A UNIFIED SCHEME FOR MODULAR INVARIANT
PARTITION FUNCTIONS OF WZW MODELS

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Abstract

We introduce a unified method which can be applied to any WZW model at arbitrary level to search systematically for modular invariant physical partition functions. Our method is based essentially on modding out a known theory on group manifold $G$ by a discrete group $\Gamma$. We apply our method to $su(n)$ with $n = 2, 3, 4, 5, 6$, and to $\tilde{g}_2$ models, and obtain all the known partition functions and some new ones, and give explicit expressions for all of them.
1. Introduction
Conformal field theories (CFT’s) play an important role in two dimensional critical statistical mechanics and string theory, and has been extensively studied in the past decade. Among these theories WZW models have attracted considerable attentions, because as two dimensional rational conformal field theories they are exactly solvable, and most known CFT’s can be obtained from them via coset construction. Moreover these models explicitly appear in some statistical models like quantum chains, and describe their critical behavior. WZW models in addition to conformal symmetry, have an infinite dimensional symmetry whose currents satisfy a Kac-Moody algebra at some level. The partition function of a WZW model takes the form

\[ Z(\tau, \bar{\tau}) = \sum \chi_\lambda(\tau) M_{\lambda, \lambda'} \chi^*_{\lambda'}(\bar{\tau}), \tag{1.1} \]

where \( \chi_\lambda \) is the character of the affine module whose highest weight (HW) is \( \lambda \), \( M_{\lambda, \lambda'} \) are positive integers which determine how many times the HW representations \( \lambda, \lambda' \) in the left and right moving sectors couple with each other, and the sum is over the finite set of integrable representations (see Subsec. 2.1. for a precise definition). For the consistency of a physical theory it is necessary that the partition function (1.1) be invariant under the modular group of the torus. Construction and classification of partition functions of WZW models has been the goal of a large body of work in the past few years. However, up to now only the classifications of \( \hat{su}(2) \) and \( \hat{su}(3) \) have been completed.

To find modular invariant partition functions of WZW models a number of methods have been used: Automorphism of Kac-Moody algebras, simple currents, conformal embedding, automorphism of the fusion rules of the extended chiral algebra, lattice method, and finally direct computer calculations. In these efforts many modular invariant partition functions have been found, and may be arranged in three broad categories:

i) Diagonal Series – For every WZW model with a simply connected group manifold, there exists a physical modular invariant theory with diagonal matrix \( M_{\lambda, \lambda'} = \delta_{\lambda, \lambda'} \). They are often designated as a member of the \( A \) series.

ii) Complementary Series – There are some nondiagonal series for every WZW model whose Kac-Moody algebra has a nontrivial centre, associated to subgroups of the centre. They are often designated as members of the \( D \) series.

iii) Exceptional Series – In addition to the above two series, WZW models have a number of nondiagonal partition functions which occur only at certain levels. They are called \( E \) series.

Some of the known \( E \) series have been found by the conformal embedding method (see e.g. Ref. 13), some by utilizing the nontrivial automorphism of the fusion rules of the extended algebra, and some others by computer calculations. Although many exceptional partition functions have been obtained by these methods, however they don’t follow from a unified method and prove to be impractical for high rank groups and high levels. Furthermore, these methods don’t answer the question of why there are exceptional partition functions only at certain levels. It must be mentioned that corresponding to any
physical theory, there exists a charge conjugation, c.c., counterpart, such that $M^{(c.c.)}_{\lambda, c} = M_{\lambda, C(\lambda')}$, where $C(\lambda)$ is the complex conjugate representation of $\lambda$.

In this paper a unified approach which we call orbifold-like method, is presented and shown to lead to all the known nondiagonal theories. The method is easily applied to high rank groups at arbitrary level. The organization of the paper is as follows: In Sec. 2, we briefly review some characteristic features of Kac-Moody algebra, such as their unitary highest weight representations and modular transformation properties of their characters. Then we present our approach. We start with some known theory whose partition function is $Z(G)$ and mod it out by a discrete group $\Gamma$. We will take $\Gamma$ to be a cyclic group $\mathbb{Z}_N$. It is not necessarily a subgroup of the centre of $\hat{g}$. In order for the modding to gives rise to a modular invariant combination with rational coefficients, certain relations must be satisfied, which will be explicated. In Sec. 3, we apply our method to $\hat{su}(n)$ models with $n = 2, 3, 4, 5, 6$ and as an example of a non-simply laced affine Lie algebra, to $\hat{g}_2$ models, and generate all the known nondiagonal theories and some new exceptional ones. In Sec. 4, we conclude with some remarks. In Appendix A, we gather some formulas and relations that are used in the body of the paper.

2. The Orbifold-like method
Observing that all modular invariant partition functions of $\hat{su}(n)$ models at level one are obtained by modding out the group $SU(n)$ by subgroups of its centre $^{15}$, the authors in Ref. 29 for the first time found that not only the $D$ series of $su(2)$ WZW models are obtained with modding out the diagonal theories by the $\mathbb{Z}_2$ centre, but also all their exceptional partition functions can be found by modding out the $A$ or $D$ series by a $\mathbb{Z}_3$ which is not obviously a subgroup of the centre.

In this paper we are going to generalize that work to WZW models with any associated affine Lie algebra at arbitrary level. We have called our approach orbifold-like method, because the finite group which one uses in modding may not be the symmetry of a target manifold. Befor going to the details of our approach, we collect in the following subsection, some facts about untwisted Kac-Moody algebras and set up our notations.

2.1. Preliminaries and Notations
A WZW model is denoted by $\hat{g}_k$ where its affine symmetry algebra is the untwisted Kac-Moody algebra $\hat{g}$ associated with a compact Lie algebra $g$, and the positive integer number $k$ is the level of the Kac-Moody algebra (see Refs. 30,31 for details). The primary fields of the model can be labelled by the highest weight representations of horizontal Lie algebra $g$. In the basis of fundamental weights $\omega_i$ of $g$ these HW representations are expressed by $\lambda = \sum_{i=1}^{r} \lambda_i \omega_i$ where $\lambda_i$’s are positive integers (Dynkin labels) and $r$ is the rank of $g$. Imposing unitarity condition, restricts the number of HW representations which appear in a theory at a given level. These representations which are usually called integrable representations satisfy the relation $\frac{2}{\Psi^2}(\lambda, \Psi) \leq k$, where $\Psi$ is the highest root of $g$. The set of integrable representations is called the fundamental domain and is denoted by $B_h$, where the height $h$ is defined by $h = \hat{h} + k$ and $\hat{h}$ is the dual Coxeter number of $g$. We choose the normalization such that $\Psi^2 = 2$. The characters of integrable representations at a given level $k$ constitute a linear unitary representation of modular group of the torus
(see e.g. Ref. 32). The character of an integrable HW representation transforms under the action of the generators of the modular group \( S : (\tau \to -1/\tau) \) and \( T : (\tau \to \tau + 1) \) as

\[
\chi_\lambda(-1/\tau) = C \sum_{\{\lambda' \in B_h\}} \sum_{\{\omega \in W(G)\}} \varepsilon(\omega)e^{(2\pi i)(\lambda,\omega(\lambda'))} \chi_{\lambda'}(\tau)
\]

\[
\chi_\lambda(\tau + 1) = e^{\pi i (\tilde{\lambda}^2/h - \rho^2/\tilde{h})} \chi_\lambda(\tau),
\]

where \( C = \frac{i^{\Delta_+}}{(k + \tilde{h})^{r/2}} \left( \frac{\text{vol. cell of } Q^*}{\text{vol. cell of } Q} \right)^{1/2} \), \( Q \) is the coroot lattice, \( Q^* \) its dual lattice, and \( \tau, \Delta_+, W, \epsilon(\omega) \) are the parameter of the torus, the number of positive roots, the Weyl group, and the determinant of the Weyl reflection \( \omega \), respectively.

2.2. Orbifold approach

It was noted in Ref. 35 that each complementary series obtained in Ref. 17 is actually the partition function of an orbifold, which is constructed via modding the covering group \( \tilde{G} \) by a subgroup of its center. For the construction of this partition function, one starts with the WZW theory defined on the group manifold \( \tilde{G} \) and impose boundary conditions on the fields up to the action of some subgroup \( \Gamma \) of the center of \( \tilde{G} \):

\[
\Phi(\sigma_1 + 2\pi, \sigma_2) = h_1 \Phi(\sigma_1, \sigma_2) ; \quad \Phi(\sigma_1, \sigma_2 + 2\pi) = h_2 \Phi(\sigma_1, \sigma_2),
\]

(2.3)

where \( \sigma_1, \sigma_2 \) are coordinates on the torus and \( h_1, h_2 \) are some elements of \( \Gamma \). Let us denote by \( (h_1, h_2) \) the contribution to the partition function from the twisted sector with boundary conditions (2.3). In order for the theory \( G/\Gamma \) to be modular invariant, all twisted sectors must be included.9,34 So the partition function of the new theory can be written in the following form

\[
Z(G/\Gamma) = \frac{1}{|\Gamma|} \sum_{h_1, h_2 \in \Gamma} (h_1, h_2),
\]

(2.4)

where \( |\Gamma| \) is the order of \( \Gamma \). For \( \Gamma = \mathbb{Z}_N \) which is the interesting group for us, eq. (2.4) reduces to:

\[
Z(G/\mathbb{Z}_N) = \frac{1}{N} \sum_{\alpha, \beta = 1}^N (h^\alpha, h^\beta).
\]

(2.5)

It can easily be seen that using the untwisted sector, defined by

\[
Z_1(G/\mathbb{Z}_N) = \frac{1}{N} \sum_{\alpha = 1}^N (1, h^\alpha),
\]

(2.6)
and acting properly on it by the generators of the modular group, $S$ and $T$, one can obtain the full partition function $Z(G/Z_N)$.\(^9\) In Ref. 35 the following formula is derived for $Z(G/Z_N)$ when $Z_N$ is a subgroup of the centre of $G$ and $N$ is prime:

$$Z(G/Z_N) = \left[ \sum_{\alpha=1}^{N} \left( T^\alpha S + 1 \right) Z_1(G/Z_N) \right] - Z(G). \quad (2.7)$$

Concerning the method mentioned above we make two crucial observations. First, the orbifold method can be similarly applied to the case where $G$ is not the covering group, and even when $Z_N$ is not a subgroup of the centre of $G$, but is a symmetry of the classical theory. However, in those cases in order for the modding to be meaningful, i.e., the sum of the terms in the bracket of eq. (2.7) and therefore the whole expression be modular invariant with real coefficients, the following relation must be satisfied

$$T^N S Z_1(G/Z_N) = S Z_1(G/Z_N). \quad (2.8)$$

Secondly, for the case when $N$ is not prime, the expression (2.7) does not completely describe an orbifold partition function and in order to generate the full partition function $Z(G/Z_N)$, some additional terms must be included into the bracket. But then, extra constraints beyond (2.8) have to be satisfied for modular invariance (see eq. (A.3)). We call the appropriate moddings for a given theory which satisfy these constraints, allowed moddings. We have collected in Appendix A, a list of formulas for the cases of interest to us.

Our strategy in finding the nondiagonal WZW theories with the affine symmetry algebra $\hat{g}_k$ is as follows. First we start with a diagonal theory and mod it out by some group $Z_N$. The untwisted part of the partition function $Z_1$, is realized by representing the action of $Z_N$ on the characters of HW representations in the left—moving sector as

$$p \cdot \chi_\lambda(\tau) = e^{\frac{2\pi i \beta (\lambda^2 - \rho^2)}{N}} \chi_\lambda(\tau) \quad (2.9)$$

where $p$ is the generator of $Z_N$, $\beta$ is the smallest rational number such that $\beta (\lambda^2 - \rho^2)$ is an integer. Therefor the the untwisted part $Z_1$ consists of left—moving representations with $\beta (\lambda^2 - \rho^2) = 0 mod N$. It must be mentioned that the realization of $Z_N$ in (2.9) is such that it gives the exact form of $D_h$ series for $\widehat{su}(2)$ models, then we have simply generalized it to account for all nondiagonal partition functions. Now we act on the untwisted part $Z_1$ by the operators $S$ and $T$ according to the bracket in the r.h.s of the corresponding formula of $Z(G/Z_N)$, and check in every step if the related condition (A.3) is satisfied, and finally we calculate the sum. Then we encounter three cases:

**Case I** All the terms $\chi_\lambda \overline{\chi}_{\lambda'}$ which appear in the sum have positive rational coefficients. This indicates that the sum is a positive linear combination of modular invariant partition functions. Sometimes after subtracting some previously known partition functions from this, a new partition function will appear. See for example, the case of modding $D_8^{(2)}$ by $Z_{16}$ in $\widehat{su}(4)$ models on page 19.

