Chiral thermodynamics in a magnetic field

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(Dated: December 27, 2001)

We study thermodynamic properties of the QCD vacuum in a magnetic field below chiral phase transition. The hadronic phase free energy in a constant homogeneous magnetic field is calculated in the framework of the chiral perturbation theory at non-zero pionic mass. It is demonstrated that the order parameter of the chiral phase transition remains constant provided temperature and magnetic field strength are related through obtained equation (the phenomenon of “quark condensate freezing”).

I. INTRODUCTION

The investigation of the vacuum state behavior under the influence of the various external factors is known to be one of the central problems in quantum field theory. In the realm of strong interactions (QCD) the main factors are the temperature and the baryon density. At temperatures below the chiral phase transition, \( T < T_c \), the dynamics of the system is characterized by confinement and spontaneous breaking of chiral symmetry (SBCS). At low temperatures, \( T < T_c \), the partition function of the system is dominated by the contribution of the lightest particles in the physical spectrum. In QCD this role is played by the \( \pi \)– meson which is the Goldstone excitation mode in chiral condensate. Therefore the low temperature physics (the hadron phase) enables an adequate description in terms of the effective chiral theory \([1, 2, 3]\). A very important problem is the behavior of the order parameter (the quark condensate \( \langle \bar{q}q \rangle \)) with the increase of the temperature. In the ideal gas approximation the contribution of the thermal pions into the \( \pi \) spectrum. In QCD this role is played by the \( \pi \)– meson which is the Goldstone excitation mode in chiral condensate.

\[ \langle \bar{q}q \rangle \sim \frac{T^4}{\pi^2} \]

In chiral perturbation theory (ChPT) the two – and three–loops contributions \( \sim T^4 \) and \( \sim T^6 \) correspondingly into \( \langle \bar{q}q \rangle \) have been found in \([4, 5]\) and \([6, 7]\). At non-zero quark mass the analytic temperature dependence of the quark condensate was found in \([8]\) in perfect agreement with the numerical calculations obtained at the three-loop level of the chiral perturbation theory with \( m_q \neq 0 \) \([6]\).

The situation with the gluon condensate \( \langle G^2 \rangle = \langle (G_{\mu\nu})^2 \rangle \) is very different. The gluon condensate is not an order parameter in phase transition and it does not lead to any spontaneous symmetry breaking (SSB). At the quantum level the trace anomaly leads to the breaking of the scale invariance and this in turn results in nonzero value of \( \langle G^2 \rangle \). However, this is not a SSB phenomenon and hence does not lead to the appearance of the Goldstone particle. The mass of the lowest excitation (dilaton) is directly connected to the gluon condensate, \( m_D \propto \langle G^2 \rangle^{1/4} \). Thus the thermal excitations of glueballs are exponentially suppressed by the Boltzmann factor \( \sim \exp\{-m_g/T\} \) and their contribution to the shift of the gluon condensate is small \( \Delta \langle G^2 \rangle / \langle G^2 \rangle \sim 0.1 \% \) at \( T = 200 \text{ MeV} \) \([9]\). Next we note that in the one-loop approximation of ChPT pions are described as a gas of massless noninteracting particles. Such a system is obviously scale-invariant and therefore does not contribute into the trace of the energy-momentum tensor and correspondingly into \( \langle G^2 \rangle \). It was shown in \([10]\) that only in the three-loop approximation of ChPT does the gluon condensate become dependent on temperature in the chiral limit. At non-zero quark mass the analytic temperature dependence of the gluon condensate was found in \([11]\). The effect of thermal excitations of massive hadrons on the properties of the quark and the gluon condensate was studied in \([12]\) within the generalized nonlinear \( \sigma \)-model invariant under conformal transformations.

Another interesting problem is the study of the vacuum phase structure under the influence of the external magnetic field \( H \). Quarks play an active role in shaping the QCD vacuum structure. Being dual carriers of both 'color' and 'electric' charges they also respond to externally applied electromagnetic fields. The vacuum of strong interactions influences some QED processes has been discussed in Ref. \([12]\). The behavior of the quark condensate in the presence of a magnetic field was studied in Nambu-Iona-Lasinio model earlier \([13]\). For QCD, the analogous investigation in the one-loop approximation of ChPT was done in \([14]\). It was found that the quark condensate grows with the increase of the magnetic field \( H \) in both cases. It implies that a naive analogy with superconductivity, where the order parameter vanishes at same critical field, is not valid here. The behavior of the gluon condensate \( \langle G^2 \rangle \) in the Abelian magnetic field is also a nontrivial effect. Gluons do not carry electric charge; nevertheless, virtual quarks produced by them interact with electromagnetic field and lead to the changes in the gluon condensate. This phenomenon was studied

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in \[13, 14\] based on the low-energy theorems in QCD. The vacuum energy density, the values of \( \langle G^2 \rangle \) and \( \langle \bar{q}q \rangle \) as functions of \( H \) have been found in the two–loop approximation of ChPT in \[16\].

