GERSTEN CONJECTURE FOR EQUIVARIANT $K$-THEORY AND APPLICATIONS

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ABSTRACT. For a reductive group scheme $G$ over a regular semi-local ring $A$, we prove the Gersten conjecture for the equivariant $K$-theory. As a consequence, we show that if $F$ is the field of fractions of $A$, then $K^G_0(A) \cong K^G_0(F)$, generalizing the analogous result for a dvr by Serre [11]. We also show the rigidity for the $K$-theory with finite coefficients of a henselian local ring in the equivariant setting. We use this rigidity theorem to compute the equivariant $K$-theory of algebraically closed fields.

1. Introduction

The classical Gersten conjecture in the algebraic $K$-theory has had tremendous amount of applications in the study of algebraic $K$-theory and algebraic cycles on smooth schemes. This conjecture was settled by Quillen [10] for regular semi-local rings which are essentially of finite type over a field. Now let $A$ be a regular semi-local ring and let $F$ denote the field of fractions of $A$. Let $G$ be a connected reductive group scheme over $A$. For any ring extension $A \to B$, let $R_B(G)$ denote the Grothendieck group of the linear representations of $G$ over the base ring $B$. The extension of scalars gives a natural map

\[
j : R_A(G) \to R_F(G)
\]

and one can now ask if this is an isomorphism. This question was asked by Grothendieck when $G$ is the general linear group over a discrete valuation ring or more generally, a dedekind domain, and was affirmatively answered by Serre (cf. [11, Théoreme 5]). One of the motivations behind the lookout for such an isomorphism is that the representation ring of an algebraic group over a field is relatively easier to compute and one can use the above to compute such rings over more general rings and this will have applications in the study of the equivariant $K$-theory of schemes with group actions. In this note, we show that the isomorphism of (1.1) is a direct consequence of the more general equivariant Gersten conjecture which we now state.

Assume that $A$ is a regular semi-local ring which is essentially of finite type over a field $k$. Let $F$ be the field of fractions of $A$. Let $G$ be a connected and reductive affine group scheme over $A$. Recall that such a group scheme is said to be split if there is a maximal torus of $G$ which is defined and split over $A$. For any ring extension $A \to B$, let $K^G(B)$ (resp. $G^G(B)$) denote the spectrum of the $K$-theory of finitely generated projective $B$-modules (resp., finitely generated $B$-modules).

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with an action of the group scheme $G_B$ over $B$. For $i \in \mathbb{Z}$, let $K^G_i(B)$ (resp. $G^G_i(B)$) denote the homotopy groups of the spectrum $K^G(B)$ (resp. $G^G(B)$). For a prime ideal $p$ of $A$, let $k(p)$ denote the residue field of $p$.

**Theorem 1.1.** For a split and connected reductive group scheme $G$ over the regular semi-local ring $A$ as above, and for any $i \in \mathbb{Z}$, there is an exact sequence

\[ 0 \to K^G_i(A) \xrightarrow{i} K^G_i(F) \xrightarrow{di} \prod_{\text{height } p=1} K^G_{i-1}(k(p)) \to \cdots. \]

**Corollary 1.2.** The natural map $j : R_A(G) \to R_F(G)$ of representation rings is an isomorphism. In particular, $R_A(G)$ is noetherian.

**Corollary 1.3.** Let $G$ be a split and connected reductive group scheme over the regular semi-local ring $A$ as above, and let $H$ be a subgroup scheme of $G$ of the same type. Then the natural map of rings $R_A(G) \to R_A(H)$ is finite.

We next turn our attention to the equivariant $K$-theory of henselian local rings with finite coefficients. Let $A$ be a henselian regular local ring over a field $k$, and let $L$ denote the residue field of $A$. It was shown by Gillet and Thomason (cf. [4, Theorem A], see also [12] and [3]) that the algebraic $K$-theory of $A$ with finite coefficients agrees with that of $L$. This was later used by Suslin (cf. [12]) in a crucial way to compute the algebraic $K$-theory of algebraically closed fields, which allowed him to settle a conjecture of Quillen. We prove here a similar rigidity theorem in the equivariant setting.

