Complex quotients by nonclosed groups and their stratifications

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Abstract

We define the notion of complex stratification by quasifolds and show that such stratified spaces occur as complex quotients by certain nonclosed subgroups of tori associated to convex polytopes. The spaces thus obtained provide a natural generalization, to the nonrational case, of the notion of toric variety associated with a rational convex polytope.

Résumé

On définit la notion de stratification complexe de quasifolds et on montre que ces espaces stratifiés se réalisent comme quotients complexes par de sousgroupes non fermés de tori, associés aux polytopes convexes. Les espaces ainsi obtenus donnent une généralisation naturelle, au cas non rationnel, de la notion de variété torique associée à un polytope convexe rationnel.

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Introduction

Our aim is to define a geometric object that naturally generalizes, to the nonrational setting, the notion of toric variety associated with a rational convex polytope. Let $\mathfrak{d}$ be a real vector space of dimension $n$ and let $\Delta$ be a polytope in $\mathfrak{d}^*$, rational with respect to a lattice $L$, with $d$ faces of codimension 1. Then the toric variety corresponding to $\Delta$ is the categorical quotient of a suitable open subset of $\mathbb{C}^d$, modulo the action of a subtorus of $(\mathbb{C}^*)^d$, as shown by Cox in [C]. On the other hand complex toric spaces corresponding to simple convex polytopes, not necessarily rational, were constructed as complex quotients in joint work with Elisa Prato [BP1], who had previously given the construction of these spaces as symplectic quotients [P]. More precisely, to each (simple) convex polytope in $\mathfrak{d}^*$, the moment polytope, there correspond a unique fan in $\mathfrak{d}$--generated by the 1-dimensional cones relative to the codimension-1 faces of the polytope--and many choices of the following data: the generators of the 1-dimensional

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cones of the fan and a quasilattice $Q$ in $\mathcal{D}$ containing such generators. A quasilattice in the vector space $\mathcal{D}$ is a $\mathbb{Z}$-submodule of $\mathcal{D}$ generated by a set of generators of $\mathcal{D}$ (cf. for example [S]). In [BP1], for each choice of data relative to the polytope $\Delta$, an $n$-dimensional toric space is constructed, given by the geometric quotient of an open subset $\mathcal{C}_\Delta^d$ of $\mathbb{C}^d$ by the action of a non necessarily closed subgroup $N_\mathcal{C}$ of $(\mathbb{C}^*)^d$. The space thus obtained has the structure of a complex quasifold and is endowed with the holomorphic action of the $n$-dimensional complex quasitorus $D_\mathcal{C} = \mathfrak{a}_\mathcal{C}/Q$. Quasitori are a first, natural example of quasifolds; they have been introduced, together with the notion of quasifold, in Prato’s article [P]. Complex quasifolds are topological spaces whose local models are open subsets of $\mathbb{C}^n$ modulo the action of finitely generated groups. Local models are glued together to give rise to a global quasifold structure. Quasifolds structures are highly singular: for example the finitely generated groups obtained are in general nonclosed, so that the corresponding spaces are non Hausdorff.

