A Commentary on Ruppeiner Metrics for Black Holes

A.J.M. Medved
Physics Department,
University of Seoul,
Seoul 130-743,
Korea
E-mail(1): allan@physics.uos.ac.kr
E-mail(2): joey_medved@yahoo.com

Abstract

There has been some recent controversy regarding the Ruppeiner metrics that are induced by Reissner–Nordstrom (and Reissner–Nordstrom-like) black holes. Most infamously, why does this family of metrics turn out to be flat, how is this outcome to be physically understood, and can/should the formalism be suitably modified to induce curvature? In the current paper, we provide a novel interpretation of this debate. For the sake of maximal analytic clarity and tractability, some supporting calculations are carried out for the relatively simple model of a rotating BTZ black hole.
I. BACKGROUND

From a historical perspective (with attention to the early seventies), the paradigm of black hole thermodynamics can be viewed as the synthesis of three main theoretical inputs: (i) a stunning analogy between black hole mechanics and the “standard” laws of thermodynamics [1], (ii) Bekenstein’s proposal of a “generalized” second law of thermodynamics [2] and (iii) Hawking’s quantum-field-based calculation of the black hole radiation spectrum [3]. (Note that the second point implies that a black hole should be assigned an entropy, while the third directly ascribes it with a temperature.) Given its rigorous nature, the Hawking calculation was particularly significant in legitimizing the overall picture. Moreover, Hawking’s result allowed an unambiguous calibration for the thermodynamic properties of interest.

Now being more explicit, let us focus on a (“non-exotic”\(^1\)) black hole of mass \(M\), charge \(Q\) and angular momentum \(J\) in a four-dimensional asymptotically flat spacetime. (Also known as a Kerr–Newman black hole.) One would find it natural to assign this object with an entropy \((S)\), a temperature \((T)\), an electrostatic potential \((\Phi)\) and a rotational velocity \((\Omega)\) in precisely the following manner:

\[
S = \frac{A}{4} = 8\pi M \left[ M - \frac{Q^2}{2M} + \sqrt{M^2 - Q^2 - \frac{J^2}{M^2}} \right],
\]

\[
T = \frac{2}{A} \sqrt{M^2 - Q^2 - \frac{J^2}{M^2}},
\]

\[
\Phi = Q \sqrt{\frac{4\pi}{A}}
\]

and

\[
\Omega = 4\pi \frac{J}{MA}.
\]

Here, we have set all fundamental constants equal to unity (these can easily be restored by way of dimensional analysis) and introduced the notion of a black hole having an event horizon with a surface area of \(A\). Actually, black hole thermodynamics is essentially based on the properties of this outer-most horizon surface, and we note, in passing, that \(T = \kappa/2\pi\) where \(\kappa\) is the surface gravity (which measures the strength of the gravitational field in the proximity of this special surface).

As alluded to by point (i) above, the concept of black holes as thermodynamic systems follows — in part — from the following quantitative statement:

\(^1\)Meaning that we are dismissing the possibility of any (so-called) “quantum hair” and other esoteric considerations from the present discussion.
\[ dM = TdS + \Phi dQ + \Omega dJ , \]  

which is clearly analogous to the first law of thermodynamics. But, strictly speaking, this analogy is only provisional: it depends on identifying the conserved mass \( M \) with the black hole internal energy and the last two terms with the external work being done on the black hole.

These identifications are standard lore, but suppose we now consider the redefined mass

\[ \tilde{M} = M - \Phi Q - \Omega J , \]  

then the first law can just as easily be written in the equivalent form

\[ d\tilde{M} = TdS - Qd\Phi - Jd\Omega . \]  

Note that the absolute value of the last two terms now represents the work that is done by the black hole on its environment.

Although Eq.(5) is the common-place form of the first law for black holes, one may ask if it is any more (or less) legitimate than its alternative representation in Eq.(7). Actually, the answer — like in most things scientific or otherwise — depends on the context. For instance, suppose some adventurous scientist is monitoring the energy fluctuations of a black hole under some sort of (yet-to-be-elaborated-on) experimental procedure. If the experiment was such that the charge and angular momentum were to be held fixed, Eq.(5) would be more appropriate for the subsequent analysis. On the other hand, our hypothetical scientist may have better control over the electrostatic potential and rotational velocity (which, if anything, seems the more realistic scenario) and would, thereby, prefer to think in terms of Eq.(7). To put it another way, the physically motivated context should determine the choice of extensive variables and, consequently, the most appropriate formulation of the first law.

