Characterization of ground-states for a system of $M$ coupled semilinear Schrödinger equations and applications

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Abstract
We focus on the study of ground-states for the system of $M$ coupled semilinear Schrödinger equations with power-type nonlinearities and couplings. General results regarding existence and characterization are derived using a variational approach. We show the usefulness of such a characterization in several particular cases, including those for which uniqueness of ground-states is already known. Finally, we apply the results to find the optimal constant for the vector-valued Gagliardo-Nirenberg inequality and we study global existence, $L^2$-concentration phenomena and blowup profile for the evolution system in the $L^2$-critical power case.

Keywords: Coupled semilinear Schrödinger equations; ground-states; qualitative properties; vector-valued Gagliardo-Nirenberg inequality.

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1 Introduction
In this work, we consider the system of $M$ coupled semilinear Schrödinger equations

$$i(v_i)_t + \Delta v_i + \binom{M}{j=1} k_{ij} |v_j|^{p+1} |v_i|^{p-1} v_i = 0, \quad i = 1, ..., M \quad (M\text{-NLS})$$

where $V = (v_1, ..., v_M) : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^M$, $k_{ij} \in \mathbb{R}$, $k_{ij} = k_{ji}$, and $0 < p < 4/(N-2)^+$ (we use the convention $4/(N-2)^+ = +\infty$, if $N = 1, 2$, and $4/(N-2)^+ = 4/(N-2)$, if $N \geq 3$). Given $1 \leq i \neq j \leq M$, if $k_{ij} > 0$, one says that the coupling between the components $v_i$ and $v_j$ is attractive; if $k_{ij} < 0$, it is repulsive. The Cauchy problem for $V_0 \in (H^1(\mathbb{R}^N))^M$ is locally well-posed and, letting $T_{\text{max}}(V_0)$ be the maximal time of existence of the solution with initial data $V_0$: if $T_{\text{max}}(V_0) < \infty$, then $\lim_{t \to T_{\text{max}}(V_0)} \|\nabla V(t)\|_2 = +\infty$.

When we look for nontrivial periodic solutions of the form $V = e^{it} U$, with $U = (u_1, ..., u_M) \in (H^1(\mathbb{R}^N))^M$ (called bound-states), we are led to the study of the system

$$\Delta u_i - u_i + \binom{M}{j=1} k_{ij} |u_j|^{p+1} |u_i|^{p-1} u_i = 0 \quad i = 1, ..., M. \quad (1.1)$$
Especially relevant, for both physical and mathematical reasons, are the bound-states which have minimal action among all bound-states, the so-called ground-states. In the scalar case, one may prove that there is a unique ground-state (modulo translations and rotations). Examples of its relevance may be found, for example, in [13], [9], [1].

In the vector-valued case, very little is known. In fact, despite several results for the existence of bound-states, there are almost no results concerning ground-states and their characterization. To our knowledge, only the papers [7], [12], [3] present advances in the characterization of ground-states, where the results obtained are quite specific. The approach for the first two is an analysis of the system of ODE’s that one obtains after proving that all ground-states are radial functions. In the third paper, the approach is variational and offers only conditions for the existence (or nonexistence) of ground-states with all components different from zero. However, each of these results display several restrictions, both on the power $p$ and on the coefficients $k_{ij}$.

Our approach is also variational, does not make restrictions on $p$ and is valid if the system (1.1) has the following property: it is possible to group the components in such a way that two components attract each other if and only if they are in the same group. This property is verifiable in all the referred papers. Intuitively, the results tell us that the attractive components have the same profile and, if there are repulsive components, one of them has to be zero: otherwise, it would be possible to move them away from each other indefinitely and therefore lowering the action, which would contradict the minimality of the ground-state.

We call the reader’s attention to theorem 4. A simple integration by parts shows that, if the matrix $K = (k_{ij})$ is such that $X^T K X \leq 0$, for any $X \in \mathbb{R}^M$ with nonnegative components, there are no bound-states. Theorem 4 claims that, if $K$ does not satisfy this property, then there exist ground-states of (M-NLS). Therefore, this is the optimal result for the existence of ground-states of (M-NLS). The main difference regarding the known existence results is that, instead of using Schwarz symmetrization (for which one needs the positivity of the coupling coefficients), one uses the concentration-compactness principle by P.-L. Lions. Notice that this approach does not say whether there exist radial ground-states or not. However, in conjunction with the characterization theorems, we prove radiality of ground-states in all the cases where it would be possible to use Schwarz symmetrization.

The structure of this work is as follows: in section 2, we define precisely the concepts of bound-state and ground-state and formulate the main results. The main lemma that allows the characterization in the case of attractive couplings can be set in a general framework, which we present in section 3. In section 4, we prove the main results. In section 5, we apply the results to some special cases, obtaining in particular the results of [7] and [12]. We also prove the uniqueness of ground-state in the case considered in [6]. Finally, in section 6, we use the characterization of ground-states to determine the optimal constant for the vector-valued Gagliardo-Nirenberg inequality and we apply the result to the study of global existence, concentration phenomena and blow-up profile for the (M-NLS) system, in the critical case $p = 2/N$.

## 2 Definitions and main results

**Definition 1.** (Bound-states and ground-states of (M-NLS))

1. We define bound-state of (M-NLS) as any element $(u_1, \ldots, u_M) \in (H^1(\mathbb{R}^N))^M \setminus \{0\}$ solution of (1.1) and define $A_{\text{(M-NLS)}}$ to be the set of all bound-states of (M-NLS).

2. A fully nontrivial bound-state is a bound-state such that $u_i \neq 0$, $\forall i$. The set of such bound-states is called $A^e_{\text{(M-NLS)}}$. 

3. Given \( U = (u_1, \ldots, u_M) \in (H^1(\mathbb{R}^N))^M \), set
\[
I_M(U) = \sum_{i=1}^{M} \left[ |\nabla u_i|^2 + \int |u_i|^2 \right], \quad J_M(U) = \sum_{i,j=1}^{M} k_{ij} \int |u_i|^{p+1} |u_j|^{p+1}
\]
and define the action of \( U \),
\[
S_M(U) = \frac{1}{2} I_M(U) - \frac{1}{2p+2} J_M(U).
\]
(2.1)

4. The set of ground-states of \((M\text{-NLS})\) is defined as
\[
G_{(M\text{-NLS})} = \{ U \in A_{(M\text{-NLS})} : S_M(U) \leq S_M(W), \forall W \in A_{(M\text{-NLS})} \} \subset A_{(M\text{-NLS})},
\]
(2.2)
and the set of fully nontrivial ground-states is
\[
G^+_{(M\text{-NLS})} = G_{(M\text{-NLS})} \cap A^+_{(M\text{-NLS})}.
\]
(2.3)

Remark 1. If \( U \in A_{(M\text{-NLS})}, I_M(U) = J_M(U) \) (one multiplies the \( i \)-th equation by \( u_i \) and integrates over \( \mathbb{R}^N \)). Therefore
\[
S_M(U) = \left[ \frac{1}{2} - \frac{1}{2p+2} \right], \quad I_M(U) = \left[ \frac{1}{2} - \frac{1}{2p+2} \right] J_M(U).
\]
(2.5)

Hence a ground-state is a bound-state with \( I_M \) (or \( J_M \)) minimal.

Remark 2. Throughout this work, we shall assume that \( \{ U \in (H^1(\mathbb{R}^N))^M : J_M(U) > 0 \} \neq \emptyset \). (P1)

This hypothesis is necessary for the existence of bound-states, since \( J_M(U) = I_M(U) > 0 \), for any \( U \in A_{(M\text{-NLS})} \).

Remark 3. Since \( M \geq 2 \) will always be fixed, to simplify notations, we write
\[
A := A_{(M\text{-NLS})}, \quad G := G_{(M\text{-NLS})}, \quad G^+ := G^+_{(M\text{-NLS})}
\]
(2.6)
and
\[
I := I_M, \quad J := J_M, \quad S := S_M.
\]
(2.7)

The following two lemmas are well-known results concerning ground-states for \((1\text{-NLS})\) (see [1]).