Let us now consider a nonsimple convex polytope, not necessarily rational; once generators of 1-cones and a quasilattice are chosen, we carry on the construction of our space as complex quotient. Both $\mathcal{C}_\Delta^d$ and $N_\mathcal{C}$ can still be defined, the open set $\mathcal{C}_\Delta^d$ depends only on the combinatorics of the polytope, or, equivalently, of the associated fan, whilst the group $N_\mathcal{C}$ depends on the choice of generators and quasilattice. The first problem we have to deal with is to make sense of the quotient $\mathcal{C}_\Delta^d//N_\mathcal{C}$: as in the rational nonsimple case, we cannot simply take the geometric quotient, hence in Section 2 we define a suitable notion of quotient. Then, in Section 3, we work out its structure. Consider a $p$-dimensional face $F$ of $\Delta$ and a point $\nu \in F$. The polytope $\Delta$, in a neighborhood of $\nu$, is given by the product of $F$ by a cone over an $n-p-1$ dimensional polytope $\Delta_F$. We denote by $X_{\Delta_F}$ the complex space associated to $\Delta_F$, together with the induced choice of generators of 1-cones and of quasilattice. The face $F$ is regular if $\Delta_F$ is a simplex, singular otherwise. As in the rational case, each $p$-dimensional face of the polytope corresponds to a $p$-dimensional orbit of the quasitorus $D_\mathcal{C}$ acting on $X_\Delta$. In particular the interior of the polytope corresponds to the dense open orbit of $D_\mathcal{C}$ in $X_\Delta$. Orbits produce a stratification of the quotient $X_\Delta$ that mirrors the structure of the associated polytope $\Delta$. The union of orbits corresponding to regular faces gives the regular set of $X_\Delta$. Orbits that corresponds to singular faces are the singular strata. Intuitively it is clear that each stratum has a natural structure of complex quasifold, since it is an orbit (or union of orbits) of the quasitorus $D_\mathcal{C}$. Furthermore, in a neighborhood of a singular orbit, the quotient $X_\Delta$ is the product, in a suitable way, of the singular stratum by a complex cone over $X_{\Delta_F}$; it is a twisted product by a finitely generated group. This group is finite in the rational case, when the decomposition is indeed locally trivial and strata are smooth, namely the usual notion of stratification is satisfied (cf. [GM]). The definitions of complex cone and complex stratification are given in Section 1. We can then state our main result: given a convex polytope $\Delta$, to each set of generators and of quasilattice containing them, there corresponds a complex quotient $X_\Delta$, endowed with an $n$-dimensional complex stratification by quasifolds and acted on holomorphically by a complex quasitorus with a dense open orbit. Moreover the space $X_\Delta$ is homeomorphic to its symplectic counterpart, obtained as symplectic quotient (cf. [BP2, B1]). Such homeomorphism respects the decompositions and its restriction to each stratum is a diffeomorphism, with respect to which the symplectic
and complex structures of strata are compatible. In particular the space $X_{\Delta}$ is compact and its strata are in fact Kähler quasifolds. Remark that, as in the simple case (cf. [P], [BP1]), the space $X_{\Delta}$ as complex space depends only on the set of generators and on the quasilattice, while its symplectic structure depends on the polytope.

These results complete the generalization of the notion of toric spaces associated to nonrational convex polytopes, in particular they shed light onto the relationship with the theory of classical toric varieties. The geometry and topology of our spaces and the relationship with the properties of the polytope are natural questions related to our work. A first step towards a better understanding of these different aspects, that will be pursued in the sequel, is to define and investigate cohomological invariants of our spaces. In this note we give the general idea of our construction and results, leaving the details of the proofs to the forthcoming paper [B2].

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1 Complex stratifications by quasifolds

For the detailed definitions of real symplectic quasifold, quasitorus and related notions we refer the reader to [P], for the complex version of these notions see [BP1]. Roughly speaking, as we have already observed, a complex quasifold of dimension $n$ is locally modelled on the topological quotient of an open subset of $\mathbb{C}^n$ by the action of a finitely generated group. A basic example of real quasifold is Prato’s quasicircle $D^1_\alpha = \mathbb{R}/\mathbb{Z} + \alpha \mathbb{Z}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, this gives also an example of quasitorus (cf. [P]). Notice that, if $\alpha$ is taken in $\mathbb{Q}$, then $D^1_\alpha$ is either an orbifold or $S^1$. The complexification $(D^1_\alpha)_\mathbb{C}$ of $D^1_\alpha$ is the complex quasitorus given by $\mathbb{C}/\mathbb{Z} + \alpha \mathbb{Z}$.

The notions of decomposition and stratification of a space $X$ were given in [B1]. We allow the pieces of a decomposition to be quasifolds and we require then the usual properties of a decomposition. The dimension of the maximal piece is by definition the dimension of $X$, say $n$. A smooth map (resp. an isomorphism) between decomposed spaces is a continuous map (resp. homeomorphism) that respect the decompositions and is smooth (resp. a diffeomorphism) when restricted to pieces. A stratification is a decomposition that locally, near to each point $t$ of an $r$-dimensional stratum $T$, is given by a twisted product of the kind $\tilde{B} \times C(L)/\Gamma$, where $\tilde{B}/\Gamma$ is a local model of $T$ around $t$ such that the finitely generated group $\Gamma$ acts freely on $\tilde{B}$; $L$ is a $(n-r-1)$-dimensional compact space, called the link of $t$, decomposed by quasifolds and endowed with an action of $\Gamma$ that preserves the decomposition; $C(L)$ is a real cone over $L$. Then the decomposition of the link $L$ is required to satisfy the above condition. Recursively we end up, after a finite number of steps, with links that are compact quasifolds.

