AW-(K) TYPE CURVES ACCORDING TO PARALLEL TRANSPORT FRAME IN EUCLIDEAN SPACE $\mathbb{E}^4$

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Abstract. In this study we consider AW(k)-type curves according to parallel transport frame in Euclidean space $\mathbb{E}^4$. We give the relations between the parallel transport curvatures of these kinds of curves.

1. Introduction

The Frenet frame is constructed for the curve of 3-time continuously differentiable non-degenerate curves. But, curvature may vanish at some points on the curve. That is, the second derivative of the curve may be zero. In this situation, we need an alternative frame in $\mathbb{E}^3$: Therefore in [4], Bishop defined a new frame for a curve and he called it Bishop frame which is well defined even if the curve has vanishing second derivative in 3-dimensional Euclidean space. In [4]–[6], the advantages of the Bishop frame and the comparison of Bishop frame with the Frenet frame in Euclidean 3-space were given. In Euclidean 4-space $\mathbb{E}^4$, we have the same problem for a curve like being in Euclidean 3-space. That is, one of the $i-th$ ($1 < i < 4$) derivative of the curve may be zero. In this situation, we need an alternative frame.

In [5] using the similar idea authors considered such curves and construct an alternative frame. They gave parallel transport frame of a curve and they introduced the relations between the frame and Frenet frame of the curve in 4-dimensional Euclidean space $\mathbb{E}^4$. They generalized the relation which is well known in Euclidean 3-space for 4-dimensional Euclidean space $\mathbb{E}^4$.

The notion of AW(k)-type submanifolds was introduced by Arslan and West in [2]. In particular, many works related to curves of AW(k)-type have been done by several authors. For example, in [3], the authors gave curvature conditions and characterizations related to these curves in $\mathbb{E}^n$. In [7], authors considered curves and surfaces of AW(k) ($k = 1, 2$

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or 3)-type. They also gave related examples of curves and surfaces satisfying $AW(k)$-type conditions.

In [8], the authors considered $AW(k)$-type curves according to the Bishop Frame in Euclidean space $\mathbb{E}^3$. They gave the relations between the Bishop curvatures $k_1, k_2$ of these types of curves in $\mathbb{E}^3$.

Furthermore, in [1], the authors considered a generalization of $AW(k)$-type ($k = 1, 2, ..., 7$) curves in Euclidean $n$-space $\mathbb{E}^n$. Also they gave curvature conditions of these types of curves.

In this study, we consider $AW(k)$-type ($k = 1, 2, ..., 7$) curves according to parallel transport frame in Euclidean space $\mathbb{E}^4$. We give the relations between the parallel transport curvatures of these kinds of curves.

2. Basic Concepts

Let $\gamma = \gamma(s) : I \to \mathbb{E}^4$ be an arbitrary curve in the Euclidean space $\mathbb{E}^4$, where $I$ is interval in $\mathbb{R}$. $\gamma$ is said to be of unit speed (parametrized by arc length function $s$) if $||\gamma'(s)|| = 1$. Then the derivatives of the Frenet frame of $\gamma$ (Frenet-Serret formula) are:

$$
\begin{bmatrix}
T' \\
N' \\
B'_1 \\
B'_2
\end{bmatrix}
= \begin{bmatrix}
0 & \kappa & 0 & 0 \\
-\kappa & 0 & \tau & 0 \\
0 & -\tau & 0 & \sigma \\
0 & 0 & -\sigma & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B_1 \\
B_2
\end{bmatrix}
$$

where $\{T, N, B_1, B_2\}$ is the Frenet frame of $\gamma$ and $\kappa, \tau$ and $\sigma$ are principal curvature functions according to Frenet frame of the curve $\gamma$, respectively.

In [5], authors used the tangent vector $T(s)$ and three relatively parallel vector fields $M_1(s), M_2(s)$, and $M_3(s)$ to construct an alternative frame. They called this frame a parallel transport frame along the curve $\gamma$. Then they gave the following theorem for a parallel transport frame.

**Theorem 2.1.** [5] Let $\{T, N, B_1, B_2\}$ be a Frenet frame along a unit speed curve $\gamma = \gamma(s) : I \to \mathbb{E}^4$ and $\{T, M_1, M_2, M_3\}$ denotes the parallel transport frame of the curve $\gamma$. The relation may be expressed as

$$
\begin{align*}
T &= T(s) \\
N &= \cos \theta(s) \cos \psi(s) M_1 + (\cos \phi(s) \sin \psi(s) + \sin \phi(s) \sin \theta(s) \cos \psi(s)) M_2 \\
&\quad + (\sin \phi(s) \sin \psi(s) + \cos \phi(s) \sin \theta(s) \cos \psi(s)) M_3 \\
B_1 &= \cos \theta(s) \sin \psi(s) M_1 + (\cos \phi(s) \cos \psi(s) + \sin \phi(s) \sin \theta(s) \sin \psi(s)) M_2 \\
&\quad + (\sin \phi(s) \cos \psi(s) + \cos \phi(s) \sin \theta(s) \sin \psi(s)) M_3 \\
B_2 &= -\sin \theta(s) M_1 + \sin \phi(s) \cos \theta(s) M_2 + \cos \phi(s) \cos \theta(s) M_3
\end{align*}
$$
where $\theta$, $\psi$ and $\phi$ are the Euler angles. Then the alternative parallel frame equations are

\[ \begin{bmatrix} T' \\ M'_1 \\ M'_2 \\ M'_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} \]

where $k_1$, $k_2$ and $k_3$ are principal curvature functions according to parallel transport frame of the curve $\gamma$ and their expressions are as follows:

\[
\begin{align*}
  k_1 &= \kappa \cos \theta \cos \psi \\
  k_2 &= \kappa (-\cos \phi \sin \psi + \sin \phi \sin \theta \cos \psi) \\
  k_3 &= \kappa (\sin \phi \sin \psi + \cos \phi \sin \theta \cos \psi),
\end{align*}
\]

where $\theta' = \frac{\sigma}{\sqrt{k_1^2 + k_2^2 + k_3^2}}$, $\psi' = -\tau - \sigma \frac{\sqrt{\sigma^2 - \theta'^2}}{\cos \theta}$ and the following equalities

\[
\begin{align*}
  \kappa &= \sqrt{k_1^2 + k_2^2 + k_3^2},  \\
  \tau &= -\psi' + \phi' \sin \theta,  \\
  \sigma &= \frac{\phi'}{\sin \psi},  \\
  \phi' \cos \theta + \theta' \cot \psi &= 0
\end{align*}
\]

hold.