Lemma 2. There exists \( Q \in H^1(\mathbb{R}^N) \setminus \{0\} \) radial, positive and strictly decreasing such that
\[
G_{(1\text{-NLS})} = \{ e^{i\theta} Q(\cdot + y) : \theta \in \mathbb{R}, \ y \in \mathbb{R}^N \}.
\]
(2.8)

Lemma 3. \( G_{(1\text{-NLS})} \) is the set of solutions of the minimization problem
\[
I_1(u) = \min_{J_1(u) = J_1(Q)} I_1(w), \quad J_1(u) = J_1(Q).
\]
(2.9)

We now state the main results of this paper.
**Theorem 4.** Under assumption (P1), $G \neq \emptyset$.

**Theorem 5.** Suppose (P1) and that $k_{ij} \geq 0$, $\forall i \neq j$. Then $U^0 \in G^+$ if and only if there exist $\theta_i \in \mathbb{R}$, $i = 1, \ldots, M$, and $y \in \mathbb{R}^N$ such that

$$U^0 = (a_i e^{i\theta_i} Q(\cdot + y))_{1 \leq i \leq M}$$

(2.10)

where

$$(a_1, \ldots, a_M) \in S^+ = \left\{ B = (b_1, \ldots, b_M) \in (\mathbb{R}^+)^M : \sum_{j=1}^M k_{ij} b_j^{p-1} b_j^{p+1} = 1, i = 1, \ldots, M \right\}$$

(2.11)

and

$$\sum_{i=1}^M a_i^2 I_1(Q) = \min_{B \in S^+} \left\{ \min_{i,j} \left\{ \sum_{i=1}^M b_i^2 I_1(Q) \right\}, \min_{U \in G^+} I(U) \right\}. \quad (2.12)$$

**Remark 4.** Even though the result only characterize, a priori, the elements of $G^+$, one may obtain the description of $G$. Simply notice that, if $U^0 \in G \cap G^+$, then $U^0$ has $L$ nonzero components, with $1 \leq L < M$. If $(U^0)^+$ is the vector formed by such components, $(U^0)^+$ has to be a ground-state of a (L-NLS) system. By theorem 5 applied with $M = L$, we find the explicit expression of $(U^0)^+$ and therefore of $U^0$.

**Theorem 6.** Suppose (P1) and there exists a partition $\{Y_k\}_{1 \leq k \leq K}$ of $\{1, \ldots, M\}$ such that, given $1 \leq i \neq j \leq M$,

$$k_{ij} \geq 0 \text{ if and only if } \exists k : i, j \in Y_k. \quad (2.13)$$

Then, if $U^0 = (u_1^0, \ldots, u_M^0) \in G$, there exists $k \in \{1, \ldots, K\}$ such that $u_k^0 = 0, \forall i \notin Y_k$.

**Remark 5.** In the conditions of theorem 5 we can also characterize the set $G$, since the vector of the nonzero components of a given ground-state of (M-NLS) is a ground-state for a (L-NLS) system, with $L < M$, where all the coupling coefficients are nonnegative. Therefore it is possible to apply theorem 5 to (L-NLS), and thus obtaining the description of the initial ground-state.

**Remark 6.** One may also consider solutions of (M-NLS) of the form $V(t) = (e^{i\omega_i t} Q_i)_{1 \leq i \leq M}$, $\omega_i > 0$, and define bound-states and ground-states by making the appropriate changes. Our results of existence and characterization of ground-states can be easily extended to such a case, since one still has the homogeneity property for the functional $I$.

### 3 A general lemma

Given a real vector space $X$, consider operators $I_1, J_1 : X \to \mathbb{R}$ and $C : X \times X \to \mathbb{R}$ such that

(H1) $I_1$ is homogeneous of degree $\alpha > 0$;

(H2) $J_1$ is homogeneous of degree $2\beta > 0$ and $J_1(w) > 0$ if $w \neq 0$;

(H3) $C(\eta w, \xi w) = \eta^\beta \xi^\beta J_1(w)$ and $C(w, z) \leq J_1(w)^{1/2} J_1(z)^{1/2}$, $\forall w, z \in X \forall \eta, \xi > 0$.

Given $c_{ij} \in \mathbb{R}, \ 1 \leq i, j \leq M$, with $c_{ij} \geq 0$ if $i \neq j$, we define

$$I(U) := \sum_{i=1}^M I_1(u_i) \text{ and } J(U) := \sum_{i,j=1}^M c_{ij} C(u_i, u_j).$$
Lemma 7. Fix $\gamma > 0$. Suppose that the family $\mathcal{M} \subset X$ of solutions of the minimization problem

$$I_1(u) = \min_{J_1(u) = \gamma} I_1(w), \quad J_1(u) = \gamma \tag{3.1}$$

is nonempty and that $U = (u_1, \ldots, u_M) \in (X \setminus \{0\})^M$ is a solution of the minimization problem

$$I(W) = \min_{J(V) \geq J(U)} I(V), \quad J(W) \geq J(U). \tag{3.2}$$

Then there exist $d_i > 0$ and $P_i \in \mathcal{M}$ such that $U = (d_iP_i)_{1 \leq i \leq M}$.

Proof. Let $R \in \mathcal{M}$. First of all, we have

$$J_1 \left( \left( \frac{J_1(R)}{J_1(u_i)} \right) \frac{d}{du_i} u_i \right) = J_1(R), \quad 1 \leq i \leq M \tag{3.3}$$

Suppose, by absurd, and without loss of generality, that $d_1u_1 \neq P, \forall d_1 > 0 \ \forall P \in \mathcal{M}$. By the minimality of $\mathcal{M}$,

$$I_1 \left( \left( \frac{J_1(R)}{J_1(u_1)} \right) \frac{d}{du_1} u_1 \right) > I_1(R) \tag{3.4}$$

and

$$I_1 \left( \left( \frac{J_1(R)}{J_1(u_i)} \right) \frac{d}{du_i} u_i \right) \geq I_1(R), \quad 2 \leq i \leq M \tag{3.5}$$

This implies that

$$I(U) = I_1(u_1) + \sum_{i=2}^{M} I_1(u_i) = I_1 \left( \left( \frac{J_1(u_1)}{J_1(R)} \right) \frac{d}{du_1} R \right) + \sum_{i=2}^{M} I_1 \left( \left( \frac{J_1(u_i)}{J_1(R)} \right) \frac{d}{du_i} R \right)$$

$$= I \left( \left( \frac{J_1(u_1)}{J_1(R)} \right) \frac{d}{du_1} R, ..., \left( \frac{J_1(u_M)}{J_1(R)} \right) \frac{d}{du_M} R \right).$$

By the minimality of $U$,

$$J \left( \left( \frac{J_1(u_1)}{J_1(R)} \right) \frac{d}{du_1} R, ..., \left( \frac{J_1(u_M)}{J_1(R)} \right) \frac{d}{du_M} R \right) < J(U). \tag{3.6}$$

Using the definition of $J$,

$$\sum_{i,j=1, i \neq j}^{M} c_{ij} C \left( \left( \frac{J_1(u_i)}{J_1(R)} \right)^{\frac{1}{2}} R, \left( \frac{J_1(u_j)}{J_1(R)} \right)^{\frac{1}{2}} R \right) < \sum_{i,j=1, i \neq j}^{M} C_{ij} C(u_i, u_j) \leq \sum_{i,j=1, i \neq j}^{M} C_{ij} J_1(u_i)^{\frac{1}{2}} J_1(u_j)^{\frac{1}{2}}. \tag{3.7}$$

However, by the homogeneity of $C$,

$$\sum_{i,j=1, i \neq j}^{M} c_{ij} C \left( \left( \frac{J_1(u_i)}{J_1(R)} \right)^{\frac{1}{2}} R, \left( \frac{J_1(u_j)}{J_1(R)} \right)^{\frac{1}{2}} R \right) = \sum_{i,j=1, i \neq j}^{M} c_{ij} J_1(u_i)^{\frac{1}{2}} J_1(u_j)^{\frac{1}{2}}, \tag{3.8}$$

which is absurd. \qed
4 Proof of the main results

In this section, we fix \( X = H^1(\mathbb{R}^N) \) and we adopt the definitions of section 2. Given \( w, z \in H^1(\mathbb{R}^N) \), define

\[
I_1(w) = \int |\nabla w|^2 + |w|^2, \quad J_1(w) = \int |w|^{2p+2}, \quad C(w, z) = \int |w|^{p+1}|z|^{p+1}.
\]

(4.1)

It is easy to check that \( I_1, J_1, C \) satisfy (H1)-(H3).

