EXTENDED TQFT ARISING FROM ENRICHED MULTI-FUSION CATEGORIES

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Abstract. We define a symmetric monoidal (4,3)-category with duals whose objects are certain enriched multi-fusion categories. For every modular tensor category $\mathcal{C}$, there is a self enriched multi-fusion category $\mathcal{C}$ giving rise to an object of this symmetric monoidal (4,3)-category. We conjecture that the extended 3D TQFT given by the fully dualizable object $\mathcal{C}$ extends the 1-2-3-dimensional Reshetikhin-Turaev TQFT associated to the modular tensor category $\mathcal{C}$ down to dimension zero.

1. Introduction

In his seminal paper [W2], Witten gave an explanation of the Jones polynomial [J1, J2] in terms of 3D Chern-Simons theory and generalized the Jones polynomial to invariants of 3-manifolds (with ribbon links inside). The corresponding mathematical theory is the Reshetikhin-Turaev invariant [RT1, RT2], which is defined for any modular tensor category $\mathcal{C}$ including those arising from Chern-Simons theory. Unlike many other topological invariants, the Reshetikhin-Turaev invariant essentially involves a framing anomaly which has already been observed in Witten’s work. But rather, the double theory of the Reshetikhin-Turaev invariant does not suffer from such an anomaly and, in fact, coincides with the Turaev-Viro invariant of 3-manifolds associated to the same modular tensor category $\mathcal{C}$ [TV, Tu]. Moreover, the Reshetikhin-Turaev invariant turns out to be a boundary of a 4D theory [BGM], the Crane-Yetter invariant of 4-manifolds [CY, CKY] associated to the modular tensor category $\mathcal{C}$.

These invariants are naturally settled in the framework of TQFT introduced in [W1, At] and extended TQFT developed in, for example, [La, F1, BD, Lu]. The Reshetikhin-Turaev invariant is incorporated in a beautiful 1-2-3-dimensional extended TQFT $Z_{\text{RT}}$ where the 1-2-dimensional theory is formulated in terms of modular functor and Moore-Seiberg data (see [TM, BK] and references therein, see also [F3]). If $\mathcal{C}$ arises from Chern-Simons theory with finite gauge group (known as Dijkgraaf-Witten theory), the theory can be extended even down to dimensional zero [F2].

Extended TQFT for Turaev-Viro invariant and Crane-Yetter invariant were built from various points of view (see, for example, [BaK, DSS1, F3]). In fact, both invariants could be extended all the way down to dimension zero. According to the cobordism hypothesis, which is proposed by Baez and Dolan [BD] and proved in sketch by Lurie [Lu] (see also [F4]), a fully extended TQFT is determined by its value on a point. By realizing the modular tensor category $\mathcal{C}$ as a fully dualizable object in a certain symmetric monoidal 3-category, one obtains a fully extended 3D
TQFT $Z_c^{TV}$ whose values on 3-manifolds give (modulo certain reasonable conjectures) the Turaev-Viro invariant $[DSS1]$. Similarly, by realizing the modular tensor category $C$ as a fully dualizable object in a certain symmetric monoidal 4-category, one obtains a fully extended 4D TQFT $A_C$ whose values on 4-manifolds should recover the Crane-Yetter invariant $[F3]$. As before, the double theory of $Z_c^{RT}$ agrees with $Z_c^{TV}$ $[Ba2]$, and $Z_c^{RT}$ is a boundary theory of $A_C$ $[F3]$. Although there still remain substantial works to be fulfilled to make many of these statements rigorous, the global picture has been fairly clear.

It is natural to ask whether the Reshetikhin-Turaev TQFT $Z_c^{RT}$ can be extended further to dimensional zero. According to the cobordism hypothesis, this is equivalent to ask what mathematical object should be assigned to a point. Experiences on TQFT told us this amounts to find a mathematical object whose “center” is the modular tensor category $C$. For example, if $C$ is the Drinfeld center of a spherical fusion category $D$, such an extension is available. In fact, the Turaev-Viro invariant is also defined for a spherical fusion category $[Oc, BW]$ and it is known that $Z_c^{TV}$ $[Ba1, Ba2, TV]$ is equivalent to $Z_c^{TV}$ $[Ba1, Ba2, TV]$. But for a general modular tensor category $C$, this question remains open. See $[PHLT]$ for a treatment where $C$ arises from Chern-Simons theory with torus gauge group. Recently, Henriques also proposed a candidate by showing that certain unitary modular tensor categories (completed by separable Hilbert spaces) are the Drinfeld centers of certain categories of solitons $[H1, H2]$.

In this paper, we propose another approach to this problem. We introduce the notions of enriched multi-fusion categories (see $[MP]$ for a definition of monoidal category enriched in a braided monoidal category) and their bimodules, and show their dualities. In particular, a modular tensor category $C$ gives rise to a self enriched multi-fusion category $C = (C, C)$ whose Drinfeld center was shown in $[KZ2]$ to be equivalent to $C$ itself. After these concrete mathematical results are established, we organize the enriched multi-fusion categories and their bimodules into a symmetric monoidal (4, 3)-category with duals (see Theorem 4.1), and argue that the extended 3D TQFT $Z_c$ given by the fully dualizable object $C$ provides a way to extend the Reshetikhin-Turaev TQFT $Z_c^{RT}$ down to dimension zero. Actually, we will argue that the 1-2-3-dimensional theory of $Z_c$ is a combination of those of $Z_c^{RT}$ and $A_C$ and show that the double theory of $Z_c$ is equivalent to $Z_c^{TV}$.

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2. Unitary categories

In this section, we recall some basic facts about unitary multi-fusion categories and unitary modular tensor categories, and set our notations.

A $*\text{-category}$ $C$ is a $C$-linear category equipped a $*$-operation on morphism, i.e. $* : \text{Hom}_{C}(x, y) \to \text{Hom}_{C}(y, x)$ is defined so that $(g \circ f)^* = f^* \circ g^*$, $(\lambda f)^* = \overline{\lambda} f^*$, $f^{**} = f$ for $f \in \text{Hom}_{C}(x, y)$, $g \in \text{Hom}_{C}(y, z)$, $\lambda \in C$. An isomorphism $f$ in $C$ is unitary if $f^* = f^{-1}$. A $*\text{-functor}$ $F : C \to D$ between two $*\text{-categories}$ is a $C$-linear functor such that $F(f^*) = F(f)^*$ for all morphisms $f$ in $C$.

An example of $*\text{-category}$ is the category of finite-dimensional Hilbert spaces, denoted by $H$. A unitary category is a $*\text{-category}$ $C$ which is equivalent via a $*$-functor to a finite direct sum of $H$. We always assume a functor $F : C \to D$ between
two unitary categories is an additive $*$-functor. The Deligne tensor product $\mathcal{C} \boxtimes \mathcal{D}$ $[\text{De}]$ of two unitary categories $\mathcal{C}$ and $\mathcal{D}$ is automatically unitary.

A unitary multi-fusion category (UMFC) is a rigid monoidal category $\mathcal{C}$ such that $\mathcal{C}$ is equipped with the structure of a unitary category, the tensor product is an additive $*$-functor $\otimes : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$ and the natural isomorphisms $1 \otimes a \simeq a$, $a \otimes 1 \simeq a$, $(a \otimes b) \otimes c \simeq a \otimes (b \otimes c)$ are unitary. In this case, the left dual and the right dual of an object $a \in \mathcal{C}$ are canonically identified, which we denote as $a^*$. We always assume a monoidal functor $F : \mathcal{C} \to \mathcal{D}$ between two UMFC’s is such that $F$ is an additive $*$-functor and the natural isomorphisms $F(1_\mathcal{C}) \simeq 1_\mathcal{D}$, $F(a \otimes b) \simeq F(a) \otimes F(b)$ are unitary.

A UMFC is called a unitary fusion category if the tensor unit 1 is a simple object. A UMFC is indecomposable (UMFC for short) if it is neither zero nor a direct sum of two nonzero UMFC’s. Given a UMFC $\mathcal{C}$, we use $\mathcal{C}^{rev}$ to denote the UMFC which has the same underlying category $\mathcal{C}$ but equipped with the reversed tensor product $a \otimes^{rev} b := b \otimes a$.

