Finding all squared integers expressible as the sum of consecutive squared integers using generalized Pell equation solutions with Chebyshev polynomials

Vladimir Pletser

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European Space Research and Technology Centre, ESA-ESTEC P.O. Box 299, NL-2200 AG Noordwijk, The Netherlands; E-mail: Vladimir.Pletser@esa.int

Abstract

Square roots $s$ of sums of $M$ consecutive integer squares starting from $a^2 \geq 1$ are integers if $M \equiv 0, 9, 24$ or $33 \pmod{72}$; or $M \equiv 1, 2$ or $16 \pmod{24}$; or $M \equiv 11 \pmod{12}$ and cannot be integers if $M \equiv 3, 5, 6, 7, 8$ or $10 \pmod{12}$. Finding all solutions with $s$ integer requires to solve a Diophantine quadratic equation in variables $a$ and $s$ with $M$ as a parameter. If $M$ is not a square integer, the Diophantine quadratic equation in variables $a$ and $s$ is transformed into a generalized Pell equation whose form depends on the $M \pmod{4}$ congruent value, and whose solutions, if existing, yield all the solutions in $a$ and $s$ for a given value of $M$. Depending on whether this generalized Pell equation admits one or several fundamental solution(s), there are one or several infinite branches of solutions in $a$ and $s$ that can be written simply in function of Chebyshev polynomials evaluated at the fundamental solutions of the related simple Pell equation. If $M$ is a square integer, it is known that $M \equiv 1 \pmod{24}$ and $M = (6n - 1)^2$ for all integers $n$; then the Diophantine quadratic equation in variables $a$ and $s$ reduces to a simple difference of integer squares which yields a finite number of solutions in $a$ and $s$ to the initial problem.

Keywords: Sum of consecutive squared integers ; Generalized Pell equation ; Chebyshev polynomials

MSC2010 : 11D09 ; 11E25 ; 33D45

1 Introduction

Lucas’ cannonball problem [17, 18] of finding a square number of cannonballs stacked in a square pyramid has only two solutions, 1 and 4900, the later corre-
sponding to the sum of the first 24 squared integers and was proven by several authors [25, 19, 36, 16, 20, 2].

More generally, finding all integers \( s \) equal to the sum of \( M \) consecutive integer squares starting from \( a^2 \geq 1 \) involves solving a single Diophantine quadratic equation in three variables, two independent \((a \text{ and } M)\) and one dependent \((s)\). Philipp [28], extending the previous work of Alfred [1], proved that there are a finite or an infinite number of solutions depending on whether \( M \) is or not a square integer and in the later case, using a form of the generalized Pell equation. Beeckmans [3], after demonstrating eight necessary conditions on \( M \) with a table of values of \( M < 1000 \) and the smallest values of \( a > 0 \), developed a method based on solving generalized Pell equations to provide all solutions. In two previous papers, this Author showed [30] that no solution exists if \( M \) is congruent to 3, 5, 6, 7, 8 or 10 \((\text{mod } 12)\) using Beeckmans necessary conditions, and that integer solutions exist if \( M \) is congruent to 0, 1, 2, 4, 9 or 11 \((\text{mod } 12)\), yielding \( M \) to be congruent to 0, 9, 24 or 33 \((\text{mod } 72)\), and \( M \) to be congruent to 1, 2 or 16 \((\text{mod } 24)\). These are called allowed values. Additional congruence conditions were demonstrated [31] on the allowed values of \( M \) using Beeckmans’ necessary conditions. Furthermore, it was shown also [30] that if \( M \) is a square itself, \( M \) must be congruent to 1 \((\text{mod } 24)\) and \((M - 1)/24\) are all pentagonal numbers, except the first two. The values of \( M \) yielding integer solutions are given in [33].

In this paper, firstly for non-square integer values of \( M \), the Diophantine quadratic equation expressing the sum of consecutive squared integers equaling a squared integer is transformed into a generalized Pell equation for which, depending on its number of fundamental solutions, one or several infinite branch(es) of solutions in \( a \) and \( s \) are found analytically, using Chebyshev polynomials. Secondly, for square values of \( M \), the quadratic equation reduces to a difference of squares for which a finite number of solutions in \( a \) and \( s \) are found analytically.

