AN INTEGRAL REPRESENTATION, SOME INEQUALITIES, AND COMPLETE MONOTONICITY OF BERNOULLI NUMBERS OF THE SECOND KIND

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Abstract. In the paper, the authors discover an integral representation, some inequalities, and complete monotonicity of Bernoulli numbers of the second kind.

1. Introduction

In number theory, Bernoulli numbers of the second kind \( b_n \) for \( n \in \{0\} \cup \mathbb{N} \) may be generated by

\[
\frac{x}{\ln(1 + x)} = \sum_{n=0}^{\infty} b_n x^n. \tag{1.1}
\]

They are also known as Cauchy numbers of the first kind (see [5, p. 294]), Gregory coefficients, or logarithmic numbers. The first few Bernoulli numbers of the second kind \( b_n \) are

\[
b_0 = 1, \quad b_1 = \frac{1}{2}, \quad b_2 = -\frac{1}{12}, \quad b_3 = \frac{1}{24}, \quad b_4 = -\frac{19}{720}, \quad b_5 = \frac{3}{160}. \tag{1.2}
\]

The first main result of this paper is the following integral representation of \( b_n \) for \( n \in \mathbb{N} \).

Theorem 1.1. Bernoulli numbers of the second kind \( b_n \) may be represented as

\[
b_n = (-1)^{n+1} \int_1^\infty \frac{1}{\left( \left[ \ln(t-1) \right] + \pi^2 \right) t^n} \, dt, \quad n \in \mathbb{N}. \tag{1.3}
\]

Recall from [17, p. 108, Definition 4] that a sequence \( \{\mu_n\}_{n \geq 0} \) is said to be completely monotonic if its elements are non-negative and its successive differences are alternatively non-negative, that is

\[
(-1)^k \Delta^k \mu_n \geq 0 \tag{1.4}
\]

for \( n, k \geq 0 \), where

\[
\Delta^k \mu_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} \mu_{n+k-m}. \tag{1.5}
\]

Recall from [17, p. 163, Definition 14a] that a completely monotonic sequence \( \{a_n\}_{n \geq 0} \) is minimal if it ceases to be completely monotonic when \( a_0 \) is decreased.
Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n \). A sequence \( \lambda \) is said to be majorized by \( \mu \) (in symbols \( \lambda \preceq \mu \)) if \( \sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i \) for \( k = 1, 2, \ldots, n-1 \) and \( \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \mu_i \), where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) and \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \) are rearrangements of \( \lambda \) and \( \mu \) in a descending order. A sequence \( \lambda \) is said to be strictly majorized by \( \mu \) (in symbols \( \lambda \prec \mu \)) if \( \lambda \) is not a permutation of \( \mu \).

Basing on Theorem 1.1, the following inequalities and properties of Bernoulli numbers of the second kind \( b_n \) are discovered.

**Theorem 1.2.** The infinite sequence \( \{(-1)^n b_{n+1}\}_{n \geq 0} \) is completely monotonic and minimal.

**Theorem 1.3.** Let \( m \in \mathbb{N} \) and let \( n \) and \( a_k \) for \( 1 \leq k \leq m \) be nonnegative integers. Then
\[
\left| (a_k + a_j) ! b_{a_k + a_j + 1} \right| \geq 0 \quad (1.6)
\]
and
\[
\left| (-1)^{a_k + a_j} (a_k + a_j) ! b_{a_k + a_j + 1} \right| \geq 0, \quad (1.7)
\]
where \( |a_k|_m \) denotes a determinant of order \( m \) with elements \( a_{kj} \).

**Theorem 1.4.** Let \( m \in \mathbb{N} \) and let \( \lambda \) and \( \mu \) be two \( m \)-tuples of nonnegative numbers such that \( \lambda \leq \mu \). Then
\[
\prod_{\ell=1}^{m} \lambda_\ell ! b_{\lambda_\ell + 1} \leq \prod_{\ell=1}^{m} \mu_\ell ! b_{\mu_\ell + 1}, \quad (1.8)
\]

**Corollary 1.1.** The infinite sequence \( \{(-1)^n n ! b_{n+1}\}_{n \geq 0} \) is logarithmically convex.

2. **Lemmas**

For prove our main results, we need the following two integral representations.

