Correction to “An optimal regularity result for Kolmogorov equations and weak uniqueness for some critical SPDEs”

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Abstract: We show uniqueness in law for the critical SPDE
\[ dX_t = AX_t dt + (-A)^{1/2} F(X_t) dt + dW_t, \quad X_0 = x \in H, \]
where \( A: \text{dom}(A) \subset H \to H \) is a negative definite self-adjoint operator on a separable Hilbert space \( H \) having \( A^{-1} \) of trace class and \( W \) is a cylindrical Wiener process on \( H \). Here \( F: H \to H \) can be locally Hölder continuous with at most linear growth (some functions \( F \) which grow more than linearly can also be considered). This leads to new uniqueness results for generalized stochastic Burgers equations and for three-dimensional stochastic Cahn-Hilliard type equations which have interesting applications. We do not know if uniqueness holds under the sole assumption of continuity of \( F \) plus growth condition as stated in Priola [37]. To get weak uniqueness we use an infinite dimensional localization principle and an optimal regularity result for the Kolmogorov equation \( \lambda u - Lu = f \) associated to the SPDE when \( F = z \in H \) is constant and \( \lambda > 0 \). This optimal result is similar to a theorem of Da Prato [7].

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1. Introduction

We establish weak uniqueness (or uniqueness in law) for critical stochastic evolution equations like
\[ dX_t = AX_t dt + (-A)^{1/2} F(X_t) dt + dW_t, \quad X_0 = x \in H. \]  
(1.1)

Here \( H \) is a separable Hilbert space, \( A: D(A) \subset H \to H \) is a self-adjoint operator of negative type such that the inverse \( A^{-1} \) is of trace class (cf. Section 1.1 and see also Remark 5). \( W = (W_t) \) is a cylindrical Wiener process on \( H \), cf. [16], [17], [26] and the references therein. We assume that there exists \( \theta \in (0, 1) \) such that

\[ F : H \to H \text{ is locally } \theta\text{-Hölder continuous and verifies } |F(x)|_H \leq C_F(1 + |x|_H), \quad x \in H, \]

(1.2)

for some constant \( C_F > 0 \). The first assumption means that \( F \) is \( \theta \)-Hölder continuous on each bounded set of \( H \). This allows to prove both weak existence and weak uniqueness for (1.1).

Assumption (1.2) can be relaxed if we assume that weak existence holds for (1.1); see Section 7 where we consider \( F \) which is only locally \( \theta \)-Hölder continuous without imposing the growth condition.

Using the analytic semigroup \( (e^{tA}) \) generated by \( A \) we consider mild solutions to (1.1), i.e.,

\[ X_t = e^{tA}x + \int_0^t e^{(t-s)A} F(X_s) ds + \int_0^t e^{(t-s)A} dW_s, \quad t \geq 0 \]

This paper is a correction of [37] which deals with SPDEs like (1) where \( F : H \to H \) is only continuous with at most linear growth. Since there is a mistake in the proof of the regularity lemma [37, Lemma 6] (see in particular the change of variable at the end of page 1319) it remains an open problem if the weak uniqueness result of [37, Theorem 1] holds under the sole hypothesis of continuity on \( F \) plus growth condition (cf. Remark 2). To prove weak uniqueness for (1) we replace the continuity condition on \( F \) with the stronger assumption that \( F \) is locally Hölder continuous. Moreover, [37, Theorem 7] is replaced by Theorem 7, which is an optimal regularity result in Hölder spaces and further [37, Sections 5.2 and 5.3] are replaced by Section 5.2. Then we basically follow the lines of [37].
(cf. Section 1.1) and prove the following result.

**Theorem 1.** Under Hypothesis 1 and assuming (1.2), for any \( x \in H \), there exists a weak mild solution defined on some filtered probability space. Moreover uniqueness in law (or weak uniqueness) holds for (1.1), for any \( x \in H \).

We do not know if weak uniqueness holds, for any \( x \in H \), when \( F : H \to H \) is only continuous with at most linear growth. It is an open problem if such more general result holds. This general result is stated in Theorem 1 of [37] but there is a mistake in the proof of Lemma 6 of [37]; see Remark 2 for more details. On the other hand existence of weak solutions holds only assuming continuity of \( F \) with at most linear growth; see Section 4.

As in [37] examples of SPDEs of the form (1.1) are considered in Section 2 (replacing the continuity of the coefficients considered in [37] with a Hölder type condition). In particular, we can deal with stochastic Burgers-type equations like

\[
du(t, \xi) = \frac{\partial^2}{\partial \xi^2}u(t, \xi)dt + h(u(t, \xi))dt + dW_t(\xi), \quad u(0, \xi) = u_0(\xi), \quad \xi \in (0, \pi),
\]

with suitable boundary conditions (cf. [25], [7] and [39]) and stochastic Cahn-Hilliard equations (cf. [19], [9], [35], [20]) like

\[
du(t, \xi) = -\Delta u(t, \xi)dt + \Delta h(u(t, \xi))dt + dW_t(\xi), \quad t > 0, \quad u(0, \xi) = u_0(\xi) \text{ on } G,
\]

with suitable boundary conditions (\( G \subset \mathbb{R}^3 \) is a regular bounded open set). We prove weak well-posedness for both SPDEs when \( h = h_1 + h_2 \) where \( h_1 \) is \( \theta \)-Hölder continuous, for some \( \theta \in (0, 1) \), and \( h_2 \) is Lipschitz continuous (see also the end of Section 2.0.1 where we consider different non-local nonlinearities like \( h(u) = u \cdot g(|u|_H) \)). Assumptions of Theorem 1 do not cover classical stochastic Burgers equations (i.e., \( h(u) = \frac{D}{2}u^2 \)) and stochastic Chan-Hilliard equations (i.e., \( h(u) = u^3 - u \)) for which strong existence and uniqueness can be proved by different methods (cf. [3] and [9]). On the other hand, in Section 7 we consider some locally Hölder continuous perturbations of classical Burgers equations (cf. Propositions 22 and 23).

We mention [41] and [2] where weak uniqueness has been investigated for stochastic evolution equations with Hölder continuous coefficients and non-degenerate multiplicative noise (the diffusion coefficient must be sufficiently close to a fixed operator). Such papers do not cover our main result. Indeed both in [41] and in [2] the term \((-A)^{1/2}F\) is replaced by \( F \) which is \( \theta \)-Hölder continuous and bounded (cf. Hypothesis 2 in [41] and hypotheses (5.4) and (5.5) in Theorem 5.6 of [2]) On the other hand, weak uniqueness for (1.1) follows by Section 4 of [7] assuming that \( F \) is \( \theta \)-Hölder continuous and bounded, \( \theta \in (0, 1) \), with \( \|F\|_{C^0} \) small enough.

To establish weak uniqueness for (1.1) we first prove an optimal regularity result for the following infinite-dimensional Kolmogorov equation

\[
\lambda u - Lu - \langle z, (-A)^{1/2}Du \rangle = f,
\]

with \( z \in H \); see Theorem 7 which is similar to a result proved in [7]. In (1.3) \( \lambda > 0, f : H \to \mathbb{R} \) is a given \( \theta \)-Hölder continuous and bounded function (i.e., \( f \in C_b^\theta (H) \)) and \( L \) is an infinite-dimensional Ornstein-Uhlenbeck operator which is formally given by

\[
Lg(x) = \frac{1}{2} \text{Tr}(D^2g(x)) + \langle Ax, Dg(x) \rangle, \quad x \in D(A),
\]

where \( Dg(x) \) and \( D^2g(x) \) denote respectively the first and second Fréchet derivatives of a regular function \( g \) at \( x \in H \) and \( \langle \cdot, \cdot \rangle \) is the inner product in \( H \) (for regularity results concerning \( L \) when \( H = \mathbb{R}^n \) see [31] and the references therein). According to Chapter 6 in [16] (see also [7] and [11]) we investigate properties of the bounded solution \( u^{(z)} : H \to \mathbb{R} \), given by

\[
u^{(z)}(x) = \int_0^\infty e^{-\lambda t}P_t^{(z)}f(x)dt, \quad x \in H;
\]
here \( P^{(z)}_t \) is an Ornstein-Uhlenbeck type semigroup associated to \( L + \langle z, (-A)^{1/2}Du \rangle \), see (1.20). When \( z = 0 \) we write \( P^{(0)}_t = P_t \) and we find the well-known Ornstein-Uhlenbeck semigroup:

\[
P_t f(x) = \mathbb{E}[f(Z^x_t)] = \mathbb{E}\left[f(e^{tA}x + \int_0^t e^{(t-s)A}dW_s)\right];
\]

\( Z^x_t \) denotes the Ornstein-Uhlenbeck process which solves (1.1) when \( F \equiv 0 \) (cf. Section 1.2).

It easy to prove that \( u^{(z)} \in C^0_b(H) \), i.e., \( u^{(z)} \) is continuous and bounded with the first Fréchet derivative \( Du^{(z)} : H \to H \) which is continuous and bounded. In Theorem 7 we prove that \( Du^{(z)}(x) \in D((-A)^{1/2}) \), for any \( x \in H, z \in H \), and there exists constants \( M_\theta > 0 \) and \( C_\theta (\lambda) > 0 \) (independent of \( z \) and \( f \)) such that

\[
\sup_{x \in H} |((-A)^{1/2}Du^{(z)}(x)|_H \leq C_\theta (\lambda) \|f\|_{C^\theta}, \quad \|((-A)^{1/2}Du^{(z)}(x)|_{C^\theta} \leq M_\theta \|f\|_{C^\theta},
\]

(1.5)

with \( \lim_{\lambda \to \infty} C_\theta (\lambda) = 0 \) (here \( [\cdot]_{C^\theta} \) stands for the Hölder seminorm, see (1.14)). The fact that in (1.5) the constant \( M_\theta \) is independent of \( \lambda \) and the fact that \( C_\theta (\lambda) \to 0 \) are important in the proof of Lemma 18; this allows to perform the localization principle. Estimates like (1.5) have not been proved in recent papers on Schauder estimates in infinite dimensions, see in particular [1], [5], [34] and the references therein. Bounds similar to (1.5) are given in Theorem 3.3 of [7]; we improve the estimates in [7] clarifying the dependence of the constants on \( \lambda \) (see the remarks before Theorem 7).

The bound

\[
\sup_{x \in H} |((-A)^{1/2}DP^{(z)}_t f(x)| = \|(-A)^{1/2}DP^{(z)}_t f\|_0 \sim \frac{t}{\gamma} \|f\|_0, \quad \text{as } t \to 0^+
\]

(1.6)

(see (3.7)) containing the singular term \( \frac{1}{\gamma} \) suggests that (1.5) cannot be improved replacing \((-A)^{1/2}\) by \((-A)^{\gamma}\), \( \gamma \in (\frac{1}{2}, 1) \); see also Chapter 6 of [16] and Remark 4.

When \( z = 0 \) we can mention related optimal regularity results in \( L^p(H, \mu) \)-spaces with respect to the Gaussian invariant measure \( \mu \) for \((P_t)\) (cf. Section 3 of [6]):

\[
\|(-A)^{1/2}Du\|_{L^p(\mu)} \leq C_p \|f\|_{L^p(\mu)}, \quad 1 < p < \infty.
\]

(1.7)

When \( f \in L^2(\mu) \) the fact that the estimate \( \|(-A)^{1/2}Du\|_{L^2(\mu)} \leq C_2 \|f\|_{L^2(\mu)} \) is sharp follows by Proposition 10.2.5 in [16].

We stress that for \( p = \infty \) in general the previous estimate (1.7) does not hold in infinite dimensions (cf. Remark 2). A counterexample is given in [18] when \((P_t)\) is associated to a stochastic heat equation in one dimension.

Concerning the SPDE (1.1) we first prove the weak existence in Section 4 (see also Remark 12). To this purpose we adapt a compactness argument already used in [23] (see also Chapter 8 in [17]). The proof of the uniqueness part of Theorem 1 is more involved and it is done in various steps (see Sections 5 and 6). In the case when \( F \in C^0_b(H, H) \) we first consider equivalence between mild solutions and solutions to the martingale problem of Stroock and Varadhan [40] (cf. Section 5.1). This allows to use some uniqueness results available for the martingale problem (cf. Theorems 15, 16 and 17). On this respect we point out that an infinite-dimensional generalization of the martingale problem is given in Chapter 4 of [21].

In Section 5.2 we prove weak uniqueness when \( F \in C^0_b(H, H) \) assuming an additional condition. More precisely, we show that there exists a constant \( \tilde{C}_0 > 0 \) such that if \( F \in C^0_b(H, H) \) verifies

\[
\sup_{x \in H} |F(x) - z|_H = \|F - z\|_0 < \tilde{C}_0
\]

(1.8)

for some \( z \in H \) then weak uniqueness holds for (1.1) for any initial condition \( x \in H \). Note that \( \tilde{C}_0 \) is a constant small enough, depending on \( \theta \) and \( \|F\|_{C^\theta} \).

Estimate (1.5) is needed in order to prove that

\[
\|\langle F - z, (-A)^{1/2}Du^{(z)}\rangle\|_{C^\theta} \leq \frac{1}{2} \|f\|_{C^\theta}, \quad f \in C^0_b(H),
\]

(1.9)

for \( \lambda \) large enough if \( F \) verifies (1.8) (see Lemma 18). We obtain weak uniqueness using (1.9) and adapting an argument used in finite dimension in [40] and [29] (see the proof of Theorem 3.3 in [29]).
This argument is simpler than the other approach to get uniqueness passing through the study of the equation \( \lambda u - Lu - \langle z, (-A)^{1/2} Du \rangle = f + \langle F - z, (-A)^{1/2} Du \rangle \) (cf. Sections 5.2 and 5.3 in [37]).

In Section 5.3 we prove uniqueness in law when \( F \in C^0_b(H, H) \) (removing condition (1.8)). To this purpose we adapt the localization principle which has been introduced in [40] (cf. Theorem 16). In Section 6 we complete the proof of Theorem 1, showing weak uniqueness under (1.2). To this purpose we truncate \( F \) and prove uniqueness for the martingale problem up to a stopping time (cf. Theorem 17). Section 7 considers the more general case of \( F \) which is only locally \( \theta \)-Hölder continuous without imposing a growth condition.

We finally mention recent papers which investigate pathwise uniqueness for SPDEs with additive noise like (1.1) when \( (-A)^{1/2} F \) is replaced by a measurable drift \( F \) (cf. [11], [12] and see also [4] for the case of semilinear stochastic heat equations). In [11] and [12] pathwise uniqueness holds for \( \mu \)-a.e. \( x \in H \). It is still not clear if pathwise uniqueness holds, for any initial \( x \in H \), when \( F \in C_0(H, H) \).

On the other hand if \( F \in C^0_b(H, H) \) then pathwise uniqueness holds, for any \( x \in H \); see [10].

**Remark 2.** As we say before in [37, Theorem 1] it is claimed weak uniqueness for (1.1), for any \( x \in H \), assuming only that \( F : H \to H \) is continuous with at most a linear growth. Actually we do not know if this result holds or not. In fact the proof of [37, Theorem 1] uses [37, Theorem 7] which shows in particular that there exists \( C > 0 \), independent of \( f \) and \( z \), such that

\[
\sup_{x \in H} \|(-A)^{1/2} Du(x)\|_H \leq C \sup_{x \in H} |f(x)|, \quad f \in C^1_b(H)
\]

(\( u(z) \) is defined in (1.4)). This estimate corresponds to the case \( p = \infty \) of (1.7). The proof of (1.10) is based on [37, Lemma 6] but there is a mistake in the proof of such lemma (see in particular the change of variable at the end of page 1319 in [37]). On the other hand, a counterexample given in [18] shows that in general the \( L^\infty \)-bound (1.10) fails to hold in infinite dimensions even with \( z = 0 \). Theorem 1 in [37] could be true with a different proof.

**Remark 3.** If we replace \( (-A)^{1/2} F \) in (1.1) with \( (-A)^{1/2-\epsilon} F \) with \( \epsilon \in (0, 1/2] \) then following Sections 5 and 6 one could prove uniqueness in law for \( F : H \to H \) continuous with at most a linear growth. To this purpose one can use that for any \( x \in H \), \( f \in B_b(H) \), one has \( Du(x) \in D((-A)^{1/2-\epsilon}) \) and

\[
\|(-A)^{1/2-\epsilon} Du(x)\| \leq c \|f\|_0
\]

(this follows by the estimate \( \|(-A)^{1/2-\epsilon} D \|_t f\|_0 \leq C\|f\|_0, \quad t > 0 \), which can be obtained in the same way we get (3.7)). However such assumption excludes the examples of Sections 2 and 7.

**Remark 4.** It is not clear if the uniqueness result holds for (1.1) when \( (-A)^{1/2} \) is replaced by \( (-A)^{\gamma} \), \( \gamma \in (1/2, 1) \). We believe that for \( \gamma \in (1/2, 1) \) there should exist a \( \theta \)-Hölder continuous and bounded drift \( F_\gamma : H \to H, \theta \in (0, 1) \), and \( x_\gamma \in H \) such that weak uniqueness fails for \( dX_t = AX_t dt + (-A)^{\gamma} F_\gamma(X_t) dt + dW_t, \quad X_0 = x_\gamma \) (on the other hand, weak existence holds, cf. Remark 12). In this sense (1.1) can be considered as a critical SPDE.

