Seiberg-Witten Theory as $d < 1$ Topological Strings

Katsushi Ito, Chuan-Sheng Xiong

Yukawa Institute for Theoretical Physics
Kyoto University, Kyoto 606-8502, Japan

Sung-Kil Yang

Institute of Physics, University of Tsukuba
Ibaraki 305-8571, Japan

Abstract

In view of two-dimensional topological gravity coupled to matter, we study the Seiberg-Witten theory for the low-energy behavior of $N = 2$ supersymmetric Yang-Mills theory with $ADE$ gauge groups. We construct a new solution of the Picard-Fuchs equations obeyed by the Seiberg-Witten periods. Our solution is expressed as the linear sum over the infinite set of one-point functions of gravitational descendants in $d < 1$ topological strings. It turns out that our solution provides the power series expansion around the origin of the quantum moduli space of the Coulomb branch. For $SU(N)$ gauge group we show how the Seiberg-Witten periods are reconstructed from the present solution.
Recently it has become clear that the Seiberg-Witten (SW) solution \[1\] of $N = 2$ supersymmetric Yang-Mills theory shares several common properties with two-dimensional topological field theory. For $ADE$ gauge groups this relation is recognized if one considers the SW solution in view of the $ADE$ singularity theory. The relevance of the $ADE$ singularity theory to the SW solution was speculated when the original $SU(2)$ SW solution was generalized to the case of other gauge groups \[2\].

To further develop the idea it was crucial that the $ADE$ singularity, which is usually expressed in terms of three complex variables, admits the description just by using a single variable \[3, 4\]. As a result the SW curve is regarded as the fibration over $\mathbb{CP}^1$ whose fiber is the single-variable version of the superpotential for the $ADE$ topological Landau-Ginzburg (LG) models in two dimensions.

An important consequence is then that the Picard-Fuchs equations obeyed by the SW period integrals are shown to be equivalent to the Gauss-Manin system for the $ADE$ topological LG models plus in addition the scaling equation \[5\]. Note that the Gauss-Manin system, on the other hand, is known to yield the topological recursion relation \[6\] when topological LG models are coupled to two-dimensional topological gravity \[7\]. Another interesting topological field theoretic aspect of the SW theory is that the holomorphic prepotential of $N = 2$ Yang-Mills theory satisfies the WDVV equations \[8\]. A simple proof based on topological LG models is given in \[9\].

Thus we have seen that fundamental properties of two-dimensional topological field theory such as LG superpotentials, the Gauss-Manin system and the WDVV equations are all exhibited by the SW solution of four-dimensional $N = 2$ Yang-Mills theory. One may then ask if there is any room in the SW theory where two-dimensional topological gravity plays a role. Our purpose in this article is to present explicitly the affirmative answer to this question. As will be shown, the effect of two-dimensional topological gravity can be captured when we study in detail the behavior of the SW period integrals near the origin of the quantum moduli space of the Coulomb branch.

For $ADE$ type gauge group $G$ with rank $r$, the underlying Riemann surface \[10\] to describe the low energy behavior of the Coulomb branch of $N = 2$ Yang-Mills theory is
given by
\[ W_G(x; t^0, \cdots, t^{r-1}) - z - \frac{\mu^2}{z} = 0, \]  
(1)
where \( W_G(x; t^0, \cdots, t^{r-1}) \) is identified as the superpotential for the LG models of type \( G \) with flat coordinates \( t^\alpha (\alpha = 0, \cdots, r - 1) \). The overall degree of \( W_G \) is equal to \( h \), the Coxeter number of \( G \), and \( \mu^2 = \Lambda^2 h / 4 \) with \( \Lambda \) being the dynamical scale. \( t^\alpha \) has the degree \( r^\alpha = h - e^\alpha + 1 \) where \( e_i \) is the \( i \)-th exponent of \( G \) \( (e_1 = 1, e_r = h - 1) \). The SW differential
\[ \lambda_{SW} = \frac{x}{2\pi i} \frac{dz}{z} \]  
(2)
is used to define the period integrals
\[ a'_I = \oint_{A_I} \lambda_{SW}, \quad a'_D = \oint_{B_I} \lambda_{SW}, \quad I = 1, \cdots, r \]  
(3)
where \( A_I \) and \( B_I \) are canonical one-cycles on the curve.

