Asymptotic analysis of a boundary-value problem with the nonlinear boundary multiphase interactions in a perforated domain

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Abstract

We consider a boundary-value problem for the second order elliptic differential operator with rapidly oscillating coefficients in a domain $\Omega_\varepsilon$ that is $\varepsilon-$periodically perforated by small holes. The holes are divided into two $\varepsilon-$periodical sets depending on the boundary interaction at their surfaces. Therefore, two different nonlinear Robin boundary conditions $\sigma_\varepsilon(u_\varepsilon) + \varepsilon \kappa_m(u_\varepsilon) = \varepsilon g_\varepsilon(m)$, $m = 1, 2$, are given on the corresponding boundaries of the small holes. The asymptotic analysis of this problem is made as $\varepsilon \to 0$, namely the convergence theorem both for the solution and for the energy integral is proved without using extension operators, the asymptotic approximations both for the solution and for the energy integral are constructed and the corresponding error estimates are obtained.

1 Introduction and statement of the problem

In recent years, a rich collection of new results on asymptotic analysis of boundary-value problems in perforated domains is appeared (see for example [1]-[10]). The classical method proposed by E. Khruslov [11] and D. Cioranescu and J. Saint Jean Paulin [12] is based on a special bounded extension of solutions in Sobolev spaces. It was established by V. Zhikov [8, 9] that the homogenization results can be obtained without using the extension technique in Sobolev spaces in periodically perforated domains. It should be mentioned the paper [2], where the homogenization results for an elliptic problem with a nonlinear boundary condition in a perforated domain were obtained with the help of a new unfolding method that does not need any extension operators as well.

In this paper we use this simple Zhikov’s approach and the scheme of the paper [13], where the full asymptotic analysis (the convergence of the solution and the energy integral, the approximation for the solution and the corresponding asymptotic error estimate in the Sobolev space $H^1$) was made for an elliptic problem with a nonlinear boundary condition in a thick junction.

Let $B$ be a finite union of smooth disjoint nontangent domains strictly lying in the unit square $\square := \{\xi \in \mathbb{R}^n : 0 < \xi_i < 1, \ i = 1, n\}$. In an arbitrary way, we divide $B$ into two sets, $B^{(1)} = \bigcup_{k=1}^{N_1} B_k^{(1)}$ and $B^{(2)} = \bigcup_{k=1}^{N_2} B_k^{(2)}$. Let us introduce the following notations:

$$Q_0 := \square \setminus B, \quad B^{(m)} := \bigcup_{z \in \mathbb{Z}^n} (z + B^{(m)}), \quad B_\varepsilon^{(m)} := \varepsilon B^{(m)} = \{x \in \mathbb{R}^n : \varepsilon^{-1} x \in B^{(m)}\}, \quad m = 1, 2,$$
where $\varepsilon$ is a small parameter. Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$. Define the following perforated domain $\Omega_\varepsilon = \Omega \setminus (B^{(1)}_\varepsilon \cup B^{(2)}_\varepsilon)$ and require the domain $\Omega_\varepsilon$ to be a domain with the Lipschitz boundary. Denote $\Gamma_\varepsilon = \partial \Omega \cap \Omega_\varepsilon$ and $\Xi^{(m)}_\varepsilon = \Omega \cap \partial B^{(m)}_\varepsilon$, $m = 1, 2$, $\Xi_\varepsilon = \Xi^{(1)}_\varepsilon \cup \Xi^{(2)}_\varepsilon$. (see Fig. 1).

Figure 1:

Let $a_{ij}(\xi), \xi \in \mathbb{R}^n, i, j = 1, \ldots, n$, be smooth 1–periodic functions such that

1) $\forall i, j = 1, \ldots, n, \forall \xi \in \mathbb{R}^n : a_{ij}(\xi) = a_{ji}(\xi)$,
2) $\exists \kappa_1 > 0 \exists \kappa_2 > 0 \forall \xi \in \mathbb{R}^n \forall \eta \in \mathbb{R}^n : \kappa_1 |\eta|^2 \leq a_{ij}(\xi) \eta_i \eta_j \leq \kappa_2 |\eta|^2. \quad (1)$

Remark 1. Here and in the sequel we adopt the Einstein convention of summation over repeated indexes.

Let $f_\varepsilon, f_0, g_\varepsilon^{(m)}, g_0^{(m)}$ be given functions such that $f_\varepsilon, f_0 \in L^2(\Omega), g_\varepsilon^{(m)}, g_0^{(m)} \in H_0^1(\Omega)$ and

$$f_\varepsilon \overset{a}{\rightarrow} f_0 \quad \text{in} \quad L^2(\Omega), \quad g_\varepsilon^{(m)} \overset{w}{\rightarrow} g_0^{(m)} \quad \text{weakly in} \quad H^1(\Omega), \quad m = 1, 2. \quad (2)$$

The given functions $\kappa_m : \mathbb{R} \rightarrow \mathbb{R}, \quad m = 1, 2,$ are Lipschitz continuous (it is equivalent that $\kappa_m \in W^{1,\infty}_{loc}(\mathbb{R})$) and such that

$$\exists c_1 > 0 \exists c_2 > 0 : \ c_1 \leq \kappa'_m \leq c_2 \quad \text{a.e. in} \ \mathbb{R} \quad (m = 1, 2). \quad (3)$$

In the perforated domain $\Omega_\varepsilon$ we consider the following nonlinear problem

$$\begin{cases}
-L_\varepsilon(u_\varepsilon) = f_\varepsilon & \text{in} \ \Omega_\varepsilon, \\
\sigma_\varepsilon(u_\varepsilon) + \varepsilon \kappa_1(u_\varepsilon) = \varepsilon g_\varepsilon^{(1)} & \text{on} \ \Xi^{(1)}_\varepsilon, \\
\sigma_\varepsilon(u_\varepsilon) + \varepsilon \kappa_2(u_\varepsilon) = \varepsilon g_\varepsilon^{(2)} & \text{on} \ \Xi^{(2)}_\varepsilon, \\
u_\varepsilon = 0 & \text{on} \ \Gamma_\varepsilon,
\end{cases} \quad (4)$$

where $L_\varepsilon(u_\varepsilon) \equiv \partial_{\xi_i} \left(a_{ij}(\xi) \partial_{\xi_j} u_\varepsilon(x)\right), \ \sigma_\varepsilon(u_\varepsilon) \equiv a_{ij}(\xi) \partial_{\xi_j} u_\varepsilon(x) \nu_i, \ a_{ij}(\xi) \equiv a_{ij}(\xi), \ \partial_{\xi_i} u = \frac{\partial u}{\partial \xi_i}.$

($\nu_1(\xi), \ldots, \nu_n(\xi)$) is the outward normal.
Recall that a function \( u_\varepsilon \) from the Sobolev space \( H^1(\Omega_\varepsilon, \Gamma_\varepsilon) = \{ u \in H^1(\Omega_\varepsilon) : u|_{\Gamma_\varepsilon} = 0 \} \) is a weak solution to problem [4] if the following integral identity

\[
\int_{\Omega_\varepsilon} a_{ij}(x) \partial_{x_j} u_\varepsilon \partial_{x_i} \phi \, dx + \varepsilon \sum_{m=1}^{2} \int_{\Xi_m} \kappa_m(u_\varepsilon) \phi \, ds_x = \int_{\Omega_\varepsilon} f_\varepsilon \phi \, dx + \varepsilon \sum_{m=1}^{2} \int_{\Xi(m)} g_{\varepsilon}^{(m)} \phi \, ds_x
\]

holds for any function \( \phi \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon) \).

Our goal is to study the asymptotic behavior of \( u_\varepsilon \) as \( \varepsilon \to 0 \). Also it will be understandable further how conduct research in the case of \( p \)-multiphase interactions in perforated domains.

