SURFACE GROUP REPRESENTATIONS AND HIGGS BUNDLES

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1. Introduction

In these notes we give an introduction to Higgs bundles and their application to the study of surface group representations. This is based on two fundamental theorems. The first is the theorem of Corlette and Donaldson on the existence of harmonic metrics in flat bundles which we treat in Lecture 1, after explaining some preliminaries on surface group representations, character varieties and flat bundles. The second is the Hitchin–Kobayashi correspondence for Higgs bundles, which goes back to the work of Hitchin and Simpson; this is the main topic of Lecture 2. Together, these two results allows the character variety for representations of the fundamental group of a Riemann surface in a Lie group $G$ to be identified with a moduli space of holomorphic objects, known as $G$-Higgs bundles. Finally, in Lecture 3, we show how the $\mathbb{C}^*$-action on the moduli space $G$-Higgs bundles can be used to study its topological properties, thus giving information about the corresponding character variety.

For lack of time and expertise, we do not treat many other important aspects of the theory of surface group representations, such as the approach using bounded cohomology (see, e.g., Burger–Iozzi–Wienhard [9,10]), higher Teichmüller theory (see, e.g., Fock–Goncharov [17]), or ideas related to geometric structures on surfaces (see, e.g., Goldman [28]). We also do not touch on the relation of Higgs bundle moduli with mirror symmetry and the Geometric Langlands Programme (see, e.g., Hausel [33] and Kapustin–Witten [39]).

In keeping with the lectures we do not give proofs of most results. For more details and full proofs, we refer to the literature. Some references that the reader may find useful are the papers of Hitchin [35,37], García-Prada [19], Goldman [25,28,27] and also the papers [3,4,21,20,6].

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Notation. Our notation is mostly standard. Smooth $p$-forms are denoted by $\Omega^p$ and smooth $(p, q)$-forms by $\Omega^{p,q}$. We shall occasionally confuse vector bundles and locally free sheaves.

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2. Lecture 1: Character varieties for surface groups and harmonic maps

In this lecture we give some basic definitions and properties of character varieties for representations of surface groups. We then explain the theorem of Corlette and Donaldson on the existence of harmonic maps in flat bundles, which is one of the two central results in the non-abelian Hodge theory correspondence (the other one being the Hitchin–Kobayashi correspondence, which will be treated in Lecture 2).

2.1. Surface group representations and character varieties. More details on the following can be found in, for example, Goldman\[25\].

Let $\Sigma$ be a closed oriented surface of genus $g$. The fundamental group of $\Sigma$ has the standard presentation

$$\pi_1 \Sigma = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod [a_i, b_i] = 1 \rangle,$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ is the commutator.

Let $G$ be a real reductive Lie group. We denote its Lie algebra by $\mathfrak{g} = \text{Lie}(G)$. Though not strictly necessary for everything that follows, we shall assume that $G$ is connected. We shall also fix a non-degenerate quadratic form on $G$, invariant under the adjoint action of $G$ (when $G$ is semisimple, the Killing form or a multiple thereof will do).

By definition a representation of $\pi_1 \Sigma$ in $G$ is a homomorphism $\rho: \pi_1 \Sigma \to G$. Let $\text{Ad}: G \to \text{Aut}(\mathfrak{g})$ be the adjoint representation of $G$ on its Lie algebra $\mathfrak{g}$. We say that $\rho$ is reductive$^1$ if the composition

$$\text{Ad} \circ \rho: \pi_1 \Sigma \to \text{Aut}(\mathfrak{g})$$

$^1$When $G$ is algebraic an alternative equivalent definition is to ask for the Zariski closure of $\rho(\Sigma) \subset G$ to be reductive.
is completely reducible. Denote by $\text{Hom}^{\text{red}}(\pi_1 \Sigma, G) \subset \text{Hom}(\pi_1 \Sigma, G)$ the subset of reductive representations.

**Definition 2.1.** The *character variety* for representations of $\pi_1 \Sigma$ in $G$ is

$$\mathcal{M}^B(\Sigma, G) = \text{Hom}^{\text{red}}(\pi_1 \Sigma, G)/G,$$

where the $G$-action is by simultaneous conjugation:

$$(g \cdot \rho)(x) = g\rho(x)g^{-1}.$$ 

The character variety is also known as the *Betti moduli space* (in Simpson’s language [48]).

Note that, using the presentation (2.1), a representation $\rho$ is given by a $2g$-tuple of elements in $G$ satisfying the relation. Hence we get an inclusion $\text{Hom}(\pi_1 \Sigma, G) \hookrightarrow G^{2g}$, which endows $\text{Hom}(\pi_1 \Sigma, G)$ with a natural topology. However, it turns out that the quotient space $\text{Hom}(\pi_1, G)/G$ is not in general Hausdorff. The restriction to reductive representations remedies this problem.

We also remark that, in case $G$ is a complex reductive algebraic group, the character variety can be constructed as an affine GIT quotient (this is classical; a nice exposition is contained in §3.1 of Casimiro–Florentino [11]).

2.2. **Review of connections and curvature in principal bundles.**

Recall that a (smooth) *principal $G$-bundle* on $\Sigma$ is a smooth fibre bundle $\pi: E \to \Sigma$ with a $G$-action (normally taken to be on the right) which is free and transitive on each fibre. Moreover, $E$ is required to admit $G$-equivariant local trivializations $E|_U \cong U \times G$ over small open sets $U \subset \Sigma$ (where $G$ acts by right multiplication on the second factor of the product $U \times G$). Note that the fibre $E_x$ over any $x \in \Sigma$ is a $G$-torsor so, choosing an element $e \in E_x$, we get a canonical identification $E_x \cong G$.

**Example 2.2.** (1) The *frame bundle* of a rank $n$ complex vector bundle $V \to \Sigma$ is a principal $\text{GL}(n, \mathbb{C})$-bundle, which has fibres

$$E_x = \{e: \mathbb{C}^n \overset{\cong}{\to} V_x \mid e \text{ is a linear isomorphism}\}.$$ 

(2) The universal covering $\tilde{\Sigma} \to \Sigma$ is a principal $\pi_1 \Sigma$-bundle over $\Sigma$. In this case the action is on the left.

Whenever we have a principal $G$-bundle $E \to \Sigma$ and a smooth $G$-space $V$ (i.e., $V$ is a smooth manifold on which $G$ acts by smooth maps), we obtain a fibre bundle $E(V)$ with fibres modeled on $V$ by taking the quotient of $E \times V$ under the diagonal $G$-action:

$$E(V) = E \times_G V \to \Sigma.$$ 

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2This is in fact the same as the compact-open topology on the mapping space $\text{Hom}(\pi_1 \Sigma, G)$, where we give $\pi_1 \Sigma$ the discrete topology.
In particular, if $V$ is a vector space with a linear $G$-action, we obtain a vector bundle $E(V)$ with fibres modeled on $V$. An important instance of this construction is when $V = g$ acted on by $G$ via the adjoint action. The resulting vector bundle $\text{Ad} E := E(g)$ is then known as the \textit{adjoint bundle} of $E$.

There is a bijective correspondence between sections $s: \Sigma \to E(V)$ of the bundle $\pi: E(V) \to \Sigma$ and $G$-equivariant maps $\tilde{s}: E \to V$, given by

$$s(x) = [e, \tilde{s}(e)]$$

for $e \in E(V)_x = \pi^{-1}(x)$ and $x \in \Sigma$. Similarly, a $G$-equivariant differential $p$-form $\alpha \in \Omega^p(E,V)$ descends to an $E(V)$-valued $p$-form $\tilde{\alpha} \in \Omega^p(\Sigma, E(V))$ if and only if it is \textit{tensorial}, i.e., it vanishes on the vertical tangent spaces $T_v E = T_e E_x$ to $E$.

