Cosmic microwave background anisotropies at second order: I

Nicola Bartolo\textsuperscript{1}, Sabino Matarrese\textsuperscript{2} and Antonio Riotto\textsuperscript{3}

\textsuperscript{1} The Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, 34100 Trieste, Italy
\textsuperscript{2} Dipartimento di Fisica “G Galilei”, Università di Padova, and INFN, Sezione di Padova, via Marzolo 8, Padova I-35131, Italy
\textsuperscript{3} CERN, Theory Division, CH-1211 Geneva 23, Switzerland
E-mail: nbartolo@ictp.trieste.it, sabino.matarrese@pd.infn.it and antonio.riotto@pd.infn.it

Received 26 April 2006
Accepted 24 May 2006
Published 28 June 2006

Online at stacks.iop.org/JCAP/2006/i=06/a=024
doi:10.1088/1475-7516/2006/06/024

Abstract. We present the computation of the full system of Boltzmann equations at second order describing the evolution of the photon, baryon and cold dark matter fluids. These equations allow one to follow the time evolution of the cosmic microwave background anisotropies at second order at all angular scales from the early epoch, when the cosmological perturbations were generated, to the present, through the recombination era. This paper sets the stage for the computation of the full second-order radiation transfer function at all scales and for a generic set of initial conditions specifying the level of primordial non-Gaussianity. In a companion paper, we will present the computation of the three-point correlation function at recombination which is very relevant for the issue of non-Gaussianity in the cosmic microwave background anisotropies.

Keywords: CMBR theory, cosmological perturbation theory, inflation
1. Introduction

Cosmological inflation [1] has become the dominant paradigm for understanding the initial conditions for the CMB anisotropies and structure formation. In the inflationary picture, the primordial cosmological perturbations are created from quantum fluctuations ‘redshifted’ out of the horizon during an early period of accelerated expansion of the universe, where they remain ‘frozen’. They are observable as temperature anisotropies in the CMB. This picture has recently received further spectacular confirmations from the Wilkinson Microwave Anisotropy Probe (WMAP) three-year set of data [2]. Since the observed cosmological perturbations are of the order of $10^{-5}$, one might think that first-order perturbation theory will be adequate for all comparisons with observations. This might not be the case, though. Present [2] and future experiments [3] may be sensitive
to the non-linearities of the cosmological perturbations at the level of second-order or higher order perturbation theory. The detection of these non-linearities through the non-Gaussianity (NG) in the CMB [4] has become one of the primary experimental targets.

There is one fundamental reason why a positive detection of NG is so relevant: it might help in discriminating among the various mechanisms for the generation of the cosmological perturbations. Indeed, various models of inflation, firmly rooted in modern particle physics theory, predict a significant amount of primordial NG generated either during or immediately after inflation when the comoving curvature perturbation becomes constant on super-horizon scales [4]. While single-field [5] and two-field (multi-field) [6] models of inflation predict a tiny level of NG, ‘curvaton-type models’, in which a significant contribution to the curvature perturbation is generated after the end of slow-roll inflation by the perturbation in a field which has a negligible effect on inflation, may predict a high level of NG [7]. Alternatives to the curvaton model are those models characterized by the curvature perturbation being generated by an inhomogeneity in the decay rate [8,9], the mass [10] or the interaction rate [11] of the particles responsible for the reheating after inflation. In that case the reheating can be the first one (caused by the scalar field(s) responsible for the energy density during inflation) or alternatively the particle species causing the reheating can be a fermion [12]. Other opportunities for generating the curvature perturbation occur at the end of inflation [13], during preheating [14] and at a phase transition producing cosmic strings [15].

Statistics like the bispectrum and the trispectrum of the CMB can then be used to assess the level of primordial NG on various cosmological scales and to distinguish it from the one induced by secondary anisotropies and systematic effects [4], [16]–[18]. A positive detection of a primordial NG in the CMB at some level might therefore confirm and/or rule out a whole class of mechanisms by which the cosmological perturbations have been generated.

Despite the importance of NG in the CMB anisotropies, little effort has been made so far to provide accurate theoretical predictions of it. On the contrary, the vast majority of the literature has been devoted to the computation of the bispectrum of either the comoving curvature perturbation or the gravitational potential on large scales within given inflationary models. These, however, are not the physical quantities which are observed. One should instead provide a full prediction for the second-order radiation transfer function. A first step towards this goal has been taken in [19] (see also [20]) where the full second-order radiation transfer function for the CMB anisotropies on large angular scales in a flat universe filled with matter and a cosmological constant was computed, including the second-order generalization of the Sachs–Wolfe effect, both the early and late integrated Sachs–Wolfe (ISW) effects and the contribution of the second-order tensor modes.

There are many sources of NG in the CMB anisotropies beyond the primordial one. The most relevant sources are the so-called secondary anisotropies, which arise after the last scattering epoch. These anisotropies can be divided into two categories: scattering secondaries, when the CMB photons scatter with electrons along the line of sight, and gravitational secondaries when effects are mediated by gravity [21]. Among the scattering secondaries we may list the thermal Sunyaev–Zeldovich effect, where hot electrons in clusters transfer energy to the CMB photons, the kinetic Sunyaev–Zeldovich effect produced by the bulk motion of the electrons in clusters, the Ostriker–Vishniac
effect, produced by bulk motions modulated by linear density perturbations, and effects due to reionization processes. The scattering secondaries are most significant on small angular scales as density inhomogeneities, bulk and thermal motions grow and become sizable on small length scales when structure formation proceeds.

Gravitational secondaries arise from the change in energy of photons when the gravitational potential is time dependent, the ISW effect and gravitational lensing. At late times, when the universe becomes dominated by the dark energy, the gravitational potential on linear scales starts to decay, causing the ISW effect mainly on large angular scales. Other secondaries that result from a time dependent potential are the Rees–Sciama effect, produced during the matter-dominated epoch at second order and by the time evolution of the potential on non-linear scales.

The fact that the potential never grows appreciably means that most second-order effects created by gravitational secondaries are generically small compared to those created by scattering ones. However, when a photon propagates from the last scattering to us, the path may be deflected because of the gravitational lensing. This effect does not create anisotropies; it only modifies existing ones. Since photons with large wavenumbers $k$ are lensed over many regions ($\sim k/H$, where $H$ is the Hubble rate) along the line of sight, the corresponding second-order effect may be sizable. The three-point function arising from the correlation of the gravitational lensing effect and the ISW effect generated by the matter distribution along the line of sight \cite{22,23} and the Sunyaev–Zeldovich effect \cite{24} are large and detectable by Planck \cite{25}.

Another relevant source of NG comes from the physics operating at the recombination. A naive estimate leads us to think that these non-linearities are tiny, being suppressed by an extra power of the gravitational potential. However, the dynamics at recombination is quite involved because all the non-linearities in the evolution of the baryon–photon fluid at recombination and the ones coming from general relativity should be accounted for. This complicated dynamics might lead to unexpected suppressions or enhancements of the NG at recombination. A step towards the evaluation of the three-point correlation function has been taken in \cite{26} where it was computed in the so-called squeezed triangle limit, when one mode has a wavelength much larger than the other two and is outside the horizon.

In this paper we present the computation of the full system of Boltzmann equations at second order describing the evolution of the photon, baryon and CDM fluids, neglecting polarization. These equations allow us to follow the time evolution of the CMB anisotropies at second order at all angular scales from the early epoch, when the cosmological perturbations were generated, to the present, through the recombination era. This paper therefore sets the stage for the computation of the full second-order radiation transfer function at all scales and for a generic set of initial conditions specifying the level of primordial non-Gaussianity. In a companion paper \cite{27}, we will present the computation of the three-point correlation function at recombination which is so relevant for the issue of NG in the CMB anisotropies. Of course on small angular scales, fully non-linear calculations of specific effects like the Sunyaev–Zel’dovich one and gravitational lensing would provide a more accurate estimate of the resulting CMB anisotropy; however, as far as the leading contribution to the second-order statistics like the bispectrum is concerned, second-order perturbation theory suffices.
The paper is organized as follows. In section 2 we provide the second-order metric and corresponding Einstein equations. In section 3 the left-hand side of the Boltzmann equation for the photon distribution function is derived at second order. The collision term is computed in section 4. In section 5, we present the second-order Boltzmann equation for the photon brightness function, its formal solution with the method of integration along the line of sight and the corresponding hierarchy equations for the multipole moments. Section 6 contains the derivation of the Boltzmann equations at second order for baryons and cold dark matter (CDM). Section 7 contains the expressions for the energy–momentum tensors and, finally, section 8 contains our summary.

In performing the computation presented in this paper, we have mainly followed [28] (in particular section 4) where an excellent derivation of the Boltzmann equations for the baryon–photon fluid at first order is given. Since the derivation at second order is straightforward, but lengthy, the reader might benefit from reading the appropriate sections of [28]. Table A.1 in the appendix summarizes the many symbols adopted that appear throughout the paper.

2. Perturbing gravity

Before tackling the problem of interest—the computation of the Boltzmann equations for the baryon–photon and CDM fluids—we first provide the necessary tools for dealing with perturbed gravity, giving the expressions for the Einstein tensor perturbed up to second order around a flat Friedmann–Robertson–Walker background, and the relevant Einstein equations. In the following we will adopt the Poisson gauge which eliminates one scalar degree of freedom from the \( g_{0i} \) component of the metric and one scalar and two vector degrees of freedom from \( g_{ij} \). We will use a metric of the form

\[
d s^2 = a^2(\eta) \left[ -e^{2\Phi} \, d\eta^2 + 2\omega_i \, dx^i \, d\eta + (e^{-2\Psi} \delta_{ij} + \chi_{ij}) \, dx^i \, dx^j \right],
\]

where \( a(\eta) \) is the scale factor as a function of the conformal time \( \eta \), and \( \omega_i \) and \( \chi_{ij} \) are the vector and tensor perturbation modes respectively. Each metric perturbation can be expanded into a linear (first-order) and a second-order part; for example, the gravitational potential \( \Phi = \Phi^{(1)} + \Phi^{(2)} / 2 \). However, in the metric (2.1) the choice of the exponentials greatly helps in computing the relevant expressions, and thus we will always keep them where it is convenient. From equation (2.1), one recovers at linear order the well-known longitudinal gauge, while at second order, one finds \( \Phi^{(2)} = \phi^{(2)} - 2(\phi^{(1)})^2 \) and \( \Psi^{(2)} = \psi^{(2)} + 2(\psi^{(1)})^2 \) where \( \phi^{(1)} \), \( \psi^{(1)} \) and \( \phi^{(2)} \), \( \psi^{(2)} \) (with \( \phi^{(1)} = \Phi^{(1)} \) and \( \psi^{(1)} = \Psi^{(1)} \)) are the first- and second-order gravitational potentials in the longitudinal (Poisson) gauge adopted in [4, 29] as far as scalar perturbations are concerned. For the vector and tensor perturbations, we will neglect linear vector modes since they are not produced in standard mechanisms for the generation of cosmological perturbations (as inflation), and we also neglect tensor modes at linear order, since they give a negligible contribution to second-order perturbations. Therefore we take \( \omega_i \) and \( \chi_{ij} \) to be second-order vector and tensor perturbations of the metric.

Let us now give our definitions for the connection coefficients and their expressions for the metric (2.1). The number of spatial dimensions is \( n = 3 \). Greek indices \( (\alpha, \beta, \ldots, \mu, \nu, \ldots) \) run from 0 to 3, while italic indices \( (a, b, \ldots, i, j, k, \ldots, m, n \ldots) \) run...
from 1 to 3. The total spacetime metric $g_{\mu\nu}$ has signature $(-, +, +, +)$. The connection coefficients are defined as

$$
\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\rho} \left( \frac{\partial g_{\rho\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\rho}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\rho} \right). \tag{2.2}
$$

The Riemann tensor is defined as

$$
R^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\lambda\mu} \Gamma^\lambda_{\beta\nu} - \Gamma^\alpha_{\lambda\nu} \Gamma^\lambda_{\beta\mu}. \tag{2.3}
$$

The Ricci tensor is a contraction of the Riemann tensor:

$$
R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}. \tag{2.4}
$$

and in terms of the connection coefficient it is given by

$$
R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\nu\alpha} + \Gamma^\alpha_{\lambda\mu} \Gamma^\lambda_{\nu\alpha} - \Gamma^\alpha_{\lambda\nu} \Gamma^\lambda_{\mu\alpha}. \tag{2.5}
$$

The Ricci scalar is given by contracting the Ricci tensor:

$$
R = R^\mu_{\mu}. \tag{2.6}
$$

The Einstein tensor is defined as

$$
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{2.7}
$$

### 2.1. The connection coefficients

For the connection coefficients we find

$$
\Gamma^0_{00} = \mathcal{H} + \Phi',
$$

$$
\Gamma^0_{0i} = \frac{\partial \Phi}{\partial x^i} + \mathcal{H} \omega_i,
$$

$$
\Gamma^i_{00} = \omega^i + \mathcal{H} \omega^i + e^{2\Psi+2\Phi} \frac{\partial \Phi}{\partial x^i},
$$

$$
\Gamma^0_{ij} = \frac{1}{2} \left( \frac{\partial \omega_j}{\partial x^i} + \frac{\partial \omega_i}{\partial x^j} \right) + e^{-2\Psi-2\Phi} (\mathcal{H} - \Psi') \delta_{ij} + \frac{1}{2} \chi'_{ij} + \mathcal{H} \chi_{ij}, \tag{2.8}
$$

$$
\Gamma^i_{0j} = (\mathcal{H} - \Psi') \delta_{ij} + \frac{1}{2} \chi'_{ij} + \frac{1}{2} \left( \frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right),
$$

$$
\Gamma^i_{jk} = -\mathcal{H} \omega^i \delta_{jk} - \frac{\partial \Psi}{\partial x^k} \delta^i_j - \frac{\partial \Psi}{\partial x^j} \delta^i_k + \frac{\partial \Psi}{\partial x^i} \delta_{jk} + \frac{1}{2} \left( \frac{\partial \chi^i_j}{\partial x^k} + \frac{\partial \chi^i_k}{\partial x^j} + \frac{\partial \chi_{jk}}{\partial x^i} \right).$$
2.2. Einstein equations

The Einstein equations are written as $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$, so $\kappa^2 = 8\pi G_N$, where $G_N$ is the usual Newtonian gravitational constant. They read

$$G^0_0 = -\frac{e^{-2\Phi}}{a^2} \left[ 3\mathcal{H}^2 - 6\mathcal{H}\Psi' + 3(\Psi')^2 - e^{2\Phi+2\Psi} \left( \partial_i \Psi \partial^i \Psi - 2\nabla^2 \Psi \right) \right] = \kappa^2 T^0_0,$$

$$G^i_0 = 2\frac{e^{2\Psi}}{a^2} \left[ \partial^i \Psi' + (\mathcal{H} - \Psi') \partial^i \Phi \right] - \frac{1}{2a^2} \nabla^2 \omega^i + \left( 4\mathcal{H}^2 - 2\frac{a''}{a} \right) \frac{\omega^i}{a^2} = \kappa^2 T^i_0,$$

$$G^i_j = 1\frac{a}{a^2} \left[ e^{-2\Phi} \left( \mathcal{H}^2 - 2\frac{a''}{a} - 2\Psi'\Phi' - 3(\Psi')^2 + 2\mathcal{H}(\Phi' + 2\Psi') + 2\Psi'' \right) \right]$$

$$+ e^{2\Phi} \left( \partial_i \Phi \partial^i \Phi + \nabla^2 \Phi - \nabla^2 \Psi \right) \delta^j_i + \frac{e^{2\Psi}}{a^2} (-\partial^i \Phi \partial_j \Phi - \partial^i \partial_j \Phi$$

$$\partial^i \partial_j \Psi - \partial^i \Phi \partial_j \Psi + \partial^i \Psi \partial_j \Phi - \partial^i \partial_j \Phi)$$

$$+ \frac{\mathcal{H}}{a^2} \left( \partial^i \omega_j + \partial_j \omega^i \right) - \frac{1}{2a^2} \left( \partial^i \omega_j + \partial_j \omega^i \right)$$

$$- \frac{1}{a^2} \left( \mathcal{H} \chi^i_j + \frac{1}{2} \chi^i_j - \frac{1}{2} \nabla^2 \chi^i_j \right) = \kappa^2 T^i_j. \quad (2.11)$$

Here $T_{\mu\nu}$ is the energy–momentum tensor accounting for different components, photons, baryons, dark matter. We will give the expressions later for each component in terms of the distribution functions.