2 Simple and generalized Pell equations

Pell equations of the general form

\[
X^2 - DY^2 = N \tag{1}
\]

with \( X, Y, N \in \mathbb{Z} \) and squarefree \( D \in \mathbb{Z}^+ \), i.e. \( \sqrt{D} \notin \mathbb{Z}^+ \), have been investigated in various forms since long (see historical accounts in [7, 15, 37, 14]) and are treated in several classical text books (see e.g. [26, 27, 38] and references therein). A simple reminder is given here and further details can be found in the references.

For \( N = 1 \), the simple Pell equation reads classically

\[
X^2 - DY^2 = 1 \tag{2}
\]

which has, beside the trivial solution \((X_t, Y_t) = (1, 0)\), a whole infinite branch
of solutions for $k \in \mathbb{Z}^+$ given by

$$X_k = \frac{(X_1 + \sqrt{D}Y_1)^k + (X_1 - \sqrt{D}Y_1)^k}{2}$$

$$Y_k = \frac{(X_1 + \sqrt{D}Y_1)^k - (X_1 - \sqrt{D}Y_1)^k}{2\sqrt{D}}$$

where $(X_1, Y_1)$ is the fundamental solution to (2), i.e. the smallest integer solution $(X_1 > 1, Y_1 > 0, \in \mathbb{Z}^+)$ different from the trivial solution. Among the five methods listed by Robertson [32] to find the fundamental solution $(X_1, Y_1)$, the classical method introduced by Lagrange [13], based on the continued fraction expansion of the quadratic irrational $\sqrt{D}$, is central to several other methods.

For $N = n^2$ an integer square, the generalized Pell equation (1) admits always integer solutions. The variable change $(X', Y') = \left(\frac{X}{n}, \frac{Y}{n}\right)$ transforms the generalized Pell equation in a simple Pell equation $X'^2 - DY'^2 = 1$ which has integer solutions $(X'_k, Y'_k)$. The integer solutions to the generalized Pell equation can then be found as $(X_k, Y_k) = (nX'_k, nY'_k)$. Note however that not all solutions in $(X, Y)$ may be found in this way (see e.g. [38]).

For the case where $N$ is not an integer square, the generalized Pell equation (1) can have either no solution at all, or one or several fundamental solutions $(X_1, Y_1)$, and all integer solutions, if they exist, can be expressed in function of the fundamental solution(s) $(X_1, Y_1)$. Several authors (see e.g. [13, 4, 26, 24, 32, 7, 21, 22] and references therein) discussed how to find the fundamental solution(s) of the generalized Pell equation, based on Lagrange’s method of continued fractions with various modifications (see e.g. [29]), and further how to find additional solutions from the fundamental solution(s).

Noting now $(x_f, y_f)$ the fundamental solutions of the related simple Pell equation (2), the other solutions $(X_k, Y_k)$ can be found from the fundamental solution(s) $(X_1, Y_1)$ by

$$X_k + \sqrt{D}Y_k = \pm \left(X_1 + \sqrt{D}Y_1\right) \left(x_f + \sqrt{D}y_f\right)^k$$

for a proper choice of sign $\pm$ [32].

It is less known that Chebyshev polynomials can be used to find the additional solutions of the generalized Pell equation once the fundamental solutions $(X_1, Y_1)$ have been found. In fact, Chebyshev polynomials $T_k(x)$ and $U_k(x)$ of the first and second kinds [35, 12] can be defined as solutions of the simple Pell equation

$$T_k(x)^2 - (x^2 - 1) U_{k-1}(x)^2 = 1$$

on a ring $R(x)$ [6, 5]. The following lemma shows how to find the additional solutions of the generalized Pell equation.
Lemma 1. For $X, Y, D, N, k \in \mathbb{Z}^+$ and $D$ not a perfect square (i.e. $\sqrt{D} \notin \mathbb{Z}$), if the generalized Pell equation

$$X^2 - DY^2 = N$$

admits one or several fundamental solution(s) $(X_1, Y_1)$, then it admits one or several infinite branch(es) of solutions and these can be written as

