Toric Representation and Positive Cone of Picard Group and Deformation Space in Mirror Symmetry of Calabi-Yau Hypersurfaces in Toric Varieties

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Abstract

We derive the combinatorial representations of Picard group and deformation space of anti-canonical hypersurfaces of a toric variety using techniques in toric geometry. The mirror cohomology correspondence in the context of mirror symmetry is established for a pair of Calabi-Yau (CY) n-spaces in toric varieties defined by reflexive polytopes for an arbitrary dimension n. We further identify the Kahler cone of the toric variety and degeneration cone of CY hypersurfaces, by which the Kahler cone and degeneration cone for a mirror CY pair are interchangeable under mirror symmetry. In particular, different degeneration cones of a CY 3-fold are corresponding to flops of its mirror 3-fold.

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1 Introduction

For a quasi-smooth Calabi-Yau (CY) n-space X in a toric variety, the Picard group Pic(X) and deformation space Def(X) in the Hodge spaces, $H^{1,1}(X)$ and $H^{n-1,1}(X)$ respectively, play an important role in the algebraic geometry study of X and its applications in string physics (see, e.g.
and references therein). In the case when the toric variety is defined by a simplicial-cone decomposition with certain triangulation conditions, the anti-canonical hypersurface $X$ is indeed a smooth manifold for $n \leq 3$. In particular for the case $n = 2$,

$$X = \text{K3 surface} : \quad H^{1,1}(X) = \text{Pic}(X) \mathbb{C} \oplus \text{Def}(X) \mathbb{C}. \quad (1.1)$$

For $n$ greater than 2, the Picard group and deformation space are the same as the Hodge spaces:

$$H^{1,1}(X) = \text{Pic}(X) \mathbb{C}, \quad H^{n-1,1}(X) = \text{Def}(X) \mathbb{C} \quad \text{for } n \geq 3. \quad (1.2)$$

By the mirror symmetry of Calabi-Yau (CY) spaces, we mean a pair of quasi-smooth CY $n$-spaces, $X$ and $X^*$, with a canonical identification of Picard group and deformation space:

$$\text{Pic}(X) \mathbb{C} \simeq \text{Def}(X^*) \mathbb{C}, \quad \text{Def}(X) \mathbb{C} \simeq \text{Pic}(X^*) \mathbb{C}, \quad (1.3)$$

which are compatible with (quantum) cohomology $n$-product. For $n = 2$, the mirror symmetry (1.3) signifies the interchangeable relation between the algebraic cycles and the transcendental cycles of K3 surfaces $X$ and $X^*$. This symmetry is linked to the Arnold duality for the 14 exceptional singularities of modality one [10, 19]. For $n \geq 3$, by (1.3), the mirror symmetry (1.3) implies

$$h^{1,1}(X) = h^{n-1,1}(X^*) \quad (n \geq 3). \quad (1.4)$$

In particular when $n = 3$, the Euler numbers of $X$ and $X^*$ are of opposite sign, $\chi(X) = -\chi(X^*)$. Indeed, the mirror symmetry was first found in [6, 26] as a pair of Fermat-type CY 3-folds corresponding to the same conformal field theory with a reversal of left $U(1)$-charge in string theory. Subsequently, the mirror cohomology correspondence (1.3) for Fermat-type CY 3-folds was shown in [18] by the method of toric geometry as anti-canonical hypersurfaces of abelian quotients of weighted projective 4-spaces. A similar discussion for higher dimensional Fermat-type CY spaces was also given in [21]. The mirror symmetry was further extended to a large class of CY spaces in [2], where the anti-canonical hypersurfaces in $n$-dimensional toric varieties defined by reflexive polytope and its dual polytope for $n(= n + 1)$ are found to satisfy the Hodge-number equality (1.4) ([2] Theorem 4.4.3). However, to the best of the author’s knowledge, the mirror correspondence (1.3) has not been established in literature till now, except the simplex polytope (i.e. Fermat-type) case for dimension $n = 4$ [18]. The importance of a canonical identification of toric representatives in (1.3) is stressed as the keys either to string conformal-field applications, or to enumerative geometry about rational-curve counting in a CY space. With a correct cohomology identification in (1.3), it is believed that the quantum cohomology computation of a CY manifold can be carried out by the method of complex-structure variation in its mirror degeneration family, as suggested by works in [3, 8, 26] about the rational-curve problem in Fermat quintic or some other specific CY 3-folds. The object of this paper is to establish the canonical isomorphism (1.3) between a mirror pair of quasi-smooth CY hypersurfaces in $n$-dimensional toric varieties defined by reflexive polytopes for $n \geq 2$. In this work, we employ techniques in toric geometry to build a special kind of combinatorial representations for Picard group and deformation space of those CY spaces, by
which the mirror correspondence is established by identifying toric representatives of the representation. Of particular interest is the effect of the topology of CY spaces on those representations in mirror symmetry. It is known that the cohomology product of Picard group in (1.3) depends on the topology of manifolds, even in the Fermat CY case. Indeed, the Fermat CY 3-folds in [6, 18] are constructed as crepant resolutions of a (singular) hypersurface of an abelian-quotient of weighted projective 4-space [14, 16, 25]. Such crepant resolutions are not unique in general. Two different ones are connected by a process of flops in birational geometry of 3-folds with different topological triple-products of Picard group (see, e.g. [20]), upon which the quantum product builds, albeit they both have the same cohomology representation. In the mirror correspondence (1.3), the effect of flop of a CY 3-fold on the deformation of its mirror has not been previously investigated in the literature, even for the Fermat-type case [24]. Indeed, it is one main purpose of this present work to search an equivalent notion on the deformation space in mirror symmetry which corresponds to the flop in birational geometry. After sorting out the toric data between mirror Fermat-type CY 3-folds, we find that the topological change due to flop of a CY 3-fold is linked to a different path of maximal unipotent degeneration in the moduli space of its mirror CY spaces. Consequently, by using techniques in toric geometry, one expects the same conclusion should also be valid for a mirror pair of anti-canonical hypersurfaces in toric varieties defined by reflexive polytope and its dual polytope. In the present paper we show that it is indeed the case. However, it turns out that the identifications in (1.3) are quite involved, especially for the positive cone structure. Various technical details need to be checked before we then explicitly verify our proposal in a precise form.

This paper is organized as follows. Section 2 is devoted to structures related to Picard group of hypersurfaces of a toric variety. We start with the homogeneous coordinate system of a toric variety in Subsection 2.1. In Subsection 2.2, we recall some facts in [23] about the Picard group of a generic anti-canonical hypersurface of a toric variety defined by reflexive polytope that will be used in the following. In Subsection 2.3, we present an effective mechanism to extract Kahler cone of a toric variety from its rational simplicial-cone toric structure. Explicit calculation of some examples are performed here. In Section 3, we investigate structures related to the deformation space of anti-canonical hypersurfaces of a toric variety. In the discussion of this section, we use some facts in toric geometry of which some detailed argument with technical complexity are provided in Appendix for easy reference. We first recall the result in [23] about a combinatorial basis of anti-canonical sections of a toric variety, then describe the coordinate form of the basis elements. In Subsection 3.1, we determine the deformation classes of anti-canonical hypersurfaces in the "minimal" toric variety. First, we find the moduli space of the anti-canonical hypersurface by Jacobian-ring technique, then identify the deformation equivalent classes modulus the transformation of toric-variety automophisms. By the cohomology relation of hypersurfaces between a toric variety and the minimal toric variety, we derive the combinatorial representation of the deformation space of an anti-canonical hypersurface in toric variety defined by reflexive polytope in Subsection 3.2. In Subsection 3.3, we define the degeneration cone in the moduli of CY hypersurfaces, a structure parallel to the Kahler cone of the toric variety in Subsection 2.3. Using the combinatorial representation of Picard group and deformation space established in Sections 2 and 3 we show the
mirror correspondence \([13]\) and the identification of Kahler and degeneration cones between the mirror anti-canonical hypersurfaces of toric varieties defined by a reflexive and its dual polytopes in Section [4]. We close in Section [5] with concluding remarks. Some results in \([23]\) are reviewed in Appendix, where some basic facts with technical arguments in toric geometry are also presented for easy use and reference of this paper (for the details, also see e.g. \([5, 9, 12]\)).

**Notation.** In this work, we use the following notations. Let \(L\) be a \(n\)-dimensional lattice \((\simeq \mathbb{Z}^n)\), and \(L^* (= \text{Hom}(L, \mathbb{Z}))\) the dual lattice of \(L\). We set \(L_{\mathbb{K}} = L \otimes \mathbb{Z} \mathbb{K}\) for \(\mathbb{K} = \mathbb{Q}, \mathbb{R}\) or \(\mathbb{C}\). Denote \(T(L) = L_{\mathbb{C}}^*/L (\simeq \mathbb{C}^n)\) the (algebraic) \(n\)-torus whose 1-parameter subgroups and characters are identified with elements in \(L\) and \(L^*\) respectively. For a complete rational fan \(\Sigma\) in \(L_{\mathbb{R}}\), i.e. a polyhedral cone decomposition of \(L_{\mathbb{Q}} \subseteq L_{\mathbb{R}}\), we shall denote \(\mathbb{P}_\Sigma (= \mathbb{P}_{(\Sigma, L)})\) the complete \(T(L)\)-toric variety associated to \(\Sigma\) \([5, 9, 12]\). In general, \(\mathbb{P}_\Sigma\) is a singular space, with at most abelian quotient singularities when \(\Sigma\) is a simplicial cone decomposition. By an integral polytope \(\triangle \subseteq L_{\mathbb{R}}\), we mean a \(n\)-dimensional convex hull generated by finitely many lattice points of \(L\), whose interior contains the origin of \(L\), and the set of vertices of \(\triangle\) will be denoted by \(V(\triangle)\). A \((n - 1)\)-dimensional face will also be called a facet of \(\triangle\). The face-decomposition of the boundary \(\partial \triangle\) of a polytope \(\triangle\) gives rise to the polyhedral-cone decomposition \(\Sigma_0\) in \(L_{\mathbb{R}}\). The \(T(L)\)-toric variety \(\mathbb{P}_{\Sigma_0}\) defined by \(\Sigma_0\) is minimal in the sense that \(\mathbb{P}_{\Sigma_0}\) is dominated by every \(T(L)\)-toric variety \(\mathbb{P}_\Sigma\) for a refinement \(\Sigma\) of \(\Sigma_0\), (equivalently, \(\Sigma \cap \partial \triangle\) is polytope decomposition of \(\partial \triangle\)). In particular, a simplicial decomposition \(\Lambda\) of \(\partial \triangle\) with \(\Lambda^{(0)} \subseteq \partial \triangle \cap L\), where \(\Lambda^{(j)}\) denotes the collection of \(j\)-simplices of \(\Lambda\) for \(0 \leq j \leq n - 1\), gives rise to a complete fan \(\Sigma (= \Sigma (\Lambda))\) of \(L_{\mathbb{R}}\) consisting of cones \(\sigma(s)(= \sum_{\delta \in s \cap \Lambda^{(0)}} \mathbb{R}_{\geq 0} \delta)\) for simplices \(s \in \Lambda\). Then \(\mathbb{P}_\Sigma (= \mathbb{P}_{\Sigma (\Lambda)})\) is a complete \(T(L)\)-toric variety dominating \(\mathbb{P}_{\Sigma_0}\) with at most abelian quotient singularities.

Denote \(\langle *, * \rangle: L_{\mathbb{R}}^* \times L_{\mathbb{R}}^* \rightarrow \mathbb{R}\) the canonical pairing with integral values on \(L^* \times L\). The dual polytope \(\triangle^*\) of \(\triangle\) in \(L_{\mathbb{R}}^*\) is defined by
\[
\triangle^* = \{ y \in L_{\mathbb{R}}^* \mid \langle y, x \rangle \geq -1 , \text{ for } x \in \triangle (\Leftrightarrow x \in V(\triangle))\}.
\]
The dual face \(F^*\) of a \(m\)-dimensional polyhedral face \(F\) of \(\triangle\) \((0 \leq m \leq n - 1)\) is the \((n - m - 1)\)-dimensional face of \(\triangle^*\) defined by
\[
F^* = \{ y \in \triangle^* \mid \langle y, x \rangle = -1 \text{ for } x \in F (\Leftrightarrow x \in F \cap V(\triangle))\}.
\]

**Definition.** \((\triangle, L)\) is a reflexive polytope iff both \((\triangle, L)\) and \((\triangle^*, L^*)\) are integral \([2]\).

It is known that the origin is the only lattice point in the interior \(\text{Int} (\triangle)\) of a reflexive polytope \((\triangle, L)\). For a reflexive polytope \((\triangle, L)\), a simplicial decomposition \(\Lambda\) of \(\partial \triangle\) will always be assumed to satisfy the following condition
\[
\Lambda^{(0)} = L \cap (\partial \triangle - \bigcup \{ \text{Int}(F) \mid F : \text{codim-1 face of } \triangle \}). \tag{1.5}
\]
Note that the toric variety \(\mathbb{P}_\Sigma\) is Gorenstein over \(\mathbb{P}_{\Sigma_0}\). Throughout this paper except in Appendix or otherwise stated, a polytope \((\triangle, L)\) will always be a reflexive polytope with a simplicial
decomposition $\Lambda$ of $\partial \Delta$ satisfying (1.5) and the relation 1

$$L = \text{the sublattice } L_0 \text{ of } L \text{ generated by } \Lambda^{(0)}. \quad (1.6)$$

The irreducible toric divisors in $\mathbb{P}_{\Sigma(\Lambda)}$ are determined by 1-skeleton of $\Sigma(\Lambda)$, parametrized by $\Lambda^{(0)}$. Denote by $e^\delta$ the toric divisor in $\mathbb{P}_{\Sigma(\Lambda)}$ associated to an element $\delta \in \Lambda^{(0)}$. The divisor lattice is defined by

$$D_{\Delta} (= D_{(\Delta, L)}):= \bigoplus_{\delta \in \Lambda^{(0)}} \mathbb{Z}e^\delta \quad (\cong \mathbb{Z}^d), \quad d := |\Lambda^{(0)}|. \quad (1.7)$$

By (1.6), we have the following exact sequence of abelian groups:

$$0 \longrightarrow n_{\Delta} \xrightarrow{\iota} D_{\Delta} \xrightarrow{\beta} L \longrightarrow 0, \quad \beta(e^\delta) := \delta, \quad (1.8)$$

where $n_{\Delta} (= n_{(\Delta, L)}) := \text{Ker}(\beta)$ is the rank-$(d - n)$ sublattice of $D_{\Delta}$ with $\iota$ the inclusion morphism. The dual sequence of (1.8) is

$$0 \longrightarrow L^* \xrightarrow{\beta^*} D^\dagger_{\Delta} \xrightarrow{\iota^*} n^\dagger_{\Delta} \longrightarrow 0, \quad \left( D^\dagger_{\Delta} := \text{Hom}(D_{\Delta}, \mathbb{Z}), \quad n^\dagger_{\Delta} := \text{Hom}(n_{\Delta}, \mathbb{Z}) \right), \quad (1.9)$$

where the basis of $D^\dagger_{\Delta}$ dual to $e^\delta$’s are denoted by

$$D^\dagger_{\Delta} = \bigoplus_{\delta \in \Lambda^{(0)}} \mathbb{Z}e^{\delta^\dagger}, \quad \langle e^{\delta^\dagger}, e^{\delta'} \rangle = e^{\delta^\dagger} \delta' := 1 \text{ if } \delta = \delta', \quad 0 \text{ otherwise}. \quad (1.10)$$

For a cone $C$ in a $\mathbb{R}$-vector space $V$, the dual cone of $C$ in the dual space will be denoted by

$$\tilde{C} = \{ x \in \text{Hom}(V, \mathbb{R}) \mid \langle x, v \rangle \geq 0 \ (v \in C) \}. \quad (1.11)$$

## 2 Picard Group of Anti-canonical Hypersurface of Toric Variety

We start with coordinate systems of toric varieties for later use, then review some facts in [23] about the Picard group of a generic anti-canonical hypersurface of a toric variety defined by reflexive polytope. This will also serve to establish the notation. The Kahler cone of toric variety will be discussed in the last subsection here.

### 2.1 Homogeneous coordinates of a toric variety

First, we describe the homogeneous coordinate system of the toric variety $\mathbb{P}_{\Sigma(\Lambda)}$ in [1, 4, 14, 23]. Regard (1.8) and (1.9) as the 1-parameter subgroups and characters of the following exact sequence of tori:

$$0 \longrightarrow T(n_{\Delta}) \longrightarrow T(D_{\Delta}) \longrightarrow T(L) \longrightarrow 0.$$

For simple notations, we assume the relation (1.5) here, without which the conclusions of this paper are still valid as argued in [23] by regarding $\mathbb{P}_{\Sigma(\Lambda)}$ as a $(L/L_0)$-quotient of $\mathbb{P}_{\Sigma(\Lambda_0)}$ with the homogeneous coordinates in Section 2.1 and identifying the anti-canonical hypersurfaces of $\mathbb{P}_{\Sigma(\Lambda)}$ with $(L/L_0)$-invariant anti-canonical hypersurfaces of $\mathbb{P}_{\Sigma(\Lambda_0)}$. 

