Self-shrinkers with bounded HA

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Abstract

We study integral and pointwise bounds on the second fundamental form of properly immersed self-shrinkers with bounded $HA$. As applications, we discuss gap and compactness results for self-shrinkers.

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1 Introduction

A hypersurface $\Sigma \hookrightarrow \mathbb{R}^{n+1}$ is said to be a self-shrinker if it is the time $t = -1$ slice of a mean curvature flow moving by rescalings with $\Sigma_t = \sqrt{-t}\Sigma$, or equivalently if it satisfies the equation

$$H = \frac{\langle x, n \rangle}{2},$$

where $n$ and $H$ denote the unit normal vector and the mean curvature, respectively. Self-shrinkers play an important role in the study of mean curvature flow, not least because they are models for type-I singularities of the flow by Huisken [8, 9].

It is interesting to compare the $HA$ tensor in mean curvature flow with the Ricci curvature in Ricci flow since they describe the corresponding metric evolution, respectively. In [2, 10] Chen-Wang and Kotschwar-Munteanu-Wang showed the Ricci curvature blows up at the rate of type-I at the first finite singularity. In [13] Sesum proved the type-I blowup of mean curvature at the finite type-I singularity.

In [16] Li-Wang studied the flow with type-I mean curvature and confirmed the multiplicity-one conjecture in this case. The present paper follows the method of [18] and can be seen as an attempt to understand more about the asymptotic behaviour of self-shrinkers in terms of $HA$.

Let $f = |x|^2/4$. By integral estimates and the Moser iteration we get the following pointwise growth estimate of the second fundamental form.
Theorem 1.1. (Theorem 3.2) Let \( x : \Sigma^n \to \mathbb{R}^{n+1} \) be a properly immersed self-shrinker with \( \sup_{\Sigma} |H A| \leq K \). Then for any \( p > \max\{n, 4\} \) there exist positive constants \( C = C(n, p, K, \int_{\Sigma} e^{-f} \int_{B(0, r_0) \cap \Sigma} |A|^p) \) where \( r_0 = c(n, p)(1 + K) \) and \( a = a(n, p, K) \) such that

\[
|A|(x) \leq C(|x| + 1)^a, \quad \forall x \in \Sigma,
\]

i.e., the second fundamental form grows at most polynomially in the distance.

Based on the polynomial growth of volume and second fundamental form, we find that a self-shrinker with sufficiently small \( |H A| \) must be a hyperplane.

Theorem 1.2. (Corollary 4.2) Let \( x : \Sigma^n \to \mathbb{R}^{n+1} \) be a smooth properly embedded self-shrinker. There exists a constant \( \varepsilon_n = \frac{1}{\sqrt{n(n+3)}} \) such that if \( \sup_{\Sigma} |H A| \leq \varepsilon_n \) then \( \Sigma \) is a hyperplane through \( 0 \).

By similar argument we find that a local energy bound implies a global energy bound.

Theorem 1.3. (Proposition 4.3) Let \( x : \Sigma^n \to \mathbb{R}^{n+1} \) be a properly immersed self-shrinker with \( n \geq 4 \) and \( \sup_{\Sigma} |H A| \leq K \). Then there exists a \( r_1 = c_n \sqrt{K} \) such that if \( \int_{B(0, r_1) \cap \Sigma} |A|^n \leq E \), then for any \( r > 0 \) we have \( \int_{B(0, r) \cap \Sigma} |A|^n \leq 3E e^{r^2/4} \).

By virtue of the energy estimate above and the \( \varepsilon \)-regularity from Li-Wang [15] we find that the space of properly embedded self-shrinkers with uniformly bounded entropy, uniformly bounded \( |H A| \) and uniformly bounded local energy is compact.

Theorem 1.4. (Theorem 4.4) Let \( \{\Sigma^n_i\} \) be a sequence of properly embedded self-shrinkers with \( n \geq 4 \) normalized by \( \int_{\Sigma^n_i} e^{-f} \leq (4\pi)^{n/2} \). Assume that \( \sup_i \sup_{\Sigma_i} |H A| \leq K \) and \( \sup_i \int_{B(0, r_1) \cap \Sigma_i} |A|^n < \infty \) where \( r_1 = c_n \sqrt{K} \) is the positive constant in Proposition 4.3. Then a subsequence of \( \{\Sigma_i\} \) converges smoothly to a smooth properly embedded self-shrinker \( \Sigma_\infty \).

The organization of this paper is as follows. In Sect.2 we recall some results on self-shrinkers and differential equations. In Sect.3 we develop \( L^p \) estimate of \( A \) and derive pointwise estimate by standard Moser iteration. In Sect.4 we obtain the gap theorem using weighted integral estimate and get the convergence result by \( \varepsilon \)-regularity.

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2 Preliminaries

Let \( x : \Sigma^n \to \mathbb{R}^{n+1} \) be a hypersurface without boundary. \( \Sigma \) is called a self-shrinker if it satisfies

\[
H = \frac{\langle x, n \rangle}{2}.
\]

Throughout this paper, we set the potential function

\[
f := \frac{|x|^2}{4}.
\]

Some related equations are listed below for later calculations. See the proof of Lemma 3.20 and Theorem 5.2 of [17] for details.
Lemma 2.1. On a self-shrinker we have

\[ \Delta f - |\nabla f|^2 = \frac{n}{2} - f, \]  
\[ HA + \nabla^2 f = \frac{1}{2} g, \]  
\[ H^2 + \Delta f = \frac{n}{2}, \]  
\[ |\nabla f|^2 + H^2 = f, \]  
\[ \nabla^2 H - \nabla f \cdot \nabla A = \frac{1}{2} A - HA^2, \]  
\[ \Delta A - \nabla f \cdot \nabla A = \left( \frac{1}{2} - |A|^2 \right) A, \]  
\[ \frac{1}{2} \left( \Delta |A|^2 - \nabla f \cdot \nabla |A|^2 \right) = |\nabla A|^2 + \left( \frac{1}{2} - |A|^2 \right) |A|^2, \]  
\[ \Delta H - \nabla f \cdot \nabla H = \left( \frac{1}{2} - |A|^2 \right) H, \]  
\[ \frac{1}{2} \left( \Delta H^2 - \nabla f \cdot \nabla H^2 \right) = |\nabla H|^2 + \left( \frac{1}{2} - |A|^2 \right) H^2. \]

By Corollary 2.8 of [7], we know

Lemma 2.2. (Corollary 2.8 of [7]) If \( \Sigma \) is a self-shrinker and \( H \equiv 0 \), then \( \Sigma \) is a minimal cone. In particular, if \( \Sigma \) is also smooth and embedded, then it is a hyperplane through 0.

