Coalescence, deformation and Bäcklund symmetries of Painlevé IV and II equations

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Received 5 May 2020, revised 7 September 2020
Accepted for publication 10 September 2020
Published 9 October 2020

Abstract
We extend Painlevé IV model by adding quadratic terms to its Hamiltonian obtaining two classes of models (coalescence and deformation) that interpolate between Painlevé IV and II equations for special limits of the underlying parameters. We derive the underlying Bäcklund transformations, symmetry structure and requirements to satisfy Painlevé property.

Keywords: Painleve equations, coalescence, symmetries, Backlund transformations

1. Introduction

The Painlevé equations are second-order differential equations whose solutions have no movable singular points except poles. This feature (pure poles are the only movable singularities) of some second order differential equations is known as Painlevé property. The Painlevé equations naturally emerge as special scaling limits of integrable models \cite{4–9} and a fundamental conjecture \cite{1} establishes connection between Painlevé property and solvability by inverse scattering. Another basic aspect of Painlevé equations and their Hamiltonian structures is invariance under extended affine Weyl symmetry groups \cite{17, 18}. Bäcklund transformations have also been extensively studied in connection with the Schlesinger transformations, see for instance references \cite{10, 15, 22} for the case of Painleve II and IV equations.

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Hybrid Painlevé equations have been a focus of several papers, e.g. [14, 20]. More recently, in reference [3] we introduced the hybrid P_{III–V} model that was obtained as reduction of a class of integrable models known as multi-boson systems [6, 7] that generalize the AKNS hierarchy [5]. The P_{III–V} model reduces to P_{III}, P_{V} and I_{12}, I_{38} and I_{49} equations from Ince’s list [2, 12] for special limits of its parameters while for remaining finite values of its parameters preserves enough symmetry under remaining Bäcklund transformations of the extended affine Weyl symmetry group to satisfy Painlevé property [3].

We will conduct here a similar investigation for the hybrid of P_{II} and P_{IV} models and point out how the presence of remaining Bäcklund transformations symmetries influences the outcome of the Painlevé test. Starting from the symmetric Painlevé IV equations, in section 2, we enlarge its parameter space to allow for extension of symmetry structure by additional automorphisms $\pi_i, \rho_i, i = 0, 1, 2$. We derive algebraic relations between these automorphisms and $A^{(1)}_2$ Bäcklund transformations.

We present two different limiting procedures leading to Painlevé II equation. One way, described in section 3, is to formulate coalescence/degeneracy in a framework of symmetric Painlevé IV equations augmented by a non-zero integration constant. This generalization of P_{IV} equation remains invariant under the additional automorphism $\rho_2$. The underlying Weyl group symmetry reduces from $A^{(1)}_2$ down to $A^{(1)}_1$ in the appropriate limit and we are able to obtain close expressions for the Bäcklund transformations of P_{II} from their P_{IV} counterparts. In the P_{II} limit the automorphism $\rho_2$ toggles between two copies of P_{II} equation each with its own $A^{(1)}_1$ symmetry.

In another scheme, presented in section 5, the $A^{(1)}_2$ symmetry group of symmetric Painlevé IV equation is explicitly broken by addition of a deformation parameter before the limit resulting in Painlevé II equation is taken. The deformed model is formulated in such a way that it is invariant under additional automorphisms $\pi_2, \rho_2$. We point out a connection between existence of residual symmetry of the deformed model (invariance under one of the original three Bäcklund transformations of $A^{(1)}_2$) and passing of the Kovalevskaya–Painlevé test by this model. Such deformed model provides another example of hybrid Painlevé equations with properties that they pass Painlevé test, retain invariance under residual Bäcklund transformations and reduce down to underlying Painlevé or Ince equations for special values of their parameters.

In section 4 we will introduce and study a generalization of P_{IV} Hamiltonian structure of the form:

$$H = H_0 + \frac{1}{\epsilon} (f_0 + f_1) \left( k_1 \sigma z - \frac{k_2}{2} (f_0 + f_1) \right),$$

(1.1)

where $\epsilon, \sigma, k_1, k_2$ are complex parameters and

$$H_0 = -f_0 f_1 f_2 + \frac{-\alpha_1 + \alpha_2}{3} f_0 + \frac{-\alpha_1 - 2 \alpha_2}{3} f_1 + \frac{2 \alpha_1 + \alpha_2}{3} f_2,$$

(1.2)

is the well-known Okamoto’s P_{IV} Hamiltonian [19]. The two basic conditions that guide our construction of such generalization are: (1) that the original cubic Hamiltonian is augmented only by terms of dimensions lower than three and (2) the Hamilton equations remain finite and do not violate the Painlevé property. These conditions restrict the allowed generalization of P_{IV} Hamiltonian structure to be of the form given in equation (1.1). As we will see below the combination $f_0 + f_1$ appearing in the above expression ensures invariance under a pair of Bäcklund transformations $s_2, \rho_2$, if we used $f_0 + f_2$ or $f_1 + f_2$ we would encounter invariance under $s_1, \rho_1$ or $s_0, \rho_0$ with all these transformations being defined in the forthcoming sections.
We show that this natural generalization (1.1) represents either coalescence/degeneracy or \( A_2^{(1)} \) deformation of \( \text{PIV} \) and we present arguments that those two approaches are the only ones leading from \( \text{PIV} \) model to \( \text{PII} \) model under the above conditions.

We summarize the novel features of our formalism and reiterate rationale for expanding the parameter space of Painlevé IV model by additional parameters in section 6.

2. The structure of \( \text{PIV} \) model, definition and symmetries

This section is devoted to a summary of relevant results on \( \text{PIV} \) equations, Bäcklund transformations and coalescence between \( \text{PIV} \) and \( \text{PII} \) available in the literature (e.g. [11, 18]).

We also generalize the conventional symmetric Painlevé IV model by adding the new parameter \( \sigma \) in a way that makes the generalized model invariant under additional automorphisms \( \pi_i, \rho_i, i = 0, 1, 2 \) satisfying the braid relations.

2.1. \( \text{PIV} \) symmetric equations

The starting point of subsection is the Okamoto Hamiltonian (1.2) for \( \text{PIV} \) equation. In the literature the parameters \( \alpha_i, i = 0, 1, 2 \) satisfy the condition

\[
\alpha_1 + \alpha_2 + \alpha_0 = 1.
\]

Here we find that our discussion of symmetries and coalescence limits will profit from working instead with conditions:

\[
\alpha_1 + \alpha_2 + \alpha_0 = \sigma, \quad f_2 = \sigma z - f_1 - f_0.
\] (2.1)

Here we introduced \( \sigma \) as an additional parameter for the \( \text{PIV} \) model that enables us to extend symmetry group of the model. The advantages of introducing the \( \sigma \) parameter will be summarized in the concluding section 6.

The corresponding Hamilton’s equations can be cast in a form of the so-called symmetric \( \text{PIV} \) system described by e.g. [18]:

\[
f_0' = f_0 (f_1 - f_2) + \alpha_0,
\]

\[
f_1' = f_1 (f_2 - f_0) + \alpha_1,
\]

\[
f_2' = f_2 (f_0 - f_1) + \alpha_2,
\] (2.2)

where \( f_i = f_i(z) \) and \( ' = d/dz \).

Eliminating \( f_2 = \sigma z - f_0 - f_1 \) from (2.2) we obtain:

\[
f_0'(z) = f_0 (-\sigma z + f_0 + 2f_1) + \alpha_0,
\]

\[
f_1'(z) = f_1 (\sigma z - 2f_0 - f_1) + \alpha_1,
\] (2.3)

while the third equation in (2.2) can be obtained by summing the above two equations.

By further eliminating \( f_1 \) or \( f_0 \) from (2.3) we get for the remaining component:

\[
f_i'''(z) = \frac{f_i'^2}{2f_i} - \frac{\alpha_i^2}{2f_i} + \left(\frac{1}{2}\sigma^2 z^2 + (-1)(2\alpha_0 + 2\alpha_1 - \alpha_i - \sigma)\right)
\]

\[
f_i - 2\sigma zf_i^2 + \frac{3}{2} f_i^3, \quad i = 0, 1.
\] (2.4)

Both equations are equivalent to the standard \( \text{PIV} \) equation [5, 11]:

\[
w_{xx} = \frac{w_x^2}{2w} + \frac{3w^3}{2} + 4xw^2 + 2 \left(x^2 - A \right)w + \frac{B}{w}
\] (2.5)
by setting $\sigma \to 1$ followed by transformations
\[ f_0(z) = \frac{v(x)}{\sqrt{-2}}, \quad z = x \sqrt{-2}, \quad \alpha_1 = \frac{1}{2} (1 + A - \alpha_0), \quad \alpha_0 = \sqrt{-\frac{B}{2}} \quad (2.6) \]
and a similar transformation for $f_i$ with the appropriate changes.

Equation (2.4) will be referred to as $P_{IV}$ equations throughout this document while equation (2.2) will be referred to as symmetric $P_{IV}$ equations.

2.2. Bäcklund and auto-Bäcklund transformations

Equation (2.2) are manifestly invariant under Bäcklund transformations $s_i$ ($i = 0, 1, 2$) and automorphism $\pi$ defined as follows (see e.g. [18]):

\[
\begin{array}{|c|ccc|ccc|}
\hline
& \alpha_0 & \alpha_1 & \alpha_2 & f_0 & f_1 & f_2 \\
\hline
s_0 & -\alpha_0 & \alpha_1 + \alpha_0 & \alpha_2 + \alpha_0 & f_0 & f_1 + \frac{\alpha_0}{f_0} & f_2 - \frac{\alpha_0}{f_0} \\
\hline
s_1 & \alpha_0 + \alpha_1 & -\alpha_1 & \alpha_2 + \alpha_1 & f_0 - \frac{\alpha_1}{f_1} & f_1 & f_2 + \frac{\alpha_1}{f_1} \\
\hline
s_2 & \alpha_0 + \alpha_2 & \alpha_1 + \alpha_2 & -\alpha_2 & f_0 + \frac{\alpha_2}{f_2} & f_1 - \frac{\alpha_2}{f_2} & f_2 \\
\hline
\pi & \alpha_1 & \alpha_2 & \alpha_0 & f_1 & f_2 & f_0 \\
\hline
\end{array}
\quad (2.7)
\]

These transformations satisfy
\[ s_i^2 = 1, \quad (s_is_{i+1})^3 = 1, \quad \pi^3 = 1, \quad \pi s_i = s_{i+1}\pi, \quad i = 0, 1, 2, \quad (2.8) \]

and thus $(s_0, s_1, s_2, \pi)$ form the extended affine Weyl group $A^{(1)}_2$ [18].

