COMBINATORIAL BASES OF FEIGN-STOYANOVSKY’S TYPE
SUBSPACES OF LEVEL 1 STANDARD MODULES FOR \(\tilde{\mathfrak{sl}}(\ell + 1, \mathbb{C})\)

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Abstract. Let \(\tilde{\mathfrak{g}}\) be an affine Lie algebra of type \(A_\ell^{(1)}\). Suppose we’re given a \(\mathbb{Z}\)-gradation of the corresponding simple finite-dimensional Lie algebra \(\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1\); then we also have the induced \(\mathbb{Z}\)-gradation of the affine Lie algebra

\[
\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1.
\]

Let \(L(\Lambda)\) be a standard module of level 1. Feigin-Stoyanovsky’s type subspace \(W(\Lambda)\) is the \(\tilde{\mathfrak{g}}_1\)-submodule of \(L(\Lambda)\) generated by the highest-weight vector \(v_\Lambda\),

\[
W(\Lambda) = U(\tilde{\mathfrak{g}}_1) \cdot v_\Lambda \subset L(\Lambda).
\]

We find a combinatorial basis of \(W(\Lambda)\) given in terms of difference and initial conditions. Linear independence of the generating set is proved inductively by using coefficients of intertwining operators. A basis of \(L(\Lambda)\) is obtained as an “inductive limit” of the basis of \(W(\Lambda)\).

1. Introduction

Let \(\mathfrak{g}\) be a simple complex Lie algebra, \(\mathfrak{h} \subset \mathfrak{g}\) its Cartan subalgebra, \(R\) the corresponding root system. Then one has a root decomposition

\[
\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha.
\]

Fix root vectors \(x_\alpha \in \mathfrak{g}_\alpha\). Let

\[
\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1
\]

be a \(\mathbb{Z}\)-gradation of \(\mathfrak{g}\), where \(\mathfrak{h} \subset \mathfrak{g}_0\). All such gradations are obtained by choosing some minuscule coweight \(\omega \in \mathfrak{h}\). Denote by \(\Gamma \subset R\) a set of roots such that \(\mathfrak{g}_1 = \sum_{\alpha \in \Gamma} \mathfrak{g}_\alpha = \sum_{\omega(\alpha) = 1} \mathfrak{g}_\alpha\).

Affine Lie algebra associated with \(\mathfrak{g}\) is \(\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} c \oplus \mathbb{C} d\), where \(c\) is the canonical central element, and \(d\) the degree operator. Elements \(x_\alpha(n) = x_\alpha \otimes t^n\) are fixed real root vectors. Gradation of \(\mathfrak{g}\) induces analogous \(\mathbb{Z}\)-gradation of \(\tilde{\mathfrak{g}}\):

\[
\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1,
\]

where \(\tilde{\mathfrak{g}}_1 = \mathfrak{g}_1 \otimes \mathbb{C}[t, t^{-1}]\) is a commutative Lie subalgebra with a basis

\[
\{x_\gamma(j) \mid j \in \mathbb{Z}, \gamma \in \Gamma\}.
\]

Let \(L(\Lambda)\) be a standard \(\tilde{\mathfrak{g}}\)-module of level \(k = \Lambda(c)\), with a fixed highest weight vector \(v_\Lambda\). A Feigin-Stojanovsky’s type subspace is a \(\tilde{\mathfrak{g}}_1\)-submodule of \(L(\Lambda)\) generated with \(v_\Lambda\),

\[
W(\Lambda) = U(\tilde{\mathfrak{g}}_1) \cdot v_\Lambda \subset L(\Lambda).
\]

This is similar to the notion of principal subspace introduced in [FS] where, instead of \(\mathbb{Z}\)-gradation \(\mathfrak{g}\), one considers triangular decomposition of \(\mathfrak{g}\) and from it derived decomposition of \(\tilde{\mathfrak{g}}\); in the case \(\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})\), these two definitions are equivalent.

We would like to find a monomial basis of \(W(\Lambda)\), i.e. a basis consisting of vectors \(x(\pi)v_\Lambda\), where \(x(\pi)\) are monomials in basis elements \(\{x_\gamma(-j) \mid j \in \mathbb{N}, \gamma \in \Gamma\}\).

2000 Mathematics Subject Classification. Primary 17B67; Secondary 17B69, 05A19.

Partially supported by the Ministry of Science and Technology of the Republic of Croatia, Project ID 037-0372794-2806.
The problem of finding monomial bases is a part of Lepowsky-Wilson’s program to study representations of affine Lie algebras by means of vertex-operators and to obtain Rogers-Ramanujan-type combinatorial bases of these representations ([LW], [LP], [MP]).

Principal subspaces of standard \( \widehat{\mathfrak{g}} \)-modules were introduced in [FS]. These subspaces are generated by the affinization of the nilpotent subalgebra \( \mathfrak{n}_+ \) of \( \mathfrak{g} \) from the triangular decomposition \( \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \). B. Feigin and A. Stoyanovsky described the dual space of the principal subspace for \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \) and \( \mathfrak{sl}(3, \mathbb{C}) \) in terms of symmetric polynomial forms satisfying certain conditions, and calculated its character. In the \( \mathfrak{sl}(2, \mathbb{C}) \)-case, they also described the dual in a geometric way, recovering in this way the Rogers-Ramanujan and Gordon identities.

Principal subspaces were studied further by G. Georgiev in [G]. He constructed combinatorial bases and calculated characters of principal subspaces for certain representations of \( \mathfrak{sl}(\ell + 1, \mathbb{C}) \). In the proof of linear independence, Georgiev used intertwining operators from [DL].

Also by using intertwining operators, S. Capparelli, J. Lepowsky and A. Milas in [CLM1,2] obtained Rogers-Ramanujan and Rogers-Selberg recursions for characters of principal subspaces for \( \mathfrak{sl}(2, \mathbb{C}) \). As a continuation of the program laid out in [CLM1,2], C. Calinescu obtained systems of recursions for characters of principal subspaces of level 1 standard modules for \( \mathfrak{sl}(\ell + 1, \mathbb{C}) \) ([C1]) and of certain higher-level standard modules for \( \mathfrak{sl}(3, \mathbb{C}) \) ([C2]). By solving these recursions they also established formulas for characters of these subspaces. Furthermore, in [CalLM1,2], Calinescu, Lepowsky and Milas provided new proofs of presentations of principal subspaces for \( \mathfrak{sl}(2, \mathbb{C}) \).

Feigin-Stoyanovsky’s type subspace \( W(\Lambda) \) was implicitly studied in [P1] and [P2], where M. Primc constructed a combinatorial basis of this subspace. By translating the basis of \( W(\Lambda) \) by a certain Weyl group element, and then taking a inductive limit, he obtained a basis of the whole \( L(\Lambda) \). This was done in [P1] for \( \mathfrak{g} = \mathfrak{sl}(\ell + 1, \mathbb{C}) \) and a particular choice of gradation \( \mathfrak{h} \), and for any dominant integral weight \( \Lambda \). For any classical simple Lie algebra and any possible gradation \( \mathfrak{h} \), combinatorial bases were constructed in [P2], but only for basic modules \( L(\Lambda_0) \).

In the particular \( \mathfrak{sl}(\ell + 1, \mathbb{C}) \) case studied in [P1], the basis of \( W(\Lambda) \) is parameterized by combinatorial objects called \( (k, \ell + 1) \)-admissible configurations. These objects were introduced and further studied in [FJLMM] and [FJMMT], where different formulas for the character of \( W(\Lambda) \) were obtained.

The hardest part of constructing the combinatorial basis of \( W(\Lambda) \) is a proof of linear independence of a reduced spanning set. This was proved in [P1] by using Schur functions, while in [P2] this was proved by using the crystal base character formula [KKMNN]. In [P3], Primc used Capparelli-Lepowsky-Milas’ approach via intertwining operators and a description of the basis from [FJLMM] to give a simpler proof of linear independence of the basis of \( W(\Lambda) \) constructed in [P1]. It seems that this should be the way to obtain a proof in other cases as well.

In this paper we extend these results to any possible \( \mathbb{Z} \)-gradation of \( \mathfrak{g} = \mathfrak{sl}(\ell + 1, \mathbb{C}) \) and all level 1 standard modules. In [T] we will further extend this to standard modules of any higher level, obtaining a combinatorial basis parameterized by a certain generalization of \( (k, \ell + 1) \)-admissible configurations.

Let \( \delta = \{ \alpha_1, \ldots, \alpha_\ell \} \) be a basis of the root system \( \mathfrak{R} \) for \( \mathfrak{g} = \mathfrak{sl}(\ell + 1, \mathbb{C}) \), and \( \{ \omega_1, \ldots, \omega_\ell \} \) the corresponding set of fundamental weights. We identify \( \mathfrak{h} \) and \( \mathfrak{h}^* \) in the usual way and fix a fundamental weight \( \omega = \omega_m \). Set

\[
\Gamma = \{ \gamma \in \mathfrak{R} \mid \langle \gamma, \omega \rangle = 1 \} = \{ \gamma_{ij} \mid i = 1, \ldots, m; j = m, \ldots, \ell \},
\]

where

\[
\gamma_{ij} = \alpha_i + \cdots + \alpha_m + \cdots + \alpha_j.
\]
Set
\[ g_{\pm 1} = \sum_{\alpha \in \pm \Gamma} g_{\alpha}, \quad g_0 = h \oplus \sum_{(\alpha, \omega) = 0} g_{\alpha}, \]
then
\[ g = g_{-1} \oplus g_0 \oplus g_1 \]
is a \( \mathbb{Z} \)-gradation of \( g \). The set \( \Gamma \) is called the set of colors. For \( \gamma \in \Gamma \), we say that a fixed basis element \( x\gamma \in g_\gamma \) is of the color \( \gamma \). The set of colors \( \Gamma \) can be pictured as a rectangle with row indices \( 1, \ldots, m \) and column indices \( m, \ldots, \ell \) (see figure 1).

