Abstract

BPS wall solutions are obtained for $\mathcal{N} = 2$ supersymmetric nonlinear sigma model with Eguchi-Hanson target manifold in a manifestly supersymmetric manner. The model is constructed by a massive hyper-Kähler quotient method both in the $\mathcal{N} = 1$ superfield and in the $\mathcal{N} = 2$ superfield (harmonic superspace). We describe the model in simple terms and give relations between various parameterizations which are useful to describe the model and the solution. Some more details can be found in our previous paper [1] [hep-th/0211103]. This article is dedicated to Professor Hiroshi Ezawa on the occasion of his seventieth birthday.
1 Introduction

Supersymmetry (SUSY) has been a most promising guiding principle to construct realistic unified models beyond the standard model. In recent years there have been vigorous studies on models with extra dimensions, where our world is assumed to be realized on an extended topological defects such as domain walls or various branes. Supersymmetry can be combined with this brane-world scenario and helps the construction of the extended topological defects.

Solitons saturating an energy bound, called the BPS bound, have played a crucial role also in non-perturbative studies of supersymmetric (SUSY) field theories in four dimensions. BPS domain walls are topological solitons of co-dimension one, which depend on one spatial coordinate and connect two SUSY vacua. Since they preserve half of the original SUSY, they are called \( \frac{1}{2} \) BPS states. Such BPS domain walls were well studied in various models with global \( \mathcal{N} = 1 \) SUSY in four dimensions. Non-BPS multi-wall solutions were also studied to understand the SUSY breaking mechanism on the brane due to the coexistence of the other brane. More recently we have constructed an exact BPS wall solution as well as non-BPS multi-wall solutions in the supergravity theory in four dimensions.

In order to consider models with extra dimensions, we need to discuss supersymmetric theories in spacetime with dimensions higher than four. They should have at least eight supercharges. The simplest field theory with eight SUSY is based on hypermultiplets containing only scalar and spinor as physical fields. Recently we have formulated \( \frac{1}{2} \) BPS domain walls in an eight SUSY model in four dimensions. Moreover we have also succeeded in constructing the \( \frac{1}{2} \) BPS wall consistently in five-dimensional supergravity. Before discussing the SUSY five-dimensional theories, it is useful to consider models with eight SUSY in four dimensions.

The rest of our paper is organized as follows. Sec. 2 explains how to obtain nonlinear sigma models of hypermultiplets with eight SUSY. Secs. 3, 4 and 5 are devoted to \( \mathcal{N} = 1 \) superfield formulation of the model. In Sec. 3, we present the model using the \( U(1) \) gauge field. We give the bosonic part of the action and eliminate auxiliary fields in the Wess-Zumino gauge. The hyper-Kähler quotient method in terms of the so-called moment map becomes very clear. In Sec. 4, we eliminate auxiliary superfields in the superfield level, taking a gauge compatible with SUSY rather than the Wess-Zumino gauge. This has the advantage because we obtain the Lagrangian in terms of independent superfields. In Sec. 5, we use the \( O(2) \) gauge field to formulate the model instead of the \( U(1) \) gauge field. Sec. 6 is devoted to a brief review of harmonic superspace formalism (HSF). In Sec. 7, we formulate the model in HSF and eliminate auxiliary fields in the Wess-Zumino gauge. In Sec. 8, the constraints are solved by independent fields. We close our paper by Sec. 9, in which the BPS equation and the domain wall solution are given.
2 \( \mathcal{N} = 2 \) Model with Hypermultiplets in 4 Dimensions

If we take two free chiral scalar supermultiplets \( \phi \) and \( \chi \) with a complex mass term \( m \) between them, they together become a free massive hypermultiplet with \( \mathcal{N} = 2 \) SUSY.\cite{18}

\[
\mathcal{L} = [\phi^* \phi + \chi^* \chi] \partial_{\bar{\theta} \theta} + (m [\chi \phi] \partial_{\bar{\theta} \theta} + \text{c.c.}) - \partial_{\mu} \phi^* \partial^{\mu} \phi - \partial_{\mu} \chi^* \partial^{\mu} \chi + F^*_\phi F_\phi + F^*_\chi F_\chi + (m (F_\phi \chi + F_\chi \phi) + \text{c.c.}) + \text{fermionic terms}
\]

where complex conjugate is denoted as c.c., and the scalar components are denoted by the same letter as the superfields\(^1\) \( \phi, \chi \). Since four real scalar fields \( \text{Re}\phi, \text{Im}\phi, \text{Re}\chi, \text{Im}\chi \) are symmetric, we can form three complex fields using any one of the fields, say \( \text{Re}\phi \) with any one of the other three fields: \( \text{Re}\phi + i\text{Re}\chi \), and \( \text{Re}\phi + i\text{Im}\chi \) beside the ordinary \( \text{Re}\phi + i\text{Im}\phi = \phi \). These three complex structures are completely symmetric and serve as a characterization of \( \mathcal{N} = 2 \) SUSY for hypermultiplets. It has been shown that any nonlinear sigma model consisting of hypermultiplets should have a triplet of complex structures, and the target manifold should be hyper-Kähler \( \text{HK} \) in contrast to Kähler of the \( \mathcal{N} = 1 \) SUSY nonlinear sigma model.\cite{21}

Theories with eight SUSY are so restrictive that the nontrivial interactions require the nonlinearity of kinetic term (nonlinear sigma model) if there are only hypermultiplets. In order to obtain a wall solution, we need to have a nontrivial potential. In the case of \( \mathcal{N} = 2 \) SUSY nonlinear sigma model containing only hypermultiplets, one can introduce a nontrivial potential which is the square of the Killing vector acting on the HK manifold multiplied by a mass parameter. Moreover the Killing vector has to be holomorphic with respect to the three complex structures (tri-holomorphic)\cite{22}. These models are called “massive HK nonlinear sigma models”.

Let us now explain a mechanism to obtain a nontrivial potential as a Sherk-Schwarz reduction\cite{23} from six or five dimensions.\cite{24} It is usually best to start from a model in spacetime with maximal dimensions which is allowed by the postulated number of SUSY charges. In the present case of eight SUSY, we should consider hypermultiplets in six dimensions. Let us first illustrate the dimensional reduction by a free massless hypermultiplet in six dimensions, since a mass term is forbidden by SUSY. If two (one) spatial dimensions are compactified, a nonvanishing momentum in these compactified dimensions gives a complex (real) mass term resulting in Eq.\( \text{(1)} \). This mass parameter gives rise to a central term \( Z = -i (\partial_5 + i \partial_6) \) in the \( \mathcal{N} = 2 \) SUSY algebra in four dimensions. In the case of a nonlinear sigma model with the target space metric \( g_{ij*} \) in six dimensions, the kinetic term (of bosonic part of the Lagrangian) reads

\[
\mathcal{L} = -g_{ij*} \partial_M \phi^i \partial_N \phi^{j*} \eta^{MN},
\]

\(^1\)We follow mostly the notation of Ref.\cite{19}, except that \( \mu, \nu, \ldots \) denote space time in four dimensions,
where $\eta^{MN} = \text{diag}(-1,1,1,1,1)$, $M,N = 0,\cdots,5$. Then we can twist the boundary condition for the compactified directions, say $x^4$ and $x^5$ using a Killing vector $k^i(\phi,\phi^*)$ for the isometry of the target space metric $g_{ij^*}$

$$-i(\partial_5+i\partial_6)\phi^i = \mu k^i(\phi,\phi^*), \quad \mu \in \mathbb{C}.$$  \tag{3}

Then we obtain a nontrivial potential term $V(\phi)$ from the Lagrangian 

$$\mathcal{L} = -g_{ij^*}\partial_\mu \phi^i\partial_\nu \phi^{j^*}\eta^{\mu\nu} - V(\phi,\phi^*),$$  \tag{4}

$$V(\phi,\phi^*) = |\mu|^2 g_{ij^*}k^i(\phi,\phi^*)k^{j^*}(\phi,\phi^*).$$  \tag{5}

This is the Sherk-Schwarz dimensional reduction. Since theories with eight SUSY have three complex structures, the Killing vector $k^i$ has to be holomorphic with respect to all three complex structures (tri-holomorphic).

