Dynamics and oscillations of a predator–prey model with modified Leslie–Gower Holling-type II schemes time-dependent delays

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Abstract

A predator–prey system is investigated in this research, which is based on a modified version of the Leslie–Gower scheme and a Holling-type II scheme with time-dependent delays. Using Schauder’s fixed-point theorem, we studied the existence of pseudo almost periodic solution for the suggested model. Based on the suitable Lyapunov functional, sufficient conditions are established for the globally attractive pseudo almost periodic solution. At the end, two numerical examples are presented to demonstrate the effectiveness of our results.

keywords: Prey-predator model, Pseudo almost periodic solution, continuous delays, Global exponential stability.

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1 Introduction

One of the most attractive and significant issues in mathematical ecology is the connection between prey and predator. The predator–prey interaction is heavily influenced by functional responses. As a result, both mathematicians and ecologists have looked at predator–prey systems with a wide variety of functional responses. In particular, Aziz-Alaoui and Daher [3] designed and investigated the following predator–prey model with modified Leslie–Gower and Holling-type II:

\[
\begin{align*}
\frac{du}{dt} &= \left(a_1 - bu - \frac{c_1 v}{k_1 + u}\right) u, \\
\frac{dv}{dt} &= \left(a_2 - \frac{c_2 v}{k_2 + u}\right) v,
\end{align*}
\]

(1)

with initial value \(u(0) = u_0 > 0, v(0) = v_0 > 0\), where \(u(t)\) and \(v(t)\) stand for the prey population size and the predator population size respectively, and \(a_1, a_2, b, c_1, c_2, k_1\) and \(k_2\) are all positives with the ecology meaning as follows:

- \(a_1\): the growth rate of prey
- \(a_2\): the growth rate of predator
- \(b\): measures the strength of competition among individuals of species \(u\)
- \(c_1\): is the maximum value which per capita reduction rate of \(u\) can attain
- \(c_2\): is the maximum value which per capita reduction rate of \(v\) can attain
- \(k_1\): measures the extent to which environment provides protection to prey \(u\)
- \(k_2\): measures the extent to which environment provides protection to predator \(v\)

This model can treat the interaction prey-predator which based on the following assumptions

1. the prey growth following the logistic equation (i.e. \(u'(t) = (a_1 - bu(t))u(t)\)) in the absence of his predator.

2. the link between the attack rate and predator size describe following the Holling type II \(\left(\frac{c_1 v}{k_1 + u}\right)\) which a predator’s rate of prey consumption grows as prey density grows, but finally reaches a plateau (or asymptote) where the rate of consumption remains constant regardless of prey density increases.

3. the Leslie–Gower formula is built on the premise that a predator population’s decrease is proportional to the availability of its favorite food per capita. It is \(\frac{dv}{dt} = a_2 v \left(1 - \frac{u}{C}\right)\), in which the growth of the predator population is of logistic form i.e. \(\frac{dv}{dt} = a_2 v \left(1 - \frac{v}{C}\right)\). Here, "C" measures the carry capacity set by the environmental resources.

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and is proportional to prey abundance, \( C = \alpha u \), where is the conversion factor of prey into predator \([14, 13, 20]\). In the case of severe scarcity, \( v \) can switch over to other populations but its growth will be limited by the fact that its most favorite food \( u \) is not available in abundance. This case can be taken care of by adding a positive constant \( k \) to the denominator, see \([2, 3]\).

Many natural and man-made processes in biology, medicine, and other fields now incorporate time-delays, according to current research. Time delays occur so often in nearly every circumstance that ignoring them is ignoring reality. Kuang \([12]\) mentioned that animals must take time to digest their food before further activities and responses take place, and hence any model of species dynamics without delays is an approximation at best. It is now beyond question that the influence of time-delay owing to the time necessary to transition from egg to adult stage, gestation duration, and other factors must be considered in a better study. The famous works of Macdonald \([17]\), Gopalsmy \([11]\), and Kuang \([12]\) include detailed arguments for the relevance and use of time-delays in realistic models. As a result, the “Ordinary Differential Equation”, which is at the heart of Mathematical Biology, should be replaced by the “Delay Differential Equation”.

Furthermore, the occurrence of almost periodic solutions is one of the most fascinating subjects in qualitative differential equations since they may be used to dynamic of prey-predator system \([15, 18, 21, 22]\). An extension of the almost periodic function is the pseudo almost periodic function. It was defined in \([23]\). It is worth noting that due to their potential applicability in a wide range of fields, almost periodic and pseudo almost periodic solutions have received a lot of attention in the last decade \([11, 4, 6]\).

Roughly speaking, we shall consider the following differential system of predator-prey model which incorporates the Holling type II and a modified Leslie-Gower functional response:

\[
\begin{align*}
u'(t) &= \left( a_1(t) - b(t)u(t) - \frac{c_1(t)v(t - \tau_1(t))}{u(t - \sigma_1(t)) + k_1(t)} \right) u(t); \\
\phi'(t) &= \left( a_2(t) - \frac{c_2(t)v(t - \tau_2(t))}{u(t - \sigma_2(t)) + k_2(t)} \right) v(t),
\end{align*}
\]

where \(a_i, b, c_i, k_i, \tau_i, \sigma_i : [0, +\infty] \rightarrow [0, +\infty], i = 1, 2\), are continuous functions. The term \(\frac{c_1(t)v(t - \tau_1(t))}{u(t - \sigma_1(t)) + k_1(t)}\) is of this equation is called the Leslie-Gower term and the term \(\frac{c_2(t)v(t - \tau_2(t))}{u(t - \sigma_2(t)) + k_2(t)}\) is the Holling II functional response. Pose

\[ r = \sup_{t \in \mathbb{R}} (\tau_i(t), \sigma_i(t); i = 1, 2). \]

Denote by \(BC([-r, 0], \mathbb{R}_+^2)\) the set of bounded continuous functions from \([-r, 0]\) to \(\mathbb{R}_+^2\). If \( z(\cdot) \) is defined on \([-\tau_0, \rho]\) with \(\tau_0, \rho \in \mathbb{R}\), then we define \( z_t \in C([-r, 0], \mathbb{R}_+^2) \) where \( z_t(\theta) = z(t + \theta) \) for all \( \theta \in [-r, 0] \). Notice that we restrict our selves to \(\mathbb{R}_+^2\)-valued functions since only non-negative solutions of \((2)\) are biologically meaningful. So, let us consider the following initial condition

\[ z_{t_0} = \phi, \quad \phi = (\phi_1, \phi_2) \in BC([-r, 0], \mathbb{R}_+^2) \quad \text{and} \quad \phi_1(0), \phi_2(0) > 0. \]

We write \(z_t(t_0, \phi)\) for a solution of the admissible initial value problem \((2)\) and \((3)\). Also, let \([t_0, \eta(\phi)]\) be the maximal right-interval of existence of \(z_t(t_0, \phi)\).

## 2 Preliminaries and definitions

Throughout this paper, for all functions \(f \in BC(\mathbb{R}, \mathbb{R})\), we note:

\[ f^* = \sup_{x \in \mathbb{R}} |f(x)| \quad \text{and} \quad f^i = \inf_{x \in \mathbb{R}} |f(x)|. \]

**Definition 2.1.** \([27]\)**

Let \(f \in BC(\mathbb{R}, \mathbb{X})\). \(f\) is said almost periodic \((a.p)\) if for any \(\epsilon > 0\), there exists \(l_\epsilon > 0\), such that

\[ \exists \tau \in [a, a + l_\epsilon], \forall a \in \mathbb{R}, \quad ||f(x + \tau) - f(x)||_\mathbb{X} < \epsilon, \]

As well know \(\tau\) is called \(\epsilon\) – period of \(f\). We denote by \(AP(\mathbb{R}, \mathbb{X})\) the set of such functions.

It is well known that the set \(AP(\mathbb{R}, \mathbb{X})\) is a Banach space with the supremum norm:

\[ ||f||_\infty = \sup_{x \in \mathbb{R}} |f(x)|. \]
In the early 1990’s, the concept of pseudo almost periodicity (p.a.p) was introduced by Zhang (see [7]). It is a generalization of the almost periodicity. Define the class of functions $PAP_0(\mathbb{R},X)$ as follows:

$$PAP_0(\mathbb{R},X) = \left\{ f \in BC(\mathbb{R},X); \lim_{t \to +\infty} \int_{-T}^{T} ||f(t)||_X dt = 0 \right\}.$$

**Definition 2.2.** ([7])

A function $f \in BC(\mathbb{R},X)$ is called pseudo almost-periodic if it can be expressed as

$$f = f_1 + f_2,$$

where $f_1, f_2 \in AP(\mathbb{R},X) \times PAP_0(\mathbb{R},X)$.

**Proposition 2.3.** ([7]) 1. $(PAP(\mathbb{R},X), ||.||_\infty)$ is Banach space and

$$AP(\mathbb{R},X) \subseteq PAP(\mathbb{R},X) \subseteq BC(\mathbb{R},X).$$

2. For $f \in PAP(\mathbb{R},X)$ and $g \in PAP(\mathbb{R},X)$ with inf$_{t \in \mathbb{R}} |g(t)| > 0$, then $\frac{f}{g} \in PAP(\mathbb{R},X)$.