3. The collisionless Boltzmann equation for photons

We are now interested in the anisotropies in the cosmic distribution of photons and inhomogeneities in the matter. Photons are affected by gravity and by Compton scattering with free electrons. The latter are tightly coupled to protons. Both are, of course, affected by gravity. The metric which determines the gravitational forces is influenced by all of these components plus CDM (and neutrinos). Our plan is to write down Boltzmann equations for the phase-space distributions of each species in the universe.

The phase-space distribution of particles $g(x^i, P^\mu, \eta)$ is a function of the spatial coordinates $x^i$, conformal time $\eta$ and momentum of the particle $P^\mu = dx^\mu/d\lambda$ where $\lambda$ parametrizes the particle path. Through the constraint $P^2 \equiv g_{\mu\nu} P^\mu P^\nu = -m^2$, where $m$ is the mass of the particle, one can eliminate $P^0$ and $g(x^i, P^j, \eta)$ gives the number of particles in a differential volume $dx^1 dx^2 dx^3 dP^1 dP^2 dP^3$ in phase space. In the following we will indicate the distribution function for photons with $f$.

The photon distribution evolves according to the Boltzmann equation

$$\frac{df}{d\eta} = a C[f], \quad (3.1)$$

where the collision term is due to scattering of photons off free electrons. In the following we will derive the left-hand side of equation (3.1) while in the next section we will compute the collision term.
For photons we can impose $P^2 \equiv g_{\mu\nu} P^\mu P^\nu = 0$ and using the metric (2.1) in the conformal time $\eta$ we find
\[ P^2 = a^2 \left[ -e^{2\Phi}(P^0)^2 + \frac{P^2}{a^2} + 2\omega_i P^0 P^i \right] = 0, \] (3.2)
where we define
\[ P^2 = g_{ij} P^i P^j. \] (3.3)
From the constraint (3.2)
\[ P^0 = e^{-\Phi} \left( \frac{P^2}{a^2} + 2\omega_i P^0 P^i \right)^{1/2}, \] (3.4)
Notice that we immediately recover the usual zero-and first-order relations $P^0 = p/a$ and $P^0 = p(1 - \Phi^{(1)})/a$.

The components $P^i$ are proportional to $p n^i$, where $n^i$ is a unit vector with $n^i n_i = \delta_{ij} n^i n^j = 1$. We can write $P^i = C n^i$, where $C$ is determined by
\[ g_{ij} P^i P^j = C^2 a^2 (e^{-2\Psi} + \chi_{km} n^k n^m) = p^2, \] (3.5)
and so
\[ P^i = \frac{p}{a} n^i \left( e^{-2\Psi} + \chi_{km} n^k n^m \right)^{-1/2} = \frac{p}{a} n^i e^\Psi \left( 1 - \frac{1}{2} \chi_{km} n^k n^m \right), \] (3.6)
where the last equality holds up to second order in the perturbations. Again we recover the zero-and first-order relations $P^i = pn^i/a$ and $P^i = pn^i(1 + \Psi^{(1)})/a$ respectively. Thus up to second order we can write
\[ P^0 = e^{-\Phi} \frac{p}{a} \left( 1 + \omega_i n^i \right). \] (3.7)

Equations (3.6) and (3.7) allow us to replace $P^0$ and $P^i$ by expressions in terms of the variables $p$ and $n^i$. Therefore, as is standard in the literature, from now on we will consider the phase-space distribution $f$ as a function of the momentum $p = pn^i$ with magnitude $p$ and angular direction $n^i$, $f \equiv f(x^i, p, n^i, \eta)$.

Thus in terms of these variables the total time derivative of the distribution function reads
\[ \frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial x^i} \frac{dx^i}{d\eta} + \frac{\partial f}{\partial p} \frac{dp}{d\eta} + \frac{\partial f}{\partial n^i} \frac{dn^i}{d\eta}. \] (3.8)

In the following we will compute $dx^i/d\eta$, $dp/d\eta$ and $dn^i/d\eta$:
(a) $dx^i/d\eta$:
From
\[ P^i = \frac{dx^i}{d\lambda} = \frac{dx^i}{d\eta} \frac{d\eta}{d\lambda} = \frac{dx^i}{d\eta} P^0 \] (3.9)
and from equations (3.6) and (3.7),
\[ \frac{dx^i}{d\eta} = n^i e^\Phi + \Psi \left( 1 - \omega_j n^j - \frac{1}{2} \chi_{km} n^k n^m \right). \] (3.10)
(b) \( dp/d\eta \):

For \( dp/d\eta \) we make use of the time component of the geodesic equation \( dP^\alpha/d\lambda = -\Gamma^\alpha_{\alpha\beta} P^\beta P^\beta \) where we can replace the derivative \( d/d\lambda \) with \( (d\eta/d\lambda) d/d\eta \), giving

\[
\frac{dP^0}{d\eta} = -\Gamma^0_{\alpha\beta} \frac{P^\alpha P^\beta}{P^0}.
\]  

(3.11)

Using the metric (2.1) we find

\[
2\Gamma^0_{\alpha\beta} P^\alpha P^\beta = g^{0\nu} \left[ 2 \frac{\partial g_{\nu\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right] P^\alpha P^\beta
\]

\[
= 2(\mathcal{H} + \Phi') \left( P^0 \right)^2 + 4\Phi_i P^0 P^i + 4\mathcal{H}\omega_i P^0 P^i
\]

\[
+ 2e^{-2\Phi} \left[ (\mathcal{H} - \Psi')e^{-\Phi} \delta_{ij} - \omega_{ij} + \frac{1}{2} \chi'_{ij} + \mathcal{H} \chi_{ij} \right] P^i P^j.
\]  

(3.12)

On the other hand, the expression (3.7) for \( P^0 \) in terms of \( p \) and \( n^i \) is

\[
\frac{dP^0}{d\eta} = -\frac{p}{a} \frac{dp}{d\eta} e^{-\Phi} \left( 1 + \omega_i n^i \right) + e^{-\Phi} \left( 1 + \omega_i n^i \right) \frac{d(p/a)}{d\eta} + \frac{p}{a} e^{-\Phi} \frac{d(\omega_i n^i)}{d\eta}.
\]  

(3.13)

Thus equation (3.11) allows us express \( dp/d\eta \) as

\[
\frac{1}{p} \frac{dp}{d\eta} = -\mathcal{H} + \Psi' - \Phi_i n^i e^{\Phi + \Psi} - \omega'_i n^i - \frac{1}{2} \chi'_{ij} n^i n^j,
\]

(3.14)

where in equation (3.12) we have replaced \( P^0 \) and \( P^i \) by equations (3.7) and (3.6).

Notice that in order to obtain equation (3.14) we have used the following expressions for the total time derivatives of the metric perturbations:

\[
\frac{d\Phi}{d\eta} = \frac{\partial \Phi}{\partial \eta} + \frac{\partial \Phi}{\partial x^i} \frac{dx^i}{d\eta} = \frac{\partial \Phi}{\partial \eta} + \frac{\partial \Phi}{\partial x^i} n^i e^{\Phi + \Psi} \left( 1 - \omega_j n^j - \frac{1}{2} \chi_{km} n^k n^m \right),
\]

(3.15)

and

\[
\frac{d(\omega_i n^i)}{d\eta} = n^i \left( \frac{\partial \omega_i}{\partial \eta} + \frac{\partial \omega_i}{\partial x^j} \frac{dx^j}{d\eta} \right) = \frac{\partial \omega_i}{\partial \eta} n^i + \frac{\partial \omega_i}{\partial x^j} n^j n^i,
\]

(3.16)

where we have taken into account that \( \omega_i \) is already a second-order perturbation so that we can neglect \( dn^i/d\eta \) which is at least a first-order quantity, and we can take the zero-order expression in equation (3.10), \( dx^i/d\eta = n^i \). In fact there is also an alternative expression for \( dp/d\eta \) which turns out to be useful later and which can be obtained by applying once more equation (3.15):

\[
\frac{1}{p} \frac{dp}{d\eta} = -\mathcal{H} - \frac{d\Phi}{d\eta} + \Phi' + \Psi' - \omega'_i n^i - \frac{1}{2} \chi'_{ij} n^i n^j.
\]  

(3.17)

(c) \( dn^i/d\eta \):

We can proceed in a similar way to compute \( dn^i/d\eta \). Notice that since in equation (3.8) it multiplies \( \partial f/\partial n^i \) which is first order, we need only the first-order perturbation of \( dn^i/d\eta \). We use the spatial components of the geodesic equations
\[
\frac{dP^i}{d\eta} = -\Gamma^i_{\alpha\beta} P^\alpha P^\beta \text{ written as }
\]
\[
\frac{dP^i}{d\eta} = -\Gamma^i_{\alpha\beta} \frac{P^\alpha P^\beta}{P^0}.
\]

For the right-hand side we find up to second order
\[
2\Gamma^i_{\alpha\beta} P^\alpha P^\beta = g^{i\nu} \left[ \frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right] P^\alpha P^\beta
\]
\[
= 4(H - \Psi') P^i P^0 + 2 \left( \chi'^i_k + \omega'^i_k - \omega'^i_k \right) P^0 P^k
\]
\[
+ \left( 2 \frac{\partial \Phi}{\partial x^i} e^{2\Phi + 2\Psi} + 2\omega'' + 2H\omega^i \right)(P^0)^2 - 4 \frac{\partial \Psi}{\partial x^k} P^i P^k
\]
\[
+ 2 \frac{\partial \Psi}{\partial x^i} \delta_{km} P^k P^m - 2H\omega^i \delta_{jk} \left( \frac{\partial \chi^i_j}{\partial x^k} + \frac{\partial \chi^i_k}{\partial x^j} + \frac{\partial \chi^j_k}{\partial x^i} \right) P^j P^k,
\]

while putting the expression (3.6) for \( P^i \) in terms of our variables \( p \) and \( n^i \) in the left-hand side of equation (3.18) yields
\[
\frac{dP^i}{d\eta} = \frac{p}{a} e^\Psi \left[ \frac{d\Psi}{d\eta} n^i + \frac{a}{p} \frac{d(p/a)}{d\eta} n^i + \frac{dn^i}{d\eta} \left( 1 - \frac{1}{2} \chi_{km} n^k n^m \right) - \frac{p}{a} n^i e^\Psi \frac{1}{2} \frac{d(\chi_{km} n^k n^m)}{d\eta} \right].
\]

Thus, using the expression (3.6) for \( P^i \) and (3.4) for \( P^0 \) in equation (3.19), together with the previous result (3.14), the geodesic equation (3.18) gives the following expression \( d\eta / d\eta \) (valid up to first order):
\[
\frac{dn^i}{d\eta} = (\Phi, k + \Psi, k) n^k n^i - \Phi^i - \Psi^i.
\]

To proceed further we now expand the distribution function for photons around the zero-order value \( f^{(0)} \) which is that of a Bose–Einstein distribution:
\[
f^{(0)}(p, \eta) = 2 \frac{1}{\exp \{p/T(\eta) \} - 1}.
\]

where \( T(\eta) \) is the average (zero-order) temperature and the factor 2 comes from the spin degrees of photons. The perturbed distribution of photons will depend also on \( x^i \) and on the propagation direction \( n^i \) so as to account for inhomogeneities and anisotropies:
\[
f(x^i, p, n^i, \eta) = f^{(0)}(p, \eta) + f^{(1)}(x^i, p, n^i, \eta) + \frac{1}{2} f^{(2)}(x^i, p, n^i, \eta),
\]

where we split the perturbation of the distribution function into first- and second-order parts. The Boltzmann equation up to second order can be written in a straightforward way by recalling that the total time derivative of a given \( i \) th perturbation, e.g. \( df^{(i)}/d\eta \), is at least a quantity of the \( i \) th order. Thus it is easy to see, looking at equation (3.8),
that the left-hand side of Boltzmann equation can be written up to second order as

$$\frac{df}{d\eta} = \frac{df^{(1)}}{d\eta} + \frac{1}{2} \frac{df^{(2)}}{d\eta} - p \frac{\partial f^{(0)}}{\partial p} \frac{d\Phi^{(1)}}{d\eta} + \frac{1}{2} \frac{\partial f^{(0)}}{\partial p} \frac{d\Phi^{(2)}}{d\eta} + p \frac{\partial f^{(0)}}{\partial p} \frac{\partial \omega_i}{\partial \eta} n_i n_j,$$

where for simplicity in equation (3.24) we have already used the background Boltzmann equation $df/d\eta|^{(0)} = 0$. In equation (3.24) there are all the terms which will give rise to the integrated Sachs–Wolfe effects (corresponding to the terms which explicitly depend on the gravitational perturbations), while other effects, such as the gravitational lensing, are still hidden in the (second-order part) of the first term. In fact in order to obtain equation (3.24) we just need for the time being to know the expression for $dp/d\eta$, equation (3.17).

4. The collision term

4.1. The collision integral

In this section we focus on the collision term due to Compton scattering

$$e(q)\gamma(p) \longleftrightarrow e(q')\gamma(p'),$$

where we have indicated the momentum of the photons and electrons involved in the collisions. The collision term will be important for small scale anisotropies and spectral distortions. The important point for computing the collision term is that for the epoch of interest very little energy is transferred. Therefore one can proceed by expanding the right-hand side of equation (3.1) both in the small perturbation, equation (3.23), and in the small energy transfer. Part of the computation up to second order has been done in \[30\]–\[32\] (see also \[33\]). In particular \[30,31\] are focused on the effects of reionization on the CMB anisotropies thus keeping in the collision term those contributions relevant for the small scale effects due to reionization and neglecting the effects of the metric perturbations on the left-hand side of equation (3.1). We will mainly follow the formalism of \[31\] and we will keep all the terms arising from the expansion of the collision term up to second order.

