\(U_q(sl(n))\)-covariant quantization of symmetric coadjoint orbits via reflection equation algebra

J. Donin and A. Mudrov

Abstract. We study relations between the two-parameter \(U_q(sl(n))\)-covariant deformation quantization on \(sl^*(n)\) and the reflection equation algebra. The latter is described by a quantum permutation on \(\text{End}(\mathbb{C}^n)\) given explicitly. The reflection equation algebra is used for constructing the one-parameter quantization on coadjoint orbits, including symmetric, certain bisymmetric and nilpotent ones. Our approach is based on embedding the quantized function algebras on orbits into the algebra of functions on the quantum group \(SL_q(n)\) by means of reflection equation algebra characters.

1. Introduction

This paper is devoted to a particular case of the following problem. Let \(M\) be a Poisson manifold with an action of a Lie group \(G\). Let \(\mathfrak{g}\) be the Lie algebra of the group \(G\) and \(U_h(\mathfrak{g})\) the corresponding quantized universal enveloping algebra. The problem is to construct a quantization \(\mathcal{A}_h\) of the function algebra \(\mathcal{A}\) on \(M\) covariant with respect to the action of \(U_h(\mathfrak{g})\). In this paper, we consider the case \(\mathfrak{g} = sl(n)\) or \(gl(n)\) and \(M\) is either \(\mathfrak{g}^*\) or an orbit in \(\mathfrak{g}^*\). We quantize the algebra of polynomial functions on \(M\) and present it as a quotient of the corresponding tensor algebra by very natural relations, which are similar to those in the classical case. We prove the flatness of the deformations obtained. In that way, we quantize all symmetric and some bisymmetric orbits. Note that the analogous relations can be written for all semisimple orbits but there arises the question whether the quotient algebras by those relations are flat.

Let us recall some facts about the quantization on \(\mathfrak{g}^*\) and its orbits. It was shown in \([\text{Do}]\) that in the case \(\mathfrak{g} = sl(n)\) there exists a two-parameter deformation of the polynomial algebra \(S(\mathfrak{g})\) on \(\mathfrak{g}^*\) which can be viewed as a \(U_h(\mathfrak{g})\)-covariant quantization of the Lie-Poisson bracket on \(\mathfrak{g}^*\). Recall that in the classical case a natural one-parameter \(U(\mathfrak{g})\)-covariant quantization of \(S(\mathfrak{g})\) is given by the family \(S_t(\mathfrak{g}) = T(\mathfrak{g})[t]/J_t\), where \(T(\mathfrak{g})\) is the tensor algebra of the vector space \(\mathfrak{g}\) and the ideal \(J_t\) is generated by the elements \(x \otimes y - \tau(x \otimes y) - t[x, y]\), \(x, y \in \mathfrak{g}\). Here, \(\tau\) is the flip operator on \(\mathfrak{g} \otimes \mathfrak{g}\). So, the algebra \(S_t(\mathfrak{g})\) is quadratic-linear. By the Poincaré-Birkhoff-Witt theorem, \(S_t(\mathfrak{g})\) is a free module over \(\mathbb{C}[t]\). As was proven

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in [Do], that picture can be extended to the quantum case for \( g = sl(n) \). Namely, there exist deformations \( \tau_0 \) and \([\cdot, \cdot]_h\) of both maps \( \tau \) and \([\cdot, \cdot]_L\) such that the ideal \( J_{h,t} \) generated by the elements \( x \otimes y - \tau_0(x \otimes y) - t[x, y]_h, \ x, y \in g \), gives the two-parameter \( U_h(g)\)-covariant quantization \( S_{h,t}(g) = T(g)([h])[[t]]/J_{h,t} \) of the Lie-Poisson bracket on \( g^* \). It was also shown that such a family does not exist for \( g \neq sl(n) \).

It is possible to prove that \( S_{h,t}(g) \) can be restricted to any semisimple orbit to provide a two-parameter \( U_h(g)\)-covariant quantization on it. On the other hand, one can obtain a one-parameter covariant quantization on a semisimple orbit \( M \) as the subalgebra of quantized functions on \( G \) invariant under the action of the quantized stabilizer, [DoG]. Note that none of these statements imply any explicit form of the quantized algebras. In this paper, we explicitly describe \( S_{h,t}(g) \) and the quantum symmetric orbits in \( g^* \). We also describe quantizations of certain nilpotent and bisymmetric orbits. We conjecture that all bisymmetric orbits can be quantized within our approach.

The paper is organized as follows. Sections 2 and 3 contain some basic material about the quantum group \( U_q(g) \), \( g = sl(n) \), and its fundamental representation in the vector space \( V = \mathbb{C}^n \). In this paper, we work with the quantum group \( U_q(g) \) in the sense of Lusztig, [L], instead of \( U_h(g) \). The latter may be considered as the completion of \( U_q(g) \) at the point \( q = 1 \). Correspondingly, instead of \( S_{h,t}(g) \) we deal with the two-parameter family \( S_{q,t}(g) \) as a free module over \( \mathbb{C}(q) \). In Section 4, we study two different algebra structures on \( T(V) \otimes T(V^*) \) covariant with respect to the actions of either \( U_{q}^{\otimes 2}(g) \) or \( U_q(g) \). As a result, we come to inequivalent embeddings of \( T(V \otimes V^*) \) into \( T(V) \otimes T(V^*) \). That, in its turn, leads to different quantizations of the polynomial algebra on \( V \otimes V^* \) studied in Sections 5 and 6. We prove that they are the reflection equation (RE) algebra (see [KS]) and the algebra of functions on the quantum group known as the FRT algebra (after Faddeev, Reshetikhin, and Takhtajan), [FRT]. It turns out that both algebras can be described uniformly by quantum permutations on \( (V \otimes V^*)^{\otimes 2} \) but within different categories. In Section 7 we evaluate the lowest weight vectors of the irreducible \( U_q(g)\)-modules in \( M^{\otimes 2} \), where \( M \) is the algebra of \( n \times n \) complex matrices. Using this information, we derive the involutive permutation \( \tau_{RE} \) giving the commutation relations in the reflection equation algebra. That is done in Section 8. In Section 9, we reduce this permutation to the submodule \( g^{\otimes 2} \subset M^{\otimes 2}, \ g = sl(n) \).

In Section 7, we formulate a relation between \( S_{q,t}(g) \) and the RE algebra, following [Do]. That relation involves the reduction of \( \tau_{RE} \) from \( M \) to \( g \), and a quantum deformation of the Lie commutator \( g^{\otimes 2} \to g \) determined by \( \tau_{RE} \) up to a scalar factor. Specifying values of the quantum trace in the RE algebra, we obtain one-parameter subfamilies of \( S_{q,t}(g) \) as quotients of the RE algebra.

Section 11 is devoted to the quantization on coadjoint orbits in \( g^* \). We work, actually, with the adjoint orbits in the matrix space using the isomorphism between adjoint and coadjoint modules and employing the relation between the RE algebra and the two-parameter family \( S_{q,t}(g) \).

The RE algebra appeared as an abstraction of algebraic constructions, [AFS], [KS], arising from the theory of integrable models, [Ch]. It is also related to the braid group of a solid handlebody, [K]. In this paper, we consider the RE algebra as the \( U_q(g)\)-covariant quantization of the polynomial algebra on the matrix
space. It is generated by elements arranged in the matrix $L$ and subject to a set of quadratic relations called the reflection equation.

The conjugation transformation of $L$ with $T$, the generating matrix of the FRT algebra whose entries commute with the entries of $L$, is again an RE matrix, $[K_S]$. It follows from this observation that any character of the RE algebra specifies a homomorphism to the FRT algebra. This is done in the same way as in the classical geometry, where maximal ideals of the function algebra on $M$ correspond to points of $M$. Each point defines an embedding of the function algebra on the orbit into that on the group $G$. We use this idea for constructing quantized manifolds as quotients of the quantized group. Characters of the RE algebra are exactly solutions to the numerical reflection equation, $[K_S]$. We find such solutions in the form of projectors of rank $k < n$. Thus, we present the one-parameter quantizations on symmetric orbits as quotients of the RE algebra. Simultaneously, they turn out to be subalgebras in the quantized function algebra on the group, and this proves flatness of the quantizations. There are also solutions other than projectors, corresponding to different paths in the parameter space of the two-parameter quantization $S_{n,l}(g)$.

We prove that, for the standard quantum linear groups, there is an epimorphism from the RE algebra defined on $n \times n$ matrices onto the RE algebra defined on $k \times k$ matrices for $k < n$. That enables us to build new solutions to the matrix RE by embedding a given solution into a bigger matrix as the left upper block and extending it to the whole matrix with zeros. In particular, starting from a semisimple non-degenerate RE matrix we gain an additional, zero eigenvalue. We use this method for constructing examples of quantized bisymmetric orbits, i.e., consisting of matrices with three eigenvalues. The complete classification of solutions to the matrix RE is unknown. In particular, it is interesting to find nilpotent matrices providing quantization of nilpotent orbits. We managed to build such solutions among the matrices whose square is equal to zero.

