DYNAMICS OF A STAGE-STRUCTURED POPULATION MODEL WITH A STATE-DEPENDENT DELAY

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Abstract. This paper is devoted to the dynamics of a predator-prey model with stage structure for prey and state-dependent maturation delay. Firstly, positivity and boundedness of solutions are addressed to describe the population survival and the natural restriction of limited resources. Secondly, the existence, uniqueness, and local asymptotical stability of (boundary and coexisting) equilibria are investigated by means of degree theory and Routh-Hurwitz criteria. Thirdly, the explicit bounds for the eventual behaviors of the mature population are obtained. Finally, by means of comparison principle and two auxiliary systems, it observed that the local asymptotical stability of either of the positive interior equilibrium and the positive boundary equilibrium implies that it is also globally asymptotical stable if the derivative of the delay function around this equilibrium is small enough.

1. Introduction. Time-delay in a natural ecosystem has been widely considered for a long time (see e.g. [23, 35, 36, 37]). Recently, stage-structured models have received great attention. For example, Gurney, Blythe, and Nisbet [14] proposed a time delay growth model of blowflies and verified solutions of the time delay model are in accordance with the data in blowflies growth experiments by Nicholson (see [28]). This forces researchers to introduce time delay and stage structure into population research (see e.g. [8, 9, 10, 15, 16, 21, 22, 31, 38, 40]). The work of Aiello and Freedman [2] on a single species growth model with stage structure represents a mathematically more careful and biologically meaningful formulation approach. This model assumes an average age to maturity which appears as a constant time delay reflecting a delayed birth of immatures and a reduced survival of immatures to their maturity. The model takes the form

\[
\begin{align*}
\dot{v}(t) &= \alpha u(t) - \gamma v(t) - \alpha e^{-\gamma \tau} u(t - \tau), \\
\dot{u}(t) &= \alpha e^{-\gamma \tau} u(t - \tau) - \beta u^2(t),
\end{align*}
\]

where \(v(t)\) denotes the immature population density, \(u(t)\) represents the mature population density, \(\alpha > 0\) represents the birth rate, \(\gamma > 0\) is the immature death rate, \(\beta > 0\) is the mature death and overcrowding rate, \(\tau\) is the time to maturity. The term \(\alpha e^{-\gamma \tau} u(t - \tau)\) represents the immatures who were born at time \(t - \tau\) and survive at time \(t\) (with the immature death rate \(\gamma\)), and therefore represents the...
transformation of immatures to matures. Following the way of Aiello and Freedman [2], a number of authors studied different kinds of stage-structured models and carried out some significant works (see e.g. [3, 4, 5, 12, 30, 32, 33, 34]). Due to the influence of circumstances such as resources and interaction, however, the constant time delay is not reasonable any more (see e.g. [1, 5, 11, 13, 18, 42, 43]). Motivated by their knowledge about whale and seal populations, Aiello, Freedman and Wu [3] considered a system with a state-dependent delay and two stages, and proposed the following state-dependent delay model

\[
\begin{align*}
\dot{v}(t) &= \alpha u(t) - \gamma v(t) - \alpha e^{-\gamma(t)u(t)}(t - \tau(v + u)), \\
\dot{u}(t) &= \alpha e^{-\gamma(t)u(t)}(t - \tau(v + u)) - \beta u^2(t),
\end{align*}
\]

(2)

where the state-dependent time delay \(\tau(v + u)\) is taken to be an increasing differentiable function of the total population \((v + u)\) so that \(\tau'(v + u) \geq 0\) and \(\tau_m \leq \tau(v + u) \leq \tau_M\) with \(\tau(0) = \tau_m\) and \(\tau(+\infty) = \tau_M\). These assumptions imply that the maturation time for the species depends on the total number of them (matures plus immatures) around. The term \(\alpha e^{-\gamma(t)u(t)}(t - \tau(v + u))\) appearing in both equations of system (2) represents the density of individuals survive to leave the immature and just enter the mature class.

Lotka-Volterra type predator-prey systems are very important in the models of multi-species populations interactions and have been studied widely. For a number of animals, it seems reasonable to assume that the predator population feed on the mature prey because immature prey population are concealed in the mountain caves and are raised by their parents, and that the rate of predators attack at immature preys can be ignored (see e.g. [26]). Song and Chen [33] considered the exploitation of a predator-prey population with stage structure and harvesting for prey, and proposed the following system of delay differential equations:

\[
\begin{align*}
\dot{X}_1(t) &= aX_2(t) - \gamma X_1(t) - \alpha e^{-\gamma(t)}X_2(t - \tau), \\
\dot{X}_2(t) &= \alpha e^{-\gamma(t)}X_2(t - \tau) - \beta X_2(t) - aX_2(t)Y(t) - E X_2(t), \\
\dot{Y}(t) &= Y(t) \left[-r + cX_2(t) - bY(t)\right], \\
X_1(0) &= 0, \quad Y(0) > 0, \quad X_2(t) = \varphi(t) \geq 0, \quad -\tau \leq t \leq 0,
\end{align*}
\]

(3)

where \(X_1(t)\) and \(X_2(t)\) represent the immature population density and mature prey population density, respectively; \(Y(t)\) represents the density of predator population; \(a\) is the transformation coefficient of mature predator population; \(\tau\) represents the transformation of immatures to matures; \(\alpha e^{-\gamma(t)}X_2(t - \tau)\) represents the immatures who were born at \(t - \tau\) and survive at \(t\) with the immature death rate \(\gamma\); \(a\) is the birth rate of the immature prey population; \(\beta\) represents the mature death and overcrowding rate; \(E\) is the harvesting effort. Song and Chen [33] investigated the global asymptotical stability of three nonnegative equilibria and a threshold of harvesting for the mature prey population. The effect of the delay on the populations at positive equilibrium and the optimal harvesting of the mature prey population are also considered. Since then, more and more researchers have worked on two-species predator-prey systems with different stage structure on predator or prey, see, for example, [19, 20, 24, 25, 30, 32, 34, 39]. Furthermore, Qu and Wei [30] investigated the stability and Hopf bifurcation of the interior equilibrium. Considering a window in maturation time delay parameter that generates sustainable oscillatory dynamics, Gourley and Kuang [12] formulated a general and robust predator-prey model.
with stage-structure and constant maturation time delay. And then by following Gourley and Kuang [12], further results have been investigated (see e.g. [4]).

Motivated by the works of Aiello, Freedman, and Wu [3] and Song and Chen [33], in this paper, we are concerned with the effect of stage structure for prey and stage-structure and constant maturation time delay. And then by following Gourley and Kuang [12], further results have been investigated (see e.g. [4]).

where \( \alpha \), \( \beta \), \( \gamma \), \( a \), \( b \), \( c \), \( r \) are all positive constant, the state-dependent time delay \( \tau(X) \) is taken to be an increasing, concave downwards, twice differentiable function on \([0, +\infty)\) of the total population \( X = X_1 + X_2 \), so that \( \tau'(X) \geq 0 \) and \( \tau''(X) \leq 0 \) for all \( X \geq 0 \), and we shall also assume that \( \tau(0) \triangleq \tau_m \leq \tau(X) \leq \tau_M \triangleq \tau(\infty) \). The initial value \( X_1(0) \) and \( \varphi(x) \) with \( x \in [\tau_m, 0] \) of (4) satisfies

\[
X_1(0) = \int_{-\tau_s}^0 \alpha \varphi(s) e^{\gamma s} ds,
\]

which is the number of immatures that have survived to time \( t = 0 \). Here, \( \tau_s \) is the maturation time of the prey at \( t = 0 \), and the lower limit on the integral is \( -\tau_s \) because any prey born before that time will have matured before time \( t = 0 \). It follows that \( \tau_s \) is given by \( \tau_s = \tau(X_1(0) + X_2(0)) \), i.e.,

\[
\tau_s = \tau \left( \varphi(0) + \int_{-\tau_s}^0 \alpha \varphi(s) e^{\gamma s} ds \right).
\]

Note that \( \tau_s \) appears on both the left- and right-hand sides of the above equation, so that \( \tau_s \) is determined implicitly. For our model to make sense, i.e., to exclude the possibility of adults becoming immatures except by birth, we need to find conditions ensuring that \( t - \tau(X(t)) \) is an increasing function of \( t \) as \( t \) increases. Namely, we need \( \tau'(X_1 + X_2)(X_1 + X_2) < 1 \), which is equivalent to

\[
\tau'(X_1 + X_2)(X_1 + X_2 - \gamma X_1 - \beta X_2^2 - aX_2Y) < 1.
\]

In this case, we have \( \alpha X_2 - \gamma X_1 - \beta X_2^2 - aX_2Y \leq \alpha X_2 - \beta X_2^2 \leq \alpha^2/4\beta \). Therefore, if \( \tau'(X) < 4\beta/\alpha^2 \) then \( t - \tau(X(t)) \) is strictly increasing.

Compared with [3] and [33], our theoretical results are more accurate and our method is completely different. More precisely, we shall employ degree theory and Routh-Hurwitz criteria to investigate the existence and linearized stability of equilibria. Moreover, we find out the relationship among uniqueness, local asymptotical stability and global asymptotical stability of equilibria. Two auxiliary systems and comparison principles are introduced to prove the global asymptotical stability of nonnegative equilibria.

This paper is organized as follows: the positivity and boundedness of all solutions of (4) are obtained in section 2. In section 3, we first investigate the existence of positive coexisting equilibria by means of degree theory. Then we discuss the uniqueness of positive coexisting equilibria. Section 4 is devoted to the stability of equilibria, especially the positive ones, through Routh-Hurwitz criteria. In section 5, we investigate the state bounds on the eventual behaviour of \( X_2(t) \) and also introduce two auxiliary systems in order to overcome the difficulties caused by the
presence of state-dependent delay during the investigation of the global stability of system (4). Section 6 is devoted to the global asymptotical stability. Moreover, we illustrate our results with some numerical simulations in Section 7.

2. Positivity and boundedness. Since the solutions of system (4) represent populations and we also anticipate that limited resources will place a natural restriction to how many individuals can survive, we need address positivity and boundedness of the solution of the solution of the system.

Theorem 2.1. If \( \varphi(t) > 0 \) for \( -\tau_M \leq t \leq 0 \), then \( X_2(t) > 0 \) for all \( t > 0 \).

Proof. Suppose that \( X_2(t) = 0 \) for some value of \( t \). Since \( X_2(t) = \varphi(t) > 0 \) for \( -\tau_M \leq t \leq 0 \), if such a value of \( t \) exists, it is positive. Let \( t^* = \inf \{ t | t > 0, X_2(t) = 0 \} \). Note that \( \tau(X) > 0 \) and \( t^* - \tau(X(t^*)) < t^* \), then it follows from the definition of \( t^* \) that \( X_2(t^* - \tau(X(t^*))) > 0 \). Then from system (4) we have \( X_2(t^*) = \alpha e^{-\gamma \tau(X(t^*))} X_2(t^* - \tau(X(t^*))) > 0 \), giving a contradiction. Therefore, no such \( t^* \) exists.

Theorem 2.2. If \( Y(0) > 0 \) and \( \varphi(t) > 0 \) for \( -\tau_M \leq t \leq 0 \), then \( Y(t) > 0 \) for all \( t > 0 \).

Proof. It follows from Theorem 2.1 that if \( \varphi(t) > 0 \) for \( -\tau_M \leq t \leq 0 \), \( X_2(t) > 0 \) for all \( t > 0 \). Thus we have \( \dot{Y}(t) \geq Y(t)[-r - bY(t)] \). Then \( Y(t) \geq u(t) \), where \( u(t) \) is a solution of \( \dot{u}(t) = u(t)[-r - bu(t)] \), \( u(0) = Y(0) > 0 \). Note that the solution \( u(t) \) satisfies \( \frac{1}{u(t)} = -\frac{b}{\tau} + \left( \frac{1}{u(0)} + \frac{b}{\tau} \right) e^{rt} > 0 \) for all \( t > 0 \). Then we complete the proof of this theorem.

It follows from (4) that

\[
\begin{align*}
\dot{X}_1(t) &= \alpha X_2(t) - \gamma X_1(t) - \alpha e^{-\gamma \tau(X)} X_2(t - \tau(X)), \\
\dot{X}_2(t) &\leq \alpha e^{-\gamma \tau(X)} X_2(t - \tau(X)) - \beta X_2^2(t),
\end{align*}
\]

and hence that \( X_1(t) \leq x_1(t) \), \( X_2(t) \leq x_2(t) \), where \( (x_1(t), x_2(t)) \) is a solution of the following system with the initial value \( x_1(s) = X_1(s) \) and \( x_2(0) = X_2(0) \) for \( -\tau_M \leq s \leq 0 \),

\[
\begin{align*}
\dot{x}_1(t) &= \alpha x_2(t) - \gamma x_1(t) - \alpha e^{-\gamma \tau(x_1(t) + x_2(t))} x_2(t - \tau(x_1(t) + x_2(t))), \\
\dot{x}_2(t) &= \alpha e^{-\gamma \tau(x_1(t) + x_2(t))} x_2(t - \tau(x_1(t) + x_2(t))) - \beta x_2^2(t).
\end{align*}
\]

It follows from Theorem 5.1 in Aiello, Freedman and Wu [3] that \( x_1(t) \) and \( x_2(t) \) are bounded, and that we have the following theorem immediately.

