STRICT SINGULARITY OF VOLterra type OPERATORS ON HARDY SPACES

QINGZE LIN, JUNMING LIU*, YUTIAN WU

Abstract. In this paper, we first characterize the boundedness and compactness of Volterra type operator $S_g f(z) = \int_0^z f'(\omega) g(\omega) d\omega$, $z \in \mathbb{D}$, defined on Hardy spaces $H^p$, $0 < p < \infty$. The spectrum of $S_g$ is also obtained. Then we prove that $S_g$ fixes an isomorphic copy of $\ell^p$ if the operator $S_g$ is not compact on $H^p$. In particular, this implies that the strict singularity of the operator $S_g$ coincides with the compactness of the operator $S_g$ on $H^p$. Moreover, when $p \neq 2$, we show that $S_g$, when acting on $H^p$, does not fix any isomorphic copy of $\ell^2$ when $g$ satisfies certain condition. At last, we post an open question related to the symbol function $g$.

1. Introduction

Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space consisting of all analytic functions on $\mathbb{D}$. Then for $0 < p < \infty$, the Hardy space $H^p$ on $\mathbb{D}$ consists of all analytic functions $f \in H(\mathbb{D})$ satisfying

$$H^p := \{ f \in H(\mathbb{D}) : \|f\|_{H^p} = \left( \lim_{r \to 1^-} \int_{\partial \mathbb{D}} |f(r\xi)|^p dm(\xi) \right)^{1/p} < \infty \},$$

where $m$ is the normalized Lebesgue measure on $\partial \mathbb{D}$. By [12 Theorem 2.6], this norm is equal to the following norm:

$$H^p := \{ f \in H(\mathbb{D}) : \|f\|_{H^p} = \left( \int_{\partial \mathbb{D}} |f(\xi)|^p dm(\xi) \right)^{1/p} < \infty \},$$

where for any $\xi \in \partial \mathbb{D}$, $f(\xi)$ is the radial limit which exists almost every (see [32 Theorem 9.4]).

When $p = \infty$, the space $H^\infty$ is defined by

$$H^\infty = \{ f \in H(\mathbb{D}) : \|f\|_\infty := \sup_{z \in \mathbb{D}} \{|f(z)|\} < \infty \}.$$

For any analytic function $g \in H(\mathbb{D})$, there are two kinds of Volterra type operators defined, respectively, by

$$(T_g f)(z) = \int_0^z f(\omega) g(\omega) d\omega, \quad z \in \mathbb{D}, f \in H(\mathbb{D}),$$

and

$$(S_g f)(z) = \int_0^z f'(\omega) g(\omega) d\omega, \quad z \in \mathbb{D}, f \in H(\mathbb{D}).$$

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*Corresponding author.
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The boundedness and compactness of these two operators on some spaces of analytic functions were extensively studied. Pommerenke [28] firstly studied the boundedness of $T_g$ on Hardy-Hilbert space $H^2$. After his work, Aleman, Siskakis and Cima [2, 3] systematically studied the boundedness and compactness of $T_g$ on Hardy space $H^p$, in which they showed that $T_g$ is bounded (or compact) on $H^p$, $0 < p < \infty$, if and only if $g \in BMOA$ (or $g \in VMOA$). What’s more, Aleman and Siskakis [4] studied the boundedness and compactness of $T_g$ on Bergman spaces while Galanopoulos, Girela and Peláez [13, 14] investigated the boundedness of $T_g$ and $S_g$ on Dirichlet type spaces and Xiao [20] studied $T_g$ and $S_g$ on $Q_p$ spaces.

Recently, Lin, et al [20] studied the boundedness of $T_g$ and $S_g$ acting on the derivative Hardy spaces $S^p$. For these operators on other spaces like Fock spaces and weighted Banach spaces, see [5, 7, 9, 21, 22, 23, 29] and the references therein.

