Representation theory of W-algebras and Higgs branch conjecture

ICM 2018 Session “Lie Theory and Generalizations”

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August 2, 2018
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What are W-algebras?

W-algebras are certain generalizations of infinite-dimensional Lie algebras such as affine Kac-Moody algebras and the Virasoro algebra.

W-algebras can be also considered as affinizations of finite W-algebras ([Premet '02]) which are quantizations of Slodowy slices ([De-Sole-Kac '06]).

W-algebras appeared in '80s in physics in the study of two-dimensional conformal field theories.

W-algebras are closely connected with integrable systems, (quantum) geometric Langlands program (e.g. [T.A.-Frenkel '18]), four-dimensional gauge theory ([Alday-Gaiotto-Tachikawa '10]), etc.
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The Zamolodchikov $W_3$-algebra

An example

W-algebras are not Lie algebras in general but vertex algebras.
The Zamolodchikov $W_3$-algebra

generators: $L_n \ (n \in \mathbb{Z})$, $W_n \ (n \in \mathbb{Z})$, $c$,

relations: $[c, W_3] = 0$, $[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c$, $c^2 = 0$. 

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$[W_m, W_n]$
    $= (m - n)\left(\frac{1}{15}(m + n + 3)(m + n + 2) - \frac{1}{6}(m + 2)(n + 2)\right)L_{m+n}$
    $+ \frac{16}{22+5c}(m - n)\Lambda_{m+n} + \frac{1}{360}m(m^2 - 1)(m^2 - 4)\delta_{m+n,0}c$,

where $\Lambda_n = \sum_{k \geq 0} L_{n-k}L_k + \sum_{k < 0} L_kL_{n-k} - \frac{3}{10}(n + 2)(n + 3)L_n$. 

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Representations of $W_3$-algebra

A representation of $W_3$ on a ($\mathbb{C}$-)vector space $M$ makes sense by imposing the conditions

$$L_nm = W_nm = 0 \ (n \gg 0, \ \forall m \in M).$$
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A highest weight representation of $W_3$ is a representation $M$ that is generated by a vector $v$ satisfying

$$L_n v = W_n v = 0 \ (n > 0),$$

$$L_0 v = a_1 v, \ W_0 v = a_2 v, \ c v = c v, \ \exists (a_1, a_2, c) \in \mathbb{C}^3.$$
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For a highest weight representation $M$ of $W_3$ the (normalized) character

$$\chi_M(q) = \text{tr}_M(q^{L_0 - \frac{c}{24}})$$

makes sense.
In general, a W-algebra is defined by means of the (quantized) Drinfeld-Sokolov reduction ([Feigin-Frenkel ’90, . . . , Kac-Roan-Wakimoto ’03]).
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$\rightsquigarrow \mathcal{W}^k(\mathfrak{g}, f) = H^0_{DS,f}(V^k(\mathfrak{g}))$: the $W$-algebra associated with $(\mathfrak{g}, f)$ at level $k \in \mathbb{C}$. 
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$H^\bullet_{DS,f}(M)$: the BRST cohomology of the Drinfeld-Sokolov reduction associated with $(\mathfrak{g}, f)$ with coefficient in $M$;
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\( H^\bullet_{DS,f}(M) : \text{the BRST cohomology of the Drinfeld-Sokolov reduction associated with } (\mathfrak{g}, f) \text{ with coefficient in } M; \)

\( V^k(\mathfrak{g}) : \text{the universal affine vertex algebra associated with } \mathfrak{g} \text{ at level } k \) (vertex algebra associated with the affine Kac-Moody algebra \( \hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \)).
Examples of $\mathcal{W}^k(g, f)$

1). $\mathcal{W}^k(g, 0) = V^k(g) = U(\hat{g}) \otimes U(g[t] + \mathbb{C}K) \mathbb{C}_k$

(a $V^k(g)$-module = a smooth $\hat{g}$-module of level $k$).

2). $\mathcal{W}^k(sl_2; f_{\text{prin}}) = \text{the Virasoro vertex algebra of central charge } 1 - \frac{6(k + 1)}{2} = \frac{(k + 2)}{2}$ (if $k$ is not critical, i.e., $k \neq 2$).

