A field theoretic generalization of Hajicek and Kuchar’s quantization scheme in 3+1 canonical quantum gravity

Evangelos Melas
National and Capodistrian University of Athens, 8 Pesmazoglou Street, 105 59 Athens
E-mail: evangelosmelas@yahoo.co.uk

Abstract. The 3+1 (canonical) decomposition of all geometries admitting two-dimensional space-like surfaces is exhibited as a generalization of a previous work. A proposal, consisting of a specific re-normalization Assumption and an accompanying Requirement, which has been put forward in the 2+1 case is now generalized to 3+1 dimensions. This enables the canonical quantization of these geometries through a generalization of Kuchar’s quantization scheme in the case of infinite degrees of freedom. The resulting Wheeler-deWitt equation is based on a re-normalized manifold parameterized by three smooth scalar functionals. The entire space of solutions to this equation is analytically given, a fact that is entirely new to the present case. This is made possible by exploiting the freedom left by the imposition of the Requirement and contained in the third functional.

1. Introduction

In a previous work of ours [4], a method has been developed which enables a generalization of Kuchar’s quantization scheme [2], [3] to be applied to a system of infinite degrees of freedom. The system consisted of a midi-superspace axially symmetric model in 2+1 dimensions plus a \( \Lambda \) term. In our present work [?], we apply the same method to the case of a 3+1 midi-superspace model under the assumption of the existence of maximally symmetric two dimensional subsurfaces, i.e. spheres (constant positive curvature), planes (zero curvature) and Gauss-Bolyai-Lobachevsky (henceforth GBL) spaces (constant negative curvature).

The structure of the paper is as follows: In the second section the reduced metrics, the space of classical solutions and the Hamiltonian formulation of the reduced Einstein-Hilbert action principle, resulting in one (quadratic) Hamiltonian and one (linear) momentum first class constraint, are presented. In the third section we consider the quantization of this constraint system by implementing the quantum operator constraints as conditions annihilating the wave-function [5], according to a conceptual generalization of the quantization scheme developed in [2], [3] for constraint systems with finite degrees of freedom. This generalization is possible through the use of a specific re-normalization Assumption and an associated Requirement. After the symmetry reduction, the system still represents an one-dimensional field theory since all remaining metric components depend on time and the radial coordinate. Nevertheless, we manage to extract a Wheeler-deWitt equation in terms of three smooth scalar functionals of the reduced geometries. The exploitation of a residual freedom left by the imposition of the
Requirement enables us to acquire the entire solution space to this equation. Finally, some concluding remarks are included in the Discussion.

2. Classical configuration space and Hamiltonian Formulation

Our starting point is the two-dimensional spaces of positive, zero and negative constant curvature. Their line elements are respectively:

\[ ds^2 = d\theta^2 + \sin^2\theta\,d\phi^2, \quad ds^2 = d\theta^2 + \theta^2\,d\phi^2, \quad ds^2 = d\theta^2 + \sinh^2\theta\,d\phi^2 \]  

(1)

with an obvious range of the coordinates for each case. The corresponding (maximal) symmetry groups are generated by the following KVF's:

\[ \xi_1 = \frac{\partial}{\partial\phi}, \quad \xi_2 = -\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}, \quad \xi_3 = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi} \]  

(2)

\[ \xi_1 = \frac{\partial}{\partial\phi}, \quad \xi_2 = -\phi\frac{\partial}{\partial\theta} + \frac{\sin\phi}{\theta}\frac{\partial}{\partial\phi}, \quad \xi_3 = \sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\theta}\frac{\partial}{\partial\phi} \]  

(3)

\[ \xi_1 = \frac{\partial}{\partial\phi}, \quad \xi_2 = -\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}, \quad \xi_3 = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi} \]  

(4)

We next promote these KVF’s to four-dimensional fields by adding to each and every of them the zero-sum \( \frac{\partial}{\partial t} + \theta\frac{\partial}{\partial r} \). We then enforce these vector fields as symmetries of a generic space-time metric \( g_{\alpha\beta}(t,r,\theta,\phi) \), i.e. we demand that its Lie derivative with respect to all three fields for each family vanishes. We thus arrive at three classes of metrics, collectively described by the following line element:

\[ ds^2 = \left( -\alpha(t,r)^2 + \frac{\beta(t,r)^2}{\gamma(t,r)^2} \right) dt^2 + 2\beta(t,r) dt\,dr + \gamma(t,r)^2 dr^2 + \psi(t,r)^2 d\theta^2 + \psi(t,r)^2 f(\theta)^2 d\phi^2 \]  

(5)

where \( f(\theta) = \sin\theta \) in the case of spherical symmetry, \( f(\theta) = \theta \) in the case of plane symmetry, and \( f(\theta) = \sinh\theta \) in the case of the GBL symmetry.

For each type of symmetry, there exists one solution in which the metric components depend only on the radial coordinate (“point-like”) and another with only time dependence (“cosmological”) [7]. For example, in the case of spherical symmetry and \( \Lambda = 0 \), we have the well known Schwarzschild and Kantowski-Sachs metrics respectively.

