The Absolute Grothendieck Conjecture is false for Fargues-Fontaine Curves

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And in its time the spell was snapt: once more
I viewed the ocean green,
I look’d far-forth, but little saw
Of what else had been seen.

Samuel Taylor Coleridge [Col97]

1 Introduction

Let $E, E'$ be fields. Following [Jos20, Section 2], I say that $E$ and $E'$ are anabelomorphic (denoted as $E \leftrightarrow E'$) if there exists a topological isomorphism of their absolute Galois groups $G_E \simeq G_{E'}$ and refer to a topological isomorphism $\alpha : G_E \simeq G_{E'}$ as an anabelomorphism $\alpha : E \leftrightarrow E'$ between $E$ and $E'$. I will say that an anabelomorphism $E \leftrightarrow E'$ is a strict anabelomorphism if $E$ is not isomorphic to $E'$. Anabelomorphism of fields is an equivalence relation and in loc. cit. the invariants of the anabelomorphism class of a field are called amphoric, for example if $E$ is a $p$-adic field (here and elsewhere in this paper a $p$-adic field will mean a finite extension of $\mathbb{Q}_p$) then the residue characteristic $p$ of $E$ and the degree $[E : \mathbb{Q}_p]$ are amphoric (see loc. cit. and its bibliography for a list of amphoric quantities).

The notion of anabelomorphisms of fields can be extended to a large class of smooth schemes by replacing the absolute Galois group by the étale fundamental group in the definition. More precisely consider the class of geometrically connected, smooth varieties over fields. Let $E$ be a field and $X/E$ be a geometrically connected, smooth scheme and let $\pi_1(X/E)$ be its étale fundamental group. I say that $X/E$ and $X'/E'$ are anabelomorphic schemes if their étale fundamental groups are topologically isomorphic.

As mentioned earlier anabelomorphy of fields (resp. schemes) is an equivalence relation on the respective classes. Note that isomorphism of schemes is another (tautological) equivalence relation on schemes and isomorphic schemes are evidently anabelomorphic.

The extraordinary absolute Grothendieck Conjecture (see [Gro97]) asserts that in some situations these two equivalence relations coincide:

Any two geometrically connected, smooth hyperbolic curves over number fields are anabelomorphic if and only if they are isomorphic.
This is known to be true and has been extended to include finite and $p$-adic fields thanks to the remarkable works of Hiroaki Nakamura [Nak90] (the genus zero hyperbolic case), [Pop94] (the birational case), Akio Tamagawa [Tam97], Shinichi Mochizuki [Moc96] (finite fields and number fields) and [Moc99] ($p$-adic fields). The formulation of Grothendieck’s conjecture considered above is the simplest (and is adequate for the present paper), but let me say that there are other variants of the (absolute) Grothendieck Conjecture which are also considered in the literature on this subject and the aforementioned papers will serve as a starting point for the interested reader.

Let me point out that while Mochizuki has established (see [Moc99]) that the relative Grothendieck conjecture holds for smooth, hyperbolic curves over a $p$-adic field, and has also proved that the absolute Grothendieck Conjecture is true for smooth, hyperbolic curves of strict Belyi Type (see [Moc07, Corollary 2.12]). Note that a hyperbolic curve of strict Belyi Type ([Moc07, Definition 2.9]) is necessarily an affine hyperbolic curve by definition, while the examples of this paper (Theorem 2.1, Theorem 2.3) are complete curves (in the sense that the degree of the divisor of any non-constant meromorphic function is zero—see [FF18, Définition 5.1.3] and [FF18, Théorème 5.2.7]).

In [Moc04, Remark 1.3.5.1] Mochizuki has suggested that the absolute Grothendieck conjecture may be false for hyperbolic curves over $p$-adic fields—however, in recent correspondence, Mochizuki has emphatically reminded me that his assertion ([Moc04, Remark 1.3.5.1]) was made in relation to what was known at the time that paper was written, subsequent work (such as [Moc07, Corollary 2.12], or [Mur19]) on the other hand have raised the possibility that the absolute Grothendieck conjecture for hyperbolic curves over $p$-adic fields may very well be true.

