Relative Yamabe Invariant

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Abstract

We define a relative Yamabe invariant of a smooth manifold with given conformal class on its boundary. In the case of empty boundary the invariant coincides with the classic Yamabe invariant. We develop approximation technique which leads to gluing theorems of two manifolds along their boundaries for the relative Yamabe invariant. We show that there are many examples of manifolds with both positive and non-positive relative Yamabe invariants.

1 Introduction

1.1. General setting. Let \( W \) be a compact smooth manifold with boundary, \( \partial W = M \neq \emptyset \), and \( n = \dim W \geq 3 \). Let \( \mathcal{Riem}(W) \) be the space of all Riemannian metrics on \( W \). For a metric \( \bar{g} \in \mathcal{Riem}(W) \) we denote \( H_{\bar{g}} \) the mean curvature along the boundary \( \partial W = M \), and \( g = \bar{g}|_M \). We also denote \([\bar{g}]\) and \([g]\) the corresponding conformal classes, and \( \mathcal{C}(M) \) and \( \mathcal{C}(W) \) the space of conformal classes on \( M \) and \( W \) respectively. Let \( \bar{C} \) and \( C \) be conformal classes of metrics on \( W \) and \( M \) respectively. We say that \( C \) is the boundary of \( \bar{C} \) or \( \bar{C} \) is a coboundary of \( C \) if \( \bar{C}|_M = C \). We use notation \( \partial \bar{C} = C \) in this case.

Let \( \mathcal{C}(W,M) \) be the space of pairs \((\bar{C},C)\) such that \( \partial \bar{C} = C \). Denote \( \bar{C}^0 = \{ \bar{g} \in \bar{C} \mid H_{\bar{g}} = 0 \} \). We call \( \bar{C}^0 \subset \bar{C} \) the normalized conformal class. Let \( \mathcal{C}^0(W,M) \) be the space of pairs \((\bar{C}^0,C)\), so that \( \bar{C}^0 \subset \bar{C} \), and \((\bar{C},C) \in \mathcal{C}(W,M)\). It is easy to observe (see [5, formula (1.4)]) that for any conformal class \( \bar{C} \in \mathcal{C}(W) \) the subclass \( \bar{C}^0 \) is not empty. Thus there is a natural bijection between the spaces \( \mathcal{C}^0(W,M) \) and \( \mathcal{C}(W,M) \). Let \( \bar{g} \in \bar{C}^0 \) be a metric. Then \( \bar{C}^0 \) may be described as follows:

\[
\bar{C}^0 = \left\{ u^{-\frac{4}{n-2}} \bar{g} \mid u \in C^\infty_+(W) \text{ such that } \partial_{\nu} u = 0 \text{ along } M \right\}.
\]

Here \( \nu \) is a normal unit (inward) vector field along the boundary, and \( C^\infty_+(W) \) is the space of positive smooth functions on \( W \).

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1.2. The Einstein-Hilbert functional. Let $C \in \mathcal{C}(M)$ be given. We define the following subspaces of metrics:

\[ \mathcal{Riem}_C(W, M) = \{ \bar{g} \in \mathcal{Riem}(W) \mid \partial[\bar{g}] = C \}, \]

\[ \mathcal{Riem}_C^0(W, M) = \{ \bar{g} \in \mathcal{Riem}_C(W) \mid H_{\bar{g}} = 0 \}. \]

We consider the normalized Einstein-Hilbert functional $I : \mathcal{Riem}_C^0(W, M) \to \mathbb{R}$ given by

\[ I(\bar{g}) = \frac{\int_W R_{\bar{g}} d\sigma_{\bar{g}}}{\text{Vol}_{\bar{g}}(W)^{\frac{n-2}{n}}}, \]

where $R_{\bar{g}}$ and is the scalar curvature, and $d\sigma_{\bar{g}}$ is the volume element. As in the case of closed manifolds, we have the following result.

**Theorem 1.1.** Critical points of the functional $I$ on the space $\mathcal{Riem}_C^0(W, M)$ coincide with the set of Einstein metrics $\bar{g}$ on $W$ with $\partial[\bar{g}] = C$, and $H_{\bar{g}} = 0$.

We postpone the proof of Theorem 1.1 to Section 3.

1.3. Relative Yamabe invariants. Similarly to the case of closed manifolds, the functional $I$ is not bounded. Precisely, it is easy to prove that for any manifold $W$, $\partial W = M$, $\dim W \geq 3$, and any conformal class $C \in \mathcal{C}(M)$

\[ \inf_{\bar{g} \in \mathcal{Riem}_C^0(W, M)} I(\bar{g}) = -\infty, \quad \text{and} \quad \sup_{\bar{g} \in \mathcal{Riem}_C^0(W, M)} I(\bar{g}) = \infty. \]

Let $(\bar{C}, C) \in \mathcal{C}(W, M)$. We define the **relative Yamabe constant of** $(\bar{C}, C)$ as

\[ Y_{\bar{C}}(W, M; C) = \inf_{\bar{g} \in \mathcal{C}_0} I(\bar{g}). \]

**Remark.** We notice that the Yamabe constant $Y_{\bar{C}}(W, M; C)$ coincides with the constant $Q(W)$ (up to a universal positive factor depending only on the dimension of $W$) defined by J. Escobar [5] for each pair of conformal classes $(\bar{C}, C) \in \mathcal{C}(W, M)$.

Clearly the Yamabe constant $Y_{\bar{C}}(W, M; C)$ must be related to the Yamabe problem on a manifold with boundary, which was solved by P. Cherrier [4] and J. Escobar [5] under some restrictions. Indeed, P. Cherrier proved the existence of a minimizer for the Yamabe functional $|C_0|$ provided

\[ Y_{\bar{C}}(W, M; C) < Y_{[\bar{g}_0]}(S^n_+, S^{n-1}; [g_0]). \quad (1) \]

Here $S^n_+$ is a round hemisphere with standard metric $\bar{g}_0$, and $S^{n-1} \subset S^n_+$ the equator with $g_0 = \bar{g}_0|_{S^{n-1}}$. More generally, J. Escobar [5] solved the Yamabe problem under restrictions we list below. We emphasize that the Escobar’s result includes the case when the inequality (1) is satisfied. Here is the list of restrictions given in [5]:

(a) $n \geq 6$
(b) $M = \partial W$ is umbilic in $W$
(c) the Weyl tensor $W_C = 0$ on $M$
(d) the Weyl tensor $\bar{W}_C \neq 0$ on $W$. 

(2)
Notice that the conditions (2) are conformally invariant. We denote
\[ C^\text{Esc}(W, M) = \left\{ (\bar{C}, C) \in C(W, M) \mid \text{at least one of the conditions (a)-(d) from (2) is not satisfied} \right\} \]

**Remark.** It is easy to see that \( C^\text{Esc}(W, M) \subset C(W, M) \) is open dense.

We state the Escobar’s result using the terms introduced above.

**Theorem 1.2.** (Escobar, [5, Theorem 6.1]) Let \((W, M)\) be a compact manifold with boundary, and \((\bar{C}, C) \in C^\text{Esc}(W, M)\). Then there exists a metric \(\bar{g} \in \bar{C}0\) such that \(Y_{\bar{C}}(W, M; C) = I(\bar{g})\). Such metric \(\bar{g}\) is called a relative Yamabe metric.

**Remark.** A relative Yamabe metric \(\bar{g} \in \bar{C}0\), of course, has constant scalar curvature \(R_{\bar{g}} = Y_{\bar{C}}(W, M; C) \cdot \text{Vol}_{\bar{g}}(W)^{-\frac{2}{n}}\).

We define the relative Yamabe invariant with respect to a conformal class \(C \in C(M)\):
\[ Y(W, M, C) = \sup_{C, \partial C = C} Y_{\bar{C}}(W, M; C). \]

Then we would like to define the following relative Yamabe invariants:
\[ Y(M; C) = \sup_{W, \partial W = M} Y(W, M; C), \]
\[ Y(W, M) = \sup_{C \in C(M)} Y(W, M; C), \]
\[ Y(M) = \sup_{W, \partial W = M} Y(W; M). \]

The invariants \(Y(W, M; C), Y(M; C)\) have clear geometric meaning in terms of positive scalar curvature (abbreviated as psc). We call a conformal class \(C \in C(M)\) positive if the Yamabe constant \(Y_C(M) > 0\). Of course, it means that a Yamabe metric \(g \in C\) has positive constant scalar curvature. The following statement follows from the above definitions.

**Claim 1.3.**
(1) Let \(C \in C(M)\) be a positive conformal class. Then
- \(Y(W, M; C) > 0\) if and only if any psc-metric \(g \in C\) may be extended conformally to a psc-metric \(\bar{g}\) on \(W\) with \(H_{\bar{g}} = 0\).
- \(Y(M; C) > 0\) if and only if for any psc-metric \(g \in C\) there exists a manifold \(W\), so that \(g\) may be extended conformally to a psc-metric \(\bar{g}\) on \(W\) with \(H_{\bar{g}} = 0\).

(2) The invariants \(Y(W, M), Y(M)\) are diffeomorphism invariants.

We present our main results on the relative Yamabe invariant in the next section.

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2 Overview of the results

2.1. Minimal boundary condition and approximation theorems. First, one can notice that the minimal boundary condition \( H_g = 0 \) is too weak for applications. For instance, to apply a relative index theory, one needs much stronger condition that a metric \( \bar{g} \) is a product metric near the boundary.

The closest differential-geometric approximation to the product metric near the boundary is when a boundary is totally geodesic. In more detail, let \( \bar{g} \in \text{Riem}_{\partial W}^0(W, M) \), \( g = \bar{g}|_M \), and, as above \( A_{\bar{g}} = (A_{ij}) \) be the second fundamental form of \( M = \partial W \) with respect to \( \bar{g} \). The boundary \( M = \partial W \) is said to be totally geodesic if \( A_{\bar{g}} \) vanishes identically on \( M \). Clearly any metric from the normalized conformal class \([\bar{g}]^0\) is totally geodesic if \( \bar{g} \) is. We call the conformal class \([\bar{g}]\) of such metric \( \bar{g} \) umbilic. We denote \( \mathcal{C}_{\text{um}}(W, M) := \{ \bar{C} \in \mathcal{C}(W) \mid \bar{C}|_M = C \} \) the subspace of umbilic conformal classes.

