ON SOME COHOMOLOGICAL PROPERTIES OF ALMOST COMPLEX MANIFOLDS

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Abstract. We study a special type of almost complex structures, called pure and full and introduced by T.J. Li and W. Zhang in [16], in relation to symplectic structures and Hard Lefschetz condition. We provide sufficient conditions to the existence of the above type of almost complex structures on compact quotients of Lie groups by discrete subgroups. We obtain families of pure and full almost complex structures on compact nilmanifolds and solvmanifolds. Some of these families are parametrized by real 2-forms which are anti-invariant with respect to the almost complex structures.

1. Introduction

Let $M$ be a compact oriented manifold of dimension $2n$. A symplectic form $\omega$ compatible with the orientation is a closed 2-form $\omega$ such that the $2n$-form $\omega^n$ is a volume form compatible with the orientation. An almost complex structure $J$ on a symplectic manifold $(M, \omega)$ is said to be tamed by $\omega$ if $\omega_x(u, Ju) > 0$, for every $x \in M$ and every tangent vector $u \neq 0 \in T_x M$. $J$ is called calibrated by $\omega$ or, equivalently, $\omega$ is said to be compatible with $J$ if in addition $\omega_x(Ju, Jv) = \omega_x(u, v)$, for any pair of tangent vectors $u$ and $v$. In this case the pair $(\omega, J)$ is an almost-Kähler structure or, equivalently, $J$ is said to be almost-Kähler.

Let $C(M)$ be the symplectic cone of $M$, i.e. the image of the space of symplectic forms on $M$ compatible with the orientation under the projection to the de Rham cohomology $H^2(M, \mathbb{R})$. In [16] T. J. Li and W. Zhang have studied the following subcones of $C(M)$: the $J$-tamed symplectic cone $K^t_J(M) = \{ [\omega] \in H^2(M, \mathbb{R}) | \omega$ is tamed by $J \}$ and the $J$-compatible symplectic cone $K^c_J(M) = \{ [\omega] \in H^2(M, \mathbb{R}) | \omega$ is compatible with $J \}$. An almost complex structure $J$ is integrable if its Nijenhuis tensor vanishes. For almost-Kähler manifolds $(M, J, \omega)$, one has that the cone $K^c_J(M)$ is not empty and, when $J$ is integrable, $K^c_J(M)$ is equal to the usual Kähler cone.

In [16] it was studied the relation between the $J$-tamed symplectic cone and the $J$-compatible symplectic cone in the case of an integrable almost complex structure $J$, showing that if $K^c_J(M)$ is non-empty, then one has the splitting $K^c_J(M) = K^c_J(M) + \left( H^{2,0}_\partial(M, \mathbb{R}) \oplus H^{0,2}_\partial(M) \right) \cap H^2(M, \mathbb{R})$, where $H^{p,q}_\partial(M)$ denotes the $(p,q)$-Dolbeault cohomology of the complex manifold $(M, J)$. In order to generalize the previous result to the case of non-integrable almost complex structures, they

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study differential forms and currents on almost complex manifolds. The complex of currents on an almost complex manifold \((M, J)\) is the dual to the complex of differential forms and vice versa. On an almost complex manifold \((M, J)\), the space of real \(k\)-currents \(\mathcal{E}_k(M)\) has a type decomposition:

\[
\mathcal{E}_k(M) = \bigoplus_{p+q=k} \mathcal{E}^p_q(M).
\]

Denote by \(Z^1_{1,1}\) and \(B^1_{1,1}\) respectively the space of real closed bidimension \((1, 1)\) currents and the one of real boundary bidimension \((1, 1)\) currents. Consider the space \(Z^2_{(2,0),(0,2)}\) (respectively \(B^2_{(2,0),(0,2)}\)) of closed (resp. boundary) real 2-currents which are sums of currents of bidimension \(2, 0)\) and \((0, 2)\). Then, by using the results of [12], in [16] the notions of pure and full almost complex structure have been introduced. More precisely, an almost complex structure \(J\) on \((M, J)\) is pure if

\[
\frac{Z^1_{1,1}}{B^1_{1,1}} \cap \left( \frac{Z^2_{(2,0),(0,2)}}{B^2_{(2,0),(0,2)}} \right) = 0
\]

and it is full if

\[
\frac{Z_2}{B_2} = \frac{Z^1_{1,1}}{B^1_{1,1}} + \left( \frac{Z^2_{(2,0),(0,2)}}{B^2_{(2,0),(0,2)}} \right),
\]

where \(Z_2\) (resp. \(B_2\)) are the space of real closed (boundary) 2-currents.

One can give similar notions of \(C^\infty\) pure and full almost complex structures by considering differential forms instead of currents, i.e. \(J\) is \(C^\infty\) pure and full if and only if

\[
H^2(M, \mathbb{R}) = H^{1,1}_J(M) \oplus H^{(2,0),(0,2)}_J(M),
\]

where

\[
H^{1,1}_J(M) = \{ [\alpha] \mid \alpha \in Z^{1,1}_J \}, \quad H^{(2,0),(0,2)}_J(M) = \{ [\alpha] \mid \alpha \in Z^{(2,0),(0,2)}_J \}
\]

and \(Z^{1,1}_J, Z^{(2,0),(0,2)}_J\) are defined in a similar way as before, i.e. are respectively the closed \(J\)-invariant and the \(J\)-anti-invariant forms. In general, there is no relation between the two notions. If \(J\) a smooth closed almost complex structure, i.e. it is such that the image of the space \(B_2\) of real boundary 2-currents under the projection \(\pi_{1,1}\) is a (weakly) closed subspace of \(\mathcal{E}^1_{1,1}(M)\), then some relations between the spaces \(H^{1,1}_J(M)\) and \(Z^{1,1}_J\) are found in [10]. Moreover, on a compact manifold of real dimension 4 any almost complex structure is \(C^\infty\) pure and full by [7, Theorem 2.3].

In Section 2 we review some known facts about calibrated almost complex structures and some properties of almost-Kähler manifolds. In Section 3 we study \(C^\infty\) full and pure almost complex structures on compact quotients \(M = \Gamma\backslash G\), where \(\Gamma\) is a uniform discrete subgroup of a Lie group \(G\), such that the de Rham cohomology \(H^2(M, \mathbb{R})\) of \(M\) is isomorphic to the Chevalley-Eilenberg cohomology \(H^2(\mathfrak{g})\) of the Lie algebra \(\mathfrak{g}\) of \(G\). In this case we show that \(H^{1,1}_J(M)\) and \(H^{(2,0),(0,2)}_J(M)\) can be determined by using invariant forms (Theorem 3.4).

If \(\omega\) is a non-degenerate 2-form on a 2n-dimensional compact manifold \(M\), then we prove that a \(C^\infty\) pure and full almost complex structure \(J\) calibrated by \(\omega\) is pure. Moreover, if, in addition, either \(n = 2\) or if any cohomology class in \(H^{1,1}_J(M)\) (\(H^{(2,0),(0,2)}_J(M)\) respectively) has a harmonic representative in \(Z^{1,1}_J\) (\(Z^{(2,0),(0,2)}_J\) respectively) with respect to the metric induced by \(\omega\) and \(J\), then \(J\) is pure and full (Theorem 3.7). We give examples of compact non-Kähler solvmanifolds, i.e. compact quotients of solvable Lie groups by discrete subgroups, satisfying the previous conditions.

In Section 4 we prove that on a compact symplectic manifold which satisfies the Hard Lefschetz condition a \(C^\infty\) pure and full almost complex structure is pure and full.
An integrable almost complex structure \( J \) is closed but, in general, it is not necessarily \((C^\infty)\) pure and full. We show that a complex parallelizable manifold \((\Gamma/G, J)\) for which \( H^2(M, \mathbb{R}) \) is isomorphic to the Chevalley-Eilenberg cohomology \( H^2(g) \), then \( J \) is \( C^\infty \) full and it is pure (Theorem 5.1). We provide an example of complex parallelizable manifold, endowed with a pure and full complex structure, namely the Nakamura manifold, for which the de Rham cohomology is not isomorphic to the Chevalley cohomology of the corresponding solvable Lie algebra (see [18] and [4]).

By using [3, Proposition 1.16] in section 6 we provide a family of pure and full almost complex structures on a 3-step nilmanifold of real dimension 4 and on two 2-step solvmanifolds of real dimension 4 and 6.

