THE NATURAL MATROID OF AN INTEGER POLYMATROID

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ABSTRACT. The natural matroid of an integer polymatroid was introduced to show that a simple construction of integer polymatroids from matroids yields all integer polymatroids. As we illustrate, the natural matroid can shed much more light on integer polymatroids. We focus on characterizations of integer polymatroids using their bases, their circuits, and their cyclic flats along with the rank of each cyclic flat and each element; we offer some new characterizations and insights into known characterizations.

1. INTRODUCTION

A polymatroid is a pair $P = (E, \rho)$ where $E$ is a finite set and the real-valued function $\rho : 2^E \to \mathbb{R}$, the rank function of $P$, has the following properties:

1. $\rho$ is normalized, that is, $\rho(\emptyset) = 0$,
2. $\rho$ is non-decreasing, that is, if $A \subseteq B \subseteq E$, then $\rho(A) \leq \rho(B)$, and
3. $\rho$ is submodular, that is, $\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$ for all $A, B \subseteq E$.

Less formally, we often talk about a polymatroid $\rho$ on $E$. A $k$-polymatroid, where $k \in \mathbb{R}$ and $k > 0$, is a polymatroid $(E, \rho)$ for which $\rho(e) \leq k$ for all $e \in E$. For much of this paper, we are concerned with integer polymatroids (also called discrete polymatroids), that is, polymatroids $\rho$ where the rank $\rho(A)$ of each set $A$ is in the set $\mathbb{N}$ of nonnegative integers.

Intuitively, a matroid (an integer 1-polymatroid) can be thought of as a configuration of points, lines, planes, and so on, in which each of the elements that make up these objects has rank 0 (loops) or 1 (points). An integer polymatroid is the natural generalization in which the elements are not limited to points and loops; we also allow, as the elements, lines (elements of rank 2), planes (elements of rank 3), and so on. Not surprisingly, every integer polymatroid comes from a matroid, as the following result of Helgason [14] states.

Theorem 1.1. A function $\rho : 2^E \to \mathbb{N}$ is an integer polymatroid if and only if there is a matroid $M$ on a set $E'$ and a function $\phi : E \to 2^{E'}$ with $\rho(A) = r_M(\bigcup_{e \in A} \phi(e))$ for all $A \subseteq E$.

Helgason [14] introduced the natural matroid to prove this result. Geometrically, we get the natural matroid by, for each element $e$ of $E$, replacing $e$ by a set $\phi(e)$ of $\rho(e)$ points that are placed freely in $e$; thus, a line is replaced by two points that are put freely on the line, and a plane by three points that are placed freely on the plane, and so on. (Section 2 has a precise definition of the natural matroid.) Many important properties of integer polymatroids are closely linked to properties of its natural matroid. For instance, Oxley, Semple, and Whittle [22] showed that an integer 2-polymatroid is 3-connected if and only if it has no loops and its natural matroid is 3-connected. We study the natural matroid in its own right.

In Section 2, we review the definition of the natural matroid and prove two results that make it easy to verify that a matroid $M$ is the natural matroid of an integer polymatroid $\rho$. 

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We show that, for an integer polymatroid \( \rho \) on \( E \) that is the sum of the rank functions of matroids \( M_1, M_2, \ldots, M_k \) on \( E \), the natural matroid of \( \rho \) is the matroid union of certain extensions of \( M_1, M_2, \ldots, M_k \) by loops and elements parallel to those in these matroids.

Herzog and Hibi [15] treat characterizations of integer polymatroids using bases and exchange properties. In Section 3, we show how these results follow easily by observing that the bases of an integer polymatroid are the type vectors of the bases of its natural matroid.

Viewing the bases of an integer polymatroid as the type vectors of the bases of its natural matroid suggests developing an analogous theory for circuits. We do this in Section 4, where, in Theorem 4.3, we introduce circuit axioms for integer polymatroids.

Cyclic flats of matroids, along with their ranks, provide relatively compact descriptions of matroids that allow one to focus on crucial features when, for instance, defining certain matroid constructions (e.g., see [1, 2, 7, 11]); this perspective is also useful in applications such as coding theory (e.g., see [13]). In Section 5, we show that some results about cyclic flats lift from the natural matroid of an integer polymatroid to the polymatroid. In the case of integer polymatroids, this gives another perspective on recent work by Csirmaz [10] characterizing all polymatroids via their cyclic flats and the ranks of these flats and of singleton sets. A key result behind this characterization is the formula that gives the rank of a polymatroid \( \rho \) on \( E \) using only its values on cyclic flats and singleton sets, namely,

\[
\rho(A) = \min \{ \rho(X) + \sum_{i \in A - X} \rho(i) : X \in \mathcal{Z}_\rho \},
\]

where \( \mathcal{Z}_\rho \) is the lattice of cyclic flats of \( \rho \). For a subset \( A \) of \( E \), we consider the set \( \mathcal{R}_\rho(A) \) of cyclic flats \( X \) that yield this minimum. We show that \( \mathcal{R}_\rho(A) \) is a sublattice of \( \mathcal{Z}_\rho \), we identify its least and greatest elements, and we show that each pair of flats in \( \mathcal{R}_\rho(A) \) is a modular pair.

Our matroid notation follows Oxley [21]. For a positive integer \( n \), let \([n]\) be the set \( \{1, 2, \ldots, n\} \). We often take the ground set of an integer polymatroid \( \rho \) to be \([n]\) since this provides a natural correspondence between the elements of \( \rho \) and the entries in \( n \)-tuples. For \( n \in \mathbb{N} \), the set of nonnegative integers, let \([n]_0\) be the set \( \{0, 1, 2, \ldots, n\} \).

For a polymatroid \( \rho \) on \( E \) and for \( A \subseteq E \), the deletion \( \rho - A \) and contraction \( \rho_A \), both on \( E - A \), are defined by \( \rho - A(X) = \rho(X) \) and \( \rho_A(X) = \rho(X \cup A) - \rho(A) \) for all \( X \subseteq E - A \). The minors of \( \rho \) are the polymatroids of the form \( (\rho_A)_B \) for disjoint subsets \( A \) and \( B \) of \( E \). The \( k \)-dual \( \rho^* \) of a \( k \)-polymatroid \( \rho \) on \( E \) is the \( k \)-polymatroid that is given by \( \rho^*(X) = k|X| - \rho(E) + \rho(E - X) \) for all \( X \subseteq E \). The direct sum \( \rho_1 \oplus \rho_2 \) of polymatroids \( \rho_1 \) and \( \rho_2 \) on disjoint sets \( E_1 \) and \( E_2 \) is defined by \( (\rho_1 \oplus \rho_2)(X) = \rho_1(X \cap E_1) + \rho_2(X \cap E_2) \) for \( X \subseteq E_1 \cup E_2 \). A polymatroid that is not a direct sum of two polymatroids on nonempty sets is connected.

2. THE NATURAL MATROID OF AN INTEGER POLYMATROID

The construction of the natural matroid uses Theorem 2.1 below, due to McDiarmid [20], which strengthens an earlier result of Edmonds and Rota. (Theorem 2.1 is treated in [21, 28].) Consider a collection \( L \) of subsets of a set \( E \) that includes \( E \) and \( \emptyset \), and that is closed under intersection; thus, under inclusion, \( L \) is a lattice, and for \( A, B \in L \), their meet \( A \wedge B \) is \( A \cap B \), but their join \( A \vee B \) need not be \( A \cup B \). For such a lattice \( L \), a function \( \sigma : L \to \mathbb{N} \) is submodular if \( \sigma(A \cup B) \leq \sigma(A) + \sigma(B) \) for all \( A, B \in L \).
Theorem 2.1. Let \( L \) be a lattice of subsets of \( E \) that contains \( \emptyset \) and \( E \), and is closed under intersection. Let \( \sigma : L \to \mathbb{N} \) be submodular with \( \sigma(\emptyset) = 0 \). Define \( r : 2^E \to \mathbb{N} \) by
\[
(2.1) \quad r(Y) = \min\{\sigma(S) + |Y - S| : S \in L\},
\]
for \( Y \subseteq E \). The function \( r \) is the rank function of a matroid on \( E \); its independent sets are the subsets \( I \) of \( E \) for which \( |I \cap S| \leq \sigma(S) \) for all \( S \in L \).

Given an integer polymatroid \( \rho \) on a set \( E \), its natural matroid \( M_\rho \) is defined as follows. For each \( i \in E \), let \( X_i \) be a set of \( \rho(i) \) elements so that the sets \( X_i \), for all \( i \in E \), are pairwise disjoint. For \( A \subseteq E \), set
\[
X_A = \bigcup_{i \in A} X_i
\]
and let \( E' = X_E \). Let \( L = \{X_A : A \subseteq E \} \). Now \( L \) is a lattice of subsets of \( E' \) with \( \emptyset, E' \in L, X_A \vee X_B = X_A \cup X_B = X_{A \cup B}, X_A \wedge X_B = X_A \cap X_B = X_{A \cap B} \). Define \( \sigma : L \to \mathbb{N} \) by \( \sigma(X_A) = \rho(A) \). Since \( \rho \) is submodular, so is \( \sigma \). The natural matroid of \( \rho \), denoted \( M_\rho \), is the matroid on \( E' \) whose rank function is given by Equation (2.1). The choice of the sets \( X_i \) is not unique, but the natural matroid is well-defined up to relabeling the elements in \( E' \).

Corollary 2.2. A subset \( I \) of \( E' \) is independent in \( M_\rho \) if and only if \( |I \cap X_A| \leq \rho(A) \) for every \( A \subseteq E \).

