Vector valued Macdonald polynomials

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Abstract

This paper defines and investigates nonsymmetric Macdonald polynomials with values in an irreducible module of the Hecke algebra of type $A_{N-1}$. These polynomials appear as simultaneous eigenfunctions of Cherednik operators. Several objects and properties are analyzed, such as the canonical bilinear form which pairs polynomials with those arising from reciprocals of the original parameters, and the symmetrization of the Macdonald polynomials. The main tool of the study is the Yang-Baxter graph. We show that these Macdonald polynomials can be easily computed following this graph. We give also an interpretation of the symmetrization and the bilinear forms applied to the Macdonald polynomials in terms of the Yang-Baxter graph.

1 Introduction

For each partition $\lambda$ of $N$ there is an irreducible module of the Hecke algebra of type $A_{N-1}$ whose basis is labeled by standard tableaux of shape $\lambda$. This paper defines and analyzes nonsymmetric Macdonald polynomials with values in such modules. The double affine Hecke algebra generated by multiplication by coordinate functions, $q$-type Dunkl operators, the Hecke algebra and a $q$-shift acts on these polynomials. They appear as simultaneous eigenfunctions of the associated Cherednik operators. There is a canonical bilinear

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form which pairs these polynomials with those arising from the reciprocals of
the original parameters. The Macdonald polynomials and their reciprocal-
parameter versions form a biorthogonal set of the form. The values of the
form are found explicitly.

There are symmetric Macdonald polynomials in this structure. They are
labeled by column-strict tableaux of shape \( \lambda \) (non-decreasing entries in each
row, strictly increasing in each column). Formulae for these polynomials in
terms of nonsymmetric Macdonald polynomials are derived and the values
of the bilinear form are obtained in this case. There are analogous results
for antisymmetric Macdonald polynomials, which are labeled by row-strict
tableaux. There is a hook-length type formula for the bilinear form evaluated
at the minimal symmetric polynomial associated with \( \lambda \).

In the study of one-variable orthogonal polynomials the very simple graph
\[ 0 \rightarrow 1 \rightarrow 2 \rightarrow \ldots \] symbolizes the Gram-Schmidt process used to produce the
polynomials. In the present multi-variable setting the Yang-Baxter graph
displays how each Macdonald polynomial is produced. Each arrow corre-
sponds to either an adjacent transposition or an affine step \((u_1, \ldots, u_N) \rightarrow
(u_2, \ldots, u_N, u_1 + 1)\). This idea is developed in Section 4.

In Section 2 we give the basic definitions of the Hecke algebra, its modules,
and the machinery necessary to describe the leading terms of Macdonald
polynomials. Section 3 begins with the simplest two-dimensional module
associated to the partition \((2, 1)\) of \(N = 3\). We describe how the basic
operations arise in this situation and thus motivate our general definitions.
The rest of the section gives the definitions and proves the fundamental
relations, notably the braid relations, for the vector-valued situation. A key
part is played by the triangularity property of the Cherednik operators with
respect to a natural partial order on monomials.

Section 4 contains the description of the simultaneous eigenfunctions, the
spectral vectors, the transformation rules for the action of the generators of
the Hecke algebra on the polynomials, and the Yang-Baxter graph.

Section 5 concerns the connected components of the Yang-Baxter graph
modified by the removal of the affine edges. Here we find the conditions
under which the component contains a unique symmetric or antisymmetric
polynomial.

The bilinear form is defined and evaluated in Section 6. The method
of evaluation relies on relatively simple calculations of the effects of a sin-
gle arrow in the Yang-Baxter graph. The minimal symmetric polynomials
are studied in this section. The hook-length formula for the bilinear form
gives some information about aspherical modules of the double affine Hecke algebra, a topic to be pursued in future work.

The paper concludes with a symbol index and a list of basic relations for quick reference.

2 Double affine Hecke algebra

2.1 Definitions and basic properties

Consider the elements $T_i$ and $w$ verifying:

1. $(T_i + t_1)(T_i + t_2) = 0$
2. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$
3. $T_i T_j = T_j T_i$ for $|i-j| > 1$
4. $T_i w = w T_{i-1}$

These operators act on $\mathbb{C}(t_1, t_2, q)[x_1, \ldots, x_N]$ by

1. $T_i := \pi_i(t_1 + t_2) - t_2 s_i$
2. $w := \tau_1 s_1 \ldots s_{N-1}$

where $\pi_i = \partial_i x_{i+1}$, $\partial_i$ is the divided difference defined by

$$\partial_i = (1 - s_i) \frac{1}{x_i - x_{i+1}},$$

$s_i$ the transposition $(i, i+1)$ and $f(x_1, \ldots, x_N) \tau_i = f(x_1, \ldots, x_{i-1}, qx_i, x_{i+1}, \ldots, x_N)$.

Note that the parameter $t_1$ should be omitted since, dividing each $T_i$ by $t_1$ we obtain

$$\frac{1}{t_1} T_i^{t_1, t_2, q} = T_i^{1, q}$$

For simplicity we will use the parameters: $t_1 = 1$ and $t_2 = -s$.

Then, the quadratic relation is $(T_i + 1)(T_i - s) = 0$ and $T_i := \pi_i(1 - s) + ss_i$.

Note that these operators have interesting commutation properties w.r.t. the multiplications by $x_i$:

$$x_i T_i - T_i x_{i+1} - (1 - s)x_{i+1} = 0 \quad (1)$$
$$x_{i+1} T_i - T_i x_i + (1 - s)x_{i+1} = 0. \quad (2)$$
The double affine Hecke algebra is defined as
\[ \mathcal{H}_N(q, s) := \mathbb{C}(s, q)[T_1, \ldots, T_{N-1}, w^\pm 1, x_1^ \pm 1, \ldots, x_N^\pm 1]. \]

The double affine Hecke algebra admits a maximal commutative subalgebra generated by the Cherednik elements:
\[ \xi_i := s^{i-N}T_{i-1}^{-1} \ldots T_1^{-1}wT_{N-1} \ldots T_i. \]

The (nonsymmetric) Macdonald polynomials are the simultaneous eigenfunctions of the Cherednik operators. This implies that one can compute them using the Yang-Baxter graphs: the spectral vector of 1 is \( \zeta = [(\frac{1}{q^{1-i}})]_{1 \leq i \leq N}. \)

The nonaffine edges act by \( s_i \) on the spectral vector and by \( T_i - \frac{T_i - 1}{q^{i-s_i} - 1} \) on the polynomials. The affine edges act by \( w \) on the spectral vector and by \( \Phi_q := T_1^{-1} \ldots T_{N-1}^{-1}x_N \) on the polynomial. Note that there exists a shifted version. All of that is contained in the papers [10, 1].

From [1], we define a \((q, s)\)-version of the Dunkl operator:

1. \( D_N := (1 - s^{N-1}\xi_N)x_N^{-1} \)
2. \( D_i := \frac{1}{s}T_iD_{i+1}T_i \)

These operators generalize the Dunkl operator for the double affine Hecke algebra. For instance one has
\[ D_{i+1}T_i = -sT_i^{-1}D_i, \quad -T_iD_{i+1} + (1 - s)D_i + D_iT_i = 0 \]
\[ -D_{i+1}T_i^{-1} - (1 - \frac{1}{s})D_{i+1} + T_i^{-1}D_i = 0 \quad (3) \]
\[ [D_i, T_j] = 0 \text{ when } |i - j| > 2. \]

The \((q, s)\)-Dunkl operators have also interesting commutation properties w.r.t. the operator \( w \)
\[ D_{i+1}w = wD_i, \quad 1 \leq i \leq N - 1 \quad (4) \]
\[ qD_1w = wD_N \quad (5) \]

Note also that the operators \( D_i \) commute with each other.
\[ [D_i, D_j] = 0, \quad 1 \leq i, j \leq N. \quad (6) \]
2.2 Modules for the Hecke algebra

Definition 2.1 A tableau of shape $\lambda$ is a filling with integers weakly increasing in each row and in each column. In the sequel row-strict means (strictly) increasing in each row and column-strict means (strictly) increasing in each column.

A reverse standard tableau (RST) is obtained by filling the shape $\lambda$ with integers $1, \ldots, N$ and with the conditions of strictly decreasing in the line and the column. We will denote by $\text{Tab}_\lambda$, the set of the RST with shape $\lambda$.

Let $T$ be a RST, we define the vector of contents of $\tau$ as the vector $CT_T$ such that $CT_T[i]$ is the content of $i$ in $T$ (The coordinates of the cell are $(\text{ROW}_T[i], \text{COL}_T[i])$, row and column; $CT_T[i] = \text{COL}_T[i] - \text{ROW}_T[i]$.)

Example 2.2

$$CT_T = \begin{pmatrix} 2 & 5 & 4 \\ 6 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 2, -2, 1, 0, -1, 0 \end{pmatrix}$$

As in [3, 4] (see also [9]), let us introduce the pairwise commuting Murphy elements

$$L_N := 0,$$
$$L_i := T_i + \frac{1}{s}T_iL_{i+1}T_i, 1 \leq i < N.$$

Let $V_\lambda$ be the vector space spanned by (independent) $\{T : T \in \text{Tab}(\lambda)\}$. The action of $H_N(q, s)$ on $V_\lambda$ satisfies

$$TL_i = \frac{1 - s^{CT_T[i]}}{1 - s}T, 1 \leq i \leq N.$$

These equations determine $\{T\}$ up to scalar multiplication. There is a modification of the Murphy elements which is actually more useful for our applications.

Definition 2.3 For $1 \leq i \leq N$ let $\phi_i := s^{i-N}T_iT_{i+1}T_{N-1}\ldots T_i$, or equivalently, $\phi_N = 1$ and $\phi_i = \frac{1}{s}T_i\phi_{i+1}T_i$ for $1 \leq i < N$.

Proposition 2.4 $\phi_i = 1 + \frac{s-1}{s}L_i$ for $1 \leq i \leq N$, and if $T \in \text{Tab}(\lambda)$ then $v_T\phi_i = s^{CT_T[i]}T$. 


Proof. Use downward induction; the statement is true for $i = N$. Suppose the statement is true for $\phi_{i+1}$ then

$$
\phi_i = \frac{1}{s} T_i \left( 1 + \frac{s-1}{s} L_{i+1} \right) T_i 
= \frac{1}{s} \left( T_i^2 + \frac{s-1}{s} T_i L_{i+1} T_i \right) 
= \frac{1}{s} \left( (s-1) T_i + s + \frac{s-1}{s} T_i L_{i+1} T_i \right) 
= 1 + \frac{s-1}{s} L_i.
$$

Thus $T\phi_i = \left( 1 + \frac{s-1}{s} \left( 1 - s^{-1} C_{T[i]} \right) \right) T = s C_{T[i]} T$. □

There is an important commutation relation.

Lemma 2.5 Suppose $1 \leq i, j \leq N - 1$ and $i \neq j, j + 1$ then $T_{j-1}^{-1} \phi_i T_j = \phi_i$.

Proof. If $j < i - 1$ the result follows from $T_k T_j = T_j T_k$ for $|i - j| \geq 2$. Suppose $j > i$ then (note $T_{j-1} T_j = T_j T_{j-1}$)

$$
s^{N-i} T_{j-1}^{-1} \phi_i T_j = T_{j-1} T_i \ldots T_{N-1} T_{N-1} \ldots T_i T_j 
= T_i \ldots T_{j-2} T_{j-1} T_{j-1} T_j \ldots T_j T_{j-1} T_j \ldots T_i 
= T_i \ldots T_{j-2} T_{j-1} T_{j-1} T_{j-1} T_{j-1} \ldots T_j T_{j-1} T_j \ldots T_i 
= T_i \ldots T_{j-2} T_{j-1} T_{j-1} T_{j-1} T_{j-1} \ldots T_{j+1} T_{j-1} T_j T_{j-1} \ldots T_i 
= s^{N-i} \phi_i.
$$

□

We describe the action of $T_i$ on $T$. There are two special cases:

$$
\text{ROW}_T[i] = \text{ROW}_T[i+1] \implies TT_i = sT, \\
\text{COL}_T[i] = \text{COL}_T[i+1] \implies TT_i = -T.
$$

Otherwise, if we denote by $T^{(i,j)}$ the tableau $T$ where the entries $i$ and $j$ have been permuted, the tableaux $T^{(i,j+1)}$ is a RST. If $\text{ROW}_T[i] < \text{ROW}_T[i+1]$ (implying $\text{COL}_T[i] > \text{COL}_T[i+1]$) then

$$
TT_i = T^{(i,i+1)} - \frac{1-s}{1-s^{-1} \text{COL}_T[i+1]-\text{COL}_T[i]} T,
$$

(7)

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note this is a formula for $T^{(i,i+1)}$. If $\text{ROW}_T[i] > \text{ROW}_T[i + 1]$ (implying $\text{COL}_T[i] < \text{COL}_T[i + 1]$) then set $m := \text{CT}_T[i + 1] - \text{CT}_T[i]$ ($> 0$ by the hypothesis)

$$TT_i = \frac{s - 1}{1 - s^m} + \frac{s (1 - s^{m+1}) (1 - s^{m-1})}{(1 - s^m)^2} T^{(i,i+1)}.$$ \hspace{1cm} (8)

Formally this gives the special cases; $m = 1$ when $\text{COL}_T[i] = \text{COL}_T[i + 1]$ and $m = -1$ when $\text{ROW}_T[i] = \text{ROW}_T[i + 1]$.

### 2.3 Hecke elements associated to a multi-index

Denote $S := T_1 \ldots T_{N-1}$ and $\theta = s_1 \ldots s_{N-1}$ . Observe that if $i > 1$

$$T_i S = ST_{i-1}$$

and $s_i \theta = \theta s_{i-1}$. \hspace{1cm} (9)

For each multi-index $u = [u_1, \ldots, u_N]$ we define

$$T_u = \begin{cases} 
1 & \text{if } u = [0, \ldots, 0] \\
T_{[u_{N-1},u_1,\ldots,u_{N-1}]} S & \text{if } u_N > 0 \\
T_{[u_1,\ldots,u_{i-1},0,u_i,0,\ldots]} T_i & \text{if } u_i > 0.
\end{cases}$$ \hspace{1cm} (10)

Example 2.6 Let $u = [0, 1, 0, 2]$ then $T_u = ST_3T_2ST_3S$:

\[
\begin{array}{c}
[0, 1, 0, 2] \\
[0, 0, 0, 0] \\
[1, 0, 1, 0] \\
[0, 1, 0, 0] \\
[0, 0, 1, 0] \\
[0, 0, 0, 0] \\
\end{array}
\]

Since we use only braid relations and commutations, if $u[j] > u[j + 1]$ one has

$$T_u = T_{us} T_j.$$ \hspace{1cm} (11)

Hence, the vector $T_u$ can be obtained by any product of the type $A_1 \ldots A_k$ where $A_i \in \{S\} \cup \{T_i : i = 1..N-1\}$ such that
1. We obtain $u$ from $[0, \ldots, 0]$ by applying $a_1 \ldots a_k$ where $a_i = s_j$ if $A_i = T_j$ and $a_i = \theta$ if $A_i = S$.

2. If $a_i = s_j$ then $u' := u.a_1 \ldots a_{i-1}$ verifies $u'[j] < u'[j+1]$.

Example 2.7 One has

$$T_{[0102]} = ST_3 T_3 ST_3$$
$$= ST_3 T_3 T_2 T_2 T_1 T_2 T_3 T_3$$
$$= ST_3 T_3 T_2 T_1 T_3 T_1 T_2 T_3$$
$$= ST_3 T_3 T_2 T_1 T_3 T_2 T_3 T_2$$
$$= ST_3 S ST_1 S T_2$$

graphically:

![Graphical representation of the example](image)

Remark 2.8 The construction of $T_u$ can be illustrated in terms of braids. The generators $T_i$ and $S$ are interpreted as

$$T_i = \begin{array}{c}
\vdots \\
u_i \\
u_{i+1}
\end{array}$$

$$S = \begin{array}{c}
\vdots \\
u_i \\
u_{i+1}
\end{array}$$
For instance for \( u = [0, 1, 0, 2] \) one obtains the braid:

We introduce the creation operator

\[
\mathcal{C}_i := (ST_{N-1} \ldots T_i)^i
\]

This operator is such that if \( v = [v[1], \ldots, v[N]] \) is partition, then

\[
T_v \mathcal{C}_i = T_{[v[1]+1, v[i]+1, v[i+1], \ldots, v[N]]}
\]

is the partition obtained from \( v \) by adding 1 to the \( i \) first entries. As a consequence, the element associated to a partition is a product of creation operators

\[
T_{[v_1, \ldots, v_N]} = \mathcal{C}_{1}^{v_1-v_2} \ldots \mathcal{C}_{N-1}^{v_{N-1}-v_N} \mathcal{C}_{N}^{v_N}
\]

**Example 2.9** Consider the computation of \( T_{[2,1,0]} \) in the following figure.
Setting $\tilde{\phi}_i := s^{N-i} \phi_i = T_i \ldots T_{N-1} T_{N-1} \ldots T_i$, one has

**Proposition 2.10**

$$c_i = \tilde{\phi}_1 \ldots \tilde{\phi}_i.$$

We need the following lemma

**Lemma 2.11** Let $i - k > 1$, one has

$$(T_{i-k} \ldots T_i) (ST_{N-1} \ldots T_i) = (ST_{N-1} \ldots T_{i+1}) (T_{i-k-1} \ldots T_i)$$

**Proof.** By equation (9), one has

$$T_i (ST_{N-1} \ldots T_i) = ST_{i-1} (T_{N-1} \ldots T_i) = (ST_{N-1} \ldots T_{i+1}) (T_{i-1} T_i)$$

Hence, using successively equation (9), one obtains

$$(T_{i-k} \ldots T_i) (ST_{N-1} \ldots T_i) = (T_{i-k} \ldots T_{i-1}) (ST_{N-1} \ldots T_{i+1}) (T_{i-1} T_i) = S (T_{i-k-1} \ldots T_{i-2}) (T_{N-1} \ldots T_{i+1}) (T_{i-1} T_i) = (ST_{N-1} \ldots T_{i+1}) (T_{i-k-1} \ldots T_i),$$

as expected. ■

**Proof.** (Proposition 2.10)

Applying successively lemma 2.11, one has

$$\tilde{\phi}_1 \tilde{\phi}_2 \ldots \tilde{\phi}_i = (ST_{N-1} \ldots T_i) (T_{i-1} \ldots T_2) (ST_{N-1} \ldots T_2) \tilde{\phi}_3 \ldots \tilde{\phi}_i = (ST_{N-1} \ldots T_i)^2 (T_{i-2} T_{i-1}) \ldots (T_1 T_2) \tilde{\phi}_3 \ldots \tilde{\phi}_i = (ST_{N-1} \ldots T_i)^2 (T_{i-2} T_{i-1}) \ldots (T_2 T_3) ST_{N-1} \ldots T_3 \tilde{\phi}_4 \ldots \tilde{\phi}_i = (ST_{N-1} \ldots T_i)^3 (T_{i-3} T_{i-2} T_{i-1}) \ldots (T_1 T_2 T_3) \tilde{\phi}_4 \ldots \tilde{\phi}_i = (ST_{N-1} \ldots T_i)^4 (T_{i-4} T_{i-3} T_{i-2} T_{i-1}) \ldots (T_1 T_2 T_3 T_4) \tilde{\phi}_5 \ldots \tilde{\phi}_i = \ldots = (ST_{N-1} \ldots T_{N-i})^i$$

As a consequence, if $\mathbb{T}$ is a RST and $v$ is a partition, one has

$$\mathbb{T} v = s^* \mathbb{T},$$

where $*$ denotes an integer which depends only on $v$ and $\mathbb{T}$. 

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2.4 Rank function

There is a unique element of $\mathcal{H}_N (q, s)$ associated to each $\sigma \in S_N$. The length of $\sigma \in S_N$ is

$$\ell (\sigma) = \# \{(i, j) : 1 \leq i < j \leq N, i.\sigma > j.\sigma\}.$$ 

There is a shortest expression $\sigma = s_{i_1} \ldots s_{i_{\ell (\sigma)}}$ and a unique element $\tilde{T}_\sigma \in \mathcal{H}_N (q, s)$ defined by

$$\tilde{T}_\sigma = T_{i_1} \ldots T_{i_{\ell (\sigma)}}. \tag{13}$$

For any $s_i \ell (s_i \sigma) = \ell (\sigma) + 1$; if $\ell (s_i \sigma) = \ell (\sigma) + 1$ then $\tilde{T}_{s_i \sigma} = T_i \tilde{T}_\sigma$ and if $\ell (s_i \sigma) = \ell (\sigma) - 1$ then $\tilde{T}_{s_i \sigma} = T_i^{-1} \tilde{T}_\sigma$. Similarly, if $\ell (s_i \sigma) = \ell (\sigma) + 1$ then $\tilde{T}_{s_{i_1}} = \tilde{T}_{\sigma} T_i$, or if $\ell (s_i \sigma) = \ell (\sigma) - 1$ then $\tilde{T}_{s_{i_1}} = \tilde{T}_{\sigma} T_i^{-1}$. The following will be used in the analysis of the raising operator for polynomials.

**Proposition 2.12** Suppose $\sigma \in S_N$ then $\tilde{T}_{\sigma}^{-1} \tilde{T}_{\theta^{-1} \sigma} = s^{N-1.\sigma} \phi_{1.\sigma}$.

**Proof.** Use induction on $\ell (\sigma)$. The statement is true for $\ell (\sigma) = 0$, $\sigma = 1$, because $\tilde{T}_{\sigma} = T_1 \ldots T_{N-1} T_{N-1} \ldots T_1 = s^{N-1} \phi_1$. Suppose the statement is true for all $\sigma'$ with $\ell (\sigma') \leq n$ and $\ell (\sigma) = n + 1$. For some $k$ one has $\ell (\sigma s_k) = \ell (\sigma) - 1$. Set $\sigma' := \sigma s_k$ and $i := 1.\sigma'$, then $\tilde{T}_\sigma = \tilde{T}_{\sigma'} T_k$. If $\ell (\theta^{-1} \sigma' s_k) = \ell (\theta^{-1} \sigma') + 1$ then $\tilde{T}_k = \tilde{T}_{\sigma'} T_k$ and

$$\tilde{T}_{\sigma}^{-1} \tilde{T}_{\theta^{-1} \sigma} = T_k^{-1} \tilde{T}_{\sigma'}^{-1} \tilde{T}_k \tilde{T}_{\theta^{-1} \sigma'} T_k^{-1} = s^{N-i} \tilde{T}_k^{-1} \phi_i T_k^{-1},$$

by the inductive hypothesis. If $\ell (\theta^{-1} \sigma' s_k) = \ell (\theta^{-1} \sigma') + 1$ then $\tilde{T}_{\sigma}^{-1} \tilde{T}_{\theta^{-1} \sigma} = \tilde{T}_{\theta^{-1} \sigma'} T_k$ and $\tilde{T}_{\sigma}^{-1} \tilde{T}_{\theta^{-1} \sigma} = s^{N-i} \tilde{T}_k^{-1} \phi_i T_k$ by a similar argument. Let $i_1 = k.\sigma' - 1$ and $i_2 = (k + 1).\sigma' - 1$, by hypothesis $i_1 < i_2$. Let $j_1 = k. (\theta^{-1} \sigma') - 1 = i_1.\theta$ and $j_2 = (k + 1). (\theta^{-1} \sigma') - 1 = i_2.\theta$. Then $\ell (\theta^{-1} \sigma' s_k) = \ell (\theta^{-1} \sigma') + 1$ if and only if $j_1 < j_2$. (Note $j.\theta = j - 1$ if $j > 1$ and $1.\theta = N$.) Since $i_2 > i_1 \geq 1$ it follows that $j_2 = i_2 - 1$. If $i_1 = 1$ then $j_1 = N > j_2$ and so $\ell (\theta^{-1} \sigma' s_k) = \ell (\theta^{-1} \sigma') - 1$, $k = 1.\sigma' = i$. This implies $1.\sigma = i + 1$ and $\tilde{T}_{\sigma}^{-1} \tilde{T}_{\theta^{-1} \sigma} = s^{N-i} \tilde{T}_k^{-1} \phi_i T_k^{-1} = s^{N-i} \phi_{i+1}$. If $i_1 > 1$ then $j_1 = i_1 - 1 < j_2$ and $\ell (\theta^{-1} \sigma' s_k) = \ell (\theta^{-1} \sigma') + 1$. In this case $1.\sigma' \neq k, k + 1$ and so $s^{N-i} \tilde{T}_k^{-1} \phi_i T_k = s^{N-i} \phi_i$, by Lemma 2.5; also $1.\sigma = 1.\sigma' = i$; and this completes the induction. ■
Consider the rank function of a multi-index $v = [v[1], \ldots, v[N]]$ as an element of $S_N$

$$r_v[i] := \# \{ j : 1 \leq j \leq i, v[j] \geq v[i] \} + \# \{ j : i < j \leq N, v[j] > v[i] \}.$$ 

Example 2.13

1. If $v = [4, 2, 2, 3, 2, 1, 4, 4]$ then $r_v = [1, 5, 6, 4, 7, 8, 2, 3]$.

2. If $v$ is a (decreasing) partition $r_v = id$.

The length of $r_v$ is

$$\ell(r_v) := \#inv(v)$$

with $inv(v) := \{(i, j) : 1 \leq i \leq N, v[i] < v[j]\}$, the number of inversions in $v$ (note for $i < j$ that $r_v[i] > r_v[j]$ if and only if $v[i] < v[j]$). There is a shortest expression $r_v = s_{i_1} \cdots s_{i_{\ell(r_v)}}$ and an element $R_v \in H_N(q, s)$ defined by

$$R_v := T^{-1}_{i_{\ell(r_v)}} \cdots T^{-1}_{i_1} = \tilde{T}_{r_v}^{-1}.$$ 

One has

Lemma 2.14

1. If $v[i] > v[i+1]$ then $R_{vs_l} = R_v T^{-1}_{i_l}$.

2. If $v[i] < v[i+1]$ then $R_{vs_l} = R_v T_i$.

3. If $v[i] = v[i+1]$ then $R_v T_i = T_{r_v[i]} R_v$.

Proof.