**Case II** Some of the terms in the sum have negative coefficients. However after adding or subtracting some known partition functions or/and some modular invariant
combinations of characters that have been found in the process of the previous moddings, a new modular invariant partition function will be found. See for example, the case of modding $D_8$ by $\mathbb{Z}_9$ in $\widehat{su}(3)$ models on page 15.

**Case III** Some of the terms in the *bracket* have negative coefficients but after subtracting from it some known physical partitions or/and some modular invariant combinations of characters, at most a new modular invariant combination will be obtained, which is not a partition function. See for example, the case of modding $D_{24}$ by $\mathbb{Z}_2$ in $\widehat{su}(3)$ models on page 16.

3. **Applications**

In this section we apply our method to $\widehat{su}(n)$ WZW models with $n = 2, 3, 4, 5, 6$, and as an example of a nonsimply-laced affine Lie algebra to $\widehat{\mathfrak{g}}_2$ WZW models.

3.1. $\widehat{su}(2)$ WZW models

For $\widehat{su}(2)$ models besides the usual diagonal series $A_h = \sum_{\{\lambda \in B_h\}} |\chi_{\lambda}|^2$ at each level, where

$$B_h = \{\lambda = m \omega \mid 1 \leq m < h = k + 2\} \quad (3.1)$$

is the fundamental domain and $\omega$ is the fundamental weight of $su(2)$, and a nondiagonal $D$ series at even levels, three exceptional modular invariant partition functions have been found at levels $k = 10, 16, 28$; and it has been shown that this set completes the classification of $\widehat{su}(2)$ WZW models.\textsuperscript{10} In what follows we will review the results of Ref. 29, where it was shown that the exceptional partition functions can be obtained by our orbifold method, and present some further calculations. The action of the $\mathbb{Z}_2$ group on the characters of the left–moving HW representations of $su(2)$ is defined due to eq. (2.9) by

$$p \cdot \chi_m = e^{2\pi i (m^2 - 1)} \chi_m, \quad (3.2)$$

where $p$ is the generator of $\mathbb{Z}_N$. Hereafter the HW representation $\lambda$ is designated by its Dynkin label $m$. Thus the untwisted part of a partition function, consists of HW representations $\lambda = m \omega$ with $m^2 = 1 \mod N$.

**D - Series**

These modular invariant partition functions are obtained by modding out the diagonal series by $\mathbb{Z}_2$, the centre of $SU(2)$. The untwisted part of the partition function $Z_1$ is given by

$$Z_1(A_h/\mathbb{Z}_2) = \sum_{\{\lambda \mid m \text{ odd}\}} |\chi_{\lambda}|^2. \quad (3.3)$$

The partition function $Z(G/\mathbb{Z}_2)$ can be calculated using eq. (2.7) with $N = 2$. The calculation is straightforward. First the untwisted part (3.3) is written in the following form

$$Z_1(A_h/\mathbb{Z}_2) = \frac{1}{|W|} \sum_{\{\lambda \in W B_h : m \text{ odd}\}} |\chi_{\lambda}|^2, \quad (3.4)$$
using the identity $\chi_\omega(\lambda) = \epsilon(\omega)\chi_\lambda$, where $WB_h$ is the Weyl reflection of the fundamental domain $B_h$. Then we rewrite (3.4) in the following form

$$Z_1(A_h/\mathbb{Z}_2) = \frac{1}{|W|} \sum_{\{\lambda \in (Q^*/hQ): m \text{ odd}\}} |\chi_\lambda|^2,$$

(3.5)

noting that $\chi_0 = \chi_h = 0$. On the other hand the lattice $Q^*/hQ$ consists of the following HW representations:

$$Q^*/hQ = \{ \lambda = m \omega \mid 1 \leq m \leq 2h \}.$$ 

(3.6)

Now acting by the operator $S$ in (2.1) on eq. (3.5) we obtain:

$$SZ_1 = \frac{1}{2h} \frac{1}{2!} \sum_{\lambda \in Q^*/hQ} \sum_{m \text{ odd}} \varepsilon(\omega'\omega'') \exp \left[ \frac{2\pi i}{h} \left( \lambda, \omega'(\lambda') - \omega''(\lambda'') \right) \right] \chi_{\lambda'} \bar{\chi}_{\lambda''}.$$ 

(3.7)

The sum over $\lambda$ is easily done using eq. (3.6), and it appears that the sum over one of the two Weyl groups can be factored out and give an overall factor equal to the order of Weyl group. Finally the sum over the other Weyl group must be done. Substituting (3.7) in eq. (2.7), we easily do the sum and find a nondiagonal partition function at every even level given by

$$D_h \equiv Z(A_h/\mathbb{Z}_2) = \sum_{m \text{ odd}=1}^{h-1} |\chi_m|^2 + \sum_{m \text{ odd}=1}^{h/2-2} (\chi_{m\bar{\chi}_{h-m}} + c.c.) + 2|\chi_{h/2}|^2,$$

(3.8a)

for $h = 2 \mod 4$, and

$$D_h \equiv Z(A_h/\mathbb{Z}_2) = \sum_{m \text{ odd}=1}^{h-1} |\chi_m|^2 + \sum_{m \text{ even}=2}^{h/2-2} (\chi_{m\bar{\chi}_{h-m}} + c.c.) + |\chi_{h/2}|^2,$$

(3.8b)

for $h = 0 \mod 4$. These are exactly the $D$ series given in Ref. 10.

E - Series

One expects to find possibly, exceptional partition functions at levels $k = 4, 10, 28$ according to the following conformal embeddings:\textsuperscript{20}

$$\widehat{su}(2)_{k=4} \subset \widehat{su}(3)_{k=1}, \quad \widehat{su}(2)_{k=10} \subset (B_2)_{k=1}, \quad \widehat{su}(2)_{k=28} \subset (g_2)_{k=1}. $$

(3.9)

1. At level $k = 4$ ($h = 6$), we start with $A_6$ and mod it out by $\mathbb{Z}_3$, and check that the eq. (2.8) is satisfied; then we do the sum in the \textit{bracket} of eq. (2.7) with $N = 3$, and finally get

$$\left[ \sum_{\alpha=1}^{3} T^\alpha SZ_1 + Z_1 \right] = 4A_6 - D_6.$$ 

(3.10)
We continue modding by allowed $Z_N$’s up to $N = 48$, but nothing more than $A_6$ and $D_6$ appears. For example, in the case $N = 6$ we obtain

$$\left[ \left( \sum_{\alpha=1}^{6} T^{\alpha} + \sum_{\alpha=1}^{3} T^{\alpha} ST^2 + \sum_{\alpha=1}^{2} T^{\alpha} ST^3 \right) SZ_1 + Z_1 \right] = 4A_6 + D_6. \quad (3.11)$$

Then we start with $D_6$, mod it out by allowed moddings but also nothing more is found. For example, in modding by $Z_3$ we find

$$\left[ \sum_{\alpha=1}^{3} T^{\alpha} SZ_1 + Z_1 \right] = 2D_6. \quad (3.12)$$

This is not surprising, since it can easily be seen that $D_6$ given by

$$D_6 = |\chi_1 + \chi_5|^2 + 2|\chi_3|^2,$$

exactly corresponding to conformal embedding $\widehat{su(2)}_{k=4} \subset \widehat{su(3)}_{k=1}$.

2. At level $k = 10$ ($h = 12$), we start with $A_{12}$ and mod it out by $Z_6$ which is allowed. After doing the sum in the bracket of eq. (A.5), we encounter the case I of Subsec. 2.2., which after subtracting the known partition functions $A_{12}$ and $D_{12}$ each with multiplicity 2, we obtain the exceptional partition function $E_{12}$, which in our notation is described by $E_{12}$

$$\left[ \left( \sum_{\alpha=1}^{6} T^{\alpha} + \sum_{\alpha=1}^{3} T^{\alpha} ST^2 + \sum_{\alpha=1}^{2} T^{\alpha} ST^3 \right) SZ_1 + Z_1 \right] = 2A_{12} + 2D_{12} + 2E_{12},$$

where

$$E_{12} = |\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2.$$ \hspace{1cm} (3.14)

In Ref. 29, $E_{12}$ was obtained by modding $A_{12}$ or $D_{12}$ by $Z_6$, but there, we encounter the case II of Subsec. 2.2. We also get $E_{12}$ in modding $D_{12}$ by $Z_6$; the result is exactly the same as eq. (3.14).

3. At level $k = 16$ ($h = 18$), we start with $A_{18}$ and mod it out by $Z_6$ which is allowed. After doing the sum in the bracket of eq. (A.5), we encounter the case I of Subsec. 2.2., which after subtraction $A_{18}$ of multiplicity 3, the exceptional partition function $E_{18}$ is obtained:

$$\left[ \left( \sum_{\alpha=1}^{6} T^{\alpha} + \sum_{\alpha=1}^{3} T^{\alpha} ST^2 + \sum_{\alpha=1}^{2} T^{\alpha} ST^3 \right) SZ_1 + Z_1 \right] = 3A_{18} + 3E_{18},$$

where

$$E_{18} = |\chi_1 + \chi_{17}|^2 + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{11}|^2 + |\chi_9|^2 + (\chi_9(\chi_3 + \chi_{15}) + c.c.). \quad (3.17)$$
We also find $E_{18}$ in modding $D_{18}$ by the allowed modding like $Z_3$:

$$\left[ \sum_{\alpha=1}^{3} T^\alpha S Z_1 + Z_1 \right] = D_{18} + 2E_{18}. \quad (3.18)$$

4. At level $k = 28$ ($h = 30$), we start with $D_{30}$ and mod it out by $Z_3$, which is allowed. After doing the sum in the bracket of eq. (2.7), we encounter the case I of Subsec. 2.2, which after subtraction $D_{12}$ of multiplicity 2, leads to the exceptional partition function $E_{30}$:

$$\left[ \sum_{\alpha=1}^{3} T^\alpha S Z_1 + Z_1 \right] = 2D_{30} + E_{30}, \quad (3.19)$$

where

$$E_{30} = |\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}|^2 + |\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}|^2. \quad (3.20)$$

So in this subsection we have generated, by orbifold method, not only the partition functions which correspond to a conformal embedding like $E_{12}$ and $E_{30}$, but also the one which follows from a nontrivial automorphism of the fusion rules of the extended algebra i.e. $E_{18}$. It is interesting to notice that all the nondiagonal partition functions of $su(2)$ are obtained from moddings by $Z_N$’s, with $N$ a divisor of $2h$ and $h = k + 2$.

3.2. $\tilde{su}(3)$ WZW models

For $\tilde{su}(3)$ models besides the usual diagonal series $A_h = \sum_{\{\lambda \in B_h\}} |\chi_\lambda|^2$ at each level, where

$$B_h = \{ \lambda \mid 2 \leq \sum_{i=1}^{2} m_i < h = k + 3 \} \quad (3.21)$$

and $\lambda = \sum_{i=1}^{2} m_i \omega_i$, and a nondiagonal $D_h$ series at each level; four exceptional modular invariant partition functions have been found at levels $k = 5, 9, 21, 12, 13$. Recently, it was shown that this set completes the classification of $\tilde{su}(3)$ WZW models. In what follows we obtain all of these by the orbifold method. We define the action of the $Z_N$ group on the characters of the left–moving HW representations of $su(3)$ by

$$p \cdot \chi_{(m_1,m_2)} = e^{\frac{2\pi i}{N}(m_1^2 + m_2^2 + m_1 m_2 - 1)} \chi_{(m_1,m_2)} \quad (3.22)$$

where $p$ is the generator of $Z_N$. Thus the untwisted part of a partition function, consists of left–moving HW representations which satisfy: $m_1^2 + m_2^2 + m_1 m_2 = 1 \mod N$.

