A “RARE” PLANE SET WITH HAUSDORFF DIMENSION 2

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Abstract. We prove that for every at most countable family \( \{ f_k(x) \} \) of real functions on \([0, 1)\) there is a single-valued real function \( F(x) \), \( x \in [0, 1) \), such that the Hausdorff dimension of the graph \( \Gamma \) of \( F(x) \) equals 2, and for every \( C \in \mathbb{R} \) and every \( k \), the intersection of \( \Gamma \) with the graph of the function \( f_k(x) + C \) consists of at most one point.

1. Introduction

The motivation of this note comes from the following question by Sergei Treil (August 2018, private communication). Let \( E \) be a set in \( \mathbb{R}^n \), and let \( K \) be an \( n \)-dimensional cone in \( \mathbb{R}^n \). Suppose that for every line \( l \) in \( K \) and for every vector \( b \), the intersection \( E \cap (l + b) \) is at most countable. Does it follow that the Hausdorff dimension of \( E \) is less than \( n \)?

We consider the case \( n = 2 \) and try to approach the question from the other end: for which sets \( K \) of directions (not necessarily \( n \)-dimensional) is the answer negative? The case when \( K \) consists of only one direction is known—see for example [1]. Namely, there exists a function \( F(x) \) (which can even be continuous!) whose graph has Hausdorff dimension 2. So, intersection of the graph of \( F(x) \) with every vertical line consists of at most one point.

We show that the answer to Treil’s question is negative in the case of any countable set of directions. In fact we prove a much more general assertion.

Theorem. For every at most countable family \( \mathcal{F} \) of real functions on \([0, 1)\) there is a (single-valued) function \( F(x) \), \( x \in [0, 1) \), such that

(i) the Hausdorff dimension of the graph \( \Gamma \) of \( F(x) \) equals 2;

(ii) the intersection of \( \Gamma \) with the graph of any function \( f_k(x) + C \), where \( f_k(x) \in \mathcal{F} \), \( C \in \mathbb{R} \), consists of at most one point.

Recall that the Hausdorff measure \( H^s(E) \), \( s \geq 0 \), and the Hausdorff dimension \( \dim_H(E) \) of a set \( E \) are defined by the equalities

\[
H^s(E) = \lim \inf_{\delta \to 0} \sum_{r_i < \delta} r_i^s.
\]

\[
\dim_H(E) = \sup \{ s : H^s(E) = \infty \} = \inf \{ s : H^s(E) = 0 \},
\]

where the \( \inf_{r_i < \delta} \) is taken over all at most countable covers of \( E \) by disks with radii \( r_i < \delta \).

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T. Keleti [2] constructed a compact subset of $\mathbb{R}$ with Hausdorff dimension 1 that intersects each of its non-identical translates in at most one point. This result and our theorem have a similar flavor, but their proofs are completely different.

2. Proof of the Theorem

Every $x \in \mathbb{R}$ can be written in the form

$$x = [x] + \{x\} = \lfloor x \rfloor + \sum_{i=1}^{\infty} x_i 2^{-i},$$

where $\lfloor x \rfloor$, $\{x\}$ are the integer and the fractional parts of $x$ correspondingly, and each $x_i$ is either 0 or 1. We write $(0100 \cdots 0 \cdots)$ instead of $(0011 \cdots 1 \cdots)$. In other words, the binary expansion of every number $\{x\}$ in $[0, 1)$ contains infinitely many zeros. Such a representation is unique.

We partition the set $\mathbb{N}$ of positive integers into a set $T$ and a collection of 3-element sets $S_{ij}$ indexed by ordered pairs $(i, j)$ of positive integers in such a way that the following statements hold:

1. The density of $T$ in the positive integers is 1.
2. Each $S_{ij}$ is of the form $\{s_{ij}, s_{ij} + 1, s_{ij} + 2\}$ for some positive integer $s_{ij}$.
3. All sets $S_{ij}$ and $T$ are mutually disjoint.

If $x = \sum_{i} x_i 2^{-i} \in [0, 1)$ and $s$ is a positive integer, we define

$$g_s(x) := x_s 2^{-s} + x_{s+1} 2^{-s-1}.$$

We extend $g_s(x)$ to a function on $\mathbb{R}$ by imposing periodicity: $g_s(x+1) = g_s(x)$. In other words, we set $g(x) := g(\{x\})$.