The collision term is given by

$$C(p) = \frac{1}{E(p)} \int \frac{dq}{(2\pi)^3 2E(q)} \frac{dq'}{(2\pi)^3 2E(q')} \frac{dp'}{(2\pi)^3 2E(p')} (2\pi)^4 \delta^4(q + p - q' - p') |M|^2$$

$$\times \{ g(q') f(p') [1 + f(p)] - g(q) f(p) [1 + f(p')] \}$$

$$= \frac{1}{E(p)} \int \frac{dq}{(2\pi)^3 2E(q)} \frac{dq'}{(2\pi)^3 2E(q')} \frac{dp'}{(2\pi)^3 2E(p')} (2\pi)^4 \delta^4(q + p - q' - p') |M|^2$$

$$\times \{ g(q') f(p') [1 + f(p)] - g(q) f(p) [1 + f(p')] \}$$

$$= \frac{1}{E(p)} \int \frac{dq}{(2\pi)^3 2E(q)} \frac{dq'}{(2\pi)^3 2E(q')} \frac{dp'}{(2\pi)^3 2E(p')} (2\pi)^4 \delta^4(q + p - q' - p') |M|^2$$

$$\times \{ g(q') f(p') [1 + f(p)] - g(q) f(p) [1 + f(p')] \}$$

where $E(q) = (q^2 + m_e^2)^{1/2}$, $M$ is the amplitude of the scattering process, $\delta^4(q + p - q' - p') = \delta^4(q + p - q' - p')$ ensures the energy–momentum conservation and $g$ is the distribution function for electrons. The Pauli suppression factors $(1 - g)$ have been dropped since for the epoch of interest the density of electrons $n_e$ is low. The electrons are kept in thermal equilibrium by the Coulomb interactions with protons and...
they are non-relativistic; thus we can take a Boltzmann distribution around some bulk velocity \( v \):

\[
g(\mathbf{q}) = n_e \left( \frac{2\pi}{m_e T_e} \right)^{3/2} \exp \left\{ -\frac{(\mathbf{q} - m_e \mathbf{v})^2}{2m_e T_e} \right\}. \tag{4.3}
\]

Using the three-dimensional delta function, the energy transfer is given by \( E(\mathbf{q}) - E(\mathbf{q} + \mathbf{p} - \mathbf{p}') \) and it turns out to be small compared to the typical thermal energies

\[
E(\mathbf{q}) - E(\mathbf{q} + \mathbf{p} - \mathbf{p}') \simeq \frac{(\mathbf{p} - \mathbf{p}') \cdot \mathbf{q}}{m_e} = \mathcal{O}(T q/m_e). \tag{4.4}
\]

In equation (4.4) we have used \( E(\mathbf{q}) = m_e + q^2/2m_e \) and the fact that since the scattering is almost elastic \( (p \simeq p') \), \( \mathbf{p} - \mathbf{p}' \) is of order \( p \sim T \), with \( q \) much bigger than \( \mathbf{p} - \mathbf{p}' \). In general, the electron momentum has two contributions, the bulk velocity \( (q = m_e v) \) and the thermal motion \( (q \sim (m_e T)^{1/2}) \) and thus the parameter expansion \( q/m_e \) includes the small bulk velocity \( v \) and the ratio \( (T/m_e)^{1/2} \) which is small because the electrons are non-relativistic.

The expansion of all the quantities entering the collision term in the energy transfer parameter and the integration over the momenta \( \mathbf{q} \) and \( \mathbf{q}' \) is described in detail in [31]. Here we just note that it is easy to see that we just need the scattering amplitude up to first order since at zero order \( g(\mathbf{q}) = g(\mathbf{q} + \mathbf{p} - \mathbf{p}') = g(\mathbf{q}) \) and \( \delta(E(\mathbf{q}) + p - E(\mathbf{q}') - p') = \delta(p - p') \) so all the zero-order quantities multiplying \( |M|^2 \) vanish. To first order

\[
|M|^2 = 6\pi\sigma_T m_e^2 [(1 + \cos^2 \theta) - 2 \cos \theta(1 - \cos \theta) \mathbf{q} \cdot (\mathbf{p} + \mathbf{p}')/m_e], \tag{4.5}
\]

where \( \cos \theta = \mathbf{n} \cdot \mathbf{n}' \) is the scattering angle and \( \sigma_T \) the Thompson cross section. The resulting collision term up to second order is given by [31]

\[
C(\mathbf{p}) = \frac{3n_e \sigma_T}{4p} \int d\mathbf{p}' p' \frac{d\Omega'}{4\pi} [c^{(1)}_v(\mathbf{p}, \mathbf{p}') + c^{(2)}_v(\mathbf{p}, \mathbf{p}') + c^{(2)}_\Delta(\mathbf{p}, \mathbf{p}')] + c^{(2)}_{\Delta e}(\mathbf{p}, \mathbf{p}') + c^{(2)}_{\text{em}}(\mathbf{p}, \mathbf{p}') + c^{(2)}_K(\mathbf{p}, \mathbf{p}'), \tag{4.6}
\]

where we arrange the different contributions following [31]. The first-order term reads

\[
c^{(1)}(\mathbf{p}, \mathbf{p}') = (1 + \cos^2 \theta) \left[ \delta(p - p')(f^{(1)}(\mathbf{p}') - f^{(1)}(\mathbf{p})) \right.
\]

\[
+ \left. (f^{(0)}(\mathbf{p}') - f^{(0)}(\mathbf{p}))(\mathbf{p} - \mathbf{p}') \cdot \mathbf{v} \frac{\partial\delta(p - p')}{\partial p'} \right], \tag{4.7}
\]

while the second-order terms have been separated into four parts. There is the so-called anisotropy suppression term

\[
c^{(2)}_\Delta(\mathbf{p}, \mathbf{p}') = \frac{1}{2} \left( 1 + \cos^2 \theta \right) \delta(p - p')(f^{(2)}(\mathbf{p}') - f^{(2)}(\mathbf{p})); \tag{4.8}
\]

a term which depends on the second-order velocity perturbation defined by the expansion of the bulk flow as \( \mathbf{v} = \mathbf{v}^{(1)} + \mathbf{v}^{(2)}/2 \):

\[
c^{(2)}_v(\mathbf{p}, \mathbf{p}') = \frac{1}{2} \left( 1 + \cos^2 \theta \right)(f^{(0)}(\mathbf{p}') - f^{(0)}(\mathbf{p}))(\mathbf{p} - \mathbf{p}') \cdot \mathbf{v}^{(2)} \frac{\partial\delta(p - p')}{\partial p'}; \tag{4.9}
\]
a set of terms coupling the photon perturbation to the velocity

\[
c^{(2)}_{\Delta v}(\mathbf{p}, \mathbf{p}') = (f^{(1)}(\mathbf{p}') - f^{(1)}(\mathbf{p})) \left[ (1 + \cos^2 \theta) (\mathbf{p} - \mathbf{p}') \cdot \mathbf{v} \frac{\partial \delta(p - p')}{\partial p'} - 2 \cos \theta (1 - \cos \theta) \delta(p - p')(\mathbf{n} + \mathbf{n}') \cdot \mathbf{v} \right];
\]

and a set of source terms quadratic in the velocity

\[
c^{(2)}_{\text{sv}}(\mathbf{p}, \mathbf{p}') = (f^{(0)}(\mathbf{p}') - f^{(0)}(\mathbf{p})) \cdot \mathbf{v} \left[ (1 + \cos^2 \theta) \frac{(\mathbf{p} - \mathbf{p}') \cdot \mathbf{v}}{2} \frac{\partial^2 \delta(p - p')}{\partial p'^2} - 2 \cos \theta (1 - \cos \theta) (\mathbf{n} + \mathbf{n}') \cdot \mathbf{v} \frac{\partial \delta(p - p')}{\partial p'} \right].
\]

The last contribution is from the Kompaneets terms describing spectral distortions to the CMB

\[
c^{(2)}_K(\mathbf{p}, \mathbf{p}') = (1 + \cos^2 \theta) \frac{(\mathbf{p} - \mathbf{p}')^2}{2m_e} \left[ (f^{(0)}(\mathbf{p}') - f^{(0)}(\mathbf{p})) T_e \frac{\partial^2 \delta(p - p')}{\partial p'^2} - (f^{(0)}(\mathbf{p}') + f^{(0)}(\mathbf{p}) + 2f^{(0)}(\mathbf{p}') f^{(0)}(\mathbf{p})) \frac{\partial \delta(p - p')}{\partial p'} \right]
+ \frac{2(p - p') \cos \theta (1 - \cos^2 \theta)}{m_e}
\times \left[ \delta(p - p') f^{(0)}(\mathbf{p}') (1 + f^{(0)}(\mathbf{p})) f^{(0)}(\mathbf{p}') - f^{(0)}(\mathbf{p}) \frac{\partial \delta(p - p')}{\partial p'} \right].
\]

Let us make a couple of comments about the various contributions to the collision term. First, notice the term \( c^{(2)}_{\Delta v}(\mathbf{p}, \mathbf{p}') \) due to second-order perturbations in the velocity of electrons which is absent in [31]. In standard cosmological scenarios (like inflation), vector perturbations are not generated at linear order, so linear velocities are irrotational, \( v^{(1)i} = \partial^i v^{(1)} \). However, at second order, vector perturbations are generated after horizon crossing as non-linear combinations of primordial scalar modes. Thus we must take into account also a transverse (divergence free) component, \( v^{(2)i} = \partial^i v^{(2)} + v^{(2)}_T \) with \( \partial_i v^{(2)}_T = 0 \). As we will see, such vector perturbations will break the azimuthal symmetry of the collision term with respect to a given mode \( \mathbf{k} \), which instead usually holds at linear order. Secondly, notice that the number density of electrons appearing in equation (4.6) must be expanded as \( n_e = \bar{n}_e (1 + \delta_e) \) and then

\[
\delta_e^{(1)} c^{(1)}(\mathbf{p}, \mathbf{p}')
\]

gives rise to second-order contributions in addition to the list above, where we split \( \delta_e = \delta_e^{(1)} + \delta_e^{(2)} / 2 \) into first- and second-order parts. In particular, the combination with the term proportional to \( \mathbf{v} \) in \( c^{(1)}(\mathbf{p}, \mathbf{p}') \) gives rise to the so-called Vishniac effect as discussed in [31].
4.2. Computation of different contributions to the collision term

In the integral (4.6) over the momentum $p'$ the first-order term gives the usual collision term

$$C^{(1)}(p) = n_e \sigma_T \left[ f_0^{(1)}(p) + \frac{1}{2} f_2^{(1)}(p) P_2(\hat{v} \cdot \hat{n}) - f^{(1)} - p \frac{\partial f^{(0)}}{\partial p} \hat{v} \cdot \hat{n} \right], \quad (4.14)$$

where one uses the decomposition in Legendre polynomials:

$$f^{(1)}(x, p, n) = \sum_{\ell} (2\ell + 1) f_{\ell m}(p) P_{\ell}(\cos \vartheta), \quad (4.15)$$

where $\vartheta$ is the polar angle of $n$, $\cos \vartheta = n \cdot \hat{v}$.

In the following we compute the second-order collision term separately for the different contributions, using the notation $C(p) = C^{(1)}(p) + C^{(2)}(p)/2$. We have not reported the details of the calculation of the first-order term because for its second-order analogue, $c^{(2)}_{\Delta}(p, p') + c^{(2)}_{v}(p, p')$, the procedure is the same. The important difference is that the second-order velocity term includes a vector part, and this leads to a generic angular decomposition of the distribution function (for simplicity we drop the time dependence)

$$f^{(i)}(x, p, n) = \sum_{\ell} \sum_{m=-\ell} \sum_{m=\ell} f_{\ell m}^{(i)}(x, p)(-i)^{\ell} \sqrt{\frac{4\pi}{2\ell + 1}} Y_{\ell m}(n), \quad (4.16)$$

such that

$$f_{\ell m}^{(i)} = (-i)^{-\ell} (2\ell + 1) \delta_{m0} f_{\ell}^{(i)}. \quad (4.17)$$

Such a decomposition holds also in Fourier space [34]. The notation at this stage is a bit confusing, so let us restate it: superscripts denote the order of the perturbation; the subscripts refer to the moments of the distribution. Indeed at first order one can drop the dependence on $m$ setting $m = 0$, using the fact that the distribution function does not depend on the azimuthal angle $\phi$. In this case the relation with $f_{\ell}^{(1)}$ is

$$f_{\ell m}^{(1)} = (-i)^{-\ell} (2\ell + 1) \delta_{m0} f_{\ell}^{(1)}. \quad (4.18)$$

(a) $c^{(2)}_{\Delta}(p, p')$:

The integral over $p'$ yields

$$C^{(2)}_{\Delta}(p) = \frac{3n_e \sigma_T}{4p} \int dp' \frac{d\Omega'}{4\pi} c^{(2)}_{\Delta}(p, p')$$

$$= \frac{3n_e \sigma_T}{4p} \int dp' \delta(p - p') \int \frac{d\Omega'}{4\pi} [1 + (n \cdot n')^2] [f^{(2)}(p') - f^{(2)}(p)]. \quad (4.19)$$

To do the angular integration we write the angular dependence on the scattering angle $\cos \theta = n \cdot n'$ in terms of the Legendre polynomials:

$$[1 + (n \cdot n')^2] = \frac{4}{3} \left[ 1 + \frac{1}{2} P_2(n \cdot n') \right] = \left[ 1 + \frac{1}{2} \sum_{m=-2}^{2} Y_{2m}(n) Y_{2m}^*(n') \frac{4\pi}{2\ell + 1} \right], \quad (4.20)$$
where in the last step we used the addition theorem of spherical harmonics:

\[ P_t = \frac{4\pi}{2\ell + 1} \sum_{m=-2}^{2} Y_{\ell m}(n)Y_{\ell m}^*(n'). \]  

(4.21)

Using the decomposition (4.17) and the orthonormality of the spherical harmonics we find

\[ C_\Delta^{(2)}(p) = n_e\sigma_T \left[ f^{(2)}_{00}(p) - f^{(2)}(p) - \frac{1}{2} \sum_{m=-2}^{2} \sqrt{\frac{4\pi}{5}} f_{2m}^{(2)}(p) Y_{2m}(n) \right]. \]  

(4.22)

It is easy to recover the result for the corresponding first-order contribution in equation (4.14) by using equation (4.18).

