Symplectic analysis of three dimensional Abelian topological gravity

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A detailed Faddeev-Jackiw quantization of an Abelian topological gravity is performed; we show
that this formalism is equivalent and more economical than Dirac’s method. In particular, we
identify the complete set of constraints of the theory, from which the number of physical degrees
of freedom is explicitly computed. We prove that the generalized Faddeev-Jackiw brackets and the
Dirac ones coincide to each other. Moreover, we perform the Faddeev-Jackiw analysis of the theory
at the chiral point, and the full set of constraints and the generalized Faddeev-Jackiw brackets are
constructed. Finally we compare our results with those found in the literature and we discuss some
remarks and prospects.

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I. INTRODUCTION

It is well-known that the unification between quantum mechanics and gravity is a difficult task
to perform. The two promising approaches for solving the problems that emerge in the quantum
formulation of gravity, namely, string theory and loop quantum gravity are, unless still now, in
progress [1–4]. Due to the difficulties found in the quantum formulation of gravity, it is convenient
try to study the classical and the quantum formulation of toy models for testing ideas about actual
gravity theories. In this respect, there are a lot of interesting models that have been useful for
this aim, as for instance, three dimensional gravity [5–8], topological formulations of gravity in
three and four dimensions and the well-known Abelian gravity theories [10–13]. With respect to
three dimensional gravity, there is a toy model very close to pure gravity, the so-called topologically
massive gravity [TMG] [14–20]. This model describes the propagation of a massive graviton and
black hole solutions obeying the laws of black hole thermodynamics, is Poincaré gauge invariant and
also provides at the chiral point a description of the structure of an $AdS_3$ asymptotic spacetime
[21]. However, in spite of the works found in the literature on this model, the classical and quantum
analysis is still a subject of research. In fact, the classical and quantum study of this model is also
a difficult task to perform; from the point of view of Dirac’s approach, the correct identification of
the constraints between first and second class is a difficult work to carry out [22] and we know the
correct identification of the constraints is both the best guideline to carry out the correct counting of physical degrees of freedom and the first step to perform the quantum treatment of the theory. From the quantum point of view TMG with negative cosmological constant is unstable, massive gravitons and black holes have negative energy. An exception is present when the theory is analyzed at the chiral point, where the black holes and gravitons have non-negative masses and the linearized gravitational excitations around $AdS_3$ is stable [22]. In this manner, new developments focussed to the classical and the quantum analysis of TMG by using a different framework to the Dirac one come to be relevant. In this respect, in the present paper we study an Abelian analog of TMG by using the Faddeev-Jackiw [FJ] method being a powerful alternative approach for studying singular systems [23–30]. In fact, the FJ method is a symplectic approach, namely, all relevant information of the theory can be obtained through an invertible symplectic tensor, which is constructed by means of the symplectic variables that are identified as the degrees of freedom. Because of the theory is singular there will be constraints, and FJ has the advantage that all constraints are at the same footing, since it is not necessary to perform the classification of the constraints in primary, secondary, first class or second class as in Dirac’s method is done [31–42]. When the symplectic tensor is obtained, then its components are identified with the FJ generalized brackets; Dirac’s brackets and FJ brackets coincide to each other. On the other hand, we have chosen to analyze the Abelian analog of TMG because it is well-known that Abelian models of gravity in four and three dimensions have interesting features. In fact, in the four dimensional case, there exists the so-called $SU(2)$ gravity (or Husain-Kuchar gravity) [43]; the model itself is closely related to Ashtekar’s formulation of general relativity in terms of the new canonical variables. Moreover, $SU(2)$ gravity has three degrees of freedom per space-time point, and is devoid of the Hamiltonian constraint, and this fact makes the theory more easy to quantize where the physical states for general relativity form a subset of the states of the theory under study. On the other hand, an alternative description for Abelian gravity has been proposed in [44]. In fact, in [44] the model is a $U(1)$ gauge invariant theory, with zero degrees of freedom and reducibility among the constraints, and the quantum formulation shares the same path-integral formula with that of $BF$ theory. Furthermore, an Abelian description of gravity is obtained in the limit $G \to 0$ of the four dimensional Palatini’s theory [45, 46]. In fact, the theory presents reducible constraints and lacks physical degrees of freedom. With respect to the three dimensional case, we find in the literature the so-called 2+1 gravity without dynamics [47], describing a generally covariant topological theory without Hamiltonian constraint. In fact, the theory only presents the spatial diffeomorphism constraint and the quantum description can be carried out in an elegant form by using the loop representation. Finally, we find also in the literature the 2+1 Abelian topological massive gauge theories, where the so-called SD model and Maxwell Chern-Simons theory are the subjects of several works (see [48] and cites there in). In this manner, with the antecedents commented above, in this paper the FJ analysis of an Abelian version of TMG is performed. We find the full set of constraints and the generalized FJ brackets are constructed. Furthermore, we study the theory at the chiral point and by using the symplectic approach the symmetries of the theory are revealed. Finally, a pure Dirac’s method applied to TMG
at the chiral point is added. We report the complete set of first and second class constraints, then we construct the fundamental Dirac’s brackets and we show that the Dirac and FJ brackets coincide to each other.

II. SYMPLECTIC FORMALISM OF THREE DIMENSIONAL ABELIAN TOPOLOGICAL GRAVITY

As it was commented above, the model that we shall study in this section is an Abelian version of TMG. In spite of TMG has been analyzed in the context of Dirac’s approach, in the process one finds several problems in order to identify which constraints of the theory are first class or second class [22]. In this manner, in this section we will study a toy model for testing ideas about classical gravity that could be useful either in the three dimensional or four-dimensional gravity theory. Hence, our laboratory is given by an Abelian version of TMG, and our tool is the symplectic approach developed by FJ for revealing the fundamental symmetries and the constraints of the theory. We start from the well-known TMG action given by [20]

\[ S[A, e, \lambda] = \int_{\mathcal{M}} \left[ 2e^i \wedge F_i[A] + \lambda^i \wedge T_i + \frac{1}{\mu} A^i \wedge \left( dA_i + \frac{G}{3} f_{ijk} A^j \wedge A^k \right) \right], \]  

(1)

where \( A^i = A^i_\mu dx^\mu \) is a connection 1-form valued on the adjoint representation of the Lie group \( SO(2,1) \), which admits an invariant totally anti-symmetric tensor \( f_{ijk} \), \( e^i = e^i_\mu dx^\mu \) is a triad 1-form that represents the gravitational field and \( F^i \) is the curvature 2-form of the connection \( A^i \), i.e., \( F_i = da_i + \frac{G}{3} f_{ijk} A^j \wedge A^k \) and \( G \) is the gravitational coupling constant. Finally, \( \lambda^i \) are Lagrange multiplier 1-forms that ensure that the torsion vanishes \( T_i \equiv de_i + G f_{ijk} A^j \wedge e^k = 0 \); \( x^\mu \) are the coordinates that label the points of the 3-dimensional manifold \( \mathcal{M} \). In our notation, Greek letters are indices for the spacetime and run from 0 to 2, and \( a, b, c = 1, 2 \) are space indices, the middle latin alphabet letters \( (i,j,k,...) \) are associated with the internal group \( SO(2,1) \) and run from 1 to 3.