**Figure 1.**

\[ \Gamma: \]

\[ \begin{array}{|c|c|c|c|c|}
\hline
1 & & & & \\
\hline
2 & & & & \\
\hline
& & & & \\
\hline
& & & & \\
\hline
& & & & \\
\hline
m & m+1 & j & \ell & \\
\hline
\end{array} \]

Fix a fundamental weight \( \Lambda_i, i = 0, \ldots, \ell \) of \( \tilde{g} \). Let \( L(\Lambda_i) \) be the standard module with highest weight \( \Lambda_i \), and \( v_i \) the highest weight vector of \( L(\Lambda_i) \).

We find a basis of the Feigin-Stoyanovsky’s type subspace \( W(\Lambda_i) \) consisting of monomial vectors
\[ \{ x_{\gamma_1}(-n_1) \cdots x_{\gamma_t}(-n_t)v_i \mid t \in \mathbb{Z}_+, \gamma_j \in \Gamma, n_j \in \mathbb{N} \} \]
whose monomial parts
\begin{equation}
(2) \quad x_{\gamma_1}(-n_1) \cdots x_{\gamma_t}(-n_t)
\end{equation}
satisfy certain combinatorial conditions, called difference and initial conditions. By difference conditions, colors of elements of degree \( -j \) and \( -j - 1 \) in a monomial (2) lie on a diagonal path in \( \Gamma \) as pictured on the figure 2.

**Figure 2.**
So, if a monomial \((2)\) has elements of degrees \(-j\) and \(-j-1\) of colors \(\gamma_{r_1 s_1}, \ldots, \gamma_{r_t s_t}\) and \(\gamma'_{r'_1 s'_1}, \ldots, \gamma'_{r'_t s'_t}\), respectively, then
\[
 r_1 < r_2 < \cdots < r_t \quad \text{and} \quad s_1 > s_2 > \cdots > s_t,
\]
and, similarly,
\[
 r'_1 < r'_2 < \cdots < r'_t \quad \text{and} \quad s'_1 > s'_2 > \cdots > s'_t.
\]
Also,
\[
 r_t < r'_1 \quad \text{or} \quad s_t > s'_1.
\]

Initial conditions on monomials \((2)\) require that diagonal path of colors of elements of degree \(-1\) lie below the \(i\)-th row, in case \(1 \leq i \leq m\), or left of the \(i\)-th column, in case of \(m \leq i \leq \ell\), as it is pictured on the figure 3.

**Figure 3.**

Difference conditions on monomials are obtained by observing relations between fields \(x_\gamma(z), \gamma \in \Gamma\) on \(L(\Lambda_i)\), while initial conditions follow from the obvious requirement that elements of degree \(-1\) mustn’t annihilate the highest weight vector \(v_i\).

By observing configurations of colors of elements of degrees \(-1\) and \(-2\), one is able to construct coefficients of suitable intertwining operators between standard modules that would either send basis elements of one module to basis elements of the other module, or it would annihilate them. These operators are then used for the inductive proof of linear independence.

Thus we are able to prove the main result of this work

**Theorem** Let \(L(\Lambda_i)\) be a standard module of level 1. Then the set of monomial vectors \(x_{\gamma_1}(-n_1) \cdots x_{\gamma_t}(-n_t)v_i\) whose monomial part satisfies difference and initial conditions, is a basis of \(W(\Lambda_i)\).

2. **Affine Lie algebras**

For \(\ell \in \mathbb{N}\), let
\[
g = sl(\ell + 1, \mathbb{C}),
\]
a simple Lie algebra of the type \(A_\ell\). Let \(\mathfrak{h} \subset \mathfrak{g}\) be a Cartan subalgebra of \(\mathfrak{g}\) and \(R\) the corresponding root system. Fix a basis \(\Pi = \{\alpha_1, \ldots, \alpha_\ell\}\) of \(R\). Then we have the triangular decomposition \(\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+\). By \(R_+\) and \(R_-\) we denote sets of positive and negative roots, and let \(\theta\) be the maximal root. Let \(\langle x, y \rangle = \text{tr} \, xy\) be a normalized invariant bilinear form on \(\mathfrak{g}\); via \(\langle \cdot, \cdot \rangle\) we have an identification \(\nu : \mathfrak{h} \to \mathfrak{h}^*\). For each root \(\alpha\) fix a root vector \(x_\alpha \in \mathfrak{g}_\alpha\).
Let \( \{\omega_1, \ldots, \omega_\ell\} \) be the set of fundamental weights of \( \mathfrak{g} \), \( \langle \omega_i, \alpha_j \rangle = \delta_{ij}, i, j = 1, \ldots, \ell \). Denote by \( Q = \sum_{i=1}^\ell \mathbb{Z} \alpha_i \) the root lattice, and by \( P = \sum_{i=1}^\ell \mathbb{Z} \omega_i \) the weight lattice of \( \mathfrak{g} \).

Denote by \( \hat{\mathfrak{g}} \) the associated affine Lie algebra

\[ \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d. \]

Set \( x(j) = x \otimes t^j \) for \( x \in \mathfrak{g}, j \in \mathbb{Z} \). Commutation relations are then given by

\[
\begin{align*}
[c, \hat{\mathfrak{g}}] &= 0, \\
[d, x(j)] &= jx(j), \\
[x(i), y(j)] &= [x, y](i + j) + i(x, y) \delta_{i+j, 0} c.
\end{align*}
\]

Set \( \mathfrak{h}^c = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d, \mathfrak{n}_\pm = \mathfrak{g} \otimes t^{\pm 1} \mathbb{C}[t^{\pm 1}] \oplus \mathfrak{n}_\pm \). Then we also have the triangular decomposition \( \hat{\mathfrak{g}} = \mathfrak{n}_- \oplus \mathfrak{h}^c \oplus \mathfrak{n}_+ \).

Let \( \Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\} \subset (\mathfrak{h}^c)^* \) be the set of simple roots of \( \hat{\mathfrak{g}} \). Usual extensions of bilinear forms \( \langle \cdot, \cdot \rangle \) onto \( \mathfrak{h}^c \) and \( (\mathfrak{h}^c)^* \) are denoted by the same symbols (we take \( \langle c, d \rangle = 1 \)). Define fundamental weights \( \Lambda_i \in (\mathfrak{h}^c)^* \) by \( \langle \Lambda_i, \alpha_j \rangle = \delta_{ij} \) and \( \Lambda_i(\mathfrak{d}) = 0 \), \( i, j = 0, \ldots, \ell \).

Let \( V \) be a highest weight module for affine Lie algebra \( \hat{\mathfrak{g}} \). Then \( V \) is generated by a highest weight vector \( v_\Lambda \) such that

\[
h \cdot v_\Lambda = \Lambda(h)v_\Lambda, \quad \text{for } h \in \mathfrak{h}^c, \\
x \cdot v_\Lambda = 0, \quad \text{for } x \in \mathfrak{n}_+,
\]

for \( \Lambda \in (\mathfrak{h}^c)^* \). Module \( V \) is a direct sum of weight subspaces \( V_\mu = \{v \in V \mid h \cdot V = \mu(h)v \text{ for } h \in \mathfrak{h}^c\}, \mu \in \mathfrak{h}^c \).

Standard (i.e. integrable highest weight) \( \hat{\mathfrak{g}} \)-module \( L(\Lambda) \) is an irreducible highest weight module, with the highest weight \( \Lambda \) being dominant integral, i.e.

\[
\Lambda = k_0 \Lambda_0 + k_1 \Lambda_1 + \cdots + k_\ell \Lambda_\ell,
\]

where \( k_i \in \mathbb{Z}_+, i = 0, \ldots, \ell \). The central element \( c \) acts on \( L(\Lambda) \) as multiplication

\[
k = \Lambda(c) = k_0 + k_1 + \cdots + k_\ell,
\]

which is called the level of module \( L(\Lambda) \).

### 3. Feigin-Stoyanovsky’s Type Subspace

Vector \( v \in \mathfrak{h} \) is said to be **cominuscule** if

\[
\{\alpha(v) \mid \alpha \in R\} = \{-1, 0, 1\}.
\]

Similarly, weight \( \omega \in P \) is said to be **minuscule** if

\[
\{\langle \omega, \alpha \rangle \mid \alpha \in R\} = \{-1, 0, 1\}.
\]

One immediately sees that a dominant integral weight \( \omega \in P^+ \) is minuscule if and only if

\[
\langle \omega, \theta \rangle = 1.
\]

So, there exist a finite number of minuscule weights. Furthermore, a vector \( v \in \mathfrak{h} \) is cominuscule if and only if it is dual to some minuscule fundamental weight \( \omega \), in the sense that

\[
v = \nu^{-1}(\omega),
\]

for some choice of positive roots.

Fix a cominuscule vector \( v \in \mathfrak{h} \). In the case of \( \mathfrak{g} = \mathfrak{sl}(\ell + 1, \mathbb{C}) \), all fundamental weights are minuscule. Then we can assume that the cominuscule vector \( v \) is dual to a fundamental weight

\[
\omega = \omega_m,
\]
for some $m \in \{1, \ldots, \ell\}$. Set
\[
\Gamma = \{ \alpha \in R \mid \alpha(v) = 1 \} = \{ \alpha \in R \mid \langle \omega, \alpha \rangle = 1 \}.
\]
Then we have the induced $\mathbb{Z}$-gradation of $\mathfrak{g}$:
\[
\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,
\]
where
\[
\mathfrak{g}_0 = \mathfrak{h} \oplus \sum_{\alpha(v) = 0} \mathfrak{g}_\alpha
\]
\[
\mathfrak{g}_{\pm 1} = \sum_{\alpha \in \pm \Gamma} \mathfrak{g}_\alpha.
\]
Subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are commutative, and $\mathfrak{g}_0$ acts on them by adjoint action. The subalgebra $\mathfrak{g}_0$ is reductive with semisimple part $l_0 = [\mathfrak{g}_0, \mathfrak{g}_0]$ of the type $A_{m-1} \times A_{\ell-m}$; as a root basis one can take $\{\alpha_1, \ldots, \alpha_{m-1}\} \cup \{\alpha_{m+1}, \ldots, \alpha_\ell\}$, and the center is equal to $\mathbb{C}v$.

We illustrate decomposition (3) on the picture 4, which corresponds to the usual realization of $\mathfrak{g}$ as matrices of trace 0. In this case the subalgebra $\mathfrak{g}_0$ consists of block-diagonal matrices, while $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ consist of matrices with non-zero entries only in the upper right or lower-left block, respectively.