Since \( \rho \) is submodular and non-decreasing, if \( A, S \subseteq E \), then
\[
\rho(A) \leq \rho(A \cap S) + \sum_{i \in A-S} \rho(i) \leq \rho(S) + \sum_{i \in A-S} \rho(i).
\]
It follows that \( r_{M_\rho}(X_A) = \rho(A) \) for all \( A \subseteq E \). Theorem 1.1 follows by letting \( M \) be \( M_\rho \) and defining \( \phi : E \to 2^{E'} \) by \( \phi(i) = X_i \).

The next lemma simplifies proving that a matroid is the natural matroid of \( \rho \). Recall that two elements \( a, b \) of a matroid \( M \) on \( E \) are clones if the permutation of \( E \) given by the 2-cycle \((a, b)\) (i.e., switching \( a \) and \( b \)) is an automorphism of \( M \). We say that \( X \subseteq E \) is a set of clones if \( a, b \in X \) are clones whenever \( a \neq b \). A cyclic set of \( M \) is a set \( X \) that is a union of circuits, that is, \( M|X \) has no coloops. A cyclic flat is a flat that is cyclic. It is easy to prove that, for the set \( Z_M \) of cyclic flats of \( M \), we have, for \( Y \subseteq E \),
\[
(2.2) \quad r_M(Y) = \min\{r_M(Z) + |Y - Z| : Z \in Z_M\}.
\]

Lemma 2.3. Let \( \rho, E, E', X_i, \) and \( X_A \) be as above. A matroid \( M \) on \( E' \) is the natural matroid \( M_\rho \) of \( \rho \) if and only if \( Z_M \subseteq \{X_A : A \subseteq E \} \) and \( r_M(X_A) = \rho(A) \) whenever \( X_A \in Z_M \).

Proof. Above we showed that \( r_{M_\rho}(X_A) = \rho(A) \) for all \( A \subseteq E \). Also, \( X_i \) is a set of clones of \( M_\rho \), so if \( C \) is a circuit of \( M_\rho \) and \( a \in C \cap X_i \), then \((C - a) \cup b \), for each \( b \in X_i - C \), is a circuit of \( M_\rho \) and so \( X_i \subseteq \cl_M(C) \). Thus, \( Z_{M_\rho} \subseteq \{X_A : A \subseteq E \} \).

To prove the converse, assume that \( Z_M \subseteq \{X_A : A \subseteq E \} \) and \( r_M(X_A) = \rho(A) \) whenever \( X_A \in Z_M \). By construction, the rank function of \( M_\rho \) is given by
\[
r_{M_\rho}(Y) = \min\{\rho(A) + |Y - X_A| : A \subseteq E \}
= \min\{r_M(X_A) + |Y - X_A| : A \subseteq E \}
\]
for all \( Y \subseteq E' \). Now
- if \( Y, W \subseteq E' \), then \( r_M(Y) \leq r_M(W) + |Y - W| \),
for each $Y$, some cyclic flat $W$ yields equality in that inequality, and

- $Z_M \subseteq \{X_A : A \subseteq E\}$.

Thus, Equation (2.2) gives

$$r_M(Y) = \min\{r_M(X_A) + |Y - X_A| : A \subseteq E\}.$$ 

Thus, $M$ and $M_{\rho}$ have the same rank function and so are equal, as claimed. \hfill $\Box$

The containment $Z_{M_{\rho}} \subseteq \{X_A : A \subseteq E\}$ is proper since each set $X_i$ is independent.

Two elements are clones in $M$ if and only if they are in exactly the same cyclic flats of $M$, so we get the following corollary.

**Corollary 2.4.** Let $\rho, E, E', X_i$, and $X_A$ be as above. A matroid $M$ on $E'$ is $M_{\rho}$ if and only if each set $X_i$ is a set of clones and $r_M(X_A) = \rho(A)$ for all $X_A \in Z_{M_{\rho}}$.

With Corollary 2.4, it follows that the natural matroid defined above is the same as that obtained by the construction of iterated principal extensions followed by deletion that is given in the proof of [21, Theorem 11.1.9], and which justifies the geometric view of the natural matroid that is mentioned after Theorem 1.1.

It follows easily from Corollary 2.4, or from the rank functions, that the operations of deletion and taking the natural matroid commute: if $i \in E$, then $M_{\rho_{i'} \setminus i} = M_{\rho\setminus i}$. The same is not true of contraction. For $i \in E$ and each $j \in E - i$, fix a subset $Y_j$ of any $\rho(\{i, j\}) - \rho(i)$ elements of $X_i$, and let $E'_j$ be the union of all such sets $Y_j$. It follows from Corollary 2.4 that $M_{\rho_{i}} = M_{\rho}/X_i[E'_j]$. From Corollary 2.4, we also get $M_{\rho_{i} \oplus \rho_{j}} = M_{\rho_{i}} \oplus M_{\rho_{j}}$ for integer polymatroids $\rho_{i}$ and $\rho_{j}$; so an integer polymatroid $\rho$ on $E$ with $|E| > 1$ is connected if and only if $\rho$ has no loops and $M_{\rho}$ is connected. The number of elements in the natural matroid is the sum of all terms $\rho(i)$ for $i \in E$, so, for a positive integer $k$, the natural matroid of the $k$-dual of an integer $k$-polymatroid $\rho$ can have fewer, the same number of, or more elements compared to the natural matroid of $\rho$.

**Theorem 2.1**, which we used to construct the natural matroid, is the key to defining an important matroid operation, namely, matroid union (see [21, 28]). Let $M_1, M_2, \ldots, M_k$ be matroids on $E$. Their **matroid union**, denoted $M_1 \lor M_2 \lor \cdots \lor M_k$, is the matroid on $E$ having the rank function $r'$ where, for $Y \subseteq E$,

$$r'(Y) = \min\{r_{M_1}(X) + r_{M_2}(X) + \cdots + r_{M_k}(X) + |Y - X| : X \subseteq Y\}.$$ 

The independent sets of $M_1 \lor M_2 \lor \cdots \lor M_k$ are the sets of the form $I_1 \cup I_2 \cup \cdots \cup I_k$ where $I_j$ is independent in $M_j$. The matroids $M_1, M_2, \ldots, M_k$ also give an integer polymatroid on $E$: the function $\rho$ on $2^E$ where, for $X \subseteq E$,

$$\rho(X) = r_{M_1}(X) + r_{M_2}(X) + \cdots + r_{M_k}(X),$$ 

is an integer $k$-polymatroid on $E$. We write this as $\rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k}$ for brevity. We call the multiset $\{M_1, M_2, \ldots, M_k\}$ a decomposition of $\rho$ and we say that $\rho$ is **decomposable**. Not all integer polymatroids are decomposable. (See [3] for more on this topic.) The next theorem identifies the natural matroid of a decomposable integer polymatroid as a particular matroid union.

**Theorem 2.5.** Let $\{M_1, M_2, \ldots, M_k\}$ be a decomposition of an integer polymatroid $\rho$ on $E$. Let the sets $E'$, $X_i$, and $X_A$ be as above. For each $j \in [k]$, construct $M'_j$ from $M_j$ by, for each $i \in E$, adding the elements of $X_i$ parallel to $i$, or as loops if $r_{M_j}(i) = 0$, and then deleting $i$. Then the natural matroid $M_{\rho}$ is the matroid union $M'_1 \lor M'_2 \lor \cdots \lor M'_k$. 

Proof. For \( X, Y \subseteq E' \), note that \( r_{M'_j}(Y \cap X) \leq r_{M'_j}(X) \) for each \( j \in [k] \), and that \( |Y - (Y \cap X)| = |Y - X| \). Given how \( M'_j \) is defined, if \( X_A \) is the union of all sets \( X_i \) such that \( X_i \cap X \neq \emptyset \), then \( r_{M'_j}(X_A) = r_{M'_j}(X) \), for each \( j \in [k] \); also, \( |Y - X_A| \leq |Y - X| \).

It follows that the rank \( r'(Y) \) of \( Y \) in \( M'_1 \cup M'_2 \cup \cdots \cup M'_k \) is given by

\[
r'(Y) = \min \{ r_{M'_j}(X_A) + r_{M'_j}(X_A) + \cdots + r_{M'_j}(X_A) + |Y - X_A| : A \subseteq E \}.
\]

Since \( \rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k} \), we get \( r'(Y) = \min \{ \rho(A) + |Y - X_A| : A \subseteq E \} \), that is, \( r'(Y) = r_\rho(Y) \). Thus, \( M_\rho \) is \( M'_1 \cup M'_2 \cup \cdots \cup M'_k \).

An integer polymatroid and its natural matroid may have very different connections to important classes of matroids. For instance, for the binary integer polymatroid on the set of seven lines of the projective plane \( P_{G}(2, 2) \) using the construction in Theorem 1.1, the natural matroid is \( U_{3,14} \), which is not binary. (The integer 2-polymatroids having natural matroids that are binary are characterized in [8].) In contrast, the next example and result give links between transversal, or Boolean, polymatroids and transversal matroids.

Example 1. If \( \rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k} \) where \( r(M_h) \leq 1 \) for all \( h \in [k] \), then \( \rho \) is called a Boolean polymatroid. Helgason [14] introduced Boolean polymatroids, calling them covering hypermatroids. Some authors call them transversal polymatroids [9, 16, 27]. The class of Boolean polymatroids is closed under minors; Matuš [19] found their excluded minors. By Theorem 2.5 and the result that a matroid is transversal if and only if it is a matroid union of rank-1 matroids (see, e.g., [21, Proposition 11.3.7]), it follows that the natural matroid of a Boolean polymatroid is transversal.