The gravitational coupling constant has been introduced in order to take the \( G \rightarrow 0 \) limit, and to obtain an Abelian version of TMG, something similar can be found in [46] where the FJ analysis of an Abelian analog of four-dimensional Palatini’s theory was reported. By taking the \( G \rightarrow 0 \) limit we obtain the following action

\[ S[A, e, \lambda] = \int_{\mathcal{M}} \left[ 2e^i \wedge F_i[A] + \lambda^i \wedge T_i + \frac{1}{\mu} A^i \wedge dA_i \right], \]  

(2)

where \( F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a \), \( T^i_{ab} = \partial_a e^i_b - \partial_b e^i_a \) and now the dynamical variables are a collection of \( U(1) \) gauge invariant fields.

The equations of motion obtained from the action [22] are given by

\[ e^{\alpha\nu\rho} (2F_{\nu\rho}^i + \partial_{\nu\lambda}^i a^i = 0, \]

\[ e^{\alpha\nu\rho} (2T_{\nu\rho}^i + \epsilon_{ijk}^i \lambda_{\nu}^j e_{\rho}^k + 2\mu^{-1} F_{\nu\rho}^i) = 0, \]

\[ e^{\alpha\nu\rho} T_{\nu\rho}^i = 0, \]  

(3)
we can see that the equation of motion related with the torsion $\partial_\alpha e^i_\beta - \partial_\beta e^i_\alpha = 0$, implies that $e^i_\alpha = \partial_\alpha f^i$, thus, the background scenario corresponds locally to Minkowski spacetime, the model shares similarities with the Abelian version of Palatini’s theory reported in [43].

On the other hand, by performing the $2+1$ decomposition of the fields, the action takes the following form

$$\mathcal{L}^{(0)} = \int \left( 2 \varepsilon^{ab} e^i_b \dot{A}_{ai} + \frac{1}{\mu} \varepsilon^{ab} A^i_b \dot{A}_{ai} + \varepsilon^{ab} \lambda_{ibi} \dot{e}^i_a - V^{(0)} \right) dx^3,$$

(4)

where $\varepsilon^{0ab} \equiv \varepsilon^{ab}$ is the antisymmetric tensor and $V^{(0)} = -e^i_0 \left[ F_{abi} e^{ab} + \varepsilon^{ab} \partial_a \lambda_{bi} \right] - A^{i0} [ \varepsilon^{ab} T_{abi} + \frac{1}{\mu} \varepsilon^{ab} F_{abi} ] - \frac{\Lambda^0}{2} [ \varepsilon^{ab} T_{abi} ]$ is identified as the symplectic potential (see the appendix B). Hence, in the following lines we will study the action [44] within the context of Faddeev-Jackiw. In order to perform this aim, from [44] we identify the following symplectic variables $(\xi^A) = (A^i_a, A^{i0}, e^i_a, e^0, \lambda^i_a, \lambda^0_a)$ and the 1-form $\omega_{AB} = \left( 2 \varepsilon^{ab} e^i_b + \frac{1}{\mu} \varepsilon^{ab} A^i_b, 0, \varepsilon^{ab} \lambda_{ibi}, 0, 0, 0 \right)$, here $A, B, C = 1, 2, 3$,... label the number of field variables, for instance, there are 27 field variables and all of them are represented by $(\xi^A)$. Thus, by taking into account these symplectic variables the equations of motion of the action [44] are given by

$$f^{(0)}_{AB} = \frac{\partial V^{(0)}(\xi)}{\partial \xi^A},$$

(5)

where the symplectic matrix $f^{(0)}_{AB}$ takes the form

$$f^{(0)}_{AB}(x, y) = \frac{\delta a_B(y)}{\delta \xi^A(x)} - \frac{\delta a_A(x)}{\delta \xi^B(y)},$$

(6)

and it is given explicitly by

$$f^{(0)}_{AB} = \begin{pmatrix}
  2 \mu \varepsilon^{ab} \eta_{ij} & 0 & -2 \varepsilon^{ab} \eta_{ij} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  2 \varepsilon^{ab} \eta_{ij} & 0 & 0 & 0 & -\varepsilon^{ab} \eta_{ij} & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & \varepsilon^{ab} \eta_{ij} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \delta^3(x - y),$$

(7)

we can observe that this matrix is singular and therefore there will constraints. Because of the symplectic matrix is singular, it has the following null-vectors $\nu_1 = \left( 0, v^{A_0}, 0, 0, 0, 0 \right)$, $\nu_2 = \left( 0, 0, 0, v^{e_0}, 0, 0 \right)$ and $\nu_3 = \left( 0, 0, 0, 0, v^{\lambda_0}, 0 \right)$, where $v^{A_0}, v^{e_0}, v^{\lambda_0}$ are arbitrary functions. From those modes, we obtain the following constraints [45]

$$\Omega_i^{(0)} = \int dx^2 \nu_i \frac{\delta}{\delta \xi^A} \int dy^2 V^{(0)}(\xi) = \varepsilon^{ab} T_{abi} + \frac{1}{\mu} \varepsilon^{ab} F_{abi} = 0,$$

(8)

$$\beta_i^{(0)} = \int dx^2 \nu_i \frac{\delta}{\delta \xi^A} \int dy^2 V^{(0)}(\xi) = \varepsilon^{ab} F_{abi} + \varepsilon^{ab} \partial_a \lambda_{bi} = 0,$$

(9)

$$\Sigma_i^{(0)} = \int dx^2 \nu_i \frac{\delta}{\delta \xi^A} \int dy^2 V^{(0)}(\xi) = \varepsilon^{ab} T_{abi} = 0.$$  

(10)

