A Hamilton-Jacobi equation for the continuum limit of non-dominated sorting

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Summary
Non-dominated sorting is the process of arranging points in Euclidean space into fronts by repeated removal of the set of minimal elements with respect to the component-wise partial order. We prove that in the (random) large sample size limit, the fronts converge almost surely to the level sets of a function $U$ that satisfies the Hamilton-Jacobi equation (P) in the viscosity sense.

$$\{p\} \begin{cases} U_{x_1} \ldots U_{x_d} = \frac{1}{\alpha} f & \text{in } \mathbb{R}^d_+ \\ U = 0 & \text{on } \partial \mathbb{R}^d_+. \end{cases}$$

Applications of non-dominated sorting:
- Polynomial growth model (crystals)
- Multi-criteria optimization
- Combinatorics and probability

Non-dominated sorting
Let $X_1, \ldots, X_n$ be points in $\mathbb{R}^d$.
Set $S = \{X_1, \ldots, X_n\}$ and define, for $x, y \in \mathbb{R}^d$,
$x \leq y \iff x_i \leq y_i$ for all $i \in \{1, \ldots, d\}$.
Non-dominated sorting arranges $S$ into layers $S = \bigcup F_j$
where $F_1, F_2, \ldots$, are defined recursively by $F_1$ = minimal elements of $S$, $F_k$ = minimal elements of $S \setminus \bigcup_{j \leq k-1} F_j$.

Illustration of fronts $F_1, F_2, F_3, \ldots$

Application: Polynuclear growth
Random model for layer by layer growth [1]
1. Initially a flat crystal in contact with supersaturated vapor.
2. At a random later time, a supercritical nucleus forms, and spreads laterally via attachment of particles at the perimeter.
3. New layers are randomly nucleated upon existing layers.
4. When islands within the same layer collide, they coalesce.

Illustration of 1D polynuclear growth.

Connection to 2D non-dominated sorting
- Let $h(x, t)$ be the height at position $x$ at time $t$.
- Place a point in the plane at $(x, t)$ if a new layer nucleated at position $x$ at time $t$.
- Finding $h(x, t)$ is equivalent to non-dominated sorting rotated by $\pi/2$.

Comparison of level sets of $U$ to fronts $F_1, F_2, \ldots$

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Main Results
Assumptions
Let $X_1, \ldots, X_n$ be i.i.d. random variables with density $f: \mathbb{R}^d \rightarrow \mathbb{R}$.
(H1) $f: \mathbb{R}^d \rightarrow [0, \infty)$ is uniformly continuous on $\Omega$ and $f = 0$ on $\mathbb{R}^d \setminus \Omega$.
(H2) $\Omega \subset (0, 1)^d$ is open with Lipschitz boundary.

Notation
Let $u_\ast: \mathbb{R}^d \rightarrow \mathbb{N}_0$ be the function that “counts” the layers $F_1, F_2, \ldots$ generated by $X_1, \ldots, X_n$.

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References
[1] Prähofer, M. and Spohn, H. (2000). Universal distributions for growth processes in $1+1$ dimensions and random matrices. Physical Review Letters, 84(21):4882–4885.
[2] Spohn, H. (2006) Exact solutions for KPZ-type growth processes, random matrices, and equilibrium shapes of crystals. Physica A: Statistical Mechanics and its Applications, 369(1):71–99.
[3] Calder, J., Esedoğlu, S., and Hero, A. O. (2013). A Hamilton-Jacobi equation for the continuum limit of non-dominated sorting. Preprint at arXiv:1302.5828.

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