Hamiltonian analysis of the double null 2+2 decomposition of Ashtekar variables

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Abstract
We derive a canonical analysis of a double null 2+2 Hamiltonian description of general relativity in terms of complex self-dual 2-forms and the associated \( SO(3) \) connection variables. The algebra of first class constraints is obtained and forms a Lie algebra that consists of two constraints that generate diffeomorphisms in the 2-surface, a constraint that generates diffeomorphisms along the null generators and a constraint that generates self-dual spin and boost transformations.

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1. Introduction

Early attempts at using the methods of canonical quantization to provide a theory of quantum gravity were based on the ADM formalism [1], however, it was soon realized that the non-polynomial nature of the constraints prevented one from moving from the classical canonical analysis to a quantum description. A significant advance was made by Ashtekar [2] with the introduction of a new formulation of general relativity in which the variables are taken to be the components of a complex \( SO(3) \) connection with conjugate momenta the components of a triad of vector densities. This had the effect of presenting general relativity in a form similar to Yang–Mills theory and also making the constraints polynomial in the variables. However, despite substantial progress in developing a quantum theory of gravity based on this approach [3] there are still some difficulties that arise from the Hamiltonian constraint and the fact that the first class constraint algebra is not a Lie algebra. (Although see the work of Thiemann [4] for recent progress with this problem.)

It was pointed out by Goldberg et al [5] that if one works with a null foliation the Hamiltonian constraint becomes second class and that the algebra of first class constraints then becomes a Lie algebra. However, if one chooses a null 3+1 decomposition there is no natural projection operator associated with the foliation. A further problem is that the transformations...
which preserve the foliation are the group of null rotations which is algebraically awkward to work with. Both these problems are avoided if one works with a double null 2+2 description in which the projection is well defined and the relevant group is that of spin and boost transformations whose group structure is simply multiplication of non-zero complex numbers. An additional reason for working with a 2+2 formalism is that an analysis of the field equations shows that the gravitational degrees of freedom may be chosen to lie in the conformal structure of the induced 2-metric [6].

In this paper, we will derive a Hamiltonian description of a double null 2+2 decomposition of general relativity given in terms of Ashtekar’s formulation using a complex $SO(3)$ connection. We then use the Dirac theory of constraints to construct the first class algebra and give a geometric interpretation to the four first class constraints. The formulation will be based on an earlier paper by d’Inverno and Vickers [7] (which we will refer to as paper I) in which we presented a double null 2+2 Lagrangian description of general relativity using a version of Ashtekar variables based on self-dual 2-forms and the corresponding first order action given by Jacobson and Smolin [8] and also Samuel [9]. Unlike some recent treatments using real connection variables we use the original Ashtekar formulation in which the manifold is real but we allow complex solutions of the field equations. However, once the canonical analysis has been carried out reality conditions are imposed to limit the solutions to real solutions of Einstein’s equations. In section 2, we review the basic variables used in paper I and in the following section we briefly describe the Lagrangian approach. The Hamiltonian is then introduced and in the subsequent sections we demonstrate that this gives us all the Einstein equations as well as the structure equations for the $SO(3)$ connection. In section 7, we use the Dirac theory of constraints to obtain the first class algebra and interpret the constraints geometrically.

2. The 2+2 tetrad

We start by briefly reviewing the notation and geometric variables employed in [7]. Throughout the paper Greek indices run from 0 to 3, early Latin indices $(a, b, \ldots)$ run from 0 to 1, middle Latin indices $(i, j, \ldots)$ run from 2 to 3, uppercase Latin indices $(A, B, \ldots)$ run from 1 to 3 and tetrad indices will be written in bold. Let $M$ be a four-dimensional orientable manifold with metric $g$ of signature $(+1, -1, -1, -1)$. A foliation of codimension two can be described by two closed 1-forms $n^a$ and $n^b$.

\begin{equation}
\begin{aligned}
dn^a &= 0 \iff n^a = d\phi^a. \\
\end{aligned}
\end{equation}

The two 1-forms generate hypersurfaces defined by

\begin{equation}
\begin{aligned}
\{\Sigma_0\} : \phi^0(x^a) &= \text{constant}, \\
\{\Sigma_1\} : \phi^1(x^a) &= \text{constant},
\end{aligned}
\end{equation}

respectively. These hypersurfaces define a family of 2-surfaces $\{S\}$ by

\begin{equation}
\{S\} = \{\Sigma_0\} \cap \{\Sigma_1\}.
\end{equation}

We restrict attention to the case when $\{S\}$ is spacelike and denote the family of two-dimensional timelike spaces orthogonal to $\{S\}$ at each point by $\{T\}$. Let $n_a$ be the dyad basis of vectors dual to $n^a$ in $\{T\}$, so that

\begin{equation}
n_a n^b = \delta^b_a.
\end{equation}
We define projection operators into \{S\} and \{T\} by
\[ B^\alpha_\beta = \delta^\alpha_\beta - n^\alpha_\alpha n^\beta_\beta, \tag{2.5a} \]
\[ T^\alpha_\beta = n^\alpha_\alpha n^\beta_\beta. \tag{2.5b} \]

The 2-metric induced on \{S\} is given by the projection
\[ \hat{2} g_{\alpha\beta} = B^\alpha_\gamma B^\gamma_\beta g_{\gamma\delta} = B^\alpha_\delta B^\delta_\beta. \tag{2.6} \]
and we use the \( n^\alpha_\alpha \) to define a \( 2 \times 2 \) matrix of scalars \( N_{ab} \) by
\[ N_{ab} = g_{\alpha\beta} n^\alpha_a n^\beta_b. \tag{2.7} \]

The elements \( N_{00} \) and \( N_{11} \) define the lapses of \{S\} in \{\Sigma_0\} and \{\Sigma_1\}, respectively.