Employing the technique of topological LG models, one can obtain the Picard-Fuchs equations for the period integrals \( \Pi = (a'_I, a'_D) \) \cite{5}. For example, from the operator product expansions for the primary fields \( O_\alpha = \partial_{t^\alpha} W(x) \) with \( \partial_{t^\alpha} = \frac{\partial}{\partial t^\alpha} \):
\[ O_\alpha(x) O_\beta(x) = \sum_{\gamma=0}^{r-1} C_{\alpha\beta}^\gamma(t) O_\gamma(x) + Q_{\alpha\beta}(x) \partial_x W(x), \]  
(4)
where \( \partial_x Q_{\alpha\beta} = \partial_{t^\alpha} \partial_{t^\beta} W \), one gets the Gauss-Manin system
\[ \partial_{t^\alpha} \partial_{t^\beta} \Pi - \sum_{\gamma=0}^{r-1} C_{\alpha\beta}^\gamma(t) \partial_{t^\gamma} \partial_{t^\beta} \Pi = 0. \]  
(5)
Similarly, from the scaling relation for the LG superpotential:
\[ x \partial_x W + \sum_{\alpha=0}^{r-1} r^\alpha t^\alpha \frac{\partial W}{\partial t^\alpha} = h W, \]  
(6)
one obtains
\[ \left( \sum_{\alpha=0}^{r-1} r^\alpha t^\alpha \partial_{t^\alpha} + h \mu \partial_{t^\beta} - 1 \right) \Pi = 0. \]  
(7)
\(^1\)Note that the flat coordinates are labeled differently from \cite{3}. Here we follow the convention customary in two-dimensional topological gravity.
The remaining equation is obtained by regarding the LHS of the spectral curve (1) as the superpotential of topological CP\(^1\) model [11]. Namely, from the superpotential

\[
W_{\text{CP}^1} = z + \frac{\mu^2}{z} - t^0,
\]

one may obtain the differential equation

\[
\left( (\mu \partial_\mu)^2 - 4\mu^2 \partial_\mu^2 \right) \Pi = 0.
\]

Note that the log \(\mu^2\) and \(t^0\) play a role of flat coordinates of the topological CP\(^1\) model. The Picard-Fuchs equations (5), (7) and (9) completely characterize the structure of the periods. The weak-coupling (\(\mu \sim 0\)) analysis shows that the prepotential agrees with the microscopic calculations up to one-instanton level [12, 13].

We now discuss the topological gravity coupled to the ADE topological minimal model at genus zero [3, 14]. The topological minimal model associated with the ADE type Lie group \(G\) with rank \(r\) is obtained by twisting the \(N = 2\) minimal models with the central charge \(c = 3d\) with \(d = (h - 2)/h\). The primary field \(\mathcal{O}_\alpha (\alpha = 0, \cdots, r - 1)\) is a BRST invariant observable, which has the ghost number charge \(q_\alpha = (e_{\alpha+1} - 1)/h\). In particular, \(\mathcal{O}_0\) is the identity operator.

Coupling to topological gravity, we obtain gravitational descendants \(\sigma_n(\mathcal{O}_\alpha) \ (n = 0, 1, 2, \cdots)\) of \(\mathcal{O}_\alpha\). Here we define \(\sigma_0(\mathcal{O}_\alpha) = \mathcal{O}_\alpha\). Note that the identity operator \(\mathcal{O}_0\) and its first descendant \(\sigma_1(\mathcal{O}_0)\) are identified with the puncture operator \(P\) and the dilaton operator, respectively. To each \(\sigma_n(\mathcal{O}_\alpha)\) we may associate a coupling constant \(t^n_\alpha\). We refer to the space spanned by \(t^n_\alpha\) as the large phase space. The small phase space is defined by putting \(t^n_\alpha = 0 \ (n \geq 1)\) except \(t^0_1 = -1\) and \(t^0_0 \neq 0\). \(t^0_0\) is actually the flat coordinate \(t^0_\alpha\).

Then the correlation functions are given by the derivatives of the free energy \(F_0[t]\):

\[
\langle \prod_i \sigma_{n_i}(\mathcal{O}_{\alpha_i}) \rangle = \left( \prod_i \partial_{\alpha_i}^{n_i} \right) F_0[t].
\]

In particular, in the small phase space, we have

\[
\langle P \mathcal{O}_\alpha \mathcal{O}_\beta \rangle = \eta_{\alpha\beta},
\]

\(^2\)Notice that the ghost number charge is equal to the degree divided by \(h\).
where \( \eta_{\alpha\beta} = \delta_{\epsilon_{\alpha+1}+\epsilon_{\beta+1}} \) is the flat metric. Structure constants \( C_{\alpha\beta}^\gamma(t) \) are given by the three point functions:

\[
\langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\gamma \rangle = C_{\alpha\beta}^\gamma(t),
\]

where \( \mathcal{O}_\gamma = \eta^{\alpha\beta} \mathcal{O}_\beta \) and \( \eta^{\alpha\beta} \) is the inverse matrix of \( \eta_{\alpha\beta} \). In the large phase space, one can show that \( F_0[t] \) obeys the following equations:

- **the dilaton equation**
  \[
  2F_0[t] = \sum_{n,\alpha} t_n^\alpha \partial_{t_n} F_0[t].
  \]

- **the ghost-number conservation equation (or \( L_0 \)-constraint)**
  \[
  \sum_{n,\alpha} \left( n + b_\alpha \right) t_n^\alpha \partial_{t_n} F_0[t] = 0,
  \]
  where \( b_\alpha = q_\alpha - (d - 1)/2 = \epsilon_{\alpha+1}/h \).

