Symmetry Algebras of Large-$N$ Matrix Models for Open Strings

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Abstract

We have discovered that the gauge invariant observables of matrix models invariant under $U(N)$ form a Lie algebra, in the planar large-$N$ limit. These models include Quantum Chromodynamics and the M(atrix)-Theory of strings. We study here the gauge invariant states corresponding to open strings (‘mesons’). We find that the algebra is an extension of a remarkable new Lie algebra $\mathcal{V}_A$ by a product of more well-known algebras such as $gl_{+\infty}$ and the Cuntz algebra. $\mathcal{V}_A$ appears to be a generalization of the Lie algebra of vector fields on the circle to non-commutative geometry. We also use a representation of our Lie algebra to establish an isomorphism between certain matrix models (those that preserve ‘gluon number’) and open quantum spin chains. Using known results from quantum spin chains, we are able to identify some exactly solvable matrix models. Finally, the Hamiltonian of a dimensionally reduced QCD model is expressed explicitly as an element of our Lie algebra.

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1 Introduction

One of the major problems in physics is to understand Yang-Mills theories such as Quantum ChromoDynamics (QCD). It is natural to look for a classical approximation to this difficult problem. The obvious classical limit is to let Planck’s constant $\hbar$ go to 0. Unfortunately, this limit is ill-suited for describing long-distance phenomena such as quark confinement. Another, less obvious classical limit is to let the number of colors $N$ go to $\infty$. This large-$N$ limit smooths out quantum fluctuations of gauge-invariant observables, and is believed to manifest not only asymptotic freedom but also quark confinement and so should describe long-distance interactions among quarks better.

We obtained the Poisson algebra of observables in this large-$N$ limit in a previous paper [1]. It was highly nonlinear, reflecting the dynamical fact that the large-$N$ limit of QCD is a nonlinear classical theory. A further approximation is needed. This is the theory of small oscillations around the ground state, which can also be understood as the sum of planar diagrams in perturbation theory. In this paper we will study this ‘planar large-$N$ limit’ of matrix models such as QCD. We will not use directly the results of Ref.[1]; instead we will give a direct argument starting with the canonical (anti)-commutation relations.

In a previous Letter [2], we reported on a Lie algebra describing the symmetry of closed string states like glueballs. The present paper serves as a mathematical foundation for the results in that Letter. Here we will deal exclusively with open string states. Building upon the mathematical results here, we will turn to closed string states, the mathematics of which turns out to be more tricky, and give a detailed exposition of the results in that Letter in the next paper [3].

We will not study the problems of Yang-Mills theories directly here; instead we will look at some matrix models for Yang-Mills theories. In this sense the present work fits into the program of studying Universal Yang-Mills Theories [4]. The case with fermionic matter fields in Universal Yang-Mills theory was studied in Ref.[5]. In our present language Universal Yang-Mills Theory corresponds to the case $\Lambda = 1$ with just one matrix degree of freedom. This thus generalizes the considerations of Ref.[3] to the case where there are several matrices modeling gluons. For an earlier attempt to consider multi-matrix models from this point of view, see Ref.[4].
It is well known [7, 8] that in the planar limit mesons and glueballs are stable, non-interacting particles. Gauge-invariant observables, such as the Hamiltonian, act as linear operators on these states in the light-cone formalism [9] in which (unlike in some other quantization schemes) the ground state is the same as that in free theory — we do not need to worry about non-perturbative zero mode contributions. We find that the gauge-invariant observables form a Lie algebra in this limit. Just as the representation theories of Kac-Moody and Virasoro algebras allow us to solve conformal field theories, we believe that the study of our Lie algebra will yield a solution to Yang-Mills theory in the planar limit. Indeed any attempt to understand the spectrum of QCD analytically has to begin with a solution of the planar limit. This means that we need to identify the symmetries peculiar to this limit and learn to exploit them to the fullest.

The idea that a dynamical symmetry will help solve for the spectrum of a quantum theory is familiar from other problems of theoretical physics. Perhaps the oldest examples are the hydrogen atom and the harmonic oscillator. In the former, the dynamical symmetry is SO(4) and the Hamiltonian can be written as a rational function of the generators of this Lie algebra. The representation theory of SO(4) then yields the spectrum. (This was Pauli’s solution of the hydrogen atom, done even before the Schrödinger equation was discovered.) In the case of the harmonic oscillator the symmetry is the symplectic Lie algebra $sp(2)$, which is the algebra of derivations of the canonical commutation relations. The Hamiltonian is an element of $sp(2)$. The familiar solution of the harmonic oscillator in terms of creation-annihilation operators has exactly this meaning. If we understand the dynamical symmetry of planar QCD equally well, we would be able to solve for its spectrum by a similar method. We will give an example of this in a simplified model of QCD, in a variational approximation.

Conformal field theory provides more recent examples of the solution of a quantum theory by Lie algebraic methods [10]. The underlying symmetry of the Wess-Zumino-Witten model (for example) is the semi-direct product of the Kac-Moody algebra by the Virasoro algebra; the latter acts as derivations on the former. The Hamiltonian is an element of the Virasoro algebra. The partition function of the WZW model can then be obtained from the Kac-Weyl character formula: this determines the eigenvalues of the Hamiltonian. The corresponding wavefunctions can be understood as sections of the determinant line on coset spaces of the Loop Group [11].
We are at the moment a long way from such a complete understanding of planar QCD. In this paper we take a crucial step in this direction: we identify its symmetry algebra. Somewhat surprisingly we find that it has a structure reminiscent of conformal field theory. Indeed it turns out that planar QCD is a non-commutative generalization of conformal field theory. The analogue of Kac-Moody algebra is the Cuntz algebra \[12, 13\]: it is the current algebra of planar QCD\(^1\).

There is even a natural chiral structure for the current algebra we find. There are two commuting (up to extensions) copies of the Cuntz algebra that act on the left end (the end containing an antiquark) and right end (the one with a quark) of a meson state. We discover the analogue of the Virasoro algebra \[16\] which we will call the centrix algebra. It consists of operators that act on the gluons in the meson state. It acts as derivations on the Cuntz algebra, exactly like the action of the Virasoro algebra on the Kac-Moody algebra. Indeed, in the special case of one degree of freedom for the gluons, our algebras just reduce to the Kac-Moody-Virasoro algebra. Even this special case is useful: it determines the spectrum of some models for QCD within a variational approximation.

We discover that the symmetry algebra of planar QCD has ideals isomorphic to the algebra of finite rank matrices. This shows that the essential spectrum of the planar QCD Hamiltonian is determined by the quotient algebra. Thus the essential spectrum may be easier to determine; this is also suggested by the analogy with the theory of Toeplitz operators. Thus our mathematical study suggests some strategies for the kind of physical approximations we should make in solving planar QCD.

It has been suspected at least since ‘t Hooft’s paper \[7\] that planar Yang-Mills theory is closely related to a string theory. Indeed, recent works \[17, 18\] indicates that M-theory, the latest incarnation of string theory, is a Matrix model. Thus our Lie algebra is also the symmetry algebra of M-theory. A special case of our Lie algebra is just the Virasoro algebra, suggesting that our theory is indeed a generalization of string theory.

We will find that the Lie algebra of observables has several ideals isomorphic to well-known algebras such as \(gl_{+\infty}\) and Cuntz algebras and their

\(^1\)Previously, Gopakumar and Gross found that the Cuntz algebra was relevant to the master field \[14\]. Even in this special case, our discussion will be more complete and direct, being in the Hamiltonian rather than the perturbative formalism. Another previous work on relating the Cuntz algebra and the large-\(N\) limit was done by Turgut \[15\].
products. After we quotient by these ideals we will find that what is left over is a remarkable new Lie algebra $\mathcal{V}_\Lambda$. When $\Lambda = 1$ this is just the Lie algebra of (polynomial ) vector fields on a circle, the Virasoro algebra, up to an extension. In the general case it is, roughly speaking, the Lie algebra of outer derivations of the Cuntz algebra. Now, the Cuntz algebra is a non-commutative generalization of the algebra of functions on a circle; and derivations are non-commutative generalizations of vector fields. Thus we can think of $\mathcal{V}_\Lambda$ as the algebra of vector fields on a non-commutative manifold, whose algebra of functions is Cuntz algebra. This interpretation is particularly striking in the context of M-theory and its connection to quantum theories of gravity.

If we have found an essential new symmetry algebra, it ought to help us solve exactly some matrix models. We are in fact able to do this by using our Lie algebra to prove an isomorphism between certain classes of matrix models (those that preserve ‘gluon number’) and quantum spin chain systems. By translating known exactly solvable spin chains into the language of matrix model we identify some solvable matrix models. In the case of open spin chains these are solutions to the Yang-Baxter and Sklyanin relations [19, 20, 21].

The synopsis of this paper is as follows. In Section 2, we will give a formulation of the physical states and the operators in the large-$N$ limit. Then, in Section 3, we will describe the current algebra in the large-$N$ limit. It has two parts — the leftix and rightix algebras of operators — that act on either end of a meson. In the following Section 4, we will talk about the algebra of the operators that act on the gluons, hence in the middle portion of a meson state. This will be the centrix algebra. Finally, in Section 5, we will piece together all the algebras we will have studied so far, together with the finite-dimensional matrix algebras for quarks and antiquarks, to form a grand Lie algebra.

We discuss a couple of illustrative examples of our algebra in the last sections. In Section 6, we will apply our algebra to study the relationship between quantum spin chain systems and matrix models. In Section 7, we will show that QCD can be formulated in terms of the grand algebra; we will also show to determine an approximate spectrum analytically.

In Appendix A we will summarize the notations used throughout this article.
2 Physical States and Operators

We begin by introducing the basic microscopic degrees of freedom of our theory, which satisfy the Canonical (Anti-)Commutation Relations:

\[
\begin{align*}
[a_{\mu_1}^{\dagger}(k_1), a_{\mu_4}^{\dagger}(k_2)] &= \delta_{k_1k_2} \delta_{\mu_4}^{\mu_3}; \\
[a_{\mu_2}^{\dagger}(k_1), a_{\mu_3}^{\dagger}(k_2)] &= 0; \\
[a_{\mu_1}^{\dagger}(k_1), a_{\mu_3}^{\dagger}(k_2)] &= 0; \\
[q_{\mu_1}^{\dagger}(k_1), q_{\mu_2}^{\dagger}(k_2)] &= \delta_{k_1k_2} \delta_{\mu_1}^{\mu_2}; \\
[q_{\mu_1}(k_1), q_{\mu_2}(k_2)] &= 0; \\
[q_{\mu_1}^{\dagger}(k_1), q_{\mu_2}(k_2)] &= 0; \\
[q_{\mu_1}(k_1), q_{\mu_2}^{\dagger}(k_2)] &= \delta_{k_1k_2} \delta_{\mu_2}^{\mu_1}; \\
[q_{\mu_1}^{\dagger}(k_1), q_{\mu_2}^{\dagger}(k_2)] &= 0; \\
[\bar{q}_{\mu_1}(k_1), \bar{q}_{\mu_2}(k_2)] &= \delta_{k_1k_2} \delta_{\mu_2}^{\mu_1}; \\
[\bar{q}_{\mu_1}^{\dagger}(k_1), \bar{q}_{\mu_2}^{\dagger}(k_2)] &= 0; \\
[\bar{q}_{\mu_1}^{\dagger}(k_1), \bar{q}_{\mu_2}(k_2)] &= 0; \\
[\bar{q}_{\mu_1}(k_1), \bar{q}_{\mu_2}^{\dagger}(k_2)] &= 0.
\end{align*}
\]

Here, \([A, B] = AB - BA\) is a commutator and \([A, B]_+ \equiv AB + BA\) an anti-commutator.

The bosonic operators \(a, a^{\dagger}\) create and annihilate fundamental degrees of freedom that we will call ‘gluons’. This comes from the application of our theory to regularized QCD, in which case these are gluon operators. (But other interpretations in terms of string bits \([22]\), D-branes \([23]\) or M-theory \([18]\) are also possible.) The indices \(\mu, \nu = 1, 2, \cdots, N\) label a degree of freedom we will call ‘color’ following this analogy. In the same way we will call the states created by \(q^{\dagger}\) ‘quarks’ and \(\bar{q}^{\dagger}\) ‘antiquarks’. To simplify the following discussion, we will mostly assume that gluons, quarks and antiquarks have only a finite number of distinct quantum states other than color. This can be done, for instance, by discretizing the momentum and other quantum numbers with continuous ranges of values. Let \(\Lambda\) be the possible number of distinct quantum states of a gluon (not counting color), and let us use the numbers 1, 2, \(\cdots\), \(\Lambda\) to denote these quantum states. Likewise, let us use \(\lambda, \rho = 1, 2, \cdots, \Lambda_F\) to label collectively the quantum numbers (other than color) of quarks or antiquarks.

