ON HILBERT’S GRAVITATIONAL REPULSION  
( A HISTORICAL NOTE )

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Abstract. In the literature on general relativity no mention is made of a remarkable result contained in Hilbert’s memoir Die Grundlagen der Physik, according to which in particular instances and in particular regions the Einsteinian gravity exerts a repulsive action. We give here a concise illustration of this peculiar phenomenon.

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1. – Papers and treatises on general relativity do not mention an important and peculiar result by Hilbert (1917, 1924) [1], which concerns a repulsive gravitational effect. We have made various applications of it [2]. In the present Note, which has a historical character, we illustrate the essential of the Hilbertian argument, following faithfully the treatment of our Author (sect. 2), and adding (sects. 3, 4) some remarks of a physical nature that intend to clarify the merits of the choice of coordinate time $t = x^4$ as evolution parameter for particular geodesics of Schwarzschild manifold.

2. – Hilbert’s fundamental memoir Die Grundlagen der Physik [1] ends with a detailed treatment of various properties of Schwarzschild manifold created by a point mass $m$ at rest [3]. As it is well known [4], the general expression of the relevant $ds^2$ is, if we employ spherical polar coordinates $r, \vartheta, \varphi$:

$$ds^2 = \left[ \frac{f(r)}{f(r) - 2m} \right] (df(r))^2 + [f(r)]^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) - \frac{f(r) - 2m}{f(r)} dt^2,$$

(1)

where $f(r)$ is any regular function of $r$ (of course, under the condition that the $ds^2$ becomes pseudo-Euclidean at spatial infinity). Schwarzschild’s original $ds^2$ [3] can be obtained by eq. (1) putting $f(r) \equiv \left[ r^3 + (2m)^3 \right]^{1/3}$. Hilbert – and, independently, Droste [5] and Weyl [6] – found a $ds^2$ for which $f(r) \equiv r$, a form of solution that became the standard form, owing to its apparent simplicity. Schwarzschild’s field is maximally extended and diffeomorphic to the “exterior part” $r > 2m$ of the standard field. If we write, with Schwarzschild and Hilbert, $\alpha \equiv 2m$, the standard form is:
\[ (1') \quad ds^2 = \frac{r}{r - \alpha} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2) - \frac{r - \alpha}{r} \, dt^2 . \]

The geodesic lines of test-particles and light-rays in the field \( g_{jk}, (j, k = 1, 2, 3, 4), \) which characterizes the \( ds^2 \) of eq. \((1')\), are exhaustively investigated by Hilbert. Equations of motion and first integrals are mainly, but not exclusively, written assuming an affine parameter \( p \), in lieu of the proper time \( s \), as evolution parameter; this allows a unified treatment of particles and light-rays. For special purposes – and pour cause – he assigns to coordinate time \( t = x^4 \) the role of evolution parameter.

The existence of gravitational action of a repulsive nature reveals itself in a very simple way for the circular orbits. Hilbert’s result – which was rediscovered by Einstein (1939, [7]) in a different context – affirms that the value of the radial coordinate \( r \) of a circular trajectory of a particle must satisfy the following inequality:

\[ (2) \quad r > \frac{3\alpha}{2} ; \]

then, the passage from the affine parameter \( p \) to coordinate time \( t = x^4 \) as evolution parameter gives for the speed \( v(t) \) a formula which coincides with Newton’s (remember that \( c = G = 1 \)):

\[ (3) \quad v^2 = \left( r \frac{d\varphi}{dt} \right)^2 = \frac{\alpha}{2r} ; \]

by virtue of \((2)\), we have:

\[ (4) \quad v < \frac{1}{\sqrt{3}} . \]

For the circular geodesics of the light-rays, Hilbert found that they satisfy necessarily the following equations:

\[ (5) \quad r = \frac{3\alpha}{2} , \]

\[ (6) \quad v = \frac{1}{\sqrt{3}} . \]

The above formulae are a clear expression of the gravitational repulsion exerted by our gravitating mass \( m \).