Continuing along these lines, let us take notice of the relative signs of the work terms in Eqs.(5) and (7). Comparing with textbook thermodynamic relations, one can observe that the mass \( M \) is actually indicative of an “enthalpic-type” of potential, while it is the quantity \( \tilde{M} \) that is more suggestive of an internal energy. (Such an observation, although in stronger terms, was recently made by Shen et al [4]. Consult the cited paper for further justification of this interpretation.) To clarify this last point, it is (in spite of many statements to the contrary) not \textit{a priori} clear what should be regarded as the definitive internal energy of a black hole. That is to say, Hawking’s calculation pinpoints the temperature, which then vicariously fixes the entropy given the functional form that was proposed by Bekenstein. After that, it is an open question as to how the other terms in the first law should be divided up between work and internal energy. Really, only for the Schwarzschild case \((Q = J = 0)\) can this division be regarded as unambiguous.

What does all this mean? If \( \tilde{M} \) — and not \( M \) — is the true measure of a black hole’s internal energy, it might be fair to argue on behalf of Eq.(7) as being the more “fundamental” of the two realizations. On the other hand, most researchers would probably regard \( Q \) and \( J \) (rather than \( \Phi \) and \( \Omega \)) as being the “natural” choice for the extensive work variables; thus implicating Eq.(5) as the more fundamental statement. Either way, the current author would rather allow the two formulations to have an equal status; with a particular choice
being (as stressed above) predicated on the situational or experimental context.²

Let us now slightly alter course and talk about what is known as Ruppeiner geometries
and, in particular, how these can relate to black hole thermodynamics. The story begins,
some time ago, with Weinhold [5] proposing a metrical structure and, hence, a geometrical
description for a given thermodynamic system. More specifically, the proposed metric is
based upon the Hessian (or second-order partial derivatives) of the internal energy with
respect to the entropy and any other extensive variables of the system. A few years later,
Ruppeiner [6] made a similar proposal, but now with the metric being defined in terms of
the Hessian of the entropy. To be more formal, let \( U \) be the internal energy, \( S \) be (as always)
the entropy and let \( X_i \) [\( Y_i \)] collectively denote all of the system’s extensive variables except
for \( U \) [\( S \)]. Then the Weinhold and Ruppeiner metrics can respectively be represented as
follows:

\[
ds^2_W = +\partial_i\partial_j U \; dX_i dX_j
\]

(8)

and

\[
ds^2_R = -\partial_i\partial_j S \; dY_i dY_j .
\]

(9)

For future reference, as well as a point of interest in its own right, it can be shown that
[7,8]

\[
ds^2_R = \frac{1}{T} ds^2_W ,
\]

(10)

where \( T = \partial S \frac{M}{M} \) is (of course) the temperature.

It is Ruppeiner’s contention — and convincingly supported [9] — that Eq.(9) describes
a Riemannian geometry which provides a substantial amount of information about the cor-
responding thermodynamic system and its statistical-mechanical counterpart.³ For in-
stance, he has demonstrated that a flat-space metric indicates a non-interacting statistical-
mechanical system (with the converse having been validated as well). Moreover, the resul-
tant scalar curvature has been shown to provide significant information about the system’s
thermodynamic stability. In particular, curvature singularities are expected to be in one-
to-one correspondence with critical or phase-transition points. Also, localized zeroes should
indicate thermodynamically special points where the interactions are “turned off”.

It is natural to apply the Ruppeiner metrical formalism to black holes, given their current
status as “full-fledged” thermodynamic systems (as discussed above ⁴). Indeed, much has

²As far as favoring \( \tilde{M} \) over \( M \) with the assignment of internal energy (as argued for in [4]), the
current author chooses to remain agnostic. Hence, we will adopt the term “alternative energy”
when verbally referencing \( \tilde{M} \).

³In this regard, the Weinhold geometry has some utility of its own, but it will not be considered
here except as an intermediary calculational tool.