**Figure 4.**

![Figure 4](image)

Basis of a subalgebra $\mathfrak{g}_1$ can be identified with the set of roots $\Gamma$. We will call elements $\gamma \in \Gamma$ colors and the set $\Gamma$ the set of colors. In the case of $\mathfrak{g} = \mathfrak{sl}(\ell+1, \mathbb{C})$, $\omega = \omega_m$, the set of colors is
\[
\Gamma = \{ \gamma_{ij} \mid i = 1, \ldots, m; j = m, \ldots, \ell \}
\]
where
\[
\gamma_{ij} = \alpha_i + \cdots + \alpha_m + \cdots + \alpha_j.
\]
The maximal root $\theta$ is equal to $\gamma_{1\ell}$.

We picture the set of colors $\Gamma$ as a rectangle with row-indices $1, \ldots, m$ and column-indices $m, \ldots, \ell$ (see figure 4).

Similarly, one also has the induced $\mathbb{Z}$-gradation of affine Lie algebra $\tilde{\mathfrak{g}}$:
\[
\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,
\]
\[
\tilde{\mathfrak{g}}_{\pm 1} = \mathfrak{g}_{\pm 1} \otimes \mathbb{C}[t, t^{-1}],
\]
\[
\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} + \tilde{\mathfrak{g}}_0 + \tilde{\mathfrak{g}}_1.
\]
As above, $\tilde{\mathfrak{g}}_{-1}$ and $\tilde{\mathfrak{g}}_1$ are commutative subalgebras, and $\tilde{\mathfrak{g}}_1$ is a $\tilde{\mathfrak{g}}_0$-module.
For a dominant integral weight $\Lambda$, we define a Feigin-Stoyanovsky’s type subspace

$$W(\Lambda) = U(\tilde{g}^-_1) \cdot v_\Lambda \subset L(\Lambda).$$

Our objective is to find a combinatorial basis of $W(\Lambda)$. Set

$$\tilde{g}^+_1 = \tilde{g}_1 \cap \tilde{n}^+, \tilde{g}^-_1 = \tilde{g}_1 \cap \tilde{n}^-.$$ 

Then we have

$$W(\Lambda) = U(\tilde{g}^-_1) \cdot v_\Lambda.$$

By Poincaré-Birkhoff-Witt theorem, we have a spanning set of $W(\Lambda)$ consisting of monomial vectors

$$\{x_{\gamma_1}(-n_1)x_{\gamma_2}(-n_2)\cdots x_{\gamma_r}(-n_r)v_\Lambda \mid r \in \mathbb{Z}_+, \gamma_j \in \Gamma, n_j \in \mathbb{N}\}. \quad (5)$$

In the end, we’ll say a few words about notation. Elements of the spanning set $S$ can be identified with monomials from $U(\tilde{g}_1) = S(\tilde{g}_1)$. With this in mind, we’ll often refer to elements of $\{x_\gamma(-j) \mid \gamma \in \Gamma, j \in \mathbb{Z}\}$ in $\tilde{g}_1$ as to variables, elements or factors of a monomial.

We can also identify monomials from $S(\tilde{g}_1)$ with colored partitions. From the beginnings of the representation-theoretic approach to Rogers-Ramanujan identities, combinatorial basis of certain representations were parameterized by partitions satisfying certain conditions (cf. [LW], [LP]). Let $\pi : \{x_\gamma(-j) \mid \gamma \in \Gamma, j \in \mathbb{Z}\} \to \mathbb{Z}_+$ be a colored partition (cf. [P1], section 3). The corresponding monomial $x(\pi) \in S(\tilde{g}_1)$ is

$$x(\pi) = x_{\gamma_1}(-j_1)^{\pi(x_{\gamma_1}(-j_1))} \cdots x_{\gamma_r}(-j_r)^{\pi(x_{\gamma_r}(-j_r))}.$$ 

From this identification we’ll take notation $x(\pi)$ for the monomials from $S(\tilde{g}_1)$. It will be convenient to define some new monomials by using this identification. Also, our combinatorial conditions for the basis elements can be written in terms of exponents $\pi(x_\gamma(-j))$, which gives a parametrization of the basis by a certain generalization of $(k, \ell+1)$-admissible configurations from [FJKLM]. This will prove to be useful in a higher-level case (cf. [T]).

4. ORDER ON THE SET OF MONOMIALS

We introduce a linear order on the set of monomials.

On the weight and root lattice, we have an order $\prec$ defined in the standard fashion: for $\mu, \nu \in P$ set $\mu \prec \nu$ if $\mu - \nu$ is an integral linear combination of simple roots $\alpha_i, i = 1, \ldots, \ell$, with non-negative coefficients.

Next, we define a linear order $\prec$ on the set of colors $\Gamma$ which is an extension of the order $\prec$. For elements of $\Gamma$, $\gamma_{i'}j' \prec \gamma_{ij}$ is equivalent to saying that $i' > i$ and $j' \leq j$. The order $\prec$ on $\Gamma$ is defined in the following way:

$$\gamma_{i'j'} \prec \gamma_{ij} \quad \text{if} \quad \begin{cases} i' > i \\ i' = i, \ j' < j. \end{cases}$$

It is clear that this is a linear order on the set of colors.

On the set of variables $\{x_\gamma(-n) \mid \gamma \in \Gamma, n \in \mathbb{Z}\} \subset \tilde{g}_1$ we define a linear order $\prec$ so that we compare degrees first, and then colors of variables:

$$x_\alpha(-i) < x_\beta(-j) \quad \text{if} \quad \begin{cases} -i < -j, \\ i = j \quad \text{and} \quad \alpha < \beta. \end{cases}$$

Since the algebra $\tilde{g}_1$ is commutative, we can assume that the variables in monomials from $S(\tilde{g}_1)$ are sorted ascendingly from left to right. The order $\prec$ on the set of monomials is defined as a lexicographic order, where we compare variables
from right to left (from the greatest to the lowest one). If \( x(\pi) \) and \( x(\pi') \) are two monomials,

\[
x(\pi) = x_{\gamma_1}(-n_1)x_{\gamma_2}(-n_2-1) \cdots x_{\gamma_r}(-n_r-1),
\]
\[
x(\pi') = x_{\gamma'_1}(-n'_1)x_{\gamma'_2}(-n'_2-1) \cdots x_{\gamma'_s}(-n'_s-1),
\]
then \( x(\pi) < x(\pi') \) if there exist \( i_0 \in \mathbb{N} \) so that \( x_{\gamma_i}(-n_i) = x_{\gamma'_i}(-n'_i) \), for all \( i < i_0 \), and either \( i_0 = r + 1 \leq s \) or \( x_{\gamma_i}(-n_i) < x_{\gamma'_i}(-n'_i) \).

This monomial order is compatible with multiplication:

**Proposition 1.** Let

\[
x(\pi_1) \leq x(\mu_1) \text{ and } x(\pi_2) \leq x(\mu_2).
\]

Then

\[
x(\pi_1)x(\pi_2) \leq x(\mu_1)x(\mu_2),
\]

and if one of the first two inequalities is strict, then the last one is also strict.

**Proof:** By the definition of the order \( < \), we compare two monomials so that we compare their greatest elements first. Let \( x_{\alpha_1}(-j_1), x_{\alpha_2}(-j_2), x_{\beta_1}(-i_1), x_{\beta_2}(-i_2) \) be the greatest variables in \( x(\pi_1), x(\pi_2), x(\mu_1), x(\mu_2) \) respectively. Then \( x_{\alpha_1}(-j_1) \leq x_{\beta_1}(-i_1) \) and \( x_{\alpha_2}(-j_2) \leq x_{\beta_2}(-i_2) \). The greatest element in \( x(\pi) \) will be greater of the two \( x_{\alpha_1}(-j_1) \) and \( x_{\alpha_2}(-j_2) \); one can assume it to be \( x_{\alpha_1}(-j_1) \). Similarly, greatest element in \( x(\mu) \) will be greater of the two \( x_{\beta_1}(-i_1) \) and \( x_{\beta_2}(-i_2) \). There are two possibilities:

(i) the greatest element of \( x(\mu) \) is strictly greater than the greatest element of \( x(\pi) \). In that case \( x(\pi) < x(\mu) \).

(ii) the greatest element of \( x(\mu) \) is equal to the greatest element of \( x(\pi) \). Then \( x_{\alpha_1}(-j_1) = x_{\beta_1}(-i_1) \) and we can take \( x_{\beta_2}(-i_2) \) for the greatest element of \( x(\mu) \). We proceed by induction: let \( x(\pi'_1) \) and \( x(\mu'_1) \) be monomials gotten from \( x(\pi_1) \) and \( x(\mu_1) \), respectively, by omitting \( x_{\alpha_1}(-j_1) = x_{\beta_1}(-i_1) \). Then \( x(\pi'_1) \leq x(\mu'_1) \) and we can continue to apply the same procedure to monomials \( x(\pi'_1), x(\pi'_2), x(\mu'_1), x(\mu'_2) \). After a finite number of steps either case (i) will occur, or we’ll exhaust monomials \( x(\pi_1) \) and \( x(\pi_2) \). Both these cases imply \( x(\pi) \leq x(\mu) \), and the equality occurs only if both initial inequalities were in fact equalities.

For monomials from \( S(\tilde{\mathfrak{h}}_1) \), we also define degree and shape of a monomial.

**Degree** of a monomial is equal to the sum of degrees of its variables. For

\[
x(\pi) = x_{\gamma_1}(-n_1)x_{\gamma_2}(-n_2-1) \cdots x_{\gamma_r}(-n_r-1),
\]
its degree is equal to \( -n_1 - n_2 - \cdots - n_r \). A **shape** of a monomial is gotten from its colored partition by forgetting colors and considering only degrees of factors. More precisely, for a monomial \( x(\pi) \) and its partition \( \pi : \{ x_\gamma(-n) \mid \gamma \in \Gamma, n \in \mathbb{Z} \} \to \mathbb{Z}_+ \), the corresponding shape will be

\[
s_\pi : \mathbb{Z} \to \mathbb{Z}_+,
\]
\[
s_\pi(j) = \sum_{\gamma \in \Gamma} \pi(x_\gamma(-j)).
\]
A linear order can also be defined on the set of shapes; we’ll say that \( s_\pi < s_{\pi'} \) if there exists \( j_0 \in \mathbb{Z} \) such that \( s_\pi(j) = s_{\pi'}(j) \) for \( j < j_0 \) and either \( s_\pi(j_0) < s_{\pi'}(j_0) \) and \( s_\pi(j') \neq 0 \) for some \( j' > j_0 \), or \( s_\pi(j_0) > s_{\pi'}(j_0) \) and \( s_\pi(j) = 0 \) for \( j > j_0 \).