- **the topological recursion relation**
  \[
  \langle \sigma_n(\mathcal{O}_\alpha) \mathcal{O}_\beta \mathcal{O}_\gamma \rangle = \sum_{\beta'} \langle \sigma_{n-1}(\mathcal{O}_\alpha) \mathcal{O}_\beta \rangle \langle \mathcal{O}_\beta \mathcal{O}_\gamma \sigma_{n-1}(\mathcal{O}_\alpha) \rangle.
  \]

In the small phase space, we have the puncture equations

\[
\langle P \prod_{i=1}^s \sigma_{n_i}(\mathcal{O}_{\alpha_i}) \rangle = \sum_{i=1}^s \langle \prod_{j=1}^s \sigma_{n_j-\delta_{ij}}(\mathcal{O}_{\alpha_j}) \rangle.
\]

From the topological recursion relation and the puncture equation, it is easy to show

\[
\langle \mathcal{O}_\alpha \mathcal{O}_\beta \sigma_n(\mathcal{O}_\gamma) \rangle = \sum_{\beta'} \langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_{\beta'} \rangle \langle P \mathcal{O}_{\beta'} \sigma_n(\mathcal{O}_\gamma) \rangle.
\]

This is equivalent to the differential equation

\[
\left( \partial_{t_{\alpha}} \partial_{t_{\beta}} - \sum_{\beta'} C_{\alpha\beta}^{\beta'}(t) \partial_{t_{\beta'}} \partial_{t_{\beta}} \right) \langle \sigma_n(\mathcal{O}_\gamma) \rangle = 0,
\]

which is nothing but the Gauss-Manin system. Then one can introduce the power series

\[
\tilde{\Pi}_\alpha = \sum_{n \geq 0} c_{n,\alpha} \langle \sigma_n(\mathcal{O}_\alpha) \rangle \mu^{\theta_{\alpha} n}.
\]
which satisfies the Gauss-Manin system. Here \( c_{n,\alpha}, \rho \) and \( \theta_{\alpha} \) are constants which are determined by requiring that \( \tilde{\Pi}_{\alpha} \) obeys the scaling relation (7) and the \( \mathbb{CP}^1 \) relation (9).

From the dilaton equation (13) and the ghost number conservation (14) in the small phase space, we obtain

\[
\langle \sigma_m(O_\beta) \rangle + \langle \sigma_1(P)\sigma_m(O_\beta) \rangle - \sum_{\alpha} t^\alpha \langle O_\alpha \sigma_m(O_\beta) \rangle = 0, \tag{20}
\]

\[
(m + b_\beta)\langle \sigma_m(O_\beta) \rangle - (b_0 + 1)\langle \sigma_1(P)\sigma_m(O_\beta) \rangle + \sum_{\alpha} b_\alpha t^\alpha \langle O_\alpha \sigma_m(O_\beta) \rangle = 0, \tag{21}
\]

respectively. Eliminating the term \( \langle \sigma_1(P)\sigma_m(O_\beta) \rangle \) in the above two equations, we get

\[
\sum_{\alpha} (q_\alpha - 1)t^\alpha \partial_{\alpha} \langle \sigma_n(O_\beta) \rangle + (n + b_\beta + b_0 + 1)\langle \sigma_n(O_\beta) \rangle = 0. \tag{22}
\]

From this relation and (4) we find \( \rho = -1 \) and \( \theta_{\alpha} = b_\alpha + 1 \). We next check the \( \mathbb{CP}^1 \) relation. From the puncture equation (16), we have

\[
\langle P^2 O_\alpha \rangle = \langle \sigma_{n-2}(O_\alpha) \rangle. \tag{23}
\]

Thus the \( \mathbb{CP}^1 \) relation yields the recursion relation for \( c_{n,\alpha} \):

\[
c_{n+2,\alpha} = \frac{1}{4} (n + b_\alpha + 1)^2 c_{n,\alpha}. \tag{24}
\]

We need two extra terms \( 4c_{-2,\alpha} \eta_{0,\alpha} \mu^{1-b_\alpha} / (1 - b_\alpha)^2 + 4c_{-1,\alpha} \eta_{0,\alpha} \mu^{b_\alpha} \) in addition to \( \tilde{\Pi}_\alpha \) in order to satisfy the \( \mathbb{CP}^1 \) relation. To summarize the solution to the Picard-Fuchs equations is given by

\[
\Pi_\alpha = A_\alpha \sum_{n \geq 0} \frac{\Gamma(n + \frac{b_\alpha}{2})^2}{\Gamma(1 + \frac{b_\alpha}{2})^2} \langle \sigma_{2n-1}(O_\alpha) \rangle \mu^{-(2n+b_\alpha)} + B_\alpha \sum_{n \geq -1} \frac{\Gamma(n + \frac{b_\alpha+1}{2})^2}{\Gamma(\frac{b_\alpha+1}{2})^2} \langle \sigma_{2n}(O_\alpha) \rangle \mu^{-(2n+b_\alpha+1)}, \tag{25}
\]

where \( \langle \sigma_{-1}(O_\alpha) \rangle \equiv \langle PO_\alpha \rangle = \eta_{0,\alpha} t^\beta, \langle \sigma_{-2}(O_\alpha) \rangle \equiv \langle P^2 O_\alpha \rangle = \eta_{0,\alpha} \), \( \Gamma(x) \) is the Gamma function and \( A_\alpha, B_\beta \) are certain constants.