Now consider the class $\mathcal{D}_{\text{irrat}}$ of separated schemes $X$ satisfying the following:

\begin{enumerate}[label=\textbf{(D.)}]
  \item $X$ is a Dedekind scheme i.e. one dimensional, Noetherian, and regular.
  \item $X = \text{Proj}(R)$ for a graded ring $R = \oplus_{n \geq 0} R_n$ generated by degree one elements over $R_0$, and let $\mathcal{O}_X(1)$ be the tautological line bundle given by this grading.
  \item $H^1(X, \mathcal{O}_X(-1)) \neq 0$.
\end{enumerate}

Note that $\mathbb{P}^1 \notin \mathcal{D}_{\text{irrat}}$ as $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$ (here $\mathcal{O}_{\mathbb{P}^1}(1)$ is the tautological line bundle of degree one given by the construction of $\mathbb{P}^1$) and $\mathcal{D}_{\text{irrat}} \supset \mathcal{D}_{\text{irrat}}^{\text{hyp}}$ where $\mathcal{D}_{\text{irrat}}^{\text{hyp}}$ is the subclass of schemes in $\mathcal{D}_{\text{irrat}}$ which corresponds to smooth, complete and hyperbolic curves (i.e. of genus at least two) over fields. Note that $\mathcal{D}_{\text{irrat}}^{\text{hyp}}$ contains the class of smooth, proper non-rational curves over fields (hence the superscript).

The purpose of this note is to record the proof of the following:

\textbf{Theorem 1.1.}

\begin{enumerate}[label=\textbf{(1)}]
  \item The class of Fargues-Fontaine curves is contained in $\mathcal{D}_{\text{irrat}}$, and
  \item the absolute Grothendieck conjecture is false for Fargues-Fontaine curves and hence the conjecture is false in general for the class $\mathcal{D}_{\text{irrat}}^{\text{hyp}}$.
\end{enumerate}

Fargues-Fontaine curves, which were constructed in [FF18], are not contained in $\mathcal{D}_{\text{irrat}}^{\text{hyp}}$ as they are not of finite type over their base fields but as mentioned above these curves are complete in the sense of function theory of curves: the divisor of zeros and poles of any meromorphic function on these curves is of degree zero (see [FF18, Définition 5.1.3] and [FF18, Théorème 5.2.7]). As was also established in [FF18], these curves play a fundamental role in
$p$-adic Hodge Theory so these curves form a natural class of examples from the point of view of the theory of $p$-adic representations. Notably, in Theorem 2.3, I show that Fargues-Fontaine curves also provide examples of strictly anabelomorphic curves (of class $D_{irrat}$) whose étale fundamental group is not isomorphic to the absolute Galois group of their respective base fields. For explicit examples of Theorem 1.1, Theorem 2.1 and Theorem 2.3 see Remark 2.2 and Remark 2.4.

The assertion Theorem 1.1(1) is proved, amongst many other beautiful results, in [FF18] (see below for precise references). So the main result of this paper is Theorem 1.1(2) and this assertion will be immediate from the more precise Theorem 2.1 which is proved in the next section.

Given the fundamental role which $p$-adic Hodge Theory plays in Mochizuki’s work on Grothendieck’s conjecture (see [Moc96], [Moc99] and his subsequent works on related questions) some readers may perhaps find it surprising that the absolute Grothendieck conjecture fails for the fundamental curves of $p$-adic Hodge Theory! In some sense the point is that these curves themselves have distinguishable $p$-adic Hodge theories. Let me also say that this paper grew out of my philosophy of combining Anabelian Geometry, Perfectoid Geometry and Modern $p$-adic Hodge Theory. This philosophy has recently led me to a construction of arithmetic and adelic Teichmuller spaces (see [Jos21]).