**Theorem 1.4.** Let $A$ be the strict henselization of a ring which is either a discrete valuation ring or the local ring at a smooth point of a variety over a field $k$, and let $L$ denote the residue field of $A$. Let $G$ be a connected and reductive group scheme over $A$. Then for any positive integer $n$ prime to the characteristic of $L$, the natural map

\[ K^G_x(A, \mathbb{Z}/n) \to K^G_x(L, \mathbb{Z}/n) \]

is an isomorphism.

In the special case when $G$ is an algebraic group over a field $k$ and $A$ is the henselization of a $k$-rational point of a smooth variety over $k$ with a trivial $G$-action, this result was also shown by Ostvaer and Yagunov (cf. [2]) by a different method. We also remark that we can replace the strict henselization in the above theorem by the henselization, if one assumes that $G$ is split.

We finally prove the following equivariant analogue of the main results of Suslin in [13] and [12]. As a consequence, we obtain an explicit computation of the equivariant $K$-theory for reductive groups over all algebraically closed fields.

**Theorem 1.5.** Let $G$ be a split connected and reductive group scheme over $\mathbb{Z}$. Let $L$ be an algebraically closed field of characteristic $p > 0$ and let $A = W(L)$ denote the ring of Witt vectors over $L$. Let $E$ denote the algebraic closure of the quotient field $F$ of $A$. Then for any positive integer $n$ prime to $p$, there is a canonical isomorphism $K^G_x(L, \mathbb{Z}/n) \cong K^G_x(E, \mathbb{Z}/n)$.

**Corollary 1.6.** For a connected reductive group $G$ over an algebraically closed field $k$, and any $n$ prime to the characteristic of $k$, $K^G_x(k, \mathbb{Z}/n)$ is zero (if $i$ is odd) or isomorphic to $R_k(G)/n$ (if $i$ is even).
2. Some Preliminaries

We give some preparatory results in this section in order to prove our equivariant Gersten conjecture. We fix a regular semi-local ring $A$ which is essentially of finite type over a field $k$. Let $F$ denote the field of fractions of $A$. Put $S = \text{Spec}(A)$. Let $G$ be an affine group scheme over $A$ and let $G \to S$ be the structure map.

Recall from [2] that the group scheme $G$ is said to be reductive if all the geometric fibers of $\pi$ are connected and reductive algebraic groups. In particular, $G$ is smooth over $S$ if and only if the order of the torsion subgroup of $M$ is prime to all the residue characteristics of $A$. The group scheme $G$ is called a Torus if it is isomorphic to $D_S(M)$ in the fpqc topology on $S$, where $M$ is torsion-free. A closed subgroup scheme $T$ of $G$ is called a maximal torus if it is a torus and every geometric fiber of $T$ is a maximal torus of the corresponding fiber of $G$.

We shall say that $G$ is split (déployé) over $S$ if it has a maximal torus $T$ which is of the form $D_S(M)$ for a torsion-free abelian group $M$ such that $G$ is defined by the root datum $(G, T, M, R)$ over $S$. Here $R$ is the set of constant functions from $S$ to $M$ (cf. [2, Chapter XXII]). We also recall that a subgroup scheme $B$ of $G$ is called a Borel if every geometric fiber of $B$ is a Borel subgroup of the corresponding fiber of $G$. Such a Borel subgroup scheme is called split if its unipotent radical is split. It is known that a reductive group over a field is split in the above sense if it contains a split maximal torus. But this is false over a general base. However, the following result will show that this is indeed the case over semi-local rings.

**Proposition 2.1.** Let $G \to S$ be a reductive group scheme as above. Assume that $G$ contains a split maximal torus over $S$. Then $G$ is split over $S$. In other words, it is given by a root system.

*Proof.* Cf. [2, Chapter XXII, Proposition 2.2]. □

**Corollary 2.2.** Assume that $A$ is a henselian local ring and $G \to S$ is a reductive group scheme. Then there is a finite Galois extension $S' \to S$ such that $G_{S'}$ is split over $S'$. In particular, every reductive group scheme over a strictly henselian local ring is split in the above sense.