We now choose a pair of vectors \( E^a \) which connect neighbouring 2-surfaces in \{S\}. We choose them such that
\[ n^\alpha_a E^a_\alpha = \delta^a_\alpha, \tag{2.8} \]
which defines \( E^a \) up to an arbitrary shift vector \( b_a \), i.e.
\[ E^a = n^a_a + b_a \tag{2.9} \]
with
\[ n^\alpha_a b^\alpha_c = 0. \tag{2.10} \]

In suitably adapted coordinates this results in the 2+2 decomposition of the contravariant metric
\[ g^{\alpha\beta} = \begin{pmatrix} N^{ab} & -N^{ij}b^i_j \\ -N^{ij}b^j_i & 2g^{ij} + N^{ab}b^i_j b^j_i \end{pmatrix}. \tag{2.11} \]

In order to give a 2+2 description of Ashtekar variables we start with a four-dimensional description of general relativity in terms of self-dual 2-forms and a complex \( SO(3) \) connection (see e.g. [10]) and then project this into \{S\} and \{T\}. Our starting point is a Newman–Penrose null tetrad \( (e_a) \) for the metric with dual basis of 1-forms \( (\theta^a) \) so that
\[ dx^2 = \eta_{\alpha\beta} \theta^\alpha \otimes \theta^\beta. \tag{2.12} \]
where
\[ \eta_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{2.13} \]

(For real Lorentzian metrics \( \theta^2 \) and \( \theta^4 \) are real and \( \theta^3 \) is the complex conjugate of \( \theta^2 \).) This tetrad is then used to define a basis of self-dual 2-forms given by
\[ S^1 = \frac{1}{2} (\theta^4 \wedge \theta^8 + \theta^3 \wedge \theta^2), \]
\[ S^2 = \theta^1 \wedge \theta^2, \]
\[ S^3 = \theta^3 \wedge \theta^8. \tag{2.14} \]

It was shown in paper I that a general basis of 1-forms with 2+2 decomposition is given by
\[ \theta^a = \mu^a_i dx^i + \alpha^a_i (dx^i + s^i_j dx^j), \tag{2.15a} \]
\[ \theta^4 = v^i_j (dx^i + s^i_j dx^j). \tag{2.15b} \]
The four $2 \times 2$ matrix variables $\mu^a_b$, $\nu^i_j$, $s'^a_i$, and $\alpha^a_i$ constitute the 16 degrees of freedom corresponding to the 10 metric variables and the 6 Lorentz degrees of freedom. The dual basis is then given by

\begin{align}
e_a &= u_a^b \left( \frac{\partial}{\partial x^b} - s'^b_j \frac{\partial}{\partial x^j} \right), \quad (2.16a) \\
e_i &= v_i^j \frac{\partial}{\partial x^j} + \alpha^a_i v^j_i \left( u_a^b s'^b_j \frac{\partial}{\partial x^j} - u_a^b \frac{\partial}{\partial x^b} \right), \quad (2.16b)
\end{align}

where the $2 \times 2$ matrices $u_a^b$ and $v_i^j$ are defined to be inverses of $\mu^a_b$ and $\nu^i_j$, respectively, so that

\begin{align}
u^i_j \nu^j_i &= \delta^i_i, \quad \nu^i_i \nu^i_i = \delta^i_i, \quad (2.17) \\
u^i_i &= \delta^i_i, \quad \nu^i_j &= \delta^j_j. \quad (2.18)
\end{align}

Considerable simplification can be obtained by working in an adapted frame in which the $e_i$ are tangent to $\{S\}$. With our choice of frame this requires that the $\alpha^a_i$ vanish, i.e.

$$\alpha^a_i = 0. \quad (2.19)$$

This is not a restriction on the metric but purely on the choice of frame and reduces the full six-parameter group of Lorentz transformations to the two-parameter subgroup of spin and boost transformations. It is worth pointing out that this is different from the 3+1 null formulation of Goldberg et al [5] where choosing an adapted frame also leads to coordinate conditions and so must be imposed using a Lagrange multiplier if one is to obtain all the Einstein equations. With our choice of an adapted frame, $\mu^{a_b}$ and $s'^a_i$ generate the lapses and shifts while $\nu^i_j$ generates the 2-metric.

The next step is to impose the condition that both $x^0$ and $x^1$ are null coordinates. In an adapted frame this is simply the condition that

\begin{align}
g^{00} = g^{0\theta^0} \theta^0_\theta^\beta = 2\mu^0_0 \mu^0_1 = 0, \quad (2.20a) \\
g^{11} = g^{1\theta^1} \theta^1_\theta^\beta = 2\mu^1_0 \mu^1_1 = 0. \quad (2.20b)
\end{align}

The volume form is given by

$$V = -\mu^0_0 \wedge \theta^0_1 \wedge \theta^1_2 \wedge \theta^3 = -i\mu_0 \nu_0 \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \quad (2.21)$$

(where $\mu = \det(\mu^a_b)$ and $\nu = \det(\nu^i_j)$) which implies that $\mu$ and $\nu$ are non-zero. Therefore, in order to satisfy the double null slicing conditions as well as the condition that $\mu = \mu^0_0 \mu^1_1 - \mu^0_1 \mu^1_0$ is non-zero, we require that either

$$\mu^0_0 = \mu^1_1 = 0 \quad (2.22)$$

or

$$\mu^0_1 = \mu^1_0 = 0 \quad (2.23)$$

are satisfied. Although we choose to require (2.23), there is no loss of generality because a change to the other condition (2.22) is equivalent to interchanging the coordinates $x^0$ and $x^1$.

We are now in a position to write down the self-dual 2-forms in terms of the metric variables. These are given as follows:

\begin{align}\nonumber S^1 &= \frac{1}{2} (\theta^1_0 \wedge \theta^0_0 + \theta^3_0 \wedge \theta^2_0) \wedge dx^a \wedge dx^b - (\nu^2_0 s'^2_0 \nu^2_1 \nu^3_1 - \nu^3_0 s'^3_0 \nu^3_1 \nu^2_1) \wedge dx^a \wedge dx^b \wedge dx^1. \quad (2.24a)
\end{align}
\[ S^2 = \theta^1 \wedge \theta^2 \]
\[ = (\mu^1_a v^i_a) dx^a \wedge dx^b + (\mu^1_i v^j_i) dx^a \wedge dx^b + (v^3_i v^j_i) dx^a \wedge dx^b. \]
\[ S^3 = \theta^3 \wedge \theta^0 \]
\[ = v^3_i (dx^a + s^i_a dx^a) \wedge (\mu^0_b dx^b) = (v^3_i s^i_a \mu^0_b) dx^a \wedge dx^b - (v^3_i \mu^0_a) dx^a \wedge dx^b. \]

(2.24b)

(2.24c)

As is usual in the Ashtekar formalism we define a densitized version of \( S^A \) by introducing the quantities
\[ \tilde{\Sigma}^A_{\alpha\beta} = \frac{1}{2} e^{\alpha\beta\gamma} S^B_{\gamma \delta} g_{AB}, \]
where \( g_{AB} \) is the \( SO(3) \) invariant metric (defined by equation (4.3a) in paper I). We may now express the sigma variables in terms of the tetrad variables using (2.24c).