The low-energy theorems, playing an important role in the understanding of the vacuum state properties in quantum field theory, were discovered almost at the same time as quantum field methods appeared in particle physics (see, for example Low theorems \[17\]). In QCD, these theorems were obtained in the beginning of eighties \[18\]. These theorems, being derived from the very general symmetrical considerations and not depending on the details of confinement mechanism, sometimes give information which is not easy to obtain in another way. Also, they can be used as ”physically sensible” restrictions in the constructing of effective theories. An important step was made in \[19\], where low-energy theorems for gluodynamics were generalized to finite temperature case. A relation between the trace anomaly and thermodynamic pressure in pure-glue QCD was obtained in \[20\] by making use of the dimensional regularization in the framework of the renormalization group (RG) method. Also within the RG method, but by employing slightly different techniques an analogous relation was derived in the theory with quarks in \[21\]. The QCD phase structure in a magnetic field at finite temperature was investigated in \[22\]. Within the ChPT framework the dependence of the quark condensate on \( T \) and \( H \) was studied and it was shown that the shift of the condensate is not the simple sum of the temperature (\( \sim T^2/F_{\pi}^2 \)) and magnetic (\( \sim H/(4\pi F_{\pi})^2 \)) contributions. There appears an additional term which physically stems from the orbital diamagnetism of the charged pion gas.

The QCD Euclidean partition function with two quark flavors in external Abelian field \( A_{\mu} \) has the following form \((T = 1/\beta)\)

\[
Z = \exp \left\{ -\frac{1}{4g_0^2} \int_0^\beta dx_4 \int_V d^3x F_{\mu\nu}^2 \right\} \int |DB||D\bar{q}|Dq\exp \left\{ -\int_0^\beta dx_4 \int_V d^3x \right. (1)

Here the QCD Lagrangian in the background field is

\[
\mathcal{L} = \frac{1}{4g_0^2} (G_{\mu\nu}^a)^2 + \sum_{q=u,d} \bar{q}[\gamma_\mu (\partial_\mu - iQ_q A_\mu - \frac{i}{2}B_\mu^a) + m_q]q,
\]

(2)

where \( Q_q \) – is the matrix of the quark charges for the quarks \( q = (u, d) \), and for the simplicity the ghost terms have been omitted.

The free energy density is given by the relation \( \beta V F \quad (T, H, m_{0u}, m_{0d}) = -\ln Z \). Eq. \([2]\) yields the following expression for the gluon condensate \( \langle (G^2) \rangle \equiv \langle |G_{\mu\nu}^a|^2 \rangle \)

\[
\langle G^2 \rangle(T, H, m_{0u}, m_{0d}) = 4 \frac{\partial F}{\partial (1/g_0^2)}.
\]

(3)

1 At zero quark mass and in the absence of magnetic field the gas of massless noninteracting pions is obviously scale-invariant and therefore does not contribute to the trace of the energy-momentum tensor and correspondingly to the gluon condensate \( \langle (G_{\mu\nu}^a)^2 \rangle \).
The system described by the partition function $Z$ is characterized by the set of dimensionful parameters $M, T, H, m_{qg}(M)$ and dimensionless charge $g_0^2(M)$, where $M$ is the ultraviolet cutoff. On the other hand one can consider the renormalized free energy $F_R$ and by using the dimensional and renormalization-group properties of $F_R$ recast (3) into the form containing derivatives with respect to the physical parameter $T, H$ and renormalized masses $m_q$.