3). $\mathcal{W}^k(sl_3; f_{\text{prin}}) = W_3$ with $c = \frac{24(k + 2)}{2} = \frac{(k + 3)}{2}$ (for a non-critical $k$).

4). $\mathcal{W}^k(sl_n; f_{\text{prin}})$ is the Fateev-Lukyanov $W_n$-algebra.

5). Almost all superconformal algebras are realized as the $W$-algebra $\mathcal{W}^k(g, f_{\text{min}})$ associated with some Lie superalgebra $g$ and a minimal nilpotent element $f_{\text{min}}$ (Kac-Roan-Wakimoto '03).

Presentation of $\mathcal{W}^k(g, f)$ by generators and relations are not known in general.
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$O_k$: the category $O$ of $\hat{g}$ at level $k$.

$L(\lambda) \in O_k$: the irreducible highest weight representation of $\hat{g}$ with highest weight $\lambda$ of level $k$. 
Representation theory of minimal W-algebras

Theorem (T.A. ’05, \( f = f_{\text{min}} = \text{minimal nilpotent element} \))

1. \( H_i \neq 0 \) for any \( M \in \mathcal{O}_k \). Therefore, the functor \( \mathcal{O}_k \to \mathcal{W}_k(g; f_{\text{min}}) \text{-Mod} \), \( M \mapsto H_0 \mathcal{D}_S(f_{\text{min}})(M) \), is exact.

2. \( H_0 \mathcal{D}_S(f_{\text{min}})(L(M)) \) is zero or simple. Moreover, any irreducible highest weight representation of \( \mathcal{W}_k(g; f_{\text{min}}) \) arises in this way.

Remark

The above theorem holds for Lie superalgebras as well. This in particular proves the Kac-Roan-Wakimoto conjecture ’03.
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### Remark

The above theorem holds for Lie superalgebras as well. This in particular proves the Kac-Roan-Wakimoto conjecture ’03.
One can extend the previous results for more general nilpotent elements by modifying the DS functor following Frenkel-Kac-Wakimoto ’92. As a result, we obtain characters of all irreducible highest weight representations of principal $W$-algebras $W^k (g; f_{prin})$ ([T.A. ’07]), which in particular proves the conjecture of Frenkel-Kac-Wakimoto ’92 on the existence and construction of modular invariant representations of principal $W$-algebras; characters of all (ordinary) representations of $W$-algebras $W^k (sl_n; f)$ of type $A$ ([T.A.’12]), which in particular proves the similar conjecture of Kac-Wakimoto ’08.
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Theorem (Zhu '96)

Let $V$ be a "nice" vertex (operator) algebra. Then the character $M(e^{2\pi i})$ converges to a holomorphic function on the upper half plane for any $M \in \text{Irrep}(V)$. Moreover, the space spanned by the characters $M(e^{2\pi i}), M \in \text{Irrep}(V)$, is invariant under the natural action of $\text{SL}_2(\mathbb{Z})$.

Here a vertex operator algebra $V$ is called "nice" if $V$ is lisse (or $\mathbb{C}$-co-finite), that is, $\text{Spec}(\text{gr} V) = \{0\}$. $V$ is rational, that is, any positively graded $V$-modules are completely reducible.
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Let $V$ be a “nice” vertex (operator) algebra. Then the character $\chi_M(e^{2\pi i \tau})$ converges to a holomorphic function on the upper half plane for any $M \in \text{Irrep}(V)$. Moreover, the space spanned by the characters $\chi_M(e^{2\pi i \tau})$, $M \in \text{Irrep}(V)$, is invariant under the natural action of $SL_2(\mathbb{Z})$. 

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Example of a “nice” vertex algebra

The universal affine vertex algebra $V^k(g)$ is not lisse.

Indeed, $V^k(g) = U(t^1 g [t^1])$, and we have $\text{gr } V^k(g) = S(t^1 g [t^1]) = \mathbb{C}[J^1 g]$. Here $J^1 X$ is the arc space of $X$.

Let $L^k(g)$ be the simple (graded) quotient $L(k^0) of \ V^k(g)$ (simple affine vertex algebra).