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2 The main theorem

Let $F$ be an algebraically closed perfectoid field of characteristic $p > 0$. Let $E$ be a $p$-adic field i.e. $E/\mathbb{Q}_p$ is a finite extension. Following [Jos20], I say that two $p$-adic fields are anabelomorphic if there exists a topological isomorphism $G_E \simeq G_{E'}$ of their absolute Galois groups; I write this as $E \leftrightarrow E'$. As is remarked in loc. cit. anabelomorphism (of $p$-adic fields) is an equivalence relation on $p$-adic fields. The notion of anabelomorphism extends to schemes: two schemes are anabelomorphic if their étale fundamental groups are isomorphic and two anabelomorphic schemes are strictly anabelomorphic schemes if they are anabelomorphic but not isomorphic.

Note that there exist $p$-adic fields which are not isomorphic but are anabelomorphic (see for instance [JR79] or [Jos20] for examples) and hence the Grothendieck conjecture is already false for $p$-adic fields; on the other hand Mochizuki has established (see [Moc99]) that the Grothendieck conjecture holds for smooth, hyperbolic curves over isomorphic $p$-adic fields. In [Moc04, Remark 1.3.5.1] Mochizuki has suggested that the Grothendieck conjecture may be false for hyperbolic curves over arbitrary (i.e. non-isomorphic) $p$-adic fields.

Now suppose that $E$, $E'$ are $p$-adic fields. Let $\mathcal{X}_{F,E}$ (resp. $\mathcal{X}_{F,E'}$) be the Fargues-Fontaine curve [FF18, Chap 6] associated to $(F, E)$ and $(F, E')$ respectively. Note that in loc. cit. this curve is denoted by $X_{F,E,\pi}$ where $\pi$ is a uniformizer for $E$. I will suppress $\pi$ from the notation in the present paper as it is irrelevant to what is done here. Let me remark that $\mathcal{X}_{F,E}$ is not of finite type and while it is supposed to have many properties similar to $\mathbb{P}^1$ (see [FF18, Chap 5]), $\mathcal{X}_{F,E}$ also shares some properties of curves of genus $\geq 2$. As mentioned in the Introduction, the Fargues-Fontaine curves $\mathcal{X}_{F,E}$ are complete curves in the sense of [FF18, Definition 5.1.3 and Theorem 5.2.7] i.e. the divisor of any meromorphic function on $\mathcal{X}_{F,E}$ has degree zero.

Proof of Theorem 1.1. Let me note $\mathcal{X}_{F,E} \in \mathcal{D}^{irrat}$ for every $p$-adic field and every algebraically
closed perfectoid field $F$. This is proved in [FF18]: by [FF18, Definition 5.1.1 and Theorem 6.5.2] $\mathcal{X}_{F,E}$ satisfies (D.1); by [FF18, Definition 6.1.1 and Theorem 6.5.2], $\mathcal{X}_{F,E}$ satisfies (D.2). That (D.3) holds follows from the computation of the cohomology of the tautological line bundle $\mathcal{O}_{\mathcal{X}_{F,E}}(1)$ on $\mathcal{X}_{F,E}$ is computed in [FF18, Section 8.2.1.1]. This proves the assertion Theorem 1.1(1). The assertion Theorem 1.1(2) is evident from Theorem 2.1 proved below.

The main theorem is the following:

**Theorem 2.1.** Assume $F$ is an algebraically closed perfectoid field of characteristic $p > 0$, $E, E'$ are $p$-adic fields. Let $\alpha : E' \longleftrightarrow E$ be an anabelomorphism (i.e. one has an isomorphism $\alpha : G_{E'} \overset{\simeq}{\longrightarrow} G_E$ of topological groups). Then one has the following assertions.

1. There is an isomorphism of topological groups
   \[ \pi_1(\mathcal{X}_{F,E}/E) \simeq G_E \simeq G_{E'} \simeq \pi_1(\mathcal{X}_{F,E'}/E'). \]

2. Hence $\mathcal{X}_{F,E}/E$ and $\mathcal{X}_{F,E'}/E'$ are anabelomorphic, one dimensional Dedekind schemes over anabelomorphic $p$-adic fields $E \longleftrightarrow E'$.

3. If $E' \longleftrightarrow E$ is a strict anabelomorphism (i.e. $E'$ is not isomorphic to $E$) then $\mathcal{X}_{F,E}$ and $\mathcal{X}_{F,E'}$ are not isomorphic as schemes.