1. If $v[i] > v[i+1]$ then $r_{vs_l} = s_i r_v$ and $\#inv(vs_l) = \#inv(v) + 1$ so $R_{vs_l} = R_v T^{-1}_i$.

2. Similarly if $v[i] < v[i+1]$ then $R_{vs_l} = R_v T_i$.

3. If $v[i] = v[i+1]$ and $k = r_v[i]$ then $s_i r_v = r_v s_k$ and $\ell(s_i r_v) = \ell(r_v) + 1$ (one extra inverted pair $(k+1, k)$); thus $\tilde{T}_{vs_l} = T_i \tilde{T}_{r_v}$ and $\tilde{T}_{r_v s_k} = \tilde{T}_{r_v} T_k$. Hence, $R_v T_i = T_k R_v$.

We compare the elements $T_v$ and $R_v$ in terms of $T_v R_v^{-1}$. We need to consider three cases:
1. If \( T_{[0,\ldots,0]} = I, \) then let \( \eta = T_{[0,\ldots,0]} \).

2. In the case \( T_{[v_1,0,\ldots,v_i,0,\ldots]} = T_{[v_1,0,\ldots,v_i,0,\ldots]}T_i \) for \( i \geq 1, i < N \) we see that \( \#\text{inv}(v.s_i) = \#\text{inv}(v) + 1 \), hence \( r_{v.s_i} = r_{v} \) (see Lemma 2.14 (1)) and \( \tilde{T}_{v.s_i} = T_i\tilde{T}_{v}, R_{v.s_i} = R_{v}T_i^{-1} \). So we have

\[
T_{v.s_i}R_{v.s_i}^{-1} = T_{v}R_{v}^{-1}. \tag{14}
\]

3. If \( T_{v} = T_{v}S \) \( (v_{s} := (v_2, v_3, \ldots, v_N, v_1 + 1)) \), then we have \( r_{v_{s}} = s_{N-1}s_{N-2} \ldots s_1r_{v} = \theta^{-1}r_{v} \), where \( \theta = s_1s_2 \ldots s_{N-1} \). By Proposition 2.12 (let \( k = r_{v} \) [1])

\[
\tilde{T}_{v}^{-1}\tilde{T}_{v}\theta^{-1} = s_{N-k}\phi_k,
\]

\[
s_{N+k}\phi_k^{-1}R_{v}S = R_{v_{s}},
\]

and thus

\[
T_{v}R_{v}^{-1} = s_{N-k}T_{v}SS^{-1}R_{v}^{-1}\phi_k = s_{N-k}T_{v}R_{v}^{-1}\phi_k. \tag{15}
\]

As a consequence:

**Proposition 2.15** \( T_{v}R_{v}^{-1} \) is in the commutative algebra generated by \( \{ \phi_i : 1 \leq i \leq N \} \) for each \( v \), and acts by scalar multiplication (by powers of \( s \)) on each \( \mathbb{T} \) (recall \( \mathbb{T}\phi_i = s^{CT(i,\mathbb{T})}\mathbb{T}, \ 1 \leq i \leq N \)). Furthermore :

\[
T_{v} = \prod_{i=1}^{N} (s^{N-i}\phi_i)^{u_i} R_{v}.
\]

**Proof.** By equation (14) if the formula is true for \( v \) with \( v_j = 0 \) for \( j > i \) and \( v_i \geq 1 \) then it is true for \( v.s_i \) (note \((v.s_i)^+ = v^+\)). Using induction, suppose the formula is true for all \( v \) with \( |v| \leq n \), for some \( n \geq 0 \) (the case \( n = 0 \) is trivially satisfied). Let \( |v| = n + 1 \). Using the case 2 step as often as necessary assume \( v_N \geq 1 \). Thus \( v = u_{s} \) with \( |u| = n \), and \( r_{v} = \theta^{-1}r_{u} \), in particular, let \( k = r_{v} [N] = r_{u} [1] \). Then \( v^+ = (u^+_1, \ldots, u^+_k + 1, \ldots, u^+_N) \) (\( u \) has exactly \( k-1 \) entries \( > u_1 \), and thus \( v \) has exactly \( k \) entries \( \geq v_N = u_1 + 1 \), including \( v_N \); hence \( v_k^+ = v_N = u_1 + 1 = u_k^+ + 1 \)). By equation (15) and the inductive hypothesis

\[
T_{v}R_{v}^{-1} = (s^{N-k}\phi_k) T_{u}R_{u}^{-1} = (s^{N-k}\phi_k) \prod_{i=1}^{N} (s^{N-i}\phi_i)^{u_i^+},
\]

and this proves the claim.

In particular if \( v \) is a partition then \( T_{v} = \prod_{i=1}^{N} (s^{N-i}\phi_i)^{v_i} \).
3 Vector valued polynomials

3.1 First Examples

To motivate our definitions we consider the simplest two-dimensional situation: \( N = 3 \), isotype \( \lambda = (2, 1) \). A basis for the representation of \( \{T_1, T_2\} \) is

\[
\begin{align*}
    f_1 &= s x_1 - \frac{1}{s+1} (x_2 + x_3), \\
    f_2 &= x_2 - \frac{1}{s} x_3.
\end{align*}
\]

Then \( f_1 T_2 = s f_1, f_2 T_2 = -f_2 \) and

\[
\begin{align*}
    f_1 T_1 &= -\frac{1}{s+1} f_1 + \frac{s(1+s+s^2)}{(1+s)^2} f_2, \\
    f_2 T_1 &= f_1 + \frac{s^2}{1+s} f_2.
\end{align*}
\]

We aim to set up a Macdonald-type structure in \( \{p_1(x) f_1 + p_2(x) f_2\} \). Firstly define operators \( T'_i \) acting on pairs \( [p_1, p_2] \) so that

\[
[p_1, p_2] T'_i [f_1, f_2] = (p_1 f_1 + p_2 f_2) T_i, i = 1, 2,
\]

where \( [a_1, a_2], [b_1, b_2] := a_1 b_1 + a_2 b_2 \). Indeed

\[
[p_1, p_2] T'_1 = \begin{bmatrix}
    p_1 T_1 - \frac{1+s+s^2}{1+s} p_1 s_1 + p_2 s_1, p_2 T_1 - \frac{s}{1+s} p_2 s_1 + \frac{s(1+s+s^2)}{(1+s)^2} p_1 s_1
\end{bmatrix}
\]

\[
[p_1, p_2] T'_2 = [p_1 T_2, p_2 T_2 - (s+1) p_2 s_2]
\]

The inverses follow from the quadratic relation: \( T'_i^{-1} = \frac{1}{s} (T'_i + 1 - s) \).

Secondly we need a definition of \( w \) (to be generalized in the sequel). The relation \( w T_1 = T_2 w \) must be satisfied. The braid relation gives a solution \( T_2 (T_1 T_2) = (T_1 T_2) T_1 \). Using \( w' = T_1 T_2 \) let

\[
\begin{align*}
    f_1 w' &= -\frac{s}{1+s} f_1 - \frac{s(1+s+s^2)}{(1+s)^2} f_2, \\
    f_2 w' &= s f_1 - \frac{s^2}{1+s} f_2.
\end{align*}
\]
Then $w'T_1 = T_2w'$ acting on $\text{span}\{f_1, f_2\}$. Now define

$$[p_1, p_2] w = \left[ -\frac{s}{1+s}p_1 w + sp_2w, \frac{s (1+s + s^2)}{(1+s)^2}p_1w - \frac{s^2}{1+s}p_2w \right].$$

Set

$$\xi_1 = s^{-2}w'T_2'T_1',
\xi_2 = s^{-1}T'_1 T^{-1}w'T_2',
\xi_3 = T'_2 T^{-1}_1 w.$$

These operators commute. Here are the degree 1 simultaneous eigenfunctions:

$$[- (1+s) x_3, sx_3],
[x_3, \frac{1+s+s^2}{1+s} x_3],
[(s+1)x_2 + \frac{q(1-s^2)}{1-q s}x_3, x_2 - \frac{sq (1-s)}{1-q s}x_3],
[x_2 - \frac{q(1-s)}{s(q-s)}x_3, \frac{1+s+s^2}{s(1+s)} \left\{ x_2 + \frac{q(1-s)}{q-s}x_3 \right\}],
[q(1-s) \left\{ sx_2 - x_3 \right\}, x_1 + \frac{sq(1-s)}{(1+s)(1-q^2 s)} \left\{ x_2 + x_3 \right\}],
[x_1 + \frac{qs(1-s)}{(1+s)(q-s^2)} \left\{ x_2 + x_3 \right\}, -\frac{q(1+s+s^2)(1-s)}{(1+s)^2(q-s^2)} \left\{ x_2 - sx_3 \right\}].$$

To generalize to an arbitrary irreducible module $V_\lambda$ (basis corresponding to Tab$_\lambda$) we need to define $w$; a necessary condition is that there be an intertwining operator $S$ on $V$ so that $ST_i = T_{i+1}S$ for $1 \leq i < N$. The correct definition is $S = T_1 T_2 \ldots T_{N-1}$. Indeed

$$ST_i = T_1 \ldots T_{i-1}T_i T_{i+1}T_i T_{i+2} \ldots T_{N-1} = T_1 \ldots T_{i-1}T_{i+1}T_{i+1}T_i T_{i+2} \ldots T_{N-1} = T_{i+1}S.$$

**Definition 3.1** The space of vector valued polynomials for the isotype $\lambda$ (partition of $N$) we be denoted by $\mathcal{M}_\lambda := \mathbb{C}[x_1, \ldots, x_N] \otimes V_\lambda$. 
The elements of $\mathcal{M}_\lambda$ are linear combinations of $x^\nu \mathbb{T}$ where $x^\nu := x_1^{\nu[1]} \cdots x_N^{\nu[N]}$. We will denote by ‘normal symbols’ ($s_i, T_i, w, \xi_i$ etc.) the operators acting only on the tableaux. The operator acting only on the letters will be denoted with superscript $x$ ($s^x_i, T^x_i, w^x, \xi^x_i$ etc.). The operators acting on both letters and tableaux will be denoted by bold symbols ($s_i, T_i, w, \xi_i$ etc.).

3.2 Action of the double Hecke algebra on vectors

Denote $\delta^x_i := \partial^x_i x_{i+1}(1 - s) + s^x_i x_i - s^x_i x_{i+1}(1 - s)$. We have:

**Lemma 3.2** The operator $T_i$ satisfies the quadratic relation:

$$(T_i + 1)(T_i - s) = 0$$ (16)

**Proof.** From

$$\partial^x_i x_{i+1} \partial^x_i = \partial^x_i \partial^x_i x_{i+1} + \partial^x_i s^x_i (x_{i+1} \partial^x_i) = -\partial^x_i,$$

we deduce

$$\delta^x_i \partial^x_i x_{i+1}(1 - s) = -(1 - s) \delta^x_i.$$ (17)

And from

$$\partial^x_i x_{i+1} s^x_i + s^x_i \partial^x_i x_{i+1} = \partial^x_i (x_i - x_{i+1}) = 1 - s^x_i,$$

one obtains

$$\delta^x_i s^x_i + s^x_i \delta^x_i = (1 - s)(1 - s^x_i).$$ (18)

Now, expanding $(T_i + 1)(T_i - s)$ we observe

$$(T_i + 1)(T_i - s) = (\delta^x_i \partial^x_i + (1 - s) \partial^x_i) + (\delta^x_i s^x_i + s^x_i \delta^x_i + (1 - s)(s^x_i - 1))T_i$$

$$= 0,$$

from equations (17) and (18). □

We found also commutations:

**Lemma 3.3** If $|i - j| > 1$ we have

$$T_i T_j = T_j T_i.$$ (19)
Proof. First we expand

$$T_i T_j = \delta_i^x \delta_j^x + \delta_i^x s_j^x T_j + s_i^x \delta_j^x T_i + s_i^x s_j^x T_i T_j.$$  \hfill (20)

But since $|i - j| > 1$, one has straightforwardly $s_i^x s_j^x = s_j^x s_i^x$, $T_i T_j = T_j T_i$, $\delta_i^x s_j^x = s_j^x \delta_i^x$ and $\delta_i^x \delta_j^x = \delta_j^x \delta_i^x$. Using these relations in equation (20), we find the result. ■

To prove the braid relations, we need the following preliminary results.

Lemma 3.4 1. $s_i^x s_{i+1}^x s_i^x T_i T_{i+1} T_i = s_i^x s_{i+1}^x s_i^x T_{i+1} T_i T_{i+1}$

2. $\delta_i^x \delta_{i+1}^x \delta_i^x = \delta_{i+1}^x \delta_i^x \delta_i^x$

3. $\delta_i^x s_i^x \delta_{i+1}^x = s_i^x \delta_{i+1}^x \delta_i^x + \delta_i^x \delta_{i+1}^x s_i^x + (s - 1) s_i^x \delta_{i+1}^x s_i^x$

4. $\delta_i^x s_{i+1}^x s_i^x = s_i^x \delta_{i+1}^x \delta_i^x + \delta_{i+1}^x \delta_i^x s_{i+1}^x + (s - 1) s_i^x \delta_{i+1}^x s_{i+1}^x$

5. $\delta_i^x s_{i+1}^x s_i^x = s_i^x s_{i+1}^x \delta_i^x$

6. $\delta_i^x s_i^x s_{i+1}^x = s_i^x s_i^x \delta_i^x$

7. $s_i^x \delta_i^x s_i^x = s_i^x s_i^x \delta_i^x$

Proof. The first identity is trivial. But the others need to be proved. The simplest way to check these formulæ is the direct verification on monomial $x_i^a x_{i+1}^b x_{i+2}^c$. For instance, the second equality follows from

$$x_i^a x_{i+1}^b x_{i+2}^c \delta_i^x \delta_{i+1}^x = (1 - s)^3 \frac{x_{i+2}^2 x_{i+1}}{V(x_1, x_{i+1}, x_{i+2})} \det \begin{pmatrix} x_i^a & x_{i+1}^a & x_{i+2}^a \\ x_i^b & x_{i+1}^b & x_{i+2}^b \\ x_i^c & x_{i+1}^c & x_{i+2}^c \end{pmatrix}$$

$$= x_i^a + x_{i+1}^b x_{i+2}^c \delta_i^x \delta_{i+1}^x,$$

where $V(x_1, x_2, x_3) := \prod_{0 < i < j < 3} (x_i - x_j)$ denotes the Vandermonde determinant. The others identities give (for simplicity we omit the superscript $x$ on $\delta$ and $s$)

3.

$$x_i^a x_{i+1}^b x_{i+2}^c \delta_i s_{i+1} \delta_i = (1 - s)^3 \frac{x_{i+2}^2 x_{i+1}}{V(x_1, x_{i+1}, x_{i+2})} \left[ x_i^a \left( x_{i+1}^b x_{i+2}^c x_{i+1}^a - x_i^b x_{i+1}^c - x_i^c x_{i+1}^b \right) - x_{i+2}^b \left( x_{i+1}^a x_{i+2}^c x_{i+1}^a - x_{i+1}^b x_{i+1}^c - x_{i+1}^c x_{i+1}^a \right) \right]$$

$$x_i^a x_{i+1}^b x_{i+2}^c \left( s_{i+1} \delta_i s_{i+1} + \delta_i s_{i+1} + (s - 1) s_{i+1} \delta_i s_{i+1} \right).$$
Next we show that the operators \( \{ T_i \} \) satisfy the braid relations.

**Proposition 3.5** For each \( i < N - 1 \), one has

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}
\]

**Proof.** Expanding the braid relation \( T_i T_{i+1} T_i \) we obtain

\[
T_i T_{i+1} T_i = \delta^x_i \delta^x_{i+1} T_i + \delta^x_i s^x_{i+1} \delta^x_{i+1} T_i + (s^y_i \delta^x_{i+1} s^x_i + s^x_i \delta^x_{i} s^x_{i+1} + (s-1) s^x_i \delta^x_{i+1} s^x_i)
\]

Using the fact that \( T_i^2 = (s-i) T_i + s \) we obtain

\[
T_i T_{i+1} T_i = \delta^x_i \delta^x_{i+1} T_i + \delta^x_i s^x_{i+1} \delta^x_{i+1} T_i + (s^y_i \delta^x_{i+1} s^x_i + s^x_i \delta^x_{i} s^x_{i+1} + (s-1) s^x_i \delta^x_{i+1} s^x_i)
\]

Now applying lemma \( \text{[34]} \) we show the desired result. ■

Now, examine the relation between the generators \( T_i \) and the multiplication by an indeterminate \( x_i \). One has to show three identities:
Proposition 3.6

1. \( x_i T_i - T_i x_{i+1} - (1 - s) x_{i+1} = 0 \)

2. \( x_{i+1} T_i - T_i x_i + (1 - s) x_{i+1} = 0 \)

3. \( x_i T_j = T_j x_i \) when \(|i - j| > 1\).

Proof.

1. One has
   \[
   x_i \delta^x_i = (1 - s) x_i \partial^x_i x_{i+1}
   = (1 - s) \partial^x_i x_{i+1}^2 + (1 - s) x_{i+1}
   = \delta^x_i x_{i+1} + (1 - s) x_{i+1}
   \]
   Hence
   \[
   x_i T_i = [\delta^x_i + s^x_i T_i] x_{i+1} + (1 - s) x_{i+1}
   = T_i x_{i+1} + (1 - s) x_{i+1}
   \]
   as expected.

2. The second equality is proved in the same way remarking that
   \[
   x_{i+1} \delta^x_i = (1 - s) x_{i+1} \partial^x_i x_{i+1}
   = (1 - s) \partial^x_i x_{i+1} x_i - (1 - s) x_{i+1}
   = \delta^x_i x_i - (1 - s) x_{i+1}.
   \]

3. The third equality is straightforward.

Now, we examine the affine action and set
   \[
   w = \tau^x_1 \theta^x S
   \]
   where \( \theta^x = s^x_1 \ldots s^x_{N-1} \) and \( S = T_1 \ldots T_{N-1} \). When \( i < N - 1 \) one has
   \[
   w T_i = (\tau^x_1 \theta^x S)(\delta^x_i + s^x_i T_i)
   \]
   But since \( i < N - 1 \), one has
   \[
   \tau^x_1 \theta^x \partial^x_i x_{i+1} = \tau^x_1 \partial^x_{i+1} x_{i+2} \theta^x
   \]
   and \( i + 1 > 1 \) implies \( \tau^x_1 \partial^x_{i+1} x_{i+2} = \partial^x_{i+1} x_{i+2} \tau^x_1 \). Hence,
   \[
   \tau^x_1 \theta^x \delta^x_i = \delta^x_{i+1} \tau^x_1 \theta^x.
   \]
   One easily obtains \( \tau^x_1 \theta^x s^x_i = s^x_{i+1} \tau^x_1 \theta^x \) and \( S T_i = T_{i+1} S \). We deduce
Lemma 3.7 \( wT_i = T_{i+1}w \).

From lemmas 3.2, 3.3, 3.7, propositions 3.5 and 3.6, we obtain:

**Theorem 3.8** The algebra \( \mathbb{C}(s, q)[x_1^\pm 1, \ldots, x_N^\pm 1, T_1, \ldots, T_{N-1}, w^\pm 1] \) is isomorphic to \( \mathcal{H}_N(s, q) \). More precisely, the morphism sends \( T_i \) to \( T_i \), \( w \) to \( w \) and \( x_i \) to \( x_i \).

### 3.3 Cherednik and Dunkl operators

**Definition 3.9** In this context, the (vector valued) Cherednik operators are defined as

\[
\xi_i = s^{i-N}T_{i-1}^{-1} \cdots T_1^{-1}wT_{N-1} \cdots T_i,
\]

where

\[
T_i^{-1} = \frac{1}{s}(T_i + (1 - s)) = \frac{1}{s}((1 - s)(\partial_i x_{i+1} + 1) + s_i T_i).
\]

It follows immediately that

\[
[\xi_i, \xi_j] = 0 \quad (22)
\]

since, from Theorem 3.8, the operators \( \xi_i \) are the image of the Cherednik operators \( \xi_i \).

Furthermore the tableaux are simultaneous eigenfunctions of the Cherednik elements and the associated spectral vectors can be expressed in terms of contents.

**Proposition 3.10** For each tableau \( T \), one has

\[
T\xi_i = s^{CT_i[i]}T.
\]

**Proof.** Since,

1. \( TT_i = T T_i \),
2. \( TT_i^{-1} = T T_i^{-1} \),
3. \( Tw = T S \),

one has \( T\xi_i = T\phi_i \). Hence, the result follows from proposition 2.4. \( \square \)

In the aim to define the Dunkl-Cherednik operators, we set \( F_N = 1 - \xi_N \).
Proposition 3.11 The operator $F_N$ is divisible by $x_N$, that is, for each $P \in \mathbb{C}[x_1, \ldots, x_N] \otimes V$, $PF_N = x_NQ$ with $Q \in \mathbb{C}[x_1, \ldots, x_N] \otimes V$.

Proof. We prove the result by induction on $N$. Suppose first that $N = 2$, our operator is

$$F_2 = 1 - \frac{1}{s}((1-s)(\partial_1^x x_2 + 1) + s^x_1 T_1)(\tau^x_1 s^x_1 T_1) = 1 - \frac{1}{s}((1-s)(\partial_1^x x_2 + 1)\tau^x_1 s^x_1 T_1 + s^x_1 \tau^x_1 s^x_1 T_1^2).$$

From $T_1^2 = (s-1)T_1 + s$ and $s_1 \tau_1 s_1 = \tau_2$ one obtains

$$F_2 = 1 - \frac{1}{s}((1-s)(\partial_1^x x_2 + 1 - s_1)\tau^x_1 s^x_1 T_1 + s\tau^x_2).$$

Note that

$$\partial_1^x x_2 + 1 - s^x_1 = \partial_1^x x_1$$

implies

$$F_2 = \frac{s-1}{s}g\partial_1^x \tau^x_1 s^x_1 T_1 x_2 + 1 - \tau^x_2.$$

But for any polynomial $P$, one has

$$P(x_1)x_2^b(1 - \tau^x_2) = \begin{cases} 0 & \text{if } b = 0 \\ P(x_1)x_2^b(1 - q^b) & \text{if } b > 0 \end{cases}$$

This proves the result for $N = 2$.

