Abstract

We study the two dimensional XY model with quenched random phases and its Coulomb gas formulation. A novel renormalization group (RG) method is developed which allows to study perturbatively the glassy low temperature XY phase and the transition at which frozen topological defects (vortices) proliferate. This RG approach is constructed both from the replicated Coulomb gas and, equivalently without the use of replicas, using the probability distribution of the local disorder (random defect core energy). By taking into account the fusion of environments (i.e charge fusion in the replicated Coulomb gas) this distribution is shown to obey a Kolmogorov’s type (KPP) non linear RG equation which admits travelling wave solutions and exhibits a freezing phenomenon analogous to glassy freezing in Derrida’s random energy models. The resulting physical picture is that the distribution of local disorder becomes broad below a freezing temperature and that the transition is controlled by rare favorable regions for the defects, the density of which can be used as the new perturbative parameter. The determination of marginal directions at the disorder induced transition is shown to be related to the well studied front velocity selection problem in the KPP equation and the universality of the novel critical behaviour obtained here to the known universality of the corrections to the front velocity. Applications to other two dimensional problems are mentionned at the end.

1 Introduction

1.1 Overview

Topological phase transitions in two dimensions, which are induced by the proliferation of topological defects, are naturally described via Coulomb gas
formulations. In the description of the Kosterlitz-Thouless transition of the XY model, the corresponding Coulomb gas (CG) with integer charges is obtained as an effective theory for the topological defects of the model: the vortices [1,2]. Similarly, a collection of dislocations whose proliferation induce the melting transition in a two-dimensional elastic lattice can be described by a Coulomb gas with vector charges which belong to the reciprocal lattice [3,4]. Finally, most of two-dimensional statistical models, such as the Ising, Potts and Askin-Teller models, can be transformed into Coulomb gases [5] (see also the review [6]). In all these cases, the transitions can be studied by renormalization in considering the screened interaction between two test Coulomb charges. This interaction is logarithmic at large distance $r$, and the renormalization procedure consists in neglecting the higher terms in an expansion in $1/r$ as being irrelevant [6,7]. This renormalization is thus valid for a dilute gas of charges, and is usually implemented by an expansion of the Coulomb gas partition function in powers of the charge fugacity $y$. This fugacity $y = \exp(-E_c/T)$ is related to the core energy $E_c$ of the charges (i.e. the local energy to create a defect or their chemical potential). Thus this Coulomb gas renormalization procedure is perturbatively controlled in the limit of large $E_c$, or equivalently at finite temperature in the limit of small fugacity. Several topological transitions in two dimensions have been successfully described using this CG technique [2,3,6].

Soon afterwards, several authors attempted to extend these techniques to models with quenched disorder [8–10]. The randomness in the original statistical models translates into random fields in the Coulomb gas formulation. The averaged free energy (instead of the partition function) of this Coulomb gas is then expanded, as in the pure case, in powers of the fugacity $y$ of the charges. Usual scaling techniques then allow to study the topological transitions in these disordered systems.

It does not seem to have been realized at that time that these approaches rely on the crucial assumption of a uniform fugacity for the charges in the sample, or equivalently of a core energy spatially uniform over the sample. Although this assumption is natural for pure models, it is at least questionable in the presence of a random potential [11]. The randomness may favor the appearance of topological defects (the charges of the CG) in some sites where the core energy will be effectively lower than on other sites. At high temperature, the randomness of the core energy $E_c$ is irrelevant as it is averaged out by thermal phase fluctuations. In that case the large scale behaviour of the model can indeed be described by simply considering the averaged fugacity over the sample $\bar{y}$, as in [8–10]. However, as we show here, when the temperature is lowered, the spatial inhomogeneities in the local core energy become more and more relevant. As a result, approaches based on a single uniform fugacity are doomed to fail. A correct detailed description of the scaling behaviour of the site-dependent fugacities (or core energies) becomes then necessary to de-
termine the phase diagram and describe quantitatively the transitions in the random model. It is the purpose of this work to define a novel renormalization method which allows this description in a consistent and perturbatively controlled manner. We present here a detailed analysis, a shorter account of the method and results has appeared in [12]. Although we focus here on the random phase XY model, these techniques can be applied to a much wider class of systems which can be formulated as disordered Coulomb gases (e.g. see Section 7 for a random Sine-Gordon formulation). In particular, extensions to the melting of two dimensional crystals in presence of disorder are studied in [13–15] and applications to the statistics of localized wave functions of electrons in random magnetic fields as well as entropic transitions in Liouville theory and other models are studied in [16].

1.2 Random phase XY model

In this paper, we focus on the 2d XY model with randomness in the phase (see eq. (1)), originally studied in [9] in the context of XY magnets with random Dzyaloshinskii-Moriya interactions. The topological defects (vortices) of this model are represented by integer charges \( n_r \) at sites \( r \). Two charges \( n_r, n_{r'} \) interact for a large separation via the usual Coulomb interaction of strength \( J \), with the corresponding energy \( -2J n_r n_{r'} \ln(|r - r'|/a) \) where \( a \) is the typical size of the core region of the vortices, i.e the short distance cut-off (as discussed in section 2). These charges also couple to a random potential \( V_r \), which arises from the randomness of the XY model, via a term \( n_r V_r \) which depends on the sign of the charge. The crucial property of this random potential is that it has long range logarithmic correlations:

\[
(V_r - V_{r'})^2 = 4 \sigma J^2 \ln |r - r'| + O(1)
\]

the on-site variance being \( V_r^2 = \Delta \sim 2 \sigma J^2 \ln L \) where \( L \) is the system size.

Rubinstein, Schraiman and Nelson [9] identified the most relevant vortices as the \( n = \pm 1 \) charges. They derived scaling equations for the stiffness \( J(l) \), the disorder strength \( \sigma(l) \) and a single fugacity for these charges \( y(l) \). As in the pure case, the nature of the phase is determined by the behaviour of the fugacity : in a quasi-ordered phase, \( y(l) \) decreases with the scale while in a disordered phase it increases. In this last case, vortices appear at the scale \( ae^{l^*} \) at which their renormalized core energy \( E_c(l) \) is about zero, which corresponds to a fugacity of order 1 : \( y(l^*) \simeq 1 \) (see e.g [17]). Interestingly the divergence of the correlation length at criticality found in [9], \( \xi \sim \exp(cste/|T - T_c|^{1/2}) \), is identical to the result of Kosterlitz-Thouless for the pure system. The phase diagram was found to be reentrant at low temperature, and is shown as a dashed line in fig. 1 as a function of the renormalized value of \( \sigma \) and \( J \).
Recently, several authors [18,19] have proposed a modified phase diagram for the same model, where the reentrance of the phase transition of fig. 1 disappears. The main point, further developed in [20], is an energy argument for a single vortex at zero temperature in a finite size sample, in the spirit of the Kosterlitz-Thouless argument for the transition in the pure model [2]. The energy to create a single defect has two main components (neglecting the bare core energy) : the elastic energy $E_{el} \approx J \ln L$ and the disorder energy. Indeed the defect can take advantage of the spatial variations of $V_r$ and choose the minimal value of $V_r$ in the $N = (L/a)^2$ sites of a sample of size $L$. To estimate this minimum $V_{min}$ of $V_r$ over $N$ sites, the above authors neglected the spatial correlations of $V_r$. Under that hypothesis, the problem of this single particle in the random potential $V_r$ becomes identical to the Random Energy Model (REM) defined and solved in the seminal work of Derrida [21]. In particular, the averaged $\overline{V_{min}}$ of the minimum behaves for large $N$ as $\overline{V_{min}} \simeq -\sqrt{2\Delta (\ln N)^{\frac{3}{2}}} \simeq -\sqrt{8\sigma J \ln L}$, where $\Delta$ is the onsite variance (see above). Adding the elastic contribution $E_{el}$ yields a creation energy $E \simeq J(1 - \sqrt{8\sigma}) \ln L$. Hence this simple single vortex argument points towards the existence of a topological phase transition at $\sigma = \sigma_c = \frac{1}{8}, T = 0$ where vortices proliferate. The modified phase diagram is shown as a solid line on figure 1.

Although appealing, this single vortex argument is not sufficient by itself to prove the existence of a phase transition in the system of many interacting charges, and even less so to describe the critical behaviour. Screening by mutual interactions usually plays an important role in Coulomb systems, and screening of the disorder could also in principle modify the results. The single vortex argument can only become valid asymptotically (at infinite scaling length) if there indeed exists a phase with finite renormalized values for $J$ and $\sigma$ and vanishing fugacity at zero temperature. The very existence of the XY phase in the present problem has been questioned in recent works [22] and a
further analysis is thus necessary. Thus, we cannot avoid the use of a renormalization approach to determine the true phase diagram. Such a controlled RG description should correctly take into account the correlations of the random potential, neglected in the above argument. It should allow to precisely characterize the universal features of the critical behaviour at the disorder driven transition.

Several attempts to construct a RG method were proposed [19,23–25] prior to this work, but no agreement was found between the results of these different approaches. As we discuss below, this is largely because none of these approaches was internally consistent. In particular, they were all based on expansions in dipole fugacities while considering single charge fugacities not only appears more natural but is essential in constructing a consistent renormalization procedure. As a result these approaches missed the very important contribution of the fusion of environments (see below). Korshunov and Nattermann started by approximating the free energy of a given disordered Coulomb gas by the sum of the free energies of independent dipoles \(^1\) of charges [23]. They found no renormalization of the disorder strength \(\sigma\) and a modified phase diagram in agreement with the figure 1. An interesting replica approach was developed by Scheidl [24], with the use of a Coulomb gas of \(m\)-component charges \(\mathbf{n}\), where \(m\) is the number of replica. Scheidl noticed that, in the renormalization procedure, one must a priori take into account charges with a number \(p > 1\) of non zero components, contrarily to [9] where only single component charges were considered. His RG equations are not compatible with the one of [23]: since the disorder strength \(\sigma\) is renormalized, it leads to a different phase diagram when expressed in the bare constants. Upon closer examination in Section 8 and interpretation within our formalism, the assumption implicitly underlying Scheidl’s work [24] appears to be the gaussian nature of the renormalized local disorder, while we are able to prove that the local disorder does not remain gaussian upon coarse-graining. Although we are also able to show a posteriori that in the XY phase some results are compatible between the two approaches, the non gaussian nature of the local disorder becomes crucial to determine the critical behaviour at the transition. Finally Tang [25] developed an appealing physical picture for the single vortex freezing phenomena in this model beyond the REM approximation mentionned above, and correctly foresaw the existence of a connection [25,26] with the problem of the Directed Polymer on the Cayley Tree (DPCT). However, no precise Coulomb gas renormalization procedure was developed following these ideas.

Finding a controlled RG procedure which allows to describe this physics of freezing in a collection of interacting Coulomb charges in a random environments is thus a non trivial and challenging problem. As we will see (and as can be expected from the above single charge argument), it requires a quan-

\(^1\) We will come back to this expansion in section 8
tative description of the scale dependence of the probability distribution of the fugacities associated with the rare sites were the charges are frozen.

Before embarking in the (sometimes technical) remainder of the paper, it is useful to depict the spirit of the method and summarize the main results.

1.3 Description of the methods

To capture the physics of freezing in a gas of Coulomb charges we find it crucial to start by realizing that upon increasing the size of the charges (cut-off $a$), their core energy acquires a random component from the potential $V_r$. Increasing $a$ to $\tilde{a} = a(1 + dl)$, this potential $V_r$ with logarithmic correlation splits into the rescaled logarithmically correlated potential $V^\triangledown_{\tilde{a}}$ at scale $\tilde{a}$, and a random part $v_r$ uncorrelated at scales larger than $\tilde{a}$. This local potential is naturally incorporated in the (renormalized) core energy $E_c$, which thus becomes random and site dependent: $E_c(a) \rightarrow E_c(\tilde{a}) = E_c(a) \pm v_r$. Thus, upon coarse graining, the fugacities of the $\pm 1$ charges of the Coulomb gas become random (and dependent on the sign of the charge): $z_{\pm} = y e^{\pm \beta v_r}$. Let us stress that the definition of $v_r$ (and thus of $z_{\pm}$) depends on details of the cutoff procedure. However the RG procedure developed here depends only on the logarithmic behaviour of the correlator of $V^\triangledown_{\tilde{a}}$ at large distances, which is cutoff independent. As a result, a remarkable universality will emerge at the end of the procedure, which happens to be related to universality of front solutions in non-linear equations (see below).

We are now faced with the description of the scale-dependence of a Coulomb gas with random fugacities, and thus a priori of the full corresponding fugacity distribution. Since we find that, at low temperature, the distribution of the local core energy does not remain gaussian under coarse graining we cannot restrict a priori the renormalization study to follow a small number of variables (such as the mean and the variance) as illustrated in figure 2. Instead, we must study the scale dependence of the full distribution $\tilde{P}(E_c)$ and especially the precise form of its tails around $E_c \lesssim 0$ (i.e $y \sim 1$) which control the low temperature physics (at $T = 0$ defects appear where the local core energy is negative). This requires new techniques in two dimensions.

Before developing a new RG procedure, we have to find a correct perturbative parameter to study the disorder driven transitions at low temperature. Around the pure topological transition, this parameter corresponds to the charge fugacity $y$ (for the most relevant charges $n = \pm 1$). However, we expect that, upon lowering the temperature, a freezing of the defects occurs. In that case, most sites have a large core energy ($y \sim 0$) while only a few sites with $E_c \lesssim 0$ are favorable to the defects. A natural choice for a perturbative pa-
An essential contribution to the renormalization of $P_l(z_+, z_-)$, absent in previous approaches, originates from what we call the fusion of random environments. The main idea is the following. Each random fugacities $z_{r\pm}$ is associated with a region of size $a$ around $r$. Upon increasing the cut-off, the size of these regions increases, and we thus have to merge two regions distant from less than the new cutoff $\tilde{a}$ (see fig. 3). The probability distribution of the fugacities in the new region can be deduced from the one in the two merged regions by a fusion rule (see fig. 3). The final renormalization of $P_l(z_+, z_-)$ can be formulated into a single differential equation for the distribution function $P_l(z_+, z_-)$ (given in Section 3). Most interestingly we find that the universal features at the transition of the model ($\sigma^R = \frac{1}{2}$) do not depend on the precise definition of these regions. The fusion of environments corresponds to a non-linear term in the differential equation, which can thus be affected by a change of cut-off procedure. Quite remarkably, the universality class at the new transition is determined by the properties of the front solutions of this differential equation (i.e their velocity), that do not depend on the precise form of this non-linear term. Hence universality in the usual RG sense finally appears from non trivial

2 Actually, the random fugacities $y_r$ depends on the sign of the charge: $z_{r\pm}$, and the perturbative parameter is the density $P(z_+ \sim 1) = P(z_- \sim 1)$ of sites favorable either to $+1$ charges or to $-1$ charges. One must thus follow the RG flow of the full probability distribution $P_l(z_+, z_-)$.

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**Fig. 2.** Schematic representation of the probability distribution $P(y)$ of the local fugacities $y$, plotted as a function of $-\ln y = \beta E_c$ (where $E_c$ is the renormalized core energy). At high temperature (left) it is narrow and a description in terms of the averaged fugacity is correct. At low temperature the distribution of $\beta E_c$ is broad and non gaussian and information about the whole distribution is a priori needed.
Fig. 3. Schematic representation of the fusion of two local environments upon coarse-graining. Two regions, distant from less than the new cut-off $\tilde{a}$, are merged when the cut-off is increased. The corresponding fusion rule for the distribution function of the local variables $z^r_{\pm}$ is shown on the figure.

results of front propagation in non-linear equations.

This differential RG equation for $P_l(z_+^r, z_-^r)$, together with the screening equations for $J$ and $\sigma$, can be obtained by a systematic and perturbatively controlled RG procedure. One formulation consists in using the replica trick to average the free energy over disorder, leading to a Coulomb gas with $m$-component charges. In this formulation, the infinite set of fugacities, associated with the vector charges $Y[n]$, encode the distribution function $P_l(z_+^r, z_-^r)$ of the random fugacities $z_{\pm}^r$. Hence, since this last distribution cannot be a pri-ori described by a finite number of moments at low temperature, we have to consider the scaling behaviour of the fugacities of all the replica charges with components $0, \pm 1$. The $m \to 0$ limit of this infinite set of RG equations is then taken by using an appropriate parametrization in terms of $P_l(z_+^r, z_-^r)$ which yields the non linear RG equation for $P_l(z_+^r, z_-^r)$. The second, equivalent formulation, does not rely on replica. It is constructed using a new expansion of physical quantities in the “number of independent regions”, introduced in this paper.

1.4 Disorder induced topological transitions and connection with DP

After some transformations, we reduce the non-linear RG equation that governs the scale dependence of the distribution $P_l(z_+^r, z_-^r)$ to the celebrated Kolmogorov-Petrovskii-Piscounov (KPP) equation (also named the Fisher equation). The known results on this equation allow us to derive the form of the tails of the distribution and thus to obtain the phase diagram. In particular, the velocity of the front solutions of the KPP equation directly determines whether the small parameter $P_l(y \sim 1)$ increases or decreases. Hence using known results on the selection of this velocity, we obtain the phase diagram of the figure 1 expressed in renormalized variables. The critical behaviour at the transition $\sigma = \frac{1}{8}$, follows from the finite size corrections to the velocity cor-
rections of the KPP front and defines a new universality class. In particular, the correlation length $\xi$ diverges at the $\sigma = \sigma_c = \frac{1}{8}, T = 0$ transition as

$$\xi \sim \exp\left(\frac{\text{cte}}{|\sigma - \sigma_c|}\right)$$

in contrast with the KT behaviour found in [9]. Note that recent numerical simulations [27,28] of this model seem to agree with the phase diagram of fig. 1 and it would be interesting to also determine numerically the precise critical behaviour at low temperature.

Besides the characterization of the new critical behaviour our work also enables to study the freezing of vortices, which occurs below a temperature $T_g$ (see figure 1). This freezing corresponds to a transition for the single charge problem, whose study with our new RG technique is presented in [16]. In particular, this renormalization approach draws a precise connection between the physics of freezing of the XY defects and the problem of branched processes (or of random directed polymers on Cayley trees (DPCT) for a discrete version) studied by Derrida and Spohn [29]. This connection naturally emerges from our Coulomb gas RG equations via the KPP equation which has also appeared in the exact solution of the DPCT problem [29]. It does not rely on any ad-hoc construction.

The paper is organized as follows: in Section 2, the random XY model is defined and its CG formulation is carefully derived. In particular, the relation between the continuum limit and the decomposition of the random potential $V_r$ into a local part (random core energy) and a long-range potential is discussed in section 2.2. Part 3 describes the renormalization method of the replicated Coulomb gas, and while a direct method (without replica), which consists in expanding the free energy into the number of independent regions (random environments) is presented in Section 4. The RG equations are analyzed in Section 5 using results on the propagation of KPP-like fronts. The consequences for the determination of the phase diagram and the critical behaviour is detailed in Section 6. A formulation in terms of a Sine-Gordon model is given in Section 7. The comparison with previous approaches is postponed to Section 8, and most of the technical details can be found in the appendices.

2 XY Model with random phases
2.1 lattice Coulomb gas

In this Section and in the following we study the XY model with random phases [9]. We start with the model defined on the 2D square lattice by its partition function:

$$Z[A] = \prod_i \int_{-\pi}^{\pi} d\theta_i \ e^{-\beta H[\theta_i, A]} \quad \text{with} \quad H[\theta, A] = \sum_{\langle i,j \rangle} V(\theta_i - \theta_j - A_{ij})$$ (1)

where the sum is over pairs of nearest neighbors (i.e over bonds) on the lattice and $\beta = 1/T$ is the inverse temperature. The $A_{ij}$ are random gauge fields, independent from bond to bond, each with gaussian distribution of variance $\overline{A_{ij}^2} = \pi \sigma$. The periodic potential $V(\theta)$ is defined for the XY model by $V(\theta) = -\frac{1}{\pi} J \cos(\theta)$ where $J$ is the stiffness. In the limit $\sigma \to \infty$ this model corresponds to the 2D “gauge glass model” [30,31]. For finite $\sigma$ it was studied in Ref. [9,25,24,23].

The standard way to study this model is to decompose it into spin waves and vortex degrees of freedom $Z[A] = Z_{sw}Z_{CG}$. This decomposition can be performed exactly (see Appendix A) for the corresponding Villain model defined by the potential

$$e^{-\beta V(\theta)} = \sum_{p=-\infty}^{+\infty} e^{-\frac{2\pi^2}{\sigma^2}(\theta - 2\pi p)^2}$$

Technically this is the model which we study here. It is reasonable however to expect that this Villain model and the XY model should be in the same universality class (as is the case without disorder). The RG analysis contained in the following Sections is consistent with this assumption [3]. The vortex part is described in terms of a Coulomb gas with integer charges $n_r$ defined on the sites $r$ of the dual infinite (square) lattice:

$$Z_{CG} = Z_{latt} = \sum_{\{n_r\}} e^{-\beta H}$$

$$H = -\frac{1}{2} \sum_{r \neq r'} 2J n_r G_{r-r'} n_{r'} - \sum_r n_r V_r$$

where $G_{r-r'} = \int_{\mathbf{k}} G_{\mathbf{k}} (1 - e^{i\mathbf{k}(r-r')})$ is the 2D Coulomb potential with $G_{r-r'} \approx \ln |r - r'|$ at large distance. $G_{\mathbf{k}}^{-1} = \frac{1}{\pi} [2 - \cos(k_x a_o) - \cos(k_y a_o)]$ is the lattice Laplacian and we denote $f_{\mathbf{k}} \equiv \int_{-\pi/a_o}^{\pi/a_o} \int_{-\pi/a_o}^{\pi/a_o} \frac{dk_x dk_y}{(2\pi)^2}$ where $a_o$ is the lattice spacing. Note that the energy associated with a dipole of unit charge of size $r$ is

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3 It can be shown to lead to unimportant additional random terms in the bare fugacity.
As discussed in the Appendix A we can consider a neutral CG (since $\int q = 0$) with $\sum r n_r = 0$, and neutrality has already been used to arrive at (2).

In the vortex representation the random gauge fields $A$ translate into random dipoles (along $z$) $q_r' = \frac{1}{2\pi} \nabla r' \times A$ which couple to the vortex charges via the Coulomb potential. As detailed in Appendix A this results in a gaussian bare disorder potential $V_r = -2J \sum_{r'} G_{r-r'} q_{r'}$. An important feature of this problem is that the disorder potential seen by the charges (the vortices) has logarithmic long range correlations

$$\left( V_r - V_{r'} \right)^2 = 4\sigma J^2 \ln |r - r'| + O(1) \quad (3)$$

for $r \neq r'$, since $V_k V_{-k} = 2\sigma J^2 G_k$. Note that at a given point $V_r^2 \approx 2\sigma J^2 \ln L$ where $L$ is the system size and that up to now, these are exact transformations.

Before defining the continuum limit of this Coulomb gas, we note that an alternative definition to this lattice Coulomb gas consists in labelling the non zero charges in (2) by their positions $r_i$ and their corresponding charge $n_i$. Instead of the integer field $n_r$ of (2) defined at each site of the lattice, a configuration is represented by the set $\{n_i, r_i\}$. We obviously have $n_r = \sum_i n_i \delta_{r, r_i}$. With this representation, the partition function of (2) reads

$$Z_{\text{latt}} = 1 + \sum_{p>0} \sum_{\{n_1, \ldots, n_p\}} \sum_{r_1 \neq \cdots \neq r_p} e^{\beta J \sum_{r_i \neq r_j} n_i G_{r_i-r_j} n_j + \beta \sum_i n_i V_{r_i}} \quad (4)$$

where $G_{r_i-r_j}$ and $V_{r_i}$ have been defined after eq. (2) and the sum over the charge configurations (primed sum) counts each distinct neutral configuration of non zero charges only once\footnote{This leads to the factor $1/\prod_n N(n)!$ in the definition of the configuration sum of the CG, where $N(n)$ is the total number of particles of charge $n$}.

### 2.2 continuum limit and decomposition of the disorder

In order to implement a renormalization procedure one first needs to introduce a continuum version of this model. In the usual approach to 2D Coulomb gas the continuum limit is obtained by replacing the lattice Coulomb interaction by the approximation $[2,32]$

$$G_{r-r'} \approx G_{r-r'}^{\text{app}} = \left( \ln \left( \frac{|r-r'|}{a_o} \right) + \gamma \right) \left( 1 - \delta^{(a_o)}_{r-r'} \right) \quad (5)$$
where $\delta_{r,r'}^{(a_o)} = 1$ for $|r - r'| < a_o$ and 0 otherwise, and $\gamma = \ln(2\sqrt{2}e^C)$ with $C = 0.577216$ is the Euler constant. This approximation is excellent [33] on the lattice for $|r - r'| \gtrsim a_o$. In the standard method the continuum CG model is then defined by considering a gas of integer hard core charges $n_r$ at point $r_i$ of diameter $a_o$ which interact with $G_{r-r'}^{app}$. Using neutrality $\sum_r n_r = 0$ and (5), allows to rewrite [2,32] the hamiltonian (4) as a sum of a simple logarithmic interaction $-J \sum_{i \neq j} n_r \ln \left( \frac{|r_i - r_j|}{a_o} \right) n_{r'}$ between the hard core charges and a fugacity term $\ln y \sum_r n_r^2$ of bare value $y = e^{-\beta E_c}$ where $E_c = \gamma J$ can be interpreted as the bare core energy for the defects of the model (2).

