A Brownian Particle and Fields I: 
Construction of Kinematics and Dynamics

Keita Seto*

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Extreme Light Infrastructure – Nuclear Physics (ELI-NP) / 
Horia Hulubei National Institute for R&D in Physics and Nuclear Engineering (IFIN-HH), 
30 Reactorului St., Bucharest-Magurele, jud. Ilfov, P.O.B. MG-6, RO-077125, Romania.

Abstract

Tracking a “real” trajectory of a quantum particle still has been treated as the interpretation problem. 
It shall be expressed by a Brownian (stochastic) motion suggested by E. Nelson, however, the well-defined 
mechanism of field generation from a stochastic particle hasn’t been proposed yet. For the improvement 
of this, I propose the extension of Nelson’s quantum dynamics, for describing a relativistic scalar electron 
with its radiation equivalent to the Klein-Gordon particle and field system.

Keyword:

[Physics] Stochastic quantum dynamics, relativistic motion, field generation
[mathematics] Applications of stochastic analysis

*keita.seto@eli-np.ro
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### Notation and Conventions

| Symbol | Description |
|--------|-------------|
| $c$    | Speed of light |
| $\hbar$ | Planck's constant |
| $m_0$  | Rest mass of an electron |
| $e$    | Charge of an electron |
| $\mathbb{V}^4_M$ | 4-dimensional standard vector space for the metric affine space |
| $g$    | Metric on $\mathbb{V}^4_M$: $g := \text{diag}(+1, -1, -1, -1)$ |
| $\mathbb{A}^4(\mathbb{V}^4_M, g)$ | 4-dimensional metric affine space with respect to $\mathbb{V}^4_M$ and $g$ |
| $\mathcal{B}(I)$ | Borel $\sigma$-algebra of a topological space $I$ |
| $\varphi_E := \{\varphi^A_E | A \in \text{set of indexes}\}$ | Coordinate mapping on $E$; $\varphi^A_E : E \to \mathbb{R}$ |
| $(\Omega, \mathcal{F}, \mathcal{P})$ | Probability space |
| $\mathbb{E}[\hat{X}(\bullet)] := \int_{\Omega} d\mathcal{P}(\omega) \hat{X}(\omega)$ | Expectation of $\hat{X}(\bullet) := \{\hat{X}(\omega) | \omega \in \Omega\}$ |
| $\mathbb{E}[\hat{X}(\bullet)|\mathcal{C}]$ | Conditional expectation of $\hat{X}(\bullet)$ given $\mathcal{C} \subset \mathcal{F}$ |
| $\mathcal{P}_{\tau} \subset \mathcal{F}$ | Sub-$\sigma$-algebra in the family of "the past", $\{\mathcal{P}_{\tau}\}_{\tau \in \mathbb{R}}$ |
| $\mathcal{F}_{\tau} \subset \mathcal{F}$ | Sub-$\sigma$-algebra in the family of "the future", $\{\mathcal{F}_{\tau}\}_{\tau \in \mathbb{R}}$ |
| $\hat{x}(\circ, \bullet)$ | Dual progressively measurable process (D-progressive, D-process) |
| $\forall \in \mathbb{V}^4_M \oplus i\mathbb{V}^4_M$ | Complex velocity; $\forall^\alpha(x) := i\lambda^2 \times \partial^\alpha \ln \phi(x) + e/m_0 \times A^\alpha(x)$ |
1 Introduction

This series of the papers proposes quantum dynamics coupled with stochastic kinematics of a scalar (spinless) electron and its radiation mechanism as the extension from the model by E. Nelson [1, 2, 3]. Especially, our main purpose by using Nelson’s stochastic quantization is the investigation of radiation reaction which is the kicked-back effect acting on an electron by its radiation [4]. Many works of radiation reaction have been discussed in classical dynamics from early 1900’s, however, the corrections by non-linear quantum electrodynamics (QED) becomes important in high-intensity field physics produced by the state-of-the-art $O(10^{15})$ lasers [5, 6, 7, 8, 9]. Comparing the radiation formulas in classical dynamics and non-linear QED, the factor of $q(\chi)$ can be found [10, 11, 12]:

$$\frac{dW_{\text{QED}}}{dt} = q(\chi) \times \frac{dW_{\text{classical}}}{dt}$$

Where, $dW_{\text{classical}}/dt = e^2/6\pi\varepsilon_0 c^3 \times g_{\alpha\beta} dv^\alpha/d\tau dv^\beta/d\tau$ denotes Larmor’s formula for the energy loss of radiation in classical dynamics [13] with respect to the Minkowski metric $g := \text{diag}(+1, -1, -1, -1)$ and the 4-velocity $v$. The uniqueness of radiation reaction in high-intensity field physics is the dependence of the field strength $F_{\text{ex}}$ and $\gamma$ the normalized energy of an electron via this factor $q(\chi)$, since

$$\chi := \frac{3}{2} \frac{\hbar}{m_0^2 c^3} \sqrt{-g_{\mu\nu}(-eF_{\text{ex}}^{\mu\alpha} v^\alpha)(-eF_{\text{ex}}^{\nu\beta} v^\beta)} = O(F_{\text{ex}} \times \gamma).$$

However as we will discuss the detail in Volume II [14], this is the formula applied only in the case of an electron interacting with an external laser field of a plane wave [15, 16] in order to the formulation of the non-linear Compton scattering [17, 18, 19]. Thus, it is natural to consider how to generalize it for the case of focused or superpositioned light. In addition, what is the origin of this factor $q(\chi)$?

Due to this reason, to identify the origin of $q(\chi)$ under general external fields is the strong research motivation on the issue of radiation reaction. For this aspect, we choose Nelson’s quantum dynamics by using Brownian motions in this article series. Nelson’s scheme has a very high-compatibility between equations of motion in classical and quantum dynamics. Here, let us summarize Nelson’s stochastic quantization.

E. Nelson proposed a quantization via a Brownian kinematics. A certain quanta draws a 3-dimensional Brownian motion as its trajectory in the non-relativistic regime [1, 2, 3]. By employing the kinematics $d\hat{x}(t, \omega) = V_+\hat{x}(t, \omega), dt + \sqrt{\hbar/2m_0} \times d\hat{w}_\pm(t, \omega)$ and its Fokker-Planck equations, he succeeded to demonstrate not only (A) his classical-like dynamics with the sub-equations

$$m_0 \left[ \partial_t v(x, t) + v(x, t) \cdot \nabla v(x, t) - u(x, t) \cdot \nabla u(x, t) - \frac{\hbar}{2m_0} \nabla^2 u(x, t) \right] = -\nabla \psi'(x, t)$$

$$v(x, t) = \frac{V_+(x, t) + V_-(x, t)}{2} = \text{Im} \left\{ \frac{\hbar}{m_0} \nabla \ln \psi(x, t) \right\}$$

$$u(x, t) = \frac{V_+(x, t) - V_-(x, t)}{2} = \text{Re} \left\{ \frac{\hbar}{m_0} \nabla \ln \psi(x, t) \right\}$$

is equivalent to the Schrödinger equation

$$i\hbar \partial_t \psi(x, t) = \left[ -\frac{\hbar^2}{2m_0} \nabla^2 + \psi'(x, t) \right] \psi(x, t),$$
but also (B) he answered why the square of the wave function should be regarded as the probability density \( \rho(x,t) = \psi^*(x,t)\psi(x,t) \) \([1, 2]\). The biggest advantage of his method is (C) the ability to draw the “real” trajectory of a quantum particle by a stochastic process. However, the feasibility of the coupled system between a stochastic particle and fields has not been established enough. Hence, the realization of the field-generation mechanism on it is the milestone for the description of radiation reaction.

Thus, this first volume dedicates the fundamental construction of a stochastic kinematics and dynamics of a scalar electron (a Klein-Gordon particle) including the mechanism of its light emission. Then, Volume II \([14]\) focuses on the formulation of radiation reaction on its stochastic trajectory.

In Section 2 of this paper, we introduce the kinematics of a scalar electron. At first, we define a stochastic differential equation \( d\hat{x}(\tau, \omega) = \mathcal{V}_\pm(\hat{x}(\tau, \omega))d\tau + \lambda \times dW_\pm(\tau, \omega) \) as a relativistic kinematics of a scalar electron in the Minkowski spacetime. Where, \( W_+(\circ, \bullet) \) and \( W_-(\circ, \bullet) \) are defined as a special classes of Wiener processes for this relativistic kinematics. Then, its complex velocity \( \mathcal{V}(\hat{x}(\tau, \omega)) := (1 - i)/2 \times \mathcal{V}_+(\hat{x}(\tau, \omega)) + (1 + i)/2 \times \mathcal{V}_-(\hat{x}(\tau, \omega)) \) \([20]\) is introduced. It can be found that this \( \mathcal{V} \) plays a role of the main cast in the present dynamics of a scalar electron in Section 3. In order to this kinematics, its Fokker-Planck equation is derived. In the end of Section 2 we discuss the most delicate problem how to define the proper time holding the transition between classical physics and quantum physics.

For the calculation of \( d\hat{x}(\tau, \omega) = \mathcal{V}_\pm(\hat{x}(\tau, \omega))d\tau + \lambda \times dW_\pm(\tau, \omega) \), it requires us to define the evolution of the velocities \( \mathcal{V}_\pm(\hat{x}(\tau, \omega)) \) or \( \mathcal{V}(\hat{x}(\tau, \omega)) \). Therefore, the dynamics of a scalar electron interacting with fields is discussed in Section 3. The mechanism of the field generation from a stochastic particle is discussed at here, too. Thus, our attention is devoted to the equivalency between the present model and the well-known system of the Klein-Gordon equation and the Maxwell equation via an action integral (the functional). Hereby, the following dynamics of a stochastic scalar electron and fields is realized.

\[
m_0 \partial_\tau \mathcal{V}^\mu(\hat{x}(\tau, \omega)) = -e \hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega)) F^\mu\nu(\hat{x}(\tau, \omega))
\]

\[
\partial_\mu [F^{\mu\nu}(x) + \delta f^{\mu\nu}] = \mu_0 \times \text{E} \left[ -e c \int_\text{R} \text{Re} \{\mathcal{V}^\nu(x)\} \delta^4(x - \hat{x}(\tau, \bullet)) \right]
\]

Finally, we summarize this Volume I and propose the motivation to Volume II \([14]\) in Section 4. Where, the transition between quantum and classical dynamics is discussed by using Ehrenfest’s theorem \([21]\) which is main tool to consider radiation reaction in Volume II.

2 Kinematics of a scalar electron

The first part is a stochastic and relativistic kinematics of a scalar electron. Let \( \mathbb{A}^4(\mathbb{V}_M^4, g) \) be a 4-dimensional metric affine space with respect to a 4-dimensional standard vector space \( \mathbb{V}_M^4 \) and its \( g \) on \( \mathbb{V}_M^4 \) \([22]\). Defining the measure space \( (\mathbb{A}^4(\mathbb{V}_M^4, g), \mathcal{B}(\mathbb{A}^4(\mathbb{V}_M^4, g)), \mu) \), we consider this as the Minkowski spacetime. Where, \( \mathcal{B}(I) \) denotes the Borel \( \sigma \)-algebra of a topological space \( I \). The relativistic kinematics of a stochastic scalar electron is characterized by the complex velocity \( \mathcal{V} \), the Fokker-Planck equations and the proper time \( \tau \).

Let us regard the coordinate mapping \( \varphi(x) := (x^0, x^1, x^2, x^3) \) has been already introduced for \( \forall x \in \mathbb{V}_M^4 \) or \( \forall x \in \mathbb{A}^4(\mathbb{V}_M^4, g) \) with its origin even if we do not declare it explicitly.
2.1 Stochastic process

For a certain abstract non-empty set \( \Omega \), consider the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Where, \( \mathcal{F} \) is a \( \sigma \)-algebra of \( \Omega \) and \( \mathbb{P} \) denotes the probability measure on the measurable space \( (\Omega, \mathcal{F}) \). Let us define the continuous stochastic process \( \hat{x}(\omega) := \{ \hat{x}(t, \omega) \in \mathbb{H}^4(V^4_M, g) \mid t \in [0, \infty), \omega \in \Omega \} \) as a \( \mathbb{D}([0, \infty]) \times \mathcal{F}((\mathbb{H}^4(V^4_M, g)) \)-measurable mapping. Where, by considering two measurable spaces \((X, \mathcal{X})\) and \((Y, \mathcal{Y})\), a \( \mathcal{X}/\mathcal{Y} \)-measurable mapping \( f \) is a mapping \( f : X \to Y \) satisfying \( f^{-1}(A) := \{ x \in X \mid f(x) \in A \} \subset \mathcal{X} \) for all \( A \in \mathcal{Y} \). Our milestone at here is the construction of the 4-dimensional stochastic differential equation

\[
\sqrt{\hbar} \sum_{i=1}^{6} \left( \frac{\partial}{\partial x_i} \right) \hat{x}(t, \omega) = \sqrt{\hbar} \sum_{i=1}^{6} \left( \frac{\partial}{\partial x_i} \right) \hat{\omega}(t, \omega) + \lambda \times dW^\mu_\omega(t, \omega)
\]

as a relativistic kinematics, extended from Nelson’s model \( d\hat{x}^{1, 2, 3}(t, \omega) = V^i_\omega(\hat{x}(t, \omega), t)dt + \sqrt{\hbar/2m_0} \times dw^i_\omega(t, \omega) \).

2.1.1 Forward evolution

Let us start from the definition of the usual 1-dimensional Wiener process \( w(\omega, \cdot) \):

**Definition 1** (Wiener process). When a 1-dimensional stochastic process \( w(\omega, \cdot) := \{ w(t, \omega) \in \mathbb{R} \mid t \in [0, \infty), \omega \in \Omega \} \) satisfies below, it is named a 1-dimensional **Wiener process** or a 1-dimensional **Brownian motion**.

1. \( w(0, \omega) = 0 \ a.s. \)
2. \( t \mapsto w(t, \omega) \) is continuous for each \( t, \omega \in \mathbb{R} \times \Omega \).
3. For all times \( 0 = t_0 < t_1 < \cdots < t_n \) \((n \in \mathbb{Z})\), \( \{ w(t_i, \omega) - w(t_{i-1}, \omega) \}_{i=1}^n \) are independent and each of them follows the normal distribution \( \{ N(0, t_i - t_{i-1}) \}_{i=1}^n \).