Employing the Hamiltonian formulation (presented, e.g., in chapter 9 of [6]) for the system (5) we get the primary constraints \( P_{\alpha} \equiv \delta L/\delta \dot{\alpha} \approx 0 \), \( P^{\beta} \equiv \delta L/\delta \dot{\beta} \approx 0 \) and

\[ \mathcal{H}_o = \frac{1}{2} G^{\alpha\beta}\pi_\alpha \pi_\beta + V, \]  

(6)

\[ \mathcal{H}_1 = -\gamma\pi'_\gamma + \psi\pi_\psi, \quad \mathcal{H}_2 = 0, \quad \mathcal{H}_3 = 0, \]  

(7)

where \( ' = \frac{d}{dr} \), the indices \( \{\alpha, \beta\} \) take the values \( \{\gamma, \psi\} \) and \( H = \int \left( N^\alpha \mathcal{H}_o + N^i \mathcal{H}_i \right) dr \) is the Hamiltonian of our system with \( N^\alpha = \alpha(t, r) \) and \( N^i = \left( \frac{\beta(t, r)}{\gamma(t, r)^2}, 0, 0 \right) \) the lapse and shift functions, respectively. The reduced Wheeler-deWitt super-metric \( G^\alpha{}^\beta \) appearing in (6) reads

\[ G^{\alpha\beta} = \begin{pmatrix}
\frac{\gamma}{\psi^2} & -\frac{1}{\psi} \\
\frac{1}{\psi} & 0
\end{pmatrix}, \]

while the potential \( V \) is

\[ V = -2\epsilon\gamma + 2\Lambda\gamma\psi^2 - 2\frac{\psi'^2}{\gamma} + 4\left( \frac{\psi\psi'}{\gamma} \right)', \]  

(9)
with \( \epsilon = \{1, 0, -1\} \) for the families (5) of two-dimensional subspaces with positive, zero or negative constant curvature, respectively. The open Poisson bracket algebra of the secondary constraints

\[
\mathcal{H}_o \approx 0, \quad \mathcal{H}_1 \approx 0
\]

produced by demanding the time preservation of the primary constraints is the following

\[
\{\mathcal{H}_o(r), \mathcal{H}_o(\tilde{r})\} = \left[ \frac{1}{\gamma'(r)} \mathcal{H}_1(r) + \frac{1}{\gamma'(\tilde{r})} \mathcal{H}_1(\tilde{r}) \right] \delta'(r, \tilde{r}),
\]

\[
\{\mathcal{H}_1(r), \mathcal{H}_o(\tilde{r})\} = \mathcal{H}_o(r) \delta'(r, \tilde{r}),
\]

\[
\{\mathcal{H}_1(r), \mathcal{H}_1(\tilde{r})\} = \mathcal{H}_1(r) \delta'(r, \tilde{r}) - \mathcal{H}_1(\tilde{r}) \delta(r, \tilde{r})'.
\]

This means that (10) are first class and that Dirac’s algorithm is terminated.

In order to unveil the nature of the action of \( \mathcal{H}_1 \) on \( \gamma(t, r), \psi(t, r) \) we have to investigate their behavior under \( r \) re-parameterizations of the form \( r \rightarrow \tilde{r} = h(r) \). By observing (5) it can be readily deduced that \( \psi \) is a scalar and \( \gamma, \psi' \) are covariant rank 1 tensors (i.e. one-forms).

A very important consequence of this feature is that the scalar derivative is given by \( \gamma^{-1} d/dr \) and not by \( d/dr \). By considering now an infinitesimal \( r \) re-parametrization \( r \rightarrow \tilde{r} = r - \eta(r) \) and by using the aforementioned results we can easily infer that the corresponding changes on the fields \( \gamma(t, r), \psi(t, r) \) are nothing but the one-dimensional analogues of the appropriate Lie derivatives, i.e. \( \delta \gamma(r) = (\gamma(r) \eta(r))' \) and \( \delta \psi(r) = \psi'(r) \eta(r) \), respectively. (Notice that the time dependence has been suppress for the sake of brevity.) One is in position now to specify the nature of the action of the linear constraint on our basic configuration variables as that of the generators of spatial diffeomorphisms:

\[
\left\{ \gamma(r), \int d\tilde{r} \eta(\tilde{r}) \mathcal{H}_1(\tilde{r}) \right\} = (\gamma(r) \eta(r))', \quad \left\{ \psi(r), \int d\tilde{r} \eta(\tilde{r}) \mathcal{H}_1(\tilde{r}) \right\} = \psi'(r) \eta(r).
\]

With this knowledge in hand we are legitimated to consider \( \mathcal{H}_1 \) as the representative, in phase-space, of an arbitrary infinitesimal re-parametrization of the radial coordinate. In the same manner we can also see that the action of the quadratic constraint \( \mathcal{H}_o \) on the basic configuration space variables is identical to an infinitesimal change of the time coordinate.