**Remark 7.** Given \( \lambda > 0 \), let

\[
I^\lambda = \inf_{J(U)=\lambda} I(U) > 0.
\]

(4.2)

By the homogenous property of \( I \) and \( J \), one easily checks that \( I^\lambda = \lambda^{\frac{p+1}{p}} I^1 \).

Let

\[
\lambda_G := \left( \inf_{J(U)=1} I(U) \right)^{\frac{p+1}{p}}.
\]

(4.3)

**Lemma 8.** The minimization problems

\[
I(U) = \min_{J(W) = \lambda_G} I(W), \quad J(U) = \lambda_G
\]

(4.4)

and

\[
I(U) = \min_{J(W) \geq \lambda_G} I(W), \quad J(U) \geq \lambda_G
\]

(4.5)

are equivalent.

**Proof.** Let \( U^0 \) be a solution of (4.5). If \( J(U^0) > \lambda_G \), there would exist \( c < 1 \) such that \( J(cU^0) = \lambda_G \) and \( I(cU^0) = c^2 I(U^0) < I(U^0) \), contradicting the minimality of \( U^0 \). Hence \( U^0 \) is a solution of (4.4).

Now let \( U^0 \) be a solution of (4.4). If there existed \( W \) with \( J(W) \geq \lambda_G \) and \( I(W) < I(U^0) \), then, for some \( c \leq 1 \), \( J(cW) = \lambda_G \) and, from the minimality of \( U^0 \), \( I(U^0) \leq I(cW) \leq I(W) < I(U^0) \), which is absurd.

**Lemma 9.** Suppose that there exists a solution of the problem (4.4). Then \( G \) is the set of solutions for (4.4).

**Proof.** Let \( U \) be a minimizer of (4.4). Then, for some \( \mu \in \mathbb{R} \) and any \( H = (h_1, \ldots, h_M) \in (H^1(\mathbb{R}^N))^M \),

\[
\langle -\Delta u_i + u_i, h_i \rangle_{H^{-1}, H^1} = \mu(p+1) \sum_{j=1}^M k_{ij} |u_j|^{p+1} |u_i|^{p-1} u_i \langle h_i \rangle_{H^{-1}, H^1}, \quad 1 \leq i \leq M.
\]

(4.6)

Taking \( H = U \),

\[
\lambda_G^{\frac{1}{p+1}} I^1 = I^\lambda = I(U) = \mu(p+1) J(U) = \mu(p+1) \lambda_G
\]

(4.7)

The definition of \( \lambda_G \) implies that \( \mu(p+1) = 1 \) and so \( U \in A \). Therefore

\[
I(U) = \lambda_G \quad \text{and} \quad S(U) = \left( \frac{1}{2} - \frac{1}{2p+2} \right) \lambda_G.
\]

(4.8)
Now we take $W \in A$. We want to see that $S(W) \geq S(U)$. Let $\gamma = J(W)$. Then
\begin{equation}
I(W) = \gamma \text{ and } S(W) = \left(\frac{1}{2} - \frac{1}{2p+2}\right) \gamma.
\end{equation}

Set $X = \left(\frac{\lambda_G}{\gamma}\right)^{\frac{p}{p+1}} W$. Then $J(X) = \lambda_G$. Since $U$ is a minimizer of (4.3),
\begin{equation}
\lambda_G^{\frac{p}{p+1}} I^1 = I(U) \leq I(X) = \left(\frac{\lambda_G}{\gamma}\right)^{\frac{p}{p+1}} I(W) = \left(\frac{\lambda_G}{\gamma}\right)^{\frac{p}{p+1}} \gamma
\end{equation}
and so $\gamma \geq (I^1)^{\frac{p+1}{p}} = \lambda_G$. Hence
\begin{equation}
S(W) = \left(\frac{1}{2} - \frac{1}{2p+2}\right) \gamma \geq \left(\frac{1}{2} - \frac{1}{2p+2}\right) \lambda_G = S(U),
\end{equation}
which implies $U \in G$. If $W \in G$, one must have equality in the above inequality. Then $J(W) = \lambda_G$ and, since $U, W \in A$, $I(W) \geq J(U) = I(U)$. Therefore $W$ is a minimizer of $I^{\lambda_G}$.

**Proof of theorem 4**

By lemma 9 it suffices to prove that (4.3) has a solution.

Let $\{U_n\}$ be a minimizing sequence of (4.3). Fix $\epsilon = \frac{1}{m}, m \in \mathbb{N}$. In what follows, $\delta(\epsilon)$ shall be a function that goes to 0 when $\epsilon \to 0$. Through the concentration-compactness principle of P.L. Lions (4, 5), up to a subsequence, it is possible to associate to each $(U_n)_i, 1 \leq i \leq M$, a set of functions $\{(U_l)_i, (W_n)_i\}_{1 \leq l \leq L} \subset H^1(\mathbb{R}^N)$ (a set of bubbles plus a remainder), such that

1. Each $(U_n)_i$ has support in a ball of radius $R$ and the distance between the supports of $(U_n)_i$ and $(U_n)_{j, j \neq i}$, goes to $\infty$ as $n \to \infty$;
2. One has
\begin{equation}
\left|\|U_n\|_{2p+2}^{2p+2} - \sum_{l=1}^{L} \|U_l\|_{2p+2}^{2p+2}\right| < \delta(\epsilon)
\end{equation}
and
\begin{equation}
\|\nabla(U_n)_i\|^2 \geq \sum_{l=1}^{L} \|\nabla(U_l)_i\|^2 - \delta(\epsilon), \quad \|U_n\|_2 \geq \sum_{l=1}^{L} \|U_l\|_2 - \delta(\epsilon)
\end{equation}

Essentially, one applies successively the concentration-compactness principle to each sequence $\{(U_n)_i\}$ to obtain the various bubbles. This process ends since the total $L^2$ norm is finite and because one always picks up the bubble with greater $L^2$ norm, which implies that, after $L_i$ steps, the remainder $W_n$ has $L^{2p+2}$ smaller than $\epsilon$. Setting $L = \text{max}\{L_i\}$, we define, for each $i$, $(U_n)_i = 0$ if $L_i < l \leq L$.

One easily sees that, up to a subsequence, it is possible to group the bubbles into several clusters in such a way that: each cluster has one and only one bubble from each sequence $\{(U_n)_i\}$; if the supports of two bubbles have a nonempty intersection, then they must belong to the same cluster. Obviously, we shall end up with $L$ clusters. Define $U^l_n$ as the vector of bubbles from the cluster $l$. Then
\begin{equation}
\sum_{i=1}^{M} \|U_n\|_2 \geq \sum_{l=1}^{L} \sum_{i=1}^{M} \|U^l_n\|_2 - \delta(\epsilon)
\end{equation}
and
\[
\sum_{i=1}^{M} \| (\nabla U_n)_i \|_2^2 \geq \sum_{i=1}^{L} \sum_{i=1}^{M} \| (\nabla U_n)_i \|_2^2 - \delta(\varepsilon). 
\]  
(4.15)

Due to the way we grouped the bubbles, we have
\[
\left| J(U_n) - \sum_{l=1}^{L_n} J(U_n^l) \right| \leq \delta(\varepsilon). 
\]  
(4.16)

Up to a subsequence, we can define \( \lambda_l := \lim J(U_n^l), \, 1 \leq l \leq L \). Using a diagonalization process, we obtain, for each \( n \), a decomposition of \( \{U_n\} \) in \( L_n \) bubbles (where \( L_n \to T \in \mathbb{N} \cup \{\infty\} \)) such that
\[
\sum_{i=1}^{M} \| (U_n)_i \|_2^2 \geq \sum_{i=1}^{L} \sum_{i=1}^{M} \| (U_n^l)_i \|_2^2 - \delta \left( \frac{1}{n} \right), \quad \sum_{i=1}^{M} \| (\nabla U_n)_i \|_2^2 \geq \sum_{i=1}^{L} \sum_{i=1}^{M} \| \nabla (U_n)_i \|_2^2 - \delta \left( \frac{1}{n} \right), 
\]  
(4.17)

\[
\left| J(U_n) - \sum_{l=1}^{L_n} J(U_n^l) \right| \leq \delta \left( \frac{1}{n} \right) 
\]  
(4.18)

and
\[
\lambda_G = \sum_{l=1}^{T} \lambda_l. 
\]  
(4.19)