D- Series

These modular invariant partition functions are obtained by modding out the diagonal series $A_h$ by $Z_3$, the centre of $SU(3)$. The untwisted part of the partition function $Z_1$ is given by

$$Z_1(A_h/Z_3) = \sum_{\{\lambda | m_1 - m_2 = 0 \mod 3\}} |\chi_\lambda|^2. \quad (3.23)$$
The partition function \(Z(G/\mathbb{Z}_3)\) can be calculated using eq. (2.7) with \(N = 3\). Following the same recipe mentioned in section 3.1, first we write the untwisted part (3.23) in the following form

\[
Z_1(A_h/\mathbb{Z}_3) = \frac{1}{|W|} \sum_{\{\lambda \in W \mid m_1 - m_2 \equiv 0 \mod 3\}} |\chi_\lambda|^2.
\]  

(3.24)

It is more convenient to write HW representations in the basis consisting of the simple root \(\alpha_1\) and the corresponding fundamental weight \(\omega_1\). In this basis

\[
\frac{Q^*}{hQ} = \left\{ \lambda = m \omega_1 + m' \alpha_1 \mid 1 \leq m \leq 3h + 2; -1 \leq m' \leq h - 2 \right\}.
\]  

(3.25)

so that, just as in eq. (3.5) we can rewrite eq. (3.24) in the form,

\[
Z_1(A_h/\mathbb{Z}_3) = \frac{1}{3!} \sum_{\{\lambda = m \omega_1 + m' \alpha_1 \mid 3 \leq m \leq 3h + 2, -1 \leq m' \leq h - 2; m \equiv 0 \mod 3\}} |\chi_\lambda|^2.
\]  

(3.26)

Now the action of the operator \(S\) in eq. (2.1) on eq. (3.26) can be calculated as mentioned in Subsec. 3.1. Substituting the untwisted part (3.26) in eq. (2.7) with \(N = 3\), and doing the sum in the bracket, finally we obtain at each level a nondiagonal partition function denoted by \(D_h\):

\[
D_h = Z(A_h/\mathbb{Z}_3) = \sum_{\{\lambda \mid m_1 - m_2 \equiv 0 \mod 3\}} |\chi_\lambda|^2 + \sum_{\{\lambda \mid m_1 - m_2 \equiv 2k \mod 3\}} \chi_\lambda \tilde{\chi}_\sigma(\lambda)
\]

\[
+ \sum_{\{\lambda \mid m_1 - m_2 \equiv k \mod 3\}} \chi_\lambda \tilde{\chi}_\sigma^2(\lambda),
\]  

(3.27)

where \(\sigma(\lambda) = m_2 \omega_1 + (m_1 - m_2) \omega_2\). The eq. (3.27) agrees with the result of Ref. 17. *

E- Series

We expect to find nondiagonal partition functions at levels \(k = 5, 9, 21\), due to the following conformal embeddings \(^{20}\)

\[
\hat{su}(3)_{k=3} \subset (\hat{D}_4)_{k=1}, \quad \hat{su}(3)_{k=5} \subset \hat{su}(6)_{k=1}
\]

\[
\hat{su}(3)_{k=9} \subset (\hat{e}_6)_{k=1}, \quad \hat{su}(3)_{k=21} \subset (\hat{e}_7)_{k=1}.
\]  

(3.28)

Thus, we begin from these levels.

1. At level \(k = 3\) \((h = 6)\), we start with \(A_6\) and mod it out by allowed \(Z_N\)’s up to \(N = 18\), but all of them results only in a combination of \(A_6\) and \(D_6\). For example, in the case \(N = 6\) after doing the sum according to eq. (A.5) we obtain

\[
\left[ (\sum_{\alpha=1}^6 T^\alpha + \sum_{\alpha=1}^3 T^\alpha ST^2 + \sum_{\alpha=1}^2 T^\alpha ST^3)SZ_1 + Z_1 \right] = 3A_6 + D_6.
\]  

(3.29)

* There is a minus sign error in Ref. 17. Defining \(\sigma(\lambda)\) the same as in our case, the two terms in the exponent of eq. (6.4) of Ref. 17 must both have negative signs, in order for \(M_{\lambda,\lambda'}\) to commute with the operator \(T\).
Then, we start with $D_6$, mod it out by allowed moddings but also nothing more is found. For example, in modding by $Z$ we find
\[
\left[ \sum_{\alpha=1}^{6} T^\alpha + \sum_{\alpha=1}^{3} T^\alpha ST^2 + \sum_{\alpha=1}^{2} T^\alpha ST^3 \right] SZ_1 + Z_1 = \frac{9}{2} D_6. \tag{3.30}
\]
This is not surprising, since it can easily be seen that $D_6$ given by
\[
D_6 = \left| \chi_{1,1} + \chi_{1,4} + \chi_{4,1} \right|^2 + 3 \left| \chi_{2,2} \right|^2,
\]
exactly corresponding to conformal embedding $su(3)_{k=3} \subset \widehat{(D_4)}_{k=1}$.

2. At level $k = 5$ ($h = 8$), starting with $D_8$ and modding by $Z_8$ according to eq. (A.6), we see that the sum of terms in the bracket leads to the case I of section 2.2 which after subtracting the known partition functions $A^{c,c}_k$ and $D_8$ each with mutiplicity 2, we obtain an exceptional partition function, which is called $E^{c,c}_8$,
\[
\left[ \sum_{\alpha=1}^{8} T^\alpha + \sum_{\alpha=1}^{2} T^\alpha ST^2 + ST^4 \right] SZ_1 + Z_1 = 2D_8 + 2A^{c,c}_8 + E^{c,c}_8. \tag{3.32}
\]
with
\[
E^{c,c}_8 = \left| \chi_{1,1} + \chi_{3,3} \right|^2 + \left| \chi_{1,4} + \chi_{4,1} \right|^2
+ \left( (\chi_{3,1} + \chi_{3,4})(\chi_{1,3} + \chi_{4,3}) + (\chi_{2,3} + \chi_{6,1})(\chi_{3,2} + \chi_{1,6}) + c.c. \right). \tag{3.33}
\]
Thus,
\[
E_8 = \left| \chi_{1,1} + \chi_{3,3} \right|^2 + \left| \chi_{1,3} + \chi_{4,3} \right|^2 + \left| \chi_{3,1} + \chi_{3,4} \right|^2
+ \left| \chi_{1,4} + \chi_{4,1} \right|^2 + \left| \chi_{2,3} + \chi_{6,1} \right|^2 + \left| \chi_{3,2} + \chi_{1,6} \right|^2, \tag{3.34}
\]
where $\chi_{(m_1,m_2)}$ denotes the character of HW representation $\lambda = m_1 \omega_1 + m_2 \omega_2$. One can easily see that $E_8$ corresponds to conformal embedding $su(3)_{k=5} \subset su(6)_{k=1}^{13}$. It must be mentioned that we also find $E_8$ in modding by $Z_2$ and $Z_4$, but in these cases we encounter the case II of Subsec. 2.2., and we obtain
\[
\left[ \sum_{\alpha=1}^{2} T^\alpha SZ_1 + Z_1 \right] = \frac{1}{2} (5D_8 - A^{c,c}_8 + E_8) \tag{3.35}
\]
\[
\left[ \sum_{\alpha=1}^{4} T^\alpha + ST^2 \right] SZ_1 + Z_1 \right] = \frac{1}{2} (7D_8 + A^{c,c}_8 - E^{c,c}_8). \tag{3.36}
\]
Modding by $Z_3$ gives $D_8$ itself, but modding for example by $Z_5$ or $Z_7$ are not allowed because the condition (2.8) is not satisfied. We have continued modding, up to $Z_{48}$ but no other exceptional partition function is found.
3. At level \( k = 9 \) \((h = 12)\), we start with \( D_{12} \) and mod it out by \( \mathbb{Z}_9 \) and do the sum according to the \textit{bracket} of eq. (A.7) with \( N = 9 \), finally we encounter the case II of Subsec. 2.2., which after subtraction \( D_{12} \) of multiplicity 12, we obtain an exceptional partition function, denoted by \( E_{12}^{(1)} \) with an overal multiplicity \(-3\):

\[
\left[ (\sum_{\alpha=1}^{9} T^\alpha + ST^3 + ST^6)SZ_1 + Z_1 \right] = 12D_{12} - 3E_{12}^{(1)},
\]

where

\[
E_{12}^{(1)} = |\chi_{1,1} + \chi_{1,10} + \chi_{10,1} + \chi_{2,5} + \chi_{5,2} + \chi_{5,5}|^2 + 2|\chi_{3,3} + \chi_{3,6} + \chi_{6,3}|^2.
\]

This partition function corresponds to conformal embedding \( su(3)_{k=9} \subset (e_6)_{k=1} \). In modding \( D_{12} \) by \( \mathbb{Z}_2 \), after doing the sum in the \textit{bracket} of eq. (2.7) with \( N = 2 \), one encounters case II, however in this case the trace of another modular invariant can easily be seen. Actually subtracting \( D_{12} \), its charge conjugation counterpart \( D_{12}^{c,c} \) and \( E_{12}^{(1)} \) with multiplicities 5/2, \(-1/2\), and \(1/2\) respectively, we obtain another exceptional partition function which we denote by \( E_{12}^{(2)} \)

\[
\left[ \sum_{\alpha=1}^{2} T^\alpha SZ_1 + Z_1 \right] = \frac{1}{2}(5D_{12} - D_{12}^{c,c} + E_{12}^{(1)} + E_{12}^{(2)}),
\]

with

\[
E_{12}^{(2)} = |\chi_{1,1} + \chi_{1,10} + \chi_{10,1}|^2 + |\chi_{2,5} + \chi_{5,2} + \chi_{5,5}|^2 + |\chi_{3,3} + \chi_{3,6} + \chi_{6,3}|^2
\]

\[
+ |\chi_{1,4} + \chi_{4,7} + \chi_{7,1}|^2 + |\chi_{4,1} + \chi_{7,4} + \chi_{1,7}|^2
\]

\[
+ 2|\chi_{4,4}|^2 + \left( (\chi_{2,2} + \chi_{2,8} + \chi_{8,2}) + c.c. \right),
\]

which does not correspond with a conformal embedding; and was found using a nontrivial automorphism of the fusion rules of the extended algebra.\textsuperscript{24} We continued modding up to \( \mathbb{Z}_{72} \), but no other exceptional theory appears at this level.