We always assume a left module $M$ (see $[\text{Os}]$) over a UMFC $\mathcal{C}$ is such that $M$ is a unitary category, the left $\mathcal{C}$-action is an additive $*$-functor $\otimes : \mathcal{C} \boxtimes M \to M$ and the natural isomorphisms $1 \otimes x \simeq x$, $(a \otimes b) \otimes x \simeq a \otimes (b \otimes x)$ are unitary; and assume a left $\mathcal{C}$-module functor $F : M \to N$ is such that $F$ is an additive $*$-functor and the natural isomorphisms $F(a \otimes x) \simeq a \otimes F(x)$ are unitary; and make similar assumptions on right modules and bimodules over UMFC’s.

Let $\mathcal{C}, \mathcal{D}$ be UMFC’s. Given left $\mathcal{C}$-modules $M$ and $N$, we use $\text{Fun}_\mathcal{C}(M, N)$ to denote the category of left $\mathcal{C}$-module functors. It is also a unitary category $[\text{ENO}], [\text{GHR}]$. Moreover, $\text{Fun}_\mathcal{C}(M, \mathcal{D})$ is a UMFC $[\text{ENO}], [\text{GHR}]$. In the special case $\mathcal{C} = \mathcal{H}$, we simply denote $\text{Fun}_\mathcal{H}(M, N)$ as $\text{Fun}(M, N)$. Given $\mathcal{C}$-$\mathcal{D}$-bimodules (equivalently, left $\mathcal{C} \boxtimes \mathcal{D}^{rev}$-modules) $M$ and $N$, we use $\text{Fun}_{\mathcal{C}\mathcal{D}}(M, N)$ to denote the category of $\mathcal{C}$-$\mathcal{D}$-bimodule functors.

Given a $\mathcal{C}$-$\mathcal{D}$-bimodule $M$, the opposite category $M^{op}$ is automatically a $\mathcal{D}$-$\mathcal{C}$-bimodule with the induced action $b \circ x \circ a := a^* \otimes x \circ b^*$ for $a \in \mathcal{C}$, $x \in M^{op}$, $b \in \mathcal{D}$. Note that the functor $a \mapsto a^*$ defines an equivalence of $\mathcal{C}$-$\mathcal{C}$-bimodules $\mathcal{C} \simeq \mathcal{C}^{op}$.

For a left module $M$ over a UMFC $\mathcal{C}$, the internal hom $[x, -] : \mathcal{M} \to \mathcal{C}$ for $x \in \mathcal{M}$ is defined to be the right adjoint functor of $- \otimes x : \mathcal{C} \to \mathcal{M}$, i.e. $\text{Hom}_\mathcal{M}(a \otimes x, y) \simeq \text{Hom}_\mathcal{C}(a, [x, y]_\mathcal{C})$ for $a \in \mathcal{C}$, $y \in \mathcal{M}$. Since unitary categories are semisimple, such an internal hom always exists.

Let $\mathcal{C}$ be a UMFC and let $M$ be a right $\mathcal{C}$-module, $N$ a left $\mathcal{C}$-module. The tensor product $M \boxtimes_e N$ is the universal unitary category that is equipped with a functor $\boxtimes_e : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{M} \boxtimes_e \mathcal{N}$ intertwining the $\mathcal{C}$-actions (see, for example, $[\text{Ta}, \text{ENO}^3, \text{DSS2}]$). In the special case $\mathcal{C} = \mathcal{H}$, $\boxtimes_e \mathcal{N}$ is just the Deligne tensor product $\boxtimes_e \mathcal{N}$.

It turns out that the tensor product $M \boxtimes_e N$ always exists and is equivalent to $\text{Fun}_\mathcal{C}(M^{op}, N)$ $[\text{ENO}^3]$. More explicitly, there is a canonical equivalence for left $\mathcal{C}$-modules $M, N$ $[\text{KZ}], \text{Corollary 2.2.5(1)}$:

$$M^{op} \boxtimes_e N \simeq \text{Fun}_\mathcal{C}(M, N) \text{ defined by } x \boxtimes_e y \mapsto [-, x]_\mathcal{C} \circ y. \quad (2.1)$$

Moreover, there is a canonical equivalence for a right $\mathcal{C}$-module $M$ and a left $\mathcal{C}$-module $N$ $[\text{KZ}], \text{Corollary 2.2.5(3)}$:

$$(M \boxtimes_e N)^{op} \simeq N^{op} \boxtimes_e M^{op} \text{ defined by } x \boxtimes_e y \mapsto y \boxtimes_e x. \quad (2.2)$$
Let \( \mathcal{C}, \mathcal{D} \) be UMFC’s and let \( M \) be a \( \mathcal{C}, \mathcal{D} \)-bimodule, \( N \) a \( \mathcal{D}, \mathcal{C} \)-bimodule. We say that \( M \) is left dual to \( N \) and \( N \) is right dual to \( M \), if there exist a \( \mathcal{C}, \mathcal{D} \)-bimodule functor \( u : \mathcal{D} \to N \boxtimes_\mathcal{C} M \) and a \( \mathcal{D}, \mathcal{C} \)-bimodule functor \( v : M \boxtimes_\mathcal{D} N \to \mathcal{C} \) such that the composite bimodule functors

\[
M \simeq M \boxtimes_\mathcal{D} \mathcal{D} \xrightarrow{\Id_M \boxtimes_\mathcal{D} v} M \boxtimes_\mathcal{D} N \boxtimes_\mathcal{C} M \xrightarrow{\varepsilon_\mathcal{C} \Id_M} \mathcal{C} \boxtimes_\mathcal{C} M \simeq M,
\]

\[
N \simeq \mathcal{D} \boxtimes_\mathcal{D} N \xrightarrow{\varepsilon_\mathcal{D} \Id_N} N \boxtimes_\mathcal{C} M \boxtimes_\mathcal{D} N \xrightarrow{\Id_N \varepsilon_\mathcal{C}} N \boxtimes_\mathcal{C} \mathcal{C} \simeq N
\]

are isomorphic to the identity functors. In the special case where \( u, v \) are equivalences, we say that the bimodules \( M, N \) are invertible and say that the UMFC’s \( \mathcal{C}, \mathcal{D} \) are Morita equivalent.

It was shown in \([\text{DSS1}]\) that the left dual and the right dual of a \( \mathcal{C}, \mathcal{D} \)-bimodule \( M \) are given by \( \text{Fun}_{\text{rev}}(\mathcal{M}, \mathcal{D}) \) and \( \text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{C}) \), respectively. Given below is another form of this result which might be useful for computation (see \([\text{AKZ}, \text{Theorem 4.6}]\)).

**Theorem 2.1.** Let \( \mathcal{C}, \mathcal{D} \) be UMFC’s and let \( M \) be a \( \mathcal{C}, \mathcal{D} \)-bimodule. Then the \( \mathcal{D}, \mathcal{C} \)-bimodule \( \mathcal{M}^{\text{op}} \) is right dual to \( M \) with two duality maps \( u \) and \( v \) defined as follows:

\[
u : D \to \text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M}) \simeq \mathcal{M}^{\text{op}} \boxtimes_{\mathcal{C}} \mathcal{M}, \quad d \mapsto \cdot \circ d,
\]

\[
v : M \boxtimes_{\mathcal{D}} \mathcal{M}^{\text{op}} \to \mathcal{C}, \quad x \boxtimes \mathcal{D} y \mapsto [x, y]_\mathcal{C}^\epsilon.
\]

Since \( (\mathcal{M}^{\text{op}})^{\text{op}} \simeq \mathcal{M} \) as \( \mathcal{C}, \mathcal{D} \)-bimodules, \( \mathcal{M}^{\text{op}} \) is also left dual to \( M \).

A unitary braided fusion category is a unitary fusion category \( \mathcal{C} \) equipped with a braiding \( c_{a,b} : a \otimes b \to b \otimes a \) such that the isomorphisms \( c_{a,b} \) are unitary. We use \( \tilde{\mathcal{C}} \) to denote the same unitary fusion category \( \mathcal{C} \) but equipped with the anti-braiding \( \tilde{c}_{a,b} := c_{b,a}^{-1} \).

