ON A PROPERTY OF FERMI CURVES OF 2-DIMENSIONAL PERIODIC SCHRÖDINGER OPERATORS

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Abstract. We consider a compact Riemann surface with a holomorphic involution, two marked fixed points of the involution and a divisor obeying an equation up to linear equivalence of divisors involving all this data. Examples of such data are Fermi curves of 2-dimensional periodic Schrödinger operators. We show that the equation has a solution if and only if the two marked points are the only fixed points of the involution.

1. Introduction

Let $X$ be a compact Riemann surface of genus $g < \infty$ and $\sigma : X \to X$ a holomorphic involution with two fixed points $P_1$ and $P_2$. Then the linear equivalence $D + \sigma(D) \simeq K + P_1 + P_2$ is solvable by a divisor $D$ of degree $g$ if and only if $P_1$ and $P_2$ are the only fixed points of $\sigma$. It is known that on Fermi curves of 2-dimensional periodic Schrödinger operators, there exists a holomorphic involution $\sigma$ with two fixed points such that the pole divisor of the corresponding normalized eigenfunctions obeys this linear equivalence, see [N-V]. It was remarked in [N-V] without proof that I.R. Shafarevich and V. V. Shokurov pointed out that $D + \sigma(D) \simeq K + P_1 + P_2$ can hold if and only if $P_1$ and $P_2$ are the only fixed points of $\sigma$. To prove this assertion, we basically use the results reflecting the connection between the Jacobian variety and the Prym variety which was shown in [Mu]. Since we are mainly using the ideas shown there and not the whole concept, we will explain later how this connection involves here.

2. A two-sheeted covering

Let $X$ be a compact Riemann surface, $\sigma$ a holomorphic involution on $X$ and $P_1, P_2 \in X$ fixed points of $\sigma$, i.e. $\sigma(P_i) = P_i$ for $i = 1, 2$. For $p, q \in X$ let $p \sim q :\iff (p = q \lor p = \sigma(q))$ and define $X_\sigma := X/\sim$. Let $\pi : X \to X_\sigma$ be the canonical two-sheeted covering map. Since the subgroup of $\text{Aut}(X)$ generated by $\sigma$ is $\mathbb{Z}_2$, $X_\sigma$ is a compact Riemann surface and $\pi : X \to X_\sigma$ is holomorphic, compare [Mu, Theorem III.3.4]. Due to the construction of $X_\sigma$, the fixed points of $\sigma$ coincide with the ramification points of $\pi$. The set of ramification points of $\pi$ we denote by $r_\pi$. Then the map $\pi$ is locally

1991 Mathematics Subject Classification. Primary 14H81; Secondary 14H40.

Key words and phrases. Fermi curves, divisors, Jacobian variety.

Date: June 8, 2016.
bigholomorphic on \( X \setminus r_\pi \), see \cite{Fo} Corollary I.2.5. We define the ramification divisor of \( \pi \) on \( X \) as \( R_\pi := \sum_{p \in r_\pi} p \). In general, the ramification divisor is defined as \( R_\pi := \sum_{p \in X} (\text{mult}_p(\pi) - 1) \cdot p \), where the multiplicity \( \text{mult}_p(\pi) \) of \( \pi \) in \( p \) denotes the number of sheets which meet in \( p \), compare \cite{Mir} Definition II.4.2. Since \( \text{mult}_p(\pi) = 1 \) for \( p \in X \setminus r_\pi \) and \( \text{mult}_p(\pi) = 2 \) for \( p \in r_\pi \), this coincides with the above definition. Furthermore, \( b_\pi := \pi[r_\pi] \) is the set of branchpoints of \( \pi \) on \( X_\sigma \). The involution \( \sigma \) extends to an involution on the divisors on \( X \) by \( \sigma(\sum_{p \in X} a(p)p) := \sum_{p \in X} a(p)\sigma(p) \) which we also denote as \( \sigma \). So the degree of a divisor is conserved under \( \sigma \). We define the pullback of a point \( p_\sigma \in X_\sigma \) as

\[
\pi^*p_\sigma := \sum_{p \in \{\pi^{-1}\{p_\sigma\}\}} \text{mult}_p(\pi)p.
\]

With this definition, the pullback of a divisor \( D := \sum_{p_\sigma \in X_\sigma} a(p_\sigma)p_\sigma \) on \( X_\sigma \) is defined as \( \pi^*D := \sum_{p_\sigma \in X_\sigma} a(p_\sigma)\pi^*p_\sigma \). Since \( \pi \) is a non-constant holomorphic map between two Riemann surfaces, every meromorphic 1-form on \( X_\sigma \) can be pulled back to a meromorphic 1-form \( \omega := \pi^*\omega_\sigma \) on \( X \), compare for example \cite{Mir} Section IV.2.

**Lemma 2.1.** Let \( X, X_\sigma \) and \( \pi \) be given as above and let \( \omega_\sigma \) be a non-constant meromorphic 1-form on \( X_\sigma \).

(a) The divisor of \( \pi^*\omega_\sigma \) on \( X \) is given by \( (\pi^*\omega_\sigma) = \pi^*(\omega_\sigma) + R_\pi \).

(b) Let \( g_\sigma \) be the genus of \( X_\sigma \). Then there exists a divisor \( \tilde{K} \) on \( X \) with \( \text{deg}(\tilde{K}) = 2g_\sigma - 2 \) such that \( (\pi^*\omega_\sigma) = \tilde{K} + \sigma(\tilde{K}) + R_\pi \).