A \( J \)-holomorphic map into an almost complex manifold \((M, J)\) is a map \( f : (\Sigma, j) \rightarrow (M, J)\) from a compact Riemann surface \((\Sigma, j)\) whose differential is complex linear. Gromov-Witten invariants of a symplectic manifold are counts of holomorphic curves with respect to an almost complex structure compatible with the symplectic structure (see e.g. [15]). In [14] it was shown that, when \( M \) is 4-dimensional, there is a natural infinite-dimensional family of almost complex structures \( J_\alpha \) parametrized by the \( J \)-anti invariant real 2-forms on \( M \).

In the last section we construct an explicit family \( J_\alpha \) of pure and full almost complex structures on a 4-dimensional nilmanifold and on a 4-dimensional solvmanifold.

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2. Calibrated almost complex structures

Let \((V, \omega)\) be a \(2n\)-dimensional symplectic real vector space. We recall the following

**Definition 2.1.** A (linear) complex structure \( J \) on \((V, \omega)\) is said to be \( \omega \)-calibrated if:

1. \( \omega(Jv, Jw) = \omega(v, w) \),
2. \( g_J(v, w) := \omega(v, Jw) \) is positive definite,

for every \( v, w \in V \).

Denote by \( \mathcal{C}_\omega(V) \) the set of \( \omega \)-calibrated linear complex structures on \( V \). Then, it is well known (see e.g. [3]) that:

\[
\mathcal{C}_\omega(V) \simeq \mathcal{C}(n) := \text{Sp}(n, \mathbb{R})/U(n)
\]

and

\[
\mathcal{C}(n) := \{ X \in M_{2n,2n}(\mathbb{R}) \mid X = {}^tX, \ XJ_n + J_nX = 0 \},
\]

where \( M_{2n,2n}(\mathbb{R}) \) denotes the set of real matrices of order \( 2n \) and \( J_n \) is the standard complex structure on \( \mathbb{R}^{2n} \). Therefore, \( \mathcal{C}_\omega(V) \) is homeomorphic to an \((n^2 + n)\)-dimensional cell.

For any integer \( k, 0 \leq k \leq n \), let \( L^k : \wedge^{n-k}V^* \rightarrow \wedge^{n+k}V^* \) be the linear map defined by

\[
L^k(\alpha) = \alpha \wedge \omega^k.
\]

We will need the following (see e.g. [20, Corollary 2.7])

**Lemma 2.2.** For any integer \( k, 0 \leq k \leq n \), the map \( L^k : \wedge^{n-k}V^* \rightarrow \wedge^{n+k}V^* \) is an isomorphism.

Let \((M, \omega)\) be a symplectic manifold.
Definition 2.3. An almost complex structure $J$ is said to be tamed by $\omega$ if $\omega_x(u, Ju) > 0$, for any $x \in M$ and any tangent vector $u \neq 0 \in T_x M$. $J$ is called calibrated by $\omega$ if in addition $\omega_x(Ju, Ju) = \omega_x(u, v)$, for any pair of tangent vectors $u$ and $v$.

For a fixed non-degenerate closed 2-form $\omega$ on $\mathbb{R}^{2n} = \mathbb{C}^n$, denote by $\mathcal{J}_c(\omega)$ (resp. $\mathcal{J}_t(\omega)$) the set of almost-complex structures calibrated (respectively tamed) by $\omega$.

By [3, Proposition 1.16], if on $\mathbb{C}^n$ one considers the canonical standard symplectic structure $(J_0, \omega)$, then the map

$$J \mapsto (J + J_0)^{-1} \circ (J - J_0)$$

is a diffeomorphism from $\mathcal{J}_c(\omega)$ (resp. $\mathcal{J}_t(\omega)$) onto the open unit ball in the vector space of matrices (resp. symmetric matrices) $L$ such that $J_0L = -LJ_0$.

Indeed, if $L$ is matrix such that $\|L\| < 1$, then the endomorphism

$$J_0 \circ (I + L) \circ (I - L)^{-1},$$

where $I$ is the identity matrix, is an almost complex structure if and only if $J_0L = -LJ_0$ and it is tamed by $\omega$. Moreover, the almost complex structure

$$J_0 \circ (I + L) \circ (I - L)^{-1}$$

is calibrated if and only if $L$ is symmetric.

Then, if $J_0$ is a almost complex structure calibrated by $\omega$ and $L$ is a symmetric matrix such that $\|L\| < 1$, $J_0L = -LJ_0$, then the new endomorphism

$$(I + L) \circ J_0 \circ (I + L)^{-1}$$

is still an almost complex structure calibrated by $\omega$.

Let again $(M, \omega)$ be a $2n$-dimensional symplectic manifold and let $Sp_\omega(M)$ be the principal $Sp(n, \mathbb{R})$-bundle of symplectic frames on $M$; then $Sp_\omega(M)/U(n)$ is the bundle of $\omega$-calibrated almost complex structures on $M$. Since $Sp_\omega(M)$ has contractible fiber, there exist many global sections. Denote by $\mathcal{E}_\omega(M)$ the space of such sections.

Given an $\omega$-calibrated almost complex structure $J \in \mathcal{E}_\omega(M)$, the space $\Omega^k(M)$ of complex smooth differential $k$-forms has a type decomposition:

$$\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}_J(M),$$

where $\Omega^{p,q}_J(M)$ denotes the space of complex forms of type $(p, q)$ with respect to $J$. We have that:

$$d : \Omega^{p,q}_J(M) \to \Omega^{p+2,q-1}_J(M) \oplus \Omega^{p+1,q+1}_J(M) \oplus \Omega^{p,q+2}_J(M) \oplus \Omega^{p-1,q+2}_J(M)$$

and so $d$ splits accordingly as

$$d = A_J + \partial_J + \bar{\partial}_J + \tilde{A}_J$$

where all the pieces are graded algebra derivations, $A_J$, $\bar{\partial}_J$ are 0-order differential operators, and, in particular, for $\alpha \in \Omega^1(M)$ we have

$$(A_J(\alpha) + \tilde{A}_J(\alpha))(X, Y) = \frac{1}{4}\alpha(N_J(X, Y)),$$
$N_J$ being the Nijenhuis tensor of $J$. From $d^2 = 0$ we obtain

\[ \begin{cases} 
A_J^2 = 0, \\
A_J \partial_J + \partial_J A_J = 0, \\
\partial_J^2 + A_J \partial_J + \partial_J A_J = 0, \\
\partial_J \partial_J + \partial_J A_J + A_J \partial_J + A_J A_J = 0, \\
\partial_J^2 + \partial_J A_J + \partial_J A_J = 0, \\
\partial_J A_J + \partial_J A_J = 0, \\
\overline{\partial}_J^2 = 0. 
\end{cases} \]

If $N_J = 0$, then $J$ is called integrable and $(\omega, J)$ defines a Kähler structure. In general

**Definition 2.4.** An almost-Kähler structure on 2n-dimensional manifold $M$ is a pair $(\omega, J)$ where $\omega$ is a symplectic form and $J$ is an almost complex structure calibrated by $\omega$.

If $(M, \omega, J)$ is an almost-Kähler manifold, then

$$g(X, Y) = \omega(X, JY)$$

is a $J$-Hermitian metric, i.e. $g(JX, JY) = g(X, Y)$, for any $X, Y$.

### 3. Pure and Full Almost Complex Structures and Differential Forms

Let $J$ be a smooth almost complex structure on a compact 2n-dimensional manifold $M$. The space $\Omega^k(M)_{\mathbb{R}}$ of real smooth differential $k$-forms has a type decomposition:

$$\Omega^k(M)_{\mathbb{R}} = \bigoplus_{p+q=k} \Omega^{p,q}_J(M)_{\mathbb{R}},$$

where

$$\Omega^{p,q}_J(M)_{\mathbb{R}} = \{ \alpha \in \Omega^p_J(M) \oplus \Omega^q_J(M) \mid \alpha = \overline{\alpha} \}.$$

For a finite set $S$ of pairs of integers, let

$$Z^S_J = \bigoplus_{(p, q) \in S} Z^{p,q}_J, \quad B^S_J = \bigoplus_{(p, q) \in S} B^{p,q}_J,$$

where the spaces $Z^{p,q}_J$ and $B^{p,q}_J$ are respectively the space of real $d$-closed $(p, q)$-forms and those one of $d$-exacts $(p, q)$-forms. There is a natural map

$$\rho_S : Z^S_J / B^S_J \to Z^S_J / B$$

where $B$ is the space of $d$-exact forms. As in [16] we will write $\rho_S(Z^S_J / B^S_J)$ simply as $Z^S_J / B^S_J$ and we may define the cohomology spaces

$$H^S_J(M)_{\mathbb{R}} = \{ [\alpha] \mid \alpha \in Z^S_J \} = \frac{Z^S_J}{B^S_J}.$$