In presence of disorder, special care has to be taken to define properly the continuum limit for the random potential, since its correlator is logarithmic. Since on the lattice the correlator of the disorder $G$ is the same as the Coulomb interaction, it is consistent to use the same $G^{app}$ for the disorder in the continuum model. This immediately leads to the fact that the disorder $V_r$ must be separated in two parts, using (5): a long range correlated gaussian part $V_r^>$ and a local part $v_r$:

\[
V_r = V_r^> + v_r \\
(V_r^> - V_r'^>)^2 = 4\sigma J^2 \ln \left( \frac{|r - r'|}{a_o} \right) (1 - \delta_{r,r'}^{(a_o)}) \\
v_r v_r' = 2\sigma J^2 \gamma \delta_{r,r'}^{(a_o)}
\]

with no cross correlation. Using this decomposition we can now write the partition function of the continuum model:

\[
Z_{\text{cont}} = 1 + \sum_{p=2}^{\infty} \sum_{n_1, \ldots, n_p} \int_{|r_i - r_j| \geq a_o} \frac{d^2 r_1}{a_o^2} \ldots \frac{d^2 r_p}{a_o^2} e^{-\beta H[n,r]} \\
H[n, r] = -J \sum_{i \neq j} n_i \ln \left( \frac{|r_i - r_j|}{a_o} \right) n_j - \sum_i n_i V_i^> - \sum_i \ln Y[n_i, r_i]
\]

where the primed configuration sum counts only once each distinct neutral charge configuration. We have introduced the spatially dependent fugacity, of bare value, from (5,6):

\[
\ln Y[n, r] = -\gamma \beta J n^2 + \beta n v_r
\]

Thus we find that disorder favors some regions resulting in a local fugacity for $\pm 1$ charges $y(r) = e^{-\beta E_c(r)}$ with a core energy $E_c(r) = E_c \pm v_r$ which now varies from point to point. Thus one anticipates that problems will arise in the conventional fugacity expansion if $y(r)$ varies strongly from point to point.
In addition, note that the local fugacities are different for $+$ charges and $-$ charges (although there is still a statistical $+|-$ symmetry).

We have thus defined a continuum model with a particular cutoff procedure. This is apparent, e.g. in the decomposition of the disorder that we have used. This decomposition is cutoff dependent and we have taken care to choose the same cutoff procedure for the disorder term and the interaction. We have chosen here a real space hard cutoff procedure, which is often used in CG studies. Other cutoff procedures can be used. It is natural to expect, and we will partially verify that the large scale results will not depend on the particular procedure chosen.

The alert reader will have already noticed that the disorders $V_r^>$ and $v_r$ as defined above in (6) are not strictly speaking physical, since the Fourier transform of their respective correlators is not positive. This is an artefact of this particular choice of a real space hard cutoff. It is not a serious problem, and is easily cured by choosing instead a cutoff in Fourier space. The resulting decomposition of disorder is then completely legitimate. This is further explained in Appendix B. For simplicity, we will however proceed using the above cutoff choice, which has illustrative value, keeping in mind that the more legitimate choice detailed in Appendix B can be used instead, completely equivalently at all stages, for technical rigor.

We now turn to the renormalization of the model.

## 3 Renormalization using replica

In this Section we study, using replica, the renormalization group properties of the disordered Coulomb gas defined by (2). In the present case the replica method is particularly convenient in order to perform the combinatorics necessary to renormalize consistently the model.

The strategy is first in (3.1) to transform the model (2) into a vector Coulomb gas with $m$-replica charges. The fugacities of these $m$-vector charges will then naturally encode the distribution of the spatially dependent fugacities defined above (see (9)). The renormalization group (RG) equations for these fugacities are derived in (3.2) for fixed $m$. By a suitable parametrization, in (3.3) we then extract in the $m \to 0$ limit the RG equations which describe the scale dependence of the full distribution $P(z_+, z_-)$ of the local fugacities of the topological defects in the original disordered model (2), as well as the scale dependence of the stiffness $J$ and long wavelength disorder $\sigma$. 
3.1 Replica Coulomb gas, continuum limit and fugacity expansion

We start again from the model (2) on a square lattice. As is well known, disordered averaged correlation functions and free energy can be obtained by studying the replicated partition function (generating function) $Z_{\text{lat}}^m$ in the limit $m \to 0$. For integer $m$, $Z_{\text{lat}}^m$ can then be written exactly as a CG with $m$-component vector charges $n_a^r$, $a = 1,..,m$ (each $n_a^r$ is integer), living on the sites of the lattice (note that $a$ denotes a replica index while ",a_o" denotes the cutoff). Averaging over the bare disorder one obtains the partition sum of a fully coupled, translationally invariant, vector Coulomb gas on a square lattice: $Z_{\text{lat}}^m = \sum \{n_a^r\} e^{-\beta H_{\text{lat}}^{(m)}}$ with

$$\beta H_{\text{lat}}^{(m)} = - \sum_{r \neq r'} K_{ab} n_a^r G_{r-r'} n_b^{r'}$$

(10)

where $K_{ab} = \beta J \delta_{ab} - \sigma \beta^2 J^2$ and summation over repeated replica indices is assumed unless otherwise specified.

The next step is to approximate the lattice replica model (10) by a continuum Coulomb gas with $m$-component vector charges. In the following we consider a hard core cutoff in real space. The problem of the choice and consequences of the cutoff procedure is rather subtle here and will be discussed below. Using the approximate propagator (5) we obtain the continuum hamiltonian

$$\beta H_{\text{cont}}^{(m)}[n, r] = - \sum_{i \neq j} K_{ab} n_a^i G_{r_i-r_j} n_j^b$$

$$= - \sum_{i \neq j} K_{ab} n_a^i \ln \left( \frac{|r_i-r_j|}{a_o} \right) n_j^b - \sum_i \ln Y[n_i]$$

(11a-b)

where the charge $n_i$ is located in $r_i$. In the second equality we have used the neutrality of the Coulomb gas, i.e. $\sum_r n_a^r = 0$ for each $a = 1,..,m$, to introduce the local fugacity $Y[n]$ which is a function of the whole set of components of the vector charge $n = (n^1,..,n^m)$. Its bare value from (5) is a simple quadratic function [24]:

$$Y[n]_{\text{bare}} = e^{-n_a \gamma K_{ab} n_b}$$

(12)

This quadratic form results from the Gaussian nature of the bare local disorder and corresponds to (9) in the unreplicated version. If this form was preserved by the RG, as was implicitly assumed in [24], one would be able to study the model using only two coupling constants. However this is not the case. As shown below, the vector charge fugacity $Y[n]$ has a non trivial flow under RG
and does not remain purely quadratic. The local disorder does not remain gaussian and we will have to follow its full probability distribution.

We now study the scale dependent properties of the $m$-component vector Coulomb gas using the expansion in the vector charge fugacity $Y[n]$. Although it is the natural way to study the renormalization of a vector Coulomb gas, it may seem at this stage somewhat formal. This is not so however since, as will become clear below, it turns out to correspond exactly to the expansion in the number of rare favorable regions of local disorder, which is the physically relevant (and novel) expansion for this model. For the replicated partition function in the continuum model this vector fugacity expansion reads:

$$Z^m = 1 + \sum_{p=2}^{\infty} \sum_{n_1, \ldots, n_p} \int_{|r_i - r_j| \geq a_o} d^2r_1 \ldots \frac{d^2r_p}{a_o^2} e^{-\beta H^{(m)}_{\text{cont}}[n,r]}$$  \hspace{1cm} (13)

which contains fugacities via (11) and the primed sum over charge configurations counts each distinct neutral configuration only once. In presence of disorder, infrared divergences appear everywhere in the low temperature XY phase [34]. To treat these divergences we now turn to the RG method.

### 3.2 RG equations

We perform the RG analysis of the present $m$-component vector Coulomb gas on the partition function $Z^m$. It is a simple extension of the analysis for the scalar Coulomb gas [2,6]. Details are presented in the Appendix C and we only sketch the method here. For any fixed $m$ it is possible to leave the form of the expansion (13) unchanged under the increase of the hard core cut-off $a_o \rightarrow a_o e^{dl}$ provided one defines scale dependent coupling constants $K^{ab}_l$ and fugacities $Y_l[n]$. This corresponds to the (one loop) renormalizability of the $m$-component vector model, which we checked here to order $Y[n]^2$. The RG flow equations which determine these couplings are found as (see Appendix C):

$$\partial_t (K_l^{-1})_{ab} = c_1 \sum_{n \neq 0} n^a n^b Y[n] Y[-n]$$  \hspace{1cm} (14a)

$$\partial_t Y[n] = (2 - n^a K_{ab} n^b) Y[n] + c_2 \sum_{n' + n'' = n, n', n'' \neq 0} Y[n'] Y[n'']$$  \hspace{1cm} (14b)

with $c_1 = 2\pi^2$, $c_2 = \pi$ for our hard cut-off procedure, and the second equation (14b) is defined only for $n \neq 0$. The first equation (14a) comes from the annihilation of dipoles of opposite replica vector charges separated by $a_o \leq |r_i - r_j| \leq a_o e^{dl}$. It gives the renormalization of the interaction (screening by
small dipoles) and of the disorder. Simple rescaling gives the first term of the second equation (14b), i.e. the naive scaling dimension of $Y^{|n|}$.

The second contribution in (14b) comes from the possibility of fusion of two replica vector charges upon coarse graining (see fig. 4). While such term can be neglected in the (pure) scalar Coulomb gas (as it yields less relevant operators) it is usually crucial when studying most vector CG models, as e.g. in the analysis of two dimensional melting transition [3]. Indeed following too closely the analysis for the pure XY model has led previous studies [9,19,25,24,23] to miss the possibility of fusion and thus such a contribution. As we see in the following it has important and non trivial consequences for the physics of the low T phase and the transition. For the present disordered Coulomb gas, contrarily to the conventional analysis [3], one cannot hope to capture the most relevant operators by restricting to single component charges (e.g. $n^a = \pm \delta_{aa_1}$). This was recently emphasized by Scheidl [24]. However, since this leads to considering multicomponent vector charges, it is thus crucial to treat properly this fusion term, which was not done previously (e.g. in [24]). Moreover, discarding this term in (14b) leads to a set of RG equations which is not consistent to their lowest order $O(Y^{|n|^2})$. This term may a priori modify the scaling dimensions in a non trivial way in the $m \to 0$ limit, and it is thus crucial to study carefully its effect.

Before doing so in the next Section, let us give for completeness the renormal-
ization of the free energy density per replica, defined as $f_m = a^2 L^{-2 \frac{1}{m}} \ln Z^m$. It reads:

$$\partial_t f_m = 2f - \pi^2 \frac{T}{m} \sum_{n' \neq 0} Y[n']Y[-n']$$

from which the flow of the free energy density can be obtained as $f = \lim_{m \to 0} f_m$.

### 3.3 limit $m \to 0$ and fusion-diffusion formalism

We now have to find an analytical continuation to $m \to 0$ of the whole set of RG equations (14). This is a priori a formidable task because (14) are in fact, for arbitrary $m$, an infinite set of coupled equations. Remarkably, in the process of performing this analytical continuation, an appealing physical interpretation in terms of probability distributions of local fugacities (local disorder: see Section 2.2) will emerge naturally and be our guide in the following.

#### 3.3.1 parametrization of the fugacities and analytical continuation

As a first step, we will consider only charges with components $n^a = 0, \pm 1$ in each replica. How to incorporate higher charges (e.g. $n^a = \pm 2 \ldots$) is discussed in Appendix I, where it is shown that they are less relevant in the region of the phase diagram studied here. We first remark that the possible forms of the $Y[n]$ are severely constrained. Replica permutation symmetry, which we will assume here and is preserved by the RG, together with $n^a = 0, \pm 1$ implies that $Y[n]$ depends only on the number $n_+ = n_+^{+1/−1}$ of +1/−1 components of $n$. A natural possible parametrisation of $Y[n] \equiv Y[n_+, n_-]$ consists in introducing a function of two arguments $\Phi(z_+, z_-)$ such that:

$$Y[n] = \langle z_+^{n_+} z_-^{n_-} \rangle = \langle \prod_a [\delta_{n^a, 0} + z_+ \delta_{n^a, +1} + z_- \delta_{n^a, −1}] \rangle \Phi$$

where we denote $\langle A \rangle\Phi(z) = \int_{z_+, z_-} A \Phi(z_+, z_-)$. Our strategy is to establish an RG equation for $\Phi(z_+, z_-)$ (in the limit $m \to 0$) whose solutions $\Phi_t(z_+, z_-)$ will parametrize solutions $Y_t[n]$ of (14a, 14b).

Let us now examine how (14b) can be transformed in an integro-differential equation for $\Phi(z_+, z_-)$. The technical details are given in Appendix D. The first terms in the r.h.s. of (14b) translate into a differential operator $(2 + O)\Phi$ where:
Using new “core energy” variables \( u, v \) such that \( z_\pm = e^{\beta(u \pm v)} \), and the corresponding function \( \tilde{\Phi}(u, v) \) such that \( \tilde{\Phi}(u, v) dudv = \Phi(z_+, z_-) dz_+ dz_- \), it can be interpreted as a diffusion process since the first term of (14b) now translates into \( 2 + \tilde{\mathcal{O}} \) with \( \tilde{\mathcal{O}} = J \partial_u + \sigma J^2 \partial_v^2 \).

To deal with the second term we first extend the RG equation (14b) so as to allow for zero charge \( n = 0 \), since it is easier to continue analytically unrestricted sums. After some combinatorics (see appendix D), we find that using the representation (16), (14b) can be rewritten completely equivalently in terms of \( \Phi \) as:

\[
\partial_l \Phi(z_+, z_-) = (2 + \mathcal{O}) \Phi(z_+, z_-) - 2c_2 N \Phi(z_+, z_-) + c_2 N^2 \delta(z_+) \delta(z_-) 
\]

(18)

\[
+ c_2 \int_{z_1, z_2, z'_+, z'_-} \Phi(z'_+, z'_-) \Phi(z''_+, z''_-) 
\times \delta \left( z_+ - \frac{z'_+ + z''_+}{1 + z'_+ z''_+ + z'_- z''_-} \right) \delta \left( z_- - \frac{z'_- + z''_-}{1 + z'_- z''_- + z'_+ z''_-} \right)
\]

where \( N = \int_{z_+, z_-} \Phi(z_+, z_-) \). This equation describes the scale dependence (\( l \)) of the function \( \Phi_l(z_+, z_-) \) which parametrizes the whole set of scale dependent fugacities \( Y_l[n] \) in the limit \( m \to 0 \).

### 3.3.2 RG equations and fugacity distribution

Up to now the function \( \Phi(z_+, z_-) \) has been introduced as a generating function to parametrize the fugacities \( Y[n] \). It is a priori an arbitrary function and in particular \( \mathcal{N} = \int \Phi \) is still undetermined. In this paragraph we will exchange \( \Phi(z_+, z_-) \) for a physical function \( P(z_+, z_-) \) of norm unity, which will be interpreted in the following as the probability distribution for the local fugacities \( z_+, z_- \) of \( \pm 1 \) charges. We start from the above equation (18) for \( \Phi(z_+, z_-) \) which can be simply interpreted as describing the sum of two processes. Defining from the random fugacities \( z_\pm = e^{\beta(u \pm v)} \) the random core energy variables \( E_c^\pm = -(u \pm v) \), the first process in (18) corresponds to a brownian diffusion for \( v \) (i.e the local disorder potential as in (9)) together with a convection for \( u \). The second process involves a fusion, with a rate \( c_2 \) upon increase of the cutoff, of two sets of random variables \( (z'_+, z'_-), (z''_+, z''_-) \) into a single one \( (z_+, z_-) \) according to the transformation law:

\[
(\{z'_+\}; \{z''_+\}) \rightarrow z_\pm = \frac{z'_+ + z''_+}{1 + z'_- z''_- + z'_+ z''_-} \]

(19)

as in a \( A + A \to A \) reaction. The term \(-2c_2 N \Phi \) in (18) corresponds to a loss of two charges, while the last term corresponds to a creation of the fused
one. The term \( \delta(z_-)\delta(z_+) \) keeps track of the “dead charges” which disappear by setting them to 0 (since they decouple from the system). It is in a sense only a counting device, since by construction \( \mathcal{N} = \int \Phi \) is unchanged upon fusion. We thus introduce \( \Phi_>(z_+, z_-) \) restricted to \( z_+ > 0, z_- > 0 \), such that \( \Phi = \Phi_> + (\mathcal{N} - \mathcal{N}_>)\delta(z_-)\delta(z_+) \) where \( \mathcal{N}_> = \int_{z_+ > 0, z_- > 0} \Phi \) is the total weight of non zero charges. Integrating (18) over \( z_+ > 0, z_- > 0 \) we obtain that:

\[
\partial_l \mathcal{N}_> = 2\mathcal{N}_> - c_2 \mathcal{N}_>^2
\]  

(20)

Thus in the presence of fusion it converges quickly towards \( \mathcal{N}_>^* = 2/c_2 \).

Since \( \mathcal{N}_> = \int \Phi_> \) converges to a constant, this suggests to introduce a normalized function \( P(z_+, z_-) = \Phi_> / \mathcal{N}_>^* \). As shown below it is natural to interpret \( P(z_+, z_-) \) as a probability distribution. From (18) we find that it obeys the following RG equation:

\[
\partial_l P(z_+, z_-) = \mathcal{O}P - 2P(z_+, z_-)
\]

\[
+ 2 \left( \delta(z_+) \frac{z'_+ + z''_+}{1 + z'_+ z''_+ + z'_+ z''_-} \right) \delta(z_-) \frac{z'_- + z''_-}{1 + z'_- z''_- + z'_+ z''_-}
\]

(21)

where \( \langle .. \rangle_{PP'} \) denotes \( \int_{z'_+, z'_-, z''_+, z''_-} P(z'_+, z'_-)P(z''_+, z''_-) \) and the probability conserving diffusion operator \( \mathcal{O} \) has been defined in (17).

The limit \( m \to 0 \) of the other RG equations (14a) which give the renormalization of the stiffness \( J \) and disorder strength \( \sigma \) is performed in Appendix D using \( \Phi \). Reexpressed in terms of \( P \) they read:

\[
T \frac{dTJ^{-1}}{dl} = \frac{4c_1}{c_2^2} \left( \frac{z'_+ z''_- + z'_- z''_+ + 4z'_+ z''_+ z'_- z''_-}{(1 + z'_+ z''_+ + z'_- z''_-)^2} \right)_{PP}
\]

(22a)

\[
\frac{d\sigma}{dl} = \frac{4c_1}{c_2^2} \left( \frac{(z'_+ z''_- - z'_- z''_+)^2}{(1 + z'_+ z''_+ + z'_- z''_-)^2} \right)_{PP}
\]

(22b)

where we have chosen (arbitrarily) to keep \( T \) fixed and renormalize \( J \) (only the combination \( K = \beta J \) flows).

The above formulae (21,22) forms our complete set of RG equations. As will become clear in the following Sections, \( P(z_+, z_-) \) represents the distribution of the fugacities \( z_+, z_- \) of local environments and the last term in (21) corresponds to fusion of environments upon coarse graining. Remarkably, once expressed in terms of \( P \) the coefficients of the above RG equations exhibit some universality. The factor of 2 in the last term in (21) arises from the fraction of environments \( \partial V/V = 2 \) which are fused when increasing the cutoff. Note also that the coefficient \( c_1/c_2^2 \) which naturally appears in (22) is not affected by a
uniform rescaling of the fugacities. Note that some features of these equations are cutoff dependent, as will be discussed in a following Section.

Finally, the flow of the free energy density is found to be:

$$\partial_t f = 2f - T\pi^2 \left\langle \ln(1 + z'_+ z''_+ + z'_- z''_-) \right\rangle_{\Phi}$$

$$= 2f - 4T \left\langle \ln(1 + z'_+ z''_+ + z'_- z''_-) \right\rangle_{PP}$$

(23)

These RG equations will be studied in Section 5. First we will present another renormalisation procedure, without replicas. Although it is technically more difficult to implement, it allows for a more direct physical interpretation, which in turns sheds some light on the more systematic replica method presented above. In addition it may be more appealing to the replicaphobic Section of the community.

3.4 validity of the method and universality

The above method which relies on an expansion in the vector fugacities $Y[n]$ can be justified provided there are few vector charges in the system i.e in the dilute limit. Even though the $Y[n]$ may appear as formal fugacities, the above RG equations can be justified in an expansion of the exact renormalised potential $G^R(r)$ seen by two test charges distant of $r$ in $\alpha_o/r$, as was emphasized by Nienhuis [6]. It is even claimed to be exact [6] in that limit. As will become apparent in the following Section the physical meaning of this diluted limit of vector charges exactly corresponds to the limit of a small density of regions favorable to the creation of frozen defects which is the physically relevant limit in the regimes studied in this paper.

To understand how cutoff dependence comes in the method used here, it is instructive to study the limit of zero disorder. One can indeed check that one recover the usual results in the limit of the pure case $\sigma = 0$. This however, requires some careful consideration of the cutoff procedure for the vector CG representation. As discussed in the Appendix F, even in the pure case the distribution $\Phi(z_+, z_-)$ which parametrizes the vector fugacities solution of the CG RG equations can be non trivial. It does satisfy $P(z_+, z_-) \sim \delta(z_+ - z_-)$ but the fugacity $z_+ = z_- = y$ still has a non trivial ”distribution” $\Phi(y)$ for a generic choice of cutoff. There is no paradox there and it is compatible with the

$$-hY[n] \sum_{n' \neq 0} Y[n']Y[-n']$$

with $h = \pi(\frac{3}{2}\pi + \sqrt{3})/2$ in the second equation. We omitted this term as we found that it vanishes in the limit $m \to 0$.
standard Korsterlitz-Thouless RG equation of the pure case, as is explained in Appendix F, universality being recovered at small fugacity.

4 Direct method of renormalisation

In this Section we introduce a method to study the model (2) and more generally Coulomb gas with disorder without using replicas. We start in Section 4.1 from the general motivation and rederive the RG equation for $P(z_+, z_-)$ in (21) in a more physical way. We also identify the small parameter which allows to study perturbatively the present problem. In Section 4.2 we introduce a quantitative and systematic method to expand in this small parameter. The direct renormalization approach using this expansion is performed in Section 4.3. The connection with the replica method of Section 3 is presented in Section 4.4.

4.1 RG method for broad disorder: physical derivation

Let us first explain the spirit of the direct method and illustrate how one is led to the RG equation (21), derived more quantitatively in the next Sections. We have seen in Section 2.2 that the local disorder $v_r$ defines the site dependent fugacities. We concentrate on $\pm 1$ charges for which these fugacity variables read (see (9))

$$z_r^\pm = y_r \exp(\pm \beta v_r)$$  \hspace{1cm} (24)

and are quenched random variables with only short range spatial correlations. One now studies the system under a change of cutoff $a_o \rightarrow a_o e^{dl}$ (coarse graining) which includes an integration over the corresponding degrees of freedom. We find that the coarse grained model remains of the same form as the original one, with a renormalized stiffness $J_l$, a renormalized gaussian long range disorder strength $\sigma_l J_l^2$ and a local disorder distribution $P_l(z_+, z_-)$. Note that, although the bare local disorder $v_r$ is gaussian, it becomes non gaussian under coarse graining. This is a novel feature of the present approach, at variance with previous attempts at renormalizing the model [23,25,24]. It complicates the analysis but is necessary to capture correctly the physics of the model which is driven by the rare events.

The RG equations (21, 22) for the fugacity distribution $P_l(z_+, z_-)$ of local environments (higher charges are less important and considered later), for the stiffness $J_l$ and for the correlated disorder strength $\sigma_l$ can be understood
from the following considerations. The correction to the fugacity distribution $P_l(z_+, z_-)$ is the sum of two contributions:

(i) \textit{rescaling}: the first observation is that upon changing the cutoff, as can be seen from its correlator (6) and is detailed below, the long range disorder $V^>$ produces an additional local disorder contribution which can be written as a renormalisation of the local charge fugacity:

$$z_r^\pm \rightarrow z_r^\pm e^{\beta(-Jdl \pm dv_r)} \tag{25}$$

where $dv_r$ is a gaussian random variable, uncorrelated from site to site and with $<dv_r dv_{r'}>=2J^2\sigma dl \delta_{r,r'}$. This contribution leads to an effective diffusion and drift in the random core energy variables $E_c^\pm = -(u \pm v) = -T \ln z^\pm$ as $\partial_l E_c^\pm = Jdl \mp dv$ and thus produces the first terms in (21).

(ii) \textit{fusion of environments}. The second contribution comes from the fusion of environments. Upon a change of cutoff, any two regions located around $r_1$ and $r_2$ with $a_0 < |r_1 - r_2| < a_0 e^{dl}$ have to be considered as a single region in the system with the rescaled cutoff. As a consequence the two corresponding pairs of fugacities $z^\pm_{r_1}$ and $z^\pm_{r_2}$ must be combined and replaced by a single pair of effective fugacity variables $z_+, z_-$ associated with the new region at $r = \frac{1}{2}(r_1 + r_2)$, as illustrated in Fig. 5. $z_+$ can be determined by estimating the relative Boltzman weight $W_+ / W_0$ to have a configuration with charge 1 (which lies either in $r_1$ or $r_2$) versus a neutral one (either no charges or a dipole in $r_1, r_2$), and similarly for $z_-$. This gives the fusion rule:

$$z_r^\pm = \frac{z_{r_1}^\pm + z_{r_2}^\pm}{1 + z_{r_1}^+ z_{r_2}^- + z_{r_1}^- z_{r_2}^+} \tag{26}$$

The corresponding correction to the distribution $P_l(z_+, z_-)$ produces the last two terms of (21).

Finally, the RG equation for $J_l$ and $\sigma_l$ can be obtained from the screening by small dipoles of the effective interaction and disorder between two infinitesimal test charges as described in Section 4.3.3.

Several comments are in order concerning this RG procedure. First we note that in defining local fugacity variables (24) we have added an explicit spatial dependence to the part $y_r$ of the fugacity which does not distinguish between a $+1$ and a $-1$ charge. This dependence is not explicitly present in the bare model formula (9) (although it is present if an additional small disorder in the local stiffness $J_{ij}$ is included) but, as we can see from (26) it appears as soon as fusion takes place (the fusion rule is not compatible with a uniform $y_r = y$). Second, there are some assumptions underlying the RG procedure:
Fig. 5. Schematic representation of the fusion of two local environments: when the
cutoff is increased the two regions centered around \( r_1 \) (with local fugacities \( z_1^\prime, z_1^\prime \))
and \( r_2 \) (with local fugacities \( z_2^\prime, z_2^\prime \)) with \( a_0 < |r_1 - r_2| < a_0 e \) are fused into a single
region around \( r = \frac{1}{2}(r_1 + r_2) \) with fugacity \( z_+, z_- \)

technically we treat the local regions as independent from point to point, we
restrict \( V_{\mathbf{r}}^\triangleright \) to be strictly gaussian, together with the usual assumptions (e.g.
short distance expansions) of the CG renormalization. These assumptions are
consistent and amount to discard less relevant operators. These irrelevant op-
erators can be identified within the method using replica of Section 3 where
the above assumptions appear as standard in the RG of the \( m \)-component
vector CG. For instance, the separation of the disorder into the two compo-
ents \( V_{\mathbf{r}}^\triangleright \) and \( v_{\mathbf{r}} \) corresponds in the replica method to the natural splitting
in (11) between the vector fugacity local operator \( Y_{\mathbf{n}} \) (originating from \( v_{\mathbf{r}} \))
and the off-diagonal replica Coulomb interaction \( K_{a \neq b} \) (from \( V_{\mathbf{r}}^\triangleright \)). This will be
further apparent on the equivalent Sine Gordon representation of the problem
presented in Section 7. Accordingly, the definition of the independent local
regions (and thus of the local disorder environments and of the detailed form
of the distribution \( P(z^+, z^-) \)) is clearly cutoff dependent. So is the detailed
form of the fusion rule (26). However, universality of the physical results will
be recovered in a remarkable way in Section 5, independently of the details
of the cutoff procedure. As we will see, this is because, in the low tempe-
ration limit, the above definitions and fusion rules capture correctly (to order
\( P(z \sim 1)^2 \)) the rare events which dominates the physics. They correctly eva-
luate the universal part of \( P(z^+, z^-) \) (its tails in the low temperature region
where they dominate the physics) while they also correctly describe the weak
disorder regime at higher temperatures.

