Supplement to the article “Approximating multi-dimensional Hamiltonian flows by billiards”:
Proof of $C^0$ and $C^r$ - closeness Theorems

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Theorem 1. Let the potential $V(q;\varepsilon)$ in the equation

$$H = \frac{p^2}{2} + V(q;\varepsilon),$$  \hspace{1cm} (1)

satisfy Conditions I-IV stated above. Let $h_\varepsilon$ be the Hamiltonian flow defined by (1) on an energy surface $H = H^* < E$, and $b_T$ be the billiard flow in $D$. Let $\rho_0$ and $\rho_T = b_T(\rho_0)$ be two inner phase points. Assume that on the time interval $[0,T]$ the billiard trajectory of $\rho_0$ has a finite number of collisions, and all of them are either regular reflections or non-degenerate tangencies. Then $h_\varepsilon \to b_T$, uniformly for all $\rho$ close to $\rho_0$ and all $t$ close to $T$.

Theorem 2. In addition to the conditions of Theorem 1, assume that the billiard trajectory of $\rho_0$ has no tangencies to the boundary on the time interval $[0,T]$. Then $h_\varepsilon \to b_T$ in the $C^r$-topology in a small neighborhood of $\rho_0$, and for all $t$ close to $T$.

Proof. By Condition I the Hamiltonian flow is $C^r$-close to the billiard flow outside an arbitrarily small boundary layer. So we will concentrate our attention on the behavior of the Hamiltonian flow inside such a layer.

Let the initial conditions correspond to the billiard orbit which hits a boundary surface $\Gamma_i$ at a (non-corner) point $q_c$. By Condition IIa, the surface $\Gamma_i$ is given by the equation $Q(q;0) = Q_i$, hence the boundary layer near $\Gamma_i$ can be defined as $N_\delta = \{ |Q(q;\varepsilon) - Q_i| \leq \delta \}$, where $\delta$ tends to zero sufficiently slowly as $\varepsilon \to +0$. Take $\varepsilon$ sufficiently small. The smooth trajectory enters $N_\delta$ at some time $t_{in}(\delta,\varepsilon)$ at a point $q_{in}(\delta,\varepsilon)$ which is close to the collision point $q_c$ with the velocity $p_{in}(\delta,\varepsilon)$ which is close to the initial velocity $p_0$. See Figure 1. The same trajectory exits from $N_\delta$ at the time $t_{out}(\delta,\varepsilon)$ at a point $q_{out}(\delta,\varepsilon)$ with velocity $p_{out}(\delta,\varepsilon)$. In these settings, the theorems are equivalent ($r = 0$ corresponds to Theorem 1, while $r > 0$ corresponds to

\footnote{Hereafter, $T$ always denotes a finite number.}
Figure 1: Hamiltonian flow inside small boundary layer.

Theorem 2) to proving the following statements:

\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0^+} \left\| (q_{\text{out}}(\delta, \varepsilon), t_{\text{out}}(\delta, \varepsilon)) - (q_{\text{in}}(\delta, \varepsilon), t_{\text{in}}(\delta, \varepsilon)) \right\|_{C^r} = 0, \tag{2}
\]

which guarantees that the trajectory does not travel along the boundary, and

\[
\lim_{\delta \to 0} \lim_{\varepsilon \to 0^+} \left\| p_{\text{out}}(\delta, \varepsilon) - p_{\text{in}}(\delta, \varepsilon) + 2n(q_{\text{in}})(p_{\text{in}}(\delta, \varepsilon), n(q_{\text{in}})) \right\|_{C^r} = 0, \tag{3}
\]

where \( p_{\text{out}} = p_{\text{in}} - 2\langle p_{\text{in}}, n(q) \rangle n(q) \) and \( n(q) \) is the unit inward normal to the level surface of \( Q \) at the point \( q \).

