A class of fractional $p(\cdot)$-Kirchhoff type systems

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Abstract
This paper is concerned with an elliptic system of Kirchhoff type, driven by the fractional $p(x)$-operator. By means of the direct variational method and Ekeland variational principle, we show the existence of a weak solution for the fractional $p(x)$-Kirchhoff system. This is our first attempt to study the elliptic system with fractional variable exponents. Our main theorem extends previous results in several directions.

Keywords: Elliptic system; Variational method; Fractional $p(\cdot)$-Kirchhoff type; Ekeland variational principle.

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1. Introduction
In this article, we discuss the following fractional $p(\cdot)$-Kirchhoff type system

$$
\begin{align*}
M_1 \left( \int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + p(x,y)s}} \, dx \right) (-\Delta)^s_{p(x)} u(x) &= f(u,v) + a(x) \quad \text{in } \Omega, \\
M_2 \left( \int_{\mathbb{R}^N} \frac{1}{p(x,y)} \frac{|v(x) - v(y)|^{p(x,y)}}{|x - y|^{N + p(x,y)s}} \, dx \right) (-\Delta)^s_{p(x)} v(x) &= g(u,v) + b(x) \quad \text{in } \Omega, \\
u = v = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{align*}
$$

(1.1)

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain with $N > p(x,y)s$ for any $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. Here, the main operator $(-\Delta)^s_{p(x)}$ is the fractional $p(\cdot)$-Laplacian given by

$$
(-\Delta)^s_{p(x)} \varphi(x) = P.V. \int_{\mathbb{R}^N} \frac{|\varphi(x) - \varphi(y)|^{p(x,y)s - 2}(\varphi(x) - \varphi(y))}{|x - y|^{N + p(x,y)s}} \, dy, \quad x \in \mathbb{R}^N,
$$

(1.2)

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along any $\varphi \in C_0^\infty(\mathbb{R}^N)$, where P.V. denotes the Cauchy principle value. It consists with the classical non-local fractional $p$-Laplacian i.e., $(-\Delta)^s_p ((-\Delta)^s)$ in case $p(x) \equiv p(\text{or } p = 2)$. In recent years, more and more attention has been focused on the study of Kirchhoff type system involving fractional operators due to the nonlocal nature of Kirchhoff equation (see [18]). We refer the interested readers to [9, 10, 11, 12, 13, 14, 15] and references therein. Meanwhile, when $s \equiv 1$, the operators in (1.1) reduce to the integer order, i.e., $p(\cdot)$-Laplacian $\Delta_{p(\cdot)}$. This kind of variable exponents problem has a wide range of real applications, such as electrorheological fluids (see [1]), elastic mechanics ([2]), image restoration ([3]) and so on. For this kind of operator combined with kirchhoff function system problem, we recall [4, 5, 6, 7, 8]. For example, S. Boulaaras ([4]) at al.

Inspired by the above works, we consider a new fractional Kirchoff system (1.1). To our best knowledge, this is the first attempt on fractional situations to study a bi-non-local problem with variable exponent. In order to show the existence of weak solution, we use the direct variational method and Ekeland variational principle to deal with it. Our result is new to the fractional system of variable exponent.

From now on, in order to simplify the notation, we denote

$$p^- = \min_{(x,y) \in \mathbb{R}^{2N}} p(x,y), \quad p^+ = \max_{(x,y) \in \mathbb{R}^{2N}} p(x,y).$$

We will assume that $M_1, M_2 : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions satisfying the condition

(M) : there exist $m > 0$ and $\gamma > \frac{1}{p^-}$ such that

$$M_1(t), M_2(t) > mt^{\gamma - 1}, \text{ for all } t > 0.$$

Note that the Kirchoff functions $M_1, M_2$ may be singular at $t = 0$ for $\gamma \in (0, 1)$;