### 1.1. Notations and preliminaries

Let \( H \) be a real separable Hilbert space. Denote its norm and inner product by \( \|\cdot\|_H \) and \( \langle \cdot, \cdot \rangle \) respectively. Moreover \( B(H) \) indicates its Borel \( \sigma \)-algebra. Concerning (1.1) as in [7], [11] and [12] we assume

**Hypothesis 1.** \( A : D(A) \subseteq H \to H \) is a negative definite self-adjoint operator with domain \( D(A) \) (i.e., there exists \( \omega > 0 \) such that \( \langle Ax, x \rangle \leq -\omega \|x\|^2_H, \quad x \in D(A) \)). Moreover \( A^{-1} \) is a trace class operator.

In the sequel we will concentrate on an infinite dimensional Hilbert space \( H \). Since \( A^{-1} \) is compact, there exists an orthonormal basis \( (e_k) \) in \( H \) and an infinite sequence of positive numbers \( (\lambda_k) \) such that

\[
Ae_k = -\lambda_k e_k, \quad k \geq 1, \quad \text{and} \quad \sum_{k \geq 1} \lambda_k^{-1} < \infty.
\]

(1.11)
Note that $D(A)$ is dense in $H$. We denote by $\mathcal{L}(H)$ the Banach space of bounded and linear operators $T : H \to H$ endowed with the operator norm $\| \cdot \|_\mathcal{L}$. The operator $A$ generates an analytic semigroup $(e^{tA})$ on $H$ such that $e^{tA}e_k = e^{-\lambda_k t}e_k, \ t \geq 0$. Remark that
\[
\|(-A)^{1/2}e^{tA}\|_\mathcal{L} = \sup_{k \geq 1} \{(\lambda_k)^{1/2}e^{-\lambda_k t}\} \leq \frac{c}{t^{1/2}}, \quad t > 0,
\]
with $c = \sup_{u \geq 0} u e^{-u^2} = (2e)^{-1/2}$. We will also use orthogonal projections with respect to $(e_k)$:
\[
\pi_m = \sum_{j=1}^m e_j \otimes e_j, \quad \pi_m x = \sum_{k=1}^m x^{(k)}e_k, \quad \text{where } x^{(k)} = \langle x, e_k \rangle, \ x \in H, \ m \geq 1.
\]

Let $(E, | \cdot |_E)$ be a real separable Banach space. We denote by $B_b(H, E)$ the Banach space of all real, bounded and Borel functions on $H$ with values in $E$, endowed with the supremum norm $\|f\|_0 = \sup_{x \in H} |f(x)|_E, f \in B_b(H, E)$. Moreover $C_b(H, E) \subset B_b(H, E)$ indicates the subspace of all bounded and continuous functions. We denote by $C_b^k(H, E) \subset B_b(H, E), k \geq 1$, the space of all functions $f : H \to E$ which are bounded and Fréchet differentiable on $H$ up to the order $k \geq 1$ with all the derivatives $D^i f$ bounded and continuous on $H, 1 \leq j \leq k$.

Moreover $C_b^0(H, E), \ \theta \in (0, 1)$, denotes the Banach space of all functions $f : H \to E$ which are $\theta$-Hölder continuous and bounded endowed with the norm
\[
\|f\|_{C^\theta} = \|f\|_0 + |f|_{C^\theta},
\]
where $|f|_{C^\theta} = \sup_{x \neq x' \in H} \{(f(x) - f(x'))|_E |x - x'|_E^{-\theta}\}$.

We also set $B_b(H) = B_b(H, \mathbb{R}), C_b(H) = C_b(H, \mathbb{R}), C_b^0(H) = C_b^0(H, \mathbb{R})$ and $C_b^k(H) = C_b^k(H, \mathbb{R})$.

Let $\theta \in (0, 1).$ We say that $F : H \to H$ is locally $\theta$-Hölder continuous if for any bounded set $B \subset H$, we have that $F : B \to H$ is $\theta$-Hölder continuous. Note that this condition implies that $F$ is is bounded on each bounded set of $H$.

We will deal with the SPDE (1.1) where $W = (W_t) = (W(t))$ is a \textit{cylindrical Wiener} process on $H$. Thus $W$ is formally given by “$W_t = \sum_{k \geq 1} W_t^{(k)}e_k$” where $(W^{(k)})_{k \geq 1}$ are independent real Wiener processes and $(e_k)$ is the basis of eigenvectors of $A$ (cf. [16, 26] and [17]). The next definition is meaningful for $F : H \to H$ which is only continuous because of (1.12).

A \textit{weak mild solution} to (1.1) is a sequence $(\Omega, F, (\mathcal{F}_t), \mathbb{P}, W, X)$, where $(\Omega, F, (\mathcal{F}_t), \mathbb{P})$ is a filtered probability space on which it is defined a cylindrical Wiener process $W$ and an $\mathcal{F}_t$-adapted, $H$-valued continuous process $X = (X_t) = (X_t)_{t \geq 0}$ such that, $\mathbb{P}$-a.s.,
\[
X_t = e^{tA}x + \int_0^t (-A)^{1/2}e^{(t-s)A}F(X_s)ds + \int_0^t e^{(t-s)A}dW_s, \quad t \geq 0.
\]
(hence $X_0 = x, \ \mathbb{P}$-a.s.). We say that \textit{uniqueness in law holds for (1.1) for any $x \in H$ if given two weak mild solutions $X$ and $Y$ (possibly defined on different filtered probability spaces and starting at $x \in H$), we have that $X$ and $Y$ have the same law on $B(C([0, \infty); H))$ which is the Borel $\sigma$-algebra of $C([0, \infty); H)$ (this is the Polish space of all continuous functions from $[0, \infty)$ into $H$ endowed with the metric of the uniform convergence on bounded intervals; cf. [28] and [17]). Note that the stochastic convolution
\[
W_A(t) = \int_0^t e^{-(t-s)A}dW_s = \sum_{k \geq 1} \int_0^t e^{-(t-s)\lambda_k}e_kW^{(k)}(s)
\]
is well defined since the series converges in $L^2(\Omega; H)$, for any $t \geq 0$. Moreover $W_A(t)$ is a Gaussian random variable with values in $H$ with distribution $N(0, Q_t)$ where
\[
Q_t = \int_0^t e^{2sA}ds = (-2A)^{-1}(I - e^{2tA}), \quad t \geq 0,
\]
is the covariance operator (see also Chapter 1 in [16]). Note that $W_A$ has a continuous version with values in $H$ (see Corollary 2 in [27]); if we assume in addition that $(-A)^{-1+\delta}$ is of trace class, for some $\delta \in (0, 1)$, then this fact follows by Theorem 5.11 in [17].
Equivalence between different notions of solutions for (1.1) are clarified in [16] and [26] (see also [30] for a more general setting). If we write \( X^{(k)}(t) = X^{(k)}_t = \langle X(t), e_k \rangle, k \geq 1 \), (1.1) is equivalent to the system
\[
X^{(k)}_1 = x^{(k)} - \lambda_k \int_0^t X^{(k)}_s ds + \lambda_k^{1/2} \int_0^t F^{(k)}(X_s) ds + W^{(k)}_t, \quad k \geq 1,
\]
(1.17)
or to \( X^{(k)}_t = e^{-\lambda_k t} x^{(k)} + \int_0^t e^{-\lambda_k (t-s)} \lambda_k^{1/2} F^{(k)}(X_s) ds + \int_0^t e^{-\lambda_k (t-s)} dW^{(k)}_s \), for \( k \geq 1, t \geq 0 \), with
\[ F(x) = \sum_{k \geq 1} F^{(k)}(x) e_k, \quad x \in H. \]

We will also use the natural filtration of \( X \) which is denoted by \( (\mathcal{F}^X_t) \): \( \mathcal{F}^X_t = \sigma(X_s : 0 \leq s \leq t) \) is the \( \sigma \)-algebra generated by the r.v. \( X_s, 0 \leq s \leq t \) (cf. Chapter 2 in [21]).

**Remark 5.** We point out that Theorem 1 holds under the following more general hypothesis: \( A : D(A) \subseteq H \to H \) is self-adjoint, \( \langle Ax, x \rangle \leq 0, x \in D(A) \), and \( (I - A)^{-1} \) is of trace class, with \( I = I_H \). Indeed in this case one can rewrite equation (1.1) in the form
\[
\frac{dX_t}{dt} = (A - I)X_t dt + (I - A)^{-1/2}(I - A)^{-1/2}X_t + (-A)^{-1/2}(I - A)^{-1/2}F(X_t) dt + dW_t,
\]
\( X_0 = x \). Now the linear operator \( \hat{A} = I - A \) and the nonlinear term \( \hat{F}(x) = [(I - A)^{-1/2}x + (-A)^{1/2}(I - A)^{-1/2}F(x)], x \in H \), verify Hypothesis 1 and condition (1.2) respectively.

### 1.2. A generalised Ornstein-Uhlenbeck semigroup

Let us fix \( z \in H \). We will consider generalised Ornstein-Uhlenbeck operators like
\[
L^{(z)} g(x) = \frac{1}{2} \tr(D^2 g(x)) + \langle x, ADg(x) \rangle + \langle z, (I - A)^{1/2}Dg(x) \rangle, \quad x \in H, \quad g \in C^2_{ci}(H).
\]
(1.18)

Here \( C^2_{ci}(H) \) denotes the space of regular cylindrical functions. We say that \( g : H \to \mathbb{R} \) belongs to \( C^2_{ci}(H) \) if there exist elements \( e_{i_1}, \ldots, e_{i_n} \) of the basis \( (e_k) \) of eigenvectors of \( A \) and a \( C^2 \)-function \( \tilde{g} : \mathbb{R}^n \to \mathbb{R} \) with compact support such that
\[
g(x) = \tilde{g}(\langle x, e_{i_1} \rangle, \ldots, \langle x, e_{i_n} \rangle), \quad x \in H.
\]
(1.19)

By writing the stochastic equation \( dX_t = AX_t dt + (I - A)^{1/2}dW_t, X_0 = x \) in mild form as \( X_t = e^{tA}x + \int_0^t e^{(t-s)A}dW_s + \int_0^t (I - A)^{1/2}e^{(t-s)A}z ds \), one can easily check that the Markov semigroup associated to \( L^{(z)} \) is a generalised Ornstein-Uhlenbeck semigroup \( (P^{(z)}_t) \):
\[
P^{(z)}_t f(x) = \int_H f(e^{tA}x + y + \Gamma_t z) N(0, Q_t)(dy), \quad f \in B_b(H), \quad x \in H,
\]
setting \( \Gamma_t = (A)^{1/2} \int_0^t e^{sA} ds, \quad \Gamma_t z = (A)^{-1/2}z - e^{tA}z = \sum_{k \geq 1} \frac{(1 - e^{-\lambda_k t})}{(\lambda_k)^{3/2}} z^{(k)} e_k. \)
(1.20)

The case \( z = 0 \). i.e., \( (P^{(0)}_t) = (P_t) \) corresponds to the well-known Ornstein-Uhlenbeck semigroup (see, for instance, [16], [17], [7], [11] and [12]) which has a unique invariant measure \( \mu = N(0, S) \) where \( S = -\frac{1}{4} A^{-1} \). It is also well-known (see, for instance, [16] and [17]) that under Hypothesis 1, \( (P_t) \) is strong Feller, i.e., \( P_t(B_b(H)) \subseteq C_b(H), t > 0 \). Indeed we have \( e^{tA}(H) \subset Q_1^{1/2}(H), t > 0 \), or, equivalently,
\[
\Lambda_t = Q_t^{-1/2} e^{tA} = \sqrt{2} (I - A)^{-1/2} e^{tA}(I - e^{2tA})^{-1/2} \in \mathcal{L}(H), \quad t > 0.
\]
(1.21)
Moreover \( P_t(B_b(H)) \subseteq C_b^k(H), t > 0 \), for any \( k \geq 1 \). Following the same proof of Theorem 6.2.2 in [16] one can show that under Hypothesis 1, for any \( z \in H \), we have \( P^{(z)}_t(B_b(H)) \subseteq C_b^k(H), t > 0 \), for any \( k \geq 1 \). Moreover, for any \( f \in C_b(H), t > 0 \), the following formula for the directional derivative along a direction \( h \) holds:
\[
D_h P^{(z)}_t f(x) = \langle DP^{(z)}_t f(x), h \rangle = \int_H \langle \Lambda_t h, Q_t^{-1/2} y \rangle f(e^{tA}x + y + \Gamma_t z) \mu_t(dy), \quad x, h \in H.
\]
(1.22)
where $\mu_t = N(0, Q_t)$ (cf. $(1.16)$) and the mapping: $y \mapsto \langle \Lambda_t h, Q^{-1}_t y \rangle$ is a centered Gaussian random variable on $(H, \mathcal{B}(H), \mu_t)$ with variance $|\Lambda_t h|^2$ (cf. Theorem 6.2.2 in [16]).

We deduce that, for $t > 0$, $g \in C_b(H), h, k \in H$,

$$
\|D_h P_t^{(z)} g\|_0 \leq |\Lambda_t h|_H \|g\|_0, \quad \|D^2_{hk} P_t^{(z)} g\|_0 \leq |\Lambda_t h|_H |\Lambda_t k|_H \|g\|_0,
$$

(1.23)

where $D_h P_t^{(z)} g = (D_t P_t^{(z)} g)(h)$, $D^2_{hk} P_t^{(z)} g = (D^2_t P_t^{(z)} g(h, k))$. We have

$$
\Lambda_t e_k = \sqrt{2} (\lambda_k)^{1/2} e^{-t \lambda_k} (1 - e^{-2t \lambda_k})^{-1/2} e_k, \quad \|\Lambda_t\|_{L^\infty} \leq C_1 t^{-\frac{1}{2}}, \quad t > 0, \quad C_1 = \sqrt{2} \sup_{u \geq 0} [ue^{-u^2}(1 - e^{-2u^2})^{-1/2}].
$$

(1.24)

and so $\|D_h P_t^{(z)} g\|_0 \leq \sqrt{2} \|h\|_H \|g\|_0, \|D^2_{hk} P_t^{(z)} g\|_0 \leq \sqrt{2} \|h\|_H \|g\|_0 |h|_H |k|_H$.

To study equation (1.3) we will investigate regularity properties of the continuous function

$$
u^{(z)}(x) = \int_0^\infty e^{-t \lambda} P_t^{(z)} f(x) dt, \quad x \in H, \ f \in C_b(H)
$$

(1.25)

we drop the dependence of $\nu^{(z)}(x)$ on $\lambda$; see also the remark below.

**Remark 6.** Let us fix $z \in H$. For any $\lambda > 0$, $u^{(z)} : H \to \mathbb{R}$ given in (1.25) belongs to $C_b(H)$. Moreover, also the mapping: $t \mapsto D_h P_t^{(z)} f(x)$ is right-continuous on $(0, \infty)$, for $x, h \in H$.

Since $\sup_{x \in H} |D_t P_t^{(z)} f(x)|_H \leq \frac{c(f)_0}{\sqrt{t}}, \ t > 0$, differentiating under the integral sign, one shows that there exists the directional derivative $D_h u^{(z)}(x)$ at any point $x \in H$ along any direction $h \in H$. Moreover, it is not difficult to prove that there exists the first Frechet derivative $D_h u^{(z)}(x)$ at any $x \in H$ and $D_h u^{(z)} : H \to H$ is continuous and bounded (cf. the proof of Lemma 9 in [11]). Finally we have the formula

$$
D_h u^{(z)}(x) = \int_0^\infty e^{-t \lambda} D_h P_t^{(z)} f(x) dt, \quad x, h \in H
$$

(2.1)

and the straightforward estimate $\|D_h u^{(z)}\|_0 \leq c(\lambda) \|f\|_0$ with $c(\lambda)$ independent of $z \in H$. We will prove a better regularity result for $D_h u^{(z)}$ in Section 3.

**2. Examples**

**2.0.1. One-dimensional stochastic Burgers-type equations**

We consider

$$
du(t, \xi) = \frac{\partial^2}{\partial \xi^2} u(t, \xi) dt + \frac{\partial}{\partial \xi} h(\xi, u(t, \xi)) dt + dW_t(\xi), \quad u(0, \xi) = u_0(\xi), \ \xi \in (0, \pi),
$$

(2.1)

with Dirichlet boundary condition $u(t, 0) = u(t, \pi) = 0, \ t > 0$ (cf. [25] and [7] and see the references therein). Here $u_0 \in H = L^2(0, \pi)$ and $A = \frac{\partial^2}{\partial \xi^2}$ with Dirichlet boundary conditions, i.e. $D(A) = H^2(0, \pi) \cap H^1_0(0, \pi)$. It is well-known that $A$ verifies Hypothesis 1. The eigenfunctions are $e_k(\xi) = \sqrt{2/\pi} \sin(k \xi), \ \xi \in \mathbb{R}, \ k \geq 1$.

The eigenvalues are $-\lambda_k$, where $\lambda_k = k^2$. The cylindrical noise is $W_t(\xi) = \sum_{k \geq 1} W_t^{(k)} e_k(\xi)$ (cf. [17]). Classical stochastic Burgers equations with $h(\xi, u) = \frac{u^2}{2}$ are examples of locally monotone SPDEs and strong uniqueness holds (cf. [3]). In [25] strong uniqueness is proved assuming that $h(\xi, \cdot)$ is locally Lipschitz with a linearly growing Lipschitz constant.