2. The quantum universal enveloping algebra $\mathcal{U}_q(sl(n))$

By $g$ we mean the complex Lie algebra $sl(n)$, and $\mathcal{U}_q(g)$ is the corresponding quantum group. The latter is understood in the sense of Lusztig $[L]$, i.e., a free module over the field of rational functions in $q$. In the classical limit $q \to 1$, $\mathcal{U}_q(g)$ turns into the universal enveloping algebra $\mathcal{U}(g)$. The quantum group $\mathcal{U}_q(g)$ is generated by the elements $H_i, X^\pm_i, i = 1, \ldots, n - 1$, satisfying the commutation relations

$$[H_i, X^\pm_i] = \pm 2X^\pm_i, \quad [H_i, X^\pm_{i \pm 1}] = \mp X^\pm_{i \pm 1},$$

$$[H_i, X^\pm_j] = 0, \quad |i - j| > 1,$$

$$[X^+_i, X^-_j] = \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}.$$

Besides, the Serre relations hold:

$$(X^\kappa_i)^2 X^\kappa_{i \pm 1} - 2_q X^\kappa_i X^\kappa_{i \pm 1} X^\kappa_i + X^\kappa_{i \pm 1}(X^\kappa_i)^2 = 0, \quad [X^\kappa_i, X^\kappa_j] = 0, \quad |i - j| > 1,$$

where $\kappa = \pm$. Here $2_q = q + q^{-1}$. In general, the quantum integer numbers are defined as $n_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. The coproduct $\Delta$, counit $\epsilon$, and antipode $\gamma$ are

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta(X^\pm_i) = X^\pm_i \otimes q^{-\frac{H_i}{2}} + q^{\frac{H_i}{2}} \otimes X^\pm_i.$$
The elements of $\mathcal{M}$ are considered as right endomorphisms of $V = \mathbb{C}^n$ with the action $\rho = \delta^k_i e_i \in \mathcal{V}$. Formulas
\[ \rho(H_i) = e_i - e_{i+1}; \quad \rho(X_i^+) = e_{i+1}; \quad \rho(X_i^-) = e_i, \quad i = 1, \ldots, n - 1. \]
define a homomorphism $\rho$ of the algebras $\mathcal{U}_q(\mathfrak{g})$ into $\mathcal{M}$. On the Chevalley generators, it is given by the same formulas as in the classical limit $q \to 1$.

The homomorphism $\rho$ defines the right adjoint and left coadjoint actions of $\mathcal{U}_q(\mathfrak{g})$ on $\mathcal{M}$:
\[
(3.1) \quad A \triangleleft \text{ad}_\rho(x) = \rho(x(1)) A \rho(x(2)), \\
(3.2) \quad \text{ad}_\rho^*(x) \triangleright A = \rho(x(2)) A \rho(x(1))
\]
for $x \in \mathcal{U}_q(\mathfrak{g})$ and $A \in \mathcal{M}$. These actions are conjugate via the trace pairing on $\mathcal{M}$:
\[
\text{Tr} \left( (A \triangleleft \text{ad}_\rho(x)) B \right) = \text{Tr} \left( A \ (\text{ad}_\rho^*(x) \triangleright B) \right).
\]
The left coadjoint module $\mathcal{M}$ equipped with action (3.2) will be denoted $\mathcal{M}^*$. The element
\[
(3.3) \quad R = q^{-\frac{1}{2}}(q \sum_i e_i \otimes e_i + \sum_{i \neq j} e_i \otimes e_j + \omega \sum_{i < k} e_k \otimes e_i) \in \mathcal{M} \otimes \mathcal{M},
\]
where $\omega = q - q^{-1}$, satisfies the Yang-Baxter equation
\[
R_{12} R_{23} = R_{23} R_{12}.
\]
It differs from the R-matrix used in [FRT] by the overall factor $q^{-\frac{1}{2}}$. In this form, this is the image of the universal R-matrix $\mathcal{R}$ under the homomorphism $\rho \otimes 2$. Let $\sigma_{V \otimes V}$ be the ordinary (classical) permutation on $V \otimes V$. The braid matrix $S = q^{\frac{3}{2}} \sigma_{V \otimes V} R$ satisfies the Hecke condition $S^2 = \omega S + 1$ and is represented by the sum of two orthogonal projectors
\[
(3.4) \quad P_q^+ = \frac{1}{2q}(q - S), \quad P_q^- = \frac{1}{2q}(q^{-1} + S),
\]
The matrix $S$ commutes with all elements $(\rho \otimes \rho) \Delta(x)$, $x \in \mathcal{U}_q(\mathfrak{g})$.

Let $V^*$ be the space of linear functionals on $V$. We choose a basis $\{\epsilon_i\}$ in $V$ and denote $\{f^i\}$ its dual with respect to the canonical pairing between $V$ and $V^*$. 

\[
\varepsilon(H_i) = \varepsilon(X_i^\pm) = 0, \quad \gamma(H_i) = -H_i, \quad \gamma(X_i^\pm) = -q^{\mp 1} X_i^\pm.
\]
PROPOSITION 3.1. The homomorphism $\rho: U_q(g) \to M$ induces the right and left actions of $U_q(g)$ on the tensor algebra $T(V \oplus V^*)$. They are given on the basis elements $e_i \in V$ and $f^i \in V^*$ by

$$e_i \triangleleft x = \sum_{\alpha=1}^{n} e_{\alpha} \rho(x)_{\alpha}^{i}, \quad f^i \triangleleft x = \sum_{\alpha=1}^{n} f^\alpha \rho(\bar{x})_{\alpha}^{i},$$

$$x \triangleright e_i = \sum_{\alpha=1}^{n} e_{\alpha} \rho(x)_{\alpha}^{i}, \quad x \triangleright f^i = \sum_{\alpha=1}^{n} f^\alpha \rho(x)_{\alpha}^{i},$$

for $x \in U_q(g)$. Formula $(e_i, f^j) = \delta_i^j$ defines a $U_q(g)$-invariant right pairing between $V$ and $V^*$. The right module $V^* \otimes V$ with the action induced by (3.6) is isomorphic to the adjoint module $\mathcal{M}$ via the correspondence $f^i \otimes e_j \to e_j^i$. The left module $V \otimes V^*$ with the action induced by (3.6) is isomorphic to the coadjoint module $\mathcal{M}^*$ via the correspondence $e_j \otimes f^i \to e_j^i$.

PROOF. Direct verification. $\square$

One can introduce the following $U_q(g)$-invariant involutive permutations among the elements of $V$ and $V^*$.

PROPOSITION 3.2. The involutive permutations

1. $\tau_{V \otimes V}(e_i \otimes e_j) = \begin{cases} \text{sign}(j-i) \frac{1-q^2}{1+q^2} e_i \otimes e_j + \frac{2q}{1+q^2} e_j \otimes e_i, & i \neq j, \\ e_i \otimes e_i, & i = j, \end{cases}$

2. $\tau_{V^* \otimes V^*}(f^i \otimes f^j) = \begin{cases} \text{sign}(i-j) \frac{1-q^2}{1+q^2} f^i \otimes f^j + \frac{2q}{1+q^2} f^j \otimes f^i, & i \neq j, \\ f^i \otimes f^j, & i = j, \end{cases}$

3. $\tau_{V \otimes V^*}(e_j \otimes f^i) = \begin{cases} q^{-\frac{i}{2}} (q f^i \otimes e_j + \omega \sum_{k>0} f^k \otimes e_k), & i = j, \\ q^{-\frac{i}{2}} f^i \otimes e_j, & i \neq j, \end{cases}$

are invariant with respect to the left $U_q(g)$-action (3.4). The permutation

4. $\tau_{V^* \otimes V^*}(e_i \otimes e_j) = \begin{cases} \text{sign}(i-j) \frac{1-q^2}{1+q^2} e_i \otimes e_j + \frac{2q}{1+q^2} e_j \otimes e_i, & i \neq j, \\ e_i \otimes e_i, & i = j, \end{cases}$

is invariant with respect to the right $U_q(g)$-action (3.3).

PROOF. As a linear operator, composition of $\mathcal{R}$ acting on $V \otimes V$ and $V^* \otimes V^*$ with the ordinary flip has two eigenspaces. They are deformations of symmetric and antisymmetric 2-tensors, and involutions (3.7), (3.8), and (3.10) are defined as multiplication by $\pm 1$ on those subspaces. Operation (3.9) is readily obtained from the action of the universal R-matrix: $\tau_{V \otimes V^*}(f \otimes e) = \mathcal{R}_2 e \otimes \mathcal{R}_1 f, \quad \tau_{V^* \otimes V^*}(e \otimes f) = \mathcal{R}_1^{-1} f \otimes \mathcal{R}_2^{-1} e, \quad e \in V, f \in V^*$. To compute $\tau_{V \otimes V^*}$, it is enough to know only the image (3.3) of $\mathcal{R}$ in $\mathcal{M}^\otimes 2$. $\square$

4. Algebra $T(V) \otimes T(V^*)$ in braided categories

In this section, we consider the tensor algebras $T(V)$ and $T(V^*)$ from the different points of view: as objects from the category of either $U_q(g)$- or $U_q^2(g)$-representations. Correspondingly, the algebra structure on $T(V) \otimes T(V^*)$ may be introduced in different ways, as the tensor product of algebras in the those categories. A particular choice of the category determines embedding of $T(V \otimes V^*)$, ...
the tensor algebra of the space $V \otimes V^*$, into $T(V) \otimes T(V^*)$. That leads to different quantizations of the polynomial algebra on $V \otimes V^*$.