$$X_k = X_1 T_{k-1} (x_f) + D Y_1 y_f U_{k-2} (x_f)$$  
$$Y_k = X_1 y_f U_{k-2} (x_f) + Y_1 T_{k-1} (x_f)$$

in function of the fundamental solution(s) $(X_1, Y_1)$ and of Chebyshev polynomials of the first and second kinds, $T_{k-1} (x_f)$ and $U_{k-2} (x_f)$ evaluated at the fundamental solution $(x_f, y_f)$ of the related simple Pell equation $X^2 - DY^2 = 1$.

Proof. For $X, Y, D, N, k, i \in \mathbb{Z}^+$ and square free $D$, let $(X_1, Y_1)$ be one of the fundamental solutions of (7) if they exist, and let $(x_f, y_f)$ be the fundamental solution of the related simple Pell equation $X^2 - DY^2 = 1$ (i.e. $x_f > 1, y_f > 0$).

(i) Additional solutions $(X_k, Y_k)$ of (7) can then be found by the recurrence relations

$$X_k = x_f X_{k-1} + D y_f Y_{k-1}$$  
$$Y_k = x_f Y_{k-1} + y_f X_{k-1}$$

which can be demonstrated by induction.

For $k = 2$, as $(X_1, Y_1)$ is a fundamental solution of (7), $(X_2, Y_2)$ obtained from (10) and (11) verify also (7) as $x_f^2 - D y_f^2 = 1$.

Let $(X_{k-1}, Y_{k-1})$ be a solution of (7), i.e. $X_{k-1}^2 - D Y_{k-1}^2 = N$. Then multiplying the two terms on the left of this equation by $1 = x_f^2 - D y_f^2$, adding and subtracting $2 D x_f y_f X_{k-1} Y_{k-1}$, rearranging and replacing by (10) and (11) yield $X_k^2 - D Y_k^2 = N$, i.e. $(X_k, Y_k)$ is also a solution of (7).

(ii) Further, to express $X_k$ and $Y_k$ in function of $X_1, Y_1, x_f$ and $y_f$ only, one replaces successively for $3 \leq i \leq k$, $X_{i-1}$ and $Y_{i-1}$ in function of $X_1$ and $Y_1$ in the expressions (10) and (11) of $X_i, Y_i$ (with the substitution $x_f^2 + D y_f^2 = 2x_f^2 - 1$ whenever needed) to obtain successively Chebyshev polynomials of the first and second kinds evaluated at $x_f$ and of increasing indices, respectively $i - 1$ and $i - 2$, i.e. $T_{i-1} (x_f)$ and $U_{i-2} (x_f)$, yielding eventually (8) and (9).

One can verify by induction that (8) and (9) yield all solutions to (7).

As $(X_1, Y_1)$ is a fundamental solution of (7), for $k = 2$, one has $T_1 (x_f) = x_f$ and $U_0 (x_f) = 1$ in (8) and (9), yielding directly (10) and (11).

Further, let us assume that $(X_{k-1}, Y_{k-1})$ with

$$X_{k-1} = X_1 T_{k-2} (x_f) + D Y_1 y_f U_{k-3} (x_f)$$
$$Y_{k-1} = X_1 y_f U_{k-3} (x_f) + Y_1 T_{k-2} (x_f)$$

are a solution of (7); then replacing (12) and (13) in (10) and (11) yield
where $Dy_f^2$ has been replaced by $Dy_f^2 = x_f^2 - 1$ in (14) and (15). As

$$T_{k-1}(x_f) = x_f T_{k-2}(x_f) + (x_f^2 - 1) U_{k-3}(x_f)$$

$$U_{k-2}(x_f) = x_f U_{k-3}(x_f) + T_{k-2}(x_f)$$

(see e.g. [35]), (14) and (15) yield directly (8) and (9). Replacing now (8) and (9) in (7) gives

$$X_k^2 - Dy_k^2 = (X_1^2 - Dy_1^2) \left( (T_{k-1}(x_f))^2 - Dy_f^2 U_{k-2}(x_f)^2 \right) = N$$

by (6) with $Dy_f^2 = x_f^2 - 1$, showing that $(X_k, Y_k)$ (8, 9) also solve (7).

