Wilson Renormalization Group and Continuum Effective Field Theories

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Abstract

This is an elementary introduction to Wilson renormalization group and continuum effective field theories. We first review the idea of Wilsonian effective theory and derive the flow equation in a form that allows multiple insertion of operators in Green functions. Then, based on this formalism, we prove decoupling and heavy-mass factorization theorems, and discuss how the continuum effective field theory is formulated in this approach.

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I. INTRODUCTION

The purpose of this lecture is twofold. First, we will give an elementary introduction to Wilson renormalization group in field theories developed recently. Then based on this formalism, we discuss some basic aspects of continuum effective field theories.

The concept of effective field theories has played an important role in modern theoretical physics and it acquires its natural physical interpretation in the Wilson renormalization group formalism [1]. In the latter, one integrates out the high frequency modes scanned by a cutoff and then considers lowering the cutoff. This generates the renormalization group flow, and the differential equation governing this flow is called the exact renormalization group equation or the flow equation. After Wilson’s formulation, this flow equation has been applied to wide area of physics, from condensed matter physics to particle physics [2], especially for nonperturbative problems which are difficult to treat in the other approaches. It has also greatly enhanced the understanding of renormalization in quantum field theory.

In spite of the understanding of renormalization with Wilsonian approach, the precise connection to the conventional approach has, however, not been explicitly given until Polchinski was able to prove the perturbative renormalizability of a renormalizable quantum field theory within this framework [3], taking the $\phi^4$-theory as his model field theory. His method was remarkably simple and could diffuse all difficulties of the conventional approach. More recently, direct physical meaning of Wilsonian effective action was given in the framework of conventional field theory that it is nothing but the generating functional of Green functions with an infrared cutoff [4]. After this important observation, Wilsonian effective action and its renormalization group equation became a powerful tool to study wide range of nuclear and high-energy physics problems [5]-[26].

If we take seriously the above point of view on effective field theory and renormalization, it should be natural also to have a simple connection to the conventional formulation of effective theory. Here, one of the most basic results is the decoupling theorem [27] which states that, in a generic renormalizable quantum field theory with heavy particles of mass $M$, heavy-particle effects decouple from low-energy light-particle physics except for renormalization of couplings involving light fields and corrections of order $1/M$. Thus the resulting renormalizable effective field theory describes low energy physics to the accuracy $1/M$. In other words, to the zeroth order in $1/M$ there are no observable effects due to the existence of virtual heavy particles. If one wishes to understand the low-energy manifestations of heavy particles, then irrelevant (nonrenormalizable) terms should be considered to incorporate higher order effects in $1/M$. Investigation of this issue shows that the virtual heavy particle effects can be isolated via a set of effective local vertices with calculable couplings. It implies that low-energy light-particle physics can be described by a suitable local effective field theory when combined with appropriate calculation rules to deal with irrelevant vertices. This heavy-mass factorization was proved to the lowest in $1/M$ adapting Zimmermann’s algebraic identities [28] in the BPHZ formulation [29], and, recently, to all orders in perturbation theory and to any given order in powers of $1/M$ using the flow equation [7] (see also Refs. [23,24]). Actually the proof is almost as simple as the renormalizability proof itself and the whole scheme allows a natural physical interpretation from the viewpoint of the renormalization group flow as we will see later.

There is, however, an issue to clarify regarding the difference between Wilsonian effective
theories and effective theories in continuum. We have three widely separated scales here—light particle mass $m$, heavy particle mass $M$, and the ultraviolet (UV) cutoff $\Lambda_0$, with $m \ll M \ll \Lambda_0$. And our hope is to find a low-energy effective theory with the UV cutoff $\Lambda_0$ which involves only the light field of mass $m$ and describes low-energy light-particle physics accurately up to the order $1/MN$ ($N$ is any fixed integer). Then the effective theory must include a finite number of irrelevant terms (restricted by the dimensional argument). Here arises a problem. In the original Wilson’s view, the cutoff scale of the effective theory may well be identified with the heavy mass scale $M$, above which it is no more effective. The presence of irrelevant terms in the effective Lagrangian at scale $M$ is then very natural as discussed above and there is no need to worry about their presence in particular. The “natural scale” of those terms will be around $M$, i.e., they are order one at scale $M$, and among them we may choose to keep explicitly some minimal number of irrelevant terms in our effective Lagrangian for the accuracy of order $1/MN$. But, in the conventional discussion of quantum field theory, the UV cutoff $\Lambda_0$ is supposed to go to infinity eventually. So, to connect Wilson’s view with this conventional formulation, we may suppose scaling up the cutoff of the above Wilsonian effective Lagrangian from $M$ to $\Lambda_0$. It will then generate infinitely many irrelevant terms which are unnaturally large. Also, during the scaling, all of them get mixed together and so we are forced to work with the Lagrangian consisting of infinitely many terms all the time. Any kind of truncation for the bare Lagrangian to some finite number of terms would yield divergences in physical quantities as $\Lambda_0 \to \infty$, because the unnaturally large coefficients would be amplified by some positive power of $\Lambda_0/M$ as the cutoff is scaled down. This is nothing but the statement of nonrenormalizability in the language of renormalization group flow. To avoid this problem we need to deal with irrelevant terms carefully, i.e., give suitable rules to obtain unambiguous finite results with only a finite number of terms included in the bare Lagrangian. It is achieved through the modification of the flow equation when irrelevant vertices are inserted, in the more-or-less same way as one treats the renormalization of composite operators and their normal products in the flow equation approach [28]. We will discuss this procedure later.

The plan of the paper is as follows. In section 2, starting from the generating functional, we identify the Wilsonian effective action and give a rather lengthy derivation of the Wilson renormalization group equation [31,21]. This derivation will show clearly how the Wilsonian effective action is interpreted as a generating functional of Green functions. Then we derive the flow equation in a form applicable to the case that multiple insertion of composite operators is allowed in Green functions. In section 3, after a review on perturbative renormalizability, we show decoupling and heavy-mass factorization taking as our model a scalar theory involving two real scalar fields of masses $m$ and $M$ with $m \ll M$. Then we discuss how the continuum effective field theory is formulated to any desired order of accuracy in the flow equation approach. We conclude in section 4.

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1For example, the continuum version of Symanzik’s improved action [30] in lattice theory have been discussed in [31] by adding suitable irrelevant terms in Lagrangian.
II. THE FLOW EQUATION

For simplicity let us consider a theory of single scalar field in four Euclidean dimensions with a momentum space cutoff $\Lambda_0$. Generalization to several fields is straightforward. The bare action is written as

$$S^{\Lambda_0}[\phi] = \frac{1}{2} \langle \phi, P^{\Lambda_0^{-1}} \phi \rangle + S^\text{int}_{\Lambda_0}[\phi],$$

(1)

where $P^{\Lambda_0}$ is the free-particle cutoff propagator and $\langle f, g \rangle$ is defined by the momentum-space integral of $f$ and $g$,

$$\langle f, g \rangle \equiv \int \frac{d^4p}{(2\pi)^4} f(p)g(-p),$$

(2)

and $S^\text{int}_{\Lambda_0}[\phi]$ represents the interaction part of the bare action. For later use, we will also allow the insertions of some additional local vertices or certain composite operators in Green functions of the theory. To account for this, we define $S^\Lambda_{\Lambda_0}^\text{tot}[\phi]$ as a formal power series in $\alpha$, which has $S^\text{int}_{\Lambda_0}[\phi]$ as its zeroth order term, viz.,

$$S^\Lambda_{\Lambda_0}^\text{tot}[\phi] = \sum_{|N|\geq0} \frac{\alpha^N}{N!} S^\text{int}_N[\phi].$$

(3)

Also, for later conveniences, we will denote

$$S^\Lambda_{\Lambda_0}^\text{tot}[\phi] = S^\Lambda_{\Lambda_0}^\text{tot}[\phi].$$

(4)

$S^\text{int}_N$’s may be regarded as additional local vertices appended to the original Lagrangian $S^\text{int}_{\Lambda_0}$ or as composite operators in which one is interested. For example, if one wishes to consider a single or twice insertion of a composite operator $O(x)$, one may consider

$$S^\text{int}_{\Lambda_0}^{1} = \int d^4x \chi(x) O(x),$$

$$S^\text{int}_{\Lambda_0}^{2} = \int d^4xd^4y \chi(x) \chi(y) O'(x,y),$$

(5)

where $\chi(x)$ is the source for $O(x)$ and $O'(x,y)$ is a suitable counterterm for the product of operators $O(x)O(y)$, which can be determined through renormalization conditions and the flow equation derived below \[11\]. In this case, $S^\text{int}_{\Lambda_0}^\text{tot}$ is equal to the original interaction part of the action plus the composite-operator source terms.