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Denote
\[ D_\triangle \mathbb{K} = \bigoplus_{\delta \in \Lambda(0)} \mathbb{K} e^\delta \quad (\simeq \mathbb{K}^d) \text{ for } \mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}, \]
and the first quadrant of \( D_\triangle \mathbb{R} \) by
\[ \Omega(= \Omega_\triangle) := \sum_{\delta \in \Lambda(0)} \mathbb{R} \geq 0 e^\delta. \]
The \( T(D_\triangle) \)-toric variety associated to the face-decomposition of \( \Omega \) is \( D_\triangle \mathbb{C} \) with coordinates \( \sum_\delta z_\delta e^\delta \) for \( z_\delta \) corresponding to \( e^\delta \) in (1.10):
\[ T(D_\triangle) \simeq \prod_{\delta \in \Lambda(0)} \mathbb{C}^* \subseteq D_\triangle \mathbb{C} \ni z = (z_\delta)_{\delta \in \Lambda(0)}. \]
Associated to a triangulation \( \Lambda \) of \( \partial \Delta \), there is the (integral) simplicial fan \( \bar{\Sigma} \subseteq \partial \Omega \) in \( D_\triangle \mathbb{R} \) lying over the fan \( \Sigma(\Lambda) \) of \( L \):
\[ \bar{\Sigma} (= \bar{\Sigma}(\Lambda)) = \{ \bar{\sigma}(s)|s : \text{simplex in } \Lambda \}, \quad \bar{\sigma}(s) := \sum_{\delta \in \sigma \cap \Lambda(0)} \mathbb{R} \geq 0 e^\delta \subset D_\triangle \mathbb{R}. \]
Then \( \bar{\Sigma} \) gives rise to a \( T(D_\triangle) \)-toric variety, denoted by \( \mathfrak{C}_{\bar{\Sigma}}(= \mathfrak{C}_{\bar{\Sigma}(\Lambda)}) \), which is an affine open subset of \( D_\triangle \mathbb{C} \) given by
\[ T(D_\triangle) \subset \mathfrak{C}_{\bar{\Sigma}} = D_\triangle \mathbb{C} - \bigcup_I \{ \sum_I z_\delta e^\delta \mid z_\delta = 0 \text{ for } \delta \in I \} \tag{2.1} \]
where the index \( I \) runs over subsets of \( \Lambda(0) \) which are not in the form \( s \cap \Lambda(0) \) for some \( s \in \Lambda \). Since \( \beta \) in (1.8) induces a fan-correspondence from \( \bar{\Sigma} \) to \( \Sigma \), sending \( \bar{\sigma} \) to \( \sigma \), it gives rise to the ”principal \( T(\mathfrak{n}_\triangle) \)-bundle”
\[ \pi : \mathfrak{C}_{\bar{\Sigma}} \rightarrow \mathbb{P}_{\Sigma} \quad (= \mathfrak{C}_{\bar{\Sigma}}/T(\mathfrak{n}_\triangle)) \tag{2.2} \]
The coordinates \( z = (z_\delta)_\delta \) of \( \mathfrak{C}_{\bar{\Sigma}} \) will be called the homogeneous coordinates of \( \mathbb{P}_{\Sigma} \). Similarly, there is the coordinate system of the minimal toric variety \( \mathbb{P}_{\Sigma_0} \) defined by the face-decomposition of \( \partial \Delta \), by considering the following exact sequences as (1.9) and (1.10):
\[ 0 \rightarrow \mathfrak{n}_\triangle \rightarrow D_\triangle \rightarrow \mathbb{L} \rightarrow 0, \quad 0 \rightarrow \mathbb{L}^* \rightarrow D_\triangle^* \rightarrow \mathfrak{n}_\triangle^* \rightarrow 0 \tag{2.3} \]
where \( D_\triangle := \bigoplus_{v \in \mathfrak{V}(\Delta)} \mathbb{Z} e^v \), \( \beta_0(e^v) := v \), and \( \mathfrak{n}_\triangle := \text{Ker}(\beta_0) \). As in (2.2), there is the homogeneous coordinates, \( \zeta = (\zeta_v)_{v \in \mathfrak{V}(\Delta)} \), of \( \mathbb{P}_{\Sigma_0} \) with the ”\( T(\mathfrak{n}_\triangle) \)-principal bundle”
\[ \pi_{\Sigma_0} : \mathfrak{C}_{\bar{\Sigma}_0} \rightarrow \mathbb{P}_{\Sigma_0} \tag{2.4} \]
where \( \mathfrak{C}_{\bar{\Sigma}_0} \) is the complement of \( \bigcup_{I_0=\{F\in\mathfrak{V}(\Delta)\mid F: \text{face of } \Delta \}} \{ \sum_v \zeta_v e^v \mid \zeta_v = 0 \text{ for } v \in I_0 \} \) in \( D_\triangle \mathbb{C} \). The fibrations (2.2) and (2.4) are related by the following morphisms between \( D_\triangle \mathbb{C} \) and \( D_\triangle \mathbb{C} \):
\[ j : D_\triangle \hookrightarrow D_\triangle, \quad j(e^v) = e^v \quad \text{for } v \in \mathfrak{V}(\Delta), \]
\[ p : D_\triangle \rightarrow D_\triangle, \quad p(e^\delta) = \sum_{v \in F \cap \mathfrak{V}(\Delta)} r_{v, \delta} e^v \quad \text{for } \delta \in \Lambda(0), \tag{2.5} \]
where $F$ is a face of $\Delta$ whose interior $\text{Int}(F)$ contains $\delta$, and $r_{\nu}^{F,\delta}$s are the $F$-face expression of $\delta$, i.e. $\delta = \sum_{\nu \in F \cap V(\Delta)} r_{\nu}^{F,\delta} \nu$ for $r_{\nu}^{F,\delta} \in \mathbb{Q}_{>0}$ with $\sum_{\nu' \in F \cap V(\Delta)} r_{\nu'}^{F,\delta} = 1$. Then
\[ p \cdot j = \text{id}_{D_\Delta^\perp}, \quad \beta \cdot j = \beta_0, \quad \beta_0 \cdot p = \beta, \]
where $\beta, \beta_0$ are morphisms in (1.8) and (2.3) respectively. Then $j, p$ induce an embedding and projection between $n_\Delta$ and $n_\Delta$. The morphism
\[ p^* : D_{\Delta}^\perp \rightarrow D_{\Delta}^\perp, \quad p^*(e^{\nu}) = \sum_{\nu \in F, \delta \in \text{Int}(F) \cap \Lambda(0)} r_{\nu}^{F,\delta} e^{\delta \nu} \quad \text{for} \quad \nu \in V(\Delta) \quad (2.6) \]
induced by $p$ in (2.5) gives rise to a regular map between fiber spaces of (2.2) and (2.4):
\[ \varphi : \mathbb{C}_\Sigma \rightarrow \mathbb{C}_{\Sigma_0}, \quad z = (z_\delta)_{\delta \in \Lambda(0)} \mapsto \zeta = (\zeta_\nu)_{\nu \in V(\Delta)} \quad \zeta_\nu = \prod_{\nu \in F, \delta \in \text{Int}(F) \cap \Lambda(0)} z_{\nu}^{\delta} e^{\delta \nu}, \quad (2.7) \]
which induces the dominating morphism $\varphi : \mathbb{P}_\Sigma \rightarrow \mathbb{P}_{\Sigma_0}$ with exceptional divisors $E_\delta$ labeled by $\delta \in \Lambda(0) \setminus V(\Delta)$. Note the morphism $j^* : D_{\Delta}^\perp \rightarrow D_{\Delta 0}^\perp$ induced by $j$ in (2.5) gives rise to the biregular morphism outside exceptional divisors: $\mathbb{P}_{\Sigma_0} \setminus (\cup \varphi(E_\delta)) \sim \mathbb{P}_{\Sigma} \setminus (\cup E_\delta)$. 

### 2.2 Picard group of toric variety and anti-canonical hypersurfaces

We shall identify an element $\nu \in n_\Delta^\perp$ with the character function $\chi_\nu : T(n_\Delta) \rightarrow \mathbb{C}^*(= \mathbb{C}/\mathbb{Z})$. For $\nu \in n_\Delta^\perp$, the (orbifold) line bundle $\mathcal{O}(\nu)$ over $\mathbb{P}_\Sigma$ is defined through (2.2) as the $T(n_\Delta)$-quotient of $\mathbb{C}_\Sigma \times \mathbb{C}$ via the action: $(p, \zeta) \cdot t = (p \cdot t, \chi_\nu(t) \zeta)$: $\mathcal{O}(\nu) = \mathbb{C}_\Sigma \times T(n_\Delta) \mathbb{C}$. Identifying $\nu$ with $\mathcal{O}(\nu)$, one may regard $n_\Delta^\perp$ as the Picard group Pic($\mathbb{P}_\Sigma$) of $\mathbb{P}_\Sigma$. Indeed, we have the following result (see, e.g. (11) and Theorem 1 in [23]):
\[ n_\Delta^\perp \mathbb{C} \simeq \text{Pic}(\mathbb{P}_\Sigma)_\mathbb{C} \simeq H^2(\mathbb{P}_\Sigma, \mathbb{C}) \quad , \quad \nu \leftrightarrow \mathcal{O}(\nu) \leftrightarrow \text{Chern class of } \mathcal{O}(\nu). \quad (2.8) \]

An element $\varrho$ in $D_\perp$ can be regarded as a divisor of $\mathbb{P}_\Sigma$, whose image under $\iota^*$ in (1.9) defines the line bundle $\mathcal{O}(\iota^* \varrho)$ over $\mathbb{P}_\Sigma$. The canonical bundle of $\mathbb{P}_\Sigma$ is equal to $\mathcal{O}(\iota^* \kappa)$, where
\[ \kappa := - \sum_{\delta \in \Lambda(0)} e^{\delta \nu} \in D_{\Delta}^\perp \quad (2.9) \]
and $e^{\delta \nu}$ are basis elements in (1.10). Similarly, the canonical bundle of the minimal toric variety $\mathbb{P}_{\Sigma_0}$ is equal to $\mathcal{O}(\iota_{0*}^* \kappa_0)$ with
\[ \kappa_0 = - \sum_{\nu \in V(\Delta)} e^{\nu \perp} \in D_{\Delta 0}^\perp \quad (2.10) \]

Under the morphism $p^*$ in (2.6), $p^*(\kappa_0) = \kappa$, hence $\mathbb{P}_\Sigma$ is Gorenstein and dominating over $\mathbb{P}_{\Sigma_0}$. Let $X$ be a generic anti-canonical hypersurface $X$ in $\mathbb{P}_\Sigma$. Then $X$ is quasi-smooth with the

\footnote{Here we mean $\pi^*(X)$ is smooth hypersurface of $\mathbb{G}_\delta$ in (2.2), which is equivalent to the smoothness of $X$ when $r \leq 4$ (23 Proposition 6).}
Picard group described by (23) Proposition 3, or Proposition 5.3 with \( \rho = -\kappa \) in Appendix; for the simplex-\( \Delta \) case, see [18] Lemma 7 (ii)

\[
\text{Pic}(X)_{\mathbb{Q}} \simeq n^\dagger_{\Delta} \mathbb{Q} \oplus \bigoplus_{F, \nu_F} \mathbb{C} \nu_F
\]  

(2.11)

where \( F \) runs over codimensional 2 faces of \( \Delta \) with its dual face \( F^* \), and \( \nu_F \in (\text{Int}(F) \cap L) \times (\text{Int}(F^*) \cap L^*) \). In (2.11), the line bundles in \( n^\dagger_{\Delta} \mathbb{Q} \) are those inherited from \( \mathbb{P}_\Sigma \), and elements in the second term are the contribution from extract exceptional divisors lying over 1-tori determined by codim-2 faces \( F \) in \( \mathbb{P}_{\Sigma_0} \) with elements in \( \text{Int}(F^*) \cap L^* \) representing primitive 0-cohomology basis of ”blow-up-centers” in the 1-torus (see, Lemma 5.2 Remark (1) in Appendix). Indeed, \( X \) is a pull-back of a generic hypersurface \( X_0 \) of \( \mathbb{P}_{\Sigma_0} \), in which the closure \( \overline{O}_F \) of \( T(L) \)-orbit \( O_F \) fixed by \( L \cap \sigma(F) \) is isomorphic to \( \mathbb{P}^1 \). The line bundle \( \mathcal{O}(i^0_k \kappa_0) \) over \( \mathbb{P}_{\Sigma_0} \) restricting on \( \overline{O}_F \) is equivalent to \( \mathcal{O}_{\mathbb{P}^1}(d) \) \((d = |\text{Int}(F^*)| + 1)\) over \( \mathbb{P}^1 \) with the homogeneous coordinates \( z = [z_1, z_2] \). Then \( z^k \) := \( z_1^k z_2^{d-k} \) \((0 \leq k \leq d)\) form the monomial basis of \( \mathcal{O}_{\mathbb{P}^1}(d) \) so that \( \{z^k\}_{k=1}^{d-1} \) are in one-to-one correspondence with \( \text{Int}(F^*) \cap L^* \). The set \( X_0 \cap \overline{O}_F \) consists of \( d \) generic elements \( x_j \in \mathbb{P}^1 \) \((1 \leq j \leq d)\). Let \( x_j^* \) \((1 \leq j \leq d)\) be a basis of \( H^0(X_0 \cap \overline{O}_F) \) dual to \( x_j \)'s with \( \langle x_j^*, x_k \rangle = \delta_{jk} \). Then \( p^k = \sum_{j=1}^{d} e^{2\pi i jk/d} x_j^* \) \((0 \leq k \leq d-1)\) form a basis of \( H^0(X_0 \cap \overline{O}_F) \), which contains the \((d-1)\)-dimensional primitive cohomology \( H^{0,0}(X_0 \cap \overline{O}_F) \) spanned by \( p^k \) \((1 \leq k \leq d-1)\). Then all \( \nu_F \)'s in (2.11) for a fixed \((n-2)\)-face \( F \) are indexed by \( (\delta_F, p^k) \) for \( \delta_F \in \text{Int}(F) \cap L \) and \( 1 \leq k \leq d-1 \). Note that \( p^0 = \sum_{j=1}^{d} x_j^* \) and \( (\delta_F, p^0) \) is the toric divisor of \( \mathbb{P}_\Sigma \) determined by \( \delta_F \), i.e. \( e^{\delta_F^*} \) \( \in D^\dagger_{\Delta} \) in (1.9), which gives rise to the line bundle \( \mathcal{O}(\iota^* e^{\delta_F^*}) \in n^\dagger_{\Delta} \mathbb{Q} \) in (2.11).

Remark. \( \text{Pic}(X)_{\mathbb{Q}} \) in (2.11) depends only on \( \Lambda^{(0)} \), not on the detailed simplicial structure of \( \Lambda \).

The structure in Picard group which is relevant to \( \Lambda \) appears only in the part \( n^\dagger_{\Delta} \mathbb{Q} \) in (2.11), reflected in the positive divisor cone discussed in the next subsection.

2.3 Kahler cone of a toric variety

In this subsection, we discuss the positive cone structure in (2.11) related to \( n^\dagger_{\Delta} \mathbb{Q} \) which is a part of Kahler cone of \( X \), consisting of line bundles induced by positive divisors of \( \mathbb{P}_\Sigma \). An element in \( \varrho \in D^\dagger (\dagger D^\dagger_{\Delta} \mathbb{Q}) \) will be denoted by

\[\varrho = \sum_{\delta \in \Lambda^{(0)}} \varrho^\delta e^{\delta^*}, \quad \varrho^\delta = \langle \varrho, e^\delta \rangle,\]

regarded as a (toric) divisor whose \( \iota^* \)-image in (1.9) defines the line bundle \( \mathcal{O}(\iota^* \varrho) \) over \( \mathbb{P}_\Sigma \) in (2.8).

In the first quadrant cone of \( D^\dagger \),

\[\text{Int}(D^\dagger_{\Delta}) = \{ \varrho \in D^\dagger \mid \langle \varrho, e^\delta \rangle > 0 \text{ for } \delta \in \Lambda^{(0)} \} \subseteq D^\dagger_+ = \{ \varrho \in D^\dagger \mid \langle \varrho, e^\delta \rangle \geq 0 \text{ for } \delta \in \Lambda^{(0)} \}\]

consists of effective divisors of \( \mathbb{P}_\Sigma \). First we define the following elements in \( n_{\Delta} \mathbb{Q} \). For \( \delta \in \Lambda^{(0)} \), there is a (unique) simplex \( \mathbf{s}^\dagger_\delta \in \Lambda \) so that \( -\delta \in \text{Int}(\mathbf{s}^\dagger_\delta) \), equivalently, \( -\delta = \sum_{\delta^* \in \mathbf{s}^\dagger_\delta} n^\delta_{\delta^*} \delta^* \) for

\footnote{The formula (2.11) was stated in Proposition 5 of [23] for cases \( n \geq 4 \), but the formula is still valid for \( n \leq 3 \).}
positive rational numbers $n^\delta$. Define
\begin{equation}
n^\delta = n_\Delta^\delta = e^\delta + \sum_{\delta' \in \sigma^\delta} n_{\delta'}^\delta e^{\delta'} \in n_\Delta \mathcal{Q} \cap \Omega, \quad (n_{\delta'}^\delta, \in \mathcal{Q}_{>0}),
\end{equation}
whose one-parameter group $s^\delta$ acts on $T(D)$ in (2.11) with $(s = 0)$-limit outside $C_\Sigma$. Note that the above $n^\delta$'s in (2.12) depend on the triangulation $\Lambda$, and generate a convex cone in $\mathbf{n}_\Delta \mathcal{Q}$,
\[n_{\Lambda,0} := \sum_{\delta \in \Lambda(0)} \mathbf{Q}_{\geq 0} n^\delta \subseteq \mathbf{n}_\Delta \mathcal{Q} \cap \Omega.
\]
Using $n^\delta$'s in (2.12), we are going to describe a system of local coordinates of the principal bundle (2.2). The following lemma is obvious:

**Lemma 2.1** For a $(n-1)$-simplex $s \in \Lambda^{(n-1)}$, the $(d-n)$ elements, $n^\delta$ for $\delta' \in \Lambda^{(0)} \setminus s$, form a $\mathbf{n}_\Delta \mathcal{Q}$-basis with the expression,
\[n^\delta = \sum_{\delta \in s \cap \Lambda(0)} n_{\delta'}^\delta e^\delta + \sum_{\delta'' \in \Lambda^{(0)} \setminus s} n_{\delta''}^\delta e^{\delta''} \text{ with } n_{\delta'}^\delta, n_{\delta''}^\delta \geq 0, \quad n_{\delta''}^\delta = 1, \quad (\delta' \in \Lambda^{(0)} \setminus s).
\]
The above $\mathbf{n}_\Delta \mathcal{Q}$-basis, together with $e^\delta$ ($\delta \in s \cap \Lambda^{(0)}$), form a basis of $D_\Delta \mathcal{Q}$. The basis $\{e_{\delta, s}\} \cup \{n_{\delta''}^\delta, s\}$ of $D^1$ dual to $\{e^\delta\} \cup \{n^\delta\}$ is related to the basis $\{e^{\delta\dagger}\} \cup \{e^{\delta\dagger\dagger}\}$ in (1.10) by
\begin{align*}
e_{\delta, s} &= e^{\delta\dagger} + \sum_{\delta'' \in \Lambda^{(0)} \setminus s} g^{\delta''}_{\delta'} e^{\delta''\dagger}, \quad n_{\delta''}^\delta, s = \sum_{\delta'' \in \Lambda^{(0)} \setminus s} g^{\delta''}_{\delta'} e^{\delta''\dagger}; \\
e^{\delta\dagger} &= e_{\delta, s} + \sum_{\delta'' \in \Lambda^{(0)} \setminus s} n_{\delta''}^\delta e^{\delta\dagger\dagger}, \quad e^{\delta\dagger\dagger} = \sum_{\delta'' \in \Lambda^{(0)} \setminus s} n_{\delta''}^\delta n_{\delta''}^\delta,
\end{align*}
where the index $\delta \in s \cap \Lambda^{(0)}$, $\delta', \delta'' \in \Lambda^{(0)} \setminus s$, and the coefficients satisfy the relations, $(g^{\delta''}_{\delta'}) = -(n_{\delta''}^\delta/n_{\delta'}^\delta)^{-1}$ and $(g^{\delta''}_{\delta''}) = (n_{\delta''}^\delta/n_{\delta''}^\delta)^{-1}$. Furthermore, $e_{\delta, s}$ ($\delta \in s \cap \Lambda^{(0)}$) form a rational basis of the subspace $\beta^*(L^*_\mathcal{Q})$ of $D^1$ in (1.9).