**Lemma 2.1** ([3, p. 2130]). Let \( \mathbb{C} \) be the set of all complex numbers and let
\[
\ln z = \ln |z| + i \arg z, \quad (2.1)
\]
be the principal branch of the holomorphic extension of \( \ln x \) from the open half-line \((0, \infty)\) to the cut plane \( \mathcal{A} = \mathbb{C} \setminus (-\infty, 0] \), where \(-\pi < \arg z < \pi \) and \( i = \sqrt{-1} \). The function \( \frac{1}{\ln(1+z)} \) for \( z \in \mathbb{C} \setminus (-\infty, 0] \) has the integral representation
\[
\frac{1}{\ln(1+z)} = \frac{1}{z} + \int_{1}^{\infty} \frac{1}{\ln(t-1)^2 + \pi^2} \frac{dt}{z + t}, \quad (2.2)
\]

**Lemma 2.2.** Let \( \mathbb{C} \) be the set of all complex numbers. The function
\[
F(z) = \begin{cases} 
\frac{z}{(1+z) \ln(1+z)}, & z \in \mathbb{C} \setminus (-\infty, -1] \setminus \{0\} \\
1, & z = 0
\end{cases} \quad (2.3)
\]
has the integral representation
\[
F(z) = \int_{0}^{\infty} \frac{t+1}{t[(\ln t)^2 + \pi^2]} \frac{dt}{t+1+z}, \quad z \in \mathbb{C} \setminus (-\infty, -1]. \quad (2.4)
\]

The proof of Lemma 2.2 will be carried out in Section 4 below.
3. Proofs of theorems

Now we start out to prove Theorems 1.1 to 1.4 and Corollary 1.1 as follows.

First proof of Theorem 1.1. By (2.2), we have
\[
x \ln(1 + x) = 1 + \int_1^\infty \frac{1}{[\ln(t-1)]^2 + \pi^2} \frac{x}{t} \, dt \tag{3.1}
\]
and
\[
\left[ \frac{x}{\ln(1 + x)} \right]^{(k)} = \int_1^\infty \frac{1}{[\ln(t-1)]^2 + \pi^2} \left( \frac{x}{t} \right)^{(k)} \, dt \\
= \int_1^\infty \frac{1}{[\ln(t-1)]^2 + \pi^2} \left( 1 - \frac{t}{x+t} \right)^{(k)} \, dt \\
= (-1)^{k+1} k! \int_1^\infty \frac{1}{[\ln(t-1)]^2 + \pi^2} \frac{1}{(x+t)^{k+1}} \, dt \tag{3.2}
\]
for \( k \in \mathbb{N} \). On the other hand, by (1.1), we also have
\[
\left[ \frac{x}{\ln(1 + x)} \right]^{(k)} = \sum_{n=k}^{\infty} b_n \frac{n!}{(n-k)!} x^{n-k} \tag{3.3}
\]
Combining (3.2) with (3.3) leads to
\[
\sum_{n=k}^{\infty} b_n \frac{n!}{(n-k)!} x^{n-k} = (-1)^{k+1} k! \int_1^\infty \frac{1}{[\ln(t-1)]^2 + \pi^2} \frac{1}{(x+t)^{k+1}} \, dt.
\]
Letting \( x \to 0^+ \) on both sides of the above equation produces
\[
k! b_k = (-1)^{k+1} k! \int_1^\infty \frac{1}{[\ln(t-1)]^2 + \pi^2} \frac{1}{t^k} \, dt.
\]
Thus, the formula (1.3) is proved. \( \square \)