In [1], the authors obtained the higher order derivatives of $\gamma$ in $E^4$ as follows:

\[
\begin{align*}
  \gamma''(s) &= \kappa N \\
  \gamma'''(s) &= -\kappa^2 T + \kappa' N + \kappa \tau B_1 \\
  \gamma^{(iv)}(s) &= -3\kappa \kappa' T + (\kappa'' - \kappa^3 - \kappa \tau^2) N + (2\kappa' \tau + \kappa \tau') B_1 + \kappa \tau \sigma B_2 \\
  \gamma^{(v)}(s) &= (-3\kappa'^2 - 4\kappa \kappa'' + \kappa^4 + \kappa^2 \tau^2) T + (\kappa'''' - 6\kappa \kappa'' + 3\kappa' \tau^2 - 3\kappa \tau \tau') N + (3\kappa'' \tau + 3\kappa' \tau' - \kappa^3 \tau - \kappa \tau^3 + \kappa \tau' \tau' - \kappa \tau s^2) B_1 \\
  &\quad + (3\kappa' \tau \sigma + 2\kappa \tau' \sigma + \kappa \tau \sigma') B_2.
\end{align*}
\]

And also they gave the following notation and definition:

**Notation:**

\[
\begin{align*}
  N_1 &= \kappa N \\
  N_2 &= \kappa' N + \kappa \tau B_1 \\
  N_3 &= \lambda N + \lambda_1 B_1 + \lambda_2 B_2 \\
  N_4 &= \mu N + \mu_1 B_1 + \mu_2 B_2,
\end{align*}
\]
where

\[ \lambda = \kappa'' - \kappa^3 - \kappa \tau^2 \]
\[ \lambda_1 = 2\kappa' \tau + \kappa \tau' \]
\[ \lambda_2 = \kappa \tau \sigma \]

and

\[ \mu = \kappa''' - 6\kappa^2 \kappa' - 3\kappa' \tau^2 - 3\kappa \tau \tau' \]
\[ \mu_1 = 3\kappa'' \tau + 3\kappa' \tau' - \kappa^3 \tau - \kappa \tau^3 + \kappa \tau'' - \kappa \tau \sigma^2 \]
\[ \mu_2 = 3\kappa' \tau \sigma + 2\kappa \tau' \sigma + \kappa \tau \sigma' \]

are differentiable functions.

**Definition 2.2.** Frenet curves are

1) of $GAW$ (1) type if they satisfy

\[ N_4 = 0 \]

2) of $GAW$ (2) type if they satisfy

\[ \|N_2\|^2 N_4 = \langle N_2, N_4 \rangle N_2 \]

3) of $GAW$ (3) type if they satisfy

\[ \|N_3\|^2 N_4 = \langle N_3, N_4 \rangle N_3 \]

4) of $GAW$ (4) type if they satisfy

\[ \langle N_3, N_4 \rangle N_3 \]

5) of $GAW$ (5) type if they satisfy

\[ N_4 = a_1 N_1 + b_1 N_2 \]

6) of $GAW$ (6) type if they satisfy

\[ N_4 = a_2 N_1 + b_2 N_3 \]

7) of $GAW$ (7) type if they satisfy

\[ N_4 = a_3 N_2 + b_3 N_3, \]

where $a_i, b_i \ (1 \leq i \leq 3)$ are non-zero real valued differentiable functions.
3. AW(κ)-Type Curves with Parallel Transport Frame in $\mathbb{E}^4$

In this section, we consider $GAW(k)$-type curves according to the Parallel Transport Frame in Euclidean space $\mathbb{E}^4$.

Let $\gamma : I \subseteq \mathbb{R} \longrightarrow \mathbb{E}^4$ be a unit speed curve in $\mathbb{E}^4$. By the use of Parallel Transport Frame formulas (1), we obtain the higher order derivatives of $\gamma$ as follows:

\[
\begin{align*}
\gamma''(s) &= T'(s) = k_1 M_1 + k_2 M_2 + k_3 M_3 \\
\gamma'''(s) &= \{-k_1^2 - k_2^2 - k_3^2\} T + k'_1 M_1 + k'_2 M_2 + k'_3 M_3 \\
\gamma^{(iv)}(s) &= \{-3k_1k'_1 - 3k_2k'_2 - 3k_3k'_3\} T \\
&+ \{k''_1 - k'_1^2 - k'_1 k'_2 - k'_1 k'_3\} M_1 \\
&+ \{k''_2 - k'_2^2 - k'_2 k'_3\} M_2 \\
&+ \{k''_3 - k'_3^2 - k'_3 k'_1 - k'_3 k'_2\} M_3 \\
\gamma^{(v)}(s) &= \left\{-3k_1^2 - 3k_2^2 - 3k_3^2 - 4k_1k''_1 - 4k_2k''_2 - 4k_3k''_3 + k''_1^2 + k''_2^2 + 2k''_1k''_2 + 2k''_2k''_3 + 2k''_3k''_3\right\} T \\
&+ \{-6k_1^2k'_1 - 5k_1k_2k'_2 - 5k_1k_3k'_3 + k''_1 - k'_1^2 - k'_1 k'_2 - k'_1 k'_3\} M_1 \\
&+ \{-6k_2^2k'_2 - 5k_1k_2k'_1 - 5k_2k_3k'_3 + k''_2 - k'_2^2 - k'_2 k'_3 - k'_2 k'_3\} M_2 \\
&+ \{-6k_3^2k'_3 - 5k_1k_3k'_1 - 5k_2k_3k'_2 + k''_3 - k'_3^2 - k'_3 k'_1 - k'_3 k'_2\} M_3.
\end{align*}
\]

Then we give the following notation:

**Notation:**

\[
\begin{align*}
\overline{N}_1 &= k'_1 M_1 + k'_2 M_2 + k'_3 M_3 \\
\overline{N}_2 &= k''_1 M_1 + k''_2 M_2 + k''_3 M_3 \\
\overline{N}_3 &= \phi_1 M_1 + \phi_2 M_2 + \phi_3 M_3 \\
\overline{N}_4 &= \psi_1 M_1 + \psi_2 M_2 + \psi_3 M_3,
\end{align*}
\]

where

\[
\begin{align*}
\phi_1 &= k''_1 - k'_1^2 - k'_1 k'_2 - k'_1 k'_3 \\
\phi_2 &= k''_2 - k'_2^2 - k'_2 k'_3 \\
\phi_3 &= k''_3 - k'_3^2 - k'_3 k'_1 - k'_3 k'_2
\end{align*}
\]

and

\[
\begin{align*}
\psi_1 &= -6k_1^2k'_1 - 5k_1k_2k'_2 - 5k_1k_3k'_3 + k''_1 - k'_1^2 - k'_1 k'_2 - k'_1 k'_3 \\
\psi_2 &= -6k_2^2k'_2 - 5k_1k_2k'_1 - 5k_2k_3k'_3 + k''_2 - k'_2^2 - k'_2 k'_3 \\
\psi_3 &= -6k_3^2k'_3 - 5k_1k_3k'_1 - 5k_2k_3k'_2 + k''_3 - k'_3^2 - k'_3 k'_1 - k'_3 k'_2
\end{align*}
\]
are differentiable functions.

**Definition 3.1.** Let \( \gamma \) be a unit speed curve in \( \mathbb{E}^4 \). According to its *Parallel Transport Frame*, \( \gamma \) is

- **i)** of PAW(1) type if it satisfies

\[
\mathbf{N}_4 = 0
\]

- **ii)** of PAW(2) type if it satisfies

\[
\| \mathbf{N}_2 \|^2 \mathbf{N}_4 = \langle \mathbf{N}_2, \mathbf{N}_4 \rangle \mathbf{N}_2
\]

- **iii)** of PAW(3) type if it satisfies

\[
\| \mathbf{N}_1 \|^2 \mathbf{N}_4 = \langle \mathbf{N}_1, \mathbf{N}_4 \rangle \mathbf{N}_1
\]

- **iv)** of PAW(4) type if it satisfies

\[
\| \mathbf{N}_3 \|^2 \mathbf{N}_4 = \langle \mathbf{N}_3, \mathbf{N}_4 \rangle \mathbf{N}_3
\]

- **v)** of PAW(5) type if it satisfies

\[
\mathbf{N}_4 = a_1 \mathbf{N}_1 + b_1 \mathbf{N}_2
\]

- **vi)** of PAW(6) type if it satisfies

\[
\mathbf{N}_4 = a_2 \mathbf{N}_1 + b_2 \mathbf{N}_3
\]

- **vii)** of PAW(7) type if it satisfies

\[
\mathbf{N}_4 = a_3 \mathbf{N}_2 + b_3 \mathbf{N}_3
\]

where \( a_i, b_i \) (\( 1 \leq i \leq 3 \)) are non-zero real valued differentiable functions.

**Theorem 3.2.** Let \( \gamma \) be a unit speed curve in \( \mathbb{E}^4 \). According to its *Parallel Transport Frame*, \( \gamma \) is

- **i)** of PAW(1) type if and only if

\[
-6k_1^2k_1' - 5k_1k_2k_2' - 5k_1k_3k_3' + k_1'' - k_1'k_2' - k_1'k_3' = 0
\]

- **ii)** of PAW(2) type if and only if

\[
\begin{align*}
\psi_1 & = k_1'(k_2'\psi_2 + k_3'\psi_3) \\
\psi_2 & = k_2'(k_1'\psi_1 + k_3'\psi_3) \\
\psi_3 & = k_3'(k_1'\psi_1 + k_2'\psi_2)
\end{align*}
\]
iii) of PAW(3) type if and only if
\[ (k_2^2 + k_3^2) \psi_1 = k_1(k_2 \psi_2 + k_3 \psi_3) \quad (14) \]
\[ (k_1^2 + k_3^2) \psi_2 = k_2(k_1 \psi_1 + k_3 \psi_3) \]
\[ (k_3^2 + k_2^2) \psi_3 = k_3(k_1 \psi_1 + k_2 \psi_2). \]

iv) of PAW(4) type if and only if
\[ (\phi_2^2 + \phi_3^2) \psi_1 = \phi_1(\phi_2 \psi_2 + \phi_3 \psi_3) \quad (15) \]
\[ (\phi_1^2 + \phi_3^2) \psi_2 = \phi_2(\phi_1 \psi_1 + \phi_3 \psi_3) \]
\[ (\phi_3^2 + \phi_2^2) \psi_3 = \phi_3(\phi_1 \psi_1 + \phi_2 \psi_2). \]
v) of PAW(5) type if and only if
\[ \psi_1 = a_1 k_1 + b_1 k_1' \quad (16) \]
\[ \psi_2 = a_1 k_2 + b_1 k_2' \]
\[ \psi_3 = a_1 k_3 + b_1 k_3'. \]
vii) of PAW(7) type if and only if
\[ \psi_1 = a_3 k_1' + b_3 \phi_1 \quad (18) \]
\[ \psi_2 = a_3 k_2' + b_3 \phi_2 \]
\[ \psi_3 = a_3 k_3' + b_3 \phi_3. \]

Proof. i) Let \( \gamma \) be of PAW(1)-type. Then from the equations (2) and (5), we have \( \overrightarrow{N_4} = \psi_1 M_1 + \psi_2 M_2 + \psi_3 M_3 = 0. \) Since \( M_1, M_2, M_3 \) are linearly independent, we obtain \( \psi_1 = \psi_2 = \psi_3 = 0, \) which means
\[ -6k_1^2 k_1' - 5k_1 k_2 k_2' - 5k_1 k_3 k_3' + k''_1 - k'_1 k_2^2 - k'_1 k_3^2 = 0 \quad (19) \]
\[ -6k_2^2 k_2' - 5k_1 k_2 k_3' - 5k_2 k_3 k_3' + k''_2 - k'_2 k_1^2 - k'_2 k_3^2 = 0 \]
\[ -6k_3^2 k_3' - 5k_1 k_3 k_1' - 5k_2 k_3 k_2' + k''_3 - k'_3 k_1^2 - k'_3 k_2^2 = 0. \]
The sufficiency is trivial.

