Baryon as a Quantum Hall Droplet and the Cheshire Cat Principle

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We show that the recent proposal to describe the \( N_f = 1 \) baryon in the large number of color limit as a quantum Hall droplet, can be understood as a chiral bag in a 1+2 dimensional strip using the Cheshire cat principle. For a small bag radius, the bag reduces to a vortex line which is the smile of the cat with flowing gapless quarks all spinning in the same direction. The disc enclosed by the smile is described by a topological field theory due to the Callan-Harvey anomaly out-flow. The chiral bag carries naturally unit baryon number and spin \( \frac{1}{2} N_c \). The generalization to arbitrary \( N_f \) is discussed.

I. INTRODUCTION

In the large number of color limit, ’t Hooft suggested that QCD is dominated by planar diagrams, with infinitely many weakly interacting mesons and glueballs [1]. Witten argued that in this limit, baryons are heavy solitons made out of the interacting mesons. The coupling of the mesons is weak and of order \( 1/N_c \), while the coupling of the baryons is strong and of order \( N_c \) [2].

Chiral solitons made solely of non-linearly interacting pions are prototype of these solitons, an idea put forth decades ago by Skyrme [3] well before the advent of QCD. Chiral solitons are topologically protected in 1+3 dimensions, and their quantum numbers emerge through semi-classical quantization. However, their masses and “charges” depend sensitively on the truncated chiral effective action, and somehow less through the more elaborate chiral holographic constructions [4].

Recently, Komargodski [5] pointed at the peculiar character of the QCD baryons for \( N_f = 1 \) where the chiral effective theory is dominated by the axial U(1) anomaly, and where the soliton construction no longer applies since, for instance, the topological charge cannot be identified. He noted that the presence of stable superselection rules in the QCD vacuum (instanton tunneling between vacua with different Chern-Simons number) implies the existence of 1+2 dimensional domain walls. These walls connect vacua with different Chern-Simons number and are observed to be stable at large \( N_c \).

Remarkably, when these sheets are finite dimensional with a boundary, Komargodski noted that they can carry massless edge excitations with baryon quantum numbers. They are identified with fast spinning baryons. These sheets are described by a topological field theory through a level-rank duality argument [6], much like in the fractional quantum Hall (FQH) effect [7]. The baryons are analogous to the gapless edge excitations in quantum Hall (QH) droplets. Arguments were put forth for their generalization to arbitrary \( N_f \).

In this note, we suggest that these baryonic QH droplets can be understood using the Cheshire cat principle (CCP) [8]. More specifically, we show that a chiral bag with a single quark species of charge \( e \) (electric charge or fermion number) confined to a 1+2 dimensional annulus, leaks most quantum numbers. For all purposes the bag radius is immaterial thanks to the CCP. In particular, when the bag radius is shrunk to zero, only the smile of the cat is left with spinning gapless quarks running longitudinally, explaining the edge modes and their chirality [9].

A current transverse to the smile is shown to appear, embodying the Callan-Harvey anomaly out-flow [10]. This transverse current is shown to be analogous to the Hall current typical of the QH effect through the emergence of an effective U(1) gauge field. This U(1) gauge field lives in the disc enclosed by the Cheshire cat smile, and is described by a purely topological field theory in 1+2 dimensions. The quantum numbers of this baryon as a QH droplet follow readily from the chiral bag construction. The generalization to many species is discussed.

II. BAG IN A DOMAIN WALL

Consider a 1+2 dimensional chiral bag in the form of an annulus of radius \( R \) lying in the \( xy \)-plane and clouded by an \( \eta \)-field with a monodromy of \( 2\pi \) or a U(1) winding number of 1. We will refer to \( x \) as the radial direction and to \( y \) as the tangential direction as illustrated in Fig. 1. The bag consists of free 2-dimensional quarks, say of charge \( e \), and subject to a chiral bag boundary condition along the radial \( x \)-direction. We now suggest that this 1+2 dimensional U(1) chiral bag in the limit of zero bag radius is the pancake baryon suggested by Komargodski thanks to the CCP. Note that in the limit of zero bag radius, the chiral bag reduces to a vortex line!
The essence of the CCP lies in the fact that the charge $e$ of the chiral bag leaks through an anomaly. This leakage is best described by noting that in the presence of the $\eta'$-cloud along the $x$-direction, the Dirac spectrum in the bag undergoes a spectral flow. Since the discussion is about leakage of charge along the $x$-direction and flow of charge along the $y$-direction, the shape of the bag as an annulus is topologically equivalent to an infinite strip along the $y$-direction with periodic boundary condition, and $U(1)$ chiral boundary condition along the $x$-direction as illustrated in Fig. 2.