Finally, as $k$ is unbound, there is an infinity of solutions (8) and (9). \qed

### 3 General method to find all solutions

The sum of $M > 1$ consecutive integer squares starting from $a^2 \geq 1$ being equal to an integer square $s^2$ can be written in all generality as [30]

$$\sum_{i=0}^{M-1} (a + i)^2 = M \left[ \left( a + \frac{M-1}{2} \right)^2 + \frac{M^2-1}{12} \right] = s^2$$

(19)

where $M$ are allowed values (see [30, 31]). To find all integer solutions of (19), two cases are considered and treated separately: first, $M$ is not a squared integer, and second, $M$ is a squared integer.

#### 3.1 $M$ not a squared integer

The next theorem allows to find all the solutions to (19) in $a$ and $s$ for allowed values of $M$ not being squared integers.

**Theorem 2.** For $M > 1, \sigma, j, k, a_{k,j}, s_{k,j}, x_f, y_f \in \mathbb{Z}^+, \lambda \in \mathbb{Q}$, for all allowed square free values of $M$ (i.e. $\sqrt{M} \notin \mathbb{Z}$), there is a number $\sigma \geq 1$ of infinite branch(es) of values of $a_{k,j}$, $1 \leq j \leq \sigma$, such that the sums of squares of $M$ consecutive integers starting from $a_{k,j}$ are equal to squared positive integers $s_{k,j}^2$.
and these can be written in function of Chebyshev polynomials of the first and second kinds, $T_{k-1}(x_f)$ and $U_{k-2}(x_f)$ as

$$a_{k,j} = \frac{2\lambda s_{1,j} y_f U_{k-2}(x_f) + (2a_{1,j} + M - 1) T_{k-1}(x_f) - (M - 1)}{2}$$

$$s_{k,j} = s_{1,j} T_{k-1}(x_f) + \frac{\lambda M}{2} y_f (2a_{1,j} + M - 1) U_{k-2}(x_f)$$

with $\lambda = 1$ for $M \equiv 1 (mod\ 2)$ or $M \equiv 2 (mod\ 4)$, and $\lambda = 1/2$ for $M \equiv 0 (mod\ 4)$, and where $(a_{1,j}, s_{1,j})$ are the smallest positive values of $(a_{k,j}, s_{k,j})$ solutions of (19) and $(x_f, y_f)$ is the fundamental solution of the simple Pell equation $X^2 - (X^2 M) Y^2 = 1$.

Proof. For $M > 1, \sigma, j, k, a, s, a_{k,j}, s_{k,j}, x_f, y_f, X, Y, N, D \in \mathbb{Z}^+, \lambda \in \mathbb{Q}$, for the allowed square free values of $M$, rewriting (19) for $M \equiv 1 (mod\ 2)$ as

$$(s^2 - M (a + M - 1)^2 = M (M^2 - 1)$$

or for $M \equiv 0 (mod\ 4)$ as

$$s^2 - \frac{M}{4} (2a + M - 1)^2 = \frac{M (M^2 - 1)}{12}$$

or for $M \equiv 2 (mod\ 4)$ as

$$(2s)^2 - M (2a + M - 1)^2 = \frac{M (M^2 - 1)}{3}$$

transform (19) in generalized Pell equations (1) in $X = s$ or $2s$ and $Y = (a + (M - 1)/2)$ or $(2a + M - 1)$, with $N = M (M^2 - 1)/12$ or $M (M^2 - 1)/3$ and $D = M$ or $M/4$.