A \textit{connection} in a principal $G$-bundle $E \to \Sigma$ is given by a smooth $G$-invariant Lie algebra valued 1-form $A \in \Omega^1(E, g)$ which restricts to the identity on the vertical tangent spaces $T_v E$ under the natural identification $T_v E \cong g$ given by the choice of $e \in E_x$. Equivalently, a connection corresponds to the choice of a horizontal complement $T_h E = \ker(A(e): T_e E \to g)$ to $T_v E$ in each $T_e E$. Moreover, the $G$-invariance means that these complements correspond under the $G$-action. The difference of two connections is a tensorial form, so it follows that the space $\mathcal{A}$ of connections on $E$ is an affine space modeled on $\Omega^1(\Sigma, \text{Ad} E)$.

Given a connection $A$ in a principal bundle $E$, we obtain a covariant derivative

$$d_A: \Omega^0(\Sigma, E(V)) \to \Omega^1(\Sigma, E(V))$$

on sections in any associated vector bundle $E(V)$ as follows. Let $s \in \Omega^0(\Sigma, E(V))$ and let $\tilde{s}: E \to V$ be the corresponding $G$-equivariant map as above. Then we define a tensorial one-form $d_A(s)$ on $E$ by composing $d\tilde{s}$ with the projection $TE \to T_h E$ defined by $A$, and let $d_A(s) \in \Omega^1(\Sigma, E(V))$ be the corresponding $E(V)$-valued one-form.

Given a connection in $E$, the horizontal subspaces define a $G$-invariant distribution on the total space of $E$. The obstruction to integrability of this horizontal distribution is given by the \textit{curvature}

$$F(A) = dA + \frac{1}{2} [A, A] \in \Omega^2(E, g)$$

of the connection $A$, where the bracket $[A, A]$ is defined by combining the wedge product on forms with the Lie bracket on $g$. One checks that $F(A)$ is in fact a tensorial form and therefore descends to a 2-form on $\Sigma$ with values in the adjoint bundle, which we denote by the same symbol,

$$F(A) \in \Omega^2(\Sigma, E(g)).$$

A connection $A$ is \textit{flat} if $F(A) = 0$. A principal $G$-bundle $E \to \Sigma$ with a flat connection is called a \textit{flat bundle}. Equivalently, a flat bundle is one for

4 P. B. GOTHEN
which the structure group $G$ is discrete. The Frobenius Theorem has the following immediate consequence.

**Proposition 2.3.** Let $E \to \Sigma$ be a flat bundle and let $e_0 \in E_{x_0}$ for some $x_0 \in \Sigma$. Then, for any sufficiently small neighbourhood $U \subseteq \Sigma$, there is a unique section $s \in \Omega^0(U, E|_U)$ such that $d_A(s) = 0$ and $s(x_0) = e_0$.

### 2.3. Surface group representations and flat bundles.

Given a $G$-bundle $E$ on $\Sigma$ with a connection $A$, it follows from the existence and uniqueness theorem for ordinary differential equations that we can lift any loop $\gamma$ in $\Sigma$ to a covariantly constant loop in $E$ (i.e., one whose tangent vectors are horizontal for the connection). In this way we obtain a well-defined parallel transport $E_x \to E_x$, which is given by multiplication by a unique group element, the **holonomy** of $A$ along $\gamma$, denoted by $h_A(\gamma) \in G$.

Moreover, if the connection $A$ is flat, it follows from Proposition 2.3 that the holonomy only depends on the homotopy class of $\gamma$ and thus we obtain the **holonomy representation** of $\pi_1 \Sigma$:

$$\rho_A : \pi_1 \Sigma \to G$$

defined by $\rho_A([\gamma]) = h_A(\gamma)$. We say that a flat connection $A$ is **reductive** if its holonomy representation is a reductive representation of $\pi_1 \Sigma$ in $G$.

On the other hand, let $\rho : \pi_1 \Sigma \to G$ be representation. We can then define a principal $G$-bundle $E_\rho$ by taking the quotient

$$E_\rho = \tilde{\Sigma} \times_{\pi_1 \Sigma} G,$$

where $\pi_1 \Sigma$ acts on the universal cover $\tilde{\Sigma} \to \Sigma$ by deck transformations and on $G$ by left multiplication via $\rho$. Moreover, since $\tilde{\Sigma} \to \Sigma$ is a covering, there is a natural choice of horizontal subspaces in $E_\rho$. Therefore this bundle has a naturally defined connection which is evidently flat.

One sees that these two constructions are inverses of each other. Next we shall introduce the natural equivalence relation on (flat) connections and promote this correspondence to a bijection between equivalence classes of flat connections and points in the character variety.

### 2.4. Flat bundles and gauge equivalence.

The **gauge group** is the automorphism group

$$\mathcal{G} = \Omega^0(\Sigma, \text{Aut}(E))$$

where $\text{Aut}(E) = E \times_{\text{Ad}} G \to \Sigma$ is the bundle of automorphisms of $E$. The gauge group acts on the space of connections $\mathcal{A}_E$ via

$$g \cdot A = gAg^{-1} + gdg^{-1}.$$  

Moreover, the corresponding action on the curvature is

$$(2.3) \quad F(g \cdot A) = gF(A)g^{-1}$$  

This is the mathematician’s definition. To a physicist the gauge group is the structure group $G$. 

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and hence $G$ preserves the subspace of flat connections on $E$.

Recall that principal $G$-bundles on $\Sigma$ are classified (up to smooth isomorphism) by a characteristic class

$$c(E) \in H^2(\Sigma, \pi_1 G) \cong \pi_1 G.$$  

Here we are using the fact that $G$ is connected and that $\Sigma$ is a closed oriented surface. Fix $d \in \pi_1 G$ and let $E \to \Sigma$ be a principal $G$-bundle with $c(E) = d$. We can then consider the quotient space

$$\mathcal{M}_d^{BR}(\Sigma, G) = \{ A \in \mathcal{A} \mid F(A) = 0 \text{ and } A \text{ is reductive} \}/G,$$

which is known as the de Rham moduli space (recall that $\mathcal{A}$ denotes the space of connections).

**Proposition 2.4.** If flat connections $B_i$ correspond to representations $\rho_i : \pi_1 \Sigma \to G$ for $i = 1, 2$, then $B_1$ and $B_2$ are gauge equivalent if and only if there is a $g \in G$ such that $\rho_1 = g \rho_2 g^{-1}$.

This proposition implies that there is a bijection

$$\mathcal{M}_d^{BR}(\Sigma, G) \cong \mathcal{M}_d^{B}(\Sigma, G),$$

where we denote by $\mathcal{M}_d^{B}(\Sigma, G) \subset \mathcal{M}^{B}(\Sigma, G)$ the subspace of representations with characteristic class $d$.

2.5. Harmonic metrics in flat bundles. Let $G' \subset G$ be a Lie subgroup. Recall that a reduction of structure group in a principal $G$-bundle $E \to \Sigma$ to $G' \subset G$ is a section

$$h : \Sigma \to E/G'$$

of the bundle $E/G' = E \times_G (G/G')$, picking out a $G'$-orbit in each fibre $E_x$.

Let us now fix a maximal compact subgroup $H \subset G$. This choice, together with the invariant inner product on $\mathfrak{g}$, gives rise to a Cartan decomposition:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m},$$

where $\mathfrak{h}$ is the Lie algebra of $H$ and $\mathfrak{m}$ is its orthogonal complement.

**Definition 2.5.** A metric in a principal $G$-bundle $E \to \Sigma$ is a reduction of structure group to $H \subset G$.