The generating functional, with the insertion of the operator $e^{S^\text{int}_{\Lambda_0}^\text{tot}[\phi]}$, is

\[ \text{More generally, one can consider power series of many parameters } \alpha_1, \ldots, \alpha_k. \text{ Equation (3) and subsequent discussion can accommodate this case with the interpretation that } \alpha = (\alpha_1, \ldots, \alpha_k), \text{ } N \text{ is a multi-index } N = (N_1, \ldots, N_k), |N| = \sum_{i=1}^{k} N_i, N! = N_1! \cdots N_k!, \text{ and } \alpha^N = \alpha_1^{N_1} \cdots \alpha_k^{N_k}. \]
Following the idea of Wilson and Polchinski \[1,3\], we wish to integrate out the high-momentum components of \( \phi \) and reduce the cutoff \( \Lambda_0 \) to a lower scale \( \Lambda \). For this, we divide the propagator \( P^{\Lambda_0} \) into the high-frequency part \( P^{\Lambda_0}_\Lambda \) and the low-frequency part \( P^\Lambda \), the borderline being set by momentum \( p = \Lambda \):

\[
P^{\Lambda_0} = P^\Lambda + P^{\Lambda_0}_\Lambda ,
\]

Then it is not difficult to show that the generating functional may be written in terms of two fields \( \phi_H \) and \( \phi_L \) rather than \( \phi \) alone as

\[
Z[J] = \int \mathcal{D}\phi_H \mathcal{D}\phi_L e^{-\frac{1}{2}(\phi_H P^{\Lambda_0}_\Lambda \phi_H) - \frac{1}{2}(\phi_L P^\Lambda \phi_L) - S^{\Lambda_0}_{\text{tot}}[\phi_H+\phi_L]+\langle J,\phi_H+\phi_L \rangle},
\]

up to a multiplicative factor. (The equivalence of Eq. (6) and Eq. (8) may be seen if one substitutes \( \phi_L = \phi - \phi_H \) into Eq. (8) and performs the integral over \( \phi_H \) which is Gaussian.) Since the field \( \phi_L(\phi_H) \) has \( P^\Lambda(P^{\Lambda_0}_\Lambda) \) as its propagator, only the low-(high-)frequency modes of \( \phi_L(\phi_H) \) will now propagate effectively. The integral over \( \phi_H \) may be performed to obtain

\[
Z[J] = \int \mathcal{D}\phi_L e^{-\frac{1}{2}(\phi_L P^\Lambda \phi_L) - S^\Lambda_{\text{tot}}[\phi_H+\phi_L]+\langle J,\phi_H+\phi_L \rangle},
\]

where \( W^\Lambda_{\text{tot}} \) is given by

\[
e^{-W^\Lambda_{\text{tot}}[\phi_L,J]} \equiv \int \mathcal{D}\phi_H e^{-\frac{1}{2}(\phi_H P^{\Lambda_0}_\Lambda \phi_H) - S^{\Lambda_0}_{\text{tot}}[\phi_H+\phi_L]+\langle J,\phi_H+\phi_L \rangle}.
\]

Notice that \( W^\Lambda_{\text{tot}}[0,J] \) is nothing but the generating functional of connected Green functions (with the operator \( e^{S^{\Lambda_0}[\phi]} \) inserted) in the theory with both UV cutoff \( \Lambda_0 \) and IR cutoff \( \Lambda \). Substituting \( \phi_H = \phi - \phi_L \) into Eq. (10), we may write

\[
e^{-W^\Lambda_{\text{tot}}[\phi_L,J]} = e^{-\frac{1}{2}(\phi_L P^{\Lambda_0}_\Lambda \phi_L)} \int \mathcal{D}\phi_L e^{-\frac{1}{2}(\phi,P^{\Lambda_0}_\Lambda \phi) - S^{\Lambda_0}_{\text{tot}}[\phi] + \langle J,\phi \rangle}
\]

\[
= e^{\frac{1}{2}(\langle J,\phi_L \rangle + \langle J,\phi \rangle)} e^{-\frac{1}{2}(\langle J,\phi \rangle - \langle J+P^{\Lambda_0}_\Lambda \phi_L \rangle)}
\]

\[
\times e^{-S^{\Lambda_0}_{\text{tot}}[\phi]} \int \mathcal{D}\phi \phi^{\Lambda_0}_\Lambda (J+P^{\Lambda_0}_\Lambda \phi_L)
\]

\[
= e^{-W^\Lambda_{\text{tot}}[0,0]} e^{\frac{1}{2}(\langle J,\phi_L \rangle + \langle J,\phi \rangle)} e^{-S^{\Lambda_0}_{\text{tot}}[\phi]},
\]

for some \( S^{\Lambda_0}_{\text{tot}}[\phi] \) satisfying \( S^{\Lambda_0}_{\text{tot}}[0] = 0 \). (Here we factored out the field independent part as \( e^{-W^\Lambda_{\text{tot}}[0,0]} \) which goes to 1 as \( \Lambda \to \Lambda_0 \).) Therefore, the generating functional can be written, up to a multiplicative factor, as

\[3\]There will be infinite ways in doing the separation; choosing one way of separation may be considered as choosing a “renormalization scheme” in the Wilson renormalization group approach, and physical quantities are independent of such choices. It is not necessary to specify a particular scheme at this stage.
\[ Z[J] = \int \mathcal{D}\phi e^{-\frac{1}{2}\langle \phi, P^{\Lambda_0} \phi \rangle - S_{\text{tot}}^{\Lambda_0}[P^{\Lambda_0}_\phi J + \phi] + \langle J, \phi \rangle + \frac{i}{2}\langle J, P^{\Lambda_0}_\phi J \rangle}, \]  

(12)

where we have replaced \( \phi_L \) by \( \phi \). Suppose that \( J(p) = 0 \) for \( p > \Lambda \) so that \( J \) couples to the low-frequency modes only. Then, since \( P^{\Lambda_0}_\phi \) has only high-frequency modes, all \( J \)'s drop out from the expression except for \( \langle J, \phi \rangle \) and \( S_{\text{tot}}^{\Lambda_0} \) coincides with the Wilsonian effective action \( \Pi \). However, we do not have to insist on such a restriction for \( J(p) \); for general \( J(p) \), Eq. (12) connects directly \( S_{\text{tot}}^{\Lambda_0} \) with Green functions as will be discussed shortly.

Now we derive the flow equation \( \Pi \) satisfied by \( S_{\text{tot}}^{\Lambda} \). Differentiating Eq. (10) with respect to \( \Lambda \) while choosing \( \phi_L = 0 \), we have the result

\[
\partial_\Lambda e^{-W_{\text{tot}}^0[0,J]} = -\frac{1}{2} \left( \frac{\delta}{\delta J} \right)_{\text{field-dep. part}} e^{-W_{\text{tot}}^0[0,J]},
\]

(13)

which, on using Eq. (11) for \( e^{-W_{\text{tot}}^0[0,J]} \), gives the flow equation,

\[
\partial_\Lambda S_{\text{tot}}^{\Lambda}[\phi] = -\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \partial_\Lambda P^{\Lambda}(p) \left[ \frac{\delta^2 S_{\text{tot}}^{\Lambda}}{\delta \phi(p) \delta \phi(-p)} - \frac{\delta S_{\text{tot}}^{\Lambda}}{\delta \phi(p)} \frac{\delta S_{\text{tot}}^{\Lambda}}{\delta \phi(-p)} \right]_{\text{field-dep. part}}.
\]

(14)

Equation (14) is integrable; the formal solution with the appropriate boundary condition satisfied at \( \Lambda = \Lambda_0 \) is

\[
e^{-S_{\text{tot}}^{\Lambda}[\phi] - I^\Lambda} = e^{-\frac{1}{2}\langle \phi, P^{\Lambda_0} \phi \rangle} \left. e^{S_{\text{tot}}^{\Lambda}[\phi]} \right|_{\phi = 0},
\]

(15)

where \( I^\Lambda \) is supposed to collect precisely all \( \phi \)-independent pieces from the right hand side so that we may have \( S_{\text{tot}}^{\Lambda}[0] = 0 \) as we required in (11). Equation (15), together with Eq. (11), reproduces a well-known expression for the generating functional \( e^{W_{\text{tot}}^{\Lambda}} \). In fact, \( S_{\text{tot}}^{\Lambda} \) bears a simple connection with physical amplitudes. To see this, notice that we have, from Eq. (11),

\[
W_{\text{tot}}^{\Lambda}[0, J] - W_{\text{tot}}^{\Lambda}[0, 0] = -\frac{1}{2} \langle J, P^{\Lambda_0}_\phi J \rangle + S_{\text{tot}}^{\Lambda}[P^{\Lambda_0}_\phi J],
\]

(16)

and hence

\[
\left. \frac{\delta^n S_{\text{tot}}^{\Lambda}[\phi]}{\delta \phi(p_1) \cdots \delta \phi(p_n)} \right|_{\phi = 0} = \prod_{i=1}^n P^{\Lambda_0-1}(p_i) \left. \frac{\delta^n W_{\text{tot}}^{\Lambda}}{\delta J(p_1) \cdots \delta J(p_n)} \right|_{J = 0}, \quad n > 2.
\]

(17)

