Weighted Hardy inequality with higher dimensional singularity on the boundary

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Received: 26 September 2012 / Accepted: 15 July 2013
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Abstract Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 3$ and let $\Sigma_k$ be a closed smooth submanifold of $\partial \Omega$ of dimension $1 \leq k \leq N - 2$. In this paper we study the weighted Hardy inequality with weight function singular on $\Sigma_k$. In particular we provide necessary and sufficient conditions for existence of minimizers.

Mathematics Subject Classification 35J20 · 35J57 · 35J75 · 35B33 · 35A01

1 Introduction

Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^N$, $N \geq 2$ and let $\Sigma_k$ be a smooth closed submanifold of $\partial \Omega$ with dimension $0 \leq k \leq N - 1$. Here $\Sigma_0$ is a single point and $\Sigma_{N-1} = \partial \Omega$. For $\lambda \in \mathbb{R}$, consider the problem of finding minimizers for the quotient:

$$
\mu_\lambda (\Omega , \Sigma_k) := \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 p \, dx - \lambda \int_\Omega \delta^{-2} |u|^2 q \, dx}{\int_\Omega \delta^{-2} |u|^2 q \, dx},
$$

where $\delta(x) := \text{dist}(x, \Sigma_k)$ is the distance function to $\Sigma_k$ and where the weights $p$, $q$ and $\eta$ satisfy

$$
p, q \in C^2(\overline{\Omega}), \quad p, q > 0 \quad \text{in} \quad \overline{\Omega}, \quad \eta > 0 \quad \text{in} \quad \overline{\Omega} \setminus \Sigma_k, \quad \eta \in \text{Lip}(\overline{\Omega})
$$

Communicated by A. Malchiodi.

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and
\[ \max_{\Sigma_k} \frac{q}{p} = 1, \quad \eta = 0 \quad \text{on } \Sigma_k. \] (3)

We put
\[ I_k = \int_{\Sigma_k} \frac{d\sigma}{\sqrt{1 - (q(\sigma)/p(\sigma))}}, \quad 1 \leq k \leq N - 1 \quad \text{and} \quad I_0 = \infty. \] (4)

It was shown by Brezis and Marcus [4] that there exists \( \lambda^* \) such that if \( \lambda > \lambda^* \) then \( \mu(\Omega, \Sigma, N-1) < \frac{1}{4} \) and it is attained while for \( \lambda \leq \lambda^* \), \( \mu(\Omega, \Sigma, N-1) = \frac{1}{4} \) and it is not achieved for every \( \lambda < \lambda^* \). The critical case \( \lambda = \lambda^* \) was studied by Brezis, Marcus and Shafrir [5], where they proved that \( \mu_{\lambda^*}(\Omega, \Sigma, N-1) \) admits a minimizer if and only if \( I_{N-1} < \infty \). The case where \( k = 0 \) (\( \Sigma_0 \) is reduced to a point on the boundary) was treated by the first author in [11] and the same conclusions hold true.

Here we obtain the following

**Theorem 1.1** Let \( \Omega \) be a smooth bounded domain of \( \mathbb{R}^N \), \( N \geq 3 \) and let \( \Sigma_k \subset \partial \Omega \) be a closed submanifold of dimension \( k \in [1, N - 2] \). Assume that the weight functions \( p, q \) and \( \eta \) satisfy (2) and (3). Then, there exists \( \lambda^* = \lambda^*(p, q, \eta, \Omega, \Sigma_k) \) such that

\[ \mu_{\lambda}(\Omega, \Sigma_k) = \frac{(N-k)^2}{4}, \quad \forall \lambda \leq \lambda^*, \]
\[ \mu_{\lambda}(\Omega, \Sigma_k) < \frac{(N-k)^2}{4}, \quad \forall \lambda > \lambda^*. \]

The infimum \( \mu_{\lambda}(\Omega, \Sigma_k) \) is attained if \( \lambda > \lambda^* \) and it is not attained when \( \lambda < \lambda^* \).

Concerning the critical case we get

**Theorem 1.2** Let \( \lambda^* \) be given by Theorem 1.1 and consider \( I_k \) defined in (4). Then \( \mu_{\lambda^*}(\Omega, \Sigma_k) \) is achieved if and only if \( I_k < \infty \).

By choosing \( p = q \equiv 1 \) and \( \eta = \delta^2 \), we obtain the following consequence of the above theorems.

**Corollary 1.3** Let \( \Omega \) be a smooth bounded domain of \( \mathbb{R}^N \), \( N \geq 3 \) and \( \Sigma_k \subset \partial \Omega \) be a closed submanifold of dimension \( k \in \{1, \ldots, N - 2\} \). For \( \lambda \in \mathbb{R} \), put

\[ v_{\lambda}(\Omega, \Sigma_k) = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 \, dx - \lambda \int_\Omega |u|^2 \, dx}{\int_\Omega \delta^{-2} |u|^2 \, dx}. \]

Then, there exists \( \tilde{\lambda} = \tilde{\lambda}(\Omega, \Sigma_k) \) such that

\[ v_{\lambda}(\Omega, \Sigma_k) = \frac{(N-k)^2}{4}, \quad \forall \lambda \leq \tilde{\lambda}, \]
\[ v_{\lambda}(\Omega, \Sigma_k) < \frac{(N-k)^2}{4}, \quad \forall \lambda > \tilde{\lambda}. \]

Moreover \( v_{\lambda}(\Omega, \Sigma_k) \) is attained if and only if \( \lambda > \tilde{\lambda} \).
The proof of the above theorems are mainly based on the construction of appropriate sharp $H^1$-subsolution and $H^1$-supersolutions for the corresponding operator

$$L_\lambda := -\Delta - \frac{(N - k)^2}{4} q\delta^{-2} + \lambda \delta^{-2} \eta$$

(with $p \equiv 1$). These super and sub-solutions are perturbations of an approximate “virtual” ground-state for the Hardy constant $\frac{(N - k)^2}{4}$ near $\Sigma_k$. For that we will consider the projection distance function $\tilde{\delta}$ defined near $\Sigma_k$ as

$$\tilde{\delta}(x) := \sqrt{|\text{dist}_{\partial \Omega}(x, \Sigma_k)|^2 + |x - \bar{x}|^2},$$

where $\bar{x}$ is the orthogonal projection of $x$ on $\partial \Omega$ and $\text{dist}_{\partial \Omega}(x, \Sigma_k)$ is the geodesic distance to $\Sigma_k$ on $\partial \Omega$ endowed with the induced metric. While the distances $\delta$ and $\tilde{\delta}$ are equivalent, $\delta$ and $\tilde{\delta}$ differ and $\delta$ does not, in general, provide the right approximate solution for $k \leq N - 2$.

Letting $d_{\partial \Omega} = \text{dist}(\cdot, \partial \Omega)$, we have

$$\tilde{\delta}(x) := \sqrt{|\text{dist}_{\partial \Omega}(x, \Sigma_k)|^2 + d_{\partial \Omega}(x)^2}.$$ 

Our approximate virtual ground-state near $\Sigma_k$ reads then as

$$x \mapsto d_{\partial \Omega}(x) \tilde{\delta}^{\frac{k-N}{2}}(x).$$ (5)

In some appropriate Fermi coordinates $y = (y^1, y^2, \ldots, y^{N-k}, y^{N-k+1}, \ldots, y^N) = (\tilde{y}, \bar{y}) \in \mathbb{R}^N$ with $\tilde{y} = (y^1, y^2, \ldots, y^{N-k}) \in \mathbb{R}^{N-k}$ and $\bar{y} = (y^{N-k+1}, \ldots, y^N)$ (see next section for a precise definition), the function in (5) then becomes

$$y \mapsto y^1 |\tilde{y}|^{\frac{k-N}{2}}$$

which is the “virtual” ground-state for the Hardy constant $\frac{(N - k)^2}{4}$ in the flat case $\Sigma_k = \mathbb{R}^k$ and $\Omega = \mathbb{R}^N$. We refer to Sect. 2 for more details about the constructions of the super and sub-solutions.

The proof of the existence part in Theorem 1.2 is inspired from [5]. It amounts to obtain a uniform control of a specific minimizing sequence for $\mu_{\lambda^*}(\Omega, \Sigma_k)$ near $\Sigma_k$ via the $H^1$-super-solution constructed.

We recall that the existence and non-existence of extremals for (1) and related problems were studied in [1,6–9,12–14,16,19–21] and some references therein. We would like to mention that some of the results in this paper can be useful in the study of semilinear equations with a Hardy potential singular at a submanifold of the boundary. We refer to [2,3,10], where existence and nonexistence for semilinear problems were studied via the method of super/sub-solutions.

2 Preliminaries and notations

In this section we collect some notations and conventions we are going to use throughout the paper.

Let $U$ be an open subset of $\mathbb{R}^N$, $N \geq 3$, whose boundary $M := \partial U$ is a smooth closed hypersurface of $\mathbb{R}^N$. Assume that $M$ contains a smooth closed submanifold $\Sigma_k$ of dimension $1 \leq k \leq N - 2$. In the following, for $x \in \mathbb{R}^N$, we let $d(x)$ be the distance function of $M$ and $\delta(x)$ the distance function of $\Sigma_k$. We denote by $N_M$ the unit normal vector field of $M$ pointed into $U$. 