We are now ready to construct open string states, or meson states. This is analogous to the way closed string states, or glueball states, are constructed...
by Thorn [24]. A meson consists of one quark, one antiquark and an arbitrary number of gluons. Hence, a typical colorless state of a meson can be written down as a linear combination of

\[ \bar{\phi}^{\rho_1} \otimes s^K \otimes \phi^{\rho_2} \equiv N^{-(c+1)/2} \bar{q}^{\mu_1} (\rho_1) a^{\nu_1}_{\mu_2} (k_1) \cdots a^{\nu_{c+1}}_{\mu_{c+2}} (k_{c+1}) q^{\nu_{c+2}} (\rho_2) |0\rangle, \]

where \( c = \#(\hat{K}) \). Note that it is possible for \( \hat{K} \) to be empty, which corresponds to a meson state containing no gluons. Moreover, the summation convention for color indices is implicit in the above and following expressions. The factor of \( N^{-(c+1)/2} \) is inserted so that the state has a finite norm in the large-\( N \) limit. The justification of the use of the symbol \( \otimes \) on the left hand side will be given shortly. A meson state given by Eq.(4) is depicted in Figs. (a) and (b).

Let us turn our attention to operators representing dynamical variables. As in models studied before (e.g., [25, 26, 22]), the only operators we will look at are those which can propagate single meson states to single meson states, in the leading order of the planar large-\( N \) limit. For a finite value of \( N \), they can also convert a single meson to a state with more than one meson or glueball. However, these terms are of order \( 1/N \) and so are suppressed in the planar large-\( N \) limit. This fact has been well known for a long time [24], and is closely related to the planarity of Feynman diagrams of perturbation theory [27]. A non-rigorous diagrammatic proof of this fact is given in Ref.[28].

There are four kinds of such gauge invariant operators that survive in the large-\( N \) limit. The simplest, operators of the first kind are finite linear combinations of operators of the form

\[ \Xi^{\lambda_1}_{\lambda_2} \otimes f^i_j \otimes \Xi^{\lambda_3}_{\lambda_4} \equiv N^{-(a+b+2)/2} \bar{q}^{\mu_1} (\lambda_1) a^{\nu_1}_{\mu_2} (i_1) \cdots a^{\nu_{a+b+1}}_{\mu_{a+b+2}} (i_{a+b+1}) q^{\nu_{a+b+2}} (\lambda_3) \cdot \]

\[ q_{\nu_1} (\lambda_2) a^{\nu_1}_{\nu_2} (j_1) \cdots a^{\nu_{b+1}}_{\nu_{b+2}} (j_{b+1}) q^{\nu_{b+2}} (\lambda_4), \]

where \( a = \#(\hat{I}) \) and \( b = \#(\hat{J}) \). In the planar limit, an operator of the first kind propagates a single meson state to either zero or another single meson state:

\[ \Xi^{\lambda_1}_{\lambda_2} \otimes f^i_j \otimes \Xi^{\lambda_3}_{\lambda_4} \left( \bar{\phi}^{\rho_1} \otimes s^K \otimes \phi^{\rho_2} \right) = \delta^{\lambda_1}_{\lambda_2} \delta^{\lambda_3}_{\lambda_4} \bar{\phi}^{\rho_1} \otimes s^K \otimes \phi^{\rho_2}. \]

It is now clear why we are using the direct product symbol \( \otimes \) — the single meson state can be regarded as a direct product of \( \bar{\phi}^{\rho_1}, s^K \) and \( \phi^{\rho_2} \).
Figure 1: (a) A typical single meson state with 3 gluons in detail. The solid square at the top is the creation operator of an antiquark of Quantum State \( \rho_1 \). Following the antiquark is a series of 3 gluons of Quantum States \( k_1 \), \( k_2 \), and \( k_3 \) respectively. The creation operators of these gluons are represented by solid circles. The solid square at the bottom is the creation operators of a quark of Quantum State \( \rho_2 \). Note that all the creation operators carry color indices. A solid line, no matter how thick it is, connecting two circles, or a circle and a square, is used to mean that the two corresponding creation operators share a color index, and this color index is being summed over. The arrow indicates the direction of the integer sequence \( \dot{K} \). (b) A single meson state in brief. The quark and aniquark are neglected. They will be consistently ignored in all brief diagrammatic representations. The gluon series is represented by the integer sequence \( \dot{K} \). (c) A typical operator of the first kind. The solid squares and circles are creation operators of a quark, an antiquark and gluons. The hollow squares are annihilation operators of an antiquark of Quantum State \( \lambda_2 \) and a quark of Quantum State \( \lambda_4 \), and the hollow circles are annihilation operators of gluons. In this particular example, there are 2 creation and 4 annihilation operators of gluons. The creation operators are joined by thick lines, whereas the annihilation operators are joined by thin lines. Note that the sequence \( \dot{J} \) is in reverse. (d) An operator of the first kind in brief. The thick line represents a sequence of creation operators of gluons, whereas the thin line represents a sequence of annihilation operators of them. \( \dot{J} \) carries an asterisk to signify the fact that \( \dot{J} \) is put in reverse. Note that the lengths of the two lines have no bearing on the numbers of creation or annihilation operators they represent.
The set of all $\tilde{\phi}^{\rho_1}$'s, where $\rho_1 = 1, 2, \ldots$, and $\Lambda_F$, form a basis of a $\Lambda_F$-dimensional vector space. The set of all $\phi^{\rho_2}$'s, where again $\rho_2 = 1, 2, \ldots$, and $\Lambda_F$, form a basis of another $\Lambda_F$-dimensional vector space. The operator of the first kind can be regarded as a direct product of the operators $\Xi_{\lambda_2}^{\lambda_1}$, $f_j^I$ and $\Xi_{\lambda_2}^{\lambda_1}$. The first operator acts as a $\Lambda_F \times \Lambda_F$ matrix on $\tilde{\phi}^{\rho_1}$, the second one acts on $s^K$, whereas the last one acts as another $\Lambda_F \times \Lambda_F$ matrix on $\phi^{\rho_2}$.

It is therefore clear that that a meson state lies within a direct product of two $\Lambda_F$-dimensional vector spaces (labelling the quark states) and a countably infinite-dimensional vector space spanned by all $s^K$'s labelling the gluon states (including the state containing no gluons). An operator of the first kind lies within the direct product $gl(\Lambda_F) \otimes F_\Lambda \otimes gl(\Lambda_F)$. Here, $gl(\Lambda_F)$ is the Lie algebra of the general linear group $GL(\Lambda_F)$. Also the infinite-dimensional Lie algebra $F_\Lambda$ is spanned by $f_j^I$. We will see that $F_\Lambda$ is isomorphic to the inductive limit $gl_{+\infty}$ of the $gl(n)$'s as $n \to \infty$.

Operators of the second kind are finite linear combinations of operators of the form

$$\Xi_{\lambda_2}^{\lambda_1} \otimes I_j \otimes 1 = N^{-(a+b)/2}q_{\mu_1}^\lambda (\lambda_1)a_{\mu_2}^1 (i_1)a_{\mu_3}^2 (i_2) \cdots a_{\mu_k}^{i_{a-1}} (i_a)$$

$$= a_{\mu_{a+1}}^\nu_b (j_b) a_{\nu_{b-1}} (j_{b-1}) \cdots a_{\nu_2}^\nu_1 (j_1) q_\nu (\lambda_2)$$

(7)

whereas an operator of the third kind can be written as a linear combination of operators of the form

$$1 \otimes I_j \otimes \Xi_{\lambda_2}^{\lambda_1} = N^{-(a+b)/2}q_{\mu_1}^\lambda (\lambda_1)a_{\mu_2}^{i_{a+1}} (i_a) a_{\mu_{a-1}}^{i_{a-1}} (i_{a-1}) \cdots$$

$$a_{\mu_1}^\lambda (i_1) a_{\nu_1}^\nu_1 (j_1) a_{\nu_2}^\nu_2 (j_2) \cdots a_{\nu_b}^{i_{b-1}} (j_b) q_\nu (\lambda_2).$$

(8)

Both the second and third kinds are regularized versions of the ‘current density’ operators of QCD. There are two components to the currents, namely the left-handed component $\bar{q}^a(x)q^b(x)$ and right-handed component $q^a(x)\bar{q}^b(x)$, where $a,b$ are spin and flavor indices. (They act on the left and right ends respectively of a meson.) These operators however are not well defined due to the divergences of quantum field theory. We must instead consider the point-split form, where the fields are evaluated at different points in space. But then gauge invariance forces us to insert a parallel transport operator along a path $P$ that connects the two points. Then the current operators become functions of an open path, i.e., string-like operators:

$$\bar{q}^a(x)e^{i \int_P A dx} q^b(y), \quad q^a(x)e^{i \int_P A dx} \bar{q}^b(y).$$

(9)
We can now expand the field in terms of creation–annihilation operators. Also, we must introduce a regularization that allows only a finite number of values for the momenta of quarks and gluons, in order to get a rigorous theory. These are precisely our operators of the first three kinds. What we discover is that in the large $N$ limit these operators form a closed Lie algebra. Thus we have found the current algebra for QCD in the planar large-$N$ limit.

Let us see how these operators act on a meson state, in the large $N$ limit. For the second kind,

$$\bar{\Xi}^{\lambda_1}_2 \otimes I^I_j \otimes 1 \left( \bar{\phi}^{\rho_1} \otimes s^K \otimes \phi^{\rho_2} \right) = \delta^{\rho_1}_2 \sum_{K_1, K_2 = K} \delta^{K_1}_j \bar{\phi}^{\lambda_1} \otimes s^{iK_2} \otimes \phi^{\rho_2}. \quad (10)$$

There will only be a finite number of non-zero terms in this sum (bounded by the number of ways of splitting $K$ into sub-sequences), so there are no problems of convergence. For example, if $K$ is shorter than $J$, the right hand side will be zero. The action of an operator of the third kind is similar except that it acts on the quark end:

$$1 \otimes r^I_j \otimes \Xi^{\lambda_1}_2 \left( \bar{\phi}^{\rho_1} \otimes s^K \otimes \phi^{\rho_2} \right) = \delta^{\rho_2}_2 \sum_{K_1, K_2 = K} \delta^{K_2}_j \bar{\phi}^{\rho_1} \otimes s^{iK_1} \otimes \phi^{\lambda_1}. \quad (11)$$

We saw that a properly regularized version of the current operators involves the gluon field: This means that we must also consider analoges of current densities that are made up entirely of gluons. In any case such terms will be present in the hamiltonian of QCD. They will dominate the dynamics of QCD in the large-$N$ limit. These lead us to operators of the fourth kind, or **gluonic operators**, which are finite linear combinations of operators of the form:

$$1 \otimes \sigma^I_J \otimes 1 \equiv N^{-(a+b-2)/2} a^{i\mu_1}_{\mu_2}(i_1) a^{i\mu_3}_{\mu_4}(i_2) \cdots a^{i\mu_a}_{\mu_1}(i_a)$$

$$a^{i\nu_b}_{\nu_{b-1}}(j_b) a^{i\nu_{b-2}}_{\nu_{b-1}}(j_{b-1}) \cdots a^{i\nu_1}_{\nu_1}(j_1), \quad (12)$$

where $a = \#(I)$ and $b = \#(J)$. Unlike the operators of the first three kinds, both $I$ and $J$ in the operators of the fourth kind must be non-empty sequences. In the large-$N$ limit, a gluonic operator propagates a single meson state to a linear combination of single meson states:

$$1 \otimes \sigma^I_J \otimes 1 \left( \bar{\phi}^{\rho_1} \otimes s^K \otimes \phi^{\rho_2} \right) = \bar{\phi}^{\rho_1} \otimes \left( \sum_{K_1, K_2, K_3 = K} \delta^{K_2}_J s^{iK_1} \otimes \phi^{\rho_2} \right). \quad (13)$$
We see clearly from this equation that the gluonic operator acts trivially on the quark and antiquark. It acts on a central portion of a meson, i.e., the gluons lying in between the quark/antiquark pair, or on a gluon segment of a glueball.