**Definition 2.4.** (Definition 2.12, [7])

Let $\Omega \subseteq Y$. A continuous function $f : \mathbb{R} \times \Omega \longrightarrow X$ is called pseudo almost periodic (p.a.p) in $t$ uniformly with respect to $x \in \Omega$ if the following conditions are satisfied:

1. $\forall x \in \Omega, f(.,x) \in PAP(\mathbb{R},X)$,

2. for all compact $K$ of $\Omega, \forall \epsilon > 0, \exists \delta > 0, \forall t \in \mathbb{R}, \forall x_1, x_2 \in K,$

$$||x_1 - x_2||_Y \leq \delta \Rightarrow ||f(t,x_1) - f(t,x_2)||_X \leq \epsilon.$$

Denote by $PAP_0(\mathbb{R} \times \Omega;X)$ the set of all such functions.

## 3 Positivity and Bounded of the solution

**Theorem 3.1.** Let $(u,v) \in \mathbb{R}^2$ solution of system (2). If the initial condition (3) is satisfied, then the solution $(u,v)$ is strictly positive.

**Proof.** By integration from $t_0$ into $t$ of the system (2), we have

$$u(t) = \phi_1(0)exp \left( \int_{t_0}^{t} a_1(s) - b(s)u(s) - \frac{c_1(s)v(s - \tau_1(s))}{u(s - \sigma_1(s)) + k_1(s)} ds \right),$$

$$v(t) = \phi_2(0)exp \left( \int_{t_0}^{t} a_2(s) - \frac{c_2(s)v(s - \tau_2(s))}{u(s - \sigma_2(s)) + k_2(s)} ds \right).$$

then it is clear that the solution $(u,v)$ has the same sign as the initial condition (3). Hence, the solution is strictly positive. \[\Box\]

**Definition 3.2.** ([15])

We will say that the solution $(u,v)$ of (2) is:

1. **permanent** if there exists $M_1, M_2 > 0$ such that

$$0 < \liminf_{t \to +\infty} u(t) \leq \limsup_{t \to +\infty} u(t) \leq M_1 \quad \& \quad 0 < \liminf_{t \to +\infty} v(t) \leq \limsup_{t \to +\infty} v(t) \leq M_2$$

2. **uniformly permanent** if there exists $M_1 > m_1 > 0$ and $M_2 > m_2 > 0$ such that

$$m_1 \geq \liminf_{t \to +\infty} u(t) \leq \limsup_{t \to +\infty} u(t) \leq M_1 \quad \& \quad m_2 \leq \liminf_{t \to +\infty} v(t) \leq \limsup_{t \to +\infty} v(t) \leq M_2$$

**Lemme 3.3.** ([5]) Let $a > 0, b > 0$.

1. If $\frac{dx}{dt} \geq x(b - ax)$, then $\lim_{t \to +\infty} x(t) \geq b/a$ for $t \geq 0$ and $x(0) > 0$;

2. If $\frac{dx}{dt} \leq x(b - ax)$, then $\lim_{t \to +\infty} x(t) \leq b/a$ for $t \geq 0$ and $x(0) > 0$.

**Theorem 3.4.** [Uniform permanent]
If
\[
(C0) \quad a_1^t k_1^i - M_2 c_i^a > 0
\]
holds, then any positive solution \((u(t), v(t))^T\) of differential system (2) satisfies
\[
m_1 \leq \lim \inf_{t \to +\infty} u(t) \leq \lim \sup_{t \to +\infty} u(t) \leq M_1; \quad m_2 \leq \lim \inf_{t \to +\infty} v(t) \leq \lim \sup_{t \to +\infty} v(t) \leq M_2.
\]

where
\[
\begin{align*}
M_1 &:= \frac{a_1^t k_1^i}{b^t}, \quad m_1 := \frac{a_1^t k_1^i - M_2 c_i^a}{b^t k_1^i}, \\
M_2 &:= \frac{a_2^t (M_1 + k_2^a)}{c_2^a}, \quad m_2 := \frac{a_2^t (m_1 + k_2^a)}{c_2^a (k_2^a + m_1)}.
\end{align*}
\]

Proof. Let \((u(t), v(t))^T\) be any positive solution of system (2). It follows from the first equation system (2) that
\[
\frac{du(t)}{dt} \leq u(t) \left[ a_1^t - b^t u(t) \right]; \tag{4}
\]
From Lemma 3.3, we get
\[
\lim \sup_{t \to +\infty} u(t) \leq \frac{a_1^t}{b^t} := M_1. \tag{5}
\]
From the second equation in system (2), we have
\[
\frac{dv(t)}{dt} \leq a_2^t v(t). \tag{6}
\]
By integrating (6) from \(t - \tau_2(t)\) to \(t\), we have
\[
v(t) \leq v(t - \tau_2(t)) e^{a_2^t \tau_2(t)} \leq v(t - \tau_2(t)) e^{a_2^t \tau_2}. \tag{7}
\]
Then,
\[
v(t - \tau_2(t)) \geq v(t) e^{(-a_2^t \tau_2)}. \tag{8}
\]
By (8) and the second equation of system (2)
\[
\frac{dv(t)}{dt} \leq v(t) \left[ a_2^t - \frac{c_2^a e^{(-a_2^t \tau_2)}}{M_1 + k_2^a} v(t) \right]. \tag{9}
\]
From Lemma 3.3, we get
\[
v(t) \leq \frac{a_2^t (M_1 + k_2^a)}{c_2^a e^{-a_2^t \tau_2}} := M_2. \tag{10}
\]
By positivity of the solution \((u, v)\) and from (10), we have
\[
\frac{du(t)}{dt} \geq u(t) \left[ a_1^t - \frac{M_2 c_i^a}{k_1^i} - b^t u(t) \right]. \tag{11}
\]
From Lemma 3.3, we obtain
\[
\lim \inf_{t \to +\infty} u(t) \geq \frac{a_1^t k_1^i - M_2 c_i^a}{b^t k_1^i} := m_1. \tag{12}
\]
By integrating second inequality of system (2) from \(t\) to \(t - \tau_2(t)\), we have
\[
v(t) \geq v(t - \tau_2(t)) e^{(-c_2^a M_2 \tau_2(t))}. \tag{12}
\]
Then,
\[
v(t - \tau_2(t)) \leq v(t) e^{(c_2^a M_2 \tau_2)} \leq v(t) e^{(c_2^a M_2 \tau_2)}. \tag{13}
\]
On the other hand, from the 2nd equation of (2), one has
\[
\frac{dv(t)}{dt} \geq v(t) \left[ a_2^t - \frac{c_2^a e^{(c_2^a M_2 \tau_2)}}{M_2^2 + m_1} v(t) \right]. \tag{14}
\]
which yields

\[
\lim_{t \to +\infty} v(t) \geq \frac{a^2_m (m + k^2_j)}{c^2_2 \exp \left( \frac{c^2_2 M^2 r^2_2}{k^2_2 + m_1} \right)} := m_2.
\]

**Remark:** If the condition (C0) is not satisfied, every solution \((u, v)\) of system (2) is permanent. So, at this case we consider \(m_1 = 0\).

## 4 Existence the p.a.p solution

We will offer here adequate criteria that assure the existence of the pseudo almost periodic solution of (2), as stated in the introduction. The following lemmas will be stated in order to show this conclusion.

**Lemma 4.1.** (Theorem 2.17, [3]) If \(f \in PAP_0(\mathbb{R} \times \mathbb{X}; \mathbb{X})\) and for each bounded subset \(B\) of \(\mathbb{X}\), \(f\) is bounded on \(\mathbb{R} \times B\), then the Nymetskii operator

\[N_f : PAP(\mathbb{R}, \mathbb{X}) \to PAP(\mathbb{R}, \mathbb{X})\]

with \(N_f(u) = f(\cdot; u(\cdot))\)

is well defined.

**Lemma 4.2.** Let \(\psi, \tau \in PAP(\mathbb{R}, \mathbb{R})\). Then \(\psi(\cdot - \tau(\cdot)) \in PAP(\mathbb{R}, \mathbb{R})\).

**Proof.** Let \(h(t, z) = \psi(t - z)\), since the numerical application \(\psi\) is continuous function and the space \(PAP(\mathbb{R}, \mathbb{R})\) is a translation invariant then

i. for any \(z \in \mathbb{R}\), the function \(h(\cdot, z) \in PAP(\mathbb{R}; \mathbb{R})\).

ii. for all compact \(K\) of \(\mathbb{R}\), \(\forall \varepsilon > 0, \exists \delta > 0, \forall t \in \mathbb{R}, \forall z_1, z_2 \in K\),

\[|z_1 - z_2| \leq \delta \Rightarrow |h(t, z_1) - h(t, z_2)| \leq \varepsilon.\]

Furthermore \(\psi\) is bounded \((PAP(\mathbb{R}, \mathbb{R}) \subset BC(\mathbb{R}; \mathbb{R}))\), then \(h\) is bounded on \(\mathbb{R} \times B\) where \(B\) is bounded interval. By the lemma 4.1 the Nymetskii operator

\[N_f : PAP(\mathbb{R}, \mathbb{R}) \longrightarrow PAP(\mathbb{R}, \mathbb{R})\]

\[\tau_i \longrightarrow h(\cdot, \tau(\cdot))\]

is well defined for \(\tau \in PAP(\mathbb{R}, \mathbb{R})\). Consequently, \([t \longmapsto \psi(t - \tau(t))] \in PAP(\mathbb{R}, \mathbb{R})\).