**Case 1:** If \( \lambda_l \geq 0 \), for any \( l \), one has
\[
J \left( \frac{\lambda_l}{J(U_n^l)} \right) \frac{U_n^l}{\lambda_l} = \lambda_l 
\]  
(4.20)

and so
\[
I^{\lambda_G} = \lim I(U_n) \geq \lim \sup_{l=1}^{L_n} \frac{J(U_n^l)}{\lambda_l} I \left( \frac{\lambda_l}{J(U_n^l)} \right) \frac{U_n^l}{\lambda_l} \geq \lim \sup_{l=1}^{L_n} I^{\lambda_l} = \sum_{l=1}^{T} I^{\lambda_l}. 
\]  
(4.21)

However, the function
\[
\lambda \mapsto I^\lambda = \lambda^\frac{1}{p+2} I^1 
\]  
(4.22)

is strictly concave in \( \mathbb{R}^+ \), which implies that there exists \( l_0 \) such that \( \lambda_l = 0 \), for \( l \neq l_0 \). By \( \text{supp} \), \( \lambda_{l_0} = \lambda_G \). Therefore, defining
\[
W_n = \left( \frac{\lambda_G}{J(U_n^{l_0})} \right) \frac{U_n^{l_0}}{\lambda_{l_0}}, 
\]  
(4.23)

one has
\[
\lim \inf I(U_n) - I(W_n) \geq 0, \quad J(W_n) = \lambda_G 
\]  
(4.24)

and so \( \{W_n\} \) is a minimizing sequence for \( I \), for which the compactness alternative from the concentration-compactness principle is verified (recall that \( W_n \), as it is up to a multiplicative factor, the vector of a group of bubbles of \( U_n \)). Since \( \{W_n\} \) is bounded in \( (H^1(\mathbb{R}^N))^M \), there exists \( W \in (H^1(\mathbb{R}^N))^M \) such that \( W_n \to W \) and, from the compactness alternative, it follows that \( W_n \to W \) in \( (L^2(\mathbb{R}^N) \cap L^{2p+2}(\mathbb{R}^N))^M \). In particular,
\[
I(W) \leq \lim I(W_n) = I^{\lambda_G}, \quad J(W) = \lim J(W_n) = \lambda_G. 
\]  
(4.25)
Therefore $W$ is a minimizer of (4.4).

**Case 2:** Now suppose that $L^{-} = \{ i : \lambda^{i} < 0 \} \neq \emptyset$. Define $L^{+}$ to be the complementary set of $L^{-}$ and

$$
\eta_{l} := \sum_{i \in L^{+}} \lambda_{i}^{-1} \eta_{i}.
$$

(4.27)

Notice that (4.19) implies $L^{+} \neq \emptyset$. Furthermore,

$$
\lambda_{G} = \sum_{i \in L^{+}} \eta_{i}.
$$

(4.28)

Since

$$
J\left( \left( \frac{\eta_{i}}{J(U_{n}^{i})} \right)^{\frac{p}{p-1}} U_{n}^{i} \right) = \eta_{l}, \ l \in L^{+},
$$

(4.29)

one has

$$
I^{\lambda_{G}} = \lim I(U_{n}) \geq \lim \sup \sum_{i \in L^{+}} \frac{J(U_{n}^{i})}{\eta_{l}} J\left( \left( \frac{\eta_{i}}{J(U_{n}^{i})} \right)^{\frac{p}{p-1}} U_{n}^{i} \right) \geq \lim \sup \sum_{i=1}^{L^{+}} I_{n}^{i} = \sum_{i=1}^{T} I_{n}^{i}. \quad (4.30)
$$

We now conclude in the same way as the previous case. □

For the case where all components attract each other, one may improve the above result using Schwarz symmetrization. This fact is not new (see [3]), however we display the following result for the sake of completeness.

**Proposition 10.** If $k_{ij} \geq 0, \forall 1 \leq i \neq j \leq M$, then (4.4) has a positive, radial, decreasing solution.

**Proof.** Let $\{ U_{n} \}$ be a minimizing sequence of (4.4). Defining $|W| := (|w_{1}|, \ldots, |w_{M}|)$, clearly $\{|U_{n}|\}$ is also a minimizing sequence. Let $|W|^{*} = (|w_{1}|^{*}, \ldots, |w_{M}|^{*})$ be the vector of the Schwarz symmetrizations of the components of $|W|$. The properties of the symmetrization imply that $\{|U_{n}|^{*}\}$ satisfies

$$
J(|U_{n}|^{*}) \geq \lambda_{G}, \quad I^{\lambda_{G}} \leq \lim \inf I(|U_{n}|^{*}) \leq \lim I(U_{n}) = I^{\lambda_{G}}. \quad (4.31)
$$

Using a compactness result for Schwarz symmetrizations, up to a subsequence, $|U_{n}|^{*} \rightarrow U$ in $(H^{1}(\mathbb{R}^{N}))^{M}$ and $|U_{n}|^{*} \rightarrow U$ in $(L^{2}(\mathbb{R}^{N}) \cap L^{2p+2}(\mathbb{R}^{N}))^{M}$. Hence

$$
J(U) = \lim J(|U_{n}|^{*}) \geq \lambda_{G}, \quad I^{\lambda_{G}} \leq I(U) \leq \lim \inf I(|U_{n}|^{*}) = I^{\lambda_{G}}. \quad (4.32)
$$

Therefore $U$ is a solution of (4.3) and, by lemma [8] it is a solution of (4.4). □

**Proof of Theorem [5]** We divide the proof in three steps:

**Step 1:** $U^{0} \in G^{+}$ satisfies (2.10), with $A^{0} = (a_{1}, \ldots, a_{M}) \in S^{+}$. Let $U_{0} \in G^{+}$. By lemmata [2] [3] and [10] we may apply lemma [7] to $I_{1}, J_{1}$ and $C$ and therefore we conclude that there exist, for each $1 \leq i \leq M$, $a_{i} > 0, \theta_{i} \in \mathbb{R}$ and $y_{i} \in \mathbb{R}^{N}$ such that

$$
U^{0} = (a_{i} e^{\theta_{i}} Q(y_{i}))_{1 \leq i \leq M}. \quad (4.33)
$$
If there exist \( i_0, j_0 \) such that \( y_{i_0} \neq y_{j_0} \), one easily sees that there exists \( D \subset \mathbb{R}^N \) of positive measure such that, for all \( x \in D \), \( Q(x + y_{i_0}) \neq Q(x + y_{j_0}) \) and so, using Young’s inequality,

\[
Q(x + y_i)^{p+1}Q(x + y_j)^{p+1} < \frac{1}{2}Q(x + y_i)^{2p+2} + \frac{1}{2}Q(x + y_j)^{2p+2}, \quad x \in D. \tag{4.34}
\]

On the other hand, we have in general

\[
Q(x + y_i)^{p+1}Q(x + y_j)^{p+1} \leq \frac{1}{2}Q(x + y_i)^{2p+2} + \frac{1}{2}Q(x + y_j)^{2p+2}, \quad x \in \mathbb{R}^N, \quad 1 \leq i, j \leq M. \tag{4.35}
\]

Consequently,

\[
\int (a_i Q(x + y_i))^{p+1}(a_j Q(x + y_j))^{p+1} \leq a_i^{p+1}a_j^{p+1} \left( \frac{1}{2} \int Q(x + y_i)^{2p+2} + \frac{1}{2} \int Q(x + y_j)^{2p+2} \right) = a_i^{p+1}a_j^{p+1} \int Q^{2p+2} = \int (a_i Q^p(a_j Q)^{p+1},
\]

with strict inequality if \( i = i_0 \) and \( j = j_0 \). Therefore, \( \lambda_G = J(U_0) < J((a_i Q)_{1 \leq i \leq M}) =: \lambda \). Hence

\[
J \left( \left( \frac{\lambda_G}{\lambda} \right)^{\frac{1}{p+1}} (a_i Q)_{1 \leq i \leq M} \right) = \lambda_G \tag{4.36}
\]

and

\[
I \left( \left( \frac{\lambda_G}{\lambda} \right)^{\frac{1}{p+1}} (a_i Q)_{1 \leq i \leq M} \right) < I ((a_i Q)_{1 \leq i \leq M}) = I(U_0), \tag{4.37}
\]

which contradicts the minimality of \( U_0 \). Therefore \( y_i = y_j \), for any \( 1 \leq i, j \leq M \) and so \( U_0 \) is of the form \( (2.10) \).