The operators defined in Eqs. (7), (8), and (12) are depicted in Figs. 1(c), (d) and 2.

As can be seen from the four kinds of operators, the essentially new algebraic structures are contained in the gluonic indices. In the succeeding sections, we are going to confine ourselves to the gluonic parts of the physical states and operators. The quark degrees of freedom will be put back later when we discuss the overall algebra, or the ‘grand’ algebra.

3 Current Algebra in the Large-\(N\) Limit

In this section, we are going to study the actions of the operators of the first three kinds on the gluon segment of a meson. This can be viewed as providing a representation of an associative algebra among these operators. We will see that the algebra of the first kind is a direct product of 2 finite-dimensional algebras and a well-known infinite-dimensional algebra, \(gl_{+\infty}\) \([29, 30]\): it is thus not an essentially new object.

Then we will turn our attention to current operators; i.e., those of the second and third kinds. We will see that the associative algebra among the current operators is a direct product of a finite-dimensional algebra and an infinite-dimensional associative algebra closely related to the Cuntz algebra \([12, 13]\). Then we will focus on this infinite-dimensional algebra and study its structure. We will particularly be interested in the Lie algebra defined by the commutator in this new infinite-dimensional Lie algebra: it is a kind of current algebra in our theory. We will relate this current algebra with the Cuntz algebra \([12]\) and Kac-Moody algebra \([29]\).

Let us consider operators of the first kind but ignoring the action on the quark indices; this corresponds to setting \(\Lambda_F = 1\) temporarily. Then Eq. (11) tells us that

\[
f^I_J s^K_s = \delta^K_s \delta^I_J.
\]

(14)

This equation is diagrammatically illustrated in Fig. 3(a). In particular, \(f^\emptyset_s\), where \(\emptyset\) is the null sequence, is nothing but the projection operator to the
Figure 2: (a) A typical operator of the second kind acting on the antiquark end in detail. There are 5 creation and 4 annihilation operators of gluons. The solid square is a creation operator of an antiquark, whereas the hollow square is an annihilation operator of it. (b) An operator of the second kind acting on the antiquark end in brief. (c) A typical operator of the third kind acting on the quark end in detail. There are 4 creation and 3 annihilation operators of gluons in this current operator. This time the solid square is a creation operator of a quark, whereas the hollow square is an annihilation operator of it. Compare the orientations of the arrows with those in (a). The choices of the orientations reflect the fact that the color indices are contracted differently in Eqs. (3) and (5). (d) An operator of the third kind acting on the quark end in brief. (e) A typical gluonic operator in detail. There are 2 creation and 2 annihilation operators of gluons. There are no operators acting on a quark or an antiquark. (f) A gluonic operator in brief.
gluonless state. Define the composite operator \( f^I_J f^K_L \) by the requirement that its action on \( s^M \) give \( f^I_J (f^K_L s^M) \). Then it follows that
\[
f^I_J f^K_L = \delta^K_J f^I_L. \tag{15}\]
Thus the \( f \)'s form an algebra. Moreover, it is obvious that this algebra is the associative algebra of finite-rank matrices on the infinite dimensional space spanned by the \( s^I \). The \( f^I_J \) are Weyl matrices in this orthonormal basis.

Let us define a Lie algebra \( F_\Lambda \) spanned by the \( f^I_J \), with the usual commutator. Then
\[
[f^I_J, f^K_L] = \delta^K_J f^I_L - \delta^I_J f^K_L. \tag{16}\]
Hence it is clear that \( F_\Lambda \) is isomorphic to \( gl_{+\infty} \), which is the Lie algebra, with the usual bracket, of all complex matrices \((a_{ij})_{i,j} \in Z_+\) such that the number of nonzero \( a_{ij} \)'s is finite \([29]\). The isomorphism is given by a one-one correspondence between the multi-indices \( I \) and the set of natural numbers \( Z_+ \).

Let us now turn to current operators. Ignoring the action on the (anti)-quark indices, we can simplify Eqs.(10) and (11) to
\[
\begin{align*}
l^I_J s^K & = \sum_{K_1 K_2 = K} \delta^K_{K_1} s^I_{K_2}; \quad \text{and} \quad \tag{17} \\
r^I_J s^K & = \sum_{K_1 K_2 = K} \delta^K_{K_2} s^K_{K_1} \tag{18},
\end{align*}
\]
The reader can appreciate the meanings of these equations better by looking at Fig. 3(b) and (c). Note that \( l^0_0 = r^0_0 = 1 \), the identity operator.

Let us consider the multiplication rule for the \( l^I_J \)'s. The properties of the algebra of the set of all \( r^I_J \)'s are completely analogous. Let us define the composite operator \( l^I_J l^K_L \) by the requirement that its action on \( s^M \) give \( l^I_J (l^K_L s^M) \). Then it follows that
\[
l^I_J l^K_L = \delta^K_I l^I_J + \sum_{J_1 J_2 = J} \delta^K_{J_1} l^I_{J_1 J_2} + \sum_{K_1 K_2 = K} \delta^K_{K_1} l^I_{K_2}. \tag{19}\]
Here we sum over all possible sequences except that \( J_2 \) and \( K_2 \) are required to be non-empty. \( j_2 \) being empty and \( K_2 \) being empty describe really the
Figure 3: (a) The action of an operator of the first kind on the gluon portion of a single meson state (Eq. (14)). The dotted lines connect the line segments to be ‘annihilated’ together. The figure on the right of the arrow is the resultant meson state. (b) and (c) The actions of current operators on the gluon portion of a single meson state. In (b) we have the diagrammatic representation of the R.H.S. of Eq. (17), whereas in (c) we have the representation of the R.H.S. of Eq. (18). (d) The action of a gluonic operator on a single meson state.

First term on the R.H.S. and we want to make sure that this term is counted only once in the sums.

Thus the $l$'s form an algebra. This algebra is associative [28]. Moreover, this algebra has an intimate relationship with the extended Cuntz algebra [12, 13]. Indeed, let $\mathcal{H}$ be a $\Lambda$-dimensional Hilbert space spanned by $v^1, v^2, \ldots, v^\Lambda$, and $\mathcal{F}(\mathcal{H})$ be the full Fock space $\bigoplus_{r=0}^{\infty} (\otimes^r \mathcal{H})$, where $(\otimes^0 \mathcal{H})$ is a one-dimensional Hilbert space spanned by a unit vector $\Omega$, the ‘vacuum’ [2]. We will see that $\mathcal{F}(\mathcal{H})$ is a representation space for the extended Cuntz algebra. There is a bijective mapping between the Fock space of physical states spanned by single mesons and a vector in the space $\mathcal{F}(\mathcal{H})$. The state

---

2For us the state $\Omega$ is not exactly the vacuum, but rather is the gluonless state, which still contains a quark and an antiquark.
s^∅ is mapped to the Ω, and the state s^K is mapped to \(v^{k_1} \otimes v^{k_2} \otimes \cdots \otimes v^{k_c}\).

Define the operators \(a^{i\dagger}, a^{2\dagger}, \ldots, a^{\Lambda\dagger}\) as follows:

\[
a^{i\dagger} v^{k_1} \otimes v^{k_2} \otimes \cdots \otimes v^{k_c} = v^i \otimes v^{k_1} \otimes v^{k_2} \otimes \cdots \otimes v^{k_c} \\
a^{i\dagger} \Omega = v^i
\]

(20)

where \(i\) is an integer between 1 and \(\Lambda\) inclusive. The corresponding adjoint operators \(a_1, a_2, \ldots, a_\Lambda\) have the following properties:

\[
a_i v^{k_1} \otimes v^{k_2} \otimes \cdots \otimes v^{k_c} = \delta_{k_1}^i v^{k_2} \otimes \cdots \otimes v^{k_c}; \text{ and} \\
a_i \Omega = 0.
\]

(21)

As a result, the \(a\)'s and \(a^{\dagger}\)'s satisfy the following properties:

\[
a_i a^{j\dagger} = \delta_{j}^i; \text{ and} \\
\sum_{i=1}^{\Lambda} a^{i\dagger} a_i = 1 - P_\Omega,
\]

(22)

(23)

where \(P_\Omega\) is the projection operator to the vacuum \(\Omega\). Thus the \(a\)'s and \(a^{\dagger}\)'s are the annihilation and creation operators of the extended Cuntz algebra respectively. Furthermore,

\[
v^{k_1} \otimes v^{k_2} \otimes \cdots \otimes v^{k_c} = a^{k_1\dagger} a^{k_2\dagger} \cdots a^{k_c\dagger} \Omega.
\]

(24)

It is now straightforward to see that there is a one-to-one correspondence among the operators characterized by Eq.(7) and the operators acting on \(\mathcal{F}(\mathcal{H})\). A current operator \(l^I_J\) corresponds to \(a^{i\dagger} a^{j\dagger} \cdots a^{K\dagger} a_{j_{b-1}} \cdots a_1; l^I_0\) corresponds to \(a_{j_0} a_{j_{b-1}} \cdots a_1\); and \(l^I_0\) to the identity operator.

Define the Lie algebra between the \(l\)'s as follows:

\[
[l^I_J, l^K_L] = \sum_{J_1, J_2 = J}^{J} \delta_{J}^{K} l^I_{J_1, J_2} + \sum_{K_1, K_2 = K}^{K} \delta_{K}^{I} l^{K_1, K_2}_L \\
- \delta_{L}^{I} l^K_{J} - \sum_{L_1, L_2 = L}^{L} \delta_{L}^{I} l^{K}_{L_1, L_2} - \sum_{I_1, I_2 = I}^{I} \delta_{I}^{I_1} l^{K}_{I_1, I_2}.
\]

(25)

The first three terms on the R.H.S. of Eq.(25) are diagrammatically represented in Fig. 3(a) and (b). Let us call the Lie algebra defined by Eq.(25) the
leftix algebra or \( \hat{L}_\Lambda \). (We justify this name as an abbreviation of ‘left multi-matrix’.) The analogous algebra spanned by \( r^I_j \) will be called the rightix algebra \( \hat{R}_\Lambda \).

Now we note the following identity:

\[
\left( l^K_L - \sum_{i=1}^{\Lambda} l^I_{Li} \right) s^M = \delta^K_L s^K.
\]  

(26)

Thus these are nothing but \( f^K_L \) we saw earlier:

\[
f^K_L \equiv l^K_L - \sum_{j=1}^{\Lambda} l^I_{Lj}.
\]  

(27)

We can obtain from Eqs. (14) and (17) the following relation:

\[
\left[ l^i_J, f^K_L \right] = \sum_{\kappa_1, \kappa_2 = K} \delta^K_J f^i_{L\kappa_2} - \sum_{\bar{L}_1, \bar{L}_2 = L} \delta^i_{L_1} f^K_{\bar{L}_2}. \]

(28)

This formula is depicted in Figs. 4(c).

As a result of Eq. (28), \( F_\Lambda \) forms an ideal of the algebra \( \hat{L}_\Lambda \). This is a proper ideal as it is obvious that finite linear combinations of \( f^i_J \) do not
span the whole leftix algebra. The quotient $L_\Lambda = \hat{L}_\Lambda/F_\Lambda$ is thus also a Lie algebra. Put it another way, $\hat{L}_\Lambda$ is the extension of $L_\Lambda$ by $F_\Lambda$: we have the exact sequence of Lie algebras

$$0 \to F_\Lambda \to \hat{L}_\Lambda \to L_\Lambda \to 0. \quad (29)$$

At the moment we have only this indirect construction of the Lie algebra $L_\Lambda$. It would be exciting to find new representations for this leftix algebra.