3. The 2+2 Lagrangian

In the place of the usual Einstein–Hilbert action, we work with the complex first-order action appropriate to self-dual 2-forms used by Jacobson and Smolin [8],
\[ L = \int R^A \wedge S^B g_{AB}, \]

(3.1)

where \( R^A \) is the curvature 2-form of the \( SO(3) \) connection \( \Gamma^A \). The connection 1-forms \( \Gamma^A \) have 2+2 decomposition
\[ \Gamma^A = \Gamma^A_\mu dx^\mu = A^A_i dx^i + B^A_a dx^a. \]

(3.2)

The curvature 2-forms \( R^A \) are defined by
\[ R^A = d\Gamma^A + \eta^A_{BC} \Gamma^B \wedge \Gamma^C. \]

(3.3)

When written in terms of the \( SO(3) \) covariant derivative \( D \) these have 2+2 decomposition
\[ R^A_{ab} = B^A_{b,a} - D_b B^A_a, \]
\[ R^A_{ai} = A^A_{i,a} - D_i B^A_a, \]
\[ R^A_{ij} = A^A_{j,i} - D_j A^A_i. \]

(3.4a)

(3.4b)

(3.4c)

We may now write the \( SO(3) \) action (3.1) in terms of our variables as
\[ L = \int (R^A_{01} \tilde{\Sigma}^A_{01} + R^A_{23} \tilde{\Sigma}^A_{23} + R^A_{ai} \tilde{\Sigma}^A_{ai}) dx^4. \]

(3.5)

Note that in an adapted frame
\[ (\tilde{\Sigma}^A_{01}, \tilde{\Sigma}^A_{23}, \tilde{\Sigma}^A_{ai}) = (-\nu, 0, 0) \]
so that the term \( R^A_{01} \tilde{\Sigma}^A_{01} \) simplifies to \( -\nu R^A_{01} \).

In paper I we adopted a description similar to that of Goldberg et al [5] in their 3+1 null formulation and considered the configuration space to be given in terms of \( \mu^0_a \) and \( s^i_a \) (the lapse and shift parts of the frame) but replaced the \( v^i_j \) variables by the mixed terms \( \tilde{\Sigma}^A_{ai} \) in the densitized 2-forms. As shown in paper I these variables are not independent but must satisfy eight constraints which are given by
\[ C^i \equiv \mu^0_a \tilde{\Sigma}^A_{ai} = 0, \]
\[ \tilde{C}^i \equiv \mu^1_a \tilde{\Sigma}^A_{ai} = 0. \]

(3.7)

(3.8)
and
\[ C^i_a \equiv s^i_a \hat{\Sigma}_A^{01} - \epsilon_{ab} \hat{\Sigma}_A^{bi} = 0. \]  
(3.9)

As is explained in [6] a further simplification in the 2+2 formalism is obtained by introducing the conformal factor \( \nu \) of the 2-metric as an explicit variable. This results in one further constraint
\[ \hat{C} \equiv \hat{\Sigma}_2^a \hat{\Sigma}_3^b \epsilon_{ab} \delta_{ij} - \mu \nu = 0. \]  
(3.10)

We are now in a position to write down the primary Lagrangian. It is obtained from the action given above. The double null slicing conditions are imposed through the Lagrange multipliers \( \mu \) and \( \rho \), the adapted frame conditions, which require that \( \hat{\Sigma}_2^{01} \) and \( \hat{\Sigma}_3^{01} \) vanish, are imposed through Lagrange multipliers \( \tau^2 \) and \( \tau^3 \) while the constraints (3.7)–(3.10) are imposed through the corresponding \( \lambda \) Lagrange multipliers. The end result is
\[
L = \int \left( \hat{\Sigma}_A^a A_i^A_a + \hat{\Sigma}_A^{01} B_A^{01} + B_0^A D_j \hat{\Sigma}_A^{01} + B_0^A D_i \hat{\Sigma}_A^{01} - \mu R_{12}^1 - s^i_j R_A^{ij} \hat{\Sigma}_2^{01} - s^i_j R_A^{ij} \hat{\Sigma}_3^{01} - s^i_j \right) \sum_{i=1}^{3} \left( \hat{\Sigma}_i^a \hat{\Sigma}_i^{01} - \epsilon_{ab} \hat{\Sigma}_i^{bi} \right) d^2x,
\]

(3.11)

where we have explicitly written out the curvature terms \( R_{A_i}^{ij} \) and \( R_A^{ij} \) since they contain time derivatives of the connection. It is worth noting that if one imposes the double null slicing condition the constraints \( C^i \) and \( \hat{C}^i \) simplify and become
\[
C^i = \hat{\Sigma}_2^{0i} = 0, \quad \hat{C}^i = \hat{\Sigma}_3^{0i} = 0. \]  
(3.12)

In this formulation, the configuration space consists of the variables \( \nu, \mu^a, s^i, \hat{\Sigma}_A^a, A_i^A_a \) and \( B_i^A \), which are required to satisfy a total of 13 constraints which are imposed through the Lagrange multipliers \( \hat{\lambda}, \lambda_i, \hat{\lambda}_i, \lambda_{ij}, \mu, \rho \), and \( \mu^i \). It was shown in paper I that variation of this Lagrangian with respect to the variables \( A_i^A_a \) and \( B_i^A \) produces the structure equations of the \( SO(3) \) connection while variation with respect to the other variables allows one to eliminate the Lagrange multipliers and obtain all the Einstein field equations. In this paper, we will give the corresponding Hamiltonian description and carry out an analysis of the constraints to obtain a full canonical analysis of the \( 2+2 \) Hamiltonian description in terms of Ashtekar-type variables.