A bounded operator $T: X \to Y$ between Banach spaces is strictly singular if its restriction to any infinite-dimensional closed subspace is not an isomorphism onto its image. This notion was introduced by Kato [15]. The obvious example of strictly singular non-compact operators are inclusion mappings $i_{p,q}: \ell^p \hookrightarrow \ell^q$, when $1 \leq p < q < \infty$.

A bounded operator $T: X \to Y$ between Banach spaces is said to fix a copy of the given Banach space $E$ if there is a closed subspace $M \subseteq X$, linearly isomorphic to $E$, such that the restriction $T|_M$ defines an isomorphism from $M$ onto $T(M)$. The bounded operator $T: X \to Y$ is called $\ell^p$-singular if it does not fix any copy of $\ell^p$.

Mihkînen [24] studied the strict singularity of $T_g$ on Hardy space $H^p$ and showed that the strict singularity of $T_g$ coincides with its compactness on $H^p$, $1 \leq p < \infty$, whose main ideas come from the recent paper [18] where the corresponding questions are investigated for composition operators.

Although the boundedness and compactness of the operator $T_g$ on $H^p$ had been studied, from the literature that we have looked at so far, the proofs of the boundedness and compactness for the operator $S_g$ on $H^p$ are still not been shown in detail, except for the case $p = 2$ whose study seems to be elementary (see [19]). Thus, in this paper, We first characterize the boundedness and compactness of Volterra type operator $S_g f$ defined on Hardy spaces $H^p$ for $0 < p < \infty$. Base on the characterization of the boundedness for the operator $S_g$ on $H^p$, we are able to characterize the spectrum of $S_g$ on $H^p$, inspired by the idea in the papers [7, 8]. Then we prove that the bounded operator $S_g$ fixes an isomorphic copy of $\ell^p$ if the operator $S_g$ is not compact on $H^p$. In particular, this implies that the strict singularity of the operator $S_g$ coincides with the compactness of the operator $S_g$ on $H^p$. Moreover, when $p \neq 2$, we show that $S_g$, when acting on $H^p$, does not fix any isomorphic copy of $\ell^2$ satisfies some conditions. And in the last section, we post an open question related to the conditions that we force on the symbol function $g$.

Our main results are as follows:

**Proposition 1.** Let $g \in H(\mathbb{D})$ and $0 < p < \infty$. Then the operator $S_g: H^p \to H^p$ is bounded if and only if $g \in H^\infty$.

**Proposition 2.** Let $g \in H(\mathbb{D})$ and $0 < p < \infty$. Then the operator $S_g: H^p \to H^p$ is compact if and only if $g = 0$.

The following proposition characterize the spectrum of the bounded operator $S_g: H^p \to H^p$. 
Proposition 3. Let $g \in H(D)$ and $0 < p < \infty$. Then the spectrum of the bounded operator $S_g : H^p \to H^p$ is $\sigma(S_g) = \{0\} \cup g(D)$.

Theorem 1. Let $1 \leq p < \infty$ and suppose that $S_g : H^p \to H^p$ is bounded but not compact. Then the operator $S_g : H^p \to H^p$ fixes an isomorphic copy of $\ell^p$. In particular, the operator $S_g$ is not strictly singular, that is, strict singularity of bounded operator $S_g$ coincides with its compactness.

For any $\xi \in \partial D$, define the Stolz domain $S(\xi)$ in $D$ with vertex at $\xi$ as the interior of the convex hull of the set $\{z : |z| < 1/2\} \cup \{\xi\}$. For a compact subset $K \subset \partial D$, let $\Lambda_K = \cup_{\xi \in K} S(\xi)$ and $\varphi$ be a Riemann map from $D$ onto $\Lambda_K$ with $\varphi(0) = 0$. To any $g \in H(D)$, we associate the positive Borel measure $\mu_g$ on $D$ defined by
$$d\mu_g(z) = |g(z)|^2(1 - |z|^2)dA(z),$$
where $A$ is the normalized Lebesgue measure on $D$. Denote $\chi_{\Lambda_K}$ as the characterization function on $\Lambda_K$. Then we have