⁴In spite of our previous discussion, such a claim is still somewhat open to debate [20].
been done in this field, with Aman and collaborators at the forefront [10–15]. (For some other relevant work, see [16–19].) However, even if black holes are thermodynamic entities in the truest sense, there are both conceptual and technical barriers that can impede such an application. Most notoriously, black hole thermodynamics still lacks a universally acceptable statistical explanation. ⑤ Furthermore, many black hole systems are dangerously unstable due to negative heat capacities. And let us not overlook, even if only recently having come to light [4], what choice should be made for the internal energy?

In view of such caveats, it should probably not be too surprising that the Ruppeiner metric can yield some perplexing — and perhaps even disturbing — results in a black hole context. Most infamously, a Reissner–Nordstrom black hole (i.e., a charged but non-rotating black hole in an asymptotically flat spacetime) produces a perfectly flat Ruppeiner geometry [10]. (Importantly, this result persists for any applicable dimensionality [11] and for a family of closely related dimensionally reduced models [13].) Aman et al consistently argue that this flatness should not necessarily have gone unanticipated; rather, they attribute the flat Ruppeiner metric to the scale invariance of the Einstein–Maxwell action [12]. The implication then being that such scale invariance is (somehow) the black hole analogue of a non-interacting statistical system. If we put this rationalization aside, what is perhaps most discomforting is that a flat-space metric means (quite obviously) an everywhere-vanishing scalar curvature. Hence, the curvature is rendered incapable of providing any information about the interesting points of the Reissner–Nordstrom phase space; namely, the extremal point where the inner and outer horizon coalesce and the Davies critical point [22] where the heat capacity diverges. (The status of the Davies point as a meaningful transition point is highly controversial [23]. Those who argue on behalf of the dissenting side would most likely view the vanishing curvature as further support for their case. Still, it seems rather bothersome that a singularity in the formalism could be completely overlooked by a function that aspires to provide detailed information about the system’s stability.)

Whether or not this Ruppeiner flatness (and any implications thereof) is a tenable state of affairs would be up to each individual reader to decide. But, suffice it to say, there has been substantial backlash to this outcome in the form of proposed “resolutions”. That is, formal modifications that are able to induce curved Ruppeiner metrics ⑥ for the Reissner–Nordstrom family after all. These approaches include the following: (i) adding a physically motivated regulator (in the guise of a spacetime curvature and/or an angular momentum) that is taken to zero only at the end of the calculation [18], (ii) taking quantum effects such as thermal fluctuations into account [17], (iii) redefining the thermodynamic metric on the basis of invariance under canonical transformations [19] and, as already mentioned, (iv) using the extensive variables as prescribed by the “reformulated” first law of Eq.(7) [4].

Later on, we will comment briefly on the first three of these proposals, but our focus will be mainly on the fourth. In the next section, we will present two distinct calculations of the Ruppeiner metric and its associated scalar curvature. The distinction to be made is between

⑤There are, however, model-specific explanations such as (for instance) certain string-theoretic extremal and near-extremal black holes [21].

⑥In the case of proposal (iv), the metric is no longer Ruppeiner per se but rather “Ruppeiner-like”.
the two (presumably) valid choices for the black hole internal energy; that is, $M$ and $\tilde{M}$. This will allow the reader a clear picture of how the talked-about results actually transpire. An interpretation and further discussion will then be provided in the final section.

For an analytic illustration that optimizes clarity (and so that no symbolic computing is required), we will adopt the rotating BTZ black hole [24,25] as our system of choice. In its most literal interpretation, the BTZ black hole is a “toy model” of a three-dimensional black hole in a (necessarily) negatively curved spacetime. It mimics many of the features of higher-dimensional (and, presumably, more realistic) black holes, while providing much more tractable calculations. But, in spite of this simplicity, the BTZ black hole does play an important role in the near-horizon limit of many string-theoretic scenarios [27] and — by virtue of the AdS–CFT correspondence [28] — can be viewed as a holographic dual to an almost extremal Reissner–Nordstrom black hole [29].