3. Quantization

In this section a quantization scheme for the Hamiltonian system described in the preceding chapter is proposed. The main motivation behind our approach closely can be traced in Dirac’s desire to construct a quantum theory manifestly invariant under the “gauge” generated by the quadratic and linear (first class) constraints (10). In order to achieve this goal we must seek i) a proper realization of our first class constraints as quantum operators and ii) an appropriate wave function that will be annihilated by them. To begin with, let us note that, despite the simplification brought by the imposition of the symmetries (2), (3), (4), the system is still a field theory as all configuration variables and their canonical conjugate momenta depend not only on time, but also on the radial coordinate \( r \). Thus, to canonically quantize the system in the Schrödinger representation, we have to realize the classical dynamical variables as operators in the following sense

\[
\hat{\gamma}(r) = \gamma(r), \quad \hat{\psi}(r) = \psi(r), \quad \hat{\pi}_\gamma(r) = -i \hbar \frac{\delta}{\delta \gamma(r)}, \quad \hat{\pi}_\psi(r) = -i \hbar \frac{\delta}{\delta \psi(r)}.
\]

In the rest of the article we will assume that \( \hbar = 1 \). These operators obey the canonical commutation relations

\[
[\hat{q}^k(r), \hat{p}_j(\tilde{r})] = i \delta^k_j \delta(r, \tilde{r}),
\]
where $\hat{q}^k = \{\hat{\gamma}, \hat{\psi}\}$ and $\hat{p}_j = \{\hat{\pi}_\gamma, \hat{\pi}_\psi\}$. Our next step is to specify the initial space of state vectors in a way that the action of the operators (12) be well-defined. By this is meant that their action does not produce $\delta(0)$ terms. Consider, for example, the action of a momentum operator on some function of the configuration field variables, say

$$\hat{\pi}_\gamma (r) f(\gamma(\tilde{r})) = -i \frac{\partial f(\gamma(\tilde{r}))}{\partial \gamma(\tilde{r})} \delta(\tilde{r}, r).$$

Obviously, when the momentum operator acts at the same point as the function, i.e. $\tilde{r} = r$, the Dirac delta-function that appears on the r.h.s of the above equation produces a $\delta(0)$. In addition, the action of the momentum operator produced a distribution instead of a function. A way to bypass both of these unwanted features is to choose as our initial collection of states all smooth functionals (i.e., integrals over $r$) of the configuration variables $\gamma(r), \psi(r)$ and their derivatives of any order:

$$\Phi \equiv \int f(\gamma, \psi, + \text{derivatives of any order}) \, d\tilde{r}. \quad (13)$$

By applying this choice to the above example, i.e. by replacing $f(\gamma(\tilde{r}))$ with $\int f(\gamma(\tilde{r})) \, d\tilde{r}$ the action of the momentum operator becomes well-defined; no $\delta(0)$ terms and no distributions appear at the r.h.s. Unfortunately, the above choice of initial states can not prevent the occurrence of $\delta(0)s$ when local expressions quadratic in momenta are considered, i.e. the action of $\hat{\pi}_\gamma (r) \hat{\pi}_\gamma (r)$ on $\int f(\gamma(\tilde{r})) \, d\tilde{r}$ apparently produces $\delta(0)$ terms.

An other problem of equal, if not greater, importance has to do with the number of derivatives (with respect to $r$) considered: A momentum operator acting on a smooth functional of degree $n$ in derivatives of $\gamma(r), \psi(r)$ will, in general, produce a function of degree $2n$. Thus, clearly, more and more derivatives must be included if we desire the action of momentum operators to keep us inside the space of integrands corresponding to the initial collection of smooth functionals; eventually, we have to consider $n \to \infty$. This, in a sense, can be considered as the reflection to the canonical approach, of the non-re-normalizability results existing in the so-called covariant approach. Loosely speaking, the way to deal with these problems is to regularize (i.e., render finite) the infinite distribution limits, and re-normalize the theory by, somehow, enforcing $n$ to terminate at some finite value.

Despite the fact that the choice (13) prevents the occurrence of $\delta(0)$ terms when expressions linear in momenta are considered, we still have to find a way to exclude their appearance when expressions quadratic in momenta are included. In addition, we have to take care of the unceasing occurrence of higher and higher derivatives emanating from the action of the momentum operators. With the motivation to dispose off these unwanted features, we are going, in the rest of this article, to present a quantization scheme of our system which manages to:

(a) avoid the occurrence of $\delta(0)s$,

(b) reveal the value $n = 1$ as the only natural (i.e. without ad-hoc cut-offs) possibility to obtain a self-contained space of state vectors,

(c) extract a finite-dimensional Wheeler-deWitt equation governing the quantum dynamics.