**Remark 4.3.** In addition, many authors suppose that \(\phi\) is bounded or uniformly continuous or in \(f_1 \in \mathbb{R}(1 - \tau'(t)) > 0\) for given the result. So this is the first one give the proof for any \(\phi, \tau \in PAP(\mathbb{R}, \mathbb{R})\), we have \(\phi(\cdot - \tau(\cdot)) \in PAP(\mathbb{R}, \mathbb{R})\).

**Lemma 4.4.**

If \(a, b : t \in \mathbb{R} \to \mathbb{R}\) continuous functions, then

\[e^{- \int^t_s a(\alpha + \alpha)du} - e^{- \int^t_s b(\alpha)du} = \int^t_s e^{- \int^r_s a(\alpha + \alpha)du} e^{- \int^r_s b(\alpha)du} (b(r) - a(r + \alpha))dr, \forall \alpha \in \mathbb{R}.\]  

(15)

**Proof.** Let us denote by \(h\) the function defined by \(h(t, s) = g_a(t, s) - g_b(t, s)\), where \(g_a(t, s) = e^{- \int^t_s a(\alpha + \alpha)du}\) and \(g_b(t, s) = e^{- \int^t_s b(\alpha)du}\). Then, by partial differentiation of \(h\) with respect to the first variable, we get

\[d_t h(t, s) = -a(t + \alpha)g_a(t, s) + b(t)g_b(t, s),\]

\[= -a(t + \alpha)h(t, s) + g_b(t, s)(b(t) - a(t + \alpha)).\]

Now, by multiplying by \(g_a(t, r)\) and simplifying, we deduce that

\[g_b(r, s)(b(r) - a(r + \alpha))g_a(t, r) = d_r h(r, s)g_a(t, r) + a(r + \alpha)g_a(t, r)h(r, s),\]

\[= d_r h(r, s)g_a(t, r) + d_r g_a(t, r)h(r, s),\]

\[= d_r (h(r, s)g_a(t, r)).\]

Thus, by integration over the interval \([s, t]\), we have that

\[h(t, s)g_a(t, t) - g_a(t, s)h(s, s) = \int^t_s g_b(r, s)g_a(t, r)(b(r) - a(r + \alpha))dr,\]

which implies the result by noticing that \(g_a(t, t) = 1\) and \(h(s, s) = 0\).
Corollary 4.5. If $a, b : t \in \mathbb{R} \to \mathbb{R}$ continuous functions, then

$$e^{\int_s^t a(u)du} - e^{\int_s^t b(u)du} = \int_s^t e^{\int_s^r a(u)du} e^{\int_r^t b(u)du} (a(r) - b(r)) dr.$$  \hspace{1cm} (16)

Proof.

We just replace $a$ (resp.) with $-a_1$ ($-b_1$ resp.) in lemma 4.4.

Lemma 4.6. If $a, f \in AP(\mathbb{R}, \mathbb{R})$, then the following function

$$F_\infty(t) = \int_{-\infty}^t e^{\int_s^t a(u)du} f(s) ds, \quad F^\infty(t) = \int_t^{+\infty} e^{\int_s^t a(u)du} f(s) ds \in AP(\mathbb{R}), \quad \forall t \in \mathbb{R}$$

Proof. For $a \in AP(\mathbb{R}^+)$ and $f \in AP(\mathbb{R}, \mathbb{R})$, for $\epsilon > 0; \exists \epsilon > 0; \exists \tau \in [\epsilon, n, \epsilon + n]$ where $n \in \mathbb{R}$, then

$$\left|f(t + \tau) - f(t)\right| \leq \epsilon \quad \text{and} \quad \left|a(t + \tau) - a(t)\right| \leq \epsilon, \quad \forall t \in \mathbb{R}.$$  

We pose $\epsilon = \epsilon' \frac{2a^2}{\|f\|_\infty + a}$ where $\epsilon' > 0$. Therefore, applying the lemma 4.4 we get

$$|F_\infty(t + \tau) - F_\infty(t)| = \left|\int_{-\infty}^t e^{\int_s^{t+\tau} a(u-du)du} f(s) ds - \int_{-\infty}^t e^{\int_s^t a(u-du)du} f(s) ds\right|$$

$$= \left|\int_{-\infty}^t e^{\int_s^t a(u-du)du} f(s) ds - \int_{-\infty}^t e^{\int_s^t a(u-du)du} f(s) ds\right| + \int_t^t e^{\int_s^t a(u-du)du} f(s) ds - \int_t^t e^{\int_s^t a(u-du)du} f(s) ds$$

$$\leq \int_{-\infty}^t e^{\int_s^t a(u-du)du} - e^{\int_s^t a(u-du)du} ds \|f\|_\infty + \int_{-\infty}^t e^{\int_s^t a(u-du)du} f(s) ds$$

$$\leq \|f\|_\infty \int_{-\infty}^t \int_{-\infty}^s e^{\int_s^t a(u-du)du} e^{\int_t^t a(u-du)du} |a(r - \tau) - a(r)| dr ds + \frac{\epsilon}{2a}$$

$$\leq \epsilon \|f\|_\infty \int_{-\infty}^t e^{-2a(t-s)} (t-s) ds + \frac{\epsilon}{2a}$$

$$\leq \epsilon \|f\|_\infty \frac{e}{2a^2} + \frac{\epsilon}{2a} = \epsilon'.$$

and, here, we pose $\epsilon = \epsilon'' \frac{2\pi^2}{\|f\|_\infty + a}$ where $\epsilon'' > 0$. Therefore, applying the corollary 4.5 we get

$$|F^\infty(t + \tau) - F^\infty(t)| \leq \|f\|_\infty \frac{\epsilon}{2\pi^2} + \frac{\epsilon}{2a} = \epsilon''.$$  \hspace{1cm} \hfill \Box

Define the non-linear operator $\Upsilon$ as follows, for each $(\phi, \psi) \in PAP(\mathbb{R}, \mathbb{R}) \times PAP(\mathbb{R}, \mathbb{R})$,

$$\Upsilon(\phi, \psi) = (\Upsilon_1(\phi, \psi), \Upsilon_2(\phi, \psi))$$

where

$$\begin{align*}
\Upsilon_1(\phi, \psi)(t) &= \left( \int_t^{+\infty} e^{\int_r^t a_1(\tau)du} \left[ b_1(s) \phi^2(s) + \frac{c_1(s)\psi(s - \tau_1(s))\phi(s)}{\psi(s - \tau_2(s)) + k_1(s)} \right] ds \right) \\
\Upsilon_2(\phi, \psi)(t) &= \left( \int_t^{+\infty} e^{\int_r^t a_2(\tau)du} \left[ c_2(s) \psi(s - \tau_2(s)) \psi(s) \right] ds \right)
\end{align*}$$

and Pose

$$\begin{align*}
f_1(\phi(t), \phi(t - \sigma_1(t)), \psi(t - \tau_1(t))) &= \left( b_1(t) \phi^2(t) + \frac{c_1(t)\psi(t - \tau_1(t))\phi(t)}{\psi(t - \tau_2(t)) + k_1(t)} \right) \\
f_2(\phi(t - \sigma_2(t)), \psi(t), \psi(t - \tau_2(t))) &= \left( c_2(t) \psi(t - \tau_2(t)) \psi(t) \right)
\end{align*}$$

Lemma 4.7.

If all the functions $a_{1,2}, b, c, k_{1,2}, \tau_{1,2}$ and $\sigma_{1,2}$ are p.a.p and positives, Then $\Upsilon$ maps $PAP^2(\mathbb{R}, \mathbb{R}^+)$ into itself.
Proof. Fix $\phi, \psi \in PAP(\mathbb{R}, \mathbb{R}^+)$. It follows from Lemma 4.2 that $\phi(\cdot - \sigma_1(\cdot)), \psi(\cdot - \tau_1(\cdot)) \in PAP(\mathbb{R}, \mathbb{R}^+)$. From $\inf_{s \in \mathbb{R}} \phi(s - \sigma_1(s)) + k_1(s) > 0$ and from properties of space $PAP(\mathbb{R}, \mathbb{R})$, we infer that

$$f_1(\phi(s), \phi(s - \sigma_1(s)), \psi(s - \tau_1(s))) = b_1(s)\phi^2(s) + \frac{c_1(s)\psi(s - \tau_1(s))\phi(s)}{\phi(s - \sigma_1(s)) + k_1(s)} \in PAP(\mathbb{R}_+) \text{.}$$

Similarly stages,

$$f_2(\phi(s - \sigma_2(s)), \psi(s), \psi(s - \tau_2(s))) = \frac{c_2(s)\psi(s - \tau_2(s))\psi(s)}{\phi(s - \sigma_2(s)) + k_2(s)} \in PAP(\mathbb{R}, \mathbb{R}_+) \text{.}$$