Due to the presence of parameter $\sigma$ introduced in equation (2.1) in the setting of symmetric $P_{IV}$ equation (2.2) we have additional automorphisms $\pi_i$ and $\rho_i, i = 0, 1, 2$:

\[
\begin{array}{|c|ccc|c|}
\hline
& \alpha_0 & \alpha_1 & \alpha_2 & f_0 & f_1 & f_2 & \sigma \\
\hline
\pi_0 & -\alpha_0 & -\alpha_2 & -\alpha_1 & -f_0 & -f_2 & -f_1 & -\sigma \\
\hline
\pi_1 & -\alpha_2 & -\alpha_1 & -\alpha_0 & -f_2 & -f_1 & -f_0 & -\sigma \\
\hline
\pi_2 & -\alpha_1 & -\alpha_0 & -\alpha_2 & -f_1 & -f_0 & -f_2 & -\sigma \\
\hline
\end{array}
\quad (2.9)
\]

and

\[
\begin{array}{|c|ccc|c|}
\hline
& \alpha_0 & \alpha_1 & \alpha_2 & f_0 & f_1 & f_2 & \sigma & z \\
\hline
\rho_0 & -\alpha_0 & -\alpha_2 & -\alpha_1 & f_0 & f_2 & f_1 & -\sigma & -z \\
\hline
\rho_1 & -\alpha_2 & -\alpha_1 & -\alpha_0 & f_2 & f_1 & f_0 & -\sigma & -z \\
\hline
\rho_2 & -\alpha_1 & -\alpha_0 & -\alpha_2 & f_1 & f_0 & f_2 & -\sigma & -z \\
\hline
\end{array}
\quad (2.10)
\]

that keep equation (2.2) invariant. The automorphisms $\pi_i$ and $\rho_i$ square to one
\[ \pi_i^2 = 1, \quad \rho_i^2 = 1, \quad i = 0, 1, 2, \quad (2.11) \]

and satisfy the so-called braid relations
\[ \pi_i \pi_j \pi_i = \pi_j \pi_i \pi_i, \quad \rho_i \rho_j \rho_i = \rho_j \rho_i \rho_j, \quad i \neq j. \quad (2.12) \]
The automorphisms $\pi_i$ and $\rho_i$ are related to automorphism $\pi$ from (2.7) via

$$\pi = \pi_2\pi_0 = \pi_1\pi_2 = \pi_0\pi_1 = \rho_2\rho_0 = \rho_1\rho_2 = \rho_0\rho_1$$  \hspace{1cm} (2.13)

and satisfy the following commutation relations with the Bäcklund transformations $s_i$:

$$\pi_is_i = s_i\pi_i, \quad \pi_is_j = s_i\pi_j, \quad \rho_is_i = s_i\rho_0, \quad \rho_is_j = s_i\rho_0, \quad i \neq j, \quad k \neq j, \quad i \neq k.$$  \hspace{1cm} (2.14)

We will now describe the Bäcklund transformations for the second order $P_{IV}$ equation (2.4). The procedure will be illustrated by considering the $s_2$ transformation only. Generalizations to other generators follow easily.

First, we consider $s_2(\alpha_0), s_2(f_0)$ from (2.9) and eliminate $\alpha_2 = \sigma - \alpha_0 - \alpha_1$ and $f_2 = \sigma z - f_0 - f_1$ to obtain:

$$s_2(\alpha_0) = \sigma - \alpha_1, \quad s_2(\alpha_1) = \sigma - \alpha_0,$$  \hspace{1cm} (2.15)

$$s_2(f_0) = f_0 + \frac{\sigma - \alpha_0 - \alpha_1}{\sigma z - f_0 - f_1},$$  \hspace{1cm} (2.16)

$$s_2(f_1) = f_1 - \frac{\sigma - \alpha_0 - \alpha_1}{\sigma z - f_0 - f_1}.$$  \hspace{1cm} (2.17)

Equation (2.3) allows us to write down the following relations between $f_0$ and $f_1$:

$$f_1 = \frac{-\alpha_0 + \sigma zf_0 + f_0^2 - f_1^2}{2f_0}$$  \hspace{1cm} (2.18)

$$f_0 = \frac{\alpha_1 + \sigma zf_1 - f_1^2 - f_0^2}{2f_1}$$  \hspace{1cm} (2.19)

used below to realize $s_2$ as (1) Bäcklund and (2) auto-Bäcklund transformations, respectively as shown below:

(1) Eliminating $f_0$ from the rhs of equation (2.16) and $f_1$ from the rhs of equation (2.17) yields:

$$s_2(f_0) = -\frac{2f_1(\sigma - \alpha_0 - \alpha_1)}{\alpha_1 + \sigma zf_1 + f_1^2 - f_1^2} + \frac{\alpha_1 + \sigma zf_1 - f_1^2 - f_1^2}{2f_1},$$  \hspace{1cm} (2.20)

$$s_2(f_1) = -\frac{2f_0(\sigma - \alpha_0 - \alpha_1)}{\alpha_0 + \sigma zf_0 - f_0^2 - f_0^2} - \frac{\alpha_0 - \sigma zf_0 + f_0^2 + f_0^2}{2f_0}.$$  \hspace{1cm} (2.21)

(2) Inversely, eliminating $f_0$ from the rhs of equation (2.17) and $f_1$ from the rhs of equations and (2.16) yields (note that in this case we denote $s_2$ by $\tilde{s}_2$):

$$\tilde{s}_2(f_0) = f_0 + \frac{2(\alpha_0 f_0 + \alpha_1 f_0 - \sigma f_0)}{-\alpha_0 + \sigma zf_0 + f_0^2 + f_0^2},$$  \hspace{1cm} (2.22)

$$\tilde{s}_2(f_1) = f_1 - \frac{2(-\alpha_0 f_1 - \alpha_1 f_1 + \sigma f_1)}{-\alpha_1 + \sigma zf_1 + f_1^2 - f_1^2}.$$  \hspace{1cm} (2.23)

Acting with $\rho_2$ connects relations (2.20) and (2.21) as well as relations (2.22) and (2.23):

$$\rho_2(s_2(f_0)) = s_2(f_1), \quad \rho_2(\tilde{s}_2(f_0)) = \tilde{s}_2(f_1).$$  \hspace{1cm} (2.24)
As we saw above in items (1) and (2), $s_2(f_i)$, $i = 0, 1$ could either be expressed in terms of $f_i$, $i = 0, 1$ or $f_j$, $i \neq j$ by simple substitutions (2.18) or (2.19). The transformation $s_2$ that maps $f_0 \rightarrow f_1$ and $f_1 \rightarrow f_0$ is referred to by us as Bäcklund transformation of the system of second order PIV equation (2.4) and maps equation (2.4) with $i = 0$ to that with $i = 1$ and vice versa.

The corresponding transformation that maps $f_0 \rightarrow f_0$ and $f_1 \rightarrow f_1$ is denoted by as $\tilde{s}_2$ and is referred to as an auto-Bäcklund transformation of the second order PIV equation (2.4) with either $i = 0$ or $i = 1$.

### 3. Coalescence in the setting of symmetric PIV equations

In this section we look at coalescence in the setting of symmetric PIV equations. Such framework makes it easier to see what happens with the Bäcklund symmetries in the $\epsilon \rightarrow 0$ limit.

Here we formulate coalescence in a setting of the symmetric PIV equation (2.2) through the following transformations:

$$f_i(z) \rightarrow f_i(z) + \frac{1}{\epsilon} z \rightarrow z + \frac{2}{\sigma \epsilon^2},$$

$$\alpha_0 \rightarrow \epsilon \alpha_0 - \frac{1}{\epsilon^2}, \quad \alpha_1 \rightarrow \epsilon \alpha_1 + \frac{1}{\epsilon^2}, \quad \alpha_2 \rightarrow \epsilon \alpha_2.$$  \hspace{1cm} (3.1)

Applying the above transformation to the first order equation (2.2) yields:

$$f_i'(z) = f_i(f_{i-1} - f_{i+1}) + f_i - f_i' = \frac{1}{\epsilon}$$

$$f_i'(z) = f_i(f_{i-1} - f_{i+1}) + f_i - f_i' = \frac{1}{\epsilon}$$

$$f_i'(z) = f_i(f_{i-1} - f_{i+1}) + f_i - f_i' = \frac{1}{\epsilon}$$

Now we proceed by the same steps as in the preceding sections. Summing the equations above we get:

$$\epsilon \alpha_0 + \epsilon \alpha_1 + \epsilon \alpha_2 = \epsilon \sigma, \quad f_i' + f_{i-1}' + f_{i+1}' = \epsilon \sigma.$$  \hspace{1cm} (3.2)

Integrating equation $\sum f_i' = \epsilon \sigma$ yields $\sum f_i = \epsilon \sigma z + C$, where $C$ is an arbitrary constant of integration. Initially $C$ is set to zero but after applying transformation (3.1) on $f_i$ and $z$ we obtain:

$$f_0 + f_1 + f_2 + \frac{3}{\epsilon} = \epsilon \sigma z + \frac{2}{\epsilon} \quad \rightarrow \quad f_0 + f_1 + f_2 = \epsilon \sigma z - \frac{1}{\epsilon}.$$  \hspace{1cm} (3.3)

with $C = -1/\epsilon$. Note that the presence of the non-zero integration constant does not affect the symmetry of the symmetric PIV equations since we can always work with symmetry transformations acting on redefined $f_i$'s as will be done below.

Eliminating $f_2$ and $\alpha_2$ from (3.2), we get:

$$f_i'(z) = \epsilon (\alpha_0 - \sigma f_0) + \frac{2f_0 + 2f_1}{\epsilon} + f_i' + 2f_1f_0 - \sigma z,$$

$$f_i'(z) = \epsilon (\alpha_1 + \sigma f_1) + \frac{-2f_0 - 2f_1}{\epsilon} - f_i' - 2f_0f_1 + \sigma z.$$  \hspace{1cm} (3.3)
Substituting \( a_0 = a_0/\epsilon, a_1 = a_1/\epsilon, \sigma = \sigma_0/\epsilon \) with finite \( a_0, a_1, \sigma_0 \) and taking \( \epsilon \to \infty \) limit we recover \( P_N \) equation (2.3).