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Given $P \in \Sigma_k$, the tangent space $T_P \mathcal{M}$ of $\mathcal{M}$ at $P$ splits naturally as

$$T_P \mathcal{M} = T_P \Sigma_k \oplus N_P \Sigma_k,$$

where $T_P \Sigma_k$ is the tangent space of $\Sigma_k$ and $N_P \Sigma_k$ stands for the normal space of $T_P \Sigma_k$ at $P$. We assume that these subspaces are spanned respectively by $(E_a)_{a=N-k+1,\ldots,N}$ and $(E_i)_{i=2,\ldots,N-k}$. We will assume that $N_{\mathcal{M}}(P) = E_1$.

A neighborhood of $P$ in $\Sigma_k$ can be parameterized via the map

$$\tilde{y} \mapsto f^P(\tilde{y}) = \text{Exp}_P^{\Sigma_k} \left( \sum_{a=N-k+1}^N \chi^a E_a \right),$$

where, $\tilde{y} = (y^{N-k+1}, \ldots, y^N)$ and where $\text{Exp}_P^{\Sigma_k}$ is the exponential map at $P$ in $\Sigma_k$ endowed with the metric induced by $\mathcal{M}$. Next we extend $(E_i)_{i=2,\ldots,N-k}$ to an orthonormal frame $(X_i)_{i=2,\ldots,N-k}$ in a neighborhood of $P$. We can therefore define the parameterization of a neighborhood of $P$ in $\mathcal{M}$ via the mapping

$$(\tilde{y}, \bar{y}) \mapsto h_{\mathcal{M}}^P(\tilde{y}, \bar{y}) := \text{Exp}_P^{\mathcal{M}} \left( \sum_{i=2}^{N-k} y^i X_i \right),$$

with $\bar{y} = (y^2, \ldots, y^{N-k})$ and $\text{Exp}_P^{\mathcal{M}}$ is the exponential map at $P$ in $\mathcal{M}$ endowed with the metric induced by $\mathcal{M}$ defined via the above Fermi coordinates by the map

$$y = (y^1, \tilde{y}, \bar{y}) \mapsto F_{\mathcal{M}}^P(y^1, \tilde{y}, \bar{y}) = h_{\mathcal{M}}^P(\tilde{y}, \bar{y}) + y^1 N_{\mathcal{M}}(h_{\mathcal{M}}^P(\tilde{y}, \bar{y})).$$

Next we denote by $g$ the metric induced by $F_{\mathcal{M}}^P$ whose components are defined by

$$g_{\alpha\beta}(y) = (\partial_\alpha F_{\mathcal{M}}^P(y), \partial_\beta F_{\mathcal{M}}^P(y)).$$

Then we have the following expansions (see for instance [15])

$$g_{11}(y) = 1$$
$$g_{1\beta}(y) = 0, \quad \text{for } \beta = 2, \ldots, N$$
$$g_{\alpha\beta}(y) = \delta_{\alpha\beta} + \mathcal{O}(|\tilde{y}|), \quad \text{for } \alpha, \beta = 2, \ldots, N,$$

where $\bar{y} = (y^1, \tilde{y})$ and $\mathcal{O}(r^m)$ is a smooth function in the variable $y$ which is uniformly bounded by a constant (depending only $\mathcal{M}$ and $\Sigma_k$) times $r^m$.

In concordance to the above coordinates, we will consider the “half”-geodesic neighborhood contained in $\mathcal{U}$ around $\Sigma_k$ of radius $\rho$

$$\mathcal{U}_\rho(\Sigma_k) := \{ x \in \mathcal{U} : \tilde{\delta}(x) < \rho \},$$

where $\tilde{\delta}$ is the projection distance function given by

$$\tilde{\delta}(x) := \sqrt{|\text{dist}^{\mathcal{M}}(\bar{x}, \Sigma_k)|^2 + |x - \bar{x}|^2},$$

where $\bar{x}$ is the orthogonal projection of $x$ on $\mathcal{M}$ and $\text{dist}^{\mathcal{M}}(\cdot, \Sigma_k)$ is the geodesic distance to $\Sigma_k$ on $\mathcal{M}$ with the induced metric. Observe that

$$\tilde{\delta}(F_{\mathcal{M}}^P(y)) = |\tilde{y}|,$$
where \( \tilde{y} = (y^1, \tilde{y}) \). We also define \( \sigma(\tilde{x}) \) to be the orthogonal projection of \( \tilde{x} \) on \( \Sigma_k \) within \( \mathcal{M} \). Letting
\[
\tilde{\delta}(\tilde{x}) := \text{dist}^{\mathcal{M}}(\tilde{x}, \Sigma_k),
\]
one has
\[
\tilde{x} = \text{Exp}^{\mathcal{M}}_{\sigma(\tilde{x})}(\tilde{\delta} \nabla \tilde{\delta}) \quad \text{or equivalently} \quad \sigma(\tilde{x}) = \text{Exp}^{\mathcal{M}}(\tilde{\delta} \nabla \tilde{\delta}).
\]
Next we observe that
\[
\tilde{\delta}(x) = \sqrt{\tilde{\delta}^2(x) + d^2(x)}.
\]
In addition it can be easily checked via the implicit function theorem that there exists a positive constant \( \beta_0 = \beta_0(\Sigma_k, \Omega) \) such that \( \tilde{\delta} \in C^{\infty}(U_{\beta_0}(\Sigma_k)) \).

It is clear that for \( \rho \) sufficiently small, there exists a finite number of Lipschitz open sets \( (T_i)_{1 \leq i \leq N_0} \) such that
\[
T_i \cap T_j = \emptyset \quad \text{for} \quad i \neq j \quad \text{and} \quad U_{\rho}(\Sigma_k) = \bigcup_{i=1}^{N_0} T_i.
\]
We may assume that each \( T_i \) is chosen, using the above coordinates, so that
\[
T_i = F_{\mathcal{M}}^{p_i}(B^N_{+\rho}(0, \rho) \times D_i) \quad \text{with} \quad p_i \in \Sigma_k,
\]
where the \( D_i \)'s are Lipschitz disjoint open sets of \( \mathbb{R}^k \) such that
\[
\bigcup_{i=1}^{N_0} f^{p_i}(D_i) = \Sigma_k.
\]
In the above setting we have

**Lemma 2.1** As \( \tilde{\delta} \to 0 \), the following expansions hold

1. \( \delta^2 = \tilde{\delta}^2(1 + O(\tilde{\delta})) \),
2. \( \nabla \tilde{\delta} \cdot \nabla d = \frac{d}{\tilde{\delta}} \),
3. \( |\nabla \tilde{\delta}| = 1 + O(\tilde{\delta}) \),
4. \( \Delta \tilde{\delta} = \frac{N-k-1}{\tilde{\delta}} + O(1) \),

where \( O(r^m) \) is a function for which there exists a constant \( C = C(\mathcal{M}, \Sigma_k) \) such that
\[
|O(r^m)| \leq Cr^m.
\]

**Proof** (1) Let \( P \in \Sigma_k \). With an abuse of notation, we write \( x(y) = F_{\mathcal{M}}^{P}(y) \) and we set
\[
\vartheta(y) := \frac{1}{2} \delta^2(x(y)).
\]
The function \( \vartheta \) is smooth in a small neighborhood of the origin in \( \mathbb{R}^N \) and a Taylor expansion yields
\[
\vartheta(y) = \vartheta(0, \tilde{y}) \tilde{y} + \nabla \vartheta(0, \tilde{y})[\tilde{y}] + \frac{1}{2} \nabla^2 \vartheta(0, \tilde{y})[\tilde{y}, \tilde{y}] + O(\|\tilde{y}\|^3)
\]
\[
= \frac{1}{2} \nabla^2 \vartheta(0, \tilde{y})[\tilde{y}, \tilde{y}] + O(\|\tilde{y}\|^3).
\]
Here we have used the fact that \( x(0, \bar{y}) \in \Sigma_k \) so that \( \delta(x(0, \bar{y})) = 0 \). We write

\[
\nabla^2 \vartheta(0, \bar{y})[\tilde{y}, \bar{y}] = \sum_{i, l = 1}^{N-k} \Lambda_{il} y^i y^l,
\]

with

\[
\Lambda_{il} := \frac{\partial^2 \vartheta}{\partial y^i \partial y^l} / \tilde{y} = 0
\]

\[
= \frac{\partial}{\partial y^l} \left( \frac{\partial}{\partial x^j} \left( \frac{1}{2} \delta^2(x) \frac{\partial x^j}{\partial y^l} \right) \right) / \tilde{y} = 0
\]

\[
= \frac{\partial^2}{\partial x^l \partial x^s} \left( \frac{1}{2} \delta^2 \right)(x) \frac{\partial x^j}{\partial y^l} \frac{\partial x^s}{\partial y^l} / \tilde{y} = 0 + \frac{\partial}{\partial x^j}(\delta^2(x)) \frac{\partial^2 x^s}{\partial y^l \partial y^l} / \tilde{y} = 0.
\]