4. In particular the absolute Grothendieck Conjecture is false for Fargues-Fontaine curves in general.

**Proof.** The first and the second assertion follows from the computation of the fundamental group of $\mathcal{X}_{F,E}$ and $\mathcal{X}_{F,E'}$ (for $F$ algebraically closed) in [FF18], [FF12, Prop. 5.2.1]. So it remains to prove the third assertion (which obviously implies the fourth assertion). So let me prove the third assertion.

I provide two different proofs of this.

Suppose
\[ \alpha : \pi_1(\mathcal{X}_{F,E}/E) \simeq \pi_1(\mathcal{X}_{F,E'}/E') \]
is an anabelomorphism of $\mathcal{X}_{F,E}/E \longleftrightarrow \mathcal{X}_{F,E'}/E'$. By the identification of $\pi_1(\mathcal{X}_{F,E}/E) \simeq G_E$ one sees that $\alpha$ induces an isomorphism $\alpha : G_{E'} \rightarrow G_E$ hence the fields $E'$ and $E$ are anabelomorphic.

Let me note a useful consequence of the fact that one has in the present case an anabelomorphism $E \longleftrightarrow E'$. Let $E \supseteq E_0$ (resp. $E \supseteq E_0'$) be the maximal unramified subextensions of $E$ (resp. $E'$). Then the two extensions $E_0, E_0'$ of $\mathbb{Q}_p$ are isomorphic $E_0 \simeq E_0'$. This is because there is a unique unramified extension of $\mathbb{Q}_p$ of a given degree and as $E \longleftrightarrow E'$ by [JR79] the degree of the maximal unramified subextensions is amphoric (i.e. determined by the topological group $G_{E'} \simeq G_E$).

Now returning to the proof of the assertion, assume that the Grothendieck conjecture is true in this context: this means the anabelomorphism
\[ \alpha : \pi_1(\mathcal{X}_{F,E}/E) \simeq \pi_1(\mathcal{X}_{F,E'}/E') \]
induces an isomorphism of schemes
\[ \alpha : \mathcal{X}_{F,E'} \overset{\simeq}{\longrightarrow} \mathcal{X}_{F,E}. \]
By [FF18] one has $H^0(\mathcal{X}_{F,E}, \mathcal{O}_{\mathcal{X}_{F,E}}) = E$. This is a part of the more general assertion (see [FF18, Chap 8, 8.2.1.1]) that the graded ring $P = \bigoplus_{d \geq 0} P_d$ is identified with the graded ring

$$P = \bigoplus_{d \in \mathbb{Z}} H^0(\mathcal{X}_{F,E}, \mathcal{O}_{\mathcal{X}_{F,E}}(d)),$$

with $P_d = H^0(\mathcal{X}_{F,E}, \mathcal{O}_{\mathcal{X}_{F,E}}(d))$.

Thus the isomorphism $\mathcal{X}_{F,E} \simeq \mathcal{X}_{F,E'}$ of schemes provides an isomorphism

$$H^0(\mathcal{X}_{F,E}, \mathcal{O}_{\mathcal{X}_{F,E}}) \simeq H^0(\mathcal{X}_{F,E'}, \mathcal{O}_{\mathcal{X}_{F,E'}}).$$

Hence this gives us an isomorphism of rings

$$E \simeq H^0(\mathcal{X}_{F,E}, \mathcal{O}_{\mathcal{X}_{F,E}}) \simeq H^0(\mathcal{X}_{F,E'}, \mathcal{O}_{\mathcal{X}_{F,E'}}) \simeq E',$$

and this evidently extends to an isomorphism of these fields and by [Sch33] any (arbitrary) isomorphism of fields equipped with a discrete valuations and complete with respect to the respective valuation topologies, is in fact an isomorphism of discretely valued fields. On the other hand I have assumed in my hypothesis (3) that the anabelomorphism $E \leftrightarrow E'$ is strict i.e. $E$ is not isomorphic to $E'$ and so one has arrived at a contradiction. \hfill \Box

**Second Proof of Theorem 2.1.** Let me provide a second more natural proof which illustrates precisely how $p$-adic Galois representations are responsible for the failure of Grothendieck conjecture for Fargues-Fontaine curves.