*Proof.* Since $G$ is reductive, it has a maximal torus $T$ in the étale topology on $S$ by [2, XXII, Théoreme 1.7]. Since $A$ is henselian, this $T$ is split over a finite Galois extension of $S$ by [1, X, Corollary 4.6]. Hence $G$ is given by a root system over a finite Galois extension of $S$ by Proposition 2.1. In particular, if $A$ is strictly henselian, such a group scheme must be split over $S$ itself. □

All the affine group schemes in sight will be assumed to be connected and reductive although most of the results of this paper hold for all smooth and affine group schemes if the base field is of characteristic zero.

Let $G \to S$ be a reductive group scheme as before. The coordinate ring $A[G]$ of $G$ is a Hopf algebra over $A$ such that $G$ is the spectrum of $A[G]$. In this case, the category of $G$-equivariant finitely generated $A$-modules with the trivial action of $G$ on $S$ is same as the category of finitely generated $A[G]$-comodules...
and is an abelian category (cf. [11]). Recall here that a finitely generated \(A[G]\)-comodule means an \(A[G]\)-comodule which is finitely generated as an \(A\)-module. In the same way, the exact category of \(G\)-equivariant vector bundles over \(S\) with the trivial \(G\)-action on \(S\) is same as the category of finitely generated \(A[G]\)-comodules which are projective (and hence free) as \(A\)-modules. We refer to loc. cit. for the further details on this equivalence. Let \(G^G(A)\) denote the spectrum of the \(K\)-theory of the abelian category of finitely generated \(A[G]\)-comodules, and let \(K^G(A)\) denote the \(K\)-theory spectrum of the exact category of finitely generated \(A[G]\)-comodules which are projective over \(A\). Then there is a natural map of spectra \(K^G(A) \to G^G(A)\). We shall need the following result repeatedly in this paper.

**Proposition 2.3.** Let \(A\) be a regular semi-local ring (or a field) and let \(G\) be either a split reductive group or a torus over \(S = \text{Spec}(A)\). Let \(X\) be a smooth quasi-projective \(S\)-scheme with a \(G\)-action. Then the map \(K^G(X) \to G^G(X)\) is a weak equivalence. In particular, if \(G\) is any reductive group over a strictly henselian regular local ring, then the map \(K^G(A) \to G^G(A)\) is a weak equivalence.

**Proof.** Since \(A\) is regular and \(G\) is split reductive or torus, every finitely generated \(G\)-equivariant \(A\)-module has a \(G\)-equivariant resolution by \(G\)-equivariant vector bundles, as shown in [14, Corollary 2.9]. In particular, \((G, S, S)\) has the resolution property in the notation of [14]. Now since \(X\) is smooth and quasi-projective over \(S\), we conclude from [loc. cit., Lemma 2.10] that every \(G\)-equivariant coherent sheaf on \(X\) has a finite \(G\)-equivariant resolution by \(G\)-equivariant vector bundles. In particular, the map \(K^G(X) \to G^G(X)\) is a weak equivalence. The case of strictly henselian ring now follows from this and Corollary 2.2. \(\Box\)

Let \(\mathcal{M}^G(A)\) (or \(\mathcal{M}^G(S)\)) denote the abelian category of \(G\)-equivariant finitely generated \(A\)-modules. For \(p \geq 0\), let \(\mathcal{M}^G_p(A)\) denote the Serre subcategory of those \(G\)-equivariant \(A\) modules which are supported on a closed subscheme of codimension at least \(p\) on \(S\). Recall that since \(G\) acts trivially on \(S\), every subscheme of \(S\) is \(G\)-invariant. The following lemma is now elementary.

**Lemma 2.4.** Let \(i : Z \hookrightarrow S\) be a closed subscheme and let \(M\) be a coherent \(\mathcal{O}_Z\)-module such that \(i_* M \in \mathcal{M}^G(S)\). Then \(M \in \mathcal{M}^G(Z)\).