In order to observe if there are more constraints, we construct the following matrix (see Appendix B)

$$\tilde{f}_{AB} \xi^A = Z_B(\xi),$$

(11)
where

\[ \tilde{f}_{AB} = \left( \frac{f^{(0)}_{AB}}{\delta \xi^A} \right), \quad (12) \]

and

\[ Z_A(\xi) = \left( \frac{\partial V^{(0)}}{\partial \xi^A} \right), \]

\[ \quad (13) \]

hence \( \tilde{f}_{AB} \) is given by

\[ \tilde{f}_{AB} = \begin{pmatrix}
\frac{2}{\mu} \varepsilon^{ab} \eta_{ij} & 0 & -2 \varepsilon^{ab} \eta_{ij} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
2 \varepsilon^{ab} \eta_{ij} & 0 & 0 & 0 & -\varepsilon^{ab} \eta_{ij} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \varepsilon^{ab} \eta_{ij} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{\mu} \eta_{ij} \varepsilon^{ab} \partial_a & 0 & 2 \eta_{ij} \varepsilon^{ab} \partial_a & 0 & 0 & 0 \\
2 \varepsilon^{ab} \partial_a \delta_{ij} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \delta_{ij} 2 \varepsilon^{ab} \partial_a & 0 & 0 & 0
\end{pmatrix} \delta^2(x - y), \quad (14) \]

the matrix (14) is not a square matrix and still has null vectors. The modes of the matrix (14) are given by

\[ \tilde{V}^A_1 = \left( \partial_a V^\lambda, V^\lambda \epsilon_{i}, 0, 0, 0, V^\lambda, 0, 0 \right), \quad (15) \]
\[ \tilde{V}^A_2 = \left( 0, 0, 0, V^\lambda \epsilon_{i}, 2 \partial_a V^\lambda, 0, 0, 0, -V^\lambda \right), \quad (16) \]
\[ \tilde{V}^A_3 = \left( 0, 0, \partial_a V^\lambda, 0, 0, V^\lambda \epsilon_{i}, 0, V^\lambda, 0 \right). \quad (17) \]

On the other hand, \( Z_A \) takes the form

\[ Z_A = \begin{pmatrix}
\varepsilon^{ab} \partial_a e_{0i} + \frac{1}{\mu} \varepsilon^{ab} \partial_a A_{0i} \\
- \varepsilon^{ab} T_{abi} + \frac{1}{\mu} \varepsilon^{ab} F_{abi} \\
\varepsilon^{ab} \partial_a A_{0i} + \varepsilon^{ab} \partial_a \lambda_{0i} \\
- F_{abi} \varepsilon^{ab} + \varepsilon^{ab} \partial_a \lambda_{bi} \\
\varepsilon^{ab} \partial_a e_{0i} \\
- \frac{\varepsilon^{ab}}{2} T_{abi} \\
0 \\
0 \\
0
\end{pmatrix}. \]
The contraction of the null vectors with $Z_A$, namely, $\tilde{V}^A Z_A|_{\Omega^{(0)}, \beta_i^{(0)}, \Sigma^{(0)} = 0} = 0$ yields identities. Hence, there are not more constraints in the theory (see the Appendix B). With this information, we construct a new symplectic Lagrangian given by

$$
\mathcal{L}^{(1)} = [2\varepsilon^{ab}e_{bi} + \frac{1}{\mu}\varepsilon^{ab} A_{bi}] \dot{A}_i^a + \varepsilon^{ab}\lambda_{bi} \dot{e}_i^a - [\varepsilon^{ab} F_{abi} + \varepsilon^{ab} \partial_a \lambda_{bi}] \dot{\alpha}^i - [\varepsilon^{ab} T_{abi} + \frac{1}{\mu}\varepsilon^{ab} F_{abi}] \dot{\beta}^i - [\varepsilon^{ab} T_{ab}] \dot{\Gamma}^i - V^{(1)},
$$

where we have taken $e_i^0 = \dot{\alpha}^i, A_i^0 = \dot{\beta}^i, \lambda_i^0 = \dot{\Gamma}^i$ as a set of Lagrange multipliers enforcing the constraints with a vanishing potential $V^{(1)} = V^{(0)}|_{\Omega^{(0)} = 0, \beta_i^{(0)} = 0, \Sigma^{(0)} = 0} = 0$, which reflects the general covariance of the theory. In this manner, from (18) we identify the following symplectic variables $(1) \xi^A = (A_i^a, \beta^i, e_i^a, \alpha^i, \lambda_i^a, \Gamma^i)$ and the 1-forms $(1) A^a_i = \left(2\varepsilon^{ab}e_{bi} + \frac{1}{\mu}\varepsilon^{ab} A_{bi}, -\varepsilon^{ab} T_{abi} - \frac{1}{\mu}\varepsilon^{ab} F_{abi}, \varepsilon^{ab}\lambda_{bi}, -\theta e^{ab} F_{abi} - \varepsilon^{ab} \partial_a \lambda_{bi}, 0, \varepsilon^{ab} T_{ab} \right)$. By using these symplectic variables, we find the following symplectic matrix

$$
f_{AB}^{(1)} = 
\begin{pmatrix}
2\varepsilon^{ab}\eta_{ij} & -2\varepsilon^{ab}\eta_{ij} & -2\varepsilon^{ab}\eta_{ij} & 0 & 0 \\
-2\varepsilon^{ab}\eta_{ij} & 0 & -2\eta_{ij} \varepsilon^{ab} \partial_a & 0 & 0 \\
2\eta_{ij} \varepsilon^{ab} \partial_a & 0 & 0 & \eta_{ij} \varepsilon^{ab} \partial_a & 0 \\
0 & 0 & -\varepsilon^{ab}\eta_{ij} & -\eta_{ij} \varepsilon^{ab} \partial_a & 0 \\
0 & 0 & -2\eta_{ij} \varepsilon^{ab} \partial_a & 0 & 0 \\
\end{pmatrix}
\times \delta^2(x - y). \tag{19}
$$

