Mathematical model of DBH uncertainty from the shape of the wood cross-section

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Abstract. We investigate the uncertainty of Diameter at Breast Height, or DBH, of woods via a simple mathematical model. Usually, the DBH is not measured directly. The measurement is done on the Girth at Breast Height (GBH), which, by definition, is the circumference of wood at 130 cm above the ground. The DBH can then be calculated, assuming that the wood cross-section is a circle. This can introduce uncertainty to the DBH due to the irregularity of the shape of the wood cross-section. In this work, we approximate the wood cross-section by an ellipse, which can be fitted to the cross-section better than the circle. However, the eccentricity of the fitted ellipse is unknown, so we treat it as a random variable. The discrepancy between the diameter of the circle with the same area as the fitted ellipse, and the diameter of the circle with the same circumference as the GBH, is calculated. Monte Carlo simulations are performed to obtain the uncertainty of the DBH.

1. Introduction
Diameter at Breast Height, or DBH, is an important parameter in forestry measurement. Together with the tree height, we can evaluate the mass of the tree using the allometric equation. This is very important in many forestry activities, e.g. evaluation of wood product or carbon credit estimation [1].

In practice, the DBH is not measured directly. The measurement is done on the Girth at Breast Height (GBH), which, by definition, is the circumference of a tree at 130 cm above the ground. The DBH can then be calculated, assuming that the cross-section is a circle. At this point that we raise a question, is it reasonable that any tree with the same GBH will have the same DBH regardless of their shape?

In our work, we are mainly interested in biomass evaluation for a certain kind of tree that has a straight, cone-like shape, such as Eucalyptus and Teak. There are suggestions, for example in [2], that biomass of the tree may be obtained from its volume and density. This method has an advantage that we need fewer trees cut down to gain the data for the biomass evaluation model. However, it is obvious that tree cross-sectional shape is crucial in the volume calculation. Since the conventional DBH described above does not depend on the tree cross-sectional shape, it’s needed to be modified to be used in tree volume determination.

In the following, we will propose an alternative definition of DBH, based on the tree cross-sectional area at breast height. This new DBH can be directly used to estimate the tree volume. In principle, the discrepancy of the tree volume from new and conventional DBH can be analyzed. However, it is hard to obtain an analytical formula for this discrepancy, since the shape of the tree
cross-section is irregular. We propose a model that uses an ellipse as an approximation of a tree cross-section. The ellipse eccentricity is treated as a random variable. The uncertainty component due to the tree cross-section shape can then be assigned to the DBH, and thus to the tree volume.

2. Area-based DBH
For a solid cone of height $h$ and base area $A$, it’s volume can be obtained as,

$$ V = \frac{Ah}{3}. \quad (1) $$

If we assume that a tree has a cone-like shape of height $h$ and cross-sectional area at breast height of $A_D$, it’s volume will be,

$$ V = \frac{A_Dh^2}{3(h-h_D)^2}, \quad (2) $$

where $h_D = 130 \text{ cm}$ is the breast height. Let us define the area based DBH $D_A$ of a tree as the diameter of a circle that has an area equal to $A_D$. From this definition, the tree volume can be expressed as,

$$ V = \frac{\pi D_A^2h^2}{12(h-h_D)}. \quad (3) $$

We can see that for trees with the same height and GBH, the value of $D_A$ will depend on the tree cross-section shape at breast height. If we assume a circular cross-section at breast height, the tree volume will be,

$$ V_c = \frac{\pi D_c^2h^2}{12(h-h_D)}, \quad (4) $$

where $D_c$ is the conventional DBH, obtained from GBH as described above. For the same value of circumference, we know that $D_c \geq D_A$, so we have $V_c \geq V$. The discrepancy,

$$ \delta_V = V - V_c = \frac{\pi (D_c^2 - D_A^2)h^2}{12(h-h_D)}, \quad (5) $$

can be used as a correction to $V_c$, the tree volume calculated from field measurement, to obtain $V$ as,

$$ V = V_c + \delta_V. \quad (6) $$

Irregularity of tree cross-section shape will introduce the variation of $D_A$, and thus the variation of $V$. We can rewrite equation (6), taking into account the uncertainty due to the irregularity of cross-section shape as [3],

$$ V \pm u(V) = V_c + (\delta_V)_{\text{average}} \pm u(\delta_V), \quad (7) $$

where,
3. Elliptic approximation of wood cross section

The next question is that, how can we determine the uncertainty \( u(D_A) \)? What is the parameter that can be used to represent the irregularity of a tree cross-section shape?