We then give the notion of complex structure of the stratified space. Our requirements are very strong and are not usually satisfied by complex stratified spaces, for example the local trivialization is usually far from being holomorphic, however they are verified by toric varieties as well as by our toric spaces. More precisely we require that for each link $L$ there exists a compact space $Y$ decomposed by quasifolds and a smooth surjective map $s : L \rightarrow Y$, with fibers diffeomorphic to a fixed 1-parameter subgroup.
2 Complex quotients by nonclosed groups

Our goal here is to generalize the construction given by Cox: we need to produce, in association to $\Delta$, an open subset of $\mathbb{C}^d$ and a subgroup of the torus $(\mathbb{C}^*)^d$. We first take care of the open subset. Consider the open faces of $\Delta$. They can be described as follows. Write the polytope as intersection of half spaces: $\Delta = \cap_{j=1}^d \{ \mu \in \mathfrak{d}^* \mid \langle \mu, X_j \rangle \geq \lambda_j \}$ for inward pointing vectors $X_1, \ldots, X_d$ in $\mathfrak{d}$ and the real numbers $\lambda_1, \ldots, \lambda_d$ determined by our choice of $X_i$. Notice that the vectors $X_i$ are generators of the 1-dimensional cones of the fan in $\mathfrak{d}$ dual to the polytope. For each face $F$ there exists a subset $I_F \subset \{1, \ldots, d\}$ such that $F = \{ \mu \in \Delta \mid \langle \mu, X_j \rangle = \lambda_j \}$ if and only if $j \in I_F$. The $n$-dimensional open face of $\Delta$ corresponds to the empty subset. Define the open set $\tilde{V}_F = \{(z_1, \ldots, z_d) \in \mathbb{C}^d \mid z_j \neq 0 \text{ if } j \notin I_F\}$ and denote by $\mathbb{C}^d_\Delta$ the open subset of $\mathbb{C}^d$ given by $\mathbb{C}^d_\Delta = \cup_{F \in \Delta} \tilde{V}_F$. Notice that, in the definition of the open subset $\mathbb{C}^d_\Delta$, only the combinatorics of the polytope intervenes. Moreover, the open subset $\mathbb{C}^d_\Delta$ coincides with the one defined in $[Q]$ for the rational case. It is in the definition of the group acting on $\mathbb{C}^d_\Delta$ that nonrationality intervenes. In order to define the group $N_C$ we adopt the following procedure: it is an extension of the procedure introduced by Delzant in $[D]$, extended to the nonrational case first in $[P]$ and then in $[BP1, B1]$. Let us fix a quasilattice $Q$ in the space $\mathfrak{d}$ containing the elements $X_j$ (for example $\text{Span}_\mathbb{Z}\{X_1, \ldots, X_d\}$). Consider the surjective linear mappings $\pi: \mathbb{R}^d \rightarrow \mathfrak{d}$ (respect. $\pi_C: \mathbb{C}^d \rightarrow \mathfrak{d}_C$) defined by $\pi(e_j) = X_j$ (respect. $\pi_C(e_j) = X_j$), with $\{e_1, \ldots, e_d\}$ the standard basis. Consider the quasitorus $\mathfrak{d}/Q$ and its complexification $\mathfrak{d}_C/Q$. The mappings $\pi$ and $\pi_C$ induce the group homomorphisms $\Pi: (S^1)^d = \mathbb{R}^d/\mathbb{Z}^d \rightarrow \mathfrak{d}/Q$ and $\Pi_C: (\mathbb{C}^*)^d = \mathbb{C}^d/\mathbb{Z}^d \rightarrow \mathfrak{d}_C/Q$ respectively. We define $N$ (respect. $N_C$) to be the kernel of the mapping $\Pi$ (respect. $\Pi_C$). The group $N$ has dimension $(d-n)$. Let $n = \text{Ker} \pi$ the Lie algebra of $N$, then for the complexified group $N_C$ the polar decomposition holds, namely

$$N_C = NA,$$

where $A = \exp(in)$. If $Q$ is an honest lattice then $N$ is a compact real torus.