Where we can observe that this matrix is singular. However, we have shown that there are not more constraints, hence the system has a gauge symmetry. In fact, the gauge symmetry is encoded in the null vectors of the matrix (19). It is straightforward to see that the following Abelian gauge symmetries are obtained from the null vectors of the above matrix

$$
A_\mu^i \rightarrow A_\mu^i + \partial_\mu \kappa^i, \\
e^i_\mu \rightarrow e^i_\mu + \partial_\mu \tilde{\kappa}^i, \tag{20}
$$

where $\kappa^i$ and $\tilde{\kappa}^i$ are gauge parameters. Hence, in order to obtain a symplectic tensor, we will fix the gauge, and we choose the temporal gauge

$$
A_i^0 = 0, \\
e_i^0 = 0, \\
\lambda_i^0 = 0,
$$

this means that $\beta^i = cte, \alpha^i = cte$ and $\Gamma^i = cte$. In this manner, with that information we construct the following new symplectic Lagrangian

$$
\mathcal{L}^{(2)} = [2\varepsilon^{ab}e_{bi} + \frac{1}{\mu}\varepsilon^{ab} A_{bi}] \dot{A}_i^a + \varepsilon^{ab}\lambda_{bi} \dot{e}_i^a - [\varepsilon^{ab} F_{abi} + \varepsilon^{ab} \partial_a \lambda_{bi}] \dot{\alpha}^i + \alpha^i \dot{\rho}_i \\
- [\varepsilon^{ab} T_{abi} + \frac{1}{\mu}\varepsilon^{ab} F_{abi}] \dot{\beta}^i - \beta^i \dot{\varphi}_i - [\varepsilon^{ab} T_{ab}] \dot{\Gamma}^i - \Gamma^i \dot{\lambda}^i, \tag{21}
$$
where we have introduced new Lagrange multipliers enforcing the gauge fixing, namely, \( \varpi_i, \Gamma^i, r^i \). Hence, from (21) we identify the following symplectic variables \( \xi^A \) and the 1-forms \( \alpha^B = (2\varepsilon^{ab}e_{bi} + \frac{1}{\mu}e^{ab}A_{bi} - \varepsilon^{ab}T_{abi} - \frac{1}{\mu}e^{ab}F_{abi} + \varpi_i, e^{ab}\lambda_{bi} - e^{ab}F_{abi} - e^{ab}\partial_a\lambda_{bi} + \rho_i, 0, -e^{ab}T_{abi} + r, 0, 0, 0, 0) \). In this manner, the symplectic matrix reads

\[
\begin{align*}
\left( \begin{array}{cccccc}
\frac{2}{\mu}e^{ab}\eta_{ij} & -\frac{2}{\mu}\delta_{ij}e^{ab}\partial_a & -2\varepsilon^{ab}\eta_{ij} & -2\eta_{ij}e^{ab}\partial_a & 0 & 0 & 0 & 0 & 0 \\
2\eta_{ij}e^{ab}\partial_a & 0 & -2\eta_{ij}e^{ab} & 0 & 0 & 0 & -\eta^j_i & 0 \\
2\eta_{ij}e^{ab} & 2\eta_{ij}e^{ab} & 0 & 0 & -\varepsilon^{ab}\eta_{ij} & -2\varepsilon^{ab}\partial_a & 0 & 0 & 0 \\
0 & 0 & \varepsilon^{ab}\eta_{ij} & -\eta_{ij}e^{ab}\partial_a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2\varepsilon^{ab}\partial_a & 0 & 0 & 0 & 0 & -\eta^i_j & 0 \\
0 & 0 & 0 & \eta^i_j & 0 & 0 & 0 & 0 & 0 \\
0 & \eta^i_j & 0 & 0 & 0 & 0 & \eta^i_j & 0 & 0 \\
0 & 0 & 0 & 0 & \eta^i_j & 0 & 0 & \eta^i_j & 0 \\
\end{array} \right) \times \delta^2(x - y).
\end{align*}
\]

We can observe that this matrix is not singular, hence, it is a symplectic tensor. The inverse of \( f_{AB} \) is given by

\[
\begin{align*}
\left( \begin{array}{cccccc}
\frac{1}{2}\varepsilon_{ab}\eta^{ij} & 0 & 0 & 0 & -\mu\eta^j_i\varepsilon_{ab} & 0 & 0 & \eta^j_i\partial_a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta^j_i & 0 \\
0 & 0 & 0 & 0 & \varepsilon_{ab}\eta^j_i & 0 & \eta^j_i\partial_a & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta^j_i & 0 \\
\mu\eta^j_i\varepsilon_{ab} & 0 & -\varepsilon_{ab}\eta^j_i & 0 & 2\mu\varepsilon_{ab}\eta^{ij} & 0 & 0 & 2\eta^j_i\partial_a & -2\eta^j_i\partial_a \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta^j_i & 0 \\
0 & 0 & \eta^j_i\partial_a & -\eta^j_i & 0 & 0 & 0 & 2\varepsilon^{ab}\partial_a & 0 \\
-\eta^j_i\partial_a & -\eta^j_i & 0 & 0 & -2\eta^j_i\partial_a & 0 & 2\eta^j_i\varepsilon_{ab}\partial_a & 0 & 0 \\
0 & 0 & 0 & 0 & 2\eta^j_i\partial_a & -\eta^j_i & 0 & 0 & 0 \\
\end{array} \right) \times \delta^2(x - y).
\end{align*}
\]

(23)

Where we can identify the following generalized FJ brackets

\[
\{\xi^A(x), \xi^B(y)\}_{FD} = [f_{AB}^{(2)}(x, y)]^{-1},
\]

(24)
thus, we obtain
\[
\begin{align*}
\{e^i_a, e^j_b\}_{FJ} &= 0, \\
\{A^i_a, A^j_b\}_{FJ} &= \frac{\mu}{2} \varepsilon_{ab} \eta^{ij} \delta^2(x - y), \\
\{A^i_a, \lambda^j_b\}_{FJ} &= -\mu \eta^{ij} \varepsilon_{ab} \delta^2(x - y), \\
\{\lambda^i_a, \lambda^j_b\}_{FJ} &= 2 \mu \varepsilon_{ab} \eta^{ij} \delta^2(x - y), \\
\{e^i_a, \lambda^j_b\}_{FJ} &= \varepsilon_{ab} \eta^{ij} \delta^2(x - y), \\
\{A^i_a, e^j_b\}_{FJ} &= 0.
\end{align*}
\]
\]
\[(25)\]

It is important to remark that the algebraic structure of these brackets coincide with those reported in [20], where the non-Abelian case was studied. Furthermore, the constraints are not reducible, which makes a difference with other models reported in the literature [44]. Moreover, the theory under study is a topological one. In fact, because in the FJ framework there is no difference between the constraints, namely, there does not exist a classification of the constraints in first class and second class as in Dirac’s method, in the FJ framework the counting of physical degrees of freedom is performed in the usual way; \(DF = \text{dynamical variables} - \text{FJ constraints}\). Thus, for the theory under study there are 18 canonical variables given by \((A^i_a, e^i_a, \lambda^i_a)\) and the following 18 FJ constraints \((\Omega_i^{(0)}, \beta_i^{(0)}, \Sigma_i^{(0)}, A^i_0, e^i_0, \lambda^i_0)\), thus, the theory is devoid of physical degrees of freedom.