Second proof of Theorem 1.1. By the integral representation (2.4), we have
\[
x \ln(1 + x) = \int_1^\infty \frac{t}{(t-1)\left([\ln(t-1)]^2 + \pi^2\right)} \frac{1+x}{x+t} \, dt \tag{3.4}
\]
and
\[
\left[ \frac{x}{\ln(1 + x)} \right]^{(k)} = \int_1^\infty \frac{t}{(t-1)\left([\ln(t-1)]^2 + \pi^2\right)} \left( \frac{1+x}{x+t} \right)^{(k)} \, dt \\
= \int_1^\infty \frac{t}{(t-1)\left([\ln(t-1)]^2 + \pi^2\right)} \left( 1 + \frac{1-t}{x+t} \right)^{(k)} \, dt \\
= (-1)^{k+1} k! \int_1^\infty \frac{t}{[\ln(t-1)]^2 + \pi^2} \frac{1}{(x+t)^{k+1}} \, dt \tag{3.5}
\]
for \( k \in \mathbb{N} \). Combining (3.5) with (3.3) leads to
\[
\sum_{n=k}^{\infty} b_n \frac{n!}{(n-k)!} x^{n-k} = (-1)^{k+1} k! \int_1^\infty \frac{t}{[\ln(t-1)]^2 + \pi^2} \frac{1}{(x+t)^{k+1}} \, dt \tag{3.6}
\]
Letting \( x \to 0^+ \) on both sides of (3.6) yields the formula (1.3). The proof of Theorem 1.1 is complete. \( \square \)
First proof of Theorem 1.2. Theorem 4a in [17, p. 108] reads that a necessary and sufficient condition that the sequence \( \{\mu_n\}_0^\infty \) should have the expression
\[
\mu_n = \int_0^1 t^n \, d\alpha(t)
\]  
(3.7)
for \( n \geq 0 \), where \( \alpha(t) \) is non-decreasing and bounded for \( 0 \leq t \leq 1 \), is that it should be completely monotonic. Theorem 14a in [17, p. 164] states that a completely monotonic sequence \( \{\mu_n\}_{n \geq 0} \) is minimal if and only if the equality (3.7) is valid for \( n \geq 0 \) and \( \alpha(t) \) is a non-decreasing bounded function continuous at \( t = 0 \).

Setting in the equality (3.7)
\[
\alpha(t) = \int_0^t \frac{1}{s \{\ln(1/s - 1)^2 + \pi^2\}} \, ds
\]  
(3.8)
for \( t \in [0, 1] \) and \( \alpha(1) = b_1 = \frac{1}{2} \) yields the required complete monotonicity and minimality. \( \square \)

Second proof of Theorem 1.2. From (1.3), it follows that for \( n \in \mathbb{N} \)
\[
(-1)^{n+1} b_n = \int_1^\infty \frac{1}{\{\ln(t-1)^2 + \pi^2\}t^n} \, dt
\]
\[
= \int_0^1 \frac{1}{\{\ln(1/s - 1)^2 + \pi^2\}} s^n \, d\left(\frac{1}{s}\right)
\]
\[
= \int_0^1 \frac{1}{\{\ln(1/s - 1)^2 + \pi^2\}} s^{n-2} \, ds
\]
\[
= \int_0^1 \frac{1}{s \{\ln(1/s - 1)^2 + \pi^2\}} s^{n-1} \, ds
\]
\[\triangleq c_{n-1}.
\]
Since \( c_0 = b_1 = \frac{1}{2} \) and the function \( \frac{1}{s \{\ln(1/s - 1)^2 + \pi^2\}} \) is positive on \( (0, 1) \), then the complete monotonicity and minimality of the sequence \( \{c_n\}_0^\infty \) is readily obtained. The proof of Theorem 1.2 is complete. \( \square \)

Proof of Theorem 1.3. A function \( f \) is said to be completely monotonic on an interval \( I \) if \( f \) has derivatives of all orders on \( I \) and \((-1)^nf^{(n)}(x) \geq 0 \) for \( x \in I \) and \( n \geq 0 \). See [9, Chapter XIII] and [17, Chapter IV].

From the proofs of Theorem 1.1, we observe that
\[
h_n = (-1)^{n+1} \lim_{x \to 0^+} h_n(x)
\]  
(3.9)
and
\[
h_n(x) = \int_1^\infty \frac{1}{\{\ln(t-1)^2 + \pi^2\} (t+x)^n} \, dt
\]  
(3.10)
is completely monotonic on \( [0, \infty) \).

