GRAPHS WITH $\alpha_1$ AND $\tau_1$ BOTH LARGE

GREGORY J. PULEO

Abstract. Given a graph $G$, let $\tau_1(G)$ denote the smallest size of a set of edges whose deletion makes $G$ triangle-free, and let $\alpha_1(G)$ denote the largest size of an edge set containing at most one edge from each triangle of $G$. Erdős, Gallai, and Tuza introduced several problems with the unifying theme that $\alpha_1(G)$ and $\tau_1(G)$ cannot both be “very large”; the most well-known such problem is their conjecture that $\alpha_1(G) + \tau_1(G) \leq \frac{|V(G)|^2}{4}$, which was proved by Norin and Sun. We consider three other problems within this theme (two introduced by Erdős, Gallai, and Tuza, the other by Norin and Sun), all of which request an upper bound either on $\min\{\alpha_1(G), \tau_1(G)\}$ or on $\alpha_1(G) + k\tau_1(G)$ for some constant $k$, and prove the existence of graphs for which these quantities are “large”.

1. Introduction

A triangle independent set in a graph $G$ is a set of edges containing at most one edge from each triangle of $G$, while a triangle edge cover in a graph $G$ is a set of edges containing at least one edge from each triangle of $G$. Equivalently, a triangle edge cover is a set of edges whose deletion from $G$ results in a triangle-free graph. We write $\alpha_1(G)$ to denote the size of a largest triangle independent set in $G$ and we write $\tau_1(G)$ for the size of a smallest triangle edge cover in $G$. Erdős, Gallai, and Tuza [3] considered several problems relating the quantities $\alpha_1(G)$ and $\tau_1(G)$, with the unifying theme that $\alpha_1(G)$ and $\tau_1(G)$ should not both be “large”: informally, if it is easy for an edge set to avoid all triangles, in the sense of $\alpha_1(G)$ being large, then it should also be easy to destroy all triangles, so that $\tau_1(G)$ should be small. In particular, they posed the following conjecture.

Conjecture 1.1 (Erdős–Gallai–Tuza [3]). If $G$ is an $n$-vertex graph, then $\alpha_1(G) + \tau_1(G) \leq \frac{n^2}{4}$.

The complete graph $K_n$ and the complete bipartite graph $K_{n/2,n/2}$ both satisfy $\alpha_1(G) + \tau_1(G) = \frac{n^2}{4}$, but a different part of the sum dominates for each graph. Conjecture 1.1 was proved (in a somewhat stronger form) by Norin and Sun [9], but many interesting problems relating $\alpha_1(G)$ and $\tau_1(G)$ remain open.

Norin and Sun [9] posed the following problem:

Problem 1.2 (Question 8 of [9]). Determine the largest constant $c$ such that $\alpha_1(G) + c\tau_1(G) \leq |E(G)|$ for every graph $G$.

Erdős, Gallai, and Tuza [8] proved that $\alpha_1(G) + \tau_1(G) \leq |E(G)|$ for every graph $G$, which gives a lower bound of $c \geq 1$ in Problem 1.2. As Norin and Sun [9] observed, a conjecture of Tuza mentioned in [2] is equivalent to the claim that we can take $c \geq 5/3$ in Problem 1.2. In Section 2 we prove that $c = 1$ is the correct answer to Problem 1.2, which refutes the conjecture of Tuza.
Erdős, Gallai, and Tuza also posed the following closely related problems. A triangular graph is a graph such that every edge lies in some triangle.

**Problem 1.3** (Problem 13 of [3]). Determine the largest constant $c$ for which there exists a triangular graph $G$ such that $\min(\alpha_1(G), \tau_1(G)) \geq c|E(G)|$.

**Problem 1.4** ([3]). Determine the largest constant $c'$ for which there exists a triangular graph $G$ such that $\alpha_1(G) + 2\tau_1(G) \geq c'|E(G)|$.

Since $\alpha_1(G) + \tau_1(G) \leq |E(G)|$ and $\tau_1(G) \leq \frac{1}{4}|E(G)|$ for all $G$, we have upper bounds of $c \leq 1/2$ in Problem 1.3 and $c' \leq 3/2$ in Problem 1.4. If we ignore the “triangular” restriction, then by taking a disjoint union of appropriately sized $K_4$ and $K_{1, t}$ one can easily get $c \geq 1/3 - \varepsilon$ in Problem 1.3 likewise, taking any triangle-free graph yields $c' \geq 1$ in Problem 1.4.

In Section 3 we give a construction yielding triangular graphs with $c$ bounds of $\alpha$ for this proof; we only need $3\sqrt{3} - \sqrt{2} - \varepsilon$, as desired. □
3. A Lower Bound on \( \min \{ \alpha_1(G), \tau_1(G) \} \)

**Lemma 3.1.** Let \( \varepsilon, \theta \in (0, 1) \) be fixed constants, let \( p(n) = n^{-\theta} \), and let \( G \sim G(n, p) \). With high probability,

\[
|E(G[S])| \geq (1 - \varepsilon)p\frac{|S|^2}{2}
\]

for all \( S \subseteq V(G) \) such that \( |S| \geq \varepsilon n \).

