Polynomial-like semi-conjugates of the shift map. *

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Abstract

In this paper I prove that for a polynomial of degree $d$ with a Cantor Julia set $J$, the Julia set can be understood as the simplest possible quotient of the one sided shift space $\Sigma_d$ with dynamics given by the shift. Here simplest possible means that, the projection $\pi: \Sigma_d \longrightarrow J$ is as injective as possible.

1 Introduction

Denote by $\Sigma_d = \{0, \ldots, d-1\}^\mathbb{N}$ the set of one-sided infinite sequences of symbols the $0, \ldots, d-1$ equipped with the natural product topology. And denote by $\sigma: \Sigma_d \longrightarrow \Sigma_d$ the shift map:

$$\sigma((\epsilon_i)_i) = (\epsilon_{i+1})_i = (\epsilon_2, \epsilon_3, \ldots).$$

Douady and Hubbard introduced in [DH] the notion of polynomial-like maps. Here we shall use a slightly generalized version of such maps (see also [L-V]):

Let $f: U' \longrightarrow U$ be a proper holomorphic map where $U \simeq \mathbb{D}, U' \subset U$, $U' = U_1 \cup U_1 \cup \ldots \cup U_N$, $U_i \simeq \mathbb{D}$ for each $i$ and $U_i \cap U_j = \emptyset$ for $i \neq j$.

The filled-in Julia set $K_f$ for $f$ is the set of points:

$$K_f = \{ z \in U'| f^n(z) \in U', \forall \ n \in \mathbb{N} \},$$

and the Julia set is its topological boundary $J_f = \partial K_f$.

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For such a map the degree \( d \) is the sum of the degrees \( d_i \) of the restrictions \( f_i : U_i \to U \). By the Riemann-Hurwitz formula \( f \) has counting multiplicity \( d_i' = d_i - 1 \) critical points in \( U_i \). In particular if \( f \) does not have any critical point in some \( U_i \), then \( f \) has a globally defined inverse branch \( f_i^{-1} : U \to U_i \). In particular if \( f \) has no critical points at all then \( d = N \) and \( f \) has \( d \) distinct globally defined inverse branches. In this case it follows that \( K_f = J_f \) is a Cantor set and an elementary proof going back to Fatou shows that in the later case there is a homeomorphism \( \pi : \Sigma_d \to J_f \) such that \( \pi \circ \sigma = f \circ \pi \).

If no critical point of \( f \) is periodic then the function \( \chi : K_f \to \mathbb{N} \) given by the maximal local degree of iterates of \( f \) near \( z \):

\[
\chi(z) = \sup_{n \in \mathbb{N}} \deg(f^n, z)
\]

is bounded by the product of the local degrees of \( f \) at its critical points and satisfies

\[
\chi(z) = \deg(f, z) \cdot \chi(f(z)).
\]

Since \( f \) has \( d' = d - N = \sum d_i' \) critical points the function \( \chi \) is bounded by \( 2^d \). Note that any periodic critical point is surrounded by an open attracted basin, and thus belongs to the interior of \( K_f \).

The main theorem of this paper is:

**Theorem 1.1.** Let \( f : U' \to U \) be a degree \( d > 1 \) generalized polynomial-like map in the sense above. If \( K_f = J_f \) is a Cantor set containing all critical points of \( f \). Then there is a semi-conjugacy \( \pi : \Sigma_d \to J_f \), \( \pi \circ \sigma = f \circ \pi \) such that:

\[
\forall z \in J_f : \#\pi^{-1}(z) = \chi(z).
\]

**Remark 1)** Branner and Hubbard proved in [B-H] that there are many cubic polynomials with a generalized polynomial-like restriction as above satisfying the hypothesis and thus the conclusion of the above Theorem. Moreover recently this Branner-Hubbard Theorem has been extend to all degrees and all orders of critical points. See e.g. [T-Y], [Y] and [K-S].

**Remark 2)** The main theorem is related to the structure of the complement of the cubic connectedness locus through ther paper [DGK] of Devaney, Goldberg and Keen.

**Remark 3)** The injectivity statement of the main Theorem is best possible, since if for some point \( z : \#\pi^{-1}(P(z)) = l \) and if the local degree of \( f \) at \( z \) is \( m \geq 1 \). Then \( \#\pi^{-1}(z) = ml \).

**Remark 4)** \( N \geq 2 \) since if not \( J_f \) would be connected and not a Cantor set.

**Remark 5)** The hypothesis that \( J_f \) is Cantor set is equivalent to asking that
the diameters of the connected components of $f^{-n}(U)$ converge to zero as $n$
tends to infinity.