Here we assume that $h : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ is continuous in both variables; moreover $h = h_1 + h_2$ where $h_1$ is $\theta$-H"older continuous in the second variable, for some $\theta \in (0, 1)$, and $h_2$ is Lipschitz continuous in the second variable, uniformly with respect to the first variable. Hence we assume that there exists $C_\theta > 0$ such that

$$
|h_1(\xi, s) - h_1(\xi, s')| \leq C_\theta |s - s'|^\theta,
$$
To check that \( \partial \) introduces the Sobolev spaces \( H \), we assume \( u \) verifies (1.13). Using that \((-A)^{1/2}\) is self-adjoint and integrating by parts we find (we use inner product in \( L^2(0, \pi) \) and the fact that \( y(0) = y(\pi) = 0 \))

\[
\langle (-A)^{-1/2} \partial \xi y, x_N \rangle = \langle \partial \xi y, (-A)^{-1/2} x_N \rangle = -\langle y, \partial \xi (-A)^{-1/2} x_N \rangle.
\]

Now \( \partial \xi (-A)^{-1/2} x_N(\xi) = \sqrt{2/\pi} \sum_{k=1}^{N} x^{(k)}(\xi) \cos(k\xi) \) and so \( |\partial \xi (-A)^{-1/2} x_N|^2_{L^2(0, \pi)} = |x_N|^2_{L^2(0, \pi)} \).

It follows that, for any \( N \geq 1 \), \( \langle (-A)^{-1/2} \partial \xi y, x_N \rangle \leq |y|_{L^2(0, \pi)} |x|_{L^2(0, \pi)} \) and we easily get the assertion. Hence \( F = T \circ S \) verifies (1.2) and SPDE (2.1) is well-posed in weak sense, for any initial condition \( u_0 \in L^2(0, \pi) \).

Note that instead of \( h(\xi, u) \) one can consider different non-local nonlinearities like, for instance, \( u g(|u|) \) assuming that \( g : \mathbb{R} \to \mathbb{R} \) is bounded and locally \( \theta \)-Hölder continuous, for some \( \theta \in (0, 1) \).

Indeed let \( M > 0 \); if \( u, v \in B = \{ x \in H : |x|_H \leq M \} \) we have

\[
\int_0^\pi |u(t)g(|u|_H) - v(t)g(|v|_H)|^2 dt \\
\leq 2 \int_0^\pi |u(t)|^2 |g(|u|_H) - g(|v|_H)|^2 dt + 2 \int_0^\pi |g(|v|_H)|^2 |u(t) - v(t)|^2 dt \\
\leq 2C_M,\theta \int_0^\pi |u(t)|^2 dt |u - v|_{L^2(\mathbb{R})}^2 + 2\|g\|_0^2 \int_0^\pi |u(t) - v(t)|^2 dt \\
\leq K |u - v|_{L^2(\mathbb{R})}^2,
\]

for some constant \( K \) possibly depending on \( M, g \) and \( \theta \), and assumption (1.2) follows easily.

### 2.0.2. Three-dimensional stochastic Cahn-Hilliard equations

The Cahn-Hilliard equation is a model to describe phase separation in a binary alloy and some other media, in the presence of thermal fluctuations; we refer to [35] for a survey on this model. The stochastic Cahn-Hilliard equation has been recently much investigated under monotonicity conditions on \( h \) which allow to prove pathwise uniqueness; in one dimension a typical example is \( h(s) = s^3 - s \) (see [19], [9], [35], [20] and the references therein).

We can treat such SPDE in one, two or three dimensions. Let us consider Neumann boundary conditions in a regular bounded open set \( G \subset \mathbb{R}^3 \). For the sake of simplicity we concentrate on the cube \( G = (0, \pi)^3 \). The equation has the form

\[
\begin{aligned}
\left\{
\begin{array}{ll}
du(t, \xi) = -\triangle^2 u(t, \xi) dt + \Delta \xi h(u(t, \xi)) dt + dW_t(\xi), & t > 0, \\
\frac{\partial}{\partial n}u = \frac{\partial}{\partial n}(\Delta u) = 0 & \text{on } \partial G,
\end{array}
\right.
\end{aligned}
\]

(2.3)

where \( \triangle^2 \) is the bilaplacian and \( n \) is the outward unit normal vector on the boundary \( \partial G \). Let us introduce the Sobolev spaces \( H^j(G) = W^{j,2}(G) \) and the Hilbert space \( H \),

\[
H = \{ f \in L^2(G) : \int_G f(\xi) d\xi = 0 \}.
\]

We assume \( u_0 \in H \) and define \( D(A) = \{ f \in H^4(G) \cap H : \frac{\partial}{\partial n} f = \frac{\partial}{\partial n}(\Delta f) = 0 \text{ on } \partial G \} \), \( Af = -\triangle^2 f \), \( f \in D(A) \). Using also the divergence theorem, we have \( A : D(A) \to H \).
The square root has domain \( D((−A)^{1/2}) = \{ f \in H^2(G) \cap H : \frac{∂}{∂n} f = 0 \text{ on } ∂G \} \): \( (−A)^{1/2} f = \triangle f \), \( f \in D((−A)^{1/2}) \). Note that \( A \) is self-adjoint with compact resolvent and it is negative definite with \( \omega = 1 \) (cf. Hypothesis 1). The eigenfunctions are

\[
e_k(\xi_1, \xi_2, \xi_3) = (\sqrt{2/\pi})^3 \cos(k_1 \xi_1) \cos(k_2 \xi_2) \cos(k_3 \xi_3), \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3,
\]

\( k = (k_1, k_2, k_3) \in \mathbb{N}^3, \) \( k \neq (0, 0, 0) = 0^* \). The corresponding eigenvalues are \( −\lambda_k \), where \( \lambda_k = (k_1^2 + k_2^2 + k_3^2)^{1/2} \). Since \( \sum_{k \in \mathbb{N}^3, k \neq 0^*} \lambda_k^{-1} < +\infty \) we see that \( A \) verifies Hypothesis 1. The cylindrical Wiener process is \( W_t(\xi) = \sum_{k \in \mathbb{N}^3, k \neq 0^*} W_t^{(k)} e_k(\xi). \) Note that \( \triangle h(u(t, \xi)) = \triangle \left[ h(u(t, \xi)) - \int_0^t h(u(s, \xi))d\xi \right]. \)

Assuming that \( h = h_1 + h_2 \), with \( h_1, h_2 : \mathbb{R} \rightarrow \mathbb{R} \), where \( h_2 \) is Lipschitz continuous and \( h_1 \) is \( \theta \)-Hölder continuous, \( \theta \in (0, 1) \), we can define \( F : H \rightarrow H \) as follows:

\[
F(x)(\xi) = h(x(\xi)) - \int_0^t h(x(\xi))d\xi, \quad x \in H, \xi \in G.
\]

It is not difficult to prove that \( F \) verifies (1.2). Thus SPDE (2.3) is well-posed in weak sense, for any initial condition \( u_0 \in H \).

### 3. An optimal regularity result

Let \( f \in C^0_b(H), \theta \in (0, 1), \) and fix \( z \in H \). Here we are interested in the regularity property of the function \( u(z) : H \rightarrow \mathbb{R} \) given in (1.25). By Remark 6 we know that \( u(z) \in C^1_b(H) \) and we have a formula for the directional derivative:

\[
D_h u(z)(x) = \langle Du(z)(x), h \rangle = \int_0^\infty e^{-\lambda t} D_h P^z_t f(x)dt, \quad x, h, \in H, \lambda > 0. \tag{3.1}
\]

The next assertion (i) is similar to Theorem 3.3 in [7] (recall that Section 3 of [7] considers a little different generalized OU operator \( L(z) \) where the term \( \langle z, (−A)^{1/2} Dg(x) \rangle \) in (1.18) is replaced by \( \langle z, Dg(x) \rangle \)). Such theorem shows that \( \|(-A)^{1/2} D_u(z)\|_{C^0} \leq \|(-A)^{1/2} D_u(z)\|_{C^0} + \|(-A)^{1/2} D_u(z)\|_0 \leq K_0(\lambda)\|\|f\|_{C^\theta} \) with \( K_0(\lambda) \) which is independent of \( f \) (see also Remark 9). Below we will improve the result by clarifying the dependence of \( K_0(\lambda) \) on \( \lambda \). Indeed the fact that in (3.2) we have \( C_0(\lambda) \rightarrow 0 \) will be important in the proof of Lemma 18.

**Theorem 7.** Let \( f \in C^0_b(H), \lambda > 0, z \in H \) and consider \( u(z) \in C^1_b(H) \) given in (1.25). The following assertions hold.

(i) For any \( x \in H \), \( D_h u(z)(x) \in D((-A)^{1/2}) \) and \( (-A)^{1/2} D_h u(z) \in C^0_b(H, H) \). There exist constants \( C_0(\lambda) \) and \( M_0 > 0 \) with \( C_0(\lambda) \) which is decreasing in \( \lambda \in (0, \infty) \), \( \lim_{\lambda \rightarrow \infty} C_0(\lambda) = 0 \), such that

\[
\sup_{x \in H} \left[ \sum_{k \geq 1} \lambda_k(D_{ek} u(z)(x))^2 \right]^{1/2} \leq \|(-A)^{1/2} D u(z)\|_0 \leq C_0(\lambda)\|f\|_{C^\theta}; \tag{3.2}
\]

\[
\|(-A)^{1/2} D u(z)\|_{C^0} \leq M_0\|f\|_{C^\theta}. \tag{3.3}
\]

(ii) Let \( (f_n) \subset C^0_b(H) \) be such that \( \sup_{n \geq 1} \|f_n\|_{C^\theta} \leq C < \infty \) and \( f_n(x) \rightarrow f(x), \) \( x \in H \). Define

\[
u_n(z)(x) = \int_0^\infty e^{-\lambda t} P^z_t f_n(x)dt.
\]

Then

\[
\langle (-A)^{1/2} D u_n(z)(x), h \rangle \rightarrow \langle (-A)^{1/2} D u(z)(x), h \rangle \quad \text{as } n \rightarrow \infty, \quad h, x, z \in H. \tag{3.4}
\]

**Proof.** Let us prove (i). The first preliminary step of the proof uses an interpolation argument similar to the one used in [7]. In the second step we argue similarly to [38].

**I step.** We first prove there there exists \( C_0 \) > 0 such that

\[
\|(-A)^{1/2} D P^z_t f\|_0 \leq \frac{C_0}{t^{1-\theta}}\|f\|_{C^\theta}, \quad t > 0, \quad f \in C^0_b(H) \tag{3.5}
\]
Similarly one can prove that, for any $g \in C_b(H)$, we have (3.5) with $c_\theta = c_2^{1-\theta}(2\epsilon)^{-\theta/2}k_\theta$; here $k_\theta > 0$ verifies

$$k_\theta^{-1} \|\varphi\|_{C^\theta} \leq \|\varphi\|_{C_b(H), C_b^\theta(0, \infty)} \leq k_\theta \|\varphi\|_{C^\theta}, \quad \varphi \in C_b^\theta(H).$$

Similarly one can prove that, for any $h \in H$, $l \in D((-A)^{1/2})$, $f \in C_b^\theta(H)$,

$$\|\langle D^2P_t^{(z)} f(-A)^{1/2}l, h\rangle\|_0 \leq \frac{\tilde{c} \epsilon}{t^{\frac{\theta}{2}-\frac{\theta}{2}}} \|l\|_H \|h\|_H \|f\|_{C^\theta}, \quad t > 0.$$

To prove the previous estimate, let us first consider $g \in C_b(H)$; we have

$$\|\langle D^2P_t^{(z)} g(-A)^{1/2}l, h\rangle\|_0 \leq \|(-A)^{1/2} \Lambda t l\|_H \|\Lambda t h\|_H \|g\|_0$$

$$\leq c_2C_1 \|l\|_H \|h\|_H \frac{1}{t^{1/2}} \|g\|_0, \quad t > 0$$

(cf. (1.23)). If $g \in C_b^\theta(H)$ then

$$\langle D^2P_t^{(z)} g(x)(-A)^{1/2}l, h\rangle = \langle DP_t^{(z)} ((Dg(\cdot)(-A)^{1/2} e^{tA})\rangle(x), h\rangle$$

and

$$\|\langle D^2P_t^{(z)} g(-A)^{1/2}l, h\rangle\|_0 \leq \|(-A)^{1/2} e^{tA}l\|_H \|\Lambda t h\|_H \|Dg\|_0$$

$$\leq c_1 \|l\|_H \|h\|_H \frac{1}{t} \|Dg\|_0, \quad t > 0.$$
To prove (3.3) we argue as in the proof of Theorem 4.2 in [38]. We fix \( x, h \in H \). We have
\[
\left| (-A)^{1/2} Du^{(z)}(x+h) - (-A)^{1/2} Du^{(z)}(x) \right|_H \leq u_h^{(z)}(x) + v_h^{(z)}(x),
\]
where
\[
u_h^{(z)}(x) = \int_0^{\frac{|h|^2}{|h|}} e^{-\lambda t} \left| (-A)^{1/2} DP_t^{(z)} f(x+h) - (-A)^{1/2} DP_t^{(z)} f(x) \right|_H dt;
\]
\[
\nu_h^{(z)}(x) = \int_0^{\infty} e^{-\lambda t} \left| (-A)^{1/2} DP_t^{(z)} f(x+h) - (-A)^{1/2} DP_t^{(z)} f(x) \right|_H dt.
\]
In order to estimate \( u_h^{(z)}(x) \) we use (3.5). We find
\[
\sup_{x \in H} u_h^{(z)}(x) \leq c_0 \left\| f \right\|_{C^0} \int_0^{\frac{|h|^2}{|h|}} t^{\frac{\theta}{2} - 1} dt \leq C_0 \left\| f \right\|_{C^0} |h|^\theta_H.
\]
Concerning \( v_h^{(z)}(x) \) we will use estimate (3.9). Let \( B_1 = \{ x \in H : |x|_H \leq 1 \} \). Recall that \( D((-A)^{1/2}) \cap B_1 \) is dense in \( B_1 \). For \( t > 0 \), we have:
\[
\left| (-A)^{1/2} DP_t^{(z)} f(x+h) - (-A)^{1/2} DP_t^{(z)} f(x) \right|_H = \sup_{l \in D((-A)^{1/2})} \left| \langle DP_t^{(z)} f(x+h) - DP_t^{(z)} f(x), (-A)^{1/2} I \rangle \right|.
\]
Let us consider \( l \in D((-A)^{1/2}) \), with \( |l|_H \leq 1 \). We write
\[
\left| \langle DP_t^{(z)} f(x+h) - DP_t^{(z)} f(x), (-A)^{1/2} I \rangle \right| = \left| \int_0^1 \langle D^2 P_t^{(z)} f(x+sh) h, (-A)^{1/2} I \rangle ds \right|
\]
\[
\leq c_0 t^{\frac{\theta}{2} - \frac{\theta}{2}} |l|_H \left\| f \right\|_{C^0}, \quad t > 0.
\]
and so \( \left| (-A)^{1/2} DP_t^{(z)} f(x+h) - (-A)^{1/2} DP_t^{(z)} f(x) \right|_H \leq c_0 t^{\frac{\theta}{2} - \frac{\theta}{2}} |h|_H \left\| f \right\|_{C^0}, \quad t > 0 \). We obtain
\[
\sup_{x \in H} v_h^{(z)}(x) \leq c_0 \left\| f \right\|_{C^0} |h|_H \int_0^{\infty} \left( t^{\frac{\theta}{2} - \frac{\theta}{2}} \right) dt \leq C_0'' \left\| f \right\|_{C^0} |h|_H \left\| f \right\|_{C^0} = C_0'' \left\| f \right\|_{C^0}.
\]
By the previous estimates on \( u_h^{(z)} \) and \( v_h^{(z)} \) we deduce easily (3.3).

To prove (ii) we fix \( x \in H \) and \( l \in D((-A)^{1/2}) \). We write (see (1.22)) for \( t > 0 \)
\[
D_{(-A)^{1/2}} P_t^{(z)} f(x) = \langle DP_t^{(z)} f_n(x), (-A)^{1/2} I \rangle = \int_H \langle A_t(-A)^{1/2} I, Q_t^{\frac{\theta}{2}} y \rangle f_n(e^{tA} x + y + \Gamma_t z) \mu_t(dy).
\]
We can pass to the limit as \( n \to \infty \) by the Lebesgue convergence theorem and get \( D_{(-A)^{1/2}} P_t^{(z)} f_n(x) \to D_{(-A)^{1/2}} P_t^{(z)} f(x) \) as \( n \to \infty \).

Similarly, using also the estimate \( |DP_t^{(z)} f_n(x)|_H \leq \frac{c_0}{t^{1-\frac{\theta}{2}}} \left\| f_n \right\|_{C^0} \leq \frac{c_0 C_0''}{t^{1-\frac{\theta}{2}}} \), \( t > 0 \), we have, for any \( l \in D((-A)^{1/2}) \), \( x \in H \),
\[
\lim_{n \to \infty} \langle Du_n^{(z)}(x), (-A)^{1/2} I \rangle = \langle Du^{(z)}(x), (-A)^{1/2} I \rangle.
\]
We deduce easily that (3.4) holds.

\( \square \)

**Remark 8.** This following fact will be useful in the sequel: if \( G \in C_b(H, H) \), then (3.4) implies that
\[
\lim_{n \to \infty} \left\langle (-A)^{1/2} Du_n^{(z)}(x), G(x) \right\rangle = \left\langle (-A)^{1/2} Du^{(z)}(x), G(x) \right\rangle, \ x, z \in H.
\]

**Remark 9.** (a) Actually Theorem 3.3 in [7] shows that \( \left\| (-A)^{1/2} Du^{(z)} \right\|_{C^0} = \left[ (-A)^{1/2} Du^{(z)} \right]_{C_b^0} + \left\| (-A)^{1/2} Du^{(z)} \right\|_0 \leq K_\theta(\lambda) \left\| f \right\|_{C^0} \) with \( K_\theta(\lambda) \) independent of \( f \in C_b^0(H) \) and \( z \in H \). The fact that \( K_\theta(\lambda) \) in [7] depends also on \( \lambda \) follows from estimate (2.15) in [7]. Indeed in such estimate one has also to consider the supremum norm \( \left\| b_i \right\|_0 \).