In case $E_\rho = \Sigma \times_\rho G$ is a flat bundle, we have

$$E/H = \Sigma \times_\rho (G/H),$$

and hence a metric $h$ in $E$ corresponds to a $\pi_1 \Sigma$-equivariant map

$$\tilde{h} : \Sigma \to G/H.$$

The energy of the metric $h$ is essentially the integral over $\Sigma$ of the norm squared of the derivative of $h$. In the following we make precise this concept.
To start with we give \( \Sigma \) a Riemannian metric and note that \( G/H \) is a Riemannian manifold. Hence we can calculate the norm \( |D\tilde{h}(\tilde{x})| \) at any point \( \tilde{x} \in \tilde{X} \). Furthermore, since the group \( G \) acts on \( G/H \) by isometries, the derivative of \( \tilde{h} \) satisfies
\[
|D\tilde{h}(\tilde{x})| = |D\tilde{h}(\gamma \cdot \tilde{x})|
\]
for any \( \gamma \in \pi_1 \Sigma \). Alternatively, we may proceed as follows. Let \( T^v E \to \Sigma \) be the vertical tangent bundle of \( E \). The fact that \( E \) is flat means that there is a natural projection \( p: TE \to T^v E \) and we can define the vertical part of the derivative of \( \tilde{h} \) as the composition
\[
Dh = p \circ dh: T\Sigma \to TE \to T^v E.
\]
Clearly we have \( |Dh(x)| = |D\tilde{h}(\tilde{x})| \) for any \( \tilde{x} \in E_x \).

**Definition 2.6.** Let \( \Sigma \) be a closed oriented surface with a Riemannian metric and let \( E \to \Sigma \) be a flat principal \( G \)-bundle. The energy of a metric \( h \) in \( E \) is
\[
\mathcal{E}(h) = \int_\Sigma |Dh|^2 \text{vol}.
\]

**Remark 2.7.** Recall that on a surface the integral of a one-form is conformally invariant. Hence it suffices to give \( \Sigma \) a conformal structure in order to make the energy functional well defined.

**Definition 2.8.** A metric \( h \) in a flat \( G \)-bundle \( E \to \Sigma \) is harmonic if it is a critical point of the energy functional.

Next we want to reformulate this in terms of connections. Let \( i: E_H \to E \) be the principal \( H \)-bundle obtained by the reduction of structure group defined by the metric \( h \), and denote the flat connection on \( E \) by \( B \in \Omega^1(E,g) \). Using the Cartan decomposition (2.5) we can then write
\[
(2.6)
i^* B = A + \psi,
\]
where \( A \in \Omega^1(E_H,h) \) defines a connection \( E_H \) and \( \psi \in \Omega^1(E_H,m) \) is a tensorial 1-form which therefore descends to a section, abusively denoted by the same symbol,
\[
\psi \in \Omega^1(E_H(m)).
\]
Note that we have a canonical identification
\[
E_H(m) \cong T^v E
\]
and that under this identification we have
\[
\psi = Dh,
\]
as is easily checked. To calculate the critical points of the energy functional, take a deformation of the metric \( h \) of the form
\[
h_t = \exp(t \cdot s)h \in \Omega^0(\Sigma, E/H)
\]
for \( s \in \Omega^0(\Sigma, E_H(m)) \). One then calculates
\[
\frac{d}{dt}(E(h_t))|_{t=0} = \langle \psi, d_A s \rangle
\]
from which we deduce the following.

**Proposition 2.9.** Let \( h \) be a metric in a flat bundle \( E \to \Sigma \) and let \((A, \psi)\) be defined by (2.6). Then \( h \) is harmonic if and only if
\[
d_A^* \psi = 0.
\]

### 2.6. The Corlette–Donaldson theorem.

The following result was proved independently by Donaldson [15] (for \( G = \text{SL}(2, \mathbb{C}) \)) and Corlette [13] (for more general groups and base manifolds of dimension higher than two); see also Labourie [41]. The idea of the proof is to adapt the proof of Eells–Sampson on the existence of harmonic maps into negatively curved target manifolds to the present “twisted situation”.

**Theorem 2.10.** A flat bundle \( E \to \Sigma \) corresponding to a representation \( \rho : \pi_1 \Sigma \to G \) admits a harmonic metric if and only if \( \rho \) is reductive.

In terms of the pair \((A, \theta)\) given by (2.6), the flatness condition on \( B \) becomes
\[
F(A) + \frac{1}{2}[\theta, \theta] = 0,
\]
\[
d_A \theta = 0,
\]
(2.7)
as can be seen by considering the \( \mathfrak{h} \)- and \( m \)-valued parts of the equation \( F(B) = 0 \) separately. This motivates the following definition.

**Definition 2.11.** Let \( E_H \to X \) be a principal \( H \)-bundle on \( X \), let \( A \) be connection on \( E_H \) and let \( \theta \in \Omega^1(X, E_H(m)) \). The triple \((E_H, A, \theta)\) is called a harmonic bundle if the equations
\[
F(A) + \frac{1}{2}[\theta, \theta] = 0,
\]
\[
d_A \theta = 0,
\]
\[
d_A^* \theta = 0
\]
(2.8) (2.9) (2.10)
are satisfied.

Next we want to obtain a statement at the level of moduli spaces (analogous to (2.4)). Fix a reduction \( E_H \hookrightarrow E \) and consider the gauge groups
\[
\mathcal{H} = \text{Aut}(E_H) = \Omega^0(\Sigma, E_H \times_{\text{Ad}} H),
\]
\[
\mathcal{G} = \text{Aut}(E) = \Omega^0(\Sigma, E \times_{\text{Ad}} G).
\]
Then Theorem 2.10 can equivalently be formulated as saying that for any flat reductive connection \( B \) in \( E \), there is a gauge transformation \( g \in \mathcal{G} \) such that, writing \( g \cdot B = A + \psi \), the triple \((E_H, A, \psi)\) is a harmonic bundle.
Let $d \in \pi_1 H$ and fix $E_H$ with $c(E_H) = d$. The moduli space of harmonic bundles of topological class $d$ is

$$\mathcal{M}_d^{\text{Har}}(\Sigma, G) = \{(A, \theta) \mid (2.8) - (2.10) \text{ hold}\}/\mathcal{H}.$$  

Now Theorem 2.10 can be complemented by a suitable uniqueness statement (analogous to Proposition 2.4) which allows us to altogether obtain a bijective correspondence

$$(2.11) \quad \mathcal{M}_d^{\text{dR}}(\Sigma, G) \cong \mathcal{M}_d^{\text{Har}}(\Sigma, G).$$  

3. LECTURE 2: $G$-Higgs bundles and the Hitchin–Kobayashi correspondence

In Lecture 1 we saw that any reductive surface group representation gives rise to an essentially unique harmonic metric in the associated flat bundle. In this lecture, we shall reinterpret this in holomorphic terms, introducing $G$-Higgs bundles. Moreover we shall explain the Hitchin–Kobayashi correspondence for these.

Recall from Remark 2.7 that we equipped the surface $\Sigma$ with a conformal class of metrics. This is equivalent to having defined a Riemann surface, which we shall henceforth denote by $X = (\Sigma, J)$.

3.1. Lie theoretic preliminaries. Let $H^C$ be the complexification of the maximal compact subgroup $H \subseteq G$ and let $h^C$ and $g^C$ be the complexifications of the Lie algebras $h$ and $g$, respectively. In particular, $h^C = \text{Lie}(H^C)$. However we do not need to assume the existence of a complexification of the Lie group $G$.

The Cartan decomposition (2.5) complexifies to

$$(3.1) \quad g^C = h^C + m^C;$$

note that this is a direct sum of vector spaces but not of Lie algebras. In fact, we have

$$[h^C, h^C] \subseteq h^C, \quad [h^C, m^C] \subseteq m^C, \quad [m^C, m^C] \subseteq h^C.$$

Moreover, we have the $C$-linear Cartan involution

$$\theta: g^C \to g^C,$$

whose $\pm 1$-eigenspace decomposition is (3.1), the real structure (i.e. $C$-antilinear involution) corresponding to $g \subset g^C$

$$\sigma: g^C \to g^C$$

and the compact real structure

$$\tilde{\tau} = \theta \circ \sigma: g^C \to g^C.$$

The $+1$-eigenspace of $\tilde{\tau}$ is a maximal compact subalgebra of $g^C$ whose intersection with $h^C$ is $h$.  