For a $(n-1)$-simplex $s \in \Lambda^{(n-1)}$, the generators $s \cap \Lambda^{(0)}$ of the simplicial cone $\sigma(s)$ span a sublattice $L_s$ of $L$. It is known that the collection of $U_s = \text{Spec}[\sigma(s) \cap L^*]$ for $s \in \Lambda^{(n-1)}$ forms an affine open chart of $\mathbb{P}_\Sigma$. For a given $s$, the $T(\mathbf{n}_\Delta)$-space (2.2) over $U_s$ is given by $\pi^{-1}(U_s) = \text{Spec}[\sigma(s) \cap D^1_\Delta]$. Consider the basis $\{e^\delta\} \cup \{n^\delta\}$ of $D_\Delta \mathcal{Q}$ in Lemma 2.1. Then $e^\delta$'s and $n^\delta$'s generate a cone $\Omega_s$ in $D_\Delta \mathbb{R}$ which contains $\sigma(s)$ as a $n$-face. The relations of cones and lattices, $(\sigma(s), D_\Delta) \hookrightarrow (\Omega_s, D_\Delta) \xrightarrow{\beta} (\sigma(s), L)$, give rise to the morphisms of toric varieties: $\pi^{-1}(U_s) \rightarrow \overline{\pi^{-1}(U_s)} \rightarrow U_s$, where $\beta$ is the morphism in (1.8), and $\pi^{-1}(U_s) = \text{Spec}[\overline{\sigma(s)} \cap D^1_\Delta]$. For $\delta' \notin s$, let $k_{\delta'}$ be the positive integer characterized by the primitive property of $k_{\delta'} n^\delta$ in $\mathbf{n}_\Delta$. Consider the lattices
\[n_{\Delta, s} := \sum_{\delta'' \in \Lambda^{(0)} \setminus s} \mathbb{Z}[k_{\delta''} n^\delta] \subseteq \mathbf{n}_\Delta, \quad D_\Delta, s := \sum_{\delta \in s \cap \Lambda^{(0)}} \mathbb{Z} e^{\delta} + \mathbf{n}_{\Delta, s} \subseteq D_\Delta.
\]
The cone-lattice relation $(\Omega_s, D_{\Delta, s}) \xrightarrow{\beta} (\sigma(s), L_s)$ induces the toric morphism, $\mathcal{Q}^d = \mathcal{Q}^n \times \mathcal{Q}^{d-n} \rightarrow \mathcal{Q}^n$, with the coordinates corresponding to $e_{\delta, s}$'s and $n_{\delta''}^\delta$'s in Lemma 2.1. The morphism $\overline{\pi}$ is equivalent to the projection from $\mathcal{Q}^d/(D_\Delta/D_{\Delta, s}) = \pi^{-1}(U_s)$ to $\mathcal{Q}^n/(L/L_s) = U_s$, and we have
\[
\pi^{-1}(U_s) = (\mathbb{C}^n \times \mathbb{C}^n - \mathbb{C}^n)/(D_\Delta/D_{\Delta,s}) \quad \text{and} \quad T(n_\Delta) = \mathbb{C}^n - (n_\Delta/n_\Delta,s).
\]

Using \(e^\delta (\delta \in s \cap \Lambda(0))\), one can lift \(L\) into \(D_\Delta\) so that \(D_\Delta \simeq L \oplus n_\Delta\). Hence the inclusion \(\pi^{-1}(U_s) \hookrightarrow \pi^{-1}(U_s)\) is equivalent to
\[
\pi^{-1}(U_s) \simeq U_s \times T(n_\Delta) \hookrightarrow U_s \times \overline{T(n_\Delta)} \simeq \pi^{-1}(U_s), \quad \overline{T(n_\Delta)} := \mathbb{C}^n - (n_\Delta/n_\Delta,s),
\]
and \(\pi\) in (2.2) corresponds to the projection to the first component.

For a simplex \(s \in \Lambda^{(n-m_s)}\) for \(0 \leq m_s \leq n - 1\), it is known that the \(T(L)\)-orbit \(O_s\) fixed by \(L \cap \sigma(s)\) is isomorphic to a \(m_s\)-torus, whose closure \(\overline{O_s}\) is a \(m_s\)-dimensional toric variety \([5, 9, 12]\) (or in Appendix (A13) for the detailed structure). Consider the subspace of \(\Sigma\) to whose interior consists of all positive divisors whose \(\cap L\) over \(I P\) is a non-negative (or convex) divisor if \(\delta\) here with \(\delta^1\) \(\in \Lambda(0)\), hence \(\langle \delta, e^\delta' \rangle\) is the difference of \(e^\delta''\)-values between \(\delta\) and \(\delta'\) for \(\delta'' \notin s\).

**Definition.** \([13, 15, 23]\) Let \(\delta\) be an element in \(D^1(=D_\Delta^{+})\).

(I) \(\delta\) is a non-negative (or convex) divisor if \(\delta\) in (2.14) is an element of \(D^1_{s,+}\) for all \(s \in \Lambda^{(n-1)}\).

(II) \(\delta\) is a positive (or strictly convex) divisor if \(\delta\) in \(\text{Int}(D^1_{s,+})\) for all \(s \in \Lambda^{(n-1)}\).

Consider the positive cone in \(D^1\),
\[
C^1_+ = C^1_{\Lambda,+} = \{ \delta \in D^1 | \langle \delta, e^\delta \rangle \geq 0 \, \text{for} \, \delta' \in \Lambda(0) \setminus s, \, s \in \Lambda^{(n-1)} \},
\]
whose interior consists of all positive divisors whose \(\iota^*\)-images in \((1,9)\) are positive line bundles over \(\mathbb{P}\). We now derive an mechanism to express the positive cone \(C^1_+\). For a \((n - 1)\)-simplex \(s \in \Lambda^{(n-1)}\) and \(\delta' \in \Lambda(0) \setminus s\), we consider the following element in \(n_\Delta Q\):
\[
n^\delta_s := e^\delta + \sum_{\delta \in s \cap \Lambda(0)} n^\delta_{s,\delta} e^\delta \quad \text{with} \quad \delta' + \sum_{\delta \in s \cap \Lambda(0)} n^\delta_{s,\delta} \delta = 0 \in \mathbb{L}_Q.
\]

\[^{1}\]Here the definition is defined via the divisor lattice \(D_\Delta Q\), which is equivalent to the notion in \([13, 15, 23]\) where the convex divisor is formulated through the graph over \(\mathbb{L}_Q\). Indeed, the \(l_s, f_s\) in Definition 3 of \([23]\) are corresponding to \(\delta, \varrho\) here with \(\varrho = \sigma(s)\). Furthermore, results and definitions of Subsection \(0.2.3.1\) here can be carried over to a general toric variety \(\mathbb{P}\) defined by a simplicial cone decomposition of \(\mathbb{L}_Q\). Then \(\Lambda(0)\) is identified with the collection of all primitive \(L\)-elements representing 1-dimensional cones in \(\Sigma\), but the corresponding \(\Delta\) in \(\mathbb{L}_R\) may not be convex in general.

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Hence we obtain (I). When the vertices of \( n \) in (II) can be derived from the definition of non-negative or positive divisors. Consider the dual of \( n_{\Lambda,+} \),

\[
\mathbf{n}_{\Lambda,+}^\dagger = \text{dual cone of } n_{\Lambda,+} \text{ in } n_{\Delta, \mathbb{Q}} \cap \mathbf{H} \Rightarrow \hat{\mathbf{n}}_{\Lambda,+}(D^\dagger) = \text{dual cone of } n_{\Lambda,+} \text{ in } D_{\Delta, \mathbb{Q}}^\dagger.
\] (2.18)

Through \( \iota^* \) in (1.9), the cones in (2.18) are related by

\[
\hat{\mathbf{n}}_{\Lambda,+}(D^\dagger) = \iota^* -1(\mathbf{n}_{\Lambda,+}^\dagger), \quad \text{Int}(\hat{\mathbf{n}}_{\Lambda,+}(D^\dagger)) = \iota^* -1(\text{Int}(\mathbf{n}_{\Lambda,+}^\dagger)).
\]

**Proposition 2.1** (I) The positive cone \( C_+^\dagger \) in (2.15) is equal to \( \hat{\mathbf{n}}_{\Lambda,+}(D^\dagger) \).

(II) The non-negative divisors in the first quadrant cone of \( D^\dagger \) are given by

\[
C_+^\dagger \cap D_+^\dagger = \{ (\varrho) \in D_+^\dagger \mid \langle \varrho, n_+^\delta \rangle \geq 0 \text{ for } \delta \notin \mathbf{A}^{(n-1)}, \mathbf{s}^\dagger \mathbf{g} \mathbf{s} \neq \emptyset \},
\]

\[
\text{Int}(C_+^\dagger) \cap \text{Int}(D_+^\dagger) = \{ (\varrho) \in \text{Int}(D_+^\dagger) \mid \langle \varrho, n_+^\delta \rangle > 0 \text{ for } \delta \notin \mathbf{A}^{(n-1)}, \mathbf{s}^\dagger \mathbf{g} \mathbf{s} \neq \emptyset \},
\]

which satisfy the relations, \( \iota^*(C_+^\dagger \cap D_+^\dagger) = \mathbf{n}_{\Lambda,+}^\dagger \) and \( \iota^*(\text{Int}(C_+^\dagger) \cap \text{Int}(D_+^\dagger)) = \text{Int}(\mathbf{n}_{\Lambda,+}^\dagger) \).

**Proof.** For an element \( \delta \notin \mathbf{A}^{(n-1)} \), using Lemma 2.1 and (2.14), one finds

\[
\langle \varrho, \mathbf{e}^\delta \rangle = \sum_{\delta \in \mathbf{A}} \sum_{\delta' \notin \mathbf{A}} n_+^{\delta} g_+^{\delta'} \mathbf{e}^\delta + \mathbf{e}^\delta = \langle \varrho, \sum_{\delta \in \mathbf{A}} \sum_{\delta' \notin \mathbf{A}} n_+^{\delta} g_+^{\delta'} \mathbf{e}^\delta \rangle = \langle \varrho, \sum_{\delta \in \mathbf{A}} n_+^{\delta} g_+^{\delta} \rangle = \langle \varrho, n_+^{\delta} \rangle.
\]

The last equality in above follows from the form of \( n_+^{\delta} \in n_{\Delta, \mathbb{Q}} \) and the definition of \( n_+^{\delta} \) in (2.16). Hence we obtain (I). When \( \delta \notin \mathbf{A}^{(n-1)} \), with \( n_+^{\delta} \) as in (2.16) all non-negative, which implies the constraint \( \langle \varrho, \mathbf{e}^\delta \rangle \geq 0 \) (or > 0) for \( C_+^\dagger \cap D_+^\dagger \) (or \( \text{Int}(C_+^\dagger) \cap \text{Int}(D_+^\dagger) \) respectively) is redundant. Since \( \beta^*(L_{\mathbb{Q}}) \) is contained in \( C_+^\dagger \) with \( n_+^{\delta} \)-zero value, the relation between divisors and line bundles in (II) can be derived from the definition of non-negative or positive divisors.

By Proposition 2.1, \( n_{\Lambda,+}^\dagger \) is Kahler cone of \( \mathbb{P}_\Sigma \) consisting of all non-negative line bundles over \( \mathbb{P}_\Sigma \), regarded as a part of Kahler cone of \( X \), and will be denoted by

\[
C_{\text{Pic}}(X) = C_{\text{Pic}, \Lambda}(X) = \mathbf{n}_{\Lambda,+}^\dagger.
\]

The vertices of \( n_{\Lambda,+}^\dagger \), dual to facets of \( n_{\Lambda,+} \) in (2.17), can be described by a lifting of \( n_{\Lambda,+}^\dagger \) in \( C_+^\dagger \).

We now provide some examples as demonstration of the computation of \( n_{\Lambda,+}^\dagger \); the first two are well-known, which serve as useful 'toy'-models to illustrate the method and notations used in this subsection.

**Example 1.**

\[
\text{Rank}(n_{\Delta}) = 1 \text{ case, i.e. } (\Delta, L) = \text{a reflexive simplex}. \quad \text{Equivalently, } \mathbb{P}_\Sigma \text{ is isomorphic to an abelian quotient of weighted projective space } [223], \quad \mathbb{P}_\Sigma \text{ isomorphic to } \mathbb{P}_{\Sigma_0} = \mathbb{P}_{\Sigma_0}^{d_1} / G.
\]

\footnote{In this example, the conditions, (1.5) and (1.6), are not satisfied when \( G \neq 1 \). The positive divisors discussed here are those for the minimal toric variety \( \mathbb{P}_{\Sigma_0,L} \).}

\footnote{Some misprints occur in RIMS journal paper [223] page 832, where \( d_0 := \frac{d}{n} \text{ and dia} \{ e^{2\pi i / d_1}, \ldots, e^{2\pi i / d_5} \} \) (on line 14, 15) should be \( d_0 := \frac{d}{n} \text{ and dia} \{ e^{2\pi i / d_1}, \ldots, e^{2\pi i / d_5} \} \) respectively.}
with weights $w_i \in \mathbb{Z}_{>0}$ ($1 \leq i \leq n + 1$), and $G$ a subgroup of the diagonal finite group $SD := \{\text{diag}[t_1, \ldots, t_{n+1}] | t_i^d = 1 (1 \leq i \leq n+1)\}$, containing the element $\text{diag}[e^{2\pi i a/d_1}, \ldots, e^{2\pi i a/d_{n+1}}] \in G$, where $d = \sum_{i=1}^{n+1} w_i$ and $d_i = \frac{d}{\text{g.c.d}(d, w_i)}$. Indeed, the simplex $\Delta$ is spanned by $(n + 1)$ generators $\delta^i$s of a $n$-dimensional real vector space $V(\simeq \mathbb{R}^n)$ satisfying the relation $\sum_{i=1}^{n+1} w_i \delta^i = 0$. Consider the $n$-lattices in $V$: $L_0 = \sum_{i=1}^{n+1} \mathbb{Z}\delta^i$, and $\hat{L} = (\sum_{i=1}^{n+1} k_i \delta^i | k_i, \sum_i k_i = 0 \in \mathbb{Z} \forall i)$. Then $L$ is the lattice in $V$ satisfying the relations, $L_0 \subseteq L \subseteq \hat{L}$ and $L/L_0 \simeq G$. The sublattice $n_{\Delta, L_0}$ of $D_{\Delta, L_0}$ in (1.5) is given by $\mathbb{Z}^n$ with $w_i = \sum_{i=1}^{n+1} w_i e^{\delta^i}$, and homogeneous coordinates in (2.3) are given by $\zeta = (\zeta^i)_{i=1}^{n+1}$ with $\zeta_i$ corresponding to $e^{\delta^i} \in D^+_i$. The triangulation $\Lambda_0$ is the face-decomposition of $\Delta$ with the $(n - 1)$-simplex $s_i$ spanned by $\{\delta^i\}_{1 \neq i \leq n+1}$. Then $n_{s_i} = n$ for all $i$, hence $n_+ = \mathbb{Q}_{\geq 0} n$, and $n^t_i = \mathbb{Q}_{\geq 0} n^t$, where $n^t_i$ is the $n^t_i$-basis dual to $n$ with $\langle n^t_i, n \rangle = 1$. The element $n^t_i$ gives rise to a line bundle over $\mathbb{P}_{\Sigma_0}$, denoted by $O(1)$. The positive cone $C^+_i$ consists of $\varrho \in D^+_i$ with $\sum_i w_i \varrho^i \geq 0$, whose divisor class is the non-negative line bundle $O(\sum_i w_i)$ over $\mathbb{P}_{\Sigma_0}$, with the relation $C^+_i \cap \text{Int}(D^+_i) = \text{Int}(D^+_i)$. In particular, the divisor defined by $\zeta_i = 0$ gives rise to the bundle $O(n_i)$. 