(b) We do not know if estimates (3.2) and (3.3) hold in a stronger form with \( \left\| f \right\|_{C^0} \) replaced by \( \left\| f \right\|_{C^0} \) (see (1.14)). This happens, for instance, in the finite-dimensional case considered in [14].
4. Proof of weak existence of Theorem 1 (only assuming continuity of $F$)

In this section we require that

$$F : H \to H \text{ is continuous and verifies } |F(x)|_H \leq C_F(1 + |x|_H), \ x \in H,$$  \hspace{1cm} (4.1)

for some constant $C_F > 0$. We will prove weak existence by adapting a compactness approach of [23]. This approach is inspired by [13] (it is also explained in Chapter 8 of [17]).

Let us fix $x \in H$. To construct the solution we start with some approximating mild solutions. We introduce, for each $m \geq 1$,

$$A_m = A \circ \pi_m, \ A_m e_k = -\lambda_k e_k, \ k = 1, \ldots, m,$$

$A_m e_k = 0$, $k > m$; here $\pi_m = \sum_{j=1}^{m} e_j \otimes e_j$ ($(e_j)$ is the basis of eigenvectors of $A$; see (1.13)).

For each $m$ there exists a weak mild solution $X_m = (X_m(t))_{t \geq 0}$ on some filtered probability space, possibly depending on $m$ (such solution can also be constructed by the Girsanov theorem, see [22], [17] and [11]).

Usually the mild solutions $X^m$ are constructed on a time interval $[0, T]$. However there is a standard procedure based on the Kolmogorov extension theorem to define the solutions on $[0, \infty)$. On this respect, we refer to Remark 3.7, page 303, in [28].

We know that

$$X_m(t) = e^{tA}x + \int_0^t e^{(t-s)A}(-A_m)^{1/2}F(X_m(s))ds + \int_0^t e^{(t-s)A}dW_s, \ t \geq 0.$$  \hspace{1cm} (4.2)

Recall that, for any $t \geq 0$, the stochastic convolution $W_A(t) = \int_0^t e^{(t-s)A}dW_s$ is a Gaussian random variable with law $N(0, Q_t)$. Let $p > 2$ and $q = \frac{p}{p-1} < 2$. We find (using also (1.12) and the Hölder inequality)

$$|X_m(t)|_H^p \leq c_p(|e^{tA}x|_H^p + \left| \int_0^t e^{(t-s)A}(-A_m)^{1/2}F(X_m(s))ds \right|_H^p + |W_A(t)|_H^p)$$

$$\leq c_T|x|_H^p + c_T\left( \int_0^t (t-s)^{-q/2}ds \right)^{p/q} \cdot \int_0^t (1 + |X_m(s)|_H^p)ds + c_p|W_A(t)|_H^p$$

$$\leq C_T|x|_H^p + C_T + C_T \int_0^t |X_m(s)|_H^p ds + C_T|W_A(t)|_H^p, \ t \in [0, T].$$

By the Gronwall lemma we find the bound

$$\sup_{m \geq 1} \sup_{t \in [0, T]} \mathbb{E}|X_m(t)|_H^p = C_T < \infty.$$  \hspace{1cm} (4.3)

The mild solution $X$ will be a weak limit of solutions $(X_m)$. To this purpose we need some compactness results. The next result is proved in [23] (the proof uses that $(e^{tA})$ is a compact semigroup).

**Proposition 10.** If $0 < \frac{1}{p} < \alpha \leq 1$ then the operator $G_{\alpha} : L^p(0, T; H) \to C([0, T]; H)$

$$G_{\alpha}f(t) = \frac{\sin \pi \alpha}{\pi} \int_0^t (t-s)^{\alpha-1}e^{(t-s)A}f(s)ds, \ f \in L^p(0, T; H), \ t \in [0, T],$$

is compact.

Below we consider a variant of the previous result. In the proof we use estimate (1.12).

**Proposition 11.** Let $p > 2$. Then the operator $Q : L^p(0, T; H) \to C([0, T]; H)$,

$$Qf(t) = \int_0^t (-A)^{1/2}e^{(t-s)A}f(s)ds, \ f \in L^p(0, T; H), \ t \in [0, T],$$

is compact.

**Proof.** Since the proof is similar to the one of Proposition 10 we only give a sketch of the proof. Denote by $| \cdot |_p$ the norm in $L^p(0, T; H)$. According to the infinite dimensional version of the Ascoli-Arzelá theorem one has to show that
(i) For arbitrary $t \in [0, T]$ the sets $\{Qf(t) : |f|_p \leq 1\}$ are relatively compact in $H$.
(ii) For arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that
\[ |Qf(t) - Qf(s)|_H \leq \varepsilon, \quad \text{if} \quad |f|_p \leq 1, \quad |t - s| \leq \delta, \quad s, t \in [0, T]. \] (4.4)

To check (i) let us fix $t \in (0, T]$ and define operators $Q^t$ and $Q^{\varepsilon,t}$ from $L^p(0; T; H)$ into $H$, for $\varepsilon \in (0, t)$,
\[ Q^t f = Qf(t), \quad Q^{\varepsilon,t} f = \int_{t-\varepsilon}^t (A)^{1/2} e(t-s)A f(s) ds, \quad f \in L^p(0, T; H). \]

Since $Q^{\varepsilon,t} f = e^A \int_{t-\varepsilon}^t (A)^{1/2} e(t-s)A f(s) ds$ and $e^A, \varepsilon > 0$, is compact, the operators $Q^{\varepsilon,t}$ are compact. Moreover, by using (1.12) and the Hölder inequality (setting $q = \frac{p}{1-p} < 2$)
\[ |Q^t f - Q^{\varepsilon,t} f|_H = \left| \int_{t-\varepsilon}^t (A)^{1/2} e(t-s)A f(s) ds \right|_{H^p} \leq M \left( \int_{t-\varepsilon}^t (t-s)^{-q/2} ds \right)^{1/q} \left( \int_{t-\varepsilon}^t |f(s)|^p ds \right)^{1/p} \leq M_0 \varepsilon^{1/2+1/q} |f|_p \]
with $-\frac{1}{2} + \frac{1}{q} > 0$. Hence $Q^{\varepsilon,t} \to Q^t$, as $\varepsilon \to 0^+$, in the operator norm so that $Q^t$ is compact and (i) follows. Let us consider (ii). For $0 \leq t \leq t + u \leq T$ and $|f|_p \leq 1$, we have
\[ |Qf(t + u) - Qf(t)|_H \leq \int_0^u \left| (A)^{1/2} e(t + u - s)A - (A)^{1/2} e(t - s)A \right| \|f(s)\|_H ds \]
\[ + \int_u^{t+u} \left| (A)^{1/2} e(t + u - s)A - (A)^{1/2} e(s)A \right| \|f(s)\|_H ds \leq M_p \left( \int_0^u s^{-q/2} ds \right)^{1/q} \left( \int_0^{t+u} \left| (A)^{1/2} e(u+s)A - (A)^{1/2} e^{sA} \right|^q \|f(s)\|_H^q ds \right)^{1/q} = I_1 + I_2. \]

It is clear that $I_1 = M_p u^{1/2-1/p} \to 0$ as $u \to 0$.

Moreover, for $s > 0$, $(A)^{1/2} e^{sA}$ is compact; indeed $(A)^{1/2} e^{sA} e_k = (\lambda_k)^{1/2} e^{s\lambda_k} e_k$ and $(\lambda_k)^{1/2} e^{s\lambda_k} \to 0$ as $k \to \infty$. It follows that \[ \|e^{uA} - (A)^{1/2} e^{sA} - (A)^{1/2} e^{sA}\|_H \to 0 \] as $u \to 0$ for arbitrary $s > 0$.

Since \[ \|e^{uA} - (A)^{1/2} e^{sA}\|_H \leq \frac{2M q^q}{s^{q/2}}, \quad s > 0, \quad u \geq 0, \quad q < 2, \quad \text{by the Lebesgue's dominated convergence theorem} \]
$\|e^{uA} - (A)^{1/2} e^{sA}\|_H \to 0$ as $u \to 0$. Thus the proof of (ii) is complete. □

**Proof of the existence part of Theorem 1.** Let $x \in H$. We proceed in two steps.

**I Step.** Let $(X_m)$ be solutions of (4.2). We prove that their laws $\{\mathcal{L}(X_m)\}$ form a tight family of probability measures on $\mathcal{B}(C([0, \infty); H))$.

To this purpose it is enough to show that for each $T > 0$ the laws $\{\mathcal{L}(X_m)\}$ form a tight family of probability measures on $\mathcal{B}(C([0, T]; H))$.

Let us fix $p > 2$ and $T > 0$. We know by (4.3) that there exists a constant $c_p > 0$ such that $\mathbb{E}|X_m(t)|_H^p \leq c_p, m \geq 1, t \in [0, T]$. It follows that
\[ \sup_{m \geq 1} \mathbb{E} \int_0^T |F_m(X_m(t))|_H^p < \infty. \] (4.5)

with $\pi_m \circ F = F_m$, since $|F_m(x)|_H \leq CF(1 + |x|_H), m \geq 1$.

In order to prove the tightness of $\{\mathcal{L}(X_m)\}$ on $\mathcal{B}(C([0, T]; H))$ we note that
\[ X_m(t) = e^{tA}x + Q(F_m(X_m))(t) + W_A(t), \quad t \in [0, T]. \] (4.6)

Let $Z_m(t) = e^{tA}x + Q(F_m(X_m))(t), t \in [0, T]$. It is not difficult to prove that the tightness of $\{\mathcal{L}(Z_m)\}$ implies the tightness of $\{\mathcal{L}(Z_m + W_A)\} = \{\mathcal{L}(X_m)\}$ on $\mathcal{B}(C([0, T]; H))$.

Thus it remains to show the tightness of $\{\mathcal{L}(Z_m)\}$. By (4.5) and Chebyshev’s inequality, for $\varepsilon > 0$ one can find $r > 0$ such that for all $m \geq 1$
\[ \mathbb{P} \left( \left( \int_0^T |F_m(X_m(s))|_H^p ds \right)^{1/p} \leq r \right) > 1 - \varepsilon. \] (4.7)
By Proposition 11 (recall that $| \cdot |_p$ denotes the norm in $L^p(0, T; H)$) the set

$$K = \{ e^{(\cdot)}x + Qg(\cdot) : |g|_p \leq r \} \subset C([0, T]; H)$$

is relatively compact. Since $P(Z_m \in K) = \mathcal{L}(Z_m)(K) > 1 - \varepsilon$, for any $m \geq 1$, the tightness follows.

II Step. By the Skorokhod representation theorem, possibly passing to a subsequence of $(X_m)$ still denoted by $(X_m)$, there exists a probability space $(\hat{\Omega}, \mathcal{F}, \hat{P})$ and random variables $\hat{X}$ and $\hat{X}_m$, $m \geq 1$, defined on $\hat{\Omega}$ with values in $C(0, \infty; H)$ such that the law of $X_m$ coincide with the law of $X_m$, $m \geq 1$, and moreover

$$\hat{X}_m \rightarrow \hat{X}, \hat{P} - \text{a.s.}$$

Let us fix $k_0 \geq 1$. Let $\hat{X}_m^{(k_0)} = (\hat{X}_m, \epsilon_{k_0})$. Recall that $\pi_m \circ F = F_m$. It is not difficult to prove that the processes $(M_m^{(k_0)})_{m \geq 1}$

$$M_m^{(k_0)}(t) = \begin{cases} \hat{X}_m^{(k_0)}(t) - x^{(k_0)} + \lambda_{k_0} \int_0^t \hat{X}_m^{(k_0)}(s)ds - \lambda_{k_0}^{1/2} \int_0^t F^{(k_0)}(\hat{X}_m(s))ds, & k_0 \leq m \\ \hat{X}_m^{(k_0)}(t) - x^{(k_0)} + \lambda_{k_0} \int_0^t \hat{X}_m^{(k_0)}(s)ds, & k_0 > m, \ t \geq 0, \end{cases}$$

are square-integrable continuous $\mathcal{F}^{\hat{X}_m}$-martingales on $(\hat{\Omega}, \mathcal{F}, \hat{P})$ with $M^{(k_0)}(0) = 0$. Moreover the quadratic variation process $(M^{(k_0)})(t) = t, m \geq 1$ (cf. Section 8.4 in [17]).

Passing to the limit as $m \rightarrow \infty$ we find that

$$M^{(k)}(t) = \hat{X}^{(k)}(t) - x^{(k)} + \lambda_k \int_0^t \hat{X}^{(k)}(s)ds - \lambda_k^{1/2} \int_0^t F^{(k)}(\hat{X}(s))ds, \ t \geq 0,$$

is a square-integrable continuous $\mathcal{F}^{\hat{X}}$-martingale with $M^{(k)}(0) = 0$. To check the martingale property, let us fix $0 < s < t$. We know that $\hat{P}[M^{(k)}(t) - M^{(k)}(s)/\mathcal{F}^{\hat{X}_m}] = 0, m \geq 1$.

Consider $0 \leq s_1 < \ldots < s_n \leq s \leq t$. For any $h_j \in C_b(H)$, we have, for $m \geq k_0$,

$$\hat{P}\left[\hat{X}_m^{(k_0)}(t) - \hat{X}_m^{(k_0)}(s) + \lambda_{k_0} \int_s^t \hat{X}_m^{(k_0)}(r)dr - \lambda_{k_0}^{1/2} \int_s^t F^{(k_0)}(\hat{X}_m(r))dr \cdot \prod_{j=1}^n h_j(\hat{X}_m(s_j))\right] = 0.$$  \hfill (4.9)

Using that $|F^{(k_0)}(x)| \leq C_F(1 + |x|_H)$ and that, for any $T > 0$, sup sup $\hat{E}[|\hat{X}_m(t)|^p_H] \leq C < \infty$ (cf. (4.3)) by the Vitali convergence theorem we get easily that (4.9) holds when $\hat{X}_m$ is replaced by $\hat{X}$ (note that this assertion could be proved by using only the dominated convergence theorem). Then we obtain that $M^{(k)}$ is a square-integrable continuous $\mathcal{F}^{\hat{X}}$-martingale.

Moreover, by a limiting procedure, arguing as before, we find that $((M^{(k)}(t))^2 - t)$ is a martingale. It follows that $M^{(k)}$ is a real Wiener process on $(\hat{\Omega}, \mathcal{F}, \hat{P})$.

Hence, for any $k \geq 1$, we find that there exists a real Wiener process $M^{(k)}$ such that

$$\hat{X}^{(k)}(t) = x^{(k)} - \lambda_k \int_0^t \hat{X}^{(k)}(s)ds + \lambda_k^{1/2} \int_0^t F^{(k)}(\hat{X}(s))ds + M^{(k)}(t).$$

We prove now that $(M^{(k)})_{k \geq 1}$ are independent Wiener processes.

We fix $N \geq 2$ and introduce the processes $(S_m^{(N)})_{m \geq 1}$, $S_m^{(N)}(t) = (M_m^{(1)}(t), \ldots, M_m^{(N)}(t))_t \geq 0$, with values in $\mathbb{R}^N$. The components of $S_m^{(N)}$ are square-integrable continuous $\mathcal{F}_t^\hat{X}$-martingales. Moreover the quadratic covariation $(M_m^{(1)}, M_m^{(j)})_t = \delta_{ij}t$.

Passing to the limit as before we obtain that also the $\mathbb{R}^N$-valued process $(S^{(N)}(t)) = (M^{(1)}(t), \ldots, M^{(N)}(t))_t \geq 0$, has components which are square-integrable continuous $\mathcal{F}_t^{\hat{X}}$-martingales with quadratic covariation $(M^{(i)}, M^{(j)})_t = \delta_{ij}t$. Note that $S^{(0)}(0) = 0$, $\hat{P}$-a.s.

By the Lévy characterization of the Brownian motion (see Theorem 3.16 in [28]) we have that $(M^{(1)}(t), \ldots, M^{(N)}(t))$ is a standard Wiener process with values in $\mathbb{R}^N$. Since $N$ is arbitrary, $(M^{(k)})_{k \geq 1}$ are independent real Wiener processes and the proof is complete.
Remark 12. Following the previous method one can prove existence of weak mild solution even for
\[ dX_t = AX_t dt + (-A)^\gamma F(X_t) dt + dW_t, \quad X_0 = x \in H, \]
with \( \gamma \in (0, 1) \) and \( F : H \to H \) continuous and having at most a linear growth.

5. Proof of weak uniqueness when \( F \in C_0^\theta(H, H) \), for some \( \theta \in (0, 1) \)

To get the weak uniqueness of Theorem 1 when \( F \in C_0^\theta(H, H) \) we first show the equivalence between martingale solutions and mild solutions. Indeed for martingale problems some useful uniqueness results are available even in infinite dimensions (see, in particular, Theorems 15, 16 and 17).