Therefore, we arrive at a very interesting result. Namely, Wilsonian effective action \( S_{\text{tot}}^{\Lambda} \) with \( UV \) cutoff \( \Lambda \) can also be interpreted as the generating functional of amputated connected Green functions (with the operator \( e^{S_{\text{tot}}^{\Lambda}[\phi]} \) inserted) with \( IR \) cutoffs \( \Lambda \). Physical Green functions are then obtained in the limit \( \Lambda \to 0 \) \([21, 22, 23]\). Actually, the reason behind this result is rather simple. The generating functional of a field theory is obtained by integrating out all modes, all informations of the theory being encoded in the dependence on the external source coupled to the field. If a cutoff \( \Lambda \) is introduced and integration is performed only over high frequency modes, then we will get the generating functional with \( IR \) cutoff \( \Lambda \) because low frequency modes remain unintegrated. On the other hand, from the point of view of low frequency modes, the resulting functional gives, by definition, the Wilsonian effective action. Equation (16) precisely expresses this relation.
Now that Wilsonian effective action is essentially the same as the generating functional of connected Green functions, it is also interesting to perform a Legendre transformation to obtain the flow equation for the generating functional of one-particle irreducible Green functions with IR cutoff $\Lambda$. As usual, we define
\[
\Gamma^{\Lambda}[\varphi] \equiv W^{\Lambda}[0, J] + \langle J, \varphi \rangle = \Gamma^{\Lambda}_{\text{int}}[\varphi] + \frac{1}{2} \langle \varphi, P^{\Lambda_{0}}_{-1} \varphi \rangle,
\]
(18)
with
\[
\frac{\delta \Gamma^{\Lambda}}{\delta \varphi} = J, \quad \frac{\delta W^{\Lambda}}{\delta J} = -\varphi.
\]
(19)

Then from Eq. (13), we obtain\[17\]
\[
\partial_{\Lambda} \Gamma^{\Lambda}_{\text{int}} = \frac{1}{2} \int \frac{d^{4}p}{(2\pi)^{4}} \partial_{\Lambda} P^{\Lambda_{0}}_{-1}(p) \left[ \frac{\delta^{2} W^{\Lambda}}{\delta J(p) \delta J(-p)} \right]^{-1}.
\]
(20)
It is often more convenient to work with 1PI quantities than connected ones, especially for the purpose of practical calculations\[18,21\]. However, we will not discuss it further. See Refs. \[8,20,21\] for details.

Let us go back to $S^{\Lambda}_{\text{tot}}$. Expanding $S^{\Lambda}_{\text{tot}}$ in powers of $\alpha$, we define $S^{\Lambda,N}_{\text{int}}$'s as the expansion coefficients, i.e.,
\[
S^{\Lambda}[\phi] = \sum_{|N| \geq 0} \frac{\alpha^{N}}{N!} S^{\Lambda;N}_{\text{int}}[\phi], \quad \left( S^{\Lambda;0}_{\text{int}}[\phi] \equiv S^{\Lambda}_{\text{int}}[\phi] \right).
\]
(21)
As we mentioned above, $S^{\Lambda;N}_{\text{int}}[\phi]$ is the generating functional of amputated connected Green functions with an insertion of an operator $O_{N}$ which is identified as the $N$-th order coefficient of $e^{S^{\Lambda}_{\text{int}}[\phi]}$, i.e.,
\[
\sum_{|N| \geq 1} \frac{\alpha^{N}}{N!} O_{N} \equiv e^{S^{\Lambda}_{\text{int}}} = \exp \left( \sum_{|N| \geq 1} \frac{\alpha^{N}}{N!} S^{\Lambda;N}_{\text{int}} \right).
\]
(22)
For example, $O_{1} = S^{\Lambda;1}_{\text{int}}$, $O_{2} = S^{\Lambda;2}_{\text{int}} + (S^{\Lambda;1}_{\text{int}})^{2}$, $O_{3} = S^{\Lambda;3}_{\text{int}} + 3 S^{\Lambda;2}_{\text{int}} S^{\Lambda;1}_{\text{int}} + (S^{\Lambda;1}_{\text{int}})^{3}$ and so on\[18\]. At the zeroth order of $\alpha$, we obtain the flow equation for the effective Lagrangian $S^{\Lambda}_{\text{int}}$ from Eq. (14), which assumes an identical form as Eq. (14) other than the replacement $S^{\Lambda}_{\text{tot}} \rightarrow S^{\Lambda}_{\text{int}}$. At the $N$-th order of $\alpha$, on the other hand, we have

\[4\]Expression for general $O_{N}$ is also available in\[8\].
\[ \partial_{\Lambda} S_{\text{int}}^{\Lambda;N} = -\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \partial_{\Lambda} P^\Lambda(p) \left[ \frac{\delta^2 S_{\text{int}}^{\Lambda;N}}{\delta \phi(p) \delta \phi(-p)} - \frac{2 \delta S_{\text{int}}^{\Lambda}}{\delta \phi(p)} \frac{\delta S_{\text{int}}^{\Lambda;N}}{\delta \phi(-p)} \right. \\
- \sum_{0 < I < N} \left( \begin{array}{c} N \\ I \end{array} \right) \frac{\delta S_{\text{int}}^{\Lambda;I}}{\delta \phi(p)} \frac{\delta S_{\text{int}}^{\Lambda;N-I}}{\delta \phi(-p)} \right] \text{field-dep. part}, \] (23)

For a single insertion of operator through \( S_{\text{int}}^{\Lambda;1} \), the last term of the right hand side in the second line of Eq. (23) vanishes and so we have a linear and homogeneous equation in \( S_{\text{int}}^{\Lambda;1} \). This equation has been used in discussing the renormalization of composite operators \([10,11]\). Equation (23) with multi index \( N = (1,1) \) is useful in discussing the short-distance expansion of two composite operators \([11]\). Notice that, if \( N > 1 \), Eq. (23) contains inhomogeneous terms which may look difficult to deal with. However, in spite of those terms, it is still of the first order in \( S_{\text{int}}^{\Lambda;N} \) while the homogeneous part remains exactly the same for all \( N \). Therefore the general solution will be a sum of a particular solution and a solution to the homogeneous equation (which is just the \( N = 1 \) equation for Green functions with a single insertion of an operator). Though we will not explicitly use this property here, it is important in considering Zimmermann’s normal product and multiple insertion of local operators of which the formulation is crucial in understanding the structure of effective theories \([7]\).

A few remarks are in order.

(i) As noted above, choice of the cutoff function in the propagator is quite arbitrary. For example, the propagator \( P^\Lambda \) of a particle with mass \( m \) is given by

\[ P^\Lambda = \frac{R(\Lambda, p)}{p^2 + m^2}, \] (24)

where \( R(\Lambda, p) \) is a cutoff function. If we choose sharp cutoff, \( R(\Lambda, p) = \theta(\Lambda - p) \). Sometimes it is more convenient to choose smooth cutoff such as \( R(\Lambda, p) = 1 - e^{p^2/\Lambda^2} \). More generally, even the separation of the action \( S^\Lambda \) into the free part and the interaction part \( S_{\text{int}}^{\Lambda} \) is arbitrary. One may write, for example,

\[ e^{-W^\Lambda_{\text{tot}}[J]} \equiv \int \mathcal{D}\phi e^{-\frac{1}{2} \left( \phi, \bar{P}_\Lambda^{\Lambda_0 - 1} \phi \right) - S^{\Lambda_0}[\phi] + \langle J, \phi \rangle}, \] (25)

where \( S^{\Lambda_0} \) is the full bare action of the theory with UV cutoff \( \Lambda_0 \) and

\[ \bar{P}_\Lambda^{\Lambda_0 - 1} \equiv P_\Lambda^{\Lambda_0 - 1} - P_\Lambda^{\Lambda_0 - 1} \] (26)

is the pure cutoff term added to the action. Then all the previous equations are valid with \( S_{\text{int}}^{\Lambda} \) and \( P_\Lambda^{\Lambda_0} \) replaced by \( S^{\Lambda} \) and \( \bar{P}_\Lambda^{\Lambda_0 - 1} \), respectively. The corresponding flow equation for \( \Gamma^{\Lambda} \) is the one used by Berges in this volume.

(ii) The flow equation (14) is too complicated to solve exactly. For practical purposes, it is therefore necessary to find suitable approximation methods. An obvious way is to perform a derivative expansion

\[ S^\Lambda = \int d^4x \left[ V(\varphi, \Lambda) + \frac{1}{2} (\partial_\mu \varphi)^2 Z(\varphi, \Lambda) + O(\partial^4) \right]. \] (27)
Then the flow equation reduces to simple differential equations of coefficient functions $V(\varphi, \Lambda)$, $Z(\varphi, \Lambda)$ etc. and one can study their properties by various means. There are some problems with approximation methods. As mentioned above, physical quantities should be independent of the choice of cutoff functions. However, this scheme independence is lost after approximations. Another problem is about reparametrization invariance: physics should not depend on the reparametrization of fields in the partition function. This is reflected in the flow equation in some complicated way \[22\] and is broken in general by approximations. For the discussion of these issues see, e.g., \[12,13\].

(iii) When the system has a gauge symmetry, integrating out high momentum modes does not preserve gauge symmetry and this could be a potentially serious problem. There are a few ways with which this problem can be coped with. The simplest way is to insist on using the gauge-symmetry-violated flow equation derived here. Even if the gauge symmetry is not manifest for finite cutoff $\Lambda$, it is eventually restored at the physical point $\Lambda = 0$ up to $O(1/\Lambda_0)$ \[5,20,14,15\]. Also, one may work in the background gauge in which the background gauge invariance may be maintained in the effective action \[14\] though it may not necessarily mean quantum gauge invariance \[16\].

### III. CONSTRUCTION OF EFFECTIVE FIELD THEORIES

As discussed in section 1, one of the most natural application of the flow equation would be to understand basic results of effective field theories in continuum. Here we discuss decoupling and heavy-mass factorization, and so construct the continuum effective theory to any desired order of accuracy in flow equation approach. As a first step, let us review Polchinski’s proof of perturbative renormalizability \[3\].