Let us remind the construction of the tensor product of algebras in the category of a quantum group modules. Let $\mathcal{H}$ be a quantum group with an $R$-matrix $\mathcal{R}$. Recall that a (left) $\mathcal{H}$-module algebra $\mathcal{A}$ is an associative algebra with unit in the category of $\mathcal{H}$-representations:

$$(4.1) \quad x \triangleright (ab) = (x_{(1)} \triangleright a)(x_{(2)} \triangleright b), \quad x \triangleright 1 = \varepsilon(x)1, \quad x \in \mathcal{U}_q(\mathfrak{g}), \ a, b \in \mathcal{A}. $$

Given two module algebras $\mathcal{A}$ and $\mathcal{B}$, one can introduce their (braided) tensor product $\mathcal{A} \hat{\otimes} \mathcal{B}$, which is again an $\mathcal{H}$-module algebra.

**Proposition 4.1.** Let $\mathcal{R}_1 \otimes \mathcal{R}_2$ denote the decomposition of the universal $R$-matrix into the two tensor factors (summation suppressed). The formula

$$(4.2) \quad (a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = a_1(\mathcal{R}_2 \triangleright a_2)(\mathcal{R}_1 \triangleright b_1)b_2, \quad a_i \in \mathcal{A}, \ b_i \in \mathcal{B}$$

defines an associative multiplication on $\mathcal{A} \hat{\otimes} \mathcal{B}$ turning it into an $\mathcal{H}$-module algebra. Embeddings $\mathcal{A} \to \mathcal{A} \hat{\otimes} 1$ and $\mathcal{B} \to 1 \hat{\otimes} \mathcal{B}$ are homomorphisms of algebras.

**Proof.** Associativity of multiplication (4.2) follows from (2.1). Compatibility with the action of $\mathcal{H}$ is a consequence of (2.2). The last statement of the proposition is immediate due to the equality $(\varepsilon \otimes \text{id})(\mathcal{R}) = (\text{id} \otimes \varepsilon)(\mathcal{R}) = 1 \otimes 1$ following from the definition of the universal $R$-matrix. \hfill $\square$

Note that multiplication (4.2) is characterized by the permutation relation

$$(4.3) \quad ba = (\mathcal{R}_2 \triangleright a)(\mathcal{R}_1 \triangleright b)$$

between the elements $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

**Proposition 4.2.** Let $V$ and $U$ be $\mathcal{H}$-modules. The subalgebra in $T(V) \hat{\otimes} T(U)$ generated by the submodule $V \hat{\otimes} U$ is isomorphic to the tensor algebra $T(V \otimes U)$. It is an algebra in the category of $\mathcal{H}$-modules.

**Proof.** Embedding $V \otimes U \to T(V) \hat{\otimes} T(U)$ is extended to a homomorphism of the free algebra $T(V \otimes U)$. It is an isomorphism due to invertibility of permutation (1.3). The subalgebra generated by $V \otimes U$ is $\mathcal{H}$-invariant, so it is a module algebra. \hfill $\square$

We will use this proposition when $U = V^*$.

Let us apply the construction above to the algebras $\mathcal{A} = T(V)$, $\mathcal{B} = T(V^*)$ considered as $\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$- and $\mathcal{U}_q(\mathfrak{g})_{op} \otimes \mathcal{U}_q(\mathfrak{g})$-module algebras in the following way. First let us note that a right action of a Hopf algebra is the same as a left one for its opposite. Right and left actions (3.7) and (3.8) of the algebra $\mathcal{U}_q(\mathfrak{g})$ are extended to the left actions of the Hopf algebras $\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$- and $\mathcal{U}_q(\mathfrak{g})_{op} \otimes \mathcal{U}_q(\mathfrak{g})$ on $V$ and $V^*$:

$$(4.4) \quad (x \otimes y) \triangleright v = \varepsilon(y)x \triangleright v, \quad (x \otimes y) \triangleright \phi = \varepsilon(x)y \triangleright \phi,$$

where $v \in V$ and $\phi \in V^*$. These actions are extended to the actions on $T(V)$, $T(V^*)$, and on the tensor product $T(V) \otimes T(V^*)$.

**Proposition 4.3.** The algebra $T(V) \hat{\otimes} T(V^*)$ in the categories of $\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$- and $\mathcal{U}_q(\mathfrak{g})_{op} \otimes \mathcal{U}_q(\mathfrak{g})$-modules has the multiplication:

$$(4.5) \quad (a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = a_1a_2 \hat{\otimes} b_1b_2, \quad a_i \in T(V), \quad b_i \in T(V^*).$$
Considered as matrix elements $T$ product in the category of involutive permutations $\tau$ of them in Sections 5 and 6 and present their defining commutation relations in terms of involutive permutations $\tau_{\text{FRT}}$ and $\tau_{\text{RE}}$.

Taking into account Proposition 4.3, we reserve the symbol $\hat{\otimes}$ only for the tensor product in the category of $\mathcal{U}_q(\mathfrak{g})$-modules where the permutation (4.3) between $V$ and $V^*$ is non-classical. So, by $T(V) \hat{\otimes} T(V^*)$ we will denote the tensor product considered as $\mathcal{U}_q(\mathfrak{g})$-module.

5. Algebra $\mathcal{A}_{\text{FRT}}(\mathcal{M})$

Denote $\mathcal{A}_{\text{FRT}}(\mathcal{M})$ be the associative unital algebra over $\mathbb{C}(q)$ generated by the matrix elements $T_j$ subject to relations

$$ST_1T_2 = T_1T_2S.$$  

Here, $S$ is the Hecke matrix defined in Section 3. The algebra $\mathcal{A}_{\text{FRT}}(\mathcal{M})$ was introduced in [FRT] as a quantized polynomial algebra on the space of matrices. It is endowed with the $\mathcal{U}_q(\mathfrak{g})_{\text{op}} \otimes \mathcal{U}_q(\mathfrak{g})$-module structure coming from that on $V \otimes V^*$. Relations (5.1) are given by a quantum permutation $\tau_{\text{FRT}}$ on the space $V \otimes V^*$, which we are going to present explicitly.

Consider $T(V \otimes V^*)$ as a subalgebra of $T(V) \otimes T(V^*)$ in the tensor category of $\mathcal{U}_q(\mathfrak{g})_{\text{op}} \otimes \mathcal{U}_q(\mathfrak{g})$-modules, along the line of Proposition 4.2. Consider the permutation

$$\tau_{\text{FRT}} = \sigma_{V \otimes V^*} \circ \tau_{V \otimes V^*} \circ \sigma_{V \otimes V^*}^{-1}$$

on $(V \otimes V^*)^\otimes 2$, which is the composition of elementary permutations (5.8), (5.10), and the classical flip $\sigma_{V \otimes V^*}$. The permutation $\tau_{\text{FRT}}$ is $\mathcal{U}_q(\mathfrak{g})_{\text{op}} \otimes \mathcal{U}_q(\mathfrak{g})$-invariant. Let $(V \otimes V^*)^\otimes 2 = I^+_{\text{FRT}} \oplus I^-_{\text{FRT}}$ be the decomposition into symmetric and skew-symmetric submodules with respect to $\tau_{\text{FRT}}$. It is easy to see that $I^-_{\text{FRT}}$ is a deformation of the exterior square $(V \otimes V^*)^\wedge 2$.

**Theorem 5.1.** Let $\mathcal{J}_{\text{FRT}}$ be the ideal in $T(V \otimes V^*)$ generated by $I^-_{\text{FRT}}$. The quotient algebra $T(V \otimes V^*)/\mathcal{J}_{\text{FRT}}$ is isomorphic to $\mathcal{A}_{\text{FRT}}(\mathcal{M})$.

**Proof.** Condition (5.1) is equivalent to the pair of relations $P^\pm_q T_1 T_2 P^\mp_q = 0$, where $P^\pm_q$ are the projectors (5.4). Let us take a basis $\{e_i\} \subset V$ and its right dual $\{f^\dagger_i\} \subset V^*$, i.e., $(e_i, f^\dagger_j) = \delta^j_i$. The elements $e_i \otimes f^\dagger_j$ transform as entries of the matrix $T$ under the action of $\mathcal{U}_q(\mathfrak{g})_{\text{op}} \otimes \mathcal{U}_q(\mathfrak{g})$, so we can identify them. The product of two matrix entries can be rewritten as $T_j T_m = (e_j \otimes f^\dagger_i)(e_n \otimes f^\dagger_m) = e_j e_n \otimes f^\dagger_i f^\dagger_m$, since the elements of $V$ and $V^*$ commute. This implies that the quadratic submodule $I^-_{\text{FRT}}$ goes over into the submodule spanned by $\sum_{\alpha, \beta, \mu} (P^+_q)^{\alpha \beta}_{j \mu} e_\alpha e_\beta \otimes f^\mu f^\nu(P^+_q)^{\nu \mu}_{j \mu}$.
under the embedding $T(V \otimes V^*) \to T(V) \otimes T(V^*)$. This submodule is formed by symmetric tensors with respect to the permutation $\tau_{V \otimes V} \circ \tau_{V^* \otimes V^*}$. \hfill \qed

6. Algebra $A_{RE}(\mathcal{M})$

The reflection equation algebra $A_{RE}(\mathcal{M})$ is generated by the entries of the matrix $L$ subject to the quadratic relations, $[\mathcal{A},\mathcal{S},\mathcal{K},\mathcal{S}]_\mathcal{L}$.

\begin{equation}
L_2 S L_2 = S L_2 S L_2.
\end{equation}

Here, $S$ is the Hecke matrix introduced in Section 3. The generators $L^i_j$ form the coadjoint module $\mathcal{M}^*$ via the correspondence $L^i_j \to e^i_j$. Action (3.2) is extended over $A_{RE}(\mathcal{M})$ turning it into a $\mathcal{U}_q(\mathfrak{g})$-module algebra.