Our first aim is to prove a generalization (Proposition 4.5) of the approximation theorem due to Kobayashi [10]. We show that any metric \( \bar{g} \) with totally geodesic boundary is \( C^1 \)-close to a metric \( \tilde{g} \) which is conformally equivalent to a product metric near the boundary. Moreover, we show that the scalar curvature \( R_{\bar{g}} \) is \( C^0 \)-close to \( R_{\tilde{g}} \) of \( \tilde{g} \).

Next, we prove the approximation Theorem 4.6 under the minimal boundary condition. Theorem 4.6 gives us a fundamental tool on the relative Yamabe invariant. In particular, we prove the following result.

Theorem 2.1. For any \( \bar{C} \in \mathcal{C}(W, M) \), and any \( \varepsilon > 0 \) there exists a conformal class \( \tilde{C} \in \mathcal{C}^\text{um}(W, M) \), and a metric \( \tilde{g} \in \tilde{C}^0 \), such that

\[
\begin{cases}
\bar{C} \text{ and } \tilde{C} \text{ are } C^0\text{-close conformal classes} \\
|Y_{\bar{C}}(W, M; C) - Y_{\tilde{C}}(W, M; C)| < \varepsilon \\
\tilde{g} \sim g + dr^2 \text{ (conformally equivalent near } M),
\end{cases}
\]

where \( C = \partial \bar{C} \) and \( g = \bar{g}|_M \). More precisely,

\[
\tilde{g} = \left(1 + \frac{r^2}{2} f(x)\right)^{\frac{4}{n-2}} (g + dr^2) \text{ near } M, \text{ where}
\]

\[
f(x) = -\frac{n-2}{4(n-1)} (R_{\bar{g}} - R_g + 3|A_{\bar{g}}|^2_g) \text{ on } M.
\]

We define the “umbilic Yamabe invariant” \( Y^\text{um}(W, M; C) \) as

\[
Y^\text{um}(W, M; C) = \sup_{\tilde{C} \in \mathcal{C}^\text{um}(W, M)} Y_{\tilde{C}}(W, M; C).
\]

Theorem 2.1 leads to the following conclusion.

Corollary 2.2. \( Y^\text{um}(W, M; C) = Y(W, M; C) \).

2.2. Gluing Theorem. We analyze the gluing procedure for manifolds equipped with conformal structures. Let \( W_1, W_2 \) be two compact manifolds, \( \dim W_1 = \dim W_2 \geq 3 \), with boundaries

\[
\partial W_1 = M_1 = M_0 \sqcup M, \quad \text{and} \quad \partial W_2 = M_2 = M_0 \sqcup M'.
\]
endowed with conformal classes $C_1 = C_0 \sqcup C \in \mathcal{C}(M_1)$, $C_2 = C_0 \sqcup C' \in \mathcal{C}(M_2)$, where $C_0 \in \mathcal{C}(M_0)$, $C \in \mathcal{C}(M)$, $C' \in \mathcal{C}(M')$. Let $W = W_1 \cup_{M_0} (-W_2)$ be the boundary connected sum of $W_1$ and $W_2$ along $M_0$.

We study the case when the conformal class $C_0 \in \mathcal{C}(M_0)$ is positive, and the relative Yamabe invariants $Y(W_j, M_j; C_j)$, $j = 1, 2$ are positive as well. We essentially use the approximation Theorem 4.6 to prove the following result (Theorem 5.1):

**Theorem 2.3.** Let $C_0 \in \mathcal{C}(M_0)$ be a positive conformal class, and $Y(W_j, M_j; C_j) > 0$ for $j = 1, 2$. Then $Y(W, \partial W; C \sqcup C') > 0$.

In particular, this result allows us to construct many examples with positive relative Yamabe invariants (Theorem 5.2).

### 2.3. Non-positive Yamabe invariant

Let $W$ be a compact smooth $n$-manifold with boundary $\partial W = M \neq \emptyset$, and $M = M_0 \sqcup M_1$. (Here $M_1$ may be empty.) Let $(\bar{C}, C = C_0 \sqcup C_1)$ be a pair of conformal classes on $(W, M)$, where $C_0 \in \mathcal{C}(M_0)$, $C_1 \in \mathcal{C}(M_1)$. Let $X = W \cup_{M_0} (-W)$ be the double of $W$ along $M_0$. Then, the boundary of $X$ is $\partial X = M_1 \sqcup (-M_1)$. We prove the following result (Theorem 6.2):

**Theorem 2.4.**

1. If $Y_C(W, M; C) \leq 0$, then
   $$2^{\frac{n}{2}} Y_C(W, M; C) \leq Y(X, M_1 \sqcup (-M_1); C_1 \sqcup C_1).$$
2. If $Y_C(W, M; C) \leq 0$, then
   $$2^{\frac{n}{2}} Y(W, M; C) \leq Y(X, M_1 \sqcup (-M_1); C_1 \sqcup C_1).$$
3. If $Y(W, M) \leq 0$, then
   $$2^{\frac{n}{2}} Y(W, M) \leq Y(X, M_1 \sqcup (-M_1)).$$

When the manifolds $\partial W = M = M_0$, and $M_1$ is empty, (in this case, the boundary of $X = W \cup_M (-W)$ is empty) the following holds (Corollary 6.3):

**Corollary 2.5.**

1. If $Y_C(W, M; C) \leq 0$, then $2^{\frac{n}{2}} Y_C(W, M; C) \leq Y(X)$.
2. If $Y(W, M; C) \leq 0$, then $2^{\frac{n}{2}} Y(W, M; C) \leq Y(X)$.
3. If $Y(W, M) \leq 0$, then $2^{\frac{n}{2}} Y(W, M) \leq Y(X)$.

We use these results to show that there are many examples of manifolds with non-positive Yamabe invariant. In particular, we have (Corollary 6.4):

**Corollary 2.6.** Let $N$ be an enlargeable compact closed manifold. Then

$$2^{\frac{n}{2}} Y(N \setminus \text{int}(D^n), S^{n-1}) \leq Y(N \#(-N)).$$

In particular, $Y(N \setminus \text{int}(D^n), S^{n-1}) \leq 0$.

The rest of the paper is organized as follows. We prove Theorem 1.1 in Section 3. Then we prove the approximation theorems in Section 4. We give a gluing construction in Section 5. We analyze the Yamabe invariant for a double, and study the non-positive Yamabe invariant in Section 6. In the last Section 7 we define and study the moduli space of positive conformal classes and introduce conformal concordance and conformal cobordism.
3 Proof of Theorem 1.1

Let $\bar{g} \in \mathcal{R}_{\text{iem}}(W, M)$ be a metric, and $\{\bar{g}(t)\}$ be a variation of $\bar{g}$ in the space $\mathcal{R}_{\text{iem}}(W, M)$, i.e. $\bar{g}(0) = \bar{g}$. Thus we consider first a general variation, i.e. $\{\bar{g}(t)\}$ is not necessarily in the subspace $\mathcal{R}_{\text{iem}}^0(W, M) \subset \mathcal{R}_{\text{iem}}(W, M)$. Now we need the following notations. Let $h = \frac{d}{dt}|_{t=0} \bar{g}(t)$ be a variational vector, and

$$g(t) = \bar{g}(t)|_M, \quad \text{where } g(0) = g.$$

Remark. We observe that the condition $g(t) \in C$ implies that $h_{ij} = fg_{ij}$ on $M$, where $f \in C^\infty(M)$.

Let $r = r(t)$ be the distance function to the boundary $M$ in $W$ with respect to the metric $\bar{g}(t)$. Let $\nu = \frac{\partial}{\partial r}$ be a unit normal (inward) vector field along the boundary $\partial W = M$.

Let $p \in M$, and $\{r, x^1, \ldots, x^{n-1}\}$ be a Fermi coordinate system near $p$. We use indices $\alpha, \beta = 0, 1, \ldots, n-1$, where 0 corresponds to the normal direction, and $i, j, k = 1, \ldots, n-1$ are indices corresponding to the tangent directions (only on the boundary $\partial W = M$). We denote $(\cdot)’ = \frac{d}{dt}(\cdot)|_{t=0}$, the variational derivative evaluated at $t = 0$. In order to prove Theorem 1.1, it is enough to prove the following formula.

Claim 3.1. Let $\{\bar{g}(t)\}$ be a variation as above. Then

$$\left( \int_W R_{\bar{g}(t)}d\sigma_{\bar{g}(t)} \right)’ = -\int_W \langle Ric_{\bar{g}} - \frac{1}{2}R_{\bar{g}\bar{g}}, h \rangle_{\bar{g}}d\sigma_{\bar{g}} - \int_M (2H’_{\bar{g}} + fH_{\bar{g}}) d\sigma_{\bar{g}}.$$

Proof. We denote $\nabla$ and $\partial$ corresponding Levi-Civita connections with respect to the metrics $\bar{g}$ and $g$. Standard calculation gives:

$$\left( R_{\bar{g}(t)} \right)’ = -\nabla^\alpha \nabla_\alpha (\text{Tr}\bar{g} h) + \nabla^\alpha \nabla_\alpha h_{\alpha\beta} - \langle Ric_{\bar{g}}, h \rangle_{\bar{g}},$$

$$\left( d\sigma_{\bar{g}(t)} \right)’ = \langle \frac{1}{2}R_{\bar{g}\bar{g}}, h \rangle_{\bar{g}}d\sigma_{\bar{g}}. \quad (4)$$

The formula (4) together with Gauss divergence formula gives

$$\left( \int_W R_{\bar{g}(t)}d\sigma_{\bar{g}(t)} \right)’ = -\int_W \langle Ric_{\bar{g}} - \frac{1}{2}R_{\bar{g}\bar{g}}, h \rangle_{\bar{g}}d\sigma_{\bar{g}}$$

$$+ \int_M \langle \nabla(\text{Tr}\bar{g} h), \nu \rangle_{\bar{g}}d\sigma_{\bar{g}} - \int_M \sum_{\alpha=0}^{n-1} \langle \nabla_{e_\alpha} h \rangle(e_\alpha, \nu)d\sigma_{\bar{g}}.$$

Here $\{e_\alpha\} = \{\nu, e_1, \ldots, e_{n-1}\}$ is a local orthonormal field. We denote

$$B_I = \langle \nabla(\text{Tr}\bar{g} h), \nu \rangle_{\bar{g}}, \quad B_{II} = -\sum_{\alpha=0}^{n-1} \langle \nabla_{e_\alpha} h \rangle(e_\alpha, \nu).$$