#### A. Perturbative Renormalizability

A theory is said to be perturbatively renormalizable if Green functions of the theory are bounded and converge to finite limits at each order of perturbation as UV cutoff of the theory goes to infinity. Now knowing that Wilsonian effective action gives physical Green functions in the limit $\Lambda \to 0$ and that it is controlled by the flow equation, we may expect that the boundedness and convergence of Green functions can be easily shown by estimating the flow equation. Indeed, this line of argument can be implemented in a quite straightforward way. Since basically the same reasoning is repeatedly used throughout in discussing effective theories, we first explain the method in rather detail. It consists of the following steps:

(i) Write down the flow equation for vertex functions and identify boundary conditions which follow from the form of the bare action (at $\Lambda = \Lambda_0$) and also from the renormalization conditions on Green functions (at $\Lambda = 0$).

(ii) Integrate the flow equation over $\Lambda$ from the boundary at which boundary conditions are given (either at $\Lambda = 0$ or at $\Lambda = \Lambda_0$) and estimate the resulting expression.

(iii) Prove boundedness and convergence order by order using induction.
Let us illustrate this with a $\phi^4$-theory. The bare action with UV cutoff $\Lambda_0$ reads

$$S^{\Lambda_0}\[\phi\] = \frac{1}{2} \left\langle \phi, P^{\Lambda_0-1}_m \phi \right\rangle + S_{\text{int}}^{\Lambda_0},$$

where $P^{\Lambda_0}_m$ is the free-particle cutoff propagator of mass $m$ and $S_{\text{int}}^{\Lambda_0}$ is the interaction part of the bare action as before. Explicitly,

$$S_{\text{int}}^{\Lambda_0} = \int d^4x \left[ \frac{1}{2} \rho_1 \phi^2(x) + \frac{1}{2} \rho_2 (\partial_\mu \phi(x))^2 + \frac{1}{4!} \rho_3 \phi^4(x) \right].$$

In conventional way, the bare couplings $\rho_a$ may be written as

$$\rho_1 = Z_\phi m_0^2 - m^2, \quad \rho_2 = Z_\phi - 1, \quad \rho_3 = Z_\phi^2 g^0,$$

where $m_0$, $Z_\phi$, and $g^0$ are the bare mass, the wavefunction renormalization, and the bare coupling of the real scalar field $\phi$. Propagator $P^{\Lambda_0}_m$ is defined by

$$P^{\Lambda_0}_m = \frac{R_m(\Lambda_0, p)}{p^2 + m^2},$$

where $R_m(\Lambda, p)$ is a smoothed variant of the sharp cutoff function $\theta(\Lambda - p)$. As we indicated in section 2, choosing $R_m$ corresponds to choosing a “renormalization scheme” and physical quantities are not affected. Here, for convenience’ sake, we choose

$$R_m(\Lambda, p) = \left[ 1 - K \left( (1 + \Lambda/m)^2 \right) \right] K \left( \frac{p^2}{\Lambda^2} \right),$$

where $K(t)$ is a smooth function such that $K(t) \to 1$ (exponentially) as $t \to 1$ and $K(t) \to 0$ (also exponentially) as $t \to 4$ (detailed form of $K(t)$ is not important).

Now, the Wilsonian effective action $S_{\text{int}}^\Lambda$ is defined to satisfy the flow equation (14). Expanding $S_{\text{int}}^\Lambda$ in powers of $\phi$ in momentum space, we write

$$S_{\text{int}}^\Lambda[\phi] = \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} g^r \int \prod_{j=1}^{2n-1} \frac{d^4p_j}{(2\pi)^4} \phi(p_1) \cdots \phi(p_{2n}) G_{r,2n}^{\Lambda}(p_1, \ldots, p_{2n-1}) \cdot$$

$$\left( p_{2n} \equiv - \sum_{j=1}^{2n-1} p_j \right),$$

where $g$ is the perturbation-expansion parameter which may be identified as renormalized coupling constants. Then as we noted above, at $\Lambda = 0$, vertex functions $G_{r,2n}^{\Lambda}$ can be identified with amputated connected Green functions of the theory. Furthermore, we have $G_{r,2n}^{\Lambda} = 0$ if $n > r + 1$ because just the connected diagrams contribute to $G_{r,2n}^{\Lambda}$. Therefore the sum over $n$ in Eq. (33) actually extend only over a finitely many $n$ for given $r$.

If we insert the expansion (33) into the flow equation (14) we arrive at

$$\partial_\Lambda G_{r,2n}^{\Lambda}(p_1, \ldots, p_{2n-1})$$

$$= - \left( \begin{array}{c} 2n + 2 \\ 2 \end{array} \right) \int \frac{d^4q}{(2\pi)^4} \partial_\Lambda P_m^{\Lambda}(q) G_{r,2(n+1)}^{\Lambda}(q, -q, p_1, \ldots, p_{2n-1})$$

$$+ 2 \sum_{r=1}^{r-1} \sum_{l=1}^{n} l(n - l + 1) \partial_\Lambda P_m^{\Lambda}(P)$$

$$\times \left[ G_{r,2l}^{\Lambda}(p_1, \ldots, p_{2l-1}) G_{r'-2(n-l+1)}^{\Lambda}(P, p_{2l}, \ldots, p_{2n-1}) \right]_{\text{symm}},$$

$$\left( P = - \sum_{j=1}^{2n-1} p_j \right).$$

(34)
where \([\cdots]_{\text{symm}}\) implies symmetrization with respect to momenta \(p_1, \ldots, p_{2n}\). Denoting \(\partial_{\Lambda} P_m^A\) by a straight line, we can represent this equation by the diagram shown in Fig. 1.

If suitable boundary conditions are given, the flow equation (34) will produce the amputated connected Green functions of the theory, \(G^{\Lambda}_{r,2n} = G^{\Lambda=0}_{r,2n}\). First, we know that the bare action (29) has a simple form: at \(\Lambda = \Lambda_0\),

\[
\begin{align*}
\partial_p^w G^{\Lambda}_{r,2n} &= 0, \\
2n + |w| &> 4, \\
\partial_p^w G^{\Lambda}_{r,2n}(0) &\sim \text{bare couplings } \rho_a \text{ in Eq. (30)}, 2n + |w| \leq 4,
\end{align*}
\]

(35)

where \(w\) is the multi index \(\{w\} \equiv \{w_1, \ldots, w_{2n-1}\}\) with \(w_j = (w_{j1}, \ldots, w_{j4})\) and \(\partial_p^w \equiv \partial_{p_1}^{w_1} \cdots \partial_{p_{2n-1}}^{w_{2n-1}}\). \(\partial_p^w \equiv \prod_{\mu=1}^{4} \partial^{w_{1\mu}}/(\partial p_{1\mu})^{w_{1\mu}}\). We will also use the notations \(|w| \equiv \sum_{i,\mu} |w_{i\mu}|\).

We do not know the value of bare couplings \(\rho_a\); they are fixed by the renormalization conditions imposed on Green functions at \(\Lambda = 0\) for \(2n + |w| \leq 4\). For example, we may choose the conditions at zero momenta,

\[
G^{\Lambda=0}_{r,2}(0) = \partial_{p_1} \partial_{p_2} G^{\Lambda=0}_{r,2}(0) = 0, \quad G^{\Lambda=0}_{r,4}(0) = \frac{1}{4!} \delta_{r1}.
\]

(36)

Now that the boundary conditions have been completely specified, we integrate Eq. (34) and estimate it. For that, we introduce a set of norms \([11], \| \cdot \|(a,b)\), where \(a\) and \(b\) are positive real numbers:

\[
\|(\partial^2 f)g\|_{(a,b)} \equiv \max_{|p_j| \leq \max \{a,b\}} \max_{|w| = z} |(\partial_p^w f)g(p_1, \ldots, p_n)|.
\]

(37)

Then using the property of cutoff function in (32) one can find [7] that, for a fixed constant \(\mu\) of order \(m\) (\(\mu\) may be considered as the momentum scale),

\[
\|\partial_{\Lambda} \partial^2 G^\Lambda_{r,2n}\|_{(2\Lambda,\mu)} \leq \text{const} \left\{ (\Lambda + m) \|\partial^2 G^\Lambda_{r,2n+1}\|_{(2\Lambda,\mu)} \\
+ \sum_{r'\neq r} \frac{1}{(\Lambda + m)^{3+z_1}} \|\partial^{z_2} G^\Lambda_{r',2l}\|_{(2\Lambda,\mu)} \|\partial^{z_3} G^\Lambda_{r-r',2(n-l+1)}\|_{(2\Lambda,\mu)} \right\}.
\]

(38)
where “const” stands for some finite number which is independent of \( \Lambda \) and \( \Lambda_0 \) (but may have dependence on \( m \) and \( \mu \) through \( (\mu/m) \)). This estimate shows that the derivative of vertex functions with respect to \( \Lambda \) is well bounded for all \( \Lambda \in [0, \Lambda_0] \).