4. Hamiltonian description

The Lagrangian density is of the form \( L = \rho^i \dot{q}_i - \mathcal{H} \), and therefore we can see directly that the canonical variables are \( A_i^A \) and \( B_i^A \), and have the respective momenta \( \hat{\Sigma}_A^{0i} \) and \( \hat{\Sigma}_A^{01} \). We can therefore simply read off the Hamiltonian density which is given by
\[
\mathcal{H} = \mu R_{12}^1 + s^i_j R_A^{ij} \hat{\Sigma}_2^{01} + s^i_j \sum_{i=1}^{3} \left( \hat{\Sigma}_i^a \hat{\Sigma}_i^{01} - \epsilon_{ab} \hat{\Sigma}_i^{bi} \right) - B_0^A \left( D_i \hat{\Sigma}_A^{01} + D_i \hat{\Sigma}_A^{01} \right) + \lambda_i C^i + \hat{\lambda}_i \hat{C}^i + \hat{\lambda}_i \hat{C}^i + \lambda_{ij} C^i + \lambda_{ij} C^j + \rho \left( \mu^i \right)^2 + \rho \left( \mu^i \right)^2 + \tau^2 \hat{\Sigma}_2^{01} + \tau^3 \hat{\Sigma}_3^{01}. \]  
(4.1)

The canonical Poisson brackets are then given by
\[
\{ A_i^A(x), \hat{\Sigma}_B^{0j}(\tilde{y}) \} = \delta_{AB}^A \delta^j_0 \delta(x, \tilde{y}), \quad \{ B_i^A(x), \hat{\Sigma}_B^{0j}(\tilde{y}) \} = \delta_{AB}^A \delta(x, \tilde{y}). \]  
(4.2a)
(4.2b)
In the Hamiltonian given above the variables \( \mu^a_b, s^i_a, \Sigma_A^{ij} \) and \( B_0^A \) are cyclic variables. Because of the structure equations and the Bianchi identities the equations obtained by variation with respect to these variables are propagated by the primary Hamiltonian and we do not need to include them in the full canonical analysis but can consider them as if they were multipliers (as is done with the lapse and shift in the standard ADM treatment). This ‘shortcut’ procedure is described in section 7.2.1 in the article on canonical gravity by Isenberg and Nester [11]. A similar observation was made by Goldberg et al. [5] for the corresponding variables in the null 3+1 case. It is also important to note that some of the constraints introduced into the primary Hamiltonian are not constraints on the canonical variables, but simply on the cyclic variables, and so may be treated as multiplier equations. As a result of this we have a phase space which consists of 18 variables \( A^A_i, B^A_i, \Sigma_A^{ij} \) and \( \Sigma_A^{i0} \). Furthermore only four of the original 13 constraints are actually primary constraints:

\[
C^i = 0, \quad \Sigma_2^{0i} = 0, \quad \Sigma_3^{0i} = 0. \tag{4.3}
\]

We now start the constraint analysis algorithm by varying the Hamiltonian with respect to the cyclic variables. This leads to the equations

\[
\frac{\delta H}{\delta \mu^0} = -\mu^1_1 R^1_{23} - \mu^1_3 \Sigma^{0i}_2 \lambda - \lambda \Sigma^{00}_2,
\]

\[
\frac{\delta H}{\delta \mu^1} = -\mu^0_0 R^1_{23} - \mu^0_3 \Sigma^{0i}_2 \lambda - \lambda \Sigma^{00}_2,
\]

\[
\frac{\delta H}{\delta \mu^0} = \mu^1_0 R^1_{23} + \mu^1_3 \Sigma^{0i}_2 \lambda - \lambda \Sigma^{00}_2,
\]

\[
\frac{\delta H}{\delta \mu^0} = \mu^0_0 R^1_{23} + \mu^0_3 \Sigma^{0i}_2 \lambda - \lambda \Sigma^{00}_2,
\]

\[
\frac{\delta H}{\delta s^i_0} = R^A_{ij} \bar{\Sigma}^{0j}_A + \lambda \bar{\Sigma}^{00}_A,
\]

\[
\frac{\delta H}{\delta s^i_3} = R^A_{ij} \bar{\Sigma}^{0j}_A + \lambda \bar{\Sigma}^{00}_A,
\]

\[
\frac{\delta H}{\delta \Sigma_A^{ij}} = R^1_{1p} + \lambda^0_0,
\]

\[
\frac{\delta H}{\delta \Sigma^{ij}_A} = R^2_{1p} - R^j_{ij} \lambda^0_0 - \lambda \mu^1_0 + \bar{\lambda} \bar{\Sigma}^{0j}_A \epsilon^{pj},
\]

\[
\frac{\delta H}{\delta \Sigma^{ij}_A} = R^3_{1p} - R^j_{ij} \lambda^0_0 - \lambda \mu^1_0 + \bar{\lambda} \bar{\Sigma}^{0j}_A \epsilon^{pj},
\]

\[
\frac{\delta H}{\delta B^0_0} = D_t \Sigma^{0i}_A + D_t \Sigma^{0i}_A.
\]

We now propagate the primary constraints (4.3) using \( \hat{Z} = \{ Z, H \} \) and obtain

\[
\hat{C}^i = \mu^0_0 \bar{\Sigma}^{0i}_2,
\]

\[
\bar{\Sigma}_2^{0i} = \Sigma_2^{0i} + A^1_i \Sigma_1^{0i} + 2A^1_i \Sigma_2^{0i} + B_0^1 \Sigma_4^{0i} + B_0^1 \Sigma_2^{01},
\]

\[
\bar{\Sigma}_3^{0i} = \Sigma_3^{0i} - A^1_i \Sigma_1^{0i} - 2A^1_i \Sigma_3^{0i} - B_0^1 \Sigma_4^{0i} - B_0^1 \Sigma_3^{0i}.
\]
We must now check which of the above equations are secondary equations and which
define multipliers. We first see that \( (4.5b) \) defines the multipliers \( \lambda^1_p = -(R^2_{pj} \tilde{\Sigma}_2^{1j} + R^3_{pj} \tilde{\Sigma}_3^{1j})/\Sigma_4^{01} \). Equation \((4.4a) \) then determines \( \tilde{\lambda} \) which is given by \( \tilde{\lambda} \approx -R^1_{23}/\Sigma_4^{01} \) (where the symbol \( \approx \) indicates weak equality in which we ignore terms that vanish by virtue of the equations of motion). If this is substituted into \( (4.4b) \) then it becomes weakly zero. Also, after substituting \( \tilde{\lambda} \approx \) into \( \lambda^0_p = -R^1_{ip}, \) \((4.6c) \) to define \( \mu^1 \tilde{\lambda}_p = R^3_{ip} - R^1_{ip} \tilde{\lambda}_j + R^1_{ip} \Sigma_2^{0j}/\Sigma_4^{01}. \) We will show in the next section that \((4.10) \) gives five of the Einstein equations and are therefore automatically preserved by the
virtue of the equations of motion). If this is substituted into \( (4.10) \) then it becomes weakly zero. Also, after substituting \( \tilde{\lambda} \approx \) into equation \((4.4c) \), the multiplier equation \( \lambda^1_\Sigma_2^{0i} \approx 0 \) is obtained. We use \((4.6a) \) to define the multipliers \( \lambda^0_p = -R^1_{ip}, \) \((4.6c) \) to define \( \mu^1 \tilde{\lambda}_p = R^3_{ip} - R^1_{ip} \tilde{\lambda}_j + R^1_{ip} \Sigma_2^{0j}/\Sigma_4^{01}. \) Equations \((4.8) \) define the cyclic variables \( \tilde{\Sigma}_2^{1i}, \) while the final equations \((4.9a) \) and \((4.9b) \) define \( B^2_0 \) and \( B^3_0. \) This leaves us with eight secondary constraints \((4.4d), \) \((4.6b), \) \((4.5a) \) and \((4.7), \) which can be written as