Theorem 2. Let $1 \leq p < \infty$ and suppose that $S_g : H^p \to H^p$ is bounded. If for any $\varepsilon > 0$, there is a compact subset $K \subset \partial D$ with $m(\partial D \setminus K) < \varepsilon$ such that $\chi_{\Lambda_K} d\mu_g$ is a vanishing 3-Carleson measure. Then if $S_g$ is bounded below on an infinite-dimensional subspace $M \subset H^p$, then the restriction $S_M$ fixes an isomorphic copy of $\ell^p$ in $M$. In particular, if $p \neq 2$, the operator $S_g$ does not fix any isomorphic copy of $\ell^2$ in $H^p$.

2. BOUNDEDNESS AND COMPACTNESS OF $S_g$ ON $H^p$

In this section, we provide a detail proof for the conditions of boundedness and compactness of the operator $S_g$ on $H^p$ when $0 < p < \infty$. Although the following proof for the boundedness can be deduced from \cite[Lemma 2.1(i)], we give our proof which is useful not only to the proof of compactness of $S_g$ on $H^p$, but also to the proof of Theorem 1.

Proof of Proposition A. Assume that $S_g : H^p \to H^p$ is bounded. From \cite{31}, we know that $H^p \subset BMOA_p^{1+1/p}$, where $BMOA_p^{1+1/p}$ is the space of analytic functions $f$ satisfying
$$\sup_{a \in D} \int_D |f'(z)|^p(1 - |z|^2)^{p-1}(1 - |\varphi_a(z)|^2)dA(z) < \infty$$
in which $\varphi_a(z) = \frac{a - z}{1 - az}$ is the Möbius transformation on $D$ and $A$ is the normalized Lebesgue measure on $D$. Hence, if $S_g : H^p \to H^p$ is bounded, then $S_g : H^p \to BMOA_p^{1+1/p}$ is also bounded. Therefore, for any $f \in H^p$, we have
$$\sup_{a \in D} \int_D |(S_g f)'(z)|^p(1 - |z|^2)^{p-1}(1 - |\varphi_a(z)|^2)dA(z) \leq C\|f\|_{H^p}^p.$$ 

It is easy to verify that for any $a \in D$, the function
$$f_a(z) = \frac{(1 - |a|^2)^{2-1/p}}{(1 - az)^2}$$
is a unit vector in $H^p$. Thus,
$$\sup_{a \in D} \int_D |(S_g f_a)'(z)|^p(1 - |z|^2)^{p-1}(1 - |\varphi_a(z)|^2)dA(z) \leq C\|f_a\|_{H^p}^p.$$
or equivalently,
\[
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\bar{a}|^p |g(z)|^p \frac{(1 - |a|^2)^{2p-1}}{1 - \bar{a}z}^p (1 - |z|^2)^{p-1} (1 - |\varphi_a(z)|^2) dA(z) \leq C.
\]

Then, let \( z = \varphi_a(\omega) = \frac{a - \omega}{1 - \bar{a} \omega} \) be a Möbius transformation on \( \mathbb{D} \), we have
\[
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\bar{a}|^p |g(\varphi_a(\omega))|^p \frac{(1 - |a|^2)^{2p-1}}{1 - \bar{a} \varphi_a(\omega)}^p (1 - |\varphi_a(\omega)|^2)^{p-1} (1 - |\omega|^2)^2 |\varphi'_a(\omega)|^2 dA(z) \leq C.
\]

Note that \( |1 - \bar{a} \varphi_a(\omega)| = (1 - |a|^2)/|1 - \bar{a} \omega| \) and \( (1 - |\varphi_a(\omega)|^2) = (1 - |z|^2)/|\varphi'_a(\omega)| \), we obtain
\[
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\bar{a}|^p |g(\varphi_a(\omega))|^p |1 - \bar{a} \omega|^p (1 - |\omega|^2)^p dA(z) \leq C.
\]

