Dimensional analysis and electric potential due to a uniformly charged sheet

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Abstract
Dimensional analysis, superposition principle, and continuity of electric potential are used to study the electric potential of a uniformly charged square sheet on its plane. It is shown that knowing the electric potential on the diagonal and inside the square sheet is equivalent to knowing it everywhere on the plane of the square sheet. The behaviour of the electric potential near the centre of the square is obtained. Then an exact solution for the electric potential at any point on the plane of the square sheet is obtained. This result is used to calculate the electric potential of a right triangular sheet on its plane which can be used to find the electric potential at any point on the plane of any uniformly charged polygon sheet.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Physics problems are studied using analytical techniques, approximation methods, and simulations. Although it is interesting to have a complete solution for any problem, there are many problems in physics which cannot be solved analytically. To solve such problems, one may use other methods such as numerical techniques. For problems which cannot be handled analytically, dimensional analysis may also be a useful technique to enlighten the subject, and may give a better understanding of the subject. This technique, despite its simplicity, may give the opportunity to find some information about the solution, which of course may be incomplete. There are some texts and articles on dimensional analysis, e.g., [1–6]. Dimensional analysis is used in many different areas of physics. In [6], it is shown that many of the commonly considered applications of weak turbulence possess the incomplete self-similarity property, which can be exploited to obtain core results using a simple dimensional argument. In [7], dimensional analysis is used to study the resistance force of
the fluid that occurs when a body moves through it and the speed of propagation of waves on water.

If one uses dimensional analysis together with some other techniques, it could be more powerful. One of the standard problems in elementary electricity and magnetism is the electric field due to an infinite uniformly charged plane. A more physical problem is the electric field of a charged plane of finite size.

By direct integration, a closed form for the electrostatic potential at any arbitrary positions of a uniformly charged cube was determined [8]. See also [9] and [10]. As a two-dimensional electrostatic problem, the electrostatic potential of a uniformly charged square was also calculated in [8]. There are many interesting problems involving the electrostatics of cubic geometries, e.g., the electrostatic interaction energy of two coinciding homogeneous cubic charge distributions was considered in [11].

Here we consider the 3D potential. The electric potential of the charged sheet at any point can be written as a double integral. But we are interested in a different method. Our method is based on using dimensional analysis together with the superposition principle which comes from the linearity of Maxwell equations [12], and symmetries of the system under scaling. In the first part of this paper, elementary mathematical techniques are used. Using dimensional analysis and the superposition principle, the electric potential due to a uniformly charged square sheet on its plane is studied. In section 2, using dimensional analysis and the superposition principle, an identity is obtained between the electric potential at the centre and at the corner of a uniformly charged square sheet. In section 3, some identities are obtained for the electric potential of a uniformly charged square sheet at some different points. Using these identities, one can obtain electric potential everywhere on the plane of the square sheet in terms of the electric potential on the diagonal and inside the square sheet. In section 4, using the behaviour of the electric potential at far distances, the electric potential at the points near the centre of the square sheet is obtained. In section 5, using the behaviour of the electric potential at large distances, an equation for the electric potential is obtained. Solving this equation, the electric potential on the diagonal of the square sheet is obtained from which the electric potential at any arbitrary point on the plane of the square sheet is also obtained. It should be noted that knowing the electric potential at any arbitrary point on the plane of the square sheet is equivalent to knowing the electric field on the plane of the charged sheet. In section 6, the electric potential at any point on the plane of a uniformly charged right triangular sheet is obtained. This helps us to find the electric potential on the plane of any uniformly charged sheet polygon.

Our technique could be useful for both undergraduate and graduate physics students. The first four sections need only a background in general physics, and the next two sections need a bit more mathematics and thus are accessible to senior students who are familiar with solving the differential equation. I believe that the method used in this paper may be helpful for better understanding of dimensional analysis.