Replacing the formula of \( U_0 \) into the system \( (1.1) \), we derive

\[
\sum_{j=1}^M k_{ij}a_i^{p-1}a_j^{p+1} = 1 \quad \forall 1 \leq i \leq M. \tag{4.38}
\]

Hence \( A_0 \in S^+ \).

**Step 2:** If \( U^0 \) is of the form \( (2.10) \), with \( A_0 \in S^+ \), \( U^0 \in A \).

Simply notice that \( U^0 \) satisfies the system \( (1.1) \), using the conditions of \( S^+ \).

**Step 3:** Conclusion.

Let \( U^0 \in G^+ \). If \( A_0 \) does not satisfy \( (2.12) \), then either

\[
\min_{U \in G \cap G^+} I(U) < \sum_{i=1}^M a_i^2 I_i(Q) = I(U^0) \tag{4.39}
\]

or there exists \( B \in S^+ \) such that

\[
\sum_{i=1}^M b_i^2 I_i(Q) < \sum_{i=1}^M a_i^2 I_i(Q). \tag{4.40}
\]

In the first case, there would exist \( U \in G \cap G^+ \) with \( I(U) < I(U^0) \), which contradicts \( U^0 \in G \). In the second case, given \( \theta_i \in \mathbb{R}, \ 1 \leq i \leq M \), and \( y \in \mathbb{R}^N \),

\[
W^0 := (b \theta_i e^{i \theta_i} Q(x + y))_{1 \leq i \leq M} \tag{4.41}
\]
is in $A$. Moreover,

$$S(W^0) = \left( \frac{1}{2} - \frac{1}{2p+2} \right) I(W^0) = \left( \frac{1}{2} - \frac{1}{2p+2} \right) \sum_{i=1}^{M} b_i^2 I_1(Q)$$

$$< \left( \frac{1}{2} - \frac{1}{2p+2} \right) \sum_{i=1}^{M} a_i^2 I_1(Q) = S(U_0),$$

which contradicts $U_0 \in G$. We conclude that $A_0$ satisfies \cite{12}. It remains to prove that $W^0 \in G$. In fact,

$$S(W^0) = \left( \frac{1}{2} - \frac{1}{2p+2} \right) I(W^0) = \left( \frac{1}{2} - \frac{1}{2p+2} \right) \sum_{i=1}^{M} b_i^2 I_1(Q)$$

$$= \left( \frac{1}{2} - \frac{1}{2p+2} \right) \sum_{i=1}^{M} a_i^2 I_1(Q) = S(U^0).$$

Therefore $W_0 \in G$, which ends the proof. $\square$

**Proof of theorem 6**: The partition $\{Y_k\}_{1 \leq k \leq K}$ defines an equivalence relation in the set $\{1, ..., M\}$:

$$i \sim j \text{ if and only if } \exists k \text{ s.t. } i, j \in Y_k. \quad (4.42)$$

We claim that $(4.4)$ is equivalent to

$$\sum_{i=1}^{M} I(u_i) = \min_{B} \left( \sum_{i=1}^{M} I(w_i), \quad (u_1, ..., u_M) \in B \right) \quad (4.43)$$

where

$$B = \left\{ (w_1, ..., w_M) \in (H^1(\mathbb{R}^N))^M : \sum_{k=1}^{K} \sum_{i,j \in Y_k} k_{ij} C(w_i, w_j) = \lambda_G, \ C(w_i, w_j) = 0 \text{ if } i \neq j \right\}. \quad (4.44)$$

To see this, suppose that $U^0$ is a solution of $(4.4)$. If $C(u_i, u_j) = 0, \forall i \neq j$, then $U^0$ is a solution of $(4.43)$. By absurd, suppose that there exist $i_0 \neq j_0$ such that $C(u_{i_0}, u_{j_0}) \neq 0$. Let $U^R$ be defined by

$$(U^R)_i = (U^0)_i, \quad (U^R)_i = (U^0)_i (- + Re_1) \text{ if } i \sim j_0. \quad (4.45)$$

Then, for large $R$, $C((U^R)_i, (U^R)_j) \leq C((U^0)_i, (U^0)_j)$ if $i \neq j$ (with strict inequality if $i = i_0$, $j = j_0$) and $C((U^R)_i, (U^R)_j) = C((U^0)_i, (U^0)_j)$ if $i \sim j$. Hence, $J(U^R) > J(U^0)$. Since

$$J \left( \left( \frac{J(U^0)}{J(U^R)} \right)^{\frac{1}{p+2}} U^R \right) = J(U^R), \quad (4.46)$$

we have, by the minimality of $U^0$,

$$I(U^0) \leq I \left( \left( \frac{J(U^0)}{J(U^R)} \right)^{\frac{1}{p+2}} U^0 \right) = \left( \frac{J(U^0)}{J(U^R)} \right)^{\frac{1}{p+2}} I(U^0) < I(U^0), \quad (4.47)$$

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which is absurd. On the other hand, if \( U^0 \) is a solution of (4.4), suppose that there exists \( W \) such that \( J(W) = \lambda_G \) and \( I(W) < I(U^0) \). If \( C(w_i, w_j) = 0, \forall i \neq j \), we obtain, through the minimality of \( U^0 \), \( I(U^0) \leq I(W) \), which is absurd. If there exist \( i_0 \neq j_0 \) such that \( C(w_{i_0}, w_{j_0}) > 0 \), let \( \xi : \mathbb{R}^N \to [0, 1] \) be a smooth cutoff function with support on the unit ball and \( \xi_R(x) = \xi(x/R) \). Define \( W^R \) by

\[
(W^R)_i = \xi_R w_i(i + 2kRe_1), \quad \text{if } i \in Y_k
\]

and, for each \( n \in \mathbb{N} \), let \( R_n \) be such that

\[
|I_1((W^{R_n})_i) - I_1(w_i)| < \frac{1}{n}, \quad |J_1((W^{R_n})_i) - J_1(w_i)| < \frac{1}{n}
\]

It is clear that

\[
C((W^{R_n})_i, (W^{R_n})_j) = 0, \quad i \neq j; \quad C((W^{R_n})_i, (W^{R_n})_j) = C(W_i, W_j), \quad i \sim j
\]

and so

\[
\limsup J(W^{R_n}) \geq \lambda_G
\]

Therefore there exist \( \lambda_n \), with \( \liminf \lambda_n \leq 1 \), such that

\[
J(\lambda_n W^{R_n}) = \lambda_G
\]

and, by the minimality of \( U^0 \),

\[
I(U^0) \leq \lim I(\lambda_n W^{R_n}) = I(W) < I(U^0),
\]

which is absurd. Hence \( U^0 \) is a solution of (4.4). Thus the minimization problems (4.3) and (4.4) are equivalent.

Let

\[
K_G = \{k \in \{1, \ldots, K\} : \exists U \in (H^1(\mathbb{R}^N))^M : \sum_{i,j \in Y_k} k_{ij} C(u_i, u_j) > 0 \}.
\]

For \( Z = (z_1, \ldots, z_M) \in B \), define

\[
K^+ = \{k \in \{1, \ldots, K\} : \sum_{i,j \in Y_k} k_{ij} C(z_i, z_j) > 0 \} \subset K_G,
\]

and \( \overline{Z} \) as \( \overline{Z}_i = z_i \), \( i \in Y_k, k \in K^+ \) and \( \overline{Z}_i = 0 \), \( i \in Y_k, k \notin K^+ \).

Then

\[
W := \left( \frac{J(Z)}{J(Z)} \right)^{1/\lambda_G} \overline{Z} \in B \quad \text{and} \quad I \left( \left( \frac{J(Z)}{J(Z)} \right)^{1/\lambda_G} \overline{Z} \right) \leq I(\overline{Z}) \leq I(Z).
\]

with strict inequality if \( Z \neq \overline{Z} \).