The idea is to use (1) on one hand the correspondence established by Fargues-Fontaine in [FF18, Chap 11], [FF12] between de Rham (resp. semi-stable and crystalline) representations $\rho : G_E \rightarrow GL(V)$ (with $V/\mathbb{Q}_p$ a finite dimensional vector space) of $G_E$ and $G_E$-equivariant vector bundles of a suitable sort on $\mathcal{X}_{F,E}$ (see [FF18, Chap 8] for details). This correspondence is given by $V \mapsto \mathcal{V} = V \otimes \mathcal{O}_{\mathcal{X}_{F,E}}$ and the Fontaine functor $D_{\text{cris}}(V)$ is naturally identified as

$$H^0(\mathcal{X}_{F,E}, \mathcal{V}) = D_{\text{cris}}(V) \otimes_{E_0} E,$$

see [FF18, Chap 11], [FF12, Theorem 6.3] and by [Fon94], $V$ is crystalline if and only if

$$\dim_{\mathbb{Q}_p}(V) = \dim_{E_0} D_{\text{cris}}(V).$$

(2) on the other hand a fundamental fact implicit in the proof of [Hos13, Corollary 3.4, Remark 3.3.1] and [Hos18, Discussion on Page 3] shows that if $\alpha : E' \leftrightarrow E$ is a strict anabelomorphism then there exist a potentially crystalline representation $\rho$ of $G_E$ such that $\rho' = \rho \circ \alpha$ is not Hodge-Tate representation of $G_{E'}$ (let me note that in Hoshi’s proof, the potentially crystalline $\mathbb{Q}_p$-representation, over a suitable open subgroup, is the crystalline $\mathbb{Q}_p$-representation arising from a Lubin-Tate group over $E$). Let me give a proof now assuming that this representation is in fact crystalline (other wise one can pass to a finite extensions of $E$ over which this happens and replacing $E'$ by a suitable finite extension (denoted again by $E, E'$) such that $E' \leftrightarrow E$, $\rho$ is crystalline and $\rho' = \rho \circ \alpha$ is not Hodge-Tate). Choose such a crystalline representation $\rho$ of $G_E$.

Now the pull-back of $\mathcal{V}$ by the isomorphism $\alpha : \mathcal{X}_{F,E'} \simeq \mathcal{X}_{F,E}$, denoted $\mathcal{V}' = \alpha^*(\mathcal{V})$ (with $\mathcal{V}'$ for the underlying vector space of the corresponding representation), evidently satisfies

$$H^0(\mathcal{X}_{F,E'}, \mathcal{V}') \simeq H^0(\mathcal{X}_{F,E}, \mathcal{V}).$$
Now from the identification $H^0(\mathcal{X}_{F,E}, \mathcal{V}) \simeq D_{cris}(V) \otimes_{E_0} E$, and the fact that $E \rightsquigarrow E'$ one knows (from [JR79] or [Jos20, Theorem 3.3]) that $E_0 \simeq E'_0$ and also $[E : \mathbb{Q}_p] = [E' : \mathbb{Q}_p]$ (i.e. anabelomorphic $p$-adic fields have the same degree over $\mathbb{Q}_p$) and also $[E : E_0] = [E' : E'_0]$ (i.e. anabelomorphic $p$-adic fields have the same absolute ramification index). So the identification of the two cohomologies gives an equality of dimensions
\[
\dim_{E_0} D_{cris}(V) \cdot [E : E_0] = \dim_{E'_0} D_{cris}(V') \cdot [E' : E'_0],
\]
hence one sees that $\mathcal{V}'$ is also crystalline as $\dim_{\mathbb{Q}_p}(V)$ is the common dimension (over $E_0 \simeq E'_0$) of both of these vector spaces.