We shall also need the isotropy representation of $H^C$ on $m^C$,
\begin{equation}
\iota: H^C \rightarrow \text{GL}(m^C),
\end{equation}
which is induced by the complexification of the adjoint action of $H$ on $\mathfrak{g}$.

3.2. The Hitchin equations. We extend $\tilde{\tau}$ to
\[ \tau: \Omega^1(X, E(m^C)) \rightarrow \Omega^1(X, E(m^C)) \]
by combining it with conjugation on the form component. Locally
\[ \tau(\omega \otimes a) := \bar{\omega} \otimes \tau(a) \]
for a complex 1-form $\omega$ on $X$ and a section $a$ of $E(m^C)$. There is an isomorphism
\begin{equation}
\Omega^1(E(m)) \rightarrow \Omega^{1,0}(X, E(m^C)),
\end{equation}
\[ \theta \mapsto \theta - iJ\theta / 2 \]
where $J$ is the complex structure on the tangent bundle of $X$. The inverse given by
\begin{equation}
\theta = \varphi - \tau(\varphi).
\end{equation}
This is entirely analogous to the way in which we can write the connection $A \in \Omega^1(E_H, \mathfrak{h})$
\begin{equation}
A = A^{1,0} + A^{0,1}
\end{equation}
with $A^{p,q} \in \Omega^{p,q}(E_H(m^C))$.

Remark 3.1. Note that $E_H(m^C) = E_{H^C}(m^C)$, where $E_{H^C} = E_H \times_H H^C$ is the principal $H^C$-bundle obtained by extension of structure group.

The bijective correspondence $A \leftrightarrow A^{0,1}$ gives us a bijective correspondence between connections $A$ on $E_H$ and holomorphic structures on $E_{H^C}$ (the integrability condition is automatically satisfied because $\dim_{\mathbb{C}} X = 1$).

Correspondingly, for any complex representation $V$ of $H^C$, the vector bundle $E_{H^C}(V)$ becomes a holomorphic bundle and the covariant derivative on sections of $E_{H^C}(V)$ given by $A$ decomposes as $d_A = \bar{\partial}_A + \partial_A$, where $\bar{\partial}_A: \Omega^0(X, E_{H^C}(V)) \rightarrow \Omega^{0,1}(X, E_{H^C}(V))$.

The holomorphic sections of $E_{H^C}(V)$ are just the ones which are in the kernel of $\bar{\partial}_A$. 
With all this notation in place one sees, using the Kähler identities, that the harmonic bundle equations (2.8)–(2.10) are equivalent to the Hitchin equations

\[(3.6) \quad F(A) - [\varphi, \tau(\varphi)] = 0,\]

\[(3.7) \quad \bar{\partial}_A \varphi = 0.\]

Thus we have a canonical identification

\[(3.8) \quad \mathcal{M}^{\text{Har}}_d(X, G) = \mathcal{M}^{\text{Hit}}_d(X, G),\]

where we have introduced the moduli space

\[\mathcal{M}^{\text{Hit}}_d(X, G) = \{(A, \varphi) \mid (3.6)-(3.7) \text{ hold}\}/\mathcal{H}\]

of solutions to the Hitchin equations. This gauge theoretic point of view allows one to give the moduli space \(\mathcal{M}^{\text{Hit}}_d(X, G)\) a Kähler structure. While the metric depends on the choice of conformal structure on \(\Sigma\), the Kähler form is independent of this choice, and in fact coincides with Goldman’s symplectic form [29].

3.3. \(G\)-Higgs bundles, stability and The Hitchin–Kobayashi correspondence. The second Hitchin equation (3.7) says that \(\Phi\) is holomorphic with respect to the structure of holomorphic bundle. Write \(K = T^*X^\mathbb{C}\) for the holomorphic cotangent bundle, or canonical bundle, of \(X\) and \(H^0\) for holomorphic sections. We have thus reached the conclusion that the harmonic bundle gives rise to a holomorphic object, a so-called \(G\)-Higgs bundle, defined as follows.

**Definition 3.2.** A \(G\)-Higgs bundle on \(X\) is a pair \((E, \varphi)\), where \(E \to X\) is a holomorphic principal \(H^\mathbb{C}\)-bundle and \(\varphi \in H^0(X, E(g^\mathbb{C}) \otimes K)\).

When \(G\) is a complex group, we have that \(H^\mathbb{C} = G\) and the Cartan decomposition \(g^\mathbb{C} = g + ig^\mathbb{C}\). Hence a \(G\)-Higgs bundle is a pair \((E, \varphi)\), where \(E\) is a holomorphic principal \(G\)-bundle and \(\varphi \in H^0(X, E(g) \otimes K)\). Note that \(E(g^\mathbb{C}) = \text{Ad} E\) is just the adjoint bundle of \(E\).

Another particular case is when \(G = H\) is a compact group. Then we have \(\varphi = 0\), so a \(G\)-Higgs bundle is just a holomorphic principal bundle and the Hitchin equations simply say that \(F(A) = 0\).

In the following we give some examples of \(G\)-Higgs bundles for specific groups.

**Example 3.3.** Let \(G = \text{SU}(n, \mathbb{C})\). Then a \(G\)-Higgs bundle is just a holomorphic vector bundle \(V \to X\) with trivial determinant.

**Example 3.4.** Let \(G = \text{SL}(n, \mathbb{C})\). Then a \(G\)-Higgs bundle is a pair \((V, \varphi)\), where \(V \to X\) is a holomorphic vector bundle with trivial determinant and \(\varphi \in H^0(X, \text{End}_0(E) \otimes K)\) (where \(\text{End}_0(E)\) is the subspace of traceless endomorphisms).
Example 3.5. Let $G = \text{SU}(p, q)$. Then a $G$-Higgs bundle is a triple $(V, W, \varphi)$, where $V$ and $W$ are holomorphic vector bundles on $X$ of rank $p$ and $q$ respectively, satisfying $\det(V) \otimes \det(W) \cong \mathcal{O}$ and
\[
\varphi = (\beta, \gamma) \in H^0(X, \text{Hom}(W, V) \otimes K) \oplus H^0(X, \text{Hom}(V, W) \otimes K).
\]

Example 3.6. Let $G = \text{Sp}(2n, \mathbb{R})$. Then a $G$-Higgs bundle is a pair $(V, \varphi)$, where $V$ is a holomorphic vector bundle on $X$ of rank $n$ and
\[
\varphi = (\beta, \gamma) \in H^0(X, S^2 V \otimes K) \oplus H^0(X, S^2 V^* \otimes K).
\]

In case $G = \text{SU}(n)$, we are thus in the presence of a complex vector bundle with a flat unitary connection. Such a bundle turns out to be polystable. The Narasimhan–Seshadri Theorem \cite{43}, conversely, says that if a holomorphic vector bundle is polystable then it admits a metric such that the unique unitary connection compatible with the holomorphic structure is (projectively) flat. There is an analogous statement for other compact $G$, due to Ramanathan \cite{45}.

These results generalize to $G$-Higgs bundles. The appropriate stability condition is a bit involved to state in general. However, in the case of Higgs vector bundles it is simply the following. Recall that the slope of a vector bundle $E \to X$ is $\mu(E) = \deg(E)/\text{rk}(E)$. Also, we say that a subbundle $F \subset E$ is $\varphi$-invariant if $\varphi(F) \subset F \otimes K$.