**Proof.** (a) Due to \cite{Mir} Lemma IV.2.6] one has for \( p \in X \) that

\[
\text{ord}_p(\pi^*\omega_\sigma) = (1 + \text{ord}_p(\omega_\sigma))\text{mult}_p(\pi) - 1
\]

with \( \text{ord}_p(\pi^*\omega_\sigma) \) as defined in \cite{Mir} Section IV.1.9. Inserting this into the definition of \( (\pi^*\omega_\sigma) = \sum_{p \in X} \text{ord}_p(\pi^*\omega) \) yields the assertion.

(b) One has \( \text{deg}(K_\sigma) = \text{deg}(\omega_\sigma) = 2g_\sigma - 2 \) where \( K_\sigma \) is the canonical divisor on \( X_\sigma \). Let \( p_\sigma \in X_\sigma \) be a point in the support of \( (\omega_\sigma) \) as defined in \cite{Mir} Section V.1. For \( p_\sigma \notin b_\pi \) one has \( \pi^*p_\sigma = p + \sigma(p) \) with \( p \neq \sigma(p) \in X \) and for \( p_\sigma \in b_\pi \) it is \( \pi^*p_\sigma = 2p \) with \( p \in r_\pi \). For \( p_\sigma \notin b_\pi \), let one of the pulled back points in \( \pi^*p_\sigma \) be the contribution to \( \tilde{K} \) and for \( p_\sigma \in b_\pi \) the pulled back point is counted with multiplicity one in \( \tilde{K} \). Then \( \pi^*(K) = \tilde{K} + \sigma(\tilde{K}) \) and the claim follows from (a).

\[\square\]

Now we are going to construct a symplectic cycle basis of \( H_1(X, \mathbb{Z}) \) from a symplectic cycle basis of \( H_1(X_\sigma, \mathbb{Z}) \). The holomorphic map \( \sigma : X \to X \) induces a homomorphism of \( H_1(X, \mathbb{Z}) \) which we denote as
\[
\sigma_\sharp : H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z}), \quad \gamma \mapsto \sigma_\sharp \gamma.
\]
Let $g_\sigma$ be the genus of $X_\sigma$ and $A_{\sigma,1}, \ldots, A_{\sigma,g_\sigma}$, $B_{\sigma,1}, \ldots, B_{\sigma,g_\sigma}$ be representatives of a symplectic basis of $H_1(X_\sigma, \mathbb{Z})$, i.e.

$$A_{\sigma,i} \ast A_{\sigma,\ell} = B_{\sigma,i} \ast B_{\sigma,\ell} = 0 \quad \text{and} \quad A_{\sigma,i} \ast B_{\sigma,\ell} = \delta_{i\ell},$$

where $\ast$ is the intersection product between two cycles. From Riemann surface theory it is known that such a basis exists, compare e.g. [Mir, Section VIII.4]. Due to Hurwitz’s Formula, e.g. [Mir, Theorem II.4.16], one knows that $\sharp \pi = 2n$ is even for a two sheeted covering $\pi : X \to X_\sigma$ and that the genus $g$ of $X$ is given by $g = 2g_\sigma + n - 1$. Hence a basis of $H_1(X, \mathbb{Z})$ consists of $4g_\sigma + 2n - 2$ cycles. The aim is to construct a symplectic basis of $H_1(X, \mathbb{Z})$ which we denote as $A_i^\sigma, B_i^\sigma, C_j, D_j$ and which has the following two properties: First of all, the only non-trivial pairwise intersections between elements of the basis of $H_1(X, \mathbb{Z})$ must be given by $A_i^\sigma \ast B_i^\sigma = \sigma^\ast A_i^\sigma \ast \sigma^\ast B_i^\sigma = C_j \ast D_j = 1$.

Secondly, the involution $\sigma^\ast$ has to map $A_i^\sigma$ to $\sigma^\ast A_i^\sigma$ and vice versa, $B_i^\sigma$ to $\sigma^\ast B_i^\sigma$ and vice versa and has to act on $C_j$ and $D_j$ as $\sigma^\ast C_j = -C_j$ and $\sigma^\ast D_j = -D_j$. Here, and from now on, we consider $i, \ell \in \{1, \ldots, g_\sigma\}$ and $j, k \in \{1, \ldots, n-1\}$ as long as not pointed out differently. The difference in the notation of the cycles indicates the origin of these basis elements: the $A$- and $B$-cycles on $X$ will be constructed via lifting a certain symplectic cycle basis of $H_1(X_\sigma, \mathbb{Z})$ via $\pi$ and the $C$- and $D$-cycles originate from the branchpoints of $\pi$.

We will start by constructing the $C$- and $D$-cycles. A sketch of the idea how to do this is shown for $n = 3$ and $g_\sigma = 0$ in figure 1. We connect the points in $b_\pi$ pairwise by paths $s_j$ for $j = 1, \ldots, n$. The set of points corresponding to a path $s_j : [0,1] \to X_\sigma$ we denote by $[s_j] := \{s_j(t) \mid t \in [0,1]\}$ and use the same notation for any other path considered as a set of points in $X$ or $X_\sigma$. Let $[s_j]_\sigma$ be the corresponding set with $t \in (0,1)$. The paths $s_j$ are constructed in such a way that every branchpoint is connected with exactly one other branchpoint and such that $s_k \cap s_j = \emptyset$ for $k \neq j$. This is possible since the branchpoints lie discrete on $X_\sigma$: suppose the first two branchpoints are connected by $s_1$ such that $s_1$ contains no other branchpoint. Then one can find a small open tubular neighborhood $N(s_1)$ of $s_1$ in $X_\sigma$ with boundary $\partial N(s_1)$ in $X_\sigma$ isomorphic to $S^1$. To see that $X_\sigma \setminus [s_1]$ is