(b) \( c_v^{(2)}(p, p') \):

Let us for simplicity fix our coordinates such that the polar angle of \( n' \) is defined by \( \mu' = \hat{v}^{(2)} \cdot n' \) with \( \phi' \) the corresponding azimuthal angle. The contribution of \( c_v^{(2)}(p, p') \) to the collision term is then

\[
C_v^{(2)}(p) = \frac{3n_e\sigma_T}{4p} \int dp' p' [f^{(0)}(p') - f^{(0)}(p)] \frac{\partial\delta(p - p')}{\partial p'} \\
\times \int^{2\pi}_{-\pi} \frac{d\phi'}{2\pi} [1 + (p \cdot p')^2].
\]  

(4.23)

We can use equation (4.20) which in our coordinate system reads

\[
[1 + (n \cdot n')^2] = \frac{4}{3}[1 + \frac{1}{2}P_2(n \cdot n')]
\]

\[
= \frac{4}{3} \left[ 1 + \frac{1}{2} \sum_{m=-2}^{2} \frac{(2 - m)!}{(2 + m)!} P_2^m(n \cdot \hat{v}^{(2)}) P_2^m(n' \cdot \hat{v}^{(2)}) e^{im(\phi' - \phi)} \right],
\]  

(4.24)

and so

\[
\int \frac{d\phi'}{2\pi} P_2(n \cdot n') = P_2(n \cdot \hat{v}^{(2)}) P_2(n' \cdot \hat{v}^{(2)}) = P_2(\mu)P_2(\mu').
\]  

(4.25)

By using the orthonormality of the Legendre polynomials and integrating by parts over \( p' \) we find

\[
C_v^{(2)}(p) = -n_e\sigma_T p \frac{\partial f^{(0)}(p)}{\partial p} \hat{v}^{(2)} \cdot n.
\]  

(4.26)

As is clear from the presence of the scalar product \( \hat{v}^{(2)} \cdot p \) the final result is independent of the coordinates chosen.

(c) \( c_{\Delta v}^{(2)}(p, p') \):

Let us consider the contribution from the first term:

\[
c_{\Delta v}^{(2)}(p, p') = (1 + \cos^2 \theta)(f^{(1)}(p') - f^{(1)}(p))(p - p') \cdot v \frac{\partial\delta(p - p')}{\partial p'}.
\]  

(4.27)
where the velocity has to be considered at first order. In the integral (4.6) this provides

\[
\frac{1}{2} C^{(2)}_{\Delta v(1)} = \frac{3n_e \sigma_T v}{4p} \int dp' \frac{d\delta(p - p')}{dp'} \int_{-1}^{1} \frac{d\mu'}{2} \left[ f^{(1)}(p') - f^{(1)}(p) \right] \left( p\mu - p'\mu' \right) \int_{0}^{2\pi} \frac{d\phi'}{2\pi} (1 + \cos^2 \theta). \tag{4.28}
\]

The procedure for doing the integration is the same as above. We use the same relations as in equations (4.24) and (4.25) where now the angles are those taken with respect to the first-order velocity. This eliminates the integral over \( \phi' \), and integrating by parts over \( p' \) yields

\[
\frac{1}{2} C^{(2)}_{\Delta v(1)}(\mathbf{p}) = -\frac{3n_e \sigma_T v}{4p} \int_{-1}^{1} \frac{d\mu'}{2} \left[ \frac{2}{3} P_2(\mu) + \frac{3}{5} P_3(\mu') \right] \times \left[ p(\mu - 2\mu')(f^{(1)}(p, \mu') - f^{(1)}(p, \mu)) + p^2(\mu - \mu') \frac{\partial f^{(1)}(p, \mu')}{\partial p} \right]. \tag{4.29}
\]

We now use the decomposition (4.15) and the orthonormality of the Legendre polynomials to find

\[
\int \frac{d\mu'}{2} \mu' f^{(1)}(p, \mu') P_2(\mu') = \sum_\ell \int \frac{d\mu'}{2} \mu' P_2(\mu') f^{(1)}_{\ell}(p)
\]

\[
= \sum_\ell \int \frac{d\mu'}{2} \left[ \frac{2}{3} P_1(\mu) + \frac{3}{5} P_3(\mu') \right] P_\ell(\mu') f^{(1)}_{\ell}(p)
\]

\[
= \frac{2}{5} f^{(1)}_{1}(p) + \frac{3}{5} f^{(1)}_{3}(p), \tag{4.30}
\]

where we have used \( \mu' P_2(\mu') P_3(\mu') = \frac{2}{5} P_1(\mu') + \frac{3}{5} P_3(\mu') \), with \( P_1(\mu') = \mu' \). Thus from equation (4.29) we get

\[
\frac{1}{2} C^{(2)}_{\Delta v(1)}(\mathbf{p}) = n_e \sigma_T \left\{ \mathbf{v} \cdot \mathbf{n} \left[ f^{(1)}(\mathbf{p}) - f^{(1)}_0(p) - p \frac{\partial f^{(1)}_0(p)}{\partial p} \right] - \frac{1}{2} P_2(\mathbf{v} \cdot \mathbf{n}) \left( f^{(1)}_{2}(p) + p \frac{\partial f^{(1)}_{2}(p)}{\partial p} \right) \right. \]

\[
+ v \left[ 2 f^{(1)}_{1}(p) + p \frac{\partial f^{(1)}_{1}(p)}{\partial p} + \frac{1}{5} P_2(\mathbf{v} \cdot \mathbf{n}) \left( 2 f^{(1)}_{1}(p) + p \frac{\partial f^{(1)}_{1}(p)}{\partial p} + 3 f^{(1)}_{3}(p) \right) \right.
\]

\[
+ \left. \frac{3}{2} p \frac{\partial f^{(1)}_{3}(p)}{\partial p} \right]\} \left[ \right]. \tag{4.31}
\]

In \( C^{(2)}_{\Delta v(1)}(\mathbf{p}, \mathbf{p}') \) there is a second term

\[
C^{(2)}_{\Delta v(2)} = -2 \cos \theta (1 - \cos \theta) \left( f^{(1)}(\mathbf{p}') - f^{(1)}(\mathbf{p}) \right) \delta(p - p')(\mathbf{n} + \mathbf{n}') \cdot \mathbf{v}, \tag{4.32}
\]

whose contribution to the collision term is

\[
\frac{1}{2} C^{(2)}_{\Delta v(2)}(\mathbf{p}) = -\frac{3n_e \sigma_T v}{2p} \int dp' \delta(p - p') \int_{-1}^{1} \frac{d\mu'}{2} f^{(1)}(p') \left. \right|_{\phi' = 0} - f^{(1)}(p)(\mu + \mu') \int_{0}^{2\pi} \frac{d\phi'}{2\pi} \cos \theta (1 - \cos \theta). \tag{4.33}
\]
This integration proceeds through the same steps as for $C_{\Delta v (I)}^{(2)}(p)$. In particular, noting that $\cos \theta (1 - \cos \theta) = -1/3 + P_1(\cos \theta) - 2P_3(\cos \theta)/3$, equations (4.24) and (4.25) allow us to compute

$$\int \frac{d\phi'}{2\pi} \cos \theta (1 - \cos \theta) = -\frac{1}{3} + P_1(\mu)P_1(\mu') - \frac{2}{3}P_2(\mu)P_2(\mu'),$$

and using the decomposition (4.15) we arrive at

$$\frac{1}{2} C_{\Delta v (II)}^{(2)}(p) = -n_e \sigma_T \left\{ v \cdot n f_2^{(1)}(p) (1 - P_2(\hat{v} \cdot \hat{n})) + v \left[ \frac{1}{2} P_2(\hat{v} \cdot \hat{n}) (3f_1^{(1)}(p) - 3f_2^{(1)}(p)) \right] \right\}. \quad (4.35)$$

Thus summing equations (4.29) and (4.35) we obtain

$$\frac{1}{2} C_{\Delta v}^{(2)}(p) = n_e \sigma_T \left\{ v \cdot n \left[ f^{(1)}(p) - f_0^{(1)}(p) - p \frac{\partial f_0^{(1)}(p)}{\partial p} - f_2^{(1)}(p) \right] + \frac{1}{2} P_2(\hat{v} \cdot \hat{n}) \left( f_2^{(1)}(p) - p \frac{\partial f_2^{(1)}(p)}{\partial p} \right) \right\} + v \left[ 2f_1^{(1)}(p) + p \frac{\partial f_1^{(1)}(p)}{\partial p} + \frac{1}{5} P_2(\hat{v} \cdot \hat{n}) \left( -f_1^{(1)}(p) + p \frac{\partial f_1^{(1)}(p)}{\partial p} \right) + 6f_3^{(1)}(p) + \frac{3}{2} p \frac{\partial f_3^{(1)}(p)}{\partial p} \right] \right\}. \quad (4.36)$$

As regards the remaining terms, these have already been computed in [31] (see also [30]) and here we just report them:

(d) $c_{wv}^{(2)}(p, p')$:
The terms proportional to the velocity squared yield a contribution to the collision term

$$\frac{1}{2} C_{wv}^{(2)}(p) = n_e \sigma_T \left\{ (v \cdot n)^2 \left[ p \frac{\partial f_0^{(0)}}{\partial p} + \frac{11}{20} p^2 \frac{\partial^2 f_0^{(0)}}{\partial p^2} \right] + v^2 \left[ p \frac{\partial f_0^{(0)}}{\partial p} + \frac{3}{20} p^2 \frac{\partial^2 f_0^{(0)}}{\partial p^2} \right] \right\}. \quad (4.37)$$

(e) $c_K^{(2)}(p, p')$:
The terms responsible for the spectral distortions give

$$\frac{1}{2} C^{(2)}(p) = \frac{1}{m_e^2} \frac{\partial}{\partial p} \left\{ p^4 \left[ T_e \frac{\partial f_0^{(0)}}{\partial p} + f_0^{(0)} (1 + f_0^{(0)}) \right] \right\}. \quad (4.38)$$

Finally we write also that part of the collision term coming from equation (4.13):

$$\delta^{(1)} C^{(1)}(p, p') = \delta^{(1)} C^{(1)}(p) = n_e \sigma_T \delta^{(1)} \left[ f_0^{(1)}(p) + \frac{1}{2} f_2^{(1)} \right] P_2(\hat{v} \cdot \hat{n}) - f^{(1)} - p \frac{\partial f^{(0)}}{\partial p} \hat{v} \cdot \hat{n}. \quad (4.39)$$

JCAP06(2006)024
4.3. The final expression for the collision term

Summing all the terms we find the final expression for the collision term (4.6) up to second order:

\[ C(p) = C^{(1)}(p) + \frac{1}{2} C^{(2)}(p) \]  

(4.40)

with

\[ C^{(1)}(p) = n_e \sigma T \left( f_0^{(1)}(p) + \frac{1}{2} f_2^{(1)} P_2(\hat{v} \cdot \hat{n}) - f^{(1)} - p \frac{\partial f^{(0)}}{\partial p} \hat{v} \cdot \hat{n} \right) \]  

(4.41)

and

\[
\frac{1}{2} C^{(2)}(p) = n_e \sigma T \left( \frac{1}{2} f_0^{(2)}(p) - \frac{1}{4} \sum_{m=-2}^{2} \sqrt{\frac{4\pi}{5^{3/2}}} f_2^{(2)}(p) Y_{2m}(n) - \frac{1}{2} f^{(2)}(p) \right.

+ \delta_e^{(1)} \left[ f_0^{(1)}(p) + \frac{1}{2} f_2^{(1)} P_2(\hat{v} \cdot \hat{n}) - f^{(1)} - p \frac{\partial f^{(0)}}{\partial p} \hat{v} \cdot \hat{n} \right]

- \frac{1}{2} p^2 \frac{\partial f^{(0)}}{\partial p} \hat{v} \cdot \hat{n} + \hat{v} \cdot \hat{n} \left[ f^{(1)}(p) - f_0^{(1)}(p) - p \frac{\partial f_0^{(1)}(p)}{\partial p} - f_2^{(1)}(p) \right]

+ \frac{1}{2} P_2(\hat{v} \cdot \hat{n}) \left[ f_2^{(1)}(p) - p \frac{\partial f^{(1)}(p)}{\partial p} \right]

+ v \left[ 2 f_1^{(1)}(p) + p \frac{\partial f_1^{(1)}(p)}{\partial p} + \frac{1}{5} P_2(\hat{v} \cdot \hat{n}) \right.

\times \left( -f_1^{(1)}(p) + p \frac{\partial f_1^{(1)}(p)}{\partial p} + 6 f_3^{(1)}(p) + \frac{3}{2} p \frac{\partial f_3^{(1)}(p)}{\partial p} \right) \right]

\left. \times \left[ \frac{1}{2} (\hat{v} \cdot \hat{n})^2 \left[ p \frac{\partial f^{(0)}}{\partial p} + \frac{11}{20} p^2 \frac{\partial^2 f^{(0)}}{\partial p^2} \right] + v^2 \left[ p \frac{\partial f^{(0)}}{\partial p} + \frac{3}{20} p^2 \frac{\partial^2 f^{(0)}}{\partial p^2} \right] \right. 

\left. + \frac{1}{m_e^2} \frac{\partial}{\partial p} \left[ p^4 \left( T_e \frac{\partial f^{(0)}}{\partial p} + f^{(0)}(1 + f^{(0)}) \right) \right] \right). \]  

(4.42)

Notice that there is an internal hierarchy, with terms which do not depend on the baryon velocity \( \hat{v} \), terms proportional to \( \hat{v} \cdot \hat{n} \) and thus to \( (\hat{v} \cdot \hat{n})^2 \), \( v \) and \( v^2 \) (apart from the Kompaneets terms). In particular, notice that the term proportional to \( \delta_e^{(1)} \hat{v} \cdot \hat{n} \) is the one corresponding to the Vishniac effect. We point out that we have kept all the terms up to second order in the collision term. In [30, 31] many terms coming from \( c^{(2)}_{\Delta e} \) have been dropped mainly because these terms are proportional to the photon distribution function \( f^{(1)} \) which on very small scales (those of interest for reionization) is suppressed by the diffusion damping. Here we want to be completely general and we have to keep them.
5. The brightness equation

5.1. First order

The Boltzmann equation for photons is obtained by combining equation (3.24) with equations (4.41) and (4.42). At first order the left-hand side reads

$$\frac{df}{d\eta} = \frac{df^{(1)}}{d\eta} - p \frac{\partial f^{(0)}}{\partial p} \frac{\partial \Phi^{(1)}}{\partial x^i} \frac{dx^i}{d\eta} + p \frac{\partial f^{(0)}}{\partial p} \frac{\partial \Psi^{(1)}}{\partial \eta}. \quad (5.1)$$

At first order it is useful to characterize the perturbations to the Bose–Einstein distribution function (3.22) in terms of a perturbation to the temperature as

$$f(x^i, p, n^i, \eta) = 2 \left[ \exp \left\{ \frac{p}{T(\eta)(1 + \Theta^{(1)})} \right\} - 1 \right]^{-1}. \quad (5.2)$$

Thus it turns out that

$$f^{(1)} = -p \frac{\partial f^{(0)}}{\partial p} \Theta^{(1)}, \quad (5.3)$$

where we have used that $\frac{\partial f}{\partial \Theta}|_{\Theta=0} = -p\frac{\partial f^{(0)}}{\partial p}$. In terms of this variable $\Theta^{(1)}$ the linear collision term (4.41) will now become proportional to $-p\frac{\partial f^{(0)}}{\partial p}$ which contains the only explicit dependence on $p$, and the same happens for the left-hand side, equation (5.1). This is telling us that at first order $\Theta^{(1)}$ does not depend on $p$, only on $x^i, n^i, \eta$, $\Theta^{(1)} = \Theta^{(1)}(x^i, n^i, \tau)$. This is well known and the physical reason is that at linear order there is no energy transfer in Compton collisions between photons and electrons. Therefore the Boltzmann equation for $\Theta^{(1)}$ reads

$$\frac{\partial \Theta^{(1)}}{\partial \eta} + n^i \frac{\partial \Theta^{(1)}}{\partial x^i} + \frac{\partial \Phi^{(1)}}{\partial x^i} n^i - \frac{\partial \Psi^{(1)}}{\partial \eta} = n_c \sigma_T a \left[ \Theta_0^{(1)} + \frac{1}{2} \Theta_2^{(1)} P_2(\hat{v} \cdot \hat{n}) - \Theta^{(1)} + \hat{v} \cdot \hat{n} \right], \quad (5.4)$$

where we used that $f^{(1)} = -p\frac{\partial f^{(0)}}{\partial p} \Theta^{(1)}$ according the decomposition of equation (4.15), and we have taken the zero-order expressions for $dx^i/d\eta$, dropping the contribution from $dn^i/d\eta$ in equation (3.8) since it is already first order.

