A MINIMIZATION PROBLEM INVOLVING A FRACTIONAL HARDY-SOBOLEV TYPE INEQUALITY

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Abstract. In this work, we obtain existence results for a minimization problem involving a fractional Hardy-Sobolev type inequality. Precisely, let 0 < s < 1, n > 4s, 0 < α < 2s, and Ω ⊂ \mathbb{R}^n be a bounded domain. We find a critical values 0 < \lambda_* < \lambda^* such that for 0 < \lambda < \lambda_* there exists a solution to the problem
\[ \mu_{\alpha,\lambda}(\Omega) := \inf \left\{ [u]_{s,\Omega}^2 + \lambda \int_\Omega |u|^2 \, dx : u \in H^s(\Omega), \int_\Omega \frac{|u(x)|^{2s,\alpha}}{|x|^{\alpha}} \, dx = 1 \right\} \]
where \(2s,\alpha = \frac{2(n-\alpha)}{n-2s}\), and there is no solution for values \(\lambda > \lambda^*\). The crucial tools are the properties of \(\mu_{\alpha,\lambda}(\Omega)\) seen as a function in \(\lambda\), and a fractional Hardy-Sobolev type inequality. Both cases inner and boundary singularity are treated.

1. INTRODUCTION
1.1. Overview. Let 0 < s < 1, 0 < α < 2s and n > 2s. In [14], S. A. Marano and S. Mosconi prove the existence of an extremal function \(u_0\), solution to
\[ (1.1) \quad \mu_{\alpha} := \inf \left\{ [u]_{s}^2 : u \text{ measurable, vanishing at infinity}, \int_{\mathbb{R}^N} \frac{|u(x)|^{2s,\alpha}}{|x|^{\alpha}} \, dx = 1 \right\}, \]
where \(2_{s,\alpha} = \frac{2(n-\alpha)}{n-2s}\). See also [15]. Here, \(u\) vanishes at infinity means
\[ |\{ |u| > a \}| < \infty \text{ for every } a \in \mathbb{R}. \]
Observe that \(2_{s,0} = 2^* = \frac{2n}{n-2s}\), which is related to the non compact but continuous embedding \(H^s(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)\), and \(2_{s,2s} = 2\). The constant \(\mu_{2s}\) is calculated in [14],
\[ \mu_{2s} = \frac{2\pi \frac{n}{4} \Gamma \left( \frac{n+2s}{4} \right)^2 \Gamma \left( \frac{n+2s}{2} \right)}{\Gamma \left( \frac{n+2s}{2} \right)^2 |\Gamma(-s)|}, \]
where the authors consider not only \(p = 2\), but also \(p > 1\). In [14], the existence of extremal functions \(u\) for the Hardy-Sobolev inequality in the fractional Sobolev space \(W^{s,p}(\mathbb{R}^n)\) is established through concentration-compactness. The authors also show the asymptotic behavior of extremal functions
\[ u(x) \sim |x|^{-\frac{n-p\alpha}{p-1}}, \]
as \(|x| \to \infty\), and the summability information \(u \in W^{s,\gamma}(\mathbb{R}^n)\), for every \(\frac{n(p-1)}{n-s} < \gamma < p\). Such properties turn out to be optimal when \(s \to 1^-\), in which case optimizers are explicitly known.

In [9], the sharp constant in the Hardy inequality for fractional Sobolev spaces is calculated, by using a non-linear and non-local version of the ground state representation. Also, from the sharp Hardy inequality they deduce the sharp constant in a Sobolev embedding which is optimal in the Lorentz scale.

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Later, in [10], R. Frank and R. Seiringer give an expression for the best constant involved in the fractional Hardy-Sobolev inequality in the half space of the form

$$\kappa_{n,p,s} \int_{\mathbb{R}^n_+} \frac{|u(x)|^p}{x_n^{ps}} \, dx \leq \frac{1}{2} \int_{\mathbb{R}^n_+ \times \mathbb{R}^n_+} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}} \, dxdy,$$

$1 \leq p < \infty$. Other related work is [1], where the same outcome is done in the case $p = 2$ and by replacing $2s$ by $0 < \gamma < 2$.

Other reference in the nonlocal setting concerning unbounded domains, different from $\mathbb{R}^n$ is [7], where a variant of the fractional Hardy-Sobolev-Maz’ya inequality for half spaces is proved, applying a new version of the fractional Hardy-Sobolev inequality general unbounded John domains.

Concerning bounded domains $\Omega \subset \mathbb{R}^n$, in [6], the author considers the inequality

$$\int_{\Omega} \frac{|u(x)|^p}{\delta_{\Omega}(x)^\alpha} \, dx \leq C(\Omega, n, \alpha) \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+\alpha}} \, dxdy$$

for every $u \in C_c(\Omega)$, where $0 < \alpha, p < \infty, n \geq 1, \delta_{\Omega}(x) := \inf\{|x-y|: y \in \mathbb{R}^n \setminus \Omega\}$. Precisely, Dyda proves in [6] that (1.2) holds true in the following cases

(i) $\Omega$ is a bounded Lipschitz domain and $\alpha > 1$.
(ii) $\Omega$ is the complement of a bounded Lipschitz domain and $\alpha \neq 1, \alpha \neq n$.
(iii) $\Omega$ is a domain above the graph of a Lipschitz function $\varphi: \mathbb{R}^{n-1} \to \mathbb{R}$ and $\alpha \neq 1$.
(iv) $\Omega = \mathbb{R}^n \setminus \{x_0\}$, for $x_0 \in \mathbb{R}^n$, and $\alpha \neq n$.

In addition, it is shown some counterexamples for (1.2), encompasses

(i) $\Omega$ is a Lipschitz domain and $\alpha \leq 1, \alpha < p$.
(ii) $\Omega$ is the complement of a compact set and $n = \alpha < p$.

The case of (1.2) in where $\Omega$ is a convex set and $1 < p < \infty, 1 < sp < p$ was studied by Loss and Sloane, [13].

In [13], there is a fractional Hardy-Sobolev type inequality, for every $\Omega \subset \mathbb{R}^n$ with non empty boundary, involving functions different from $|x|^\alpha$. Given a direction $w \in S^{n-1}$, they consider

$$d_{w,\Omega}(x) := \inf\{|t|: x + tw \notin \Omega\}, \quad \delta_{w,\Omega}(x) := \sup\{|t|: x + tw \in \Omega\},$$

and define

$$M_{\alpha}(x) = \frac{\int_{S^{n-1}} |w_n|^\alpha \, dw}{\int_{S^{n-1}} \left(\frac{1}{d_{w,\Omega}(x)} + \frac{1}{\delta_{w,\Omega}(x)}\right)^\alpha \, dw}, \quad \text{and} \quad m_{\alpha}(x) = \frac{\int_{S^{n-1}} |w_n|^\alpha \, dw}{\int_{S^{n-1}} \frac{1}{d_{w,\Omega}(x)^\alpha} \, dw}.$$

The inequalities shown in [13] are

$$\int_{\Omega} \frac{|u(x)|^2}{M_{\alpha}(x)^\alpha} \, dx \leq 2k_{n,\alpha} \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+\alpha}} \, dxdy$$

and

$$\int_{\Omega} \frac{|u(x)|^2}{m_{\alpha}(x)^\alpha} \, dx \leq D_{n,\alpha} \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+\alpha}} \, dxdy$$

for every $u \in C_c(\Omega)$, where $k_{n,\alpha}$ is explicit and $D_{n,\alpha}$ is the sharp constant for the fractional Hardy-Sobolev inequality in the half-space, computed in [1, 9]. See [10] for $p > 1$.

In [8], the authors study fractional Hardy-Sobolev type inequalities where the domain has uniformly fat complement.
In the local setting, in [11], the authors show that the value and the attainability of the best Hardy-Sobolev constant on a smooth domain $\Omega \subset \mathbb{R}^n$

$$\nu_\alpha(\Omega) := \left\{ \int_\Omega |\nabla u|^2 \, dx : u \in H^1(\Omega), \int_\Omega \frac{|u(x)|^{2\alpha}}{|x|^\alpha} \, dx = 1 \right\}$$

where $2\alpha = \frac{2(n-\alpha)}{n-2}$, $n \geq 3$, $0 < \alpha < 2$, when $0 \in \partial \Omega$, are closely related to the properties of the curvature of $\partial \Omega$ at 0. For the non-singular context either $\alpha = 0$ or $0$ belonging in the interior of the domain $\Omega$, it is well-known that $\nu_\alpha(\Omega) = \nu_0(\mathbb{R}^n)$ for any domain $\Omega$.