For each \( 1 \leq k \leq K_G \), let \( Q^k \in H^1(\mathbb{R}^N)^{Y_k} \) be a ground-state of the system formed by the equations of the \( i \)-th components, with \( i \in Y_k \). Fix \( k \in K^+ \). Then, defining

\[
c_k = \left( \sum_{i,j \in Y_k} k_{ij} C(w_i, w_j) \right)^{1/\lambda_G}
\]

we have

\[
\sum_{i,j \in Y_k} k_{ij} C\left( \frac{w_i}{c_k}, \frac{w_j}{c_k} \right) = \sum_{i,j \in Y_k} k_{ij} C\left( Q_i^k, Q_j^k \right).
\]
Since \( Q^k \) is a solution of (4.13), with \( M = |Y_k| \) and \( i, j \in Y_k \), we obtain

\[
\sum_{i \in Y_k} I_1(Q^k_i) \leq \sum_{i \in Y_k} I_1(\frac{w_i}{c_k}) = \frac{1}{c_k^2} \sum_{i \in Y_k} I_1(w_i) \quad (4.58)
\]

and so

\[
\sum_{k \in K^+} c_k^2 \sum_{i \in Y_k} I_1(Q^k_i) \leq \sum_{i = 1}^M I_1(w_i).
\]

Let \( k_0 \) be such that

\[
\frac{\left( \sum_{i \in Y_{k_0}} I_1(Q^k_i) \right)^{p+1}}{\sum_{i,j \in Y_{k_0}} k_{ij} C(Q^k_i, Q^k_j)} \leq \frac{\left( \sum_{i \in Y_k} I_1(Q^k_i) \right)^{p+1}}{\sum_{i,j \in Y_k} k_{ij} C(Q^k_i, Q^k_j)}, \quad \forall k \in K_G.
\]

Let \( Q \in (H^1(\mathbb{R}^N))^M \) be defined by \( Q_i = 0 \) if \( i \notin Y_{k_0} \) and otherwise

\[
Q_i = \left( \frac{\lambda_G}{\sum_{i,j \in Y_{k_0}} k_{ij} C(Q^k_i, Q^k_j)} \right)^{\frac{1}{p+2}} I_1(Q^k_i) = \left( \frac{\sum_{i,j \in Y_{k_0}} k_{ij} C(Q^k_i, Q^k_j)}{\sum_{i,j \in Y_{k_0}} k_{ij} C(Q^k_i, Q^k_j)} \right)^{\frac{1}{p+2}} Q^k_i =: d_{k_0} Q^k_i.
\]

It is easy to see that \( \lambda_G = \frac{1}{\lambda_G} \) and, by the definition of \( k_0 \),

\[
I(Q) = d_{k_0}^2 \sum_{i \in Y_{k_0}} I_1(Q^k_i) = \left( \sum_{k \in K^+} c_k^2 \sum_{i \in Y_k} I_1(Q^k_i) \left( \frac{\sum_{i \in Y_{k_0}} I_1(Q^k_i)}{\sum_{i,j \in Y_{k_0}} k_{ij} C(Q^k_i, Q^k_j)} \right)^{p+1} \right)^{\frac{1}{p+1}} \leq \left( \sum_{k \in K^+} \left( \sum_{i \in Y_k} I_1(Q^k_i) \right)^{p+1} \right)^{\frac{1}{p+1}} = \sum_{k \in K^+} \left( \sum_{i \in Y_k} I_1(w_i) \right) \leq I(W) \leq I(Z).
\]

Therefore \( Q \) is a solution of (4.13) and, by lemma 6, \( Q \in G \). Finally, if \( Z \in G \), then \( I(Z) = I(Q) \), which implies that all of the above inequalities must be in fact equalities. From the above computation, we obtain, for some \( 1 \leq k_Z \leq K, c_k = 0, \forall k \neq k_Z \) and \( Z = Z \). Hence \( K^+ = \{ k_Z \} \) and the proof is concluded. 

## 5 Some special cases

In this section, we apply the results to some special cases, obtaining in particular the results of [7] and [12]. We shall always suppose \( k_{ij} \geq 0, i \neq j \).

We start with \( M = 2 \). Given \((u_0, v_0) \in G^+\), we note by \( a_0, b_0 \) the constants of the characterization from theorem 5

**Corollary 11.** Suppose that \( k_{11} = k_{22} \leq 0 \) and \( k_{12} > -k_{11} \). Let \((u_0, v_0) \in G^+\). Then \( a_0 = b_0 = (k_{11} + k_{12})^{-\frac{1}{2}} \).
Corollary 12. Suppose that $a_0 \neq b_0$. By the symmetry of the system, it’s enough to prove that $a_0 \geq b_0$.

Multiplying the first equation by $a_0^2$, the second by $b_0^2$ and subtracting,

$$k_{11}a_0^{2p+2} - k_{22}b_0^{2p+2} = a_0^2 - b_0^2.$$  \hfill (5.2)

If $a_0 < b_0$, the left-hand side is nonnegative and the right one is negative, which is absurd. Therefore $a_0 = b_0$. The value of $a_0$ can now be directly calculated from the system. \hfill \Box

Corollary 12. Suppose that $p = 1$ and $k_{ij} > 0$, $i, j = 1, 2$. Then

1. If $k_{11} \neq k_{22}$ and $k_{11} \leq k_{12} \leq k_{22}$, $G^+ = \emptyset$;
2. If $k_{12} \notin \{\min\{k_{11}, k_{22}\}, \max\{k_{11}, k_{22}\}\}$ and $(u_0, v_0) \in G^+$, then

$$a_0 = \sqrt{\frac{k_{22} - k_{12}}{k_{11}k_{22} - k_{12}}, \quad b_0 = \sqrt{\frac{k_{11} - k_{12}}{k_{11}k_{22} - k_{12}}}. \hfill (5.3)$$

Consequently, $G^+ = \emptyset$ if $k_{12} < \min\{k_{11}, k_{22}\}$ and $G^+ = G$ if $k_{12} > \max\{k_{11}, k_{22}\}$.

3. If $k_{11} = k_{12} = k_{22}$, $(u_0, v_0) \in G^+$ if and only if

$$(a_0, b_0) = \left(\frac{1}{\sqrt{k_{11}}} \cos \alpha, \frac{1}{\sqrt{k_{11}}} \sin \alpha \right), \quad \alpha \in [0, \pi/2]. \hfill (5.4)$$

Proof. By theorem 5 we know that

$$\begin{cases} k_{11}a_0^2 + k_{12}b_0^2 = 1 \\ k_{22}b_0^2 + k_{12}a_0^2 = 1 \end{cases} \hfill (5.5)$$

Therefore

$$\begin{cases} (k_{11}k_{22} - k_{12}^2)a_0^2 = k_{22} - k_{12} \\ (k_{11}k_{22} - k_{12}^2)b_0^2 = k_{11} - k_{12} \end{cases} \hfill (5.6)$$

1. If $k_{11} \neq k_{22}$ and $k_{11} \leq k_{12} \leq k_{22}$, suppose, without loss of generality, that $k_{11} < k_{12}$. Then

$$\frac{a_0^2}{b_0^2} = \frac{k_{22} - k_{12}}{k_{11} - k_{12}} \leq 0 \hfill (5.7)$$

which is absurd.

2. If $k_{12} \notin \{\min\{k_{11}, k_{22}\}, \max\{k_{11}, k_{22}\}\}$, one can explicitly determine the values of $a_0$ and $b_0$, thus obtaining the formulas 5.3

Suppose, w.l.o.g., that $k_{11} \leq k_{22}$. If $k_{12} < k_{22}$, then one easily checks that

$$I(a_0Q, b_0Q) = \frac{k_{22} - k_{12}}{k_{11}k_{22} - k_{12}^2} + \frac{k_{11} - k_{12}}{k_{11}k_{22} - k_{12}^2} > \frac{1}{k_{22}} = I\left(\frac{1}{\sqrt{k_{22}}}Q\right).$$

Therefore $G^+ = \emptyset$. If $k_{12} > k_{22}$, the above inequality is reversed and one obtains $G = G^+$.\hfill 14
3. If \( k_{11} = k_{12} = k_{22} \), then \( a_0^2 + b_0^2 = 1/k_{11} \) and so there exists \( \alpha \in [0, \pi/2] \) such that
\[
(a_0, b_0) = \left( \frac{1}{\sqrt{k_{11}}} \cos \alpha, \frac{1}{\sqrt{k_{11}}} \sin \alpha \right). \tag{5.8}
\]

On the other hand, any pair of this form is in \( S^+ \) and has minimal norm. The conclusion follows from theorem 5.

**Corollary 13.** Suppose that \( k_{11} = k_{22} > 0, k_{12} > 0 \) and \( pk_{11} > k_{12} \). If \((u_0, v_0) \in G^+\), then \( a_0 = b_0 = (k_{11} + k_{12})^{-\frac{1}{2}} \).

