Two-proton emission systematics

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The simultaneous emission of two protons is an exotic and complex three-body process. It is very important for experimental groups investigating the nuclear stability on the proton drip line to have a simple rule predicting the two-proton decay widths with a reasonable accuracy for transitions between ground as well as excited states in terms of relevant physical variables. In spite of its complexity, we show that the two-proton emission process obeys similar rules as for binary emission processes like proton, alpha and heavy cluster decays. It turns out that the logarithm of the decay width, corrected by the centrifugal barrier, linearly depends upon the Coulomb parameter within one order of magnitude. On the other hand, the universal linear dependence with a negative slope between the logarithm of the reduced width and the fragmentation potential, valid for any kind of binary decay process, is also fulfilled for the two-proton emission with a relative good accuracy. As a consequence of pairing correlations the two protons are simultaneously emitted from a singlet paired state. We evidence that indeed one obtains a linear dependence between the logarithm of the reduced width and pairing gap within a factor of two, giving a good predictive power to this law. It turns out that the diproton and alpha-cluster formation probabilities have similar patterns versus the pairing gap, while in the one-proton case one has a quasi-constant behavior.

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The proton drip line is mainly investigated by proton and two-proton emission processes 1–4. The two-proton emission is a very exotic mode that is energetically possible in some nuclei. In the earlier sixties Goldansky proposed two extreme mechanisms in which the particles are emitted, either simultaneously or sequentially 5. The first systematic theoretical analysis of the processes involving the inherent three-body problem was performed in Ref. 5. The theoretical description of two-proton emission was performed by using few-body formalism in terms of hyperspherical coordinates 5,8, as well as R-matrix approaches 10,11. The Feshbach reaction theory and the continuum shell model were also applied 12,13. The one-proton decay systematics reveals simple two-body features depending on the Coulomb and centrifugal parameters 14. The systematic analysis indicates that the two-proton emission has a three-body character, between the diproton and pure sequential decay 15. On the other hand, it is important to point out that the pairing interaction induces a clustering of the two protons. This is a fundamental property in α-emission, explaining the clustering of the four nucleons 16. In the last years several investigations were performed in order to describe half-lives of the two-proton emission process by using effective liquid drop model 17, Gamow coupled channel approach 18, Gamow model with variable energy 19, semiempirical four-parameter 20 or two-parameter relation 21 and the Gamow approach with square nuclear plus Coulomb potential 22.

In this paper we will show that the systematics of the two-proton emission has a similar universal feature compared to the usual binary decays, namely that the logarithm of the reduced width linearly decreases upon the increase of the fragmentation potential, defined as the difference between the top of the Coulomb barrier and Q-value. On the other hand, as a consequence of the pairing corrections between emitted protons, the same quantity is directly proportional to the pairing gap.

Let us consider the two-proton emission process

\[ P \rightarrow D + 2p \ . \] (1)

Experimental data evidenced the quasi-simultaneous detection of the emitted protons with equal energies. This allowed us to propose in Ref. 23 a simplified approach, where we have shown that the distribution of emitted protons is centered around the configuration with almost equal distances \( r_1 \sim r_2 \), corresponding to polar an angle \( \phi \sim \pi/4 \) defined by usual relations

\[ r_1 = r \cos \phi \ , \quad r_2 = r \sin \phi \ . \] (2)

This is a consequence of the initial condition given by the two-proton pairing wave function on the nuclear surface, centered around \( \phi = \pi/4 \). It is well known that the deuteron has a bound state around \( E \sim \sim 1 \text{ MeV} \). The diproton system beyond the nuclear surface becomes unbound, but a simple estimate shows that the deuteron bound state is pushed up to a resonant state inside the
inter-proton nuclear plus centrifugal plus Coulomb potential

\[ v(r) = -v_0 \exp \left( -\frac{r^2}{b^2} \right) + \frac{\hbar^2 l(l+1)}{2\mu r^2} + \frac{e^2}{r} \]

\[ v_0 = 35 \text{ MeV} , \quad b = 2 \text{ fm} , \quad (3) \]

in terms of the relative distance \( r = r_1 - r_2 \), angular momentum \( l \) and reduced proton mass \( \mu_p = m_0/2 \), at a very small energy \( E_{res} \sim 0.4 \text{ MeV} \). Therefore the diproton "cluster" is weakly bound but its center of mass (cm) radius \( R = \frac{1}{2}(r_1 + r_2) \) moves in the Coulomb field of the daughter nucleus as a real diproton particle. Thus, we suppose for the decay width a similar to the binary case expression, proportional to the scattering amplitude squared \( N_1^2 \) in some channel characterized by the angular momentum \( L \). It can be rewritten