Now using the fact that

\[
\frac{\partial x^s}{\partial y^l} / \tilde{y} = 0 = g_{ls} = \delta_{ls} \quad \text{and} \quad \frac{\partial}{\partial x^j}(\delta^2(x)) / \tilde{y} = 0 = 0,
\]

we obtain

\[
\Lambda_{il} y^i y^l = y^i y^l \frac{\partial^2}{\partial x^l \partial x^s} \left( \frac{1}{2} \delta^2 \right)(x) / \tilde{y} = 0
\]

\[
= |\tilde{y}|^2,
\]

where we have used the fact that the matrix \( \left( \frac{\partial^2}{\partial x^l \partial x^s} \left( \frac{1}{2} \delta^2 \right)(x) / \tilde{y} = 0 \right)_{1 \leq i, s \leq N} \) is the matrix of the orthogonal projection onto the normal space of \( T_{f \rho} (\tilde{y}) \Sigma_k \). Hence using (10), we get

\[
\delta^2(x(y)) = |\tilde{y}|^2 + \mathcal{O}(|\tilde{y}|^3).
\]

This together with (8) prove the first expansion.

(2) Thanks to (8) and (6), we infer that

\[
\nabla \delta \cdot \nabla d(x(y)) = \frac{\partial \delta(x(y))}{\partial y^l} = y^l / |\tilde{y}| = \frac{d(x(y))}{\delta(x(y))}
\]

as desired.

(3) We observe that

\[
\frac{\partial \delta}{\partial x^\tau} \frac{\partial \delta}{\partial x^\tau}(x(y)) = g^{\tau\alpha}(y) g^{\tau\beta}(y) \frac{\partial \delta(x(y))}{\partial y^\alpha} \frac{\partial \delta(x(y))}{\partial y^\beta},
\]

where \( g^{\alpha\beta} \) are the entries of the inverse of the matrix \( (g_{\alpha\beta})_{\alpha, \beta=1,\ldots,N} \). Therefore using again (6) and (8), we get the desired result.

(4) Finally using the expansion of the Laplace-Beltrami operator \( \Delta_g \), see Lemma 3.3 in [18], applied to (8), we get the last estimate. \( \square \)

In the following of – only – this section, the function \( q : \tilde{U} \rightarrow \mathbb{R} \) will be such that

\[
q \in C^2(\tilde{U}), \quad \text{and} \quad q \leq 1 \quad \text{on} \quad \Sigma_k.
\]
Let $M, a \in \mathbb{R}$, we consider the function
\[ W_{a,M,q}(x) = X_a(\tilde{\delta}(x)) e^{Md(x)} d(x) \tilde{\delta}(x)^{\alpha(x)}, \]
where
\[ X_a(t) = (-\log(t))^a \quad 0 < t < 1 \]
and
\[ \alpha(x) = \frac{k - N}{2} + \frac{N - k}{2} \sqrt{1 - q(\sigma(x)) + \tilde{\delta}(x)}. \]

In the above setting, the following useful result holds.

**Lemma 2.2** As the parameter $\delta \to 0$, the laplacian of the function $W_{a,M,q}$ defined in (12) can be expanded as
\[
\Delta W_{a,M,q} = -\frac{(N - k)^2}{4} q \delta^{-2} W_{a,M,q} - 2a \sqrt{\tilde{\alpha}} X_{-1}(\delta) \delta^{-2} W_{a,M,q} \\
+ a(a - 1) X_{-2}(\delta) \delta^{-2} W_{a,M,q} + \frac{h + 2M}{d} W_{a,M,q} + O(|\log(\delta)| \delta^{-\frac{3}{2}}) W_{a,M,q},
\]
where $\tilde{\alpha}(x) = \frac{(N-k)^2}{4} \left(1 - q(\sigma(x)) + \tilde{\delta}(x)\right)$ and $h = \Delta d$. Here the lower order term satisfies
\[ |O(r)| \leq C|r|, \]
where $C$ is a positive constant only depending on $a$, $M$, $\Sigma_k, \mathcal{U}$ and $\|q\|_{C^2(\mathcal{U})}$.

**Proof** We put $s = \frac{(N-k)^2}{4}$. Let $w = \tilde{\delta}(x)^{\alpha(x)}$ then the following formula can be easily verified
\[ \Delta w = w \left( \Delta \log(w) + |\nabla \log(w)|^2 \right). \tag{13} \]
Since
\[ \log(w) = \alpha \log(\tilde{\delta}), \]
we get
\[ \Delta \log(w) = \Delta \alpha \log(\tilde{\delta}) + 2 \nabla \alpha \cdot \nabla (\log(\tilde{\delta})) + \alpha \Delta \log(\tilde{\delta}). \tag{14} \]
We have
\[ \Delta \alpha = \Delta \sqrt{\tilde{\alpha}} = \sqrt{\tilde{\alpha}} \left( \frac{1}{2} \Delta \log(\tilde{\alpha}) + \frac{1}{4} |\nabla \log(\tilde{\alpha})|^2 \right), \tag{15} \]
\[ \nabla \log(\tilde{\alpha}) = \frac{\nabla \tilde{\alpha}}{\tilde{\alpha}} = \frac{-s \nabla (q \circ \sigma) + s \nabla \tilde{\delta}}{\tilde{\alpha}} \]
and using the formula (13), we obtain
\[
\Delta \log(\tilde{\alpha}) = \frac{\Delta \tilde{\alpha}}{\tilde{\alpha}} - \frac{|\nabla \tilde{\alpha}|^2}{\tilde{\alpha}^2} = \frac{-s \Delta (q \circ \sigma) + s \Delta \tilde{\delta}}{\tilde{\alpha}} - \frac{s^2 |\nabla (q \circ \sigma)|^2}{\tilde{\alpha}^2} + \frac{s^2 |\nabla \tilde{\delta}|^2}{\tilde{\alpha}^2} + 2s^2 \frac{\nabla (q \circ \sigma) \cdot \nabla \tilde{\delta}}{\tilde{\alpha}^2}. \]
Putting the above in (15), we deduce that
\[
\Delta \alpha = \frac{1}{2\sqrt{\alpha}} \left\{ -s \Delta (q \circ \sigma) + s \Delta \bar{\delta} - \frac{1}{2} s^2 |\nabla (q \circ \sigma)|^2 + \frac{s^2 |\nabla \bar{\delta}|^2 - 2s^2 \nabla (q \circ \sigma) \cdot \nabla \bar{\delta}}{\bar{\alpha}} \right\}.
\]
(16)

Using Lemma 2.1 and the fact that \( q \) is in \( C^2(U) \), together with (16) we get
\[
\Delta \alpha = O(\bar{\delta}^{-\frac{3}{2}}).
\]
(17)

On the other hand
\[
\nabla \alpha = \nabla \sqrt{\alpha} = \frac{1}{2} \frac{\nabla \alpha}{\sqrt{\alpha}} = -\frac{s}{2\sqrt{\alpha}} \nabla (q \circ \sigma) + \frac{s}{2} \frac{\nabla \bar{\delta}}{\sqrt{\alpha}}
\]
so that
\[
\nabla \alpha \cdot \nabla \bar{\delta} = -\frac{s}{2\sqrt{\alpha}} \nabla (q \circ \sigma) \cdot \nabla \bar{\delta} + \frac{s}{2} \frac{|\nabla \bar{\delta}|^2}{\sqrt{\alpha}} = O(\bar{\delta}^{-\frac{1}{2}})
\]
and from which we deduce that
\[
\nabla \alpha \cdot \nabla \log(\bar{\delta}) = \frac{1}{\bar{\delta}} \nabla \alpha \cdot \nabla \bar{\delta} = O(\bar{\delta}^{-\frac{3}{2}}).
\]
(18)

By Lemma 2.1 we have that
\[
\alpha \Delta \log(\bar{\delta}) = \alpha \frac{N - k - 2}{\bar{\delta}^2} (1 + O(\bar{\delta})).
\]
Taking back the above estimate together with (18) and (17) in (14), we get
\[
\Delta \log(w) = \alpha \frac{N - k - 2}{\bar{\delta}^2} (1 + O(\bar{\delta})) + O(|\log(\bar{\delta})| \bar{\delta}^{-\frac{3}{2}}).
\]
(19)

We also have
\[
\nabla (\log(w)) = \nabla (\alpha \log(\bar{\delta})) = \alpha \frac{\nabla \bar{\delta}}{\bar{\delta}} + \log(\bar{\delta}) \nabla \alpha
\]
and thus
\[
|\nabla (\log(w))|^2 = \frac{\alpha^2}{\bar{\delta}^2} + \frac{2\alpha \log(\bar{\delta})}{\bar{\delta}} \nabla \bar{\delta} \cdot \nabla \alpha + |\log(\bar{\delta})|^2 |\nabla \alpha|^2 = \frac{\alpha^2}{\bar{\delta}^2} + O(|\log(\bar{\delta})| \bar{\delta}^{-\frac{3}{2}}).
\]
Putting this together with (19) in (13), we conclude that
\[
\frac{\Delta w}{w} = \alpha \frac{N - k - 2}{\bar{\delta}^2} + \frac{\alpha^2}{\bar{\delta}^2} + O(|\log(\bar{\delta})| \bar{\delta}^{-\frac{3}{2}}).
\]
(20)

