Three-Neutrino Mass Matrices with Two Texture Zeros

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Abstract

Out of the fifteen $3 \times 3$ neutrino mass matrices with two texture zeros seven are compatible with the neutrino oscillation data. While 2 of them correspond to hierarchical neutrino masses and 1 to an inverted hierarchy, the remaining 4 correspond to degenerate masses. Moreover only the first 3 of the 7 mass matrices are compatible with the maximal mixing angle of atmospheric neutrino and hence favoured by data. We give compact expressions for mass matrices in terms of mass eigenvalues and study phenomenological implications for the 7 cases. Similarity of the textures of the neutrino, charged-lepton mass matrices with those of quark mass matrices is also discussed.
I. Introduction

There has been a long standing interest in the texture zeros of the $3 \times 3$ quark mass matrices as a possible source of the observed hierarchies in their masses and mixing angles. In particular the up and down quark mass matrices with 2 or 3 texture zeros are known to successfully relate the ratio of quark masses to the mixing angles [1]. In the context of neutrino physics also there is a long history of invoking texture zeros for describing the $4 \times 4$ [2] as well as the standard $3 \times 3$ neutrino [3,4,5] mass matrices. Recently Frampton, Glashow and Marfatia [6] have systematically compared the predictions of all the symmetric $3 \times 3$ neutrino mass matrices with two or more independent texture zeros with the neutrino mass and mixing parameters as derived from the oscillation data. A symmetric $3 \times 3$ mass matrix has in general 6 independent elements. They find no neutrino mass matrix with three or more texture zeros, which is compatible with the neutrino oscillation data. Moreover they find that 7 out of the 15 independent neutrino mass matrices with two texture zeros ($6C_2 = 15$) are compatible with the oscillation data. The present work carries this investigation one step further, giving explicit expressions for the mass matrices in terms of the three mass eigenvalues. We also give numerical estimates of these mass eigenvalues for the 7 cases, which are compatible with the neutrino oscillation data.

As usual the neutrino mass matrices shall be written in their flavour basis, which corresponds to the mass basis of the charged leptons. For simplicity we shall consider real mass matrices. One can easily check that the main results are unaffected by the introduction of CP violating phases. On the other hand the constraints on these phases arising from the texture zeros have been recently discussed in [7]. A $3 \times 3$ real symmetric mass matrix with two texture zeros has four independent parameters. Three of them can be determined in terms of the three mass eigenvalues using the invariance of trace and determinant, i.e.

$$\text{Tr. } M = m_1 + m_2 + m_3, \quad \text{Det. } M = m_1 m_2 m_3,$$

$$\text{Det. } M \times \text{Tr. } M^{-1} = m_1 m_2 + m_2 m_3 + m_3 m_1.$$  \hspace{1cm} (1)

But we need one experimental input to determine the remaining parameter. We shall assume the maximal mixing angle for the atmospheric neutrino oscillation [8] to provide this input, i.e.

$$t_1 = \tan \theta_{23} = 1.$$  \hspace{1cm} (2)

This represents by far the most robust result of the neutrino oscillation experiments so far. The most favoured value of this mixing parameter has remained 1 over the years, while the error bar has shrunk steadily. Therefore we treat this as a manifestation of an underlying symmetry of the neutrino mass matrix. It is conceivable of course that the physical value of this parameter may not be exactly 1. Nonetheless it is fair to assume that this point is smoothly connected to the symmetry limit. Therefore we expect the phenomenologically favoured mass matrix to be compatible with the maximal mixing angle of eq. (2). We shall see in section III below that 4 out of the 7 experimentally allowed mass matrices with two texture zeros are incompatible with this requirement.

The relation (1) was successfully used in determining the structure of the up- and down-quark mass matrices in terms of the mass eigenvalues for two and three texture zeroes [9]. The supplemental relation in this case came from the assumption that the three angles were very small.
(mass hierarchy) so that the triangular matrix technique could be implemented [9],[10]. We 
have briefly discussed this in section IV and compared their textures with those of the neutrino, 
charged-lepton system.