Consequently, for all $1 \leq j \leq 2$, $f_j$ can be expressed as

$$f_j = f_{j1} + f_{j2} \text{.}$$

where $f_{j1} \in AP(\mathbb{R}, \mathbb{R}_+) \text{ and } f_{j2} \in PAP_0(\mathbb{R}, \mathbb{R}_+)$. So,

$$\begin{align*}
\Upsilon_j(\phi, \psi)(t) &= \int_{-\infty}^{+\infty} e^{\int_{s}^{s_0} \alpha_j(r)dr} f_{j1}(\phi(s), \phi(s - \sigma_j(s)), \psi(s - \tau_j(s)))ds, \\
&= \int_{-\infty}^{+\infty} e^{\int_{s}^{s_0} \alpha_j(r)dr} \left(f_{j1}(\phi(s), \phi(s - \sigma_j(s)), \psi(s - \tau_j(s))) + f_{j2}(\phi(s), \phi(s - \sigma_j(s)), \psi(s - \tau_j(s)))\right)ds \\
&= I_j(t) + II_j(t) \text{.}
\end{align*}$$

By Lemma 4.6 $I_j(t) \in AP(\mathbb{R}, \mathbb{R}^+)$, for $j = 1, 2$. In the other hand, we prove that $II_j(t) \in PAP_0(\mathbb{R}, \mathbb{R}_+)$, for $j = 1, 2$. We have

$$\int_{-T}^{T} |II_j(t)|dt = \int_{-T}^{T} \int_{-\infty}^{+\infty} e^{\int_{s}^{s_0} \alpha_j(r)dr} f_{j2}(\phi(s), \phi(s - \sigma_j(s)), \psi(s - \tau_j(s)))ds |dt| \text{.}$$

Pose $f_{j2}(s) = f_{j2}(\phi(s), \phi(s - \sigma_j(s)), \psi(s - \tau_j(s)))$ and $\vartheta = t - s$. Then by Fubini Tonnelli’s Theorem one has

$$\frac{1}{2T} \int_{-T}^{T} |II_j(t)|dt = \frac{1}{2T} \int_{-\infty}^{+\infty} e^{\int_{s}^{s_0} \alpha_j(r)dr} f_{j2}(s)ds |dt| \geq \frac{1}{2T} \int_{-\infty}^{+\infty} e^{-\alpha_j(s)} f_{j2}(t - \vartheta) |dt| |\vartheta| \text{.}$$

Pose $s = t - \vartheta$, then

$$\frac{1}{2T} \int_{-T}^{T} |II_j(t)|dt \leq \frac{1}{2T} \int_{-\infty}^{+\infty} e^{-\alpha_j(s)} f_{j2}(s)ds |\vartheta| \text{.}$$

Since the function $f_{j2} \in PAP_0(\mathbb{R}, \mathbb{R}_+)$, then the following function $\Psi_T$

$$\Psi_T(\vartheta) = \frac{T + \vartheta}{T} \frac{1}{2(T + \vartheta)} \int_{-\vartheta}^{T + \vartheta} f_{j2}(s)ds \text{,}$$

is bounded and $\lim_{T \to +\infty} \Psi_T(\vartheta) = 0$. Hence from dominated convergence Theorem, we obtain that $II_j(t) \in PAP_0(\mathbb{R}, \mathbb{R}_+)$, for $j = 1, 2$. So for all $j = 1, 2$, $\Upsilon_j(\phi, \psi)$ belongs to $PAP(\mathbb{R}, \mathbb{R}_+)$ and consequently $\Upsilon$ belongs to $PAP^2(\mathbb{R}, \mathbb{R}_+)$. 

$\square$

Definition 4.8. Let $X \in \mathbb{R}^n$ and let $A(t)$ be a $n \times n$ continuous matrix defined on $\mathbb{R}$. The linear system

$$X'(t) = A(t)X(t) \quad (17)$$

is said to admit an exponential dichotomy on $\mathbb{R}$ if there exist positive constants $k; \ k_1; \ h_2$ a projection $P$ (i.e $P^2 = I$), and the fundamental solution matrix $Y(t)$ of $(17)$ satisfying

$$\|Y(t)PY^{-1}(s)\| \leq ke^{-h_1(t-s)}, \quad t \geq s$$

$$\|Y(t)(I - P)Y^{-1}(s)\| \leq ke^{-h_2(s-t)}, \quad s \geq t$$

where $I$ is the identity matrix.

Lemma 4.9. If the linear system $(17)$ admits an exponential dichotomy and $f \in BC(\mathbb{R}, \mathbb{R}^n)$, then the system

$$X'(t) = A(t)X(t) + f(t) \quad (18)$$

has a bounded solution $\tilde{X}(t)$, and

$$\tilde{X}(t) = \int_{-\infty}^{t} Y(t)PY^{-1}(s)f(s)ds - \int_{t}^{+\infty} Y(t)(I - P)Y^{-1}(s)f(s)ds \text{;} \quad (19)$$

where $Y(t)$ is the fundamental solution matrix of $\{17\}$. 


Theorem 4.10. (Schauder Theorem)[12]:
Let \( M \) be a non-empty convex subset of a normed space \( B \). Let \( T \) be a continuous mapping of \( M \) into a compact set \( H \subset M \). Then \( T \) has at least fixed point.

Let define the following set
\[
M = \{ \phi, \psi \in PAP(\mathbb{R}); m_1 \leq \phi \leq M_1 \text{ and } m_2 \leq \psi \leq M_2 \}.
\]

Theorem 4.11. The differential system (2) has at least pap solution in \( M \).

Proof. Fix \( \phi, \psi \in M \). Let us consider the following differential system
\[
\begin{cases}
u'(t) = a_1(t)u(t) - b(t)\phi^2(t) - c_1(t)\phi(t)\psi(t - \tau_1(t)) / k_1(t) + \phi(t - \sigma_1(t)), \\\psi'(t) = a_2(t)\psi(t) - c_2(t)\psi(t)\psi(t - \tau_2(t)) / k_2(t) + \phi(t - \tau_2(t)).
\end{cases}
\]
(20)

It is clearly that
\[
b(t)\phi^2(t) + c_1(t)\phi(t)\psi(t - \tau_1(t)) / k_1(t) + \phi(t - \sigma_1(t)) \in BC(\mathbb{R}, \mathbb{R}).
\]

By Lemma 4.9, the system (20) has a unique bounded solution given by
\[
\tilde{X}(t) = \left( \begin{array}{c}
\int_t^{+\infty} e^{f_1'(u)} n_1(u) du f_1(\phi(s), \phi(s - \sigma_1(s)), \psi(s - \tau_1(s))) ds \\
\int_t^{+\infty} e^{f_2'(u)} n_2(u) du f_2(\phi(s), \phi(s - \sigma_2(s)), \psi(s - \tau_2(s))) ds
\end{array} \right)
\]
where
\[
\left( f_1(\phi(t), \phi(t - \sigma_1(t)), \psi(t - \tau_1(t))) \right) = \left( \begin{array}{c}
b_1(t)\phi^2(t) + c_1(t)\phi(t - \tau_1(t)) / \phi(t - \sigma_1(t)) + k_1(t) \\
c_2(t)\psi(t - \tau_2(t)) / \phi(t - \sigma_2(t)) + k_2(t)
\end{array} \right).
\]

Then, we consider the operator \( \Upsilon \). By Lemma 4.7 we get \( \Upsilon \) maps \( PAP(\mathbb{R}, \mathbb{R}^+) \times PAP(\mathbb{R}, \mathbb{R}^+) \) into itself. By the way, one has the following inequality
\[
u'(t) \leq a_1^* u(t) - b^* \phi^2(t).
\]
By the same Lemma 4.9 and by definition of \( M_1 \), the equation \( z(t) = a_1^* z(t) - b^* \phi^2(t) \) has a unique bounded solution given by
\[
\tilde{z}(t) = \int_t^{+\infty} e^{z(t-s)} b^* \phi^2(s) ds \text{ and } \tilde{z}(t) \leq M_1.
\]
Using the comparison theorem, we get
\[
\Upsilon_1(\phi, \psi)(t) \leq \tilde{z}(t) \leq M_1.
\]

We consider the inequality (23)
\[
\nu'(t) \leq a_2^* \nu(t) - c_2^* \exp(-a_2^* \tau_2) \psi^2(t).
\]
By the same Lemma 4.9 and by definition of \( M_2 \), the equation \( z'(t) = a_2^* z(t) - c_2^* \exp(-a_2^* \tau_2) \psi^2(t) \) has a unique bounded solution given by
\[
\tilde{z}(t) = \int_t^{+\infty} e^{z(t-s)} c_2^* \exp(-a_2^* \tau_2) M_1 + k_2^* \psi^2(s) ds \text{ and } \tilde{z}(t) \leq M_2.
\]
Using the comparison theorem, we get
\[
\Upsilon_2(\phi, \psi)(t) \leq \tilde{z}(t) \leq M_2.
\]

From the inequality (11), one has
\[
\frac{du(t)}{dt} \geq a_1^* u(t) - \frac{M_2 c_1^*}{k_1^*} \phi(t) - b_1^* \phi^2(t).
\]
By the same Lemma 4.9 and by definition of \( m_1 \), the equation \( z'(t) = a_1^* z(t) - \frac{M_2 c_1^*}{k_1^*} \phi(t) - b_1^* \phi^2(t) \) has a unique bounded solution given by
\[
\tilde{z}(t) = \int_t^{+\infty} e^{z(t-s)} \left( \frac{M_2 c_1^*}{k_1^*} \phi(t) + b_1^* \phi^2(s) \right) ds \text{ and } \tilde{z}(t) \geq m_1.
\]
Using the comparison theorem, we get
\[
\Upsilon_1(\phi, \psi)(t) \geq \tilde{z}(t) \geq m_1.
\]
From the inequality (13), one has
\[
\frac{dv(t)}{dt} \geq a_2^v(t) - \frac{c_2^v e^{\frac{c_2^v M_2^2 t^2}{k_3^2 + m_1^2}}}{m_1 + k_2^2} \psi^2(t).
\]

Using the lemma 4.9 and by definition of \( m_2 \), the equation \( z(t) = a_2 z(t) - \frac{c_2^v e^{\frac{c_2^v M_2^2 t^2}{k_3^2 + m_1^2}}}{m_1 + k_2^2} \psi^2(t) \) has a unique bounded solution given by
\[
\zeta(t) = \int_t^{+\infty} e^{a_2(t-s)} \frac{c_2^v e^{\frac{c_2^v M_2^2 t^2}{k_3^2 + m_1^2}}}{m_1 + k_2^2} \psi^2(s) ds
\]
and \( \zeta(t) \geq m_2 \).

