Probing non-Abelian anyonic statistics with cold atoms in an optical lattice

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We propose a scheme to probe the non-Abelian statistics of the collective anyonic excitation in Kitaev's honeycomb model with cold atoms in an optical lattice. The generation of the anyonic excitation can be realized by simple rotating operation acting on an effective spin-$1/2$ system, which is encoded in the atomic hyperfine energy levels. The non-Abelian nature of the anyonic excitation is manifested by the braiding of four vortices, which leads to different operations on the subspace of degenerate ground states, and thus results in different final states. Here, by introducing an ancilla atom, the effective control over the lattice atoms can be realized and the final different states can also be imprinted on the ancilla and further distinguished by measurement.

Quantum statistics is one of the basic concepts in quantum physics [1]. It’s also the root of some condensed matter physical phenomena, such as, superfluid, Bose-Einstein condensates, etc. In three-dimension, due to the rotating and time reversal symmetries, particles obey either the Bose-Einstein or Fermi-Dirac statistics corresponding to their statistical phase \(\alpha\) \((\alpha = 0\) or \(1\)). But, there exist cases with \(\alpha \neq 0\) or \(1\) corresponding to more exotic particles called anyons, which maybe either Abelian or non-Abelian because of symmetry breaking in 2-dimension [2]. The elementary excitation with fractional charge in fractional quantum Hall effects (FQHE) is anyon satisfying exotic statistics [3, 4], which provides a good platform to understand anyonic statistics. For Abelian anyons, the anyonic wave-function acquires a phase factor coming from the winding of two quasiparticles, which is just the one-dimensional representation of the braid group \(B_N\) (for \(N\) indistinguishable particles). For non-Abelian anyons, quasiparticles exchange would be represented by a matrix acting on the subspace of the degenerate ground states, which is an exactly non-commutative representation of the braid group in higher dimension [5]. In particular, Kitaev showed [6] that non-Abelian anyon is a promising candidate for topological quantum computation, which is fault-tolerant at the hardware level.

Up to now, no experiment has unambiguously verified the existence of Abelian or non-Abelian anyons although promising progress have been made in \(\nu = 5/2\) FQHE state [7]. This is mainly due to the extremely complicated noise channels in a quantum many-body solid-state system. On the other hand, a cold atom system possesses unprecedented possibility of controlling almost all relevant physical parameters [8] and thus been recognized as an ideal system for quantum simulation. Proposals of observing the non-Abelian statistics are also presented with cold atoms in \(s\)-wave [9, 10] and \(p\)-wave [11] superfluidity. For Kitaev’s toric model, it has been proposed to simulate the anyonic interferometry in various systems [12–19] with experimental verification in optical systems [20–22] using a method of generating dynamically the ground states and the excitations of the model Hamiltonian. However, there is few reports about the verification of non-Abelian statistics in the Kitaev honeycomb lattice model [23]. Recently, theoretical implementation of this model has been proposed with cold atoms in an honeycomb lattice [24]. Meanwhile, elementary experimental verification of the Dirac physics associated has also been made with cold atoms in a honeycomb lattice [25] based on theoretical proposals in [26, 27].

Here, motivated by the above advances, we propose a concrete example of probing the exotic non-Abelian statistics in the Kitaev honeycomb model with cold atoms in optical lattice. The honeycomb lattice is constructed from cold atoms located in an optical lattice and the anisotropic spin interaction is induced from spin-dependent tunneling between the nearest neighbor of the lattice [24]. To get effective control over the lattice atoms, following Ref. [14], we introduce an auxiliary atom. Then, creation and manipulation of anyons are implemented by Rydberg gates based on dipole-dipole interaction between the ancilla and lattice atoms. Finally, the exotic non-Abelian statistics can be recorded by the ancilla during the implemented braiding process. Therefore, detect the final state of the ancilla can reveal the non-Abelian statistics. Furthermore, it is usually difficult to read out the states of the anyons due to the fact that they are degenerated and neutral. Therefore, we introduce an auxiliary atom to imprint the difference of the anyonic final states after braiding. Meanwhile, for probing the non-Abelian statistics, one needs at least four anyons. If only two anyons are involved, the final state after braiding will be different from the initial state by a phase factor. However, this difference is insufficient to
verify that the involved anyons are of non-Abelian nature since braiding of Abelian anyons may also cause such a phase difference.