ii) Let \( \gamma \) be of PAW(2)-type. If we calculate \( \| \overrightarrow{N_2} \|^2 \) and \( \langle \overrightarrow{N_2}, \overrightarrow{N_4} \rangle, \) by the use of equations (2) and (6), we get
\[ (k_1'^2 + k_2'^2 + k_3'^2)(\psi_1 M_1 + \psi_2 M_2 + \psi_3 M_3) = (k_1' \psi_1 + k_2' \psi_2 + k_3' \psi_3)(k_1' M_1 + k_2' M_2 + k_3' M_3), \]
which means
\[
(k_2^2 + k_3^2)\psi_1 = k_1'(k_2'\psi_2 + k_3'\psi_3)
\]
(21)
\[
(k_1^2 + k_3^2)\psi_2 = k_2'(k_1'\psi_1 + k_3'\psi_3)
\]
\[
(k_1^2 + k_2^2)\psi_2 = k_3'(k_1'\psi_1 + k_2'\psi_2).
\]
Conversely, if the equations (21) are satisfied, by the equation (6), \( \gamma \) is of \( PAW(2) \)-type.

iii) Let \( \gamma \) be of \( PAW(3) \)-type. If we calculate \( ||N_1||^2 \), \( \langle N_1, N_4 \rangle \) and substitute them in the equation (7), we get
\[
(k_1^2 + k_2^2 + k_3^2)(\psi_1 M_1 + \psi_2 M_2 + \psi_3 M_3) =
\]
(22)
\[
(k_1 \psi_1 + k_2 \psi_2 + k_3 \psi_3)(k_1 M_1 + k_2 M_2 + k_3 M_3)
\]
which means
\[
(k_1^2 + k_2^2)\psi_1 = k_1(k_2\psi_2 + k_3\psi_3)
\]
(23)
\[
(k_1^2 + k_2^2)\psi_2 = k_2(k_1\psi_1 + k_3\psi_3)
\]
\[
(k_1^2 + k_2^2)\psi_3 = k_3(k_1\psi_1 + k_2\psi_2).
\]
Conversely, if the equations (23) are satisfied, by the equation (7), \( \gamma \) is of \( PAW(3) \)-type.

iv) Let \( \gamma \) be of \( PAW(4) \)-type. If we calculate \( ||N_3||^2 \), \( \langle N_3, N_4 \rangle \) and substitute them in (8), we get
\[
(\phi_1^2 + \phi_2^2 + \phi_3^2)(\psi_1 M_1 + \psi_2 M_2 + \psi_3 M_3) =
\]
(24)
\[
(\phi_1 \psi_1 + \phi_2 \psi_2 + \phi_3 \psi_3)(\phi_1 M_1 + \phi_2 M_2 + \phi_3 M_3),
\]
which means
\[
(\phi_1^2 + \phi_2^2)\psi_1 = \phi_1(\phi_2\psi_2 + \phi_3\psi_3)
\]
(25)
\[
(\phi_1^2 + \phi_2^2)\psi_2 = \phi_2(\phi_1\psi_1 + \phi_3\psi_3)
\]
\[
(\phi_1^2 + \phi_2^2)\psi_3 = \phi_3(\phi_1\psi_1 + \phi_2\psi_2).
\]
Conversely, if the equations (25) are satisfied, by the equation (8), \( \gamma \) is of \( PAW(4) \)-type.

v) Let \( \gamma \) be of \( PAW(5) \)-type. In view of equations (2) and (9), we can write
\[
\psi_1 M_1 + \psi_2 M_2 + \psi_3 M_3 = a_1(k_1 M_1 + k_2 M_2 + k_3 M_3) + b_1(k_1' M_1 + k_2' M_2 + k_3' M_3),
\]
which gives us
\[
\psi_1 = a_1 k_1 + b_1 k_1'
\]
(27)
\[
\psi_2 = a_1 k_2 + b_1 k_2'
\]
\[
\psi_3 = a_1 k_3 + b_1 k_3'.
\]
Conversely, if the equations (27) are satisfied, by the equation (9), \( \gamma \) is of PAW(5)-type.

\textit{vi}) Let \( \gamma \) be of PAW(6)-type. In view of equations (2) and (10), we can write
\begin{equation}
\psi_1 M_1 + \psi_2 M_2 + \psi_3 M_3 = a_2 (k_1 M_1 + k_2 M_2 + k_3 M_3) + b_2 (\phi_1 M_1 + \phi_2 M_2 + \phi_3 M_3),
\end{equation}
that means
\begin{align*}
\psi_1 &= a_2 k_1 + b_2 \phi_1 \\
\psi_2 &= a_2 k_2 + b_2 \phi_2 \\
\psi_3 &= a_2 k_3 + b_2 \phi_3.
\end{align*}

Conversely, if the equations (29) are satisfied, by the equation (10), \( \gamma \) is of PAW(6)-type.

\textit{vii}) Let \( \gamma \) be of PAW(7)-type. In view of equations (2) and (11), we can write
\begin{equation}
\psi_1 M_1 + \psi_2 M_2 + \psi_3 M_3 = a_3 (k'_1 M_1 + k'_2 M_2 + k'_3 M_3) + b_3 (\phi_1 M_1 + \phi_2 M_2 + \phi_3 M_3),
\end{equation}
which means
\begin{align*}
\psi_1 &= a_3 k'_1 + b_3 \phi_1 \\
\psi_2 &= a_3 k'_2 + b_3 \phi_2 \\
\psi_3 &= a_3 k'_3 + b_3 \phi_3.
\end{align*}

Conversely, if the equations (31) are satisfied, by the equation (11), \( \gamma \) is of PAW(7)-type.

From now on, we consider space curves whose curvatures \( k_1 \) is non-zero constant, \( k_2 \) and \( k_3 \) are not constant. We give curvature conditions of such a curve to be of PAW(\( k \))-type. In this case, we obtain:
\begin{align*}
\overline{N}_1 &= k_1 M_1 + k_2 M_2 + k_3 M_3 \\
\overline{N}_2 &= k'_2 M_2 + k'_3 M_3 \\
\overline{N}_3 &= \phi_{11} M_1 + \phi_{21} M_2 + \phi_{31} M_3 \\
\overline{N}_4 &= \psi_{11} M_1 + \psi_{21} M_2 + \psi_{31} M_3,
\end{align*}
where
\begin{align*}
\phi_{11} &= -k'^3_1 - k_1 k'^2_2 - k_1 k'^2_3 \\
\phi_{21} &= k'^2_2 - k^2_3 - k'^2_2 k_2 - k^2_3 k_2 \\
\phi_{31} &= k'^2_3 - k^2_3 - k'^2_1 k_3 - k^2_3 k_3
\end{align*}
\[ \psi_{11} = -5k_1k_2k'_2 - 5k_1k_3k'_3 \]
\[ \psi_{21} = -6k_2^2k'_2 - 5k_2k_3k'_3 + k''_2 - k_1^2k'_2 - k_3^2k'_2 \]
\[ \psi_{31} = -6k_3^2k'_3 - 5k_2k_3k'_2 + k''_3 - k_1^2k'_3 - k_2^2k'_3. \]