For a single quark species, the chiral bag model on the strip is described by

\begin{equation}
(i\partial_t + i\sigma_2\partial_x - i\sigma_3\partial_y) q(t, x, y) = 0 \quad |x| < R
\end{equation}

\begin{equation}
e^{-i\sigma_2\theta(t,x)} - \sigma_3\epsilon(x) \right) q(t, x, y) = 0 \quad |x| = R,
\end{equation}

with $\epsilon(x) = x/|x|$ the outside normal to the bag, and $(\gamma^0, \gamma^1, \gamma^2) = (\sigma_1, i\sigma_3, i\sigma_2)$. The $\eta'$ field acts only at the boundary through the chiral angle $\theta = \eta'/f_\eta$ which is in general time-dependent but $y$-independent. $f_\eta$ is the $\eta'$ decay constant. Throughout, the reference to chirality in $1+2$ dimensions will be a slight abuse of language for a discrete parity transformation $x_1, x_2 \rightarrow -x_1, x_2$, and $q \rightarrow \sigma_2q$ with the mass term $\bar{q}q = q^+ \sigma_1q \rightarrow -\bar{q}q$ breaking parity. It only becomes chirality in $1+1$ dimensions under dimensional reduction. The anomaly in $1+2$ dimensions is the parity anomaly [10].

With this in mind, the spectral flow is seen by considering the case of a static boundary condition for the $\eta'$ field. In this case, the mode solution to Eq. (1) is of the form

\begin{equation}
g_n(t, x, y) = e^{-iE_n t + ik_y y} \varphi_n(x)
\end{equation}

with $E_n$ following from the transcendental equation

\begin{equation}
\tan \left(2R\sqrt{E_n^2 - k_y^2}\right) = \frac{1 + t_+ t_-}{t_- \sqrt{E_n + k_y} - t_+ \sqrt{E_n - k_y}}
\end{equation}

with $t _\pm = \tan(\theta(\pm R)/2)$. Note that the spectrum is now twisted through $t_+$. For the special case of $1+1$ dimensions with $k_y = 0$, the twist is manifest as Eq. (3) simplifies to

\begin{equation}
\tan(2RE_n) = \tan \left(\frac{\theta}{2} + \frac{\Delta \theta}{2}\right)
\end{equation}

with $\Delta \theta = (\theta(+R) - \theta(-R))$ as the jump of the $\eta'$-field across the chiral bag. The twisted spectrum is now

\begin{equation}
E_n = \frac{(2n + 1)\pi}{4R} + \frac{\Delta \theta}{4R}
\end{equation}

with the level $E_{-1}$ crossing zero at the magic angle $\Delta \theta = \pi$ and requiring a vacuum re-definition. This re-definition implies that the charge in the chiral bag fractionalizes with the result [11]

\begin{equation}
\Delta Q = e\frac{\Delta \theta}{2\pi}
\end{equation}

as the rest of the charge is now located in the topological charge carried by the outside $\eta'$ field. At the magic angle, half the charge is in and half is out. The in-charge is solely carried by the crossing state

\begin{equation}
q_{-1} = \frac{1}{\sqrt{4R}} \begin{pmatrix} -i \\ +i \end{pmatrix}
\end{equation}

spinning along the $y$-direction i.e. $\sigma_2q_{-1} = q_{-1}$ with fixed helicity. If the monodromy is flipped $2\pi \rightarrow -2\pi$, the charge and the helicity are flipped.

The explicit description of the present chiral bag model with space-time dependent boundaries is involved for general $R$ and finite $k_y$, but around the magic angle the
III. ANOMALY OUT-FLOW

When viewed in 1+1 dimensions, the preceding result is the consequence of an exact bosonization which captures the essence of the CCP, namely that the bag radius \( R \) is immaterial (the smile of the Cheshire cat). More specifically, \( \rho_{1+1} \) is the first of the two Abelian bosonization relations in 1+1 dimensions

\[
\rho_{1+1} = \frac{e \partial_q \theta}{2\pi} - \frac{e}{2\pi} \frac{\partial_x \eta'}{f_\eta}
\]

\[
j_{x}^{1+1} = \frac{e \partial_q \theta}{2\pi} - \frac{e}{2\pi} \frac{\partial_y \eta'}{f_\eta}
\]

These observations are now important for the 1+2 dimensional chiral bag and its mapping on the baryon as a QH droplet.