In [12], a minimization problem involving a Hardy-Sobolev type inequality is solved, where the author analyzes both inner and boundary singularity, that is, zero belongs in the interior of the bounded domain, or zero belongs to its boundary. For references on more general inequalities in the local setting, see for instance [3, 4].

Our goal is analyzing the existence of solution to a minimization problem involving a fractional Hardy-Sobolev type inequality, and a positive parameter $\lambda > 0$, in both cases inner and boundary singularity. To be precise, we first set the notation.

1.2. Main results. We start by fixing notation. Let $0 < s < 1$, $n > 4s$, $0 < \alpha < 2s$, and $\Omega \subset \mathbb{R}^n$ be a bounded domain. We introduce the fractional Sobolev space, see for instance [5],

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{n+2s}{2}}} \in L^2(\Omega \times \Omega) \right\},$$

endowed with the norm

$$\|u\|_{s,\Omega} := \left( \int_\Omega |u|^2 \, dx + \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dxdy \right)^{\frac{1}{2}}.$$ 

Denote

$$[u]_{s,\Omega} := \left( \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dxdy \right)^{\frac{1}{2}}, \quad \text{and} \quad \|u\|_{s,\alpha,\Omega} := \left( \int_\Omega \frac{|u(x)|^{2s,\alpha}}{|x|^\alpha} \, dx \right)^{\frac{1}{2s,\alpha}}.$$ 

When $\Omega = \mathbb{R}^n$, the notation becomes $[u]_s, \|u\|_{s,\alpha}$ respectively.

Let $\lambda > 0$. Consider the following problem

$$\mu_{\alpha,\lambda}(\Omega) := \inf \left\{ [u]^2_{s,\Omega} + \lambda \int_\Omega |u|^2 \, dx : u \in H^s(\Omega), \int_\Omega \frac{|u(x)|^{2s,\alpha}}{|x|^\alpha} \, dx = 1 \right\}$$

Our aim is proving existence of solution to the minimization problem (1.3), for certain values of $\lambda$, where the singularity belongs either in the interior or in the boundary of the domain $\Omega$. In addition, we find a critical value $\lambda^* > 0$ such that there is no solution to (1.3) for every $\lambda > \lambda^*$. Roughly speaking, we prove

**Theorem 1.1** (Inner singularity). Let $\lambda > 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $0 \in \Omega$. Then, there exist $0 < \lambda_s < \lambda^*$ depending only on $\Omega$, such that

1. $\mu_{\alpha,\lambda}(\Omega)$ is attained for any $0 < \lambda < \lambda_s$,
2. $\mu_{\alpha,\lambda}(\Omega)$ is not attained for any $\lambda > \lambda^*$,
3. $\mu_{\alpha,\lambda}(\Omega) = 2\mu_{\alpha,\lambda}(\Omega)$, and the following estimate for $\lambda_s$ from below holds

$$\lambda_s \geq \frac{\mu_{\alpha}}{2|\Omega|} \left( \int_\Omega \frac{1}{|x|^\alpha} \, dx \right)^{\frac{2}{2s,\alpha}}.$$ 

**Theorem 1.2** (Boundary singularity). Let $\Lambda > 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $0 \in \partial \Omega$ and $\partial \Omega$ is flat near 0. Then, $0 < \Lambda_s < \Lambda^*$ depending only on $\Omega$, such that
(1) \(\mu_{\alpha, \Lambda}(\Omega)\) is attained for any \(0 < \Lambda < \Lambda^*\).
(2) \(\mu_{\alpha, \Lambda}(\Omega)\) is not attained for any \(\Lambda > \Lambda^*\).
(3) \(\mu_{\alpha, \Lambda^*}(\Omega) = 2^{\frac{2s-s}{2(n-s)}}\mu_{\alpha, \Lambda^*}(\Omega)\), and the following estimate for \(\Lambda^*\) from below holds

\[
\Lambda^* \geq \frac{\mu_{\alpha}}{2^{\frac{2s-s}{2(n-s)}}|\Omega|^{\frac{2}{s}}}
\left(\int_{\Omega} \frac{1}{|x|^\alpha} \, dx \right)^{\frac{2^s}{s}}.
\]

The rest of the paper is organized as follows. In Section 2, we gather some preliminaries and features of the constant \(\mu_{\alpha, \lambda}(\Omega)\). Section 3 is dedicated to the proof of Theorem 1.1 (inner singularity), and Section 4 to Theorem 1.2 (boundary singularity). In both cases, the crucial ingredients are the properties of \(\mu_{\alpha, \lambda}(\Omega)\) seen as a function in \(\lambda\), and a fractional Hardy-Sobolev type inequality.

2. Preliminaries

The relation between the global constant \(\mu_{\alpha}\) and \(\mu_{\alpha, \lambda}(\Omega)\), defined in (1.1) and (1.3) respectively, will be a key element for the non-existence result (part (2)) of Theorems 1.1 and 1.2. As we mention in the introduction, some features of \(\mu_{\alpha, \lambda}(\Omega)\) seen as a function in the parameter \(\lambda\) play an important role as well. To this aim, we first need the following basic properties. Denote \(\dot{H}^s(\mathbb{R}^n) := \{u \, \text{measurable}, [u]_s < \infty\}\).

**Remark 2.1.** Let \(\Omega \subset \mathbb{R}^n\) be a bounded domain, and \(\phi \in C^\infty_c(\Omega)\). Let \(u \in \dot{H}^s(\mathbb{R}^n)\) be such that \(\|u\|_{s, \alpha} < \infty\), and \(|u(x)| \leq \frac{C}{|x|^\alpha}\) if \(|x| \geq 1\). Then, \(\phi u \in H^s(\Omega)\).

**Proof.** Let \(u \in \dot{H}^s(\mathbb{R}^n)\) be such that \(\|u\|_{s, \alpha} < \infty\), and \(\phi \in C^\infty_c(\Omega)\). We have to show that \(\phi u \in L^2(\Omega)\), and \([\phi u]_{s, \Omega} < \infty\).

Notice that \(\phi u = 0\) in \(\mathbb{R}^n \setminus \Omega\), since \(\text{supp} \phi \subset \Omega\). It is clear that \(\phi u \in L^2(\Omega)\), since the embedding \(L^{2s, \alpha} (\Omega, |x|^{-\alpha} \, dx) \hookrightarrow L^2(\Omega)\) is continuous, as a consequence of Hölder’s inequality with \(p = \frac{2s}{2s-\alpha}, p' = \frac{s}{2s-\alpha}\) and the boundedness of \(\Omega\).

On the other hand, observe that

\[
(2.1) \quad |\phi(x)u(x) - \phi(y)u(y)| \leq |u(x)||\phi(x) - \phi(y)| + |\phi(y)||u(x) - u(y)|.
\]

Therefore, by Minkowski’s inequality,

\[
[\phi u]_{s, \Omega} \leq \left(\int_{\Omega} |u(x)|^2 \left(\int_{\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} \, dydx \right)^{\frac{1}{2}} \right. \left. + \left(\int_{\Omega} |\phi(x)|^2 \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dydx \right)^{\frac{1}{2}} \right) \right. \left. =: I + C(\phi)[u]_s.\n\]
We split the integral and apply Hölder inequality in \( v \) where we have used \( \mu \). Finally, uniformly in \( \mu \),

\[
\int \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n+2s}} \, dy \leq \int \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n+2s}} \, dy
\]

\[
\leq \left[ \int_{|x-y|<1} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n+2s}} \, dy + \int_{|x-y|\geq 1} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n+2s}} \, dy \right] \leq \|\nabla \phi\|_\infty^2 \int_{|x-y|<1} \frac{1}{|x-y|^{n+2s-2}} \, dy + 2\|\phi\|_\infty^2 \int_{|x-y|\geq 1} \frac{1}{|x-y|^{n+2s}} \, dy
\]

\[
\leq |B_1(0)||\nabla \phi\|_\infty^2 \int_0^1 \frac{r^{n-1}}{r^{n+2s-2}} \, dr + 2|B_1(0)||\phi\|_\infty^2 \int_1^\infty \frac{r^{n-1}}{r^{n+2s}} \, dr
\]

\[
\leq \frac{1}{2(1-s)} |B_1(0)||\nabla \phi\|_\infty^2 + \frac{1}{s} |B_1(0)||\phi\|_\infty^2 =: C(\phi, n, s).
\]

Finally, uniformly in \( x \in \mathbb{R}^n \),

\[
(2.2) \quad \int_{\mathbb{R}^n} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n+2s}} \, dy \leq C(\phi, n, s).
\]

We split the integral and apply Hölder inequality in \(|x| < 1 \) and the behavior of \( u \) for \(|x| \geq 1 \), to obtain

\[
\int_{\Omega} |u|^2 \, dx \leq \int_{\Omega \cap \{|x|<1\}} \frac{|u|^2}{|x|^{2s, \alpha}} \, dx + \int_{\{|x|\geq 1\}} \frac{1}{|x|^{2(n-2s)}} \, dx \leq C \left( \int_{\mathbb{R}^n} \frac{|u|^{2s, \alpha}}{|x|^\alpha} \, dx \right)^2 + C < \infty.
\]

Hence, \([\phi u]_{s, \Omega} < \infty\), which finishes the proof of \( \phi u \in H^s(\Omega) \).