Finally, we note that usual charge fusion between certain types of replica
charges, represented on figure 4 corresponds, in the method without replica,
to fusion of environments.
4.2 Expansion of physical quantities in the number of local regions ("number of points")

To renormalize consistently the present model we need a method which can handle in a systematic way broad distributions of local fugacities. We have found such a method, which we now introduce. It is based on a systematic expansion of physical quantities in the number of independent points. It generalises the conventional fugacity expansion in $y$ of the pure case, but is more powerful. In effect, it amounts to a partial resummation of the conventional expansion. It is versatile since, as we will see, it yields back the conventional expansion in the pure case or at high temperature, but is also able to handle the broad distributions which arise at low temperature. The idea there is that only few rare regions (favorable to the charges) in each environment will dominate the observables and thus it becomes possible again to expand in the density of such rare regions.

The idea underlying this expansion in the number of “independent points” is that the site-dependent fugacities associated with distinct “points” can be considered as statistically independent. On a lattice, these “independent points” naturally correspond to the sites of the lattice, while in the continuum model, their definition is necessarily cut-off dependent. Upon coarse graining, we will be able to use this systematic expansion to renormalize consistently the fugacity distribution associated with each point.

For definiteness, we will show how to construct the expansion in number of points on the free energy $F[V] = -\beta^{-1} \ln Z[V]$. It can be performed in a given environment, keeping the full functional dependence in the set of potentials $\{V_r\}$. The construction can then be easily generalized to any physical quantity, such as arbitrary powers of the free energy $F[V]^p$ (which yield an expansion of all moments averaging term by term over disorder) or any correlation functions of the field $n_r$ which can be obtained from products of free energies $F^{a_1}[V_1] \cdots F^{a_k}[V_k]$ by differentiation with respect to the potentials $V_r$. In a second stage (next Section) we will use this expansion to justify the RG equation for the disorder distribution.

We recall that the Coulomb gas model we consider is defined by its partition function

$$Z[V] = 1 + \sum_{p>0} \sum_{\{n_1, \ldots, n_p\}} \sum_{r_1 \neq \cdots \neq r_p} e^{\beta J \sum_{r_i \neq r_j} n_i G_{r_i - r_j} n_j + \beta \sum_i n_i V_{r_i}}$$  \hspace{1cm} (27)

Here and below, as in (7), all formulae can be extended to the continuum model by replacing discrete sums over distinct sites $r_1 \neq r_2 \cdots \neq r_p$ by integrals with, e.g. hard core conditions $|r_1 - r_2| \geq a_o$ and introducing the uniform fugacity $y$ as was done in Section 2. Note that in the above expression
we do not make use of the decomposition (6) and \( V \) denotes the original disorder.

The expansion in the number of points of the free energy has the form:

\[
F[V] = f^{(0)} + \sum_{r_1 \neq r_2} f^{(2)}_{r_1, r_2}[V] + \sum_{r_1 \neq r_2 \neq r_3} f^{(3)}_{r_1, r_2, r_3}[V] + \ldots
\]  

(28)

where, in the disordered case, each term \( f^{(k)}_{r_1, \ldots, r_k} \) depends on the values taken by the disorder potential \( V \) exactly and only at points \( r_1, \ldots, r_k \). The explicit construction of these terms is given both for the disordered and pure case \( (V = 0) \) in Appendix G. From a practical point of view, the explicit expression of \( f^{(k)}_{r_1, \ldots, r_k} \) is in all cases:

\[
f^{(k)}_{r_1, \ldots, r_k} = \sum_{l=0}^{k} (-1)^{k-l} \sum_{i_1, \ldots, i_l \in [1, \ldots, k]} F_{r_{i_1}, \ldots, r_{i_l}}[V]
\]  

(29)

where \( F_{r_{i_1}, \ldots, r_{i_l}}[V] \) is the free energy of the Coulomb gas defined only on the set of \( l \) points \( r_{i_1}, \ldots, r_{i_l} \) (instead of the full lattice) and the summation is over all distinct subsets of the set \( r_1, \ldots, r_k \). The definition (28) is unambiguous, though subtle. Looking at the explicit expression (29) as a sum over smaller subset of points one could imagine adding other terms to the \( f^{(k)}_{r_1, \ldots, r_k} \) depending on less than \( k \) points. This is not possible in a global way, as the whole series must add up to the free energy, and the formula (29) enforces it order by order. We refer to the Appendix G for further details about the precise definition and construction. Note that the term \( f^{(1)} \) vanishes here because due to neutrality one cannot define a CG on a single site and that the expansion in the number of points of powers \( F^q[V] \) involves a rearrangement of the expansion of \( F[V] \).

It is important to stress that each term \( f^{(k)} \) of the above expansion corresponds to an infinite sum over terms of arbitrary high order in the conventional fugacity \( y \) (as defined in (9)). This can also be seen when setting disorder to zero, where we find that each \( f^{(2k)} \) \( (k \geq 1) \) and \( f^{(2k-1)} \) \( (k \geq 2) \) starts as \( y^{2k} \) plus an infinite number of additional higher order terms in \( y \) (see below). Indeed the expansion (28) corresponds to a complete resummation of the conventional fugacity expansion usually performed in Coulomb gas studies \[9\] except that it is usually performed on the partition function \( Z \) while here we perform it on the free energy. The expansion (28) using the free energy is the appropriate expansion when the fugacities (or the core energies) are random and strongly site dependent with a broad distribution (e.g. most of the sites have \( z \approx 0 \) except for a few which have \( z \approx O(1) \)). Indeed, in this case the only small parameter is the probability \( P(z \sim 1) \) that a given point is favorable for a charge. Thus the \( k \)-th term of the expansion (28) is of order \( P(z \sim 1)^k \), since it is associated with \( k \) independent points. We can thus consider (28) as a
perturbative expansion in the small parameter $P(z \sim 1)$, valid in the XY phase, which replaces the conventional expansion in $y$.

To compute the $f_{r_1, \ldots, r_k}^{(k)}$ we thus have to consider a Coulomb gas defined by the partition function (27) (in the continuum limit), restricted to the system of points $r_1, \ldots, r_k$. We will in addition restrict ourselves to charges $n_r = \pm 1, 0$, as higher charges, examined later will be less relevant. Let us closely examine the lowest order terms $k = 2$ and $k = 3$.

4.2.1 independent dipole approximation

For $k = 2$ we need only two points and the partition function (27) reads simply

$$Z_{r_1, r_2} \equiv 1 + W_{r_1, r_2} = 1 + \left[ \frac{|r_1 - r_2|}{a_o} \right]^{-2\beta J} \left( y^2 e^{V_{r_1} - V_{r_2}} + y^2 e^{V_{r_2} - V_{r_1}} \right) \quad (30)$$

The three terms of the partition function (30) corresponds respectively to no charges or a dipole in $\{r_1, r_2\}$, and the two possible positions for the dipole. This results in the Boltzmann weight for a dipole $W_{r_1, r_2}$. The first terms of (28) thus reads

$$f_{r_1, r_2}^{(2)} = -\beta^{-1} \ln(1 + W_{r_1, r_2}) \quad (31)$$

Restriction of the expansion (28) to its first nonvanishing term, i.e $F = \sum_{r_1, r_2} f_{r_1, r_2}^{(2)}$ corresponds to the so-called independent dipole approximation. This is the approximation on which the analysis of [23] was based. This approximation, which neglects all interactions between dipoles, may seem at first sight to be a good enough approximation in the XY phase and at the transition. Although this is reasonable in the pure XY model, this turns out to be incorrect here: by discarding the next term $f_{r_1, r_2, r_3}^{(3)}$, one throws away crucial statistical correlations. Indeed, when renormalizing the distribution of local fugacities, we have to take into account correlations between dipoles induced by the disorder, which arise as follows. Suppose that the site $r_1$ is favorable to the creation of a $+$ defect, i.e if $z_{r_1}^+ \sim 1$ and $r_2$ and $r_3$ are both favorable to creation of a $-$ defect (while other neighbouring sites are unfavorable). Within the independent dipole approximation the dominant configuration would be to put both a dipole in $r_1, r_2$ and one on $r_1, r_3$ to take advantage of these three favorable sites. These are the configurations (4-6) in Fig. 6. However this configuration is forbidden because of the hard core constraint (and we have restricted to $\pm 1$ charges, higher ones being less favorable energetically). Thus we need to take into account the effective correlations between dipoles which arise because of rare favorable sites. This is done by considering the second term of the expansion $f_{r_1, r_2, r_3}^{(3)}$. Furthermore, as we will see below, consistent one loop renormalization requires to examine all terms in the expansion.
Fig. 6. Possible configurations for three sites: (1-3) are the only configurations physically allowed, (4-6) are present in the independent dipole approximation but forbidden in real model (see text). The bars denote pairs of ±1 charges (28) and how they change under coarse graining, and thus to go well beyond the independent dipole approximation.

4.2.2 Third order term and fusion

Let us derive the explicit formula for $f_{r_1 r_2 r_3}^{(3)}$. The partition function with three sites reads:

$$F[V_{r_1}, V_{r_2}, V_{r_3}] = -\beta^{-1} \ln (1 + W_{r_1 r_2} + W_{r_1 r_3} + W_{r_2 r_3})$$  (32)

However, since all terms of the expansion of $f_{r_1 r_2 r_3}^{(3)}$ in terms of $W$ depend exactly and only on $r_1 r_2 r_3$, we have to substract to this free energy the terms depending on less than three sites, which can be identified as $f_{r_1 r_2}^{(2)} + f_{r_1 r_3}^{(2)} + f_{r_2 r_3}^{(2)}$. The final expression for the second term of the expansion of the free energy is thus

$$f_{r_1 r_2 r_3}^{(3)} = -\beta^{-1} \ln (1 + W_{r_1 r_2} + W_{r_1 r_3} + W_{r_2 r_3})$$  (33)

$$+ \beta^{-1} \left[ \ln (1 + W_{r_1 r_2}) + \ln (1 + W_{r_1 r_3}) + \ln (1 + W_{r_2 r_3}) \right]$$

$$= -\beta^{-1} \ln \left( \frac{1 + W_{r_1 r_2} + W_{r_1 r_3} + W_{r_2 r_3}}{(1 + W_{r_1 r_2})(1 + W_{r_1 r_3})(1 + W_{r_2 r_3})} \right)$$

Let us first notice that now the three sites component of the expansion (28) restricted to the first two terms is exactly $-\beta^{-1} \ln (1 + W_{r_1 r_2} + W_{r_1 r_3} + W_{r_2 r_3})$. Thus adding this second term has cured the problem coming from the configurations (4,5,6) of figure (6) which, as discussed above are not allowed.

It is interesting to note that the term $f_{r_1 r_2 r_3}^{(3)}$ is present (and important) in the expansion of $\ln Z$, even though there are no actual configuration with three charges in a given environment (from neutrality). Indeed each term $f^{(k)}$ in the expansion (28) contains contributions from every even number (less or equal to $k$) of charges. As can be seen from above (fig 6) it takes into account
effective correlations between the distributions of fugacities of three different sites \( r_1 r_2 r_3 \) induced by the hard core constraints. By contrast, there would be no such term involving three sites (because of neutrality) in the similar expansion carried out on \( Z \) (conventional fugacity expansion). It was thus missed in previous studies, while in fact we show in the next Section that it gives rise to a crucial contribution to the renormalization of the distribution of disordered fugacities: the fusion of environments.

The above defined expansion in number of independent points (a type of cluster-virial expansion) which we have illustrated on the first few terms, can be performed systematically to all orders. By coarse graining all terms of the expansion self-consistently, we will now obtain the renormalisation of the disorder distribution.

4.3 Direct renormalisation on the free energy

We now propose a renormalisation scheme based on the above expansion (28). From (4.1) one captures the relevant physics by defining random local charge fugacities \( z^+_r \) at each point and splitting the disorder distribution as \( P[V] = \prod_r P(z^+_r, z^-_r) P[V^>_r] \) as in (6). We know from the results of Section 3 that one can renormalize the model by defining only three scale dependent quantities: the full local disorder distribution \( P_l(z^+, z^-) \), the stiffness \( J_l \) and the second moment \( \sigma_l J^2_l \) of the long range disorder \( V^> \). We now separately obtain the renormalization of each of these quantities, in a systematic perturbative expansion in the small parameter \( P(z \sim 1) \) deduced from the expansion in the number of points. There are three types of contributions as follows.

4.3.1 Rescaling

As can be seen from (7, 6) there is an explicit dependence in the cutoff \( a_0 \) in the expression of the interaction energy between charges and of the correlator of the long range part of the disorder \( V^> \). This dependence will appear in each term \( f^{(k)} \) of the expansion (28). Upon changing the cutoff this will results in contributions of order \( O(dl) \) which can be absorbed by appropriate redefinitions as follows.

The interaction term changes as:

\[
\beta J \sum_{r \neq r'} n_r \ln \left( \frac{|r - r'|}{a_0} \right) n_{r'} = \beta J \sum_{r \neq r'} n_r \ln \left( \frac{|r - r'|}{a_0 e^{dl}} \right) n_{r'} - \beta J dl \sum_r n_r^2
\]
interaction term with the rescaled cutoff while the additional term produces an additive contribution to the $\pm 1$ charge fugacity $z^\pm_r \to z^\pm_r e^{-\beta J dl}$.

Similarly from (6) the correlator of $V^>$ can be rewritten as

$$
(V^>(\mathbf{r}) - V^>(\mathbf{r}'))^2 = 4\sigma J^2 (\ln \frac{|\mathbf{r} - \mathbf{r}'|}{\bar{a}_0}) (1 - \delta(\bar{a}_0)(\mathbf{r} - \mathbf{r'}))
+ 4\sigma J^2 dl (1 - \delta(\bar{a}_0)(\mathbf{r} - \mathbf{r'}))
$$

explicitly as the sum of a new long range disorder correlator with cutoff $\bar{a}_0 = a_0 e^{dl}$ and a short range disorder correlator (we have discarded terms of order $O(dl^2)$). Thus the original problem with cutoff $a_0$ can be rewritten as one with cutoff $\bar{a}_0$ with (i) a new gaussian long range disorder with identical form of the correlator (6) with $a_0$ replaced by $\bar{a}_0$ (ii) a new local (short range) disorder $v(\mathbf{r}) \to v(\mathbf{r}) + dv(\mathbf{r})$ with $dv(\mathbf{r}) dv(\mathbf{r}') = 2\sigma J^2 dl \delta(\bar{a}_0)(\mathbf{r} - \mathbf{r'})$ since it is clear from (35) that when $a_0 \to a_0 e^{dl}$ the LR disorder produces an additive gaussian contribution $dv$ to the SR disorder. Adding the two contributions we find that the change of cutoff produces the total rescaling contribution (25).

### 4.3.2 Fusion of environments

Having introduced the expansion (28) one can now coarse grain its continuum version by increasing the cutoff $a_0 \to \bar{a}_0 = a_0 e^{dl}$ and integrating over the corresponding degrees of freedom. Upon this increase two points $\mathbf{r}_i, \mathbf{r}_j$ from the $k$ points integral ($f^{(k)}$) will be fused if they satisfy $a_0 < |\mathbf{r}_i - \mathbf{r}_j| < a_0 e^{dl}$. This will produce a contribution which is an integral at scale $\bar{a}_0$ depending only on $k - 1$ independent points and thus, by definition of the $f^{(k)}$, produces a correction of order $O(dl)$ to the $k - 1$ terms $f^{(k-1)}$. All these corrections can be reabsorbed into a correction to the fugacity distribution (together with an additive change to $f^{(0)}$ the free energy contribution of all degrees of freedom which have been eliminated in the change of cutoff).

We now illustrate how this works on the case $k = 3$, and indicate in the Appendix H how it works for arbitrary $k$. To lowest order in $dl$, this correction is independent of $k$ and can be easily performed by considering the three points integral, using (33) with $e.g$ $a_0 < |\mathbf{r}_1 - \mathbf{r}_2| < \bar{a}_0 = a_0 e^{dl}$:

$$
\int_{a_0 < |\mathbf{r}_1 - \mathbf{r}_2| < a_0 e^{dl}} f^{(3)}_{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3} = -\beta^{-1} \int_{a_0 < |\mathbf{r}_1 - \mathbf{r}_2| < a_0 e^{dl}} \left[ \ln \left( 1 + \frac{W_{\mathbf{r}_1 \mathbf{r}_3} + W_{\mathbf{r}_2 \mathbf{r}_3}}{1 + W_{\mathbf{r}_1 \mathbf{r}_2}} \right) \right]
- \ln(1 + W_{\mathbf{r}_1 \mathbf{r}_3}) - \ln(1 + W_{\mathbf{r}_2 \mathbf{r}_3})
$$

(36)

where we assume $|\mathbf{r}_1 - \mathbf{r}_3| \geq \bar{a}_0, |\mathbf{r}_2 - \mathbf{r}_3| \geq \bar{a}_0$. Upon coarse graining, the two points $\mathbf{r}_1, \mathbf{r}_2$ are fused to a single point $\mathbf{\bar{r}} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ and one obtains a
correction to $f_{r_1 r_3}^3$.

Using the decomposition (6) of the disorder potential into a correlated component and a local part: $V_r = V_r^\triangleright + v_r$, we can rewrite the Boltzmann weight for a dipole $W_{r_1 r_2}$, defined in (30), as:

$$W_{r_1 r_2} = \left(\frac{|r_1 - r_2|}{a_o}\right)^{-2\beta J} \left(\begin{array}{c} z_{r_1}^+ z_{r_2}^- e^{\beta(V_{r_1}^\triangleright - V_{r_2}^\triangleright)} + z_{r_2}^+ z_{r_1}^- e^{\beta(V_{r_2}^\triangleright - V_{r_1}^\triangleright)} \end{array}\right)$$

(37)

where we have used the definition $z_r^\pm = ye^{\pm \beta v_r}$ for the local charge fugacities.

Several simplifications now occur in evaluating (36). We first note that in (36) the integral is of order $dl$. Thus to find the leading correction of $f_{r_1 r_3}^{(2)}$ to order $O(dl)$, we only have to consider the integrand expanded in order 0 in $dl$. To this order, from the correlator (6) of the disorder potential $V_r^\triangleright$ and using $|r_1 - r_2|/a_o = 1 + O(dl)$, we find that

$$V_{r_1}^\triangleright = V_{r_2}^\triangleright + O(dl) = V_{r_2}^\triangleright + O(dl)$$

(38)

Thus the weight of the fused pair can be simplified as $W_{r_1 r_2} = (z_{r_1}^+ z_{r_2}^- + z_{r_2}^+ z_{r_1}^-)$. Similarly, $W_{r_1 r_3}$ and $W_{r_2 r_3}$ simplify to order 0 in $dl$ using (38):

$$W_{r_1 r_3} = \left(\frac{|\bar{r} - r_3|}{a_o}\right)^{-2\beta J} \left(\begin{array}{c} z_{r_1}^+ z_{r_3}^- e^{\beta(V_{r_1}^\triangleright - V_{r_3}^\triangleright)} + z_{r_3}^+ z_{r_1}^- e^{\beta(V_{r_3}^\triangleright - V_{r_1}^\triangleright)} \end{array}\right)$$

(39a)

$$W_{r_2 r_3} = \left(\frac{|\bar{r} - r_2|}{a_o}\right)^{-2\beta J} \left(\begin{array}{c} z_{r_2}^+ z_{r_3}^- e^{\beta(V_{r_2}^\triangleright - V_{r_3}^\triangleright)} + z_{r_3}^+ z_{r_2}^- e^{\beta(V_{r_3}^\triangleright - V_{r_2}^\triangleright)} \end{array}\right)$$

(39b)

Using these simplifications, we can now rewrite the first term of (36): we obtain for the corresponding dipole weight:

$$W'_{r_1 r_3} = \frac{W_{r_1 r_3} + W_{r_2 r_3}}{1 + W_{r_1 r_2}} = \left(\frac{|\bar{r} - r_3|}{a_o}\right)^{-2\beta J} \times \left(\begin{array}{c} (z_{r_1}^+ + z_{r_2}^+) z_{r_3}^- e^{\beta(V_{r_3}^\triangleright - V_{r_3}^\triangleright)} + (z_{r_1}^- + z_{r_2}^-) z_{r_3}^+ e^{\beta(V_{r_3}^\triangleright - V_{r_3}^\triangleright)} \end{array}\right)$$

(40)

It is now simple to verify that the change in the disorder averaged $<\sum_{r_1 \neq r_2} f_{r_1 r_3}^{(2)} [V] >_V$ originating from the $f^{(3)}$ term is correctly accounted for by following rules for the random fugacity variables. The last two terms of (36) produce, with the simplification (39a,39b), the rules:

$$(z_{r_1}^+, z_{r_2}^-) \rightarrow z_{r_1}^+ = z_{r_1}^+ ; \quad (z_{r_1}^-, z_{r_2}^+) \rightarrow z_{r_2}^+ = z_{r_2}^+$$

(41)
From (41), we find the rule corresponding to the first term of (36):

\[
(z_{r_1}^+, z_{r_2}^+) \rightarrow z_{r_1}^+ = \frac{(z_{r_1}^+ + z_{r_2}^+)}{1 + z_{r_1}^+ z_{r_2}^- + z_{r_1}^- z_{r_2}^+},
\]

\[
(z_{r_1}^-, z_{r_2}^-) \rightarrow z_{r_1}^- = \frac{(z_{r_1}^- + z_{r_2}^-)}{1 + z_{r_1}^- z_{r_2}^+ + z_{r_1}^+ z_{r_2}^-}.
\]

(42)

This is true for any point \( r_3 \) different from \( r_1 \) and \( r_2 \). One can thus rewrite these fusion corrections as a correction to the (unnormalised) 'distribution' of local fugacities \( \Phi(z_+, z_-) \):

\[
\partial_l \Phi(z_+, z_-) = c_2 \int_{z_{r_1}^+}^{z_{r_2}^+} \Phi(z_{r_1}^+, z_{r_2}^-) \Phi(z_{r_1}^+ z_{r_2}^-) \times \left[ \delta \left( z^+ - \frac{(z_{r_1}^+ + z_{r_2}^-)}{1 + z_{r_1}^+ z_{r_2}^- + z_{r_1}^- z_{r_2}^+} \right) \right.
\]

\[
\left. \delta \left( z^- - \frac{(z_{r_1}^- + z_{r_2}^+)}{1 + z_{r_1}^- z_{r_2}^+ + z_{r_1}^+ z_{r_2}^-} \right) \right] - \delta \left( z^+ - z_{r_1}^+ \right) \delta \left( z^- - z_{r_1}^- \right) - \delta \left( z^+ - z_{r_2}^+ \right) \delta \left( z^- - z_{r_2}^- \right)
\]

(43)

The coefficient \( c_2 \) comes from the integration over the relative position \( r_1 - r_2 \). Thus from (43) and (25), we recover exactly the RG equation (18) for the function \( \Phi \).

One can similarly check that the above rule correctly account for the corrections to \( k - 1 \) term from the \( k \) term upon change of cutoff. To be exhaustive, we must also consider a constant term in the expansion (28) of the free energy: \( f_0(l) \) which satisfies \( \partial_l \Phi \sim \langle \ln(1 + z_+ z' + z_-' z_+') \rangle \). This term corresponds to the free energy sum of all degrees of freedom which have been eliminated up to scale \( l \). Indeed, \( (z_+, z_-) \) contains an average over all disorder configurations at smaller scales (all environments which have been eliminated). A full and systematic proof can in principle be made by considering all averages of powers \( F_q[V] \) expanded in number of points and all fusions to order \( k \) giving corrections to order \( k - 1 \). We will not attempt it here, as the systematic study to all orders is much easy to perform within the (equivalent) replica formalism.

4.3.3 Screening

To derive the scaling behaviour of both the stiffness \( J \) and the strength of the correlated disorder \( V_r > r \), we consider the screening of the interaction and the correlation of the disorder between two infinitesimal test charges \( e_1 \) and \( e_2 \) in the sample. To implement the study of this screening within our expansion in number of independent points, we first define \( F[V, e_1, e_2] \) as the free energy of the disordered Coulomb gas defined in (7) with the two additional test charges \( e_1, e_2 \) fixed in \( r_1, r_2 \). These test charges interact with the other integer charges of the Coulomb gas, which screen the interaction between them, and
the correlator of the disorder $V^>$. From $F[V, e_1, e_2]$, one can define the screened interaction and disorder by

$$\beta G^R(r_1, r_2) = \frac{1}{2} \left. \frac{\partial^2 F[V, e_1, e_2]}{\partial e_1 \partial e_2} \right|_{e_1=e_2=0}$$

(44a)

$$-\beta V^R(r_1) = \left. \frac{\partial F[V, e_1, e_2]}{\partial e_1} \right|_{e_1=e_2=0}$$

(44b)

At large distance we expect from renormalisability $\beta G^R \sim \beta J^R \ln \left( |r_1 - r_2|/a_o \right)$, which imply the definitions for the renormalized coupling constant and disorder strength

$$J_R = \lim_{k \to 0} G_k^{-1} G_k^R$$

(45a)

$$2\sigma_R J_R^2 = \lim_{k \to 0} G_k^{-1} V^R_k V^R_{-k}$$

(45b)

where the Fourier transform of the 2D laplacian $G_k^{-1}$ has been defined after equation (2). These definitions can be transformed exactly using (7) into the relation

$$J_R = J + 2\pi \beta J^2 \int d^2 \mathbf{r} (\hat{\mathbf{q}} \cdot \mathbf{r})^2 \langle n_0 n_R \rangle_c$$

(46a)

$$\sigma_R J_R^2 = \sigma J^2 - 2\pi J^2 \int d^2 \mathbf{r} (\hat{\mathbf{q}} \cdot \mathbf{r})^2 \langle n_0 \rangle \langle n_R \rangle$$

(46b)

$$+ 4\pi \sigma \beta J^3 \int d^2 \mathbf{r} (\hat{\mathbf{q}} \cdot \mathbf{r})^2 \langle n_0 n_R \rangle_c$$

Up to now, these are only standard definitions of the renormalised stiffness $J^R$ and disorder strength $\sigma^R$. From this one deduces the RG equations for $J$ and $\sigma$ since by coarse graining (46a,46b) we obtain [25,24,23]

$$T \partial_t J^{-1} = -2a_0^4 \pi^2 \langle n_0 n_{R=a} \rangle_c ; \quad \partial_t \sigma = -2a_0^4 \pi^2 \langle n_0 \rangle \langle n_{R=a} \rangle$$

where the $a_0^4$ factor arises because we are dealing with correlations of the charge density.