## 2 Auxiliary uniform estimates

Let \( H^1_{\text{per}}(Q_0) = \{ v \in H^1(Q_0) : v - 1 \text{-periodic in } \xi_1, \ldots, \xi_n \} \). Obviously, we can periodically extend every function \( v \) from \( H^1_{\text{per}}(Q_0) \) into \( H^1_{\text{loc}}(\mathbb{R}^n \setminus (B^{(1)} \cup B^{(2)}) \} \); this extension will be denoted again by \( v \). Let \( \psi_0^{(m)} \in H^1_{\text{per}}(Q_0), m = 1, 2 \), be weak solutions to the corresponding problems

\[
\begin{align*}
\mathcal{L}_{\xi \xi}(\psi_0^{(1)}) &= q_1 \text{ in } Q_0, \\
\sigma_\xi(\psi_0^{(1)}) &= 0 \text{ on } S^{(1)}, \\
\langle \psi_0^{(1)} \rangle_{Q_0} &= 0,
\end{align*}
\]

\[
\begin{align*}
\mathcal{L}_{\xi \xi}(\psi_0^{(2)}) &= q_2 \text{ in } Q_0, \\
\sigma_\xi(\psi_0^{(2)}) &= 0 \text{ on } S^{(1)}, \\
\langle \psi_0^{(2)} \rangle_{Q_0} &= 0,
\end{align*}
\]

where \( \mathcal{L}_{\xi \xi}(\psi) \equiv \partial_{\xi_i}(a_{ij}(\xi)\partial_{\xi_j}(\psi(\xi))), \sigma_\xi(\psi) = a_{ij}\partial_{\xi_j}(\psi(\xi))\nu_i(\xi), (\nu_1, \ldots, \nu_n) \) is the outward normal to \( S, S = S^{(1)} \cup S^{(2)}, S^{(m)} = \partial B^{(m)}, q_m = \frac{|S^{(m)}|}{|Q_0|}, |S^{(m)}| = \text{meas}_2 S^{(m)} (m = 1, 2), |Q_0| = \text{meas}_3 Q_0 \), \( \langle \psi \rangle_{Q_0} = \int_{Q_0} \psi(\xi) \, dx \). The existence and uniqueness of the solutions to problems [6] follows from the lemma.

**Lemma 1.** Let \( F_i \in L^2(Q_0), i = \overline{1,n} \), \( F_{n+1}^{(m)} \in L^2(S^{(m)}) \), \( m = 1, 2 \). There exists a solution \( N \in H^1_{\text{per}}(Q_0) \) to the following problem

\[
\begin{align*}
-L_{\xi \xi}(N) &= F_0 + \partial_{\xi_i} F_i \text{ in } Q_0, \\
\sigma_\xi(N) &= -F_i \nu_i + F_{n+1}^{(1)} \text{ on } S^{(1)}, \\
\sigma_\xi(N) &= -F_i \nu_i + F_{n+1}^{(2)} \text{ on } S^{(2)},
\end{align*}
\]

if and only if

\[
\langle F_0 \rangle_{Q_0} + \langle F_{n+1}^{(1)} \rangle_{S^{(1)}} + \langle F_{n+1}^{(2)} \rangle_{S^{(2)}} = 0.
\]

In addition this solution is defined up to an additive constant.

The proof is standard (see for instance [5]). Then the \( \varepsilon \)-periodic functions \( \psi_0^{(m)}(\frac{x}{\varepsilon}), x \in \Omega_\varepsilon, m = 1, 2 \), satisfy the following relations

\[
\begin{align*}
\frac{\partial}{\partial x_i}(a_{ij}(x) \frac{\partial}{\partial x_j}(\psi_0^{(1)}(\frac{x}{\varepsilon}))) &= \varepsilon^{-2}q_1 \text{ in } \Omega_\varepsilon, \\
\sigma_\varepsilon(\psi_0^{(1)}(\frac{x}{\varepsilon})) &= \varepsilon^{-1} \text{ on } \Xi_\varepsilon^{(1)}, \\
\sigma_\varepsilon(\psi_0^{(1)}(\frac{x}{\varepsilon})) &= 0 \text{ on } \Xi_\varepsilon^{(2)}, \\
\frac{\partial}{\partial x_i}(a_{ij}(x) \frac{\partial}{\partial x_j}(\psi_0^{(2)}(\frac{x}{\varepsilon}))) &= \varepsilon^{-2}q_2 \text{ in } \Omega_\varepsilon, \\
\sigma_\varepsilon(\psi_0^{(2)}(\frac{x}{\varepsilon})) &= 0 \text{ on } \Xi_\varepsilon^{(1)}, \\
\sigma_\varepsilon(\psi_0^{(2)}(\frac{x}{\varepsilon})) &= \varepsilon^{-1} \text{ on } \Xi_\varepsilon^{(2)}.
\end{align*}
\]
Multiplying with arbitrary function \( \varphi \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon) \) the corresponding differential equation, integrating over \( \Omega_\varepsilon \) and taking into account the boundary conditions, we get the following integral identities

\[
\varepsilon \int_{\Xi^{(m)}} \varphi \ ds_x = \varepsilon \int_{\Omega_\varepsilon} a^{ij}_\varepsilon(x) \partial_{x_i} \psi_0^{(m)}(\varepsilon \xi) \partial_{x_j} \varphi \ dx + q_m \int_{\Omega_\varepsilon} \varphi \ dx \quad m = 1, 2. \tag{9}
\]

Due to the regularity properties of solutions to elliptic problems we have

\[
\sup_{x \in \Omega_\varepsilon} |\nabla_\varepsilon \psi_0^{(m)}(\varepsilon \xi)|_{\xi = \frac{x}{\varepsilon}} = \sup_{\xi \in \Xi_0} |\nabla_\varepsilon \psi_0^{(m)}(\xi)| \leq C_0 \quad (m = 1, 2). \tag{10}
\]

Using Cauchy’s inequality with \( \delta \) \( (a b \leq \delta a^2 + \frac{b^2}{4\delta}; \ a, b, \delta > 0) \) and (10), we deduce from (9) the following estimates \( (m=1, 2) \)

\[
\varepsilon \int_{\Xi^{(m)}} \varphi^2 \ ds_x \leq C_1 \left( \varepsilon^2 \int_{\Omega_\varepsilon} |\nabla_x \varphi|^2 \ dx + \int_{\Omega_\varepsilon} \varphi^2 \ dx \right), \tag{11}
\]

\[
\int_{\Omega_\varepsilon} \varphi^2 \ dx \leq C_2 \left( \varepsilon^2 \int_{\Omega_\varepsilon} |\nabla_x \varphi|^2 \ dx + \varepsilon \int_{\Xi^{(m)}} \varphi^2 \ ds_x \right) \quad \forall \varphi \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon), \tag{12}
\]

where the constant \( C_1 \) and \( C_2 \) are independent of \( \varepsilon \).

**Remark 2.** In what follows all constants \( \{C_i\} \) and \( \{c_i\} \) in inequalities are independent of the parameter \( \varepsilon \).

It follows from (11) and (2) that

\[
\sqrt{\varepsilon} \sum_{m=1}^{2} \|g^{(m)}_\varepsilon\|_{L^2(\Xi^{(m)})} \leq C_3. \tag{13}
\]

Also with the help of (11) and (12) it is easy to prove that the usual norm \( \|\cdot\|_{H^1(\Omega_\varepsilon)} \) is uniformly equivalent with respect to \( \varepsilon \) to a new norm

\[
\|u\|_\varepsilon := \left( \int_{\Omega_\varepsilon} |\nabla u|^2 \ dx + \varepsilon \int_{\Xi_\varepsilon} u^2 \ ds_x \right)^{1/2}
\]

in the space \( H^1(\Omega_\varepsilon, \Gamma_\varepsilon) \), i.e., there exist constants \( C_3 > 0, \ C_4 > 0 \) and \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) and \( u \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon) \) the following relations hold

\[
C_3 \|u\|_{H^1(\Omega_\varepsilon)} \leq \|u\|_\varepsilon \leq C_4 \|u\|_{H^1(\Omega_\varepsilon)}. \tag{14}
\]

### 2.1 Existence and uniqueness of the solution to problem (4)

Associated with (4), we consider the energy functional

\[
I_\varepsilon[u] := \frac{1}{2} \int_{\Omega_\varepsilon} a^{ij}_\varepsilon(x) \partial_x_i u \partial_x_j u \ dx + \varepsilon \sum_{m=1}^{2} \left( \int_{\Xi^{(m)}} K^{(m)}(u) \ ds_x - \int_{\Xi^{(m)}} g^{(m)}_\varepsilon u \ ds_x \right) - \int_{\Omega_\varepsilon} f_\varepsilon u \ dx \tag{15}
\]

on \( H^1(\Omega_\varepsilon, \Gamma_\varepsilon) \), where

\[
K^{(m)}(z) = \int_{0}^{z} \kappa_m(t) \ dt \quad \forall \ z \in \mathbb{R}, \quad m = 1, 2. \tag{16}
\]

It is easy to prove that if \( u_\varepsilon \) is a minimizer of \( I_\varepsilon \) at a fixed value of \( \varepsilon \), then \( u_\varepsilon \) is a weak solution to problem (4).
Theorem 1. At each fixed value of $\varepsilon$ problem (4) has exactly one solution $u_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ for which the following estimate

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C_1 \left( 1 + \|f_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon} \sum_{m=1}^{2} \|g_\varepsilon^{(m)}\|_{L^2(\Xi_\varepsilon^{(m)})} \right) \leq C_2$$

holds, where the constants $C_1$ and $C_2$ are independent of $\varepsilon$, $f_\varepsilon$, $g_\varepsilon^{(m)}$ and $u_\varepsilon$.