**Proposition 3.3.** Let \( \gamma : I \subseteq \mathbb{R} \to \mathbb{E}^4 \) be a unit speed curve with non-zero constant \( k_1 \). Then \( \gamma \) is

i) \( \) of \( \text{PAW}(1) \)-type if and only if

\[ -k_2^2 = k_3^2 + c \]

and

\[ -6k_2^2k'_2 - 5k_2k_3k'_3 + k''_2 - k_1^2k'_2 - k_3^2k'_2 = 0 \]
\[ -6k_3^2k'_3 - 5k_2k_3k'_2 + k''_3 - k_1^2k'_3 - k_2^2k'_3 = 0. \]

ii) \( \) of \( \text{PAW}(2) \)-type if and only if the equations

\[ -k_2^2 = k_3^2 + c \]

and

\[ k'_3\psi_{21} = k'_2\psi_{31}. \]

iii) \( \) of \( \text{PAW}(3) \)-type if and only if

\[ (k_2^2 + k_3^2)\psi_{11} = k_1(k_2\psi_{21} + k_3\psi_{31}) \]
\[ (k_1^2 + k_3^2)\psi_{21} = k_2(k_1\psi_{11} + k_3\psi_{31}) \]
\[ (k_1^2 + k_2^2)\psi_{31} = k_3(k_1\psi_{11} + k_2\psi_{21}). \]

iv) \( \) of \( \text{PAW}(4) \)-type if and only if

\[ (\phi_{21}^2 + \phi_{31}^2) \psi_{11} = \phi_{11}(\phi_{21}\psi_{21} + \phi_{31}\psi_{31}) \]
\[ (\phi_{11}^2 + \phi_{31}^2) \psi_{21} = \phi_{21}(\phi_{11}\psi_{11} + \phi_{31}\psi_{31}) \]
\[ (\phi_{11}^2 + \phi_{21}^2) \psi_{31} = \phi_{31}(\phi_{11}\psi_{11} + \phi_{21}\psi_{21}). \]

v) \( \) of \( \text{PAW}(5) \)-type if and only if

\[ \psi_{11} = a_1k_1 \]
\[ \psi_{21} = a_1k_2 + b_1k'_2 \]
\[ \psi_{31} = a_1k_3 + b_1k'_3 \]

and

\[ a_1 = -\frac{5}{2}(\kappa^2)'. \]
vi) of $\text{PAW}(6)$-type if and only if
\begin{align*}
\psi_{11} &= a_2k_1 + b_2\phi_{11} \\
\psi_{21} &= a_2k_2 + b_2\phi_{21} \\
\psi_{31} &= a_2k_3 + b_2\phi_{31}
\end{align*}
and
\begin{align*}
a_2 - b_2\kappa^2 = -\frac{5}{2}(\kappa^2)'.
\end{align*}

vii) of $\text{PAW}(7)$-type if and only if
\begin{align*}
\psi_{11} &= b_3\phi_{11} \\
\psi_{21} &= a_3k'_2 + b_3\phi_{21} \\
\psi_{31} &= a_3k'_3 + b_3\phi_{31}
\end{align*}
and
\begin{align*}
\frac{5}{2}(\kappa^2)' = b_3\kappa^2.
\end{align*}

Proof. i) Let $\gamma$ be of $\text{PAW}(1)$-type. Using the equations (5), (32) and (34), we obtain
\begin{align*}
\psi_{11} &= \psi_{21} = \psi_{31} = 0,
\end{align*}
that means
\begin{align*}
-5k_1k_2k'_2 - 5k_1k_3k'_3 &= 0 \\
-6k_2^2k'_2 - 5k_3k_2k'_3 + k'' - k_1k'_2 - k_3k'_2 &= 0 \\
-6k_3^2k'_3 - 5k_2k_3k'_2 + k'' - k_1k'_3 - k_2k'_3 &= 0.
\end{align*}
If we solve the equation (48), we get
\begin{align*}
-k_2^2 = k_3^2 + c
\end{align*}
where $c$ is an arbitrary constant. Converse proposition is trivial.

ii) Let $\gamma$ be of $\text{PAW}(2)$-type. Using the equations (6), (32) and (34), we obtain
\begin{align*}
(k_2^2 + k_3^2)\psi_{11} &= 0 \\
k_3^2\psi_{21} &= k'_2k'_3\psi_{31} \\
k_2^2\psi_{31} &= k'_2k'_3\psi_{21}.
\end{align*}
Since $k_2$ and $k_3$ are not constant the solution of the first equation of the system (50) is
\begin{align*}
\psi_{11} &= 0,
\end{align*}
which corresponds to
\begin{align*}
-k_2^2 = k_3^2 + c.
\end{align*}
If we simplify the second and the third equations of the system \((50)\), we obtain
\[
 k'_3 \psi_{21} = k'_2 \psi_{31}.
\]

Converse proposition is trivial.

\(iii\) Let \(\gamma\) be of \(PAW(3)\)-type. Substituting the equations \((32)\) and \((34)\) in \((7)\), we get the solution. Converse proposition is trivial.

\(iv\) Let \(\gamma\) be of \(PAW(4)\)-type. Substituting the equations \((32)\), \((33)\) and \((34)\) in \((8)\), we get the solution. Converse proposition is trivial.

\(v\) Let \(\gamma\) be of \(PAW(5)\)-type. Using the equation \((9)\), \((32)\) and \((34)\), we get
\[
\begin{align*}
\psi_{11} &= a_1 k_1 \\
\psi_{21} &= a_1 k_2 + b_1 k'_2 \\
\psi_{31} &= a_1 k_3 + b_1 k'_3.
\end{align*}
\]
From the first equation of the system \((51)\), we obtain
\[
-5k_1 k_2 k'_2 - 5k_1 k_3 k'_3 = a_1 k_1,
\]
which corresponds to
\[
a_1 = -5k_2 k'_2 - 5k_3 k'_3.
\]
Using \(k_1^2 + k_2^2 + k_3^2 = \kappa^2\) and solving the last equation, we obtain
\[
a_1 = -\frac{5}{2}(\kappa^2)' = b_3.
\]
Converse proposition is trivial.