In the end, for the sake of simplicity, we introduce the following notation:

\[
x_{rs}(\pi) = x_{\gamma_{rs}}(-j),
\]
for \( \gamma_{rs} \in \Gamma \).
5. VERTEX OPERATOR CONSTRUCTION OF LEVEL 1 MODULES

We use the vertex operator algebra construction of the basic \( \tilde{\mathfrak{g}} \)-modules (i.e. the standard \( \tilde{\mathfrak{g}} \)-modules of level 1). We'll sketch this construction in this section, details can be found in [FLM], [DL] or [LL]; see also [FK], [S].

We have denoted by \( P \) and \( Q \) weight and root lattices of \( \mathfrak{g} \), respectively. There exists a central extension \( \hat{P} \) of \( P \) by the finite cyclic group \( \langle e^{\pi i/(\ell+1)^2} \rangle \) of order \( 2(\ell+1)^2 \),

\[
1 \longrightarrow \langle e^{\pi i/(\ell+1)^2} \rangle \longrightarrow \hat{P} \longrightarrow P \longrightarrow 1.
\]

By restricting, one gets a central extension \( \hat{Q} \) of \( Q \). Central extension can be chosen such that the corresponding 2-cocycle \( \epsilon : P \times P \rightarrow \langle e^{\pi i/(\ell+1)^2} \rangle \) satisfies

\[
\epsilon(\alpha, \beta)/\epsilon(\beta, \alpha) = (-1)^{\langle \alpha, \beta \rangle} \quad \text{for } \alpha, \beta \in Q.
\]

Let \( c(\lambda, \mu) = \epsilon(\lambda, \mu)/\epsilon(\mu, \lambda) \) for \( \lambda, \mu \in P \) be the corresponding bimultiplicative, alternating commutator map (cf. [FLM]).

Inside \( \tilde{\mathfrak{g}} \) there is a Heisenberg subalgebra

\[
\hat{h}_{\mathbb{Z}} = \sum_{n \in \mathbb{Z}\setminus\{0\}} \mathfrak{h} \otimes t^n \oplus \mathbb{C}c.
\]

We also introduce subalgebras

\[
\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c, \\
\hat{\mathfrak{h}}_{\pm} = \mathfrak{h} \otimes \mathbb{C}[t^\pm].
\]

and by \( \mathbb{C}[P] \) and \( \mathbb{C}[Q] \) we denote group algebras of weight and root lattices, respectively. Bases of \( \mathbb{C}[P] \) and \( \mathbb{C}[Q] \) consist of elements \( \{e^\lambda : \lambda \in P\} \) and \( \{e^\alpha : \alpha \in Q\} \), respectively.

Consider the induced \( \hat{h}_{\mathbb{Z}} \)-module

\[
M(1) = U(\hat{\mathfrak{h}}) \otimes_{\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}c} \mathbb{C},
\]

where \( \mathfrak{h} \otimes \mathbb{C}[t] \) acts trivially on \( \mathbb{C} \), and \( c \) acts as 1. Module \( M(1) \) is irreducible module for the Heisenberg subalgebra \( \hat{h}_{\mathbb{Z}} \); as a vector space, \( M(1) \) is naturally isomorphic to the symmetric algebra \( S(\hat{\mathfrak{h}}) \) (cf. [FLM]).

Consider tensor products

\[
V_P = M(1) \otimes \mathbb{C}[P], \\
V_Q = M(1) \otimes \mathbb{C}[Q];
\]

there is a natural inclusion \( V_Q \subset V_P \). For simplicity, we will often write \( e^\lambda \) instead of \( 1 \otimes e^\lambda \), and 1 instead of \( 1 \otimes 1 \).

Space \( V_P \) carries a \( \hat{\mathfrak{h}} \)-module structure: \( \hat{h}_{\mathbb{Z}} \) acts as \( \hat{h}_{\mathbb{Z}} \otimes 1 \) and \( \mathfrak{h} \otimes t^0 \) acts as \( 1 \otimes \mathfrak{h} \). Operators \( h(0), h \in \mathfrak{h} \) on \( \mathbb{C}[P] \) are defined as follows

\[
h(0) \cdot e^\lambda = (h, \lambda)e^\lambda
\]

for \( \lambda \in P \). On \( V_P \) we have also the action of the group algebra \( \mathbb{C}[P] \):

\[
e^\lambda = 1 \otimes e^\lambda, \quad \lambda \in P,
\]

where the latter operator \( e^\lambda \) is a multiplication in \( \mathbb{C}[P] \). It will be clear from the context when \( e^\lambda \) represents a multiplication operator, and when an element of \( V_P \).

Define also operators \( \epsilon_\lambda \) by

\[
\epsilon_\lambda \cdot e^\mu = \epsilon(\lambda, \mu)e^\mu,
\]

for \( \lambda, \mu \in P \).
For elements of $V_P$ define a degree: for $v = h_1(-n_1) \cdots h_r(-n_r) \otimes e^\lambda$ set
\[
\deg(v) = -n_1 - n_2 - \cdots - n_r - \frac{1}{2}(\lambda, \lambda).
\]
This gives a grading on $V_P$, which is bounded from above.

We will use independent commuting formal variables $z, z_0, z_1, z_2, \ldots$. For a vector space $V$, denote by $V[[z]]$ the space of formal series of nonnegative integral powers of $z$ with coefficients in $V$. Similarly, denote by $V[[z, z^{-1}]]$ the space of formal Laurent series, and by $V\{z\}$ the space of formal series of rational powers of $z$ with coefficients in $V$.

Define also one more family of operators $z^h \in (\text{End } V_P)\{z\}$ by
\[
z^h \cdot e^\lambda = e^\lambda z^{(h, \lambda)},
\]
for $h \in \mathfrak{h}, \lambda \in \mathbb{P}$.

Space $V_Q$ has a natural structure of vertex operator algebra and $V_P$ is a module for this algebra (cf. [FLM],[DL]). Before we define VOA-structure on $V_Q$, define operators
\[
h(z) = \sum_{j \in \mathbb{Z}} h(j)z^{-j-1},
\]
\[
E^\pm(h, z) = \exp \left( \sum_{m \geq 1} h(\pm m)\frac{z^{\mp m}}{\pm m} \right),
\]
for $h \in \mathfrak{h}$. We define vertex operators for all elements of $V_P$, rather than just for elements of $V_Q$. For the lattice elements, i.e. for the elements $1 \otimes e^\lambda = e^\lambda$ set:
\[
Y(e^\lambda, z) = E^-(-\lambda, z)E^+(\lambda, z) \otimes e^\lambda z^\lambda \epsilon_\lambda.
\]
Generally, for a homogenous vector $v \in V_P$
\[
v = h_1(-n_1) \cdots h_r(-n_r) \otimes e^\lambda,
\]
normally, $n_1, \ldots, n_r \geq 1$, set
\[
Y(v, z) = \circ \left( \left( \frac{\partial^{n_1-1}}{(n_1 - 1)!}h_1(z) \right) \cdots \left( \frac{\partial^{n_r-1}}{(n_r - 1)!}h_r(z) \right) \right) Y(e^\lambda, z) \circ,
\]
where $\circ \cdot \circ$ is a normal ordering procedure (cf. [FLM]), meaning that coefficients in the enclosed expression should be rearranged in a way that in every product all the operators $h(m), h \in \mathfrak{h}, m < 0$ are placed to the left of the operators $h(m), h \in \mathfrak{h}, m \geq 0$. This way we get a well defined linear map
\[
Y : V_P \rightarrow (\text{End } V_P)\{z\},
\]
\[
v \mapsto Y(v, z).
\]

By using vertex operators, one can define a structure of $\tilde{\mathfrak{g}}$-module on $V_P$. For $\alpha \in \mathbb{R}$ set
\[
x_\alpha(z) = \sum_{j \in \mathbb{Z}} x_\alpha(j)z^{-j-1} = Y(e^\alpha, z).
\]
Actions of $h(j)$ and $c$ have already been defined, and $d$ acts as a degree operator. Then the cosets $V_Q$ and $V_Q e^{\omega_j}, j = 1, \ldots, \ell$ become standard $\tilde{\mathfrak{g}}$-modules of level 1 with highest weight vectors $v_0 = 1$ and $v_j = e^{\omega_j}, j = 1, \ldots, \ell$, respectively (cf. [FLM,DL]). Moreover,
\[
L(\Lambda_0) \cong V_Q, \ L(\Lambda_j) \cong V_Q e^{\omega_j} \text{ for } j = 1, \ldots, \ell
\]
and
\[
V_P \cong L(\Lambda_0) \oplus L(\Lambda_1) \oplus \ldots L(\Lambda_\ell).
\]
Vertex operators \( Y(v, z), v \in V_P \) satisfy (generalized) Jacobi identity. It will be of importance to us a variant of that identity in the case when vectors \( u, v \) of type \( u = u^* \otimes e^\lambda, v = v^* \otimes e^\mu \), for \( \lambda \in Q, \mu \in P, u^*, v^* \in M(1) \), or, even more special, when \( u = 1 \otimes e^\lambda, v = 1 \otimes e^\mu \), for \( \lambda \in Q, \mu \in P \). Then one has

\[
\begin{align*}
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(u, z_1) Y(v, z_2) - (-1)^{\langle \lambda, \mu \rangle} e(\lambda, \mu) z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(v, z_2) Y(u, z_1) = \\
\quad = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(Y(u, z_0)v, z_2),
\end{align*}
\]

where \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \) is a formal delta-function (cf. [FLM], [LL]), and binomial expressions that appear in expansions of delta-functions are understood to be expanded in nonnegative terms of the second variable.