Theorem 2.3. (i): If \( \varphi(t) > 0 \) for \( -\tau_M \leq t \leq 0 \), then \( X_2(t) \leq \Delta \) for all \( t \geq 0 \), where \( \Delta = \max \{ \sup_{-\tau_M \leq t \leq 0} \varphi(t), \alpha \beta^{-1} e^{-\gamma \tau_M} \} \). In particular, \( \lim_{t \to \infty} X_2(t) \leq \alpha \beta^{-1} e^{-\gamma \tau_M} \).

(ii): If \( \varphi(t) > 0 \) for \( -\tau_m \leq t \leq 0 \), then \( X_1(t) < \mathcal{V} \) for all \( t \geq 0 \), where \( \mathcal{V} = u_c(0) + \alpha \gamma^{-1} \Delta \).

Theorem 2.4. If \( \varphi(t) > 0 \) for \( -\tau_M \leq t \leq 0 \), then there exists a positive constant \( \Theta \) depending on \( \varphi \) and \( Y(0) \) such that \( Y(t) \leq \Theta \) for all \( t \geq 0 \). In particular, \( \lim_{t \to \infty} Y(t) = 0 \) if \( \beta r > ace^{-\gamma \tau_M} \) and \( \limsup_{t \to \infty} Y(t) \leq (ace^{-\gamma \tau_M} - \beta r)/(b \beta) \) if \( \beta r \leq ace^{-\gamma \tau_M} \).
DYNAMICS OF A STAGE-STRUCTURED POPULATION MODEL 3527

Proof. It follows from Theorem 2.3 that \( \dot{Y}(t) < Y(t)[-r+c\Delta-bY(t)] \), then \( Y(t) \leq \Theta \) for all \( t \geq 0 \), where

\[
\Theta = \begin{cases} 
\max \{ Y(0), \frac{c\Delta-r}{b} \} & \text{if } c\Delta > r, \\
Y(0) & \text{if } c\Delta < r.
\end{cases}
\]

It follows from Theorem 2.3 that for every sufficiently small positive constant \( \varepsilon \) there exists \( T > 0 \) such that \( X_2(t) < \alpha\beta^{-1}e^{-\gamma\tau M} + \varepsilon \) and hence \( \dot{Y}(t) < Y(t)[A_x - bY(t)] \) for all \( t > T \), where \( A_x = \alpha\beta^{-1}e^{-\gamma\tau M} + \varepsilon - r \). If \( \beta r > \alpha c e^{-\gamma\tau M} \) then \( \varepsilon \) can be chosen small enough such that \( A_x \leq 0 \) and hence \( \lim_{t \to \infty} Y(t) = 0 \). If \( \beta r \leq \alpha c e^{-\gamma\tau M} \) then \( A_x > 0 \) for all \( \varepsilon \), and hence \( \limsup_{t \to \infty} Y(t) \leq A_x/b \). Since this is true for any \( \varepsilon > 0 \), it follows that \( \limsup_{t \to \infty} Y(t) \leq A_0/b \). This completes the proof. \( \square \)

Theorem 2.5. If \( ar\beta + ba\beta e^{-\gamma\tau M} > a ace^{-\gamma\tau M} \) and \( \varphi(t) > 0 \) for \( -\tau_M \leq t \leq 0 \), then there exists a positive constant \( \delta \) depending on \( \varphi \) such that \( X_2(t) > \delta \) for all \( t \geq 0 \).

Proof. It follows from Theorem 2.1 that \( X_2(t) > 0 \) for all \( t > 0 \). Now, we show that there exists a positive constant \( \delta \) depending on \( \varphi \) such that \( X_2(t) > \delta \) for all \( t \geq 0 \). Assume on the contrary that \( \liminf_{t \to \infty} X_2(t) = 0 \). One case is that \( X_2(t) \) is eventually oscillatory and \( \liminf_{t \to \infty} X_2(t) = 0 \). Suppose that there exists \( \{t_k\} \) such that \( t_k - t_{k-1} > \tau_M \) for all \( k \in \mathbb{N} \), and that \( X_2(t_k) \) achieves a local minimum and \( X_2(t_k) \geq X_2(t) \) for all \( t \in (t_k, t_{k+1}) \), and that \( \lim_{k \to \infty} X_2(t_k) = 0 \). If \( \beta r > \alpha c e^{-\gamma\tau M} \) then there exists \( K > 0 \) such that \( X_2(t) < \frac{\alpha}{\beta r} e^{-\gamma\tau M} \) and \( Y(t) < \frac{\alpha}{\beta r} e^{-\gamma\tau M} \) for all \( k \geq K \) and hence

\[
0 = \alpha e^{-\gamma\tau M} X_2(t_k - \tau(X(t_k))) - \beta X_2^2(t_k) - a X_2(t_k) Y(t_k)
\]

\[
> \alpha e^{-\gamma\tau M} X_2(t_k) - \beta X_2^2(t_k) - a X_2(t_k) Y(t_k)
\]

\[
> \frac{1}{2} \alpha e^{-\gamma\tau M} X_2(t_k) - \beta X_2^2(t_k) > 0,
\]

for all \( k > K \). This is a contraction. If \( \beta r \leq \alpha c e^{-\gamma\tau M} \) then for every \( 0 < \varepsilon < (ar\beta + ba\beta e^{-\gamma\tau M} - a ace^{-\gamma\tau M})/(b\beta) \) there exists \( K > 0 \) such that \( X_2(t_k) \leq \varepsilon/\beta \) and \( Y(t_k) < \varepsilon + (ace^{-\gamma\tau M} - \beta r)/(b\beta) \) for all \( k > K \). Thus, we have

\[
0 > \alpha e^{-\gamma\tau M} X_2(t_k) - \beta X_2^2(t_k) - a X_2(t_k) Y(t_k)
\]

\[
> \varepsilon X_2(t_k) - \beta X_2^2(t_k) > 0,
\]

for all \( k > K \). This is a contraction. Thus, we complete the proof of this theorem. \( \square \)

Theorem 2.5 implies that for each given positive initial function \( \varphi \), the mature population \( X_2(t) \) is uniformly bounded away from zero. We have proved that \( X_2 \) remains positive and bounded with the given initial condition. Similar to Aiello, Freedman and Wu [3], we shall give the positivity of \( X_1 \), which depend on the initial condition and our having a strictly positive lower bound \( \delta \) and an upper bound \( \Delta \) for \( X_2 \).
Theorem 2.6. Suppose that \( ar\beta + ba\beta e^{-\gamma \tau_M} > acae^{-\gamma \tau_m} \), \( \tau'(X) < 4\beta/\alpha^2 \) and that \( \tau'(X) > 0 \) is small enough so that
\[
\delta \int_{t-\tau_m}^{t} e^{\gamma s} \, ds > \Delta \int_{t-\tau_m}^{t} \frac{\alpha^2 \tau'(X)}{4\beta - \alpha^2 \tau'(X)} e^{\gamma s} \, ds
\]
for all values of \( t \). Then \( X_1(t) > 0 \) for all \( t \geq 0 \).

Remark 1. To prove \( X_1(t) > 0 \) for values of \( t \), it seems to be impossible without placing additional restrictions on either the initial conditions or on the delay \( \tau(X) \). For example, if \( \tau'(X) \equiv 0 \), it has been shown in [2] that \( X_1(t) \) is positive for all \( t \). In Theorem 2.6, we give a set of initial conditions on \( \tau'(X) \), while maintaining the essential character of the state-dependent time delay. The following is a corollary of Theorem 2.6, both of them are only sufficient conditions for the positivity of \( X_1(t) \).

Corollary 1. Suppose that \( ar\beta + ba\beta e^{-\gamma \tau_M} > acae^{-\gamma \tau_m} \) and \( e^{-\gamma \tau_m} \leq \delta/\Delta \), then \( X_1(t) > 0 \) for all \( t \geq 0 \).

3. Existence of equilibria. The purpose of this section is to investigate the existence of equilibria \((X_1, X_2, Y)\) of system (4), which satisfy
\[
\begin{align*}
\alpha X_2 - \gamma X_1 - a e^{-\gamma \tau(X)} X_2 &= 0, \\
\alpha e^{-\gamma \tau(X)} X_2 - \beta X_2^2 - a X_2 Y &= 0, \\
Y (r + c X_2 - b Y) &= 0,
\end{align*}
\]
(5)
It follows from the first two equations of (5) that \( X_1 = g(X_2, Y) \), where \( g: \mathbb{R}^2 \to \mathbb{R} \) is defined as
\[
g(x, y) = \frac{1}{\gamma} (a x - \beta x^2 - a x y)
\]
for all \( x, y \in \mathbb{R} \). Thus, system (5) can be reduced to
\[
\begin{align*}
\alpha e^{-\gamma \tau(X_2 + g(X_2, Y))} - \beta X_2 - a Y &= 0, \\
Y (r + c X_2 - b Y) &= 0.
\end{align*}
\]
(6)
Our main interest is the existence and uniqueness of positive equilibria \( E(X_1, X_2, Y) \) with \( Y = (c X_2 - r)/b \) and \((X_1, X_2)\) satisfying \( X_1 = g(X_2) \) and \( f(X_2) = 0 \), where \( g: \mathbb{R} \to \mathbb{R} \) and \( f: [0, \infty) \to \mathbb{R} \) are defined by
\[
g(x) = \frac{1}{b \gamma} [(b \alpha + a r) x - (b \beta + a c) x^2]
\]
and
\[
f(x) = \alpha e^{-\gamma \tau(x + g(x))} - \frac{1}{b} [(b \beta + a c) x - a r],
\]
(7)
respectively. Thus, it suffices to investigate the existence and uniqueness of positive zero points of \( f \). Note that
\[
f(x) \leq \alpha e^{-\gamma \tau_M} - \frac{1}{b} [(b \beta + a c) x - a r].
\]
Thus, each zero \( x^* \) of \( f \) satisfies \( 0 < x^* \leq \frac{bae^{-\gamma \tau_M} + ar}{b\beta + ac} \). Consider a set
\[
\Omega = \left\{ x \in \mathbb{R} \left| 0 < x < \frac{ba (1 + e^{-\gamma \tau_M}) + ar}{b\beta + ac} \right. \right\},
\]
Note that
\[ f'(x) = -ae^{-\gamma\tau(x+g(x))}(x+g(x))\left[\gamma + \alpha + \frac{ar}{b} - 2\left(\beta + \frac{ac}{b}\right)x\right] - \left(\beta + \frac{ac}{b}\right), \]
\[ f''(x) = ae^{-\gamma\tau(x+g(x))}\left[\gamma + \alpha + \frac{ar}{b} - 2\left(\beta + \frac{ac}{b}\right)x\right]^2 \]
\[ - \gamma^{-1}ae^{-\gamma\tau(x+g(x))}\tau''(x+g(x))\left[\gamma + \alpha + \frac{ar}{b} - 2\left(\beta + \frac{ac}{b}\right)x\right]^2 \]
\[ + 2ae^{-\gamma\tau(x+g(x))}\tau'(x+g(x))\left(\beta + \frac{ac}{b}\right) > 0 \]
for all \(x \in \overline{\Omega}\). This implies that the graph of the curve \(y = f(x)\) is concave upwards on \(\Omega\). In what follows, we shall employ the Brouwer degree to prove the existence of positive zero points of function \(f\).

**Lemma 3.1.** The function \(f\) has exactly one positive zero point.

**Proof.** First, we shall show the function \(f\) has at least one positive zero point. For this purpose, define \(H : [0, 1] \times \overline{\Omega} \to \mathbb{R}\) as
\[ H(t,x) = ae^{-\gamma\tau(x+g(x))} - \frac{(b\beta + ac)x - ar}{b} \]
for all \((t,x) \in [0, 1] \times \overline{\Omega}\). Thus, we have \(G(x) = \alpha - (\beta + \frac{ac}{b})x + \frac{ar}{b}\). For \(x \in \partial\Omega\), we have either \(x = 0\) or \(x = (\beta\beta + ac)^{-1} [ab(1 + e^{-\gamma\tau s}) + ar]\). Note that
\[ H(t,0) = ae^{-\gamma\tau(0)} + \frac{ar}{b} > 0, \quad H\left(t, \frac{bo(1 + e^{-\gamma\tau s}) + ar}{b\beta + ac}\right) \leq \alpha - (1 + e^{-\gamma\tau s}) < 0. \]
It turns out that \(H(t,x)\) is an \(\Omega\)-admissible homotopy. By the homotopy invariance,
\[ \deg(f,\Omega) = \deg(H(1,\cdot),\Omega) = \deg(H(0,\cdot),\Omega) = \deg(G,\Omega). \]
Note that \(\deg(G,\Omega) = -1\). This implies that \(f\) has at least one zero point \(X^*\) in \(\Omega\).

Finally, it follows from \(f''(x) > 0\) for all \(x \in \overline{\Omega}\) that \(f\) has exactly one or two different zeros in \(\Omega\). In fact, if \(f\) has exactly two different zeros in \(\Omega\), then \(\deg(f,\Omega) = 0\), which contradicts \(\deg(f,\Omega) = -1\). Therefore, \(f\) has exactly one zero in \(\Omega\).

In view of Lemma 3.1, system (4) has at least one nontrivial equilibrium \((X_1^*, X_2^*, Y^*)\), where \(X_1^* = g(X_2^*)\) and \(Y^* = (X_2^* - r)/b\). It follows from \(\beta X_2^* + aY^* = ae^{-\gamma\tau(s)} < \alpha\) that \(X_2^* < (ab + ar)/(\beta\beta + ac)\), which implies that \(X_1^* > 0\). Furthermore, \(Y^* > 0\) if \(X_2^* > r/c\). Note that \(X_2^* > r/c\) if and only if \(f(r/c) > 0\), which can be satisfied if \(\beta r < ace^{-\gamma\tau s}\). Thus, we have the following result.