Another way to see this is via graphs. A Boolean polymatroid \( \rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k} \) has the following reformulation using a bipartite graph \( G_\rho \). Assume that \( E \cap [k] = \emptyset \). The vertex set of \( G_\rho \) is \( E \cup [k] \), and \( G_\rho \) has an edge \( eh \) if and only if \( r_M(e) = 1 \). The rank \( \rho(A) \) of a set \( A \subseteq E \) is the cardinality of the set \( \mathcal{N}(A) \) of neighbors of \( A \). The natural matroid is the transversal matroid that is obtained from \( G_\rho \) by replacing each element \( e \in E \) by \( \rho(e) \) elements, each of which is adjacent to all neighbors of \( e \). (See Figure 1.)

Loopless Boolean 2-polymatroids have received much attention, in part due to another connection with graphs. Given such a polymatroid \( \rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k} \), the graph \( G \) has an edge \( e \in E \) incident with a vertex \( h \in [k] \) if and only if \( r_{M_h}(e) = 1 \). Then \( \rho(A) \), for \( A \subseteq E \), is the number of vertices that are incident with at least one edge in \( A \). The natural matroid of \( \rho \) is the bicircular matroid of the graph \( G' \) that is obtained from \( G \) by putting a new edge parallel to each nonloop edge of \( G \).

Using \( G_\rho \), we see that an integer polymatroid \((E, \rho)\) is Boolean if and only if, for some \( k \), there is a map \( \mathcal{N} : E \rightarrow 2^{[k]} \) with \( \rho(X) = |\bigcup_{x \in X} \mathcal{N}(x)| \) for all \( X \subseteq E \). (This is Helgason’s definition in [14].) Given a set of subsets of \([k]\), there is an isomorphism from the lattice of all unions of those sets onto the lattice of cyclic flats of a transversal matroid so that the size of each union is the rank of its image. This gives the following variant of Theorem 1.1.

Theorem 2.6. A polymatroid \( \rho \) on \( E \) is Boolean if and only if there is a transversal matroid \( M \) and map \( \phi : E \rightarrow Z_M \) with \( \rho(X) = r_M(\bigcup_{i \in X} \phi(i)) \) for all \( X \subseteq E \).

Most transversal matroids, such as \( U_{2,3} \), are not Boolean polymatroids, so the codomain of the map \( \phi : E \rightarrow Z_M \) cannot be extended to the lattice of flats of \( M \).
3. Bases of an Integer Polymatroid and Its Natural Matroid

Independent vectors and bases of integer polymatroids are discussed, for instance, by Herzog and Hibi in [15]. In this section, where we focus solely on integer polymatroids, we show how relating the bases of an integer polymatroid to the bases of its natural matroid makes transparent some characterizations of integer polymatroids that use bases.

A basis $B$ of a matroid $M$ on $E = [n]$ is a subset of $[n]$ and so can be represented by its characteristic vector $b$, the $n$-tuple of 0s and 1s in which entry $i$, denoted $b_i$, is 1 if and only if $i \in B$. No basis contains a loop, so $b_i \leq r(i)$. Let $e_i$ be the characteristic vector of the singleton $\{i\}$. For the characteristic vector $b$ of a basis $B$ and a basis $B' = (B - i) \cup j$ obtained by an exchange, the characteristic vector of $B'$ is $b - e_i + e_j$.

The norm of $v \in \mathbb{N}^n$ is $|v| = v_1 + v_2 + \cdots + v_n$. For $u$ and $v$ in $\mathbb{N}^n$, we write $u \leq v$ if $u_i \leq v_i$ for all $i \in [n]$; also, $u < v$ if $u \leq v$ and $u \neq v$. With this order, $\mathbb{N}^n$ is a lattice; meet and join are given by component-wise min and max, respectively.

A definition of an integer polymatroid that is equivalent to the definition in Section 1 is that an integer polymatroid $P$ is a nonempty finite subset $I$ of $\mathbb{N}^n$, for some $n$, for which

(I) if $v \in I$ and $u \in \mathbb{N}^n$ with $u \leq v$, then $u \in I$, and

(II) if $u, v \in I$ with $|u| < |v|$, then there is a $w$ in $I$ with $u < w \leq u \vee v$.

(To extend this to all polymatroids, replace $\mathbb{N}$ by $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ and require $I$ to be compact, rather than finite.) The vectors in $I$ are the independent vectors of $P$. A basis of $P$ is a vector $v \in I$ for which there is no $u \in I$ with $v < u$. Property (II) gives $|v| = |u|$ for all bases $v$ and $u$ of $P$.

We now relate this notion to the definition given in Section 1. Let $E = [n]$. For $v \in \mathbb{N}^n$ and $X \subseteq E$, let $|v|_X$ be $\sum_{i \in X} v_i$, the sum of the entries in $v$ that are indexed by the elements in $X$. The rank function $\rho : 2^E \to \mathbb{N}$ of an integer polymatroid $P$ on $E$ whose set of independent vectors is $I$ and whose set of bases is $B$ is given by

\[
\rho(X) = \max\{|u|_X : u \in I\} = \max\{|u|_X : u \in B\}
\]
for \( X \subseteq E \). The function \( \rho \) satisfies properties (1)-(3) in Section 1 and so is the rank function of an integer polymatroid. Conversely, given an integer polymatroid \( \rho : 2^E \to \mathbb{N} \), the set

\[
I = \{ u \in \mathbb{N}^n : |u|_X \leq \rho(X) \text{ for all } X \subseteq E \}
\]

satisfies properties (11) and (12). (For a proof, see [28, p. 340, Lemma 5].) Also, the maps \( I \mapsto \rho \) and \( \rho \mapsto I \) are inverses of each other. (See [23, Corollaries 44.3f and 44.3g].)

Given an integer polymatroid \( \rho \) on \( E = [n] \), let \( E', X_i, \) and \( X_A, \) for \( i \in E \) and \( A \subseteq E \), and the natural matroid \( M_\rho \) be defined as above. The type vector of a subset \( V \) of \( E' \) is the vector \( v \in \mathbb{N}^n \) with \( v_i = |V \cap X_i| \) for all \( i \in E \). We use \( T(V) \) to denote the type vector of \( V \). By Corollary 2.2 and Equation (3.2), a subset \( V \) of \( E' \) is independent in \( M_\rho \) if and only if \( T(V) \) is an independent vector of \( \rho \), and so \( V \) is a basis of \( M_\rho \) if and only if \( T(V) \) is a basis of \( \rho \).

**Example 2.** Consider a Boolean polymatroid \( \rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k} \) on \( [n] \) where each \( M_h \) has rank 1 and its rank-1 elements are consecutive integers \( a_h, a_h + 1, \ldots, b_h \), with \( a_1 \leq a_2 \leq \cdots \leq a_k \) and \( b_1 \leq b_2 \leq \cdots \leq b_k \). The matroids \( M_1, M_2, \ldots, M_k \) correspond to the rows in a lattice path diagram, where north steps are labeled by their first coordinate, the lower left corner is \((1, 0)\), and the upper right corner is \((n, k)\). (See Figure 2.) Bases correspond to lattice paths: entry \( u_i \) in a basis \( u \) is the number of north steps in the corresponding path that are labeled \( i \). An elementary argument (as in the proof of [5, Theorem 3.3]) shows that the correspondence between bases and lattice paths is bijective. Schweig [24, 25] introduced these **lattice path polymatroids**. The description of the natural matroid of a Boolean polymatroid in Example 1 along with the ideas in [6, Section 6.1] show that the natural matroid of a lattice path polymatroid is a lattice path matroid (see [5] for these matroids). Like the class of lattice path matroids, that of lattice path polymatroids is closed under minors; the excluded minors for lattice path polymatroids are found in [4]. Unlike the class of lattice path matroids, that of lattice path polymatroids is not closed under duality. Also, most lattice path matroids are not lattice path polymatroids.

The following characterizations of integer polymatroids by bases are known (see, e.g., [15]). We provide a transparent way to see this and similar results using the natural matroid.

**Theorem 3.1.** A nonempty set \( B \subseteq \mathbb{N}^n \) is the set of bases of an integer polymatroid on \( E = [n] \) if and only if either of the following equivalent conditions holds:

1. \( \textbf{(B)} \) if \( u, v \in B \) with \( u_i > v_i \) for some \( i \in [n] \), then there is a \( j \in [n] \) for which \( u_j < v_j \) and \( u - e_i + e_j \) is in \( B \),
2. \( \textbf{(B')} \) if \( u, v \in B \) with \( u_i > v_i \) for some \( i \in [n] \), then there is a \( j \in [n] \) for which \( u_j < v_j \) and both \( u - e_i + e_j \) and \( v - e_j + e_i \) are in \( B \).
that the equivalence of the following statements shows that
\[ E = \{ u \mid u \leq v, v \in V \} \] for the middle basis property.

\[ \text{axiom schemes for matroids that use bases or independent sets. We cite just one example, } \]
\[ \text{if } M \text{ is the set of bases of an integer polymatroid. For each } i, \]
\[ \rho \text{ is the set of clones, we may assume that } M_i \subseteq M \text{ and } \rho \text{ has the same type vector as } U. \]
\[ \text{Thus, } (V - U) \cap X_j \neq \emptyset. \]
\[ \forall y \in (V - U) \cap X_j, \text{ the set } W = (U - x) \cup y \text{ has type vector } w, \]
\[ \text{so } W \in B. \]
\[ \text{Thus, } B \text{ is the set of bases of a matroid } \]
\[ \text{on } E'. \]
\[ \text{Define } \rho : 2^E \to \mathbb{N} \text{ by } \rho(A) = r_M(X_A). \]
\[ \text{Thus, } \rho \text{ is an integer polymatroid on } E. \]
\[ \text{Also, } X_j \text{ is a set of clones in } M. \]
\[ \text{It now follows from the definition of } \rho \text{ and Corollary 2.4 that } M \text{ is the natural matroid of } \rho. \]
\[ \text{From the definition of } M \text{ and the comments before Example 2, we have that } B \text{ is the set of bases of } \rho, \text{ as needed.} \]

The strategy we used above adapts to prove integer-polymatroid counterparts of other axiom schemes for matroids that use bases or independent sets. We cite just one example, for the middle basis property.