**Proof.** This follows easily from the definition of the group action and the fact that the inverse image of \(Z\) under the action map \(G \times S \to S\) is \(G \times Z\). We skip the details. \(\Box\)

For \(p \geq 0\), let \(S_p\) denote the set of all codimension \(p\) points of \(S\). From the above lemma, we see following Quillen’s techniques (cf. [10]) that there is a finite filtration of \(\mathcal{M}^G(A)\) by Serre subcategories

\[
\mathcal{M}^G(A) = \mathcal{M}^G_0(A) \supset \mathcal{M}^G_1(A) \supset \cdots
\]

such that for each \(p \geq 0\), one has

\[
\frac{\mathcal{M}^G_p(A)}{\mathcal{M}^G_{p+1}(A)} \cong \bigcap_{s \in S_p} \bigcup_n \mathcal{M}^G(\mathcal{O}_{S,s}/m^k_{S,s}).
\]
Moreover, if \( M \in \mathcal{M}^G (\mathcal{O}_{S,s}/\mathfrak{m}_{S,s}^n) \), then each \( \mathfrak{m}_{S,s}^i/\mathfrak{m}_{S,s}^n M \) is in fact a \( G \)-equivariant submodule of \( M \) for \( i \leq n \) and hence there is a finite filtration of \( M \) by such subsheaves such that each graded quotient is a \( k(s) \)-module and hence is in \( \mathcal{M}^G (k(s)) \) by Lemma 2.4. The Devissage theorem (cf. [10, Theorem 4]) now implies that the map \( \mathcal{M}^G (k(s)) \rightarrow K (\mathcal{M}^G (\mathcal{O}_{S,s}/\mathfrak{m}_{S,s}^n)) \) is a weak equivalence and hence we have the weak equivalence

\[
(2.1) \quad K \left( \frac{\mathcal{M}^G_p (A)}{\mathcal{M}^G_{p+1} (A)} \right) \cong \prod_{s \in S_p} K^G (k(s)) \quad \forall \ p \geq 0.
\]

The equivariant version of Quillen localization sequence (cf. [16]) now gives for each \( p \geq 0 \), a fibration sequence

\[
(2.2) \quad K (\mathcal{M}^G_{p+1} (A)) \rightarrow K (\mathcal{M}^G_p (A)) \rightarrow \prod_{s \in S_p} K^G (k(s)).
\]

Thus we have shown the existence of Quillen spectral sequence in the equivariant setting.

**Proposition 2.5.** There is a strongly convergent spectral sequence

\[
E_1^{pq} = \prod_{s \in S_p} K^G_{-p-q} (k(s)) \Rightarrow G^G_{-n} (A).
\]

The following equivariant analogue of [10, Proposition] follows directly from 2.2 and the spectral sequence in Proposition 2.5.

**Corollary 2.6.** The following are equivalent.

(i) For all \( p \geq 0 \), the inclusion \( \mathcal{M}^G_{p+1} (A) \rightarrow \mathcal{M}^G_p (A) \) induces zero map on the \( K \)-groups.

(ii) For all \( i \in \mathbb{Z} \), the sequence

\[
0 \rightarrow K^G_i (A) \xrightarrow{j} K^G_i (F) \rightarrow \prod_{s \in S_1} K^G_{i-1} (k(s)) \rightarrow \cdots
\]

is exact.

3. **Equivariant Gersten Conjecture**

Before we prove our Theorem 1.1 we recall from [6] that since the group scheme \( G \) acts trivially on \( S \), there is a natural exact functor \( \mathcal{M} (S) \rightarrow \mathcal{M}^G (S) \) via the trivial action. Here \( \mathcal{M} (S) \) is the category of coherent \( S \)-modules. This induces a natural map \( G(S) \xrightarrow{f} G^G (S) \) of spectra. Since \( G^G_i (S) \) is an \( R_A (G) \)-module, we get a natural map

\[
G_i (S) \otimes \mathbb{Z} R_A (G) \rightarrow G^G_i (S)
\]

\[\alpha \otimes \rho \mapsto [\rho \cdot f(\alpha)].\]
Proof of Theorem 1.1. We first assume that $G = T$ is a split torus over $S$ and put $T = D_s(M)$, where $M$ is a free abelian group of finite type. In this case, every $T$-equivariant $A$-module $E$ is canonically identified with an $M$-graded $A$-module of finite type (cf. [11, Section 3.4]). In particular, one has a canonical decomposition $E = \bigoplus_{\lambda \in M} E_\lambda$. This shows that there is a canonical equivalence