4. At level \( k = 21 \) \((h = 24)\), starting with \( D_{24} \) and modding by \( \mathbb{Z}_2 \) leads to the case III of Subsec. 2.2., which after subtracting \( D_{24} \) of multiplicity 2, yields a modular invariant combination with some of its coefficients negative integers, which we call \( M_{24} \)

\[
\left[ \sum_{\alpha=1}^{2} T^\alpha SZ_1 + Z_1 \right] = 2D_{24} + M_{24},
\]

with

\[
M_{24} = \left[ |\chi_{[1,1]} + \chi_{[2,11]}|^2 + |\chi_{[5,5]} + \chi_{[7,7]}|^2 + |\chi_{[1,7]} + \chi_{[8,5]}|^2 + |\chi_{[7,1]} + \chi_{[5,8]}|^2 \right.
\]

\[
+ |\chi_{[3,3]} + \chi_{[6,9]}|^2 + |\chi_{[1,4]} - \chi_{[4,7]}|^2 + |\chi_{[4,1]} - \chi_{[7,4]}|^2 + 2|\chi_{[3,9]}|^2
\]

\[
+ 2|\chi_{[9,3]}|^2 - \left( |\chi_{[2,2]} + \chi_{[2,8]} + \chi_{[8,2]} - \chi_{[4,10]}|^2 + 3|\chi_{[4,4]} - \chi_{[8,8]}|^2 \right) \right] .
\]
where
\[ \chi[m_1, m_2] \equiv \chi(m_1, m_2) + \chi(m_2, h-m_1-m_2) + \chi(h-m_1-m_2, m_1). \] (3.43)
Modding by \( Z_3 \) gives \( D_{12} \) itself, and modding by \( Z_4, Z_8, Z_9, Z_{18}, Z_{24}, Z_{36} \) give rise to three extra modular invariant combinations, which we do not mention here their explicit form. Finally, modding by \( Z_{72} \) and doing the sum in the bracket of eq. (A.16), we encounter the case II of Subsec. 2.2., which after Subtracting charge conjugation counterparts \( D_{24}^{c}, M_{24}^{c} \) with multiplicities 12 and 3 respectively, we are left with an exceptional partition function which is denoted by \( E_{24} \):
\[
\begin{align*}
&\left[ \sum_{\alpha=1}^{72} T^\alpha + \sum_{\alpha=1}^{18} T^\alpha ST^2 + \sum_{\alpha=1}^{8} T^\alpha ST^3 + \sum_{\alpha=1}^{9} T^\alpha ST^4 + \sum_{\alpha=1}^{2} T^\alpha ST^6 \\
&\quad + \sum_{\alpha=1}^{9} T^\alpha ST^8 + \sum_{\alpha=1}^{8} T^\alpha ST^9 + ST^{12} + \sum_{\alpha=1}^{8} T^\alpha ST^{15} + \sum_{\alpha=1}^{2} T^\alpha ST^{18} \\
&\quad + ST^{24} + \sum_{\alpha=1}^{2} T^\alpha ST^{30} + ST^{36} + ST^{48} + ST^{60} \right]SZ_1 + Z_1 \\
&= 12D_{24} + 12D_{24}^{c} + 3M_{24} + 3M_{24}^{c} + 9E_{24},
\end{align*}
\] (3.44)
with
\[ E_{24} = \left| \chi[1,1] + \chi[5,5] + \chi[2,11] + \chi[7,7] \right|^2 + \left| \chi[1,7] + \chi[7,1] + \chi[5,8] + \chi[8,5] \right|^2. \] (3.45)
This theory corresponds to conformal embedding \( \widehat{su(3)}_{k=21} \subset \widehat{(e_7)}_{k=1} \).

So with our method we reproduce not only exceptional partition functions corresponding to a certain conformal embedding like \( E_8, E_{12}^{(1)}, \) and \( E_{24} \), but also the one which can not be obtained by a conformal embedding i.e. \( E_{12}^{(2)} \). Note that at each level the allowed modding \( Z_N \) has \( N \) a divisor of \( 3h \), where \( h = k + 3 \).

### 3.3. \( \widehat{su(4)} \) WZW models

In addition to the diagonal series \( A_h = \sum_{\{\lambda \in B_h\}} |\chi_\lambda|^2 \), where
\[ B_h = \left\{ \lambda \mid 3 \leq \sum_{i=1}^{3} m_i < h = k + 4 \right\}, \] (3.46)
and \( \lambda = \sum_{i=1}^{3} m_i \omega_i \), there exist two \( D \) series corresponding to the two subgroups of the centre of \( SU(4) \). Furthermore, up to now three exceptional partition functions have been found in levels \( k = 4, 6, 8 \). In the following we obtain all of these by orbifold approach. We define the action of the \( Z_N \) on the characters of left—moving HW representations of \( su(4) \) due to eq. (2.9) by
\[ p \cdot \chi(m_1, m_2, m_3) = e^{2\pi i (\varphi_\lambda - 20)} \chi(m_1, m_2, m_3) \] (3.47)
where
\[ \varphi_\lambda = 3m_1^2 + 4m_2^2 + 3m_3^2 + 4m_1m_2 + 2m_1m_3 + 4m_2m_3 \]
and $p$ is the generator of $\mathcal{Z}_N$. Thus, the untwisted part of a partition function, consists of left–moving HW representations $\lambda$ which satisfy: $\varphi_\lambda = 20 \mod N$.

**D - Series**

We follow the same recipe of calculation that was described in Subsec. 3.2., but without going into the details, and find the general form of $D_h$ series at each level. Starting with $A_h$, first we mod it out by a subgroup $\mathcal{Z}_2$ of the centre and obtain at every level a nondiagonal partition function which we denote by $D_h^{(2)}$,

$$D_h^{(2)} \equiv Z(A_h/\mathcal{Z}_2) = \sum_{\lambda|\Sigma_{i=1}^3 i m_i = 0 \mod 2} |\chi\lambda|^2 + \sum_{\lambda|\Sigma_{i=1}^3 i m_i = k \mod 2} \chi\lambda \bar{\chi}\mu(\lambda), \quad (3.48)$$

where $\mu(m_1, m_2, m_3) = (m_3, h - \Sigma_{i=1}^3 m_i, m_1)$. Then we mod out the $A_h$ series by $\mathcal{Z}_4$ according to eq. (A.4) and find at every even level a nondiagonal partition function $D_h^{(4)}$, which has the form

$$D_h^{(4)} \equiv Z(A_h/\mathcal{Z}_4) = \sum_{\lambda|\Sigma_{i=1}^3 i m_i = 2 \mod 4} |\chi\lambda|^2 + \sum_{\lambda|\Sigma_{i=1}^3 i m_i = 2 + \frac{1}{2} \mod 4} \chi\lambda \bar{\chi}\sigma(\lambda) + \sum_{\lambda|\Sigma_{i=1}^3 i m_i = 2 - \frac{1}{2} \mod 4} \chi\lambda \bar{\chi}\sigma^2(\lambda), \quad (3.49)$$

where $\sigma(m_1, m_2, m_3) = (m_2, m_3, h - \Sigma_{i=1}^3 m_i)$. These results agree with the ones obtained in Ref. 17 modulo the comment in the footnote of page 13.

**E - Series**

It is expected that there are nondiagonal partition functions at levels $k = 2, 4, 6, 8$, due to the following conformal embeddings:20

$$\widetilde{su}(4)_{k=2} \subset su(6)_{k=1} \subset (B_7)_{k=1}, \quad \widetilde{su}(4)_{k=4} \subset (B_7)_{k=1}$$

$$su(4)_{k=6} \subset su(10)_{k=1} \subset (D_{10})_{k=1}, \quad (D_{10})_{k=8} \subset (D_{10})_{k=1}. \quad (3.50)$$

1. At level $k = 2$ ($h = 6$), we have only $D_6^{(2)}$, and $D_6^{(4)} = A_{6c.c.}$. First, we start with $A_6$ and mod it out by all allowed $\mathcal{Z}_N$’s up to $N = 24$. Nothing other than $A_6$ and $D_6^{(2)}$ and their charge conjugation counterparts is found. For example, in the cases $N = 3, 6, 8$ we obtain

$$\left[\sum_{\alpha=1}^3 T^\alpha S Z_1 + Z_1\right] = 4A_6 - 2D_6^{(2)} \quad (3.51)$$

$$\left[\sum_{\alpha=1}^6 T^\alpha + \sum_{\alpha=1}^3 T^\alpha S T^2 + \sum_{\alpha=1}^2 T^\alpha S T^3\right] S Z_1 + Z_1 = 4A_6 + 4D_6^{(2)} \quad (3.52)$$

$$\left[\sum_{\alpha=1}^8 T^\alpha + \sum_{\alpha=1}^2 T^\alpha S T^2 + S T^4\right] S Z_1 + Z_1 = 2A_6 + 2A_{6c.c.} + D_6^{(2)}, \quad (3.53)$$

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respectively. Then we start with \( D_6^{(2)} \) and do the allowed moddings. Again, nothing more is found. For example, in modding by \( Z_3 \) we find

\[
\left[ \sum_{\alpha=1}^{3} T^\alpha SZ_1 + Z_1 \right] = 2D_6^{(2)}.
\]  

(3.54)

This is not surprising, since it can easily be seen that \( D_6^{(2)} \)

\[
D_6^{(2)} = \left| \chi_{(1,1,1)} + \chi_{(1,3,1)} \right|^2 + \left| \chi_{(1,1,3)} + \chi_{(3,1,1)} \right|^2 + 2\left| \chi_{(1,2,1)} \right|^2 + 2\left| \chi_{(2,1,2)} \right|^2,
\]  

(3.55)

exactly corresponding to conformal embedding \( \widehat{su}(4)_{k=2} \subset \widehat{su}(6)_{k=1} \).

2. At level \( k = 4 \) (\( h = 8 \)), there exist two \( D_8^{(2)} \) and \( D_8^{(4)} \) partition functions. We choose to start with \( D_8^{(2)} \). Modding by \( Z_2 \) gives \( D_8^{(2)} \) itself, and modding by \( Z_4 \) and \( Z_8 \) give a combination of \( D_8^{(2)} \) and \( D_8^{(4)} \):

\[
\left[ \sum_{\alpha=1}^{4} T^\alpha + ST^2 \right] SZ_1 + Z_1 = 2D_8^{(2)} + 2D_8^{(4)}
\]  

(3.56)

\[
\left[ \sum_{\alpha=1}^{8} T^\alpha + \sum_{\alpha=1}^{2} T^\alpha ST^2 + ST^4 \right] SZ_1 + Z_1 = 4D_8^{(2)} + 4D_8^{(4)}.
\]  

(3.57)

The next modding which satisfies the condition (A.3) is \( Z_{16} \). After doing the sum in the bracket of eq. (A.10) we encounter the case I of Subsec. 2.2., which after subtracting the \( D_8^{(2)} \) and \( D_8^{(4)} \) each with multiplicity 4, an exceptional partition function is found which we denote by \( E_8 \):

\[
\left[ \sum_{\alpha=1}^{16} T^\alpha + \sum_{\alpha=1}^{4} T^\alpha ST^2 + ST^4 + ST^8 + ST^{12} \right] SZ_1 + Z_1 = 4D_8^{(2)} + 4D_8^{(4)} + 4E_8,
\]  

(3.58)

where

\[
E_8 = \left| \chi_{(1,1,1)} + \chi_{(1,5,1)} + \chi_{(1,2,3)} + \chi_{(3,2,1)} \right|^2
+ \left| \chi_{(1,1,5)} + \chi_{(5,1,1)} + \chi_{(2,1,2)} + \chi_{(2,3,2)} \right|^2 + 4\left| \chi_{(2,2,2)} \right|^2.
\]  

(3.59)

It can easily be shown that this partition function corresponds to conformal embedding \( \widehat{su}(4)_{k=4} \subset \widehat{B}_7_{k=1} \). The above exceptional partition function was obtained in the context of fixed-point resolution of Ref. 19. We then repeat the above moddings starting with \( D_8^{(4)} \) and find exactly the same results as with \( D_8^{(2)} \).