Now, we are able to establish the main result of this section, which gives useful properties of \( \mu_{\alpha, \lambda}(\Omega) \) seen us a function in the parameter \( \lambda > 0 \). Part of the next Lemma relies on the existence of an extremal function for the global constant \( \mu_{\alpha} \), and its behavior for \(|x| \geq 1 \), given by [14].

**Lemma 2.2.** Let \( \lambda > 0 \) and \( \Omega \subset \mathbb{R}^n \) be an open bounded domain.

1. \( \mu_{\alpha, \lambda}(\Omega) \leq \mu_{\alpha} \), for every \( \lambda > 0 \).
2. \( \mu_{\alpha, \lambda}(\Omega) \) is continuous and nondecreasing with respect \( \lambda \).
3. \( \lim_{\lambda \to 0} \mu_{\alpha, \lambda}(\Omega) = 0 \),

where \( \mu_{\alpha, \lambda}(\Omega) \), and \( \mu_{\alpha} \) are defined in (1.3), and (1.1) respectively.

**Proof.** (1) Let \( \varepsilon > 0, R > 0 \) and \( \phi \in C_{c}^\infty(\Omega) \) be such that \( 0 \leq \phi \leq 1 \), \( \phi = 1 \) in \( B_R(x_0) \subset \Omega \), \( \phi = 0 \) in \( \Omega \setminus B_{2R}(x_0) \), for some \( x_0 \in \Omega \). Without loss of generality, assume \( x_0 = 0 \).

Let \( u_0 \) be a positive minimizer of \( \mu_{\alpha} \), see [14] for the existence of \( u_0 \). Consider

\[
u_{\varepsilon}(x) := \varepsilon^{-\frac{n-2s}{2s}} u_0 \left( \frac{x}{\varepsilon} \right) \phi(x), \quad u_{\varepsilon}(x) := \frac{1}{\|u_{\varepsilon}\|_{s, \alpha, \Omega}} u_{\varepsilon}(x).
\]

Then, \( u_{\varepsilon} \in H^s(\Omega) \), by Remark 2.1, since \( u_0 \) verifies the growth condition \( |u_0(x)| \leq \frac{C}{|x|^{n-2s}} \) if \(|x| \geq 1 \), given in [14, Theorem 1.1]. Moreover, \( \|u_{\varepsilon}\|_{s, \alpha, \Omega} = 1 \). Thus,

\[
(2.3) \quad \mu_{\alpha, \lambda}(\Omega) \leq \|v_{\varepsilon}\|_{s, \alpha, \Omega}^2 + \lambda \int_{\Omega} v_{\varepsilon}^2(x) \, dx.
\]

Observe that, after a change of variables,

\[
\int_{\Omega} \frac{u_{\varepsilon}^{2s, \alpha}(x)}{|x|^\alpha} \, dx = \int_{\varepsilon^{-1} \Omega} \phi^{2s, \alpha}(\varepsilon y) u_0^{2s, \alpha}(y) \frac{1}{|y|^\alpha} \, dy.
\]
Since $\phi = 1$ in $B_{R}(0) \subset \Omega$, $0 \leq \phi \leq 1$ and $\text{supp} \phi \subset B_{2R}(0)$, we get
\[
\int_{B_{2R}(0)} \frac{u_{0}^{2s,\alpha}(y)}{|y|^\alpha} dy \leq \int_{\Omega} \frac{u_{2}^{2s,\alpha}(x)}{|x|^\alpha} dx \leq \int_{B_{2R}(0)} \frac{u_{0}^{2s,\alpha}(y)}{|y|^\alpha} dy,
\]
from where we deduced,
\[
\lim_{\varepsilon \to 0} \int_{\Omega} \frac{u_{\varepsilon}^{2s,\alpha}(x)}{|x|^\alpha} dx = \int_{\mathbb{R}^n} \frac{u_{0}^{2s,\alpha}(y)}{|y|^\alpha} dy = 1.
\]

Moreover,
\[
\int_{\Omega} v_{\varepsilon}^2(x) dx = \frac{\varepsilon^{2s-n}}{\|u_{\varepsilon}\|^2_{s,\alpha,\Omega}} \int_{\Omega} \phi^2(x) u_{0} \left(\frac{x}{\varepsilon}\right) dx = \frac{\varepsilon^{2s}}{\|u_{\varepsilon}\|^2_{s,\alpha,\Omega}} \int_{B_{2\varepsilon R}(0)} \phi^2(\varepsilon y) u_{0}(y)^2 dy = O(\varepsilon^2).
\]

The last identity is due to (2.4), and the fact that
\[
\int_{B_{2\varepsilon R}(0)} \phi^2(\varepsilon y) u_{0}(y)^2 dy \leq C.
\]

Indeed, by [14, Theorem 1.1], we know that for
\[
|u_{0}(y)| \leq \frac{C}{|y|^{n-2s}}, \quad \text{for every } |y| \geq 1.
\]

Then, there exist $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ we have $\frac{2R}{\varepsilon} > 1$. Therefore, for every $0 < \varepsilon \leq \varepsilon_0$,
\[
\int_{B_{2\varepsilon R}(0)} \phi^2(\varepsilon y) u_{0}(y)^2 dy = \left(\int_{\{y < 1\}} + \int_{\{1 \leq |y| \leq \frac{2R}{\varepsilon}\}}\right) \phi^2(\varepsilon y) u_{0}(y)^2 dy =: I + II.
\]

To manage $I$, recall $0 \leq \phi \leq 1$, and apply Hölder inequality with $p = \frac{2s}{n}$, $p' = \frac{n}{2s}$, to obtain
\[
I \leq C \left(\int_{\{y < 1\}} \frac{u_{0}^{2s,\alpha}(y)}{|y|^\alpha} dy\right)^{\frac{2}{2s,\alpha}} \leq C \left(\int_{\mathbb{R}^n} \frac{u_{0}^{2s,\alpha}(y)}{|y|^\alpha} dy\right)^{\frac{2}{2s,\alpha}} = C.
\]

To control $II$, we use $0 \leq \phi \leq 1$, and (2.6), recalling $n > 4s$, to find
\[
II \leq C \int_{|y| \geq 1} \frac{1}{|y|^{2(n-2s)}} dy = C \int_{1}^{\infty} r^{-n-1+4s} dr = C.
\]

Now, we have to estimate $[v_{\varepsilon}]^2_{s,\Omega} = \|[u_{\varepsilon}]^{2}_{s,\alpha,\Omega}\|_{s,\Omega}$. Thanks to (2.4), it will be enough to analyze $[u_{\varepsilon}]^2_{s,\Omega}$. Similar to what we have done in Remark 2.1 ((2.1), Minkowski’s inequality), but changing variables, and recalling $0 \leq \phi \leq 1$, we get
\[
[u_{\varepsilon}]_{s,\Omega} \leq [u_{0}]_{s} + \left(\int_{\varepsilon^{-1}\Omega \times \varepsilon^{-1}\Omega} \frac{u_{0}(x)^2|\phi(\varepsilon x) - \phi(\varepsilon y)|^2}{|x - y|^{n+2s}} dxdy\right)^{\frac{1}{2}} = \mu_{\alpha}^{\frac{1}{2}} + \left(\int_{\varepsilon^{-1}\Omega \times \varepsilon^{-1}\Omega} \frac{u_{0}(x)^2|\phi(\varepsilon x) - \phi(\varepsilon y)|^2}{|x - y|^{n+2s}} dxdy\right)^{\frac{1}{2}}.
\]
Since $u_0$ is an extremal function for the constant $\mu_\alpha$, we obtain

\begin{equation}
[u_\varepsilon]_{s,\Omega} \leq \mu_\alpha^\frac{1}{2} + \left( \int_{\varepsilon^{-1}\Omega \times \varepsilon^{-1}\Omega} \frac{u_0(x)^2|\phi(\varepsilon x) - \phi(\varepsilon y)|^2}{|x - y|^{n+2s}} \, dx \, dy \right)^\frac{1}{2}.
\end{equation}