Now suppose $N > 2$, then

$$F_N = 1 - T_{N-1}^{-1} \cdots T_1^{-1}(s^x_1 s^x_1 \cdots s^x_{N-1} T_1 \cdots T_{N-1})$$

Similarly to the case $N = 2$, one obtains

$$F_N = 1 - \frac{1}{s}T_{N-1}^{-1} \cdots T_2^{-1}((s q \partial_1^x \tau^x_1 s^x_1 \cdots s^x_{N-1} T_1 \cdots T_{N-1})x_{N-1} + s\tau^x_2 s^x_2 \cdots s^x_{N-1} T_2 \cdots T_{N-1})$$

So it suffices to prove that the operator $1 - T_{N-1}^{-1} \cdots T_2^{-1} s^x_2 \cdots s^x_{N-1} T_2 \cdots T_{N-1}$ is divisible by $x_N$. Remarking that

$$1 - T_{N-1}^{-1} \cdots T_2^{-1}(s^x_2 \cdots s^x_{N-1} T_2 \cdots T_{N-1}) = \theta^{-1} F_{N-1} \theta$$

the result follows by induction. ■
Definition 3.12 The vector valued Dunkl operators are defined as $D_N := F_N x_N^{-1}$ and $D_i := \frac{1}{s} T_i D_{i+1} T_i$.

As for the Cherednik operators, theorem 3.8 implies that the classical relations hold. For instance one has

$$[D_i, D_j] = 0$$

and the relations w.r.t. the generators $T_i$ occur

$$D_{i+1} T_i = -s T_i^{-1} D_i, \quad -T_i D_{i+1} + (1 - s) D_i + D_i T_i = 0$$

$$-D_{i+1} T_i^{-1} - (1 - \frac{1}{s}) D_{i+1} + T_i^{-1} D_i = 0$$

$$[D_i, T_j] = 0 \text{ when } |i - j| > 1.$$

Note identities 1. and 2. of proposition 3.6 are equivalent to $x_i T_i = sx_i + 1 T_i^{-1}$ or $sx_{i+1} = T_i x_i T_i$ (these are dual to the $D_i$ relations $D_i = (1/s) T_i D_{i+1} T_i$).

3.4 Triangularity of the Cherednik operators

Let $v$ be a vector, in the sequel we will denote by $v^+$ (resp. $v^R$) the unique decreasing (resp. increasing) partition whose entries are obtained by permuting those of $v$.

Let $\pi_i^x = \partial_i^x x_{i+1} = \frac{1}{2(s-1)} \delta_i^x$, $\pi_i^x = \partial_i^x x_{i+1} + 1$ and more generally $\pi_{ij}^x = \partial_{ij}^x x_j + 1$.

Observe that if $i < j$ then one has

$$x^v \pi_{ij}^x = \sum_{v' \preceq v} (\ast) x^{v'}$$

(25)

where $(\ast)$ denotes a coefficient and $\preceq$ is the dominance order on vectors defined by

$$v \preceq v' \iff \begin{cases} v^+ \prec v'^+ & \text{when } v^+ \neq v'^+ \\ v \prec v' & \text{when } v^+ = v'^+ . \end{cases}$$

$\prec$ denoting the (classical) dominance order on partitions

$$v \prec v' \text{ iff for each } i, v[1] + \cdots + v[i] \leq v'[1] + \cdots + v'[i].$$

Indeed, it suffices to understand the computation of $x_{1}^{a} x_{2}^{b} \pi_1$. So we have three cases to consider:
1. if \( a < b \):

\[
x_1^a x_2^b \pi = - \sum_{i=1}^{b-a-1} x_i^{a+i} x_2^{b-i}
\]

In this case, one has \( x_1^a x_2^b \pi = \sum_{v'=\lfloor b,a \rfloor} (\ast) x^{v'} \).

2. if \( a = b \):

\[
x_1^a x_2^b \pi = x_1^a x_2^b
\]

3. if \( a > b \):

\[
x_1^a x_2^b \pi = \sum_{i=0}^{a-b} x_i^{a-i} x_2^{b+i}
\]

and the leading term in this expression is \( x^{[a,b]} \).

Similarly,

\[
x^{v} \pi_{ij} = \sum_{v' \leq v} (\ast) x^{v'}
\]

(26)

With these notations, write

\[
T_i = (\ast) \pi_{ij} + (\ast) s_i^x T_i
\]

and

\[
T_i^{-1} = (\ast) \pi_{ij} + (\ast) s_i^x T_i
\]

here (\ast) denotes a certain coefficient (we need not know it to follow the computation).

Observe that for each \( j \)

\[
T_1^{-1} s_i^x \ldots s_{j-1}^x = [(\ast) \pi_{11}^x + (\ast) s_1^x T_1] s_i^x \ldots s_{j-1}^x = [(\ast) \pi_{11}^x + (\ast) T_1] s_i^x \ldots s_{j-1}^x
\]

since \( \pi_1 s_1^x = \pi_{11}^x \). But \( \pi_1 s_2^x \ldots s_{j-1}^x = s_2^x \ldots s_{j-1}^x \pi_{11} \), hence:

\[
T_1^{-1} s_i^x \ldots s_{j-1}^x = s_2^x \ldots s_{j-1}^x [(\ast) \pi_{11}^x + (\ast) T_1]
\]

Iterating the process, one finds

\[
T_{j-1}^{-1} \ldots T_1^{-1} s_i^x \ldots s_{j-1}^x = [(\ast) \pi_{j-1}^x + (\ast) T_{j-1}] \ldots [(\ast) \pi_{1}^x + (\ast) T_1].
\]

(27)

One has also

\[
s_j^x \ldots s_{N-1}^x T_{N-1} = s_j^x \ldots s_{N-1}^x [(\ast) \pi_{N-1}^x + (\ast) s_{N-1}^x T_{N-1}]
\]
but $s_{N-1}^x \partial_{N-1}^x = -\partial_{N-1}^x$, hence
\[
s_j^x \ldots s_{N-1}^x T_{N-1} = s_j^x \ldots s_{N-2}^x [(\ast)\pi_{N-1}^x + (\ast)T_{N-1}]^N.
\]
Since, $s_j^x \ldots s_{N-2}^x \pi_{N-1}^x = [\pi_{j,N}^x s_j^x \ldots s_{N-2}^x \pi_{j-1}^x + (\ast)T_{N-1}]\pi_{j-1}^x s_j^x \ldots s_{N-2}^x$ one obtains
\[
s_j^x \ldots T_{N-1} = [(\ast)\pi_{j,N}^x + (\ast)T_{N-1}]s_j^x \ldots s_{N-2}^x.
\]
Iterating this process, one finds
\[
s_j^x \ldots s_{N-1}^x T_{N-1} = [\pi_{j,N}^x + (\ast)T_{N-1}^j(\ast)T_{j+1}^x] \ldots [\pi_{j,1}^x + (\ast)T_{1}^x] \ldots [\pi_{j,j}^x + (\ast)T_{j}^x] (28)
\]
Now with these notations the Cherednik operator reads
\[
\xi_j = [(\ast)\pi_{j-1,N}^x + (\ast)T_{j-1}] \ldots [(\ast)\pi_{1,N}^x + (\ast)T_{1}] \tau_{j}^x s_j^x \ldots s_{N-1}^x S
\]
Now apply eq (27) and (28):
\[
\xi_i = [(\ast)\pi_{i-1,N}^x + (\ast)T_{i-1}] \ldots [(\ast)\pi_{1,N}^x + (\ast)T_{1}] [\tau_{j}^x S]
\]
\[
\text{where } x_i \tau_{j}^x = x_i \text{ if } i \neq j \text{ and } x_j \tau_{j}^x = q x_j.
\]
From (25) and (26), we obtain
\[
T x^v \xi_i = T \left[ x^v H_v + \sum_{v' < v} x^{v'} H_{v'} \right] (29)
\]
with $H_v \in H_N(q,s)$ (we apply to $x^v$ an algebraic combination of $\pi^x$ and $\pi^x$ and the operator $\tau_{j}^x$ does not change the exponents). Finally,

**Theorem 3.13** We have
\[
x^v T \xi_j = x^v (T.H_v) + \sum_{v' < v} x^{v'} (T.H_{v'})
\]
where $H_v \in H_N(q,s)$.

**Proof.** Eq (29) gives
\[
\begin{align*}
x^v T \xi_j &= T x^v \xi_j \\
&= x^v (T H_v) + \sum_{v' < v} x^{v'} (T H_{v'})
\end{align*}
\]

4 Eigenfunctions of Cherednik operators

4.1 Yang-Baxter graph

As in [6], we construct a Yang-Baxter-type graph with vertices labeled by 4-tuples \((T, \zeta, v, \sigma)\), where \(T\) is a RST, \(\zeta\) is a vector of length \(N\) (\(\zeta\) will be called the spectral vector), \(v \in \mathbb{N}^N\) and \(\sigma \in \mathfrak{S}_N\). First, consider a RST of shape \(\lambda\) and write a vertex labeled by the 4-tuple \((T, \zeta, v, \sigma)\).

Example 4.1

\[
\begin{align*}
1. & \quad (31, 542, [s, 1, q^2, q s^2, q s^{-1}], [0, 0, 2, 1, 1], [45123]) s_2 = \\
& \quad (31, 542, [1, q^2, 1, q s^2, q s^{-1}], [0, 2, 0, 1, 1], [41523]) \\
2. & \quad (31, 542, [s, 1, q^2, q s^2, q s^{-1}], [0, 0, 2, 1, 1], [45123]) s_4 = \\
& \quad (21, 543, [s, 1, q^2, q s^{-1}, q s^2], [0, 2, 1, 1], [51234]) \\
3. & \quad (31, 542, [1, 0, 2 \alpha, \alpha + 2, \alpha - 1], [0, 0, 2, 1, 1], [45123]) s_1 = \\
& \quad (31, 542, [s, 1, q^2, q s^2, q s^{-1}], [0, 2, 1, 1], [45123]) \\
4. & \quad (31, 542, [s, 1, q^2, q s^2, q s^{-1}], [0, 0, 2, 1, 1], [45123]) \Psi = \\
& \quad (31, 542, [1, q^2, q s^2, q s^{-1}, q s], [0, 2, 1, 1], [12534]) \\
\end{align*}
\]

Definition 4.2

If \(\lambda\) is a partition, denote by \(T_\lambda\) the tableau obtained by filling the shape \(\lambda\) from bottom to top and left to right by the integers \(\{1, \ldots, N\}\)
in decreasing order.
The graph $G_{\lambda}^{q,s}$ is an infinite directed graph constructed from the 4-tuple

$$(T_{\lambda}, CT_{T_{\lambda}}^s, [0^N], [1,2,\ldots,N])$$

called the root and adding vertices and edges following the rules

1. We add an arrow labeled by $s_i$ from the vertex $(T, \zeta, v, \sigma)$ to $(T', \zeta', v', \sigma')$ if $(T, \zeta, v, \sigma)s_i = (T', \zeta', v', \sigma')$ and $v[i] < v[i+1]$ or $v[i] = v[i+1]$ and $\tau$ is obtained from $\tau'$ by interchanging the position of two integers $k < \ell$ such that $k$ is at the south-east of $\ell$ (i.e. $CT_T(k) \geq CT_T(\ell) + 2$).

2. We add an arrow labeled by $\Psi$ from the vertex $(T, \zeta, v, \sigma)$ to $(T', \zeta', v', \sigma')$ if $(T, \zeta, v, \sigma)\Psi = (T', \zeta', v', \sigma')$

3. We add an arrow $s_i$ from the vertex $(\tau, \zeta, v, \sigma)$ to $\emptyset$ if $(T, \zeta, v, \sigma)s_i = (T, \zeta, v, \sigma)$.

An arrow of the form

$$(T, \zeta, v, \sigma) \rightarrow_{s_i \text{ or } \Psi} (T', \zeta', v', \sigma')$$

will be called a step. The other arrows will be called jumps, and in particular an arrow

$$(T, \zeta, v, \sigma) \rightarrow_{s_i} \emptyset$$

will be called a fall; the other jumps will be called correct jumps.

As usual a path is a finite sequence of consecutive arrows in $G_{\lambda}$ starting from the root and is denoted by the vector of the labels of its arrows. Two paths $P_1 = (a_1, \ldots, a_k)$ and $P_2 = (b_1, \ldots, b_k)$ are said to be equivalent (denoted by $P_1 \equiv P_2$) if they lead to the same vertex.

We remark that when $v[i] = v[i+1]$, the part 1 of definition 4.2 is equivalent to the following statement: $T'$ is obtained from $T$ by interchanging $\sigma_v[i]$ and $\sigma_v[i+1] = \sigma_v[i] + 1$ where $\sigma_v[i]$ is to the south-east of $\sigma_v[i+1] + 1$, that is, $CT_T[\sigma_v[i]] - CT_T[\sigma_v[i] + 1] \geq 2$.

**Example 4.3** The following arrow is a correct jump

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
31_{543} \\
[0,0,2,1,1],[45123]
\end{array}
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
21_{543} \\
[0,0,2,1,1],[45123]
\end{array}
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
S_4
\end{array}
\end{array}
\end{array}\]
whilst

\[
\begin{array}{c}
\begin{array}{c}
\text{31}\text{, }[s, q^2, q^{-2}, q^{s^{-1}}] \\
[0.0, 2.1, 1, 45123]
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\text{31}\text{, }[s, q^2, 1, q^{-1}, q^2] \\
[0.2, 0.1, 1, 45123]
\end{array}
\end{array}
\]

is a step.

The arrows

\[
\begin{array}{c}
\begin{array}{c}
\text{31}\text{, }[s, q^2, q^{-2}, q^{s^{-1}}] \\
[0.0, 2.1, 1, 45123]
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\text{31}\text{, }[s, q, 1, q^{-1}, q^2] \\
[0.0, 2.1, 1, 45123]
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{c}
\text{31}\text{, }[s, q^2, q^{-2}, q^{s^{-1}}] \\
[0.0, 2.1, 1, 45123]
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\text{31}\text{, }[s, q^2, 1, q^{-1}, q^2] \\
[0.2, 0.1, 1, 45123]
\end{array}
\end{array}
\]

are not allowed.

The graph \(G^{q,s}_\lambda\) is very similar to the Yang Baxter graph \(G_\lambda\) described in [9]: only the spectral vectors change. Indeed, these are the same graphs but with different labels: the spectral vector of \(G^{q,s}_\lambda\) is obtained from \(G_\lambda\) by sending \(a\alpha + b\) to \(q^a s^b\). Hence, many properties are still applicable. For instance,

**Proposition 4.4** All the paths joining two given vertices in \(G_\lambda\) have the same length.

For a given 4-tuple \((T, \zeta, v, \sigma)\) the values of \(\zeta\) and \(\sigma\) are determined by those of \(T\) and \(v\), as shown by the following proposition.

**Proposition 4.5** If \((T, \zeta, v, \sigma)\) is a vertex in \(G_\lambda\), then \(\sigma = r_v\) and \(\zeta[i] = q^{v[i]} s^{CT_{\tau}[v[i]]}\). We will set \(\zeta_{v,T} := \zeta\).

**Example 4.6** Consider the RST \(\tau = 3\ 7\ 4\ 1\ 8\ 6\ 5\ 2\) and the vector \(v = [6, 2, 4, 2, 3, 1, 4]\). One has \(r_v = [1, 5, 2, 6, 7, 4, 8, 3]\) and \(CT_{\tau} = [1, 3, -2, 0, 2, 1, -1, 0]\) and then

\[
\zeta_{v,\tau} = [q^6 s, q^2 s^2, q^4 s^3, q^2 s^4, q^2 s^{-1}, q^3 s, q, q^4 s^{-2}].
\]

Hence, the 4-tuple

\[
\begin{pmatrix}
3 & 7 & 4 & 1 \\
8 & 6 & 5 & 2
\end{pmatrix}
\] \(;\) \([q^6 s, q^2 s^2, q^4 s^3, q^2 s^4, q^2 s^{-1}, q^3 s, q, q^4 s^{-2}], [6, 2, 4, 2, 3, 1, 4], [1, 5, 2, 6, 7, 4, 8, 3]\)

labels a vertex of \(G^{q,s}_{431}\).
As a consequence,

**Corollary 4.7** Let \((T, v)\) be a pair consisting of \(T \in \text{Tab}(\lambda)\) (\(\lambda\) is a partition of \(N\)) and a multi-index \(v \in \mathbb{N}^N\). Then there exists a unique vertex in \(G_{\lambda}^{q,s}\) labeled by a 4-tuple of the form \((T, \zeta, v, \sigma)\). We will denote \(\Psi_{T,\zeta,v,\sigma} := (T, v)\).

Conversely, all the information can be retrieved from the spectral vector \(\zeta\) - the exponents of \(q\) give \(v\), the rank function of \(v\) gives \(\sigma\), and the exponent of \(s\) in the spectral vector gives the content vector which does uniquely determine the RST \(T\).

For simplicity, when needed, we will label the vertices by pairs \((T, v)\) or by the associated spectral vector \(\zeta_{v,T}\).

**Example 4.8** In figure 1, the first several vertices are labeled by pairs \((T, v)\) of the graph \(G_{21}^{q,s}\) while in figure 2, the vertices are labeled by spectral vectors.

**Definition 4.9** We define the subgraph \(G_{\lambda}^{q,s}\) as the graph obtained from \(G_{\lambda}^{q,s}\) by erasing all the vertices labeled by RST other than \(T\) and the associated arrows. Such a graph is connected.

The graph \(G_{\lambda}^{q,s}\) is the union of the graphs \(G_{T}^{q,s}\) connected by jumps. Furthermore, if \(G_{T}^{q,s}\) and \(G_{T'}^{q,s}\) are connected by a succession of jumps then there is no step from \(G_{T}^{q,s}\) to \(G_{T'}^{q,s}\). Since the graphs \(G_{T}^{q,s}\) are connected graphs, we have:

**Proposition 4.10** Each vertex \((T, v)\) is obtained from \((T, 0^N)\) by a sequence of steps.

**Example 4.11** In Fig. 1 and 2, the graph \(G_{21}^{q,s}\) is constituted with the two graphs \(G_{132}^{q,s}\) and \(G_{231}^{q,s}\) connected by jumps (in blue).

### 4.2 Macdonald polynomials from scratch

Following [1], we define the operator

\[
\Phi = T_1^{-1} \ldots T_{N-1}^{-1} x_N,
\]
which satisfies

\[ \Phi \xi_j = \xi_{j+1}, 1 \leq j < N, \]
\[ \Phi \xi_N = q \xi_1 \Phi. \]

The operator \( \Phi \) is injective (kernel is \( \{0\} \)).

Let \( \lambda \) be a partition and \( G^{q,t}_\lambda \) be the associated graph. We construct the set of polynomials \( (P_\psi) \) path in \( G_\lambda \) using the following recurrence rules:

1. \( P[] := (T_\lambda) \)
2. If \( \psi = [a_1, \ldots, a_{k-1}, s_i] \) then

\[ P_\psi \psi := P_{[a_1, \ldots, a_{k-1}]} \left( T_i + \frac{1-s}{1 - \frac{\zeta(i+1)}{\zeta(i)}} \right), \]

where the vector \( \zeta \) is defined by

\[ (T_\lambda, CT_{T_\lambda}^s, 0^N, [1, 2, \ldots, N])a_1 \ldots a_{k-1} = (T, \zeta, v, \sigma), \]

3. If \( \psi = [a_1, \ldots, a_{k-1}, \Psi] \) then

\[ P_\psi = P_{[a_1, \ldots, a_{k-1}]} \Phi. \]

One has the following theorem.

**Theorem 4.12** Let \( \psi = [a_0, \ldots, a_k] \) be a path in \( G^{q,t}_\lambda \) from the root to \( (T, \zeta, v, \sigma) \) with no fall. The polynomial \( P_\psi \) is a simultaneous eigenfunction of the operators \( \xi_i \). Furthermore, the eigenvalues of \( \xi_i \) associated to \( P_\psi \) are equal to \( \zeta[i] \).

Consequently \( P_\psi \) does not depend on the path, but only on the end point \( (T, \zeta, v, \sigma) \), and will be denoted by \( P_{v,T} \) or alternatively by \( P_\zeta \). The family \( (P_{v,T})_{v,T} \) forms a basis of \( M_\lambda \) of simultaneous eigenfunctions of the Cherednik operators.

**Proof.** We will prove the result by induction on the length \( k \). If \( k = 0 \) then the result follows from proposition [3.10].

Suppose now that \( k > 0 \) and let

\[ (T', \zeta', v', r_{v'}) = (T_\lambda, CT_{T_\lambda}^{q,s}, 0^N, [1, \ldots, N])a_1 \ldots a_{k-1}. \]
By induction, $P_{[a_1,\ldots,a_{k-1}]}$ is a simultaneous eigenfunction of the operators $\xi_i$ such that the associated vector of eigenvalues is given by

$$P_{[a_1,\ldots,a_{k-1}]}\xi_i = \zeta'[i]P_{[a_1,\ldots,a_{k-1}]}.$$ 

The argument depends on the value of the last operator $a_k$.

1. If $a_k = \Psi$ is an affine arrow, then $\mathbf{T} = \mathbf{T}'$, $\zeta = [\zeta'[2], \ldots, \zeta'[N], q \zeta'[1]]$, $v = v'\Psi$, $r_v = r_{v'}[2, \ldots, N, 1]$ and $P_{\Psi} = J_{[a_1,\ldots,a_{k-1}]}\Phi$.

If $i \neq N$

$$P_{\Psi}\xi_i = P_{[a_1,\ldots,a_{k-1}]}\Phi \xi_i$$
$$= P_{[a_1,\ldots,a_{k-1}]}\xi_{i+1} \Phi$$
$$= \zeta'[i+1]P_{\Psi}$$
$$= \zeta[i]P_{\Psi}.$$ 

If $i = N$ then,

$$P_{\Psi}\xi_N = P_{[a_1,\ldots,a_{k-1}]}\Phi \xi_N$$
$$= P_{[a_1,\ldots,a_{k-1}]}q \xi_1 \Phi$$
$$= (\zeta'[1]q)P_{\Psi}$$
$$= \zeta[N]P_{\Psi}.$$ 

2. Suppose now that $a_k = s_i$ is a non-affine arrow, then $\zeta = \zeta's_i$, $v = v's_i$ and

$$P_{\Psi} = P_{[a_1,\ldots,a_{k-1}]} \left( T_i + \frac{1-s}{1-\zeta'[i+1]} \right).$$

If $j \neq i$, $i+1$ then

$$P_{\Psi}\xi_j = P_{[a_1,\ldots,a_{k-1}]} \left( T_i + \frac{1-s}{1-\zeta'[i+1]} \right) \xi_j$$
$$= P_{[a_1,\ldots,a_{k-1}]} \xi_j \left( T_i + \frac{1-s}{1-\zeta'[i+1]} \right)$$
$$= \zeta'[j]P_{\Psi}$$
$$= \zeta[j]P_{\Psi}.$$ 

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If \( j = i \) then

\[
P_P \xi_i = P_{[a_1, \ldots, a_k-1]} \left( T_i + \frac{1-s}{1-C[i+1]/C[i]} \right) \xi_i
\]

\[
= P_{[a_1, \ldots, a_k-1]} \left( \xi_{i+1} T_i + (1-s) \left( -1 + \frac{1}{1-C[i+1]/C[i]} \right) \xi_i \right)
\]

\[
= P_{[a_1, \ldots, a_k-1]} \left( \zeta'[i+1] T_i + (1-s) \left( -1 + \frac{1}{1-C[i+1]/C[i]} \right) \zeta'[i] \right)
\]

\[
= \zeta'[i+1] P_{[a_1, \ldots, a_k-1]} \left( T_i + \frac{1-s}{1-C'[i+1]/C'[i]} \right) \xi_{i+1}
\]

\[
= \zeta[i+1] P_P.
\]

If \( j = i + 1 \) then

\[
P_P \xi_{i+1} = P_{[a_1, \ldots, a_k-1]} \left( T_i + \frac{1-s}{1-C'[i+1]/C'[i]} \right) \xi_{i+1}
\]

\[
= P_{[a_1, \ldots, a_k-1]} \left( s \xi_{i+1} T_i^{-1} + \xi_{i+1} \frac{1-s}{1-C[i+1]/C[i]} \right)
\]

\[
= P_{[a_1, \ldots, a_k-1]} \left( \zeta'[i] T_i + \zeta'[i] (1-s) + \zeta'[i+1] \frac{1-s}{1-C'[i+1]/C'[i]} \right)
\]

\[
= \zeta'[i] P_{[a_1, \ldots, a_k-1]} \left( T_i + \frac{1-s}{1-C'[i+1]/C'[i]} \right)
\]

\[
= \zeta[i+1] P_P.
\]

\[\blacksquare\]

**Example 4.13** Figure 3 illustrates how to obtain the first values of the polynomial \( P_\zeta \) for isotype \((2, 1)\).

Besides \( \Phi = T_1^{-1} \ldots T_{N-1}^{-1} x_N \) there is another raising operator \( \Phi' := w_x N \).