The scheme comprises a conceptual generalization of the quantization developed in [2], [3] for finite systems with one quadratic and a finite number of linear constraints. A detailed description of the method is given in [4] (pp. 8-9). Here, we are going to give only a brief outline of the basic ideas underlying it. The central idea is to Dirac-quantize our classical system while keeping the resulting quantum theory invariant under the transformations that leave the classical theory unaffected. This means that the action of the quantum constraints on the wave function has to be chosen in such a way as to secure that the whole scheme is independent under i) the
“gauge” transformations generated by the quadratic and linear constraints, ii) the mixing of the linear constraints, iii) the gauging of the quadratic constraint with the linear constraints and iv) the scaling of the quadratic constraint. To construct a quantum theory that respects the above restrictions we have to i) realize the linear constraints with the momentum operators to the right and ii) define an induced metric and realize the quadratic in momenta part of the quadratic constraint as the conformal Laplace-Beltrami operator based on the induced metric. It is noteworthy that in our case the aforementioned procedure resolves two more problems, namely the occurrence of $\delta(0)$ terms emanating from the action of the quadratic constraint and the ordering of the operators inside the constraints. Let us now to apply the above described method to our system.

Firstly, the quantum analogue of the linear constraint $H_1(r) \approx 0$ is defined with the momentum operators to the far right, i.e.

$$\hat{H}_1(r)\Phi = 0 \Leftrightarrow -\gamma'(r) \frac{\delta \Phi}{\delta \gamma(r)} + \psi'(r) \frac{\delta \Phi}{\delta \psi(r)} = 0,$$

where $\Phi$ is given by 13. It can be proven (see [4] references therein and [7]) that the general solution to the previous equation is any scalar functional:

$$\Phi = \int \gamma(\tilde{r}) L \left( \Psi^{(0)}, \Psi^{(1)}, \ldots, \Psi^{(n)} \right) d\tilde{r},$$

$$\Psi^{(0)} \equiv \psi(\tilde{r}), \quad \Psi^{(1)} \equiv \frac{\psi'(\tilde{r})}{\gamma(\tilde{r})}, \ldots, \Psi^{(n)} \equiv \frac{1}{\gamma(\tilde{r})} \frac{d}{d\tilde{r}} \left( \cdots \psi^{(n-1)}(\tilde{r}) \right),$$

with $L$ being any arbitrary function of its arguments. Thus, the initial collection of state vectors is, by the imposition of the quantum linear constraint $\hat{H}_1(r)$, reduced to the above given smooth scalar functionals.

Secondly, we define the equivalent of Kuchař’s induced metric on the so far space of “physical” states (15). It must be pointed out that in our case these “physical” states are the analogues of Kuchař’s physical variables $q^\alpha$. We start by considering one initial candidate of the form (15). Then, the induced metric is defined as:

$$g^{\Phi \Phi} = G^{\alpha \beta} \frac{\delta \Phi}{\delta x^\alpha} \frac{\delta \Phi}{\delta x^\beta}, \quad \text{where} \quad x^\alpha = \{\gamma, \psi\}$$

and $G^{\alpha \beta}$ is given by (8). A gratifying property of this metric is the non-appearance of $\delta(0)$’s, since only first functional derivatives are employed. However, the following problem arises: $g^{\Phi \Phi}$ is a local function while the “physical” states (15) are functionals. This means that the problem of whether $g^{\Phi \Phi}$ is composed of the so far physical states cannot even be addressed. It is thus clear that, in order for such a property of $g^{\Phi \Phi}$ to be imposed, a correspondence between local functions and smooth functionals must be established as a prerequisite. A way to achieve this is to adopt the following ansatz:

**Assumption:** We assume that, as part of the re-normalization procedure, we are permitted to map local functions to their corresponding smeared expressions e.g., $\psi(r) \leftrightarrow \int d\tilde{r} \psi(\tilde{r})$.

For a detailed discussion about the nature of this Assumption the interested reader is referred to [4] (p. 12). The use of this anzatz allows us to start examining the validity of the above mentioned crucial property of $g^{\Phi \Phi}$. In the case of systems with finite degrees of freedom this feature is automatically satisfied (see [2], [3]). But, in the infinite case here discussed, things
become more complicated because of the dependence of $\gamma$ and $\psi$ on the radial coordinate $r$. This feature will cause more and more derivatives to appear as the action of a functional derivative, say $\frac{\delta}{\delta \psi(r)}$, on a functional whose integrand contains $\Psi^{(n)}$, will produce a maximum term proportional to $\Psi^{(2n)}$. Even with the use of the Assumption $g^{\Phi \Phi}$ is not automatically composed out of the initial physical states. Therefore, such a property must be enforced. This difference between the finite and infinite case can be traced to the fact that the first Poisson bracket in (2) does not vanish as in the finite case (see [4]). Thus, a non-trivial mixing of the dynamical evolution generator $\mathcal{H}_o$ and the linear generator $\mathcal{H}_1$ is brought forth in the infinite case.