Towards a proof of the main theorem we introduce some notation.

We shall in the following tacitly assume the hypotheses of the Theorem, i.e..
f : $U' \rightarrow U$ is a generalized polynomial-like map for which $K_f = J_f$
is a Cantor set containing all critical points of $f$.

Note that taking a restriction with $U$ slightly smaller if necessary we can assume
the boundaries of all disks $U$ and $U_i$ are smooth and disjoint. Let $w \in U \setminus U'$
be arbitrary and let $w_0, \ldots, w_{d-1}$ denote the $d$ distinct preimages
of $w$, and let $i = i(j)$ denote the function given by $w_j \in U_{i(j)}$. Renumbering
if necessary we can assume that $i$ is weakly increasing, i.e. we have filled-in
from below.

Let $\phi : \mathbb{D} \rightarrow U \setminus J_f$ be a universal covering with $\phi(0) = w$.

**Proposition 1.2.** There exist $d$ (univalent) lifts $g_i : \mathbb{D} \rightarrow \mathbb{D}$, $i = 0, \ldots, d-1$
of $\phi$ to $f \circ \phi$, i.e. $f \circ \phi \circ g_i = \phi$ with $\phi \circ g_i(0) = w_i$. These satisfy $\phi \circ g_j(z) \neq \phi \circ g_{j'}(z)$ for $j \neq j' \mod d$, i.e. for any $z \in \mathbb{D}$ the points $\phi \circ g_j(z)$ are the $d$
distinct preimages of $\phi(z)$ under $f$. In particular

$$f^{-1}(\phi(\mathbb{D})) = f^{-1}(U \setminus J_f) = U' \setminus J_f = \bigcup_{i=0}^{d-1} (\phi \circ g_i)(\mathbb{D}).$$

Remark that the $g_i$ are by no means unique.

**Proof.** For $0 \leq j < d$ and $i = i(j)$ let $V_i$ be a connected component of
$\phi^{-1}(U_i \setminus J_f)$. Then $V_i$ is simply connected, because $U_i \setminus J_f$ is a retract of $U \setminus J_f$.
Hence the restriction $\phi : V_i \rightarrow U_i \setminus J_f$ is a universal covering map. Since the restriction $f : U_i \setminus J_f \rightarrow U \setminus J_f$ has no critical points it is also a covering and thus each $f \circ \phi|_{V_i}$ is a universal covering. Let $x_j \in V_i$ be any point with $\phi(x_j) = w_j$. Then there is a unique lift $g_j : \mathbb{D} \rightarrow V_j$ of the universal covering $\phi$ to the (universal) covering $f \circ \phi|_{V_i}$ mapping $0$ to $x_j$. Being lifts of $f \circ \phi$ to
$\phi$, any two of the $g_j$ either agree everywhere or nowhere. They are chosen to
disagree at 0. \hfill $\Box$

Note that changing the choice of some $z_j$ to some other preimage $z'_j$ of $w_j$
amounts to post composing $g_j$ with the decktransformation for $\phi$, which
maps $z_j$ to $z'_j$. We shall think and speak of the maps $g_j$ as lifts of $f^{-1}$ though
technically they are self-maps of a different space.

For $k \geq 1$ let $\Sigma_d^k$ denote the set of $k$-blocks $\epsilon^k = (\epsilon_1, \ldots, \epsilon_k)$ in the alphabet
$\{0, \ldots, d-1\}$. Every such $\epsilon^k$ defines a “cylinder” clopen set

$$\{ (\tau_j) | \tau_j = \epsilon_j, j \leq k \}.$$
The map \( \sigma \) thus has a natural extension as a map from \( \Sigma_d^k \) to \( \Sigma_d^{k-1} \). Also for \( n < k \) there is a natural projection from \( \Sigma_d^k \) to \( \Sigma_d^n \): \( \epsilon^k \rightarrow \epsilon^n \), which simply forgets the last \( k - n \) entries.