From [3] one sees the equivalence of weighted volume finiteness, polynomial volume growth and properness of an immersed self-shrinker in Euclidean space.

Lemma 2.3. (Theorem 1.1 of [3]) Let \( \Sigma^n \) be a complete noncompact properly immersed self-shrinker in Euclidean space \( \mathbb{R}^{n+1} \). Then \( \Sigma \) has finite weighted volume

\[ \text{Vol}_f(\Sigma) = \int_{\Sigma} e^{-f} \, dv < +\infty \]

and

\[ \text{Vol}(B(0, r) \cap \Sigma) \leq C r^n, \quad \forall r > 0, \]

where \( C \) is a positive constant depending only on \( \int_{\Sigma} e^{-f} \, dv \).

Provided bounded mean curvature we can also get volume ratio lower bound. See Lemma 3.5 in Li-Wang [15].

Lemma 2.4. (Lemma 3.5 of [15]) Let \( \Sigma^n \rightarrow \mathbb{R}^{n+1} \) be a properly immersed hypersurface in \( B(x_0, r_0) \) with \( x_0 \in \Sigma \) and \( \sup_{\Sigma} |H| \leq \Lambda \). Then for any \( s \in (0, r_0) \) we have

\[ \frac{\text{Vol}_\Sigma(B(x_0, s) \cap \Sigma)}{\omega_n s^n} \leq e^{\Lambda r_0} \frac{\text{Vol}_\Sigma(B(x_0, r_0) \cap \Sigma)}{\omega_n r_0^n}. \]

In particular,

\[ \text{Vol}(B(x_0, r) \cap \Sigma) \geq e^{-\Lambda r} \omega_n r^n, \quad \forall r \in (0, r_0]. \]
In order to obtain the growth rate of second fundamental form from the $L^p$ estimate we will apply
the standard Moser iteration. Recall the Michael-Simon inequality which needs mean curvature. Here
we present a precise estimate of elliptic case derived from [12] and [14].

**Lemma 2.5 (Moser iteration).** Let $\Sigma^n \hookrightarrow \mathbb{R}^{n+1}$ be a hypersurface without boundary. Consider the
differential inequality

$$-\Delta u \leq \varphi u, \quad u \geq 0.$$  

Fix $x_0 \in \Sigma$ and denote $D_r := B(x_0, r) \cap \Sigma$. Then for any $r > 0$, $q > \frac{n}{2}$ and $\beta \geq 2$ there exists a
positive constant $C = C(n, q, \beta)$ such that

$$\|u\|_{L^\infty(D_{r/2})} \leq C r^{-\frac{2n^2}{n-2}} \left( \|\varphi\|_{L^q(D_r)} + \|H\|_{L^{n+2}(D_r)} \right)^{\frac{n^2}{(n-2)n}} \|u\|_{L^p(D_r)}.$$  

Finally we recall some technical results on interior estimates and compactness of immersed hy-
persurfaces in $\mathbb{R}^{n+1}$. Note that if $\Sigma$ is a self-shrinker then $\{\Sigma_t = \sqrt{-t} \Sigma, -\frac{3}{2} \leq t \leq -\frac{1}{2}\}$ is a mean
curvature flow, i.e., $\partial_t \mathcal{H} = -\mathcal{H}$. We obtain some kind of $\epsilon$--regularity from Corollary 3.11 and
Theorem 3.7 in Li-Wang [14] and the interior estimates of Ecker and Huisken in [17].

**Lemma 2.6 ($\epsilon$-regularity).** There exist constants $\epsilon = \epsilon(n) > 0$, $\delta = \delta(n) > 0$, $\eta = \eta(n) > 0$
and $\{D_k(n, \theta)\}_{k \geq 1}$ satisfying the following properties. Let $\Sigma^n \hookrightarrow \mathbb{R}^{n+1}$ be a properly immersed
self-shrinker satisfying $\sup_{\Sigma} |H| \leq \Lambda$. If

$$\int_{B(x_0, r) \cap \Sigma} |A|^n \leq \epsilon$$  

for some $x_0 \in \mathbb{R}^{n+1}$ and some $0 < r \leq \frac{1}{\Lambda}$, then we have

$$\sup_{B(x_0, r/2) \cap \Sigma} |A| \leq \frac{1}{r};$$  

$$\sup_{B(x_0, r/32) \cap \Sigma} |A| \leq \frac{2}{\delta r}, \quad \forall t + 1 \in \left[ -\frac{\eta r^2}{16}, \frac{\eta r^2}{16} \right] \cap \left[ -\frac{1}{2}, \frac{1}{2} \right];$$  

$$\sup_{B(x_0, \sqrt{\theta} R) \cap \Sigma} |\nabla^k A| \leq \frac{2D_k(n, \theta)}{\delta r}, \quad \forall \theta \in (0, \frac{1}{2}], \forall k \geq 1,$$

where $R := \min\{\frac{r}{32}, \frac{n}{2}\sqrt{r}, \sqrt{2n}\}$.

The following compactness result of mean curvature flow is well-known. See [1] for a detailed
proof.

**Lemma 2.7 (Compactness of mean curvature flow).** Let $\{\Sigma_i^n, x_i(t), -1 < t < 1\}$ be a sequence
of mean curvature flow properly immersed in $B(0, R) \subset \mathbb{R}^{n+1}$. Suppose that

$$\sup_{B(0, R) \cap \Sigma_i, t} |A|(\cdot, t) \leq \Lambda, \quad \forall t \in (-1, 1)$$  

for some $\Lambda > 0$. Then a subsequence of $\{(B(0, R) \cap \Sigma_i, t), -1 < t < 1\}$ converges in smooth topology
to a smooth mean curvature flow $\{\Sigma_{\infty, t}, -1 < t < 1\}$ in $B(0, R)$.  