**Theorem 3.2.** A nonempty set \( B \subseteq \mathbb{N}^n \) is the set of bases of an integer polymatroid on \( E = \{ u \} \) if and only if the following two conditions hold:

- if \( u, v \in B \) with \( u \neq v \), then \( u \neq v \) and \( v \neq u \), and
- whenever \( x, y \in \mathbb{N}^n \) with \( x \leq y \) and there are \( u, v \in B \) with \( x \leq u \) and \( v \leq y \), then there is some \( w \in B \) with \( x \leq w \leq y \).

For a positive integer \( k \), certain properties of the \( k \)-dual \( \rho^* \) of an integer \( k \)-polymatroid are significant. For instance, generalizing a result of Kung [17] for matroids, Whittle [29] showed that the map \( \rho \mapsto \rho^* \) is the only involution on the class of integer \( k \)-polymatroids that switches deletion and contraction, i.e., \( \rho_{ij} = (\rho^*)_{ji} \) and \( (\rho_{ij})^* = (\rho^*)_{ji} \) for all \( i \in E \). The set of bases of the dual of a matroid \( M \) on \( E \) is given by \( \{ E - B : B \in B \} \) where \( B \) is the set of bases of \( M \); the next result generalizes this to the \( k \)-dual of an integer \( k \)-polymatroid.

**Theorem 3.3.** Let \( k \) be a positive integer and let \( \rho \) be an integer \( k \)-polymatroid on \( E = \{ u \} \), with \( B \) its set of bases. The set \( B^* \) of bases of the \( k \)-dual \( \rho^* \) is \( \{ u^* : u \in B \} \) where
\[ u^* = (k, k, \ldots, k) - u. \]

**Proof.** Note that \( |u| = \rho(E) \) if and only if \( |u^*| = k|E| - \rho(E) = \rho^*(E) \). With this, the equivalence of the following statements shows that \( u \in B \) if and only if \( u^* \in B^* : \)

- \( |u|_A \leq \rho(A) \) for all \( A \subseteq E \),
- \( |u|_{E-A} \leq \rho(E-A) \) for all \( A \subseteq E \),
- \( \rho(E) - |u|_A \leq \rho(E-A) \) for all \( A \subseteq E \),
- \( k|A| - |u|_A \leq k|A| - \rho(E) + \rho(E-A) \) for all \( A \subseteq E \),
- \( |u^*|_A \leq \rho^*(A) \) for all \( A \subseteq E \).
1

3

2

Figure 3. The circuits of this rank-3 integer polymatroid are (2, 0, 2), (2, 1, 1), and (0, 1, 2).

4. CIRCUITS OF AN INTEGER POLYMATROID AND ITS NATURAL MATROID

We next develop a theory of circuits for integer polymatroids that is analogous to that for bases in Section 3. Just as the bases of an integer polymatroid \( \rho \) are the type vectors of the bases of the natural matroid \( M_\rho \), so the circuits of \( \rho \) are the type vectors of the circuits of \( M_\rho \), with one exception: loops of \( \rho \) map to the empty set in \( M_\rho \). This is addressed by the ambient set \( U \) that we consider below. Another issue that we must address so that the circuits determine the integer polymatroid is that we need the rank of each element \( i \) for which \( X_i \) is a set of coloops of \( M_\rho \); the set \( U \) also takes care of this. (In matroids, such elements have rank one, but in integer polymatroids, the rank could be any positive integer.) Recall that \([n]_0 = \{0, 1, \ldots, n\}\).

The circuits of an integer polymatroid \( \rho \) on \( E = [n] \) are the vectors \( u \) in the set

\[
U = [\rho(1)]_0 \times [\rho(2)]_0 \times \cdots \times [\rho(n)]_0
\]

that are not independent and each vector \( w \) with \( w < u \) is independent. Thus, from the set \( C \) of circuits of \( \rho \), the set of independent vectors of \( \rho \) is

\[
I = \{ u \in U : \text{there is no } c \in C \text{ with } c \leq u \}.
\]

By the remarks before Example 2, a vector \( u \in U \) is a circuit of \( \rho \) if and only if some (equivalently, every) set \( C \) in the natural matroid \( M_\rho \) with \( u = T(C) \) is a circuit of \( M_\rho \). See Figure 3 for an example. The set \( C \) is an antichain in \( \mathbb{N}^n \). Recall that all antichains in \( \mathbb{N}^n \) are finite.

While \( U \) gives the rank of each element, the next lemma shows how, from \( C \) alone, to get the rank of any element \( i \) for which there is a \( c \in C \) with \( c_i > 0 \).

**Lemma 4.1.** Let \( \rho \) be an integer polymatroid on \( E = [n] \). For \( i \in E \) with \( \rho(i) > 0 \), the set \( X_i \) is a subset of a circuit of the natural matroid \( M_\rho \) if and only if \( \rho(E) < \rho(E - i) + \rho(i) \). In this case, \( \rho(i) = \max \{ c_i : c \in C \} \).

**Proof.** First assume that \( \rho(E) < \rho(E - i) + \rho(i) \). Fix \( a \in X_i \). Since \( X_i \) is independent in \( M_\rho \), so is \( X_i - a \). Extend \( X_i - a \) to a basis \( B \) of \( M_\rho \setminus a \), which, by the assumed inequality, is a basis of \( M_\rho \). Let \( C \) be the fundamental circuit of \( a \) with respect to the basis \( B \) of \( M_\rho \). Then \( X_i \subseteq C \), for if \( b \in X_i - C \), the subset \( (C - a) \cup b \) of the basis \( B \) would be a circuit of \( M_\rho \) since \( a \) and \( b \) are clones, but that is a contradiction.

To prove the contrapositive of the converse, assume that \( \rho(E) = \rho(E - i) + \rho(i) \). Thus, \( r_{M_\rho}(E') = r_{M_\rho}(E' - X_i) + r_{M_\rho}(X_i) \), from which we get \( M_\rho = (M_\rho \setminus X_i) \oplus (M_\rho | X_i) \). With this direct sum decomposition and the fact that \( X_i \) is independent in \( M_\rho \), it follows that \( X_i \) is disjoint from all circuits of \( M_\rho \). \( \square \)

The following result will be used in the next section.

**Lemma 4.2.** Let \( \rho \) be an integer polymatroid on \( E = [n] \). For \( A \subseteq E \) and \( i \in A \), we have \( \rho(A) < \rho(A - i) + \rho(i) \) if and only if there is a circuit \( u \in C \) with \( u_i > 0 \) and \( u_j = 0 \) for all \( j \in E - A \).
Proof. Given \( u \in C \) with \( u_i > 0 \) and \( u_j = 0 \) for all \( j \neq i \), for each \( a \in X_i \), there is a circuit \( C \) of \( M_a \) with \( u = T(C) \) and \( a \in C \). Thus, \( a \) is not a loop of \( M/X_A \), so \( \rho(A) > \rho(A-i) + \rho(i) \). The converse follows by applying Lemma 4.1 to \( \rho_{E-A} \).

Lemma 4.1 and the remarks before it lead to the following setting for characterizations of integer polymatroids via circuits, as in Theorem 4.3: the set of circuits is a subset of a set of the form \( U = [m_1]_0 \times [m_2]_0 \times \cdots \times [m_n]_0 \) where each \( m_i \) is a nonnegative integer. Recording \( m_i \) is the only way we get \( \rho(i) \) when all elements of \( X_i \) (if there are any) are loops of \( M_p \). Two circuits in the natural matroid may have different type vectors or the same type vector; therefore there are two circuit elimination properties, (C3) and (C4).

**Theorem 4.3.** Let \( m_1, m_2, \ldots, m_n \) be nonnegative integers and let \( C \) be a subset of \([m_1]_0 \times [m_2]_0 \times \cdots \times [m_n]_0 \) where, for each \( i \in [n] \), if \( u_i > 0 \) for some \( u \in C \), then \( m_i = \max\{u_i : u \in C\} \). The set \( C \) is the set of circuits of an integer polymatroid on \( E = [n] \) if and only if \( C \) satisfies properties (C1)–(C4):

(C1) each vector in \( C \) has at least two positive entries,

(C2) if \( u, v \in C \) with \( u \neq v \), then \( u \not< v \) and \( v \not< u \),

(C3) if \( u, v \in C \) with \( u \neq v \) and \( u_i, v_i > 0 \), then there is a \( z \in C \) so that \( z < u \cup v \) and \( z_i < \max(u_i, v_i) \), and

(C4) if \( u \in C \), and \( i \in E \) with \( 0 < u_i < m_i \), and \( j \in E - i \) with \( 0 < u_j \), then there is an \( a \in C \) with \( v_i = u_i + 1 \), with \( v_i \leq u_i \) for all \( h \neq i \), and with \( v_j < u_j \).