The element $Q = R_{21} R$ defines a morphism of $\mathcal{U}_q(\mathfrak{g})$-modules $A_{RE}(\mathcal{M}) \to \mathcal{U}_q(\mathfrak{g})$ by the correspondence $L \to Q_2 \otimes \rho(Q_1)$, where $\mathcal{U}_q(\mathfrak{g})$ is considered as a left $\mathcal{U}_q(\mathfrak{g})$-module via the adjoint action. The algebra $A_{RE}(\mathcal{M})$ is a flat deformation of the polynomial algebra on the matrix space $\mathcal{M}$, $[\mathcal{M}]$.

The algebra $A_{RE}(\mathcal{M})$ can be described in the same way as $A_{FRT}(\mathcal{M})$ by a quantum permutation $\tau_{RE}$. Following Proposition 4.2, consider the tensor algebra $T(V \otimes V^*)$ as the subalgebra $T(V \otimes V^*) \subset T(V) \otimes T(V^*)$ in the category of $\mathcal{U}_q(\mathfrak{g})$-modules. Introduce the involutive operation

\begin{equation}
\tau_{RE} = \tau_{V \otimes V^*} \circ \tau_{V \otimes V} \circ \tau_{V^* \otimes V^*} \circ \tau_{V^* \otimes V^*},
\end{equation}

which is the composition of elementary permutations $[3,7,10]$. The permutation $\tau_{RE}$ is $\mathcal{U}_q(\mathfrak{g})$-invariant. Consider the decomposition $(V \otimes V^*)^{\otimes 2} = I_{RE}^R \oplus I_{RE}^L$ into the symmetric and skew-symmetric submodules with respect to $\tau_{RE}$. It is easy to see that the submodule $I_{RE}^L$ is a deformation of the exterior square $(V \otimes V^*)^{\otimes 2}$.

**THEOREM 6.1.** Let $J_{RE}$ be the ideal in $T(V \otimes V^*)$ generated by $I_{RE}$. The quotient algebra $T(V \otimes V^*)/J_{RE}$ is isomorphic to $A_{RE}(\mathcal{M})$.

**PROOF.** As in the proof of Theorem 6.1, let $\{f^i\} \subset V^*$ be the right dual to the basis $\{e^i\} \subset V$. Due to Proposition 6.1, the elements $e^i \otimes f^j$ form the coadjoint module $\mathcal{M}^*$ for $\mathcal{U}_q(\mathfrak{g})$, so we can put $L^i_j = e^i \otimes f^j$. By Proposition 1.2, the algebra $T(V \otimes V^*)$ is isomorphic to the free associative algebra generated by $L^i_j$. It remains to show that the quadratic submodule generating the ideal $J_{RE}$ is isomorphic to the submodule specified by the reflection equation. They are conjugate by the $\tau_{V \otimes V^*}$, which is the restriction to $V \otimes V^*$ of permutation (1.3). Consider the equality $e_n e_i \otimes f^k f^n \otimes e^i_n \otimes e^m_k = \sigma_{V \otimes V} L_2 S L_2$ in the algebra $T(V) \otimes T(V^*)$. Since $T(V)$ is considered as a left module for $\mathcal{U}_q(\mathfrak{g})$, the permutation $\tau_{V \otimes V}$ is $\mathcal{U}_q(\mathfrak{g})$-conjugate with the permutation $\tau_{V \otimes V}$. Therefore, the quadratic submodule of $q$-symmetric tensors in $V^{\otimes 2} \otimes V^{\otimes 2}$ is $\mathcal{U}_q(\mathfrak{g})$-conjugate with the $\mathcal{U}_q(\mathfrak{g})_{op} \otimes \mathcal{U}_q(\mathfrak{g})_{invar}$ submodule providing the algebra $A_{FRT}(\mathcal{M})$: $\sigma_{V \otimes V} S \sigma_{V \otimes V} (\sigma_{V \otimes V} L_2 S L_2) = (\sigma_{V \otimes V} L_2 S L_2 S)$. This is equivalent to the reflection equation (1.1). \hfill \qed

**REMARK 6.2.** In the classical case, the polynomial algebra on $V \otimes V^*$ can be built equivalently as a $\mathcal{U}(\mathfrak{g})$- or $\mathcal{U}^{\otimes 2}(\mathfrak{g})$-module. In the quantum case, depending on the point of view, one comes to either FRT or RE algebras. It can be shown that there is a sequence of twist transformations in the quasitensor category of $\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$-modules relating algebras $A_{FRT}(\mathcal{M})$ and $A_{RE}(\mathcal{M})$. This is a subject of our forthcoming publication.
7. Irreducible submodules of $\mathcal{M}^{\otimes 2}$

This technical section contains information about submodules of $\mathcal{M}^{\otimes 2}$ and its dual in terms of highest and lowest weight vectors. They will be used in computing the quantum permutation $\tau_{RE}$ on $(V \otimes V^*)^{\otimes 2} \sim \mathcal{M}^{\otimes 2}$. It is convenient to evaluate the conjugate operator $\tau_{RE}^*$ in the dual space first. For that reason, we consider three $\mathcal{U}_q(g)$-module structures on $\mathcal{M}^{\otimes 2}$:

\begin{align*}
(7.1) & \quad (A \otimes B) \triangleleft x = \rho(x_{(2)}) \rho(x_{(1)}) B \rho(x_{(4)}), \\
(7.2) & \quad (A \otimes B) \triangleright x = \rho(x_{(2)}) \rho(x_{(1)}) B \rho(x_{(4)}), \\
(7.3) & \quad x \triangleright (A \otimes B) = \rho(x_{(2)}) \rho(x_{(1)}) B \rho(x_{(3)}). 
\end{align*}

Actions (7.1) and (7.2) are nothing else than $\mathrm{ad}^\ast$ and it is just $\mathrm{ad}^\ast$ with respect to the action (7.3) may be calculated directly. It is:

\begin{align*}
\delta_1 & = \sum_{i=1}^{n} q^{-2i+1} e_i \otimes e_i, \\
\delta_2 & = \sum_{i=1}^{n} e_i \otimes q^{-2i+1} e_i, \\
\delta_3 & = \sum_{i=1}^{n} e_i \otimes e_i, \\
\delta_4 & = \sum_{i=1}^{n} a_i e_i \otimes e_i, \quad a_i = 1, q^{-3}, \ldots, q^{-2n+3}, q^{-2n}, \\
\delta_5 & = \sum_{i,k=1}^{n} q^{-(2i-1)} q^{-(2k-1)} e_i \otimes e_k, \\
\delta_6 & = \sum_{i,k=1}^{n} a_i e_i \otimes e_k, \quad a_i = q^{-(2n-1)}, q^{-(2n-3)}, \ldots, q^{-3}, q^{-1}, \\
\delta_7 & = q e_i \otimes e_i - q^{-1} e_i \otimes e_i - e_i \otimes e_i - e_i \otimes e_i, \\
\delta_8 & = e_i \otimes e_i, \\
\delta_9 & = e_i \otimes e_i - q^{-1} e_i \otimes e_i, \\
\delta_{10} & = e_i \otimes e_i - q^{-1} e_i \otimes e_i.
\end{align*}

The coefficients $a_i$ entering $\delta_4$ and $\delta_6$ decrease by $q^{-2}$ each step within the interval $i = 1, \ldots, n$. Here, we assume $n > 2$. The case $n = 2$ is simple and left to the reader.

The vectors $\delta_8$ and $\delta_6$ are invariant elements under action (7.3). They are symmetric with respect to the permutation $\tau_{RE}$, since that is so for the invariant elements in the classical situation. The vectors $\delta_7$ and $\delta_5$ belong to the symmetric part, too, while $\delta_8$ and $\delta_{10}$ are antisymmetric with respect to $\tau_{RE}$. Indeed, they generate submodules of multiplicity one in $\mathcal{M}^{\otimes 2}$ and are deformations of classical submodules belonging to symmetric and antisymmetric parts, respectively. The
situations is more complicated in what concerns vectors $\delta_1, \ldots, \delta_4$ because they generate the isotypic component of type $g$ and, in the limit $q = 1$, the corresponding classical component intersects the symmetric and skew-symmetric parts of $M^{\otimes 2}$ with multiplicity two. So the problem of evaluating the permutation $\tau_{RE}$ reduces to calculating its restriction to the $g$-isotypic component of $M^{\otimes 2}$.