How do we understand the above exact sequence in the context of the Cuntz algebra? The analogue of the algebra $F_\Lambda$ is the algebra $K(\mathcal{H})$ of compact operators on $\mathcal{H} = \mathbb{C}^\Lambda$; $K(\mathcal{H})$ is just the completion of $F_\Lambda$ in the topology defined by the operator norm. It follows that $K(\mathcal{H})$ is an ideal of the Banach algebra generated by $a, a^\dagger$. If we quotient by $K(\mathcal{H})$, then we get the Cuntz algebra, with the $a$’s and $a^\dagger$’s satisfying the following relations:

$$a_i a_{i}^\dagger = \delta_i^j; \text{ and } \quad \sum_{i=1}^\Lambda a_{i}^\dagger a_i = 1. \quad (30)$$

The role of the last relation is to set all $f_j^I$’s to zero; i.e., to quotient by $F_\Lambda$. Thus, we can regard the Lie algebra $\hat{L}_\Lambda$ as an extension of the Lie algebra associated with the Cuntz algebra by the subalgebra $F_\Lambda$.

We will see that in fact the presence of $F_\Lambda$ as a proper ideal is a generic feature of most of the algebras we will study. Quotienting by this ideal will get us the essentially new algebras we are interested in studying. But it is only the extension that will have interesting representations. It appears that the extension by $gl_{+\infty}$ plays a role in our theory that central extensions play in the theory of Kac-Moody and Virasoro algebras. Similar extensions have appeared in previous approaches to current algebras [31].

There is of course a parallel theory for $\hat{R}_\Lambda$ which is a Lie algebra spanned by elements of the form $r_j^I$, and $R_\Lambda = \hat{R}_\Lambda/F_\Lambda$ which is the quotient of $\hat{R}_\Lambda$ by $F_\Lambda$. Let us briefly describe it here for the sake of completeness. The Lie bracket expression between two $r$’s is:

$$[r_J^I, r_K^L] = \delta^K_J r_L^I + \sum_{J_1 J_2 = J} \delta^K_{J_2} r_{J_1}^L + \sum_{K_1 K_2 = K} \delta^K_{J_2} r_{K_1}^I - \delta^K_J r_L^I - \sum_{L_1 L_2 = L} \delta^K_{L_2} r_{L_1}^I - \sum_{I_1 I_2 = I} \delta^K_{I_2} r_{I_1}^L. \quad (32)$$
A vector in $F_\Lambda$ can be expressed as an element in $\hat{R}_\Lambda$ as well:

$$f^K_L = r^K_L - \sum_{i=1}^\Lambda r^K_{iL}. \quad (33)$$

The Lie bracket between an $r$ and an $f$ is

$$\left[ r^K_L, f^J_J \right] = \sum_{K_1K_2=K} \delta^K_{j_2} f^K_{j_1L} - \sum_{L_1L_2=L} \delta^I_{L_2} f^K_{L_1j}. \quad (34)$$

The algebra $\hat{M}_\Lambda$ spanned by the operators $l^I_J$, $r^I_J$ together is also interesting. $\hat{L}_\Lambda$ and $\hat{R}_\Lambda$ are subalgebras of $\hat{M}_\Lambda$. In addition, the Lie bracket between an element of $\hat{L}_\Lambda$ and an element of $\hat{R}_\Lambda$ is

$$\left[ l^K_J, r^K_L \right] = \sum_{J_1J_2=J} \delta^K_{J_2} f^K_{J_1L} - \sum_{L_1L_2=L} \delta^K_{L_2} f^K_{L_1J}. \quad (35)$$

A heuristic way to see that Eq.(35) is true is to verify that the action of the R.H.S. on an arbitrary state $s^M$ gives $l^K_J(r^K_Ls^M) - r^K_L(l^K_Js^M)$. This, however, does not automatically lead to the conclusion that the Jacobi identity is satisfied by this definition of the Lie bracket (Eq.(33)) because Eq.(27) and (33) imply that the set of all finite linear combinations of $l$'s and $r$'s is not linearly independent. The reader is referred to a future publication [32] for a rigorous proof of Eq.(35). A typical term in the above equation is depicted in Fig.4(d).

Therefore $\hat{L}_\Lambda$ and $\hat{R}_\Lambda$ are proper ideals of $\hat{M}_\Lambda$. In particular, $F_\Lambda$ is a proper ideal of $M_\Lambda$, hence we have the quotient algebra $M_\Lambda = \hat{M}_\Lambda/F_\Lambda$. Since the commutator of an $l$ and an $r$ is in $F_\Lambda$, they commute when thought of as elements of $M_\Lambda$.

These operators are regularized versions of the left- and right-handed current operators of QCD. We see that the chiral structure of the current algebra is reproduced beautifully in the multix algebra. The fact that only the extension of the current algebra by $gl_{+\infty}$ has a representation on the one-meson states is also reminiscent of what happens in lower dimensional theories.

Along the way we have learned an important lesson: the precisely defined left and right currents do not commute: their commutator is a finite rank
operator instead. Our naive expectations on current algebra have to be revised: our extensions by $F_\Lambda$ are just as important as the central extensions in the theory of Kac–Moody algebras.

We will see that much of this structure is embedded in the algebra of gluonic operators on open string states. We have summarized the relationship among the various algebras discussed in this section in Table 1.

### 4 Lie Algebra of Gluonic Operators

In the last section we studied the actions of the operators of the first three kinds, and obtained associative algebras closely related to known algebras such as the Cuntz algebra. It turns out that products of gluonic operators with each other cannot be written as finite linear combinations of these operators: they do not span an algebra under multiplication. However the commutator of two gluonic operators can be written as a finite linear combination of gluonic operators. Thus gluonic operators acting on meson states form a Lie algebra, which we will call the centrix algebra $\hat{\Sigma}_\Lambda$. This new Lie algebra we discover reduces to the Virasoro algebra (up to an extension by $gl_{+\infty}$) in the special case $\Lambda = 1$. Thus the centrix algebra is a sort of generalization of the Lie algebra of vector fields on a circle to non-commutative geometry.

In a previous Letter [2] we reported on the Lie algebra of gluonic operators.
acting on glueball states. This is a similar but different algebra, the cyclix algebra. The fact that the glueball states are cyclically symmetric under permutations makes the cyclix Lie algebra differ in an essential way from the centrix algebra.

We will see that there is a proper ideal $F_{\Lambda'} \equiv \mathfrak{gl}_{+\infty}$ of the Lie algebra $\hat{\Sigma}_{\Lambda}$; hence, there is a quotient Lie algebra $\Sigma_{\Lambda} = \hat{\Sigma}_{\Lambda}/F'_{\Lambda}$. Furthermore, we will see that $\hat{L}_{\Lambda}$ and $\hat{R}_{\Lambda}$ have some elements in common with $\hat{\Sigma}_{\Lambda}$. This is perhaps a bit surprising since they originally were introduced using quark operators and $\hat{\Sigma}_{\Lambda}$ is the algebra of purely gluonic operators acting on meson states. More precisely, the generators $l_{IJ}$, $r_{IJ}$ with $I, J$ non-empty are in fact some linear combinations of $\sigma_{IJ}$. Thus we can form quotients of $\hat{\Sigma}_{\Lambda}$ by the algebras of $l_{IJ}$'s and $r_{IJ}$'s. This is the essentially new object we have discovered.

In this section, we will only focus on the action of $\sigma_{IJ}$ on a segment of the gluonic sequence of a single meson state. We can capture the essence of Eq.(13) by the following 3:

$$\sigma_{IJ} s^K = \sum_{\hat{K}_1\hat{K}_2\hat{K}_3 = \hat{K}} \delta^K_{\hat{K}_2} s^{\hat{K}_1\hat{K}_3}. \quad (36)$$

This equation can be visualized in Fig. 3(d). From Eq.(36), we can see that the set of $\sigma_{IJ}$'s with all possible non-empty sequences $I$ and $J$'s is linearly independent.

Unlike the case of the algebra $\hat{L}_{\Lambda}$ considered in Section 3, we cannot define the composite operator $\sigma_{IJ}^{\hat{K}} \sigma_{L}^{\hat{K}'}$ by the requirement that its action on $\sigma_{M}^{\hat{P}}$ give $\sigma_{IJ}^{\hat{K}} (\sigma_{L}^{\hat{K}'} \sigma_{M}^{\hat{P}})$ because in general this action cannot be written as a linear combination of the $\sigma$'s 28. Nonetheless, the Lie bracket between two $\sigma$'s is well-defined by the requirement that

$$\left( [\sigma_{IJ}, \sigma_{L}^{\hat{K}}] \right) s^\hat{P} \equiv \sigma_{IJ} \left( \sigma_{L}^{\hat{K}} s^\hat{P} \right) - \sigma_{L}^{\hat{K}} \left( \sigma_{IJ} s^\hat{P} \right). \quad (37)$$

for any arbitrary sequence $\hat{P}$. Then it can be shown 28 (by a tedious but straightforward calculation) that the expression of the Lie bracket is

$$\left[ \sigma_{IJ}, \sigma_{L}^{\hat{K}} \right] = \delta^{\hat{K}}_{\hat{P}} \sigma_{L}^{\hat{I}} + \sum_{J_1, J_2 = J} \delta^{\hat{K}}_{J_1} \sigma_{J_1L}^{\hat{I}} + \sum_{K_1, K_2 = K} \delta^{\hat{K}}_{J} \sigma_{L}^{I_{K_1}K_2}$$

3Of course, $\sigma_{IJ}$ acting on the gluonless state gives zero.
\[ J_1 J_2 = J \\
\frac{K_1 K_2}{K} = K \]

\[ + \sum_{J_1 J_2 = J} \delta_{J_2}^{K_1} \sigma_{J_1 L}^{K_2} + \sum_{J_1 J_2 = J} \delta_{J_1}^{K_1} \sigma_{L J_2}^{K_2} + \sum_{K_1 K_2 = J} \delta_{K_2}^{K_1} \sigma_{K_1 L}^{K_2} \]

\[ + \sum_{J_1 J_2 = J} \delta_{J_1}^{K_2} \sigma_{K_1 J_2}^{K} + \sum_{J_1 J_2 J_3 = J} \delta_{J_2}^{K} \sigma_{J_1 L J_3}^{K} + \sum_{K_1 K_2 K_3 = K} \delta_{K_2}^{K} \sigma_{K_1 L K_3}^{K} \]

\[-(I \leftrightarrow K, J \leftrightarrow L). \]

The diagrammatic representations of the first 9 terms are given in Fig. 5.

We will call the Lie algebra defined by Eq. (38) the centrix algebra \( \hat{\Sigma}_\Lambda \).

Now let us note the following identities that follow easily from the action of \( \sigma_{I J}^{I} \):

\[
\left( \sigma_{I}^{I} - \sum_{i=1}^{\Lambda} \sigma_{i I}^{I} \right) s_{I}^{K} = \sum_{K_1 K_2 = K} \delta_{I}^{K_1} s_{I}^{K_2}
\]

and

\[
\left( \sigma_{I}^{I} - \sum_{j=1}^{\Lambda} \sigma_{j I}^{I} \right) s_{I}^{K} = \sum_{K_1 K_2 = K} \delta_{I}^{K_2} s_{I}^{K_1 I}.
\]

These are exactly the action of the operators \( l_{I}^{I} \) and \( r_{I}^{I} \) we defined previously. Thus we have,

\[
l_{I}^{I} = \sigma_{I}^{I} - \sum_{i=1}^{\Lambda} \sigma_{i I}^{I}; \text{ and } \]

\[
r_{I}^{I} = \sigma_{I}^{I} - \sum_{j=1}^{\Lambda} \sigma_{j I}^{I}.
\]

Of course we must require that \( I \) and \( J \) be non-empty for this to be true.

The operators \( l_{I}^{I} \)'s with non-empty sequences \( I \) and \( J \) form a subalgebra \( \hat{L}_{\Lambda} \) of \( \hat{L}_{\Lambda} \), the operators \( r_{I}^{I} \)'s form another subalgebra \( \hat{R}_{\Lambda} \) of \( \hat{R}_{\Lambda} \), and the operators \( l_{I}^{I} \)'s and \( r_{I}^{I} \)'s together form yet another subalgebra \( \hat{M}_{\Lambda} \) of \( \hat{M}_{\Lambda} \). What we have just seen is that \( \hat{L}_{\Lambda} \), \( \hat{R}_{\Lambda} \) and \( \hat{M}_{\Lambda} \) can also be viewed as subalgebras of \( \hat{\Sigma}_{\Lambda} \); in fact they are even ideals of \( \hat{\Sigma}_{\Lambda} \). This will be evident from the (more general) commutation relations between \( \sigma_{I}^{I} \) and \( l_{I}^{I} \), etc. given below.