Moreover, $H : \mathbb{R}^2 \to \mathbb{R}$ is a $C^1$-function verifying

(H1) : \frac{\partial H}{\partial u}(u, v) = f(u, v) \text{ and } \frac{\partial H}{\partial v}(u, v) = g(u, v) \text{ for all } (u, v) \in \mathbb{R}^2;

(H2) : there exists $K > 0$ such that

$$H(u, v) = H(u + K, v + K) \text{ for all } (u, v) \in \mathbb{R}^2;$$

Finally, we suppose that

(AB) : $a(x), b(x) \in L^{q(x)}(\Omega), \frac{1}{p(x)} + \frac{1}{q(x)} = 1, 1 < q(x) < p^*_s(x)$ where $p^*_s(x) = N\overline{p}(x)/(N - s\overline{p}(x))$, $\overline{p}(x) = p(x, x)$,

(P) : $p(\cdot) : \mathbb{R}^{2N} \to (1, \infty)$ is a continuous function fulfilling $0 < s < 1 < p^- \leq p^+, \text{ and } p(\cdot)$ is symmetric, that is, $p(x, y) = p(y, x)$ for any $(x, y) \in \mathbb{R}^{2N}$.

Now, we give the main result of this paper, our energy functional $I$ will be introduced in Section 2.
Theorem 1.1. Let $\Omega$ be a bounded smooth domain of $\mathbb{R}^N$, with $N > p(x,y)s$ for any $(x,y) \in \overline{\Omega} \times \overline{\Omega}$, where $p(\cdot)$ verify (P). Assume that (M) and (H)$_1$ – (H)$_2$ are satisfied. Then, problem (1.1) admits a weak solution if $I$ is differentiable at $(u_0, v_0)$.

The paper is organized as follows. In Section 2, we state some interesting properties of variable exponent Lebesgue spaces and fractional Sobolev spaces with variable exponent. In Section 3, we prove the functional $I$ is bounded from below and give the proof of Theorem 1.1.

2. Abstract framework

In this section, first of all, we recall some basic properties about the variable exponent Lebesgue spaces in [19] and fractional Sobolev spaces. Secondly, we give some necessary lemmas that will be used in this paper. Finally, we introduce the definition of weak solutions for problem (1.1) and build the corresponding energy functional. Consider the set

$$C_+(\overline{\Omega}) = \{ p \in C(\overline{\Omega}) : p(x) > 1 \text{ for all } x \in \overline{\Omega} \}.$$ 

For any $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space as

$$L^{p(\cdot)}(\Omega) = \left\{ u : \text{the function } u : \Omega \to \mathbb{R} \text{ is measurable, } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

the vector space endowed with the Luxemburg norm,

$$\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \frac{|u(x)|^{p(x)}}{\lambda} dx \leq 1 \right\}.$$ 

Then $(L^{p(\cdot)}(\Omega), \| \cdot \|_{p(\cdot)})$ is a separable reflexive Banach space, see [21, Theorem 2.5].

The fractional Sobolev spaces with variable exponent is defined by

$$W^{s,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + p(x,y)s}} dxdy < \infty \right\},$$

with the norm $\|u\| = \|u\|_{p(\cdot)} + [u]_{s,p(\cdot)}$, where

$$[u]_{s,p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N + p(x,y)s}} dxdy < 1 \right\}.$$ 

For a more detailed introduction of this space we can refer to [17]. For the reader’s convenience, we now list some of the results in reference [16] which will be used in our paper. We define the new variable order fractional Sobolev spaces with variable exponent

$$X = \left\{ u : \mathbb{R}^N \to \mathbb{R} : \|u\|_{L^{p(\cdot)}(\Omega)} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N + p(x,y)s}} dxdy < \infty, \text{ for some } \lambda > 0 \right\},$$

where $Q := \mathbb{R}^{2N} \setminus (\Omega \times \Omega)$. The space $X$ is endowed with the norm

$$\|u\|_X = \|u\|_{p(\cdot)} + [u]_X,$$
where

$$[u]_X = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + p(x,y)}} \, dx dy < 1 \right\}.$$ 

We know that the norms $\| \cdot \|_{L^p(\Omega)}$ and $\| \cdot \|_X$ are not the same due to the fact that $\Omega \times \Omega \subset Q$ and $\Omega \times \Omega \neq Q$. This makes the fractional Sobolev space with variable exponent $W^{s,p(x)}(\Omega) \times W^{s,p(x)}(\Omega)$ not sufficient for investigating the class of problems like (1.1).