3 $L^p$ estimate and growth rate

Throughout the section we set

$$\sup_{\Sigma} |HA| \leq K.$$  

Proposition 3.1 ($L^p$ estimate). Let $x : \Sigma^n \to \mathbb{R}^{n+1}$ be a properly immersed self-shrinker with $\sup_{\Sigma} |HA| \leq K$. Then for any $p \geq 4$ there exists positive constants $a = a(n, p, K)$ and $C = C(n, p, K, \int_{\Sigma} e^{-f} \int_{B(0, r_0)} |A|^p)$ where $r_0 = c(n, p)(1 + K)$ such that

$$\int_{\Sigma} |A|^p (|x|^2 + 1)^{-a} \leq C.$$  

Moreover, for any $x \in \Sigma$,

$$\int_{B(x, 1) \cap \Sigma} |A|^p \leq C(|x|^2 + 1)^a.$$  

Proof. We always use $c$ to denote a nonegative constant depending only on $n$ and $p$. For $a > 0$ and $p > 1$, integrating by parts we have

$$a \int_{\Sigma} |\nabla f|^2 (f + 1)^{-a-1} |A|^p \phi = - \int_{\Sigma} \nabla f \cdot \nabla (f + 1)^{-a} |A|^p \phi$$  

$$= \int_{\Sigma} \Delta f (f + 1)^{-a} |A|^p \phi + \int_{\Sigma} \nabla f \cdot \nabla |A|^p \cdot (f + 1)^{-a} \phi$$  

$$+ \int_{\Sigma} \nabla f \cdot \nabla \phi \cdot (f + 1)^{-a} |A|^p.$$  

Note that

$$a|\nabla f|^2 (f + 1)^{-a-1} - \Delta f (f + 1)^{-a} = \left(\frac{a f - H^2}{f + 1} + H^2 - \frac{n}{2}\right)(f + 1)^{-a}.$$  

Since $H$ is bounded by $\frac{n}{2} K^{\frac{1}{2}}$ there exists a positive constant

$$r_0 = \left(\frac{8}{n}(1 + n^{1/2} K) a\right)^{1/2}$$  

such that

$$a \frac{f - H^2}{f + 1} + H^2 - \frac{n}{2} \geq a - n, \quad \forall |x| \geq r_0.$$  

Let $\phi(x) := \eta(|x|^2)$ be a cutoff where $\eta : [0, \infty) \to \mathbb{R}$ is a nonnegative decreasing Lipschitz function. Thus

$$\nabla f \cdot \nabla \phi = 4 \eta' |\nabla f|^2 \leq 0.$$  

Moreover, for any $r > 0$ and $0 < \delta < 1$ fixed, let

$$\eta \equiv 1 \quad \text{on} \ [0, r^2] ; \quad \eta \equiv 0 \quad \text{on} \ [4r^2, \infty) ; \quad |\eta'| \leq \frac{1}{3 \delta r^2} \eta^{1-\delta},$$  

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which implies
\[ \phi^{-1}|\nabla \phi|^2 \leq \frac{16}{9\delta^2 r^2} \phi^{1-2\delta}. \]

Back to (3.1),

\[
(a - n) \int_{\Sigma} |A|^p (f + 1)^{-\alpha} \phi \leq \int_{\Sigma} \nabla f \cdot \nabla |A|^p (f + 1)^{-\alpha} \phi + \int_{\Sigma} 4\eta |\nabla f|^2 (f + 1)^{-\alpha} |A|^p + C_1, \tag{3.5}
\]

where

\[
C_1 = C_1(p) := \int_{\{|x| \leq r_0\} \cap \Sigma} \left( -\frac{a f - H^2}{f + 1} - H^2 + \frac{n}{2} + a - n \right) (f + 1)^{-\alpha} |A|^p \phi \leq \int_{\{|x| \leq r_0\} \cap \Sigma} \left( \frac{a - H^2}{f + 1} - \frac{n}{2} \right) (f + 1)^{-\alpha} |A|^p \leq ca(1 + K) \int_{\{|x| \leq r_0\} \cap \Sigma} |A|^p (f + 1)^{-\alpha - 1} < \infty. \tag{3.4}
\]

Using (2.5) and integrating by parts we get

\[
\int_{\Sigma} \nabla f \cdot \nabla |A|^p (f + 1)^{-\alpha} \phi = p \int_{\Sigma} \frac{1}{2} (x, \nabla A) A |A|^{p-2} (f + 1)^{-\alpha} \phi \tag{3.6}
\]

\[
\leq p \int_{\Sigma} \left( \nabla^2 H + HA^2 - \frac{1}{2} A \right) |A|^{p-2} (f + 1)^{-\alpha} \phi \leq p \int_{\Sigma} \nabla^2 H \cdot A \cdot |A|^{p-2} (f + 1)^{-\alpha} \phi + (pK - \frac{p}{2}) \int_{\Sigma} |A|^p (f + 1)^{-\alpha} \phi \leq p(p - 1) \int_{\Sigma} |\nabla H| |\nabla A| |A|^{p-2} (f + 1)^{-\alpha} \phi + ap \int_{\Sigma} |\nabla H| |\nabla f| |A|^{p-1} (f + 1)^{-\alpha - 1} \phi + p \int_{\Sigma} |\nabla H| |\nabla \phi| |A|^{p-1} (f + 1)^{-\alpha} + (pK - \frac{p}{2}) \int_{\Sigma} |A|^p (f + 1)^{-\alpha} \phi.
\]

Using the Cauchy-Schwarz inequality and the Young’s inequality, we estimate the right hand side of (3.5) above as follows:

\[
p(p-1) \int_{\Sigma} |\nabla H| |\nabla A| |A|^{p-2} (f + 1)^{-\alpha} \phi \leq cK^{-1} \int_{\Sigma} |\nabla H|^2 |A|^{p+1} (f + 1)^{-\alpha} \phi + cK \int_{\Sigma} |\nabla A|^2 |A|^{p-4} (f + 1)^{-\alpha} \phi,
\]

\[
ap \int_{\Sigma} |\nabla H| |\nabla f| |A|^{p-1} (f + 1)^{-\alpha - 1} \phi \leq cK^{-1} \int_{\Sigma} |\nabla H|^2 |A|^p (f + 1)^{-\alpha} \phi + ca^2 K \int_{\Sigma} |\nabla f|^2 |A|^{p-2} (f + 1)^{-\alpha - 2} \phi \leq cK^{-1} \int_{\Sigma} |\nabla H|^2 |A|^p (f + 1)^{-\alpha} \phi + ca^2 K \int_{\Sigma} |A|^{p-2} (f + 1)^{-\alpha - 1} \phi,
\]