Thus, an integer polymatroid on \( [n] \) is a pair \((U, C)\) where

(i) \( U = [m_1]_0 \times [m_2]_0 \times \cdots \times [m_n]_0 \) for some \( m_1, m_2, \ldots, m_n \) in \( \mathbb{N} \),

(ii) \( C \subseteq U \) and \( C \) satisfies properties (C1)–(C4), and

(iii) if \( i \in E \) and \( u_i > 0 \) for some \( u \in C \), then \( m_i = \max\{u_i : u \in C\} \).

If \( \rho \) is the rank function of the integer polymatroid given by the pair \((U, C)\), then \( \rho(i) = m_i \) for all \( i \in E \).

Proof. Let \( C \) be the set of circuits of an integer polymatroid \( \rho \) on \( E \). By Lemma 4.1, if \( u \in C \) and \( u_i > 0 \), then \( m_i = \rho(i) \). Property (C1) holds since each set \( X_i \) is independent in \( M_p \). If \( u, v \in C \) and \( u < v \), then a circuit \( V \) of \( M_p \) with \( v = T(V) \) and a subset \( U \) of \( V \) with \( u = T(U) \) would be comparable circuits of \( M_p \); this contradiction proves property (C2).

For property (C3), take \( u, v \in C \) with \( u \neq v \) and \( u_i, v_i > 0 \). Let \( U \) and \( V \) be circuits of \( M_p \) with \( u = T(U) \) and \( v = T(V) \). Each set \( X_j \) is a set of clones in \( M_p \), so we may assume that if \( u_j \leq v_j \), then \( U \cap X_j \subseteq V \cap X_j \), and if \( v_j \leq u_j \), then \( V \cap X_j \subseteq U \cap X_j \). Thus, \( U \cap V \cap X_i \neq \emptyset \). For any \( a \in U \cap V \cap X_i \), circuit elimination applied to \( U \) and \( V \) gives a circuit \( C \) of \( M_p \) with \( C \subseteq (U \cup V) - a \). The inclusions that we have assumed give \( |C \cap X_j| \leq \max(u_j, v_j) \) for all \( j \in E \), and \( |C \cap X_i| < \max(u_i, v_i) \), as needed.

For property (C4), consider \( u \in C \) and \( i \in E \) with \( 0 < u_i < \rho(i) \). Let \( C' \) be a circuit of \( M_p \) with \( C' = u \). Fix a \( a \in C \cap X_j \) and \( b \in X_i - C \), and set \( C'' = (C - a) \cup b \), which is a circuit of \( M_p \) since \( a \) and \( b \) are clones. For \( j \in E - i \) with \( u_j > 0 \), fix \( c \in C \cap X_j \). By circuit elimination, \( M_p \) has a circuit \( D \subseteq (C \cup c) - c = (C \cup b) - c \). Property (C2) applied to \( u \) and \( T(D) \) forces \( (C \cup b) \cap X_i \subseteq D \). Thus, \( |D \cap X_i| = u_i + 1 \) and \( |D \cap X_i| \leq u_i \) for all \( h \neq i \), and the inequality is strict for \( h = j \), so property (C4) holds.

For the converse, assume that the pair \((U, C)\) satisfies properties (i)–(iii). For each \( i \in E \), let \( X_i \) be a set of size \( m_i \) with \( X_i \cap X_j = \emptyset \) whenever \( i \neq j \). We use \( X_A \) and \( E' \) as above. Let \( C = \{C : C \subseteq E' \text{ and } T(C) \in C\} \). Now \( \emptyset \notin C \) by property (C1). For two sets \( C \) and \( C' \) in \( C \), either (i) \( T(C) = T(C') \), so for at least one \( i \in E \), the subsets...
that an integer polymatroid 

Thus, $\mathcal{C}$ is the set of circuits of a matroid $M$ on $E'$. As in the proof of Theorem 3.1, from $M$, we get an integer polymatroid $\rho$ whose natural matroid is $M$. Since $\mathcal{C}$ is the set of circuits of $\rho$, this completes the proof. \hfill $\square$

The set $\mathcal{C} = \{(4, 1), (2, 2)\}$ satisfies all properties except (C4), so property (C4) does not follow from properties (C1)–(C3).

Let $k$ be a positive integer. Let $\mathbf{H}$ be the set of the type vectors of the hyperplanes of the natural matroid of an integer $k$-polymatroid. Let $\mathcal{C}^* = \{(k, k, \ldots, k) - \mathbf{u} : \mathbf{u} \in \mathbf{H}\}$. In contrast to Theorem 3.3, $\mathcal{C}^*$ might not be the set of circuits of an integer polymatroid. For instance, for the integer 2-polymatroid $\rho$ in Figure 3, we have

$$\mathcal{C}^* = \{(2, 1, 0), (0, 2, 2), (1, 1, 2), (1, 2, 1)\},$$

and properties (C3) and (C4) fail.

For an integer polymatroid $\rho$ on $E = [n]$ and any $i \in E$, since $\rho_{ji}(j) = \rho\{i, j\} - \rho(i)$ for all $j \in E - i$, the circuits of the contraction $\rho_{ji}$ are contained in the Cartesian product

$$\mathbf{U}_{ji} = \prod_{j \in E - i} [\rho\{i, j\} - \rho(i)]\mathbf{a}.$$

Let $\mathcal{C}'$ be the set of circuits of $\rho$ with the $i$th entry deleted from each vector. Since the circuits of a contraction $M/Y'$ of a matroid $M$ are the minimal nonempty sets of the form $C - Y$ as $C$ ranges over the circuits of $M$, and the natural matroid $M_{\rho_{ji}}/E_{ji}'$ of $\rho_{ji}$ is $M_{\rho}/X_{ji}E_{ji}'$ (in the notation used after Corollary 2.4), it follows that the circuits of $\rho_{ji}$ are the minimal vectors in $\mathcal{C}' \cap \mathbf{U}_{ji}$ that have at least two positive entries.

We noted in Section 2 that an integer polymatroid $\rho$ on $E$ with $|E| > 1$ is connected if and only if $\rho$ has no loops and $M_{\rho}$ is connected. Thus, an integer polymatroid $\rho$ on $[n]$ is connected if and only if for each pair of distinct integers $i, j \in [n]$, there is a circuit $\mathbf{u}$ of $\rho$ with $u_i > 0$ and $u_j > 0$.

5. FLATS, CYCLIC SETS, AND CYCLIC FLATS IN POLYMATROIDS

While some results in this section apply only to integer polymatroids, many apply to all polymatroids. To describe what we do in this section, we first need some definitions. Flats in a polymatroid $\rho$ on $E$ are defined as in matroids: a subset $A$ of $E$ is a flat of $\rho$ if $\rho(A \cup i) > \rho(A)$ for all $i \in E - A$. Let $\mathcal{F}_\rho$ denote the set of flats of $\rho$. Unless we are focusing only on matroids, $\mathcal{F}_\rho$ does not determine $\rho$ since, for instance, $\rho$ and $c\rho$, for any positive real $c$, have the same flats.

There are various equivalent ways to say that a set $X$ in a matroid $M$ is cyclic, including:

(i) $X$ is a union of circuits;
(ii) $M|X$ has no coloops;
(iii) $r(X) < r(X - y) + r(y)$ for each $y \in X$ that is not a loop.

As in [10], we adapt condition (iii) to define cyclic sets in a polymatroid $\rho$ on $E$: a subset $A$ of $E$ is cyclic if $\rho(A) < \rho(A - i) + \rho(i)$ for all $i \in A$ with $\rho(i) > 0$. We let $\mathcal{Y}_\rho$ denote the set of all cyclic sets of $\rho$.

Of greatest interest are the cyclic flats, that is, the flats that are cyclic. The set of cyclic flats of $\rho$ is denoted $\mathcal{Z}_\rho$, or $Z_M$ for a matroid $M$. As in the case of matroids, $Z_\rho$ is a lattice under inclusion. (See the comment after Lemma 5.14.) The next result, from [26, 7], characterizes matroids in terms of their cyclic flats and the ranks of those sets.

**Theorem 5.1.** For a pair $(Z, r)$, where $Z \subseteq 2^E$ and $r : Z \to \mathbb{N}$, there is a matroid $M$ for which $Z = Z_M$ and $r(Z) = r_M(Z)$ for all $Z \in Z$ if and only if

(Z0) ordered by inclusion, $Z$ is a lattice,
(Z1) $r(\emptyset_Z) = 0$, where $\emptyset_Z$ is the least element of $Z$,
(Z2) $0 < r(B) - r(A) < |B - A|$ for all sets $A, B$ in $Z$ with $A \subseteq B$, and
(Z3) $r(A \vee B) + r(A \wedge B) + |(A \cap B) - (A \wedge B)| \leq r(A) + r(B)$ for all $A, B$ in $Z$.

Csirmaz [10] extended this theorem. His result, stated next, characterizes polymatroids using cyclic flats and the value of the rank function on each of those flats as well as on each singleton set. The rank of each element must be given since, while in a matroid each element that is not in the least cyclic flat (the set of loops) has rank 1, in a polymatroid, such an element may have any positive rank.