Next we introduce intertwining operators \( \mathcal{Y} \). For \( \mu \in P, v = v^* \otimes e^\mu \) define

\[
\mathcal{Y}(v, z) = Y(v, z)e^{i\gamma\mu}c(\cdot, \mu).
\]

This way we obtain a map

\[
\mathcal{Y} : V_P \rightarrow (\text{End} V_P) \{z\},
\]

\[
v \mapsto \mathcal{Y}(v, z).
\]

Then we have (ordinary) Jacobi identity

\[
\begin{align*}
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(u, z_1) \mathcal{Y}(v, z_2) - z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) \mathcal{Y}(v, z_2) Y(u, z_1) = \\
\quad = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) \mathcal{Y}(Y(u, z_0)v, z_2).
\end{align*}
\]

For \( \mu \in Q \), operators \( \mathcal{Y}(v, z) \) are equal to vertex operators \( Y(v, z) \). Restrictions of \( \mathcal{Y}(v, z) \) are in fact maps

\[
(7) \quad \mathcal{Y}(v, z) : L(\Lambda_j) \rightarrow L(\Lambda_j) \{z\},
\]

if \( \mu + \omega_j \equiv \omega_j \mod Q \). So, restrictions of \( \mathcal{Y} \) define intertwining operators between standard modules of level 1 ([DL]).

Consider now a special case when \( v = e^\mu \). It is interesting to know when the operators \( \mathcal{Y}(e^\mu, z_2) \) from (7) commute with the action of \( \hat{g}_1 \), i.e. when

\[
\langle Y(e^\gamma, z_1), \mathcal{Y}(e^\mu, z_2) \rangle = 0, \quad \gamma \in \Gamma.
\]

By the commutator formula for intertwining operators ([DL]) that is equivalent to

\[
Y(e^\gamma, z_0) e^\mu \in V_P[[z_0]],
\]

for all \( \gamma \in \Gamma \). From the definition of vertex operators (8) one gets

\[
(8) \quad Y(e^\gamma, z_0)e^\nu = Ce^{\gamma+\nu}z_0^{\langle \gamma, \nu \rangle} + \underbrace{\text{higher power terms}}_{\text{higher power terms}} \in z_0^{\langle \gamma, \nu \rangle}V_P[[z_0]],
\]

for some \( C \in \mathbb{C}^\times \). So, operators \( \mathcal{Y}(e^\mu, z_2) \) commute with \( \hat{g}_1 \) if and only if

\[
\langle \gamma, \mu \rangle \geq 0, \quad \text{for all } \gamma \in \Gamma.
\]

In the section \ref{section6} we describe all \( \mu \in P \) that satisfy this relation.

6. OPERATOR \( e(\omega) \)

For \( \lambda \in P \), \( e^\lambda \) denotes multiplication operator \( 1 \otimes e^\lambda \) in \( V_P = M(1) \otimes \mathbb{C}[P] \). Set

\[
e(\lambda) = e^\lambda c(\cdot, \lambda), \quad e(\lambda) : V_P \rightarrow V_P.
\]

Clearly, \( e(\lambda) \) is a linear bijection. Its restrictions on standard modules are bijections from one fundamental module \( L(\Lambda_i) \) onto another fundamental module \( L(\Lambda_{j'}) \). From the definition of vertex operators (8), one gets the following commutation relation

\[
Y(e^\alpha, z)e(\lambda) = e(\lambda)z^{\langle \lambda, \alpha \rangle}Y(e^\alpha, z),
\]
for $\alpha \in R$. In terms of components, we have

$$x_\alpha(n)e(\lambda) = e(\lambda)x_\alpha(n + (\lambda, \alpha)), \quad n \in \mathbb{Z}. \tag{9}$$

For $\lambda = \omega$ and $\gamma \in \Gamma$, the relation (9) becomes

$$x_\gamma(n)e(\omega) = e(\omega)x_\gamma(n + 1). \tag{10}$$

More generally, for a monomial $x(\pi) \in S(\tilde{\mathfrak{g}}_1)$, denote by $x(\pi^+) \in S(\tilde{\mathfrak{g}}_1)$ the monomial corresponding to the partition $\pi^+$, defined by $\pi^+(x_\gamma(n + 1)) = \pi(x_\gamma(n))$. We can say that $x(\pi^+)$ is obtained from $x(\pi)$ by raising degrees of all its factors by 1. Then

$$x(\pi)e(\omega) = e(\omega)x(\pi^+). \tag{11}$$

7. Difference and Initial Conditions

Initial conditions for the level 1 standard module $L(\Lambda_i)$ are consequence of a simple observation that monomials from the monomial basis can’t contain elements of degree $-1$ that act as zero on the highest weight vector $v_i$ of $L(\Lambda_i)$. So, we have to establish for which $\gamma \in \Gamma$, elements $x_\gamma(-1)$ annihilate $v_i$. Then we can exclude from the spanning set (5) all monomials $x(\pi)$ that contain such factors.

Since $v_i = e^{\omega_i}$, for $i = 1, \ldots, \ell$, and $v_0 = 1 = e^0$, relation (8) gives

$$x_\gamma(z)v_i = \left( \sum_{j \in \mathbb{Z}} x_\gamma(-j)z^{j-1} \right) v_i \in z^{(\gamma,\omega_i)}(v_iV_Q)[[z]], \tag{10}$$

and

$$x_\gamma(z)v_i = \left( \sum_{j \in \mathbb{Z}} x_\gamma(-j)z^{j-1} \right) v_i \in z^{(\gamma,\omega_i)}(v_iV_Q)[[z]], \tag{11}$$

for $i = 1, \ldots, \ell$. Since by (4), $(\gamma_r, \omega_i) = 1$ if $r \leq i \leq s$, and zero otherwise, by comparing constant terms in (10) and (11), we get

$$x_{\gamma_r}(-1)v_i = \begin{cases} 0, & r \leq i \leq s, \\ Ce^{\gamma_r}, & C \in \mathbb{C}^x, \quad i = 0, \\ Ce^{\gamma_r+\omega_i}, & C \in \mathbb{C}^x, & \text{otherwise.} \end{cases} \tag{12}$$

For a monomial $x(\pi) \in S(\tilde{\mathfrak{g}}_1)$ we say that it satisfies initial conditions for $W(\Lambda_i)$, if it doesn’t contain factors of degree $-1$ that annihilate $v_i$. We’ll often abbreviate this by saying that $x(\pi)$ satisfies IC for $W(\Lambda_i)$. From (12) we see that $x(\pi)$ satisfies initial conditions on $W(\Lambda_i)$ if the colors of elements of degree $-1$ lie below the $i$-th row (in case $i \leq m$), or, to the left of the $i$-th column (for $i \geq m$).

Difference conditions will be consequences of relations between operators $x_\gamma(z)$, and fortiori, between monomial vectors $x(\pi)v_i$.

To obtain these, consider the basic module $L(\Lambda_0)$ with highest weight vector $v_0 = 1 = e^0$ (cf. section 5). This is a vertex operator algebra, with 1 as the vacuum element, and $L(\Lambda_0)$ is a module for this algebra. We are looking for relations between vectors of type

$$x_\gamma(-1)x_{\gamma'}(-1)1, \quad \gamma, \gamma' \in \Gamma. \tag{13}$$

These will in turn induce relations between corresponding vertex operators on $L(\Lambda_i)$.

From (10) we have

$$x_\gamma(-1)x_{\gamma'}(-1)1 = x_\gamma(-1)e^{\gamma'}. \tag{14}$$
Since
\[\langle \gamma, \gamma' \rangle = \begin{cases} 2, & \gamma = \gamma', \\ 1, & \gamma \text{ and } \gamma' \text{ lie in the same row or column}, \\ 0, & \text{otherwise}, \end{cases}\]
relation (8) implies
\[x_\gamma(-1)x_{\gamma'}(-1)1 = \begin{cases} 0, & \gamma \text{ and } \gamma' \text{ lie in the same row or column}, \\ Ce^{\gamma+\gamma'}, & C \in \mathbb{C}^\times, \text{ otherwise.} \end{cases}\]

Fix two rows \(r_1 < r_2\) and two columns \(s_2 < s_1\). Note that
\[\gamma_{r_1s_1} + \gamma_{r_2s_2} = \gamma_{r_1s_2} + \gamma_{r_2s_1}.\]
This gives us
\[x_{r_2s_2}(-1)x_{r_1s_1}(-1)1 = C \cdot x_{r_2s_1}(-1)x_{r_1s_2}(-1)1,\]
for some \(C \in \mathbb{C}^\times\).

We’ve obtained two types of relations:
\[x_\gamma(-1)x_{\gamma'}(-1)1 = 0,\]
if \(\gamma \text{ and } \gamma' \text{ lie in the same row/column, and}\]
\[x_\gamma(-1)x_{\gamma'}(-1)1 = C \cdot x_{\gamma_1}(-1)x_{\gamma_1'}(-1)1,\]
if \(\gamma, \gamma_1, \gamma' \text{ are vertices of a rectangle in } \Gamma, \text{ as in the figure 5.}\)

Since the algebra \(\tilde{g}_1\) is commutative, vertex operators \(Y(x_\gamma(-1)x_{\gamma'}(-1)1, z)\) are equal to products of \(x_\gamma(z)\) and \(x_{\gamma'}(z)\) as ordinary products of Laurent series (cf. [DL],[LL]). This way we get relations between vertex operators on level 1 modules
\[(13) \quad x_\gamma(z)x_{\gamma'}(z) = 0,\]
\[(14) \quad x_\gamma(z)x_{\gamma'}(z) = C \cdot x_{\gamma_1}(z)x_{\gamma_1'}(z).\]