**Proposition 4.14** \( \Phi' = s^{N-1} \xi_1 \Phi \), and if \( v \in \mathbb{N}_0^N, T \in \text{Tab}_\lambda \) then \( P_{v,T} \Phi' = s^{N-1+CT_T[v,1]} q^{v[1]} P_{v,T} \Phi \).

**Proof.** From \( \xi_1 = s^{1-N} w T_{N-1} \ldots T_1 \) it follows that

\[
\xi_1 T_1^{-1} T_2^{-1} \ldots T_{N-1}^{-1} = s^{1-N} w,
\]

\[
\xi_1 \Phi = s^{1-N} \Phi'.
\]

Also \( P_{v,T} \xi_1 = q^{v[1]} s^{N-1+CT_T[v,1]} P_{v,T} \). \[\blacksquare\]

Note that it is easier to compute \( P \Phi' \) for a polynomial \( P \).
4.3 Leading terms

We will denote by $x^v, T := x^v T R_v$. By abuse of language $x^v, T$ will be referred to as a monomial. Note that the space $M_\lambda$ is spanned by the set of polynomials

$$M_\lambda := \{x^v, T : v \in \mathbb{N}^N, T \in \text{Tab}_\lambda\},$$

which can be naturally endowed with the order $\prec$ defined by

$$x^v, T \prec x^{v'}, T' \text{ iff } v \prec v'.$$

**Theorem 4.15** The leading term (up to constant multiple) of $P_v, T$ is $x^v, T$.

**Proof.** Theorem 3.13 shows that the leading term of $P_v, T$ is $x^v T H_v$ for some $H_v \in \mathcal{H}_N(q, s)$ (because the eigenvalues determine $q^v[i]$). Use induction on $\#\text{inv}(v) = \#\{i, j : 1 \leq i < j \leq N, v[i] < v[j]\}$. The claim is true for partitions $v$, that is, $\#\text{inv}(v) = 0$. Suppose the claim is true for all $u$ with $\#\text{inv}(u) \leq k$ and $\#\text{inv}(v) = k + 1$. There is some $i$ for which $v[i] < v[i + 1]$. By Theorem 4.12 $p := P_{v, T}(T_i + (1 - s)\zeta[i] - \zeta[i + 1])$ is a $\xi$-eigenvector with eigenvalues $[\zeta[1], \ldots, \zeta[i + 1], \zeta[i], \ldots]$, where $\zeta[j] = \zeta_{v, T}[j]$. The list of eigenvalues implies that the leading term of $p$ is $x^{v, s_i, T'}$ for some $T' \in V_{\lambda}$. In fact, $p_{\xi_j} = q^{v, s_i}[j] s C T v r_{v s_i} p$ for all $j$ and so the inductive hypothesis ($\#\text{inv}(v s_i) = \#\text{inv}(v) - 1$) implies that $p$ is a scalar multiple of $P_{v s_i, T}$ and has leading term $x^{v, s_i, T} T r_{v s_i}$. The only appearance of $x^{v, s_i}$ in $p$ comes from $x^v T H_v T_i$ (by dominance, $x^{v, s_i}$ does not appear in $P_{v, T}$).

But when $v[i] < v[i + 1]$ and $T \in V_{\lambda}$ then

$$x^v T T_i = x^v T T_i + x^{v, s_i} (T T_i) = -(1 - s)x^v T + x^{v, s_i} (T T_i) + \sum_{x^{v'} \subseteq v} x^{v'} P_{v'}.$$

Hence by (30),

$$x^v T H_v T_i = -(1 - s)x^v T H_v + x^{v, s_i} (T H_v T_i) + \sum_{x^{v'} \subseteq v} x^{v'} P_{v'}.$$
Thus $T_H v T_i = T R_{r_u s_i}$ and

$$TH v = T R_{r_u s_i} T_i^{-1} = T R v,$$

by Lemma 2.14. This completes the inductive proof. ■

As a consequence.

**Corollary 4.16** Let $\mathcal{P} = [a_1, \ldots, a_k]$ such that $a_k$ is a fall, then $P_\mathcal{P} = 0$.

**Proof.** Without loss of generality, we can suppose that $[a_1, \ldots, a_{k-1}]$ is a path without fall. From theorem 4.12, there exists a pair $(v, T)$ such that $P_{v, T} = P_{[a_1, \ldots, a_{k-1}]}$. From theorem 4.15, one has

$$P_{v, T} = x_{v, T} + \sum_{P_{v', v} \in V_\lambda} x_{v', v} P_{v'}. $$

Since, $a_k$ is a fall one has:

$$P_\mathcal{P} = x_{\mathcal{P}} + \sum_{P_{v', v} \in V_\lambda} x_{v', v} P_{v'}. $$

with $\mathcal{P} \in V_\lambda$. Since $P_\mathcal{P}$ is a simultaneous eigenfunction of the Cherednik operators, it is proportional to $P_{v, T}$. Noting that the associated eigenvectors are uniquely determined, one obtains $P_\mathcal{P} = 0$. ■

### 4.4 Action of $T_i$

We have more formulae than those exhibited in the proof of theorem 4.12. For instance:

**Proposition 4.17** Suppose $v \in \mathbb{N}^N_0$, $T \in \text{Tab}_\lambda$, $v[i] = v[i + 1]$ for some $i$, and $k := r_v[i]$, $m := CT_T[k + 1] - CT_T[k]$, then

1) if $CT_T[k + 1] = CT_T[k] - 1$ then $P_{v, T} T_i = s P_{v, T}$;

2) if $CT_T[k + 1] = CT_T[k] + 1$ then $P_{v, T} T_i = -P_{v, T}$;

3) if $CT_T[k + 1] \leq CT_T[k] - 2$ then $P_{v, T} T_i = P_{b, T}(k, k+1) - \frac{1-s}{1-s^m} P_{v, T}$;

4) if $CT_T[k + 1] \geq CT_T[k] + 2$ then $P_{v, T} T_i = \frac{s(1-s^{m+1})(1-s^{m-1})}{(1-s^m)^2} P_{v, T}(k, k+1) - \frac{1-s}{1-s^m} P_{v, T}$.

We introduce a partial order which will be used to compare eigenvalues, that is, the spectral vectors.
Definition 4.18 For integers \( n_1, m_1, n_2, m_2 \) define

\[
q^{n_1} s^{m_1} > q^{n_2} s^{m_2} \iff n_1 > n_2 \text{ or } n_1 = n_2, m_1 \leq m_2 - 2;
\]

\[
q^{n_1} s^{m_1} \sim q^{n_2} s^{m_2} \iff n_1 = n_2, |m_1 - m_2| = 1.
\]

We will write also \( q^{n_1} s^{m_1} > q^{n_2} s^{m_2} \) if \( n_1 > n_2 \).

This formulation is used to unify the various recursion relations. Note that if \( \zeta = \zeta_{v,T} \) is a spectral vector, we have necessarily \( \zeta[i] \neq \zeta[i+1] \) for each \( i \). Indeed, either \( v[i] \prec v[i+1] \) or \( v[i] \succ v[i+1] \) and the contents are different (since a RST can not have adjacent entries on a diagonal).

Here is a unified transformation formula. Theorem 4.12 is implicitly used.

Proposition 4.19 Suppose \( v \in \mathbb{N}_0^N, T \in \text{Tab}_\lambda \) and \( 1 \leq i < N \).

\[
P_\zeta \left( T_i + \frac{(1-s)\zeta_i}{\zeta_i - \zeta_{i+1}} \right) = \begin{cases} 
P_{\zeta s_i, \zeta_{i+1} > \zeta_i}, & \frac{(\zeta_i - \zeta_{i+1})(s\zeta_i - s\zeta_{i+1})}{(\zeta_i - \zeta_{i+1})} \\
0, & \zeta_i \prec \zeta_{i+1}.
\end{cases}
\]  

(31)

and

\[
P_\zeta \Phi = P_{\zeta \Psi}. \tag{32}
\]

This proposition shows that we can easily use the spectral vector \( \zeta \) instead of the pair \( (v, T) \) for labeling the Macdonald polynomials (assuming that \( \zeta = \zeta_{v,T} \) for a given vector \( v \) and a given tableau \( T \)).

Indeed, we showed that if \( \zeta \) is a spectral vector and \( \zeta[i] \succ \zeta[i+1] \) then \( \zeta_{s_i} \) is also a spectral vector. Such an action is called a permissible transposition. If \( \zeta[i] \sim \zeta[i+1] \) then \( \zeta_{s_i} \) is not a spectral vector. We use some of the ideas developed by [15] see Theorem 5.8, p.22. Let \( \mu \) be a decreasing partition. Suppose \( \mu[i] = \mu[j], i < j \) and \( CT_T[i] = CT_T[j] = a \), then \( \{a-1, a+1\} \subset \{CT_T[i+1], \ldots, CT_T[j-1]\} \). That is, there exists \( k \) with \( i < k < j \) such that \( CT_T[k] = a+1 \), and \( \mu[k] = \mu[i] \) (because of the partition property). Thus the spectral vector \( \zeta \) contains a substring (preserving the order from \( \zeta \)) \( (q^{\mu[i]} s^a, q^{\mu[i]} s^{a+1}, q^{\mu[i]} s^a) \), it is impossible to move \( q^{\mu[i]} s^a \) past \( q^{\mu[i]} s^{a+1} \) with a permissible transposition, and adjacent entries of a spectral vector can not be equal.

One description of the permissible permutations is the set of permutations of \( \zeta \) in which each pair \( (\zeta_i, \zeta_j) \) with \( \zeta_i \sim \zeta_j \) maintains its order, that is, if \( i < j \)
and $\zeta_i \sim \zeta_j$ and $(\zeta_i, \sigma)_{i=1}^N$ is a spectral vector then $i.\sigma < j.\sigma$. The structure of permissible permutations is analyzed in Section 5.1.

For example, take $\lambda = (3, 2), \mu = (1, 1, 1, 1, 0)$

\[
\begin{align*}
\mathcal{T} &= 2 \ 1 \\
\zeta &= (q, qs^{-1}, qs^2, qs, 1), \\
\zeta_1 \sim \zeta_2 \succ \zeta_3 \sim \zeta_4 \succ \zeta_5.
\end{align*}
\]

But also $\zeta_1 \sim \zeta_4$ so the order of the pairs $(\zeta_1, \zeta_2), (\zeta_3, \zeta_4), (\zeta_1, \zeta_4)$ must be preserved in the permissible permutations (of which there are 25). Observe that $\zeta$ is a maximal element, in the sense that only $\succ$ and $\sim$ occur in the comparisons of adjacent elements. Clearly there must be a minimal element (if $\zeta_i \succ \zeta_{i+1}$ then apply $s_i$ to $\zeta$). In the example this is

\[
\begin{align*}
\zeta &= (1, qs^2, q, qs, qs^{-1}) \\
&= \zeta_{(0,1,1,1,1), \mathcal{T}_1}, \\
\mathcal{T}_1 &= \begin{pmatrix} 4 & 2 \\ 5 & 3 & 1 \end{pmatrix}.
\end{align*}
\]

To finish this discussion we show that the maximal and minimal elements are unique. By the definition of $\succ$ we need only consider the possible arrangements of $\zeta_i, \zeta_{i+1}, \ldots, \zeta_j$ where $\mu[i-1] > \mu[i] = \ldots = \mu[j] > \mu[j+1]$ (or $i = 1$, or $j = N$ and $\mu[N] > 0$). Let

\[
\text{inv} (\mu, \mathcal{T}) = \{(i,j) : \mu[i] = \mu[j], i < j, \zeta_{\mu, \mathcal{T}}[i] \prec \zeta_{\mu, \mathcal{T}}[j]\};
\]

we showed there is a unique RST $\mathcal{T}_0$ where $(\zeta_{\mu, \mathcal{T}_0[i]})_{i=1}^N$ is a permissible permutation of $\zeta$ and $\#	ext{inv} (\mu, \mathcal{T}_0) = 0$. By a similar argument there is a unique RST $\mathcal{T}_1$ which maximizes $\text{inv} (\mu, \mathcal{T})$. The minimum spectral vector is $(\zeta_{\mu^R, \mathcal{T}_1[i]})_{i=1}^N$, where $\mu^R[i] = \mu[N + 1 - i], 1 \leq i \leq N$.

According to the previous remark, we will use the notations below:

**Definition 4.20** If $\zeta = \zeta_{\mu, \mathcal{T}}$

\[
\text{inv}_\sim(\zeta) := \{(i,j) : 1 \leq i < j \leq N, \zeta[i] \sim \zeta[j]\},
\]

\[
\text{inv}_\prec(\zeta) := \{(i,j) : 1 \leq i < j \leq N, \zeta[i] \prec \zeta[j]\}.
\]
for \( \triangleleft \in \{<, >, <, >\} \). If \( \zeta = \zeta_{v, T} \) then we will denote \( \zeta^+ = \zeta_{v+, T} \). Note that
\[
\zeta^+[1] \geq \zeta^+[2] \cdots \geq \zeta^+[N]
\]
and set
\[
\text{inv}(\zeta) := \text{inv}_{<}(\zeta) = \text{inv}(v).
\]

The action of the symmetric group \( S_N \) on the spectral vector is defined by
\[
\zeta s_i = \begin{cases} 
[\zeta[1], \ldots, \zeta[i - 1], \zeta[i + 1], \zeta[i], \zeta[i + 1], \ldots, \zeta[N]] & \text{if } \zeta[i] < \zeta[i + 1] \\
\zeta & \text{or } \zeta[i] \succ \zeta[i + 1] \\
\zeta & \text{otherwise}
\end{cases}
\]
(33)

Say \( \zeta' \prec \zeta \) if and only if there exists a sequence of elementary transpositions \((s_{i_1}, \ldots, s_{i_k})\) such that
\[
\zeta_0 = \zeta, \zeta_1 = \zeta_0 s_{i_1}, \ldots, \zeta_k = \zeta s_{i_1} \ldots s_{i_k} = \zeta'
\]
and for each \( j < k \), \( \zeta_[i_{j+1}] \prec \zeta[j_{i_{j+1}} + 1] \).

5 Stable subspaces

5.1 Connected components

We denote by \( H_{\lambda}^{q,s} \) the graph obtained from \( G_{\lambda}^{q,s} \) by removing the affine edges, all the falls and the vertex \( \emptyset \).

Recall that \( v^+ \) is the unique decreasing partition obtained by permuting the entries of \( v \).

**Definition 5.1** Let \( v \in \mathbb{N}^N \) and \( T \in \text{Tab}_\lambda (\lambda \text{ partition}) \). We define the filling \( T(T, v) \) obtained by replacing \( i \) by \( v^+[i] \) in \( T \) for each \( i \).

As in [6], we have

**Proposition 5.2** Two 4-tuples \((T, \zeta, v, \sigma)\) and \((T', \zeta', v', \sigma')\) are in the same connected component of \( H_{\lambda}^{q,t} \) if and only if \( T(T, v) = T(T', v') \).

This shows that the connected components of \( H_{\lambda}^{q,s} \) are indexed by the \( T(T, \mu) \) where \( \mu \) is a partition.
Definition 5.3 We will denote by $H_{q,s}^T$ the connected component associated to $T$ in $H_{q,s}^\lambda$. The component $H_{q,s}^T$ will be said to be 1-compatible if $T$ is a column-strict tableau. The component $H_{q,s}^T$ will be said to be $(-1)$-compatible if $T$ is a row-strict tableau.

Note that each connected component has a unique lower element (i.e. without antecedent) called its root and denoted by $\text{root}(T) := (T_{\text{root}(T)}, \zeta_{\text{root}(T)}, v_{\text{root}(T)}, r_{\text{root}(T)})$ and a unique maximal element called its sink and denoted by $\text{sink}(T) := (T_{\text{sink}(T)}, \zeta_{\text{sink}(T)}, v_{\text{sink}(T)}, r_{\text{sink}(T)})$.

With the notations of the previous section, we have $v_{\text{sink}}(T) = v^+$ and $T_{\text{sink}}(T) = T_0$ for any pair $(v, T) \in T$. In the same way, $v_{\text{root}}(T) = v^R$ and $T_{\text{root}}(T) = T_1$.

Example 5.4 Let $\mu = [2, 1, 1, 0, 0]$ and $\lambda = [3, 2]$. There are four connected components with vertices labeled by permutations of $\mu$ in $H_{q,s}^{\lambda}$. The possible values of $T(T, \mu)$ are

12 02 01 and 11
001’ 011’ 012 002’

The 1-compatible components are $H_{12}^{q,s}$ and $H_{11}^{q,s}$ while there is only one $(-1)$-compatible component $H_{01}^{q,s}$. The component $H_{02}^{q,s}$ is neither 1-compatible nor $(-1)$-compatible.

The component $H_{12}^{q,s}$ contains vertices of $G_{31}^{q,s}$ and $G_{21}^{q,s}$ connected by jumps. In Fig. 4 we have drawn the components $H_{11}^{q,s}$ and $H_{01}^{q,s}$.

Example 5.5 Consider the tableau $T = 01$; the graph $H_{q,s}^T$ is:

![Graph Diagram]

\[37\]
The sink is denoted by a red disk and the root by a green disk.

By abuse of language, we will write $\zeta \in T$ to mean that $\zeta$ appears in a vertex of the connected component $H^s_{T_q}.$

**Definition 5.6** In the same way, we define $\text{std}_0 T$ of $T$ is the reverse standard tableau with shape $\lambda$ obtained by the following process:

1. Denote by $|T|_i$ the number of occurrences of $i$ in $T$
2. Read the tableau $T$ from the left to the right and the bottom to the top and replace successively each occurrence of $i$ by the numbers $N - |T|_0 - \cdots - |T|_{i-1}, N - |T|_0 - \cdots - |T|_{i-1} - 1, \ldots N - |T|_0 - \cdots - |T|_i$.

Let $T$ be a filling of shape $\lambda$, $\text{std}_1 T$ of $T$ is the reverse standard tableau with shape $\lambda$ obtained by the following process:

1. Denote by $|T|_i$ the number of occurrences of $i$ in $T$
2. Read the tableau $T$ from the bottom to the top and the left to the right and replace successively each occurrence of $i$ by the numbers $N - |T|_0 - \cdots - |T|_{i-1}, N - |T|_0 - \cdots - |T|_{i-1} - 1, \ldots N - |T|_0 - \cdots - |T|_i$.

**Example 5.7** To construct $\text{std}_0 \left( \begin{array}{ccc} 0 & 1 & 2 \\ 0 & 0 & 0 \\ . & . & 1 \\ . & . & 2 \end{array} \right)$ we first write:

\[
\begin{array}{ccc|ccc}
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & . & . \\
. & . & 1 & . \\
. & . & 2 & . \\
\end{array}
\]

and we renumber in increasing order from the bottom to the top and the right to the left:

\[
\begin{array}{ccc|ccc}
0 & 0 & 0 & 1 & 2 \\
5 & 4 & 3 & . & . \\
. & . & 2 & . \\
. & . & 1 & . \\
\end{array}
\]
We obtain $\text{std}_0 \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 5 & 3 & 1 \end{pmatrix}$.

Pictorially, we construct $\text{std}_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ writing:

$\begin{array}{ccc}
0 & 0 & 2 \\
0 & 0 & . \\
. & . & . \\
. & 2 & .
\end{array}$

and renumbering in increasing order from the bottom to the top and the right to the left:

$\begin{array}{ccc}
0 & 0 & 2 \\
5 & 4 & . \\
. & . & 3 \\
. & . & 2 \\
. & 1 & .
\end{array}$

This gives $\text{std}_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 5 & 4 & 1 \end{pmatrix}$.

Alternatively, one has

\[
\text{std}_0(T)[i, j] := \# \{(k, l) : T[k, l] > T[i, j]\} + \# \{(k, l) : k > i, T[k, l] = T[i, j]\} + \# \{(i, l) : l \geq j, T[i, l] = T[i, j]\}
\]

and

\[
\text{std}_1(T)[i, j] := \# \{(k, l) : T[k, l] > T[i, j]\} + \# \{(k, l) : l > j, T[k, l] = T[i, j]\} + \# \{(k, j) : k \geq i, T[k, j] = T[i, j]\}.
\]

We can characterize the root and the sink of a connected component:

**Lemma 5.8** One has:

1. $T_{\text{root}(T)} = \text{std}_0 T$ and $T_{\text{sink}(T)} = \text{std}_1 T$.
2. $v_{\text{root}(T)} = v^R$ and $v_{\text{sink}(T)} = v^+$

**Proof.** First observe that $T(\text{std}_0(T), v) = T(\text{std}_1(T), v) = T$ by construction. So, we have $(v^R, \text{std}_0(T)), (v^+, \text{std}_1(T)) \in H_T^{sa}$. Since, $v^R$ is an increasing partition, each arrow
is a jump (i.e. \( u = v^R \)). Let \([i, j]\) be a cell of \( \text{std}_0(T) \) and \( k = \text{std}_0(T)[i, j] \). Let \([i', j']\) be the cell such that \( k + 1 = \text{std}_0(T)[i', j'] \). From the definition of \( \text{std}_0(T) \), we have either \( T[i, j] \neq T[i', j'] \) or \( j = j' \) or \( i < i' \) and \( j > j' \) (that is \( \text{CT}_{\text{std}_0(T)}[k] < \text{CT}_{\text{std}_0(T)}[k + 1] - 1 \)). Hence, such a row does not exist and \((\text{std}_0(T), v^R)\) has no antecedent in \( H^{q,s}_T \). This is equivalent to \( \text{std}_0(T) = T_{\text{root}(T)} \).

In a equivalent way, we find that there is no arrow in \( H^{q,s}_T \) of the form

\[
(\text{std}_1(T), v^+) \xrightarrow{s_i} (T, u)
\]

and then \( \text{std}_1(T) = T_{\text{sink}(T)} \).

**Example 5.9** We write the example 5.5 in terms of tableaux:

\[
\begin{array}{c|c|c|c|c}
21 & 43 & 31 & 42 & 21 \\
0001 & 0010 & 0100 & 0100 & 0100 \\
\end{array}
\]

We observe that \( \text{std}_0 \left( \begin{array}{c}
01 \\
00
\end{array} \right) = \left( \begin{array}{c}
31 \\
42
\end{array} \right) = T_{\text{root}(\begin{array}{c}
01 \\
00
\end{array})} \) and \( \text{std}_1 \left( \begin{array}{c}
01 \\
00
\end{array} \right) = \left( \begin{array}{c}
21 \\
43
\end{array} \right) = T_{\text{sink}(\begin{array}{c}
01 \\
00
\end{array})} \).

**Remark 5.10** As a consequence: Let \( m_i \) be the number of occurrences of \( i \) in the entries of \( T \),

\[
r_{\text{root}(T)} = [\ldots, m_0 + \cdots + m_i + 1, \ldots, m_0 + \cdots + m_i + 1 + 1, \ldots, m_0 + 1, \ldots, m_0 + m_i, 1, \ldots, m_0]
\]

and \( r_{\text{sink}(T)} = [1, \ldots, N] \).

The notion of \((\pm 1)\)-compatibility is easily detectable on the root and the sink:

**Lemma 5.11** If \( H^{q,s}_T \) is 1-compatible then for each \( i, i + 1 \) are not in the same column of \( T_{\text{root}(T)} \).

If \( H^{q,s}_T \) is \((-1)\)-compatible then for each \( i, i + 1 \) are not in the same row of \( T_{\text{sink}(T)} \).
Proof. From lemma 5.8, we have $T_{\text{root}(T)} = \text{std}_0(T)$ and $T_{\text{sink}(T)} = \text{std}_1(T)$. But if $k$ and $k+1$ are in the same column of $\text{std}_0(T)$, supposing $\text{std}_0(T)[i,j] = k$, then $\text{std}_0(T)[i,j+1] = k+1$ and the only possibility is that $T[i,j] = T[i,j+1]$ which contradicts the fact that $T$ is a column-strict tableau. Similarly, if $k$ and $k+1$ are in the same row of $\text{std}_1(T)$, then $T[i,j] = T[i+1,j]$ for some $(i,j)$ which contradicts the fact that $T$ is a row-strict tableau. Now, we have all the materials for an interpretation of the $(\pm 1)$-compatibility in terms of spectral vectors:

**Proposition 5.12** If $H_{T}^{q,s}$ is $1$-compatible then for each $i$, $\zeta_{\text{root}(T)}[i] \not\sim \zeta_{\text{root}(T)}[i+1]$ implies $\zeta_{\text{root}(T)}[i] = s\zeta_{\text{root}(T)}[i+1]$.

