Elementary abelian regular subgroups as hidden sums for cryptographic trapdoors

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Abstract
We study special subgroups of the affine general linear group acting on a vector space. To be more precise, we study elementary abelian regular subgroups. It turns out that these have a strong cryptographic significance, since they can be exploited to insert or detect algebraic trapdoors in block ciphers. When used in this context, they take the name of hidden sums. Our results include a convenient representation of their elements, a classification of their conjugacy classes and several combinatorial counting formulas. We also show how to use hidden sums to attack a block cipher and we hint at ongoing research, where the hidden sums are studied to obtain more sophisticated algebraic attacks.

Keywords: Translation group, hidden sum, trapdoor, block cipher.

1. Introduction
The affine general linear group acting on a vector space is a well-understood mathematical object, for any field characteristic. In this paper we aim at showing that there are some of its subgroups that play an important role in cryptography. In particular, its elementary abelian regular subgroups can be exploited to insert or detect algebraic trapdoors in some block ciphers. With trapdoors we mean a hidden algebraic structure in the cipher design that would allow an attacker with full knowledge to break it easily, while letting the rest of the cryptographic community trust the security of the cipher.

Our paper starts with recalling some preliminaries on group theory, linear algebra and the connection between the security of a family of block ciphers and properties of some permutations groups (Section 2). The following section presents the subgroups of the affine general linear groups that we consider to build our trapdoors. We call these subgroups hidden sum. We
characterize, modulo conjugations, their elements and we are able to provide their standard representation. In Section 4 we present our attack to a block cipher containing hidden sums and show its low complexity, providing also a toy cipher. In Section 5 we investigate hidden sums more deeply and we are able to provide formulae for estimating them in the binary case. Up to dimension 6, we are able to describe them precisely. Finally in Section 6 we provide some remarks and explain the connection with related studies, including some ongoing research, while in the Appendix we list some computational results.

2. Preliminaries

For any positive integer \( m \), we let \( [m] = \{1, \ldots, m\} \). We write \( \mathbb{F}_q \) to denote the finite field of \( q \) elements, where \( q \) is a power of a prime, and \( (\mathbb{F}_q)^{i \times j} \) to denote the set of all matrices with entries over \( \mathbb{F}_q \) with \( i \) rows and \( j \) columns. The identity matrix of size \( j \) is denoted by \( I_j \). Let \( N \geq 2 \), we use \( e_i = (0, \ldots, 0,1,0,\ldots,0) \in (\mathbb{F}_q)^N \) to denote the unit vector, which has 1 in the \( i \)-th position, and zeros elsewhere. The vector (sub)space generated by vectors \( v_1,\ldots,v_m \) is denoted by \( \text{Span}\{v_1,\ldots,v_m\} \), where \( m \geq 1 \).

2.1. Group Theory

Let \( V = (\mathbb{F}_q)^N \), we denote by Sym(\( V \)), Alt(\( V \)), respectively, the symmetric and the alternating group acting on \( V \). By AGL(\( V \)) and GL(\( V \)) we denote the affine and linear group of \( V \). The map \( 1_V \) will denote the identity map on \( V \). We write \( \langle g_1,\ldots,g_m \rangle \) for the group generated by \( g_1,\ldots,g_m \) in Sym(\( V \)), where \( m \geq 1 \).

Let \( G \) be a finite group acting on \( V \). We write the action of a permutation \( g \in G \) on a vector \( v \in V \) as \( v_g \).

We denote the translation with respect to a vector \( v \in V \) by \( \sigma_v : x \mapsto x + v \). We denote by T(\( V \)) the group of translations.

**Definition 2.1.** Let \( G \) be a group acting on \( V \). \( G \) is called **transitive** if for any \( x,y \in V \) there exists \( g \in G \) such that \( xg = y \).

\( G \) is called **regular** if for any \( x,y \in V \) there exists a unique \( g \in G \) such that \( xg = y \).

**Remark 2.2.** \( G \) is regular if and only if \( G \) is transitive and \( |G| = |V| \).

**Definition 2.3.** An element \( r \) of a ring \( R \) is called **nilpotent** if \( r^m = 0 \) for some \( m \geq 1 \) and it is called **unipotent** if \( r-1 \) is nilpotent, i.e. \( (r-1)^m = 0 \) for some \( m \geq 1 \).

Let \( G \subset \text{GL}(V) \) be a subgroup consisting of unipotent permutations, then \( G \) is called unipotent.
Definition 2.4. An element $\kappa \in \text{GL}(V)$ is said upper triangular in a basis $\{v_1, \ldots, v_N\}$ if and only if

$$v_i \kappa - v_i \in \text{Span}\{v_{i+1}, \ldots, v_N\}$$

for all $1 \leq i \leq N$. A linear map which is upper triangular in the canonical basis is called upper unitriangular. We denote by $U(V)$ the upper unitriangular linear map group.

Remark 2.5. Usually the definition of upper triangular matrix in a basis $v_1, \ldots, v_N$ is that $v_i \kappa - v_i \in \text{Span}\{v_1, \ldots, v_{i-1}\}$. Our definition comes from the fact that the map $\kappa$ acts on the right of $x$ also when the action is seen as a multiplication of a vector times a matrix, rather than a matrix times a vector, i.e. $x \kappa = x M$ where $M$ is the matrix associated to $\kappa$.

The following theorem is well-known (see for instance [1]).

Theorem 2.6. Let $G$ be a group consisting of unipotent matrices. Then there is a basis in which all elements of $G$ are upper triangular.

2.2. Translation based ciphers

Most modern block ciphers are iterated ciphers, i.e. they are obtained by the composition of a finite number $\ell$ of rounds.

There have been proposed several formal definitions for an iterated block cipher (e.g. substitution permutation network [2] and key-alternating block cipher [3]). Here we consider one more recent definition [4], determining a class large enough to include some common ciphers (AES [3], SERPENT [5], PRESENT [6]), but with enough algebraic structure to allow for security proofs.

Let $V = (F_2)^N$ with $N = mb$, $b \geq 2$. The vector space $V$ is a direct sum $V = V_1 \oplus \cdots \oplus V_b$,

where each $V_i$ has the same dimension $m$ (over $F_2$). For any $v \in V$, we will write $v = v_1 \oplus \cdots \oplus v_b$, where $v_i \in V_i$.

Any $\gamma \in \text{Sym}(V)$ that acts as $v \gamma = v_1 \gamma_1 \oplus \cdots \oplus v_b \gamma_b$, for some $\gamma_i$'s in $\text{Sym}(V_i)$, is a bricklayer transformation (a "parallel map") and any $\gamma_i$ is a brick. Traditionally, the maps $\gamma_i$'s are called "S-boxes" and $\gamma$ a "parallel S-box". A linear map $\lambda : V \to V$ is traditionally said a "Mixing Layer" when used in composition with parallel maps. For any $I \subset \{1, \ldots, b\}$ with $I \neq \emptyset$ and $I \neq \{b\}$, we say that $\bigoplus_{i \in I} V_i$ is a wall.
Definition 2.7. A linear map $\lambda \in \text{GL}(V)$ is a proper mixing layer if no wall is invariant under $\lambda$.

We can characterize the translation-based class by the following:

Definition 2.8. A block cipher $C = \{\varphi_k \mid k \in K\}$ over $\mathbb{F}_2$ is called translation based (tb) if:

- it is the composition of a finite number $\ell$ of rounds, such that any round $\rho_{k,h}$ can be written as $\gamma\lambda\sigma_k$, where
  - $\gamma$ is a round-dependent bricklayer transformation (but it does not depend on $k$),
  - $\lambda$ is a round-dependent linear map (but it does not depend on $k$),
  - $\bar{k}$ is in $V$ and depends on both $k$ and the round ($\bar{k}$ is called a “round key”),
- for at least one round, that we call proper round, we have (at the same time) that $\lambda$ is proper and that the map $K \to V$ given by $k \mapsto \bar{k}$ is surjective.

For a tb cipher $C$ it is possible to define the following groups. For any round $1 \leq h \leq \ell$

\[
\Gamma_h(C) = \langle \rho_{k,h} \mid k \in K \rangle \subseteq \text{Sym}(V),
\]

and the round function group is given by

\[
\Gamma_\infty(C) = \langle \Gamma_h(C) \mid h = 1, \ldots, \ell \rangle.
\]

An interesting problem is determining the properties of the permutation group $\Gamma_\infty(C) = \Gamma_\infty$ that imply weaknesses of the cipher. A trapdoor is a hidden structure of the cipher, whose knowledge allows an attacker to obtain information on the key or to decrypt certain ciphertexts.

The first paper dealing with properties of $\Gamma_\infty$ was published by Paterson [7], who showed that if this group is imprimitive, then it is possible to embed a trapdoor in the cipher. On the other hand, if the group is primitive no such trapdoor can be inserted. In [4] the authors investigates cryptographic properties for the S-boxes to guarantee that $\Gamma_\infty$ is primitive. Other results on $\Gamma_\infty$ can be found in [8, 9].

However, the primitivity of $\Gamma_\infty$ does not guarantee the absence of trapdoors. Indeed, if the group is contained in $\text{AGL}(V)$, the encryption function is affine and once we know the image of a basis of $V$ and the image of the zero vector, then we are able to reconstruct the matrix and the translation that compose the map.

\[1\text{we drop the round indices}\]
As mentioned in the Introduction, there are different structures of vector space \((V, \circ)\) that yield different copies of \(\operatorname{AGL}(V)\). Thus, we would like to understand if it is possible to embed the group \(\Gamma^\infty\) in a conjugated of the affine group \(\operatorname{AGL}(V)\) in \(\operatorname{Sym}(V)\). If it happens, the abelian additive group \((V, \circ)\) (or just \(\circ\)) is said a hidden sum.

**Remark 2.9.** Note that if \(h\) is a proper round of a tb cipher \(C\), then \(\Gamma_h(C) = \langle \lambda_h, \gamma_h, T(V) \rangle\) (see for instance [4]).

From Remark 2.9 we can identify a necessary property for an alternative operation \(\circ\) to be a hidden sum for a tb cipher: \(T(V)\) is contained in the affine group associated to the operation \(\circ\). In the following section we characterize some hidden sums that satisfy this property.

3. On translation groups contained in the affine group over prime fields

In the following, if not specified, \(V\) is a vector space over \(\mathbb{F}_p\) of dimension \(N\) and \(p\) a prime number (so \(q = p\)).