Proof. Integrating inequalities in (3), we obtain

$$c_1 t^2 + \kappa_m(0) t \leq \kappa_m(t) t \leq c_2 t^2 + \kappa_m(0) t \quad \forall t \in \mathbb{R},$$

whence it follows that

$$\frac{c_1}{2} z^2 + \kappa_m(0) z \leq K^{(m)}(z) \leq \frac{c_2}{2} z^2 + \kappa_m(0) z \quad \forall z \in \mathbb{R} \quad m = 1, 2.$$ (19)

Using (14), (18), (19), (2) and the same arguments as in Theorem 1 ([13]), we can prove the coercitivity condition on $I$, i.e., the following inequality

$$I_\varepsilon[u] \geq C_1 \|u\|^2_{H^1(\Omega_\varepsilon)} - C_2$$

holds for any function $u \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$.

With the help of (9) we can re-write the energy functional as

$$I_\varepsilon[u] = \frac{1}{2} \int_{\Omega_\varepsilon} a_{ij}^\varepsilon(x) \partial_{x_i} u \partial_{x_j} u \, dx + \sum_{m=1}^{2} \left( \varepsilon \int_{\Omega_\varepsilon} a_{ij}^\varepsilon \partial_{x_i} (\psi_0(x)) |_{\xi=x} \kappa_m(u) \partial_{x_j} u \, dx + q_\varepsilon \int_{\Omega_\varepsilon} K^{(m)}(u) \, dx - \varepsilon \int_{\Omega_\varepsilon} a_{ij}^\varepsilon \partial_{x_i} (\psi_0(x)) |_{\xi=x} (u \partial_{x_j} g_\varepsilon^{(m)} + g_\varepsilon^{(m)} \partial_{x_j} u) \, dx - q_\varepsilon \int_{\Omega_\varepsilon} g_\varepsilon^{(m)} u \, dx \right) - \int_{\Omega_\varepsilon} f_\varepsilon u \, dx.$$ (20)

Consider the function

$$L(p, t, x) = \frac{1}{2} a_{ij}^\varepsilon p_i p_j + \sum_{m=1}^{2} \left( \varepsilon a_{ij}^\varepsilon \partial_{x_i} (\psi_0(x)) |_{\xi=x} \kappa_m(t) p_i + q_m K^{(m)}(t) - \varepsilon a_{ij}^\varepsilon \partial_{x_i} (\psi_0(x)) |_{\xi=x} (t \partial_{x_j} g_\varepsilon^{(m)} + g_\varepsilon^{(m)} p_i) - q_m t g_\varepsilon^{(m)} \right) - f_\varepsilon t.$$ (21)

Since

$$\partial_{p_i p_j} L(p, t, x) \eta_i \eta_j = 2^{-1} a_{ij}^\varepsilon(x) \eta_i \eta_j \eta_j \geq \kappa_1 |\eta|^2 \quad \forall p, \eta \in \mathbb{R}^n, \quad x \in \Omega_\varepsilon,$$

the function $L$ is uniformly convex in $p$ for each $x \in \Omega_\varepsilon$. This means that $I[.]$ is weakly lower semicontinuous on $H^1(\Omega_\varepsilon, \Gamma_\varepsilon)$ and there exists at least one minimizer (see [14, Chapter 8.2]).

Thanks to (3) it is easy to prove the uniqueness of this minimizer (see Theorem 1 ([13])).

Finally, let us deduce the uniform estimate (17). Denote by $u_\varepsilon$ the solution to problem (4). Setting $\varphi = u_\varepsilon$ in (5) and taking into account (11) and the left inequality in (18), we get

$$\kappa_1 \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx + \varepsilon c_1 \int_{\Xi_\varepsilon} u_\varepsilon^2 \, ds_x + \varepsilon \kappa_m(0) \sum_{m=1}^{2} \int_{\Xi_\varepsilon^{(m)}} u_\varepsilon \, ds_x \leq \int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon \, dx + \varepsilon \sum_{m=1}^{2} \int_{\Xi_\varepsilon^{(m)}} g_\varepsilon^{(m)} u_\varepsilon \, ds_x.$$ (17)
from which

\[
| 2 \left( \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx + \varepsilon \int_{\Xi_\varepsilon} u_\varepsilon^2 \, d\sigma_x \right) \leq c_3 \varepsilon \|u_\varepsilon\|_{L^2(\Xi_\varepsilon)} + \|f_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \\
+ \varepsilon \sum_{m=1}^2 \|g_\varepsilon^{(m)}\|_{L^2(\Xi_\varepsilon^{(m)})} \|u_\varepsilon\|_{L^2(\Xi_\varepsilon^{(m)})}.
\]

Using (14) and (11), we derive the first part of the estimate (17) from the last inequality, and then the second one on the basis of (2) and (13).

\[\Box\]

3 Convergence theorem

In the sequel, \(\bar{y}\) denotes the zero-extension of a function \(y\) defined on \(\Omega_\varepsilon\) into the domain \(\Omega\).

Also we introduce the following characteristic function

\[\chi_{Q_0^0}(x) = \begin{cases} 1, & x \in Q_0, \\ 0, & x \in \Box \setminus Q_0. \end{cases}\] (21)

It is known that \(\chi_{Q_0^0}(x) := \chi_{Q_0^0}(\varepsilon^\omega) \rightharpoonup |Q_0|\) weakly in \(L^2(\Omega)\) as \(\varepsilon \to 0\).

**Lemma 2.** Let \(\{v_\varepsilon\}_{\varepsilon > 0}\) be a sequence in \(H^1(\Omega_\varepsilon, \Gamma_\varepsilon)\) uniformly bounded in \(\varepsilon\) in \(H^1(\Omega_\varepsilon, \Gamma_\varepsilon)\) and such that

\[
\kappa_\varepsilon^{(m)}(v_\varepsilon) \rightharpoonup \zeta \text{ weakly in } L^2(\Omega) \text{ as } \varepsilon \to 0 \quad (m = 1, 2).
\]

Then for any function \(\varphi \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)\)

\[
\varepsilon \int_{\Xi_\varepsilon^{(m)}} \kappa_\varepsilon^{(m)}(v_\varepsilon) \varphi \, ds_x \to g_m \int_{\Omega} \zeta(x) \varphi(x) \, dx \quad \text{as } \varepsilon \to 0 \quad (m = 1, 2).
\] (22)

**Proof.** By virtue of (9) we have

\[
\varepsilon \int_{\Xi_\varepsilon^{(m)}} \kappa_\varepsilon^{(m)}(v_\varepsilon) \varphi \, ds_x = \varepsilon \int_{\Omega_\varepsilon} a_{ij}^\varepsilon(x) \partial_x v_j^\varepsilon(\varepsilon^\omega)(\varepsilon^\omega_x \kappa^{(m)}(v_\varepsilon) \partial_x v_\varepsilon \varphi + \kappa(v_\varepsilon) \partial_x v_\varepsilon \varphi) \, dx + \\
+ q_m \int_{\Omega} \kappa_\varepsilon^{(m)}(v_\varepsilon) \varphi \, dx, \quad m = 1, 2.
\]

Thanks to the Lemma’s condition, (3) and (10), the first summand vanishes and the second one tends to \(g_m \int_{\Omega} \zeta(x) \varphi \, dx\) as \(\varepsilon \to 0\) \(m = 1, 2\).

\[\Box\]

**Remark 3.** From Lemma 2 it follows that for any sequence \(\{v_\varepsilon\}_{\varepsilon > 0} \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)\), which is uniformly bounded with respect to \(\varepsilon\), there exists a subsequence \(\{\varepsilon^{(m)}\} \subset \{\varepsilon\}\) (again denoted by \(\{\varepsilon^{(m)}\}\)) such that the convergences (22) hold.

Using (2), we can prove similarly as in Lemma 2 that for any function \(\varphi \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon)\)

\[
\varepsilon \int_{\Xi_\varepsilon^{(m)}} g_\varepsilon^{(m)}(v_\varepsilon) \varphi \, ds_x \to |S^{(m)}| \int_{\Omega} g_\varepsilon^{(m)}(x) \varphi(x) \, dx \quad \text{as } \varepsilon \to 0 \quad (m = 1, 2).
\] (23)
Consider 1-periodic solutions $T_l$, $l = 1, \ldots, n$, to the following problems

$$
\begin{cases}
L_{\xi}(T_l) = -\partial_{\xi} a_{ij} & \text{in } Q_0, \\
\sigma(T_l) = -a_{ij} \nu_i & \text{on } S, \quad \langle T_l \rangle_{Q_0} = 0.
\end{cases}
$$

(24)

From Lemma 1 it follows the existence and uniqueness of the solutions to these problems.