Many target space metrics can be embedded in higher dimensional flat space as illustrated by a sphere embedded in three dimensions in Fig.1. The nonlinear sigma models with these target space metrics can be realized by giving a constraint on hypermultiplets with minimal kinetic terms. One of the most convenient methods to impose the constraint is to introduce a vector multiplet without a kinetic term. If we integrate the vector multiplet, it acts as a Lagrange multiplier field to produce a constraint on hypermultiplets, resulting in a curved target manifold such as the Eguchi-Hanson manifold. In this process, the mass term automatically becomes a nontrivial potential which is a square of a Killing vector corresponding to the isometry of the resulting curved target space. Since the gauge field serves to identify the gauge orbit, the introduction of the gauge field without a kinetic term gives a quotient manifold. In particular, we call the method as hyper-Kähler quotient method, when the resulting manifold is hyper-Kähler. In our SUSY case, the curved manifold is a result of the constraint coming from integrating the auxiliary fields in the gauge supermultiplet as well as from the gauge orbit quotient. Since we also have a mass term as a central extension of the SUSY algebra, this procedure is called the massive hyper-Kähler quotient method. In this way, we can understand the potential term as the square.
of the tri-holomorphic Killing vector and the mass parameter multiplying the potential as the central extension of the $\mathcal{N} = 2$ SUSY algebra in four dimensions.

There have been a number of works to study nonlinear sigma models with eight supercharges. The massive nonlinear sigma model with nontrivial Kähler metric as target space was studied, and BPS equations and BPS solutions such as walls and junctions were obtained. Multi domain walls solution was also obtained and the dynamics of those walls was examined. Single or parallel domain walls in such models preserve $\frac{1}{2}$ SUSY, whereas their intersections preserve $\frac{1}{4}$ SUSY. In most papers, nonlinear sigma models were studied in terms of component fields. However, it is often useful to maintain as much SUSY as possible. Harmonic superspace formalism (HSF) is most suited to maintain the SUSY maximally, but there has been relatively few attempt to formulate the BPS equations and to obtain BPS solutions in the HSF until our recent work.

Our nonlinear sigma model can most easily be obtained by a quotient method in terms of a $U(1)$ vector multiplet without a kinetic term. In $\mathcal{N} = 1$ formalism, the massless HK sigma model on $T^*\mathbb{C}P^n$ was obtained as the HK quotient. The massive IK quotient was obtained in component level. The massless model with the Eguchi-Hanson target manifold ($T^*\mathbb{C}P^1$) has been constructed in $\mathcal{N} = 2$ formalism and its central extension was analysed. In order to obtain also a potential term, we need to perform a quotient method for a massive hypermultiplet charged under the $U(1)$ vector multiplet. When we are writing this up this work, another interesting work appeared discussing various wall and flux tube solutions in similar $\mathcal{N} = 2$ models.

3 Eguchi-Hanson Nonlinear Sigma Model in $\mathcal{N} = 1$ Superfields in the $U(1)$ Basis

The massive hyper-Kähler quotient method for the massive Eguchi-Hanson nonlinear sigma model requires two hypermultiplets: $\phi, \chi$ as doubles. The doublet $\phi(\chi)$ has charge $+1(-1)$ under an $\mathcal{N} = 2$ $U(1)$ vector multiplet $(V, \Sigma)$ without a kinetic term which serves as a Lagrange multiplier constraining hypermultiplets to form a four-dimensional (in terms of number of real degrees of freedom) target manifold. Here $V$ and $\Sigma$ are vector and chiral superfields in $\mathcal{N} = 1$ superfield formalism, respectively. Representing two doubles $\phi, \chi$ as column vectors, the action is given in terms of $\mathcal{N} = 1$ superfields as

$$L = \left[ e^V \phi^\dagger \phi + e^{-V} \chi^\dagger \chi - e^V \right]_{\bar{a}\bar{b}}^a + \left( \left[ \Sigma (\chi^T \phi - b) + \frac{\mu}{2} \chi^T \sigma_3 \phi \right]_{\bar{a}\bar{b}} + \text{c.c.} \right),$$

where we have absorbed a common mass of hypermultiplets into the field $\Sigma$ and denote $\mu$ as a complex parameter for the mass splitting. The electric and magnetic Fayet-Iliopoulos (FI) parameters are denoted as $c \in \mathbb{R}, b \in \mathbb{C}$. We see below that these parameters become the value of the triplet of the moment map for the $U(1)$ gauge symmetry.
In the limit of $\mu = 0$, the model has a global (flavor) symmetry $U(2) = SU(2) \times U(1)_A$ defined by
$$\phi \to \phi' = g\phi, \quad \chi \to \chi' = g^*\chi, \quad \Sigma \to \Sigma' = \Sigma, \quad g \in U(2),$$
(7)
and, in the case of $b = 0$, it has the additional $U(1)_D$ symmetry
$$\phi \to \phi' = e^{i\theta_1}\phi, \quad \chi \to \chi' = e^{i\theta_1}\chi, \quad \Sigma \to \Sigma' = e^{-2i\theta_1}\Sigma.$$
(8)
The $U(2)$ symmetry is consistent with the $\mathcal{N} = 2$ SUSY, but the $U(1)_D$ symmetry is only with the manifest $\mathcal{N} = 1$ SUSY. The mass splitting parameter breaks this $U(2)[\times U(1)_D]$ global symmetry (for $b = 0$) down to $U(1) \times U(1) \in U(2)$ defined by $\phi \to e^{i\theta_1} + i\theta_2\sigma_3\phi, \quad \chi \to e^{-i\theta_1} - i\theta_2\sigma_3\chi$, with $\Sigma$ unchanged. The $U(1)$ subgroup parametrized by $\theta_1$ is gauged. Since this mass splitting parameter affects only the potential term without affecting the kinetic term, the curved target manifold has Killing vectors for the isometry $SU(2)[\times U(1)_D]$ for $b = 0$. The $U(1)_A$ symmetry of $U(2) = SU(2) \times U(1)_A$ is gauged away and is absent in the target manifold. However, only the Killing vectors for the isometry $SU(2)$ are consistent with the $\mathcal{N} = 2$ SUSY and hence tri-holomorphic. The Killing vector for the $U(1)_D$ isometry for $b = 0$ is holomorphic, but not tri-holomorphic. Since we have introduced the mass parameter through the $\sigma_3$ generator, we will eventually obtain the potential term which is a square of the tri-holomorphic Killing vector corresponding to $\sigma_3$, after eliminating the vector multiplet.

In the Wess-Zumino gauge, the bosonic action is given by
$$\mathcal{L}_{\text{boson}} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{constr}} + \mathcal{L}_{\text{pot}},$$
(9)
where
$$\mathcal{L}_{\text{kin}} = \left|\left(\partial_\mu + \frac{i}{2}v_\mu\right)\phi\right|^2 - \left|\left(\partial_\mu - \frac{i}{2}v_\mu\right)\chi\right|^2,$$
$$= -|\partial_\mu\phi|^2 - |\partial_\mu\chi|^2 + i\frac{\partial^\mu}{\partial^{\mu}}\left(\phi^{\dagger}\partial_\mu\phi - \chi^{\dagger}\partial_\mu\chi\right) - \frac{1}{4}v^\mu v_\mu \left(\phi^{\dagger}\phi + \chi^{\dagger}\chi\right),$$
(10)
where $\phi^{\dagger}\partial_\mu\phi \equiv \phi^{\dagger}\left(\partial_\mu\phi\right) - \left(\partial_\mu\phi^{\dagger}\right)\phi$.

$$\mathcal{L}_{\text{constr}} = \frac{D}{2} \left(\phi^{\dagger}\phi - \chi^{\dagger}\chi - c\right) + \left(F_\Sigma \left(\chi^T \phi - b\right) + c.c.ight),$$
(11)
$$\mathcal{L}_{\text{pot}} = F_\phi^{\dagger}F_\phi + F_\chi^{\dagger}F_\chi + \left(F_\Sigma^T \left(\Sigma 1 + \frac{\mu}{2}\sigma_3\right) \phi + \chi^T \left(\Sigma 1 + \frac{\mu}{2}\sigma_3\right) F_\phi + c.c.\right)$$
$$\equiv -V(\phi, \chi, \Sigma).$$
(12)
The equation of motion for gauge field $v_\mu$ in (11) allows us to write gauge field $v_\mu$ in terms of scalar fields
$$v_\mu = i \frac{\left(\phi^{\dagger}\partial_\mu\phi - \chi^{\dagger}\partial_\mu\chi\right)}{\phi^{\dagger}\phi + \chi^{\dagger}\chi}. $$
(13)
If we eliminate the gauge field by this algebraic equation of motion, we obtain the kinetic term for hypermultiplets as

$$L_{\text{kin}} = -|\partial_\mu \phi|^2 - |\partial_\mu \chi|^2 + \frac{i \left( \phi^\dagger \partial_\mu \phi - \chi^\dagger \partial_\mu \chi \right)^2}{4(\phi^\dagger \phi + \chi^\dagger \chi)}.$$  \hspace{1cm} (14)