**Theorem 3.2.** Assume that \(\beta r < ace^{-\gamma\tau s}\), then system (4) has exactly one positive equilibrium \(E^*(X_1^*, X_2^*, Y^*)\).

Note that \(f(X_2^*) = 0\). It follows from the proof of Theorem 3.1 that \(f'(X_2^*) < 0\) and hence that
\[ \xi^*[b(\gamma + \alpha) + ar - 2(b\beta + ac) X_2^*] + (b\beta + ac)X_2^* > 0, \]
that is,
\[ b\xi^*\left[\gamma + \alpha - ae^{-\gamma\tau(s)}\right] > (\xi^* - \gamma)(b\beta + ac)X_2^*, \]
where
\[ \xi^* = \gamma ae^{-\gamma\tau(s)}\tau'(X^*)X_2^*. \]

Finally, we consider the existence of boundary equilibria. For an equilibrium \((X_1, X_2, Y)\) of system (4), if \(X_2 = 0\) then \(X_1 = 0\). It is clear that system (4) has at
Theorem 3.3. Which implies that at least two boundary equilibria \( E_0(0,0,0) \) and \( E_1(X_{10},X_{20},0) \) with \( X_{10} = \varrho(X_{20},0) \) and \( X_{20} \) satisfying \( h(X_{20}) = 0 \), where \( h: [0,\infty) \to \mathbb{R} \) is defined by
\[
h(x) = \alpha \exp \left\{ -\gamma \tau \left( \frac{(\alpha + \gamma)x - \beta x^2}{\gamma} \right) \right\} - \beta x. \tag{9}
\]
It is easy to see that each zero \( x_0 \) of \( h \) satisfies \( 0 < x_0 < \alpha \beta^{-1}e^{-\gamma \tau_m} \). Let \( \Omega_1 = \{ x \in \mathbb{R} \mid 0 < x < \alpha \beta^{-1}(1+e^{-\gamma \tau_m}) \} \). Moreover, \( h''(x) > 0 \) for all \( x \in \Omega_1 \). This implies that the graph of the curve \( y = h(u) \) is concave upwards on \( \Omega_1 \). Using a similar argument as the proof of Lemma 3.1, we can conclude that \( h \) has exactly one zero \( X_{10} \) in \( \Omega_0 \), and hence that system (4) has a nontrivial equilibrium \((X_{10},X_{20},0)\), where \( X_{10} = \varrho(X_{20},0) \). It follows from \( \beta X_{20} = \alpha e^{-\gamma \tau(X_{10}+X_{20})} < \alpha \) that \( 0 < X_{20} < \alpha/\beta \), which implies that \( X_{10} > 0 \). Thus, we have the following result.

**Theorem 3.3.** System (4) has exactly one positive boundary equilibrium \( E_1(X_{10},X_{20},0) \), where \( X_{10} = \varrho(X_{20},0) \) and \( X_{20} \) is the unique positive zero of \( h \) defined by (9).

In view of \( h(X_{20}) = 0 \), using a similar argument as the proof of Theorem 3.1, we have \( h'(X_{20}) < 0 \) and hence
\[
\rho \triangleq X_{20} \tau'(X_{10} + X_{20}) [2\beta X_{20} - (\gamma + \alpha)] < 1. \tag{10}
\]

4. Linearized stability. Linearizing (4) is not completely straightforward because the delay is a function depending on the state variables \( X_1 \) and \( X_2 \). It was shown in [7, 17] that generically the behaviour of the state-dependent delay except for its value has no effect on the stability of an equilibrium, and that a local linearization is valid by treating the delay function as a constant at the equilibrium point. Hence to study the local stability of an equilibrium \( E(X_{10}^0,X_{20}^0,Y^0) \) of (4), we linearize (4) at \( E(X_{10}^0,X_{20}^0,Y^0) \) by treating the delay \( \tau(X_1(t) + X_2(t)) \) as \( \tau(X_{10}^0 + X_{20}^0) \). The resulting linear system is a differential equation with a constant delay:
\[
\dot{U}(t) = BU(t) + B_1U(t - \tau(X^0)) \tag{11}
\]
for \( U(t) = (X_1(t),X_2(t),Y(t))^T \in C([0,\tau_M];\mathbb{R}^3) \triangleq C_{\tau_M}, \) where
\[
B = \begin{bmatrix}
-\gamma + \xi^0 & -\xi^0 & 0 \\
-\xi^0 & -2\beta X_{20}^0 - aY^0 & -aX_{20}^0 \\
0 & cY^0 & -r + cX_{20}^0 - 2bY^0
\end{bmatrix},
\]
\[
B_1 = \begin{bmatrix}
0 & -ae^{-\gamma \tau(X^0)} & 0 \\
0 & ae^{-\gamma \tau(X^0)} & 0 \\
0 & 0 & 0
\end{bmatrix},
\]
and \( \xi^0 = \alpha e^{-\gamma \tau(X^0)} \tau'(X^0)X_{20}^0, \) \( X^0 = X_{10}^0 + X_{20}^0 \). This leads to the following characteristic equation
\[
\det \left( \lambda Id_3 - B - B_1e^{-\lambda \tau(X^0)} \right) = 0. \tag{12}
\]
For the extinction equilibrium \( E_0(0,0,0) \), (12) reduces to
\[
(\lambda + \gamma)(\lambda - \alpha e^{-(\gamma + \lambda)\tau_m})(\lambda + r) = 0.
\]
Clearly \( \lambda = -\gamma \) and \( \lambda = -r \) are two of the eigenvalues. All other eigenvalues \( \lambda \) satisfy the equation \( \lambda = \alpha e^{-(\gamma + \lambda)\tau_m} \), which always has a solution with positive real part. Hence \( E_0 \) is unstable.
For the boundary equilibrium $E_1(X_{10}, X_{20}, 0)$ with its existence guaranteed by Theorem 3.3, the eigenvalues $\lambda$ are the roots of the equation
\[
(\lambda + \gamma - cX_{20}) [\lambda^2 + (\gamma + 2\beta X_{20})\lambda + \gamma(\xi_0 + 2\beta X_{20}) - 2\beta\xi_0 X_{20} + \alpha\xi_0 - \alpha(\lambda + \gamma)e^{-\tau(X_0)(\lambda+\gamma)}] = 0,
\]
where $\xi_0 = \alpha e^{-\gamma\tau(X_0)}\tau'(X_0)X_{20}$ and $X_0 = X_{10} + X_{20}$. We have the following result.

**Theorem 4.1.** The boundary equilibrium $E_1(X_{10}, X_{20}, 0)$ is locally asymptotically stable if and only if $X_{20} < r/c$.

**Proof.** Since one eigenvalue is $\lambda = cX_{20} - r$, we only need to investigate the eigenvalues given by
\[
\lambda^2 + (\gamma + 2\beta X_{20})\lambda + \gamma(\xi_0 + 2\beta X_{20}) - 2\beta\xi_0 X_{20} + \alpha\xi_0 - \alpha(\lambda + \gamma)e^{-\tau(X_0)(\lambda+\gamma)} = 0. \tag{13}
\]
It suffices to show that all zeros of (13) have negative real parts.

If $\tau'(X_0) = 0$, (13) can be rewritten as
\[
(\lambda + \gamma)^2 \left( \lambda + 2\beta X_{20} - \alpha e^{-\gamma\tau(X_0)}\right)^2 = 0.
\]
Obviously, $\lambda = -\gamma$ is an eigenvalue, and the others are given by the equation
\[
\lambda + 2\beta X_{20} - \alpha e^{-\gamma\tau(X_0)} = 0.
\]
Note that $\beta X_{20} = \alpha e^{-\gamma\tau(X_0)}$. Suppose that $\text{Re}\lambda \geq 0$ then we compute the real parts and get
\[
\text{Re} \lambda + 2\beta X_{20} = \beta X_{20} \cos(\text{Im}\lambda \tau(X_0)) e^{-\text{Re}\lambda \tau(X_0)} \leq \beta X_{20},
\]
and hence $\text{Re}\lambda \leq -\beta X_{20} < 0$, a contradiction proving the theorem.

We now consider the case where $\tau'(X_0) > 0$ and rewrite (13) as
\[
\lambda^2 + (\gamma + 2\beta X_{20})\lambda + \zeta - (\lambda + \gamma)\beta e^{-\lambda\tau(X_0)} X_{20} = 0,
\]
where $\zeta = \gamma X_{20} [\tau'(X_0)X_{20}\beta(\alpha + \gamma - 2\beta X_{20}) + 2\beta]\gamma X_{20}(2 - \rho)$ and $\rho$ is defined by (10). Let $\lambda = u + iv$, then separating it into real and imaginary parts, we get
\[
\begin{align*}
&u^2 - v^2 + (\gamma + 2\beta X_{20})u + \zeta = \beta X_{20} e^{-u\tau(X_0)} \left( (u + \gamma) \cos v\tau(X_0) + v \sin v\tau(X_0) \right), \\
&2uv + (\gamma + 2\beta X_{20})v = \beta X_{20} e^{-u\tau(X_0)} \left( v \cos v\tau(X_0) - (u + \gamma) \sin v\tau(X_0) \right).
\end{align*}
\tag{14}
\]
We have known that $E_1$ is asymptotically stable if $\tau'(X_0) = 0$. Now suppose that $\tau'(X_0) > 0$ and seek for the value of $\zeta$ such that $u = 0$, i.e., $E_1$ loses its stability. Then (14) becomes
\[
\begin{align*}
-v^2 &+ \zeta = \beta X_{20} \left( \gamma \cos v\tau(X_0) + v \sin v\tau(X_0) \right), \\
(\gamma + 2\beta X_{20})v & = \beta X_{20} \left( v \cos v\tau(X_0) - \gamma \sin v\tau(X_0) \right).
\end{align*}
\]
Squaring and adding the above two equations yield
\[
v^4 + [(\gamma + 2\beta X_{20})^2 - \beta^2 X_{20}^2 - 2\zeta] v^2 + \zeta^2 - \beta^2 X_{20}^2 \tau^2 = 0. \tag{15}
\]
For such $\zeta$ to exist, (15) must have real roots $v$. After substituting for $\zeta$ and rearranging, we see that $v$ is a zero of the following function

$$\tilde{h}(v) = \left(v^2 + \gamma \beta X_{20} + \gamma^2 + 3\gamma^2 X_{20}^2\right) v^2 + 3\gamma^2 \beta^2 X_{20}^2 - 4\rho \beta^2 \gamma^2 X_{20}^2.$$ 

If $2\beta X_{20} \leq \alpha + \gamma$ then it follows from $X_{20} > 0$ and $\tau'(X_0) > 0$ that $\rho < 0$ and hence that $\tilde{h}(v^2) > 0$. If $2\beta X_{20} > \alpha + \gamma$, then $\rho > 0$ and hence $\tilde{h}(v)$ attains its minimum value at $v = 0$, i.e.,

$$\tilde{h}(v) \geq h(0) = \gamma^2 \beta^2 X_{20}^2 \left(\rho^2 - 4\rho + 3\right).$$

It follows from (10) that $\rho < 1$ and hence that $\tilde{h}(v) > 0$ for all $v \in \mathbb{R}$. Therefore, (15) has no real solutions. Thus, (13) has no purely imaginary zeros when $\tau'(X_0) > 0$. Note that all zeros of (13) have negative real parts when $\tau'(X_0) = 0$. Then by continuity, all zeros of (13) have negative real parts when $\tau'(X_0) \geq 0$. $\square$

**Remark 2.** Note that (13) is exactly the characteristic equation described by Aiello, Freedman and Wu [3], who obtained a set of sufficient conditions ensuring that all zeros of (13) have negative real parts. In view of the proof of Theorem 4.1, we see that the sufficient conditions proposed by Aiello, Freedman and Wu [3] are unnecessary and can be removed.

It is very interesting to investigate the stability of the boundary equilibrium $E_1(X_{10}, X_{20}, 0)$ when $\beta r < ace^{-\gamma t/M}$. In this case, we have $ch(r/c) > ace^{-\gamma t/M} - \beta r > 0$ and hence $X_{20} > r/c$. In view of Theorem 4.1, we have the following result.

**Corollary 2.** Assume that $\beta r < ace^{-\gamma t/M}$, then the boundary equilibrium $E_1(X_{10}, X_{20}, 0)$ is unstable.

Now, we consider the stability at the positive equilibrium $E^*(X_1^*, X_2^*, Y^*)$ under the assumption that $\beta r < ace^{-\gamma t/M}$. The characteristic equation (12) can be rewritten as

$$P(\lambda, \xi^*) + Q(\lambda)e^{-\gamma t_0} = 0,$$

where $\tau_0 = \tau(X^*)$, $X^* = X_1^* + X_2^*$, $\xi^* = \alpha \gamma e^{-\gamma t_0} \tau'(X^*)X_2^*$, $P(\lambda, \xi) = \lambda^3 + p_1 \lambda^2 + p_2(\xi) \lambda + p_3(\xi)$, $Q(\lambda) = q_1 \lambda^2 + q_2 \lambda + q_3$, $p_1 = \gamma + 2\beta X_2^* + a Y^* + b Y^*$, $q_1 = -(\beta X_2^* + a Y^*)$, $q_2 = -(\gamma + b Y^*)/(\beta X_2^* + a Y^*)$, $q_3 = -\gamma b Y^*/(\beta X_2^* + a Y^*)$, and $p_2(\xi) = (\gamma + b Y^*)(2\beta X_2^* + a Y^*) + \xi(\alpha + \gamma - 2\beta X_2^* - a Y^*) + a X_2^* Y^* + \gamma b Y^*$, $p_3(\xi) = b Y^*[\gamma(2\beta X_2^* + a Y^*) + \xi(\alpha + \gamma - 2\beta X_2^* - a Y^*)] + (\gamma - \xi) a e X_2^* Y^*$.