As a next step, we apply this estimate to the integral form of the flow equations. For irrelevant vertices, \( 2n + z > 4 \), boundary conditions are given at \( \Lambda = \Lambda_0 \) and so we integrate the flow equation from \( \Lambda = \Lambda_0 \) down to \( \Lambda \):

\[
\left\| \partial^z \mathcal{G}^\Lambda_{r,2n} \right\|_{(2\Lambda, \mu)} \leq \left\| \partial^z (\mathcal{G}^\Lambda_{r,2n} - \mathcal{G}^{\Lambda_0}_{r,2n}) \right\|_{(2\Lambda, \mu)} \leq \int_{\Lambda_0}^\Lambda d\lambda \left\| \partial_\lambda \partial^z \mathcal{G}^\Lambda_{r,2n} \right\|_{(2\lambda, \mu)}.
\]

(39)

For the relevant vertices, i.e., in the case of \( 2n + z \leq 4 \), we integrate the flow equation from \( 0 \) to \( \Lambda \) (at \( p_i = 0 \) instead),

\[
|\partial^w \mathcal{G}^\Lambda_{r,2n}(0) - \partial^w \mathcal{G}^{\Lambda=0}_{r,2n}(0)| \leq \int_{0}^{\Lambda} d\lambda \left\| \partial_\lambda \partial^z \mathcal{G}^\Lambda_{r,2n} \right\|_{(2\lambda, \mu)}.
\]

(40)

The right hand side of (39) and (40) are now estimated and integrated using (38). Finally, applying a suitable induction argument over the perturbation order \( r \) and the number of legs \( n \), we obtain the following bound on vertex functions [7]

\[
\left\| \partial^z \mathcal{G}^\Lambda_{r,2n} \right\|_{(2\Lambda, \mu)} \leq (\Lambda + m)^{4-2n-z}\text{Plog}\left(\frac{\Lambda + m}{m}\right), \quad 0 \leq \Lambda \leq \Lambda_0,
\]

(41)

where \( \text{Plog}((\Lambda + m)/m) \) is some polynomial in \( \log((\Lambda + m)/m) \) whose coefficients are independent of \( \Lambda \) and \( \Lambda_0 \). In particular, at \( \Lambda = 0 \), (41) tells us that amputated connected Green functions and their derivatives of the theory are bounded by

\[
|\partial^w \mathcal{G}^c_{r,2n}(p_1, \ldots, p_{2n-1})| \leq \text{const} \cdot m^{4-2n-|w|}, \quad \text{for } |p_i| \leq \mu.
\]

(42)

Convergence of Green functions as \( \Lambda_0 \to \infty \) can also be proved in a similar way and the result is

\[
\left\| \partial_{\Lambda \mu} \partial^z \mathcal{G}^\Lambda_{r,2n} \right\|_{(2\Lambda, \mu)} \leq \left(\frac{\Lambda + m}{\Lambda_0}\right)^3 (\Lambda + m)^{3-2n-z}\text{Plog}\left(\frac{\Lambda_0}{m}\right), \quad 0 \leq \Lambda \leq \Lambda_0.
\]

(43)

Consequently, amputated connected Green functions \( \partial^w \mathcal{G}^c_{r,2n} \) converge to finite limits as fast as \( O\left(\frac{m}{\Lambda_0^2}\right) \) (modulo powers of \( \log(\Lambda_0/m) \)).

Before we move to the next topic we make a few remarks.

(i) When estimating the relevant vertices, we have had to integrate up from \( \Lambda = 0 \) and not just down from \( \Lambda_0 \). If we had integrated down from \( \Lambda_0 \) assuming “natural values” like the irrelevant-vertex case, we would have had simply \( \left\| \mathcal{G}^\Lambda_{r,2n} \right\|_{(2\Lambda, \mu)} \leq \Lambda_0^2\text{Plog}\left(\frac{\Lambda_0}{\Lambda + m}\right) \) etc. That is, by imposing Eq. (36) we have forced the initial bare values of \( \rho^0_\mu \) to be finely adjusted so as to produce a scalar with mass \( m \ll \Lambda_0 \). This is the famous naturalness problem in scalar theories [32].

(ii) In this proof of perturbative renormalizability, we have not encountered any complication usually found in diagrammatic methods, for example, overlapping divergences, cancellation between diagrams and so on. This is because, here, we always work with the whole Green function directly and so problems in considering only a part of a Green function do not appear.
B. The Full Theory

We are now going to discuss decoupling and heavy-mass factorization of effective field theories using the renormalization group flow equation. We will try to avoid going into technical details but explain only main line of the argument. As our model for the full theory, we consider a scalar theory (\(\phi-\psi\) theory) which involves two real scalar fields \(\phi\) and \(\psi\) of which the masses are \(m\) and \(M(\gg \Lambda_0)\), respectively, and interact via quartic couplings. Let us write the bare action as

\[
S^{(f)_{\Lambda_0}} = \frac{1}{2} \left\langle \phi, P^\Lambda_{m}^{-1} \phi \right\rangle + \frac{1}{2} \left\langle \psi, P^\Lambda_{M}^{-1} \psi \right\rangle + S^{(f)_{\Lambda_0}}_{\text{int}}. \tag{44}
\]

Here, \(P^\Lambda_{m}\) and \(P^\Lambda_{M}\) are respective free-particle cutoff propagators, and the interaction part \(S^{(f)_{\Lambda_0}}_{\text{int}}\) is given by

\[
S^{(f)_{\Lambda_0}}_{\text{int}} = \int d^4x \left[ \frac{1}{2} \rho^f_1 \phi^2(x) + \frac{1}{2} \rho^f_2 (\partial_\mu \phi(x))^2 + \frac{1}{4!} \rho^f_3 \phi^4(x) + \frac{1}{2} \rho^f_4 \psi^2(x) \right. \\
\left. + \frac{1}{2} \rho^f_5 (\partial_\mu \psi(x))^2 + \frac{1}{4!} \rho^f_6 \psi^4(x) + \frac{1}{4} \rho^f_7 \phi^2 \psi^2(x) \right]. \tag{45}
\]

The superscript \(f\) is used for couplings of the full theory. As in the previous section, the bare couplings \(\rho^f_a\) \((a = 1, 2, \ldots, 7)\) may be written as

\[
\begin{align*}
\rho^f_1 &= Z_\phi m^2_0 - m^2, \\
\rho^f_2 &= Z_\phi - 1, \\
\rho^f_3 &= Z^2_\phi g^0_1, \\
\rho^f_4 &= Z_\psi M^2_0 - M^2, \\
\rho^f_5 &= Z_\psi - 1, \\
\rho^f_6 &= Z^2_\psi g^0_2, \\
\rho^f_7 &= Z_\phi Z_\psi g^0_3,
\end{align*}
\]

where \((m^2_0, M^2_0)\), \((Z_\phi, Z_\psi)\) and \((g^0_1, g^0_2, g^0_3)\) are bare masses, wave-function renormalizations and bare coupling constants, respectively.

As pointed out in section 2, there is a freedom to choose the cutoff functions in propagators \(P^\Lambda_{m}\) and \(P^\Lambda_{M}\). Here, we take advantage of this flexibility and choose a “mass-dependent scheme,” i.e., take different cutoff functions for the light-particle propagator \(P^\Lambda_{m}\) and the heavy-particle propagator \(P^\Lambda_{M}\): \(P^\Lambda_{m}\) is chosen to be the same as Eq. \([31]\) while, for \(P^\Lambda_{M}\), we choose

\[
P^\Lambda_{M} = \frac{R_M(\Lambda_0, p)}{p^2 + M^2}, \tag{47}
\]

with

\[
R_M(\Lambda, p) = \left[ 1 - K \left( \Lambda^2/M^2 \right) \right] K \left( \frac{p^2}{\Lambda^2} \right). \tag{48}
\]

The reason for this choice is because, if \(\Lambda < M\), we have \(R_M = 0\) identically. This property is very natural for our purpose, since it implies that at the scale \(\Lambda = M\) we have all modes of the heavy field \(\psi\) integrated out and there remains only the light field \(\phi\) below the scale \(M\); it explicitly implements Wilson’s point of view on effective field theory. Therefore, in
the flow equation, terms involving the heavy-particle propagator drop out if \( \Lambda < M \) and it will look like the flow equation for the theory having the light particle field only. One may regard this property specific to our choice \((32)\) and \((48)\) as the analogy of the so-called "manifest decoupling" in the conventional approach \([33]\). On the contrary, if one chose a "mass-independent scheme", i.e. if \(R_m\) and \(R_M\) were chosen independently of their masses \(m\) and \(M\), a substantial part of heavy particle modes would not be integrated out even below the scale \(M\), thus making subsequent discussions rather complicated.