With the symbol \(+\) we refer to the usual sum over the vector space \(V\). We denote by \(T_+ = T(V, +)\), \(\operatorname{AGL}(V, +)\) and \(\operatorname{GL}(V, +)\), respectively, the translation, affine and linear groups w.r.t. \(+\). We use \(T_0\), \(\operatorname{AGL}(V, \circ)\) and \(\operatorname{GL}(V, \circ)\) to denote, respectively, the translation, affine and linear groups corresponding to an operation \(\circ\) such that \((V, \circ)\) is a vector space.

**Remark 3.1.** An elementary group acting on \(V = (\mathbb{F}_p)^N\) is obviously \(p\)-elementary. In particular, \(T_+\) is an elementary abelian regular group. Vice versa, if \(T\) is an elementary abelian regular group, there exists a vector space structure \((V, \circ)\) such that \(T\) is the related translation group, i.e \(T = T_0\). Indeed as shown in [10], from the regularity of \(T\) we have \(T = \{\tau_a \mid a \in V\}\) where \(\tau_a\) is the unique map in \(T\) such that \(0 \mapsto a\). Then, defining the sum \(x \circ a := x\tau_a\), it is easy to check that \((V, \circ)\) is an abelian group. Moreover, let the multiplication of a vector by an element of \(\mathbb{F}_p\) be defined by

\[
sv := \underbrace{v \circ \cdots \circ v}_s, \text{ for all } s \in \mathbb{F}_p,
\]

then it is easy to check that for all \(s, t \in \mathbb{F}_p\), and \(v, w \in V\)

\[
svw = sv \circ sw, \\
(s + t)v = sv \circ tv, \\
(st)v = s(tv)
\]

and being \(T\) elementary \(pv = 0\). Thus \((V, \circ)\) is a vector space over \(\mathbb{F}_p\). Observe that \((V, \circ)\) and \((V, +)\) are isomorphic vector spaces, since \(|V| < \infty\).
In [11] the authors give an easy description of the abelian regular subgroups of the affine group in terms of commutative associative algebras that one can define on the vector space \((V, +)\). Here, we summarize the principal result shown in [11]. Recall that a (Jacobson) radical ring is a ring \((V, +, \cdot)\) such that \((V, \circ)\) is a group, where the operation \(\circ\) is given by \(x \circ y = x + y + x \cdot y\). Note that in this case the \(\circ\) operation may not induce a vector space structure on \(V\).

**Theorem 3.2.** Let \(K\) be any (finite or infinite) field, and \((V, +)\) be a vector space of any dimension over \(K\).

There is a one-to-one correspondence between

1. (not necessarily elementary) abelian regular subgroups \(T\) of \(AGL(V, +)\), and
2. commutative, associative \(K\)-algebra structures \((V, +, \cdot)\) that one can impose on the vector space structure \((V, +)\), such that the resulting ring is radical.

In this correspondence, isomorphism classes of \(K\)-algebras correspond to conjugacy classes of abelian regular subgroups of \(AGL(V, +)\), where the conjugation is under the action of \(GL(V, +)\).

We write explicitly the correspondence, as follows.

Let \(T = \{\tau_a \mid a \in V\}\) be as in (1) of Theorem 3.2. Any \(\tau_a \in T\) can be written as \(\tau_a = \kappa \sigma\) with \(\kappa \in GL(V)\) and \(\sigma \in T_+\). Then for any \(a \in V\) we consider the map \(\delta_a = \kappa - 1_V\), with \(\kappa\) as before and \(1_V\) the identity map of \(V\). The product operation on \(V\) defined by \(x \cdot a = x \delta_a\) is such that the structure \((V, +, \cdot)\) is a commutative \(K\)-algebra and the resulting ring is radical.

**Remark 3.3.** From Theorem 3.2 we can note that if the characteristic is 2, algebras corresponding to elementary abelian regular subgroups of \(AGL(V, +)\) are exterior algebras or a quotient thereof.

We have that if \(\circ\) is such that \((V, \circ)\) is a vector space, then \(T_0\) is elementary abelian and regular. In other words, \(\circ\) is a hidden sum.

As noted in the remark above, since \(T_0\) is regular, we will label the group \(T_0\)

\[T_0 = \{\tau_a \mid a \in V\}\]

where \(\tau_a\) is the unique map such that \(0 \mapsto a\). The relation between \(T_0\) and \(AGL(V, \circ)\) is that \(AGL(V, \circ)\) is the normalizer of \(T_0\) in \(Sym(V)\). Indeed, \(AGL(V, +)\) is the normalizer of \(T_+\) and these are, respectively, the isomorphic images of \(AGL(V, \circ)\) and \(T_0\), via conjugation in \(Sym(V)\).

**Remark 3.4.** Being the semi-direct product \(AGL(V, +) = GL(V, +) \rtimes T_+\), then \(\tau_a \in T_0 \subset AGL(V, +)\) can be written uniquely as \(\kappa \sigma_b\) for \(\kappa \in GL(V, +)\)
and \( b \in V \), with \( T_+ = \{ \sigma_b \}_{b \in V} \). By definition, we have \( 0\tau_a = a \) and thus \( b = a \). We denote by \( \kappa_a \) the linear map \( \kappa \) corresponding to \( \tau_a \) and by \( \Omega(T_b) = \{ \kappa_a \mid a \in V \} \subset \text{GL}(V,+) \).

Let \( T \subseteq \text{AGL}(V,+ \) and define the set
\[
U(T) = \{ a \mid \sigma_a \in T \}.
\]

\( U(T) \) is a subspace of \( V \) (whenever \( T \) is a subgroup). Moreover if \( T = T_0 \) for a hidden sum \( \circ \), then \( U(T_0) \) is nontrivial for the following lemma.

**Lemma 3.5** ([4]). Let \( T \subseteq \text{AGL}(V,+) \) be an abelian regular subgroup. If \( V \) is finite, then \( T_+ \cap T \) is nontrivial.

We can bound the dimension of \( U(T_0) \) in the following.

**Proposition 3.6.** Let \( T \subseteq \text{AGL}(V,+) \) be an elementary abelian regular subgroup. If \( T \neq T_+ \), then \( 1 \leq \dim(U(T)) \leq N - 2 \).

**Proof.** From the lemma above we have \( 1 \leq \dim(U(T)) \). If \( \dim(U(T)) = N \) then \( T = T_+ \). Let \( T \neq T_+ \) and suppose that \( U(T) \) contains \( v_1, \ldots, v_{N-1} \) linear independent vectors. Let \( v_N \) be a vector linear independent from \( v_1, \ldots, v_{N-1} \). Being \( T \) elementary abelian regular subgroup, then \( T = T_0 \) for a hidden sum \( \circ \). For all \( 1 \leq i \leq n - 1 \), \( v_i \circ v_N = v_i + v_N \), thus we have \( v_i v_N = v_i \) for all \( 1 \leq i \leq N - 1 \). Moreover, \( v_N \circ v_N = 0 \) implies \( v_N \kappa v_N = v_N \). Then for all \( v \in V \) we have \( v \circ v_N = (\sum_{i<N} \alpha_i v_i + \alpha_N v_N)\kappa v_N + v_N = \sum_{i<N} \alpha_i v_i + \alpha_N v_N + v_N = v + v_N \). This implies \( \dim(U(T)) = N \), which leads to a contradiction. \( \square \)

We will see later on that \( U(T_0) \) plays an important role for the characterization of maps in the group \( T_0 \).

Let \( W \) be a subspace of \( V \), then for all \( \gamma \in \text{GL}(V) \) such that \( W \gamma = W \), the action of \( \gamma \) over \( V/W \) is well defined, via the map \( \tilde{\gamma} : [v] \mapsto [v \gamma] \) in \( \text{GL}(V/W) \). Then we can characterize groups of special affine maps in the following.

**Lemma 3.7.** Let \( V = (\mathbb{F}_p)^N \), with \( p \) prime number. Let \( T \) be a subgroup of \( \text{AGL}(V,+) \) generated by the affine maps \( t_{e_1}, \ldots, t_{e_N} \) such that:

1. \( t_{e_i} : x \mapsto x \kappa e_i + e_i \) for all \( i \) and \( \{ \kappa e_i \mid 1 \leq i \leq N \} \subseteq U(V) \)
2. the action of \( \kappa e_i \) over \( V/\text{Span}\{e_{i+1}, \ldots, e_N\} \) is the identity map for any \( 1 \leq i \leq N \).

Then \( T \) is transitive.
Proof. Note that for any \( i \) the action of \( \kappa_{e_i} \) over \( V/\text{Span}\{e_{i+1}, \ldots, e_N\} \) is well defined, and from conditions (1) and (2) when we apply the map \( t_{e_i} \) to a vector \( v \) the first \( i-1 \) entries of \( v \) do not change.

Consider two vectors \( v = (v_1, \ldots, v_N) \) and \( \bar{v} = (\bar{v}_1, \ldots, \bar{v}_N) \). We will show that there exists \( t \in T \) such that \( vt = \bar{v} \). We start with considering \( v_1 \) and \( \bar{v}_1 \). For some \( \gamma_1 \in \mathbb{F}_p \) we have \( v_1 = \bar{v}_1 + \gamma_1 \). So applying \( t_{e_1} \) for \( \gamma_1 \) times to \( v \) (if \( \gamma_1 = 0 \) by \( t_{e_1}^0 \) we mean \( 1V \)) we obtain from (1),

\[
v t_{e_1}^{\gamma_1} = v' = (\bar{v}_1, v_2 + c_2, \ldots, v_N + c_N),
\]

for some \( c_i \)'s in \( \mathbb{F}_p \).

Now we consider \( v_2' = \bar{v}_2 + \gamma_2 \) for some \( \gamma_2 \in \mathbb{F}_p \). Thus, from (1) and (2)

\[
v' t_{e_2}^{\gamma_2} = v'' = (\bar{v}_1, \bar{v}_2, v_3' + c_3', \ldots, v_N' + c_N').
\]

Iterating this process we obtain \( t = t_{e_1}^1 \cdots t_{e_N}^N \), which sends \( v \) to \( \bar{v} \).

\[\Box\]

**Corollary 3.8.** Let \( T = \langle t_{e_1}, \ldots, t_{e_N} \rangle \subseteq AGL(V, +) \) satisfying the condition (1) and (2) of Lemma 3.7. If \( T \) is an elementary abelian subgroup, then \( T \) is regular.

**Proof.** From Lemma 3.7 \( T \) is transitive, which implies \( |T| \geq |V| = p^N \). Now, \( T \) elementary and abelian implies that we can obtain from the composition of \( t_{e_1}, \ldots, t_{e_N} \) at most \( p^n \) maps. So \( T \) is also regular. \[\Box\]

**Remark 3.9.** These last two results imply that if \( T \) is an elementary abelian subgroup as above, then \( \{e_1, \ldots, e_N\} \) is a basis of the associated vector space structure \((V, \circ)\) and \( T = T_0 \).