Now we define the function
\[
v(x) := d(x) w(x),
\]
where we recall that \( d \) is the distance function to the boundary of \( U \). It is clear that
\[
\Delta v = w \Delta d + d \Delta w + 2 \nabla d \cdot \nabla w.
\]
(21)

Notice that
\[
\nabla w = w \nabla \log(w) = w \left( \log(\bar{\delta}) \nabla \alpha + \alpha \frac{\nabla \bar{\delta}}{\bar{\delta}} \right)
\]
and so
\[ \nabla d \cdot \nabla w = w \left( \log(\delta) \nabla d \cdot \nabla \alpha + \frac{\alpha}{\delta} \nabla d \cdot \nabla \delta \right). \] (22)

Recall the second assertion of Lemma 2.1 that we rewrite as
\[ \nabla d \cdot \nabla \delta = \frac{d}{\delta}. \] (23)

Therefore
\[ \nabla d \cdot \nabla \alpha = \nabla d \cdot \left( -\frac{s}{2\sqrt{\alpha}} \nabla (q \circ \sigma) + \frac{s}{\sqrt{\alpha}} \nabla \delta \right) = \frac{s}{2\sqrt{\alpha} \delta} \frac{d}{\delta} - \frac{s}{2\sqrt{\alpha} \delta} \nabla d \cdot \nabla (q \circ \sigma). \] (24)

Notice that if \( x \) is in a neighborhood of some point \( P \in \Sigma_k \) one has
\[ \nabla d \cdot \nabla (q \circ \sigma)(x) = \frac{\partial}{\partial y^1} q(\sigma(x)) = \frac{\partial}{\partial y^1} q(f^P(\bar{\Sigma})) = 0. \]

This with (24) and (23) in (22) give
\[ \nabla d \cdot \nabla w = w \left( O\left(\delta^{-\frac{3}{2}} \log(\delta)\right) \right) + \frac{\alpha}{\delta^2} \frac{d}{\delta}. \] (25)

From (20), (21) and (25) (recalling the expression of \( \alpha \) above), we get immediately
\[ \Delta v = \left( \alpha \frac{N-k}{\delta^2} + \frac{\alpha^2}{\delta^2} \right) v + O\left(\left| \log(\delta)\right| \delta^{-\frac{3}{2}} \right) v + \frac{h}{d} v \]
\[ = \left( -\frac{(N-k)^2 q(x)}{4 \delta^2} + O\left(\left| \log(\delta)\right| \delta^{-\frac{3}{2}}\right) \right) v + \frac{h}{d} v, \] (26)

where \( h = \Delta d \). Here we have used the fact that \( |q(x) - q(\sigma(x))| \leq C \delta(x) \) for \( x \) in a neighborhood of \( \Sigma_k \).

Recall the definition of \( W_{a,M,q} \)
\[ W_{a,M,q}(x) = X_a(\tilde{\delta}(x)) e^{Md(x)} v(x), \quad \text{with} \quad X_a(\tilde{\delta}(x)) := (-\log(\tilde{\delta}(x)))^a, \]
where \( M \) and \( a \) are two real numbers. We have
\[ \Delta W_{a,M,q} = X_a(\tilde{\delta}) \Delta(e^{Md} v) + 2 \nabla X_a(\tilde{\delta}) \cdot \nabla(e^{Md} v) + e^{Md} v \Delta X_a(\tilde{\delta}) \]
and thus
\[ \Delta W_{a,M,q} = X_a(\tilde{\delta}) e^{Md} \Delta v + X_a(\tilde{\delta}) \Delta(e^{Md} v) + 2 X_a(\tilde{\delta}) \nabla v \cdot \nabla(e^{Md} v) \]
\[ + 2 \nabla X_a(\tilde{\delta}) \cdot \left( v \nabla(e^{Md}) + e^{Md} \nabla v \right) + e^{Md} v \Delta X_a(\tilde{\delta}). \] (27)

We shall estimate term by term the above expression.
First we have form (26)
\[ X_a(\tilde{\delta}) e^{Md} \Delta v = -\frac{(N-k)^2 q}{4 \delta^2} W_{a,M,q} + \frac{h}{d} W_{a,M,q} + O\left(\left| \log(\delta)\right| \delta^{-\frac{3}{2}}\right) W_{a,M,q}. \] (28)

On the other hand it is plain that
\[ X_a(\tilde{\delta}) \Delta(e^{Md} v) = O(1) W_{a,M,q}. \] (29)
It is clear that
\[ \nabla v = w \nabla d + d \nabla w = w \nabla d + d \left( \log(\delta) \nabla \alpha + \frac{\nabla \delta}{\delta} \right) w. \tag{30} \]

From which and (23) we get
\[
X_a(\delta) \nabla \cdot \nabla (e^{Md}) = M X_a(\delta) e^{Md} w \left\{ |\nabla d|^2 + d \left( \log(\delta) \nabla d \cdot \nabla \alpha + \frac{\alpha}{\delta} \nabla \delta \cdot \nabla d \right) \right\}
\]
\[
= M X_a(\delta) e^{Md} w \left\{ 1 + O(\log(\delta) \delta^{-\frac{1}{2}}) d + O(\delta^{-1}) d \right\}
\]
\[
= W_{a,M,q} \left\{ \frac{M}{d} + O(\log(\delta) \delta^{-1}) \right\}. \tag{31}
\]

Observe that
\[ \nabla (X_a(\delta)) = -a \frac{\nabla \delta}{\delta} X_{a-1}(\delta). \]

This with (30) and (23) imply that
\[ \nabla X_a(\delta) \cdot \left( v \nabla (e^{Md}) + e^{Md} \nabla v \right) = -\frac{a(\alpha + 1)}{\delta^2} X_{-1} W_{a,M,q} + O(\log(\delta) \delta^{-\frac{3}{2}}) W_{a,M,q}. \tag{32} \]

By Lemma 2.1, we have
\[ \Delta (X_a(\delta)) = \frac{a}{\delta^2} X_{a-1}(\delta) [2 + k - N + O(\delta)] + \frac{a(a - 1)}{\delta^2} X_{a-2}(\delta). \]

Therefore we obtain
\[ e^{Md} v \Delta (X_a(\delta)) = \frac{a}{\delta^2} [2 + k - N + O(\delta)] X_{-1} W_{a,M,q} + \frac{a(a - 1)}{\delta^2} X_{-2} W_{a,M,q}. \tag{33} \]

Collecting (28), (29), (31), (32) and (33) in the expression (27), we get as \( \delta \to 0 \)
\[
\Delta W_{a,M,q} = -\frac{(N - k)^2}{4} \delta^{-2} W_{a,M,q} - 2a \sqrt{\alpha} X_{-1}(\delta) \delta^{-2} W_{a,M,q}
\]
\[ + a(a - 1) X_{-2}(\delta) \delta^{-2} W_{a,M,q} + \frac{h + 2M}{d} W_{a,M,q} + O(\log(\delta) \delta^{-\frac{3}{2}}) W_{a,M,q}. \]

The conclusion of the lemma follows then from the first assertion of Lemma 2.1.

2.1 Construction of a subsolution

For \( \lambda \in \mathbb{R} \) and \( \eta \in Lip(\Omega) \) with \( \eta = 0 \) on \( \Sigma_k \), we define the operator
\[ \mathcal{L}_\lambda := -\Delta - \frac{(N - k)^2}{4} q \delta^{-2} \lambda \eta \delta^{-2}, \tag{34} \]

where \( q \) is as in (11). We have the following lemma

**Lemma 2.3** There exist two positive constants \( M_0, \beta_0 \) such that for all \( \beta \in (0, \beta_0) \) the function \( V_\varepsilon := W_{-1,M_0,q} + W_{0,M_0,q-\varepsilon} \) (see (12)) satisfies
\[ \mathcal{L}_\lambda V_\varepsilon \leq 0 \quad \text{in } \Omega, \quad \text{for all } \varepsilon \in [0, 1). \tag{35} \]
Moreover $V_\varepsilon \in H^1(U_\beta)$ for any $\varepsilon \in (0, 1)$ and in addition
\[ \int_{U_\beta} \frac{V_\varepsilon^2}{\delta^2} \, dx \geq C \int_{\Sigma_k} \frac{1}{\sqrt{1 - q(\sigma)}} \, d\sigma. \] (36)