\(vi\) Let \(\gamma\) be of \(PAW(6)\)-type. Substituting the equations \((32)\), \((33)\) and \((34)\) in \((10)\), we obtain the equations \((13)\). Substituting the equations \((33)\) and \((34)\) in \((13)\), we get
\[
k_1(a_2) + k_1(-b_2 k_1^2 - b_2 k_2^2 - b_2 k_3^2) = k_1(-5k_2 k'_2 - 5k_3 k'_3),
\]
which means
\[
a_2 - b_2 k^2 = -\frac{5}{2}(\kappa^2)'.
\]
Converse proposition is trivial.

\(vii\) Let \(\gamma\) be of \(PAW(7)\)-type. Substituting the equations \((32)\), \((33)\) and \((34)\) in \((11)\), we obtain the equations \((15)\). Substituting the equations \((33)\) and \((34)\) in \((15)\), we get
\[
(-5k_1 k_2 k'_2 - 5k_1 k_3 k'_3) = b_3(-k_1^3 - k_1 k_2^2 - k_1 k_3^2).
\]
If we divide both side with \(-k_1\), we obtain
\[
5(k_2 k'_2 + k_3 k'_3 + k_1 k'_1) = b_3(k_1^2 + k_2^2 + k_3^2),
\]
which means \[ \frac{5}{2}(\kappa^2)' = b_3\kappa^2. \]

Converse proposition is trivial. \(\square\)

**Corollary 3.4.** Let \(\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^4\) be a unit speed curve with non-zero constant \(k_1\). If \(\gamma\) is of PAW(1)-type then \(\gamma\) is of PAW(2)-type.

Now, let’s assume that \(k_2\) is non-zero constant, \(k_1\) and \(k_3\) are not constant. In this case, we obtain:

\[
\begin{align*}
\overline{N}_1 &= k_1 M_1 + k_2 M_2 + k_3 M_3 \\
\overline{N}_2 &= k_1' M_1 + k_3' M_3 \\
\overline{N}_3 &= \phi_{12} M_1 + \phi_{22} M_2 + \phi_{32} M_3 \\
\overline{N}_4 &= \psi_{12} M_1 + \psi_{22} M_2 + \psi_{32} M_3
\end{align*}
\]

where

\[
\begin{align*}
\phi_{12} &= k_1'' - k_1^3 - k_1 k_2^2 - k_1 k_3^2 \\
\phi_{22} &= -k_3^2 - k_1^2 k_2 - k_2^2 k_2 \\
\phi_{32} &= k_3' - k_3^3 - k_1^2 k_3 - k_2^2 k_3
\end{align*}
\]

and

\[
\begin{align*}
\psi_{12} &= -6k_2^2 k_1' - 5k_1 k_3 k_3' + k_1'' - k_1' k_2^2 - k_1' k_3^2 \\
\psi_{22} &= -5k_1 k_2 k_1' - 5k_3 k_2 k_3' \\
\psi_{32} &= -6k_3^2 k_3' - 5k_1 k_3 k_1' + k_3'' - k_1' k_3' - k_2^2 k_3'.
\end{align*}
\]

**Proposition 3.5.** Let \(\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^4\) be a unit speed curve with non-zero constant \(k_2\). Then \(\gamma\) is

i) of PAW(1)-type if and only if

\[-k_1^2 = k_3^2 + c\]

and

\[-6k_1^2 k_1' - 5k_1 k_3 k_3' + k_1'' - k_1' k_2^2 - k_1' k_3^2 = 0\]

\[-6k_3^2 k_3' - 5k_1 k_3 k_1' + k_3'' - k_1' k_3' - k_2^2 k_3' = 0.\]

ii) of PAW(2)-type if and only if

\[-k_1^2 = k_3^2 + c\]

\[k_1' \psi_{32} = k_3' \psi_{12}.\]

iii) of PAW(3)-type if and only if

\[(k_2^2 + k_3^2) \psi_{12} = k_1 (k_2 \psi_{22} + k_3 \psi_{32})\]

\[(k_1^2 + k_3^2) \psi_{22} = k_2 (k_1 \psi_{12} + k_3 \psi_{32})\]

\[(k_1^2 + k_2^2) \psi_{32} = k_3 (k_1 \psi_{12} + k_2 \psi_{22}).\]
iv) of PAW (4)-type if and only if
\[
(\phi_{22}^2 + \phi_{32}^2)\psi_{12} = \phi_{12}(\phi_{22}\psi_{22} + \phi_{32}\psi_{32})
\]
\[
(\phi_{12}^2 + \phi_{32}^2)\psi_{22} = \phi_{22}(\phi_{12}\psi_{12} + \phi_{32}\psi_{32})
\]
\[
(\phi_{12}^2 + \phi_{22}^2)\psi_{32} = \phi_{32}(\phi_{12}\psi_{12} + \phi_{22}\psi_{22}).
\]

v) of PAW (5)-type if and only if
\[
\psi_{12} = a_1k_1 + b_1k'_1
\]
\[
\psi_{22} = a_1k_2
\]
\[
\psi_{32} = a_1k_3 + b_1k'_3
\]
and
\[
a_1 = -\frac{5}{2}(\kappa^2)'.
\]

vi) of PAW (6)-type if and only if
\[
\psi_{12} = a_2k_1 + b_2\phi_{12}
\]
\[
\psi_{22} = a_2k_2 + b_2\phi_{22}
\]
\[
\psi_{32} = a_2k_3 + b_2\phi_{32}
\]
and
\[
a_2 - b_2k^2 = -\frac{5}{2}(\kappa^2)'.
\]

vii) of PAW (7)-type if and only if
\[
\psi_{12} = a_3k'_1 + b_3\phi_{12}
\]
\[
\psi_{22} = b_3\phi_{22}
\]
\[
\psi_{32} = a_3k'_3 + b_3\phi_{32}
\]
and
\[
\frac{5}{2}(\kappa^2)' = b_3k^2.
\]