By eliminating \( f_0 \) from (3.3) we obtain:

\[
\frac{f''_1}{\epsilon f_1} + \frac{1}{\epsilon f_1 + 1} \left( \sigma - 2\alpha_0 - 2\alpha_1 - 2\sigma zf_1 + 2f_1^3 + \epsilon^3 \left( \frac{1}{2} \alpha^2 z^2 f_1^2 - \frac{\alpha^2}{2} \right) \right) + \epsilon^2 \left( -2\alpha_0 f_1^2 - \alpha_1 f_1^2 + \sigma^2 z^2 f_1 + 2\sigmazf_1 + \sigma f_1^2 \right) + \epsilon \left( -4\alpha_0 f_1 - 2\alpha_1 f_1 - 4\sigma zf_1 + 2\sigma f_1 + \frac{1}{2} f_1^2 + \frac{3}{2} f_1^4 + \frac{\sigma^2 z^2}{2} \right) = 0. \tag{3.4}
\]

Taking instead the limit \( \epsilon \to 0 \) in equation (3.4) and the corresponding equation for \( f_0 \) results in two copies of \( P_N \) equations, namely:

\[
f''_i = (\epsilon + 1)^i (-\sigma - 2\alpha_0 + 2\alpha_1 - 2\sigma zf_i + 2f_i^3), \quad i = 0, 1. \tag{3.5}
\]

The above \( P_N \) equations transform into each other under the automorphism \( \rho_2 \) from (2.10). Since transformations (3.1) are nothing but Möbius transformations on the variables \( f_i \) and \( z \), they naturally preserve the Painlevé property.

As a digression we note that equation (3.4) for \( \epsilon \to 0 \) and finite \( \epsilon \) becomes for \( w = f_i - 1/\epsilon \):

\[
w'' = \frac{w^2}{2w} + \frac{3w^3}{2} - \frac{4w^2}{\epsilon} - \frac{w \left( 2\alpha \epsilon^3 + \alpha_1 \epsilon^3 - 3 \right)}{\epsilon^2} - \frac{(\alpha_1 \epsilon^3 + 1)^2}{2w^4}. \tag{3.6}
\]

in which we recognize the equation XXX (I_{30}) of the Gambier’s classification, that is listed in the classical book of Ince [12] (see also [2] for connection between Painlevé equations with additional parameters and equations in [12]) as:

\[
I_{30}: \quad w'' = \frac{w^2}{2w} + \frac{3w^3}{2} + 4aw^2 + 2bw + \frac{c}{w}. \tag{3.7}
\]

Also, if we make transformation \( z \to z + \frac{2}{\alpha \epsilon^3} - \xi/\sigma \) in equation (3.1) (equivalent to a different choice of integration constant \( C \) in \( \sum_i f_i = \sigma z + C \)) with some new parameter \( \xi \) and take the limit \( \epsilon \to 0 \) in the corresponding second order equation for \( f_0 \) we obtain

\[
f''_0 = 2f_0^3 - 2(\sigma z - \xi)f_0 - \sigma - 2\alpha_0 + 2\alpha_1. \tag{3.8}
\]

By taking \( \sigma = 0 \) we arrive at Ince’s \( I_8 \) equation:

\[
I_8: \quad w'' = 2w^3 + aw + b. \tag{3.9}
\]

3.1. The Bäcklund transformations in the coalescence limit

In this subsection we will show how \( A_2^{(1)} \) symmetry group reduces to \( A_1^{(1)} \) symmetry in the appropriate limit.
3.1.1. $A_2^{(1)}$ symmetry is maintained in equation (3.2). Equation (3.2) are invariant under:

|    | $\alpha_0$   | $\alpha_1$   | $\alpha_2$   | $f_0$     | $f_1$     | $f_2$     |
|----|--------------|--------------|--------------|-----------|-----------|-----------|
| $s_0$ | $\frac{2}{\pi} - \alpha_0$ | $\alpha_0 + \alpha_1 - \frac{1}{\pi}$ | $\alpha_0 + \alpha_2 - \frac{1}{\pi}$ | $f_0$ | $\frac{\alpha \epsilon f_0 + f_1}{f_0 + \frac{\alpha \epsilon f_0 + f_1}{f_0 + \frac{\alpha \epsilon f_0 + f_1}{f_0 + f_1}}}$ | $f_2$ |
| $s_1$ | $\alpha_0 + \alpha_1 - \frac{1}{\pi}$ | $-\alpha_1 - \frac{1}{\pi}$ | $\alpha_1 + \alpha_2 + \frac{1}{\pi}$ | $f_0 - \frac{\alpha \epsilon f_0 + f_1}{f_0 + \frac{\alpha \epsilon f_0 + f_1}{f_0 + \frac{\alpha \epsilon f_0 + f_1}{f_0 + f_1}}}$ | $f_1$ | $\frac{\alpha \epsilon f_0 + f_1}{f_0 + \frac{\alpha \epsilon f_0 + f_1}{f_0 + \frac{\alpha \epsilon f_0 + f_1}{f_0 + f_1}}}$ |
| $s_2$ | $\alpha_0 + \alpha_2$ | $\alpha_1 + \alpha_2$ | $-\alpha_2$ | $\frac{\alpha \epsilon f_0 + f_1}{f_0 + \frac{\alpha \epsilon f_0 + f_1}{f_0 + \frac{\alpha \epsilon f_0 + f_1}{f_0 + f_1}}}$ | $f_0$ | $f_1$ | $f_2$ |
| $\pi$ | $\alpha_1 + \frac{1}{\pi}$ | $\alpha_2 - \frac{1}{\pi}$ | $\alpha_0 - \frac{1}{\pi}$ | $f_1$ | $f_2$ | $f_0$ |

(3.10)

and the automorphism $\rho_2$ from (2.10).

After we eliminate $f_2$ and $\alpha_2$, we still have invariance under $s_0$, $s_1$, but no longer under $\pi$ and the $s_2$ transformation is modified to:

$$s_2(f_0) = f_0 - \frac{\epsilon (-\alpha_0 - \alpha_1 + \sigma)}{f_0 + f_1 - \sigma \epsilon}, \quad s_2(f_1) = f_1 - \frac{\epsilon (-\alpha_0 - \alpha_1 + \sigma)}{-f_0 - f_1 + \sigma \epsilon},$$

$$s_2(\alpha_0) = -\alpha_1, \quad s_2(\alpha_1) = -\alpha_0. \quad (3.11)$$

3.1.2. Emergence of $A_1^{(1)}$ symmetry in the $\epsilon \to 0$ limit. It is now easy to see from equation (3.10) that the transformations $s_0$, $s_1$, and $\pi$ diverge in the limit $\epsilon \to 0$. Also $s_2$ becomes trivial in this limit. The way around this problem is to form the composition $s_0 s_1 s_0$ that will be shown not to diverge in the limit $\epsilon \to 0$ [21]. Similar ideas of using compositions of Bäcklund transformations to obtain reduction from $A_1^{(1)} l$ to $A_1^{(1)} l-1$ appeared in [16].

The main conclusion of this subsection is that for the $P_{IV}$ system of equation (3.2) for $f_0, f_1$ (obtained after elimination of $f_2$) the $\epsilon \to 0$ limit will yield transformations $s_0 s_1 s_0$ (or identically $s_1 s_0 s_1$) and $s_2$ as the two Bäcklund transformations that maintain $P_{II}$ invariant.

Explicitly, the action of $s_0 s_1 s_0$ on all variables is:

$$s_0 s_1 s_0(f_0) = f_0 - \frac{(\alpha_0 + \alpha_1) \epsilon (\epsilon f_0 + 1)}{\alpha_0 \epsilon^2 + \epsilon f_0 f_1 + f_0 + f_1},$$

$$s_0 s_1 s_0(f_1) = \frac{(\alpha_0 + \alpha_1) \epsilon^2}{\epsilon f_0 + 1} + \frac{\alpha_1 \epsilon^2 + \alpha_0 \alpha_1 \epsilon^2}{\epsilon f_0 + 1} \left(-\alpha_1 \epsilon^2 + \epsilon f_0 f_1 + f_0 + f_1\right) + f_1,$$

$$s_0 s_1 s_0(\alpha_0) = -\alpha_1, \quad s_0 s_1 s_0(\alpha_1) = -\alpha_0. \quad (3.12)$$

Now just looking at transformations of the parameters $\alpha_0, \alpha_1$ and using notation $\beta := \alpha_0 + \alpha_1$ (since they are always together from now on), we see that they have a $A_1^{(1)}$ group structure due to:

$$s_0 s_1 s_0 \beta = -\beta, \quad 2\beta + \alpha_2,$$

$$s_2 2\alpha_2 + \beta = -\alpha_2. \quad (3.13)$$

and

$$s_2^2 = 1, \quad (s_0 s_1 s_0)^2 = 1, \quad (s_0 s_1 s_0) s_2 = s_2 (s_0 s_1 s_0).$$
From now on we will use for brevity the following notation:

\[ S_0 = s_0 s_1 k_0, \quad S_1 = s_1. \]  

(3.14)

The relations (2.18), (2.19) obtained in section (2.2) generalize to the following relations

\[ f_0 = \frac{\alpha_1 \epsilon^2 + \sigma \epsilon f_1^2 - \epsilon f_1 - \epsilon f_1^2 - 2 f_1 + \sigma \epsilon}{2 (\epsilon f_1 + 1)}, \]

\[ f_1 = \frac{-\alpha_0 \epsilon^2 + \sigma \epsilon f_0^2 + \epsilon f_0' - \epsilon f_0^2 - 2 f_0 + \sigma \epsilon}{2 (\epsilon f_0 + 1)}, \]

(3.15)

obtained from (3.3). Using relations (3.15) in an exactly the same way as we did below equations (2.18), (2.19) we obtain two expressions for auto-Bäcklund transformations \( S_1 \) and Bäcklund transformations \( \tilde{S}_1 \) from those given in equation (3.11)

\[ \tilde{S}_1(f_0) = f_0 + \frac{2 (-\alpha_0 - \alpha_1 + \sigma)(\epsilon f_0 + 1)}{\alpha_0 \epsilon + \sigma \epsilon f_0 - f_0^2 - f_0^2 + \sigma \epsilon}, \]

\[ \tilde{S}_1(f_1) = f_1 - \frac{2 (-\alpha_0 - \alpha_1 + \sigma)(\epsilon f_1 + 1)}{-\alpha_1 \epsilon + \sigma \epsilon f_1 + f_1' - f_1' + \sigma \epsilon}. \]

S_1(f_0) = \frac{\alpha_1 \epsilon^2 + f_1 (\sigma \epsilon^2 - 2) + \epsilon (\sigma z - f_1^2) - \epsilon f_1^2}{2 \epsilon f_1 + 2}