\[\text{Figure 1.}\]
path connected, let $\gamma$ be a path in $X_\sigma$ which intersects $\partial N(s_1)$ in the two points $p_1, p_2 \in X_\sigma$. Then there is a path $\tilde{\gamma}$ such that $\tilde{\gamma}|_{X_\sigma \setminus N(s_1)} = \gamma|_{X_\sigma \setminus N(s_1)}$ and such that the points $p_1$ and $p_2$ are connected via a path of $\partial N(s_1)$. Hence $X_\sigma \setminus \{s_1\}$ is path connected. Like that one can gradually choose $s_2, \ldots, s_n$. To find a path $s_j$ not intersecting $s_1, \ldots, s_{j-1}$, consider $X_\sigma \setminus (\{s_1\} \cup \cdots \cup \{s_{j-1}\})$ which is path connected and repeat the above procedure until all branchpoints are sorted in pairs. The preimage of $s_j$ under $\pi$ yields two paths in $X$ which both connect the preimage of the connected two branchpoints. These preimages are ramification points of $\pi$ and we denote them as $b_j^1$ and $b_j^2$. A suitable linear combination of the two paths on $X$ then defines a cycle $C_j$ for $j = 1, \ldots, n$. Since $\pi$ is unbranched on $N \setminus r_\pi$, i.e. a homeomorphism, and since $\pi|_{r_\pi} = b_\pi \subset \{s_1\} \cup \cdots \cup \{s_n\}$, $\pi^{-1}[X_\sigma \setminus (\{s_1\} \cup \cdots \cup \{s_n\})]$ consists of two disjoint connected manifolds whose boundaries both equal to $\pi^{-1}[\{s_1\} \cup \cdots \cup \pi^{-1}[\{s_n\}]$ and $\sigma$ interchanges those manifolds. We will call them $M$ and $\sigma[M]$. Since the $n$ $C$-cycles are the boundary of $M$ respectively $\sigma[M]$, they are homologous to another, i.e. $C_n = -\sum_{i=1}^{n-1} C_i$, so this construction yields maximal $n-1$ $C$-cycles which are not homologous to each other. These $n$ cycles we orientate as the boundary of the Riemann surface $M$. We will see later on that, due to the intersection numbers, the cycles $C_1, \ldots, C_{n-1}$ are not homologous to each other. By construction, each cycle $C_j$ contains the two ramification points $b_j^1$ and $b_j^2$ of $\pi$ and no other ramification points.

The next step is to construct $n-1$ $D$-cycles such that one has $C_j \ast D_k = \delta_{jk}$. We will see that it is possible to connect $\pi(b_j^2)$ with $\pi(b_{j+1}^1)$ by a path $t_j$ for $j = 1, \ldots, n-1$ such that $t_j \cap t_k = \emptyset$ for $j \neq k$. Since $X_\sigma \setminus (\{s_1\} \cup \cdots \cup \{s_n\})$ is path connected, also $X_\sigma \setminus (\{s_1\} \cup \cdots \cup \{s_n\})$ is path connected. So one can connect $b_j^2$ with $b_j^1$ with a path $t_1$ in $X_\sigma$ not intersecting $s_3, \ldots, s_n$ and the path $s_1 + t_1 + s_2$ in $X_\sigma$ contains no loop. As above, one can chose a small open neighborhood $N(\{s_1\} \cup \{t_1\} \cup \{s_2\})$ with boundary isomorphic to $S^1$. Therefore, $X_\sigma \setminus (\{s_1\} \cup \cdots \cup \{s_n\} \cup \{t_1\})$ is path connected. Repeating this procedure shows that $X_\sigma \setminus (\{s_1\} \cup \cdots \cup \{s_n\} \cup \{t_1\} \cup \cdots \cup \{t_j\})$ remains path connected and that $\sum_{i=1}^{j}(s_{m} + t_{m}) + s_{j+1}$ contains no loop for $j = 1, \ldots, n-1$. This yields the desired $n-1$ paths $t_j$ in $X_\sigma$. Lifting these paths via $\pi$ yields each $n-1$ paths on $M$ and $n-1$ paths on $\sigma[M]$. The paths on $M$ and $\sigma[M]$ which result from the lift of $t_j$ both start at $b_j^2$ and end in $b_{j+1}^1$. Hence identifying these end points with each other yields a cycle on $X$ which we denote as $\bar{D}_j$. We orientate $\bar{D}_j$ such that $C_j \ast \bar{D}_j = 1$ and $C_{j+1} \ast \bar{D}_j = -1$ for $j \in \{1, \ldots, n-1\}$. Due to the construction of $\bar{D}_j$ one
has $C_i \ast \tilde{D}_j = 0$ for $i \notin \{j, j + 1\}$. Defining $D_j := \sum_{l=j}^{n-1} \tilde{D}_l$ yields for $k < j$

$$C_j \ast D_j = C_j \ast \sum_{l=j}^{n-1} \tilde{D}_l = C_j \ast \tilde{D}_j = 1, \quad C_k \ast D_j = C_k \ast \sum_{l=j}^{n-1} \tilde{D}_l = 0$$

$$C_j \ast D_k = C_j \ast \sum_{l=k}^{n-1} \tilde{D}_l = C_j \ast (\tilde{D}_j + \tilde{D}_{j-1}) = 1 - 1 = 0$$