**Example 2.** Some rank($n_\Delta$) = 2 case. We consider the blow-up of weighted projective space $\mathbb{P}^n_{(w_i)}$ at the point $\zeta^k = (\zeta^k_{n+1})_{i=1}^n$, say $k = n + 1$, denoted by $\pi : \hat{\mathbb{P}}^n_{(w_i)} \longrightarrow \mathbb{P}^n_{(w_i)}$, $\pi^{-1}(e^{n+1}) = \text{exceptional divisor } E$.

With $\delta^i (1 \leq i \leq n+1)$, $L = L_0$ in Example 1, and $\delta^{n+2} := -\delta^{n+1}$, the polytope $\Delta$ is the convex hull spanned by $\{\delta^i\}_{1 \neq i \leq n+1}$, and $\Lambda$ is simplicial decomposition of $\partial \Delta$ so that $\Lambda^{(n-1)}$ consists of the first $n$ simplices $s_i (1 \leq i \leq n)$ in Example 1, together with simplices $t_i (1 \leq i \leq n)$ given by $t_i$ = the $(n - 1)$-simplex $s_i$ spanned by $\delta^k (1 \leq k \neq i \leq n)$ and $\delta^{n+2}$. Then $\mathbb{P}_{\Sigma_0} \simeq \hat{\mathbb{P}}^n_{(w_i)}$. Since $\sum_{i=1}^{n+1} w_i \delta^i = 0$ and $\delta^{n+1} + \delta^{n+2} = 0$, the subspace $n_\Delta \mathbb{Q}$ of $D_\Delta \mathbb{Q}$ in (1.5) is generated by $n = \sum_{i=1}^{n+1} w_i e^{\delta^i}$ and $n^t = e^{\delta^{n+1}} + e^{\delta^{n+2}}$. The vectors in (2.16) are

$$n^t_{s_i} = n, \quad n^t_{s_i} = n^t_{t_i} = n^t, \quad n^t_{t_i} = n^2, \quad (1 \leq i \leq n)$$

where $n^2 = n - w_{n+1} n^1$. The positive cones in (2.17), (2.18) and Proposition 2.1 are now given by

$$_n{+} = \mathbb{Q}_{\geq 0} n^1 + \mathbb{Q}_{\geq 0} n^2, \quad n^t_i = \mathbb{Q}_{\geq 0} n^1 + \mathbb{Q}_{\geq 0} n^2,$$

$$C^+_i = \{\varrho \in D^+_i | \varrho^{n+1} + \varrho^{n+2} \geq 0, \sum_{i=1}^{n+1} w_i \varrho^i - w_{n+1} \varrho^{n+2} \geq 0\},$$

$$\text{Int}(C^+_i) \cap \text{Int}(D^+_i) = \{\varrho \in \text{Int}(D^+_i) | \sum_{i=1}^{n+1} w_i \varrho^i - w_{n+1} \varrho^{n+2} > 0\},$$

where $n^1, n^2$ are the basis of $n^t i_\Delta$ dual to $n^1, n^2$. The morphism $\pi^*$ in (1.9) is given by $\varrho \mapsto (\varrho^{n+1} + \varrho^{n+2}) n^1 + (\sum_{i=1}^{n+1} w_i \varrho^i - w_{n+2} \varrho^{n+2}) n^2$, which sends $C^+_i$ to $n^t_i$. Since $\pi^*(e^{\delta^{n+1}}) = n^1$ and $\pi^*(e^{\delta^{n+2}}) = w_1 n^2$, we find the non-negative line bundles over $\hat{\mathbb{P}}^n_{(w_i)}$: $n_1 = \pi^* O(w_{n+1})$ trivial on the exceptional divisor $E$, and $n_2 = \pi^* O(w_{n+1}) \otimes O(-w_1 E)$ trivial on fibers of the canonical projection from $\hat{\mathbb{P}}^n_{(w_i)}$ to $\mathbb{P}^n_{(w_i)}(1 \leq i \leq n)$. 

**Example 3.** Flop of hypersurfaces in 4-dimensional toric variety. Let $\mathbb{P}_{\Sigma(\Delta)}$ be a 4-dimensional toric variety associated to a triangulation $\Delta$ of $\partial \Delta$ satisfying (1.5) and (1.6). Suppose that there

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*The polytope in this example is not reflexive. As stated in footnote of the definition of positive divisors, Proposition 2.1 also holds for a general toric variety.*
are two 3-simplices \( s, t \in \Lambda^{(3)} \) with the form \( s = \delta \delta \delta \delta \delta \), \( t = \delta \delta \delta \delta \delta \), where \( \delta \) (1 \leq \delta \leq 4) are 4 elements in a 2-dimensional face of \( \triangle \) satisfying the relation \( \delta + \delta + \delta = \delta + \delta + \delta \). Then a generic anti-canonical hypersurface \( X \) of \( P_{\Sigma(A)} \) is non-singular in the open chart \( U_a \cap U_v \) of \( P_{\Sigma(A)} \). Define another two 3-simplices, \( s^2 = \delta \delta \delta \delta \delta \delta \) and \( t^2 = \delta \delta \delta \delta \delta \delta \), in \( \partial \triangle \) with the property \( s \cup t = s^2 \cup t^2 \).

We form another triangulation \( \Lambda^2 \) of \( \partial \triangle \) by replacing \( s, t \in \Lambda^{(3)} \) for \( s^2, t^2 \in \Lambda^2 \). The toric variety \( P_{\Sigma(A^2)} \) is a flop of \( P_{\Sigma(A)} \) so that the anti-canonical hypersurface \( X^2 \) of \( P_{\Sigma(A^2)} \) defined by the same equation as \( X \) is a flop in birational geometry of 3-folds. In \( D^i \), there are two positive cones, \( C^{(i)}_{\Lambda^+, a} \) for \( P_{\Sigma(A)} \) and \( C^{(i)}_{\Lambda^+, a} \) for \( P_{\Sigma(A^2)} \). Since \( \delta + \delta = \delta + \delta \), \( n := e^\delta + e^\delta - e^\delta - e^\delta \) is an element in \( n_\Lambda \), and vectors in \((2.16)\) are expressed by

\[
\Lambda : n^\delta = n^\delta = n, \quad \Lambda^3 : n^\delta = n^\delta = -n.
\]

By Proposition \([2.1]\) we find

\[
C^{(i)}_{\Lambda^+, a} \subset \{ \varrho \in D^i | \langle \varrho, n \rangle \geq 0 \}, \quad C^{(i)}_{\Lambda^+, a} \subset \{ \varrho \in D^i | \langle \varrho, n \rangle \leq 0 \},
\]

hence \( P_{\Sigma(A)} \) and \( P_{\Sigma(A^2)} \) have no common positive divisors, equivalently \( \text{Int}(n^\delta_{\Lambda^+, a}) \cap \text{Int}(n^\delta_{\Lambda^+, a}) = \emptyset \).

The process of flops indeed appears in the Gorenstein toric variety \( P_{\Sigma(A)} = P_{(w_1)} / G \) over \( P_{(w_1)} / G \) with \( \Lambda \) satisfying \((1.5)\) and \((1.6)\). Since any triangulation \( \Upsilon \) on all faces of codimension \( \leq 2 \) of a \( n \)-simplex \( \triangle \) satisfying \((1.5)\) can always be extended to a triangulation \( \Lambda \) of \( \partial \triangle \) with the same \( \Lambda^{(0)} \) one, can construct a flop process by changing the triangulation on 2-dimensional faces of the 4-dimensional simplex \( \triangle \) whenever the group \( G \) is sufficient "big", e.g. \( \hat{P}^4 / SD \) with \( G = SD \) and \( w_i = 1 \) for all \( i \). Indeed, the extended \( (n - 1) \)-simplices \( \Lambda^{(n-1)} \) of \( \partial \triangle \) can be constructed from \( \Upsilon \) as follows. Let \( \Delta_{n-1} \) be a facet of \( \triangle \). If \( \Delta_{n-1} \cap \Lambda^{(0)} \) is equal to \( \Upsilon(\Delta_{n-1}) \) (the set of vertices of \( \Delta_{n-1} \)), we set \( s = \Delta_{n-1} = \Lambda^{(n-1)} \). Otherwise, \( \Delta_{n-1} \) can be expressed as the join of a vertex \( v \) and a \( (n - 2) \)-face \( \Delta_{n-2} \) of \( \Delta_{n-1} = v * \Delta_{n-2} \), with the property \( \Lambda^{(0)} \setminus \Upsilon(\Delta_{n-2}) \neq \emptyset \). Write \( \Delta_{n-2} = \cup u, \) where \( u \in \Upsilon^{(n-2)} \) and \( u \subset \Delta_{n-2} \). Then \( \Delta_{n-1} = \cup \{ v * u \} \). We set \( s = v * u \in \Lambda^{(n-1)} \).

\[ u \in \text{Int}(\Delta_{n-2}). \]

Otherwise, \( u \) is spanned by a finite number of vertices of \( \Delta_{n-2}, v_1, \ldots, v_{n-1-k} \) \((k \geq 1)\), together with \( \delta_1, \ldots, \delta_k \in \Lambda^{(0)} \setminus \Upsilon(\Delta_{n-2}) \). Then \( v * u = u' \) \((u' \in \Upsilon^{(n-1-k)}, \) and \( v * u = \cup \{ u' * \delta_1, \ldots, \delta_k \}, \) by which we set \( s = u' * \delta_1, \ldots, \delta_k \in \Lambda^{(n-1)} \). Therefore we obtain a triangulation \( \Lambda \) of \( \partial \triangle \) as an extension of \( \Upsilon \) with the same 0-skeleton.

### 3 Deformation Space of Anti-canonical Hypersurface in Toric Variety

Consider the dual polytope \((\Delta^*, L^*) \) in \( L^*_\mathbb{R} \), equipped with a simplicial decomposition \( \Lambda^* = \{ \Lambda^{*j} \}_{j=1}^{n-1} \) of \( \partial \Delta^* \) satisfying \((1.5)\) and \((1.6)\) for \( \Lambda^* \) and \( L^* \):

\[
0 \longrightarrow n_{\Delta^*} \longrightarrow D_{\Delta^*} \longrightarrow L^* \longrightarrow 0, \quad 0 \longrightarrow L \xrightarrow{\beta^*} D^i_{\Delta^*} \xrightarrow{e^\beta^*} n^i_{\Delta^*} \longrightarrow 0, \quad (3.1)
\]

where \( \beta^*(e^\beta^*) = \delta^* \) for \( \delta^* \in \Lambda^{*0} \). Note that no relation is imposed between the triangulation \( \Lambda^* \) of \( \partial \Delta^* \) and \( \Lambda \) of \( \partial \Delta \) in Section \([2]\) We shall denote the origin of \( L^*_\mathbb{R} \) by \( 0^* \), and the collection of
vertices of \( \Delta^* \) by \( \mathcal{V}(\Delta^*) \). A simplex in \( \Lambda^* \) will be denoted by \( s^* \), and \( d^* := |\Lambda(0)| \). Corresponding to the basis \( e^\delta \)'s of \( D_{\Delta^*} \), the torus \( T(D_{\Delta^*}) \) is parametrized by coordinates \( u = (u_\delta^*)_{\delta^* \in \Lambda^*} \). The simplicial decomposition \( \Lambda^* \) of \( \partial \Delta^* \) gives rise to the \( T(D_{\Delta^*}) \)-variety \( \mathfrak{C}_{\Xi^*} \) as in (2.1):

\[
u = (u_\delta^*)_{\delta^* \in \Lambda^*(0)} \in \mathfrak{C}_{\Xi^*} = D_{\Delta^*} \mathfrak{C} - \bigcup_{K \notin \{s \in \Lambda^*(0) \mid s \in \Lambda^* \}} \{ \sum u_\delta^* e^\delta^* \mid u_\delta^* = 0 \text{ for } \delta^* \in K \}. \tag{3.2} \]

By Proposition 2 and Section 7 in [23] or (A4) in Appendix, the space of anti-canonical bundle of \( \mathbb{P}_\Sigma \) or \( \mathbb{P}_{\Sigma_0} \) is given by

\[
\Gamma(\mathbb{P}_{\Sigma_0}, \mathcal{O}(\iota_0^*(-\kappa_0))) \simeq \Gamma(\mathbb{P}_\Sigma, \mathcal{O}(\iota^*(-\kappa))) \simeq \bigoplus \{ \mathfrak{C} \nu^* \mid \nu^* \in \Delta^* \cap L^* \}. \tag{3.3} \]

By (A3) in Appendix, the above basis elements \( \nu^* \) have the following homogeneous-coordinate representation using \( \zeta = (\zeta_v)_{v \in \mathcal{V}(\Delta)} \) of \( \mathbb{P}_{\Sigma_0} \) in (2.4) or \( z = (z_\delta)_{\delta \in \Lambda(0)} \) of \( \mathbb{P}_\Sigma \) in (2.2):

\[
\zeta_v^* = \prod_{v \in \mathcal{V}(\Delta)} \zeta_v^* \quad \text{and} \quad z_\delta^* = \prod_{\delta \in \Lambda(0)} z_\delta^* \quad \text{where} \quad v_\delta^* = (v^*, \delta) + 1 \in \mathbb{Z}_{\geq 0} \quad \text{for} \quad \delta \in \Lambda(0). \tag{3.4} \]

Note that \( \zeta_v^* \) is determined by \( \zeta_v \) via the regular map \( \tilde{\varphi} \) in (2.7) : \( \tilde{\varphi}(\zeta_v^*) = z_v^* \). Furthermore, the above functions define the zero-divisor for the section \( \nu^* \) in \( \mathbb{P}_{\Sigma_0} \) and \( \mathbb{P}_\Sigma \) respectively,

\[
\sum_{v \in \mathcal{V}(\Delta)} \nu_v^* e_{v^*} \in D_{\Delta^*} \quad \text{and} \quad \sum_{\delta \in \Lambda(0)} \nu_\delta^* e_{\delta^*} \in D_{\Delta^*},
\]

which are related by \( p^* \) in (2.6). The zero-loci of a generic section in (3.3) define a quasi-smooth hypersurface \( X \) of \( \mathbb{P}_\Sigma \), and the same for \( X_0 \) in \( \mathbb{P}_{\Sigma_0} \). Then \( X = \varphi^{-1}(X_0) \) with a dominating morphism induced by \( \varphi : \mathbb{P}_\Sigma \longrightarrow \mathbb{P}_{\Sigma_0} \):

\[
\varphi(= \varphi_{\text{restriction}}) : X \longrightarrow X_0. \tag{3.5} \]

Note that both \( X \) and \( X_0 \) are disjoint with 0-dimensional torus-orbits in the toric variety \( \mathbb{P}_\Sigma \) or \( \mathbb{P}_{\Sigma_0} \) (see Appendix, the case \( m_a = m = 0 \) in Proposition 5.3 (III)). By (3.3), \( X \) is a CY \((n-1)\)-space Gorenstein over the (possibly) singular CY space \( X_0 \). By condition (1.5) and the property of reflexive polytope which contains no integral points in the interior except the origin, \( X \) is indeed non-singular when \( n \leq 4 \), hence a crepant resolution of \( X_0 \) (see, e.g., [16] Proposition 2, or [23] Proposition 6). We shall write a general element of (3.3) in the form

\[
\sum_{w^* \in \mathcal{V}(\Delta^*)} w^* + a0^* + \sum_{\nu^* \in \Delta^* \cap L^*} \alpha_{\nu^*} \nu^*, \quad a \neq 0, \quad \alpha_{\nu^*} \in \mathfrak{C} \tag{3.6} \]

regarded as the hypersurface deformation of

\[
X : \quad f(z) = \sum_{w^* \in \mathcal{V}(\Delta^*)} z^{w^*} + az^0 = 0 \subset \mathbb{P}_\Sigma; \quad \text{and} \quad \tag{3.7}
X_0 : \quad f(\zeta) = \sum_{w^* \in \mathcal{V}(\Delta^*)} \zeta^{w^*} + az^0 = 0 \subset \mathbb{P}_{\Sigma_0}.
\]

In this section, we are going to determine the deformation space of \( X \) through parameters in (3.6).
3.1 Deformation of anti-canonical hypersurfaces in minimal toric variety

In this subsection, we determine the deformation space of $X_0$ in the minimal toric variety $\mathbb{P}_{\Sigma_0}$ by using homogeneous coordinates $\zeta = (\zeta_v)_{v \in \mathcal{V}(\Delta)}$. First we will show that by the linear change of variables of $\mathbb{C}_z$ in (2.4), the parameters $\alpha_{v,*}$'s in (3.6) can be reduced to those in faces of dimension less than $n-1$ by Jacobian-ring technique, as argued in [18], for the simplicial-$\Delta$ case, i.e. Fermat hypersurfaces in weighted projective spaces. By (3.4), monomials in the expression (3.7) of $f(\zeta)$ are given by

$$\zeta_0^* = \prod_{v \in \mathcal{V}(\Delta)} \zeta_v^*, \quad \zeta^w = \prod_{v \in \mathcal{V}(\Delta)} \zeta_v^w \quad (w_v^* \geq 1) \quad \text{for } w^* \in \mathcal{V}(\Delta^*),$$

where $F(w^*)$ is the facet of $\Delta$ dual to $w^*$; and the coordinates for an element $x^*$ in the interior $\text{Int}(F^*(x))$ of a facet $F^*(x)$ of $\Delta^*$ dual to a vertex $x \in \mathcal{V}(\Delta)$ are expressed by

$$\zeta^x = \prod_{v \in \mathcal{V}(\Delta), v \neq x} \zeta_v^x, \quad x_v^* \geq 1.$$  \hspace{1cm} (3.8)

The Jacobian ring of $f(\zeta)$ in (3.7) is the ideal in $\mathbb{C}[\zeta]$ generated by partial derivatives of $f(\zeta)$: $J(f) = \langle \partial_v f(\zeta) |_{\mathcal{V}(\Delta)} \rangle$, where $\partial_v := \frac{\partial}{\partial \zeta_v}$. Note that by (3.3), $f(\zeta) = J(f)$ since $f(\zeta) = \sum_{v \in \mathcal{V}(\Delta)} \beta_v \zeta_v \partial_v f(\zeta)$, where $\beta_v$'s are positive rational numbers satisfying $\sum_{v \in \mathcal{V}(\Delta)} \beta_v = 1$, and $\sum_{v \in \mathcal{V}(\Delta)} \beta_v \zeta = 0$ in $\Delta$.

