One-loop QCD thermodynamics in a strong homogeneous and static magnetic field

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Abstract

We have studied how the equation of state of thermal QCD with two light flavours is modified in strong magnetic field by calculating the thermodynamic observables of hot QCD matter up to one-loop, where the magnetic field affects mainly the quark contribution and the gluonic part is largely unaffected except for the softening of the screening mass due to the strong magnetic field. To begin with the effect of magnetic field on the thermodynamics, we have first calculated the pressure of a thermal QCD medium in strong magnetic field limit (SML), where the pressure at fixed temperature increases with the magnetic field faster than the increase with the temperature at constant magnetic field. This can be envisaged from the dominant scale of thermal medium in SML, which is the magnetic field, like the temperature in thermal medium in absence of strong magnetic field. Thus although the presence of strong magnetic field makes the pressure of hot QCD medium harder but the increase of pressure with respect to the temperature becomes less steeper. Corroborated to the above observations, the entropy density is found to decrease with the temperature in the ambience of strong magnetic field which resonates with the fact that the strong magnetic field restricts the dynamics of quarks in two dimensions, hence the phase space gets squeezed resulting the reduction of number of microstates. Moreover the energy density is seen to decrease and the speed of sound of thermal QCD medium is increased in the presence of strong magnetic field. These crucial findings in strong magnetic field could have phenomenological implications in heavy ion collisions because the expansion dynamics of the medium produced in noncentral ultrarelativistic heavy ion collisions is effectively controlled by both the energy density and the speed of sound.

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1 Introduction

The last three decades had witnessed hectic activities towards recreating the conditions similar to those existing shortly after the big bang, known as Quark-gluon Plasma (QGP), in the terrestrial laboratory by carrying out collisions of ultra-relativistic heavy ions RHIC, BNL and LHC at CERN, where only few events are truly head-on, indeed most occur under a finite impact parameter or centrality. As a consequence, the two highly charged ions impacting with a small offset may produce extremely large magnetic fields $\sim m_\pi^2$ ($\simeq 10^{18}$ Gauss) at RHIC and $\sim 15m_\pi^2$ at LHC [1, 2]. Naive classical estimates of the lifetime of these magnetic fields indicate that they only exist for a small fraction of the lifetime of QGP [3, 4]. However depending on the transport coefficients of the medium, the magnetic field may be near to its maximum strength and also be stationary [5] - [8] in its lifetime. Moreover the magnetic field may be assumed uniform because even though the spatial distribution of the magnetic field is globally inhomogeneous but in the central region of the overlapping nuclei the magnetic field in the transverse plane varies very smoothly, which is found in the simulations of hadron-string-dynamics model [9] for Au-Au collisions at $\sqrt{s_{NN}} = 200$ GeV with impact parameter, b=10 fm. Therefore, it is worthwhile to investigate the QCD physics in a strong homogeneous and static magnetic field, such as chiral magnetic effect related to the generation of electric current parallel to magnetic field due to difference in number of right and left-handed quarks [10] - [12], axial magnetic effect due to the energy flow by the axial magnetic field [13, 14], chiral vortical effect due to an effective magnetic field in rotating QGP [15, 16], magnetic catalysis and inverse magnetic catalysis at finite temperature arising due to the breaking and restoration of chiral symmetry [17] - [21], thermodynamic properties [22] - [24], refractive indices and decay constant [25, 26] of mesons in hot magnetized medium, conformal anomaly and production of soft photons [27, 28] at RHIC and LHC, dispersion relation in a magnetized thermal QED [29], synchrotron radiation [30], dilepton production from both weakly [30] - [34] and strongly [35] coupled plasma.

A variety of studies of the effects of strong magnetic fields on QCD thermodynamics have been recently resurrected by the possibility to achieve the magnetic fields at RHIC and LHC. Since the magnetic field breaks the translational invariance in space, so the pressure becomes anisotropic arising due to the difference between the pressures that are transverse and longitudinal to the direction of background magnetic field, which is illustrated for an ensemble of spin one-half particles [24]. Recent lattice QCD calculations [36] delve into the effects of background magnetic fields on the equation of state.
(EoS) by calculating the thermodynamic observables, namely transverse and longitudinal pressure, magnetization, energy density, entropy density etc. and inferred that the transition temperature gets reduced by the magnetic field [37] - [42]. For the hadronic matter too, the phase structure and the phase transitions in strong magnetic fields and zero quark chemical potentials have been reviewed in [23] through the low-energy effective theories and models, where the thermodynamic quantities are also found to increase with the magnetic field [43]. Thus the EoS is expected to be changed due to the magnetic field and this fascinates us to study and explore the modification of the EoS in presence of strong magnetic field. In the present work, we also aim to study the thermal QCD equation of state perturbatively up to one-loop in a background of strong and homogeneous magnetic field.

For thermal medium the free energy of non-abelian gauge theories has been analytically computed up to $O(g^4)$ in [44, 45] ($g$, the coupling constant) and up to $O(g^5)$ in [46]. The values of pressure obtained by the addition of successive higher order contributions oscillate haphazardly and seem to diverge, hence the reorganization of perturbative expansion of thermodynamic quantities becomes necessary. In this process, various renormalization scales and effective field theory methods have been emerged, such as evaluation of free energy by the finite temperature effective field theory methods [47], where the contributions coming from various momentum scales, viz. $T$, $gT$, and $g^2T$ are separated in weak-coupling regime. However, in presence of magnetic field, a thermal medium possesses an additional scale related to the magnetic field and depending on the strength of magnetic field compared to the temperature of thermal medium and the quark masses, QCD thermodynamics has been studied in two scenarios: In weak field limit ($T^2 \gg |q_f B|$, $T^2 \gg m_f^2$, where $m_f$ and $|q_f|$ are the mass and the absolute charge of the quark with flavor $f$, respectively), the temperature remains the dominant scale of the system, so the hard thermal loop (HTL) perturbation theory remains the best theoretical tool in calculating the free energy of hot quark gluon plasma [48] - [52]. On the other hand, in strong magnetic field limit ($|q_f B| \gg T^2$, $|q_f B| \gg m_f^2$), we calculate the thermodynamic observables by replacing the upper limit of loop momentum by the magnetic field, which is the most prominent scale available now (precisely $\sqrt{eB}$), like the temperature in thermal medium in absence of strong magnetic field in HTL perturbation theory.

In presence of magnetic field, the quark momentum $p$ is separated into components transverse and longitudinal to the direction of magnetic field (say, $z$-direction), hence the
The dispersion relation for quarks is modified quantum mechanically into

\[ E_n(p_z) = \sqrt{p_z^2 + m_f^2 + 2n |q_f B|}, \]  

where \( n = 0, 1, 2, \ldots \) are the quantum numbers specifying the Landau levels. In strong magnetic field, the quarks are rarely excited thermally to the higher Landau levels, only the lowest Landau levels (LLL) \( (n = 0) \) are populated \( (E_0 = \sqrt{p_z^2 + m_f^2}) \). Thus, the dynamics of quarks are effectively restricted to \((1 + 1)\) dimensions. In addition, the quark propagator is also modified in the magnetic field, which was first derived in coordinate space by Schwinger [53] using the proper-time method and later by Tsai [54] in the momentum space. With the modifications discussed above in strong magnetic field, our aim will be to calculate the one-loop contribution to the thermodynamic observables of a hot strongly magnetized QCD matter to analyze the behavior of QGP phase in strong magnetic field.

Our work proceeds in the following way. In section 2, we have derived the effective quark propagator in a thermal QCD medium in strong magnetic field limit. For that purpose, we first revisit the vacuum quark propagator in the presence of a strong magnetic field and obtain both the quark and gluon propagators at finite temperature in real-time formalism (RTF), in sections 2.1 and 2.2, respectively. This helps us to compute the one-loop quark self-energy in section 2.3 at finite temperature in strong magnetic field limit. Similarly, we have derived the effective gluon propagator in the similar environment in section 3, where we first calculate the one-loop gluon self-energy in terms of screening mass in section 3.1. The effect of magnetic field enters through the screening mass, so we have calculated the screening mass in strong magnetic field limit in section 3.2 for both massless and physical quark masses by the static limit of real part of gluon self-energy. Having thus obtained the effective propagators for quarks and gluons in sections 2 and 3, respectively, we have calculated the quark and gluon free energies and then, the thermodynamic observables for a strongly magnetized QCD matter have been calculated in section 4. Finally, we conclude in section 5.
The effective quark propagator in a strongly-magnetized hot QCD medium is obtained from the Schwinger-Dyson equation:

\[ S^{-1}(P) = S_0^{-1}(P) - \Sigma(P), \quad (2) \]

where \( S_0(P) \) and \( \Sigma(P) \) are the free propagator and quark self-energy in a strongly-magnetized hot QCD medium, respectively. As mentioned earlier, the strong magnetic field affects the quark propagator via the projection operator and the dispersion relation, which will, in turn affect the quark self-energy. In addition, the QCD coupling will now run with both the magnetic field and temperature, however, in strong magnetic field limit \((eB \gg T^2)\), it runs exclusively with the magnetic field and is almost independent of the temperature, because the most dominant scale available is the magnetic field, not the temperature of medium anymore. For this purpose we closely follow the results in [55], where the coupling is split into terms dependent on the momentum parallel and perpendicular to the magnetic field, separately. In our case of magnetic field \((B = B\hat{z})\), we will use the coupling which depends on the longitudinal component only, because the energies of Landau levels for quarks in SML depend only on the longitudinal component of momentum. In fact, the coupling dependent on the transverse momentum does not depend on magnetic field at all, thus the relevant coupling is given by [55]

\[
\alpha_s^\parallel(eB) = \frac{g^2}{4\pi} = \frac{1}{\alpha_s^0(\mu_0)^{-1} + \frac{11}{12\pi} \ln \left( \frac{\Lambda_{QCD}^2 + M_B^2}{\mu_0^2} \right) + \frac{1}{3\pi} \sum_f \frac{|q_f B_l|}{\tau}, \quad (3)
\]

where

\[
\alpha_s^0(\mu_0) = \frac{12\pi}{11N_c \ln \left( \frac{\mu_0^2 + M_B^2}{\Lambda_{\overline{MS}}^2} \right)} , \quad (4)
\]

\( M_B \) is taken \( \sim 1 \text{ GeV} \) as an infrared mass and the string tension is taken as \( \tau = 0.18 \text{ GeV}^2 \).