**Theorem 5.2.** For a pair $(Z', \rho')$, where $Z \subseteq 2^E$ and $\rho' : Z \cup E \to \mathbb{R}_{\geq 0}$, there is a polymatroid $\rho$ on $E$ with $Z = Z_\rho$ and $\rho(x) = \rho'(x)$ for all $x \in Z \cup E$ if and only if

(PZ0) ordered by inclusion, $Z$ is a lattice,
(PZ1) the least element of $Z$, denoted $\emptyset_Z$, is $\{ i \in E : \rho'(i) = 0 \}$, and $\rho'(\emptyset_Z) = 0$,
(PZ2) for all sets $A, B$ in $Z$ with $A \subseteq B$,

$$0 < \rho'(B) - \rho'(A) < \sum_{i \in B - A} \rho'(i),$$

(PZ3) for all sets $A, B$ in $Z$,

$$\rho'(A \vee B) + \rho'(A \wedge B) + \sum_{i \in (A \wedge B) - (A \wedge B)} \rho'(i) \leq \rho'(A) + \rho'(B),$$

and

(PZ4) if $A \in Z$ and $i \in A$, then $\rho'(i) \leq \rho'(A)$.

We will show how, in the case of an integer polymatroid $\rho$, Theorem 5.2 follows from Theorem 5.1; we do this by relating the flats of $\rho$ to those of its natural matroid, and likewise for cyclic sets and for cyclic flats. The proof of Theorem 5.2 in [10] has the same general outline as the proof of Theorem 5.1 that Sims [26] gave. In particular, for the more involved implication, assuming that the properties above hold for $(Z, \rho')$, one defines a function $\rho : 2^E \to \mathbb{R}$, checks that the defining properties of a polymatroid hold, and shows that its cyclic flats are precisely the sets in $\mathcal{Z}$, and that $\rho$ and $\rho'$ have the same values on the sets in $\mathcal{Z}$ and the elements of $E$. The function $\rho$ is defined by

$$\rho(A) = \min\{ \rho'(X) + \sum_{i \in A - X} \rho'(i) : X \in \mathcal{Z} \}.$$
This makes it natural to consider, for a polymatroid $\rho$ on $E$ and subset $A$ of $E$, the set

\begin{equation}
\mathcal{R}_\rho(A) = \{ B \in \mathcal{Z}_\rho : \rho(A) = \rho(B) + \sum_{i \in A - B} \rho(i) \}.
\end{equation}

For a matroid $M$, we write this set as $\mathcal{R}_M(A)$. Our main new result is Theorem 5.17, where we show that $\mathcal{R}_\rho(A)$ is a sublattice of $\mathcal{Z}_\rho$, we identify its least and greatest elements, and we show that each pair of elements in $\mathcal{R}_\rho(A)$ is a modular pair. To prepare for that, we develop basic results about flats and cyclic sets, and two operators related to them. (While some of these results may be known, we include proofs for completeness.)

We start with flats. The flats of an integer polymatroid are related to those of its natural matroid in the simplest possible way, as the next lemma states.

**Lemma 5.3.** For an integer polymatroid $\rho$ on $E$, let $\hat{0}_\rho$ be $\{ i \in E : \rho(i) = 0 \}$. A subset $A$ of $E$ is a flat of $\rho$ if and only if $\hat{0}_\rho \subseteq A$ and $X_A$ is a flat of the natural matroid $M_\rho$.

**Proof.** Assume that $\hat{0}_\rho \subseteq A$ and that $X_A$ is a flat of $M_\rho$. If $i \in E - A$, then there are elements $b \in X_i$, and $r_{M_\rho}(X_{A \cup i}) \supseteq r_{M_\rho}(X_A \cup b) > r_{M_\rho}(X_A)$, so $\rho(A \cup i) > \rho(A)$, as needed. We now prove the contrapositive of the converse. If $\hat{0}_\rho \not\subseteq A$, then clearly $A$ is not a flat of $\rho$. Assume that $X_A$ is not a flat of $M_\rho$, so $r_{M_\rho}(X_A \cup b) = r_{M_\rho}(X_A)$ for some $b \in E' - X_A$; say $b \in X_i$. Then $r_{M_\rho}(X_A \cup c)$ for all $c \in X_i$ since $X_i$ is a set of clones. From this, repeatedly applying submodularity gives $r_{M_\rho}(X_{A \cup i}) = r_{M_\rho}(X_A)$, so $\rho(A \cup i) = \rho(A)$, so $A$ is not a flat of $\rho$. \( \square \)

**Corollary 5.4.** The set $\mathcal{F}_\rho$ of flats of an integer polymatroid $\rho$, ordered by inclusion, is isomorphic to a sublattice of the lattice $\mathcal{F}_{M_\rho}$. The meet of two flats of $\rho$ is their intersection.

By [7, Theorem 2.1], every finite lattice is isomorphic to the lattice of cyclic flats of a matroid. With that and the construction in Theorem 1.1, it follows that, in contrast to matroids, every finite lattice is isomorphic to the lattice of flats of an integer polymatroid.

**Lemma 5.5.** For a polymatroid $\rho$ on $E$, the intersection of two flats is a flat, so, ordered by inclusion, $\mathcal{F}_\rho$ is a lattice.

**Proof.** Fix $A, B \in \mathcal{F}_\rho$ and $e \in E - (A \cap B)$; say $e \not\in A$. From submodularity and these assumptions, $\rho((A \cap B) \cup e) - \rho(A \cap B) \geq \rho(A \cup e) - \rho(A) > 0$, as needed. \( \square \)

This lemma justifies extending the definition of the closure operator from matroids to polymatroids. The closure operator $\text{cl}_\rho : 2^E \to 2^E$ of a polymatroid $\rho$ on $E$ is given by

\begin{equation}
\text{cl}_\rho(A) = \bigcap \{ F : F \in \mathcal{F}_\rho \text{ and } A \subseteq F \}.
\end{equation}

for $A \subseteq E$; equivalently, $\text{cl}_\rho(A)$ is the minimum flat (with respect to inclusion) that is a superset of $A$. Several results follow immediately: $\text{cl}_\rho(A) \in \mathcal{F}_\rho$, by Lemma 5.5, the image of $\text{cl}_\rho$ is $\mathcal{F}_\rho$, and $\text{cl}_\rho$ is a closure operator in the general sense, that is, (i) $A \subseteq \text{cl}_\rho(A)$ for all $A \subseteq E$, (ii) if $A \subseteq B \subseteq E$, then $\text{cl}_\rho(A) \subseteq \text{cl}_\rho(B)$, and (iii) $\text{cl}_\rho(\text{cl}_\rho(A)) = \text{cl}_\rho(A)$ for all $A \subseteq E$. The MacLane-Steinitz exchange property of matroid closure operators fails for most polymatroids; for instance, in the integer polymatroid $\rho$ in Figure 1, (c), we have $e_2 \in \text{cl}_\rho(e_3) - \text{cl}_\rho(\emptyset)$ but $e_3 \not\in \text{cl}_\rho(e_2)$.

**Lemma 5.6.** Let $\rho$ be a polymatroid on $E$. If $A \subseteq E$, then $\text{cl}_\rho(A) = \{ i : \rho(A \cup i) = \rho(A) \}$ and $\rho(A) = \rho(\text{cl}_\rho(A))$. 
Proof. Let $X = \{ i : \rho(A \cup i) = \rho(A) \}$. Now $A \subseteq X$. Repeated use of submodularity gives $\rho(A) = \rho(X)$. If $i \in E - X$, then $\rho(X \cup i) \geq \rho(A \cup i) > \rho(A) = \rho(X)$, so $X$ is a flat. Let $F$ be a flat with $A \subseteq F$. Now $X \subseteq F$ since, for any $i \in X$, from $\rho(A \cup i) = \rho(A)$ we get $\rho(F \cup i) = \rho(F)$ by submodularity. Thus, $\cl_{\rho}(A) = X$. □

We next give properties of the closure operator that are special to integer polymatroids.

Lemma 5.7. For an integer polymatroid $\rho$ on $E$, let $\hat{0}_\rho$ be $\{ i : \rho(i) = 0 \}$. For $A \subseteq E$,

1. $\cl_{\rho}(A) = B$ if and only if $\hat{0}_\rho \subseteq B$ and $\cl_{M_\rho}(X_A) = X_B$, and
2. $\cl_{\rho}(A) = A \cup \hat{0}_\rho \cup C_A$ where $C_A$ is the set of all $i \in E$ for which some circuit $C$ of $\rho$ has $u_i = 1$ and $u_j = 0$ for all $j \in E - (A \cup i)$.

Proof. Each set $X_i$ is a set of clones of $M_\rho$, so $\cl_{M_\rho}(X_A)$ is the smallest flat $X_B$ that contains $X_A$. Part (1) follows from this observation and Lemma 5.3. For part (2), clearly $A \cup \hat{0}_\rho \subseteq \cl_{\rho}(A)$. Fix $i \in C_A$ and a circuit $u$ with $u_i = 1$ and $u_j = 0$ for all $j \in E - (A \cup i)$. The circuits $C$ of $M_\rho$ with $\mathbf{T}(C) = u$ show that $X_i \subseteq \cl_{M_\rho}(X_A)$; thus, $i \in \cl_{\rho}(A)$, and so $A \cup \hat{0}_\rho \cup C_A \subseteq \cl_{\rho}(A)$. For the other inclusion, fix a basis $D$ of $M_\rho|X_A$, so $D$ is also a basis of $M_\rho|\cl_{M_\rho}(X_A)$. If $i \in \cl_{\rho}(A) - (A \cup \hat{0}_\rho)$ and $a \in X_i$, then the type vector of the fundamental circuit of $a$ with respect to $D$ shows that $i \in C_A$. □

We now turn to cyclic sets. We first focus on integer polymatroids.