**Proof** Let $\beta_1$ be a positive small real number so that $d$ is smooth in $U_{\beta_1}$. We choose
\[ M_0 = \max_{x \in U_{\beta_1}} |h(x)| + 1. \]
Using this and Lemma 2.2, for some $\beta \in (0, \beta_1)$, we have
\[ \mathcal{L}_\beta W_{-1,M_0,q} \leq \left( -2\delta^{-2} X_{-2} + C |\log(\delta)| \delta^{-\frac{3}{2}} + |\lambda| \eta \delta^{-2} \right) W_{-1,M_0,q} \quad \text{in} \quad U_{\beta}. \] (37)
Using the fact that the function $\eta$ vanishes on $\Sigma_k$ (this implies in particular that $|\eta| \leq C \delta$ in $U_\beta$), we have
\[ \mathcal{L}_\beta (W_{-1,M_0,q}) \leq -\delta^{-2} X_{-2} W_{-1,M_0,q} = -\delta^{-2} X_{-3} W_{0,M_0,q} \quad \text{in} \quad U_\beta, \]
for $\beta$ sufficiently small. Again by Lemma 2.2, and similar arguments as above, we have
\[ \mathcal{L}_\beta W_{0,M_0,q-\varepsilon} \leq C |\log(\delta)| \delta^{-\frac{3}{2}} W_{0,M_0,q-\varepsilon} \leq C |\log(\delta)| \delta^{-\frac{3}{2}} W_{0,M_0,q} \quad \text{in} \quad U_\beta, \] (38)
for any $\varepsilon \in [0, 1)$. Therefore we get
\[ \mathcal{L}_\beta (W_{-1,M_0,q} + W_{0,M_0,q-\varepsilon}) \leq 0 \quad \text{in} \quad U_\beta, \]
if $\beta$ is small. This proves (35).

The proof of the fact that $W_{a,M_0,q} \in H^1(U_\beta)$, for any $a < -\frac{1}{2}$ and $W_{0,M_0,q-\varepsilon} \in H^1(U_\beta)$, for $\varepsilon > 0$ can be easily checked using polar coordinates (by assuming without any loss of generality that $M_0 = 0$ and $q \equiv 1$), we therefore skip it.

We now prove the last statement of the theorem. Using Lemma 2.1, we have
\[ \int_{U_\beta} \frac{V_0^2}{\delta^2} \, dx \geq \int_{U_\beta} \frac{W_{0,M_0,q}^2}{\delta^2} \, dx \]
\[ \geq C \int_{U_{\beta} \setminus (\Sigma_k)} d^2(x) \tilde{\delta}(x)^{2a(x) - 2} \, dx \]
\[ \geq C \sum_{i=1}^{N_0} \int_{T_i} d^2(x) \tilde{\delta}(x)^{2a(x) - 2} \, dx \]
\[ = C \sum_{i=1}^{N_0} \int_{B_{x,N-4}(0,\beta) \times D_i} (y_1)^2 |\tilde{y}|^{2a(F_{\mathcal{M}}^p_i(y)) - 2} |\text{Jac}(F_{\mathcal{M}}^p_i)(y)| \, dy \]
\[ \geq C \sum_{i=1}^{N_0} \int_{B_{x,N-4}(0,\beta) \times D_i} (y_1)^2 |\tilde{y}|^{-k-N-2+(N-k)\sqrt{1-q(f^p_i(\tilde{y}))}} |\tilde{y}|^{-\sqrt{1}|\tilde{y}|} \, dy. \]

Here we used the fact that $|\text{Jac}(F_{\mathcal{M}}^p_i)(y)| \geq C$. Observe that
\[ |\tilde{y}|^{-\sqrt{1}|\tilde{y}|} \geq C > 0 \quad \text{as} \quad |\tilde{y}| \to 0. \]
Using polar coordinates, the above integral becomes
\[ \int_{U_\beta} \frac{V_0^2}{\delta^2} \, dx \geq C \sum_{i=1}^{N_0} \int_{D_i} \int_{S^{N-k-1}_+} \left( \frac{y_1}{|\tilde{y}|} \right)^2 \, d\theta \int_0^{\beta} r^{1+(N-k)-q(f^*_i(y))} \, d\tilde{y} \]
\[ \geq C \sum_{i=1}^{N_0} \int_0^{r_i} \int_0^{\beta} r^{1+(N-k)-q(f^*_i(y))} \left| \text{Jac}(f^*_i)(\tilde{y}) \right| \, d\tilde{y} \, d\sigma. \]

We therefore obtain
\[ \int_{U_\beta} \frac{V_0^2}{\delta^2} \, dx \geq C \int_{\Sigma_k} \int_0^{\beta} r^{1+(N-k)-q(\sigma)} \, dr \, d\sigma \]
\[ \geq C \int_{\Sigma_k} \frac{1}{\sqrt{1-q(\sigma)}} \, d\sigma. \]

This concludes the proof of the lemma. \( \square \)

2.2 Construction of a supersolution

In this subsection we provide a supersolution for the operator \( \mathcal{L}_\lambda \) defined in (34). We prove

**Lemma 2.4** There exist constants \( \beta_0 > 0, M_1 < 0, M_0 > 0 \) (the constant \( M_0 \) is as in Lemma 2.3) such that for all \( \beta \in (0, \beta_0) \) the function \( U := W_{0,M_1,q} - W_{-1,M_0,q} > 0 \) in \( U_\beta \) and satisfies

\[ \mathcal{L}_\lambda U_a \geq 0 \quad \text{in} \quad U_\beta. \] (39)

Moreover \( U \in H^1(U_\beta) \) provided

\[ \int_{\Sigma_k} \frac{1}{\sqrt{1-q(\sigma)}} \, d\sigma < +\infty. \] (40)

**Proof** We consider \( \beta_1 \) as in the beginning of the proof of Lemma 2.3 and we define

\[ M_1 = -\frac{1}{2} \max_{x \in \mathcal{A}_{\beta_1}} |h(x)| - 1. \] (41)

Since

\[ U(x) = (e^{M_1 d(x)} - e^{M_0 d(x)} X_{-1} (\tilde{d}(x))) d(x) \tilde{d}(x)^a(x), \]

it follows that \( U > 0 \) in \( U_\beta \) for \( \beta > 0 \) sufficiently small. By (41) and Lemma 2.2, we get

\[ \mathcal{L}_\lambda W_{0,M_1,q} \geq \left( -C |\log(\delta)| \delta^{-\frac{3}{2}} - |\lambda| \eta \delta^{-2} \right) W_{0,M_1,q}. \]

Using (37) we have

\[ \mathcal{L}_\lambda (-W_{-1,M_0,q}) \geq \left( 2\delta^{-2} X_{-2} - C |\log(\delta)| \delta^{-\frac{3}{2}} - |\lambda| \eta \delta^{-2} \right) W_{-1,M_0,q}. \]
Taking the sum of the two above inequalities, we obtain

\[ \mathcal{L}_\lambda U \geq 0 \quad \text{in} \quad U_\beta, \]

which holds true because \(|\eta| \leq C \delta \) in \( U_\beta \). Hence we get readily (39).

Our next task is to prove that \( U \in H^1(U_\beta) \) provided (40) holds, to do so it is enough to show that \( W_{0,M_1,q} \in H^1(U_\beta) \) provided (40) holds.

We argue as in the proof of Lemma 2.3. We have

\[
\int_{U_\beta} |\nabla W_{0,M_1,q}|^2 \leq C \int_{U_\beta} d^2(x) \tilde{\delta}(x)^{2\alpha(x)-2} dx \\
\leq C \sum_{i=1}^{N_0} \int_{B_{v_+}^{N-k}(0,\beta) \times D_i} d^2(F_{p_1}^{\mathcal{M}}(y))\tilde{\delta}(F_{p_1}^{\mathcal{M}}(y))^{2\alpha(F_{p_1}^{\mathcal{M}}(y))-2} |\text{Jac}(F_{p_1}^{\mathcal{M}})|(y) dy \\
\leq C \sum_{i=1}^{N_0} \int_{B_{v_+}^{N-k}(0,\beta) \times D_i} (y^1)^2 |\tilde{y}|^{2\alpha(F_{p_1}^{\mathcal{M}}(y))-2} |\text{Jac}(F_{p_1}^{\mathcal{M}})|(y) dy \\
\leq C \sum_{i=1}^{N_0} \int_{B_{v_+}^{N-k}(0,\beta) \times D_i} (y^1)^2 |\tilde{y}|^{k-2+(N-k)\sqrt{1-q(f_{p_1}(\bar{y}))}} |\bar{y}|^{-\sqrt{|\bar{y}|}} d\bar{y}.
\]

Here we used the fact that \(|\text{Jac}(F_{p_1}^{\mathcal{M}})|(y) \leq C\). Note that

\[ |\tilde{y}|^{-\sqrt{|\bar{y}|}} \leq C \quad \text{as} \quad |\bar{y}| \to 0. \]

Using polar coordinates, it follows that

\[
\int_{U_\beta} |\nabla W_{0,M_1,q}|^2 \leq C \sum_{i=1}^{N_0} \int_{D_i} \int_{S_{v_+}^{N-k-1}} \left( \frac{y^1}{\tilde{y}} \right)^2 d\theta \int_0^\beta r^{-1+(N-k)\sqrt{1-q(f_{p_1}(\bar{y}))}} dr d\bar{y} \\
\leq C \sum_{i=1}^{N_0} \int_{D_i} \frac{1}{\sqrt{1-q(f_{p_1}(\bar{y}))}} d\bar{y}.
\]

Recalling that \(|\text{Jac}(f_{p_1})|(\bar{y}) = 1 + O(|\bar{y}|)\), we deduce that

\[
\sum_{i=1}^{N_0} \int_{D_i} \frac{1}{\sqrt{1-q(f_{p_1}(\bar{y}))}} d\bar{y} \leq C \sum_{i=1}^{N_0} \int_{D_i} \frac{1}{\sqrt{1-q(f_{p_1}(\bar{y}))}} |\text{Jac}(f)|(y) d\bar{y} \\
= C \int_{\Sigma_k} \frac{1}{\sqrt{1-q(\sigma)}} d\sigma.
\]

Therefore

\[
\int_{U_\beta} |\nabla W_{0,M_1,q}|^2 dx \leq C \int_{\Sigma_k} \frac{1}{\sqrt{1-q(\sigma)}} d\sigma
\]

and the lemma follows at once. \qed
3 Existence of $\lambda^*$

We start with the following local improved Hardy inequality.