The obvious idea for proving Theorem 1.1 would now be to iterate the \( d \) branches \( g_j : \mathbb{D} \rightarrow \mathbb{D} \) of the inverse of \( f \), project back to \( U \) and obtain sets for defining a semiconjugacy. More precisely for \( \epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k, \ldots) \in \Sigma_d \) and \( k \in \mathbb{N} \) define

\[
g_{\epsilon k} = g_{\epsilon_1} \circ \cdots \circ g_{\epsilon_k},
\]

and

\[
V_{\epsilon k} = g_{\epsilon k}(\mathbb{D}).
\]

Then for each \( \epsilon \in \Sigma_d \) the set \( \cap_{k \geq 1} \phi(V_{\epsilon k}) \) is a connected subset of the Cantor set \( J_f \) and thus a singleton \( \{z_\epsilon\} \). Define \( \Psi : \Sigma_d \rightarrow J_f \) by \( \Psi(\epsilon) = z_\epsilon \). Then \( \Psi \) is indeed a semi-conjugacy of \( \sigma : \Sigma_d \rightarrow \Sigma_d \) to \( P : J_f \rightarrow J_f \). However in general it will not have the promised injectivity properties. The problem originates in the number of connected components \( V_{\epsilon k} \), \( \epsilon_k \in \Sigma_d^k \) is growing much faster than the number of connected components of \( f^{-k}(U) \). To remove this problem we shall use decktransformations for \( \phi \) to push together the sets \( V_{\epsilon k} \) and \( V_{\beta k} \) whenever \( \phi(V_{\epsilon k}) = \phi(V_{\beta k}) \).

To fix the ideas let \( \Gamma \) denote the group of decktransformations for the universal covering \( \phi \), i.e. \( \gamma \in \Gamma \), if and only if \( \gamma \) is an automorphism of \( \mathbb{D} \) with \( \phi \circ \gamma = \phi \).

**Lemma 1.3.** Given \( \epsilon_k \in \Sigma_d^k \) and \( \gamma_1, \ldots, \gamma_k \in \Gamma \) let

\[
V = \gamma_1 \circ g_{\epsilon_1} \circ \cdots \circ g_{\epsilon_k}(\mathbb{D}) \quad \text{and} \quad V' = \gamma_2 \circ g_{\epsilon_2} \circ \cdots \circ \gamma_k \circ g_{\epsilon_k}(\mathbb{D}).
\]

Then the restrictions \( \phi : V \rightarrow \phi(V) \) and \( \phi : V' \rightarrow \phi(V') \) are universal coverings, \( \phi(V) = W \setminus J_f \), \( \phi(V') = W' \setminus J_f \), where \( W, W' \) are connected components of \( f^{-k}(U) \) and \( f^{-(k-1)}(U) \) respectively and the restriction \( f : W \rightarrow W' \) is a branched covering.

**Proof.** The first statements is an easy induction proof, based on Proposition 1.2, the details are left to the reader. The last statement follows from

\[
f \circ \phi \circ \gamma_1 \circ g_{\epsilon_1} \circ \gamma_2 \circ \cdots \circ g_{\epsilon_k} = \phi \circ \gamma_2 \circ g_{\epsilon_2} \circ \cdots \circ \gamma_k \circ g_{\epsilon_k}.
\]

Note also that

\[
V \subset \gamma_1 \circ g_{\epsilon_1} \circ \cdots \circ \gamma_{k-1} \circ g_{\epsilon_{(k-1)}}(\mathbb{D}).
\]
Proposition 1.4. There exists a sequence of families \( \{ \gamma_i^k \}_{i=1}^k \subset \Gamma \), \( e^k \in \Sigma_d \), \( k \in \mathbb{N} \) such that the family of sets \( V_{\hat{c}^k} := \gamma_1^k \circ g_{c_1} \circ \ldots \circ \gamma_l^k \circ g_{c_l}(\mathbb{D}) \) and the sequence of families of decktransformations \( \{ \gamma_i^k \}_{i=1}^k \) satisfies the following three properties:

1. For all \( k \geq 2 \) and for all \( e^k : V_{\hat{c}^k} \subset V_{\hat{c}^{k-1}}, \) where \( e^{k-1} = \epsilon_1 \ldots \epsilon_{k-1} \).
2. For all \( k \geq 2 \), for all \( e^k \) and for all \( l = 2, \ldots, k : \gamma_i^k = \gamma_i^{\sigma(e^k)} \).
3. For all \( k \geq 1 \) and for all \( e^k, \hat{e}^k : \) If \( \phi(V_{\hat{c}^k}) = \phi(V_{\hat{c}^k}) \), then \( V_{\hat{c}^k} = V_{\hat{c}^k} \).

Remark that \( 3 \) implies that there is a \( 1:1 \) correspondence between connected components of \( f^{-k}(U) \) and connected components of \( \bigcup_{k \in \mathbb{N}} V_{\hat{c}^k} \).