For \((\Omega, \mathcal{F}, \mathbb{P})\), let \( \mathbb{E}[\hat{X}(\cdot)] \) be the expectation of a random variable \( \hat{X}(\cdot) := \{ \hat{X}(\omega) \mid \omega \in \Omega \} \), i.e., \( \mathbb{E}[\hat{X}(\cdot)] := \int_{\Omega} d\mathbb{P}(\omega) \hat{X}(\omega) \). The conditional probability of \( B \) given \( A \) is described by \( \mathbb{P}_A(B) \). For \( \{ A_n \}_{n=1}^{\infty} \) of a countable decomposition of \( \Omega \), its minimum \( \sigma \)-algebra \( \mathcal{G} = \sigma(\{ A_n \}_{n=1}^{\infty}) \) is introduced. Then, \( \mathbb{E}[\hat{X}(\cdot) | \mathcal{G}](\omega) := \sum_{n=1}^{\infty} \{ \int_{\Omega} d\mathbb{P}_n(\omega') \hat{X}(\omega') \} \mathbb{1}_{A_n}(\omega) \) is defined as the conditional expectation of \( \hat{X}(\cdot) \) given \( \mathcal{G} \subset \mathcal{F} \); \( \mathbb{1}_{A_n}(\omega) \) satisfies \( \mathbb{1}_{A_n}(\omega) = 1 \) and \( \mathbb{1}_{A_n}(\omega \notin A_n) = 0 \).

**Lemma 2.** Consider \((\Omega, \mathcal{F}, \mathbb{P})\), a 1-dimensional Wiener process \( w(\omega, \cdot) \) satisfies the following relations for all \( t \):

\[
\mathbb{E}[w(t + \delta t, \omega) - w(t, \omega)] = 0 \quad (1)
\]

\[
\lim_{\delta t \to 0^+} \mathbb{E} \left[ \frac{[w(t + \delta t, \omega) - w(t, \omega)] \times [w(t + \delta t, \omega) - w(t, \omega)]}{\delta t} \right] = 1 \quad (2)
\]

For \( i = 1, 2, \cdots, N \), consider individual 1-dimensional Wiener processes \( w_i(\omega, \cdot) \) in each probability spaces \((\Omega_i, \mathcal{F}_i, \mathbb{P}_i)\). By \( \Omega := \Omega_1 \times \cdots \times \Omega_N \), \( \mathcal{F} := \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_N \) and \( \mathbb{P} := \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_N \), let \( \omega := (\omega_1, \cdots, \omega_N) \) be in \( \Omega \), an \( N \)-dimensional Wiener process \( w(\omega, \cdot) = (w_1(\omega, \omega_1), \cdots, w_N(\omega, \omega_N)) \) on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is imposed naturally, too. Let us describe it like \( w(\omega, \cdot) = (w_1(\omega, \cdot), \cdots, w_N(\omega, \cdot)) \) for emphasizing that it is on \((\Omega, \mathcal{F}, \mathbb{P})\).

\[
\mathbb{E}[w_i(t + \delta t, \omega) - w_i(t, \omega)] = 0 \quad (3)
\]

\[
\lim_{\delta t \to 0^+} \mathbb{E} \left[ \frac{[w_i(t + \delta t, \omega) - w_i(t, \omega)] \times [w_i(t + \delta t, \omega) - w_i(t, \omega)]}{\delta t} \right] = \delta^{ij} \quad (4)
\]

Where, \( i, j = 1, 2, \cdots, N \).

\(^1\text{Though} \, \hat{x}(\tau, \omega) \text{can be written as} \, \hat{x}(\tau) \text{or} \, \hat{x}_\tau, \text{however, we dare to write} \, \omega \text{to visualize the names of each paths explicitly.} \)
**Definition 3** ($\{\mathcal{P}_t\}$ and $\{\mathcal{F}_t\}$). For $(\Omega, \mathcal{F}, \mathcal{P})$, $\{\mathcal{P}_t\}_{t \in \mathbb{R}}$ is the increasing family of sub-$\sigma$-algebras of $\mathcal{F}$ such that $-\infty < s \leq t \implies \mathcal{P}_s \subset \mathcal{P}_t$. And $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ is the decreasing family fulfilling $t \leq s < \infty \implies \mathcal{F}_s \supset \mathcal{F}_t$.

For $(\Omega, \mathcal{F}, \mathcal{P})$, consider a family of $\mathcal{B}([0, t]) \times \mathcal{F}/\mathcal{B}(X)$-measurable mappings ($X$ is a $N$-dimensional topological space)

$$L^p_t(X) := \left\{ f(\omega, \bullet) \left| \int_{T \subset \mathbb{R}} |\varphi^i \circ f(t, \omega)|^p dt < \infty \text{ a.s., } i = 1, 2, \cdots, N \right\}$$

where, $\{\varphi^i\}_{i=1}^N$ is the coordinate mapping $\varphi^i : X \to \mathbb{R}$. Then, its "adapted" class is,

$$L^p_{loc}(\{\mathcal{P}_t\}; X) := \left\{ f(\omega, \bullet) \in L^p_{[0, t]}(X) \left| f(\omega, \bullet) \text{ is } \{\mathcal{P}_t\}-\text{adapted} \right\}.$$ 

Where, $\{\mathcal{P}_t\}$-adapted means a stochastic process which is $\mathcal{P}_t/\mathcal{B}(X)$-measurable for all $t$.

**Definition 4** ($\{\mathcal{P}_t\}$-Wiener process). When an $N$-dimensional Wiener process $w(\omega, \bullet)$ satisfies the following, it is called a $\{\mathcal{P}_t\}$-Wiener process.

1. $w(\omega, \bullet)$ is $\{\mathcal{P}_t\}$-adapted.
2. $w(t, \omega) - w(s, \omega)$ and $\mathcal{P}_s$ are independent for $0 \leq s \leq t$.

**Definition 5** (Itô integral). The integral,

$$\int_0^t f(t', \omega) dw^i(t', \omega), \quad i = 1, 2, \cdots, N$$

is defined as an Itô integral of a function $f \in L^2_{loc}(\{\mathcal{P}_t\}; \mathbb{R})$ for a $\{\mathcal{P}_t\}$-Wiener process $w(\omega, \bullet)$.

Let us use $w_+(\omega, \bullet)$ as a $\{\mathcal{P}_t\}$-Wiener process from here.

**Lemma 6.** For $f(\omega, \bullet), g(\omega, \bullet) \in L^2_{loc}(\{\mathcal{P}_t\}; \mathbb{R})$, the following formula is imposed $^{[23]}$:

$$\mathbb{E} \left[ \int_0^T f(t', \bullet) dw^i_+(t', \bullet) \cdot \int_0^T g(t'', \bullet) dw^j_+(t'', \bullet) \right] = \delta^{ij} \times \mathbb{E} \left[ \int_0^T f(t', \bullet) \cdot g(t', \bullet) dt' \right] \quad (5)$$

**Theorem 7** (Itô formula). For $a^i(\omega, \bullet) \in L^1_{loc}(\{\mathcal{P}_t\}; \mathbb{R})$ and $b^i_\alpha(\omega, \bullet) \in L^2_{loc}(\{\mathcal{P}_t\}; \mathbb{R})$ ($1 \leq i \leq N, 1 \leq \alpha \leq d$), consider an $N$-dimensional process $\hat{X}(\omega, \bullet)$ given by

$$\hat{X}^i(t, \omega) = \hat{X}^i(0, \omega) + \int_0^t a^i(t', \omega) dt' + \sum_{\alpha=1}^d \int_0^t b^i_\alpha(t', \omega) dw^\alpha_+(t', \omega)$$

with respect to an initial condition $\hat{X}^i(0, \omega)$. Then for a function $f \in C^2(\mathbb{R}^N)$,

$$f(\hat{X}(t, \omega)) = f(\hat{X}(0, \omega)) + \sum_{i=1}^N \int_0^t \frac{\partial f}{\partial x^i}(\hat{X}(t', \omega)) \left[ a^i(t', \omega) dt' + \sum_{\alpha=1}^d b^i_\alpha(t', \omega) dw^\alpha_+(t', \omega) \right] \quad (6)$$
Let us consider a type of a backward evolution. The point at here is to consider a decreasing family of \( J \) examples. Let us define a new stochastic process like:

\[
\begin{align*}
\tilde{X}(t', \omega) &= X(t, \omega) - \hat{X}(0, \omega) \\
\tilde{w} &= f(\tilde{X}(t', \omega)) - f(\hat{X}(0, \omega)),
\end{align*}
\]

is almost surely satisfied. By introducing \( \int_0^t d_+ \tilde{X}(t', \omega) := \hat{X}(t, \omega) - \tilde{X}(0, \omega) \) and \( \int_0^t d_+ f(\tilde{X}(t', \omega)) := f(\tilde{X}(t, \omega)) - f(\hat{X}(0, \omega)) \), it is written like

\[
d_+ f(\tilde{X}(t, \omega)) = \sum_{i=1}^N \frac{\partial f}{\partial x^i}(\hat{X}(t, \omega))d_+ \hat{X}^i(t, \omega) + \frac{1}{2} \sum_{i,j=1}^N \frac{\partial^2 f}{\partial x^i \partial x^j}(\hat{X}(t, \omega))d_+ \hat{X}^i(t, \omega)d_+ \hat{X}^j(t, \omega)
\]

Though a \( \{P_t\} \)-Wiener process is defined on \([0, \infty]\), however, it can be expand on to \( \mathbb{R} \) naturally. For example, let us define a new stochastic process like \( w'_+(t - T, \omega) \) with respect to \( T > 0 \). This new \( w'_+(t, \omega) \) is a \( \{P_t\} \)-Wiener process such that \( w'_+(t - T, \omega) = 0 \) a.s. By defining another process \( w''_+(t - T, \omega) := w'_+(t - T, \omega) - w'_+(0, \omega) \), this \( w''_+(t, \omega) \) is a \( \{P_t\} \)-Wiener process on \([-T, \infty)\), satisfying \( w''_+(0, \omega) = 0 \) a.s. By repeating this procedure, the domain of a \( \{P_t\} \)-Wiener process \( w_+(t, \omega) \) can be expanded on to \( \mathbb{R} \) with the condition \( w_+(0, \omega) = 0 \) a.s. Therefore, the differential form of (8) is imposed for all \( t \in \mathbb{R} \). From here, we select the domain of stochastic processes on \( \mathbb{R} \) for the later discussion. In order to this reason, \( L^0_{\text{loc}}(\{P_t\}; X) \) is replaced by the new definition

\[
L^0_{\text{loc}}(\{P_t\}; X) := \left\{ f(\cdot, \cdot) \in L^0_{\text{loc}}(\mathbb{R}^n) \left| f(\cdot, \cdot) \text{ is } \{P_t\}-\text{adapted} \right. \right\}.
\]

If a stochastic process \( f(\cdot, \cdot) \) is \( \mathcal{P}(\mathbb{R}^n) \times \mathcal{P}_t/\mathcal{P}(X) \)-measurable for all \( t \in \mathbb{R} \), let us name \( f(\cdot, \cdot) \) \( \{P_t\} \)-progressively measurable or \( \{P_t\} \)-progressive. Hence, \( \hat{X}(\cdot, \cdot) \) of (6) is \( \{P_t\} \)-progressive.

Consider a simple 1-dimensional \( \{P_t\} \)-progressive \( \hat{X}(\cdot, \cdot) \) such that \( d_+ \hat{X}(t, \omega) = a_+(\hat{X}(t, \omega))dt + \theta_+ \times dw_+(t, \omega) \) (\( \theta_+ > 0 \)) and \( d\mathbb{E}[f(\hat{X}(t, \cdot))] \)/dt for a function \( f \in C^2(\mathbb{R}) \). Since \( \mathbb{E}[f(\hat{X}(t, \cdot))] := \int_{\mathbb{R}} dx f(x)p(x, t) \) for the probability density \( p(x, t) \), thus, \( d\mathbb{E}[f(\hat{X}(t, \cdot))] / dt = \int_{\mathbb{R}} dx d\mathbb{E}[f(x)p(x, t)] \) and also

\[
\frac{d\mathbb{E}[f(\hat{X}(t, \cdot))]}{dt} = \lim_{\Delta t \to 0^+} \frac{\mathbb{E}[f(\hat{X}(t + \Delta t, \cdot))] - \mathbb{E}[f(\hat{X}(t, \cdot))]}{\Delta t}
= \lim_{\Delta t \to 0^+} \mathbb{E} \left[ E \left[ \frac{1}{\Delta t} \int_t^{t + \Delta t} d_+ f(\hat{X}(t, \cdot)) \bigg| P_t \right] \right],
\]

where, \( \mathbb{E}[f(\hat{X}(\tau, \cdot))] = \mathbb{E}[\mathbb{E}[f(\hat{X}(\tau, \cdot))]|P_t] \) is employed. Then, the Fokker-Planck equation is derived:

\[
\partial_t p(x, t) + \partial_x [a_+(x)p(x, t)] = \frac{\theta_+}{2} \partial_x^2 p(x, t)
\]

\[8\] 2.1.2 Backward evolution

Let us consider a type of a backward evolution. The point at here is to consider a decreasing family of \( \{F_t\} \).

**Lemma 8.** An \( N \)-dimensional \( \{F_t\} \)-Wiener process \( w_-(\cdot, \cdot) \) fulfills the following relations (\( i, j = 1, 2, \cdots, N \)):

\[
\mathbb{E}[w_i^j(t, \cdot) - w_i^j(t - \delta t, \cdot)] = 0
\]

\[
\lim_{\delta t \to 0^+} \mathbb{E} \left[ \frac{[w_i^j(t, \cdot) - w_i^j(t - \delta t, \cdot)] \times [w_i^j(t, \cdot) - w_i^j(t - \delta t, \cdot)]}{\delta t} \right] = \delta^{ij}
\]
Then, the following function family is defined:

\[
\mathcal{L}_p^{\text{loc}}(\{\mathcal{F}_t\}; X) := \left\{ f(\omega, \bullet) \in L_p^{\text{loc}}(X) \mid f(\omega, \bullet) \text{ is } \{\mathcal{F}_t\}\text{-adapted} \right\}
\]

If a stochastic process \( f(\omega, \bullet) \) is \( \mathcal{B}([t, \infty)) \times \mathcal{F}_t / \mathcal{B}(X) \)-measurable for all \( t \in \mathbb{R} \), let us call it \( \{\mathcal{F}_t\}\)-progressively measurable or \( \{\mathcal{F}_t\}\)-progressive.