Let $p \in M$ be an arbitrary point of the boundary. As before, let $\nu$ be a unit vector field normal (inward) to the boundary, such that $\nabla_\nu \nu = 0$, and $\{e_i\}$ be an orthonormal frame
near \( p \) in \( W \) such that \( \nabla_{e_i}e_j = 0 \) at \( p \) and \( t = 0 \). We emphasize that, in general, \( \nabla_{e_i}e_j \) does not vanish at \( p \). We have the second fundamental form of \( M \):

\[
A_{ij} = A(e_i, e_j) = g(\nabla_{e_i}e_j, \nu) = -\bar{g}(\nabla_e, \nu, e_j).
\]

Then we have \( H = g^{ij}A_{ij} \) the mean curvature of the boundary \( M \). We have:

\[
H = - \sum_{i=0}^{n-1} \bar{g}(\nabla_{e_i}e_i, e_i) = \sum_{\alpha=0}^{n-1} \bar{g}(\nabla_{e_{\alpha}}e_{\alpha}, e_{\alpha}) = -\nabla_{\alpha}\nu^\alpha = - \left( \partial_{\alpha}\nu^\alpha + \Gamma_{\alpha\beta}^\gamma \nu^\beta \right)
\]

Here \( \{x^\alpha\} = \{r, x^1, \ldots, x^{n-1}\} \) are a Fermi coordinate system near \( p \) in \( W \), and \( \partial_{\alpha} = \frac{\partial}{\partial x^\alpha} \) (and \( \partial_{\alpha} = e_\alpha \) at \( p \)). We have:

\[
H' = - \left[ \partial_{\alpha}(\nu')^\alpha + \Gamma_{\alpha\beta}(\nu')^\beta + (\Gamma')_{\alpha\beta}^\gamma \nu^\beta \right]
\]

\[
= - \left[ \nabla_{\alpha}(\nu')^\alpha + \frac{1}{2} (\nabla_{\alpha}h_{\beta}^\alpha + \nabla_{\beta}h_{\alpha}^\alpha - \nabla_{\alpha}h_{\alpha\beta}) \nu^\beta \right]
\]

\[
= -\nabla_{\alpha}(\nu')^\alpha - \frac{1}{2} \nabla_{\beta}(\text{Tr}_{\beta}h)\nu^\beta
\]

\[
= -\nabla_{\alpha}(\nu')^\alpha - \frac{1}{2} \langle \nabla(\text{Tr}_{\beta}h), \nu \rangle \bar{g}.
\]

Thus we obtain

\[
B_I = \langle \nabla(\text{Tr}_{\beta}h), \nu \rangle \bar{g} = - \left( 2H' + 2\nabla_{\alpha}(\nu')^\alpha \right).
\]

Now we compute the term \( B_{II} \). We have

\[
\sum_{\alpha=0}^{n-1} (\nabla_{e_\alpha}h)(e_\alpha, \nu) = \sum_{\alpha=0}^{n-1} (\nabla_{e_\alpha}h(e_\alpha, \nu) - h(\nabla_{e_\alpha}e_\alpha, \nu) - h(e_\alpha, \nabla_{e_\alpha}\nu))
\]

\[
= \nabla_{\nu}h(\nu, \nu) + \sum_{i=1}^{n-1} \nabla_{e_i}h(e_i, \nu) - h_{00}H + h^{ik}A_{ik}
\]

since

\[
\sum_{\alpha=0}^{n-1} \nabla_{e_\alpha}\nu = \nabla_{\nu} + \sum_{i=1}^{n-1} \nabla_{e_i}\nu, \quad \nabla_{\nu} = 0,
\]

and

\[
\nabla_{e_i}\nu = -\sum_{k=1}^{n-1} A_{ik}e_k, \quad \nabla_{e_\alpha}e_\alpha = \nabla_{\nu} + \sum_{i=1}^{n-1} \nabla_{e_i}e_i = H\nu.
\]

Notice that \( h^{ik} = fg^{ik} \) and \( \nabla_{e_i}h(e_j, \nu) = \nabla_{e_i}h(e_j, \nu) \). Thus we have

\[
B_{II} = -\sum_{\alpha=0}^{n-1} (\nabla_{e_\alpha}h)(e_\alpha, \nu) = -\nabla_{\nu}h(\nu, \nu) - \sum_{i=1}^{n-1} \nabla_{e_i}h(e_i, \nu) + h_{00}H - fH. \tag{6}
\]
To continue, we notice that \( \bar{g}(\nu, \nu) = 1 \) implies
\[
0 = \bar{g}'(\nu, \nu) + 2\bar{g}(\nu', \nu) = h(\nu, \nu) + 2\bar{g}(\nu', \nu).
\]
Also we have
\[
0 = \nabla_\nu h(\nu, \nu) + 2\bar{g}(\nabla_\nu \nu', \nu) + 2\bar{g}(\nu', \nabla_\nu \nu) = \nabla_\nu h(\nu, \nu) + 2\bar{g}(\nabla_\nu \nu', \nu)
\]
since \( \nabla_\nu \nu = 0 \). Thus we have that
\[
2\bar{g}(\nabla_\nu \nu', \nu) = -\nabla_\nu h(\nu, \nu).
\] (7)
Then the identity \( \bar{g}(\nu, e_i) = 0 \) implies
\[
0 = \bar{g}'(\nu, e_i) + \bar{g}(\nu', e_i) + \bar{g}(\nu, e'_i).
\]
Notice that \( e'_i \in T_p M \) since \( g(t) \in C \). Thus
\[
0 = \bar{g}'(\nu, e_i) + \bar{g}(\nu', e_i).
\]
Now it follows that
\[
0 = \sum_{i=1}^{n-1} \left( \nabla_{e_i} h(\nu, e_i) + \bar{g}(\nabla_{e_i} \nu', e_i) + \bar{g}(\nu', \nabla_{e_i} e_i) \right).
\]
Notice that \( \sum_{i=1}^{n-1} \nabla_{e_i} e_i = H\nu \), and \( H\bar{g}(\nu', \nu) = -\frac{1}{2}h(\nu, \nu) \). Thus we obtain
\[
2 \sum_{i=1}^{n-1} \bar{g}(\nabla_{e_i} \nu', e_i) = -2 \sum_{i=1}^{n-1} \nabla_{e_i} h(\nu, e_i) + h_{00} H.
\] (8)
We combine (7) and (8) to obtain
\[
2\nabla_\alpha (\nu')^\alpha = 2 \sum_{\alpha=0}^{n-1} \bar{g}(\nabla_{e_\alpha} \nu', e_\alpha) = 2 \left[ \bar{g}(\nabla_\nu \nu', \nu) + \sum_{i=1}^{n-1} \bar{g}(\nabla_{e_i} \nu', e_i) \right]
\] (9)
\[
= -\nabla_\nu h(\nu, \nu) - 2 \sum_{i=1}^{n-1} \nabla_{e_i} h(\nu, e_i) + h_{00} H.
\]
Now it follows from (5), (6) and (9) that
\[
B_I + B_{II} = -2H' - 2\nabla_\alpha (\nu')^\alpha - \nabla_\nu h(\nu, \nu) - \sum_{i=1}^{n-1} \nabla_{e_i} h(e_i, \nu) + h_{00} H - fH
\]
\[
= -2H' - \sum_{i=1}^{n-1} \nabla_{e_i} h(\nu, e_i) - fH.
\]
Denote \( \theta(\nu) = h(\nu, \nu) \) for \( \nu \in T_x M \), so \( \theta \) is a 1-form on \( M \). We notice that
\[
\nabla_{e_j} \theta(e_i) = (\nabla_{e_i} \theta)(e_j) + \theta(\nabla_{e_i} e_j) = (\nabla_{e_i} \theta)(e_j) \quad \text{since} \quad \nabla_{e_i} e_j = 0 \quad \text{at} \ p.
\]
Thus we have that
\[
B_I + B_{II} = -2H' - fH - \nabla_j \theta^i \quad \text{on} \ M.
\]
This proves Claim 3.1 and concludes the proof of Theorem 1.1. \( \square \)
4 Approximation Theorems

4.1. Kobayashi approximation lemma. First we reformulate several known facts in our terms. The following fact follows from a modification of the continuity property of the Yamabe constant due to Bérard Bergery.

Lemma 4.1. (cf. [1, Proposition 4.31]) Let $\bar{g}_t$, $\bar{g} \in \mathcal{R}iem_0(W, M)$ be Riemannian metrics, and $C_\delta = [\bar{g}_t]$, $\bar{C} = [\bar{g}]$. Assume that

(i) \( w_\delta(t) \equiv 1 \) on \([0, \varepsilon(\delta)]\), \( w_\delta(t) \equiv 0 \) on \([\delta, \infty)\),

(ii) \( |t\bar{w}_\delta(t)| < \delta \) for \( t \geq 0 \),

(iii) \( |t\bar{\bar{w}}_\delta(t)| < \delta \) for \( t \geq 0 \).

(see Fig. 4.1.)

\[
\begin{align*}
\omega_\delta(r) = \frac{1}{r} & \quad \text{for} \quad 0 < r < \varepsilon(\delta), \\
\omega_\delta(r) &= \frac{\delta}{2r^2} & \text{for} \quad \varepsilon(\delta) \leq r < \delta.
\end{align*}
\]

\[ y = \omega_\delta(r) \]

\[ r \rightarrow \delta \]

\[ y \rightarrow 1 \]

\[ \varepsilon(\delta) \]

Fig. 4.1

Lemma 4.2. (O. Kobayashi, [10]) For any \( \delta > 0 \) there exists a smooth nonnegative function \( w_\delta \), and there exists \( \varepsilon(\delta) = \frac{1}{\delta}e^{-\frac{1}{\delta}} \) such that

\[
\begin{align*}
\{ & w_\delta(t) \equiv 1 \quad \text{on} \quad [0, \varepsilon(\delta)], \\
& w_\delta(t) \equiv 0 \quad \text{on} \quad [\delta, \infty), \\
& |t\bar{w}_\delta(t)| < \delta \quad \text{for} \quad t \geq 0,
\end{align*}
\]

\[
\begin{align*}
& |t\bar{\bar{w}}_\delta(t)| < \delta \quad \text{for} \quad t \geq 0.
\end{align*}
\]

(see Fig. 4.1.)

\[
\begin{align*}
R_{\bar{g}} - R_{\bar{g}} &= P_{\bar{g}}(h) + Q_{\bar{g}}(h),
\end{align*}
\]

\[
\begin{align*}
P_{\bar{g}}(h) &= -\Delta_{\bar{g}}(\text{Tr}_{\bar{g}}h) + \bar{\nabla}^i\bar{\nabla}^j h_{ij} - \langle h, \text{Ric}_{\bar{g}} \rangle_{\bar{g}}, \\
|Q_{\bar{g}}(h)| &\leq C \left( |\bar{\nabla}h|^2 q^3 + |h| \cdot |\bar{\nabla}^2 h| q^2 + (|h| \cdot |\bar{\nabla}^2 h| + |\text{Ric}_{\bar{g}}| \cdot |h|^2) q \right),
\end{align*}
\]

where the constant \( C > 0 \) depends only on \( n = \dim W \), and \( q \in C^\infty(W) \) is a function satisfying \( q \cdot \bar{g} \geq \bar{g} \).