If these generalized Pell equations (22) to (24) admit $\sigma$ solution(s), then for $1 \leq j \leq \sigma$,

(i) for $M \equiv 1 (mod\ 2)$, let $(s_{1,j}, (a_{1,j} + (M - 1)/2))$ be the $j^{th}$ fundamental solution of (22) and let $(x_f, y_f)$ be the fundamental solution of the related simple Pell equation $X^2 - MY^2 = 1$, i.e. $x_f > 1$ and $y_f > 0$. Then, (8) and (9) yield

$$a_{k,j} = s_{1,j} y_f U_{k-2}(x_f) + \left(a_{1,j} + \frac{M - 1}{2}\right) T_{k-1}(x_f) - \left(M - 1\right)$$

$$s_{k,j} = s_{1,j} T_{k-1}(x_f) + M y_f \left(a_{1,j} + \frac{M - 1}{2}\right) U_{k-2}(x_f)$$

(ii) for $M \equiv 0 (mod\ 4)$, similarly let $(s_{1,j}, (2a_{1,j} + M - 1))$ be the $j^{th}$ fundamental solution of (23) and let $(x_f, y_f)$ be the fundamental solution of the related simple Pell equation $X^2 - (M/4) Y^2 = 1$. Then, (8) and (9) yield

$$a_{k,j} = \frac{s_{1,j} y_f U_{k-2}(x_f) + (2a_{1,j} + M - 1) T_{k-1}(x_f) - (M - 1)}{2}$$

$$s_{k,j} = s_{1,j} T_{k-1}(x_f) + \frac{M}{4} y_f (2a_{1,j} + M - 1) U_{k-2}(x_f)$$
Table 1: First solutions \((a_{k,j}, s_{k,j})\) for \(M = 11, 1 \leq j \leq 2\) and \(1 \leq k \leq 6\) of the \(\sigma = 2\) infinite branches of solutions of \(s^2 - 11(a + 5)^2 = 110\)

| \(k\) | \(a_{k,1}\) | \(s_{k,1}\) | \(a_{k,2}\) | \(s_{k,2}\) |
|---|---|---|---|---|
| 1 | -4 | 11 | 18 | 77 |
| 2 | 38 | 143 | 456 | 1529 |
| 3 | 854 | 2849 | 9192 | 30503 |
| 4 | 17132 | 56837 | 183474 | 608531 |
| 5 | 341876 | 1133891 | 3660378 | 12140117 |
| 6 | 6820478 | 22620983 | 73024176 | 242193809 |

\([a_{1,1}]:\) solution rejected as \(a_{1,1} \leq 0\)

(iii) for \(M \equiv 2 \pmod{4}\), similarly let \((2s_{1,j}, (2a_{1,j} + M - 1))\) be the \(j^{th}\) fundamental solution of (24) and let \((x_f, y_f)\) be the fundamental solution of the related simple Pell equation \(X^2 - MY^2 = 1\). Then, (8) and (9) yield

\[
a_{k,j} = \frac{2s_{1,j}y_fU_{k-2}(x_f) + (2a_{1,j} + M - 1)T_{k-1}(x_f) - (M - 1)}{2}
\]

\[
s_{k,j} = s_{1,j}T_{k-1}(x_f) + \frac{M}{2}y_f(2a_{1,j} + M - 1)U_{k-2}(x_f)
\]

Finally, as \(k\) is unbound, there is in each case and for each \(1 \leq j \leq \sigma\) an infinity of solutions \((s_{k,j}, a_{k,j})\).

Note that some of the first solutions \(a_{1,j}\) may be rejected if the \(j^{th}\) fundamental solution of (22) (or (23) or (24)) is such that \((a_{1,j} + (M - 1)/2) < (M - 1)/2\), yielding a non-positive value of \(a_{1,j}\).

In the following examples, the method indicated by Matthews [22] based on an algorithm by Frattini [9, 10, 11] using Nagell’s bounds [26, 23] is used to find the fundamental solution(s) of the generalized Pell equation.