**Theorem 9.** For \( t < \forall \alpha < 2 \), \( \{\mathcal{A}^i(\omega, \bullet)\} \in \mathcal{L}_p^{\text{loc}}(\{\mathcal{F}_t\}; \mathbb{R}) \), \( \{\mathcal{B}^\alpha(\omega, \bullet)\} \in \mathcal{L}_p^{\text{loc}}(\{\mathcal{F}_t\}; \mathbb{R}) \) and a d-dimensional \( \{\mathcal{F}_t\}\)-Wiener process \( \{w^\alpha(\omega, \omega)\} \) (\( 1 \leq i \leq N \), \( 1 \leq \alpha \leq d \)), consider an \( N \)-dimensional process \( \{\mathcal{F}_t\}\)-progressive \( \hat{X}(\omega) \) given by

\[
\hat{X}^i(t, \omega) = \hat{X}^i(T, \omega) - \int_t^T A^i(t', \omega)dt' - \sum_{\alpha=1}^d \int_t^T B^\alpha(t', \omega)dw^\alpha(t', \omega)
\]

with respect to a terminal condition \( \hat{X}(T, \omega) \). By introducing \( \int_t^T d_-\hat{X}(t', \omega) := \hat{X}(T, \omega) - \hat{X}(t, \omega) \), i.e.,

\[
d_-\hat{X}^i(t, \omega) = A^i(t, \omega)dt + \sum_{\alpha=1}^d B^\alpha(t, \omega)dw^\alpha(t, \omega)
\]

and \( \int_t^T d_-f(\hat{X}(t', \omega)) := f(\hat{X}(T, \omega)) - f(\hat{X}(t, \omega)) \) for a function \( f \in C^2(\mathbb{R}^N) \),

\[
d_-f(\hat{X}(t, \omega)) = \sum_{i=1}^N \frac{\partial f}{\partial x^i}(\hat{X}(t, \omega))d_-\hat{X}^i(t, \omega) - \frac{1}{2} \sum_{i, j=1}^N \frac{\partial^2 f}{\partial x^i \partial x^j}(\hat{X}(t, \omega))d_-\hat{X}^i(t, \omega)d_-\hat{X}^j(t, \omega) \text{ a.s.}
\]

Consider the derivation of \([15]\) by using \([8]\). The decreasing family \( \{\mathcal{F}_t\}_{t \in \mathbb{R}} \) relates to an increasing family \( \{\mathcal{P}_t\}_{t \in \mathbb{R}} \) by the replacement of its subscripts: By defining a monotonically decreasing function \( f : \mathbb{R} \rightarrow \mathbb{R} \), then, \( \{\mathcal{F}_{f(t)}\}_{t \in \mathbb{R}} \) becomes an increasing family for \( t \) since \( t \leq s \Rightarrow \mathcal{F}_{f(t)} \subset \mathcal{F}_{f(s)} \). Thus, there is a correspondence between each of \( t \) and \( t' \). \( \{\mathcal{P}_t\}_{t \in \mathbb{R}} \equiv \{\mathcal{F}_{f(t)}\}_{t \in \mathbb{R}} \). For an \( \{\mathcal{F}_t\}\)-adapted \( \hat{X}(\omega, \bullet) \), there is a \( \{\mathcal{P}_t\}\)-adapted \( \hat{X}'(\omega, \bullet) \) satisfying \( \hat{X}'(t', \omega) = \hat{X}(t, \omega) \) at a fixed \( t \). In order to the construction of \( \{\mathcal{P}_t\} \) and \( \hat{X}'(\omega, \bullet) \), \( \hat{X}'(t' + T, \omega) = \hat{X}(t, \omega) \) for all \( T \in \mathbb{R} \) via the relation \( f(t - T) := t' + T \). Since

\[
\int_{t'-T}^{t'+T} d_+\hat{X}'(t, \omega) = \hat{X}'(t' + T, \omega) - \hat{X}'(t' - T, \omega) = \hat{X}(t - T, \omega) - \hat{X}(t + T, \omega) = -f_{t'-T}^{t'+T} d_-\hat{X}(t, \omega)
\]

for \( T > 0 \), therefore, \( d_+\hat{X}'(t', \omega) = -d_-\hat{X}(t, \omega) \) is derived. And \( d_+f(\hat{X}'(t', \omega)) = -d_-f(\hat{X}(t, \omega)) \) is also satisfied for a function \( f \in C^2(\mathbb{R}^N) \) by the same way. Let us employ those relations on \([8]\), namely,

\[
d_+f(\hat{X}'(t', \omega)) = \sum_{i=1}^N \frac{\partial f}{\partial x^i}(\hat{X}'(t', \omega))d_+\hat{X}^i(t', \omega) + \frac{1}{2} \sum_{i, j=1}^N \frac{\partial^2 f}{\partial x^i \partial x^j}(\hat{X}'(t', \omega))d_+\hat{X}^i(t', \omega)d_+\hat{X}^j(t', \omega) \text{ a.s.,}
\]

the Itô formula of \([15]\) is imposed by switching from the \( \{\mathcal{P}_t\}\)-adapted \( \hat{X}'(\omega, \bullet) \) to the \( \{\mathcal{F}_t\}\)-adapted \( \hat{X}(\omega, \bullet) \). This discussion can be applied for each \( t \in \mathbb{R} \).

In the case of a 1-dimensional \( \{\mathcal{F}_t\}\)-progressive \( \hat{Y}(\omega, \bullet) \), i.e., \( d_-\hat{Y}(t, \omega) = a_-(\hat{Y}(t, \omega))dt + \theta_- \times dw_-(t, \omega) \)
the following Fokker-Planck equation is fulfilled:

\[
\frac{d\mathbb{E}[f(\hat{Y}(t,\bullet))]}{dt} = \lim_{\Delta t \to 0^+} \mathbb{E}[f(\hat{Y}(t,\bullet))] - \mathbb{E}[f(\hat{Y}(t-\Delta t,\bullet))] = \lim_{\Delta t \to 0^+} \mathbb{E} \left[ \mathbb{E} \left[ \frac{1}{\Delta t} \int_{t-\Delta t}^{t} d_{-f}(\hat{X}(t',\bullet)) \big| \mathcal{F}_t \right] \right],
\]

(17)

the following Fokker-Planck equation is fulfilled:

\[
\frac{\partial}{\partial t} p(y, t) + \frac{\partial}{\partial y} [a_- (y) p(y, t)] = -\frac{\theta^2}{2} \frac{\partial^2}{\partial y^2} p(y, t)
\]

(18)

2.1.3 Forward-backward composition

Let us rewrite 1-dimensional stochastic processes of \( \{\mathcal{P}_t\} \)-adapted \( \hat{X}(\omega, \bullet) \) and \( \{\mathcal{F}_t\} \)-adapted \( \hat{Y}(\omega, \bullet) \) on \( T \subset \mathbb{R} \):

\[
\begin{align*}
\int_{t \in T} d_{+} \hat{X}(t, \omega) &= \int_{t \in T} a_{+}(\hat{X}(t, \omega)) dt + \theta \int_{t \in T} dw_{+}(t, \omega), \quad \{\mathcal{P}_t\} \text{-adapted} \\
\int_{t \in T} d_{-} \hat{Y}(t, \omega) &= \int_{t \in T} a_{-}(\hat{Y}(t, \omega)) dt + \theta \int_{t \in T} dw_{-}(t, \omega), \quad \{\mathcal{F}_t\} \text{-adapted}
\end{align*}
\]

(19)

Where, the following Itô rule is satisfied; \( dt \cdot dt = 0, dt \cdot dw_{\pm}(t, \omega) = 0, dw_{\pm}(t, \omega) \cdot dw_{\pm}(t, \omega) = dt \) and \( dw_{\pm}(t, \omega) \cdot dw_{\mp}(t, \omega) = 0 \). By describing (19) like

\[
\begin{bmatrix}
\hat{X}(b, \omega) - \hat{X}(a, \omega) \\
\hat{Y}(b, \omega) - \hat{Y}(a, \omega)
\end{bmatrix} = \int_{a}^{b} \begin{bmatrix}
a_{+}(\hat{X}(t, \omega), \hat{Y}(t, \omega)) \\
a_{-}(\hat{X}(t, \omega), \hat{Y}(t, \omega))
\end{bmatrix} dt + \theta \times \int_{a}^{b} \begin{bmatrix}
dw_{+}(t, \omega) \\
dw_{-}(t, \omega)
\end{bmatrix}
\]

(20)

for \( a_{+}, a_{-} : \mathbb{R}^2 \to \mathbb{R} \), it can be regarded as the equation of a 2-component vector \( \hat{\gamma}(t, \omega) := (\hat{X}(t, \omega), \hat{Y}(t, \omega)) \).

Though, \( \{\mathcal{P}_t\} \) and \( \{\mathcal{F}_t\} \) are the different types of the families of sub-\( \sigma \)-algebras, however, the index of \( t \in \mathbb{R} \) is common. Therefore, the mathematical set of \( \{\mathcal{P}_t, \mathcal{F}_t\} \) can be defined for each \( t \). Let us regard this as a new sub-\( \sigma \)-algebra for a 2-dimensional stochastic process denoted by \( \mathcal{M}_t = \mathcal{P}_t \otimes \mathcal{F}_t \in \mathcal{F} \) and consider its family \( \{\mathcal{M}_t\}_{t \in \mathbb{R}} \).

**Definition 10** (Forward-backward composition). Let \( \{\mathcal{M}_t\}_{t \in \mathbb{R}} \) be a family of a sub-\( \sigma \)-algebras for a 2-dimensional stochastic process \( \hat{\gamma}(t, \bullet) := (\hat{X}(t, \bullet), \hat{Y}(t, \bullet)) \) on \( (\Omega, \mathcal{F}, \mathcal{P}) \). Then, \( \hat{\gamma}(t, \bullet) \) is called \( \mathcal{M}_t \otimes \mathcal{B}(\mathbb{R}^2) \)-measurable or \( \mathcal{P}_t \otimes \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^2) \)-measurable, when \( \hat{X}(t, \bullet) \) is \( \mathcal{P}_t \otimes \mathcal{B}(\mathbb{R}) \)-measurable and \( \hat{Y}(t, \bullet) \) is \( \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}) \)-measurable. If \( \{\hat{\gamma}(t, \bullet)\}_{t \in \mathbb{R}} \in \mathcal{M}_t \otimes \mathcal{B}(\mathbb{R}^2) \)-measurable for all \( t \), \( \hat{\gamma}(\omega, \bullet) \) is called \( \{\mathcal{M}_t\} \)-adapted or \( \{\mathcal{P}_t \otimes \mathcal{F}_t\} \)-adapted. On the contrary when a 2-dimensional \( \{\mathcal{M}_t\} \)-adapted process is given, it can be decomposed into 1-dimensional \( \{\mathcal{P}_t\} \) and \( \{\mathcal{F}_t\} \)-adapted processes.

For a \( \mathcal{B}(\mathbb{R}^2) \)-measurable function \( f : \mathbb{R}^2 \to X \) for a certain topological space \( X \), \( f(\hat{\gamma}(t, \bullet)) \) is \( \mathcal{M}_t \otimes \mathcal{B}(X) \)-measurable. Hence, \( a(\hat{\gamma}(t, \bullet), \bullet) := (a_{+}(\hat{\gamma}(t, \bullet)), a_{-}(\hat{\gamma}(t, \bullet)) \) is \( \mathcal{M}_t \otimes \mathcal{B}(\mathbb{R}^2) \)-measurable. Then, consider an Itô formula of a function \( f \in C^2(\mathbb{R}^2) \) on \( \hat{\gamma}(\omega, \bullet) \).

\[
f(\hat{\gamma}(b, \omega)) - f(\hat{\gamma}(a, \omega)) = \int_{a}^{b} \left[ \frac{\partial}{\partial x} f(\hat{\gamma}(t, \omega)) d_{+} \hat{X}(\hat{\gamma}(t, \omega)) + \frac{\partial}{\partial y} f(\hat{\gamma}(t, \omega)) d_{-} \hat{Y}(\hat{\gamma}(t, \omega)) \right] dt + \frac{\theta^2}{2} \int_{a}^{b} \left[ \frac{\partial^2}{\partial x \partial y} f(\hat{\gamma}(t, \omega)) - \frac{\partial^2}{\partial y \partial y} f(\hat{\gamma}(t, \omega)) \right] dt \text{ a.s.}
\]

(21)
As the next step, consider a \( \{ \mathcal{F}_t \otimes \mathcal{P}_t \otimes \mathcal{P}_t \otimes \mathcal{P}_t \} \)-adapted \( \dot{x}(\cdot, \cdot) \), namely,

\[
d_{[\cdot]} \dot{x}^{0,1,2,3}(t, \omega) := a^i(\dot{x}(t, \omega)) dt + \theta \times dw_i(t, \omega),
\]

(22)

Where, \( \dot{x}^{0,1,2,3} \) is \( \{ \mathcal{F}_t \} \)-adapted and \( \{ \dot{x}^i(\cdot, \cdot) \}_{i=1,2,3} \) are \( \{ \mathcal{P}_t \} \)-adapted. Hence, \( \{ w^i(\cdot, \cdot) \}_{i=0,1,2,3} \) is the set of a 1-dimensional \( \{ \mathcal{F}_t \} \)-Wiener process \( w^0(\cdot, \cdot) \) and a 3-dimensional \( \{ \mathcal{P}_t \} \)-Wiener process \( \{ w^i(\cdot, \cdot) \}_{i=1,2,3} \) (see its characteristics in Definition 11 and Theorem 12). Then, the Itô formula of a function \( f \in C^2(\mathbb{R}^2) \) with respect to \( \dot{x}(\cdot, \omega) \) is,

\[
f(\dot{x}(t, \omega)) = f(\dot{x}(t_0, \omega)) + \sum_{i=0,1,2,3} \int_{t_0}^{t} \partial_t f(\dot{x}(t', \omega)) \cdot d_{[i]} \dot{x}^i(t', \omega) - \frac{\theta^2}{2} \int_{t_0}^{t} \left[ \partial_\omega^2 f(\dot{x}(t', \omega)) \right] dt' \quad \text{a.s.}
\]

(23)