In [8], or see [9, p. 367], it was obtained that if \( f \) is a completely monotonic function on \( [0, \infty) \), then
\[
|f^{(a_1+a_j)}(x)|_m \geq 0
\]  
(3.11)
and
\[
|(-1)^{a_1+a_j} f^{(a_1+a_j)}(x)|_m \geq 0,
\]  
(3.12)
where $|a_{ij}|_m$ denotes a determinant of order $m$ with elements $a_{ij}$ and $a_i$ for $1 \leq i \leq m$ are nonnegative integers. Applying $f$ in (3.11) and (3.12) to the function $h_n(x)$ yields

$$|h_n^{(a_i+a_j)}(x)|_m \geq 0$$

(3.13)

and

$$|(-1)^{a_i+a_j} h_n^{(a_i+a_j)}(x)|_m \geq 0,$$

(3.14)

that is,

$$|(-1)^{a_i+a_j} \frac{(n+a_i+a_j-1)!}{(n-1)!} h_{n+a_i+a_j}(x)|_m \geq 0$$

(3.15)

and

$$|\frac{(n+a_i+a_j-1)!}{(n-1)!} h_{n+a_i+a_j}(x)|_m \geq 0.\quad (3.16)$$

Letting $x \to 0^+$ in (3.15) and (3.16) and making use of (3.9) produce

$$|(-1)^{a_i+a_j} \frac{(n+a_i+a_j-1)!}{(n-1)!} (-1)^{n+a_i+a_j+1} b_{n+a_i+a_j}|_m \geq 0$$

(3.17)

and

$$|\frac{(n+a_i+a_j-1)!}{(n-1)!} (-1)^{n+a_i+a_j+1} b_{n+a_i+a_j}|_m \geq 0.\quad (3.18)$$

Further simplifying (3.17) and (3.18) leads to

$$|(-1)^{n+1} (n+a_i+a_j-1)! b_{n+a_i+a_j}|_m \geq 0$$

and

$$|(-1)^{n+a_i+a_j+1} (n+a_i+a_j-1)! b_{n+a_i+a_j}|_m \geq 0,$$

which are equivalent to (1.6) and (1.7). Theorem 1.3 is thus proved. \hfill \Box

**Proof of Theorem 1.4.** In [16, p. 106, Theorem A] and [9, p. 367, Theorem 2], a minor correction of [6, Theorem 1], it was obtained that if $f$ is a completely monotonic function on $(0, \infty)$ and $\lambda \leq \mu$, then

$$\prod_{i=1}^{n} f^{(\mu)}(x) \leq \prod_{i=1}^{n} f^{(\mu)}(x).$$

(3.19)

Applying the inequality (3.19) to $h_n(x)$, defined by (3.10), creates

$$\prod_{i=1}^{m} (-1)^{\lambda_i} \frac{(n+\lambda_i-1)!}{(n-1)!} h_{n+\lambda_i}(x) \leq \prod_{i=1}^{m} (-1)^{\mu_i} \frac{(n+\mu_i-1)!}{(n-1)!} h_{n+\mu_i}(x)$$

which can be simplified as

$$\prod_{i=1}^{m} (n+\lambda_i-1)! h_{n+\lambda_i}(x) \leq \prod_{i=1}^{m} (n+\mu_i-1)! h_{n+\mu_i}(x).$$

Further taking $x \to 0^+$ and utilizing (3.9) turn out

$$\prod_{i=1}^{m} (n+\lambda_i-1)! (-1)^{n+\lambda_i+1} b_{n+\lambda_i} \leq \prod_{i=1}^{m} (n+\mu_i-1)! (-1)^{n+\mu_i+1} b_{n+\mu_i},$$

which is equivalent to (1.8). The proof of Theorem 1.4 is complete. \hfill \Box
Proof of Corollary 1.1. It is clear that \((i, i+2) \succ (i+1, i+1)\) for \(i \geq 0\). Therefore, by virtue of (1.8), we have
\[(i!|b_{i+1}|(i+2)!|b_{i+3}|) \geq [(i+1)!|b_{i+2}|]^2.\] (3.20)
This implies the required logarithmic convexity.

This conclusion can also be deduced from Theorem 1.3. The proof of Theorem 1.1 is thus complete. □

4. Proofs of Lemma 2.2

Now we are in a position to prove Lemma 2.2 as follows.