For a given polytope a choice of normals and of quasilattice $Q$ is said to be rational if $Q$ is a true lattice. There are polytopes that do not admit rational choices and that are not combinatorially equivalent to rational polytopes $[G]$. If the polytope is rational in a lattice $L$ the categorical quotient $\mathbb{C}^d_\Delta//N_C$, constructed by Cox in $[C]$, can be thought of as the quotient of $\mathbb{C}^d_\Delta$ by the following equivalence relation: two points in $\mathbb{C}^d_\Delta$ are equivalent if the closures of the $N_C$-orbits through these points have nonempty intersection. In our context this definition does not lead anywhere, since the group $N$ itself is nonclosed. Therefore we have to distinguish two different ways in which orbits
are non closed, the first is given by the fact that $N$ is nonclosed, this is peculiar to the nonrational setting and produces the quasifold structure of strata, the other is due to the fact that, as in the rational case, there are nonclosed $A$-orbits. There is absolutely no difference of behavior when it comes to $A$-orbits, as an example consider $e^{2\pi at}$, with $t \in \mathbb{R}$ and $a$ a real constant. Let $z \in \mathbb{C}$, obviously the orbit $e^{2\pi at}z$ is not influenced by having a rational or not. Therefore we are led to consider $A$-orbits. Let $z \in \mathbb{C}_\Delta$, we say that the $A$-orbit $A_\Delta z$ is closed if it is closed in $\mathbb{C}_\Delta$. Let $J$ be any set of indices in $\{1, \ldots, d\}$, we denote by $K^J = \{z \in \mathbb{C}^d \mid z_j \in K \text{ if } j \in J, z_j = 0 \text{ if } j \notin J\}$, where $K$ can be either $\mathbb{C}$ or $\mathbb{C}^*$. Consider the $(\mathbb{C}^*)^d$-orbit $(\mathbb{C}^*)^I = \{(z_1, \ldots, z_d) \in \mathbb{C}^d \mid z_j = 0 \text{ iff } j \in I\}$.

**Theorem 2.1** Let $z \in \mathbb{C}_\Delta$. Then the $A$-orbit through $z$, $A_\Delta z$, is closed if and only if there exists a face $F$ such that $z \in (\mathbb{C}^*)^F_\Delta$. Moreover, if $A_\Delta z$ is nonclosed, then it contains one and only one closed $A$-orbit.

Theorem 2.1 which holds in the rational setting too, allows us to define on the open set $\mathbb{C}_\Delta^d$ the following equivalence relation: two points $z$ and $w$ are equivalent if and only if

$$\left(N(\overline{A_\Delta z}) \cap \overline{A_\Delta w}\right) \neq \emptyset$$

where the closure is meant in $\mathbb{C}_\Delta^d$. We define the space $X_\Delta$ to be the quotient of $\mathbb{C}_\Delta^d$ by the equivalence relation just defined, we denote the quotient by $X_\Delta = \mathbb{C}_\Delta^d//N_C$. Notice that, if the polytope is simple, then $\mathbb{C}_\Delta^d = \bigcup_{F \in \Delta} (\mathbb{C}^*)^F_{\Delta}$ and the quotient $X_\Delta$ is just the geometric quotient, whilst if the polytope is nonsimple and rational the quotient $X_\Delta$ is the known categorical quotient.