With no loss of generality, assume that \( Q(q; 0) \) increases as \( q \) leaves \( D' \)'s boundary towards \( D' \)'s interior. Choose the coordinates \((x, y)\) so that the hyperplane \( x \) is tangent to the level surface \( Q(q; \varepsilon) = Q(q_c; \varepsilon) \) and the \( y \)-axis is the inward normal to this surface at \( q = q_c \). Hence, the partial derivatives of \( Q \) satisfy:

\[
Q_x|_{q_c,x} = 0, \quad Q_y|_{q_c,x} = 1 \tag{4}
\]

By (1) and Condition II, near the boundary the equations of motion have the form:

\[
\dot{x} = \frac{\partial H}{\partial p_x} = p_x \quad \dot{p}_x = -\frac{\partial H}{\partial x} = -W'(Q; \varepsilon)Q_x, \tag{5}
\]

\[
\dot{y} = \frac{\partial H}{\partial p_y} = p_y \quad \dot{p}_y = -\frac{\partial H}{\partial y} = -W'(Q; \varepsilon)Q_y. \tag{6}
\]
We start with the $C^0$ version of (2) and (3). First, we will prove that given a sufficiently slowly tending to zero $\xi(\epsilon)$, if the orbit stays in the boundary layer $N_\delta$ for all $t \in [t_m, t_m + \xi]$, then in this time interval

\begin{align}
q(t) &= q_{\text{in}}(\delta, \epsilon) + O(\xi), \\
p_x(t) &= p_x(t_m(\delta, \epsilon)) + O(\xi).
\end{align}

(7) and (8)

Note that (7) follows immediately from (5)-(6) and the fact that $|q_{\text{in}} - q_t|$ tends to zero as $O(\delta)$ for regular trajectories and $O(\sqrt{\delta})$ for non-degenerate tangent trajectories, so by assuming that $\xi(\epsilon)$ is slow enough, we extract from (7) that

\begin{equation}
q(t) = q_c + O(\xi).
\end{equation}

(10)

Now, from (9), (10), for $t \in [t_m(\delta, \epsilon), t_m(\delta, \epsilon) + \xi]$ we have

\begin{equation}
Q_x(q(t); \epsilon) = O(\xi), \quad Q_y(q(t); \epsilon) = 1 + O(\xi).
\end{equation}

(11)

Divide the interval $I = [t_m, t_m + \xi]$ into two sets: $I_\epsilon$ where $|W'(Q; \epsilon)| < 1$ and $I_\omega$ where $|W'(Q; \epsilon)| \geq 1$. In $I_\epsilon$ we have $\dot{p}_x = O(\epsilon)$ by (5), (11). In $I_\omega$, as $|W'(Q; \epsilon)| \geq 1$ and $Q_y \neq 0$, we have that $\dot{p}_x$ is bounded away from zero, so in (5) we can divide $\dot{p}_x$ by $\dot{p}_y$:

\begin{equation}
\frac{dp_x}{dp_y} = \frac{Q_x}{Q_y}.
\end{equation}

(9)

It follows that the change in $p_x$ on I can be estimated from above as $O(\epsilon^2)$ (the contribution from $I_\epsilon$) plus $O(\epsilon)$ times the total variation in $p_y$. Thus, in order to prove (8), it is enough to show that the total variation in $p_x$ on $I$ is uniformly bounded. Recall that $p_x$ is uniformly bounded ($|p_x| \leq 1$ from the energy constraint) and monotone (as $W'(Q) < 0$ and $Q_y > 0$, we have $\dot{p}_x > 0$, see (6)) everywhere on $I$, so its total variation is uniformly bounded indeed. Thus, (8) is proven. The approximate conservation law (9) follows now from (8) and the conservation of $H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + W(Q; \epsilon) \epsilon$.

Finally, we prove that $\tau_\delta$, the time the trajectory spends in the boundary layer $N_\delta$, tends to zero as $\epsilon \to 0$. This step completes the proof of Theorem 1 by plugging the time $\tau_\delta \to 0$ instead of $\xi$ in the right-hand sides of (7), (8), (9), we immediately obtain the $C^0$ version of (2) and (3).