We will show that

\begin{equation}
\lim_{\varepsilon \to 0} \int_{\varepsilon^{-1}\Omega \times \varepsilon^{-1}\Omega} \frac{u_0(x)^2|\phi(\varepsilon x) - \phi(\varepsilon y)|^2}{|x - y|^{n+2s}} \, dx \, dy = 0.
\end{equation}

That will be a consequence of the Lebesgue Dominated convergence Theorem. Clearly,

\[ \lim_{\varepsilon \to 0} \chi_{\varepsilon^{-1}\Omega \times \varepsilon^{-1}\Omega}(x,y) \frac{u_0(x)^2|\phi(\varepsilon x) - \phi(\varepsilon y)|^2}{|x - y|^{n+2s}} = 0 \quad \text{a.e. in } \mathbb{R}^n \times \mathbb{R}^n. \]

To find the dominated function in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$, we split the domain, and use (2.6). Indeed, for every $0 < \varepsilon < 1$,

\[ \frac{u_0(x)^2|\phi(\varepsilon x) - \phi(\varepsilon y)|^2}{|x - y|^{n+2s}} \leq C\psi(x,y) \left( \chi_{\{x\leq 1\}}u_0(x)^2 + \chi_{\{|x\geq 1\}}\frac{1}{|x|^{2(n-2s)}} \right) =: \Psi(x,y), \]

where $\psi(x,y) = \frac{1}{|x-y|^{n+2s}} \chi_{\{|x-y|<1\}} + \frac{1}{|x-y|^{n+2s}} \chi_{\{|x-y|\geq 1\}}$. For the previous inequality, we have used

\[ \frac{|\phi(\varepsilon x) - \phi(\varepsilon y)|^2}{|x - y|^{n+2s}} \leq \begin{cases} \frac{C\varepsilon^2}{|x-y|^{n+2s}} & \text{if } |x-y| < 1, \\ \frac{C}{|x-y|^{n+2s}} & \text{if } |x-y| \geq 1. \end{cases} \]

Let us see that $\Psi \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.

\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} \Psi(x,y) \, dx \, dy \leq C \int_{|x|<1} u_0(x)^2 \int_{\mathbb{R}^n} \psi(x,y) \, dy \, dx + C \int_{|x|\geq 1} \frac{1}{|x|^{2(n-2s)}} \int_{\mathbb{R}^n} \psi(x,y) \, dy \, dx \\
\leq C \int_{|x|<1} u_0(x)^2 \, dx + C \int_{|x|\geq 1} \frac{1}{|x|^{2(n-2s)}} \, dx \\
\leq C \int_{|x|<1} u_0(x)^2 \, dx + C
\]

Then, apply Hölder inequality with $p = \frac{2s+\alpha}{2}$, $p' = \frac{n-\alpha}{2s-\alpha}$ in the first term, to obtain

\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} \Psi(x,y) \, dx \, dy \leq C \left( \int_{\mathbb{R}^n} u_0^{2s+\alpha}(x) \, dx \right)^{\frac{2}{2s+\alpha}} + C = C.
\]

Hence, (2.8) holds. Consequently, from (2.7),

\[
\lim_{\varepsilon \to 0} \sup_\varepsilon |u_\varepsilon|_{s,\Omega}^2 \leq \mu_\alpha.
\]

Then, (2.3) becomes

\[
\mu_{\alpha,\lambda}(\Omega) \leq \frac{1}{\|u_\varepsilon\|_{s,\alpha,\Omega}^2} |u_\varepsilon|_{s,\Omega}^2 + O(\varepsilon^{2s}).
\]

Taking the limit $\varepsilon \to 0$, we conclude $\mu_{\alpha,\lambda}(\Omega) \leq \mu_\alpha$.

(2) It follows from the definition (1.3).
(3) Consider \( c := \left( \int_\Omega \frac{1}{|x|^s} \, dx \right)^{-\frac{1}{2s,\alpha}} \in H^s(\Omega) \). Then,

\[
\mu_{\alpha,\lambda}(\Omega) \leq [c]_{s,\Omega}^2 + \lambda \int_\Omega c^2 \, dx = \lambda c^2 |\Omega|.
\]

Now, take the limit \( \lambda \to 0 \) to conclude (3). \( \square \)

3. Extremal function in case of inner singularity.

Through all this section \( \Omega \subset \mathbb{R}^n \) is a bounded domain such that \( 0 \in \Omega \). We establish in the next lemma the second fundamental ingredient of the proof of Theorem 1.1, which is a Hardy-Sobolev type inequality with the inner singularity. We follow ideas from [12], where the local version is studied.

**Lemma 3.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain such that \( 0 \in \Omega \). Then, there exists a positive constant \( C_1 = C_1(\Omega, n, s) \) such that

\[
(3.1) \quad \frac{\mu_\alpha}{2} \left( \int_\Omega \frac{|u(x)|^{2s,\alpha}}{|x|^{\alpha}} \, dx \right)^{\frac{2}{2s,\alpha}} \leq [u]_{s,\Omega}^2 + C_1 \int_\Omega |u|^2 \, dx
\]

for every \( u \in H^s(\Omega) \).

**Proof.** Let \( \Omega_1 \subset \Omega_2 \subset \Omega \) be bounded sets to be determined, such that \( 0 \in \Omega_1 \). Let \( \phi \in C_c^\infty(\Omega) \) be such that \( 0 \leq \phi \leq 1 \) in \( \Omega \), \( \phi = 1 \) in \( \Omega_1 \), \( \phi = 0 \) in \( \Omega \setminus \Omega_2 \). Consider

\[
\eta_1 = \frac{\phi^2}{\phi^2 + (1-\phi)^2}, \quad \eta_2 = \frac{(1-\phi)^2}{\phi^2 + (1-\phi)^2}.
\]

Then, \( \eta_1^\frac{1}{2} \in C^1_c(\Omega), \eta_2^\frac{1}{2} \in C^1(\Omega), \eta_1 + \eta_2 = 1, \text{ supp } \eta_1 \subset \Omega_2 \subset \Omega, \text{ supp } \eta_2 \subset \mathbb{R}^n \setminus \Omega_1 \). Let \( u \in H^s(\Omega) \). We consider \( \eta_2^\frac{1}{2} u : \Omega \to \mathbb{R} \), by [5, Lemma 5.3], \( \eta_2^\frac{1}{2} u \in H^s(\Omega) \), since \( u \in H^s(\Omega) \) and \( \eta_2^\frac{1}{2} \in C^{0,1}(\Omega) \). Moreover, \( \| \eta_2^\frac{1}{2} u \|_{H^s(\Omega)} \leq C(n, s, \Omega) \| u \|_{H^s(\Omega)} \). By using the auxiliary functions \( \eta_1, \eta_2 \), we can split the main integral into two pieces and analyze them separately, as follows,

\[
\mu_\alpha \left( \int_\Omega \frac{|u(x)|^{2s,\alpha}}{|x|^{\alpha}} \, dx \right)^{\frac{2}{2s,\alpha}} \leq \mu_\alpha \left( \sum_{i=1}^2 \left( \int_\Omega \frac{|\eta_i^\frac{1}{2} u|^{2s,\alpha}}{|x|^{\alpha}} \, dx \right)^{\frac{2}{2s,\alpha}} \right) =: I_1 + I_2.
\]