2. Dimensional analysis

Let us consider a uniformly charged square sheet with length $a$. We are considering the 3D potential, but only on the $z = 0$ plane, the plane of the charged square sheet, and cgs units are used. For a given localized charge distribution, the electric potential can in principle be obtained uniquely, provided that the electric potential at a reference point is also given. Let us take the potential at infinity to be equal to zero. Then the electric potential of a uniformly charged square at the point $(x, y)$, denoted by $\Phi(x, y)$, depends on the surface charge density...
Figure 1. Using dimensional analysis together with the superposition principle, it can be shown that the electric potential at the centre of a square sheet is twice the electric potential at the corner of the uniformly charged square sheet.

\[ \sigma, \text{ the length of the square } a, \text{ and the coordinates } x \text{ and } y. \]  From these parameters, three dimensionless quantities can be constructed:

\[ \frac{\Phi}{\sigma a}, \frac{x}{a}, \frac{y}{a}. \]  (1)

Using dimensional analysis, there is a relation between these dimensionless quantities. The relation between these dimensionless quantities should be such as

\[ \frac{\Phi(x, y)}{\sigma a} = F\left(\frac{x}{a}, \frac{y}{a}\right), \]  (2)

where \( F \) is an unknown function. The electric potential at the origin \( O(x = 0, y = 0) \) is

\[ \Phi_O = C_1 \sigma a, \]  (3)

where \( C_1 = F(0, 0) \) is a constant. It should be noted that \( F \) is scale invariant, which means that if the length of the square sheet scales by a factor \( \lambda \), and the coordinates of the point of observations also scale with the same factor, \( F \) remains unaltered. So if one scales the square sheet with a factor of 2, \( \sigma \) remaining unaltered, the electric potential at the centre of the scaled square sheet becomes twice the electric potential at the centre of the unscaled square sheet. Dimensional analysis cannot tell us anymore. The electric potential at the point \( A(x = \frac{a}{2}, y = \frac{a}{2}) \) is

\[ \Phi\left(x = \frac{a}{2}, y = \frac{a}{2}\right) = C_2 \sigma a, \]  (4)

where \( C_2 = F(\frac{1}{2}, \frac{1}{2}) \) is another constant. Because of the linearity of Maxwell equations, the electric potential due to a group of charges can be obtained by using the superposition principle. The superposition principle may help us to go further. Now, let us consider a uniformly charged square with the same charge density \( \sigma \), whose length is 2\( a \). Then using (3), the electric potential at the origin of the scaled square is

\[ \Phi_{O'} = 2a \times \sigma F(0, 0) = 2C_1 \sigma a, \]  (5)

where \( \Phi_{O'} \) is the electric potential at the point \( O' \), the centre of the scaled square. But the point \( O' \) is the corner of four squares with the lengths \( a \). See figure 1.

Using the principle of superposition,

\[ \Phi_{O'} = 4C_2 \sigma a. \]  (6)

Comparing (5) and (6), one arrives at \( C_1 = 2C_2 \), which means that the electric potential at the centre of the square is twice the electric potential at the corner of the square.
\begin{align*}
(n + m)\varphi_{n,m} &= \frac{n \varphi_{1,1}}{2} + \frac{m \varphi_{1,1}}{2} + 2\psi_{n,m}

\Phi(n, m) &= (\psi_{n,m} - \psi_{n-1,m}) - (\psi_{n,m-1} - \psi_{n-1,m-1})
\end{align*}

3. Some identities on the electric potential of a uniformly charged square sheet

Consider a uniformly charged square sheet with unit length, and let us choose a point on one of its diagonals and inside the square sheet, which divides the diagonal into two parts. Let the ratio of these two parts be a rational number \( \frac{n}{m} \). We denote the electric potential at this point by \( \varphi_{n,m} \). Then the electric potential at the centre of the square is \( \varphi_{1,1} \). It is obvious that \( \varphi_{1,1} = \varphi_{n,n} \) and \( \varphi_{n,m} = \varphi_{m,n,m} \) for any positive integer \( \lambda \). By symmetry, it is also obvious that \( \varphi_{n,m} = \varphi_{m,n} \). In this section, we obtain identities between electric potential at different points. Investigating the behaviour of \( \lim_{n \to \infty} \varphi_{n,n-1} \) gives us electric potential at the points on the diagonal and near the centre of the square.