First proof. For \(z = \varepsilon e^{i\theta}\) with \(\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\) and \(\varepsilon \in (0, 1)\), by standard argument, we have
\[|zF(z-1)|^2 = \frac{|\varepsilon e^{i\theta} - 1|}{\ln(\varepsilon e^{i\theta})} = \frac{1 - 2\varepsilon \cos \theta + \varepsilon^2}{(\ln \varepsilon)^2 + \theta^2} \to 0\]
uniformly as \(\varepsilon \to 0^+\). Consequently,
\[\lim_{\varepsilon \to 0^+} [zF(z-1)] = 0\] (4.1)
uniformly.

For \(\theta \in (-\pi, \pi)\) and \(z = re^{i\theta}\), by standard argument, we have
\[|F(z-1)| = \frac{|re^{i\theta} - 1|}{|re^{i\theta}| \ln(re^{i\theta})} = \sqrt{\frac{1 + 2r \cos \theta + r^2}{r^2(\ln r)^2 + \theta^2}} \to 0\] (4.2)
uniformly as \(r \to \infty\).

For \(t \in (0, \infty)\) and \(\varepsilon \in (0, 1)\), we have
\[F(-t-1+i\varepsilon) = \frac{-t-1+i\varepsilon}{(-t+i\varepsilon) \ln(-t+i\varepsilon)} = \frac{-t-1+i\varepsilon}{(-t+i\varepsilon) \ln(-t+i\varepsilon) + i \arg(-t+i\varepsilon)}\]
\[= \frac{-t-1+i\varepsilon}{(-t+i\varepsilon) \ln(-t+i\varepsilon) + i(\pi - \arctan \frac{\varepsilon}{t})}\]
\[\to \frac{t+1}{t(\ln t + \pi i)} = \frac{(t+1)(\ln t - \pi i)}{t((\ln t)^2 + \pi^2)}\]
as \(\varepsilon \to 0^+\). In other words, for \(t \in (0, \infty)\),
\[\lim_{\varepsilon \to 0^+} \Im F(-t-1+i\varepsilon) = -\frac{\pi(t+1)}{t((\ln t)^2 + \pi^2)}.\] (4.3)

Let \(D\) be a bounded domain with piecewise smooth boundary. If \(f(z)\) is analytic on \(D\) and extendable smoothly to the boundary of \(D\), then
\[f(z) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(w)}{w-z} \, dw, \quad z \in D.\] (4.4)
In the literature, we call (4.4) Cauchy integral formula. See [7, p. 113]. For any fixed point \( z_0 = x_0 + iy_0 \in \mathbb{C} \setminus (-\infty, 0] \), choose \( \varepsilon \) and \( r \) such that
\[
\begin{cases}
0 < \varepsilon < |y_0| \leq |z_0| < r, & y_0 \neq 0, \\
0 < \varepsilon < x_0 = |z_0| < r, & y_0 = 0,
\end{cases}
\]
and consider the positively oriented contour \( C(\varepsilon, r) \) in \( \mathbb{C} \setminus (-\infty, -1] \) consisting of the half circle \( z = -1 + \varepsilon e^{\theta i} \) for \( \theta \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \) and the half lines \( z = -1 + x \pm \varepsilon i \) for \( x \leq 0 \) until they cut the circle \( |z + 1| = r \), which close the contour at the points \(-1 - r(\varepsilon) \pm \varepsilon i\), where \( 0 < r(\varepsilon) \to r \) as \( \varepsilon \to 0 \). By the formula (4.4), we have
\[
F(z_0) = \frac{1}{2\pi i} \left[ \int_{-r(\varepsilon)}^{r(\varepsilon)} \frac{F(x - 1 + \varepsilon i)}{x - 1 - \varepsilon i - z_0} \, dx + \int_{0}^{\infty} \frac{F(x - 1 - \varepsilon i)}{x - 1 + \varepsilon i - z_0} \, dx \right] + \int_{0}^{\arg[-r(\varepsilon) + \varepsilon i]} \frac{F(x - 1 + \varepsilon i) \, d\varepsilon}{x - 1 + \varepsilon i - z_0} + \int_{0}^{\arg[-r(\varepsilon) - \varepsilon i]} \frac{F(x - 1 - \varepsilon i) \, d\varepsilon}{x - 1 - \varepsilon i - z_0}. \tag{4.5}
\]
By the formula (4.1), it follows that
\[
\lim_{\varepsilon \to 0^+} \int_{-\pi/2}^{