By the functoriality of the constructions of [FF18, Chap 11], the bundle $\mathcal{V}'$ is the bundle corresponding to the pull-back via $\alpha$ of the representation $\rho$ of $G_E$ to $G_{E'}$ i.e. to the $G_{E'}$ representation $\rho'$. Hence $\rho'$ is crystalline and hence de Rham and hence Hodge-Tate. So one has arrived at a contradiction because by my assumption $\rho'$ is not Hodge-Tate. \hfill \Box

The hypothesis in Theorem 2.1 that $F$ is an algebraically closed perfectoid field of characteristic $p > 0$ can be replaced by the weaker assumption that $F$ is a perfectoid field of characteristic $p > 0$. The same proof as above also proves this general case:

**Remark 2.2.** To make Theorem 2.1 completely explicit, one can take
\[
F = \overline{\mathbb{F}_p((t))}
\]
i.e. one can take $F$ to be the completion of the algebraic closure of $\mathbb{F}_p((t))$ (this field, of characteristic $p > 0$, is a complete valued field and also perfect and hence perfectoid [Sch12]) and one can take for $p > 2$
\[
E = \mathbb{Q}_p(\zeta_p, \sqrt[p]{p}) \text{ and } E' = \mathbb{Q}_p(\zeta_p, \sqrt[p]{1+p}),
\]
where $\zeta_p$ is a primitive $p^{th}$-root of unity; and for $p = 2$ one can take
\[
E = \mathbb{Q}_2(\sqrt[4]{8}, \sqrt{4}) \text{ and } E' = \mathbb{Q}_2(\sqrt[8]{4}, \sqrt{2}),
\]
where $\zeta_8$ (resp. $\zeta_4$) is a primitive $8^{th}$-root of unity (resp. a primitive $4^{th}$-root of unity). Then $E$ and $E'$ are strictly anabelomorphic $p$-adic fields. These examples are due to [JR79]–similar examples were also found in [Yam76]; for infinitely many examples (for each prime $p$) see [Jos20, Lemma 4.4].

**Theorem 2.3.** Let $F$ be a perfectoid field of characteristic $p > 0$. Let $G_F$ be the absolute Galois group of $F$. Let $E, E'$ be $p$-adic fields. If $E \rightsquigarrow E'$ is a strict anabelomorphism of $p$-adic fields then $\mathcal{X}_{F,E}, \mathcal{X}_{F,E'}$ are anabelomorphic schemes of class $\mathcal{G}^\text{irrat}$ with
\[
\pi_1(\mathcal{X}_{F,E}) \simeq G_F \times G_E \simeq G_F \times G_{E'} \simeq \pi_1(\mathcal{X}_{F,E'})
\]
but $\mathcal{X}_{F,E}, \mathcal{X}_{F,E'}$ are not isomorphic.

**Proof.** The proof is the same as the one given above except for the assertion about the fundamental group $\pi_1(\mathcal{X}_{F,E}) \simeq G_F \times G_E$ which is can be found it [FF12, Prop. 5.2.1]. Now by [FF18, Chap 7, 7.2] one has $H^0(\mathcal{X}_{F,E}, \mathcal{O}_{\mathcal{X}_{F,E}}) = E$.

Thus any isomorphism $\mathcal{X}_{F,E} \simeq \mathcal{X}_{F,E'}$ provides an isomorphism of rings
\[
E \simeq H^0(\mathcal{X}_{F,E}, \mathcal{O}_{\mathcal{X}_{F,E}}) \simeq H^0(\mathcal{X}_{F,E'}, \mathcal{O}_{\mathcal{X}_{F,E'}}) \simeq E'.
\]
This extends to an isomorphism of (discretely valued) fields $E \simeq E'$ as before and this contradicts my assumption that $E \rightsquigarrow E'$ is a strict anabelomorphism. \hfill \Box
Remark 2.4. To make Theorem 2.3 completely explicit, one can take \( F \) one can take \( F \) to be the completion (with respect to its valuation) of the perfection \( \mathbb{F}_p((t))^{\text{perf}} \) of \( \mathbb{F}_p((t)) \) and \( E, E' \) as in Remark 2.2.