The novel and most tricky part is how to evaluate the right hand side in a systematic way. Our method is to expand these correlation functions in the number of independent points. This can be done in a systematic way using (28) and the definitions
\[
\langle n_0 n_R \rangle_c = \frac{\partial^2 F[V]}{\partial V_0 \partial V_R} \quad (47a)
\]
\[
\langle n_0 \rangle \langle n_R \rangle = \frac{\partial F[V]}{\partial V_0} \frac{\partial F[V]}{\partial V_R} \quad (47b)
\]

Note that while the expansion of the first term (connected correlation) in the number of points is readily obtained from the expansion (28), the expansion of the second term (disconnected correlation) is more subtle since we need to square the derivatives of (28) and rearrange it again in an expansion in the number of points. Fortunately, to the order we are working, we need only the two point term in the expansion. Thus we can restrict ourselves to terms with two points on both expansions. The expansion of the first term (47a) follows straightforwardly from (28):

\[
\frac{\partial^2 F[V]}{\partial V_0 \partial V_R} = \sum_{r_1 \neq r_2} \frac{\partial^2 f^{(2)}_{r_1,r_2}}{\partial V_0 \partial V_R} + \sum_{r_1 \neq r_2 \neq r_3} \frac{\partial^2 f^{(2)}_{r_1,r_2,r_3}}{\partial V_0 \partial V_R} + \ldots
\]

and thus from (31) one finds:

\[
\frac{\partial^2 F[V]}{\partial V_0 \partial V_R} \bigg|_{2\text{points}} = \frac{\partial^2 f^{(2)}_{0,R}}{\partial V_0 \partial V_R} = -\beta^{-1} \frac{\partial^2 W_{0,R}}{1 + W_{0,R} \partial V_0 \partial V_R} + \frac{\beta^{-1}}{(1 + W_{0,R})^2} \frac{\partial V_0 \partial V_R}{\partial V_0 \partial V_R}
\]

since we can keep only terms depending of the two points (0 and R) in the last equality (the other contributions vanish). Similarly one finds that:

\[
\frac{\partial F[V]}{\partial V_0} \frac{\partial F[V]}{\partial V_R} \bigg|_{2\text{points}} = \left( \sum_{r_1 \neq r_2} \frac{\partial f^{(2)}_{r_1,r_2}}{\partial V_0} + \ldots \right) \left( \sum_{r_1 \neq r_4} \frac{\partial f^{(2)}_{r_1,r_4}}{\partial V_R} + \ldots \right) \bigg|_{2\text{points}}
\]

\[
= \frac{\partial f^{(2)}_{0,R} \partial f^{(2)}_{0,R}}{\partial V_0 \partial V_R} = \frac{\beta^2}{(1 + W_{0,R})^2} \frac{\partial W_{0,R} \partial W_{0,R}}{\partial V_0 \partial V_R}
\]

where the dipole weights \( W_{0,R} \) have been defined in (30)

These expressions further simplify since it is sufficient to work to lowest order in \( dl \), i.e. to order 0 in \( dl \). We can use that \( V_R^> = V_0^> + O(dl) \) since we need only to evaluate the correlation at \( |R| = a_0 \). This yields that \( W_{0,R} = z_0^+ z_R^- + z_0^- z_R^+ \). One can also replace the derivatives with respect to the full disorder \( \partial \gamma \) by derivatives with respect to the local disorder \( \partial_v \) (see (6)) and thus we find:
By using this expansion into (47a, 47b) we find the screening RG equations (22).

Thus the expansion in the number of points has allowed us to derive consistent RG equations without using replicas. Note the difference with the previous unsuccessful attempt in [23] to derive RG equations for the stiffness and disorder without replicas. The authors of [23] were working in the independent dipole approximation. Thus they computed the correlation functions as in the first line of (49) except that they kept the full expression \( (\sum_{r_1 \neq r_2} \frac{\partial f^{(2)}}{\partial V^{0}_{r_1}})(\sum_{r_3 \neq r_4} \frac{\partial f^{(2)}}{\partial V^{R}_{r_4}}) \). This led to intractable expressions involving up to four independent points. The beauty of the present method is that we know that we can keep, in a consistent way to this order, only the terms in this expression which involve two points.

4.4 Conclusion and connection with the replica method

We have thus developed two RG procedures, one using replicas based on an expansion in the *vector fugacity* \( Y[n] \) which may have seemed somewhat formal, the other one, direct without replicas, as an expansion in the number of favorable regions \( P(z \sim 1) \) (at low temperature, while at higher temperature it smoothly crosses over into the usual expansion in the uniform fugacity \( y =< z > \)). In fact these two approaches are equivalent. Indeed, we have checked using some combinatorics detailed in Appendix E, that the fugacity expansion in \( Y[n] \) (13, 11) is identical term by term, in the limit \( m \rightarrow 0 \) to the expansion in number of points of the free energy (28). Thus, working with one or the other is equivalent \(^6\). It also shows that the limit of diluted *vector charges* physically corresponds to the limit of rare favorable regions.

5 Analysis of the RG equations

In this Section we analyse the RG equations derived in the previous Sections and we obtain the phase diagram and the critical behaviour of the XY model

\[^6\] the RG procedure used in the previous Section to derive the screening equations without replicas was found to be consistent with the more direct RG of the replicated CG explained in Appendix C.
5.1 general analysis of the full RG equations

Our task now is to study the closed set of RG equations (21,22) for the scale dependent distribution of fugacities $P_l(z_+, z_-)$, the long range disorder strength $\sigma_l$ and the stiffness $J_l$. Since we study below the zero temperature limit, we have chosen to keep $T$ fixed and renormalize $J$ (only the combination $K = \beta J$ flows).

We started by studying the set (21,22) numerically. Although we will not reproduce here the details we describe the main results. They confirmed the overall physical picture and led us to introduce an approximation, described below, which allows for an analytical solution.

We found that at low $T$ and for $\sigma$ smaller than some critical value $\sigma_c$ an XY phase exists (as in Fig. 1), where both the stiffness $J$ and $\sigma$ converge to finite non zero values at large $l$:

$$J_l \to J_R > 0 \quad \sigma_l \to \sigma_R$$

In general, at low $T$ we found that, starting at scale $l = 0$ from (9) with a small initial averaged fugacity (and small local disorder) the distribution $P_l(z_+, z_-)$ becomes broad and develops power law tails in the variables $z_+, z_-$ which quickly extend up to fugacities $z \pm \sim O(1)$. However, in the XY phase (50) we found that the typical $z \pm$ goes to zero and that the concentration of rare favorable regions, after an initial increase, ends up decreasing towards zero at large $l$. It is useful to define the single fugacity distribution:

$$P_l(z) = \int dz_+ P_l(z_+, z) = \int dz_- P_l(z, z_-)$$

which does not satisfy a closed RG equation. Then we find in the XY phase that $P_l(z \sim 1)$ decreases at large $l$ (or equivalently $\int_{z \pm > z^*} P_l(z)$ where $z^*$ is some arbitrary threshold). Thus the probability that either $z_+$ or $z_-$ is of order 1 decreases. On the other hand, for $\sigma > \sigma_c$ we find that $P_l(z \sim 1)$ ends up increasing at large scale and the whole distribution $P_l(z)$ ends up drifting towards increasing $z$ values. This corresponds to the disordered phase where $J_l$ vanishes at large scale ($J_R = 0$). The most interesting flow occurs in the critical regime near the transition. There, if one interprets the fraction of favorable regions $P_l(z \sim 1)$ as the perturbative parameter, the structure of the flow near $\sigma = \sigma_c$ is reminiscent of the RG flow a la Kosterliz-Thouless.
with a separatrix, in the plane \((P_l(1), \sigma)\) between the XY and the disordered phase. Accordingly, exactly at the transition, i.e. on the separatrix (which, to be accurate, is a critical manifold in the space of distributions and couplings \(P_l(z_+, z_-), J, \sigma\)) the distribution \(P(z_+, z_-)\) becomes very broad and develops a fixed shape, with \(P_l(z \sim 1)\) slowly decreasing to zero (which makes the critical regime perturbative in \(P_l(1)\)). A schematic representation of the distribution is given in Fig. 7.

Let us close this paragraph by noting that at higher \(T\), by contrast, we found that the distribution remains peaked and that the overall picture becomes more consistent with considering a uniform average fugacity (as in [9]). The XY transition then occurs when this uniform average fugacity ceases to flow to zero at large scale.

To quantify these (mostly numerical) observations, we now turn to the single fugacity approximation.

Fig. 7. Schematic representation of the scale dependent behavior of the distribution \(P(z_+, z_-)\) in the plane of core energy variables \((u, v)\) and the regions discussed in the text). The stiffness \(J_l\) produces a convection towards the left (negative \(u\) axis) while at the same time the long range disorder \(\sigma\) produces a diffusive spreading of the probability weight along the (local disorder) \(v\) axis (in both directions). For \(\sigma > \sigma_c\) diffusion wins and the distribution spreads. For smaller \(\sigma < \sigma_c\) the weight of the distribution remains confined within the left quadrant \(z_\pm < 1\). In general it develops a two dimensional front solution.

5.2 Single fugacity RG equation

We now argue that, given the structure of the RG flow observed numerically, we can, with no loss of accuracy in all the regimes of interest (i.e. within and
near the boundaries of the XY phase) approximate the full RG equations (21) for \( P_t(z_+, z_-) \) by a simpler equation for the single fugacity distribution \( P_t(z) \), and similarly for (22).

Let us first focus on the low \( T \) regime, where the distribution \( P_t(z_+, z_-) \) is broad and the physics is dominated by rare favorable regions. For such a broad distribution the parameter which allows to organize perturbation theory is:

\[
P_t(1) \equiv P_t(z \sim 1) \sim P_t(z_+ \sim 1, z_- \sim 0) = P_t(z_+ \sim 0, z_- \sim 1)
\]

(52)

where we distinguish symbolically the rare configurations \( z_\pm \sim 1 \) from the typical ones where \( z_\pm \sim 0 \). The first observation is that \( z_+ \) and \( z_- \) are even more rarely simultaneously \( \sim 1 \). We note symbolically:

\[
P_t(1, 1) \equiv P_t(z_+ \sim 1, z_- \sim 1) \sim P_t(1)^2
\]

(53)

Indeed such configurations, absent from the start, are generated from the fusion term in (21). To generate from the fusion rule \( z_\pm = (z'_+ + z''_+)/ (1 + z'_+ z''_+ + z'_- z''_-) \) a configuration where both \( z_+ \sim z_- \sim O(1) \) one clearly needs at least that either \( z'_+ \sim z''_+ \sim 1 \) or \( z'_- \sim z''_- \sim 1 \), which is of probability of order \( P_t(1)^2 \) (symbolically one can write \( \partial_t P_t(1, 1) \sim P_t(1)^2 + .. \)). Thus, and this can be further checked, the estimate (53) is valid (the diffusion term does not change it). One can now inspect the corrections to the stiffness \( J \) and to the long range disorder \( \sigma \) in (22) which come mainly from configurations where either \( z'_+ \sim z''_+ \sim 1 \) or \( z'_- \sim z''_- \sim 1 \). This leading contribution is thus of order \( \sim P_t(z \sim 1)^2 \). Other fugacity configurations, e.g. with both \( z'_+ \sim z'_- \sim 1 \) contribute in (22) but, from the above estimate (53), their probability is \( \sim P_t(1, 1)^2 P_t(1) \sim P_t(1)^3 \) and are thus subleading in an expansion in \( P_t(1) \). To summarize, one can see schematically the RG equations (22) as a correction of leading order \( P_t(1)^2 \) to \( J_t \) and \( \sigma_t \) and the RG equation (21) as a correction to \( P_t(1) \) of order \( P_t(1) \) (diffusion term) and \( P_t(1)^2 \) (fusion term).

Guided by these observations, we now approximate the full equation (21) by a RG equation for a function of a single variable which should correctly describe the behaviour of \( P(z_+, z_-) \) around \( z_\pm \sim 1 \). This approximation amounts to neglect the denominators in the fusion rule of (21). Indeed, in the spirit of our previous estimates, this denominator contributes only when \( z'_+ \sim z''_+ \sim 1 \) (or \( z'_- \sim z''_- \sim 1 \)) and then yields a contribution of order \( P_t(1)^2 \) but which corrects only \( P_t(1, 1) \) since this fusion produces both a \( z_+ \sim z_- \sim 1 \). This corresponds to a correction of order \( P_t(1)^3 \) to \( P_t(1) \) (which we neglect, since the leading order of the correction to \( P_t(1) \) coming from the fusion term is \( P_t(1)^2 \)). This can be seen e.g. since \( P_t(1, 1) \) itself in turns contributes to \( P_t(1) \) to order \( P_t(1)^2 \) (through integration over one of the two variables), or by more detailed arguments, not reproduced here. Neglecting the denominators gives
the equation for the probability distribution

\[ \partial_l P(z_+, z_-) = \mathcal{O} P - 2P(z_+, z_-) \]
\[ + 2 \left\langle \delta(z_+ - (z'_+ + z''_+)) \delta(z_- - (z'_- + z''_-)) \right\rangle_{P'_+P''_-} \]  

(54)

This equation can now be explicitly integrated over one variable, say \( z_- \) and yields a closed equation\footnote{Note similarly that (54) admits solutions of the form \( P_l(z_+, z_-) = P_l(z_+)P_l(z_-) \).} for the single fugacity distribution \( P_l(z_+) \) defined in (51):

\[ \partial_l P(z) = \left[ \beta J(1 + z \partial_z) + \sigma(\beta J)^2(1 + z \partial_z)^2 \right] P(z) - 2P(z) \]
\[ + 2 \int_{z', z'' > 0} dz' dz'' \delta(z - (z' + z'')) P(z')P(z'') \]  

(55)

We can now obtain the corresponding equations for the corrections to \( J \) and \( \sigma \). In (22) one keeps only the configurations corresponding to either \( z'_+ \sim z''_+ \sim 1 \) or \( z'_- \sim z''_- \sim 1 \) but discard the much less probable configurations where all four fugacities are \( \sim 1 \). The expressions then nicely factor out in terms of \( P_l(z) \) and one obtains:

\[ T \partial_l J^{-1} = \frac{8c_1}{c_2} \int_{z'_+, z''_+} P(z'_+)P(z''_-) \frac{z'_+z''_-}{(1 + z'_+z''_-)^2} \]  

(56a)

\[ \partial_l \sigma = \frac{8c_1}{c_2} \int_{z'_+, z''_+} P(z'_+)P(z''_-) \frac{(z'_+z''_-)^2}{(1 + z'_+z''_-)^2} \]  

(56b)

where the additional factor of 2 counts the equivalent integral with + and − exchanged.

We now summarize the obtained closed set of RG equations in terms of the variable \( u \) defined as:

\[ z = e^{\beta u} \]  

(57)

of distribution \( \tilde{P}(u)du = P(z)dz \). Physically the random variable \( u \) can be interpreted as the random local core energy \( u_r = -E_c(r) \). One has:
\[ \partial_t \tilde{P}(u) = (J \partial_u + \sigma \mathcal{J}^2 \partial^2_u) \tilde{P} - 2 \tilde{P} \]  
\[ + 2 \int_{u', u''} \tilde{P}(u') \tilde{P}(u'') \delta \left( u - \frac{1}{\beta} \ln(e^{\beta u'} + e^{\beta u''}) \right) \]  
(58a)

\[ \partial_t (J^{-1}) = \frac{8c_1}{c_2} \left\langle \frac{\beta e^{-\beta(u'+u'')}}{(1 + e^{-\beta(u'+u'')})^2} \right\rangle \tilde{P}(u') \tilde{P}(u'') = \frac{8c_1}{c_2} \left\langle \frac{\beta e^{-\beta u}}{(1 + e^{-\beta u})^2} \right\rangle \tilde{P} \ast_u \tilde{P} \]  
(58b)

\[ \partial_t (\sigma) = \frac{8c_1}{c_2} \left\langle \frac{1}{(1 + e^{-\beta(u'+u'')})^2} \right\rangle \tilde{P}(u') \tilde{P}(u'') = \frac{8c_1}{c_2} \left\langle \frac{1}{(1 + e^{-\beta u})^2} \right\rangle \tilde{P} \ast_u \tilde{P} \]  
(58c)

and thus the corrections to \( J \) and \( \sigma \) involve only the convolution \( \tilde{P} \ast_u \tilde{P} \).

We have thus justified, in the low temperature regime, that these approximate RG equations should describe the tails of the fugacity distribution exactly to the order \( P_l(1)^2 \) in the expansion in \( P_l(1) \) to which we are working. Indeed, since we have studied the RG in the previous Sections only to order \( Y[n]^2 \) (in the replica formulation, corresponding to \( P_l(1)^2 \) in the rare events formulation) it probably does not make sense at this stage to try to be more accurate.\(^8\)

Finally, let us note that although we have focused until now on the low \( T \) regime it is rather obvious that the approximation of discarding the denominators will be even more valid at higher temperature since in the conventional fugacity expansion (in a uniform averaged fugacity) these terms corresponds to \( O(y^4) \) terms while we work to \( O(y^2) \). So these new RG equations interpolate all limits correctly.

### 5.3 RG equation at \( T = 0 \)

One can now easily see on the form (58) that the RG equations admit a well defined \( T \to 0 \) limit. In the equation (58a), the fusion rule \( u = \frac{1}{\beta} \ln(e^{\beta u'} + e^{\beta u''}) \) now becomes a max rule \( u = \max(u', u'') \) and (58a) transforms as

\[ \partial_t \tilde{P}_1(u) = (J \partial_u + \sigma I \mathcal{J}^2 \partial^2_u) \tilde{P}_1 - 2 \tilde{P}_1 + 4 \tilde{P}_1 \int_{u''} \tilde{P}_1(u') \]  
(59)

Similarly, the function of \( u', u'' \) in (58b,58c) become respectively a delta and a theta function when \( T \to 0 \), so that in that limit one has:

\(^8\) Although we believe the arguments based on an expansion in \( P_l(1) \) are correct, only an exhaustive analysis of the two dimensional front solutions of (21), clearly a complex task which goes beyond this paper, could confirm the exactness of the approximation and the validity of the arguments based on organizing the perturbation theory using a single small parameter \( P_l(1) \).
\[ \frac{\partial_t (J^{-1})}{c_2^2} \int du \tilde{P}_l(u) \tilde{P}_l(-u) \]  
\[ \frac{\partial_t (\sigma)}{c_2^2} \int_{u' + u'' \geq 0} du' du'' \tilde{P}_l(u') \tilde{P}_l(u'') \]  

Note that the last integral exactly evaluates the probability to find two local regions with a total negative core energy, energetically favorable for a dipole, in agreement with the physical picture valid at low T.

We thus obtain a close set of equations (59,60) which describes the scaling behaviour of the system at zero temperature. The equation for the distribution (59) can be conveniently simplified using the parametrization

\[ G_l(x) = \int_{x-E_l}^{+\infty} \tilde{P}_l(u) du \]  

with \( E_l = \int_0^l J_l' dl' \). The function \( G_l(x) \) then satisfies

\[ \frac{1}{2} \partial_t G = \frac{\sigma J^2}{2} \partial_x^2 G + G(1 - G) \]

In the case where \( \sigma \) and \( J \) are \( l \)-independent, this equation is known as the Kolmogorov-Fisher equation and we will recall some results on this equation in the next Section. Before turning to its study, let us notice that the screening equations (60) can be rewritten using this new parametrization simply substituting \( \tilde{P}_l(u) = -\partial_x G(u + E_l) \).

5.4 Parametrization at \( T > 0 \)

Although (58) at finite temperature \( T > 0 \) could be in principle studied directly, it is much more convenient to introduce the generalization of the \( T = 0 \) parametrization (61) as

\[ G_l(x) = 1 - \int_{z>0} dz P_l(z) \exp(-ze^{-\beta(x-E_l)}) \]  
\[ = 1 - \int_{-\infty}^{\infty} du \tilde{P}_l(u) \exp(-\epsilon^{\beta(u-x+E_l)}) \]

with \( E_l = \int_0^l J(l') dl' \). In the limit \( \beta \to \infty \), this function reduces to the previous one (61). Using this new function and variable the integral equation (58a) can again be transformed exactly into a simpler differential equation which, interestingly, remains exactly the same as in the \( T = 0 \) case:

\[ \frac{1}{2} \partial_t G = \frac{\sigma J^2}{2} \partial_x^2 G + G(1 - G) \]
Only the initial condition explicitly depends on temperature (see below).

Thus solving the KPP equation (63) allows us in principle to obtain the scale dependent fugacity distribution $P_l(z)$ at any $T$. However the relation between this function $G_l(x)$ and the distribution $P_l(z)$ becomes much more involved at $T > 0$. As a result, and contrarily to the $T = 0$ case, the screening equations do not admit a simple expression in term of $G_l(x)$.

For any fixed $l$ one can reconstruct the integer moments $\langle z^n \rangle_P$ of the distribution $P_l(z)$ by simply expanding the generating function in powers of $e^{-\beta x}$ since:

$$G_l(x) = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n!} \langle z^n \rangle_P e^{-n\beta(x-E_l)}$$

Taking back this expansion in the KPP equation (63) and identifying the coefficient of the exponentials $e^{-n\beta x}$, we find the exact RG equations for the moments $\langle z^n \rangle$ of the distribution $P_l(z)$:

$$\partial_t \langle z^n \rangle = \left(2 - n\beta J + \sigma(n\beta J)^2\right) \langle z^n \rangle$$

$$+ \int dz' dz'' \left[ (z' + z'')^n - (z')^n - (z'')^n \right] P_l(z') P_l(z'')$$

Starting from a reasonable initial distribution with finite moments, the moments remain finite for finite $l$, but, as we now discuss, increase quickly as $l \to +\infty$. Let us examine the scaling dimension of the moments, neglecting for now the bilinear (fusion) term. We find that for fixed $T$ and $J$, each successive moment $\langle z^n \rangle$ diverges with the scale $l$ when the long range disorder $\sigma$ becomes larger than a critical disorder $\sigma > \sigma^{(n)} = (T/nJ) - 2(T/nJ)^2$ (see Fig. 8) (these values will be slightly renormalized by the screening equations but the conclusion will be similar). Putting back the fusion term simply implies that the moments with $n \geq 2$ also diverge when the first one does, as indicated in the Fig. 8.

The divergence of the first moment $y_l = \langle z \rangle_P$ which can be identified with the uniform fugacity $y$ of Rubinstein et al. yields the (incorrect) reentrant phase diagram of [9] where the XY phase is destroyed above the line $\sigma^{(1)}(T/J)$. Indeed it is already clear that the uniform fugacity approximation cannot work since we now see that higher moments diverge for even smaller disorder strengths. Thus even within the XY phase this result for the moments show that the distribution $P_l(z)$ rapidly becomes broader and broader. Atypical sites where $z$ is large appear and dominate the behaviour of the higher moments. Thus it becomes meaningless to study the scale dependence of the integer moments, but rather we must now consider the whole probability distribution $P_l(z)$ and in particular understand its tail. This can be achieved with the help
Fig. 8. Critical disorder $\sigma^{(n)}(T/J)$ above which the successive moments $\langle z^n \rangle$ diverge as the scale $l \to +\infty$.

of the known solutions of the KPP equation which we briefly review in the next Section, before coming back to describing the scaling behaviour of $P_l(z)$.

It is interesting to note at this stage that the above KPP equation also arises in the problem of the directed polymer with quenched disorder on the Cayley Tree (DPCT) [29]. There the variable $l$ corresponds to the number of generations and $\tilde{P}_l(u)$ to the distribution of free energy $-u = -T \ln z$. Thus we have demonstrated in an explicit and non trivial way, i.e at the level of the RG equations, that there are close connections between the two problems, which both exhibit a similar freezing transition. There are also notable differences between the two problems. For instance, while the diffusion coefficient $D$ is constant in the DPCT studied in [29], in the disordered CG, $D_l$ depends on the scale and, in a self consistent manner, on the solution of the KPP equation itself, via the equations (56) which describes the physics of screening, absent in the DPCT. As a result, additional phase transitions exist here as will be detailed in Section 6.

5.5 The Kolmogorov-Petrovskii-Piscounov equation: some known useful properties

We recall in this Section some known facts on the Kolmogorov-Petrovskii-Piscounov (KPP) equation (also known as the Kolmogorov-Fisher equation) which we will need in the following Sections. The equation reads, in a general form:

$$\frac{1}{2} \partial_t G = D \partial_x^2 G + f(G)$$  \hspace{1cm} (65)$$

where the diffusion coefficient is constant, the function $f(G)$ satisfies $f(0) = f(1) = 0$, $f$ positive between 0 and 1 and $f'(0) = 1$, $f'(G) \leq 1$ between 0 and 1. The usual case corresponds to $f(G) = G - G^2$. This equation has been applied to a wide range of problems, from chemistry to hydrodynamic instabilities or to the propagation of the Meissner phase into the normal phase
in a superconductor [35,36]. It is the prototype of equations describing the
diffusive invasion of an unstable state by a stable one. This can be seen by
writing it as a Landau equation $\partial_t G = -\partial F / \partial G$ whose free energy $F$ takes by
construction its local maximum in $G = 0$ (the unstable state) and its minimum
in $G = 1$ (stable state). One usually chooses an initial condition $G_{t=0}(x)$
monotonously decreasing from $G_{t=0}(x \to -\infty) = 1$ to $G_{t=0}(x \to +\infty) = 0$.
For a large class of initial conditions, the solutions of the KPP equations are
known to converge uniformly towards traveling waves solutions of the form:

\[ G_t(x) \to h(x - m_t) \]  

(66)

with $h(x \to +\infty) = 0$ and $h(x \to -\infty) = 1$. However the question of the
determination, given an initial condition, of the asymptotic traveling wave $h$
and its velocity $c = \lim_{t \to +\infty} \partial_t m_t$ has been largely debated for KPP equations
or for similar more complex non linear equations, and is still of current interest.
It is known as the front velocity selection problem.