Notice that since $\Theta^{(1)}$ is independent on $p$ it is equivalent to consider the quantity

$$\Delta^{(1)}(x^i, n^i, \tau) = \frac{\int dp^3 f^{(1)}}{\int dp^3 f^{(0)}}, \quad (5.5)$$

$$\Delta^{(1)} = 4 \Theta^{(1)}. \quad (5.6)$$

The physical meaning of $\Delta^{(1)}$ is that of a fractional energy perturbation (in a given direction). From equation (3.24) another way to write an equation for $\Delta^{(1)}$—the so-
called brightness equation—is
\[
\frac{d}{d\eta} \left[ \Delta^{(1)} + 4\Phi^{(1)} \right] - 4 \frac{\partial}{\partial \eta} \left( \Phi^{(1)} + \Psi^{(1)} \right) = n_e \sigma_T a \left[ \Delta_0^{(1)} + \frac{1}{2} \Delta_2^{(1)} P_2(\hat{v} \cdot \mathbf{n}) - \Delta^{(1)} + 4\mathbf{v} \cdot \mathbf{n} \right].
\]

(5.7)

5.2. Second order

The previous results show that at linear order the photon distribution function has a Planck spectrum with the temperature that at any point depends on the photon direction. At second order one could characterize the perturbed photon distribution function in a similar way to in equation (5.2):

\[
f(x^i, p, n^i, \eta) = 2 \left[ \exp \left\{ \frac{p}{T(\eta) \Theta - 1} \right\} - 1 \right]^{-1},
\]

(5.8)

where by expanding $\Theta = \Theta^{(1)} + \Theta^{(2)}/2 + \cdots$ as usual one recovers the first-order expression. For example, in terms of $\Theta$, the perturbation of $f^{(1)}$ is given by equation (5.3), while at second order

\[
\frac{f^{(2)}}{2} = \frac{p}{2} \frac{\partial f^{(0)}}{\partial p} \Theta^{(2)} + \frac{1}{2} \left( p^2 \frac{\partial^2 f^{(0)}}{\partial p^2} + p \frac{\partial f^{(0)}}{\partial p} \right) \left( \Theta^{(1)} \right)^2.
\]

(5.9)

However, as discussed in detail in [30,31], now the second-order perturbation $\Theta^{(2)}$ will not be momentum independent because the collision term in the equation for $\Theta^{(2)}$ does depend explicitly on $p$ (defining the combination $-(p\partial f^{(0)}/\partial p)^{-1} f^{(2)}$ does not lead to a second-order momentum independent equation as above). Such dependence is evident in, for example, the terms of $C^{(2)}(\mathbf{p})$, equation (4.42), proportional to $v$ or $v^2$, and in the Kompaneets terms. The physical reason is that at the non-linear level, photons and electrons do exchange energy during Compton collisions. As a consequence spectral distortions are generated. For example, in the isotropic limit, only the Kompaneets terms survive, giving rise to the Sunyaev–Zeldovich distortions. As discussed in [30], the Sunyaev–Zeldovich distortions can also be obtained with the correct coefficients by replacing the average over the direction electron $\langle v^2 \rangle$ with the mean squared thermal velocity $\langle v_{th}^2 \rangle = 3T_e/m_e$ in equation (4.42). This is due simply to the fact that the distinction between thermal and bulk velocity of the electrons is just for convenience. This fact also shows that spectral distortions due to the bulk flow (kinetic Sunyaev–Zeldovich ones) have the same form as the thermal effect. Thus spectral distortions can in general be described by a global Compton $y$-parameter (see [30] for a full discussion of spectral distortions). However, in the following we will not be interested in the frequency dependence, only in the anisotropies of the radiation distribution. Therefore we can integrate over the momentum $p$ and define [30,31]

\[
\Delta^{(2)}(x^i, n^i, \tau) = \frac{\int dp p^3 f^{(2)}}{\int dp p^3 f^{(0)}}.
\]

(5.10)

as in equation (5.5).
Integration over $p$ of equations (3.24)–(4.42) is straightforward using the following relations:

$$
\int dp p^3 \frac{\partial f^{(0)}}{\partial p} = -4N,
$$

$$
\int dp p^3 \frac{\partial^2 f^{(0)}}{\partial p^2} = 20N,
$$

$$
\int dp p^3 f^{(1)} = N \Delta^{(1)},
$$

$$
\int dp p^3 \frac{\partial f^{(1)}}{\partial p} = -4N \Delta^{(1)},
$$

$$
N = \int dp p^3 f^{(0)},
$$

where $N$ is the normalization factor in equation (5.10) (it is just proportional the background energy density of photons $\bar{\rho}_\gamma$. At first order, one recovers equation (5.7). At second order we find

$$
\frac{1}{2} \frac{d}{d\eta} \left[ \Delta^{(2)} + 4\Phi^{(2)} \right] + \frac{d}{d\eta} \left[ \Delta^{(1)} + 4\Phi^{(1)} \right] - 4\Delta^{(1)} \left( \Psi^{(1)} - \Phi^{(1)} n^i \right)
$$

$$
-2 \frac{\partial}{\partial \eta} \left( \Psi^{(2)} + \Phi^{(2)} \right) + 4 \frac{\partial \omega_i}{\partial \eta} n^i + 2 \frac{T}{\partial \eta} n^i n^j
$$

$$
= \frac{\tau'}{2} \left[ \Delta^{(2)} - \Delta^{(2)} - \frac{1}{2} \sum_{m=-2}^{2} \frac{\sqrt{4\pi}}{2\ell+1} \Delta^{(2)}_{2m} Y_{2m}(n) + 2\delta^{(1)}_\ell \left( \Delta^{(1)} + \frac{1}{2} \Delta^{(2)}_2 P_2(\hat{\mathbf{v}} \cdot \mathbf{n}) - \Delta^{(1)} + 4\mathbf{v} \cdot \mathbf{n} \right) + 4\mathbf{v}^{(2)} \cdot \mathbf{n}
$$

$$
+ 2(\mathbf{v} \cdot \mathbf{n})[\Delta^{(1)} + 3\Delta^{(1)}_0 - \Delta^{(1)}_2] \left( 1 - \frac{5}{2} P_2(\hat{\mathbf{v}} \cdot \mathbf{n}) \right)
$$

$$
- v \Delta^{(1)}_1 \left( 4 + 2P_2(\hat{\mathbf{v}} \cdot \mathbf{n}) \right) + 14(\mathbf{v} \cdot \mathbf{n})^2 - 2v^2 \right],
$$

where we have expanded the angular dependence of $\Delta$ as in equation (4.16)

$$
\Delta^{(i)}(x, n) = \sum_{\ell} \sum_{m=-\ell}^{\ell} \Delta^{(i)}_{\ell m}(x)(-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} Y^{*}_{\ell m}(n),
$$

with

$$
\Delta^{(i)}_{\ell m} = (-i)^{-\ell} \sqrt{\frac{2\ell+1}{4\pi}} \int d\Omega \Delta^{(i)} Y^{*}_{\ell m}(n),
$$

where we recall that the superscript stands for the order of the perturbation. At first order one can drop the dependence on $m$, setting $m = 0$ so that $\Delta^{(1)}_{\ell m} = (-i)^{-\ell}(2\ell+1)\delta_{m0} \Delta^{(1)}_{\ell}$. In equation (5.12) we have introduced the differential optical depth

$$
\tau' = -\bar{n}_e \sigma_T a.
$$
It is understood that on the left-hand side of equation (5.12) one has to pick up for the total time derivatives only those terms which contribute to second order. Thus we have to take

$$\frac{1}{2} \frac{d}{d\eta} \left[ \Delta^{(2)} + 4 \Phi^{(2)} \right] + \frac{d}{d\eta} \left[ \Delta^{(1)} + 4 \Phi^{(1)} \right] = \frac{1}{2} \left( \frac{\partial}{\partial \eta} + n^i \frac{\partial}{\partial x^i} \right) \left( \Delta^{(2)} + 4 \Phi^{(2)} \right) + n^i (\Phi^{(1)} + \Psi^{(1)}) \partial_i (\Delta^{(1)} + 4 \Phi^{(1)})
$$

$$+ \left[ (\Phi_j^{(1)} + \Psi_j^{(1)} ) n^i n^j - (\Phi^i + \Psi^i) \right] \frac{\partial \Delta^{(1)}}{\partial n^i},$$

where we used equations (3.10) and (3.21). Notice that we can write $\partial \Delta^{(1)}/\partial n^i = (\partial \Delta^{(1)}/\partial x^i)(\partial x^i/\partial n^i) = (\partial \Delta^{(1)}/\partial x^i)(\eta - \eta_i)$, from the integration in time of equation (3.10) at zero order when $n^i$ is constant in time.

### 5.3. Hierarchy equations for multipole moments

Let us now move to Fourier space. In the following for a given $k$ mode we will choose the coordinate system such that $e_3 = k$ and the polar angle of the photon momentum is $\vartheta$, with $\mu = \cos \vartheta = \hat{k} \cdot \hat{n}$. Then equation (5.12) can be written as

$$\Delta^{(2)r} + ik_\mu \Delta^{(2)} - \tau' \Delta^{(2)} = S(k, n, \eta),$$

where $S(k, n, \eta)$ can be easily read off from equation (5.12). We now expand the temperature anisotropy in the multipole moments $\Delta^{(2)}_{\ell m}$ in order to obtain a system of coupled differential equations. On applying the angular integral of equation (5.14) to (5.17) we find

$$\Delta^{(2)r}_{\ell m}(k, \eta) = k \left[ \frac{\kappa_{\ell m}}{2\ell - 1} \Delta^{(2)}_{\ell - 1, m} - \frac{\kappa_{\ell+1, m}}{2\ell + 3} \Delta^{(2)}_{\ell+1, m} \right] + \tau' \Delta^{(2)}_{\ell m} + S_{\ell m},$$

where the expansion coefficients of the source term are given by

$$S_{\ell m} = (4\Psi^{(2)r} - \tau' \Delta^{(2)}_{00}) \delta_{00} \delta_{m0} + 4k_\mu \Phi^{(2)} \delta_{\ell 1} \delta_{m0} - 4 \omega_{\pm 1} \delta_{\ell 1} - 8 \tau' \nu^{(2)} \delta_{\ell 1}
$$

$$- \frac{\tau'}{10} \Delta^{(2)}_{\ell m} \delta_{\ell 2} - 2 \chi_{\pm 2} \delta_{\ell 2} - 2 \tau' \int \frac{d^3k_1}{(2\pi)^3} [\Gamma^{(1)}(k_1)v^{(1)}(k_2)k_2 \cdot \tilde{k}_1
$$

$$+ \delta^{(1)}_e(k_1) \Delta^{(1)}_{00}(k_2) - i \frac{2}{3} v(k_1) \Delta^{(1)}_{10}(k_2)] \delta_{\ell 0} \delta_{m0}
$$

$$+ k \int \frac{d^3k_1}{(2\pi)^3} \left[ \Phi^{(1)}(k_1) \Phi^{(1)}(k_2) \right] \delta_{\ell 1} \delta_{m0}
$$

$$- 2 \left[ (\Psi^{(1)} \nabla \Phi^{(1)})_m + 8 \tau' (\delta^{(1)}_e v)_m + 6 \tau' (\Delta^{(1)}_{01} v)_m - 2 \tau' (\Delta^{(1)}_{21} v)_m \right] \delta_{\ell 1}
$$

$$+ \tau' \int \frac{d^3k_1}{(2\pi)^3} \left[ \delta^{(1)}_e(k_1) \Delta^{(1)}_{21}(k_2) - i \frac{2}{3} v(k_1) \Delta^{(1)}_{11}(k_2) \right] \delta_{\ell 2} \delta_{m0}
$$

$$+ \int \frac{d^3k_1}{(2\pi)^3} \left[ 8 \Psi^{(1)}(k_1) + 2 \tau' \delta^{(1)}_e(k_1) \right]
$$

$$- (\eta - \eta_i)(\Psi^{(1)} + \Phi^{(1)})(k_1) k_1 \cdot k_2 \Delta^{(1)}_{00}(k_2) \delta_{m0}$$
- \( i(-1)^{-\ell}(-1)^{-m}(2\ell + 1) \sum_{\ell''} \sum_{m''} (2\ell'' + 1) \)

\[
\times \left[ 8\Delta_{\ell''} \nabla \Phi^{(1)} - 2(\Phi^{(1)} + \Psi^{(1)}) \nabla \Delta_{\ell''} \right]_{m''} \left( \begin{array}{ccc} \ell'' & 1 & \ell \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} \ell'' & 1 & \ell \\ 0 & m' & -m \end{array} \right)
\]

\[
+ \tau^\prime i(-1)^{-\ell}(-1)^{-m}(2\ell + 1) \sum_{\ell''} \sum_{m''} (2\ell'' + 1) \]

\[
\times \left[ 2\Delta_{\ell''} v + 5\delta_{\ell''} \Delta_{\ell''} \right]_{m''} \left( \begin{array}{ccc} \ell'' & 1 & \ell \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} \ell'' & 1 & \ell \\ 0 & m' & -m \end{array} \right)
\]

\[
+ 14\tau^\prime(-1)^{-\ell}(-1)^{-m} \sum_{m',m''} \int \frac{d^3k_1}{(2\pi)^3} \]

\[
\times \left[ \frac{1}{k_2} v^{(1)}(k_1) \frac{1}{3} Y_{1m'}^* (\hat{k}_1) \left( kY_{1m''}^* (\hat{k}) - k_1Y_{1m''}^* (\hat{k}_1) \right) \right]
\]

\[
\times \left( \begin{array}{ccc} 1 & 1 & \ell \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} m' & 1 & m'' \\ 0 & 0 & -m \end{array} \right)
\]

\[
+ 2(\eta - \eta_i)(-1)^{-\ell}(-1)^{-m} \sqrt{\frac{2\ell + 1}{4\pi}} \]

\[
\times \sum_{L} \sum_{m',m''} \int \frac{d^3k_1}{(2\pi)^3} \sqrt{\frac{4\pi}{2L + 1}} \left( \frac{4\pi}{3} \right)^2 \Delta_{L}^{(1)}(k_1)
\]

\[
\times \left( \Phi^{(1)} + \Psi^{(1)} \right)(k_2) Y_{1m'}^* (\hat{k}_1) \left( kY_{1m''}^* (\hat{k}) - k_1Y_{1m''}^* (\hat{k}_1) \right)
\]

\[
\times \int d\Omega Y_{1m'}(\hat{n}) Y_{1m''}(\hat{n}) Y_L(n) Y_{L-m}(n),
\]  

(5.19)

where

\[ k_2 = k - k_1 \]  

(5.20)

and \( k_2 = |k_2| \). In equation (5.19) it is understood that \( |m| \leq \ell \).