II. RUPPEINER (BTZ) GEOMETRY

As a starting point for our analysis, it is useful to recall the metric for a rotating BTZ (three-dimensional anti-de Sitter) black hole [24]:

$$ds^2_{BTZ} = -N^\perp dt^2 + \frac{1}{N^\perp} dr^2 + r^2 (N^\phi dt + d\phi)^2,$$

where

$$N^\perp = \frac{r^2}{l^2} - 8GM + \frac{J^2}{4r^2}$$

and

$$N^\phi = -\frac{J}{2r^2}.$$  

Also, $G$ is the three-dimensional Newton’s constant (necessarily being of dimension length), $l$ is the radius of curvature (corresponding to a cosmological constant of $-1/l^2$), $M$ is the conserved mass and $J$ is the conserved angular momentum (which is physically equivalent to a U(1) Abelian charge 8).

Solving for the two zeros of $N^\perp$, which have significance as an outer and inner horizon $r_\pm$, one readily obtains 9

7For an earlier study on the BTZ black hole and thermodynamic geometries, see [26].

8We will assume, without loss of generality, that $J$ is non-negative.

9If $J = 8GMl$, then the two horizons coincide and we have the special case of an extremal black hole. It should be kept in mind that, for this case, the temperature is a vanishing quantity. Further note that $J > 8GMl$ would mean a naked singularity, and so this parameter range is not part of the relevant (thermodynamic) phase space.
\[ r_{\pm}^2 = 4GMI^2 \left[ 1 \pm \sqrt{1 - \left( \frac{J}{8GMI} \right)^2} \right]. \] (14)

It is also useful to note that
\[ M = \frac{r_+^2 + r_-^2}{8Gl^2} \quad \text{and} \quad J = \frac{2r_+r_-}{l}. \] (15)

Because this geometry is, locally, just anti-de Sitter space, there are many subtleties to the BTZ framework [25]. However, for our purposes, it is sufficient to remark that the spacetime can — with the appropriate identifications — be interpreted as a black hole with an outer horizon having the following thermodynamic properties: an entropy of \( S = A/4G = \pi r_+ / 2G \), a temperature of \( T = \kappa / 2\pi = \partial_r N(r = r_+) / 4\pi \) and a rotational velocity of \( \Omega = r_+/r_- \). To avoid obscuring the calculations with needless clutter, we will subsequently set \( 8G = 1 \) and \( l = 1 \). Along the same lines, Boltzmann’s constant (previously set to unity) will now be calibrated to \( k_B = 1 / 4\pi \); a choice that conveniently fixes \( S = r_+ \).

Our intent is to calculate the Ruppeiner scalar curvature for the two previously discussed choices of extensive variables. Firstly, \textbf{A.} the “conventional” set \( \{M, S, J\} \) which identifies the conserved mass \( M \) as the (black hole) internal energy and then \textbf{B.} the “unorthodox” set \( \{\tilde{M}, S, \Omega\} \) for which
\[ \tilde{M} = M - J\Omega \] (16)
can also be plausibly viewed, in accordance with our previous discussion, as the internal energy.

It turns out that, rather than perform a direct calculation, it is much easier to compute the Weinhold metric (8) and then use the conformal transformation of Eq.(10) to obtain the Ruppeiner metric (9). We will proceed accordingly.

\textbf{A.} \( \{M, S, J\} \)

Here, the first step is to find the functional form \( M = M(S, J) \). Given our choice of conventions (see above), it can readily be verified that [\textit{cf, Eq.}(15)]
\[ M = S^2 + \frac{J^2}{4S^2}. \] (17)

By taking the second-order partial derivatives of \( M \) with respect to \( S \) and \( J \), one obtains the Hessian of the mass and thus
\[ ds^2_W = \partial_S^2 M \, ds^2 + (\partial_J \partial_S + \partial_S \partial_J) \, M \, dS \, dJ + \partial_J^2 M \, dJ^2 \]
\[ = \left( 2 + \frac{3J^2}{2S^4} \right) \, ds^2 - \frac{J}{S^3} \, dS \, dJ + \frac{1}{2S^2} \, dJ^2. \] (18)

We can diagonalize the above metric by replacing \( J \) with \( x = J / S^2 \). This process yields
\[ ds^2_W = \left( 2 - \frac{x^2}{2} \right) \, ds^2 + \frac{S^2}{2} \, dx^2. \] (19)
Next, let us consider the temperature in terms of $S$ and $x$:

$$T = \partial_S M = 2S - \frac{J^2}{2S^3} = 2S - \frac{1}{2} Sx^2; \quad (20)$$

by which Eq.(10) prescribes the following for the Ruppeiner metric:

$$ds_R^2 = \frac{1}{S} dS^2 + \frac{S}{(4-x^2)} dx^2. \quad (21)$$

A further coordinate transformation to the variables $y = 2\sqrt{S}$ and $\omega$, such that $x = 2\sin \omega$, then gives us

$$ds_R^2 = dy^2 + y^2 d\omega^2. \quad (22)$$

This is easily recognizable as a flat Euclidean disc, meaning that — at least in this case — the Ruppeiner metric has an everywhere-vanishing curvature. $^{10}$