Even more striking is the behaviour of the radial geodesics. The elimination of parameter \( p \) from the Lagrangean differential equation

\[ (7) \quad \frac{d}{dp} \left( \frac{2r}{r - \alpha} \frac{dr}{dp} \right) + \frac{\alpha}{(r - \alpha)^2} \left( \frac{dr}{dp} \right)^2 + \frac{\alpha}{r^2} \left( \frac{dt}{dp} \right)^2 = 0 , \]

by means of the first integral
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\[ \frac{r - \alpha \, dt}{r \frac{dp}{d\alpha}} = 1, \]
yields the following differential equation for \( r \) as a function of \( t = x^4 \):

\[ \frac{d^2r}{dt^2} - \frac{3\alpha}{2r(r - \alpha)} \left( \frac{dr}{dt} \right)^2 + \frac{\alpha(r - \alpha)}{2r^3} = 0. \]

Further, an analogous elimination of \( p \) in the first integral

\[ \frac{r}{r - \alpha} \left( \frac{dt}{dp} \right)^2 - \frac{r - \alpha}{r} \left( \frac{dt}{dp} \right)^2 = A \]
gives:

\[ \left( \frac{dr}{dt} \right)^2 = \left( \frac{r - \alpha}{r} \right)^2 + A \left( \frac{r - \alpha}{r} \right)^3, \]
where \( A \) is equal to zero for the light-rays and is a negative constant for the material particles. Eq. (9) tells us that the acceleration is negative or positive – i.e., the gravitational force is attractive or repulsive – in accord with the following inequalities for the absolute value of the velocity:

\[ \left| \frac{dr}{dt} \right| < \frac{1}{\sqrt{3}} \frac{r - \alpha}{r} \quad \text{(attraction)}, \]
or:

\[ \left| \frac{dr}{dt} \right| > \frac{1}{\sqrt{3}} \frac{r - \alpha}{r} \quad \text{(repulsion)}, \]
as it is not difficult to prove. For the light-rays, in particular, we have from eq. (10) with \( A = 0 \) – or from \( ds^2 = 0 \) – that

\[ \left| \frac{dr}{dt} \right| = \frac{r - \alpha}{r}, \]

\[ i.e., \ by \ virtue \ of \ (13), \ always \ repulsion; \ we \ see \ that \ light \ velocity \ increases \ from \ zero \ at \ r = \alpha \ \text{to} \ 1 \ \text{ar} \ r = \infty. \]

If both \( dr/dt \) and \( \alpha \) are small, eq. (9) gives the Newton equation

\[ \frac{d^2r}{dt^2} = -\frac{\alpha}{2r^2}. \]

Of course, it is physically very important that eqs. (9) and (11) have as a consequence that for \( r = \alpha \) both velocity and acceleration are equal to zero; no particle, no light-ray can overcome this “barrier”.

3. – We have seen that the existence of a repulsive gravitational action for the circular geodesics is certified by inequality (2), which has been derived
through equations having an affine parameter as evolution parameter. But for the radial geodesics Hilbert has evidenced the repulsive action by means of equations having \( t = x^4 \) as evolution parameter.

One could object that if we employ the proper time \( s \) for the geodesics of the particles, things stand otherwise. As a matter of fact, the 4-velocity \( \frac{dr}{ds} \) and the 4-acceleration \( \frac{d^2r}{ds^2} \) of a test-particle are, as it follows easily from Hilbert’s formalism by substituting \( p \) with \( s \):

\[
\left( \frac{dr}{ds} \right)^2 = E^2 - \frac{r - \alpha}{r}, \quad E = \text{a non zero constant},
\]

\[
\frac{d^2r}{ds^2} = -\frac{1}{2} \frac{\alpha}{r^2} - \text{(attraction)},
\]

from which:

\[
\left( \frac{dr}{ds} \right)^2 \bigg|_{r=\alpha} = E^2,
\]

\[
\frac{d^2r}{ds^2} \bigg|_{r=\alpha} = -\frac{1}{2\alpha},
\]

and it seems that the “barrier” \( r = \alpha \) can be overcome. Accordingly, it is necessary to make clear – for the present case – the respective merits and defects of coordinate time \( t = x^4 \) and of proper time \( s \).

4. – First of all, we remember that in general relativity – as it was explicitly emphasized by McVittie [8] – the interpretation of the four coordinates (mere “labels”) of the point-events depends on the physical situation contemplated by the investigation – exactly as it happens in the Lagrangean formulation of Newton’s mechanics in regard to space coordinates.