3. At level \( k = 6 \) (\( h = 10 \)), there exist \( D_{10}^{(2)} \) and \( D_{10}^{(4)} \). Starting with \( D_{10}^{(2)} \) the following results are obtained. Modding by \( Z_2 \) and \( Z_4 \) gives \( D_{10}^{(2)} \) itself, but modding by \( Z_8 \) and doing the sum in the bracket of eq. (A.6) we encounter the case III of Subsec. 2.2., which
after subtraction $D^{(2)}_{10}$ of multiplicity 6 leads to a modular invariant combination, which we call $M_{10}$

$$
\left[ \sum_{\alpha=1}^{8} T^\alpha + \sum_{\alpha=1}^{2} T^\alpha ST^2 + ST^4 \right] S Z_1 + Z_1 = 6D_{10} - M_{10}
$$

where

$$
M_{10} = |(\chi_{(1,1,1)} + \chi_{(1,7,1)}) - (\chi_{(2,1,6)} + \chi_{(6,1,2)}) - (\chi_{(3,1,3)} + \chi_{(3,3,3)})|^2
$$

$$
+ |(\chi_{(1,1,7)} + \chi_{(7,1,1)}) - (\chi_{(1,2,1)} + \chi_{(1,6,1)}) - (\chi_{(3,3,3)} + \chi_{(3,3,1)})|^2
$$

$$
+ 3|\chi_{(2,2,4)} + \chi_{(4,2,2)}|^2 + 3|\chi_{(2,2,4)} + \chi_{(4,2,2)}|^2.
$$

In modding by $\mathcal{Z}_5$ after doing the sum in the bracket of eq. (2.7) with $N = 5$, we encounter the case II of Subsec. 2.2., which by subtracting $D^{(2)}_{10}$, and its charge conjugation $D^{(2)}_{10} \ c. c.$, and $M_{10}$ by multiplicities $-3/5, 4/5$, and $8/5$ respectively, an exceptional partition function is found which we call $E_{10}$

$$
\left[ \sum_{\alpha=1}^{5} T^\alpha S Z_1 + Z_1 \right] = \frac{1}{3}(4D^{(2)}_{10} \ c. c. - 3D^{(2)}_{10} + 8M_{10} + 2E_{10}),
$$

where

$$
E_{10} = |\chi_{(1,1,1)} + \chi_{(1,7,1)} + \chi_{(3,1,3)} + \chi_{(3,3,3)}|^2
$$

$$
+ |\chi_{(1,1,7)} + \chi_{(7,1,1)} + \chi_{(1,3,3)} + \chi_{(3,3,1)}|^2
$$

$$
+ |\chi_{(1,1,3)} + \chi_{(3,5,1)} + \chi_{(3,2,3)}|^2 + |\chi_{(3,1,1)} + \chi_{(1,5,3)} + \chi_{(3,2,3)}|^2
$$

$$
+ |\chi_{(1,1,5)} + \chi_{(5,3,1)} + \chi_{(2,3,2)}|^2 + |\chi_{(5,1,1)} + \chi_{(1,3,5)} + \chi_{(2,3,2)}|^2
$$

$$
+ |\chi_{(1,2,3)} + \chi_{(3,4,1)} + \chi_{(4,1,4)}|^2 + |\chi_{(1,4,3)} + \chi_{(3,2,1)} + \chi_{(4,1,4)}|^2
$$

$$
+ |\chi_{(2,1,4)} + \chi_{(4,3,2)} + \chi_{(1,4,1)}|^2 + |\chi_{(4,1,2)} + \chi_{(2,3,4)} + \chi_{(1,4,1)}|^2.
$$

It can easily be shown that $E_{10}$ actually corresponds to conformal embedding $\widehat{su(4)}_{k=6} \subset \widehat{su(10)}_{k=1}$. This exceptional partition function was found in the context of simple current method in Ref. 18. We have carried out all allowed moddings up to $\mathcal{Z}_{40}$, and except in the case of $\mathcal{Z}_5$ which gives rise to another modular invariant combination, we find nothing new other than some linear combination of $D^{(2)}_{10}, M_{10},$ and $E_{10}$ and their charge conjugations. For example, modding by $\mathcal{Z}_{40}$ according to eq. (A.15) yields

$$
\left[ \sum_{\alpha=1}^{40} T^\alpha + \sum_{\alpha=1}^{20} T^\alpha ST^2 + \sum_{\alpha=1}^{5} T^\alpha ST^4 + \sum_{\alpha=1}^{8} T^\alpha ST^5
$$

$$
+ \sum_{\alpha=1}^{5} T^\alpha ST^8 + \sum_{\alpha=1}^{2} T^\alpha ST^{10} + ST^{20} \right] S Z_1 + Z_1
$$

$$
= \frac{6}{5}(2D^{(2)}_{10} + 4D^{(2)}_{10} \ c. c. + 3M_{10} + 2E_{10})
$$

(3.64)
4. At level \( k = 8 \) \( (h = 12) \), there exist \( D^{(2)}_{12} \) and \( D^{(4)}_{12} \). We choose \( D^{(4)}_{12} \) and obtain the following results. Modding by \( \mathbb{Z}_2 \) and \( \mathbb{Z}_4 \) gives \( D^{(4)}_{12} \) itself, but in modding by \( \mathbb{Z}_8 \) we encounter the case II of Subsec. 2.2., which after subtraction \( D^{(4)}_{12} \) of multiplicity 12, an exceptional partition function is found, which we call \( E^{(1)}_{12} \)
\[
\left( \sum_{\alpha=1}^{8} T^{\alpha} + \sum_{\alpha=1}^{2} T^{\alpha} ST^2 + ST^4 \right) SZ_1 + Z_1 = 12D^{(4)}_{12} - 2E^{(1)}_{12}, \tag{3.65}
\]
where *
\[
E^{(1)}_{12} = \left| \chi(1,1,1) + \chi(1,1,9) + \chi(1,9,1) + \chi(9,1,1) + \chi(2,3,2) + \chi(2,5,2) + \chi(3,2,5) + \chi(5,2,3) \right|^2 
+ \left| \chi(1,3,1) + \chi(1,7,1) + \chi(3,1,7) + \chi(7,1,3) + \chi(1,1,4) + \chi(3,4,1) + \chi(4,1,4) + \chi(4,3,4) \right|^2 
+ 2 \left| \chi(2,2,4) + \chi(2,4,4) + \chi(4,2,2) + \chi(4,4,2) \right|^2. \tag{3.66}
\]
It can easily be seen that \( E^{(1)}_{12} \) just corresponds to conformal embedding \( \left( \widehat{D}_3 \right)_{k=8} \subset \left( \widehat{D}_{10} \right)_{k=1} \) with the following branching rules:
\[
\begin{align*}
ch_1 &= \chi(1,1,1) + \chi(1,1,9) + \chi(1,9,1) + \chi(9,1,1) + \chi(2,3,2) + \chi(2,5,2) + \chi(3,2,5) + \chi(5,2,3) \\
ch_2 &= \chi(1,3,1) + \chi(1,7,1) + \chi(3,1,7) + \chi(7,1,3) + \chi(1,1,4) + \chi(3,4,1) + \chi(4,1,4) + \chi(4,3,4) \\
ch_3 &= \chi(2,2,4) + \chi(2,4,4) + \chi(4,2,2) + \chi(4,4,2), \\
\end{align*}
\tag{3.67}
\]
where \( \chi_i \)'s are the characters of the integrable representations of \( \left( \widehat{D}_{10} \right)_{k=1} \) and \( \chi_{(m_1,m_2,m_3)} \)'s are those of \( SU(4)_{k=8} \).

Then, modding by \( \mathbb{Z}_3 \) and doing the sum in eq. (2.7) with \( N = 3 \) leads to the case I of Subsec. 2.2., which after subtraction \( D_{12}, D^{c.c.}_{12} \) and \( E^{(1)}_{12} \) each of multiplicity 1/3 another exceptional partition function is found, which we call \( E^{(2)}_{12} \)
\[
\left( \sum_{\alpha=1}^{4} T^{\alpha} + ST^2 \right) SZ_1 + Z_1 = \frac{1}{3} \left( D^{(4)}_{12} + D^{(4 \ c.c.)}_{12} + E_{12} + 4E^{(2)}_{12} \right) \tag{3.68}
\]
with
\[
E^{(2)}_{12} = \left| \chi(1,1,1) + \chi(1,1,9) + \chi(1,9,1) + \chi(9,1,1) \right|^2 + \left| \chi(2,3,2) + \chi(2,5,2) + \chi(3,2,5) + \chi(5,2,3) \right|^2 \\
+ \left| \chi(1,3,1) + \chi(1,7,1) + \chi(3,1,7) + \chi(7,1,3) \right|^2 + \left| \chi(1,1,4) + \chi(3,4,1) + \chi(4,1,4) + \chi(4,3,4) \right|^2 \\
+ \left| \chi(1,1,5) + \chi(1,5,1) + \chi(5,1,1) + \chi(5,5,1) \right|^2 + \left| \chi(1,3,5) + \chi(3,1,3) + \chi(3,5,3) + \chi(5,3,1) \right|^2 \\
+ \left| \chi(1,5,1) + \chi(5,1,5) \right|^2 + \left| \chi(2,2,4) + \chi(2,4,4) + \chi(4,2,2) + \chi(4,4,2) \right|^2 + 2 \left| \chi(3,3,3) \right|^2 \\
+ \left( \chi(1,2,3) + \chi(1,6,3) + \chi(2,1,6) + \chi(2,3,6) + \chi(3,2,1) + \chi(3,6,1) + \chi(6,1,2) + \chi(6,3,2) \right) \\
\cdot \chi(3,3,3) + \chi(1,5,1) + \chi(5,1,5) \right( \chi(1,2,7) + \chi(2,1,2) + \chi(2,7,2) + \chi(7,2,1) \right) + c.c. \right).
\tag{3.69}
\]
* We are not aware of the explicit form of this partition function in the literature.
This exceptional partition function which doesn’t correspond to a conformal embedding, was recently found by a computational method which essentially looks for the eigenvectors of matrix $S$ (the generator of the modular group) with eigenvalues equal to one, and can be shown to be a consequence of an automorphism of the fusion rules of the extended algebra. We continue the modding by allowed groups up to $\mathbb{Z}_{96}$, but no new partition function is found. For example in modding by $\mathbb{Z}_{24}$ according to eq. (A.12), we get

$$\left[\sum_{\alpha=1}^{24} T^\alpha + \sum_{\alpha=1}^{6} T^\alpha ST^2 + \sum_{\alpha=1}^{8} T^\alpha ST^3 + \sum_{\alpha=1}^{3} T^\alpha ST^4 + \sum_{\alpha=1}^{2} T^\alpha ST^6 \\
+ \sum_{\alpha=1}^{3} T^\alpha ST^8 + ST^{12}\right]SZ_1 + Z_1 = 4\left(D_{12} + D_{12}^c + E_{12}^{(1)} + 4E_{12}^{(2)}\right).$$

So again, as in the case of $\widehat{su}(3)$ models, with the orbifold method not only exceptional partition functions which correspond to conformal embeddings like $E_8$, $E_{10}$, and $E_{12}^{(1)}$ are found, but also the a partition function which does not corresponds to a conformal embedding i.e. $E_{12}^{(2)}$, is generated.

In this subsection we have been able to obtain explicitly all the exceptional partition functions which correspond to a conformal embedding and moreover the one ($E_{12}^{(2)}$) which follows from an automorphism of the fusion rules of the extended algebra. Note that at each level the allowed modding $\mathbb{Z}_N$ has $N$ a divisor of $4h$, where $h = k + 4$.

### 3.4. $\widehat{su}(5)$ WZW models

For $\widehat{su}(5)$ models besides the usual diagonal series $A_h = \sum_{\{\lambda \in B_h\}} |\chi_\lambda|^2$, at each level, where

$$B_h = \left\{ \lambda \mid 4 \leq \sum_{i=1}^{4} m_i < h = k + 5 \right\},$$

and $\lambda = \sum_{i=1}^{4} m_i \omega_i$, and one nondiagonal $D$ series at each level, up to now some exceptional partition functions have been found which we are going to obtain by our method. We define the action of the $\mathbb{Z}_N$ on the characters of HW representations of $su(5)$ due to eq. (2.9) by

$$p \cdot \chi(m_1, m_2, m_3, m_4) = e^{\frac{2\pi i}{N}(\varphi_\lambda - 50)} \chi(m_1, m_2, m_3, m_4),$$

where

$$\varphi_\lambda = 4m_1^2 + 6m_2^2 + 6m_3^2 + 4m_4^2 + 6m_1 m_2 + 4m_1 m_3 + 2m_1 m_4 + 8m_2 m_3 + 4m_2 m_4 + 6m_3 m_4,$$

and $p$ is the generator of $\mathbb{Z}_N$. Thus, the untwisted part of a partition function, consists of left–moving HW representations $\lambda$ which satisfy: $\varphi_\lambda = 50 \mod N$.

### D - Series
We start with $A_h$ series, following the same recipe mentioned in Subsec. 3.2., mod it out by $Z_5$ using the eq. (2.7) with $N = 5$. Doing the sum in the bracket, we obtain the general form of $D_h$ series:

$$D_h \equiv Z(A_h/Z_5) = \sum_{\lambda\mod 3} |\chi(\lambda)|^2 + \sum_{\lambda\mod 2} \chi(\lambda) \bar{\lambda} \bar{\lambda} \sigma(\lambda) + \sum_{\lambda\mod 1} \chi(\lambda) \bar{\lambda} \bar{\lambda} \sigma^2(\lambda) + \sum_{\lambda\mod 0} \chi(\lambda) \bar{\lambda} \bar{\lambda} \sigma^3(\lambda)$$

where $\sigma(m_1, m_2, m_3, m_4) = (m_2, m_3, m_4, h - \Sigma_{i=1}^4 m_i)$. These results agree with the ones obtained in Ref. 17, modulo the comment mentioned in the footnote of page 13.