Fix \(n \in \mathbb{N}\) and consider the coefficients of \(z^{n-2}\) in (13) and (14). From the first relation we have
\[0 = \sum_{i+j=n} x_\gamma(-i)x_{\gamma'}(-j).\]
In each such sum we can identify the minimal monomial with regard to the ordering \(<\), which is then called the leading term of the relation. This can be expressed in terms of other monomials in the sum, so we can exclude from the spanning set all monomials that contain leading terms (cf. [P1],[P2]). All the monomials appearing in the sum are of length 2 and of total degree \(-n\). Because of this, the
minimal between them has to be of the “minimal shape”, i.e. its factors have to be either of the same degree (for \( n \) even), or degrees have to differ only for 1 (for \( n \) odd). In the case of even \( n \), there is only one monomial of minimal shape,
\[
x_{\gamma}(-j)x_{\gamma'}(-j),
\]
and that’s the leading term of the sum above. For \( n \) odd, there are two monomials of minimal shape,
\[
x_{\gamma}(-j - 1)x_{\gamma}(-j), \quad x_{\gamma}(-j - 1)x_{\gamma'}(-j).
\]
By the definition of the order \(<\), next we compare colors of elements. First we compare colors of elements of degree \(-j\), and then of elements of degree \(-j - 1\). If we assume \( \gamma < \gamma' \), then the leading term will be
\[
x_{\gamma'}(-j - 1)x_{\gamma}(-j).
\]
Analogously we consider relation \((14)\); we get
\[
0 = \sum_{i+j=n} x_{\tau}(-i)x_{\tau'}(-j) - Cx_{\gamma}(-i)x_{\gamma'}(-j).
\]
Assume \( \gamma < \gamma_1 < \gamma_1' < \gamma' \), as in the figure 4. For \( n \) even we have two monomials of minimal shape
\[
x_{\gamma}(-j)x_{\gamma'}(-j), \quad x_{\gamma_1}(-j)x_{\gamma_1'}(-j),
\]
and for \( n \) odd we have four of them
\[
x_{\gamma'}(-j - 1)x_{\gamma}(-j), \quad x_{\gamma}(-j - 1)x_{\gamma'}(-j),
\]
\[
x_{\gamma_1'}(-j - 1)x_{\gamma_1}(-j), \quad x_{\gamma_1}(-j - 1)x_{\gamma_1'}(-j).
\]
The leading terms are
\[
x_{\gamma_1}(-j)x_{\gamma_1'}(-j)
\]
for \( n \) even, and
\[
x_{\gamma'}(-j - 1)x_{\gamma}(-j)
\]
for \( n \) odd.

We say that a monomial \( x(\pi) \in S(\Lambda_{i-1}) \) satisfies difference conditions if it doesn’t contain any of the leading terms \((15), (17), (18)\).

Then, by using proposition \( \text{I} \) we get the following proposition (cf. [P1, Lemma 9.4] and [P2, Theorem 5.3])

**Proposition 2.** The set
\[
\{x(\pi)v_i \mid x(\pi) \text{ satisfies IC and DC for } W(\Lambda_i)\}
\]
spans \( W(\Lambda_i) \).

Finally, let’s have a closer look at the structure of monomials that satisfy difference and initial conditions for the standard module \( L(\Lambda_i) \) of level 1. Assume that a monomial \( x(\pi) \) contains elements \( x_{r,s}(-j) \) and \( x_{r',s'}(-j) \), and \( \gamma_{r,s'} \leq \gamma_{r,s} \). Then by \((15)\), \( \gamma_{r,s'} \) and \( \gamma_{r,s} \) cannot lie in the same column or row, because otherwise \( x(\pi) \) would contain a leading term. Hence \( \gamma_{r,s'} \) and \( \gamma_{r,s} \), are opposite vertices of a rectangle in \( \Gamma \). By \((17)\), they have to be upper-right and lower-left vertices of this rectangle, otherwise \( x(\pi) \) would contain a leading term. Since \( \gamma_{r,s'} \leq \gamma_{r,s} \), we conclude that \( r' > r \) and \( s' < s \), i.e. \( \gamma_{r,s'} \) must lie in the shaded area as illustrated on the figure 6.

Next, assume that a monomial \( x(\pi) \) contains elements \( x_{r,s}(-j) \) and \( x_{r',s'}(-j - 1) \). Then, by a similar argument as above, one concludes that \( r' > r \) or \( s' < s \), which is illustrated on the figure 7.
From these observations we conclude that colors of the elements of the same degree $-j$ inside $x(\pi)$ make a descending sequence as pictured on the figure 2; appropriate row-indices strictly increase, while column-indices strictly decrease. Set of colors of elements of degree $-j - 1$ also form a decreasing sequence which is placed below or on the left of the minimal color of elements of degree $-j$.

Initial conditions for $W(\Lambda_i)$ imply that the sequence of colors of elements of degree $-1$ lies below the $i$-th row (if $0 \leq i \leq m$), or on the left of the $i$-th column (for $m \leq i \leq \ell$) (see figure 3).

These considerations also imply the following

**Proposition 3.** If $x_\gamma(-j) < x_{\gamma'}(-j') < x_{\gamma''}(-j'')$ are such that monomials $x_\gamma(-j)x_{\gamma'}(-j')$ and $x_{\gamma''}(-j'')$ satisfy difference conditions, then so does $x_\gamma(-j)x_{\gamma''}(-j'')$, and consequently $x_\gamma(-j)x_{\gamma'}(-j')x_{\gamma''}(-j'')$.

Hence, under the assumption that factors in monomials are sorted descendingly from right to left, to see if a monomial satisfies difference conditions, it is enough to check difference conditions on all pairs of successive factors in it.

### 8. Intertwining operators

As we’ve already seen in section 5 operators
\[ \mathcal{Y}(e^\lambda, z) : L(\Lambda_i) \to L(\Lambda_i')\{z\}, \]
commute with the action of $\tilde{\mathfrak{g}}_1$ if and only if
\[ \langle \lambda, \gamma \rangle \geq 0, \]
for
for all $\gamma \in \Gamma$.

Define “minimal” weights that satisfy (20):

$$
\begin{align*}
\lambda_1 &= \omega_1, & \lambda'_m &= \omega_m - \omega_{m+1}, \\
\lambda_2 &= \omega_2 - \omega_1, & \lambda'_{m+1} &= \omega_{m+1} - \omega_{m+2}, \\
\lambda_3 &= \omega_3 - \omega_2, & \vdots & \\
\lambda_m &= \omega_m - \omega_{m-1}, & \lambda'_{l-1} &= \omega_{l-1} - \omega_l, \\
& & \lambda'_l &= \omega_l.
\end{align*}
$$

Then relation (23) gives

$$
\langle \lambda_r, \gamma \rangle = \begin{cases} 
1, & \text{if } \gamma \text{ lies in the } r\text{-th row}, \\
0, & \text{otherwise,}
\end{cases}
\langle X'_s, \gamma \rangle = \begin{cases} 
1, & \text{if } \gamma \text{ lies in the } s\text{-th column}, \\
0, & \text{otherwise.}
\end{cases}
$$

It is obvious that every nonnegative $\mathbb{Z}$-linear combination $\lambda \in P$ of these weights also satisfies condition (20) and, consequently, the appropriate intertwining operator $\mathcal{Y}(e^\mu, z)$ commutes with $\tilde{g}_1$. It can easily be shown that a weight $\lambda \in P$ satisfies (20) if and only if $\lambda$ can be written in this way. For example

$$
\begin{align*}
\omega_3 &= \lambda_1 + \lambda_2 + \lambda_3, \\
\omega_r &= \lambda_r + \lambda_{r-1} + \cdots + \lambda_1, & \text{for } r \leq m, \\
\omega_s &= \lambda'_s + \lambda'_{j+1} + \cdots + \lambda'_l, & \text{for } s \geq m, \\
\omega_m &= \lambda_m + \lambda_{m-1} + \cdots + \lambda_1 = \lambda'_m + \lambda'_{m+1} + \cdots + \lambda'_l.
\end{align*}
$$

In the next section we’ll need the following lemma

**Lemma 4.** Let $\gamma_{rs} \in \Gamma$. Then

$$
\gamma_{rs} = \lambda_r + \lambda'_s.
$$

**Proof:** By the Cartan matrix of $\mathfrak{g}$, we have

$$
\begin{align*}
\alpha_1 &= 2\omega_1 - \omega_2, \\
\alpha_j &= -\omega_{j-1} + 2\omega_j - \omega_{j+1}; & j &= 2, \ldots, \ell - 1, \\
\alpha_l &= -\omega_{l-1} + 2\omega_l.
\end{align*}
$$

The claim now follows from (4) and (21). $\blacksquare$

9. **Proof of linear independence**

Write a monomial $x(\pi) \in S(\tilde{\mathfrak{g}}^-)$ as a product $x(\pi) = x(\pi_2)x(\pi_1)$, where $x(\pi_1)$ consists of elements of degree $-1$, and $x(\pi_2)$ consists of elements of lower degree. The main technical tool in the proof of linear independence is the following proposition:

**Proposition 5.** Suppose that a monomial $x(\pi)$ satisfies difference and initial conditions for a level 1 standard module $L(\Lambda_i)$. Then there exists a coefficient $w(\mu)$ of an intertwining operator $\mathcal{Y}(e^\mu, z)$

$$
w(\mu) : L(\Lambda_i) \to L(\Lambda_{i'})
$$

for some $i' \in \{0, \ldots, \ell\}$, such that:

- $w(\mu)x(\pi_1)v_i = C e(\omega)v_{i'}, \quad C \in \mathbb{C}^\times$,
- $x(\pi_2')$ satisfies IC and $\text{DC}$ for $W(\Lambda_{i'})$,
- $x(\pi_1)$ is maximal for $w(\mu)$, i.e. all the monomials $x(\pi')$ that satisfy IC and $\text{DC}$ for $L(\Lambda_i)$ and such that $w(\mu)x(\pi')v_i \neq 0$, have their $(-1)$-part $x(\pi'_1)$ smaller or equal to $x(\pi_1)$.  