If $H_{T}^{q,s}$ is $(-1)$-compatible then for each $i$, $\zeta_{\text{sink}(T)}[i] \not\sim \zeta_{\text{sink}(T)}[i+1]$ implies $\zeta_{\text{sink}(T)}[i] = s^{-1}\zeta_{\text{sink}(T)}[i+1]$.

**Proof.** This is just the translation of lemma 5.11 in terms of spectral vectors.

### 5.2 Invariant subspaces

The Yang-Baxter graph and the previous section imply that we can characterize the irreducible subspaces $U$ of polynomials invariant under $H_N(s)$ and $\{\xi_i : 1 \leq i \leq N\}$, that is, $U T_i, U \xi_i \subset U$.

**Definition 5.13** Let $T$ be a tableau with increasing row and column entries. We will denote by $\mathcal{M}_T$ the space generated by the polynomials $P_{\zeta}$ with $\zeta \in T$.

**Example 5.14** For instance, $\mathcal{M}_{\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}}$ is spanned by

\[ \{P_{[s,s^{-1},1,q]}, P_{[s,s^{-1},q,1]}, P_{[s,q,s^{-1},1]}, P_{[q,s,s^{-1},1]}, P_{[s^{-1},s,1,q]}, P_{[s^{-1},s,q,1]}, P_{[s^{-1},q,s,1]}, P_{[q,s^{-1},s,1]}\}. \]

The spaces $\mathcal{M}_T$ are the irreducible invariant subspaces.

**Proposition 5.15** We have $\mathcal{M}_T T_i, \mathcal{M}_T \xi_i \subset \mathcal{M}_T$. Furthermore, if $U$ is a proper subspace of $\mathcal{M}_T$ then $UT_i \not\subset U$ or $U \xi_i \not\subset U$.

**Proof.** Let $U$ be a subspace of $\mathcal{M}_T T_i$ such that $UT_i, U \xi_i \subset U$. The operators $\xi_i$ being simultaneously diagonalizable, $U$ is spanned by a set of polynomials $\{P_{\zeta_1}, \ldots, P_{\zeta_k}\}$ with $k \in \mathbb{N}$ and $\zeta_i \in \mathbb{T}$. But from the Yang-Baxter
construction, if there exists \( \zeta \in T \) such that \( P_\zeta \in U \) then for each \( \zeta \in T, P_\zeta \in U \). So \( U \) is not a proper subspace.

In the rest of the section, we investigate the dimension of the spaces \( \mathcal{M}_T \).

The dimension of such a space equals the number of permutations of the vector of the entries of \( T \) multiplied by the number of tableaux \( T \) appearing in \( \mathcal{H}_T \). The first number is easy to obtain but for the second we need to introduce some results of Okounkov and Ol’shanski [12]:

Suppose \( \mu, \lambda \) are partitions with \( \mu \subset \lambda \) (\( \mu[i] \leq \lambda[i] \) for all \( i \)), \( |\mu| = k, |\lambda| = n \) then the set \( \{(i, j) : 1 \leq i \leq \ell(\lambda), \mu[i] < j \leq \lambda[i]\} \) is the skew-diagram \( \lambda \setminus \mu \). The basic step in determining the dimension of a connected component is to find the number (denoted \( \dim(\lambda \setminus \mu) \)) of RST’s of shape \( \lambda \setminus \mu \), that is, the number of ways the numbers \( (n-k),(n-k-1),\ldots,1 \) can be entered in \( \lambda \setminus \mu \) so that the entries decrease in each row and in each column. There is an elegant formula due to Okounkov and Ol’shanski [12] using shifted Schur functions. Writing \( \det(a_{ij}) \) to denote the determinant of the matrix \( (a_{ij})_{i,j=1}^m \), where \( m \geq \ell(\lambda) \) (the formula is independent of \( m \))

\[
\begin{align*}
\text{dim}(\lambda \setminus \mu) &= \frac{s_\mu^*(\lambda)(n-k)!}{h(\lambda)}, \\
h(\lambda) &= \frac{\prod_{i=1}^{\ell(\lambda)} (\lambda[i] + \ell(\lambda) - i)! \prod_{1 \leq i < j \leq \ell(\lambda)} (\lambda[i] - \lambda[j] - i + j)}{\prod_{i=1}^{\ell(\lambda)} (\lambda[i] + m - i)!},
\end{align*}
\]

where \( (n)_k = n(n-1)\ldots(n-k+1) \) denotes the descending Pochhammer symbol. Note \( h(\lambda) \) is version of the hook-product formula (see [13] p.11 (4)). Also the denominator in \( s_\mu^* \) is (up to a sign) the Vandermonde determinant of \( \{\lambda[i] + m - i, 1 \leq i \leq m\} \) giving the simplified formula

\[
\begin{align*}
\frac{s_\mu^*(\lambda)}{h(\lambda)} &= \frac{\det((m + \lambda[i] - i)_{\mu[\mu] + m - j})}{\prod_{i=1}^{\ell(\lambda)} (\lambda[i] + m - i)!},
\end{align*}
\]

Now consider a tableau \( T \), let \( M \) denote the maximum entry (also of any \( v \) in this component) and let

\[
\mu_m = \{(i, j) \in T : T(i, j) \leq m \}, \ 0 \leq m \leq M.
\]
Then each $\mu_m$ is the Ferrers diagram of a partition, $\mu_m \subseteq \mu_{m+1}$ (possibly $\mu_m = \mu_{m+1}$ for some $m$ but the following formula works because $s^*_\mu(\mu) = h(\mu)$ for any partition), and $v^+[j] = m$ when $j$ is an entry in $\mu_m \setminus \mu_{m-1}$. The number of RST’s in the connected component of $T$ is

$$\frac{|\mu_0|!}{h(\mu_0)} \prod_{m=1}^M \frac{s^*_{\mu_{m-1}}(\mu_m)(|\mu_m| - |\mu_{m-1}|)!}{h(\mu_m)},$$  \hspace{1cm} (34)$$

and the number of permutations of $v^+$ is $N!/|\mu_0|! \prod_{m=1}^M (|\mu_m| - |\mu_{m-1}|)!$; the dimension of the component is

$$\frac{N!}{h(\mu_0)} \prod_{m=1}^M \frac{s^*_{\mu_{m-1}}(\mu_m)}{h(\mu_m)}.$$  \hspace{1cm} (35)$$

This product can be restricted to the values of $m$ for which $\mu_{m-1} \neq \mu_m$, that is, the set of entries of $v^+$.

**Example 5.16**

1. Consider again the tableau $T = \begin{array}{lr}0 & 1 \\0 & 0 \end{array}$. Then, $\mu_0 = [2,1]$ and $\mu_1 = [2,2]$. Hence, $h(\mu_0) = 3$, $h(\mu_1) = 12$ and

$$s^*_{\mu_0}(\mu_1) = \begin{vmatrix}6 & 0 \\3 & 2 \\3 & 2 \\1 & 1 \end{vmatrix} = 12.$$ 

Hence, from eq. (34), the number of tableaux $T$ in $T$ equals $3! \cdot \frac{12}{12} = 2$. The tableaux are $\begin{array}{l}2 \\4 \\3 \\2 \end{array}$ and $\begin{array}{l}3 \\4 \\3 \\2 \end{array}$. So the dimension of $\mathcal{M}_T$ is 8.

2. Consider the bigger example given by the tableaux $T = \begin{array}{l}1 \\2 \\0 \\0 \end{array}$ (see Fig 4). Here $\mu_0 = [2]$, $\mu_1 = [3,1]$ and $\mu_2 = [3,2]$. So we compute: $h(\mu_0) = 2$, $h(\mu_1) = 8$, $h(\mu_2) = 24$, $s^*_{\mu_0}(\mu_1) = 8$ and $s^*_{\mu_1}(\mu_2) = 8$. By eq (34) we find 2 tableaux; graphically, the graph decomposes into two parts when we remove the jump edges. The dimension of $\mathcal{M}_T$ is 60.

3. Consider $T = \begin{array}{l}0 \\0 \\1 \\2 \end{array}$ (Fig 4). One has $\mu_0 = [1,1]$, $\mu_1 = [2,2]$ and $\mu_2 = [3,2]$. Hence, we have only 1 tableau in the connected component. Graphically, there is no jump (blue arrow) in the connected component $H_T^{d,s}$. The dimension of $\mathcal{M}_T$ is 30.
5.3 Symmetrizer/Antisymmetrizer

We define the operator

\[ S_N := \sum_{\sigma \in \mathcal{S}_N} \tilde{T}_\sigma, \]

where \( \tilde{T}_\sigma = T_{i_1} \cdots T_{i_k} \) if there is a shortest expression \( \sigma = s_{i_1} \ldots s_{i_k} \).

The operator \( S_N \) is a \( s \)-deformation of the classical symmetrizer in the following sense:

**Proposition 5.17** For each \( i \) one has

\[ S_N T_i = s S_N. \]

**Proof.** It suffices to split the sum as

\[ S_N T_i = \sum_{\sigma \in \mathcal{S}_N, \ell(\sigma) \leq \ell(\sigma)} \tilde{T}_\sigma T_i + \sum_{\sigma \in \mathcal{S}_N, \ell(\sigma) > \ell(\sigma)} \tilde{T}_\sigma T_i. \tag{36} \]

We use the quadratic relation to write the second sum as

\[ \sum_{\sigma \in \mathcal{S}_N, \ell(\sigma) > \ell(\sigma)} \tilde{T}_\sigma T_i = (s - 1) \sum_{\sigma \in \mathcal{S}_N, \ell(\sigma) < \ell(\sigma)} \tilde{T}_{s \sigma} T_i + s \sum_{\sigma \in \mathcal{S}_N, \ell(\sigma) < \ell(\sigma)} \tilde{T}_{s \sigma i}. \]

But

\[ \sum_{\sigma \in \mathcal{S}_N, \ell(\sigma) < \ell(\sigma)} \tilde{T}_{s \sigma i} T_i = \sum_{\sigma \in \mathcal{S}_N, \ell(\sigma) > \ell(\sigma)} \tilde{T}_{\sigma} \]

Hence,

\[ \sum_{\sigma \in \mathcal{S}_N, \ell(\sigma) > \ell(\sigma)} \tilde{T}_\sigma T_i = (s - 1) \sum_{\sigma \in \mathcal{S}_N, \ell(\sigma) < \ell(\sigma)} \tilde{T}_\sigma + s \sum_{\sigma \in \mathcal{S}_N, \ell(\sigma) < \ell(\sigma)} \tilde{T}_{s \sigma i}. \]

Replacing it in (36), we obtain the result. \[ \square \]

As a consequence

**Corollary 5.18** \( S_N \) satisfies:

\[ S_N^2 = \phi_N(s) S_N \]

where \( \phi_N(s) := \prod_{j=2}^{N} \frac{1-s^j}{1-s} \) is the Poincaré polynomial of \( \mathcal{S}_N \).
Proof. From proposition 5.17, one obtains
\[ S_N^2 = S_N \sum_{\sigma \in S_N} T_{\sigma} = \sum_{\sigma \in S_N} s^{(\sigma)} S_N = \phi_N(s) S_N.\]

Alternatively, we define
\[ S'_N = \sum_{\sigma \in S_N} T_{i_1}^{-1} \ldots T_{i_k}^{-1}.\]

This operator satisfies
\[ S'_N T_i = sS'_N \quad (37) \]
and
\[ S'^{2}_N = \phi_N \left( \frac{1}{s} \right) S'_N. \quad (38) \]

The action of the symmetrizer on leading terms has some nice properties.

Lemma 5.19 Let \( v \) and \( T \) such that \( \text{COL}_T[r_v[i]] = \text{COL}_T[r_v[i] + 1] \) and \( v[i] = v[i + 1] \) for some \( i \). Then,
\[ x^{v,T}S_N = 0. \]

Proof. We have:
\[ x^{v,T}T_i = x^{v,T_i} + x^{v,T}R_v T_i \]
But \( v[i] = v[i + 1] \) implies \( x^{v,T_i} = 0 \) and since \( \text{COL}_T[r_v[i]] = \text{COL}_T[r_v[i] + 1] \), we have \( T_{r_v[i]} = -T \). Hence
\[ x^{v,T}T_i = x^{v,T}T_{r_v[i]}R_v = -x^{v,T}. \quad (39) \]

Now, we split the sum \( x^{v,T}S_N \) into two sums:
\[ x^{v,T}S_N = x^{v,T} \sum_{\ell(s_i) < \ell(\sigma)} T_{\sigma} + x^{v,T} \sum_{\ell(s_i) < \ell(\sigma)} \tilde{T}_{\sigma} \]
\[ = x^{v,T} \sum_{\ell(s_i) < \ell(\sigma)} T_{i_1}T_{s_i} + x^{v,T} \sum_{\ell(s_i) < \ell(\sigma)} \tilde{T}_{\sigma}. \]
From eq (39) one obtains

\[ x^v, T S_N = -x^v, T \sum_{\ell(s, \sigma) < \ell(\sigma)} \tilde{T}_{s, \sigma} + x^v, T \sum_{\ell(s, \sigma) > \ell(\sigma)} \tilde{T}_{\sigma} \]

\[ = 0. \]

In the same way, we define

\[ A_N = \sum_{\sigma \in S_N} (-s)^{\ell(\sigma)} T_{\sigma} \]

where \( T_{\sigma} = T_{i_1}^{-1} \ldots T_{i_k}^{-1} \) if there is a shortest expression \( \sigma = s_{i_1} \ldots s_{i_k} \).

One has

**Proposition 5.20** For each \( i \):

\[ A_N T_i = -A_N. \]

**Proof.** The proof is very close to the proof of proposition 5.17 and left to the reader. ■

Again, as for the operator \( S_N \) one has:

**Corollary 5.21** \( A_N \) satisfies:

\[ A_N^2 = \phi_N(s) A_N. \]

**Lemma 5.22** Let \( v \) and \( T \) such that \( \text{ROW}_T[r_v[i]] = \text{ROW}_T[r_v[i] + 1] \) and \( v[i] = v[i + 1] \) for some \( i \). Then,

\[ x^v, T A_N = 0. \]

**Lemma 5.23** Let \( v = [v[1] < \cdots < v[N]] \) and \( T \) such that for each \( i \), \( v[i] = v[i + 1] \) implies \( \text{COL}_T[r_v[i]] = \text{COL}_T[r_v[i] + 1] \). The coefficient of \( x^v, T \) in \( x^v, T A_N \) equals \( \prod_i s^{m_i} \phi_{m_i}(s) \) where \( m_i \) denotes the number of parts \( i \) in \( v \).
5.4 Symmetric/Antisymmetric polynomials

When \( \zeta = \zeta_{v,T} \) and \( \zeta s_i = \zeta_{v',T'} \), we set \( s^i_{\zeta} := P_{\zeta s_i} + \frac{\zeta[i+1]}{\zeta[i]} P_{\zeta} \) and \( a^i_{\zeta} := P_{\zeta s_i} - \frac{1-s\zeta[i+1]}{1-\zeta[i]} P_{\zeta} \).

Lemma 5.24 If \( \zeta_i+1 \succ \zeta_i \), we have:

\[
\begin{align*}
\frac{s^i_{\zeta}}{T} &= s^i_{\zeta} \\
\frac{a^i_{\zeta}}{T} &= -a^i_{\zeta}.
\end{align*}
\]

Proof. We prove only the result for \( s^i_{\zeta} \), since the proof is very similar for \( a^i_{\zeta} \).

Recall that proposition 4.19 gives

\[
P_{\zeta} T_i = P_{\zeta s_i} - \frac{1}{1-\zeta[i+1]} \frac{\zeta[i+1]}{\zeta[i]} P_{\zeta}.
\]

We prove only the result for \( s^i_{\zeta} \).

Let \( \Sigma_{\zeta \in T} b_{\zeta} \in M_T \) be a symmetric polynomial, i.e. \( \Sigma T_i = s \Sigma \) for each \( i \).

Lemma 5.25 If \( \zeta[i+1] \succ \zeta[i] \) then \( \frac{b_{\zeta}}{b_{\zeta s_i}} = \frac{\zeta[i] - \zeta[i+1]}{\zeta[i] - \zeta[i+1]} \)

Proof. Since \( \Sigma T_i = s \Sigma \) this implies:

\[
(b_{\zeta} P_{\zeta} + b_{\zeta s_i} P_{\zeta s_i}) T_i = s (b_{\zeta} P_{\zeta} + b_{\zeta s_i} P_{\zeta s_i})
\]

And then \( b_{\zeta} P_{\zeta} + b_{\zeta s_i} P_{\zeta s_i} \) is proportional to \( s^i_{\zeta} \). This ends the proof.

Since each vertex of \( T \) is connected to \( \text{sink}(T) \) by a series of edges
the polynomial \( f \) is unique up to a global multiplicative coefficient and \( b_\zeta \neq 0 \) for all \( \zeta \) if \( f \neq 0 \).

If \( T[i, j] = T[i, j + 1] \) for some \( (i, j) \) then \( \zeta_{\text{root}(T)}[k] = q^n s^m \neq \zeta_{\text{root}(T)}[k + 1] = q^n s^{m+1} \) for some \( k \). Indeed, \( T[i, j] = T[i, j + 1] \) implies \( v_{\text{root}(T)}[k] = v_{\text{root}(T)}[k + 1] \), hence \( r_{\text{root}(T)}[k] + 1 = r_{\text{root}(T)}[k + 1] \). It follows that \( m = CT_{\text{root}(T)}[\ell] \) and \( m + 1 = CT_{\text{root}(T)}[\ell + 1] \) for some \( \ell \).

**Example 5.26** If \( T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) we have

\[
\text{root}(T) = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}, [s, s^{-1}, 1, q], [0, 0, 0, 1], [2, 3, 4, 1]
\]

We have \( T[1, 1] = T[1, 2] = 0 \) the corresponding cells in the tableau \( T_{\text{root}(T)} \) are \( T_{\text{root}(T)}[1, 1] = 4 \) and \( T_{\text{root}(T)}[1, 2] = 3 \). So \( \ell = 3, k = 2 \) and \( m = -1 = CT_{\text{root}(T)}[3] = CT_{\text{root}(T)}[4] - 1 \).

From \( fT_k = sf \), one deduces

\[
b_{\text{root}(T)}[i, [s, s^{-1}, 1, q], [0, 0, 0, 1], [2, 3, 4, 1] = (3, 4, 1, 2)
\]

Finally, \( \frac{\zeta_{\text{root}(T)}[k]}{\zeta_{\text{root}(T)}[k] - \zeta_{\text{root}(T)}[k + 1]} \) implies \( b_{\zeta_{\text{root}(T)}} = 0 \) and \( f = 0 \).

In the other cases, the coefficients \( b_\zeta \) are not zero and can be computed via the recurrence given in lemma 5.25. More, precisely setting \( b_{\zeta_{\text{root}(T)}} = 1 \), and \( b_{\zeta s_i} = \frac{\zeta[i]}{s_\zeta[i] - s[i + 1]} b_\zeta \) if \( \zeta[i + 1] \succ \zeta[i] \), we define the polynomial

\[
\mathcal{M}_T = \sum_{\zeta \in T} b_\zeta P_\zeta
\]

which is the unique generator of the subspace of symmetric polynomials of \( \mathcal{M}_T \).

So one has:

**Theorem 5.27** The subspace of \( \mathcal{M}_T \) of symmetric polynomials

1. a 1-dimension space generated by \( \mathcal{M}_T \) if \( T \) is a strict-column tableau;

2. a 0-dimension space in the other cases.

**Example 5.28** Consider the graph \( H_{11}^{q,s} \)

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The polynomial
\[ M_{10}^{11} = P_{[s,1,q,qs^{-1}]} + \frac{1-q}{s-1} P_{[s,q,1,qs^{-1}]} + \frac{(1-q)}{(s^2-q)} P_{[g,s,1,qs^{-1}]} + \frac{(1-q)}{(s^2-q)} P_{[s,q,qs^{-1},1]} \]
\[ + \frac{(1-q)(s-q)}{(s^2-q)^2} P_{[q,s,qs^{-1}]} + \frac{(1-q)(s-q)}{(s^2-q)(s^3-q)} P_{[q,qs^{-1},s,1]} \]
is symmetric.

In the same way, define \( b_{\zeta_{\text{root}(T)}}^i = 1 \), and \( b_{\zeta_{s_i}}^i = -\frac{\zeta[i]-\zeta[i+1]}{\zeta[i]-s\zeta[i+1]} b_{\zeta}^i \) if \( \zeta[i+1] \succ \zeta[i] \),
and the polynomial
\[ M_{T}^2 = \sum_{\zeta \in T} b_{\zeta}^i P_{\zeta}. \]
We have

**Theorem 5.29** *The subspace of* $\mathcal{M}_T^0$ *of antisymmetric polynomials is*

1. a 1-dimension space generated by $\mathfrak{M}_T^0$ if $T$ is a strict-row tableau;
2. a 0-dimension space in the other cases.

### 5.5 The group of permutations leaving $T$ invariant

Let $T$ be a filling of shape $\lambda$ with increasing rows and strictly increasing columns.

To each $i$ we associate the pair $\text{COORD}_T[i] = (\text{COL}_{\text{std}_1(T)}[i], \text{ROW}_{\text{std}_1(T)}[i])$.

An elementary transposition $s_i$ acts on $T$ by permuting the cells $\text{COORD}_T[i]$ and $\text{COORD}_T[i+1]$.

For a tableaux $T$, we will denote by $\mathfrak{S}_T$ the maximal subgroup of $\mathfrak{S}_N$ leaving invariant the sets of entries of each line.

**Example 5.30** For instance, consider the tableau $T = \begin{array}{ccc} 3 & 2 \\ 5 & 4 & 1 \end{array}$. We have $\mathfrak{S}_T = \mathfrak{S}_{\{1,4,5\}} \times \mathfrak{S}_{\{2,3\}}$.

We will denote also by $\mathfrak{S}_T$ the maximal subgroup of $\mathfrak{S}_{\text{std}_1(T)}$ leaving $T$ invariant.

**Example 5.31** Let $T = \begin{array}{ccc} 1 & 1 \\ 0 & 0 & 1 \end{array}$ we have $\text{std}_1(T) = \begin{array}{ccc} 3 & 2 \\ 5 & 4 & 1 \end{array}$ and

$$\mathfrak{S}_T = \mathfrak{S}_{\{2,3\}} \times \mathfrak{S}_{\{4,5\}} \times \mathfrak{S}_{\{1\}} \subset \mathfrak{S}_{\text{std}_1(T)} = \mathfrak{S}_{\{1,4,5\}} \times \mathfrak{S}_{\{2,3\}}.$$ 

Let $\mathfrak{S}_r(T)$ be the subgroup of $\mathfrak{S}_N$ leaving invariant the partition $v_{\text{sink}(T)}$

**Example 5.32** Again with $T = \begin{array}{ccc} 1 & 1 \\ 0 & 0 & 1 \end{array}$, we have $v_{\text{sink}(T)} = [1,1,1,0,0]$ and

$$\mathfrak{S}_r(T) = \mathfrak{S}_{\{1,2,3\}} \times \mathfrak{S}_{\{4,5\}}.$$ 

Observe that $\mathfrak{S}_T = \mathfrak{S}_{\text{std}_1(T)} \cap \mathfrak{S}_r(T)$. This implies that for each $\sigma \in \mathfrak{S}_T$ is $(v_{\text{sink}(T)}, \text{std}_1(T))\sigma = (v_{\text{sink}(T)}, \text{std}_1(T))$
Remark 5.33 In terms of spectral vectors we have \( \zeta_{\text{sink}}(T)\sigma = \zeta_{\text{sink}}(T) \) (here we use the action defined in eq. (33)). The property of \( T \) to have only strictly increasing columns can be also interpreted in terms of spectral vector. Indeed for each \( i \), we have:

\[
\zeta_{\text{sink}}(T)[i] \succ \zeta_{\text{sink}}(T)[i+1] \quad \text{or} \quad \zeta_{\text{sink}}(T)[i] = q^n s^{m+1} \not\sim \zeta_{\text{sink}}(T)[i+1] = q^n s^m. \tag{40}
\]

Example 5.34 Consider the tableau \( T = \begin{bmatrix} 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \), we compute \( \zeta_{\text{sink}}(T) \) from the vector \( v_{\text{sink}}(T) = [1, 1, 1, 0, 0] \) and the tableau \( \text{std}_1(T) = \begin{bmatrix} 3 & 2 \\ 5 & 4 & 1 \end{bmatrix} \).