**Definition 11 (\((-g)\)-Wiener process).** When \( a = 0 \) and \( \theta = 1 \) on (22), namely, the solution of

\[
d_{[i]} \dot{x}^{0,1,2,3}(t, \omega) = dw_i(t, \omega),
\]

(24)

is called a \( \{ \mathcal{F}_t \otimes \mathcal{P}_t \otimes \mathcal{P}_t \otimes \mathcal{P}_t \} \)-\((\omega)\)-Wiener process denoted by \( w_{(\omega)}(\cdot, \cdot) \) such that \( w_{(\omega)}^0(\cdot, \cdot) \) is a 1-dimensional \( \{ \mathcal{F}_t \} \)-Wiener process and \( \{ w_{(\omega)}^i(\cdot, \cdot) \}_{i=1,2,3} \) is a 3-dimensional \( \{ \mathcal{P}_t \} \)-Wiener process, satisfying the following for \( i, j = 0, 1, 2, 3 \):

\[
\mathbb{E} \left[ \int_{t}^{t+\delta t} dw_{(\omega)}^i(t', \cdot) \right] = 0, \quad \mathbb{E} \left[ \int_{t}^{t+\delta t} dw_{(\omega)}^i(t', \cdot) \times \int_{t}^{t+\delta t} dw_{(\omega)}^j(t'', \cdot) \right] = \delta_{ij} \times \delta t
\]

(25)

The name of "\((\omega)\)" derives from the following characteristics:

**Theorem 12 (Itô formula and Fokker-Planck equation on \( w_{(\omega)}(\cdot, \cdot) \)).** For \( g = \text{diag}(+1, -1, -1, -1) \), \( w_{(\omega)}(\cdot, \cdot) \) induces the Itô formula of a function \( f \in C^2(\mathbb{R}^4) \)

\[
df(w_{(\omega)}(t, \omega)) = \sum_{i=0}^{3} \frac{\partial}{\partial x^i} f(w_{(\omega)}(t, \omega)) \cdot dw_{(\omega)}^i(t, \omega) + \lambda^2 \sum_{i,j=0}^{3} (-g^{ij}) \frac{\partial^2}{\partial x^i \partial x^j} f(w_{(\omega)}(t, \omega)) \cdot d\tau \quad \text{a.s.}
\]

(26)

and the following Fokker-Planck equation for its probability density \( p \in C^{2,1}(\mathbb{R}^4 \times \mathbb{R}) \)

\[
\partial_t p(x, t) = \frac{1}{2} \sum_{i,j=0}^{3} (-g^{ij}) \frac{\partial^2}{\partial x^i \partial x^j} p(x, t).
\]

(27)

The following is the basic idea to describe a relativistic kinematics.

**Definition 13 (\( \{ \mathcal{P}_\tau \} \) and \( \{ \mathcal{F}_\tau \} \)).** For \( (\Omega, \mathcal{F}, \mathcal{P}) \), consider \( \{ \mathcal{P}_\tau \}_{\tau \in \mathbb{R}} \) a family of composited sub-\( \sigma \)-algebras \( \mathcal{P}_\tau \in \mathcal{F} \) for a 4-dimensional stochastic process. By introducing a monotonically decreasing function \( f : \mathbb{R} \to \mathbb{R} \), \( \{ \mathcal{F}_f(\tau) \otimes \mathcal{P}_f(\tau) \otimes \mathcal{P}_f(\tau) \otimes \mathcal{P}_f(\tau) \}_{\tau \in \mathbb{R}} \) imposes \( \{ \mathcal{F}_\tau \}_{\tau \in \mathbb{R}} \), a family of \( \mathcal{F}_\tau \in \mathcal{F} \).
**Definition 14** \((W_+(\omega, \bullet)\) and \(W_-(\omega, \bullet)\)). Let \(W_+(\omega, \bullet)\) be a \(\{\mathcal{P}_\tau\}\)-(\(-g\))-Wiener process, namely, \(w_{(-g)}(\omega, \bullet)\). Then, an \(\{\mathcal{F}_\tau\}\)-(\(-g\)) Wiener process \(W_-(\omega, \bullet)\) is imposed by **Definition 13**. Its satisfies the following relations for \(\mu, \nu = 0, 1, 2, 3\) and a function \(f \in C^2(\mathbb{R}^4)\):

\[
\mathbb{E}\left[\int_\tau^\tau + \delta \tau dW^\mu_-(\tau', \bullet)\right] = 0, \quad \mathbb{E}\left[\int_\tau^\tau + \delta \tau dW^\mu_-(\tau', \bullet) \times \int_\tau^\tau + \delta \tau dW^\nu_-(\tau', \bullet)\right] = \delta_{\mu\nu} \times \delta \tau,
\]

(28)

\[
f(W_+(\tau_0, \omega)) - f(W_+(\tau_0, \omega)) = \int_{\tau_0}^{\tau_0} \partial_{\mu} f(W_+(\tau, \omega)) dW^\mu_+(\tau, \omega) + \frac{\lambda^2}{2} \int_{\tau_0}^{\tau_0} \partial_{\mu} \partial^\mu f(W_+(\tau, \omega)) d\tau \text{ a.s.}
\]

(29)

### 2.2 Dual-progressively measurable process \(\hat{x}(\omega, \bullet)\)

Let \(\hat{x}(\omega, \bullet)\) be a stochastic process as a relativistic kinematics extended from 3-dimensional Nelson’s (R0), (R1), (R2), (R3), (S1), (S2) and (S2)-processes to 4-dimensional one. For our convenience, the coordinate mappings \(\{\varphi_\omega^A\}\) is introduced again, such that the index \(A\) becomes \(A = \mu\) if \(E = V^4_M\) or \(E = \mathbb{A}^4(V^4_M, g)\) with its origin and \(A = \mu\nu\) when \(E = V^4_M \otimes V^4_M\) \((\mu, \nu = 0, 1, 2, 3)\):

**Definition 15** \((\text{(R0)-process)\}\). For \((\Omega, \mathcal{F}, \mathcal{P})\), a \(\mathcal{B}(\mathbb{R}) \times \mathcal{F}/\mathcal{B}(\mathbb{A}^4(V^4_M, g))\)-measurable \(\hat{x}(\omega, \bullet)\) is a 4-dimensional (R0)-process if each of \(\{\varphi_\omega^{\mu}(V^4_M, g) \circ \hat{x}(\tau, \bullet)\}_{\mu=0,1,2,3}\) belongs to \(L^1(\Omega, \mathcal{P})\) and the mapping \(\tau \mapsto \hat{x}(\tau, \omega)\) is almost surely continuous.

By employing **Definition 13** and \(L^p_T(E)\) a family of \(\mathcal{B}(\mathbb{R}) \times \mathcal{F}/\mathcal{B}(E)\)-measurable mappings for a topological space \(E\), let us introduce \(L^p_{\text{loc}}(\{\mathcal{P}_\tau\}; E)\) and \(L^p_{\text{loc}}(\{\mathcal{F}_\tau\}; E)\) as families of stochastic processes as follows:

\[
L^p_T(E) := \left\{ \hat{X}(\omega, \bullet) \mid \hat{X}(\omega, \bullet) : \mathbb{R} \times \Omega \to E, \sum_\mathcal{A} \int_{\tau \in \mathbb{R}} |\varphi_\omega^A \circ \hat{X}(\tau, \omega)|^p d\tau < \infty \text{ a.s.} \right\}
\]

\[
L^p_{\text{loc}}(\{\mathcal{P}_\tau\}; E) := \left\{ \hat{X}(\omega, \bullet) \in L^p_{(-\infty, \tau]}(E) \mid \hat{X}(\omega, \bullet) \text{ is } \{\mathcal{P}_\tau\}-\text{adapted} \right\}
\]

\[
L^p_{\text{loc}}(\{\mathcal{F}_\tau\}; E) := \left\{ \hat{X}(\omega, \bullet) \in L^p_{[\tau, \infty)}(E) \forall \tau \in \mathbb{R}, \hat{X}(\omega, \bullet) \text{ is } \{\mathcal{F}_\tau\}-\text{adapted} \right\}
\]

For the later discussion, \(\hat{c}\) is defined as \(\hat{c} = 1\) when a 1-dimensional \(\hat{x}^\mu(\omega, \bullet) = \varphi_\omega^{\mu}(V^4_M, g) \circ \hat{x}(\omega, \bullet)\) is \(\{\mathcal{P}_\tau\}\)-adapted and \(\hat{c} = -1\) if \(\hat{x}^\mu(\omega, \bullet)\) is \(\{\mathcal{F}_\tau\}\)-adapted.

**Definition 16** \((\text{(R1)-process)\})\). If \(\hat{x}(\omega, \bullet)\) is an (R0)-process and a following \(V_+(\hat{x}(\omega, \bullet)) \in L^1_{\text{loc}}(\{\mathcal{P}_\tau\}; V^4_M)\) exists, \(\hat{x}(\omega, \bullet)\) is named an (R1)-process.

\[
V_+(\hat{x}(\tau, \omega)) := \lim_{\delta \tau \to 0+} \mathbb{E}\left[\left(\frac{\hat{x}(\tau + \delta \tau, \bullet) - \hat{x}(\tau, \bullet)}{\delta \tau}\right) \bigg| \mathcal{P}_\tau\right](\omega)
\]

(30)

The definitions of its each components are the following:

\[
\begin{align*}
V^0_+(\hat{x}(\tau, \omega)) &= \lim_{\delta \tau \to 0+} \mathbb{E}\left[\left(\frac{\hat{x}^0(\tau, \bullet) - \hat{x}^0(\tau - \delta \tau, \bullet)}{\delta \tau}\right) \bigg| \mathcal{F}_\tau\right](\omega) \\
V^i_{1,2,3,4}(\hat{x}(\tau, \omega)) &= \lim_{\delta \tau \to 0+} \mathbb{E}\left[\left(\frac{\hat{x}^i(\tau + \delta \tau, \bullet) - \hat{x}^i(\tau, \bullet)}{\delta \tau}\right) \bigg| \mathcal{P}_\tau\right](\omega)
\end{align*}
\]

(31)
**Definition 17 ((S1)-process).** If \( \hat{x}(\cdot, \bullet) \) is an (R1)-process and a following \( \mathcal{V}_-(\hat{x}(\cdot, \bullet)) \in \mathcal{L}^1_{\text{loc}}(\{ \mathcal{F}_\tau \}; \mathbb{V}_M^4) \) exists, \( \hat{x}(\cdot, \bullet) \) is named an (S1)-process.

\[
\mathcal{V}_-(\hat{x}(\tau, \omega)) := \lim_{\delta t \to 0^+} \mathbb{E} \left[ \frac{\hat{x}(\tau + \epsilon \times \delta \tau, \bullet) - \hat{x}(\tau, \bullet)}{\epsilon \times \delta \tau} \bigg| \mathcal{F}_\tau \right] (\omega) \quad (32)
\]

Then, an (S1)-process provides us the relation of the stochastic integral on \( \tau_a \leq \tau \leq \tau_b \):

\[
\hat{x}^2(\tau, \omega) = \hat{x}^2(\tau_a, \omega) + \int_{\tau_a}^\tau d\tau' \mathcal{V}_+^4(\hat{x}(\tau', \omega)) + \int_{\tau_a}^\tau dy^4_4(\tau', \omega)
\]

\[
= \hat{x}^2(\tau_b, \omega) - \int_{\tau}^{\tau_b} d\tau' \mathcal{V}_+^4(\hat{x}(\tau', \omega)) - \int_{\tau}^{\tau_b} dy^4_4(\tau', \omega)
\quad (34)
\]

Where, \( y_+(\cdot, \bullet) \) and \( y_-(\cdot, \bullet) \) of martingales are \{ \mathcal{P}_\tau \}-adapted and \{ \mathcal{F}_\tau \}-adapted, satisfy the following basic relations:

\[
\mathbb{E} \left[ y_+(\tau + \epsilon \times \delta \tau, \bullet) - y_+(\tau, \bullet) \bigg| \mathcal{P}_\tau \right] (\omega) = 0
\quad (35)
\]

\[
\mathbb{E} \left[ y_-(\tau + \epsilon \times \delta \tau, \bullet) - y_-(\tau, \bullet) \bigg| \mathcal{F}_\tau \right] (\omega) = 0
\quad (36)
\]

**Definition 18 ((R2)-process).** When \( \hat{x}(\cdot, \bullet) \) is an (R1)-process and let \( y_+(\cdot, \bullet) \) be \( y_+(\tau + \epsilon \times \delta \tau, \bullet) - y_+(\tau, \bullet) \in \mathcal{L}^2_{\text{loc}}(\{ \mathcal{P}_\tau \}; \mathbb{V}_M^4) \), \( \hat{x}(\cdot, \bullet) \) is named an (R2)-process if

\[
\mathbb{E} \left[ y_+(\tau + \epsilon \times \delta \tau, \bullet) - y_+(\tau, \bullet) \bigg| \mathcal{P}_\tau \right] (\omega) = 0
\quad (37)
\]

and a following \( \sigma^2_+(\tau, \bullet) \in \mathcal{L}^1_{\text{loc}}(\{ \mathcal{P}_\tau \}; \mathbb{V}_M^4 \otimes \mathbb{V}_M^4) \) exists, such that \( \tau \mapsto \sigma^2_+(\tau, \omega) \) is continuous:

\[
\sigma^2_+(\tau, \omega) := \lim_{\delta t \to 0^+} \mathbb{E} \left[ \frac{[y_+(\tau + \epsilon \times \delta \tau, \bullet) - y_+(\tau, \bullet)] \otimes [y_+(\tau + \epsilon \times \delta \tau, \bullet) - y_+(\tau, \bullet)]}{\delta \tau} \bigg| \mathcal{P}_\tau \right] (\omega)
\quad (38)
\]

**Definition 19 ((S2)-process).** When \( \hat{x}(\cdot, \bullet) \) is an (R2) and (S1)-process and let \( y_-(\cdot, \bullet) \) be \( y_-(\tau + \epsilon \times \delta \tau, \bullet) - y_-(\tau, \bullet) \in \mathcal{L}^2_{\text{loc}}(\{ \mathcal{F}_\tau \}; \mathbb{V}_M^4) \), \( \hat{x}(\cdot, \bullet) \) is called an (S2)-process if

\[
\mathbb{E} \left[ y_-(\tau + \epsilon \times \delta \tau, \bullet) - y_-(\tau, \bullet) \bigg| \mathcal{F}_\tau \right] (\omega) = 0
\quad (39)
\]

and a following \( \sigma^2_-(\tau, \bullet) \in \mathcal{L}^1_{\text{loc}}(\{ \mathcal{F}_\tau \}; \mathbb{V}_M^4 \otimes \mathbb{V}_M^4) \) exists, such that \( \tau \mapsto \sigma^2_-(\tau, \omega) \) is continuous:

\[
\sigma^2_-(\tau, \omega) := \lim_{\delta t \to 0^+} \mathbb{E} \left[ \frac{[y_-(\tau + \epsilon \times \delta \tau, \bullet) - y_-(\tau, \bullet)] \otimes [y_-(\tau + \epsilon \times \delta \tau, \bullet) - y_-(\tau, \bullet)]}{\delta \tau} \bigg| \mathcal{F}_\tau \right] (\omega)
\quad (40)
\]

**Definition 20 ((R3)-process).** If \( \hat{x}(\cdot, \bullet) \) is an (R2)-process and \( \det \sigma^2_+(\tau, \omega) > 0 \) is almost surely satisfied for each \( \tau \in \mathbb{R} \), then, \( \hat{x}(\cdot, \bullet) \) is named an (R3)-process.
Definition 21 ((S3)-process). If \( \hat{x}(\omega, \cdot) \) is an (R3) and (S2)-process, \( \sigma^2(\tau, \omega) > 0 \) is almost surely satisfied for each \( t \in \mathbb{R} \), then, \( \hat{x}(\omega, \cdot) \) is called an (S3)-process.