16
Proof: Assume \( i = 0; \Lambda_i = \Lambda_0 \), and \( v_0 = 1 = e^0 \) is the highest weight vector of \( L(\Lambda_0) \). Let
\[
x(\pi_1) = x_{r_1,s_1}(-1) \cdots x_{r_2,s_2}(-1) x_{r_1,s_1}(-1),
\]
where \( 1 \leq r_1 < r_2 < \cdots < r_t \leq m, \ell \geq s_1 > s_2 > \cdots > s_t \geq m \). Then colors of elements of degree \(-2\) lie either below the \( r_t\)-th row, or left of the \( s_t\)-th column (see figure 3). Suppose that they lie below the \( r_t\)-th row. Since \( \langle \gamma_{t_1}, \gamma_{t_2} \rangle = 0 \) for \( 1 \leq p < q \leq t \), one has
\[
x(\pi_1)v_0 = x_{r_1,s_1}(-1) \cdots x_{r_1,s_1}(-1) = C_1 \cdot e^{\gamma_{r_1} + \cdots + \gamma_{r_t}},
\]
for some \( C_1 \in \mathbb{C}^\times \). By lemma 4 we have
\[
x(\pi_1)v_0 = C_1 \cdot e^{\lambda_{r_1} + \cdots + \lambda_{r_t} + \lambda_{s_1} + \cdots + \lambda_{s_t}}.
\]
Set
\[
\mu = \sum_{1 \leq r < \ell \leq m} \lambda_r + \sum_{1 \leq s > t \leq m} \lambda_s' + \sum_{t = m}^{s_t - 1} \lambda_s',
\]
Weight \( \mu \) is the sum of all \( \lambda_r \)'s, \( 1 \leq r < r_1 \), and all \( \lambda_s' \)'s, \( \ell \geq s \geq m \), such that in the appropriate rows and columns, respectively, there doesn’t lie any color of elements of \( x(\pi_1) \). Let \( w(\mu) \) be a coefficient of \( z^\mu = z^{(\mu, \omega)} \) in \( \mathcal{V}(e^\mu, z) \). For \( \gamma \in \Gamma \), \( w(\mu)e^{\gamma} \neq 0 \) if and only if \( (\mu, \gamma) = 0 \), by (8). Because of (22), for a monomial \( x(\pi_1^t) \) consisting of elements of degree \(-1\) and satisfying difference conditions for \( L(\Lambda_0) \), vector \( x(\pi_1^t)v_0 \) won’t be annihilated by \( w(\mu) \) if and only if its colors lie in the intersection of rows \( \{r_1, \ldots, r_t\} \cup \{r_1 + 1, \ldots, m\} \) and columns \( \{s_1, \ldots, s_t\} \). Clearly, \( x(\pi_1) \) is maximal among such, so if \( w(\mu)x(\pi_1^t)v_0 \neq 0 \) then \( x(\pi_1^t) \leq x(\pi_1) \).

Note that
\[
\mu + \lambda_{r_1} + \cdots + \lambda_{r_t} + \lambda_{s_1}' + \cdots + \lambda_{s_t}' = \sum_{r=1}^{r_t} \lambda_r + \sum_{s=m}^{s_t-1} \lambda_s' = \omega_{r_t} + \omega.
\]
Hence
\[
w(\mu)x(\pi_1)v_0 = C_2 e^{\omega_{r_t} + \omega} = C e^{(\omega, \omega)} v_{r_t},
\]
for some \( C_2, C \in \mathbb{C}^\times \). Since colors of elements of degree \(-2\) lie below the \( r_t\)-th row, the monomial \( x(\pi_1^t) \) satisfies difference and initial conditions for \( W(\Lambda_{r_t}) \). Hence the operator \( w(\mu) : L(\Lambda_0) \rightarrow L(\Lambda_{r_t}) \) satisfies the statement of the proposition.

If colors of elements of \( x(\pi) \) of degree \(-2\) lie on the left of the \( s_t\)-th row instead of lying below the \( r_t\)-th row, then, when constructing \( \mu \), one will replace \( \lambda_s' \)'s, for \( m \leq s < s_t \), with \( \lambda_r \)'s, for \( s_t < s \leq m \). That way, we get an operator \( w(\mu) : L(\Lambda_0) \rightarrow L(\Lambda_{s_t}) \).

Finally, assume \( 1 \leq i \leq \ell \); \( v_i = e^{\omega_i} \) is the highest weight vector of \( L(\Lambda_i) \). Colors of elements of \( x(\pi_1) \) lie either below the \( i\)-th row, or on the left of \( i\)-th column (see figure 3). Then one constructs \( \mu \in P \) similarly as before, with an exception that if \( i \leq m \), one won’t take \( \lambda_r \)'s for \( r \leq i \), and if \( i \geq m \) one won’t take \( \lambda_s' \)'s for \( s \geq i \). For instance, if \( i \leq m \) and colors of elements of degree \(-2\) in \( x(\pi) \) is on the left of the \( s_t\)-th column, we would set
\[
\mu = \sum_{i < r < r_1 \text{ or } r \notin \{r_1, \ldots, r_t\}} \lambda_r + \sum_{s=m}^{s_t} \lambda_s' + \sum_{r=r_t+1}^{m} \lambda_r.
\]
For the operator \( w(\mu) \) we take the coefficient of \( z^{(\mu, \omega_i)} \) in \( \mathcal{V}(e^\mu, z) \). Since \( \omega_i = \lambda_1 + \cdots + \lambda_i \), we have
\[
\mu + \gamma_{r_1}s_1 + \cdots + \gamma_{r_t}s_t + \omega_i = \omega + \omega_i,
\]
17
Hence
\[ w(\mu)x(\pi_1)v_i = C e(\omega)v_{s_1}, \]
as desired. ■

Proposition 6 enables us to prove linear independence of the set
\[ \{x(\pi)v_i \mid x(\pi) \text{ satisfies IC and DC for } W(\Lambda_i)\}. \]

We prove this by induction on degree and on order of monomials. The proof is
carried out simultaneously for all level 1 standard modules by using coefficients of
intertwining operators.

Assume
\[ (25) \quad \sum c_\pi x(\pi)v_i = 0, \]
where all monomials \( x(\pi) \) satisfy difference and initial conditions for \( W(\Lambda_i) \) and
are of degree greater or equal to some \( -n \in \mathbb{Z} \). Fix \( x(\pi) \) in (25) and suppose that
\[ c_\pi = 0 \quad \text{for} \quad x(\pi') < x(\pi). \]

We want to show that \( c_\pi = 0 \).

By proposition 5 there exists an operator \( w(\mu) \) such that
\[ w(\mu)x(\pi_1)v_i = C e(\omega)v_{s_1}, \quad C \in \mathbb{C}^\times, \]
\[ x(\pi_2^+) \text{ satisfies IC and DC for } W(\Lambda_{i'}), \]
\[ w(\mu)x(\pi')v_i = 0 \quad \text{if} \quad x(\pi_1) > x(\pi_1), \]
where \( \Lambda_{i'} \) is another fundamental weight of \( \tilde{\mathfrak{g}} \). Applying the operator \( w(\mu) \) to (25) gives
\[ 0 = w(\mu) \sum c_\pi' x(\pi')v_i \]
\[ = w(\mu) \sum_{\pi_1' > \pi_1} c_\pi' x(\pi')v_i + w(\mu) \sum_{\pi_1' < \pi_1} c_\pi' x(\pi')v_i + w(\mu) \sum_{\pi_1' = \pi_1} c_\pi' x(\pi')v_i \]
The first sum becomes 0 after application of \( w(\mu) \), while the second sum is also
equal to 0 by the induction hypothesis. What is left is
\[ 0 = w(\mu) \sum_{\pi_1' = \pi_1} c_\pi' x(\pi')v_i \]
\[ = \sum_{\pi_1' = \pi_1} c_\pi' x(\pi_2^+) C e(\omega)v_{s_1} \]
\[ = C e(\omega) \sum_{\pi_1' = \pi_1} c_\pi' x(\pi_2^+)v_{s_1} \]
Since \( e(\omega) \) is injection, it follows that
\[ \sum_{\pi_1' = \pi_1} c_\pi' x(\pi_2^+)v_{s_1} = 0. \]
All monomials \( x(\pi_2^+) \) satisfy difference conditions because \( x(\pi') \) were such. If
some of them doesn’t satisfy initial conditions for \( W(\Lambda_{i'}) \), then the corresponding
monomial vectors \( x(\pi_2^+)v_{s_1} \) will be equal to 0. Certainly, \( x(\pi_2^+) \) won’t be among
those. We’ve ended up with a relation of linear dependence on the standard module
\( L(\Lambda_{i'}) \) in which all monomials are of degree greater or equal to \( -n + 1 \). By the
induction hypothesis they are linearly independent, and, in particular, \( c_\pi = 0 \). We
have proven

**Theorem 6.** Let \( L(\Lambda_i) \) be a standard \( \tilde{\mathfrak{g}} \)-module of level 1. Then the set
\[ \{x(\pi)v_i \mid x(\pi) \text{ satisfies IC and DC for } W(\Lambda_i)\} \]
is a basis of \( W(\Lambda_i) \).
10. BASES OF STANDARD MODULES

Knowledge of a basis of Feigin-Stoyanovsky’s type subspace $W(\Lambda)$ was used in [P1] and [P2] to obtain a basis of the whole standard module $L(\Lambda)$. We’re following here this approach to obtain a basis of a standard level 1 module $L(\Lambda_i)$, $i = 0, \ldots, \ell$, for any choice of $\mathbb{Z}$-gradation (1).

Set

$$e = \prod_{\gamma \in \Gamma} e^\gamma = e^{\sum_{\gamma \in \Gamma} \gamma}.$$

From lemma [4] and (24), we have

$$e = e^m \sum_{i=1}^n \lambda_i + (\ell - m + 1) \sum_{i=1}^\ell \lambda_i' = e^{(\ell + 1)\omega}.$$

The following proposition was proven in [P1] and [P2] (cf. [P1, Theorem 8.2.] or [P2, Proposition 5.2.])

Proposition 7. Let $L(\Lambda_i)_\mu$ be a weight subspace of $L(\Lambda_i)$. Then there exists an integer $n_0$ such that for any fixed $n \leq n_0$ the set of vectors

$$e^n x_{\gamma_1}(j_1) \cdots x_{\gamma_s}(j_s) v_i \in L(\Lambda_i)_\mu,$$

where $s \geq 0$, $\gamma_1, \ldots, \gamma_s \in \Gamma$, $j_1, \ldots, j_s \in \mathbb{Z}$, is a spanning set of $L(\Lambda_i)_\mu$. In particular,

$$L(\Lambda_i) = \langle e \rangle U(\tilde{\mathfrak{g}}_1) v_i.$$

Theorem 8. Let $L(\Lambda_i)_\mu$ be a weight subspace of a standard level 1 $\tilde{\mathfrak{g}}$-module $L(\Lambda_i)$. Then there exists $n_0 \in \mathbb{Z}$ such that for any fixed $n \leq n_0$ the set of vectors

$$e^n x_{\geq 1}(n) v_i \in L(\Lambda_i)_\mu, \quad x(\pi) \text{ satisfies IC and DC for } W(\Lambda_i),$$

is a basis of $L(\Lambda_i)_\mu$. Moreover, for two choices of $n_1, n_2 \leq n_0$, the corresponding two bases are connected by a diagonal matrix.