We next investigate the physical meaning of the solutions (25) of the Picard-Fuchs equations. The SW period integrals \( \Pi = (a_I^I, a_I^D) \) should be expressed in terms of \( \Pi_\alpha \).

\footnote{Since \( 1 - q_\alpha = r_\alpha / h \), eq. (22) indicates that the degree of \( \langle \sigma_n(O_\beta) \rangle \) equals \( h(n + 1 + b_\beta) + 1 \). This indeed agrees with the degree we can directly read off from the integral representation (35).}
are the expansion around $\mu = 0$ and exhibit the logarithmic behavior. The solutions\(^{[25]}\) are, on the other hand, expansions around $\mu = \infty$. Hence we expect that these correspond to the period integrals expanded around the origin of the moduli space $t^\alpha = 0$. At this point, one cannot expect any massless soliton, and hence there appear no logarithmic solutions.

In the following we check this claim in the case of $A_{N-1}$ type gauge groups explicitly. Firstly we examine the simplest case, i.e. $A_1$ case, in which the superpotential is given by $W_{A_1} = x^2 - u$. The periods $(a, a_D)$ are evaluated by the elliptic integral. The solution at the weak coupling region is given in terms of the hypergeometric function $^{[12]}$

\[
a(u) = \frac{1}{\sqrt{2}} \sqrt{u} F\left(\frac{1}{4}, \frac{1}{4}; 1; \frac{\Lambda^4}{u^2}\right),
\]

\[
a_D(u) = -\frac{i\sqrt{u}}{\sqrt{2\pi}} \left\{ \frac{F\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{\Lambda^4}{u^2}\right) \log \left(\frac{\Lambda^4}{u^2}\right)}{u^2} \right. \\
+ \sum_{n=1}^{\infty} \frac{(\frac{n}{2})_n(-\frac{n}{2})_n}{(1)^2_n} \left(\frac{\Lambda^4}{u^2}\right)^2 \left[ -2\psi(n+1) + \psi(n + \frac{1}{4}) + \psi(n - \frac{1}{4}) \right],
\]

where $\Lambda^4 = \mu^2/4$, $(x)_n = \Gamma(x+n)/\Gamma(x)$ and $\psi(x) = \frac{d\log \Gamma(x)}{dx}$. The hypergeometric function $F(a, b; c; z)$ is defined by

\[
F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n.
\]

In order to make analytic continuation to the region around $u = 0$, we use the formulas for the hypergeometric functions:\(^{[15]}\)

\[
F(a, b; c; z) = \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(c-a)\Gamma(b)} (e^{\pi i} z)^{-a} F(a, a-c+1; a-b+1; z^{-1}) \\
+ \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(a)} (e^{\pi i} z)^{-b} F(b, b-c+1; b-a+1; z^{-1}),
\]

\[
\frac{\Gamma(a)}{\Gamma(c)} F(a, a; c; z) = \frac{(e^{-\pi i} z)^{-a}}{\Gamma(c-a)} \sum_{n=0}^{\infty} \frac{(a)_n(1-c+a)_n}{(1)^2_n} z^{-n} [\log (e^{-\pi i} z) + h_n],
\]

where in $^{[29]}$ we take the branch $|\arg(e^{-\pi i} z)| < \pi$. Applying $^{[28]}$ and $^{[29]}$ to the solutions $^{[29]}$, we obtain

\[
a(u) = \frac{\sqrt{u} \Gamma\left(\frac{1}{2}\right)}{4\sqrt{2}} \left( e^{\pi i} \Lambda^4 \right)^{\frac{1}{2}} \left\{ \Gamma\left(-\frac{1}{4}\right)^2 F\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{u^2}{\Lambda^4}\right) + 2 \left(e^{\pi i} \Lambda^4 \right)^{\frac{1}{2}} \Gamma\left(\frac{1}{4}\right)^2 F\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{u^2}{\Lambda^4}\right) \right\},
\]

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\[ a_D(u) = -\frac{i\Lambda}{8\pi} \left\{ \Gamma\left(-\frac{1}{4}\right)^2 F\left(-\frac{1}{4}, -\frac{1}{4}; \frac{1}{2}; \frac{u^2}{\Lambda^4}\right) - 2 \left( \frac{u}{\Lambda^2} \right) \Gamma\left(\frac{1}{4}\right)^2 F\left(\frac{1}{4}, \frac{1}{4}; \frac{3}{2}; \frac{u^2}{\Lambda^4}\right) \right\} . \tag{30} \]