Using the comparison theorem, we get
\[
\Upsilon_2(\phi, \psi)(t) \geq \zeta(t) \geq m_2.
\]

Therefore, \( \Upsilon(M) \subseteq M \). Next step, we prove that \( \Upsilon \) is continuous. For \((\phi_1, \psi_1), (\phi_2, \psi_2) \in M\), such that \( \|\phi_1 - \phi_2\|_\infty \leq \frac{\min(a_1^v, a_2^v)}{2\beta} \), and \( \|\psi_1 - \psi_2\|_\infty \leq \frac{\min(a_1^v, a_2^v)}{2\beta} \), one has
\[
|\Upsilon_1(\phi_1, \psi_1)(t) - \Upsilon_1(\phi_2, \psi_2)(t)| = \int_t^{+\infty} e^{t \Delta(t)} \|f_1(\phi_1(s), \phi_1(s - \tau_1(s)), \psi_1(s - \tau_1(s))) - f_1(\phi_2(s), \phi_2(s - \tau_1(s)), \psi_2(s - \tau_1(s)))\| ds
\]
\[
\leq \int_t^{+\infty} e^{t \Delta(t)} \|b(s)\| \|\phi_1(s) - \phi_2(s)\| ds
\]
\[
+ \int_t^{+\infty} e^{t \Delta(t)} \|c_1(s)\| \|\phi_1(s) - \phi_2(s)\| ds
\]
\[
\leq \int_t^{+\infty} e^{t \Delta(t)} \|b(s)\| \|\phi_1(s) - \phi_2(s)\| ds
\]
\[
+ \int_t^{+\infty} e^{t \Delta(t)} \|c_1(s)\| \|\phi_1(s) - \phi_2(s)\| ds
\]
\[
\leq \frac{M_2}{k_1} \{\|\phi_1 - \phi_2\| + \|\psi_1 - \psi_2\|\} + \frac{M_1}{k_1} \{\|\phi_1 - \phi_2\| + \|\psi_1 - \psi_2\|\}
\]
\[
\leq 2M_1 \|\phi_1 - \phi_2\|_\infty.
\]

For simplicity, pose that
\[
\psi_j(s, \tau_1(s)) = \psi_j(s - \tau_1(s)), \phi_j(s, \sigma_1(s)) = \phi_j(s - \sigma_1(s)), \text{ for } j = 1, 2.
\]

We have
\[
|\psi_j(s, \tau_1(s))\phi_1(s) - \phi_j(s, \sigma_1(s))\phi_2(s)| \leq \psi_j(s, \tau_1(s))|\phi_1 - \phi_2| + \phi_j(s, \sigma_1(s))|\psi_1 - \psi_2| + \frac{M_2}{k_1^2} \|\phi_1 - \phi_2\|_\infty + \frac{M_1 M_2}{(k_1^2)^2} \|\psi_1 - \psi_2\|_\infty.
\]

And we have
\[
|\phi_j^2(s) - \phi_j^2(s)| \leq 2M_1 \|\phi_1 - \phi_2\|_\infty.
\]
Therefore, we obtain
\[
|Φ_1(φ_1, ψ_1)(t) - Φ_1(φ_2, ψ_2)(t)| \leq \int_t^{+∞} e^{∫_s^t a_1(r)dr} \left\{ 2b^s N1∥φ_1 - φ_2∥ + c_1(s) \left( \frac{M_2}{k_1^2} + \frac{M_1 M_2}{(k_1^2)^2} \right) \right\} ds
+ c_1(s) \left( \frac{M_2}{k_1^2} + \frac{M_1 M_2}{(k_1^2)^2} \right) \right\} ds
\]
\leq \int_t^{+∞} e^{a_1(t-s)} \left\{ \left( \frac{2b^s M_1}{k_1} + \frac{c_1^2 M_2}{(k_1^2)^2} \right) \right\} ds
\leq \int_t^{+∞} e^{a_1(t-s)} \left\{ 3α∥φ_1 - φ_2∥ + β∥ψ_1 - ψ_2∥ \right\} ds
= \frac{3α}{a_1^2}∥φ_1 - φ_2∥ + \frac{β}{a_1^2}∥ψ_1 - ψ_2∥
\leq \frac{ε}{2} + \frac{ε}{2} = ε.

By conclusion, Φ_1 is a continuous operator.

Now, we should prove that Φ_2 is a continuous operator.

\[
|Φ_2(φ_1, ψ_1)(t) - Φ_2(φ_2, ψ_2)(t)| = \int_t^{+∞} e^{∫_s^t a_2(s)dr} f_2 (φ_1(s - σ_2(s)), ψ_1(s), ψ_1(s - τ_2(s))) ds
- \int_t^{+∞} e^{∫_s^t a_2(s)dr} f_2 (φ_2(-σ_2(s)s), ψ_2(s), ψ_2(s - τ_2(s))) ds
\]
\[
= \int_t^{+∞} e^{∫_s^t a_2(s)dr} f_2 (φ_1(s - σ_2(s)), ψ_1(s), ψ_1(s - τ_2(s))) \frac{ψ_1(s - τ_2(s))ψ_1(s)}{ψ_1(s - τ_2(s))ψ_1(s)} ds
- \int_t^{+∞} e^{∫_s^t a_2(s)dr} f_2 (φ_2(s - σ_2(s)), ψ_2(s), ψ_2(s - τ_2(s))) \frac{ψ_2(s - τ_2(s))ψ_2(s)}{ψ_2(s - τ_2(s))ψ_2(s)} ds
\]
\[
≤ \int_t^{+∞} e^{∫_s^t a_2(s)dr} \left\{ \frac{ψ_1(s - τ_2(s))ψ_1(s)}{ψ_1(s - τ_2(s))ψ_1(s)} \right\} ds
- \frac{ψ_2(s - τ_2(s))ψ_2(s)}{ψ_2(s - τ_2(s))ψ_2(s)} ds
= \frac{M_2}{k_2^2}∥ψ_1 - ψ_2∥ + \frac{(M_2)^2}{(k_2^2)^2}∥φ_1 - φ_2∥.
\]

For j = 1 or 2, pose that
\[
ψ_j(s) = ψ_j(s - τ_2(s)), φ_j(s) = φ_j(s - σ_2(s)).
\]

We get
\[
\frac{ψ_1(s - τ_2(s))ψ_1(s)}{ψ_1(s - τ_2(s))ψ_1(s)} \leq \frac{ψ_1(s - τ_2(s))ψ_1(s)}{ψ_1(s - τ_2(s))ψ_1(s)} + \frac{ψ_2(s)}{ψ_1(s - τ_2(s))ψ_1(s)} + \frac{ψ_1(s - τ_2(s))ψ_1(s)}{ψ_1(s - τ_2(s))ψ_1(s)} + \frac{ψ_2(s)}{ψ_1(s - τ_2(s))ψ_1(s)}
\]
\[
≤ \frac{2M_2}{k_2^2}∥ψ_1 - ψ_2∥ + \frac{(M_2)^2}{(k_2^2)^2}∥φ_1 - φ_2∥.
\]

Thus, we get
\[
|Φ_2(φ_1, ψ_1)(t) - Φ_2(φ_2, ψ_2)(t)| \leq \int_t^{+∞} e^{∫_s^t a_2(s)dr} c_2(s) \left\{ \frac{2M_2}{k_2^2}∥ψ_1 - ψ_2∥ + \frac{M_2^2}{(k_2)^2}∥φ_1 - φ_2∥ \right\} ds
\leq \int_t^{+∞} e^{a_2(t-s)} \left\{ \frac{2c_2 M_2}{k_2^2}∥ψ_1 - ψ_2∥ + \frac{c_2^2 M_2^2}{(k_2)^2}∥φ_1 - φ_2∥ \right\} ds
= \frac{g}{a_2^2}∥ψ_1 - ψ_2∥ + \frac{g}{a_2}∥φ_1 - φ_2∥
\leq ε.
\]

By conclusion, Φ_2 is a continuous operator.