We now first revisit the vacuum quark propagator in a strong magnetic field and then thermalize both quark and gluon propagators in a hot QCD medium, which are the ingredients to compute the quark self-energy.
2.1 Vacuum propagators in strong magnetic field

The magnetic field breaks the translational invariance of space, as a result the quark propagator becomes function of separate components of momentum transverse and longitudinal to the magnetic field direction. Schwinger’s proper-time method \[53\] computes the quark propagator in coordinate-space as

\[
S(x, y) = \phi(x, y) \int \frac{d^4K}{(2\pi)^4} e^{-iK(x-y)} S(K),
\]

(5)

where the phase factor \(\phi(x, y)\) is expressed as

\[
\phi(x, y) = e^{i|q_f| \int_0^x A^\mu(\zeta) d\zeta}. \tag{6}
\]

(6)

The above phase factor is the gauge-dependent part and is responsible for breaking of translational invariance. In a single fermion propagator, for a symmetric gauge, i.e. \(A^\mu(x) = B \hat{z}(0, -x_2, x_1, 0)\) in a magnetic field directed along the \(z\) axis \((B = B \hat{z})\), it is possible to gauge away the phase factor by an appropriate gauge transformation and one can work with the momentum-space representation of the propagator \[56\] as an integral over the proper-time \((s)\)

\[
S(K) = i \int_0^\infty ds e^{-is m_f^2} \exp \left( is k_\parallel^2 - \frac{ik_\perp^2 \tan(|q_f B| s)}{|q_f B|} \right) \\
\times \left[ (m_f + \gamma^\parallel \cdot k_\parallel) (1 + \gamma^\parallel \gamma^\perp \tan(|q_f B| s)) - \gamma^\perp \cdot k_\perp (1 + \tan^2(|q_f B| s)) \right]. \tag{7}
\]

(7)

The quantities in above equation are defined as follows

\[
k_\parallel \equiv (k_0, 0, 0, k_3), \quad k_\perp \equiv (0, k_1, k_2, 0),
\]

\[
\gamma^\parallel \equiv (\gamma^0, \gamma^3), \quad \gamma^\perp \equiv (\gamma^1, \gamma^2),
\]

\[
g^{\mu\nu} = g^{\mu\nu}_\parallel + g^{\mu\nu}_\perp,
\]

\[
g^{\mu\nu}_\parallel = \text{diag}(1, 0, 0, -1), \quad g^{\mu\nu}_\perp = \text{diag}(0, -1, -1, 0),
\]

\[
\gamma^\parallel \cdot k_\parallel = \gamma^0 k_0 - \gamma^3 k_3, \quad \gamma^\perp \cdot k_\perp = \gamma^1 k_1 + \gamma^2 k_2,
\]

\[
k_\parallel^2 \equiv k_0^2 - k_3^2, \quad k_\perp^2 \equiv k_1^2 + k_2^2.
\]

After integration over the proper-time, \(s\), \(S(K)\) can be written in discrete notation

\[
S(K) = ie^{-\frac{k_\parallel^2}{m_f}} \sum_{n=0}^\infty (-1)^n \frac{D_n(|q_f B|, K)}{k_\parallel^2 - m_f^2 - 2|q_f B| n}, \tag{8}
\]

(8)
where $D_n(|q_f B|, K)$ can be expressed in terms of generalized Laguerre polynomials labelling the Landau levels [54, 57, 58] as

$$
D_n(|q_f B|, K) = \left( \gamma^\parallel \cdot k^\parallel + m_f \right) \left[ (1 - i\gamma^1\gamma^2) L_n \left( \frac{2k^2_\perp}{|q_f B|} \right) - \left( 1 + i\gamma^1\gamma^2 \right) L_{n-1} \left( \frac{2k^2_\perp}{|q_f B|} \right) \right] 
+ 4\gamma^\perp \cdot k^\perp L^{(1)}_{n-1} \left( \frac{2k^2_\parallel}{|q_f B|} \right) .
$$

(9)

In presence of a strong magnetic field ($k^2_\parallel, k^2_\perp \ll |q_f B|$), the transitions to the higher Landau levels ($n \geq 1$) are suppressed and only LLL ($n = 0$) is occupied. Putting $n = 0$ in equation (9) yields the quark propagator in strong magnetic field in momentum-space

$$
S_{LLL}(K) = ie \frac{k^2_\parallel k^2_\perp}{|q_f B|} \left( \gamma^\parallel \cdot k^\parallel + m_f \right) \left( 1 - \gamma^0\gamma^3\gamma^5 \right) .
$$

(10)

However, the gluons remain unaffected by the presence of magnetic field, hence the form of the vacuum gluon propagator remains the same even in presence of magnetic field.

### 2.2 Thermalization of propagators in strong magnetic field

The vacuum quark and gluon propagators in presence of strong magnetic field discussed above get thermalized in a thermal QCD medium using the RTF, where the medium effects are conceived through the distribution functions. In this formalism, propagators are manifestly separated into vacuum and thermal parts and the degrees of freedom has also been doubled so the propagators acquire a $2 \times 2$ matrix structure. We also note that, in an equilibrium system, to evaluate the real part of the one-loop quark self energy, it is adequate to calculate the 11-components of the quark and gluon propagators.

#### 2.2.1 Quark propagator

Since quarks are only populated in the lowest Landau levels in strong magnetic field, so LLL quark propagator in equation (10) is used in the thermalization process. Denoting $S_{LLL}(K) = S_0(K)$, the vacuum quark propagator gets matrix structure in thermal medium as

$$
S(K) = U_F(k_0) \begin{pmatrix} S_0(K) & 0 \\ 0 & S_0^*(K) \end{pmatrix} U_F(k_0) ,
$$

(11)
where $U_F(k_0)$ is the unitary matrix which brings the temperature dependence through the distribution function and it has the following form.

$$U_F(k_0) = \begin{pmatrix} \sqrt{1 - n_F(k_0)} & -\sqrt{n_F(k_0)} \\ \sqrt{n_F(k_0)} & \sqrt{1 - n_F(k_0)} \end{pmatrix},$$

(12)

with the distribution function for quarks

$$n_F(k_0) = \frac{1}{e^{\beta |k_0|} + 1}.$$  

(13)

Substituting the unitary matrix in equation (11) and simplifying, we get

$$S(K) = \begin{pmatrix} n_2^2 S_0(K) - n_1^2 S_0^*(K) & -n_1 n_2 (S_0(K) + S_0^*(K)) \\ n_1 n_2 (S_0(K) + S_0^*(K)) & n_2^2 S_0^*(K) - n_1^2 S_0(K) \end{pmatrix},$$

(14)

where $n_1 = \sqrt{n_F(k_0)}$ and $n_2 = \sqrt{1 - n_F(k_0)}$.

Now from the above matrix, the $11$ - component of the quark propagator in a strongly magnetized thermal medium is obtained as

$$S_{11}(K) = \int e^{-\frac{k^2}{\eta^2} \left( \gamma^0 k_0 - \gamma^3 k_3 + m_f \right) \left( 1 - \gamma^0 \gamma^3 \gamma^5 \right)} \left[ \frac{1}{k_\parallel - m_f^2 + i\epsilon} + 2\pi i n_F(k_0) \delta \left( k_\parallel^2 - m_f^2 \right) \right],$$

(15)

which is found to be modified by the strong magnetic field.

**2.2.2 Gluon propagator**

For gluon, the unitary matrix needed to thermalize the propagator is of the form

$$U_B(q_0) = \begin{pmatrix} \sqrt{1 + n_B(q_0)} & \sqrt{n_B(q_0)} \\ \sqrt{n_B(q_0)} & \sqrt{1 + n_B(q_0)} \end{pmatrix},$$

(16)

with the distribution function for gluons

$$n_B(q_0) = \frac{1}{e^{\beta |q_0|} - 1}.$$  

(17)

In matrix form, the gluon propagator is expressed as

$$D^{\mu\nu}(Q) = U_B(q_0) \begin{pmatrix} D_0^{\mu\nu}(Q) & 0 \\ 0 & D_0^{\mu\nu}(Q) \end{pmatrix} U_B(q_0).$$

(18)
Now proceeding like the quark case, the 11-component of the gluon propagator can be read from the above matrix as

\[ D_{11}^{\mu \nu}(Q) = ig^{\mu \nu} \left[ \frac{1}{Q^2 + i\epsilon} - 2\pi i n_B(q_0) \delta(Q^2) \right]. \tag{19} \]

### 2.3 One-loop quark self-energy in strongly magnetized medium

Using Feynman rules and 11-components of the quark and gluon propagators (equations (15) and (19)), the one-loop quark self energy (in figure 1) in presence of a strong magnetic field is given as

\[
\Sigma(P) = -\frac{4}{3}g^2 i \int \frac{d^4K}{(2\pi)^4} \left[ \gamma_{\mu} S_{11}(K) \gamma^\mu D_{11}(P - K) \right],
\]

where the factor 4/3 is associated with the fundamental representation of SU(3)\(_c\) gauge group through the relation: \( C_F = \frac{N^2-1}{2N_c} \) and \( g \) is the running coupling constant.

![Figure 1: Quark self energy](image)

The momentum integration can be factorized into parallel and perpendicular components with respect to the direction of magnetic field

\[
\Sigma(P) = \frac{4g^2i}{3(2\pi)^4} \int d^2k_\perp d^2k_\parallel e^{-\frac{k_\perp^2}{\alpha_f B}} \left[ \gamma_{\mu} \left( \gamma^0 k_0 - \gamma^3 k_3 + m_f \right) \left( 1 - \gamma^0 \gamma^3 \gamma^5 \right) \gamma^\mu \right]
\]
\[
\times \left[ \frac{1}{k_\parallel^2 - m_f^2 + i\epsilon} + 2\pi i n_F(k_0) \delta(k_\parallel^2 - m_f^2) \right]
\]
\[
\times \left[ \frac{1}{(P - K)^2 + i\epsilon} - 2\pi i n_B(p_0 - k_0) \delta((P - K)^2) \right]. \tag{21} \]

In strong magnetic field limit, the external quark momentum \( P \) can be assumed to be purely longitudinal [59], i.e. \( p_\perp = 0 \), so the internal gluon momentum squared becomes

\[ (P - K)^2 = (p_\parallel - k_\parallel)^2 - k_\perp^2. \tag{22} \]

In LLL approximation, \( k_\parallel^2, k_\perp^2 \) are assumed to be much smaller than \( |q_f B| \), hence, the form factor \( e^{-\frac{k_\perp^2}{\alpha_f B}} \) can be set equal to 1 and the upper limits of all the momenta integrals
should be cut off at $|q_f B|$. By evaluating the product of gamma matrices in (21) as
\[
\gamma_\mu (\gamma^0 k_0 - \gamma^3 k_3 + m_f) (1 - \gamma^0 \gamma^3 \gamma^5) \gamma^\mu = -2 (\gamma^0 k_0 - \gamma^3 k_3 - 2m_f) ,
\]
the quark self energy can thus be separated into the vacuum and medium components as
\[
\Sigma(p_\parallel) = \frac{-8g^2 i}{3(2\pi)^4} \int d^2k_\perp d^2k_\parallel (\gamma^0 k_0 - \gamma^3 k_3 - 2m_f)
\times \left[ \frac{1}{k_\parallel^2 - m_f^2 + i\epsilon} + 2\pi i n_F(k_0) \delta(k_\parallel^2 - m_f^2) \right]
\times \left[ \frac{1}{(p_\parallel - k_\parallel)^2 - k_\perp^2 + i\epsilon} - 2\pi i n_B(p_0 - k_0) \delta((p_\parallel - k_\parallel)^2 - k_\perp^2) \right]
\equiv \Sigma_V(p_\parallel) + \Sigma_n(p_\parallel) + \Sigma_{n^2}(p_\parallel),
\]
where $\Sigma_V(p_\parallel)$ represents the quark self-energy in vacuum, $\Sigma_n(p_\parallel)$ is the quark self energy in a medium containing both quark and gluon distribution functions and $\Sigma_{n^2}(p_\parallel)$ represents the quark self energy in a medium containing product of quark and gluon distribution functions.