The centralizer of a subcategory \( \mathcal{E} \) in a unitary braided fusion category \( \mathcal{C} \), denoted by \( \mathcal{C}' \), is the full subcategory of \( \mathcal{C} \) consisting of those objects \( x \) such that \( c_y x \circ c_{x,y} = \Id_x \otimes y \) for all \( y \in \mathcal{E} \). The centralizer \( \mathcal{C}' \) of \( \mathcal{C} \) itself is referred to as the M"uger center of \( \mathcal{C} \). A unitary braided fusion category has a canonical spherical structure \([\text{Kop}]\).

A unitary modular tensor category (UMTC) is a unitary braided fusion category, equipped with the canonical spherical structure, such that \( \mathcal{C}' \simeq \mathcal{H} \).

If \( \mathcal{C} \) is an IUMFC, the Drinfeld center \( Z(\mathcal{C}) \) is a UMTC \([\text{Mu2}]\). Note that the UMTC \( Z(\mathcal{C}^{\text{rev}}) \) is identical to \( Z(\mathcal{C}) \). If \( \mathcal{C} \) is a UMTC, then the canonical embeddings \( \mathcal{C}, \tilde{\mathcal{C}} \hookrightarrow Z(\mathcal{C}) \) induce a braided monoidal equivalence \( \mathcal{C} \boxtimes \tilde{\mathcal{C}} \simeq Z(\mathcal{C}) \) \([\text{Mu2}]\). If \( \mathcal{C} \hookrightarrow \mathcal{C}' \) is a fully faithful braided monoidal functor between UMTC’s, the centralizer \( \mathcal{C}' \) of \( \mathcal{C} \) in \( \mathcal{C} \) is also a UMTC and we have \( \mathcal{C} \simeq \mathcal{C} \boxtimes \mathcal{C}' \) \([\text{Mu3}, \text{Theorem 4.2}]\) (see also \([\text{DGNO}]\)).

Let \( \mathcal{C}, \mathcal{D} \) be UMTC’s. A multi-fusion \( \mathcal{C}, \mathcal{D} \)-bimodule is a UMFC \( \mathcal{M} \) equipped with a braided monoidal functor \( \phi_\mathcal{M} : \tilde{\mathcal{C}} \boxtimes \mathcal{D} \to Z(\mathcal{M}) \). (The standard terminology should be a UMFC over \( \tilde{\mathcal{C}} \boxtimes \mathcal{D} \), however, this nonstandard one is sometimes more convenient.) A multi-fusion \( \mathcal{C}, \mathcal{D} \)-bimodule \( \mathcal{M} \) is said to be closed if \( \phi_\mathcal{M} \) is an equivalence. We say that two multi-fusion \( \mathcal{C}, \mathcal{D} \)-bimodules \( \mathcal{M} \) and \( \mathcal{N} \) are equivalent if there is a monoidal equivalence \( \mathcal{M} \simeq \mathcal{N} \) such that the composition of \( \phi_\mathcal{M} \) with the induced braided monoidal equivalence \( Z(\mathcal{M}) \simeq Z(\mathcal{N}) \) is isomorphic to \( \phi_\mathcal{N} \).

Let \( \mathcal{C}, \mathcal{D}, \mathcal{E} \) be UMTC’s, and let \( M \) be a multi-fusion \( \mathcal{C}, \mathcal{D} \)-bimodule, \( N \) a multi-fusion \( \mathcal{D}, \mathcal{E} \)-bimodule. The category \( \mathcal{M} \boxtimes_\mathcal{D} \mathcal{N} \) has a natural structure of a UMFC with the tensor product \((a \boxtimes_\mathcal{D} b) \otimes (c \boxtimes_\mathcal{D} d) := (a \otimes c) \boxtimes_\mathcal{D} (b \otimes d) \). Moreover, \((\mathcal{M} \boxtimes_\mathcal{D} \mathcal{N})^{\text{rev}} \) can be identified with \( \mathcal{N}^{\text{rev}} \boxtimes_\mathcal{D} \mathcal{M}^{\text{rev}} \).
Theorem 2.2 ([KZI] Theorem 3.3.6). Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be UMTC's. If $\mathcal{M}$ is a closed multi-fusion $\mathcal{C}\mathcal{D}$-bimodule and $N$ is a closed multi-fusion $\mathcal{D}\mathcal{E}$-bimodule, then $\mathcal{M} \boxtimes \mathcal{D}$ $N$ is a closed multi-fusion $\mathcal{C}\mathcal{E}$-bimodule.

We recall the main result in [KZI] that can be generalized to the unitary case automatically. Let $\text{IUMFC}$ be the category of IUMFC's with morphisms given by the equivalence classes of nonzero bimodules. Let $\text{UMTC}$ be the category of UMTC's with morphisms given by the equivalence classes of closed multi-fusion bimodules. The composition law of both categories is tensor product of bimodules.

Theorem 2.3 ([KZI] Theorem 3.3.7). There is a well-defined functor $Z: \text{IUMFC} \to \text{UMTC}$ given by $\mathcal{C} \mapsto Z(\mathcal{C})$ on object and $cM_D \mapsto \text{Fun}_{\mathcal{C}|\mathcal{D}}(M, M)$ on morphism. Moreover, the functor $Z$ is fully faithful.

Remark 2.4. The fully-faithfulness of $Z$ has essentially been given in [ENO2, ENO3, DMNO].

More explicitly, Theorem 2.3 implies the following result. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be IUMFC's. Let $\mathcal{M}$ be a nonzero $\mathcal{C}\mathcal{D}$-bimodule and $N$ a nonzero $\mathcal{D}\mathcal{E}$-bimodule. The assignment $f \boxtimes Z(\mathcal{D}) g \mapsto f \boxtimes_D g$ defines an equivalence between two closed multi-fusion $Z(\mathcal{C})-Z(\mathcal{E})$-bimodules:

$$\text{Fun}_{\mathcal{C}|\mathcal{D}}(M, M) \boxtimes_{Z(\mathcal{D})} \text{Fun}_{\mathcal{D}|\mathcal{E}}(N, N) \simeq \text{Fun}_{\mathcal{C}|\mathcal{E}}(M \boxtimes_{\mathcal{D}} N, M \boxtimes_{\mathcal{D}} N).$$  (2.3)

Corollary 2.5. Let $\mathcal{C}, \mathcal{D}$ be IUMFC's. Given a braided monoidal equivalence $Z(\mathcal{C}) \simeq Z(\mathcal{D})$, there is a unique invertible $\mathcal{C}\mathcal{D}$-bimodule $\mathcal{M}$ up to equivalence such that $\text{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M}, \mathcal{M}) \simeq Z(\mathcal{C})$ as multi-fusion $Z(\mathcal{C})-Z(\mathcal{D})$-bimodules. Moreover, $\mathcal{M}$ is the unique $\mathcal{C}\mathcal{D}$-bimodule up to equivalence such that the canonical monoidal functor $\mathcal{C} \boxtimes_{Z(\mathcal{C})} Z(\mathcal{E}) \to \text{Fun}(\mathcal{M}, \mathcal{M})$, $c \boxtimes_{Z(\mathcal{C})} d \mapsto c \odot - \odot d$ is an equivalence.