Proposition 4.4. Let \( W \) be a manifold with boundary \( \partial W = M \), and let metrics \( \bar{g}, \tilde{g} \in \mathcal{R}iem_0(W, M) \) such that \( j_M^1 \bar{g} = j_M^1 \tilde{g} \) (i.e. \( \bar{g} \) coincides with \( \tilde{g} \) up to second derivatives on \( M \)), and \( R_{\bar{g}} = R_{\tilde{g}} \) on \( M \). Then the family of metrics

\[
\tilde{g}_\delta = \bar{g} + w_\delta(r)(\bar{g} - \tilde{g}) \in \mathcal{R}iem_0(W, M)
\]

satisfies the following properties:

(i) \( \tilde{g}_\delta \to \bar{g} \) in the \( C^1 \)-topology on \( W \) (as \( \delta \to 0 \)),

(ii) \( R_{\tilde{g}_\delta} \to R_{\bar{g}} \) in the \( C^0 \)-topology on \( W \) (as \( \delta \to 0 \)),

(iii) \( \tilde{g}_\delta \equiv \bar{g} \) on the collar \( U_\delta(M, \bar{g}) = \{ x \in W \mid \text{dist}_{\bar{g}}(x, M) < \varepsilon(\delta) \} \),

(iv) \( \tilde{g}_\delta \equiv \tilde{g} \) on \( W \setminus U_{\varepsilon(\delta)}(M, \tilde{g}) \).
Proof. The statements (iii), (iv) are obvious. We prove (i) and (ii).

(i) The function \( w_\delta \) is such that \( \text{supp}(w_\delta) \subset [0, \delta] \). Then it follows

\[
g_\delta - g = w_\delta(r)(\tilde{g} - \tilde{g}) = O(r^2),
\]
thus \( g_\delta \to g \) in the \( C^0 \)-topology on \( W \). Furthermore,

\[
\partial(\tilde{g}_\delta - \tilde{g}) = \tilde{w}_\delta(r)(\tilde{g}_\delta - \tilde{g}) + w_\delta(r)\partial(\tilde{g} - \tilde{g}).
\]

By the condition on the metrics \( \tilde{g}_\delta, \tilde{g} \),

\[
\tilde{g}_\delta - \tilde{g} = O(r^2), \quad \partial(\tilde{g}_\delta - \tilde{g}) = O(r).
\]
We use Lemma 4.2 to estimate

\[
|\partial \tilde{g}_\delta - \partial \tilde{g}| \leq |\tilde{w}_\delta(r)| \cdot \frac{O(r^2)}{r} + w_\delta(r) \cdot O(r) \leq \delta O(\delta) + O(\delta).
\]
Thus \( \partial \tilde{g}_\delta \to \partial \tilde{g} \) in the \( C^0 \)-topology, i.e. \( \tilde{g}_\delta \to \tilde{g} \) in the \( C^1 \)-topology on \( W \).

(ii) We use Lemma 4.3 to write

\[
\begin{align*}
R_{\tilde{g}_\delta} - R_{\tilde{g}} & = P_\delta(w_\delta(r)(\tilde{g}_\delta - \tilde{g})) + Q_\delta(w_\delta(r)(\tilde{g}_\delta - \tilde{g})), \\
w_\delta(r)(R_{\tilde{g}_\delta} - R_{\tilde{g}}) & = w_\delta(r)P_\delta(\tilde{g}_\delta - \tilde{g}) + w_\delta(r)Q_\delta(\tilde{g}_\delta - \tilde{g}).
\end{align*}
\]
We use again Lemma 4.3:

\[
|P_\delta(w_\delta(r)(\tilde{g}_\delta - \tilde{g})) - w_\delta(r)P_\delta(\tilde{g}_\delta - \tilde{g})| \\
\leq C (|\tilde{w}_\delta(r)| \cdot |w_\delta(r)| \cdot |\tilde{g}_\delta - \tilde{g}| + |\tilde{w}_\delta(r)|^2 \cdot |\tilde{g}_\delta - \tilde{g}| + |\tilde{w}_\delta(r)| \cdot |w_\delta(r)| \cdot |\partial(\tilde{g}_\delta - \tilde{g})|)
\leq C \left( (|\tilde{w}_\delta(r)|^2 + |\tilde{w}_\delta(r)|^2) \frac{O(r^2)}{r^2} + |\tilde{w}_\delta(r)| \cdot \frac{O(r)}{r} \right) \leq C_1 \delta.
\]
Similarly we obtain

\[
\begin{align*}
|Q_\delta(w_\delta(r)(\tilde{g}_\delta - \tilde{g}))| & \leq C_2 \delta, \\
|w_\delta(r)Q_\delta(\tilde{g}_\delta - \tilde{g})| & \leq C_3 \delta.
\end{align*}
\]
Notice that

\[
|w_\delta(r)(R_{\tilde{g}_\delta} - R_{\tilde{g}})| \leq C_4 \delta
\]
since \( R_{\tilde{g}} \equiv R_{\tilde{g}} \) on \( M \). Thus we obtain:

\[
|R_{\tilde{g}_\delta} - R_{\tilde{g}}| \leq |P_\delta(w_\delta(r)(\tilde{g}_\delta - \tilde{g})) - w_\delta(r)P_\delta(\tilde{g}_\delta - \tilde{g})| + |Q_\delta(w_\delta(r)(\tilde{g}_\delta - \tilde{g}))| + |w_\delta(r)Q_\delta(\tilde{g}_\delta - \tilde{g})| + |w_\delta(r)(R_{\tilde{g}_\delta} - R_{\tilde{g}})|
\leq \left( C_1 + C_2 + C_3 + C_4 \right) \delta.
\]
Here \( C_j \) \((j = 1, \ldots , 4)\) are positive constants independent of \( \delta \). Thus \( R_{\tilde{g}_\delta} \to R_{\tilde{g}} \) in the \( C^0 \)-topology on \( W \).
**Proposition 4.5.** (Kobayashi Approximation Theorem, cf. [10, Lemma 3.2])

Let $W$ be a manifold with boundary $\partial W = M$, $C \in C(M)$. Let $\bar{g} \in \operatorname{Riem}_C(W, M)$ be a metric (respectively $\bar{g} \in \operatorname{Riem}^0_0(W, M)$). Let $g = \bar{g}|_{M}$, and $A_{\bar{g}}$ be the second fundamental form of $M = \partial W$. There exists a family of metrics $\tilde{g}_{\delta} \in \operatorname{Riem}_C(W, M)$ (respectively $\tilde{g}_{\delta} \in \operatorname{Riem}^0_0(W, M)$) such that

(i) $\tilde{g}_{\delta} \to \bar{g}$ in the $C^1$-topology on $W$ (as $\delta \to 0$),
(ii) $R_{\tilde{g}_{\delta}} \to R_{\bar{g}}$ in the $C^0$-topology on $W$ (as $\delta \to 0$),
(iii) $\tilde{g}_{\delta}$ conformally equivalent to the metric $(g - 2rA_{\bar{g}}) + dr^2$ on $U_\varepsilon(M, \bar{g})$,
(iv) $\tilde{g}_{\delta} \equiv \bar{g}$ on $W \setminus U_\delta(M, \bar{g})$.

**Proof.** First, we note that the exponential map $\exp : T^1M \to W$ sends $(x, r \cdot \nu) \in T^1M$ to $\exp_x(r \cdot \nu) = (x, r) \in W$. On $M$ we have

$$
\bar{g}_{i0} = \bar{g}(\partial_r, \partial_r) = 1, \quad \bar{g}_{i0} = \partial_r \bar{g}(\partial_r, \partial_r) = 2\bar{g}(\nabla_{\partial_r} \partial_r, \partial_r) \equiv 0,
$$

$$
\bar{g}_{ii} = \bar{g}(\partial_r, \partial_i) = 0, \quad \bar{g}_{ii} = \partial_r \bar{g}(\partial_r, \partial_i) = \bar{g}(\nabla_{\partial_r} \partial_r, \partial_i) + \bar{g}(\partial_r, \nabla_{\partial_r} \partial_i) = 0,
$$

$$
\bar{g}_{ij} = \bar{g}(\partial_i, \partial_j) = g_{ij}, \quad \bar{g}_{ij} = \bar{g}(\partial_r, \partial_i) = g(\nabla_{\partial_r} \partial_i) + \bar{g}(\partial_r, \nabla_{\partial_r} \partial_i) = -2A_{ij}.
$$

Here we used that

$$
\nabla_{\partial_r} \partial_r = 0, \quad \nabla_{\partial_r} \partial_i = -A^k_i \partial_k,
$$

which implies

$$
\bar{g}(\partial_r, \nabla_{\partial_r} \partial_i) = \bar{g}(\partial_r, -A^k_i \partial_k) = 0, \quad \text{and}
$$

$$
\bar{g}(\nabla_{\partial_r} \partial_i, \partial_j) + \bar{g}(\partial_i, \nabla_{\partial_r} \partial_j) = \bar{g}(\partial_i, -A^k_j \partial_k) + \bar{g}(\partial_i, A^k_j \partial_k) = -2A_{ij}.
$$

