result shows the following inequality

$$\sigma \leq \frac{|\langle \psi |\mathcal{B}_M|\psi\rangle|^2}{16}$$  \((22)\)

for numbers of three-qubit pure states $|\psi\rangle$, this implies that our claim is numerically true for general three-qubit pure states.

In summary, we have studied a relation between the CoA and the Mermin-inequality violation for several classes of three-qubit pure states, and have obtained an analytical result that if a three-qubit pure state in those classes has the minimal CoA greater than $1/2$ then the state violates some Mermin inequality. Furthermore, we have also found that our result numerically holds for any three-qubit pure states.

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