Analogously, we can define a vector space \( F_{\Lambda}^{I} \) spanned by \( f_{I}^{I} \) with non-empty \( I \) and \( J \). By the same counting argument as before, this is also
Figure 5: Diagrammatic representations of the first 9 terms on the R.H.S. of Eq. (38).
isomorphic to $gl_{+\infty}$. We have already seen that:

$$f^K_L = l^K_L - \sum_{j=1}^{\Lambda} l^K_{Lj}$$  \hspace{1cm} (41)$$

and

$$f^K_L = r^K_L - \sum_{i=1}^{\Lambda} r^K_{iL}.$$  \hspace{1cm} (42)

Thus we have

$$f^I_J = \sigma^I_J - \sum_{j=1}^{\Lambda} \sigma^I_{Jj} - \sum_{i=1}^{\Lambda} \sigma^I_{iJ}.$$  \hspace{1cm} (43)

Thus $F^\Lambda_\Lambda$ is an ideal of $\hat{\Sigma}_\Lambda$ as well.

Now we give the commutation relations between the centrix and multix operators:

$$\left[ \sigma^I_J, l^K_L \right] = \delta^K_I l^K_L + \sum_{j_1j_2=j} \delta^K_{j_1} l^K_{Lj_2} + \sum_{K_1K_2=K} \delta^K_{K_1} l^K_{K_1L}$$

$$+ \sum_{\substack{K_1K_2=K \\ K_1L_2=L}} \delta^K_{K_1} l^K_{L_1j_2} + \sum_{\substack{J_1J_2=J \\ K_1L_1=K}} \delta^K_{K_1} l^K_{L_2j_1} + \sum_{\substack{J_1J_2=J \\ K_1K_2=K \\ K_1L_1=K \\ K_2L_2=L}} \delta^K_{K_1} l^K_{L_1j_2}$$

$$- \delta^K_{L_2j_1} - \sum_{L_1L_2=L} \delta^K_{L_2} l^K_{L_1j_1} - \sum_{L_1L_2=L} \delta^K_{L_2} l^K_{L_1j_2};$$  \hspace{1cm} (44)

$$\left[ \sigma^I_J, r^K_L \right] = \delta^K_I r^K_L + \sum_{j_1j_2=j} \delta^K_{j_2} r^K_{j_1L} + \sum_{K_1K_2=K} \delta^K_{K_1} r^K_{L_1}$$

$$+ \sum_{\substack{K_1K_2=K \\ K_1L_2=L}} \delta^K_{K_1} r^K_{L_1L} + \sum_{\substack{J_1J_2=J \\ K_1K_2=K \\ K_1L_1=K \\ K_2L_2=L}} \delta^K_{K_1} r^K_{L_1L_2}$$

$$- \delta^K_{L_2j_1} - \sum_{L_1L_2=L} \delta^K_{L_2} r^K_{L_1j_1} - \sum_{L_1L_2=L} \delta^K_{L_2} r^K_{L_1j_2};$$
\[- \sum_{L_1L_2 = L} \delta_{L_1}^I r^{K}_{JL_2} - \sum_{L_1L_2 = L} \delta_{L_1}^J r^{K}_{JL_2} - \sum_{L_1L_2L_3 = L} \delta_{L_2}^J r^{K}_{L_3L_3} ; \text{ and (45)}\]

\[
\left[ \sigma_J^I, f^K_L \right] = \sum_{K_1K_2K_3 = K} \delta_{K_1}^J f^{K_1}_{K_2} - \sum_{L_1L_2L_3 = L} \delta_{L_2}^J f^{K}_{L_3L_3}.
\]

Eqs. (44), (45) and (46) are illustrated in Figs. 6, 7 and 8 respectively.

- Figure 6: Diagrammatic representation of Eq. (44). Only the first 6 terms on the R.H.S. of Eq. (44) are shown here.

These prove, in particular, that the algebras $\hat{L}_\Lambda$, $\hat{R}_\Lambda$ and $F'_\Lambda$ are proper ideals of the centrix algebra. In addition, Eqs. (14) and (15) show that an element which does not belong to $\hat{L}'_\Lambda$ in the centrix algebra is an outer derivation of the algebra $\hat{L}_\Lambda$, i.e., the Lie algebra associated with the extended Cuntz algebra, and an element which does not belong to $\hat{R}'_\Lambda$ in the centrix algebra is an outer derivation of the algebra $\hat{R}_\Lambda$. Let us summarize the various relationships among the Lie algebras in Table 2.

It follows from the above discussion that there are various ways to form different quotient algebras from the centrix algebra. For instance, we can make the set of all cosets $\sigma_J^I + F'_\Lambda$ into a quotient algebra $\Sigma_\Lambda$. Furthermore,
Figure 7: Diagrammatic representation of Eq. (43). Only the first 6 terms on the R.H.S. of Eq. (43) are shown here.

| operators  | extended algebra | comment                      | quotient algebra(s) |
|------------|------------------|-------------------------------|---------------------|
| $f^I_j$    | $F'_{\Lambda}$   | $F'_{\Lambda}$ $\equiv$ $gl_{+\infty}$ |                     |
| $l^I_j$    | $L'_{\Lambda}$   | $F'_{\Lambda}$ is a proper ideal of $\hat{L}'_{\Lambda}$. | $L'_{\Lambda}$ $\equiv$ $E'_{\Lambda}/F'_{\Lambda}$ |
| $f^K_{L} = l^K_{L} - l^K_{Lj}$ | $R'_{\Lambda}$   | $F'_{\Lambda}$ is a proper ideal of $\hat{R}'_{\Lambda}$. | $R'_{\Lambda}$ $\equiv$ $R'_{\Lambda}/F'_{\Lambda}$ |
| $r^K_{L} = r^K_{Li} - r^K_{LiL}$ | $M'_{\Lambda}$ | $M' \not= E'_{\Lambda} \oplus R'_{\Lambda}$ | $M'_{\Lambda}$ $\equiv$ $M'_{\Lambda}/F'_{\Lambda}$ |
| $l^I_j$ and $r^I_j$ | $\Sigma_{\Lambda}$ | $\Sigma_{\Lambda}$, $L'_{\Lambda}$ and $R'_{\Lambda}$ are proper ideals of $\hat{\Sigma}_{\Lambda}$. | $\Sigma_{\Lambda}$ $\equiv$ $\Sigma_{\Lambda}/F'_{\Lambda}$ |
| $\sigma^I_j$ | $\Sigma_{\Lambda}$ | $\Sigma_{\Lambda}$, $L'_{\Lambda}$ and $R'_{\Lambda}$ are proper ideals of $\hat{\Sigma}_{\Lambda}$. | $\Sigma_{\Lambda}$ $\equiv$ $\Sigma_{\Lambda}/F'_{\Lambda}$ |
| $\sigma^I_j$ | $\Sigma_{\Lambda}$ | $\Sigma_{\Lambda}$, $L'_{\Lambda}$ and $R'_{\Lambda}$ are proper ideals of $\hat{\Sigma}_{\Lambda}$. | $\Sigma_{\Lambda}$ $\equiv$ $\Sigma_{\Lambda}/F'_{\Lambda}$ |

Table 2: $\Sigma_{\Lambda}$, its subalgebras and quotient algebras. The summation convention is adopted for repeated indices in this table.

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within this quotient algebra, all cosets of the form $l_j^* + F_\Lambda$ span a proper ideal $L_\Lambda'$, and all cosets of the form $r_j^* + F_\Lambda$ span another ideal $R_\Lambda'$. Thus we can extract the quotient Lie algebra

$$\mathcal{V}_\Lambda \equiv \hat{\Sigma}_\Lambda/M'_\Lambda = \Sigma_\Lambda/M'_\Lambda.$$  \hfill (47)

In the simplest case $\Lambda = 1$, $\mathcal{V}_\Lambda = \mathcal{V}_1$ is just the algebra of vector fields on a circle. Indeed, now all integer sequences are just repetitions of the number 1 a number of times. Write $s^K$ as $s^\#(K)$, $l_j^*$ as $l_j^\#(I)$, and $\sigma_j^*$ as $\sigma_j^\#(I)$. (Note that $r_j^\#(I) = l_j^\#(I)$.) It can be easily seen that $M'_1$ is spanned by the cosets of the forms $l_a^1 + F'_1$, $l_b^1 + F'_1$, and $\Sigma_1$ is spanned by the cosets $l_a^1 + F'_1$, $l_b^1 + F'_1$, $\sigma_a^1 + F'_1$ and $\sigma_b^1 + F'_1$, where $a$ and $b$ run over all positive integers. It is a straightforward matter to verify the following Lie brackets, modulo $F'_1$:

\[
\begin{align*}
[l_1^a, l_1^c] &= 0 = [l_1^a, l_d^1] = [l_1^1, l_d^1]; \\
[\sigma_1^a, \sigma_1^c] &= (c - a)\sigma_1^{a+c-1}; \\
[\sigma_1^a, \sigma_1^d] &= \begin{cases} 
(2 - a - d)\sigma_1^{a+d+1} & \text{if } d \leq a, \text{ or} \\
(2 - a - d)\sigma_1^{a-d+1} & \text{if } a \leq d; 
\end{cases} \\
\sigma_1^a, \sigma_1^d &= (b - d)\sigma_1^{b+d-1}; \\
[\sigma_1^a, l_1^1] &= (c - 1)l_1^{a+c-1}; \\
\sigma_1^a, l_1^c &= \begin{cases} 
(c - 1)l_1^{c-b+1} & \text{if } b \leq c, \text{ or} \\
(c - 1)l_1^{c-b+1} & \text{if } c \leq b; 
\end{cases} \\
\sigma_1^a, l_1^d &= -(d - 1)\sigma_1^{b+d-1}; \text{ and} \\
\sigma_1^a, l_1^d &= \begin{cases} 
-(d - 1)l_1^{d-a+1} & \text{if } a \leq d, \text{ or} \\
-(d - 1)l_1^{a-d+1} & \text{if } d \leq a. 
\end{cases}
\end{align*}
\hfill (48)
\]
Let us make the following identifications:

\begin{align*}
l_1^a & \rightarrow z^{a-1}; \\
l_1^b & \rightarrow z^{-(b-1)}; \\
\sigma_1^a & \rightarrow z^{a-1} \frac{d}{dz} = -L_{a-1}; \quad \text{and} \quad \sigma_1^b \rightarrow z^{-(b-1)} \frac{d}{dz} = -L_{1-b}. \end{align*}

Here \( z \) is a complex number with \(|z| = 1\). Note that the \( L \) here is not an integer sequence. Then Eq. (48) becomes

\begin{align*}
[z^p, z^q] &= 0; \\
[L_p, L_q] &= (p - q)L_{p+q}; \quad \text{and} \\
[L_p, z^q] &= -qz^{p+q-1}, \end{align*}

where \( p \) and \( q \) are integers which may be positive, negative or zero. Thus this is the extension of the Virasoro algebra without any central element by the functions on a unit circle on the complex plane. We also see that the algebra \( \mathcal{V}_1 \) is just the Lie algebra of vector fields on the circle, spanned by all \( L_p \)'s for \( p \) being all integers.

More generally, the generators of \( \hat{\Sigma}_\Lambda \) act as derivations on the Cuntz algebra. When we quotient out \( \hat{\Sigma}_\Lambda \) by the ideal \( \hat{M'}_\Lambda \), we are basically extracting the part which corresponds to ‘inner derivations’ (up to some extensions by standard algebras such as \( gl_{+\infty} \)). Thus the vectrix algebra \( \mathcal{V}_\Lambda \) should be viewed as a generalization of the algebra of vector fields of a circle. It is the Lie algebra of vector fields on a sort of non-commutative generalization of the circle, which has the Cuntz algebra as its algebra of functions. It would be interesting to investigate this further.

5 Grand Algebra Acting on Open Strings

Now we piece together the algebras of current and gluonic operators to form a grand Lie algebra. In general the Hamiltonian of a matrix model in the large-\( N \) limit will be an element of this algebra. We are going to write down the Lie brackets of various operators, and we will see that this grand Lie
algebra is a sum, though not a direct sum, of four subalgebras, and there are two proper ideals in this grand Lie algebra.