\int_{-r(\varepsilon)}^{r(\varepsilon)} \frac{F(x - 1 + \varepsilon i) \, d\varepsilon}{x - 1 - \varepsilon i - z_0} + \int_{0}^{\arg[-r(\varepsilon) + \varepsilon i]} \frac{F(x - 1 + \varepsilon i) \, d\varepsilon}{x - 1 + \varepsilon i - z_0} + \int_{0}^{\arg[-r(\varepsilon) - \varepsilon i]} \frac{F(x - 1 - \varepsilon i) \, d\varepsilon}{x - 1 - \varepsilon i - z_0}. \tag{4.5}
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\int_{-r(\varepsilon)}^{r(\varepsilon)} \frac{F(x - 1 + \varepsilon i) \, d\varepsilon}{x - 1 - \varepsilon i - z_0} + \int_{0}^{\arg[-r(\varepsilon) + \varepsilon i]} \frac{F(x - 1 + \varepsilon i) \, d\varepsilon}{x - 1 + \varepsilon i - z_0} + \int_{0}^{\arg[-r(\varepsilon) - \varepsilon i]} \frac{F(x - 1 - \varepsilon i) \, d\varepsilon}{x - 1 - \varepsilon i - z_0}. \tag{4.5}
\]
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\[
\lim_{\varepsilon \to 0^+} \int_{-\pi/2}^{

\int_{-r(\varepsilon)}^{r(\varepsilon)} \frac{F(x - 1 + \varepsilon i) \, d\varepsilon}{x - 1 - \varepsilon i - z_0} + \int_{0}^{\arg[-r(\varepsilon) + \varepsilon i]} \frac{F(x - 1 + \varepsilon i) \, d\varepsilon}{x - 1 + \varepsilon i - z_0} + \int_{0}^{\arg[-r(\varepsilon) - \varepsilon i]} \frac{F(x - 1 - \varepsilon i) \, d\varepsilon}{x - 1 - \varepsilon i - z_0}. \tag{4.5}
\]
By the formula (4.1), it follows that
\[
\lim_{\varepsilon \to 0^+} \int_{-\pi/2}^{

\int_{-r(\varepsilon)}^{r(\varepsilon)} \frac{F(x - 1 + \varepsilon i) \, d\varepsilon}{x - 1 - \varepsilon i - z_0} + \int_{0}^{\arg[-r(\varepsilon) + \varepsilon i]} \frac{F(x - 1 + \varepsilon i) \, d\varepsilon}{x - 1 + \varepsilon i - z_0} + \int_{0}^{\arg[-r(\varepsilon) - \varepsilon i]} \frac{F(x - 1 - \varepsilon i) \, d\varepsilon}{x - 1 - \varepsilon i - z_0}. \tag{4.5}
\]
as $\varepsilon \to 0^+$ and $r \to \infty$. Substituting equations (4.6), (4.7), and (4.8) into (4.5) and simplifying produce the integral representation (2.4). The proof of Lemma 2.2 is complete. □

Second proof. In all treatments of Pick functions, a main example is the principal logarithm $\ln$ defined in the cut plane $\mathcal{A}$ as well as

$$
- \frac{1}{\ln z} = - \frac{1}{z - 1} + \int_{-\infty}^{0} \frac{1}{(t - z)((\ln t)^2 + \pi^2)} \, dt.
$$

(4.9)

This formula is equivalent to [2, (1.4)]. Multiplying the identity

$$
\int_{0}^{\infty} \frac{1}{t((\ln t)^2 + \pi^2)} \, dt = 1
$$

(4.10)

by $\frac{1}{z}$ and inserting it in the previous formula yield

$$
\frac{z - 1}{z \ln z} = \int_{0}^{\infty} \left[ \frac{1}{tz} + \frac{z - 1}{z(t + z)} \right] \frac{dt}{(\ln t)^2 + \pi^2} = \int_{0}^{\infty} \frac{1 + t}{(t + z)((\ln t)^2 + \pi^2)} \, dt,
$$

(4.11)

which is the formula (2.4). The proof of Lemma 2.2 is complete. □

5. Remarks

Finally, we would like to give some remarks on something related to the integral representations (2.2) and (2.4).