Then there is a natural inclusion

$$H^{1,1}_J(M)_{\mathbb{R}} + H^{2,0,0,2}(M)_{\mathbb{R}} \subseteq H^2(M, \mathbb{R}).$$

As in [16] Definition 4.12] we set the following

**Definition 3.1.** A smooth almost complex structure $J$ on $M$ is said to be $C^\infty$ pure and full if

$$H^2(M, \mathbb{R}) = H^{1,1}_J(M)_{\mathbb{R}} \oplus H^{2,0,0,2}_J(M)_{\mathbb{R}}.$$
In particular $J$ is $C^\infty$ pure if and only if
\[
\frac{Z_j^{1,1}}{B_j^{1,1}} \cap \frac{Z_j^{(2,0),(0,2)}}{B_j^{(2,0),(0,2)}} = 0
\]
and $J$ is $C^\infty$ full if and only if
\[
\frac{Z^2}{B^2} = \frac{Z_j^{1,1}}{B_j^{1,1}} + \frac{Z_j^{(2,0),(0,2)}}{B_j^{(2,0),(0,2)}},
\]
where $Z^2$ and $B^2$ denote respectively the space of 2-forms which are $d$-closed and exact. Let $\pi_{1,1} : \Omega^2(M)_R \to \Omega_j^{1,1}(M)_R$ be the natural projection. If $J$ is $C^\infty$ pure and full, then the natural homomorphism
\[
\frac{Z_j^{1,1}}{B_j^{1,1}} \to \pi_{1,1} \frac{Z^2}{B^2}
\]
is an isomorphism (see \cite[Lemma 4.9]{16}).

**Proposition 3.2.** Let $\omega$ be a symplectic form on a $2n$-dimensional compact manifold $M$. If $J$ is an almost complex structure on $M$ calibrated by $\omega$, then $J$ is $C^\infty$ pure.

**Proof.** Let $a \in Z_j^{1,1}/B_j^{1,1} \cap Z_j^{(2,0),(0,2)}/B_j^{(2,0),(0,2)}$. Then $a = [\alpha] = [\beta]$, where $\alpha \in Z_j^{1,1}$, $\beta \in Z_j^{(2,0),(0,2)}$ respectively.

We have to show that $a = 0$. By $[\alpha] = [\beta]$, it follows that $\alpha = \beta + d\gamma$, for $\gamma \in \Omega^1(M)_R$.

If $(, )$ denotes the $L^2$-product on $\Omega^k(M)_R$, then we get:
\[
(1) \quad (\alpha, \omega) = (\beta + d\gamma, \omega) = (d\gamma, \omega) = (\gamma, d^*\omega) = 0,
\]
where $*$ is the star Hodge operator with respect to the Riemannian metric associated with $(\omega, J)$.

The 2-form $\omega$ determines an $U(n)$-equivariant map
\[
L : \Omega^{p-1,q-1}_j(M) \to \Omega^p_q(M),
\]
and the orthogonal decomposition
\[
\Omega^p_q(M) = \Omega_0^p_q(M) \oplus L(\Omega^{p-1,q-1}_j(M)),
\]
where $\Omega_0^p_q(M)$ denotes the space of primitive forms of type $(p, q)$, i.e. the space of $(p, q)$-forms $\beta$ such that $\beta \wedge \omega = 0$. In particular
\[
\Omega^2(M)_R = \Omega_0^{1,1}(M)_R \oplus <\omega> \oplus \Omega_j^{(2,0),(0,2)}(M)_R.
\]

Since $\alpha$ is a real form of type $(1, 1)$ and by \cite{11} it is orthogonal to $\omega$, it follows that $\alpha$ is primitive and, consequently, if $n > 2$
\[
\alpha \wedge \omega^{n-1} = 0.
\]

Then, by Lemma \cite{22} it follows that $\alpha = 0$. Hence $a = 0$.

If $n = 2$ we have that the spaces $\Omega_0^{1,1}(M)_R$ and $<\omega> \oplus \Omega_j^{(2,0),(0,2)}(M)_R$ are respectively the spaces of self-dual and anti-self-dual 2-forms.

Therefore, both $\alpha$ and $\beta$ are harmonic forms. By the assumption,
\[
[\alpha - \beta] = 0,
\]
and by the harmonicity of $\alpha$ and $\beta$, we get $\alpha = \beta$. Therefore, $\alpha = \beta = 0$, i.e. $a = 0$. \hfill $\Box$

By the last proposition, we obtain at once that, if $(\omega, J)$ is an almost-Kähler structure then $J$ is $C^\infty$ pure. This result has been also proved in \cite{7}.

By \cite[Theorem 2.3]{7} on a compact manifold of real dimension 4 any almost complex structure is $C^\infty$ pure and full. We will show in the next example that a compact manifold of real dimension 6 may admit non $C^\infty$ pure almost structures.
Example 3.3. Consider the 6-dimensional nilmanifold $M$, compact quotient of the 6-dimensional real nilpotent Lie group with structure equations
\[
\begin{align*}
\text{de}^j &= 0, & j &= 1, \ldots, 4, \nonumber \\
\text{de}^5 &= e_1^2, \\
\text{de}^6 &= e_1^3,
\end{align*}
\]
by a uniform discrete subgroup. The left-invariant almost complex structure on $M$, defined by the $(1,0)$-forms
\[
\eta^1 = e^1 + ie^2, \quad \eta^2 = e^3 + ie^4, \quad \eta^3 = e^5 + ie^6,
\]
is not $C^\infty$ pure, since one has that
\[
[\text{Re}(\eta^1 \wedge \eta^2)] = [e_1^3 + e_2^4] = [e_1^3 - e_2^4].
\]

If $(\omega, J)$ is a Kähler structure, then of course $J$ is $C^\infty$ pure and full. Therefore, the interesting case is to find examples of $C^\infty$ full and pure almost complex structures, not associated to Kähler structures.

Theorem 3.4. If $J$ is a $C^\infty$ full and pure almost complex structure on a compact quotient $M = \Gamma \backslash G$, where $\Gamma$ is a uniform discrete subgroup of a Lie group $G$, such that the de Rham cohomology $H^2(M, \mathbb{R})$ of $M$ is isomorphic to the Chevalley-Eilenberg cohomology $H^2(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of $G$, then the following isomorphisms hold:
\[
H^{1,1}_j(M) \cong \frac{\pi_{1,1} Z_{\text{inv}}^2}{\pi_{1,1} B^2}, \quad H^{(2,0), (0,2)}_j(M) \cong \frac{\pi_{(2,0), (0,2)} Z_{\text{inv}}^2}{\pi_{(2,0), (0,2)} B^2},
\]
where $Z_{\text{inv}}^2$ denotes the space of closed left-invariant 2-forms on $M$.

Proof. By the assumption on the de Rham cohomology of $M$, we have
\[
H^2(M, \mathbb{R}) = \frac{Z^2}{B^2} \cong H^2_{\text{inv}}(M, \mathbb{R}) = \frac{Z_{\text{inv}}^2}{B_{\text{inv}}^2},
\]
where $H^2_{\text{inv}}(M, \mathbb{R}) \cong H^2(\mathfrak{g})$ is the cohomology of complex of left-invariant 2-forms on $M$. Let $[\alpha] \in H^{1,1}_j(M)$, then $[\alpha] = [\bar{\alpha}]$, with $\bar{\alpha}$ a left-invariant closed 2-form. Therefore
\[
\alpha = \pi_{1,1} \bar{\alpha} + \pi_{1,1} \text{d}\gamma,
\]
with $\bar{\alpha}$ a left-invariant form closed 2-form i.e. $[\alpha] = [\pi_{1,1} \bar{\alpha}]$ in $\pi_{1,1} Z_{\text{inv}}^2 / \pi_{1,1} B^2$ and similarly for an element of $H^{(2,0), (0,2)}_j(M)$. Since $J$ is $C^\infty$ pure and full we have that
\[
H^{1,1}_j(M) \cong \frac{Z_{\text{inv}}^2}{\pi_{1,1} B^2}, \quad H^{(2,0), (0,2)}_j(M) \cong \frac{Z_{\text{inv}}^2}{\pi_{(2,0), (0,2)} B^2}.
\]
Then the theorem follows by
\[
H^2(M, \mathbb{R}) = \frac{\pi_{1,1} Z_{\text{inv}}^2}{\pi_{1,1} B^2} \oplus \frac{\pi_{(2,0), (0,2)} Z_{\text{inv}}^2}{\pi_{(2,0), (0,2)} B^2}.
\]

The previous assumption on the de Rham cohomology of $\Gamma \backslash G$ is not so restrictive, since it is satisfied when $G$ is completely solvable, so, in particular, if $G$ is nilpotent.