Let us begin by considering operators of the first kind. It is easy to see that all $\Xi_{\lambda_2}^\lambda \otimes f_j^i \otimes \Xi_{\lambda_4}^{\lambda_3}$’s span a Lie algebra $\hat{F}_{\Lambda \Lambda_F}$ which is isomorphic to $gl(\Lambda_F) \otimes \hat{F}_{\Lambda} \otimes gl(\Lambda_F)$. Of course this in turn is just isomorphic to $gl_{+,\infty}$, by an appropriate one-to-one correspondence with natural numbers.

Next, let us move to operators of the second kind. We deduce from Eqs.(11) and (32) that the Lie bracket between two operators of this form is

$$
\left[\Xi_{\lambda_2}^\lambda \otimes l_j^i \otimes 1, \Xi_{\lambda_4}^{\lambda_3} \otimes f_L^K \otimes 1\right] = \delta_{\lambda_2}^\lambda \Xi_{\lambda_4}^{\lambda_3} \otimes \left(\delta^K_j l_j^i \sum_{j_1, j_2 = j} \delta^K_{j_1} l_{j, j_2} + \delta^K_{j_2} l_{j, j_2} \right) \otimes 1
$$

Thus operators of the second kind form another subalgebra $\hat{L}_{\Lambda \Lambda_F}$ of the grand algebra.

The Lie bracket between two operators of the third kind can be similarly derived from Eqs.(11) and (32):

$$
\left[1 \otimes r_j^i \otimes \Xi_{\lambda_2}^\lambda, 1 \otimes r_L^K \otimes \Xi_{\lambda_4}^{\lambda_3}\right] = \delta_{\lambda_2}^\lambda 1 \otimes \left(\delta^K_j r_j^i \sum_{j_1, j_2 = j} \delta^K_{j_1} r_{j, j_2} + \delta^K_{j_2} r_{j, j_2} \right) \otimes \Xi_{\lambda_4}^{\lambda_3}
$$

This equation shows clearly that all operators of the third kind form yet another subalgebra of the grand algebra.

Let us consider the Lie bracket relations between operators of different kinds. We can derive the Lie bracket relations between an operator of the first kind, and either an operator of the second kind or third kind with the help of Eqs.(28) and (34) as follows:

$$
\left[\Xi_{\lambda_2}^\lambda \otimes l_j^i \otimes 1, \Xi_{\lambda_4}^{\lambda_3} \otimes f_L^K \otimes \Xi_{\lambda_6}^{\lambda_5}\right] = \delta_{\lambda_2}^\lambda \Xi_{\lambda_4}^{\lambda_3} \otimes \left(\delta^K_j l_j^i \sum_{j_1, j_2 = j} \delta^K_{j_1} l_{j, j_2} + \delta^K_{j_2} l_{j, j_2} \right) \otimes \Xi_{\lambda_6}^{\lambda_5}
$$
\[ \delta^{\lambda_1}_{\lambda_2} \Xi^{\lambda_2}_{\lambda_4} \otimes \sum_{K_1 K_2 = K} \delta^{\lambda_1}_{\lambda_2} f_{L_{1 L_2}} \otimes \Xi^{\lambda_5}_{\lambda_6} - \delta^{\lambda_1}_{\lambda_2} \Xi^{\lambda_3}_{\lambda_6} \otimes \sum_{L_1 L_2 = L} \delta^{\lambda_2}_{L_{1 L_2}} f_{L_{1 L_2}} \otimes \Xi^{\lambda_5}_{\lambda_6} (53) \]

and

\[ [1 \otimes r^I_j \otimes \Xi^{\lambda_1}_{\lambda_2}, \Xi^{\lambda_3}_{\lambda_4} \otimes f^L_K \otimes \Xi^{\lambda_5}_{\lambda_6}] = \delta^{\lambda_1}_{\lambda_2} \Xi^{\lambda_3}_{\lambda_4} \otimes \sum_{K_1 K_2 = K} \delta^{\lambda_1}_{\lambda_2} f_{K_{1 K_2}} \otimes \Xi^{\lambda_1}_{\lambda_6} - \delta^{\lambda_1}_{\lambda_2} \Xi^{\lambda_3}_{\lambda_4} \otimes \sum_{L_1 L_2 = L} \delta^{\lambda_2}_{L_{1 L_2}} f_{L_{1 L_2}} \otimes \Xi^{\lambda_5}_{\lambda_6} (54) \]

These equations show that the subalgebra of the operators of the first kind form a proper ideal of the Lie algebra spanned by the operators of the first, second and third kinds. The Lie bracket between a typical operator of the second and that of the third kind is

\[ [\Xi^{\lambda_1}_{\lambda_2} \otimes l^I_j \otimes 1, 1 \otimes r^K_L \otimes \Xi^{\lambda_3}_{\lambda_4}] = \Xi^{\lambda_1}_{\lambda_2} \otimes [l^I_j, r^K_L] \otimes \Xi^{\lambda_3}_{\lambda_4}; \quad (55) \]

where \([l^I_j, r^K_L]\) is given by the R.H.S. of Eq. (53). As a result, this Lie bracket gives a linear combination of operators of the first kind.

Finally, consider operators of the fourth kind. The Lie bracket relations between an operator of the fourth kind, and an operator of any kind, are as follows:

\[ [1 \otimes \sigma^I_j \otimes 1, 1 \otimes \sigma^K_L \otimes 1] = 1 \otimes [\sigma^I_j, \sigma^K_L] \otimes 1; \quad (56) \]
\[ [1 \otimes \sigma^I_j \otimes 1, 1 \otimes r^K_L \otimes \Xi^{\lambda_1}_{\lambda_2}] = 1 \otimes [\sigma^I_j, r^K_L] \otimes \Xi^{\lambda_2}_{\lambda_4}; \quad (57) \]
\[ [1 \otimes \sigma^I_j \otimes 1, \Xi^{\lambda_1}_{\lambda_2} \otimes l^K_L \otimes 1] = \Xi^{\lambda_1}_{\lambda_2} \otimes [\sigma^I_j, l^K_L] \otimes 1; \quad \text{and} \quad (58) \]
\[ [1 \otimes \sigma^I_j \otimes 1, \Xi^{\lambda_1}_{\lambda_2} \otimes f^K_L \otimes \Xi^{\lambda_3}_{\lambda_4}] = \Xi^{\lambda_1}_{\lambda_2} \otimes [\sigma^I_j, f^K_L] \otimes \Xi^{\lambda_3}_{\lambda_4}. \quad (59) \]

From the above four Lie brackets and Eqs. (53), (54), (55) and (56), we conclude that all operators of the fourth kind form a subalgebra of the grand algebra, and that the Lie bracket between an operator of the fourth kind and an operator of the third, second or first kind produces an operator of the third, second or first kind respectively. Operators of these four kinds span the whole grand algebra of operators acting on mesons or open strings. Operators of the first, second and third kinds together span a proper ideal of the grand algebra. We have summarized these results in Table 3.
operators & algebra & comment \\
\Xi_\lambda^1 \otimes f_I^f \otimes \Xi_\lambda^3 & F_{\Lambda,\Lambda_F} & F_{\Lambda,\Lambda_F} \equiv gl(\Lambda_F) \otimes F_{\Lambda} \otimes gl(\Lambda_F) \\
\Xi_\lambda^1 \otimes l_I^f \otimes 1 & gl(\Lambda_F) \otimes L_\Lambda & \\
1 \otimes r_I^f \otimes \Xi_\lambda^3 & \hat{R}_\Lambda \otimes gl(\Lambda_F) & \\
\Xi_\lambda^1 \otimes f_I^f \otimes \Xi_\lambda^3, \Xi_\lambda^1 \otimes l_I^f \otimes 1, \text{ and} & M_{\Lambda,\Lambda_F} & F_{\Lambda,\Lambda_F} \text{ is a proper ideal of } M_{\Lambda,\Lambda_F}. \\
1 \otimes r_I^f \otimes \Xi_\lambda^3 & & \\
\sigma_f^I & \Sigma_\Lambda & \\
all of the above & G_{\Lambda,\Lambda_F} & F_{\Lambda,\Lambda_F} \text{ and } M_{\Lambda,\Lambda_F} \text{ are proper ideals of } \hat{G}_{\Lambda,\Lambda_F}.

Table 3: The grand algebra, its subalgebras and its ideals.

6 Spin Chain Models

Now that we have understood the algebraic properties of current and gluonic operators and their actions on open string states, let us give some physical models in which these operators and states show up. We will discuss two families of models in detail: multi-matrix models integrable in the large-\( N \) limit associated with integrable quantum spin chain models with open boundary conditions in this section, and quantum chromodynamics in the large-\( N \) limit in the next section.

In a previous Letter [2], we described the procedure for transcribing quantum spin chain models satisfying the periodic boundary condition into the corresponding multi-matrix models. The same procedure can be used to establish a one-to-one correspondence between quantum spin chain models satisfying open boundary conditions and multi-matrix models. The major modification is that boundary terms in the Hamiltonian of a spin chain correspond to elements in \( \hat{L}_\Lambda \) and \( \hat{R}_\Lambda \). Suppose a Hamiltonian of a matrix model is a linear combination \( H = \sum_{I,J} (h_I^f \sigma_f^J + a_I^f l_I^f + b_I^f r_I^f) \) where \( h_I^f, a_I^f \) and \( b_I^f \) are non-zero coefficients only if \( I \) and \( J \) have the same number of indices. Such linear combinations form a subalgebra of \( \hat{\Sigma}_\Lambda \); let us call it \( \hat{\Sigma}_\Lambda^0 \). Then there is an isomorphism between multi-matrix models whose Hamiltonians are in \( \hat{\Sigma}_\Lambda^0 \) and quantum spin chains with interactions involving neighborhoods of spins, and with open boundary conditions. Furthermore, if the quantum spin chain
model is exactly solvable, so is the associated multi-matrix model, and vice versa.

These matrix models preserve the number of ‘gluons’ or string bits. Such models can be used to study interesting string phenomena whenever the string tension becomes so large that the ground state favors as many string bits as possible \[33\]. Since matrix models are appearing in many different contexts, the ideas in Ref.\[33\] are probably interesting even outside string theory.

Our work also suggests a deep relationship between integrable spin chain models and our Lie algebras: to each solution of the Yang-Baxter-Sklyanin relations, there is a maximal abelian subalgebra of the centrix algebra. This is just the subalgebra spanned by the conserved quantities of the corresponding spin chain: each conserved quantity is an element of our centrix algebra. Thus it emerges that the Lie algebras we have found are the underlying symmetries of many integrable models. We intend to develop these ideas in the future into a systematic theory. For now we content ourselves with a list of integrable multi-matrix models.

- **the six-vertex model and the corresponding integrable open quantum spin-$\frac{1}{2}$ chain model** \[20, 21, 34\]. A typical quantum state $\Phi$ can be characterized by a sequence of $c$ 2-dimensional column vectors, where $c$ is the number of sites in the open chain. Spin-up and spin-down states at the $j$-th site are characterized by the $j$-th column vectors being

$$
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$

respectively. The Hamiltonian $H_{\text{spin}}^{\text{XXZ}}$ of this spin chain model is

$$
H_{\text{XXZ}}^{\text{spin}} = \frac{1}{2 \sin \gamma} \sum_{j=1}^{c-1} (\tau_j^x \tau_{j+1}^x + \tau_j^y \tau_{j+1}^y + \cos \gamma \tau_j^z \tau_{j+1}^z) + i \sin \gamma (\coth \xi_- \tau_1^z + \coth \xi_+ \tau_c^z)
$$

where $\gamma \in (0, \pi)$ and both $\xi_-$ and $\xi_+$ are arbitrary constants. In Eq.\,(60), $\tau_j^x$, $\tau_j^y$ and $\tau_j^z$ are Pauli matrices at site $j$. They act on the $j$-th column vector. Two Pauli matrices at different sites (i.e., with different subscripts) are commuting. This is actually an integrable XXZ model with an open boundary condition.
We can paraphrase the model in terms of the states and operators of the matrix model introduced in Section 2 as follows. A single meson state corresponds to an open spin chain. We allow the existence of only one possible quantum state other than color for both quarks and antiquarks. Thus the notations for quarks and antiquarks in the expressions for single meson states can be suppressed. Each gluon corresponds to a spin. There are 2 possible quantum states (i.e., \( \Lambda = 2 \)) other than color for both quarks and antiquarks.