A first example for the case \(M \equiv 11 \pmod{12}\), let \(M = 11\). Then, (22) reads \(s^2 - 11(a + 5)^2 = 110\), which has \(\sigma = 2\) fundamental solutions, yielding, with \(1 \leq j \leq 2\), \((s_{1,j}, (a_{1,j} + 5)) = (11,1), (77,23)\) and the fundamental solution of the related simple Pell equation \(X^2 - 11Y^2 = 1\) is \((x_f,y_f) = (10,3)\). Replacing in (25) and (26) yield then the solutions given in Table 1. The first solution \((a_{1,1},s_{1,1})\) is rejected as \(a_{1,1} < 0\). The solutions are then ordered as \(a_{1,2} < a_{2,1} < a_{2,2} < a_{3,1} < \ldots\)

A second example for the case \(M \equiv 0 \pmod{24}\), let \(M = 24\). Then, (23) reads \(s^2 - 6(2a + 23)^2 = 1150\), having \(\sigma = 6\) fundamental solutions, \((s_{1,j}, (2a_{1,j} + 23)) = (34,1), (38,7), (50,15), (70,25), (106,41), (158,63)\) and the fundamental solution of the related simple Pell equation \(X^2 - 6Y^2 = 1\) is \((x_f,y_f) = (5,2)\). Replacing in (27) and (28) yield then the solutions given in Table 2. The first three solutions \((a_{1,j},s_{1,j})\) for \(1 \leq j \leq 3\) are rejected as \(a_{1,j} < 0\). The solutions are then ordered as \(a_{1,4} < a_{1,5} < a_{1,6} < a_{2,1} < a_{2,3} < a_{2,4} < \ldots\). Note that the first solution of the fourth branch \((a_{1,4} = 1, s_{1,4} = 70)\) gives the second solution of Lucas’ cannonball problem.
Table 2: First solutions \((a_{k,j}, s_{k,j})\) for \(M = 24\), \(1 \leq j \leq 6\) and \(1 \leq k \leq 6\) of the \(\sigma = 6\) infinite branches of solutions of \(s^2 - 6(2a + 23)^2 = 1150\)

| \(k\) | \(a_{k,1}\) | \(s_{k,1}\) | \(a_{k,2}\) | \(s_{k,2}\) | \(a_{k,3}\) | \(s_{k,3}\) |
|------|-----------|----------|-----------|----------|-----------|----------|
| 1    | -11       | 34       | -8        | 38       | -4        | 50       |
| 2    | 25        | 182      | 44        | 274      | 76        | 430      |
| 3    | 353       | 1786     | 540       | 2702     | 856       | 4250     |
| 4    | 3597      | 17678    | 5448      | 26746    | 8576      | 42070    |
| 5    | 35709     | 174994   | 54032     | 264758   | 84996     | 416450   |
| 6    | 353585    | 1732262  | 534964    | 2620834  | 841476    | 4122430  |

\([a_{1,j}]\): solutions rejected as \(a_{1,j} \leq 0\) for \(1 \leq j \leq 3\)

A third example for the case \(M \equiv 2 \pmod{24}\), let \(M = 2\). Then, (24) reads \((2s)^2 - 2(2a + 1)^2 = 2\), having \(\sigma = 1\) fundamental solution \((2s_{1,1}, (2a_{1,1} + 1)) = (2, 1)\) and the fundamental solution of the related simple Pell equation \(X^2 - 2Y^2 = 1\) is \((x_f, y_f) = (3, 2)\). Replacing in (29) and (30) yield then the solutions given in Table 3, where the first solution is again to be rejected (it corresponds to the identity relation \(0^2 + 1^2 = 1^2\)) and the second solution is the Pythagorean relation \(3^2 + 4^2 = 5^2\).

Still for the case \(M = 2 \pmod{24}\), let \(M = 842\) which does not yield solutions to (19). Indeed, although the related simple Pell equation \(X^2 - 842Y^2 = 1\) has the fundamental solution \((x_f, y_f) = (1683, 58)\), the generalized Pell equation from (24) \((2s)^2 - 842(2a + 841)^2 = 198982282\) has no fundamental solution.

Table 3: First solutions \((a_{k,1}, s_{k,1})\) for \(M = 2\) and \(1 \leq k \leq 6\) of the single infinite branch of solutions of \((2s)^2 - 2(2a + 1)^2 = 2\)

| \(k\) | \(a_{k,1}\) | \(s_{k,1}\) |
|------|------------|----------|
| 1    | [0]        | [1]      |
| 2    | 3          | 5        |
| 3    | 20         | 29       |
| 4    | 119        | 169      |
| 5    | 696        | 985      |
| 6    | 4059       | 5741     |

\([a_{1,1}]\): solution rejected as \(a_{1,1} \leq 0\)
\( \phi \) is finite, there is a finite number \( F \) or \( M \) where
\[ u = (2N + 1) \]

An integer square, the above method with solutions of the Pell equation
\[ \sigma = 0 \]. This case was already signaled by Beeckmans \cite{3}: the value of \( M = 842 = 24 \times 35 + 2 \), although complying with Beeckmans' conditions does not yield solutions to (19) (see also \cite{31}).