5.1. Mild solutions and martingale problem

We formulate the martingale problem of Stroock and Varadhan [40] for the operator \( L \) given below in (5.9) and associated to (1.1). We stress that an infinite-dimensional generalization of the martingale problem is proposed in Chapter 4 of [21]. Here we follow Appendix of [36]. In such appendix some extensions and modifications of theorems given in Sections 4.5 and 4.6 of [21] are proved.

The results of this section hold more generally when \( F \in C_b(H, H) \) in (1.1).

We use the space \( C_{2d}^\gamma(H) \) of regular cylindrical functions (cf. (1.19)). We deal with the following linear operator \( L : D(L) \subset C_b(H) \to C_b(H) \), with \( D(L) = C_{2d}^\gamma(H) \):

\[
L f(x) = \frac{1}{2} \text{Tr}(D^2 f(x)) + \langle x, AD f(x) \rangle + \langle F(x), (-A)^{1/2} D f(x) \rangle \\
= L f(x) + \langle F(x), (-A)^{1/2} D f(x) \rangle, \quad f \in D(L), \ x \in H.
\]

Remark 13. We stress that the linear operator \((L, D(L))\) in (5.9) is countably pointwise determined, i.e., it verifies Hypothesis 17 in [36]. Indeed, arguing as in Remark 8 of [36], one shows that there exists a countable set \( \mathcal{H}_0 \subset D(L) \) such that for any \( f \in D(L) \), there exists a sequence \((f_n) \subset \mathcal{H}_0\) satisfying

\[
\lim_{n \to \infty} (\| f - f_n \|_0 + \| L f_n - L f \|_0) = 0.
\]

Let \( x \in H \). An \( H \)-valued stochastic process \( X = (X_t) = (X_t)_{t \geq 0} \) defined on some probability space \((\Omega, F, \mathbb{P})\) with continuous trajectories is a solution of the martingale problem for \((L, \delta_x)\) if, for any \( f \in D(L) \),

\[
M_t(f) = f(X_t) - \int_0^t L f(X_s) ds, \quad t \geq 0, \quad \text{is a martingale}
\]

(with respect to the natural filtration \((\mathcal{F}_t^X)\)) and, moreover, \( X_0 = x, \mathbb{P}\)-a.s..

If we do not assume that \( F \) is bounded then in general \( M_t(f) \) is only a local martingale because in general \( L f \) is not a bounded function.

We say that the martingale problem for \( L \) is well-posed if, for any \( x \in H \), there exists a martingale solution for \((L, \delta_x)\) and, moreover, uniqueness in law holds for the martingale problem for \((L, \delta_x)\).

Equivalence between mild solutions and martingale solutions has been proved in a general setting in [30] even for SPDEs in Banach spaces. We only give a sketch of the proof of the next result for the sake of completeness (see also Chapter 8 in [17]).

Proposition 14. Let \( F \in C_b(H, H) \) and \( x \in H \).

(i) If \( X \) is a weak mild solution to (1.1) with \( X_0 = x, \mathbb{P}\)-a.s., then \( X \) is also a solution of the martingale problem for \((L, \delta_x)\).

(ii) Viceversa, if \( X = (X_t) \) is a solution of the martingale problem for \((L, \delta_x)\) on some probability space \((\Omega, F, \mathbb{P})\) then there exists a cylindrical Wiener process on \((\Omega, F, (\mathcal{F}_t^X), \mathbb{P})\) such that \( X \) is a weak mild solution to (1.1) on \((\Omega, F, (\mathcal{F}_t^X), \mathbb{P})\) with initial condition \( x \).
Proof. (i) Let $X$ be a weak mild solution to (1.1) with $X_0 = x$, $\mathbb{P}$-a.s. defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Let $f \in D(\mathcal{L})$. Since $f$ depends only on a finite number of variables by the Itô formula we obtain that $f(X_t) - \int_0^t \mathcal{L} f(X_s)ds$ is an $\mathcal{F}_t$-martingale, for any $f \in D(\mathcal{L})$. We get easily the assertion since $\mathcal{F}_t^X \subset \mathcal{F}_t$, $t \geq 0$.

(ii) Let $X$ be a solution to the martingale problem for $(\mathcal{L}, \delta_x)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

I Step. Let $X^{(k)}_t = (X_t, e_k)$ and $F(x) = \sum_{k \geq 1} F^{(k)}(x)e_k$. We show that, for any $k \geq 1$,

$$X^{(k)}_t - x^{(k)}_t + \lambda_k \int_0^t X^{(k)}_s ds - \int_0^t (\lambda_k)^{1/2} F^{(k)}(X_s)ds$$

is a one-dimensional Wiener process $W^{(k)} = (W^{(k)}_t)$.

Let $k \geq 1$. We will modify a well known argument (see, for instance, the proof of Proposition 5.3.1 in [21]). By the definition of martingale solution, it follows easily that if $f(x) = x^{(k)} = \langle x, e_k \rangle$, $x \in H$, the process

$$M^{(k)}_t = X^{(k)}_t - x^{(k)}_t + \lambda_k \int_0^t X^{(k)}_s ds - \int_0^t b_k(s)ds$$

is a continuous local martingale, (5.3)

which is $\mathcal{F}_t^X$-adapted, with $b_k(s) = (\lambda_k)^{1/2} F^{(k)}(X_s)$ (to this purpose one has to approximate the unbounded function $k(x) = \langle x, e_k \rangle$ by functions $k_n(x) = \eta_n(x)e_k$, $n \geq 1$, where $\eta \in C_0^\infty(\mathbb{R})$ is such that $\eta(s) = 1$ for $|s| \leq 1$). Then using $f(x) = \langle x, e_k \rangle^2$, $x \in H$, we find that

$$N^{(k)}_t = (X^{(k)}_t)^2 - (x^{(k)}_t)^2 + 2\lambda_k \int_0^t (X^{(k)}_s)^2ds - 2 \int_0^t b_k(s)X^{(k)}_s ds - t,$$

is also a continuous local martingale. On the other hand, starting from (5.3) and applying the Itô formula (cf. Theorem 5.2.9 in [21]), we get

$$(X^{(k)}_t)^2 = (x^{(k)}_t)^2 - 2\lambda_k \int_0^t (X^{(k)}_s)^2ds + 2 \int_0^t b_k(s)X^{(k)}_s ds + 2 \int_0^t b_k(s)dM^{(k)}_s + \langle M^{(k)} \rangle_t,$$

where $(\langle M^{(k)} \rangle_t)$ is the variation process of $M^{(k)}$. Comparing this identity with (5.4) we deduce: $N^{(k)}_t - 2 \int_0^t b_k(s)dM^{(k)}_s = \langle M^{(k)} \rangle_t - t$ and so $\langle M^{(k)} \rangle_t = t$ (a continuous local martingale of bounded variation is constant). By the Lévy martingale characterization of the Wiener process (see Theorem 5.2.12 in [21]) we get that $M^{(k)}$ is a real Wiener process.

II Step. We prove that the previous Wiener processes $W^{(k)} = M^{(k)}$ are independent.

We fix any $N \geq 2$ and prove that $W^{(k)}$, $k = 1, \ldots, N$ are independent. We will argue similarly to the first step. By using functions $f(x) = x^jx^k$, $i, k \in \{1, \ldots, N\}$, we get that $(W^{(j)}, W^{(k)})_t = \delta_{jk} t$. Again by the Lévy martingale characterization of the Wiener process (cf. Theorem 3.16 in [28]) we get that $(W^{(1)}, \ldots, W^{(N)})$ is an $N$-dimensional standard Wiener process. It follows that $\{W^{(k)}\}_{k=1,\ldots,N}$ are independent real Wiener processes. \[\square\]

For the martingale problem for $\mathcal{L}$ in (5.9) we have the following uniqueness result (we refer to Corollary 21 in [36]; see also Theorem 4.4.6 in [21] and Theorem 2.2 in [30]).

**Theorem 15.** Suppose the following two conditions:

(i) for any $x \in H$, there exists a martingale solution for $(\mathcal{L}, \delta_x)$;

(ii) for any $x \in H$, any two martingale solutions $X$ and $Y$ for $(\mathcal{L}, \delta_x)$ have the same one-dimensional marginal laws (i.e., for $t \geq 0$, the law of $X_t$ is the same as $Y_t$ on $\mathcal{B}(H)$).

Then the martingale problem for $\mathcal{L}$ is well-posed.

Throughout Section 5 we will apply the previous result and also the next localization principle for $\mathcal{L}$ (cf. Theorem 26 in [36]).

**Theorem 16.** Suppose that for any $x \in H$ there exists a martingale solution for $(\mathcal{L}, \delta_x)$. Suppose that there exists a family $\{U_j\}_{j \in J}$ of open sets $U_j \subset H$ with $\cup_{j \in J} U_j = H$ and linear operators $\mathcal{L}_j$ with the same domain of $\mathcal{L}$, i.e., $\mathcal{L}_j : D(\mathcal{L}) \subset C_b(H) \to C_b(H)$, $j \in J$ such that

i) for any $j \in J$, the martingale problem for $\mathcal{L}_j$ is well-posed.

ii) for any $j \in J$, $f \in D(\mathcal{L})$, we have $\mathcal{L}_j f(x) = \mathcal{L} f(x)$, $x \in U_j$.

Then the martingale problem for $\mathcal{L}$ is well-posed.
In Sections 6 and 7 we treat \textit{possibly unbounded} $F$; we will prove uniqueness by truncating $F$ and using uniqueness for the martingale problem up to a stopping time. According to Section 4.6 of [21] this leads to the concept of \textit{stopped martingale problem} for $\mathcal{L}$ which we define now.

Let us fix an open set $U \subset H$ and consider the Kolmogorov operator $\mathcal{L}$ in (5.9) with $F \in C^b_b(H, H)$.

Let $x \in H$. A stochastic process $Y = (Y_t)_{t \geq 0}$ with values in $H$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with continuous paths is a solution of the stopped martingale problem for $(\mathcal{L}, \delta_x, U)$ if $Y_0 = x$, $\mathbb{P}$-a.s. and the following conditions hold:

(i) $Y_t = Y_{t\wedge \tau}$, $t \geq 0$, $\mathbb{P}$-a.s., where
\[ \tau = \tau^U = \inf\{t \geq 0 : Y_t \not\in U\} \] (5.5)
($\tau = +\infty$ if the set is empty; this exit time $\tau$ is an $\mathcal{F}^Y_t$-stopping time);

(ii) for any $f \in D(\mathcal{L}) = C^2_{c\mathbb{R}}(H)$, $f(Y_t) - \int_0^{t\wedge \tau} \mathcal{L}f(Y_s)ds$, $t \geq 0$, is a $\mathcal{F}^Y_t$-martingale.

A key result says, roughly speaking, that if the (global) martingale problem for an operator is well-posed then also the stopped martingale problem for such operator is well-posed for any choice of the open set $U$ and for any initial condition $x$ (we refer to Theorem 22 in [36]; see also the beginning of Section A.3 for a comparison between this result and Theorem 4.6.1 in [21]). We state this result for the operator $\mathcal{L}$ in (5.9).

\textbf{Theorem 17.} Suppose that the martingale problem for $\mathcal{L}$ is well-posed.

Then also the stopped martingale problem for $(\mathcal{L}, \delta_x, U)$ is well-posed for any $x \in H$ and for any open set $U$ of $H$. In particular uniqueness in law holds for the stopped martingale problem for $(\mathcal{L}, \delta_x, U)$, for any $x \in H$ and $U$ open set in $H$.

\section{Weak uniqueness when $F \in C^b_b(H, H)$ and $\|F - z\|_0 < \tilde{C}_0$}

Here we show that there exists a constant $\tilde{C}_0 > 0$ small enough (depending on $\theta$ and $\|F\|_{C^g}$) such that if $F \in C^b_b(H, H)$ verifies in addition
\[ \sup_{x \in H} |F(x) - z|_H = \|F - z\|_0 < \tilde{C}_0 \] (5.6)
for some $z \in H$, then uniqueness in law holds for (1.1) for any initial $x \in H$.

To this purpose we will also use Proposition 14 on equivalence between mild solutions and martingale solutions. We start with a lemma which is a consequence of Theorem 7. We consider the resolvent
\[ R^{(z)}(x) = u^{(z)}(x) = \int_0^\infty e^{-\lambda t} F_t^{(z)} f(x)dt, \quad f \in B_0(H), \quad x \in H, \quad \lambda > 0. \] (5.7)

\textbf{Lemma 18.} There exists $\lambda_0 = \lambda_0(\|F\|_{C^g}, \theta) > 0$ large enough and $\tilde{C}_0 = \tilde{C}_0(\|F\|_{C^g}, \theta) > 0$ small enough such that if $F \in C^b_b(H, H)$ verifies (5.6) for some $z \in H$ then for any $\lambda \geq \lambda_0$, $g \in C^b_b(H)$,
\[ \|\langle F - z, A^{1/2}DR^{(z)}_{\lambda}g \rangle\|_{C^g} \leq \frac{1}{2} \|g\|_{C^g}. \] (5.8)

\textbf{Proof.} Let $g \in C^b_b(H)$, $z \in H$ and $\lambda > 0$. We are considering the map: $x \mapsto \langle F(x) - z, A^{1/2}DR^{(z)}_{\lambda}g(x) \rangle$ from $H$ into $\mathbb{R}$. By Theorem 7 we know that $A^{1/2}DR^{(z)}_{\lambda}g \in C^b_b(H, H)$. Moreover,
\[ \|\langle F - z, (A)^{1/2}DR^{(z)}_{\lambda}g \rangle\|_{C^g} = \|\langle F - z, (A)^{1/2}DR^{(z)}_{\lambda}g \rangle\|_0 \]
\[ + \|\langle F - z, (A)^{1/2}DR^{(z)}_{\lambda}g \rangle\|_{C^g}. \]

Recall that given two $\theta$-Hölder continuous and bounded functions $l, m$ we have $[lm]_{C^g} \leq \|l\|_0[m]_{C^g} + \|m\|_0[l]_{C^g}$. We get
\[ \|\langle F - z, (A)^{1/2}DR^{(z)}_{\lambda}g \rangle\|_{C^g} \leq \tilde{C}_0 \|\langle (A)^{1/2}DR^{(z)}_{\lambda}g \rangle\|_{C^g} \]
\[ + \|F\|_{C^g} \|\langle (A)^{1/2}DR^{(z)}_{\lambda}g \rangle\|_0. \]
By Theorem 7 there exist $M_\theta$ and $C_\theta(\lambda) \downarrow 0$ as $\lambda \to \infty$ such that
\[ \|(-A)^{1/2}DR_{\lambda}^{(z)}g\|_{C^*} \leq C_\theta(\lambda)\|g\|_{C^*}, \quad \|(-A)^{1/2}DR_{\lambda}^{(z)}g\|_{C^*} \leq M_\theta\|g\|_{C^*}, \quad \lambda > 0. \]

It follows that
\[ \|([-F - z], (-A)^{1/2}DR_{\lambda}^{(z)}g)\|_{C^*} \leq (\tilde{C}_0M_\theta + C_\theta(\lambda)\|F\|_{C^*})\|g\|_{C^*}, \quad \lambda > 0. \]

On the other hand $\|([-F - z], (-A)^{1/2}DR_{\lambda}^{(z)}g)\|_{C^*} \leq \tilde{C}_0\|g\|_{C^*}, \lambda > 0$. By choosing $\lambda \geq \lambda_0$ with $\lambda_0$ large enough and $\tilde{C}_0$ small enough, we obtain the assertion. \hfill \Box

**Lemma 19.** Let $x \in H$ and consider the SPDE (1.1). If there exists $z \in H$ such that (5.6) holds (the constant $\tilde{C}_0$ is given in Lemma 18) then we have uniqueness in law for (1.1).

**Proof.** By Section 4, for any $x \in H$, there exists a weak mild solution starting at $x$. Equivalently, by Proposition 14, for any $x \in H$, there exists a solution to the martingale problem for $(\mathcal{L}, \delta_x)$.

Taking into account Proposition 14, we will prove that given two martingale solutions $X^1$ and $X^2$ for $(\mathcal{L}, \delta_x)$ we have that the law of $X^1_t$ coincides with the law of $X^2_t$ on $\mathcal{B}(H)$, for any $t \geq 0$. By Theorem 15 we will deduce that $X^1$ and $X^2$ have the same law on $\mathcal{B}(C([0, \infty); H))$.

Thanks to Lemma 18 we will adapt an argument used in finite dimension in [40] and [29] (see the proof of Theorem 3.3 in [29]).

Let us fix $x_0 \in H$ and let $X = (X_t)$ be a martingale solution for $(\mathcal{L}, \delta_{x_0})$ (defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$); recall that
\[ Lf(x) = Lf(x) + \langle F(x), (-A)^{1/2}Df(x) \rangle, \quad f \in D(\mathcal{L}), \quad x \in H, \] (5.9)
where $Lf(x) = \frac{1}{2}\text{Tr}(D^2f(x)) + \langle x, ADf(x) \rangle$ and $D(\mathcal{L}) = C^2_{cil}(H)$ the space of all regular cylindrical functions (cf. (1.19)). By the martingale property we have, for $f \in D(\mathcal{L})$,
\[ \mathbb{E}[f(X_t)] = f(x_0) + \mathbb{E}\int_0^t \mathcal{L}f(X_t)dt. \]

Integrating over $[0, \infty)$ with respect to $e^{-\lambda t}$ and using the Fubini theorem we get
\[ \int_0^\infty e^{-\lambda t}\mathbb{E}[f(X_t)]dt = \frac{f(x_0)}{\lambda} + \frac{1}{\lambda}\int_0^\infty e^{-\lambda t}\mathbb{E}[\mathcal{L}f(X_t)]dt. \]

Hence, introducing the bounded linear operators $G^{\lambda, x_0} : B_b(H) \to \mathbb{R}$:
\[ G^{\lambda, x_0}h = \int_0^\infty e^{-\lambda t}\mathbb{E}[h(X_t)]dt, \quad h \in B_b(H), \quad \lambda > 0, \] (5.10)
we can write
\[ \lambda G^{\lambda, x_0} f = f(x_0) + G^{\lambda, x_0}(\mathcal{L}f), \quad \lambda > 0. \] (5.11)

Next we proceed in some steps.