Let us explain the notation that we have adopted in writing equation (5.19). The baryon velocity at linear order is irrotational, meaning that it is the gradient of a potential, and thus in Fourier space it is parallel to \( \hat{\mathbf{k}} \), and following the conventions of [37], we write

\[ \mathbf{v}^{(1)}(\mathbf{k}) = -i v^{(1)}_0(\mathbf{k}) \hat{\mathbf{k}}. \]  

(5.21)

For the second-order velocity perturbation, it will contain a transverse (divergence free) part whose components are orthogonal to \( \hat{\mathbf{k}} = \mathbf{e}_3 \), and we can write

\[ \mathbf{v}^{(2)}(\mathbf{k}) = -i v^{(2)}_0(\mathbf{k}) \mathbf{e}_3 + \sum_{m=\pm 1} v^{(2)}_m(\mathbf{k}) \mathbf{e}_m \pm \mathbf{e}_1 \sqrt{2}, \]  

(5.22)

where the \( \mathbf{e}_i \) form an orthonormal basis with \( \hat{\mathbf{k}} \). The second-order perturbation \( \omega_i \) is decomposed in a similar way, and the \( \omega_{\pm 1} \) are the corresponding components (in this case
in the Poisson gauge there is no scalar component). Similarly for the tensor perturbation \( \chi_{ij} \) we have indicated its amplitudes as \( \chi_{\pm 2} \) in the decomposition [36]

\[
\chi_{ij} = \sum_{m=\pm 2} -\sqrt{2\over 3} \chi_m (e_1 \pm ie_2)_i (e_1 \pm ie_2)_j.
\] (5.23)

We have taken into account that in the gravitational part of the Boltzmann equation and in the collision term there are some terms, like \( \delta_0^{(1)} \mathbf{v} \), which can still be decomposed into the scalar and transverse parts in Fourier space as in equation (5.22). For a generic quantity \( f(x)v \) we have indicated the corresponding scalar and vortical components with \( (f\mathbf{v})_m \) and their explicit expression is easily found by projecting the Fourier modes of \( f(x)v \) along the \( \mathbf{k} = \mathbf{e}_3 \) and \( (\mathbf{e}_2 \mp i\mathbf{e}_1) \) directions:

\[
(f\mathbf{v})_m(k) = \int {d^3k_1 \over (2\pi)^3} \gamma_0^{(1)}(k_1)f(k_2)Y_{lm}^*(\hat{k}_1)\sqrt{4\pi \over 3}.
\] (5.24)

Similarly, for a term like \( f(x)\nabla g(x) \) we used the notation

\[
(f\nabla g)_m(k) = -\int {d^3k_1 \over (2\pi)^3} k_1 g(k_1)f(k_2)Y_{lm}^*(\hat{k}_1)\sqrt{4\pi \over 3}.
\] (5.25)

Finally, the first term on the right-hand side of equation (5.18) has been obtained by using the relation

\[
\mathbf{i}k \cdot \mathbf{n} \Delta^{(2)}_\ell(k) = \sum_{\ell m} \Delta^{(2)}_{\ell m}(k) {k \over 2\ell + 1} [\kappa_{\ell m} \tilde{G}_{\ell - 1, m} - \kappa_{\ell + 1, m} \tilde{G}_{\ell + 1, m}]
\]

\[
= k \sum_{\ell m} \left[ {\kappa_{\ell m} \over 2\ell - 1} \Delta^{(2)}_{\ell - 1, m} - {\kappa_{\ell m} \over 2\ell + 3} \Delta^{(2)}_{\ell + 1, m} \right] \tilde{G}_{\ell m},
\] (5.26)

where \( \tilde{G}_{\ell m} = (-i)^\ell \sqrt{4\pi/(2\ell + 1)}Y_{\ell m}(\mathbf{n}) \) is the angular mode for the decomposition (5.13) and

\[
\kappa_{\ell m} = \sqrt{l^2 - m^2}.
\] (5.27)

This relation has been discussed in [34, 36] and corresponds to the term \( n^i \partial \Delta^{(2)/\partial x^i} \) in equation (5.12).

As expected, at second order we recover some intrinsic effects which are characteristic of the linear regime. In equation (5.18) the relation (5.26) represents the free streaming effect: when the radiation free streams the inhomogeneities of the photon distribution are seen by the observer as angular anisotropies. At first order this is responsible for the hierarchy of Boltzmann equations coupling the different \( \ell \) modes, and it represents a projection effect of fluctuations on a scale \( k \) onto the angular scale \( \ell \sim k\eta \). The term \( \tau_i \Delta^{(2)}_{\ell m} \) causes an exponential suppression of anisotropies in the absence of the source term \( S_{\ell m} \). The first line of the source term (5.19) just reproduces the expression for the first-order case. Of course the dynamics of the second-order metric and baryon velocity perturbations which appear will be different and governed by the second-order Einstein equations and continuity equations. The remaining terms in the source are second-order effects generated as non-linear combinations of the primordial (first-order) perturbations. We have ordered them according to the increasing number of \( \ell \) modes that they contribute to. Notice in
particular that they involve the first-order anisotropies $\Delta^{(1)}$ and as a consequence such terms contribute to generating the hierarchy of equations (apart from the free streaming effect). The source term contains additional scattering processes and gravitational effects. For large scales (above the horizon at recombination) we can say that the main effects are due to gravity, and they include the Sachs–Wolfe and the (late and early) Sachs–Wolfe effect due to the redshift that photons suffer when travelling through the second-order gravitational potentials. These, together with the contribution due to the second-order tensor modes, have already been studied in detail in [19]. Another important gravitational effect is that of lensing of photons as they travel from the last scattering surface to us. A contribution of this type is given by the last term of equation (5.19). On small scales scattering effects around the epoch of recombination are important and we will study them in detail in a companion paper [27].

### 5.4. Integral solution to the second-order Boltzmann equation

As in linear theory, one can derive an integral solution to the Boltzmann equation (5.12) in terms of the source term $S$. Following the standard procedure (see e.g. [28,35]) for linear perturbations, we write the left-hand side as $\Delta^{(2)} + i k \mu \Delta^{(2)} - \tau \Delta^{(2)} = e^{-i k \mu + \tau} d[\Delta^{(2)} e^{i k \mu \eta - \tau}] / d\eta$ in order to derive the integral solution

$$\Delta^{(2)}(k, n, \eta_0) = \int_{0}^{\eta_0} d\eta S(k, n, \eta) e^{i k \mu (\eta - \eta_0) - \tau},$$

(5.28)

where $\eta_0$ stands for the present time. The expression for the photon moments $\Delta^{(2)}_{\ell m}$ can be obtained as usual from equation (5.14). In the previous section we have already found the coefficients for the decomposition of the source term $S$:

$$S(k, n, \eta) = \sum_{\ell} \sum_{m=-\ell}^{\ell} S_{\ell m}(k, \eta)(-i) \sqrt{\frac{4\pi}{2\ell + 1}} Y_{\ell m}(n).$$

(5.29)

In equation (5.28) there is an additional angular dependence in the exponential. It is easy to take it into account by recalling that

$$e^{i k \cdot x} = \sum_{\ell} (i)^{\ell} (2\ell + 1) j_{\ell}(k x) P_{\ell}(\hat{k} \cdot \hat{x}).$$

(5.30)

Thus the angular integral (5.14) is computed by using the decomposition of the source term (5.29) and equation (5.30):

$$\Delta^{(2)}_{\ell m}(k, \eta_0) = (-1)^{-m} (-i)^{-\ell} (2\ell + 1) \int_{0}^{\eta_0} d\eta e^{-\tau(\eta)} \sum_{\ell_2} \sum_{m_2=-\ell_2}^{\ell_2} (-i)^{\ell_2} S_{\ell_2 m_2}
\times \sum_{\ell_1} i^{\ell_1} (2\ell_1 + 1) j_{\ell_1}(k(\eta - \eta_0))(2\ell_1 + 1) \begin{pmatrix} \ell_1 & 0 & \ell_2 & \ell \\ 0 & 0 & m_2 & -m \end{pmatrix},$$

(5.31)
where the Wigner 3j symbols appear because of the Gaunt integrals
\[
\mathcal{G}_{l_1, l_2, l_3}^{m_1, m_2, m_3} \equiv \int d^2 \hat{n} Y_{l_1 m_1} (\hat{n}) Y_{l_2 m_2} (\hat{n}) Y_{l_3 m_3} (\hat{n})
\]
\[
= \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} m_1 & m_2 & m_3 \end{pmatrix}.
\]
Since the second of the Wigner 3j symbols in equation (5.31) is non-zero only if \( m = m_2 \), our solution can be rewritten to recover the corresponding expression found for linear anisotropies in [34,36]:
\[
\Delta^{(2)}_{\ell m}(k, \eta_0) = \int_0^\eta \frac{d\eta e^{-\tau(\eta)}}{} \sum_{\ell_2} \sum_{m_2=-\ell_2}^{\ell_2} S_{\ell_2 m_2} j_{l_2}(l_2, m_2) [k(\eta_0 - \eta)],
\]
where \( j_{l_2}(l_2, m_2) [k(\eta_0 - \eta)] \) are the so-called radial functions. Of course the main information at second order is included in the source term containing different effects due to the non-linearity of the perturbations. In the total angular momentum method of [34,36], equation (5.32) is interpreted just as the integral over the radial coordinate \( \chi = \eta_0 - \eta \) of the projected source term. Another important comment is that, as in linear theory, the integral solution (5.31) is in fact just a formal solution, since the source term \( S \) contains itself the second-order photon moments up to \( l = 2 \) (see equation (5.19)). This means that one has anyway to resort to the hierarchy equations for photons, equation (5.18), to solve for these moments. Nevertheless, as in linear theory [35], one expects to need just a few moments beyond \( \ell = 2 \) in the hierarchy equations, and once the moments entering in the source function are computed the higher moments are obtained from the integral solution. Thus the integral solution should in fact be more advantageous than solving the system of coupled equations (5.18).

6. The Boltzmann equation for baryons and CDM

In this section we will derive the Boltzmann equation for massive particles, which is the case of interest for baryons and dark matter. These equations are necessary for finding the time evolution of number densities and velocities of the baryon fluid which appear in the brightness equation, thus allowing one to close the system of equations. Let us start from the baryon component. Electrons are tightly coupled to protons via Coulomb interactions. This forces the relative energy density contrasts and the velocities to a common value, \( \delta_e = \delta_p \equiv \delta_b \) and \( v_e = v_p \equiv v \), so we can identify electrons and protons collectively as ‘baryonic’ matter.

To derive the Boltzmann equation for baryons let us first focus on the collisionless equation and compute therefore \( dg/d\eta \), where \( g \) is the distribution function for a massive species with mass \( m \). One of the differences with respect to photons is just that baryons are non-relativistic for the epochs of interest. Thus the first step is to generalize the formulae in section 2 up to equation (3.21) to the case of a massive particle. In this case one enforces the constraint \( Q^2 = g_{\mu\nu} Q^\mu Q^\nu = -m^2 \) and it also useful to use the particle energy
\[
E = \sqrt{q^2 + m^2},
\]
where $q$ is defined as in equation (3.3). Moreover in this case it is very convenient to take the distribution function as a function of the variables $q^i = qn^i$, the position $x^i$ and time $\eta$, without using the explicit splitting into the magnitude of the momentum $q$ (or the energy $E$) and its direction $n^i$. Thus the total time derivative of the distribution functions reads

$$\frac{dg}{d\eta} = \frac{\partial g}{\partial \eta} + \frac{\partial g}{\partial x^i} \frac{dx^i}{d\eta} + \frac{\partial g}{\partial q^i} \frac{dq^i}{d\eta}. \quad (6.2)$$

We will not give the details of the calculation since we just need to replicate the same computation as we did for the photons. For the 4-momentum of the particle notice that $Q^i$ has the same form as equation (3.6), while for $Q^0$ we find

$$Q^0 = \frac{e^{-\Phi}}{a} E \left(1 + \omega^i \frac{q^i}{E}\right). \quad (6.3)$$

In the following we give the expressions for $dx^i/d\eta$ and $dq^i/d\eta$.