**Example 3.10.** If the conditions in Corollary 3.8 are not satisfied, then the canonical basis may not be a basis w.r.t. a new sum \( \circ \). Let \( V = (\mathbb{F}_2)^3 \) and

\[
T_0 = \langle \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} + (1,0,1), \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + (0,1,1), 1V + (1,1,1) \rangle.
\]

The translations \( \tau_{e_1}, \tau_{e_2}, \tau_{e_3} \) are given by

\[
\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + e_1, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + e_2, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + e_3.
\]

Then \( e_1 \circ e_2 = e_1 \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + e_2 = e_3 \).

We come back to the more general situation (see Lemma 4 in [10]).
Lemma 3.11. Let $V = (ℙ_p)^N$ and $T_o \subseteq AGL(V,+)$ be an elementary abelian regular subgroup. Then for each $a \in V$, $\kappa_a \in GL(V,+)$ has order $p$ and it is unipotent. In particular $\Omega(T_o)$ is a unipotent subgroup of $GL(V,+)$.

Proof. We know that $\tau_a$ has order $p$, because $T_o$ is elementary. Then $\tau_a^p = 1_V$ implies $a\tau_a^{p-1} = 0$. So for all $x \in V$

$$x = x\tau_a^p = (x\kappa_a + a)\tau_a^{p-1} = (x\kappa_a^2 + a\tau_a)\tau_a^{p-2} = ... = x\kappa_a^p + a\tau_a^{p-1} = x.$$ 

This implies $(\kappa_a - 1_V)^p = \kappa_a^p - 1_V = 0$.

Lemma 3.12. Let $V = K^N$, with $K$ any field. Let $G \subseteq GL(V)$ be a unipotent subgroup and let $W \subseteq V$ be a subspace such that for all $v \in W$ and $g \in G$ $vg = v$, i.e. $G$ is contained in the stabilizer of $W$. Let $d$ be the dimension of $W$ and $N = n + d$. Then all elements of $G$ are upper triangular in a basis $\{v_1,\ldots,v_n,v_{n+1},\ldots,v_{n+d}\}$, where $\{v_{n+1},\ldots,v_{n+d}\}$ is any basis of $W$.

Proof. The vectors of $W$ are fixed by all elements of $G$. So, $G$ acts by unipotent maps on $V/W$. From Theorem 2.7 there exists a basis $\{v_1,\ldots,v_n\}$ of $V/W$, such that $[v_1][g] - [v_1]$ lies in Span$\{[v_{i+1}],\ldots,[v_n]\}$ for all elements of $G$. Then all elements of $G$ are upper triangular in the basis $v_1,\ldots,v_n,v_{n+1},\ldots,v_{n+d}$, since $v_ig - v_i = 0$ for all $n + 1 \leq i \leq n + d$.

Corollary 3.13. Let $V = K^N$ and $T_o \subseteq AGL(V,+)$ be an abelian regular subgroup such that $\dim(U(T_o)) = d$ and $\Omega(T_o)$ is a unipotent group. Let $N = n + d$, then all elements of $\Omega(T_o)$ are upper triangular in a basis $\{v_1,\ldots,v_n,v_{n+1},\ldots,v_{n+d}\}$, with $\{v_{n+1},\ldots,v_{n+d}\}$ any basis of $U(T)$.

Proof. By definition, for all $v \in U(T_o)$ and $\kappa \in \Omega(T_o)$, $v\kappa = v$. So from Lemma 3.12 we have our claim.

Let $V = (ℙ_p)^N$ then any elementary abelian regular subgroup $T \subseteq AGL(V,+)$ is unipotent. Thus we can summarize our results so far obtained in the following theorem.

Theorem 3.14. Let $V = (ℙ_p)^N$ and $T \subseteq AGL(V,+)$ be an elementary abelian regular subgroup. Then there exists a subgroup $T'$ conjugated to $T$ such that $\Omega(T') \subseteq U(V)$ and $U(T') = \text{Span}\{e_{n+1},\ldots,e_{n+d}\}$, where $d = \dim(U(T))$ and $n$ is such that $N = n + d$.

Proof. From Corollary 3.13 we have that all the elements of $\Omega(T)$ are upper triangular with respect to a basis $v_1,\ldots,v_N$, with the last $d$ vectors which form a basis of $U(T)$. Consider $g \in GL(V)$ such that $v_ig = e_i$ for all $i$'s. Since $\Omega(g^{-1}Tg) = g^{-1}\Omega(T)g$, for all $\kappa \in \Omega(T)$ we have

$$e_ig^{-1}\kappa g - e_i = v_i\kappa g - v_ig = (v_i\kappa - v_i)g.$$ 

So, being $v_i\kappa - v_i \in \text{Span}\{v_{i+1},\ldots,v_N\}$ it results $(v_i\kappa - v_i)g \in \text{Span}\{e_{i+1},\ldots,e_N\}$. In conclusion, from the fact that $g^{-1}\tau_v g : x \mapsto xg^{-1}\kappa xg + vg$, we have also $U(g^{-1}Tg) = U(T)g = \text{Span}\{e_{n+1},\ldots,e_{n+d}\}$. 

Now, we want to characterize the translations in $T_o$ such that $T_o \subseteq AGL(V, \circ)$.

We report the following lemma from \cite{11}.

**Lemma 3.15.** Let $V$ be a vector space over any field $\mathbb{K}$ and $T \subseteq AGL(V, +)$ be an abelian regular subgroup. Then for all $\sigma_x \in T_+$ and $\tau_y \in T$

$$[\sigma_x, \tau_y] = \sigma_{xy}.$$ 

Where $x \cdot y$ is the product of the $\mathbb{K}$-algebra related to $T$ as in Theorem 3.2 and $[\sigma_x, \tau_y] = \sigma_x^{-1} \tau_y^{-1} \sigma_x \tau_y$.

From Lemma 3.15, we obtain that $T_+$ normalizes $T_+ \subseteq AGL(V, +)$ if and only if $\sigma_{xy} \in T_o$ for all $x, y \in V$. Indeed, if $T_+$ normalizes $T_o$ for all $\sigma_x \in T_+$, $\sigma_x^{-1} T_o \sigma_x = T_o$, thus $\sigma_{xy} = \sigma_x^{-1} \tau_y^{-1} \sigma_x \tau_y \in T_o$.

Conversely if $\sigma_{xy} \in T_o$ then

$$\sigma_{xy} \tau_y^{-1} = \sigma_x^{-1} \tau_y^{-1} \sigma_x \in T_o.$$ 

**Remark 3.16.** Let $V = \mathbb{K}^N$, with $\mathbb{K}$ any field. Let $T \subseteq AGL(V, +)$ be an abelian regular subgroup such that $T_+$ is in the normalizer of $T$. Then any conjugate to $T$ in $AGL(V, +)$ is conjugated under the action of $GL(V, +)$. In fact, let $\tau \in AGL(V, +)$ with $\tau = \kappa \sigma$ for some $\kappa \in GL(V, +)$ and $\sigma \in T_+$ and let $T' = \tau T \tau^{-1}$. Because $T_+$ normalizes $T$, we have

$$\kappa \sigma T \sigma^{-1} \kappa^{-1} = \kappa T.$$ 

The following theorem is reported for any finite field $\mathbb{F}_p$.

**Theorem 3.17.** Let $V = (\mathbb{F}_p)^{n+d}$, with $n \geq 2$, $d \geq 1$, and $T_o \subseteq AGL(V, +)$ be such that $U(T_o) = \text{Span}\{e_{n+1}, \ldots, e_{n+d}\}$. Then, $T_+ \subseteq AGL(V, \circ)$ if and only if for all $\kappa_y \in \Omega(T_o)$ there exists a matrix $B_y \in (\mathbb{F}_p)^{n \times d}$ such that

$$\kappa_y = \begin{bmatrix} I_n & B_y \\ 0 & I_d \end{bmatrix}.$$ 

Proof. Let $T_o$ conjugated to $T_o$ in $AGL(V, +)$ be such that $U(T_o) = U(T_o)$ and $\Omega(T_o) \subseteq U(V)$, such a group exists for Theorem 3.14. Let $y \in V$ and

$$\kappa_y = \begin{bmatrix} A_y & B_y \\ 0 & I_d \end{bmatrix},$$ 

for some $B_y \in (\mathbb{F}_p)^{n \times d}$ and $A_y \in (\mathbb{F}_p)^{n \times n}$ ($A_y$ is also upper unitriangular). The identity matrix in the right bottom part is obtained by the fact that $y \circ e_i = e_i \kappa_y + y = y + e_i$ for $i = n + 1, \ldots, n + d$. 

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Lemma 3.15 implies $T_+ \subseteq \text{AGL}(V, \circ)$ if and only if $x \cdot y \in U(T_0)$ for all $x, y \in V$. Recall that $x \cdot y = x \kappa_y - x$ for all $x, y \in V$. Thus $x \cdot y \in U(T_0)$ if and only if $x \kappa_y - x \in U(T_0)$. Consider $W = \text{Span}\{e_1, \ldots, e_n\}$, then for all $x \in W$ we have that $x \kappa_y - x \in U(T_0)$ if and only if $A_y = I_n$.

Now, we need to prove only that any conjugate (in $\text{AGL}(V, +)$) $T_0$ of $T_0$ is such that all the matrices in the group $\Omega(T_0)$ have this form, whenever the space $U(T_0)$ is spanned by the last $d$ elements of the canonical basis. From Remark 3.16 we can consider the conjugates obtained from maps in $\text{GL}(V, +)$.

Let $g \in \text{GL}(V, +)$ be such that

$$U(g^{-1}T_0g) = U(T_0) = \text{Span}\{e_{n+1}, \ldots, e_{n+d}\}.$$  

This implies $U(T_0)g = U(T_0)$ and also $U(T_0)g^{-1} = U(T_0)$, so

$$g = \begin{bmatrix} G_1 & G_2 \\ 0 & G_3 \end{bmatrix}, \quad g^{-1} = \begin{bmatrix} G_1^{-1} & G_2' \\ 0 & G_3' \end{bmatrix},$$

for some $G_1 \in (\mathbb{F}_p)^{n \times n}, G_2, G_2' \in (\mathbb{F}_p)^{n \times d}$ and $G_3 \in (\mathbb{F}_p)^{d \times d}$. Then for all $\kappa \in \Omega(T_0)$ we have

$$g^{-1} \kappa g = \begin{bmatrix} G_1^{-1} & G_2' \\ 0 & G_3^{-1} \end{bmatrix} \begin{bmatrix} I_n & B_{n \times d} \\ 0 & I_d \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ 0 & G_3 \end{bmatrix} = \begin{bmatrix} I_n & B'_{n \times d} \\ 0 & I_d \end{bmatrix}.$$

\[\square\]

4. Attack to tb cipher based on hidden sum

In this section we will show that starting from a group $T_0$ whose maps are affine, if the round function group of a tb cipher lies in the affine group associated to this hidden sum $\circ$, then we can implement a practical attack.