### 2.3.1 Vacuum part

The vacuum contribution to the one-loop quark self energy is given by
\[
\Sigma_V(p_\parallel) = \frac{-8g^2 i}{3(2\pi)^4} \int d^2k_\perp d^2k_\parallel (\gamma^0 k_0 - \gamma^3 k_3 - 2m_f)
\times \left[ \frac{1}{k_\parallel^2 - m_f^2 + i\epsilon} \right]
\left[ \frac{1}{(p_\parallel - k_\parallel)^2 - k_\perp^2 + i\epsilon} \right].
\]
Separating the (real) principal value and imaginary part by the identity:
\[
\frac{1}{x \pm y \pm i\epsilon} = P \left( \frac{1}{x \pm y} \right) \mp i\pi \delta(x \pm y) ,
\]
we obtain the real part of vacuum contribution, $\Sigma_V(p_\parallel)$ with the following form
\[
\Sigma_V(p_\parallel) = \frac{-8g^2 i}{3(2\pi)^4} \int d^2k_\perp d^2k_\parallel (\gamma_\parallel \cdot k_\parallel - 2m_f)
\times \left[ \frac{1}{k_\parallel^2 - m_f^2} \right]
\left[ \frac{1}{(p_\parallel - k_\parallel)^2 - k_\perp^2} \right].
\]
Using the Feynman parametrization and Wick rotation, the parallel momentum ($k_\parallel$) integration can be recast into the form

$$I = i \int_0^1 dz \ d^2k' \ \frac{z\gamma^\parallel \cdot p_\parallel - 2m_f}{\left[k'^2 - z(1-z)p_\parallel^2 + zk_\perp^2 + (1-z)m_f^2\right]^{2}}$$

$$= -\frac{i\pi}{|q_f B|} \int_0^1 dz \ (z\gamma^\parallel \cdot p_\parallel - 2m_f) + i\pi \int_0^1 dz \ \frac{z\gamma^\parallel \cdot p_\parallel - 2m_f}{-z(1-z)p_\parallel^2 + zk_\perp^2 + (1-z)m_f^2}$$

$$\equiv I_1 + I_2 , \quad (28)$$

where $I_1$ is solved as

$$I_1 = -\frac{i\pi}{|q_f B|} \int_0^1 dz \ (z\gamma^\parallel \cdot p_\parallel - 2m_f)$$

$$= -\frac{i\pi}{2|q_f B|} (\gamma^\parallel \cdot p_\parallel - 4m_f) . \quad (29)$$

For solving $I_2$, we expand it in a Taylor series around the mass-shell condition: $\gamma^\parallel \cdot p_\parallel = m_f$ in a strong magnetic field,

$$I_2 = A + B (\gamma^\parallel \cdot p_\parallel - m_f) + C (\gamma^\parallel \cdot p_\parallel - m_f)^2 + \cdots \quad (30)$$

Dropping the higher-order terms, the integral, $I_2$ is given by

$$I_2 = A + B (\gamma^\parallel \cdot p_\parallel - m_f) , \quad (31)$$

where $A$ and $B$ are given by the following expressions

$$A = I_2 |_{\gamma^\parallel \cdot p_\parallel = m_f}$$

$$= i\pi m_f \int_0^1 dz \frac{z - 2}{m_f^2(1-z)^2 + zk_\perp^2} \quad (32)$$

$$B = \left. \frac{\partial I_2}{\partial (\gamma^\parallel \cdot p_\parallel)} \right|_{\gamma^\parallel \cdot p_\parallel = m_f}$$

$$= i\pi \int_0^1 dz \frac{z}{(1-z)^2m_f^2 + zk_\perp^2} + i\pi \int_0^1 dz \frac{2m_f^2 z(1-z)(z-2)}{((1-z)^2m_f^2 + zk_\perp^2)^2} , \quad (33)$$

respectively. At least, for light flavours we may drop terms proportional to $m_f^2$ and higher orders in the solutions of above integrals to get $A$ and $B$ as

$$A = \frac{i\pi}{2m_f} \ln \left(\frac{k_\perp^2}{m_f^2}\right) , \quad (34)$$

$$B = -\frac{i\pi}{2m_f^2} \ln \left(\frac{k_\perp^2}{m_f^2}\right) . \quad (35)$$
Substituting the values for A and B in equation (31), it yields

\[ I_2 = \frac{i\pi}{2m_f} \ln \left( \frac{k_\perp^2}{m_f^2} \right) - \frac{i\pi}{2m_f} \left( \gamma^\parallel \cdot p_\parallel - m_f \right) \frac{1}{2m_f} \ln \left( \frac{k_\perp^2}{m_f^2} \right) \]

\[ = \frac{i\pi}{m_f} \ln \left( \frac{k_\perp^2}{m_f^2} \right) - \left( \gamma^\parallel \cdot p_\parallel \right) \frac{i\pi}{2m_f^2} \ln \left( \frac{k_\perp^2}{m_f^2} \right). \] (36)

Now the integral involving the \(k_\parallel\) integration (28) yields

\[ I = -\frac{i\pi}{2|q_fB|} \left( \gamma^\parallel \cdot p_\parallel - 4m_f \right) + \frac{i\pi}{m_f} \ln \left( \frac{k_\perp^2}{m_f^2} \right) - \left( \gamma^\parallel \cdot p_\parallel \right) \frac{i\pi}{2m_f^2} \ln \left( \frac{k_\perp^2}{m_f^2} \right). \] (37)

Finally inserting the integral, I in (27) and then performing the remaining \(k_\perp\) integration, we get the real part of the vacuum contribution of one-loop quark self-energy,

\[ \Sigma_V(p_\parallel) = \frac{(\gamma^\parallel \cdot p_\parallel) g^2}{6\pi^2} \left[ -\frac{1}{2} - \frac{|q_fB|}{2m_f^2} \left\{ \ln \left( \frac{|q_fB|}{m_f^2} \right) - 1 \right\} \right] \]

\[ + \frac{g^2}{6\pi^2} \left[ 2m_f + \frac{|q_fB|}{m_f} \left\{ \ln \left( \frac{|q_fB|}{m_f^2} \right) - 1 \right\} \right]. \] (38)

2.3.2 Medium part

In a medium, both \(\Sigma_n(p_\parallel)\) and \(\Sigma_{n^2}(p_\parallel)\), which contain single quark and gluon distribution and product of quark and gluon distribution functions, contribute to the one-loop quark self-energy. Now, using the identity (26), we obtain the real-part of one-loop quark self-energy due to single distribution function,

\[ \Sigma_n(p_\parallel) = \frac{8g^2}{3(2\pi)^3} \int d^2k_\perp dk_3 dk_0 \left( \gamma^0 k_0 - \gamma^3 k_3 - 2m_f \right) \]

\[ \times \delta \left( k_\parallel^2 - m_f^2 \right) n_F(k_0) + \delta \left( (p_\parallel - k_\parallel)^2 - k_\perp^2 \right) \left\{ -n_B(p_0 - k_0) \right\} \]

\[ \equiv \Sigma_{n_F}(p_\parallel) + \Sigma_{n_B}(p_\parallel), \] (39)

with the quark and gluon contributions are now separable and the quark part is

\[ \Sigma_{n_F}(p_\parallel) = \frac{8g^2}{3(2\pi)^3} \int d^2k_\perp dk_3 dk_0 \left( \gamma^0 k_0 - \gamma^3 k_3 - 2m_f \right) \frac{\delta \left( k_0^2 - \omega_f^2 \right) n_F(k_0)}{(p_\parallel - k_\parallel)^2 - k_\perp^2}, \] (40)
with $\omega_k^2 = k_3^2 + m_f^2$. After performing the $k_0$ integration using the property of Dirac delta function, we find
\[ \Sigma_{n_F}(p_\parallel) = -\frac{8g^2}{3(2\pi)^3} \int dk_3 \frac{n_F(\omega_k)}{\omega_k} \left( \gamma^3 k_3 + 2m_f \right) \int d^2k_\perp \frac{1}{(p_\parallel - m_f)^2 - k_\perp^2}, \quad (41) \]
which involves two independent integrations over $k_3$ and $k_\perp$ momenta. The integral involving $k_3$ integration is solved into
\[ I_{k_3} = \int_{-\infty}^{+\infty} dk_3 \frac{n_F(\omega_k)}{\omega_k} \left( \gamma^3 k_3 + 2m_f \right) \]
\[ = 4m_f \left[ -\frac{1}{2} \ln \left( \frac{m_f}{\pi T} \right) - \frac{1}{2} \gamma_E + \mathcal{O} \left( \frac{m_f^2}{T^2} \right) \right], \quad (42) \]
where $\gamma_E$ is the Euler-Mascheroni constant. For a thermal medium considered here, $m_f^2$ for light flavours is much less than $T^2$, so the term, $\mathcal{O} \left( \frac{m_f^2}{T^2} \right)$ can be dropped and $I_{k_3}$ turns out to be
\[ I_{k_3} = -2m_f \left[ \ln \left( \frac{m_f}{\pi T} \right) + \gamma_E \right]. \quad (43) \]
Now the $k_\perp$ integration in equation (41) is performed after taking the upper limit of the integration by $|q_f B|$ compatible to LLL approximation
\[ I_{k_\perp} = -\pi \left[ i\pi + \ln \left( \frac{|q_f B|}{(p_\parallel - m_f)^2} \right) \right]. \quad (44) \]
Substituting the values of $I_{k_3}$ and $I_{k_\perp}$ integrations in equation (41) and keeping the real-part only, we obtain
\[ \Sigma_{n_F}(p_\parallel) = -\frac{2g^2m_f}{3\pi^2} \ln \left( \frac{|q_f B|}{(p_\parallel - m_f)^2} \right) \left[ \ln \left( \frac{m_f}{\pi T} \right) + \gamma_E \right]. \quad (45) \]
Similarly the part involving gluon distribution in equation (39) is
\[ \Sigma_{n_B}(p_\parallel) = -\frac{8g^2}{3(2\pi)^3} \int d^2k_\perp dk_3 dk_0 \left( \gamma^0 k_0 - \gamma^3 k_3 - 2m_f \right) \]
\[ \times \frac{\delta \left( (p_\parallel - k_\parallel)^2 - k_\perp^2 \right)}{k_0^2 - \omega_k^2} n_B(p_0 - k_0). \quad (46) \]
Simplifying the argument of Dirac delta function in the above integration for small $k_\perp$, the $k_0$ and $k_\perp$ integrations have been facilitated to yield the contribution of gluon distribution
as

\[
\Sigma_{n_B}(p_{||}) = -\frac{4\pi|q_f B|^2}{3(2\pi)^3} \int_{-\infty}^{+\infty} dk_3 \frac{n_B(p_3 - k_3)}{(p_3 - k_3)} \left[ \frac{\gamma^0 p_0 - 2m_f}{(p_0 + p_3 - k_3)^2 - \omega_k^2} \right. \\
+ \frac{\gamma^0 (p_3 - k_3)}{(p_0 + p_3 - k_3)^2 - \omega_k^2} - \frac{\gamma^3 k_3}{(p_0 + p_3 - k_3)^2 - \omega_k^2} + \frac{\gamma^0 p_0 - 2m_f}{(p_0 - p_3 + k_3)^2 - \omega_k^2} \\
- \frac{\gamma^0 (p_3 + k_3)}{(p_0 - p_3 + k_3)^2 - \omega_k^2} - \frac{\gamma^3 k_3}{(p_0 - p_3 + k_3)^2 - \omega_k^2} \left. \right].
\]