If we integrate the Lagrange multiplier fields $D$ and $F_{\Sigma}$ in (11), we obtain two constraints

$$\phi^\dagger \phi - \chi^\dagger \chi = c, \quad \chi^T \phi = b.$$  \hspace{1cm} (15)

The Lagrangian (12) gives algebraic equations of motion for the auxiliary fields $F_\phi, F_\chi$:

$$F_\phi^\dagger = -\chi^T \left( \Sigma \mathbf{1} + \frac{\mu}{2} \sigma_3 \right), \quad F_\chi^\dagger = -\phi^T \left( \Sigma \mathbf{1} + \frac{\mu}{2} \sigma_3 \right).$$  \hspace{1cm} (16)

After eliminating the auxiliary fields $F_\phi, F_\chi$ by these algebraic equations of motion, we obtain the potential term

$$V(\phi, \chi, \Sigma) = |F_\phi|^2 + |F_\chi|^2 = \left| \left( \Sigma \mathbf{1} + \frac{\mu}{2} \sigma_3 \right) \phi \right|^2 + \left| \left( \Sigma \mathbf{1} + \frac{\mu}{2} \sigma_3 \right) \chi \right|^2 = \left( \frac{\mu}{2} \right)^2 \left| \phi \right|^2 + \left| \chi \right|^2 + \left( \frac{\mu}{2} \Sigma + \frac{\mu}{2} * \Sigma \right) \left( \phi^\dagger \sigma_3 \phi + \chi^\dagger \sigma_3 \chi \right)^2 = \left( \frac{\mu}{2} \right)^2 \left[ \left| \phi \right|^2 + \left| \chi \right|^2 - \frac{\left( \phi^\dagger \sigma_3 \phi + \chi^\dagger \sigma_3 \chi \right)^2}{\left| \phi \right|^2 + \left| \chi \right|^2} \right].$$  \hspace{1cm} (17)

In the last line, we have eliminated the scalar field $\Sigma$ in the $\mathcal{N} = 2$ vector multiplet $(V, \Sigma)$ by its algebraic equation of motion

$$\Sigma = -\frac{\mu}{2} \frac{\phi^\dagger \sigma_3 \phi + \chi^\dagger \sigma_3 \chi}{\left| \phi \right|^2 + \left| \chi \right|^2}.$$  \hspace{1cm} (18)

In Eq. (15), the left hand sides constitute the triplet of the moment map (Killing potential) for the $U(1)$ gauge symmetry. Hence we see that these values are fixed to the FI parameters by integrating the auxiliary fields $D$ and $F_{\Sigma}$, and that the hyper-Kähler quotient is obtained together with the $U(1)$ quotient (13). In the limit of $b = c = 0$ the singularity appears and the manifold becomes the orbifold $\mathbb{C}^2/\mathbb{Z}_2$, whereas the non-zero values of $b$ and $c$ resolve the orbifold singularity through the deformation of the complex structure and the blow up, respectively.

4 Eguchi-Hanson Nonlinear Sigma Model After Integrating Vector Multiplet in the $U(1)$ Basis

Instead of taking the Wess-Zumino gauge in the component level, we can eliminate the auxiliary superfields $V$ and $\Sigma$ directly in the superfield formalism. Their equations of motion read from Eq. (6)
as
\[
\frac{\partial L}{\partial V} = e^V |\phi|^2 - e^{-V} |\chi|^2 - c = 0 ,
\]
(19)
\[
\frac{\partial L}{\partial \Sigma} = \chi^T \phi - b = 0 ,
\]
(20)
in which \( V \) can be solved immediately to give
\[
e^V = \frac{(c \pm \sqrt{c^2 + 4|\phi|^2|\chi|^2})/2|\phi|^2}.
\]
We thus obtain the Kähler potential
\[
K = c\sqrt{1 + \frac{4}{c^2}|\phi|^2|\chi|^2} - c \log \left(1 + \frac{1}{c^2} \sqrt{1 + \frac{4}{c^2}|\phi|^2|\chi|^2}\right) + c \log |\phi|^2 ,
\]
(21)
where we have chosen the plus sign of the solution for the positivity of the metric.

Fixing the complexified \( U(1) \) gauge symmetry and solving \( 20 \), we can obtain the Lagrangian of the nonlinear sigma model in terms of independent superfields. We have presented some gauge fixing \( 1 \) applied to \( T^* \mathbf{C}P^n \) for general \( n \). Here we give a more symmetric gauge for \( T^* \mathbf{C}P^1 \) in the case of \( b = 0 \). In this case, we can fix the gauge as \( \chi_1/\phi_2 = 1 \), and \( 20 \) can be solved as \( \phi^T = (x, y) \) and \( \chi^T = (y, -x) \) (and hence \( |\phi|^2 = |\chi|^2 = |x|^2 + |y|^2 \)). The Kähler potential becomes \( 16 \)
\[
K = c\sqrt{1 + \frac{4}{c^2}|\phi|^4} - c \log \left(1 + \frac{1}{c^2} \sqrt{1 + \frac{4}{c^2}|\phi|^4}\right) ,
\]
(22)
and the superpotential is given by
\[
W = \mu xy .
\]
(23)
The metric and its inverse can be calculated to give
\[
g_{ij} = \frac{c}{|\phi|^4} \sqrt{1 + \frac{4}{c^2}|\phi|^4} \begin{pmatrix} |y|^2 + \frac{4}{c^2}|\phi|^6 & -x^* y \\ -y^* x & |x|^2 + \frac{4}{c^2}|\phi|^6 \end{pmatrix} ,
\]
\[
g^{ij} = \frac{c}{4|\phi|^4} \sqrt{1 + \frac{4}{c^2}|\phi|^4} \begin{pmatrix} |x|^2 + \frac{4}{c^2}|\phi|^6 & y^* x \\ x^* y & |y|^2 + \frac{4}{c^2}|\phi|^6 \end{pmatrix} ,
\]
(24)
where \( g_{ij} = \partial^2 K/\partial \phi^i \partial \phi^j \) and \( \phi^i = (x, y) \). The scalar potential can be calculated as
\[
V = g^{ij*} \partial_i W \partial_j W^* = \frac{c|\mu|^2}{|\phi|^4} \sqrt{1 + \frac{4}{c^2}|\phi|^4} \left( \frac{1}{c^2} |\phi|^8 + |x|^2 |y|^2 \right) .
\]
(25)

The manifold admits the tri-holomorphic isometry \( SU(2) \), defined in Eq. 7.\(^2\) The Killing vectors

\(^2\)The \( SU(2) \) transformation law of \( \phi^i \) as the coordinates of the quotient target manifold is unchanged from the one in Eq. 8 and hence is still \( linear \), because the gauge fixing condition is invariant under the \( SU(2) \) action. This is the advantage of our gauge fixing condition. To define the \( SU(2) \) action in the cases of the other gauge condition \( \Pi \), we need an appropriate \( U(1) \) gauge action to compensate the variation of the gauge condition, which makes the transformation law of \( \phi^i \) \( nonlinear \).

The diagonal \( U(1)_D \) isometry defined in Eq. 5 of \( U(2) \times U(1)_D = SU(2) \times U(1)_A \times U(1)_D \) isometry is holomorphic but not tri-holomorphic.
for this action $k_A^j = \frac{1}{\epsilon} \delta_c \phi^a = \frac{i}{2} (\sigma_A)^{ij} \phi^j$ \((A = 1, 2, 3)\) are given as

$$\left( k_1, k_2, k_3 \right) = \frac{i}{2} \begin{pmatrix} y \\ x \end{pmatrix}, \begin{pmatrix} -iy \\ ix \end{pmatrix}, \begin{pmatrix} x \\ -y \end{pmatrix}. \tag{26}$$

The Killing potential $D_A(\phi, \phi^*)$ for these vectors, defined by $k_A^j = ig^{ij*} \partial_j D_A$, are given as

$$D_1, D_2, D_3 = \frac{c}{2} \sqrt{\frac{1 + \frac{4}{3} |\phi|^4}{|\phi|^2}} \left( xy^* + yx^*, i(xy^* - yx^*), |x|^2 - |y|^2 \right). \tag{27}$$

Using these geometric quantities, the scalar potential can be rewritten by the square of the Killing vector

$$V = |\mu|^2 g_{ij} k_3^{*j} = |\mu|^2 g^{ij*} \partial_i \partial_j D_3 \tag{28}$$

as was shown in Eq. (5). It is now manifest that only the action of $k_3$ among three Killing vectors preserves the potential term and therefore is the symmetry of the whole Lagrangian as expected from the mass term in (6). The vacua are fixed points of the Killing vectors $k_3$ or the critical points of the Killing potential $D_3$.