4.1. Affine maps normalized by the translation group

In this subsection we want to explain the reason why we concentrate our studies on translation groups coming from subgroups in $\text{AGL}(V, +)$, that are normalized by the usual translation group $T_+$.

To embed a hidden sum trapdoor in a block cipher we need $\Gamma_\infty \subseteq \text{AGL}(V, \circ)$ for some hidden sum $\circ$, thus a first condition is $T_+ \subseteq \text{AGL}(V, \circ)$, as $T_+ \subseteq \Gamma_\infty$. Now, let $T_0 \subseteq \text{AGL}(V, +)$ be such that $T_+ \subseteq \text{AGL}(V, \circ)$. Consider the vector space $U(T_0)$, which has dimension $d$ for some $d \geq 1$. Let $g \in \text{GL}(V, +)$ be such that $U(T_0)g = \text{Span}\{e_{n+1}, \ldots, e_{n+d}\} = U(T_0)$, with $T_0 = g^{-1}T_0g$. $g$ is an isomorphism of vector space between $(V, \circ)$ and $(V, \circ)$. From Theorem 3.17 we have that the maps relatives to the canonical basis are

$$\kappa_{e_i} e_i = \begin{bmatrix} I_n & B_{e_i} \\ 0 & I_d \end{bmatrix} + e_i.$$
for some \( B_{e_i} \in (\mathbb{F}_2)^{n \times d} \). Moreover from Lemma 3.7, we have also that \( e_1, \ldots, e_N \) is a basis of \((V, \circ)\) and to write \( \mathbf{v} \in V \) as a linear combination of these w.r.t. to the sum \( \circ \), i.e., \( \mathbf{v} = \lambda_1 e_1 \circ \cdots \circ \lambda_N e_N \), we can use the Algorithm 4.1.

**Algorithm 4.1.**

**Input:** vector \( \mathbf{v} = (v_1, \ldots, v_N) \in V \)

**Output:** coefficients \( \lambda_1, \ldots, \lambda_N \).

1. \( \lambda_i \leftarrow v_i \) for \( 1 \leq i \leq n \);
2. \( \mathbf{v}' \leftarrow \mathbf{v} \tau_{e_1}^{\lambda_1} \cdots \tau_{e_n}^{\lambda_n} \);
3. \( \lambda_i \leftarrow v'_i \) for \( n + 1 \leq i \leq n + d \);

return \( \lambda_1, \ldots, \lambda_N \),

where \( \tau_{e_i} \) is the translation \( x \mapsto x \circ e_i \) and the notation \( x \tau_{\mathbf{v}}^b \), with \( b \in \mathbb{F}_2 \), denote either \( x \tau_{\mathbf{v}} \) (when \( b = 1 \)) or \( x \) (when \( b = 0 \)). Thus, let \( \mathbf{v}_i = e_i g^{-1} \) for all \( i \), applying Algorithm 4.1 to \( \mathbf{v}_g \) we can obtain the combination of \( \mathbf{v}_g \)'s w.r.t the sum \( \circ \) of the vector \( \mathbf{v} \). The complexity of this procedure is \( O(N^3) \).

Indeed, we multiply a vector of length \( N \) for an \( N \times N \) matrix (which has complexity \( O(N^3) \)) for \( n \leq N \) times.

**Remark 4.1.** If \( T_+ \subseteq \text{AGL}(V, \circ) \), but \( T_0 \nsubseteq \text{AGL}(V, +) \), then for any basis of \((V, \circ)\) there exists a vector \( \mathbf{v} \) such that \( \tau_{\mathbf{v}} \notin \text{AGL}(V, +) \), thus we need to apply a non-linear map to vectors of length \( n \), which might implies a huge quantity of memory making infeasible the computation of the combination with respect to the new operation \( \circ \).

### 4.2. Hidden sum attack

Let \( \mathcal{C} = \{ \varphi_k \mid k \in \mathcal{K} \} \) be a tb cipher such that \( \Gamma_{\infty} \subseteq \text{AGL}(V, \circ) \) for some operation \( \circ \), and also \( T_0 \subseteq \text{AGL}(V, +) \). Let \( \dim(U(T_0)) = d \). Let \( g \in \text{GL}(V, +) \) be a linear permutation such that \( U(T_0)g = \text{Span}\{ e_{n+1}, \ldots, e_{n+d} \} \). Denote by

\[
[\mathbf{v}] = [\lambda_1, \ldots, \lambda_N]
\]

the vector with the coefficients obtained from Algorithm 4.1. Let \( \varphi = \varphi_K \) be the encryption function, with a given unknown session key \( K \). We are able to mount an attack, computing the matrix \( M \) and the translation vector \( t \) defining \( \varphi \in \text{AGL}(V, \circ) \).

Choose the plaintext \( 0 \varphi, \mathbf{v}_1 \varphi, \ldots, \mathbf{v}_N \varphi \), where \( \mathbf{v}_i = e_i g^{-1} \), and compute \( [0 \varphi g], [\mathbf{v}_1 \varphi g], \ldots, [\mathbf{v}_N \varphi g] \), since the translation vector is \( [t] = [0 \varphi g] \) and the \( [e_i \varphi g] + [t]'s \) are the matrix rows. In other words, we will have

\[
[\mathbf{v} \varphi g] = [\mathbf{v} g] \cdot M + [t], \quad [\mathbf{v} \varphi^{-1} g] = ([\mathbf{v} g] + [t]) \cdot M^{-1},
\]

for all \( \mathbf{v} \in V \), where the product row by column is the standard scalar product. The knowledge of \( M \) and \( M^{-1} \) provides a global deduction (reconstruction), since it becomes trivial to encrypt and decrypt. Moreover from
We obtain that $v = 0_{\tau_{v_1}} \cdots \tau_{v_N}^N$, where $\tau_{v_i} : x \mapsto x \circ v_i$. So, we need only $N + 1$ plaintext to reconstruct the cipher and the cost of this attack is given from the algorithm above to compute the combinations plus the cost of $n + 1$ encryptions.

Our discussion has thus proved the following result.

**Theorem 4.2.** Hidden sum trapdoors coming from translation groups such that $T_\circ \subseteq AGL(V, +)$ are trapdoors, that allows for any key to perform a global deduction attack in $O(N^3)$ encryptions.

4.3. A toy-block cipher with a hidden sum

In this section we give an example, in a small dimension, of a translation based block cipher in which it is possible to embed a hidden-sum trapdoor.

Let $m = 3$, $b = 2$, then $N = 6$ and we have the message space $V = (F_2)^6$.

The mixing layer of our toy cipher is given by the matrix

$$
\lambda = \begin{bmatrix}
0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}
$$

The bricklayer transformation $\gamma = (\gamma_1, \gamma_2)$ of our toy cipher is given by two identical S-boxes

$$
\gamma_1 = \gamma_2 = \alpha^5 x_6^3 + \alpha x_5^3 + \alpha^2 x_4 + \alpha^5 x_3^3 + \alpha x_2 + \alpha x
$$

where $\alpha$ is a primitive element of $F_{2^3}$ such that $\alpha^3 = \alpha + 1$.

We show now the existence of a hidden-sum trapdoor for our toy cipher. We consider the hidden sum $\circ$ over $V_1 = V_2 = (F_2)^3$ induced by the elementary abelian regular group $T_0 = \langle \tau_1, \tau_2, \tau_3 \rangle$, where

$$
\tau_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix} + e_1, \quad \tau_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} + e_2, \quad \tau_3 = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} + e_3. \quad (1)
$$

Obviously, $T = T_0 \times T_0$ is an elementary abelian group inducing the hidden sum $(x_1, x_2) \circ' (y_1, y_2) = (x_1 \circ y_1, x_2 \circ y_2)$ on $V = V_1 \times V_2$.

**Theorem 4.3.** $\langle T_+ , \gamma \lambda \rangle \subseteq AGL(V, \circ')$.

**Proof.** By a computer check $\gamma \lambda \in AGL(V, \circ')$, and from Theorem 4.1, $T_+ \subseteq AGL(V, \circ')$. \qed

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Theorem 4.3 implies that \( \phi' \) is a hidden sum for our toy cipher. We show, now, how it is possible to use it to attack the toy cipher with an attack that costs less than brute force. We note that the cost of our attack (expressed in number of encryptions) does not depend on either the number of round or the key-schedule. Therefore, the hidden sum will actually break the cipher only if the attack that we build will cost significantly less than 64 encryptions, considering that the key space is \( (\mathbb{F}_2)^6 \).

We denote by \([w]\) the vector with the coefficients obtained from the Algorithm 4.1.

Let \( \varphi = \varphi_K \) be the encryption function, with a given unknown session key \( K \). We want to mount two attacks by computing the matrix \( M \) and the translation vector \( t \) defining \( \varphi \in AGL(V, \circ') \).

Assuming that we can call the encryption oracle, then \( M \) can be computed from the 7 ciphertexts \( 0\varphi, e_1\varphi, \ldots, e_6\varphi \) as seen before. In other words, we will have

\[
[w_{\varphi}] = [w] \cdot M + [t], \quad [w_{\varphi}^{-1}] = ([w] + [t]) \cdot M^{-1},
\]

for all \( w \in V \). However, we have an alternative, depending on how we compute \( \varphi^{-1} \):

- if we compute \( M^{-1} \) from \( M \), by applying Gaussian reduction, we will need only our 7 initial encryptions;
- else we can compute \( M^{-1} \) from the action of \( \varphi^{-1} \), assuming we can call the decryption oracle. Indeed, performing the 7 decryptions \( e_i\varphi^{-1} \) and \( 0\varphi^{-1} \), the rows of \( M^{-1} \) will obviously be \([e_i\varphi^{-1}] + [0\varphi^{-1}]\).

The first attack (of type chosen-plaintext attack) requires only 7 encryptions but more binary operations. The second (of type chosen-plaintext/chosen-ciphertext attack) requires both 7 encryptions and 7 decryptions, but the binary operations are less. Both obtain the same goal, that is, the complete reconstruction of the encryption and decryption functions.