and hence $n - 1$ cycles which obey $C_k \ast D_j = \delta_{kj}$. Two cycles can not be homologous to each other if the intersection number of each one of those cycles with a third cycle is not equal, hence $C_k \ast D_j = \delta_{kj}$ implies that the above construction yields $2n - 2$ cycles $C_j$ and $D_j$ which are not homologous to each other. To construct the missing $4g_\sigma$ cycles, we choose a symplectic cycle basis $A_{\sigma,i}, B_{\sigma,i}$ of $H_1(X_\sigma, \mathbb{Z})$ such that they intersect none of the paths $s_1, \ldots, s_n$ and $t_1, \ldots, t_{n-1}$. This is possible since all of these paths in $X_\sigma$ are contractible and hence can be contracted to a point. On the preimage of $X_\sigma \setminus \bigcup_{j=1}^{n-1} ([s_j] \cup [t_j]) \cup [s_n])$, the map $\pi$ is a homeomorphism. So each of the cycles in $H_1(X_\sigma, \mathbb{Z})$ is lifted to one cycle in $M$ and one cycle in $\sigma[M]$ via $\pi$ and those two cycles are interchanged by $\sigma$. Thus lifting the whole basis yields $4g_\sigma$ cycles on $X$ where we denote the $2g_\sigma$ cycles lifted to $M$ as $A_i$ and $B_i$ and the corresponding cycles lifted to $\sigma[M]$ as $\sigma_1 A_i$ and $\sigma_2 B_i$. Then these cycles obey the desired transformation behavior under $\sigma$. Since $M$ and $\sigma[M]$ are disjoint, the intersection number of the lifted cycles on $X$ stays the same as the intersection number of the corresponding cycles on $X_\sigma$ if two cycles are lifted to the same sheet $M$ respectively $\sigma[M]$ or equals zero if they are lifted to different sheets. Furthermore, the construction of these cycles ensured that the lifted $A$- and $B$-cycles do not intersect any of the $C$- and $D$-cycles on $X$. Hence $A_i, \sigma_1 A_i, B_i, \sigma_2 B_i, C_j$ and $D_j$ are in total $4g_\sigma + 2n - 2$ cycles which obey condition (1). So by Hurwitz Formula, they represent a symplectic basis of $H_1(X, \mathbb{Z})$ and the $A$- and $C$-cycles are disjoint. That the $C$- and $D$-cycles constructed like this have the desired transformation behavior under $\sigma$ is shown in the next lemma.

**Lemma 2.2.** For $C_j, D_j \in H_1(X, \mathbb{Z})$ as defined above one has $\sigma_2 C_j = -C_j$ and $\sigma_1 D_j = -D_j$.

**Proof.** Every cycle $C_j$ is the preimage of a path in $X_\sigma$ and $X_\sigma$ is invariant under $\sigma$. So $\sigma[C_j] = [C_j]$ and the two points $b_j^1$ and $b_j^2$ stay fixed. Therefore $\sigma_2 C_j = \pm C_j$. Since $\sigma$ commutes the two lifts of the path $s_j$ in $X_\sigma$, i.e. $b_j^1$ and $b_j^2$ are the only fixed points of $\sigma$ on $C_j$, one has $\sigma_2 C_j = -C_j$. By the same means, since $D_j$ also consists of the two lifts of $t_j$ which are interchanged by $\sigma$, one also has $\sigma_1 D_j = -D_j$.

3. **Decomposition of $H_1(X, \mathbb{Z})$**

With help of the Abel map $\text{Ab}$ one can identify the elements of $H_1(X, \mathbb{Z})$ with a lattice in $\mathbb{C}^g$ such that $\text{Jac}(X) \simeq \mathbb{C}^g / \Lambda$, compare [Mir, Section VIII.2].
To do so, let $\omega_1, \ldots, \omega_g \in H^0(X, \Omega)$ be a basis of the $g = 2g_\sigma + n - 1$ holomorphic differential forms on $X$ which are normalized with respect to the $A$, $\sigma x A$- and $C$-cycles, i.e.

$$\oint_{A_i} \omega_\ell = \delta_\ell i, \quad \oint_{\sigma x A_i} \omega_{g_\sigma + \ell} = \delta_\ell i, \quad \oint_{C_j} \omega_{2g_\sigma + k} = \delta_{jk}$$

and all other integrals over one of the $A$- and $C$-cycles with another element of the basis of $H^0(X, \Omega)$ are equal to zero. Furthermore, note that the construction of the $A$-cycles yields $\sigma^* \omega_i = \omega_{g_\sigma + i}$ for $i = 1, \ldots, g_\sigma$ and that Lemma 2.2 implies $\sigma^* \omega_{2g_\sigma + j} = -\omega_{2g_\sigma + j}$. We define

$$\omega_i^\pm := \frac{1}{2}(\omega_i \pm \omega_{g_\sigma + i}) \quad \text{and} \quad \omega_{g_\sigma + j}^- := \omega_{2g_\sigma + j}.$$

Direct calculation shows that these differential forms also yield a basis of $H^0(X, \Omega)$. For a path $\gamma$ in $X$, we define the vectors

$$\Omega_\gamma := \left( \int_\gamma \omega_k \right)_{k=1}^g, \quad \Omega_\gamma^+ := \left( \int_\gamma \omega_k^+ \right)_{k=1}^{g_\sigma}, \quad \Omega_\gamma^- := \left( \int_\gamma \omega_k^- \right)_{k=1}^{g_\sigma + n - 1}$$

and the following lattices generated over $\mathbb{Z}$ as

$$\Lambda := \langle \Omega_{A_1}, \Omega_{\sigma x A_1}, \Omega_{C_j}, \Omega_{B_1}, \Omega_{\sigma x B_1}, \Omega_{D_j} \rangle_{i=1, \ldots, g_\sigma}^{j=1, \ldots, n-1},$$

$$\Lambda^+ := \langle \Omega_{A_1 + \sigma x A_1}, \Omega_{B_1 + \sigma x B_1} \rangle_{i=1, \ldots, g_\sigma}^{j=1, \ldots, n-1},$$

$$\Lambda^- := \langle \Omega_{A_1 - \sigma x A_1}, \Omega_{B_1 - \sigma x B_1} \rangle_{i=1, \ldots, g_\sigma}^{j=1, \ldots, n-1}.$$