Now, consider the analytic function \( G_a(\omega) := \bar{a}^p g(\varphi_a(\omega))^p (1 - \bar{a} \omega)^{-p-2} \), we get that
\[
\sup_{a \in \mathbb{D}} |G_a(0)| \leq \sup_{a \in \mathbb{D}} |\bar{a}|^p |g(a)|^p \leq \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\bar{a}|^p |g(\varphi_a(\omega))|^p |1 - \bar{a} \omega|^p (1 - |\omega|^2)^p dA(z) \leq C
\]
which implies that
\[
\sup_{|a| \to 1^-} |g(a)|^p \leq C,
\]
that is, \( g \in H^\infty \).

Conversely, assume that \( g \in H^\infty \), then by \[3\,11\], the operator \( T_g : H^p \to H^p \) and the multiplication operator \( M_g : H^p \to H^p \) are both bounded. Therefore, it follows from the obvious equality \( (M_g f)(z) - (M_g f)(0) = (T_g f)(z) + (S_g f)(z) \) that \( S_g : H^p \to H^p \) is also bounded. Accordingly, the proof is complete. \( \square \)

**Remark 1.** We note that the sufficiency of Proposition \([1] \) can also be proven directly by using the following equivalent norms for \( H^p \) (see \([1] \) p. 125):
\[
\|f\|_{H^p}^p \asymp \int_{\mathbb{D}} \left( \int_{S(\xi)} |f^{(n)}(z)|^2 (1 - |z|^2)^{2n-2} dA(z) \right)^{p/2} dm(\xi) + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p.
\]

**Proof of Proposition 2.** It is obvious that if \( g = 0 \), then \( S_g : H^p \to H^p \) is compact. Conversely, if \( S_g : H^p \to H^p \) is compact, \( S_g : H^p \to BMOA^p_{1+1/p} \) is also compact. Since for any sequence \( \{a_n\}_{n=1}^\infty \) such that \( \lim_{n \to \infty} |a_n| = 1 \), \( f_{a_n} \) converges to 0 uniformly on compact subsets of \( \mathbb{D} \), it holds that
\[
\lim_{n \to \infty} \int_{\mathbb{D}} |(S_g f_{a_n})'(z)|^p (1 - |z|^2)^{p-1} (1 - |\varphi_{a_n}(z)|^2)^2 dA(z) = 0,
\]
then similar to the arguments in the proof of Proposition \([1] \) we obtain
\[
\lim_{n \to \infty} |g(a_n)|^p = 0.
\]
That is, \( g = 0 \). Accordingly, the proof is complete. \( \square \)
3. The spectrum of $S_g$ on $H^p$

In this section, we characterize the the spectrum of the bounded operator $S_g$ on $H^p$.

Proof of Proposition 3. Since for any $f \in S^p \setminus \{0\}$, the function $S_gf$ has a zero at $z = 0$, it holds that $0 \in \sigma(S_g)$.

Now, we assume that $\lambda \in \mathbb{C} \setminus \{0\}$. For any $h \in H(\mathbb{D})$, it is easy to show that the equation

$$f - \frac{1}{\lambda} S_g f = h$$

has the unique solution $f$ in $H(\mathbb{D})$ and the solution is

$$f(z) = R_{\lambda,g} h(z) := \int_0^z \frac{h'(\zeta)}{1 - \frac{1}{\lambda} g(\zeta)} d\zeta + h(0).$$

Therefore, the resolvent set $\rho(S_g)$ of the bounded operator $S_g$ consists precisely of all points $\lambda \in \mathbb{C}$ for which $R_{\lambda,g}$ is a bounded operator on $H^p$.