With the help of $T_l$, $l = 1, \ldots, n$, we define the coefficients of the homogenized matrix $\{\hat{a}_{ij}\}$ by the formula

$$
\hat{a}_{ij} = \langle a_{ij} + a_{ik} \partial_{\xi k} T_j \rangle_{Q_0}, \quad i, j \in \{1, 2, \ldots, n\}.
$$

(25)

It is easy to see that

$$
\hat{a}_{ij} = \langle a_{kl} \partial_{\xi k} (\xi_i + T_l) \partial_{\xi l} (\xi_j + T_j) \rangle_{Q_0}
$$

(26)
i.e., the matrix $\{\hat{a}_{ij}\}$ is symmetric and it is well known that it is elliptic (see for instance \cite{5}).

**Theorem 2.** For the solution $u_\varepsilon$ to problem (4) there exists the following convergences

$$
\begin{align*}
\tilde{u}_\varepsilon & \rightharpoonup w_{\varepsilon} & |Q_0| v_0 & \text{weakly in } L^2(\Omega), \\
\hat{a}_{ij} \partial_{x_j} u_\varepsilon & \rightharpoonup w_{\varepsilon} & \partial_{x_j} v_0 & \text{weakly in } L^2(\Omega), & i = 1, \ldots, n,
\end{align*}
$$

as $\varepsilon \to 0$, 

(27)

where $v_0$ is a unique weak solution to the following problem

$$
\begin{cases}
-\hat{a}_{ij} \partial_{x_i x_j} v_0(x) + \sum_{m=1}^{\infty} |S^{(m)}| \kappa_m(v_0(x)) = \sum_{m=1}^{\infty} |S^{(m)}| g_0^{(m)}(x) + |Q_0| f_0(x), & x \in \Omega, \\
v_0(x) = 0, & x \in \partial \Omega,
\end{cases}
$$

(28)

which is called homogenized problem for (4).

Furthermore, the following energy convergence holds as $\varepsilon \to 0$:

$$
E_{\varepsilon}(u_\varepsilon) := \int_{\Omega_\varepsilon} a_{ij}^\varepsilon \partial_{x_i} u_\varepsilon \partial_{x_j} u_\varepsilon \, dx + \varepsilon \sum_{m=1}^{\infty} \int_{\varepsilon^{(m)}_\varepsilon} \kappa_m(u_\varepsilon) \, u_\varepsilon \, ds_x
\rightarrow
\int_{\Omega} \hat{a}_{ij} \partial_{x_i} v_0 \partial_{x_j} v_0 \, dx + \sum_{m=1}^{\infty} |S^{(m)}| \int_{\Omega} \kappa_m(v_0) \, v_0 \, dx =: E_0(v_0).
$$

(29)

**Proof.** 1. It follows from \cite{17} and \cite{3} that the values

$$
\|\tilde{u}_\varepsilon\|_{L^2(\Omega)}, \quad \|a_{ij}^\varepsilon \partial_{x_j} u_\varepsilon\|_{L^2(\Omega)}, \quad i = 1, \ldots, n, \quad \|\kappa_m(u_\varepsilon)\|_{L^2(\Omega)}, \quad m = 1, 2,
$$

are uniformly bounded with respect to $\varepsilon$. Hence there exists a subsequence $\{\varepsilon'\} \subset \{\varepsilon\}$, again denoted by $\{\varepsilon\}$, such that

$$
\begin{align*}
\tilde{u}_\varepsilon & \rightharpoonup w_{\varepsilon} & |Q_0| v_0 & \text{weakly in } L^2(\Omega), \\
a_{ij}^\varepsilon \partial_{x_j} u_\varepsilon & \rightharpoonup \gamma_i & \text{weakly in } L^2(\Omega), & i = 1, \ldots, n, \\
\kappa_m(u_\varepsilon) & \rightharpoonup \zeta_m & \text{weakly in } L^2(\Omega), & m = 1, 2,
\end{align*}
$$

as $\varepsilon \to 0$, 

(30)

where $v_0$, $\gamma_i$, $i = 1, \ldots, n$, $\zeta_m$, $m = 1, 2$, are some functions which will be determined in what follows.
2. Obviously the $\varepsilon$-periodic functions $T_l(\frac{z}{\varepsilon})$, $l = 1, \ldots, n$, defined in [24] satisfy the following relations

$$
\begin{aligned}
&\partial_{x_i} \left( a_{ij}(\xi) \partial_{\xi_j} T_l(\xi) |_{\xi = \frac{z}{\varepsilon}} \right) + \partial_{x_i} a_{il}(x) = 0 \quad \forall x \in \Omega_{\varepsilon}, \\
&\left( a_{ij}(\xi) \partial_{\xi_j} T_l(\xi) \nu_i(\xi) + a_{il}(\xi) \nu_i(\xi) \right) |_{\xi = \frac{z}{\varepsilon}} = 0 \quad \forall x \in \Xi_{\varepsilon}.
\end{aligned}
$$

Multiplying the first relation by $u_\varepsilon \phi$, where $\phi$ is arbitrary function from $C_0^\infty(\Omega)$, and integrating over $\Omega_{\varepsilon}$, we obtain

$$
\int_{\Omega_{\varepsilon}} \left( a_{ij}(\xi) \partial_{\xi_j} T_l(\xi) + a_{il}(\xi) \right) |_{\xi = \frac{z}{\varepsilon}} \left( u_\varepsilon \partial_{x_i} \phi + \phi \partial_{x_i} u_\varepsilon \right) \, dx = 0, \quad l = 1, n. \tag{31}
$$

Put the following test-function $\varphi(x) = \varepsilon T_l(\frac{x}{\varepsilon}) \phi(x)$, $x \in \Omega_{\varepsilon}$, into the integral identity (5). The result is as follows

$$
\int_{\Omega_{\varepsilon}} a_{ij}(x) \partial_{x_j} u_\varepsilon \partial_{\xi_i} T_l(\xi) |_{\xi = \frac{x}{\varepsilon}} \phi(x) \, dx + \varepsilon \int_{\Omega_{\varepsilon}} a_{ij}(x) \partial_{x_j} u_\varepsilon T_l(\frac{x}{\varepsilon}) \partial_{x_i} \phi(x) \, dx + \\
+ \varepsilon^2 \sum_{m=1}^{2 m} \int_{\Xi_{\varepsilon}} \kappa_m(u_\varepsilon) T_l(\Omega_{\varepsilon}) \partial_{x_i} \phi(\xi) \, d\xi = \varepsilon \int_{\Omega_{\varepsilon}} f_i T_l \phi \, dx + \varepsilon^2 \sum_{m=1}^{2 m} \int_{\Xi_{\varepsilon}} \gamma_{m}^{(\varepsilon)} T_l \phi \, ds. \tag{32}
$$

Using [2], [3] and the identities [9], it follows from (32) that

$$
\int_{\Omega_{\varepsilon}} a_{ij}(x) \partial_{x_j} u_\varepsilon \partial_{\xi_i} T_l(\xi) |_{\xi = \frac{x}{\varepsilon}} \phi(x) \, dx = \mathcal{O}(\varepsilon) \quad \text{as} \quad \varepsilon \to 0, \quad l = 1, n. \tag{33}
$$

Subtracting (32) from (31), we get

$$
\int_{\Omega} \left( a_{ij}(\xi) \partial_{\xi_j} T_l(\xi) + a_{il}(\xi) \right) |_{\xi = \frac{z}{\varepsilon}} \tilde{u}_\varepsilon \partial_{x_i} \phi \, dx + \int_{\Omega} a_{il}(x) \partial_{x_i} \tilde{u}_\varepsilon \phi \, dx = \mathcal{O}(\varepsilon), \quad l = 1, n. \tag{34}
$$

In (34) we regard that the functions $a_{ij} \partial_{\xi_j} T_l + a_{il}$, $l = 1, \ldots, n$, are equal to zero on $B$.