**Lemma 2.1.** Let $s$ be a positive integer, $U$ a subset of the positive integers which is disjoint from $\{s, s+1, s+2\}$, and $a \in \mathbb{R}$. Let

$$A := g_s(a) + \sum_{i \in U} 2^{-i} - a; \quad B := g_s(2^{-s} + a) + \sum_{i \in U} 2^{-i} - a.$$

Then

$$\{2^{s-1} A\} \in [0, 1/8] \cup [3/4, 1); \quad \{2^{s-1} B\} \in [1/4, 5/8].$$

**Proof.** We partition $U$ into $U^+ := U \cap [1, s-1]$ and $U^- := U \cap [s+3, \infty)$. Thus

$$\sum_{i \in U} 2^{-i} = \sum_{i \in U^+} 2^{-i} + \sum_{i \in U^-} 2^{-i} = \frac{m}{2^{s-1}} + \delta$$

for some $m \in \mathbb{Z}$, $\delta \in [0, 2^{-s-2}]$. As $\{2^{s-1} A\}$ and $\{2^{s-1} B\}$ only depend on $\{a\}$ and $U^-$, we may assume $a \in [0, 1)$. We can therefore write

$$a - g_s(a) = \sum_{i \in A^+} 2^{-i} + \sum_{i \in A^-} 2^{-i},$$

where $A^+ \subset [1, s-1]$ and $A^- \subset [s+2, \infty)$ are sets of integers. Thus,

$$a - g_s(a) = \frac{n}{2^{s-1}} + \varepsilon.$$
for some $n \in \mathbb{Z}, \varepsilon \in [0, 2^{-s-1}]$. It follows that 
\[ 2^{s-1}A = m - n + 2^{s-1}(\delta - \varepsilon) \in [m - n - 1/4, m - n + 1/8]. \]
Likewise,
\[ a - g_{s}2^{-s} + a = (2^{-s} + a) - g_{s}(2^{-s} + a) - 2^{-s} = \frac{n - 1/2}{2^{s-1}} + \varepsilon \]
for some integer $n$ and $\varepsilon \in [0, 2^{-s-1}]$, so
\[ 2^{s-1}B = m - n + 1/2 + 2^{s-1}(\delta - \varepsilon) \in [m - n + 1/4, m - n + 5/8]. \]
Lemma 2.1 is proved. \(\square\)

For positive integers $i$ and $j$, we define 
\[ h_{ij}(x) = g_{s_{ij}}(f_{i}(x) + x_{j}2^{-s_{ij}}), \quad f_{i} \in \mathcal{F}. \]
Define $F(x), x \in [0, 1)$, by the equality
\[ F(x) = F\left(\sum_{i=1}^{\infty} x_{i}2^{-i}\right) = \sum_{i \in T} x_{i}2^{-i} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} h_{ij}(x). \]  

(2.1)

**Lemma 2.2.** The function $F(x) - f_{i}(x)$ is one-to-one on $[0, 1)$ for every $f_{i} \in \mathcal{F}$.

*Proof.* Fix $i$, $j$, and $x \in [0, 1)$, and observe that $h_{ij}(x)$ is a sum of $2^{-k}$ as $k$ ranges over some subset of $\{s_{ij}, s_{ij} + 1\}$, while $\sum_{k \in T} x_{k}2^{-k}$ and $\sum_{(k,l) \neq (i,j)} h_{kl}(x)$ are sums of $2^{-k}$ over some subsets of $T$ and of $\mathbb{N} \setminus T$ which are both disjoint from $S_{ij}$. Thus,
\[ F(x) - f_{i}(x) = \begin{cases} 
  g_{s_{ij}}(f_{i}(x)) + \sum_{i \in U} 2^{-i} - f_{i}(x), & \text{if } x_{j} = 0, \\
  g_{s_{ij}}(f_{i}(x) + 2^{-s_{ij}}) + \sum_{i \in U} 2^{-i} - f_{i}(x), & \text{if } x_{j} = 1,
\end{cases} \]
where $U$ is a set of positive integers which is disjoint from $S_{ij}$.