Finally the above \( k_3 \) integration can be integrated out to give

\[
\Sigma_{n_B}(p_{||}) = -\frac{4\pi|q_f B|^2}{3(2\pi)^3} \left( I^1 + I^2 + I^3 + I^4 + I^5 + I^6 \right),
\]

where \( I^1, I^2, I^3, I^4, I^5 \) and \( I^6 \) are found as follows

\[
I^1 = \frac{i\pi(\gamma^0 p_0 - 2m_f)}{2(p_0 + p_3)} \left[ \frac{\beta(a - p_3) - 2}{2\beta(p_3 - a)^2} + \frac{n_B(a - p_3)}{a - p_3} \right],
\]

\[
I^2 = -\frac{i\pi\gamma^0}{2(p_0 + p_3)} \left[ \frac{1}{\beta(p_3 - a) + n_B(a - p_3)} \right],
\]

\[
I^3 = -\frac{i\pi\gamma^3}{2(p_0 + p_3)} \left[ \frac{-2a - \beta p_3(p_3 - a)}{2\beta(p_3 - a)^2} + \frac{an_B(a - p_3)}{a - p_3} \right],
\]

\[
I^4 = -\frac{i\pi(\gamma^0 p_0 - 2m_f)}{2(p_0 - p_3)} \left[ \frac{\beta(b - p_3) - 2}{2\beta(p_3 - b)^2} + \frac{n_B(b - p_3)}{b - p_3} \right],
\]

\[
I^5 = \frac{-i\pi\gamma^0}{2(p_0 - p_3)} \left[ \frac{1}{\beta(p_3 - b) + n_B(b - p_3)} \right],
\]

\[
I^6 = -\frac{i\pi\gamma^3}{2(p_0 - p_3)} \left[ \frac{-2b - \beta p_3(p_3 - b)}{2\beta(p_3 - b)^2} + \frac{bn_B(b - p_3)}{b - p_3} \right],
\]

where \( a \) and \( b \) are given by

\[
a = \frac{(p_0 + p_3)^2 - m_f^2}{2(p_0 + p_3)},
\]

\[
b = \frac{(p_0 - p_3)^2 - m_f^2}{2(p_0 - p_3)}.
\]

Therefore, \( \Sigma_{n_B}(p_{||}) \) cannot contribute to the real part of one-loop quark self-energy.

Finally, the medium contribution to the quark self energy involving product of quark and gluon distribution functions (from equation (24)) is

\[
\Sigma_{n^2}(p_{||}) = -\frac{8g^2i}{3(2\pi)^4} \int d^2k_\perp d^2k_{||} 4\pi^2 n_F(k_0) n_B(p_0 - k_0) \left( \gamma^0 k_0 - \gamma^3 k_3 - 2m_f \right) \times \left[ \delta(k^2_{||} - m_f^2) \delta((p_{||} - k_{||})^2 - k^2_\perp) \right],
\]

(57)
which however does not contribute to the real part of quark self energy.

Thus the vacuum \((38)\) and the medium contributions \((45)\) are added together to give the real part of one-loop quark self energy of a thermal QCD medium in strong magnetic field

\[
\Sigma(p_{||}) = \frac{(\gamma_{||} \cdot p_{||}) g^2}{6\pi^2} \left[ -\frac{1}{2} - \frac{|q_f B|}{2m_f^2} \left\{ \ln \left( \frac{|q_f B|}{m_f^2} \right) - 1 \right\} \right] \\
+ \frac{g^2}{6\pi^2} \left[ 2m_f + \frac{|q_f B|}{m_f} \left\{ \ln \left( \frac{|q_f B|}{m_f^2} \right) - 1 \right\} \right] \\
- \frac{2g^2 m_f}{3\pi^2} \ln \left( \frac{|q_f B|}{(p_{||} - m_f)^2} \right) \left[ \ln \left( \frac{m_f}{\pi T} \right) + \gamma_E \right],
\]

which enables us to compute the effective quark propagator from Dyson-Schwinger equation \((2)\).

### 3 Thermalized effective gluon propagator in a strongly-magnetized hot QCD medium

This section is attributed to the evaluation of effective gluon propagator in a thermal medium in presence of a strong magnetic field. In general, the effective gluon propagator can be obtained from the Schwinger-Dyson equation

\[
D_{\mu\nu}^{-1}(P) = D_{\mu\nu}^{-1}(P_0) + \Pi_{\mu\nu}(P).
\]

At finite temperature, the effective gluon propagator gets decomposed into longitudinal and transverse components in thermal medium. Although gluons are not affected directly by the presence of magnetic field but the dependence of magnetic field enters directly through the Debye mass and indirectly through the running strong coupling. To evaluate the components of gluon propagator, first we revisit how to decompose the gluon self-energy in a thermal medium in the coming subsection.

#### 3.1 One-loop gluon self-energy in a hot QCD medium

In vacuum, the gluon self-energy tensor is the linear combination of available four-momentum of the particle \((P_\mu)\) and the metric tensor \((g_{\mu\nu})\). Being a Lorentz invariant quantity, self-energy depends on \(P^2\) and further restriction by the Ward identity : \(P^\mu \Pi_{\mu\nu}(P) = 0\)
imposes the structure of the tensor as
\[
\Pi_{\mu\nu}(P) = \left( g_{\mu\nu} - \frac{P_\mu P_\nu}{P^2} \right) \Pi(P^2) \\
\equiv P^{\mu\nu}_T \Pi(P^2),
\]  
where \( P^{\mu\nu}_T \) is the transverse projection operator. However, at finite temperature, the Lorentz invariance is broken due to the direction of heat bath, which is introduced in terms of a four-velocity, \( u_\mu \) in the rest frame of heat bath. Now with the available four vectors, \( P_\mu, u_\mu \) and the tensor, \( g_{\mu\nu} \), two orthogonal tensors, which are most adopted to the physical degrees of freedom and project on the subspace transverse and parallel to the three momentum, \( \mathbf{p} \), respectively, are constructed
\[
P^{\mu\nu}_T = g_{\mu\nu} - \frac{P_\mu P_\nu}{P^2} - \frac{P^\mu L \cdot P^\nu L}{N^2}, \tag{61}
\]
\[
P^{\mu\nu}_L = -N_\mu N_\nu, \text{with } N_\mu = \frac{P_\mu (P_\mu u) - u_\mu P^2}{(P_\mu u)^2 - P^2}, \tag{62}
\]
as the tensorial basis to decompose the gluon self-energy tensor in thermal medium
\[
\Pi_{\mu\nu}(p_0, \mathbf{p}) = P^{\mu\nu}_T \Pi_T(p_0, \mathbf{p}) + P^{\mu\nu}_L \Pi_L(p_0, \mathbf{p}), \tag{63}
\]

The above functions \( \Pi_T \) and \( \Pi_L \) are known as transverse and longitudinal self-energies, respectively, which depends on both energy and three momentum in the rest frame of medium due to lack of Lorentz invariance as
\[
p_0 = u_\mu P_\mu, \tag{64}
\]
\[
|\mathbf{p}| = \sqrt{(u_\mu P_\mu)^2 - P^2}. \tag{65}
\]

By using the properties of the above projection operators, \( P^{\mu\nu}_L \) and \( P^{\mu\nu}_T \), the transverse and longitudinal self energies can be obtained as
\[
\Pi_L(P) = -\Pi_00(P), \tag{66}
\]
\[
\Pi_T(P) = \frac{1}{2} \left( \Pi^\mu_\mu(P) - \frac{P^2}{P^2} \Pi_L(P) \right), \tag{67}
\]
which are obtained in HTL perturbation theory [50]. The HTL gluon self-energy tensor in a thermal medium determined by the angular average over the spatial directions of light-like vectors is
\[
\Pi_{\mu\nu}(P) = m_D^2 \int \frac{d\Omega}{4\pi} K_\mu K_\nu \left( \frac{P_\mu \cdot u^\mu}{P_\mu \cdot K_\mu} - u_\mu u_\nu \right) \\
= m_D^2 \int \frac{d\Omega}{4\pi} K_\mu K_\nu \left( \frac{p_0}{p_0 + \mathbf{p} \cdot \mathbf{k}} - u_\mu u_\nu \right), \tag{68}
\]
where $K_\mu = (1, \mathbf{k})$ is a light like four-vector and $m_D$ is the Debye mass. Therefore, the transverse and longitudinal components, $\Pi_T(P)$ and $\Pi_L(P)$ become

$$
\Pi_T(P) = \frac{m_D^2 p_0^2}{2 p^2} + \frac{m_D^2 p_0}{4 p} \left(1 - \frac{p_0^2}{p^2}\right) \ln \left(\frac{p_0 + p}{p_0 - p}\right), 
$$

(69)

$$
\Pi_L(P) = m_D^2 - \frac{m_D^2 p_0}{2 p} \ln \left(\frac{p_0 + p}{p_0 - p}\right), 
$$

(70)

respectively. Thus the Schwinger-Dyson equation gives the dressed thermal gluon propagator as

$$
D_{\mu\nu} = P^T_{\mu\nu} \Delta_T + P^L_{\mu\nu} \frac{p^2}{p^2} \Delta_L, 
$$

(71)

where the transverse and longitudinal components of gluon propagator are given by

$$
\Delta_T = \frac{-1}{p^2 + \Pi_T(P)}, 
$$

(72)

$$
\Delta_L = \frac{1}{p^2 + \Pi_L(P)}.
$$

(73)

Physically, $\Delta_T$ describes the propagation of two transverse vacuum modes in thermal medium whereas $\Delta_L$ does not exist in the vacuum and thus represents collective modes of medium.