\[6\]
\[ ca^2 \int_\Sigma |A|^{p-2}(f + 1)^{-a-1} \phi = \int_\Sigma |A|^{p-2}(f + 1)^{-\frac{2}{p} - a} \cdot ca^2(f + 1)^{-\frac{2}{p}a - 1} \cdot \phi \]
\[ \leq c \int_\Sigma |A|^p(f + 1)^{-a} \phi + ca^p \int_\Sigma (f + 1)^{-a-\frac{2}{p}a} \phi, \]

\[ p \int_\Sigma |\nabla H| |\nabla \phi| |A|^{p-1}(f + 1)^{-a} \]
\[ \leq cK^{-1} \int_\Sigma |\nabla H|^2 |A|^p(f + 1)^{-a} \phi + cK \int_\Sigma \phi^{-1} |\nabla \phi|^2 |A|^{p-2}(f + 1)^{-a} \]
\[ \leq cK^{-1} \int_\Sigma |\nabla H|^2 |A|^p(f + 1)^{-a} \phi + \frac{cK}{\delta^2 r^2} \int_\Sigma \phi^{1-2\delta} |A|^{p-2}(f + 1)^{-a}, \]

\[ \frac{c}{\delta^2 r^2} \int_\Sigma \phi^{1-2\delta} |A|^{p-1}(f + 1)^{-a} = \int_\Sigma |A|^{p-1} \phi^{-1} \frac{2}{p} - \frac{c}{\delta^2 r^2} \phi^{1-2p\delta} \cdot (f + 1)^{-a} \]
\[ \leq c \int_\Sigma |A|^p(f + 1)^{-a} \phi + \frac{c}{\delta^2 r^2} \int_\Sigma \phi^{1-2p\delta}(f + 1)^{-a}. \]

Let \( \delta = \frac{1}{4p} \) so that \( 1 - 2p\delta = \frac{1}{2} > 0 \). Then plugging the estimates above into (3.5) yidles

\[ \int_\Sigma \nabla f \cdot \nabla |A|^p \cdot (f + 1)^{-a} \phi \]
\[ \leq cK^{-1} \int_\Sigma |\nabla H|^2 |A|^p(f + 1)^{-a} \phi + cK \int_\Sigma |\nabla A|^2 |A|^{p-4}(f + 1)^{-a} \phi \]
\[ + cK \int_\Sigma |A|^p(f + 1)^{-a} \phi + c(a^p + r^{-2p}) K \int_\Sigma (f + 1)^{-a}. \]

Furthermore, using (2.9) we have

\[ \int_\Sigma |\nabla H|^2 |A|^p(f + 1)^{-a} \phi \]
\[ = \int_\Sigma \left( \frac{1}{2} \Delta H^2 - \frac{1}{2} \nabla f \cdot \nabla H^2 + |A|^2 - \frac{1}{2} H^2 \right) |A|^p(f + 1)^{-a} \phi \]
\[ \leq - \frac{p}{2} \int_\Sigma \nabla H^2 \cdot \nabla A \cdot A \cdot |A|^{p-2}(f + 1)^{-a} \phi + \frac{a}{2} \int_\Sigma \nabla H^2 \cdot \nabla f \cdot |A|^p(f + 1)^{-a} \phi \]
\[ - \frac{1}{2} \int_\Sigma \nabla H^2 \cdot \nabla \phi \cdot |A|^p(f + 1)^{-a} - \frac{1}{2} \int_\Sigma \nabla f \cdot \nabla H^2 \cdot |A|^p(f + 1)^{-a} \phi \]
\[ + K^2 \int_\Sigma |A|^p(f + 1)^{-a} \phi. \]
Then,
\[
\int_\Sigma |\nabla H|^2 |A|^p(f + 1)^{-a} \phi
\]
\[
\leq pK \int_\Sigma |\nabla H||\nabla A||A|^{p-2}(f + 1)^{-a} \phi + (a + 1)K \int_\Sigma |\nabla H||\nabla f||A|^{p-1}(f + 1)^{-a} \phi
\]
\[
+ K \int_\Sigma |\nabla H||\nabla \phi||A|^{p-1}(f + 1)^{-a} + K^2 \int_\Sigma |A|^p(f + 1)^{-a} \phi
\]
\[
\leq \frac{1}{4} \int_\Sigma |\nabla H|^2 |A|^p(f + 1)^{-a} \phi + p^2 K^2 \int_\Sigma |\nabla A|^2 |A|^{p-4}(f + 1)^{-a} \phi
\]
\[
+ \frac{1}{4} \int_\Sigma |\nabla H|^2 |A|^p(f + 1)^{-a} \phi + (a + 1)^2 K^2 \int_\Sigma |\nabla f|^2 |A|^{p-2}(f + 1)^{-a} \phi
\]
\[
+ \frac{1}{4} \int_\Sigma |\nabla H|^2 |A|^p(f + 1)^{-a} \phi + K^2 \int_\Sigma \phi^{-1} |\nabla \phi|^2 |A|^{p-2}(f + 1)^{-a}
\]
\[
+ K^2 \int_\Sigma |A|^p(f + 1)^{-a} \phi.
\]

Thus,
\[
\int_\Sigma |\nabla H|^2 |A|^p(f + 1)^{-a} \phi
\]
\[
\leq 4p^2 K^2 \int_\Sigma |\nabla A|^2 |A|^{p-4}(f + 1)^{-a} \phi + 4(a + 1)^2 K^2 \int_\Sigma |A|^{p-2}(f + 1)^{-a+1} \phi
\]
\[
+ cr^{-2}K^2 \int_\Sigma \phi^{1-2\delta}|A|^{p-2}(f + 1)^{-a} + 4K^2 \int_\Sigma |A|^p(f + 1)^{-a} \phi
\]
\[
\leq cK^2 \int_\Sigma |\nabla A|^2 |A|^{p-4}(f + 1)^{-a} \phi + 4K^2 \int_\Sigma |A|^p(f + 1)^{-a} \phi
\]
\[
+ cK^2 \int_\Sigma |A|^p(f + 1)^{-a} \phi + c(a + 1)^pK^2 \int_\Sigma (f + 1)^{-a+\frac{p}{2}}
\]
\[
+ cK^2 \int_\Sigma |A|^p(f + 1)^{-a} \phi + cr^{-p}K^2 \int_\Sigma (f + 1)^{-a}
\]
\[
\leq cK^2 \int_\Sigma |\nabla A|^2 |A|^{p-4}(f + 1)^{-a} \phi + cK^2 \int_\Sigma |A|^p(f + 1)^{-a} \phi
\]
\[
+ c((a + 1)^p + r^{-p})K^2 \int_\Sigma (f + 1)^{-a+\frac{p}{2}}.
\]