Lemma 5.8. Let $\rho$ be an integer polymatroid on $E$. For $A \subseteq E$, statements (1)–(3) are equivalent:

1. $A$ is a cyclic set of $\rho$,
2. $X_A$ is a cyclic set of $M_\rho$,
3. for each $i \in A$, either $\rho(i) = 0$ or there is a circuit $u$ of $\rho$ for which $u_i > 0$ and $u_j = 0$ for all $j \in E - A$.

Proof. Assume that statement (1) holds. For any $a \in X_A$, there is an $i \in A$ with $\rho(i) > 0$ and $a \in X_i$. If $a$ were a coloop of $M_\rho|X_A$, then all elements of $X_i$ would be coloops of $M_\rho|X_A$, contrary to having $\rho(A) < \rho(A - i) + \rho(i)$. Thus, statement (2) holds.

Assume that statement (2) holds. Fix $i \in A$ with $\rho(i) > 0$. No $a \in X_i$ is a coloop of $M_\rho|X_A$, so some circuit $C$ of $M_\rho$ has $a \in C \subseteq X_A$. Statement (3) then follows.

By Lemma 4.2, statement (3) implies statement (1). □

We can expand the list of equivalent conditions for $X$ being a cyclic set of a matroid $M$ (items (i)–(iii) in the second paragraph of this section):

1. $X$ is a union of cocircuits of the dual $M^*$,
2. $E - X$ is an intersection of hyperplanes of $M^*$, and
3. $E - X$ is a flat of $M^*$.

The flats of $M^*$, ordered by inclusion, form a geometric lattice, so the cyclic sets of $M$, ordered by inclusion, form a lattice, the order-quotient of which is geometric. Thus, we have the following corollary of Lemma 5.8.

Corollary 5.9. For an integer polymatroid $\rho$ on $E$, its set $\mathcal{Y}_\rho$ of cyclic sets, ordered by inclusion, is a lattice. The join of two cyclic sets is their union.

If $\rho$ is not a matroid, then the order dual of $\mathcal{Y}_\rho$ need not be a geometric lattice, as one can see from the 2-polymatroid counterpart of the Vámos matroid shown in Figure 4, where the only sets not in $\mathcal{Y}_\rho$ are the singleton sets and $\{ a, d \}$. 
Lemma 1 of [10] shows that every flat of a polymatroid contains a maximum cyclic flat. By the next result and the discussion below it, a similar statement holds for all sets, and it comes from a property that generalizes Corollary 5.9.

**Lemma 5.10.** Let $\rho$ be a polymatroid on $E$.

1. If $X, Y \in \mathcal{Y}_\rho$, then $X \cup Y \in \mathcal{Y}_\rho$. Thus, under inclusion, $\mathcal{Y}_\rho$ is a lattice.
2. If $X \in \mathcal{Y}_\rho$, then $\text{cl}_\rho(X) \in \mathcal{Y}_\rho$, and so $\text{cl}_\rho(X) \in \mathcal{Z}_\rho$.

**Proof.** To prove part (1), fix $X, Y \in \mathcal{Y}_\rho$ and $i \in X \cup Y$ with $\rho(i) > 0$; say $i \in X$. By the assumptions and submodularity, $\rho(X \cup Y) - \rho((X \cup Y) - i) \leq \rho(X) - \rho(X - i) < \rho(i)$, so $X \cup Y \in \mathcal{Y}_\rho$. For part (2), take $X \in \mathcal{Y}_\rho$ and $i \in \text{cl}_\rho(X)$ with $\rho(i) > 0$. Then, as needed, $\rho(\text{cl}_\rho(X)) - \rho(\text{cl}_\rho(X) - i) < \rho(i)$ since the left side is 0 if $i \not\in X$ (since $\rho(X) = \rho(X \cup i)$), and at most $\rho(X) - \rho(X - i)$ if $i \in X$ (by submodularity). \[\Box\]

This lemma justifies making the following definition. For a polymatroid $\rho$ on $E$, its cyclic operator $\text{cy}_\rho : 2^E \rightarrow 2^E$ is given by, for $A \subseteq E$,

$$\text{cy}_\rho(A) = \bigcup \{ D : D \in \mathcal{Y}_\rho \text{ and } D \subseteq A \}.$$ 

Thus, $\text{cy}_\rho(A)$ is the maximum cyclic subset of $A$. If $A \in \mathcal{F}_\rho$, then $\text{cy}_\rho(A) \in \mathcal{F}_\rho$, since $\text{cy}_\rho(A) \subseteq \text{cl}_\rho(\text{cy}_\rho(A)) \subseteq A$ and $\text{cl}_\rho(\text{cy}_\rho(A))$ is cyclic (by part (2) of Lemma 5.10) and so must be $\text{cy}_\rho(A)$. For a matroid $M$, the cyclic set $\text{cy}_M(A)$ is the union of the circuits that are subsets of $A$. The operator $\text{cy}_M$ plays roles in recent papers, such as [12]. Note that the image of $\text{cy}_\rho$ is precisely $\mathcal{Y}_\rho$. Also, (i) if $A \subseteq E$, then $\text{cy}_\rho(A) \subseteq A$, (ii) if $A \subseteq B \subseteq E$, then $\text{cy}_\rho(A) \subseteq \text{cy}_\rho(B)$, and (iii) if $A \subseteq E$, then $\text{cy}_\rho(\text{cy}_\rho(A)) = \text{cy}_\rho(A)$.

**Lemma 5.11.** Let $\rho$ be an integer polymatroid on $E$. For any set $A \subseteq E$, the set $\text{cy}_\rho(A)$ is the union of all subsets of $A$ of either of the following forms:

1. $\{ i \}$ with $\rho(i) = 0$, or
2. $S(u) = \{ i : u_i \neq 0 \}$ where $u$ is a circuit of $\rho$ and $u_j = 0$ for all $j \in E - A$.

Also, $\text{cy}_\rho(A) = B$ if and only if $B$ is the maximum subset of $A$ with $\text{cy}_{M_B}(X_A) = X_B$.

**Proof.** The first assertion follows from Lemma 5.8 and the definition of $\text{cy}_\rho$. That and the connection between the circuits of $\rho$ and those of $M_\rho$ give the second assertion. \[\Box\]

We state the next lemma, which is basic and well known, so that we can cite it.

**Lemma 5.12.** Let $\rho$ be a polymatroid on $E$. Assume that $A \subseteq E$, that $i \in A$, and that $\rho(A) = \rho(A - i) + \rho(i)$. If $Y \subseteq A$ and $i \in Y$, then $\rho(Y) = \rho(Y - i) + \rho(i)$.

**Proof.** By submodularity, $\rho(i) \geq \rho(Y) - \rho(Y - i) \geq \rho(A) - \rho(A - i) = \rho(i)$. \[\Box\]
The next lemma identifies the elements in $A - \text{cy}_p(A)$ as the counterparts of coloops in the deletion $\rho_{E-A}$.

**Lemma 5.13.** Let $\rho$ be a polymatroid on $E$. For any set $A \subseteq E$, 
\begin{enumerate}
\item $\text{cy}_p(A) = A - \{i \in A : \rho(i) > 0 \text{ and } \rho(A) = \rho(A - i) + \rho(i)\}$, and
\item $\rho(A) = \rho(\text{cy}_p(A)) + \sum_{i \in A-\text{cy}_p(A)} \rho(i)$.
\end{enumerate}

**Proof.** Let $X = \{i \in A : \rho(i) > 0 \text{ and } \rho(A) = \rho(A - i) + \rho(i)\}$. By Lemma 5.12, no cyclic subset of $A$ contains any $i \in X$, so $\text{cy}_p(A) \subseteq A - X$. Part (1) will follow by showing that $A - X$ is cyclic. First note that repeatedly applying Lemma 5.12, adding one element at a time to go from $A - X$ to $A$, gives
\begin{equation}
\rho(A) = \rho(A - X) + \sum_{i \in X} \rho(i).
\end{equation}
If there were a $j \in A - X$ with $\rho(j) > 0$ and $\rho(A - X) = \rho((A - X) - j) + \rho(j)$, then this equality, Equation (5.3), and submodularity would give
\[\rho(A) = \rho((A - X) - j) + \sum_{i \in X} \rho(i) \geq \rho(A - j).\]
This inequality is contrary to having $j \notin X$, so $A - X$ is cyclic. Part (2) follows from part (1) and Equation (5.3). \hfill \Box

The next lemma is like part (2) of Lemma 5.10, but switches flats and cyclic sets.

**Lemma 5.14.** For a polymatroid $\rho$ on $E$, if $A \in F_\rho$, then $\text{cy}_p(A) \in F_\rho$, so $\text{cy}_p(A) \in Z_\rho$.

**Proof.** Fix $A \in F_\rho$ and $i \notin \text{cy}_p(A)$. We must show that $\rho(\text{cy}_p(A) \cup i) > \rho(\text{cy}_p(A))$. This holds by Lemmas 5.13 and 5.12 if $i \in A - \text{cy}_p(A)$. If $i \notin A$, then the assumption $A \in F_\rho$ and submodularity give $\rho(\text{cy}_p(A) \cup i) - \rho(\text{cy}_p(A)) \geq \rho(A \cup i) - \rho(A) > 0$. \hfill \Box

With Lemmas 5.10 and 5.14, we see that $Z_\rho$ is a lattice: for $A, B \in Z_\rho$, their meet is $A \wedge B = \text{cy}_p(A \cap B)$ and their join is $A \vee B = \text{cl}_p(A \cup B)$.

The next lemma, along with Lemma 5.13, is a basic tool for investigating the sets $R_\rho(A)$, which we defined in Equation (5.1).