\section{Eguchi-Hanson Nonlinear Sigma Model in $\mathcal{N} = 1$ Superfields in the $O(2)$ Basis}

It is also useful to rewrite the model in terms of $O(2)$ gauge group instead of $U(1)$, since $O(2)$ basis is most frequently employed in the harmonic superspace formalism as given in the next section. Introducing $O(2)$ doublets $\tilde{\phi}^a$ as a column vector and $\tilde{\chi}_a$ as a row vector, superspace action in the $O(2)$ basis is given by

$$L^{O(2)} = \left[ \tilde{\phi}^a \left( e^{V_T} \right)^a_b \tilde{\phi}^b + \tilde{\chi}_a \left( e^{-V_T} \right)^a_b \tilde{\chi}^{b*} - cV \right]_{\theta^a \bar{\theta}^b} + \left( \left[ \Sigma \left( \tilde{\chi}_a T^a {\tilde{\phi}^b}^* - b \right) + \frac{\mu}{2} \tilde{\chi}_a \tilde{\phi}^a \right]_{\theta^a \bar{\theta}^b} + \text{c.c.} \right), \tag{29}$$

where the hermitian $O(2)$ generator is given from a diagonal generator $\sigma_3$ by

$$T \equiv \sigma_2 = e^{\frac{i\sigma_3}{4}} \sigma_3 e^{-\frac{i\sigma_3}{4}} = \frac{1 + i\sigma_1}{\sqrt{2}} \sigma_3 - \frac{i\sigma_1}{\sqrt{2}}. \tag{30}$$

In order to establish a relation between superspace action in the $U(1)$ and $O(2)$ bases, it is convenient to define fields $\tilde{\phi}', \tilde{\chi}'$ with definite $U(1)$ charge by means of a rotation from the $O(2)$ basis

$$\tilde{\phi}' \equiv \frac{1 - i\sigma_1}{\sqrt{2}} \tilde{\phi}, \quad \tilde{\chi}' \equiv \frac{1 - i\sigma_1}{\sqrt{2}} \tilde{\chi}. \tag{31}$$

In terms of these superfields, the action becomes

$$L^{O(2)} = \left[ \tilde{\phi}^a \left( e^{V} \sigma_3 \right)^a_b \tilde{\phi}^b + \tilde{\chi}_a \left( e^{-V} \sigma_3 \right)^a_b \tilde{\chi}^{b*} - cV \right]_{\theta^a \bar{\theta}^b} + \left( \left[ \Sigma \left( \tilde{\chi}_a \left( \sigma_2 \right)^a_b {\tilde{\phi}^b}^* - b \right) + \frac{\mu}{2} \tilde{\chi}_a \tilde{\phi}^a \right]_{\theta^a \bar{\theta}^b} + \text{c.c.} \right). \tag{32}$$
By identifying the $O(2)$ and $U(1)$ gauge fields, we find that the superfields $\phi_i, \chi_i$ in the $U(1)$ basis should be related to the superfields $\tilde{\phi}_i, \tilde{\chi}_i$ in the $O(2)$ basis as

$$|\tilde{\phi}'_{1}|^2 + |\tilde{\chi}'_{1}|^2 = |\phi_1|^2 + |\phi_2|^2,$$  \hspace{1cm} (33)

$$|\tilde{\phi}'_{2}|^2 + |\tilde{\chi}'_{2}|^2 = |\chi_1|^2 + |\chi_2|^2,$$  \hspace{1cm} (34)

$$i\tilde{\chi}'_{2}\tilde{\phi}'_{1} - i\tilde{\chi}'_{1}\tilde{\phi}'_{2} = \chi_1\phi_1 + \chi_2\phi_2;$$  \hspace{1cm} (35)

$$i\tilde{\chi}'_{2}\tilde{\phi}'_{1} + i\tilde{\chi}'_{1}\tilde{\phi}'_{2} = \chi_1\phi_1 - \chi_2\phi_2.$$  \hspace{1cm} (36)

The most general solution of the conditions (33) and (34) is given in terms of two unitary matrices $U, V$

$$
\begin{pmatrix}
\phi'_1 \\
\tilde{\chi}'_1
\end{pmatrix} =
U
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}, \quad U^\dagger U = 1,
$$

$$
\begin{pmatrix}
\tilde{\phi}'_2 \\
\tilde{\chi}'_2
\end{pmatrix} =
V
\begin{pmatrix}
\chi_1 \\
\chi_2
\end{pmatrix}, \quad V^\dagger V = 1.
$$

The constraints (35) and (36) give conditions

$$VT\sigma_2 U = 1, \quad V^T i\sigma_1 U = \sigma_3,$$

respectively. The first condition gives $VT = U^\dagger \sigma_2$. By substituting it to the second condition we obtain

$$U^\dagger \sigma_3 U = \sigma_3.$$  \hspace{1cm} (40)

The most general solution of these conditions is now given in terms of two arbitrary angle parameters $\alpha, \beta$ by

$$U = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix}, \quad V = -\sigma_2 \begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & e^{-i\beta} \end{pmatrix}. $$  \hspace{1cm} (41)

These angles $\alpha, \beta$ clearly represent the $U(1) \times U(1)$ symmetry of our Lagrangian. Therefore a general solution for the identification of superfields $\phi_i, \chi_i$ in the $U(1)$ basis and superfields $\tilde{\phi}_a, \tilde{\chi}_a$ in the $O(2)$ superfields is given by

$$
\phi =
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix} =
\begin{pmatrix}
eg^\dagger \tilde{\phi}'_1 \\
eg^\dagger \tilde{\chi}'_1
\end{pmatrix} =
\frac{1}{\sqrt{2}}
\begin{pmatrix}
eg^\dagger (\tilde{\phi}'_1 - i\tilde{\phi}'_2) \\
eg^\dagger (\tilde{\chi}'_1 - i\tilde{\chi}'_2)
\end{pmatrix},
$$

$$
\chi =
\begin{pmatrix}
\chi_1 \\
\chi_2
\end{pmatrix} =
\begin{pmatrix}
eg^\dagger \tilde{\phi}'_2 \\
eg^\dagger \tilde{\chi}'_2
\end{pmatrix} =
\frac{1}{\sqrt{2}}
\begin{pmatrix}
eg^\dagger (\tilde{\chi}'_1 + i\tilde{\chi}'_2) \\
eg^\dagger (-\tilde{\phi}'_1 - i\tilde{\phi}'_2)
\end{pmatrix}. $$  \hspace{1cm} (42)

From now on, we shall take $\alpha = \beta = 0$ case as a representative choice which is given by

$$
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix} =
\frac{1}{\sqrt{2}}
\begin{pmatrix}
\tilde{\phi}'_1 - i\tilde{\phi}'_2 \\
\tilde{\chi}'_1 - i\tilde{\chi}'_2
\end{pmatrix}, \quad
\begin{pmatrix}
\chi_1 \\
\chi_2
\end{pmatrix} =
\frac{1}{\sqrt{2}}
\begin{pmatrix}
\tilde{\chi}'_1 + i\tilde{\chi}'_2 \\
\tilde{\phi}'_1 - i\tilde{\phi}'_2
\end{pmatrix}. $$  \hspace{1cm} (43)

Bosonic part of the Lagrangian in the $O(2)$ basis is given in the Wess-Zumino gauge

$$L_{\text{boson}}^{O(2)} = L_{\text{kin}}^{O(2)} + L_{\text{constr}}^{O(2)} + L_{\text{pot}}^{O(2)},$$  \hspace{1cm} (45)
Eliminating the algebraic equations of motion for the auxiliary fields $D$

Integrating the Lagrange multiplier fields

The equation of motion for gauge field $v_\mu$ is given by

Eliminating the gauge field by this algebraic equation of motion, we obtain the kinetic term for hypermultiplets as

Integrating the Lagrange multiplier fields $D$ and $F_\Sigma$ in (47), we obtain two constraints

Eliminating the algebraic equations of motion for the auxiliary fields $F_\phi, F_\chi$