The discussion up to here is the simple extension from the original idea by Nelson to a 4-dimensional composited process of \( \{ \mathcal{P}_\tau \} \) and \( \{ \mathcal{F}_\tau \} \). Where, \( y_\pm(\tau, \omega) := \lambda \times W_\pm(\tau, \omega) \) for \( \lambda > 0 \) satisfies the above (S3) processes \cite{2} \cite{3}. Let \( \{ \mathcal{P}_\tau \} \)-progressive be a \( \mathcal{B}((\tau, \infty)) \times \mathcal{P}_\tau / \mathcal{B}(X) \)-measurable process and \( \{ \mathcal{F}_\tau \} \)-progressive be a \( \mathcal{B}([\tau, \infty)) \times \mathcal{F}_\tau / \mathcal{B}(X) \)-measurable process for all \( \tau \in \mathbb{R} \).

Definition 22 (D-progresssive \( \hat{x}(\omega, \cdot) \)). A 4-dimensional (S3)-process \( \hat{x}(\omega, \cdot) \) on \( (\Omega, \mathcal{F}, \mathcal{P}) \) is named "the dual-progressively measurable process", or by shortening "D-progresssive" and also "the D-process" when \( y_\pm(\omega, \cdot) := \lambda \times W_\pm(\omega, \cdot) \) with respect to \( \lambda > 0 \). For \( \tau_0 \leq \tau \leq \tau_n \), \( \hat{x}(\omega, \cdot) \) is expressed by the following \( \{ \mathcal{P}_\tau \} \)-progressive and \( \{ \mathcal{F}_\tau \} \)-progressive process:

\[
\hat{x}^\mu(\tau, \omega) = \hat{x}^\mu(\tau_0, \omega) + \int_{\tau_0}^{\tau} d\tau' \nu_\mu^\mu(\hat{x}(\tau', \omega)) + \lambda \times \int_{\tau_0}^{\tau} dW_\pm^\mu(\tau', \omega) (41)
\]

\[
= \hat{x}^\mu(\tau_n, \omega) - \int_{\tau}^{\tau_n} d\tau' \nu_\mu^-\mu(\hat{x}(\tau', \omega)) - \lambda \times \int_{\tau}^{\tau_n} dW_\pm^\mu(\tau', \omega) (42)
\]

These (41)(42) is regarded as the solution of the following stochastic differential equation:

\[
d\hat{x}^\mu(\tau, \omega) = \nu_\mu^\mu(\hat{x}(\tau, \omega)) d\tau + \lambda \times dW_\pm^\mu(\tau, \omega) (43)
\]

For \( \delta \tau > 0 \), let \( d_\pm \hat{x}^\mu(\tau, \omega) \) be defined by \( \int_{\tau}^{\tau + \delta\tau} d\tau' d_\pm \hat{x}^\mu(\tau', \omega) := \hat{x}^\mu(\tau + \epsilon \delta \tau, \omega) - \hat{x}^\mu(\tau, \omega) \) with the signature \( \epsilon = \pm \). Since D-progressive \( \hat{x}(\tau, \cdot) \) is \( \mathcal{P}_\tau / \mathcal{B}(\mathbb{H}^4(\mathbb{V}_M^4, g)) \) and \( \mathcal{F}_\tau / \mathcal{B}(\mathbb{H}^4(\mathbb{V}_M^4, g)) \)-measurable for all \( \tau \), the following theorem is imposed:

Theorem 23. A D-progressive \( \hat{x}(\omega, \cdot) \) is \( \{ \mathcal{P}_\tau \cap \mathcal{F}_\tau \} \)-adapted.

By using this Definition 22 let us propose the following conjecture.

Conjecture 24. A D-progressive \( \hat{x}(\omega, \cdot) \) is a trajectory of a scalar electron satisfying the Klein-Gordon equation when \( \lambda = \sqrt{\hbar / m_0} \).

The demonstration of Conjecture 24 is the center of our main issue in this paper and its feasibility is shown in Section 3.

Let \( \hat{\xi}_\pm(\tau, \omega) \) be the white noise as the time derivatives of \( W_\pm(\tau, \omega) \) in means of the generalized-function satisfying \( \int_\mathbb{R} d\tau d\Phi / d\tau(\tau, \omega) \cdot W_\pm^\mu(\tau, \omega) = - \int_\mathbb{R} d\tau \Phi(\tau, \omega) \cdot \hat{\xi}_\pm^\mu(\tau, \omega) \) with respect to a test function \( \Phi \) for all \( \omega \). By introducing \( d_\pm \hat{\xi}(\tau, \omega) \) as the RHS in (43), (43) is recognized as the summation of the drift velocity \( \nu_\pm^\mu(\hat{x}(\tau, \omega)) \) and the randomness \( \lambda \times \hat{\xi}_\pm^\mu(\tau, \omega) = \lambda \times dW_\pm^\mu / d\tau(\tau, \omega) \),

\[
\frac{d_\pm \hat{x}^\mu(\tau, \omega)}{d\tau} = \nu_\pm^\mu(\hat{x}(\tau, \omega)) + \lambda \times \hat{\xi}_\pm^\mu(\tau, \omega) (44)
\]

Since \( \mathbb{E}[\hat{\xi}_+^\mu(\tau, \cdot) | \mathcal{P}_\tau] = 0 \) and \( \mathbb{E}[\hat{\xi}_-^\mu(\tau, \cdot) | \mathcal{F}_\tau] = 0 \), the conditional expectation of (44) (the mean-derivative)
imposes $\mathcal{V}_\pm$ its drift velocity:

$$
\mathbb{E}\left[ \left. \frac{d_+\hat{x}_\mu}{d\tau}(\tau, \bullet) \right| \mathcal{F}_\tau \right] (\omega) = \mathcal{V}_+^\mu(\hat{x}(\tau, \omega)), \quad \mathbb{E}\left[ \left. \frac{d_-\hat{x}_\mu}{d\tau}(\tau, \bullet) \right| \mathcal{F}_\tau \right] (\omega) = \mathcal{V}_-^\mu(\hat{x}(\tau, \omega))
$$

(45)

In general, a D-progressive $\hat{x}(\cdot, \cdot)$ imposes the following Itô formula [22, 23].

**Lemma 25** (Itô formula). Consider a $C^2$-function $f : \mathcal{A}^4(\mathcal{V}_M^4, g) \to \mathbb{C}$, the following Itô formula with respect to a D-progressive $\hat{x}(\cdot, \cdot)$ is found:

$$
d_\pm f(\hat{x}(\tau, \omega)) = \partial_\mu f(\hat{x}(\tau, \omega)) d_\pm \hat{x}_\mu(\tau, \omega) \mp \frac{\lambda^2}{2} \partial_\mu \partial_\eta f(\hat{x}(\tau, \omega)) d\tau \text{ a.s.}
$$

(46)

This is given by the following stochastic integral, too:

$$
f(\hat{x}(\tau, \omega)) - f(\hat{x}(\tau_\alpha, \omega)) = \int_{\tau_\alpha}^{\tau_\beta} d_\pm f(\hat{x}(\tau, \omega))
$$

(47)

$$
= \int_{\tau_\alpha}^{\tau_\beta} d_\pm \hat{x}_\mu(\tau, \omega) \partial_\mu f(\hat{x}(\tau, \omega)) \mp \frac{\lambda^2}{2} \int_{\tau_\alpha}^{\tau_\beta} d\tau \partial_\mu \partial_\eta f(\hat{x}(\tau, \omega)) \text{ a.s.}
$$

(48)

### 2.3 Complex velocity

In order to **Theorem 23** a D-progressively measurable process is $\{\mathcal{P}_\tau \cap \mathcal{F}_\tau\}$-adapted. Therefore, the superposition of $d_+$ and $d_-$ is introduced. L. Nottale introduces the following complex differential $d$ and the complex velocity $\mathcal{V}(\hat{x}(\cdot, \cdot))$ as the essential manners of quantum dynamics [20].

**Definition 26** (Complex differential and velocity). Consider a $C^2$-function $f : \mathcal{A}^4(\mathcal{V}_M^4, g) \to \mathbb{C}$ and its Itô formula $d_\pm f$ characterized by **Lemma 25**. Let $\hat{d}$ be the complex differential on a given D-progressive $\hat{x}(\cdot, \cdot)$:

$$
\hat{d} := \frac{1-i}{2} d_+ + \frac{1+i}{2} d_-
$$

(49)

$$
\hat{d} f(\hat{x}(\tau, \omega)) = \partial_\mu f(\hat{x}(\tau, \omega)) \hat{d} \hat{x}_\mu(\tau, \omega) + \frac{i\lambda^2}{2} \partial^\eta \partial_\eta f(\hat{x}(\tau, \omega)) d\tau \text{ a.s.}
$$

(50)

Then, consider a conditional expectation of the derivative given $\gamma_\tau := \mathcal{P}_\tau \cap \mathcal{F}_\tau \subset \mathcal{F}$ is denoted by

$$
\mathbb{E}\left[ \left. \frac{\hat{d} \hat{x}_\mu}{d\tau}(\tau, \bullet) \right| \gamma_\tau \right] (\omega) = \mathcal{V}_+^\mu(\hat{x}(\tau, \omega)) \partial_\eta f(\hat{x}(\tau, \omega)) + \frac{i\lambda^2}{2} \partial^\eta \partial_\eta f(\hat{x}(\tau, \omega)),
$$

(51)

especially when $f(\hat{x}(\tau, \omega)) = \hat{x}(\tau, \omega)$, it derives the complex velocity $\mathcal{V} \in \mathcal{V}_M^4 \oplus i\mathcal{V}_M^4$,

$$
\mathcal{V}_+^\mu(\hat{x}(\tau, \omega)) := \mathbb{E}\left[ \left. \frac{\hat{d} \hat{x}_\mu}{d\tau}(\tau, \bullet) \right| \gamma_\tau \right] (\omega) = \frac{1-i}{2} \mathcal{V}_+^\mu(\hat{x}(\tau, \omega)) + \frac{1+i}{2} \mathcal{V}_-^\mu(\hat{x}(\tau, \omega)).
$$

(52)

By choosing a $C^2$-function $\phi : \mathcal{A}^4(\mathcal{V}_M^4, g) \to \mathbb{C}$ like Ref. [20], the following is introduced:

$$
\mathcal{V}_\alpha(x) := i\lambda^2 \times \partial^\alpha \ln \phi(x) + \frac{e}{m_0} A^\alpha(\tau), \quad x \in \mathcal{V}(\tau, \Omega) \text{ for each } \tau
$$

(53)
The the gauge invariance of scalar QED is found easily.

**Theorem 27** (Gauge invariance of $\mathcal{V}$). For a given $C^1$-function $A : \Lambda^4(\mathbb{V}_M^4, g) \to \mathbb{R}$, the complex velocity $\mathcal{V}\alpha(x)$ satisfies the local $U(1)$-gauge symmetry in the transformation $(\phi, A) \to (\phi', A')$:

$$
\phi'(x) = e^{-i\varepsilon A(x)/\hbar} \times \phi(x), \quad A\alpha'(x) = A\alpha(x) - \partial^\alpha A(x)
$$

### 2.4 Fokker-Planck equations

Let us consider a $C^2$-function $f : \Lambda^4(\mathbb{V}_M^4, g) \to \mathbb{C}$ on $(\Lambda^4(\mathbb{V}_M^4, g), \mathcal{B}(\Lambda^4(\mathbb{V}_M^4, g)), \mu)$ and its expectation $\mathbb{E}[f(\hat{x}(\tau, \bullet))]$ at $\tau$. Where, $\mu : \Lambda^4(\mathbb{V}_M^4, g) \to [0, \infty)$. The probability density $p : \Lambda^4(\mathbb{V}_M^4, g) \times \mathbb{R} \to [0, \infty)$ with respect to $\hat{x}(\tau, \bullet)$ is introduced by the following relation at $\tau \in \mathbb{R}$:

$$
\mathcal{P}(\Omega) := \int_{\hat{x}(\tau, \Omega) \in \Lambda^4(\mathbb{V}_M^4, g)} d\mu(x) p(x, \tau) = 1
$$

Where, $\hat{x}(\tau, \Omega) := \text{supp}(p(\tau, \tau))$ and it leads $p(\Lambda^4(\mathbb{V}_M^4, g) \backslash \hat{x}(\tau, \Omega), \tau) = 0$. Since

$$
\mathbb{E}[f(\hat{x}(\tau, \bullet))] := \int_{\Omega} d\mathcal{P}(\omega) f(\hat{x}(\tau, \omega)) = \int_{\Lambda^4(\mathbb{V}_M^4, g)} d\mu(x) f(x)p(x, \tau),
$$

the probability density

$$
p(x, \tau) = \mathbb{E} [\delta^4(x - \hat{x}(\tau, \bullet))]
$$

is regarded as the kernel of a linear functional $\{\mathbb{E}[f(\hat{x}(\tau, \bullet))]\}_{\mathcal{F}, f}$. Consider the derivative of it with respect to $\tau$,

$$
\frac{d}{d\tau} \mathbb{E}[f(\hat{x}(\tau, \bullet))] = \int_{\Lambda^4(\mathbb{V}_M^4, g)} d\mu(x) f(x)\partial_\tau p(x, \tau).
$$

The LHS of this equation (58) along the evolution $d_\pm \hat{x}(\tau, \omega)$ is considered as follows;

$$
\frac{d}{d\tau} \mathbb{E}[f(\hat{x}(\tau, \bullet))] = \mathbb{E} \left[ \mathcal{V}_\mu^\pm(\hat{x}(\tau, \bullet))\partial_\mu f(\hat{x}(\tau, \bullet)) \pm \frac{\lambda^2}{2} \partial^\mu \partial_\mu f(\hat{x}(\tau, \bullet)) \right]
$$

$$
= \int_{\Lambda^4(\mathbb{V}_M^4, g)} d\mu(x) f(x) \left\{ -\partial_\mu [\mathcal{V}_\mu^\pm(x)p(x, \tau)] \pm \frac{\lambda^2}{2} \partial^\mu \partial_\mu p(x, \tau) \right\}.
$$

For an arbitrary $C^2$-function $f$, the following Fokker-Planck equations of a D-progressive $\hat{x}(\tau, \bullet)$ are derived.