Let us consider a uniformly charged square with length \((n + m)\). As it is \((m + n)\) times a square with unit length with the same charge density, the electric potential at its centre is \((n + m)\varphi_{1,1}\), and at a point \(M\) on its diagonal which divides it into the ratio of \(n\) to \(m\) the potential is \((n + m)\varphi_{n,m}\). See figure 2. Using the superposition principle, the electric potential at the point \(M\) can be written as the scalar sum of

1. the electric potential of a uniformly charged square sheet with length \(n\) at its corner, which is \(\frac{n}{2} \varphi_{1,1}\);
2. the electric potential of a uniformly charged square sheet with length \(m\) at its corner, which is \(\frac{m}{2} \varphi_{1,1}\);
3. the electric potential of two uniformly charged rectangular sheets with lengths \(m\) and \(n\) at their corners, where each of them is denoted by \(\psi_{n,m}\).
Then one arrives at
\[(n + m)\varphi_{n,m} = \frac{1}{2}(n\varphi_{1,1} + m\varphi_{1,1} + \varphi_{n,m}),\]
\[\Rightarrow \varphi_{n,m} = \frac{1}{2}(\frac{n + m}{2})\varphi_{n,m} - \frac{\varphi_{1,1}}{2}. \quad (7)\]

We are now ready to write the electric potential due to a uniformly charged square with unit length at any point on a lattice with the same length on the plane of the square sheet. Take the origin at the corner of the square. Let us denote the electric potential due to a square sheet with unit length at a point with the coordinates \((n, m)\) (for \(n, m > 1\)), by \(\Phi_{1}(n, m)\). Then, using the superposition principle, \(\Phi(n, m)\) is (see figure 2)
\[\Phi(n, m) = (\varphi_{n,m} - \varphi_{n-1,m}) - (\varphi_{n,m-1} - \varphi_{n-1,m-1}). \]
\[\Phi(n, m) = (n + m)\varphi_{n,m} - (n + m - 1)\varphi_{n-1,m} + \varphi_{n,m-1} + (n + m - 2)\varphi_{n-1,m-1}. \quad (8)\]

Now, we want to show that the electric potential at any arbitrary point with rational coordinates can also be written in terms of \(\varphi_{s}\), the electric potential on the diagonal of the square sheet. First consider the points in the square sheet. Let us consider a square with length \(k\), and let \(m\) and \(n\) be two integers less than \(k\). See figure 3.

It can be divided into four rectangular sheets of the sizes \(m \times n\), \(n \times (k - m)\), \(m \times (k - n)\), and \((k - m) \times (k - n)\). The electric potential at the point \((n, m)\) of a uniformly charged square sheet with length \(k\), denoted by \(\Phi(n, m)\), is the sum of the electric potential at the corners of these four rectangular uniformly charged sheets:
\[\Phi(n, m) = (\varphi_{n,m} + \varphi_{n,k-m} + \varphi_{k-n,m} + \varphi_{k-m,k-n}) \]
\[= (k - n + m)\varphi_{k-n,m} + \frac{(n + m)}{2}\varphi_{n,m} + \frac{(k - m + n)}{2}\varphi_{k-m,n} + \frac{2k - n - m}{2}\varphi_{k-n,k-m} - k\varphi_{1,1}. \quad (9)\]
So the electric potential at the point with the coordinate \((\frac{n}{k}, \frac{m}{k})\) in a square with unit length \((m, n < k)\) is
\[
\Phi(\frac{n}{k}, \frac{m}{k}) = \frac{1}{2k} ((k - n + m)\psi_{k-n,m} + (k - m + n)\psi_{k,m,n} \\
+ (2k - n - m)\psi_{k-n,k-m} + (n + m)\psi_{n,m} - 2k\psi_{1,1}).
\] (10)
This result can be generalized to any point on the plane of a square sheet of unit length (inside or outside it). For this case, \(m\) or \(n\) could be greater than \(k\). Then the electric potential at any point with the coordinate \((\frac{n}{k}, \frac{m}{k})\) \((m, n > 0)\) is
\[
\Phi(\frac{n}{k}, \frac{m}{k}) = \frac{1}{k} \left[ \psi_{n,m} + \text{sgn}[k - m] \psi_{|k-m|,n} + \text{sgn}[k - n] \psi_{|k-n|,m} \\
+ \text{sgn}[(k - n)(k - m)]\psi_{|k-m|,|k-n|} \right]
\] (11)
where \(\text{sgn}[x]\) is the sign function defined through
\[
\text{sgn}[x] := \begin{cases} +1 & x > 0 \\ -1 & x < 0. \end{cases}
\] (12)
Because of the symmetry, the electric potential for the points with negative \(m\) or \(n\) can be obtained easily. It should be noted that although this solution is obtained for the points whose coordinates are rational numbers, because of the continuity of the electric potential, it will also be true for points with real coordinates. So if the electric potential of the square sheet at its diagonal is known, the electric potential at any other point on the plane of the sheet can be obtained through the above-mentioned identities.