It can be illustrated as follows. A family of possible front solutions exists,
parametrized by the velocity $c$ and noted $h_c(x - m_t)$, as can be seen by substi-
tuting (66) in (65). Constraints exist for the velocity. Indeed, one can linearize
the KPP equation in the region ahead of the front for large positive $x - m_t$
where $h$ is very small. As discussed below it is in fact this region which deter-
mines the velocity and is universal. In this region one has:

\[ Dh'' + \frac{c}{2} h' + h = 0 \]  

(67)

and thus $h$ is a superposition of two exponentials $e^{-\mu(x-c)}$ with (see figure 9)

\[ c(\mu) = 2(\mu^{-1} + D\mu) \]  

(68)

with $c \geq c^* = 4\sqrt{D}$. The large $x$ behaviour is dominated by the smaller $\mu$ and
thus correspond to the left branch of the curve $c(\mu)$ in figure 9 when $c > c^*$.
In the marginal case where $\mu = \mu_c = 1/\sqrt{D}$ and $c = c^*$, the two eigenvalues
are degenerate and the front thus has the asymptotic behaviour for $x \to +\infty$:

\[ h(x) \sim (ax + b)e^{-\mu_c x} \]  

(69)

Let us now give the known results for the selection of the asymptotic front
among the family of possible $h_c$.

(i) Velocity selection.

Although for more complex equations one relies on stability analysis and a
"marginal stability criterion" [37], in the case of the KPP equation a rigorous
result is available. A theorem due to Bramson [38] shows that the asymptotic traveling wave is determined by the behaviour at \( x \to +\infty \) of the initial condition \( G_{t=0}(x) \) in the following manner. If \( G_{t=0}(x) \) decays fast enough, as \( G_{t=0}(x) \sim e^{-\mu x} \) for \( x \to +\infty \) with \( \mu \geq \mu_c = 1/\sqrt{D} \) (or faster) theorem B of [38] states that \( G_t(x) \) uniformly converges towards the traveling wave solution \( h_{c^*}(x-m_l) \) of velocity \( c = \lim_{x \to +\infty} \partial_t m_l = c^* = 4\sqrt{D} \). If \( G_{t=0}(x) \) decays slower, as \( G_{t=0}(x) \sim e^{-\mu x} \) with \( \mu < \mu_c = 1/\sqrt{D} \) then \( G_t(x) \) uniformly converges towards \( h_{c(\mu)}(x-m_l) \) of velocity \( c(\mu) \) continuously depending on \( \mu \) and given by (68). The asymptotic velocity is thus given by \( c = 4\sqrt{D} \) for steep enough initial condition \( \mu \geq \mu_c = 1/\sqrt{D} \) and \( c = 2(D\mu + \mu^{-1}) \) otherwise.

Moreover, the leading corrections to the velocity are also given by the theorem of Bramson and are independent of the function \( f(G) \) in (65). The corresponding position of the front \( m_l \) is given by [38–40]

\[
m_l = \sqrt{D} \left( 4l - \frac{3}{2} \ln l + O(1) \right) \quad \text{for} \quad \mu > D^{-\frac{1}{2}} \tag{70a}
\]

\[
= \sqrt{D} \left( 4l - \frac{1}{2} \ln l + O(1) \right) \quad \text{for} \quad \mu = D^{-\frac{1}{2}} \tag{70b}
\]

\[
= 2(D\mu + \mu^{-1})l \quad \text{for} \quad \mu < D^{-\frac{1}{2}} \tag{70c}
\]

Note that cases (70a) and (70b) differ only by the velocity corrections but not by the asymptotic front shape which is \( h_{c^*} \) in both case.

Let us emphasize again the remarkable universality which arises in this problem. Clearly the detailed shape of the asymptotic front depends on the detailed form of the non linear term \( f(G) \) in the KPP equation. However, the selection itself, the selected velocity, its corrections, and the tail of the selected front function \( h \) are all independent of \( f(G) \). It is also natural physically that it is the region in which the front penetrates (region ahead of the front \( x \to +\infty \)) which determines the selection. This universality has been explored further [40,41] and it has even been shown that the next leading corrections to \( \partial_t m_l \) are also universal. Finally the marginal case which interpolates between (70a)
and (70b) has been also explored\textsuperscript{9}. The asymptotic front shape is the same for all these cases, but not the detailed finite \( l \) shape as we now discuss.

(ii) \textit{Shape of the front for finite \( l \)}

The problem has also been studied for finite but large \( l \) \[39–41\]. We start with the case where \( G_{\ell=0}(x) \) decays fast for \( x \to +\infty \) as \( G_{\ell=0}(x) \sim e^{-\mu x} \) with \( \mu > \mu_c = 1/\sqrt{D} \), most relevant for the following Sections. Then one must distinguish two regions in the traveling wave solution as illustrated in figure 12. The central region is the “bulk” or “interior front” for \( |x - m_l| \) fixed and finite. There, the shape of the front corresponds to its asymptotic form \( G_{\ell}(x) \approx h_{c^*}(x-m_l) \) and the center moves as \( \partial_l m_l = c^* - 3\sqrt{D}/2l + O(l^{-3/2}) \) with \( c^* = 4\sqrt{D} \).

Far ahead of this “interior front”, \( x - m_l \gg 1, \) \( G_{\ell}(x) \) still decays faster than in the asymptotic solution \( h_{c^*}(x - m_l) \). Thus there exist an intermediate region (which we call the “intermediate front region”) which matches between the interior region and the region at infinity. In this intermediate region which corresponds to \( x - m_l \sim \sqrt{Dl} \) one must take into account diffusion, and one can solve the linearized KPP equation without assuming a front solution, but assuming a scaling form. It is found \[40,39\] that in this scaling “far front” region \( x - m_l \sim \sqrt{Dl} \), the solution \( G_{\ell}(x) \) behaves as\textsuperscript{10}:

\[
G_{\ell}(x) \approx g_l(x - m_l) \quad \text{(71a)}
\]

\[
g_l(x) = A \left( \frac{x}{\sqrt{D}} + \text{cte} + O(l^{-\frac{1}{2}}) \right) e^{-\mu_c x} e^{-\frac{x^2}{8l}} \quad \text{(71b)}
\]

Note that later on we will need to distinguish a region even further ahead of the front for \( x - m_l \sim l \).

Let us close this Section by returning to the question of the dependence of the RG equation of the random XY model in the cutoff procedure. We can show that the cutoff procedure will determine the function \( f(G) \). Let us consider for instance again the limit of zero disorder, discussed in Appendix F. Since in that case \( D = 0 \) one should have:

\[
\frac{1}{2} \partial_l G_{\ell}(x) = f(G) \quad \text{(72)}
\]

with in that case \( 1 - G_{\ell}(x) = \langle \exp(ye^{-\beta(x-E_l)}) \rangle_{P_l(y)} \) and \( P_l \) characterizes the “disorder” associated with the choice of cutoff procedure. An interesting choice

\textsuperscript{9} It was found in \[39\] that when \( G_{\ell=0}(x) \sim x^a e^{-\mu_c x} \) with \(-2 < a < 0\) one has: \( m_l = \sqrt{D} \left( 4l - \frac{a}{2} \ln l + O(1) \right) \)

\textsuperscript{10} Similar results were obtained for the marginal cases \( G_{\ell=0} \sim x^a e^{-\mu_c x} \) with \(-2 < a < 0\) with a scaling function noted \( G_{\ell=0}((x - m(l))/(2Dl)) \) in \[39\] (for \( a = -2 \) one gets the above result, but the form for general \( a \) is more complicated).
corresponds to \( P_l(y) = \delta(y - y_l) \) where \( y_l \) satisfies the Kosterlitz Thouless RG equation \( \partial_l y_l = (2 - K) y_l \). It is easy to see that in that case \( G_l(x) \) satisfies (72) with \( f(G) = -(1 - G) \ln(1 - G) \). This is further discussed in [16]

6 XY phase and critical behaviour at low temperature

6.1 Phase diagram from the scale dependent fugacity distribution

We now use the solutions of the KPP equation discussed in the previous Section to determine the phase diagram of the XY model with random phase shift. Let us first look for the XY phase where we expect that \( J_l \) and \( D_l = \frac{1}{2} \sigma_l J_l^2 \) reach limits, respectively \( J_R \) and \( D_R = \frac{1}{2} \sigma_R J_R^2 \) at large \( l \). Thus in the XY phase at large \( l \) the KPP equation with constant \( D = D_R \) can be used. Precise behaviour near the phase transition away from the XY phase, as well as intermediate scale dependence inside the XY phase \textit{a priori} requires taking into account the \( l \) dependence of \( D_l \) which will be done in the following Sections. Note that \( l \) dependence of \( J_l \) itself results only in a shift that can always be trivially taken into account.

First we note that the phase diagram will be entirely determined by the velocity selected in the KPP equation. Indeed, we know from the previous section that \( G_l(x) \) converges to a traveling front solution \( G_l(x) \rightarrow h_c(x - m_l) \) of velocity \( c \). The parametrization (62) then implies that the distribution \( P_l(z) \) of vortex fugacity for the random XY model, itself converges to a traveling front solution, more conveniently expressed in the random core energy variable \( u = \ln z \) as:

\[
P_l(u) \rightarrow_{l \rightarrow +\infty} \tilde{p}(u - X_l), \quad X_l = m_l - E_l
\]

(73)

where \( m_l \) is given in (70) and \( E_l = \int_0^1 J_{l'} dl' \). The center of the front of \( \tilde{P}_l(u) \), located in \( u_{typ} \approx X_l + O(1) \) corresponds to the maximum of the distribution \( \tilde{P}(u) \) (as can be easily seen on (61)) and to the typical values of the random variable \( u \). The front shape of \( \tilde{P}_l(u) \) is simply related to the KPP front solution \( h_c \) through \( \int_u \tilde{p}(u) \exp(e^{\beta(u-x)}) = 1 - h_c(x) \). The asymptotic velocity of the front of \( \tilde{P}_l(u) \) is thus:

\[- \partial_t E_{typ}^c = \partial_t u_{typ} = \partial_t X_l = \partial_t m_l - J_l \rightarrow_{l \rightarrow +\infty} c - J_R \]

(74)

The total velocity in the \( u \) variable is thus the KPP velocity minus the stiffness. The former comes from the spread of the distribution due to disorder, while the latter from the effect of interactions. In previous Sections we have explained
Fig. 10. Schematic representation of the connection between the sign of the total velocity of the front \( v = \partial_t X_t = c - J \) and the increase or decrease of \( P_l(1) \).

that the XY phase corresponds to a decrease of the density of favorable regions \( P_l(1) = P_l(z \sim 1) \) (i.e \( u \sim 0 \)) and to the absence of topological defects at large scale. In particular, for \( J \) and \( D \) to reach finite asymptotic values \( J_R \) and \( D_R \), it is necessary that \( P_l(z \sim 1) \) decreases fast enough. \( P_l(1) \) can be estimated crudely\[11\] as \( \tilde{\rho}(-X_t) \). Thus, as shown in fig. 10, when the total velocity \( \partial_t X_t \) is negative (XY phase), the front moves to the left (large \( z \) or large \( u \)) and thus the probability of events \( z \sim 1 \) decreases while if \( \partial_t X_t \) is positive, \( P_l(1) \) increases asymptotically (disordered phase). The XY phase thus corresponds to the region where the total velocity is negative and thus to:

\[
c - J_R > 0
\] (75)

in which case the whole probability distribution of the core energy \( E_{c_{typ}}^c = -u \), its typical and average values drift to \( +\infty \) at larger scale. The transition line between the disordered and the XY phase can be located, in the plane \( J_R, \sigma_R \), by finding the line where this relative velocity \( \partial_t X_t \) vanishes.

The phase diagram can now be obtained by determining the velocity \( c \) as a function of \( T \) and \( \sigma \). The crucial observation is that by construction (as is the case in [29]) the initial condition \( G_{l=0}(x) \) decays for large \( x \) as:

\[
G_{l=0}(x) \sim_{x \to +\infty} < z >_{P_0} e^{-\beta x}
\] (76)

thus, since \( < z >_{P_0} \) is finite, it decays exponentially and one can apply Bramson’s results (70) detailed in the previous Sections with the identification:

\[
\mu = \beta
\] (77)

Thus in effect it is the temperature which selects the velocity \( c \). We find that, as depicted in fig 11:

\[\text{\cite{11} this estimate turns out to be accurate only near the transition as discussed below, but this does not change the conclusions}\]
(i) for $\beta < \beta_g = \mu_c = 1/\sqrt{D_R}$, i.e at high enough temperature $T > T_g = J_R \sqrt{\sigma_R/2}$, the velocity continuously depends on temperature:

$$c = c(\beta) = 2(D_R \beta + \beta^{-1}) = T \left( 2 + \frac{\sigma_R J_R^2}{T^2} \right)$$  \hspace{1cm} (78)

and thus, in that regime, the XY phase exists for:

$$\partial_t X_I = T \left( 2 - \frac{J_R}{T} + \frac{\sigma_R J_R^2}{T^2} \right) < 0$$  \hspace{1cm} (79)

which is exactly the condition which would be obtained from the averaged fugacity (at least when expressed in the renormalized parameters) and leads to the transition line of Rubinstein et al. as we have discussed earlier (see eq.(64)).

(ii) the velocity of the front freezes at $\beta = \beta_g$ and for $\beta \geq \beta_g$, i.e at low temperature $T \leq T_g$, where it becomes temperature independent:

$$c = c^* = 4 \sqrt{D_R} = J_R \sqrt{8 \sigma_R}$$  \hspace{1cm} (80)

and the total velocity of the front $\bar{P}_I(u)$ is now

$$\partial_t X_I = J_R (\sqrt{8 \sigma_R} - 1)$$  \hspace{1cm} (81)

Thus we obtain that below this freezing temperature at:

$$T_g = J_R \sqrt{\frac{\sigma_R}{2}}$$  \hspace{1cm} (82)

the transition between the XY phase and the disordered phases occurs at: $\sigma_R = \sigma_c = \frac{1}{8}$. For $\sigma_R > \frac{1}{8}$ the system is unstable to the proliferation of topological defects induced by disorder.

6.2 study of the XY phase

To describe the XY phase it is important to understand the various regions of the scale dependent fugacity distribution $P_I(z)$. Indeed the fugacity distribution also gives the distribution of the charge correlation functions in the Coulomb gas from the relations (49a).
Fig. 11. Phase diagram as a function of the renormalized disorder strength $\sigma_R$ and the temperature $T/J_R$. $T_g$ corresponds to the freezing temperature and the transition line for $T < T_g$ is strictly horizontal when drawn in the renormalized parameters (and slightly curved when drawn in the bare parameters).

6.2.1 shape of the fugacity distribution in the XY phase

Fig. 12. Structure of the scale dependent fugacity distribution $P_l(z)$ in the XY phase at low $T$ and the several regions discussed in the text.

In the XY phase one can describe the large scale behaviour of the fugacity distribution by neglecting the slow scale dependence of $\sigma$ and $J$ (adiabatic approximation). The behaviour of $\tilde{P}_l(u)$ can be obtained by studying $\tilde{G}_l(x) = G_l(x + E_l)$ since one has $1 - \tilde{G}_l(x) = \int_u^\infty \tilde{P}_l(u) \exp(e^{\beta(u-x)})$ with $E_l = \int_0^l dl' J_{l'}$.

At $T = 0$ this simplifies into $\tilde{P}_l(u) = -\tilde{G}_l(u)$. It is also useful to note that the Laplace transform of $P_l(z)$ is $\tilde{P}_l(s) = 1 - \tilde{G}_l(-T \ln s)$.

At low $T$ in the XY phase $P_l(z)$ becomes very broad and one must distinguish for large but finite $l$ several different regions represented in fig. 12. Using the results of the preceding Sections on the KPP equation we now describe these regions in details:

*interior front:* the first region corresponds to the bulk of the probability distri-
bution centered around the typical value of the fugacity $z_{\text{typ}} \sim e^{\beta X_l}$. From the results for the front velocity $(81,79)$ one thus obtains that in the XY phase at high temperature $T > T_g = J\sqrt{\sigma/2}$ the typical renormalized fugacity decays with the scale as:

$$z_{\text{typ}}^l \sim e^{(2 - \frac{J}{T} + \frac{\sigma J^2}{T^2})l}$$

(83)

which in that case is also the scaling of the average fugacity $\langle z \rangle_{P_l}$. On the other hand, below the freezing temperature $T < T_g$ it decays as:

$$z_{\text{typ}}^l \sim e^{-\beta J(1 - \sqrt{\frac{8}{\sigma}})l}$$

(84)

The bulk of the distribution is thus well described by the asymptotic front of the KPP equation as $\tilde{G}_l(x) \approx h(x - X_l)$ and $\tilde{P}_l(u) \approx \tilde{p}(u - X_l)$. Its precise shape is of course non universal as it depends on the details of the definition of the fugacity at the level of the cutoff (e.g. the function $f(G)$ in (65)). However, since the physically interesting region $z > z_{\text{typ}}$ corresponds to the region ahead of the KPP front (even though the total velocity $v = c - J$ is negative, see fig. 12) universal results about the KPP equation can be used. In particular the near tail (i.e the scaling region $u - X_l$ fixed but large) is universal and can be obtained since $h(x) \sim xe^{-x/T_g}$. Through Laplace inversion one gets that the fugacity distribution in that near tail region behaves as a power law:

$$P_l(z)dz \sim \left(\frac{z_{\text{typ}}^l}{z}\right)^{1 + \frac{T}{T_g}} \ln\left(\frac{z}{z_{\text{typ}}^l}\right) \frac{dz}{z_{\text{typ}}^l}$$

(85)

which is valid for $0 < T < T_g$ and in the regime $z/z_{\text{typ}}^l$ fixed (as $l$ grows) but large. The freezing which occurs at $T_g$ thus concerns typical regions which, deep in the XY phase have a very small fugacity $z_{\text{typ}}^l$. It corresponds to the temperature at which the first moment of the distribution on the r.h.s. of (85) becomes infinite. Thus for $T < T_g$ the true average fugacity $\langle z \rangle_{P_l} > z_{\text{typ}}^l$ and becomes dominated by rare local environments corresponding to values of $z$ outside of the bulk of $P_l(z)$ (where (85) is invalid).

intermediate front: the second region is the “intermediate front region” (see fig. 12) and corresponds to $u - X_l \sim \sqrt{l}$. There we know from (71a) that $\tilde{G}_l(x) \approx g_l(x - X_l)$ with $g_l(y) \sim \frac{y}{\sqrt{D}} \exp(-\frac{y^2}{4D} - \frac{y^2}{8DL})$. This region will be crucial to describe the critical behaviour in the following Section.

far tail region: finally the most important region to describe the XY phase is the “far tail region” far ahead of the front with $u - X_l \sim l$ (see fig. 12). Indeed, to obtain the renormalization of $J$ and $\sigma$ (via the screening equations (56)) we are interested in the rare events $z \sim 1$ of small probability $P_l(1)$, but which
dominate the average correlations. These events correspond to the region of fixed $u$ and thus to $u - X_l \sim -X_l \sim (J - c)$. Fortunately, this region is so far ahead of the front that $G_l(x)$ can be obtained with excellent accuracy by solving the linearized KPP equation:

$$\frac{1}{2} \partial_t G = D \partial_x^2 G + G$$

by straightforward integration from from $l = 0$ to $l$. This leads to:

$$G_l(x) = \frac{e^{2l}}{\sqrt{8\pi \int_0^l D_l \, dl}} \int x' e^{-\frac{(x-x')^2}{8\int_0^l D_l \, dl}} G_{l=0}(x')$$

We also notice that in this regime $\tilde{P}_l(u)$ can also be obtained explicitly. Indeed, since the relation between $\tilde{P}_l(u)$ and $G_l(x)$ is a simple linear ($l$-dependent) convolution it is straightforward to show (e.g. via $x$ Fourier space) that $G_l(x)$ obeying the linearized KPP equation (86) is equivalent to $\tilde{P}_l(u)$ satisfying:

$$\partial_t \tilde{P}(u) = \left( J \partial_u + \sigma J^2 \partial_u^2 \right) \tilde{P} + 2 \tilde{P}$$

This linearized equation was not entirely obvious to guess directly from (58) and justifies, even in this simplest regime, the detour through the rigorous results for the KPP equation. Using $E_l = \int_0^l D_l \, J_l$ it yields:

$$\tilde{P}_l(u) = \frac{e^{2l}}{\sqrt{8\pi \int_0^l D_l \, dl}} \int u' e^{-\frac{(u+E_l-u')^2}{8\int_0^l D_l \, dl}} \tilde{P}_{l=0}(u')$$

It is interesting to note that for fixed $x$ and $u$, $\tilde{G}_l(x)$ and $\tilde{P}_l(u)$ can be estimated in the large $l$ limit as simple exponentials:

$$\tilde{P}_l(u) \sim_{l \to \infty} C e^{2l(1 - \frac{1}{12})} e^{-\frac{u}{2\sigma J}}$$

with $C = \int x' e^{\frac{x'^2}{2\sigma J}} P_{l=0}(x')$ and an identical expression for $\tilde{G}_l(x = u)$ with a different prefactor $C' = \int x' e^{\frac{x'^2}{2\sigma J}} G_{l=0}(x')$. These forms, which are valid in the region of fixed fixed $u \sim O(1)$ (and fixed $x \sim O(1)$ respectively) will be useful to estimate the renormalization of $J$ and $\sigma$ below\[\textsuperscript{12}\]

\[\textsuperscript{12}\] they were obtained from (87) and (89) by approximating $\exp(-x^2 +
Thus at low temperature we find the following decay of the probability of rare favorable local environments:

\[ P_l(z \sim 1) \sim A l^{-1/2} e^{2l(1 - \frac{1}{8\sigma})} \]  

(91)

with \( A = C/\sqrt{4\pi\sigma J^2} \). We also find that the distribution \( P_l(z) \) has an algebraic tail at low temperature in the region \( z \sim 1 \) (i.e. \( z \) fixed as \( l \to \infty \)) as:

\[ P_l(z) \approx P_l(1) \frac{1}{z^{1+T/T^*}}, \quad T^* = 2\sigma J \]  

(92)

with is a different power law behaviour than the one which characterizes typical fugacities (85). These two different power law behaviours are represented in figure 12.

Finally, one can check that the three regions match properly and thus we have a fairly complete description of \( P_l(u) \). For instance, using the expression valid in the region (ii) for fixed \( x \) at large \( l \) one gets:

\[ \tilde{G}_l(x) \approx g_l(x - X_l) \sim l^{-1} \frac{AX_l x_{l/2} - X_l^2}{\sqrt{D}} e^{-\frac{x}{\sqrt{D}}} e^{2l(1 - \frac{1}{8\sigma})} \]

which always gives the result (90) up to the logarithmic corrections \( l \) prefactors factors, and reproduces even the \( l \) prefactors correctly for \( X_l/l \to 0 \), which is when we expect the matching to become exact\(^{13}\).

6.2.2 Screening and correlations in the XY phase

To study the screening equations (56) in the XY phase we need the distribution \( \tilde{P}_l^d(u_d) = \tilde{P}_l^* u \tilde{P}_l \) of dipole core energy \( u_d = u' + u'' \). Fortunately in this regime it can be computed simply from (89) and is given simply by:

\[ x^2/\langle 8DL \rangle \approx 1, \text{ and similarly for } u \] (to be precise the \( D \) and \( J \) which appear here are \( 1/l \int_0^l D_l \) and \( 1/l \int_0^l J_l \) respectively). They are valid only as long as the integrals which defines \( C \) and \( C' \) are convergent. Since \( G_{l=0}(x') \sim e^{-\beta x'} \) for large positive \( x' \) the estimate for \( \tilde{G}_l(x') \) is valid only as long as \( T < T^* = 2\sigma J \). For \( T > T^* \) one must instead perform saddle point estimate, see below

\(^{13}\) note that for \( T > T^* \) the averaged fugacity \( \langle z \rangle_{P_l} \) is controlled by \( z \ll 1 \), while for \( T < T^* \) is controlled by \( z \gg 1 \)

\(^{14}\) this type of matching has been used to derive the \( 3/2 \) coefficient in the logarithmic correction to the velocity \([40]\).
\[
\tilde{P}^d_l(u) = \frac{e^{4l}}{\sqrt{16\pi}} \int_{u'} e^{-\frac{(u-u')^2}{16}} \tilde{P}^d_0(u') \sim \frac{C''e^{4l(1-\frac{1}{8\sigma})}}{\sqrt{8\pi\sigma} J^2} e^{-\frac{u^2}{2\sigma J}}
\]

the last equality being valid for fixed \(u\) and \(l \to +\infty\), and \(C'' = \int_{u'} \exp\left(\frac{u'}{2\sigma J}\right) \tilde{P}_0^d(u')\) in terms of the initial dipole core energy distribution. Substituting this last form in (56) we obtain, in the large \(l\) limit:

\[
\partial_l (J^{-1}) = \frac{8c_1}{c_2^2} \sqrt{8\pi\sigma J^2} C'' \int_{-\infty}^{+\infty} \frac{\beta e^{-\beta u} e^{-\beta^* u}}{(1 + e^{-\beta u})^2} = C_s I_J(\beta) l^{\frac{1}{2}} P_l(1)^2 \tag{94a}
\]

\[
\partial_l \sigma = \frac{8c_1}{c_2^2} \sqrt{8\pi\sigma J^2} C'' \int_{-\infty}^{+\infty} \frac{e^{-\beta^* u}}{(1 + e^{-\beta u})^2} = C_s I_\sigma(\beta) l^{\frac{1}{2}} P_l(1)^2 \tag{94b}
\]

with \(C_s = 8c_1 C'' J\sqrt{\pi\sigma}/(\sqrt{2c_2^2} C^2)\) a constant roughly independent of the temperature. The two above integrals

\[
I_J(\beta) = \frac{T}{T^*} \frac{\pi}{\sin\left(\frac{T}{T^*} \pi\right)}, \quad I_\sigma(\beta) = \frac{T(1 - \frac{T}{T^*})\pi}{\sin\left(\frac{T}{T^*} \pi\right)} \tag{95}
\]

are convergent respectively only for \(T < T^*\) and \(T < 2T^*\), in which cases it is indeed legitimate to replace \(\tilde{P}^d_l(u)\) by its above asymptotic form. Note that the \(T = 0\) limit is well defined since in that case \(I_J = 1\) and \(I_\sigma = T^* = 2\sigma J\).