Proof. The first claim is an immediate consequence of the fully-faithfulness of $Z$. Moreover, applying (2.3) we have

$$\mathcal{C} \boxtimes_{Z(\mathcal{C})} \mathcal{D} \simeq \text{Fun}_{\mathcal{H}|\mathcal{C}}(\mathcal{C}, \mathcal{C}) \boxtimes_{Z(\mathcal{C})} \text{Fun}_{\mathcal{C}|\mathcal{D}}(M, M) \boxtimes_{Z(\mathcal{D})} \text{Fun}_{\mathcal{D}|\mathcal{H}}(\mathcal{D}, \mathcal{D})$$

$$\simeq \text{Fun}_{\mathcal{H}|\mathcal{C}}(\mathcal{C} \boxtimes_{\mathcal{D}} M \boxtimes_{\mathcal{D}} \mathcal{D}, \mathcal{E} \boxtimes_{\mathcal{C}} M \boxtimes_{\mathcal{D}} \mathcal{D})$$

$$\simeq \text{Fun}(\mathcal{M}, \mathcal{M}).$$

If $N$ is another $\mathcal{C}\mathcal{D}$-bimodule such that $\mathcal{C} \boxtimes_{Z(\mathcal{C})} \mathcal{D} \simeq \text{Fun}(N, N)$, then the monoidal equivalence $\text{Fun}(\mathcal{M}, \mathcal{M}) \simeq \text{Fun}(N, N)$ induces an equivalence $\mathcal{M} \simeq N$ which is automatically a bimodule equivalence. \hfill $\Box$

Corollary 2.6. Let $\mathcal{C}, \mathcal{D}$ be UMTC's and let $\mathcal{M}$ be a closed multi-fusion $\mathcal{C}\mathcal{D}$-bimodule. We have the following assertions:

1. The canonical monoidal functor $\mathcal{M} \boxtimes_{\mathcal{C}\mathcal{D}} \mathcal{M}^{\text{rev}} \to \text{Fun}(\mathcal{M}, \mathcal{M})$, $x \boxtimes_{\mathcal{C}\mathcal{D}} y \mapsto x \odot - \odot y$ is an equivalence.

2. The $\mathcal{M} \boxtimes_{\mathcal{C}\mathcal{D}} \mathcal{M}^{\text{rev}}$-bimodule $\mathcal{M}$ is invertible and $\text{Fun}_{M\boxtimes_{\mathcal{D}}}^{\text{rev}}(\mathcal{C}, \mathcal{M}) \simeq \mathcal{C} \boxtimes_{\mathcal{D}} \mathcal{C}$ as multi-fusion $Z(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}^{\text{rev}})-Z(\mathcal{C})$-bimodules.

3. The $\mathcal{D}^{\text{rev}} \boxtimes_{\mathcal{C}} \mathcal{M}$ bimodule $\mathcal{M}$ is invertible and $\text{Fun}_{\mathcal{D}|\mathcal{C}}^{\text{rev}}(\mathcal{C}, \mathcal{M}) \simeq \mathcal{D} \boxtimes_{\mathcal{C}} \mathcal{D}$ as multi-fusion $Z(\mathcal{D})-Z(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}^{\text{rev}})$-bimodules.

Proof. (1) is an easy consequence of Corollary 2.3. Note that $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}^{\text{rev}}$ is a closed multi-fusion $\mathcal{C}\mathcal{D}$-bimodule by Theorem 2.2. Moreover, $(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}^{\text{rev}}) \boxtimes_{\mathcal{C}} \mathcal{C} \simeq \mathcal{M} \boxtimes_{\mathcal{C}\mathcal{D}} (\mathcal{M}^{\text{rev}} \boxtimes_{\mathcal{C}} \mathcal{C}) \simeq \mathcal{M} \boxtimes_{\mathcal{C}\mathcal{D}} (\mathcal{C} \boxtimes_{\mathcal{D}} \mathcal{M}^{\text{rev}}) \simeq \mathcal{M} \boxtimes_{\mathcal{C}\mathcal{D}} \mathcal{M}^{\text{rev}} \simeq \text{Fun}(\mathcal{M}, \mathcal{M})$. Applying Corollary 2.5 again, we obtain (2). (3) is proved similarly. \hfill $\Box$
Corollary 2.7. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be UMTC’s and let $\mathcal{M}, \mathcal{N}$ be closed multi-fusion $\mathcal{C}$-$\mathcal{D}$-bimodules, $\mathcal{M}', \mathcal{N}'$ be closed multi-fusion $\mathcal{D}$-$\mathcal{E}$-bimodules. Suppose $\mathcal{F}$ is an invertible $\mathcal{M}$-$\mathcal{N}$-bimodule such that $\text{Fun}_{\mathcal{M}|\mathcal{N}}(\mathcal{F}, \mathcal{F}) \simeq \mathcal{C} \boxtimes \mathcal{D}$ as multi-fusion $Z(\mathcal{M})$-$Z(\mathcal{N})$-bimodules, and $\mathcal{G}$ is an invertible $\mathcal{M}'$-$\mathcal{N}'$-bimodule such that $\text{Fun}_{\mathcal{M}'|\mathcal{N}'}(\mathcal{G}, \mathcal{G}) \simeq \mathcal{D} \boxtimes \mathcal{E}$ as multi-fusion $Z(\mathcal{M}')$-$Z(\mathcal{N}')$-bimodules. We have the following assertions:

1. The canonical monoidal functor $\mathcal{M} \boxtimes \mathcal{E} \mathcal{N}^{\text{rev}} \to \text{Fun}_{\mathcal{D}^{\text{rev}}}(\mathcal{F}, \mathcal{F})$, $x \boxtimes y \mapsto x \odot - y$ is an equivalence.
2. The canonical monoidal functor $\mathcal{M}' \boxtimes \mathcal{E} \mathcal{N}^{\text{rev}} \to \text{Fun}_{\mathcal{D}^{\text{rev}}}(\mathcal{G}, \mathcal{G})$, $x \boxtimes y \mapsto x \odot - y$ is an equivalence.
3. The $\mathcal{M} \boxtimes \mathcal{M}'$-$\mathcal{N} \boxtimes \mathcal{N}'$-bimodule $\mathcal{F} \boxtimes \mathcal{D}$ is invertible and $\text{Fun}_{\mathcal{M} \boxtimes \mathcal{M}'|\mathcal{N} \boxtimes \mathcal{N}'}(\mathcal{F} \boxtimes \mathcal{D}, \mathcal{F} \boxtimes \mathcal{D}, \mathcal{G}) \simeq \mathcal{C} \boxtimes \mathcal{E}$ as multi-fusion $Z(\mathcal{M} \boxtimes \mathcal{M}')$-$Z(\mathcal{N} \boxtimes \mathcal{N}')$-bimodules.

Proof. Using a similar augment as the proof of Corollary 2.6 we deduce that the $\mathcal{D}^{\text{rev}}$-$(\mathcal{M} \boxtimes \mathcal{E} \mathcal{N}^{\text{rev}})^{\text{rev}}$-bimodule $\mathcal{F}$ and the $\mathcal{D}$-$(\mathcal{M}' \boxtimes \mathcal{E} \mathcal{N}^{\text{rev}})^{\text{rev}}$-bimodule $\mathcal{G}$ are invertible. Applying Theorem 2.1 we obtain (1) and (2). Moreover,

\[
\begin{align*}
\text{(1)} & \qquad (\mathcal{M} \boxtimes \mathcal{M}') \boxtimes \mathcal{E} \mathcal{N}^{\text{rev}} \simeq (\mathcal{M} \boxtimes \mathcal{E} \mathcal{N}^{\text{rev}}) \boxtimes \mathcal{D} \boxtimes \mathcal{G} \\
& \qquad \simeq \text{Fun}_{\mathcal{D}^{\text{rev}}}(\mathcal{F}, \mathcal{F}) \boxtimes Z(\mathcal{D}) \text{Fun}_{\mathcal{D}^{\text{rev}}}(\mathcal{G}, \mathcal{G}) \\
& \qquad \simeq \text{Fun}(\mathcal{F} \boxtimes \mathcal{D}, \mathcal{F} \boxtimes \mathcal{D}, \mathcal{G})
\end{align*}
\]

where the last $\simeq$ is due to (2.3). Applying Corollary 2.5 we obtain (3). \hfill $\square$

3. Enriched IUMFC’s and Bimodules

In this section, we introduced the notions of enriched IUMFC’s, bimodules and bimodule functors, etc. For reader’s convenience, we draw several figures to illustrate these concepts.

Definition 3.1. An enriched IUMFC is a pair $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$ where $\mathcal{C}_2$ is a UMTC and $\mathcal{C}_1$ is an IUMFC equipped with a fully faithful braided monoidal functor $\mathcal{C}_2 \hookrightarrow Z(\mathcal{C}_1)$. See Figure 1(a). We use $\mathcal{C}^{\text{rev}}$ to denote the enriched IUMFC $(\mathcal{C}_1^{\text{rev}}, \mathcal{C}_2)$. See Figure 1(b). By abusing notation, we use $\mathcal{H}$ to denote the enriched IUMFC $(\mathcal{H}, \mathcal{H})$.