### 3 The Stratification

The decomposition in pieces of the quotient $X_\Delta$ reflects the geometry of the polytope $\Delta$. A partial order on the set of all faces of $\Delta$ is defined by setting $F \leq F'$ if $F \subseteq \overline{F'}$. The polytope $\Delta$ is the disjoint union of its faces. Let $F$ be a $p$-dimensional face and let $r_F = \text{card}(I_F)$, clearly $r_F \geq n - p$. When the equality holds the face $F$ is said to be regular, singular otherwise. Consider $\mathcal{D}_F = \text{Span}\{X_j \mid j \in I_F\}$, the natural injection $j_F : \mathcal{D}_F \hookrightarrow \mathcal{D}$ and the subset $\Sigma_F^* =\bigcap_{j \in I_F} \{\xi \in \mathcal{D}^* \mid \langle \xi, X_j \rangle \geq \lambda_j\}$. Then $j_F^*(\Sigma_F^*) = \Sigma_F$ is a polyhedral cone with vertex $j_F^*(F)$. By cutting this cone with an affine hyperplane, transversal to its codimension 1 faces, we obtain an $(n - p - 1)$-dimensional polytope, that we call $\Delta_F$, which of course depends on the choice of the hyperplane. Each $q$-dimensional face $G$ of $\Delta$ greater than $F$ gives a face in $\Delta_F$ of dimension $q - p - 1$, singular if and only if $G$ is singular in $\Delta$. Near to $F$ the polytope $\Delta$ is the product of $F$ by a cone over $\Delta_F$, and this is exactly the stratified structure of the toric space that we have constructed. The maximal stratum is $T_{\text{max}} = \cup_F \text{reg}((\mathbb{C}^*)^F_{\Delta})/N_C$, whilst there is a piece $T_F$ for each singular face $F$ of $\Delta$, which can be identified with the orbit space $T_F = (\mathbb{C}^*)^F_{\Delta}/N_C$. Theorem 2.1 implies that the space $X_\Delta$ is given by the union of the strata just defined. The structure of $X_\Delta$ as decomposed space and the properties that characterize $X_\Delta$ as toric space associated to $\Delta$ are given in the following statement:
Theorem 3.1 The subset $T_F$ of $X_\Delta$ corresponding to each $p$-dimensional singular face of $\Delta$ is a $p$-dimensional complex quasi-fold. The open subset $T_{\text{max}}$ is an $n$-dimensional complex quasi-fold. These subsets give a decomposition by complex quasi-folds of $X_\Delta$. Moreover there is a continuous action of $D_C$ on $X_\Delta$, with the dense open orbit $(\mathbb{C}^*)^d/N_C$. Such action is holomorphic when restricted to pieces.

The maximal stratum is said to be the regular stratum, it is the analogue of the open set of rationally smooth points in the rational case, whilst the strata corresponding to singular faces are singular (see Lemma 3.2).

Now consider the cone $\Sigma_F$ in the space $\mathfrak{d}_F$, with induced normal vectors $X_j \in \mathfrak{d}_F$, $j \in I_F$, and quasilattice $Q \cap \mathfrak{d}_F$. Then let $X_0 = \sum_{j=1}^d s_j X_j$, with $s_j \in (0, 1)$ suitably chosen for $j \in I_F$ and $s_j = 0$ for $j \notin I_F$. We consider $X_0$ in $\mathfrak{d}_F$ and we denote by $\text{ann}(X_0)$ the annihilator of $X_0$. We can view $\Delta_F$ as a convex polytope lying in the linear hyperplane $\text{ann}(X_0) \subset \mathfrak{d}_F$, with induced normal vectors $k_X(X_j) \subset \text{ann}(X_0)^* \simeq \mathfrak{d}_F/(X_0)$, $j \in I_F$, and quasilattice $k_X(Q \cap \mathfrak{d}_F)$. The spaces corresponding to the convex sets $\Sigma_F$ and $\Delta_F$ are $X_{\Sigma_F} = \mathbb{C}^d / N_{\mathbb{C}}$ and $X_{\Delta_F} = \mathbb{C}^d / (N_F)_C$, where the $(r_F - n + p)$-dimensional group $N_{\mathbb{C}}^F$ is given by $N_C \cap (\mathbb{C}^*)^I_F$ and $(N_F)^C/N_{\mathbb{C}}^F \simeq \exp(\mathfrak{s} + i\mathfrak{s})$, with $\mathfrak{s} = \text{Span}\{s_1, \ldots, s_d\}$.

We first prove that the stratification satisfies the local triviality condition:

Lemma 3.2 Let $F$ be a singular face, then the singular stratum $T_F$ can be identified with $(\mathbb{C}^*)^p/\Gamma_F$, where $\Gamma_F$ is a finitely generated subgroup of $(\mathbb{C}^*)^p$ acting on $X_{\Sigma_F}$ and freely on $(\mathbb{C}^*)^p$. There is a mapping from $(\mathbb{C}^*)^p \times X_{\Sigma_F}/\Gamma_F$ onto the open subset $(\mathbb{C}^*)^p \times (\mathbb{C}^*)^p/\Gamma_F$ of $X_\Delta$, which is a homeomorphism and a biholomorphism when restricted to the pieces of the respective decompositions.

Now, in order to complete the proof that the space $X_\Delta$ is a stratified space, it remains to show that:

(*) for each singular face $F$ in $\Delta$ there is a link $L_F$ which satisfies the definition, moreover the cone $X_{\Sigma_F}$, the link $L_F$ and the toric space $Y_F = X_{\Delta_F}$ have the properties required for a stratification to be complex.