Let us find the limit of the first summand in the left-hand side of (34). At first we note that the limit function $v_0$ in (30) belongs to $H_0^1(\Omega)$ because of the connectedness of the domain $\mathbb{R}^n \setminus \left( B^{(1)}(\Omega) \cup B^{(2)}(\Omega) \right)$ (see [8]-[10]). Since $(a_{ij}(\xi) \partial_{\xi_j} T_l(\xi) + a_{il}(\xi)) \nu_i(\xi) = 0$ at $\xi \in S$ and the vector-functions

$$
F_l = (a_{ij}(\xi) \partial_{\xi_j} T_l(\xi) + a_{il}(\xi)), \ldots, a_{nj}(\xi) \partial_{\xi_j} T_l(\xi) + a_{nl}(\xi)), \quad l = 1, \ldots, n, \tag{35}
$$

are solenoidal in $Q_0$ (see [24]), their zero-extensions into $\square \setminus Q_0$ are also solenoidal in weak sense, i.e.,

$$
\int_{Q_0} F_l(\xi) \cdot \nabla \psi(\xi) \, d\xi = \int_{\square} F_l(\xi) \cdot \nabla \psi(\xi) \, d\xi = 0 \quad \forall \psi \in C_0^\infty(\square), \quad l = 1, \ldots, n.
$$

Then using results by V.V. Zhikov (see [8] Th. 2.1]), we get that

$$
\lim_{\varepsilon \to 0} \int_{\Omega} \left( a_{ij}(\xi) \partial_{\xi_j} T_l(\xi) + a_{il}(\xi) \right) |_{\xi = \frac{z}{\varepsilon}} \tilde{u}_\varepsilon \partial_{x_i} \phi \, dx = \int_{\Omega} a_{il} v_0 \partial_{x_i} \phi \, dx.
$$
As a result, it follows from (34) in the limit passage as $\varepsilon \to 0$ that
\[
\int_\Omega \tilde{a}_{ij} \partial_{x_j} v_0 \partial_{x_i} \phi \, dx + \int_\Omega \gamma_l \phi \, dx = 0 \quad \forall \phi \in C^\infty_0(\Omega), \quad (l = 1, \ldots, n),
\]
i.e.,
\[
\gamma_l(x) = \tilde{a}_{ij} \partial_{x_j} v_0(x) \quad \text{for a.e. } x \in \Omega \quad (l = 1, \ldots, n). \tag{36}
\]

4. Using the extension by zero and the identities (9), we rewrite the integral identity (5) in the following way
\[
\int_\Omega a_{ij} \tilde{\partial}_{x_j} u_\varepsilon \partial_{x_i} \varphi \, dx + 
+ \sum_{m=1}^2 \left( \varepsilon \int_{\Omega_\varepsilon} a_{ij}(x) \partial_{x_j} \psi_0^{(m)}(\xi) |_{\xi = \varepsilon} \left( \kappa_m(u_\varepsilon) \partial_{x_j} u_\varepsilon \varphi + \kappa_m(u_\varepsilon) \partial_{x_i} \varphi \right) \, dx + q_m \int_\Omega \kappa_m(u_\varepsilon) \varphi \, dx \right) = 
= \int \chi_{Q_0} f_\varepsilon \varphi \, dx + \sum_{m=1}^2 \left( \varepsilon \int_{\Omega_\varepsilon} a_{ij}(x) \partial_{x_j} \psi_0^{(m)}(\xi) |_{\xi = \varepsilon} \left( \partial_{x_j} g_\varepsilon^{(m)} \varphi + g_\varepsilon^{(m)} \partial_{x_i} \varphi \right) \, dx + \right.
\left. + q_m \int \chi_{Q_0} g_\varepsilon^{(m)} \varphi \, dx \right) \quad \forall \varphi \in C^\infty_0(\Omega). \tag{37}
\]

It is easy to see that the pointed summands in (37) vanish as $\varepsilon \to 0$; the first one due to (3), (10) and (17), the second one due to (10) and (2).

Taking into account (30), (36) and (2), we pass to the limit in (37) as $\varepsilon \to 0$. As a result we get the identity
\[
\int_\Omega \tilde{a}_{ij} \partial_{x_j} v_0 \partial_{x_i} \varphi \, dx + \sum_{m=1}^2 q_m \int_\Omega \zeta_m \varphi \, dx = |Q_0| \int_\Omega f_0 \varphi \, dx + \sum_{m=1}^2 |S^{(m)}| \int_\Omega g_0^{(m)} \varphi \, dx \tag{38}
\]
for any function $\varphi \in C^\infty_0(\Omega)$. Since the space $C^\infty_0(\Omega)$ is dense in $H_0^1(\Omega)$, identity (38) is valid for any function $\varphi \in H_0^1(\Omega)$.

5. With the help of (2), (5) and (38) we can find that
\[
\lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon) = \lim_{\varepsilon \to 0} \left( \int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon \, dx + \sum_{m=1}^2 \int_{\Omega_\varepsilon} g_\varepsilon^{(m)} u_\varepsilon \, dx \right) = \lim_{\varepsilon \to 0} \left( \int_{\Omega} f_\varepsilon \tilde{u}_\varepsilon \, dx + \right.
\left. + \sum_{m=1}^2 \left( \varepsilon \int_{\Omega_\varepsilon} a_{ij}(x) \partial_{x_j} \psi_0^{(m)} |_{\xi = \varepsilon} \left( \partial_{x_j} g_\varepsilon^{(m)} \tilde{u}_\varepsilon + g_\varepsilon^{(m)} \partial_{x_i} \tilde{u}_\varepsilon \right) \, dx + q_m \int_{\Omega} g_\varepsilon^{(m)} \tilde{u}_\varepsilon \, dx \right) = 
= |Q_0| \int_{\Omega} f_0 v_0 \, dx + \sum_{m=1}^2 |S^{(m)}| \int_{\Omega} g_0^{(m)} v_0 \, dx = \int_{\Omega} \tilde{a}_{ij} \partial_{x_j} v_0 \partial_{x_i} v_0 \, dx + \sum_{m=1}^2 q_m \int_\Omega \zeta_m(x) v_0 \, dx. \tag{39}
\]

6. Now it remains to determine the last summand in (39). For this we will use the method of Browder and Minty, a remarkable technique which somehow applies to the corresponding inequality of monotonicity to justify passing to a weak limit within a nonlinearity.
Thanks to (1) and (3), the inequality of monotonicity in our case reads as follows

\[ \int_{\Omega} a_{ij} \partial_{x_j}(u_\varepsilon - \varphi - \varepsilon T_p \partial_{x_p} \varphi) \partial_{x_i}(u_\varepsilon - \varphi - \varepsilon T_q \partial_{x_q} \varphi) \, dx + \varepsilon \sum_{m=1}^{2} \int_{\Xi_{x}^{m}} (\kappa_{m}(u_\varepsilon) - \kappa_{m}(\varphi))(u_\varepsilon - \varphi) \, ds_x \geq 0 \quad \forall \varphi \in C^\infty_0(\Omega), \] 

(40)

which is equivalent to

\[ \int_{\Omega} a_{ij} \partial_{x_j} u_\varepsilon \partial_{x_i} u_\varepsilon \, dx + \varepsilon \sum_{m=1}^{2} \int_{\Xi_{x}^{m}} \kappa_{m}(u_\varepsilon) u_\varepsilon \, ds_x + \]

\[ + \int_{\Omega} a_{ij} \left( \partial_{x_j} \varphi + \partial_{x_i} T_p \partial_{x_p} \varphi \right) \left( \partial_{x_i} \varphi + \partial_{x_i} T_q \partial_{x_q} \varphi \right) \, dx - 2 \int_{\Omega} a_{ij} \partial_{x_j} u_\varepsilon \partial_{x_i} \varphi \, dx - \]

\[ - 2 \int_{\Omega} a_{ij} \partial_{x_j} u_\varepsilon \partial_{x_i} T_q \partial_{x_q} \varphi \, dx - 2 \varepsilon \int_{\Omega} a_{ij} \left( \partial_{x_i} u_\varepsilon - \partial_{x_j} \varphi - \partial_{x_i} T_p \partial_{x_q} \varphi \right) T_q \partial_{x_i} T_q \partial_{x_q} \varphi \, dx + \]

\[ + \varepsilon^2 \int_{\Omega} a_{ij} T_p T_q \partial_{x_i} \varphi \partial_{x_q} \varphi \, dx - \]

\[ - \varepsilon \sum_{m=1}^{2} \int_{\Xi_{x}^{m}} (\kappa_{m}(\varphi) u_\varepsilon + \kappa_{m}(u_\varepsilon) \varphi - \kappa_{m}(\varphi) \varphi) \, ds_x \geq 0 \quad \forall \varphi \in C^\infty_0(\Omega). \] 

(41)

The limit of the first line in (41) is equal to the right-hand side in (39). The first integral in the second line can be re-written in the form

\[ \int_{\Omega} \left( a_{ij}(\xi) \partial_{\xi_i}(\xi_p + T_p) \partial_{\xi_i}(\xi_q + T_q) \right) \big|_{\xi = \varepsilon \varphi} \partial_{x_p} \varphi \partial_{x_q} \varphi \, dx. \] 

(42)

It follows from (8) that its limit equals \( \int_{\Omega} \widetilde{a}_{pq} \partial_{x_p} \varphi \partial_{x_q} \varphi \, dx \). Due to (33) the integral in third line vanishes. Obviously, the limits of summands in the fourth line are equal to zero. The limits of the integrals in the last line can be found with the help of Lemma 2. As a results we have

\[ \int_{\Omega} \widetilde{a}_{ij} \partial_{x_i} (v_0 - \varphi) \partial_{x_i} (v_0 - \varphi) \, dx + \sum_{m=1}^{2} q_m \int_{\Omega} (\zeta_{m} - |Q_0|\kappa_{m}(\varphi)) (v_0 - \varphi) \, dx \geq 0. \] 

(43)

Evidently, this inequality holds for any function \( \varphi \in H^1_0(\Omega) \).