**B. \{ \tilde{M}, S, \Omega \}**

This time around, it is the functional form of the alternative energy, $\tilde{M} = \tilde{M}(S, \Omega)$, that is required. By way of Eqs.(15-16) and $\Omega = r_-/r_+ = J/2S^2$ (also keeping our conventions in mind), it follows that

$$\tilde{M} = S^2 \left( 1 - \Omega^2 \right). \quad (23)$$

A calculation of the relevant Hessian (i.e., $\tilde{M}$ varied by $S$ and $\Omega$) now leads to

$$\tilde{d}s_W^2 = 2 \left( 1 - \Omega^2 \right) dS^2 - 8S \Omega dS d\Omega - 2S^2 d\Omega^2. \quad (24)$$

As before, we will transform coordinates so as to diagonalize the metric. This can be achieved by eliminating $\Omega$ in favor of $X = \Omega S^2$, and one then obtains

$$\tilde{d}s_W^2 = 2 \left( \frac{X^2}{S^4} + 1 \right) dS^2 - \frac{2}{S^2} dX^2. \quad (25)$$

In terms of $x$ and $S$, the temperature is expressible as $^{11}$

$$T = \partial_S \tilde{M} = 2S \left( 1 - \Omega^2 \right) = \frac{2}{S^3} \left( S^4 - X^2 \right); \quad (26)$$

---

$^{10}$This outcome is already known [10]. We have, however, documented the calculation for completeness and illustrative purposes.

$^{11}$As a consistency check, one can readily confirm the equivalency of our various forms for $T$; namely, Eq.(20), Eq.(26) and [recalling that $k_B = 1/4\pi$ ] $\partial_r N^\perp(r = r_+)$. 

8
and so it follows that
\[ \hat{\bar{R}}_R = \frac{1}{S} \frac{(3X^2 + S^4) dS^2}{(S^4 - X^2)} - \frac{S}{(S^4 - X^2)} dX^2. \] (27)

Given a two-dimensional diagonal metric of the generic form \( ds^2 = -A d\theta^2 + B d\phi^2 \) (with \( A \) and \( B \) both positive), the scalar curvature can be computed by way of \( \frac{1}{12} R_{\text{R}} = \frac{1}{2A} \left[ \partial^2_a \ln B + (\partial_a \ln B)^2 - \partial_a \ln A \partial_a \ln B \right] - \frac{1}{2B} \left[ \partial^2_b \ln A + (\partial_b \ln A)^2 - \partial_b \ln A \partial_b \ln B \right]. \) (28)

Directly plugging in the Ruppeiner metric of Eq.(27), one finds (after some simplification) that
\[ \hat{\bar{R}}_R = 2 \frac{S^3 (S^4 - X^2)}{(S^4 + 3X^2)^2}. \] (29)

It is illustrative to re-express the above in terms of more familiar parameters, such as the horizon radii,
\[ \hat{\bar{R}}_R = 2 \frac{r_+ \left( r_+^2 - r_-^2 \right)}{\left( r_+^2 + 3r_-^2 \right)^2}. \] (30)

Unlike the prior (orthodox) case, we now have a decidedly non-flat Ruppeiner geometry. Moreover, the curvature is of just such a form that it provides us with a clear signal for each one of the anticipated pair of thermodynamically “special” points. Specifically, there is (first of all) the extremal point defined by the coincidence of the horizons or \( r_- = r_+ \). At precisely this point, we observe a vanishing curvature; quite sensibly, inasmuch as all of the degrees of freedom should be “frozen” at extremality [cf, Footnote 9].