Therefore a typical quantum state of a spin chain can be denoted by \( s \). There are 2 possible quantum states other than color for both quarks and antiquarks. Thus the notations for quarks and antiquarks in the expressions for single meson states can be suppressed. Each gluon corresponds to an open spin chain. We allow the existence of only one possible quantum state other than color for both quarks and antiquarks. Thus the notations for quarks and antiquarks in the expressions for single meson states can be suppressed. Each gluon corresponds to an open spin chain. We allow the existence of only one possible quantum state other than color for both quarks and antiquarks.

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The bulk term of the Hamiltonian \( H_{\text{XXZ}}^{\text{matrix}} \) of the corresponding integrable matrix model in the large-\( N \) limit can be determined by the almost same procedure outlined in Ref.[2], except a minor change in the ranges of summations in some formulae there. The boundary terms on the left can be obtained from the correspondences \( \tau^+_1 \leftrightarrow l^+_1 - l^+_2 \), \( \tau^-_1 \leftrightarrow 2l^+_2 \) and \( \tau^+_1 - \tau^-_1 \leftrightarrow 2l^+_2 \). The boundary terms on the right can be determined analogously. Consequently,

\[
H_{\text{XXZ}}^{\text{matrix}} = \frac{1}{2 \sin \gamma} \left\{ 2 (\sigma^{21}_{12} + \sigma^{12}_{21} + \cos \gamma (\sigma^{11}_{11} - \sigma^{12}_{12} - \sigma^{21}_{21} + \sigma^{22}_{22}) + i \sin \gamma [\coth \xi_- (l^+_1 - l^+_2) + \coth \xi_+ (r^+_1 - r^+_2)] \right\}. \tag{61}
\]

We can further rewrite this formula in terms of the creation and annihilation operators \( a^\dagger \) and \( a^\dagger \):

\[
H_{\text{XXZ}}^{\text{matrix}} = \frac{1}{2N \sin \gamma} \left\{ 2 (a^\dagger_{\mu_1} a_{\mu_2} (1) a^\dagger_{\nu_1} a_{\nu_2} (1) a_{\mu_1} (2)
+ a^\dagger_{\mu_1} (2) a^\dagger_{\mu_2} (1) a^\dagger_{\nu_2} (2) a_{\nu_1} (1))
+ \cos \gamma (a^\dagger_{\mu_1} a_{\mu_2} (1) a^\dagger_{\nu_1} a_{\nu_2} (1) a_{\mu_1} (1) - a^\dagger_{\mu_1} (1) a^\dagger_{\mu_2} (1) a^\dagger_{\nu_2} (2) a_{\nu_1} (2) a_{\mu_1} (1)
- a^\dagger_{\mu_1} (2) a^\dagger_{\mu_2} (1) a^\dagger_{\nu_1} (2) a_{\nu_2} (2) a_{\mu_1} (2))
+ i \sin \gamma [\coth \xi_- (q^\dagger_{\mu_1} a^\dagger_{\mu_2} (1) a_{\mu_3} (1) q_{\mu_3} - q^\dagger_{\mu_1} a^\dagger_{\mu_2} (2) a_{\mu_3} (2) q_{\mu_3})
\coth \xi_+ (q_{\mu_1} a_{\mu_2} (1) a_{\mu_3} (1) q_{\mu_3} - q_{\mu_1} a_{\mu_2} (2) a_{\mu_3} (2) q_{\mu_3})] \right\}. \tag{62}
\]
integrable spin-$\frac{1}{2}$ XYZ model \[35\]. The Hamiltonian is
\[ H_{\text{spin}}^{\text{XYZ}} = \sum_{j=1}^{c-1} \left\{ \left[ (1 + \Gamma) \tau_j^x \tau_{j+1}^x + (1 - \Gamma) \tau_j^y \tau_{j+1}^y + \Delta \tau_j^z \tau_{j+1}^z \right] + \text{sn} \left( A_+ \tau_1^+ + B_+ \tau_1^+ + C_+ \tau_1^+ \right) \right\}, \]
where
\[ \tau^\pm = \tau^x \pm i \tau^y \]
\[ \Gamma = k \text{sn}^2 \eta, \quad \Delta = c \eta \text{dn} \eta, \]
\[ A_\pm = \frac{c \eta \pm d \eta}{\text{sn} \eta}, \quad B_\pm = \frac{2 \mu_\pm (\lambda_\pm + 1)}{\text{sn} \eta}, \quad C_\pm = \frac{2 \mu_\pm (\lambda_\pm - 1)}{\text{sn} \eta}, \]
and \( \text{sn} u = \text{sn}(u; k) \), the Jacobi elliptic function of modulus \( 0 \leq k \leq 1 \). \( \eta, \xi_\pm \) and \( \lambda_\pm \) are arbitrary constants. The corresponding Hamiltonian in the matrix model is
\[ H_{\text{matrix}}^{\text{XYZ}} = 2(\sigma_{21}^1 + \sigma_{12}^2) + 2\Gamma(\sigma_{11}^{22} + \sigma_{22}^{11}) + \Delta(\sigma_{11}^{11} - \sigma_{12}^{21} - \sigma_{21}^{12} + \sigma_{22}^{22}) + \text{sn} \left[ A_-(l_1^1 - l_2^1) + B_- l_2^1 + C_- l_1^1 \right] + A_+(r_1^1 - r_2^2) + B_+ r_2^1 + C_+ r_1^1. \]

integrable spin-1 XXZ model, or Fateev-Zamolodchikov model \[36\]. The Hamiltonian is
\[ H_{\text{spin}}^{\text{FZ}} = \sum_{j=1}^{c-1} \left\{ S_j \cdot S_{j+1} - (S_j \cdot S_{j+1})^2 + \frac{1}{2}(q - q^{-1})^2 S_j^z S_{j+1}^z \right\} + \frac{1}{2}(q - q^{-1})^2 (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + S_j^z S_{j+1}^z (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + \frac{1}{2}(q - q^{-1})^2 ([S_j^x]^2 + [S_j^z]^2) \]
where \( q \) is an arbitrary constant, and \( S^x, S^y \) and \( S^z \) are spin-1 matrices:
\[ S^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \] and
We identify $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ in the spin-1 chain model by the numbers 1, 2 and 3 respectively in the matrix model. Then the corresponding Hamiltonian in the matrix model is

$$H_{\text{FZ}}^{\text{matrix}} = (-\sigma_{12}^{12} + \sigma_{21}^{12} + \sigma_{21}^{12} - \sigma_{21}^{21}) + (-\sigma_{23}^{23} + \sigma_{23}^{32} + \sigma_{32}^{23} - \sigma_{32}^{32})$$

$$-2(\sigma_{13}^{13} + \sigma_{31}^{31}) - (\sigma_{13}^{13} + \sigma_{31}^{13} + 2\sigma_{22}^{22})$$

$$-\frac{1}{2}(q^2 + q^{-2})(\sigma_{13}^{13} + \sigma_{31}^{31}) + (q + q^{-1})(\sigma_{13}^{22} + \sigma_{31}^{22} + \sigma_{22}^{13} + \sigma_{22}^{31})$$

$$+\frac{1}{2}(q - q^{-1})^2(2\sigma_{11}^{11} + 2\sigma_{33}^{33} - l_1^1 - r_1^1 - l_3^3 - r_3^3)$$

$$+(q^2 - q^{-2})(r_1^1 - r_3^3 + l_1^1 - l_3^3).$$

(66)

- integrable spin-1 XYZ model. The Hamiltonian was worked out by Mezincescu and Nepomechie [37]. We can use the same trick as above to work out the corresponding integrable matrix model in the large-$N$ limit.

7 Quantum Chromodynamics in Two Dimensions

The second family of models in which our symmetry algebras are applicable are QCD models in various dimensions, dimensionally reduced or not. We are going to explain briefly how to obtain the approximate Hamiltonian of a (2+1)-dimensional QCD model with 1 flavor of quarks and antiquarks by dimensional reduction from (2+1) to (1+1), which is actually a model of mesons studied by Antonuccio and Dalley [26], and to write down the
Hamiltonian as an element of the grand algebra. We will give explicit formulae for the Hamiltonian. We will then make a different approximation from ref.[26] (not assuming that the gluon number is conserved) which will give us a partial analytical understanding of the spectrum.

In this section we will let the regulator \( \Lambda \to \infty \), so that the momentum indices will take an infinite number of values. In order that the previous discussions of our algebra apply directly here, we will need to regularize the field theory such that the momentum variables can take only a finite number \( \Lambda \) of distinct values. But this is mostly a technicality, since we will only talk about field theories without divergences for which the limit \( \Lambda \to \infty \) should exist.

Consider an SU(\( N \)) gauge theory in (2+1) dimensions with one flavor of quarks of mass \( m \). Let \( \tilde{g} \) be the strong coupling constant, \( \alpha \) and \( \beta \) be ordinary spacetime indices (\( \alpha \) and \( \beta \in \{0, 1, 3\} \)), \( A_\alpha \) be a gauge potential and \( \Psi \) be a quark field in the fundamental representation of the gauge group U(\( N \)). \( A_\alpha \) is a traceless \( N \times N \) Hermitian matrix fields whereas \( \Psi \) is a column vector of \( N \) Grassman fields. Let the covariant derivative \( D_\alpha \Psi = \partial_\alpha \Psi + iA_\alpha \Psi \) and the Yang-Mills field be \( F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + i[A_\alpha, A_\beta] \) as usual. Then the action is

\[
S = \int d^4x \left[ -\frac{1}{4\tilde{g}^2} \mathrm{Tr} F_{\alpha \beta} F^{\alpha \beta} + i\bar{\Psi} \gamma^\alpha D_\alpha \Psi - m\bar{\Psi} \Psi \right]
\]

in the Weyl representation

\[
\gamma^0 = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \quad \text{and} \quad \gamma^i = \left( \begin{array}{cc} 0 & \tau^i \\ -\tau^i & 0 \end{array} \right)
\]

for \( i = 1 \) or 3. Following the same procedure in Ref.[23], we can get the following expressions for the light-front momentum \( P^- \) and energy \( P^+ \):

\[
P^+ = \int_0^\infty dk \left[ 1 \otimes \sigma_k^1 \otimes 1 + \sum_{j=+,\,-} (\Xi_{kj}^i \otimes l_\phi^i \otimes 1
\right.

\left. + 1 \otimes r_\phi^i \otimes \Xi_{kj}^i ) \right]
\]

\[
P^- = \int_0^\infty dk h_{IV}(k) 1 \otimes \sigma_k^1 \otimes 1 + \int_0^\infty dk_1 dk_2 dk_3 dk_4.
\]

\[^4\text{In fact our equation Eq.(70) do not agree with Eq.(14) for the self-energy in Ref.[26]. But since these self-energy terms will anyway be absorbed into a redefinition of the mass of the scalar, they don't affect the answer.}\]
\[
\begin{align*}
&\left[h_{IV}(k_1, k_2; k_3, k_4)\delta(k_1 + k_2 - k_3 - k_4)1 \otimes \sigma^{k_3 k_4}_{k_1 k_2} \otimes 1 \\
&+ h_{IV}(k_1; k_2, k_3, k_4)\delta(k_1 - k_2 - k_3 - k_4) \left(1 \otimes \sigma^{k_2 k_3 k_4}_{k_1} \otimes 1 \right. \\
&\left. + 1 \otimes \sigma^{k_1}_{k_2 k_3 k_4} \otimes 1 \right) \right] \\
&+ \sum_{j=+,-} \int_{0}^{\infty} dk h_{II}(k) \left(\Xi_{k_j}^j \otimes t_0^0 \otimes 1 + 1 \otimes r_0^0 \otimes \Xi_{k_j}^j \right) \\
&+ \int_{0}^{\infty} dk_1 dk_2 dk_3 h_{II}(k_1; k_2, k_3)\delta(k_1 - k_2 - k_3) \left(-\Xi_{k_3}^{-1} \otimes t_{k_2}^0 \otimes 1 \\
&+ \Xi_{k_3}^{1} \otimes t_{k_2}^0 \otimes 1 - \Xi_{k_3}^{-1} \otimes t_{k_2}^0 \otimes 1 + \Xi_{k_3}^{1} \otimes t_{k_2}^0 \otimes 1 \\
&- 1 \otimes r_{k_2}^0 \otimes \Xi_{k_3}^{-1} + 1 \otimes r_{k_2}^0 \otimes \Xi_{k_3}^{1} - 1 \otimes r_{k_2}^0 \otimes \Xi_{k_3}^{1} \\
&+ 1 \otimes r_{k_2}^0 \otimes \Xi_{k_3}^{-1} \right) \\
&+ \sum_{j=+,-} \int_{0}^{\infty} dk_1 dk_2 dk_3 dk_4 h_{II}(k_1, k_2; k_3, k_4)\delta(k_1 + k_2 - k_3 - k_4) \cdot \\
&\left(\Xi_{k_4}^{1, j} \otimes t_{k_3}^0 \otimes 1 + 1 \otimes r_{k_3}^0 \otimes \Xi_{k_4}^{1, j} \right) \\
&+ h_{II}(k_1; k_2, k_3, k_4)\delta(k_1 - k_2 - k_3 - k_4) \left(\Xi_{k_4}^{1, j} \otimes t_{k_3}^0 \otimes 1 + 1 \otimes r_{k_3}^0 \otimes \Xi_{k_4}^{1, j} \right) \\
&+ \int_{0}^{\infty} dk_1 dk_2 dk_3 dk_4 h_1(k_1, k_2; k_3, k_4)\delta(k_1 + k_2 - k_3 - k_4) \cdot \\
&\left(\Xi_{k_3}^{-1} \otimes f_0^0 \otimes \Xi_{k_4}^{1, j} + \Xi_{k_3}^{1, j} \otimes f_0^0 \otimes \Xi_{k_4}^{-1} \right) \right) \quad (70)
\end{align*}
\]