**Lemma 3.1** \hspace{0.5cm} \hspace{0.5cm} Let $X_0$ be defined by $f(\zeta)$ in (3.7) with a generic $a$, and $x^*$ be a $L^*$-integral element in interior $\text{Int}(F^*(x))$ of a facet $F^*(x)$ of $\Delta^*$ in (3.8). Then

$$x^* \equiv \sum \{ \Phi^x | \Phi^x \in \text{I} \chi \cap (\partial \Delta^* \cup \bigcup_{F^*: \text{facet of } \Delta^*} \text{Int}(F^*)) \} \quad \text{ (mod } J(f)) \quad \hspace{1cm} (3.9)$$

**Proof.** For an element $x^*$ in (3.8), let $m(= m(x^*)) := \min \{ x_v^* | v \in \mathcal{V}(\Delta) \} \geq 1$, and write $x^* = \eta^x(\prod_{v \in \mathcal{V}(\Delta)} \zeta_v)^m$, where $\eta^x = \prod_{v \in \mathcal{V}(\Delta) \setminus \{ x \}} \zeta_v^x$ with $x_v^* = x_v^* - m \geq 0$. Since $\partial_v f \in J(f)$,

$$\zeta^x = \eta^x(\prod_{v \in \mathcal{V}(\Delta) \setminus \{ x \}} \zeta_v)^m \equiv (-a)^m \eta^x \sum \{ \partial_v \zeta_v^w | w \in \mathcal{V}(\Delta^*) \} \sum \eta_{m,w,*}^x \quad \text{(mod } J(f)) \quad \hspace{1cm} (3.10)$$

where the index $\{ m_{w,*} \}$ runs $m_{w,*} \in \mathbb{Z}_{\geq 0}$ with $\sum_{w \in \mathcal{V}(\Delta^*)} m_{w,*} = m$, and

$$\eta_{m,w,*}^x := \eta^x \prod_{w \in \mathcal{V}(\Delta^*) \setminus \{ x \}} \left( \partial_v \zeta_v^w \right)^{m_{w,*}} = \prod_{w \in \mathcal{V}(\Delta^*) \setminus \{ x \}} w_v^{m_{w,*}} \prod_{v \in \mathcal{V}(\Delta)} \zeta_v^{m_{w,*}} \zeta_v^x.$$

Note that $\partial_v \zeta_v^w = 0$ when $w \in F^*(x)$, (here we use the convention $0^0 := 1$). Using $\frac{\partial_v \zeta_v^w}{\partial_v \zeta_v^w} = w_v^* \zeta_v^w$, and $\eta^x = \frac{\zeta^x \prod_{w \in \mathcal{V}(\Delta^*) \setminus \{ x \}} \left( \partial_v \zeta_v^w \right)^{m_{w,*}}}{(\partial_v \zeta_v^w)^m}$, one finds

$$\eta_{m,w,*}^x = \zeta^x \prod_{w \in \mathcal{V}(\Delta^*) \setminus \{ x \}} (\partial_v \zeta_v^w)^{m_{w,*}} = (\prod_{w \in \mathcal{V}(\Delta^*)} w_v^{m_{w,*}})^x \zeta^x \prod_{w \in \mathcal{V}(\Delta^*)} (\zeta_v^w)^{m_{w,*}}.$$  \hspace{1cm} (3.11)

---

\footnote{This lemma and the relation (3.11) in this subsection correspond to results in $n$-simplex $\Delta$ case through the Jacobian-ring calculation of a Fermat-type polynomial, which is easily performed (see, e.g. [18] or [21] Theorem 1), where the coefficients $a_{\ell,v}$'s in the proof of Lemma 3.1 here are zeros when $n \geq 4$, but could be non-zeros in the case $n = 3$.}
Then $\eta_{\{m_w^*\}}^* \neq 0$ iff $m_{w^*} = 0$ for all $w^* \in F^*(x)$, in which case, zeros of the monomial $\eta_{\{m_w^*\}}^*$ define a toric divisor of $\mathbb{P}_{\Sigma_0}$ linear equivalent to the zero-divisor of $\zeta^*$, equivalently $x^*\left(\{m_{w^*}\}\right) := x^* + \sum_{w^* \in \mathcal{V}(\Delta^*)} m_{w^*} w^* \in L^* \cap \Delta^*$ by (3.3). By $x^* \in \text{Int}(F^*(x))$ and $0^* \in \text{Int}(\Delta^*)$, one finds $x^*(\{m_{w^*}\}) \neq x^*$. Therefore

$$x^* \equiv (-a)^{m(x^*)} \sum_{\{m_{w^*}\} \in \mathcal{V}(\Delta^*)} x^*(\{m_{w^*}\}) \prod_{w^* \in \mathcal{V}(\Delta^*)} w^* m_{w^*} \pmod{J(f)},$$

where $m_{w^*} \in \mathbb{Z}_{\geq 0}$ and $m_{w^*} = 0$ for $w^* \in F^*(x)$ with $\sum_{w^* \in \mathcal{V}(\Delta^*)} m_{w^*} = m(x^*) (\geq 1)$, $x^*(\{m_{w^*}\}) \in L^* \cap \Delta^* \setminus \{x^*\}$. Denote the vector space on the right side of (3.9) by $V$, and label elements in $L^* \cap \left(\cup_{F^* \text{ facet of } \Delta^* \text{ Int}(F^*)}\right)$ by $x^*_\ell$ for $1 \leq \ell \leq r$. By $f(\zeta) \in J(f)$, $0^* \equiv V \pmod{J(f)}$. Hence the above relation implies that for each $\ell$, there exist a positive integer $m(x^*_\ell)$ and $a_{\ell, \ell'} \in \mathbb{Z}$ so that

$$x^*_\ell - (-a)^{m(x^*_\ell)} \sum_{\ell' \neq \ell} a_{\ell, \ell'} x^*_\ell' \equiv V \pmod{J(f)}.$$

Therefore for generic $a$, $x^*_\ell \equiv V \pmod{J(f)}$ for all $\ell$. This shows the relation (3.9).

By Lemma 3.14, we may assume $\alpha_{v^*}$'s appeared in the parameters of (3.6) only for $v^*$ in faces of $\Delta^*$ of dimension less than $n - 1$, i.e. $v^* = \delta^* \in \Lambda^*(0)$ by condition (1.3) for the triangulation $\Lambda^*$ of $\partial \Delta^*$. However, there are still constraints (modulus $J(f)$) remained among the coefficients $a$ and $\alpha_{v^*}$'s. Since $\mathcal{V}(\Delta^*) \subseteq \Lambda^*(0)$, the monomials $w^*$'s in $f(\zeta)$ can be absorbed into the rest parameters of (3.6), expressed in the form of moduli space using variables in (3.2):

$$\zeta^{0*} + \sum_{\delta^* \in \Lambda^*(0)} u_{\delta^*} \zeta^{\delta^*}, \; u = (u_{\delta^*}) \in \mathfrak{F}_{\Sigma^*}. \tag{3.10}$$

Note that the above space depends on the triangulation $\Lambda^*$, which might not contain the defining equation of $X_0$ in (3.7), but with sufficient variables to describe the deformation near $X_0$. We are going to describe a parametrization of (3.10) using the lattice $D^1_{\Delta^*}$ in (3.11). Choose a base element $b = (b_{\delta^*})$ in $\mathfrak{F}_{\Sigma^*}$. An element $\mu \in D^1_{\Delta^*} \mathfrak{Q}^*$ denoted by

$$\mu = \sum_{\delta^* \in \Lambda^*(0)} \mu_{\delta^*} e^{\delta^*} \in D^1_{\Delta^*} \mathfrak{Q}^*, \tag{3.11}$$

gives rise to an one-parameter family in (3.10):

$$X_0(b; \mu) : \zeta^{0*} + s^\mu \cdot b = 0, \quad s^\mu \cdot b := \sum_{\delta^* \in \Lambda^*(0)} s^{\mu_{\delta^*}} b_{\delta^*} \zeta^{\delta^*}, \quad (s \in \mathfrak{Q}^*). \tag{3.12}$$

The families $X_0(b; \alpha \mu)$ and $X_0(b; \mu)$ for $\alpha \in \mathfrak{Q}_{>0}$ are related by the change of parameter $s \mapsto s^\alpha$, hence essentially are the same. In particular, $X_0(b; \delta^* e^{\delta^*}) = \zeta^{0*} + s_{\delta^*} b_{\delta^*} \zeta^{\delta^*} + \sum_{\delta^* \neq \delta^* \in \Lambda^*(0)} b_{\delta^*} \zeta^{\delta^*}$, by which we introduce the multi-variable deformation,

$$X_0(b; \{e^{\delta^*}|\delta^* \in \Lambda^*(0)\}) : \zeta^{0*} + \sum_{\delta^* \in \Lambda^*(0)} s_{\delta^*} b_{\delta^*} \zeta^{\delta^*} = 0. \tag{3.13}$$
Note that the above expression provides a parametrization of $\mathbf{T}(D_{\Delta^*})$-orbit in $\mathbf{C}_{\Sigma^*}$ containing the base point $b$ in (3.10), and (3.12) is obtained by the substitution $s_{\delta^*} = s_{\mu_{\delta^*}}$ in (3.13). With $b$'s in all $\mathbf{T}(D_{\Delta^*})$-orbits, (3.13) provides a simultaneous parametrization of the deformation space (3.10). However among families in (3.12), those induced from automorphisms of toric variety $\mathbb{P}_{\Sigma_0}$ are redundant ones. Indeed by (Remark of) Lemma 5.1 in Appendix, an one-parameter group of $\mathbf{T}(L)$, i.e. $\beta^*\xi(\xi)$ in (3.1) for $\xi \in L$, acts on $v^*$ in (3.3) by $(s, v^*) \mapsto s(v^* \xi) v^*$, where $s \in \mathbb{C}^* (= \mathbb{C}/\mathbb{Z})$ is the independent variable of one-parameter group $\xi$ in the expression of $X_0(b; \beta^*\xi(\xi))$ (3.12) with $\beta^*\xi(\xi) = \sum_{\delta^* \in \Lambda^*(0)} (\delta^*, \xi) e^{s_{\delta^*}}$. Any two deformations in (3.10) differ by $X_0(b; \beta^*\xi(\xi))$ for $\xi \in L$ are equivalent up to a family of transformations of $\mathbb{P}_{\Sigma_0}$-automorphism. Hence deformations of $X_0$ modulo $\mathbb{P}_{\Sigma_0}$-automorphism are identified with $D_{\Delta^*}^\dagger \mathbf{Q}$ modulo $L\mathbf{Q}$, which is the same as $n_{\Delta^*}^\dagger \mathbf{Q}$ by (3.1). Hence we obtain the following description of deformation classes:

$$\text{Def}(X_0)_{\mathbf{Q}} \simeq n_{\Delta^*}^\dagger \mathbf{Q}. \quad (3.14)$$

Note that $n_{\Delta^*}^\dagger \mathbf{Q}$ is defined through integral points in the faces of $\Delta^*$, which does not depend on the detailed triangulation structure of $\Lambda^*$. A realization of $\text{Def}(X_0)_{\mathbf{Q}}$ is provided by a lifting of $n_{\Delta^*}^\dagger \mathbf{Q}$ in $D_{\Delta^*}^\dagger \mathbf{Q}$ through $t^*$ in (3.11).

### 3.2 Deformation of anti-canonical hypersurface in toric variety defined by reflexive polytope

In this subsection, we describe a combinatorial representation of the deformation space of anti-canonical hypersurface $X$ of $\mathbb{P}_\Sigma$ in (3.7). By (1.1) and (1.3), $\text{Def}(X)_{\mathbf{Q}}$ can be regarded a subspace of the Hodge space $H^{n-2,1}(X)$ in $H^{n-1}(X)$, similarly $\text{Def}(X_0)_{\mathbf{Q}} \subseteq H^{n-2,1}(X_0) \subseteq H^{n-1}(X_0)$. Using Mayer-Vietoris cohomology sequences, one finds the cohomology relation between $X$ and $X_0$ through $\varphi$ in (3.5):

$$H^{n-1}(X) \simeq H^{n-1}(X_0) \oplus \left( \oplus_E H^{n-1}(E) \right), \quad H^{n-2,1}(X) \simeq H^{n-2,1}(X_0) \oplus \left( \oplus_E H^{n-3,0}(E) \right)$$

where $E$'s run the exceptional divisors in $X$, and $H^{n-1}(X)$ is related to $H^{n-1}(E)$ by restriction of cohomology class, equivalent to the cup-product of (1,1)-form representing the Chern class of $\mathcal{O}(E)$, (see, e.g. [17] Theorem 2). By Proposition 5.3 (III) (with $\rho = -\kappa, m_{\delta} = n - 1$) in Appendix, $H^{n-3,0}(E) = 0$ except $\varphi(E)$ is a hypersurface of a $(n - 2)$-dimensional toric subvariety of $\mathbb{P}_{\Sigma_0}$, in which case $E = X \cap \overline{\mathcal{O}_s}$ for a vertex $s \in \Lambda^{(0)} \cap \text{Int}(F)$ with $F$ a 1-dimensional face of $\Delta$, and $H^{n-3,0}(E)$ is represented by (A19) with $F^* = F^*, m = n - 2$. Indeed, the exceptional divisor $E$ is generic $\mathbb{P}^1$-bundle over $\varphi(E)$, and the contribution $H^{n-3,0}(E)$ in the deformation space are induced from canonical sections of $\varphi(E)$. Hence $\text{Def}(X)_{\mathbf{Q}}$ is the direct sum of $\text{Def}(X_0)_{\mathbf{Q}}$ with all those $H^{n-3,0}(E)$'s. By (3.14), we obtain the combinational representation of deformation space of $X$:

$$\text{Def}(X)_{\mathbf{Q}} \simeq n_{\Delta^*}^\dagger \mathbf{Q} \oplus \bigoplus_{F^*, \nu_{F^*}} \mathbf{Q} \nu_{F^*}. \quad (3.15)$$

where $F^*$ runs over codim. 2 faces of $\Delta^*$, and $\nu_{F^*} \in (\text{Int}(F^*) \cap L^*) \times (\text{Int}(F) \cap L)$. For $\mu \in D_{\Delta^*}^\dagger \mathbf{Q}$ in (3.11), the deformation $t^* \mu \in n_{\Delta^*}^\dagger \mathbf{Q}$ in $\text{Def}(X)_{\mathbf{Q}}$ is induced from $X_0(b; \mu)$ in (3.14), hence by
which is a degeneration with $\mu$-degenerations. Note that any such deformation consists of non-negative (one-parameter) degeneration classes. The interior of $\mathfrak{h}(\Lambda^*)$ in (3.13), is represented by

$$\{x = \sum_{s \in \Lambda^*} \langle \mu, e_s^* \rangle e_s^* \in \mathfrak{h}(\Lambda^*) \mid \langle \mu, e_s^* \rangle \geq 0, (\delta_s^* \notin s^* \in \Lambda^{(s(n-1))})\},$$

$$\mathfrak{h}(\Lambda^*)$$

3.3 Degeneration cone in moduli space of anti-canonical hypersurface of a toric variety

In this subsection, we discuss the degeneration of anti-canonical hypersurfaces of a toric variety in the moduli space (3.10), whose structure depends on the simplicial decomposition $\Lambda^*$ of $\partial \Delta^*$. In (3.13), we use the one-parameter families of $e_s^*$’s to parameterize the moduli space (3.10). However, the structure of $n^\dagger_{\Delta^*} \mathfrak{q}$ in (3.15) will be better represented by a $D^\dagger_{\Delta^*} \mathfrak{q}$-basis dual to a basis of $D_{\Delta^*} \mathfrak{q}$ which contains a $n_{\Delta^*} \mathfrak{q}$-basis. For $(n-1)$-simplex $s^* \in \Lambda^{(n-1)}$, let $\{e^s_{\delta} \}_{\delta \in \mathbb{R}^{n-1} \cap \Lambda^{(n-1)}} \cup \{n^\dagger \} = \{e^s_{\delta} \} \cup \{n^\dagger \}$ be the basis of $D_{\Delta^*} \mathfrak{q}$ in Lemma 2.1 with respective to $(L^*, \Lambda^*)$, and $\{e^s_{\delta^*} \} \cup \{n^\dagger \}$ is the dual basis. As in (2.14), any element $\mu \in D^\dagger_{\Delta^*} \mathfrak{q}$ can be expressed by

$$\mu = \mu^\dagger + \mu_{s^*}, \quad \mu^\dagger = \sum_{\delta \in \mathbb{R}^{n-1}} \langle \mu, e^s_{\delta} \rangle e^s_{\delta} \in \mathfrak{h}(\Lambda^*), \quad \mu_{s^*} = \sum_{\delta \notin \mathbb{R}^{n-1}} \langle \mu, n^\dagger \rangle n^\dagger \in D^\dagger_{s^*}. \quad (3.17)$$

Parallel to (2.15), Proposition 2.1 and (2.17), we define the positive cone in $D^\dagger_{\Delta^*} \mathfrak{q}$,

$$C_{s^*}^+ (= C_{s^*}^+ \mathfrak{q}) = \{\mu \in D^\dagger_{\Delta^*} \mathfrak{q} \mid \langle n^\dagger, e^s_{\delta} \rangle (= \langle \mu, e^s_{\delta} \rangle) \geq 0 (\delta \notin s^* \in \Lambda^{(n-1)})\},$$

where $n^\dagger \in n_{\Lambda^*} \mathfrak{q}$ are defined by (2.16) for a $(n-1)$-simplex $s^* \subset L^*$. All $n^\dagger$’s generate the cone $n_{\Lambda^*}^+ = \sum_{\delta \notin \mathbb{R}^{n-1}} \langle \mu, n^\dagger \rangle n^\dagger \in n_{\Lambda^*} \mathfrak{q}$. As in (2.18), we define the degeneration cone

$$C_{\text{Deg}}(X) (= C_{\text{Deg}, \Lambda^*}(X)) = n_{\Lambda^*}^+ = \text{dual cone of } n_{\Lambda^*}^+ \subseteq n_{\Lambda^*} \mathfrak{q}, \quad (3.18)$$

which consists of non-negative (one-parameter) degeneration classes. The interior of $C_{s^*}^+$ consists of all positive degeneration of anti-canonical hypersurfaces in $\mathbb{P}_{\Lambda^*}$. In particular, elements in $\text{Int}(C_{s^*}^+) \cap \text{Int}(D^\dagger_{\Delta^*} \mathfrak{q})$ are CY degenerations with $(s = 0)$-limit $z_0^s$, called the maximal unipotent degenerations. Note that any such deformation $\mu$ is equivalent to $\mu_{s^*}$ in (3.17) for $s^* \in \Lambda^{(n-1)}$, which is a degeneration with $(s = 0)$-limit $\sum_{\delta \notin \mathbb{R}^{n-1}} \langle \mu, e^s_{\delta} \rangle e^s_{\delta} z^s$.