Now, we define \(S_{\text{int}}^{(f)\Lambda}[\phi, \psi]\) following the general line discussed in section 2. It will then satisfy the flow equation,

\[
\partial_\Lambda S_{\text{int}}^{(f)\Lambda} = -\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \left\{ \partial_\Lambda P_m^{(f)\Lambda}(p) \left[ \frac{\delta^2 S_{\text{int}}^{(f)\Lambda}}{\delta \phi(p) \delta \phi(-p)} - \frac{\delta S_{\text{int}}^{(f)\Lambda}}{\delta \phi(p)} \frac{\delta S_{\text{int}}^{(f)\Lambda}}{\delta \phi(-p)} \right] \right\}_{\text{field-dep. part}}.
\]

Expanding \(S_{\text{int}}^{(f)\Lambda}\) in powers of \(\phi\) and \(\psi\) in momentum space, we write

\[
S_{\text{int}}^{(f)\Lambda}[\phi, \psi] = \sum_{|r| \geq 0} \sum_{|n| \geq 1} g^r \int \prod_{j=1}^{2n_1} \frac{d^4p_j}{(2\pi)^4} \prod_{j=1}^{2n_2-1} \frac{d^4p'_j}{(2\pi)^4} \phi(p_1) \cdots \phi(p_{2n_1}) \psi(p'_1) \cdots \psi(p'_{2n_2-1})
\]

\[
\times \mathcal{G}_{r,2n}^{(f)\Lambda}(p_1, \ldots, p_{2n_1}; p'_1, \ldots, p'_{2n_2-1})
\]

\[
(p'_{2n_2} \equiv - \sum_{j=1}^{2n_1} p_j - \sum_{j=1}^{2n_2-1} p'_j)
\]

Some explanations on our notations are in order. \(g \equiv (g_1, g_2, g_3)\) are perturbation-expansion parameters which may be identified as renormalized coupling constants, \(r\) and \(n\) represent \((r_1, r_2, r_3)\) and \((n_1, n_2)\), respectively, with \(|n| \equiv n_1 + n_2\), \(|r| \equiv r_1 + r_2 + r_3\) and finally, \(g^r \equiv g_1^{r_1} g_2^{r_2} g_3^{r_3}\). The vertex functions satisfy the flow equation

\[
\partial_\Lambda \mathcal{G}_{r,2n}^{(f)\Lambda}(p_1, \ldots, p_{2n_1}; p'_1, \ldots, p'_{2n_2-1})
\]

\[
= - \left( \frac{2n_1 + 2}{2} \right) \int \frac{d^4q}{(2\pi)^4} \partial_\Lambda P_m^{(f)\Lambda}(q) \mathcal{G}_{r,2(n_1+1,n_2)}^{(f)\Lambda}(p_1, \ldots, p_{2n_1+1}, q, -q; p'_1, \ldots, p'_{2n_2-1})
\]

\[
+ 2 \sum_{\ell, \ell', \ell'' = n_1 + 1} l_1 l'_1 \partial_\Lambda P_m^{(f)\Lambda}(P) \left[ \mathcal{G}_{r',2(l_1,l_2)}^{(f)\Lambda}(p_1, \ldots, p_{2l_1-1}, P; p'_1, \ldots, p'_{2l_2-1}) \right]
\]

\[
\times \mathcal{G}_{r'',2(l'_1,l'_2)}^{(f)\Lambda}(p_{2l_1}, \ldots, p_{2l_1+1}, P; p'_{2l_2+1}, \ldots, p'_{2n_2-1})
\]

\[
\left. \right|_{\text{symm}}
\]

\[
- \left( \frac{2n_2 + 2}{2} \right) \int \frac{d^4q}{(2\pi)^4} \partial_\Lambda P_M^{(f)\Lambda}(q) \mathcal{G}_{r,2(n_1,n_2+1)}^{(f)\Lambda}(p_1, \ldots, p_{2n_1}; p'_1, \ldots, p'_{2n_2+1}, q)
\]

\[
+ 2 \sum_{\ell, \ell', \ell'' = n_1 + 1} l_2 l'_2 \partial_\Lambda P_M^{(f)\Lambda}(P') \left[ \mathcal{G}_{r',2(l_1,l_2)}^{(f)\Lambda}(p_1, \ldots, p_{2l_1-1}, P'; p'_1, \ldots, p'_{2l_2-1}) \right]
\]

\[
\times \mathcal{G}_{r'',2(l'_1,l'_2)}^{(f)\Lambda}(p_{2l_1+1}, \ldots, p_{2n_1}; p'_{2l_2+1}, \ldots, p'_{2n_2})
\]

\[
\left. \right|_{\text{symm}} ,
\]

where \(P = - \sum_{j=1}^{2l_1-1} p_j - \sum_{j=1}^{2l_2} p'_j\), \(P' = - \sum_{j=1}^{2l_1-1} p_j - \sum_{j=1}^{2l_2-1} p'_j\). This equation can be represented by the diagram shown in Fig. 2, where we denote \(\phi(\psi)\)-legs by thin(thick)-lines. Also, With our choice of propagators, (see Eq. (45)), the last two terms in Eq. (51) vanish.
FIG. 2. Graphical representation of the flow equation (51). Thin (thick) lines represents light (heavy) field.

for $\Lambda < M$. Therefore the form of this flow equation coincides with that of a field theory for a single scalar field. Finally, in relation with the low-energy effective theory, it should be noted that, for $G_{r,2(n_1,0)}^{(f)\Lambda}$ (i.e., no external heavy particles), the last term in Eq. (51) is identically zero and the heavy field $\psi$ enters the flow equation only through the $\Lambda$-differentiated propagator in the third term.

The boundary conditions which $G_{r,2n}$'s obey are given as in the previous section. First, at $\Lambda = \Lambda_0$, irrelevant vertices vanish, i.e.,

$$\partial^w G^{(f)\Lambda_0}_{r,2n} = 0,$$

At $\Lambda = 0$, we impose renormalization conditions on relevant terms:

$$G_{r,(2,0)}^{(f)\Lambda=0}(0) = 0,$$
$$\partial_{p_1} \partial_{p_2} G_{r,(2,0)}^{(f)\Lambda=0}(0) = 0,$$
$$G_{r,(4,0)}^{(f)\Lambda=0}(0) = \frac{1}{4!} \delta_{r,(1,0,0)},$$
$$G_{r,(0,2)}^{(f)\Lambda=0}(\vec{p}_1') = 0,$$
$$\partial_{p_1} \partial_{p_2} G_{r,(0,2)}^{(f)\Lambda=0}(\vec{p}_1') = 0,$$
$$G_{r,(0,4)}^{(f)\Lambda=0}(\vec{p}_2, \vec{p}_3, \vec{p}_4) = \frac{1}{4!} \delta_{r,(0,1,0)},$$
$$G_{r,(2,2)}^{(f)\Lambda=0}(0, 0; \vec{p}_5') = \frac{1}{4!} \delta_{r,(0,0,1)}.$$

Here, normalization momenta $\vec{p}_i'$ ($i = 1, \ldots, 5$) are chosen to be constants of magnitude $M$; i.e., we have chosen the renormalization points for the light-particle Green functions at zero momentum, and those for the heavy particles at momenta of magnitude $M$.

With the flow equation (51) and the boundary conditions (52) and (53), now one can demonstrate the perturbative renormalizability (i.e., boundedness and convergence of Green functions as $\Lambda_0 \to \infty$) of the full theory following the line of arguments given in section 3.1, with the help of our choice of cutoff functions. The result is (54): for vertices with no external heavy-particle leg,

$$\| \partial^2 G^{(f)\Lambda}_{r,2(n_1,0)} \|_{(2\Lambda,\mu,M)} \leq (\Lambda + m)^{4 - 2n_1 - z} \log \left( \frac{\Lambda + m}{m} \right),$$

15
while, for irrelevant vertices \((2|n| + z > 4)\) with external heavy-particle legs,

\[
\| \partial^z G_{r,2n}^{(f)\Lambda} \|_{(2\Lambda,\mu, M)} \leq (\Lambda + m)^{4-2|n|-z} \text{Plog} \left( \frac{\Lambda + M}{m} \right), \tag{55}
\]

and, finally, for relevant vertices \((2|n| + 4 \leq 4)\) with external heavy-particle legs,

\[
\| \partial^z G_{r,2n}^{(f)\Lambda} \|_{(2\Lambda,\mu, M)} \leq (\Lambda + M)^{4-2|n|-z} \text{Plog} \left( \frac{\Lambda + M}{m} \right), \tag{56}
\]

where the coefficients in polynomials of logarithms are independent of \(M, \Lambda, \) and \(\Lambda_0\), and the norm \(\|\cdot\|_{(2\Lambda,\mu, M)}\) is defined by

\[
\| \partial^z G_{r,2(n_1,n_2)}^{(f)\Lambda} \|_{(a,b,a',b')} \equiv \max_{|p_i| \leq \max \{a,b\}} \max_{|p'_i| \leq \max \{a',b'\}} \max_{z = |a|} |\partial^np G_{r,2(n_1,n_2)}(p_1,\ldots,p_{2n_1};p'_1,\ldots,p'_{2n_2-1})|. \tag{57}
\]

(If there is no external heavy-particle leg \((n_2 = 0)\), this definition reduces to that of the norm \(\|\cdot\|_{(a,b)}\) in Eq. (37).)

Actually these bounds are not unexpected — they just show the right scaling behaviors. That is, the effect of imposing the renormalization conditions at momenta of magnitude \(M\) for heavy-particle legs shows up with appropriate powers of \(M\) (up to logarithmic corrections) in the flow equation of vertices with external heavy-particle legs. On the other hand, in the light-particle sector, the bounds have the same form as those of the single scalar theory (see Eq. (41)): the large \(M\) corrections do not show up in the bounds because the light-particle mass \(m\) is forced to be (unnaturally) small by hand. Also, it should be noted that the bounds (54) on the light particle sector have exactly the same form as that of the theory with a single scalar field. Of course, this should be the case for decoupling to occur in the first place.