The complex of currents is dual to the complex of forms and vice versa. Since the smooth $k$-forms can be considered as $(2n - k)$-currents, the $k$-th de Rham homology group $H_k(M, \mathbb{R})$ is isomorphic to the $(2n - k)$-th de Rham cohomology group $H^{2n-k}(M, \mathbb{R})$ of $M$ (see [3]). Moreover, a $k$-current is a boundary if and only if it vanishes on the space of closed $k$-forms.
Indeed, on an almost complex manifold \((M, J)\) the space of real \(k\)-currents \(\mathcal{E}_k(M)_\mathbb{R}\) has a decomposition:

\[
\mathcal{E}_k(M)_\mathbb{R} = \bigoplus_{p+q=k} \mathcal{E}^J_{p,q}(M)_\mathbb{R},
\]

where \(\mathcal{E}^J_{p,q}(M)_\mathbb{R}\) denotes the space of real \(k\)-currents of bidimension \((p, q)\).

If \(S\) is a finite set of pairs of integers, then let

\[
\mathcal{E}^J_S = \bigoplus_{(p,q)\in S} \mathcal{E}^J_{p,q}, \quad \mathcal{B}^J_S = \bigoplus_{(p,q)\in S} \mathcal{B}^J_{p,q},
\]

where the spaces \(\mathcal{E}^J_S\) and \(\mathcal{B}^J_S\) are respectively the space of real closed bidimension \((p, q)\) currents and the one of real exact bidimension \((p, q)\) currents. Let, as in [16],

\[
H^J_S(M)_\mathbb{R} = \{[\alpha] | \alpha \in \mathcal{E}^J_S\} = \frac{\mathcal{E}^J_S}{\mathcal{B}^J_S},
\]

where \(\mathcal{B}\) denotes the space of currents which are boundaries.

Denote by \(\mathcal{E}^J_2\) and \(\mathcal{B}^J_2\) respectively the space of real 2-currents which are closed and boundaries.

We recall the following (see [16] Definitions 4.3 and 4.4)

**Definition 3.5.** An almost complex structure \(J\) is said to be pure if

\[
\frac{\mathcal{E}^J_{1,1}}{\mathcal{B}^J_{1,1}} \cap \frac{\mathcal{E}^{2,(0),(0,2)}_{(2,0),(0,2)}}{\mathcal{B}^{2,(0),(0,2)}_{(2,0),(0,2)}} = 0,
\]

equivalently, if and only if \(\pi_{1,1} \mathcal{B}^J_2 \cap \mathcal{Z}^J_{1,1} = \mathcal{B}^J_{1,1}\).

\(J\) is said to be full if

\[
\frac{\mathcal{E}^J_2}{\mathcal{B}^J_2} = \frac{\mathcal{E}^J_{1,1}}{\mathcal{B}^J_{1,1}} + \frac{\mathcal{E}^{2,(0),(0,2)}_{(2,0),(0,2)}}{\mathcal{B}^{2,(0),(0,2)}_{(2,0),(0,2)}}.
\]

Therefore, an almost complex structure \(J\) is pure and full if and only if

\[
H^J_2(M, \mathbb{R}) = H^J_{1,1}(M)_\mathbb{R} \oplus H^J_{(2,0),(0,2)}(M)_\mathbb{R},
\]

where \(H^J_2(M, \mathbb{R})\) is the 2-nd de Rham homology group. In [16] a link between the two notions of pure and full in terms of differential forms and currents was found.

**Definition 3.6.** An almost complex structure \(J\) is said to be \(C^\infty\) closed if the image of the operator

\[
d_{1,1} : \Omega^{1,1}_J(M)_\mathbb{R} \to \Omega^2(M)_\mathbb{R}
\]

is closed.

By [16] an almost complex structure \(J\) is closed if and only if \(J\) is \(C^\infty\) closed. Moreover, in [16] relations between the spaces \(H^J_{1,1}(M)_\mathbb{R}\) and \(H^J_{1,1}(M)_\mathbb{R}\) have been determined.

If a 2-form \(\omega\) on a 2\(n\)-dimensional manifold is not necessarily closed but it is only non-degenerate, the manifold \((M, \omega)\) is called almost symplectic manifold. We can prove the following

**Theorem 3.7.** Let \((M, \omega)\) be an almost symplectic 2\(n\)-dimensional compact manifold and \(J\) be a \(C^\infty\) pure and full almost complex structure calibrated by \(\omega\). Then \(J\) is pure.

If, in addition, either \(n = 2\) or if any cohomology class in \(H^J_{1,1}(M)_\mathbb{R}\) \((H^J_{(2,0),(0,2)}(M)_\mathbb{R}\) respectively) has a harmonic representative in \(\mathcal{Z}^J_{1,1}\) \((\mathcal{Z}^J_{(2,0),(0,2)}\) respectively) with respect to the metric induced by \(\omega\) and \(J\), then \(J\) is pure and full.

**Proof.** We start to prove that \(J\) is pure. We show that \(\pi_{1,1} \mathcal{B}^J_2 \cap \mathcal{Z}^J_{1,1} = \mathcal{B}^J_{1,1}\).

Since \(\pi_{1,1} \mathcal{B}^J_2 \cap \mathcal{Z}^J_{1,1} \supset \mathcal{B}^J_{1,1}\), it will be enough to prove the other inclusion.

Let \(T \in \pi_{1,1} \mathcal{B}^J_2 \cap \mathcal{Z}^J_{1,1}\); then \(T = \pi_{1,1} dS\), where \(S\) is a real 3-current and \(d(\pi_{1,1} dS) = 0\). We have to show that \(T = \pi_{1,1} dS\) is a boundary, i.e. that it vanishes on any closed real 2-form \(\alpha\).
If $\alpha$ is exact, then one has immediately that $(\pi_{1,1} dS)(\alpha) = 0$. Suppose that $[\alpha] \neq 0 \in H^2(M, \mathbb{R})$, then since $J$ is $C^\infty$ pure and full, we can write

$$\alpha = \alpha_1 + \alpha_2 + d\gamma,$$

with $\alpha_1 \in Z_j^{1,1}$, $\alpha_2 \in Z_j^{(2,0),(0,2)}$ and $\gamma \in \Omega^1(M)$. Therefore,

$$T(\alpha) = (\pi_{1,1} dS)(\alpha) = (\pi_{1,1} dS)(\alpha_1 + \alpha_2) = (dS)(\alpha_1) = 0,$$

since $\alpha_1$ is closed.

If $n = 2$ in order to prove that $J$ is also full, we have to show that (2) holds.

Let $[T] \in H_2(M, \mathbb{R})$; then there exists a smooth closed 2-form $\beta$ on $M$ such that $[T] = [\beta]$. Since $J$ is $C^\infty$ full, we have that $[\alpha] = [\alpha_1] + [\alpha_2]$, with $\alpha_1 \in Z_j^{1,1}$ and $\alpha_2 \in Z_j^{(2,0),(0,2)}$, then $\alpha_1$ and $\alpha_2$ can be viewed as elements of $Z_j^{1,1}$ and $Z_j^{(2,0),(0,2)}$ respectively.

If $n > 2$ and for any cohomology class in $H_j^{1,1}(M)_{\mathbb{R}}$ ($H_j^{(2,0),(0,2)}(M)_{\mathbb{R}}$ respectively) one can find a harmonic representative in $Z_j^{1,1}$ ($Z_j^{(2,0),(0,2)}$ respectively), then in order to prove that $J$ is full we can argue in the following way.

Let $[T] \in H_2(M, \mathbb{R})$, then there exists a smooth harmonic $(2n-2)$-form $\beta$ on $M$ such that $[T] = [\beta]$. Then, the 2-form $\gamma = **\beta$ is closed and defines a cohomology class $[\gamma] \in H^2(M, \mathbb{R})$. By the assumption, thus there exist real harmonic forms $\gamma_1 \in \Omega_j^{1,1}(M)_{\mathbb{R}}$ and $\gamma_2 \in \Omega_j^{(2,0),(0,2)}(M)_{\mathbb{R}}$ such that

$$[\gamma] = [\gamma_1] + [\gamma_2].$$

The $(2n-2)$-forms $\beta_1 = **\gamma_1$ and $\beta_2 = **\gamma_2$ then can be viewed as elements respectively of $Z_j^{1,1}$ and $Z_j^{(2,0),(0,2)}$, i.e.

$$[T] = [\beta_1] + [\beta_2].$$

\[\square\]

**Remark 3.8.** The assumption in Theorem 3.7 that any $(1,1)$ (respectively $(2,0) + (0,2)$) de Rham class contains a harmonic representative seems quite strong. Nevertheless, we will provide several examples of compact non-Kähler solvmanifolds satisfying the above assumption. In order to get the pureness of $J$, it is enough to assume that $J$ is $C^\infty$ full (see also [16]).