**Proof.** Fix \( S \subseteq V(G) \) with \( |S| \geq \varepsilon n \) and let the random variable \( X \) denote the number of edges in \( S \). We have \( X \sim \text{Bin}(\binom{|S|}{2}, p) \). We may assume that \( n \) is large enough that \( \binom{\varepsilon n}{2} \geq \varepsilon^2 n^2 / 3 \). By Chernoff’s inequality (as formulated in Corollary 2.3 of [1]),

\[
P[X < (1 - \varepsilon/2)\mathbb{E}[X]] \leq 2 \exp \left(-\frac{\varepsilon^2}{3} p \varepsilon^2 n^2 / 2 \right) \leq 2 \exp \left(-\frac{\varepsilon^4}{9} n^{2-\theta} \right) \ll 2^{-n}.
\]

For sufficiently large \( n \), we have \( (1 - \varepsilon/2)\mathbb{E}[X] \geq (1 - \varepsilon)p\frac{|S|^2}{2} \). The desired claim therefore follows by applying the union bound. \( \square \)

**Lemma 3.2.** Let \( d, \varepsilon > 0 \) be fixed constants with \( \varepsilon < 1 \), let \( p = n^{-\theta} \), where \( 0 < \theta < 1 \), and let \( G \sim G(n, p) \). Let \( k = dnp \). If \( d \geq 2 \varepsilon (1 - \varepsilon) \), then with high probability, \( \max_{S \subseteq V(G)} \phi_k(S) \leq \frac{k^2}{2(1 - \varepsilon)p} \).

**Proof.** If \( |S| < \varepsilon n \), then we have

\[
\phi_k(S) \leq k |S| < k \varepsilon n = \frac{k^2 \varepsilon}{dp} \leq \frac{k^2}{2(1 - \varepsilon)p},
\]

as desired. Thus, it suffices to consider \( S \) with \( |S| \geq \varepsilon n \). By Lemma 3.1, we may assume that \( |E(G[S])| \geq (1 - \varepsilon)\frac{|S|^2}{2} \) for all \( S \subseteq V(G) \) with \( |S| \geq \varepsilon n \). Thus, for all such \( S \) and for \( n \) sufficiently large, we have

\[
\phi_k(S) \leq k |S| - (1 - \varepsilon)p\frac{|S|^2}{2}.
\]

Letting \( f(x) = kx - (1 - \varepsilon)p\frac{x^2}{2} \), we see that \( f \) is maximized at \( x = \frac{k}{(1 - \varepsilon)p} \), attaining a maximum value of \( k^2/(2(1 - \varepsilon)p) \). The conclusion follows. \( \square \)

**Lemma 3.3.** Let \( \varepsilon \in (0, 1) \) a fixed constant and let \( d \) be a fixed constant with \( d \geq 2 \varepsilon (1 - \varepsilon) \). Let \( p(n) = n^{-\theta} \). For sufficiently large \( n \), there exists a triangle-free graph \( G \) with no isolated vertices such that:

- \( |E(G)| \leq (1 + \varepsilon)p\frac{n^2}{2} \),
- \( |E(G)| \geq (1 - \varepsilon)p\frac{n^2}{2} \), and
- \( \max_{S \subseteq V(G)} \phi_k(S) \leq \frac{k^2}{2(1 - \varepsilon)p} \).
Proof. Consider a random graph \( G_0 \) drawn from \( G(n, p) \). Since \( pn^2 \gg n \), Chernoff’s inequality implies that with high probability,

\[
(1 - \varepsilon)p \frac{n^2}{2} + n \leq |E(G_0)| \leq (1 + \varepsilon)p \frac{n^2}{2}.
\]

Furthermore, Lemma 3.2 implies that with high probability,

\[
\max_{S \subseteq V(G)} \phi_k(S) \leq \frac{k^2}{2(1 - \varepsilon/2)p}.
\]

Furthermore, as the expected number of triangles in \( G_0 \) is at most \( n^{3/4} \), Markov’s inequality implies that with high probability, \( G_0 \) has at most \( n \) triangles. Similarly, with high probability \( G_0 \) has no isolated vertices.

Thus, for sufficiently large \( n \), there is a graph \( G_0 \) with at most \( n \) triangles and with no isolated vertices for which Inequalities (1) and (2) both hold. Fix such a graph \( G_0 \), and let \( X \) be a smallest set of edges such that \( G - X \) is triangle-free. Observe that \( |X| \leq n \), since \( G \) has at most \( n \) triangles, and that \( G_0 - X \) has no isolated vertices, since if \( v \) is an isolated vertex in \( G_0 - X \), then as \( v \) is not isolated in \( G_0 \), there is some edge \( vw \in X \), and \( G_0 - (X - vw) \) is also triangle-free, contradicting the minimality of \( X \).