**Definition 3.7.** A Higgs vector bundle $(E, \varphi)$ is semistable if
\[
\mu(F) \leq \mu(E)
\]
for any subbundle $\varphi$-invariant subbundle $F \subset E$ and it is stable if, moreover, strict inequality holds whenever $F$ is proper and non-zero. A Higgs vector bundle $(E, \varphi)$ is polystable if it is isomorphic to a direct sum of stable Higgs bundles, all of the same slope.

The stability conditions for $G$-Higgs bundles can be obtained as a special case of a general stability conditions for pairs and we refer the reader to \cite{21} for the detailed formulation. It is worth noting that poly- and semistability of a $G$-Higgs bundle $(E, \varphi)$ are equivalent to poly- and semistability of the Higgs vector bundle $(E(g^C), \text{ad}(\varphi))$.

The general Hitchin–Kobayashi correspondence for principal pairs \cite{38} \cite{7} now has as a consequence the following Hitchin–Kobayashi correspondence for $G$-Higgs bundles. (see \cite{21} for the full extension to polystable pairs, as well as a detailed analysis of the case of $G$-Higgs bundles.)

**Theorem 3.8.** Assume that $G$ is semisimple. A $G$-Higgs bundle $(E, \varphi)$ is polystable if and only if it admits a reduction of structure group to the maximal compact $H \subset H^C$, unique up to isomorphism of $H$-bundles, such that the following holds: denoting by $A$ the unique $H$-connection compatible with the reduction and $\bar{\partial}_A$ the $\bar{\partial}$-operator induced form the holomorphic structure, the pair $(A, \varphi)$ satisfies the Hitchin equations (3.6) and (3.7).
In the context of Higgs bundles, the Hitchin–Kobayashi correspondence goes back to the work of Hitchin [35] and Simpson [47].

Remark 3.9. The assumption that \( G \) be semisimple is not essential. Indeed, as can be expected from the situation for usual \( G \)-bundles, for reductive \( G \) an analogous statement holds, if one adds a suitable central term to the right hand side of the first of the Hitchin equations. For the correspondence with representations, one must then consider homomorphisms of a central extension of the fundamental group. We refer to [21] for more details on this.

Just as for vector bundles, stability of \( G \)-Higgs bundles has a dual importance. Namely, apart from its role in the Hitchin–Kobayashi correspondence, it is also the appropriate notion for constructing moduli spaces using GIT. The constructions Schmitt (see the book [46]) are in fact sufficiently general to also cover many cases of \( G \)-Higgs bundles. Thus we have yet another moduli space at our disposal, namely the moduli space

\[
\mathcal{M}^\text{Dol}_d(X, G)
\]

of semistable \( G \)-Higgs bundles of topological class \( d \in \pi_1 H \). Alternatively, this moduli space can be constructed using a Kuranishi slice method. From this point of view, we fix a principal \( H^C \)-bundle \( E \to \Sigma \) and consider the complex configuration space

\[
C^C = \{(A^{0,1}, \varphi) \mid \bar{\partial}_A \varphi = 0\}.
\]

The complex gauge group \( H^C \) acts naturally on this space and on the subspace \( C^C_{\text{polystable}} \) of pairs \( (A^{0,1}, \varphi) \) which define the structure of a polystable \( G \)-Higgs bundle on \( E \). The moduli space is then

\[
\mathcal{M}^\text{Dol}_d(X, G) = C^C_{\text{polystable}}/H^C.
\]

Either way, Theorem 3.10 implies that we have an identification

\[
(3.9) \quad \mathcal{M}^\text{Dol}_d(X, G) \cong \mathcal{M}^\text{Hit}_d(X, G).
\]

Putting together this with the previous identifications (2.4), (2.11) and (3.8) we finally obtain the non-abelian Hodge Theorem.

**Theorem 3.10.** Let \( X \) be a closed Riemann surface of genus \( g \). Then there is a homeomorphism

\[
\mathcal{M}^B_d(X, G) \cong \mathcal{M}^\text{Dol}_d(X, G).
\]

**Remark 3.11.** The fact that the identification of Theorem 3.10 is a homeomorphism is not too hard to see, but more is true: outside of the singular loci, the identification is in fact an analytic isomorphism. On the other hand, it is definitely not algebraic. In this respect, it is instructive to consider the example \( G = \mathbb{C}^* \).
3.4. The Hitchin map. We end this Lecture by recalling the definition of the Hitchin map which plays a central role in the theory of Higgs bundles.

Take a basis \( \{ p_1, \ldots, p_r \} \) for the invariant polynomials on the Lie algebra \( g^C \) and let \( d_i = \deg(p_i) \). Given a \( G \)-Higgs bundle \((E, \varphi)\), evaluating \( p_i \) on \( \varphi \) gives a section \( p_i(\varphi) \in H^0(X, K^{d_i}) \). The Hitchin map is defined to be

\[
\begin{align*}
&h : \mathcal{M}_{\text{Dol}}^d \to B, \\
&(E, \varphi) \mapsto (p_1(\varphi), \ldots, p_r(\varphi)),
\end{align*}
\]

where the Hitchin base is

\[
B := \bigoplus H^0(X, K^{d_i}).
\]

The Hitchin map is proper and, for \( G \) complex, defines an algebraically completely integrable system known as the Hitchin system (see Hitchin [36]).

3.5. The moduli space of \( SU(p, q) \)-Higgs bundles. We end this section by illustrating how the Higgs bundle point of view allows for easy proofs of strong results by proving the Milnor–Wood inequality for \( SU(p, q) \)-Higgs bundles, and discussing a closely related rigidity result.

Recall that an \( SU(p, q) \)-Higgs bundle is a quadruple \((V, W, \beta, \gamma)\), where \( V \) and \( W \) are vector bundles on \( X \) of rank \( p \) and \( q \) respectively, satisfying \( \det(V) \otimes \det(W) \cong \mathcal{O} \), and where \( \beta \in H^0(X, \Hom(W, V) \otimes K) \) and \( \gamma \in H^0(X, \Hom(V, W) \otimes K) \). The topological classification of such bundles is given by \( \deg(V) = -\deg(W) \in \mathbb{Z} \). Denote by \( \mathcal{M}_d \) the moduli space of \( SU(p, q) \)-Higgs bundles with \( \deg(V) = d \).

In the case \( p = q = 1 \), we have \( SU(1, 1) = SL(2, \mathbb{R}) \) and the degree \( d \) is just the Euler class of the corresponding flat \( SL(2, \mathbb{R}) \)-bundle. In 1957 Milnor [42] proved that it satisfies the bound

\[
|d| \leq g - 1.
\]

Much more generally, whenever \( G \) is non-compact of Hermitian type, one can define an integer invariant, the Toledo invariant, of representations \( \rho : \pi_1 \Sigma \to G \) and there is a bound on the Toledo invariant, usually known as a Milnor–Wood inequality. In various degrees of generality this is due to, among others, Domic–Toledo [14], Dupont [16], Toledo [49, 50] and Turaev [51]. In the case of \( G = SU(p, q) \), the Milnor–Wood inequality is

\[
|d| \leq \min\{p, q\}(g - 1).
\]