![Figure 1](image_url)

**Figure 1.** (a) depicts an enriched IUMFC $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$ where $\mathcal{C}_1$ admits a central action by $\mathcal{C}_2$. (b) depicts its reverse $\mathcal{C}^{\text{rev}} = (\mathcal{C}_1^{\text{rev}}, \mathcal{C}_2)$. Keep in mind that the categorical constructions $\mathcal{C} \mapsto \mathcal{C}$, $\mathcal{C} \mapsto \mathcal{C}^{\text{rev}}$, $\mathcal{C} \mapsto \mathcal{C}^{\text{op}}$ correspond to changing the orientations of 2,1,0-cells in figures.
Remark 3.2. It was shown in [KZ2] that an enriched IUMFC $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2)$ is naturally associated with a monoidal category $\mathcal{E}^\sharp$ enriched in the sense of [MP] such that $\mathcal{E}^\sharp$ takes $\mathcal{E}_1$ as underlying category and $\text{Hom}_{\mathcal{E}^\sharp}(x, y) = [x, y]_{\mathcal{E}^\sharp}$. Moreover, the Drinfeld center $Z(\mathcal{E}^\sharp)$ was shown to be equivalent to the centralizer of $\mathcal{E}_2$ in $Z(\mathcal{E}_1)$. In particular, $Z(\mathcal{E}_1) \simeq \mathcal{E}_2 \boxtimes Z(\mathcal{E}^\sharp)$. In the special case $\mathcal{E} = (\mathcal{E}, \bar{\mathcal{E}})$ where $\mathcal{E}$ is a UMTC, note that $Z(\mathcal{E}^\sharp) \simeq \mathcal{E}$.

Definition 3.3. Let $\mathcal{C}$ and $\mathcal{D}$ be enriched IUMFC’s. The Deligne tensor product $\mathcal{C} \boxtimes \mathcal{D}$ is defined to be the enriched IUMFC $(\mathcal{C}_1 \boxtimes \mathcal{D}_1, \mathcal{C}_2 \boxtimes \mathcal{D}_2)$.

Figure 2. (a) depicts a $\mathcal{C}$-$\mathcal{D}$-bimodule $\mathcal{M} = (M_0, M_1)$ where $M_1$ admits actions by $\mathcal{C}_2, \mathcal{D}_2$, and $M_0$ admits actions by $\mathcal{C}_2, \mathcal{M}_1, \mathcal{D}_2, \mathcal{C}_1, \mathcal{D}_1$. (b) depicts the opposite $\mathcal{D}$-$\mathcal{C}$-bimodule $\mathcal{M}^{\text{op}} = (M_0^{\text{op}}, M_1^{\text{rev}})$.

Definition 3.4. Let $\mathcal{C}$ and $\mathcal{D}$ be enriched IUMFC’s. A $\mathcal{C}$-$\mathcal{D}$-bimodule is a pair $\mathcal{M} = (M_0, M_1)$ where $M_1$ is a closed multi-fusion $\mathcal{C}_2$-$\mathcal{D}_2$-bimodule and $M_0$ is a left $\mathcal{C}_1 \boxtimes \mathcal{C}_2 \mathcal{M}_1 \boxtimes \mathcal{D}_2$ module. See Figure 2(a). We use $\mathcal{M}^{\text{op}}$ to denote the $\mathcal{D}$-$\mathcal{C}$-bimodule $(M_0^{\text{op}}, M_1^{\text{rev}})$. See Figure 2(b).

Remark 3.5. A $\mathcal{C}$-$\mathcal{D}$-bimodule is automatically a $\mathcal{D}^{\text{rev}}$-$\mathcal{C}^{\text{rev}}$-bimodule, a $\mathcal{C} \boxtimes \mathcal{D}^{\text{rev}}$-$\mathcal{H}$-bimodule as well as an $\mathcal{H}$-$\mathcal{C}^{\text{rev}} \boxtimes \mathcal{D}$-bimodule.

Figure 3. The tensor product $\mathcal{M} \boxtimes_\mathcal{D} \mathcal{N} = (M_0 \boxtimes_{\mathcal{D}_1} N_0, M_1 \boxtimes_{\mathcal{D}_2} N_1)$ is depicted by either of these two equivalent figures. The right one is obtained from the left one by contracting the region labelled by $\mathcal{D}_2$ and $\mathcal{D}_1$. Such an equivalence under contraction will be implicitly used in the following figures.

Definition 3.6. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be enriched IUMFC’s and let $\mathcal{M}$ be a $\mathcal{C}$-$\mathcal{D}$-bimodule, $\mathcal{N}$ be a $\mathcal{D}$-$\mathcal{E}$-bimodule. The tensor product $\mathcal{M} \boxtimes_\mathcal{D} \mathcal{N}$ is defined to be the $\mathcal{C}$-$\mathcal{E}$-bimodule $(M_0 \boxtimes_{\mathcal{D}_1} N_0, M_1 \boxtimes_{\mathcal{D}_2} N_1)$. See Figure 3. This is well defined because $M_1 \boxtimes_{\mathcal{D}_2} N_1$ is a closed multi-fusion $\mathcal{C}_2$-$\mathcal{E}_2$-bimodule by Theorem 2.2.
Remark 3.8. See Figure 4.

Remark 3.9. A bimodule functor $\tilde{\mathfrak{F}}: \mathfrak{M} \to \mathfrak{N}$ is a pair $\tilde{\mathfrak{F}} = (F, \mathcal{F})$ where $\mathcal{F}$ is an invertible $M_1$-$N_1$-bimodule such that $\text{Fun}_{M_1|N_1}(\mathfrak{M}, \mathfrak{N}) \simeq \tilde{\mathfrak{F}}_0 \boxtimes \tilde{\mathfrak{F}}_2$ as multi-fusion $Z(\mathfrak{M})$-$Z(\mathfrak{N})$-bimodules and $F$ is a left $\tilde{\mathfrak{C}}_1 \boxtimes \tilde{\mathfrak{C}}_2 M_1 \boxtimes \tilde{\mathfrak{C}}_2 D_1^{\text{rev}}$-module functor $\mathcal{F}: \mathfrak{M}_0 \to \mathcal{F} \boxtimes \mathcal{N}_1 \mathfrak{N}_0$. See Figure 4.

Remark 3.10. According to Theorem 2.3 an invertible $M_1$-$N_1$-bimodule $\mathcal{F}$ in the definition always exists and is unique up to equivalence.

Definition 3.11. Let $\mathfrak{M}, \mathfrak{N}, \mathfrak{L}$ be $\mathfrak{C}$-$\mathfrak{D}$-bimodules and let $\tilde{\mathfrak{F}}: \mathfrak{M} \to \mathfrak{N}$ and $\mathfrak{G} : \mathfrak{N} \to \mathfrak{L}$ be bimodule functors. The composition $\mathfrak{G} \circ \tilde{\mathfrak{F}} : \mathfrak{M} \to \mathfrak{L}$ is defined to be the bimodule functor given by the composite left module functor

$$
\mathfrak{M}_0 \xrightarrow{F} \mathfrak{F} \boxtimes \mathfrak{N}_1 \mathfrak{N}_0 \xrightarrow{\mathfrak{G}} (\mathfrak{F} \boxtimes \mathfrak{G}) \boxtimes \mathfrak{L}_1 \mathfrak{L}_0.
$$
This is well defined due to Theorem 2.3, i.e. $\mathfrak{F} \boxtimes \mathfrak{G}$ satisfies the condition required by a bimodule functor.

Definition 3.12. Let $\mathfrak{M}, \mathfrak{N}$ be $\mathfrak{C}$-$\mathfrak{D}$-bimodules, $\mathfrak{M}', \mathfrak{N}'$ be $\mathfrak{D}$-$\mathfrak{E}$-bimodules, and let $\tilde{\mathfrak{F}}, \mathfrak{G} : \mathfrak{M} \to \mathfrak{N}$ be bimodule functors. The tensor product $\tilde{\mathfrak{F}} \boxtimes \mathfrak{D} \mathfrak{G} : \mathfrak{M} \boxtimes \mathfrak{D} \mathfrak{M}' \to \mathfrak{N} \boxtimes \mathfrak{E} \mathfrak{N}'$ is defined to be the bimodule functor $(\mathfrak{F} \boxtimes \mathfrak{D}, \mathfrak{F} \boxtimes \mathfrak{E})$. This is well defined in view of Corollary 2.13.