**Proof.** Again by theorem 5.
\[
\begin{align*}
\left\{ \begin{array}{l}
k_{11}a_0^{2p} + k_{12}a_0^{p-1}b_0^{p+1} = 1 \\
k_{22}b_0^{2p} + k_{12}b_0^{p-1}a_0^{p+1} = 1
\end{array} \right.
\tag{5.9}
\end{align*}
\]

Taking the difference between the two equations and dividing by \( b_0^{2p} \),
\[
k_{11} \left( \frac{a_0}{b_0} \right)^{2p} - k_{11} + k_{12} \left( \left( \frac{a_0}{b_0} \right)^{p-1} - \left( \frac{a_0}{b_0} \right)^{p+1} \right) = 0. \tag{5.10}
\]

Consider the function \( f(x) = k_{11}x^{2p} - k_{11} + k_{12}(x^{p-1} - x^{p+1}), \ x > 0 \). It is clear that \( f(1) = 0 \) and \( f(0) < 0 \). We want to see that \( f \) does not have zeroes on both sides of 1. One has
\[
f'(x) = 2pk_{11}x^{2p-1} + k_{12}((p-1)x^{p-2} - (p+1)x^p) = x^{p-2}(2pk_{11}x^{p+1} + k_{12}((p-1) - (p+1)x^2))
\]
and
\[
g'(x) = 2p(p+1)k_{11}x^p - 2(p+1)k_{12}x. \tag{5.11}
\]

Clearly
\[
g'(x) = 0 \iff x = \left( \frac{k_{12}}{pk_{11}} \right)^{\frac{1}{p-1}}. \tag{5.12}
\]

Since \( g' \) has a unique zero, \( f \) has at most three (counting multiplicities), one of which \( x = 1 \). Since \( f(x) \to \infty \) when \( x \to \infty \), and \( f'(1) = g(1) = 2(pk_{11} - k_{12}) > 0 \), all the zeroes of \( f \) have to be on the same side with respect to \( x = 1 \), as we wanted.

Suppose, without loss of generality, that \( f \) has no zeroes on \([0,1]\). It follows that \( f(x) = 0 \) implies \( x \leq 1 \) and so \( a_0 \leq b_0 \). By the symmetry of the system, \( a_0 \geq b_0 \). Hence \( a_0 = b_0 \). The value of \( a_0 \) can now be determined from the system.

To conclude this section, we prove the following result:

**Proposition 14.** Fix \( M \geq 2 \) and suppose that, for each \( 1 \leq i \leq M, \ k_{ii} > 0 \) and \( k_{ij} \geq 0, j \neq i \).
If \( \beta = \max_{i,j} k_{ij} \) is sufficiently small, then, letting \( \iota_0 \) be such that \( k_{\iota_0}^{-\frac{1}{p+1}} = \min_{i,j} k_{ij}^{-\frac{1}{p+1}} \) and \( Q \in (H^1(\mathbb{R}^N))^M \) defined by \((Q)_i = 0 \) if \( i \neq \iota_0 \) and \((Q)_{\iota_0} = k_{\iota_0}^{-\frac{1}{p+1}} Q \) (recall lemma 2), one has
\[
G = \{ e^{i\theta} Q(\cdot + y), \ \theta \in \mathbb{R}, y \in \mathbb{R}^N \}. \tag{5.13}
\]
Proof. Set $A^1 = (k_{ij}^{-1})_{1 \leq i < M}$ and $S_0$ the vector space of symmetrical matrices $M \times M$ with zero diagonal, equipped with the $t^{\infty}$ norm. Consider $F : S_0 \times \mathbb{R}^M \to \mathbb{R}^M,$

$$F_i(D, A) = k_{ii} a_i^{2p+2} + \sum_{j=1, j \neq i}^{M} d_{ij} a_i^{p+1} a_j^{p+1}, \quad D = (d_{ij}), \quad A = (a_i).$$  

(5.14)

Then $F(0, A^1) = 0,$ $F$ is $C^1$ and it is easy to see that the jacobian of $F$ with respect to $A$ in $A^1$ is nonzero. By the implicit function theorem, if $|D|_{S_0} < \delta,$ there exists a unique solution of $F(D, A) = 1,$ called $A(D),$ and there exists $\epsilon > 0$ small enough such that $\|A(D) - A^1\|_{S_0} < \epsilon.\) Consequently

$$\sum_{i=1}^{M} (A(D))_i^2 \geq \sum_{i=1}^{M} (A^1)_i^2 - \epsilon = \sum_{i=1}^{M} k_{ii}^{-\frac{2}{p+1}} - \epsilon > \min_i \{k_{ii}^{-\frac{2}{p+1}}\},$$

(5.15)

for $\epsilon$ small. If there existed $U^0 \in G^+,$ by theorem $U^0$ would be of the form

$$U^0 = (a_i e^{i\theta_i} q_0 + y_{1, \ldots, 1})_{1 \leq i < M}, \quad A^0 = (a_i)_{1 \leq i < M} \in S^+$$

(5.16)

and $A^0$ would be a solution of (6.12). By uniqueness, $A_0 = A(D),$ if $|D|_{S_0} = \max_{i \neq j} \{|k_{ij}|\} < \delta.$ Therefore

$$I(\mathcal{Q}) < \sum_{i=1}^{M} (A(D))_i^2 I_1(q) = I(U^0)$$

(5.17)

which contradicts $U^0 \in G.$ Therefore $G^+$ is empty.

If there exists $U^0 \in G$ with at least two nonzero components, the vector of nonzero components of $U^0,$ $(U^0)^+,$ has to be a fully nontrivial ground-state for a (L-NLS) system, with $2 \leq L \leq M.$ Applying the above argument, we obtain a contradiction.

6 Applications to the (M-NLS) system

We recall that we are always assuming (P1). Define

$$D = \{ U \in (H^1(\mathbb{R}^N))^M : J(U) > 0 \}$$

(6.1)

and, for each $U \in (H^1(\mathbb{R}^N))^M,$

$$M(U) = \sum_{i=1}^{M} \|u_i\|^2_2, \quad T(U) = \sum_{i=1}^{M} \|\nabla u_i\|^2_2, \quad E(U) = \frac{1}{2} T(U) - \frac{1}{2p+2} J(U),$$

(6.2)

$$GN(U) = \frac{M(U)^{p+1} - \frac{2p}{p+2} T(U)^{\frac{2p}{p+2}}}{J(U)}.$$  

(6.3)

Proposition 15. The set of solutions for the minimization problem

$$GN(U) = \min_{W \in D} GN(W), \quad U \in D$$

(6.4)

is $G,$ up to scalar multiplication and scaling.

Proof. By lemma $\|$ and theorem $\|$ we know that $G = \emptyset$ is the set of solutions of

$$I(U) = \min_{J(W) = \lambda_G} I(W), \quad J(U) = \lambda_G.$$  

(6.5)

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Let \( Q \in G \) and \( W \in D \). Since \( I(Q) = J(Q) > 0 \), we have \( Q \in D \). Define

\[
\nu = \left( \frac{J(Q)M(W)}{M(Q)J(W)} \right)^{\frac{1}{p}}
\]

and

\[
\zeta = \left( \frac{\nu^2 \left( \frac{M(W)}{M(Q)} \right)}{\nu^2 \left( \frac{M(W)}{M(Q)} \right)} \right)^{\frac{1}{p}}.
\]

Then \( Z(x) = \nu W(\zeta x) \) satisfies

\[
J(Z) = J(Q), \quad M(Z) = M(Q), \quad G(Z) = G(W).
\]

By the minimality of \( Q \), \( I_p Q q \leq I_p Z q \), which implies that \( G_N p Q q \leq G_N p Z q \). Therefore \( Q \) is a solution of (6.4). On the other hand, if \( W \) is a solution of (6.4), then one has necessarily \( G_N p Z q \leq G_N p Q q \), which implies that \( I_p Z q \leq I_p Q q \). Therefore \( Z \in G \), which concludes our proof.