According to the above reasoning, it clear now that if we want to proceed further with the generalization of Kuchâr’s method, the property of $g^{\Phi \Phi}$ being composed out of the physical states must be enforced. Consequently, we demand that:

**Requirement:** $L \left( \Psi^{(0)}, \ldots, \Psi^{(n)} \right)$ must be such that $g^{\Phi \Phi}$ becomes a general function, say $F \left( \gamma(r) L(\Psi^{(0)}, \ldots, \Psi^{(n)}) \right)$ of the integrand of $\Phi$, so that it can be considered a function of this state: $g^{\Phi \Phi}$ Assumption $\equiv F \left( \int \gamma(\vec{r}) L(\Psi^{(0)}, \ldots, \Psi^{(n)}) d\vec{r} \right) = F(\Phi)$.

At this point, it must emphasized that the application of the Requirement in the subsequent development of our quantum theory will result in very severe restrictions on the form of (15). Essentially, all higher derivatives of $\psi(r)$ (i.e. $\Psi^{(2)} \ldots \Psi^{(n)})$ are eliminated from $\Phi$ (see (18), (19) below). This might, at first sight, strike as odd; indeed, the common belief is that all the derivatives of the configuration variables should enter the physical states. However, before the imposition of both the linear and the quadratic constrains there are no truly physical states. Thus, no physical states are lost by the imposition of the Requirement; ultimately the only true physical states are the solutions to (26).

Let us now see how the Assumption and the Requirement further restrict our “physical” states (15). As we argued before, the functional derivatives $\frac{\delta}{\delta \psi(r)}$ and $\frac{\delta}{\delta \gamma(r)}$ acting on a functional containing in its integrand $\Psi^{(n)}$ will, upon partial integration of the $n$th derivative of the Dirac delta function, produce a term proportional to $\Psi^{(2n)}$ and $\Psi^{(2n-1)}$ respectively. Thus, by inserting (15) in (17) one can, after a somewhat tedious calculation, get

$$g^{\Phi \Phi} = \ldots + 2G^{\Phi \Phi} \frac{\partial L}{\partial \gamma(r)} \frac{\partial \psi}{\partial \gamma(r)} =$$

$$= \ldots + 2G^{\Phi \Phi} (-1)^{n-1} \frac{\partial^2 L}{\partial \left( \Psi^{(n)} \right)^2} \Psi^{(2n-1)} L^{(1)}(-1)^n \gamma \frac{\partial^2 L}{\partial \left( \Psi^{(n)} \right)^2} \Psi^{(2n)} =$$

$$= \ldots - \frac{\gamma}{2\Psi} (-1)^{2n-1} \left( \frac{\partial^2 L}{\partial \left( \Psi^{(n)} \right)^2} \right)^2 \Psi^{(1)} \Psi^{(2n-1)} \Psi^{(2n)},$$

where $\ldots$ stand for all other terms, not involving $\Psi^{(2n)}$. According to the Requirement we demand $g^{\Phi \Phi} = F(\gamma L)$. For this to happen the coefficient of $\Psi^{(2n)}$ must vanish, i.e. $\partial^2 L / \partial \left( \Psi^{(n)} \right)^2 = 0$. Thus, by integrating this necessary condition, $L$ is reduced to the form

$$L = L_1 \left( \Psi^{(0)}, \ldots, \Psi^{(n-1)} \right) \Psi^{(n)} + L_2 \left( \Psi^{(0)}, \ldots, \Psi^{(n-1)} \right).$$

Notice that the term of $\Phi$ corresponding to $L_1$ is, up to a surface term, equivalent to a general term depending on $\Psi^{(0)}, \ldots, \Psi^{(n-1)}$ only:

$$\Phi_1 = \int \gamma(\vec{r}) L_1 \frac{1}{\gamma(r)} \frac{d}{dr} \Psi^{(n-1)} d\vec{r},$$
which upon subtraction of the surface term

\[ A = \int d\bar{r} \frac{d}{d\bar{r}} \left( \int d\Psi^{(n-1)} L_1 \right) \]

produces a smooth functional with arguments up to \( \Psi^{(n-1)} \) only. Since a surface term in \( \Phi \) does not affect the outcome of the variational derivatives \( \frac{\delta \Phi}{\delta \psi(\bar{r})} \) and \( \frac{\delta \Phi}{\delta \gamma(\bar{r})} \), we conclude that only \( L_2 \) is important for the local part of \( \Phi \). By repeating the entire argument successively for \( n - 1, n - 2, \ldots, 2 \) we can show that all \( \Psi^{(n)} \)s are suppressed from \( L \) except when \( n = 1 \). Thus, we have

\[ \Phi \equiv \int \gamma(\bar{r}) L \left( \psi, \Psi^{(1)} \right) d\bar{r}. \]  

(18)

It can be shown \[ ? \], with the use of the re-normalization Assumption and the accompanying Requirement, that the reduced re-normalized manifold of the physical states \( \Phi \) is finally parameterized by the following three smooth scalar functionals:

\[ y^1 = \int \gamma(\bar{r}) d\bar{r}, \quad y^2 = \int \gamma(\bar{r}) \psi(\bar{r})^2 d\bar{r}, \quad y^3 = \int \gamma(\bar{r}) L(\Psi^{(1)}) d\bar{r}. \]  