$$G(S)[M] = \bigoplus_{\lambda \in M} G(S)_\lambda \to G^G(S). \quad (3.2)$$

Since $K_* (G(S)[M]) = G_* (S)[M] = G_* (S) \otimes \mathbb{Z} R A (G)$, we conclude from (3.2) that the map in (3.1) is an isomorphism. Furthermore, we can use Proposition 2.3 to replace these $G$-groups with $K$-groups. By the same reason, we have for any point $s \in S$, a canonical isomorphism $K_i (k(s)) \otimes \mathbb{Z} R k(s)(G) \cong K^G_i (k(s))$. Since $R A (G) \cong \mathbb{Z}[M] \cong R k(s)(G)$, we conclude that the equivariant Gersten sequence in the split torus case is simply the tensor product of the non-equivariant Gersten sequence with $\mathbb{Z}[M]$. Now the non-equivariant Gersten conjecture together with the flatness of $\mathbb{Z}[M]$ as $\mathbb{Z}$-module imply that the equivariant Gersten sequence is exact. This proves the case of split torus.

We now prove the general case. In view of Corollary 2.6, it suffices to show that the map $\mathcal{M}^G_{p+1}(A) \to \mathcal{M}^G_p(A)$ induces zero map on the $K$-groups for all $p \geq 0$. We choose a split maximal torus $T$ inside $G$. Then we get the following commutative diagram.

$$
\begin{array}{ccc}
K_* (\mathcal{M}^G_{p+1}(A)) & \longrightarrow & K_* (\mathcal{M}^T_{p+1}(A)) \\
\downarrow & & \downarrow \\
K_* (\mathcal{M}^G_p(A)) & \longrightarrow & K_* (\mathcal{M}^T_p(A))
\end{array} \quad (3.3)
$$

By the proof of the theorem for the torus case and Corollary 2.6, we see that the right vertical map is zero. Thus we only need to show that for all $p \geq 0$, the restriction map

$$K_* (\mathcal{M}^G_p(A)) \to K_* (\mathcal{M}^T_p(A)) \quad (3.4)$$

is injective.

By Lemma 2.4 and [10, 5.1], one has

$$K_* (\mathcal{M}^G_p(A)) = \lim_{\mathcal{A} \to \mathcal{A}', \text{ codim}_A (\mathcal{A}') \geq p} G^G_* (\mathcal{A}).$$

Since the direct limit is an exact functor, it suffices to show that for any such quotient $\mathcal{A}'$, the natural map

$$G^G_* (\mathcal{A}') \to G^T_* (\mathcal{A}') \quad (3.5)$$

is injective. The proof of the theorem is now completed by Lemma 3.1. \qed
Lemma 3.1. Let $A$ be a noetherian commutative ring such that either it is regular or essentially of finite type over a field. Let $G$ be a connected and split reductive group scheme over $A$ with a split maximal torus $T$. Then the restriction map $G^G_*(A) \to G^T_*(A)$ is split injective.

Proof. Put $S = \text{Spec}(A)$. Since $G$ is split reductive, it is given by a root system and has a split Borel subgroup scheme $B$ containing $T$ and $G/B$ is a projective $S$-scheme (cf. [2, Chapter XXII, Proposition 5.5.1]). We also have maps

$$G^G_*(A) \to G^B_*(A) \xrightarrow{=} G^G_*(G/B) \to G^T_*(A).$$

Here the last map is the push-forward map induced by the proper map $G/B \to S$. Moreover, under our assumption on $A$, we can use [15, Remark 1.9(d) and Theorem 1.10] to see that the second map above is an isomorphism. By the same reason, the map $G^T_*(A) \xrightarrow{=} G^B_*(B/T)$ is also an isomorphism.

Using the projection formula for the smooth and proper map $G/B \to S$, we see that for any $\alpha \in G^G_*(A)$, one has $f_* f^*(\alpha) = [f_* (\mathcal{O}_{G/B})] \cdot \alpha$. On the other hand, one has $f_* (\mathcal{O}_{G/B}) = \mathcal{O}_S$ by [5, 13.2]. This shows that the composite map in (3.6) is identity. On the other hand, the isomorphism $G^T_*(A) \xrightarrow{=} G^B_*(B/T)$ and the homotopy invariance implies that the restriction map $G^B_*(A) \to G^T_*(A)$ is an isomorphism, proving the lemma.