\[ \Gamma_L = \hbar vN_1^2 = \hbar v \frac{\gamma_L^2(R)}{G_L^2(\chi, \rho)} = \gamma_L^2(R)P_L(\chi, \rho) , \quad (4) \]

in terms of the reduced width \( \gamma_L^2(R) \) and Coulomb penetrability \( P_L(\chi, \rho) \). Let us mention that these quantities differs by a constant factor with respect to the standard definitions in Ref. [25]. The penetrability is defined by the irregular Coulomb wave function which has the following semiclassical representation

\[ G_L(\chi, \rho) = G_0(\chi, \rho)C_L(\chi, \rho) . \quad (5) \]

Let us mention that the monopole and centrifugal terms are respectively given

\[ G_0(\chi, \rho) = (\cot \alpha)^{1/2} \exp \left[ \chi (\alpha - \sin \alpha \cos \alpha) \right] \]

\[ C_L(\chi, \rho) = \exp \left[ \frac{L(L+1)}{\chi} \tan \alpha \right] , \quad (6) \]

in terms of the dimensionless parameter

\[ \cos^2 \alpha \equiv \frac{\rho}{\chi} = \frac{Q}{V_C(R)} = \frac{QR}{4ZDE^2} , \quad (7) \]

depending upon the Coulomb parameter and reduced radius

\[ \chi = \frac{4ZDE^2}{h\nu} , \quad \rho = \kappa R \]

\[ v = \sqrt{\frac{2Q}{\mu}}, \quad \hbar \kappa = \mu v . \quad (8) \]

The reduced width \( \gamma_L^2 \) is also called diproton preformation probability. We will estimate it on the nuclear surface at the "geometrical touching configuration"

\[ R = 1.2(A_1^{1/3} + A_2^{1/3}) . \quad (9) \]

Let us stress that this quantity also includes the dissociation probability of the diproton system. In order to prove the validity of this "binary representation" we investigated the available experimental data, given in Table 1.

**Table 1. Parameters of two-proton emission**

| no. | \( Z_P \) | \( N_P \) | \( A_P \) | \( L \) | \( \beta \) | \( Q_{frag} \) (MeV) | \( V_{frag} \) (MeV) | \( \chi \) | \( \rho \) | \( \cos^2 \alpha \) | \( \log_{10} \gamma_{exp}^2 \) | \( \log_{10} \Gamma_{exp} \) (MeV) | \( \log_{10} \Gamma_1 \) (MeV) | \( \log_{10} \Gamma_2 \) (MeV) | Ref. |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   |  4  |  2  |  6  |  0  |  0.000 | 1.370 | 0.316 | 1.248 | 1.014 | 0.813  | -1.625  | -1.041  | -0.562  | -1.230 | [26] |
| 2   |  6  |  2  |  8  |  0  |  0.000 | 2.101 | 1.010 | 1.233 | 1.443 | 0.676  | -1.500  | -0.881  | -1.211  | -1.322 | [27] |
| 3   |  7  |  3  | 10  |  1  |  0.250 | 1.300 | 2.381 | 3.508 | 1.239 | 0.353  | -2.686  | -3.701  | -3.254  | -3.207 | [17] |
| 4   |  8  |  4  | 12  |  0  |  0.000 | 1.640 | 2.577 | 3.825 | 1.488 | 0.393  | -0.933  | -1.241  | -2.825  | -2.536 | [28] |
| 5   | 12  |  7  | 19  |  0  | -0.240 | 0.750 | 5.514 | 9.769 | 1.170 | 0.120  | -3.749  | -10.121 | -9.439  | -9.137 | [29] |
| 6   | 17  | 11  | 28  |  2  |  0.300 | 1.970 | 6.556 | 9.211 | 2.128 | 0.231  | -3.409  | -8.391  | -8.395  | -7.937 | [17] |
| 7   | 19  | 13  | 32  |  2  | -0.120 | 2.080 | 7.262 | 10.208 | 2.273 | 0.223  | -3.508  | -9.091  | -9.097  | -8.667 | [17] |
| 8   | 26  | 19  | 45  |  0  |  0.000 | 1.154 | 10.938 | 19.533 | 1.864 | 0.095  | -3.942  | -18.941 | -18.738 | -18.755 | [30] |
| 9   | 28  | 20  | 48  |  0  |  0.000 | 1.350 | 11.535 | 19.593 | 2.053 | 0.105  | -4.699  | -19.261 | -18.338 | -18.426 | [31] |
| 10  | 29  | 23  | 52  |  4  |  0.190 | 0.770 | 12.337 | 26.986 | 1.585 | 0.059  | -3.918  | -30.701 | -30.604 | -30.794 | [17] |
| 11  | 30  | 24  | 54  |  0  |  0.270 | 1.480 | 11.980 | 20.201 | 2.221 | 0.110  | -4.125  | -18.911 | -18.627 | -18.732 | [32] |
| 12  | 31  | 26  | 57  |  2  |  0.250 | 2.050 | 11.697 | 17.795 | 2.654 | 0.149  | -4.047  | -16.041 | -15.864 | -15.889 | [17] |
| 13  | 36  | 31  | 67  |  0  | -0.270 | 1.690 | 13.763 | 23.041 | 2.520 | 0.109  | -2.581  | -19.641 | -21.011 | -21.332 | [33] |