Secondly, there happens to be another type of extremal point — or let us rather say "quasi-extremal" point — associated with the BTZ black hole. To elaborate, let us begin by considering the near-horizon limit of a near-extremal Reissner–Nordstrom black hole. After some degree of scrutiny, one finds that the resultant geometry asymptotes to that of a nearly massless BTZ black hole [29]. Then, by virtue of this duality, one could certainly anticipate a BTZ black hole to be thermodynamically special at the point of vanishing mass. This expectation is, indeed, realized by the above result; with the Ruppeiner curvature “blowing up” as this limit of zero mass is approached (equivalently, \( \hat{\bar{R}}_R \to \infty \) as \( r_\pm \to 0 \)).

---

12This equation can be obtained from the standard Riemannian-curvature formulae via the famed “brute-force” method.

13To be absolutely precise, the Reissner–Nordstrom geometry asymptotes (in this limiting case) to a dimensionally reduced version of the (near-massless) BTZ black hole [30,31].

14It might be thought that a divergence in this limit is trivial for reasons of dimensionality: \( [R] \sim \text{length}^{-2} \). The reader should, however, keep in mind that there is another length scale in the problem; namely, the curvature radius \( l \). Hence, it is a definitively non-trivial result that a massless black hole induces a divergence in the curvature.
Interestingly, one could argue that, from the perspective of the BTZ model, the Reissner–Nordstrom extremal point represents some type of phase transition [32].

III. INTERPRETATION

To summarize the prior section, we have explicitly demonstrated (for the rotating BTZ model) a remarkable result: Given a black hole as a thermodynamic system, how one chooses to identify the extensive variables can have a very dramatic effect on the induced Ruppeiner geometry. As anticipated, the “conventional” choice of \{M, S, J\} results in a completely flat metric with an everywhere-vanishing curvature. Conversely, the “unorthodox” set of variables \{\tilde{M}, S, \Omega\} induces a curved geometry that is undoubtedly reflective of the “special” points of the thermodynamic system.

Before attempting an interpretation, it is useful to bring the discussion back around to the more-publicized Reissner–Nordstrom case. Here, one similarly finds that

$$R_R = 0 \text{ everywhere} \quad (31)$$

for the orthodox choice of \{M, S, Q\}. Meanwhile, a set of variables based on the “alternative” (internal) energy \{\tilde{M}, S, \Phi\} will yield the following [4] (up to an irrelevant numerical factor):

$$\tilde{R}_R \sim \frac{r_+ - r_-}{r_+ (r_+ - 3r_-)^2} . \quad (32)$$

It should be noted that there is a zero in Eq.(32) at the extremal point \(r_- = r_+\) and a singularity at the so-called Davies point [22] \(r_+ = 3r_-\). The former is thermodynamically special because of a vanishing Davies point, while the latter corresponds to a divergent heat capacity (and, at least naively, a transition in phase). Let us again remind the reader that the status of the Davies point as a “legitimate” critical point is notably controversial [23]; this, in spite of the singular specific heat.

So what exactly do we have here? The former (conventional) choice of extensive variables tells us that the underlying system is trivially simple: whatever, the black hole analogue of “non-interacting” means. (According to Aman et al., this means the scale invariance of the Einstein–Maxwell action [12].) On the other hand, the latter (unorthodox) choice induces a curved geometry that is indicative of a highly interacting system. What is more, the pole in the scalar curvature legitimizes the status of the Davies point. [It is interesting to note that, when taken together, Eqs.(31) and (32) serve to both encapsulate and perpetuate the very controversy this formalism should be resolving.] At the very least, we appear to have a disturbing contradiction.

But let us not be so quick to rush to a despairing judgment. The prominent mantra of the introductory section is that the correct choice of energy (\(M\) or \(\tilde{M}\)) should be dependent on the particular situation at hand; that is, the choice should really depend on what type of physical question is being posed. If this is a valid assessment, then it would be fair to say that each of the above results could have validity within a certain context. On this basis, let us conjecture as follows: If we want to know about the underlying statistical system, then \(M\) should be selected as the energy, which leads us to the flat-curvature result of Eq.(31).
However, if we want to know about the thermodynamic phase space, then $\tilde{M}$ is the best choice for the energy, thus arriving at the non-trivial geometric picture of Eq.(32).