**Theorem 28** (Fokker-Planck equations). Consider a D-progressive $\hat{x}(\tau, \bullet)$ on $(\Omega, \mathcal{F}, \mathcal{P})$, there is the $C^{2,1}$-probability density of $\hat{x}(\tau, \bullet)$ such that $p : \Lambda^4(\mathbb{V}_M^4, g) \times \mathbb{R} \to [0, \infty)$ satisfying the following Fokker-Planck equation:

$$
\partial_\tau p(x, \tau) + \partial_\mu [\mathcal{V}_\mu^\pm(x)p(x, \tau)] \pm \frac{\lambda^2}{2} \partial^\mu \partial_\mu p(x, \tau) = 0
$$

By using the definition of $\mathcal{V} \in \mathbb{V}_M^4 \oplus i\mathbb{V}_M^4$ (see (52)), the superpositions of the “$\pm$”-Fokker-Planck equations are found:

$$
\partial_\tau p(x, \tau) + \partial_\mu \text{Re}[\mathcal{V}_\mu(x)p(x, \tau)] = 0
$$
**2.5 Proper time**

One of the delicate problems in this paper is the definition of the proper time of a stochastic quanta on $(A^4(V_M^4, g), \mathcal{B}(A^4(V_M^4, g)), \mu)$. Since we want to consider the correspondence between a D-process and a classical kinematics, the limit $h \to 0$ of the proper time in the present model has to imply one in classical dynamics. The well-known proper time in classical dynamics is

$$d\tau_{\text{classical}} = \frac{1}{c} \times \sqrt{dx_\mu(\tau)dx^\mu(\tau)},$$

where the metric $g = \text{diag}(+1, -1, -1, -1)$ is selected. Let us recall the following relation in advance;

$$[\hat{d} \hat{x}_\mu(\tau, \omega) - \lambda \times \hat{d} W_\mu(\tau, \omega)] \cdot [\hat{d} \hat{x}^\mu(\tau, \omega) - \lambda \times \hat{d} W^\mu(\tau, \omega)] = \nu^\mu_\nu(\hat{x}(\tau, \omega)) \nu^\nu_\mu(\hat{x}(\tau, \omega)) d\tau^2.$$  

(64)

Where, $\hat{d} W(\tau, \omega) := (1-i)/2 \times dW_+(\tau, \omega) + (1+i)/2 \times dW_-(\tau, \omega)$, $\hat{d} W(\tau, \omega)$ stands for the complex conjugate of $A$. Again, remind the definition of the complex velocity

$$\nu^\mu_\mu(x) = \frac{1}{m_0} \times \frac{i\hbar\partial^\mu \phi(x) + eA^\mu(x)\phi(x)}{\phi(x)} = \frac{1}{m_0} \times \frac{i\hbar \mathbf{D}^\mu \phi(x)}{\phi(x)}$$

(65)

with respect to $x \in \hat{x}(\tau, \Omega)$, $\nu^\mu_\nu(x)\nu^\nu_\mu(x)$ becomes

$$\nu^\mu_\nu(x)\nu^\nu_\mu(x) = \frac{1}{2m_0^2} \times \frac{\phi(x)\cdot(-i\hbar \mathbf{D}^\mu)^* \cdot (-i\hbar \mathbf{D}^\nu)^* \phi^* (x) + \phi^*(x)(i\hbar \mathbf{D}_\mu) \cdot (i\hbar \mathbf{D}_\nu) \phi(x)}{\phi^*(x)\phi(x)}$$

$$+ \frac{\hbar^2}{2m_0^2} \times \frac{\partial_\mu \partial_\nu [\phi(x) \cdot \phi^*(x)]}{\phi^*(x)\phi(x)}.$$  

(66)

Let the $C^2$-function $\phi(x) : A^4(V_M^4, g) \to \mathbb{C}$ be the wave function of the complex Klein-Gordon equation, $(i\hbar \mathbf{D}_\mu)^* (i\hbar \mathbf{D}^\mu) \phi(x) - m_0^2 c^2 \phi(x) = 0$. Due to this assumption, the first term in the RHS of (66) is a constant of $c^2$. Then, the issue is the behavior of $\hbar^2/m_0^2 \times \partial_\mu \partial_\nu [\phi^*(x)\phi(x)]/\phi^*(x)\phi(x)$. Where, we follow the proposal by T. Zastawniak in Ref. [23]. By defining the function $\phi(x) := \exp[R(x)/\hbar + iS(x)/\hbar]$ with respect to real valued functions of $R$ and $S$, $\phi^*(x)\phi(x) = \exp[2R(x)/\hbar]$ is satisfied. Due to the definition of (65), $\partial^\mu R(x) = \text{Im}\{m_0 \nu^\mu_\nu(x)\} = \hbar/2 \times \partial^\mu \ln p(x, \tau)$ on $x \in \hat{x}(\tau, \Omega)$ (see (62));

$$\frac{\hbar^2}{2m_0^2} \times \frac{\partial_\mu \partial_\nu [\phi(x) \cdot \phi^*(x)]}{\phi^*(x)\phi(x)} = \frac{\hbar^2}{2m_0^2} \times \frac{\partial_\mu \partial_\nu [p(x, \tau)]}{p(x, \tau)}.$$  

(67)
Hence, the second term of the RHS in (66) is non-zero. However, let us introduce the expectation of (67) after the substitution of \( x = \hat{x}(\tau, \omega) \),

\[
\mathbb{E} \left[ \frac{\hbar^2}{2m_0^2} \times \frac{\partial \mu \partial^\mu p(\hat{x}(\tau, \bullet), \tau)}{p(\hat{x}(\tau, \bullet), \tau)} \right] = \frac{\hbar^2}{2m_0^2} \times \int_{\hat{x}(\tau, \Omega)} \mu(x) \left[ \frac{\partial \mu \partial^\mu p(x, \tau)}{p(x, \tau)} \right] p(x, \tau) = \frac{\hbar^2}{2m_0^2} \times \int_{\hat{x}(\tau, \Omega)} \mu(x) \partial \mu \partial^\mu p(x, \tau) = 0, \tag{68}
\]

by employing \( \partial \mu p(x, \tau) \big|_{x \in \partial \hat{x}(\tau, \Omega)} = 0 \), where \( \partial \hat{x}(\tau, \Omega) \) denotes the boundary of \( \hat{x}(\tau, \Omega) \). Therefore the following relation is realized.

**Lemma 29** (Lorentz invariant). Consider a D-progressive \( \hat{x}(\sigma, \bullet) \). A stochastic kinematics of a scalar electron satisfies the following Lorentz invariant for all \( \tau \in \mathbb{R} \) \[28\].

\[
\mathbb{E} \left[ V^*_\mu(\hat{x}(\tau, \bullet)) V^\mu(\hat{x}(\tau, \bullet)) \right] = c^2 \tag{69}
\]

Due to this **Lemma 29**, the proper time is defined as the mimic of classical dynamics \[63\].

**Definition 30** (Proper time). For all \( \tau \in \mathbb{R} \), the proper time for a stochastic kinematics is defined by the following invariant parameter;

\[
d\tau := \frac{1}{c} \sqrt{\mathbb{E}\left[ \left[ d^\mu \hat{x}_\mu(\tau, \bullet) - \lambda \times dW^\mu(\tau, \bullet) \right] \cdot \left[ d^\mu \hat{x}_\mu(\tau, \bullet) - \lambda \times dW^\mu(\tau, \bullet) \right] \right]} . \tag{70}
\]

### 3 Dynamics of a scalar electron and fields

In order to the realization of the kinematics \[43\], i.e., \( d\hat{x}(\tau, \omega) = \mathcal{V}_\pm(\hat{x}(\tau, \omega))d\tau + \lambda \times dW_\pm(\tau, \omega) \), we need to investigate the behavior of the complex velocity \( \mathcal{V}^\mu(\hat{x}(\tau, \omega)) \in \mathcal{V}_M^4 \oplus i\mathcal{V}_M^4 \). For the derivation of \( \mathcal{V}(\hat{x}(\tau, \omega)) \), the action integral (the functional) along a stochastic trajectory is required. Before entering to the main body, we consider the variational calculus associated with a stochastic particle briefly. After this explanation, let us proceed the concrete definition of the action integral and fields corresponding to the styles in classical dynamics.

#### 3.1 Euler-Lagrange (Yasue) equation

In this small section, we focus the action integral on a stochastic process. Concerning the complex velocity \( \mathcal{V} \in \mathcal{V}_M^4 \oplus i\mathcal{V}_M^4 \), L. Nottale suggests the following Lagrangian due to its forward and backward evolution, i.e., \( d_\pm \hat{x}(\sigma, \bullet) \); \( L_0(\tau, \hat{x}, \mathcal{V}_+ , \mathcal{V}_-) = L(\tau, \hat{x}, \mathcal{V}) \) \[20\]. However, we propose its extension, namely, \( L_0(\tau, \hat{x}, \mathcal{V}_+ , \mathcal{V}_-) = L(\tau, \hat{x}, \mathcal{V}_+ , \mathcal{V}_-^* ) \). Here, \( \mathcal{V}_+^* \in \mathcal{V}_M^4 \oplus i\mathcal{V}_M^4 \) is the complex conjugate of \( \mathcal{V} \). Recalling the definition of \( \mathcal{V} \) in \[52\], it is found

\[
L_0(\tau, \hat{x}, \mathcal{V}_+ , \mathcal{V}_-) = \left( \tau, \hat{x}, \frac{1-i}{2} \mathcal{V}_+ + \frac{1+i}{2} \mathcal{V}_- , \frac{1+i}{2} \mathcal{V}_+ + \frac{1-i}{2} \mathcal{V}_- \right) \tag{71}
\]
Consider the following relation at first:

\[ \gamma \tau := (Nelson\'s\ partial\ integral) \]

\[ \hat{\tau} \]

By using these expressions, \( D^\pm \hat{x}_\mu (\tau, \omega) = V^L_M (\hat{x}(\tau, \omega)) \) is obviously satisfied (the mean derivatives). The variation of the functional \( \int_{\tau_1}^{\tau_2} d\tau \ E [L_0(\tau, \hat{x}, V_+, V_-)] \) with respect to \( \hat{x} \) is

\[
\delta \int_{\tau_1}^{\tau_2} d\tau \ E [L_0(\tau, \hat{x}, V_+, V_-)] = \int_{\tau_1}^{\tau_2} d\tau \ E \left[ \frac{\partial L_0}{\partial \hat{x}_\mu} \delta \hat{x}_\mu + \frac{\partial L_0}{\partial V^\mu_+} \delta V^\mu_+ + \frac{\partial L_0}{\partial V^{*\mu}_-} \delta V^{*\mu}_- \right] = \int_{\tau_1}^{\tau_2} d\tau \ E \left[ \frac{\partial L_0}{\partial \hat{x}_\mu} \delta \hat{x}_\mu + \frac{\partial L_0}{\partial V^\mu_+} \delta \hat{x}_\mu + \frac{\partial L_0}{\partial V^{*\mu}_-} \delta \hat{x}_\mu \right],
\]

where, the following relations are introduced.

\[
\frac{\partial L_0}{\partial \hat{x}_\mu} = \frac{\partial L}{\partial \hat{x}_\mu}
\]

\[
\frac{\partial L_0}{\partial V^\mu_+} = \frac{1 + i}{2} \frac{\partial L}{\partial V^\mu} + \frac{1 - i}{2} \frac{\partial L}{\partial V^{*\mu}_-}
\]

\[
\frac{\partial L_0}{\partial V^{*\mu}_-} = \frac{1 - i}{2} \frac{\partial L}{\partial V^\mu} + \frac{1 + i}{2} \frac{\partial L}{\partial V^{*\mu}}
\]

Then, we need to recall the following Nelson’s partial integral [1225].