A proof of this Milnor–Wood inequality using Higgs bundles is very easy to give. For this it is convenient (though not essential) to pass through usual Higgs vector bundles: since \( SU(p, q) \) is a subgroup of \( SL(p + q, \mathbb{C}) \), to any \( SU(p, q) \)-Higgs bundle we can associate an \( SL(p + q, \mathbb{C}) \)-Higgs bundle

\[
(E, \Phi) = (V \oplus W, \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}).
\]
Now, if \((V, W, \beta, \gamma)\) is polystable, then so is \((E, \Phi)\) (this follows immediately from the fact that a solution to the \(\text{SU}(p,q)\)-Hitchin equations on \((V, W, \beta, \gamma)\) induces a solution on \((E, \Phi)\)). Let \(N \subset V\) be the kernel of \(\gamma: V \to W \otimes K\), viewed as a subbundle, and let \(I \otimes K \subset W \otimes K\) be the subbundle obtained by saturating the image subsheaf. Thus, \(\gamma\) induces a bundle map of maximal rank \(\bar{\gamma}: V/N \to I \otimes K\), from which we deduce that

\[
\deg(N) - \deg(V) + \deg(I) + (2g - 2) \text{rk}(\gamma) \geq 0
\]

with equality if and only if \(\bar{\gamma}\) is an isomorphism. Moreover, the subbundles \(N \subset E\) and \(V \oplus I \subset E\) are \(\Phi\)-invariant, so polystability of \((E, \Phi)\) implies that

\[
\begin{align*}
\deg(N) &\leq 0, \\
\deg(V) + \deg(I) &\leq 0.
\end{align*}
\]

Putting together equations (3.12)–(3.14) we obtain

\[
\deg(V) \leq \text{rk}(\gamma)(g - 1)
\]

from which the Milnor–Wood inequality (3.11) is immediate for \(d \geq 0\). When \(d \leq 0\) a similar argument involving \(\beta\) instead of \(\gamma\) gives the result.

But our arguments in fact give more information in the case when equality holds in (3.11). Assume for definiteness that \(d \geq 0\) and that \(p \leq q\). Then, if equality holds in (3.11), we conclude immediately that \(\text{rk}(\gamma) = p\) and that \(\gamma: V \to I \otimes K\) is an isomorphism. Hence, by polystability of \((E, \Phi)\), there is a decomposition \(W = I \oplus Q\) and \(\beta|_Q = 0\). In other words, the \(\text{SU}(p,q)\)-Higgs bundle \((V, W, \beta, \gamma)\) decomposes into the \(\text{U}(p,p)\)-Higgs bundle \((V, I, \beta, \gamma)\) and the \(\text{U}(q-p)\)-Higgs bundle \(Q\).

From the point of view of representations of the fundamental group this can be viewed as a rigidity result which was first proved by Toledo [50] for \(p = 1\) and by Hernández [34] for \(p = 2\). A more general result valid in the context of arbitrary groups of Hermitian type has been proved by Burger–Ioossi–Wienhard [10, 8]. From the point of view of Higgs bundles, the results for \(\text{U}(p,q)\) appeared in [3] and a survey of the situation for other classical groups can be found in [4], while the PhD thesis of Rubio [44] treats the question for general groups using a general Lie theoretic approach.

4. Lecture 3: Morse–Bott theory of the moduli space of \(G\)-Higgs bundles

In this final lecture we consider the \(\mathbb{C}^*\)-action on the moduli space of \(G\)-Higgs bundles and explain how to use it to study its topology. We shall consider the Dolbeault moduli space and occasionally use the identification with the gauge theory moduli space of solutions to Hitchin’s equation. For simplicity we shall denote it simply by \(\mathcal{M}_d\). Again, though not strictly necessary, we shall assume that \(G\) is semisimple. To get started we need to review some of the deformation theory of \(G\)-Higgs bundles.
4.1. Simple and infinitesimally simple $G$-Higgs bundles. Let $E$ be a principal $H^C$-bundle on $X$. An automorphism of $E$ is an equivariant holomorphic bundle map $g: E \to E$ which admits a holomorphic inverse. We denote the group of automorphisms of $E$ by $\text{Aut}(E)$. Equivalently, we may define $\text{Aut}(E)$ to be the space of holomorphic sections of the bundle of automorphisms $E \times_{\text{Ad}} G \to X$. Let $(E, \varphi)$ be a $G$-Higgs bundle. We denote by $\text{Aut}(E)$ the group of automorphisms of $(E, \varphi)$:
\[
\text{Aut}(E, \varphi) = \{ g \in \text{Aut}(E) \mid \text{Ad}(g)(\varphi) = \varphi \}.
\]
We also introduce the infinitesimal automorphism space (which, at least formally, is the Lie algebra of the automorphism group), defining
\[
\text{aut}(E, \varphi) = \{ Y \in H^0(X, E(h^C)) \mid [Y, \varphi] = 0 \}.
\]
A $G$-Higgs bundle $(E, \varphi)$ is simple if its automorphism group is smallest possible, i.e.,
\[
\text{Aut}(E, \varphi) = Z(H^C) \cap \ker(\iota).
\]
Also, we say that $(E, \varphi)$ is infinitesimally simple if
\[
\text{aut}(E, \varphi) = Z(h^C) \cap \ker(\text{ad}).
\]
Note that for Higgs vector bundles, these two notions are equivalent. This is, however, not true in general, as Example 4.2 below shows.

The following result is the $G$-Higgs bundle version of the well known fact that a stable vector bundle only has scalar automorphisms.

**Proposition 4.1.** Let $(E, \varphi)$ be a stable $G$-Higgs bundle. Then it is infinitesimally simple.

**Example 4.2.** Let $M_1$ and $M_2$ be line bundles on $X$ with $M_1^2 = K$ and $M_1 \neq M_2$. Define $V = M_1 \oplus M_2$ and let $\beta = 0 \in H^0(X, S^2V \otimes K)$ and $\gamma = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \in H^0(X, S^2V^* \otimes K)$. Then it is easy to see that the $\text{Sp}(2, \mathbb{R})$-Higgs bundle $(V, \beta, \gamma)$ is stable and hence infinitesimally simple. However, it is not simple since it has the automorphism $\left( \begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$.

4.2. Deformation theory of $G$-Higgs bundles. Next we outline the deformation theory of $G$-Higgs bundles. A useful reference for the following material is Biswas–Ramanan [2].

**Definition 4.3.** The deformation complex of a $G$-Higgs bundle $(E, \varphi)$ is the complex of sheaves
\[
C^\bullet(E, \varphi): E(h^C) \xrightarrow{[-, \varphi]} E(m^C) \otimes K.
\]
The deformation theory of a $G$-Higgs bundle $(E, \varphi)$ is governed by the hypercohomology groups of the deformation complex. Thus, we have the following standard results.

**Proposition 4.4.** Let $(E, \varphi)$ be a $G$-Higgs bundle.
(1) There is a canonical identification between the space of infinitesimal deformations of \((E, \varphi)\) and the hypercohomology group
\[ H^1(C^\bullet(E, \varphi)) \]

(2) There is a long exact sequence
\[ 0 \to H^0(C^\bullet(E, \varphi)) \to H^0(E(h^C)) \to H^0(E(m^C) \otimes K) \]
\[ \to H^1(C^\bullet(E, \varphi)) \to H^1(E(h^C)) \to H^1(E(m^C) \otimes K) \]
\[ \to H^2(C^\bullet(E, \varphi)) \to 0. \]

Note that the long exact sequence in (2) of the previous Proposition immediately implies that there is a canonical identification
\[ \text{aut}(E, \Phi) = H^0(C^\bullet(E, \varphi)). \]

One way of proving the following result is to consider the Kuranishi slice method for constructing the moduli space mentioned in Section 3.3.

**Proposition 4.5.** Assume that \((E, \varphi)\) is a stable and simple \(G\)-Higgs bundle and that the vanishing \(H^2(C^\bullet(E, \varphi)) = 0\) holds. Then \((E, \varphi)\) represents a smooth point of the moduli space \(M_d\).

**Remark 4.6.** If \(G\) is a reductive group which is not necessarily semisimple, one should consider the reduced deformation complex, obtained by dividing out by \(Z(g^C)\); equivalently, this is the deformation complex of the \(PG\)-Higgs bundle obtained from the \(G\)-Higgs bundle.