Proof. The proof is by induction on \( k \). For this it is convenient to let \( \emptyset \) denote the empty tuple of length 0 and define \( \sigma(e^1) = \emptyset \). Also we shall then extend the above properties 1 and 2 to \( k = 1 \) and property 3 to \( k = 0 \).

We then define \( V_{\emptyset} = \mathbb{D} \). This takes care of \( k = 0 \). For \( k = 1 \) we have already chosen the branches \( g_j \) of the lifted inverse of \( f \) so that \( g_{j}(\mathbb{D}) = g_{j'}(\mathbb{D}) \) whenever \( \phi(g_j(\mathbb{D})) = \phi(g_{j'}(\mathbb{D})) \). Thus we can simply take each \( \gamma_1^1 = \text{id} \).

This then complies with all three properties. For the inductive step suppose families \( \{ \gamma_i^n \}_{i=1}^n \subset \Gamma, \epsilon^n \in \Sigma_d \) satisfying the three properties have been constructed. For any \( e^k \in \Sigma_d \) define \( \gamma_i^k = \gamma_i^{\sigma(e^k)} \) for \( 1 < l < k \). Moreover define \( \gamma_1^k = \gamma_1^{k-1} \) and \( V_{\hat{c}^k} = \gamma_1^k \circ g_{c_1}(V_{\sigma(e^k)}) \) as preliminary candidates for \( \gamma_1^k \) and \( V_{\hat{c}^k} \).

With this choice \( 2 \) is immediately satisfied and hence so is \( 1 \), because

\[
V_{\sigma(e^k)} = \gamma_1^{\sigma(e^k)} \circ g_{c_1} \circ \gamma_2^{\sigma(e^k)} \circ g_{c_2} \circ \ldots \circ \gamma_{k-1}^{\sigma(e^k)} \circ g_{c_{k-1}}(\mathbb{D})
\]

\[
= \gamma_1^k \circ g_{c_1} \circ \gamma_2^k \circ g_{c_2} \circ \ldots \circ \gamma_{k-1}^k \circ g_{c_{k-1}}(\mathbb{D})
\]

and by the induction hypothesis \( V_{\sigma(e^{k-1})} \supset V_{\sigma(e^k)} \), so that

\[
\hat{V}_{c^k} = \gamma_1^{k-1} \circ g_{c_1}(V_{\sigma(e^k)}) \supset \gamma_1^{k-1} \circ g_{c_1}(V_{\sigma(e^{k-1})}) = V_{\hat{c}^{k-1}}.
\]

To complete the inductive step suppose \( \phi(\hat{V}_{c^k}) = \phi(\hat{V}_{c^k}) \). Then by the above \( \hat{V}_{c^k} \subset V_{c^k-1}, \hat{V}_{c^k} \subset V_{\hat{c}^{k-1}} \) so that

\[
\phi(V_{\hat{c}^{k-1}}) = \phi(V_{\hat{c}^{k-1}}).
\]
And thus $V_{k-1} = V_{k-1}$ by property 3. applied to $\epsilon^{k-1}$ and $\hat{\epsilon}^{k-1}$. That is $\phi(V_{\epsilon k}) = \phi(V_{\hat{\epsilon} k})$ implies

$$\hat{V}_{\epsilon k}, \hat{V}_{\epsilon k} \subset V_{\epsilon^{k-1}} = V_{\hat{\epsilon}^{k-1}}.$$ 

Define an equivalence relation $\sim$ on $\Sigma_d^k$ by $\epsilon^k \sim \hat{\epsilon}^k \iff \phi(V_{\epsilon k}) = \phi(V_{\hat{\epsilon} k})$.

For each equivalence class of $\sim$ choose a preferred representative $\epsilon^k$, e.g. the one which is minimal with respect to the lexicographic ordering, and define $\gamma^k = \gamma_1^k$, $V_{\epsilon k} = \hat{V}_{\epsilon k}$. For any other element $\hat{\epsilon}^k \in [\epsilon^k]$ choose $\gamma_1^k \in \Gamma$ so that

$$V_{\epsilon k} = \gamma_1^k \circ g_{\epsilon 1}(V_{\sigma(\epsilon k)}) = V_{\epsilon k}.$$ 

Then also property 3. is satisfied.