II. Experimental Constraints

We shall impose the following experimental constraints on the neutrino masses and mixing 
angles.

i) The atmospheric neutrino data implies

\[ \Delta_a = |m_3^2 - m_{1,2}^2| = (1.7 - 4) \times 10^{-3} \text{ eV}^2, \]

at 90% CL [8].

\[ \tan^2 \theta_{23} = t_1^2 = 0.9 - 1, \]

(4)

ii) The solar neutrino data admits LMS (LOW) solution at 90 (99.7%) CL with [11]

\[ \Delta_s = |m_2^2 - m_1^2| \simeq 6 \times 10^{-5} \times (1 \times 10^{-7}) \text{ eV}^2, \]

(5)

\[ t_3^2 = \tan^2 \theta_{12} \simeq 0.4 \times (0.6). \]

(6)

iii) The CHOOZ and Paoloverde atomic reactor experiments give the 90\% CL limit [12]

\[ s_2 = \sin \theta_{13} < 0.16. \]

(7)

Thus we shall require the ratio of the solar and atmospheric neutrino mass differences to satisfy 
the inequality

\[ \Delta_s/\Delta_a = |m_2^2 - m_1^2|/|m_3^2 - m_{1,2}^2| \lesssim 2 \times 10^{-2}. \]

(8)

III. Neutrino Mass Matrices

The mass matrix can be expressed in terms of the above masses and mixing angles as

\[ \mathcal{M} = U \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} U^T, \]

(9)

where

\[ U = \begin{pmatrix} c_2 c_3 & c_2 s_3 & s_2 \\ -c_1 s_3 - s_1 s_2 c_3 & c_1 c_3 - s_1 s_2 s_3 & s_1 c_2 \\ s_1 s_3 - c_1 s_2 c_3 & -s_1 c_3 - c_1 s_2 s_3 & c_1 c_2 \end{pmatrix}. \]

(10)
Using the maximal mixing constraint (2) we get the following expressions for the matrix elements of $\mathcal{M}$, which are valid up to the first order terms in $s_2$.

$$\mathcal{M}_{11} = m_1 c_3^2 + m_2 s_3^2,$$

$$\mathcal{M}_{12} = -(m_1 - m_2)s_3 c_3/\sqrt{2} + m_3 s_2/\sqrt{2} - (m_1 c_3^2 + m_2 s_3^2)s_2/\sqrt{2},$$

$$\mathcal{M}_{13} = (m_1 - m_2)s_3 c_3/\sqrt{2} + m_3 s_2/\sqrt{2} - (m_1 c_3^2 + m_2 s_3^2)s_2/\sqrt{2},$$

$$\mathcal{M}_{22} = (m_1 s_3^2 + m_2 c_3^2)/2 + m_3/2 + (m_1 - m_2)s_2 s_3 c_3,$$

$$\mathcal{M}_{33} = (m_1 s_3^2 + m_2 c_3^2)/2 + m_3/2 - (m_1 - m_2)s_2 s_3 c_3,$$

$$\mathcal{M}_{23} = -(m_1 s_3^2 + m_2 c_3^2)/2 + m_3/2. \quad (11)$$

From these expressions one can study the implications of setting different pairs of matrix elements to zero, as we see below.

**(A) Hierarchical Solutions**

1) $\mathcal{M}_{11} = 0, \mathcal{M}_{12} = 0$: They imply

$$t_3^2 = -m_1/m_2, \quad s_2 = \sqrt{-m_1 m_2}/m_3. \quad (12)$$

Combining these with eqs. (6) and (7) we get $|m_1| < |m_2| < m_3$, i.e. hierarchical masses. From eqs. (3) and (5) we get $|m_3| \simeq .05, |m_2| \simeq .009, |m_1| \simeq .004$ eV for the LMA solution; while the $m_1$ and $m_2$ values are each suppressed by a little over an order of magnitude for the LOW solution. In the former case we get $s_2 \simeq 0.12$, i.e. close to the CHOOZ limit (7); while it is over an order of magnitude smaller in the latter case. Note that for this solution the $\nu_e$ Majorana mass,

$$\mathcal{M}_{11} = \sum U_{ei}^2 m_i, \quad (13)$$

is zero. Thus it predicts no $0\nu\beta\beta$ signal even at the level of .001 eV, which corresponds to the highest level of sensitivity expected at the proposed GENIUS experiment [13].