\[
\begin{align*}
\frac{\delta H}{\delta \mu^0_0} & \approx \tilde{\Sigma}_3^{0p} \left( R^3_{1p} \tilde{\Sigma}_4^{01} + R^3_{ip} \tilde{\Sigma}_4^{0i} + R^1_{ip} \tilde{\Sigma}_2^{0i} \right), \\
\frac{\delta H}{\delta \Sigma_2^{1p}} & \approx R^2_{1p} \tilde{\Sigma}_4^{01} + R^1_{ip} \tilde{\Sigma}_4^{0i} + R^1_{ip} \tilde{\Sigma}_2^{0i}, \\
\frac{\delta H}{\delta \delta^0_0} & \approx -R^1_{pj} \tilde{\Sigma}_A^{0j} + R^1_{ip} \Sigma_2^{0i}, \\
\frac{\delta H}{\delta B^0_0} & \approx D_i \tilde{\Sigma}_A^{0i} + D_i \tilde{\Sigma}_A^{0i}.
\end{align*}
\]

Therefore at this point we have a phase space of 18 variables, with 4 primary constraints \((4.3) \) and 8 secondary constraints \((4.10a)-(4.10d). \) We now propagate the secondary constraints to check for any tertiary constraints. We will show in the next section that \((4.10a)-(4.10c) \) give five of the Einstein equations and are therefore automatically preserved by the Bianchi identities. When we propagate \((4.10d) \) we find that one component is identically zero on the reduced phase space, whereas the other two components define the multipliers \( \tau^2 \) and \( \tau^3 \). No further constraints are therefore obtained by propagating the secondary constraints.

Now that we have obtained all the constraints we obtain the evolution equations by making variations with respect to the canonical variables. This gives

\[
\begin{align*}
\dot{A}_p^1 & = D_p B_1^0 + R^1_{ip} s^l_0 - R^2_{pj} \tilde{\Sigma}_2^{1j} \left( \Sigma_4^{01} \right)^{-1}, \\
\dot{A}_p^2 & = D_p B_2^0 + R^2_{ip} s^l_0 + \mu_{ip} \tilde{\lambda}_p - R^1_{pj} \tilde{\Sigma}_3^{1j} \left( \Sigma_4^{01} \right)^{-1}, \\
\dot{A}_p^3 & = D_p B_3^0 + R^3_{ip} s^l_0 - R^1_{pj} \tilde{\Sigma}_2^{1j} \left( \Sigma_4^{01} \right)^{-1}, \\
\dot{B}_1^0 & = D_1 B_1^0 + \mu \tilde{\lambda} + \lambda^j_0 s^j_0, \\
\dot{B}_2^0 & = D_2 B_2^0 + \tau^2, \\
\dot{B}_3^0 & = D_3 B_3^0 + \tau^3, \\
\dot{\Sigma}_1^{0i} & = 2 D_j \left( \tilde{\Sigma}_1^{0l} s^l_0 \right) - D_j (\tilde{\Sigma}_1^{0j}) + \epsilon_{ij} (\mu - s \tilde{\Sigma}_1^{0i}) j + 2 \eta^C B_1^0 B_1^0 \Sigma_4^{0i}, \\
\dot{\Sigma}_2^{0i} & = 2 D_j \left( \tilde{\Sigma}_2^{0l} s^l_0 \right) - D_j (\tilde{\Sigma}_2^{0j}) + \epsilon_{ij} A_1^j (\mu - s \tilde{\Sigma}_4^{0i}) + 2 \eta^C B_2^0 B_2^0 \Sigma_4^{0i}, \\
\dot{\Sigma}_3^{0i} & = 2 D_j \left( \tilde{\Sigma}_3^{0l} s^l_0 \right) - D_j (\tilde{\Sigma}_3^{0j}) + \epsilon_{ij} A_2^j (\mu - s \tilde{\Sigma}_4^{0i}) + 2 \eta^C B_3^0 B_3^0 \Sigma_4^{0i}, \\
\dot{\Sigma}_A^{0l} & = D_j \tilde{\Sigma}_A^{0j} + 2 \eta^C B_0^0 B_0^0 \Sigma_4^{0i}.
\end{align*}
\]

5. Einstein equations

We now show that the equations which we have obtained so far contain the ten Einstein equations. In order to do this we first represent the Einstein equations in terms of the variables used in the Hamiltonian description:

\[
\begin{align*}
\dot{\Sigma}_1{}^0 G^9_0 &\approx 2uv(R^2_{ij} \dot{\Sigma}_1{}^0 + R^2_{ij} \dot{\Sigma}_3{}^0 + R^1_{ij} \dot{\Sigma}_4{}^0) \dot{\Sigma}_2{}^{ij}, \\
\dot{\Sigma}_1{}^0 G^9_1 &\approx -2(u^1) \dot{v}(R^3_{ij} \dot{\Sigma}_1{}^0 + \dot{\Sigma}_4{}^0) \dot{\Sigma}_3{}^{ij}, \\
\dot{\Sigma}_1{}^0 G^9_2 &\approx -2uv(R^1_{ij} \dot{\Sigma}_3{}^0 + R^1_{ij} \dot{\Sigma}_4{}^0) \dot{\Sigma}_2{}^{ij}, \\
\dot{\Sigma}_1{}^0 G^9_3 &\approx -2(u^1) \dot{v}(R^1_{ij} \dot{\Sigma}_3{}^0 + R^1_{ij} \dot{\Sigma}_4{}^0) \dot{\Sigma}_2{}^{ij}, \\
\dot{\Sigma}_1{}^0 G^3_3 &\approx 2uv[(R^1_{ij} \dot{\Sigma}_3{}^0 + R^1_{ij} \dot{\Sigma}_4{}^0 + R^2_{ij} \dot{\Sigma}_2{}^0) \dot{\Sigma}_1{}^0 \\
&+ (R^2_{ij} \dot{\Sigma}_2{}^0 - R^1_{ij} \dot{\Sigma}_3{}^0 + R^1_{ij} \dot{\Sigma}_4{}^0) \dot{\Sigma}_1{}^0] \dot{\Sigma}_2{}^{ij}. 
\end{align*}
\]  