Let us begin with a brief review of the Kitaev honeycomb lattice model, which is an anisotropic spin-1/2 model with different nearest neighbor interactions in different bonds [23]

\[
H = -J_x \sum_{x-\text{links}} \sigma^x_j \sigma^x_k - J_y \sum_{y-\text{links}} \sigma^y_j \sigma^y_k - J_z \sum_{z-\text{links}} \sigma^z_j \sigma^z_k,
\]

where $\sigma^\nu_j$ with $\nu \in \{x, y, z\}$ being the pauli matrix on the site $j$, $J_\nu$ are coupling strength for different links as indicated in Fig. 1. To get the ground states, diagonalization of the model $H$ is needed, which can be greatly simplified by constructing a set of integrals of motion $W_p = \sigma^x_j \sigma^x_k \sigma^y_j \sigma^y_k \sigma^z_j \sigma^z_k$ with $p$ being a label of a plaquette of the lattice and the spin index is labeled as in Fig. 1(a). Expanding every spin into a four Majorana operators notation, the model Hamiltonian in the enlarged space reads [23]

\[
H = \frac{i}{4} \sum_{j,k} \hat{A}_{jk} c_j c_k,
\]

where $\hat{A}_{jk} = 2J_\nu \hat{u}_{jk}$ if spins $j$ and $k$ are connected, otherwise $\hat{A}_{jk} = 0$. Here, $\hat{u}_{jk} = i\theta_j \theta_k$ with $b_j$ and $c_j$ being Majorana operators, $[H, \hat{u}_{jk}] = 0$, $\hat{u}^2_{jk} = \hat{u}_{jk}$, and $\hat{u}^3_{jk} = 1$. We first divide the total original Hilbert space into different sector which is eigenspace of $W_p$. So $W_p$ can be transformed to the eigenspaces of a set of states given by a certain configuration of eigenvalue of $\hat{u}_{jk}$. The connection between $w_p$ (eigenvalue of the $W_p$) and $u_{jk}$ (eigenvalue of $\hat{u}_{jk}$) is $w_p = \prod_{(j,k) \in \text{boundary}(p)} u_{jk}$. The variables $u_{jk}$ and the numbers $w_p$ can be interpreted as a $\mathbb{Z}_2$ gauge field and the magnetic flux through the plaquette $p$, respectively. We say that the plaquette $p$ carries a vortex when $w_p = -1$ and the ground states are denoted by the vortex-free sector, i.e. all $w_p = 1$. Then go to the momentum representation via the Fourier transformation, we can get the ground energy, ground states and the phase diagram. In the triangle region of $|J_x| \leq |J_y| + |J_z|$, $|J_y| \leq |J_x| + |J_z|$ and $|J_z| \leq |J_x| + |J_y|$, it’s gapless phase B with non-Abelian anyonic excitation [23]; otherwise, it’s gapped phase A with Abelian anyonic excitation. A magnetic field $V = -\sum_j (h_x \sigma^x_j + h_y \sigma^y_j + h_z \sigma^z_j)$ is introduced to open a gap between the ground and the excited states in phase B [23]. This gap will exponentially suppress the interaction between vortices and fermionic modes near the singularity of the spectrum, and thus makes the statistics of non-Abelian anyon well-defined.

Now, we turn to construct the honeycomb model of Kitaev using ultracold fermionic atoms in an optical lattice. Following Ref. [24], we consider a cloud of ultracold Rb atoms trapped in an optical lattice which are formed by interfering standing laser beams. First, completely suppress the tunneling and spin interchange in the vertical $Z$ direction by raising the corresponding barrier potential, and thus the atoms distribution form a two-dimensional configuration in the $X-Y$ plane. Assuming that the system in the Mott insulator regime and the atomic density is roughly one atom per site. Each atom is treated as a two-level system, labeled with the effective spin index $\sigma = \uparrow, \downarrow$, with an energy splitting of $6834 \text{ MHz}$. To form the honeycomb lattice, we apply trapping potentials $V_{j}(x, y) = V_0\sin^2(k_0(x \cos \theta_j + y \sin \theta_j) + \phi_0)$ formed by a pair of blue-detuned traveling laser beams above the X-Y plane with an angle $\phi_{||j}$, where $j = 1, 2, 3$, $\theta_{1(2,3)} = \pi/6(\pi/2, -\pi/6)$, $\phi_{||j} = 2 \arcsin(1/\sqrt{3})$, and $\phi_0 = \pi/2$. To engineer the wanted interaction in different directions, applying the following three blue-detuned standing-wave laser beams $V_{\nu j}(x, y) = V_0\sin^2(k_0(x \cos \theta_{\nu j} + y \sin \theta_{\nu j}))$ in the X-Y plane along certain directions, where $\nu$ denote the tunneling direction x-x, y-y, z-z, $\theta_{\nu x-x} = -\pi$, $\theta_{\nu y-y} = \pi$, $\theta_{\nu z-z} = \pi/3$. These laser beams will introduce spin-dependent potentials $V_{\nu j} = V_{\nu j}(+\nu)\langle +\nu| + V_{\nu j}(-\nu)\langle -\nu|$, where $\langle +\nu|$ and $\langle -\nu|$ are the eigenvectors of the eigenvalues 1 or -1 of Pauli operator $\sigma^\nu$ (cf. Fig. 1), respectively. That is to say, different spin directions will feel different effective potential, and thus have different tunneling rates. The potential depth between the nearest neighbor along the $\nu$ direction can be changed from $V_0/4$ to $V_0/4 + V_{\nu j}$ [24]. The parameters $V_{\nu \pm}$ can be tuned by varying the laser intensity of $L_1$ and $L_2$ in the $\nu$ direction. Here, we are interested in the regime $t_{+u} \gg t_{-\nu}$, i.e., the atom can virtually tunnel only when it is in the eigenstate of $|+\nu\rangle$. Then, the spin-dependent potentials will lead to anisotropic Ising exchange interaction for each tunneling direction, and the total interaction is in the form of [24]