Next, we need to prove that Φ is a compact operator. In fact, we show that the following two statements are true:

1) \{ Φ(ϕ, ψ); (ϕ, ψ) ∈ M \} ⊆ BC(ℝ, ℝ^2) is equi-continuous.

2) \{ Φ(ϕ, ψ)(t); (ϕ, ψ) ∈ M \} is relatively compact subset of ℝ^2 for each t ∈ ℝ.

To prove 1), given u, v ∈ ℝ, such that u < v and |u - v| < ε, we have
\[
\left| \Upsilon_1(\phi, \psi)(u) - \Upsilon_1(\phi, \psi)(v) \right| = \int_u^v e^{f_u^r a_1(r) dr} \left[ \frac{b(s) \phi^2(s) + c_1(s) \psi(s - \tau_1(s)) \psi(s)}{\phi(s - \sigma_1(s)) + k_1(s)} \right] ds \\
- \int_v^u e^{f_u^r a_1(r) dr} \left[ \frac{b(s) \phi^2(s) + c_1(s) \psi(s - \tau_1(s)) \psi(s)}{\phi(s - \sigma_1(s)) + k_1(s)} \right] ds
\]

\[
= \int_u^v e^{f_u^r a_1(r) dr} \left[ b(s) \phi^2(s) + c_1(s) \psi(s - \tau_1(s)) \psi(s) \right] ds \\
- \int_v^u e^{f_u^r a_1(r) dr} \left[ b(s) \phi^2(s) + c_1(s) \psi(s - \tau_1(s)) \psi(s) \right] ds
\]

\[
\leq \left| 1 - e^{f_u^v a_1(r) dr} \right| \int_u^v e^{f_u^r a_1(r) dr} \left[ b(s) \phi^2(s) + c_1(s) \psi(s - \tau_1(s)) \psi(s) \right] ds \\
+ \int_u^v e^{f_u^r a_1(r) dr} \left[ b(s) \phi^2(s) + c_1(s) \psi(s - \tau_1(s)) \psi(s) \right] ds
\]

Or if \( f_u^v a_1(r) dr > 0 \Rightarrow e^{f_u^v a_1(r) dr} > 1 \). Therefore

\[
\left| \Upsilon_1(\phi, \psi)(u) - \Upsilon_1(\phi, \psi)(v) \right| \leq \left( e^{a_1^v(v-u)} - 1 \right) \int_u^v e^{f_u^r a_1(r) dr} \left[ b(s) \phi^2(s) + c_1(s) \psi(s - \tau_1(s)) \psi(s) \right] ds \\
+ \int_u^v e^{f_u^r a_1(r) dr} \left[ b(s) \phi^2(s) + c_1(s) \psi(s - \tau_1(s)) \psi(s) \right] ds
\]

\[
\leq \left( e^{a_1^v(v-u)} - 1 \right) \int_u^v e^{f_u^r a_1(r) dr} \left[ b(s) \phi^2(s) + c_1(s) \psi(s - \tau_1(s)) \psi(s) \right] ds \\
+ \int_u^v e^{f_u^r a_1(r) dr} \left[ b(s) \phi^2(s) + c_1(s) \psi(s - \tau_1(s)) \psi(s) \right] ds
\]

\[
\leq \frac{e^{a_1^v(v-u)} - 1}{a_1^v} \left[ b^* M_1^2 + c_1^2 M_1 M_2 \right] + \frac{e^{a_1^v(v-u)} - 1}{a_1^v} \left[ b^* M_1^2 + c_1^2 M_1 M_2 \right].
\]

Or

\[
e^{a_1^v(v-u)} - 1 \to v \to 0.
\]

Then

\[
\left| \Upsilon_1(\phi, \psi)(u) - \Upsilon_1(\phi, \psi)(v) \right| \to 0.
\]

And

\[
\left| \Upsilon_2(\phi, \psi)(u) - \Upsilon_2(\phi, \psi)(v) \right| = \int_u^v e^{f_u^r a_2(r) dr} \left[ c_2(s) \psi(s - \tau_2(s)) \psi(s) \right] ds \\
- \int_v^u e^{f_u^r a_2(r) dr} \left[ c_2(s) \psi(s - \tau_2(s)) \psi(s) \right] ds
\]

\[
\leq \left| 1 - e^{f_u^v a_2(r) dr} \right| \int_u^v e^{f_u^r a_2(r) dr} \left[ c_2(s) \psi(s - \tau_2(s)) \psi(s) \right] ds \\
+ \int_u^v e^{f_u^r a_2(r) dr} \left[ c_2(s) \psi(s - \tau_2(s)) \psi(s) \right] ds
\]

\[
\leq 2 \left( e^{a_2^v(v-u)} - 1 \right) \frac{c_2^2 M_1 M_2}{a_2^v k_2^2}.
\]

Then from (21)

\[
\left| \Upsilon_2(\phi, \psi)(u) - \Upsilon_2(\phi, \psi)(v) \right| \to 0.
\]

which shows that 1) holds.

For the 2) statement, given any \( t \in \mathbb{R} \) and for any \((\phi, \psi) \in M\), we get

\[
\Upsilon_1(\phi, \psi)(t) \leq M_1 \text{ And } \Upsilon_2(\phi, \psi)(t) \leq M_2.
\]
5 Stability of the p.a.p solution

Before the stability theorem, we need the following lemma

Lemma 5.1. [7] Let \( f \) be a non-negative function defined on \([0; +\infty[\) such that \( f \) is integrable on \([0; +\infty[\) and is uniformly continuous on \([0; +\infty[\). Then

\[
\lim_{t \to +\infty} f(t) = 0.
\]

Definition 5.2. If \((u^*, v^*)\) is a pseudo almost periodic solution of system (2), and \((u, v)\) is any solution of (2) satisfying

\[
\lim_{t \to +\infty} |u(t) - u^*(t)| = \lim_{t \to +\infty} |v(t) - v^*(t)| = 0,
\]

then we call this solution \((u^*, v^*)\) is globally attractive.

Theorem 5.3. Assume that

\[
\liminf \alpha(t), \liminf \beta(t) > 0.
\]

where \(\alpha(t)\) and \(\beta(t)\) are defined in 28 and 29 The pseudo almost periodic solution \((u^*, v^*)\) is globally attractive.

Proof.

Suppose \((u, v)\) is any solution of the system (2). Define the Lyapunov functional as follows

\[
W_1(t) = \left| \ln u(t) - \ln u^*(t) \right| + \left| \ln v(t) - \ln v^*(t) \right|.
\]

Pose that \(w_1 = u - u^*, w_2 = v - v^*\). And define \(\zeta_i^{-1}, \zeta_i^{-1}\) are the inverse functions of \(\zeta_i = t - \sigma_i(t), \zeta_i = t - \tau_i(t)\), respectively \(i = 1, 2\).

By calculating Dini derivative of \(V\) along system (2), we get

\[
D^+W_1(t) = sgn(w_1(t)) \left[ \frac{u'(t)}{u(t)} - \frac{u^*(t)}{u^*(t)} \right] + sgn(w_2(t)) \left[ \frac{v'(t)}{v(t)} - \frac{v^*(t)}{v^*(t)} \right]
\]

\[
= sgn(w_1(t)) \left[ -b_1(t)w_1(t) - c_1(t) \left( \frac{v(\zeta_1(t))}{u(\zeta_1(t)) + k_1(t)} - \frac{v^*(\zeta_1(t))}{u^*(\zeta_1(t)) + k_1(t)} \right) \right]
\]

\[
- sgn(w_2(t))c_2(t) \left[ \frac{v(\zeta_2(t))}{u(\zeta_2(t)) + k_2(t)} - \frac{v^*(\zeta_2(t))}{u^*(\zeta_2(t)) + k_2(t)} \right]
\]

\[
= -b_1(t) |w_1(t)| - c_1(t) sgn(w_1(t))w_2(\zeta_1(t)) + c_1(t) sgn(w_1(t))v^*(\zeta_1(t))w_1(\zeta_1(t))
\]

\[
- c_2(t) sgn(w_2(t))w_2(\zeta_2(t)) + c_2(t) sgn(w_2(t))v^*(\zeta_2(t))w_1(\zeta_2(t))
\]

\[
= -b_1(t) |w_1(t)| - c_1(t) sgn(w_1(t))w_2(t) + c_1(t) sgn(w_1(t))w_1(t) + c_1(t) sgn(w_1(t))w_2(t) - \int_{\zeta_1(t)}^{t} w_2^*(s) ds
\]

\[
+ c_1(t) v^*(\zeta_1(t)) sgn(w_1(t)) \left[ w_1(t) - \int_{\zeta_1(t)}^{t} w_1'(s) ds \right]
\]

\[
- c_2(t) sgn(w_2(t))w_2(t) + c_2(t) sgn(w_2(t))w_2(t) - \int_{\zeta_2(t)}^{t} w_2'(s) ds
\]

\[
+ c_2(t) v^*(\zeta_2(t)) sgn(w_2(t)) \left[ w_1(t) - \int_{\zeta_2(t)}^{t} w_1'(s) ds \right]
\]
Integrating both sides of $w'_1(t)$ on the interval $[\zeta_j(t), t]$ where $j=1$ or $=2$, we have