E - Series

One expects to find, possibly, the exceptional series at levels $k = 3, 5, 7$ according to the following conformal embeddings:

$$\widehat{su}(5)_{k=3} \subset \widehat{su}(10)_{k=1}, \quad \widehat{su}(5)_{k=5} \subset (\widehat{D}_{12})_{k=1}, \quad \widehat{su}(5)_{k=7} \subset \widehat{su}(15)_{k=1}.$$  \hspace{1cm} (3.74)

We will limit ourselves only to the first two cases in this work.

1. At level $k = 3$ ($h = 8$), we start with $A_8$, doing the allowed moddings and obtain the following results. Modding by $Z_2$ gives $A_8$ itself, but modding by $Z_4$ we encounter the case II of section 2.2., which after subtracting $A_8$ and $D_{8,c}^{c,c}$ with multiplicities 5 and $-1$ respectively, an exceptional partition function is found which we call $E_8$

$$[(\sum_{a=1}^{\alpha} T^a + ST^2)SZ_1 + Z_1] = 5A_8 - D_{8,c}^{c,c} + E_8$$

with

$$E_8 = |\chi(1,1,1,1) + \chi(1,2,2,1)|^2 + |\chi(1,1,1,4) + \chi(2,2,1,2)|^2 + |\chi(1,1,2,1) + \chi(1,3,1,2)|^2 + |\chi(1,1,3,1) + \chi(3,1,1,2)|^2 + |\chi(1,1,4,1) + \chi(2,1,2,1)|^2 + |\chi(1,2,1,3) + \chi(3,1,2,1)|^2 + |\chi(1,2,1,2) + \chi(1,4,1,1)|^2 + |\chi(1,3,1,1) + \chi(2,1,1,3)|^2 + |\chi(1,2,1,1) + \chi(2,1,3,1)|^2 + |\chi(2,1,2,2) + \chi(4,1,1,1)|^2.$$  \hspace{1cm} (3.76)

We have checked that $E_8$ exactly corresponds to conformal embedding $\widehat{su}(5)_{k=3} \subset \widehat{su}(10)_{k=1}$. We carried out all the allowed moddings up to $Z_{40}$ and no other exceptional partition function was found. Then, we start with $D_8$ and mod it out by allowed moddings up to $Z_{40}$, but again no more exceptional partition function is found. For example, modding
by $Z_{16}$ after doing the sum in the bracket of eq. (A.10), we encounter the case I of Subsec. 2.2., which after subtraction $D_8$ and $A_{8}^{c,c}$ each of multiplicity 4, results in $E_{8}^{c,c}$:

$$\left[(\sum_{\alpha=1}^{16} T^{\alpha} + \sum_{\alpha=1}^{4} T^{\alpha}ST^2 + ST^4 + ST^8 + ST^{12})SZ_1 + Z_1\right] = 4D_8 + 4A_{8}^{c,c} + 2E_{8}^{c,c}. \quad (3.77)$$

2. At level $k = 5 (h = 10)$, we start with $D_{10}$ which has the form

$$D_{10} = |\chi_1|^2 + |\chi_2|^2 + |\chi_3|^2 + |\chi_4|^2 + |\chi_5|^2 + 5|\chi_6|^2, \quad (3.78)$$

here following the Ref. 19, we have used the following abbreviations

$$\chi_1 = \chi(1,1,1,1) + \chi(1,1,1,6) + \chi(1,1,6,1) + \chi(1,6,1,1) + \chi(6,1,1,1)$$
$$\chi_2 = \chi(1,2,1,3) + \chi(2,1,3,3) + \chi(1,3,3,1) + \chi(3,3,1,2) + \chi(3,1,2,1)$$
$$\chi_3 = \chi(1,1,2,4) + \chi(1,2,4,2) + \chi(2,4,2,1) + \chi(4,2,1,1) + \chi(2,1,1,2)$$
$$\chi_4 = \chi(1,1,3,2) + \chi(1,3,2,3) + \chi(3,2,3,1) + \chi(2,3,1,1) + \chi(3,1,1,3)$$
$$\chi_5 = \chi(1,2,2,1) + \chi(2,2,1,4) + \chi(2,1,4,1) + \chi(1,4,1,2) + \chi(4,1,2,2)$$
$$\chi_6 = \chi(2,2,2,2). \quad (3.79)$$

and do the allowed moddings. Then the following results are obtained. Modding by $Z_2$ gives $D_{10}$ itself, but modding by $Z_{25}$ according to eq. (A.13), after doing the sum in the bracket, we encounter the case I which after subtraction $D_{10}$ of multiplicity 10, results in a modular invariant partition function, which we denote by $E_{10}^{(1)}$:

$$\left[(\sum_{\alpha=1}^{25} T^{\alpha} + ST^5 + ST^{10} + ST^{15} + ST^{20})SZ_1 + Z_1\right] = 10D_{10} + 10E_{10}^{(1)}, \quad (3.80)$$

where

$$E_{10}^{(1)} = |\chi_1 + \chi_2|^2 + 2|\chi_3|^2 + 10|\chi_6|^2. \quad (3.81)$$

Next, we mod out $D_{10}$ by $Z_4$ and encounter case II of Subsec. 2.2., which after subtraction $D_{10}$ and $E_{10}^{(1)}$ of multiply 7/2 and $-3/2$ respectively, leads to another modular invariant partition function denoted by $E_{10}^{(2)}$:

$$\left[(\sum_{\alpha=1}^{4} T^{\alpha} + ST^2)SZ_1 + Z_1\right] = \frac{1}{2} (7D_{10} - 3E_{10}^{(1)} + 5E_{10}^{(2)}) \quad (3.82)$$

with

$$E_{10}^{(2)} = |\chi_1|^2 + |\chi_2|^2 + |\chi_4|^2 + |\chi_5|^2 + 4|\chi_6|^2 + \chi_3 \overline{\chi_6} + c.c. \quad (3.83)$$

What is interesting here is that starting with $E_{10}^{(1)}$ and modding by $Z_4$ gives rise to the case II, which after subtracting $E_{10}^{(1)}$ from the sum in the bracket of eq. (A.4) with
multiplicity $-2$, we find another modular invariant partition function, which we denote by $E_{10}^{(3)}$:

$$\left[ \sum_{\alpha=1}^{4} T^\alpha + ST^2 \right]SZ_1 + Z_1 = -2E_{10}^{(1)} + 5E_{10}^{(3)}, \quad (3.84)$$

where

$$E_{10}^{(3)} = |\chi_1 + \chi_2|^2 + |\chi_3 + \chi_6|^2 + 2|2\chi_6|^2. \quad (3.85)$$

It must be mentioned that modding by $Z_2$ gives $E_{10}^{(1)}$ itself.

Then we start with $E_{10}^{(2)}$ and do the same moddings. We find that moddings by $Z_2$ gives $E_{10}^{(2)}$ itself; but modding by $Z_4$ gives rise to the case II which after subtraction of $E_{10}^{(1)}$, $E_{10}^{(2)}$, and $E_{10}^{(3)}$ of multiplicities $-1/2, 6,$ and $1/2$, leads to yet another modular invariant partition function, which we call $E_{10}^{(4)}$:

$$\left[ \sum_{\alpha=1}^{4} T^\alpha + ST^2 \right]SZ_1 + Z_1 = \frac{1}{2}(E_{10}^{(3)} - E_{10}^{(1)} + 12E_{10}^{(2)} - 3E_{10}^{(4)}), \quad (3.86)$$

where

$$E_{10}^{(4)} = |\chi_1 + \chi_2|^2 + 8|\chi_6|^2 + 2(\chi_3 \chi_6 \text{ c.c.}). \quad (3.87)$$

One can easily see that among these exceptional partition functions, only $E_{10}^{(3)}$ exactly corresponds to a conformal embedding $\overline{su(5)}_{k=5} \subset (\overline{D}_{12})_{k=1}$, and the others were obtained only in the context of the automorphism of fusion rules techniques. So in this subsection we have generated by orbifold method not only the partition functions which can be obtained by conformal embedding like $E_8$ and $E_{10}^{(3)}$, but also the others which do not correspond to a conformal embedding like $E_{10}^{(1)}$, $E_{10}^{(2)}$, and $E_{10}^{(4)}$.

### 3.5. $\overline{su(6)}$ WZW models

In addition to the diagonal series $A_h = \sum_{\{\lambda \in B_h\}} |\chi_\lambda|^2$ where

$$B_h = \left\{ \lambda \mid 5 \leq \sum_{i=1}^{5} m_i < h = k + 6 \right\}, \quad (3.88)$$

where $\lambda = \sum_{i=1}^{5} m_i \omega_i$ there exist three $D_h$ series corresponding to the three subgroups of the centre of $SU(6)$. So far no exceptional series has been found explicitly in the literature, however by applying our method we are able to find, at the first step, a new exceptional partition functions, which could in principle be interpreted as a conformal embedding. We define the action of the $\mathcal{Z}_N$ on the characters of HW representations of $su(6)$ due to eq. (2.9) by

$$p \cdot X(m_1, m_2, m_3, m_4, m_5) = e^{\frac{2\pi i}{N} \varphi_{\lambda} - 105} X(m_1, m_2, m_3, m_4, m_5) \quad (3.89)$$
where
\[
\varphi_\lambda = 5m_1^2 + 8m_2^2 + 9m_3^2 + 8m_4^2 + 5m_5^2 + 8m_1m_2 + 6m_1m_3 + 4m_1m_4 + 2m_1m_5 \\
+ 12m_2m_3 + 8m_2m_4 + 4m_3m_5 + 12m_3m_4 + 6m_3m_5 + 8m_4m_5,
\]
and $p$ is the generator of $\mathbb{Z}_N$. Thus the untwisted part of a partition function, consists of left-moving HW representations $\lambda$ which satisfy: $\varphi_\lambda = 105 \mod N$.

**D - Series**

Just as in previous cases, we start with $A_h$ and mod it out by subgroup $\mathbb{Z}_2$ and obtain a nondiagonal series which we denote by $D_h^{(2)}$:

\[
D_h^{(2)} \equiv Z(A_h/\mathbb{Z}_2) = \sum_{\{\lambda|\Sigma i=1 \imath m_i=1 \mod 2\}} |\chi_\lambda|^2 + \sum_{\{\lambda|\Sigma i=1 \imath m_i=1+\mod 2\}} \chi_\lambda \bar{\chi}_\mu(\lambda)
\] (3.90)

for even levels, where $\mu(m_1, m_2, m_3, m_4, m_5) = (m_4, m_5, h - \Sigma_{i=1}^5 m_i, m_1, m_2)$. Then, modding by subgroup $\mathbb{Z}_3$ yields at any level a nondiagonal partition function, which we call $D_h^{(3)}$

\[
D_h^{(3)} \equiv Z(A_h/\mathbb{Z}_3) = \sum_{\{\lambda|\Sigma i=1 \imath m_i=0 \mod 3\}} |\chi_\lambda|^2 + \sum_{\{\lambda|\Sigma i=1 \imath m_i=k \mod 3\}} \chi_\lambda \bar{\chi}_\nu(\lambda) \\
+ \sum_{\{\lambda|\Sigma i=1 \imath m_i=2k \mod 3\}} \chi_\lambda \bar{\chi}_{\nu^2}(\lambda)
\] (3.91)

where $\nu(m_1, m_2, m_3, m_4, m_5) = (m_3, m_4, m_5, h - \Sigma_{i=1}^5 m_i, m_1)$, and finally, modding by $\mathbb{Z}_6$ gives rise at any even level to a $D_h^{(6)}$ partition function:

\[
D_h^{(6)} = \sum_{\{\lambda|\Sigma i=1 \imath m_i=3 \mod 6\}} |\chi_\lambda|^2 + \sum_{\{\lambda|\Sigma i=1 \imath m_i=3+\frac{1}{2} \mod 6\}} \chi_\lambda \bar{\chi}_\sigma(\lambda) \\
+ \sum_{\{\lambda|\Sigma i=1 \imath m_i=4k \mod 6\}} \chi_\lambda \bar{\chi}_{\sigma^2}(\lambda) + \sum_{\{\lambda|\Sigma i=1 \imath m_i=3+\frac{3}{2} \mod 6\}} \chi_\lambda \bar{\chi}_{\sigma^3}(\lambda) \\
+ \sum_{\{\lambda|\Sigma i=1 \imath m_i=5k \mod 6\}} \chi_\lambda \bar{\chi}_{\sigma^4}(\lambda) + \sum_{\{\lambda|\Sigma i=1 \imath m_i=3+\frac{5}{2} \mod 6\}} \chi_\lambda \bar{\chi}_{\sigma^5}(\lambda)
\] (3.92)

where $\sigma(m_1, m_2, m_3, m_4, m_5) = (m_2, m_3, m_4, m_5, h - \Sigma_{i=1}^5 m_i)$.