If $\lambda \in \mathbb{C} \setminus ((0) \cup \{0\})$, then $1 - \frac{1}{\lambda} g(\zeta)$ is bounded away from 0, that is, $\frac{1}{1 - \frac{1}{\lambda} g(\zeta)}$ is bounded. Thus,

$$f = S_{(1 - \frac{1}{\lambda} g)} h + h(0) \in H^p$$

by Proposition 1, which implies that the operator $R_{\lambda,g}$ is a bounded operator on $H^p$. Accordingly, $\mathbb{C} \setminus ((0) \cup \{0\}) \subset \rho(S_g)$, that is, $\sigma(S_g) \subset ((0) \cup \{0\})$.

Conversely, if $\lambda \in g(\mathbb{D})$ and $\lambda \neq 0$, then $\frac{1}{1 - \frac{1}{\lambda} g(\zeta)}$ is not bounded, which implies that the operator $R_{\lambda,g}$ is not bounded on $H^p$, hence we have $g(\mathbb{D}) \setminus \{0\} \subset \sigma(S_g)$.

Thus, in conjunction with the fact that $0 \in \sigma(S_g)$, it holds that

$$g(\mathbb{D}) \cup \{0\} \subset \sigma(S_g) \subset g(\mathbb{D}) \cup \{0\}.$$ 

Since the spectrum $\sigma(S_g)$ is closed, we obtain that $\sigma(S_g) = g(\mathbb{D}) \cup \{0\}$. \qed

4. Proof of Theorem 1

First, we note that Theorem 1 holds for $p = 2$ due to the fact that a bounded linear operator on $H^2$ is compact if and only if it is strict singular, if and only if it does not fix any copy of $\ell^2$ (see [27] 5.1-5.2).

From the proof in Proposition 2, it can be easily checked that, if the bounded operator $S_g: H^p \to H^p$ is not compact, then there exists a sequence $(a_n) \subset \mathbb{D}$ with $0 < |a_1| < |a_2| < \ldots < 1$ and $a_n \to \omega \in \partial \mathbb{D}$, such that there is a positive constant $h$ such that

$$\|S_g(f_{a_n})\|_{H^p} \geq h > 0$$

holds for all $n \in \mathbb{N}$ and $f_{a_n}$ defined in the previous section. We may assume without loss of generality that $a_n \to 1$ as $n \to \infty$ by utilizing a suitable rotation.

Lemma 1. Let $(a_n) \subset \mathbb{D}$ be a sequence as above. Let $A_\varepsilon = \{e^{i\theta} : |e^{i\theta} - 1| < \varepsilon\}$ for each $\varepsilon > 0$. Then for bounded operator $S_g: H^p \to H^p$, we have

$$(1) \lim_{\varepsilon \to 0} \int_{A_\varepsilon} |S_g f_{a_n}|^p dm = 0 \text{ for every } n \in \mathbb{N}. $$

$$(2) \lim_{n \to \infty} \int_{\partial \mathbb{D} \setminus A_\varepsilon} |S_g f_{a_n}|^p dm = 0 \text{ for every } \varepsilon > 0.$$
Proof. (1) For each fixed $n$, this follows immediately from the absolute continuity of Lebesgue measure and the boundedness of operator $S_g$: $H^p \to H^p$.

(2) For given $\varepsilon > 0$, it is easy to see that there is a positive $\gamma > 0$ such that $|1 - \bar{a}_n re^{i\theta}| \geq \gamma$ for all $n \in \mathbb{N}$, $0 \leq r < 1$ and $\varepsilon \leq \theta \leq \pi$. Therefore, for these $r$ and $\theta$, we get that

$$|f'_{a_n}(re^{i\theta})|^p = \frac{|\bar{a}_n|p(1 - |a|^2)^{2p-1}}{|1 - \bar{a}_n re^{i\theta}|^{3p}} \leq \frac{|\bar{a}_n|p(1 - |a_n|^2)^{2p-1}}{\gamma^{3p}},$$

for all $n \in \mathbb{N}$. Then, for any $\xi \in \partial \mathbb{D} \setminus A_\varepsilon$, we have

$$|(S_g f_{a_n})(\xi)|^p = \left| \int_0^1 f'_{a_n}(r\xi)g(r\xi)\xi dr \right|^p \leq \left( \int_0^1 |f'_{a_n}(r\xi)g(r\xi)|dr \right)^p \leq \|g\|^p_{L^\infty} \left( \int_0^1 |f'_{a_n}(r\xi)|dr \right)^p \leq \|g\|^p_{L^\infty} \frac{|\bar{a}_n|p(1 - |a_n|^2)^{2p-1}}{\gamma^{3p}}.$$