Fix any \( \psi \in C^\infty_0(\Omega) \) and set \( \varphi := v_0 - \lambda \psi \) (\( \lambda > 0 \)) in (43). We get then

\[ \lambda \int_{\Omega} \widetilde{a}_{ij} \partial_{x_j} \psi \partial_{x_i} \psi \, dx + \sum_{m=1}^{2} q_m \int_{\Omega} (\zeta_{m} - |Q_0|\kappa_{m}(v_0 - \lambda \psi)) \psi \, dx \geq 0 \quad \forall \psi \in C^\infty_0(\Omega). \]

In the limit (as \( \lambda \to 0 \)) we obtain

\[ \int_{\Omega} \sum_{m=1}^{2} q_m (\zeta_{m} - |Q_0|\kappa_{m}(v_0)) \psi \, dx \geq 0. \]
Replacing $\psi$ by $-\psi$, we deduce that in fact quality holds above. Thus

$$\sum_{m=1}^{2} q_m \zeta_m(x) = \sum_{m=1}^{2} |S^{(m)}| \kappa_m(v_0(x)) \quad \text{for a.e. } x \in \Omega. \quad (44)$$

7. Returning to (38), we see that the function $v_0$ satisfies the following integral identity

$$\int_{\Omega} \hat{a}_{ij} \partial_{x_j} v_0 \partial_{x_i} \varphi \, dx + \sum_{m=1}^{2} |S^{(m)}| \int_{\Omega} \kappa_m(v_0) \varphi \, dx = |Q_0| \int_{\Omega} f_0 \varphi \, dx + \sum_{m=1}^{2} |S^{(m)}| \int_{\Omega} g_0^{(m)} \varphi \, dx \quad (45)$$

for any function $\varphi \in H^1_0(\Omega)$. Hence $v_0$ is a weak solution to the limit problem (28). Thanks to (3), this solution is unique.

Due to the uniqueness of the solution to problem (28), the above argumentations hold for any subsequence of $\{\varepsilon\}$ chosen at the beginning of the proof. By replacing (44) in (39), one obtains the convergence of energies (29). \qed

4 Asymptotic approximation to the solution and the energy integral

We take the following approximation

$$\bar{u}_\varepsilon := v_0(x) + \varepsilon T_k(\frac{x}{\varepsilon}) \partial_{x_k} v_0(x) \quad (46)$$

to the solution $u_\varepsilon$. Substituting the difference $u_\varepsilon - \bar{u}_\varepsilon$, we find the residuals both in the differential equation and boundary conditions. Straightforward calculation show that

$$-\mathcal{L}_\varepsilon (u_\varepsilon - \bar{u}_\varepsilon) = f_\varepsilon(x) - f_0(x) - \sum_{m=1}^{2} q_m (g_0^{(m)}(x) - \kappa_m(v_0(x))) +$$

$$\left( a_{ij}(\xi) + a_{ik}(\xi) \partial_{x_j} T_j(\xi) - \frac{1}{|Q_0|} \hat{a}_{ij} \right) \kappa_\varepsilon \partial_{x_i} \partial_{x_j} v_0 + \varepsilon \partial_{x_i} (F_1^\varepsilon(x)), \quad x \in \Omega_\varepsilon; \quad (47)$$

$$\sigma_\varepsilon (u_\varepsilon - \bar{u}_\varepsilon) = -\varepsilon \kappa_m(u_\varepsilon) + \varepsilon g_0^{(m)}(x) - F_i^\varepsilon(x) \nu_i, \quad x \in \Xi^{(m)}_\varepsilon (m = 1, 2), \quad (48)$$

where

$$F_i^\varepsilon(x) = a_{ij}(\xi) T_j(\xi) \partial_{x_i} v_0(x), \quad i = 1, \ldots, n,$$

and

$$(u_\varepsilon - \bar{u}_\varepsilon)|_{\Gamma_\varepsilon} = -\varepsilon T_k(\frac{x}{\varepsilon}) \partial_{x_k} v_0(x). \quad (49)$$

Let $\varphi_\varepsilon$ be a smooth function in $\overline{\Omega}$ such that $0 \leq \varphi_\varepsilon \leq 1$, $\varphi_\varepsilon(x) = 1$ if $\text{dist}(x, \partial \Omega) \leq \varepsilon$, and $\varphi_\varepsilon(x) = 0$ if $\text{dist}(x, \partial \Omega) \geq 2\varepsilon$. Obviously,

$$|\nabla_x \varphi_\varepsilon| \leq c \varepsilon^{-1} \quad \text{in } \overline{\Omega}. \quad (50)$$

With the help of $\varphi_\varepsilon$ we define the following functions

$$\psi_\varepsilon(x) = -\varepsilon \varphi_\varepsilon(x) T_k(\frac{x}{\varepsilon}) \partial_{x_k} v_0(x) \quad \text{and} \quad w_\varepsilon(x) = u_\varepsilon(x) - \bar{u}_\varepsilon(x) - \psi_\varepsilon(x), \quad x \in \overline{\Omega}_\varepsilon.$$
It is easy to verify that \( \text{supp} (\psi) \subset \mathcal{U}_\varepsilon = \{ x \in \Omega : \ \text{dist}(x, \partial \Omega) \leq 2\varepsilon \} \) and \( w_\varepsilon \) is a solution to the following problem

\[
\begin{cases}
-L_\varepsilon (w_\varepsilon) = f_\varepsilon - f_0 - \sum_{m=1}^{2} q_m (g_\varepsilon^{(m)}(x) - \kappa_m(v_0)) + \varepsilon \partial_{\xi_j} (F_i^\psi(x)) + L_\varepsilon (\psi) + \\
+ (a_{ij}(\xi) + a_{ik}(\xi) \partial_{\xi_k} T_j(\xi) - |Q_0|^{-1} \hat{a}_{ij}) \big|_{\xi = \varepsilon z} \partial^2_{\xi_i \xi_j} v_0 \quad \text{in} \ \Omega_\varepsilon; \\
\sigma_\varepsilon (w_\varepsilon) = -\varepsilon \kappa_m(u_\varepsilon) + \varepsilon g_\varepsilon^{(m)}(x) - F_i^\psi(x) \nu_i - \sigma_\varepsilon(\psi_\varepsilon) \quad \text{on} \ \Xi^{(m)} (m = 1, 2); \\
w_\varepsilon = 0 \quad \text{on} \ \Gamma_\varepsilon.
\end{cases}
\]

Multiplying the equation of this problem by \( w_\varepsilon \), then integrating by parts and subtracting identities \([9]\) for \( \varphi_m = \kappa_m(v_0) w_\varepsilon, \ m = 1, 2 \), we get

\[
\varepsilon \sum_{m=1}^{2} \int_{\Xi^{(m)}} (\kappa_m(u_\varepsilon) - \kappa_m(v_0)) w_\varepsilon \, ds_x + \int_{\Omega_\varepsilon} a_{ij}^\varepsilon \partial_{\xi_j} w_\varepsilon \partial_{\xi_i} w_\varepsilon \, dx =
\]

\[
= \int_{\Omega_\varepsilon} (f_\varepsilon - f_0) \varphi \, dx + \sum_{m=1}^{2} \left( \int_{\Xi^{(m)}} g_\varepsilon^{(m)} w_\varepsilon \, ds_x - q_m \int_{\Omega_\varepsilon} g_0^{(m)} w_\varepsilon \, dx \right) - \\
- \varepsilon \sum_{m=1}^{2} \int_{\Omega_\varepsilon} a_{ij}^\varepsilon(x) \partial_{\xi_j} \psi_\varepsilon |_{\xi = \varepsilon z} \partial_{\xi_i} (\kappa_m(v_0) w_\varepsilon) \, dx + \\
+ \int_{\Omega_\varepsilon} (a_{ij}(\xi) + a_{ik}(\xi) \partial_{\xi_k} T_j(\xi) - |Q_0|^{-1} \hat{a}_{ij}) |_{\xi = \varepsilon z} \partial^2_{\xi_i \xi_j} v_0 \, w_\varepsilon \, dx + \\
+ \varepsilon \int_{\Omega_\varepsilon} F_i^\psi \partial_{\xi_i} w_\varepsilon \, dx - \int_{\Omega_\varepsilon} a_{ij}^\varepsilon \partial_{\xi_j} \psi_\varepsilon \partial_{\xi_i} w_\varepsilon \, dx. \quad (51)
\]