(19)

The components of the induced metric \( g^{AB} \) based on the functionals (19) read

\[ g^{AB}_{\text{ren}}(y^1, y^2, y^3) = -\frac{1}{4} \begin{pmatrix} -\frac{(y^1)^2}{y^2} & y^1 & -\frac{y^1 y^3}{y^2} \\ y^1 & 3 y^2 & -y^3 \\ -\frac{y^1 y^3}{y^2} & y^3 & -\frac{(y^3)^2}{y^2} \end{pmatrix}. \]  

(20)

To demonstrate the complexity of these calculations we present the computation of the trickiest component, namely of \( g^{33} \):

\[ g^{33} = \frac{\gamma}{4 \psi^2} \left( L - \Psi^{(1)} L' \right)^2 + \frac{\gamma}{2 \psi} \left( L - \Psi^{(1)} L' \right) \Psi^{(2)} L'' \text{ Assumption} \]

\[ g^{33}_{\text{ren}} = \int d\bar{r} \left[ \frac{\gamma}{4 \psi^2} \left( L - \Psi^{(1)} L' \right)^2 + \frac{\gamma}{2 \psi} \left( L - \Psi^{(1)} L' \right) \Psi^{(2)} L'' \right] - \int d\bar{r} \frac{d}{d\bar{r}} \int d\Psi^{(1)} \left[ \frac{(L - \Psi^{(1)} L') L''}{2 \psi} \right] \equiv \]

\[ g^{33}_{\text{ren}} = \int \frac{\gamma}{4 \psi^2} \left[ \left( L - \Psi^{(1)} L' \right)^2 - \Psi^{(1)} \int d\Psi^{(1)} \frac{\partial}{\partial \Psi^{(1)}} \left( L - \Psi^{(1)} L' \right)^2 \right] d\bar{r}. \]  

(21)

The expression inside the square brackets of \( g^{33}_{\text{ren}} \) above, being a generic function of \( \Psi^{(1)} \), can also be considered as a function of \( L \), say \( W(L(\Psi^{(1)})) \). It is thus clear that the Requirement is satisfied for any \( L(\Psi^{(1)}) \):

\[ g^{33}_{\text{ren}} = \frac{\left( \int \gamma d\bar{r} \right)^2}{4 \int \gamma \psi^2 d\bar{r}} W\left( \int \gamma L d\bar{r} \right) = \frac{(y^1)^2}{4 y^2} W\left( \frac{y^3}{y^1} \right). \]

We are legitimated to parameterize the expression \( W(L(\Psi^{(1)})) \) in the following way

\[ L \left( \Psi^{(1)} \right)^2 - \frac{4 F[L(\Psi^{(1))]}^2}{3 F[l[F[L(\Psi^{(1))]]]^2}. \]  

(22)
This “peculiar” parametrization of the arbitrariness in \( L (\Psi^{(1)}) \) has been chosen in order to facilitate the subsequent proof that the freedom in the choice of \( L \) (left by the imposition of the Requirement) is a pure general coordinate transformation (gct) of the induced re-normalized metric. Taking into account (22) the components of the induced metric (20) read

\[
g_{A\text{ren}}^{\text{AB}}(y^1, y^2, y^3) = -\frac{1}{4} \begin{pmatrix}
\frac{(y^1)^2}{y^2} & y^1 & \frac{y^1 y^3}{y^2} \\
y^1 & 3 y^2 & y^3 \\
\frac{y^1 y^3}{y^2} & y^3 & -\frac{(y^3)^2}{y^2} + \frac{4 (y^1)^2 F \left( \frac{y^3}{y^2} \right)^2}{3 y^2 F'} \left( F \left( \frac{y^3}{y^2} \right)^2 \right)
\end{pmatrix}, \quad \text{and}
\]

\[
g_{A\text{ren}}^{\text{AB}}(y^1, y^2, y^3) = \begin{pmatrix}
\frac{3 y^2 (y^1)^2}{(y^1)^3} & \frac{3 y^2 y^3 (y^1)^2}{(y^1)^3} & \frac{3 y^2 y^3 F' \left( \frac{y^3}{y^2} \right)^2}{(y^1)^3 F \left( \frac{y^3}{y^2} \right)^2} \\
-\frac{1}{y^2} & \frac{1}{y^2} & 0 \\
\frac{3 y^2 y^3 F' \left( \frac{y^3}{y^2} \right)^2}{(y^1)^3 F \left( \frac{y^3}{y^2} \right)^2} & 0 & \frac{3 y^2 F' \left( \frac{y^3}{y^2} \right)^2}{(y^1)^2 F \left( \frac{y^3}{y^2} \right)^2}
\end{pmatrix}.
\]

One can readily show that any function \( \Psi(y^1, y^2, y^3) \) on this manifold is annihilated by the quantum linear constraint, i.e.

\[
\hat{H}_1 \Psi(y^1, y^2, y^3) = \frac{\partial \Psi(y^1, y^2, y^3)}{\partial y^1} \hat{H}_1 y^1 + \frac{\partial \Psi(y^1, y^2, y^3)}{\partial y^2} \hat{H}_1 y^2 + \frac{\partial \Psi(y^1, y^2, y^3)}{\partial y^3} \hat{H}_1 y^3 = 0
\]

since the derivatives with respect to \( r \) are transparent to the partial derivatives of \( \Psi \) (which are, just like the \( y^A \)'s, r-numbers).