After eliminating the auxiliary fields $F_\phi, F_\chi$ by these algebraic equations of motion, we obtain the potential term
In the last line, we have eliminated the scalar field $\Sigma$ in the $\mathcal{N} = 2$ vector multiplet $(V, \Sigma)$ by its algebraic equation of motion

$$\Sigma = -\frac{\mu}{2} \frac{\bar{\phi} T \phi + \bar{\chi} T \chi}{|\phi|^2 + |\chi|^2}. \quad (54)$$

6 A Brief Survey of Harmonic Superspace Formalism

Harmonic superspace is defined as $\mathcal{H}(x^\mu, \theta^i, \bar{\theta}^i, u^\pm_i)$ which is called the central basis. The $u^\pm_i$ are called the harmonic variables which parameterize the coset $SU(2)_R/U(1)_r \sim S^2$, where $i = 1, 2$ is $SU(2)_R$ index and $\pm$ denotes $U(1)_r$ charge. The superfield in the Harmonic Superspace Formalism (HSF) is not defined in the central basis but in the subspace which is called the analytic subspace

$$\{\zeta_A, u^\pm_i | x^\mu = x^\mu - 2i \theta^i (\sigma^\mu \bar{\theta}^j) u^+_i u^-_j, \theta^+ = \theta^i u^+_i, \theta^+ = \bar{\theta}^i u^+_i, u^\pm_i\}, \quad (55)$$

where parentheses for indices $i, j$ mean symmetrization, for instance,

$$u^+_i u^-_j = (u^+_i u^-_j + u^-_j u^+_i) / 2. \quad (56)$$

Hypermultiplet and vector multiplet superfields are defined as the function in the analytic subspace as $q^+(\zeta_A, u)$ and $V^{++}(\zeta_A, u)$, respectively, which are called the analytic superfields.

To describe the real action in terms of the analytic superfield, the star conjugation must be introduced in addition to the usual complex conjugation. The complex conjugation rules for the coefficients in the harmonic expansions $f^i_{1\cdots i_n}$ (see (71)), the Grassmann variables $\theta_{ia}$ and the harmonic variables $u^\pm_i$ are defined as

$$\overline{f^i_{1\cdots i_n}} = \bar{f}^i_{1\cdots i_n}, \quad \overline{f^i_{1\cdots i_n}} = (-1)^n \bar{f}^i_{1\cdots i_n}, \quad (57)$$

$$\overline{\theta^i_{\alpha}} = \bar{\theta}^i_{\alpha}, \quad \overline{\theta^i_{\alpha}} = -\bar{\theta}^i_{\alpha}, \quad (58)$$

$$\overline{u^{+i}} = u^-_i, \quad \overline{u^{+i}} = -u^-_i, \quad (59)$$

respectively. The star conjugation rules are defined as

$$(f^i_{1\cdots i_n})^* = f^i_{1\cdots i_n}, \quad (60)$$

$$(\theta^i_{\alpha})^* = \theta^i_{\alpha}, \quad (61)$$

$$(u^{+i})^* = u^{-i}, \quad (u^{+i})^* = u^{-i}, \quad (u^{-i})^* = -u^{+i}, \quad (62)$$

$$(u^{+i})^{**} = -u^{-i}. \quad (63)$$

Note that the star conjugate acts only on the quantity having $U(1)_r$ charge. We write the combination of the complex and the star conjugation as

$$\overline{q^+(\zeta_A, u)} \equiv \tilde{q}^+(\zeta_A, u). \quad (64)$$
The combined conjugation rules are defined by
\[ \tilde{f}^{i_1 \cdots i_n} = f^{i_1 \cdots i_n} \equiv \tilde{f}_{i_1 \cdots i_n}, \]
\[ \tilde{\theta}^+ = \tilde{\theta}^+, \quad \tilde{\theta}^- = \tilde{\theta}^-, \quad \tilde{\theta'}^+ = -\theta^+, \quad \tilde{\theta'}^- = -\theta^-, \]
\[ (u^+_i) = u^{+i}, \quad (u^{-i}) = -u^+_i. \]  

The simple example of the real action is the free massless action of the Fayet-Sohnius hypermultiplet;
\[ S = -\int d\xi^{(-4)}_A du \tilde{\phi}^+ D^{++} \phi^+ \]  
where \( D^{++} \) is defined by
\[ D^{++} = \partial^{++} - 2i\theta^+ \sigma^\mu \tilde{\partial}^\mu A^+ - (\theta^{+2} \tilde{Z} - \tilde{\theta}^{+2} Z), \]
with \( Z = 0 \) for a massless hypermultiplet. Suffix \( A \) in the spacetime derivative \( \partial^A \) denotes the variable appropriate in the analytic superspace. \[ ^{\text{II}} \] \( \partial^{++} \) is the harmonic differential defined by \( \partial^{++} = u^+_i \partial_\partial u_i \).

For details of notation in HSF, we refer to our paper \[ ^{\text{II}} \] or a textbook. \[ ^{\text{III}} \]

The action is real in the sense of ordinary complex conjugation \( \bar{S} = S \). This property follows from the fact that \( q^+ = -q^- \).

Analytic superfields for the hypermultiplet \( q^+_a(x_A, \theta^\pm, u) \) can be expanded in powers of Grassmann numbers \( \theta \) as
\[ q^+_a(x_A, \theta^\pm, u) = F^+_a(x_A, u) + \sqrt{2}\theta^+ \psi_a(x_A, u) + \sqrt{2}\tilde{\theta}^+ \varphi_a(x_A, u) \]
\[ + i\theta^+ \sigma^\mu \tilde{\partial}^\mu A^-_a(x_A, u) + \theta^+ \theta^+ M^-_a(x_A, u) + \tilde{\theta}^+ \tilde{\theta}^+ N^-_a(x_A, u) \]
\[ + \sqrt{2}\theta^+ \theta^+ \tilde{\chi}^-_a(x_A, u) + \sqrt{2}\tilde{\theta}^+ \theta^+ \tilde{\xi}^-_a(x_A, u) \]
\[ + \theta^+ \tilde{\theta}^+ \tilde{\theta}^+ \tilde{D}^-_-_-_- (x_A, u), \]
where \( a \) is a flavor index. Note that each component in the hypermultiplet analytic superfield \[ ^{(70)} \] is a function of \( x_A \), and the harmonic variables \( u^+_i \). Therefore it includes infinite series of functions of \( x_A \) when expanded by the harmonic variables \( u^+_i \) (harmonic expansions), for instance,
\[ F^+_a(x_A, u) = \sum_{n=0}^{\infty} f^{(i_1 \cdots i_n+1j_1 \cdots j_n)}(x_A) u^+_{(i_1} \cdots u^+_{i_n+1} u^-_{j_1} \cdots u^-_{j_n}). \]  

Thus, the hypermultiplet includes infinitely many auxiliary fields in addition to physical fields.

We also use the convention to raise and lower the \( SU(2) \) indices by means of \( \epsilon_{ij} \) and \( \epsilon^{ij} \),
\[ \epsilon_{21} = \epsilon^{12} = 1, \quad \epsilon_{12} = \epsilon^{21} = -1. \]  

For instance the scalar fields for hypermultiplet \( f^i_a, i = 1, 2 \) have the property:
\[ f^i_a = \epsilon^{ij} f^j_a, \quad f_{ai} = \epsilon_{ij} f^j_a, \]
\[ f_{ai} = \epsilon_{ij} f^j_a, \quad f^i_a = \epsilon^{ij} f_{aj}. \]
where lower index $a$ denotes fundamental representation in flavor symmetry group. Therefore the scalar fields for hypermultiplet has the following reality property in conformity with our convention of complex conjugate in HSF:
\[(f^i_a)^* = f^i_{a} \equiv \bar{f}^i_a.\] (75)

Namely we have $\bar{f}^i_{a1} = -\bar{f}^2_a$, $\bar{f}^i_{a2} = \bar{f}^1_a$. We shall use $f^i_a$ and its complex conjugate field $\bar{f}^i_a$ as much as possible instead of $f^i_a = \epsilon^{ij} \bar{f}^j_a$.