Proof: From proposition [7] and theorem [6] it follows that the set above indeed is a basis of $L(\Lambda_i)_\mu$. It is left to prove the second part of theorem.

In order to see this, we’ll find a monomial $x(\mu) \in U(\tilde{\mathfrak{g}}_1)$ and $f \in \mathbb{N}$ such that the following holds

(i) $e(\omega)^f v_i = C x(\mu) v_i$, for some $C \in \mathbb{N}$

(ii) $f$ divides $\ell + 1$,

(iii) $x(\mu) \text{ satisfies difference and initial conditions for } W(\Lambda_i)$,

(iv) if a monomial $x(\pi)$ satisfies difference and initial conditions for $W(\Lambda_i)$, then so does a monomial $x(\pi^{-f}) x(\mu)$, where $\pi^{-f}$ is a partition defined by $\pi^{-f}(x, (-n-f)) = \pi(x, (-n))$, $x, \pi \in \Gamma$, $n \in \mathbb{Z}$.

Then we’ll have

$$e(\omega)^f x(\pi) v_i = x(\pi^{-f} e(\omega)^f v_i = C x(\pi^{-f}) x(\mu) v_i.$$

Since $e^n x(\pi) v_i$ and $e(\omega) x(\pi) v_i$ are proportional, the second part of the theorem follows.

Let $x(\mu) \in U(\tilde{\mathfrak{g}}_1)$ be the maximal monomial satisfying difference and initial conditions for $W(\Lambda_i)$ such that its factors are of degree greater or equal to $-f$; we’ll determine the exact value of $f$ later. Let

$$x(\mu) = x_{p_1, q_1}(-n_1) x_{p_0, q_0}(-n_0-1) x_{p_2, q_2}(-n_2) x_{p_3, q_3}(-n_3) \cdots x_{p_\ell, q_\ell}(-n_\ell),$$

where factors are decreasing from right to left. The initial conditions imply

$$x_{p_1, q_1}(-n_1) = \begin{cases} x_{1, \ell}(-1), & \text{if } i = 0, \\ x_{1, \ell}(-2), & \text{if } i = m, \\ x_{i+1, \ell}(-1), & \text{if } 0 < i < m, \\ x_{1, i-1}(-1), & \text{if } m < i \leq \ell. \end{cases}$$
Difference conditions between \( x_{p_t,q_r}(-n_t) \) and \( x_{p_{t-1},q_{r-1}}(-n_{t-1}) \) give

\[
x_{p_t,q_r}(-n_t) = \begin{cases} 
  x_{p_{t+1},q_r}(-n_{t+1}), & \text{if } 1 \leq p_{t+1} < m < q_r - 1 \leq \ell, \\
  x_{1,q_r}(-n_{t+1} - 1), & \text{if } p_{t+1} = m < q_r - 1 \leq \ell, \\
  x_{p_{t+1},\ell}(-n_{t+1} - 1), & \text{if } 1 \leq p_{t+1} < m = q_r - 1, \\
  x_{1,\ell}(-n_{t+1} - 2), & \text{if } p_{t+1} = m = q_r - 1.
\end{cases}
\]

for \( 1 \leq t \leq r \).

Degrees of elements of \( x(\mu) \) are \(-1, -2, \ldots, -f\), respectively from right to left. Of course, it is possible that some successive elements are of the same degree, or that elements of a certain degree do not occur; according to the initial and difference conditions.

From the above observation we also see that row-indices of colors of elements are moving cyclicly over the set

\((1, 2, \ldots, m),\)

and column-indices are moving cyclicly over the set

\((\ell, \ell - 1, \ldots, m).\)

We'll choose \( f \) so that we stop when we make a “full circle” over both sets of indices. More precisely, we choose \( f \) so that the last element \( x_{p_r,q_r}(-n_r) \) of \( x(\mu) \) is

\[
\begin{align*}
x_{m,m}(-f + 1), & \quad \text{if } i = 0, \\
x_{m,m}(-f), & \quad \text{if } i = m, \\
x_{m,i}(-f), & \quad \text{if } i > m, \\
x_{i,m}(-f), & \quad \text{if } 0 < i < m.
\end{align*}
\]

Then \( r \) is equal to the smallest common multiple of \( m \) and \( \ell - m + 1 \). From \( (26) \) and proposition \( [3] \) it is clear that a monomial \( x(\pi) \) satisfies difference and initial conditions for \( W(A_t) \) if and only if \( x(\pi^{-1})x(\mu) \) satisfies them.

Denote by \( x(\mu_j) \) the \((-j)\)-part of \( x(\mu) \) if there are elements of degree \(-j\) in \( x(\mu) \), put \( x(\mu_j) = 1 \) otherwise. Suppose that \( x(\mu_j) \neq 1 \). Let \( \gamma_j \) be the color of the smallest element of \( x(\mu_j) \). Then at least one of the indices of \( \gamma_j \) is equal to \( m \). Denote by \( i_j \) the other index of \( \gamma_j \) (of course, \( i_j = m \) if \( \gamma_j = \gamma_{m,m} \)). In case \( x(\mu_j) = 1 \) set \( i_j = 0 \). From \( (26) \) it is obvious that \( i = i_j \). The same calculation as in the proof of proposition \( [3] \) shows that

\[
x(\mu_1)v_i = C_1 e(\omega)v_i,
\]

and

\[
x(\mu_1^{j-1})v_{i_{j-1}} = C_j e(\omega)v_i,
\]

for some \( C_1, \ldots, C_j \in \mathbb{C}^{\times} \). Hence

\[
x(\mu)v_i = x(\mu_1)f \cdots x(\mu_1)v_i = x(\mu_1)f \cdots x(\mu_2)v_i = \cdots = C e(\omega)^f v_i,
\]

for some \( C, C_1, C_j \in \mathbb{C}^{\times} \).

It remains to determine \( f \). If \( x(\mu_j) \neq 1 \) then \( x(\mu_j) \) contains exactly one element whose one of the indices is equal to \( m \). If \( x(\mu_j) = 1 \) then \( x(\mu_{j-1}) \) contains \( x_{m,m}(-j + 1) \). Hence number \( f \) counts how many times we’ve crossed over \( m \) while cyclicly moving over the sets of indices \((1, 2, \ldots, m)\) and \((\ell, \ell - 1, \ldots, m)\), i.e. \( f \) is equal to the total number of cycles we’ve made (over both sets of indices). Hence

\[
f = \frac{r}{m} + \frac{r}{\ell - m + 1} = \frac{\ell + 1}{m(\ell - m + 1)} = \frac{\ell + 1}{r'},
\]
where \( r' = \frac{m(\ell - m + 1)}{r} \in \mathbb{N} \). In particular, \( f \) divides \( \ell + 1 \).

As an illustration, we can take a closer look at the case \( m = 1 \). This is the case that was studied in [P1], for arbitrary level, and combinatorial conditions obtained there are the same as the ones that we’ve got. Here, \( \Gamma \) is a rectangle with 1 row and \( \ell \) columns, consisting of elements \( \gamma_1, \ldots, \gamma_\ell \). Fix a fundamental weight \( \Lambda_i \). A monomial \( x(\pi) \) satisfies initial conditions on \( W(\Lambda_i) \) if it doesn’t contain elements \( x_{1,i}(-1), \ldots, x_{\ell,i}(-1) \). If we assume that elements of \( x(\pi) \) are decreasing from right to left, then we can say that \( x(\pi) \) satisfies difference conditions on \( W(\Lambda_i) \) if for any two successive factors \( x_{1,i}(-j)x_{1,i}(-j') \) of \( x(\pi) \) we either have \( j \geq j' + 2 \), or \( j = j' + 1 \) and \( s < s' \). If we would write these conditions in terms of exponentials \( \pi(x_{1,i}(-j)), \gamma \in \Gamma, j \in \mathbb{N} \), we would obtain a special case of \((k, \ell + 1)\)-admissible configurations, for \( k = 1 \) (cf. [FJLMM], [T]).

We construct a periodic tail \( x(\mu) \) as in the proof of theorem\(^8\). We obtain

\[
x(\mu) = \begin{cases} 
  x_{1,1}(-\ell) \cdots x_{1,\ell-1}(-2)x_{1,\ell}(-1), & \text{if } i = 0, \\
  x_{1,1}(-\ell - 1) \cdots x_{1,\ell-1}(-3)x_{1,\ell}(-2), & \text{if } i = 1, \\
  x_{1,i}(-\ell - 1) \cdots x_{1,\ell}(-i - 1)x_{1,1}(-i + 1) \cdots x_{1,i-1}(-1), & \text{if } 2 \leq i \leq \ell.
\end{cases}
\]

which is the maximal monomial that satisfies initial and difference conditions on \( W(\Lambda_i) \) and has elements of degree greater or equal to \(-\ell - 1\). Since

\[
  x_{1,\ell}(-1)v_0 = C_0 e(\omega)v_\ell,
\]

\[
  x_{1,j-1}(-1)v_j = C_j e(\omega)v_{j-1}, \quad j = 2, \ldots, \ell,
\]

\[
  v_1 = C_1 e(\omega)v_0,
\]

for some \( C_0, \ldots, C_\ell \in \mathbb{C} \), we see that

\[
  x(\mu)v_i = Ce(\omega)^{\ell+1}v_i,
\]

for some \( C \in \mathbb{C} \).

Also, it is clear that a monomial \( x(\pi) \) satisfies initial and difference conditions on \( W(\Lambda_i) \) if and only if \( x(\pi - \ell - 1)x(\mu) \) satisfies initial and difference conditions on \( W(\Lambda_i) \).

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