4. Electric potential near the centre of the square

The electric potential at the centre of a uniformly charged square with unit length, \(\Phi\left(\frac{1}{2}, \frac{1}{2}\right)\), can be calculated analytically:
\[
\Phi\left(\frac{1}{2}, \frac{1}{2}\right) = \int_0^1 \int_0^1 \frac{\sigma \ dx \ dy}{\sqrt{(x - 1/2)^2 + (y - 1/2)^2}} = 4\sigma \sinh^{-1}(1).
\] (13)
So \(\psi_{1,1} = 4\sigma \sinh^{-1}(1)\), and the electric potential at the corner of the square sheet is \(2\sigma \sinh^{-1}(1)\). Let us investigate the electric potential near the centre of the square. The electric potential due to a uniformly charged square sheet at the point \((n, n)\) (for far distances, \(n \gg 1\)) is \(\frac{\sigma}{n\sqrt{2}}\). So using equation (8), for \(n = m\), one arrives at
\[
\Phi(n, n) = (2n - 1)\psi_{1,1} - (2n - 1)\psi_{n-1,n} \\
\approx \frac{\sigma}{n\sqrt{2}}.
\] (14)
Here we have used \(\psi_{n,n} = \psi_{n-1,n-1} = \psi_{1,1}\) and \(\psi_{n,n-1} = \psi_{n-1,n}\). The quantity \(\psi_{n-1,n}\) for large \(n\) is the electric potential at a point \(A\) near the centre of the square and on its diagonal
\[
\psi_{n-1,n} \approx \psi_{1,1} = \frac{\sigma}{2\sqrt{2} n^2} \\
\Rightarrow \quad \Phi\left(\frac{1}{2} + \epsilon, \frac{1}{2} + \epsilon\right) \approx 4\sigma \sinh^{-1}(1) - 4\sigma \epsilon^2 \sqrt{2}.
\] (15)
The point \(A\) is at a distance \(\epsilon \sqrt{2}\) from the centre of the square sheet. The electric potential at a point on the plane of the square sheet and near its centre can be obtained using the Taylor
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expansion
\[
\Phi \left( \frac{1}{2} + x', \frac{x}{2} + y' \right) = \Phi \left( \frac{1}{2} + \frac{1}{2} \right) + x' \frac{\partial \Phi}{\partial x'} + y' \frac{\partial \Phi}{\partial x'} + \frac{\partial^2 \Phi}{\partial y'^2} + \frac{\partial^2 \Phi}{\partial y'^2} + \ldots
\]
\[
= \Phi \left( \frac{1}{2} \right) + \alpha x'^2 + \beta y'^2 + \gamma x'y' + \ldots. \tag{16}
\]

In the above equation, we have used the fact that the electric field at the centre of the square sheet is equal to zero. \(\alpha, \beta, \text{ and } \gamma\) are three constant parameters which are related to the second derivatives of the electric potential with respect to \(x'\) and \(y'\). Because of the symmetry of the problem, the points \((x' + x', \frac{1}{2} + y')\) and \((x' + x', \frac{1}{2} - y')\) have the same electric potential, so \(\gamma\) should be equal to zero. The points \((x' + x', \frac{1}{2} + y')\) and \((x' + y', \frac{1}{2} + x')\) also have the same electric potential, so \(\alpha = \beta\). So
\[
\Phi \left( \frac{1}{2} + x', \frac{1}{2} + y' \right) = \Phi \left( \frac{1}{2}, \frac{1}{2} \right) + \alpha(x'^2 + y'^2) + \ldots. \tag{17}
\]

Using (15), \(\alpha\) should be \(-2\sigma \sqrt{2}\). In summary, the electric potential near the centre of the square has azimuthal symmetry. So at a point distance \(\epsilon\) from the centre of the square sheet, the electric potential is given by
\[
\Phi \approx 4\sigma \sinh^{-1}(1) - 2\epsilon \sqrt{2}. \tag{18}
\]