7 The Eguchi-Hanson Nonlinear Sigma Model in HSF

The massive HK sigma model on Eguchi-Hanson manifold \([43]\) \((T^*\mathbb{C}P^1)\) is described in terms of harmonic superfields \([41]\) integrated over the analytic subspace \(d\zeta^A (-4)\)\): 
\[S = -\int d\zeta^A (-4) du \left( \tilde{q}^1_1 D^{+++} q^+_1 + \tilde{q}^2_1 D^{+++} q^+_2 + V^{+++}(\tilde{q}^+_1 q^-_2 - \tilde{q}^+_2 q^-_1 + \xi^{++}) \right)\] (76)

where the covariant derivative $D^{+++}$ defined in \((69)\) contains the central charge $Z$ satisfying the following eigenvalue equation
\[Z q^+_a (\zeta, u) = \frac{\mu}{2} q^+_a (\zeta, u).\] (77)

This mass parameter can be attributed to the Sherk-Schwarz reduction from six dimensions \([44]\) : $Z = -i(\bar{\partial}_5 + i\partial_6)$.

The Lagrangian \((76)\) is invariant under $O(2)$ gauge transformation
\[
\begin{align*}
\delta q^+_1 (\zeta, u) &= -\lambda (\zeta, u) q^+_2 (\zeta, u), \\
\delta q^+_2 (\zeta, u) &= \lambda (\zeta, u) q^+_1 (\zeta, u), \\
\delta V^{+++} (\zeta, u) &= D^{+++} \lambda (\zeta, u).
\end{align*}
\] (78) (79) (80)

Similarly to the Grassmann expansion of hypermultiplets \((70)\), the vector multiplet $V(\zeta, u)$ can also be expanded into infinitely many components when expanded in powers of Grassmann numbers $\theta$. These components can then be expanded into power series in harmonic variables $u_1^\pm$. However, we can exploit the gauge transformation \((80)\) to eliminate most of the auxiliary components in powers of Grassmann number $\theta$ and also in powers of harmonic variables $u_1^\pm$ in the vector multiplet. After eliminating infinitely many auxiliary fields by the gauge transformations, we obtain a gauge fixing
\[
\begin{align*}
V^{++}_{WZ} (x_A, \theta^\pm, u) &= \theta^+ \theta^+ M_v (x_A) + \bar{\theta}^+ \bar{\theta}^+ M_v (x_A) - 2i \theta^+ \sigma^\mu \bar{\theta}^+ V_{\mu} (x_A) \\
&\quad + \sqrt{2} \theta^+ \theta^+ \bar{\lambda}^i (x_A) u^-_i + \sqrt{2} \bar{\theta}^+ \bar{\theta}^+ \theta^+ \bar{\lambda}^i (x_A) u^-_i \\
&\quad + \theta^+ \theta^+ \bar{\theta}^+ \bar{\theta}^+ D^{(ij)}_v (x_A) u^-_i u^-_j.
\end{align*}\] (81)
which is called the Wess-Zumino gauge and is denoted by the suffix WZ. As a result, the remaining fields $M_v(x_A)$, $V_\mu(x_A)$, $\lambda^i(x_A)$ in HSF, are physical fields except $D_\nu(x_A)^{(ij)}$, if there is a kinetic term for vector multiplet. The field $D_\nu(x_A)^{(ij)}$ is the usual SUSY auxiliary field. However, we will use here a vector multiplet with no kinetic term. Therefore we will eventually eliminate all these component fields in the vector multiplet giving rise to constraints for hypermultiplets.

After integrating Grassmann variables and the harmonic variables, and eliminating infinitely many auxiliary fields of the hypermultiplet expanded in powers of harmonic variables $u_i^\pm$, and taking the Wess-Zumino gauge for the vector multiplet in HSF, we obtain the bosonic part of the action as

$$L_{\text{HSF}}^{\text{boson}} = - (\partial^A f_1^a + V^\mu f_1^a) \left( \partial_\mu f_{1i} + V_\mu f_{2i} \right) - (\partial^a f_2^i + V^\mu f_2^i) \left( \partial_\mu f_{2i} + V_\mu f_{3i} \right)$$

$$- \frac{1}{2} \left( \tilde{M}_v f_{1i} - \frac{\mu}{2} f_{2i} \right) \left( M_v f_{1i} - \frac{\mu}{2} f_{2i} \right) - \frac{1}{2} \left( \tilde{M}_v f_{2i} + \frac{\mu}{2} f_{3i} \right) \left( M_v f_{2i} + \frac{\mu}{2} f_{3i} \right)$$

$$- \frac{1}{2} \left( M_v f_{1i} + \frac{\mu}{2} f_{2i} \right) \left( \tilde{M}_v f_{1i} + \frac{\mu}{2} f_{2i} \right) - \frac{1}{2} \left( M_v f_{2i} - \frac{\mu}{2} f_{3i} \right) \left( \tilde{M}_v f_{2i} - \frac{\mu}{2} f_{3i} \right)$$

$$- \frac{1}{3} D_{v(ij)} (- f_{1i} f_{2j} + f_{2i} f_{1j} + \xi^{(ij)})$$

$$= L_{\text{HSF}}^{\text{kin}} + L_{\text{HSF}}^{\text{constr}} + L_{\text{HSF}}^{\text{pot}}, \quad (82)$$

where $a = 1, 2$ denotes fundamental representation in $O(2)$ gauge group. The scalar potential $V(f, \tilde{f})$ is given by

$$- L_{\text{HSF}}^{\text{pot}} = V(f, \tilde{f})$$

$$= \left( \left| \frac{\mu}{2} \right|^2 + |M_v|^2 \right) (f_1^i \bar{f}_{1i} + f_2^i \bar{f}_{2i})$$

$$+ \left( \frac{\mu}{2} \tilde{M}_v - \frac{\mu}{2} M_v \right) (\tilde{f}_1^i \bar{f}_{1i} - f_2^i \bar{f}_{2i}) \quad (85)$$

Let us stress once again that we adopt a convention for complex conjugation of complex scalar fields $(f_1^a)^* \equiv \bar{f}_{ai} = \epsilon_{ij} \bar{f}_a^j$, and use $f_a^i$ and $\bar{f}_{ai}$ to denote a complex conjugate pair.

There are still auxiliary fields $M_v$ and $V^\mu$ and $D_{v(ij)}$ of the vector multiplet. By changing variables, we can also introduce the most frequently used parameterization given by Curtright and Freedman [32] : four complex fields $\phi_\alpha^\alpha, \alpha = 1, 2, i = 1, 2$

$$\phi_1^\alpha = \frac{1}{\sqrt{2}} (f_1^{2, \alpha} + i f_2^{2, \alpha}), \quad \phi_2^\alpha = \frac{1}{\sqrt{2}} (f_1^{1, \alpha} + i f_2^{1, \alpha}), \quad (86)$$
where \( f^1_a = f^1_i \) and \( f^{1,2}_a = \bar{f}^i_a \).

The Lagrangian in the HSF can be related to the component Lagrangian in terms of \( \mathcal{N} = 1 \) superfields in the \( O(2) \) basis in the Wess-Zumino gauge by the following identification:

\[
M_v = i \Sigma, \quad V_\mu = \frac{v_\mu}{2},
\]

and fields \( f^i_a \) of HSF can be identified with \( \mathcal{N} = 1 \) fields \( \bar{\phi}^a, \tilde{\chi}^a \) in the \( O(2) \) basis as

\[
f^1_a = \left( \bar{\phi}^a \right)^*, \quad f^{2}_a = \tilde{\chi}^a.
\]

The Fayet-Iliopoulos parameters \( \xi^{(ij)} \) in HSF are identified with Fayet-Iliopoulos parameters \( c, b, b^* \) in the \( \mathcal{N} = 1 \) superfield formalism as

\[
\xi^{11} = -ib^*, \quad \xi^{22} = ib, \quad \xi^{12} = \xi^{21} = \frac{ic}{2}.
\]

The auxiliary fields \( D_v^{(ij)} \) in HSF are identified with the auxiliary fields \( D, F_\Sigma, F_\Sigma^* \) in the \( \mathcal{N} = 1 \) superfield formalism as

\[
D_v^{(11)} = 3i F_\Sigma^*, \quad D_v^{(22)} = -3i F_\Sigma, \quad D_v^{(12)} = -\frac{3i D}{2}.
\]

These results are in conformity with the reality property of the Fayet-Iliopoulos parameters \( \xi^{(ij)} \) and the auxiliary fields \( D_v^{(ij)} \) in HSF and those in \( \mathcal{N} = 1 \) superfield formalism

\[
\xi^{(ij)} = \epsilon^{ik} \epsilon^{jl} \left( \xi^{(kl)} \right)^*, \quad D_v^{(ij)} = \epsilon^{ik} \epsilon^{jl} \left( D_v^{(kl)} \right)^*.
\]

The identification implies that the complex scalar fields \( f^1_a \) belong to anti-chiral scalar superfields, and \( f_2^2 \) to chiral scalar superfields. The suffix \( a \) denotes fundamental representation of the gauge group \( O(2) \).