4.3. The \(\mathbb{C}^*\)-action and topology of moduli spaces. In order to avoid the problems arising from the presence of singularities, throughout this section we shall make the assumption that we are in a situation where the moduli space \(M_d\) is smooth.

It is a very important feature of the moduli space of Higgs bundles that it admits an action of the multiplicative group of non-zero complex numbers:
\[ \mathbb{C}^* \times M_d \to M_d \]
\[ (z, (E, \varphi)) \mapsto (E, z\varphi). \]

There are two distinct ways of using this action to obtain topological information about the moduli space, as we shall now explain. However, a theorem of Kirwan ensures that they give essentially equivalent information.

We start by a Morse theoretic point of view. For this we use the identification between the Dolbeault moduli space and the moduli space of solutions to the Hitchin equations given by Theorem 3.8. Observe that the subgroup \(S^1 \subset \mathbb{C}^*\) acts on the moduli space of solutions to the Hitchin equations. With respect to the (symplectic) Kähler form on
the moduli space this action is Hamiltonian and it has a moment map (up to some fixed scaling) given by
\[ f: \mathcal{M}_d \to \mathbb{R} \]
\[ (A, \varphi) \mapsto \|\varphi\|^2 := \int_X |\varphi|^2 \text{vol}. \]

Hitchin [35] showed, using Uhlenbeck’s weak compactness theorem, that \( f \) is a proper map. Moreover, it follows from a theorem of Frankel [18] that \( f \) is a perfect Bott–Morse function. That \( f: M \to \mathbb{R} \) is Bott–Morse means that its critical points form smooth (connected) submanifolds \( N_\lambda \subseteq M \) such that the Hessian of \( f \) is non-degenerate along the normal bundle to \( N_\lambda \) in \( M \). That \( f \) is perfect means that the Poincaré polynomial
\[ P_t(M) := \sum t^i \dim H^i(M, \mathbb{Q}) \]
can be determined as
\[ (4.2) \]
\[ P_t(M) = \sum_{\lambda} t^\text{Index}(N_\lambda) P_t(N_\lambda); \]
here \( \text{Index}(N_\lambda) \) is the index of the critical submanifold \( N_\lambda \), i.e., the real rank of the subbundle of the normal bundle on which the Hessian of \( f \) is negative definite.

The condition for \( f \) to be a moment map for the Hamiltonian \( S^1 \)-action on \( \mathcal{M}_d \) is
\[ \text{grad } f = i\xi, \]
where \( \xi \) is the vector field generating the \( S^1 \)-action. In particular, the critical submanifolds of \( f \) are just the components of the fixed locus of the \( S^1 \)-action. Moreover, if we denote by \( N^+_\lambda \) the stable manifold of \( N_\lambda \), we obtain a Morse stratification
\[ \mathcal{M}_d = \bigcup_\lambda N^+_\lambda. \]

Note that the fact that \( f \) is proper and bounded below guarantees that every point in \( \mathcal{M}_d \) belongs to one of the \( N^+_\lambda \).

The more algebraic point of view comes about by looking at the full \( \mathbb{C}^* \)-action on \( \mathcal{M}_{\text{Dol}}^d \). It is a general result of Bialynicki-Birula [1] that there is an algebraic stratification defined as follows: let \( \tilde{N}_\lambda \) be the components of the fixed locus and define
\[ \tilde{N}^+_\lambda = \{ m \in \mathcal{M}_d \mid \lim_{z \to 0} z \cdot m \in \tilde{N}_\lambda \}. \]
Then the Bialynicki-Birula stratification is
\[ \mathcal{M}_d = \bigcup_\lambda \tilde{N}^+_\lambda. \]
It is perhaps not immediately clear that every point in \( \mathcal{M}_d \) lie in one of the \( \tilde{N}^+_\lambda \). It follows, however, from the properness and equivariance (with
respect to the suitable weighted \(\mathbb{C}^\ast\)-action on the Hitchin base \(B\) of the Hitchin map (3.10).

The whole picture fits into the general setup of \(\mathbb{C}^\ast\)-actions on Kähler manifolds arising from hamiltonian circle actions. In particular, it follows from the results of Kirwan [40] that the Morse and Bialynicki-Birula stratifications coincide.

From either point of view, one can now obtain topological information on the moduli space, as pioneered by Hitchin [35] in his calculation of the Poincaré polynomial of the moduli space of the moduli space of rank 2 Higgs bundles. In general, the success of this approach depends crucially on having a good understanding of the topology of the fixed loci \(N_\lambda\). We remark that the role played by the underlying geometric decomposition of the moduli space is perhaps best brought out by studying in the first place the class of the spaces under study in the \(K\)-theory of varieties, and then obtaining from this information such as Hodge and Poincaré polynomials. For examples of this point of view we refer to Chuang–Diaconescu–Pan [12] or García-Prada–Heinloth–Schmitt [23].

4.4. Calculation of Morse indices. Let us consider the fixed points of the circle action on \(\mathcal{M}_d\). For simplicity we start out with an ordinary Higgs vector bundle \((E, \varphi)\), where \(E\) is a vector bundle and \(\varphi \in H^0(X, \text{End}(E) \otimes K)\).

The following is easily proved (see Hitchin [35] or Simpson [48]).

**Proposition 4.7.** The Higgs bundle \((E, \varphi)\) is a fixed point of the circle action on \(\mathcal{M}^\text{Dol}_d\) if and only if it is a Hodge bundle, i.e., there is a decomposition

\[
E = E_0 \oplus \ldots \oplus E_p
\]

and, with respect to this decomposition, \(\varphi\) has weight one, by which we mean that \(\varphi(E_k) \subseteq E_{k+1} \otimes K\).

The basic idea is that the weight \(k\) subbundle \(E_k \subset E\) is the \(ik\)-eigenbundle of the infinitesimal automorphism \(\psi = \lim_{\theta \to 0} g(\theta)\) counter-acting the circle action, where

\[
(E, e^{i\theta} \varphi) = g(\theta) \cdot (E, \varphi).
\]

For \(G\)-Higgs bundles in general, the simplest procedure is to work out the shape of the Hodge bundles (fixed under the circle action) in each individual case. Note that if the \(G\)-Higgs bundle \((E, \varphi)\) is fixed, then so is the the adjoint Higgs vector bundle \((E(g^C), \text{ad}(\varphi))\) and therefore it is a Hodge bundle. Moreover, since the infinitesimal automorphism \(\psi\) lies in \(E(\mathfrak{h})\), the decomposition of \(E(g^C)\) in eigenbundles is compatible with the decomposition \(E(g^C) = E(\mathfrak{h}^C) \oplus E(\mathfrak{m}^C)\). It follows that there are
decompositions

\[ E(\mathfrak{h}_C) = \bigoplus E(\mathfrak{h}_C)_k, \]
\[ E(\mathfrak{m}_C) = \bigoplus E(\mathfrak{m}_C)_k, \]

and that with respect to these we have

\[ \text{ad}(\varphi): E(\mathfrak{h}_C)_k \to E(\mathfrak{m}_C)_{k+1} \otimes K, \]
\[ \text{ad}(\varphi): E(\mathfrak{m}_C)_k \to E(\mathfrak{h}_C)_{k+1} \otimes K. \]

In particular, the deformation complex of \((E, \varphi)\) decomposes as

\[ C^\bullet(E, \varphi) = \bigoplus C^\bullet_k(E, \varphi), \]

where the weight \(k\) piece of the deformation complex is given by

\[ C^\bullet_k(E, \varphi): E(\mathfrak{h}_C)_k \xrightarrow{[-, \varphi]} E(\mathfrak{m}_C)_{k+1} \otimes K. \]

An easy calculation (see for example [22] for the case of ordinary parabolic Higgs bundles which is essentially the same as the present one) now shows the following.