Proof. i) Let \(\gamma\) be of PAW (1)-type. Using the equations (5), (52) and (54), we obtain
\[
\psi_{11} = \psi_{22} = \psi_{32} = 0,
\]
that means
\[
-6k_1^2k_1' - 5k_1k_3k_3' + k_1'' - k_1'k_2^2 - k_1'k_3^2 = 0
\]
\[
-5k_1k_2k_1' - 5k_2k_3k_3' = 0
\]
\[
-6k_3^2k_3' - 5k_1k_3k_1' + k_3'' - k_1^2k_3' - k_2^2k_3' = 0.
\]
If we solve the equation (58), we get
\[
-k_1^2 = k_3^2 + c,
\]
where \(c\) is an arbitrary constant. Converse proposition is trivial.
ii) Let \( \gamma \) be of \( PAW(2) \)-type. Using the equations (6), (52) and (54), we obtain

\[
\begin{align*}
k_3'^2 \psi_{12} &= k_1' k_3' \psi_{32} \\
(k_1'^2 + k_3'^2) \psi_{22} &= 0 \\
k_1'^2 \psi_{32} &= k_1' k_3' \psi_{12}.
\end{align*}
\]

\[(59)\]

Since \( k_1 \) and \( k_3 \) are not constant the solution of the second equation of the system \((59)\) is

\[\psi_{22} = 0,\]

which corresponds to

\[-k_1^2 = k_3^2 + c.\]

If we simplify the first and the third equations of the system \((59)\), we obtain

\[k_1' \psi_{32} = k_3' \psi_{12}.\]

Converse proposition is trivial.

iii) Let \( \gamma \) be of \( PAW(3) \)-type. Substituting the equations (52) and (54) in (7), we get the solution. Converse proposition is trivial.

iv) Let \( \gamma \) be of \( PAW(4) \)-type. Substituting the equations (52), (53) and (54) in (8), we get the solution. Converse proposition is trivial.

v) Let \( \gamma \) be of \( PAW(5) \)-type. Using (9), equations (52) and (54), we get

\[
\begin{align*}
\psi_{12} &= a_1 k_1 + b_1 k_1' \\
\psi_{22} &= a_1 k_2 \\
\psi_{32} &= a_1 k_3 + b_1 k_3'.
\end{align*}
\]

\[(60)\]

From the second equation of the system \((60)\), we obtain

\[-5 k_1 k_2 k_1' - 5 k_3 k_2 k_3' = a_1 k_2,\]

which corresponds to

\[a_1 = -5 k_1 k_1' - 5 k_3 k_3'.\]

Using \( k_1^2 + k_2^2 + k_3^2 = \kappa^2 \) and solving the last equation, we obtain

\[a_1 = -\frac{5}{2}(\kappa^2).\]

Converse proposition is trivial.

vi) Let \( \gamma \) be of \( PAW(6) \)-type. Substituting the equations (52), (53) and (54) in (10), we obtain the equations (55). Substituting the equations (53) and (54) in (55), we get

\[k_2 (a_2) + k_2 (-b_2 k_2^2 - b_2 k_1^2 - b_2 k_3^2) = k_2 (-5 k_1 k_1' - 5 k_3 k_3').\]
which means
\[ a_2 - b_2 \kappa^2 = -\frac{5}{2}(\kappa^2)' \].

Converse proposition is trivial.

\textit{vii)} Let \( \gamma \) be of PAW(7)-type. Substituting the equations (52), (53) and (54) in (11), we obtain the equations

\begin{equation}
\psi_{22} = b_3 \phi_{22},
\end{equation}

(56) and (57). Substituting the equations (53) and (54) in the last equation, we get

\[ -5k_1k_2k'_1 - 5k_3k_2k'_3 = -b_3k_2(k_2^2 + k_1^2 + k_3^2). \]

If we divide both side with \( k_2 \), we obtain

\[ 5(k_1k'_1 + k_3k'_3 + k_2k'_2) = b_3 \kappa^2 \]

which means

\[ \frac{5}{2}(\kappa^2)' = b_3 \kappa^2. \]

Converse proposition is trivial. \( \Box \)

\textbf{Corollary 3.6.} Let \( \gamma : I \subseteq \mathbb{R} \rightarrow \mathbb{E}^4 \) be a unit speed curve with non-zero constant \( k_2 \). If \( \gamma \) is of PAW(1)-type then \( \gamma \) is of PAW(2)-type.

Now, let’s assume that \( k_3 \) is non-zero constant, \( k_1 \) and \( k_2 \) are not constant. In this case, we obtain;

\begin{align*}
\overline{N}_1 &= k_1 M_1 + k_2 M_2 + k_3 M_3 \\
\overline{N}_2 &= k'_1 M_1 + k'_2 M_2 \\
\overline{N}_3 &= \phi_{13} M_1 + \phi_{23} M_2 + \phi_{33} M_3 \\
\overline{N}_4 &= \psi_{13} M_1 + \psi_{23} M_2 + \psi_{33} M_3
\end{align*}

(62)

where

\begin{align*}
\phi_{13} &= k''_1 - k^3_1 - k_1 k_2^2 - k_1 k_3^2 \\
\phi_{23} &= k''_2 - k^3_2 - k_1^2 k_2 - k_3^2 k_2 \\
\phi_{33} &= -k^3_3 - k_1^2 k_3 - k_2^2 k_3
\end{align*}

(63)

and

\begin{align*}
\psi_{13} &= -6k_1'^2 k_1 - 5k_1 k_2 k_2' + k''_1 - k_1' k_1^2 - k_1' k_3^2 \\
\psi_{23} &= -6k_2'^2 k_2 - 5k_1 k_2 k_1' + k''_2 - k_2' k_2^2 - k_3^2 k_2 \\
\psi_{33} &= -5k_1 k_3 k_1' - 5k_2 k_3 k_2'.
\end{align*}