where

\[
\begin{align*}
\frac{h_{IV}(k)}{4\pi k} &= \frac{g^2 N}{4\pi k} \int_{0}^{k} dp \left(\frac{k + p}{p(k - p)}\right)^2; \\
\frac{h_{IV}(k_1, k_2; k_3, k_4)}{8\pi} &= \frac{1}{\sqrt{k_1 k_2 k_3 k_4}} \cdot \\
&\left[\frac{(k_2 - k_1)(k_4 - k_3)}{(k_3 + k_1)(k_4 + k_2)} - \frac{(k_3 + k_1)(k_4 + k_2)}{(k_4 - k_2)^2} \right] \\
\frac{h_{IV}(k_1; k_2, k_3, k_4)}{8\pi} &= \frac{1}{\sqrt{k_1 k_2 k_3 k_4}} \cdot \\
&\left[\frac{(k_2 + k_1)(k_4 - k_3)}{(k_4 + k_3)^2} - \frac{(k_4 + k_1)(k_3 - k_2)}{(k_3 + k_2)^2} \right];
\end{align*}
\]

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\[ h_{II}(k) = \frac{m^2}{2k} + \frac{g^2 N}{4\pi} \int_0^k dp \left[ \frac{1}{p^2} + \frac{1}{2p(k-p)} + \frac{1}{(k-p)^2} \right]; \]

\[ h_{II}(k_1; k_2, k_3) = \frac{g}{4} \sqrt{\frac{N}{\pi \sqrt{k_2}}} \left( \frac{1}{k_1} - \frac{1}{k_3} \right); \]

\[ h_{II}(k_1, k_2; k_3, k_4) = \frac{g^2 N}{8\pi} \frac{1}{\sqrt{k_2 k_3}} \left[ -2(k_2 + k_3) \right] \frac{1}{(k_2 - k_3)^2} + \frac{1}{k_1 + k_2}; \]

\[ h_{II}(k_1; k_2, k_3, k_4) = \frac{g^2 N}{8\pi} \frac{1}{\sqrt{k_2 k_3}} \left[ \frac{2(k_2 - k_3)}{(k_2 + k_3)^2} + \frac{1}{k_3 + k_4} \right]; \]

\[ h_{I}(k_1, k_2; k_3, k_4) = -\frac{g^2 N}{2\pi} \frac{1}{(k_1 - k_3)^2}. \]

Remark: \( h_{IV}(k), h_{IV}(k_1, k_2; k_3, k_4) \) and \( h_{IV}(k_1; k_2, k_3, k_4) \) are closely related to the coefficients \( A, B \) and \( C \) in [25]. The Roman numerals \( I \) and \( IV \) carried by some \( h \)’s refer to the fact that these are coefficients of operators of the first and fourth kinds respectively, whereas the Roman number \( II \) carried by other \( h \)’s signify that these are coefficients of operators of the second and third kinds. \( \text{(Eq.(70) corresponds to Eq.(14) in Ref.[26], which shows terms corresponding to operators of the second kind only.)} \)

In the limit of a large number of gluons, we can regard each gluon as moving in the mean field created by the others. Hence it should be a sensible approximation to consider them all as occupying the same state with wave function \( u(k) \). In this case we effectively have a one-dimensional Hilbert space for the gluons, and our algebras simplify to the case \( \Lambda = 1 \).

Suppose we choose an orthonormal basis where the first element is \( u(k) \). Then we can split our Hamiltonian as

\[ P^- = P_0^- + P_1^- \quad (71) \]

where \( P_0^- \) is the contribution when all the indices correspond to \( u(k) \) and \( P_1^- \) is the rest of the terms. In fact,

\[ P_0^- = \mu \sigma_1^1 + \alpha \sigma_2^1 + \beta \sigma_3^1 + \beta^* \sigma_1^3. \quad (72) \]

The last two terms in \( P_0^- \) describe the processes where a gluon decays into three gluons or three of them combine into one gluon. The coefficients are integrals over the wavefunction \( u(k) \):

\[ \mu = \int_0^\infty dk h_{IV}(k) u^*(k) u(k); \quad (73) \]
\[ \alpha = \int_{0}^{\infty} dk_1 dk_2 dk_3 dk_4 h_{IV}(k_1, k_2, k_3, k_4) u^*(k_3) u^*(k_4) u(k_1) u(k_2); \quad (74) \]

and

\[ \beta = \int_{\infty}^{\infty} dk_1 dk_2 dk_3 dk_4 h_{IV}(k_1, k_2, k_3, k_4) u^*(k_1) u(k_2) u(k_3) u(k_4). \quad (75) \]

We can then determine the spectrum of \( P_{0}^- \) for a given \( u(k) \) using the \( \Lambda = 1 \) special case of our commutation relations. It amounts to solving some recursion relations which we omit for the sake of brevity.

The function \( u(k) \) can be determined by mean field theory. This is the same as picking the \( u(k) \) that minimizes the expectation value \( \langle P^- \rangle \), in each sector with a given average number of gluons. A moments thought will show that \( \langle P_1^- \rangle = 0 \). Hence we must find the spectrum of \( P_{0}^- \) and minimize the energy over all \( u(k) \).

Clearly the sectors with even and odd numbers of gluons do not mix with each other. The spectrum of \( P_{0}^- \) in the even sector is

\[ \lambda_n = \mu + (1 + 2n) \sqrt{(\mu + \alpha)^2 - 4\beta^2}, \quad n = 0, 1, 2, \ldots \quad (76) \]

The ‘principal quantum number’ \( n \) is approximately the average number of gluons. The wave function is a linear combination with differing numbers of gluons:

\[ |\psi_n\rangle = \sum_{r=1}^{\infty} \xi_{nr} |2r\rangle \quad (77) \]

where the coefficients \( \xi_{nr} \) depend on \( \mu, \alpha \) and \( \beta \). The lowest energy state has an exponentially decreasing probability to have several gluons, in agreement with previous studies. But as \( n \) grows, the wave function has contributions from states with different numbers of gluons.

The linearly rising eigenvalues of \( P^- \) correspond to a mass spectrum that is also approximately linearly rising:

\[ M_n^2 = \lambda_n P_n^+ \quad (78) \]

where

\[ P_n^+ = \int_{0}^{\infty} dk k u^*(k) u(k). \quad (79) \]
We can then determine \( u(k) \) by minimizing the energy for each fixed \( n \). We will report more detailed information on the spectrum in later papers.

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**Appendix**

**A On Multi-Indices**

Much of our work involves manipulating tensors carrying multiple indices. For the convenience of the reader, we give here a summary of the notations used in this paper for multi-indices.

We will use lower case Latin letters such as \( i, j, i_1, j_2 \) to denote indices with values 1, 2, \ldots, or \( \Lambda \). Here \( \Lambda \) itself is a fixed positive integer, denoting the number of degrees of freedom of gluons. Often we will have to deal with a whole sequence of indices \( i_1 i_2 i_3 \ldots i_a \), which we will denote by the corresponding uppercase letter \( I \) as the collective index of this sequence. The length of the sequence \( I \) will be denoted by \( \#(I) \).

Thus if

\[
I = i_1 i_2 \ldots i_a
\]

and

\[
J = j_1 j_2 \ldots j_b,
\]

then we have \( \#(I) = a \) and \( \#(J) = b \). The composition of these two sequences will be denoted by

\[
I J = i_1 i_2 \ldots i_a j_1 j_2 \ldots j_b.
\]

In particular, we have

\[
I j = i_1 i_2 \ldots i_a j,
\]  

(80)
when only a single index is added to the end. This operation of composing sequences is associative but not commutative:

\[ IJ \neq JI, \quad I(JK) = (IJ)K = IJK. \]  

Often we will have to allow the null sequence \( \emptyset \) among the range of values of a collective index. A collective index that is allowed to take the null sequence as its value will have a dot over it. Thus the possible values of \( \dot{I} \) are

\[ \dot{I} = \emptyset, 1, 2, \ldots \Lambda, 11, 12, \ldots 1\Lambda, 21, \ldots, \Lambda\Lambda, \ldots \]  

(82)

while those of \( I \) do not include the empty set:

\[ I = 1, 2, \ldots \Lambda, 11, 12, \ldots 1\Lambda, 21, \ldots, \Lambda\Lambda, \ldots \]  

(83)

Of course the length of the null sequence is zero. It is the identity element of the composition law above,

\[ \emptyset \dot{I} = \dot{I} \emptyset = \dot{I}. \]  

(84)

In fact the inclusion of the empty sequence in the set of all sequences turns it into a semi-group under the above composition law.

The equation \( I = J \) means that they have the same length \( a \) (say) and

\[ i_1 = j_1; i_2 = j_2; \ldots i_a = j_a. \]

In the same way we can define \( \dot{I} = \dot{J} \) either if they are both the empty sequence, or if they are equal in the above sense.

The Kronecker delta function for integer sequences is defined as follows:

\[ \delta^I_J \equiv \begin{cases} 1 & \text{if } I = J; \\
0 & \text{if } I \neq J. \end{cases} \]

and similarly for dotted indices. The summation sign in an expression such as

\[ \sum_I X^I_J, \]

means that all possible distinct sequences, excluding the empty sequence, are summed over. (In all practical cases, it turns out that there is only a finite
number of I's such that $X'_I \neq 0$ so there will be no convergence problems.) On the other hand, the summation sign over a dotted index

$$\sum_{\hat{I}} X'_{\hat{J}}$$

means that all possible distinct sequences for $\hat{I}$, including the empty sequence, are summed over.

Often we will have to sum over all the ways of splitting a sequence into subsequences. For example, the summation sign on the L.H.S. of the equation

$$\sum_{I_1, I_2} X^I_1 Y^I_2 \equiv \sum_{I_1, I_2} \delta^I_{I_2} X^{I_1} Y^{I_2}$$

(85)

denotes the sum over all the ways in which a given index $I$ can be split into two nonempty subsequences $I_1$ and $I_2$. If there is no way to split $I$ as required, then the sum simply yields 0. For example, Eq.(85) yields 0 if $I$ has only one integer. If the first subsequence is allowed to be empty in the sum, we would write instead

$$\sum_{I_1, I_2} X^{I_1} Y^{I_2}.$$  

(86)

When we talk of an algebra spanned by operators such as $f^I_{\hat{I}}, l^I_{\hat{J}}, r^I_{\hat{J}}$ or $\sigma^I_{\hat{J}}$, the underlying vector space is that of finite linear combinations. For example, in the linear combination $\sum_{I,J} c^I_{\hat{I}} \sigma^J_{\hat{J}}$, although $I,J$ can take an infinite number of values, only a finite number of the complex numbers $c^I_{\hat{I}}$ can be non-zero. Thus there are never any issues of convergence in the sums of interest to us: they are all finite sums.

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