Definition 3.13. Let $\mathfrak{M}, \mathfrak{N}$ be $\mathfrak{C}$-$\mathfrak{D}$-bimodules and let $\tilde{\mathfrak{G}}, \mathfrak{G} : \mathfrak{M} \to \mathfrak{N}$ be bimodule functors. A natural transformation $\xi : \tilde{\mathfrak{G}} \to \mathfrak{G}$ consists of an $M_1$-$N_1$-bimodule equivalence $\xi' : \mathfrak{F} \to \mathfrak{G}$ and a natural transformation of left module functors $\xi : \mathfrak{C}_1 \boxtimes \mathfrak{C}_2 M_1 \boxtimes \mathfrak{C}_2 D_1 \mathfrak{F} \to \mathcal{G}$ such that $\xi = \xi' \circ ((\mathfrak{F} \boxtimes \mathfrak{G}) \circ \mathfrak{G})$.

Proposition 3.14. Let $\xi : \tilde{\mathfrak{G}} \to \mathfrak{G}$ be a natural transformation between two bimodule functors $\tilde{\mathfrak{G}}, \mathfrak{G} : \mathfrak{M} \to \mathfrak{N}$. The following conditions are equivalent:

(1) $\xi$ is an equivalence, i.e. there is a natural transformation $\zeta : \mathfrak{G} \to \tilde{\mathfrak{G}}$ such that $\xi \circ \zeta \simeq \mathfrak{G}$ and $\zeta \circ \xi \simeq \tilde{\mathfrak{G}}$.

(2) The natural transformation of left module functors $\xi : (\mathfrak{C}_1 \boxtimes \mathfrak{C}_2 M_1 \boxtimes \mathfrak{C}_2 D_1 \mathfrak{F} \simeq \mathfrak{G})$ is an isomorphism.
Proposition 3.14. Let $\mathcal{F} : \mathcal{M} \rightarrow \mathcal{N}$ be a bimodule functor between $\mathcal{C}$-$\mathcal{D}$-bimodules. The following conditions are equivalent:

1. $\mathcal{F}$ is a bimodule equivalence, i.e., there is a bimodule functor $\mathcal{G} : \mathcal{N} \rightarrow \mathcal{M}$ such that $\mathcal{G} \circ \mathcal{F} \simeq \text{Id}_{\mathcal{M}}$ and $\mathcal{F} \circ \mathcal{G} \simeq \text{Id}_{\mathcal{N}}$.
2. The left module functor $F : M_0 \rightarrow \mathcal{F} \otimes_{\mathcal{N}} N_0$ is an equivalence.

Proof. (1) $\Rightarrow$ (2). The left module functor $\mathcal{F} \otimes_{\mathcal{N}} N_0 \rightarrow \mathcal{D} \otimes_{\mathcal{D}} \mathcal{M}_0$ is invertible.

(2) $\Rightarrow$ (1). The desired bimodule functor $\mathcal{G} : \mathcal{N} \rightarrow \mathcal{M}$ is given by the left module functor $N_0 \simeq \mathcal{D}op \otimes_{\mathcal{D}} \mathcal{M}_1, \mathcal{F} \otimes_{\mathcal{N}} N_0 \rightarrow \mathcal{D}op \otimes_{\mathcal{D}} \mathcal{M}_1$.

Example 3.15. Let $\mathcal{E}$ be a unitary category. We have an $\mathcal{H}$-$\mathcal{H}$-bimodule equivalence $(\mathcal{E}, \Fun(\mathcal{E}, \mathcal{E})) \simeq \mathcal{H}$ given by the left $\Fun(\mathcal{E}, \mathcal{E})$-module equivalence $\mathcal{E} \simeq \mathcal{E} \otimes_{\mathcal{H}} \mathcal{H}$.

![Diagram](image.png)

**Figure 5.** A $\mathcal{C}$-$\mathcal{D}$-bimodule $\mathcal{M} = (\mathcal{E}_1 \otimes_{\mathcal{E}_2} \mathcal{M}_1, \mathcal{M}_1)$ is depicted on the left where $\mathcal{C} = (\mathcal{E}_1, \mathcal{E}_2)$ and $\mathcal{D} = (\mathcal{D}_1 \otimes_{\mathcal{D}_2} \mathcal{M}_1, \mathcal{D}_2)$. The $\mathcal{D}$-$\mathcal{C}$-bimodule $\mathcal{M}op \simeq (\mathcal{M}_1op \otimes_{\mathcal{E}_2} \mathcal{E}_1op, \mathcal{M}_1rev)$ is depicted on the right.

Proposition 3.16. Let $\mathcal{M}$ be a $\mathcal{C}$-$\mathcal{D}$-bimodule. Suppose $M_0 = D_1 = \mathcal{E}_1 \otimes_{\mathcal{E}_2} \mathcal{M}_1$ (see Figure 5). Then the $\mathcal{D}$-$\mathcal{C}$-bimodule $\mathcal{M}op \simeq \mathcal{E}$ as $\mathcal{C}$-$\mathcal{D}$-bimodules and $\mathcal{M}op \otimes_{\mathcal{E}} \mathcal{M} \simeq \mathcal{D}$ as $\mathcal{D}$-$\mathcal{D}$-bimodules.

Proof. We have $\mathcal{M} \otimes_{\mathcal{D}} \mathcal{M}op = (\mathcal{M}_0 \otimes_{\mathcal{D}} \mathcal{M}_0op, \mathcal{M}_1 \otimes_{\mathcal{D}_2} \mathcal{M}_1rev) \simeq (\mathcal{D}_1op, \mathcal{M}_1 \otimes_{\mathcal{D}_2} \mathcal{M}_1rev)$. According to Corollary 2.6 and Proposition 3.14, the invertible $\mathcal{M}_1 \otimes_{\mathcal{D}_2} \mathcal{M}_1rev$-bimodule $\mathcal{M}_1$ and the left $\mathcal{D}_1 \otimes_{\mathcal{D}_2} \mathcal{D}_1rev$-module equivalence $\alpha : \mathcal{D}_1op = (\mathcal{E}_1 \otimes_{\mathcal{E}_2} \mathcal{M}_1)op \simeq \mathcal{M}_1op \otimes_{\mathcal{E}_2} \mathcal{E}_1op \simeq \mathcal{M}_1 \otimes_{\mathcal{E}_2} \mathcal{E}_1$ defines a $\mathcal{C}$-$\mathcal{C}$-bimodule equivalence $\mathcal{M} \otimes \mathcal{D}op \simeq \mathcal{E}$.

We have $\mathcal{M}op \otimes_{\mathcal{E}} \mathcal{M} = (\mathcal{M}_0op \otimes_{\mathcal{E}} \mathcal{M}_0, \mathcal{M}_1 \otimes_{\mathcal{E}_2} \mathcal{E}_1) \simeq (\mathcal{M}_1 \otimes_{\mathcal{E}_2} \mathcal{E}_1op, \mathcal{M}_1 \otimes_{\mathcal{E}_2} \mathcal{E}_1).$ According to Corollary 2.6(3) and Proposition 3.14, the invertible $\mathcal{D}_2 \otimes_{\mathcal{E}_2} \mathcal{M}_1$-bimodule $\mathcal{M}_1$ and the left $\mathcal{D}_1 \otimes_{\mathcal{D}_2} \mathcal{D}_1rev$-module equivalence $\beta : \mathcal{D}_1 \simeq \mathcal{D}_1op \simeq \mathcal{M}_1 \otimes_{\mathcal{E}_2} \mathcal{E}_1op \simeq \mathcal{M}_1 \otimes_{\mathcal{E}_2} \mathcal{E}_1op \simeq \mathcal{M}_1 \otimes_{\mathcal{E}_2} \mathcal{E}_1$ defines a $\mathcal{D}$-$\mathcal{D}$-bimodule equivalence $\mathcal{D} \simeq \mathcal{M}op \otimes_{\mathcal{E}} \mathcal{M}$. See the bottom figure in Figure 6.