The phenomenon of dimensional transmutation results in the appearance of a nonperturbative dimensionful parameter

$$\Lambda = M \exp \left\{ \int_{\alpha_s(M)}^{\infty} \frac{d\alpha_s}{\beta(\alpha_s)} \right\},$$

where $\alpha_s = g_0^2/4\pi$, and $\beta(\alpha_s) = d\alpha_s(M)/d \ln M$ is the Gell-Mann-Low function. Furthermore, as it is well known, the quark mass has anomalous dimension and depends on the scale $M$. The renormalization-group equation for $m_0(M)$, the running mass, is $d\ln m_0/d \ln M = -\gamma_m$ and we use the $\overline{MS}$ scheme for which $\beta$ and $\gamma_m$ are independent of the quark mass $\overline{[27]}$. Upon integration the renormalization-group invariant mass is given by

$$m_q = m_{qg}(M) \exp \left\{ \int_{\alpha_s(M)}^{\infty} \frac{\gamma_m}{\beta(\alpha_s)} d\alpha_s \right\},$$

where the indefinite integral is evaluated at $\alpha_s(M)$. Next we note that since free energy is renormalization-group invariant quantity its anomalous dimension is zero. Thus $F_R$ has only a normal (canonical) dimension equal to 4. Making use of the renorm-invariance of $\Lambda$, one can write in the most general form

$$F_R = \Lambda^4 f\left(\frac{T}{\Lambda}, \frac{H}{\Lambda^2}, \frac{m_u}{\Lambda}, \frac{m_d}{\Lambda}\right),$$

where $f$ is some function. From (11) and (3) one gets

$$\frac{\partial F_R}{\partial (1/g_0^2)} = \frac{\partial F_R}{\partial \Lambda} \frac{\partial \Lambda}{\partial (1/g_0^2)} + \sum_q \frac{\partial F_R}{\partial m_q} \frac{\partial m_q}{\partial (1/g_0^2)},$$

$$\frac{\partial m_q}{\partial (1/g_0^2)} = -4\pi \alpha_s^2 m_q \frac{\gamma_m}{\beta(\alpha_s)}.$$

With the account of (3) the gluon condensate is given by

$$\langle G^2 \rangle(T, H, m_u, m_d) = \frac{16\pi \alpha_s^2 (4 - T \frac{\partial}{\partial T} - 2H \frac{\partial}{\partial H} - \sum_q (1 + \gamma_m) m_q \frac{\partial}{\partial m_q})F_R.}$$

It is convenient to choose such a large scale that one can take the lowest order expressions, $\beta(\alpha_s) \to -b\alpha_s^2/2\pi$, where $b = (11N_c - 2N_f)/3$ and $1 + \gamma_m \to 1$. Thus, we have the following equations for condensates

$$\langle G^2 \rangle(T, H, m_u, m_d) = -\frac{32\pi^2}{b} (4 - T \frac{\partial}{\partial T} - 2H \frac{\partial}{\partial H} - \sum_q m_q \frac{\partial}{\partial m_q})F_R \equiv -\hat{D}F_R,$$

$$\langle \bar{q}q \rangle(T, H, m_u, m_d) = \frac{\partial F_R}{\partial m_q}. $$

III. FREE ENERGY OF THE QCD VACUUM AT LOW TEMPERATURE IN A MAGNETIC FIELD

The above equations enable to obtain the values of the condensates as functions of $T$ and $H$ provided the free energy density is known. To get the latter the ChPT will be used. At low temperatures $T < T_c(T_c$ is the chiral phase transition temperature) and for weak fields $H < \mu_{had}^2 \sim (4\pi F_\pi)^2$ the characteristic momenta in the vacuum loops are small and theory is adequately described by the low-energy effective chiral Lagrangian $L_{\text{eff}}\overline{[3, 2]}$. This Lagrangian can be represented as a series expansion over the momenta (derivatives) and quark masses

$$L_{\text{eff}} = L^{(2)} + L^{(4)} + L^{(6)} + ...$$

(12)
The leading term in [12] is similar to the Lagrangian of the non-linear sigma model in the external field

\[ L^{(2)} = \frac{F^2}{4} Tr(\nabla_\mu U^+ \nabla_\mu U) + \Sigma Re \, Tr(MU^+), \]

\[ \nabla_\mu U = \partial_\mu U - i[U, V_\mu]. \]  \hspace{1cm} (13)

Here \( U \) is a unitary \( SU(2) \) matrix, \( F_\pi = 93 \text{MeV} \) is the pion decay constant, and \( \Sigma \) has the meaning of the quark condensate \( \Sigma = |\langle \bar{u}u \rangle| = |\langle \bar{d}d \rangle| \). The external Abelian magnetic field \( H \) is aligned along the \( z \)-axis and corresponds to \( V_\mu(x) = (r^2/2)A_\mu(x) \) with the vector-potential \( A_\mu \), chosen as \( A_\mu(x) = \delta_{\mu 3} H x_3 \). The mass difference between the \( u \) and \( d \) quarks appears in the effective chiral Lagrangian only quadratically. Further, to obtain an expression for the quark condensate in the chiral limit we use only the first derivative with respect to the mass of one of the quarks. Therefore, we can neglect the mass difference between the \( u \) and \( d \) quarks and assume the mass matrix to be diagonal \( M = m\hat{I} \).