It is helpful to consider $\tau'(X^*)$ (or equivalently, $\xi^*$) as a parameter. If the stability of the positive equilibrium point switches as the value of $\tau'(X^*)$ (or equivalently, $\xi^*$) increases starting from 0, there must be a value of $\tau'(X^*)$ (respectively, $\xi^*$) at which there are eigenvalues on the imaginary axis.

**Theorem 4.2.** Assume that $\beta r < ace^{-\gamma t/M}$, then the positive equilibrium $E^*(X_1^*, X_2^*, Y^*)$ is locally asymptotically stable.

**Proof.** We start with the case where $\tau'(X^*)$. In this case, $\xi^* = 0$ and the characteristic equation (16) will be of the form

$$(\lambda + \gamma) \left[(\lambda + b Y^*) (\lambda + 2\beta X_2^* + a Y^* - \alpha e^{-(\lambda + \gamma) t_0}) + a X_2^* Y^*\right] = 0,$$

which obviously has a zero $\lambda = -\gamma < 0$. Thus, we only need to consider the zeros of the following equation:

$$(\lambda + b Y^*) (\lambda + 2\beta X_2^* + a Y^* - \alpha e^{-(\lambda + \gamma) t_0}) + a X_2^* Y^* = 0. \quad (17)$$
Thus, we obtain the equation with respect to $y$:

$$y^2 + a^* y + b^* = 0.$$  \hspace{1cm} (18)

where

$$a^* = (bY^*)^2 + \beta X_2^* (3\beta X_2^* + 2aY^*) - 2acX_2Y^*$$

$$b^* = [bY^* (3\beta X_2^* + 2aY^*) + acX_2Y^*] [bY^* \beta X_2^* + acX_2Y^*] > 0.$$  

The discriminant of (18) is of the form

$$\Upsilon = [(bY^*)^2 + \beta X_2^* (3\beta X_2^* + 2aY^*) - 2acX_2Y^*]^2$$

$$- 4[bY^* (3\beta X_2^* + 2aY^*) + acX_2Y^*] [bY^* \beta X_2^* + acX_2Y^*].$$

If $a^* < 0$ then $2acX_2Y^* > (bY^*)^2 + \beta X_2^* (3\beta X_2^* + 2aY^*)$ and hence $\Upsilon < 0$, which implies that equation (18) has no positive real roots. If $a^* \geq 0$ then $y^2 + a^* y + b^* > 0$ for all $y \geq 0$ and hence equation (18) has no positive real roots. When $\tau_0 = 0$ and $\tau'(X^*) = 0$, equation (17) becomes

$$\lambda^2 + (bY^* + \beta X_2^*) \lambda + (bY^*) \beta X_2^* + acX_2 Y^* = 0,$$

which has exactly two roots with negative real parts. Hence, the positive equilibrium $E^*$ is locally asymptotically stable when $\tau_0 = 0$ and $\tau'(X^*) = 0$. Thus, we conclude that the positive equilibrium $E^*$ is locally asymptotically stable for all $\tau_0 \geq 0$ when $\tau'(X^*) = 0$.

Note that $E^*$ is locally asymptotically stable when $\tau'(X^*) = 0$ (or equivalently, $\xi^*$). As $\tau'(X^*)$ (or equivalently, $\xi^*$) increases, the stability of the steady state can be lost only if purely imaginary roots of (16) appear. Substituting $\lambda = i\sqrt{w}$ ($w > 0$) into (16) and then separating the real and imaginary parts, we have

$$p_1 w - p_1(\xi^*) = q_2 \sqrt{w} \sin(\tau_0 \sqrt{w}) - (q_1 w - q_3) \cos(\tau_0 \sqrt{w}),$$

$$\sqrt{w^3} - p_2(\xi^*) \sqrt{w} = (q_1 w - q_3) \sin(\tau_0 \sqrt{w}) + q_2 \sqrt{w} \cos(\tau_0 \sqrt{w}).$$  \hspace{1cm} (19)

Similarly, we square and add the above two equations and obtain that

$$F(w, \xi^*) \equiv w^3 + C_1(\xi^*)w^2 + C_2(\xi^*)w + C_3(\xi^*) = 0,$$ \hspace{1cm} (20)

where $C_1(\xi^*) = p_1^2 - 2p_2(\xi^*) - q_3$, $C_2(\xi^*) = p_2^2(\xi^*) - 2p_1 p_3(\xi^*) - q_3^2 + 2q_1 q_3$, $C_3 = p_3^2(\xi^*) - q_3^3$. It follows from (8) that $p_3(\xi^*) > -q_3$. This, together with $p_3(\xi^*) > 0$, implies that $C_3(\xi^*) = [p_3(\xi^*) + q_3][p_3(\xi^*) - q_3] > 0$. Note that

$$C_1(0) = \gamma^2 + a^*, \quad C_2(0) = \gamma^2 a^* + b^*, \quad C_3(0) = \gamma^2 b^*$$

then we have $F(w, 0) = (w + \gamma^2)(w^2 + a^* w + b^*)$. It follows from the discussion about (18) that $F(w, 0) > 0$ for all $w \geq 0$. Note that

$$C_1(\xi) = \gamma^2 + a^* - 2(\alpha + \gamma - 2\beta X_2^* - aY^*)$$

$$= \gamma^2 + (bY^*)^2 + \beta X_2^* (3\beta X_2^* + 2aY^*) - 2acX_2 Y^* - 2v,$$
It is easy to see that \( \iota < \upsilon \), thus no stability switches can occur. This completes the proof of the theorem.

It follows from Routh-Hurwitz criteria [27] that no positive solution to (20) exists.

\[
C_2(\xi) = \gamma^2 a^* + b^* + \xi^2(\alpha + \gamma - 2\beta X_2^* - aY^*)^2
+ 2\xi(\alpha + \gamma - 2\beta X_2^* - aY^*)[(\gamma + bY^*)(2\beta X_2^* + aY^*) + acX_2^*Y^* + \gamma bY^*]
- 2\xi(\gamma + 2\beta X_2^* + aY^* + bY^*)[bY^*(\alpha + \gamma - 2\beta X_2^* - aY^*) - acX_2^*Y^*]
\]

\[
> \gamma^2 a^* + b^* + \upsilon^2 + 2\upsilon[(\gamma + bY^*)(2\beta X_2^* + aY^*) + acX_2^*Y^* + \gamma bY^*]
- 2\upsilon(\gamma + 2\beta X_2^* + aY^* + bY^*)bY^*
\]

\[
= \gamma^2 a^* + b^* + \upsilon^2 + 2\upsilon[\gamma(2\beta X_2^* + aY^* + bY^*) + \iota]
- 2\upsilon(\gamma + 2\beta X_2^* + aY^* + bY^*)bY^*
\]

\[
= \gamma^2 a^* + b^* + \upsilon^2 + 2\upsilon[\gamma(2\beta X_2^* + aY^*) + acX_2^*Y^* - (bY^*)^2]
\]

\[
\geq \gamma^2 a^* + b^* - \gamma^2(2\beta X_2^* + aY^*)^2 + 2\upsilon[acX_2^*Y^* - (bY^*)^2]
\]

\[
= \gamma^2[(bY^*)^2 - 2acX_2^*Y^* - (\beta X_2^* + aY^*)^2] + b^* + 2\upsilon[acX_2^*Y^* - (bY^*)^2]
\]

\[
C_3(\xi) = \gamma^2 b^* + \xi^2[bY^*(\alpha + \gamma - 2\beta X_2^* - aY^*) - acX_2^*Y^*] + 2\xi[\gamma(2\beta X_2^* + aY^* - bY^*) - acX_2^*Y^*]
\]

\[
\geq \gamma^2 b^* + (b\beta + ac)^2\gamma^2(X_2^*Y^*)^2 - 2\gamma^2(b\beta + ac)X_2^*Y^*
\]

and

\[
F(w, \xi) = F(w, 0) - 2w^2\xi(\alpha + \gamma - 2\beta X_2^* - aY^*) + w^2[(\alpha + \gamma - 2\beta X_2^* - aY^*)^2]
+ 2w\xi[\gamma(2\beta X_2^* + aY^*) + \upsilon^2(\alpha + \gamma - 2\beta X_2^* - aY^*)]
- 2w\xi[bY^*(\alpha + \gamma - 2\beta X_2^* - aY^*) - acX_2^*Y^*](\gamma + 2\beta X_2^* + aY^* + bY^*)
+ 2\xi^2[bY^*(\alpha + \gamma - 2\beta X_2^* - aY^*) - acX_2^*Y^*] + \xi^2[\gamma(2\beta X_2^* + aY^* + bY^*) + \iota]
\]

where \( \iota = bY^*(2\beta X_2^* + aY^*) + acX_2^*Y^* \), \( \upsilon = \xi(\alpha + \gamma - 2\beta X_2^* - aY^*) \), and \( \gamma = \xi[bY^*(\alpha + \gamma - 2\beta X_2^* - aY^*) - acX_2^*Y^*] > -(b\beta + ac)\gamma X_2^*Y^* \). If \( \alpha + \gamma - 2\beta X_2^* - aY^* \leq 0 \) then \( \upsilon < 0 \), \( \gamma < 0 \) and hence

\[
F(w, \xi) > w^2 + 2w\upsilon[(\gamma + bY^*)(2\beta X_2^* + aY^*) + acX_2^*Y^* + \gamma bY^*]
+ \gamma^2 + 2\gamma\xi[bY^*(2\beta X_2^* + aY^*) + acX_2^*Y^*] > 0.
\]

It follows from Routh-Hurwitz criteria [27] that no positive solution to (20) exists. Thus no stability switches can occur. This completes the proof of the theorem.

5. Global behaviors. In order to investigate the global asymptotical stability of the positive interior equilibrium \( F^*(X_1^*, X_2^*, Y^*) \) and the positive boundary equilibrium \( E_1(X_{10}, X_{20}, 0) \) of (4), we first investigate the state bounds on the eventual behaviour of \( X_2(t) \). We introduce the following two quantities depending on \( a \):

\[
X_a^- = \frac{bae^{-\gamma t} + ar}{\beta b + ac}, \quad X_a^+ = \frac{bae^{-\gamma t} + ar}{\beta b + ac}.
\]

It is easy to see that \( X_a^- \) is increasing (respectively, decreasing) with respect to \( a \) if \( \beta r > ace^{-\gamma t} \) (respectively, \( \beta r < ace^{-\gamma t} \)), and that \( X_a^+ \) is increasing (respectively, decreasing) with respect to \( a \) if \( \beta r > ace^{-\gamma t} \) (respectively, \( \beta r < ace^{-\gamma t} \)).
The following result gives the state bounds on the eventual behaviour of $X_2(t)$, independent of admissible initial conditions.

**Lemma 5.1.** Assume that $ar\beta + ba\beta e^{-\gamma T_{\mu}} > ace^{-\gamma T_{\mu}}$ and $\varphi(t) > 0$ for $-\tau_{\mu} \leq t \leq 0$. Let $(X_1(t), X_2(t), Y(t))$ be a solution of (4). If $\beta r < ace^{-\gamma T_{\mu}}$ then

$$X_a^- \leq \liminf_{t \to \infty} X_2(t) \leq \limsup_{t \to \infty} X_2(t) \leq X_a^+;$$

If $\beta r \geq ace^{-\gamma T_{\mu}}$ then

$$X_a^- \leq \liminf_{t \to \infty} X_2(t) \leq \limsup_{t \to \infty} X_2(t) \leq X_a^+.$$

**Proof.** We distinguish two cases to complete the proof of this lemma. The first case is that $X_2(t)$ is eventually monotonic and bounded. In this case, there exists $\xi > 0$ such that $X_2(t) \to \xi$ and $\dot{X}_2(t) \to 0$ as $t \to \infty$. Hence from (4), taking the limit superior as $t \to \infty$, we have

$$0 = \xi \left[ \alpha \exp \left\{ -\gamma \left( \limsup_{t \to \infty} X_1(t) + \xi \right) \right\} - \beta \xi - \alpha \limsup_{t \to \infty} Y(t) \right],$$

which implies that $\xi \leq X_a^+$. It follows from Lemma 5.7 that $\limsup_{t \to \infty} Y(t) = (-r + c\xi)/b$ if $c\xi > r$, that $\limsup_{t \to \infty} Y(t) = 0$ if $c\xi \leq r$, and hence that

$$(\beta b + ac)\xi = \beta a \exp \left\{ -\gamma \left( \limsup_{t \to \infty} X_1(t) + \xi \right) \right\} + ar$$

if $c\xi > r$, and

$$\beta \xi = \alpha \exp \left\{ -\gamma \left( \limsup_{t \to \infty} X_1(t) + \xi \right) \right\}$$

if $c\xi \leq r$. Now, we distinguish three subcases to prove the conclusion.