To elaborate, it is difficult to say anything definitive about the statistical framework underlying the thermodynamics of a Reissner--Nordstrom black hole. Unlike asymptotically anti-de Sitter spacetimes, there is no convenient duality (vis-a-vis, the AdS–CFT correspondence [28]) to “hang one’s hat on”. Given that a pure Reissner–Nordstrom black hole is an idealized fiction (the uncertainty principle forbids an exactly vanishing angular momentum and, depending on the exact origin of the dark energy, possibly a vanishing cosmological constant as well), it is quite feasible that the statistical origin of its thermodynamics is either trivial or even non-existent. Perhaps, this is precisely what Eq.(31) is trying to tell us!

But, irrespective of the statistical mechanics, it is clear that eventful things take place in the thermodynamic phase space of a Reissner–Nordstrom black hole. The temperature can certainly vanish (at the extremal point), the heat capacity can diverge (at the Davies point) and both of these quantities (as well as many others) can vary greatly as the extensive parameters are being tuned. We would suggest these to be the very type of features that Eq.(32) is telling us about!

The inquiring reader might now ask: does this interpretation mean the Davies point is a legitimate phase transition point after all? In our opinion, the answer is a qualified no. We reiterate that, standing on its own, the Reissner-Nordstrom black hole is a toy model with no real validity as a thermodynamic system. After all, the temperature is calculated on the basis of a quantum process (i.e., the Hawking Radiation effect [3]), and, once quantum mechanics enters into the discussion, then we must allow for fluctuations in all other parameters — including the angular momentum and anything else of pertinence (such as any relevant “quantum hair”). Meaning, in a truly realistic system, it is quite feasible that the Davies point is elevated to a phase-transition point; assuming a point of divergent heat capacity survives at all — this is certainly not guaranteed.

Let us now briefly comment upon some other proposals for “resolving” the flat Ruppeiner geometry for the Reissner–Nordstrom class of black holes. Let us first consider the work of Mirza and Zamani-Nasab [18]. These authors regulate the Reissner–Nordstrom calculation by starting with a “fully loaded” Kerr–Newman–AdS black hole 15 and then return to the Reissner–Nordstrom case only at the very end of the calculation. In this way, they manage to obtain a curved Ruppeiner metric in what can be argued as being a more physically motivated calculation. Also with the intent of steering in the direction of realism, Sarkar et al [17] consider some quantum-inspired modifications to the BTZ model. For instance, these authors find that the inclusion of thermal fluctuations does indeed induce a curved Ruppeiner metric. Both of these findings are certainly copacetic with some of the comments made above.

Very recently, Alvarez et al [19] 16 more directly address the $M$ versus $\tilde{M}$ debate. They

15That is, a black hole inside of an asymptotically (negatively) curved spacetime and having, in general, a non-vanishing charge and momentum. The Reissner–Nordstrom black hole can, in some sense, be regarded as a special case of this more general model.

16So recent, in fact, that this paper was released on to the archives while the current paper was
derive a Ruppeiner-like metric that is based on the principle of invariance under canonical transformations. (Meaning, that it is independent of the choice made for the extensive variables.) Their final outcome for the Reissner–Nordstrom black hole is qualitatively very similar to that of Eq.(32) and certainly nothing like that of Eq.(31). We would, however, question whether their formalism can be viewed as a replacement — as they strongly imply — as opposed to an alternative to the Ruppeiner metric. Moreover, their result — if taken in isolation — would miss out completely on whatever lesson is to be learned from the flat-space metric of Eq.(31). Like detectives assembling information during a crime investigation, we suggest that any pertinent resource should be utilized before reaching a conclusion.

Assuming there is some degree of validity to our interpretations and overview, the next logical step would be to understand this critical distinction that arises between the two choices of extensive variables. That is, why does the conventional choice and the unorthodox choice lead, respectively, to Eq.(31) and Eq.(32), and not vice versa? Or to put it yet another way, why is the conserved mass $M$ more sensitive to the statistical mechanics and the alternative energy $\tilde{M}$, to the thermodynamics of the black hole system? We hope to address this issue in due course.