Let us start with the non-tangent case, i.e. with the trajectories such that $p_x(t_m)$ is bounded away from zero. From Condition III it follows that the value of $W_m = W_{\text{out}} = W(Q = \delta, \epsilon)$ vanishes as $\epsilon \to +0$. Hence, by (9) the momentum $p_x(t)$ stays bounded away from zero as long as the potential $W(Q; \epsilon)$ remains small. Choose some small $\nu$, and divide $N_\delta$ into two parts $N_{<} := \{W : W(Q; \epsilon) \leq \nu\}$ and $N_{>} := \{W : W(Q; \epsilon) > \nu\}$. First, the trajectory enters $N_{<}$. Since the value of $\frac{d}{dt} Q(q) = p_x Q_x + p_y Q_y$ is negative and bounded away from zero in $N_{<}$ (because $Q_x$ is small, and $p_x$ and $Q_y$ are non-zero), the trajectory must reach the inner part $N_{>}$ by a time proportional to the width of $N_{<}$,
which is $O(\delta)$. Also, we can conclude that if the trajectory leaves $N_\delta$ after some time $t_\delta$, it must have $p_y > 0$ and, arguing as above, we obtain that $t_{\text{out}} - t_{\text{in}} = O(\delta + t_\delta)$. Let us show that $t_\delta \to 0$ as $\epsilon \to +0$. Using (6), the fact that the total variation of $p_y$ is bounded, and Condition IV, we obtain

$$|t_\delta| \leq \frac{C}{\min_{N_\delta} |W'(Q;\epsilon)|} = C \max_{N_\delta} |Q'(W;\epsilon)| \to 0 \text{ as } \epsilon \to +0.$$  

So, in the non-tangent case, the collision time is $O(\delta + t_\delta)$, i.e. it tends to zero indeed.

This result holds true for $p_{\nu,\text{in}}$ bounded away from zero, and it remains valid for $p_{\nu,\text{in}}$ trending to zero sufficiently slowly. Hence, we are left with the case where $p_{\nu,\text{in}}$ tends to zero as $\epsilon \to 0$ (the case of nearly tangent trajectories). Inside $N_\delta$, since $W$ is monotone by Condition IIc, we have $W(Q;\epsilon) > W_{\text{in}} = W(\delta;\epsilon)$. Therefore, by (9), $p_{\nu}(t)$ stays small unless the trajectory leaves $N_\delta$ or $t - t_{\text{in}}$ becomes larger than a certain bounded away from zero value. From (8) it follows then that $p_{\nu}(t)$ remains bounded away from zero. By (5), (6),

$$\dot{Q} := \frac{d}{dt} Q(q(t);\epsilon) = Q_x p_x + Q_y p_y$$

so $\dot{Q}$ is small, yet

$$\frac{d^2}{dt^2} Q(q(t);\epsilon) = p^T Q_{xx} p_x + 2 Q_{xy} p_x p_y + Q_{yy} p_y^2 - W'(Q;\epsilon)(Q_x^2 + Q_y^2).$$

For a non-degenerate tangency, $p^T Q_{xx} p_x$ is positive and bounded away from zero. Therefore, as $p_x$ is small and $W'(Q;\epsilon)$ is negative, we obtain that $\frac{d}{dt} Q(q(t);\epsilon)$ is positive and bounded away from zero for a bounded away from zero interval of time (starting with $t_{\text{in}}$). It follows that

$$Q(q(t);\epsilon) \geq Q(q_{\nu,\text{in}};\epsilon) + \dot{Q}(t_{\text{in}})(t - t_{\text{in}}) + C(t - t_{\text{in}})^2$$

on this interval, for some constant $C > 0$. We see from (12), that the trajectory has to leave the boundary layer $N_\delta = \{ |Q(q;\epsilon) - Q| \leq \delta = |Q(q_{\nu,\text{in}};\epsilon) - Q| \}$ in a time of order $O(\dot{Q}(t_{\text{in}})) = O(Q_x(q_{\nu,\text{in}})) + O(p_{\nu,\text{in}}) = O(q_{\nu,\text{in}} - q_{\epsilon}) + O(p_{\nu,\text{in}})$. As $q_{\nu,\text{in}} - q_{\epsilon} = O(\sqrt{\delta})$ for a non-degenerate tangency, we see that the time the nearly-tangent orbit may spend in the boundary layer is $O(\sqrt{\delta} + p_{\nu,\text{in}})$, i.e. in this case it tends to zero as well. This completes the proof of Theorem[11].