### III. SYMPLECTIC FORMALISM FOR ABELIAN TOPOLOGICAL GRAVITY AT THE CHIRAL POINT

As it has been commented above, at the chiral point TMG is a theory with interesting properties: In fact, either black holes or gravitons have non-negative masses and the theory is dual to a holomorphic boundary CFT. The analysis of TMG at the chiral point, has been reported in [21] where a linearized perturbation around an \(AdS_3\) background has been used. Moreover, in [18] the theory beyond the linearized approximation was studied and a non-perturbative canonical analysis was performed. With the same spirit, in this work we will not use a perturbative analysis. We will show that the FJ analysis is equivalent and more economical than the Dirac one. At the end of the paper an Appendix developing a detailed Dirac’s analysis of Abelian TMG at the chiral point has been added, and we show that the Dirac and the generalized FJ brackets coincide to each other.

The action describing TMG can be written in an alternative way [18]
\[
I[e, \omega, \lambda] = \int \left[ 2e^i \wedge F_i[\omega] + \frac{1}{3!^2} \varepsilon_{ijk} e^i \wedge e^j \wedge e^k - \frac{1}{\mu} [\omega^i \wedge d\omega_i + \frac{1}{3} \varepsilon_{ijk} \omega^j \wedge \omega^j \wedge \omega^k] + (\lambda^i - \frac{e^i}{\mu^2}) \wedge T_i \right],
\]
\[(26)\]

where \(F_i[A] = dA_i + \frac{1}{2} f_{ijk} A^j \wedge A^k\) and \(T_i \equiv \text{de}_i + f_{ijk} A^j \wedge e^k\). Furthermore, it is well-known that under the following redefinition
\[
A^i = \omega^i + \frac{e^i}{T}, \quad \tilde{A}^i = \omega^i - \frac{e^i}{T},
\]
\[(27)\]
the action can be written in the following way

\[
I[A, \tilde{A}, \lambda] = \left(1 - \frac{1}{\mu^2}\right) \int \left[ A^i \wedge dA_i + \frac{1}{3} \epsilon_{ijk} A^i \wedge A^j \wedge A^k \right]
+ \int 2\lambda^i \wedge F_i[A] - \left(1 + \frac{1}{\mu^2}\right) \int \left[ \tilde{A}^i \wedge d\tilde{A}_i + \frac{1}{3} \epsilon_{ijk} \tilde{A}^i \wedge \tilde{A}^j \wedge \tilde{A}^k \right]
- \frac{1}{2} \int 2\lambda^i \wedge F_i[\tilde{A}].
\]

(28)

Now at the chiral point we take \( \mu^2 \ell^2 = 1 \), thus the Chern-Simons term related with the dynamical field \( A \) is removed and the action is simplified to

\[
I[A, \tilde{A}, \lambda] = \frac{1}{2} \int 2\lambda^i \wedge F_i[A] - \int \left[ \tilde{A}^i \wedge d\tilde{A}_i + \frac{1}{3} \epsilon_{ijk} \tilde{A}^i \wedge \tilde{A}^j \wedge \tilde{A}^k \right] - \frac{1}{2} \int 2\lambda^i \wedge F_i[\tilde{A}],
\]

(29)

where the \( \lambda \) field has taken the role of the tetrad-like field coupling two Einstein-Hilbert copies depending on the connections \( A \) and \( \tilde{A} \) or also it can be viewed as two \( BF \)-like copies, where the field \( \lambda \) take the role of the \( B \) field. The action (29) has been analyzed in [18] by using the Dirac method and it has been shown the the action describes the propagation of a physical bulk degree of freedom corresponding, at the linearized level, to the topologically massive graviton. Hence, in this section we will analyze the Abelian analog of the action (29) from the symplectic perspective in order to obtain in an easy way its symmetries.

The Abelian analog of action (29) is given by

\[
I[A, \tilde{A}, \lambda] = \frac{1}{2} \int 2\lambda^i \wedge F_i[A] - \int \left[ \tilde{A}^i \wedge d\tilde{A}_i + \frac{1}{3} \epsilon_{ijk} \tilde{A}^i \wedge \tilde{A}^j \wedge \tilde{A}^k \right] - \frac{1}{2} \int 2\lambda^i \wedge F_i[\tilde{A}],
\]

(30)

thus, by performing the 2+1 decomposition we obtain the following Lagrangian density

\[
\mathcal{L}^{(0)} = l\lambda_{ib}\dot{A}_a^i \epsilon^{0ab} - l(\lambda_{ib} \epsilon^{0ab} + \tilde{A}_a^i) \dot{A}_a^i - V^{(0)},
\]

(31)

where \( F_{iab} \) \( \det \) is defined as above \( F_{i\dot{a}b} = \partial_a A_i^b - \partial_b A_i^a \), \( \dot{F}_{iab} = \partial_a \dot{A}_b^i - \partial_b \dot{A}_a^i \) and \( V^{(0)} = -lA_0 \epsilon_{aib} \epsilon^{0ab} + l\dot{A}_0 \epsilon_{aib} \epsilon^{0ab} - \frac{1}{2} \lambda_{\dot{a}0} [F_{i\dot{a}b} - \dot{F}_{i\dot{a}b}] \epsilon^{0ab} \) is the symplectic potential of the theory. Hence, from (31) we identify the following symplectic variables \( \xi^A = (A_i^a, A_0^i, \dot{A}_a^i, \dot{A}_0^i, \lambda_{ia}, \lambda_{\dot{a}0}) \) and the 1-forms \( \alpha^A_B = (l\lambda_{ib} \epsilon^{0ab}, 0, -l(\lambda_{ib} \epsilon^{0ab} + \dot{A}_a^i) \lambda_{\dot{a}0}, 0, 0, 0) \). By using these symplectic variables, the symplectic matrix has the form

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & -l \epsilon^{0ab} \delta^i_j & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \epsilon^{0ab} \delta^i_j & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
l \epsilon^{0ab} \delta^i_j & 0 & -l \epsilon^{0ab} \delta^i_j & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\delta^2(x - y).
\]