4 Mirror Symmetry of Anti-canonical Hypersurfaces in Toric Varieties of Reflexive Polytopes

We start with a triangulation $\Lambda$ of a reflexive polytope $(\Delta, L)$ and a triangulation $\Lambda^*$ of the dual polytope $(\Delta^*, L^*)$. In Sections 2 and 3, we derive the combinatorial representation of $\text{Pic}(X)_{\mathfrak{q}}$ and
Def\((X)\) for a generic anti-canonical hypersurface \(X\) of \(\mathbb{P}_{\Sigma(\Lambda)}\) in (2.11) and (3.15) respectively, then find an explicit form of Kahler cone \(n^\Lambda_{A,+}\) of \(\mathbb{P}_{\Sigma(\Lambda)}\) in (2.19), and degeneration cone \(n^\Lambda_{A,+}\) in the moduli space (3.10) of \(X\) in (3.18). On the other hand, \((\Delta^*, L^*)\) gives rise to a \(T(L^*)\)-toric variety \(\mathbb{P}_{\Sigma(\Lambda^*)}\), whose anti-canonical global sections possess a basis represented by \(\Delta \cap L\) as in (3.3). Let \(X^*\) be a generic anti-canonical hypersurface of \(\mathbb{P}_{\Sigma(\Lambda^*)}\). By regarding \(L^*\) as the group of one-parameter subgroups of \(T(L^*)\) with the character group \(L\), one finds the combinatorial representatives of Pic\((X^*)\) are the same as Def\((X)\) in (3.15), and a similar identification also for Def\((X^*)\) and Pic\((X)\). Therefore \(X\) and \(X^*\) constitute a mirror CY pair so that the relation (1.3) holds. Furthermore, the Kahler cone \(C_{Pic(\Lambda^*)}(X^*)\) of \(\mathbb{P}_{\Sigma(\Lambda^*)}\) is identified with the degeneration cone \(C_{Deg(\Lambda^*)}(X)\) for \(X\) in (3.18), and the same for the equality between \(C_{Deg(\Lambda^*)}(X^*)\) and the Kahler cone \(C_{Pic(\Lambda)}(X)\) of \(\mathbb{P}_{\Sigma(\Lambda)}\). In summary, the following structures between a mirror anti-canonical hypersurfaces of two toric varieties defined by a reflexive and its dual polytopes are interchangeable by identifying their toric representatives:

\[
\begin{align*}
\text{Picard group} : \text{Pic}\mathcal{C} & \quad \iff \quad \text{Deformation space} : \text{Def}\mathcal{C}, \\
\text{A line bundle} & \quad \iff \quad \text{One – parameter deformation}, \\
\text{Principal bundle space over toric variety} & \quad \iff \quad \text{Moduli space of anti – canonical hypersurface}, \\
\text{Kahler cone} : \mathcal{C}_{Pic} & \quad \iff \quad \text{Degeneration cone} : \mathcal{C}_{Deg}, \\
\text{A positive effective divisor class} & \quad \iff \quad \text{a maximal unipotent degeneration}.
\end{align*}
\]

The \(C_{Pic(\Lambda)}(X)\) of \(X\) in above describes the Kahler cone of \(\mathbb{P}_{\Sigma(\Lambda)}\), which is equal to degeneration cone \(C_{Deg(\Lambda^*)}(X^*)\) of its mirror \(X^*\), both depending on the triangulation \(\Lambda\) of \(\partial \Delta\). In particular for a CY 3-fold \(X\), there is the flop process for certain \(\mathbb{P}_{\Sigma(\Lambda)}\) as described in Example 3 of Section 2.3 where \(X\) is birational, but not biregular, to another CY 3-fold \(X^2\) with the same combinatorial representation of Picard group (2.11) and deformation space (3.15). \(X\) and \(X^2\) also share the same degeneration cone (3.18), but with different Kahler cones by \(\text{Int}(\mathcal{C}_{Pic}(X)) \cap \text{Int}(\mathcal{C}_{Pic}(X^2)) = \emptyset\). The mirror \(X^*\) and \(X^2\) of \(X\) and \(X^2\) respectively are biregular, but their degeneration cones are different with \(\text{Int}(\mathcal{C}_{Deg}(X^*)) \cap \text{Int}(\mathcal{C}_{Deg}(X^2 *)) = \emptyset\), which serves the mirror of flop of CY 3-folds.

5 Concluding Remarks

From the framework of toric geometry, we find a combinatorial representation of Picard group and deformation space of anti-canonical hypersurfaces \(X\) of a toric variety defined by reflexive polytope. By identifying the toric representatives of the representation, we establish the cohomology correspondence between the mirror pair previously found in [2], as an extension of the mirror correspondence in [18]. This paper can be regarded as a continuation of the work in [23] about Picard group of hypersurfaces in toric varieties. Here we have further identified the structure of Kahler cone and degeneration cone of mirror CY hypersurfaces using toric techniques. As a consequence, a different parametrization of the moduli space in CY degeneration provides the mirror notion equivalent to flops in CY 3-folds. For the purpose of exploring the essential role of simplicial-cone structure of toric variety in mirror symmetry, we concentrate here only on the
positive line bundles of \(X\) induced from the Kahler cone of the ambient toric variety. However in the most general situation, the complete Kahler cone or positive deformation cone of \(X\) should add the extra contribution of \(\nu_F\)'s in (2.11) or \(\nu_{F^*}\)'s in (3.15), into (2.19) or (3.18), the relation of which not immediately apparent, and not yet studied in the present article. These structures are essential for the understanding of quantum cohomology product in mirror symmetry. Work along these lines is under consideration. Here, just to keep things simple, we restrict our attention only on the mirror cone-structure with no contribution from \(\nu_F\) or \(\nu_{F^*}\)'s. We leave the further discussion of our results, and possible generalizations to future work.

**Appendix: Some Basic Facts in Toric Geometry**

In this Appendix, we summarize results in [23] and some basic facts in toric geometry used in this paper for easy reference. Some should be well-known (but we could not find an explicit reference). For completeness, we include some technical arguments for certain statements here. Results in this section hold for a general polytope \((\triangle, L)\) with a simplicial decomposition \(\Lambda\) of \(\partial \triangle\) with \(\Lambda^{(0)} \subseteq L \cap \partial \triangle\), i.e. condition (1.5) or (1.10) not necessary required. For convenience of notions, we consider the cases when the rational toric divisor class \(O(\iota^* \rho)\) over \(\mathbb{P}_\Sigma\) is effective and convex, i.e.

\[
\rho = \sum_{\delta \in \Lambda^{(0)}} \rho^\delta e^{\delta \iota} \in D_\triangle^+ \quad (\rho^\delta \in \mathbb{Z}_{>0}), \quad (A1)
\]

and the piecewise linear functional on \(L_{\mathbb{R}}\), linear on every simplicial cone \(\sigma(s)\) of \(\Sigma\) with value \(\rho^\delta\) on \(\delta \in \Lambda^{(0)}\), satisfies the convex property [13, 15, 23], a condition equivalent to the non-negative condition in Section 2.3 of this paper. Define the \(\rho\)-dual polytope of \(\triangle\) by

\[
\triangle_\rho^* = \{y \in L_{\mathbb{R}}^* \mid \langle y, \delta \rangle \geq -\rho^\delta \text{ for } \delta \in \Lambda^{(0)}\}. \quad (A2)
\]

Note that the effective-convex condition of \(\rho\) is equivalent to the convex-cone property of \(\triangle_\rho^*\) (see, e.g. Proposition 1 and (24) in [23]). By the convex property of \(\rho\), each \(m\)-face \(\rho^*\) of \(\partial \triangle_\rho^*\) for \(0 \leq m \leq n - 1\) gives rise to a \((n - 1 - m)\)-polytope \(F_\rho\) in \(\partial \triangle\) generated by finite elements in \(L_{\mathbb{Q}}\). We shall assume \(F_\rho\) is generated by a finite number of elements in \(\Lambda^{(0)} \cap F\) for some \((n - 1 - m)\)-face \(F\) of \(\partial \triangle\), so that all \(F_\rho\)'s form a polytope refinement of the face-decomposition of \(\partial \triangle\). For convenience, \(F_\rho\) will be called a \(\rho\)-face of \(\partial \triangle\), and the set of \(0\)-dimensional \(\rho\)-faces will be denoted by \(V_\rho(\triangle)\).

In general, the \(\rho\)-face \(F_\rho\) may not be equal to the face \(F\) (for the details, see the relation between \(\rho\)-graph-cone \(C_\rho\) and its dual cone \(C_\rho^*\) in [23] Section 3). Similar to the construction of the minimal toric variety \(\mathbb{P}_{\Sigma_0}\), there exists the “\(\rho\)-minimal” toric variety \(\mathbb{P}_{\Sigma_\rho 0}\) for a convex divisor \(\rho\) in (A1), where \(\Sigma_\rho 0\) is the complete polyhedral cone of \(L_{\mathbb{R}}\) induced by the \(\rho\)-face-decomposition of \(\partial \triangle\). Since \(\Lambda\) is a triangulation of \(\rho\)-faces and \(\rho\) is determined by those \(\rho^\delta\)'s for \(\delta \in V_\rho(\triangle)\) in (A1), the toric variety \(\mathbb{P}_\Sigma\) dominates \(\mathbb{P}_{\Sigma_\rho 0}\):

\[
\varphi(= \varphi_\rho) : \mathbb{P}_{\Sigma(\Lambda)} \longrightarrow \mathbb{P}_{\Sigma_\rho 0}, \quad O(\iota^* \rho) = \varphi^* O(\iota_0^* \rho_0), \quad (A3)
\]

\[\text{[The \(\triangle_\rho^*\) here is denoted by } \triangle(L^*)_\rho \text{ in [23] (24).]}
\]
where \( \rho_0 \) is the toric divisor of \( \Sigma_{\rho_0} \) defined by \( \rho_0 := \sum_{v \in \mathcal{V}_{\rho}(\Delta)} \rho^v e^v \). Hence sections of \( O(t^* \rho) \) over \( \mathbb{P}_{\Sigma(\Lambda)} \) are the \( \varphi \)-pull-back of \( O(t^*_0 \rho_0) \)-sections over \( \mathbb{P}_{\Sigma_0} \). Note that when \( \rho = -\kappa \) in (2.9), \( \Delta^*-\kappa \) is the same as the dual polytope \( \Delta^* \) of \( \Delta \), and \( (-\kappa) \)-face \( F^* \) is equal to the face \( F \) with \( \mathbb{P}_{\Sigma^* \rho_0} = \mathbb{P}_{\Sigma_0}, \rho_0 = -\kappa_0 \) in (2.10).

Now we describe a combinatorial basis of sections of \( O(t^* \rho) \). First we consider the case when all elements in \( \Lambda^{(0)} \) are primitive in \( L \). By Proposition 2 in [23], a basis of global sections of \( O(t^* \rho) \) over \( \mathbb{P}_{\Sigma} \) consists of a finite number of integral elements in \( \Delta^*_\rho \) as follows:

\[
\Gamma(\mathbb{P}_{\Sigma}, O(t^* \rho)) \quad \left( \simeq \Gamma(\mathbb{P}_{\Sigma^* \rho}, O(t^* \rho_0)) \right) \simeq \bigoplus \{ \mathcal{C} v^* \mid v^* \in \Delta^*_\rho \cap L^* \}.
\]

(A4)

Indeed by the exact sequences (21) and (22) in [23], the above basis elements \( v^* \)-s are identified with monomial functions of \( \mathcal{C}_{\Sigma} \) in (2.1) expressed by the homogeneous coordinates \( z = (z_\delta)_\delta \) of \( \mathbb{P}_{\Sigma} \) as

\[
z^{v^*} = \prod_{\delta \in \Lambda^{(0)}} z^{v^*_\delta}, \quad v^*_\delta := \langle v^*, \delta \rangle + \rho^\delta \in \mathbb{Z} \geq 0,
\]

(A5)

which are characterized as monomials \( z^k \) \((k = \sum_{\delta} k^\delta e^\delta \in D^1_{\Delta}, k^\delta \geq 0)\) with the "degree" property: \( \langle k, \iota(n) \rangle = (\rho, \iota(n)) \) for \( n \in \mathbf{n}_\Delta \). In particular, the zero-section \( 0^* \) corresponds to \( z^0 = \prod_{\delta \in \Lambda^{(0)}} z^{0^\delta} \) with \( \rho \) as its zero-divisor. Furthermore, as rational functions of the fiber space in (2.2), \( z^{v^*} \) is identified with the character function of \( T(L) \) induced from \( v^*, \chi_{v^*} : T(L) \rightarrow \mathcal{C}^* \left( = \mathbb{C}/\mathbb{Z} \right) \), regarded as a rational function of \( \mathbb{P}_{\Sigma} \). Therefore through the expression (A5), the space of sections in (A4) is identified with a subspace of polynomial functions of \( \mathcal{C}_{\Sigma} \). It is known that the translation \( m_{T^*} \) of \( T(L) \) by an element \( T \in T(L) \) extends an automorphism of \( \mathbb{P}_{\Sigma} \) which preserves all toric divisors, hence the line bundle \( O(t^* \rho) \). Furthermore, one may lift the translation \( m_{T^*} \) to the bundle \( O(t^* \rho) \) fixing the section \( z^{0^*} \). Indeed, we have the following result:

Lemma 5.1 The \( T(L) \)-action of \( \mathbb{P}_{\Sigma} \) can be lifted to the line bundle \( O(t^* \rho) \) so that \( T(L) \) acts on the space (A4) by

\[
(\overline{T}, v^*) \mapsto \chi_{v^*}(\overline{T})v^*, \quad \overline{T} \in T(L), v^* \in \Delta^*_\rho \cap L^*,
\]

where \( \chi_{v^*} \) is the character function of \( T(L) \) determined by \( v^* \).