The first subcase is that $\beta r < ace^{-\gamma T_{\mu}}$. In this case, $r/c < X_a^+ \leq X_a^-$. If $c\xi \leq r$ then it follows from (24) that $X_a^- \leq \xi \leq X_a^+$, which contradicts $\xi \leq r/c = X_a^-$. Therefore, $c\xi > r$, which together with (23) implies that $X_a^- \leq \xi \leq X_a^+$.

The second subcase is that $\beta r \geq ace^{-\gamma T_{\mu}}$. In this case, $r/c > X_a^- \geq X_a^+$. If $c\xi > r$ then it follows from (23) that $X_a^- \leq \xi \leq X_a^+$, which contradicts $\xi > r/c > X_a^-$. Therefore, $c\xi \leq r$, which together with (24) implies that $X_a^- \leq \xi \leq X_a^+$.

The last subcase is that $ace^{-\gamma T_{\mu}} \leq \beta r < ace^{-\gamma T_{\mu}}$. In this case, $X_a^- \leq r/c < X_a^+$ and $X_a^- \leq X_a^- \leq X_a^+ \leq X_a^+$. Thus, if $c\xi > r$ then it follows from (23) that $X_a^- \leq \xi \leq X_a^+$ and hence $X_a^- \leq \xi \leq X_a^+$. If $c\xi \leq r$ then it follows from (24) that $X_a^- \leq \xi \leq X_a^+$.

Next, we consider the case where $X_2(t)$ is oscillatory. We only show that

$$\limsup_{t \to \infty} X_2(t) \leq X_a^+$$

when $\beta r < ace^{-\gamma T_{\mu}}$, because the other inequalities and other cases follow analogously. Note that $\beta r < ace^{-\gamma T_{\mu}}$ implies that

$$X_a^+ \geq X_a^- > r/c$$

(25)
Define the sequence \( \{ t_k \} \) as those times for which \( X_2(t) \) achieves a maximum, i.e., \( \dot{X}_2(t_k) = 0 \) and \( \ddot{X}_2(t_k) < 0 \). Define

\[
\xi = \limsup_{k \to \infty} X_2(t_k).
\]

Then \( 0 < \xi < \infty \) and \( \limsup_{t \to \infty} X_2(t) = \xi \). We claim that \( \xi \leq X_2^+ \). In fact, if \( c\xi \leq r \) then it follows from (25) that \( \xi \leq r/c < X_2^- \leq X_2^+ \). If \( c\xi > r \) then \( \lim_{k \to \infty} Y(t_k) = (-r + c\xi)/b \). Suppose on the contrary that \( \xi > X_2^+ \). We now choose a subsequence of \( \{ t_k \} \), relabelled as \( \{ t_k \} \) such that \( t_{k+1} \geq t_k + \tau_M \) and \( X_2(t_k) \to \xi \) as \( k \to \infty \). Let \( \tilde{X} = \limsup_{k \to \infty} X(t_k) \), where \( X(t) = X_1(t) + X_2(t) \). We then choose a further subsequence of \( \{ t_k \} \), again relabelled by \( \{ t_k \} \) such that \( \lim_{k \to \infty} X(t_k) = \tilde{X} \). Now let \( \xi' = \limsup_{k \to \infty} X_2(t_k - \tau(X(t_k))) \) for this subsequence \( \{ t_k \} \). We choose a final subsequence of \( \{ t_k \} \), once again relabelled by \( \{ t_k \} \), such that \( \lim_{k \to \infty} X_2(t_k - \tau(X(t_k))) = \xi' \). It follows from the definition of \( \xi \) that \( \xi' \leq \xi \). Then from (4) and Lemma 5.7, taking the limit as \( k \to \infty \), we obtain

\[
0 = a e^{-\gamma \tau(\tilde{X})} \xi' - \beta \xi^2 - a\xi (-r + c\xi)/b
\]

\[
< a e^{-\gamma \tau(\tilde{X})} \xi' - (\beta + ac/b)\xi X_2^+ + ar\xi/b
\]

\[
\leq a e^{-\gamma \tau_m} (\xi' - \xi) \leq 0,
\]

a contradiction. Therefore, \( \xi \leq X_2^+ \). This completes the proof. \( \Box \)

In order to overcome the difficulties caused by the presence of state-dependent delay during the investigation of the global stability of system (4), we introduce two auxiliary systems, one of which takes the form

\[
\begin{align*}
X_1'(t) &= \alpha X_2 - \gamma X_1 - s\alpha e^{-\gamma \tau(X_1 + X_2)}, \\
X_2'(t) &= d\alpha e^{-\gamma \tau(X_1 + X_2)} - \beta X_2 - k\alpha X_2.
\end{align*}
\]

(26)

Here, \( \tau(\cdot) \) is the same to that in system (4), \( k \geq 0 \), \( \alpha, \beta, \gamma \) are positive constants, \( s, d \in [k_-, k_+] \), and

\[ k_+ = \frac{\alpha e^{-\gamma \tau_m} - ka}{\beta}, \quad k_- = \frac{\alpha e^{-\gamma \tau_m} - ka}{\beta}. \]

Note that (26) is a mixed quasi-monotone system (see [29, 41]), then we have the following observations.

**Lemma 5.2.** The set \( \{ (X_1, X_2) \in \mathbb{R}^2 \mid k_- \leq X_2 \leq k_+ \} \) is positively invariant for the semiflow generated by system (26).

**Lemma 5.3.** Assume that \( s < d \) or \( s > d \) and \( 2\tau_m \geq \tau_M \). Then system (26) has a positive equilibrium point \( (\tilde{X}_1(s, d, k), \tilde{X}_2(s, d, k)) \). More precisely,

(i): \( k_- < \tilde{X}_2(s, d, k) < k_+ \);

(ii): The positive equilibrium point \( (\tilde{X}_1(s, d, k), \tilde{X}_2(s, d, k)) \) is unique if \( \alpha k_+ \tau'(k_-) e^{-\gamma \tau(k_-)} < 1 \);

(iii): The positive equilibrium point \( (\tilde{X}_1(s, d, k), \tilde{X}_2(s, d, k)) \) is locally asymptotically stable and attracts all of the positive solutions of system (26) if \( \alpha k_+ \tau'(k_-) e^{-\gamma \tau(k_-)} < 1 \).

**Proof.** Obviously, the equilibria \( (X_1, X_2) \) of system (26) satisfy

\[
X_1 = \varpi(X_2, s, d, k), \quad d\alpha e^{-\gamma \tau(X_1 + X_2)} - \beta X_2 - k\alpha X_2 = 0.
\]

(27)
Define $F: \mathbb{R} \times [k_-, k_+] \times [k_-, k_+] \to \mathbb{R}$ as
\[
F(x, s, d, k) = d\alpha \exp \{-\gamma \tau (x + \varpi(x, s, d, k))\} - \beta x^2 - kax.
\]
Note that
\[
F(x, a, d, k) > d\alpha e^{-\gamma \tau M} - k_- \alpha e^{-\gamma \tau M} = (d - k_-)\alpha e^{-\gamma \tau M} \geq 0
\]
for all $x \in [0, k_-]$, and
\[
F(k_+, s, d, k) \leq d\alpha e^{-\gamma \tau m} - \beta k_+^2 - k_+ k_+ = \alpha(d - k_+^2)\alpha e^{-\gamma \tau m} \leq 0.
\]
Thus, there exists some $\tilde{X}_2(s, d, k) \in (k_-, k_+)$ such that $F(\tilde{X}_2(s, d, k), s, d, k) = 0$. It follows from (27) that $\dot{X}_1(s, d, k) = \varpi(\tilde{X}_2(s, d, k), s, d, k)$. If $s < d$ or $s > d$ and $2\tau_m \geq \tau_M$ then
\[
\frac{d\alpha}{s} - ka - \beta k_+ > \alpha \left[ \frac{\alpha e^{-\gamma \tau M} - ka}{\alpha e^{-\gamma \tau m} - ka} - e^{-\gamma \tau m} \right] \geq \alpha \left[ e^{\gamma(\tau_m - \tau_M)} - e^{-\gamma \tau m} \right] \geq 0,
\]
and hence
\[
k_- < \tilde{X}_2(s, d, k) < k_+ < \frac{d\alpha - aks}{a\beta}, \quad \tilde{X}_1(s, d, k) > 0.
\]
Therefore, system (26) has a positive equilibrium point $(\tilde{X}_1(s, d, k), \tilde{X}_2(s, d, k))$. If $ak_+ \tau'(k_-)e^{-\gamma \tau(k_-)} < 1$ then $a\alpha \tau'(x)e^{-\gamma \tau(x)} < 1$ for all $x > k_-$, and hence
\[
F(x, s, d, k) \leq (-ka - 2\beta x)\left[ 1 - a\alpha \tau'(x)e^{-\gamma \tau(x)} \right] < 0
\]
for all $x \in (k_-, k_+)$. This implies that system (26) has exactly one positive equilibrium point $(\tilde{X}_1(s, d, k), \tilde{X}_2(s, d, k))$.

If $a\alpha \tau'(\tilde{X}_1(s, d, k) + \tilde{X}_2(s, d, k))e^{-\gamma \tau(\tilde{X}_1(s, d, k) + \tilde{X}_2(s, d, k))} < 1$, then it is easy to see that the trace and determinant of the linearized matrix of (26) at the positive equilibrium point $(\tilde{X}_1(s, d, k), \tilde{X}_2(s, d, k))$ are negative and positive, respectively. Therefore, the positive equilibrium point $(\tilde{X}_1(s, d, k), \tilde{X}_2(s, d, k))$ is locally asymptotically stable if $ak_+ \tau'(k_-)e^{-\gamma \tau(k_-)} < 1$. Note that the divergence of the vector field associated with (26) is
\[
-\gamma + (s - d)\alpha \gamma \tau'(X_1 + X_2)e^{-\gamma \tau(X_1 + X_2)} - 2\beta X_2 - ka,
\]
which is negative if $a\alpha \tau'(X_1 + X_2)e^{-\gamma \tau(X_1 + X_2)} < 1$. This implies that system (26) has no periodic solutions lying in the set $\{(X_1, X_2) \in \mathbb{R}^2 \mid k_- < X_2 < k_+ \}$. Thus, the global attractivity of the positive equilibrium point $(\tilde{X}_1(s, d, k), \tilde{X}_2(s, d, k))$ follows from Lemma 5.2.

Note that $F(d, s, d, k) = d\dot{f}(s, d, k)$, where
\[
\dot{f}(s, d, k) \triangleq \alpha \exp \{-\gamma \tau (d + \varpi(d, s, d, k))\} - \beta d - ka
\]
If $ak_+ \tau'(k_-)e^{-\gamma \tau(k_-)} < 1$ then it is easy to see that $\dot{f}_d(d, s, d, k) < 0$ and $\dot{f}_s(s, d, k) > 0$ for all $s, d \in [k_-, k_+]$. Thus, $f(s, s, k)$ is strictly decreasing in $s \in (k_-, k_+)$ and satisfies $\dot{f}_d(k_-, k_-, k) > 0 > \dot{f}_d(k_+, k_+, k)$. Thus, there exists exactly one $\eta(k) \in (k_-, k_+)$ such that $f(\eta(k), \eta(k), k) = 0$. Therefore, we have the following observation.
Lemma 5.4. (i): If $k_- < s < \eta(k) < d < k_+$ and $\alpha k_+ \tau'(k_-) e^{-\gamma \tau_m} < 1$ then $s < \tilde{X}_2(d, s, k) < d$ and $s < \tilde{X}_2(s, d, k) < d$;

(ii): If $k_- < s < \eta(k), k_- < s < d < k_+$, and $\tau'(\eta(k)) \eta(k)(\beta \eta(k) + ka) < \frac{(\eta(k) - s) k_-}{(d - s) \eta(s)}$, then $\eta(k) > \tilde{X}_2(d, s, k)$;

(iii): If $k_- < s < d < k_+$ and $\tau'(\eta(k)) \eta(k)(\beta \eta(k) + ka) < \frac{d - \eta(k)}{d - s}$, then $\eta(k) < \tilde{X}_2(s, d, k)$;

(iv): If $k_- < s < d < k_+$ and $\alpha k_+ \tau'(k_-) e^{-\gamma \tau(k_-)} < \frac{1}{2}$ then $k_- < \tilde{X}_2(d, s, k) < \tilde{X}_2(s, d, k) < k_+$;

(v): If $k_- < s < d < \eta(k)$ then $\tilde{X}_2(s, d, k) < \eta(k)$;

(vi): If $\eta(k) < s < d < k_+$ then $\tilde{X}_2(d, s, k) > \eta(k)$;