Now when the medium becomes strongly magnetized, the dependence of strong magnetic field in $\Pi_T(P)$ and $\Pi_L(P)$ originates from the magnetic field dependence of the Debye mass. Therefore we are going to derive the Debye mass in strong magnetic field by the static limit of the longitudinal component of gluon self energy in the next subsection.

### 3.2 Screening mass in strong magnetic field

The Debye screening manifests in the collective oscillation of the medium via the dispersion relation and is obtained by the static limit of the longitudinal part ("00" component) of gluon self-energy, i.e.

$$
\Pi_L(p_0 = 0, p \rightarrow 0) = m_D^2. 
$$

(74)

Out of four contributing diagrams (tadpole, gluon loop, ghost loop and quark loop) of the gluon self energy, only quark-loop (in figure 2) gets influenced by the magnetic field.
In real-time formalism, using Schwinger’s proper-time propagator in strong magnetic field (15) for internal quark line, the 11-component of the gluon self-energy for the quark-loop is written as

$$\Pi_{\mu\nu}(P) = -\frac{ig^2}{2} \int \frac{d^4K}{(2\pi)^4} \text{tr} \left[ \gamma^\mu S_{11}(K) \gamma^\nu S_{11}(K - P) \right]$$

$$= \frac{ig^2}{2(2\pi)^4} \sum_f \int d^2k_\perp d^{2}k_\parallel \text{tr} \left[ \gamma^\mu \left( \gamma^0 k_0 - \gamma^3 k_3 + m_f \right) \gamma^\nu \left( \gamma^0 q_0 - \gamma^3 q_3 + m_f \right) \right]$$

$$\times \left[ \frac{1}{k_\parallel^2 - m_f^2 + i\epsilon} + 2\pi i n_F(k_0) \delta \left( k_\parallel^2 - m_f^2 \right) \right] e^{-\frac{k_\perp^2}{|q_f B|}}$$

$$\times \left[ \frac{1}{q_\parallel^2 - m_f^2 + i\epsilon} + 2\pi i n_F(q_0) \delta \left( q_\parallel^2 - m_f^2 \right) \right] e^{-\frac{q_\perp^2}{|q_f B|}}, \hspace{1cm} (75)$$

where the factor $1/2$ enters due to the trace over colour indices and we use $Q = (q_0, q)$ in place of $K - P = (k_0 - p_0, k - p)$. The momentum integration is factorized into parallel and perpendicular components with respect to the direction of magnetic field, where the $k_\perp$ integration, $\Pi_{k_\perp}(p_\perp)$ is separated out to give

$$\Pi_{k_\perp}(p_\perp) = \int dk_1 dk_2 e^{-\frac{k_1^2}{|q_f B|}} e^{-\frac{k_2^2}{|q_f B|}}$$

$$= \frac{\pi |q_f B|}{2} e^{-\frac{p_\perp^2}{|q_f B|}}. \hspace{1cm} (76)$$

Thus the gluon self energy becomes completely separable into the components of momentum which are parallel and perpendicular to the magnetic field as

$$\Pi^{\mu\nu}(P) = \frac{\pi |q_f B|}{2} e^{-\frac{p_\perp^2}{|q_f B|}} \Pi^{\mu\nu}(p_\parallel). \hspace{1cm} (77)$$

Denoting the trace over gamma matrices by $L^{\mu\nu}$ as

$$L^{\mu\nu} = 8 \left[ k_\parallel^\mu \cdot q_\parallel^\nu + k_\parallel^\nu \cdot q_\parallel^\mu - g^{\mu\nu}_\parallel \left( k_\parallel^\mu \cdot q_\parallel^\mu - m_f^2 \right) \right], \hspace{1cm} (78)$$
the self energy depending only on the parallel component of momentum is decomposed into vacuum and thermal contributions as

\[ \Pi^\mu\nu(p_\parallel) = \frac{ig^2}{2(2\pi)^4} \sum_f \int dk_0 dk_3 L^{\mu\nu} \left[ \frac{1}{k_\parallel^2 - m_f^2 + i\epsilon} + 2\pi i n_F(k_0) \delta(k_\parallel^2 - m_f^2) \right] \]

\[ \times \left[ \frac{1}{q_\parallel^2 - m_f^2 + i\epsilon} + 2\pi i n_F(q_0) \delta(q_\parallel^2 - m_f^2) \right] \]

\[ = \Pi_{V}^{\mu\nu}(p_\parallel) + \Pi_{n}^{\mu\nu}(p_\parallel) + \Pi_{n^2}^{\mu\nu}(p_\parallel), \quad (79) \]

where \( \Pi_{V}^{\mu\nu}(p_\parallel) \), \( \Pi_{n}^{\mu\nu}(p_\parallel) \) and \( \Pi_{n^2}^{\mu\nu}(p_\parallel) \) are the vacuum and medium contributions to the gluon self energy containing single and double distribution functions, respectively, and are given by

\[ \Pi_{V}^{\mu\nu}(p_\parallel) = \frac{ig^2}{2(2\pi)^4} \int dk_0 dk_3 L^{\mu\nu} \left[ \frac{1}{(k_\parallel^2 - m_f^2 + i\epsilon)(q_\parallel^2 - m_f^2 + i\epsilon)} \right], \quad (80) \]

\[ \Pi_{n}^{\mu\nu}(p_\parallel) = -\frac{g^2}{2(2\pi)^3} \int dk_0 dk_3 L^{\mu\nu} \left[ \frac{n_F(k_0)\delta(k_\parallel^2 - m_f^2)}{(q_\parallel^2 - m_f^2 + i\epsilon)} + \frac{n_F(q_0)\delta(q_\parallel^2 - m_f^2)}{(k_\parallel^2 - m_f^2 + i\epsilon)} \right], \quad (81) \]

\[ \Pi_{n^2}^{\mu\nu}(p_\parallel) = -\frac{ig^2}{2(2\pi)^2} \int dk_0 dk_3 L^{\mu\nu} \left[ n_F(k_0)n_F(q_0)\delta(k_\parallel^2 - m_f^2)\delta(q_\parallel^2 - m_f^2) \right]. \quad (82) \]

We will now first evaluate the vacuum part (80) for which the real part (using the identity (26)) is given by

\[ \Pi_{V}^{\mu\nu}(p_\parallel) = \left( g_\mu^{\nu} - \frac{p_\parallel^{\nu} p_\parallel^{\mu}}{p_\parallel^2} \right) \Pi(p_\parallel^2), \quad (83) \]

where \( \Pi(p_\parallel^2) \) is given by

\[ \Pi(p_\parallel^2) = \frac{g^2}{2\pi^3} \sum_f \left[ \frac{2m_f^2}{p_\parallel^2} \left( 1 - \frac{4m_f^2}{p_\parallel^2} \right)^{-1/2} \ln \left\{ \left( 1 - \frac{4m_f^2}{p_\parallel^2} \right)^{1/2} + 1 \right\} + 1 \right]. \quad (84) \]

The real part of 00 - component of vacuum contribution to the one-loop gluon self-energy tensor becomes

\[ \Pi_{V}^{00}(p_0, p_\parallel) = -\frac{p_\parallel^2}{p_\parallel^2} \Pi(p_\parallel^2). \quad (85) \]

Thus multiplying the transverse component (76) of gluon self-energy, the vacuum part of one-loop gluon self energy for massless flavours in the static limit \( (p_0 = 0, p_1, p_2, p_3 \to 0) \)

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is simplified into
\[ \Pi_{00}^V = \frac{g^2}{4\pi^2} \sum_f |q_f B|, \tag{86} \]
whereas for physical quark masses, it vanishes in the static limit,
\[ \Pi_{00}^0 = 0. \tag{87} \]

Similarly the real part of 00 - component of the thermal contribution to the gluon self-energy, \( \Pi_{0n}^{\mu\nu}(p_\parallel) \) with a single distribution function for \( p_0 = 0 \) is given by
\[ \Pi_{0n}^{00}(p_0 = 0, p_3) = -\frac{g^2}{2(2\pi)^3} \sum f \int dk_3 \left[ \frac{L^{00}(k_0 = \omega_k) n_F(k_0 = \omega_k)}{2\omega_k(\omega_k^2 - \omega_q^2)} + \frac{L^{00}(k_0 = -\omega_k) n_F(k_0 = -\omega_k)}{2\omega_k(-\omega_k^2 - \omega_q^2)} \right. \]
\[ \left. + \frac{L^{00}(k_0 = -\omega_q) n_F(k_0 = -\omega_q)}{2\omega_k(-\omega_q^2 - \omega_k^2)} \right], \tag{88} \]
where the different factors in above equation are given by
\[ L^{00} = 8 \left( k^0 q^0 + k_3 q_3 + m_f^2 \right), \]
\[ \omega_k^2 = k_3^2 + m_f^2, \omega_q^2 = q_3^2 + m_f^2, \]
\[ L^{00}(k_0 = \omega_k) = 8 \left( 2\omega_k^2 - k_3 p_3 \right), \]
\[ L^{00}(k_0 = -\omega_k) = 8 \left( 2\omega_k^2 - k_3 p_3 \right), \]
\[ L^{00}(k_0 = \omega_q) = 8 \left( \omega_k^2 + \omega_q^2 - k_3 p_3 \right), \]
\[ L^{00}(k_0 = -\omega_q) = 8 \left( \omega_k^2 + \omega_q^2 - k_3 p_3 \right), \]
\[ n_F(k_0 = \omega_k) = n_F(k_0 = -\omega_k) = \frac{1}{e^{\beta|\omega_k|} + 1}, \]
\[ n_F(k_0 = \omega_q) = n_F(k_0 = -\omega_q) = \frac{1}{e^{\beta|\omega_q|} + 1}. \]

For massless quarks, the medium contribution, \( \Pi_{0n}^{00}(p_0 = 0, p_3) \) reduces to
\[ \Pi_{0n}^{00}(p_0 = 0, p_3) = \frac{8g^2}{2(2\pi)^3} \left[ -1 - \frac{T}{p_3} \ln(2) + \frac{T}{p_3} \ln \left( 1 + e^{\frac{\omega_q}{T}} \right) \right], \tag{89} \]
whereas for the physical quark masses, it becomes
\[ \Pi_{0n}^{00}(p_0 = 0, p_3) = -\frac{g^2}{2(2\pi)^3} \int dk_3 \left[ \frac{8k_3 n_F(\omega_k)}{\omega_k p_3} \right. \]
\[ \left. + \frac{16m_f^2 n_F(\omega_k)}{\omega_q p_3 (2k_3 - p_3)} - \frac{16m_f^2 n_F(\omega_q)}{\omega_q p_3 (2k_3 - p_3)} \right]. \tag{90} \]
The medium contribution to the self-energy with the square of quark distribution function \((82)\) is purely imaginary so it does not have the real-part, i.e.