+ \frac{2 (-\alpha_0 - \alpha_1 + \sigma)(\epsilon f_1 + 1)}{-\alpha_1 \epsilon + \sigma \epsilon f_1 + f_1' - f_1' + \sigma \epsilon}. \]

\[ S_1(f_1) = \frac{-\alpha_0 \epsilon^2 + f_0 \sigma \epsilon^2 - 2) + \epsilon (f_0' + \sigma z) + \epsilon (-f_0^2)}{2 \epsilon f_0 + 2}

- \frac{2 (-\alpha_0 - \alpha_1 + \sigma)(\epsilon f_0 + 1)}{\alpha_0 \epsilon + \sigma \epsilon f_0 - f_0^2 - f_0^2 + \sigma \epsilon}. \]

As in relations (2.24), these two Bäcklund transformations \( S_1 \) and \( \tilde{S}_1 \) are related by the automorphism \( \rho_2 \). Repeating the same steps for \( S_0 \) we obtain:

\[ S_0(f_0) = f_0 - \frac{2 \alpha_0 + \alpha_1 + \alpha_0 \epsilon f_0 + \alpha_1 \epsilon f_0}{\alpha_0 \epsilon + \sigma \epsilon f_0 - f_0^2 - f_0^2 + \sigma \epsilon}, \]

\[ \tilde{S}_0(f_0) = -\frac{2 \left(\alpha_0 \epsilon^2 + \alpha_0 \epsilon f_0 - \sigma - \alpha_0 \right)}{(\epsilon f_1 + 1) (-2 \alpha_0 \epsilon - \alpha_1 \epsilon - \sigma \epsilon f_1 + f_1' + f_1^2 - \sigma \epsilon)}

+ \frac{-2 \alpha_0 \epsilon^2 - \alpha_1 \epsilon + \sigma \epsilon f_0 - \epsilon f_0^2 - 2 f_0 + \sigma \epsilon}{2 (\epsilon f_1 + 1)} - \frac{\epsilon f_0'}{2 (\epsilon f_1 + 1)}. \]

The Bäcklund transformations obtained in this way have non trivial limits for \( \epsilon \to 0 \):

\[ \tilde{S}_1(f_0) = \frac{2 (-\alpha_0 - \alpha_1 + \sigma)}{-f_0' - f_0^2 + \sigma \epsilon} + f_0, \quad \tilde{S}_1(\beta) = 2 \sigma - \beta, \]

\[ S_1(f_0) = \frac{2 (-\alpha_0 - \alpha_1 + \sigma)}{f_0'} - f_1, \quad S_1(\beta) = 2 \sigma - \beta. \]

(3.16)

(3.17)
We will act with its of appropriate

\[ S_0(f_0) = -\frac{2(\alpha_0 + \alpha_1)}{f_0' - f_0^2 + \sigma z} + f_0, \quad S_0(\beta) = -\beta, \quad (3.18) \]

\[ S_0(f_0) = -\frac{2(-\alpha_0 - \alpha_1)}{f_1' + f_1^2 - \sigma z} - f_1, \quad S_0(\beta) = -\beta. \quad (3.19) \]

These expressions agree with Bäcklund transformations for \( P_\Pi \) equation and they obey the \( A_1^{(1)} \) group structure described in the literature [11, 13] although the whole \( A_1^{(1)} \) group structure requires presence of an additional automorphism to be introduced below.

### 3.1.3. The \( \Pi \) automorphism for \( P_\Pi \) model

In this subsection we will construct automorphisms \( \Pi, \tilde{\Pi} \) of \( P_\Pi \) equation that satisfy \( A_1^{(1)} \)-type relations:

\[ \Pi(f_0) = f_1, \quad \Pi(f_1) = f_0, \quad \Pi(\beta) = \sigma - \beta, \]

\[ \Pi S_i = S_j \Pi, \quad i, j = 0, 1, \quad \Pi^2 = 1 \quad (3.20) \]

and

\[ \tilde{\Pi}(f_0) = -f_0, \quad \tilde{\Pi}(f_1) = -f_1, \quad \tilde{\Pi}(\beta) = \sigma - \beta \]

\[ \tilde{\Pi} \tilde{S}_i = \tilde{S}_j \tilde{\Pi}, \quad i, j = 0, 1, \quad \tilde{\Pi}^2 = 1, \quad (3.21) \]

with \( A_1^{(1)} \) transformations \( S_i, \tilde{S}_i, i = 0, 1 \) defined in equations (3.16)–(3.19) as coalescence limits of appropriate \( A_1^{(1)} \) transformations to be defined below. Note that \( f_i, i = 0, 1 \) in the above relations satisfy \( P_\Pi \) equation (3.5).

We now return to \( P_{IV} \) model where we define \( \mathcal{P} \) and \( \mathcal{P}^{-1} \):

\[ \mathcal{P} := \pi S_0 = s_1 \pi, \quad \mathcal{P}^{-1} := s_0 \pi^2 = \pi^2 s_1, \quad (3.22) \]

with \( \pi \) and \( s_1 \) defined by relation (2.7) from \( P_{IV} \) model. The actions of \( \mathcal{P} \) and \( \mathcal{P}^{-1} \) on Bäcklund transformations \( S_i \) (3.14) satisfy the following relations:

\[ S_0 \mathcal{P} = \mathcal{P} S_1 : \{ \alpha_0 \rightarrow \alpha_0, \quad \alpha_1 \rightarrow \alpha_1 + \sigma \}, \]

\[ \mathcal{P} S_0 = S_1 \mathcal{P} : \{ \alpha_0 \rightarrow \alpha_0 - \sigma, \quad \alpha_1 \rightarrow \alpha_1 \}, \]

\[ S_0 \mathcal{P}^{-1} = \mathcal{P}^{-1} S_1 : \{ \alpha_0 \rightarrow \alpha_0 + \sigma, \quad \alpha_1 \rightarrow \alpha_1 \}, \]

\[ \mathcal{P}^{-1} S_0 = S_1 \mathcal{P}^{-1} : \{ \alpha_0 \rightarrow \alpha_0, \quad \alpha_1 \rightarrow \alpha_1 - \sigma \}. \quad (3.23) \]

Accordingly \( \mathcal{P} \) and \( \mathcal{P}^{-1} \) satisfy the product rules with \( S_i \) identical to those given in relations (3.20) and (3.21) although valid in the context of \( P_{IV} \) model.

Further one finds using the table (3.10) and relations (3.15) to calculate the actions \( \mathcal{P} \) and \( \mathcal{P}^{-1} \) on \( f_i, i = 0, 1 \) that they both converge to \( \Pi \) and \( \tilde{\Pi} \) in the \( \epsilon \rightarrow 0 \) coalescence limit. To illustrate this we will act with \( \mathcal{P} \) on \( f_i, i = 0, 1 \) to obtain according to the table (3.10):

\[ \mathcal{P}(f_0) = \pi(f_0) = f_1, \quad (3.24) \]

\[ \mathcal{P}(f_1) = \pi \left( f_1 + \frac{\alpha_0 \epsilon - 1/\epsilon^2}{f_0 + 1/\epsilon} \right) = f_2 + \frac{\alpha_1 \epsilon + 1/\epsilon^2}{f_1 + 1/\epsilon}, \quad (3.25) \]

where as we recall \( f_2 = \sigma \epsilon z - f_1 - f_0 - 1/\epsilon \). The relations (3.15) can now be used to substitute \( f_1 \) by \( f_0 \) on the right-hand side of equation (3.24) and \( f_0 \) by \( f_1 \) on the right-hand side of equation (3.25) giving in the limit \( \epsilon \rightarrow 0 \) the result (3.21). Using relation (3.15) to eliminate
\( f_1 \) and substitute it by \( f_0 \) on the right-hand side of equation (3.25) gives in the limit \( \epsilon \to 0 \) the result (3.20).

4. The mixed P_{II–IV} equations and its Hamiltonian

We will now consider the following class of generalizations of P_{IV} equation (2.3) by adding nontrivial terms parameterized by constants \( k_1, k_2 \):

\[
f_0' = \alpha_0 - \sigma zf_0 + f_0^2 + 2f_0f_1 + \frac{1}{\epsilon} (-k_1\sigma z + k_2(f_0 + f_1)), \tag{4.1}
\]
\[
f_1' = \alpha_1 + \sigma zf_1 - f_1^2 - 2f_0f_1 + \frac{1}{\epsilon} (k_1\sigma z - k_2(f_0 + f_1)). \tag{4.2}
\]

We will determine values of constants \( k_1, k_2 \) for which the above equations reproduce P_{II} equation in the \( \epsilon \to 0 \) limit.

Note that we can write the equations (4.1) and (4.2) as Hamilton equations with the Hamilton function (1.1), which generalized the cubic P_{IV} Hamiltonian (1.2) due to addition of quadratic terms with constants \( k_1, k_2 \).

First let us comment on how general are such extensions of P_{IV} model. Replace the term \( k_2(f_0 + f_1) \) on the right-hand sides of equations (4.1) and (4.2) with a more general combination \( k_2 f_0 + k_3 f_1 \) such that \( k_2 \neq k_3 \). In such case the resulting second order equation for \( f_0 \) and \( f_1 \) would be divergent in the limit \( \epsilon \to 0 \). For example, \( f_0'' \) would contain the term \( (k_3 - k_2)f_0^2/(2f_0\epsilon^2 + k_3\epsilon) \) that would go to infinity for \( \epsilon \to 0 \) unless \( k_2 = k_3 \). Thus, we have to set \( k_2 = k_3 \) as we did in equations (4.1) and (4.2). The addition of terms proportional to \( z f_i \) is also forbidden for the same reason.

For \( \alpha_i = a_i \epsilon, i = 0, 1, 2, \sigma = \sigma_0 \epsilon \) the second order equation for \( f_0 \) in the \( \epsilon \to 0 \) limit is:

\[
f_0'' = 2f_0^3 + 2(k_1 - k_2)\sigma_0 f_0 + k_2 a_1 + k_2 a_0 - k_1 \sigma_0. \tag{4.3}
\]

Thus as long as

\[
k_1 \neq k_2, \tag{4.4}
\]

the system (4.1) and (4.2) will have P_{II} equation as a limit.

We now discuss the conditions for the system (4.1) and (4.2) to remain invariant under the \( A_{2}^{(1)} \) symmetry.