**Remark 3.9.** In view of [7, Theorem 2.3], if $n = 2$, then any almost complex structure $J$ is $C^\infty$ pure and full; therefore by Theorem 3.7 $J$ is pure and full.

### 4. Hard Lefschetz condition

Let $(M, \omega)$ be a symplectic manifold of dimension $2n$. For any pair of (real or complex) $k$-forms $\alpha, \beta$ on $M$, denote by $\omega(\alpha, \beta)$ the bilinear form induced by the symplectic form $\omega$ on the space $\Omega^k(M)$ of $k$-forms. Then if $\beta \in \Omega^k(M)$, for any $\alpha \in \Omega^k(M)$, the following

$$\alpha \wedge *_{\omega} \beta = \omega(\alpha, \beta) \frac{\omega^n}{n!}$$

defines the *symplectic Hodge operator* $*_{\omega} : \Omega^k(M) \to \Omega^{2n-k}(M)$ on $(M, \omega)$. Then, according to the above definition of the symplectic Hodge operator, the *symplectic codifferential* is the operator $d^{*_{\omega}} : \Omega^k(M) \to \Omega^{k-1}(M)$ defined by

$$d^{*_{\omega}} = (-1)^{k+1} *_{\omega} d *_{\omega} \alpha,$$

for any $\alpha \in \Omega^k(M)$. By definition, a $k$-form $\alpha$ is said to be *symplectic harmonic* if it satisfies

$$d \alpha = d^{*_{\omega}} \alpha = 0.$$
I) for any \((p, q)\)-form \(\alpha\), \(*_\omega(\alpha) = i^{p-q}*_g(\alpha)\), where \(*_\omega\) denotes the symplectic Hodge operator;

II) if \(\alpha\) is harmonic of type \((p, q)\), then also \(*_\omega d*_\omega(\alpha) = 0\), and, consequently, \(\alpha\) is symplectic harmonic.

Since \(d(*_\omega) = 0\), the symplectic form \(\omega\) on an almost-Kähler manifold \((M, \omega, J, g)\) is harmonic.

We recall that a 2\(n\)-dimensional symplectic manifold \((M, \omega)\) satisfies the **Hard Lefschetz condition** if the cup-product maps:

\[
[\omega]^k : H^{n-k}(M) \to H^{n+k}(M), \quad 0 \leq k \leq n
\]

are isomorphisms.

By [17], a compact symplectic manifold \((M, \omega)\) satisfies the Hard Lefschetz condition if and only if any de Rham cohomology class has a symplectic harmonic representative.

A classical result by [6] states that, if \((M, \omega, J)\) is a compact Kähler, i.e. if \(J\) is an integrable almost-Kähler structure on a compact manifold, then it satisfies the Hard Lefschetz condition.

A natural question is to see if there is any relation in the case of an almost-Kähler structure \((\omega, J)\) between the condition for \(J\) to be pure and full and the condition for \(\omega\) to satisfy the Hard Lefschetz property. We can prove the following

**Theorem 4.1.** Let \((M, \omega)\) be a 2\(n\)-dimensional compact symplectic manifold which satisfies the Hard Lefschetz condition and \(J\) be a \(C^\infty\) pure and full almost complex structure calibrated by \(\omega\). Then \(J\) is pure and full.

**Proof.** If \(n = 2\) the result follows by Theorem 3.7. If \(n > 2\), we know already that \(J\) is pure and we have to show that

\[
H_2(M, \mathbb{R}) = H^1_{C,1}(M) \oplus H^J_{(2,0),(0,2)}(M) \mathbb{R}.
\]

Let \(a = [T] \in H_2(M, \mathbb{R})\). Then we can represent \(a\) as a de Rham class \(a = [\alpha]\), where \(\alpha \in \Omega^{2n-2}(M)\) is \(d\)-closed. By the assumptions, \((M, \omega)\) satisfies the Hard Lefschetz condition. Therefore, there exists \(b \in H^2(M, \mathbb{R})\), \(b = [\beta]\) such that \(a = b \cup [\omega]^{n-2}\), i.e.

\[
[\beta \wedge \omega^{n-2}] = [\alpha].
\]

Since \(J\) is \(C^\infty\) pure and full, it follows that

\[
[\beta] = [\varphi] + [\psi],
\]

where \([\varphi] \in H^1_{C,1}(M)\mathbb{R}\) and \([\psi] \in H^J_{(2,0),(0,2)}(M)\mathbb{R}\) respectively. Then we obtain

\[
a = [T] = [\beta \wedge \omega^{n-2}] = [\varphi \wedge \omega^{n-2}] + [\psi \wedge \omega^{n-2}] = [R] + [S],
\]

where \(R \in H^1_{C,1}(M)\mathbb{R}\) and \(S \in H^J_{(2,0),(0,2)}(M)\mathbb{R}\). Hence, \(J\) is pure and full. \(\square\)

Since if \(n = 2\), then it follows that every almost complex structure is \(C^\infty\) pure and full (see [7, Theorem 2.3]), it would be interesting to find for \(n > 2\) an example of compact symplectic manifold \((M, \omega)\) which satisfies Hard Lefschetz condition and with a non pure and full almost complex structure calibrated by \(\omega\).
5. Integrable pure and full almost-complex structures

An integrable almost complex structure is closed (see [12]), but in general it is not necessarily \((C^\infty)\) pure and full. By [16] if \(J\) is an integrable almost complex structure and the Frölicher spectral sequence degenerates at \(E_1\), then \(J\) is pure and full. Any complex surface has this property, so in real dimension 4 the interesting case to study is the one of non integrable almost complex structures.

For compact complex parallelizable manifolds, i.e. compact quotients of complex Lie groups by discrete Lie subgroups, we can prove the following

**Theorem 5.1.** If \((M = \Gamma \backslash G, J)\) is a complex parallelizable manifold and for the de Rham cohomology we have the isomorphism \(H^2(M, \mathbb{R}) \cong H^2(\mathfrak{g})\), then \(J\) is \(C^\infty\) full and therefore it is pure. Moreover,

\[
H^2(M, \mathbb{R}) \cong Z^{1,1}_\text{inv} \oplus Z^{(2,0) + (0,2)}_\text{inv} / B^{(2,0),(0,2)}._\text{inv}.
\]

**Proof.** Denote by \(\Omega^p,q_{\text{inv}}(M)\) the space of left-invariant complex forms of type \((p,q)\). Since \(J\) is bi-invariant, we have at the level of left-invariant forms

\[
d(\Omega^{1,1}_{\text{inv}}(M)_{\mathbb{R}}) \subseteq (\Omega^{1,1}_{\text{inv}}(M) + \Omega^{1,2}_{\text{inv}}(M))_{\mathbb{R}},
\]

\[
d(\Omega^{2,0}_{\text{inv}}(M) + \Omega^{0,2}_{\text{inv}}(M))_{\mathbb{R}} \subseteq (\Omega^{3,0}_{\text{inv}}(M) + \Omega^{0,3}_{\text{inv}}(M))_{\mathbb{R}}.
\]

Therefore

\[
H^2(M, \mathbb{R}) \cong \left(Z^{1,1}_{\text{inv}} + Z^{(2,0)+(0,2)}_{\text{inv}} / B_{\text{inv}}\right),
\]

since \(B^{1,1}_{\text{inv}} = 0\). This implies that \(J\) is \(C^\infty\) full, since any \([\alpha] \in H^2(M, \mathbb{R})\) can be written as

\[
[\alpha] = [\alpha_1] + [\alpha_2]
\]

with \(\alpha_1\) and \(\alpha_2\) left-invariant and respectively belonging to \(Z^{1,1}_{\text{inv}}\) and to \(Z^{(2,0)+(0,2)}_{\text{inv}}\). As a consequence of Theorem 3.7 we get that \(J\) is pure. \(\square\)

Therefore we have the following

**Corollary 5.2.** Let \((M, J)\) be a complex parallelizable nilmanifold. Then \(J\) is \(C^\infty\) full and it is pure.

A classification of compact complex parallelizable solvmanifolds of complex dimension 3, 4, 5 was obtained by Nakamura in [18]. Therefore, for instance, we can apply Corollary 5.2 to the Iwasawa manifold endowed with the natural bi-invariant complex structure. The other complex solvmanifold of real dimension 6 is the Nakamura manifold. In this case we can also show that it admits pure and full almost complex structures, even if its de Rham cohomology is not isomorphic to the Chevalley cohomology of the corresponding solvable Lie algebra.