Remark 5.1. In [1, p. 230, 5.1.32], it is listed that

$$
\ln \frac{b}{a} = \int_{0}^{\infty} \frac{e^{-au} - e^{-bu}}{u} \, du.
$$

(5.1)

As a result, we have

$$
\ln[\ln(1 + x)] = \int_{0}^{\infty} \frac{e^{-u} - e^{-u \ln(1 + x)}}{u} \, du = \int_{0}^{\infty} \frac{e^{-u} - (1 + x)^{-u}}{u} \, du
$$

(5.2)

and, by a differentiation,

$$
\frac{1}{(1 + x) \ln(1 + x)} = \int_{0}^{\infty} \frac{1}{(1 + x)^{u+1}} \, du
$$

$$
= \int_{0}^{\infty} \left[ \frac{1}{\Gamma(1 + u)} \int_{0}^{\infty} t^u e^{-(1 + x)t} \, dt \right] \, du
$$

(5.3)

$$
= \int_{0}^{\infty} \left[ \int_{0}^{\infty} \frac{t^u}{\Gamma(1 + u)} \, du \right] e^{-(1 + x)t} \, dt,
$$

where $\Gamma(z)$ is the classical gamma function which may be defined for $\Re(z) > 0$ by Euler’s integral

$$
\Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} \, dt.
$$

(5.4)

The integral representation (5.3) means that $\frac{1}{(1 + x) \ln(1 + x)}$ is a completely monotonic function on $(0, \infty)$. In other words, the function $\frac{1}{\ln(1 + x)}$ is logarithmically completely monotonic on $(0, \infty)$. More strongly, it was claimed in [3, p. 2130, (34)] and [4, p. 12, (33)] that the function $\frac{1}{\ln(1 + x)}$ is a Stieltjes transform. For information on the notions “logarithmically completely monotonic function” and “Stieltjes transform”, please refer to [11, Remark 8], [12, Section 1], [13, Remark 4.7], the monograph [15] and plenty of closely-related references therein.
From (5.3) and by integration by part, it is not difficult to obtain that
\[
\frac{1}{\ln(1+x)} = \int_0^\infty \left[ \int_0^\infty \frac{t^{u-1}}{\Gamma(u)} \, d\gamma \right] e^{-(1+x)t} \, dt, \quad x > 0. \tag{5.5}
\]
By induction and integration by part, we can obtain
\[
\frac{(1+x)^k}{\ln(1+x)} = \int_0^\infty \left[ \int_0^\infty \frac{t^{u-k-1}}{\Gamma(u-k)} \, d\gamma \right] e^{-(1+x)t} \, dt
= \int_0^\infty \left[ \int_{-k}^\infty \frac{t^{u-1}}{\Gamma(u)} \, d\gamma \right] e^{-(1+x)t} \, dt \tag{5.6}
\]
for \(x > 0\) and \(k \in \mathbb{Z}\), where \(\mathbb{Z}\) denotes the set of all integers and the classical gamma function \(\Gamma(z)\) may be defined for \(z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}\) by Euler's formula
\[
\Gamma(z) = \lim_{n \to \infty} \frac{n!n^z}{z(z+1)\cdots(z+n)}. \tag{5.7}
\]
**Remark 5.2.** By the way, the term \(\frac{1}{x}\) in (2.2) was lost in [3, p. 2130, (34)] and [4, p. 12, (33)] and was corrected in [14, 18].

**Remark 5.3.** A function \(f : I \subseteq (0, \infty) \to [0, \infty)\) is called a Bernstein function on \(I\) if \(f(x)\) has derivatives of all orders and \(f'(x)\) is completely monotonic on \(I\). See the monograph [15]. We claim that the generating function \(\frac{x}{\ln(1+x)}\) of Bernoulli numbers of the second kind \(b_k\) is a Bernstein function on \((0, \infty)\). This can be proved by two approaches below.

The integral representation (3.1) shows us that the function \(\frac{x}{\ln(1+x)}\) is positive and increasing on \((0, \infty)\). The integral representation (3.2) reveals that the first derivative of \(\frac{x}{\ln(1+x)}\) is completely monotonic on \((0, \infty)\). So the function \(\frac{x}{\ln(1+x)}\) is a Bernstein function on \((0, \infty)\).