Insert \( f_2 = \sigma z - f_1 - f_0 \) back into equations (4.1), (4.2) and rewrite them as:

\[
f_0' = f_0(f_1 - f_2) + \alpha_0 + \frac{1}{\epsilon}(k_2(f_0 + f_1) - k_1 \sigma z),
\]
\[
f_1' = f_1(f_2 - f_0) + \alpha_1 - \frac{1}{\epsilon}(k_2(f_0 + f_1) - k_1 \sigma z),
\]
\[
f_2' = f_2(f_0 - f_1) + \alpha_2, \tag{4.5}
\]

with \( \alpha_2 = \sigma - \alpha_0 - \alpha_1 \).

Following appendix A we now introduce

\[
\bar{f}_0 = f_0 + \frac{d}{\epsilon}, \quad \bar{f}_1 = f_1 + \frac{d}{\epsilon}, \quad \bar{f}_2 = f_2 \tag{4.6}
\]

\[\text{v11}\]
in an effort to remove through this shift of $f_i$’s the extra terms with $k_1, k_2$ constants from the
generalized PIV equations (4.1) and (4.2). In this way we obtain

\[
\begin{align*}
\bar{f}_0' &= \bar{f}_0 (\bar{f}_1 - \bar{f}_2) + \alpha_0 + \frac{d}{\epsilon^2} - \frac{d}{\epsilon} (\bar{f}_0 + \bar{f}_1 - \bar{f}_2) \\
&\quad + \frac{1}{\epsilon} \left( k_2 \left( \bar{f}_0 + \bar{f}_1 - \frac{2d}{\epsilon} \right) - k_1 \sigma_z \right), \\
\bar{f}_1' &= \bar{f}_1 (\bar{f}_2 - \bar{f}_0) + \alpha_1 - \frac{d}{\epsilon^2} + \frac{d}{\epsilon} (\bar{f}_0 + \bar{f}_1 - \bar{f}_2) \\
&\quad - \frac{1}{\epsilon} \left( k_2 \left( \bar{f}_0 + \bar{f}_1 - \frac{2d}{\epsilon} \right) - k_1 \sigma_z \right), \\
\bar{f}_2' &= \bar{f}_2 (\bar{f}_0 - \bar{f}_1) + \alpha_2.
\end{align*}
\] (4.7)

In the first equation in (4.7) the terms with $(\bar{f}_0 + \bar{f}_1)$ and the terms with $\sigma_z$ will appear as

\[
-(\frac{2d - k_2}{\epsilon})(\bar{f}_0 + \bar{f}_1) + \sigma_z \frac{\epsilon}{d - k_1},
\] (4.8)

after eliminating $f_2$ from this equation. The same terms but with the opposite sign will appear in the second equation in (4.7).

With condition (4.4) satisfied we now describe two possible cases, the first case coincides with the PIV coalescence model discussed in section 3 and the second defines deformation of PIV model to be discussed in section 5.

Case 1. Both terms in equation (4.8) vanish. This can only occur for

\[2d = k_2, \quad d = k_1,\]

which requires

\[k_2 = 2k_1.\] (4.9)

Condition (4.9) allows to restore the full $A_2^{(1)}$ symmetry in the generalized PIV equations (4.1) and (4.2). Recall that such mechanism took place in the PIV coalescence model. For example, for $k_1 = 1, k_2 = 2, \sigma = \epsilon \sigma_0$ we recognize the coalescence case of (3.3).

Case 2. Only one term in equation (4.8) vanishes. Accordingly, we consider $k_2 \neq 2k_1$ and $k_1 \neq k_2$ (preserving (4.4)). Setting the variable $d$ to eliminate one of the two terms in (4.8), say

\[d = k_1,\]

results in $2d - k_2 = 2k_1 - k_2 \neq 0$. Consequently the only non-zero extra term in the first equation in (4.7) is

\[-\frac{(2k_1 - k_2)}{\epsilon}(\bar{f}_0 + \bar{f}_1).\] (4.10)

Such system will be referred to as a deformed PIV model and will be discussed in the subsequent section. One easily verifies that choosing $d = k_2/2$ will result in a similar model.
5. Deformation of PIV model

As we have seen in section 4, PII equation can also be obtained from deformation of PIV that changes its symmetry structure even before the limit is taken.

Following derivation presented in section 4 we now propose the following PIV model:

\[ H = -f_0 f_1 f_2 + \frac{-\alpha_1 + \alpha_2}{3} f_0 + \frac{-\alpha_1 - 2\alpha_2}{3} f_1 + \frac{2\alpha_1 + \alpha_2}{3} f_2 + \sum_{i,j,k} \frac{1}{2}\eta (f_j + f_k)^2, \quad (5.1) \]

as a generalization of the structure in (1.2). The summation in (5.1) is over all three indices \( i, j, k \) being distinct. The parameters \( \eta_i, i = 0, 1, 2 \) are referred to as deformation parameters.

The corresponding equations are

\[
\begin{align*}
\bar{f}_0, z &= f_0 (f_1 - f_2) + \alpha_0 - \eta_1 (f_0 + f_2) + \eta_2 (f_0 + f_1), \\
\bar{f}_1, z &= f_1 (f_2 - f_0) + \alpha_1 + \eta_0 (f_1 + f_2) - \eta_2 (f_0 + f_1), \\
\bar{f}_2, z &= f_2 (f_0 - f_1) + \alpha_2 - \eta_0 (f_1 + f_2) + \eta_1 (f_0 + f_2).
\end{align*}
\quad (5.2)
\]

Equation (5.2) are invariant under automorphisms (2.9) and (2.10) augmented by

\[
\begin{align*}
\pi_i(\eta_i) &= -\eta_i, \quad \pi_i(\eta_j) = -\eta_k \quad i, j, k \text{ distinct}.
\end{align*}
\]

Introduce

\[
\bar{f}_i = f_i + \xi_i, \quad i = 0, 1, 2,
\quad (5.3)
\]

with

\[
\xi_i = \frac{1}{2}(\eta_j + \eta_k), \quad i, j, k \text{ distinct}.
\quad (5.4)
\]

Note, that

\[
\sum_i \bar{f}_i = \sum_i f_i + \sum_i \xi_i = \sigma z + \eta_0 + \eta_1 + \eta_2.
\quad (5.5)
\]

The equation (5.2) can then be recast back into the original form of PIV symmetric equations:

\[
\begin{align*}
\bar{f}_0, z &= \bar{f}_0 (\bar{f}_1 - \bar{f}_2) + \bar{\alpha}_0, \\
\bar{f}_1, z &= \bar{f}_1 (\bar{f}_2 - \bar{f}_0) + \bar{\alpha}_1, \\
\bar{f}_2, z &= \bar{f}_2 (\bar{f}_0 - \bar{f}_1) + \bar{\alpha}_2.
\end{align*}
\quad (5.6)
\]

but with the \( z \)-dependent coefficients:

\[
\begin{align*}
\bar{\alpha}_0 &= \alpha_0 + \frac{1}{4}(\eta_1^2 - \eta_2^2) + \frac{1}{2}(\eta_2 - \eta_1)\sigma z, \\
\bar{\alpha}_1 &= \alpha_1 + \frac{1}{4}(\eta_2^2 - \eta_0^2) + \frac{1}{2}(\eta_0 - \eta_2)\sigma z, \\
\bar{\alpha}_2 &= \alpha_2 + \frac{1}{4}(\eta_0^2 - \eta_1^2) + \frac{1}{2}(\eta_1 - \eta_0)\sigma z.
\end{align*}
\quad (5.7)
\]
that still satisfy \( \sum \bar{\alpha}_i = \sum \alpha_i = \sigma \).

For \( \eta_i = \eta_j, i \neq j \) the \( z \)-dependence will disappear from \( \bar{\alpha}_k = \alpha_k, k \neq i, k \neq j \) and the system will become invariant under one specific Bäcklund transformation \( \bar{s}_k \) defined as one of the following transformations:

\[
\begin{array}{c|ccc|ccc}
\bar{s}_0 & -\bar{\alpha}_0 & \bar{\alpha}_1 + \bar{\alpha}_0 & \bar{\alpha}_2 & \bar{\alpha}_0 & \bar{\alpha}_1 & \bar{\alpha}_2 \\
\bar{s}_1 & \bar{\alpha}_0 + \bar{\alpha}_1 & -\bar{\alpha}_2 + \bar{\alpha}_1 & \bar{\alpha}_2 & \bar{\alpha}_0 & \bar{\alpha}_1 & \bar{\alpha}_2 \\
\bar{s}_2 & \bar{\alpha}_0 + \bar{\alpha}_2 & \bar{\alpha}_1 + \bar{\alpha}_2 & -\bar{\alpha}_2 & \bar{\alpha}_0 & \bar{\alpha}_1 & \bar{\alpha}_2 \\
\end{array}
\]

For \( \eta_i = \eta_j \) the \( z \)-dependence will disappear from \( \bar{\alpha}_k = \alpha_k, k \neq i, k \neq j \) and the system will become invariant under one specific \( \bar{s}_k \) transformation defined as one of the following transformations:

\[
(5.8)
\]

Now set \( \eta_0 = \eta_1 = 0 \) and \( \eta_2 = 2/\epsilon \) in (5.2). We see that in such case (5.2) becomes (4.5) with \( k_1 = 0 \) and \( k_2 = 2 \) and since \( k_1 \neq k_2 \) we know from equation (4.3) that the limit will still be \( P_{\Pi} \).

The condition \( \eta_i = \eta_j \) for \( i \neq j \) and corresponding invariance under \( s_k \) transformation turns out to be a condition for the model to pass Kovalevskaya–Painlevé test as we will now explain.

5.1. Kovalevskaya–Painlevé test of the deformed model (5.1)

Assume that solutions of the extended PIV (5.2) equations have the form

\[
f_i = a_i z + b_i + c_i z^2 + d_i z^3 + \cdots, \quad i = 0, 1, 2.
\]

(5.9)

Substituting into (5.2) yields

\[
0 = a_i (a_i + 1 - a_{i-1}) + a_i, \quad (5.10)
\]

\[
0 = a_i (b_i + 1 - b_{i-1}) + b_i (a_{i+1} - a_{i-1}) - \eta_{i+1} (a_i + a_{i-1}) + \eta_{i-1} (a_i + a_{i+1}), \quad (5.11)
\]

\[
c_i = a_i (c_i + c_{i-1}) + a_i + c_i (a_{i+1} - a_{i-1}) + b_i (b_i - b_{i-1}) + \eta_{i-1} (b_i + b_{i-1}) - \eta_{i+1} (b_i + b_{i-1}), \quad (5.12)
\]

\[
2d_i = b_i (c_i + c_{i+1}) + b_i (b_i + b_{i-1}) + a_i (a_{i+1} - a_{i-1}) + a_i (d_{i+1} - d_{i-1}) + \eta_{i-1} (c_i + c_{i+1}) - \eta_{i+1} (c_i + c_{i-1}), \quad (5.13)
\]

and etc for \( i = 0, 1, 2 \). Since \( \sum a_i = 0 \) there are three (up to a sign and an overall constant) possible nontrivial solutions of the top equation in (5.10)

\[
(a_0, a_1, a_2) = (0, 1, -1), \quad (5.14)
\]

\[
(a_0, a_1, a_2) = (-1, 0, 1), \quad (5.15)
\]

\[
(a_0, a_1, a_2) = (1, -1, 0), \quad (5.16)
\]

which correspond to \( a_i = 0 \) for \( i = 0 \) or \( i = 1 \) or \( i = 2 \). The automorphism \( \pi_j \) will take the configuration with \( a_i = 0 \) into the one with \( a_k = 0 \) for the three distinct indices \( i, j, k \).