General results, for example on bundles, are often not readily applicable to our spaces because of their topology, therefore, although a description of the spaces in statement (*) can be given within our set up (see the first row of diagram 2), we make use of the interplay with the symplectic quotients in order to give a neat description of the link $L_F$ and of $X_{\Sigma_F}$ as a real cone over it. Hence, before going on to describe the cone $X_{\Sigma_F}$, let us briefly recall from [31] the symplectic construction. Let $\Psi_{\Delta}: \mathbb{C}^d \rightarrow \mathfrak{n}^*$ be the moment mapping with respect to the $N$-action such that $\Psi_{\Delta}(0) = \sum_{j=1}^d \lambda_j x^*(e_j^*)$, where $\iota: \mathfrak{n} \rightarrow \mathbb{R}^d$ is the inclusion. The quotient $M_{\Delta} = \Psi_{\Delta}^{-1}(0)/N$ is endowed with a symplectic stratification by quasi-folds and is acted on effectively by the quasitorus $D = \mathfrak{d}/Q$. Moreover there is a continuous mapping $\Phi : M_{\Delta} \rightarrow \mathfrak{d}^*$ whose restriction to each stratum, with its symplectic structure, is a moment mapping with respect to the action of $D$, the image of $\Phi$ is exactly $\Delta$. The inclusion $\Psi_{\Delta}^{-1}(0) \hookrightarrow \mathbb{C}^d$ induces a continuous mapping $\chi_{\Delta} : \Psi_{\Delta}^{-1}(0)/N \rightarrow \mathbb{C}^d//N_C$. It was proved in [31] that, when the
polytope is *simple*, the mapping \( \chi_\Delta \) is a diffeomorphism inducing a Kahler structure on \( X_\Delta \). For any convex polytope, it remains to prove that:

\[ (\diamondsuit) \text{ the mapping } \chi_\Delta \text{ identifies the symplectic and complex quotients as stratified spaces.} \]

Statements (\( \ast \)) and (\( \diamondsuit \)) are strictly intertwined and they can be proved together by induction on the depth of the polytope \( \Delta \), which is the maximum length that a chain of singular faces can attain in \( \Delta \). The key diagram is the following

\[
\begin{array}{cccccc}
C^d_F /\left\{(N_C^F)\backslash\{0\}\right\} & \xrightarrow{q_2} & C^d_F /\left\{N_C^F \exp(i\theta)\right\} & \xrightarrow{q_1} & C^d_F /\left\{N_0^F\right\} \\
\chi_F^2 \uparrow & & \chi_F^3 \downarrow & & \chi_F^0 \downarrow \\
(\Psi_{\Sigma_F}^{-1}(0)/N^F)\backslash\{0\} & \xrightarrow{p_2} & (\Psi_{\Delta_F})^{-1}(0)/N^F & \xrightarrow{p_1} & (\Psi_{\Delta_F})^{-1}(0)/N_0^F \\
\end{array}
\]

The mappings \( \chi_F^j \) are all diffeomorphisms of decomposed spaces, both diagrams commute, the projections \( q_2, p_2 \) allows us to identify the space \( X_{\Sigma_F} \) to a real cone over the link \( L_F \), which is the space in the mid column. The projections \( p_1, q_1 \) are fibrations of the link \( L_F \) over the compact Kahler space \( X_{\Delta_F} \), with fibre \( \exp(\theta) \), the projections \( s, s' \) are fibrations of the complex space \( X_{\Sigma_F} \backslash\{\text{cone pt}\} \) over the compact space \( X_{\Delta_F} \), with fibre \( \exp(\theta + i\theta) \). All mappings are natural, preserve the decompositions and the structure of the strata. We finally state our main result:

**Theorem 3.3** Let \( d \) be a vector space of dimension \( n \), and let \( \Delta \subset d^* \) be a convex polytope. Choose inward-pointing normals to the facets of \( \Delta \), \( X_1, \ldots, X_d \in d \), and let \( Q \) be a quasilattice containing them. The corresponding quotient \( X_\Delta = C^d_\Delta /N_C \) is a complex stratified space. The mapping \( \chi_\Delta : \Psi_{\Delta}^{-1}(0)/N \to C^d_\Delta /N_C \) is an equivariant homeomorphism whose restriction to each stratum is a diffeomorphism of quasifolds, with respect to which the symplectic and complex structure are compatible, so that strata have the structure of Kähler quasifolds.

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