Due to \([1], [3] \text{ and } [14] \) the left-hand side of (51) is estimated by the following way

\[
\int_{\Omega_\varepsilon} a_{ij}^\varepsilon \partial_{\xi_j} w_\varepsilon \partial_{\xi_i} w_\varepsilon \, dx + \varepsilon \sum_{m=1}^{2} \int_{\Xi^{(m)}} (\kappa_m(u_\varepsilon) - \kappa_m(v_0)) w_\varepsilon \, ds_x \geq \\
\geq c_1 \left( \int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 \, dx + \varepsilon \int_{\Xi^{(m)}} \psi_\varepsilon^2 \, ds_x \right) - c_2 \varepsilon \int_{\Xi^{(m)}} \left( \varepsilon T_k \partial_{\xi_k} v_0 + \psi_\varepsilon \right) w_\varepsilon \, ds_x \geq \\
\geq c_3 \| w_\varepsilon \|^2_{H^1(\Omega_\varepsilon)} - c_2 \varepsilon \int_{\Xi^{(m)}} \left( \varepsilon T_k \partial_{\xi_k} v_0 + \psi_\varepsilon \right) w_\varepsilon \, ds_x. \quad (52)
\]

Now estimate the summands in the right-hand side of (51). Evidently, \( |\int_{\Omega_\varepsilon} (f_\varepsilon - f_0) \psi_\varepsilon \, dx| \leq \| f_\varepsilon - f_0 \|_{L^2(\Omega_\varepsilon)} \| \psi_\varepsilon \|_{H^1(\Omega_\varepsilon)} \). With the help of \([9], [2] \text{ and } [10] \), we bound the second and third terms:

\[
\left| \sum_{m=1}^{2} \left( \int_{\Xi^{(m)}} g_\varepsilon^{(m)} w_\varepsilon \, ds_x - q_m \int_{\Omega_\varepsilon} g_0^{(m)} w_\varepsilon \, dx \right) \right| = \sum_{m=1}^{2} \left( \int_{\Omega_\varepsilon} a_{ij}^\varepsilon \partial_{\xi_j} \psi_\varepsilon (\xi) |_{\xi = \varepsilon z} \partial_{\xi_i} (g_\varepsilon^{(m)} w_\varepsilon) \, dx \right) + \\
+ q_m \int_{\Omega_\varepsilon} g_\varepsilon^{(m)} w_\varepsilon \, dx - \int_{\Omega_\varepsilon} g_0^{(m)} w_\varepsilon \, dx \right| = c_4 \varepsilon \| w_\varepsilon \|_{H^1(\Omega_\varepsilon)} + c_2 \sum_{m=1}^{2} \| g_\varepsilon^{(m)} - g_0^{(m)} \|_{L^2(\Omega_\varepsilon)} \| w_\varepsilon \|_{H^1(\Omega_\varepsilon)};
\]

\[
\varepsilon \sum_{m=1}^{2} \left| \int_{\Omega_\varepsilon} a_{ij}^\varepsilon(x) \partial_{\xi_j} \psi_\varepsilon |_{\xi = \varepsilon z} \partial_{\xi_i} (\kappa_m(v_0) w_\varepsilon) \, dx \right| \leq
\]

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\[ \leq \varepsilon \int_{\Omega_\varepsilon} |\nabla v_0| |w_\varepsilon| \, dx + \varepsilon c_4 \sum_{m=1}^{2} \int_{\Omega_\varepsilon} |\kappa_m(v_0)| |\nabla w_\varepsilon| \, dx \leq \varepsilon c_5 \|w_\varepsilon\|_{H^1(\Omega_\varepsilon)}. \]

Thank to (25) and the fact that the vector-functions (35) are weak solenoidal in □, it follows from Lemma 16.4 (3) that

\[ \left| \int_{\Omega_\varepsilon} \left( a_{ij}(\xi) + a_{ik}(\xi) \partial_{x_k} T_j(\xi) - \frac{1}{|\xi|^{2\sigma}} \hat{a}_{ij} \right) |\xi|^2 \partial_{x_k} x_j v_0 w_\varepsilon \, dx \right| \leq \varepsilon c_6 \|w_\varepsilon\|_{H^1(\Omega_\varepsilon)}. \]

It is easy to see that \( \varepsilon \int_{\Omega_\varepsilon} \int_{\Omega_\varepsilon} |\nabla v_0| |\nabla w_\varepsilon| \, dx \leq \varepsilon c_6 \|w_\varepsilon\|_{H^1(\Omega_\varepsilon)}. \) The last summand in (51) is estimated with the help of Lemma 1.5 (5) and (50):

\[ \varepsilon \int_{\Omega_\varepsilon} a_{ij}^\varepsilon \partial_{x_j} \psi \partial_{x_j} w_\varepsilon \, dx \leq \varepsilon \int_{\Omega_\varepsilon} a_{ij}^\varepsilon \partial_{x_j, \psi} \partial_{x_k} \psi \partial_{x_j} w_\varepsilon \, dx \leq c_7 \int_{\Omega_\varepsilon} |\nabla v_0| |\nabla w_\varepsilon| \, dx + \varepsilon^2 c_8 \int_{\Omega_\varepsilon} |D^2 v_0| |\nabla w_\varepsilon| \, dx \leq c_7 \|v_0\|_{H^1(\Omega_\varepsilon)} \|w_\varepsilon\|_{H^1(\Omega_\varepsilon)} + \varepsilon^2 c_8 \|v_0\|_{H^2(\Omega)} \|w_\varepsilon\|_{H^1(\Omega_\varepsilon)} \|w_\varepsilon\|_{H^1(\Omega_\varepsilon)}. \]

It is remain to bound the last term in (52). Thanks to (9) and (10) we have

\[ \varepsilon^2 \int_{\Sigma_\varepsilon} |T_k \partial_{x_k} v_0 | w_\varepsilon | \, ds_x \leq 2 \varepsilon^2 \int_{\Omega_\varepsilon} |a_{ij}^\varepsilon \partial_{x_j} \psi \partial_{x_\psi}(T_k \partial_{x_k} \psi \partial_{x_j} v_0) | \partial_{x_k} v_0 \, | w_\varepsilon | \, | \partial_{x_k} v_0 | \, ds_x \leq c_{11} \varepsilon \|v_0\|_{H^2(\Omega)} \|w_\varepsilon\|_{H^1(\Omega_\varepsilon)}. \]

Finally, we conclude from (51), (52) and estimates obtained above that

\[ \|w_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C \left( \varepsilon^\frac{1}{2} + \|f_\varepsilon - f_0\|_{L^2(\Omega_\varepsilon)} + \sum_{m=1}^{2} \|g_\varepsilon^{(m)} - g_0^{(m)}\|_{L^2(\Omega_\varepsilon)} \right). \]

Since \( \|w_\varepsilon\|_{H^1(\Omega_\varepsilon)} \) is bounded above by \( C_2 \varepsilon^\frac{1}{2}, \) we have from (55) that

\[ \|u_\varepsilon - \overline{u}_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C \left( \varepsilon^\frac{1}{2} + \|f_\varepsilon - f_0\|_{L^2(\Omega_\varepsilon)} + \sum_{m=1}^{2} \|g_\varepsilon^{(m)} - g_0^{(m)}\|_{L^2(\Omega_\varepsilon)} \right), \]

where the constant \( C_3 \) is independent of \( \varepsilon. \)

Thus, we have proved the following result.

**Theorem 3.** Between the solution \( u_\varepsilon \) to problem (4) and the approximation function (46) the estimate (56) holds.

With the help of the approximation function (46) and estimate (56) we can obtain an estimate for the energy integrals.
Corollary 1. The following estimate

$$|E_\varepsilon(u_\varepsilon) - E_0(v_0)| \leq C \left( \varepsilon^\frac{1}{2} + \|f_\varepsilon - f_0\|_{L^2(\Omega_\varepsilon)} + \sum_{m=1}^2 \|g_\varepsilon^{(m)} - g_0^{(m)}\|_{L^2(\Omega_\varepsilon)} \right).$$

(57)

is satisfied, where the energy integrals $E_\varepsilon(u_\varepsilon)$ and $E_0(v_0)$ are defined in (29).