Let \( G = G_0 - X \). As we have removed at most \( n \) edges from \( G_0 \), clearly

\[
(1 - \varepsilon)p \frac{n^2}{2} \leq |E(G)| \leq (1 + \varepsilon)p \frac{n^2}{2}.
\]

Furthermore, for each \( S \subseteq V(G) \), the value of \( \phi_k(S) \) has increased by at most \( n \) relative to its value in \( G_0 \), so that

\[
\max_{S \subseteq V(G)} \phi_k(S) \leq n + \frac{k^2}{2(1 - \varepsilon/2)p} \leq \frac{k^2}{2(1 - \varepsilon)p},
\]

where the last inequality holds provided that \( n \) is sufficiently large, as the gap between \( \frac{k^2}{2(1 - \varepsilon/2)p} \) and \( \frac{k^2}{2(1 - \varepsilon)p} \) is a constant factor of \( k^2/p \), where \( k^2/p \gg n \). Thus, for sufficiently large \( n \), the graph \( G \) produced in this manner has the desired properties. \( \square \)

Lemma 3.4. Let \( d, \varepsilon > 0 \) be fixed constants. If \( d \geq 2\varepsilon(1 - \varepsilon) \), then there is a triangular graph \( H \) such that \( \tau_1(H) \geq \frac{2(1 - \varepsilon)d - d^2}{(2d + 1)(1 + \varepsilon)(1 - \varepsilon)} \) and \( \tau_1(H) \geq \frac{1 - \varepsilon}{(2d + 1)(1 + \varepsilon)}. \)

Proof. Let \( G \) be a graph satisfying the conclusion of Lemma 3.3 for the given values of \( d \) and \( \varepsilon \), let \( n = |V(G)| \), and let \( H = K_k \lor G \).

Observe that

\[
|E(H)| = nk + |E(G)| \leq nk + (1 + \varepsilon)p \frac{n^2}{2} \leq (1 + \varepsilon)n^2p \frac{2d + 1}{2}.
\]

Since \( G \) is triangle-free, Lemma 2.4 yields

\[
\tau_1(H) = nk - \max_{S \subseteq V(G)} \phi_k(S) \geq nk - \frac{k^2}{2(1 - \varepsilon)p} = \frac{n^2pd[2(1 - \varepsilon) - d]}{2(1 - \varepsilon)p}.
\]

Combining this with the upper bound on \( |E(H)| \) and simplifying, we have

\[
\frac{\tau_1(H)}{|E(H)|} \geq \frac{d[2(1 - \varepsilon) - d]}{(1 + \varepsilon)(1 - \varepsilon)(2d + 1)} = \frac{2(1 - \varepsilon)d - d^2}{(2d + 1)(1 + \varepsilon)(1 - \varepsilon)}.
\]
This establishes the desired lower bound on $\tau_1(H)$. For the bound on $\alpha_1(H)$, observe that $G$ is a triangle-independent subgraph of $H$, so that

$$\alpha_1(H) \geq |E(G)| \geq (1-\varepsilon)p\frac{n^2}{2}.$$ 

Therefore, using the upper bound on $|E(H)|$, we have

$$\frac{\alpha_1(H)}{|E(H)|} \geq \frac{1-\varepsilon}{(2d+1)(1+\varepsilon)}.$$ 

Finally, since $G$ has no isolated vertices, it is easy to see that $H$ is triangular. 

**Corollary 3.5.** For every $d > 0$, and every $\gamma > 0$, there is a graph $H$ with $\frac{\tau_1(H)}{|E(H)|} \geq \frac{2d-d^2}{2d+1} - \gamma$ and $\frac{\alpha_1(H)}{|E(H)|} \geq \frac{1}{2d+1} - \gamma$.

Choosing $d$ to maximize $\frac{2d-d^2}{2d+1}$ yields the following partial answer to Problem 1.3:

**Corollary 3.6.** For all $\gamma > 0$, there is a graph $H$ with $\frac{\tau_1(H)}{|E(H)|} \geq \frac{3-\sqrt{3}}{2} - \gamma > 0.38$ and $\frac{\alpha_1(H)}{|E(H)|} \geq \frac{1}{\sqrt{3}} - \gamma > 0.44$.

**Proof.** Take $d = \frac{-1+\sqrt{3}}{2}$ in Corollary 3.5. 

Similarly, choosing $d$ to maximize $\frac{1+2(2d-d^3)}{2d+1}$ yields the following partial answer to Problem 1.3:

**Corollary 3.7.** For all $\gamma > 0$, there is a graph $H$ with $\frac{\alpha_1(H)+2\tau_1(H)}{|E(H)|} \geq 3 - \sqrt{3} - \gamma$.

**Proof.** Take $d = \frac{-1+\sqrt{7}}{2}$ in Corollary 3.5.

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