The covariant metric (23) describes a three dimensional conformally flat geometry, since the corresponding Cotton-York tensor vanishes. The Ricci scalar is \( R = \frac{3}{8 y^2} \), indicating that the arbitrariness in \( F \) (and thus also in \( L \)) is a pure gauge. Indeed, the change of coordinates

\[
(y^1, y^2, y^3) = (e^{-\frac{1}{8} (5 Y^1 + 3 Y^3)}, e^{Y^1 + Y^2 + Y^3}, e^{-\frac{1}{8} (5 Y^1 + 3 Y^3)} F^{-1} (e^{\frac{1}{8} (-9 Y^1 + 8 Y^2 - 15 Y^3}))
\]

(24)

(24)

(24)

where \( F^{-1} \) denotes the function inverse to \( F \), i.e \( F^{-1}(F(x)) = x \) brings the metric to the manifestly conformally flat form:

\[
g_{A\text{ren}}^{\text{AB}}(Y^1, Y^2, Y^3) = \begin{pmatrix}
e^{Y^1 + Y^2 + Y^3} & 0 & 0 \\
0 & -\frac{4}{3} e^{Y^1 + Y^2 + Y^3} & 0 \\
0 & 0 & -e^{Y^1 + Y^2 + Y^3}
\end{pmatrix},
\]

(25)

in which all the \( F \) dependence has indeed disappeared.
We are now finally in a position to impose the final restriction on the form of the wave function $\Psi$ by the quantum analog of the quadratic constraint $\hat{H}_Q$: We define that the quantum gravity of the geometries given by (5) be described by the following partial differential equation (in terms of the $Y^A$):
\[
\hat{H}_Q \Psi \equiv \left[ -\frac{1}{2} c + V_{ren} \right] \Psi(Y^1, Y^2, Y^3) = 0 \tag{26}
\]
with
\[
c = c = \frac{d - 2}{4(d - 1)} R \tag{27}
\]
being the conformal Laplacian based on $g_{ABren}(Y^1, Y^2, Y^3)$, $R$ the Ricci scalar, and $d$ the dimensions of $g_{ABren}$. The metric (25) is conformally flat with Ricci scalar $R = \frac{1}{2} e^{-Y_1-Y^2-Y^3}$, and its dimension is $d = 3$. The re-normalized form of the potential (9) offers us the possibility to introduce, in a dynamical way, topological effects into our wave functional: Indeed, under our \textbf{Assumption}, the first two terms become $-2 \epsilon y^1$ and $2 \Lambda y^2$, respectively, while the last, being a total derivative, becomes $A_T \equiv 4 \frac{\psi_{\beta'}}{\gamma} (\text{if } \alpha < r < \beta)$. In the spirit previously explained we should drop this term, however one could also keep it. The re-normalized form of the remaining, third, term of the potential can be obtained as follows:
\[
y^3 = \gamma L(\Psi(1)) \Leftrightarrow L(\Psi(1)) = \frac{y^3}{\gamma} \Rightarrow L(\Psi(1)) = \Psi(1) = L^{-1} \left( \frac{y^3}{y^1} \right),
\]
thus finally
\[
\frac{y^3}{\gamma} = L^{-1} \left( \frac{y^3}{y^1} \right)
\]
and the third term becomes $-2 y^1 \left[ L^{-1} \left( \frac{y^3}{y^1} \right) \right]^2$. Finally, effecting the transformation (24) the form of the re-normalized potential is
\[
V_{ren} = -2 \epsilon e^{\frac{1}{2}(5Y^1+3Y^3)} - 2 \epsilon e^{-\frac{1}{2}(5Y^1+3Y^3)} \left[ L^{-1} \left( F^{-1}(e^{\frac{1}{2}(9Y^1+8Y^2-15Y^3)}) \right) \right]^2 + 2 \Lambda e^{Y^1+Y^2+Y^3} + A_T \tag{28}
\]
and the Wheeler-deWitt equation is given as
\[
\begin{align*}
-2 \epsilon e^{\frac{1}{2}(3Y^1+5Y^3)+Y^2} & \Psi(Y^1, Y^2, Y^3) + 2 \Lambda e^{2(Y^1+Y^2+Y^3)} \Psi(Y^1, Y^2, Y^3) - 2 \epsilon e^{\frac{1}{2}(3Y^1+5Y^3)+Y^2} \left[ L^{-1} \left( F^{-1}(e^{\frac{1}{2}(9Y^1+8Y^2-15Y^3)}) \right) \right]^2 \Psi(Y^1, Y^2, Y^3) + A_T e^{Y^1+Y^2+Y^3} \Psi(Y^1, Y^2, Y^3) & - \frac{3}{128} \Psi(Y^1, Y^2, Y^3) - \frac{1}{4} \frac{\partial \Psi(Y^1, Y^2, Y^3)}{\partial Y^2} + \frac{1}{16} \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial Y^2^2} + \frac{1}{2} \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial Y^1^2} + \frac{3}{8} \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial Y^2^2} + \frac{1}{2} \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial Y^3^2} = 0.
\end{align*}
\]
Since $F^{-1}$ is an arbitrary function of its arguments, we may contemplate the choice:
\[
F^{-1} \left( e^{\frac{1}{2}(9Y^1+8Y^2-15Y^3)} \right) = L \left( \sqrt{e^{\frac{1}{2}(9Y^1+8Y^2-15Y^3)} - \epsilon} \right). \tag{29}
\]
Of course there is a question of existence for such a choice: since $F$ which appears in (22) is nothing but a convenient parametrization of the expression inside the square brackets of (21), any demand that $F$ has a specified form (much more in terms of $L$) constitutes an implicit restriction on the form of $L$ itself. Subsequently, at least the existence of such an $L$ must be secured. Indeed, it can be shown [?] that an appropriate $L$ exists; its form is given by:

$$L(\Psi^{(1)}) = m + \int \frac{(\Psi^{(1)})^{3/2}}{((\Psi^{(1)})^2 - \epsilon)^{13/16}} e^{\frac{a_n}{16((\Psi^{(1)})^2 - \epsilon)^{2}} - e^{\frac{a_n}{16((\Psi^{(1)})^2 - \epsilon)^{2}}}} d\Psi^{(1)}$$

where $c_1 m + c_2 + c_3 e^k = 0$.

This choice for $F$ reduces the Wheeler-deWitt equation to the final almost-separable form

$$2 \Lambda e^{2(1+Y^2+Y^3)} \Psi(Y^1, Y^2, Y^3) - 2 \epsilon e^\frac{4}{3} Y^2 \Psi(Y^1, Y^2, Y^3) + A_T e^{Y^1+Y^2+Y^3} \Psi(Y^1, Y^2, Y^3) - 128 \Psi(Y^1, Y^2, Y^3) - \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial (Y^1)^2} + \frac{12 \partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial (Y^2)^2} + \frac{\partial^2 \Psi(Y^1, Y^2, Y^3)}{\partial (Y^3)^2} = 0. \quad (30)$$

This equation is separable for $\Lambda = 0$ and $A_T = 0$. In this case it can readily be solved: assuming $\Psi(Y^1, Y^2, Y^3) = \Psi^1(Y^1) \Psi^2(Y^2) \Psi^3(Y^3)$ and dividing (30) by $\Psi$ we get the three ordinary differential equations:

$$\frac{1}{4 \Psi^1(Y^1)} \frac{d \Psi^1(Y^1)}{d Y^1} + \frac{2}{\Psi^1(Y^1)} \frac{d^2 \Psi^1(Y^1)}{d (Y^1)^2} = m, \quad \frac{1}{16 \Psi^2(Y^2)} \frac{d \Psi^2(Y^2)}{d Y^2} + \frac{3}{8 \Psi^2(Y^2)} \frac{d^2 \Psi^2(Y^2)}{d (Y^2)^2} = 2 \epsilon e^\frac{4}{3} Y^2 = n, \quad \frac{1}{4 \Psi^3(Y^3)} \frac{d \Psi^3(Y^3)}{d Y^3} + \frac{1}{2 \Psi^3(Y^3)} \frac{d^2 \Psi^3(Y^3)}{d (Y^3)^2} = \frac{3}{128} = m - n,$$

where $m$ and $n$ are separation constants. Their solutions are:

$$\Psi^1(Y^1) = c_1 e^{\frac{1}{Y^1}-\sqrt{1+32m}} + c_2 e^{\frac{1}{Y^1}+\sqrt{1+32m}},$$

$$\Psi^2(Y^2) = c_3 e^{-Y^2/4} I_{\frac{\epsilon Y^2}{\sqrt{3}+128n}} \left(2 \epsilon e^{\frac{3}{2} Y^2/3} \right) + c_4 e^{-Y^2/4} I_{\frac{\epsilon Y^2}{\sqrt{3}+128n}} \left(2 \epsilon e^{\frac{3}{2} Y^2/3} \right),$$

$$\Psi^3(Y^3) = c_5 e^{\frac{1}{Y^3}-(2+Y^2) Y^3} + c_6 e^{\frac{1}{Y^3}-(2+Y^2) Y^3},$$

where $I_{i \sqrt{3}+128n} \left(2 \sqrt{3} e^{Y^2/3} \right)$ are modified Bessel functions of the first kind and non-integer order.

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