At higher temperature \(T > T^*\) the full front solution controls the renormalization of \(J\). Using the simple above gaussian form yields that the \(u\) integral is dominated by the saddle point \(u = u' - 2Jl + 4\sigma J^2 l\) which gives:

\[
\partial_l (J^{-1}) \sim e^{2(2l - \frac{1}{4} + \frac{\sigma J^2}{T^*})} \quad T > T^* \tag{96}
\]

which corresponds for \(T > T^*\), to the behaviour of \(< z >^2\) (and for \(T > T_g\) of \(z_{typ}^2\)), as expected. A similar, though more involved, analysis can be performed for \(\sigma\).

Thus we have obtained the equations (94) for the renormalization of \(J\) and \(\sigma\) in the XY phase. Since we have shown that \(P_l(1)\) decreases exponentially as (91) we conclude that \(J\) and \(\sigma\) reach their finite limits \(J_R > 0\) and \(\sigma_R < +\infty\) as power laws of the systems size.

This analysis also yields the full distribution of the correlation function of the charges in the XY phase. Indeed, let us recall that:
\[-\tilde{a}^4_0 \langle n_0 n_{R=\tilde{a}_0} \rangle_c = \frac{z^+_R \bar{z}^-_R + z^+_R \bar{z}^-_0 + 4 z^+_0 \bar{z}^-_R \bar{z}^-_0}{(1 + z^-_0 \bar{z}^-_R + z^+_R \bar{z}^-_0)^2} \]  
(97a)
\[-\tilde{a}^4_0 \langle n_0 \rangle \langle n_{R=\tilde{a}_0} \rangle = \frac{(z^+_0 \bar{z}^-_R - z^+_R \bar{z}^-_0)^2}{(1 + z^-_0 \bar{z}^-_R + z^+_R \bar{z}^-_0)^2} \]  
(97b)

Thus by following the RG up to the scale \( l^* = \ln(R/a_0) = \ln(\tilde{a}_0/a_0) \) we can obtain from \( P_l(z) \) the distribution of the charge correlations at large \( R \).

For instance, from the above and (84) one finds the following decay of the typical thermal and disorder correlations:

\[-\langle n_0 n_R \rangle_{c|_{typ}} \sim \left( \frac{a_0}{R} \right)^{4 + \frac{2}{T} (1 - \sqrt{8 \sigma})} \]  
(98a)
\[-\langle n_0 \rangle \langle n_R \rangle_{c|_{typ}} \sim \left( \frac{a_0}{R} \right)^{4 + \frac{4}{T} (1 - \sqrt{8 \sigma})} \]  
(98b)

for \( \sigma < \sigma_c \) and \( T < T_g \). For \( T > T_g \) one has instead \( \langle n_0 n_R \rangle_{c|_{typ}} \sim \left( \frac{a_0}{R} \right)^{2/T - 2 \sigma J^2 / T^2} \).

The averaged moments can be obtained as above by substituting the exponential form of the distribution \( P_d^d(u) \) for \( u \sim 0 \). Performing the corresponding integrals, one gets:

\[ \langle n_0 n_R \rangle_c^p \sim A_p(T) \left( \ln \frac{R}{a_0} \right)^{-\frac{1}{2}} \left( \frac{a_0}{R} \right)^{4p + 4(1 - \sqrt{8 \sigma})} \]  
(99a)
\[ \langle n_0 \rangle^p \langle n_R \rangle^p \sim B_p(T) \left( \ln \frac{R}{a_0} \right)^{-\frac{1}{2}} \left( \frac{a_0}{R} \right)^{4p + 4(1 - \sqrt{8 \sigma})} \]  
(99b)

with

\[ A_p(T) = \frac{C''}{\sqrt{8 \pi \sigma J^2}} T \frac{\Gamma \left[ p + \frac{T}{T_g} \right] \Gamma \left[ p - \frac{T}{T_g} \right]}{\Gamma[2p]} \]  
(100a)
\[ B_p(T) = \frac{C''}{\sqrt{8 \pi \sigma J^2}} T \frac{\Gamma \left[ \frac{T}{T_g} \right] \Gamma \left[ 2p - \frac{T}{T_g} \right]}{\Gamma[2p]} \]  
(100b)

These formulae are valid only at low enough temperature \( T < T^*(p) \).

Finally, although we have not attempted a precise RG calculation of the XY order correlation functions, the following behaviour should hold in the quasi ordered XY phase:

\[ \langle e^{i(\theta_R - \theta_0)} \rangle \sim R^{-n} \]  
(101a)
\[ \langle e^{i \theta_R} > < e^{-i \theta_0} > \sim R^{-\eta} \]  
(101b)
with \( \eta = \frac{1}{2}(\sigma_R + \frac{T}{\mu_R}) \) and \( \eta = \frac{1}{2}\sigma_R \). At the zero temperature transition point \( \sigma_R = \sigma_c = 1/8, T = 0 \), the value of these exponents are universal \( \eta = \eta = 1/16 \).

### 6.3 Critical behaviour at zero temperature

We now study the transition from the XY to the disordered phases at low temperature at \( \sigma_R = 1/8 \). From the previous Sections we know that it occurs when the total velocity of the front of the distribution \( \tilde{P}_l(u) \) vanishes, i.e the critical region is defined by

\[
\partial_t X_l = J(\sqrt{8\sigma} - 1) \approx 0
\]

in a large \( l \) regime. While in the XY phase the physics was dominated by the far tail of the traveling front \( u - X_l \sim l \) in the critical regime it is the interior front \( u - X_l \sim O(1) \), as well as the intermediate front region \( u - X_l \sim \sqrt{l} \) (see fig. 12) which controls the transition. Thus the correct description of the transition requires the knowledge of the KPP physics in an essential way, and is thus entirely novel.

For simplicity, we will only present the analysis at \( T = 0 \). We will also work to first order in \( \sigma - \sigma_c \). We first assume that the coefficient \( D = \frac{1}{2}\sigma J^2 \) varies sufficiently slowly near the transition so that the results from the KPP equation with a constant \( D \) can be used. This assumption will be self consistently verified at the end.

Near the transition the center of the front is located at \( u = X_l \) with \( X_l \approx (4\sqrt{D} - J)l - 3\sqrt{D}\ln l + X_0 \) as indicated by (70). Thus in the critical regime one still has \( X_l \to -\infty \), although logarithmically in \( l \) exactly at the transition. Thus this critical regime can still be studied perturbatively, as \( P_l(1) \) remains very small. The front velocity has the following scale dependence:

\[
\partial_t X_l = 4\sqrt{D} - J - \frac{3\sqrt{D}}{2l} + h.o.t
\]

where the h.o.t. contains the universal [40] \( O(l^{-3/2}) \) subdominant corrections to velocity in the KPP equation as well as additional subdominant corrections originating from the slowly varying \( D_l \). In the critical region at \( T = 0 \) we can use the KPP front solution \( \tilde{P}_l(u) = \tilde{p}_l(u - X_l) \) with, from (71a):

\[
\tilde{p}_l(x) = -g_l'(x) \sim_{x \gg 1} A \frac{x}{D} e^{-\sqrt{D}e^{-\frac{x^2}{8dl}}}
\]
an expression valid as long as \(x/l \rightarrow 0\). The RG equations for \(J\) and \(\sigma\) in the critical region thus read:

\[
\partial_l (J^{-1}) = \frac{8c_1}{c_2} \int du p_l(u - X_l)p_l(-u - X_l) \tag{105a}
\]
\[
\partial_l \sigma = \frac{8c_1}{c_2} \int _{u+u'>-2X_l} p_l(u)p_l(u') \tag{105b}
\]

We need to evaluate these expressions for \(X_l\) large and negative (typically either as \(\sim -\ln l\) on the critical manifold or as \(\sim (\sigma - \sigma_c)/l\) very close to it). At criticality they behave approximately as \(\exp(2X_l/\sqrt{D})\), which can be guessed by setting \(u \sim 0\) in the first expression. A more accurate estimate of the above integrals is performed in the Appendix J. The end result is that, in the regime of interest (where we can discard terms of order \((\sigma - \sigma_c)^2\)) one has:

\[
\partial_l (J^{-1}) \sim \frac{C}{\sqrt{D}} X_l^3 e^{2X_l/\sqrt{D}} , \quad \partial_l \sigma \sim CX_l^3 e^{2X_l/\sqrt{D}} \tag{106}
\]

where \(C\) is a constant. Introducing the small parameter:

\[
g_l = e^{X_l/\sqrt{D}} \tag{107}
\]

the density of favorable regions reads:

\[
P_l(1) \sim A \frac{X_l}{D} e^{X_l/\sqrt{D}} e^{-\frac{x^2}{4\sigma}} \sim X_l g_l \tag{108}
\]

since we can discard terms of order \((\sigma - \sigma_c)^2\). From (103) one finds that \(g_l\) satisfies precisely the RG flow equation:

\[
\partial_t g = \left(16(\sigma - \sigma_c) - \frac{3}{2l} + h.o.t\right) g \tag{109}
\]

to lowest order in \(\sigma - \sigma_c\). Note the \(\frac{3}{2}\) universal factor which arises from the universal velocity corrections in the KPP equation (70).

A natural choice of parameters to describe the universal behaviour around the transition is thus \(g\) (which up to logarithmic corrections is equal to the density of favorable sites \(P_l(1)\)) and \(\sigma\) which satisfies:

\[
\partial_t \sigma \sim C \left(\ln \frac{1}{g}\right)^3 g^2 \tag{110}
\]
These two equations (109, 110) form our complete set of RG equations projected on the plane $\sigma, g$. They are somewhat analogous to the one describing a Kosterlitz-Thouless type transition, with an important difference. Here one readily finds that the separatrix is vertical. Introducing the deviation from criticality, $\epsilon = \sigma_c - \sigma_\infty$ (where $\sigma_\infty \equiv \sigma_R$) one has for $\epsilon = 0$ the flow:

$$g_l \sim l^{-3/2}, \quad \sigma = \sigma_R - \gamma (\ln l)^3 l^{-2}$$

and thus satisfy $\sigma_\infty - \sigma_l \sim g_l^{4/3} (\ln (1/g_l))^3$. Thus our RG equations yields $g_l \sim l^{-3/2}$ at criticality and a correlation length:

$$\xi \sim \exp\left(\frac{\text{cst}}{\sigma - \sigma_c}\right)$$

since starting away from criticality $\epsilon > 0$ one finds $g_l \sim l^{-3/2} e^{16\epsilon l}$. This critical behaviour correspond to a new universality class which is different from Kosterlitz-Thouless and from the prediction of [24,25]. Note the crucial role of the $\frac{3}{2}$ universal factor. Replacing it by any number less than 1 would have led to usual KT behaviour.

We can now check that the variations of $D$ should be unimportant at the transition. Indeed one finds that $D_\infty - D_l \sim (\ln l)^2 l^{-2}$ at most. On the other hand we can estimate that if $D_\infty - D_l$ varies faster than $\partial_l m_l \sim 1/l$ the KPP results should not be affected (see the scaling with $l$ of all terms in e.g. Eq (A7) in [39])

Let us close by noting again how universality appears in the derivation of the critical behaviour. Although most details of the fugacity distribution $P(z_\pm)$ (e.g. its bulk, the fusion rule..) depend on the cutoff procedure the universality in the XY transition appears in a remarkable way. It arises from the independence [38–40] of the velocity, the velocity corrections, and front tail on the precise form $f[G]$ of the non linear term in (65). This gives us confidence in the method developped here.

---

15. This indicates that at $T = 0 \langle n_{00} n_{R} \rangle \sim R^{-4} (\ln R)^{-3}$ at criticality.

16. The critical finite size corrections to $\sigma$ read $\sigma_\infty - \sigma_l = \epsilon^2 F[\epsilon l]$ with $F[x] = \int_{+\infty}^x dx x^{-3} e^{-32x}$ up to logarithmic corrections.

17. Note the following interesting cancellation in the variations of $D_l$. Writing $\partial_l \sigma = A_l[X_l]$ as a function of $X_l$, we get that $\partial_l (J^{-1}) = \frac{1}{2} A'[X_l]$ since $X_l$ appears only explicitly in the bound of the integral in (105). Thus one finds $\partial_l D = \frac{1}{2} J^2 A[X_l] (1 - J \Sigma \frac{d}{dX_l} \ln A[X_l])$. Using that $\frac{d}{dX_l} \ln A[X_l] = 2/\sqrt{D} + o(1)$ one gets that the leading variations cancel at the transition point $\sigma = 1/8$. 

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7 Equivalent Sine Gordon model

As is well known the Coulomb gas can also be equivalently formulated as a Sine Gordon model (see e.g. [42]). In this section we identify the random version of the Sine Gordon model to which our analysis applies.

The scalar Sine Gordon model, of partition function $Z_{sg} = \int D\phi e^{-S(\phi)}$ is defined, in the absence of disorder by the action:

$$S(\phi) = \int d^2r \left( \frac{1}{8\pi K} (\nabla_r \phi)^2 - g \cos(\phi) \right)$$  \hspace{1cm} (113)

where $\phi(\mathbf{r}) \in [-\infty, +\infty]$. As is well known it is equivalent to a Coulomb gas since, expanding in $g$ one has:

$$Z_{sg} = \sum_{p=0}^{+\infty} \frac{1}{(2p)!} g^{2p} \langle \left( \int d^2r \cos(\phi(\mathbf{r})) \right)^{2p} \rangle_0$$  \hspace{1cm} (114)

$$= \sum_{p=0}^{+\infty} \frac{1}{(2p)!} \Gamma(0) \frac{g^{2p}}{2^p} \prod_{\alpha=1}^p \int d^2r_\alpha d^2\tilde{r}_\alpha e^{i(\phi(\mathbf{r}_\alpha) - \phi(\tilde{\mathbf{r}}_\alpha))} \langle \rangle_0$$

$$= \sum_{p=0}^{+\infty} \frac{1}{(2p)!} \frac{g^{2p}}{2^p} \prod_{\alpha=1}^p \int d^2r_\alpha d^2\tilde{r}_\alpha e^{iK \sum_{\gamma \neq \delta \gamma \neq \delta} n_\gamma n_\delta G(\mathbf{r}_\gamma - \mathbf{r}_\delta)}$$

where $\langle \rangle_0$ denotes averages with respect only to the quadratic gradient part. The interaction $G(r) = \Gamma(0) - \Gamma(r) \sim \ln |r|$ has been defined in (2). The above partition sums involve only $p$ neutral pairs of dipoles since, in the large size limit $\Gamma(0) \to \infty$. Finally $\gamma = 1, \ldots, 2p$ in the last line involves a summation over all distinct charges with $n_\gamma = 1$ for $\gamma = 1, \ldots, p$ and $n_\gamma = -1$ for $\gamma = p + 1, \ldots, 2p$.

Let us turn to the disordered version of the model. To reproduce the bare version (7) of the model, one must first add a short range correlated random imaginary field as follows:

$$S(\phi, \mathbf{A}) = \int d^2r \left( \frac{1}{8\pi K} (\nabla_r \phi)^2 - i \frac{1}{2\pi} \mathbf{A} \cdot \nabla_r \phi - g \cos(\phi) \right)$$  \hspace{1cm} (115)

with correlations $\overline{\mathbf{A}_q \mathbf{A}_{-q}} = \pi \sigma$. Since it imposes now that $\langle i \nabla_r \phi \rangle_0 = 2KA$, each factor $\exp(i\mathbf{A} \cdot \nabla_r \phi)$ in the $g$ expansion in (114) yields an additional $\exp(n_\gamma \beta V(\mathbf{r}))$ where $\beta^2 V_q V_{-q} = \frac{4\pi \sigma K^2}{q^2}$ and thus reproduces the bare CG version (7) of our model.

Note that the above model still contains a uniform "fugacity" $g$. It thus corresponds to the version studied in [9] by Rubinstein et al. However we know
that this cannot be the correct form under renormalization. The first obvious idea is to generalize $g \to g(r)$, i.e a disordered fugacity (the above expansion can be immediately generalized to this case). This term will be generated, but is not the end of the story. Indeed let us consider the symmetries of the above action $S(\phi)$ (even in the presence of a $g(r)$). When $A(r) = 0$ the action is real and invariant through $\phi(r) \to -\phi(r)$. In the presence of the random field this symmetry is broken (as $\phi(r)$ acquires a non zero, disorder environment dependent average) and the action is complex. Thus nothing prevents that under coarse graining the action will become:

$$S(\phi, A, z_+, z_-) = \int d^2 r \left( \frac{1}{8\pi K} (\nabla r \phi)^2 - i \frac{1}{2\pi} A \cdot \nabla r \phi - z_+(r)e^{i\phi(r)} - z_-(r)e^{-i\phi(r)} \right)$$

(116)

and this is indeed precisely what we have found in the CG formulation, i.e each sign of charge acquires a different local random fugacity.

There is however a symmetry constraint on the distribution of the local random fugacities. Indeed in the above bare model (115) the full partition sum is real, as there is still a statistical symmetry:

$$S(\phi, A) = S(\phi, -A)^*$$

(117)

for any configuration $\phi(r)$ and environment $A(r)$. Since the probabilities of $A(r)$ and $-A(r)$ are the same, all physical averages will be real. In the coarse grained model (116), since we have that

$$S(\phi, A, z_+, z_-) = S(\phi, -A, z_-, z_+)^*$$

(118)

the probability (over environments) of the random fugacity disorder configuration $\{z_+(r), z_-(r)\}$ should be equal to the probability of $\{z_-(r), z_+(r)\}$.

We can now check that this random Sine Gordon model (116) is indeed equivalent to the coarse grained disordered CG considered in this paper by expanding its the partition sum in a given environment, which yields:

$$Z_{sg}(A, z_+, z_-) = \sum_{p=0}^{+\infty} \frac{1}{p!^2} \prod_{\alpha=1}^p \int d^2 r_\alpha d^2 \tilde{r}_\alpha z_+(r_\alpha) z_-(\tilde{r}_\alpha)$$

$$\times \exp \left( K \sum_{\gamma \neq \delta} n_\gamma n_\delta G(r_\gamma - r_\delta) + \beta V^>(r_\gamma) n_\gamma \right)$$

(119)

and thus the $A$ random field disorder is associated to the long range part $V^>$ of the disorder in the CG (see (6)) while the random couplings constants $z_\pm(r)$ in (116) are associated to the local random fugacities in the CG.

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Similarly one can establish a correspondence directly on the replicated versions of both models. Replicating the above SG model and averaging over disorder yields (the limit \( m \to 0 \) is implicit):

\[
Z_{\text{sg}}^m = \int \prod_a D\phi_a e^{-\int d^2 r \frac{1}{8\pi} \sum_{a,b} \left( \frac{1}{K} \delta_{ab} + \sigma \right) \nabla \phi_a \cdot \nabla \phi_b } \\
\times \langle e^{i \int d^2 r (r) \sum_a e^{i \phi_a(r)} - z_+(r) \sum_a e^{-i \phi_a(r)} } \rangle_{z_+,z_-} \tag{120}
\]

Expanded to first order and treating the \( z_+(r) \) and \( z_-(r) \) as uncorrelated in space and reexponentiating yields the replicated SG model defined as:

\[
Z_{\text{sg}}^m = \int \prod_a D\phi_a e^{-S_{\text{rep}}(\phi_a)} \tag{121a}
\]

\[
S_{\text{rep}}(\phi_a) = \int d^2 r \left( \frac{1}{8\pi} \sum_{a,b} \left( \frac{1}{K} \delta_{ab} + \sigma \right) \nabla \phi_a \cdot \nabla \phi_b \right) - \sum_{n \neq 0} Y[n] e^{in \cdot \phi} \tag{121b}
\]

where \( n \cdot \phi = \sum_a n^a \phi_a \) is the scalar product in replica space and \( Y[n] \) are analogous to the vector fugacities introduced previously in the replicated vector CG model. Although the bare model obtained from (120) contains only single component replicated charges, all multicomponent charges will be generated upon coarse graining, as in the replicated vector CG. As we have seen all these charges should be taken into account and thus the generic replicated SG model should contain all possible \( Y[n] \) with \( n \neq 0 \).

This SG model can also be studied using RG, either in its replicated form (121b), or directly (116), very similarly to the vector CG studied in this paper. Although variations in definitions of the fugacities \( Y[n] \) and different cutoff procedures can induce some irrelevant differences in the details of the renormalization (and different RG equations), the two models have the same physics. The Sine gordon formulation has several advantages, such as exhibiting by construction the decomposition of the disorder in two physically very different components (see 6). It is also more amenable to replica variational methods than the CG, left for future study.

8 comparison with previous approaches

Let us compare our method and results with the work of Scheidl [24]. In this work the multicomponent charges (restricted to 0, \( \pm 1 \) in each replica) were considered, but the fusion was not taken into account. Also only dipole fugacities were introduced.
We can recover the RG equations and results of Scheidl within our approach by (i) artificially setting the fusion coefficient $c_2$ to 0 in (18) (ii) assuming a log-normal distribution $P = \Phi/\int \Phi$ for the fugacities (which is consistent only when fusion is neglected) (iii) defining random “dipole fugacities” as $z_+ = z'_+ z''_+$ and $z_\tau = z'_+ z''_\tau$ (or equivalently $u = u' + u''$) identical to the “dummy gaussian variables” introduced by Scheidl. Within our diffusion formalism (see Section(3.3)) we find that the norm $\int \Phi$ diverges exponentially, which allows to recover the extra factors $e^{4t}$ appearing in Scheidl’s screening equations. Going from our diffusion formalism back to the replica formulation yields back the RG equations of [24].

In presence of a fusion term $c_2 > 0$, the equations become a priori very different. The fugacity distribution does not remain log-normal as we do not assume a priori the form of the distribution. Interestingly, although the log-normal does not reproduce correctly the true distribution of fugacities (and misses connections such as the one with the freezing of the DPCT via the KPP equation), some of our results deep in the XY phase, such as the renormalization of $K$ and $\sigma$ (see (94)), agree with the one of Scheidl. It was not obvious a priori that the approach without fusion in the XY phase did not miss extra relevant physics in this model, and indeed near the transition fusion appears to be crucial and must be taken into account.

9 Conclusion

To conclude, we have constructed in this paper a novel renormalization group method which allows to study perturbatively, and in a consistent way, a large class of disordered models which can be formulated as two dimensional Coulomb gases with quenched disorder. We have applied it specifically to the XY model with quenched random phases. We have obtained the phase diagram for this model, confirmed the existence of a low temperature XY phase, and elucidated the critical behaviour at the transition where topological defects proliferate. It would be interesting to check our predictions in numerical simulations.

The present RG method is not based on the conventional perturbative expansion in a vortex fugacity $y$ spatially uniform over the system, which, as we have shown, is only justified for pure models or for disordered models at high enough temperature. Instead, it is constructed by first defining the local random vortex fugacity (or core energy) and then following its full probability distribution under coarse graining. Below a freezing temperature this

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18 in particular the full distribution of local fugacities for finite sizes should be directly measurable in simulations.
distribution becomes very broad and cannot be followed by conventional CG methods. Our renormalization procedure allows to follow this broad distribution in a controlled way, by defining the new perturbative parameter as the concentration $P_l(z \sim 1)$ of rare regions favorable to vortices. This running parameter flows to zero in the XY phase and at the transition (marginal flow). Hence both can be studied perturbatively. The underlying physical picture obtained here is that the transition is controlled, as the scale increases, by the proliferation of vortices in less and less rare favorable regions. We find that it has some features reminiscent of a Kosterlitz-Thouless transition, with important differences, such as the scaling of the correlation length.

To derive the RG equation for the distribution of vortex fugacities, we have introduced two equivalent methods. One is based on the replicated vector Coulomb gas version of the model, and in an expansion in the vector fugacities. The second one is direct, with no use of replicas, and is based on a systematic expansion of all physical quantities in the number of points, i.e. in independent local regions and thus, in the end, in powers of the concentration $P_l(z \sim 1)$ of rare favorable regions. As we have shown these two methods are fully equivalent: the first one being more systematic and the second one allowing for a clear physical understanding of the problem in terms of probability distributions. These two expansion methods are highly non perturbative in the original uniform fugacity variable $y$. Since they are constructed from charge fugacities they can be made fully consistent (by contrast with previous approaches based on dipole fugacities).

Our method sheds light on the broader issue of universality in random systems. The spirit of the RG method is that at large scale most information about the system is irrelevant and can be discarded. In constructing our RG procedure we have first shown, using the very special properties of logarithmic interactions, that it is enough to follow, in addition to the two parameters $K$ and $\sigma$, the distribution of only one or two local (i.e. uncorrelated) random variables $P_l(z_+, z_-)$. At this stage it is clear that we still keep too much information. Indeed we have found that the precise form the non linear RG equation obeyed by $P_l(z_+, z_-)$ as well as the detailed shape of this distribution are largely cutoff dependent. However, this complicated looking RG equation can be generally recast, up to irrelevant terms, as a well known non linear front propagation of the KPP type. Using the known remarkable universality property of this type of non linear equations, we found that all the information needed to describe the universal properties of the XY phase and of the transition is indeed independent of the detailed shape of the fugacity distribution, or of the precise form of its RG equation. Only its tails, and the finite size corrections to the front velocity seem to be needed to determine the physical quantities and the critical behaviour.

Since we are following a full distribution of local disorder, the present method
could be termed a functional RG. We can indeed draw some parallel between this functional RG procedure for the disordered CG and two other known examples where the universal behaviour of a disordered finite-range system is extracted from a functional RG equation. The first one is the asymptotically exact real space RG in $d = 1$ [43,44] well suited to “infinite disorder” fixed points. As emphasized in [16] there are indeed similarities when treating the single vortex, $d = 1$ version of the present model. The method of [44] can be applied for the Sinai potential which has correlations growing as a power of distance. For the present case of weaker, logarithmic correlations, these methods can also be applied in principle (at least around zero temperature where disorder is still very broad) but become very hard to work out analytically. The other example of functional RG appears in a dimensional expansion for the problem of an interface in a random potential [45], which has infinite number of marginal directions at the upper critical dimension $d = d_c$. An important question is whether in the problem studied in this paper there is also an infinite number of marginal directions. As discussed above, the results extracted from the KPP equation seem to suggest that a smaller amount of information than the exact full distribution may be needed here. Thus one can speculate that the critical theory studied here could be equivalently formulated as a more conventional field theory, yet to be identified, with a small number of marginal or relevant operators. In any case the RG method developed here should provide a physically transparent guide to study the system. Given the wide applications of Coulomb gases in two dimensions it is likely that other two dimensional disordered models can be studied using methods similar to the one introduced here.