The complex fields \( \phi_i^a \) in the Curtright-Freedman basis are more directly related to the complex scalar fields of the \( \mathcal{N} = 1 \) superfields \( \phi_i, \chi_i \) in the \( U(1) \) basis as

\[
\phi_1^1 = \frac{1}{\sqrt{2}} \left( f^2_1 + i f^2_2 \right) = \frac{1}{\sqrt{2}} \left( \bar{\chi}_1 + i \bar{\chi}_2 \right) = \chi_1,
\]

\[
\phi_2^1 = \frac{1}{\sqrt{2}} \left( f^1_1 + i f^1_2 \right) = \frac{1}{\sqrt{2}} \left( \bar{\phi}_1 + i \bar{\phi}_2 \right) = \left( \phi_1 \right)^*,
\]

\[
\phi_1^2 = \frac{1}{\sqrt{2}} \left( f^2_1 + i f^2_2 \right) = \frac{1}{\sqrt{2}} \left( -\bar{f}_{11} - i \bar{f}_{21} \right) = \frac{1}{\sqrt{2}} \left( -\bar{\phi}^1 - i \bar{\phi}^2 \right) = \chi_2,
\]

\[
\phi_2^2 = \frac{1}{\sqrt{2}} \left( f^1_1 + i f^1_2 \right) = \frac{1}{\sqrt{2}} \left( \bar{f}_{12} + i \bar{f}_{22} \right) = \frac{1}{\sqrt{2}} \left( \bar{\chi}_1 + i \bar{\chi}_2 \right) = \left( \phi_2 \right)^*.
\]
Therefore the complex scalar fields \( \phi^i_1 \) are identified as those of chiral scalar superfield, and \( \phi^i_2 \) are identified as those of anti-chiral scalar superfield. We also notice that all the complex fields \( \phi^\alpha_i, i = 1, 2, \alpha = 1, 2 \) in the Curtright-Freedman basis have \( U(1) \) charge \(-1\) in conformity with the charge assignment obtained in the model constructed by the tensor calculus for supergravity. This supergravity model shows that our model can be embedded into supergravity. Moreover it explicitly demonstrates that our model can be extended to a model in five dimensions.

The equation of motion for the gauge field \( V_\mu \) gives

\[
V_\mu = \frac{\epsilon_{ab} \left( \partial_\mu f^i_{ai} \bar{f}_i - f^i_{ai} \partial_\mu \bar{f}_i \right)}{2f^i_{ai} f^i_{ai}}.
\]

(97)

After eliminating the vector field \( V^\mu \), we obtain the kinetic term for the scalar fields \( f^i_{ai} \) in the hypermultiplets

\[
\mathcal{L}_{\text{kin}}^\text{HSF} = - \partial_A f^i_{ai} \partial^A \bar{f}_i \partial_A f^j_{aj} - \partial_A f^2 \partial^A \bar{f}_2 + \frac{(f^i_{ai} \partial_A \bar{f}_i - f^i_{ai} \partial_A \bar{f}_i)^2}{4(f^i_{ai} f^i_{ai} + f^j_{aj} f^j_{aj})}.
\]

(98)

Integrating over scalar \( M_v \) and the auxiliary fields \( D_v(ij) \) in the vector multiplet, we obtain constraints

\[
- \bar{f}_1 f^j_2 + f^i_1 f^j_1 + \xi^{(ij)} = 0.
\]

(99)

This constraint makes the target space of the massive nonlinear sigma model into the Eguchi-Hanson manifold. In the case of massless (without potential) model, the target metric for the four independent real bosonic fields has been shown to be just the Eguchi-Hanson metric.

The equation of motion for scalar field \( M_v \) gives

\[
M_v = \mu \frac{\epsilon_{ab} f^i_{ai} \bar{f}_i}{f^i_{ai} f^i_{ai}}.
\]

(100)

where the flavor indices are summed. Integrating over \( M_v \) gives the potential term as

\[
V(f_1, f_2) = \left| \frac{\mu}{2} \right|^2 \frac{1}{f^i_{ai} f^j_{aj}} \left\{ -|f^i_{ai} \bar{f}_i| f^j_{aj} + (f^j_{aj} f^i_{ai} + f^i_{ai} f^j_{aj})^2 \right\}.
\]

(101)

The parameters \( \xi^{(ij)} \) have mass dimensions two and represent the scale of the curvature of the target manifold.

The bosonic action becomes in the Curtright-Freedman basis as

\[
\mathcal{L}_{\text{boson}} = - \partial_\mu f_1 \partial^\mu \bar{f}_1 - \partial_\mu f_2 \partial^\mu \bar{f}_2 - \frac{(f_1 \partial_A \bar{f}_1 + f_2 \partial_A \bar{f}_2)^2}{4(|\phi_1|^2 + |\phi_2|^2)}
\]

\[
- \frac{\mu^2}{4(|\phi_1|^2 + |\phi_2|^2)} \left( - (\phi_1 \sigma^3 \bar{\phi}_1 + \phi_2 \sigma^3 \bar{\phi}_2)^2 + (|\phi_1|^2 + |\phi_2|^2)^2 \right).
\]

(102)
Nonlinear Sigma Model in Independent Fields: Spherical Coordinates and Gibbons-Hawking Parameterization

Here we shall describe the model in terms of independent fields in several parameterizations by solving the constraints (99). In the following we shall take
\[ \xi^{(12)} = -i\xi, \quad \xi^{(11)} = \xi^{(22)} = 0 \] (103)
for simplicity. Then the constraints (99) become
\[ |\phi_1|^2 - |\phi_2|^2 = 2\xi, \quad \phi_1^*\phi_2 = \phi_2^*\phi_1 = 0. \] (104)

It is convenient to introduce independent fields \(z^\alpha, \bar{z}^\alpha, \alpha = 1, 2\) through the following Ansatz [22, 45]
\[ \phi_1^\alpha = g(r) \frac{z^\alpha}{\sqrt{r}}, \quad \phi_2^\alpha = f(r)i\sigma_2^\alpha \bar{z}^\beta \sqrt{r}, \] (105)
\[ r = z^1\bar{z}^1 + z^2\bar{z}^2, \] (106)
where \(z^\alpha\) are complex fields satisfying
\[ \phi_1^1\phi_2^1 - \phi_1^2\phi_2^2 = -z^1\bar{z}^1 - z^2\bar{z}^2. \] (107)
The real functions \(f(r)\) and \(g(r)\) are uniquely determined by the constraints (99) and (107) as
\[ f(r)^2 = -\xi + \sqrt{r^2 + \xi^2}, \quad g(r)^2 = \xi + \sqrt{r^2 + \xi^2}. \] (108)
The action can be described without constraint by the independent complex fields \(z^\alpha\). These fields \(z^\alpha\) are invariant under the \(O(2)(U(1))\) gauge transformations in (80) which is used to take the quotient of the target manifold.

Another useful parameterization of the model is given by the spherical coordinates which are invariant under the \(U(1)\) gauge transformations
\[ z^1 = \sqrt{r} \cos \frac{\Theta}{2} \exp \frac{i}{2}(\Psi + \Phi), \] (109)
\[ z^2 = \sqrt{r} \sin \frac{\Theta}{2} \exp \frac{i}{2}(\Psi - \Phi), \] (110)
\[ 0 \leq r \leq \infty, \quad 0 \leq \Theta \leq \pi, \quad 0 \leq \Phi \leq 2\pi, \quad 0 \leq \Psi \leq 2\pi. \] (111)