Remark 2.5. Theorem 2.3 provides counter examples to the absolute Grothendieck conjecture exists wherein the étale fundamental group is strictly larger than the absolute Galois groups of the base field.

Remark 2.6. Let me remark that the proof of Theorem 2.1 also provides a proof of the following assertion. Assume \( X/E, Y/E' \) are geometrically connected, smooth, proper, hyperbolic curves over \( p \)-adic fields. Suppose that the absolute Grothendieck conjecture is true for \( X/E, Y/E' \) i.e. one has a bijection of sets

\[
\text{Isom}_{Z-sch}(X, Y) \simeq \text{Isom}^{\text{Out}}(\pi_1(X/E), \pi_1(Y/E')).
\]

Then one has the following dichotomy:

1. Either one (and hence both) of the above \( \text{Isom} \)-sets is an empty set i.e. the absolute Grothendieck Conjecture holds vacuously for \( X/E, Y/E' \), or

2. one has an isomorphism \( E \simeq E' \) of discretely valued fields.

In particular it follows that if \( E, E' \) are strictly anabelomorphic \( p \)-adic fields, then for geometrically connected, smooth, proper, hyperbolic curves over \( E \) (resp. \( E' \)), the absolute Grothendieck conjecture either holds vacuously or it does not hold i.e. there exist some (compact) hyperbolic curves \( X/E, Y/E' \) such that

\[
\text{Isom}^{\text{Out}}(\pi_1(X/E), \pi_1(Y/E')) \neq \emptyset
\]

while

\[
\text{Isom}_{Z-sch}(X, Y) = \emptyset.
\]

Remark 2.7. Let \( E, E' \) be strictly anabelomorphic \( p \)-adic fields (see Remark 2.2). For any is a geometrically connected, smooth, projective scheme \( X/\mathbb{Q}_p \), write \( X_E = X \times_{\mathbb{Q}_p} E \) (resp. \( X_{E'} = X \times_{\mathbb{Q}_p} E' \)). Now let me note that the examples of Theorem 2.1 arise as follows. Let \( \mathcal{X}_{F, Q_p} \) be the complete Fargues-Fontaine curve for the datum \( F, \mathbb{Q}_p \). Let \( \mathcal{X}_{F, E}, \mathcal{X}_{F, E'} \) be as in Theorem 2.1. Then by [FF18, Théorème 6.5.2(2)], one has isomorphisms of \( E \)-schemes (resp. \( E' \)-schemes)

\[
\mathcal{X}_{F, E} \simeq \mathcal{X}_{F, Q_p} \times_{\mathbb{Q}_p} E \quad \text{and} \quad \mathcal{X}_{F, E'} \simeq \mathcal{X}_{F, Q_p} \times_{\mathbb{Q}_p} E',
\]

and by Theorem 2.1, a strict anabelomorphism \( \pi_1(\mathcal{X}_{F, E}) \simeq \pi_1(\mathcal{X}_{F, E'}) \). Hence one has

\[
\text{Isom}^{\text{Out}}(\pi_1(\mathcal{X}_{F, E}), \pi_1(\mathcal{X}_{F, E'})) \neq \emptyset.
\]

On the other hand by Theorem 2.1 one has

\[
\text{Isom}_{Z-sch}(\mathcal{X}_{F, E}, \mathcal{X}_{F, E'}) = \emptyset.
\]

Remark 2.6 and the examples of Theorem 2.1 thus suggests the following question:

Question 2.10. Is it true that for every pair of strictly anabelomorphic \( p \)-adic fields \( E, E' \) and for every geometrically connected, smooth, proper, hyperbolic curve \( X/\mathbb{Q}_p \) one has

\[
\text{Isom}^{\text{Out}}(\pi_1(X_E), \pi_1(X_{E'})) = \emptyset?
\]

If the answer to Question 2.10 is false for some hyperbolic curve \( X/\mathbb{Q}_p \) (or over a finite extension of \( \mathbb{Q}_p \)) then one has arrived at examples of proper hyperbolic curves of finite type providing counter examples to the absolute Grothendieck conjecture (by the method of proof of 2.1).
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