We will show that for a given \( i \) such that \( a_i = 0 \) the solution (5.9) will pass the Kovalevskaya–Painlevé test [23, 24] as long as \( \eta_j = \eta_k \).
We will illustrate the argument for $a_0 = 0$ as in (5.14). Plugging the sequence from (5.14) into (5.11) we find that
\[ b_0 = -\frac{1}{2}(\eta_1 + \eta_2), \quad b_2 = b_1 + \frac{1}{2}(\eta_2 - \eta_1). \] (5.17)

Thus, in the case of (5.14) all the parameters $b_i$ are determined with exception of one, either $b_1$ or $b_2$. For (5.15) the determined coefficient in term of $\eta$-coefficients will be $b_1$ with one of $b_0$ or $b_2$ coefficients being undetermined. For (5.16) the determined coefficient will be $b_2$ while one of the two other coefficients remaining undetermined. This is a general feature which is present independently of whether the $\eta$ deformation terms are present or not.

From (5.12) we find that all the coefficients $c_i$ multiplying $z$ are determined in terms of the lower coefficients:
\[ c_0 = -\alpha_0 + (\eta_1 + \eta_2)^2/4 - \eta_1^2 + b_1(\eta_1 - \eta_2). \] (5.18)
\[ c_1 = \frac{1}{3}(3\alpha_0 + 2\alpha_1 + \alpha_2) + \frac{1}{3}b_1(\eta_2 - 2\eta_0 - \eta_1 - b_1) + \frac{1}{6}\eta_1(\eta_0 - 2\eta_2) + (\eta_1^2 - \eta_2^2)/4 - \frac{1}{6}\eta_0\eta_2, \] (5.19)
\[ c_2 = \frac{1}{3}(3\alpha_0 + 2\alpha_1 + \alpha_2) + \frac{1}{3}b_1(2\eta_2 + 2\eta_0 - 2\eta_1 + b_1) + \frac{1}{6}(\eta_0\eta_2 - \eta_1\eta_2 - \eta_0\eta_1) + \eta_1\eta_2/4. \] (5.20)

By summing the above coefficients one confirms that they satisfy the condition
\[ c_0 + c_1 + c_2 = \alpha_0 + \alpha_1 + \alpha_2 = \sigma, \] (5.21)
as expected from their definition in (5.9).

Let us rewrite equation (5.13) as
\[ 2d_i - d_i(a_{i+1} - a_{i-1}) - a_i(d_{i+1} - d_{i-1}) = b_i(c_i + c_{i+1} - c_{i-1}) + c_i(b_{i+1} - b_{i-1}) + \eta_{i-1}(c_i + c_{i+1}) - \eta_{i+1}(c_i + c_{i-1}), \] (5.22)
where we have grouped the terms with $d_i$ on the left-hand side of the equation. In all three (5.14), (5.15) and (5.16) cases summing the left-hand side of (5.22) over $i = 0, 1, 2$ gives $2(d_0 + d_1 + d_2)$ while the sum of the right-hand side of (5.22) over $i = 0, 1, 2$ vanishes as all the terms cancel each other. This confirms that $\sum_i d_i = 0$ as expected from the definition in (5.9).

For the choice (5.14) the left-hand side of equation (5.22) vanishes for $i = 0$ while the right-hand side is equal to
\[ \frac{1}{2}(c_0 + c_1 + c_2)(\eta_2 - \eta_1) = \frac{1}{2}\sigma(\eta_2 - \eta_1). \]

Thus consistency requires in the case of (5.14) that $\eta_2 = \eta_1$. Similarly for the case (5.15) we find the left-hand side of equation (5.22) vanishes for $i = 1$ while the right-hand side is equal to $\sigma(\eta_2 - \eta_0)/2$ and for (5.16) we find the left-hand side of equation (5.22) vanishes for $i = 2$ while the right-hand side is equal to $\sigma(\eta_1 - \eta_0)/2$.

Thus the condition for consistency is such that $\eta_j = \eta_k$ for the case of $a_i = 0$ with $d_i$ being the only undetermined coefficient among $d_1, d_2, d_2$. Generalizing the equation (5.22) to
coefficient $f^{(k)}_i$ of $z^k$ gives an equation with a left-hand side: $k f^{(k)}_i - f^{(k)}_i(a_{i+1} - a_{i-1}) - a_i(f^{(k)}_i - f^{(k)}_{i+1})$. This relation can be cast in terms of the $3 \times 3$ matrix with a determinant $k(k - 2)(2k + 1)$. Correspondingly, the undetermined coefficients only appear for $k = 0$ and $k = 2$ as one of $b_i$ and $d_i$ coefficients consistent with what we have seen above. Together with a position of the pole this leaves exactly three parameters as arbitrary with all the remaining coefficients fully determined. This demonstrates existence of a solutions with simple pole structure and dependence on three arbitrary constants that are consistent when two of the deformations parameters are equal to each other.

Thus we have connected the integrability property associated with the fact of passing the Kovalevskaya–Painlevé test to presence of the Bäcklund symmetry under $s_i$ emerging from the consistency condition $\eta_j = \eta_k$.

5.2. $P_{II}$ limit of the deformed symmetric $P_{IV}$ equation

The starting point here are equations

$$
egin{align*}
\bar{f}_{0z} &= f_0 (f_1 - f_2) + \alpha_0 + \eta (f_0 + f_1), \\
\bar{f}_{1z} &= f_1 (f_2 - f_0) + \alpha_1 - \eta (f_0 + f_1), \\
\bar{f}_{2z} &= f_2 (f_0 - f_1) + \alpha_2.
\end{align*}
$$

(5.23)

of the deformed $P_{IV}$ obtained from (5.2) by setting $\eta_0 = \eta_1 = \eta_0 = 0$. The parameter $\eta$ is equal to the constant $-(2k_1 - k_2)/\epsilon$ in equation (4.10) and as we have learned in section 4 equation (5.23) will have $P_{II}$ limit which we elaborate in this section in greater details including application of the Painlevé test.

We recall that for $\eta_2 = \eta, \eta_1 = \eta_0 = 0$ equation (5.23) is invariant under $s_2$ Bäcklund symmetry, and automorphism from the table (2.9) with $\pi_2(\eta) = -\eta$ and $\rho_2$ from the table (2.10) with $\rho_2(\eta) = \eta$.

Using association $f_1 = -q$ with $f_0 + f_1 + f_2 = \sigma z$ we get from (5.23) the following equation for $q$:

$$
q_{z} = \frac{q_{z}^2}{2q - \eta} + \frac{1}{2q - \eta} \left(3q^4 + 2q^2(2\sigma z - \eta) + q^2(2\alpha_1 + 4\alpha_2 - 2\sigma - 5\eta \sigma z + \sigma^2 z^2) + q(3\sigma \eta - 2\alpha_1 \eta - \eta \sigma^2 z^2 + 2q^2 \sigma z - 4\alpha_2 \eta) - \eta \sigma z - \alpha_1 - \alpha_2 \eta^2 + \eta \alpha_1 \sigma z. \right)
$$

(5.24)

For $\eta \to 0$ we obtain $P_{IV}$ equation:

$$
q_{z} = \frac{q_{z}^2}{2q} + \frac{3q^4}{2q} + 2q^2 \sigma z + q \left(\alpha_1 + 2\alpha_2 - \sigma + \frac{1}{2} \sigma^2 z^2 \right) - \frac{\alpha_2^2}{2q},
$$

(5.25)

that agrees with equation (2.4) for $f_1 = -q$.

For $\sigma \to 0$ and $Q = q - \eta/2$:

$$
Q_{z} = \frac{Q_{z}^2}{2Q} + \frac{3}{2} Q^3 + 2Q^2 \eta + Q \left(\alpha_1 + 2\alpha_2 + \frac{3}{4} \eta^2 \right) - \frac{1}{2Q} \left(\alpha_1 + \frac{1}{4} \eta^2 \right)^2,
$$

(5.26)

which is $I_{40}$ for $\eta \neq 0$.

For $\sigma = \sigma_0/\eta, \alpha_1 = \alpha_1/\eta, \alpha_2 = \alpha_2/\eta$ and in the limit $\eta \to \infty$ we get:

$$
q_{z} = 2q^3 - 2q \sigma_0 z - a_2 + \sigma_0 = 2q^3 - 2q \sigma_0 z + (a_0 + a_1).
$$

(5.27)
More generally for $f_0$ and $f_1$ from equation (5.23) we obtain in the limit $\eta \to \infty$;

$$f_i^{(n)}(z) = (-1)^i(\alpha_0 + \alpha_1) - 2\sigma_0 z f_i + 2 f_i^3, \quad i = 0, 1. \quad (5.28)$$

in which we recognize two PII equations for $i = 0$ and $i = 1$ that again are transformed into each other under the automorphism $p_2$ from (2.10) but differ from PII equations in (3.5) by the values of the constant coefficients on the right-hand sides.

Because of the presence of deformation parameter $\eta$ in the denominator in relation (5.24) it appears that the three cases $\eta \ll 1, \eta \gg 1$ and $\eta$-finite need to be considered separately. For the first two cases we are in PIV and PII regimes, respectively but for finite $\eta$ it makes sense to make a change of variables $q \to Q = q - \eta/2$ with corresponding equation

$$Q_{zz} = \frac{Q^2}{2Q} + \frac{3}{2} Q^3 + 2 \mathcal{O}(\sigma z + \eta) + \mathcal{O} \left( \frac{1}{2} \sigma \beta^2 - \frac{1}{2} \eta \sigma z + \alpha_1 + 2 \alpha_2 - \sigma \frac{3}{2} \eta \right)$$

$$+ \frac{1}{2} \eta \sigma \alpha_1 - \frac{1}{2} \eta^2 \alpha_1 - \frac{1}{4} \eta^3 \sigma z - \frac{1}{16} \eta^4 \right). \quad (5.29)$$

In appendix B we provide details of the Painlevé test applied on equation (5.29). That equation (5.29) passes the direct Painlevé test agrees with the result of the Kovalevskaya–Painlevé test that established the consistency of the extended PIV (5.2) as long as two out three $\eta$ parameters are equal (which is the case here).