Finally substituting (12) in (11) gives the explicit form of the mass matrix to first order in $s_2$, i.e.

$$\mathcal{M} = \begin{pmatrix}
0 & 0 & \sqrt{-2m_1 m_2} \\
0 & m_1 + m_2 + m_3 & m_3 - m_1 - m_2 \\
\sqrt{-2m_1 m_2} & m_3 - m_1 - m_2 & m_1 + m_2 + m_3
\end{pmatrix} \quad (14)$$

One can also derive this directly from eqs. (1) and (2). A special case of this mass matrix corresponding to bimaximal mixing, $t_3 = -m_1/m_2 = 1$, has been obtained in a $U(1)$ gauge extension of the standard model, corresponding to the gauge charge $L_\mu - L_e$ [14]. However we could find no simple dynamical model for the general case of $m_1 \neq -m_2$. Some explorations in this direction can be found in refs. [5] and [15].

2) $\mathcal{M}_{11} = 0, \mathcal{M}_{13} = 0$: The phenomenological predictions for this case are identical to the previous one except for the change of sign of $s_2$. The mass matrix is simply obtained from (14) by interchanging the 12 and 13 elements.
(B) Degenerate Solutions

1) $\mathcal{M}_{22} = 0, \mathcal{M}_{13} = 0$: Substituting these in eq. (11) imply

$$m_1^2 = m_3^2 + 4s_2m_3^2/t_3, \quad m_2^2 = m_3^2 - 4s_2m_3^2t_3,$$

i.e. $m_1^2, m_3^2, m_2^2$ are nearly degenerate and occur in that order. Thus $|m_2^2 - m_1^2| > |m_3^2 - m_2^2|$, in gross contradiction with the observed inequality of the solar and atmospheric neutrino masses (eq. 8).

As shown in [6], this mass matrix is compatible with the experimental constraint of eq. (8) away from the maximal mixing points (2), i.e. for $t_1 \neq 1$. In this case one gets

$$m_1^2 = m_3^2[t_1^4 + 2s_2t_1^2(t_1 + 1/t_1)/t_3],$$

$$m_2^2 = m_3^2[t_1^4 - 2s_2t_1^2(t_1 + 1/t_1)t_3].$$

Compatibiity with the experimental constraint of eq. (8) can be achieved for $s_2 \sim 0.5 \times 10^{-2}(1 - t_1^4)/(t_3 + 1/t_3) \lesssim 4 \times 10^{-4}$,

$$s_2 \sim 0.5 \times 10^{-2}(1 - t_1^4)/(t_3 + 1/t_3) \lesssim 4 \times 10^{-4},$$

(17)

corresponding to the $t_1^2 \geq 0.9$ range of eq. (4). Thus the $s_2$ angle in this case is too small to be observed at the future long base line experiments. On the other hand the $\nu_e$ Majorana mass

$$\mathcal{M}_{11} \simeq t_1^2 \sqrt{\frac{\Delta m^2}{1 - t_1^2}} \geq 0.10 \text{ eV}.$$ 

(18)

This is not too far below the present experimental upper limit of 0.2 eV [16], and will surely be measurable at GENIUS [13]. Note that this represents the common mass scale of degenerate neutrinos and it is cosmologically significant. Nonetheless we consider the incompatibility of this solution with the maximal mixing region ($t_1 = 1$), favoured by the atmospheric neutrino data, to be a serious drawback of this model. The expressions for the matrix elements are rather long for $t_1 \neq 1$. Therefore we are not displaying the explicit form of the mass matrix.

2) $\mathcal{M}_{33} = 0, \mathcal{M}_{12} = 0$: Substituting these in eq. (11) imply

$$m_1^2 = m_3^2 - 4s_2m_3^2/t_3, \quad m_2^2 = m_3^2 + 4s_2m_3^2t_3,$$

i.e. nearly degenerate $m_1^2, m_3^2, m_2^2$ with $|m_2^2 - m_1^2| > |m_3^2 - m_2^2|$ as in the previous case. In fact the magnitudes of $s_2$ and $t_3$ are the same as above. The results for $t_1 \neq 1$ are similar to the previous case with $t_1$ replaced by $-1/t_1$. Consequently the predicted $s_2$ and $\nu_e$ Majorana mass are very close to those of eqs. (17) and (18).

3) $\mathcal{M}_{22} = 0, \mathcal{M}_{13} = 0$: The results are practically the same as in 2).