Here \( r_{\text{sink}}(T) = [1, 2, 3, 4, 5] \), hence \( \zeta_{\text{sink}}(T) = [s^2 q, q, s^{-1} q, s, 1] \). Observe that \( \zeta_{\text{sink}}(T)[1] \succ \zeta_{\text{sink}}(T)[2], \zeta_{\text{sink}}(T)[2] \not\sim \zeta_{\text{sink}}(T)[3] \) with \( \frac{\zeta_{\text{sink}}(T)[2]}{\zeta_{\text{sink}}(T)[3]} = s \), \( \zeta_{\text{sink}}(T)[3] \succ \zeta_{\text{sink}}(T)[4] \) and \( \zeta_{\text{sink}}(T)[4] \not\sim \zeta_{\text{sink}}(T)[5] \) with \( \frac{\zeta_{\text{sink}}(T)[4]}{\zeta_{\text{sink}}(T)[5]} = s \).

Let \( \sigma_T \) be the minimal permutation such that \( \zeta_{\text{root}} T \sigma_T = \zeta_{\text{sink}}(T) \).

As a consequence, one has:

Lemma 5.35 The group \( \mathfrak{S}_T \) is the subgroup of \( \mathfrak{S}_N \) consisting of the permutations \( \sigma \) such that \( \ell(\sigma_T \sigma) = \ell(\sigma_T) + \ell(\sigma) \)

Furthermore, we will use the following result

Lemma 5.36 For each permutation \( \sigma \) one has:

\[
P_{\zeta_{\text{root}}(T)} \tilde{T}_\sigma = P_{\zeta_{\text{root}}(T)} \sigma + \sum_{\zeta' \prec \zeta_{\text{root}}(T) \sigma} (\ast) P_{\zeta'}
\]

**Proof.** We will prove the result by induction on the length of \( \sigma \). If \( \sigma = \text{Id} \) then the result is obvious. Now suppose \( \sigma = \sigma' s_j \) with \( \ell(\sigma) = \ell(\sigma') + 1 \) and \( \zeta_{\text{root}}(T) \sigma \prec \zeta_{\text{root}}(T) \sigma s_j \) or \( \zeta_{\text{root}}(T) \sigma \not\prec \zeta_{\text{root}}(T) \sigma s_j \). Then \( \tilde{T}_\sigma = \tilde{T}_{\sigma'} T_j \) and using the induction hypothesis:

\[
P_{\zeta_{\text{root}}(T)} \tilde{T}_\sigma = P_{\zeta_{\text{root}}(T)} \sigma' T_j + \sum_{\zeta' \prec \zeta_{\text{root}}(T) \sigma} (\ast) P_{\zeta'} T_j. \tag{41}
\]

But if \( P_{\zeta_{\text{root}}(T) \sigma' T_j} = P_{\zeta_{\text{root}}(T) \sigma s_j} + (\ast) P_{\zeta_{\text{root}}(T) \sigma} \). Furthermore since \( \zeta' \prec \zeta_{\text{root}}(T) \sigma' \) we have \( \zeta' s_j \prec \zeta_{\text{root}}(T) \sigma \). But

\[
P_{\zeta'} T_j = (\ast) P_{\zeta' s_j} + (\ast) P_{\zeta'}.
\]

Hence, replacing it in (41) we find the result. \( \blacksquare \) We deduce
Lemma 5.37 Denote by $\beta_T^\sigma$ the coefficient of $P_{\varsigma\text{ink}(T)}$ in $P_{\text{root}(T)} T_\sigma$. We have:

1. If $\sigma^{-1}_T \sigma \notin \mathcal{S}_T$ then $\beta_T^\sigma = 0$.
2. If $\sigma^{-1}_T \sigma \in \mathcal{S}_T$ then $\beta_T^\sigma = s^{\ell(\sigma) - \ell(\sigma_T)}$.

Proof. The part (1) is a direct consequence of lemma 5.36. To show the part (2), we first use lemma 5.36 and write $P_{\text{root}(T)} T_\sigma = P_{\varsigma\text{ink}(T)} + \sum \varsigma \varsigma_{\varsigma\varsigma\text{ink}(T)} (\ast) P_\varsigma$. Now, set $\tau := \sigma^{-1}_T \sigma \in \mathcal{S}_T$ and observe that for each element $\tau' \in \mathcal{S}_T$, $\varsigma \varsigma' = \varsigma\varsigma\text{ink}(T)$ implies $\varsigma = \varsigma\varsigma\text{ink}(T)$. Hence, the coefficient of $\varsigma\varsigma\text{ink}(T)$ in $\sum \varsigma \varsigma_{\varsigma\varsigma\text{ink}(T)} (\ast) P_\varsigma T_\varsigma$ is 0. It follows that $\beta_T^\sigma$ equals the coefficient of $\varsigma\varsigma\text{ink}(T)$ in $P_{\varsigma\text{ink}(T)} T_\varsigma$. But $\mathcal{S}_T$ is generated by transposition $s_i$ such that $\varsigma\varsigma\text{ink}(T)[i] = q^{s^{m+1}} \varsigma\varsigma\text{ink}(T)[i+1] = q^n s^m$ (see eq (40)). This implies $P_{\varsigma\text{ink}(T)} s_i = s P_{\varsigma\text{ink}(T)}$. Hence, $P_{\varsigma\text{ink}(T)} \tau = s^{\ell(\tau)} P_{\varsigma\text{ink}(T)}$. Since, from lemma 5.35, $\ell(\tau) = \ell(\sigma) - \ell(\sigma_T)$, we recover the result. 

Proposition 5.38 The coefficient of $P_{\varsigma\text{ink}(T)}$ in $P_{\text{root}(T)} S_N$ equals the Poincaré polynomial $\phi_T(s)$ of $\mathcal{S}_T$.

Proof. We write

$$P_{\text{root}(T)} S_N = P_{\text{root}(T)} \sum_{\sigma \in \mathcal{S}_T} T_{\sigma T} T_\sigma + P_{\varsigma\text{ink}(T)} \sum_{\ell(\sigma_T) < \ell(\sigma_T) + \ell(\sigma)} T_\sigma.$$ 

From lemma 5.37 the coefficient of $P_{\varsigma\text{ink}(T)}$ in

$$P_{\text{root}(T)} \sum_{\ell(\sigma_T) < \ell(\sigma_T) + \ell(\sigma)} T_\sigma = P_{\text{root}(T)} \sum_{\sigma^{-1}_T \sigma \notin \mathcal{S}_T} T_\sigma$$

is 0. Furthermore lemma 5.37 implies

$$P_{\text{root}(T)} \sum_{\sigma \in \mathcal{S}_T} T_{\sigma T} T_\sigma = P_{\varsigma\text{ink}(T)} \sum_{\sigma \in \mathcal{S}_T} T_\sigma + \sum_{\varsigma \varsigma_{\varsigma\varsigma\text{ink}(T)}} (\ast) P_\varsigma T_\varsigma.$$

But the since $\varsigma \neq \varsigma\varsigma\text{ink}(T)$, the coefficient of $P_{\varsigma\text{ink}(T)}$ in $P_\varsigma T_\varsigma$ is 0. Hence, the coefficient of $P_{\varsigma\text{ink}(T)}$ in $P_{\text{root}(T)} S_N$ equals the coefficient of $P_{\varsigma\text{ink}(T)}$ in $P_{\varsigma\text{ink}(T)} \sum_{\sigma \in \mathcal{S}_T} T_\sigma$. The result follows from lemma 5.37.

The polynomial $M_T$ is proportional to any $P_\varsigma S_N$ for $\varsigma \in T$. In fact, we can compute the coefficient:
**Theorem 5.39** We have

\[ \mathcal{M}_T = \frac{b_{\text{sink}(T)}}{\phi_T(s)} P_{\zeta_{\text{root}(T)}} S_N \]

**Proof.** It suffices to compare the coefficient of \( P_{\zeta_{\text{sink}(T)}} \) in \( \mathcal{M}_T \) (given by theorem 5.27) and in \( P_{\zeta} S_N \) (given by proposition 5.38).

**Example 5.40** Consider the tableau \( T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \). Here, \( \zeta_{\text{root}(T)} = [s, 1, qs^2, qs^{-1}] \) and \( \zeta_{\text{sink}(T)} = [qs^{-1}, qs^2, s, 1] \). The images of \( \zeta_{\text{root}(T)} \) by

\[ \mathcal{G}_4 = \{[1, 2, 3, 4], [1, 2, 4, 3], [1, 3, 2, 4], [1, 3, 4, 2], [1, 4, 2, 3], [1, 4, 3, 2], [2, 1, 3, 4], [2, 1, 4, 3], [2, 3, 1, 4], [2, 3, 4, 1], [2, 4, 1, 3], [2, 4, 3, 1], [3, 1, 2, 4], [3, 1, 4, 2], [3, 2, 1, 4], [3, 2, 4, 1], [3, 4, 2, 3], [3, 4, 2, 1], [4, 1, 2, 3], [4, 1, 3, 2], [4, 2, 1, 3], [4, 2, 3, 1], [4, 3, 1, 2], [4, 3, 2, 1] \}

are respectively

\begin{align*}
[&s, 1, qs^2, qs^{-1}], \quad [s, 1, qs^{-1}, qs^2], \quad [s, qs^{-1}, 1, qs^2], \quad [s, qs^{-1}, qs^2, 1], \quad [s, qs^2, 1, qs^{-1}], \quad [s, qs^2, qs^{-1}, 1], \\
[&s, 1, qs^2, qs^{-1}], \quad [s, 1, qs^{-1}, qs^2], \quad [s, qs^{-1}, 1, qs^2], \quad [s, qs^{-1}, qs^2, 1], \quad [s, qs^2, 1, qs^{-1}], \quad [s, qs^2, qs^{-1}, 1], \\
[&qs^2, s, 1, qs^{-1}], \quad [qs^2, s, qs^{-1}, 1], \quad [qs^2, s, 1, qs^2], \quad [qs^2, s, qs^2, 1], \quad [qs^2, qs^{-1}, s, 1], \quad [qs^2, qs^{-1}, s, 1], \\
[&qs^{-1}, s, 1, qs^2], \quad [qs^{-1}, s, qs^2, 1], \quad [qs^{-1}, s, qs^2, 1], \quad [qs^{-1}, s, qs^2, 1], \quad [qs^{-1}, s, qs^2, 1], \quad [qs^{-1}, s, qs^2, 1], \quad [qs^{-1}, s, qs^2, 1].
\end{align*}

Only two permutations give \( \zeta_{\text{sink}(T)} \): \([4, 3, 1, 2]\) and \([4, 3, 2, 1]\). Indeed, one computes \( \sigma_T \) by choosing a maximal path in the Yang-Baxter graph: \( \sigma_T = s_2 s_3 s_1 s_2 s_1 = [4, 3, 1, 2] \). The group \( \mathcal{G}_T \) is the order-two group \( \mathcal{G}_T = \mathcal{G}_{\{3, 4\}} \).

We see that acting by \( T_3 \) on \( P_{[qs^{-1}, qs^2, s, 1]} \) gives \( s P_{[qs^{-1}, qs^2, s, 1]} \). Hence,

\[ P_{[qs^{-1}, qs^2, s, 1]} (1 + T_3) = (1 + s) P_{[qs^{-1}, qs^2, s, 1]} = \phi_T(s) P_{[qs^{-1}, qs^2, s, 1]} \]

Note that, \( \phi_T(s) \) is the product of the \( \phi_{\lambda}(s) \) for each row \( \lambda = [a^m_1, \ldots, a^m_k] \) of \( T \) where \( \phi_{\lambda}(s) = \prod_i \phi_{m_i}(s) \).

In the same way, we prove a similar formula for antisymmetric polynomials:

**Theorem 5.41** We have

\[ \mathcal{M}_T^a = \frac{b_{\text{sink}(T)}}{\phi_T(s)} P_{\zeta_{\text{root}(T)}} A_N, \]

where \( T \) denotes the conjugate of \( T \) (that is the tableau obtained exchanging rows and columns).
\textbf{Proof.} Similarly to lemma 5.37 we denote by $\overline{\beta}'_T$ the coefficient of $P_{\zeta_{\text{sink}}(T)}$ in $P_{\zeta_{\text{root}}(T)} \mathbf{T}_\sigma$ and we obtain:

1. If $\sigma^1_\mathcal{T} \sigma \not\in \mathcal{S}_T$ then $\overline{\beta}'_T = 0$.

2. If $\sigma^1_\mathcal{T} \sigma \in \mathcal{S}_T$ then $\overline{\beta}'_T = (-1)^{\ell(\sigma') - \ell(\sigma)}$.

Using these properties we prove as in proposition 5.38 that the coefficient of $P_{\zeta_{\text{sink}}(T)}$ in $P_{\zeta_{\text{root}}(T)} \mathbf{A}_N$ equals the Poincaré polynomial $\phi_\mathcal{T}(s)$. The result follows. \hfill $\blacksquare$

\textbf{Example 5.42} Consider the tableau $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Here, $\zeta_{\text{root}}(T) = [s^{-1}, 1, qs, qs^{-2}]$ and $\zeta_{\text{sink}}(T) = [qs^{-2}, qs, s^{-1}, 1]$. The images of $\zeta_{\text{root}}(T)$ by $\mathcal{S}_4$ are:

\begin{align*}
[s^{-1}, 1, sq, \frac{1}{q}], & \quad [s^{-1}, 1, \frac{1}{q}, sq], & \quad [s^{-1}, sq, 1, \frac{1}{q}], & \quad [s^{-1}, sq, \frac{1}{q}, 1], & \quad [s^{-1}, \frac{1}{q}, 1, sq], & \quad [s^{-1}, \frac{1}{q}, sq, 1], \\
[s^{-1}, 1, \frac{1}{q}, sq], & \quad [s^{-1}, 1, sq, \frac{1}{q}], & \quad [s^{-1}, sq, 1, \frac{1}{q}], & \quad [s^{-1}, sq, \frac{1}{q}, 1], & \quad [s^{-1}, \frac{1}{q}, 1, sq], & \quad [s^{-1}, \frac{1}{q}, sq, 1], \\
[sq, s^{-1}, 1, \frac{1}{q}], & \quad [sq, s^{-1}, \frac{1}{q}, 1], & \quad [sq, \frac{1}{q}, s^{-1}, 1], & \quad [sq, \frac{1}{q}, 1, s^{-1}], & \quad [sq, 1, s^{-1}, \frac{1}{q}], & \quad [sq, 1, \frac{1}{q}, s^{-1}].
\end{align*}

Only two permutations give $\zeta_{\text{sink}}(T)$: $[4, 3, 1, 2]$ and $[4, 3, 2, 1]$. These permutations generate $\mathcal{S}_T$ with $T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

5.6 Minimal symmetric/antisymmetric polynomials

We have seen that for a given isotype $\lambda$ the symmetric polynomials are indexed by column-strict $T$ tableaux of shape $\lambda$. There exists only one tableau filling $\lambda$ such that the sum of its entries is minimal. This tableau is obtained by filling the first row with 0, the second with 1 \textit{etc.}. Let

\[ T_\lambda := \begin{pmatrix} m - 1 & \ldots & m - 1 \\ \vdots & \vdots \\ 1 & \ldots & \ldots & 1 \\ 0 & \ldots & \ldots & 0 \end{pmatrix} \]

if $\lambda = [\lambda_1, \ldots, \lambda_m]$ with $\lambda_1 \geq \cdots \geq \lambda_m$ and the number of $i$ in the entries of $T$ equals $\lambda_i$. 

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Example 5.43 Let $\lambda = [5, 3, 2, 2, 1]$, then

\[
\begin{array}{ccccc}
4 & 3 & 3 \\
2 & 2 & 2 \\
1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

$T_\lambda = \ldots$

We have

Corollary 5.44 The space of the minimal symmetric polynomials for isotype $\lambda$ is spanned by $M_{T_\lambda}$ and similarly the space of minimal antisymmetric polynomials is spanned by $M_{T_{\overline{\lambda}}}$ where $\overline{\lambda}$ denotes the conjugate partition of $\lambda$.

Example 5.45 Consider the isotype $\lambda = [5, 3, 2, 2, 1]$ then $\overline{\lambda} = [5, 4, 2, 1, 1]$

\[
\begin{array}{ccccc}
4 & 3 \\
2 & 2 \\
1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

$T_{\overline{\lambda}} = \ldots$

Hence, the space of minimal antisymmetric polynomials for isotype $\lambda$ is spanned by

\[
\begin{array}{cccc}
0 & 1 \\
0 & 1 \\
0 & 1 & 2 \\
0 & 1 & 2 & 3 & 4 \\
\end{array}
\]

6 Bilinear form

6.1 Bilinear form on the space $V_\lambda$

To define a pairing for $V_\lambda$ introduce the dual Hecke algebra $\mathcal{H}_N (q^{-1}, s^{-1})$; we use $*$ to indicate objects associated with $\mathcal{H}_N (q^{-1}, s^{-1})$, e.g. $T_i^*$, $(c_0 + c_1 s) = c_0 + \frac{\phi_i}{2}$. Recall that when acting on $V_\lambda$, $T_i = T_i$. There is a bilinear form $V_\lambda^* \times V_\lambda : (u^*, v) \mapsto \langle u^*, v \rangle \in \mathbb{Q} (s)$ such that $\langle u^* T_i^*, v T_i \rangle = \langle u^*, v \rangle$ for $1 \leq i < N$ and $T_1, T_2 \in \text{Tab}_\lambda, T_1 \neq T_2$ implies $\langle T_1^*, T_2 \rangle = 0$; the latter property follows from the eigenvalues of $L_i$, since $\langle u^* \phi_i^*, v \phi_i \rangle = \langle u^*, v \rangle$. We establish a formula for $\langle T_i^*, T \rangle$ after the following recurrence relation:
Lemma 6.1 If $T \in \text{Tab}_\lambda$ and $m := CT_T[i] - CT_T[i + 1] \geq 2$ then $T^{(i,i+1)} \in \text{Tab}_\lambda$ and

$$\langle (T^{(i,i+1)*}, T^{(i,i+1)}) = \frac{(1 - s^m)}{(1 - s^m)} \langle T^*, T \rangle.$$

Proof. The equation $TT_i = T^{(i,i+1)} - \frac{1-s}{1-s^m} T$ implies

$$\langle T^*, T \rangle = \langle T^{(i,i+1)*}, T^{(i,i+1)} \rangle + \frac{(1 - s^{-1})(1 - s)}{(1 - s^m)(1 - s^{-m})} \langle T^*, T \rangle,$$

thus

$$\langle T^{(i,i+1)*}, T^{(i,i+1)} \rangle = \left(1 - \frac{s^{m-1}(1 - s)}{(1 - s^m)^2}\right) \langle T^*, T \rangle.$$

Definition 6.2 For $T \in \text{Tab}_\lambda$ let

$$\nu(T) := \prod_{1 \leq i < j \leq N, CT_T[i] - CT_T[j] \leq -2} \frac{(1 - s^{CT_T[i] - CT_T[j]}) (1 - s^{CT_T[i] - CT_T[j] + 1})}{(1 - s^{CT_T[i] - CT_T[j]})^2}.$$

Proposition 6.3 The bilinear form defined by $\langle T^*, T_2 \rangle = 0$ for $T_1 \neq T_2$ and $\langle T^*, T \rangle = \nu(T)$ (for $T, T_1, T_2$) and extended by linearity satisfies $\langle P^* T^*_i, Q T_i \rangle = \langle P^*, Q \rangle$ for all $P^*, Q, i$.

Proof. It suffices to show $\langle T^* T^*_i, T T_i \rangle = \langle T^*, T \rangle$ for all $T$. If $TT_i = sT$ then $T_i = s^{-1}T^*$ and $\langle T^* T^*_i, T T_i \rangle = s^{-1}s \langle T^*, T \rangle$. The case $TT_i = -T$ is treated similarly. Otherwise consider the pair $(T, T^{(i,i+1)}$) with $CT_T[i] - CT_T[i + 1] \geq 2$. There is only one factor in $\nu(T^{(i,i+1)})$ different from $\nu(T)$, the one corresponding to $j = i + 1$. The proof follows from Lemma 6.1 and $CT_T[i + 1][i] = CT_T[i + 1], CT_T[i + i + 1][i + 1] = CT_T[i]$.

Any other bilinear form satisfying $\langle P^* T^*_i, Q T_i \rangle = \langle P^*, Q \rangle$ is a constant multiple of the above form.
6.2 Bilinear form on the space $\mathcal{M}_\lambda$

Consider the bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$\langle T_1^*, T_2 \rangle = \delta_{T_1, T_2} \nu(T_1)$$

(42)

and

$$\langle P_{x_i}, Q \rangle = \langle P, Q D_i \rangle$$

(43)

One has:

**Proposition 6.4**

$$\langle P(T_i^*)^{\pm 1}, Q \rangle = \langle P, QT_i^{\mp 1} \rangle$$

**Proof.** We proceed by induction on the degree of the polynomials. The initial case is given by the inner product on the tableaux.

Using the induction, we have from eq (23) and proposition 3.6

$$\langle P_{x_i} T_i^*, Q \rangle = \langle P_{x_i}, QT_i^{-1} \rangle,$$

(44)

$$\langle P_{x_i+1}(T_i^*)^{-1}, Q \rangle = \langle P_{x_i+1}, QT_i \rangle,$$

(45)

and

$$\langle P_{x_j}(T_i^*)^{\pm 1}, Q \rangle = \langle P_{x_j}, QT_i^{\mp 1} \rangle = \langle P, \text{when } |i - j| > 1 \rangle.$$

(46)

Indeed, one has

$$\langle P_{x_{i+1}}(T_i^*)^{-1}, Q \rangle = \frac{1}{s} \langle PT_i^{i+1} x_{i+1}, Q \rangle = \frac{1}{s} \langle PT_i^{i+1}, Q D_{i+1} \rangle = \frac{1}{s} \langle P, Q D_{i+1} T_i \rangle,$$

using the induction hypothesis. Hence,

$$\langle P_{x_{i+1}}(T_i^*)^{-1}, Q \rangle = \langle P, Q D_{i+1} T_i \rangle = \langle P, QT_i^{-1} D_{i} \rangle = \langle P_{x_i}, QT_i^{-1} \rangle$$

which gives (44). The proofs of (45) and (46) are similar.

Now by proposition 3.6, one has

$$\langle P_{x_{i+1}}(T_i^*)^{-1}, Q \rangle = \langle P, QT_i^{-1} D_{i+1} \rangle = \langle P_{x_i+1}, QT_i^{-1} \rangle$$

by induction. Hence, by (23) one obtains

$$\langle P_{x_{i+1}}(T_i^*)^*, Q \rangle = \langle P, QT_i^{-1} D_{i+1} \rangle = \langle P_{x_{i+1}}, QT_i^{-1} \rangle.$$

(47)
Similarly, one has
\[
\langle Px_i T_i^{-1}, Q \rangle = \langle Px_i, QT_i \rangle
\] (48)

Equations (44), (45), (46), (47) and (48) give the result. □

Now one has also the following equalities involving the operator \( w \):
\[
D_{i+1}w^{-1} = w^{-1}D_i, \quad x_{i+1}w = wx_{i+1} \neq 1
\]
and
\[
D_N w^{-1} = q w^{-1} D_1, \quad x_1 w = qw_N.
\]

It follows

**Proposition 6.5**
\[
\langle P w_{\pm 1}^*, Q \rangle = \langle P, Qw_{\mp 1} \rangle
\]

From propositions (46) and (6.5) one deduces

**Theorem 6.6**
1. \( \langle P \xi_i^*, Q \rangle = \langle P, Q \xi_i^{-1} \rangle \)
2. \( \langle P_\xi^*, P_{\xi'} \rangle = (*) \delta_{\xi, \xi'} \)
   where \((*)\) denotes a certain coefficient which remains to be computed.