To estimate \( I_1 \), notice that we can use the fractional Hardy-Sobolev inequality given by \( \mu_\alpha \) for \( \eta_1^\frac{1}{2} u \), see (1.1). Thus,

\[
(3.2) \quad I_1 = \mu_\alpha \left( \int_\Omega \frac{|\eta_1^\frac{1}{2} u|^{2s,\alpha}}{|x|^{\alpha}} \, dx \right)^{\frac{2}{2s,\alpha}} = \mu_\alpha \left( \int_{\mathbb{R}^n} \frac{|\eta_1^\frac{1}{2} u|^{2s,\alpha}}{|x|^{\alpha}} \, dx \right)^{\frac{2}{2s,\alpha}} \leq [\eta_1^\frac{1}{2} u]_s^2
\]

Notice that \( \text{ supp } \eta_1 \subset \Omega \). Similarly to (2.7), we obtain

\[
[\eta_1^\frac{1}{2} u]_s^2 \leq \left( \int_{\Omega \times \Omega} \frac{|\eta_1^\frac{1}{2}(x) u(x) - \eta_1^\frac{1}{2}(y) u(y)|^2}{|x - y|^{n+2s}} \, dx dy + 2 \int_{(\mathbb{R}^n \setminus \Omega) \times \Omega} \frac{\eta_1(x) |u(x)|^2}{|x - y|^{n+2s}} \, dx dy \right)^{\frac{1}{2}}
\]
For the first term, we use (2.1) for \( \eta_1^\frac{1}{2} u \) and Minkowski’s inequality. For the second term, we proceed similar to Remark 2.1 (2.2), to get

\[
[\eta_1^\frac{1}{2} u]_s \leq \left( \int_{\Omega \times \Omega} \frac{\eta_1(y) |u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \right)^{\frac{1}{2}} + C(\phi, n, s) \left( \int_{\Omega} |u|^2 \, dx \right)^{\frac{1}{2}}
\]

which implies,

\[
(3.3) \quad [\eta_1^\frac{1}{2} u]^2_s \leq 2 \int_{\Omega \times \Omega} \frac{\eta_1(y) |u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy + C(\phi, n, s) \int_{\Omega} |u|^2 \, dx.
\]

Therefore, taking into account (3.2)-(3.3), we obtain

\[
(3.4) \quad I_1 \leq 2 \int_{\Omega \times \Omega} \frac{\eta_1(y) |u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy + C(\phi, n, s) \int_{\Omega} |u(x)|^2 \, dx
\]

To analyze \( I_2 \), notice that \( \eta_2 = 0 \) in \( \Omega_1 \), so that

\[
I_2 = \mu_\alpha \left( \int_{\Omega \setminus \tilde{\Omega}} \frac{\eta_2^\frac{1}{2} u |2_s, \alpha}{|x|^\alpha} \, dx \right)^{\frac{2}{2s, \alpha}} = \mu_\alpha \left( \int_{\Omega \setminus \Omega_1} \frac{\eta_2^\frac{1}{2} u |2_s, \alpha}{|x|^\alpha} \, dx \right)^{\frac{2}{2s, \alpha}}.
\]

Observe that \( 0 \not\in \text{supp} \eta_2 \). Denote by \( d_1 := \text{dist}(0, \partial \Omega_1) \). Thus, by Hölder’s inequality with \( p = \frac{n}{n-\alpha}, p' = \frac{n}{\alpha} \),

\[
I_2 \leq \mu_\alpha d_1^{\frac{2\alpha}{2s, \alpha}} \left( \int_{\Omega \setminus \Omega_1} \frac{\eta_2^\frac{1}{2} u |2_s, \alpha}{|x|^\alpha} \, dx \right)^{\frac{2}{2s, \alpha}}
\]

\[
\leq \mu_\alpha d_1^{\frac{2\alpha}{2s, \alpha}} \left( |\Omega \setminus \Omega_1|^{\frac{\alpha}{n}} \left( \int_{\Omega \setminus \Omega_1} \frac{\eta_2^\frac{1}{2} u |2_s, \alpha}{|x|^\alpha} \, dx \right)^{\frac{n-\alpha}{2}} \right)^{\frac{2}{2s, \alpha}}
\]

\[
\leq \mu_\alpha d_1^{\frac{2\alpha}{2s, \alpha}} |\Omega \setminus \Omega_1|^{\frac{\alpha}{n}} \kappa_{\Omega_1} \left( \int_{\Omega \setminus \Omega_1} \frac{\eta_2^\frac{1}{2} u |2_s, \alpha}{|x|^\alpha} \, dx \right)^{\frac{2}{2s, \alpha}}
\]

where \( \kappa_{\Omega_1} \) is given by

\[
\kappa_{\Omega_1} := \inf \left\{ [v]_{s, \Omega}^2 : v \in H^s(\Omega \setminus \Omega_1), v = 0 \text{ in } \Omega_1, \int_{\Omega \setminus \Omega_1} v^{2s} \, dx = 1 \right\}.
\]

It will be enough to prove that

\[
(3.5) \quad \mu_\alpha d_1^{\frac{2\alpha}{2s, \alpha}} |\Omega \setminus \Omega_1|^{\frac{2\alpha}{n}} \kappa_{\Omega_1}^{-1} \leq 1.
\]

Indeed, given \( \delta > 0 \), choose \( \Omega_1 \subset \Omega \) such that \( 0 \in \Omega_1 \) and \( |\Omega \setminus \Omega_1| < \delta \). Let \( \Omega_0 \subset \Omega \) be an open bounded set such that \( 0 \in \Omega_0 \subset \Omega_1 \). Then, \( d_1 \geq d_0 := \text{dist}(0, \partial \Omega_0) \). Also, notice that \( \kappa_{\Omega_0} \leq \kappa_{\Omega_1} \),
since
\[
\left\{ v \in H^s(\Omega \setminus \Omega_1) : v = 0 \text{ in } \Omega_1, \int_{\Omega \setminus \Omega_1} |v|^2 dx = 1 \right\}
\subset \left\{ w \in H^s(\Omega \setminus \Omega_0) : w = 0 \text{ in } \Omega_0, \int_{\Omega \setminus \Omega_0} |w|^2 dx = 1 \right\}
\]
with the identification
\[
v \in H^s(\Omega \setminus \Omega_1) \mapsto w = \begin{cases} v & \text{in } \Omega \setminus \Omega_1, \\ 0 & \text{in } \Omega_1 \setminus \Omega_0. \end{cases}
\]
Therefore,
\[
\mu_{\alpha} d_1^{-\frac{2\alpha}{n+2\alpha}} |\Omega \setminus \Omega_1|^{\frac{2\alpha}{n+2\alpha}} \kappa_{\Omega_1}^{-1} \leq \mu_{\alpha} d_0^{-\frac{2\alpha}{n+2\alpha}} |\Omega \setminus \Omega_1|^{\frac{2\alpha}{n+2\alpha}} \kappa_{\Omega_1}^{-1}
\leq C(\Omega_0) |\Omega \setminus \Omega_1|^{\frac{2\alpha}{n+2\alpha}} \leq C(\Omega_0) \delta^{\frac{2\alpha}{n+2\alpha}}.
\]
Let $\delta > 0$ be such that $C(\Omega_0)\delta^{\frac{2\alpha}{n+2\alpha}} < 1$. Consequently, proceeding similar to the estimate of $[\eta_1^2 u]_s$, but with $[\eta_2^2 u]_{s,\Omega}$, we obtain
\[
(I_2) \leq [\eta_2^2 u]_{s,\Omega}^2 \leq 2 \int_{\Omega \times \Omega} \frac{\eta_2(y)|u(x) - u(y)|^2}{|x - y|^{n+2s}} dxdy + C(\phi, n, s) \int_\Omega |u(x)|^2 dx.
\]
By (3.4), (3.6) and the fact that $\eta_1 + \eta_2 = 1$, we conclude (3.1), where the constant only depends on $\Omega_0, \phi, n$ and $s$, so that $C = C(\Omega, n, s)$.

As a consequence of the Hardy-Sobolev type inequality in the inner case (Lemma 3.1), and (1) of Lemma 2.2, we are able to show the next result related to the attainability of the constant $\mu_{\alpha,\lambda}(\Omega)$, which will be used in Theorem 1.1 as well.

**Lemma 3.2.** Let $\lambda > 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $0 \in \Omega$.

1. If $\mu_{\alpha,\lambda}(\Omega) < \frac{\mu_{\alpha}}{2}$, then $\mu_{\alpha,\lambda}(\Omega)$ is attained.
2. If there exists a $\tilde{\lambda} > 0$ such that $\mu_{\alpha,\tilde{\lambda}}(\Omega) = \mu_{\alpha}$, then for every $\lambda > \tilde{\lambda}$, $\mu_{\alpha,\lambda}(\Omega)$ is not attained.