Proof of Corollary 1.2: Since $R_A(G) = K^G_0(A)$ and similarly for $F$, the corollary follows directly by using Theorem 1.4 for $i = 0$ and by noting that $K^G_i(k(s)) = 0$ as $\mathcal{M}^G(k(s))$ is an abelian category. To show that $R_A(G)$ is noetherian, it suffices now to show that $R_F(G)$ is so. Now it follows directly from [11, Théorème 5] that $R_F(G) = R_F(T)^W$, where $T$ is a maximal split Torus of $G$ and $W$ is the Weyl group. Since $R_F(T)$ is a truncated polynomial ring over $\mathbb{Z}$ and hence noetherian, we deduce from [8, Lemma 4.4] that $R_F(G)$ is noetherian too.

Proof of Corollary 1.3: By Corollary 1.2, we can replace $A$ by $F$, and then it is already shown in [7, Proposition 2.1].

4. Rigidity for Equivariant $K$-theory

Let $A$ be the strict henselization of a ring which is either a discrete valuation ring or the local ring of a smooth point of a variety over a field. Let $L$ denote the residue field of $A$. We fix a positive integer $n$ which is prime to the characteristic of $L$. Let $G$ be a connected and reductive group scheme over $A$. By Corollary 2.2, $G$ is split and hence has a split maximal torus $T$. Moreover, we have seen before that $G$ contains a split Borel subgroup scheme $B$ containing $T$. We have seen in the proof of Lemma 3.1 that there are maps

$$G^G_*(A) \to G^B_*(A) \xrightarrow{=} G^G_*(G/B) \to G^T_*(A).$$
such that the composite is identity. Put $X = G/B$ and let $X \xrightarrow{f} S$ be the smooth and proper map as before. Note then that $G$ naturally acts on $X$ by left multiplication and $X$ is a homogeneous $G$-space. Let $i : \text{Spec}(L) \hookrightarrow S$ be the inclusion of the closed point, and let $X_L$ denote the closed fiber of $X$.

**Lemma 4.1.** The diagram

$$
\begin{array}{ccc}
G^*_* (X, \mathbb{Z}/n) & \xrightarrow{i^*} & G^*_* (X_L, \mathbb{Z}/n) \\
\downarrow f_* & & \downarrow f^*_L \\
G^*_* (A, \mathbb{Z}/n) & \xrightarrow{i^*} & G^*_* (L, \mathbb{Z}/n)
\end{array}
$$

is commutative.

**Proof.** We consider the following Cartesian diagram.

$$
\begin{array}{ccc}
X_L & \xrightarrow{i} & X \\
\downarrow f^L & & \downarrow f \\
\text{Spec}(L) & \xrightarrow{i} & S
\end{array}
$$

Now $f$ and $f^L$ are clearly smooth and proper maps. Since $A$ is the strict henselization of the local ring of the smooth point of a $k$-variety, we see that $A$ is a regular local ring. Hence $i$ and $i^*$ are complete intersection maps. In particular, they are of finite tor-dimension and the pull-back maps $i^*$ and $i^*$ are defined. Moreover, since $f$ is smooth, we see that $X$ and $\text{Spec}(L)$ are Tor-independent over $S$. The lemma now follows from the equivariant version of [10, Proposition 2.11] (see also [17, Proof of Theorem 3.2]).

**Proof of Theorem 1.4.** Put $\Lambda = \mathbb{Z}/n$ and write $R_A(G)/n$ by $R_A (G, \Lambda)$. As in the proof of Theorem 1.1 we first prove the case when $G = T$ is a split torus. In this case, we have seen before that $K^*_X (A) \cong K_* (A)[M]$. Using this and the natural short exact sequence

$$
0 \to K^*_X (A)/n \to K^*_X (A, \Lambda) \to \text{Tor}^1_{\mathbb{Z}} (K^*_{X-1} (A), \Lambda) \to 0
$$

(cf. [6, Lemma 6.3]), we see that the natural map $K_1 (A, \Lambda) \otimes_A R_A (G, \Lambda) \to K^*_X (A, \Lambda)$ is an isomorphism of $R_A (G, \Lambda)$-modules. The same conclusion also holds for the $K$-theory of $L$. Now the torus case of the theorem follows from the non-equivariant rigidity (cf. [12] Corollary 2.5, Corollary 3.9) plus the canonical isomorphism $R_A (G, \Lambda) \cong \Lambda[M] \cong R_L (G, \Lambda)$.