At \( T < T_c, H < \mu_{\text{had}}^2 \), the QCD partition function coincides with the partition function of the effective chiral theory

\[ Z_{\text{eff}}[T, H] = e^{-\beta V_{F,\text{eff}}[T, H]} = Z_0[H] \int [DU] \exp\{- \int_0^\beta dx_4 \int_V d^3x L_{\text{eff}}[U, A] \} \]  \hspace{1cm} (14)

At the one-loop level it is sufficient to restrict the expansion of \( L_{\text{eff}} \) by the quadratic terms with respect to the pion field. Using the exponential parameterization of the matrix \( U(x) = \exp\{i\tau^a\pi^a(x)/F_\pi \} \) one finds

\[ L^{(2)} = \frac{1}{2}(\partial_\mu \pi^0) + \frac{1}{2} M_\pi^2(\pi^0)^2 + (\partial_\mu \pi^+ + i A_\mu \pi^+)(\partial_\mu \pi^- - i A_\mu \pi^-) + M_\pi^2 \pi^+ \pi^- , \]  \hspace{1cm} (15)

where the charged \( \pi^\pm \) and neutral \( \pi^0 \) meson fields are introduced

\[ \pi^\pm = (\pi^1 \pm i \pi^2)/\sqrt{2}, \hspace{0.5cm} \pi^0 = \pi^3 \]  \hspace{1cm} (16)

Thus [14] can be recasted into the form \(^2\)

\[ Z_{\text{eff}}^{R}[T, H] = Z_{P.T.}^{-1} Z_0[H] \int [D\pi^0][D\pi^+] [D\pi^-] \exp\{- \int_0^\beta dx_4 \int_V d^3x L^{(2)}[\pi, A] \} \]  \hspace{1cm} (17)

where partition function is normalized for the case of perturbation theory at \( T = 0, H = 0 \)

\[ Z_{P.T.} = [\det(\partial_\mu^2 + M_\pi^2)]^{-3/2}. \]  \hspace{1cm} (18)

Integration of (17) over \( \pi \)-fields leads to

\[ Z_{\text{eff}}^{R} = Z_{P.T.}^{-1} Z_0[H] [\det_T(\partial_\mu^2 + M_\pi^2)]^{-1/2} [\det_T(-|D_\mu|^2 + M_\pi^2)]^{-1}, \]  \hspace{1cm} (19)

where \( D_\mu = \partial_\mu - i A_\mu \) is a covariant derivative and a symbol "\( T \)" means that the determinant is calculated at finite temperature \( T \) according to standard Matsubara rules. Taking (18) into account and regrouping multipliers in (19) one gets the following expression for \( Z_{\text{eff}}^{R} \)

\[ Z_{\text{eff}}^{R}[T, H] = Z_0[H] \left[ \frac{\det_T(\partial_\mu^2 + M_\pi^2)}{\det(\partial_\mu^2 + M_\pi^2)} \right]^{-1/2} \left[ \frac{\det_T(-|D_\mu|^2 + M_\pi^2)}{\det(\partial_\mu^2 + M_\pi^2)} \right]^{-1} \times \left[ \frac{\det_T(-|D_\mu|^2 + M_\pi^2)}{\det(-|D_\mu|^2 + M_\pi^2)} \right]^{-1} \]  \hspace{1cm} (20)

\(^2\) The partition function \( Z_{\text{eff}}^{R} \) describes charged \( \pi^\pm \) and neutral \( \pi^0 \) ideal Bose gas in magnetic field. Relativistic charged Bose gas in magnetic field at finite temperature and density with application to Bose-Einstein condensation and Meissner effect was studied in Refs. [28], [29], [30].
Then the effective free energy can be written in the form

\[ F_{\text{eff}}^R(T, H) = -\frac{1}{\beta V} \ln Z_{\text{eff}}^R = \frac{H^2}{2e^2} + F_{\pi^0}(T) + F_{\pi^\pm}(H) + F_{\text{dia}}(T, H). \]  

(21)