First, we compute the dual conjugate operation to $\tau_{RE}$ on the right module $M^{\otimes 2}$ with action (7.2). A basis of lowest weight vectors has the dual consisting of highest weight ones; by that reason we also consider the highest weight vectors

$$\alpha^1 = 1 \otimes e_n^1,$$

$$\alpha^2 = \sum_{i=1}^{n} a_i e_n^i \otimes e_n^i, \quad a_i = q, 1, \ldots, 1, q^{-1},$$

$$\alpha^3 = \sum_{i=1}^{n} a_i e_n^i \otimes e_n^i, \quad a_i = q, 1, \ldots, 1, q^{-1},$$

$$\alpha^4 = \sum_{i=1}^{n} a_i e_n^i \otimes e_n^1, \quad a_i = 1, \ldots, 1, q^{-1},$$

$$\beta^1 = 1 \otimes e_n^1,$$

$$\beta^2 = e_n^1 \otimes 1,$$

$$\beta^3 = \sum_{i=1}^{n} a_i e_n^i \otimes e_n^i, \quad a_i = q^{-(2n-3)}, q^{-(2n-3)}, \ldots, q^{-3}, q^{-1},$$

$$\beta^4 = \sum_{i=1}^{n} a_i e_n^i \otimes e_n^1, \quad a_i = q, 1, \ldots, 1, q^{-1}. \tag{7.4}$$

Let us denote the isotypic $g$-type components with respect to actions (7.1), (7.2), and (7.3) by $M_{\alpha}^{\otimes 2}$, $M_{\beta}^{\otimes 2}$, and $M_{\delta}^{\otimes 2}$. They are generated by the vectors $\alpha^i, \beta^i$, and $\delta_i$, $i = 1, \ldots, 4$, respectively.

### 8. The quantum permutation $\tau_{RE}$

In this section, we calculate the permutation $\tau_{RE}$ in terms of the lowest weight vectors introduced in the previous section. The permutations $\tau_{FRT}$ and $\tau_{RE}$ define quadratic relations in the deformed algebras of functions on matrices. In both cases, relations (7.1) and (7.3) are formulated using the algebra structure on $M^{\otimes 2}$, i.e., in the dual setting. Thus, the FRT and RE algebra permutations are introduced through the dual operations $\tau_{FRT}^*$ and $\tau_{RE}^*$ conjugate to $\tau_{FRT}^*$ and $\tau_{RE}^*$ by the trace pairing. The involutions $\tau_{FRT}^*$ and $\tau_{RE}^*$ are invariant with respect to right actions (7.1) and (7.2). It is natural to compute them first, using the algebra structure on $M^{\otimes 2}$, and then evaluate $\tau_{RE}$ by duality.

The subspace of $\Omega \in M^{\otimes 2}$ such that $S\Omega = \Omega S$ is the annihilator of the submodule defining FRT relations (5.1). Therefore, the involution $\tau_{FRT}^*$ is determined by the conjugation transformation with the Hecke matrix $S$. Solutions to the equation $S\sigma(\Omega) = \sigma(\Omega)S$, where the map $\sigma$ is defined as $\Omega \to (1 \otimes \Omega_1)S(1 \otimes \Omega_2)$, $\Omega = \Omega_1 \otimes \Omega_2 \in M^{\otimes 2}$, form the submodule annihilating the RE relations. Hence, the involutions $\tau_{RE}^*$ and $\tau_{FRT}^*$ are $\sigma$-conjugate: $\tau_{RE}^* = \sigma^{-1} \tau_{FRT}^* \sigma$. Since the Hecke
matrix $S$ is invariant under the adjoint action $\text{ad}_{\rho \otimes \rho}$, the map $\sigma$ intertwines actions (7.2) and (7.1). It is an isomorphism of modules, being a deformation of the classical flip $\sigma_{M \otimes M}$.

As was noted in Section 7.2 only the restriction of $\tau_{RE}^*$ to the isotypic $\mathfrak{g}$-type component requires a special consideration. Restricted to the $\mathfrak{g}$-component $M_{\alpha;+}^\otimes$, the conjugation operation with the matrix $S$ has eigenvalues $+1$ and $-q^{\pm 2}$. The eigenspace $M_{\alpha;+}^\otimes$ corresponding to the eigenvalue $+1$ consists of symmetric tensors, in the sense of the permutation $\tau_{FRT}$. Eigenvectors of the eigenvalues $-q^{\pm 2}$ may be called $\tau_{FRT}^*$-antisymmetric. We denote this submodule $M_{\alpha;+}^\otimes$. Thus, the conjugation with the matrix $S$ determines decomposition

\[ M_{\alpha;+}^\otimes = M_{\alpha;+}^+ \oplus M_{\alpha;+}^- , \]

where both $M_{\alpha;+}^\pm$ contain irreducible submodule $\mathfrak{g}$ with multiplicity two. Decomposition \((8.1)\) induces decompositions

\[ M_{\beta;+}^\otimes = M_{\beta;+}^+ \oplus M_{\beta;+}^- , \]
\[ M_{\delta;+}^\otimes = M_{\delta;+}^+ \oplus M_{\delta;+}^- , \]

into the symmetric and antisymmetric parts. The subspaces $M_{\beta;+}^\otimes$ are the images of $M_{\alpha;+}^\otimes$ via the inverse transformation $\sigma^{-1}$. They yield the involution $\tau_{RE}^*$. The subspaces $M_{\beta;+}^\otimes$ are the annihilators of $M_{\beta;+}^\otimes$ with respect to the trace pairing between $M^\otimes$ and $M^*^\otimes$. The permutation $\tau_{RE}$ is determined by the submodules $M_{\delta;+}^\otimes$.

**Proposition 8.1.** The $\mathfrak{g}$-type submodules of $\tau_{RE}$-symmetric and antisymmetric tensors are generated by the following highest and lowest weight vectors:

\[
M_{\alpha;+}^\otimes : \quad \alpha^3 + \alpha^4, \quad \alpha^1 + \alpha^2 + \omega \alpha^3, \quad M_{\alpha;+}^- : \quad q^{\pm 1} \alpha^1 - q^{\mp 1} \alpha^2 \mp \alpha^3 \mp \alpha^4, \\
M_{\beta;+}^\otimes : \quad \beta^1 + \beta^2 - \omega \beta^3, \quad \beta^3 + \beta^4 + \omega \beta^1, \quad M_{\beta;+}^- : \quad \beta^1 - \beta^2, \quad \beta^3 - \beta^4, \\
M_{\delta;+}^\otimes : \quad \delta_3 + \delta_4 - \omega \delta_2, \quad \delta_1 + \delta_2 + \omega \delta_3, \quad M_{\delta;+}^- : \quad \delta_1 - \delta_2, \quad \delta_3 - \delta_4.
\]

**Proof.** The eigenvectors of the conjugation by $S$ restricted to $M_{\alpha;+}^\otimes$ are computed directly. Thus we obtain \((8.1)\). The transformation $\sigma$ acts on the highest weight vectors of the $\mathfrak{g}$-type components as

\[
\sigma : \begin{pmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \\ \beta^4 \end{pmatrix} \rightarrow \begin{pmatrix} \alpha^3 \\ \alpha^1 + \omega \alpha^3 \\ \alpha^1 \\ \alpha^2 \end{pmatrix} .
\]

Decomposition \((8.2)\) is the $\sigma$-preimage of \((8.1)\). Decomposition \((8.3)\) is computed by pairing $\delta_i$ with $\beta^k$ via the ordinary trace. \hfill $\Box$

**Corollary 8.2.** The submodule of $\tau_{RE}$-symmetric tensors in $M^\otimes$ is generated by the lowest weight vectors

\[
\{ \delta_1 + \delta_2 + \omega \delta_3, \delta_3 + \delta_4 - \omega \delta_2, \delta_5, \delta_6, \delta_7, \delta_8 \}
\]

The submodule of $\tau_{RE}$-antisymmetric tensors in $M^\otimes$ is generated by the lowest weight vectors

\[
\{ \delta_1 - \delta_2, \delta_3 - \delta_4, \delta_9, \delta_{10} \} .
\]
The permutation \( \tau_{RE} \) can be reduced to the submodule \( \mathfrak{g}^{\otimes 2} \subset M^{* \otimes 2} \). We study this problem in the next section.

9. Reducing \( \tau_{RE} \) to \( \mathfrak{g}^{\otimes 2} \)

The left coadjoint module \( M^* \) contains a one-dimensional submodule \( \mathfrak{m}_0 \), which is spanned by the invariant element \( \sum_{i=1}^{n} q^{-2i+2} e_i \) and the \( \mathfrak{g} \)-type submodule of \( q \)-traceless matrices (definition (10.2) of the quantum trace will be given in the next section). The decomposition \( M^* = \mathfrak{m}_0 \oplus \mathfrak{g} \) leads to the decomposition

\[
M^* \otimes M^* = \mathfrak{m}_0 \otimes \mathfrak{m}_0 \oplus \mathfrak{m}_0 \otimes \mathfrak{g} \oplus \mathfrak{g} \otimes \mathfrak{m}_0 \oplus \mathfrak{g} \otimes \mathfrak{g}.
\]

The lowest weight vectors \( \delta_3 \) and \( \delta_4 \) belonging to \( \mathfrak{g} \otimes \mathfrak{g} \) are obtained by subtracting proper linear combinations of \( \delta_1 \) and \( \delta_2 \) from \( \delta_3 \) and \( \delta_4 \):

\[
(9.1) \quad \delta_3 = \delta_3 - \frac{\omega}{1-q^{-2}} \delta_1 - \frac{\omega}{1-q^{-2}} \delta_2, \quad \delta_4 = \delta_4 - \frac{\omega}{q^{n-1}} \delta_1 - \frac{\omega}{q^{n-1}} \delta_2.
\]

Now we can evaluate the reduction \( \tilde{\tau}_{RE} \) of the permutation \( \tau_{RE} \) to the submodule \( \mathfrak{g}^{\otimes 2} \).