Proof. By the definition of \( O(t^* \rho) \) and (A5), it suffices to construct a lifting of the \( T(L) \)-action of \( \mathbb{P}_{\Sigma} \) to the fiber space \( \mathcal{C}_{\Sigma} \) of (2.2) which fixes the monomial \( z^{0^*} \). Let \( \delta^j \) \((1 \leq j \leq n)\) be \( n \) elements in \( \Lambda^{(0)} \) which generate a \((n-1)\)-simplex contained in a \( \rho \)-facet \( F^\rho \) of \( \Delta \). Then \( \delta^j \)'s form a basis of \( L_{\mathbf{Q}} \) with the dual basis \( \delta_j \in L_{\mathbf{Q}}^* \) \( \langle \delta_j, \delta^k \rangle = \varepsilon_{jk} \). For \( \delta \in \Lambda^{(0)} \setminus \{ \delta^j \}_{j=1}^n \), one has

\[
\delta^j = -\sum_{j=1}^n a^j_{\delta} \delta^j \quad \text{for } a^j_{\delta} \in \mathbf{Q},
\]

(A6)

by which we define \( n^\delta = e^\delta + \sum_{j=1}^n a^j_{\delta} e^\delta \in \mathbf{n}_{\Delta \mathbf{Q}} \). Then \( n^\delta \)'s form a basis of \( \mathbf{n}_{\Delta \mathbf{Q}} \); and together with \( \{ e^\delta \}_{j=1}^n \), they form a basis of \( L_{\Delta \mathbf{Q}} \). The basis \( \{ n^\delta \}_{j=1}^n \cup \{ n^\delta \}_{j=1}^n \) of \( L_{\Delta \mathbf{Q}} \) dual to

\footnote{In [23], we assume the polytope \( \Delta \) contains the origin as the only lattice point in its interior. Indeed by the same argument, the results in Proposition 2 and Theorem 2 of [23] are still valid for integral polytopes discussed in the appendix here.}
\{e^{\delta_j}\} \cup \{n^{\delta_j}\} \text{ is related to } e^{\delta_j}\text{'s in (1.10) by }
\[ e_{\delta_j} = e^{\delta_j} - \sum_{\delta_j' \in \Lambda(0) \setminus \{\delta_j\}} a^{\delta_j}_{\delta_j'} e^{\delta_j'}, \quad n_{\delta_j} = e^{\delta_j}, \quad e^{\delta_j} = e_{\delta_j} + \sum_{\delta_j' \in \Lambda(0) \setminus \{\delta_j\}} a^{\delta_j}_{\delta_j'} n_{\delta_j}. \] (A7)

Under $\beta^\ast$ in (1.9), one has $\beta^\ast(\delta_j) = e_{\delta_j}$ for $1 \leq j \leq n$. We now use the basis $\delta_j$'s of $I^\ast Q$ to express an element $w^\ast \in L^\ast$ by $w^\ast = \sum_{j=1}^n w^j \delta_j$ for $w^j \in Q$, and write $\mathfrak{w} = u + L \in T(L)(= L_Q/L)$ with $u = \sum_{j=1}^n u_j \delta_j \in L_Q$ for $u_j \in C$. Then the character $\chi^\ast_w$ of $T(L)$ is expressed by $\chi^\ast_w(\mathfrak{w}) = \prod_{j=1}^n e^{2\pi i u_j \delta_j}$. In particular, $\chi^\ast_{\delta_j}(\mathfrak{w}) = e^{2\pi i u_j}$, hence $\chi^\ast_{\delta_j}(\mathfrak{w}) e_{\delta_j} = e^{2\pi i u_j} e_{\delta_j}$ for $1 \leq j \leq n$. Since $z^{0^\ast}$ is identified with $\rho \in D_{\Delta}^\ast$, expressed by $\rho = \sum_{j=1}^n \rho^{\delta_j} e_{\delta_j} + \sum_{\delta_j' \in \Lambda(0) \setminus \{\delta_j\}} (\rho^{\delta_j'} + \sum_{j=1}^n \rho^{\delta_j} a^{\delta_j}_{\delta_j'}) n_{\delta_j'}$ in terms of the basis $\{e_{\delta_j}\} \cup \{n_{\delta_j}\}$, a $T(L)$-lifting of $P_Q \Sigma$ to $C_{\Sigma}$ in (2.2) fixing $z^{0^\ast}$ is equivalent to a collection of $(d - n)$ linear functionals, $v_{\delta_j} \in \text{Hom}(L_Q, C)$ for $\delta_j \in \Lambda(0) \setminus \{\delta_j\}$, satisfying the equation
\[ \sum_{j=1}^n \rho^{\delta_j} u_j + \sum_{\delta_j' \in \Lambda(0) \setminus \{\delta_j\}} (\rho^{\delta_j'} + \sum_{j=1}^n \rho^{\delta_j} a^{\delta_j}_{\delta_j'}) v_{\delta_j}(u) = 0, \]
which by (A7), the $T(L)$-action of $C_{\Sigma}$, $(\mathfrak{w}, (z_{\delta_j})_{\delta_j \in \Lambda(0)}) \mapsto (z_{\delta_j}(\mathfrak{w}))_{\delta_j \in \Lambda(0)}$, is described by
\[ z_{\delta_j}(\mathfrak{w}) = e^{2\pi i u_j \delta_j} + \sum_{\delta_j' \in \Lambda(0) \setminus \{\delta_j\}} (\rho^{\delta_j'} + \sum_{j=1}^n \rho^{\delta_j} a^{\delta_j}_{\delta_j'}) v_{\delta_j}(u) \]
By (A6), the vanishing condition, $\rho^{\delta_j'} + \sum_{j=1}^n \rho^{\delta_j} a^{\delta_j}_{\delta_j'} = 0$, is equivalent to $\delta_j$ lies in the hypersurface of $L_R$ containing the facet $F_{\rho}$. For a given element $\delta^0 \in \Lambda(0)$ outside the hypersurface containing $F_{\rho}$, an action in (A8) is given by the following $v^\delta_j$: \[ v_{\delta_j}(u) = -\sum_{j=1}^n \frac{\rho^{\delta_j}}{\rho^{\delta_j} + \sum_{j=1}^n \rho^{\delta_j} a^{\delta_j}_{\delta_j'}} u_j, \quad v_{\delta_j}(u) = 0 \text{ for } \delta_j \neq \delta^0, \]
which provides a $T(L)$-lifting to the line bundle $O(t^\ast \rho)$.  

**Remark.** As in (2.2) and (A5), there is a homogeneous coordinate system $\zeta = (\zeta_v)_{v \in V_{\rho}(\Delta)}$ for $P_{\Sigma_{\rho^0}}$, by which the section $v^\ast$ of $O(t^\ast \rho_0)$ in (A4) is identified with $\zeta^v$ (for the case $\rho = -\kappa$, see (2.4) and (3.4)). By the same argument for the expression $\zeta^v$, one can show that Lemma 5.1 also holds for the line bundle $O(t^\ast \rho_0)$ over $P_{\Sigma_{0}}$.

Suppose $X$ is a quasi-smooth hypersurface $X$ of $P_{\Sigma}$ defined by zeros of a section in (A4). By using the standard residual argument in algebraic geometry, a basis for sections of the canonical bundle $K_X$ of $X$ can be derived from those in (A4) as follows:
\[ H^{n-1,0}(X)(= \Gamma(X, K_X)) \sim \bigoplus \{ Cw^\ast \mid w^\ast \in \text{Int}(\Delta^\ast_{\rho}) \cap L^\ast \}. \] (A9)

We now consider the general case for an integral polytope $(\Delta, L)$ with a simplicial decomposition $\Lambda$ of $\partial\Delta$ with $\Lambda(0) \subset L \cap \partial\Delta$. For each $\rho \in \Lambda(0)$, let $m_{\rho}$ be the positive integer so that $\mathfrak{p} := \frac{\rho}{m_{\rho}}$ is a primitive element in $L$. Instead of $\rho$'s, the primitive toric divisors in $P_{\Sigma}$ are represented by $\mathfrak{p}$'s. An integral toric divisor is an element in the lattice $D_{\Delta}^\ast$ dual to $D_{\Delta}$, where
\[ D_{\Delta}^\ast := \bigoplus_{\delta \in \Lambda(0)} \mathbb{Z} \mathfrak{e}^\delta \subset D_{\Delta} Q, \quad (\mathfrak{e}^\delta = \frac{e^\delta}{m_{\rho}}), \]
\[ \mathfrak{e}^\delta := \bigoplus_{\delta \in \Lambda(0)} \mathbb{Z} \mathfrak{e}^\delta \subset D_{\Delta}^\ast Q, \quad (\mathfrak{e}^\delta = m_{\rho} e^\delta), \]

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by which elements in $D^\dagger_\Delta$ are rational toric divisors for $\overline{D}$. In particular, $\rho$ is expressed by $\rho = \sum_{\delta \in \Lambda(0)} \rho^\delta m^\rho$. Note that $\Delta^*_\rho$ can also be described by

$$\Delta^*_\rho = \{ y \in L^\dagger_\mathbb{R} \mid \langle y, \overline{\nu} \rangle \geq -\frac{\rho^\delta}{m^\rho} \text{ for } \delta \in \Lambda(0) \}. \quad (A10)$$

The homogeneous coordinates of $\mathbb{P}_\Sigma$ corresponding to the $\overline{D}$-structure are $z = (z_\delta)^\dagger = \sum_{\delta \in \Lambda(0)} z_\delta^\delta e^\overline{\nu}$, related to $z = (z_\delta)$ in (2.1) by $z_\delta = z_\delta^m$. Sections of an integral toric-divisor class of $\mathbb{P}_\Sigma$ are presented by $z$-polynomials, and those for a rational toric-divisor class of $\mathbb{P}_\Sigma$ are expressed by polynomials of rational powers of $z_\delta$'s. In particular, the rational toric-divisor classes induced by elements in $D^\dagger_\Delta$ are described by monomials $z^{\rho^\delta}$, by which elements in (A4) are presented by polynomials, and those for a rational toric-divisor class of $\mathbb{P}_\Sigma$ are expressed by polynomials of rational powers of $z_\delta$'s. In particular, the rational toric-divisor classes induced by elements in $D^\dagger_\Delta$ are described by monomials $z^{\rho^\delta}$.

**Remark.** (1) When $X$ has a basis in (A4) with the coordinate representatives in (A5).

(2) A section $\rho^{\dagger_\rho}$ in (A11) defines the effective divisor $\rho^{\dagger_\rho} = \sum_{\delta \in \Lambda(0)} w_\delta^{\rho^\delta} e^\overline{\nu}$, which differs from $\rho$ by $i^*(\ell^*)$ for some $\ell^* \in L^*$, again satisfying the convex property. The $(\rho^{\dagger_\rho})$-dual polytope $\Delta^{*\rho^{\dagger_\rho}}$ is the translation of $\Delta^{*\rho}$: $\Delta^{*\rho^{\dagger_\rho}} = \Delta^{*\rho} - \overline{w}^\ast$.

(3) By the discussion in [23], the relations (A11) and (A9) are also valid for a convex toric divisor with $\rho^\delta \geq 0$. In this situation, the origin may not be in the interior of the polytope $\Delta^{*\rho}$. Furthermore, $\Delta^{*\rho}$ could be a polytope of dimension less than $n$, in which case we set $\text{Int}(\Delta^{*\rho}) := \emptyset$ in (A9).

Next we extend (A4) and (A9) to formulas held for a face of $\Delta^{*\rho}$. For a $(n - m - 1)$-face $F$ of $\Delta$ $(0 \leq m \leq n - 1)$, $L \cap F$ spans a $(n - m)$-dimensional subspace of $L_0\mathbb{Q}$, whose intersection with $L$ gives rise to a $(n - m)$-sublattice $L^\bot_F$ of $L$. Let $L_F := \langle L / L^\bot_F \rangle$ be the quotient lattice of rank $m$ with the exact sequence of lattices:

$$0 \rightarrow L^\bot_F \hookrightarrow L \xrightarrow{\nu} L_F \rightarrow 0, \quad (A11)$$
where $\varphi$ is the natural projection. A simplex $s$ of a triangulation $\Lambda$ of $\partial \Delta$ whose interior is contained in the interior of $F$ must be in the $(n-m_s-1)$-skeleton $\Lambda^{(n-m_s-1)}$ with $m \leq m_s \leq n-1$. We define 

$$s^* = \{ y \in \Delta^*_{\rho} \mid \langle y, \delta \rangle = -\rho^\delta \text{ for } \delta \in s \cap \Lambda^{(0)} \},$$

which is a $m$-face $F^*_\rho$ in $\partial \Delta^*_\rho$ with its dual $(n-m-1)$-$\rho$-face $F^*_\rho$ contained in $F$. The $(n-m_s)$-dimensional subspace of $L^*_\Theta$ spanned by one-parameter subgroups in the cone $\sigma(s)$, i.e. $L \cap \sigma(s)$, intersects $L$ on a $(n-m_s)$-sublattice $L^\perp_{s}$, whose quotient $L_s := L/L^\perp_{s}$ in $L$ is a $m_s$-lattice:

$$0 \rightarrow L^\perp_s \hookrightarrow L \xrightarrow{\rho_s} L_s \rightarrow 0, \quad \text{with } L^\perp_s \subseteq L^\perp_F, \quad L_s/L_F \simeq L^\perp_F/L^\perp_s. \quad (A12)$$

The $T(L)$-orbit $O_s$ fixed by $L \cap \sigma(s)$ is isomorphic to a $m_s$-torus, whose closure $\overline{O}_s$ is a subvariety of $\mathbb{P}_\Sigma$. First we describe the toric structure of $\overline{O}_s$. Indeed, $\overline{O}_s = \bigcup_{s' \in \Lambda} O_{s'}$. For a $s' \in \Lambda^{(n-m_s+k)}$ with $s \subset s'$ ($k \geq 0$), the vertices of $s'$ outside $s$ form a $k$-simplex $s'' \in \Lambda^{(k)}$. Denote

$$\text{Star}(s) = \bigcup_{k=0}^{m_s-1} \{ s'' \in \Lambda^{(k)} | s * s'' \in \Lambda^{(n-m_s+k)} \}. $$

Under the projection $\varphi_s$ in (A12), $\varphi_s(s)$ is the origin $\overline{0}$ of $(L_s)_{\mathbb{R}}$, and $\varphi_s(\text{Star}(s))$ becomes a star of $\overline{0}$ in $(L_s)_{\mathbb{R}}$. Then $\overline{s''} := \varphi_s(s''))$ is a $k$-simplex with vertices in $L_s$ for $s'' \in \text{Star}(s)^{(k)}$. The convex hull of all $\overline{s''}$, denoted by $\overline{\Delta}(s)$, is a $m_s$-polytope in $(L_s)_{\mathbb{R}}$ containing the origin so that $\varphi_s(\text{Star}(s))$ form a triangulation $\Lambda(s)$ of the boundary $\partial \Delta(s)$. Through $\varphi_s$ in (A11), the $T(L)$-space $\overline{O}_s$ becomes a $T(L_s)$-toric variety:

$$\overline{O}_s = T(L_s)-\text{toric variety } \mathbb{P}_{\Sigma,(\Lambda_s)}, \quad \Lambda(s) : \text{triangulation of } \partial \Delta(s). \quad (A13)$$

Then $O(t^* \rho)$ over $\mathbb{P}_\Sigma$ restricting on $\overline{O}_s$ gives rise to a line bundle over $\mathbb{P}_{\Sigma,(\Lambda_s)}$, whose divisor class we are now going to derive. Let $F^*_\rho, F$ be the $(n-m-1)$-$\rho$-face and $(n-m-1)$-face of $\Delta$ which contain the interior of $s$. Since $s^* = F^*_\rho$, using the coordinate form (A5), one finds that the section corresponding to $v^*$ in (A4) vanish on $\overline{O}_s$ when $v^* \notin F^*_\rho$. So one needs only to consider those $O(t^* \rho)$-sections corresponding to elements in $F^*_\rho \cap L^*$. Note that vertices in $\Lambda(0)^{(0)}$ are of the form $\overline{\delta''}$ for $\delta'' \in \text{Star}(s)^{(0)}$, equivalently $s * \delta''$ is a $(n-m)$-simplex in $\Lambda^{(n-m)}$. Hence $(s * \delta'')^*$ is a facet of $F^*_\rho$ with the expression $(s * \delta'')^* = \{ y \in F^*_\rho | \langle y, \delta'' \rangle = -\rho^{\delta''} \}$. The facets of $F^*_\rho$ are characterized by $(m-1)$-faces $F'' \rho$ with $F'' \rho \subset F^*_\rho$, equivalently, $(n-m)$-$\rho$-faces $F'' \rho$ with $F'' \rho \supset F^*_\rho$. Since $(\text{Star}(s)^{(0)} \setminus F^*_\rho) \cap F'' \rho \neq \emptyset$ when $F'' \rho \supset F^*_\rho$, we obtain

$$F^*_\rho = \{ y \in L^*_\mathbb{R} | \langle y, \delta \rangle = -\rho^\delta \text{ (} \delta \in (F^*_\rho \cap \Lambda^{(0)}) \}, \quad \langle y, \delta'' \rangle \geq -\rho^{\delta''} \text{ (} \delta'' \in \text{Star}(s)^{(0)} \setminus F^*_\rho) \}. \quad (A14)$$

If $F^*_\rho \cap L^* = \emptyset$, the zeros of every section in (A4) contain $\overline{O}_s$. Otherwise, we choose an element $v_0^*$ in $F^*_\rho \cap L^*$ as the base element. For convenience, $v_0^*$ will be chosen in the interior of $F^*_\rho$ if $\operatorname{Int}(F^*_\rho) \cap L^* = \emptyset$. Consider the $m$-dimensional subspace $\langle F^*_\rho - v_0^* \rangle$ of $L^*_\Theta$ spanned by $F^*_\rho - v_0^*$. There is the $m$-lattice $L^*_\rho(F^*_\rho, v_0^*) = L^* \cap \langle F^*_\rho - v_0^* \rangle$ so that $L^*_\rho(F^*_\rho, v_0^*) = L^*_\rho(F^*_\rho, v_0^*)$. Since $L^*_\rho(F^*_\rho, v_0^*)$

\footnote{Indeed, the conclusion can also be derived from the relation between the $\rho$-graph cone $C^*_\rho$ and its dual cone $C^*_\rho$ in the discussion of [23] Section 3.}
vanishes on $F_\rho$, hence on $L_\rho^\perp$, one can identify $L_{(F_\rho^*,v_0^*)}^*$ with the dual lattice $L_F^*$ of $L_F$ in (A11). Then $F_\rho^* - v_0^*$ corresponds to an $m$-polytope in $L_F^* \mathbb{R}$, denoted by $\Delta_{F_\rho}^+$:

$$\phi : L_F^* \xrightarrow{\sim} L_{(F_\rho^*,v_0^*)}^*, \quad \Delta_{F_\rho}^+ \leftrightarrow F_\rho^* - v_0^*, \quad \text{(A15)}$$

where the isomorphism $\phi$ satisfies the relation $\langle y, \phi(x) \rangle = \langle \phi(y), x \rangle$ for $y \in L_F^*$ and $x \in L$. The integral elements in $\Delta_{F_\rho}^+$ are given by $y_{\nu^*}$ ($\nu^* \in F_\rho^* \cap L^*$), where $y_{\nu^*}$'s are defined by the relation $\phi(y_{\nu^*}) = \nu^* - v_0^*$. Define the divisor $\overline{p}$ of the $T(L_a)$-toric variety $\mathbb{P}_{\Sigma(\Lambda(a))}$ in (A13):

$$\overline{p} = \sum_{\delta'' \in \text{Star}(s)^{(0)}} \overline{p}^{\delta''} e^{\delta''} \in \overline{D}_{\Lambda(a)}^1, \quad \overline{p}^{\delta''} := \rho^{\delta''} + (v_0^*, \delta'') \in \mathbb{Z}_{\geq 0} \quad \text{for} \quad \delta'' \in \text{Star}(s)^{(0)}. \quad \text{(A16)}$$

Note that $\overline{p}^{\delta''} = 0$ when $\delta'' \in \text{Star}(s)^{(0)} \cap F_\rho$. By (A14) and (A15), one finds the equality of polytopes in $L_F^* \mathbb{R}$:

$$\Delta_{F_\rho}^+ = \Delta(s)_{\overline{p}} = \{ y \in L_F^* \mathbb{R} | \langle y, \delta'' \rangle \geq -\overline{p}^{\delta''} \text{ for } \delta'' \in \text{Star}(s)^{(0)} \}. \quad \text{(A17)}$$

The convex property of $F_\rho^*$ implies the convex condition of the divisor $\overline{p}$, which is effective when $\text{Int}(F_\rho^*) \cap L^* \neq \emptyset$. Using (A15), (A16), (A17) and Lemma 5.2 (I), together with Remark (3) of Lemma 5.2, one obtains the following results:

**Proposition 5.1** Let $s \in \Lambda^{(n-m_a-1)}$ be a $(n-m_a-1)$-simplex whose interior is contained in the interior of a $(n-m-1)$-face $F_\rho$ and $(n-m-1)$-face $F$ (with $F_\rho \subseteq F$ and $0 \leq m \leq m_a \leq n-1$). Denote by $\overline{O}_s$ the closure of the $T(L)$-orbit associated to $s$ in $\mathbb{P}_{\Sigma(\Lambda)}$.