5. Exact solution for a uniformly charged square sheet

Here, we want to obtain an exact value of electric potential at any point on the plane of the square sheet. It should be noted that knowing the electric potential on the plane of the charged sheet is equivalent to knowing the tangential component of the electric field on the plane of the charged sheet. The normal component of the electric field in the charged sheet is \(\sigma/2\), and out of the charged sheet, it is pure tangential. The electric potential on the diagonal of the square sheet can be obtained. Let us consider equation (8) for the points \(n = N \cos \theta\) and \(m = N \sin \theta\). For large \(N\) or large distances, the electric potential behaves as \(\frac{1}{r}\). On the right-hand side of (8), there are four terms. Let us consider each term separately, for large \(N\), and up to \(\frac{1}{N}\) expansion
\[
\frac{n + m}{2} \varphi_{n,m} = \frac{N}{2} (\sin \theta + \cos \theta) \varphi_{N \cos \theta, N \sin \theta}
\]
\[
= \frac{N}{2} (\sin \theta + \cos \theta) \varphi_{1, \tan \theta}, \tag{19}
\]
\[
\frac{n + m - 1}{2} \varphi_{n,m-1} = \frac{N}{2} (\sin \theta + \cos \theta - 1) \varphi_{N \cos \theta - 1, N \sin \theta - 1}
\]
\[
= \frac{N}{2} (\sin \theta + \cos \theta - 1) \varphi_{1, \tan \theta - \frac{1}{N}}
\]
\[
= \frac{N}{2} (\sin \theta + \cos \theta - 1) \left[ \varphi_{1, \tan \theta} - \frac{\varphi'}{N \cos \theta} + \frac{\varphi''}{2N^2 \cos^2 \theta} + \cdots \right]. \tag{20}
\]
\[
\frac{n + m - 1}{2} \varphi_{n-1,m} = \frac{N}{2} (\sin \theta + \cos \theta - 1) \varphi_{N \cos \theta - 1, N \sin \theta}
\]
\[
= \frac{N}{2} (\sin \theta + \cos \theta - 1) \varphi_{1, \tan \theta + \frac{N \tan \theta}{N \cos \theta} + \cdots}
\]
\[
= \frac{N}{2} (\sin \theta + \cos \theta - 1) \left[ \varphi_{1, \tan \theta} + \tan \theta \frac{\varphi'}{N \cos \theta} + \frac{\tan^2 \theta \varphi''}{2N^2 \cos^2 \theta}ight.
\]
\[
+ \left. \frac{\tan \theta \varphi'}{N^2 \cos^2 \theta} + \cdots \right]. \tag{21}
\]
where prime means differentiation with respect to \( \tan \theta \), and \( \phi_{1, \tan \theta} \) stands for the potential at a point on the diagonal of the square sheet which divides it in the ratio \( 1 : \tan \theta \). Gathering all these together up to the term \( \frac{1}{N} \), one arrives at

\[
\frac{\tan \theta}{\cos \theta} \phi'_{1,u} + \frac{\tan \theta (\tan \theta + 1)}{2 \cos \theta} \phi''_{1,u} = -\sigma. \tag{23}
\]

Defining \( u := \tan \theta \), the above equation can be recast as

\[
\frac{u(u + 1)}{2} \phi''_{1,u} + u \phi'_{1,u} = -\frac{\sigma}{\sqrt{u^2 + 1}} \tag{24}
\]

or

\[
\frac{d}{du} \left( \frac{(u + 1)^2}{2} \phi'_{1,u} \right) = -\frac{\sigma (u + 1)}{u \sqrt{u^2 + 1}} \tag{25}
\]

which can be integrated to

\[
\phi'_{1,u} = \frac{2 \sigma}{(u + 1)^2} \left\{ \sinh^{-1}(u) - \sinh^{-1} \left( \frac{1}{u} \right) + C \right\}, \tag{26}
\]

where \( C \) is constant. Noting that \( \phi' \) is proportional to the electric field, and the electric field at the centre of the square sheet (where \( u = 1 \) vanishes, gives \( C = 0 \). Because of the symmetry, the electric field on the diagonal and on the plane of the square sheet has no component perpendicular to the diagonal of the square sheet. Its component along the diagonal is proportional to \( \phi' \), and its component in the \( z \) direction is \( \sigma / 2 \). Then the electric field on the diagonal of the square sheet is exactly obtained. Now, the electric potential on the diagonal of the square sheet can be obtained through an integration,

\[
\phi_{1,u} = \int du \frac{2 \sigma}{(u + 1)^2} \left\{ \sinh^{-1}(u) - \sinh^{-1} \left( \frac{1}{u} \right) \right\}. \tag{27}
\]