**Proposition 4.8.** Let \((E, \varphi)\) be a stable \(G\)-Higgs bundle which is fixed under the circle action and represents a smooth point of the moduli space. With the notation introduced above, we have

\[ \dim N^+_\lambda = \dim H^1(C^{\bullet}_{\leq 0}), \]
\[ \dim N^-\lambda = \dim H^1(C^{\bullet}_{\geq 0}), \]
\[ \dim N_\lambda = \dim H^1(C^{\bullet}_0). \]

Hence the Morse index of the critical submanifold \(N_\lambda\) is

\[ \text{index}(N_\lambda) = 2 \dim H^1(C^{\bullet}_{>0}). \]

Bott–Morse theory shows that the number of connected components of the moduli space equals that of the subspace of local minima of \(f\). Thus, for the determination of this most basic of topological invariants it is important to have a convenient criterion for the Morse index to be zero. This is provided by the following result ([3, Proposition 4.14]; see [5, Lemma 3.11] for a corrected proof).

**Proposition 4.9.** Let \((E, \varphi)\) represent a critical point of \(f\). Then \((E, \varphi)\) represents a local minimum of \(f\) if and only if the map

\[ [-, \varphi]: E(\mathfrak{h}_C) \to E(\mathfrak{m}_C) \otimes K \]

is an isomorphism for all \(k > 0\).
4.5. **The moduli space of Sp(2n, R)-Higgs bundles.** We end by illustrating how the ideas explained in this section work, by considering the case $G = \text{Sp}(2n, \mathbb{R})$.

Recall that an Sp(2n, R)-Higgs bundle is a triple $\left( V, \beta, \gamma \right)$, where $V \to X$ is a rank $n$ vector bundle, $\beta \in H^0(X, S^2V) \otimes K$ and $\gamma \in H^0(X, S^2V^*)$. The topological classification of such bundles is given by $\text{deg}(V) \in \mathbb{Z}$. Denote by $M_d$ the moduli space of Sp(2n, R)-Higgs bundles with $\text{deg}(V) = d$.

In the following we outline the application of the Morse theoretic point of view for determining the number of connected components of $M_d$. We should point out that $M_d$ is not a smooth variety, so that care must be taken in dealing with singularities in applying the theory. We shall ignore this issue for reasons of space, and in order to bring out more clearly the main ideas. We refer to [20] for full details.

Note that a Sp(2n, R)-Higgs bundle is in particular an SU(n,n)-Higgs bundle. Hence we have from (3.11) that the Milnor–Wood inequality

$$|d| \leq n(g - 1)$$

holds. Say that a Sp(2n, R)-Higgs bundle is *maximal* if equality holds.

Note that taking $V$ to its dual and interchanging $\beta$ and $\gamma$ defines an isomorphism $M_d \cong M_{-d}$. Hence we shall assume without loss of generality that $d \geq 0$ for the remainder of this section.

Denote by $N_0 \subset M_d$ the subspace of local minima of $f$. In the non-maximal case, Proposition 4.9 leads to the following result.

**Proposition 4.10.** Assume that $0 < d < n(g - 1)$. Then the subspace of local minima $N_0 \subset M_d$ consists of all $(V, \beta, \gamma)$ with $\beta = 0$. If $d = 0$, the subspace of local minima $N_0 \subset M_0$ consists of all $(V, \beta, \gamma)$ with $\beta = 0$ and $\gamma = 0$.

Thus for $d = 0$, the subspace $N_0$ can be identified with the moduli space of polystable vector bundles of degree zero. Since this moduli space is known to be connected, we conclude that $M_0$ is also connected.

For $0 < d < n(g - 1)$, the moduli space $M_d$ is known to be connected only for $n = 1$ (by the results of Goldman [26], reproved by Hitchin [35] using Higgs bundles) and for $n = 2$ by Garcia-Prada–Mundet [24] (see also [31]). However, for $n \geq 3$, the connectedness of $N_0$ — and hence $M_d$ — appears to be difficult to establish.

On the other hand, when $d = n(g - 1)$ is maximal, the complete answer is known from the work of Goldman and Hitchin cited above when $n = 1$, from [30] when $n = 2$, and from [20] when $n \geq 3$. It is as follows.

**Theorem 4.11.** Let $M_{\text{max}}$ be the moduli space of Sp(2n, R)-Higgs bundles $(V, \beta, \gamma)$ with $\text{deg}(V) = n(g - 1)$. Then

1. $\# \pi_0 M_{\text{max}} = 2^{2g}$ for $n = 1$,
2. $\# \pi_0 M_{\text{max}} = 3 \cdot 2^{2g} + 2g - 4$ for $n = 2$, and
(3) \( \#\pi_0\mathcal{M}_{\text{max}} = 3 \cdot 2^{2g} \) for \( n \geq 3 \).

We end by briefly explaining how this result comes about. Hitchin [37] showed that whenever \( G \) is a split real form, the moduli space of \( G \)-Higgs bundles has a distinguished component, now known as the Hitchin component, which can be concisely described in terms of representations of the fundamental group: it consists of \( G \)-Higgs bundles corresponding to representations which factor through a Fuchsian representation of the fundamental group in \( \text{SL}(2, \mathbb{R}) \), where \( \text{SL}(2, \mathbb{R}) \hookrightarrow G \) is embedded as a so-called principal three-dimensional subgroup (when \( G = \text{Sp}(2n, \mathbb{R}) \) this is just the irreducible representation of \( \text{SL}(2n, \mathbb{R}) \) on \( \mathbb{R}^{2n} \)). For every \( n \), the moduli space \( \mathcal{M}_{\text{max}} \) has \( 2^{2g} \) Hitchin components which, however, become identified if the pass to the projective group \( \text{PSp}(2n, \mathbb{R}) \).

To explain the appearance of the remaining components, recall the argument used to prove the Milnor–Wood inequality (3.11) in Section 3.5. This shows that for a maximal \( \text{Sp}(2n, \mathbb{R}) \)-Higgs bundle \((V, \beta, \gamma)\) (with \( d \geq 0 \)), we have an isomorphism

\[ \gamma: V \to V^* \otimes K. \]

Hence, since \( \gamma \) is symmetric, \( V \) admits a \( K \)-valued everywhere non-degenerate quadratic form. Defining \( W = V \otimes K^{-n/2} \) and \( Q = \gamma \otimes 1_{K^{-n/2}} \) we obtain an \( O(n, \mathbb{C}) \)-bundle \((W, Q)\), meaning that we obtain new topological invariants defined by the Stiefel–Whitney classes \( w_1 \) and \( w_2 \) of \((W, Q)\). These then give rise to new subspaces \( \mathcal{M}_{w_1, w_2} \) and, using the Morse theoretic approach, one shows that they are in fact connected components. When \( n = 2 \), even more components appear since, when \( w_1 = 0 \), there is a reduction to the circle \( \text{SO}(2, \mathbb{C}) \subset O(2, \mathbb{C}) \) and this give rise to an integer invariant because \( \text{SO}(2) = S^1 \).

Remark 4.12. These new invariants have been studied (and generalized) from the point of view surface group representations in the work of Guichard–Wienhard [32].

Remark 4.13. Let \((W, Q)\) be the \( O(n, \mathbb{C}) \)-bundle arising from a maximal \( \text{Sp}(2n, \mathbb{R}) \)-Higgs bundle as above and define \( \theta = (\beta \otimes 1_{K^{-n/2}}) \circ Q: W \to W \otimes K^2 \). Then \((W, Q, \theta)\) is a \( \text{GL}(n, \mathbb{R}) \)-Higgs bundle, except for the fact the twisting is by the square of the canonical bundle rather than the canonical bundle itself. This observation is the beginning of an interesting story known as the “Cayley correspondence”; for more on this we refer to [4] and Rubio [44].

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