This matrix is singular and has the following modes \( \mathcal{V}_1^{(0)} = (0, v^{A_i^a}, 0, 0, 0, 0) \), \( \mathcal{V}_2^{(0)} = (0, 0, 0, v^{A_0^i}, 0, 0) \), \( \mathcal{V}_3^{(0)} = (0, 0, 0, 0, 0, v^{\lambda_{\dot{a}0}}) \); by using these null vectors, we can identify the
following constraints
\[ \Omega^{(0)}_1 = \int dx^2 \left( V^{(0)}_1 \right)^A \frac{\delta}{\delta \xi^A} \int d^2 y V^{(0)}(\xi) = l \partial_a \lambda_{ib} \epsilon^{0ab} = 0, \]
\[ \Omega^{(0)}_2 = \int dx^2 \left( V^{(0)}_2 \right)^A \frac{\delta}{\delta \xi^A} \int d^2 y V^{(0)}(\xi) = l \{ \epsilon^{0ab} \tilde{F}_{iab} + \partial_a \lambda_{ib} \epsilon^{0ab} \} = 0, \]
\[ \Omega^{(0)}_3 = \int dx^2 \left( V^{(0)}_3 \right)^A \frac{\delta}{\delta \xi^A} \int d^2 y V^{(0)}(\xi) = \frac{\epsilon^{0ab}}{2} \left[ F_{iab} - \tilde{F}_{iab} \right] = 0. \]

Now, just as we have done in the above section, we will observe if there are more constraints. In order to archive this aim, we construct the system given in \([11], [12] \) and \([13] \), where \( \tilde{f}_{AB} \) is given by
\[
\tilde{f}_{AB} = \begin{pmatrix}
0 & 0 & 0 & 0 & -l \epsilon^{0ab} \delta^i_j & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2l \epsilon^{0ab} \eta_{ij} & 0 & l \epsilon^{0ab} \delta^i_j & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
l \epsilon^{0ab} \delta^i_j & 0 & -l \epsilon^{0ab} \delta^i_j & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & l \delta^i_j \epsilon^{0ab} \partial_a \\
0 & 0 & 0 & 0 & l \delta^i_j \epsilon^{0ab} \partial_a \\
-l \delta^i_j \epsilon^{0ab} \partial_b & 0 & l \epsilon^{0ab} \delta^i_j \partial_b & 0 & 0 & 0 \\
\end{pmatrix} \delta^2(x - y).
\]

This matrix is not a square matrix, but it still has zero modes. These zero modes are given by
\[
\tilde{V}^A_1 = \left( 0, V^{A',0}, 0, V^{A_i,0}, \partial_a V^i, V^{\lambda,0}, 0, 0, -V^j \right),
\]
\[
\tilde{V}^A_2 = \left( 0, V^{A',0}, -\partial_a V^j, V^{A_j,0}, 0, V^{\lambda,0}, 0, V^j, 0 \right),
\]
\[
\tilde{V}^A_3 = \left( \partial_a V^j, V^{A',0}, 0, V^{A_i,0}, 0, V^{\lambda,0}, V^j, 0, 0 \right).
\]

By performing the contraction of these zero modes with \( Z_A \) given by
\[
Z_A = \begin{pmatrix}
l \epsilon^{0ab} \partial_a \lambda_{i0} \\
l \epsilon^{0ab} \partial_a \lambda_{ib} \\
-\epsilon^{0ab} l \partial_a \lambda_{i0} + 2l \epsilon^{0ab} \partial_a \tilde{A}^i_0 \\
-\{ \tilde{F}_{iab} \epsilon^{0ab} + \partial_a \lambda_{ib} \epsilon^{0ab} \} \\
l \partial_a A^i_0 \epsilon^{0ab} + \partial_a \tilde{A}^i_0 \epsilon^{0ab} \\
\frac{1}{2} \left( F_{iab} - \tilde{F}_{iab} \right) \epsilon^{0ab} \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]
we find that the contraction yields identities. Hence, there are not more constraints. We can
observe that in the FJ formulation there are less constraints than in Dirac’s approach (see the
Appendix A), however, if in the Dirac method we eliminate the second class constraints, then the
FJ and Dirac approaches share equivalent results, the advantage of FJ method is that it is more
economical. The following step is to add all previous information in order to construct a new
symplectic Lagrangian

\[ L^{(1)} = l\lambda_{ib} \epsilon^{0ab} \dot{A}_i^a - l \left( \lambda_{ib} \epsilon^{0ab} + \dot{A}_b\epsilon^{0ab} \right) \dot{A}_a^i - \frac{l}{2} \dot{\Gamma}_i \left[ F_{iab} - \tilde{F}_{iab} \right] \epsilon^{0ab} - l\dot{\beta}^i \partial_a \lambda_{ib} \epsilon^{0ab} 
+ l\alpha^i \left[ \dot{F}_{iab} \epsilon^{0ab} + \partial_a \lambda_{ib} \epsilon^{0ab} \right] - V^{(1)}, \] (32)

where, \( \beta^i, \Gamma_i \) and \( \alpha^i \) are Lagrange multipliers enforcing the constraints and the
symplectic potential \( V^{(1)} = V^{(0)} \bigg|_{\Omega^{(0)}=0,\Omega^{(2)}=0,\Omega^{(3)}=0} = 0 \). By following
with the method, from the Lagrangian we identify the new set of symplectic variables given by \( \xi^A \)
\( (A^i_a, \beta^i, \tilde{A}_i^a, \alpha^i, \lambda_{ia}, \Gamma_i) \) and the 1-forms \( (\xi^A_B = \tilde{A}^i_a, \beta^i, \tilde{A}_i^a, \alpha^i, \lambda_{ia}, \Gamma_i) \)
and \( \tilde{F}_{iab} \epsilon^{0ab} + \partial_a \lambda_{ib} \epsilon^{0ab}, 0, -\frac{l}{2} \left( F_{iab} - \tilde{F}_{iab} \right) \epsilon^{0ab} \). By using these symplectic variables, we construct a new symplectic matrix, namely \( f^{(1)}_{ab} \), however, we will find that this symplectic matrix is still singular: again, this suggests that the theory has a
gauge symmetry. In fact, the gauge symmetry is given by the following Abelian transformations

\[ A^i_\mu \rightarrow A^i_\mu + \partial_\mu \kappa^i, \]
\[ \tilde{A}_\mu^i \rightarrow \tilde{A}_\mu^i + \partial_\mu \tilde{\kappa}^i, \] (33)

where \( \kappa^i \) and \( \tilde{\kappa}^i \) are gauge parameters. Furthermore, in order to obtain a symplectic tensor, we fix the following gauge

\[ A^i_0 = 0, \]
\[ \tilde{A}^i_0 = 0, \]
\[ \lambda_{i0} = 0, \]

this implies that \( \beta^i, \alpha^i \) and \( \Gamma_i \) are constants. By introducing the gauge fixing in the Lagrangian, we
construct a new symplectic Lagrangian