$$\int_{\zeta_j(t)}^{t} w'_1(s)ds = \int_{\zeta_j(t)}^{t} u(s) \left[ a_1(s) - b_1(s)u(s) - \frac{c_1(s)v'_{\zeta_1}(s)}{u(\zeta_1(s)) + k_1(s)} \right] ds$$

$$- \int_{\zeta_j(t)}^{t} \left[ a_1(s) - b_1(s)u(s) - \frac{c_1(s)v'_{\zeta_1}(s)}{u(\zeta_1(s)) + k_1(s)} \right] ds$$

$$= \int_{\zeta_j(t)}^{t} a_1(s)w_1(s) - b_1(s)(u(s) + u*(s))w_1(s) - \frac{c_1(t)u(s)}{u(\zeta_1(s)) + k_1(s)} w_2(\zeta_1(s))$$

$$- \frac{c_1(s)v'_{\zeta_1}(s)}{u(\zeta_1(s)) + k_1(s)} w_1(s) + \frac{c_1(s)v'_{\zeta_1}(s)}{u(\zeta_1(s)) + k_1(s)} w_1(\zeta_1(s))ds$$

and Integrating both sides of $w'_2(t)$ on the interval $[\zeta_j(t), t]$ where $j=1$ or $=2$, we have

$$\int_{\zeta_j(t)}^{t} w'_2(s)ds = \int_{\zeta_j(t)}^{t} v(s) \left[ a_2(s) - \frac{c_2(s)v'_{\zeta_2}(s)v(s)}{u(\zeta_2(s)) + k_2(s)} \right] - v*(s) \left[ a_2(s) - \frac{c_2(s)v'_{\zeta_2}(s)v(s)}{u(\zeta_2(s)) + k_2(s)} \right] ds$$

$$= \int_{\zeta_j(t)}^{t} a_2(s)w_2(s) - \frac{c_2(t)v(s)}{u(\zeta_2(s)) + k_2(s)} w_2(\zeta_1(s))$$

$$- \frac{c_2(s)v'_{\zeta_2}(s)}{u(\zeta_2(s)) + k_2(s)} w_2(s) + \frac{c_2(s)v'_{\zeta_2}(s)v(s)}{u(\zeta_2(s)) + k_2(s)} w_1(\zeta_2(s))ds$$

From theorem 3.4 we have for all solution $(u, v)$ of system (2) $\exists M_1 > m_1 \geq 0$, $M_2 > m_2 > 0$ such that

$m_1 \leq u(t) \leq M_1; \quad m_2 \leq v(t) \leq M_2 \quad \forall t \in \mathbb{R}$

Therefore, By substituting (23)–(24) into (22), we get

$$D^+ W_1(t) \leq - \left( b'_1 - \frac{c'_1 M_2}{(m_1 + k_1)^2} - \frac{c'_2 M_2}{(m_1 + k_1)^2} \right) |w_1(t)| - \left( \frac{c'_2}{M_1 + k_1^2} - \frac{c'_1}{m_1 + k_1^2} \right) |w_2(t)|$$

$$+ \frac{c'_1}{m_1 + k_1^2} \left[ a'_2 + \frac{c'_2 M_2}{m_1 + k_1^2} \right] \int_{\zeta_j(t)}^{t} |w_2(s)| ds + \frac{c'_1 c'_2 M_2}{(m_1 + k_1^2)(m_1 + k_1^2)} \int_{\zeta_j(t)}^{t} |w_2(\zeta_1(s))| ds$$

$$+ c'_1 c'_2 M_2^2 \int_{\zeta_j(t)}^{t} |w_1(\zeta_2(s))| ds + (c'_1)^2 M_1 M_2 \int_{\zeta_j(t)}^{t} |w_1(\zeta_1(s))| ds$$

$$+ \left[ \frac{c'_1 a'_1 M_2}{(m_1 + k_1)^2} + \frac{2c'_2 b'_1 M_2}{(m_1 + k_1)^2} + \frac{(c'_1)^2 M_2^2}{(m_1 + k_1)^2} \right] \int_{\zeta_j(t)}^{t} |w_1(s)| ds$$

$$+ \left[ \frac{2c'_1 b'_1 M_2}{(m_1 + k_1)^2} + \frac{c'_2 c'_1 M_2^2}{(m_1 + k_1)^2} + \frac{c'_2 a'_1 M_2}{(m_1 + k_1)^2} \right] \int_{\zeta_j(t)}^{t} |w_2(\zeta_2(s))| ds$$

$$+ \frac{c'_1 M_1 M_2}{(m_1 + k_1)^2} \int_{\zeta_j(t)}^{t} |w_2(\zeta_1(s))| ds + \frac{c'_1 c'_2 M_1 M_2^2}{(m_1 + k_1)^2} \int_{\zeta_j(t)}^{t} |w_1(\zeta_2(s))| ds$$

$$+ \left( \frac{(c'_1)^2 M_2^2}{(m_1 + k_1)^2} \right) \int_{\zeta_j(t)}^{t} |w_1(\zeta_1(s))| ds$$

$$+ \left( \frac{(c'_1)^2 M_2^2}{(m_1 + k_1)^2} \right) \int_{\zeta_j(t)}^{t} |w_1(\zeta_2(s))| ds$$

$$+ \left( \frac{(c'_1)^2 M_2^2}{(m_1 + k_1)^2} \right) \int_{\zeta_j(t)}^{t} |w_2(\zeta_1(s))| ds$$

$$+ \left( \frac{(c'_1)^2 M_2^2}{(m_1 + k_1)^2} \right) \int_{\zeta_j(t)}^{t} |w_2(\zeta_2(s))| ds$$

$$+ \left( \frac{(c'_1)^2 M_2^2}{(m_1 + k_1)^2} \right) \int_{\zeta_j(t)}^{t} |w_2(\zeta_1(s))| ds$$

$$+ \left( \frac{(c'_1)^2 M_2^2}{(m_1 + k_1)^2} \right) \int_{\zeta_j(t)}^{t} |w_2(\zeta_2(s))| ds$$
Now, Let

\[ W_2(t) = \frac{c^s_t}{m_1 + k_1} \left[ a^s + \frac{c^s_M M_2}{m_1 + k_2} \right] \int_t^{\zeta^{-1}(t)} \int_{\zeta_{1(u)}}^u \left| w_2(s) \right| ds du + \frac{c^s_{M_1} M_2}{(m_1 + k_2)^2} \int_t^{\zeta^{-1}(t)} \int_{\zeta_{2(u)}}^u \left| w_1(s) \right| ds du \]

\[ + \frac{c^s_t c^s_M M_2}{(m_1 + k_1)(m_1 + k_2)^2} \int_t^{\zeta^{-1}(t)} \int_{\zeta_{1(u)}}^u \left| w_2(c_2(s)) \right| ds du + \frac{(c^s_t)^2 M_1 M_2}{(m_1 + k_1)^3} \int_t^{\zeta^{-1}(t)} \int_{\zeta_{1(u)}}^u \left| w_1(s) \right| ds du \]

\[ + \left[ \frac{c^s_t a^s_M M_2}{(m_1 + k_1)^2} + \frac{2 c^s_t b^s_t M_1 M_2}{(m_1 + k_1)^2} + \frac{(c^s_t)^2 M_2}{(m_1 + k_1)^3} \right] \int_t^{\zeta^{-1}(t)} \int_{\zeta_{2(u)}}^u \left| w_1(s) \right| ds du \]

then its derivative is as follows:

\[ W'_2(t) = \frac{c^s_t}{m_1 + k_1} \left[ a^s + \frac{c^s_M M_2}{m_1 + k_2} \right] \left[ \left| w_2(t) \right| (\zeta^{-1}(t) - t) - \int_{\zeta_{1(t)}}^t \left| w_2(s) \right| ds \right] \]

\[ + \frac{c^s_t c^s_M M_2}{(m_1 + k_1)(m_1 + k_2)^2} \left[ \left| w_2(\zeta_1(t)) \right| (\zeta^{-1}(t) - t) - \int_{\zeta_{1(t)}}^t \left| w_2(\zeta_1(s)) \right| ds \right] \]

\[ + \frac{c^s_t c^s_M M_2}{(m_1 + k_1)(m_1 + k_2)^2} \left[ \left| w_1(\zeta_2(t)) \right| (\zeta^{-1}(t) - t) - \int_{\zeta_{1(t)}}^t \left| w_1(\zeta_1(s)) \right| ds \right] \]

\[ + \left[ \frac{c^s_t a^s_M M_2}{(m_1 + k_1)^2} + \frac{2 c^s_t b^s_t M_1 M_2}{(m_1 + k_1)^2} + \frac{(c^s_t)^2 M_2}{(m_1 + k_1)^3} \right] \left[ \left| w_1(t) \right| (\zeta^{-1}(t) - t) - \int_{\zeta_{1(t)}}^t \left| w_1(s) \right| ds \right] \]

\[ + \left[ \frac{c^s_t c^s_M M_2}{(m_1 + k_1)(m_1 + k_2)^2} + \frac{(c^s_t)^2 M_1 M_2}{(m_1 + k_1)^3} \right] \left[ \left| w_2(\zeta_1(t)) \right| (\zeta^{-1}(t) - t) - \int_{\zeta_{2(t)}}^t \left| w_2(\zeta_2(s)) \right| ds \right] \]