Fact (Frenkel-Zhu '92, Zhu '96, Dong-Mason '06) $L^k(g)$ is lisse if this is the case, $L^k(g)$ -Mod = $f$ integrable $b$ g-modules of level $k g$. Thus, $L^k(g)$ is rational as well.
The universal affine vertex algebra $V^k(g)$ is not lisse.

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Lisse condition and associated varieties

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$R_V = V/C_2(V)$: Zhu’s $C_2$-algebra (a Poisson algebra)
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Lisse condition and associated varieties

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1). $X_{V^k(g)} = \mathfrak{g}^*$, and so $X_{L_k(g)} \subset \mathfrak{g}^*$, $G$-invariant and conic.
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1). $X_{V^k(g)} = g^*$, and so $X_{L^k(g)} \subset g^*$, $G$-invariant and conic.

2). $X_{W^k(g,f)} \cong S_f := f + g^e \subset g = g^*$, the Slodowy slice at $f$ ([De-Sole-Kac ’06]), where \( \{e, f, h\} \) is an $\mathfrak{sl}_2$-triple.
Let \( W_k(g; f) \) be the simple quotient of \( \text{S}_f \), invariant under the natural \( C \)-action which contracts to \( f \).

So \( W_k(g; f) \) is lisse iff \( X_{W_k(g; f)} = f \).

One can show that \( W_k(g; f) \) is a quotient of the vertex algebra \( H_0 DS; f(\mathcal{L}_k(g)) \), provided that it is nonzero ([T.A. '16]).

Theorem (T.A. '16)

We have \( X_{H_0 DS; f(\mathcal{L}_k(g))} = X_{\mathcal{L}_k(g)} \ S_f \) (holds as schemes).

Hence,

(i). \( H_0 DS; f(\mathcal{L}_k(g)) \neq 0 \) iff \( X_{\mathcal{L}_k(g)} G: f; \)

(ii). If \( X_{\mathcal{L}_k(g)} = G: f; \) then \( X_{H_0 DS; f(\mathcal{L}_k(g))} = f f g \). Hence \( H_0 DS; f(\mathcal{L}_k(g)) \) is lisse, and so is its quotient \( W_k(g; f) \).
Associated varieties of $W$-algebras

Let $\mathcal{W}_k(g, f)$ be the simple quotient of $\mathcal{W}^k(g, f)$. 

One can show that $\mathcal{W}_k(g, f)$ is lisse iff $X_{\mathcal{W}_k(g, f)} = \mathcal{V}_k(g, f)$.

Theorem (T.A. '16)

We have $X_{\mathcal{H}_0 DS; f(L_k(g))} = X_{L_k(g)} \cdot S_f$ (holds as schemes).

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Admissible representations of affine Kac-Moody algebras

Note that $H_{DS,f}^0(L_k(g)) = 0$ if $L_k(g)$ is integrable.
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Note that $H^0_{DS,f}(L_k(\mathfrak{g})) = 0$ if $L_k(\mathfrak{g})$ is integrable. Therefore we need to study more general representations of $\hat{\mathfrak{g}}$ to obtain lisse $W$-algebras using the previous result.
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Note that $H^0_{DS,f}(L_k(g)) = 0$ if $L_k(g)$ is integrable. Therefore we need to study more general representations of $\hat{g}$ to obtain lisse $W$-algebras using the previous result.

There is a nice class of representations of $\hat{g}$ which are called admissible representations (Kac-Wakimoto ’88):

$$\{\text{integrable rep.}\} \subsetneq \{\text{admissible rep.}\} \subsetneq \{\text{highest weight rep.}\}$$
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The simple affine vertex algebra $L_k(\mathfrak{g})$ is admissible as a $\hat{\mathfrak{g}}$-module iff

$$k + h^\vee = \frac{p}{q}, \quad p, q \in \mathbb{N}, \quad (p, q) = 1, \quad p \geq \begin{cases} h^\vee & \text{if } (q, r^\vee) = 1, \\ h & \text{if } (q, r^\vee) = r^\vee. \end{cases}$$

Here $h$ is the Coxeter number of $\mathfrak{g}$ and $r^\vee$ is the lacity of $\mathfrak{g}$. 
**Theorem (T.A. ’16)**

Let $L_k(g)$ be an admissible affine vertex algebra.
Theorem (T.A. ’16)

Let $L_k(\mathfrak{g})$ be an admissible affine vertex algebra.