\[ L^{(2)} = l\lambda_{ib} \epsilon^{0ab} \dot{A}_i^a - l \left( \lambda_{ib} \epsilon^{0ab} + \dot{A}_b\epsilon^{0ab} \right) \dot{A}_a^i - \frac{l}{2} \dot{\Gamma}_i \left[ F_{iab} - \tilde{F}_{iab} \right] \epsilon^{0ab} - l\dot{\beta}^i \partial_a \lambda_{ib} \epsilon^{0ab} 
- \beta^i \dot{\omega}^i + l\dot{\alpha}^i \left( \dot{F}_{iab} \epsilon^{0ab} + \partial_a \lambda_{ib} \epsilon^{0ab} \right) - \alpha^i \dot{\rho}_i, \] (34)

where \( \dot{r}^i, \dot{\omega}_i \) and \( \dot{\rho}_i \) are Lagrange multipliers enforcing the gauge conditions. Now, we identify the
following symplectic variables \( \xi^A \) and the one-forms \( \xi^B = \tilde{A}^i_\mu, \beta^i, \tilde{A}_i^a, \alpha^i, \lambda_{ia}, \Gamma_i, \dot{r}^i, \dot{\omega}_i, \dot{\rho}_i \)
and \( \tilde{F}_{iab} \epsilon^{0ab} + \partial_a \lambda_{ib} \epsilon^{0ab}, 0, -\frac{l}{2} \left( F_{iab} - \tilde{F}_{iab} \right) \epsilon^{0ab}, -\Gamma_i, -\beta^i, -\alpha^i \). By using these symplectic variables, we obtain the following symplectic matrix
the theory lacks physical degrees of freedom and it is topological. Given by (35), we carry out the counting of physical degrees of freedom. There are 18 canonical variables for TMG and for TMG at the chiral point, in general, are different, as expected. On the other hand, in [18] the Dirac brackets between the dynamical variables for TMG were not reported, hence, from our results we expect that the generalized FJ brackets of TMG will share a similar structure just like that given in (35).

Finally, we carry out the counting of physical degrees of freedom. There are 18 canonical variables given by \((A_i^a, \tilde{A}_i^a, \lambda_{ab})\) and the following 18 FJ constraints \((\Omega_1^{(0)}, \Omega_2^{(0)}, \Omega_3^{(0)}, A_i^0, \tilde{A}_i^0, \lambda_{ab})\), therefore the theory lacks physical degrees of freedom and it is topological.
IV. CONCLUSIONS AND PROSPECTS

In this paper a full FJ approach for an Abelian analog of TMG and TMG at the chiral point has
been performed. For both theories the complete set of FJ constraints were found and the quantum
brackets identified with the FJ brackets have been constructed. Moreover, we observed that in
the FJ framework there are less constraints than in the conventional canonical formalism, and the
construction of the FJ brackets is more economical and gives the desired Dirac brackets. In addition,
we have calculated the number of physical degrees of freedom concluding that the theories are devoid
of physical degrees of freedom and therefore the theories are topological. The results of this paper
are preliminary for performing the quantization because the principal cornerstone in future works
will be the symplectic study of non-abelian TMG with cosmological constant at the chiral point.
However, that work is still in progress and will be the subject of forthcoming works.

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V. APPENDIX A

In this appendix, we will summarize the Dirac analysis of the Lagrangian \( \mathcal{L} \). By performing a
full Dirac’s analysis, we find the following results; there are 18 first class constraints

\[
\begin{align*}
\gamma^0_i : & \quad \tilde{\Pi}^0_i \approx 0, \\
\gamma^0_i : & \quad \Pi^0_i \approx 0, \\
\gamma^{00} : & \quad \gamma^{00} \approx 0, \\
\gamma^i : & \quad \left[ F^{ab}_{ijab} - \tilde{F}^{ab}_{ijab} \right] \epsilon^{ab} + \frac{2}{l} \partial_a P^{ia} + 2 \partial_a \chi^{ia} - \frac{1}{2l} \partial_a \tilde{\chi}^{ia} \approx 0, \\
\rho_i : & \quad \frac{1}{2} \tilde{F}_{iab} \epsilon^{ab} - \frac{1}{l} \partial_a \Pi^a_i + \frac{1}{7} \partial_a P^a_i - \frac{1}{2l} \partial_a \chi^a_i - \frac{1}{l} \partial_a \tilde{\chi}^a_i \approx 0, \\
\beta_i : & \quad \partial_a \Pi^a_i \approx 0,
\end{align*}
\]

(36)

and the following 12 second class constraints

\[
\begin{align*}
\chi^a_i : & \quad \tilde{\Pi}^a_i + l \left( \lambda_{ib} \epsilon^{ab} + \tilde{A}_{ib} \epsilon^{ab} \right) \approx 0, \\
\chi^a_i : & \quad \Pi^a_i - 2l \lambda_{ib} \epsilon^{ab} \approx 0, \\
\tilde{\chi}^{ib} : & \quad P^{ib} \approx 0,
\end{align*}
\]

(37)

where \( \left( \Pi^0_i, \tilde{\Pi}^0_i, P^{ia}, \tilde{P}^{ia}, \Pi^a_i, \tilde{\Pi}^a_i \right) \) are the canonical momenta of the dynamical variables
\( \left( A^0_i, \tilde{A}^0_i, \lambda_{ib}, \lambda_{ia}, A^a_i, \tilde{A}^a_i \right) \) respectively. It is important to comment, that in the Dirac framework there are always more constraints than in the FJ formalism. However, if we eliminate the
second class constraints by introducing the Dirac brackets, then equivalent results are obtained. Hence, we will construct the Dirac brackets by eliminating only the second class constraints. In order to perform this aim, we construct the following matrix whose entries are given by the Poisson brackets between the second class constraints, say, $C^{\lambda\nu}$

$$
C^{\lambda\nu} = \begin{pmatrix}
2\eta_{ij}\epsilon^{ab} & 0 & l\epsilon^{ag}\delta^i_j \\
0 & 0 & -l\epsilon^{ag}\delta^i_j \\
-l\epsilon^{ab}\delta^i_j & l\epsilon^{ab}\delta^i_j & 0
\end{pmatrix} \delta^2(x - y),
$$

and the inverse is given by

$$
(C^{\alpha\nu})^{-1} = \begin{pmatrix}
\eta^{ij}/2l\epsilon_{ab} & \eta^{ij}/2l\epsilon_{ab} & 0 \\
-\eta^{ij}/2l\epsilon_{ab} & \eta^{ij}/2l\epsilon_{ab} & \frac{1}{l}\epsilon_{ab}\delta^i_j \\
0 & -\frac{1}{l}\epsilon_{ab}\delta^i_j & 0
\end{pmatrix} \delta^2(x - y).
$$