When the bag radius is increasingly small, the chiral bag is more like a vortex line. At the magic angle, a gapless mode with half fermion number (the other half is sitting on the wall) and momentum \( k_y \) flows along the \( y \)-direction. More importantly, the vortex line carries a charge per unit length \( \rho \) and is **leaking radially** a current \( j_x \) as given by \( \delta \) irrespective of how small is \( R \)! An observer along the vortex line will see charge increasing or decreasing and would conclude that his tangential current \( j_y \) is not conserved or anomalous! In other words,

\[
\partial^i \rho + \partial^y j_y = \frac{e^2}{2\pi} E_y
\]

as if an emergent \( U(1) \) effective electric field \( E_y \) was acting on his vacuum. However, this increase or decrease is caused by the leaking current in the radial direction noted earlier, so that \( j_x = j_y \) in magnitude leading to the identification of the emergent electric field \( E_y \) as

\[
\frac{e^2}{2\pi} E_y = \frac{j_{x}^{1+1}}{L_y} = \frac{e}{2\pi} \frac{\partial_\theta}{f_\eta}
\]

after using Eqs. (8) and (9) with \( L_y \) the \( y \)-length of the chiral bag as a strip. A close reading of (10) shows that \( j_x \sim E_y \), which is reminiscent of the Hall current, hence the immediate analogy of the present chiral bag construction with the QH effect. This is the Callan-Harvey mechanism for anomaly out-flow \( \delta \), now realized for a proposed baryon. It is a physical realisation of the descent equation between anomalies in even and odd dimensions \( \delta \) (and references therein).

IV. EMERGENT EFFECTIVE ACTION

The emergent out-flow to the outside disc formed by the chiral bag as an annulus, can be captured in a 1+2 dimensional effective action describing the outside of the bag. Indeed, since the leaking and radial current to the chiral bag is \( j_x \), its extension in 1+2 dimensions defines the variation of the effective action \( S_{1+2} \) with respect to the emergent \( U(1) \) gauge field \( A_y \) as

\[
\frac{\delta S_{1+2}}{\delta A_y} = \frac{j_{x}^{1+1}}{L_y}.
\]

Inserting (10) into (11) and solving gives

\[
S_{1+2} = \int_{1+2} \frac{e^2}{4\pi} A_y E_y = \frac{e^2}{4\pi} \int_{1+2} AdA,
\]

where covariance was subsumed in the 3-form. This is the topological field theory describing the FQH droplet outside the bag illustrated in Fig. 1. One of the chief purposes of the emergent \( U(1) \) field \( A_y \) in the 1+2 dimensional droplet is to enforce the anomaly out-flow, hence its topological rather than dynamical character.

In (12) the Chern-Simons coupling or flux attachment factor is \( \kappa = e^2/2\pi \). A coupling of a charge \( e \) to the emergent \( U(1) \) field in 1+2 dimensions, amounts to a flux attachment of \( e/\kappa \). The exchange of any pair of particles will generate a statistical phase \( e^2/2\kappa = \pi \) through the Aharonov-Bohm interaction. A charged boson coupled to the emergent \( U(1) \) gauge field in 1+2 dimensions, transmutes to a fermion and vice versa!

The generalization of these results to many quark species, say of different colors \( N_c \), requires the use of non-Abelian bosonization, but the CCP still holds \( \delta \). However, in our case this is not needed. Indeed, ordinary quarks carry baryon or fermion number \( 1/N_c \) (instead of the integer \( e^2 \rightarrow 1 \) discussed here), hence a fraction \( \pi/N_c \) of the fermion statistics. This statistics is readily enforced through a flux attachment factor \( \kappa = N_c/2\pi \), leading to the emergent Abelian Chern-Simons contribution

\[
\frac{N_c}{4\pi} \int_{1+2} AdA.
\]

This is the QH droplet suggested by Komargodski for baryons made of \( N_c \) quarks and \( N_f = 1 \). For vanishingly small radius, the Cheshire cat smile reduces to a vortex line with running gapless quarks all spinning in concert in the \( y \)-direction (recall the magic angle), naturally explaining the large spin \( 1/2N_c \). Baryon number is still 1 and now lodged in the \( \eta' \) field through the \( 2\pi \) monodromy. Anti-baryons follow from a \(-2\pi\) monodromy, with the out-flaw turning to an in-flow.

For arbitrary \( N_f \) the spin and statistics arguments do not change as they are solely fixed by \( N_c \). However, the leaking flavor currents lead to a \( U(N_f) \) flavor-valued emergent gauge field \( A_{\mu} \). Again, the CCP applies *mutatis mutandis*. In particular, the emergent non-Abelian Chern-Simons action \( \delta \) is now
\[
\frac{N_c}{4\pi} \int_{1+2} \text{Tr} \left( A \partial A + \frac{2}{3} A^3 \right)
\] (14)

V. CONCLUSIONS

QCD in the large number of colors and \( N_f = 1 \) does not admit a representation of baryons as chiral solitons since \( \pi_3(U(1)) = 0 \). In this limit, Komargodski suggested that baryons are edge excitations of a \( 1+2 \) dimensional QH droplet, and concluded that these baryons are heavy and highly spinning.