We define new metrics $\dot{g}$ and $G$ near $M$ and compare them with the metric $\bar{g}$:

$$
\dot{g}(x, r) = (g_{ij}(x) - 2rA_{ij}(x) + O(r^2))dx^i dx^j + O(r^2)dr dx^i + dr^2,
$$

$$
G(x, r) := (g_{ij}(x) - 2rA_{ij}(x))dx^i dx^j + dr^2,
$$

$$
\dot{g}(x, r) := g_{ij}(x)dx^i dx^j + dr^2.
$$

Clearly $j^1_M \bar{g} = j^1_M \dot{g}$, and, in general, $j^1_M \bar{g} \neq j^1_M G$. We notice

$$
R_{\dot{g}}|_M = R_G + 2\text{Ric}_{\dot{g}}(\nu, \nu) + |A_{\dot{g}}|^2 - H^2_{\dot{g}},
$$

$$
= R_G + 2\text{Ric}_{\dot{g}}(\nu, \nu) + |A_{\dot{g}}|^2 - H^2_{\dot{g}}.
$$

We define a metric

$$
\dot{g} = (\dot{g}_{ij}) := (g_{ij}(x) - 2rA_{ij}(x))
$$
on each hypersurface $M \times \{r\} \subset W$ (for small $r$). Then we have

$$R_g = R_g + \frac{3}{4} |\partial_r \tilde{g}_{ij}|^2 - \tilde{g}^{ij} \cdot \partial^2_r \tilde{g}_{ij} - \frac{1}{4} |\tilde{g}_{ij} \cdot \partial_r \tilde{g}_{ij}|^2$$

$$= R_g + 3|A_{\bar{g}}|^2 - H_{\bar{g}}^2 + O(r) \text{ near } M, \text{ and}$$

$$R_{\tilde{g}} = R_g + 3|A_{\tilde{g}}|^2 - H_{\tilde{g}}^2 \text{ on } M.$$ 

We choose the conformal metric $\tilde{g}(x, r) = u(x, r)^{\frac{n-4}{2}} \cdot \hat{g}$ such that $\hat{g}_{ij}^1 \tilde{g} = j^1_M \tilde{g}$ by giving $u$ the boundary conditions:

$$u(x, 0) \equiv 1, \quad \partial_r u(x, 0) \equiv 0 \text{ on } M.$$ 

We have

$$-\frac{4(n-1)}{n-2} \Delta_{\tilde{g}} u + R_{\tilde{g}} u = R_{\tilde{g}} u^{\frac{n+2}{2}} - R_{\tilde{g}} u,$$

$$\Delta_{\tilde{g}} u = -\frac{n-2}{4(n-1)} \left(R_{\tilde{g}} u^{\frac{n+2}{2}} - R_{\tilde{g}} u\right).$$

We specify $\Delta_{\tilde{g}} u$ on $M$:

$$\Delta_{\tilde{g}} u = \nabla^\alpha \partial_\alpha u = \tilde{g}^{\alpha\beta} \left(\partial_\alpha \partial_\beta u - \hat{\Gamma}^{\gamma}_{\alpha\beta} \partial_\gamma u\right)$$

$$= \partial^2_r u + g^{ij} \partial_i \partial_j u - \hat{\Gamma}^0_{00} \partial_r u - \hat{\Gamma}^{ij}_{00} \partial_i u - g^{ij} \left(\hat{\Gamma}^0_j \partial_r u - \hat{\Gamma}^k_j \partial_k u\right) = \partial^2_r u$$

since $\partial_i \partial_j u = 0$, $\partial_r u = 0$, and $\partial_i u = 0$ on $M$. Here we use the $u(x, 0) \equiv 1, \partial_r u(x, 0) \equiv 0$ on $M$. Thus we obtain that on $M$

$$\partial^2_r u = \Delta_{\tilde{g}} u = -\frac{n-2}{4(n-1)} \left(R_{\tilde{g}} - R_{\tilde{g}}\right)$$

$$= -\frac{n-2}{4(n-1)} \left(R_{\tilde{g}} - (R_g + 3|A_{\tilde{g}}|^2 - H_{\tilde{g}}^2)) \right).$$

We let $u(x, r) := 1 + \frac{1}{2} r^2 \varphi(x)$ near $M$, where

$$\varphi(x) = -\frac{n-2}{4(n-1)} \left(R_{\tilde{g}}|_M - (R_g + 3|A_{\tilde{g}}|^2 - H_{\tilde{g}}^2)\right)$$

$$= -\frac{n-2}{4(n-1)} \left(R_{\tilde{g}}|_M - R_{\tilde{g}}\right).$$

(10)

Then the metric

$$\tilde{g} = u^{\frac{4}{n-2}} \cdot \hat{g} = \left(1 + \frac{1}{2} r^2 \varphi(x)\right)^{\frac{1}{n-2}} \left[(g - 2rA) + dr^2\right]$$
is such that \( j_M^1 \bar{g} = j_M^1 \bar{g} \), and \( R_{\bar{g}} = R_{\bar{g}} \) on \( M \). We use Proposition 4.4 to define a family of metrics \( \bar{g}_\delta \):

\[
\bar{g}_\delta = \bar{g} + w_\delta(r) \cdot (\bar{g} - \bar{g}) \in \mathcal{Riem}_C(W, M).
\]

We also notice that

\[
C = [\bar{g}|_M] = [\bar{g}|_M] = [\bar{g}|_M] = [\bar{g}_\delta|_M], \quad \text{and}
\]

\[
H_{\bar{g}} = 0 \implies H_{\bar{g}} = 0 \implies H_{\bar{g}} = 0 \implies H_{\bar{g}_\delta} = 0
\]

since \( A_{\bar{g}} = A_{\bar{g}} \), \( \partial_r u = 0 \) on \( M \), and \( \bar{g} = \bar{g}_\delta \) near \( M \). Then \( \bar{g} \in \mathcal{Riem}_C^0(W, M) \) implies that \( \bar{g}_\delta \in \mathcal{Riem}_C^0(W, M) \).

4.2. The approximation trick under minimal boundary condition. One notices that the above results do not allow to use a metric which is conformally equivalent to a product metric near the boundary to approximate the relative Yamabe constant \( Y_C(W, M; C) \). This is the problem which we address and solve here.

**Theorem 4.6.** (Approximation Trick)

Let \( W \) be a manifold with boundary \( \partial W = M \), \( C \in C(M) \). Let \( \bar{g} \in \mathcal{Riem}_C^0(W, M) \) be a metric. Let \( g = \bar{g}|_M \), and \( A_{\bar{g}} \) be the second fundamental form of \( M = \partial W \). There exists a family of metrics \( \bar{g}_\delta \in \mathcal{Riem}_C^0(W, M) \) such that

(i) \( \bar{g}_\delta \to \bar{g} \) in the \( C^0 \)-topology on \( W \) (as \( \delta \to 0 \)),

(ii) \( R_{\bar{g}_\delta} \to R_{\bar{g}} \) in the \( C^0 \)-topology on \( W \) (as \( \delta \to 0 \)),

(iii) \( \bar{g}_\delta \) conformally equivalent to the metric \( g + dr^2 \) on \( U_{\varepsilon(\delta)}(M, \bar{g}) \),

(iv) \( \bar{g}_\delta \equiv \bar{g} \) on \( W \setminus U_{\delta}(M, \bar{g}) \).

**Remark.** In order to control the scalar curvature without the minimal boundary condition, one needs the \( C^1 \)-convergence of metrics as in Proposition 4.5. Furthermore, when \( \bar{g} \) is not totally umbilic on \( M \), the metric \( \bar{g} \) can never be approximated in the \( C^1 \)-topology to a metric which conformally is a product metric near the boundary. However, we emphasize that the convergence in (i) of Theorem 4.6 is the \( C^0 \)-convergence only. The minimal boundary condition plays a crucial role to achieve the \( C^0 \)-convergence for scalar curvatures in (ii).

**Proof.** There are two steps in the proof.

**Step 1.** First, Proposition 4.5 allows us to assume that the metric \( \bar{g} \) is such that

\[
\bar{g} = \left( 1 + \frac{r^2}{2} \varphi(x) \right)^{\frac{n-2}{2}} \left[ (g(x) - 2rA_{\bar{g}}(x)) + dr^2 \right] \quad \text{on a collar } U_{\delta_0}(M, \bar{g}),
\]

where \( \{x, r\} = \{x^1, \ldots, x^{n-1}, r\} \) denotes a Fermi coordinate system near each point of \( M \), and \( \varphi(x) \) is the \( C^\infty \)-function on \( M \) defined by (10).

For each positive \( \delta < \delta_0 \), let \( G_\delta \in \mathcal{Riem}_C^0(W, M) \) be a metric defined by

\[
G_\delta(x, r) = \bar{g}(x, r) + w_\delta(r) \cdot (G(x, r) - \bar{g}(x, r)) = g(x) - 2r(1 - w_\delta(r)) \cdot A_{\bar{g}} + dr^2.
\]
Here $\tilde{g}(x, r)$ and $G(x, r)$ are given by
\[
\begin{align*}
\tilde{g}(x, r) &= (g(x) - 2rA_g(x)) + dr^2, \\
G(x, r) &= g(x) + dr^2
\end{align*}
\]
\[
\text{on } U_\delta(M, \bar{g}).
\]
We also let $\tilde{g}_\delta(x, r) = g(x) - 2r(1 - w_\delta(r)) \cdot A_g(x)$ on $U_\delta(M, \bar{g})$. It follows then from Lemma 4.2 that near $M$ the scalar curvature of the metric $G_\delta$ satisfies
\[
R_{G_\delta} = R_g + 3(1 - w_\delta(t))^2|A_g|^2_g - (1 - w_\delta(t))^2H_g^2 - (4\dot{w}_\delta(r) + 2r \cdot \ddot{w}_\delta(r))H_g + O(\delta).
\]
\[R_{G_\delta} = R_g + 3(1 - w_\delta(r))^2|A_g|^2_{\bar{g}} + O(\delta) \quad \text{near } M.
\]
\textbf{Step 2.} We define now the metric $\tilde{g}_\delta \in \mathcal{R}iem^0_C(W, M)$ as follows:
\[
\tilde{g}_\delta(x, r) = \left(1 + \frac{r^2}{2}\varphi_\delta(x, r)\right)^{-\frac{4}{n-2}} \cdot G_\delta(x, r)
\]
on $U_\delta(M, \bar{g})$, with
\[
\varphi_\delta(x, r) = \varphi(x) - \frac{3(n - 2)}{4(n - 1)}(2 - w_\delta(r))w_\delta(r)|A_g|^2_g.
\]
We obtain that the assertions (iii) and (iv) hold since $G_\delta = g + dr^2$ on the collar $U_\varepsilon(\delta)(M, \bar{g})$, and $G_\delta = \tilde{g}$, and $\varphi_\delta = \varphi$ outside of the collar $U_\delta(M, \bar{g})$. By construction
\[
G_\delta \rightarrow \tilde{g}, \quad \text{and} \quad \frac{r^2}{2}\varphi_\delta(x, r) \rightarrow \frac{r^2}{2}\varphi(x)
\]
in the $C^0$-topology on $W$ as $\delta \rightarrow 0$. Thus the assertion (i) holds. Finally, the scalar curvature $R_{\tilde{g}_\delta}$ is given by
\[
R_{\tilde{g}_\delta} = (1 + \frac{r^2}{2}\varphi_\delta(x, r))^{-\frac{4n-2}{n-4}} \left[-\frac{4(n-2)}{n-1} \Delta_{G_\delta}(1 + \frac{r^2}{2}\varphi_\delta(x, r)) + R_{G_\delta}(1 + \frac{r^2}{2}\varphi_\delta(x, r))\right]
\]
\[
= (1 + O(\delta^2)) \left[-\frac{4(n-2)}{n-1} \varphi + 3(2 - w_\delta(r)) \cdot w_\delta(r) \cdot |A_g|^2_g + R_{G_\delta} + O(\delta)\right]
\]
\[
= R_{G_\delta} + (R_g - R_g - 3|A_g|^2_g) + 3(2 - w_\delta(r)) \cdot w_\delta(r) \cdot |A_g|^2_g + O(\delta)
\]
\[
= R_g + 3(1 - w_\delta(r))^2|A_g|^2_g + (R_g - R_g - 3|A_g|^2_g) + 3(2 - w_\delta(r))w_\delta(r)|A_g|^2_g + O(\delta)
\]
\[
= R_g + O(\delta) \quad \text{on } W \text{ as } \delta \rightarrow 0.
\]
This implies the assertion (ii) and completes the proof of Theorem 4.6. \hfill \Box
5 Gluing Theorems