Accordingly,

$$\lim_{n \to \infty} \int_{\partial \mathbb{D} \setminus A_\varepsilon} |S_g f_{a_n}|^p dm \leq \lim_{n \to \infty} \|g\|^p_{L^\infty} \frac{|\bar{a}_n|p(1 - |a_n|^2)^{2p-1}}{\gamma^{3p}} = 0.$$

The proof is complete. \[\square\]

Now, we are prepared to give a proof of Theorem 1.

Proof of Theorem 1. First, as noted above, there exists a sequence $(a_n) \subset \mathbb{D}$ with $0 < |a_1| < |a_2| < \ldots < 1$ and $a_n \to 1$, such that there is a positive constant $h$ such that $\|S_g(f_{a_n})\|_{H^p} \geq h > 0$ holds for all $n \in \mathbb{N}$.

Then by Lemma 1 and induction method, we can find a decreasing positive sequence $(\varepsilon_n)$ such that $A_{\varepsilon_1} = \partial \mathbb{D}$ and $\lim_{n \to \infty} \varepsilon_n = 0$, and a subsequence $(b_n) \subset (a_n)$ such that the following three conditions hold:

1. \[\left( \int_{A_n} |S_g f_{b_k}|^p dm \right)^{1/p} < 4^{-n}\delta h, \quad k = 1, \ldots, n - 1;\]
2. \[\left( \int_{\partial \mathbb{D} \setminus A_n} |S_g f_{b_k}|^p dm \right)^{1/p} < 4^{-n}\delta h;\]
3. \[\left( \int_{A_n} |S_g f_{b_n}|^p dm \right)^{1/p} > \frac{h}{2}\]

for every $n \in \mathbb{N}$, where $A_n = A_{\varepsilon_n}$ and $\delta > 0$ is a small constant whose value will be determined later.

Now we are ready to prove that $\| \sum_{j=1}^\infty c_j S_g(f_{b_j}) \|_{H^p} \geq C \| (c_j) \|_{L^p}$, where the constant $C > 0$ may depend on $p$.

\[
\| \sum_{j=1}^\infty c_j S_g(f_{b_j}) \|_{H^p} \geq \sum_{n=1}^\infty \int_{A_n \setminus A_{n+1}} |S_g f_{b_n}|^p dm \geq \sum_{n=1}^\infty \left| c_n \right| \left( \int_{A_n \setminus A_{n+1}} |S_g f_{b_n}|^p dm \right)^{1/p} \left( \sum_{j \neq n} |c_j| \left( \int_{A_n \setminus A_{n+1}} |S_g f_{b_j}|^p dm \right)^{1/p} \right)^p.
\]
Observe that for every $n \in \mathbb{N}$, we have
\[
\left( \int_{A_n \setminus A_{n+1}} |S_g f_{b_n}|^p \, dm \right)^{1/p} = \left( \int_{A_n} |S_g f_{b_n}|^p \, dm - \int_{A_{n+1}} |S_g f_{b_n}|^p \, dm \right)^{1/p} \\
\geq \left( \frac{h}{2} - (4^{-n-1} \delta h)^p \right)^{1/p} \geq \frac{h}{2} - 4^{-n-1} \delta h
\]
according to conditions (1) and (3) above, where the last estimate holds for $1 \leq p < \infty$.