**Remark 4.4.** We focus on the case \( T_+ \subseteq AGL(V, \circ) \), since it permits to implement a hidden sum trapdoor independently from the action of the key-schedule. However, if the translation group \( T_+ \) is not properly contained in the affine group of the hidden sum, but the intersection \( T_+ \cap AGL(V, \circ) \) is non-trivial, then the translations in that intersection represent a set of weak keys for the cipher. The set of weak keys can be huge and for any key there exist different hidden sums which linearize it. That permits to have a high probability to break the cipher with the hidden sum trapdoor. Thus, it could be possible to create a partial trapdoor.
5. Some combinatorial counting results

In this section we specialize to the cryptographically-important case of the binary field. In this case we are able to give an upper bound on the number of the elementary abelian regular subgroups of the form as in Theorem 3.17. Moreover when the co-dimension of $U(T_o)$ is 2 or 3 we can compute the number of these groups. In the last part we report the full classification of the elementary abelian regular subgroups of AGL($V$,$+$) up to dimension 6.

A first result in characteristic 2 is that, if the co-dimension of the space $U(T_o)$ is at most 5, then the group $T_o$ normalizes $T_o$.

**Proposition 5.1.** Let $V = (F_2)^{n+d}$. If $T_o \subseteq \text{AGL}(V,+)$ is an elementary abelian regular subgroup with $\dim U(T_o) = d$ and $2 \leq n \leq 5$, then $T_o \subseteq \text{AGL}(V,o)$.

**Proof.** We will show that for each pair $x, y \in V$ the product $xy$ lies in $U(T_o)$. Let $x, y \in V$ and suppose $xy \not\in U(T_o)$, thus there exists $z \in V \setminus U(T_o)$ such that $xyz \neq 0$. This implies, also, $xz \neq 0$ and $yz \neq 0$.

Thus the vectors $x, y, z, xy, xz, yz$ and $xyz$, they are all non-zero. Suppose now that there exist $\lambda_x, \lambda_y, \lambda_z, \lambda_{xy}, \lambda_{xz}, \lambda_{yz}, \lambda_{xyz} \in F_2$ such that

$$\lambda_x x + \lambda_y y + \lambda_z z + \lambda_{xy} xy + \lambda_{xz} xz + \lambda_{yz} yz + \lambda_{xyz} xyz = 0. \quad (2)$$

Multiplying by $yz$ Equation (2), and recalling that $a^2 = 0$ for all $a \in V$, we have $\lambda_{xyz} xyz = 0$, that implies $\lambda_x = 0$. Analogously multiplying by $xz, xy, x, y$ and $z$ we obtain $\lambda_y = \lambda_z = \lambda_{xy} = \lambda_{xz} = \lambda_{yz} = 0$. So, it results $\lambda_{xyz} xyz = 0$, that implies $\lambda_{xyz} = 0$.

Then $x, y, z, xy, xz, yz$ and $xyz$ are linear independent, and none of $x, y, z, xy, xz, yz$ belong to $U(T_o)$ (because $xyz \neq 0$). It is possible to prove with similar argument also that $\text{Span}\{x, y, z, xy, xz, yz\} \cap U(T_o) = \{0\}$ and this implies $n \geq 6$ which leads to a contradiction.

**Example 5.2.** Proposition 5.1 does not hold, in general, for $n \geq 6$. Let $(V,+,$ $.)$ be the exterior algebra over a vector space of dimension three, spanned by $e_1, e_2, e_3$. That is, $V$ has basis

$$e_1, e_2, e_3, e_4 = e_1 \wedge e_2, e_5 = e_1 \wedge e_3, e_6 = e_2 \wedge e_3, e_7 = e_1 \wedge e_2 \wedge e_3.$$  

The associated translation group $T_o$ is such that $U(T_o) = \text{Span}\{e_7\}$, but we
have

\[
\kappa_{e_i} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

From Theorem 3.17, $\text{AGL}(V, \circ)$ does not contain the translation group $T_+$.  

**Remark 5.3.** Let $T \subseteq \text{AGL}(V, +)$ be an abelian regular group and $\tau_{e_i}, \tau_{e_j} \in T$ be the maps corresponding to the canonical vectors $e_i, e_j$ (see Remark 3.4). Then from

\[
e_i \circ e_j = e_i \kappa_{e_j} + e_j = e_j \kappa_{e_i} + e_i = e_j \circ e_i
\]

we obtain that the $i$-th row of $\kappa_{e_j}$ and the $j$-th row of $\kappa_{e_i}$ differ only in the positions $i$ and $j$.

Vice versa, if we have that the $i$-th row of $\kappa_{e_j}$ and the $j$-th row of $\kappa_{e_i}$ differ only in the positions $i$ and $j$, we obtain that $\tau_{e_i}$ and $\tau_{e_j}$ commute and in particular the group generated by the maps $\tau_{e_i}$’s is abelian.

**Lemma 5.4.** Let $V = (\mathbb{F}_2)^{n+d}$. Let $T_0$ be as in Theorem 3.17 with $U(T_0) = \text{Span}\{e_{n+1}, \ldots, e_{n+d}\}$. Let the matrices

\[
\kappa_{e_i} = \begin{bmatrix}
I_n & B_{e_i} \\
0 & I_d
\end{bmatrix}
\]

for all $i$. Let $x \in V$ with $x = x_1 e_1 \circ \cdots \circ x_{n+d} e_{n+d}$ for some $x_i \in \mathbb{F}_2$. Then

\[
\kappa_x = \begin{bmatrix}
I_n & \sum_{i=1}^n x_i B_{e_i} \\
0 & I_d
\end{bmatrix}.
\]

**Proof.** We have that the canonical basis is a basis also for the space $(V, \circ)$ and we have the blocks $B_{e_i} \neq 0$ for $1 \leq i \leq n$ and $B_{e_i} = 0$ for $n+1 \leq i \leq n+d$. Let us suppose $x = e_i \circ e_j$ for some $i$ and $j$. For all $y \in V$

\[
y \circ x = y \circ e_i \circ e_j
= (y \kappa_{e_i} + e_i) \tau_{e_j}
= y \kappa_{e_i} \kappa_{e_j} + e_i \kappa_{e_j} + e_j
= y \kappa_{e_i} \kappa_{e_j} + x = y \kappa_x + x.
\]

Thus

\[
\kappa_x = \kappa_{e_i} \kappa_{e_j} = \begin{bmatrix}
I_n & B_{e_i} + B_{e_j} \\
0 & I_d
\end{bmatrix}.
\]
Suppose now that this is true for the sum (with respect to \(\circ\)) of \(m \geq 2\) \(e_i\)’s. Consider \(x\) the sum of \(m + 1\) \(e_i\)’s, then \(x = \bigcup_{1 \leq j \leq m+1} \cdot e_i\). By induction we have

\[
y \circ x = \left(\bigcup_{1 \leq j \leq m} e_i\right) \circ e_{m+1}
\]

\[
= \left(y \prod_{1 \leq j \leq m} k_{e_ij} + \left(\bigcup_{1 \leq j \leq m} e_i\right)\right) \tau e_{m+1}
\]

\[
= y \prod_{1 \leq j \leq m+1} k_{e_i j} + \left(\bigcup_{1 \leq j \leq m} e_i\right) k_{e_i m+1} + e_i
\]

\[
= y \prod_{1 \leq j \leq m+1} k_{e_i j} + x = y \kappa x + x.
\]

From the block matrix form we obtain

\[
\kappa x = \begin{bmatrix} I_n & \sum_{j=1}^{m+1} B e_j \\ 0 & I_d \end{bmatrix}.
\]

\[
\square
\]

**Theorem 5.5.** Let \(N = n + d\) and \(V = (\mathbb{F}_2)^N\), with \(n \geq 2\) and \(d \geq 1\). The elementary abelian regular subgroups \(T_0 \subseteq AGL(V, +)\) such that \(\dim(U(T_0)) = d\) and \(T_+ \subseteq AGL(V, \cdot)\) are

\[
\begin{bmatrix} N \\ d \end{bmatrix} : |\mathcal{V}(\mathcal{I}_{n,d})|
\]

where \(\mathcal{I}_{n,d}\) is the ideal in \(\mathbb{F}_2 \left[ b_{i,j}^{(s)} \mid i, s \in [n], j \in [d] \right] \) generated by

\[
S_0 \cup S_1 \cup S_2 \cup S_3
\]

with

\[
S_0 = \left\{ b_{i,j}^{(s)} \mid i, s \in [n], j \in [d] \right\}
\]

\[
S_1 = \left\{ \prod_{i=1}^{n} \prod_{j=1}^{d} \left(1 + \sum_{s \in S} b_{i,j}^{(s)} \right) \mid S \subseteq [n], S \neq \emptyset \right\},
\]

\[
S_2 = \left\{ b_{i,j}^{(s)} - b_{s,j}^{(i)} \mid i, s \in [n], j \in [d] \right\},
\]

\[
S_3 = \left\{ b_{i,j}^{(i)} \mid i \in [n], j \in [d] \right\},
\]

\(\mathcal{V}(\mathcal{I}_{n,d})\) is the variety of \(\mathcal{I}_{n,d}\) and \(\begin{bmatrix} N \\ d \end{bmatrix}_q = \prod_{i=0}^{k-1} q^{N_i - 1} \) is the Gaussian Binomial.
Proof. As seen in Theorem 3.14 we have that for any group $T_0$ there exists a conjugated in the form given in Theorem 3.17. Thus, first we need to compute the number of the groups as in Theorem 3.17 and then all the conjugates obtained from these.

Let $T_0 \subseteq AGL(V, +)$ such that $U(T_0)$ is generated by the last $d$ elements of the canonical basis and $T_+ \subseteq AGL(V, \circ)$. From Theorem 3.17 we have that the matrices $\kappa_{e_i}$’s for $1 \leq i \leq n$ are

$$
\kappa_{e_i} = \begin{bmatrix}
    b^{(i)}_{1,1} & \ldots & b^{(i)}_{1,d} \\
    I_n & \vdots & \vdots \\
    b^{(i)}_{n,1} & \ldots & b^{(i)}_{n,d} \\
    0 & \ldots & 0
\end{bmatrix}.
$$

(4)

To individuate such a group $T_0$ we need to individuate only the matrices $\kappa_{e_1}, \ldots, \kappa_{e_n}$ (and thus $B_{e_1}, \ldots, B_{e_n}$), since the matrices corresponding to the last $k$ vectors $e_{n+1}, \ldots, e_{n+d}$ are all equal to the identity. We will show that to a set of binary matrices $\{B_{e_1}, \ldots, B_{e_n}\}$ there corresponds one point in $V(I_{n,d})$ and, vice versa, from a point of $V(I_{n,d})$ we can obtain one set of binary matrices $\{B_{e_1}, \ldots, B_{e_n}\}$.