\[ \Pi_{\alpha\beta}(p_0) = 0 . \] (91)

Again multiplying the transverse component \((76)\) to parallel component \((89)\) gives the real part of 00 - component of medium contribution to one-loop gluon self-energy tensor for massless quarks, which, in the static limit \((p_0 = 0, p_3 \to 0)\), is simplified into

\[ \Pi_{00}^m = -\frac{g^2}{4\pi^2} \sum_f |q_f B| + \frac{g^2}{8\pi^2} \sum_f |q_f B| , \] (92)

whereas the real part of 00 - component of medium contribution to one-loop gluon self-energy tensor for physical quark masses in the static limit is reduced to

\[ \Pi_{00}^m = \frac{g^2}{4\pi^2T} \sum_f |q_f B| \int_0^\infty dk_3 \frac{e^{\beta\omega_k}}{(1 + e^{\beta\omega_k})^2} . \] (93)

Finally the vacuum \((86)\) and medium contributions \((92)\) are added together to give the real part of the 00 - component of one-loop gluon self-energy in static limit for massless quarks

\[ \Pi_{00} = \frac{g^2}{8\pi^2} \sum_f |q_f B| , \] (94)

whereas the vacuum \((87)\) and medium contributions \((93)\) yield the real part of the 00 - component of one-loop gluon self-energy tensor in static limit for physical quark masses

\[ \Pi_{00} = \frac{g^2}{4\pi^2T} \sum_f |q_f B| \int_0^\infty dk_3 \frac{e^{\beta\omega_k}}{(1 + e^{\beta\omega_k})^2} . \] (95)

Therefore the definition \((74)\) gives the Debye mass for the massless quarks

\[ m_D^2 = \frac{g^2}{8\pi^2} \sum_f |q_f B| , \] (96)

which was recently obtained by one of us \([62]\) and by others by different approach \([61, 33]\). It is found that the Debye mass of thermal QCD medium in the presence of strong magnetic field depends solely on the magnetic field and is independent of temperature, therefore the collective behaviour of the medium gets strongly affected by the presence of strong magnetic field. However, for physical quark masses, the Debye mass is obtained as

\[ m_D^2 = \frac{g^2}{4\pi^2T} \sum_f |q_f B| \int_0^\infty dk_3 \frac{e^{\beta\sqrt{k_3^2 + m_f^2}}}{(1 + e^{\beta\sqrt{k_3^2 + m_f^2}})^2} , \] (97)
which depends now on both magnetic field and temperature. However it becomes independent of temperature beyond a certain temperature for a particular strong magnetic field [62].

4 Thermodynamic observables of QCD matter in strong magnetic field

4.1 Free energy and pressure

The one loop free energy for \(N_f\) quarks with \(N_c\) colours in hot QCD medium in a static and homogeneous strong magnetic field is given by the sum of free energies due to quarks \((\mathcal{F}_q)\) and gluons \((\mathcal{F}_g)\), which are obtained by the functional determinant of effective quark and gluon propagators, respectively. Finally the pressure for the quark matter is obtained by the negative of free energy in the thermodynamic limit. We are now in a position to calculate the free energies and hence the pressures due to quarks and gluons using their respective one-loop propagators.

4.1.1 Quark contribution

The free energy due to \(N_f\) quarks with \(N_c\) colours is obtained by the effective quark propagator, \(S(P)\) from equation (2)

\[
\mathcal{F}_q = N_c N_f \int \frac{d^4 P}{(2\pi)^4} \ln \left[ \det \left( S(P) \right) \right] = -N_c N_f \int \frac{d^3 P}{(2\pi)^4} \ln \left[ \det \left( \gamma^\parallel \cdot p^\parallel - m_f - \Sigma(p^\parallel) \right) \right].
\]

Due to the external magnetic field in \(z\) direction, the momentum integration in the quark free energy is also factorized into the momentum parallel and perpendicular to the magnetic field, which is facilitated by the finding that the integrand (i.e. the effective propagator) depends only on the longitudinal momentum component. For the sake of simplicity, we first express the quark self-energy into dependent and independent terms on (parallel)
momentum from equation (58)

\[
\Sigma(p_\parallel) = (\gamma \cdot p_\parallel)C + D + E, \quad \text{with}
\]

\[
C = \frac{g^2}{6\pi^2} \left[ \frac{1}{2} - \frac{|q_f B|}{2m_f^2} \left\{ \ln \left( \frac{|q_f B|}{m_f^2} \right) - 1 \right\} \right], \tag{100}
\]

\[
D = \frac{g^2}{6\pi^2} \left[ 2m_f + \frac{|q_f B|}{m_f} \left\{ \ln \left( \frac{|q_f B|}{m_f^2} \right) - 1 \right\} \right], \tag{101}
\]

\[
E = -\frac{2g^2 m_f}{3\pi^2} \ln \left( \frac{|q_f B|}{(p_\parallel - m_f)^2} \right) \left[ \ln \left( \frac{m_f}{\pi T} \right) + \gamma_E \right], \tag{102}
\]

and then evaluate the determinant,

\[
\det [\gamma \cdot p_\parallel - m_f - \Sigma(p_\parallel)] = \left[ p_\parallel^2 (1 - C)^2 - (m_f + D + E)^2 \right]^2. \tag{103}
\]

Thus after plugging the determinant into the integration (98), the thermodynamic free energy of QCD matter due to quark contribution is expressed as

\[
F_q = -2N_c N_f \int \frac{d^2 p_\perp}{(2\pi)^2} \int \frac{d^2 p_\parallel}{(2\pi)^2} \ln \left[ p_\parallel^2 (1 - C)^2 - (m_f + D + E)^2 \right]
\]

\[
= -2N_c N_f \int \frac{d^2 p_\perp}{(2\pi)^2} \left[ \int \frac{d^2 p_\parallel}{(2\pi)^2} \ln \left( p_\parallel^2 \right) + \int \frac{d^2 p_\parallel}{(2\pi)^2} \ln \left[ (1 - C)^2 - \frac{1}{p_\parallel^2} (m_f + D + E)^2 \right] \right]
\]

\[
\equiv -\frac{N_c N_f |q_f B|}{2\pi} \left( I_{1p_\parallel} + I_{2p_\parallel} \right). \tag{104}
\]

At finite temperature, the integrals of type \( I_{1p_\parallel} \) have been frequently solved in 4-dimension employing Matsubara frequency sum method, where the continuous energy integrals are replaced by discrete frequency sums. Thus we solve this analytically as

\[
I_{1p_\parallel} = \int \frac{dp_\parallel}{2\pi} \int \frac{dp_\perp}{2\pi} \ln \left( p_\perp^2 - p_\parallel^2 \right)
\]

\[
= \frac{\pi T^2}{6}. \tag{105}
\]

The integral, \( I_{2p_\parallel} \) cannot be evaluated analytically, so we compute it numerically

\[
I_{2p_\parallel} = \int \frac{d^2 p_\parallel}{(2\pi)^2} \ln \left[ (1 - C)^2 - \frac{1}{p_\parallel^2} (m_f + D + E)^2 \right]
\]

\[
= \frac{1}{4\pi} \int_0^{|q_f B|} dp_\parallel^2 \ln \left[ (1 - C)^2 - \frac{1}{p_\parallel^2} \left\{ m_f + D + W \left( \frac{|q_f B|}{(p_\parallel - m_f)^2} \right) \right\} \right] \tag{106}
\]
by reexpressing the (parallel) momentum-dependent term, \( E \) as

\[
E = W \ln \left( \frac{|q_f B|}{(p_\parallel - m_f)^2} \right), \quad \text{with}
\]

\[
W = -\frac{2g^2 m_f}{3\pi^2} \left[ \ln \left( \frac{m_f}{\pi T} \right) + \gamma_E \right].
\]  

(107)

(108)

Now, substituting the integrals \( I_{1p_\parallel} \) and \( I_{2p_\parallel} \) in equation (104), the free energy due to quark contribution of hot QCD matter in strong magnetic field is obtained as

\[
\mathcal{F}_q = -\frac{N_c N_f |q_f B|}{4} \left[ \frac{T^2}{3} + \frac{1}{2\pi^2} \int_0^{[q_f B]} dp_\parallel^2 \ln [(1 - C)^2]
\right.

\[
- \frac{1}{p_\parallel^2} \left\{ m_f + D + W \ln \left( \frac{|q_f B|}{(p_\parallel - m_f)^2} \right)^2 \right\}. \]

(109)

Hence the negative of the above free energy in the thermodynamic limit gives the quark contribution to the thermodynamic pressure

\[
P_q = \frac{N_c N_f |q_f B|}{4} \left[ \frac{T^2}{3} + \frac{1}{2\pi^2} \int_0^{[q_f B]} dp_\parallel^2 \ln [(1 - C)^2]
\right.

\[
- \frac{1}{p_\parallel^2} \left\{ m_f + D + W \ln \left( \frac{|q_f B|}{(p_\parallel - m_f)^2} \right)^2 \right\}. \]

(110)

After substituting the values of \( C, D \) and \( W \) (equations (100), (101) and (108)) in above equation, we get the pressure due to quark contribution

\[
P_q = \frac{N_c N_f |q_f B|}{4} \left[ \frac{T^2}{3} + \int_0^{[q_f B]} \frac{dp_\parallel^2}{2\pi^2} \ln \left[ \left( 1 - \frac{g^2}{6\pi^2} \left\{ -\frac{1}{2} - \frac{|q_f B|}{2m_f^2} \left( \ln \left( \frac{|q_f B|}{m_f^2} \right) - 1 \right) \right) \right] \right]
\]

\[
- \frac{1}{p_\parallel^2} \left\{ m_f + \frac{g^2}{6\pi^2} \left( 2m_f + \frac{|q_f B|}{m_f^2} \left( \ln \left( \frac{|q_f B|}{m_f^2} \right) - 1 \right) \right) \right\}
\]

\[
- \frac{2g^2 m_f}{3\pi^2} \left\{ \ln \left( \frac{m_f}{\pi T} \right) + \gamma_E \right\} \ln \left( \frac{|q_f B|}{(p_\parallel - m_f)^2} \right)^2 \right\}. \]

(111)
4.1.2 Gluon contribution

The free energy due to gluons in adjoint representation of $SU(N_c)$ gauge theory is given by both transverse and longitudinal modes

$$F_g = (N_c^2 - 1) \left[ 2F_g^T + F_g^L \right]$$

$$= (N_c^2 - 1) \left[ \int \frac{d^4P}{(2\pi)^4} \ln \left( -\Delta_T(P) \right) - \frac{1}{2} \int \frac{d^4P}{(2\pi)^4} \ln \left( -\Delta_L(P) \right) \right]$$

$$= - (N_c^2 - 1) \left[ \int \frac{d^4P}{(2\pi)^4} \ln \left( -\Delta_T(P) \right) + \frac{1}{2} \int \frac{d^4P}{(2\pi)^4} \ln \left( -\Delta_L(P) \right) \right] , \quad (112)$$

where $\Delta_T(P)$ and $\Delta_L(P)$ are the transverse and longitudinal parts of the hard thermal loop gluon propagator, respectively, obtained earlier in equations (72,73).