### 6.3 Computation of \( \langle P_\xi^*, P_\zeta \rangle \)

First we establish some recurrences:

**Proposition 6.7** Let \( \zeta = \zeta_{v,T} \) for some \( T \in \text{Tab}_\lambda \) and \( v \in \mathbb{N}^N \). Suppose \( \zeta[i+1] \succ \zeta[i] \) for some \( i \). Then
\[
\langle P_{\zeta_{s_i}}^*, P_{\zeta_{s_i}} \rangle = \left( \frac{1 - s \frac{c[i+1]}{c[i]}}{1 - \frac{c[i+1]}{c[i]}} \right) \left( \frac{s - \frac{c[i+1]}{c[i]}}{s - \frac{c[i+1]}{c[i]}} \right)^2 \langle P_{\zeta}^*, P_{\zeta} \rangle.
\]

**Proof.** From equation (31) \( P_{\zeta} T_i = \frac{1 - s}{1 - \frac{c[i+1]}{c[i]}} P_{\zeta} + P_{\zeta_{s_i}} \). Thus
\[
\langle P_{\zeta}^*, P_{\zeta} \rangle = \langle P_{\zeta}^* T_i^*, P_{\zeta} T_i \rangle
\]
\[
= \left( \frac{1 - s}{1 - \frac{c[i+1]}{c[i]}} \right) \left( \frac{1 - s}{1 - \frac{c[i+1]}{c[i]}} \right)^* \langle P_{\zeta}^*, P_{\zeta} \rangle + \langle P_{\zeta_{s_i}}^*, P_{\zeta_{s_i}} \rangle.
\]

Hence \( \langle P_{\zeta_{s_i}}^*, P_{\zeta_{s_i}} \rangle = \left( 1 - \frac{(1-s)(1-s^{-1})}{(1 - \frac{c[i+1]}{c[i]})(1 - \frac{c[i+1]}{c[i]})} \right) \langle P_{\zeta}^*, P_{\zeta} \rangle = \left( 1 - \frac{s \frac{c[i+1]}{c[i]}}{s - \frac{c[i+1]}{c[i]}} \right) \langle P_{\zeta}^*, P_{\zeta} \rangle.
\) □
Definition 6.8 We define
\[ E_a(\zeta) = \prod_{(i,j) \in \mathrm{inv}(\zeta)} \frac{1 - s^a \frac{\zeta[j]}{\zeta[i]}}{1 - \frac{\zeta[j]}{\zeta[i]}} \]
and
\[ E(\zeta) = E_1(\zeta) E_{-1}(\zeta). \]

Proposition 6.9 Let \( \zeta = \zeta_{v,T} \) for some \( v \in \mathbb{N}^N \) and \( T \in \mathrm{Tab}_\lambda \). One has
\[ \langle P^*_{\zeta}, P_{\zeta} \rangle = E(\zeta)^{-1} \langle P^*_{\zeta_s}, P_{\zeta_s} \rangle. \]

Proof. Argue by induction on \#\mathrm{inv}(\zeta). The statement is trivially true for \#\mathrm{inv}(\zeta) = 0, that is, \( \zeta = \zeta^+ \). Suppose the statement is true for all \( \zeta' = \zeta_{v',T'} \) with \#\mathrm{inv}(\zeta') \leq n \) and \#\mathrm{inv}(\zeta) = n+1. Thus \( \zeta[i] < \zeta[i+1] \) for some \( i < N \).

By Proposition 6.7 \( \langle P^*_{\zeta}, P_{\zeta} \rangle = \left( \frac{1 - \frac{\zeta[i+1]}{\zeta[i]}}{1 - s \frac{\zeta[i+1]}{\zeta[i]}} \right) \langle P^*_{\zeta_s}, P_{\zeta_s} \rangle \); thus
\[ \frac{\langle P^*_{\zeta}, P_{\zeta} \rangle}{\langle P^*_{\zeta_s}, P_{\zeta_s} \rangle} = \frac{E(\zeta,s_i)}{E(\zeta)}. \]

This completes the induction since \#\mathrm{inv}(\zeta_s) = \#\mathrm{inv}(\zeta) - 1. ■

Alternatively, the computation of \( \langle P^*_{\zeta}, P_{\zeta} \rangle \) can be related to the root or the sink of the connected component of \( \zeta \).

Proposition 6.10 Let \( \zeta = \zeta_{v,T} \) for some \( v \) and \( T \). Let \( H^*_T \) be the connected component of \( \zeta \). We define the values:
\[ S(\zeta) = \prod_{(i,j) \in \mathrm{inv}_<(\zeta)} \frac{(1 - s \frac{\zeta[j]}{\zeta[i]})(1 - s^{-1} \frac{\zeta[j]}{\zeta[i]})}{(1 - \frac{\zeta[j]}{\zeta[i]})^2} \]
and
\[ R(\zeta) = \prod_{(i,j) \in \mathrm{inv}_>(\zeta)} \frac{(1 - s \frac{\zeta[j]}{\zeta[i]})(1 - s^{-1} \frac{\zeta[j]}{\zeta[i]})}{(1 - \frac{\zeta[j]}{\zeta[i]})^2}. \]

One has
1. \( \langle P^*_{\zeta}, P_{\zeta} \rangle = S(\zeta)^{-1} \langle P^*_{\zeta_{\mathrm{sink}(T)}}, P_{\zeta_{\mathrm{sink}(T)}} \rangle \)

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2. \( \langle P^*_\zeta, P_\zeta \rangle = \mathcal{R}(\zeta) \left( P^*_{\zeta_{\text{root}(T)}} P_{\zeta_{\text{root}(T)}} \right) \)

**Proof.** The proof goes as in proposition \ref{proposition6.9} using an induction on \( \sharp \text{inv}_\prec(\zeta) \) (\( \prec \in \{\prec, \succ\} \)) since there is a unique maximal (resp. minimal) element in the connected component: the sink (resp. the root). These elements are connected by a sequence of steps or jumps to \( \zeta \). ■

Also we have

**Proposition 6.11** Let \( \zeta = \zeta_v \), \( T \in \text{Tab}_\lambda \). One has

\[ \langle P^*_{\Psi q}, P_{\Psi q} \rangle = (1 - q[1]) \langle P^*_\zeta, P_\zeta \rangle \]

**Proof.** From proposition \ref{proposition4.19} one has

\[ \langle P^*_{\Psi q}, P_{\Psi q} \rangle = \langle P^*_\zeta \Phi^*, P_{\zeta \Phi} \rangle = \langle P^*_\zeta (T_1^{-1} \cdots T_{N-1}^{-1})^* x_N, P_{\zeta T_1^{-1} \cdots T_{N-1}^{-1} x_N} \rangle. \]

But proposition \ref{proposition6.4} implies

\[ \langle P^*_\zeta (T_1^{-1} \cdots T_{N-1}^{-1})^* x_N, P_{\zeta T_1^{-1} \cdots T_{N-1}^{-1} x_N} \rangle = \langle P^*_\zeta, P_{\zeta T_1^{-1} \cdots T_{N-1}^{-1} x_N D_N T_{N-1} \cdots T_1} \rangle, \]

and by \( D_N = (1 - \xi_N)x_N^{-1} \), we obtain

\[ \langle P^*_{\Psi q}, P_{\Psi q} \rangle = \langle P^*_\zeta, P_\zeta \rangle - \langle P^*_\zeta, P_{\Psi q \xi_N x_N^{-1} T_{N-1} \cdots T_1} \rangle \]

Using again proposition \ref{proposition4.19} we found

\[ \langle P^*_{\Psi q}, P_{\Psi q} \rangle = (1 - (\Psi^q)[N]) \langle P^*_\zeta, P_\zeta \rangle. \]

Since \( (\Psi^q)[N] = q[1] \), we recover the result. ■

**Definition 6.12** We introduce the symbol \( \chi(i, j) = 1 \) if \( j < i \) and \( = 0 \) when \( j \geq i \).

Let \( \rho(a, b) = \frac{(a-s^{-1}b)(a-sb)}{(a-b)^2} \) and

\[ \Delta(\zeta) := \prod_{j=1}^{N} \prod_{\xi[1], \xi[j] \in \zeta^k \atop k \geq \chi(i,j)} \rho(\zeta[j]q^k, \zeta[i]). \]

Let \( \square(q^n a^m) = (q; qa^m)_{n} \) with \( (a; q)_n = (1 - a)(1 - qa) \cdots (1 - q^{n-1}a) \) and

\[ \square(\zeta) = \prod_{i=1}^{N} \square(\zeta[i]). \]
Example 6.13 Let $\zeta := [q^2s^{-1}, qs^2, qs, q]$ be the spectral vector associated to $v = [2, 1, 1, 1]$ and $T = \begin{pmatrix} 1 & 3 & 2 \end{pmatrix}$. We have

$$\square(\zeta) = \square(q^2s^{-1})\square(qs^2)\square(qs)\square(q) = (1-qs^{-1})(1-q^2s^{-1})(1-qs^2)(1-qs)(1-q).$$

With the aim of computing $\triangle(\zeta)$, we list the triplets $(i, j, k)$ such that $\zeta[i] \succ \zeta[j]q^k$. Here we find 6 triplets:

$$(1, 2, 0), (1, 2, 1), (1, 3, 0), (1, 3, 1), (1, 4, 0), (4, 2, 0).$$

Note $(1, 4, 1)$ does not occur in the list since $q^2s^{-1} \not\sim q^2$. Furthermore, there is no factor corresponding to $(4, 2, 0)$ in $\triangle(\zeta)$ because $\chi(4, 2) = 1$. Hence, $\triangle(\zeta)$ is a product of 5 factors:

$$\triangle(\zeta) = \rho(qs^2, q^2s^{-1})\rho(q^2s^2, q^2s^{-1})\rho(qs, q^2s^{-1})\rho(q^2s, q^2s^{-1})\rho(q, q^2s^{-1})$$

$$= \frac{(q-s^2)(s^2+1)(-1+q)}{(-s^4+q)(s^2+1+s)(q-s)}.$$

With these notations, one has

Lemma 6.14

1. If $\zeta = \zeta_{0N,T}$ then $\triangle(\zeta) = \nu(T)$ and $\square(\zeta) = 1$.

2. If $\zeta = \zeta_{v,T}$ with $\zeta[\ell+1] \succ \zeta[\ell]$ then $\triangle(\zeta s_\ell) = \rho(\zeta[\ell], \zeta[\ell+1])\triangle(\zeta)$ and $\square(\zeta s_\ell) = \square(\zeta)$.

3. If $\zeta = \zeta_{v,T}$ then $\triangle(\zeta \Psi^q) = \triangle(\zeta)$ and $\square(\zeta \Psi^q) = (1-q\zeta[1])\square(\zeta)$.

Proof.

1. First note that when $\zeta = \zeta_{0N,T}$ there is no occurrence of $q$ in $\zeta$, so we have $\square(\zeta) = 1$. Also, we have:

$$\nu(T) = \prod_{1 \leq i < j \leq N}^{CT_T[i] - CT_T[j] \leq -2} \rho(\zeta[i], \zeta[j])$$

$$= \prod_{j=1}^{N} \prod_{\zeta[i] \succ \zeta[j]} \rho(\zeta[j], \zeta[i]).$$
2. Obviously we have \( \Box(\zeta_s) = \Box(\zeta) \). Furthermore,

\[
\frac{\Delta(\zeta_s)}{\Delta(\zeta)} = \frac{\prod_{\zeta \neq [\ell + 1] \zeta_s \rho(\zeta_s[\ell]q^k, \zeta_s[\ell])} {\prod_{\zeta \neq [\ell + 1] \zeta_s \rho(\zeta[\ell]q^k, \zeta[\ell + 1])}}
\]

This prove the result.

3. One has \( \Box(\zeta \Psi^q) = (1 - (\zeta \Psi^q)[N]) \Box(\zeta) = (1 - q\zeta[1]) \Box(\zeta) \). Furthermore,

\[
\frac{\Delta(\zeta \Psi^q)}{\Delta(\zeta)} = \frac{\prod_{\zeta \neq [\ell + 1] \zeta \Psi^q \rho((\zeta \Psi^q)[\ell]q^k, (\zeta \Psi^q)[\ell])} {\prod_{\zeta \neq [\ell + 1] \zeta \Psi^q \rho(\zeta[\ell]q^k, \zeta[\ell + 1])}}
\]

But \((\zeta \Psi^q)[N] = q\zeta[1]\) and \((\zeta \Psi^q)[i] = \zeta[i + 1]\). Hence, \((\zeta \Psi^q)[i] > (\zeta \Psi^q)[N]q^k\) for \( k \geq 0 \) implies \( \zeta[i + 1] = \zeta[1]q^{k + 1} \). In the same way \((\zeta \Psi^q)[N] > \zeta[i]q^k\) for \( k \geq 1 \) implies \( \zeta[1] = \zeta[i + 1]q^{k - 1} \). Hence, the quotient simplifies to

\[
\frac{\Delta(\zeta \Psi^q)}{\Delta(\zeta)} = 1,
\]

as expected.

\[\blacksquare\]

We deduce the following result.

**Theorem 6.15** Let \( \zeta = \zeta_{v,T} \), the value of the square \( \langle P^*_\zeta, P^*_\zeta \rangle \) is

\[
\langle P^*_\zeta, P^*_\zeta \rangle = \Box(\zeta) \Delta(\zeta).
\]

**Proof.** Comparing the statement of lemma [6.14] to proposition [6.7, 6.11 and 6.3], we show that \( \langle P^*_\zeta, P^*_\zeta \rangle \) and \( \Box(\zeta) \Delta(\zeta) \) satisfy the same recurrence rules and have the same values when \( \zeta = \zeta_{0N,T} \). \(\blacksquare\)
6.4 Computation of \(\langle \mathcal{M}_T^*, \mathcal{M}_T \rangle\)

First observe that
\[
\langle P, QS_N \rangle = \langle PS_N^*, Q \rangle. \tag{49}
\]
We use theorem 5.39 to write:
\[
\langle \mathcal{M}_T^*, \mathcal{M}_T \rangle = \frac{b_{\text{sink}(T)}}{\phi_T(s)} \langle \mathcal{M}_T^*, P_{\text{root}(T)} S_N \rangle.
\]
Hence, from eq (49) we have,
\[
\langle \mathcal{M}_T^*, \mathcal{M}_T \rangle = \frac{b_{\text{sink}(T)}}{\phi_T(s)} \langle \mathcal{M}_T^* S_N^*, P_{\text{root}(T)} \rangle.
\]
Since \(\mathcal{M}_T\) is symmetric eq (38) gives
\[
\langle \mathcal{M}_T^*, \mathcal{M}_T \rangle = \frac{\phi_N(s)}{\phi_T(s)} \langle \mathcal{M}_T^*, P_{\text{root}(T)} \rangle.
\]
Hence,
\[
\langle \mathcal{M}_T^*, \mathcal{M}_T \rangle = \frac{\phi_N(s)}{\phi_T(s)} b_{\text{sink}(T)} b_{\text{root}(T)} \langle P_{\text{root}(T)}^*, P_{\text{root}(T)} \rangle.
\]

Using the normalization described in section 5.4, \(b_{\text{root}(T)} = 1\).

**Theorem 6.16**
\[
\langle \mathcal{M}_T^*, \mathcal{M}_T \rangle = \frac{\phi_N(s)}{\phi_T(s)} b_{\text{sink}(T)} \langle P_{\text{root}(T)}^*, P_{\text{root}(T)} \rangle.
\]

In the same way, for antisymmetric polynomials, we have:

**Theorem 6.17**
\[
\langle \mathcal{M}_T^a, \mathcal{M}_T^a \rangle = \frac{\phi_N(s)}{\phi_T(s)} b_{\text{sink}(T)} \langle P_{\text{root}(T)}^a, P_{\text{root}(T)}^a \rangle.
\]

**Proof.** The proof goes as in the symmetric case, but using the operator \(A_N^\prime\) such that
\[
\langle P, QA_N \rangle = \langle PA_N^\prime^*, Q \rangle.
\]
This operator is the antisymmetrizer:
\[
A_N^\prime = \sum_{\sigma \in \mathcal{S}_N} (-s)^{\ell(T)} T_\sigma
\]
verifying
\[
A_N^\prime^2 = \phi_N \left( \frac{1}{s} \right) A_N^\prime.
\]
Hence, by a similar reasoning we find the proof.  \(\blacksquare\)
6.5 Hook-length type formula for minimal polynomials

The topic of this section is simpler formulae for \( \langle M^*_T, M_T \rangle \) for a decreasing partition \( \lambda \) in the situation where the entries of \( T \) are constant in each row. The formulae are then specialized to the minimal symmetric/antisymmetric polynomials. In this case they are expressions in terms of hook-lengths.

First consider a partition \( \mu \) verifying \( \mu = [\mu_1^{\lambda[1]}, \ldots, \mu_m^{\lambda[m]}] \) with \( \mu_1 > \cdots > \mu_m \).

Let

\[
T = \begin{array}{cccc}
\lambda[m] & \ldots & 1 \\
\lambda[m-1] + \lambda[m] & \ldots & \ldots & \lambda[m] + 1 \\
\vdots & & \vdots & \\
\lambda[1] + \cdots + \lambda[m] & \ldots & \ldots & \ldots & \lambda[2] + \cdots + \lambda[m] + 1
\end{array}
\]

be the RST obtained by filling the shape \( \lambda \) with \( 1, \ldots, N(= \lambda[1] + \cdots + \lambda[N]) \) row by row and

\[
\begin{array}{cccc}
\mu_1 & \ldots & \mu_m \\
\vdots & & \vdots \\
\mu_{m-1} & \ldots & \ldots & \mu_{m-1} \\
\mu_m & \ldots & \ldots & \ldots & \mu_m
\end{array}
\]

be the column strict tableau obtained by filling the shape \( \lambda \) with the entries of \( \mu \) row by row. Then \( \mu = v_{\text{sink}(T)} \) and \( T = T_{\text{sink}(T)} \). Hence,

\[
\zeta_{\text{sink}(T)} = [q^{\mu_1 s^{\lambda[m]-m}}, \ldots, q^{\mu_1 s^{1-m}}, q^{\mu_2 s^{1-m+\lambda[m]-1}}, \ldots, q^{\mu_2 s^{2-m}}, \ldots, q^{\mu_m s^{1-\lambda[1]}}, \ldots, q^{\mu_m}],
\]

(50)

Example 6.18 Let \( \lambda = [3, 3, 2] \) and \( \mu = [3, 3, 2, 2, 2, 1, 1, 1] \). We construct

\[
T = \begin{array}{ccc}
2 & 1 \\
5 & 4 & 3 \\
8 & 7 & 6
\end{array}
\]

and

\[
T = \begin{array}{ccc}
3 & 3 \\
2 & 2 & 2 \\
1 & 1 & 1
\end{array}
\]

Here \( \zeta_{\text{sink}(T)} = [q^3 s^{-1}, q^3 s^{-2}, q^2 s, q^2, q^2 s^{-1}, q s^2, q s, q] \)
We have

\[
\langle P_\root(T)^*, P_\sink(T)^* \rangle = S(\zeta_\root(T))^{-1} \langle P_\sink(T)^*, P_\sink(T)^* \rangle = S(\zeta_\root(T))^{-1} \triangle(\zeta_\sink(T)) \triangle (\zeta_\sink(T)).
\]

where

\[
\zeta_\root(T) = [q^{\mu_1} s^{-1+\lambda[1]}, \ldots, q^{\mu_m}, \ldots, q^{\mu_2} s^{1-m+\lambda[m-1]}, \ldots, q^{\mu_1} s^{1-m}] .
\]

By telescoping we find

\[
S(\zeta_\root(T)) = \prod_{1 \leq i < j \leq m} \frac{(1 - q^{\mu_j - \mu_i} s_j - i - \lambda[m-i+1]) (1 - q^{\mu_j - \mu_i} s_j - i + \lambda[m-j+1])}{(1 - q^{\mu_j - \mu_i} s_j - i + \lambda[m-j+1] - \lambda[i-m+1])}.
\]

(51)

First we compute \(\triangle(\zeta_\sink(T))\) and following eq (50) we write

\[
\triangle(\zeta_\sink(T)) = (\mathbb{T}^*, \mathbb{T}) \triangle.
\]

with

\[
\triangle = \prod_{1 \leq i < j \leq m} \prod_{k=0}^{\mu_j - \mu_i - 1} \prod_{a=1}^{\lambda[m-j+1]} \prod_{b=1}^{\lambda[m-i+1]} \frac{1 - q^{\mu_j - \mu_i} s_j - i + b - a - 1}{1 - q^{\mu_j - \mu_i} s_j - i + b - a - 2}.
\]

(52)

Indeed, \(\langle P_\sink(T)^*, P_\sink(T)^* \rangle\) splits into two factors: the first factor \(\langle \mathbb{T}^*, \mathbb{T} \rangle\) does not depend on \(q\), all the factors of the second factor \(\triangle \triangle (\zeta_\sink(T))\) involve \(q\).

By telescoping we have

\[
\prod_{b=1}^{\lambda[m-j+1]} \frac{(1 - q^{\mu_j - \mu_i} s_j - i + b - a - 1)}{(1 - q^{\mu_j - \mu_i} s_j - i + b - a - 2)} = \frac{(1 - q^{\mu_j - \mu_i} s_j - i - a)}{(1 - q^{\mu_j - \mu_i} s_j - i - a - 1)} \cdot \frac{(1 - q^{\mu_j - \mu_i} s_j - i + \lambda[m-j+1] - a + 1)}{(1 - q^{\mu_j - \mu_i} s_j - i + \lambda[m-j+1] - a)}.
\]

(53)

\[
\prod_{a=1}^{\lambda[m-i+1]} \frac{1 - q^{\mu_j - \mu_i} s_j - i - a}{1 - q^{\mu_j - \mu_i} s_j - i - a + 1} = \frac{1 - q^{\mu_j - \mu_i} s_j - i - \lambda[m-i+1]}{1 - q^{\mu_j - \mu_i} s_j - i}.
\]

(54)

and

\[
\prod_{a=1}^{\lambda[m-i+1]} \frac{1 - q^{\mu_j - \mu_i} s_j - i + \lambda[m-j+1] - a + 1}{1 - q^{\mu_j - \mu_i} s_j - i + \lambda[m-j+1] - a} = \frac{1 - q^{\mu_j - \mu_i} s_j - i + \lambda[m-j+1]}{1 - q^{\mu_j - \mu_i} s_j - i + \lambda[m-j+1] - \lambda[m-i+1]}.
\]

(55)
So, equalities (53), (54) and (55) give
\[
\Diamond = \prod_{1 \leq i < j \leq m} \prod_{k=0}^{\mu_i-\mu_j-1} \frac{(1 - q^{\mu_j - \mu_i + k s_{j-i}})(1 - q^{\mu_j - \mu_i + k s_{j-i} - 1})}{(1 - q^{\mu_j - \mu_i + k s_{j-i}})(1 - q^{\mu_j - \mu_i + k s_{j-i} - 1})}.
\]

Note from equality (52),
\[
S(\zeta_{\text{root}(T)})^{-1} \Diamond = \prod_{1 \leq i < j \leq m} \prod_{k=1}^{\mu_i-\mu_j-1} \frac{(1 - q^{\mu_j - \mu_i + k s_{j-i} - 1})}{(1 - q^{\mu_j - \mu_i + k s_{j-i}})} \
\quad \times \prod_{1 \leq i < j \leq m} \frac{(q s^{i-j} + \lambda_{m-i+1}, q)_{\mu_i-\mu_j-1}(q s^{i-j} - \lambda_{m-j+1}, q)_{\mu_i-\mu_j-1}}{(q s^{i-j} + \lambda_{m-i+1} - \lambda_{m-j+1}, q)_{\mu_i-\mu_j-1}(q s^{i-j}, q)_{\mu_i-\mu_j-1}}.
\]

Furthermore,
\[
\Box(\zeta_{\text{sink}(T)}) = \prod_{i=1}^{m} \prod_{j=1}^{\lambda_{m-i+1}} (q s^{j-m-i+1}, q)_{\mu_i}.
\]

Hence,
\[
\langle P_{\text{root}(T)}^*, P_{\text{root}(T)} \rangle = \langle T^*, T \rangle \prod_{i=1}^{m} \prod_{j=1}^{\lambda_{m-i+1}} (q s^{j-m-i+1}, q)_{\mu_i} \times \prod_{1 \leq i < j \leq m} \frac{(q s^{i-j} + \lambda_{m-i+1}, q)_{\mu_i-\mu_j-1}(q s^{i-j} - \lambda_{m-j+1}, q)_{\mu_i-\mu_j-1}}{(q s^{i-j} + \lambda_{m-i+1} - \lambda_{m-j+1}, q)_{\mu_i-\mu_j-1}(q s^{i-j}, q)_{\mu_i-\mu_j-1}}.
\]

We find also
\[
b_{\zeta_{\text{sink}(T)}} = \prod_{1 \leq i < j \leq m} \lambda_{m-i+1} \prod_{a=1}^{\mu_i-\mu_j-1} \frac{1 - q^{\mu_j - \mu_i + s_{j-i+a}}}{1 - q^{\mu_j - \mu_i + s_{j-i+a-1}}}. \]

Now, we specialize to \( \mu = m - i \). The tableau \( T \) becomes
\[
T = \begin{array}{cccc}
m - 1 & \ldots & m - 1 \\
\vdots & \ddots & \vdots \\
1 & \ldots & 1 \\
0 & \ldots & \ldots & 0
\end{array}
\]

For convenience, consider the normalization:
\[
\widetilde{\mathcal{M}}_T := b_{\zeta_{\text{sink}(T)}}^{-1} \mathcal{M}_T
\]
and we set \( \nabla_{\lambda} := \frac{\phi_T(s)}{\phi_N(s)} \frac{(\tilde{M}_r^*, \tilde{M}_r)}{\langle T^*, T \rangle} \) So, we have

\[
\nabla_{\lambda} = \left(b_{\text{sink}(T)}^{-1}\right)^* \frac{P_{\text{root}(T)}^* P_{\text{root}(T)}}{\langle T^*, T \rangle}.
\]

From equality (56), we obtain

\[
\nabla_{\lambda} = \prod_{i=1}^{m} \lambda[m-i+1] \prod_{j=1}^{i-1} (q s_{j-m+i-1}; q)_{i-1} \times
\]

\[
\prod_{1 \leq i < j \leq m} \frac{(q s_{j-i+\lambda[m-i+1]}; q)_{j-i-1}(q s_{j-i-\lambda[m-j+1]}; q)_{j-i-1}}{(q s_{j-i+\lambda[m-i+1]-\lambda[m-j+1]}; q)_{j-i-1}(q s_{j-i}; q)_{j-i-1}} \times \lambda[m-i+1] \prod_{a=1}^{1} 1 - q^{i-j}s_{i-j+a-\lambda[m-j+1]-1} \]

\[
\prod_{a=1}^{1} 1 - q^{i-j}s_{i-j+a-1}
\]

\[(57)\]

Note that this formula remains valid when \( \lambda[m] = 0 \):

\[
\nabla_{[\lambda[1], \ldots, \lambda[m-1], 0]} := \nabla_{[\lambda[1], \ldots, \lambda[m-1]]}.
\]

Let \( \lambda' = [\lambda[1], \lambda[2], \ldots, \lambda[m-1], \lambda[m] - 1] \) be the partition obtained from \( \lambda \) by subtracting 1 from its last part. We will denote by \( T' \) and \( T'' \) the associated tableaux.