(a) As in equation (3.10), $dx^i/d\eta = Q^i/Q^0$ and it turns out to be

$$\frac{dx^i}{d\eta} = \frac{q^i}{E} n^i e^{\Phi+\Psi} \left(1 - \omega^i n^i \frac{q}{E}\right) \left(1 - \frac{1}{2} \chi_{km} n^k n^m\right). \quad (6.4)$$

(b) For $dq^i/d\eta$ we need the expression for $Q^i$ which is the same as that of equation (3.6):

$$Q^i = \frac{q^i}{a} e^{\Psi} \left(1 - \frac{1}{2} \chi_{km} n^k n^m\right). \quad (6.5)$$

The spatial component of the geodesic equation up to second order reads

$$\frac{dQ^i}{d\eta} = -2(\mathcal{H} - \Psi') \left(1 - \frac{1}{2} \chi_{km} n^k n^m\right) \frac{q^i a}{E} n^i e^{\Psi} + 2 \frac{\partial \Psi}{\partial x^k} \frac{q^2}{a E} n^i n^k e^{\Phi+2\Psi}
- \frac{\partial \Phi}{\partial x^i} \frac{E}{a} e^{\Phi+2\Psi} - \frac{\partial \Psi}{\partial x^i} \frac{q^2}{a E} e^{\Phi+2\Psi} - (\omega^i + \mathcal{H} \omega^i) \frac{E}{a}
- (\chi^i_{,k} + \omega^i_k - \omega_k^i) \frac{E}{a} + \left[\mathcal{H} \omega^i \delta_{jk} - (\chi^i_{,jk} + \chi^i_{k,j} + \chi^i_{jk})\right] \frac{q^j q^k}{E a}. \quad (6.6)$$

Proceeding as in the massless case we now take the total time derivative of equation (6.5) and using equation (6.6) we find

$$\frac{dq^i}{d\eta} = -(\mathcal{H} - \Psi') q^i + \Psi^k q^i q^k \frac{E}{a} e^{\Phi+\Psi} - \Phi^i E e^{\Phi+\Psi}
- \Psi^i q^2 \frac{E}{a} e^{\Phi+\Psi} - E (\omega^i + \mathcal{H} \omega^i) - (\chi^i_{,k} + \omega^i_k - \omega_k^i) E
+ \left[\mathcal{H} \omega^i \delta_{jk} - (\chi^i_{,jk} + \chi^i_{k,j} + \chi^i_{jk})\right] \frac{q^j q^k}{E}. \quad (6.7)$$
We can now write the total time derivative of the distribution function as
\[
\frac{dg}{d\eta} = \frac{\partial g}{\partial \eta} + \frac{q}{E} n^i e^{\Phi+\Psi} \left( 1 - \omega_i n^i - \frac{1}{2} \chi_{km} n^k n^m \right) \frac{\partial g}{\partial x^i} \\
+ \left[ -\left( \mathcal{H} - \Psi' \right) q^i + \Psi_k q^k \left( \frac{q^i q^k}{E} e^{\Phi+\Psi} - \Phi^i E e^{\Phi+\Psi} - \Psi_i q^2 E e^{\Phi+\Psi} \right) \right] \frac{\partial g}{\partial q^i} \\
- E(\omega' + \mathcal{H} \omega^i) \left( \chi_{k,k} - \omega_{k}^i \right) \frac{\partial g}{\partial q^i} \\
+ \left( \mathcal{H} \omega^i \delta_{jk} - \left( \chi_{j,k}^i + \chi_{k,j}^i + \chi_{j,k}^i \right) \right) \frac{q^i q^k}{E} \frac{\partial g}{\partial q^i}. \tag{6.8}
\]

This equation is completely general since we have just solved for the kinematics of massive particles. As far as the collision terms are concerned, for the system of electrons and protons we consider the processes of Coulomb scattering between the electrons and protons and the Compton scattering between photons and electrons:
\[
\frac{dg_e}{d\eta}(x, q, \eta) = \langle c_{ep} \rangle_{qq'} q q' + \langle c_{e\gamma} \rangle_{pp'} p p' \tag{6.9}
\]
\[
\frac{dg_p}{d\eta}(x, Q, \eta) = \langle c_{ep} \rangle_{qq} q q, \tag{6.10}
\]
where we have adopted the same formalism as [28] with \( p \) and \( p' \) the initial and final momenta of the photons, \( q \) and \( q' \) the corresponding quantities for the electrons and for protons \( Q \) and \( Q' \). The integral over different momenta is indicated by
\[
\langle \cdots \rangle_{pp'} \equiv \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 p'}{(2\pi)^3} \int \frac{d^3 q'}{(2\pi)^3} \cdots, \tag{6.11}
\]
and thus one can read \( c_{e\gamma} \) as the unintegrated part of equation (4.2), and similarly for \( c_{ep} \) (with the appropriate amplitude \( |M|^2 \)). In equation (6.9) Compton scatterings between protons and photons can be safely neglected because this process has a much smaller amplitude than Compton scatterings, with electrons being weighted by the inverse squared mass of the particles.

At this point for the photons we considered the perturbations around the zero-order Bose–Einstein distribution function (which are the unknown quantities). For the electrons (and protons) we can take the thermal distribution described by equation (4.3). Moreover we will take the moments of equations (6.9) and (6.10) in order to find the energy–momentum continuity equations.

### 6.1. The energy continuity equation

We now integrate equation (6.8) over \( d^3 q/(2\pi)^3 \). Let us recall that in terms of the distribution function the number density \( n_e \) and the bulk velocity \( v \) are given by
\[
n_e = \int \frac{d^3 q}{(2\pi)^3} g, \tag{6.12}
\]
and
\[
v^i = \frac{1}{n_e} \int \frac{d^3 q}{(2\pi)^3} g \frac{q n^i}{E}, \tag{6.13}
\]
where one can set $E \simeq m_e$ since we are considering non-relativistic particles. We will also make use of the following relations when integrating over the solid angle $d\Omega$:

$$
\int d\Omega n^i = \int d\Omega n^i n^k = 0, \quad \int \frac{d\Omega}{4\pi} n^i n^j = \frac{1}{3} \delta^{ij}.
$$

Finally notice that $dE/dq = q/E$ and $\partial g/\partial q = (q/E) \partial g/\partial E$.

Thus the first two integrals just provide $n_e$ and $(n_e v')_i$. Notice that all the terms proportional to the second-order vector and tensor perturbations of the metric give a vanishing contribution at second order since in this case we can take the zero-order distribution function which depends only on $\eta$ and $E$, integrate over the direction and use the fact that $\delta^{ij} \chi_{ij} = 0$. The trick for solving the remaining integrals is an integration by parts over $q^i$. We have an integral like (the one multiplying $(\Psi^i - \mathcal{H})$)

$$
\int \frac{d^3q}{(2\pi)^3} q^i \frac{\partial g}{\partial q^i} = -3 \int \frac{d^3q}{(2\pi)^3} \frac{q^i}{E} \frac{\partial g}{\partial q^i} = -3 n_e,
$$

after an integration by parts over $q^i$. The remaining integrals can be solved still by integrating by parts over $q^i$. The integral proportional to $\Phi^i$ in equation (6.8) gives

$$
e^{\Phi+\Psi \Phi^i} \int \frac{d^3q}{(2\pi)^3} \frac{q^i}{E} \frac{\partial g}{\partial q^i} = e^{\Phi+\Psi \Phi^i} v_i,
$$

where we have used that $dE/dq^i = q^i/E$. For the integral

$$
e^{\Phi+\Psi \Psi_k} \int \frac{d^3q}{(2\pi)^3} \frac{q^i q^k}{E} \frac{\partial g}{\partial q^i} = e^{\Phi+\Psi \Psi_k} v_i v_k,
$$

the integration by parts provides two pieces, one from the derivation of $q^i q^k$ and one from the derivation of the energy $E$:

$$
-4 e^{\Phi+\Psi \Psi_k} \int \frac{d^3q}{(2\pi)^3} \frac{q^i q^k}{E} + e^{\Phi+\Psi \Psi_k} \int \frac{d^3q}{(2\pi)^3} \frac{q^i}{E} \frac{q^k}{E} = -4 e^{\Phi+\Psi \Psi_k} v_i v_k
$$

$$
+ e^{\Phi+\Psi \Psi_k} \int \frac{d^3q}{(2\pi)^3} \frac{q^i q^k}{E} \frac{\partial g}{\partial q^i}.
$$

The last integral in equation (6.18) can indeed be neglected. To check this, one makes use of the explicit expression (4.3) for the distribution function $g$ to derive

$$
\frac{\partial g}{\partial v^i} = g \frac{q_i}{T_e} - \frac{n_e}{T_e} v_i q,
$$

and

$$
\int \frac{d^3q}{(2\pi)^3} q^i q^j = \delta^{ij} n_e m_e T_e + n_e m_e^2 v^i v^j.
$$

Thus it is easy to compute

$$
e^{\Phi+\Psi \Psi_k} \int \frac{d^3q}{(2\pi)^3} g q^k = -e^{\Phi+\Psi \Psi_k} v_i v_k \frac{T_e}{m_e} + 3 e^{\Phi+\Psi \Psi_k} v_i v_k n_e \frac{T_e}{m_e} + e^{\Phi+\Psi \Psi_k} v_k v^2,
$$

which is negligible, taking into account that $T_e/m_e$ is of the order of a thermal velocity squared.
With these results we are able to compute the left-hand side of the Boltzmann equation (6.9) integrated over \(d^3q/(2\pi)^3\). The same operation must be done for the collision terms on the right-hand side. For example for the first of the equations in (6.9) this provides the integrals \(\langle c_{\nu p} \rangle Q_i q_{\nu} q_{\nu} + \langle c_{\nu p} \rangle p_{\nu} q_{\nu}\). However, looking at equation (4.2) one realizes that \(\langle c_{\nu p} \rangle p_{\nu} q_{\nu}\) vanishes because the integrand is antisymmetric under the change \(q \leftrightarrow q'\) and \(p \leftrightarrow p'\). In fact this is simply a consequence of the fact that the electron number is conserved for this process. The same argument holds for the other term \(\langle c_{\nu p} \rangle Q_i q_{\nu} q_{\nu}\). Therefore the right-hand side of equation (6.9) integrated over \(d^3q/(2\pi)^3\) vanishes and we can give the evolution equation for \(n_e\). Collecting the results from equation (6.15) to (6.21) we find

\[
\frac{\partial n_e}{\partial \eta} + e^{\Phi + \Psi} \frac{\partial (v'n_e)}{\partial x^i} + 3(H - \Psi)n_e - 2e^{\Phi + \Psi} \Psi K v^k + e^{\Phi + \Psi} \Phi K v^k = 0. \tag{6.22}
\]

Similarly, for CDM particles, we find

\[
\frac{\partial n_{CDM}}{\partial \eta} + e^{\Phi + \Psi} \frac{\partial (v'n_{CDM})}{\partial x^i} + 3(H - \Psi)n_{CDM} - 2e^{\Phi + \Psi} \Psi K_{CDM} v^k + e^{\Phi + \Psi} \Phi K_{CDM} v^k = 0. \tag{6.23}
\]

### 6.2. The momentum continuity equation

Let us now multiply equation (6.8) by \((q'/E)/(2\pi)^3\) and integrate over \(d^3q\). In this way we will find the continuity equation for the momentum of baryons. The first term just gives \((n_e v')\). The second integral is of the type

\[
\frac{\partial}{\partial x^i} \int \frac{d^3q}{(2\pi)^3} q_i q_i q_i = \frac{\partial}{\partial x^j} \left( n_e T_e m_e \delta^{ij} + n_e v^i v^j \right), \tag{6.24}
\]

where we have used equation (6.20) and \(E = m_e\). The third term proportional to \((H - \Psi)\) is

\[
\int \frac{d^3q}{(2\pi)^3} q_i q_i \frac{\partial q}{\partial q_k} = 4n_e + \int \frac{d^3q}{(2\pi)^3} \frac{q^2}{E^2} q^i, \tag{6.25}
\]

where we have integrated by parts over \(q^i\). Notice that the last term in equation (6.25) is negligible, being the same integral as we discussed above, from equation (6.21). By the same arguments as led us to neglect the term of equation (6.21) it is easy to check that all the remaining integrals proportional to the gravitational potentials are negligible except for

\[
-e^{\Phi + \Psi} \Phi_k \int \frac{d^3q}{(2\pi)^3} \frac{\partial q}{\partial q_k} q^i = n_e e^{\Phi + \Psi} \Phi^i. \tag{6.26}
\]

The integrals proportional to the second-order vector and tensor perturbations vanish, as vector and tensor perturbations are traceless and divergence free. The only one which survives is the term proportional to \(\omega^v + \omega^\nu\) in equation (6.8).

Therefore for the integral over \(d^3q q_i/E\) of the left-hand side of the Boltzmann equation (6.8) for a massive particle with mass \(m_e\) (\(m_p\)) and distribution function (4.3),
we find
\[
\int \frac{d^3q}{(2\pi)^3} q_i \frac{dg_e}{d\eta} = \frac{\partial(n_e v^i)}{\partial \eta} + 4(\mathcal{H} - \Psi)n_e v^i + \Phi^i e^{\Phi + \Psi} n_e \\
+ e^{\Phi + \Psi} \left( \frac{T_e}{m_e} \right)^i + e^{\Phi + \Psi} \frac{\partial}{\partial x^j} (n_e v^j v^i) + \frac{\partial \omega^i}{\partial \eta} n_e + \mathcal{H} \omega^i n_e.
\]  
(6.27)

Now, in order to derive the momentum conservation equation for baryons, we take the first moments of both equations (6.9) and (6.10), multiplying them by \( q \) and \( Q \) respectively and integrating over the momenta. Since previously we integrated the left-hand side of these equations over \( d^3q^j / E \), we just need to multiply the previous integrals by \( m_e \) for the electrons and for \( m_p \) for the protons. Therefore if we sum the first moments of equations (6.9) and (6.10), the dominant contribution on the left-hand side will be that of the protons:
\[
\int \frac{d^3Q}{(2\pi)^3} Q_i \frac{dg_p}{d\eta} = \langle c_{ep}(q^i + Q^i) \rangle_{QQ'qq'} + \langle c_{ep}q^i \rangle_{pp'qq'}.
\]  
(6.28)

Notice that the integral of the Coulomb collision term \( c_{ep}(q^i + Q^i) \) over all momenta vanishes simply because of momentum conservation (due to the Dirac function \( \delta^i(q + Q - q' - Q') \)). As far as the Compton scattering is concerned we have that, following [28],
\[
\langle c_{ep}q^i \rangle_{pp'qq'} = -\langle c_{ep}p^i \rangle_{pp'qq'},
\]  
(6.29)

still because of the total momentum conservation. Therefore we can compute now is the integral over all momenta of \( c_{ep}p^i \). Notice however that this is equivalent just to multiplying the Compton collision term \( C(p) \) of equation (4.2) by \( p^i \) and integrating over \( d^3p/(2\pi)^3 \):
\[
\langle c_{ep}p^i \rangle_{pp'qq'} = \int \frac{d^3p}{(2\pi)^3} p^i C(p)
\]  
(6.30)

where \( C(p) \) has been already computed in equations (4.41) and (4.42).