Let $T_0$ be a hidden sum and $\{B_{e_1}, \ldots, B_{e_n}\}$ be the related set of matrices (Theorem 3.17). Let $S$ be a non-empty subset of $[n]$ and $x = \bigcap_{i \in S} e_i$, then (see Lemma 3.4)

$$
\kappa_x = \begin{bmatrix}
    \sum_{s \in S} b^{(s)}_{1,1} & \ldots & \sum_{s \in S} b^{(s)}_{1,d} \\
    I_n & \vdots & \vdots \\
    \sum_{s \in S} b^{(s)}_{n,1} & \ldots & \sum_{s \in S} b^{(s)}_{n,d} \\
    0 & \ldots & 0
\end{bmatrix}.
$$

If $U(T_0)$ is generated only by $e_{n+1}, \ldots, e_{n+d}$, then we have that $\kappa_x \neq 1_V$ otherwise the vector $x$ lies in $U(T_0)$. Thus there exist $i, j$ such that

$$
\sum_{s \in S} b^{(s)}_{i,j} = 1.
$$

This happens if and only if

$$
\prod_{i=1}^{n} \prod_{j=1}^{d} \left( 1 + \sum_{s \in S} b^{(s)}_{i,j} \right) = 0.
$$

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Moreover, from the fact that \( \kappa \) fixes \( e_i \), we have a solution also for set \( S_3 \). \( S_0 \) is obviously satisfied, since the matrices are binary.

Vice versa, consider a solution of the ideal \( I_{n,d} \), from this we obtain some matrices \( B_0, \ldots, B_{e_n} \) constructed as in (1). Then, we consider the group \( T \) generated by the affine maps \( \tau_{e_i} = \kappa_{e_i} \sigma_{e_i} \) with

\[
\kappa_{e_i} = \begin{bmatrix} I_n & B_{e_i} \\ 0 & I_d \end{bmatrix}
\]

for \( 1 \leq i \leq n \), and \( \kappa_{e_i} = I_N \) for \( n + 1 \leq i \leq n + d \).

We can note that these maps satisfy the conditions of Lemma 3.7, thus such a group is transitive. Thanks to the expressions given by set \( S \) generated by the affine maps \( B_{e_i} \)'s commute (Remark 5.3). Thus \( T \) is abelian. Now, let \( x \in V \) and \( e_i \) fixed. Then

\[
x \kappa_{e_i}^2 = (x \kappa_{e_i}^2 + e_i \kappa_{e_i} + e_i).
\]

From the form of block matrix, we have

\[
\kappa_{e_i}^2 = \begin{bmatrix} I_n & B_{e_i} + B_{e_i} \\ 0 & I_d \end{bmatrix} = I_N.
\]

Moreover, the conditions given by set \( S_3 \) implies \( e_i \kappa_{e_i} = e_i \). Thus we have obtained \( \tau_{e_i}^2 = 1_V \), which implies \( T \) is elementary. From Corollary 3.8 \( T \) is also regular. And so our claim is proved and we have a one-to-one correspondence between the points of \( V(I_{n,d}) \) and the set of the subgroups \( T \) such that \( U(T \circ g) = \text{Span} \{ e_{n+1}, \ldots, e_{n+d} \} \) and \( T_+ \subseteq \text{AGL}(V, \circ) \).

Finally, consider \( d \)-dimensional vector subspace \( W \subseteq V \) and let \( \Delta = |V(I_{n,d})| \). From Remark 3.10 we can consider only the conjugated obtained with elements of \( \text{GL}(V, +) \). Let \( g \in \text{GL}(V, +) \) be such that \( W g = \text{Span} \{ e_{n+1}, \ldots, e_{n+d} \} \). Denote by \( T_1, \ldots, T_\Delta \) the distinct elementary abelian regular groups with \( U(T_1) = \text{Span} \{ e_{n+1}, \ldots, e_{n+d} \} \), obtained above. Then the groups \( T_1, \ldots, T_\Delta \), with \( T_j' = g T_j g^{-1} \) are all distinct and \( U(T_j') = W \).

Now, let \( T_0 \) be an elementary abelian regular subgroup such that \( U(T_0) = W \). Thus \( U(g^{-1} T_0 g) = U(T_0) g = \text{Span} \{ e_{n+1}, \ldots, e_{n+d} \} \). This implies \( g^{-1} T_0 g = T_i \) for some \( i \), and so \( T_0 = T_i \). Being the number of \( d \)-dimensional vector subspace of \( V \) given by \( \binom{N}{d} \), we immediately have our claimed formula (3).

**Proposition 5.6.** Let \( I_{n,d} \) defined as in Theorem 5.6 then

\[
|V(I_{n,d})| \leq 2^{inom{n(n-1)}{2}} - 1 - \sum_{r=1}^{n-2} \binom{n}{r} \left( 2^d - 1 \right) \binom{n-r}{2}.
\]

**Proof.** Let us consider the vector

\[
\mathbf{B} = (b_1^{(1)}, \ldots, b_n^{(1)}, b_1^{(2)}, \ldots, b_n^{(2)}, \ldots, b_1^{(n)}, \ldots, b_n^{(n)}) \in V(I_{n,d}),
\]

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where \( b_i^{(s)} = (b_i^{(s)}; \ldots; b_i^{(s)}) \in (\mathbb{F}_2)^d \) for all \( i, j \) as in (4). That is, \( b_i^{(s)} \) is the \( i \)th row of the matrix \( B_{e_i} \).

We would like to count the vectors \( \mathbf{B} \) that satisfy the constrains given by set \( S_1, S_2 \) and \( S_3 \). We are unable to do that in the general case, so we proceed in two steps. First we consider all the solutions for \( S_2 \) and \( S_3 \). Second, we subtract some of them for which the equations of \( S_1 \) are not satisfied.

**First step:** From the conditions in \( S_3 \) we have \( b_i^{(i)} = 0 \) for all \( i \), and from \( S_2 \), \( b_j^{(i)} = b_i^{(j)} \) for all \( i, j \). So the matrix \( B_{e_1} \) is determined only by the rows \( b_2^{(1)}, \ldots, b_n^{(1)} \), the first row \( b_1^{(1)} \) equal to zero. Then, \( B_{e_2} \) is determined only by the rows \( b_3^{(2)}, \ldots, b_n^{(2)} \) and \( b_2^{(1)} \), being the first row of \( B_{e_2} \) equal to the second row of \( B_{e_1} \) and being the second row of \( B_{e_2} \) equal to zero.

Iterating this argument we can consider only the vector formed by

\[
\mathbf{B} = (b_2^{(1)}, \ldots, b_n^{(1)}, b_3^{(2)}, \ldots, b_n^{(2)}, \ldots, b_{n-1}^{(n-2)}, b_n^{(n-2)}, b_n^{(n-1)})
\]

and thus we have \( 2^{d-\binom{n}{2}} \) solutions of the equations in \( S_2 \cup S_3 \).

**Second step:** The entries of \( \mathbf{B} \) must satisfy also the constrains given by \( S_1 \), so for any subset \( S \) of \([n]\) we can exclude some of the cases for which

\[
B_{e_i} = \begin{cases} 
0 & \text{if } i \in S \\
\neq 0 & \text{if } i \notin S.
\end{cases}
\]

In particular we want to count when the entries of the matrices \( B_{e_i} \) with \( i \in S \) are all zeros and the remaining entries of the matrices \( B_{e_i} \) with \( i \notin S \) are all non-zero.

We start considering those vectors \( \mathbf{B} \) (that we obtain) when exactly one \( B_{e_i} \) is zero and other \( B_{e_j} \)'s with all the remaining entries non-zero, that is, we consider any set \( S \) of cardinality 1. In this case we have that \( n-1 \) entries \((b_i^{(s)})\)s of \( \mathbf{B} \) are zero and the others are all non-zero.

Similarly, if any pair \( B_{e_s}, B_{e_t} \) are equal to zero and the others are not (with all non-zero entries) then \( n-1+n-2 \) entries of \( \mathbf{B} \) are zero and the others are all non-zero. In fact, suppose \( s < t \) then the zero entries of \( \mathbf{B} \) must be \( b_s^{(1)}, \ldots, b_s^{(s-1)}, b_{s+1}^{(s)}, \ldots, b_n^{(s)} \) to have \( B_{e_s} = 0 \) and \( b_t^{(1)}, \ldots, b_t^{(t-1)}, b_{t+1}^{(t)}, \ldots, b_n^{(t)} \) to have \( B_{e_t} = 0 \). Considering that \( b_i^{(s)} \) is already zero from the condition on \( B_{e_2} \) we have \( n-1+n-2 \) entries of \( \mathbf{B} \) are zero. Iterating, if we consider \( r \) matrices that have to be zero then \( \sum_{i=1}^{r} n-i \) entries of \( \mathbf{B} \) are zero and the others are all non-zero.

We have \( \binom{n}{r} \) possible choices of these \( r \) matrices and, any time, we have
\(2^d - 1\) non zero elements to fill each of the other entries of \(B\), that are

\[
\frac{n(n-1)}{2} - \sum_{i=1}^{r} n - i = \binom{n}{2} - \sum_{i=n-r}^{n-1} i
\]

\[
= \binom{n}{2} - \sum_{i=1}^{n-r} i + \sum_{i=1}^{n-r} i
\]

\[
= \binom{n}{2} - \binom{n}{2} + \binom{n-r}{2}
\]

\[
= \binom{n-r}{2}.
\]

The last case is when \(n - 1\) matrices \(B_{e_i}\) are zero. By the conditions of \(S_2 \cup S_3\) also the last one is zero, and this happens only when \(B\) is zero. □

When \(U(T_0)\) has co-dimension 2 and 3 we have the following results.

**Corollary 5.7.** Let \(V = (\mathbb{F}_2)^N\). There exist

\[
\left[ \begin{array}{c} N \\ N-3 \end{array} \right] \cdot \left( 2^{2(N-3)} - 7(2^{N-3} - 1) - 1 \right)
\]

distinct elementary abelian regular subgroups of \(AGL(V, +)\) such that \(\dim(U(T)) = N - 3\).

