NON-SINGULAR STIFF FLUIDS

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In this talk the possibility of constructing geodesically complete inhomogeneous stiff fluid cosmologies is discussed. A family with infinite parameters is derived. A wide and easy to implement sufficient condition for geodesic completeness is shown.

1. Introduction

The interest in geodesically complete cosmologies arises in 1990 with the publication of the first non-singular cosmological model by Senovilla. Since then, the number of new regular inhomogeneous cosmologies has increased very little and therefore it has been suggested that they may be a negligible subset among cosmological models.

In this talk we want to show that it is possible to obtain general singularity-free cosmological models depending on two nearly arbitrary functions.

2. Equations for $G_2$ stiff models

We shall consider spacetimes with an Abelian orthogonally transitive $G_2$ group of isometries acting on timelike surfaces. The Killing fields are chosen to be mutually orthogonal. The metric is written as

$$ds^2 = e^{2K}(-dt^2 + dr^2) + e^{-2U}dz^2 + \rho^2 e^{2U}d\phi^2,$$

$$-\infty < t, z < \infty, \ 0 < r < \infty, \ 0 < \phi < 2\pi,$$

interpreting the isometries as cylindrical symmetry. Metric functions depend just on $r$ and $t$. The matter content is a stiff fluid, that is, a perfect fluid with pressure equal to energy density, $\mu = p$. 

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The Einstein equations can be reduced to a simple set in convenient coordinates, $r = \rho$,

\begin{align}
0 &= U_{tt} - U_{rr} - \frac{U_r}{r}, \\
K_t &= U_t + 2rU_tU_r, \\
K_r &= U_r + r(U_t^2 + U_r^2) + \alpha r, \\
p &= \alpha e^{-2K}, \quad \alpha > 0,
\end{align}

(3a) (3b) (3c) (3d)

a two-dimensional homogeneous reduced wave equation in polar coordinates and a quadrature for $K$, which can be integrated provided we have a solution of the wave equation. Since the solution of the Cauchy problem for this equation is known\cite{3}, the problem is completely solved,

$$U(r, t) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^1 \frac{\tau}{\sqrt{1 - \tau^2}} \left\{ \tau g(v) + f(v) + tf'(v) \frac{t\tau^2 + r\tau \cos \phi}{v} \right\},$$

(4)

where $v = \sqrt{r^2 + t^2 + 4rt \cos \phi}$, in terms of two functions $f$ and $g$. These solutions are always regular at the axis $r = 0$.

3. Geodesic Completeness

In order to check geodesic completeness of these spacetimes we resort to the available theorems\cite{4}. The only non-trivial condition on the metric functions is,

$$U(0, t) \geq -\frac{1}{2} \ln |t| + b,$$

(5)

for a constant $b$.

4. Polynomial Metric Functions

A simple and wide family of functions that satisfy (5) can be written in terms of polynomials. Consider

$$f(r) = \sum_{i=0}^n a_i r^i, \quad g(r) = \sum_{i=0}^m b_i r^i.$$

(6)

The leading terms in $U(0, t)$ depending on $f$ and $g$ are

$$U_f(t) = \sqrt{\pi} \frac{\Gamma((n+2)/2)}{\Gamma((n+1)/2)} |t|^n, \quad U_g(t) = \sqrt{\pi} \frac{\Gamma((m+2)/2)}{2 \Gamma((m+3)/2)} |t|^m t.$$

(7)

There are two ways of generating a geodesically complete model:

- If $f, g$ are polynomials in $r$ respectively of degree $n, m$ and $n > m + 1$, we have a non-singular model if $a_n$ is positive.
• If \( f, g \) are polynomials in \( r \) respectively of degree \( n, n - 1 \), \( U_f \) and \( U_g \) at the axis are polynomials of degree \( n \) and we have a non-singular model if
\[(n + 1/2) a_n > |b_n - 1|.
\]

Therefore, if we restrict ourselves to spacetimes with a metric for which \( U|_{r=0} \) is a polynomial of degree equal or lower than \( n \), we find that the subset of singularity-free spacetimes is an open set.

A simple regular model\(^5\) may be generated for \( f(x) = \beta x^2/2, \ g(x) = 0, \ \beta > 0 \).

5. Conclusions

We have reduced the problem of checking the geodesic completeness of \( G_2 \) orthogonally transitive stiff fluid cosmologies to a simple condition on the behaviour of one of the metric functions at the axis. The case of polynomial functions has been used to generate a large family of singularity-free spacetimes depending on two functions, which are related to the initial value problem of a homogeneous 2D-wave equation. These results seem to point out that regular cosmologies cannot be considered as a negligible set.

Since these models fulfill every energy and causality condition\(^6\), the reason for their lack of singularities lies on the absence of trapped sets, such as closed trapped surfaces or compact achronous sets without edge.

Pressure is determinant for preventing the formation singularities in these models. From the point of view of exact solutions, it is interesting to mention that models obtained with a separability Ansatz\(^7\) are singular. This suggests a reason for the limited number of regular models in the literature that have been found so far.

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