**Proposition 9.1.** The composition

\[
\mathfrak{g} \otimes \mathfrak{g} \to M^* \otimes M^* \xrightarrow{\tau_{RE}} M^* \otimes M^* \to \mathfrak{g} \otimes \mathfrak{g},
\]

where the left arrow is embedding and the right one is the projection along \( \mathfrak{m}_0 \otimes M^* + M^* \otimes \mathfrak{m}_0 \), defines an involutive \( \mathcal{U}_q(\mathfrak{g}) \)-invariant permutation \( \tilde{\tau}_{RE} \) on \( \mathfrak{g}^{\otimes 2} \). The lowest weight vectors

\[
(9.2) \quad \tilde{\delta}_+ = (1 - \frac{\omega q^{-n}}{n_q})\delta_3 + (1 + \frac{\omega q^n}{n_q})\delta_4, \quad \tilde{\delta}_- = \delta_4 - \delta_3,
\]

where \( \tilde{\delta}_i \) are introduced by (9.3), generate the \( \mathfrak{g} \)-type submodules in \( \mathfrak{g}^{\otimes 2} \) of symmetric and antisymmetric tensors with respect to \( \tilde{\tau}_{RE} \), correspondingly.

**Proof.** The first statement is immediate. It easy to check that the vector \( \tilde{\delta}_+ \in \mathfrak{g}^{\otimes 2} \subset M^{* \otimes 2} \) is \( \tau_{RE} \)-symmetric. As to the vector \( \tilde{\delta}_- \), it is the image of \( \tau_{RE} \)-antisymmetric vector \( \delta_4 - \delta_3 \) under the projection \( M^{* \otimes 2} \to \mathfrak{g}^{\otimes 2} \).

**Corollary 9.2.** The submodules of symmetric and antisymmetric tensors are generated by the sets

\[
\{ \delta_6 - \frac{\omega}{1-q^{-2n}} \delta_5, \delta_7, \tilde{\delta}_+ \}, \quad \{ \delta_9, \delta_10, \tilde{\delta}_- \},
\]

of lowest weight vectors, respectively.

**Proof.** The vector \( \delta_5 \) is the square of the invariant element \( \sum_{i=1}^{n} q^{-2i+2} e_i \) spanning \( \mathfrak{m}_0 \). The combination \( \delta_6 - \frac{\omega}{q^{n-1}} \delta_5 \) is the Casimir element of \( \mathfrak{g}^{\otimes 2} \subset M^{* \otimes 2} \). The \( \mathfrak{g} \)-type vectors \( \delta_\pm \) were considered in the above proposition. Concerning the other lowest weight vectors, they already lie in \( \mathfrak{g}^{\otimes 2} \subset M^{* \otimes 2} \), so the proof is an immediate consequence of Corollary 8.2.

**Remark 9.3.** Unlike in the classical case, the submodule \( \mathfrak{g}^{\otimes 2} \subset M^{* \otimes 2} \) is not preserved by the permutation \( \tau_{RE} \). By this reason, \( \tilde{\tau}_{RE} \) is not a simple restriction of \( \tau_{RE} \) to \( \mathfrak{g}^{\otimes 2} \).
As a left $\mathcal{U}_q(\mathfrak{g})$-module, the tensor square $\mathcal{M}^\otimes 2$ is endowed with another permutation coming from the universal R-matrix representation: $A \otimes B \rightarrow \mathcal{R}_2 \triangleright B \otimes \mathcal{R}_1 \triangleright A$, $A \otimes B \in \mathcal{M}^\otimes 2$. It can be shown that its restriction to the $\mathfrak{g}$-component $\mathcal{M}_g^\otimes 2 \subset \mathcal{M}^\otimes 2$ acts according to the rule
\[
\begin{pmatrix}
\delta_1 \\
\delta_2 \\
\delta_3 \\
\delta_4
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\delta_2 \\
\delta_1 \\
\delta_4 \\
q^{-2n}\delta_3
\end{pmatrix}.
\]
It is easy to see, that this operator has eigensubspaces distinct from those of $\tau_{\mathcal{R}E}$.

With this remark, we complete the study of the RE algebra. In the remainder of the paper, we investigate relations between $\mathcal{A}_{\mathcal{R}E}(\mathcal{M})$ and the quantization on $\mathfrak{g}^*$ and its orbits.

10. Algebra $\mathcal{A}_{\mathcal{R}E}$ and the quantization on $\mathfrak{g}^*$

It was shown in [Do1] that a two-parameter $\mathcal{U}_q(\mathfrak{g})$-covariant quantization $S_{q,t}(\mathfrak{g})$ of the Lie-Poisson bracket on $\mathfrak{g}^*$ exists only in the $\mathfrak{sl}(n)$ case. It is realized as the quotient algebra of $T(\mathfrak{g})[t](q)$ by the quadratic-linear relations
\[
(10.1) \quad x \otimes y - \tau_{\mathcal{R}E}(x \otimes y) = t[x, y]_q,
\]
where $[\cdot, \cdot]_q: \mathfrak{g}^\otimes 2 \rightarrow \mathfrak{g}$ is a deformed Lie bracket. This is a $\mathcal{U}_q(\mathfrak{g})$-equivariant map $\mathfrak{g}^\otimes 2 \rightarrow \mathfrak{g}$ sending the submodule of $\tau_{\mathcal{R}E}$-symmetric tensors to zero. The commutator $[\cdot, \cdot]_q$ is uniquely defined up to a factor, since the module of $\tau_{\mathcal{R}E}$-antisymmetric tensors contains the submodule isomorphic to $\mathfrak{g}$ with multiplicity one. In the limit $q \rightarrow 1$, the commutator turns into the classical Lie bracket. In this way, one recovers the algebra $U(\mathfrak{g})[t]$ viewed as a quantization of the Lie-Poisson bracket. Another limit $t \rightarrow 0$ yields a one-parameter family $S_{q,0}(\mathfrak{g})$, which can be interpreted as a $\tau_{\mathcal{R}E}$-commutative algebra of polynomials on $\mathfrak{g}^*$.

The RE algebra is closely related to $S_{q,t}(\mathfrak{g})$. To formulate that relation, let us introduce the quantum trace of the RE matrix $L$:
\[
(10.2) \quad \text{Tr}_q(L) = \sum_{i=1}^{n} q^{-2i+2} L_i^i = \text{Tr}(DL), \quad \text{where} \quad D = \sum_{i=1}^{n} q^{-2i+2} e_i^i.
\]
Remark that this definition makes sense for matrices with entries being elements of any associative algebra $\mathcal{A}$ over $\mathbb{C}(q)$. The quantum trace is the map $\text{id} \otimes \text{Tr}_q: \mathcal{A} \otimes \mathcal{M} \rightarrow \mathcal{A} \otimes \mathbb{C}(q) \sim \mathcal{A}$. For the generating matrix $L$ of the RE algebra, the elements $\text{Tr}_q(L^k)$, $k \in \mathbb{N}$, are invariant. It follows from the commutation relations (5.1) that they belong to the center of $\mathcal{A}_{\mathcal{R}E}(\mathcal{M})$.

**Theorem 10.1.** The quotient algebra of $\mathcal{A}_{\mathcal{R}E}(\mathcal{M})$ by the ideal $(\text{Tr}_q(L) - \lambda)$, where $\lambda \in \mathbb{C}$, is isomorphic to the one-parameter sub-family $t = \lambda q^{-2n-1} \text{Tr}_q(D)$ in $S_{q,t}(\mathfrak{g})$.

**Proof.** In its essential part, this theorem was proven in [Do1]. Here we specify the exact relation between the parameter $t$ and the value of $\lambda$, using the explicit expression for the permutation $\tau_{\mathcal{R}E}$ derived in Section 8 in terms of the lowest weight vectors. By Proposition 8.3, the $\tau_{\mathcal{R}E}$-symmetric $\mathfrak{g}$-type submodules are generated by the vectors $\delta_1 - \delta_2$ and $\delta_4 - \delta_3$. The first one is irrelevant because it turns zero when projected to $\mathfrak{g}^\otimes 2 \subset \mathcal{M}^\otimes 2$. The second vector is represented as $\delta_4 - \delta_3 = \delta_4 - \delta_3 - \omega \delta_1$, where $\delta_i$ belong to $\mathfrak{g}^\otimes 2$. Vector $\delta_1$ is expressed through
the matrix \( D \) introduced in (10.2) and the lowest weight vector \( e^n_1 \) of the adjoint representation in \( g \): \( \delta_1 = q^{-1} D \otimes e^n_1 \). Comparing this with (10.1), we find the value of the parameter \( t \). 

11. The quantization of coadjoint orbits

It can be shown that the two-parameter family \( S_{q,t}(g) \) can be restricted to any semisimple orbit in \( g^* \). In this connection, there arises the problem of explicit description of the quantized manifolds in terms of ideals in \( S_{q,t}(g) \). We solve this problem (in a one-parameter setting) for certain classes of orbits including the symmetric ones, using the relation between \( A_{RE} \) and \( S_{q,t}(g) \) (Theorem 10.1). Let \( O_A \subset M \) be the orbit passing through the matrix \( A \). Consider the two commutative diagrams

\[
\begin{array}{cc}
G \times M & \rightarrow & M \\
\uparrow & & \downarrow \\
G \times \{A\} & \rightarrow & O_A \\
\end{array}
\]

Fun(\( G \)) \otimes Fun(M) \leftarrow Fun(M)

where the horizontal arrows correspond to the action of the group \( G \). The right square represents morphisms of the polynomial algebras induced by the maps of manifolds depicted on the left. The map \( \chi^A \) is the character of the algebra Fun(\( M \)) corresponding to the point \( A \in M \). We are going to quantize the right square; that will give us realization of the quantized algebra \( \text{Fun}_q(O_A) \), on the one hand, as a quotient of \( A_{RE}(M) \) and as a subalgebra in the quantized function algebra on \( G \), on the other.