Example 3.17. (1) If $\mathcal{E}$ is an IUMFC, $(\mathcal{E}, \mathcal{E})$ is an invertible $\mathcal{H}$-$(\mathcal{E}, Z(\mathcal{E}))$-bimodule.
(2) If $\mathcal{E}$ is a UMTC, $(\mathcal{E}, \mathcal{E})$ is an invertible $(\mathcal{E} \otimes \mathcal{E}, \mathcal{E} \otimes \mathcal{E})$-(\mathcal{E}, \mathcal{H})$-bimodule.
Figure 6. These figures depict two bimodule equivalences $(\alpha, M_1) : M \boxtimes D \to C$ and $(\beta, M_1) : D \to M^{\text{op}} \boxtimes C$.

4. A symmetric monoidal $(4,3)$-category

We have a symmetric monoidal $(4,3)$-category $\text{IUMFC}^{en}$ constructed as follows. An object is an enriched $\text{IUMFC} C$. A morphism $C \leftarrow D$ is a $C$-$D$-bimodule. A 2-morphism is a bimodule functor. A 3-morphism is a natural transformation of bimodule functors. A 4-isomorphism is a modification. The tensor unit is $H$ and the tensor product is Deligne tensor product $\boxtimes$.

**Theorem 4.1.** The symmetric monoidal $(4,3)$-category $\text{IUMFC}^{en}$ has duals.

**Proof.** We need to show that every object, morphism, 2-morphism has left dual and right dual.

1. The dual of an object $C$ is $C^{\text{rev}}$. The unit map $C^{\text{rev}} \boxtimes C \leftarrow H$ and the counit map $H \leftarrow C \boxtimes C^{\text{rev}}$ are given by the trivial $C$-$C$-bimodule $C$ (see Remark 3.5).

2. Let $\mathfrak{M} : C \leftarrow D$ be a morphism. We claim that $\mathfrak{M}^{\text{op}} : D \leftarrow C$ is right dual to $\mathfrak{M}$. Indeed, in the special case where $C_2 = M_1 = D_2$, it suffices to take $(u, D_2) : D \to M^{\text{op}} \boxtimes \mathfrak{C}$ as unit map and take $(v, D_2) : \mathfrak{M} \boxtimes D^{\text{op}} \to C$ as counit map to exhibit the duality (see Figure 7), where $u : D_1 \to M_0^{\text{op}} \boxtimes C_1, M_0$ and $v : C_0 \boxtimes D_1, M_0^{op} \to C_1$ are the ordinary bimodule functors given in Theorem 2.1. For the general case, we have a commutative diagram:

Moreover, $\mathfrak{M}^{\text{op}}$ is inverse to $\mathfrak{M}$ by Proposition 3.16. Replacing $\mathfrak{C}$ and $\mathfrak{M}$ by $\mathfrak{C}'$ and $\mathfrak{M}'$, respectively, we reduce the problem to the special case. Since $(\mathfrak{M}^{\text{op}})^{\text{op}} = \mathfrak{M}$, $\mathfrak{M}^{\text{op}}$ is also left dual to $\mathfrak{M}$.
is the 2-morphism which by definition is a left module functor $F$.

**Corollary 4.2.** A morphism $\mathcal{F} : \mathcal{C} \leftrightarrow \mathcal{D}$ is the right adjoint functor to $F$. The unit map $\text{Id} \circ \mathcal{F}$ and the counit map $\mathcal{F} \circ \text{Id}$ are induced by the unit map $\text{Id} \to F^* \circ F$ and the counit map $F \circ F^* \to \text{Id}$, respectively. Similarly, $\mathcal{F}$ has a left dual. \qed

**Corollary 4.2.** A morphism $\mathcal{M} : \mathcal{E} \leftrightarrow \mathcal{D}$ is an equivalence if and only if the $\mathcal{C}$-$\mathcal{C}$-$\mathcal{M}$-$\mathcal{D}$-bimodule $M_0$ is invertible. In this case, $\mathcal{M}^{op}$ is inverse to $\mathcal{M}$.

**Proof.** This is clear from Part (2) of the proof of Theorem 4.1. \qed

The following corollary generalizes a similar result for unitary fusion categories [Mu1, ENO2].

**Corollary 4.3.** Two enriched IUMFC’s $\mathcal{C}, \mathcal{D}$ are equivalent in IUMFC$^{un}$ if and only if $Z(\mathcal{C}^f) \simeq Z(\mathcal{D}^f)$ as UMTC’s, where $Z(\mathcal{C}^f)$, $Z(\mathcal{D}^f)$ are the centralizers of $\mathcal{E}_2, \mathcal{D}_2$ in $Z(\mathcal{E}_1), Z(\mathcal{D}_1)$, respectively (see Remark 3.2).

**Proof.** Suppose there is an equivalence $\mathcal{M} : \mathcal{C} \leftrightarrow \mathcal{D}$. Note that $\mathcal{C}_1$ is a closed multi-fusion $Z(\mathcal{C}^f)$-$\mathcal{E}_2$-bimodule, thus $Z(\mathcal{C}_1 \boxtimes \mathcal{E}_2 \mathcal{M}_1) \simeq Z(\mathcal{C}^f) \boxtimes \mathcal{D}_2$. The invertible $\mathcal{C}_1 \boxtimes \mathcal{E}_2 \mathcal{M}_1$-$\mathcal{D}_1$-bimodule $M_0$ induces an equivalence $Z(\mathcal{C}^f) \boxtimes \mathcal{D}_2 \simeq Z(\mathcal{D}_1)$ which preserves $\mathcal{D}_2$. Thus $Z(\mathcal{C}^f) \simeq Z(\mathcal{D}^f)$.

Conversely, suppose $Z(\mathcal{C}^f) \simeq Z(\mathcal{D}^f)$. Let $\mathcal{M}_1 = \mathcal{C}_1^{rev} \boxtimes_{Z(\mathcal{C}^f)} \mathcal{D}_1$. Note that $\mathcal{M}_1$ is a closed multi-fusion $\mathcal{E}_2$-$\mathcal{D}_2$-bimodule and $Z(\mathcal{C}_1 \boxtimes \mathcal{E}_2 \mathcal{M}_1) \simeq Z(\mathcal{C}^f) \boxtimes \mathcal{D}_2 \simeq Z(\mathcal{D}_1)$. Therefore, there exists an invertible $\mathcal{C}_1 \boxtimes \mathcal{E}_2 \mathcal{M}_1$-$\mathcal{D}_1$-bimodule $M_0$ such that the pair $(M_0, \mathcal{M}_1)$ defines an equivalence $\mathcal{C} \simeq \mathcal{D}$. \qed
Remark 4.4. Theorem [4.1] and its corollaries can be formulated and proved in terms of 1-category theory without referring to the higher category \textit{IUMFC}$_{en}$.

According to the cobordism hypothesis [BD, Lu], every object \( \mathcal{C} \) of \textit{IUMFC}$_{en}$ gives rise to an extended framed 3D TQFT \( Z_\mathcal{C} \). We would like to examine its values on closed manifolds.

The value \( Z_\mathcal{C}(S^1) \) on a circle is the \( \text{H-H} \)-bimodule
\[
\mathcal{C}^{\text{op}} \boxtimes_{\mathcal{C}} \mathcal{C}^{\text{rev}} \mathcal{C} = (\mathcal{C}_1^{\text{op}}, \mathcal{C}_2^{\text{rev}}) \boxtimes_{(\mathcal{C}_1, \mathcal{C}_2)} (\mathcal{C}_1, \mathcal{C}_2)
\]
\[
= (\mathcal{C}_1^{\text{op}} \boxtimes_{\mathcal{C}} \mathcal{C}_1^{\text{rev}}, \mathcal{C}_2^{\text{rev}}) \boxtimes_{(\mathcal{C}_1, \mathcal{C}_2)} (\mathcal{C}_1, \mathcal{C}_2)
\]
\[
\simeq (\text{Fun}_{\mathcal{C}}(\mathcal{C}_1, \mathcal{C}_1), \text{Fun}(\mathcal{C}_2, \mathcal{C}_2))
\]
\[
\simeq (Z(\mathcal{C}_1), \text{Fun}(\mathcal{C}_2, \mathcal{C}_2))
\]
\[
\simeq (Z(\mathcal{C}_1), \mathcal{H}) \boxtimes (\mathcal{C}_2, \text{Fun}(\mathcal{C}_2, \mathcal{C}_2))
\]
\[
\simeq (Z(\mathcal{C}_1), \mathcal{H})
\]
where the first \( \simeq \) is due to (2.1) and Corollary 2.6(1), and the last \( \simeq \) is due to Example 3.15. This result agrees with the general phenomenon that the value of an extended TQFT on a circle is the “center” of that on a point.