\[
H_{\text{eff}} = -\sum_{\langle i,j\rangle, \nu} J_{\nu i} \sigma^\nu_i \sigma^\nu_j - h_{\nu} \sum_{j\nu} \sigma^\nu_j.
\]
In the case of $V_{-}/V_{+} \gg 1$ and $U_{-} \approx U_{-} \approx U$, where $U_{-}$ and $U_{+}$ are the intra- and interspin coupling energy, $J_{v} = t_{+}^{2}/(2U)$ and $h_{v} = 4t_{+}^{2}/U$. Then, we get a Hamiltonian of Eq. (1) with additional effective magnetic field terms $h_{v} \sum_{j} \sigma_{v}^{x}$, which can be used to open the gap in phase B. Therefore, this implementation naturally leads to the effective magnetic field terms, and thus makes it more easier for realization.

To probe the non-Abelian statistics, we need at least two pairs of non-Abelian anyons due to their superselection rule. The generation of a pair of anyons can be realized by a simple $\sigma^{z}$ rotation acting on a single spin, which is in a vortex-free configuration. But, the generation process will accompany with fermionic excitations in the mean time. Fortunately, the unwanted fermionic excitations can be exponentially suppressed by the energy gap opened by the effective magnetic field. The $\sigma^{z}$ rotation may be implemented by laser-atom interaction, but single atom addressing is difficult experimentally. To avoid such difficulty [14], we consider an ancilla atom that is trapped above $X - Y$ plane by three standing wave laser beams along three direction with its internal Zeeman levels $|0\rangle_{a}$ and $|1\rangle_{a}$. The $\sigma^{z}$ rotation can be realized by Rydberg gates based on dipole-dipole interactions between the ancilla and a lattice atom [28, 29], which maximally reduce the addressing difficulty and eliminates the need for cooling both the lattice and ancilla to the physical ground states in the cold collisions implementation [30]. On the other hand, this single ancilla atom can be moved by trap potential without decoherence [31] and this is important to guarantee our implementation. As a result, 2-qubit unitary operations $U_{i}^{a} = |0\rangle_{a} \otimes I + |1\rangle_{a} |1\rangle \otimes \sigma_{v}^{x}$ acting on site atom $i$ and the ancilla atom are implemented, which are sufficient to create and manipulate the anyons [23], as shown in Fig. 2. Experimentally, an arbitrary qubit hopping time is $8\mu s$ with a nearly unit fidelity [32]. On the condition that the ancilla atom is in the $|1\rangle_{a}$ state, $\sigma_{v}^{x}$ rotations can be implemented on site $i$ atom. The generation of two vortex configuration can be represented as

$$H_{2-v} = \sigma_{i}^{z} H_{v} \sigma_{i}^{z} = H + 2J_{x} \sigma_{i}^{x} \sigma_{j}^{x} + 2J_{y} \sigma_{i}^{y} \sigma_{j}^{y},$$

where $\sigma_{v}^{z}$ denotes a $\sigma^{z}$ rotation on site $i$, $(i, j)$ label a x-link, and $(i, k)$ labels a y-link. Meanwhile, we can also get two vortices via a $\sigma_{i}^{y}$ or $\sigma_{i}^{y}$ rotation but with different anyons configuration. This simple configuration due to $u_{i} = -1$ at the site $i$ along the $z - z$ direction with the neighbor honeycomb lattice $\omega_{p} = -1$, and the vortex-free configuration as it gives $\omega_{p} = 1$ for all plaquettes $p$. The exotic collective excitation are non-Abelian anyons bound to the vortices, and the non-Abelian anyons are Majorana fermions [23]. At low temperature, the initial state of the pair of Majorana fermions is typically a vacuum state $|0\rangle_{v}$ [11].