Proof. For convenience, let $\tilde{X}_2 = \tilde{X}_2(s, d, k)$, $\tilde{X}_2 = \tilde{X}_2(d, s, k)$, and $\eta = \eta(k)$. It follows from $\alpha k_+ \tau'(k_-) e^{-\gamma \tau_m} < 1$ that $\alpha k_+ \tau'(k_-) e^{-\gamma \tau(k_-)} < 1$ and hence $\tilde{f}_s(d, s, k) > 0$ and $\tilde{f}_d(s, d, k) < 0$. If $k_- < s < \eta < d < k_+$ then $\tilde{f}(s, d, k) < \tilde{f}(d, s, k) < 0$ and hence $\mathcal{F}(d, s, d, k) < 0 < \mathcal{F}(s, s, d, k)$, which implies that $\tilde{X}_2 < d$ and $s < \tilde{X}_2$. Note that

$$\mathcal{F}_d(x, s, d, k) > \left[1 + x(-ka - \beta x) \tau'(k_-)\right] \alpha \exp\left\{ -\gamma \tau (x + \varpi(x, s, d, k))\right\}$$

$$> \left[1 - \alpha k_+ \tau'(k_-) e^{-\gamma \tau_m}\right] \alpha \exp\left\{ -\gamma \tau (x + \varpi(x, s, d, k))\right\} > 0$$

for all $x \in [k_-, k_+]$. This implies that

$$\mathcal{F}(s, s, d, k) > \mathcal{F}(s, s, s, k) = s \tilde{f}(s, s, k) > 0,$n$$

$$\mathcal{F}(d, d, s, k) < \mathcal{F}(d, d, d, k) = d \tilde{f}(d, d, k) < 0,$n$$

and hence that $\tilde{X}_2 > s$ and $\tilde{X}_2 < d$. Note that

$$\mathcal{F}(\eta, s, d, k)$$

$$= \alpha \exp\left\{ -\gamma \tau (\eta + \varpi(\eta, s, d, k))\right\} - \alpha \eta \exp\left\{ -\gamma \tau (\eta + \varpi(\eta, \eta, \eta, k))\right\}$$

$$\geq \left[ \alpha \exp\left\{ -\tau'(\eta) \left(1 - \frac{s}{d}\right) \eta(-ka - \beta \eta)\right\} - \eta \right] \alpha \exp\left\{ -\gamma \tau (\eta + \varpi(\eta, \eta, \eta, k))\right\}$$

$$\geq \left[ d - \bar{d} - (d - s) \tau'(\eta) \eta(\beta \eta + ka)\right] \alpha \exp\left\{ -\gamma \tau (\eta + \varpi(\eta, \eta, \eta, k))\right\},$$

and that

$$\mathcal{F}(\eta, d, s, k)$$

$$= \alpha \exp\left\{ -\gamma \tau (\eta + \varpi(\eta, d, s, k))\right\} - \alpha \eta \exp\left\{ -\gamma \tau (\eta + \varpi(\eta, \eta, \eta, k))\right\}$$

$$\leq \left[ \alpha \exp\left\{ -\tau'(\eta) \left(1 - \frac{d}{s}\right) \eta(-ka - \beta \eta)\right\} \right] \alpha \exp\left\{ -\gamma \tau (\eta + \varpi(\eta, d, s, k))\right\}$$

$$\leq \left[ s - \eta + \frac{\eta}{s} (d - s) \tau'(\eta) \eta(\beta \eta + ka)\right] \alpha \exp\left\{ -\gamma \tau (\eta + \varpi(\eta, d, s, k))\right\},$$

Thus, if $k_- < s < \eta$ and $k_- < s < d < k_+$ and

$$\tau'(\eta) \eta(\beta \eta + ka) < \frac{(\eta - s)s}{(d - s)\eta},$$

then $\mathcal{F}(\eta, d, s, k) < 0$ and hence $\eta > \tilde{X}_2$. If $k_- < s < d < k_+$ and

$$\tau'(\eta) \eta(\beta \eta + ka) < \frac{d - \eta}{d - s},$$

then $\mathcal{F}(\eta, d, s, k) < 0$ and hence $\eta > \tilde{X}_2$.
then \( F(\eta, s, d, k) > 0 \) and hence \( \eta < \tilde{X}_2 \). Note that

\[
F(\tilde{X}_2, s, d, k) = d \alpha \exp \left\{ - \gamma \left( X_2 + \varpi(X_2, s, d, k) \right) \right\} - s \alpha \exp \left\{ - \gamma \left( X_2 + \varpi(X_2, d, s, k) \right) \right\}
\]

\[
\geq \left[ d \exp \left\{ \tau'(X_2) \left( \frac{s}{d} - \frac{s}{d} \right) (ka + \beta X_2) X_2 \right\} - s \right] \alpha \exp \left\{ - \gamma \left( X_2 + \varpi(X_2, d, s, k) \right) \right\}
\]

\[
\geq \left[ d + \tau'(X_2) \left( \frac{s}{d} - s \right) (kaX_2 - \beta X_2^2) \right] \alpha \exp \left\{ - \gamma \left( X_2 + \varpi(X_2, d, s, k) \right) \right\}
\]

\[
\geq (d - s) \left[ 1 + 2 \tau'(X_2)(kaX_2 - \beta X_2^2) \right] \alpha \exp \left\{ - \gamma \left( X_2 + \varpi(X_2, d, s, k) \right) \right\}.
\]

Thus, if \( k_- < s < d < k_+ \) and \( 2ak_+ \tau'(k_-) e^{-\gamma \tau} < 1 \) then \( F(\tilde{X}_2, s, d, k) > 0 \), which implies that \( \tilde{X}_2 < \tilde{X}_2 \). Then

\[
\tilde{F}(\eta, s, d, k) \leq \alpha \eta \exp \left\{ - \gamma \left( \eta + \varpi(\eta, s, d, k) \right) \right\} - \alpha \eta \exp \left\{ - \gamma \left( \eta + \varpi(\eta, \eta, \eta, k) \right) \right\}
\]

\[
\leq \left[ \exp \left\{ \tau'(\theta) \left( 1 - \frac{s}{d} \right) \eta (ka - \beta \eta) \right\} - 1 \right] \alpha \eta \exp \left\{ - \gamma \left( \eta + \varpi(\eta, \eta, \eta, k) \right) \right\} < 0,
\]

where \( \theta \) lies between \([d\tau + d\alpha - s\alpha k\eta - s\beta \eta^2]/(d\tau)\) and \([\gamma + \alpha \eta + (-ka - \beta \eta)\eta]/\gamma\). This implies that \( \tilde{X}_2(s, d, k) < \eta \). Using a similar argument as above, we see that \( \tilde{X}_2(d, s, k) > \eta \) if \( \eta < s < d < k_+ \). This completes the proof of this lemma. \( \square \)

Now, we consider some properties of the function \( \eta(k) \) for \( k \geq 0 \). Note that \( \eta(k) \) is the unique zero of the function \( \vartheta(k, \cdot) \), where \( \vartheta: [0, \infty) \times [\mu^-, \mu^+] \to \mathbb{R} \) is defined as

\[
\vartheta(k, x) = \alpha \exp \left\{ - \gamma \left( \frac{(\gamma + \alpha - ka - \beta x)x}{\gamma} \right) \right\} - \beta x - ka
\]

for \( k \geq 0 \) and \( x \in [\mu^-, \mu^+] \), where \( \mu^\pm = X^\pm_a \) (respectively, \( \mu^\pm = X^\pm_b \)) if \( \beta r < ace^{-\gamma \tau} \) (respectively, \( \beta r \geq ace^{-\gamma \tau} \)), and \( X^\pm_a \) is defined as (21). Note that \( h(x) = \vartheta(0, x) = \bar{f}(0, x, x) \), where function \( h \) is defined as (9). It follows from the proof of Theorem 3.3 that \( h \) has exactly one positive zero \( X_{20} \). Namely, \( \eta(0) = X_{20} \). If \( \alpha \mu^+ \tau'(\mu^-) e^{-\gamma \tau(\mu^-)} < 1 \) then it is easy to see that \( \vartheta_x(k, x) < 0 \) and \( \vartheta_x(k, x) > 0 \), and hence \( \eta(k) \) is decreasing with respect to \( k \). Note that \( f(x) = \vartheta(\chi(x), x) = \bar{f}(x, x, \chi(x)) \), where function \( f \) is defined as (7) and \( \chi(x) = (cx - r)/b \) for \( x \geq 0 \). It follows from the proof of Theorem 3.1 that \( h \) has exactly one positive zero \( X^*_2 \). Namely, \( \eta(Y^*) = X^*_2 \) and \( \chi(X^*_2) = Y^* \). Thus, \( f(x) > 0 \) for all \( x \in [0, X^*_2] \), and \( f(x) < 0 \) for all \( x \in [X^*_2, \alpha b + \tau (\mu^+) \beta b + \alpha \beta k] \). Therefore, we have the following observation.

**Lemma 5.5.**

(i): \( \eta(0) = X_{20} \).

(ii): If \( \alpha \mu^+ \tau'(\mu^-) e^{-\gamma \tau(\mu^-)} < 1 \), then \( \eta(k) \) is decreasing with respect to \( k \geq 0 \). Moreover, if \( k > Y^* \) then \( \eta(k) < X^*_2 \); if \( k < Y^* \) then \( \eta(k) > X^*_2 \).

(iii): \( \chi(x) \) is increasing with respect to \( x \). Moreover, if \( x < X^*_2 \) then \( \chi(x) < Y^* \); if \( x > X^*_2 \) then \( \chi(x) > Y^* \).

**Lemma 5.6.** Define two sequences \( \{B^u_n\}_{n=1}^\infty \) and \( \{B^l_n\}_{n=1}^\infty \) as follows:

\[
B^u_1 = \tilde{X}_2(k_-, k_+, k), \quad B^l_1 = \tilde{X}_2(k_+, k_-, k),
\]

and

\[
B^u_2 = \tilde{X}_2(B^u_{n-1}, B^u_{n-1}, k), \quad B^l_2 = \tilde{X}_2(B^l_{n-1}, B^l_{n-1}, k)
\]

for all \( n > 2 \). If

\[
\alpha k_+ \tau'(k_-) e^{-\gamma \tau(k_-)} < \frac{1}{2}
\]

(29)
then \( \lim_{n \to \infty} B_n^u = \lim_{n \to \infty} B_n^l = \eta(k) \).

Proof. It follows from Lemma 5.4(i)(iv) that \( k_- < B_n^l < B_n^u < k_+ \) for all \( n \in \mathbb{N} \).

Let

\[
B_n^l = \liminf_{n \to \infty} B_n^l, \quad B_n^l = \limsup_{n \to \infty} B_n^l, \quad B_n^u = \liminf_{n \to \infty} B_n^u, \quad B_n^u = \limsup_{n \to \infty} B_n^u.
\]

Obviously, we have \( B_n^l \leq B_n^l \leq B_n^u \). It follows from (28) that

\[
B_n^l \leq \tilde{X}_2(B_n^l, B_n^l, k), \quad B_n^l \geq \tilde{X}_2(B_n^u, B_n^l, k),
\]

and hence that

\[
\mathcal{F}(B_n^u, B_n^l, B_n^l, B_n^l, k) \geq 0 \geq \mathcal{F}(B_n^l, B_n^u, B_n^l, k),
\]

that is,

\[
\hat{f}(B_n^u, B_n^l, k) \geq \hat{f}(B_n^l, B_n^l, k) \geq 0 \geq \hat{f}(B_n^u, B_n^l, k) \geq \hat{f}(B_n^l, B_n^l, k).
\]

Thus, we have \( B_n^l \leq \eta(k) \leq B_n^u \). Therefore, \( B_n^l = B_n^l = B_n^u = B_n^u = \eta(k) \). This completes the proof of this lemma.

The following result is trivial and the proof can be found in Chen [6].

Lemma 5.7. Consider the following logistic equation

\[
v'(t) = v(t)(a - bv(t)), \quad b > 0.
\]

Then, \( \lim_{t \to \infty} v(t) = a/b \) if \( a > 0 \) and \( \lim_{t \to \infty} v(t) = 0 \) if \( a \leq 0 \).

The other auxiliary system is

\[
\begin{align*}
X_1'(t) &= \alpha X_2 - \gamma X_1 - \alpha e^{-\tau(t)}(X_1 + X_2) X_2(t - \tau(t) + X_2)), \\
X_2'(t) &= \alpha e^{-\tau(t)}(X_1 + X_2) X_2(t - \tau(t) + X_2) - \beta X_2^2 - ka X_2,
\end{align*}
\]

where \( \gamma \) are positive constants, \( k \geq 0 \), and function \( \tau(\cdot) \) is the same to that of system (4). We have the following result on the existence, uniqueness, and global attractivity of a positive equilibrium point of system (30).

Lemma 5.8. System (30) has a positive equilibrium point \((\zeta(k), \eta(k))\), where \( \eta(k) \in (k_-, k_+) \) satisfies

\[
\hat{f}(\eta(k), \eta(k), k) = 0 \quad \text{and} \quad \zeta(k) = [\alpha - \beta \eta(k)]\eta(k)/\gamma.
\]

Assume further that (29) holds, then the positive equilibrium point \((\zeta(k), \eta(k))\) is unique and attracts all of the positive solutions of system (30). In particular, \( \zeta(0) = X_{10} \) and \( \eta(0) = X_{20} \).

Proof. It follows from the proof of Lemma 5.4 that we can obtain the existence and uniqueness of the positive equilibrium point \((\zeta(k), \eta(k))\). In what follows, we only need to prove the global attractivity of the positive equilibrium point \((\zeta(k), \eta(k))\).