4) $\mathcal{M}_{33} = 0, \mathcal{M}_{12} = 0$: The results are practically the same as in 1).

Matrices $B_1$ through $B_4$ in terms of mass eigenvalues are listed in Appendix I.
(C) Inverted Hierarchy

\( M_{22} = 0, M_{33} = 0 \): Substituting these in eq. (11) gives

\[ s_2 = 0, \quad t_3^2 = -(m_2 + m_3)/(m_1 + m_3). \] (20)

The first equality implies of no observable CP violation in the neutrino sector. The consistency of the second with eqs. (6) and (8) implies an inverted hierarchy of masses, i.e.

\[-m_2 \simeq m_1 \simeq (2 - 4)m_3.\] (21)

Substituting this in eq. (3) gives \(-m_2 \simeq m_1 \simeq 0.05 \text{ eV} \) and \(m_3 \simeq 0.02 \text{ eV}\). The predicted \(\nu_e\) Majorana mass is

\[ M_{11} \simeq \frac{1 - t_3^2}{2t_3} \sqrt{\Delta_a} \simeq (0.025 - 0.013) \text{ eV}.\] (22)

This is an order of magnitude below the present upper limit of 0.2 eV [15], but will be measurable at GENIUS [13]. Substituting (20) in (11) gives the explicit form of the mass matrix

\[ M = \begin{pmatrix}
    m_1 + m_2 + m_3 & -\sqrt{(m_1 + m_3)(m_2 + m_3)/2} & \sqrt{(m_1 + m_3)(m_2 + m_3)/2} \\
    \sqrt{(m_1 + m_3)(m_2 + m_3)/2} & 0 & m_3 \\
    \sqrt{(m_1 + m_3)(m_2 + m_3)/2} & m_3 & 0
\end{pmatrix}.\] (23)

(D) Disallowed Cases

Finally let us briefly indicate why the remaining 8 mass matrices are experimentally disallowed.

1) \( M_{12} = 0, M_{13} = 0 \): \( M_{12} - M_{13} = 0 \) implies

\[ (m_1 - m_2) \sin 2\theta_{12} = 0, \] (24)

i.e. no solar neutrino oscillation.

2) \( M_{12} or M_{13} = 0, M_{23} = 0 \): \( \sqrt{2}M_{12(13)} - 2s_2M_{23} = 0 \) implies

\[ |\tan 2\theta_{12}| = \frac{2|t_3|}{1 - t_3^2} = 2|s_2| < 0.32, \] (25)

in gross disagreement with the solar neutrino result (6).

3) \( M_{11} = 0, M_{22} or M_{33} \) or \( M_{23} = 0 \): In each case one gets (neglecting \(O(s_2)\) terms)

\[ |m_2^2 - m_3^2| = (1 - t_3^2)m_2^2, \quad |m_2^2 - m_3^2| = (2t_3^2 - t_3^4)m_2^2, \] (26)

in conflict with eqs. (6) and (8).

4) \( M_{22} or M_{33} = 0, M_{23} = 0 \): \( M_{22(33)} + M_{23} = 0 \) implies

\[ m_3 = -(m_1 - m_2)s_2s_3c_3, \quad \text{i.e. } m_1 \simeq -m_2 \gg m_3 \] (27)

using eqs. (7) and (8). Substituting this in \( M_{23} = 0 \) implies \(t_3^2 \simeq 1\), in conflict with eq. (6).

Matrices \(D_1\) through \(D_8\) in terms of mass eigenvalues are listed in Appendix I.
IV. Comparison with Quark Mass Matrices

As stated earlier, equation (1) was used in obtaining the structure of the up- and down-quark mass matrices, U and D, with two and three texture zeroes, in terms of the mass eigenvalues [9]. In determining their matrix elements it was further assumed that the mixing angles were very small so as to be compatible with mass hierarchy [9], [10]. Among the available choices, the structures that fit the experiment the best, with two texture zeroes, were [9]

$$U = \begin{pmatrix} 0 & 0 & \sqrt{-m_1 m_2} \\ 0 & m_2 & \sqrt{m_1 m_3} \\ \sqrt{-m_1 m_2} & \sqrt{m_1 m_3} & m_3 \end{pmatrix}$$ \hspace{1cm} (28)$$

with $m_1 = m_u$, $m_2 = m_c$, $m_3 = m_t$

and

$$D = \begin{pmatrix} 0 & \sqrt{-m_1 m_2} & 0 \\ \sqrt{m_1 m_2} & 0 & \sqrt{m_1 m_3} \\ 0 & \sqrt{m_1 m_3} & m_3 \end{pmatrix}$$ \hspace{1cm} (29)$$

with $m_1 = m_d$, $m_2 = m_s$, $m_3 = m_b$.