1). (Feigin-Frenkel conjecture) $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$, the nilpotent cone of $\mathfrak{g}$. 

2). $X_{L_k(\mathfrak{g})}$ is irreducible, that is, for any nilpotent orbit $O_k$ of $\mathfrak{g}$ such that $X_{L_k(\mathfrak{g})} = O_k$. 

By previous theorems we immediately obtain the following assertion, which was (essentially) conjectured by Kac-Wakimoto '08.
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By previous theorems we immediately obtain the following assertion, which was (essentially) conjectured by Kac-Wakimoto ’08.

Theorem (T.A. ’16)

Let $L_k(g)$ be an admissible affine vertex algebra, and let $f \in O_k$. Then the simple affine $W$-algebra $\mathcal{W}_k(g, f)$ is lisse.
An admissible affine vertex algebra $L_k(\mathfrak{g})$ is called non-degenerate if

$$X_{L_k(\mathfrak{g})} = \mathcal{N} = \overline{G.f_{\text{prin}}}.$$ 

If this is the case $k$ is called a non-degenerate admissible number for $\hat{\mathfrak{g}}$. 

Theorem (T.A. '15, Frenkel-Kac-Wakimoto conjecture '92)

Let $k$ be a non-degenerate admissible number. Then the simple principal $W$-algebra $W_k(\mathfrak{g}; f_{\text{prin}})$ is lisse by the previous theorem.

For $\mathfrak{g} = \mathfrak{sl}_2$, the corresponding rational $W$-algebras are exactly the minimal series of the Virasoro (vertex) algebra.
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The proof of the previous theorem is based on the following assertion on admissible affine vertex algebras.
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**Theorem (T.A. ’16, Adamović-Milas conjecture ’95 )**

Let $L_k(\mathfrak{g})$ be an admissible affine vertex algebra. Then $L_k(\mathfrak{g})$ is rational in the category $\mathcal{O}$, that is, any $L_k(\mathfrak{g})$-module that belongs to $\mathcal{O}$ is completely reducible.
Recently, Beem, Lemos, Liendo, Peelaers, Rastelli, and van Rees '15 have constructed a remarkable map:

\[ f_{4d} \mathcal{N} = 2 \text{ SCFTs} \rightarrow f_{2d} \text{ vertex algebras} \]

such that, among other things, the character of the vertex algebra \((T)\) coincides with the Schur index of the corresponding 4d \(\mathcal{N} = 2\) SCFT \(T\), which is an important invariant of the theory \(T\).
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How do vertex algebras coming from 4d $N = 2$ SCFTs look like?
VOAs coming from $4d$ theory

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The main examples of vertex algebras considered by Rastelli et al. ’15. are the simple affine vertex algebras $L_k(g)$ of types $D_4, E_6, E_7, E_8$ at level $k = -h^\vee/6 - 1$, which are non-rational, non-admissible affine vertex algebras at negative integer levels.
Higgs branch conjecture

There is another important invariant of a 4d $N = 2$ SCFT $\mathcal{T}$, called the Higgs branch. The Higgs branch $Higgs_\mathcal{T}$ is an affine algebraic variety that has a hyperKähler structure in its smooth part. In particular, $Higgs_\mathcal{T}$ is a (possibly singular) symplectic variety.
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Let $\mathcal{T}$ be one of the 4d $\mathcal{N} = 2$ SCFTs such that $\Phi(\mathcal{T}) = L_k(\mathfrak{g})$ with $k = h^\vee / 6 - 1$ for types $D_4$, $E_6$, $E_7$, $E_8$ appeared previously. It is known that $Higgs_{\mathcal{T}} = \mathcal{O}_{\text{min}}$, and it turned out that this equals to the associated variety $X_{\Phi(\mathcal{T})}$ ([T.A.-Moreau ’18]).
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**Conjecture (Beem and Rastelli ’17)**
There is another important invariant of a 4d $N = 2$ SCFT $\mathcal{T}$, called the **Higgs branch**. The Higgs branch $Higgs_{\mathcal{T}}$ is an affine algebraic variety that has a hyperKähler structure in its smooth part. In particular, $Higgs_{\mathcal{T}}$ is a (possibly singular) symplectic variety.