Furthermore, the mapping

$$\Phi : \mathbb{C}^g \rightarrow \mathbb{C}^{g_\sigma + n - 1}, \quad \left( \begin{array}{c} v_1 \\ \vdots \\ v_{g_\sigma} \end{array} \right) \mapsto \left( \begin{array}{c} \frac{1}{2}(v_1 + v_{g_\sigma + 1}) \\ \vdots \\ \frac{1}{2}(v_{g_\sigma} + v_{2g_\sigma}) \end{array} \right) \oplus \left( \begin{array}{c} \frac{1}{2}(v_1 - v_{g_\sigma + 1}) \\ \vdots \\ \frac{1}{2}(v_{g_\sigma} - v_{2g_\sigma}) \end{array} \right) \oplus \left( \begin{array}{c} v_{g_\sigma + 1} \\ \vdots \\ v_{2g_\sigma + n - 1} \end{array} \right)$$

is obviously linear and bijective. Hence $\Phi$ is a vector space isomorphism.

**Lemma 3.1.** For every path $\gamma$ on $X$ one has

$$\Phi(\Omega_\gamma) = \Omega_\gamma^+ \oplus \Omega_\gamma^- = \Omega_{\frac{1}{2}(\gamma + \sigma x \gamma)}^+ \oplus \Omega_{\frac{1}{2}(\gamma - \sigma x \gamma)}^-.$$
Proof. The first equality follows immediately from the definition of \( \Phi \) and the differential forms in \( \Omega(A) \):

\[
\Phi(\Omega_A) = \Phi \left( \begin{array}{c}
\int_{\gamma} \omega_1 \\
\vdots \\
\int_{\gamma} \omega_g
\end{array} \right) = \left( \begin{array}{c}
\frac{1}{2} (\int_{\gamma} \omega_1 + \omega_{g+1}) \\
\vdots \\
\frac{1}{2} (\int_{\gamma} \omega_g + \omega_{2g})
\end{array} \right) \oplus \left( \begin{array}{c}
\frac{1}{2} (\int_{\gamma} \omega_1 - \omega_{g+1}) \\
\vdots \\
\frac{1}{2} (\int_{\gamma} \omega_g - \omega_{2g})
\end{array} \right)
\]

Since \( \omega^+_k = \sigma^* \omega_k^+ \) for \( k = 1, \ldots, g \)\(_\sigma\) and \( \omega^-_k = -\sigma^* \omega_k^- \) for \( k = 1, \ldots, g \sigma + n - 1 \) one has

\[
\int_{\gamma} \omega_k^+ = \frac{1}{2} \left( \int_{\gamma} \omega_k^+ + \sigma^* \omega_k^+ \right) = \int_{\frac{1}{2} (\gamma + \sigma_2 \gamma)} \omega_k^+
\]

as well as

\[
\int_{\gamma} \omega_k^- = \frac{1}{2} \left( \int_{\gamma} \omega_k^- - \sigma^* \omega_k^- \right) = \int_{\frac{1}{2} (\gamma - \sigma_2 \gamma)} \omega_k^-
\]

which implies the second equality.

\[\square\]

Corollary 3.2. The generators of \( \Phi^{-1}(\Lambda_+ \oplus \Lambda_-) \) span a basis of \( \mathbb{C}^g \) over \( \mathbb{R} \), the generators of \( \Lambda_+ \) span a basis of \( \mathbb{C}^{g_+} \) over \( \mathbb{R} \) and the generators of \( \Lambda_- \) span a basis of \( \mathbb{C}^{g_- + n - 1} \) over \( \mathbb{R} \).

Proof. Since \( \text{Jac}(X) = \mathbb{C}^g / \Lambda \) is a complex torus, the generators of \( \Lambda \) given in \( \Omega(A) \) are a basis of \( \mathbb{C}^g \) over \( \mathbb{R} \), compare for example \[L3\], Section II.2. Basis transformation yields that \( \Omega_{A_i + \sigma_2 A_i}, \Omega_{A_i - \sigma_2 A_i}, \Omega_{B_i + \sigma_2 B_i}, \Omega_{B_i - \sigma_2 B_i}, \Omega_{C_j} \) and \( \Omega_{D_j} \) are also a basis of \( \mathbb{C}^g \) over \( \mathbb{R} \). Since \( \Phi \) is a vector space isomorphism with

\[
\Phi(\Omega_{A_i + \sigma_2 A_i}) = \Omega_{A_i + \sigma_2 A_i}^+ \oplus 0, \quad \Phi(\Omega_{A_i - \sigma_2 A_i}) = 0 \oplus \Omega_{A_i - \sigma_2 A_i}^+,
\]

\[
\Phi(\Omega_{B_i + \sigma_2 B_i}) = \Omega_{B_i + \sigma_2 B_i}^+ \oplus 0, \quad \Phi(\Omega_{B_i - \sigma_2 B_i}) = 0 \oplus \Omega_{B_i - \sigma_2 B_i}^+,
\]

\[
\Phi(\Omega_{C_j}) = 0 \oplus \Omega_{C_j}^+, \quad \Phi(\Omega_{D_j}) = 0 \oplus \Omega_{D_j}^+
\]

the generators of \( \Phi^{-1}(\Lambda_+ \oplus \Lambda_-) \) yield a basis of \( \mathbb{C}^g \) over \( \mathbb{R} \). Since \( \Phi \) is an isomorphism, the generators of \( \Lambda_+ \) are a basis of \( \mathbb{C}^{g_+} \) and the generators of \( \Lambda_- \) of \( \mathbb{C}^{g_- + n - 1} \) over \( \mathbb{R} \). \[\square\]