**Proof.** From Theorem 5.5 we need to compute the number of groups such that \(U(T) = \text{Span}\{e_1, \ldots, e_N\}\). To do this we count the impossible case when the \(\prod_{i \in S} \kappa_{e_i} = 1\) for \(S \subseteq \{1, 2, 3\}\). Using the notation in Proposition 5.6 we have

\[
\kappa_{e_1} = \begin{bmatrix} 1 & 0 & 0 & b_2^{(1)} \\ 0 & b_2^{(1)} & 1 \\ 0 & 1 & b_3^{(1)} \\ 1 & 0 & b_3^{(1)} \end{bmatrix}, \kappa_{e_2} = \begin{bmatrix} 1 & 0 & 0 & b_2^{(1)} \\ 0 & b_2^{(1)} & 1 \\ 0 & 1 & b_3^{(1)} \\ 1 & 0 & b_3^{(1)} \end{bmatrix}
\]

\[
\kappa_{e_3} = \begin{bmatrix} 1 & 0 & 0 & b_2^{(1)} \\ 0 & b_2^{(1)} & 1 \\ 0 & 1 & b_3^{(1)} \\ 1 & 0 & b_3^{(1)} \end{bmatrix}, \kappa_{e_1 \kappa_{e_2}} = \begin{bmatrix} 1 & 0 & 0 & b_2^{(1)} \\ 0 & b_2^{(1)} & 1 \\ 0 & 1 & b_3^{(1)} + b_3^{(2)} \\ 1 & 0 & b_3^{(1)} \end{bmatrix}
\]

\[
\kappa_{e_1 \kappa_{e_3}} = \begin{bmatrix} 1 & 0 & 0 & b_2^{(1)} + b_3^{(1)} \\ 0 & b_2^{(1)} & 1 \\ 0 & 1 & b_3^{(1)} + b_3^{(2)} \\ 1 & 0 & b_3^{(1)} \end{bmatrix}, \kappa_{e_2 \kappa_{e_3}} = \begin{bmatrix} 1 & 0 & 0 & b_2^{(1)} + b_3^{(1)} \\ 0 & b_2^{(1)} & 1 \\ 0 & 1 & b_3^{(1)} + b_3^{(2)} \\ 1 & 0 & b_3^{(1)} \end{bmatrix}
\]

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We have the following cases

1. $\kappa e_1 = 1 \iff b_2^{(1)} = 0$ and $b_3^{(1)} = 0$;
2. $\kappa e_2 = 1 \iff b_2^{(1)} = 0$ and $b_3^{(2)} = 0$;
3. $\kappa e_3 = 1 \iff b_3^{(1)} = 0$ and $b_3^{(2)} = 0$;
4. $\kappa e_1 \kappa e_2 = 1 \iff b_2^{(1)} = 0$ and $b_3^{(1)} = b_3^{(2)}$;
5. $\kappa e_1 \kappa e_3 = 1 \iff b_3^{(1)} = 0$ and $b_2^{(1)} = b_3^{(2)}$;
6. $\kappa e_2 \kappa e_3 = 1 \iff b_2^{(1)} = b_3^{(1)}$ and $b_3^{(2)} = 0$;
7. $\kappa e_1 \kappa e_2 \kappa e_3 = 1 \iff b_2^{(1)} = b_3^{(1)}$, $b_2^{(1)} = b_3^{(2)}$ and $b_3^{(1)} = b_3^{(2)}$.

Each case admits $2^{N-3}$ values for $\mathbf{B} = (b_2^{(1)}, b_3^{(1)}, b_3^{(2)})$ and the only common solution is $(b_2^{(1)}, b_3^{(1)}, b_3^{(2)}) = (0, 0, 0)$. Thus we have $2^{2(N-3)} - 7(2^{N-3} - 1) - 1$ subgroups with $U(T) = \text{Span}\{e_1, \ldots, e_N\}$.

**Corollary 5.8.** Let $V = (\mathbb{F}_2)^N$. There exist $\left[\begin{array}{c} N \\ N-2 \end{array}\right] \cdot (2^{N-2} - 1)$ distinct elementary abelian regular subgroups of $\text{AGL}(V, +)$ such that $\text{dim}(U(T_0)) = N - 2$.

**Proof.** In this case, using the same notation of Proposition 5.6 we have to find the possible values of the vector $\mathbf{B} = (b_2^{(1)})$, that are those for which $\mathbf{B} \neq 0$. Then there exist $2^{N-2} - 1$ elementary abelian regular subgroups with $U(T_0) = \text{Span}\{e_3, \ldots, e_N\}$, and so $\left[\begin{array}{c} N \\ N-2 \end{array}\right] \cdot (2^{N-2} - 1)$ subgroups of $\text{AGL}(V, +)$ such that $\text{dim}(U(T_0)) = N - 2$.

We recall that the Hamming weight of a vector $\mathbf{b}$ is the number of its non-zero entries and it is denoted by $w_H(\mathbf{b})$.

**Proposition 5.9.** The groups of Corollary 5.8 are all conjugated.

**Proof.** We need to prove that the groups such that $U(T_0) = \text{Span}\{e_3, \ldots, e_N\}$ are all conjugated. Each of those groups corresponds to a vector $b_2^{(1)} \in (\mathbb{F}_2)^{N-2} \setminus \{0\}$ as in Corollary 5.8. We first consider a special case, that
is, two groups, \( T = \langle \tau_{e_1}, \ldots, \tau_{e_N} \rangle \) and \( T' = \langle \tau'_{e_1}, \ldots, \tau'_{e_N} \rangle \) corresponding to \( b = (b_{2,1}^{(1)}, \ldots, b_{2,N-2}^{(1)}) \) and \( b' = (b_{2,1}^{(1)}, \ldots, b_{2,N-2}^{(1)}) \), with same Hamming weight \( w_H(b) = w_H(b') \). Then, there exists a permutation matrix \( P \in (\mathbb{F}_2)^{N-2 \times N-2} \) such that \( bP = b' \). Let \( P' \in (\mathbb{F}_2)^{N \times N} \) be the permutation matrix given by

\[
P' = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}.
\]

Note that when we multiply a matrix \( M \) by \( P' \) on the right, i.e. \( MP' \), we are permuting the last \( N-2 \) columns of \( M \). On other hands when we multiply \( M \) by \( P'^{-1} \) on the left, we are permuting the last \( N-2 \) rows of \( M \). So, we have

\[
P'^{-1} \tau_{e_i} P' = P'^{-1} \kappa_{e_i} P' \sigma_{e_{i}} P' = \tau'_{e_{i}} P' = \tau'_{e_{(i)}}
\]

where \( \pi \) is the permutation on the indices related to \( P' \), thus \( P'^{-1} TP' = T' \). This implies that two groups corresponding to vectors with the same weight are conjugated.

Now, we consider consider another special case, that is, two vectors in \((\mathbb{F}_2)^{N-2}\)

\[
b = (1, \ldots, 1, 0, \ldots, 0) \quad \text{and} \quad b' = (1, \ldots, 1, 0, \ldots, 0)
\]

and the corresponding groups \( T = \langle \tau_{e_i}, \ldots, \tau_{e_N} \rangle \), \( T' = \langle \tau'_{e_i}, \ldots, \tau'_{e_N} \rangle \).

Let \( P \in (\mathbb{F}_2)^{N \times N} \) be the matrix with rows \( P_j = e_j \) if \( j \neq i + 2 \) and \( P_{i+2} = e_{i+2} + e_{i+3} \),

\[
P = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}.
\]

Note that \( P^{-1} = P \). Note also that when we multiply a matrix \( M \) by \( P \) on the right, we are changing the \( i+3 \)-th column of \( M \) summing to it the \( i+2 \)-th column. On the other hand, when we multiply a matrix \( M \) by \( P^{-1} = P \) on the left, we are changing the \( i+2 \)-th row of \( M \) summing the \( i+3 \)-th row to it. So, we have

\[
P \tau_{e_j} P = P \kappa_{e_j} P \sigma_{e_j} P = \tau'_{e_j}
\]

for \( j \neq i + 2 \) and

\[
P \tau_{(e_{i+2} + e_{i+3})} P = \tau'_{e_{i+2}}.
\]

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Note that if we consider the group generated by the translation
\[ \tau_{e_1}, \ldots, \tau_{e_{i+1}}, \tau_{(e_{i+2}+e_{i+3})}, \tau_{e_{i+3}}, \ldots, \tau_{e_N} \]
we obtain again the group \( T \), as \( \tau_{(e_{i+2}+e_{i+3})}\tau_{e_{i+3}} = \tau_{e_{i+2}} \), implying \( PTP = T' \).

Then, if \( b \) and \( b' \) are such that \( w_H(b) = w_H(b') + 1 \), composing the two arguments we have that the associated groups \( T \) and \( T' \) are conjugated.

In conclusion, if we have \( T \) and \( T' \) associated to two vectors \( b \) and \( b' \), with \( w_H(b) < w_H(b') \), iterating the process above and using the two special cases, we obtain that \( T \) and \( T' \) are conjugated.

**Remark 5.10.** Note that for the cases of Corollary 5.7 and Corollary 5.8, we are counting all the elementary abelian subgroup \( T \circ \subseteq \text{AGL}(V,\circ) \) with co-dimension of \( U(T) \) less or equal to 3. In fact, the condition \( T_+ \subseteq \text{AGL}(V,\cdot) \) is always guaranteed by Proposition 5.1.

### 5.1. Classes in small dimension

In the case of the binary field \( \mathbb{F}_2 \), from Proposition 5.1 we obtain the following corollary.

**Corollary 5.11.** If \( \text{dim}(V) \leq 6 \), then \( T_+ \subseteq \text{AGL}(V,\cdot) \) if and only if \( T_0 \subseteq \text{AGL}(V,\circ) \).

The bound \( \text{dim}(V) \leq 6 \) is actually tight for \( \mathbb{K} = \mathbb{F}_2 \), as shown below.

**Theorem 5.12.** If \( \text{dim}(V) \geq 7 \), then there exists \( T_0 \subseteq \text{AGL}(V,\circ) \) such that \( T_+ \nsubseteq \text{AGL}(V,\cdot) \).

**Proof.** Let \( N \) be the dimension of \( V \), then \( V = V_1 \oplus V_2 \) where
\[
V_1 = \text{Span}\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}
\]
and
\[
V_2 = \text{Span}\{e_8, \ldots, e_N\}.
\]
If \( N = 7 \) then we consider only \( V_1 \).

Over \( V_1 \) we impose the algebra structure induced by the exterior algebra over a vector space of dimension 3, which is
\[
e_1 \wedge e_2 = e_4, e_1 \wedge e_3 = e_5, e_2 \wedge e_3 = e_6, e_1 \wedge e_2 \wedge e_3 = e_7,
\]
and over \( V_2 \) we impose the algebra structure given by the trivial product \( x \cdot y = 0 \) for any \( x, y \in V_2 \).

Over \( V \) we can define the product
\[
v \cdot w = (v_1 + v_2) \cdot (w_1 + w_2) = (v_1 \wedge w_1 + v_2 \cdot w_2) = v_1 \wedge w_1
\]
where $v_1, w_1 \in V_1$ and $v_2, w_2 \in V_2$.