Moreover, we have
\[
\left( \int_{A_n \setminus A_{n+1}} |S_g f_{b_j}|^p \, dm \right)^{1/p} \leq \left( \int_{A_n} |S_g f_{b_j}|^p \, dm \right)^{1/p} < 4^{-n} \delta h
\]
for $j < n$ by condition (1) and
\[
\left( \int_{A_n \setminus A_{n+1}} |S_g f_{b_j}|^p \, dm \right)^{1/p} \leq \left( \int_{\partial D \setminus A_j} |S_g f_{b_j}|^p \, dm \right)^{1/p} < 4^{-j} \delta h
\]
for $j > n$ by condition (2).

Thus it always holds that
\[
\left( \int_{A_n \setminus A_{n+1}} |S_g f_{b_j}|^p \, dm \right)^{1/p} < 2^{-n-j} \delta h \quad \text{for } j \neq n.
\]

Consequently, by the triangle inequality in $L^p$, we obtain that
\[
\| \sum_{j=1}^{\infty} c_j S_g(f_{b_j}) \|_{H^p} \geq \left( \sum_{n=1}^{\infty} \left( |c_n| \left( \frac{h}{2} - 4^{-n-1} \delta h \right) - 2^{-n} \delta h \| (c_j) \|_{\ell^p} \right)^p \right)^{1/p} \\
\geq \left( \sum_{n=1}^{\infty} \left( |c_n| \left( \frac{h}{2} - 2^{-n+1} \delta h \| (c_j) \|_{\ell^p} \right)^p \right)^{1/p} \\
\geq \frac{h}{2} \| (c_j) \|_{\ell^p} - \delta h \| (c_j) \|_{\ell^p} \left( \sum_{n=1}^{\infty} 2^{-(n-1)p} \right)^{1/p} \\
\geq h \left( \frac{1}{2} - \delta \left( 1 - 2^{-p} \right)^{-1/p} \right) \| (c_j) \|_{\ell^p} \geq C \| (c_j) \|_{\ell^p},
\]
where the last inequality holds when we choose $\delta$ small enough.

A straightforward variant of the above procedure also gives
\[
\| \sum_{j=1}^{\infty} c_j S_g(f_{b_j}) \|_{H^p} \leq C_1 \| (c_j) \|_{\ell^p},
\]
where the constant $C_1 > 0$ may depend on $p$.

By choosing $g = 1$ and the fact that $\lim_{n \to \infty} f_{a_n}(0) = 0$, we obtain that
\[
C_2 \| (c_j) \|_{\ell^p} \leq \| \sum_{j=1}^{\infty} c_j f_{b_j} \|_{H^p} \leq C_3 \| (c_j) \|_{\ell^p}.
\]
Thus, we have
\[
\left\| \sum_{j=1}^{\infty} c_j S_g(f_{b_j}) \right\|_{H^p} \geq C \left\| (c_j) \right\|_{\ell^p} \geq CC_3^{-1} \left\| \sum_{j=1}^{\infty} c_j f_{b_j} \right\|_{H^p}
\]

The proof is complete. \qed

5. Proof of Theorem 2

In this section, we give the proof of Theorem 2.

Proof of Theorem 2. For bounded operator $S_g : H^p \to H^p$, we first show that the product of the composition operator $C_\varphi$ and $S_g$ (that is, $C_\varphi S_g$) is compact on $H^p$. We first consider the case $p = 2$. By the Littlewood-Paley identity (see [10, Theorem 2.30]), we get that for any $f \in H^2$,
\[
\left\| C_\varphi S_g f \right\|_{H^2} \approx \int_{\Lambda_K} |f'(\omega)|^2 |g(\omega)|^2 \left(1 - |\varphi^{-1}(\omega)|^2\right) dA(\omega).
\]
By Schwarz's Lemma, it holds that $|\varphi^{-1}(\omega)| \geq \omega$ for all $\omega \in \Lambda_K$, thus, if $\chi_{E_k}dm_\mu$ is a vanishing 3-Carleson measure, it follows from [26, Theorem 1(b)] that the derivative embedding $H^2 \to L^2(\mathbb{D}, \chi_{E_k}dm_\mu)$ is a compact operator, which implies that $C_\varphi S_g$ is also compact on $H^p$.