Let \( F_q(G) \) be the Hopf algebra of quantized polynomial functions on the group \( G \). It is a quotient of the FRT algebra by the additional relation \( \det_q(T) = 1 \), where \( \det_q \) is the quantum determinant. Since \( U_q(g) \) acts on the RE algebra \( A_{RE}(M) \), this action generates the coaction of \( F_q(G) \) on \( A_{RE}(M) \).

**Theorem 11.1** (KS). Let \( T^{-1} \) be the matrix with entries \( \gamma(T^i_j) \), where \( T \) is the generating matrix of the algebra \( F_q(G) \) and \( \gamma \) the antipode. The conjugation transformation \( L \rightarrow T^{-1}LT \) of an RE matrix \( L \) with the FRT matrix \( T \) whose entries commute with the entries of \( L \) is again an RE matrix.

It follows that the correspondence \( L \rightarrow T^{-1}LT \) extends to an homomorphism \( A_{RE}(M) \rightarrow F_q \otimes A_{RE}(M) \). Theorem 11.1 yields the quantization of the upper arrow of the right square on the above diagram. To quantize the other maps, we replace \( \chi^A \) by \( \chi^A_q \), a character of the reflection equation algebra. It is determined by the correspondence \( L^i_j \rightarrow \chi^A_q \) such that the numeric matrix \( A \) satisfies the reflection equation, \( \text{KSS} \).

\[
A_2 S A_2 S = S A_2 S A_2
\]

supported in \( M^{\otimes 2} \). Any solution to this equation gives rise to the algebra \( \text{Fun}_q(O_A) \) closing the commutative diagram

\[
\begin{array}{cc}
F_q(G) \otimes A_{RE}(M) & \leftarrow A_{RE}(M) \\
\text{id} \otimes \chi^A_q \downarrow & \downarrow \\
F_q(G) \otimes \mathbb{C}(q) & \leftarrow \text{Fun}_q(O_A)
\end{array}
\]

so that the bottom arrow is embedding. Thus, we obtain

\[
A_2 S A_2 S = S A_2 S A_2
\]
THEOREM 11.2. Let $A$ be a solution of the numeric RE (11.1). Then the algebra $\text{Fun}_q(O_A)$ in the diagram above is the quantization of the polynomial algebra on the orbit $O_A$ passing through the matrix $A$.

PROOF. It is clear that at $q = 1$ the algebra $\text{Fun}_q(O_A)$ coincides with the polynomial algebra on the orbit $O_A$ passing through the matrix $A$. The flatness of $\text{Fun}_q(O_A)$ over $q$ follows from the fact that it is simultaneously a quotient and a subalgebra of the flat $\mathbb{C}(q)$-algebras $A_{RE}(\mathcal{M})$ and $\mathcal{F}_q(G)$. □

The matrix $L(A) = T^{-1}AT$ possesses the following properties:

**Lemma 11.3.** For any polynomial function $P$ in one variable,

\begin{align}
P(L(A)) &= L(P(A)). \tag{11.2}
\end{align}

The quantum trace is invariant under the conjugation:

\begin{align}
\text{Tr}_q(L(A)) &= \text{Tr}_q(A). \tag{11.3}
\end{align}

**Proof.** The first statement is evident. The second one is checked using the commutation relations in the algebra $\mathcal{F}_q(G)$. □

Note that solutions $A$ of (11.1) yield quantizations which are quotients of one-parameter subfamilies in $S_{q,t}(g)$. Those subfamilies are defined by paths in the parameter space $(q,t)$ crossing the axis $q = 1$ at the point $t = 0$. In particular, the solutions with $\text{Tr}_q(A) = 0$ correspond to the path $t = 0$. The limit $q \to 1$ for $t$ separated from 0 cannot be reached within our approach. In the last subsections, we consider quantizations of various types of orbits, along the line of Theorem 11.2.

11.1. Quantized symmetric orbits. To begin with, let us prove a statement relating reflection equation algebras in different dimensions. Let $\mathcal{M}(n)$ and $\mathcal{M}(k)$ be the matrix algebra of $n \times n$ and $k \times k$ matrices, $0 < k < n$. The homomorphism $\mathcal{M}(k) \to \mathcal{M}(n)$ of embedding as the left upper block induces a contravariant epimorphism of the function algebras. The same holds in the quantum situation:

**Proposition 11.4.** The quotient algebra of $A_{RE}(\mathcal{M}(n))$ by the relations $L_{ij} = 0$, $i > k$ or $j > k$, is isomorphic to $A_{RE}(\mathcal{M}(k))$.

**Proof.** Let $P^+$ and $P^-$ be the projectors from $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^{n-k}$ to the first and the second addends, respectively. Denote $R^{++}$ the projection of the R-matrix (3.3) to $\mathcal{M}(k) \otimes \mathcal{M}(k)$. Up to a nonzero scalar factor, this is the R-matrix in dimension $k$. The equalities

\begin{align}
(P^+ \otimes 1)R &= R^{++} + q^{-\frac{1}{k}}P^+ \otimes P^-, \tag{11.4}
R(1 \otimes P^+) &= R^{++} + q^{-\frac{1}{k}}P^- \otimes P^+ \tag{11.5}
\end{align}

follows directly from (3.3). The matrix $L$ is equal to $L^+ = P^+LP^+$ modulo the ideal specified in the hypothesis. In terms of the matrix $R$ instead of the Hecke matrix $S$, the reflection equation is rewritten as

\begin{align}
R_{21}L_1^+RL_2^+ &= L_2^+R_{21}L_1^+R
\end{align}

modulo the relations of concern. Using (11.4) and (11.5), the matrix $R$ can be replaced by $R^{++}$ thus leading to the RE in dimension $k$. □
Proposition 11.4 suggests a method of building solutions to (11.1) by those in smaller dimensions extending them by zero matrix elements. For example, the projectors $P_k = \sum_{i=1}^{k} e_i^i$ of rank $k = 1, \ldots, n$ satisfy (11.1). Indeed, each can be obtained from the $k \times k$ unit matrix (which is apparently a solution to the RE) by extending it with zeros to the $n \times n$ matrix.

Let us introduce the quantum integer numbers

$$ (11.6) \hat{k}_q = \sum_{i=1}^{k} q^{-2i+2} = \text{Tr}_q(P_k). $$

**Theorem 11.5.** The quotient algebra of $A_{RE}(\mathcal{M})$ by the relations

$$ (11.7) \quad L^2 = L, \quad \text{Tr}_q(L) = \hat{k}_q $$

yields a $U_q(g)$-covariant quantization $\text{Fun}_q(\mathcal{O}_{P_k})$ of the symmetric orbit $\mathcal{O}_{P_k}$ passing through the projector $P_k$.

**Proof.** Relations (11.7) follow from identities (11.2) and (11.3) as applied to $A = P_k$. The projectors $P_k$ are stable under the right adjoint action (3.1) of $U_q(sl(k)) \otimes U_q(sl(n-k)) \otimes U_q(sl(1))$ as a quantum subgroup in $U_q(sl(n))$. Therefore, the subalgebra in $F_q(G)$ generated by the RE matrix $L(P_k)$ is invariant under the right action of $U_q(sl(k)) \otimes U_q(sl(n-k)) \otimes U_q(sl(1))$. Thus $\mathcal{O}_{P_k} = \mathcal{O}_{P_1}$.

**Remark 11.6.** The quantum $\mathcal{O}_{P_k}$ may be obtained directly from the description of $A_{RE}$ given in Section 4. Indeed, imposing the additional conditions $V^\wedge 2 = 0$ and $V^* \wedge 2 = 0$ one comes to the subalgebra in $\text{Sym}_q(V) \otimes \text{Sym}_q(V^*)$ generated by $e_i \otimes f^j$ with $e_i$ and $f^j$ commutative in the sense of permutations (3.7–3.9). The matrix elements $L^i_j = e_i f^j$ satisfy the equality $L^i_j L^j_i = e_i (f^i e_j) f^j = \text{Tr}_q(L) L^j_i$ following from the commutation relations. So, conditions (11.7) for the case $k = 1$ hold simultaneously in this algebra.

As in the classical situation, we can consider solution $\lambda P_k$ with arbitrary $\lambda \neq 0$. This will lead to the relation $L^2 = \lambda L$ and the corresponding rescaling of the quantum trace. The resulting conditions give the deformation quantization of the orbit passing through $\lambda P_k$. It may be regarded as the quantization of the same manifold but along the path $t = \lambda \omega^{(k)}_{\lambda_k}$ in the two parameter space of the universal family $S_{q,t}(g)$ (Theorem 10.1).

11.2. The Cayley-Hamilton identity. The goal of this subsection is to exhibit correspondence between Theorem 11.5 with identities in $A_{RE}(\mathcal{M})$ of the Cayley-Hamiltonian type, [PS]. In the algebra $A_{RE}(\mathcal{M})$, the $k$-th powers $L^k$ of the generating matrix form the left coadjoint module $\mathcal{M}^*$. The quantum traces $\text{Tr}_q(L^k)$, $k = 1, 2, \ldots$ are $U_q(sl(n))$-invariant and central, (see, e.g., [PS]).