**I step.** We check that (5.11) holds also for any $f \in C^2_{cil, b}(H)$ with $D(\mathcal{L}) \subset C^2_{cil, b}(H)$.

We say that $f : H \to \mathbb{R}$ belongs to $C^2_{cil, b}(H)$ if there exist elements $e_{i_1}, \ldots, e_{i_m}$ of the basis $(e_k)$ of eigenvectors of $A$ and a bounded $C^2$-function $\tilde{f} : \mathbb{R}^m \to \mathbb{R}$ with all bounded derivatives such that
\[ f(x) = \tilde{f}(x, e_{i_1}, \ldots, x, e_{i_m}), \quad x \in H. \] (5.12)
(cf. (1.19)). To check that (5.11) holds for $f \in C^2_{cil, b}(H)$ we start from (5.12) and first use a standard argument to approximate $\tilde{f} \in C^2_b(\mathbb{R}^m)$ with a sequence of functions $(\tilde{f}_n) \subset C^2_b(\mathbb{R}^m)$ having compact support. To this purpose let $\phi \in C^\infty(\mathbb{R}^m)$ be such that $0 \leq \phi \leq 1$, $\phi(x) = 1$, $|x| \leq 1$ and $\phi(x) = 0$, $|x| > 2$. Define $\tilde{f}_n(y) = \tilde{f}(y) \cdot \phi_n(y)$, $y \in \mathbb{R}^m$, where $\phi_n(y) = \phi\left(\frac{y}{n}\right)$, for $n \geq 1$. 
Let us consider the resolvent associated to $z$

Set $f_n(x) = \hat{f}_n(\langle x, e_{i_1} \rangle, \ldots, \langle x, e_{i_m} \rangle), x \in H$. We know that

$$\lambda G^{\lambda, x_0} f_n = f_n(x_0) + G^{\lambda, x_0}(L f_n), \ n \geq 1. \tag{5.13}$$

By Proposition 14 $X_t$ is also a weak mild solution. Then we obtain easily that $\mathbb{E}|X_t|_H \leq (ct + 1), t \geq 0$, for some constant $c > 0$. On the other hand, using that first and second derivatives of $(f_n)$ are uniformly bounded, we have

$$\mathbb{E}|L f_n(X_t)| \leq (C + 1) \mathbb{E}|X_t|_H \leq Mt + 1, \ n \geq 1,$$

where $C$ and $M$ are positive constants independent of $t$ and $n$ ($C$ and $M$ may depend on $m$). Passing to the limit in (5.13) by the dominated convergence theorem we get easily the assertion. It follows that

$$G^{\lambda, x_0}(\lambda f - L f) = f(x_0), \ \lambda > 0, \ f \in C^2_{cil, b}(H).$$

Let $z \in H$ be such that (5.6) holds, we write

$$Lf(y) = Lf(y) + \langle (-A)^{1/2} D f(y), z \rangle + \langle (-A)^{1/2} D f(y), F(y) - z \rangle \tag{5.14}$$

$$= L(z)f(y) + Sf(y),$$

$$Sf(y) = \langle (-A)^{1/2} D f(y), F(y) - z \rangle, \ y \in H,$$

$$L(z)f(y) = Lf(y) + \langle (-A)^{1/2} D f(y), z \rangle \ (\text{cf. (1.18)}).$$

Hence

$$G^{\lambda, x_0}(\lambda f - L(z)f) = f(x_0) + G^{\lambda, x_0}(Sf), \ \lambda > 0, \ f \in C^2_{cil, b}(H). \tag{5.15}$$

Let us consider the resolvent associated to $L(z)$:

$$R^{(z)}_{\lambda}(g(x)) = \int_0^\infty e^{-\lambda t} P_t^{(z)} g(x) dt, \ g \in B_b(H), \ x \in H \tag{cf. (5.7)}.$$ 

Note that $f = R^{(z)}_{\lambda} g \in C^2_{cil, b}(H)$ if $g \in C^2_{cil, b}(H)$.

Inserting in (5.15) $f = R^{(z)}_{\lambda} g$ with $g \in C^2_{cil, b}(H)$, using that $(\lambda I - L(z))(R^{(z)}_{\lambda} g) = g$, we obtain

$$G^{\lambda, x_0} g = R^{(z)}_{\lambda} g(x_0) + G^{\lambda, x_0}(S R^{(z)}_{\lambda} g), \ g \in C^2_{cil, b}(H), \ \lambda > 0. \tag{5.16}$$

From now on we consider $\lambda \geq \lambda_0$ (where $\lambda_0 > 0$ is given in Lemma 18).

II step. We prove that (5.16) holds even for $g \in C^2_0(H), \ \lambda \geq \lambda_0$.

Let us first consider $g \in C^2_0(H)$ which is cylindrical, i.e., $g(x) = \tilde{g}(\langle x, e_{i_1} \rangle, \ldots, \langle x, e_{i_m} \rangle), x \in H$, with $\tilde{g} \in C^2_b(\mathbb{R}^m)$. A standard argument based on convolution with mollifiers, shows that there exists $(\tilde{g}_n) \subset C^2_b(\mathbb{R}^m)$ such that $\tilde{g}_n(y) \to \tilde{g}(y)$, as $n \to \infty$, $y \in \mathbb{R}^m$, and

$$\|\tilde{g}_n\|_{C^0} \leq \|\tilde{g}\|_{C^0}, \ n \geq 1.$$ 

Define $g_n(x) = \tilde{g}_n(\langle x, e_{i_1} \rangle, \ldots, \langle x, e_{i_m} \rangle), x \in H$.

Applying (ii) of Theorem 7 with $f_n = g_n$ and $f = g$ (see also (3.13)) we deduce that, for $\lambda \geq \lambda_0$,

$$\lim_{n \to \infty} \langle (-A)^{1/2} D R^{(z)}_{\lambda} g_n(x), F(x) - z \rangle = \langle (-A)^{1/2} D R^{(z)}_{\lambda} g(x), F(x) - z \rangle, \ x \in H, \tag{5.17}$$

i.e., $SR^{(z)}_{\lambda} g_n(x) \to SR^{(z)}_{\lambda} g(x)$, as $n \to \infty$, for any $x \in H$. We write (5.16) with $g_n$:

$$\int_0^\infty e^{-\lambda t} \mathbb{E}[g_n(X_t)] dt = R^{(z)}_{\lambda} g_n(x_0) + \int_0^\infty e^{-\lambda t} \mathbb{E}[S R^{(z)}_{\lambda} g_n(X_t)] dt. \tag{5.18}$$

Since by (3.2)

$$\|SR^{(z)}_{\lambda} g_n\| \leq \tilde{C}_0 C_\theta(\lambda) \|g_n\|_{C^0} \leq \tilde{C}_0 C_\theta(\lambda_0) \|g\|_{C^0}, \ n \geq 1, \ \lambda \geq \lambda_0,$$
we can pass to the limit as \( n \to \infty \) in (5.18) by the dominated convergence theorem and get that (5.16) folds with \( g \in C_0^0(H) \) cylindrical.

If \( g \in C_0^0(H) \) we consider the cylindrical functions \( g_k(x) = g(\pi_k x), x \in H, k \geq 1 \), where \( \pi_k \) are the orthogonal projections considered in (1.13). We have that \( g_k \in C_0^0(H) \) and
\[
\lim_{k \to \infty} g_k(x) = g(x), \quad x \in H, \quad \|g_k\|_{C^\alpha} \leq \|g\|_{C^\alpha}, \quad k \geq 1.
\]

Arguing as before, applying Theorem 7, we pass to the limit as \( k \to \infty \) in
\[
G^{\lambda,x_0}g_k = R_\lambda(x_0)g_k(x_0) + G^{\lambda,x_0}(SR_\lambda g_k);
\]
we finally obtain that (5.16) folds for any \( g \in C_0^0(H) \).

**III step.** Given two martingale solutions \( X^1 \) and \( X^2 \) for \((\mathcal{L}, \delta_{x_0})\). We show that they have the same law.

\( X^1 \) and \( X^2 \) are defined respectively on the probability spaces \((\Omega^1, \mathcal{F}^1, \mathbb{P}^1)\) and \((\Omega^2, \mathcal{F}^2, \mathbb{P}^2)\); we consider
\[
G^{\lambda,x_0}_1 f = \int_0^\infty e^{-\lambda t}\mathbb{E}^1[f(X^1_t)]dt, \quad f \in B_b(H), \quad i = 1, 2
\]
(using the expectation on each probability space). By (5.16) we infer with \( g \in C_0^0(H), \lambda \geq \lambda_0 \),
\[
(G^{\lambda,x_0}_1 - G^{\lambda,x_0}_2)g = (G^{\lambda,x_0}_1 - G^{\lambda,x_0}_2)(SR_\lambda g).
\]
Define \( T^{\lambda,x_0} = G^{\lambda,x_0}_1 - G^{\lambda,x_0}_2 \). Clearly, \( T^{\lambda,x_0} : C_0^0(H) \to \mathbb{R} \) is a bounded linear functional (we denote by \( \|T^{\lambda,x_0}\|_{\mathcal{L}(C_0^0(H), \mathbb{R})} = \|T^{\lambda,x_0}\|_{\mathcal{L}} \) its norm). We have, for \( \lambda \geq \lambda_0 \),
\[
\|T^{\lambda,x_0}\|_{\mathcal{L}} = \sup_{g \in C_0^0(H), \|g\|_{C^\alpha} \leq 1} |T^{\lambda,x_0}g| = \sup_{g \in C_0^0(H), \|g\|_{C^\alpha} \leq 1} |T^{\lambda,x_0}(SR_\lambda g)|
\]
\[
\leq \|T^{\lambda,x_0}\|_{\mathcal{L}} \sup_{g \in C_0^0(H), \|g\|_{C^\alpha} \leq 1} \|SR_\lambda g\|_{C_0^0}.
\]

Using Lemma 18 we know that \( \|SR_\lambda g\|_{C_0^0} = \|\langle F - z, (-A)^{1/2}D\rho(z)g \rangle\|_{C^\alpha} \leq \frac{1}{2} \|g\|_{C^\alpha} \). This shows that \( \|T^{\lambda,x_0}\|_{\mathcal{L}} \leq \frac{1}{2} \|T^{\lambda,x_0}\|_{\mathcal{L}} \) and so \( T^{\lambda,x_0} = 0 \) for \( \lambda \geq \lambda_0 \).

We obtain, for any \( g \in C_0^0(H), \lambda \geq \lambda_0 > 0 \),
\[
\int_0^\infty e^{-\lambda s}\mathbb{E}^1[g(X^1_s)]ds = \int_0^\infty e^{-\lambda s}\mathbb{E}^2[g(X^2_s)]ds.
\]
By the uniqueness of the Laplace transform and using an approximation argument we find that \( \mathbb{E}^1[g(X^1_s)] = \mathbb{E}^2[g(X^2_s)] \) for any \( g \in C_0(H), s \geq 0 \). Applying Theorem 15 we find that \( X^1 \) and \( X^2 \) have the same law on \( \mathcal{B}(C([0, \infty); H)) \). This finishes the proof.

\[\Box\]

### 5.3. Weak uniqueness when \( F \in C_0^0(H, H) \)

Here we prove uniqueness using the localization principle (cf. Theorem 16) and Lemma 19. We will use the constant \( C_0 \) introduced in Section 5.2.

**Lemma 20.** Let \( x \in H \) and consider the SPDE (1.1). If \( F \in C_0^0(H, H) \) then we have uniqueness in law for (1.1).

**Proof.** By Proposition 14 it is enough to show that the martingale problem for \( \mathcal{L} \) is well-posed (cf. (5.9)). By Section 4, for any \( x \in H \), there exists a solution to the martingale problem for \((\mathcal{L}, \delta_x)\).

In order to apply Theorem 16 we proceed into two steps. In the first step we construct a suitable covering of \( H \); in the second step we define suitable operators \( \mathcal{L}_j \) such that according to Lemma 19 the martingale problem associated to each \( \mathcal{L}_j \) is well-posed.
I Step. There exists a countable set of points \((x_j) \subset H, j \geq 1\), and numbers \(r_j > 0\) with the following properties:

(i) the open balls \(U_j = B(x_j, \frac{r_j}{2})\) = \(\{x \in H : |x - x_j|_H < r_j/2\}\) form a covering for \(H\);
(ii) we have: \(|F(x) - F(x_j)|_H < \tilde{C}_0, \ x \in B(x_j, r_j)\).

Using the continuity of \(F\): for any \(x\) we find \(r(x) > 0\) such that

\[|F(y) - F(x)|_H < \tilde{C}_0, \ y \in B(x, r(x)).\]

We have a covering \(\{U_x\}_{x \in H}\) with \(U_x = B(x, \frac{r(x)}{2})\). Since \(H\) is a separable Hilbert space we can choose a countable subcovering \((U_j)_{j \geq 1}\), with \(U_j = B(x_j, \frac{r(x_j)}{2}) = B(x_j, \frac{r_j}{2})\).

II Step. We construct \(L_j\) in order to apply the localization principle.

Let us consider the previous covering \((B(x_j, r_j/2))\). We take \(\rho \in C_0^\infty(\mathbb{R}_+)\), \(0 \leq \rho \leq 1\), \(\rho(s) = 1, 0 \leq s \leq 1\), \(\rho(s) = 0\) for \(s \geq 2\). Define

\[\rho_j(x) = \rho(4r_j^{-2}|x - x_j|_H^2), \ x \in H.\]

Now \(\rho_j = 1\) in \(B(x_j, \frac{r_j}{2})\) and \(\rho_j = 0\) outside \(B(x_j, r_j)\). Set

\[F_j(x) := \rho_j(x)F(x) + (1 - \rho_j(x))F(x_j), \ x \in H.\]

It is easy to prove that

\[F_j \in C^0_b(H, H).\]  \quad (5.19)

We also have

\[
\sup_{x \in H} |F_j(x) - F(x_j)|_H = \sup_{x \in B(x_j, r_j)} |F(x) - F(x_j)|_H < \tilde{C}_0.
\]

Moreover \(F_j(x) = F(x), x \in B(x_j, \frac{r_j}{2}) = U_j\). Define \(D(L_j) = C^2_{cil}(H), j \geq 1, \text{ and } L_j f(x) = \frac{1}{2}Tr(D^2 f(x)) + \langle x, A D f(x) \rangle + \langle (-A)^{1/2} F_j(x), D f(x) \rangle, f \in C^2_{cil}(H), x \in H.\)

We have \(L_j f(x) = L f(x), x \in U_j, f \in C^2_{cil}(H)\) and the martingale problem for each \(L_j\), is well-posed by Lemma 19 (with \(F = F_j\) an \(z = F(x_j))\). By Theorem 16 we find the assertion.

\[\square\]

6. Proof of weak uniqueness of Theorem 1

Here we prove uniqueness in law for (1.1) assuming that \(F : H \rightarrow H\) is locally \(\theta\)-Hölder continuous and has at most linear growth, i.e., it verifies (1.2). To this purpose we will use Lemma 20 and Theorem 17.

Let \(X = (X_t)_{t \geq 0}\) be a weak mild solution of (1.1) starting at \(x \in H\) (under the assumption (1.2)) defined on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) on which it is defined a cylindrical \(\mathcal{F}_t\)-Wiener process \(W_t\); see Section 4. For a cylindrical function \(f \in C^2_{cil}(H)\) in general \(\mathcal{L} f\) (see (5.9)) is not a bounded function on \(H\) because \(F\) can be unbounded. However we know by a finite-dimensional Itô’s formula that

\[
M_t(f) = f(X_t) - \int_0^t \mathcal{L} f(X_s) ds = f(x) + \int_0^t \langle D f(X_s), dW_s \rangle
\]

is still a continuous square integrable \(\mathcal{F}_t\)-martingale. Note that we can apply Itô’s formula because there exists \(N \geq 1\) such that \(f(x) = f(\pi_N x), x \in H,\) with \(\pi_N = \sum_{j=1}^N e_j \otimes e_j\) (cf. (1.13)) and so \(f(X_t) = f(\pi_N X_t)\). Moreover \(X_{t,N} = \pi_N X_t\) verifies

\[
X_{t,N} = e^{tA} \pi_N x + \int_0^t e^{(t-s)A} (-A_N)^{1/2} F(X(s)) ds + \int_0^t \pi_N e^{(t-s)A} dW_s, \ t \geq 0,
\]
where $A_N = A\pi_N$; writing $\pi_N W_t = \sum_{k=1}^{N} W_t^{(k)} e_k$ it follows that $X_{t,N}$ is an Itô process given by

$$X_{t,N} = \pi_N x + \int_0^t A_N X_{s,N} ds + \int_0^t (-A_N)^{1/2} F(X_s) ds + \pi_N W_t.$$  (6.2)

Now let us consider $B(0, n) = \{ x \in H : |x|_H < n \}$ and define $F_n \in C^\theta_0(H, H)$ such that $F_n(y) = F(y)$, $y \in B(0, n)$, $n \geq 1$.