Using integration by parts, one arrives at

\[
\phi_{1,u} = \frac{2 \sigma}{u + 1} \left\{ \sinh^{-1}(u) - \sinh^{-1} \left( \frac{1}{u} \right) \right\} - 2 \sigma \int \frac{du}{u \sqrt{u^2 + 1}}
\]

\[
= 2 \sigma \left\{ \sinh^{-1}(u) \left( \frac{u + 1}{u + 1 + \frac{1}{u}} \right) \right\} + C', \tag{28}
\]

where \( C' \) can be determined using the electric potential at the centre of the square sheet, \( \phi_{1,1} = 4 \sigma \sinh^{-1}(1) \). Then the electric potential is

\[
\phi_{1,u} = 2 \sigma \left\{ \sinh^{-1}(u) \left( \frac{u + 1}{u + 1 + \frac{1}{u}} \right) \right\} + 2 \sigma \sinh^{-1}(1). \tag{29}
\]
As is seen, the electric potential at the corner ($u \to \infty$ or $u \to 0$) is $2\sigma \sinh^{-1}(1)$, which is consistent with our previous result. For the point $(x, x)$, $u = \frac{x}{1-x}$; then the electric potential at any point on the diagonal inside the square sheet and with the coordinates $(x, x)$ is

$$\Phi(x, x) = 2\sigma \left\{ (1-x) \sinh^{-1}\left( \frac{x}{1-x} \right) + x \sinh^{-1}\left( \frac{1-x}{x} \right) \right\} + 2\sigma \sinh^{-1}(1).$$

(30)

Using (10), (29), and putting $x := \frac{n}{2}, y := \frac{m}{2}$, the electric potential at any point, $(x, y)$, in the square sheet ($0 < x, y < 1$), can be written as

$$\Phi(x, y) = \frac{1}{2}[(1-x+y)\varphi_{1-x,y} + (1+x-y)\varphi_{1-y,x} + (2-x-y)\varphi_{1-x,1-y}$$

$$+ (x+y)\varphi_{x,y} - 2\varphi_{1,1}]$$

$$\Phi(x, y) = \sigma \left\{ (1-x) \sinh^{-1}\left( \frac{y}{1-y} \right) + y \sinh^{-1}\left( \frac{1-y}{1-x} \right) \right\}$$

$$+ (1-y) \sinh^{-1}\left( \frac{x}{1-y} \right) + x \sinh^{-1}\left( \frac{1-x}{1-y} \right)$$

$$+ (1-y) \sinh^{-1}\left( \frac{1-x}{1-y} \right) + (1-x) \sinh^{-1}\left( \frac{1-y}{1-x} \right)$$

$$+ x \sinh^{-1}\left( \frac{y}{x} \right) + y \sinh^{-1}\left( \frac{x}{y} \right) \right\}. \quad (31)$$

Using (11), the electric potential at any arbitrary point, $(x, y)$ $(x, y > 0)$, on the plane of the square sheet, is given by

$$\Phi(x, y) = \sigma \left\{ |1-x| \sinh^{-1}\left( \frac{y}{1-x} \right) + y \sinh^{-1}\left( \frac{1-x}{y} \right) \right\}$$

$$+ |1-y| \sinh^{-1}\left( \frac{x}{1-y} \right) + x \sinh^{-1}\left( \frac{1-y}{1-x} \right)$$

$$+ |1-y| \sinh^{-1}\left( \frac{1-x}{1-y} \right) + |1-x| \sinh^{-1}\left( \frac{1-y}{1-x} \right)$$

$$+ x \sinh^{-1}\left( \frac{y}{x} \right) + y \sinh^{-1}\left( \frac{x}{y} \right) \right\}. \quad (32)$$

Because of the symmetry, the electric potential for the points with negative $x$ or $y$ can be obtained easily.