(5.1)

We now see that the first five equations are determined by the secondary constraints as follows:

\[
\begin{align*}
\dot{\Sigma}_1{}^0 G^9_0 &\approx 2uv \dot{\Sigma}_2{}^{ij} \frac{\delta H}{\delta \Sigma_2{}^{ij}} = 0, \\
\dot{\Sigma}_1{}^0 G^9_1 &\approx -2(u^1) \dot{v} \frac{\delta H}{\delta \mu_1} = 0, \\
\dot{\Sigma}_1{}^0 G^9_2 &\approx -2uv \dot{\Sigma}_2{}^{ij} \frac{\delta H}{\delta s_0} = 0, \\
\dot{\Sigma}_1{}^0 G^3_3 &\approx 2(u^1) \dot{v} \dot{\Sigma}_3{}^0 \frac{\delta H}{\delta s_0} = 0, \\
\dot{\Sigma}_1{}^0 G^3_3 &\approx 2(u^1) \dot{v} \dot{\Sigma}_3{}^0 \frac{\delta H}{\delta \Sigma_2{}^{ij}} = 0. 
\end{align*}
\]  

(5.2)

Note that \( \dot{\Sigma}_1{}^0 \equiv v \neq 0 \) so that these do indeed imply the vacuum Einstein equations.

We now show that the equations of motion (4.11a)–(4.12a) express the remaining Einstein equations. Writing equation (4.11a) in the form

\[
-\dot{A}^1_p + D_p B^1_0 + R^1_{ip} \dot{x}_0 - R^2_{pj} \dot{\Sigma}_2{}^{ij} (\dot{\Sigma}_1{}^0)^{-1} = 0
\]  

(5.3)

and using the definition of \( R^1_{00} \) and the constraints \( C^i_0 \), we find

\[
-R^1_{ip} \dot{\Sigma}_1{}^{01} + R^1_{ip} \dot{\Sigma}_1{}^{1i} - R^2_{pj} \dot{\Sigma}_2{}^{ij} \approx 0,
\]  

(5.4)
which implies that $G^1_{2} \approx 0$ and $G^3_{3} \approx 0$. In a similar way, we rewrite the remaining equations (4.11b), (4.11c) and (4.12a) to obtain

$$\left(-R^2_{0 p} + R^2_{i p} \Sigma^i_0 \right) \Sigma^1_0 \approx 0,$$

(5.5)

$$-R^3_{0 p} \Sigma^i_0 + R^3_{i p} \Sigma^i_1 - R^3_{1 p} \Sigma^1_1 \approx 0,$$

(5.6)

$$R^1_{0 i} \Sigma^0_1 - R^1_{1 i} \Sigma^1_1 + R^2_{1 i} \Sigma^2_1 \approx 0.$$  

(5.7)

Equation (5.5) gives $G^1_{0} \approx 0$, whilst (5.6) gives $G^3_{3} \approx 0$. The final Einstein equation $G^3_{3} \approx 0$ follows from (5.4) and (5.7). We have therefore shown that the constraint equations and evolution equations imply the Einstein equations.

6. Structure equations

From the self-dual Lagrangian approach one obtains not only the Einstein equations but also the structure equations. These are derived through the variation of the connection variables and when written in terms of the $SO(3)$ basis give the structure equations, $\delta S^A_{ij} + 2\eta^A_{BC} \Gamma^B_{ij} \wedge S^C = 0$. When this is expressed in terms of the sigma variables we obtain the equations $D_p \Sigma_{ja} = 0$ and we should expect to obtain these equations as well as the Einstein equations from our Hamiltonian analysis.

We would normally expect the structure equations to come from the equations of motion, but this is not completely true in this case. The equations of motion (4.13a)–(4.13c) and (4.14) can be rewritten as $-D_p \Sigma_{a} = 0$ and $D_p \Sigma_{i} = 0$, respectively. The remaining structure equations are not found in the equations of motion but in the constraint equation (4.7) which can be rewritten as $D_p \Sigma_{a} = 0$; this is a result of using the shortcut method. Combining these equations we obtain $D_p \Sigma_{a} = 0$ which when written in terms of $S^A$ gives us the structure equations.

7. First class constraints

The constraints obtained so far are not necessarily first class. We need to take linear combinations of the four primary and eight secondary constraints to construct a first class algebra. It is possible to do this by following the Dirac–Bergmann algorithm but in practice it is easier to use geometric insight to construct the appropriate variables. For example in the 2+2 formalism we would expect that two of the first class constraints $\psi_p$ would generate diffeomorphisms in the 2-surface $\{S\}$. These will come from the shift terms, so we start by considering the secondary constraints that arise from the variation of the multipliers $s^i_\alpha$. By calculating the Poisson brackets with the connection we find we need to adapt them by the addition of the constraint (4.7), multiplied with the canonical variables $A^A_p$. This gives the constraint

$$\psi_p := R^A_{1 p} \Sigma_{A}^0 + R^A_{1 p} \Sigma_{A}^{01} + A^A_p \left(D_l \Sigma_{A}^{01} + D_l \Sigma_{A}^0 \right)$$

$$= B^A_{1 p} \Sigma_{A}^0 + A^A_p \Sigma_{A}^0 - \left(A^A_p \Sigma_{A}^0 \right)_j = 0.$$  

(7.1)

Another first class constraint $\psi_1$ should correspond to the re-parametrization of the null generators of $\Sigma_{a}$. In the 2+2 formalism this will come from the lapse of $\{S\}$ in $\Sigma_{a}$, so we start by considering the constraint generated by $\mu^1_{ij}$. Then, to obtain the first class constraint we adapt it in a similar manner to the previous constraint to obtain

$$\psi_1 := R^A_{1 j} \Sigma_{A}^0 + B^A \left(D_l \Sigma_{A}^0 + D_l \Sigma_{A}^0 \right)$$

$$= B^A_{1 j} \Sigma_{A}^0 + A^A_p \Sigma_{A}^0 - \left(B^A \Sigma_{A}^0 \right)_j = 0.$$  

(7.2)
The final first class constraint is a Gauss type equation obtained from $B^1_0$,

$$G_1 := D_1 \hat{\Sigma}_1^{01} + D_1 \hat{\Sigma}_1^{10} = 0. \tag{7.3}$$

Based on the usual $3+1$ timelike decomposition of the action one might expect a further first class scalar Hamiltonian constraint. However it has been observed in a number of studies [5, 12, 13] that in a null formulation of Einstein’s equations the Hamiltonian constraint is second class. Geometrically this is because a null surface is special since there are no compact infinitesimal mappings from one null surface to another null surface [14]. In fact one can show that there are no further first class constraints, so at the end of this analysis we have a phase space of 18 functions subject to 4 first class and 8 second class constraints leaving 2 dynamical degrees of freedom per hypersurface point as is appropriate on a null surface [13, 15–17].