These matrices together give the correct CKM angles [9]

It is interesting that the texture of the above $U$-matrix is similar to the best choice for the neutrino mass matrix ($M$) for $s_2 > 0$, and hierarchical mass given by (14). One may conjecture that in that case the texture of the charged-lepton mass matrix ($L$) should be the same as the $D$-matrix. Indeed, since the hierarchy of masses in $L$ is similar to $D$, the entire structure of the two matrices may be similar.

In other words, the best choice for the neutrino and charged-lepton mass matrices would be (14) and

$$L = \begin{pmatrix} 0 & \sqrt{-m_e m_\mu} & \sqrt{-m_\tau m_\mu} \\ \sqrt{-m_e m_\mu} & m_\mu & 0 \\ 0 & \sqrt{m_e m_\tau} & m_\tau \end{pmatrix}$$ \hspace{1cm} (30)$$

We realize that the neutrino mass matrices were written above in the basis in which $L$ is diagonal whereas the above $L$ is not. One could diagonalize $L$ by rotating it through appropriate angles. The new $M$ matrix will then be different from matrix (14) by terms proportional to the rotation angles which, however, are very small compared to their neutrino counterparts since they are proportional to the ratios of the charged lepton masses which are, indeed, very small. Even for $M_{13}$, which is proportional to $s_2$, and, therefore, small, the new $M_{13}$ will not be drastically different since the relevant rotation angle here is proportional to $L_{13}$ which is actually zero. Thus we expect the new $M$ not to be very different from (14), inspite of the fact that (30) is not diagonal.
V. Conclusion

Only a limited number of texture patterns for the neutrino mass matrices are consistent with experiment, particularly if one takes the experimentally favoured value of \( t_1 = 1 \). These are given by \( A_1, A_2, \) and \( C \). In particular, for hierarchical masses and \( s_2 > 0 \) there is only one possible matrix \( A_1 \) given by (14). We also note that this particular matrix has a texture which is similar to the up-quark matrix, that is found compatible with experiments, which leads to the possibility that the quark and lepton mass matrices may have similar texture patterns.

We thank Anjan Joshipura and Ernest Ma for discussions. This work was supported in part by the U.S. Department of Energy under Grant No. DE-FG03-94ER40837.
Appendix I

For the sake of completeness we compile below the neutrino mass matrices for $t_1 = 1$ for the cases B and D, although, as discussed earlier, they are in conflict with the neutrino oscillation data.

\[
\begin{bmatrix}
\frac{m_1 + m_2 + m_3}{\sqrt{-2(m_1 + m_3)(m_2 + m_3)}} & \sqrt{-2(m_1 + m_3)(m_2 + m_3)} & 0 \\
0 & 0 & m_3 \\
m_3 & 0 & 0
\end{bmatrix}
\]

(Note: extra texture zero develops since $O(s_2^2)$ are neglected.)

\[
\begin{bmatrix}
\frac{m_1 + m_2 + m_3}{\sqrt{-2(m_1 + m_3)(m_2 + m_3)}} & 0 & \sqrt{-2(m_1 + m_3)(m_2 + m_3)} \\
0 & 0 & m_3 \\
m_3 & 0 & 0
\end{bmatrix}
\]

(Note: extra texture zero develops since $O(s_2^2)$ are neglected.)