In the sequel, we will apply \( \Phi \) and \( \Phi^{-1} \) to lattices. Note that, for shortage of notation, we abuse the notation in the sense that \( \Phi(\Lambda) \) denotes the lattice in \( \mathbb{C}^{g_+} \oplus \mathbb{C}^{g_- + n - 1} \) spanned by the image of the generators of \( \Lambda \) under \( \Phi \) and analogously for \( \Phi^{-1} \) applied to lattices.
Lemma 3.3. (a) $\Lambda_+ \oplus 0 = \Phi(\Lambda) \cap (\mathbb{C}^{g_0} \oplus 0)$, $0 \oplus \Lambda_+ = \Phi(\Lambda) \cap (0 \oplus \mathbb{C}^{g_0+n-1})$.

(b)

$$\Phi(\Lambda) = (\Lambda_+ \oplus \Lambda_-) + M$$

with

$$M := \left\{ \sum_{i=1}^{g_0} \left( \frac{a_i}{2} \Omega_{A_i+\sigma_i A_i}^+ + \frac{b_i}{2} \Omega_{B_i+\sigma_i B_i}^+ \right) \oplus \sum_{i=1}^{g_0} \left( \frac{a_i}{2} \Omega_{A_i-\sigma_i A_i}^- + \frac{b_i}{2} \Omega_{B_i-\sigma_i B_i}^- \right) \mid a_i, b_i \in \{0, 1\} \right\}.$$ 

(c) $M \cap (\Lambda_+ \oplus \Lambda_-) = \{0\}$

Proof. Obviously, $\Lambda_+ \oplus 0$ is contained in $\Phi(\Lambda) \cap (\mathbb{C}^{g_0} \oplus 0)$. To see that $\Phi(\Lambda) \cap (\mathbb{C}^{g_0} \oplus 0)$ is also a subset of $\Lambda_+ \oplus 0$, note that for every $\gamma \in \Lambda$ there exists coefficients $a_i, a_{\sigma_i}, b_i, b_{\sigma_i}, c_j, d_j \in \mathbb{Z}$ such that

$$\gamma = \sum_{i=1}^{g_0} a_i \Omega_{A_i} + a_{\sigma_i} \Omega_{\sigma_2 A_i} + b_i \Omega_{B_i} + b_{\sigma_i} \Omega_{\sigma_2 B_i} + c_j \Omega_{C_j} + d_j \Omega_{D_j}.$$ 

The generators of $\Lambda_+$ and $\Lambda_-$ are linearly independent, compare Corollary 3.2. So the second equality in Lemma 3.1 shows that $\Phi(\gamma) \in \mathbb{C}^{g_0} \oplus 0$ can only hold if $c_j = d_j = 0$, $a_i = a_{\sigma_i}$ and $b_i = b_{\sigma_i}$. Then for such $\gamma$ it is

$$\Phi(\gamma) = 2a_i \Omega_{A_i}^+ + 2b_i \Omega_{B_i}^+ \oplus 0 = a_i \Omega_{A_i+\sigma_i A_i}^+ + b_i \Omega_{B_i+\sigma_i B_i}^- \oplus 0 \in \Lambda_+ \oplus 0.$$ 

The equality $0 \oplus \Lambda_- = \Phi(\Lambda) \cap (0 \oplus \mathbb{C}^{g_0+n-1})$ follows in the same manner. So the first part holds.

To get insight into the second part, we will show that for the set of cosets one has

$$\Phi(\Lambda)/\Lambda_+ \oplus \Lambda_- = \{ \lambda + (\Lambda_+ \oplus \Lambda_-) \mid \Phi(\lambda) \in \Lambda \}$$

$$= \{ \lambda + (\Lambda_+ \oplus \Lambda_-) \mid \Phi(\lambda) \in M \}.$$ 

The lattice $\Lambda$ is a finitely generated abelian group, so also $\Phi(\Lambda), \Lambda_+$ and $\Lambda_-$ are finitely generated abelian groups and $\Phi(2\Lambda) \subset \Lambda_+ \oplus \Lambda_- \subset \Phi(\Lambda)$, where the second inclusion is obvious and the first inclusion holds since any element $2\gamma$ of $2\Lambda$ be decomposed as $2\gamma = 2\left( \frac{1}{2}(\gamma + \sigma_2 \gamma) + \frac{1}{2}(\gamma - \sigma_2 \gamma) \right)$. Therefore, $\Phi(\Lambda)/(\Lambda_+ \oplus \Lambda_-) \subset \Phi(\Lambda)/\Phi(2\Lambda)$ and the set of the $(\Lambda : 2\Lambda) = 2^{2g}$ elements contained in $\Phi(\Lambda)/\Phi(2\Lambda)$ is the maximal set of points which are not contained in $\Lambda_+ \oplus \Lambda_-$. In $\Phi(\Lambda)$. One has $\Phi(\Omega_{C_j}), \Phi(\Omega_{D_j}) \in \Lambda_+ \oplus \Lambda_- \subset \Phi(\Lambda)$. Therefore, all points in $M$ are linear combinations of $\Phi(\Omega_{A_i}), \Phi(\Omega_{\sigma_2 A_i}), \Phi(\Omega_{B_i})$ and $\Phi(\Omega_{\sigma_2 B_i})$ with coefficients in $\{0, 1\}$. Since $\Omega_{A_i} = \Omega_{A_i+\sigma_2 A_i} - \sigma_{\sigma_2 A_i}$, one has that $[\Phi(\Omega_{A_i})] = [\Phi(\Omega_{\sigma_2 A_i})]$ and $[\Phi(\Omega_{B_i})] = \ldots$
Furthermore, \( \deg(\tilde{\sigma}) \) since \( \deg \) acts linear on divisors and is invariant under \( \deg \). Due to Corollary 3.2, these representations of \( \Phi(\Omega_A) \) as a vector in \( \mathbb{C}^9 \) in the basis given by the generators of \( \Lambda_+ \oplus \Lambda_- \) is unique, i.e. \( \Phi(\Omega_A) \not\in \Lambda_+ \oplus \Lambda_- \) and by the same means \( \Phi(\Omega_{\sigma A}), \Phi(\Omega_{B_i}), \Phi(\Omega_{\sigma B_i}) \not\in \Lambda_+ \oplus \Lambda_- \). The linear independence of the generators of \( \Lambda \) then yields equality in (6). Hence \( (\Phi(\Lambda) : \Lambda \oplus A) \) means \( \Phi(\Omega_A) \), but that the quotient of this direct sum divided by a finite set of \( \sum_{j=1}^{g_2} b_j^1 - b_j^2 \). The explicit calculations in Lemmata 3.1 and 3.3 are mirroring this connection and the finite set of points which are divided out of the direct sum in [Mu] Section 2, Data II] are exactly the points in \( M \).