It is not difficult to see that
\[
\frac{x}{\ln(1+x)} = \int_0^1 (1+x)^t \, dt \tag{5.8}
\]
and the function \((1+x)^t\) for \(t \in (0, 1)\) is a Bernstein function.

**Remark 5.4.** This paper is a combined and revised version of the preprints [10, 14] and Chapter 5 of the thesis [18].

**REFERENCES**

[1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series 55, 9th printing, Washington, 1970.

[2] C. Berg, *A Pick function related to an inequality for the entropy function*, J. Inequal. Pure Appl. Math. 2 (2001), no. 2, Art. 26; Available online at http://www.emis.de/journals/JIPAM/article142.html.

[3] C. Berg and H. L. Pedersen, *A one-parameter family of Pick functions defined by the Gamma function and related to the volume of the unit ball in n-space*, Proc. Amer. Math. Soc. 139 (2011), no. 6, 2121–2132; Available online at http://dx.doi.org/10.1090/S0002-9939-2010-10636-6.

[4] C. Berg and H. L. Pedersen, *A Pick function related to the sequence of volumes of the unit ball in n-space*, available online at http://arxiv.org/abs/0912.2185.

[5] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel Publishing Co., Dordrecht and Boston, 1974.
[6] A. M. Fink, *Kolmogorov-Landau inequalities for monotone functions*, J. Math. Anal. Appl. **90** (1982), 251–258; Available online at [http://dx.doi.org/10.1016/0022-247X(82)90057-9](http://dx.doi.org/10.1016/0022-247X(82)90057-9).

[7] T. W. Gamelin, *Complex Analysis*, Undergraduate Texts in Mathematics, Springer, New York-Berlin-Heidelberg, 2001.

[8] D. S. Mitrinović and J. E. Pečarić, *On two-place completely monotonic functions*, Anziger Öster. Akad. Wiss. Math.-Naturwiss. Kl. **126** (1989), 85–88.

[9] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993.

[10] F. Qi, *An integral representation and properties of Bernoulli numbers of the second kind*, available online at [http://arxiv.org/abs/1301.7181](http://arxiv.org/abs/1301.7181).

[11] F. Qi, S. Guo, and B.-N. Guo, *Complete monotonicity of some functions involving polygamma functions*, J. Comput. Appl. Math. **233** (2010), no. 9, 2149–2160; Available online at [http://dx.doi.org/10.1016/j.cam.2009.09.044](http://dx.doi.org/10.1016/j.cam.2009.09.044).

[12] F. Qi, Q.-M. Luo, and B.-N. Guo, *Complete monotonicity of a function involving the divided difference of digamma functions*, Sci. China Math. **56** (2013), no. 11, 2315–2325; Available online at [http://dx.doi.org/10.1007/s11425-012-4562-0](http://dx.doi.org/10.1007/s11425-012-4562-0).

[13] F. Qi, C.-F. Wei, and B.-N. Guo, *Complete monotonicity of a function involving the ratio of gamma functions and applications*, Banach J. Math. Anal. **6** (2012), no. 1, 35–44.

[14] F. Qi and X.-J. Zhang, *A Stieltjes function involving the logarithmic function and an application*, available online at [http://arxiv.org/abs/1301.6425](http://arxiv.org/abs/1301.6425).

[15] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions—Theory and Applications*, 2nd ed., de Gruyter Studies in Mathematics **37**, Walter de Gruyter, Berlin, Germany, 2012; Available online at [http://dx.doi.org/10.1515/9783110269338](http://dx.doi.org/10.1515/9783110269338).

[16] H. van Haeringen, *Completely monotonic and related functions*, J. Math. Anal. Appl. **204** (1996), no. 2, 389–408; Available online at [http://dx.doi.org/10.1006/jmaa.1996.0443](http://dx.doi.org/10.1006/jmaa.1996.0443).

[17] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1946.

[18] X.-J. Zhang, *Integral Representations, Properties, and Applications of Three Classes of Functions*, Thesis supervised by Professor Feng Qi and submitted for the Master Degree of Science in Mathematics at Tianjin Polytechnic University in January 2013. (Chinese)

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