\footnote{To quote the authors on their own findings: “This result finishes the controversy regarding the application of geometric structures in black hole thermodynamics” \cite{19}.}
ACKNOWLEDGMENTS

Research is financially supported by the University of Seoul. The author thanks Yun Soo Myung for his inputs and valued discussions, and CQUeST at Sogang University for their hospitality.
REFERENCES

[1] S.W. Hawking, Comm. Math. Phys. 25, 152 (1972); J.M. Bardeen, B. Carter and S.W. Hawking, Comm. Math. Phys. 31, 161 (1973).
[2] J.D. Bekenstein, Lett. Nuovo. Cim. 4, 737 (1972); Phys. Rev. D 7, 2333 (1973); Phys. Rev. D 9, 3292 (1974).
[3] S.W. Hawking, Nature 248, 30 (1974); Commun. Math. Phys. 43, 199 (1975).
[4] J. Shen, R.-G. Cai, B. Wang, R-K. Su, Int. J. Mod. Phys. A 22, 11 (2007) [arXiv:gr-qc/0512035].
[5] F. Weinhold, J. Chem. Phys. 63, 2479 (1975).
[6] G. Ruppeiner, Phys. Rev. A 20, 1608 (1979).
[7] R. Mrugala, Physica A 12, 631 (1984).
[8] P. Salamon, J.D. Nulton and E. Ihrig, J. Chem. Phys. 80, 436 (1984).
[9] G. Ruppeiner, Rev. Mod. Phys. 67, 605 (1995).
[10] J. Aman, I. Bengtsson and N. Pidokrajt, Gen. Rel. Grav. 35, 1733 (2003) [arXiv:gr-qc/0304015].
[11] J. Aman, and N. Pidokrajt, Phys. Rev. D 73, 024017 (2006) [arXiv:hep-th/0510139].
[12] J. Aman, I. Bengtsson and N. Pidokrajt, Gen. Rel. Grav. 38, 1305 (2006) [arXiv:gr-qc/0601119].
[13] J. Aman, J. Bedford, D. Grumiller, N. Pidokrajt and J. Ward, J. Phys.: Conf. Ser. 66, 010207 (2007) [arXiv:gr-qc/0611119].
[14] J. Aman, N. Pidokrajt and J. Ward, “On Geometro-thermodynamics of Dilaton Black Holes”, arXiv:0711.2201 (2007).
[15] J. Aman and N. Pidokrajt, “Ruppeiner Geometry of Black Hole Thermodynamics”, arXiv:0801.0016 (2008).
[16] H. Quevedo, J. Math. Phys. 48, 013506 (2007) [arXiv:physics/0604164]; “Geometrothermodynamics of black holes”, arXiv:0704.3102 (2007).
[17] T. Sarkar, G. Sengupta and B. N. Tiwari, JHEP 0611, 015 (2006) [arXiv:hep-th/0606084].
[18] B. Mirza and M. Zamani-Nasab, JHEP 06, 059 (2007) [arXiv:0706.3450].
[19] J.L. Alvarez, H. Quevedo and A. Sanchez, “Unified geometric description of black hole thermodynamics”, arXiv:0801.2279 (2008).
[20] A.D. Helfer, Rept. Prog. Phys. 66, 943 (2003) [arXiv:gr-qc/0304042].
[21] See, for instance, A.W. Peet, “TASI lectures on black holes in string theory”, arXiv:hep-th/0008241 (2000).
[22] P.C.W. Davies, Proc. Roy. Soc. A 353, 499 (1977).
[23] D. Tranah and P.T. Landsberg, Collective Phenomena 3, 73 (1980).
[24] M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992) [arXiv:hep-th/9204099].
[25] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D 48, 1506 (1993) [arXiv:hep-th/9302012].
[26] R.-G. Cai and J.-H. Cho, Phys. Rev. D 60, 067502 (1999) [arXiv:hep-th/9803261].
[27] S. Hyun, J. Korean Phys. Soc. 33, 532 (1998) [arXiv:hep-th/9704005].
[28] J.M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) [arXiv:hep-th/9711200].
[29] J. Navarro-Salas and P. Navarro, Nucl. Phys. B 579, 250 (2000) [arXiv:hep-th/9910076].
[30] A. Achucarro and M.E. Ortiz, Phys. Rev. D 48, 3600 (1993) [arXiv:hep-th/9304068].
[31] Y.S. Myung, Y.-W. Kim and Y.-J. Park, “Thermodynamic duality between RN black hole and 2D dilaton gravity”, arXiv:0707.3314 and to appear in MPLA (2007).
[32] Y.S. Myung, Phys. Lett. B 638, 515 (2006) [arXiv:gr-qc/0603051].