Let us consider the topological gravity. In this case, the observables are the gravitational descendants \( \sigma_n(P) \) of the puncture operator \( P \). The one-point function \( \langle \sigma_n(P) \rangle \) is given by

\[ \langle \sigma_n(P) \rangle = \frac{(-u)^{n+2}}{(n+2)!} . \tag{31} \]

Thus the formula \( \Pi_\alpha \) for \( \alpha = 0 \) in (25) may be evaluated explicitly. We get

\[ \sum_{n \geq 0} \frac{\Gamma(k + \frac{b_0}{2})^2}{\Gamma(1 + \frac{b_0}{2})^2} \langle \sigma_{2n-1}(P) \rangle \mu^{-2(n+b_0)} = 16 -\frac{u}{\mu} F\left(-\frac{1}{4}, \frac{1}{4}; \frac{3}{2}; \frac{u^2}{4\mu^2}\right), \]

\[ \sum_{n \geq 1} \frac{\Gamma(k + \frac{b_0+1}{2})^2}{\Gamma(\frac{b_0+1}{2})^2} \langle \sigma_{2n}(P) \rangle \mu^{-(2n+b_0+1)} = 16\mu^2 F\left(-\frac{1}{4}, -\frac{1}{4}; \frac{1}{2}; \frac{u^2}{4\mu^2}\right), \tag{32} \]

where \( b_0 = \frac{1}{2} \). Let us define \( I_0(u, \mu) \) by choosing \( A_0 = \Gamma(1+b_0/2)/\Gamma(b_0) \) and \( B_0 = \Gamma((1+b_0)/2)^2/\Gamma(b_0) \) for \( \Pi_0 \) in (25);

\[ I_0(u, \mu) = \frac{1}{\Gamma(\frac{1}{2})^2} \mu^\frac{1}{2} \left\{ \Gamma\left(-\frac{1}{4}\right)^2 F\left(-\frac{1}{4}, -\frac{1}{4}; \frac{1}{2}; \frac{u^2}{4\mu^2}\right) - \frac{u}{\mu} \Gamma\left(\frac{1}{4}\right)^2 F\left(\frac{1}{4}, \frac{1}{4}; \frac{3}{2}; \frac{u^2}{4\mu^2}\right) \right\} . \tag{33} \]

\( I_0(u, \mu) \) and \( I_0(u, -\mu) \) are two independent solutions to the Picard-Fuchs equation for \( SU(2) \) gauge group. Comparing (30) with (33), we obtain

\[ a = \frac{1}{8\sqrt{2\pi}} (I_0(u, \mu) - I_0(u, -\mu)), \]

\[ a_D = -\frac{i\sqrt{2}}{8\pi} I_0(u, \mu) . \tag{34} \]

We now generalize the \( A_1 \) result to the \( A_{N-1} \) case. We consider the topological gravity coupled with \( A_{N-1} \)-type topological minimal models [14]. Let \( W(x) = x^N - \sum_{i=2}^N s_i x^{N-i} \) be the LG superpotential of \( A_{N-1} \) type. The one-point function of the \( n \)-th gravitational descendant \( \sigma_n(O_\alpha) \) of a primary field \( O_\alpha \) is given by

\[ \langle \sigma_n(O_\alpha) \rangle = \frac{\Gamma(b_\alpha)}{\Gamma(b_\alpha + n + 2)} \int_{x=\infty} dx \frac{dx}{2\pi i} W(x)^{n+1+b_\alpha}, \quad n \geq -2, \tag{35} \]

where \( b_\alpha = \frac{\alpha+1}{N} \). Expanding \( W(x)^{n+1+b_\alpha} \) around \( x = \infty \), we have

\[ W(x)^{n+1+b_\alpha} = x^{N(n+1+b_\alpha)} \sum_{m_2 \geq 0, \ldots, m_N \geq 0} \frac{(-1)^{\sum_{i=2}^N m_i}}{\Gamma(n+2+b_\alpha-\sum_{i=2}^N m_i)} \prod_{i=2}^N s_i^{m_i} x^{-\sum_{i=2}^N m_i} . \tag{36} \]
Taking the residue at \( x = \infty \), the non-zero contribution in the sum (35) occurs for the case \( \sum_{i=2}^{N} i m_i = N(n + 1) + \alpha + 2 \). Since \( m_N = n + 1 + \frac{\alpha + 2 - \sum_{i=2}^{N-1} i m_i}{N} \), the one-point function becomes

\[
\langle \sigma_n(O_{\alpha}) \rangle = \sum_{m_N = n + 1 + b N} \frac{\Gamma(b)}{\Gamma(1 - a_{\{m\}} + b')_{\{m\}}} (-1)^{a_{\{m\}}} \left( \prod_{i=2}^{N-1} \frac{s_i^{m_i}}{m_i!} \right) \frac{s_{m_N}}{N!},
\]