(64)
Proposition 3.7. Let $\gamma : I \subseteq \mathbb{R} \rightarrow \mathbb{E}^4$ be a unit speed curve with non-zero constant $k_3$. Then $\gamma$ is

i) of PAW(1)-type if and only if
\[-k_1^2 = k_2^2 + c\]
and
\[-6k_1^2k_1' - 5k_1k_2k_2' + k_1'' - k_1'k_2^2 - k_1'k_3^2 = 0\]
\[-6k_2^2k_2' - 5k_1k_2k_1' + k_2'' - k_1k_2' - k_2'k_3^2 = 0.\]

ii) of PAW(2)-type if and only if
\[k_2'\psi_{13} = k_1'\psi_{23}\]
\[-k_1^2 = k_2^2 + c.\]

iii) of PAW(3)-type if and only if
\[(k_2^2 + k_3^2)\psi_{13} = k_1(k_2\psi_{23} + k_3\psi_{33})\]
\[(k_1^2 + k_3^2)\psi_{23} = k_2(k_1\psi_{13} + k_3\psi_{33})\]
\[(k_1^2 + k_2^2)\psi_{33} = k_3(k_1\psi_{13} + k_2\psi_{23}).\]

vi) of PAW(4)-type if and only if
\[(\phi_1^2 + \phi_2^2)\psi_{13} = \phi_1(\phi_2\psi_{23} + \phi_3\psi_{33})\]
\[(\phi_1^2 + \phi_3^2)\psi_{23} = \phi_2(\phi_1\psi_{13} + \phi_3\psi_{33})\]
\[(\phi_2^2 + \phi_3^2)\psi_{33} = \phi_3(\phi_1\psi_{13} + \phi_2\psi_{23}).\]

v) of PAW(5)-type if and only if
\[\psi_{13} = a_1k_1 + b_1k_1'\]
\[\psi_{23} = a_1k_2 + b_1k_2'\]
\[\psi_{33} = a_1k_3\]
and
\[a_1 = -\frac{5}{2}(\kappa^2)'\].

vi) of PAW(6)-type if and only if
\[\psi_{13} = a_2k_1 + b_2\phi_{13}\]
\[\psi_{23} = a_2k_2 + b_2\phi_{23}\]
\[\psi_{33} = a_2k_3 + b_2\phi_{33}\]
and
\[a_2 - b_2\kappa^2 = -\frac{5}{2}(\kappa^2)'\].
\[ \psi_{13} = a_3 k'_1 + b_3 \phi_{13} \]
\[ \psi_{23} = a_3 k'_2 + b_3 \phi_{23} \]
\[ \psi_{33} = b_3 \phi_{33} \]

and

\[ \frac{5}{2} (\kappa^2)' = b_3 \kappa^2. \]

**Proof.**

**i)** Let \( \gamma \) be of PAW(1)-type. Using the equations (5), (62) and (64), we obtain

\[ \psi_{13} = \psi_{23} = \psi_{33} = 0, \]

that means

\[ -6k_1^2 k'_1 - 5k_1 k_2 k'_2 + k''_1 - k'_1 k''_2 - k'_1 k''_3 = 0 \]
\[ -6k_2^2 k'_2 - 5k_1 k_2 k'_1 + k''_2 - k'_1 k''_2 - k'_2 k''_3 = 0 \]
\[ -5k_1 k_3 k'_1 - 5k_2 k_3 k'_2 = 0. \]

If we solve the equation (67), we get

\[ -k_1^2 = k_2^2 + c, \]

where \( c \) is an arbitrary constant. Converse proposition is trivial.

**ii)** Let \( \gamma \) be of PAW(2)-type. Using the equations (6), (62) and (64), we obtain

\[ k_2^2 \psi_{13} = k'_2 k_2 \psi_{23} \]
\[ k_2^2 \psi_{23} = k'_2 k'_{23} \psi_{13} \]
\[ (k'_1 + k'_2) \psi_{33} = 0 \]

Since \( k_1 \) and \( k_2 \) are not constant the solution of the third equation of the system (68) is

\[ \psi_{33} = 0, \]

which corresponds to

\[ -k_1^2 = k_2^2 + c. \]

If we simplify the first and the second equations of the system (68), we obtain

\[ k'_2 \psi_{13} = k'_1 \psi_{23}. \]

Converse proposition is trivial.

**iii)** Let \( \gamma \) be of PAW(3)-type. Substituting the equations (62) and (64) in (7), we get the solution. Converse proposition is trivial.

**iv)** Let \( \gamma \) be of PAW(4)-type. Substituting the equations (62), (63) and (64) in (8), we get the solution. Converse proposition is trivial.
v) Let $\gamma$ be of $PAW(5)$-type. Using the equations (9), (62) and (64), we get

$$
\begin{align*}
\psi_{13} &= a_1 k_1 + b_1 k'_1 \\
\psi_{23} &= a_1 k_2 + b_1 k'_2 \\
\psi_{33} &= a_1 k_3.
\end{align*}
$$

(69)

From the third equation of the system (69), we obtain

$$-5k_1 k_3 k'_1 - 5k_2 k_3 k'_2 = a_1 k_3,$$

which corresponds to

$$a_1 = -5k_1 k'_1 - 5k_2 k'_2.$$

Using $k_1^2 + k_2^2 + k_3^2 = \kappa^2$ and solving the last equation, we obtain

$$a_1 = -\frac{5}{2}(\kappa^2)' .$$

Converse proposition is trivial.

vi) Let $\gamma$ be of $PAW(6)$-type. Substituting the equations (62), (63) and (64) in (10), we obtain the equations (65). Substituting the equations (63) and (64) in (65), we get

$$k_3(a_2) + k_3\left(-b_2 k_2^2 - b_2 k_1^2 - b_2 k_3^2\right) = k_3\left(-5k_1 k'_1 - 5k_2 k'_2\right),$$

which means

$$a_2 - b_2 \kappa^2 = -\frac{5}{2}(\kappa^2)' .$$

Converse proposition is trivial.

vii) Let $\gamma$ be of $PAW(7)$-type. Since $k_3$ is non-zero constant, substituting the equations (62), (63) and (64) in (11), we obtain the equations

$$\psi_{33} = b_3 \phi_{33},$$

and (66). Substituting the equations (63) and (64) in the last equation, we get

$$-5k_1 k_2 k'_1 - 5k_2 k_3 k'_2 = -b_3 k_3(k_2^2 + k_1^2 + k_3^2).$$

If we divide both side with $k_3$, we obtain

$$5(k_1 k'_1 + k_3 k'_3 + k_2 k'_2) = b_3 \kappa^2,$$

which means

$$\frac{5}{2}(\kappa^2)' = b_3 \kappa^2 .$$

Converse proposition is trivial. \qed

**Corollary 3.8.** Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^4$ be a unit speed curve with non-zero constant $k_3$. If $\gamma$ is of $PAW(1)$-type then $\gamma$ is of $PAW(2)$-type.
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