Set

\[
C_M = GN(Q)^{-1}, \quad Q \in G.
\]

**Corollary 16.** The optimal constant for the vector-valued Gagliardo-Nirenberg inequality

\[
\sum_{i,j=1}^{M} k_{ij} \| u_i u_j \|_{p+1} \leq C \left( \sum_{i=1}^{M} \| u_i \|^2_2 \right)^{p+1} \left( \sum_{i=1}^{M} \nabla u_i \|^2_2 \right)^{\frac{p}{2}}, \quad U = (u_1, ..., u_M) \in (H^1(\mathbb{R}^N))^M
\]

is \( C_M \).

**Remark 8.** Using proposition 14, we can determine, in particular, the constant \( C_M \) presented by Nguyen et al. ([10]).

We now focus on the critical case \( p = 2/N \).

**Remark 9.** Let \( Q \in A \). The Pohozaev identity

\[
\frac{N-2}{2} T(Q) + \frac{N}{2} M(Q) = \frac{N}{2p+2} J(Q),
\]

together with \( T(Q) + M(Q) = I(Q) = J(Q) \) and \( p = 2/N \), implies that

\[
E(Q) = 0.
\]

Therefore

\[
GN(Q) = \frac{M(Q)^{\frac{1}{p}}}{p + 1}, \quad \forall Q \in G.
\]

From the vector-valued Gagliardo-Nirenberg inequality, we have the following optimal global existence result for (M-NLS):

**Proposition 17.** Suppose that \( V_0 \in (H^1(\mathbb{R}^N))^M \) is such that

\[
M(V_0) < \left( \frac{p+1}{C_M} \right)^\frac{1}{p} = M(Q),
\]

with \( Q \in G \). Then \( T_{\max}(V_0) = \infty \).
Proof. It is a well-known fact that the functionals $M$ and $E$ are preserved by the flow generated by $(\text{M-NLS})$. Hence, if $V$ is the solution of $(\text{M-NLS})$ with initial data $V_0$, we have, by (6.10),

$$E(V_0) = E(V(t)) = \frac{1}{2} T(V(t)) - \frac{1}{2p + 2} J(V(t)) \geq \left( \frac{1}{2} - \frac{1}{2p + 2} C_M M(V_0) \right) T(V(t)).$$

(6.15)

Therefore $T(V(t))$ is bounded and so $T_{\text{max}}(V_0) = \infty$. \hfill \Box

Remark 10. It is easy to see that, for any $U \in (H^1(\mathbb{R}^N))^M$,

$$\left( 1 - \frac{1}{p + 1} C_M \left( \sum_{i=1}^M |u_i|^2 \right)^\frac{p}{2} \right) \left( \sum_{i=1}^M \|\nabla u_i\|^2 \right) \leq 2E(U).$$

(6.16)

This inequality will be used later.

The following result is an adaptation of the result in [2] to the vector case.

Lemma 18. Suppose that $\{U_n\}_{n \in \mathbb{N}} \subset (H^1(\mathbb{R}^N))^M$ verifies

1. $M(U_n) = C \forall n \in \mathbb{N}$, for some $C > 0$;
2. $T(U_n) = C' \forall n \in \mathbb{N}$, for some $C' > 0$;
3. $E(U_n) \rightarrow 0$.

Then, given $\delta_0 > 0$, there exists a subsequence $\{U_{n_k}\}$, $y_k \in \mathbb{R}^N$ and $R > 0$ such that

$$\sum_{i=1}^M \int_{y_k + B_R} |(U_{n_k})_i|^2 \, dx \geq M(Q) - \delta_0,$$

(6.17)

where $Q \in G$. If $C = M(Q)$ and $C' = T(Q)$, then $U_n \rightarrow U$ in $(H^1(\mathbb{R}^N))^M$ and $U \in G$.

Sketch of the proof: The main ideas are the same as in the proof of theorem [4]. Fixed $\epsilon > 0$ and $1 \leq i \leq M$, one splits each sequence $\{U_{n_i}\}$ into a set of bubbles $\{(U_{n_i})_i\}_{1 \leq i \leq L_i}$, using the concentration-compactness principle. Setting $L := \max L_i$, define, for each $i$, $(U_{n_i})^l_i = 0$ if $L_i < l \leq L_i$. Afterwards, one groups the bubbles into $L$ clusters in the same way as before and define $U_{n_i}^l$ as the vector of bubbles from cluster $L$. From the concentration-compactness principle and from the way we grouped the bubbles, we have

$$|M(U_n) - \sum_{l=1}^L M(U_{n_i}^l)| < \delta(\epsilon), \quad T(U_n) \geq \sum_{l=1}^L T(U_{n_i}^l) - \delta(\epsilon),$$

(6.18)

and

$$|J(U_n) - \sum_{l=1}^L J(U_{n_i}^l)| \leq \delta(\epsilon).$$

(6.19)

Now suppose that $M(U_{n_i}^l) < \left( \frac{p+1}{2pM} \right)^\frac{p}{2} - \delta_0$, for any $l$. Then, by the vector-valued Gagliardo-Nirenberg inequality and by remark [10]

$$J(U_n) \leq \sum_{l=1}^L J(U_{n_i}^l) + \delta(\epsilon) \leq \sum_{l=1}^L T(U_{n_i}^l) + \delta(\epsilon) \leq \sum_{l=1}^L \left( 1 - \frac{1}{p+1} C_M M(U_{n_i}^l)^\frac{p}{2} \right)^{-1} E(U_{n_i}^l) + \delta(\epsilon) \leq E(U_n) + \delta(\epsilon).$$

(6.18)
However, $E(U_n) \to 0$ and $J(U_n) \to (p+1)C'$, which is absurd. This proves the first part of the result.

If $C = \left( \frac{p+1}{C_M} \right)^{\frac{1}{p}}$, then the above argument shows that there can only exist one cluster and that all the components of the sequence $U_n$ verify the compactness alternative from the concentration-compactness principle. Since $\{U_n\}$ is bounded in $(H^1(\mathbb{R}^N))^M$, there exists $U \in (H^1(\mathbb{R}^N))^M$ such that $U_n \rightharpoonup U$ and, from the compactness alternative, it follows that $U_n \to U$ in $(L^2(\mathbb{R}^N) \cap L^{2p+2}(\mathbb{R}^N))^M$. In particular $M(U) = M(Q)$, $T(U) \leq T(Q)$ and

$$J(U) = \lim J(U_n) = (p+1)T(Q) = J(Q).$$  \hspace{1cm} (6.20)

By the minimality of $Q$, we conclude that $U \in G$. Moreover, $T(U) = T(Q) = \lim T(U_n)$, which implies that $U_n \to U$ in $(H^1(\mathbb{R}^N))^M$.

Using the previous lemma, one may prove the following results in the same way as in the scalar case $M = 1$:

**Proposition 19** ($L^2$ concentration). Let $V_0 \in (H^1(\mathbb{R}^N))^M$ be such that $T_{max}(V_0) < \infty$. Then, if $V$ is the corresponding solution of (M-NLS), there exists $x : [0, T(V_0)) \to \mathbb{R}^N$ such that, for any $R > 0$,

$$\liminf_{t \to T_{max}(V_0)} \sum_{i=1}^M \int_{|x-x(t)|<R} |(V(t))_i|^2 \geq M(Q), \quad Q \in G. \hspace{1cm} (6.21)$$

**Proposition 20** (Blowup profile). Let $V_0 \in (H^1(\mathbb{R}^N))^M$ be such that $T_{max}(V_0) < \infty$ and $M(V_0) = M(Q)$, where $Q \in G$. Let $V$ be the corresponding solution of (M-NLS). Then, for any sequence $t_n \to T_{max}(V_0)$, there exists $Q_0 \in G$ and $y_n \in \mathbb{R}^N$ such that

$$\left( \frac{T(Q_0)}{T(V(t_n))} \right)^{\frac{1}{2}} V \left( \frac{T(Q_0)}{T(V(t_n))} \right)^{\frac{1}{2}} \cdot y_n \to Q_0 \text{ in } (H^1(\mathbb{R}^N))^M \hspace{1cm} (6.22)$$

**Remark 11.** We call the reader’s attention to the fact that, throughout this section, we have only assumed (P1). If (P1) is false, then the left hand side of the vector-valued Gagliardo-Nirenberg inequality is not positive (which gives $C_M = 0$), and, using proposition 17, one sees that all solutions of (M-NLS) are global. This implies that, in some sense, our results regarding the vector-valued Gagliardo-Nirenberg inequality, the $L^2$-concentration phenomena and the blowup profile are optimal.

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