where \( a_{\{m\}} = \sum_{i=2}^{N-1} m_i, b'_{\{m\}} = (\sum_{i=2}^{N-1} i m_i - 1)/N \) and \( \{m_i\} \) denotes \( m_2 \geq 0, \ldots, m_{N-1} \geq 0 \). As in the case of \( A_1 \), we define \( I_\alpha(s_i, \mu) \) by choosing \( A_\alpha = \Gamma(1 + b_\alpha/2)/\Gamma(b_\alpha) \) and \( B_\alpha = \Gamma((1 + b_\alpha)/2)^2/\Gamma(b_\alpha) \) in (25):

\[
I_\alpha(s_i, \mu) = \frac{1}{\Gamma(b_\alpha)} \sum_{n \geq 0} \Gamma\left( \frac{n + b_\alpha - 1}{2} \right) \langle \sigma_{n-2}(O_{\alpha}) \rangle \mu^{-n+b_\alpha-1}.
\]

One may regard \( I_\alpha(s_i, \mu) \) and \( I_\alpha(s_i, -\mu) \) as independent solutions of the Picard-Fuchs equations. Combining (37) and (38), we obtain

\[
I_\alpha(s_i, \mu) = \sum_{\{m\}} \frac{(-1)^{a_{\{m\}}}}{\Gamma(1 - a_{\{m\}} + b'_{\{m\}})} \left( \prod_{i=2}^{N-1} \frac{s_i^{m_i}}{m_i!} \right) \mu^{-b'_{\{m\}}} 
\]

\[
\times \left\{ \Gamma\left( \frac{b'_{\{m\}}}{2} \right) \binom{b'_{\{m\}}}{2}, \left( \frac{1}{2}; \frac{s_{N}^2}{4\mu^2} \right) - \Gamma\left( \frac{b'_{\{m\}}}{2} + 1 \right) \frac{s_{N}}{2} \binom{b'_{\{m\}} + 1}{2}, \left( \frac{1}{2}; \frac{s_{N}^2}{4\mu^2} \right) \right\}.
\]

(39)

We next calculate the period integral of the SW differential around the origin \( s_i = 0 \) in the quantum moduli space. We make analytic continuation from the weak coupling region \( s_2 = \cdots = s_{N-1} = 0 \) and \( s_N = \infty \) to the region around \( s_2 = \cdots = s_N = 0 \). To evaluate the period integral in the weak coupling region, we will use the formulas obtained by Masuda-Suzuki [16]. From the Barnes type integral formulas, the period integrals \( (a^k, a_D^k) \) \((k = 1, \cdots, N)\), \(^4\) which are subject to the constraint \( \sum_{k=1}^{N} a^k = \sum_{k=1}^{N} a_D^k = 0 \), are given by

\[
a^k = \frac{s_{N}^{1/N}}{N} \sum_{\{m\}} e^{-2\pi i k b'_{\{m\}}} \frac{\Gamma(a_{\{m\}} - b'_{\{m\}})}{\Gamma(1 - b'_{\{m\}})} \left( \prod_{i=2}^{N-1} \frac{\alpha_i^{m_i}}{m_i!} \right) F\left( \frac{b'_{\{m\}}}{2}, \frac{b'_{\{m\}} + 1}{2}; 1; \frac{\lambda^{2N}}{s_{N}^2} \right),
\]

\[
a_D^k = \frac{s_{N}^{1/N}}{\pi i N} \sum_{\{m\}} e^{-2\pi i k b'_{\{m\}}} \Gamma(a_{\{m\}} - b'_{\{m\}}) \left( -2 b'_{\{m\}} \sin \pi b'_{\{m\}} \right) \left( \prod_{i=2}^{N-1} \frac{\alpha_i^{m_i}}{m_i!} \right) \left( \prod_{i=2}^{N-1} \frac{b'_{\{m\}}}{m_i!} \right) F\left( \frac{b'_{\{m\}}}{2}, \frac{b'_{\{m\}} + 1}{2}; 1; \frac{\lambda^{2N}}{s_{N}^2} \right).
\]

\(^4\)For the \( A_1 \) case \((N = 2)\), we have \( a^1 = -\sqrt{2} a(u) \) and \( a_D^1 = -\sqrt{2} a_D(u) \).
\[ \frac{\Lambda^{2N}}{s_N^2} \times \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{b_{(m)}'}{2}) \Gamma(n + \frac{b_{(m)}'}{2} + 1)}{\Gamma(n + 1)^2} \left( \frac{\Lambda^{2N}}{s_N^2} \right)^n \times \left\{ -2\psi(n + 1) + \psi(n + \frac{b_{(m)}'}{2}) + \psi(n + \frac{b_{(m)}'}{2} + 1) + \log \left( \frac{\Lambda^{2N}}{s_N^2} \right) + 2\pi \cot \pi b_{(m)}' \right\}, \]