Define the Lyapunov functional by \( W(t) = W_1(t) + W_2(t) \), then

\[ D^+ W(t) = D^+ W_1(t) + W_2(t) \].
Substitution of (25)–(26) into (27) gives:

\[ D^+ W(t) \leq - \left[ b_1 - \frac{c_1^2 M_2}{(m_1 + k_1^2)^2} - \frac{c_2^2 M_2}{(m_1 + k_1^2)^2} - \frac{c_1^2 c_2^2 M_2}{(m_1 + k_1^2)^2(m_1 + k_2^2)^2}(\zeta_1^{-1}(t) - t) \right. \]

\[ - \left. \left( \frac{c_1^2 a_1 M_2}{(m_1 + k_1^2)^2} + \frac{2c_1 b_1 M_1 M_2}{(m_1 + k_1^2)^2} + \frac{(c_1^2)M_1 M_2^2}{(m_1 + k_1^2)^3} + \frac{(c_1^2)^2 M_1 M_2^2}{(m_1 + k_1^2)^4} \right) \right. \]

\[ - \left. \left( \frac{c_1^2 M_2}{M_1 + k_2^2} - \frac{c_1^2}{m_1 + k_1} - \frac{c_1}{m_1 + k_1} \left( a_2 + \frac{c_2^2 M_2}{(m_1 + k_2^2)^2} + \frac{c_1^2 c_2^2 M_2}{(m_1 + k_1)^2(m_1 + k_2^2)^2} \right) \right) \right. \]

\[ - \left. \left( \frac{c_1^2 M_2}{(m_1 + k_1)^2} + \frac{2c_1 b_1 M_1 M_2}{(m_1 + k_1^2)^2} + \frac{(c_1^2)M_1 M_2^2}{(m_1 + k_1^2)^3} + \frac{(c_1^2)^2 M_1 M_2^2}{(m_1 + k_1^2)^4} \right) \right. \]

Let’s denote

\[ \alpha(t) = b_1 - \frac{c_1^2 M_2}{(m_1 + k_1^2)^2} - \frac{c_2^2 M_2}{(m_1 + k_1^2)^2} - \frac{c_1^2 c_2^2 M_2}{(m_1 + k_1^2)^2(m_1 + k_2^2)^2}(\zeta_1^{-1}(t) - t) \]

\[ - \left( \frac{c_1^2 a_1 M_2}{(m_1 + k_1^2)^2} + \frac{2c_1 b_1 M_1 M_2}{(m_1 + k_1^2)^2} + \frac{(c_1^2)M_1 M_2^2}{(m_1 + k_1^2)^3} + \frac{(c_1^2)^2 M_1 M_2^2}{(m_1 + k_1^2)^4} \right) \left( \zeta_1^{-1}(t) - t \right) \]

\[ - \left( \frac{c_1^2 M_2}{M_1 + k_2^2} - \frac{c_1^2}{m_1 + k_1} - \frac{c_1}{m_1 + k_1} \left( a_2 + \frac{c_2^2 M_2}{(m_1 + k_2^2)^2} + \frac{c_1^2 c_2^2 M_2}{(m_1 + k_1)^2(m_1 + k_2^2)^2} \right) \right) \left( \zeta_1^{-1}(t) - t \right) \]

(28)

And

\[ \beta(t) = \frac{c_1^2}{M_2 + k_2^2} - \frac{c_1^2}{M_2 + k_1^2} - \frac{c_1^2}{M_1 + k_1^2} \left( a_2 + \frac{c_2^2 M_2}{M_1 + k_2^2} + \frac{c_1^2 c_2^2 M_2}{M_1 + k_1^2} \right) \left( \zeta_1^{-1}(t) - t \right) \]

\[ - \left( \frac{c_1^2 M_2}{M_1 + k_1^2} + \frac{2c_1 b_1 M_1 M_2}{M_1 + k_1^2} + \frac{(c_1^2)M_1 M_2^2}{M_1 + k_1^2} + \frac{(c_1^2)^2 M_1 M_2^2}{(M_1 + k_1^2)^2} \right) \left( \zeta_1^{-1}(t) - t \right) \]

(29)

From hypothesis of the theorem, one has \( \alpha^i = \liminf \alpha(t) \) and \( \beta^i = \liminf \beta(t) \) verified for sufficiently large T, we obtained

\[ D^+ W(t) \leq -\alpha^i |w_1(t)| - \beta^i |w_2(t)| < 0 \]

which implies \( W(t) \) is non-increasing on \( [T, +\infty[. \) An integration of above inequality from \( T \) to \( t \) yields

\[ W(t) + \alpha^i \int_t^T |u(s) - u^*(s)|ds + \beta^i \int_t^T |v(s) - v^*(s)|ds \leq W(T) < +\infty, \quad \forall t > T. \]

Then

\[ \limsup_{t \to +\infty} \int_t^T |u(s) - u^*(s)|ds \leq \frac{W(T)}{\alpha^i} < +\infty \text{ and } \limsup_{t \to +\infty} \int_t^T |v(s) - v^*(s)|ds \leq \frac{W(T)}{\beta^i} < +\infty. \]

Thus we have

\[ \lim_{t \to +\infty} |u(t) - u^*(t)| = \lim_{t \to +\infty} |v(t) - v^*(t)| = 0. \]

\[ \square \]

6 Example and Simulation

In order to illustrate some feature of our main results, in this section, we will apply our main results to some special prey-predator systems and demonstrate the efficiencies of our criteria.
6.1 Example 1:

In this example, we consider a system without the condition (C0). Then, \(m_1 = 0\). The system is considered

\[
\begin{align*}
    u'(t) &= \left(0.04 + 0.125 \cos(\sqrt{2}t)\right) + 0.125 \exp(-t) - (2.6 + 0.5 \cos(t)) u(t) \\
    &\quad - \frac{3.2 v(t - 0.75)}{u(t - 0.75) + 17} u(t); \\
    v'(t) &= \left(0.01 + 0.25 \sin(\sqrt{7}t)\right) - \frac{3.5 v(t - 0.75)}{u(t - 0.75) + 3.4} v(t),
\end{align*}
\]

(30)

By a direct calculation, ones have the following table

| \( j \) | \( a_j^i \) | \( a_j^a \) | \( b_j \) | \( b_j^a \) | \( c_j^i \) | \( c_j^a \) | \( k_j^i \) | \( k_j^a \) | \( \sigma_j^a \) | \( \tau_j^a \) |
|-------|--------|--------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1     | 0.04   | 0.29   | 2.6   | 3.1   | 3.2   | 3.2   | 17    | 17    | 0.75  | 0.75  |
| 2     | 0.01   | 0.26   | —     | —     | 3.5   | 3.5   | 3.4   | 3.4   | 0.75  | 0.75  |

And \( \alpha^i \approx 2.4; \beta^i \approx 0.012; M_1 = 0.7085; m_1 = 0; M_2 = 0.6506; \) and \( m_2 = 0.0829\). The theorem (4.11) is verified and the conditions of theorem (5.3) are satisfied. Therefore, there exist at least a pseudo almost periodic which is globally attractive.
6.2 Example 2:

In this example, the condition (C0) holds. Let us consider

\[
\begin{align*}
    u'(t) &= \left(4.8 + 0.125\left(|\cos(\sqrt{2}t)| + |\cos(\sqrt{2}t)|\right) - (0.25|\cos(t)|
    + \frac{33.72 + 32.72t^2}{4 + 4t^2})u(t) - \frac{0.32v(t - 0.92)}{u(t - 0.92) + 16.7}\right)u(t); \\
    v'(t) &= \left(0.03 + 0.125(|\sin(\sqrt{2}t)| + |\cos(\sqrt{5}t)|) - \frac{3.6v(t - 0.92)}{u(t - 0.92) + 5.7}\right)v(t).
\end{align*}
\]

By a direct calculation, ones have the following table

| \( j \) | \( a_j^1 \) | \( a_j^2 \) | \( b^1 \) | \( b^2 \) | \( c_j^1 \) | \( c_j^2 \) | \( k_j^1 \) | \( k_j^2 \) | \( \sigma_j^1 \) | \( \sigma_j^2 \) | \( \tau_j^1 \) | \( \tau_j^2 \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4.8 | 5.05 | 8.1 | 8.6 | 0.32 | 0.32 | 16.7 | 16.7 | 0.92 | 0.92 | 0.92 |
| 2 | 0.03 | 0.28 | — | — | 3.6 | 3.6 | 5.7 | 5.7 | 0.92 | 0.92 | 0.92 |

And \( a^1 \approx 7.25; \beta^i \approx 0.009; M_1 = 0.6226; m_1 = 0.5567; M_2 = 0.6403; \) and \( m_2 = 0.0408. \) The theorem (4.11) is verified and the conditions of theorem (5.3) are satisfied. Therefore, there exist at least a pseudo almost periodic which is globally attractive.
7 Conclusion

The aim of this paper is to prove the existence of positive almost periodic solution in a Leslie-Gower predator-prey model with continuous delays. Based on new conditions, the global attractivity of the above model is obtained by building a suitable Lyapunov functional. Moreover, some numerical examples show that the our theoretical results are effective.

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