To make our scheme more robust to decoherence and avoid the problem of single atom addressing, we use the conditional generation of anyons using $U_{i}^{v}$. When $\alpha = z$, the ancilla atom, vortex-free ground state $|gs\rangle$ and 2-vortex state $|0\rangle$ are entangled as shown in Fig. 2, i.e. $|0\rangle_{a}|gs\rangle + |1\rangle_{a}|0\rangle$. Therefore, the exotic non-Abelian statistics can be recorded by the ancilla atom, i.e., we can also use the state of the ancilla atom to manifest the non-Abelian statistics of anyon. The state of ancilla atom can be addressed independently, and thus can be measured by standard probing technique.

We next consider the simplest vortices configuration that can experimentally reveal the non-Abelian statistics as shown in Fig. 3, which consists of two pairs of vortices. To reveal the non-Abelian statistics, we would like to move one anyon (Majorana fermion 3) around the other (Majorana fermion 2), where the two come from different Majorana fermion pairs. Indeed, this braiding can be implemented by the operator $s_{23} = \sigma_{3}^{z} \sigma_{5}^{z} \sigma_{4}^{z} \sigma_{6}^{z}$ [23]. For a system with 4 vortices, the degeneracy of the ground states is 4. For an adiabatic braiding of vortices 2 and 3 once, the effect is a unitary operation acting
implement a "6" atom so that their collision interaction is in effect, ie., with the ancilla in the state $|v_0\rangle$. Because the braiding will be implemented between anyons 2 and 3 and the anyons 1 and 4 are not surrounded in the braiding pathway, so it is convenient to work in the subspace of vortex 2 and 3 on which operation $R_2$ act. Thirdly, move the ancilla close to the target atom "6" again to realize $U_6^z$ gate operation, and then "1" to "5" successively to realize the braiding operation $S_{23} = U_6^y U_4^y U_3^y U_2^y U_1^y U_0^1$ when the ancilla atom is in the $|1\rangle_a$ state (cf. Fig. 3. ). After braiding of anyons 2 and 3 twice, we get $|0\rangle_a|v_2\rangle + e^{-i\frac{\pi}{2}}|1\rangle_a R_2^2|v_4\rangle$ with $R_2^2|v_4\rangle$ is a degenerate ground state with a fermionic occupation. The vortex pair state of 2 and 3 changes from the vacuum to a fermionic state with a fermion occupation. Finally, repeat the second and first steps to bring the system back to the vortex-free state, the final state of the system is $|0\rangle_a + e^{-i\frac{\pi}{2}}|1\rangle_a|gs\rangle$ with the ancilla in the state of $|f\rangle \propto |0\rangle_a + e^{-i\frac{\pi}{2}}|1\rangle_a$. Then, compare the initial and the final states of the ancilla one can find that a phase factor $-\pi/2$ is generated due to the non-Abelian nature of anyons. The recorded interferometry result of the non-Abelian statistics can be read from the ancilla atom by a local projective measurement after a $\pi/4$ phase gate. The single atom gate and measurement can be realized with very high efficiency.

Note that while the anyon obeys Abelian statistics, our procedure will bring the final state of the ancilla to $|0\rangle - |1\rangle$ (braiding once); for bosonic or fermionic statistics, the ancilla’s state will stay unchanged. Therefore, probe the final state of the ancilla will verify the no-Abelian statistics. On the other hand, for no-Abelian anyons, when the braiding is repeated twice and four times, the final states of the ancilla atom will be the same as the Abelian anyon case (braiding once) and be identical to the initial state (up to an overall $\pi$ phase as $R_2^4 = -1$), respectively. For Abelian anyon, two successive identical braiding will lead the ancilla’s final state to be the same as its initial state. This can serve as further distinction of the Abelian and no-Abelian anyons in this particular model.

In summary, we propose a scheme to probe the non-Abelian statistics of Majorana fermions bounding to the vortex excitation in the Kitaev honeycomb lattice model with effective magnetic field using cold atom in an optical lattice. The exotic non-Abelian statistics can be recorded and detected by the internal state of the ancilla atom, the measurement of which can be realized with very high efficiency by standard techniques.

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