Direct Improvement of Hamiltonian Lattice Gauge Theory

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Abstract

We demonstrate that a direct approach to improving Hamiltonian lattice gauge theory is possible. Our approach is to correct errors in the Kogut-Susskind Hamiltonian by incorporating additional gauge invariant terms. The coefficients of these terms are chosen so that the order $a^2$ classical errors vanish. We conclude with a brief discussion of tadpole improvement in Hamiltonian lattice gauge theory.

1 Introduction

The idea of using the lattice as an ultra-violet regulator for quantum chromodynamics (QCD) was proposed by Wilson in 1974 in his action formulation of lattice gauge theory [1]. Soon after, Kogut and Susskind formulated the corresponding Hamiltonian version of lattice gauge theory [2]. Both approaches were developed by demanding the correct continuum limit be obtained in the limit of vanishing lattice spacing $a$. Creutz showed that the Kogut-Susskind Hamiltonian could be derived from the Wilson action using the transfer matrix method [3]. Later, Kogut demonstrated that the same could be done by taking the continuous time limit of the Wilson action and performing a canonical Legendre transformation [4].

To date, the majority of work in lattice QCD has been performed in the action formulation. An advantage of this approach is that it readily lends itself to Monte Carlo techniques. Working in the Hamiltonian approach brings a different intuition to the problem and serves as a check of universality. An advantage of Hamiltonian lattice gauge theory is in the applicability of techniques from many body physics [5]. Also, it appears that in finite density QCD, a Hamiltonian approach is favourable due to the so-called complex action problem which rules out the use of standard Monte Carlo techniques in the action formulation [6].

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Much work in the past decade has been devoted to improving lattice actions \cite{5}. Initiated by Symanzik in 1983 \cite{6}, the aim of the improvement programme is to reduce the deviation between lattice and continuum QCD. For pure gauge theory on a lattice, the deviations between continuum and lattice theories start at order $a^2$. The motivation for improvement lies in the fact that the computational cost of a lattice QCD simulation is proportional to $a^{-k}$, where $6 < k < 7$. It is by far more efficient to build an improved theory than it is to work on finer lattices. The improvement programme has allowed accurate calculations to be performed on relatively coarse lattices and brought the most complicated calculations within the reach of today’s most powerful computers.

In contrast, the improvement of lattice Hamiltonians has only recently begun. Perhaps the most extensive treatment to date is due to Luo, Guo, Kröger and Schüttler \cite{9} who discussed the improvement of Hamiltonian lattice gauge theory for gluons. In their study it was discovered that deriving an improved Hamiltonian from a Symanzik improved action, whether by transfer matrix or canonical Legendre transformation, results in a kinetic Hamiltonian with an infinite number of terms coupling lattice sites which are arbitrarily far apart. To derive a local kinetic Hamiltonian coupling only nearest neighbour lattice sites it was found necessary to start with an improved action with an infinite number of terms, coupling distant lattice sites.

With this technique the order $a^2$ errors are removed from the Kogut-Susskind Hamiltonian. However, generating Hamiltonians with further improvement would seem exceedingly difficult. This is because one would need to start from a Lüscher-Weisz improved action with non-planar terms \cite{10}. For this reason we propose a move to the Symanzik approach, as applied to the Hamiltonian. That is, to construct improved Hamiltonians directly by adding appropriate gauge invariant terms and fixing their coefficients so that errors are cancelled.

To date we have implemented Symanzik improvement to order $a^2$ in the pure lattice gauge theory Hamiltonian (and to order $a^4$ in the kinetic part). We report those results here.\footnote{The $O(a^2)$ results were given in preliminary form in \cite{11}}

\section{Errors in Lattice Gauge Theory}

Before discussing Hamiltonian improvement we must first understand how deviations between lattice gauge theory and its continuum counterpart arise. The deviations can be separated into two classes, classical and quantum errors, which will be described in what follows.

Rather than being constructed from gluon fields, pure gauge theory on the lattice is built from link operators

\[ U_\mu(x) = \exp \left\{ ig \int_{\text{Link}} dx \cdot A \right\}. \]  

(1)
Figure 1: The diagrammatic representation of the link operator $U_\mu(x)$.

Here $g$ is the QCD coupling, $\mu = 0, \ldots, 3$ is the Dirac index labelling the time-like and space-like directions, $x$ labels a lattice site and $A$ is the gluon field. The integral runs along the link joining the lattice sites $x$ and $x + a\mu$. This leads to the diagrammatic representation of the link operator shown in figure 1. Lattice gauge theories are built from link operators rather than gluon fields in order to maintain manifest gauge invariance.