For this, we set space as $X_0 = \{ u \in X : \lambda u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}$. The space $X_0$ is a separable reflexive Banach space, see [16, Lemma 2.3], with respect to the norm

$$\|u\|_{X_0} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + p(x,y)}} \, dx dy = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + p(x,y)}} \, dx dy < 1 \right\},$$

where last equality is a consequence of the fact that $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$.

In the following Lemma, we give a compact embedding result. For the proof we refer the reader to [20].

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $s \in (0, 1)$. Let $p(x, y)$ be continuous variable exponents with $sp(x, y) < N$ for $(x, y) \in \overline{\Omega} \times \overline{\Omega}$. Assume that $q : \overline{\Omega} \rightarrow (1, \infty)$ is a continuous function such that $p_1^s(x) > q(x) \geq q^- > 1$, for all $x \in \overline{\Omega}$. Then, there exists a constant $C = C(N, s, p, q, \Omega)$ such that for every $u \in X_0$, it holds that $\|u\|_{L^{q(x)}} \leq C \|u\|_{X_0}$. The space $X_0$ is continuously embedded in $L^{q(x)}(\Omega)$. Moreover, this embedding is compact.

We define the fractional modular function $\Phi_{p(x)} : X_0 \rightarrow \mathbb{R}$, by

$$\Phi_{p(x)}(u) = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + p(x,y)}} \, dx dy.$$

We also have the next result of [16, Lemma 2.2].

**Lemma 2.2.** Assume that $u \in X_0$ and $\{u_j\} \subset X_0$, then

1. $\|u\|_{X_0} < 1$ (resp. $= 1$, $> 1$) $\Leftrightarrow \Phi_{p(x)}(u) < 1$ (resp. $= 1$, $> 1$),
2. $\|u\|_{X_0} < 1 \Rightarrow \|u\|_{X_0}^p \leq \Phi_{p(x)}(u) \leq \|u\|_{X_0}^p$,
3. $\|u\|_{X_0} > 1 \Rightarrow \|u\|_{X_0}^p \leq \Phi_{p(x)}(u) \leq \|u\|_{X_0}^p$.

Finally, we define our workspace $S = X_0 \times X_0$, which is endowed with the norm $\|(u, v)\|_S = \|u\|_{X_0} + \|v\|_{X_0}$. We say that a pair of functions $(u, v) \in S$ is called weak solution of problem (1.1), if for all $(\phi, \varphi) \in S$ one has

$$M_1(\delta_{p(x)}(u)) \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2}(v(x) - v(y))(\phi(x) - \phi(y))}{|x - y|^{N + p(x,y)}} \, dx dy = \int_{\Omega} ((f(u, v) + a(x)) \phi dx,$$