$(V, +, \cdot)$ is a commutative associative $\mathbb{F}_2$-algebra such that the resulting ring is radical. This algebra corresponds to an elementary abelian regular subgroup $T_0$ of $AGL(V, +)$ by Theorem 3.2 and from the obvious fact $x \circ x = 0$ for all $x \in V$. From Lemma 3.15 we have our claim, in fact $e_1 \cdot e_2 \cdot e_3 \neq 0$.

Thanks to the Corollary 5.11 result we can classify all elementary abelian regular subgroups of $AGL(V, +)$ up to dimension 6. We take into account only $\dim(V) = 3, 4, 5, 6$, as the case 1 and 2 are trivial.

We report these cases in Table 5.1 with the number of classes ($C$’s), their cardinality ($|C|$) and the dimension of the space $U(T)$ ($\dim(U)$).

| n | C’s | |C| | dim(U) |
|---|---|---|---|
| 3 | 2 | $|C_1| = 1$ | 3 |
|   |   | $|C_2| = 7$ | 1 |
| 4 | 2 | $|C_1| = 1$ | 4 |
|   |   | $|C_2| = 105$ | 2 |
| 5 | 4 | $|C_1| = 1$ | 5 |
|   |   | $|C_2| = 1085$ | 3 |
|   |   | $|C_3| = 6510$ | 2 |
|   |   | $|C_4| = 868$ | 1 |
| 6 | 8 | $|C_1| = 1$ | 6 |
|   |   | $|C_2| = 9765$ | 4 |
|   |   | $|C_3| = 234360$ | 3 |
|   |   | $|C_4| = 410130$ | 3 |
|   |   | $|C_5| = 820260$ | 2 |
|   |   | $|C_6| = 218736$ | 2 |
|   |   | $|C_7| = 54684$ | 2 |
|   |   | $|C_8| = 1093680$ | 2 |

Table 1: Classes table

**Remark 5.13.** The cases $\dim(V) = 3, 4$ are consequence of Corollary 5.7 and Corollary 5.8. For the other two cases we used MAGMA to obtain the classification.

### 6. Final remarks and related works

In this paper we have characterized the maps generating some alternative translation groups, that may be used to embed a trapdoor in some block ciphers. We have presented an example of trapdoor on a toy cipher to show a possible attack using these algebraic structures. In [12] the authors give a cryptographic characteristic for the S-boxes (called Anti-crookedness in [13] and studied in [14]), that permits to avoid the case $\Gamma_\infty \subseteq AGL(V, \circ)$ for any operation $\circ$. Recently, in [15] the authors individuate another property to avoid this vulnerability. Both these properties are only sufficient conditions thwarting the trapdoor.
It is also interesting to study the properties of the maps contained in a copy of $\text{AGL}(V)$. In particular, in [10] the authors investigate the differential uniformity of the maps contained in an affine group $\text{AGL}(V, \circ)$.

In [16] it is provided a non-trivial lower bound on the cardinality of the variety $\mathcal{V}(I_{n,k})$ of Theorem 5.5. The author shows, also, that the ratio between the upper bound of Proposition 5.6 and this lower bound reaches 1 asymptotically. Moreover, he produces an algorithm, that for a fixed linear map $\lambda$ it is able to find a set of hidden sums for which $\lambda$ remains linear. Therefore, the results in [10] help significantly an attacker to build a tb cipher with a trapdoor.

A class of operations defined here is used in [17] to weaken the non-linearity of well-known APN S-boxes. In [17], also a differential attack with respect to hidden sums is presented.

From Remark 4.4 we have that it might be possible also to have a partial trapdoor. In that case, the cryptographic properties given in [12] and [13] do not guarantee anymore the security against this attack. Thus, it would also be interesting to study the particular case of groups $T_0 \subseteq \text{AGL}(V, +)$ for which $T_+ \not\subseteq \text{AGL}(V, \circ)$. On the other hand, in [10] the differential uniformity of the maps contained in these affine group $\text{AGL}(V, \circ)$ is also studied.

In conclusion, we believe that the investigation of hidden sums introduced in [13] for the first time (and now well under way with many authors involved) is important both for individuating new cryptographic criteria (to design block ciphers) and for proposing new attacks on them.

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7. Appendix

In this appendix we report the representatives of each class given in Table 5.1.

The MAGMA code used to classify the elementary abelian regular subgroups of \( AGL(V) \) is given below.

```magma
//create the spaces
n:=6; //dimension of V
fix:=2; //dimension of U(T)
Vn:=VectorSpace(GF(2),n);
e:=[v:v in Vn| Weight(v) eq 1]; //canonical basis
Vn:={v:v in Vn};
Sn:=Sym(Vn); //symmetric group
Id:=IdentityMatrix(GF(2),n);
t:=sub<Sn|[v*Id+e[t]: v in Vn]: t in [1..n]>; //translation group with respect +
V:=VectorSpace(GF(2),(n-fix-1)*fix);
v0:=V!0;

//////////////B_ei,M_ei
Matrix_ei:=function(i,v,n_fix)
//given the element e_i and a vector v of length (n-k)*k
//return a matrix in blocks form of type 
// [ I B]
// [ 0 I]
//and B (the entries of B are filled using v)
l:=Eltseq(v);
l0:=[GF(2)!0:j in [1..n_fix]];
Insert(~l,n_fix*(i-1)+1, n_fix*(i-1),l0);
dimV:=Degree(e[i]);
B:=Matrix(GF(2),dimV-n_fix,n_fix,l);
I:=IdentityMatrix(GF(2),dimV);
return B,InsertBlock(I,B,1,dimV-n_fix+1);
end function;

/////////////
control:=function(i,v,B,n_fix,v_null)
//given the matrix constructed before it verifies
//if the rows match the rows of precedent matrices constructed (that is e_i*B ej=ej*B ei)
l:=&cat[Eltseq(B[j][i]):j in [1..i-1]];
return (v ne v_null) and (l eq Eltseq([v[j]:j in [1..(i-1)*fix]])); //control on v_null because if v is zero then e_i lies in U(T)
end function;

N_e:=[Id:j in [1..n]]; B_e:=[ZeroMatrix(GF(2),n-fix,fix);j in [1..n]]; //lists of matrices associated to e_i’s
Gr:=[]; //list of the groups that fix the last "fix" elements of canonical basis
```
Representatives class in $\text{AGL}((\mathbb{F}_2)^3,\cdot)$:

In this case we have two classes of elementary abelian regular subgroups (including the group $T_+^3$). The representative of these two classes are:

$C_1 \rightarrow T_+^3$

$C_2 \rightarrow T_0^3 = \langle \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} + e_1, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} + e_2, 1V + e_3 \rangle$

Representatives class in $\text{AGL}((\mathbb{F}_2)^4,\cdot)$:

In this case we have two classes of elementary abelian regular subgroups. The representative these classes are:

$C_1 \rightarrow T_+^4$

$C_2 \rightarrow T_0^4 = \langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} + e_1, \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} + e_2, 1V + e_3, 1V + e_4 \rangle$

Representatives class in $\text{AGL}((\mathbb{F}_2)^5,\cdot)$:

In this case we have four classes of elementary abelian regular subgroups.
The representative of these classes are:

\[ C_1 \rightarrow T_+ \]

\[ C_2 \rightarrow T_0 = \langle \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 \\ 1 \end{bmatrix} + e_1, \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \end{bmatrix} + e_2, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 \end{bmatrix} + e_3, 1 \rangle \]

\[ C_3 \rightarrow T_0 = \langle \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 \\ 1 \end{bmatrix} + e_1, \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \end{bmatrix} + e_2, \begin{bmatrix} 1 & 0 \\ 1 \end{bmatrix} + e_3, 1 \rangle \]

\[ C_4 \rightarrow T_0 = \langle \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 \\ 1 \end{bmatrix} + e_1, \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} + e_2, \begin{bmatrix} 1 & 0 \\ 1 \end{bmatrix} + e_3, 1 \rangle \]

Representatives class in \( \text{AGL}(\mathbb{F}_2^6, +) \):

We have eight classes of elementary abelian regular subgroups. The rep-
resentative of these classes are:

\[ C_1 \rightarrow T_+ \]

\[ C_2 \rightarrow T_0 = \langle \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 \\ 1 \end{bmatrix} + e_1, \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 \\ 1 \end{bmatrix} + e_2, 1_V + e_3, 1_V + e_4, 1_V + e_5, 1_V + e_6 \rangle \]

\[ C_3 \rightarrow T_0 = \langle \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 \\ 1 \end{bmatrix} + e_1, \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} + e_2, 1_V + e_3, 1_V + e_4, 1_V + e_5, 1_V + e_6 \rangle \]

\[ C_4 \rightarrow T_0 = \langle \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 \\ 1 \end{bmatrix} + e_1, \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} + e_2, 1_V + e_3, 1_V + e_4, 1_V + e_5, 1_V + e_6 \rangle \]

\[ C_5 \rightarrow T_0 = \langle \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 \\ 1 \\ 1 \end{bmatrix} + e_1, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} + e_2, 1_V + e_3, 1_V + e_4, 1_V + e_5, 1_V + e_6 \rangle \]
\[
\mathcal{C}_5 \rightarrow T_6 = \{ 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 1 & 1 \\
1 & 0 \\
1
\end{bmatrix} + e_1, \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 \\
1
\end{bmatrix} + e_2, \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 \\
1
\end{bmatrix} + e_3, \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 \\
1
\end{bmatrix} + e_4 \cdot \mathbf{1}_6 + e_5, \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 \\
1
\end{bmatrix} + e_6 \}
\]

\[
\mathcal{C}_7 \rightarrow T_6 = \{ 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 0 \\
1
\end{bmatrix} + e_1, \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
1 & 1 & 1 \\
1 & 0 \\
1
\end{bmatrix} + e_2, \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 \\
1
\end{bmatrix} + e_3, \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 \\
1
\end{bmatrix} + e_4 \cdot \mathbf{1}_6 + e_5, \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 \\
1
\end{bmatrix} + e_6 \}
\]

\[
\mathcal{C}_8 \rightarrow T_6 = \{ 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 1 & 1 \\
1 & 1 \\
1
\end{bmatrix} + e_1, \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 1 & 1 \\
1 & 0 \\
1
\end{bmatrix} + e_2, \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 1 & 1 \\
1 & 0 \\
1
\end{bmatrix} + e_3, \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 1 & 1 \\
1 & 0 \\
1
\end{bmatrix} + e_4 \cdot \mathbf{1}_6 + e_5, \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 1 & 1 \\
1 & 0 \\
1
\end{bmatrix} + e_6 \}
\]

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