(40)

where \( \alpha_i = s_i/s_{N}^{i/N} \) and \( 4\mu^2 = \Lambda^{2N} \). By using the formula (28), \( a^k \) becomes

\[ a^k = \frac{1}{N} \sum_{\{m_i\}} e^{-2\pi i b_{(m)}'} \Gamma(a_{(m)} - b_{(m)}') \left( \sum_{i=2}^{N} \frac{s_{i/m_i}^{m_i}}{m_i!} \right) \mu^{-b_{(m)}'} e^{-\pi i b_{(m)}'} \sin \frac{\pi b_{(m)}'}{\mu} \]

\[ \times \left\{ \frac{\pi b_{(m)}'}{2} \Gamma\left( \frac{b_{(m)}'}{2} \right) 2F\left( \frac{b_{(m)}'}{2}, \frac{b_{(m)}'}{2}; 1; \frac{s_N^2}{4\mu^2} \right) \right\} \]

\[ + i \cos \frac{\pi b_{(m)}'}{2} \Gamma\left( \frac{b_{(m)}'}{2} \right) 2F\left( \frac{b_{(m)}'}{2}, \frac{b_{(m)}'}{2}; 1; \frac{s_N^2}{4\mu^2} \right) \}

(41)

This turns out to be

\[ a^k = -\frac{1}{4\pi i N} \sum_{\alpha} e^{-2\pi i k b_{(m)}'} \left( I_{\alpha}(s_i, \mu) - I_{\alpha}(s_i, -\mu) \right). \]

(42)

For \( a_D^k \), applying the formula (28) and (29), we get

\[ a_D^k = \frac{1}{\pi i N} \sum_{\{m_i\}} e^{-2\pi i b_{(m)}'} \Gamma(a_{(m)} - b_{(m)}') \left( \sum_{i=2}^{N} \frac{s_{i/m_i}^{m_i}}{m_i!} \right) \pi (-1)^{\frac{1}{2} - b_{(m)}'} \mu^{-b_{(m)}'} \]

\[ \times \left\{ \frac{\sin^2 \pi b_{(m)}'}{2} \Gamma\left( \frac{b_{(m)}'}{2} \right) 2F\left( \frac{b_{(m)}'}{2}, \frac{b_{(m)}'}{2}; 1; \frac{s_N^2}{4\mu^2} \right) \right\} \]

\[ - \cos^2 \frac{\pi b_{(m)}'}{2} \Gamma\left( \frac{b_{(m)}'}{2} \right) 2F\left( \frac{b_{(m)}'}{2}, \frac{b_{(m)}'}{2}; 1; \frac{s_N^2}{4\mu^2} \right) \}

(43)

This is shown to be

\[ a_D^k = -\frac{1}{4N\sqrt{\pi}} \sum_{\alpha} e^{-2\pi i k b_{(m)}'} \left( \frac{e^{-\pi i k a}}{\sin \pi b_{(m)}} I_{\alpha}(s_i, \mu) - \cot \pi b_{(m)} I_{\alpha}(s_i, -\mu) \right). \]

(44)

So far we have calculated the period integrals for \( A_{N-1} \) case. It would be a straightforward task to generalize the present results to \( D \)-type gauge groups.

In the present work, we have seen that the period integrals of the Seiberg-Witten differential at the origin of the quantum moduli space are described by two-dimensional topological gravity coupled to topological \( ADE \) minimal models. How do we then interpret the dependence on \( \mu \)? It is tempting to regard (I) as the superpotential for the \( ADE \)
minimal models combined with the topological $\mathbb{CP}^1$ model. In view of the $\mathbb{CP}^1$ model the power of $\mu^2$ counts the degree of holomorphic maps from the Riemann surface to $\mathbb{CP}^1$, and hence has to be non-negative integers. It is thus curious to have negative fractional powers of $\mu^2$ for $n \geq 1$ in (25). On the other hand, there is another interpretation of (1) in terms of topological sigma models. According to [17], the LG models with superpotentials

$$W(x, y) = W_G(x; 0) - \rho y^{-h}$$

(45)
correspond to sigma models with the ALE target space of $ADE$-type singularities; $W_G(x; 0) = \rho$. Therefore, setting $z = y^h$, we may think of the superpotential (1) as deformations of (45). The LG formulation of the sigma models on the ALE space describes the Type IIB string on $K3 \times \mathbb{R}^6$ when the $K3$ surface is near the $ADE$ singularities. Note here that Type IIB on ALE is equivalent to Type IIA on the NS fivebrane [17]. Hence, identifying the Riemann surface (1) with the superpotential in the LG description of deformed ALE sigma models seems consistent with the M-theory fivebrane interpretation of the SW solution [18]. Connections between $ADE d < 1$ topological strings and the ALE sigma models have been discussed on the basis of superconformal theories [17]. Clarifying the relation between [17] and our present observation of SW theory as $d < 1$ topological strings will be important for deeper understanding.