**Example 5.3.** (Nakamura manifold) Consider the solvable Lie group \(G\) with structure equations

\[
\begin{align*}
de^1 &= 0, \\
de^2 &= e^{12} - e^{45}, \\
de^3 &= -e^{13} + e^{46}, \\
de^4 &= 0, \\
de^5 &= e^{15} - e^{24}, \\
de^6 &= e^{16} + e^{34}.
\end{align*}
\]
The Lie group $G$ is isomorphic to $\mathbb{C}^3$ with product $\ast$, defined in terms of the complex coordinates on $\mathbb{C}^3$ $(z_1 = x_1 + ix_4, z_2 = x_2 + ix_5, z_3 = x_3 + ix_6)$ by

$$^t(z_1, z_2, z_3) \ast ^t(w_1, w_2, w_3) = ^t(z_1 + w_1, e^{-w_1}z_2 + w_2, e^{w_1}z_3 + w_3),$$

for any $^t(z_1, z_2, z_3), ^t(w_1, w_2, w_3) \in \mathbb{C}^3$.

The Nakamura manifold is the compact quotient $X = \Gamma \backslash G$ of $G$ by a uniform discrete subgroup $\Gamma$.

By [4] Corollary 4.2] we have

$$H^2(X, \mathbb{R}) = \mathbb{R} < [e^{14}], [e^{26} - e^{35}], [e^{23} - e^{56}], [\cos(2x_4)(e^{23} + e^{56}) - \sin(2x_4)(e^{26} + e^{35})],$$

$$[\sin(2x_4)(e^{23} + e^{56}) - \cos(2x_4)(e^{26} + e^{35})],$$

i.e. in this case the de Rham cohomology of $M$ is not isomorphic to $H^*(g)$. The previous representatives are all harmonic forms. The complex solvmanifold $X$ admits a left-invariant almost complex structure $J$ defined by the $(1,0)$-forms:

$$\eta^1 = e^1 + ie^4, \quad \eta^2 = e^3 + ie^5, \quad \eta^3 = e^6 + ie^2$$

calibrated by the symplectic form

$$\omega = e^{14} + e^{35} + e^{62}.$$

We have that the forms

$$e^{14}, \quad e^{26} - e^{35},$$

$$\cos(2x_4)(e^{23} + e^{56}) - \sin(2x_4)(e^{26} + e^{35}), \quad \sin(2x_4)(e^{23} + e^{56}) - \cos(2x_4)(e^{26} + e^{35})$$

are all of type $(1,1)$ with respect to $J$ and $e^{23} - e^{56}$ is of type $(2,0)$ with respect to $J$. Then, by applying Theorem [3.7] we get that $J$ is pure and full.

Moreover, $X$ admits the bi-invariant complex structure $\tilde{J}$ defined by the $(1,0)$-forms

$$\tilde{\eta}^1 = e^1 + ie^4, \quad \tilde{\eta}^2 = e^2 + ie^5, \quad \tilde{\eta}^3 = e^3 + ie^6.$$

By Theorem [3.7] $J$ is pure and full.

6. A FAMILY OF PURE AND FULL ALMOST COMPLEX STRUCTURES

In this section we will provide a family of pure and full almost complex structures on a nilmanifold of real dimension 4 and on 2-step solvmanifolds of real dimension 4 and 6.

From now on we will use the convention that the almost complex structure $J$ acts on 1-forms as $(J\alpha)(X) = -\alpha(JX)$, for any 1-form $\alpha$ and vector field $X$.

6.1. A 4-dimensional solvmanifold $M^4$. Consider the 3-dimensional completely solvable Lie group $\text{Sol}_3 = \mathbb{R} \ltimes \phi \mathbb{R}^2$, where $\phi(t)$ is the one-parameter subgroup defined by $\phi(t) = \text{diag}(e^t, e^{-t})$. By [9] a compact quotient $M^4$ of the Lie group $\text{Sol}_3 \times \mathbb{R}$ with structure equations

$$\begin{cases}
    de^1 = 0, \\
    de^2 = 0, \\
    de^3 = -e^{13}, \\
    de^4 = e^{14},
\end{cases}$$

by a uniform discrete subgroup admits a symplectic structure which satisfies the Hard Lefschetz condition. The symplectic structure is given by

$$\omega = e^{12} + e^{34}.$$
The compact symplectic manifold is cohomologically Kähler (in fact, it has the same minimal model as $\mathbb{T}^2 \times S^2$) and it does not carry complex structure. The $\omega$-calibrated almost complex structure $J_0$ defined by the $(1,0)$-forms

$$\eta^1 = e^1 + ie^2, \eta^2 = e^3 + ie^4$$

is pure and full, since by [13] we have that

$$H^2(M^4, \mathbb{R}) = \mathbb{R} < [e^{12}], [e^{34}] >$$

and both the forms $e^{12}$ and $e^{34}$ are of type $(1,1)$. We can show that this compact symplectic manifold $(M^4, \omega)$ admits a family of $\omega$-calibrated almost complex structures which are pure and full. Indeed, let us construct a $C^\infty$ pure and full deformation corresponding to the matrix

$$J_t = (I + L_t)J_0(I + L_t)^{-1}$$

with respect to the coframe $(e^1, \ldots, e^4)$, where $I$ is the identity matrix of order 4, $J_0$ is the matrix

$$J_0 = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}$$

and

$$L_t = \begin{pmatrix}
0 & 0 & ta & 0 \\
0 & 0 & 0 & -ta \\
ta & 0 & 0 & 0 \\
0 & -ta & 0 & 0
\end{pmatrix},$$

with $t, a \in \mathbb{R}$ satisfying the condition

$$4t^2a^2 < 1.$$ 

A direct computation gives

$$J_t = \frac{1}{(t^2a^2 - 1)} \begin{pmatrix}
0 & 1 + t^2a^2 & 0 & 2ta \\
-1 - t^2a^2 & 0 & 2ta & 0 \\
0 & 2ta & 0 & 1 + t^2a^2 \\
2ta & 0 & -1 - t^2a^2 & 0
\end{pmatrix}.$$ 

Therefore, by [3] (see Section 2) the almost complex structure $J_t$ is $\omega$-calibrated. A basis of $(1,0)$-forms for $J_t$ is given by

$$\varphi^1_t = e^1 + i \left( -\frac{1+t^2a^2}{t^2a^2-1}e^2 + \frac{2ta}{t^2a^2-1}e^4 \right),$$

$$\varphi^2_t = e^3 + i \left( -\frac{2ta}{t^2a^2-1}e^2 - \frac{1+t^2a^2}{t^2a^2-1}e^4 \right).$$

Thus, we get that $J_t$ is a curve of pure and full almost complex structures on $M^4$, calibrated by the symplectic form $\omega$. 
6.2. A 4-dimensional nilmanifold $\tilde{M}^4$. The only two nilmanifolds of dimension 4 which admit an almost-Kähler structure $(J, \omega)$ are the Kodaira-Thurston manifold and the 3-step nilmanifold $\hat{M}^4 = \Gamma \setminus N$, compact quotient of the 3-step nilpotent Lie group $N$, whose nilpotent Lie algebra $n$ has structure equations

$\begin{cases}
    de^j = 0, & j = 1, 2, \\
    de^3 = e^{14}, \\
    de^4 = e^{12},
\end{cases}$

(5)

where $e^{ij}$ stands for $e^i \wedge e^j$.

The Kodaira-Thurston manifold has a pure and full integrable almost complex structure since in this case the Frölicher spectral sequence degenerates at $E_1$. Since $b_1(\hat{M}^4) = 2$, $\hat{M}^4$ does not admit any integrable almost complex structure (see [8]).

Proposition 6.1. The 3-step nilmanifold $\hat{M}^4$ admits a pure and full almost-Kähler almost complex structure and a pure and full (non almost-Kähler) almost complex structure.

Proof. Consider the almost complex structure $J_0$ defined by the $(1, 0)$-forms

$\eta^1 = e^1 + ie^3, \quad \eta^2 = e^2 + ie^4$.

Then $J_0$ is almost-Kähler with respect to

$\omega = e^{13} + e^{24}$.

By Nomizu’s Theorem [19], one has that the de Rham cohomology of the nilmanifold $\tilde{M}^4$ is isomorphic to the Chevalley-Eilenberg cohomology of the Lie algebra $n$. In particular, we have

$H^1(\tilde{M}^4, \mathbb{R}) = \mathbb{R} < [e^1], [e^2] >, \quad H^2(\tilde{M}^4, \mathbb{R}) = \mathbb{R} < [e^{13}], [e^{24}] >$.