**Proof.**
(i) Let $\{u_k\}_{k \in \mathbb{N}} \subset H^s(\Omega)$ be a minimizing sequence for $\mu_{\alpha,\lambda}(\Omega)$, that is,
\[
\int_\Omega \frac{|u_k|^{2s,\alpha}}{|x|^{\alpha}} dx = 1 \text{ for every } k \in \mathbb{N}, \text{ and } \lim_{k \to \infty} \left( |u_k|^{2s,\alpha} + \lambda \int_\Omega |u_k|^2 dx \right) = \mu_{\alpha,\lambda}(\Omega).
\]
Then, $\{u_k\}_{k \in \mathbb{N}}$ is bounded in $H^s(\Omega)$. Therefore, up to a subsequence, we can assume that
\[
u_k \rightharpoonup u \text{ weakly in } H^s(\Omega),
\nu_k \rightharpoonup u \text{ strongly in } L^p(\Omega) \text{ for } 1 \leq p < 2^*_s = \frac{2n}{n-2s}, \text{ see [7, Theorem 4.54]},
u_k \to u \text{ a.e. in } \Omega.
\]
Let us see that $u \neq 0$. Indeed, we proceed by contradiction. Assume $u \equiv 0$ a.e. in $\Omega$. By (3.1), we get
\[
\frac{\mu_{\alpha}}{2} = \frac{\mu_{\alpha}}{2} \left( \int_\Omega \frac{|u_k|^{2s,\alpha}}{|x|^{\alpha}} dx \right)^{\frac{2}{2s,\alpha}} \leq |u_k|^{2s,\Omega} + C \int_\Omega |u_k|^2 dx
\]
which implies
\[
\frac{\mu_{\alpha}}{2} \leq \mu_{\alpha,\lambda}(\Omega) + o(1) + (C - \lambda) \int_\Omega |u_k|^2 dx.
\]
By taking the limit, we get \( \frac{\mu_\alpha}{\alpha} \leq \mu_{\alpha,\lambda}(\Omega) \), which contradicts the hypothesis. Therefore, \( u \neq 0 \) in \( \Omega \). By Brezis-Lieb Theorem \( [2] \), we know that

\[
\int_\Omega \frac{|u_k|^{2s,\alpha}}{|x|^\alpha} \, dx = \int_\Omega \frac{|u|^{2s,\alpha}}{|x|^\alpha} \, dx + \int_\Omega \frac{|u_k - u|^{2s,\alpha}}{|x|^\alpha} \, dx + o(1),
\]

from it follows that

\[
1 = \left( \int_\Omega \frac{|u_k|^{2s,\alpha}}{|x|^\alpha} \, dx \right)^{\frac{2}{2s,\alpha}} \\
= \left( \int_\Omega \frac{|u|^{2s,\alpha}}{|x|^\alpha} \, dx + \int_\Omega \frac{|u_k - u|^{2s,\alpha}}{|x|^\alpha} \, dx + o(1) \right)^{\frac{2}{2s,\alpha}} \\
\leq \left( \int_\Omega \frac{|u|^{2s,\alpha}}{|x|^\alpha} \, dx \right)^{\frac{2}{2s,\alpha}} + \left( \int_\Omega \frac{|u_k - u|^{2s,\alpha}}{|x|^\alpha} \, dx \right)^{\frac{2}{2s,\alpha}} + o(1) \\
\leq \frac{1}{\mu_{\alpha,\lambda}(\Omega)} \left( |u|_{s,\Omega}^2 + \lambda \int_\Omega |u|^2 \, dx \right) \\
+ \frac{1}{\mu_{\alpha,\lambda}(\Omega)} \left( |u_k - u|_{s,\Omega}^2 + \lambda \int_\Omega |u_k - u|^2 \, dx \right) + o(1) \\
= \frac{1}{\mu_{\alpha,\lambda}(\Omega)} \left( |u|_{s,\Omega}^2 + \lambda \int_\Omega |u|^2 \, dx \right) + o(1) \\
= 1 + o(1).
\]

Notice that we have used that

\[
|(u_k - u)(x) - (u_k - u)(y)|^2 = |u_k(x) - u_k(y)|^2 + |u(x) - u(y)|^2 \\
- 2(u_k(x) - u_k(y))(u(x) - u(y)),
\]

implies that

\[
|u|_{s,\Omega}^2 + |u_k - u|_{s,\Omega}^2 \leq |u|_{s,\Omega}^2 + 2|u|_{s,\Omega}^2 - 2 \int_{\Omega \times \Omega} \frac{(u_k(x) - u_k(y))(u(x) - u(y))}{|x - y|^{n+2s}} \, dx dy \\
= |u|_{s,\Omega}^2 + o(1),
\]

due to the weakly convergence \( u_k \rightharpoonup u \) in \( H^s(\Omega) \). As a consequence, there exists the following limit

\[
1 = \lim_{k \to \infty} \left( \int_\Omega \frac{|u|^{2s,\alpha}}{|x|^\alpha} \, dx + \int_\Omega \frac{|u_k - u|^{2s,\alpha}}{|x|^\alpha} \, dx \right)^{\frac{2}{2s,\alpha}} \\
= \lim_{k \to \infty} \left[ \left( \int_\Omega \frac{|u|^{2s,\alpha}}{|x|^\alpha} \, dx \right)^{\frac{2}{2s,\alpha}} + \left( \int_\Omega \frac{|u_k - u|^{2s,\alpha}}{|x|^\alpha} \, dx \right)^{\frac{2}{2s,\alpha}} \right].
\]

Since \( u \neq 0 \), we conclude that \( u_k \rightharpoonup u \) strongly in \( L^{2s,\alpha}(\Omega, |x|^{-\alpha} \, dx) \), and

\[
\int_\Omega \frac{|u|^{2s,\alpha}}{|x|^\alpha} \, dx = 1,
\]

which implies that \( \mu_{\alpha,\lambda}(\Omega) \) is attained by \( u \).
(ii) Let $\lambda > \bar{\lambda}$. Assume that there exists a function $u \in H^s(\Omega)$, which is a minimizer to $\mu_{\alpha, \lambda}(\Omega)$. Then,

$$\mu_{\alpha, \lambda}(\Omega) = \|u\|_{s, \Omega}^2 + \lambda \int_{\Omega} |u|^2 \, dx > \|u\|_{s, \Omega}^2 + \bar{\lambda} \int_{\Omega} |u|^2 \, dx \geq \mu_{\alpha, \lambda}(\Omega) = \mu_\alpha \geq \mu_{\alpha, \lambda}(\Omega),$$

where we have used (1) from Lemma 2.2 in the last inequality. This contradiction finishes the proof. □

Now, we are in condition to prove Theorem 1.1, which is the main result of this section.

**Proof of Theorem 1.1.** By Lemma 2.2, we know that $\mu_{\alpha, \lambda}(\Omega)$ is a nondecreasing and continuous function in the parameter $\lambda > 0$, and $\lim_{\lambda \to 0} \mu_{\alpha, \lambda}(\Omega) = 0$. Therefore, there exists $\lambda_* > 0$ such that $\mu_{\alpha, \lambda_*}(\Omega) = \mu_\alpha$. Notice that $\lambda_* := \inf\{C > 0 : (3.1) \text{ holds.}\}$

Thus, by (1) from Lemma 3.2, we conclude (1) of Theorem 1.1 for every $0 < \lambda < \lambda_*$, since in that case, $\mu_{\alpha, \lambda}(\Omega) < \frac{\mu_\alpha}{2}$. Analogously, by Lemma 2.2, there exists $\lambda^* > 0$ such that $\mu_{\alpha, \lambda^*}(\Omega) = \mu_\alpha$. Hence, $\mu_{\alpha, \lambda}(\Omega) = \mu_\alpha$ for every $\lambda \geq \lambda_*$. By (2) from Lemma 3.2, we conclude (2) of Theorem 1.1 for every $\lambda > \lambda^*$.