**Lemma 3.1** Let $\Omega$ be a smooth domain and assume that $\partial \Omega$ contains a smooth closed submanifold $\Sigma_k$ of dimension $1 \leq k \leq N - 2$. Assume that $p, q$ and $\eta$ satisfy (2) and (3). Then there exist constants $\beta_0 > 0$ and $c > 0$ depending only on $\Omega$, $\Sigma_k$, $q$, $\eta$ and $p$ such that for all $\beta \in (0, \beta_0)$ the inequality

$$\int_{\Omega_{\beta}} p |\nabla u|^2 \, dx - \frac{(N - k)^2}{4} \int_{\Omega_{\beta}} q |u|^2 \, dx \geq c \int_{\Omega_{\beta}} |u|^2 |\log(\delta)|^2 \, dx$$

holds for all $u \in H^1_0(\Omega_{\beta})$.

**Proof** We use the notations in Sect. 2 with $\mathcal{U} = \Omega$ and $\mathcal{M} = \partial \Omega$. Fix $\beta_1 > 0$ small and

$$M_2 = -\frac{1}{2} \max_{x \in \Omega_{\beta_1}} (|h(x)| + |\nabla p \cdot \nabla d|) - 1. \quad (42)$$

Since $\frac{p}{q} \in C^1(\Omega),$ there exists $C > 0$ such that

$$\frac{|p(x) - p(\sigma(x))|}{q(x)} \leq C\delta(x) \quad \forall x \in \Omega_{\beta}, \quad (43)$$

for small $\beta > 0$. Hence by (3) there exits a constant $C' > 0$ such that

$$p(x) \geq q(x) - C'\delta(x) \quad \forall x \in \Omega_{\beta}. \quad (44)$$

Consider $W_{1, M_2, 1}$ (in Lemma 2.2 with $q \equiv 1$). For all $\beta > 0$ small, we set

$$\tilde{w}(x) = W_{1, M_2, 1}(x), \quad \forall x \in \Omega_{\beta}. \quad (45)$$

Notice that $\text{div}(p \nabla \tilde{w}) = p \Delta \tilde{w} + \nabla p \cdot \nabla \tilde{w}$. By Lemma 2.2, we have

$$- \frac{\text{div}(p \nabla \tilde{w})}{\tilde{w}} \geq \frac{(N - k)^2}{4} p\delta^{-2} + \frac{p}{4} \delta^{-2} X_{-2}(\delta) + O(|\log(\delta)|\delta^{-\frac{3}{2}}) \quad \text{in} \ \Omega_{\beta}. \quad (46)$$

This together with (44) yields

$$- \frac{\text{div}(p \nabla \tilde{w})}{\tilde{w}} \geq \frac{(N - k)^2}{4} q\delta^{-2} + \frac{c_0}{4} \delta^{-2} X_{-2}(\delta) + O(|\log(\delta)|\delta^{-\frac{3}{2}}) \quad \text{in} \ \Omega_{\beta},$$

with $c_0 = \min_{\Omega_{\beta_1}} p > 0$. Therefore

$$- \frac{\text{div}(p \nabla \tilde{w})}{\tilde{w}} \geq \frac{(N - k)^2}{4} q\delta^{-2} + c \delta^{-2} X_{-2}(\delta) \quad \text{in} \ \Omega_{\beta}, \quad (46)$$

for some positive constant $c$ depending only on $\Omega$, $\Sigma_k$, $q$, $\eta$ and $p$.

Let $u \in C^{\infty}_c(\Omega_{\beta})$ and put $\psi = \frac{u}{\tilde{w}}$. Then one has $|\nabla u|^2 = |\tilde{w} \nabla \psi|^2 + |\psi \nabla \tilde{w}|^2 + \nabla(\psi^2) \cdot \tilde{w} \nabla \tilde{w}$. Therefore $|\nabla u|^2 = |\tilde{w} \nabla \psi|^2 + p \nabla \tilde{w} \cdot \nabla(\tilde{w} \psi^2)$. Integrating by parts, we get

$$\int_{\Omega_{\beta}} |\nabla u|^2 \, dx = \int_{\Omega_{\beta}} |\tilde{w} \nabla \psi|^2 \, dx + \int_{\Omega_{\beta}} \left( - \frac{\text{div}(p \nabla \tilde{w})}{\tilde{w}} \right) u^2 \, dx.$$

Putting (46) in the above equality, we get the desired result. \qed

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We next prove the following result

**Lemma 3.2** Let \( \Omega \) be a smooth bounded domain and assume that \( \partial \Omega \) contains a smooth closed submanifold \( \Sigma_k \) of dimension \( 1 \leq k \leq N - 2 \). Assume that (2) and (3) hold. Then there exists \( \lambda^* = \lambda^*(\Omega, \Sigma_k, p, q, \eta) \in \mathbb{R} \) such that

\[
\mu_\lambda(\Omega, \Sigma_k) = \frac{(N - k)^2}{4}, \quad \forall \lambda \leq \lambda^*,
\]

\[
\mu_\lambda(\Omega, \Sigma_k) < \frac{(N - k)^2}{4}, \quad \forall \lambda > \lambda^*.
\]

**Proof** We device the proof in two steps

**Step 1:** We claim that:

\[
\sup_{\lambda \in \mathbb{R}} \mu_\lambda(\Omega, \Sigma_k) \leq \frac{(N - k)^2}{4}.
\]  

(47)

Indeed, we know that \( v_0(\mathbb{R}^N_+, \mathbb{R}^k) = \frac{(N-k)^2}{4} \), see [17] for instance. Given \( \tau > 0 \), we let \( u_\tau \in C_\infty^\infty(\mathbb{R}^N_+) \) be such that

\[
\int_{\mathbb{R}^N_+} |\nabla u_\tau|^2 \, dy \leq \left( \frac{(N - k)^2}{4} + \tau \right) \int_{\mathbb{R}^N_+} |\tilde{y}|^{-2} u_\tau^2 \, dy.
\]

(48)

By (3), we can let \( \sigma_0 \in \Sigma_k \) be such that

\[ q(\sigma_0) = p(\sigma_0). \]

Now, given \( r > 0 \), we let \( \rho_r > 0 \) such that for all \( x \in B(\sigma_0, \rho_r) \cap \Omega \)

\[
p(x) \leq (1 + r)q(\sigma_0), \quad q(x) \geq (1 - r)q(\sigma_0) \quad \text{and} \quad \eta(x) \leq r.
\]

(49)

We choose Fermi coordinates near \( \sigma_0 \in \Sigma_k \) given by the map \( F_{\sigma_0}^{\sigma_0} \) (as in Sect. 2) and we choose \( \varepsilon_0 > 0 \) small such that, for all \( \varepsilon \in (0, \varepsilon_0), \)

\[
\Lambda_{\varepsilon, \rho, r, \tau} := F_{\sigma_0}^{\sigma_0}(\varepsilon \, \text{Supp}(u_\tau)) \subset B(\sigma_0, \rho_r) \cap \Omega
\]

and we define the following test function

\[
v(x) = \varepsilon^{2-N} u_\tau \left( \varepsilon^{-1}(F_{\sigma_0}^{\sigma_0})^{-1}(x) \right), \quad x \in \Lambda_{\varepsilon, \rho, r, \tau}.
\]