Choose $x \neq y$. There exists $j$ such that $x_{j} \neq y_{j}$. According to Lemma 2.1
\[ \{2^{s_{ij}-1}(F(x) - f_{i}(x))\} \neq \{2^{s_{ij}-1}(F(y) - f_{i}(y))\} \]
for every positive integer $i$. Hence, $F(x) - f_{i}(x) \neq F(y) - f_{i}(y)$. \(\square\)

Note that Lemma 2.2 implies (ii) of the main theorem.
The following lemma establishes the validity of (i).

**Lemma 2.3.** For every choice of functions $h_{ij}(x)$,
\[ \dim_{H}(\Gamma) = 2. \]

The proof of Lemma 2.3 is based on the following assertion.

**Lemma 2.4.** Let functions $h_{ij}(x)$ be fixed. For every $\alpha < 2$ and $\varepsilon > 0$ there exists $\delta = \delta(\alpha, \varepsilon) > 0$ with the following property. For every disk $D(r)$ with radius $r < \delta$, the length of the projection of $\Gamma \cap D(r)$ onto the $x$-axis is less than $\varepsilon r^{\alpha}$.

Let us show that Lemma 2.4 implies Lemma 2.3.
Proof of Lemma 2.3. Suppose that \( \dim_H(\Gamma) < 2 \). Choose \( \beta \) so that \( \dim_H(\Gamma) < \beta < 2 \). Let \( \delta := \delta(\beta, 1) \) be the number in Lemma 2.4. There is an at most countable family of disks \( D_i(r_i) \) such that \( r_i < \delta \), \( \Gamma \subset \bigcup_i D_i(r_i) \), and \( \sum_i r_i^\beta < 1 \). We have
\[
|\Pr(\Gamma)| \leq \sum_i |\Pr(\Gamma \cap D_i(r_i))| \leq \sum_i r_i^\beta < 1,
\]
where \( |\Pr(A)| \) denotes the length of the projection of a set \( A \) onto the \( x \)-axis. Since \( \Gamma \) projects onto the whole of \( [0, 1) \), this contradiction proves that \( \dim_H(\Gamma) = 2 \). □

Proof of Lemma 2.4. By (2.1), for every \( x \in [0, 1) \), \( y := F(x) \) can be written as \( \sum_{i \in U} 2^{-i} \) for some set \( U \) of positive integers which does not contain any integer of the form \( s_{ij} + 2 \) but does contain \( i \) whenever \( i \in T \) and \( x_i^2 = 1 \). Since there are infinitely many integers of the form \( s_{ij} + 2 \), we have \( y_i = 1 \) if and only if \( i \in U \). Hence, \( y_i = x_i^2 \) for \( i \in T \).

We may assume that \( \delta < 1/2 \). Let \( N \) be such that \( 2^{-N-1} \leq r < 2^{-N} \). Then a disk \( D(r) \) intersects at most nine dyadic squares with side length \( 2^{-N} \). Hence, it suffices to prove the existence of \( N_0 \) such that for every open dyadic square \( Q \) with side length less than \( 2^{-N} \), where \( N > N_0 \),
\[
|\Pr(Q \cap \Gamma)| < \varepsilon 2^{-N\alpha}, \quad N > N_0.
\]
Fix \( Q \). For all points \( (x, y) \in Q \), the first \( N \) digits \( x_1, \ldots, x_N \) in the binary representations of \( x \) are determined by \( Q \), and likewise for \( y_1, \ldots, y_N \). Let \( M = M(N) \) be the number of positive integers in the set \( [1, N] \cap T \). By (2.1), for all \( (x, y) \in \Gamma \) and \( n \in T \), we have \( x_n^2 = y_n \). Therefore, for all \( (x, y) \in \Gamma \cap Q \), \( x_m \) is constant for \( m \in [1, N] \) and also for \( m \in \{n^2 \mid n \in T \cap [1, N]\} \). The union of these two sets has at least \( M + N - \sqrt{N} \) elements. Since \( \lim_{N \to \infty} M/N = 1 \) and \( \alpha < 2 \), we have
\[
|\Pr(Q \cap \Gamma)| \leq 2^{-(N+M-\sqrt{N})} = 2^{-N(1+M/N-\alpha-\sqrt{N/N})} 2^{-N\alpha} < \varepsilon 2^{-N\alpha},
\]
if \( N \) is sufficiently large. Lemma 2.4 is proved □

References

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