We have now laid the grounds for the projection $\pi : \Sigma_d \rightarrow J_f$ of the Main Theorem: Define the projection mapping $\pi = \pi_f : \Sigma_d \rightarrow J_f$ by

$$\pi((\epsilon_j)_j) = \bigcap_{k=1}^{\infty} \phi(V_{\epsilon k}).$$ 

Then by construction the map $\pi$ is continuous and semi-conjugates the shift $\sigma$ on $\Sigma_d$ to $f$ on $J_f$

$$\pi \circ \sigma = f \circ \pi.$$

The rest of the paper is devoted to proving that $\pi$ is as stated in the theorem: i.e. is surjective, is injective above any non-(pre)critical point $z$ and for any $z \in J_f$ satisfies

$$\#\pi^{-1}(z) = \deg(f, z) \cdot \#\pi^{-1}(f(z)).$$ 

Let us first address the issue of surjectivity.

**Proposition 1.5.** For any $k \in \mathbb{N}$:

$$f^{-k}(U \setminus J_f) = \bigcup_{\epsilon^k \in \Sigma_d^k} \phi(V_{\epsilon k})$$ 

**Proof.** This is an elementary induction proof based on Proposition 1.2 and the observation that for any $j$ and any decktransformation $\gamma \in \Gamma : \phi \circ \gamma \circ g_j = \phi \circ g_j$. Combining the observation with Proposition 1.2 shows that the
statement holds for $k = 1$. Now suppose the statement holds for some $k$. Then by Proposition 1.2

$$f^{-(k+1)}(U \setminus J_f) = \bigcup_{j=0}^{d-1} \bigcup_{\epsilon \in \Sigma_d^k} \phi \circ g_j(V_{\epsilon^k}) = \bigcup_{\epsilon \in \Sigma_d^{k+1}} \phi(V_{\epsilon^{(k+1)}})$$

And thus the inductive step follows from the observation.

**Proposition 1.6.** The map $\pi = \pi_f : \Sigma_d \rightarrow J_f$ is surjective.

**Proof.** Let $z \in J_f$ be arbitrary and let $W_k$ denote the connected component of $f^{-k}(U)$ containing $z$. Then by Proposition 1.5, there exists a sequence $(\epsilon_k)_{k \in \mathbb{N}}$ for each $k$ such that $\phi(V_{\epsilon_k}) = W_k \setminus J_f$. By compactness of $\Sigma_d$ there exists at least one accumulation point $\xi \in \Sigma_d$ of $(\epsilon_k)_k$. That is for every $N \in \mathbb{N}$ there exists a $k > N$ such that $\epsilon_1, \ldots, \epsilon_N = \epsilon^1_k, \ldots, \epsilon^N_k$.

But then $\pi(\xi) \in W_k \subset W_N$ and thus $\pi(\xi) \in W_N$ for every $N$. That is $\pi(\xi) = z$, because $J_f$ is a Cantor set and thus diam$(W_N) \rightarrow 0$ as $N \rightarrow \infty$.

**Proposition 1.7.** If $V_{\epsilon_k} = V_{\hat{\epsilon}_k}$ and $\epsilon_1 \neq \hat{\epsilon}_1$. Then the component $W$ of $f^{-k}(U)$ with $\phi(V_{\epsilon_k}) = W \setminus J_f$ contains at least one critical point.

Moreover if $\pi(\xi) = \pi(\hat{\xi}) = z$ and and $\epsilon_1 \neq \hat{\epsilon}_1$. Then $z$ is a critical point for $f$.

**Proof.** Let $W' = f(W)$ then $\phi(V_{\sigma(\epsilon_k)}) = \phi(V_{\sigma(\hat{\epsilon}_k)}) = W'$ and thus $V_{\sigma(\epsilon_k)} = V_{\sigma(\hat{\epsilon}_k)}$ by property 3 of Proposition 1.4. Let $x$ be any point of the later set, then $\phi(g_{\epsilon_1})$ and $\phi(g_{\hat{\epsilon}_1})$ are two distinct preimages in $W$ of the point $\phi(x) \in W'$. Hence the degree of the restriction $\tau : W \rightarrow W'$ is at least 2 and thus $W$ contains at least one critical point by the Riemann-Hurwitz formula. This proves the first statement of the Lemma. The second is an immediate consequence of Proposition 1.4 and the first statement:

$$\pi(\xi) = \pi(\hat{\xi})$$

$$\bigcap_{k=1}^{\infty} \phi(V_{\epsilon_k}) = \bigcap_{k=1}^{\infty} \phi(V_{\hat{\epsilon}_k})$$