Then substituting the values of $\Delta_T(P)$ and $\Delta_L(P)$ in equation (112), we obtain the gluon contribution to the thermodynamic free energy of QCD matter in strong magnetic field

$$F_g = - (N_c^2 - 1) \left[ \int \frac{d^4P}{(2\pi)^4} \ln \left( \frac{1}{P^2 + \Pi_T(P)} \right) \right]$$

$$+ \frac{1}{2} \int \frac{d^4P}{(2\pi)^4} \ln \left( \frac{1}{p^2 + \Pi_L(P)} \right) , \quad (113)$$

where $\Pi_L$ and $\Pi_T$ depend on the magnetic field through the screening (Debye) mass (97).

For gluon, the loop momenta may be hard (i.e. order of $T$) or soft (i.e. order of $gT$). In imaginary-time, the gluon energy $p_0$ is an integer multiple of $2\pi T$, so the soft region requires $p_0 = 0$. But for quark, the energy, $p_0 = (2n + 1)\pi T$ can never be zero even for $n = 0$, so, the quark loop is always hard. Since the gluons are not affected by the presence of magnetic field, the highest scale for them in a medium is still the temperature. For the hard loop momenta, the self energy components act like perturbative corrections and thereby making the effective propagators to expand around them. In case of soft loop momenta, in the zero frequency (static) mode, $\Pi_T$ becomes zero and only $\Pi_L$ exists. Since this is as large as the inverse of free propagator, the effective propagator can not be expanded. The total free energy due to gluon contribution is therefore obtained by adding the free energies determined from both hard and soft scales. Up to $O(g^4)$ in strong running coupling, the gluon free energy is calculated in [50]

$$F_g = - (N_c^2 - 1) \left[ \frac{\pi^2 T^4}{45} - \frac{T^2 m_D^2}{24} + \frac{T m_D^3}{12 \pi} \right.$$

$$\left. + \left\{ \frac{1}{\epsilon} + 2 \ln \left( \frac{\Lambda}{4\pi T} \right) - 7 + 2\gamma_E + \frac{2\pi^2}{3} \right\} \frac{m_D^4}{128 \pi^2} \right] , \quad (114)$$
where $\Lambda$ is the renormalization scale. The divergent term $(1/\epsilon)$ is isolated by the dimensional regularization, which is taken care of by the vacuum counter term as

$$\Delta \mathcal{E}_0 = \frac{N_c^2 - 1}{128 \pi^2 \epsilon} m_D^4.$$  \hspace{1cm} (115)

After adding the counter term, the renormalized gluon free energy for a strongly magnetized thermal QCD matter takes the following form

$$\mathcal{F}_g = - (N_c^2 - 1) \left[ \frac{\pi^2 T^4}{45} - \frac{T^2 m_D^2}{24} + \frac{T m_D^3}{12 \pi} \right] + \left\{ 2 \ln \left( \frac{\Lambda}{4 \pi T} \right) - 7 + 2 \gamma_E + \frac{2 \pi^2}{3} \right\} \frac{m_D^4}{128 \pi^2}. \hspace{1cm} (116)$$

Hence the pressure is obtained from the negative of the above free energy as

$$P_g = \left( N_c^2 - 1 \right) \left[ \frac{\pi^2 T^4}{45} - \frac{T^2 m_D^2}{24} + \frac{T m_D^3}{12 \pi} \right] + \left\{ 2 \ln \left( \frac{\Lambda}{4 \pi T} \right) - 7 + 2 \gamma_E + \frac{2 \pi^2}{3} \right\} \frac{m_D^4}{128 \pi^2}. \hspace{1cm} (117)$$

After plugging the Debye mass in strong magnetic field limit (96) for light flavours, the magnetic field dependence in the gluon contribution will be explicitly seen as

$$P_g = \left( N_c^2 - 1 \right) \left[ \frac{\pi^2 T^4}{45} - g^2 T^2 eB \frac{192 \pi^2}{192 \sqrt{2} \pi^2} + g^3 T (eB)^{\frac{3}{2}} \frac{192 \sqrt{2} \pi^2}{192 \sqrt{2} \pi^2} \right] + \frac{g^4 (eB)^2}{8192 \pi^6} \left\{ 2 \ln \left( \frac{\Lambda}{4 \pi T} \right) - 7 + 2 \gamma_E + \frac{2 \pi^2}{3} \right\}. \hspace{1cm} (118)$$
4.1.3 Total pressure

The total one-loop pressure of hot QCD matter in a strong magnetic field is obtained by adding both quark and gluonic contributions and has the following expression.

\[
P(T, eB) = \frac{N_c N_f |q_f B|}{4} \left[ \frac{T^2}{3} + \int_0^{|q_f B|} \frac{dp_\parallel^2}{2\pi^2} \ln \left( 1 - \frac{g^2}{6\pi^2} \left\{ \frac{1}{2} \right\} \right) \right.
\]

\[
- \frac{|q_f B|}{2m_f^2} \left( \ln \left( \frac{|q_f B|}{m_f^2} \right) - 1 \right) \right) \right)^2
\]

\[
- \frac{1}{p_\parallel^2} \left( m_f + \frac{g^2}{6\pi^2} \left\{ 2m_f + \frac{|q_f B|}{m_f} \left( \ln \left( \frac{|q_f B|}{m_f^2} \right) - 1 \right) \right\} \right)
\]

\[
- \frac{2g^2 m_f}{3\pi^2} \left\{ \ln \left( \frac{m_f}{\pi T} \right) + \gamma_E \right\} \ln \left( \frac{|q_f B|}{(p_\parallel - m_f)^2} \right) \right]\]

\[
+ (N_c^2 - 1) \left[ \frac{\pi^2 T^4}{45} - \frac{T^2 m_D^2}{24} + \frac{Tm_D^3}{12\pi} \right.
\]

\[
+ \left\{ 2\ln \left( \frac{\Lambda}{4\pi T} \right) - 7 + 2\gamma_E + \frac{2\pi^2}{3} \frac{m_D^4}{128\pi^2} \right\} \]

\[
\left. \right] \right], \quad (119)
\]

where the renormalization scale \( \Lambda \) is set at \( 2\pi T \). The ideal component can thus be read as

\[
P_{\text{ideal}}(T, eB) = N_c N_f \frac{|q_f B| T^2}{12} + (N_c^2 - 1) \frac{\pi^2 T^4}{45}. \quad (120)
\]

Before calculating the thermodynamic observables in the strong magnetic field limit, we should be careful in choosing the range of temperatures and magnetic fields compatible with the limit. *For example*, to observe the variation of pressure with magnetic field at a temperature \( T = 300 \text{ MeV} \), the starting value of magnetic field has to be greater than \( \sim 4.60 \, m_\pi^2 \), however, we have taken the starting magnetic field, \( eB = 10 \, m_\pi^2 \), which is almost twice the above marginal value. Similarly for calculating the pressure as a function of temperature up to \( T = 400 \text{ MeV} \), we have fixed the magnetic fields at \( eB = 15 \, m_\pi^2, 25 \, m_\pi^2, \) and \( 50 \, m_\pi^2 \).

To see how the pressure of hot QCD matter with two light flavours is affected by the ambient strong magnetic fields quantitatively, we have computed the pressure as a function of magnetic field at fixed temperatures, \( T = 200 \text{ MeV} \) and \( 300 \text{ MeV} \) of the medium in figure 3a, whereas the figure 3b denotes the variation of pressure with the temperature at different strong magnetic fields, \( eB = 15 \, m_\pi^2, 25 \, m_\pi^2, \) and \( 50 \, m_\pi^2 \). It is
found that the pressure increases rapidly with the magnetic field (figure 3a) compared to its much slower variation with the temperature (figure 3b). The above contrast in the behaviour of pressure with magnetic field compared to the temperature reflects the fact that the dominant scale of thermal medium in strong magnetic field limit is the magnetic field, not the temperature anymore happened to be in thermal medium in absence of magnetic field. To be more precise, although the variation of pressure with temperature is not steeper in the presence of strong magnetic field but the pressure for thermal QCD in strong magnetic field is larger than the pressure of thermal medium in absence of strong magnetic field (denoted by solid line in figure 3b), hence the strong magnetic field makes the equation of state for a thermal medium harder. These observations will facilitate in understanding the effects of strong magnetic field on the entropy density (figure 5). Recently, the thermodynamic pressure and other observables arising due to the presence of magnetic fields have also been studied extensively in lattice QCD with 2 + 1 flavour [36], where they have noticed the similar increasing trend of the longitudinal pressure with the increase of magnetic field strengths. To see how the pressure of interacting quarks and gluons in a hot QCD medium in presence of strong magnetic field approaches to the noninteracting (ideal) limit asymptotically both with the magnetic field and the temperature, we have computed the pressure in units of ideal pressure (120) as function of magnetic fields (in figure 4a) and temperatures as well (in figure 4b). From the figure 4a, it is interesting to know that the deviation of pressure from its ideal value increases with the strong magnetic field, i.e. the thermal QCD medium never achieve its ideal
limit asymptotically in the presence of strong magnetic field. On the other hand, it is found from figure 4b that, the pressure of the thermal medium at a fixed magnetic field approaches its ideal limit as expected, but in strong magnetic field limit one cannot arbitrarily increase the temperature due to its constraint \((eB \gg T^2)\).