As shown in [17], $(\tilde{M}^4, \omega)$ does not satisfies hard Lefschetz property since

$e^1 \wedge \omega = d(e^{23}), \quad e^2 \wedge \omega = d(e^{34})$.

Since $e^{13}$ and $e^{24}$ are forms of type $(1, 1)$ with respect to $J_0$ or by [7, Theorem 2.3], we have that $J_0$ is $C^\infty$ pure and full with

$H^2(\tilde{M}^4, \mathbb{R}) = H^{1,1}_{J_0}(\tilde{M}^4, \mathbb{R})$.

Thus, by Theorem 3.7 $J_0$ is pure and full.

The nilmanifold $\hat{M}^4$ admits also an almost complex structure $\tilde{J}_0$ which is pure and full, but not almost-Kähler. Consider on $\hat{M}^4$ the left-invariant almost complex structure defined by the $(1, 0)$-forms

$\tilde{\eta}_1 = e^1 + ie^2, \quad \tilde{\eta}_2 = e^3 + ie^4$.

In this case we have

$H^{1,1}_{\tilde{J}_0}(\hat{M}^4, \mathbb{R}) = \mathbb{R} < [e^{13} + e^{24}] >, \quad H^{(2,0) + (0,2)}_{\tilde{J}_0}(\hat{M}^4, \mathbb{R}) = \mathbb{R} < [e^{13} - e^{24}] >$

and, again by using Theorem 5.7, we have that $\tilde{J}_0$ is full and pure.

In order to prove that $\tilde{J}_0$ is not almost-Kähler, we start to show that there is no symplectic left-invariant 2-form $\hat{\omega}$ on $\hat{M}^4$ such that $\tilde{J}_0$ is calibrated by $\hat{\omega}$. Indeed, such a 2-form has to be closed and of type $(1, 1)$ with respect to $\tilde{J}_0$, i.e. of the form

$\hat{\omega} = ae^{12} + b(e^{13} + e^{24})$,

with $a, b \in \mathbb{R}$, but then $\hat{\omega}(e_3, e_4) = 0$. 

By using the same argument as in [11], we then show that there exists no symplectic 2-form $\omega$ on $\bar{M}^4$ such that $\bar{J}_0$ is $\omega$-calibrated. Indeed, if such a non left-invariant symplectic 2-form $\omega$ exists, we can consider the left-invariant 2-form $\tilde{\omega}$, defined by

$$\tilde{\omega}(X_1, X_2) = \int_{x \in \mathcal{G}^4} \omega_x(X_1|_x, X_2|_x)d\mu,$$

where $X_1, X_2$ are projections of left-invariant vector fields from $G_2$ to $\bar{M}^4$ and $d\mu$ is a bi-invariant volume form on $\bar{M}^4$. The form $\tilde{\omega}$ is then a closed $(1,1)$-form and $\bar{J}_0$ is calibrated by $\tilde{\omega}$, but this is a contradiction.

We can construct on $\bar{M}^4$ a family of pure and full almost-Kähler structures. In order to do this, we will change a little bit our notation. We change the basis of 1-forms so that, in the new coframe $\{f^1, \ldots, f^4\}$, the structure equations are

$$\begin{align*}
df^1 &= 0, \\
df^2 &= f^{14}, \\
df^3 &= 0, \\
df^4 &= f^{13},
\end{align*}$$

Then the almost-Kähler $(\omega, \bar{J}_0)$ structure is given by

$$\omega = f^{12} + f^{34}, \quad \varphi^1 = f^1 + if^2, \quad \varphi^2 = f^3 + if^4.$$

Hence

$$H^2(\bar{M}^4, \mathbb{R}) = \mathbb{R} < [f^{12}] > = H^{1,1}_{\bar{J}_0}(\bar{M}^4) \mathbb{R}.$$

Let us construct a $C^\infty$ pure and full deformation of $(\omega, \bar{J}_0)$ depending on four real parameters and corresponding to the matrix

$$J_t = (I + L_t)J_0(I + L_t)^{-1}$$

with respect to the dual basis $\{f^1, \ldots, f^4\}$, where $I$ is the identity matrix of order 4, $J_0$ is the matrix

$$J_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and

$$L_t = \begin{pmatrix} ta & tb & 0 & 0 \\ tb & -ta & 0 & 0 \\ 0 & 0 & tp & tq \\ 0 & 0 & tq & -tp \end{pmatrix},$$

with $t, a, b, p, q \in \mathbb{R}$ and satisfying

$$2t^2(a^2 + b^2 + p^2 + q^2) < 1.$$

A direct computation gives

$$J_t = \begin{pmatrix} -\frac{2tb}{t^2(a^2 + b^2) - 1} & \frac{(ta+1)^2 + t^2b^2}{t^2(a^2 + b^2) - 1} & 0 & 0 \\ \frac{(ta+1)^2 + t^2b^2}{t^2(a^2 + b^2) - 1} & -\frac{2tb}{t^2(a^2 + b^2) - 1} & 0 & 0 \\ 0 & 0 & -\frac{2tq}{t^2(p^2 + q^2) - 1} & \frac{(tp+1)^2 + t^2q^2}{t^2(p^2 + q^2) - 1} \\ 0 & 0 & \frac{(tp+1)^2 + t^2q^2}{t^2(p^2 + q^2) - 1} & -\frac{2tq}{t^2(p^2 + q^2) - 1} \end{pmatrix}.$$
Then, by [3] (see Section 2) the almost complex structure $J_t$ is $\omega$-calibrated. Moreover, we obtain

$$H^2(\tilde{M}^4, \mathbb{R}) = H^{1,1}_{J_t}(\tilde{M}^4)_{\mathbb{R}},$$

i.e. $J_t$ is a curve of $C^\infty$ pure and full almost complex structures on $\tilde{M}^4$, calibrated by the symplectic form $\omega$. As a consequence of Theorem 3.7, $J_t$ is a curve of pure and full almost complex structures.

6.3. A 6-dimensional solvmanifold. Consider the completely solvable Lie algebra $\mathfrak{s}$ with structure equations

$$\begin{align*}
    df^1 &= 0, \\
    df^2 &= -f^{12}, \\
    df^3 &= f^{34}, \\
    df^4 &= 0, \\
    df^5 &= f^{15}, \\
    df^6 &= f^{46}.
\end{align*}$$

The Lie algebra is a direct sum of two copies of the 3-dimensional solvable Lie algebra $\mathfrak{so}(3)$ and by [1] p. 20] the corresponding simply connected Lie group $S$ admits a compact quotient $\tilde{M}^6 = \Gamma \backslash S$.

By Hattori’s Theorem [13] we have that the de Rham cohomology of the solvmanifold $\tilde{M}^6$ is isomorphic to the Chevalley-Eilenberg cohomology of the Lie algebra $\mathfrak{s}$. In particular

$$H^2(\tilde{M}^6, \mathbb{R}) = \mathbb{R} \langle [f^{14}], [f^{25}], [f^{36}] \rangle.$$ 

Consider the almost complex structure $J_0$ defined by the $(1,0)$-forms

$$\varphi^1 = f^1 + if^{4}, \varphi^2 = f^2 + if^{5}, \varphi^3 = f^3 + if^{6}.$$ 

Then $J_0$ is almost-Kähler with respect to

$$\omega = f^{14} + f^{25} + f^{36}$$

and by [10] $(\tilde{M}^6, J_0, \omega)$ satisfies the Hard Lefschetz condition. Moreover, $H^2(\tilde{M}^6, \mathbb{R}) = H^{1,1}_{J_0}(\tilde{M}^6)_{\mathbb{R}}$, since $f^{14}, f^{25}, f^{46}$ are of type $(1,1)$ with respect to $J_0$.