For other values of $p$, the claim can be deduced from the case $p = 2$ above by the identification $H^p = (H^{p_0}, H^{p_1})_{\theta, p}$ in terms of real interpolation spaces (see [16]) and one-sided Krasnoselskii-type interpolation of compactness for operators (see [17] and [6, Theorem 3.1]).

Then we proceed exactly as [25, Proof of Proposition 3.2, pp.9-10], we obtain that for any $\varepsilon > 0$, there is a compact subset $E \subset \partial \mathbb{D}$ with $m(\partial \mathbb{D} \setminus E) < \varepsilon$ such that $\chi_{E_k}S_g$ is compact from $H^p$ to $L^p(\partial \mathbb{D})$. In particular, for any bounded sequence $(F_n) \subset H^p$, such that $F_n \to 0$ uniformly on compact subsets in $\mathbb{D}$, it holds that $\lim_{k \to \infty} \|\chi_{E_k}S_g F_n\|_{L^p(\partial \mathbb{D})} = 0$.

Therefore, we can find a sequence of compact subsets $E_1 \subset E_2 \subset \ldots \subset \partial \mathbb{D}$ with $m(\partial \mathbb{D} \setminus E_k) \to 0$ as $k \to \infty$ such that for each $k$,
\[
\lim_{n \to \infty} \|\chi_{E_k}S_g(F_n)\|_{L^p(\partial \mathbb{D})} = \int_{E_k} |S_g F_n|^p dm = 0.
\]

On the other hand, for any fixed $n$, by the absolute continuity of Lebesgue measure, we have
\[
\lim_{k \to \infty} \int_{\partial \mathbb{D} \setminus E_k} |S_g F_n|^p dm = 0.
\]

Now, Since $M$ is the infinite-dimensional subspace of $H^p$, there exists a sequence $(F_n)$ of unit vectors in $M$ such that $F_n$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. since $S_g$ is bounded below on $M \subset H^p$, there exists $h > 0$ such that
\[
\int_{\partial \mathbb{D}} |S_g F_n|^p dm > h^p,
\]
for all $n \in \mathbb{N}$.

The remainder of the proof is a straightforward gliding hump type argument that goes exactly as the proof of Theorem 1 so we omit it. Accordingly, the proof is complete. \qed
6. Open question

In Theorem 2, we force a condition on the symbol function $g$: if for an $\varepsilon > 0$, there is a compact subset $K \subset \partial D$ with $m(\partial D \setminus K) < \varepsilon$ such that $\chi_{\Lambda K} d\mu_g$ is a vanishing 3-Carleson measure. Indeed, since we suppose that the operator $S_g$ is bounded on $H^p$, it follows from Proposition 1 that $g \in H^\infty$. We do not know whether or not the boundedness of $g \in H^\infty$ guarantees this condition. So we post it as an open question as follows:

Let $g \in H^\infty$. For any $\varepsilon > 0$, is there a compact subset $K \subset \partial D$ with $m(\partial D \setminus K) < \varepsilon$ such that $\chi_{\Lambda K} d\mu_g$ is a vanishing 3-Carleson measure?

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School of Applied Mathematics, Guangdong University of Technology, Guangzhou, Guangdong, 510520, P. R. China

E-mail address: gdlqz@gdut.edu.cn

School of Applied Mathematics, Guangdong University of Technology, Guangzhou, Guangdong, 510520, P. R. China

E-mail address: jmliu@gdut.edu.cn

School of Financial Mathematics & Statistics, Guangdong University of Finance, Guangzhou, Guangdong, 510521, P. R. China

E-mail address: 26-080@gduf.edu.cn