5.1. Setting. Here we would like to analyze the gluing procedure for manifolds equipped with conformal structures. Let $W_1, W_2$ be two compact manifolds, $\dim W_1 = \dim W_2 \geq 3$, with boundaries

$$\partial W_1 = M_1 = M_0 \sqcup M, \quad \text{and} \quad \partial W_2 = M_2 = M_0 \sqcup M'$$

endowed with conformal classes $C_1 = C_0 \sqcup C \in \mathcal{C}(M_1), C_2 = C_0 \sqcup C' \in \mathcal{C}(M_2)$, where $C_0 \in \mathcal{C}(M_0), C \in \mathcal{C}(M), C' \in \mathcal{C}(M')$. Let $W = W_1 \cup_{M_0} (-W_2)$ be the boundary connected sum of $W_1$ and $W_2$ along $M_0$ (see Fig. 5.1).

**Remark.** The boundary of the manifold $W$ is $\partial W = M \sqcup M'$ with appropriate orientation. We consider both cases when $\partial W = \emptyset$ and $\partial W \neq \emptyset$.

Recall that a conformal class $C \in \mathcal{C}(M)$ is positive if $Y_C(M) > 0$.

![Fig. 5.1. Manifold $W = W_1 \cup_{M_0} (-W_2)$](image)

**Theorem 5.1.** Let $C_0 \in \mathcal{C}(M_0)$ be a positive conformal class, and $Y(W_j, M_j; C_j) > 0$ for $j = 1, 2$. Then $Y(W, \partial W; C \sqcup C') > 0$.

**Remark.** We do not assume that the conformal classes $C \in \mathcal{C}(M), C' \in \mathcal{C}(M')$ are positive.

5.2. Proof of Theorem 5.1. There are four steps in the proof.

**Step 1.** First we notice that since $C_0 \in \mathcal{C}(M_0)$ is a positive conformal class, there exists a metric $h \in C_0$ on $M_0$ with $R_h > 0$. The metric $h$ do not have to be a Yamabe metric. We fix the metric $h$. The condition $Y(W_j, M_j; C_j) > 0$ (for $j = 1, 2$) implies that there exist conformal classes $\bar{C}_j$ on $W_j$ so that $\partial \bar{C}_j = C_j$, i.e. $(\bar{C}_j, C_j) \in \mathcal{C}(W_j, M_j)$. We denote

$$Y_{\bar{C}_j} = Y_{\bar{C}_j}(W_j, M_j; C_j) > 0, \quad j = 1, 2.$$ 

Let $\bar{g}_j \in \bar{C}_j$ be such that $\bar{g}_j|_{M_0} = h$. Moreover, we may assume that $\bar{g}_j \in \bar{C}_j^0$ (i.e. that $H_{\bar{g}_j} \equiv 0$ on $M_j$).
Remarks. (1) The metrics $\bar{g}_j \in \bar{C}^0_j$ do not have to be relative Yamabe metrics, moreover, their scalar curvature $R_{\bar{g}_j}$ is not positive, in general.

(2) The union $\bar{C}^0_1 \cup C^0_2$ does not make sense as a conformal class on $W$ since this union, in general, fails to be smooth along $M_0$.

Step 2. Theorem 4.6 and (11), (12) imply that for any $\varepsilon > 0$ there exist conformal classes $\hat{C}_j$ on $W_j$ and metrics $\hat{g}_j \in \hat{C}_j \ (j = 1, 2)$ such that

\[
\begin{align*}
\partial \hat{C}_j &= C_j, \\
\hat{g}_j &\sim \bar{g}_j, \\
R_{\hat{g}_j} &\sim R_{\bar{g}_j},
\end{align*}
\]

$C^0$-close on $W_j$, which implies $|Y_{\hat{C}_j} - Y_{\bar{C}_j}| < \varepsilon$.

\[
\hat{g}_j = \left(1 + \frac{r^2}{2} f_j \right)^{\frac{n-2}{n-4}} \cdot (h + dr^2) \text{ near } M_j \text{ in } W_j.
\]

Here the function $f_j$ is defined by

\[
f_j = -\frac{n-2}{4(n-1)} \left( R_{g_j}|_{M_0} - R_h + 3|A_{g_j}|_h^2 \right) \text{ on } M_0 \text{ in each } W_j.
\]

From now on we only need the conditions $Y_{\hat{C}_j} > 0 \ (j = 1, 2)$. Therefore we may assume that $f_j < 0$ on $M_0$ since the relative Yamabe constant $Y_{\hat{C}_j}$ is invariant under pointwise conformal change.

![Fig. 5.2. Manifold $X = W_1 \cup_{M_0} (M_0 \times [0, \ell]) \cup_{M_0} (-W_2)$.](image)

Let $\ell$ be a positive constant. We define the manifold $X$ which is diffeomorphic to $W$ as follows (see Fig. 5.2):

\[
X = W_1 \cup_{M_0} (M_0 \times [0, \ell]) \cup_{M_0} (-W_2)
\]

Now we need the cut-off function $w_\delta$ defined in Lemma 4.2. Then for each $\delta$, $0 < \delta < \ell$, we define a metric $\tilde{g}$ on $X$ as follows.

\[
\tilde{g} = \begin{cases} 
\hat{g}_1 & \text{on } W_1, \\
\hat{g}_2 & \text{on } W_2, \\
(1 + \frac{r^2}{2} w_\delta(r) f_1)^{\frac{n-4}{n-2}} (h + dr^2) & \text{on } M_0 \times [0, \delta], \\
h + dr^2 & \text{on } M_0 \times [\delta, \ell - \delta], \\
(1 + \frac{(\ell-r)^2}{2} w_\delta(\ell - r) f_2)^{\frac{n-4}{n-2}} (h + dr^2) & \text{on } M_0 \times [\ell - \delta, \ell].
\end{cases}
\]

Clearly $\tilde{g}$ is a $C^\infty$-metric on $X \cong W$. Let $\tilde{C} = [\tilde{g}] \in \mathcal{C}(W)$.

Remark. The metric $\bar{g}$ does not have, in general, positive scalar curvature.
Step 3. Let $j = 1, 2$. Denote $\nu_j$ the first eigenvalue of the Yamabe operator on $W_j$ for the Neumann boundary condition. Then

$$\nu_j = \inf_{u \in C^\infty(W_j)} \frac{\int_{W_j} \left( \frac{4(n-1)}{n-2} |du|^2_{\tilde{g}} + R_{\tilde{g}} u^2 \right) d\sigma_{\tilde{g}}}{\int_{W_j} u^2 d\sigma_{\tilde{g}}}.$$ 

The relative Yamabe constants $Y_{\tilde{C}_j} > 0$ since $Y_{\tilde{C}_j} > Y_{\tilde{C}_j} - \varepsilon$, $j = 1, 2$. Thus it follows that $\nu_j > 0$. Notice that the conditions $R_h > 0$ on $M$ and $f_j < 0$ ($j = 1, 2$) imply that $R_{\tilde{g}} > 0$ on the cylinder $M_0 \times [0, \ell]$ for small $\delta > 0$.

Let $\nu_{cyl}$ be the first eigenvalue of the Yamabe operator on $M_0 \times [0, \ell]$ for the Neumann boundary condition. We have

$$\nu_{cyl} = \inf_{u \in C^\infty(M_0 \times [0, \ell])} \frac{\int_{M_0 \times [0, \ell]} \left( \frac{4(n-1)}{n-2} |du|^2_{\tilde{g}} + R_{\tilde{g}} u^2 \right) d\sigma_{\tilde{g}}}{\int_{M_0 \times [0, \ell]} u^2 d\sigma_{\tilde{g}}}.$$ 

It follows that $\nu_{cyl} > 0$ since $R_{\tilde{g}} > 0$.

Step 4. Let $\nu$ be the first eigenvalue of the Yamabe operator on $X \cong W$ for the Neumann boundary condition, which is equal to

$$\nu = \inf_{u \in C^\infty(X)} \frac{\int_X \left( \frac{4(n-1)}{n-2} |du|^2_{\tilde{g}} + R_{\tilde{g}} u^2 \right) d\sigma_{\tilde{g}}}{\int_X u^2 d\sigma_{\tilde{g}}}.$$ 

We conclude that $\nu \geq \min \{\nu_1, \nu_2, \nu_{cyl}\} > 0$ by [3, pp. 18-19]. The condition $\nu > 0$ is equivalent that there exists a metric $\tilde{g} \in \mathcal{C}$ such that $R_{\tilde{g}} > 0$ on $X \cong W$ and $H_{\tilde{g}} = 0$ on $\partial X = \partial W$. Thus $Y_{\tilde{g}}(X, \partial X; C \cup C') > 0$, and this implies $Y(W, M \sqcup M'; C \cup C') > 0$. □

Remark. We emphasize that we can choose $\ell > 0$ to be small by choosing small $\delta > 0$.

5.3. Manifolds with positive Yamabe invariant. Here we would like to show that there are many examples of manifolds with positive relative Yamabe invariant.