Clearly, for every \( \varepsilon \in (0, \varepsilon_0) \), we have that \( v \in C_\infty^\infty(\Omega) \) and thus by a change of variable, (49) and Lemma 2.1, we have

\[
\mu_\lambda(\Omega, \Sigma_k) \leq \frac{\int_{\Omega} p |\nabla v|^2 \, dx + \lambda \int_{\Omega} \delta^{-2} \eta v^2 \, dx}{\int_{\Omega} q(x) \delta^{-2} v^2 \, dx}
\]

\[
\leq \frac{\int_{\Lambda_{\varepsilon, \rho, r, \tau}} |\nabla v|^2 \, dx}{(1 + r) \int_{\Lambda_{\varepsilon, \rho, r, \tau}} \delta^{-2} v^2 \, dx + \frac{r|\lambda|}{(1 - r)q(\sigma_0)}}
\]

\[
\leq \frac{\int_{\Lambda_{\varepsilon, \rho, r, \tau}} |\nabla v|^2 \, dx}{(1 - cr) \int_{\Lambda_{\varepsilon, \rho, r, \tau}} \delta^{-2} v^2 \, dx + \frac{r|\lambda|}{(1 - r)q(\sigma_0)}}
\]

\[
\leq \frac{(1 + r)\varepsilon^{2-N} \int_{\mathbb{R}^N_+} \varepsilon^{-2} (g_{ij}^\varepsilon \delta_{ij} + \varepsilon\sqrt{\Delta g_{ij}^\varepsilon}(y) \, dy}{(1 - cr) \int_{\mathbb{R}^N_+} \varepsilon^{2-N} |\varepsilon \tilde{y}|^{-2} u_\tau^2 \sqrt{|\Delta g_{ij}^\varepsilon}(y) \, dy + \frac{cr}{1 - r}}.
\]
where $g^\varepsilon$ is the scaled metric with components

$$g_{\alpha\beta}^\varepsilon(y) = \varepsilon^{-2} (\partial_\alpha F_{\alpha\beta}(\varepsilon y), \partial_\beta F_{\alpha\beta}(\varepsilon y))$$

for $\alpha, \beta = 1, \ldots, N$ and where we have used the fact that $\bar{\delta}(F_{\alpha\beta}(\varepsilon y)) = |\varepsilon \bar{y}|^2$ for every $\bar{y}$ in the support of $u_\varepsilon$. Since the scaled metric $g^\varepsilon$ expands a $g^\varepsilon = I + O(\varepsilon)$ on the support of $u_\varepsilon$, we deduce that

$$\mu_\lambda(\Omega, \Sigma_k) \leq \frac{1 + r}{1 - cr} \frac{1 + c\varepsilon}{1 - c\varepsilon} \left( \frac{(N - k)^2}{4} + \tau \right) + \frac{cr}{1 - r},$$

where $c$ is a positive constant depending only on $\Omega, \Sigma_k$. Hence by (48) we conclude

$$\mu_\lambda(\Omega, \Sigma_k) \leq \frac{1 + r}{1 - cr} \frac{1 + c\varepsilon}{1 - c\varepsilon} \left( \frac{(N - k)^2}{4} + \tau \right) + \frac{cr}{1 - r}.$$

Taking the limit in $\varepsilon$, then in $r$ and then in $\tau$, the claim follows.

**Step 2:** We claim that there exists $\tilde{\lambda} \in \mathbb{R}$ such that $\mu_\tilde{\lambda}(\Omega, \Sigma_k) \geq \frac{(N - k)^2}{4}$.

Thanks to Lemma 3.1, the proof uses a standard argument of cut-off function and integration by parts (see [4]) and we can obtain

$$\int_{\Omega} \delta^{-2} u^2 q \, dx \leq \int_{\Omega} |\nabla u|^2 \, p \, dx + C \int_{\Omega} \delta^{-2} u^2 \eta \, dx \quad \forall u \in C^\infty_c(\Omega),$$

for some constant $C > 0$. We skip the details. The claim now follows by choosing $\tilde{\lambda} = -C$

Finally, noticing that $\mu_\lambda(\Omega, \Sigma_k)$ is decreasing in $\lambda$, we can set

$$\lambda^* := \sup \left\{ \lambda \in \mathbb{R} : \mu_\lambda(\Omega, \Sigma_k) = \frac{(N - k)^2}{4} \right\}$$

(50)

so that $\mu_\lambda(\Omega, \Sigma_k) < \frac{(N - k)^2}{4}$ for all $\lambda > \lambda^*$. $\Box$

### 4 Non-existence result

**Lemma 4.1** Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^N$, $N \geq 3$, and let $\Sigma_k$ be a smooth closed submanifold of $\partial \Omega$ of dimension $k$ with $1 \leq k \leq N - 2$. Then, there exist bounded smooth domains $\Omega^\pm$ such that $\Omega^+ \subset \Omega \subset \Omega^-$ and

$$\partial \Omega^+ \cap \partial \Omega = \partial \Omega^- \cap \partial \Omega = \Sigma_k.$$ 

**Proof** For $\beta > 0$ small, let $\Gamma_\beta$ be a neighborhood of $\Sigma_k$ in $\mathbb{R}^N$. Define $\Omega^+_{\beta}$ by $\Omega^+_{\beta} := \Gamma_\beta \cap \Omega$ and $\Omega^-_{\beta} := \Gamma_\beta \cap (\mathbb{R}^N \setminus \Omega)$. Consider the maps defined in $\Omega^\pm_{\beta}$ by

$$x \mapsto g^\pm(x) := \tilde{d}_\beta(x) + \frac{1}{2} \delta^2(x),$$

where $\tilde{d}_\beta$ is the signed distance function to $\partial \Omega$ and we recall the notations in Sect. 2. We observe that for a point $P \in \Sigma_k$, recalling once again the local coordinates defined in Sect. 2, we can see that

$$g^+(F_{\beta\Omega}(y^1, \bar{y})) = y^1 - \frac{1}{2} |\bar{y}|^2.$$
for $y^1 > 0$ and also
\[
g^-(F_{\partial\Omega}(y^1, \bar{y}, \bar{y})) = y^1 + \frac{1}{2}|\bar{y}|^2.
\]
for $y^1 < 0$. It is clear that for small $\beta$, we have $|\nabla g^\pm| \geq C > 0$ in $\Omega^\pm_\beta$. Therefore the sets
\[
\{x \in \Omega^\pm_\beta : g^\pm = 0\},
\]
containing $\Sigma_k$, are smooth $(N - 1)$-dimensional submanifolds of $\mathbb{R}^N$. In addition, by construction, they can be taken to be part of the boundaries of smooth bounded domains $\Omega^\pm$ with $\Omega^+ \subset \Omega \subset \Omega^-$ and such that
\[
\partial \Omega^+ \cap \partial \Omega = \partial \Omega^- \cap \partial \Omega = \Sigma_k.
\]
The proof then follows at once. $\square$

Now, we prove the following non-existence result.

**Theorem 4.2** Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^N$ and let $\Sigma_k$ be a smooth closed submanifold of $\partial \Omega$ of dimension $k$ with $1 \leq k \leq N - 2$ and let $\lambda \geq 0$. Assume that $p, q$ and $\eta$ satisfy (2) and (3). Suppose that $u \in H^1_0(\Omega) \cap C(\Omega)$ is a non-negative function satisfying
\[
-\text{div}(p \nabla u) - \frac{(N - k)^2}{4} q \delta^{-2} u \geq -\lambda \eta \delta^{-2} u \text{ in } \Omega.
\] (51)

If $\int_{\Sigma_k} \frac{1}{\sqrt{1 - p(\sigma)/q(\sigma)}} d\sigma = +\infty$ then $u \equiv 0$.

**Proof** We first assume that $p \equiv 1$. Let $\Omega^+$ be the set given by Lemma 4.1. We will use the notations in Sect. 2 with $\mathcal{U} = \Omega^+$ and $\mathcal{M} = \partial \Omega^+$. For $\beta > 0$ small we define
\[
\Omega^+_\beta := \{x \in \Omega^+ : \delta(x) < \beta\}.
\]
We suppose by contradiction that $u$ does not vanish identically near $\Sigma_k$ and satisfies (51) so that $u > 0$ in $\Omega^+_\beta$ by the maximum principle, for some $\beta > 0$ small.