Now, the \( g_{44} \) of eq. (1) vanishes for \( r = \alpha \). “This means that a clock kept at this place would go at the rate zero.” [7]. This fact explains the results (16) and (17). Indeed, a clock aboard a test-particle goes gradually more and more slowly with its approaching to \( r = \alpha \).

On the contrary, the coordinate time \( t = x^4 \) of eq. (1) is a global time (“kosmische Zeit” [6]) of a static system: as it is well known [9], when \( g_{00} = 0, (\alpha = 1, 2, 3) \), all the clocks in all the spatial points \((x^1, x^2, x^3)\) can be synchronized. Accordingly, the rate of these clocks is physically more significant than the rate of the proper time of a test-particle, which suffers the influence of motion.

We think that these considerations give a physical justification of Hilbert’s use of \( t = x^4 \) as evolution parameter.

5. – A final remark. The original Schwarzschild’s form of solution [3], for which \( f(r) \equiv (r^3 + \alpha^3)^{1/3} \), \( (\alpha \equiv 2m) \), or the very simple Brillouin form
for which \( f(r) \equiv r + \alpha \), are maximally extended. They are a proof of the essential superfluity of Kruskal-Szekeres form of \( ds^2 \). Further, this interval has various defects, in particular:

i) It entails a coordinate transformation from the standard coordinates \((t, r, \vartheta, \varphi)\) to the new coordinates \((v, u, \vartheta, \varphi)\), “whose derivatives happen to be singular at \( r = \alpha \) in just the appropriate way for providing a transformed metric that is regular there.”

ii) It gives a time-dependent solution to a static problem –

iii) In a diagram \((u, v)\) the light-cones are “open”, as in special relativity.

This means the abolition of any gravitational action on light-rays: a consequence of i)!

(Quite generally, the existence of singularities in a coordinate transformation or/and in its derivatives can give origin to a misrepresentation, or/and to a partial suppression of the gravitational actions). –

In reality, Kruskal-Szekeres \( ds^2 \) has been mainly utilized for a justification of the (meaningless) role inversion of coordinate \( r \) and coordinate \( t \) in the “interior region” \( r < \alpha \) of Hilbert-Droste-Weyl potential \( g_{jk} \) of eq. \( [11] \).

Now, the mere existence of the Hilbertian gravitational repulsion is sufficient to exclude any physical meaning for this space region.

**APPENDIX**

\( \alpha \) Hilbert’s treatment of the geodesic lines of test-particles and light-rays can be immediately extended to the general form of \( ds^2 \) given by eq. \( [11] \).

For instance, eq. \( [2] \) becomes \((\alpha = 2m)\):

(A1) \[ f(r) > \frac{3}{2} \alpha \equiv 3m \quad ; \]

thus, if \( f(r) \equiv [r^3 + (2m)^3]^{1/3} \) (see \([3]\)), we obtain:

(A2) \[ r > 19^{1/3} m \approx 2.6684 m \quad ; \]

and if \( f(r) \equiv r + 2m \) (see \([10]\)):

(A3) \[ r > 2m \quad ; \]

for the isotropic coordinates we have \( f(r) \equiv (1 + \frac{m^2}{2r})^2 r \), from which (see \([7]\)):

(A4) \[ r > (2 + \sqrt{3}) \frac{m}{2} \approx 1.86603 m \quad , \]

if we employ with Fock \([13]\) harmonic coordinates, \( f(r) \equiv r + m \), from which:
It is physically interesting that inequality (A1) is stronger than inequality \( f(r) > 2m \).

Velocity \( \frac{df(r)}{dt} \) and acceleration \( \frac{d^2f(r)}{dt^2} \) of test-particles and light-rays which travel along radial geodesics are equal to zero when \( f(r) = 2m \).

\[ \beta \]

As it was emphasized by McVittie many years ago [14], only solutions to Einstein field equations concerning a single mass at rest are known. Therefore the Newtonian analogy is not sufficient for asserting that a punctual mass could be a component of a binary system, or that two punctual masses could collide. Indeed, an existence theorem would first be needed to show that Einstein equations contain solutions which describe such configurations. Accordingly, e.g., the interpretation of the observational data concerning the quasar SDSSJ153636.22+044127 put forward by Boroson and Lauer [15] cannot be correct.

\[ \text{References} \]

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