Using a similar argument as that of [3], we see that

\[
k_- < \liminf_{t \to \infty} X_2(t) \leq \limsup_{t \to \infty} X_2(t) < k_+,
\]

then for all large enough \( t \), we have

\[
\begin{align*}
X_1'(t) &> \alpha X_2 - \gamma X_1 - \alpha k_+ e^{-\gamma(t)}(X_1 + X_2), \\
X_2'(t) &> \alpha k_- e^{-\gamma(t)}(X_1 + X_2) - \beta X_2^2 - ka X_2,
\end{align*}
\]

\[
\begin{align*}
X_1'(t) &< \alpha X_2 - \gamma X_1 - \alpha k_- e^{-\gamma(t)}(X_1 + X_2), \\
X_2'(t) &< \alpha k_+ e^{-\gamma(t)}(X_1 + X_2) - \beta X_2^2 - ka X_2.
\end{align*}
\]
Thus, we have
\[ x(t) < X_1(t) < \bar{x}(t), \quad y(t) < X_2(t) < \bar{y}(t), \quad (31) \]
where \((\bar{x}, \bar{y}, \bar{z}, \bar{y})\) is a solution to the following system
\[
\begin{align*}
\dot{x}(t) &= \alpha y - \gamma x - \alpha k_+ e^{-\gamma \tau(x+y)}, \\
\dot{y}(t) &= \alpha k_+ e^{-\gamma \tau(x+y)} - \beta y^2 - k_+ y, \\
\dot{z}(t) &= \alpha \bar{y} - \gamma \bar{x} - \alpha k_- e^{-\gamma \tau(x+y)}, \\
\dot{y}(t) &= \alpha k_+ e^{-\gamma \tau(x+y)} - \beta \bar{y}^2 - k_\bar{y}.
\end{align*}
\]
It follows from (5.3) that
\[
\lim_{t \to \infty} (\bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{y}(t)) = (A_1, B_1, A_1', B_1'),
\]
where
\[
A_1 = \bar{X}_1(k_-, k_+, k), \quad B_1 = \bar{X}_2(k_-, k_+, k), \\
A_1' = \bar{X}_1(k_+, k_-, k), \quad B_1' = \bar{X}_2(k_+, k_-, k).
\]
It follows from (31) and Lemma 5.4 that
\[ B_1^l < \liminf_{t \to \infty} X_2(t) \leq \limsup_{t \to \infty} X_2(t) < B_1^u, \quad A_1' < \liminf_{t \to \infty} X_1(t) \leq \limsup_{t \to \infty} X_1(t) < A_1. \]
and
\[ k_- < B_1^l < \eta(k) < B_1^u < k_+. \]
This process can be continued to construct four sequences \(\{A_n^u\}_{n=1}^{\infty}, \{B_n^u\}_{n=1}^{\infty}, \{A_n^l\}_{n=1}^{\infty}, \{B_n^l\}_{n=1}^{\infty}\), as follows:
\[
\begin{align*}
A_n^u &= \bar{X}_1(B_{n-1}^l, B_{n-1}^u, k), \quad B_n^u = \bar{X}_2(B_{n-1}^l, B_{n-1}^u, k), \\
A_n^l &= \bar{X}_1(B_{n-1}^u, B_{n-1}^l, k), \quad B_n^l = \bar{X}_2(B_{n-1}^u, B_{n-1}^l, k).
\end{align*}
\]
It follows from Lemma 5.6 that
\[ \liminf_{t \to \infty} X_2(t) = \limsup_{t \to \infty} X_2(t) = \eta(k), \quad \liminf_{t \to \infty} X_1(t) = \limsup_{t \to \infty} X_1(t) = \zeta(k). \]
Thus, the positive equilibrium point \((\zeta(k), \eta(k))\) attracts all of the positive solutions of system (30). This completes the proof of this lemma. \(\square\)

6. **Global asymptotical stability.** In this section, we shall investigate the global asymptotical stability of the positive equilibrium \(E^*(X_1^*, X_2^*, Y^*)\) and the positive boundary equilibrium \(E_1(X_{10}, X_{20}, 0)\) of (4). Theorem 3.2 says that system (4) has exactly one positive interior equilibrium \(E^*\) if \(\beta r < \alpha c e^{-\gamma M}\). In Theorem 4.2, we see that this equilibrium \(E^*\) is locally asymptotically stable. Furthermore, the following theorem says that this equilibrium \(E^*\) is also globally asymptotically stable if we further assume that the derivative of the delay function is small enough.

**Theorem 6.1.** Assume that \(\beta r < \alpha c e^{-\gamma M}\) and \(\alpha e^{-\gamma \tau(x_\tau)} \tau'(X_\tau^+) X_\tau^+ < \frac{1}{2}\), then positive equilibrium \(E^*(X_1^*, X_2^*, Y^*)\) is globally asymptotically stable.

**Proof.** It follows from \(\beta r < \alpha c e^{-\gamma M}\) that \(X_\tau^+ > X_\tau^- > r/c\). If \(\alpha e^{-\gamma \tau(x_\tau)} \tau'(X_\tau^-) X_\tau^+ < \frac{1}{2}\) then there exists \(\sigma \in (0, X_\tau^-)\) such that
\[
\alpha e^{-\gamma \tau(x_\tau^+)} \tau'(X_\tau^- - \epsilon)(X_\tau^+ + \epsilon) < \frac{1}{2}
\]
for all $\varepsilon \in (0, \sigma)$. Furthermore, it follows from Theorems 3.2 and 4.2 that $E^*$ is unique and locally asymptotically stable. So we only need to prove the global attractivity of $E^*$. For each $i = 1, 2$, let
\[
X_i^u = \limsup_{t \to \infty} X_i(t), \quad X_i^l = \liminf_{t \to \infty} X_i(t),
\]
and
\[
Y^u = \limsup_{t \to \infty} Y(t), \quad Y^l = \liminf_{t \to \infty} Y(t).
\]
In view of Lemma 5.1, we obtain $X_1^- \leq X_2^l \leq X_2^u \leq X_1^+$. For each $\varepsilon \in (0, \sigma)$, there exists $t_0 > \tau_M$ such that $X_2(t) < X_2^+ + \varepsilon$ for all $t > t_0$. Thus, we have
\[
\dot{Y}(t) < Y(t) \left[ \gamma - c(X_0^+ + \varepsilon) - b\dot{Y}(t) \right].
\]
In view of Lemma 5.7, we have $Y^u \leq \chi(X_2^+ + \varepsilon)$. Since this is true for any $\varepsilon \in (0, \sigma)$, it follows that $Y^u \leq N_{0,1}$, where $N_{0,1} = \chi(X_2^+)$.

It follows from Lemma 5.5 that $Y^* \leq N_{0,1}$.

Thus, there exists $t_1 > t_0 + \tau_M$ such that for all $t > t_1$,
\[
\dot{X}_1(t) = \alpha X_2(t) - \gamma X_1(t) - \alpha e^{-\gamma t} X_2(t - \tau(X)),
\]
\[
\dot{X}_2(t) = \alpha e^{-\gamma t} X_2(t - \tau(X)) - \beta X_2^2(t) - a X_2(t)(N_{0,1} + \varepsilon).
\]

By comparison principle, $X_2(t) \geq y(t)$ and $X_1(t) \geq x(t)$ for all $t > t_1$, where $(x(t), y(t))$ is the solution to the following equations with the initial values $x(t_1) = X_1(t_1)$ and $y(t) = \max\{X_2(t) | t \in [t_1 - \tau_M, t_1]\}$ for all $t \in [t_1 - \tau_M, t_1]$,
\[
\dot{x}(t) = ay(t) - \gamma x(t) - \alpha e^{-\gamma(t+x+y)} y(t - \tau(x+y)),
\]
\[
\dot{y}(t) = \alpha e^{-\gamma(t+x+y)} y(t - \tau(x+y)) - \beta y^2(t) - a y(t)(N_{0,1} + \varepsilon).
\]

Thus, it follows from Lemma 5.8 that $X_1^l \geq \zeta(N_0, 1 + \varepsilon)$ and $X_2^l \geq \eta(N_0, 1 + \varepsilon)$. Since this is true for any $\varepsilon \in (0, \sigma)$, it follows that $X_1^l \geq N_{1,1}$ and $X_2^l \geq N_{2,1}$.

It follows from Lemma 5.5 that
\[
0 < N_{2,1} < X_2^+.
\]
On the other hand, there exists $t_2 > t_1 + \tau_M$ such that for all $t > t_2$,
\[
\dot{Y}(t) > Y(t) \left[ -r + c(N_{2,1} - \varepsilon) - b\dot{Y}(t) \right].
\]
It follows that $Y^l \geq M_{0,1}$, where $M_{0,1} = \chi(N_{2,1})$. Similarly, it follows from Lemmas 5.7 and 5.5 that $Y^* > M_{0,1}$.

Furthermore, there exists $t_3 > t_2 + \tau_M$ such that for all $t > t_3$,
\[
\dot{X}_1(t) = \alpha X_2(t) - \gamma X_1(t) - \alpha e^{-\gamma t} X_2(t - \tau(X)),
\]
\[
\dot{X}_2(t) < \alpha e^{-\gamma t} X_2(t - \tau(X)) - \beta X_2^2(t) - a X_2(t)(M_{0,1} - \varepsilon).
\]

Also, it follows from Lemma 5.8 that $X_1^u \leq \zeta(M_{0,1} + \varepsilon)$ and $X_2^u \leq \eta(M_{0,1} + \varepsilon)$. Since this is true for any $\varepsilon \in (0, \sigma)$, it follows that $X_1^u \leq M_{1,1} \triangleq \zeta(M_{0,1})$ and $X_2^u \leq M_{2,1} \triangleq \eta(M_{0,1})$. It follows from Lemma 5.5 that
\[
X_2^* < M_{2,1} < X_0^+.
\]

For each $\varepsilon \in (0, \sigma)$, then it follows from (32) that there exists $t_4 > t_3$ such that $X_2(t) < M_{2,1} + \varepsilon$ for all $t > t_4$.

Thus,
\[
\dot{Y}(t) < Y(t) \left[ -r + c(M_{2,1} + \varepsilon) - b\dot{Y}(t) \right].
\]
Again, by Lemmas 5.7 and 5.5, 
\[ N_{0,1} > N_{0,2} > Y^*, \]
where \( N_{0,2} = \chi(M_{2,1}) \).

This process can be continued to construct six sequences \( \{N_{j,n}\}_{n=1}^{\infty} \) and \( \{M_{j,n}\}_{n=1}^{\infty} \) \((j \in \{0, 1, 2\})\) as follows
\[
N_{0,n} = \chi(M_{2,n-1}), \quad N_{1,n} = \zeta(N_{0,n}), \quad N_{2,n} = \eta(N_{0,n}), \\
M_{0,n} = \chi(M_{2,n}), \quad M_{1,n} = \zeta(M_{0,n}), \quad M_{2,n} = \eta(M_{0,n}), \\
\]
such that
\[
N_{1,n} \leq X_{1}^{l} \leq X_{1}^{u} \leq M_{1,n}, \quad N_{2,n} \leq X_{2}^{l} \leq X_{2}^{u} \leq M_{2,n}, \quad N_{0,n} \leq Y^{l} \leq Y^{u} \leq M_{0,n}.
\]

It follows from Lemma 5.5 that
\[
N_{0,n} > N_{0,0} > Y^*, \\
N_{2,n} < N_{2,0} < X_{2}^{*}, \\
M_{0,n} < M_{0,0} < Y^*, \\
M_{2,n} > M_{2,0} > X_{2}^{*}.
\]

Hence, both \( \{N_{0,n}\}_{n=1}^{\infty} \) and \( \{M_{2,n}\}_{n=1}^{\infty} \) are monotonically decreasing and bounded, both \( \{N_{2,n}\}_{n=1}^{\infty} \) and \( \{M_{0,n}\}_{n=1}^{\infty} \) are monotonically increasing and bounded. It follows that \( N_0 = \lim_{n \to \infty} N_{0,n}, \quad N_2 = \lim_{n \to \infty} N_{2,n}, \quad M_0 = \lim_{n \to \infty} M_{0,n}, \quad M_2 = \lim_{n \to \infty} M_{2,n} \). By continuity of \( \chi(x) \) and \( \eta(k) \), we have \( N_0 = \chi(M_2), \quad N_2 = \eta(N_0), \quad M_0 = \chi(N_2), \quad M_2 = \eta(M_0) \), which, together with Lemma 5.5, implies that
\[
N_0 = M_0 = Y^*, \quad N_2 = M_2 = X_2^*.
\]

It follows from (33) and (34) that
\[
\lim_{n \to \infty} N_{i,n} = \lim_{n \to \infty} M_{i,n} = X_i^*, \quad \lim_{n \to \infty} N_{0,n} = \lim_{n \to \infty} M_{0,n} = Y^*, \quad i = 1, 2.
\]

Therefore,
\[
\liminf_{t \to \infty} X_i(t) = \sup_{t \to \infty} X_i(t) = X_i^*, \quad \liminf_{t \to \infty} Y(t) = \sup_{t \to \infty} Y(t) = Y^*, \quad i = 1, 2,
\]
that is,
\[
\lim_{t \to \infty} X_i(t) = X_i^*, \quad \lim_{t \to \infty} Y(t) = Y^*, \quad i = 1, 2.
\]
This completes the proof. \(\Box\)

In what follows, we investigate the global asymptotical stability of the positive boundary equilibrium \( E_1(X_{10}, X_{20}, 0) \) of (4).

**Theorem 6.2.** Assume that \( \beta r > \alpha c e^{-\gamma r} \) and \( \alpha e^{-\gamma r}(\bar{X}^0)\sigma'(X^0)X^0 < \frac{1}{2} \), then positive boundary equilibrium \( E_1(X_{10}, X_{20}, 0) \) is globally asymptotically stable.