We now prove the general case. By Proposition 2.3 we can replace $K$-theory by $G$-theory. Since $A$ is strictly henselian, the group scheme $G$ has a split maximal torus $T$ and a Borel subgroup scheme $B$ containing $T$. We put $X = G/B$ and follow the notations of Lemma 4.1. We consider the following diagram.
The left square is clearly commutative since it is just the pull-back diagram. The right square commutes by Lemma 4.1. We can now combine the exact sequence 4.3 and Lemma 3.1 to see that both the composite horizontal maps in the above diagram are identity. In particular, the left vertical map is a retract of the middle vertical map. On the other hand, we have just shown that the middle vertical map is an isomorphism. We conclude that the left (and the right) vertical map is also an isomorphism. □

We prove Theorem 1.5 along the lines of the Suslin’s proof of a similar result in the non-equivariant setting. As such, we begin with the following.

**Lemma 4.2.** Let $F$ be a henselian discretely valued field with the valuation ring $A$ and residue field $L$ of positive characteristic $p$. Let $G$ be a split reductive group scheme over $A$. For any $i \geq 0$, there is a short exact sequence

$$0 \to K^G_i (L, \Lambda) \to K^G_i (F, \Lambda) \to K^G_{i-1} (L, \Lambda) \to 0.$$ 

**Proof.** By Theorem 1.4, we have isomorphism $K^G_i (A, \Lambda) \cong K^G_i (L, \Lambda)$. We note here that the statement of Theorem 1.4 requires $A$ to strictly henselian. However, as we have remarked before, the strictness was used only to ensure that $G$ is split. Now using the equivariant localization sequence, we only need to show that the map $K^G_i (A, \Lambda) \to K^G_i (F, \Lambda)$ is injective. We can replace $K$-groups by $G$-groups using Proposition 2.3.

If $G$ is a split torus, we have shown the isomorphism $K^G_i (A, \Lambda) \cong K^G_i (F, \Lambda)$. Now the claim follows from Corollary 1.2 and the fact that $G_i (A, \Lambda) \to G_i (F, \Lambda)$ is split injective as shown by Suslin (cf. [12, Corollary 3.11]). If $G$ is split reductive with a split maximal torus $T$, we have seen above that the map $G^G_i (A, \Lambda) \to G^T_i (A, \Lambda)$ is split injective. The lemma now follows from the torus case. □

**Proof of Theorem 1.5:** If $E'/F$ is a finite subextension of $E/F$, then $E'$ is a complete discretely valued field with residue field $L$. Since $G$ is split over $\mathbb{Z}$, it is so over the valuation ring of $E'$. Now we apply Lemma 4.2 to get a short exact sequence

$$0 \to K^G_i (L, \Lambda) \to K^G_i (E', \Lambda) \to K^G_{i-1} (L, \Lambda) \to 0.$$ 

Moreover, the exact sequence 4.3 implies that $K^G_{i-1} (L, \Lambda)$ is of a bounded exponent. Now the proof of Suslin (cf. [12, Proposition 3.12]) goes through verbatim in the equivariant setting which completes the proof of the theorem. □

**Proof of Corollary 1.6:** If $k$ is an algebraically closed field of positive characteristic and if $G$ is a connected reductive group over $k$, then $G$ is split and hence is given by a root system. Chevalley’s theorem then implies that such a group $G$ can be lifted to a split group scheme over $\mathbb{Z}$. Now we can use Theorem 1.5 to
reduce to the case when $k$ is of characteristic zero. In that case, we can use [9] Theorem 2] to assume that $k$ is the field of complex numbers. If $G$ is now a torus, then the corollary follows from the isomorphism $K_i^G (k, \Lambda) \otimes_{\Lambda} R_k (G, \Lambda) \cong K_i^G (k, \Lambda)$ and [12, Corollary 3.13]. If $G$ is any connected reductive group, the result follows again from the corresponding non-equivariant version of Suslin and the fact that $K^T_i^G (k, \mathbb{Z}/n)$ is a retract of $K^T_i (k, \mathbb{Z}/n)$ as shown above.

□

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