Here \( F_{\pi^0} \) is the free energy of massive scalar boson

\[ F_{\pi^0}(T) = T \int \frac{d^3p}{(2\pi)^3} \ln(1 - \exp(-\sqrt{p^2 + M_{\pi}^2}/T)), \]  

(22)

\( F_{\pi^\pm} \) is a Schwinger result for the vacuum energy density of charged scalar particles in the magnetic field

\[ F_{\pi^\pm}(H) = \frac{1}{16\pi^2} \int_0^\infty ds s^3 e^{-M_{\pi}^2 s^2} \left[ \frac{H s}{\sinh(H s)} - 1 \right], \]  

(23)

and \( F_{\text{dia}} \) is the diamagnetic free energy of relativistic charged Bose gas

\[ F_{\text{dia}}(T, H) = \frac{HT}{\pi^2} \sum_{n=0}^\infty \int_0^\infty dk \ln(1 - \exp(-\omega_n/T)), \]  

(24)

where \( \omega_n \) are Landau levels of the \( \pi^\pm \) mesons in constant field \( H \).

By expanding integrand in (22), (24) in the series, one obtains the following expressions

\[ F_{\pi^0} = -\frac{M_{\pi}^2 T^2}{2\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} K_2(n M_{\pi}/T) \]  

(25)

and

\[ F_{\text{dia}} = -\frac{HT}{\pi^2} \sum_{n=0}^\infty \sqrt{M_{\pi}^2 + H(2n+1)} \sum_{k=1}^\infty \frac{1}{k} K_1 \left( \frac{k}{T} \sqrt{M_{\pi}^2 + H(2n+1)} \right) \]  

(26)

where \( K_n \) is the Mackdonald function.

### IV. QUARK CONDENSATE AT \( T \neq 0 \) AND \( H \neq 0 \)

The free energy \( F_{\text{eff}}^R \) determines the thermodynamical properties and the phase structure of the QCD vacuum state below the temperature of the chiral phase transition, i.e. in the phase of confinement.

In order to get the dependence of the quark condensate upon \( T \) and \( H \) use is made of the Gell-Mann-Oakes–Renner relation

\[ F_{\pi^0}^2 M_{\pi}^2 = -\frac{1}{2} (m_u + m_d) (\bar{u}u + \bar{d}d) = 2m \Sigma \]  

(27)

The quark condensate shift is then determined by

\[ \frac{\Delta \Sigma(T, H, M_{\pi})}{\Sigma} = -\frac{1}{F_{\pi^0}^2} \frac{\partial F_{\text{eff}}^R}{\partial M_{\pi}} \]  

(28)

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3 Technically, a transition for the free energy \( F = \frac{1}{T} \text{Tr} \ln[p_4^2 + \omega_k^2(p)] \) from the vacuum case \( (H = 0, T = 0) \) to the case of \( H \neq 0, T \neq 0 \) is straightforward. Omitting the details of the calculations, we note that, eventually, this transition reduces to the substitutions

\( p_4 \to \omega_k = 2\pi k T \) \( (k = 0, \pm 1, \ldots) \), \( \omega_0 = \sqrt{p^2 + M_{\pi}^2} \to \omega_n = \sqrt{p^2 + M_{\pi}^2 + H(2n+1)} \) and \( T \to \frac{M_{\pi}}{2\pi} \sum_{n=0}^\infty \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dp_4 \omega_k \omega_n \),

where the degeneracy multiplicity of \( H/2\pi \) has been taken into account for the Landau levels. Performing summation over Matsubara frequencies, we obtain \( \omega_n \).
Expression for quark condensate as function of $T$ and $H$ at $M_{\pi} \neq 0$ is then given by

$$
\frac{\langle \bar{q} q \rangle (T, H, M_\pi)}{\langle \bar{q} q \rangle} = 1 + \frac{\Delta \Sigma_{\pi^0} (T, M_\pi)}{\Sigma} + \frac{\Delta \Sigma_{\pi^\pm} (H, M_\pi)}{\Sigma} + \frac{\Delta \Sigma_{\text{dia}} (T, H, M_\pi)}{\Sigma}
$$

(29)

where the contribution of thermal $\pi^0$-meson to the shift of $\langle \bar{q} q \rangle$ is

$$
\frac{\Delta \Sigma_{\pi^0}}{\Sigma} = -\frac{T M_\pi}{4 \pi^2 F_\pi^2} \sum_{n=1}^\infty \frac{1}{n} K_1(n \frac{M_\pi}{T}),
$$