**E - Series**

According to the following conformal embeddings:

\[
\widetilde{su(6)}_{k=4} \subset \widetilde{su(15)}_{k=1}, \quad \widetilde{su(6)}_{k=6} \subset (\widetilde{B_{17}})_{k=1}
\]

\[
\widetilde{su(6)}_{k=6} \subset (\widetilde{C_{10}})_{k=1}, \quad \widetilde{su(6)}_{k=8} \subset \widetilde{su(21)}_{k=1},
\] (3.93)

we expect to find exceptional partition functions at those levels. We will limit ourselves to only the first case in this work.

At level $k = 4 \ (h = 10)$, we choose to start with $D^{(2)}_{10}$. Modding by $\mathbb{Z}_2$ and $\mathbb{Z}_4$ gives $D^{(2)}_{10}$ itself, and modding by $\mathbb{Z}_3$, results in a combination of $D^{(2)}_{10}$ and $D^{(6)}_{10}$. However
modding by $Z_8$ after subtracting $D_{10}^{(2)}$ with multiplicity 12, leads to the case III of Subsec. 2.2., which after subtraction $D_{10}^{(2)}$ of multiplicity 12 from the sum, we find a modular invariant combination (having some negative integer coefficients), which we denote by $M_{10}$:

$$
\left[ \left( \sum_{\alpha=1}^{8} T^\alpha + \sum_{\alpha=1}^{2} T^\alpha ST^2 + ST^4 \right) SZ_1 + Z_1 \right] = 12D_{10}^{(2)} - 2M_{10}, \quad (3.94)
$$

where

$$
M_{10} = \left| (\chi(1,1,1,1)+\chi(1,1,5,11)) - (\chi(1,1,1,3,3)+\chi(3,3,1,1,1)) - (\chi(1,2,1,2,1) + \chi(2,1,3,1,2)) \right|^2 \\
+ \left| (\chi(1,1,1,5,1)+\chi(5,1,1,1,1)) - (\chi(1,1,1,1,3)+\chi(1,3,3,1,1)) - (\chi(1,2,1,3,1) + \chi(3,1,2,1,2)) \right|^2 \\
+ \left| (\chi(1,1,1,1,5)+\chi(1,5,1,1,1)) - (\chi(1,1,3,3,1)+\chi(3,1,1,1,1)) - (\chi(1,3,1,2,1) + \chi(2,1,2,1,3)) \right|^2 \\
+ 3\left| \chi(1,1,2,2,1) + \chi(2,2,2,1,1) \right|^2 + 3\left| \chi(1,2,2,2,1) + \chi(2,2,1,1,2) \right|^2 \\
+ 3\left| \chi(2,1,1,2,2) + \chi(2,2,2,2,1) \right|^2. \quad (3.95)
$$

Then, modding by $Z_5$ gives rise to the case II, which after subtracting $D_{10}^{(2)}$, $D_{10}^{(6)\text{ c.c.}}$, and $M_{10}$ with multiplicities 2/5, 4/5, and 8/5 we find a modular invariant partition function which we denote by $E_{10}$:

$$
\left[ \sum_{\alpha=1}^{5} T^\alpha SZ_1 + Z_1 \right] = \frac{2}{5} \left( D_{10}^{(2)} + 2D_{10}^{(6)\text{ c.c.}} + 4M_{10} + E_{10} \right), \quad (3.96)
$$

where

$$
E_{10} = \left| \chi(1,1,1,1,1)+\chi(1,1,5,1,1)+\chi(1,2,1,2,1) + \chi(2,1,3,1,2) \right|^2 \\
+ \left| \chi(1,1,1,1,5)+\chi(1,5,1,1,1)+\chi(1,3,1,2,1)+\chi(2,1,2,1,3) \right|^2 \\
+ \left| \chi(1,1,1,1,5)+\chi(5,1,1,1,1)+\chi(1,2,1,3,1)+\chi(3,1,2,1,2) \right|^2 \\
+ \left| \chi(1,1,3,1,1)+\chi(1,2,1,1,3)+\chi(1,3,2,1,2) \right|^2 + \left| \chi(1,1,1,3,1)+\chi(2,1,2,3,1)+\chi(3,1,1,2,1) \right|^2 \\
+ \left| \chi(2,2,1,2,2)+\chi(1,1,1,4,1)+\chi(4,1,2,1,1) \right|^2 + \left| \chi(2,2,1,2,2)+\chi(1,1,2,1,4)+\chi(1,4,1,1,1) \right|^2 \\
+ \left| \chi(2,2,1,2,1)+\chi(2,1,4,1,1)+\chi(1,1,1,2,1) \right|^2 + \left| \chi(2,2,1,2,1)+\chi(2,1,1,4,1)+\chi(1,4,1,2,1) \right|^2 \\
+ \left| \chi(2,1,2,2,1)+\chi(1,1,4,1,2)+\chi(1,2,1,1,1) \right|^2 + \left| \chi(2,1,2,2,1)+\chi(1,2,1,4,1)+\chi(4,1,1,1,2) \right|^2 \\
+ \left| \chi(1,3,1,1,3)+\chi(1,2,3,1,1)+\chi(1,1,2,1,2) \right|^2 + \left| \chi(1,3,1,1,3)+\chi(3,2,1,2,1)+\chi(2,1,1,3,2) \right|^2 \\
+ \left| \chi(3,1,1,3,1)+\chi(1,1,3,2,1)+\chi(2,1,2,1,1) \right|^2 + \left| \chi(3,1,1,3,1)+\chi(1,2,1,2,3)+\chi(2,3,1,1,2) \right|^2. \quad (3.97)
$$

We have checked that that $E_{10}$ actually corresponds to conformal embedding $\widehat{su(6)}_{k=4} \subset \widehat{su(15)}_{k=1}$. However its explicit form was unknown, because of impracticity of other methods in high rank and high level models. We have carried out the calculation on modding out $D_{10}^{(2)}$ up to $Z_{40}$, but no more exceptional partition function is found. Then,
we start with $D_{10}^{(6)}$, mod it out by allowed groups up to $Z_{40}$, but we get the same results as above. For example, modding $D_{10}^{(6)}$ by $Z_5$ gives rise to the following result:

\[
\left[\sum_{\alpha=1}^{5} T^\alpha S Z_1 + Z_1\right] = \frac{2}{5} \left( D_{10}^{(2)} \text{ c.c.} + 2 D_{10}^{(6)} + 4 M_{10} \text{ c.c.} + E_{10} \text{ c.c.} \right), 
\]

(3.98)

so we will not go into the details any further.

3.6 $\hat{g}_2$ WZW models

As usual there exists at each level a diagonal theory $A_h = \sum_{\lambda \in B_h} |\chi_\lambda|^2$ where

\[
B_h = \left\{ \lambda \mid 3 \leq 2m_1 + m_2 < h = k + 4 \right\},
\]

(3.99)

is the fundamental domain, and $\lambda = m_1 \omega_1 + m_2 \omega_2$, but since the centre of $G_2$ is trivial these models have no $D$ series. However, some exceptional partition functions have been found,\textsuperscript{13,28} which we are going to obtain by our orbifold method. For $\hat{g}_2$ theories the factor \((\text{vol cell of } Q^*/\text{vol cell of } Q)\) in eq. (2.1) for the operator $S$ is equal to $1/3$, and $\hat{h} = 4$. The $Z_N$ action on the characters of left–moving HW representations is defined due to eq. (2.9) by

\[
p \cdot \chi_{(m_1,m_2)} = e^{\frac{2\pi i}{N}(6m_1^2 + 2m_2^2 + 6m_1m_2 - 14)} \chi_{(m_1,m_2)}
\]

(3.100)

where $p$ is the generator of $Z_N$. Thus the untwisted part of a partition function, consists of left–moving HW representations which satisfy: \[6m_1^2 + 2m_2^2 + 6m_1m_2 = 14 \mod N.\]

E - Series

One expects to find exceptional partition functions at levels $k = 3, 4$ corresponding to the following conformal embeddings [20]:

\[
(\hat{g}_2)_{k=3} \subset (e_6)_{k=1}, \quad (\hat{g}_2)_{k=4} \subset (D_7)_{k=1}.
\]

(3.101)

1. At level $k = 3$ ($h = 7$), we start with $A_7$ and mod it out by $Z_7$ according to eq. (2.7) with $N = 7$; after doing the sum in the \textit{bracket}, we encounter the case I of Subsec. 2.2., which after subtraction $A_7$ of multiplicity 2, an exceptional partition function appears, which we denote by $E_7$:

\[
\left[\sum_{\alpha=1}^{7} T^\alpha S Z_1 + Z_1\right] = 2A_7 + 3E_7,
\]

(3.102)

where

\[
E_7 = |\chi_{(1,1)} + \chi_{(2,2)}|^2 + 2|\chi_{(1,3)}|^2.
\]

(3.103)

It actually corresponds to conformal embedding \((\hat{g}_2)_{k=3} \subset (e_6)_{k=1}^{13}\). We also obtained $E_7$ in modding by $Z_3$ but in the latter case we encounter the case II which after subtracting $A_7$ with multiplicity 4 we get $E_7$

\[
\left[\sum_{\alpha=1}^{3} T^\alpha S Z_1 + Z_1\right] = 4A_7 - E_7.
\]

(3.104)
We have worked out all the allowed moddings up to $\mathbb{Z}_{42}$ but no other exceptional theories is found.

2. At level $k = 4$ ($h = 8$), starting with $A_8$ and modding by $\mathbb{Z}_8$ leads to the case I, which after subtraction $A_8$ of mutiplicity 6, an exceptional modular invariant partition function is obtained, which we call $E_8^{(1)}$

$$\left[ \left( \sum_{\alpha=1}^{8} T^\alpha + 2 \sum_{\alpha=1}^{2} T^\alpha ST^2 + ST^4 \right) SZ_1 + Z_1 \right] = 6A_8 + E_8^{(1)},$$

(3.105)

where

$$E_8^{(1)} = |\chi(1,1) + \chi(1,4)|^2 + |\chi(1,5) + \chi(2,1)|^2 + 2|\chi(2,2)|^2.$$  

(3.106)

It can easily be shown that $E_8$ actually corresponds to conformal embedding $(\hat{g}_2)_{k=4} \subset (\hat{D}_7)_{k=1}$. In modding $A_8$ by $\mathbb{Z}_3$ we encounter the case II in which we can recognize a new exceptional partition function, which we call $E_8^{(2)}$

$$\left[ \sum_{\alpha=1}^{3} T^\alpha SZ_1 + Z_1 \right] = 3A_8 + E_8^{(1)} - E_8^{(2)},$$

(3.107)

where

$$E_8^{(2)} = |\chi(1,1)|^2 + |\chi(1,4)|^2 + |\chi(1,5)|^2 + |\chi(2,2)|^2 + |\chi(1,3)|^2 + |\chi(2,3)|^2$$

$$\left( \left( \chi(1,2)\bar{x}(3,1) + \chi(1,5)\bar{x}(2,1) \right) + c.c. \right).$$

(3.108)

This exceptional partition function, which doesn’t correspond to a conformal embedding or simple currents, was found for the first time in Ref. 28. We have worked out all the allowed moddings up to $\mathbb{Z}_{48}$ but no more exceptional partition function is found.