6. Exact solution for a uniformly charged polygon

Using the results for the square sheet, we will obtain some interesting results for any uniformly charged polygon. Let us first consider a uniformly charged right triangular sheet. It is seen from (32) that the electric potential at the corners of a uniformly charged rectangular sheet of lengths $a$ and $b$ is

$$\Phi_{A}(a,b) = \sigma a \sinh^{-1}\left( \frac{b}{a} \right) + b \sinh^{-1}\left( \frac{a}{b} \right).$$

(33)

Using (33), the superposition principle and dimensional analysis, the electric potential at the corners of a uniformly charged right triangular sheet of length $a$ is

$$\Phi_{(a,b)}^{A} = \sigma a \sinh^{-1}\left( \frac{b}{a} \right) + \sigma a F\left( \frac{b}{a} \right).$$

(34)
Figure 4. Electric potential at the corner of a uniformly charged right triangular sheet.

where $\Phi_{A(a,b)}$ is the electric potential at the point $A$ of a uniformly charged right triangular sheet of lengths $a$ and $b$ (see figure 4), and $F$ is an unknown function.

Using the result for the electric potential at the corner of a uniformly charged square sheet of lengths 1, it can be easily shown that $F(1) = 0$. Now let us consider a right triangle with lengths $a$ and $b(1+\epsilon)$. The electric potential at the point $A'$ is the addition of potential due to the triangle with lengths $a$ and $b$ and the electric potential due to a sector. The electric potential due to a sector of a circle with radius $R$ and the angle $\delta \theta$ at the centre of the circle is $\sigma R \delta \theta$. So the electric potential at the point $A'$, the corner of a right triangle with lengths $a$ and $b(1+\epsilon)$ up to first order in $\epsilon$, is

$$\Phi_{A'(a,b(1+\epsilon))} \approx \sigma a \sinh^{-1} \left( \frac{b}{a} \right) + \sigma a F \left( \frac{b}{a} \right) + \epsilon \sigma \left( \frac{ab}{\sqrt{a^2 + b^2}} \right).$$

(35)

Using (34), the electric potential at the point $A'$, the corner of the right triangle, is

$$\Phi_{A'(a,b(1+\epsilon))} = \sigma a \sinh^{-1} \left( \frac{b(1+\epsilon)}{a} \right) + \sigma a F \left( \frac{b(1+\epsilon)}{a} \right),$$

$$\approx \sigma a \sinh^{-1} \left( \frac{b}{a} \right) + \epsilon \sigma \frac{ab}{\sqrt{a^2 + b^2}} + \sigma a F \left( \frac{b}{a} \right) + \epsilon \sigma b F' \left( \frac{b}{a} \right).$$

(36)

where $F'$ stands for the differentiation of $F$ with respect to its argument. Comparing (35) and (36) gives $F' = 0$ and $F$ is a constant. But $F(1) = 0$, so $F$ should be zero. Then the electric potential at the corner of a right triangle is $\sigma a \sinh^{-1}(\frac{b}{a})$ (see figure 4). The electric potential at the vertices of any arbitrary triangle can be obtained easily. See figure 5. The electric potential at the point $A$ is the superposition of the electric potential of two right triangular sheets:

$$\Phi_A = \sigma h [\sinh^{-1}(\cot \theta_1) + \sinh^{-1}(\cot \theta_2)]$$

$$= \sigma h \ln \left( \cot \left( \frac{\theta_1}{2} \right) \cot \left( \frac{\theta_2}{2} \right) \right),$$

(37)

where we have used $\sinh^{-1}(u) = \ln(u + \sqrt{u^2 + 1})$. The electric potential at any point on the plane of a uniformly charged triangular sheet can be written as a superposition of the electric potential of three (or four) triangular sheets (if the point is out of the triangle) on the plane of the sheet. Similarly the electric potential at any point on the plane of an arbitrary polygon can be written in terms of the electric potential of some triangles. This gives us also an upper and a lower bound on the magnitude of the electric potential at any point on the plane of a uniformly charged sheet of any shape. The lower bound comes from the electric potential due
Figure 5. (a) Electric potential at the corner of a uniformly charged triangular sheet can be written as a superposition of the electric potential of two right triangular sheets. (b) Electric potential at the point B on the plane of a uniformly charged triangular sheet can be written as a superposition of the electric potential of three triangular sheets, and the electric potential at the point C can be written as the superposition of the electric potential of four triangular sheets.

to a polygon which is surrounded by the charged area and the upper bound comes from the electric potential due to a polygon which surrounds the charged area.

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