**Remark 3.4.** In [Mu] it was shown that \( \text{Jac}(X_\sigma) \simeq \mathbb{C}^{g_2}/\Lambda_+ \) and that the Prym variety \( P(X, \sigma) \) can be identified with \( \mathbb{C}^{g_2+n-1}/\Lambda_- \). Furthermore, it was also shown that the direct sum \( \text{Jac}(X_\sigma) \oplus P(X, \sigma) \) is only isogenous to \( \text{Jac}(X) \), but that the quotient of this direct sum divided by a finite set of points is isomorphic to \( \text{Jac}(X) \). The explicit calculations in Lemmata 3.1 and 3.3 are mirroring this connection and the finite set of points which are divided out of the direct sum in [Mu] Section 2, Data II] are exactly the points in \( M \).

**4. The fixed points of \( \sigma \) and the linear equivalence**

**Theorem 4.1.** Let \( X \) be a Riemann surface of genus \( g \), \( K \) a canonical divisor on \( X \), \( \sigma : X \to X \) a holomorphic involution and \( P_1, P_2 \in X \) fixed points of \( \sigma \). Then there exists a divisor \( D \) of degree \( g \) on \( X \) which solves

\[
D + \sigma(D) \simeq K + P_1 + P_2
\]

if and only if \( \sigma \) has exactly the two fixed points \( P_1 \) and \( P_2 \).

**Proof.** Assume that \( \sigma \) has more fixed points then \( P_1 \) and \( P_2 \), i.e. \( n > 1 \), and that (7) holds. Due to Lemma 2.1 there exists a divisor \( \tilde{K} \) of degree \( 2g_2 - 2 \) on \( X \) such that \( K = \tilde{K} + \sigma(\tilde{K}) \). Theorem 4.1 yields \( D - \tilde{K} + \sigma(D - \tilde{K}) \simeq R_\sigma + P_1 + P_2 \). We sort the \( 2n \) ramification points in \( r_\sigma \) into pairs as it was done in the construction of the \( C \)-cycles and denote the two fixed points on \( C_\sigma \) as \( P_1 \) and \( P_2 \). Then equation (7) reads as \( D - \tilde{K} + \sigma(D - \tilde{K}) \simeq \sum_{j=1}^{n-1} (b_j^1 + b_j^2) + 2P_1 + 2P_2 \). With \( \bar{D} := D - \tilde{K} + \sum_{j=1}^{n-1} b_j^1 - P_1 - P_2 \) this is equivalent to

\[
\bar{D} + \sigma(\bar{D}) + \sum_{j=1}^{n-1} (b_j^1 - b_j^2) \simeq 0.
\]

Furthermore, \( \deg(\bar{D} + \sigma(\bar{D}) + \sum_{j=1}^{n-1} (b_j^1 - b_j^2)) = 0 \) and \( \deg(\sum_{j=1}^{n-1} (b_j^1 - b_j^2)) = 0 \). Since \( \deg \) acts linear on divisors and is invariant under \( \sigma \), this yields \( \deg(\bar{D}) = 0 \). So counted without multiplicity, there are as many points

\[
[\Phi(\Omega_{\sigma B_i})] \text{ in } \Phi(\Lambda)/\Lambda_+ \oplus \Lambda_-
\]

and thus

\[
M \subseteq \left\{ \sum_{i=1}^{g_2} a_i \Phi(\Omega_{A_i}) + b_i \Phi(\Omega_{B_i}) \mid a_i, b_i \in \{0, 1\} \right\}.
\]
with positive sign as with negative sign in $\tilde{D}$, i.e. 
$\tilde{D} = \sum_{k=1}^{f}(p_{1}^{k} - p_{2}^{k})$.
Let $\gamma_{k} : [0, 1] \to X$ be a path with $\gamma_{k}(0) = p_{1}^{k}$ and $\gamma_{k}(1) = p_{2}^{k}$. Then $\sigma \gamma_{k} : [0, 1] \to X$ is a path with $\sigma(\gamma_{k}(0)) = \sigma(p_{1}^{k})$ and $\sigma(\gamma_{k}(1)) = \sigma(p_{2}^{k})$. Then define $\gamma_{D} := \sum_{k=1}^{f} \gamma_{k}$ and $\sigma \gamma_{D} := \sum_{k=1}^{f} \sigma \gamma_{k}$. Analogously, let $\gamma_{R,j}$ be defined as the paths $\gamma_{R,j} : [0, 1] \to X$ such that $\gamma_{R,j}(0) = b_{j}^{1}$ and $\gamma_{R,j}(1) = b_{j}^{2}$ for $j = 1, \ldots, n-1$. Then, due to the construction of the $C$-cycles, one has $\gamma_{R,j} - \sigma \gamma_{R,j} = C_{j}$. We define $\gamma_{R} := \sum_{j=1}^{n-1} \gamma_{R,j}$. Set $\gamma := \gamma_{D} + \sigma \gamma_{D} + \gamma_{R}$ and let $\omega_{1}, \ldots, \omega_{g}$ be the canonical basis of $H^{0}(X, \Omega)$ normalized with respect to $A$- and $C$-cycles as in (2). Again we use the identification $\text{Jac}(X) = \mathbb{C}^{g}/\Lambda$ via the Abel map $\text{Ab}$ with the basis of holomorphic 1-forms on $X$ normalized as in (2). Due to (5), the linear equivalence can also be expressed as