### 3.2 \( M \) is a squared integer

It was demonstrated \cite{30} that, if \( M \) is a square integer, then for the sums of \( M \) consecutive squared integers to equal integer squares, \( M \equiv 1 (\text{mod} \ 24) \) and \( \exists g_n \in \mathbb{Z}^+ \) such that \( M = 24g_n + 1 \) where \( g_n = n(3n-1)/2 \) are all generalized pentagonal numbers \( \forall n \in \mathbb{Z} [8, 39] \), yielding \( M = (6n-1)^2 \), i.e. \( g_n = 0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, 57, \ldots \) \cite{34}, yielding \( M = 1, 25, 49, 121, 169, 289, 361, 529, 625, 841, 961, 1225, 1369, \ldots \) of which the first two \( M = 1, 25 \), should be rejected as \( M > 1 \) and \( \alpha > 0 \) (see further).

For \( M \) an integer square, the above method with solutions of the Pell equation can clearly not be followed as Pell equations are not defined for \( D = M \) being a squared integer. Instead, another method (see e.g. \cite{4} p. 486, and \cite{24}) is used in the following theorem showing how to find the finite number of solutions for \( M \) being a squared integer.

**Theorem 3.** For \( M > 1, \varphi, k, a_k, s_k \in \mathbb{Z}^+ \), \( n \in \mathbb{Z} \), for all allowed squared integer values of \( M = (6n-1)^2 \), there is a finite number \( \varphi \) of values of \( a_k \) such that the sums of squares of \( M \) consecutive integers starting from \( a_k \) are equal to squared positive integers \( s_k^2 \), that can be written as

\[
\begin{align*}
    s_k &= (6n-1) \left( \frac{v_k + u_k}{2} \right) \\
    a_k &= \frac{v_k - u_k}{2} - 6n(3n-1)
\end{align*}
\]

(31)

(32)

where \( u_k \) and \( v_k \) are the factor and co-factor of \( [2n(3n-1)(6n(3n-1) + 1)] \), with \( u_k < v_k \), \( u_k \equiv v_k \equiv 0 (\text{mod} \ 2) \) and \( 1 \leq k \leq \varphi \).

**Proof.** For \( M > 1, \varphi, k, a, s, a_k, s_k \in \mathbb{Z}^+ \), \( n \in \mathbb{Z} \), from (19), \( s \) must be such as \( s \equiv 0 (\text{mod} \ (6n-1)) \). Replacing in (22) yields then

\[
\left( \frac{s}{6n-1} \right)^2 - (a + 6n(3n-1))^2 = 2n(3n-1)(6n(3n-1) + 1)
\]

(33)

i.e. the difference of two integer squares must be an even integer.

One has then to determine all the integer values of \( X_k \) and \( Y_k \) solutions of the equation \( X^2 - Y^2 = N \), with \( X = s/(6n-1), Y = (a + 6n(3n-1)) \) and \( N = 2n(3n-1)(6n(3n-1) + 1) \). For this, let \( N = u_k v_k \) and only both even factor and co-factor \( u_k \) and \( v_k \) are considered as \( N \equiv 0 (\text{mod} \ 4) \) \cite{24}. As \( N \) is finite, there is a finite number \( \varphi \) of ways of decomposing \( N \) in product of two even factors. Then, with \( u_k < v_k \) and \( 1 \leq k \leq \varphi \), \( X_k = (v_k + u_k)/2 \) and \( Y_k = (v_k - u_k)/2 \), yielding \( s_k = (6n-1)(v_k + u_k)/2 \) and \( a_k = ((v_k - u_k)/2) - 6n(3n-1) \).
Table 4: All $\varphi = 12$ solutions for $M = 289$ with $N = u_k v_k = 6960$