In this manner, the Dirac bracket between two functionals, $F$ and $B$, is given by

$$
\{F, B\}_D = \{F, B\} - \int \{F, \chi^\alpha(u)\}C^{-1}_{\alpha\nu}\{\chi^\nu(v), B\}dudv,
$$

where $\chi^\alpha$ represent the set of second class constraints. Thus, the Dirac brackets between the dynamical variables are given by

$$
\begin{align*}
\{\bar{A}^i_a(x), \bar{A}^i_b(y)\}_D &= \frac{\eta^{ij}}{2l}\epsilon_{ab}\delta^2(x - y), \\
\{A^i_a(x), A^j_b(y)\}_D &= \frac{\eta^{ij}}{2l}\epsilon_{ab}\delta^2(x - y), \\
\{A^i_a, \lambda^j_b(y)\}_D &= \delta^i_j\epsilon_{ab}\delta^2(x - y), \\
\{A^i_a(x), \bar{A}^j_b(y)\}_D &= -\frac{\eta^{ij}}{2l}\epsilon_{ab}\delta^2(x - y), \\
\{\lambda^i_a(x), \lambda^j_b(y)\}_D &= 0,
\end{align*}
$$

where we can see that the Dirac brackets and the FJ brackets given in (35) coincide to each other.

VI. APPENDIX B

In this appendix we will resume the FJ approach for singular theories. We start by writing the first order Lagrangian density in the following form

$$
L^{(0)} = a_A \dot{\xi}^A - V^{(0)},
$$

here, $a_A$ are functions of the field variables $\xi^A$, $V^{(0)} = V^{(0)}(\xi^A)$ is called the symplectic potential and $A = 1, 2, 3...$ labels the number of dynamical variables. Hence, from the Lagrangian density it is straightforward to show that the equations of motion are given by

$$
\dot{f}^{(0)}_{AB} = \frac{\delta V^{(0)}}{\delta \xi^A},
$$

(39)
where
\begin{equation}
    f^{(0)}_{AB} = \frac{\delta a_A}{\delta \xi^B} - \frac{\delta a_B}{\delta \xi^A},
\end{equation}
is the antisymmetric symplectic matrix. Since the theory is a singular system, there will be constraints. In this manner, the symplectic matrix (41) is not invertible and that means that there are null vectors. In fact, we call to $\mathcal{V}^{(0)}_A$ the set of null vectors of the symplectic matrix. Hence, from the contraction of the null vectors with the equation (40) the following constraints arise
\begin{equation}
    \Omega^{(0)} = \mathcal{V}^{(0)}_A \frac{\delta \mathcal{V}^{(0)}}{\delta \xi^A} = 0.
\end{equation}

In analogy with the Dirac method, we demand consistency, this means that the constraints must be preserved in time, namely
\begin{equation}
    \dot{\Omega}^{(0)} = \frac{\delta \Omega^{(0)}}{\delta \xi^A} \dot{\xi}^A = 0,
\end{equation}
hence, by considering (40) and (43) we can form a system of linear equations with all information found, say
\begin{equation}
    \begin{cases}
        \frac{\delta \mathcal{V}^{(0)}}{\delta \xi^A} \dot{\xi}^B = \delta \mathcal{V}^{(0)}_A, \\
        \frac{\delta \Omega^{(0)}}{\delta \xi^A} \dot{\xi}^A = 0,
    \end{cases}
\end{equation}
and to rewrite (44) as
\begin{equation}
    \tilde{f}_{AB} \dot{\xi}^B = Z_A(\xi),
\end{equation}
with
\begin{equation}
    \tilde{f}_{AB} = \left( \begin{array}{c}
        f^{(0)}_{AB} \\
        \frac{\delta \mathcal{V}^{(0)}}{\delta \xi^A}
    \end{array} \right),
\end{equation}
and
\begin{equation}
    Z_A(\xi) = \left( \begin{array}{c}
        \frac{\delta \mathcal{V}^{(0)}}{\delta \xi^A} \\
        0 \\
        0
    \end{array} \right).
\end{equation}

Hence, we repeat the algorithm, we calculate the null vectors of the matrix (46), say $\mathcal{V}^{(1)}_A$, and we perform the contraction of these null vectors with $Z_A(\xi)$ in order to identify new constraints,
\begin{equation}
    \Omega^{(1)} = \mathcal{V}^{(1)}_A Z_A = 0.
\end{equation}

Similarly, by demanding the consistency condition
\begin{equation}
    \dot{\Omega}^{(1)} = \frac{\delta \Omega^{(1)}}{\delta \xi^A} \dot{\xi}^A = 0,
\end{equation}
we can combine (40) with equation (43) to construct a new set of linear equations. By using these linear equations one verifies step by step whether there are new constraints, until there are no more
and we get the identity.

On the other hand, we will assume that all FJ constraints have been identified, hence the final symplectic Lagrangian can be written as

\[ L = a(\xi) A^A - \dot{\gamma}^C \Omega^C - V(\xi), \]

(50)

where \( \gamma^C \) are Lagrange multipliers enforcing the FJ constraints \( \Omega^s \) and \( V(\xi) = V^{(0)}_{\Omega^C=0} \). In this manner, if we consider the field variables and the Lagrange multipliers as our new set of symplectic variables, say \( \tilde{\xi}^A = (\xi^A, \gamma^B) \), then we can construct the symplectic matrix of the Lagrangian (50)

\[ \tilde{f}_{AB} = \frac{\delta\tilde{a}_A}{\delta\xi^B} - \frac{\delta\tilde{a}_B}{\delta\xi^A}, \]

(51)

where \( \tilde{a}_A = a^A + \gamma^C \frac{\delta \Omega^C}{\delta \xi^A} \). If the symplectic matrix (51) is not singular, then we can calculate its inverse, namely \( \tilde{f}_{AB}^{-1} \), thus we can find all the velocities \( \dot{\xi}^A \) and the problem is finished. On the other hand, if the symplectic matrix (51) is still singular, this means that the theory has a gauge symmetry, then in order to obtain a symplectic tensor it is necessary to fix the gauge just as in the above sections it was performed. In any case, if \( \tilde{f}_{AB} \) is invertible, then we can identify the generalized FJ brackets defining

\[ \{\tilde{\xi}^A(x), \tilde{\xi}^B(y)\}_{FD} = [\tilde{f}_{AB}(x,y)]^{-1}, \]

(52)

which allow to write the equations of motion (40) in Hamiltonian form.

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