4. Conclusions

In this paper we have introduced an orbifold-like approach, as a unified method for finding all nondiagonal partition functions of a WZW model. For a WZW theory based on Lie group $G$, first we start with a known theory e.g. a member of $A_h$ or $D_h$ series and divide it by some cyclic group $\mathbb{Z}_N$ acting on quantum states. In this procedure only certain group $\mathbb{Z}_N$’s are allowed for a specific $\hat{g}_k$ theory and they lead to a modular invariant combination. Furthermore, if all the coefficients in the combination are positive integers, then we are asured of a physical theory corresponding to an orbifold. What we have learned from applying our method to $\hat{su}(n)$ and $\hat{g}_2$ in Sec. 3 is that, the allowed moddings are usually the ones for which $N$ is a divisor or multiple of $m h$, where $m$ is the factor of $Q$ in the operator $S$ in eq. (2.1), $h = k + \tilde{h}$, and $\tilde{h}$ is the dual coxeter number of $g$. All partition functions which may exist at a certain level are just found by some finite set of allowed moddings, so if no partition function appears after some finite set of moddings, one can infer that there does not exist any partition function at that level. With the aid of this method we have found all the known partition functions and some new ones for $\hat{su}(n)$ with $n = 2, 3, 4, 5, 6$, and also $\hat{g}_2$ models.
An important feature of our method is that one can systematically search for exceptional partition functions, in theories with high rank groups or/and high levels. As an example we applied our approach to $\widehat{SU}(6)$ WZW in Subsec. 3.5., and found a new exceptional partition function. Work is in progress for finding exceptional partition functions of $\hat{B}, \hat{C}, \hat{D}, \hat{F}_4$ models. Although in this paper we were concerned with WZW theories with affine symmetry $\hat{g} \otimes \hat{g}$, i.e., with the symmetry algebras of left−moving and right−moving sectors being the same, however our approach is also applicable to heterotic WZW models with different algebras in their left−moving and right−moving sectors.

Finally, we have observed that every nondiagonal theory comes from some specific moddings, and the question arises to underlying principle behind these moddings. We think that in this way, it is possible to address the basic question of classification of a WZW model.

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**Appendix A**

As it was mentioned in Subsec. 2.2., starting with a theory defined on a group manifold $G$ and modding by a cyclic group $\mathbb{Z}_N$, the partition function of $G/\mathbb{Z}_N$ theory can be obtained, using the untwisted part of partition function $Z_1$, and the generators $S$ and $T$ of the modular group. For $N$ prime, the following equation is obtained:

$$Z(G/\mathbb{Z}_N) = \sum_{\alpha=1}^{N} T^\alpha S Z_1 + Z_1 - Z(G). \quad (A.1)$$

However, it soon becomes apparent that when $N$ is not prime, for generating the full partition function (A.1), at least for each divisor $m$ of $N$, terms in the form

$$\sum_{\alpha=1}^{\beta_m} T^\alpha S T^m S Z_1(G/\mathbb{Z}_N)$$

must be added into the *bracket* of (A.1), where

$$\beta_m = \frac{(N/[m,N])}{[m, (N/[m,N])]} \quad (A.2)$$

and $[ , ]$ denotes the biggest common divisor. But in some cases, as can be seen in some of the following examples, more terms are required, which have the general form of $\sum_{\alpha=1}^{\beta_p} T^\alpha S T^p S Z_1(G/\mathbb{Z}_N)$, where $p$ has some common divisor with $N$, and is always
smaller than it. In order for a modding to lead to a modular invariant combination, for any sum: \( \sum_{\alpha=1}^{\beta m} T^\alpha ST^m SZ_1(G/\mathbb{Z}_N) \), the following relation must be satisfied

\[
T^{\beta m} ST^m SZ_1(G/\mathbb{Z}_N) = ST^m SZ_1(G/\mathbb{Z}_N). \tag{A.3}
\]

Here we gather a list of formulas which have been used in our present work in modding a theory by \( \mathbb{Z}_N \)'s with \( N \) nonprime.

\[
Z(G/\mathbb{Z}_4) = \left[ \left( \sum_{\alpha=1}^{4} T^\alpha + ST^2 \right) SZ_1 + Z_1 \right] - \frac{1}{2} Z(G/\mathbb{Z}_2) - Z(G) \tag{A.4}
\]

\[
Z(G/\mathbb{Z}_6) = \left[ \left( \sum_{\alpha=1}^{6} T^\alpha + \sum_{\alpha=1}^{3} T^\alpha ST^2 + \sum_{\alpha=1}^{2} T^\alpha ST^3 \right) SZ_1 + Z_1 \right] - Z(G/\mathbb{Z}_3) - Z(G/\mathbb{Z}_2) - Z(G) \tag{A.5}
\]

\[
Z(G/\mathbb{Z}_8) = \left[ \left( \sum_{\alpha=1}^{8} T^\alpha + \sum_{\alpha=1}^{2} T^\alpha ST^2 + ST^4 \right) SZ_1 + Z_1 \right] - \frac{1}{2} Z(G/\mathbb{Z}_4) - \frac{1}{2} Z(G/\mathbb{Z}_2) - Z(G) \tag{A.6}
\]

\[
Z(G/\mathbb{Z}_9) = \left[ \left( \sum_{\alpha=1}^{9} T^\alpha + ST^3 + ST^6 \right) SZ_1 + Z_1 \right] - \frac{2}{3} Z(G/\mathbb{Z}_4) - Z(G) \tag{A.7}
\]

\[
Z(G/\mathbb{Z}_{10}) = \left[ \left( \sum_{\alpha=1}^{10} T^\alpha + \sum_{\alpha=1}^{5} T^\alpha ST^2 + \sum_{\alpha=1}^{2} T^\alpha ST^5 \right) SZ_1 + Z_1 \right] - Z(G/\mathbb{Z}_5) - Z(G/\mathbb{Z}_2) - Z(G) \tag{A.8}
\]

\[
Z(G/\mathbb{Z}_{12}) = \left[ \left( \sum_{\alpha=1}^{12} T^\alpha + \sum_{\alpha=1}^{3} T^\alpha ST^2 + \sum_{\alpha=1}^{4} T^\alpha ST^3 + \sum_{\alpha=1}^{3} ST^4 + ST^6 \right) SZ_1 + Z_1 \right] - \frac{1}{2} Z(G/\mathbb{Z}_6) - Z(G/\mathbb{Z}_4) - Z(G/\mathbb{Z}_3) - \frac{1}{2} Z(G/\mathbb{Z}_2) - Z(G) \tag{A.9}
\]

27
\[
Z(G/Z_{16}) = \left\{(\sum_{\alpha=1}^{16} T^{\alpha} + \sum_{\alpha=1}^{4} T^{\alpha} ST^2 + ST^4 + ST^8 + ST^{12})SZ_1 + Z_1\right\} - \frac{1}{2}Z(G/Z_8) - \frac{1}{2}Z(G/Z_4) - \frac{1}{2}Z(G/Z_2) - Z(G) \tag{A.10}
\]

\[
Z(G/Z_{18}) = \left\{(\sum_{\alpha=1}^{18} T^{\alpha} + \sum_{\alpha=1}^{9} T^{\alpha} ST^2 + \sum_{\alpha=1}^{2} T^{\alpha} ST^3 + ST^6 + \sum_{\alpha=1}^{2} T^{\alpha} ST^9 + ST^{12} + \sum_{\alpha=1}^{2} T^{\alpha} ST^{15})SZ_1 + Z_1\right\} - Z(G/Z_9) - \frac{2}{3}Z(G/Z_6) - \frac{2}{3}Z(G/Z_3) - Z(G/Z_2) - Z(G) \tag{A.11}
\]

\[
Z(G/Z_{24}) = \left\{(\sum_{\alpha=1}^{24} T^{\alpha} + \sum_{\alpha=1}^{6} T^{\alpha} ST^2 + \sum_{\alpha=1}^{8} T^{\alpha} ST^3 + \sum_{\alpha=1}^{3} T^{\alpha} ST^4 + \sum_{\alpha=1}^{2} T^{\alpha} ST^6 + \sum_{\alpha=1}^{3} T^{\alpha} ST^8 + ST^{12} + \sum_{\alpha=1}^{2} T^{\alpha} ST^{15})SZ_1 + Z_1\right\} - \frac{1}{2}Z(G/Z_{12}) - \frac{1}{2}Z(G/Z_8) - \frac{1}{2}Z(G/Z_6) - \frac{1}{2}Z(G/Z_4) - \frac{1}{2}Z(G/Z_2) - Z(G) \tag{A.12}
\]

\[
Z(G/Z_{25}) = \left\{(\sum_{\alpha=1}^{25} T^{\alpha} + ST^5 + ST^{10} + ST^{15} + ST^{20})SZ_1 + Z_1\right\} - \frac{4}{5}Z(G/Z_5) - Z(G) \tag{A.13}
\]

\[
Z(G/Z_{36}) = \left\{(\sum_{\alpha=1}^{36} T^{\alpha} + \sum_{\alpha=1}^{9} T^{\alpha} ST^2 + \sum_{\alpha=1}^{4} T^{\alpha} ST^3 + \sum_{\alpha=1}^{9} T^{\alpha} ST^4 + ST^6 + \sum_{\alpha=1}^{4} T^{\alpha} ST^9 + ST^{12} + \sum_{\alpha=1}^{4} T^{\alpha} ST^{15} + ST^{18} + ST^{24} + ST^{30} + \sum_{\alpha=1}^{3} T^{\alpha} ST^{18} + ST^{24} + ST^{30})SZ_1 + Z_1\right\} - \frac{1}{2}Z(G/Z_{18}) - \frac{2}{3}Z(G/Z_{12}) - Z(G/Z_9) - \frac{1}{3}Z(G/Z_6) - \frac{2}{3}Z(G/Z_3) - Z(G) \tag{A.14}
\]
\[ Z(G/Z_{40}) = \left[ \left( \sum_{\alpha=1}^{40} T^\alpha + \sum_{\alpha=1}^{10} T^\alpha ST^2 + \sum_{\alpha=1}^{5} T^\alpha ST^4 + \sum_{\alpha=1}^{8} T^\alpha ST^5 + \sum_{\alpha=1}^{5} T^\alpha ST^8 + \sum_{\alpha=1}^{2} T^\alpha ST^{10} + ST^{20} \right) SZ_1 + Z_1 \right] - \frac{1}{2} Z(G/Z_{20}) - Z(G/Z_{10}) - Z(G/Z_8) - Z(G/Z_6) - \frac{1}{2} Z(G/Z_4) - \frac{1}{2} Z(G/Z_2) - Z(G) \]  

\[ Z(G/Z_{72}) = \left[ \left( \sum_{\alpha=1}^{72} T^\alpha + \sum_{\alpha=1}^{18} T^\alpha ST^2 + \sum_{\alpha=1}^{8} T^\alpha ST^3 + \sum_{\alpha=1}^{9} T^\alpha ST^4 + \sum_{\alpha=1}^{2} T^\alpha ST^6 + \sum_{\alpha=1}^{9} T^\alpha ST^8 + \sum_{\alpha=1}^{8} T^\alpha ST^9 + ST^{12} + \sum_{\alpha=1}^{8} T^\alpha ST^{15} + \sum_{\alpha=1}^{2} T^\alpha ST^{18} + ST^{24} + \sum_{\alpha=1}^{2} T^\alpha ST^{30} + ST^{36} + ST^{48} + ST^{60} \right) SZ_1 + Z_1 \right] - \frac{1}{2} Z(G/Z_{36}) - \frac{2}{3} Z(G/Z_{24}) - \frac{1}{2} Z(G/Z_{18}) - \frac{1}{3} Z(G/Z_{12}) - Z(G/Z_9) - Z(G/Z_6) - \frac{1}{3} Z(G/Z_6) - \frac{1}{2} Z(G/Z_4) - \frac{2}{3} Z(G/Z_3) - \frac{1}{2} Z(G/Z_2) - Z(G) \]

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