### C. Low-Energy Effective Theory

Now suppose that we are interested in physics at scale much smaller than \(M\). What is the low-energy effective theory? Decoupling theorem states that, to the zeroth order of \(1/M\), it is simply given by the \(\phi^4\)-theory with heavy field \(\psi\) removed from the Lagrangian of the full theory. To establish this, we should show that the difference between Green functions of the \(\phi-\psi\) theory and those of the \(\phi\) theory are at most of order \(1/M^2\) (no \(O(1/M)\) term because of the \(Z_2\) symmetry). In our approach, it is given by the following bound:

\[
\| \partial^z (G_{r,2(n_0,0)}^{(f)\Lambda} - G_{r_1,2n,0}\delta_{r,(r_1,0,0)}) \|_{(2\Lambda,\mu)} \leq \begin{cases} 
\left( \frac{\Lambda + m}{M} \right)^2 (\Lambda + m)^{4-2n-z} \text{Plog} \left( \frac{M}{m} \right), & 0 \leq \Lambda \leq M \\
\Lambda^{4-2n-z} \text{Plog} \left( \frac{\Lambda}{m} \right), & M \leq \Lambda \leq \Lambda_0,
\end{cases} \tag{58}
\]

where \(r = (r_1, r_2, r_3)\). Notice that the difference in the vertex functions is no longer small if \(\Lambda\) becomes comparable to the heavy particle mass \(M\), which implies that low-energy effective theory is not useful above the heavy particle mass scale.
The strategy to show this bound is almost straightforward. As we have seen, the form of the flow equation of $\phi$-$\psi$ theory is the same as that of $\phi$ theory for $\Lambda < M$. Moreover, the boundary conditions of the two theories are the same by definition. Consequently, considering the flow of the difference of vertex functions,

$$D^A_{r,2n} \equiv G_{r;2n}^{(f);A} - G_{r,2n}^A \delta_{r,\Lambda,0}$$

we can expect that $D^A_{r,2n}$ is almost zero since boundary conditions for $D^A_{r,2n}$ is zero and the flow equation for $D^A_{r,2n}$ is homogeneous for $\Lambda < M$. Following the steps explained in the previous section, one can easily establish the bound (58). As we expect, decoupling theorem is proved more or less trivially in this approach.

The zeroth order effective theory is just renormalizable $\phi$ theory. In other words, it describes the light particle physics accurately to the zeroth order of $1/M$ at low energy. Since it is perturbatively renormalizable, the theory itself does not know the scale below which the theory is effective. In a sense the effective theory nature becomes manifest only when we raise the accuracy to higher order in $1/M$. It is then necessary to include irrelevant (nonrenormalizable) operators in the effective Lagrangian. In addition, we must also supply appropriate calculation rules to obtain unambiguous finite results because the presence of such irrelevant terms can make physical quantities diverge if naively calculated, as we mentioned before. If these tasks are systematically performed with local operators, we will have a local effective quantum field theory where virtual heavy-particle effects are isolated into the coefficients of irrelevant operators in the Lagrangian to any desired order of accuracy. Let us suppose that we want to describe low-energy physics in the full theory accurately up to order $1/M^2N_0$. ($N_0$ is some positive integer.) We will show below that we can then factorize all heavy-particle effects to the given order theory by making use of the flow equation (23) and by appropriately choosing the operators $S_{int}^{A_0;N}$, $N = 1, 2, \ldots, N_0$.

Here, $S_{int}^{A_0;N}$'s may be assumed to have the form

$$S_{int}^{A_0;N} = \int d^4x \left( \text{local, Euclidean invariant, even polynomials in } \phi \text{ and its derivatives, of dim.} \leq 4 + 2N \right), \quad N = 1, 2, \ldots, N_0.$$  \hspace{1cm} (60)

The coefficient of the polynomials will be chosen later so that $S_{int}^{A_0;N}$ carries information appropriate to $O(1/M^{2N})$-effects from the full theory. Then we claim that the sum of $S_{int}^{A_0;N}$'s, viz.,

$$S_{int}^{A_0;N_0} = S_{int}^{A_0} + \sum_{N=1}^{N_0} \frac{S_{int}^{A_0;N}}{N!} \left( = S_{int}^{A_0;N_0-1} + \frac{S_{int}^{A_0;N_0}}{N_0!} \right),$$

will reproduce the predictions of the full theory at low energy with an accuracy of order $1/M^{2N_0}$. As we noted in section 2, $S_{int}^{A_0;N}$ will describe amputated connected Green functions with the insertion of the operator $O_N$ defined in Eq. (23). As usual, the quantity $S_{int}^{A_0;N}$, which satisfies the flow equation (23), may be expanded in powers of $\phi$:

$$S_{int}^{A_0;N} [\phi] = \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} g_r^N \int \frac{d^4p_1}{(2\pi)^4} \cdots \frac{d^4p_{2n}}{(2\pi)^4} \phi(p_1) \cdots \phi(p_{2n}) G_{r;2n}^{A_0;N}(p_1, \ldots, p_{2n-1}),$$

$$\left( p_{2n} \equiv - \sum_{j=1}^{2n-1} p_j \right).$$

(62)
We may also define $\tilde{G}_{r,2n}^{\Lambda_0;N}$'s through the same kind of relation as in Eq. (61). To complete the definition of $S_{int}^{\Lambda_0;N}$, we must also state the boundary conditions for $G_{r,2n}^{\Lambda_0;N}$. At $\Lambda = \Lambda_0$, Eq. (60) implies that

$$\partial_p^w G_{r,2n}^{\Lambda_0;N} = 0, \quad \text{for } 2n + |w| > 4 + 2N. \quad (63)$$

For $2n + |w| \leq 4 + 2N$, we impose the following condition, recursively in $N$:

$$\partial_p^w G_{r,2n}^{\Lambda_0;N} = N! \partial_p^w (\tilde{G}_{r,2n}^{(f)\Lambda=0}(0) - \tilde{G}_{r,2n}^{\Lambda_0;N-1}(0)), \quad (64)$$

where $\tilde{G}$ is the vertex function obtained by summing over the couplings involving heavy fields in Eq. (50), i.e.,

$$G_{r,2n}^{(f)\Lambda} = \sum_{r_2,r_3} g_{r_2}^r g_{r_3}^r G_{r_1,r_2,r_3,2n}^{(f)\Lambda}. \quad (65)$$

As a consequence, $\partial_p^w G_{r,2n}^{\Lambda_0;N}(0) = 0$ for the case $2n + |w| < 4 + 2N$. The meaning of this boundary condition is clear. To obtain the $N$-th order effective theory, we add new (higher dimensional) operators to the effective action which compensate the difference between the full theory and the $N-1$-th order effective theory.

As we discussed in section 1, this seemingly natural procedure has a potential problem because new operators are nonrenormalizable and naive treatment of those operators make Green functions divergent (rather than making the effective theory more accurate). However one can show that, if one demands the flow of the effective theory follow Eq. (23), divergences are automatically taken care of and virtual heavy particle effects are correctly incorporated. Indeed, straightforward extension of the argument used in proving decoupling shows that the difference between two vertex functions, $G_{r,2n}^{(f)\Lambda}$ of the $\phi$-$\psi$ theory and $\tilde{G}_{r,2n}^{\Lambda_0;N}$ of the $\phi$ theory, satisfies the bound

$$\left\| \partial^Z (G_{r,2n}^{(f)\Lambda} - \tilde{G}_{r,2n}^{\Lambda_0;N}) \right\|_{(2\Lambda,\mu)} \leq \begin{cases} \left( \frac{\Lambda + m}{M} \right)^{2N+2}, & 0 \leq \Lambda \leq M \\ \left( \frac{\Lambda}{M} \right)^{2N}, & M \leq \Lambda \leq \Lambda_0 \end{cases} \quad (66)$$

which establishes the factorization of virtual heavy particle effects to any given order of accuracy in $1/M$,

$$|\partial_p^w (G_{r,2n}^{(f)c}(p_1, \ldots, p_{2n-1}) - \tilde{G}_{r,2n}^{\Lambda_0;N}(p_1, \ldots, p_{2n-1}))| \leq \left( \frac{m}{M} \right)^{2N_0+2} m^{4-2n-z} \log \left( \frac{M}{m} \right), \quad (\text{for } |p_i| \leq \mu). \quad (67)$$

If $\Lambda$ is larger than $M$, Eq. (64) says that the vertex function $\partial_p^w G_{r,2n}^{\Lambda_0;N}$ is larger than $\partial_p^w \tilde{G}_{r,2n}^{\Lambda_0;N=0}$ by a factor $(\Lambda/M)^{2N}$. Therefore, in this scale, increasing $N$ (or adding more irrelevant terms, equivalently) makes the vertex functions unnaturally large and the theory becomes
“less effective”. This is because we have imposed unnatural renormalization conditions on irrelevant vertices. If we chose natural values as the bare irrelevant couplings, they would give only \( O((m/\Lambda_0)^{2N}) \) contributions to Green functions; but, in Eq. (64), we imposed the values of order \((m/M)^{2N}\) to \( G_{c_{2n}}^{\Lambda_0} \) as the renormalization conditions, which are natural only if the cutoff is around \( M \). This affirms that the effective theory is useful only below the heavy particle mass scale \( M \).