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**Conjecture (Beem and Rastelli ’17)**

For any 4d $N = 2$ SCFT $\mathcal{T}$, we have

$$Higgs_{\mathcal{T}} = X_{\Phi(\mathcal{T})}.$$
So we are expected to recover the Higgs branch of a 4d $\mathcal{N} = 2$ SCFT from the corresponding vertex algebra, which is purely an algebraic object!

Remark 1. Higgs branch conjecture is a physical conjecture since the Higgs branch is not mathematically defined in general. The Schur index is not a mathematically defined object in general, either.

Remark 2. There is a close relationship between the Higgs branches of 4d $\mathcal{N} = 2$ SCFTs and the Coulomb branches of three-dimensional $\mathcal{N} = 4$ gauge theories whose mathematical definition has been given by Braverman-Finkelberg-Nakajima '16 (4d-3d duality).
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Quasi-lisse vertex algebras

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- Physical intuition expects that vertex algebras that come from 4d $N = 2$ SCFTs via the map $\Phi$ are quasi-lisse.
**Theorem (T.A.-Kawasetsu’16)**

Let $V$ be a quasi-lisse vertex (operator) algebra (of CFT type). Then there are only finitely many simple ordinary $V$-modules. Moreover, for a finitely generated ordinary $V$-module $M$, the character $\chi_M(q)$ satisfies a modular linear differential equation (MLDE).
Modularity of Schur indices

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Since the space of solutions of a MLDE is invariant under the action of \( SL_2(\mathbb{Z}) \), the above theorem implies that a quasi-lisse vertex algebra possesses a certain modular invariance property, although we do not claim that the normalized characters of ordinary \( V \)-modules span the space of the solutions.
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Rastelli et al. ’15 conjectured that chiral algebras of class $S$ can be also described in terms of 2d TQFT (see [Tachikawa] for a mathematical exposition of their conjecture and the background).
Let $V$ be the following category (the category of vertex algebras)

- **Objects**: complex semisimple groups;
- **Morphisms**: $\text{Hom}(G_1; G_2) = V_{\text{VOAs}}(V)$ with a VA hom. $V_{\cdot}(g_1); V_{\cdot}(g_2)! = V_{g}$.

For $V_{12}; \text{Hom}(G_1; G_2), V_{23}; \text{Hom}(G_2; G_3), V_1 \circ V_2 = H_{12} + (b_{g_2}; g_2; V_1 V_2)$:

From a result of Arkhipov-Gaitsgory one finds that the identity morphism $\text{id}_G$ is the algebra $D_{\text{ch}}G$ of chiral differential operators on $G$ at the critical level, whose associated variety is canonically isomorphic to $T_G$. 
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From a result of Arkhipov-Gaitsgory one finds that the identity morphism $\text{id}_G$ is the algebra $\mathcal{D}_{G}^{ch}$ of chiral differential operators on $G$ at the critical level, whose associated variety is canonically isomorphic to $T^*G$. 
Higgs branch conjecture for class $S$ theory
Theorem (T.A., to appear, conjectured by Rastelli et al.)

Let $\mathcal{B}_2$ be the category of 2-bordisms. For each semisimple group $G$, there exists a unique monoidal functor

$$\eta_G : \mathcal{B}_2 \to \mathcal{V}$$

which sends (1) the object $S^1$ to $G$, (2) the cylinder, which is the identity morphism $\text{id}_{S^1}$, to the identity morphism $\text{id}_G = \mathcal{D}^\text{ch}_G$, and (3) the cap to $H^0_{DS,f_{\text{prin}}} (\mathcal{D}^\text{ch}_G)$. 
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$$X_{\eta_G(B)} \cong \eta_G^{\text{BFN}}(B)$$

for any 2-bordism $B$, where $\eta_G^{\text{BFN}}$ is the functor form $\mathcal{B}_2$ to the category of symplectic varieties constructed by Braverman-Finkelberg-Nakajima ’17.
Thank you!