**Theorem 11.7 ([PS]).** The generating matrix of the reflection equation algebra $A_{RE}(\mathcal{M})$ obeys the relation

$$ \sum_{j=0}^{n} \sigma_q^{L}(-L)^{n-j} = 0, $$
where $\sigma_j^q(L)$ are central elements expressed through the quantum traces of powers in $L$ by the recursive formula

$$
\hat{k}_q \sigma^q_k(L) = \sum_{j=0}^{k-1} \sigma^q_j(L)(-1)^{k-j+1} \text{Tr}_q(L^{k-j}), \quad \sigma^q_0(L) = 1.
$$

In particular, $\sigma^{n+1}_q(L) = 0$.

It follows that imposing the projector condition on the matrix $L$ specifies the quantum trace modulo the finite set of values.

**Corollary 11.8.** The condition $L^2 = L$ on the generating matrix of the $\mathcal{A}_{RE}(\mathcal{M})$ implies the discrete set $\{\hat{0}_q, \hat{1}_q, \ldots, \hat{n}_q\}$ of values for the quantum trace.

**Proof.** The statement follows from the formula (11.8) since, under the hypothesis made, one has

$$
\sigma^{n+1}_q(L) = \frac{\text{Tr}_q(L)(\text{Tr}_q(L) - \hat{1}_q) \ldots (\text{Tr}_q(L) - \hat{n}_q)}{(n+1)_q!} = 0.
$$

□

All the possible values of $\text{Tr}_q(L)$ are realized, by Theorem 11.5, giving flat deformations of symmetric spaces. Note that the Cayley-Hamiltonian identity in the RE algebra was directly used for construction of the quantum sphere $S^2_q$ in [GS].

### 11.3. On the quantization of non-semisimple orbits.

As was already mentioned, the two-parameter quantization $S^q_t(sl(n))$ may be “restricted” to every semisimple orbits. There are no definite assertions of that kind concerning orbits passing through nilpotent elements. In this section, we prove

**Theorem 11.9.** There exists a one-parameter $\mathcal{U}_q(g)$-covariant quantization of nilpotent orbits in $\mathcal{M}$ satisfying the matrix equation $A^2 = 0$. It is a restriction of the one-parameter subfamily $t = 0$ in the universal two-parameter quantization $S^q_t(g)$.

**Proof.** Like in the case of symmetric orbits, we seek for a solution to the numeric reflection equation, which will realize the quantized algebra as a subalgebra in $\mathcal{F}_q(G)$. Specializing to the skew-diagonal matrices,

$$
A = \sum_{i=1}^{n} \lambda_i e_i^{i'}, \quad i' = n + 1 - i,
$$

we find it in the form $\lambda_i \lambda_{i'} = \lambda^2$, $i = 1, \ldots, n$. In case of non-zero $\lambda$ we come to nondegenerate matrices with two eigenvalues $\pm \lambda$. These solutions were found in [KSS]. They lead to other realization of quantum symmetric orbits than by means of projectors. In the case of $\lambda = 0$, these matrices being squared are zero. By re-enumerating the basis elements, they can be brought to the sum of $2 \times 2$ jordanian blocks yielding all such matrices. The last statement of the theorem holds because $\text{Tr}_q(A) = 0$. □
11.4. On the quantization of bisymmetric orbits. To quantize the symmetric orbits in \( g^* \), we used the projectors, i.e., the semisimple elements with the eigenvalues 1 and 0. There are non-degenerate semisimple solutions to the matrix reflection equation with two eigenvalues. For example, one can take 
\[
\sum_{i=1}^{n} e_i^{n+1-i}.
\]
Such matrices can be used for constructing solutions with three eigenvalues by the embedding method, along the line of Proposition 11.4. In this way, one gains the additional zero eigenvalue. One might have expected that this will provide a tool for quantizing all the semisimple orbits, which are classified as homogeneous spaces by the number of eigenvalues and their multiplicities. However, that is probably impossible, due to certain indications. Indeed, in [KSS], there were written out all non-degenerate solutions up to \( n = 4 \). They have at most two eigenvalues and lead to different quantizations for the symmetric orbits than by means of projectors. Among them, there are also q-traceless matrices. We conjecture that non-degenerate matrix solutions to the reflection equation with arbitrary value of the quantum trace do exist for each symmetric orbit. This implies the existence of the quantization via the RE algebra characters for every bisymmetric orbit, i.e., consisting of matrices with three eigenvalues. Let us present, following [KSS], the non-degenerate q-traceless RE matrices up to \( n = 4 \).

\[
A^{1,1} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad A^{2,1} = \begin{pmatrix} q^{-2} & -\hat{2}q \\ -\hat{2}q & 1 \end{pmatrix}, \\
A^{2,2} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad A^{3,1} = \begin{pmatrix} -q^{-2\hat{2}}q & 1 \\ -q^{-2\hat{2}}q & 1 \\ -q^{-2\hat{2}}q & 1 \end{pmatrix}.
\]

The matrix \( A^{2,2} \) is interesting from the physical point of view because it yields the quantization of the twistor space. Using these RE matrices together with the projectors \( P_k \), we can cover all the semisimple orbits for \( n \leq 4 \), excepting the maximal orbit in dimension \( n = 4 \). But the maximal orbits can be quantized by specifying values of the Casimir elements, i.e., \( \text{Tr}(L_i) \), \( i = 1, \ldots, n - 1 \). So we conclude that, at least to dimension \( n = 4 \), all the semisimple orbits can be quantized explicitly as quotients of the reflection equation algebra.

References

[AFS] A. Alekseev, L. Faddeev, M. Semenov-Tian-Shansky, *Hidden quantum group inside Kac-Moody algebra*, Proceedings of the Euler International Mathematical Institute on Quantum Groups, Lect. Notes Math. **1510** (Springer, Berlin, 1992) 148.

[Cher] I. V. Cherednik, *Factorizing particles on a half line, and root systems*, Teoret. Mat. Fiz. **61** (1984), # 1, 35–44.

[D] V. G. Drinfeld, *Quantum Groups*, in Proc. Int. Congress of Mathematicians, Berkeley, 1986, ed. A.V. Gleason, AMS, Providence (1987) 798.

[Do] J. Donin, *Double quantization on the coadjoint representation of \( sl(n) \)*, Czech J. of Physics, **47**, n.11, 1997, 1115-1122.

[Do1] J. Donin, *U_h(g)-invariant quantization of coadjoint orbits and vector bundles over them*, Preprint 54 (2000), Max-Plank-Institute, J. of Geometry and Physics, **38** (2001) 54.

[DoG] J. Donin, D. Gurevich, *Some Poisson structures associated to Drinfeld-Jimbo R-matrices and their quantization*, Israel J. of Math. **92** (1995) 23.

[DGK] J. Donin, D. Gurevich, S. Khoroshkin, *Double quantization of \( \mathbb{C}P^n \) type by generalized Verma modules*, Preprint [math.QA/9803154], J. of Geom. and Phys., v.28, 1998, 384.
$U_q(sl(n))$-covariant quantization of symmetric coadjoint orbits

L. Faddeev, N. Reshetikhin, and L. Takhtajan, *Quantization of Lie groups and Lie algebras* Leningrad Math. J. 1 (1990) 193.

D. Gurevich, *Algebraic aspects of quantum Yang-Baxter equation*, Leningrad Math. J. 2 (1991) 802.

D. Gurevich, P. A. Saponov *Quantum sphere via reflection equation algebra*, Preprint math.QA/9911141.

S. M. Khoroshkin, V. N. Tolstoy *Universal R-matrix for quantized (super) algebras*, Comm. Math. Phys. 141 (1991) 559.

A. N. Kirillov, N. Yu. Reshetikhin, *q-Weyl group an multiplicative formula for R-matrices*, Comm. Math. Phys. 130 (1990) 421.

P. P. Kulish, *Quantum groups, q-oscillators, and covariant algebras*, Theor. Math. Phys. 94 (1993) 193.

P. P. Kulish, E. K. Sklyanin *Algebraic structure related to the reflection equation*, J. Phys. A 25 (1992) 5963.

P. P. Kulish, R. Sasaki *Covariance properties of reflection equation algebras*, Prog. Theor. Phys. 89 #3 (1993) 741.

P. P. Kulish, R. Sasaki, and C. Schweibert, *Constant solutions of reflection equations and quantum groups*, J. Math. Phys 34 #1 (1993) 286.

G. Lusztig, “Introduction to quantum groups”, Progress in Mathematics, 110. Birkh. Boston, Inc., Boston, MA, 1993.

S. Majid, “Foundations of quantum group theory”, Cambridge University Press, 1995.

P. N. Pyatov, P. A. Saponov *Characteristic relations for quantum matrices*, J. Phys. A 28 #3 (1995) 4415.

M. Rosso, *An analog of P.B.W. theorem and the universal $U_q(sl(N + 1)$, Comm. Math. Phys. 124 (1991) 307.

Max-Planck-Institute für Mathematik, Vivatsgasse 7, 53111 Bonn

Current address: Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel

E-mail address: domin@macs.biu.ac.il

Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel

E-mail address: mudrova@macs.biu.ac.il