Proof. By virtue of (56) we have

$$\partial_x u_\varepsilon = \partial_x v_0 + \partial_{\xi} T_k(\xi)|_{\xi=\varepsilon} \partial_{x_k} v_0 + r_\varepsilon^i(x),$$

where

$$\|r_\varepsilon^i\|_{L^2(\Omega_\varepsilon)} \leq C_1 \left( \varepsilon^\frac{1}{2} + \|f_\varepsilon - f_0\|_{L^2(\Omega_\varepsilon)} + \sum_{m=1}^2 \|g_\varepsilon^{(m)} - g_0^{(m)}\|_{L^2(\Omega_\varepsilon)} \right).$$

Then

$$\int_{\Omega_\varepsilon} a_{ij}^\varepsilon \partial_{x_j} u_\varepsilon \partial_{x_i} u_\varepsilon \, dx = \int_{\Omega_\varepsilon} a_{ij}^\varepsilon \left( \partial_{x_j} v_0 + \partial_{\xi} T_k(\xi)|_{\xi=\varepsilon} \partial_{x_k} v_0 \right) \left( \partial_{x_j} v_0 + \partial_{\xi} T_l(\xi)|_{\xi=\varepsilon} \partial_{x_l} v_0 \right) \, dx + p_\varepsilon,$$

(58)

where

$$p_\varepsilon = 2 \int_{\Omega_\varepsilon} a_{ij}^\varepsilon \left( \partial_{x_j} v_0 + \partial_{\xi} T_k(\xi)|_{\xi=\varepsilon} \partial_{x_k} v_0 \right) r_\varepsilon^j \, dx + \int_{\Omega_\varepsilon} a_{ij}^\varepsilon r_\varepsilon^i \, dx.$$

Taking into account the boundedness of $a_{ij}^\varepsilon$ and $\partial_{\xi} T_l(\xi)$ and estimate (56), we get

$$|p_\varepsilon| \leq c_1 \left( \|v_0\|_{H^1(\Omega)} \left( \int_{\Omega_\varepsilon} r_\varepsilon^i r_\varepsilon^j \, dx \right)^\frac{1}{2} + \int_{\Omega_\varepsilon} r_\varepsilon^i r_\varepsilon^j \, dx \right) \leq$$

$$\leq c_2 \left( \varepsilon^\frac{1}{2} + \|f_\varepsilon - f_0\|_{L^2(\Omega_\varepsilon)} + \sum_{m=1}^2 \|g_\varepsilon^{(m)} - g_0^{(m)}\|_{L^2(\Omega_\varepsilon)} \right).$$

(59)

Due to (2) we can regard here that $\|f_\varepsilon - f_0\|_{L^2(\Omega_\varepsilon)} \leq \|f_\varepsilon - f_0\|_{L^2(\Omega_\varepsilon)}$, similar for other summands.

Let us introduce the following functions

$$H_{kl}(\xi) \equiv a_{ij}(\xi) \partial_{\xi_i} (T_k(\xi) + \xi_k) \partial_{\xi_j} (T_l(\xi) + \xi_l) - \frac{1}{|Q_0|} \hat{a}_{kl}, \quad k, l = 1, \ldots, n.$$

After extending the functions $a_{ij}$, $T_k$, $\partial_{\xi} T_k$, $k = 1, \ldots, n$, by zero to $\square \backslash Q_0$, the functions $H_{kl}$, $k, l = 1, \ldots, n$, will be 1-periodic with zero average over $\square$.

By the same way as we rewrote a summand in (41) (see (42)) and using the functions $H_{kl}$, $k, l = 1, \ldots, n$, and (58), we obtain

$$\begin{align*}
\int_{\Omega_\varepsilon} a_{ij}^\varepsilon \partial_{x_j} u_\varepsilon \partial_{x_i} u_\varepsilon \, dx &- \int_{\Omega} \hat{a}_{ij} \partial_{x_j} v_0 \partial_{x_i} v_0 \, dx = \int_{\Omega} H_{kl}(\xi)|_{\xi=\varepsilon} \partial_{x_k} v_0 \partial_{x_l} v_0 \, dx + \\
&+ \int_{\Omega} \left( \frac{1}{|Q_0|} \chi_{Q_0}(\xi) - 1 \right) \hat{a}_{ij} \partial_{x_j} v_0 \partial_{x_i} v_0 \, dx + p_\varepsilon =: I_1 + I_1 + p_\varepsilon,
\end{align*}$$

(60)

where $\chi_{Q_0}$ is the characteristic function defined in (21). The summand $I_1$ can be estimated by the same way as in the proof of Theorem 1.3 (see [21], Ch. 2). As a result, we have $|I_1| \leq c_1 \varepsilon \|v_0\|_{H^2(\Omega)}^2$. 

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To estimate $I_2$ we note that $\int_\mathbb{R} \left( \frac{1}{|Q_0|} \chi_{Q_0}(\xi) - 1 \right) d\xi = 0$. Therefore, with the help of Lemma 1.1 ([3]) we get $|I_2| \leq c_2 \varepsilon \|v_0\|^2_{H^2(\Omega)}$.

Summarizing (59) and estimates for $I_1$ and $I_2$, from (60) we deduce as follows

$$
\int_{\Omega_\varepsilon} a_{ij}^\varepsilon \partial_\varepsilon \chi_0(\xi) \partial_\varepsilon u \partial_\varepsilon x \, dx - \int_{\Omega} \hat{a}_{ij} \partial_\varepsilon \chi_0 \partial_\varepsilon v_0 \partial_\varepsilon x \, dx \leq C_1 \left( \varepsilon^T + \varepsilon \|f_0\|^2_{L^2(\Omega)} + \|f - f_0\|_{L^2(\Omega_\varepsilon)} + \sum_{m=1}^2 \|g_m^{(m)} - g_0^{(m)}\|_{L^2(\Omega_\varepsilon)} \right). \quad (61)
$$

Now consider the difference

$$
I_3 := \varepsilon \sum_{m=1}^2 \int_{\Omega_\varepsilon} k_m(u_\varepsilon) u_\varepsilon \, dx - \sum_{m=1}^2 |S_m| \int_{\Omega} k_m(v_0) v_0 \, dx.
$$

With the help of integral identities (9) we re-write it in the form

$$
I_3 = \varepsilon \sum_{m=1}^2 \int_{\Omega_\varepsilon} a_{ij}^\varepsilon \partial_\varepsilon \chi_0^{(m)}(\xi) \partial_\varepsilon (k_m(u_\varepsilon) u_\varepsilon) \, dx + \sum_{m=1}^2 q_m \int_{\Omega_\varepsilon} k_m(u_\varepsilon) u_\varepsilon \, dx - \sum_{m=1}^2 |S_m| \int_{\Omega} k_m(v_0) v_0 \, dx.
$$

Due (3), (10) and (17) the first term is not greater than $\varepsilon c_1$. Since

$$
\left| \int_{\Omega_\varepsilon} k_m(u_\varepsilon) u_\varepsilon \, dx - \int_{\Omega_\varepsilon} k_m(v_0) v_0 \, dx \right| \leq c_2 \|u_\varepsilon - v_0\|_{L^2(\Omega_\varepsilon)},
$$

it remains to estimate the following difference

$$
\sum_{m=1}^2 \left| q_m \int_{\Omega_\varepsilon} k_m(v_0) v_0 \, dx - \sum_{m=1}^2 |S_m| \int_{\Omega} k_m(v_0) v_0 \, dx \right| = \sum_{m=1}^2 \left| q_m \chi_{Q_0}(\varepsilon) - \sum_{m=1}^2 |S_m| \right| \int_{\Omega} (q_m^* \chi_{Q_0}(\varepsilon) - |S_m^*|) k_m(v_0) v_0 \, dx.
$$

Thanks to the equality $\int_{\mathbb{R}} (q_m \chi_{Q_0}(\xi) - |S_m^*|) d\xi = 0$ ($q_m = |S_m^*|/|Q_0|$) and Lemma 1.1 ([3]), this difference is bounded by $c_3 \varepsilon \|v_0\|_{H^1(\Omega)}$. Thus, $|I_3| \leq c_4 \varepsilon + c_2 \|u_\varepsilon - v_0\|_{L^2(\Omega_\varepsilon)}$.

Finally, taking into account the previous estimate, (61), (56) and noting that $E_\varepsilon(u_\varepsilon) - E_0(v_0) = I_1 + I_2 + I_3 + p_\varepsilon$, we arrive to (57). $\Box$

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