(30)

vacuum contribution of $\pi^{\pm}$-mesons (Shwinger term) is

$$
\frac{\Delta \Sigma_{\pi^\pm}}{\Sigma} = -\frac{H}{(4\pi F_\pi^2)^2} \int_0^\infty \frac{dx}{x^2} \left[ \frac{x}{\sinh(x)} - 1 \right] \exp \left( -\frac{M_\pi^2}{H} x \right)
$$

(31)

and "diamagnetic contribution" of charged Bose gas of $\pi^{\pm}$-mesons to the quark condensate is given by

$$
\frac{\Delta \Sigma_{\text{dia}}}{\Sigma} = -\frac{H}{2 \pi^2 F_\pi^2} \sum_{n=0}^\infty \sum_{k=1}^\infty K_0 \left( \frac{k}{T} \sqrt{M_\pi^2 + H(2n+1)} \right)
$$

(32)

In the limiting case of weak magnetic field $H \ll M_\pi^2$ and low temperature $T \ll M_\pi$ one gets

$$
\frac{\Delta \Sigma_{\pi^0}}{\Sigma} = -\sqrt{8\pi} \frac{M_\pi^{1/2} T^{3/2}}{(4\pi F_\pi^2)^2} e^{-M_\pi/T} + O(e^{-2M_\pi/T})
$$

(33)

and

$$
\frac{\Delta \Sigma_{\pi^\pm}}{\Sigma} = \frac{H^2}{96 \pi^2 F_\pi^2 M_\pi^3} + O\left( \frac{H^4}{M_\pi^4} \right)
$$

(34)

Let us now consider diamagnetic term $\Delta \Sigma_{\text{dia}}$ for various limiting cases. In the case of weak magnetic field, the interval between Landau levels $\sim \sqrt{H}$ is much less than average thermal energy of $\pi^{\pm}$-mesons, $\sqrt{H} \ll T$, and we can use Euler-MacLaren summation formula. We then find for $\Delta \Sigma_{\text{dia}}$

$$
\frac{\Delta \Sigma_{\text{dia}}}{\Sigma} = 2 \frac{\Delta \Sigma_{\pi^0}}{\Sigma} + \frac{H^2}{48 \pi^2 F_\pi^2 M_\pi T} \sum_{n=1}^\infty n K_1 \left( \frac{M_\pi}{T} \right), \quad \sqrt{H} \ll T
$$

(35)

For the case of low temperature $T \ll M_\pi$ one obtains

$$
\frac{\Delta \Sigma_{\text{dia}}}{\Sigma} = 2 \frac{\Delta \Sigma_{\pi^0}}{\Sigma} + \frac{H^2}{48 \sqrt{2\pi}^{3/2} F_\pi^2 M_\pi^{3/2} T^{1/2}} e^{-M_\pi/T}, \quad \sqrt{H} \ll T \ll M_\pi
$$

(36)

In the opposite limit of low temperature and strong magnetic field (at $M_\pi \neq 0$) asymptotic behavior of $\Delta \Sigma_{\text{dia}}$ is determined by $n = 0, k = 1$ in the sum (32)

$$
\frac{\Delta \Sigma_{\text{dia}}}{\Sigma} = -\frac{H}{2 \pi^2 F_\pi^2} K_0 \left( \frac{\sqrt{M_\pi^2 + H}}{T} \right) \rightarrow \frac{1}{(2\pi)^{3/2} F_\pi^2} \frac{HT^{1/2}}{(M_\pi^2 + H)^{1/4}} e^{-\sqrt{M_\pi^2 + H}/T}, \quad T \ll \sqrt{H} \ll M_\pi
$$

(37)

An interesting phenomenon reveals itself in the vacuum QCD phase structure under consideration. One can find from (29) such a function $H(T)$ that the chiral condensate $\langle \bar{q} q \rangle (T, H, M_\pi)$ remains unchanged when the temperature and magnetic field change in accordance with $H_\ast = H(T)$. Then $H_\ast$ is found by solving the following equation $\langle \bar{q} q \rangle (T, H_\ast, M_\pi) - \langle \bar{q} q \rangle = 0$ (see (29)). At low temperature, $T \ll M_\pi$, and weak field, $\sqrt{H} \ll M_\pi$, we have

$$
H_\ast (T, M_\pi) = \sqrt{2A} \left( \frac{\pi}{2} \right)^{1/4} M_\pi^{7/4} T^{1/4} e^{-M_\pi/2T} + O(e^{-M_\pi/T})
$$