We try to answer this question by doing an approximation of the tree cross-section. We assume that when we use a measuring tape to measure the GBH, the tape acts as a fitting curve to the tree cross-section edge. We approximate this curve by an ellipse. Obviously, an ellipse can fit the tree cross-section better than a circle. The area of the ellipse can be obtained as,

\[
A_E = \pi ab.
\]  

While its circumference can be obtained as [4],

\[
C_E = 4aE(e^2).
\]  

Here, \( a \) and \( b \) are the lengths of the semi-major and minor axis, respectively. The eccentricity \( e \) is defined as,

\[
e = \sqrt{1 - \left(\frac{b}{a}\right)^2}^{1/2}.
\]  

The function \( E \) is the complete elliptic integral of the second kind,

\[
E(k) = \int_0^{\pi/2} \left[1 - k \sin^2(\theta)\right]^{1/2} d\theta
\]  

For the specific values of \( C_E \) and \( e \), we can calculate \( a \) from equation (10). Then from equations (9) and (11), we can obtain \( A_E \), which can be used to determine \( D_A \) as,

\[
D_A = 2\left(\frac{A_E}{\pi}\right)^{1/2}.
\]  

From the above calculation, the irregularity of cross-section shape can be taken into account by assigning random value to the ratio \( b/a \). The eccentricity is treated as a random variable. For a fixed value of \( C_E \), the uncertainty of \( D_A \) can be obtained as,

\[
u(D_A) = \left(\frac{1}{A_E \pi}\right)^{1/2} u(A_E).
\]  

We can also express \( u(D_A) \) as the uncertainty of the correction term \( \delta_D \) as,

\[
D_A \pm u(D_A) = D_C + (\delta_D)_{\text{average}} \pm u(\delta_D).
\]  

where,
4. Evaluation of DBH uncertainty

Due to the presence of the elliptic integral in equation (10), the propagation of uncertainty technique is not applicable to evaluate the uncertainty of $D_A$. We have performed a Monte Carlo simulation [5] to evaluate the uncertainty. In our simulation, we set $C_E = 1$, which means that all lengths are represented in the unit of circumference length. For simplicity, we assume that the ratio $b/a$ has a uniform distribution with a width of 0.1.

Figure 1(a) shows the relation between the correction $\delta_D / D_C$ and the ratio $b/a$. The uncertainties are represented as error bars. As expected, $\delta_D / D_C$ increases with the decrease of $b/a$. The more distortion from a circular shape, the more correction of DBH is needed. Figure 1(b) shows the relation between the correction $\delta_V / V_C$ and the ratio $b/a$. The ratio $\delta_V / V_C$ increases with the decreasing of $b/a$. Note that the correction can be as large as 5% of $V_C$ when $b/a$ is 0.7. This is significant to the biomass estimation. When combining with other components of uncertainty this ratio can be even larger.

![Figure 1](image1.png)

(a) | (b)
--- | ---

Figure 1. Relation between the correction (a) $\delta_D / D_C$ and (b) $\delta_V / V_C$ to the ratio $b/a$.

5. Conclusion

From the analysis above we can see that it may not reasonable to assume the same DBH for the trees that have the same GBH, regardless of their shape. From our model the correction of tree DBH and volume due to shape irregularity is significant. In the real measurement, this can be even more important, since the roughness of the tree cross-section edge prevents the measuring tape to touch some part of the tree surface, the actual cross-section area, and thus DBH, is even smaller than the above ellipse approximation. Further study is needed to get a better model of the fitting curve for the field measurement.

References

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