Now we prove Theorem[12], the $C^r$-convergence for the non-tangent case. Again, divide $N_\delta$ into $N_<\nu$ and $N_>\nu$ for a small $\nu$ and consider the limit $\lim_{\delta \to 0} \lim_{\epsilon \to 0} \lim_{\nu \to 0}$. As we have shown above, $\dot{Q} \neq 0$ in $N_<\nu$, thus we can divide the equations of motion (5), (6) by $Q$:

$$\frac{dq}{dQ} = \frac{p}{Q_x p_x + p_y Q_y}$$

$$\frac{dp}{dQ} = \frac{-W'(Q;\epsilon) \nabla Q}{Q_x p_x + p_y Q_y},$$

$$\frac{dt}{dQ} = \frac{1}{Q_x p_x + p_y Q_y}$$

(13)
Equations (13) can be rewritten in an integral form:

\[ q(Q_2) - q(Q_1) = \int_{Q_1}^{Q_2} F_q(q,p) dQ, \]

\[ p(Q_2) - p(Q_1) = -\int_{W(Q_1)}^{W(Q_2)} F_p(q,p) dW(Q), \]

\[ t'(Q_2) - t'(Q_1) = \int_{Q_1}^{Q_2} F_t(q,p) dQ, \]  

where \( F_q, F_p, F_t \) denote some functions of \((q,p)\) which are uniformly bounded along with all derivatives. In \( N_C \), the change in \( Q \) is bounded by \( \delta \) and the change in \( W \) is bounded by \( \nu \). Hence, the integrals on the right-hand side are small. Applying the successive approximation method, we obtain that the Poincaré map (the solution to (14)) from \( Q = Q_1 \) to \( Q = Q_2 \) limits to the identity map (along with all derivatives with respect to initial conditions) as \( \delta, \nu \to 0 \). It follows that in order to prove the theorem, i.e. to prove (2), we need to prove (14) from (13).

Therefore, we can divide the equations of motion (5), (5) by \( \nu \) and obtain from (19) that

\[ \text{Condition IV implies that the} \]

\[ \text{region} \]

\[ \text{we obtain that in the limit} \]

\[ \text{Since the change in} \]

\[ \text{bounded, the solution of this system is the} \]

\[ \text{with all derivatives. In} \]

\[ \text{successive approximation method, we obtain that} \]

\[ \text{as} \]

\[ \text{uniformly along with all its derivatives in the region} \]

\[ \text{bounded away from zero. Therefore, the same holds true for a sufficiently} \]

\[ \text{where} \]

\[ \text{denote some functions of} \]

\[ \text{hence, by (6), the derivative} \]

\[ \text{beyond zero as well. Therefore, we can divide the equations of motion (5), (5) by} \]

\[ \text{where} \]

\[ \text{Condition IV implies that the} \]

(19) is

\[ \frac{d(q,t)}{dp_y} = 0, \quad \frac{dp_x}{dp_y} = \frac{Q_x}{Q_y}, \]

Since the change in \( p_y \) is finite and the functions on the right-hand side of (17) are all bounded, the solution of this system is the \( C' \)-limit of the solution of (17). From (19) we obtain that in the limit \( \nu \to 0 \) \((q_{\text{in}}, t_{\text{in}}) = (q_{\text{out}}, t_{\text{out}})\), so (15) is proved. Second, we obtain from (19) that

\[(p_{x,\text{out}} - p_{x,\text{in}}) Q_y(q_{\text{in}}; \nu) = (p_{y,\text{out}} - p_{y,\text{in}}) Q_y(q_{\text{in}}; \nu)\]
in the limit $\varepsilon \to 0$, which, in the coordinate independent vector notation

$$
\begin{align*}
\vec{p}_y &= \langle \vec{n}(\vec{q}), \vec{p} \rangle, \\
\vec{p}_x &= \vec{p} - \vec{p}_y \vec{n}(\vec{q}).
\end{align*}
$$

(20)

and by using $(\vec{q}_{in}, t_{in}) = (\vec{q}_{out}, t_{out})$, amounts to the correct reflection law. \qed