On the other hand, for any \( p \geq 4 \) by (2.7) we have
\[
\int_\Sigma |\nabla A|^2 |A|^{p-4}(f + 1)^{-a} \phi
\]
\[
= \int_\Sigma \left( \frac{1}{2} \Delta |A|^2 - \frac{1}{2} \nabla f \cdot \nabla |A|^2 + (|A|^2 - \frac{1}{2})|A|^2 \right) |A|^{p-4}(f + 1)^{-a} \phi
\]
\[
\leq \frac{a}{2} \int_\Sigma \nabla |A|^2 \cdot \nabla f \cdot |A|^{p-4}(f + 1)^{-a-1} \phi - \frac{1}{2} \int_\Sigma \nabla |A|^2 \cdot \nabla \phi \cdot |A|^{p-4}(f + 1)^{-a}
\]
\[
- \frac{1}{2} \int_\Sigma \nabla f \cdot \nabla |A|^2 \cdot |A|^{p-4}(f + 1)^{-a} \phi + \int_\Sigma |A|^p(f + 1)^{-a} \phi.
\]
Then,
\[
\int_{\Sigma} |\nabla A|^2|A|^{p-4}(f+1)^{-a}\phi
\]
\[\leq (a+1) \int_{\Sigma} |\nabla A||\nabla f||A|^{p-3}(f+1)^{-a}\phi + \int_{\Sigma} |\nabla A||\nabla \phi||A|^{p-3}(f+1)^{-a}
\]
\[+ \int_{\Sigma} |A|^p(f+1)^{-a}\phi
\]
\[\leq \frac{1}{4} \int_{\Sigma} |\nabla A|^2|A|^{p-4}(f+1)^{-a}\phi + (a+1)^2 \int_{\Sigma} |\nabla f|^2|A|^{p-2}(f+1)^{-a}\phi
\]
\[+ \frac{1}{4} \int_{\Sigma} |\nabla A|^2|A|^{p-4}(f+1)^{-a}\phi + \int_{\Sigma} \phi^{-1}|\nabla \phi|^2|A|^{p-2}(f+1)^{-a}
\]
\[+ \int_{\Sigma} |A|^p(f+1)^{-a}\phi.
\]
\[\int_{\Sigma} |\nabla A|^2|A|^{p-4}(f+1)^{-a}\phi \]
\[\leq 2(a+1)^2 \int_{\Sigma} |A|^{p-2}(f+1)^{-a+1}\phi + cr^{-2} \int_{\Sigma} \phi^{1-2\delta}|A|^{p-2}(f+1)^{-a}
\]
\[+ \int_{\Sigma} |A|^p(f+1)^{-a}\phi
\]
\[= \int_{\Sigma} |A|^{p-2}(f+1)^{-\frac{p-2}{p}\cdot 2(a+1)^2 \int_{\Sigma} \left(\frac{x^2}{4} + 1\right)^{-\frac{p}{2}+1}\cdot \phi
\]
\[+ \int_{\Sigma} |A|^{p-2}\phi \cdot cr^{-2} \phi^{\frac{2(1-p\delta)}{p} \cdot (f+1)^{-a} + \int_{\Sigma} |A|^p(f+1)^{-a}\phi
\]
\[\leq c \int_{\Sigma} |A|^p(f+1)^{-a}\phi + c(a+1)^p \int_{\Sigma} (f+1)^{-a+\frac{p}{2}\phi
\]
\[+c \int_{\Sigma} |A|^p(f+1)^{-a}\phi + cr^{-p} \int_{\Sigma} (f+1)^{-a}\phi^{1-p\delta}
\]
\[+ \int_{\Sigma} |A|^p(f+1)^{-a}\phi
\]
\[\leq c \int_{\Sigma} |A|^p(f+1)^{-a}\phi + c \left((a+1)^p + r^{-p}\right) \int_{\Sigma} (f+1)^{-a+\frac{p}{2}}.
\]
Combining (3.6), (3.7) and (3.8) we conclude
\[
\int_{\Sigma} \nabla f \cdot \nabla |A|^p \cdot (f+1)^{-a}\phi
\]
\[\leq cK \int_{\Sigma} |\nabla A|^2|A|^{p-4}(f+1)^{-a}\phi + cK \int_{\Sigma} |A|^p(f+1)^{-a}\phi
\]
\[+ c \left(r^{-2p} + r^{-p} + (a+1)^p\right) K \int_{\Sigma} (f+1)^{-a+\frac{p}{2}}
\]
\[\leq cK \int_{\Sigma} |A|^p(f+1)^{-a}\phi + c \left(r^{-2p} + r^{-p} + (a+1)^p\right) K \int_{\Sigma} (f+1)^{-a+\frac{p}{2}},
\]
which together with (3.3) implies
\[
(a - n - cK) \int_{\Sigma} |A|^p(f+1)^{-a}\phi \leq c \left(r^{-2p} + r^{-p} + (a+1)^p\right) K \int_{\Sigma} (f+1)^{-a+\frac{p}{2}} + C_1.
\]
Recall the volume estimate in Lemma 2.3. Take
\[ a = n + p + cK + 1 \]
so that the right hand side above makes sense. Recall the settings (3.2) and (3.4) one sees \( r_0 \leq c(1 + K) \),
\[ C_1(p) \leq c(1 + K)^2 \int_{\{|x| \leq r_0\} \cap \Sigma} |A|^p. \]
Letting \( r \to \infty \) yields
\[ \int_{\Sigma} |A|^p (f + 1)^{-a} \leq C(n, p, K) \int_{\Sigma} (f + 1)^{-a + \frac{p}{2}} + c(1 + K)^2 \int_{\{|x| \leq r_0\} \cap \Sigma} |A|^p < \infty. \]
In particular, we restrict the integration on \( B(x_0, 1) \cap \Sigma \) for any \( x_0 \in \Sigma \), then
\[ \left( \frac{|x_0| + 1}{4} + 1 \right)^{-a} \int_{B(x_0, 1) \cap \Sigma} |A|^p \leq \int_{B(x_0, 1) \cap \Sigma} |A|^p (f + 1)^{-a} \phi \leq C, \]
i.e.,
\[ \int_{B(x_0, 1) \cap \Sigma} |A|^p \leq C(|x_0|^2 + 1)^a, \]
where
\[ a = a(n, p, K), \quad C = C(n, p, K, \int_{\Sigma} e^{-f}, \int_{B(0, r_0) \cap \Sigma} |A|^p). \]