Finally, another outcome of the present work has been to unveil some interesting connections between the renormalization of a disordered system and the universal features of the propagation of invading fronts in non linear systems. The existence of intriguing relations between freezing transitions and velocity selection in non linear fronts was noticed previously by Derrida and Spohn [29] in their study of the DPCT. Here, we found an even deeper and puzzling connection, between the “universality” in these selection mechanisms and the universality in the critical phenomena captured by the renormalization group [29]. Since attempts have been made to construct renormalization methods in order to extract the universal features of such non linear equations [46], this suggests that a common framework could be developed in connexion with two dimensional disordered models.

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19 and statistically translational invariant by opposition to e.g. Migdal Kadanoff RG, or Cayley tree recursion relations, which are exact only on very special hierachical lattices.

20 the dimensions of operators, in the RG sense, being directly related to the selected velocity of non linear fronts
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A Disordered lattice Coulomb Gas

In this appendix we derive, by exact transformations, a lattice disordered Coulomb gas formulation of the Villain form of (1), extending to this disordered case the approach of Kadanoff [5]. An alternative route to [5] in the pure case in the method used in [32]. For simplicity we first turn to the Villain version of (1) before making the duality transformations, although the inverse procedure could be used (see [32]). This Villain model corresponding to (2) is [47]:

$$Z_{\text{Villain}} = \sum_{[l_{i,x},l_{i,y}] \in \mathbb{Z}} \prod_i \int_{-\pi}^{+\pi} \frac{d\theta_i}{2\pi} \exp \left[ -\frac{K}{2\pi} \sum_{\langle i,j \rangle} (\theta_i - \theta_j - 2\pi l_{ij} - A_{ij})^2 \right]$$

(A.1)

where \( K = \beta J \) and we used the notation \( l_{i,x/y} = l_{i,i} + \hat{e}_x/y \) (see fig. A.1). This Villain partition function indeed corresponds to a \( Z \)-gauge theory, since the action is invariant under \( \theta_i \rightarrow \theta'_i = \theta_i + 2\pi q_i \), \( l_{ij} \rightarrow l'_{ij} = l_{ij} + q_i - q_j \) with \( q_i \in \mathbb{Z} \). Choosing the gauge field \( q_i \) such that for each horizontal link \( l_{i,x} = q_i + \hat{e}_x \) (i.e. \( l'_{i,x} = 0 \)) (free boundary conditions along \( x \)) we obtain a partition function of a gaussian field \( \phi_i = \theta_i/2\pi \) (\( \sum_{i,x} \int_{-\pi}^{+\pi} \frac{d\theta_i}{2\pi} = \int_{-\infty}^{\infty} d\phi_i \)):

$$Z_{\text{Villain}} = \sum_{[l_{i,y}]} \prod_i \int_{-\infty}^{+\infty} d\phi_i e^{-2\pi J \sum_i \left( (\nabla_{i,y} \phi + \frac{1}{2\pi} A_{i,y})^2 + (\nabla_{i,x} \phi - \frac{1}{2\pi} A_{i,x})^2 \right)}$$

with the notation for the discrete gradient \( \nabla_{i,y} \phi = \phi_{i+\hat{e}_y} - \phi_i \). Integrating over this gaussian field yields the Coulomb gas action

$$Z_{\text{Villain}} = \sum_{[l_{i,y}]} e^{-\beta J \sum_{\langle i,j \rangle} \left( (l_{i,y} - l_{j,y} - \frac{1}{2\pi} \nabla_i A) G_{ij} (l_{j,y} - l_{i,y} - \frac{1}{2\pi} \nabla_j A) - \frac{1}{2\pi} l_{i,y}^2 \right)}$$

(A.2)

where we have discarded the \( A/A \) self interaction and we defined the dual lattice charges \( n_\alpha \) with \( \alpha = i + (-\frac{1}{2}, \frac{1}{2}) \) (see fig. A.1):

$$n_\alpha = \hat{e}_z \cdot (\nabla_i \times l) = l_{i,y} - l_{i-x,y}$$

and the quenched random dipoles

$$q_\alpha = \frac{1}{2\pi} \nabla_i A = \frac{1}{2\pi} \left( A_i^y - A_{i-\hat{e}_y}^y + A_i^x - A_{i-\hat{e}_x}^x \right)$$

Note that the neutrality of disorder charges \( \sum_\alpha q_\alpha = 0 \) follows directly from the definition of \( q \) (e.g. taking \( A \) to vanish at the boundary) and implies neutrality
for the integer charges $\sum_\alpha n_\alpha = 0$ (from the divergence of $\Gamma(0)$). The Coulomb potential $\Gamma$ is defined on the lattice by

$$\Delta_i \Gamma = \sum_{j=i\pm \hat{e}_x,i\pm \hat{e}_y} \Gamma_j - 4\Gamma_i = \nabla_x \nabla_x \Gamma_{i-\hat{e}_x} + \nabla_y \nabla_y \Gamma_{i-\hat{e}_y} = 2\pi \delta_{i,0} \quad (A.3)$$

which gives the expression

$$\Gamma_{i-j} = -2\pi \int \frac{d^2q}{(2\pi)^2} \frac{e^{iq(i-j)}}{4 - 2\cos(q_x) - 2\cos(q_y)} \quad (A.4)$$

$$= \Gamma(0) + \ln |i-j| + \ln(2\sqrt{2e\gamma}) + O\left(\frac{1}{|i-j|}\right) \quad (A.5)$$

where the $\gamma$ is the Euler constant. Using the neutrality of the charges and the definition $G_{\alpha-\beta} = \Gamma_0 - \Gamma_{\alpha-\beta}$, and the definition of the disorder potential $V_\alpha = -2J \sum_\beta G_{\alpha\beta} q_\beta$, the lattice disordered CG action reads

$$A_{CG} = -\beta H_{CG} = \beta J \sum_{\alpha\beta} n_\alpha G_{\alpha\beta} n_\beta + \beta \sum_\alpha n_\alpha V_\alpha \quad (A.6)$$

### B Smooth cutoff procedure

The Coulomb gas, or its equivalent Sine Gordon version, can also be renormalized using smooth cutoff procedures. We will not give details here but refer the reader to [48] (see also [14]). Let us simply point out how, in that case, the full disorder $V_r$ can indeed be decomposed, as in (6), into two bona fide disorders.

In the case of a soft cutoff, the continuum approximation of the lattice Coulomb interaction reads (instead of (5)): 
\begin{equation}
G_r^{(a)} = 2\pi \int \frac{d^2q}{(2\pi)^2} \frac{\phi(aq)}{q^2} (1 - \cos(q \cdot r))
\end{equation}

where \( \phi(0) = 1 \) and we will choose \( \phi(x) \) a positive monotonously decreasing function of \( x \). One has the asymptotic large \( r \) behaviour \( G_r^{(a)} \sim \ln r + C(\phi) + O(1/r) \) (see e.g. [14]). One also has:

\begin{equation}
G_r^{(a)} = G_r^{(\bar{a})} - 2\pi dl \int \frac{d^2q}{(2\pi)^2} \frac{aq\phi'(aq)}{q^2} (1 - \cos(q \cdot r)) + O(dl^2)
\end{equation}

where \( \bar{a} = ae^{dl} \). Thus one can write the cutoff dependent decomposition \( V_r = V_r^{>,(a)} + v_r^{(a)} \) and one gets upon increase of cutoff:

\begin{equation}
\overline{(V_r^{>,(a)} - V_r^{>,(\bar{a})})^2} = 4\sigma J^2 G_{r-r'}^{(\bar{a})}
\end{equation}

and \( v_r^{(\bar{a})} = v_r^{(a)} + dv_r \) with:

\begin{equation}
\overline{dv_r dv_r'} = 4\pi \sigma J^2 dl \int \frac{d^2q}{(2\pi)^2} \frac{-aq\phi'(aq)}{q^2} e^{iq \cdot r}
\end{equation}

Thus both \( V_r^{>} \) and \( dv_r \) are well defined physical gaussian disorders since their correlators have positive Fourier transform. In addition \( dv_r \) is short range correlated. For instance, taking \( \phi(x) = e^{-x^2/2} \) one finds:

\begin{equation}
\overline{dv_r dv_r'} = \frac{2\sigma J^2}{(2\pi)^2} dl e^{-(r-r')^2/(2a^2)}
\end{equation}

\section{Renormalisation of the replicated Coulomb gas}

In this appendix, for completeness, we explicitly renormalize the vector Coulomb gas defined by (13). This amounts to extend to \( m \)-component vector charges the renormalisation of scalar CG [6]. The partition function reads:

\begin{equation}
Z_{\text{latt}} = \sum_{N \geq 2} \sum_{\{n_1,...,n_N\}} \prod_{i=1}^{N} \int_{h.c} \frac{d^2r_i}{a_0^2} Y[n_i] e^{A_{n_0}[n_1...n_N]}
\end{equation}

where as usual the primed sum is over all distinct neutral charge configurations and the notation h.c. stand for all the hard core constraints \( |r_i - r_j| \geq a_0 \) for
all pairs $i,j$, implicit in the following. The action $A_{a_o}$ is defined by

$$
A_{a_o}[n_1 \ldots n_N] = \frac{1}{2} \sum_{i \neq j} 2K_{ab} n_i^a \ln \left( \frac{|r_i - r_j|}{a_o} \right) n_j^b \quad (C.2)
$$

A common way to renormalise usual Coulomb gas consists in coarse graining the partition function, leaving the expansion in number of fugacity (here $Y$) unchanged. This amounts to define scale dependent replica stiffness $K_{ab}(l)$ and fugacities $Y_l[n]$. We will follow this scheme in this appendix. Note that another equivalent way would be to renormalise the correlation function directly, following e.g. [49]. As in the scalar case [2,6], renormalisation of the generalised vector Coulomb gas [50,51] proceeds in the same three steps: rescaling, fusion and annihilation (screening) of small dipoles. We now turn to the description of these three contributions: first we increase the hard-core cut-off $a_o \rightarrow \tilde{a}_o = a_o e^{dl}$ with $dl \ll 1$.

**C.1 Rescaling**

This increase of cut-off produces a naive rescaling:

$$
\prod_{i=1}^N \int \frac{d^2 r_i}{\tilde{a}_o^2} Y[n_i] e^{A_{a_o}[n_1 \ldots n_N]} = \prod_{i=1}^N \int \frac{d^2 r_i}{\tilde{a}_o^2} Y[n_i] e^{A_{a_o}[n_1 \ldots n_N]} e^{dl(2-K_{ab} n_i^a n_i^b)} \quad (C.3)
$$

where we used the neutrality $\sum_i n_i^a = 0$ to express the correction coming from the action. We can absorb the extra factor in (C.3) to all order in $Y$ by the change of fugacities corresponding to the equation

$$
\partial_l Y[n] = (2 - K_{ab} n_i^a n_i^b) Y[n] \quad (C.4)
$$

**C.2 Annihilation (screening) and Fusion of charges**

Upon the increase of cut-off, two charges $n_p$ and $n_q$ have to be coarse grained if they are located in $r_p$ and $r_q$ with $a_o \leq |r_p - r_q| \leq a_o e^{dl}$. Within the small charge density hypothesis, we consider only one such pair. For a dipole, these two charges have to be integrated out at scale $\tilde{a}_o$: this corresponds to the annihilation, while for a non neutral pair, the coarse grained charge is simply the sum of the two charges at scale $a_o$ (fusion). In both cases the partition function splits into $Z = Z' + Z_{p,q}$ where $Z_{p,q} \sim O(dl)$ involves configurations with one pair of charges $n_p,n_q$ distant of less than $\tilde{a}_o$ while $Z'$ doesn’t. $Z_{p,q}$
can be written as

\[ Z_{p,q} = \sum_{\{n_1 \ldots n_N\}} \prod_{i=1, \ldots, N, i \neq p, q} \int \frac{d^2 r_i}{a_o^2} Y[n_i] \]
\[ \times \sum_{n_p, n_q} \int_{a_o \leq |r_p - r_q| \leq a_o e^d} \frac{d^2 r_p}{a_o^2} \frac{d^2 r_q}{a_o^2} Y[n_p] Y[n_q] e^{A_{p,q}[n_1 \ldots n_N]} \]  

(C.5)

where, with the notation \( \alpha_{i,j} = \sum_{a,b} n_a \alpha_{i} n_b \gamma_{a,b} n_j \), the action reads

\[ e^{A_{p,q}} = \left( \frac{|r_i - r_p|}{a_o} \right)^{\alpha_{pq}} \prod_{i \neq p, q} \left( \frac{|r_i - r_q|}{a_o} \right)^{\alpha_{iq}} \prod_{i<j \neq p, q} \left( \frac{|r_i - r_j|}{a_o} \right)^{\alpha_{ij}} \]  

(C.6)

We must now distinguish between a neutral pair and a non neutral one.

### C.2.1 Fusion

In this case the small pair of charges \((n_p, n_q)\) at scale \(a_o\) gives an effective charge \(n_p + n_q\) at scale \(\tilde{a}_o\), located in \(R = (r_p + r_q)/2\). Thus we must integrate over the relative position of the two charges \(\rho = r_p - r_q\) when coarse graining the Coulomb Gas. To obtain the corresponding correction to \(Z'\) (of order one in \(dl\)), it is enough to expand (C.6) to order 0 in \(\rho\). This expansion reads

\[ \sum_{\{n_1 \ldots n_N\}} \prod_{i=1, \ldots, N, i \neq p, q} \int \frac{d^2 r_i}{a_o^2} Y[n_i] \sum_{n_p, n_q} \int \frac{d^2 R}{a_o^2} Y[n_p] Y[n_q] \left( \int_{a_o \leq \rho \leq a_o e^d} \frac{d^2 \rho}{a_o^2} \right) \]
\[ \times \prod_{i \neq p, q} \left( \frac{|r_i - R|}{a_o} \right)^{\alpha_{ip} + \alpha_{iq}} \prod_{i<j \neq p, q} \left( \frac{|r_i - r_j|}{a_o} \right)^{\alpha_{ij}} \]

Using \(\alpha_{i,p} + \alpha_{i,q} = \alpha_{i,p+q}\), we can rewrite this correction to \(Z'\) as a single contribution to the fugacity \(Y[n_p + n_q]\) of the non zero charge \(n_p + n_q\):

\[ \partial_l Y[n_p + n_q] = 2\pi \sum_{n_p, n_q} Y[n_p] Y[n_q] \]  

(C.7)

Note that in this expression \(n_p\) and \(n_q\) are distinguishable charges: this explains the factor 2 between (C.7) and (14b).

### C.2.2 Annihilation

In this case the small dipole of size \(\rho\) is integrated out at scale \(\tilde{a}_o\). When coarse graining, we sum over both \(\rho\) and \(R\), yielding a factor \(a_o^{-4}\) from the integration measure. Thus diverging contributions correspond to the expansion of (C.6)
to order 2 in $\rho$. Using $\alpha_{i,p} = -\alpha_{i,q}$, this expansion of $Z_{p,q}$ can be expressed as

$$
\sum'_{\{n_1, \ldots, n_N\}} \prod_{i=1, N, i \neq p, q} \int \frac{d^2 r_i}{a_o^2} Y[n_i] \prod_{i, j \neq p, q} \left( \frac{|r_i - r_j|}{a_o} \right)^{\alpha_{ij}} \sum_{n_p} Y[n_p] Y[-n_p] 
\times \int_{a_o \leq \rho \leq a_o e^{dl}} \int \frac{d^2 R}{a_o^2} \left( 1 + \sum_{i, j \neq p, q} \alpha_{i,p} \alpha_{j,p} \frac{1}{4} \rho \cdot \nabla \ln(r_i - R) \rho \cdot \nabla \ln(r_j - R) \right)
$$

Performing the integral and reexponentiating the last term using the neutrality of the configuration $\{n_1, \ldots, n_N\}$, we get the correction to $Z'$ coming from $Z_{p,-p}$:

$$
C \sum'_{\{n_1, \ldots, n_N\}} \prod_{i=1, N, i \neq p, q} \int \frac{d^2 r_i}{a_o^2} Y[n_i] 
\times \prod_{i, j \neq p, q} \left( \frac{|r_i - r_j|}{a_o} \right)^{\alpha_{ij} - \pi^2 d} \sum_{n_p} \alpha_{i,p} \alpha_{j,p} Y[n_p] Y[-n_p]
$$

The constant $C$ corrects the intensive free energy while the second term can be absorbed in a correction to the coupling constant

$$
\partial_t K^{-1}_{ab} = 2\pi^2 \sum_{|n|} n^c n^d Y[n] Y[-n]
$$

The three above contributions can be summarized into the set of RG equations given in the text (14) valid for all non zero vector charge, and for all $m$. The coefficient $c_1$ and $c_2$ depends on the IR regularisation. For our hard cut-off, we find within our procedure that $c_1 = 2\pi^2$ and $c_2 = \pi$. Note that the ratio $c_1/c_2$ is independent of a uniform rescaling of the fugacities and is known, in the case of single component charges to be universal at a transition [49]. For a discussion of regularisation of replicated Coulomb Gas, see discussion in appendix F.

**D m → 0 limit of the replica RG equations**

In this Appendix we explicitly perform the $m \rightarrow 0$ limit of the whole set of RG equations (14a,14b) with the restriction that the vector charges have only 0, $\pm 1$ components. This limit is taken using the parametrisation (16) of the fugacities $Y[n]$ in terms of the function $\Phi(z_+, z_-)$. In the three different terms of (14a,14b), corresponding to rescaling, annihilation and fusion contributions (see appendix C), we first modify sums to include fugacities for null charge, translate the expression in terms of $\Phi$ and naturally take the $m \rightarrow 0$ limit. We will use the notation $\langle A \rangle_{\Phi} = \int_{z_+, z_-} A \Phi(z_+, z_-).$
D.1 Rescaling

The term corresponding to rescaling in (14b) is

$$\partial_t Y[n] = (2 - n_a K^{ab} n_b) Y[n]$$  \hspace{1cm} (D.1)

which holds for any vector charge $n$. The second term can be expressed in terms of $\Phi$ using (16) and $K^{ab} = K\delta^{ab} - \sigma K^2$ with $K = \beta J$. However the expression is much simpler if one uses, instead of (16), the equivalent parametrisation

$$Y[n] = \int_{u,v} e^{\beta p u} e^{\beta q v} \tilde{\Phi}(u,v)$$

with $p = n^+ + n^-$, $q = n^+ - n^-$ and $z_{\pm} = e^{\beta(u \pm v)}$. With $\tilde{\Phi}$, this yields

$$\sum_{a,b} n_a K^{ab} n_b Y[n] = (\beta J p - \sigma \beta^2 J^2 q^2) \int_{u,v} \tilde{\Phi}(u,v) e^{\beta p u} e^{\beta q v}$$  \hspace{1cm} (D.2)

$$= \int_{u,v} \tilde{\Phi}(u,v) \left( J \frac{\partial}{\partial u} - \sigma J^2 \frac{\partial^2}{\partial v^2} \right) e^{\beta p u} e^{\beta q v}$$  \hspace{1cm} (D.3)

$$= - \int_{u,v} e^{\beta p u} e^{\beta q v} \left( J \frac{\partial}{\partial u} + \sigma J^2 \frac{\partial^2}{\partial v^2} \right) \tilde{\Phi}(u,v)$$  \hspace{1cm} (D.4)

As this is true for any vector charge $n$ satisfying $n^a = 0, \pm 1$, i.e. for any $p,q$ or equivalently $n_+, n_-$, to satisfy (D.1) we can search for a function $\Phi$ such that:

$$\partial_t \tilde{\Phi}(u,v) = \left( 2 + J \frac{\partial}{\partial u} + \sigma J^2 \frac{\partial^2}{\partial v^2} \right) \tilde{\Phi}(u,v)$$

In terms of $\Phi(z_+, z_-)$, this corresponds to the equation:

$$\partial_t \Phi(z_+, z_-) = (2 + \mathcal{O}) \Phi(z_+, z_-)$$  \hspace{1cm} (D.5)

$$\equiv \left( 2 + \beta J (2 + z_+ \partial_{z_+} + z_- \partial_{z_-}) + \sigma \beta^2 J^2 (z_+ \partial_{z_+} - z_- \partial_{z_-})^2 \right) \Phi(z_+, z_-)$$

where we have used that $\tilde{\Phi}(u,v) = 2z_+ z_- \Phi(z_+, z_-)$. Note that for this process, the integral of $\Phi$, $\mathcal{N} = \int \Phi$ satisfies simply $\partial_t \mathcal{N} = 2\mathcal{N}$.

D.2 Fusion

In (14b), the fusion term of two charges $n' + n'' = n$ is restricted to $n' \neq 0$ and $n'' \neq 0$. Furthermore since $n = 0$ corresponds to the annihilation which is treated separately in (14a) (see below) and must not be counted twice, the equation (14b) can be used only for $n \neq 0$. It is convenient to extend the equation (14b) to include $Y[n = 0] = \int \Phi = \mathcal{N}$ for which the fusion
Using permutation symmetry we find that (D.6) can be written as:

$$\partial_t Y[n] = c_2 \left( \sum_{n'+n''=n \atop n',n'' \neq 0} Y[n']Y[n''] - \delta_{n,0} \sum_{n' \neq 0} Y[n']Y[-n'] \right)$$

$$= c_2 \left( \sum_{n'+n''=n} Y[n']Y[n''] - 2Y[0]Y[n] - \delta_{n,0} \left( \sum_{n'} Y[n']Y[-n'] - 2Y[0]^2 \right) \right)$$

where in the second equality we allow for $n' = 0$ or $n'' = 0$. Turning to the representation in terms of $n_+, n_-$ we have:

$$\sum_{n'+n''=n} Y[n']Y[n''] = \left\langle \prod_a \left[ (1 + z_-'z_+'' + z_+z_-')\delta_{n_a,0} + (z_+'' + z_-')\delta_{n_a,1} + (z_-' + z_-')\delta_{n_a,-1} \right] \right\rangle_{\Phi,\Phi}$$

Using permutation symmetry we find that (D.6) can be written as:

$$\partial_t Y[n] = c_2 \left\langle (1 + z_-'z_+'' + z_+z_-')^m \left( \frac{z_+ + z_-'}{1 + z_-'z_+'' + z_+z_-'} \right)^{n_+} \left( \frac{z_-' + z_-'}{1 + z_-'z_+'' + z_+z_-'} \right)^{n_-} - 2c_2N \left\langle z_+^{n_+} z_-^{n_-} \right\rangle_\Phi - c_2\delta_{n_+,0}\delta_{n_-,0} \left( \left\langle (1 + z_-'z_+'' + z_+z_-')^m - 1 \right\rangle_\Phi - N^2 \right) \right\rangle_{\Phi,\Phi}$$

The following choice for $\partial_t \Phi(z_+, z_-)$ allows to satisfy the above equation for all $n$ (up to now $m$ is still arbitrary):

$$\partial_t \Phi(z_+, z_-) = c_2 \left\langle (1 + z_-'z_+'' + z_+z_-')^m \delta \left( z_+ - \frac{z_+ + z_-'}{1 + z_-'z_+'' + z_+z_-'} \right) \delta \left( z_- - \frac{z_-' + z_-'}{1 + z_-'z_+'' + z_+z_-'} \right) \right\rangle_{\Phi,\Phi}$$

$$-2c_2N \Phi(z_+, z_-) - c_2\delta(z_+\delta(z_-) \left( \left\langle (1 + z_-'z_+'' + z_+z_-')^m - 1 \right\rangle_\Phi - N^2 \right) \right\rangle_{\Phi,\Phi}$$

We can now take the limit $m \to 0$ explicitly on this equation, which yields:

$$\partial_t \Phi(z_+, z_-) = c_2 \left\langle \delta \left( z_+ - \frac{z_+ + z_-'}{1 + z_-'z_+'' + z_+z_-'} \right) \delta \left( z_- - \frac{z_-' + z_-'}{1 + z_-'z_+'' + z_+z_-'} \right) \right\rangle_{\Phi,\Phi}$$

$$-2c_2N \Phi(z_+, z_-) + c_2\delta(z_+)\delta(z_-)N^2$$

(D.8)
D.3 Annihilation

The screening equation (14a) for the replica coupling constant reads

\[ \partial_t (K^{-1})^{ab} = c_1 \sum_{n \neq 0} n^a n^b Y[n] Y[-n] \quad (D.9) \]

which corresponds to the RG equations for \( \sigma \) and \( J \)

\[ \partial_t (TJ^{-1} + \sigma) = \frac{c_1}{m} \sum_{a=1}^{m} \sum_{n \neq 0} n^a n^b Y[n] Y[-n] \quad (D.10) \]

\[ \partial_t \sigma = \frac{c_1}{m(m-1)} \sum_{a \neq b=1}^{m} \sum_{n \neq 0} n^a n^b Y[n] Y[-n] \quad (D.11) \]

The sums over the charge \( n \) in (D.10,D.11) can be easily expressed in terms of \( \Phi \) using the variable \( n_+, n_- \). The sum in (D.11) thus reads

\[
\begin{align*}
\sum_{a \neq b=1}^{m} \sum_{n \neq 0} n^a n^b Y[n] Y[-n] \\
= \sum_{0 \leq n_+ + n_- \leq m} C_{m,n_+,n_-}^{m,n_+} \left[ (n_+ - n_-)^2 - (n_+ + n_-) \right] \left\langle (z_+^{n_+})^{n_+} (z_-^{n_-})^{n_-} \right\rangle_{\Phi(z') \Phi(z'')} \\
= \left\langle \left[ (z_+^{n_+})^{n_+} (z_-^{n_-})^{n_-} \right] \right\rangle_{\Phi(z') \Phi(z'')} \left( (z_+^{n_+})^{n_+} (z_-^{n_-})^{n_-} \right) \end{align*}
\]

where \( C_{m,n_+}^{n_+,n_-} = m! n_+! n_-! (m - n_+ - n_-)! \). Taking the \( m \to 0 \) limit of the last equation yields directly with (D.11)

\[ \partial_t \sigma = c_1 \left\langle \left( \frac{z_+^{n_+} z_-^{n_-} - z_+^{n_-} z_-^{n_+}}{1 + z_+^{n_+} z_-^{n_-} + z_+^{n_-} z_-^{n_+}} \right)^2 \right\rangle_{\Phi(z') \Phi(z'')} \quad (D.12) \]

By the same method we get for the scaling equation of \( K^{-1} \):

\[ \partial_t (TJ^{-1}) = c_1 \left\langle \frac{z_+^{n_+} z_-^{n_-} + z_+^{n_-} z_-^{n_+} + 4z_+^{n_+} z_-^{n_-} z_+^{n_-} z_-^{n_+}}{(1 + z_+^{n_+} z_-^{n_-} + z_+^{n_-} z_-^{n_+})^2} \right\rangle_{\Phi(z') \Phi(z'')} \quad (D.13) \]
Putting all these contributions together, we obtain the scaling equations for the coupling constant $J$, the correlated disorder strength $\sigma$ and the distribution of local disorder $\Phi(z_+, z_-)$:

\[ \partial \sigma = c_1 \left\langle \left( \frac{z'_+ z''_+ - z'_- z''_-}{1 + z'_+ z''_+ + z'_- z''_-} \right)^2 \right\rangle \Phi(z') \Phi(z'') \]  
\[ \partial K^{-1} = c_1 \left\langle \frac{z'_+ z''_+ + z'_- z''_- + 4 z'_+ z''_- z'_- z''_+}{(1 + z'_+ z''_+ + z'_- z''_-)^2} \right\rangle \Phi(z') \Phi(z'') \]  
\[ \partial \Phi(z_+, z_-) = (2 + O) \Phi(z_+, z_-) \]  
\[ + c_2 \left\langle \delta \left( z_+ - \frac{z'_+ + z''_+}{1 + z'_- z''_+ + z'_- z''_-} \right) \delta \left( z_- - \frac{z'_- + z''_-}{1 + z'_- z''_+ + z'_- z''_-} \right) \right\rangle \Phi \Phi \]  
\[ - 2c_2 N \Phi(z_+, z_-) + c_2 \delta(z_+) \delta(z_-) N^2 \]

where the diffusion operator has been defined in (D.6).