**Lemma 31** (Nelson’s partial integral). Let \( \alpha, \beta : \mathcal{A}^2(V^\mu_M, g) \to V^\mu_M \oplus iV^{\mu}_M \) be a \( C^2 \)-functions on a D-progressive \( \hat{x}(\cdot, \cdot) \), the following partial integral formula is fulfilled;

\[
\int_{\tau_1}^{\tau_2} d\tau \ E \left[ D^\pm_{\tau_1} \alpha_\mu(\hat{x}(\tau, \cdot)) \cdot \beta^\mu(\hat{x}(\tau, \cdot)) + \alpha_\mu(\hat{x}(\tau, \cdot)) \cdot D^\pm_{\tau_1} \beta^\mu(\hat{x}(\tau, \cdot)) \right] = E [\alpha_\mu(\hat{x}(\tau_2, \cdot)) \beta^\mu(\hat{x}(\tau_2, \cdot)) - \alpha_\mu(\hat{x}(\tau_1, \cdot)) \beta^\mu(\hat{x}(\tau_1, \cdot))] ,
\]

or its differential form,

\[
\frac{d}{d\tau} E [\alpha_\mu(\hat{x}(\tau, \cdot)) \beta^\mu(\hat{x}(\tau, \cdot))] = E \left[ D^\pm_{\tau_1} \alpha_\mu(\hat{x}(\tau, \cdot)) \cdot \beta^\mu(\hat{x}(\tau, \cdot)) + \alpha_\mu(\hat{x}(\tau, \cdot)) \cdot D^\pm_{\tau_1} \beta^\mu(\hat{x}(\tau, \cdot)) \right].
\]

By using the superposition of the above “±” formulas, it can be switched to the formula by the complex derivatives.

\[
\frac{d}{d\tau} E [\alpha_\mu(\hat{x}(\tau, \cdot)) \beta^\mu(\hat{x}(\tau, \cdot))] = E \left[ D^\pm_{\tau_1} \alpha_\mu(\hat{x}(\tau, \cdot)) \cdot \beta^\mu(\hat{x}(\tau, \cdot)) + \alpha_\mu(\hat{x}(\tau, \cdot)) \cdot D^\pm_{\tau_1} \beta^\mu(\hat{x}(\tau, \cdot)) \right]
\]

or

\[
\frac{d}{d\tau} E [\alpha_\mu(\hat{x}(\tau, \cdot)) \beta^\mu(\hat{x}(\tau, \cdot))] = E \left[ D^\pm_{\tau_1} \alpha_\mu(\hat{x}(\tau, \cdot)) \cdot \beta^\mu(\hat{x}(\tau, \cdot)) + \alpha_\mu(\hat{x}(\tau, \cdot)) \cdot D^\pm_{\tau_1} \beta^\mu(\hat{x}(\tau, \cdot)) \right].
\]

**Proof.** Consider the following relation at first:
Let us consider the action integral of “classical” dynamics on $3.2$ Action integral

It is derived by using the Itô formula of (46) and (62), equation (79) is demonstrated. (80) is also imposed by the superposition of (79) for “$\pm$.”

By considering (80), (74) becomes

\[
E \left[ D_+^\mu \alpha_\mu (\dot{x}(\tau, \bullet)) \right. \beta^\mu (\dot{x}(\tau, \bullet)) + \alpha_\mu (\dot{x}(\tau, \bullet)) \cdot D_-^\mu \beta^\mu (\dot{x}(\tau, \bullet)) \right] \\
= E \left[ D_-^\mu \alpha_\mu (\dot{x}(\tau, \bullet)) \right. \beta^\mu (\dot{x}(\tau, \bullet)) + \alpha_\mu (\dot{x}(\tau, \bullet)) \cdot D_+^\mu \beta^\mu (\dot{x}(\tau, \bullet)) \right] \\
\] (81)

Then, by recalling the Fokker-Planck equation (60),

\[
E \left[ D_+^\mu \alpha_\mu (\dot{x}(\tau, \bullet)) \right. \beta^\mu (\dot{x}(\tau, \bullet)) + \alpha_\mu (\dot{x}(\tau, \bullet)) \cdot D_-^\mu \beta^\mu (\dot{x}(\tau, \bullet)) \right] \\
- E \left[ D_-^\mu \alpha_\mu (\dot{x}(\tau, \bullet)) \right. \beta^\mu (\dot{x}(\tau, \bullet)) + \alpha_\mu (\dot{x}(\tau, \bullet)) \cdot D_+^\mu \beta^\mu (\dot{x}(\tau, \bullet)) \right] \\
= -\lambda^2 \times \int_{\lambda^4(V_M, g)} d\mu(x) \partial^\nu \left\{ p(x, \tau) \left[ -\alpha_\mu (x) \cdot \partial_\nu \beta^\mu (x) \right] \right\} = 0. (82)
\]

By considering (80), (74) becomes

\[
\frac{d}{d\tau} E [\alpha_\mu (\dot{x}(\tau, \bullet)) \beta^\mu (\dot{x}(\tau, \bullet))] = \int_{\lambda^4(V_M, g)} d\mu(x) \alpha_\mu (x) \beta^\mu (x) \partial_x p(x, \tau) \\
= \frac{1}{2} \times E \left[ (D_+^\mu + D_-^\mu) \alpha_\mu (\dot{x}(\tau, \bullet)) \cdot \beta^\mu (\dot{x}(\tau, \bullet)) \right. + \alpha_\mu (\dot{x}(\tau, \bullet)) \cdot (D_-^\mu + D_+^\mu) \beta^\mu (\dot{x}(\tau, \bullet)) \right], (83)
\]

Theorem 32 is derived with the help of $\delta \dot{x}^\mu (\tau_i, \bullet) = 0 (i = 1, 2)$.

**Theorem 32** (Euler-Lagrange (Yasue) equation). Let the functional

\[
\mathcal{G}[\dot{x}, \dot{V}, V^*] = \int_{\tau_1}^{\tau_2} d\tau E \left[ L (\tau, \dot{x}(\tau, \bullet), \dot{V}(\dot{x}(\tau, \bullet)), V^*(\dot{x}(\tau, \bullet))) \right] \\
\] (85)

be the action integral on a D-progressive $\dot{x}(\alpha, \bullet)$. By its variation with respect to $\dot{x}(\alpha, \bullet)$, the following Euler-Lagrange (Yasue) equation is induced:

\[
\frac{\partial L}{\partial \dot{x}^\mu} - D_+^\mu \frac{\partial L}{\partial V^\mu} - D_-^\mu \frac{\partial L}{\partial V^*\mu} = 0 \\
\] (86)

3.2 Action integral

Let us consider the action integral of “classical” dynamics on $(\lambda^4(V_M, g), \mathcal{B}(\lambda^4(V_M, g)), \mu)$:

\[
S_{\text{classical}} = \int_{\mathbb{R}} d\tau \frac{m_0}{2} v_\alpha (\tau) v_\alpha (\tau) - \int_{\mathbb{R}} d\tau e A_\alpha (x(\tau)) v_\alpha (\tau) + \int_{\lambda^4(V_M, g)} d\mu(x) \frac{1}{4\mu_0 c} F_{\alpha\beta}(x) F^{\alpha\beta}(x). (87)
\]
Corresponding to (87), a new action integral of a stochastic particle and a field is proposed via the introduction of the mass measure and the charge measure:

**Definition 33** (Mass and charge measures). Let \( \mathcal{M} \) and \( \mathcal{E} \) be the mass measure and the charge measure of a stochastic scalar electron. For the positive constants \( m_0 \) and \( e \), \( \mathcal{M} \) and \( \mathcal{E} \) are characterized by

\[
\int_{\mathcal{A}(V^s_{\text{M}}, g)^4} d\mathcal{M}(x, \tau) := m_0 \times \int_{\mathcal{A}(V^s_{\text{M}}, g)^4} d\mu(x) E \left[ \delta^4(x - \hat{x}(\tau, \bullet)) \right], \quad (88)
\]

\[
\int_{\mathcal{A}(V^s_{\text{M}}, g)^4} d\mathcal{E}(x, \tau) := e \times \int_{\mathcal{A}(V^s_{\text{M}}, g)^4} d\mu(x) E \left[ \delta^4(x - \hat{x}(\tau, \bullet)) \right]. \quad (89)
\]

The key of this definition is the appearance of the smeared distribution \( \delta^4(x - \hat{x}(\tau, \bullet)) d\mu(x) \) in classical dynamics.

**Definition 34** (Action integral). The following functional \( \mathcal{G} \) is the action integral deriving the dynamics of a “stochastic” scalar electron and a field characterized by \( \mathcal{V} \in V^s_{\text{M}} \oplus iV^s_{\text{M}}, A \in V^s_{\text{M}} \) with the help by \( F \in V^s_{\text{M}} \otimes V^s_{\text{M}} \) and a given tensor \( \delta f \in V^s_{\text{M}} \otimes V^s_{\text{M}} \):

\[
\mathcal{G}[\hat{x}, \mathcal{V}, \mathcal{V}^*, A] = \int_{\mathcal{A}(V^s_{\text{M}}, g)^4} d\mathcal{M}(x, \tau) \frac{1}{2} \mathcal{V}^*_\alpha(x) \mathcal{V}^\alpha(x) \]

\[
- \int_{\mathcal{A}(V^s_{\text{M}}, g)^4} d\mathcal{E}(x, \tau) A_\alpha(x) \text{Re} \{ \mathcal{V}^\alpha(x) \}
\]

\[
+ \int_{\mathcal{A}(V^s_{\text{M}}, g)^4} d\mu(x) \frac{1}{4\mu_0c} [F_{\alpha\beta}(x) + \delta f_{\alpha\beta}(x)] \cdot [F^{\alpha\beta}(x) + \delta f^{\alpha\beta}(x)] \quad (90)
\]

Where, \( F^{\alpha\beta}(x) := \partial^\mu A^\nu - \partial^\nu A^\mu \). By writing the detail of the measures explicitly,

\[
\mathcal{G}[\hat{x}, \mathcal{V}, \mathcal{V}^*, A] = E \left[ \int_{\mathcal{A}(V^s_{\text{M}}, g)^4} d\mathcal{M}(x, \tau) \frac{1}{2} \mathcal{V}^*_\alpha(x) \mathcal{V}^\alpha(x) \hat{x}(\tau, \bullet) \right] \]

\[
+ E \left[ - \int_{\mathcal{A}(V^s_{\text{M}}, g)^4} d\tau e A_\alpha(\hat{x}(\tau, \bullet)) \text{Re} \{ \mathcal{V}^\alpha(x) \} \right] 
\]

\[
+ \frac{1}{4\mu_0c} \int_{\mathcal{A}(V^s_{\text{M}}, g)^4} d\mu(x) [F_{\alpha\beta}(x) + \delta f_{\alpha\beta}(x)] \cdot [F^{\alpha\beta}(x) + \delta f^{\alpha\beta}(x)] \quad (91)
\]

Hence, the Lagrangian density \( \mathcal{L} \) is also introduced;

\[
\mathcal{L}(x, \hat{x}, \mathcal{V}, \mathcal{V}^*, A) = \int_{\mathcal{A}(V^s_{\text{M}}, g)^4} d\mu(x) \left[ \frac{1}{2} \frac{d\mathcal{M}}{d\mu}(x, \tau) \mathcal{V}^*_\alpha(x) \mathcal{V}^\alpha(x) - \frac{d\mathcal{E}}{d\mu}(x, \tau) A_\alpha(x) \text{Re} \{ \mathcal{V}^\alpha(x) \} \right] 
\]

\[
+ \frac{1}{4\mu_0c} [F_{\alpha\beta}(x) + \delta f_{\alpha\beta}(x)] \cdot [F^{\alpha\beta}(x) + \delta f^{\alpha\beta}(x)] \quad (92)
\]

such that \( \mathcal{G}[\hat{x}, \mathcal{V}, \mathcal{V}^*, A] = \int_{\mathcal{A}(V^s_{\text{M}}, g)^4} d\mu(x) \mathcal{L}(x, \hat{x}, \mathcal{V}, \mathcal{V}^*, A) \)
3.3 Dynamics of a scalar electron

The Lagrangian of a stochastic scalar electron with its interaction is

\[
L_{\text{particle}}[\hat{x}, \mathcal{V}, \mathcal{V}^*] := \int_{\mathcal{A}^4(\mathcal{V}_M^4, g)} d\mathcal{M}(x, \tau) \left( \frac{1}{2} \mathcal{V}^\alpha(x) \mathcal{V}_\alpha(x) - \int_{\mathcal{A}^4(\mathcal{V}_M^4, g)} d\mathcal{E}(x, \tau) A_\alpha(x) \text{Re} \{ \mathcal{V}^\alpha(x) \} \right).
\]

(93)

Substituting this for (86),

\[
\text{Re} \left\{ m_0 \mathcal{D}_\tau \mathcal{V}^\mu(\hat{x}(\tau, \omega)) + e \hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega)) F^{\mu\nu}(\hat{x}(\tau, \omega)) \right\} = 0.
\]

(94)

Where, the following are introduced with the Lorenz gauge \( \partial_\mu A^\mu = 0 \) [20]:

\[
\hat{\mathcal{V}}^\mu(x) := \mathcal{V}^\mu(x) + \frac{i\lambda^2}{2} \partial^\mu f(\hat{x}(\tau, \omega), \tau)
\]

(95)

\[
\mathcal{D}_\tau = \hat{\mathcal{V}}^\mu(x) \partial_\mu
\]

(96)

\[
\mathcal{D}_\tau A_\mu(\hat{x}(\tau, \omega)) = \hat{\mathcal{V}}^\nu(\hat{x}(\tau, \omega)) \partial_\nu A_\mu(\hat{x}(\tau, \omega))
\]

(97)

\[
\mathcal{D}_\tau^* A_\mu(\hat{x}(\tau, \omega)) = \hat{\mathcal{V}}^{*\nu}(\hat{x}(\tau, \omega)) \partial_\nu A_\mu(\hat{x}(\tau, \omega))
\]

(98)

Theorem 35 (Equation of stochastic motion). The equation of “stochastic” motion of a scalar electron interacting with a field is

\[
\frac{d\mathcal{M}(x, \tau)}{d\mathcal{D}_\tau} \mathcal{V}^\mu(x) = -d\mathcal{E}(x, \tau) \hat{\mathcal{V}}_\nu(x) F^{\mu\nu}(x)
\]

(99)

or

\[
m_0 \mathcal{D}_\tau \mathcal{V}^\mu(\hat{x}(\tau, \omega)) = -e \hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega)) F^{\mu\nu}(\hat{x}(\tau, \omega))
\]

(100)

derived from the action integral [90][91]. This is equivalent to the Klein-Gordon equation.