**Lemma 5.15.** Let $\rho$ be a polymatroid on $E$. For any subsets $A$ and $B$ of $E$, the equality
\begin{equation}
\rho(A) = \rho(B) + \sum_{i \in A-B} \rho(i)
\end{equation}
holds if and only if
\begin{enumerate}
\item $\rho(A) = \rho(A - i) + \rho(i)$ for all $i \in A - B$, and
\item $\rho(A \cap B) = \rho(B)$ (equivalently, $\text{cl}_p(A \cap B) = \text{cl}_p(B)$).
\end{enumerate}

**Proof.** First assume that properties (1) and (2) hold. Applying Lemma 5.12 to add one element at a time going from $A \cap B$ to $A$ gives
\[\rho(A) = \rho(A \cap B) + \sum_{i \in A-B} \rho(i)\]
and replacing $\rho(A \cap B)$ by $\rho(B)$, as (2) justifies, yields Equation (5.4).

Now assume that Equation (5.4) holds. Repeated uses of submodularity give
\[\rho(A) \leq \rho(A \cap B) + \sum_{i \in A-B} \rho(i).\]
Also, \( \rho(A \cap B) \leq \rho(B) \). These inequalities and Equation (5.4) give \( \rho(A \cap B) = \rho(B) \), so property (2) holds. With this, for any \( i \in A - B \), we have

\[
\rho(A) = \rho(A \cap B) + \left( \sum_{j \in A - B, j \neq i} \rho(j) \right) + \rho(i) \geq \rho(A - i) + \rho(i) \geq \rho(A),
\]

from which we get \( \rho(A) = \rho(A - i) + \rho(i) \), so property (1) holds.

We now consider the operators \( \text{cl} \) and \( \text{cy} \) together. Note that if \( B \) is a basis of a matroid \( M \) that has neither loops nor coloops, then \( \text{cl}(\text{cy}(B)) = \emptyset \) but \( \text{cy}(\text{cl}(B)) = E(M) \); thus, \( \text{cl} \) and \( \text{cy} \) need not commute. Lemmas 5.3 and 5.8 give the following result.

**Corollary 5.16.** For an integer polymatroid \( \rho \) on \( E \), let \( \hat{0}_\rho \) be \( \{ i \in E : \rho(i) = 0 \} \). For \( A \subseteq E \), we have \( A \in \mathcal{Z}_\rho \) if and only if \( \hat{0}_\rho \subseteq A \) and \( X_A \in \mathcal{Z}_{M^\rho} \).

For an integer polymatroid \( \rho \), since all cyclic flats of \( M_\rho \) have the form \( X_A \) for some \( A \subseteq E \) and the map \( \phi : \mathcal{Z}_\rho \rightarrow \mathcal{Z}_{M_\rho} \) where \( \phi(A) = X_A \) is a bijection, properties that can be described via cyclic flats lift from matroids to integer polymatroids. With these ideas, the case of Theorem 5.2 for integer polymatroids follows from Theorem 5.1.

Not all properties of cyclic flats for matroids extend to polymatroids. For instance, for matroids, the cyclic flats of the dual \( M^* \) are the set complements of the cyclic flats of \( M \), so \( \mathcal{Z}_{M^*} \) is isomorphic to the order dual of \( \mathcal{Z}_M \). The same is not true for \( k \)-polymatroids and their \( k \)-duals, as one can check using the example in Figure 3 or 4.

To conclude, we use Lemmas 5.12, 5.13, and 5.15 to show that \( \mathcal{R}_\rho(A) \) is a sublattice of \( \mathcal{Z}_\rho \) (so the meet and join operations are the same as in \( \mathcal{Z}_\rho \)), identify the least and greatest elements of \( \mathcal{R}_\rho(A) \), and show that each pair \( (B, B') \) of cyclic flats in \( \mathcal{R}_\rho(A) \) is a modular pair of flats, that is, \( \rho(B) + \rho(B') = \rho(B \cup B') + \rho(B \cap B') \). (That equality can fail if only one of \( B \) or \( B' \) is in \( \mathcal{R}_\rho(A) \).)

**Theorem 5.17.** Let \( \rho \) be a polymatroid on \( E \). For any subset \( A \) of \( E \),

(1) \( \text{cl}_\rho(\text{cy}_\rho(A)) \) and \( \text{cy}_\rho(\text{cl}_\rho(A)) \) are in \( \mathcal{R}_\rho(A) \),

(2) if \( B \in \mathcal{R}_\rho(A) \), then \( \text{cl}_\rho(\text{cy}_\rho(A)) \subseteq B \subseteq \text{cy}_\rho(\text{cl}_\rho(A)) \),

(3) \( \mathcal{R}_\rho(A) \) is a sublattice of \( \mathcal{Z}_\rho \), and

(4) if \( B, B' \in \mathcal{R}_\rho(A) \), then \( (B, B') \) is a modular pair of flats.

**Proof.** When \( B \) is \( \text{cy}_\rho(A) \), property (1) in Lemma 5.15 holds by Lemma 5.13, as does property (2) since \( B \subseteq A \). Those properties then follow when \( B = \text{cl}_\rho(\text{cy}_\rho(A)) \) since \( \left( \text{cl}_\rho(\text{cy}_\rho(A)) \right) \cap A = \text{cy}_\rho(A) \), so \( \text{cl}_\rho(\text{cy}_\rho(A)) \in \mathcal{R}_\rho(A) \). Those properties clearly also hold when \( B = \text{cl}_\rho(A) \). From this, when \( B \) is \( \text{cy}_\rho(\text{cl}_\rho(A)) \), we get property (1) by Lemma 5.12, and property (2) by applying Lemma 5.12 as elements of \( A - B \) are removed from \( A \) and \( \text{cl}_\rho(A) \). Thus, \( \rho(\text{cl}_\rho(A)) \in \mathcal{R}_\rho(A) \), so part (I) holds.

Assume that \( B \in \mathcal{R}_\rho(A) \). Property (1) of Lemma 5.15 gives \( \text{cy}_\rho(A) \subseteq B \), so, since \( B \) is a flat, \( \text{cl}_\rho(\text{cy}_\rho(A)) \subseteq B \). Property (2) of Lemma 5.15 and the fact that \( B \) is a flat give \( B = \text{cl}_\rho(A \cap B) \subseteq \text{cl}_\rho(A) \), so, since \( B \) is cyclic, \( B \subseteq \text{cy}_\rho(\text{cl}_\rho(A)) \). Thus, part (II) holds.

For assertion (III), we start with an inequality that we will use below. Let \( A \) be any subset of \( E \) and let \( B \) and \( B' \) be in \( \mathcal{Z}_\rho \). We claim that

\[
\sum_{i \in (B \cup B') \setminus (B \cap B')} \rho(i) + \sum_{i \in A - B} \rho(i) + \sum_{i \in A - B'} \rho(i) \geq \sum_{i \in A - (B \cup B')} \rho(i) + \sum_{i \in A - (B \cap B')} \rho(i).
\]

This inequality holds since

- \( A - (B \cup B') \) is a subset of each of \( A - (B \cap B'), A - B, \) and \( A - B' \) (so terms \( \rho(i) \) coming from its elements appear twice on each side of the inequality), and
\( (A - (B \land B')) - (A - (B \lor B')) \subseteq ((B \land B') - (B \lor B')) \cup (A - B) \cup (A - B') \)

(since terms \( \rho(i) \) that appear once on the right side also appear on the left side).

Now assume that \( B, B' \in \mathcal{R}_\rho(A) \), so

\[
\rho(B) + \sum_{i \in A - B} \rho(i) = \rho(A) = \rho(B') + \sum_{i \in A - B'} \rho(i).
\]

Then, using submodularity as formulated in property (PZ3) of Theorem 5.2, along with the inequality above, we have

\[
2 \rho(A) = \rho(B) + \rho(B') + \sum_{i \in A - B} \rho(i) + \sum_{i \in A - B'} \rho(i) \geq \rho(B \lor B') + \rho(B \land B') + \sum_{i \in (B \land B') \setminus (B \lor B')} \rho(i) + \sum_{i \in A - (B \lor B')} \rho(i) + \sum_{i \in A - (B \land B')} \rho(i).
\]

Since

\[
\rho(B \lor B') + \sum_{i \in A - (B \lor B')} \rho(i) \geq \rho(A) \quad \text{and} \quad \rho(B \land B') + \sum_{i \in A - (B \land B')} \rho(i) \geq \rho(A),
\]

the inequality above forces these inequalities to be equalities, which proves assertion (III).

Moreover, all inequalities in the argument above must be equalities, so equality holds in (PZ3) for \( B \) and \( B' \). Now \( \rho(B \cup B') = \rho(\text{cl}_\rho(B \cup B')) = \rho(B \lor B') \) and

\[
\rho(B \cap B') = \rho(B \land B') + \sum_{i \in (B \land B') \setminus (B \lor B')} \rho(i)
\]

by part (I) since \( B \cap B' \in \mathcal{F}_\rho \) and \( B \land B' = \text{cy}_\rho(B \cap B') \in \mathcal{R}_\rho(B \cap B') \), so assertion (IV) follows. \( \square \)

While \( \mathcal{R}_\rho(A) \) is a sublattice of \( \mathcal{Z}_\rho \), it might not be an interval in \( \mathcal{Z}_\rho \), as taking \( A \) to be a basis of the Fano plane shows. The corollary below is immediate from property (II).

**Corollary 5.18.** If \( A \in \mathcal{Z}_\rho \), then \( \mathcal{R}_\rho(A) = \{A\} \).

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