So far we have shown that, at low energy, connected amputated Green functions of the full theory are reproduced by considering those of the \( \phi \) theory plus those with insertion of operators \( O_N, N = 1, \ldots, N_0 \), defined in Eq. (23). Such operator insertions have a simple interpretation if one uses the notion of renormalization of composite operators and their normal product notation. As mentioned in section 2 this subject can also be easily described using the flow equation but we will not discuss it in this lecture. Interested reader can consult Refs. [4,7]. Results are the following. For \( N_0 = 1 \), \( S_{int}^{\Lambda_0:1} \) can be written as

\[
S_{int}^{\Lambda_0:1} = \sum_{2n+|w|=6} b_{2n,\{w\}}^{R:1}[M_{2n,\{w\}}],
\]

where \( b_{2n,\{w\}}^{R:1} = \frac{1}{N_{\{w\}}} \partial_p^w (G_{c_{2n}}^{(f)c} - G_{c_{2n}}^{c})(0), N_{\{w\}} \) is a combinatorial factor defined by

\[
\partial_p^w [(ip_1)^{w_1} \cdots (ip_{2n})^{w_{2n}}]_{symm}|_{p_1=\ldots=p_{2n}=0} = \delta_{\{w\}}{\{w'\}N_{\{w\}}},
\]

and \( [M_{2n,\{w\}}] \) is the normal product of a local vertex \( M_{2n,\{w\}} = \int d^4 x \partial^{w_1}_x \phi \cdots \partial^{w_{2n-1}}_x \phi \phi(x) \). In terms of the local Euclidean-invariant vertices of dimension six, the above equation can be cast into a more attractive form:

\[
S_{int}^{\Lambda_0:1} = a_1^{(6)} \int d^4 x [\phi(\partial^2)^2 \phi(x)] + a_2^{(6)} \int d^4 x [\phi^3 \partial^2 \phi(x)] + a_3^{(6)} \int d^4 x [\phi^6 (x)]
\]

\[
\equiv \sum_{i=1}^3 a_i^{(6)}[\Omega_i^{(6)}],
\]

with

\[
a_1^{(6)} = \frac{1}{8 \cdot 41} (\partial^2)^2 (G_{c_{2n}}^{(f)c} - G_{c_{2n}}^{c})(0),
\]

\[
a_2^{(6)} = -\frac{1}{8} \partial^2 (G_{c_{2n}}^{(f)c} - G_{c_{2n}}^{c})(0),
\]

\[
a_3^{(6)} = (G_{c_{2n}}^{(f)c} - G_{c_{2n}}^{c})(0).
\]

The \( 1/M^2 \)-order effects of the full theory are then described by the insertion of the operator \( O_1 = \sum_i a_i^{(6)}[\Omega_i^{(6)}] \), viz.,

\[
G_{c_{2n}}^{(f)c} = G_{c_{2n}}^{c} + \sum_{i=1}^3 a_i^{(6)} G_{c_{2n}}^{c_1}([\Omega_i^{(6)}]) + O \left( \frac{m^{8-2n}}{M^4} \text{Plog} \left( \frac{M}{m} \right) \right),
\]

where \( G_{c_{2n}}^{c_1}([\Omega_i^{(6)}]) \) denotes the Green function with \([\Omega_i^{(6)}]\) inserted. For general \( N_0 \), \( S_{int}^{\Lambda_0:N} \) \((N = 1, \ldots, N_0)\) can be identified as
\[ S_{\text{int}}^{\Lambda_0;N} = \sum_{2n+|w|=4+2N} b_{2n,\{w\}}^{R[N]}[M_{2n,\{w\}}] + S_{CT}^{\Lambda_0;N}, \] 

(73)

where

\[ b_{2n,\{w\}}^{R[N]} = \frac{N!}{n!} \sum_{2n+|w|=4+2N} \partial_{w} \left( G^{(f)c}_{2(N,0)} - \tilde{G}_{2n}^{c[N-1]}(0) \right), \]

(74)

and \( S_{CT}^{\Lambda_0;N} \) denote counterterms which are needed to cancel the new divergences due to the multiple insertion of \( S_{\text{int}}^{\Lambda_0;I} \)'s, \( I = 1, \ldots, N - 1 \). (See [7] for explicit forms of \( S_{CT}^{\Lambda_0;N} \).) Now let \( [\Omega_i^{(4+2N)}] \)'s form a complete set of mutually independent Euclidean-invariant local vertices of dimension \((4+2N)\) and let \( a_i^{(4+2N)} \)'s be appropriate expansion coefficients as in Eq. (70), so that we may write

\[ S_{\text{int}}^{\Lambda_0;N} = \sum_i a_i^{(4+2N)}[\Omega_i^{(4+2N)}] + S_{CT}^{\Lambda_0;N}. \]

(75)

Then the effective action \( \bar{S}_{\text{int}}^{\Lambda_0;N_0} \) may be written as

\[ \bar{S}_{\text{int}}^{\Lambda_0;N_0} = S_{\text{int}}^{\Lambda_0} + \sum_{N=1}^{N_0} \sum_i \frac{a_i^{(4+2N)}}{N!} [\Omega_i^{(4+2N)}] + \sum_{N=2}^{N_0} \frac{1}{N!} S_{CT}^{\Lambda_0;N} \]

\[ = \bar{S}_{\text{int}}^{\Lambda_0;N_0-1} + \sum_i \frac{a_i^{(4+2N_0)}}{N_0!} [\Omega_i^{(4+2N_0)}] + \frac{1}{N_0!} S_{CT}^{\Lambda_0;N_0}, \]

(76)

where \( a_i^{(4+2N)} \)'s are appropriate linear combinations of \( b_{2n,\{w\}}^{R[N]} \)'s (all of which are \( O(1/M^{2N}) \)). Thus the \( 1/M^{2N} \)-order information of the full theory is factorized into the coefficient \( a_i^{(4+2N)} \)'s of local vertices of dimension \((4+2N)\). If we want to increase the accuracy from \( O(1/M^{2N_0}) \) to \( O(1/M^{2(N_0+1)}) \), we have only to include in \( S_{\text{int}}^{\Lambda_0;N_0} \) local vertices \([\Omega_i^{(4+2(N_0+1))}] \) of dimension \( 4+2(N_0+1) \). (The lower dimensional vertices need not be modified.) The coefficients \( a_i^{(4+2(N_0+1))} \) are calculable as the difference of appropriate Green functions of the full theory and those of the effective theory (i.e., \( \bar{S}_{\text{int}}^{\Lambda_0;N_0} \)) at renormalization point. New divergences are subtracted away by \( S_{CT}^{\Lambda_0;N_0+1} \). Given this bare Lagrangian, the flow equation guarantees that all Green functions are finite and accurate up to order \( 1/M^{2(N_0+1)} \).

The interpretation of this result is clear from the viewpoint of the renormalization group flow in the infinite dimensional space of possible Lagrangians. As we reduce the cutoff, the bare Lagrangian of the full theory, which is specified by the boundary conditions (52) and (53), flows down to a submanifold parametrized by relevant couplings only. It has both heavy and light operators. Now the flow may be projected onto the subspace of operators consisting of the light field only. Then finding a local low-energy effective field theory of light particles is equivalent to finding a renormalization group flow with a few (relevant or irrelevant) local operators which can best approximate the projected flow of the full theory. To the zeroth order in \( 1/M^2 \), it is done with relevant terms only by choosing the same renormalization conditions as those of the full theory. The deviation from the full-theory flow is corrected by reading off values of the remaining irrelevant coordinates at the \( \Lambda = 0 \) point. At first, components of dimension six operators may be read, which tell us the
effects of order $1/M^2$; the flow of the full theory is now approximated to the order $1/M^2$ at $\Lambda < M$. If one wants to increase the accuracy, it is necessary to read more and more irrelevant coordinates (at $\Lambda = 0$) of the projected flow of the full theory and modify the effective-theory flow with the corresponding flow equation given in Eq. (23). Our equation (23) automatically takes care of possible divergences due to the unnaturally large irrelevant components in such a way that the bare Lagrangian may contain only a finite number of irrelevant terms.

IV. CONCLUSIONS

In the context of a simple scalar field theory, we have demonstrated that, at low energy, virtual heavy particle effects on light-particle Green functions are completely factorized via effective local vertices to any desired order. For this, we have used the powerful flow equation approach which is a differential version of Wilson’s renormalization group transformation. In applying this method, we have seen that irrelevant terms in the Wilsonian effective Lagrangian (which has a finite UV cutoff $\Lambda = M$, a characteristic scale representing heavy-particle thresholds) are replaceable by the corresponding higher dimensional composite operators plus counterterms for their products in the continuum limit (where UV cutoff is supposed to go to infinity). The latter can be dealt with the help of the normal product concept. Once this fact is noticed, factorization is straightforward: all $1/M^{2N}$-order effects can be isolated in terms of local vertices involving operators of dimension $(4 + 2N)$. Thereby we arrive at a local continuum effective field theory which describes low-energy light-particle physics accurately up to any desired order in $1/M^2$ with appropriate calculation rules for irrelevant (nonrenormalizable) vertices. Since the arguments here are essentially dimensional and largely theory-independent, it should not be difficult to generalize them to different field theories such as gauge theories or theories with spontaneous symmetry breaking. Indeed, decoupling was proved for these theories in [23] within this approach.

Since the couplings for effective local vertices contain powers of $\log (M/m)$ in perturbation theory, it is often desirable to sum such logarithms in a systematic way. This has been achieved by utilizing a set of improved Callan-Symanzik equations [29]. Also, here we limited our attention to vertex functions at momentum range $\mu = O(m)$. As one increases $\mu$ to sufficiently high energy scale, the $(\mu/m)$-dependence we neglected will become important. Keeping such $\mu$-dependence and establishing bounds more carefully, one may study the high-momentum behavior of Green functions (even in the exceptional momentum region) within the effective field theory context. This bound has been obtained in [24] for zeroth order effective theory (corresponding to decoupling).

In summary, the flow equation, which is based on Wilsonian viewpoint of effective theory, provides clear understanding of effective theories in continuum.

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