**Example 6.19** For instance, if \( \lambda = [6, 3, 2] \) then

\[
T = \begin{array}{ccc}
2 & 2 & 2 \\
1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array} \quad \text{and} \quad T' = \begin{array}{ccc}
5 & 4 & 3 \\
11 & 10 & 9 & 8 & 7 & 6
\end{array}.
\]

In this case \( \lambda' = [6, 3, 1] \) and

\[
T = \begin{array}{ccc}
2 & 2 & 2 \\
1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array} \quad \text{and} \quad T' = \begin{array}{ccc}
1 & 1 & 1 \\
4 & 3 & 2 \\
10 & 9 & 8 & 7 & 6 & 5
\end{array}.
\]

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One has
\[
\frac{\nabla_{\lambda'}}{\nabla_{\lambda'}} = (qs^{\lambda[m] - m}; q)_{m-1} \times \\
\prod_{j=2}^{m} \left[ \frac{(qs^{1-j+\lambda[m]}; q)_{j-2} (qs^{\lambda[m] - \lambda[m-j+1] - j}; q)_{j-2}}{(qs^{\lambda[m]}; q)_{j-2} (qs^{1-j+\lambda[m]-\lambda[m-j+1]}; q)_{j-2}} \times \frac{1 - q^{j-1}s^{\lambda[m]-\lambda[m-j+1]-j}}{1 - q^{j-1}s^{\lambda[m]-j}} \right]^{1-q^{j-1}s^{\lambda[m]-j}}
\]
\[= (qs^{\lambda[m]-m}; q)_{m-1} \times \\
\prod_{j=2}^{m} \left[ \frac{(qs^{\lambda[m]-j+1}; q)_{j-2} (qs^{\lambda[m] - \lambda[m-j+1] - j}; q)_{j-1}}{(qs^{\lambda[m]-j}; q)_{j-1} (qs^{1-j+\lambda[m]-\lambda[m-j+1]}; q)_{j-2}} \right]^{1-q^{j-1}s^{\lambda[m]-j}}
\]
\[\text{eq (58) gives}
\]
\[
\frac{\nabla_{\lambda'}}{\nabla_{\lambda'}} = \prod_{j=2}^{m} \left[ \frac{(qs^{\lambda[m]-\lambda[j+1] - j}; q)_{j-1}}{(qs^{1-j+\lambda[m]-\lambda[m-j+1]}; q)_{j-2}} \right]^{1-q^{j-1}s^{\lambda[j+1]-j}}
\]
\[= \prod_{i=1}^{m-1} \left[ \frac{(qs^{\lambda[i+1]-i-m+1}; q)_{m-i}}{(qs^{\lambda[i]-i-m}; q)_{m-i-1}} \right]^{1-q^{i+m-1}}.
\]
\[\text{As usual, we define the arm, leg and hook length a node } (x, y) \in \lambda \text{ respectively by}
\]
\[
\lambda[x, y] = \lambda[y] - x, \quad \lambda[x, y] = \lambda[x] - y \text{ and } \lambda[x, y] = \lambda[y] + \lambda[x] + 1,
\]
\[\text{where } \lambda \text{ is the conjugate of } \lambda.
\]

\textbf{Remark 6.20} Note we use French notations for Ferrers diagram. For instance, the Ferrers diagram } \lambda = [4, 2, 1] \text{ is}

\[
\begin{array}{cccccc}
& & & 3 & & \\
2 & & & & & \\
1 & & 1 & & & \\
y/x & 1 & 2 & 3 & 4
\end{array}
\]
The coordinates of the node \( \times \) in the diagram

\[
\begin{array}{c}
\begin{array}{c}
\square \\
\square \\
\square \\
\times \\
\square \\
\end{array}
\end{array}
\]

are \([2, 1]\). We have

\[
\lambda[2, 1] = \lambda[2] - 2 = 2, \quad \int_\lambda[2, 1] = \lambda[1] - 1 = 1 \quad \text{and} \quad \overline{\lambda}[2, 1] = 4.
\]

Let

\[
H_\lambda := \ell(\lambda) - 1 \prod_{y=1}^{\lambda[i]} \prod_{x=1}^{\overline{\lambda}[x,y]} (qs^{\lambda[2,y]}; q)^{\overline{\lambda}[x,y]}.
\]

The changes from \( H_\lambda \) to \( H_{\lambda'} \) come from the node \( \{ (\lambda[m], y) : 1 \leq i \leq m - 1 \} \); each hook-length and each leg-length increases by 1, thus

\[
\frac{H_\lambda}{H_{\lambda'}} = \prod_{i=1}^{m-1} \frac{(s^{\lambda[m]-\lambda[i]+m-1}; q)_{m-i}}{(qs^{\lambda[m]-\lambda[i]+m-1}; q)_{m-i-1}}.
\]

Hence,

\[
\nabla_\lambda = \frac{H_\lambda}{H_{\lambda'}}.
\]

Using eq (61) we show:

\[
H_\lambda = \nabla_\lambda.
\]

It remains to compute \( \langle T^*, T \rangle \). We start from

\[
\langle T^*, T \rangle = \prod_{1 \leq i < j \leq N} \frac{(1 - s^{CT[i]-CT[j]-1})(1 - s^{CT[i]-CT[j]+1})}{(1 - s^{CT[i]-CT[j]})^2},
\]

and we analyze this product in terms of nodes:

\[
\langle T^*, T \rangle = \prod_{(x,y) \in \lambda} \prod_{1 \leq t \leq y - 1 \leq z \leq \lambda[y]} \frac{(1 - s^{(x-y-t)-(z-t)+1})(1 - s^{(x-y-t)-(z-t)+1})}{(1 - s^{(x-y-t)-(z-t) - 2})^2}.
\]
Indeed, consider the set $I_{\lambda}$ of the pairs $[(x, y), (z, t)]$ of nodes verifying $T[x, y] < T[z, t]$ and $(x - y) \leq z - t - 2$. This set splits into $N$ disjoint (possibly empty) sets:

$$E_{(x,y)} := \{[(x, y+t), (z, y) : 1 \leq t \leq \lambda[x] - y, 1 \leq z \leq \lambda[y], (x-y-t)-(z-t) \leq -2\}.$$

**Example 6.21** Consider the partition $\lambda = [3, 2]$ then

$$T = \begin{pmatrix} 2 & 1 \\ 5 & 4 & 3 \end{pmatrix} \quad \text{with contents} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

Hence, $I_{\lambda} = \{[(2, 2), (3, 1)], [(1, 2), (3, 1)], [(1, 2), (2, 1)]\}$, $E_{(1,1)} = \{[(1, 2), (2, 1)], [(1, 2), (3, 1)]\}$, $E_{(2,1)} = \{[(2, 2), (3, 1)]\}$ and $E_{(3,1)} = E_{(1,2)} = E_{(2,2)} = \emptyset$.

Hence,

$$\langle T^*, T \rangle = \prod_{[(x,y),(z,t)] \in I_{\lambda}} (1 - s^{x-y-z+t-1})(1 - s^{x-y-z+t+1}) \quad \frac{(1 - s^{x-y-z+t})}{(1 - s^{x-y-z})^2},$$

and we recover (63). Let us compute the products

$$E_{(x,y)} := \prod_{[(z_1,t_1),(z_2,t_2)] \in E_{(x,y)}} (1 - s^{z_1-t_1+t_2-z_2-1})(1 - s^{z_1-t_1+t_2-z_2+1}) \quad \frac{(1 - s^{z_1-t_1+t_2-z_2})}{(1 - s^{z_1-t_1+t_2})^2}.$$

Remark if $[(x, y + t), (z, y)] \in E_{(x,y)}$ then $t$ and $z$ have bounds $1 \leq z \leq \lambda[y]$, $1 \leq t \leq \lambda[x] - y$, $z + t - x - 2 \geq 0$. Hence,

$$E_{(x,y)} := \prod_{t=1}^{\lambda[y]} \prod_{z = \max\{1,x+2-t\}}^{\lambda[x]-y} (1 - s^{x-t-z+1})(1 - s^{x-t-z-1}) \quad \frac{(1 - s^{x-t-z})}{(1 - s^{x-t-z})^2}.$$

By telescoping, we find

$$E_{(x,y)} = \prod_{t=1}^{\lambda[y]} (1 - s^{\max\{1,x+2-t\}-x+t-1})(1 - s^{\lambda[y]-x+t+1}) \quad \frac{(1 - s^{\lambda[y]-x+t})}{(1 - s^{\lambda[y]-x+t})}, \quad (64)$$

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We find also
\[
\prod_{t=1}^{\lambda(x,y) - y - x + 1} \frac{1 - s^{\lambda(y) - x + t}}{1 - s^{\lambda(y) - x + t - 1}} = \frac{1 - s^{\lambda(y) - x + \lambda(x,y) - y + 1}}{1 - s^{\lambda(y) - x + 1}} = \frac{1 - s^{x \lambda(x,y)}}{1 - s^{\lambda(x,y)}}. 
\] (65)

But if $\lambda[x] - y \leq x$ then $\max\{1, x + 2 - t\} = x + 2 - t$ for $1 \leq t \leq \lambda[x] - y$
\[
\prod_{t=1}^{\lambda(x,y) - y} \frac{1 - s^{\max\{1,x+2-t\}-x+t-1}}{1 - s^{\max\{1,y+2-t\}-x+t}} = \left(\frac{1 - s}{1 - s^2}\right)^x. 
\] (66)

If $\lambda[x] - y > x$ then we use telescoping to show
\[
\prod_{t=1}^{\lambda(x,y) - y} \frac{1 - s^{\max\{1,x+2-t\}-x+t-1}}{1 - s^{\max\{1,y+2-t\}-x+t}} = \left(\frac{1 - s}{1 - s^2}\right)^x \cdot \frac{1 - s}{1 - s^{\lambda[x] - a - b + 1}}. 
\] (67)

Eq (66) and (67) give
\[
\prod_{t=1}^{\lambda(x,y) - y} \frac{1 - s^{\max\{1,x+2-t\}-x+t-1}}{1 - s^{\max\{1,y+2-t\}-x+t}} = \left(\frac{1 - s}{1 - s^2}\right)^x \cdot \frac{1 - s}{1 - s^{\lambda[x] - a - b + 1}}. 
\] (68)

Hence, from (65) and (68) we obtain
\[
E(x,y) = \left(\frac{1 - s}{1 - s^2}\right)^x \cdot \frac{1 - s^{\lambda(x,y)}}{1 - s^{\lambda(x,y) - x + 1}} \cdot \frac{1 - s^{\lambda(x,y)})}{1 - s^{\lambda(x,y) - x + 1}}. 
\] (69)

Finally (62), (63) and (69) give
Theorem 6.22

\[ \langle \widetilde{\mathcal{M}}_T^*, \widetilde{\mathcal{M}}_T \rangle = \prod_{(x,y) \in \lambda} \left( \frac{1 - s}{1 - s^2} \right)^{\min \{ x, \lambda[x,y] \}} \times \]

\[ \frac{(1 - s)(-s)^{\lambda[1][x,y]} \left( s^{-\lambda[1][x,y]} q \right)^{\lambda[x,y] + 1}}{\left( 1 - s \right)^{\max \{ 1, \lambda[x,y] - x + 1 \}} \left( 1 - s^2 \right) \lambda[x,y]^{x}} \]

For a rational expression \( f(s) \) let \( \iota f(s) = f(s^{-1}) \). Here are some immediate consequences:

\[ \iota \nu(T) = \nu(T), \]
\[ CT_T(i) = -CT_T(i), 1 \leq i \leq N, \]
\[ \zeta_{v,T} = q^{v[i]} s^{CT_T(i)} = q^{v[i]} s^{-CT_T(i)} = \iota \zeta_{v,T}. \]

If \( T_1, T_2 \in \text{Tab}_\lambda \) then

\[ \nu(T_1) \frac{\nu(T_2)}{\nu(T_1)} = \nu(T_2) \frac{\nu(T_1)}{\nu(T_2)}. \]

If \( \rho_q(m,n) = \frac{(qs^n; q)_m(qs^{n+1}; q)_m}{(qs^n; q)_m^2} \) then \( \iota \rho_q(m,n) = \rho_q(m,-n) \). Using this in the formula for \( \langle P^*_v, P^*_v \rangle \) we obtain

\[ \iota \left( \langle P^*_v, P^*_v \rangle \right) = \frac{\langle P^*_v, P^*_v \rangle}{\nu(T)} \frac{\nu(T)}{\nu(T)}. \]

Now suppose \( \lambda \) is a partition of \( N \) and \( T, T \) are the tableaux corresponding to the minimal antisymmetric polynomial.

Example 6.23 For example \( \lambda = (3, 2) \), then

\[ T = \begin{bmatrix} 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad \overline{T} = \begin{bmatrix} 4 & 2 \\ 5 & 3 & 1 \end{bmatrix}. \]
As for symmetric polynomials, we set
\[ \tilde{\mathcal{M}}_T^a = (b_\nu^T)^{-1} \mathcal{M}_T^a. \]

Our formulae show that
\[ \iota \left( \frac{\langle \tilde{\mathcal{M}}_T^{a*}, \tilde{\mathcal{M}}_T^a \rangle}{\nu(T)} \right) = \frac{\langle \tilde{\mathcal{M}}_T^{a*}, \tilde{\mathcal{M}}_T^a \rangle}{\nu(T)} = \frac{\phi_N(s)}{\prod_{i=1}^{\ell(\lambda)} \phi_{\lambda[i]}(s)} \prod_{(i,j) \in \lambda} \left( q s^{\frac{[\lambda[i,j]]}{2}} ; q \right) \prod_{(i,j) \in \lambda} \left( q s^{\lambda[i,j]} ; q \right) \]

and thus

**Theorem 6.24**
\[ \langle \tilde{\mathcal{M}}_T^{a*}, \tilde{\mathcal{M}}_T^a \rangle = \nu(T) \frac{\phi_N(s^{-1})}{\prod_{i=1}^{\lambda[i]} \phi_{\lambda[i]}(s^{-1})} \prod_{(i,j) \in \lambda} \left( q s^{\lambda[i,j]} ; q \right) \]

**Example 6.25** For the example \( \lambda = (3, 2) \),
\[ \langle \tilde{\mathcal{M}}_T^{a*}, \tilde{\mathcal{M}}_T^a \rangle = \phi_5(s^{-1}) \phi_2(s^{-1}) \left( q s^4 ; q \right) \left( q s^3 ; q \right) \left( q s^2 ; q \right) \]
\[ = s^{-8} \phi_5(s) \left( 1 - q s^2 \right) \left( 1 - q^2 s^4 \right) \left( 1 - q s^3 \right) \left( 1 - q s^4 \right). \]

Note \( \nu(T) \) does not always equal 1. For instance,
\[ \nu \left( \begin{array}{cccc}
6 & 7 & 8 & 4 & 2 \\
8 & 5 & 3 & 1 & 9 \end{array} \right) = \frac{1 + s^2}{(1 + s)^2}. \]
7 Conclusion

Throughout this paper, we have constructed and analyzed a Macdonald type structure for vector valued polynomials, that is polynomials whose coefficients belong to an irreducible module of the Hecke algebra. The "classical" Macdonald polynomials are recovered for the trivial representation and then correspond to the shapes $\lambda = (n)$, $n \in \mathbb{N}$. Thanks to the Yang-Baxter graph we have found algorithms and some explicit formulæ for computing the Macdonald polynomials, their (anti)symmetrizations, their scalar products etc. and give graphical interpretations of these properties.

We remark that almost everything works as for vector valued Jack polynomials [6] and that the Jack polynomials are recovered as a limit case of Macdonald polynomials as expected (setting $q = s^\alpha$ and sending $s$ to 1).

It remains to consider some constructions that could illuminate this theory. For instance, the shifted Macdonald polynomials could be defined by slightly changing the raising operators. For the trivial representation, shifted Macdonald polynomials are easier to manipulate than the homogeneous ones since they can be defined by vanishing properties [10, 11]. We have seen [6], that it is no longer the case for shifted vector valued Jack polynomials for a generic irreducible module. But this research is not completed, and we speculate that the vanishing properties arise when considering some polynomial representations of the Hecke algebra.

Comparing the results in [5] and [8], we find similarities between the concepts of singular non-symmetric Macdonald polynomials and highest weight symmetric Macdonald polynomials. We hope that this similarity extends to vector valued polynomials. In this context, minimal symmetric polynomials should play a special role and, perhaps, provide applications to the study of the fractional quantum Hall effect. The fractional quantum Hall effect is a state of matter with elusive physical properties whose theoretical study was pioneered by Laughlin based on wave functions describing the many-body state of the interacting electrons. Some of these wave functions (called Read-Rezayi states [14]) are multivariate symmetric polynomials with special vanishing properties and it was shown, combining minimality of the polynomials for the vanishing properties and result of [7], that they are Jack polynomials for a specialization of the parameter $\alpha$ (see eg [2]). It would be interesting to know if we can identify other relevant wave functions from vector valued Jack or Macdonald polynomials.
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Figure 1: The first vertices labeled by pairs $(\mathbb{T}, v)$ of the graph $G_{21}^{q,s}$ where we omit to write the vertex $\emptyset$ and the associated arrows.
Figure 2: The first vertices labeled by spectral vector of the graph $G_{21}^{q,s}$. 
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A Some useful formulæ for affine double Hecke Algebra

A.1 Hecke algebra of type $A_{N-1}$

The generators of $H_N(s)$ are $T_1, T_2, \ldots, T_{N-1}$ with $s^n \neq 1$ for $1 \leq n \leq N$. The generators satisfy the relations:

$$(T_i - s) (T_i + 1) = 0, \quad T_i^2 = (s - 1) T_i + s,$$

$$T_i^{-1} = \frac{1}{s} (T_i - s + 1),$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, 1 \leq i < N,$$

$$T_i T_j = T_j T_i, |i - j| > 1.$$

Let $S = T_1 T_2 \ldots T_{N-1}$ then $T_i S = S T_{i+1}$ for $1 < i < N - 1$ and $T_j S^N = S^N T_j$ for $1 \leq j < N$. Indeed

$$T_i S = T_1 \ldots T_{i-2} T_i T_{i-1} T_i T_{i+1} \ldots T_{N-1}$$

$$= T_1 \ldots T_{i-2} T_{i-1} T_i T_{i+1} \ldots T_{N-1}$$

$$= S T_{i-1},$$

and

$$T_j S^N = S^{j-1} T_1 S^{N-j+1} = S^{j-1} ((s - 1) T_1 + s) (T_2 \ldots T_{N-1} S) S^{N-j-1}$$

$$= (s - 1) S^N + s S^{j-1} (S T_1 \ldots T_{N-2}) S^{N-j-1},$$

$$S^N T_j = S^{j+1} T_{N-1} S^{N-j-1} = S^j T_1 T_2 \ldots T_{N-2} ((s - 1) T_{N-1} + s) S^{N-j-1}$$

$$= (s - 1) S^N + s S^j T_1 \ldots T_{N-2} S^{N-j-1}.$$}

A consequence of the derivation is

$$T_1 S^2 = S^2 T_{N-1}.$$
multiply both sides on the left by $S_{i-1} = T_{i-1} \ldots T_{N-1}$ and use the inductive hypothesis:

$$S_{i-1}^{N+2-i} = T_{i-1} \ldots T_{N-1}T_{N-1} \ldots T_{i-1}S_i^{N+1-i}$$

$$= \phi_i^{-1}S_i^{N+1-i} = \phi_i^{-1}\phi_i' \ldots \phi_N^{-1}.$$ 

Thus $S_N = s^{N(N-1)/2}\phi_1\phi_2 \ldots \phi_N^{-1}$.

Adjoin an invertible operator $w$ with relation:

$$wT_i = T_{i+1}w, 1 \leq i < N - 1,$$

$$w^2T_{N-1} = T_1w^2,$$

$$w^NT_i = T_iw^N, 1 \leq i < N.$$

### A.2 Action on polynomials

Let $\mathcal{P} = \mathbb{K}[x_1, \ldots, x_N]$ where $\mathbb{K}$ is an extension field of $\mathbb{Q}(s,q)$; on $\mathcal{P}$ there is a representation of $\mathcal{H}_N(s)$:

$$p(x)T_i = (1 - s) \frac{p(x) - p(xs_i)}{x_i - x_{i+1}} + sp(xs_i), 1 \leq i < N,$$

where $xs_i = (x_1, \ldots, x_{i+1}, x_i, \ldots)$ ($s_i$ is the transposition $(i, i+1)$);

$$p(x)w = p(qx_N, x_1, x_2, \ldots, x_{N-1}).$$

Denote the multiplication operator $p(x) \mapsto x_ip(x)$ by $x_i$, $1 \leq i \leq N$, then

$$x_iT_j = T_jx_i, j \neq i, i - 1,$$

$$x_iT_i = sT_i^{-1}x_{i+1}, x_i = sT_i^{-1}x_{i+1}T_i^{-1},$$

$$x_{i+1}w = wx_i, 1 \leq i < N,$$

$$x_1w = qwx_N.$$ 

### A.3 $q$-Dunkl operators

There are pairwise commuting operators $D_1, \ldots, D_N$ (dual to the multiplication operators) with relations:

$$D_iT_j = T_jD_i, j \neq i, i - 1,$$

$$sT_i^{-1}D_i = D_{i+1}T_i, D_i = \frac{1}{s}T_iD_{i+1}T_i,$$

$$D_{i+1}w = wD_i, 1 \leq i < N,$$

$$qD_1w = wD_N.$$ 

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They act on polynomials by

\[ p(x) D_N = \left( p(x) - s^{N-1} p(x) T_{N-1}^{-1} T_{N-2}^{-1} \cdots T_1^{-1} w \right) x_N^{-1}, \]

\[ D_i = \frac{1}{s} T_i D_{i+1} T_i = w^{-1} D_{i+1} w, \quad 1 \leq i < N. \]

The Cherednik operators satisfy:

\[ \xi_N = s^{1-N} (1 - D_N x_N), \]

\[ \xi_i = \frac{1}{s} T_i \xi_{i+1} T_i, \quad 1 \leq i < N. \]