Here we give the charge \( Z_P \), neutron \( N_P \) and mass number \( A_P \) of the parent nucleus, angular momentum of the emitter diproton \( L \), parent quadrupole deformation \( \beta \), Q-value, fragmentation potential

\[ V_{frag} = V_C(R) - Q , \quad (10) \]

Coulomb parameter \( \chi \), reduced radius \( \rho = \kappa R \), \( \cos^2 \alpha \),
FIG. 1. (a) Reduced radius versus Coulomb parameter. (b) Logarithm of the monopole decay width versus the Coulomb parameter. The fit parameters are given in the first line of the Table 2.

Let us stress that this is a rather large error, corresponding to more than one order of magnitude. Therefore this law has an approximate character and a poor predictive power. This is due to the variation of the reduced width $\gamma^2_L$ along the analyzed emitters.

In order to further analyze this feature let us mention that in all binary emission processes we evidenced in Refs. [15, 34] an analytical universal law for reduced widths

$$\log_{10} \gamma^2 = -\frac{\pi}{\hbar \omega_1} V_{\text{frag}} + \log_{10} s, \quad (14)$$

in terms of the harmonic oscillator frequency of the internal nuclear interaction approximated by an inverted parabola $\hbar \omega_1$ and the spectroscopic factor $s$. We plot in Fig. 2 (a) the logarithm of the experimental reduced width versus the fragmentation potential. Notice that indeed this dependence can satisfactorily be fitted by a straight line

$$\log_{10} \gamma^2_{\text{exp}} \sim a_1 V_{\text{frag}} + b_1, \quad (15)$$

![FIG. 2. Logarithm of the experimental reduced width at the geometrical touching radius $R = 1.2 \left( A_1^{1/3} + 2^{1/3} \right)$ versus the fragmentation potential (a) and pairing gap (b). The fit parameters are given in the second line of the Table 2. The rms error in parentheses corresponds to the analysis without considering cases 4 and 13.](image)

### TABLE 2. Parameters of the fit lines

| $k$ | $a_k$ | $b_k$ | $\sigma_k$ | $\sigma'_k$ |
|-----|-------|-------|------------|-------------|
| 0   | -1.009 | 1.272 | 1.238      | 1.010       |
| 1   | -0.183 | -1.757| 0.742      | 0.396       |
| 2   | 0.622  | -4.876| 0.702      | 0.333       |

where the monopole width $\Gamma_0$ is defined below by Eq. (12), and the logarithm of the experimental decay width $\Gamma_{\text{exp}} = \hbar \ln 2/T_{\text{exp}}$. In the last two columns are given two versions for computed decay widths, as described below.

In order to investigate the relation between the two parameters of the Coulomb function we plot in Fig. 1 (a) the dependence between the reduced radius and Coulomb parameter. Notice their linear correlation except two lower points, corresponding to the cases 4 and 13 in Table 1. Then we analyzed to what extent the Geiger-Nuttall law, expressing the linear relation between the logarithm of the decay width and Coulomb parameter, is fulfilled. To this purpose we extracted the influence of the centrifugal barrier by defining the monopole decay width

$$\Gamma_0 = \Gamma_{\text{exp}} C_L^2(\chi, \rho), \quad (12)$$

In Fig. 1(b) we notice such a linear dependence

$$\log_{10} \Gamma_0 \sim a_0 \chi + b_0, \quad (13)$$

on a wide interval of almost 30 orders of magnitude.