E Connection between the direct and the replica method

In this appendix we explore the connections between the expansion in number of sites of the free energy, and the expansion of the replicated partition function in power of composite charge fugacities $Y[n]$. As we will see, both expansions coincide exactly, and we can thus consider the expansion of the replicated partition function as a generating functional of the site-expansions.

E.1 The 2 point free energy $f^{(2)}$

First we consider the first term $f^{(2)}$ given by (30) of the expansion in number of independent sites (28). This term corresponds exactly to the approximation of independent dipoles considered in [23]: dipoles do not interact with each other, and the free energy is thus the sum over all positions of the pairs of the free energy of a pair: $F = \sum_{r, r'} f^{(2)}_{r, r'}$, and the free energy of a dipole is simply

\[ -\beta f^{(2)}_{r, r'} = \ln(1 + W_{r, r'}) \]  
\[ W_{r, r'} = y^2 \left( \frac{|r - r'|}{a_o} \right)^{-2\beta J} (w_{r, r'} + w_{r', r}) \]  
\[ w_{r', r} = e^{\beta (V_r - V_{r'})} \]
Using the decomposition of the disorder (section 2.2) $V_r = V_r^\varphi + v_r$, and averaging over the correlated disorder $V_r^\varphi$ using replica, we obtain

$$\ln(1 + W_{r, r'})^{V_r^\varphi} \simeq_{m \to 0} (1 + W_{r, r'})^{-1} (1 + W_{r', r''})^{-1}$$

(E.1)

$$= \sum_{1 \leq p + q \leq m} (z_r^p z_r'^q) (z_r^p z_r'^q)^{-2(p+q)\beta J + 2(p-q)\sigma^2 J^2}$$

(E.2)

where we used the non local correlation of the disorder $(V_r^\varphi - V_r'^q)^2 = 4\sigma J^2 \ln (|r - r'|/a_o)$ and the definition $z_r^p = ye^{\pm \beta v_r}$. With the bare definition of the replica charge fugacities, this finally gives after average over $v_r$, the expected first term of the expansion of $Z^m$:

$$\ln(1 + W_{r, r'})^{V_r^\varphi} \simeq_{m \to 0} \sum_{|n| \neq 0} Y[n] Y[-n] \left(\frac{|r - r'|}{a_o}\right)^{-2\mu K_{ab} n_b}$$

(E.3)

E.2 The 3 points free energy $f^{(3)}$

The methods is the same as in previous Section, but the calculations are slightly more tedious. We consider the second term of the free energy expansion:

$$\ln \left(\frac{1 + W_{r, r'} + W_{r, r''} + W_{r', r''}}{(1 + W_{r, r'}) (1 + W_{r, r''}) (1 + W_{r', r''})}\right)$$

(E.4)

With the help of the previous Section, it is enough to consider only

$$\ln (1 + W_{r, r'} + W_{r, r''} + W_{r', r''})$$

Using the decomposition of figure (E.1), we obtain the sum

$$\ln (1 + W_{r, r'} + W_{r, r''} + W_{r', r''}) \simeq_{m \to 0} \sum_{|p| + q + s \leq m} \sum_{p_1 + q_2 = p} \sum_{q_1 + q_2 = q} \sum_{s_1 + s_2 = s} w_{r, r'}^{p_1} w_{r', r''}^{p_2} w_{r''}^{q_1} w_{r'}^{q_2} w_{r, r''}^{s_1} w_{r', r'}^{s_2} w_{r'}^{s_2} w_{r'}^{s_2} \times y^{2(p+q+s)} \left(\frac{|r - r'|}{a_o}\right)^{-2\beta J p} \left(\frac{|r - r''|}{a_o}\right)^{-2\beta J q} \left(\frac{|r' - r''|}{a_o}\right)^{-2\beta J s}$$

By averaging the second term over $V_r^\varphi$, and using $z_r^p = ye^{\pm \beta v_r}$, we get
Fig. E.1. possible configurations of neutral triplet of vector charges and parametrisation of this triplet by the numbers of pairs of components (+1, -1): (p,q,s)

\[ u_{r,r'}^{p_1} u_{r',r''}^{p_2} u_{r''}^{p_3} u_{r''}^{q_1} u_{r''}^{q_2} u_{r''}^{s_1} u_{r''}^{s_2} \]

\[ \langle \mathcal{V} \rangle \]

\[ = \left( z_+^r z_+^{r'} \right)^{p_1} \left( z_-^r z_-^{r'} \right)^{p_2} \left( z_+^r z_-^{r''} \right)^{q_1} \left( z_-^r z_+^{r''} \right)^{q_2} \left( z_+^r z_-^{r''} \right)^{s_1} \left( z_-^r z_+^{r''} \right)^{s_2} \]

\[ \times \left( \frac{\left| r - r' \right|}{a_o} \right)^{2\sigma \beta^2 J^2 (p_1 - p_2)^2} \left( \frac{\left| r - r'' \right|}{a_o} \right)^{2\sigma \beta^2 J^2 (q_1 - q_2)^2} \left( \frac{\left| r' - r'' \right|}{a_o} \right)^{2\sigma \beta^2 J^2 (s_1 - s_2)^2} \]

Inserting this result in the above expression averaged over \( \mathcal{V} \rangle \), the sum over the partitions of the interval \([0, m]\) in \( 0 \leq p_1 + p_2 + q_1 + q_2 + s_1 + s_2 \leq m \) can be exactly rewritten as a sum over all neutral triplets \( n, n', n'' \) of \( m \) component vector charges with components 0, ±1 (see figure E.1). Thus one recovers the expression

\[ \sum_{n + n' + n'' = 0} Y[n] Y[n'] Y[n''] e^{-S_{n,n',n''}} - 1 \]

We now note that the last three logarithmic terms in (E.4) just give additional restrictions on this sum (as they correspond respectively to \( n = 0, n' = 0, n'' = 0 \)). Thus, averaging over the local fugacities, we end up with the expected second term (see 11) of the expansion of \( Z^m \) in power of the vector fugacities \( Y[n] \):

\[ \sum_{n + n' + n'' = 0} Y[n] Y[n'] Y[n''] e^{-\beta H_{n,n',n''}} \]  

(E.5)

These simple combinatorics can be done on higher order terms: it gives the equivalence term by term between the \( m \to 0 \) limit of the expansion in \( Y[n] \) and the expansion in the number of independent sites of the moments of the free energy.
Regularization of replicated Coulomb gases

In this Appendix we discuss the consequences on the RG equations of the choice of cutoff made in this paper for the replicated Coulomb gas. We illustrate for simplicity only the case of zero disorder. Although we will be mainly concerned with Coulomb gas, most of this discussion can be applied to other general replica field theory.

Let us first recall the results for a single component CG \((m = 1)\). We restrict to the most relevant charges \(\pm 1\) for simplicity. It is defined by the action

\[-\beta H = K \sum_{|r-r'| \geq a_0} n(r) G(|r - r'|) n(r') + \ln(y) \sum r n^2(r)\]  

(F.1)

The corresponding RG equation was derived by Kosterlitz:

\[\partial_l y(l) = (2 - K) y + \mathcal{O}(y^3)\]  

(F.2)

We now consider \(m\) copies of this model in the absence of disorder, as illustrated in Fig. (F.1). They are a priori physically completely uncoupled. The most natural cutoff procedure in that case would be independent cutoffs (e.g. hard core for each) (left figure). Another procedure, which becomes much more convenient in the presence of disorder, is to reformulate the \(m\) copies as a single Coulomb gas of vector charges with \(m\) components. However in that case the cutoff is by definition columnar (right figure) (e.g hard core vector charge are a hard columnar disk) and in a sense the copies are coupled, via the cutoff. We now check that in the pure case the ensuing vector CG RG equations are still perfectly compatible with (F.2) as they should. They read:

\[\partial_l Y[n] = (2 - K n \cdot n) Y[n] + c_2 \sum_{n' \neq 0, n} Y[n'] Y[n - n'] + \text{higher order terms}\]

Let us first illustrate the case of two copies \(m = 2\). We can choose \(Y[1, 0] = Y[-1, 0] = Y[0, 1] = Y[0, -1] = Y_1\) and \(Y[1, 1] = Y[-1, 1] = Y[1, -1] = Y[-1, -1] = Y_2\). Then considering all the possible fusions within this set the RG equations read:

\[\partial_l Y_1 = (2 - K) Y_1 + 4c_2 Y_1 Y_2\]
\[\partial_l Y_2 = (2 - 2K) Y_2 + 2c_2 Y_1^2\]

A solution of these equations, to the order \(o(y^3)\) at which we are working, is:

\[Y_1[1, 0] = \lambda_1 y_l\quad Y_1[1, 1] = \lambda_2 y_l^2\]
with $\lambda_2 = c_2 \lambda_1^2$, where $y_1$ satisfies the single copy equation (F.2). This can be
generalized to higher charges so that in general one can find solutions of the

$$
Y_p = Y_{[1,1,\ldots,1,0,\ldots,0]} = \lambda_i y_i^j
$$

with coefficients $\lambda_p$ which can be determined for a given regularization proce-
dure. They depend on both the initial ($m = 1$) and the replica regularization
(see fig F.1).

![Fig. F.1. Schematic representation of the definition of the replicated vector charges
and their cutoff](image)

This example illustrates how a generic cutoff procedure will produce non trivial
coefficients $\lambda_i$. Thus if one reinterprets formally $Y[n] = \langle y_n^+ + n^- \rangle > \Phi(y)$ in
terms of a “disorder” $\Phi(y)$ as done in this paper [21], one must keep in mind
that even the pure system corresponds to a non trivial “bare disorder” in the
fugacities, which solely originates from the (convenient) choice of a columnar
cutoff [22]. It is thus clear that the definition of the local fugacity distribution
is strongly cutoff dependent.

## G Explicit derivation of the expansion in the number of points

In this appendix we derive formula (29) for the expansion of the free energy
in the number of points. The same method can be applied to other physical
observables.

To organize this expansion we start by introducing fictitious site dependent
fugacities for the charges : $\zeta_{ri}$. These are introduced only as a trick in this
Appendix and should not be confused with the real disordered fugacities of
(6). Indeed, to recover the original model, we will set them back at the end
either to $\zeta_{ri} = 1$ (for the lattice model (4)) or $\zeta_{ri} = y$ for the continuum model,
where $y$ is the corresponding fugacity for the charge in the pure case. These

\[ z^+ = z^- = y \]

\[ z = \zeta_{ri} = y \]

in the interpretation as a branching process, it does correspond to the disorder
in the tree structure. See the end of Section 5.5 for a particular choice which does
not introduce additional disorder.

\[ z = \zeta_{ri} = y \]

\[ z = \zeta_{ri} = y \]
fugacities are introduced by writing a more general form of the the partition function (4) as

\[
Z[V, \zeta] = \sum_{p} \sum_{\{n_1, \ldots, n_p\}} \sum_{r_1 \neq \cdots \neq r_p} \left( \prod_{i} \frac{n_i!}{r_i!} \right) e^{\beta \sum_{i} \sum_{r_i \neq r_j} n_i G_{r_i - r_j} n_j + \beta \sum_i n_i V_{r_i}} \tag{G.1}
\]

Here and below, as in (7), all formulae can be extended to the continuum model by replacing discrete sums over distincts sites \( r_1 \neq r_2 \cdots \neq r_p \) by integrals with, e.g. hard core conditions \( |r_1 - r_2| \geq a_0 \). Note that in the above expression (G.1) we do not make use of the decomposition (6) and \( V \) denotes the original disorder.

Let us consider for simplicity a system of \( N \) distinct points \( r_1, \ldots r_N \). The free energy functional \( F[V, \zeta] = -T \ln Z[V, \zeta] \) is a function of the \( N \) variables \( \zeta_{r_1}, \ldots, \zeta_{r_N} \). Let us write the conventional Taylor expansion of the free energy around \( \zeta = 0 \):

\[
F[V, \zeta] = \sum_{p_1, p_2, \ldots, p_N = 1}^{\infty} \frac{1}{p_1! \cdots p_N!} \frac{\partial^{p_1 + \cdots + p_N} F[V, \zeta]}{\partial \zeta_{r_1}^{p_1} \cdots \partial \zeta_{r_N}^{p_N}} \mid_{\zeta = 0} \times \zeta_{r_1}^{p_1} \cdots \zeta_{r_N}^{p_N} \tag{G.2}
\]

We now separate in this sum the terms which have only one non zero \( p_i \), then only two non zero \( p_i \), etc.. Thus we can rewrite:

\[
F[V, \zeta] = \sum_{k=1}^{\infty} \sum_{\{r_{1}, \ldots, r_{ik}\}} \sum_{q_1, \ldots, q_k = 1}^{\infty} \frac{1}{q_1! \cdots q_k!} \frac{\partial^{q_1 + \cdots + q_k} F[V, \zeta]}{\partial \zeta_{r_{1}}^{q_1} \cdots \partial \zeta_{r_{k}}^{q_k}} \mid_{\zeta = 0} \times \zeta_{r_{1}}^{q_1} \cdots \zeta_{r_{k}}^{q_k} \tag{G.3}
\]

where the sum is over all distincts sets \( \{r_{1}, \ldots, r_{ik}\} \) of \( k \) distincts points (among the \( N \) points) and the sum over each \( q_i \) goes from 1 to \( +\infty \).

We have thus obtained an expansion of \( F[V, \zeta] \) as a sum of terms of the form \( f_{r_{1}, \ldots, r_{ik}}^{(k)} \) which depends exactly and only on the variables \( \zeta \) (and thus also only on the variables \( V \)) evaluated at the \( k \) distinct points \( r_{1}, \ldots, r_{ik} \). For a neutral Coulomb gas it starts with \( k = 2 \) and reads:

\[
F[V, \zeta] = \sum_{\{r_{1}, r_{2}\}, \{r_{1}, r_{2}, r_{3}\}} f_{r_{1}, r_{2}}^{(2)} [V, \zeta] + \sum_{\{r_{1}, r_{2}, r_{3}\}} f_{r_{1}, r_{2}, r_{3}}^{(3)} [V, \zeta] + \ldots \tag{G.4}
\]

with the definition

\[
f_{r_{1}, \ldots, r_{ik}}^{(k)} [V, \zeta] = \sum_{q_{1}, \ldots, q_{k} = 1}^{\infty} \frac{1}{q_{1}! \cdots q_{k}!} \frac{\partial^{q_{1} + \cdots + q_{k}} F[V, \zeta]}{\partial \zeta_{r_{1}}^{q_1} \cdots \partial \zeta_{r_{k}}^{q_k}} \mid_{\zeta = 0} \times \zeta_{r_{1}}^{q_1} \cdots \zeta_{r_{k}}^{q_k} \tag{G.5}
\]
Note that in this last expression we can drop out the dependence of \( F[V, \zeta] \) on the fugacities \( \zeta_r \) and the potential \( V_r \) at points different from \( r_{i_1}, \ldots r_{i_k} \).

A more explicit expression can be obtained by summing over the \( q_i \) in (G.5):

\[
f^{(k)}_{r_{i_1}, \ldots r_{i_k}}[V, \zeta] = \sum_{l=0}^{k} (-1)^{k-l} \sum_{i_1, \ldots i_l \in \{1, \ldots k\}} F_{r_{i_1}, \ldots r_{i_l}}[V, \zeta] \tag{G.6}
\]

where \( F_{r_{i_1}, \ldots r_{i_l}}[V, \zeta] \) is the free energy associated with the system of sites \( r_{i_1}, \ldots r_{i_l} \) (instead of the full lattice). Equivalently, it does depend only on the fugacities \( \zeta_r \) (or potential \( V_r \)) at points \( r_{i_1}, \ldots r_{i_l} \). After setting \( \zeta_r = 1 \) (lattice model) or \( \zeta_r = y \) (continuum model) this gives the definition of the \( f^{(k)}[V] \) that we use throughout this paper, namely the equation (29). This last expression (G.6) allows to explicitly compute the coefficient \( f^{(k)}[V] \) of the expansion (G.4) for arbitrary order \( k \). In the Section (4.2) we use it to give explicit expressions for the first few terms of the expansion of the free energy.

### H Fusion corrections to the free energy expansion

As shown in the previous Section the free energy expansion involves the partition function \( Z^{(p)}_{r_{i_1}, \ldots r_{i_p}} \) of a system of finite number of sites. We illustrate here how the rule of fusion of environments described in Section 4.3 works more generally. Let us consider a system of \( p \geq 4 \) sites, with charges restricted to \( 0, \pm 1 \) and of energy and fugacities given by (7). Upon increase of the cutoff one must take into account fusion of the sites \( r_1 \) and \( r_2 \) into the site \( \tilde{r} = \frac{1}{2}(r_1 + r_2) \), one finds that:

\[
Z^{(p)}_{r_1, r_2 \ldots r_p} \rightarrow \tilde{Z}^{(p-1)}_{\tilde{r}, r_3 \ldots r_p} \tag{H.1}
\]

with:

\[
\tilde{Z}^{(p-1)}_{\tilde{r}, r_3 \ldots r_p} = Z^{(p-2)}_{r_3 \ldots r_p} + \tilde{W}_{12} Z^{(p-2)}_{r_3 \ldots r_p} + \sum_{q=2}^{q=p-1} \sum_{q=2 \text{ even}}^{q \text{ even}} (W^{(q)}_{1, r_2 \ldots r_q} + W^{(q)}_{2, r_2 \ldots r_q}) \tag{H.2}
\]

where \( \tilde{W}_{12} = z^+ z^- + z^+ z^- \) and

\[
W^{(q)}_{1, r_2 \ldots r_q} = \sum_{n_1=\pm 1} \sum_{n_{r_i}=\pm 1} e^{-\beta H[n, r]} \tag{H.3}
\]
where $H$ has been defined in (7) and the charge $n_1$ is located in $\tilde{r}$. Note that $W_{1, r_3}^{(1)} = W_{\tilde{r}, r_3}$, the dipole weight defined in Section 4.3.2. In (H.2) the first term corresponds to the total weight of configurations with no charge in $r_1$ and $r_2$. The second term correspond to configurations with a dipole in $(r_1, r_2)$. Its original expression is complicated but simplifies when $r_1$ and $r_2$ are fused into $\tilde{r}$ (the interaction energy of any given other charge $n_\alpha$ in $r_\alpha$ with $n_{r_1}$ becomes opposite to the one with $n_{r_2}$ (up to higher order terms) and $W_{r_1, r_2}$ simplifies into $\tilde{W}_{12}$ to lowest order in $dl$ as explained in Section 4.3). The last two terms correspond to all configurations with one charge either in $r_1$ or in $r_2$.

The important property with respect to the free energy expansion is that one can factor the term $1 + \tilde{W}_{12}$ and rewrite:

$$\ln \tilde{Z}_{r, r_3, \ldots, r_p}^{(p-1)} = \ln(1 + \tilde{W}_{12}) + \ln(\tilde{Z}_{r, r_3, \ldots, r_p}^{(p-2)} + \sum_{q=2}^{p-1} \sum_{q \text{ even} \in [3, p]} \tilde{W}_{r, r_{i_2}, \ldots, r_{i_q}}^{(q)}$$

where $\tilde{W}^{(q)}$ as the same definition as $W^{(q)}$ except that the fugacity at $\tilde{r}$ has been modified according to the fusion rule for fugacities (42). The total factor inside the last logarithm is exactly the partition function $\tilde{Z}_{r, r_3, \ldots, r_p}^{(p-1)}$ of a system of $p-1$ sites with the new fugacity given by (42) on the site $\tilde{r}$.

I Higher charges and extensions

In this Appendix we briefly indicate how higher charges can be included in the same (fusion-diffusion) formalism and why including them does not affect any of our results. Let us consider for instance the charges $\pm 2$ and define $n_{++}$ and $n_{--}$ respectively as the number of component charges $+2$ and $-2$ in the vector charge $n$. The parametrization (16) can be readily extended to encode also for the charges $\pm 2$ simply by writing the vector fugacity $Y[n]$ as an average over a function $\Phi(z_+, z_-, z_{++}, z_{--})$ as

$$Y[n_+, n_-, n_{++}, n_{--}] = \int_{z_+, z_-, z_{++}, z_{--}} \Phi(z_+, z_-, z_{++}, z_{--}) z_+^{n_+} z_-^{n_-} z_{++}^{n_{++}} z_{--}^{n_{--}}$$

where the random variables $z_{++}$ and $z_{--}$ represent the random local fugacities for the charges $\pm 2$. Similar manipulations as in Appendix (D) lead to a RG equation for a normalized $P(z_+, z_-, z_{++}, z_{--})$ which we will not write explicitly. It does contain a diffusion operator as well as a fusion term. For illustration we simply give the diffusion operator, expressed using more con-
venient variables $z_\pm = e^{\beta (u \pm v)}$ and $z_{++} = z_+^2 e^{\beta (u' + v')}$ and $z_{--} = z_-^2 e^{\beta (u' - v')}$. It reads, for the corresponding probability distribution $\tilde{P}(u, u', v, v')$:

$$2 - J (\partial_u + 2 \partial_{u'}) + \sigma J^2 \partial_v^2$$

(I.1)

The detailed study of the corresponding RG equation, together with the fusion term will not be reported here. Instead let us indicate how the irrelevance of the higher charges can be justified. Since one has:

$$2 - n.K.n = 2 - \beta J(n_+ + n_- + 4(n_{++} + n_{--})) + \sigma \beta^2 J^2 (n_+ - n_- + 2(n_{++} - n_{--}))^2$$

(I.2)

it is clear that the reduced probability

$$Q(z_{++}, z_{--}) = \int_{z_+, z_-} P(z_+, z_-, z_{++}, z_{--})$$

(I.3)

satisfies the same diffusion equation as $P(z_+, z_-)$ in (21) but with the change $J \rightarrow 4J$ and $\sigma \rightarrow \sigma/4$. It becomes relevant by power counting only at $\sigma = 4\sigma_c = 1/2$. Thus it is less relevant than $P(z_+, z_-)$ in the region of the phase diagram of interest and we can rightly neglect the new fugacities $z_{++}$ and $z_{--}$. One can also check that the fusion terms imply that $P(z_{++} \sim 1)$ is of order $P(z_+ \sim 1)^2$ when this parameter is small (since at lowest order the RG equation contains a term proportional to $\delta(z_{++} - z_+')$). The rare events which involve the charges $\pm 2$ are thus subdominant.

J Evaluation of screening integrals at criticality

To evaluate the integral (105a) in the screening equation for $J$ we consider separately three intervals $u - X_l < b, X_l + b < u < -X_l - b$ and $u > -X_l - b$, where $b$ is a number such that $\tilde{p}_l(x)$ can be replaced by its asymptotic expression for $x > b$ with good accuracy. Only in the middle interval can we use the universal tail expressions for both factors $p_l$. Thus we have, using symetries:

$$\int du p_l(u - X_l) p_l(-u - X_l) = 2 \int_{-\infty}^b dv p_l(v) p_l(-v - 2X_l)$$

$$+ \int_{-X_l - b}^{-X_l - b} p_l(u - X_l) p_l(-u - X_l)$$

(J.1)
In the first integral we can replace the second function by its asymptotic form and get:

\[
\frac{2A}{D}e^{2X_l/\sqrt{D}}\int_{-\infty}^{b} dv dvp_l(v)(-2X_l - v)e^{\frac{(u+2X_l)^2}{4v}}e^{-v/\sqrt{D}} \sim CX_le^{2X_l} \tag{1.2}
\]

since, remembering that the integral \( \int_{-\infty}^{+\infty} dv dvp_l(v) = 1 \), the above integral is at most a finite number. The second integral in (1.1) is estimated as:

\[
\frac{A^2}{D^2}e^{2X_l/\sqrt{D}}\int_{X_l+b}^{-X_l-b} du (X_l^2 - u^2)e^{-u^2/4D} \]

and gives the leading contribution. For \( X_l/l \rightarrow 0 \) it behaves as \( \sim X_l^3e^{2X_l/\sqrt{D}} \) and gives the estimate in the text. It is encouraging to note that the leading behaviour precisely originates from the interval where \( p_l(u) \) can be replaced everywhere by its universal asymptotic form.

The integral (105b) in the screening equation for \( \sigma \) can also be estimated by considering for each variable \( u, u' \) two intervals \( u < b \) and \( u > b \). The end result is that we find to leading order:

\[
\partial_l J \sim \frac{1}{\sqrt{D}}\partial_l \sigma \sim \frac{A}{D^2}(-X_l)^3e^{2X_l/\sqrt{D}}\int_{-1}^{+1} dv (1 - v^2)e^{-\frac{X_l^2}{4v}(1+v^2)} \tag{1.3}
\]

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