**Theorem 3.2 (growth rate).** Let \( \mathbf{x} : \Sigma^n \to \mathbb{R}^{n+1} \) be a properly immersed self-shrinker with \( \sup_{\Sigma} |HA| \leq K \). Then for any \( p > \max\{n, 4\} \) there exist positive constants \( C = C(n, p, K, \int_{\Sigma} e^{-f}, \int_{B(0, r_0) \cap \Sigma} |A|^p) \) where \( r_0 = c(n, p)(1 + K) \) and \( a = a(n, p, K) \) such that
\[ |A|(x) \leq C(|x| + 1)^a, \quad \forall x \in \Sigma, \]
i.e., the second fundamental form grows at most polynomially in the distance.

**Proof.** Fix \( q > \max\{n/2, 2\} \). From (2.7) we know
\[ \Delta |A|^2 = \nabla f \cdot \nabla |A|^2 + 2|\nabla A|^2 + (1 - 2|A|^2)|A|^2 \]
\[ \geq \left( -\frac{1}{2} |\nabla f|^2 + 1 - 2|A|^2 \right) |A|^2 \]
\[ \geq -\left( \frac{|x|^2}{8} + 2|A|^2 \right) |A|^2. \]
If we set \( \varphi := \frac{|x|^2}{8} + 2|A|^2 \), then
\[ -\Delta |A|^2 \leq \varphi |A|^2. \]
Recall that $\sup_{\Sigma} |H| \leq c_n K^{1/2}$. Fix $x_0 \in \Sigma$. Applying the standard Moser iteration Lemma 2.5 yields that for any $\beta = \frac{n}{2} < q \leq 2$,

$$\sup_{B(x_0,1) \cap \Sigma} |A|^2 \leq C(n, q) \left( \|\varphi\|^{\frac{2q}{n}}_{L^q(B(x_0,1) \cap \Sigma)} + \|H\|^{\frac{n+2}{2n+2}}_{L^{n+2}(B(x_0,1) \cap \Sigma)} \right)^{2n} \|A\|^2_{L^\frac{q}{n}(B(x_0,1) \cap \Sigma)},$$

where by Lemma 2.3 and Proposition 3.1

$$\|\varphi\|_{L^q(B(x_0,1) \cap \Sigma)} \leq \frac{1}{8} \|x_0^2\|_{L^q(B(x_0,1) \cap \Sigma)} + 2 \|A\|_{L^q(B(x_0,1) \cap \Sigma)} + C(\|x_0^2\|_{L^q(B(x_0,1) \cap \Sigma)} + 1)^{1 + \frac{1}{2q}} + C(\|x_0^2\|_{L^q(B(x_0,1) \cap \Sigma)} + 1)^{\frac{1}{2q}},$$

$$\|H\|^{\frac{n+2}{2n+2}}_{L^{n+2}(B(x_0,1) \cap \Sigma)} \leq CK^{\frac{n+2}{2}} (\|x_0^2\|_{L^q(B(x_0,1) \cap \Sigma)} + 1)^{\frac{n}{2}},$$

$$\|\varphi\|^2_{L^\frac{q}{n}(B(x_0,1) \cap \Sigma)} \leq C(\|x_0^2\|_{L^q(B(x_0,1) \cap \Sigma)} + 1)^{\frac{2n''}{n}}.$$ 

Finally we conclude for any $x \in \Sigma$,

$$|A|(x) \leq C(|x| + 1)^a,$$

for constants

$$a = a(n, q, K),$$

$$C = C(n, q, K, \int_{\Sigma} e^{-f}, C_1(2q), C_1(n)) = C(n, q, K, \int_{\Sigma} e^{-f}, \int_{B(0,r_0) \cap \Sigma} |A|^{2q}),$$

where $r_0 = c(n, q)(1 + K)$. So is Theorem 3.2 proved. \qed

### 4 Gap and compactness theorems

Since Lemma 2.3 and Theorem 3.2 show the polynomial growth, now we can consider global integrations with the natural weight $e^{-f}$, which leads to simpler calculations. As in the previous section, we set

$$\sup_{\Sigma} |HA| \leq K.$$

We will use the notation $\Delta_f = \Delta - \nabla f \cdot \nabla$ which is self adjoint with respect to the weighted volume $e^{-f} dv$.

**Theorem 4.1 (gap theorem).** Let $x : \Sigma^n \to \mathbb{R}^{n+1}$ be a properly immersed self-shrinker. If $\sup_{\Sigma} |HA| \leq \frac{1}{\sqrt{n(n+5)}}$, then $A \equiv 0$.

**Proof.** By virtue of (2.3) and (2.4), we have

$$\int_{\Sigma} (f - \frac{n}{2}) |A|^p e^{-f} = \int_{\Sigma} \left( |\nabla f|^2 - \Delta f \right) |A|^p e^{-f} = \int_{\Sigma} \Delta (e^{-f}) |A|^p - \int_{\Sigma} \nabla (e^{-f}) \cdot \nabla |A|^p = \int_{\Sigma} \nabla f \cdot \nabla |A|^p e^{-f}. \quad (4.1)$$
Note that \( \sup_{\Sigma} |HA| \leq K \). Using (2.5) and integrating by parts, we have

\[
\int_{\Sigma} \nabla f \cdot \nabla |A|^p e^{-f} = p \int_{\Sigma} \nabla f \cdot \nabla A \cdot |A|^{p-2} e^{-f}
\]

\[
= p \int_{\Sigma} (\nabla^2 H + HA^2 - \frac{1}{2} A) |A|^{p-2} e^{-f}
\]

\[
\leq p \int_{\Sigma} \nabla^2 H \cdot A |A|^{p-2} e^{-f} + (pK - \frac{p}{2}) \int_{\Sigma} |A|^p e^{-f}
\]

\[
\leq p(p - 1) \int_{\Sigma} |\nabla H||\nabla A||A|^{p-2} e^{-f} + p \int_{\Sigma} |\nabla H||\nabla f||A|^{p-1} e^{-f}
\]

\[
+ (pK - \frac{p}{2}) \int_{\Sigma} |A|^p e^{-f}.
\]