We have shown that the nature of these baryons follows from an anomaly out-flow (in-flow for anti-baryons) in a \( 1+2 \) dimensional chiral bag model as an annulus of shrinking size thanks to the CCP. The out-flow from the bag is captured by an emergent U(1) gauge field and described by a topological field theory. The normalization of the latter is fixed by the quark fractional statistics. The emergence of a QH description in the outside of the bag is an illustration of the Callan-Harvey mechanism for the parity anomaly in \( 1+2 \) dimensions.

When the bag is shrunk to zero size, the baryonic charge 1 is lodged in the \( 2\pi \) monodromy. The chiral bag reduces to a vortex line (smile of the Cheshire cat), with running gapless modes of fixed helicity as edge excitations carrying net spin \( \frac{1}{2} N_c \). These observations generalize to arbitrary \( N_f \).

These facts prompt us to ask about the possible relationship of these highly spinning baryons, with the lowest spining skyrmions in the spin-isospin tower \( J = I = 1/2, ..., N_c/2 \). As both descriptions rely on QCD in the large number of colors, a dynamical relation may be at work that selects one from the other. Also, domains of various forms and shapes made of \( \eta' \) or even the lighter \( \pi^0 \), are likely to form at few times nuclear matter density, say in the crust of neutron stars or deeper, making the baryons as QH droplets potential candidates.

The present interplay between the QH effect and QCD baryons, is much in line with the recent suggestion between quantum magnetism and QCD confinement \( [13] \), showing the intricate interplay between concepts of particle physics and condensed matter physics at strong coupling. More insights can be achieved by using perhaps holography, since for instance baryons and the QH states find a common ground for explanation \( [4, 16] \).

Finally and more speculatively, axion quark nuggets are suggested as candidates for dark matter \( [17] \). In the cosmic QCD phase transition, axion domain walls are argued to form copiously and decay, trapping anti-matter in the form of \( 1+3 \) dimensional nuggets. It is tempting to suggest, that breaking cosmic axion domain walls can also result in \( 1+2 \) dimensional pancakes much like the ones discussed here, trapping topological fields instead, with confined hypothetical quark fields circling the boundary. Both the axion (boundary) and the topological fields (disc) are topologically stable and carry energy but are in so far invisible, a good combination for dark matter. Conversely, \( 1+3 \) dimensional \( \eta' \) or even neutral \( \pi^0 \) domain walls instead of axions can be used to trap few quarks in the more standard baryon configuration with low spin, or in the superconducting diquark phase in QCD matter at moderately high density, with tangible consequences for the neutron star equation of state. We hope to return to these and some other issues next.

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APPENDIX

In Ref. \( [14] \) a chiral bag model was constructed to prevent the charge from leaking from the bag following the CCP. In other words a boundary term was added to the chiral bag to seal the leaking charge. This boundary term can be readily obtained by noting that for \( g \)-independent fields, \( (12) \) describes the outside of the bag as a line segment in \( 1+1 \) dimensions with

\[
\frac{\epsilon}{2\pi} \int_{1+1} A d\theta = \frac{\epsilon}{2\pi} \int_{1+1} \theta F - \frac{\epsilon}{2\pi} \int_{B} A_0 d\theta \tag{15}
\]

after an integration by parts, clearly showing the leaking of the \( e \)-charge through the boundary. To seal the leak, the inside of the bag has to be supplemented by the opposite boundary term

\[
\frac{\epsilon}{2\pi} \int_{B} n A_\nu \frac{n^\nu}{f} \equiv -\frac{\epsilon}{2\pi} \int_{B} e^{\mu\nu} n_\nu A_\mu \frac{n^\nu}{f} \tag{16}
\]

with \( n^\nu \) the spatial normal to the bag boundary, after enforcing covariance on the 2-form. This is exactly the surface term suggested in the Cheshire cat construction in \( [12] \) (see Eq. (8.24)) and in \( [13] \). The present arguments illustrate the subtle relationship between the chiral bag in \( [14] \) and the present chiral bag for the baryon as a FQH droplet. In the former the \( e \)-charge is absolutely confined, while in the latter the \( e \)-charge is allowed to flow transversely, both making use of a Chern-Simons term. This is the tale of two hotels: the infinite hotel in our world for the confined anomaly, and the finite hotel in the other world for the flowing anomaly. This tale is highly relevant for nuclear and astrophysical processes involving
hadron-quark continuity \cite{18}. For instance, the role of the $\eta'$ for the color charge conservation is responsible for the Cheshire Cat mechanism for the tiny flavor singlet axial charge for the proton $g_A^{(0)}$. Furthermore since the $\eta'$ is expected to become light at high density, it could have a strong impact on the stiffness of the equation of state in compact-star matter required for the observed massive $\gtrsim 2M_\odot$ stars.

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