\[
\begin{bmatrix}
m_1 + m_2 - m_3 & 0 & -\sqrt{2}\sqrt{m_3(m_1 + m_2 - 2m_3) - (m_1m_2 - m_3^2)} \\
-\sqrt{2}\sqrt{m_3(m_1 + m_2 - 2m_3) - (m_1m_2 - m_3^2)} & 0 & 2m_3(m_1 + m_2 - 2m_3) - (m_1m_2 - m_3^2) \\
0 & 2m_3(m_1 + m_2 - 2m_3) - (m_1m_2 - m_3^2) & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
m_1 + m_2 - m_3 & \sqrt{2}\sqrt{m_3(m_1 + m_2 - 2m_3) - (m_1m_2 - m_3^2)} & 0 \\
\sqrt{2}\sqrt{m_3(m_1 + m_2 - 2m_3) - (m_1m_2 - m_3^2)} & 0 & 0 \\
0 & 0 & \frac{m_1m_2 - m_3^2}{m_1 + m_2 - 2m_3}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -2m_2m_1 + m_3(m_1 + m_2 + m_3) & 2m_2m_1 + m_3(m_1 + m_2 + m_3) \\
-2m_2m_1 + m_3(m_1 + m_2 + m_3) & 0 & \frac{1}{2}(-m_1 - m_2 + m_3) \\
2\sqrt{-2m_1 m_2} & \frac{1}{2}(-m_1 - m_2 + m_3) & m_1 + m_2 + m_3
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -2m_2m_1 + m_3(m_1 + m_2 + m_3) & 2m_2m_1 - m_3(m_1 + m_2 + m_3) \\
-2m_2m_1 - m_3(m_1 + m_2 + m_3) & 0 & \frac{1}{2}(-m_1 - m_2 + m_3) \\
2\sqrt{-2m_1 m_2} & \frac{1}{2}(-m_1 - m_2 + m_3) & m_1 + m_2 + m_3
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & \sqrt{-\frac{m_1m_2}{2}} & -\sqrt{-\frac{m_3m_2}{2}} \\
-\sqrt{-\frac{m_1m_2}{2}} & m_1 + m_2 & 0 \\
\sqrt{-\frac{m_3m_2}{2}} & 0 & m_1 + m_2
\end{bmatrix}
\]
\[
D_7 = \begin{pmatrix}
\frac{m_1 + m_2 - m_3}{\sqrt{2} \sqrt{m_3 (m_1 + m_2 - 2m_3) - (m_1 m_2 - m_3^2)}} & \frac{m_1 m_2 - m_3^2}{\sqrt{2} \sqrt{m_3 (m_1 + m_2 - 2m_3) - (m_1 m_2 - m_3^2)}} & \frac{2m_3 (m_1 + m_2 - 2m_3) - (m_1 m_2 - m_3^2)}{\sqrt{2} \sqrt{m_3 (m_1 + m_2 - 2m_3) - (m_1 m_2 - m_3^2)}} \\
\frac{m_1 m_2 - m_3^2}{\sqrt{2} \sqrt{m_3 (m_1 + m_2 - 2m_3) - (m_1 m_2 - m_3^2)}} & 0 & 0 \\
\frac{2m_3 (m_1 + m_2 - 2m_3) - (m_1 m_2 - m_3^2)}{\sqrt{2} \sqrt{m_3 (m_1 + m_2 - 2m_3) - (m_1 m_2 - m_3^2)}} & 0 & 2m_3
\end{pmatrix}
\]

\[
D_8 = \begin{pmatrix}
\frac{m_1 + m_2 - m_3}{\sqrt{2} \sqrt{m_3 (m_1 + m_2 - 2m_3) - (m_1 m_2 - m_3^2)}} & \frac{-2m_3 (m_1 + m_2 - 2m_3) + (m_1 m_2 - m_3^2)}{\sqrt{2} \sqrt{m_3 (m_1 + m_2 - 2m_3) - (m_1 m_2 - m_3^2)}} & \frac{m_3^2 - m_1 m_2}{\sqrt{2} \sqrt{m_3 (m_1 + m_2 - 2m_3) - (m_1 m_2 - m_3^2)}} \\
\frac{-2m_3 (m_1 + m_2 - 2m_3) + (m_1 m_2 - m_3^2)}{\sqrt{2} \sqrt{m_3 (m_1 + m_2 - 2m_3) - (m_1 m_2 - m_3^2)}} & 0 & 0 \\
\frac{m_3^2 - m_1 m_2}{\sqrt{2} \sqrt{m_3 (m_1 + m_2 - 2m_3) - (m_1 m_2 - m_3^2)}} & 0 & 2m_3
\end{pmatrix}
\]
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