**Proof.** It follows from \( \beta r > \alpha c e^{-\gamma r} \) that \( X^0_0 < X^*_0 < r/c \). It follows from Theorems 3.3 and 4.1 that \( E_1 \) is unique and locally asymptotically stable. So we only need to prove the global attractivity of \( E_1 \). For each \( i = 1, 2 \), let
\[
X_i^* = \limsup_{t \to \infty} X_i(t), \quad X_i^l = \liminf_{t \to \infty} X_i(t),
\]
and
\[
Y^u = \limsup_{t \to \infty} Y(t).
\]
In view of Lemma 5.1, we obtain $X_0^r \leq X_2^r \leq X_2^u \leq X_0^+ < r/c$. For each $\varepsilon \in (0, \frac{r}{c} - X_0^+)$, there exists $t_0 > \tau_M$ such that $X_2(t) < X_0^+ + \varepsilon$ for all $t > t_0$. Thus, we have

$$\dot{Y}(t) < Y(t) \left[ -r + c(X_0^+ + \varepsilon) - bY(t) \right].$$

In view of Lemma 5.7, we have $Y^u = 0$ and hence

$$\lim_{t \to \infty} Y(t) = 0.$$

Thus, there exists $t_1 > t_0 + \tau_M$ such that for all $t > t_1$,

$$\dot{X}_1(t) = \alpha X_2(t) - \gamma X_1(t) - \alpha e^{-\gamma \tau(X)} X_2(t - \tau(X)),$$

$$\dot{X}_2(t) > \alpha e^{-\gamma \tau(X)} X_2(t - \tau(X)) - \beta X_2^2(t) - \varepsilon a X_2(t).$$

By comparison principle, $X_2(t) \geq y(t)$ and $X_1(t) \geq x(t)$ for all $t > t_1$, where $(x(t), y(t))$ is the solution to the following equation

$$\begin{align*}
\dot{x}(t) &= a y(t) - \gamma x(t) - \alpha e^{-\gamma \tau(x+y)} y(t - \tau(x+y)) \quad \text{in } [t_1 - \tau_M, t_1] \\
\dot{y}(t) &= \alpha e^{-\gamma \tau(x+y)} y(t - \tau(x+y)) - \beta y^2(t) - \varepsilon a y(t),
\end{align*}$$

with the initial values $x(t_1) = X_1(t_1)$ and $y(t_1) = \max\{X_2(t) \mid t \in [t_1 - \tau_M, t_1]\}$ for all $t \in [t_1 - \tau_M, t_1]$. Thus, it follows from Lemma 5.8 that $X_1^+ \geq \zeta(\varepsilon)$ and $X_2^+ \geq \eta(\varepsilon)$. Since this is true for any sufficiently small $\varepsilon$, it follows that

$$X_1^+ \geq \zeta(0) = X_{10}, \quad X_2^+ \geq \eta(0) = X_{20}. \quad (36)$$

It follows from Theorem 2.2 that

$$\begin{align*}
\dot{X}_1(t) &= \alpha X_2(t) - \gamma X_1(t) - \alpha e^{-\gamma \tau(X)} X_2(t - \tau(X)) \\
\dot{X}_2(t) &= \alpha e^{-\gamma \tau(X)} X_2(t - \tau(X)) - \beta X_2^2(t).
\end{align*}$$

Also, by comparison principle and Lemma 5.8, we get

$$X_1^u \leq \zeta(0) = X_{10}, \quad X_2^u \leq \eta(0) = X_{20}. \quad (37)$$

It follows from (36) and (37) that

$$\lim_{t \to \infty} X_i(t) = X_{i0}, \quad i = 1, 2.$$

This completes the proof.

Theorem 6.2 means that the boundary equilibrium $E_1$ is globally asymptotically stable if $\beta r > \alpha e^{-\gamma \tau_m}$ and the derivative of the delay function is small enough. It would be very interesting to see what happens to the case where $\alpha e^{-\gamma \tau_m} < \beta r \leq \alpha e^{-\gamma \tau_m}$. In this case, we have $X_0^r \leq X_0^- \leq X_0^+ \leq X_0^+$ and $X_0^- < \frac{r}{c} \leq X_0^+$. We have the following results.

**Theorem 6.3.** Assume that $\alpha e^{-\gamma \tau_m} < \beta r \leq \alpha e^{-\gamma \tau_m}$, $c X_2 < r$, and $\alpha e^{-\gamma (X_0^-)} \tau' (X_0^-) X_0^+ < \frac{r}{c}$, then positive boundary equilibrium $E_1(X_{10}, X_{20}, 0)$ is globally asymptotically stable.

**Proof.** It follows from Theorems 3.3 and 4.1 that $E_1$ is unique and locally asymptotically stable. So we only need to prove the global attractivity of $E_1$. Define $X_1^u$, $X_2^u$, $X_1^l$, $X_2^l$, and $Y^u$ as those in the proof of Theorem 6.2. Similarly, we have

$$X_1^u \leq X_{10}, \quad X_2^u \leq X_{20} < r/c. \quad (38)$$

Thus, there exists $t_0 > \tau_M$ such that $X_2(t) < r/c$ for all $t > t_0$. Thus, we have $\dot{Y}(t) < -b Y^2(t)$. In view of Lemma 5.7, we have $Y^u = 0$ and hence

$$\lim_{t \to \infty} Y(t) = 0. \quad (39)$$
Thus, for every given \( \varepsilon \) there exists \( t_1 > t_0 + \tau_M \) such that \( 0 < Y(t) < \varepsilon \) for all \( t > t_1 \), and hence that for all \( t > t_1 \),

\[
\begin{align*}
\dot{X}_1(t) &= \alpha X_2(t) - \gamma X_1(t) - \alpha e^{-\gamma \tau(X)} X_2(t - \tau(X)), \\
\dot{X}_2(t) &= \alpha e^{-\gamma \tau(X)} X_2(t - \tau(X)) - \beta X_2^2(t) - \varepsilon aX_2(t).
\end{align*}
\]

Using a similar argument as the proof of Theorem 6.2, we have

\[
X_1^i \geq \zeta(0) = X_{10}, \quad X_2^i \geq \eta(0) = X_{20}. \tag{40}
\]

It follows from (38) and (40) that \( \lim_{t \to \infty} X_i(t) = X_{i0}, \, i = 1, 2 \) This completes the proof.

Theorems 6.2 and 6.3 means that if the derivative of the delay function is small enough then the local asymptotical stability of the boundary equilibrium \( E_1 \) implies that it is also globally asymptotically stable. It is easy to see that \( \{(X_1, X_2, 0) \mid X_1 \geq 0, \, X_2 \geq 0\} \) is a positively invariant set of system (4), that is to say, every trajectory of system (4) with the initial conditions in \( \{(X_1, X_2, 0) \mid X_1 \geq 0, \, X_2 \geq 0\} \) always stays in \( \{(X_1, X_2, 0) \mid X_1 \geq 0, \, X_2 \geq 0\} \). Furthermore, using a similar argument as the proof of Theorem 6.2, we see that \( E_1 \) attracts every solution with initial conditions in \( \{(X_1, X_2, 0) \mid X_1 \geq 0, \, X_2 \geq 0\} \). Thus, we conjecture that some solutions of system (4) can fluctuate around the positive boundary equilibrium \( E_1 \) when \( E_1 \) is unstable.

Our theoretical results are of course applicable to system (4) with constant delay. In view of Theorems 3.2, 6.1, 4.1, and 6.2, we have the following results.

**Corollary 3.**

(i): If \( \beta r < \alpha c e^{-\gamma \tau} \) then system (4) with \( \tau(\cdot) \equiv \tau \) has exactly one positive interior equilibrium \( E^* \), which is globally asymptotically stable.

(ii): System (4) with \( \tau(\cdot) \equiv \tau \) has exactly one positive boundary equilibrium \( E_1 \), which is globally asymptotically stable if \( \beta r > \alpha c e^{-\gamma \tau} \) and unstable if \( \beta r < \alpha c e^{-\gamma \tau} \).

In view of Corollary 3, we see that system (4) with \( \tau(\cdot) \equiv \tau \) undergoes transcritical bifurcation at \( \beta r = \alpha c e^{-\gamma \tau} \). Namely, as \( \beta r - \alpha c e^{-\gamma \tau} \) increases and passes through 0, the two equilibria \( E^* \) and \( E_1 \) of system (4) with \( \tau(\cdot) \equiv \tau \) collide and exchange their stability. However, when \( \beta r > \alpha c e^{-\gamma \tau} \), the third component of \( E^* \) of system (4) with \( \tau(\cdot) \equiv \tau \) is negative and hence this kind of equilibria should be ignored. In view of Corollary 3, it is important to find that the local asymptotical stability of each nonnegative equilibrium (either \( E^* \) or \( E_1 \)) implies that it is also globally asymptotically stable.

7. Conclusions and simulations. The mathematical model we have proposed consists of three nonlinear ordinary differential equations, corresponding to an immature population, a mature population, and their predator. The predator feed only on the mature prey. We have established the stability conditions for nonnegative equilibria. It is also observed that the system is locally asymptotically stable around the positive interior equilibrium \( E^* \) (respectively, the positive boundary equilibrium \( E_1 \) if \( \beta r < \alpha c e^{-\gamma \tau} \)) if \( \beta r < \alpha c e^{-\gamma \tau} \) (respectively, \( c X_{20} < r \)). In particular, if the derivative of the delay function is small enough, then the local asymptotical stability of either of the positive interior equilibrium \( E^* \) and the positive boundary equilibrium \( E_1 \) implies that it is also globally asymptotically stable. Let us now give some numerical simulations to illustrate our theoretical results. Consider system (4) with
\( \alpha = 2.8, \gamma = 0.3, \beta = 0.4, a = 0.5, \) and \( b = 0.1. \) Namely, we consider the following system
\[
\begin{align*}
\dot{X}_1(t) & = 2.8X_2(t) - 0.3X_1(t) - 2.8e^{-0.3\tau(X)}X_1(t - \tau(X)) , \\
\dot{X}_2(t) & = 2.8e^{-0.3\tau(X)}X_2(t - \tau(X)) - 0.4X_2(t) - 0.5X_2(t)Y(t) , \\
\dot{Y}(t) & = Y(t) \left[ -r + cX_2(t) - 0.1Y(t) \right] , \\
X_1(0) & > 0, \quad Y(0) > 0, \quad X_2(t) = \varphi(t) \geq 0, \quad -\tau \leq t \leq 0 ,
\end{align*}
\]
where the delay function is \( \tau(x) = 4 - 2e^{-0.1x}. \) It is easy to check that \( \tau'(x) > 0, \) \( \tau''(x) < 0, \) \( \tau_m = 2, \) and \( \tau_M = 4. \)

We first study system (41) with \( r = 0.1 \) and \( c = 1. \) Note that
\[
\tau'(x) < 4\beta/\alpha^2, \quad \beta r < \alpha e^{-\gamma M}.
\]
Then it follows from Theorems 3.2 and 4.2 that system (41) has exactly one positive equilibrium \( E^*(X_1^*, X_2^*, Y^*) , \) which is locally asymptotically stable. In addition, we can figure out that \( 2.63 < X_1^* < 2.64, 0.44 < X_2^* < 0.45 \) and \( 3.47 < Y^* < 3.48, \) and that \( \alpha e^{-\gamma M}X_1^*X_2^* < 0.1594 < \frac{1}{2}. \) Thus, it follows from Theorem 6.1 that \( E^* \) is globally asymptotically stable, as depicted in Figure 1, from which we see that the solution \( (X_1(t), X_2(t), Y(t)) \) of system (41) with \( \tau(x) = 4 - 2e^{-0.1x} \) and \( (r, c) = (0.1, 1) \) tends to the positive equilibrium \( E^* \) as \( t \to \infty \) after some initial transients.

Consider system (41) with \( r = 0.6 \) and \( c = 0.1. \) It follows from Theorem 3.3 that system (41) has exactly one positive boundary equilibrium \( E_1(X_{10}, X_{20}, 0) \) satisfying \( 32.2 < X_{10} < 32.3 \) and \( 3.8 < X_{20} < 3.9. \) Note that \( X_{20} < r/c = 6. \) It follows from Theorem 4.1 that \( E_1 \) is locally asymptotically stable. In this case, \( \beta r > \alpha e^{-\gamma M} \) and \( \alpha e^{-\gamma M}\tau'(X_0)X_{20} < 0.0402 < \frac{1}{2}, \) which together with Theorem 6.2 implies that \( E_1 \) is globally asymptotically stable, as depicted in Figure 2, from which we see that the solution \( (X_1(t), X_2(t), Y(t)) \) of system (41) with \( \tau(x) = 4 - 2e^{-0.1x} \) and \( (r, c) = (0.6, 0.1) \) tends to the boundary equilibrium \( E_1 \) as \( t \to \infty. \)

Finally, we consider system (41) with \( \tau(x) \equiv 4 \) and \( r = 0.1, c = 1. \) Corollary 3 implies that this system has exactly a globally asymptotically stable positive equilibrium \( E^*(X_1^*, X_2^*, Y^*), \) as depicted in Figure 3, from which we see that the solution \( (X_1(t), X_2(t), Y(t)) \) of system (41) with \( \tau(x) \equiv 4 \) and \( (r, c) = (0.1, 1) \) tends to the positive equilibrium \( E^* \) as \( t \to \infty \) after some initial transients.

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Figure 1. Simulations of system (41) with $\tau(x) = 4 - 2e^{-0.1x}$ and $(r,c) = (0.1, 1)$ illustrate that the positive equilibrium is globally asymptotically stable.
Figure 2. Simulations of system (41) with $\tau(x) = 4 - 2e^{-0.1x}$ and $(r,c) = (0.6, 0.1)$ illustrate that the positive equilibrium is globally asymptotically stable.
Figure 3. Simulations of system (41) with $\tau(x) \equiv 4$ and $(r,c) = (0.1,1)$ illustrate that the positive equilibrium is globally asymptotically stable.
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