We now wish to consider the geometric interpretation of the first class constraints. In order to do this we calculate the infinitesimal transformations of the canonical variables generated by the constraints. We start by considering $\psi_i$. Let $\hat{F}$ be a vector field with components tangent to $\{S\}$ and define a smeared version of the $\psi_i$ constraint by

$$\hat{\psi} (\hat{F}) = \int \hat{F}^i \psi_i d^3 x. \tag{7.4}$$

Then taking the commutator of this with the connection gives

$$\delta A^A_i = \{ A^A_i, \hat{\psi} (\hat{F}) \} = \mathcal{L}_F A^A_i, \tag{7.5a}$$
$$\delta B^A_i = \{ B^A_i, \hat{\psi} (\hat{F}) \} = \mathcal{L}_F B^A_i, \tag{7.5b}$$

(where $\mathcal{L}$ denotes the Lie derivative in $\Sigma_0$) which shows that $\psi_i$ generates diffeomorphisms within the 2-surface $\{S\}$.

We next consider $\psi_1$. This time we let $\hat{F}$ be a vector field with components tangent to the null generators of $\Sigma_0$ and defined a smeared version of the $\psi_1$ constraint by

$$\hat{\psi} (\hat{F}) = \int \hat{F}^1 \psi_1 d^3 x. \tag{7.6}$$

Taking the commutator of this constraint with the connection then gives

$$\delta A^A_i = \{ A^A_i, \hat{\psi} (\hat{F}) \} = \mathcal{L}_F A^A_i, \tag{7.7a}$$
$$\delta B^A_1 = \{ B^A_1, \hat{\psi} (\hat{F}) \} = \mathcal{L}_F B^A_1, \tag{7.7b}$$

which shows that $\psi_1$ generates diffeomorphisms along the null generators of $\Sigma_0$.

Finally we consider the constraint $G_1$. We smear this constraint with the scalar field $f$ and define

$$G(f) = \int f G_1 d^3 x. \tag{7.8}$$

The commutator of this constraint with the connection is then given by (7.3)

$$\delta A^A_i = \{ A^A_i, G(f) \} = -f, \delta A^A_i - 2 f A^A_2 \delta A^A_2 + 2 f A^A_3 \delta A^A_3, \tag{7.9a}$$
$$\delta B^A_1 = \{ B^A_1, G(f) \} = -f, \delta B^A_1 - 2 f B^A_2 \delta B^A_2 + 2 f B^A_3 \delta B^A_3. \tag{7.9b}$$

To understand the transformation generated by this constraint we need to consider the effect of spin and boost transformations. A complex spin and boost transformation is given by

$$\theta^0 \rightarrow \rho \theta^0,$$
$$\theta^1 \rightarrow \rho^{-1} r^{-1} \theta^1,$$
$$\theta^2 \rightarrow \rho^{-1} \theta^2,$$
$$\theta^3 \rightarrow \rho^{-1} r \theta^3, \tag{7.10}$$
where \( r = \bar{\rho} \) in the real case. This transformation induces the following change in the self-dual 2-forms \( S^A \):

\[
S^A \rightarrow (\Lambda^{-1})^A_B S^B.
\]

(7.11)

where

\[
(\Lambda)^A_B = \begin{bmatrix}
1 & 0 & 0 \\
0 & r^2 & 0 \\
0 & 0 & r^{-2}
\end{bmatrix}.
\]

(7.12)

Note that this only depends upon \( r \) and not on \( \rho \) which reflects the fact that \( r \) represents the self-dual part and \( \rho \) the anti-self-dual part of the spin and boost freedom.

Under a gauge transformation (7.12) the connection transforms according to

\[
\Gamma^A \rightarrow \eta^{AC}_B (\Lambda^{-1})^D_C + \eta^{AC}_B \eta^{DE}_F (\Lambda^{-1})^D_C \Gamma^E.
\]

(7.13)

Using this we find the infinitesimal transformations of the connection variables, \( A^i_A \) and \( B^a_A \), are given by

\[
\begin{align*}
\delta A^i_A & \rightarrow \delta r^i, \\
\delta B^a_A & \rightarrow \delta r^a, \\
\delta A^3_A & \rightarrow -2A^3 A^a \delta r^a, \\
\delta B^3_A & \rightarrow -2B^3 A^a \delta r^a, \\
\delta A^1_A & \rightarrow 2A^1 A^a \delta r^a, \\
\delta B^1_A & \rightarrow 2B^1 A^a \delta r^a.
\end{align*}
\]

(7.14)

Comparing this to (7.9a) and (7.9b) we see that \( G_1 \) generates the self-dual spin and boost transformations.

We are now in a position to display the structure of the first class algebra. We do this by calculating the Poisson brackets of all the smeared first class constraints with each other. This has the following structure:

\[
\{ \tilde{\Psi}(\tilde{P}), \tilde{\Psi}(\tilde{Q}) \} = \tilde{\Psi}(\mathcal{L}_P \tilde{Q}),
\]

(7.15a)

\[
\{ \hat{\Psi}(\hat{P}), \hat{\Psi}(\hat{Q}) \} = \hat{\Psi}(\mathcal{L}_P \hat{Q}),
\]

(7.15b)

\[
\{ \tilde{\Psi}(\tilde{P}), \hat{\Psi}(\hat{Q}) \} = \hat{\Psi}(\mathcal{L}_P \hat{Q}),
\]

(7.15c)

\[
\{ \tilde{\Psi}(\tilde{P}), G(q) \} = G(\mathcal{L}_P q),
\]

(7.15d)

\[
\{ \hat{\Psi}(\hat{P}), G(q) \} = G(\mathcal{L}_P q),
\]

(7.15e)

\[
\{ G(p), G(q) \} = 0.
\]

(7.15f)

We have chosen to keep \( \psi_1 \) and \( \psi_2 \) separate to illustrate the 2+2 structure of the constraint algebra. However they may be combined to give \( \psi_A \), where \( (\psi_A) = (\psi_1, \psi_2, \psi_3) \). This may be smeared with a general vector field \( F \) on \( \Sigma_0 \) to give

\[
\Psi(F) = \int F^A \psi_A \, d^3x.
\]

(7.16)

The constraint algebra then has the more compact form

\[
\{ \Psi(P), \Psi(Q) \} = \Psi(\mathcal{L}_P Q),
\]

(7.17a)
\[
\begin{align*}
\{\Psi(P), G(q)\} &= G(L_P q), \\ 
\{G(p), G(q)\} &= 0.
\end{align*}
\] (7.17b)

This algebra has a similar form to that obtained by Goldberg et al \[5\] but is simpler because in our 2+2 case \( G \) generates the self-dual spin and boost transformations rather than the more complicated null rotations which are needed in the null 3+1 setting.