$$\forall k \in \mathbb{N} : \phi(V_{\epsilon_k}) = \phi(V_{\hat{\epsilon}_k})$$

$$\forall k \in \mathbb{N} : V_{\epsilon_k} = V_{\hat{\epsilon}_k}$$
Thus if $\epsilon_1 \neq \hat{\epsilon}_1$ and $W_k \setminus J_f = \phi(V_{\epsilon_k})$. Then each $W_k$ contains a critical point, $W_{k+1} \subset W_k$ for all $k$ and
\[
z = \pi(\epsilon) = \cap_{k=1}^{\infty} \phi(V_{\epsilon_k}) = \cap_{k=1}^{\infty} W_k.
\]
Hence $z$ is a critical point. \hfill \qed

**Corollary 1.8.** Let $z \in J_f$ be any point whose orbit $(f^n(z))_{n \geq 0}$ does not contain a critical point. Then
\[
\# \pi^{-1}(z) = 1.
\]

**Proof.** Suppose $\pi(\epsilon) = \pi(\hat{\epsilon}) = z$. We shall show that $\epsilon = \hat{\epsilon}$. As a start $\epsilon_1 = \hat{\epsilon}_1$ by Proposition 1.7. The Corollary now follows by induction since by the conjugacy property of $\pi$
\[
\pi(\sigma^n(\epsilon)) = \pi(\sigma^n(\hat{\epsilon})) = f^n(z)
\]
and by assumption this point is not critical, so that $\epsilon_n = \hat{\epsilon}_n$ for all $n$ by Proposition 1.7. \hfill \qed

To shorten notation let us write $d_z = \deg(f, z)$ for any $z \in U'$.

**Proposition 1.9.** For any $z \in J_f$
\[
\# \pi^{-1}(z) = d_z \# \pi^{-1}(f(z)).
\]
More precisely there are $d_z$ distinct numbers $j_1, \ldots, j_{d_z} \in \{0, \ldots, d-1\}$ depending only on $z$ such that $\pi(\epsilon) = z$ if and only if $\pi(\sigma^n(\epsilon)) = f^n(z)$ and
\[
\epsilon_1 \in \{j_1, \ldots, j_{d_z}\}.
\]

**Proof.** Given $z \in J_f$ let $W_k$ denote the connected component of $f^{-k}(U)$ containing $z$ and let $W'_k$ denote the connected component of $f^{-(k-1)}(U)$ containing $f(z)$. Then the degree of the restrictions $f : W_k \to W'_{k-1}$ equals $d_z$ for $k$ sufficiently large, because $z$ is the only point in the nested intersection of the $W_k$. Fix any such $k_0$, let $k \geq k_0$ and let $V'_{k-1}$ denote the connected component of $\phi^{-1}(W'_{k-1} \setminus J_f)$ such that $V'_{k-1} = V_{k-1}$ for any $k \geq k_0$ with $\phi(V_{k-1}) = W'_{k-1} \setminus J_f$. Let $j_1, \ldots, j_{d_z} \in \{0, \ldots, d-1\}$ be the $d_z$ values of $j$ for which $\phi(g_j(V'_{k-1})) = W_k \setminus J_f$ as provided by Proposition 1.7. Then the index set $\{j_1, \ldots, j_{d_z}\}$ does not depend on the value of $k \geq k_0$ by nestedness of the sets $V'_{k-1}$. Hence $\pi(\epsilon) = z$ if and only if $\epsilon_1 \in \{j_1, \ldots, j_{d_z}\}$ and $\pi(\sigma(\epsilon)) = f(z)$. \hfill \qed
Proof. (of Theorem 1.1) By the above Propositions $\pi$ is a continuous and surjective semiconjugacy. In particular

$$\forall z \in J_f : \#\pi^{-1}(z) \geq 1.$$ 

Since no critical point of $f$ is periodic and there are finitely many critical points counted with multiplicity, the total branching $\chi(z)$ along the orbit of an arbitrary point is uniformly bounded. In particular for any $z \in J_f$ there exists $N \in \mathbb{N}$ such that the orbit of $f^N(z)$ does not contain any critical point. Thus by Proposition 1.7

$$\#(\pi^{-1}(f^N(z))) = 1.$$ 

Finally we have $\chi(z) = d_z \cdot d_{f(z)} \cdot \ldots \cdot d_{f^{N-1}(z)}$, so that

$$\forall z \in J_f : \#\pi^{-1}(z) = \chi(z).$$ 

by induction on Proposition 1.9. \qed

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