It remains to prove (3). Clearly, $\mu_{\alpha, \lambda}(\Omega) = 2\mu_{\alpha, \lambda_*}(\Omega)$. The estimate from below of $\lambda_*$ is a straightforward consequence of its definition. Indeed, $\lambda_*$ can be written as

$$\lambda_* = \sup\left\{ \frac{\mu_\alpha}{2} \left( \int_{\Omega} \frac{|u|^{2s-\alpha}}{|x|^{\alpha}} \, dx \right)^{2s-\alpha} - \frac{|u|_{s, \Omega}^2}{\int_{\Omega} |u|^2 \, dx} : u \in H^s(\Omega) \setminus \{0\} \right\}$$

$$\geq \frac{\mu_\alpha}{2|\Omega|} \left( \int_{\Omega} \frac{1}{|x|^{\alpha}} \, dx \right)^{\frac{2}{2s-\alpha}},$$

where we have consider the function $c = \left( \int_{\Omega} \frac{1}{|x|^{\alpha}} \, dx \right)^{-\frac{1}{2s-\alpha}}$. □

4. Extremal function in case of boundary singularity

In this section, assume $0 \in \partial \Omega$, and $\partial \Omega$ is flat near 0, that is, up to a rotation, there exists $\delta > 0$ such that $B_\delta(0) \cap \Omega = B_\delta^+(0) := B_\delta(0) \cap \mathbb{R}_+^n$, where $\mathbb{R}_+^n = \{x_n > 0\}$. Given $\Lambda > 0$, we denote by $\mu_{\alpha, \Lambda}(\Omega)$ the constant defined in (1.3). The strategy is analogous to the one of Theorem 1.1. Next Lemma is the counterpart of Lemma 3.1, and it can be seen as a Hardy-Sobolev type inequality with the boundary singularity. We follow ideas from [12], where the local version is studied.

**Lemma 4.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $0 \in \partial \Omega$, and $\partial \Omega$ is flat near 0. Then, there exists a positive constant $C_2 := C_2(\Omega, n, s) > 0$ such that

$$\mu_{\alpha, \Lambda}(\Omega) \geq \mu_\alpha \geq \mu_{\alpha, \Lambda}(\Omega),$$

where we have used (1) from Lemma 2.2 in the last inequality. This contradiction finishes the proof. □

Now, we are in condition to prove Theorem 1.1, which is the main result of this section.

**Proof of Theorem 1.1.** By Lemma 2.2, we know that $\mu_{\alpha, \lambda}(\Omega)$ is a nondecreasing and continuous function in the parameter $\lambda > 0$, and $\lim_{\lambda \to 0} \mu_{\alpha, \lambda}(\Omega) = 0$. Therefore, there exists $\lambda_* > 0$ such that $\mu_{\alpha, \lambda_*}(\Omega) = \mu_\alpha$. Notice that $\lambda_* := \inf\{C > 0 : (3.1) \text{ holds.}\}$

Thus, by (1) from Lemma 3.2, we conclude (1) of Theorem 1.1 for every $0 < \lambda < \lambda_*$, since in that case, $\mu_{\alpha, \lambda}(\Omega) < \frac{\mu_\alpha}{2}$. Analogously, by Lemma 2.2, there exists $\lambda^* > 0$ such that $\mu_{\alpha, \lambda^*}(\Omega) = \mu_\alpha$. Hence, $\mu_{\alpha, \lambda}(\Omega) = \mu_\alpha$ for every $\lambda \geq \lambda_*$. By (2) from Lemma 3.2, we conclude (2) of Theorem 1.1 for every $\lambda > \lambda^*$.

It remains to prove (3). Clearly, $\mu_{\alpha, \lambda}(\Omega) = 2\mu_{\alpha, \lambda_*}(\Omega)$. The estimate from below of $\lambda_*$ is a straightforward consequence of its definition. Indeed, $\lambda_*$ can be written as

$$\lambda_* = \sup\left\{ \frac{\mu_\alpha}{2} \left( \int_{\Omega} \frac{|u|^{2s-\alpha}}{|x|^{\alpha}} \, dx \right)^{2s-\alpha} - \frac{|u|_{s, \Omega}^2}{\int_{\Omega} |u|^2 \, dx} : u \in H^s(\Omega) \setminus \{0\} \right\}$$

$$\geq \frac{\mu_\alpha}{2|\Omega|} \left( \int_{\Omega} \frac{1}{|x|^{\alpha}} \, dx \right)^{\frac{2}{2s-\alpha}},$$

where we have consider the function $c = \left( \int_{\Omega} \frac{1}{|x|^{\alpha}} \, dx \right)^{-\frac{1}{2s-\alpha}}$. □

4. Extremal function in case of boundary singularity

In this section, assume $0 \in \partial \Omega$, and $\partial \Omega$ is flat near 0, that is, up to a rotation, there exists $\delta > 0$ such that $B_\delta(0) \cap \Omega = B_\delta^+(0) := B_\delta(0) \cap \mathbb{R}_+^n$, where $\mathbb{R}_+^n = \{x_n > 0\}$ is the half space.

Given $\Lambda > 0$, we denote by $\mu_{\alpha, \Lambda}(\Omega)$ the constant defined in (1.3). The strategy is analogous to the one of Theorem 1.1. Next Lemma is the counterpart of Lemma 3.1, and it can be seen as a Hardy-Sobolev type inequality with the boundary singularity. We follow ideas from [12], where the local version is studied.

**Lemma 4.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $0 \in \partial \Omega$, and $\partial \Omega$ is flat near 0. Then, there exists a positive constant $C_2 := C_2(\Omega, n, s) > 0$ such that

$$\mu_{\alpha, \Lambda}(\Omega) \geq \mu_\alpha \geq \mu_{\alpha, \Lambda}(\Omega),$$

where we have used (1) from Lemma 2.2 in the last inequality. This contradiction finishes the proof. □
Proof. By hypothesis, there exists $\delta > 0$ such that $B_\delta(0) \cap \Omega = B_\delta^+(0) := B_\delta(0) \cap \mathbb{R}_+^n$, up to a rotation of $\Omega$. Let $u \in H^s(\Omega)$. Then,

$$
\left( \int_{\Omega} \frac{|u|^{2s,\alpha}}{|x|^\alpha} \, dx \right)^{\frac{2}{2s,\alpha}} = \left( \int_{B_\delta^+(0)} \frac{|u|^{2s,\alpha}}{|x|^\alpha} \, dx + \int_{\Omega \setminus B_\delta^+(0)} \frac{|u|^{2s,\alpha}}{|x|^\alpha} \, dx \right)^{\frac{2}{2s,\alpha}} \\
\leq \left( \int_{B_\delta^+(0)} \frac{|u|^{2s,\alpha}}{|x|^\alpha} \, dx \right)^{\frac{2}{2s,\alpha}} + \left( \int_{\Omega \setminus B_\delta^+(0)} \frac{|u|^{2s,\alpha}}{|x|^\alpha} \, dx \right)^{\frac{2}{2s,\alpha}} \\
=: I_1 + I_2.
$$

To estimate the first term, we consider the even reflexion of $u$ in the whole ball $B_\delta(0)$, that is

$$
\tilde{u}(x', x_n) := \begin{cases} u(x', x_n) & \text{if } x \in B_\delta^+(0) \\
u(x', -x_n) & \text{if } x \in B_\delta(0) \setminus B_\delta^+(0),
\end{cases}
$$

therefore, $\tilde{u} \in H^s(B_\delta(0))$, and $[\tilde{u}]^2_{B_\delta^+(0)} \leq 4[u]^2_{s,\Omega}$, see for instance, [5, Lemma 5.2].