To this purpose one can take $\eta \in C_0^\infty(\mathbb{R})$ such that $0 \leq \eta(s) \leq 1$, $s \in \mathbb{R}$, $\eta(s) = 1$ for $|s| \leq 1$ and $\eta(s) = 0$ for $|s| \geq 2$, and set $F_n(y) = F(y) \eta(\frac{|y|_H}{n})$, $y \in H$ (one can easily check that $F_n \in C^\theta_0(H, H)$).

Define

$$L_n f(y) = \frac{1}{2} Tr(D^2 f(y)) + \langle y, ADf(y) \rangle + \langle F_n(y), (-A)^{1/2} Df(y) \rangle,$$ 

$f \in C^2_{\text{cl}}(H)$, $y \in H$.

Let us introduce the exit time $\tau_n^X = \inf\{ t \geq 0 : |X_t|_H \geq n \}$ $(\tau_n^X = +\infty$ if the set is empty; cf. (5.5)) for each $n \geq 1$. It is an $\mathcal{F}_t$-stopping time (cf. Proposition II.1.5 in [21]). Applying the optional sampling theorem (cf. Theorem II.2.13 in [21]) to (6.1) we deduce that

$$M_{t \wedge \tau_n^X}(f) = f(X_{t \wedge \tau_n^X}) - \int_0^{t \wedge \tau_n^X} Lf(X_s) ds = f(X_{t \wedge \tau_n^X}) - \int_0^{t \wedge \tau_n^X} L_n f(X_{s \wedge \tau_n^X}) ds, \ t \geq 0,$$

is a martingale with respect to the filtration $(\mathcal{F}_{t \wedge \tau_n^X})_{t \geq 0}$; note that the process $(X_{t \wedge \tau_n^X})_{t \geq 0}$ is adapted with respect to $(\mathcal{F}_{t \wedge \tau_n^X})$ (see Proposition II.1.4 in [21]).

Thus $(X_{t \wedge \tau_n^X})_{t \geq 0}$ is a solution to the stopped martingale problem for $(L_n, \delta_x, B(0, n))$. By Lemma 20 the martingale problem for each $L_n$ is well-posed because $F_n \in C^\theta_0(H, H)$. By Theorem 17 also the stopped martingale problem for $(L_n, \delta_x, B(0, n))$ is well-posed, $n \geq 1$.

Let $Y$ be another mild solution starting at $x \in H$. Then $(Y_{t \wedge \tau_n^X})_{t \geq 0}$ also solves the stopped martingale problem for $(L_n, \delta_x, B(0, n))$. By weak uniqueness of the stopped martingale problem it follows that, for any $n \geq 1$, $(X_{t \wedge \tau_n^X})_{t \geq 0}$ and $(Y_{t \wedge \tau_n^X})_{t \geq 0}$ have the same law. Now it is not difficult to prove that $X$ and $Y$ have the same law on $B(C([0, \infty); H))$ and this finishes the proof.

7. An extension to functions $F : H \to H$ without imposing a growth condition

Assuming weak existence for (1.1) one can obtain the following extension of Theorem 1.

**Theorem 21.** Let us consider (1.1) under Hypothesis 1. Assume that

H1) $F : H \to H$ is $\theta$-Hölder continuous on each bounded set of $H$, for some $\theta \in (0, 1)$;

H2) for any $x \in H$, there exists a weak mild solution $(X_t)_{t \geq 0}$ of (1.1) starting at $x$.

Under the previous assumptions weak uniqueness holds, i.e., for any $x \in H$, all weak mild solutions starting at $x$ have the same law on $B(C([0, \infty); H))$.

**Proof.** The proof is similar to the one of Section 6. We give some details for the sake of completeness. Let $X = (X_t)_{t \geq 0}$ be a weak mild solution of (1.1) starting at $x \in H$ (defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$). We have that $M_t(f)$ in (6.1) is a continuous square integrable $\mathcal{F}_t$-martingale, for any $f \in C^2_{\text{cl}}(H)$. Using that $F$ is locally $\theta$-Hölder continuous, we obtain that the bounded functions $F_n(y) = F(y) \eta(\frac{|y|_H}{n})$, $y \in H$, belong to $C^\theta_0(H, H)$.

By the optional sampling theorem we find that $(X_{t \wedge \tau_n^X})_{t \geq 0}$ is a solution to the stopped martingale problem for $(L_n, \delta_x, B(0, n))$. Using Lemma 20 and Theorem 17 we know that the stopped martingale problem for $(L_n, \delta_x, B(0, n))$ is well-posed, $n \geq 1$. Proceeding as in the final part of Section 6 we obtain the assertion. 

□
7.1. Singular perturbations of classical stochastic Burgers equations

Here we show that Theorem 21 can be applied to SPDEs (1.1) in cases when $F$ grows more than linearly. As an example we consider

\[ du(t, \xi) = \frac{\partial^2}{\partial \xi^2} u(t, \xi) dt + h(u(t, \xi)) \cdot g\big( |u(t, \cdot)|_{H^s_0} \big) dt + \frac{1}{2} \frac{\partial}{\partial \xi} \left( u^2(t, \xi) \right) dt + \sum_{k \geq 1} \frac{1}{k} dW^t_k(e_k(\xi)), \]

\[ u(0, \xi) = u_0(\xi), \quad \xi \in (0, \pi), \quad (7.1) \]

$u(t, 0) = u(t, \pi) = 0$, $t > 0$, $u_0 \in H^s_0(0, \pi)$; $g : \mathbb{R} \to \mathbb{R}$ is locally $\theta$-Hölder continuous, for some $\theta \in (0, 1)$, and $h : \mathbb{R} \to \mathbb{R}$ is a $C^1$-function with derivative $h'$ which is locally $\theta$-Hölder continuous. Moreover to get existence of solutions we require

\[ \sup_{s \in \mathbb{R}} |h'(s)| \cdot \sup_{s \in \mathbb{R}} |g(s)| \leq 1. \quad (7.2) \]

For instance, we can consider $h(u(t, \xi)) \cdot g\big( |u(t, \cdot)|_{H^s_0} \big) = u(t, \xi) \cdot \left( \sqrt{|u(t, \cdot)|_{H^s_0}} \wedge 1 \right)$.

Recall that $e_k(\xi) = \sqrt{2/\pi} \sin(k\xi)$, $\xi \in [0, \pi]$, $k \geq 1$ (cf. Section 2; note that in (7.1) the noise is "more regular" than the one in (2.1)).

We first establish existence of mild solutions with values in $H^s_0(0, \pi)$ when $g = 0$ (see Proposition 22). This is needed in order to show that the classical Burgers equation can be considered in the form (1.1) with a suitable $F = F_0 : H^s_0(0, \pi) \to H^s_0(0, \pi)$ locally $\theta$-Hölder continuous (see (7.20)) To this purpose we follow the approach in Chapter 14 of [15].

Then to get well-posedness of (7.1) (see Proposition 23) we will apply the Girsanov theorem using an exponential estimate proved in [8]. Such Girsanov theorem provides existence of weak solutions (cf. Remark 24). Uniqueness in law is obtained directly using Theorem 21.

We need to review basic facts about fractional powers of the operator $A = -\frac{\partial^2}{\partial \xi^2}$ with Dirichlet boundary conditions, i.e., $D(A) = H^2(0, \pi) \cap H^1_0(0, \pi)$ (cf. Section 2). The eigenfunctions are $e_k(\xi)$, $k \geq 1$, with eigenvalues $-k^2$ (we set $\lambda_k = k^2$). For $v \in L^2(0, \pi)$ we write $v_k = \langle v, e_k \rangle = \int_0^\pi v(x) e_k(x) dx$, $k \geq 1$.

We introduce for $s > 0$ the Hilbert spaces

\[ \mathcal{H}_s = D((-A)^s) = \{ u \in L^2(0, \pi) : \sum_{k \geq 1} \lambda_k^{2s} u_k^2 = \sum_{k \geq 1} k^{4s} u_k^2 < \infty \}. \quad (7.3) \]

Moreover, for any $u \in \mathcal{H}_s$, $(-A)^s u = \sum_{k \geq 1} k^{2s} u_k e_k$. He also set $H_0 = L^2(0, \pi)$. We have $\langle u, v \rangle_{\mathcal{H}_s} = \sum_{k \geq 1} k^{4s} u_k v_k$ (note that $|u|_{L^2} \leq |(-A)^s u|_{L^2} = |u|_{\mathcal{H}_s}$, $u \in \mathcal{H}_s$, $s > 0$).

If $u \in H^1_0(0, \pi)$, $|u|_{H^s_0(0, \pi)} = |u'|_{L^2(0, \pi)}$, where $u'$ is the weak derivative of $u$. We have

\[ \mathcal{H}_{1/2} = H^s_0(0, \pi) \ 	ext{with equivalence of norms;} \]

\[ \mathcal{H}_{1/8} \subset L^4(0, \pi) \quad (7.4) \]

(\text{with continuous inclusion, i.e., there exists $C > 0$ such that $|u|_{L^4} \leq C|u|_{\mathcal{H}_{1/8}}$, $u \in \mathcal{H}_{1/8}$). Assertion (7.5) follows by a classical Sobolev embedding theorem (cf. Theorem 6.16 and Remark 6.17 in [26]). We only note that if $u \in \mathcal{H}_{1/8}$ one can consider the odd extension $\tilde{u}$ of $u$ to $(-\pi, \pi)$; it is easy to check that $\tilde{u}$ belongs to the space $H^{1/4}(-\pi, \pi)$ considered in [26].

We also have with continuous inclusion (cf. Lemma 6.13 in [26])

\[ \mathcal{H}_s \subset \{ u \in C([0, \pi]), \ u(0) = u(\pi) = 0 \}, \quad s > 1/4. \quad (7.6) \]

Now let us consider the linear bounded operator $T : \mathcal{H}_{1/2} \to \mathcal{H}_{1/2}$, $Tu = (-A)^{-1/2} \partial \xi u$, $u \in \mathcal{H}_{1/2}$; $T$ can be extended to a linear and bounded operator $T : \mathcal{H}_0 = L^2(0, \pi) \to \mathcal{H}_0$, see Section 2.0.1. By interpolation it follows that

\[ T = (-A)^{-1/2} \partial \xi \] is a bounded linear operator from $\mathcal{H}_s$ into $\mathcal{H}_s$, $s \in [0, 1/2]$. \quad (7.7)
Indeed by Theorem 4.36 in [33] we know that \( \mathcal{H}_{s/2} \) can be identified with the real interpolation space \((\mathcal{H}_0, \mathcal{H}_{1/2})_{s,2}, s \in (0,1)\). Applying Theorem 1.6 in [33] we deduce (7.7).

Let \( T > 0 \). For \( g \in C([0,T]; \mathcal{H}_0) \) we define \((Sg)(t) = \int_0^t e^{(t-s)A}g(s)ds, t \in [0,T] \). One can prove that \( Sg \in C([0,T]; \mathcal{H}_s) \), for any \( s \in [0,1) \). More precisely,

\[
S \text{ is a bounded linear operator from } C([0,T]; \mathcal{H}_0) \text{ into } C([0,T]; \mathcal{H}_s), s \in [0,1).
\]  

(7.8)

This result can be also deduced from Proposition 5.9 in [17] with \( \alpha = 1 \), \( E_1 = \mathcal{H}_s \) and \( E_2 = \mathcal{H}_0 \). We only remark that, for any \( p > 1 \), \( L^p(0,T; \mathcal{H}_0) \subset C([0,T]; \mathcal{H}_0) \) (with continuous inclusion) and \(|(-A)^{\alpha} e^{tA}x|_{\mathcal{H}_s} = |e^{tA}x|_{\mathcal{H}_s} \leq \frac{C}{t^{\alpha}} |x|_{\mathcal{H}_0} \) (see Proposition 4.37 in [26]).

In the next proposition, assertion (i) extends a result of [15] which actually shows the existence of a mild solution to the stochastic Burgers equations with continuous path in \( \mathcal{H}_s \), \( s \in (0,1/4) \). Assertion (ii) is proved in [8].

**Proposition 22.** Let us consider (7.1) with \( g = 0 \). Then the following assertions hold:

i) for any \( u_0 \in \mathcal{H}_{1/2} \) there exists a pathwise unique mild solution \( Y = (Y_t) = (Y_t)_{t \geq 0} \) with continuous paths in \( \mathcal{H}_{1/2} \).

ii) The following estimate holds, for any \( T > 0 \),

\[
\mathbb{E}\left[ \exp\left( \frac{1}{2} \int_0^T |Y_{t1/2}|^2 ds \right) \right] < \infty.
\]

(7.9)

**Proof.** According to [15] and [8], setting \( u(t,\cdot) = Y_t \) we write (7.1) with \( g = 0 \) as

\[
Y_t = e^{tA}u_0 + \frac{1}{2} \int_0^t e^{(t-s)A} \partial_x(Y_s^2)ds + \int_0^t e^{(t-s)A} \sqrt{c} dW_s, \quad t \geq 0,
\]

(7.10)

where \( W_t = \sum_{k \geq 1} W^{(k)}_t e_k \) is a cylindrical Wiener process on \( \mathcal{H}_0 = L^2(0, \pi) \) and \( C = (-A)^{-1} : \mathcal{H}_0 \to \mathcal{H}_0 \) is symmetric, non-negative and of trace class, \( C e_k = \frac{1}{k^2} e_k, k \geq 1 \).

(i) In Theorem 12.4 of [15] (see also the references therein) it is proved that, for any \( T > 0 \), there exists a pathwise unique solution \( Y \) to (7.10) on \([0,T]\) such that, \( \mathbb{P}\)-a.s., \( Y \in C([0,T]; \mathcal{H}_0) \cap L^2(0,T; \mathcal{H}_{1/2}) \) (i.e., \( \mathbb{P}\)-a.s, the paths of \( Y \) are continuous with values in \( \mathcal{H}_0 \) and square-integrable with values in \( \mathcal{H}_{1/2} \)); such result holds even if we replace \( C \) by identity \( I \). By a standard argument based on the pathwise uniqueness, we get a solution \( Y \) defined on \([0,\infty)\) which verifies \( Y \in C([0,\infty); \mathcal{H}_0) \cap L^2_{loc}(0,\infty; \mathcal{H}_{1/2}), \mathbb{P}\)-a.s.

Let us fix \( T > 0 \). To prove our assertion, we will show that

\[
Y \in C([0,T]; \mathcal{H}_{1/2}), \quad \mathbb{P}\text{-a.s.}
\]

(7.11)

Note that the stochastic convolution \( W_{A}(t) = \int_0^t e^{(t-s)A} \sqrt{c} dW_s \) has a modification with continuous paths with values in \( \mathcal{H}_{1/2} \) (to this purpose one can use Theorem 5.11 in [17]). Moreover in Lemma 14.2.1 of [15] it is proved that the operator \( R \),

\[
(Rv)(t) = \int_0^t e^{(t-s)A} \partial_x v(s)ds, \quad t \in [0,T], \quad v \in C([0,T]; \mathcal{H}_{1/2}),
\]

(7.12)

can be extended to a linear and bounded operator from \( C([0,T]; L^1(0,\pi)) \) into \( C([0,T]; \mathcal{H}_s), s \in (0,1/4) \). It is straightforward to check that the mapping: \( h \mapsto h^2 \) is continuous from \( C([0,T]; \mathcal{H}_0) \) into \( C([0,T]; L^1(0,\pi)) \). It follows that

\[
u \mapsto R(u^2) \quad \text{continuous from } C([0,T]; \mathcal{H}_0) \text{ into } C([0,T]; \mathcal{H}_s).
\]

We deduce from Lemma 14.2.1 of [15] that the solution \( Y \in C([0,T]; \mathcal{H}_s), \mathbb{P}\text{-a.s.}, s \in (0,1/4) \). To get more spatial regularity for \( Y \) we proceed in two steps.

**I Step.** We show that, \( \mathbb{P}\)-a.s, \( Y \in C([0,T]; \mathcal{H}_s), s \in (0,1/2) \).

Let us fix \( s = 1/8 \). By (7.5) we know that the mapping: \( h \mapsto h^2 \) is continuous from \( C([0,T]; \mathcal{H}_{1/8}) \) into \( C([0,T]; \mathcal{H}_0) \). Moreover, using (7.7) we can write, for \( w \in C([0,T]; \mathcal{H}_0) \),

\[
(Rw)(t) = \int_0^t e^{(t-s)A} \partial_x w(s)ds = \int_0^t e^{(t-s)A} ((-A)^{1/2} \partial_x [\partial_x] w(s)ds, \quad t \in [0,T].
\]
Note that \[ (-A)^{-1/2} \partial \xi | w \in C([0, T]; \mathcal{H}_0). \] By (7.8) we know that, for any \( \epsilon \in (0, 1) \),
\[
t \mapsto (-A)^{1-\epsilon} \int_0^t e^{(t-s)A} \left[ (-A)^{-1/2} \partial \xi \right] w(s) ds
\]
belongs to \( C([0, T]; \mathcal{H}_0) \). Hence
\[
(-A)^s Rw \in C([0, T]; \mathcal{H}_0), \quad s \in (0, 1/2), \text{ i.e., } \quad Rw \in C([0, T]; \mathcal{H}_s), \quad s \in (0, 1/2).
\]
Using this fact we easily obtain that, \( \mathbb{P} \)-a.s., \( Y \in C([0, T]; \mathcal{H}_s), \text{ } s \in (0, 1/2). \)

**II Step.** We show that \( Y \in C([0, T]; \mathcal{H}_{1/2}), \text{ } \mathbb{P} \)-a.s.

Let us fix \( s \in (1/4, 1/2) \) and recall (7.6). According to [24] the space \( \mathcal{H}_s \) can be identified with \( \{ u \in W^{2s, 2}(0, \pi) : u(0) = u(\pi) = 0 \} \), where
\[
W^{2s, 2}(0, \pi) = \left\{ u \in \mathcal{H}_0 : |u|^2_{W^{2s, 2}(0, \pi)} = \int_0^\pi \int_0^\pi |u(x) - u(y)|^2 |x - y|^{-1/4s} dx dy < \infty \right\}
\]
is a Sobolev-Slobodeckij space; the norm \( |u|_{W^{2s, 2}(0, \pi)} = |u|_{\mathcal{H}_0} + |u|_{W^{2s, 2}(0, \pi)} \) is equivalent to \( |u|_{\mathcal{H}_s} \) (see also Theorem 3.2.3 in [32], taking into account that \( \mathcal{H}_s \) can be identified with the real interpolation space \( (\mathcal{H}_0, D(A))_{s, 2} \) by Theorem 4.36 in [33]).