\[ f_1^1 = \frac{\phi_1^1 - \bar{\phi}_2^2}{\sqrt{2}} = \frac{g(r) - f(r)}{\sqrt{2r}} z^2 = \frac{g(r) - f(r)}{\sqrt{2}} \sin \left( \frac{\Theta}{2} \right) e^{\frac{i}{2}(-\Psi + \Phi)}, \] (112)
\[ f_1^2 = \frac{\phi_1^2 + \bar{\phi}_2^1}{i\sqrt{2}} = \frac{g(r) + f(r)}{i\sqrt{2r}} z^2 = \frac{g(r) + f(r)}{i\sqrt{2}} \sin \left( \frac{\Theta}{2} \right) e^{\frac{i}{2}(\Psi + \Phi)}, \] (113)
\[ f_2^1 = \frac{\phi_1^1 - \bar{\phi}_2^2}{\sqrt{2}} = \frac{g(r) - f(r)}{\sqrt{2r}} z^1 = \frac{g(r) - f(r)}{\sqrt{2}} \cos \left( \frac{\Theta}{2} \right) e^{\frac{i}{2}(\Psi + \Phi)}, \] (114)
\[ f_2^2 = \frac{\phi_1^2 + \bar{\phi}_2^1}{i\sqrt{2}} = \frac{g(r) + f(r)}{i\sqrt{2r}} z^1 = \frac{g(r) + f(r)}{i\sqrt{2}} \cos \left( \frac{\Theta}{2} \right) e^{\frac{i}{2}(\Psi + \Phi)}. \] (115)
The bosonic action becomes in the spherical coordinates as
\[
\mathcal{L}_{\text{boson}} = \frac{1}{2\sqrt{\xi^2 + r^2}} \left[ -\partial_\mu r \partial^\mu r - (r^2 + \xi^2) \partial^A \Theta \partial_\mu \Theta \\
- (r^2 + \xi^2 \sin^2 \Theta) \partial^A \Phi \partial_\mu \Phi - r^2 \partial^A \Psi \partial_\mu \Psi - 2r^2 \cos \Theta \partial^A \Phi \partial_\mu \Psi \\
- |\mu|^2 (r^2 + \xi^2 \sin^2 \Theta) \right].
\] (120)

It is also useful to change variables into the following parameterization appropriate to describe the Gibbons-Hawking multi-center metric \[47\]
\[
X^1 = r \sin \Theta \cos \Psi, \quad (121)
\]
\[
X^2 = r \sin \Theta \sin \Psi, \quad (122)
\]
\[
X^3 = \sqrt{r^2 + \xi^2} \cos \Theta, \quad (123)
\]
\[
\varphi = \Phi + \Psi. \quad (124)
\]

By using the parameterisation (86)-(110) and (121)-(124), the bosonic part of the action (98) can be rewritten as
\[
\mathcal{L} = -\frac{1}{2} \left\{ U \partial_\mu X \cdot \partial^\mu X + U^{-1} \mathcal{D}_\mu \varphi \mathcal{D}^\mu \varphi + \mu^2 U^{-1} \right\},
\] (125)

where \( \mathcal{D}_\mu \varphi = \partial_\mu \varphi + A \cdot \partial_\mu X \) and
\[
\nabla \times A = \nabla U.
\] (126)

The harmonic function \( U \) can be described
\[
U = \frac{1}{2} \left[ \frac{1}{|X - \xi n|} + \frac{1}{|X + \xi n|} \right],
\] (127)

where \( n \) is a unit three vector, which is given by \( n = (0, 0, 1) \). \( A \) is a potential whose solution is given as
\[
A_1 = \frac{1}{2} \left\{ \frac{X^2}{|X - \xi n|(X^3 - \xi + |X - \xi n|)} + \frac{X^2}{|X + \xi n|(X^3 + \xi + |X + \xi n|)} \right\},
\]
\[
A_2 = \frac{1}{2} \left\{ -\frac{X^1}{|X - \xi n|(X^3 - \xi + |X - \xi n|)} + \frac{X^1}{|X + \xi n|(X^3 + \xi + |X + \xi n|)} \right\},
\]
\[
A_3 = 0.
\] (128)

It is found that the target metric of the action (125) is just the Eguchi-Hanson metric \[32, 40, 45\].
9 BPS Equation and Domain Wall Solution

In this Sec., we give the BPS domain wall solution in our model. In the following the complex mass parameter $\mu$ is taken to be real for simplicity. By requiring that the fermions conserve half of SUSY we obtain the BPS equations in HSF

\begin{align}
(M_v + V_2)f_2^1 + \left(\frac{\mu}{2} + \partial_2^A\right)f_1^1 &= 0, \\
(M_v - V_2)f_2^2 + \left(\frac{\mu}{2} - \partial_2^A\right)f_1^2 &= 0, \\
-(M_v + V_2)f_1^1 + \left(\frac{\mu}{2} + \partial_2^A\right)f_2^1 &= 0, \\
-(M_v - V_2)f_1^2 + \left(\frac{\mu}{2} - \partial_2^A\right)f_2^2 &= 0.
\end{align}

(129) (130) (131) (132)

BPS wall solution should approach the supersymmetric discrete vacua as $y \to \pm \infty$. From the trivial solution of BPS equation (translational invariant solution), we find \cite{1} that there are only two vacua : $(r, \Theta) = (0, 0), (0, \pi)$ in terms of the spherical coordinates \cite{109}, \cite{110}. Another way of understanding these vacua is to observe from Eq.\,(120) that these two points are the minima of the scalar potential

\begin{equation}
V(r, \Theta, \Phi, \Psi) = \frac{|\mu|^2 (r^2 + \xi^2 \sin^2 \Theta)}{2\sqrt{\xi^2 + r^2}}
\end{equation}

(133)

with vanishing vacuum energy

\begin{equation}
V(r = 0, \Theta = 0) = V(r = 0, \Theta = \pi) = 0.
\end{equation}

(134)

Therefore we consider the domain wall solution connects these vacua, and we can expect that $\Theta$ has nontrivial configuration. After some algebra, we obtain four independent differential equations in terms of the spherical coordinates \cite{1}

\begin{align}
r' &= \mu \cos \Theta \cdot r, \quad r \cdot \Psi' = 0, \\
\Theta' &= -\mu \sin \Theta, \quad \sin \Theta \cdot \Phi' = 0.
\end{align}

(135) (136)

The boundary condition of $r = 0$ at $y = -\infty$ dictates the solution of (135) to be $r = 0$ and $\Psi = 0$. The other two equations in (136) gives a nontrivial dependence in $y$ resulting in the following BPS solutions

\begin{equation}
\Theta = \arccos[\tanh \mu(y + y_0)], \quad \Phi = \varphi_0,
\end{equation}

(137)

where $y_0$ and $\varphi_0$ are real constants: $y_0$ determines the position of the domain wall along $y$ direction and $\varphi_0$ corresponds to the Nambu-Goldstone (NG) mode of $U(1)$ isometry of target space. The BPS wall solution is illustrated in Fig.\,2

Our BPS wall solution is obtained from $\mathcal{N} = 2$ SUSY theory in four dimensions. However, subsequent study\cite{17} revealed that our model can be extended to an $\mathcal{N} = 2$ SUSY theory in five dimensions. In
In terms of harmonic superfields (102) and their bosonic components (103), the BPS solution is given by

$$q_1^+ = f_1^i u_i^+ = \sqrt{\frac{\xi}{2}} e^{1/2} e^{i \varphi_0} \begin{pmatrix} -\sqrt{1 - \tanh(\mu (y + y_0))} u_1^+ \\ \sqrt{1 + \tanh(\mu (y + y_0))} u_2^+ \end{pmatrix},$$

(138)

$$q_2^+ = f_2^i u_i^+ = -i \sqrt{\frac{\xi}{2}} e^{1/2} e^{i \varphi_0} \begin{pmatrix} \sqrt{1 - \tanh(\mu (y + y_0))} u_1^+ \\ \sqrt{1 + \tanh(\mu (y + y_0))} u_2^+ \end{pmatrix}.$$

(139)

In terms of the fields in the Curtright-Freedman basis (102), the BPS wall solution is given by

$$\phi_1^1 = \sqrt{\xi} (1 + \tanh \mu (y + y_0)) e^{\frac{i}{2} \varphi_0},$$

(140)

$$\phi_1^2 = \sqrt{\xi} (1 - \tanh \mu (y + y_0)) e^{-\frac{i}{2} \varphi_0},$$

(141)

$$\phi_2^1 = \phi_2^2 = 0.$$

(142)

The BPS wall solution in the Gibbons-Hawking multi-center metric parameterization (125) is given by

$$X^1 = X^2 = 0,$$

(143)

$$X^3 = \xi \tanh \mu (y + y_0),$$

(144)

$$\varphi = \varphi_0.$$

(145)

Acknowledgements

One of the authors (NS) thanks Yoshiaki Tanii for useful discussion on hypermultiplets in six dimensions. This work is supported in part by Grant-in-Aid for Scientific Research from the Japan Ministry of
Education, Science and Culture 13640269. The work of M. Naganuma is supported by JSPS Fellowship. The work of M. Nitta is supported by the U. S. Department of Energy under grant DE-FG02-91ER40681 (Task B).

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