$$\text{Ab} \left( \tilde{D} + \sigma(\tilde{D}) + \sum_{i=1}^{n-1} (b_{1}^{i} - b_{2}^{i}) \right) = 0 \mod \Lambda.$$ 

This equation can only hold if $\Omega_{\gamma} \in \Lambda$. Due to Lemma 3.1 we can split $\Omega_{\gamma} \in \mathbb{C}^{g}$ uniquely by considering $\Phi(\Omega_{\gamma}) = \Omega_{\gamma}^{+} \oplus \Omega_{\gamma}^{-}$ Due to the decomposition of $\Lambda$ in (3), $\Omega_{\gamma} \in \Lambda$ is equivalent to $\Omega_{\gamma}^{+} \oplus \Omega_{\gamma}^{-} \in (\Lambda_{+} \oplus \Lambda_{-}) + M$ as defined in Lemma 3.3. So we want to show that $\Omega_{\gamma}^{+} \oplus \Omega_{\gamma}^{-}$ is not contained in any of the translated copies of $\Lambda_{+} \oplus \Lambda_{-}$ if $n > 1$. Since it will turn out that it is $\Omega_{\gamma}^{-}$ which leads to this assertion, we determine the explicit form of $\Omega_{\gamma}^{-}$. For every $\omega^{-} \in H^{0}(X, \Omega)$ such that $\sigma^{*} \omega^{-} = -\omega^{-}$ one has

$$\int_{\gamma_{D}} \omega^{-} = \int_{\gamma_{D}} \omega^{-} + \int_{\gamma_{D}} \sigma^{*} \omega^{-} = \int_{\gamma_{D}} \omega^{-} - \int_{\gamma_{D}} \omega^{-} = 0$$

as well as

$$2 \int_{\gamma_{R,i}} \omega^{-} = \int_{\gamma_{R,i}} \omega^{-} - \int_{\gamma_{R,i}} \sigma^{*} \omega^{-} = \int_{\gamma_{R,i}} \omega^{-} - \int_{\gamma_{R,i}} \omega^{-} = \int_{\gamma_{R,i}} \omega^{-} - \int_{C_{i}} \omega^{-},$$

i.e. $\int_{\gamma_{R,i}} \omega^{-} = \frac{1}{2} \int_{C_{i}} \omega^{-}$. Since $\gamma - \sigma \gamma = 2 \gamma_{R}$ one has

$$\Omega_{\gamma}^{-} = \left( \int_{\gamma_{R,i}} \omega^{-} \right)^{g_{a}+n-1}_{k=1} = \frac{1}{2} \left( \sum_{k=1}^{n} \int_{C_{k}} \omega^{-}_{k} \right)^{g_{a}+n-1}_{k=1}.$$

Due to the normalization of the holomorphic 1-forms defined in (2) one has $\Omega_{\gamma}^{-} = \frac{1}{2} \sum_{k=1}^{n} \Omega_{C_{k}}$. If $\Omega_{\gamma}^{+} \oplus \Omega_{\gamma}^{-}$ would be contained in one of the translated copies of $\Lambda_{+} \oplus \Lambda_{-}$, then $\Omega_{\gamma}^{-}$ would be contained in the second component of the direct sum in one of the translated copies of $\Lambda_{-}$ introduced in Lemma 3.3. This is not possible, since the generators of $\Lambda_{+} \oplus \Lambda_{-}$ are linearly independent and only integer linear combinations of $C$-cycles are contained in all translated lattices. Therefore, $\Omega_{\gamma} \notin \Lambda$ for $n > 1$. For $n \leq 1$, there are no $C$-cycles in $H_{1}(X, \mathbb{Z})$ and equation (5) would read as $D + \sigma(D) \simeq 0$. So equation (7) can only hold if $n \leq 1$. Since $P_{1}$ and $P_{2}$ are fixed points of $\sigma$
one has $n = 1$.
Let now $P_1$ and $P_2$ be the only fixed points of $\sigma$. Then Lemma 2.1 yields that there exists a divisor $\tilde{K}$ on $X$ with $\deg(\tilde{K}) = 2g_{\sigma} - 2$ such that

$$K = \tilde{K} + \sigma(\tilde{K}) + P_1 + P_2.$$

Define $D := \tilde{K} + P_1 + P_2$. The Hurwitz Formula for $n = 1$ yields $\deg(D) = 2g_{\sigma} = g$, compare e.g. [Mir, Theorem II.4.16], and one has

$$D + \sigma(D) = \tilde{K} + \sigma(\tilde{K}) + 2P_1 + 2P_2 \simeq K + P_1 + P_2.$$

\[ \square \]

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