$$M_2(\delta_{p(x)}(v)) \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^{p(x,y)-2}(v(x) - v(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + p(x,y)}} \, dx dy = \int_{\Omega} ((g(u, v) + b(x)) \varphi dx,$$
Let us consider the following functional associated to problem (1.1), defined by
\[ I(u, v) = \overline{M}_1(\delta_{p,\gamma}(u)) - \overline{M}_2(\delta_{p,\gamma}(v)) - \int_\Omega H(u, v)dx - \int_\Omega a(x)udx - \int_\Omega b(x)vdx, \]
for all \((u, v) \in S\), where \(\overline{M}_i(t) = \int_0^t M_i(\tau)d\tau\). Obviously, the continuity of \(M\) yields that \(I\) is well defined and of class \(C^1\) on \(S \setminus \{0, 0\}\). Furthermore, for every \((u, v) \in S \setminus \{0, 0\}\), the derivative of \(I\) is given by
\[
\langle I'(u, v), (\phi, \varphi) \rangle = M_1(\delta_{p,\gamma}(u)) \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N + p(x,y)}} dxdy
+ M_2(\delta_{p,\gamma}(v)) \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^{p(x,y)-2}(v(x) - v(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + p(x,y)}} dxdy
- \int_\Omega ((f(u, v) + a(x)) \phi dx - \int_\Omega ((g(u, v) + b(x)) \varphi dx,
\]
for any \((\phi, \varphi) \in S\). Therefore, the weak solution \((u, v) \in S \setminus \{0, 0\}\) of problem (1.1) is a nontrivial critical point of \(I\).

Throughout the paper, for simplicity, we use \(\{c_i, i \in \mathbb{N}\}\) to denote different non-negative or positive constant.

3. The main result

**Lemma 3.1.** Under the same assumptions of Theorem 1.1, then \(I\) is coercive and bounded from below.

**Proof.** Firstly, we know that functional \(I\) is well defined. Indeed, it is sufficient to prove that the functional \(T : S \to \mathbb{R}, T(u, v) = \int_\Omega H(u, v)dx\), is well defined. Since \(H\) is continuous on \([0, K] \times [0, K]\) and \(H(u, v) = H(u + K, v + K)\) for all \((u, v) \in \mathbb{R}^2\), we get \(|H(u, v)| \leq c_1\) for all \((u, v) \in \mathbb{R}^2\). Thus,
\[
|T(u, v)| \leq \int_\Omega |H(u, v)|dx \leq c_1|\Omega|, \text{ for all } (u, v) \in \mathbb{R}^2,
\]
i.e., \(T\) is well defined, where \(|\Omega|\) is the Lebesgue measure of \(\Omega\). Next, we will prove that \(I\) is coercive and bounded from below. Let \((u, v) \in S\), we have
\[
I(u, v) = \overline{M}_1(\delta_{p,\gamma}(u)) - \overline{M}_2(\delta_{p,\gamma}(v)) - \int_\Omega H(u, v)dx - \int_\Omega a(x)udx - \int_\Omega b(x)vdx
\geq \overline{M}_1(\delta_{p,\gamma}(u)) - \overline{M}_2(\delta_{p,\gamma}(v)) - c_1|\Omega| - \int_\Omega a(x)udx - \int_\Omega b(x)vdx.
\]
By the condition \((AB)\) and the Hölder inequality, which can be found in [21, Theorem 2.1], we get
\[
I(u, v) \geq M_1(\delta_{p(x)}(u)) - M_2(\delta_{p(x)}(v)) - c_1|\Omega| - 2\|a(x)\|_{L_p}\|u\|_{L_p} - 2\|b(x)\|_{L_p}\|v\|_{L_p}.
\]
It follows from \((M)\) and Lemmas [2.1, 2.2] that
\[
I(u, v) \geq m \int_0^{1/p} \frac{\gamma(u)^{p'}}{p'} d\tau + m \int_0^{1/p} \frac{\gamma(v)^{p'}}{p'} d\tau - c_3\|u\|_{L_p} - c_4\|v\|_{L_p} - c_2
\]
\[
= \frac{m}{\gamma(p^+)^{p'}} \left( \left( \frac{\gamma^{p'}(u)}{p'} \right)^p + \left( \frac{\gamma^{p'}(v)}{p'} \right)^p \right) - c_3\|u\|_{L_p} - c_4\|v\|_{L_p} - c_2
\]
\[
\geq \frac{m}{\gamma(p^+)^{p'}} \left( \min\{\|u\|_{L_p}, \|v\|_{L_p}\} + \min\{\|u\|_{L_p}, \|v\|_{L_p}\} \right) - \max\{c_3, c_4\}(\|u\|_{L_p} + \|v\|_{L_p}) - c_2,
\]
(3.1)
since \(\gamma p^+ > \gamma p^- > 1\), when \(|\langle u, v\rangle| \to +\infty\), at least one of \(\|u\|_{L_p}\) and \(\|v\|_{L_p}\) converges to infinity. So, \(I\) is coercive and bounded from below. The proof of Lemma [3.1] is complete. □