Consider the subsolution $V_\varepsilon$ defined in Lemma 2.3 which satisfies
\[
\mathcal{L}_\lambda V_\varepsilon \leq 0 \text{ in } \Omega^+_\beta, \quad \forall \varepsilon \in (0, 1). \tag{52}
\]

Notice that $\overline{\partial \Omega^+_\beta \cap \Omega^+} \subset \Omega$ thus, for $\beta > 0$ small, we can choose $R > 0$ (independent on $\varepsilon$) so that
\[
R V_\varepsilon \leq R V_0 \leq u \text{ on } \overline{\partial \Omega^+_\beta \cap \Omega^+} \forall \varepsilon \in (0, 1).
\]

Again by Lemma 2.3, setting $v_\varepsilon = R V_\varepsilon - u$, it turns out that $v_\varepsilon^+ = \max(v_\varepsilon, 0) \in H^1_0(\Omega^+_\beta)$ because $V_\varepsilon = 0$ on $\overline{\partial \Omega^+_\beta \setminus \partial \Omega^+_\beta \cap \Omega^+}$. Moreover by (51) and (52),
\[
\mathcal{L}_\lambda v_\varepsilon \leq 0 \text{ in } \Omega^+_\beta, \quad \forall \varepsilon \in (0, 1).
\]

Multiplying the above inequality by $v_\varepsilon^+$ and integrating by parts yields
\[
\int_{\Omega^+_\beta} |\nabla v_\varepsilon^+|^2 dx - \frac{(N - k)^2}{4} \int_{\Omega^+_\beta} \delta^{-2} q |v_\varepsilon^+|^2 dx + \lambda \int_{\Omega^+_\beta} \eta \delta^{-2} |v_\varepsilon^+|^2 dx \leq 0.
\]
But then Lemma 3.1 implies that $v^\pm = 0$ in $\Omega_\beta^+$ provided $\beta$ small enough because $|\eta| \leq C\delta$ near $\Sigma_k$. Therefore $u \geq R V_\varepsilon$ for every $\varepsilon \in (0, 1)$. In particular $u \geq R V_0$. Hence we obtain from Lemma 2.3 that
\[
\int_{\Omega_\beta^+} u^2 \geq R^2 \int_{\Omega_\beta^+} \frac{V_0^2}{\delta^2} \geq \int_{\Sigma_k} \frac{1}{\sqrt{1 - q(\sigma)}} d\sigma
\]
which leads to a contradiction. We deduce that $u \equiv 0$ in $\Omega_\beta^+$. Thus by the maximum principle $u \equiv 0$ in $\Omega$.

For the general case $p \neq 1$, we argue as in [5] by setting $\tilde{u} = \sqrt{p} u$. (53)

This function satisfies
\[
-\Delta \tilde{u} - \frac{(N - k)^2 q}{4p} \eta^{-2} \tilde{u} \geq -\frac{\lambda}{p} \delta^{-2} \tilde{u} + \left( -\frac{\Delta p}{2p} + \frac{\|\nabla p\|^2}{4p^2} \right) \tilde{u} \quad \text{in } \Omega.
\]

Hence since $p \in C^2(\overline{\Omega})$ and $p > 0$ in $\overline{\Omega}$, we get the same conclusions as in the case $p \equiv 1$ and $q$ replaced by $q/p$.

5 Existence of minimizers for $\mu_\lambda(\Omega, \Sigma_k)$

**Theorem 5.1** Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^N$ and let $\Sigma_k$ be a smooth closed submanifold of $\partial \Omega$ of dimension $k$ with $1 \leq k \leq N - 2$. Assume that $p$, $q$ and $\eta$ satisfy (2) and (3). Then $\mu_\lambda(\Omega, \Sigma_k)$ is achieved for every $\lambda < \lambda^*$.

**Proof** The proof follows the same argument of [4] by taking into account the fact that $\eta = 0$ on $\Sigma_k$ so we skip it.

Next, we prove the existence of minimizers in the critical case $\lambda = \lambda^*$.

**Theorem 5.2** Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^N$ and let $\Sigma_k$ be a smooth closed submanifold of $\partial \Omega$ of dimension $k$ with $1 \leq k \leq N - 2$. Assume that $p$, $q$ and $\eta$ satisfy (2) and (3). If $\int_{\Sigma_k} \frac{1}{\sqrt{1 - p(\sigma)/q(\sigma)}} d\sigma < \infty$ then $\mu_{\lambda^*} = \mu_{\lambda^*}(\Omega, \Sigma_k)$ is achieved.

**Proof** We first consider the case $p \equiv 1$.

Let $\lambda_n$ be a sequence of real numbers decreasing to $\lambda^*$. By Theorem 5.1, there exits $u_n$ minimizers for $\mu_{\lambda_n} = \mu_{\lambda_n}(\Omega, \Sigma_k)$ so that
\[
-\Delta u_n - \mu_{\lambda_n} \delta^{-2} u_n = -\lambda_n \delta^{-2} \eta u_n \quad \text{in } \Omega.
\]

We may assume that $u_n \geq 0$ in $\Omega$. We may also assume that $\|\nabla u_n\|_{L^2(\Omega)} = 1$. Hence $u_n \to u$ in $H_0^1(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$ and pointwise. Let $\Omega^- \supset \Omega$ be the set given by Lemma 4.1. We will use the notations in Sect. 2 with $\mathcal{U} = \Omega^-$ and $\mathcal{M} = \partial \Omega^-$. It will be understood that $q$ is extended to a function in $C^2(\overline{\Omega^-})$. For $\beta > 0$ small we define
\[
\Omega^-_\beta := \{ x \in \Omega^- : \delta(x) < \beta \}.
\]

We have that
\[
\Delta u_n + b_n(x) u_n = 0 \quad \text{in } \Omega,
\]
with $|b_n| \leq C$ in $\Omega \setminus \Omega_{\beta}^{\frac{1}{2}}$ for all integer $n$. Thus by standard elliptic regularity theory,

$$u_n \leq C \quad \text{in} \quad \Omega \setminus \Omega_{\beta}^{\frac{1}{2}}. \quad (55)$$

We consider the supersolution $U$ in Lemma 2.4. We shall show that there exits a constant $C > 0$ such that for all $n \in \mathbb{N}$

$$u_n \leq C \quad \text{in} \quad \Omega_{\beta}^{-}.$$ 

(56)

Notice that $\Omega \cap \partial \Omega_{\beta}^{-} \subset \Omega^{-}$ thus by (55), we can choose $C > 0$ so that for any $n$

$$u_n \leq C \quad \text{on} \quad \Omega \cap \partial \Omega_{\beta}^{-}.$$ 

Again by Lemma 2.4, setting $v_n = u_n - C U$, it turns out that $v_n^+ = \max(v_n, 0) \in H^1_0(\Omega_{\beta}^{-})$ because $u_n = 0$ on $\partial \Omega \cap \Omega_{\beta}^{-}$. Hence we have

$$\mathcal{L}_{\lambda_n} v_n \leq -C (\mu_{\lambda^*} - \mu_n) q U - C (\lambda^* - \lambda_n) \eta U \leq 0 \quad \text{in} \quad \Omega_{\beta}^{-} \cap \Omega.$$ 

Multiplying the above inequality by $v_n^+$ and integrating by parts yields

$$\int_{\Omega_{\beta}^{-}} |\nabla v_n^+|^2 \, dx - \mu_{\lambda_n} \int_{\Omega_{\beta}^{-}} \delta^{-2} q |v_n^+|^2 \, dx + \lambda_n \int_{\Omega_{\beta}^{-}} \eta \delta^{-2} |v_n^+|^2 \, dx \leq 0.$$ 

Hence Lemma 3.1 implies that

$$C \int_{\Omega_{\beta}^{-}} \delta^{-2} X_{-2} |v_n^+|^2 \, dx + \lambda_n \int_{\Omega_{\beta}^{-}} \eta \delta^{-2} |v_n^+|^2 \, dx \leq 0.$$ 

Since $\lambda_n$ is bounded, we can choose $\beta > 0$ small (independent of $n$) such that $v_n^+ \equiv 0$ on $\Omega_{\beta}^{-}$ (recall that $|\eta| \leq C \delta$). Thus we obtain (56).

Now since $u_n \to u$ in $L^2(\Omega)$, we get by the dominated convergence theorem and (56), that

$$\delta^{-1} u_n \to \delta^{-1} u \quad \text{in} \quad L^2(\Omega).$$ 

Since $u_n$ satisfies

$$1 = \int_{\Omega} |\nabla u_n|^2 = \mu_{\lambda_n} \int_{\Omega} \delta^{-2} q u_n^2 + \lambda_n \int_{\Omega} \delta^{-2} \eta u_n^2,$$ 

taking the limit, we have $1 = \mu_{\lambda^*} \int_{\Omega} \delta^{-2} q u^2 + \lambda^* \int_{\Omega} \delta^{-2} \eta u_n^2$. Hence $u \neq 0$ and it is a minimizer for $\mu_{\lambda^*} = \left(\frac{(N-k)^2}{4}\right)$.

For the general case $p \neq 1$, we can use the same transformation as in (53). So (56) holds and the same argument as above carries over.

\[\Box\]

### 6 Proof of Theorem 1.1 and Theorem 1.2

**Proof of Theorem 1.1** Combining Lemma 3.2 and Theorem 5.1, it remains only to check the case $\lambda < \lambda^*$. But this is an easy consequence of the definition of $\lambda^*$ and of $\mu_{\lambda}(\Omega, \Sigma_k)$, see [4, Section 3].
Proof of Theorem 1.2  Existence is proved in Theorem 5.2 for $I_k < \infty$. Since the absolute value of any minimizer for $\mu_k(\Omega, \Sigma_k)$ is also a minimizer, we can apply Theorem 4.2 to infer that $\mu_k^*(\Omega, \Sigma_k)$ is never achieved as soon as $I_k = \infty$.

Acknowledgments  This work started when the first author was visiting CMM, Universidad de Chile. He is grateful for their kind hospitality. M. M. Fall is supported by the Alexander-von-Humboldt Foundation. F. Mahmoudi is supported by the Fondecyt project n: 1100164 and Fondo Basal CMM.

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