The value \( Z_\mathcal{C}(\Sigma) \) on a closed surface \( \Sigma \) is an \( \text{H-H} \)-bimodule functor \( \text{H} \rightarrow \text{H} \).

By definition, it is encoded by a pair \((u_\Sigma, H_\Sigma)\) where \( H_\Sigma \) is a unitary category equivalent to \( \text{H} \) and \( u_\Sigma : \text{H} \rightarrow H_\Sigma \) is a functor or, equivalently, an object of \( H_\Sigma \).

The value \( Z_\mathcal{C}(\Sigma) \) on a closed 3-manifold \( M \) is a natural transformation \( \text{Id}_H \rightarrow \text{Id}_H \) of \( \text{H-H} \)-bimodule functors. By definition, it is encoded by a pair \((v_M, \mathcal{C}_M)\) where \( \mathcal{C}_M : \text{H} \rightarrow \text{H} \) is an equivalence or, equivalently, a one-dimensional Hilbert space and \( v_M : \text{Id}_H \rightarrow \mathcal{C}_M \) is a natural transformation or, equivalently, a vector in \( \mathcal{C}_M \).

Note that, a unitary fusion category \( \mathcal{C} \) can be regarded as an enriched IUMFC \((\mathcal{C}, \mathcal{H})\). So, \textit{IUMFC}$_{en}$ contains the symmetric monoidal 3-category formed by unitary fusion categories, their ordinary bimodules, bimodule functors and natural transformations. Moreover, this symmetric monoidal 3-subcategory is closed under duality. Therefore, \( Z(\mathcal{C}, \mathcal{H}) \) is nothing but the extended TQFT given in [DSS1] which is believed to be an extension of the Turaev-Viro invariant associated to \( \mathcal{C} \).

In another word, the extended Turaev-Viro TQFT \( Z_\mathcal{C}^{TV} \) is included here as a special case.

Now we focus on the special case \( \mathcal{C} = (\overline{\mathcal{C}}, \mathcal{C}) \) where \( \mathcal{C} \) is a UMTC. On the one hand side, we have an extended framed 3D TQFT \( Z_\mathcal{C} \) given by the fully dualizable object \( \mathcal{C} \). On the other hand side, there is a 1-2-3-dimensional Reshetikhin-Turaev TQFT \( Z_\mathcal{C}^{RT} \) associated to the UMTC \( \mathcal{C} \) in [BL] as well as an invertible extended 4D TQFT \( A_\mathcal{C} \) associated to the same UMTC \( \mathcal{C} \) in [F3].

We conjecture that the 1-2-3-dimensional theory of \( Z_\mathcal{C} \) is a combination of those of \( Z_\mathcal{C}^{RT} \) and \( A_\mathcal{C} \). In particular, this provides a way to extend the Reshetikhin-Turaev TQFT down to dimensional zero. In fact, as we have seen, the values of \( Z_\mathcal{C} \) on these dimensions are always encoded by pairs, and the examples we have examined match perfectly with \( Z_\mathcal{C}^{RT} \) and \( A_\mathcal{C} \). That is, the values \( \text{Fun}(\mathcal{C}, \mathcal{C}), H_\Sigma, \mathcal{C}_M \) have the same form as those of \( A_\mathcal{C} \) and the values \( Z(\mathcal{C}, \mathcal{H}) \) have the same form as those of \( Z_\mathcal{C}^{RT} \).

Another evidence of this conjecture arises from the double theory \( Z_{\mathcal{C} \boxtimes \bar{\mathcal{C}}} \). By Example 3.17(2), \( (\mathcal{C} \boxtimes \bar{\mathcal{C}}, \mathcal{C} \boxtimes \bar{\mathcal{C}}) \simeq (\mathcal{C}, \mathcal{H}) \) as objects of \textit{IUMFC}$_{en}$. Therefore,
the double theory $Z_{\mathcal{E} \boxtimes \bar{\mathcal{E}}}$ is equivalent to the extended Turaev-Viro TQFT $Z_{\mathcal{E}}^{TV}$ associated to $\mathcal{E}$. This also matches perfectly with the fact that the double theory $Z_{\mathcal{E} \boxtimes \bar{\mathcal{E}}}^{RT}$ is equivalent to $Z_{\mathcal{E}}^{TV}$ [Ba2] (while the double theory $A_{\mathcal{E} \boxtimes \bar{\mathcal{E}}}$ is trivial).

Moreover, we conjecture that every object of $\text{IUMFC}_{\text{en}}$ can be promoted canonically to a $\text{SO}(3)$-fixed point. That is, $Z_{\mathcal{E}}$ defines an extended 3D TQFT without framing anomaly. There are several evidences for this conjecture. First, in the special case $Z_{\mathcal{E}}$, there have already been such data as $H_{\Sigma}$, $\mathcal{C}_M$ to address the framing anomaly of the Reshetikhin-Turaev TQFT.

Secondly, recall that, although $Z_{\mathcal{E}}(S^1)$ is equivalent to one of its factor $(Z(\mathcal{E}^2), \mathcal{H})$, it contains another factor $(\mathcal{C}_2, \text{Fun}(\mathcal{C}_2, \mathcal{C}_2)) \simeq \mathcal{H}$ which does contribute to higher dimensional cobordisms and yields such data as $H_{\Sigma}$, $\mathcal{C}_M$. From the physical point of view, these two factors of $Z_{\mathcal{E}}(S^1)$ constitute the chiral and anti-chiral parts of the whole theory so that their framing anomalies are canceled.

Thirdly, the symmetric monoidal $(4,3)$-category $\text{IUMFC}_{\text{en}}$ has a very nice behavior under duality, due to the unitarity condition we imposed in the construction. For example, the dual of a $\mathcal{C}$-$\mathcal{D}$-bimodule $\mathcal{M}$ is given by $\mathcal{M}^{op}$ no matter we treat it as a $\mathcal{C}$-$\mathcal{D}$-bimodule or an $\mathcal{H}$-$\mathcal{C}_\text{rev} \boxtimes \mathcal{D}$-bimodule instead. This fact would have sufficed to promote every object of $\text{IUMFC}_{\text{en}}$ canonically to a $\text{SO}(2)$-fixed point. However, promoting to a $\text{SO}(3)$-fixed point is a much more subtle problem. See [DSS1] for a similar conjecture and discussions therein.

**Remark 4.5.** The unitarity is only used to eliminate the ambiguity of left/right dual of an object in a multi-fusion category. All results are equally true if we replace UMFC’s and UMTC’s by spherical multi-fusion categories and modular tensor categories. It is also possible to generalize these results to multi-fusion categories and nondegenerate braided fusion categories by taking care of the difference between left/right dual.

**Remark 4.6.** In a joint work with Liang Kong, we will explain how to use enriched fusion categories to describe both gapped edges and gapless edges of 2+1D topological orders. Roughly speaking, a gapped edge supports a topological field theory which is described mathematically by a unitary fusion category or, equivalently, a unitary fusion category enriched in $\mathcal{H}$. However, a gapless edge supports a conformal field theory which is described mathematically by a unitary fusion category enriched in the module category of a unitary vertex operator algebra.

This also shed light on Chern-Simons theory with compact gauge group which is used to study the effective theory of 2+1D topological orders. If the gauge group is finite, the boundary theory is topological hence is described by a unitary fusion category as it was done in Freed’s work [F2] (see also [FHLT]). If the gauge group is not finite, the boundary theory is a conformal field theory hence might be described by an enriched unitary fusion category. This idea is very close to that in Henriques’ work [H1, H2]. It would be very interesting to clarify the relation between categories of solitons arising from conformal net and enriched fusion categories.

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