5.3. First order PII equations as a limit of the deformed model

Here we will show how starting from equation (5.23) to obtain the first order system of equations underlying the PII equation (5.28) and their $A_{11}^{(1)}$ Bäcklund transformations in a limit $\eta \to \infty$. We set $\sigma = \sigma_0/\eta, \alpha_1 = \alpha_1/\eta, \alpha_2 = \alpha_2/\eta$ with constants $\sigma_0, \alpha_1, \alpha_2$ and represent $f_1, f_2$ as

$$f_1 = -q, \quad f_2 = -\frac{2}{\eta} p, \quad f_0 = \frac{\sigma_0}{\eta} z + \frac{2}{\eta} p + q. \quad (5.30)$$

Plugging these substitutions into (5.23) we obtain

$$-\frac{2}{\eta} p_i = -\frac{2}{\eta} p \left( \frac{\sigma_0}{\eta} z + \frac{2}{\eta} p + 2q \right) + \frac{\alpha_2}{\eta}, \quad (5.31)$$

$$-q_z = -q \left( -\frac{\sigma_0}{\eta} z - q - \frac{4}{\eta} p \right) + \frac{\alpha_1}{\eta} - \eta \left( \frac{\sigma_0}{\eta} z + \frac{2}{\eta} p \right). \quad (5.32)$$

Considering large $\eta$ and neglecting the terms of order $O(1/\eta^2)$ in the first equation and the terms of order $O(1/\eta)$ one obtains in such limit equations

$$p_z = 2pq - \frac{1}{2} \alpha_2, \quad (5.33)$$

$$q_z = -q^2 + \sigma_0 z + 2p. \quad (5.34)$$

Taking the derivative with respect to $z$ on both sides of (5.34) gives PII equation (5.27) (or (5.28) with $f_1 = -q$).

Let us now repeat the above analysis to obtain the PII equation (5.28) with a different sign of the constant term. We consider

$$f_0 = -y, \quad f_2 = -\frac{2}{\eta} h, \quad f_1 = \frac{\sigma_0}{\eta} z + \frac{2}{\eta} h + y. \quad (5.35)$$
that follows from identification (5.30) via acting with \(\rho_2\) automorphism and replacing \(q, p\) with \(y, h\) to emphasize that we are working with a different PII equation. Plugging substitutions (5.35) into (5.23) like in (5.32) and considering large \(\eta\) we arrive at the system of first order equations:

\[
\begin{align*}
  h_z &= -2hy - \frac{1}{2}a_2, \\
  y_z &= y^2 - \sigma_0 z - 2h,
\end{align*}
\]

that lead to the second PII equation namely (5.28) with \(f_1 = -y\).

From equation (5.34) we derive

\[
p = \frac{1}{2}(qz + q^2 - \sigma_0 z).
\]

In order to conveniently introduce all the \(A^{(1)}_1\) symmetry generators in the setting of equations (5.33), (5.34) let us define an auxiliary quantity \(v\) obtained from \(p\) given in (5.34) by transformation \(q \rightarrow -q\):

\[
v = \frac{1}{2}(-qz + q^2 - \sigma_0 z) = -p + q^2 - \sigma_0 z.
\]

Taking a derivative on both sides of equation (5.39) we obtain the counterparts of equations (5.33), (5.34) valid for \(v, q\):

\[
\begin{align*}
  v_z &= -2\nu q + \frac{1}{2}(-\sigma_0 - a_0 - a_1), \\
  q_z &= q^2 - \sigma_0 z - 2v,
\end{align*}
\]

Note that the transformation:

\[
\Pi : q \rightarrow -q, \quad p \rightarrow v, \quad \sigma_0 \rightarrow \sigma_0, \quad a_0 + a_1 \rightarrow -a_0 - a_1,
\]

takes equations (5.33), (5.34) into equations (5.40), (5.41) and does not change PII equation (5.27). Also note that the transformation:

\[
\bar{\rho} : z \rightarrow -z, \quad \sigma_0 \rightarrow -\sigma_0, \quad p \rightarrow v, \quad q \rightarrow q,
\]

will have the same effect.

Alternatively, equations (5.40), (5.41) can be obtained directly from symmetric deformed PIV equation (5.23) through the following substitution of \(f_1, f_2\):

\[
\begin{align*}
  f_1 &= q, \\
  f_2 &= \frac{2}{\eta}v, \\
  f_0 &= \frac{\sigma_0}{\eta}z - \frac{2}{\eta}v - q, \\
  a_2 &= -(\sigma_0 + a_0 + a_1),
\end{align*}
\]

for large \(\eta\) values.

Recall that for the deformed PIV equation (5.2) with \(\eta = \eta_2 \neq 0 \text{ and } \eta_1 = \eta_0 = 0\) the surviving symmetry generator is \(s_2\):

\[
\begin{align*}
  f_1 \xrightarrow{s_2} f_1 - \frac{a_2}{f_2}, \\
  f_2 \xrightarrow{s_2} f_2, \\
  \alpha_2 \xrightarrow{s_2} - \alpha_2,
\end{align*}
\]

or in terms of variables \(p, q\) used above:

\[
\begin{align*}
  q \xrightarrow{s_2} q - \frac{a_2}{2p}, \\
  p \xrightarrow{s_2} p, \\
  a_2 \xrightarrow{s_2} - a_2,
\end{align*}
\]
after cancellation of $\eta$. One easily checks that indeed equations (5.33), (5.34) are invariant under $s_2$ transformation as shown in (5.46). Note that $s_2 : a_0 + a_1 \rightarrow -(a_0 + a_1) + 2\sigma_0$.

Similarly inserting representation (5.44) into expression for the $s_2$-transformation (5.45) produces after cancellation of $\eta$

$q \xrightarrow{s_2} q + \frac{a_0 + a_1}{2v}, \quad v \xrightarrow{s_2} v, \quad a_0 + a_1 \xrightarrow{s_2} -(a_0 + a_1) = 2\sigma_0, \quad (5.47)$

where we denoted $s_2$ by $\tilde{s}_2$ when it acts on $q, v$ system to distinguish it from $s_2$ as defined in relations (5.46). Both transformations $s_2$ and $\tilde{s}_2$ defined in (5.46) and (5.47) keep the PII equation (5.27) invariant and square to one. It is interesting to compare action of $\tilde{s}_2$ to that of the automorphism $\tilde{\Pi}^0 = s_0 s_1 s_0$ from equation (3.18). Introducing $\gamma = \sigma_0 + a_0 + a_1$ we can rewrite the nontrivial part of transformation (5.47) as

$q \xrightarrow{\tilde{s}_2} q + \frac{\gamma}{-q - q^2 - \sigma_0 z}, \quad \gamma \xrightarrow{\tilde{s}_2} -\gamma, \quad (5.48)$

where we inserted the definition of $v$ from equation (5.39). Comparing with expression (3.18) we see that the action of $\tilde{s}_2$ almost agree with the limit of $s_0 s_1 s_0$ and the difference is only due to the difference between constant terms of PII equations given in (5.28) versus (3.5).

Using relation (5.39) between $v$ and $p$ one also derives formulas for actions of $\tilde{s}_2$ on $p$ and $s_2$ on $v$:

$v \xrightarrow{s_2} v - \frac{q}{p} a_2 + \frac{a_2^2}{4p^2}, \quad (5.49)$

$p \xrightarrow{\tilde{s}_2} p + \frac{q}{v} (\sigma_0 + a_0 + a_1) + \frac{(\sigma_0 + a_0 + a_1)^2}{4v^2}. \quad (5.50)$

These completes all the information on the Bäcklund transformations of the $A^{(1)}_{1}$ symmetry group consisted of $s_2, \tilde{s}_2, \Pi$ of PII equation obtained as a limit of the deformed PIV model.

Note that equations (5.33), (5.34) and equations (5.40), (5.41) can be compactly summarized as a system of equations

$q_t = p - v, \quad p_t = 2pq - \frac{1}{2} a_2, \quad (5.51)$

$v_t = -2vq + \frac{1}{2} (-\sigma_0 - a_0 - a_1).$

manifestly invariant under $\Pi$ and $\tilde{\Pi}$.

Similarly, equations (5.33) and (5.34) would enter into a system of equations

$y_t = u - h, \quad h_t = -2hy - \frac{1}{2} a_2, \quad u_t = 2uy + \frac{1}{2} (-\sigma_0 - a_0 - a_1). \quad (5.52)$

that lead to the other copy of PII equations in (5.28) with its own $A^{(1)}_{1}$ symmetry.
6. Concluding comments

We would like to make few comments on special novel features of our formalism.

By enlarging a parameter space of PIV model we extended the $A_{2}^{(1)}$ symmetry structure by additional automorphisms $\pi_i, \rho_i, i = 0, 1, 2$. In particular, the presence of the automorphism $\rho_2$ facilitated the reduction process from $A_{2}^{(1)}$ to $A_{1}^{(1)}$. The automorphism $\rho_2$ together with the Bäcklund transformation $s_2$ remain a symmetry for PII–IV and survive the PII limit while they also commute with each other. A crucial feature of PII limit of PIV generalized models is that it consists of two PII equations (see (3.5), (5.33), (5.34) and (5.36), (5.37)) connected via automorphism $\rho_2$. Each of the two PII equations is invariant under $A_{1}^{(1)}$ symmetry. Thus the presence of $\rho_2$ is critically important for the full understanding of all features of the formalism. Note that in order to define the action of automorphisms $\pi_i, \rho_i, i = 0, 1, 2$ they need to be formulated on an enlarged parameter space that includes $\sigma$ that transforms nontrivially under these automorphisms, see tables (2.9) and (2.10). The presence of $\sigma$ affords us also an opportunity to include in the formalism the solvable Painlevé equations (classified by Gambier) that appear on Ince’s list [12] (see also [2]). In particular, equations I30, I8 given in equations (3.7) and (3.9) were obtained here in the $\sigma \to 0$ limit.