8. Second class constraints and reality conditions

In this section we briefly examine the remaining constraints. We start by looking at the geometric origin of the second class constraints. See Goldberg and Robinson \[14\] for a similar analysis. The double null slicing condition together with the use of an adapted frame results in four constraints
\[
\tilde{\Sigma}_2^{0i} = 0, \quad \tilde{\Sigma}_2^{01} = 0, \quad \tilde{\Sigma}_3^{01} = 0.
\] (8.1)

The invariance of null directions in the hypersurface under spin and boost transformations gives a further two second class constraints given by equation (4.10d) with \( A = 2, 3 \), which we denote by
\[
\mathcal{G}_2 = 0, \quad \mathcal{G}_3 = 0.
\] (8.2)

The remaining two second class constraints are given by (4.10b) with \( p = 2, 3 \). However rather than work with these constraints we use instead the linearly independent combinations
\[
\mathcal{H}_0 = \tilde{\Sigma}_2^{0i} \frac{\delta H}{\delta \tilde{\Sigma}_2^{0i}} = 0,
\] (8.3)

and
\[
\phi = \tilde{\Sigma}_3^{0i} \frac{\delta H}{\delta \tilde{\Sigma}_2^{0i}} = 0.
\] (8.4)

We see from the analysis of section 5 that according to (5.2a) \( \mathcal{H}_0 \) generates the \( G^0_0 \) component of the Einstein equations and hence corresponds to the usual scalar Hamiltonian constraint which as expected is second class when using a null evolution \([5, 12, 13, 21]\). We also see from (5.2e) that \( \phi \) corresponds to the Einstein equations in the 2-surface \( S \).

The Poisson bracket algebra of the second class constraints is quite complicated and not very illuminating. However the general structure of the algebra can be seen by defining a vector of second class constraints by
\[
C_I = (\mathcal{G}_2, \mathcal{G}_3, \mathcal{H}_0, \phi, \tilde{\Sigma}_2^{01}, \tilde{\Sigma}_2^{02}, \tilde{\Sigma}_2^{03}, \tilde{\Sigma}_3^{01})
\] (8.5)

for \( I = 1, \ldots, 8 \). The Poisson bracket matrix then has the structure
\[
C = \begin{pmatrix} Q & R \\ -R & 0 \end{pmatrix},
\] (8.6)

where \( Q \) and \( R \) are \( 4 \times 4 \) matrices. Note this has a similar form to that of Goldberg and Robinson \[14\].

We now turn to the reality conditions. Since we are using an adapted frame the relevant conditions are
\[
\mu^2_b = \mu^3_b, \quad \nu^2_j = \nu^3_j.
\] (8.7)

However rather than use the \( \nu^i_j \) variables we have chosen to use instead the mixed terms of the densitized 2-forms. We therefore also require the reality conditions for the 2-forms which are given by
\[
\epsilon_{\alpha\beta\gamma\delta} \tilde{\Sigma}_A^{\alpha\beta} \tilde{\Sigma}_B^{\gamma\delta} = 0.
\] (8.8)
9. Conclusion

In this paper we have applied a canonical analysis to a double null description of general relativity formulated in terms of Ashtekar-type variables. We started from a first order action written in terms of self-dual 2-forms and the curvature of a complex \( SO(3) \) connection and used this to obtain a Lagrangian density in terms of our variables. From this we calculated the Hamiltonian, on which we performed the canonical analysis. We obtained four primary constraints and eight linearly independent secondary constraints. By taking particular linear combinations of these twelve constraints, we revealed four first class constraints. Two of these constraints, \( \psi_p \), generate the diffeomorphisms within the spatial hypersurface \( \{ S \} \); while one constraint, \( \psi_1 \), generates the diffeomorphisms along the null generators of \( \Sigma_0 \). The final first class constraint is the Gauss constraint which generates the self-dual spin and boost transformations. Unlike the case in the standard 3+1 description, the constraint algebra forms a Lie algebra. This results from using a null formulation in which the Hamiltonian constraint (which causes all the difficulties) is no longer a first class constraint, but because of the null formulation, is now second class.

The next step of the canonical quantization process would be to pass to a reduced phase space which represents the true degrees of freedom of the theory. This involves restricting to the phase space where the second class constraints are satisfied but replacing the Poisson brackets by Dirac brackets \([18]\) (see also Isenberg and Nester \([11]\)). These are modified versions of the Poisson brackets such that the Dirac bracket between any of the second class constraints and any other variable vanishes identically. Given two functions \( F \) and \( G \) on the phase space the Dirac bracket is given by

\[
\{ F, G \}_D = \{ F, G \} - \sum_{J,K} \{ F, C_J \} C^{-1JK} \{ C_K, G \},
\]

where \( C^{-1} \) is the inverse of the matrix given by \((8.6)\).

A similar result is obtained by using instead the ‘starred variables’ of Bergmann and Komar \([19]\) which are constructed so as to have vanishing Poisson bracket with all the second class constraints. An alternative approach would be to fix the gauge in some appropriate way and explicitly solve for the constraints to obtain a Hamiltonian in terms of the independent degrees of freedom. This procedure has been carried out for \( D \)-dimensional gravity in the light-cone gauge by Goroff and Schwartz \([20]\) (see also \([21]\)), but the complicated nature of the second class constraints in our case makes it unlikely that one can do this at all simply in the present formalism.

In the formulation of general relativity used here some of the variables, \( \mu^a_b \), contain an anti-self-dual part. Because the action is written in terms of self-dual 2-forms this could lead to complications. In the canonical analysis given above we were able to avoid this difficulty because these variables were also cyclic, so we could treat them as multipliers rather than canonical variables. However this problem is likely to be more serious when one moves on to the next step of the process. For this reason it is desirable to eliminate the frame variables entirely (as has already been done with the \( v^i_j \) terms) and work exclusively with the components of \( \Sigma_A \) which are manifestly self-dual. This approach has been developed in a subsequent paper \([22]\).

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