| $k$ | $u_k \times v_k$ | $X_k$ | $Y_k$ | $s_k$ | $a_k$ |
|-----|------------------|-------|-------|-------|-------|
| 1   | $60 \times 116$  | 88    | 28    | 1496  | -116  |
| 2   | $58 \times 120$  | 89    | 31    | 1513  | -113  |
| 3   | $40 \times 174$  | 107   | 67    | 1819  | -77   |
| 4   | $30 \times 232$  | 131   | 101   | 2227  | -43   |
| 5   | $24 \times 290$  | 157   | 133   | 2669  | -11   |
| 6   | $20 \times 348$  | 184   | 164   | 3128  | 20    |
| 7   | $12 \times 580$  | 296   | 284   | 5032  | 140   |
| 8   | $10 \times 696$  | 353   | 343   | 6001  | 199   |
| 9   | $8 \times 870$   | 439   | 431   | 7463  | 287   |
| 10  | $6 \times 1160$  | 583   | 577   | 9911  | 433   |
| 11  | $4 \times 1740$  | 872   | 868   | 14824 | 724   |
| 12  | $2 \times 3480$  | 1741  | 1739  | 29597 | 1595  |

$[a_k]$: solutions to be rejected as $a_k < 0$

Note that here also some of the first solutions $a_k$ may be rejected if half the difference of the factor and co factor of $N$ is such that $(v_k - u_k)/2 < 6n(3n − 1)$, yielding a non-positive value of $a_k$.

As a first example, let $M = 25$ with $n = 1$. Then $X = s/5$, $Y = a + 12$ and there is only one way to decompose $N = 52$ in the product of two even integer factors, $N = 52 = 2 \times 26 = u_1 v_1$, yielding then $\varphi = 1$ and there is only one solution, given by $X_1 = 14$ and $Y_1 = 12$, or $s_1 = 70$ and $a_1 = 0$. This case for $M = 25$ must be rejected as it has no solution with $a > 0$. Note however that this solution with $s = 70$ and $a = 0$ for the case $M = 25$ is obviously equivalent to the solution with $s = 70$ and $a = 1$ for the case $M = 24$ of Lucas’ cannonball problem.

A second example, let $M = 289$ with $n = 3$. Then $X = s/17$, $Y = a + 144$ and $N = 6960$. As there are twelve ways to decompose $N = 6960$ in products of two even integer factors, there are $\varphi = 12$ solutions in $X$ and $Y$ given in Table 4, five of which have to be rejected as the corresponding values of $a_k$ are negative.

4 Conclusion

The problem of finding all the integer solutions of the sum of $M$ consecutive integer squares starting at $a^2 \geq 1$ being equal to a squared integer $s^2$ can be written as a Diophantine quadratic equation $M \left( \frac{a + (M - 1)/2}{2} \right)^2 + \frac{(M^2 - 1)/12}{2} = s^2$ in variables $a$ and $s$. Based on previous results, it is known that integer solutions exist only if $M \equiv 0, 9, 24$ or $33 (mod \ 72)$; or $M \equiv 1, 2$ or $16 (mod \ 24)$; or $M \equiv 11 (mod \ 12)$.

If $M$ is different from a square integer, the Diophantine quadratic equation is solved generally by transforming it into a generalized Pell equation whose form depends on the $(mod \ 4)$ congruent value of $M$, and whose solutions, if
existing, yield all the solutions in \(a\) and \(s\) for a given value of \(M\). Depending on whether this generalized Pell equation admits one or several fundamental solution(s), there are one or several infinite branches of solutions in \(a\) and \(s\) that can be written simply in function of Chebyshev polynomials evaluated at the fundamental solutions of the related simple Pell equation.

If \(M\) is a square integer, for \(M \equiv 1 (mod 24)\) and \(M = (6n - 1)^2, \forall n \in \mathbb{Z}\), then the Diophantine quadratic equation in variables \(a\) and \(s\) reduces to a simple difference of integer squares which admits a finite number of solutions, yielding a finite number solutions in \(a\) and \(s\) to the initial problem.

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