There are several issues to be considered further in this direction. We have investigated the two-dimensional theory at genus zero and in the small phase space. Hence it is interesting to generalize the present approach to the case of arbitrary genus and general background. After appropriate change of normalization of $t_n$, the power series solutions $I_\alpha(s_i, \pm\mu)$ correspond to loop operators in two-dimensional theory and are realized by $Z_N$ twisted bosons. The Virasoro or $W$-constraints in two-dimensional theory impose the non-trivial conditions on the prepotential in four dimensions. In the four-dimensional sense, these generalizations might correspond to including the higher derivative terms or the coupling to four-dimensional gravity. It is also interesting to investigate the relation between the present formalism and the approach based on the Whitham hierarchy [10, 19].

The work of S.K.Y. was supported in part by Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (No. 09640335). S.K.Y. would like to thank A. Tsuchiya for stimulating discussions.
References

[1] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, hep-th/9407087; Nucl. Phys. B431 (1994) 484, hep-th/9408099

[2] For a review, see, W. Lerche, *Introduction to Seiberg-Witten Geometry and its Stringy Origin*, hep-th/9611190

[3] W. Lerche and N.P. Warner, Phys. Lett. B423 (1998) 79, hep-th/9608183

[4] T. Eguchi and S.-K. Yang, Phys. Lett. B394 (1997) 315, hep-th/9612086

[5] K. Ito and S.-K. Yang, Phys. Lett. B415 (1997) 45, hep-th/9708017

[6] E. Witten, Nucl. Phys. B340 (1989); Surv. Diff. Geom. 1 (1991) 243; R. Dijkgraaf and E. Witten, Nucl. Phys. B342 (1990) 486

[7] T. Eguchi, Y. Yamada and S.-K. Yang, Mod. Phys. Lett. A8 (1993) 1627, hep-th/9302048

[8] A. Marshakov, A. Mironov and A. Morozov, Phys. Lett. B389 (1996) 43, hep-th/9607109; Mod. Phys. Lett. A12 (1997) 773, hep-th/9701014; *More Evidence for the WDVV Equations in N = 2 SUSY Yang-Mills Theories*, hep-th/9701123; G. Bertoldi and M. Matone, Phys. Rev. D57 (1998) 6483, hep-th/9712103

[9] K. Ito and S.-K. Yang, *The WDVV Equations in N = 2 Supersymmetric Yang-Mills Theory*, hep-th/9803126, to appear in Phys. Lett. B

[10] A. Gorskiia, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, Phys. Lett. B355 (1995) 466, hep-th/9505035; E.J. Martinec and N.P. Warner, Nucl. Phys. B459 (1996) 97, hep-th/9509161

[11] B. Dubrovin, *Geometry of 2D Topological Field Theories*, hep-th/9407018; T. Eguchi and S.-K. Yang, Mod. Phys. Lett. A9 (1994) 2893, hep-th/9407134; T. Eguchi, K. Hori and S.-K. Yang, Int. J. Mod. Phys. A10 (1995) 4203, hep-th/9503017
A. Takahashi, *Primitive Forms, Topological LG Models Coupled to Gravity and Mirror Symmetry*, math.AG/9802059

[12] A. Klemm, W. Lerche and S. Theisen, Int. J. Mod. Phys. **A11** (1996) 1929, hep-th/9505150

[13] K. Ito and N. Sasakura, Nucl. Phys. **B484** (1997) 141, hep-th/9608054, E. D’ Hoker, I.M. Krichever and D.H. Phong, Nucl. Phys. **B489** (1997) 179, hep-th/9609041; **B489** (1997) 211, hep-th/9609143; T. Masuda and H. Suzuki, Int. J. Mod. Phys. **A13** (1998) 1495, hep-th/9609065; A.M. Ghezelbash, Phys. Lett. **B423** (1998) 87, hep-th/9710068; K. Ito and S.-K. Yang, *A-D-E Singularity and Prepotential in N = 2 Supersymmetric Yang-Mills Theory*, hep-th/9712018, to appear in Int. J. Mod. Phys. A

[14] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. **B352** (1991) 59; *Notes on Topological String Theory and 2D Quantum Gravity*, PUPT-1217, IASSNS-HEP-90/80 (November, 1990); K. Li, Nucl. Phys. **B354** (1991) 711; 725

[15] A. Erdélyi et al., *Higher Transcendental Functions*, vol. 1, MacGraw Hill, New York, 1953

[16] T. Masuda and H. Suzuki, Nucl. Phys. **B495** (1997) 149, hep-th/9612240

[17] H. Ooguri and C. Vafa, Nucl. Phys. **B463** (1996) 55, hep-th/9511164

[18] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, Nucl. Phys. **B477** (1996) 746, hep-th/9604034; E. Witten, Nucl. Phys. **B500** (1997) 3, hep-th/9703166

[19] T. Nakatsu and K. Takasaki, Mod. Phys. Lett. **A11** (1996) 417, hep-th/9509162; H. Itoyama and A. Morozov, Nucl. Phys. **B491** (1997) 529, hep-th/9512161; A. Gorsky, A. Marshakov, A. Mironov and A. Morozov, *RG Equation from Whitam Hierarchy*, hep-th/9802007