(38)

Let us now consider quark condensate as function of $T$ and $H$ in chiral limit $M_\pi = 0$ [22].
Substituting (21) into (28), calculating the derivative over \( M_\pi^2 \) and then taking the limit \( M_\pi^2 \to 0 \) one gets
\[
\langle \bar{q}q \rangle(T, H) = \langle \bar{q}q \rangle(1 - \frac{1}{3} \cdot \frac{T^2}{8F_\pi^2} + \frac{H}{(4\pi F_\pi)^2} \ln 2 - \frac{H}{2\pi^2 F_\pi^2} \varphi(\sqrt{H}/T))
\]
where
\[
\varphi(\lambda) = \sum_{n=0}^{\infty} \int_0^\infty \frac{dx}{\omega_n(x)(\exp(\lambda \omega_n(x)) - 1)}, \quad \omega_n(x) = x^2 + 2n + 1 \tag{39}
\]
Now we consider various limiting cases. In the strong field, \( \sqrt{H} \gg T (\lambda \gg 1) \), the lowest Landau level \( n = 0 \) gives the main contribution to the sum (39)
\[
\varphi(\lambda \gg 1) = \sqrt{\frac{\pi}{2\lambda}} e^{-\lambda} + O(e^{-\sqrt{3}\lambda}). \tag{40}
\]
In the opposite limit of weak field, \( \sqrt{H} \ll T (\lambda \ll 1) \), the sum in (39) is calculated with required accuracy using the Euler-MacLaren formula. Furthermore, one gets the following result with the use of the asymptotic expansion of integral (39) at \( \lambda \ll 1 \)
\[
\varphi(\lambda \ll 1) = \frac{\pi^2}{6} \frac{1}{\lambda^2} + \frac{7\pi}{24} \lambda + \frac{1}{4} \ln \lambda + C + \frac{\zeta(3)}{4\pi^2} \lambda^2 + O(\lambda^4), \tag{41}
\]
here \( C = \frac{1}{2}(\gamma - \ln 4\pi - \frac{1}{2}) \), \( \gamma = 0.577... \) is Euler’s constant and \( \zeta(3) = 1.202 \) is Riemann zeta function. Thus, one obtains the following limiting expressions for the quark condensate in the chiral limit in a magnetic field at \( T \neq 0 \)
\[
\frac{\langle \bar{q}q \rangle(T, H)}{\langle \bar{q}q \rangle} = 1 - \frac{1}{3} \cdot \frac{T^2}{8F_\pi^2} + \frac{H}{(4\pi F_\pi)^2} \ln 2 - \frac{H^{3/4}T^{1/2}}{(2\pi)^{3/2} F_\pi^2} e^{-\sqrt{H}/T}, \quad \sqrt{H} \gg T \tag{42}
\]
and
\[
\frac{\langle \bar{q}q \rangle(T, H)}{\langle \bar{q}q \rangle} = 1 - \frac{T^2}{8F_\pi^2} + \frac{H}{(4\pi F_\pi)^2} \frac{A}{\ln \frac{H}{T^2}}, \quad \sqrt{H} \ll T \tag{43}
\]
where \( A = \ln 2 - 8C \simeq 4.93 \).

In the framework of ChPT the quark condensate (33) at \( H \neq 0, T \neq 0 \) is determined by three dimensionless parameters \( H/(4\pi F_\pi)^2 \), \( T^2/F_\pi^2 \) and \( \lambda = \sqrt{H}/T \). The quantity \( \lambda \) is a natural dimensionless parameter in this approach. The motion of a particle (massless pion) in the field is characterized by the curvature radius of it’s trajectory, and in the magnetic field this is the Larmor radius \( R_L = 1/\sqrt{H} \). On the other hand, there is another length \( l_T = 1/T \) - ”temperature length” at \( T \neq 0 \). Therefore, charged \( \pi^\pm \)-mesons in magnetic field effectively acquire ”mass”, \( m_{\text{eff}} = \sqrt{H} \), determined by the lowest Landau level, when Larmor radius of a particle in the field is much less than \( l_T (\lambda \gg 1) \). Correspondingly, their contribution to the shift of the chiral condensate is suppressed by the Boltzmann factor \( \exp(- m_{\text{eff}}/T) \). In the weak field limit \( \pi^\pm \)-mesons give standard temperature one-loop approximation ChPT contribution to \( \langle \bar{q}q \rangle \). Besides, additional temperature and magnetic corrections appear. Neutral \( \pi^0 \)-meson contributes to \( \langle \bar{q}q \rangle(T, H) \) as usual massless scalar particle.