We start with a closed compact manifold $N$, $\dim N \geq 3$ with $Y(N) > 0$. We choose a small disk $D^n \subset N$ centered at $x_0 \in N$, then $\partial(N \setminus \text{int}(D^n)) = S^{n-1}$. Let $C_0 \in \mathcal{C}(S^{n-1})$ be the standard conformal class. (In particular, $C_0$ is a positive class.)

Theorem 5.2. Let $N$ be a closed compact manifold, $\dim N \geq 3$ with $Y(N) > 0$. Then $Y(N \setminus \text{int}(D^n), S^{n-1}; C_0) > 0$.

Proof. We use [10, Corollary 3.5] to choose a conformal class $\bar{C} \in \mathcal{C}(N)$ with the Yamabe constant $Y_{\bar{C}}(N) > 0$, and a metric $\bar{g} \in \bar{C}$, such that

- $\bar{g}$ is conformally flat near $x_0 \in N$,
- $R_{\bar{g}} > 0$ on $N$.

Thus (as it was observed, say, by Gromov-Lawson [7]), there exists a metric $\hat{g}$ on the manifold $N \setminus \text{int}(D^n)$ such that

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\[ \partial [\hat{g}] = C_0 \in \mathcal{C}(S^{n-1}), \]
\[ R_{\hat{g}} > 0 \text{ on } N \setminus \text{int}(D^n), \]
\[ \hat{g} = g_{S^{n-1}} + dr^2 \text{ near } S^{n-1} = \partial(N \setminus \text{int}(D^n)), \quad [g_{S^{n-1}}] = C_0. \]

Thus \( Y_{\hat{g}}(N \setminus \text{int}(D^n), S^{n-1}; C_0) > 0 \) and \( Y(N \setminus \text{int}(D^n), S^{n-1}; C_0) > 0. \)

5.4. The double. Let \( W \) be a compact manifold with \( \partial W = M = M_0 \sqcup M_1 \). We define the manifold \( X = W \cup_{M_0} (-W) \), the double of \( W \) along \( M_0 \) (see Fig. 5.3).

Remark. Here the other boundary component \( M_1 \) of \( \partial W \) may be empty or not.

![Fig. 5.3.](image)

Theorem 5.1 immediately implies the following.

Corollary 5.3. Let \( C = C_0 \sqcup C_1 \in \mathcal{C}(\partial W) \) \( (C_0 \in \mathcal{C}(M_0)) \) and \( C_1 \in \mathcal{C}(M_1) \). Let \( Y(W, \partial W; C) > 0 \). Then \( Y(X, M_1 \sqcup (-M_1); C_1 \sqcup C_1) > 0. \)

6 Non-positive Yamabe invariant

6.1. Setting. Let \( W \) be a compact smooth \( n \)-manifold with boundary \( \partial W = M \neq \emptyset \), and \( M = M_0 \sqcup M_1 \). (Here \( M_1 \) may be empty.) Let \( (\hat{C}, C = C_0 \sqcup C_1) \) be a pair of conformal classes on \( (W, M) \), where \( C_0 \in \mathcal{C}(M_0) \), \( C_1 \in \mathcal{C}(M_1) \). Similar to the case of closed manifolds, we first notice the following.

Proposition 6.1. (cf. [10, Lemma 1.6]) Suppose \( Y_{\hat{C}}(W, M; C) \leq 0 \). Then, for any \( \hat{g} \in \hat{C}^0 \),

\[ (\min R_{\hat{g}})\text{Vol}_{\hat{g}}(W)^{2/n} \leq Y_{\hat{C}}(W, M; C) \leq (\max R_{\hat{g}})\text{Vol}_{\hat{g}}(W)^{2/n}. \]

Let \( X = W \cup_{M_0} (-W) \) be the double of \( W \) along \( M_0 \). Then, the boundary of \( X \) is \( \partial X = M_1 \sqcup (-M_1) \). We use Proposition 6.1 and Theorem 4.6 to prove the following result.

Theorem 6.2.

1. If \( Y_C(W, M; C) \leq 0 \), then

\[ 2^{\frac{2}{n}} Y_C(W, M; C) \leq Y(X, M_1 \sqcup (-M_1); C_1 \sqcup C_1). \]

2. If \( Y_C(W, M; C) \leq 0 \), then

\[ 2^{\frac{2}{n}} Y(W, M; C) \leq Y(X, M_1 \sqcup (-M_1); C_1 \sqcup C_1). \]

3. If \( Y(W, M) \leq 0 \), then

\[ 2^{\frac{2}{n}} Y(W, M) \leq Y(X, M_1 \sqcup (-M_1)). \]
Proof. Clearly (1) $\implies$ (2) $\implies$ (3). Thus it is enough to prove (1). Notice that if $Y(X, M_1 \sqcup (-M_1); C_1 \sqcup C_1) > 0$ then (1) holds trivially.

Let $Y(X, M_1 \sqcup (-M_1); C_1 \sqcup C_1) \leq 0$. We choose a conformal class $\bar{C} \in \mathcal{C}(W)$. Then for a generic class $\bar{C}$ there is a relative Yamabe metric $\bar{g} \in \bar{C}^0$. Theorem 4.6 implies that, for any $\varepsilon > 0$, there exists a metric $\hat{g} \in \bar{C}$ such that $g := \hat{g}|_M = \hat{g}|_M$, and

\[
\begin{align*}
\begin{cases}
\bar{g} \sim \hat{g} \text{ on a } C^0\text{-close manifold on } W, \\
|R_{\hat{g}} - R_{\bar{g}}| < \varepsilon \text{ on } W,
\end{cases}
\end{align*}
\]

where $\hat{g} := \bar{g} \cup \hat{g}$ on $X = W \sqcup M_0 (-W)$. The metric $\hat{g}$ is smooth by construction.

Let $C := \bar{C} \sqcup \hat{C} \in \mathcal{C}(X)$. By Proposition 6.1 we have

\[
\begin{align*}
Y_C(X, M_1 \sqcup (-M_1); C_1 \sqcup C_1) &\geq \left( \min_{X} R_{\hat{g}} \right) \text{Vol}_{\hat{g}}(X)^{\frac{2}{n}} \geq 2^{\frac{2}{n}} \left( \min_{W} R_{\bar{g}} \right) \text{Vol}_{\bar{g}}(W)^{\frac{2}{n}} \\
&\geq 2^{\frac{2}{n}} (R_{\bar{g}} - \varepsilon)(\text{Vol}_{\bar{g}}(W)^{\frac{2}{n}} - \varepsilon) \geq 2^{\frac{2}{n}} R_{\bar{g}} \text{Vol}_{\bar{g}}(W)^{\frac{2}{n}} - K\varepsilon \\
&= 2^{\frac{2}{n}} Y_C(W, M; ; C) - K\varepsilon.
\end{align*}
\]

Here the constant $K > 0$ is independent of $\varepsilon$. Thus

\[
Y(X, M_1 \sqcup (-M_1); C_1 \sqcup C_1) \geq 2^{\frac{2}{n}} Y_C(W, M; ; C).
\]

Theorem 4.6 implies that, for any $\varepsilon > 0$, there exists a metric $\hat{g} \in \bar{C}$ such that $g := \hat{g}|_M = \hat{g}|_M$, and

\[
\begin{align*}
\begin{cases}
\bar{g} \sim \hat{g} \text{ on a } C^0\text{-close manifold on } W, \\
|R_{\hat{g}} - R_{\bar{g}}| < \varepsilon \text{ on } W,
\end{cases}
\end{align*}
\]

Proof of $\mathbf{Corollary 6.3.}$

1. If $Y_C(W, M; C) \leq 0$, then $2^{\frac{2}{n}} Y_C(W, M; C) \leq Y(X)$.
2. If $Y(W, M; C) \leq 0$, then $2^{\frac{2}{n}} Y(W, M; C) \leq Y(X)$.
3. If $Y(W, M) \leq 0$, then $2^{\frac{2}{n}} Y(W, M) \leq Y(X)$.

Now let $N$ be a smooth compact closed manifold, $\dim N = n$, and $D^n \subset N$ be an embedded disk. Let $W = N \setminus \text{int}(D^n)$, with $\partial W = S^{n-1}$.

Remark. Let $N$ be an enlargeable manifold (see [8]). Then the manifold $N\#(-N) = W \sqcup S^{n-1} (-W)$ is also enlargeable. Thus $Y(N\#(-N)) \leq 0$.

**Corollary 6.4.** Let $N$ be an enlargeable compact closed manifold. Then

\[
2^{\frac{2}{n}} Y(N \setminus \text{int}(D^n), S^{n-1}) \leq Y(N\#(-N)).
\]

In particular, $Y(N \setminus \text{int}(D^n), S^{n-1}) \leq 0$.

**Examples.** Let $T^n$ be a torus, $H^n$ be a hyperbolic space, and $\Gamma$ be a discrete group acting freely on $H^n$, so that $H^n/\Gamma$ is a compact manifold. Then we have

\[
Y(T^n \setminus \text{int}(D^n), S^{n-1}) \leq 0, \quad Y((H^n/\Gamma) \setminus \text{int}(D^n), S^{n-1}) \leq 0.
\]

Remark. Let $W_j$ be a compact smooth $n$-manifold with boundary $\partial W_j = M_j$, for $j = 1, 2$. Let $(W, M) = (W_1, M_1) \sqcup (W_2, M_2)$ be the disjoint union of $W_1$ and $W_2$. Let $C = C_1 \sqcup C_2$ be a conformal class on $M_1 \sqcup M_2$. Similar to the case of closed manifolds, we can show that the same inequalities as those of [10, Corollary 1.11] hold for the relative Yamabe invariants $Y(W, M; C)$ and $Y(W_j, M_j; C_j)$ ($j = 1, 2$).
7 Notes on moduli spaces

7.1. Moduli space of positive scalar curvature metrics. Let $M$ be a closed manifold admitting a positive scalar curvature metric. Then one has the space of psc-metrics

$$\mathcal{Riem}^+(M) = \{ g \in \mathcal{Riem}(M) \mid R_g > 0 \}.$$ 

It is known that this space has, in general, many connective components, and that its homotopy groups are nontrivial. For simplicity we assume that $M$ is an oriented manifold. We denote $\text{Diff}_+(M)$ the group of diffeomorphisms preserving the orientation. Then the group $\text{Diff}_+(M)$ naturally acts on the space of metrics by pulling back a metric via a diffeomorphism. Clearly this action preserves the space $\mathcal{Riem}^+(M)$. Then the moduli space of psc-metrics is defined as

$$\mathcal{M}^+(M) = \mathcal{Riem}^+(M)/\text{Diff}_+(M).$$

It is very challenging problem to describe (in some reasonable terms) the topology of the moduli space $\mathcal{M}^+(M)$. We suggest here to give an alternative model of the moduli space of psc-metrics. First, we suggest to start with the space $\mathcal{C}^+(M)$ of positive conformal classes.