**Proof of Theorem 1.1**

Obviously, since \(I \in C^{1}(S, \mathbb{R})\) is weakly lower semi-continuous and bounded from below, by means of Ekeland variational principle in [7] we have \((u_j, v_j) \subset S\) such that
\[
I(u_j, v_j) \to \inf_{S} I \text{ and } \dot{I}(u_j, v_j) \to 0.
\]
(3.2)
Furthermore, by the above expression, we get \(|I(u_j, v_j)| \leq c_5\). Thus, it follows from (3.1) that \(c_6 \leq |I(u_j, v_j)| \leq c_5\), which implies that the sequences \([u_j]\) and \([u_j]\) are bounded in \(X_0\). So, without loss of generality, there exist subsequences \([u_j]\) and \([u_j]\) such that \(u_j \to u_0\) and \(v_j \to v_0\) in \(X_0\), and thus,
\[
\int_{\Omega} a(x) u_j dx \to \int_{\Omega} a(x) u_0 dx \text{ and } \int_{\Omega} b(x) v_jdx \to \int_{\Omega} a(x) v_0 dx.
\]
According to compact embedding theorem, which is lemma [2.1], we obtain \(u_j(x) \to u_0(x) \text{ and } v_j(x) \to v_0(x) \text{ a.e. \space } x \in \Omega\). Again, by continuity of \(H\), we get \(H(u_j(x), v_j(x)) \to H(u_0(x), v_0(x)) \text{ a.e. \space } x \in \Omega\).
And because \(H\) is bounded, we get the following convergence from the Lebesgue dominated convergence theorem,
\[
\int_{\Omega} H(u_j, v_j)dx \to \int_{\Omega} H(u_0, v_0)dx.
\]
By (3.2), we note that
\[
\inf_{S} I = \lim_{S} I(u_j, v_j)
\]
\[
= \lim \left( M_1(\delta_{p(x)}(u_j)) - M_2(\delta_{p(x)}(v_j)) - \int_{\Omega} H(u_j, v_j)dx - \int_{\Omega} a(x) u_j dx - \int_{\Omega} b(x) v_j dx \right).
\]
In view of Fatou’s Lemma, we have \(\delta_{p(x)}(u_0) \leq \lim \inf \delta_{p(x)}(u_j) \text{ and } \delta_{p(x)}(v_0) \leq \lim \inf \delta_{p(x)}(v_j)\). By the continuous monotone increasing property of \(\bar{M}_1\) and \(\bar{M}_2\), we get
\[
\bar{M}_1(\delta_{p(x)}(u_0)) \leq \lim \bar{M}_1(\delta_{p(x)}(u_j)) \text{ and } \bar{M}_2(\delta_{p(x)}(v_0)) \leq \lim \bar{M}_2(\delta_{p(x)}(v_j)).
\]
In conclusion,

\[
\inf_{\tilde{S}} I \geq \tilde{M}_1(\delta_{p^+}(u_0)) - \tilde{M}_2(\delta_{p^+}(v_0)) - \int_{\Omega} H(u_0, v_0) dx - \int_{\Omega} a(x) u_0 dx - \int_{\Omega} b(x) v_0 dx = I(u_0, v_0),
\]

which implies \( I(u_0, v_0) = \inf_{\tilde{S}} I \). Thus, \((u_0, v_0) \in \tilde{S}\) is a weak solution of problem (1.1) if \( I \) is differentiable at \((u_0, v_0)\). The proof is complete.

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