In the first line of Table 2 we give the parameters of the fit line with an overall root mean square (rms) error $\sigma_0 = 1.238$. Notice a slightly smaller rms error $\sigma_0 = 1.010$ by excluding the two above mentioned cases 4 and 13.
FIG. 3. Logarithm of the experimental reduced width versus the pairing gap for proton emission (a) and alpha decay from even-even emitters (b).

with a negative slope \( a_1 < 0 \). The overall rms error in the second line of Table 1 corresponds to a factor of 5 and to a factor of 2.5 if one excludes the cases 4 and 13, corresponding to the upper two points. The fit parameters lead to the following values in Eq. (14): \( \hbar \omega_1 = 7.456 \text{ MeV}, \ s = 0.017 \). Notice that we obtained the same order of magnitude, namely \( \hbar \omega_1 = 11.389 \text{ MeV (} A < 145\), 12.580 MeV (\( A > 145 \)) in the case of the Woods-Saxon potential used to describe one-proton emission and \( \hbar \omega_1 = 9.080 \text{ MeV for double-folding potential describing alpha-decay}^{[35]} \).

Let us mention here that in Ref. [23] we estimated the two-proton formation probability on the nuclear surface within the pairing BCS approach for \( ^{56}\text{Fe (corresponding to no. 8 in Table 1) as being } \gamma_{BCS}^2 \sim 10^{-2} \). From Fig. 2 (a) we notice that for \( V_{\text{frag}} = 10.938 \text{ MeV one obtains } \gamma_{\text{exp}}^2 \sim 10^{-4} \). The missing two orders of magnitude we can ascribe to the penetration of the diproton system through the inter-proton potential [33].

In the same reference we evidenced that the two-proton formation probability quadratically depends upon the pairing gap

\[
\gamma_{BCS}^2 \sim (u \nu)^2 \sim \Delta^2 ,
\]

where \( u \) and \( \nu \) denote the standard BCS amplitudes. Anyway, the systematic analysis of two-proton emitters in Table 1 evidenced that the exponential dependence upon the pairing gap provides a significantly smaller rms error. Thus, we plotted in Fig. 2 (b) the dependence between the logarithm of the experimental reduced width and the systematic rule of the pairing gap \( \Delta = 12A^{-1/2} \)

\[
\log_{10} \gamma_{\text{exp}} \sim a_2 \Delta + b_2 .
\]  (17)

Notice the linear correlation with a rms error in the third line of Table 2 corresponding to a factor of three. One obtains a better agreement within a factor of two if one exclude the two mentioned case 4 (magic in neutrons) and 13.

Let us mention in this context that the situation is quite different in the case of one-proton emission. In Fig. 3 (a) we plotted the experimental reduced width defined by Eq. (11) versus the pairing gap. One sees that the proton formation probability has an one-particle character, being proportional to the BCS amplitude squared \( u_F^2 \) at the Fermi level. In the case of the alpha-decay from even-even emitters the behavior is similar to the two-proton emission, as seen in the panel (b) of the same figure, where we notice two parallel linear dependencies divided by the doubly magic nucleus \( ^{208}\text{Pb}. \) Therefore the formation probability of the diproton and alpha cluster has a common collective pairing nature, in spite of the fact that the first system is unbound, while the second one is strongly bound.

The influence of the quadrupole deformation can be estimated by using the Fröman method as in Ref. [30]. Thus, the influence of the non-diagonal matrix elements...
of the Frohman matrix is rather small, being about 15% for two-proton emitters at \( \beta=0.3 \).

Finally we analyzed the experimental decay width by plotting in Fig. 2 its logarithm as a function of the theoretical width for the panels (a) and (b), respectively

\[
\log_{10} \Gamma_1 = \log_{10} P_L + a_1 V_{frag} + b_1, \quad (a)
\]
\[
\log_{10} \Gamma_2 = \log_{10} P_L + a_2 \Delta + b_2, \quad (b).
\]  

The values of the theoretical predictions are given in the last two columns of Table 1 and the fit parameters in the last two lines of Table 2. Notice the very good linear correlation between these quantities along the first bisectrice plotted by a dashed line.

Concluding, in spite of the fact that the two-proton emission is a three-body process, we evidenced the binary character of laws connecting the logarithm of the decay widths in terms some physical quantities. Thus, we evidenced the linear correlation between the logarithm of the reduced width and the fragmentation potential with a negative slope, predicted as an analytical universal rule for binary emission processes like proton emission, alpha and heavy cluster decays. On the other hand, we also evidenced the role played by the pairing interaction, given by the linear correlation between the logarithm of the reduced width and pairing gap, as predicted by microscopic estimates of the two-proton formation probability. The relative small rms error give a powerfull predictive power to this last rule. The diproton and alpha-cluster formation probabilities have similar patterns versus the pairing gap, while in the one-proton emission one has a quasi-constant behavior.

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