By (2.4) and Schwarz's inequality,

\[
\int_{\Sigma} \nabla f \cdot \nabla |A|^p e^{-f}
\]

\[
\leq p(p - 1) \int_{\Sigma} |\nabla H||\nabla A||A|^{p-2} e^{-f} + \frac{1}{2} \int_{\Sigma} |\nabla f|^2 |A|^{p-2} e^{-f}
\]

\[
+ \frac{p^2}{2} \int_{\Sigma} |\nabla H|^2 |A|^{p-2} e^{-f} + (pK - \frac{p}{2}) \int_{\Sigma} |A|^p e^{-f}
\]

\[
\leq p^2(1 + \frac{\sqrt{n}}{2}) \int_{\Sigma} |\nabla H||\nabla A||A|^{p-2} e^{-f} + \frac{1}{2} \int_{\Sigma} |\nabla f|^2 |A|^{p-2} e^{-f}
\]

\[
+ (pK - \frac{p}{2}) \int_{\Sigma} |A|^p e^{-f}
\]

\[
\leq p^2(1 + \frac{\sqrt{n}}{2}) K \int_{\Sigma} |\nabla A|^2 |A|^{p-4} e^{-f} + \frac{1}{4} p^2(1 + \frac{\sqrt{n}}{2}) K^{-1} \int_{\Sigma} |\nabla H|^2 |A|^{p-2} e^{-f}
\]

\[
+ \frac{1}{2} \int_{\Sigma} f |A|^p e^{-f} + (pK - \frac{p}{2}) \int_{\Sigma} |A|^p e^{-f}.
\]

Plugging (4.2) into (4.1) yields

\[
\frac{1}{2} \int_{\Sigma} f |A|^p e^{-f} \leq (pK + \frac{n}{2} - \frac{p}{2}) \int_{\Sigma} |A|^p e^{-f} + p^2(1 + \frac{\sqrt{n}}{2}) K \int_{\Sigma} |\nabla A|^2 |A|^{p-4} e^{-f}
\]

\[
+ p^2(1 + \frac{\sqrt{n}}{2}) K^{-1} \int_{\Sigma} |\nabla H|^2 |A|^{p-2} e^{-f}.
\]

Furthermore, by (2.7) we get, for \( p \geq 4 \),

\[
\int_{\Sigma} |\nabla A|^2 |A|^{p-4} e^{-f} = \int_{\Sigma} \left( \frac{1}{2} \Delta f |A|^2 - \frac{1}{2} - |A|^2 |A|^2 \right) |A|^{p-4} e^{-f}
\]

\[
\leq -\frac{1}{2} \int_{\Sigma} |\nabla A|^2 \cdot |A|^{p-4} e^{-f} + \int_{\Sigma} (|A|^p - \frac{1}{2} |A|^{p-2}) e^{-f} \leq \int_{\Sigma} |A|^p e^{-f},
\]

i.e.,

\[
\int_{\Sigma} |\nabla A|^2 |A|^{p-4} e^{-f} \leq \int_{\Sigma} |A|^p e^{-f}. \tag{4.4}
\]
Similarly, by (2.9) we get
\[ \int_{\Sigma} |\nabla H|^2 |A|^p e^{-f} = \int_{\Sigma} \left( \frac{1}{2} \Delta_f H^2 + (|A|^2 - \frac{1}{2} H^2) \right) |A|^p e^{-f} \]
\[ \leq \frac{1}{2} \int_{\Sigma} \nabla H \cdot \nabla |A|^p e^{-f} + \int_{\Sigma} (H^2 |A|^{p+2} - \frac{1}{2} H^2 |A|^{p+1}) e^{-f} \]
\[ \leq p \int_{\Sigma} |\nabla H| |A| |H||A|^{p-1} e^{-f} + K^2 \int_{\Sigma} |A|^p e^{-f} - \frac{1}{2} \int_{\Sigma} H^2 |A|^p e^{-f} \]
\[ \leq \frac{1}{2} \int_{\Sigma} |\nabla H|^2 |A|^p e^{-f} + \frac{p^2}{2} \int_{\Sigma} |\nabla A|^2 H^2 |A|^{p-2} e^{-f} + K^2 \int_{\Sigma} |A|^p e^{-f} \]
which together with (4.3) implies
\[ \int_{\Sigma} |\nabla H|^2 |A|^p e^{-f} \leq (p^2 + 2) K^2 \int_{\Sigma} |A|^p e^{-f}. \]  
(4.5)
Finally, combining (4.3), (4.4) and (4.5), we conclude
\[ \frac{1}{2} \int_{\Sigma} f |A|^p e^{-f} \leq (cK + \frac{n}{2} - \frac{p}{2}) \int_{\Sigma} |A|^p e^{-f}, \]  
(4.6)
where
\[ c = p + p^2 (p^2 + 3)(1 + \frac{\sqrt{n}}{2}). \]
Now we take \( p = n + 4 \). Then \( c \leq 2\sqrt{n(n+5)} \). If \( K \leq \frac{1}{\sqrt{n(n+5)}} \), i.e., a upper bound which depends only on \( n \), then the above inequality implies that \( A \equiv 0 \).

**Corollary 4.2.** Let \( \mathbf{x} : \Sigma^n \to \mathbb{R}^{n+1} \) be a smooth properly embedded self-shrinker. There exists a constant \( \varepsilon_n = \frac{1}{\sqrt{n(n+5)}} \) such that if \( \sup_{\Sigma} |HA| \leq \varepsilon_n \) then \( \Sigma \) is a hyperplane through 0.

**Proof.** Combining Theorem 4.1 and Lemma 2.2 we immediately obtain Corollary 4.2. 

From the proof of Theorem 4.1 we derive the following energy estimate.