(I) $\overline{O}_s$ is isomorphic to the $m_a$-dimensional $T(L_a)$-toric variety $\mathbb{P}_{\Sigma(\Lambda(a))}$ in (A13).

(II) If the zero-loci of a generic section in (A14) does not contain $\overline{O}_s$, the restriction of $O(v^* \rho)$ on $\overline{O}_s$ is equivalent to the divisor class $O(\overline{p})$ over $\mathbb{P}_{\Sigma(\Lambda(a))}$, where $\overline{p}$ is defined in (A16) using a base element $v_0^* \in F_\rho^* \cap L^*$. The space of sections of $O(\overline{p})$ over $\mathbb{P}_{\Sigma(\Lambda(a))}$ is described by

$$\Gamma(\mathbb{P}_{\Sigma(\Lambda(a))}, O(\overline{p})) \simeq \bigoplus \{ \mathcal{O}w^* | v^* \in F_\rho^* \cap L^* \} \quad \text{(A18)}$$

so that the section with zero-divisor $\overline{p}$ corresponds to $v_0^*$.

Next we explore the relation between $L$-integral points in $\text{Int}(F_\rho^*)$ and hypersurfaces of $\overline{O}_s$ under the condition (II) of Proposition 5.1. Suppose $\overline{p}$ in Proposition 5.1 (II) is an integral toric divisor of $\mathbb{P}_{\Sigma(\Lambda(a))}$ and $\overline{X}_s$ is a quasi-smooth hypersurface in $\overline{O}_s$ defined by a generic section of $O(\overline{p})$.

First we consider the case when $m_a = m$, i.e. $s \in \Lambda^{(n-m-1)}$ with $s \subseteq F_\rho \subseteq F$ for a $(n-m-1)$-face $F_\rho$ and $(n-m-1)$-face $F$. In this situation, by (A3) we have $L_a^+ = L_F^+$ and $L_a = L_F$. By Lemma 5.2 (II), together with Remark (3) of Lemma 5.2, one finds the combinatorial representation of canonical forms of $\overline{X}_s$:

$$H^{m-1,0}(\overline{X}_s) \simeq \bigoplus \{ \mathcal{O}w^* | w^* \in \text{Int}(F_\rho^*) \cap L^* \}. \quad \text{(A19)}$$

\textsuperscript{13}In general, $\overline{p}$ is a rational, not integral, toric divisor in $\mathbb{P}_{\Sigma(\Lambda(a))}$ even when $\rho$ is an integral divisor of $\mathbb{P}_{\Sigma}$. 

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Note that in the above formula, the data on the right are the same for all \( \overline{X}_s \) whenever \( s \in \Lambda^{(n-m-1)} \) with \( s \subseteq F_\rho \). Indeed under the dominating morphism (A3), \( \varphi(\overline{O}_s) \) is equal to the closure \( \overline{O}_{F_\rho} \) of \( T(L) \)-orbit \( O_{F_\rho} \) in the \( \rho \)-minimal toric variety \( \mathbb{P}_{\Sigma_{\rho \cdot 0}} \). With the same argument as the toric structure of \( \overline{O}_s \) in (A13), one finds \( \overline{O}_{F_\rho} \) is isomorphic to a complete \( T(L_F) \)-toric variety \( \mathbb{P}_{\Sigma(\Lambda(F_\rho))} \), where \( L_F \) is the lattice in (A11), and \( \Lambda(F_\rho) \) is a polytope decomposition of the boundary of a \( m \)-polytope in \( L_F \mathbb{R} \) obtained by the \( \varphi \)-projection of all \( \rho \)-faces \( F_{\rho \cdot} \supset F_\rho \). Furthermore, as the relation between \( \rho \) and \( \rho_0 \) in (A3), the divisor \( \overline{\tau} \) in \( \overline{O}_s \) is the \( \varphi \)-pull-back of a toric divisor \( \overline{\tau}_0 \) in \( \overline{O}_{F_\rho} \): \( \mathcal{O}(\overline{\tau}_F) = \varphi^*_s \mathcal{O}(\overline{\tau}_0) \). Here \( \varphi_s \) denotes the restriction of \( \varphi \) on \( \overline{O}_s \). Hence \( \Gamma(\mathcal{P}_{\Sigma(\Lambda(F_\rho))}, \mathcal{O}(\overline{\tau} \overline{\tau}_0)) \) is isomorphic to vector spaces in (A18), and \( \overline{X}_s \) in \( \overline{O}_s \) is indeed the pull-back of a hypersurface \( \overline{X}_{F_\rho} \) in \( \overline{O}_{F_\rho} \): \( \overline{X}_s = \varphi^*(\overline{X}_{F_\rho}) \). Therefore the combinatorial data in (A19) can be regarded as those for \( \overline{X}_{F_\rho} \). A similar consideration can be carried over to the general case for \( m_s \geq m \) in Proposition 5.1 (II). Indeed, \( \varphi(\overline{O}_s) = \overline{O}_{F_\rho} \), equivalently the restriction of (A3) defines a surjective morphism

\[
\varphi_s : \mathbb{P}_{\Sigma(\Lambda(s))} \longrightarrow \mathbb{P}_{\Sigma(\Lambda(F_\rho))}, \quad \mathcal{O}(\overline{\tau}_F) = \varphi^*_s \mathcal{O}(\overline{\tau}_0) \quad \text{(A20)}
\]

so that \( \overline{X}_s \) is the pull-back of a hypersurface \( \overline{X}_{F_\rho} \) in \( \mathbb{P}_{\Sigma(\Lambda(F_\rho))} \) defined by zeros of a \( \mathcal{O}(\overline{\tau}_0) \)-section: \( \overline{X}_s = \varphi_\rho^{-1}(\overline{X}_{F_\rho}) \). By the toric data associated to the fibration (A20), the generic fiber \( f_s \) of \( \varphi_s \) is isomorphic a complete \( T(L_F/L_s) \)-toric variety \( \mathbb{P}_{\Sigma(\Lambda(s,F_\rho))} \), where \( L_F/L_s \) are lattices in (A12), and \( \Lambda(s,F_\rho) \) is a triangulation of the boundary of a \( (m_s - m) \)-polytope in \( L_F \mathbb{R} \mathbb{R} \mathbb{R} \mathbb{R} \) generated by \( \text{Star}(s)(0) \cap F_\rho \). Hence \( \overline{X}_s \) is a fibration over \( \overline{X}_{F_\rho} \) with the general fiber \( f_s \). Since \( H^{k,0}(\overline{X}_s) = 0 \) for \( 1 \leq j \leq m_s - m, H^{k,0}(\overline{X}_s) \) vanishes for \( m \leq k \leq m_s - 1 \) and \( (m - 1,0) \)-forms of \( \overline{X}_s \) are \( \varphi_s \)-pull-back of canonical forms of \( \overline{X}_{F_\rho} \) given by (A19). We summarize the result as follows:

**Proposition 5.2** Let a simplex \( s \in \Lambda^{(n-m_s-1)}, (n - m - 1) \)-\( \rho \)-face \( F_\rho \), \( (n - m - 1) \)-face \( F \), \( \overline{O}_s = \mathbb{P}_{\Sigma(\Lambda(s))} \), and \( \overline{\tau} \) be the same as in Proposition 5.1 (II). Suppose \( \overline{\tau} \) is an integral toric divisor of \( \mathbb{P}_{\Sigma(\Lambda(s))} \), and \( \overline{X}_s \) is a quasi-smooth hypersurface in \( \overline{O}_s \) defined by a generic section of \( \mathcal{O}(\overline{\tau}_F) \). Then \( H^{k,0}(\overline{X}_s) = 0 \) for \( k \geq m \), and \( H^{m-1,0}(\overline{X}_s) \) is isomorphic to the space in (A19).

**Remark.** Indeed, the \( \overline{X}_s \) in the above proposition is a fibration over a quasi-smooth hypersurface \( \overline{X}_{F_\rho} \) of \( \mathbb{P}_{\Sigma(\Lambda(F_\rho))} \) in (A20) with the general fiber \( f_s \) being a complete toric variety of dimension \( m_s - m \). The elements in (A19) are induced from the canonical forms of the base \( \overline{X}_{F_\rho} \).

We now discuss the relation between the homogeneous coordinates of \( \mathbb{P}_{\Sigma} \) in (A22) and \( \overline{O}_s \) in (A13):

\[
\pi_s : \mathbb{P}_{\Sigma(\Lambda(s))} \longrightarrow \mathbb{P}_{\Sigma(\Lambda(s))} \quad (= \overline{O}_s). \quad \text{(A21)}
\]

As in (1.8) and (1.9), we have the exact sequences

\[
0 \longrightarrow \mathbf{n}_{\Delta(a)} \overset{\iota_\mathbf{n}}{\longrightarrow} D_{\Delta(a)} \overset{\beta_s}{\longrightarrow} L_{F \cdot 0} \longrightarrow 0, \quad 0 \longrightarrow L_{F \cdot 0} \overset{\beta_s^*}{\longrightarrow} D_{\Delta(s)} \overset{\iota^*}{\longrightarrow} \mathbf{n}_{\Delta(s)} \longrightarrow 0, \quad \text{(A22)}
\]

where \( L_{F \cdot 0} \) is the \( m \)-sublattice of \( L_F \) generated by \( \Lambda(s)(0) \), and \( \beta_s(e^{s'}) := s'' \) for \( s'' \in \Lambda(s)(0) \). By (A5), (A15) and (A16), \( \mathbf{v}^* \) in (A18) can be expressed in terms of the homogeneous coordinates.
Proposition 5.3 Let \( \tilde{I} \subseteq \mathbb{P} \) be a special toric variety. Theorem 1 and 2 in Lemma 5.2 (II), Proposition 5.1 (II), and Proposition 5.2 are all satisfied. Furthermore, by [23] Theorem 2, the boundary of \( \Lambda \) in this paper is different from that in [23] Theorem 2, where \( \Lambda \) denotes the polyhedral cone of \( \rho \)-graph cone over \( \Sigma \).

On the other hand, the coordinates (A23) of \( \nu^* \in F_\rho^* \cap L^* \) in (A4) are given by

\[
\tilde{z}^{\nu^*} = \prod_{\delta'' \in \text{Star}(s)^{(0)}} \tilde{z}^{\nu_{\delta''}}, \quad \nu_{\delta''} := \langle y_{\nu^*}, \tilde{\nu}^{\nu^*} \rangle_F + \rho^{\nu^*} = \nu_{\delta''} \quad \text{for} \ \delta'' \in \text{Star}(s)^{(0)}.
\]

(A23)

since \( \nu_{\delta} = 0 \) for \( \delta \in \Lambda^{(0)} \cap F_\rho \). The coordinates (A23) are obtained from (A24) by setting \( z_{\delta} = 1 \) for \( \delta \notin \text{Star}(s)^{(0)} \), which gives rise to the regular embedding from (A21) into (1.8):

\[
\mathcal{C}_{\Sigma(\Lambda(s))} \longrightarrow \mathcal{C}_{\Sigma}: \quad \tilde{z} = (\tilde{z}_{\delta''})_{\delta'' \in \text{Star}(s)^{(0)}} \mapsto z = (z_{\delta})_{\delta \in \Lambda^{(0)}} \quad z_{\delta} = \begin{cases} \tilde{z}_{\delta} & \text{if } \delta \in \text{Star}(s)^{(0)} \\ 1 & \text{otherwise,} \end{cases}
\]

induced by the divisor-embedding of (A22) into (1.8):

\[
D_{\Delta(s)} \hookrightarrow D_\Delta, \quad \delta'' \mapsto \delta'' \quad \text{for} \ \delta'' \in \text{Star}(s)^{(0)}.
\]

Hence the lattice \( n_{\Delta(s)} \) in (A22) is given by \( n_{\Delta(s)} = \text{proj}_{\Delta(s)}(\beta^{-1}(L_1)), \) where \( \beta, L_1 \) are defined in (1.8), (A11) respectively, and \( \text{proj}_{\Delta(s)} \) is the projection of \( D_\Delta \) onto \( D_{\Delta(s)} \).

Consider the case when the \( \rho \)-dual polytope \( \Delta_{\rho}^* \) is \( L^* \)-integral. Note that in the case \( \rho = -\kappa \), the integral \( \Delta_{\rho}^* \) is the same as the reflexive polytope condition of \( (\Delta, L) \). By (A10), conditions in Lemma 5.2 (II), Proposition 5.1 (II), and Proposition 5.2 are all satisfied. Furthermore, by [23] Theorem 1 and 2, a combinatorial basis for the Picard group of a generic hypersurface of some special toric variety \( \mathbb{P}_{\Sigma} \) is described as follows:

Proposition 5.3 Let \( \Lambda \) be a simplicial decomposition of \( \partial \Delta \) of an integral polytope \( (\Delta, L) \) with \( \Lambda^{(0)} \subseteq L \cap \partial \Delta \), and \( \rho \) be an effective convex toric divisor in (A1). Suppose the \( \rho \)-dual polytope \( \Delta_{\rho}^* \) is \( L^* \)-integral, and \( \Lambda \) satisfies the following condition

\[
\Lambda^{(0)} = L \bigcap (\partial \Delta - \bigcup \{\text{Int}(F_\rho) \mid F_\rho : \text{codim} - 1 \ \rho-\text{face of } \Delta\}).
\]

Let \( s \in \Lambda^{(n-m_{\Delta}-1)} \) be a \( (n-m_{\Delta}-1) \)-simplex whose interior is contained in interior of a \( (n-m-1) \)-\( \rho \)-face \( F_\rho \) and \( (n-m-1) \)-face \( F \) (with \( F_\rho \subseteq F \) and \( 0 \leq m \leq m_{\Delta} \leq n - 1 \)), and \( \overline{O}_s \) be the closure of \( T(L) \)-orbit associated to \( s \). Consider a hypersurface \( X \) of \( \mathbb{P}_\Sigma \) defined by a generic section of \( \mathcal{O}(t_{\rho}^*) \). Then

(1) \( X \) is quasi-smooth, and we have the following combinatorial representation for the Picard group of \( X \):

\[
\text{Pic}(X)_\mathbb{Q} \simeq n_{\Delta}^+ \mathcal{Q} \bigoplus \bigoplus_{F_\rho, \rho F_\rho} \mathcal{Q}_{\rho F_\rho}.
\]

\( \text{The convention } \Lambda \text{ in this paper is different from that in [23] Theorem 2, where } \Lambda \text{ denotes the polyhedral cone of the boundary of } \rho \text{-graph cone over } \Sigma. \)
where the index $F_\rho$ runs over the codimensional 2 $\rho$-faces of $\partial \Delta$, and $\nu_{F_\rho} \in (\text{Int}(F_\rho) \cap L) \times (\text{Int}(F_\rho^*) \cap L^*)$.

(II) $\overline{O}_a$ is isomorphic to a $m_a$-dimensional toric variety $\mathbb{P}_{\Sigma(\Lambda)}$ in (A13). The restriction of $\mathcal{O}(\tau^{*} \overline{p})$ on $\overline{O}_a$ is equivalent to the divisor class $\mathcal{O}(\tau^{*} \overline{p})$ over $\mathbb{P}_{\Sigma(\Lambda(a))}$ where $\overline{p}$ is an integral toric divisor of $\mathbb{P}_{\Sigma(\Lambda(a))}$ defined in (A10), and the space of sections of $\mathcal{O}(\tau^{*} \overline{p})$ over $\mathbb{P}_{\Sigma(\Lambda(a))}$ has a basis represented by $L^{*}$-integral points of $F_\rho^*$ in (A18).

(III) $X$ intersects $\overline{O}_a$ on a quasi-smooth hypersurface $\overline{X}_a$ of $\mathbb{P}_{\Sigma(\Lambda)}$ with $H^{k,0}(\overline{X}_a) = 0$ for $k \geq m$, and $H^{m-1,0}(\overline{X}_a)$ isomorphic to the space in (A19).

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