### 4.2 Entropy density

To see how the available microstates to a given macrostate of a thermal QCD medium are affected due to the presence of strong magnetic field, we calculate the entropy density of hot QCD matter in a strong magnetic field by partially differentiating the pressure (119) with respect to the temperature

\[
S = \frac{\partial P}{\partial T} \equiv S_q + S_g ,
\] (121)
where the entropy density due to quark contribution is calculated as

\[
S_q = \frac{N_c N_f |q_f B|}{6} \left[ T - \frac{m_f g^2}{T \pi^4} \int_0^{|q_f B|} dp_\parallel \frac{1}{p_\parallel^4} \ln \left( \frac{|q_f B|}{(p_\parallel - m_f)^2} \right) \right] \times \left[ \left( m_f + \frac{g^2}{6 \pi^2} \right) \left( 2 m_f + \frac{|q_f B|}{m_f} \left( \ln \left( \frac{|q_f B|}{m_f^2} \right) - 1 \right) \right) \right. \\
- \frac{2 g^2 m_f}{3 \pi^2} \left\{ \ln \left( \frac{m_f}{\pi T} \right) + \gamma_E \right\} \ln \left( \frac{|q_f B|}{(p_\parallel - m_f)^2} \right) \\
/ \left( 1 - \frac{g^2}{6 \pi^2} \right) \left\{ -1 + \frac{|q_f B|}{2 m_f^2} \left( \ln \left( \frac{|q_f B|}{m_f^2} \right) - 1 \right) \right\}^2 \\
- \frac{1}{p_\parallel^2} \left( m_f + \frac{g^2}{6 \pi^2} \right) \left( 2 m_f + \frac{|q_f B|}{m_f} \left( \ln \left( \frac{|q_f B|}{m_f^2} \right) - 1 \right) \right) \\
- \frac{2 g^2 m_f}{3 \pi^2} \left\{ \ln \left( \frac{m_f}{\pi T} \right) + \gamma_E \right\} \ln \left( \frac{|q_f B|}{(p_\parallel - m_f)^2} \right) \left( \ln \left( \frac{|q_f B|}{m_f^2} \right) - 1 \right) \right] \right]. 
\tag{122}
\]

Similarly, the entropy density due to gluonic contribution is calculated as

\[
S_g = (N_c^2 - 1) \left[ \frac{4 \pi^2 T^3}{45} - \frac{T m_D^2}{12} + \frac{m_D^3}{12 \pi} \right], 
\tag{123}
\]

which will be seen to depend on magnetic field after substituting the Debye mass for two light flavours from (96) as

\[
S_g = (N_c^2 - 1) \left[ \frac{4 \pi^2 T^3}{45} - \frac{T m_D^2}{12} + \frac{m_D^3}{12 \pi} \right]. 
\tag{124}
\]

In calculating the above expression, we set the partial derivative of the square of screening mass with respect to the temperature to zero, i.e. \( \frac{\partial (m_D^2)}{\partial T} \approx 0 \), because the screening mass in strong magnetic field limit solely depends on the magnetic field for massless flavours, it may however depend on both magnetic field and temperature for realistic physical quark masses but the temperature dependence is still negligible.

In figure 5a, we have shown how the entropy density of a hot QCD medium at fixed temperatures, \( T = 200 \text{ MeV} \) and \( 300 \text{ MeV} \) has been affected by the stronger magnetic fields, where we found that it increases almost linearly with the magnetic field. Similarly figure 5b depicts the variation of entropy density of a hot QCD medium with the increasing temperatures in strong magnetic field backgrounds, \( eB = 15m_n^2 \), \( 25m_n^2 \), and \( 50m_n^2 \). It
Figure 5: The variation of entropy density as a function of magnetic field at different temperatures (a) and as a function of temperature at different strong magnetic fields (b).

is also seen in figure 5b that both the entropy density and its rate of increase with temperature in presence of strong magnetic field get waned compared to the medium in absence of strong magnetic field (denoted by the solid line, $B=0$). This crucial observation can be understood qualitatively: As we know that the strong magnetic field restricts the dynamics of quarks in momentum space from 4-dimension to 2-dimension, hence the phase space gets shrunk to 2-dimension only. Since the entropy is a measure of number of possible microstates in the phase space, so the entropy of thermal medium gets decreased in the presence of strong magnetic field. Thus both figures of the entropy density corroborate the observations in the variation of pressure with magnetic field and temperature (figures 3a and 3b), respectively.

4.3 Energy density

We are now in a position to calculate the energy density in baryonless ($\mu_q = 0$) hot QCD medium in a strong magnetic field. This can be obtained from the following thermodynamic relation,

$$\varepsilon = -P + TS$$

$$\equiv \varepsilon_q + \varepsilon_g \ , \quad (125)$$
where the energy density due to quark contribution is calculated as

\[
\varepsilon_q = -P_g + TS_g \\
= \frac{N_c N_f |q_f B|}{6} \left[ -\frac{T^2}{2} - \frac{3}{4\pi^2} \int_0^{\frac{|q_f B|}{m_f^2}} dp_{\parallel}^2 \ln \left( 1 - \frac{g^2}{6\pi^2} \right) \left\{ -\frac{1}{2} \right. \right.
- \frac{|q_f B|}{2m_f^2} \left( \ln \left( \frac{|q_f B|^2}{m_f^2} \right) - 1 \right) \left. \right\}^2 \\
- \frac{1}{p_{\parallel}^2} \left( m_f + \frac{g^2}{6\pi^2} \left\{ 2m_f + \frac{|q_f B|}{m_f} \left( \ln \left( \frac{|q_f B|^2}{m_f^2} \right) - 1 \right) \right\} \right)
- \frac{2g^2m_f}{3\pi^2} \left\{ \ln \left( \frac{m_f}{\pi T} \right) + \gamma_E \right\} \ln \left( \frac{|q_f B|^2}{(p_{\parallel} - m_f^2)^2} \right) \right]\]
- \frac{m_fg^2}{\pi^4} \int_0^{\frac{|q_f B|}{m_f^2}} dp_{\parallel}^2 \frac{1}{p_{\parallel}^2} \ln \left( \frac{|q_f B|^2}{(p_{\parallel} - m_f^2)^2} \right)
\times \left[ \left( m_f + \frac{g^2}{6\pi^2} \left\{ 2m_f + \frac{|q_f B|}{m_f} \left( \ln \left( \frac{|q_f B|^2}{m_f^2} \right) - 1 \right) \right\} \right)
- \frac{2g^2m_f}{3\pi^2} \left\{ \ln \left( \frac{m_f}{\pi T} \right) + \gamma_E \right\} \ln \left( \frac{|q_f B|^2}{(p_{\parallel} - m_f^2)^2} \right) \right)
/ \left( 1 - \frac{g^2}{6\pi^2} \left\{ 1 - \frac{|q_f B|^2}{2m_f^2} \left( \ln \left( \frac{|q_f B|^2}{m_f^2} \right) - 1 \right) \right\} \right)^2
- \frac{1}{p_{\parallel}^2} \left( m_f + \frac{g^2}{6\pi^2} \left\{ 2m_f + \frac{|q_f B|}{m_f} \left( \ln \left( \frac{|q_f B|^2}{m_f^2} \right) - 1 \right) \right\} \right)
- \frac{2g^2m_f}{3\pi^2} \left\{ \ln \left( \frac{m_f}{\pi T} \right) + \gamma_E \right\} \ln \left( \frac{|q_f B|^2}{(p_{\parallel} - m_f^2)^2} \right) \right]\right]. \tag{126}
\]

Similarly, the energy density due to gluonic contribution has been calculated as

\[
\varepsilon_g = -P_g + TS_g \\
= (N_c^2 - 1) \left[ \frac{\pi^2 T^4}{15} - \frac{T^2 m_D^2}{24} - \left\{ 2 \ln \left( \frac{\Lambda}{4\pi T} \right) - 7 + 2\gamma_E + \frac{2\pi^2}{3} \right\} \frac{m_D^4}{128\pi^2} \right]. \tag{127}
\]
The magnetic field dependence can be seen after replacing the Debye mass for massless flavours from (96)

\[
\varepsilon_g = (N_c^2 - 1) \left[ \frac{\pi^2 T^4}{15} - \frac{g_2^2 T^2 eB}{192\pi^2} - \frac{g_4^4 (eB)^2}{8192\pi^6} \left\{ 2\ln \left( \frac{\Lambda}{4\pi T} \right) - 7 + 2\gamma + \frac{2\pi^2}{3} \right\} \right].
\]

(128)

Figure 6: The variation of energy density with temperature at different magnetic fields.

To see how the energy density of a hot QCD medium has been affected by the presence of external strong magnetic field, we have computed the energy density as a function of temperature at different (strong) magnetic fields in figure 6, where the energy density increases with the temperature as expected but it increases with the temperature much faster for the medium in absence of magnetic field (B=0, denoted by solid line) which resonates with the observation of entropy density with temperature in figure 5b. In brief, the strong magnetic field reduces the energy density of thermal QCD medium.

4.4 Speed of sound

The speed of sound in a medium depends on the nature of equation of state, whether it is soft or hard and is related to the thermodynamic pressure and energy density through the following equation

\[
C_s^2 = \frac{\partial P}{\partial \varepsilon} = \frac{\partial P/\partial T}{\partial \varepsilon/\partial T},
\]

(129)

where the partial derivatives of pressure and energy density with respect to the temperature are obtained from equations (119) and (125), respectively. Since the existence
Figure 7: The variation of the square of the speed of sound with the strong magnetic field at various temperatures (a) and the variation with temperature in presence of strong magnetic fields of different strengths (b).

of strong magnetic field modifies the thermodynamic observables, $c_s^2$ is also expected to deviate from its value in the presence of strong magnetic field. To see the effects of the magnetic fields on the equation of state, we have computed the speed of sound of a hot QCD medium as a function of external magnetic fields in figure 7a. It is found that the speed of sound at a fixed temperature increases with the strength of magnetic fields, which can be understood by the fact that, as the strength of magnetic field increases, the energy density decreases and the pressures increases, hence the speed of sound gets increased. From the original perspective of how the speed of sound of a thermal QCD is now modified in the presence of magnetic field, we have computed $c_s^2$ as a function of temperature at different strengths of magnetic fields in figure 7b. We found that $c_s^2$ decreases with the temperature as expected and reaches asymptotically to the ideal value $1/3$ for the case when there is no magnetic field.

Interestingly when we plot the speed of sound in terms of ideal limit in strong magnetic field in figure 8, we have found that the speed of sound shows a dip in specific temperature-magnetic field combination. The above crucial observations in strong magnetic field could have phenomenological implications in heavy ion collisions, because the speed of sound modulates the hydrodynamic expansion of the medium (QGP) produced in noncentral ultrarelativistic heavy ion collisions.
Figure 8: The variation of $c_s^2$ normalized by its ideal value as a function of magnetic field at various temperatures (a) and as a function of temperature in presence of varying strong magnetic field strengths (b).

5 Conclusions

In this work, we have explored how the thermodynamic observables of a hot QCD medium in one-loop have been affected in an ambience of very strong magnetic field, which may be produced in noncentral events of ultrarelativistic heavy ion collisions. All thermodynamic observables have been contributed both by quarks and gluons through their respective one-loop self energies, where the quark contribution has been affected strongly by the strong magnetic field whereas, the gluonic part is largely unaffected except for the softening of the screening mass in strong magnetic field. As a result, even the pressure for the noninteracting quarks in thermal medium gets enhanced in strong magnetic field and overall an increase in total pressure of thermal medium is observed compared to the thermal medium in the absence of strong magnetic field. As a consequence, the entropy density gets decreased due to the presence of strong magnetic field, so the energy density too decreases with respect to pure thermal medium. Finally we obtain the equation of state by calculating the speed of sound of thermal QCD medium, which is found to increase due to the presence of strong magnetic field and shows a dip in a specific range of temperatures and strong magnetic fields. The above observations could have interesting implications on the expansion dynamics of the medium produced at RHIC and LHC in presence of strong magnetic field.
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