Theorem 7.1. Let $M$ be a closed compact manifold with $\dim M \geq 3$. Then the natural projection map $p : \mathcal{Riem}^+(M) \longrightarrow \mathcal{C}^+(M)$ is weak homotopy equivalence.

Proof. We start with the following easy observation.

Lemma 7.2. Let $C \in \mathcal{C}^+(M)$, and $g_0, g_1 \in C$ be psc-metrics. Then $g_0$ and $g_1$ are psc-homotopic, i.e. there exists a smooth family $\{g(t)\}_{t \in [0,1]} \subset C$ of psc-metrics with $g(0) = g_0$, $g(1) = g_1$.

Proof. Indeed, we have that $R_{g_0} > 0$, and $g_1 = u^{\frac{4}{n-2}} g_0$ for $u \in C^\infty_+(M)$ with the scalar curvature

$$R_{g_1} = u^{-\frac{n+2}{n-2}} \left( -\frac{4(n-1)}{(n-2)} \Delta u + R_{g_0} u \right) > 0.$$ 

Then the curve of metrics $g(t) = u(t)^{\frac{4}{n-2}} g_0 \in C$ with $u(t) = ut + (1-t) > 0$ is such that

$$R_{g(t)} = u(t)^{-\frac{n+2}{n-2}} \left( -\frac{4(n-1)}{(n-2)} \Delta u(t) + R_{g_0} u(t) \right)$$

$$= u(t)^{-\frac{n+2}{n-2}} \left( t \left[ -\frac{4(n-1)}{n-2} \Delta u + R_{g_0} u \right] + (1-t)R_{g_0} \right)$$

$$= u(t)^{-\frac{n+2}{n-2}} \left( tR_{g_1} u^{\frac{n+2}{n-2}} + (1-t)R_{g_0} \right) > 0$$

since the functions $R_{g_1} u^{\frac{n+2}{n-2}}$ and $R_{g_0}$ are both positive. \hfill $\square$

Now let $P(C) = \{ g \in C \mid R_g > 0 \}$. Clearly $C \cong C^\infty_+(M)$ is a convex set.

Lemma 7.3. The subset $P(C) \subset C$ is a convex and contractible set.
Proof. First we check that $P(C)$ is convex. Indeed, let $\hat{g} \in C$ be a Yamabe metric, with $R_{\hat{g}} = \text{const.} > 0$. Then for any $g \in P(C)$ there exists a unique function $u \in C^\infty_+(M)$ so that $g = u^{\frac{4}{n-2}}\hat{g}$. Thus we identify $P(C)$ with the following subspace of positive smooth functions

$$P(C) \equiv \left\{ u \in C^\infty_+(M) \mid -\Delta u + \frac{(n-2)}{4(n-1)}R_{\hat{g}}u > 0 \right\}.$$

A homotopy $F_t : P(C) \longrightarrow P(C)$ given by

$$F_t(g) = u^{\frac{4}{n-2}}(t)\hat{g} \quad \text{with} \quad u(t) = ut + (1-t)$$

is well defined by Lemma 7.2. Furthermore, $F_1 = Id$, and $F_0$ sends the set $P(C)$ to a single point $\hat{g} \in P(C)$. Therefore $P(C)$ is convex, and since $P(C)$ is a subspace of the convex space $C^\infty_+(M)$, it is and contractible. \[\square\]

We notice that both spaces $\mathcal{R}_{\text{Riem}^+}(M)$ and $\mathcal{C}^+(M)$ have homotopy types of $CW$-complexes. Thus we can assume (up to homotopy equivalence) that $p : \mathcal{R}_{\text{Riem}^+}(M) \longrightarrow \mathcal{C}^+(M)$ is a fibration. Since $p^{-1}(C)$ is contractible for any conformal class $C$, we obtain that $p$ induces isomorphism in homotopy groups $p_* : \pi_k(\mathcal{R}_{\text{Riem}^+}(M)) \cong \pi_k(\mathcal{C}^+(M))$. \[\square\]

Thus in the homotopy category one does not loose any information by replacing the space $\mathcal{R}_{\text{Riem}^+}(M)$ by the space of positive conformal classes $\mathcal{C}^+(M)$.

The space $\mathcal{C}(M)$ is the orbit space of the action (left multiplication) of the group $C^\infty_+(M)$ on the space of metrics $\mathcal{R}_{\text{Riem}}(M)$. It is convenient to refine this construction (as it is done in [11]) for manifolds with a base point.

Let $x_0 \in M$ be a base point. We consider the following subspace of $C^\infty_+(M)$:

$$C^\infty_{+,x_0}(M) = \left\{ u \in C^\infty_+(M) \mid u(x_0) = 1 \right\}.$$

Then let $\mathcal{C}_{x_0}(M)$ be the orbit space of the induced action of $C^\infty_{+,x_0}(M)$ on $\mathcal{R}_{\text{Riem}}(M)$. Clearly there is a canonical map $p_1 : \mathcal{C}_{x_0}(M) \longrightarrow \mathcal{C}(M)$ which is a homotopy equivalence since $p_1^{-1}(C) \cong \mathbb{R}$. Let

$$\mathcal{C}^+_{x_0}(M) = p_1^{-1}\left(\mathcal{C}^+(M)\right).$$

To construct an appropriate moduli space we assume that $M$ is a connected manifold, and consider the following subgroup of the diffeomorphism group $\text{Diff}^+_+(M)$:

$$\text{Diff}_{x_0,+}(M) = \{ \varphi \in \text{Diff}^+_+(M) \mid \varphi(x_0) = x_0, \quad d\varphi = Id : TM_{x_0} \rightarrow TM_{x_0} \}.$$

The group $\text{Diff}_{x_0,+}(M)$ inherits the action on the spaces $\mathcal{C}(M)$ and $\mathcal{C}_{x_0}(M)$. It is easy to prove the group $\text{Diff}_{x_0,+}(M)$ acts freely on the space $\mathcal{C}_{x_0}(M)$ (perhaps, it is important that $M$ is connected).

Clearly the space $C^+_{x_0}(M)$ of positive conformal classes is invariant under this action. We define the moduli space $\mathcal{M}_{x_0,\text{conf}}^+(M)$ of positive conformal structures as the orbit space...
of the action of $\text{Diff}_{x_0,+}(M)$ on $\mathcal{C}_{x_0}^+(M)$. One obtains the diagram of Serre fiber bundles

\[
\begin{array}{ccc}
\mathcal{C}_{x_0}^+(M) & \xrightarrow{i} & \mathcal{C}_{x_0}(M) \\
\pi^+ & \downarrow & \pi \\
\mathcal{M}_{x_0,\text{conf}}^+(M) & \xrightarrow{i^+} & B\text{Diff}_{x_0,+}(M)
\end{array}
\]

Here $B\text{Diff}_{x_0,+}(M)$ is the classifying space of the group $\text{Diff}_{x_0,+}(M)$ which we identify with the orbit space $\mathcal{C}_{x_0}(M)/\text{Diff}_{x_0,+}(M)$ (since the action is free, and the space $\mathcal{C}_{x_0}(M)$ is contractible). We address the following problem.

**Problem 7.4.** What is the rational homotopy type of the space $\mathcal{M}_{x_0,\text{conf}}^+(M)$?

### 7.2. Conformal isotopy and concordance.

It is well-known that isotopic psc-metrics are concordant, see [7] and [6]. It is still not known if the converse is true; (we quote [12]) “indeed, there is no known method to distinguish between isotopy classes of positive scalar curvature which is not based on distinguishing concordance classes.” We would like to address the “conformal analogue” of this problem.

Let $C_0, C_1 \in \mathcal{C}^+(M)$ be two positive conformal classes. One defines an isotopy of positive conformal classes in the obvious way. We say that the conformal classes $C_0$ and $C_1$ are *conformally concordant* if

\[Y(M \times [0,1], M \times \{0,1\}; C_0 \sqcup C_1) > 0.\]

Theorem 5.1 implies the following result:

**Corollary 7.5.** Conformal concordance is an equivalence relation on $\mathcal{C}^+(M)$.

We would like to spell out the following conjecture:

**Conjecture 7.6.** Let $M$ be a closed compact manifold admitting a psc-metric, $n \geq 5$. If $C_0, C_1 \in \mathcal{C}^+(M)$ are conformally concordant, the the classes $C_0, C_1$ are isotopic in $\mathcal{C}^+(M)$.

### 7.3. Conformal cobordism.

Once we would like to describe the whole world of manifolds equipped with psc-metrics, we are led to a concept of cobordism. Two manifolds $(M_0, g_0)$, $(M_1, g_1)$ with psc-metrics $g_0, g_1$, are said to be psc-cobordant if there exists a manifold $(W, \bar{g})$ with $\partial W = M_0 \sqcup (-M_1)$, and a psc-metric $\bar{g}$, so that:

1. $\bar{g}|_{M_j} = g_j$, $j = 1, 2$,
2. $\bar{g} = g_j + d\tau^2$ near $M_j$.

We emphasize that the metric $\bar{g}$ must be a product metric near the boundary. The psc-cobordisms was used in several papers [2], [6], [9], [13]. For instance, S. Stolz described an adequate psc-cobordism category where given manifold $M$ fits in (see [13]). This category is determined by the fundamental group $\pi_1(M)$ and the first two Stiefel-Whitney classes of $M$. 

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We define the conformal analogue of the psc-cobordism relation by means of the relative Yamabe invariant. Let \((M_0, C_0), (M_1, C_1)\) be two manifolds equipped with positive conformal classes. We call such manifolds positive conformal manifolds. Then \((M_0, C_0), (M_1, C_1)\) are conformally cobordant if there is a manifold \(W\), with \(\partial W = M_0 \sqcup (-M_1)\), and such that the relative Yamabe invariant

\[ Y(W; M_0 \sqcup (-M_1); C_0 \sqcup C_1) > 0. \]

Again, Theorem 5.1 implies the following result:

**Corollary 7.7.** Conformal cobordism is an equivalence relation on the category of positive conformal manifolds.

**Remark.** The definition of the conformal cobordism may be essentially refined in the way suggested by S. Stolz [13]. This leads to the corresponding conformal cobordism groups. We are studying these cobordism groups in another paper.
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