Using the previous characterization and (7.6) it is easy to prove that if \( u \in \mathcal{H}_s \) and \( v \in \mathcal{H}_s \) then the pointwise product \( w \in \mathcal{H}_s \). Indeed we have
\[
|u(x)v(x) - u(y)v(y)| \leq \|u\|_0 |v(x) - v(y)| + \|v\|_0 |u(x) - u(y)|, \quad x, y \in [0, \pi],
\]
and so \( |w|_{W^{2s, 2}(0, \pi)} \leq C |u|_{W^{2s, 2}(0, \pi)} |v|_{W^{2s, 2}(0, \pi)} \). It follows that \( |w|_{\mathcal{H}_s} \leq C |u|_{\mathcal{H}_s} |v|_{\mathcal{H}_s}. \)

Let now \( u \in C([0, T]; \mathcal{H}_s). \) Using that \( |u^2(t) - u^2(r)|_{\mathcal{H}_s} \leq |u(t) - u(r)|_{\mathcal{H}_s} |u(t) + u(r)|_{\mathcal{H}_s} \leq 2|u|_{C([0, T]; \mathcal{H}_s)} |u(t) - u(r)|_{\mathcal{H}_s}, \quad t, r \in [0, T], \) we see that the mapping:
\[
u \mapsto u^2 \quad \text{is continuous from } C([0, T]; \mathcal{H}_s) \text{ into } C([0, T]; \mathcal{H}_s).
\]
Hence, taking into account I Step, to get the assertion it is enough to prove that
\[
R \eta \in C([0, T]; \mathcal{H}_{1/2}) \text{ if } \eta \in C([0, T]; \mathcal{H}_s), \quad s \in (1/4, 1/2).
\]
This would imply \( R(\eta^2) \in C([0, T]; \mathcal{H}_{1/2}) \) if \( \eta \in C([0, T]; \mathcal{H}_s) \) and so \( Y \in C([0, T]; \mathcal{H}_{1/2}), \text{ } \mathbb{P} \)-a.s.. Let us fix \( \eta \in C([0, T]; \mathcal{H}_s). \) Using (7.7) we can write
\[
(R \eta)(t) = \int_0^t e^{(t-s)A} \partial \xi \eta(s) = \int_0^t e^{(t-s)A} (-A)^{1/2} \left[ (-A)^{-1/2} \partial \xi \right] \eta(s) ds, \quad t \in [0, T],
\]
where \( \left[ (-A)^{-1/2} \partial \xi \right] \eta \in C([0, T]; \mathcal{H}_s). \) Hence \( \theta(t) = (-A)^s \left[ (-A)^{-1/2} \partial \xi \right] \eta \in C([0, T]; \mathcal{H}_0). \) Writing
\[
(R \eta)(t) = \int_0^t e^{(t-r)A} (-A)^{1/2} \theta(r) dr, \quad t \in [0, T],
\]
and using (7.8), we find that \( (-A)^{1/2} R \eta \in C([0, T]; \mathcal{H}_0) \) and this shows (7.14).

(ii) A similar estimate is proved in Propositions 2.2 and 2.3 in [8]. However in [8] equation (7.10) is considered in \( L^2(0, 1) \) (instead of \( L^2(0, \pi) \)); the authors prove that \( \mathbb{E} \left[ e^{\mathbb{E} \left[ \int_0^T |\nabla \xi|^2_{H_0^1(0, 1)} ds \right]} \right] < \infty \) if \( \epsilon \leq \epsilon_0 = \pi^2/2 ||C|| \) (using the operator norm \( ||C|| \) of \( C \)).

The condition \( \epsilon \leq \epsilon_0 \) is used in the proof of Proposition 2.2 in order to get the inequality \( -|x|^2_{H_0^1} + 2c \sqrt{C x}|_{L^2}^2 \leq 0, \quad x \in H_0^1. \) In our case \( \epsilon_0 = 1/2 \) since \( ||C|| = 1. \)

In the remaining part we consider
\[
\mathcal{H} = H_0^1(0, \pi) = \mathcal{H}_{1/2}
\]
as the reference Hilbert space and study the SPDE (7.1) in \( \mathcal{H}. \)
We will consider the following restriction of $A$:
\[
\mathcal{A} = \frac{d^2}{dt^2} \quad \text{with} \quad D(\mathcal{A}) = \{u \in H^3(0, \pi) : u, \frac{d^2 u}{dx^2} \in H_0^1(0, \pi)\}; \quad \mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}.
\] (7.15)

Eigenfunctions of $\mathcal{A}$ are $\hat{e}_k(\xi) = \sqrt{2/\pi} \frac{1}{k} \sin(k\xi) = \frac{1}{k} \hat{e}_k(\xi)$ with eigenvalues $-k^2$, $k \geq 1$.

It is clear that $\mathcal{A}$ verifies Hypothesis 1 when $\mathcal{H} = \mathcal{H}_\gamma$. Moreover $(\hat{e}_k)$ forms an orthonormal basis in $\mathcal{H}$. The noise in (7.1) will be indicated by $\mathcal{W}$; it is a cylindrical Wiener process on $\mathcal{H}$:
\[
\mathcal{W}_t(\xi) = \sum_{k \geq 1} \frac{1}{k} W_t^{(k)} \hat{e}_k(\xi) = \sum_{k \geq 1} W_t^{(k)} \hat{e}_k(\xi), \quad t \geq 0, \xi \in [0, \pi].
\] (7.16)

Let $D_0$ be the space of infinitely differentiable functions vanishing in a neighborhood of 0 and $\pi$. Such functions are dense in $\mathcal{H}$. The operator
\[
(-\mathcal{A})^{1/2} \partial_\xi : D_0 \to \mathcal{H}
\]
can be extended to a bounded linear operator from $\mathcal{H}$ into $\mathcal{H}$. (7.17)

To check this fact we consider $y \in D_0$ and $x \in \mathcal{H}$. Define $x_N = \sum_{k=1}^N x_k \hat{e}_k$, with $x_k = \langle x, \hat{e}_k \rangle_{\mathcal{H}}$, $N \geq 1$. Using that $(-\mathcal{A})^{1/2}$ is self-adjoint on $\mathcal{H}$ and integrating by parts we find
\[
\langle (-\mathcal{A})^{-1/2} \partial_\xi y, x_N \rangle_{\mathcal{H}} = \langle \partial_\xi y, (-\mathcal{A})^{-1/2} x_N \rangle_{\mathcal{H}} = \langle \partial_\xi y, \partial_\xi \sum_{k=1}^N \frac{x}{k} \hat{e}_k \rangle_{L^2(0, \pi)}
\]
\[
= -\langle \partial_\xi y, \partial_\xi^2 \sum_{k=1}^N \frac{x}{k} \hat{e}_k \rangle_{L^2(0, \pi)} = \langle \partial_\xi y, \sum_{k=1}^N x_k \sin(k\xi) \rangle_{L^2(0, \pi)}.
\]
Hence $|\langle (-\mathcal{A})^{-1/2} \partial_\xi y, x_N \rangle_{\mathcal{H}}| \leq \|y\|_{\mathcal{H}} \left(\sum_{k=1}^N x_k^2\right)^{1/2} \leq \|y\|_{\mathcal{H}} x_{\mathcal{H}}$ and we get the assertion. Let us introduce, for any $x \in \mathcal{H}$,
\[
F_0(x) = \frac{1}{2} A^{-1/2} \partial_\xi |x|^2.
\] (7.18)

Since the mapping $x \mapsto x^2$ is locally Lipschitz from $\mathcal{H}$ into $\mathcal{H}$ (i.e., it is Lipschitz continuous on bounded sets of $\mathcal{H}$ with values in $\mathcal{H}$; recall that $|x^2|_{\mathcal{H}} = 2|x|_{\mathcal{H}}$) it is clear that also
\[
F_0 : \mathcal{H} \to \mathcal{H}
\]
is locally Lipschitz. (7.19)

The mild solution $Y$ of Proposition 22 with paths in $C([0, \infty); \mathcal{H})$ verifies, $\mathbb{P}$-a.s.,
\[
Y_t = e^{tA}x + \int_0^t e^{(t-s)A} F_0(Y_s) ds + \int_0^t e^{(t-s)A} d\mathcal{W}_s; \quad t \geq 0,
\] (7.20)
where $A$ is defined in (7.15) and we have set $u_0 = x \in \mathcal{H}$.

We consider the following SPDE which includes (7.1) as a special case:
\[
X_t = e^{tA}x + \int_0^t (-A)^{1/2} e^{(t-s)A} F_0(X_s) ds + \int_0^t e^{(t-s)A} B(X_s) ds + \int_0^t e^{(t-s)A} d\mathcal{W}_s,
\] (7.21)
$t \geq 0$, where
\[
B : \mathcal{H} \to \mathcal{H}
\]
is locally $\theta$-Hölder continuous and $|B(x)|_{\mathcal{H}} \leq c_0 + |x|_{\mathcal{H}}, \quad x \in \mathcal{H},
\] (7.22)
for some $\theta \in (0, 1)$ and $c_0 \geq 0$. Note that (7.1) can be written in the form (7.21) by choosing
\[
B(x) = h(x)g(|x|_{\mathcal{H}}), \quad x \in \mathcal{H}.
\] (7.23)

To check that such $B$ is locally $\theta$-Hölder continuous we argue similarly to (2.2). Let $u, v \in B = \{x \in \mathcal{H} : |x|_{\mathcal{H}} \leq M\}$, for some $M > 0$. Recall (7.6). There exists $C > 0$ such that if $u \in B$ then
\[ \sup_{r \in [0, \pi]} |u(r)| \leq CM. \]

We have
\[
\int_0^\pi \left| h'(u(t))u'(t)g(u_H) - h'(v(t))v'(t)g(u_H) \right|^2 dt \\
\leq 3 \int_0^\pi \left| h'(u(t)) - h'(v(t)) \right|^2 |u'(t)|^2 g(u_H)^2 dt + 3 \int_0^\pi \left| h'(v(t)) \right|^2 |u'(t) - v'(t)|^2 g(u_H)^2 dt \\
+ 3 \int_0^\pi \left| h'(v(t)) \right|^2 |v'(t)|^2 g(u_H) - g(u_H)|^2 dt \\
\leq 3c_1|u - v|_H^2 + 3c_1 \int_0^\pi |u'(t) - v'(t)|^2 dt + 3c_1|u - v|_H^2, \\
\]

for some constants \( c_1 \) and \( c_2 \) possibly depending on \( M, g, h \) and \( \theta \).

The function \( B \) in (7.23) verifies (7.22) with \( c_0 = 0 \) (we only note that \( |B(x)|_H^2 \leq \|g\|^2_\theta \int_0^\pi |h'(x(\xi))| d\xi \)).

We have used condition (7.2) to guarantee the bound in (7.22). This is used to check the Novikov condition (7.24) and prove the existence part in the following result.

**Proposition 23.** Let us consider (7.21) on \( \mathcal{H} = H^1_0(0, \pi) \) with \( \mathcal{A} \) given in (7.15) and the cylindrical Wiener process \( W \) on \( \mathcal{H} \) given in (7.16) (\( \langle W^{(k)} \rangle_{k \geq 1} \) are independent real Wiener processes). Let \( F_0 \) as in (7.18) and suppose that \( B : \mathcal{H} \to \mathcal{H} \) verifies (7.22). Then the following assertions hold.

i) For any \( x \in \mathcal{H} \), there exists a weak mild solution \( (X_t)_{t \geq 0} \).

ii) Weak uniqueness holds for (7.21) for any \( x \in \mathcal{H} \).

**Proof.** i) Let us fix \( x \in \mathcal{H} \). We will use the Girsanov theorem as in Appendix A.1 of [11], using the reference Hilbert space \( \mathcal{H} \).

Let \( Y = (Y_t) \) be the unique solution to the Burgers equation (7.20) with values in \( \mathcal{H} \) and such that \( Y_0 = x \). This is defined on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) on which it is defined the cylindrical Wiener process \( W \) on \( \mathcal{H} \). Set
\[
b(s) = B(Y_s), \quad s \geq 0,
\]
and note that \( |b(s)|_H \leq c_0 + |Y_s|_H \), \( s \geq 0 \), by (7.22). The process \( (b(s)) \) is progressively measurable and verifies \( \mathbb{E} \int_0^T |b(s)|_H^2 ds < \infty \), \( T > 0 \) (see (7.9) and recall that \( e^r \geq 1 + r \)). Moreover, by (7.9) and (7.22) it follows that, for any \( T > 0 \),
\[
\mathbb{E} \left[ e^{\frac{T}{2} \int_0^T |b(s)|^2_H ds} \right] \leq C_T \mathbb{E} \left[ e^{\frac{T}{2} \int_0^T |Y_s|^2_H ds} \right] < \infty. \tag{7.24}
\]

Let \( U_t = \sum_{k \geq 1} \int_0^T \langle b(s), \tilde{e}_k \rangle_H dW_s^{(k)} \), \( t \geq 0 \), and fix \( T > 0 \). By Proposition 17 in [11] we know that \( \tilde{W}_t^{(k)} = W_t^{(k)} - \int_t^T \langle \tilde{e}_k, b(s) \rangle_H ds \), \( t \in [0, T] \), \( k \geq 1 \), are independent real Wiener processes on \((\Omega, \mathcal{F}, (\mathcal{F}_t), \tilde{\mathbb{P}})\), where the probability measure
\[
\tilde{\mathbb{P}} = \mathbb{P} \left[ \mathbb{E} \left[ e^{\frac{T}{2} \int_0^T |b(s)|^2_H ds} \right] \right. \mathbb{P}.
\]

is equivalent to \( \mathbb{P} \) (the quadratic variation process \( \langle U \rangle_t = \int_0^T |b(s)|_H^2 ds \), \( t \in [0, T] \)).

Hence \( \tilde{W}_t = \sum_{k \geq 1} \tilde{W}_t^{(k)} \tilde{e}_k \), \( t \in [0, T] \), is a cylindrical Wiener process on \( \mathcal{H} \) defined on \((\Omega, \mathcal{F}, (\mathcal{F}_t), \tilde{\mathbb{P}})\). Arguing as in Proposition 21 of [11] we obtain that
\[
Y_t = e^{\tilde{A}X_t} + \int_0^T (-A)^{1/2} e^{(t-s)A} F_0(Y_s) ds + \int_0^T e^{(t-s)A} B(Y_s) ds + \int_0^T e^{(t-s)A} d\tilde{W}_s, \quad t \in [0, T],
\]
\( \mathbb{P} \)-a.s.. Thus \( Y \) a mild solution on \([0, T] \) to (7.21) defined on \((\Omega, \mathcal{F}, (\mathcal{F}_t), \tilde{\mathbb{P}})\).

Since \( T > 0 \) is arbitrary, using a standard procedure based on the Kolmogorov extension theorem, one can prove the existence of a weak mild solution \( X \) to (7.21) on \([0, \infty) \). On this respect, we refer to Remark 3.7, page 303, in [28] (cf. the beginning of Section 4).
ii) We use Theorem 21 with $H = \mathcal{H}$, $A = A$ and $W = W$. Indeed, (7.21) can be rewritten as

$$X_t = e^{tA}x + \int_0^t (-A)^{1/2} e^{(t-s)A} F(X_s) ds + \int_0^t e^{(t-s)A} dW_s, \quad t \geq 0,$$

where $F(x) = F_0(x) + (-A)^{-1/2} B(x)$, $x \in \mathcal{H}$. The function $F : \mathcal{H} \to \mathcal{H}$ is locally $\theta$-Hölder continuous (cf. (7.19) and (7.22)).

\textbf{Remark 24.} Assertion (ii) in Proposition 23 cannot be deduced directly from the Girsanov theorem as in Appendix A.1 of [11]. To this purpose, one should prove that $\mathbb{E}\left[ e^{\frac{1}{2} \int_0^T |B(X_s)|^2 ds} \right] < \infty$, for any weak mild solution $X$ to (7.21) starting at $x \in \mathcal{H}$. A sufficient condition would be $\mathbb{E}\left[ e^{\frac{1}{2} \int_0^T |X_s|^2 ds} \right] < \infty$. It seems that such estimate does not hold under our assumptions. Note that since the nonlinearity of the Burgers equation grows quadratically one cannot follow the proof of Proposition 22 of [11] to derive $\mathbb{E}\left[ e^{\frac{1}{2} \int_0^T |X_s|^2 ds} \right] < \infty$.

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