In the chiral limit the equation for \( H_\ast(T) \) takes the form
\[
1 - \frac{3}{2\pi^2} \lambda^2 \ln 2 + \frac{12}{\pi^2} \lambda^2 \varphi(\lambda) = 0 \tag{44}
\]
The numerical solution of (44) yields \( \lambda_\ast = 3.48... \). Thus, quark condensate stays unchanged when \( T \) and \( H \) are increased according to \( H = 12.11 \cdot T^2 \). Hence it is possible to say that the order parameter \( \langle \bar{q}q \rangle \) of the chiral phase transition is ”frozen” by the magnetic field.

Note that \( H(T_c)/(4\pi F_\pi)^2 \simeq 0.2 \ll 1 \) at \( T = T_c \simeq 150\text{MeV} \) and therefore the above relations remain valid up to the deconfining phase transition point \( T_c \). In the vicinity of \( T_c \) the effective low energy chiral Lagrangian fails to provide an adequate description of the QCD vacuum thermodynamical properties, and strictly speaking becomes physically invalid. The following is worth noting. In deriving (33), at the first step the physical quantity as functions of \( M_\pi \) where obtained, and only then the chiral limit \( M_\pi \to 0 \) was taken. Acting in the reversed sequence we would have obtained all temperature corrections to condensate identically equal to zero. This points to the fundamental difference of the two cases: the exactly massless particle and the particle with infinitesimal small mass.

\[\text{4 There was an arithmetical mistake in paper [22], and the value of } \lambda_\ast \text{ was underestimated. However, this mistake does not affect any physical consequences.}\]
V. CONCLUSION

It has been shown in the present letter that the quark condensate is "frozen" by the magnetic field when both temperature $T$ and magnetic field $H$ are increased according to the obtained relation $H = H_*(T, M_{\pi})$. This points to the fact that the direct analogy between the quark condensate in QCD and the theory of superconductivity is untenable. In the BCS theory the Cooper pairs condensate is extinguished by the temperature and magnetic field. The "freezing" phenomenon can be understood in terms of the general Le Chatelier–Braun principle. The external field contributes into the system an additional energy density $H^2/2$. The system tends to compensate this energy change and to decrease the free energy by increasing the absolute value of the quark condensate: $\Delta \varepsilon_v = -m|\Sigma(H) - \Sigma(0)| < 0$. On the other hand, if the temperature of the system is increased (by bringing some heat into it) the processes with heat absorption by damping the condensate are switched on. The interplay of these processes is at the origin of the above "freezing" of $\Sigma(T, H)$.

In $N_f = 2$ QCD, the temperature phase transition restoring chiral $SU(2)_L \times SU(2)_R$ symmetry is a second-order phase transition. As the temperature is increased, the order parameter $\Sigma(T)$ decreases monotonically, vanishing at the critical point $T_c$. The existence of the jump $\Delta \Sigma(T_c) \neq 0$ would indicate the occurrence of a first-order phase transition. The order parameter at any non-zero magnetic field $H$ is greater than that at $H = 0$; hence, $\Sigma(T, H) - \Sigma(T, H = 0) > 0$. Because of this, it may turn out that the chiral phase transition in a magnetic field becomes a first-order phase transition, and critical temperature of phase transition is larger at $H \neq 0$, $T_c(H) > T_c(H = 0)$. In order to clarify the character of the phase transition, it is necessary to investigate the behavior of the system and of the order parameter in the fluctuation region near $T_c$; however, the effects of $\pi\pi$ interaction are substantial in this region. It follows that, within the approach used here, it is impossible to establish the change in the order of transition, although it seems highly probable. This phenomenon may prove to be of use in investigating various cosmological scenarios after the Big Bang. For this reason, it would be interesting to continue studying the chiral phase transition in a magnetic field.

ACKNOWLEDGMENTS

The author is grateful to B.L. Ioffe, V.A. Novikov, V.A. Rubakov, Yu.A. Simonov and A.V. Smilga for comments and discussions. The financial support of RFFI grant 00-02-17836 and INTAS grant N 110 is gratefully acknowledged.

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\footnote{The external action disturbing the system from the equilibrium state induces processes in this system which tend to reduce the result of this action.}

\footnote{The presence of an external field (not necessarily a magnetic field) transforms a second-order phase transition into a first-order phase transition.}
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