Proof. Let an arbitrary smooth \( C^{1,0}\)-function \( f : \mathcal{A}^4(\mathcal{V}_M^4, g) \times \mathbb{R} \rightarrow \mathbb{R} \) be a degree of freedom of the imaginary part of (94), namely,

\[
m_0 \mathcal{D}_\tau \mathcal{V}^\mu(\hat{x}(\tau, \omega)) = -e \hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega)) F^{\mu\nu}(\hat{x}(\tau, \omega)) + \frac{i}{2m_0} \partial^\mu f(\hat{x}(\tau, \omega), \tau).
\]

(101)

Transforming \( \mathcal{D}_\tau \mathcal{V}^\mu + e/m_0 \hat{\mathcal{V}}_\nu F^{\mu\nu} \) by employing (96),

\[
\mathcal{D}_\tau \mathcal{V}^\nu + \frac{e}{m_0} \hat{\mathcal{V}}_\nu F^{\mu\nu} = \hat{\mathcal{V}}_\nu \left[ \partial^\nu \mathcal{V}^\mu + \frac{e}{m_0} F^{\mu\nu} \right] = \hat{\mathcal{V}}_\nu \partial^\mu \mathcal{V}^\nu,
\]

(102)

since the identity

\[
\partial^\alpha \mathcal{V}^\beta - \partial^\beta \mathcal{V}^\alpha = \frac{e}{m_0} F^{\alpha\beta}
\]

(103)
derived from (53) (also see Ref. [20]). By substituting (53) and the above for (101),
\[ \hat{V}_\nu \partial^\mu \nu - \frac{i}{2m_0^2} \partial^\mu f = \left[ i \lambda^2 \times \partial^\nu \ln \phi + \frac{e}{m_0} A^\nu + \frac{\lambda^2}{2} \partial_A \nu \right] \times \partial^\mu \left[ i \lambda^2 \times \partial^\nu \ln \phi + \frac{e}{m_0} A^\nu \right] - \frac{i}{2m_0^2} \partial^\mu f \]
\[ = \frac{1}{2} \partial^\mu \left[ \left( i h \partial^\nu + e A^\nu \right) \left( i h \partial^\nu + e A^\nu \right) \phi - i f \phi \right] = 0. \] (104)

Thus, the Klein-Gordon equation is found by putting an arbitrary constant $c^2$,
\[ (i h \partial^\nu + e A^\nu) \left( i h \partial^\nu + e A^\nu \right) \phi - (m_0^2 c^2 + i f) \phi = 0. \] (105)

Thus, the imaginary force of $i/2m_0 \times \partial^\mu f$ implies a non-electromagnetic interaction. This interaction should be removed or be rounded into the free-propagation term of a scalar electron as the mass, hence, $f \equiv 0$ is feasible in physics. Then, the normal Klein-Gordon equation is derived;
\[ (i h \partial^\nu + e A^\nu) \left( i h \partial^\nu + e A^\nu \right) \phi - m_0^2 c^2 \phi = 0. \] (106)

Therefore, the equation of motion (99 or 100) is equivalent to the Klein-Gordon equation.

Concerning Theorem 27, the following theorem is essential and obviously fulfilled since $V$, $\hat{V}$, $\mathcal{D}_\tau$ and $F$ are $U(1)$-gauge invariant which the Klein-Gordon equation satisfies, too.

**Theorem 36** (Gauge symmetry). The equation of motion (99 or 100) satisfies the $U(1)$-gauge symmetry.

Equations (99) or (100) are very similar style to classical dynamics, namely, $m_0 d\nu^\mu/d\tau = -e v_\nu F^\mu\nu$. Ehrenfest’s theorem [21] of it implies the average behavior of this stochastic scalar electron like this classical form. It is discussed at Section 4.

### 3.4 Dynamics of fields

Let us proceed the dynamics of the field radiated from a stochastic scalar electron. The Maxwell equation is derived by the variation of (91) with respect to $A \in \mathbb{V}_4^4 M$, namely, $\partial^\mu \left[ \partial \mathcal{L}_\text{field} / \partial (\partial^\mu A_\nu) \right] - \partial \mathcal{L}_\text{field} / \partial A_\nu = 0$. Where, the Lagrangian density for a field is,
\[ \mathcal{L}_\text{field}[\hat{x}, A] = -\int \limits_{R} d\tau \frac{d\xi}{d\mu}(x, \tau) A_\alpha(x) \text{Re} \{ \nu^\alpha(x) \} \]
\[ + \frac{1}{4\mu_0} [F_{\alpha\beta}(x) + \delta f_{\alpha\beta}(x)] \cdot [F_{\alpha\beta}(x) + \delta f_{\alpha\beta}(x)] . \] (107)

**Theorem 37** (Maxwell equation). Let a D-progressive $\hat{x}(\cdot, \cdot)$ be the trajectory of a stochastic scalar electron. The variation of (91) with respect to a field $A \in \mathbb{V}_M^4$ derives the following Maxwell equation:
\[ \partial^\mu [F^{\mu\nu}(x) + \delta f^{\mu\nu}(x)] = \mu_0 \times j^\mu_{\text{stochastic}}(x) \] (108)
Where, $\delta f \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4$ is a given field. The current of a stochastic scalar electron
\[ j^\mu_{\text{stochastic}}(x) := \mathbb{E} \left[ -e c \int \limits_{R} d\tau \text{Re} \{ \nu^\mu(x) \} \delta^4(x - \hat{x}(\tau, \cdot)) \right] \] (109)
is equivalent to the current of Klein-Gordon particle

\[ j^\mu_{\text{K-G}}(x) = -\frac{i\hbar c\lambda}{2} \times g^{\mu\nu} \left[ \phi^*(x) \partial_\nu \phi(x) - \phi(x) \partial_\nu \phi^*(x) \right]. \]  

(110)

Where, \( \partial^\mu := \partial^\mu - ie/\hbar \times A^\mu(x) \).

Remark 38. The tensor \( \delta f^{\mu\nu}(x) \) is introduced to remove the field singularity at the point of an electron. For the discussion of radiation reaction in Volume II [14], the generated field has to be separated into the homogeneous solution \( F \in V^4_M \otimes V^4_M \) such that \( \partial_\mu F^{\mu\nu} = 0 \), and the singularity as a Coulomb field by \( \partial_\mu \delta f^{\mu\nu} = \mu_0 \times j^\mu_{\text{stochastic}} \). We regards (108) as the superposition of these two equations. The detail of this discussion is in Volume II [14].

Proof. The derivation of the Maxwell equation (108) is obvious. The current \( j^\mu_{\text{stochastic}}(x) \) is calculated by using (53):

\[ j^\mu_{\text{stochastic}}(x) = -e \int d\tau \Re \{ \mathcal{V}^\alpha(x) \} p(x, \tau) \]

\[ = \int d\tau \frac{p(x, \tau)}{\phi^*(x)\phi(x)} \times j^\mu_{\text{K-G}}(x), \quad x \in \bigcup_{\tau \in \mathbb{R}} \hat{x}(\tau, \Omega) \]  

(111)

Hereby, \( j_{\text{stochastic}}(x) \in V^4_M \) satisfies \( \partial_\mu j^\mu_{\text{stochastic}}(x) = -e\partial_\mu [\Re \{ \mathcal{V}^\mu(x) \} \int d\tau p(x, \tau)] = 0 \) due to (61) with its boundary condition \( p(x, \tau = \partial \mathbb{R}) = 0 \). Of cause, \( \partial_\mu j^\mu_{\text{K-G}}(x) = 0 \) is held, too. Thus,

\[ \int d\tau \frac{p(x, \tau)}{\phi^*(x)\phi(x)} = \text{Constant} \]  

(112)

has to be imposed, \( \partial_\mu [F^{\mu\nu}(x) + \delta f^{\mu\nu}(x)] = \mu_0 j^\mu_{\text{K-G}} \) is realized by \( \int d\tau p(x, \tau)/\phi^*(x)\phi(x) = 1 \).

\[ \square \]

Proposition 39. For the realization of the Klein-Gordon equation and the Maxwell equation from the action integral (90) (91), the following is required with respect to \( x \in \bigcup_{\tau \in \mathbb{R}} \hat{x}(\tau, \Omega) \):

\[ \phi^*(x)\phi(x) := \int d\tau \mathbb{E} [\delta^4(x - \hat{x}(\tau, \bullet))] \]

\[ = \int d\tau p(x, \tau) \]  

(113)

The plot of \( \int d\tau \mathbb{E} [\delta^4(x - \hat{x}(\tau, \bullet))] \) denotes the distribution of a scalar electron in \( \hat{x}(\tau, \Omega) \subset \mathbb{R}^4(V^4_M, g) \).

4 Conclusion and discussion

In this Volume I, we discussed the relativistic and stochastic kinematics of a scalar electron with its dynamics and a field. For the kinematics of a particle, the D-progressive \( \hat{x}(\bullet, \bullet) \) on the Minkowski spacetime \( (\mathbb{R}^4(V^4_M, g), \mathcal{B}(\mathbb{R}^4(V^4_M, g)), \mu) \) was defined as the extension from Nelson’s (S3)-process [2] at Definition 22 in Section 2. It imposed the two types of the velocities \( V^4_M(\hat{x}(\bullet, \bullet)) \). We needed to consider the probability density \( p : \mathbb{R}^4(V^4_M, g) \times \mathbb{R} \rightarrow [0, \infty) \) characterized by Theorem 28 and Proposition 39. The definition of
the proper time $d\tau$ (70) corresponding to one in classical dynamics is discussed, too. The complex differential $\tilde{\dot{x}}$ (49) and the complex velocity $\tilde{\mathbf{V}}$ (53) [20] which are the main casts of the present model were also introduced. In Section 3 the dynamics of a stochastic particle was proposed. We introduced the new action integral (90-91) corresponding to the form in classical dynamics. Hence, we could obtain the dynamics of a stochastic particle and fields by the variational calculus of this action integral. The restriction of an external field is only Lorenz’s gauge of $\partial_\mu A^\mu = 0$.

**Conclusion 40** (System of a scalar electron and a field). Consider $(\mathcal{A}^4(\mathbb{V}_M^4, g), \mathcal{B}(\mathcal{A}^4(\mathbb{V}_M^4, g)), \mu)$ and $(\Omega, \mathcal{F}, \mathcal{B})$. A D-progressive $\tilde{x}(\bullet, \tau) := \{\tilde{x}(\tau, \omega) \in \mathcal{A}^4(\mathbb{V}_M^4, g)|\tau \in \mathbb{R}, \omega \in \Omega\}$ characterized by

$$d\tilde{x}^\mu(\tau, \omega) = \mathbb{V}^\mu_+(\tilde{x}(\tau, \omega))d\tau + \lambda \times dW^\mu_+(\tau, \omega) \quad (114)$$

is defined as the kinematics of a stochastic scalar electron [Definition 34]. The following action integral [Definition 34]

$$\mathcal{S}[\tilde{x}, \mathcal{V}^\tau, \mathcal{A}] = \int_\mathbb{R} d\tau \int_{\mathcal{A}^4(\mathbb{V}_M^4, g)} d\mathcal{R}(x, \tau) \frac{1}{2} \mathbb{V}^\alpha_+(x)\mathbb{V}^\alpha_+(x) - \int_\mathbb{R} d\tau \int_{\mathcal{A}^4(\mathbb{V}_M^4, g)} d\mathcal{E}(x, \tau) A_\alpha(x) \Re \{\mathbb{V}^\alpha_+(x)\} + \int_\mathcal{A}^4(\mathbb{V}_M^4, g) d\mu(x) \frac{1}{4\mu_0} \left[F_{\alpha\beta}^\mu(x) + \delta f_{\alpha\beta}^\mu(x)\right] \cdot \left[F_{\alpha\beta}^\mu(\tau, \omega) + \delta f_{\alpha\beta}^\mu(\tau, \omega)\right] \quad (115)$$

provides the following dynamics of a stochastic scalar electron [Theorem 35] and a field [Theorem 37] characterized by $\mathcal{V} := (1 - i)/2 \times \mathcal{V}_+ + (1 + i)/2 \times \mathcal{V}_- \in \mathbb{V}_M^4 \oplus i\mathbb{V}_M^4$ and $F \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4$:

$$m_0 \mathcal{D}_x \mathcal{V}^\mu(\tilde{x}(\tau, \omega)) = -\epsilon \mathcal{V}_\nu(\tilde{x}(\tau, \omega)) \mathcal{F}^\mu_{\nu}(\tilde{x}(\tau, \omega)) \quad (116)$$

$$\partial_\mu \left[F^\mu_{\nu}(x) + \delta f^\mu_{\nu}(x)\right] = \mu_0 \times \Re \left[ -2 \mathcal{E} \int_\mathbb{R} d\tau' \Re \{\mathcal{V}^\mu(x)\} \delta^\mu(x - \tilde{x}(\tau', \omega)) \right] \quad (117)$$

Here, the dynamics of (116) is equivalent to the Klein-Gordon equation. These dynamics fulfill the $U(1)$-gauge symmetry such that

$$\phi'(x) = e^{-i\epsilon A(x)/\hbar} \times \phi(x), \quad A^\alpha(x) = A^\alpha(x) - \partial^\alpha A(x). \quad (118)$$

We could hereby conclude Conjecture 24 is demonstrated, however, how does this equation correspond to classical behavior? It can be described by Ehrenfest’s theorem and it is one of the key idea for the investigation of radiation reaction in Volume II.

**Theorem 41** (Ehrenfest’s theorem). The expectation of (116) derives Ehrenfest’s theorem of the Klein-Gordon equation.

**Proof.** Due to the identity $\mathbb{E}[dW^\mu_+(\tau, \bullet)] = 0$, then, $\mathbb{E}[\mathbb{V}^\mu_+(\tilde{x}(\tau, \bullet))] = \mathbb{E}[\mathbb{V}^\mu(\tilde{x}(\tau, \bullet))]$ is satisfied. Considering the expectation of the equation of motion (116),

$$m_0 \frac{d}{d\tau} \mathbb{E}[\mathbb{V}^\mu(\tilde{x}(\tau, \bullet))] = m_0 \frac{d}{d\tau} \mathbb{E}[\Re(\mathbb{V}^\mu(\tilde{x}(\tau, \bullet)))]$$

...
The trajectory of a stochastic scalar electron satisfies the following relation;

\[
\frac{d}{d\tau} \mathbb{E} \{ \mathcal{V}_\mu^{\ast}(\hat{x}(\tau, \bullet)) \mathcal{V}_\mu(\hat{x}(\tau, \bullet)) \} = 0.
\]

(123)

For \( \Omega_{\text{ave}} \) := \{ \omega | \hat{x}(\tau, \omega) = \mathbb{E}[\hat{x}(\tau, \bullet)] \} \subset \Omega \), we will quantize the LAD equation and demonstrate and the existence of the following formula instead the radiation formula \( dW_{\text{QED}}/dt = q(\chi) \times dW_{\text{classical}}/dt \) by using the present model in **Volume II**:

\[
\frac{dW_{\text{Stochastic}}}{dt}(\mathbb{E}[\hat{x}(\tau, \bullet)]) = \mathcal{P}(\Omega_{\text{ave}}) \times \frac{dW_{\text{classical}}}{dt}(\mathbb{E}[\hat{x}(\tau, \bullet)])
\]

(124)

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