We will do the integration (6.30) in the following. First let us introduce the definition of the velocity of photons in terms of the distribution function
\[
(\rho_\gamma + p_\gamma) v^i_\gamma = \int \frac{d^3p}{(2\pi)^3} fp^i,
\]  
(6.31)

where \( p_\gamma = \rho_\gamma / 3 \) is the photon pressure and \( \rho_\gamma \) the energy density. At first order we get
\[
\frac{4}{3} v_\gamma^{(1)i} = \int \frac{d\Omega}{4\pi} \Delta^{(1)} n^i,
\]  
(6.32)

where \( \Delta \) is the photon distribution anisotropies defined in equation (5.10). At second order we find instead
\[
\frac{4}{3} v_\gamma^{(2)i} = \frac{1}{2} \int \frac{d\Omega}{4\pi} \Delta^{(2)} n^i - \frac{4}{3} \delta_\gamma^{(1)i} v_\gamma^{(1)i}.
\]  
(6.33)

Therefore the terms in equations (4.41) and (4.42) proportional to \( f^{(1)}(p) \) and \( f^{(2)}(p) \) will give rise to terms containing the velocity of the photons. On the other hand the terms proportional to \( f_0^{(1)}(p) \) and \( f_0^{(2)}(p) \), once integrated, vanish because of the
integral over the momentum direction \( n^i \), \( \int d\Omega n^i = 0 \). Also the integrals involving \( P_2(\mathbf{v} \cdot \mathbf{n}) = (3(\mathbf{v} \cdot \mathbf{n})^2 - 1)/2 \) in the first lines of equations (4.41) and (4.42) vanish since

\[
\int d\Omega P_2(\mathbf{v} \cdot \mathbf{n}) n^i = \hat{v}^k \hat{v}^j \int d\Omega n_k n_j n^i = 0, \tag{6.34}
\]

where we are using the relations (6.14). Similarly all the terms proportional to \( v \), \((\mathbf{v} \cdot \mathbf{n})^2\) and \( v^2 \) give no contributions to equation (6.30) and, in the second-order collision term, one can check that \( \int d\Omega Y_2(\mathbf{n}) n^i = 0 \). Then there are terms proportional to \((\mathbf{v} \cdot \mathbf{n}) p \partial f^{(0)} / \partial p\) and \((\mathbf{v} \cdot \mathbf{n}) p \partial f^{(1)} / \partial p\) for which we can use the rules (5.11) when integrating over \( p \) while the integration over the momentum direction is

\[
\int \frac{d\Omega}{4\pi} (\mathbf{v} \cdot \mathbf{n}) n^i = v_k \int \frac{d\Omega}{4\pi} n^k n^i = \frac{1}{3} v^i. \tag{6.35}
\]

Finally from the third and fourth lines of equation (4.42) we get three integrals. One is

\[
\int \frac{d^3p}{(2\pi)^3} p^i (v \cdot n) P_2(\mathbf{v} \cdot \mathbf{n}) \left( f_2^{(1)}(p) - p \frac{\partial f_2^{(1)}(p)}{\partial p} \right) = \frac{5}{4} \bar{\rho}_\gamma \Delta_2^{(1)} \left[ 3\hat{v}_k \hat{v}_l \int \frac{d\Omega}{4\pi} n^i n^j n^k n^l - v_j \int \frac{d\Omega}{4\pi} n^i n^j \right]
\]

\[
= \frac{1}{3} \bar{\rho}_\gamma \Delta_2^{(1)} \hat{v}^i, \tag{6.37}
\]

where we have used the rules (5.11), equation (6.14) and \( \int (d\Omega / 4\pi) n^i n^j n^k n^l = (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk})/15 \). In fact the third integral

\[
- \int \frac{d^3p}{(2\pi)^3} p^i (v \cdot n) f_2^{(1)}(p), \tag{6.38}
\]

exactly cancels the previous one. Summing the various integrals we find

\[
\int \frac{d\mathbf{p}}{(2\pi)^3} C(\mathbf{p}) \mathbf{p} = n_\gamma \sigma_T \bar{\rho}_\gamma \left[ \frac{4}{3} (v^{(1)} - v_\gamma^{(1)}) - \frac{\int d\Omega \Delta_2^{(2)}}{4\pi} + \frac{4}{3} v^{(2)} + \frac{4}{3} \delta_\gamma^{(1)} (v^{(1)} - v_\gamma^{(1)}) + \int \frac{d\Omega}{4\pi} \Delta_2^{(1)} (v \cdot n) n + \Delta_0^{(1)} v \right]. \tag{6.39}
\]

Equation (6.39) can be further simplified. Recalling that \( \delta_\gamma^{(1)} = \Delta_0^{(1)} \) we use equation (6.33) and notice that

\[
\int \frac{d\Omega}{4\pi} \Delta^{(1)} (v \cdot n) n^i = v_j^{(1)} \Pi_j^{ij} + \frac{1}{3} v^i \Delta_0^{(1)}, \tag{6.40}
\]

where \( \Pi_j^{ij} \) is the quantity defined in equation (7.5).
Thus our final expression for the integrated collision term (6.30) reads

\[
\int \frac{d^3p}{(2\pi)^3} C(p)p^i = n_e \sigma_T \bar{\rho}_\gamma \left[ \frac{4}{3} \left( v^{(1)i} - v^{(1)\gamma}_i \right) + \frac{4}{3} \left( \frac{v^{(2)i}}{2} - \frac{v^{(2)\gamma}_i}{2} \right) \right. \\
\left. + 4 \frac{\delta^1_\gamma}{3} \left( v^{(1)i}_\gamma - v^{(1)\gamma}_i \right) + v^{(1)\gamma}_j \Pi^{ji}_\gamma \right].
\]  

(6.41)

We are now able to give the momentum continuity equation for baryons by combining \( m_p \frac{dg_p}{d\eta} \) from equation (6.27) with the collision term (6.41)

\[
\frac{\partial (\rho_b v^i)}{\partial \eta} + 4 (\mathcal{H} - \Psi^i) \rho_b v^i + \Phi^i e^{\Phi + \Psi} \rho_b + e^{\Phi + \Psi} \left( \rho_b \frac{T_b}{m_p} \right)^j = - n_e \sigma_T a \bar{\rho}_\gamma \left[ \frac{4}{3} \left( v^{(1)i}_\gamma - v^{(1)\gamma}_i \right) + \frac{4}{3} \left( \frac{v^{(2)i}}{2} - \frac{v^{(2)\gamma}_i}{2} \right) \right. \\
\left. + 4 \frac{\delta^1_\gamma}{3} \left( v^{(1)i}_\gamma - v^{(1)\gamma}_i \right) + v^{(1)\gamma}_j \Pi^{ji}_\gamma \right],
\]  

(6.42)

where \( \rho_b \) is the baryon energy density and, as we previously explained, we took into account that to a good approximation the electrons do not contribute to the mass of baryons. In the following we will expand explicitly at first and second order equation (6.42).

**The first-order momentum continuity equation for baryons**

At first order we find

\[
\frac{\partial v^{(1)i}}{\partial \eta} + \mathcal{H} v^{(1)i} + \Phi^{(1)i} = 4 \tau' \bar{\rho}_\gamma \left( v^{(1)i}_\gamma - v^{(1)\gamma}_i \right).
\]  

(6.43)

**The second-order momentum continuity equation for baryons**

At second order there are various simplifications. In particular, notice that the term on the right-hand side of equation (6.42) which is proportional to \( \delta_b \) vanishes when matched to the expansion of the left-hand side by virtue of the first-order equation (6.45). Thus at the end we find a very simple equation:

\[
\frac{1}{2} \left( \frac{\partial v^{(2)i}}{\partial \eta} + \mathcal{H} v^{(2)i} + 2 \frac{\partial \omega^j}{\partial \eta} + 2 \mathcal{H} \omega_i + \Phi^{(2)i} \right) \\
- \frac{\partial \Phi^{(1)}}{\partial \eta} v^{(1)i} + \frac{1}{2} \frac{\partial v^{(1)i}}{\partial \eta} \frac{\partial v^{(1)i}}{\partial \eta} + (\Phi^{(1)} + \Psi^{(1)}) \Phi^{(1)i} + \left( \rho_b \frac{T_b}{m_p} \right)^j = 4 \tau' \bar{\rho}_\gamma \left[ \frac{v^{(2)i}}{2} - \frac{v^{(2)\gamma}_i}{2} \right] + \frac{3}{4} v^{(2)i}_\gamma \Pi^{ji}_\gamma 
\]  

(6.44)

with \( \tau' = - n_e \sigma_T a \).
The first-order momentum continuity equation for CDM

Since CDM particles are collisionless, at first order we find

\[
\frac{\partial v_{\text{CDM}}^{(1)i}}{\partial \eta} + \mathcal{H} v_{\text{CDM}}^{(1)i} + \Phi^{(1),i} = 0. \tag{6.45}
\]

The second-order momentum continuity equation for CDM

At second order we find

\[
\frac{1}{2} \left( \frac{\partial v_{\text{CDM}}^{(2)ij}}{\partial \eta} + \mathcal{H} v_{\text{CDM}}^{(2)ij} + 2 \frac{\partial \omega^j}{\partial \eta} + 2 \mathcal{H} \omega^j + \Phi^{(2),ij} \right) - \frac{\partial \Psi^{(1)i}}{\partial \eta} v_{\text{CDM}}^{(1)j} + v_{\text{CDM}}^{(1)i} \partial_j v_{\text{CDM}}^{(1)j} + (\Phi^{(1)} + \Psi^{(1)}) \Phi^{(1),i} + \left( \frac{T_{\text{CDM}}}{m_{\text{CDM}}} \right)^i = 0. \tag{6.46}
\]

7. Energy–momentum tensors

In this section we provide the expressions for the energy–momentum tensors for photons and massive particles in terms of their distribution functions.

7.1. Energy–momentum tensors for photons

The energy–momentum tensor for photons is defined as

\[
T_{\gamma \nu}^\mu = \frac{2}{\sqrt{-g}} \int \frac{d^3 p}{(2\pi)^3} \frac{P^\mu}{P^0} P_\nu \tilde{f}, \tag{7.1}
\]

where \( g \) is the determinant of the metric (2.1) and \( f \) is the distribution function. We thus obtain

\[
T_{\gamma 0}^0 = -\bar{\rho}_\gamma \left( 1 + \Delta^{(1)}_{00} + \frac{\Delta^{(2)}_{00}}{2} \right), \tag{7.2}
\]

\[
T_{\gamma i}^0 = -\frac{4}{3} \bar{\rho}_\gamma \Phi \gamma (v_{\gamma}^{(1)i} + \frac{1}{2} v_{\gamma}^{(2)ii} + \Delta^{(1)}_{00} v_{\gamma}^{(1)i}) + \frac{1}{3} \bar{\rho}_\gamma e^{\Psi - \Phi} \omega^i, \tag{7.3}
\]

\[
T_{\gamma j}^i = \bar{\rho}_\gamma \left( \Pi_{\gamma j}^i + \frac{1}{3} \delta_{\gamma j} \left( \Delta^{(1)}_{00} + \frac{\Delta^{(2)}_{00}}{2} \right) \right), \tag{7.4}
\]

where \( \bar{\rho}_\gamma \) is the background energy density of photons and

\[
\Pi_{\gamma j}^i = \int \frac{d^3 \Omega}{4\pi} \left( n^i n^j - \frac{1}{3} \delta_{ij} \right) \left( \Delta^{(1)} + \frac{\Delta^{(2)}}{2} \right), \tag{7.5}
\]

are the quadrupole moments of the photons. The photon velocity is defined in equations (6.31)–(6.33).
7.2. Energy–momentum tensors for massive particles

The energy–momentum tensor for massive particles of mass $m$, number density $n$ and degrees of freedom $g_{d}$ is

$$
T_{\mu\nu} = g_{d} \frac{1}{\sqrt{-g}} \int \frac{d^{3}Q}{(2\pi)^{3}} \frac{Q_{\mu}Q_{\nu}}{Q^{0}} g(q^{i}, x^{\mu}, \eta),
$$

(7.6)

where $g$ is the distribution function. For electrons (or protons) $g_{d} = 2$ (we are not counting antiparticles). We obtain

$$
T_{0}^{0} = -\rho_{m} = -\bar{\rho}_{m}(1 + \delta_{m}^{(1)} + \frac{1}{2}\delta_{m}^{(2)}),
$$

(7.7)

$$
T_{0}^{i} = -e^{\Psi+\Phi} \rho_{m} v_{m}^{i} = -e^{\Psi+\Phi} \bar{\rho}_{m}(v_{m}^{(1)i} + \frac{1}{2}v_{m}^{(2)i} + \delta_{m}^{(1)}v_{m}^{(1)i}),
$$

(7.8)

$$
T_{i}^{j} = \rho_{m} \left( \delta_{i}^{j} \frac{T_{m}^{m}}{m} + v_{m}^{i}v_{m}^{j} \right) = \bar{\rho}_{m} \left( \delta_{i}^{j} \frac{T_{m}^{m}}{m} + v_{m}^{(1)i}v_{m}^{(1)j} \right)
$$

(7.9)

where $\bar{\rho}_{m}$ is the background energy density of the massive particles and we have included the equilibrium temperature $T_{m}$. The velocities are defined in terms of the distribution function in equation (6.13).

8. Summary

In this paper we took the first step towards the evaluation of the full radiation transfer at second order in perturbation theory by computing the second-order Boltzmann equations describing the evolution of the baryon–photon fluid. They allow us to follow the time evolution of the CMB anisotropies at second order and at all scales from the early epoch, when the cosmological perturbations were generated, to the present, through the recombination era. The dynamics at the second order is particularly important when dealing with the issue of NG in the CMB anisotropies: many mechanisms for the generation of the primordial inhomogeneities predict a level of NG in the curvature perturbation which might be detectable by present and future experiments. Having an accurate theoretical prediction of the CMB anisotropy NG in terms of the primordial non-Gaussian seeds is somewhat mandatory. This paper will be followed by a companion one where we will present the computation of the three-point correlation function at recombination, making use of the set of Boltzmann equations derived here.

Acknowledgments

AR is on leave of absence from INFN, Padova. NB is partially supported by INFN.
Appendix

Table A.1. Definition of symbols used.

| Symbol     | Definition                                                                 | Equation |
|------------|----------------------------------------------------------------------------|----------|
| $\Phi$, $\Psi$ | Gravitational potentials in Poisson gauge                                  | (2.1)    |
| $\omega_i$   | Second-order vector perturbation in Poisson gauge                          | (2.1)    |
| $\chi_{ij}$  | Second-order tensor perturbation in Poisson gauge                          | (2.1)    |
| $\eta$       | Conformal time                                                             | (2.1)    |
| $f$           | Photon distribution function                                               | (3.8)    |
| $g$           | Distribution function for massive particles                                | (4.3) and (6.2)|
| $f^{(i)}$     | $i$th-order perturbation of photon distribution function                   | (3.23)   |
| $f_{\ell m}$ | Moments of photon distribution function                                     | (4.17)   |
| $C(p)$       | Collision term                                                             | (4.2) and (4.6)|
| $p$           | Magnitude of photon momentum ($p = p n^i$)                                | (3.3)    |
| $n^i$         | Propagation direction                                                     | (3.6)    |
| $\Delta^{(1)}(x^i, n^i, \eta)$ | First-order fractional energy photon fluctuations                        | (5.5)    |
| $\Delta^{(2)}(x^i, n^i, \eta)$ | Second-order fractional energy photon fluctuations                      | (5.10)   |
| $n_e$         | Electron number density                                                    | (6.12)   |
| $\delta_e$, $\delta_b$ | Electron (baryon) density perturbation                                    | (4.13)   |
| $k$           | Wavenumber                                                                | (5.17)   |
| $v_n$         | Baryon velocity perturbation                                              | (5.21) and (5.22)|
| $v^{(2)}_n$   | Cold dark matter velocity                                                  | (6.46)   |
| $v^{(2)}_{CDM}$ | Second-order dark matter velocity                                       | (6.33)   |
| $S_{\ell m}$ | Temperature source term                                                   | (5.18)   |
| $\tau$        | Optical depth                                                             | (5.15)   |
| $\bar{\rho}_\gamma$, $\bar{\rho}_b$ | Background photon (baryon) energy density                              | (6.45)   |

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