**Proposition 4.3.** Let \( \mathbf{x} : \Sigma^n \to \mathbb{R}^{n+1} \) be a properly immersed self-shrinker with \( n \geq 4 \) and \( \sup_{\Sigma} |HA| \leq K \). Then there exists a \( r_1 = c_n \sqrt{K} \) such that if
\[ \int_{B(0,r_1) \cap \Sigma} |A|^n \leq E, \]
then for any \( r > 0 \) we have
\[ \int_{B(0,r) \cap \Sigma} |A|^n \leq 3E e^{r^2/4}. \]

**Proof.** Set \( b = cK + \frac{n}{2} - \frac{p}{2} \) as long as it is positive. Dividing the integration in (4.6) into two parts, we see
\[ \int_{\{f \leq 3b\} \cap \Sigma} f |A|^p e^{-f} + 3b \int_{\{f \geq 3b\} \cap \Sigma} |A|^p e^{-f} \]
\[ \leq 2b \int_{\{f \leq 3b\} \cap \Sigma} |A|^p e^{-f} + 2b \int_{\{f \geq 3b\} \cap \Sigma} |A|^p e^{-f}, \]
which implies

\[ \int_{\{f \geq 3b\} \cap \Sigma} |A|^p e^{-f} \leq 2 \int_{\{f \leq 3b\} \cap \Sigma} |A|^p. \]

Moreover, for any \( r > 0 \),

\[ \int_{\{|x| \leq 2r\} \cap \Sigma} |A|^p = \int_{\{f \geq r^2\} \cap \Sigma} |A|^p \]
\[ \leq e^2 \int_{\{f \geq 3b\} \cap \Sigma} |A|^p e^{-f} + \int_{\{f \leq 3b\} \cap \Sigma} |A|^p \]
\[ \leq (2e^2 + 1) \int_{\{f \leq 3b\} \cap \Sigma} |A|^p, \]

i.e.,

\[ \int_{\{|x| \leq r\} \cap \Sigma} |A|^p \leq (2e^{r^2/4} + 1) \int_{\{|x| \leq 2\sqrt{3b}\} \cap \Sigma} |A|^p, \]
\[ \leq 3e^{r^2/4} \int_{\{|x| \leq 2\sqrt{3eK + 3(n-p)/2}\} \cap \Sigma} |A|^p. \]

Letting \( p = n \) yields the energy estimate we want.

Combining the volume estimate and the energy bound, we derive the following compactness theorem for self-shrinkers, which largely follows the techniques on minimal surfaces. Remark that here we only need a local energy bound instead of a global bound.

**Theorem 4.4 (Compactness).** Let \( \{\Sigma_i^n\} \) be a sequence of properly embedded self-shrinkers with \( n \geq 4 \) normalized by \( \int_{\Sigma_i^n} e^{-f} \leq (4\pi)^{n/2} \). Assume that \( \sup_i \sup_{\Sigma_i^n} |HA| \leq K \) and \( \sup_i \int_{B(0, r_1) \cap \Sigma_i^n} |A|^n < \infty \) where \( r_1 = c_n \sqrt{K} \) is the positive constant in Proposition 4.3. Then a subsequence of \( \{\Sigma_i^n\} \) converges smoothly to a smooth properly embedded self-shrinker \( \Sigma_\infty \).

**Proof.** Fix \( B(0, \rho) \subset \mathbb{R}^{n+1} \). Assume that \( \int_{B(0, r_1) \cap \Sigma_i^n} |A|^n < E < \infty \). By Proposition 4.3 we can define the measures \( \nu_i \) on \( B(0, \rho) \) by

\[ \nu_i(U) := \int_{U \cap B(0, \rho) \cap \Sigma_i^n} |A|^n \leq 3Ee^{\rho^2/4}, \quad \forall U \subset B(0, \rho). \]

Then a subsequence converges weakly to a Radon measure \( \nu \) with \( \nu(B(0, \rho)) \leq 3Ee^{\rho^2/4} \). We define the set

\[ S_\rho := \{ x \in B(0, \rho) \mid \nu(x) \geq \epsilon \}, \]

where \( \epsilon = \epsilon(n) \) is the positive constant in Lemma 2.6 and see the number estimate \( \sharp \{S_\rho\} \leq \frac{3Ee^{\rho^2/4}}{\epsilon} \). For any \( x_0 \in B(0, \rho) \setminus S_\rho \) there exists some \( r \in (0, \frac{1}{n^{1/4}\sqrt{K}}) \) such that \( B(x_0, r) \subset B(0, \rho) \setminus S_\rho \) with \( \nu(B(x_0, r)) < \epsilon \). For \( i \) sufficiently large we have

\[ \int_{B(x_0, r) \cap \Sigma_i^n} |A|^n \leq \epsilon. \]
Applying Lemma 2.6 yields interior estimates

\[ \sup_{B(x_0, R/2) \cap \Sigma_i} |A| \leq \frac{D_k}{r}, \quad \forall k \geq 0, \]

where \( R = R(r, n) < \frac{r}{32} \) and \( D_k = D_k(n) \). By Lemma 2.7 and a diagonal sequence argument we find a subsequence of \( \{B(0, \rho) \cap \Sigma_i\} \) converges smoothly, away from \( S_\rho \), to a properly embedded self-shrinker \( \Sigma_\infty \) in \( B(0, \rho) \). Furthermore, we find a subsequence of \( \{\Sigma_i\} \) converges in smooth topology, away from \( S := \bigcup_{\rho > 0} S_\rho \), to a properly embedded self-shrinker \( \Sigma_\infty \). By Lemma 2.3 and Lemma 2.4 the multiplicity of the convergence is bounded by \( N_0 = N_0(n, K) < \infty \). Note that \( \Sigma_\infty \) is a minimal hypersurface with some conformal metric. Following the argument in the proof of Proposition 7.14 of [5] we see that \( \Sigma_\infty \cup S \) is a smooth properly embedded self-shrinker and the convergence is also in Hausdorff distance.

Finally it boils down to the multiplicity-one convergence as in [6]. Suppose for the sake of contradiction that \( u_i \) denotes the normalized height-difference between the top and bottom sheets. In fact \( \{u_i\} \) satisfies \( Lu_i = 0 \) up to higher order correction terms and converges on \( \Sigma_\infty \setminus S \) to a solution \( u \) with \( Lu = 0 \). Applying the foliation argument and local maximum principle in a cylindrical neighbourhood of a singularity \( y \in S \), we see that \( u_i \) is bounded on a neighbourhood of \( y \) by a multiple of its supremum on the boundary. Hence \( u \) is bounded over each \( y \in S \) and then extends to a smooth positive solution on \( \Sigma_\infty \cup S \) which actually implies L-stability. However, there are no L-stable smooth properly embedded self-shrinkers according to Lemma 2.3 and Theorem 0.5 of [6]. See more details in [6] [20].

\[ \square \]

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