The term $I_1$ can be rewritten in terms of $\tilde{u}$. By Lemma 3.1 for $\tilde{u} \in H^s(B_\delta(0))$, we obtain

$$
I_1 = \left( \frac{1}{2} \right)^{\frac{2}{2s,\alpha}} \int_{B_\delta^+(0)} \frac{|\tilde{u}|^{2s,\alpha}}{|x|^\alpha} \, dx \leq \left( \frac{1}{2} \right)^{\frac{2}{2s,\alpha}} \frac{2}{\mu_\alpha} \left( [\tilde{u}]^2_{s,B_\delta(0)} + C_1 \int_{B_\delta^+(0)} |\tilde{u}|^2 \, dx \right) \\
\leq \frac{2^{1-n-2s}}{\mu_\alpha} \left( 4[u]^2_{s,B_\delta^+(0)} + 2C_1 \int_{B_\delta^+(0)} |u|^2 \, dx \right).
$$

Consequently,

(4.2)

$$
I_1 \leq \frac{2^{n-2s}}{\mu_\alpha} \left( [u]_{s,\Omega}^2 + C \int_{\Omega} |u|^2 \, dx \right).
$$

To find the bound for $I_2$, let $\eta > 0$ and $\{\phi_i\}_{i \in I}$ be a partition of unity on $\overline{\Omega \setminus B_\delta^+(0)}$ such that $\phi_i^\perp \in C^1(\Omega \setminus B_\delta^+(0))$, $|\text{supp } \phi_i| \leq \eta$ for every $i \in I$. Notice that $\frac{1}{|x|^\alpha} \leq \frac{1}{\delta^{2s}}$ in $\Omega \setminus B_\delta^+(0)$. Then,

$$
I_2 \leq \sum_{i \in I} \left( \int_{\Omega \setminus B_\delta^+(0)} \frac{|\phi_i^\perp u|^{2s,\alpha}}{|x|^\alpha} \, dx \right)^{\frac{2}{2s,\alpha}} \leq \frac{1}{\delta^{2s,\alpha}} \sum_{i \in I} \left( \int_{\Omega \setminus B_\delta^+(0)} \frac{1}{\delta^{2s}} |\phi_i^\perp u|^{2s,\alpha} \, dx \right)^{\frac{2}{2s,\alpha}}
$$

By Hölder’s inequality with $p = \frac{2^s}{2s,\alpha} = \frac{n-\alpha}{n-\alpha}$, $p' = \frac{n}{\alpha}$, we get

$$
I_2 \leq \frac{1}{\delta^{2s,\alpha}} \sum_{i \in I} \left( |\text{supp } \phi_i|^{\frac{n}{n-\alpha}} \left( \int_{\text{supp } \phi_i} \frac{1}{\delta^{2s}} |\phi_i^\perp u|^{2s,\alpha} \, dx \right)^{\frac{n-\alpha}{n}} \right)^{\frac{2}{2s,\alpha}} \\
\leq \frac{\eta^{\frac{(n-2s)}{n-\alpha}}}{\delta^{2s,\alpha}} \sum_{i \in I} \left( \int_{\text{supp } \phi_i} \frac{1}{\delta^{2s,\alpha}} |\phi_i^\perp u|^{2s,\alpha} \, dx \right)^{\frac{2}{2s,\alpha}} \leq \frac{\eta^{\frac{(n-2s)}{n-\alpha}}}{\delta^{2s,\alpha}} \sum_{i \in I} \left( \int_{\Omega \setminus B_\delta^+(0)} \frac{1}{\delta^{2s,\alpha}} |\phi_i^\perp u|^{2s,\alpha} \, dx \right)^{\frac{2}{2s,\alpha}}.
$$
Since $\phi_i^{1/2} u \in H^s(\Omega \setminus B_\delta^+(0))$ for every $i \in I$, denoting by $S_{\Omega \setminus B_\delta^+(0)}$ the best Sobolev constant of the embedding $H^s(\Omega \setminus B_\delta^+(0)) \hookrightarrow L^2(\Omega \setminus B_\delta^+(0))$, we get
\[
I_2 \leq \frac{\eta^{n(\alpha-2\alpha)}}{\delta^{2n\alpha}} S_{\Omega \setminus B_\delta^+(0)}^{-1} \sum_{i \in I} \left( [\phi_i^{1/2} u]_{s,\Omega \setminus B_\delta^+(0)}^2 + \|\phi_i u\|_{L^2(\Omega \setminus B_\delta^+(0))}^2 \right).
\]

Using (2.1) for $\phi_i^{1/2} u$, and Minkowski’s inequality, as we have done in Lemma 3.2, we obtain
\[
[\phi_i^{1/2} u]_{s,\Omega \setminus B_\delta^+(0)}^2 \leq 2 \int_{\Omega \setminus B_\delta^+(0) \times \Omega \setminus B_\delta^+(0)} \frac{\phi_i(x)|u(x) - u(y)|^2}{|x - y|^{n + 2s}} dxdy + C_{i,n,s} \int_{\Omega \setminus B_\delta^+(0)} |u(x)|^2 dx,
\]
where $C_{i,n,s} = C(\phi_i, n, s)$. Therefore, going back to the estimate of $I_2$, we get
\[
I_2 \leq \eta^{n(\alpha-2\alpha)} S_{\Omega \setminus B_\delta^+(0)}^{-1} \sum_{i \in I} \int_{\Omega \setminus B_\delta^+(0)} \frac{\phi_i(x)|u(x) - u(y)|^2}{|x - y|^{n + 2s}} dxdy + \left( \sum_{i \in I} C_{i,n,s} \int_{\Omega \setminus B_\delta^+(0)} |u(x)|^2 dx + \sum_{i \in I} \|\phi_i u\|_{L^2(\Omega)}^2 \right)
\]
\[
\leq \frac{\eta^{n(\alpha-2\alpha)}}{\delta^{2n\alpha}} S_{\Omega \setminus B_\delta^+(0)}^{-1} \sum_{i \in I} \left( [u]_{s,\Omega}^2 + C(\Omega, n, s) \int_{\Omega} |u(x)|^2 dx \right),
\]
recall that $\sum_{i \in I} \phi_i = 1$. Hence, by choosing $\eta > 0$ small enough, we get
\[
I_2 \leq \frac{\eta^{n(\alpha-2\alpha)}}{\delta^{2n\alpha}} S_{\Omega \setminus B_\delta^+(0)}^{-1} \sum_{i \in I} \left( [u]_{s,\Omega}^2 + C \int_{\Omega} |u(x)|^2 dx \right) \leq \frac{2^{2n-\alpha+2}}{\mu_\alpha} \frac{\eta^{\alpha-2\alpha}}{\delta^{2n\alpha}} \left( [u]_{s,\Omega}^2 + C \int_{\Omega} |u(x)|^2 dx \right).
\]

Finally, gathering estimates for (4.2) and (4.3), we get (4.1). \hfill \Box

We conclude this section announcing the counterpart of Lemma 3.2, that allows us to prove Theorem 1.2. We skip the proof due to the similarity with Theorem 1.1, where

\[
\Lambda_* := \inf \{ C > 0 : (4.1) \text{ holds} \},
\]
and Lemmas 4.1 and 4.2 are used, instead of Lemmas 3.1 and 3.2.

**Lemma 4.2.** Let $\Lambda > 0$, $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $0 \in \partial \Omega$ and $\partial \Omega$ is flat near 0.

(1) If $\mu_{a,\Lambda}(\Omega) < \frac{\mu_\alpha}{2^{n-\alpha+2}}$, then $\mu_{a,\Lambda}(\Omega)$ is attained.

(2) If there exists a $\bar{\Lambda} > 0$ such that $\mu_{a,\Lambda}(\Omega) = \mu_\alpha$, then $\mu_{a,\Lambda}(\Omega)$ is not attained for every $\Lambda > \bar{\Lambda}$.

**Remark 4.3.** We would like to emphasize that the fractional Hardy-Sobolev type inequality of Lemma 3.1 (inner singularity case) allows us to prove that the limit function of a minimizing sequence for the constant $\mu_{a,\Lambda}(\Omega)$ is not the trivial function, for those values $\lambda > 0$ such that

\[
\mu_{a,\lambda}(\Omega) < \frac{\mu_\alpha}{2},
\]
see (3.7) and (3.8) in Lemma 3.2. As a consequence of the factor $\frac{1}{2}$ in (3.1), the gap $[\lambda_*, \lambda^*]$ from Theorem 1.1 arises. With the techniques used here, we cannot deduce an existence result in $[\lambda_*, \lambda^*]$, which is a difference with the local case [12, Lemma 2.2, Theorem 1.1].
However, dealing with the boundary singularity, we use the same argument to prove that the limit function of a minimizing sequence for the constant $\mu_{\alpha,\Lambda}(\Omega)$ is not trivial, for those values $\Lambda > 0$ such that $\mu_{\alpha,\Lambda}(\Omega) < \frac{\mu_{\alpha}}{2^{\frac{2s}{n-s}+2}}$, see Lemma 4.2. As in the inner singularity, that generates the gap $[\Lambda^{*}, \Lambda^{*}]$ from Theorem 1.2 as well. In the local case [12, Theorem 4.1], we do not find it completely clear how the author discards having a gap. In [12, Lemma 4.2], the corresponding Hardy-Sobolev type inequality involves a factor $2^{\frac{2s}{n-s}}$, which is less than one for $n > s$ and $0 < s < 2$. Through the techniques presented in [12], it should be a gap in the existence result [12, Theorem 4.1].

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