As long as $\sigma$ remains non-zero there exists a transformation

$$\alpha_i = \tilde{\alpha}_i \sigma, \quad f(z) = \sqrt{\sigma} \tilde{f}(\tilde{z}), \quad z = \tilde{z}/\sqrt{\sigma}$$

given in terms of quantities entering equations (2.2) that allows for absorbing $\sigma$ in the formalism and thus effectively setting it to 1. Note however that the possibility of redefining $\sigma = \sigma_0/\eta$ is essential for ability of taking PII limit for $\eta \to \infty$ in the deformed PIV model. Likewise, setting $\sigma = \sigma_0/\epsilon$ and taking $\epsilon \to \infty$ limit was crucial for recovering PIV equations from equation (3.3). Since we are interested in models that interpolate between PII and PIV and in automorphisms $\pi_i, \rho_i, i = 0, 1, 2$ we work here with the formalism depending explicitly on $\sigma$.

As pointed out below the definition of the Hamiltonian $H$ in (1.1) the choice of additional terms in $H$ ensured invariance under $\rho_2$. As explained below (1.1), equivalent theories with invariance under $\rho_0$ or $\rho_1$ could be introduced via simple redefinitions of the additional terms. We have provided arguments that there is one unique (up to simple redefinition of such additional terms) generalization of PIV model allowing addition of quadratic terms to the Hamiltonian and requiring finite limits and Painlevé property.

To summarize, in this work we focused on the symmetry properties for the two generalizations, namely coalescence and deformation, of PIV model contained in PII–IV Hamiltonian of (1.1). We derived Bäcklund transformations of the PII–IV model and uncovered a connection between presence of symmetry and passing of Painlevé property test. This work raises an interesting question whether other Painlevé/Ince equations can be unified within some mixed model similar to the one presented in this paper.

Acknowledgments

JFG and AHZ thank CNPq and FAPESP for financial support. VCCA thanks São Paulo Research Foundation (FAPESP) for financial support by Grants 2016/22122-9 and 2019/03092-0.
Appendix A. About introducing two integration constants into the PIV system

Above, we have studied the transformation (4.6) with \( \sum_{i} \bar{f}_i = \sum_{i} f_i + 2d/\eta \). We would therefore now investigate the PIV systems that generally allow for \( \sum_{i} f_i = \sigma z + C \).

Given is the PIV system

\[
f'_i = f_i (f_{i+1} - f_{i+2}) + \alpha_i, \quad i = 0, 1, 2
\]

(A.1)

invariant under Bäcklund symmetries \( s_i, i = 0, 1, 2 \) and \( \pi \). The two constraints of the PIV system

\[
\sum_{i} \alpha_i = \sigma, \quad \sum_{i} f_i = \sigma z + C
\]

(A.2)

define two possible integration constants \( \sigma, C \) of the PIV system. Customarily, people set \( C = 0 \) and \( \sigma = 1 \). Recall that setting \( \sigma = 0 \) reduces PIV to Ince’s XXX equation (see also [2]).

The integration constant \( C \) can be absorbed by redefining \( f_i \)’s: \( f_i \rightarrow g_i \) so that \( \sum_{i} g_i = \sigma z \) and the system is obviously still invariant under Bäcklund symmetries \( s_i, i = 0, 1, 2 \) and \( \pi \).

There appear (at least) three ways of changing variables to eliminate \( C \) from the constraint \( \sum_{i} f_i \). In each of these cases, the constant \( C \) will appear explicitly in the resulting differential equations.

(a)

\[
g_i = f_i + \eta, \quad i = 0, 1, 2, \quad 3\eta = -C
\]

(A.3)

with the shifted PIV system:

\[
g'_i = g_i (g_{i+1} - g_{i+2}) + \alpha_i - \eta (g_{i+1} - g_{i+2}), \quad i = 0, 1, 2
\]

(A.4)

(b)

\[
h_i = f_i + \eta, \quad i = 0, 1, h_2 = f_2, \quad 2\eta = -C
\]

(A.5)

with the shifted PIV system:

\[
h_{0, z} = h_0 (h_1 - h_2) + \alpha_0 + \eta^2 - \eta (h_0 + h_1 - f_2),
\]

\[
h_{1, z} = h_1 (f_2 - h_0) + \alpha_1 - \eta^2 - \eta (f_2 - h_0 - h_1)
\]

\[
f_{2, z} = f_2 (h_0 - h_1) + \alpha_2
\]

(A.6)

with \( f_2 = \sigma z - h_0 - h_1 - 2\eta = \sigma z - h_0 - h_1 + C \).

(c)

\[
d_0 = f_0 + \eta, \quad \eta = -C
\]

(A.7)

with the shifted PIV system:

\[
d_{0, z} = d_0 (f_1 - f_2) + \alpha_0 - \eta (f_1 - f_2),
\]

\[
f_{1, z} = f_1 (f_2 - d_0) + \alpha_1 + \eta f_1
\]

\[
f_{2, z} = f_2 (d_0 - f_1) + \alpha_2 - \eta f_2
\]

(A.8)

We refer the reader to section 4 where the above scheme (b) was employed to transform accordingly generalized PIV equations.
Appendix B. Painlevé test of equation (5.29)

In this appendix we will apply the Painlevé test to equation (5.29). Following the standard procedure of this test we first insert

$$Q(z) = a_0(z - z_0)^\mu$$

and focus on the dominant behavior near singularity on both sides of equation (5.29) to obtain

$$\mu(\mu - 1)a_0(z - z_0)^{\mu - 2} = \frac{a_0^2\mu^2(z - z_0)^2}{2a_0(z - z_0)^\mu} + \frac{3}{2}a_0(z - z_0)^3$$

with contributions on the right-hand side originating from the first and the second term of the right-hand side of equation (5.29). This way we obtain:

$$a_0^2 = 1, \quad \mu = -1$$

consistent with the Painlevé requirement that $\mu$ is a negative integer for a movable pole with no branching. Next, to check the resonance condition we plug

$$Q(z) = a_0(z - z_0)^{-1} + \eta(z - z_0)^{-1 + R}$$

into equation (5.29) and keep only the terms linear in $\eta$ to obtain the resonance equation for $R$:

$$(R + 1)(R - 3) = 0$$

This resonance structure suggests that a Laurent expansion

$$q(z) = \sum_{j=0}^{\infty} a_j(z - z_0)^{j-1}$$

$$= a_0(z - z_0)^{-1} + a_1 + a_2(z - z_0) + a_3(z - z_0)^2 + a_4(z - z_0)^3 + \cdots \quad (B.1)$$

expresses expansion around an arbitrary pole at $z_0$ where we identified $a_1 = h$ as the single arbitrary coefficient. Inserting expression (B.1) into (5.29) and looking on coefficients of power of $\eta = z - z_0$ we get:

$$0 = -a_0^3 + a_0^4$$

$$0 = 3a_0^3a_1 + a_0^3\eta - a_0a_1 + a_0^3\sigma z_0$$

$$0 = 12a_0^2a_1\sigma z_0 + 12a_0^2a_2 - 6a_0a_0 + a_0^2\eta^2 z_0 + 3a_0^2\eta^2/2 + 4a_0^2\sigma + 4a_0^2\sigma_0 + 2a_0^2\sigma + 12a_0^2\sigma_0 z_0 + 12a_0^2a_1\eta$$

$$0 = \sigma_0 a_0 + 12a_0^2a_1^2\sigma z_0 + 8a_0a_1+2 + 12a_2a_0^2\sigma z_0 + 12a_0^2a_0 + 12a_0^2a_1 + 12a_0^2a_1 + 12a_0^2a_1 + 12a_0^2a_1\sigma + 12a_0^2a_1^3 + 36a_2a_0^2a_1 + 2a_0a_1z_0 \sigma + 3a_0a_1 \eta^2 + 4a_1a_1 + 12a_0a_1^2 \eta + 4a_0a_1\sigma + 2a_0^2 \sigma^2 z_0 - 12a_0a_0 + a_0^2 \eta \sigma + 2a_0a_1 \sigma^2 z_0 + 12a_0^2 \eta$$

$$\cdots$$

(B.2)

The top equation gives two possible non-zero solutions

$$a_0 = \pm 1$$
The second equation gives:

\[ a_1 = -\frac{1}{2}(\eta + \sigma z_0) \]  

for both values \( a_0 = 1 \) and \( a_0 = -1 \).

The third equation gives two values for \( a_2 \):

\[ a_2 = \frac{\eta \sigma z_0}{3} + \frac{\sigma^2 z_0^2}{12} - \frac{2\alpha_2}{3} - \frac{\sigma}{3} - \frac{\alpha_1}{3} \]  

for \( a_0 = 1 \) and \( a_1 \) as given in (B.3) and

\[ a_2 = -\frac{\eta \sigma z_0}{3} - \frac{\sigma^2 z_0^2}{12} + \frac{2\alpha_2}{3} - \sigma + \frac{\alpha_1}{3} \]  

for \( a_0 = -1 \) and \( a_1 \) as given in (B.3).

Consider now the fourth equation. The coefficient \( a_3 \) drops from this equation for both values of \( a_0 = \pm 1 \) as it should (resonance \( R = 3 \)). The solutions of the fourth equation for \( a_2 \) are

\[ a_2 = \frac{\eta \sigma z_0}{3} + \frac{\sigma^2 z_0^2}{12} - \frac{2\alpha_2}{3} - \frac{\sigma}{3} - \frac{\alpha_1}{3} \]  

for \( a_0 = 1 \) and \( a_1 \) as given in (B.3) and

\[ a_2 = -\frac{1}{12(\eta + \sigma z_0)} \left( 5\sigma^2 z_0^2 \eta + \sigma^3 z_0^3 + 16\sigma \eta + 12\sigma^2 z_0 + 4\sigma z_0 \eta^2 
- 4\alpha_1 \eta - 4\alpha_1 \sigma z_0 - 8\alpha_2 \eta - 8\alpha_2 \sigma z_0 \right) \]  

for \( a_0 = -1 \) and \( a_1 \) as given in (B.3). We see that \( a_2 \) given in (B.4) and (B.6) are equal so the solution to the recursive problem is in this case consistent. In addition \( a_3 \) and higher coefficients will depend on \( a_3 \) but \( a_3 \) is not fixed by the scheme and can be taken to any value including zero.

However \( a_2 \) given in (B.5) and (B.7) differ by \( \sigma \eta/(3(\sigma z_0 + \eta)) \). Hence the second solution for \( a_0 = -1 \) is only consistent for \( \eta = 0, \sigma \neq 0 \) or \( \sigma = 0, \eta \neq 0 \).

It seems therefore that as long as \( a_0 = 1 \) there is a solution to the recurrence relations that does not fail the Painlevé test.

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