Define the family of almost complex structure corresponding to the matrix

$$J_t = (I + L_t)J_0(I + L_t)^{-1}$$

with respect to the basis $(f^1, \ldots, f^6)$, where $I$ is the identity matrix of order 3, $J_0$ is the matrix

$$J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

and $L_t$ is the real symmetric matrix of order 6 given by

$$L_t = \begin{pmatrix} 0 & tI \\ tI & 0 \end{pmatrix}$$

and such that $6t^2 < 1$. Then, by [3] (see Section 2) the almost complex structure $J_t$ is $\omega$-calibrated. Consequently,

$$J_t = \begin{pmatrix} 2t \frac{1}{1-t^2}I & -\frac{1+t^2}{1-t^2}I \\ \frac{1-t^2}{1+t^2}I & -2t \frac{1}{1-t^2}I \end{pmatrix}$$

and

$$\omega = \frac{1}{1-t^2}I.$$
is a curve of almost-Kähler structures on $M$. Any almost complex structure $J_t$ is $C^\infty$ pure, since it is $\omega$-calibrated. Moreover, a basis of $(1,0)$-forms for $J_t$ is given by

$$
\varphi_t^1 = f^1 + i \left( \frac{2i}{1-t^2} f^1 + \frac{1+t^2}{1-t^2} f^4 \right),
$$

$$
\varphi_t^2 = f^2 + i \left( \frac{2i}{1-t^2} f^2 + \frac{1+t^2}{1-t^2} f^5 \right),
$$

$$
\varphi_t^3 = f^3 + i \left( \frac{2i}{1-t^2} f^3 + \frac{1+t^2}{1-t^2} f^6 \right).
$$

Then $J_t$ is a family of $C^\infty$ pure and full almost complex structures, that it is actually pure and full, since $\varphi_t^1 \wedge \overline{\varphi_t^1}, \varphi_t^2 \wedge \overline{\varphi_t^2}, \varphi_t^3 \wedge \overline{\varphi_t^3}$ are harmonic (see Theorem 3.7).

Consider the family $\tilde{J}_t$ of almost complex structures defined by

$$
\tilde{J}_t = J_0(I + \bar{L}_t)(I - L_t)^{-1},
$$

where $I$ is the identity matrix of order 6 and $\bar{L}_t$ is the non symmetric matrix given by

$$
\bar{L}_t = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & t & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
t & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix}
$$

Since $L_t J_0 = - J_0 \bar{L}_t$, we have that $\tilde{J}_t$ is a family of $\omega$-tamed almost complex structures defined by the $(1,0)$-forms

$$
\tilde{\varphi}_t^1 = f^1 + i (-2t f^2 + f^4),
$$

$$
\tilde{\varphi}_t^2 = f^2 + if^5,
$$

$$
\tilde{\varphi}_t^3 = f^3 + if^6.
$$

Moreover, $\tilde{J}_t$ is a family of pure and full almost complex structures.

7. The deformations $J_\alpha$

In [14] it was shown that on a almost-Kähler manifold $M$ of real dimension $2n$ there is natural infinite-dimensional family of almost complex structures $J_\alpha$ parametrized by the $J$-anti invariant real 2-forms on $M$. Indeed, given a $2n$-dimensional almost-Kähler manifold $(M, J, \omega, g)$ one may consider as in [14] a natural infinite dimensional family of almost complex structures parametrized by differential forms in $\Omega^{(2,0),(0,2)}(M)_\mathbb{R}$. Each form $\alpha$ in $\Omega^{(2,0),(0,2)}(M)_\mathbb{R}$ defines an endomorphism $K_\alpha$ of the tangent bundle $TM$ by

$$
g(X, K_\alpha Y) = \alpha(X, Y),
$$

for any pair of vector fields $X$ and $Y$. Moreover, we have (see [14])

$$
g(K_\alpha X, Y) = - g(X, K_\alpha Y), \quad JK_\alpha = - K_\alpha J, \quad g(JX, K_\alpha X) = 0,
$$

for any $X$ and $Y$. Since $JK_\alpha$ is skew-adjoint for each $\alpha \in \Omega^{(2,0),(0,2)}(M)_\mathbb{R}$, we have that $Id + JK_\alpha$ is invertible and hence

$$
J_\alpha = (Id + JK_\alpha)^{-1} J(Id + JK_\alpha)
$$

(6)
is an almost complex structure on $M$. Moreover, by [14] Proposition 1.5, $J_\alpha$ satisfies the following condition:

(7) $g(J_\alpha X, J_\alpha Y) = g(X, Y)$.

Moreover, if $n = 2$, since in this case $K_\alpha^2 = -||\alpha||^2 Id$, then we have that

$$J_\alpha = \frac{1 - ||\alpha||^2}{1 + ||\alpha||^2} J - \frac{2}{1 + ||\alpha||^2} K_\alpha$$

(see [14] Proposition 1.5) and if we denote by $J_\alpha^*$ the adjoint of $J_\alpha$ we have that the symmetric part of $J_\alpha^*J$ is $\frac{1-||\alpha||^2}{1+||\alpha||^2}Id$ and

$$J_\alpha^*J = J + 4 \frac{||\alpha||^2 - 1}{(1 + ||\alpha||^2)^2} K_\alpha.$$

Therefore

1. $J_\alpha$ is $\omega$-tamed if and only $||\alpha||^2 < 1$;
2. $J_\alpha$ is $\omega$-calibrated if and only $\alpha = 0$ or $||\alpha||^2 = 1$.

In this section we want to study the deformations $J_\alpha$ of a pure and full almost-Kähler structure $J$. Following [14], let us recall some general computations in dimension 4 in order to apply these to the solvmanifold $M^4$ and to the nilmanifold $\tilde{M}^4$ constructed respectively in section 6.1 and 6.2.

Both these examples have an almost-Kähler structure $(J_0, \omega)$ with $J_0$ pure and full.

Let $M$ be a 4-dimensional almost-Kähler manifold $(M, J, g, \omega)$. Locally, we may assume that the almost complex structure $J$ is given with respect to an orthonormal coframe $(e_1^1, \ldots, e_4^4)$ by

$$J = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}.$$

Let $\alpha$ be any real form in the space $\Omega^{(2,0),(0,2)}(M)$. Then the 2-form

$$\alpha = a(e^{13} - e^{24}) + b(e^{14} + e^{23})$$

defines the endomorphism $K_\alpha$ given with respect to the orthonormal coframe $(e_1^1, \ldots, e_4^4)$ by

$$K_\alpha = \begin{pmatrix}
0 & 0 & a & b \\
0 & 0 & b & -a \\
-a & -b & 0 & 0 \\
-b & a & 0 & 0
\end{pmatrix}$$

with $a, b \in \mathbb{R}$. Then we can consider on $M$ the almost complex structure $J_\alpha$ given by (8):

$$J_\alpha = \frac{1}{1 + ||\alpha||^2} \begin{pmatrix}
0 & -1 + ||\alpha||^2 & -2a & -2b \\
1 - ||\alpha||^2 & 0 & -2b & 2a \\
2a & 2b & 0 & -1 + ||\alpha||^2 \\
2b & -2a & 1 - ||\alpha||^2 & 0
\end{pmatrix},$$

where $||\alpha||^2 = a^2 + b^2$.

The associated $(1,0)$-forms are:

$$\varphi^1 = e^1 + \frac{1}{1 + ||\alpha||^2} \left( (1 - ||\alpha||^2)e^2 + 2ae^3 + 2be^4 \right),$$

$$\varphi^2 = e^3 + \frac{1}{1 + ||\alpha||^2} \left( -2ae^1 + 2be^2 + (1 - ||\alpha||^2)e^4 \right).$$

Therefore, we can prove the following
Proposition 7.1. The almost-Kähler solvmanifold \((M^4, J_0, \omega)\) and the almost-Kähler nilmanifold \((\tilde{M}^4, J_0, \omega)\) constructed in section 6.1 and 6.2 admit a family of pure and full almost complex structures \(J_\alpha\) parametrized by the left-invariant forms \(a(e_{13} - e_{24})\), with \(a \notin \{-1, 0, 1\}\), where \((e_i)\) is the basis of left-invariant forms satisfying (4) in the case of \(M^4\) and

\[
\begin{align*}
\text{in the case of } \tilde{M}^4.
\end{align*}
\]

Proof. For both almost-Kähler manifolds \(M^4\) and \(\tilde{M}^4\) we have

\[
\omega = e_{12} + e_{34}
\]

and \(J_0 e_1 = e_2, J_0 e_3 = e_4\).

Consider the almost complex structure \(J_\alpha\), given by (8) with \(b = 0, a \neq \pm 1, a \neq 0\), then it turns out that \(J_\alpha\) is pure and full in both cases.

In the case of \(M^4\), we have

\[
i(\varphi^1 \wedge \varphi^3) = \frac{1}{(1+|a|^2)}(-4ae^{13} + 2(1 - |a|^2)e^{12})
i(\varphi^2 \wedge \varphi^2) = \frac{1}{(1+|a|^2)}(-4ae^{13} - 2(1 - |a|^2)e^{34})
\]

and for the nilmanifold \(\tilde{M}^4\)

\[
i(\varphi^1 \wedge \varphi^2 + \varphi^1 \wedge \varphi^2) = \frac{2(-1+||a||^2)}{(1+||a||^2)}(e^{23} - e^{14})
i(\varphi^1 \wedge \varphi^2 - \varphi^2 \wedge \varphi^2) = \frac{2(-1+||a||^2)}{(1+||a||^2)}(e^{23} + e^{14})
\]

□

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