We define the lattice gluon field $A^L_\mu(x')$ to be the average of the continuum gluon field $A$ along the link joining $x$ and $x + a\mu$:

$$A^L_\mu(x') = \frac{1}{a} \int_{\text{Link}} dx \cdot A \Rightarrow U_\mu(x) = e^{igaA^L_\mu(x')},$$

(2)

where $x'$ is a point near the points $x$ and $x + a\mu$. On the lattice the gluon field is defined at only one point along (or nearby) a link. This leads to interpolation errors in the integral in eqn. 2. For instance, by choosing to evaluate the gluon field at the midpoint of the link, the lattice and continuum fields are related by

$$A^L_\mu(x) = A_\mu(x) + \frac{a^2}{24} \partial^2_\mu A_\mu(x) + \frac{a^4}{1920} \partial^4_\mu A_\mu(x) + \ldots$$

(3)

We see that the lattice gluon field reduces to its continuum counterpart in the continuum limit ($a \to 0$), but that they differ by terms of order $a^2$, which may be called interpolation errors.

Having discussed classical errors we now move on to quantum errors in lattice gauge theory. Quantum errors arise in two different contexts. Firstly, the lattice acts as an ultraviolet regulator allowing the simulation of only those states with momenta less than $\pi/a$. The absence of high momentum states results in a deviation between lattice and continuum theories. Secondly, non-physical interactions arise due to the use of the link operator in constructing the lattice theory. To demonstrate this we expand the link operator in powers of $g$,

$$U_\mu(x) = 1 + igA_\mu(x) - \frac{g^2 a^2}{2!} A_\mu(x)A_\mu(x) + \ldots,$$

(4)

and note that the interaction of any number of gluons is allowed. Naively, the unphysical interactions are suppressed by powers of $a$. However, when contracted, products of pairs
of gluon fields produce ultraviolet divergences ($\propto 1/a^2$) which exactly cancel the $a$ dependence of the expansion. These terms can be uncomfortably large and result in what are known as tadpole errors.

In the last decade the improvement programme has led to a good understanding of both classical and quantum errors in quark and gluon actions (See reference [8] and references within). In contrast, only the lowest order classical errors have been corrected in the Kogut-Susskind Hamiltonian. Conjectures have been made about the structure of a quantum improved Hamiltonian [9], but a perturbative study has not yet been carried out.

We briefly describe how quantum improved Hamiltonian can be constructed in § 4.

3 Symanzik Improvement of the Lattice Hamiltonian

3.1 Introduction

In this section we derive an improved Hamiltonian directly using the Symanzik approach of adding irrelevant terms and fixing their coefficients in order to cancel errors. As a first step we aim to correct the classical order $a^2$ errors arising in the lattice Hamiltonian for pure SU($N$) gauge theory.

The Kogut-Susskind Hamiltonian for pure SU($N$) gauge theory on the lattice is given by

$$H^{(0)} = K^{(0)} + V^{(0)},$$

where the kinetic and potential terms are given respectively by,

$$K^{(0)} = \frac{a^3}{2} \sum_{x,i} \text{Tr} \left\{ E^L_i(x) E^L_i(x) \right\},$$

$$V^{(0)} = \frac{2N}{ag^2} \sum_{x,i<j} P_{ij}(x).$$

Here $E^L$ is the lattice chromo-electric field, $N$ is the dimension of the gauge group, and $P_{ij}(x)$ is the plaquette operator in the $(i, j)$ plane,

$$P_{ij}(x) = 1 - \frac{1}{N} \text{ReTr} \left\{ \begin{array}{c|c} i & j \\ \hline i & j \end{array} \right\}. $$

We discuss the improvement of the kinetic and potential terms separately in what follows.

3.2 Improving the Potential Term

To improve the potential part of the Kogut-Susskind Hamiltonian, we follow the process of improving the Wilson action. Our first step is to expand the potential term in powers of $a$. This is done using Stokes’ theorem to expand the plaquette operator,

$$P_{ij}(x) = 1 - \frac{1}{N} \text{ReTr} \int A \cdot dx$$
\[
\approx 1 - \frac{1}{N} \text{Re} \text{Tr} \exp \left\{ i g \int_{-a/2}^{a/2} \int_{a/2}^{a/2} dx' dx'' [D_i A_j(x + x') - D_j A_i(x + x')] \right\}
\]
\[
\approx \frac{g^2 a^4}{2N} \text{Tr} \left\{ F_{ij}(x) F_{ij}(x) \right\} + \frac{g^2 a^6}{12N} \text{Tr} \left\{ F_{ij}(x) (D_i^2 + D_j^2) F_{ij}(x) \right\} + \ldots
\] (9)

Here \(D_i = \partial_i - igA_i\) is the gauge covariant derivative, and \(F_{ij}\) are the spatial component of the gluon field tensor. The potential term then becomes:

\[
V^{(0)} = \frac{2N a g^2}{N} \sum_{x,i<j} P_{ij}(x)
\]
\[
= a^3 \sum_{x,i<j} \left\{ \text{Tr} F_{ij}(x) F_{ij}(x) + \frac{a^2}{6} \text{Tr} F_{ij}(x) (D_i^2 + D_j^2) F_{ij}(x) + \ldots \right\}
\]
\[
\approx \frac{1}{2} \int d^3x \sum_{ij} \text{Tr} \left\{ F_{ij}(x) F_{ij}(x) + \frac{a^2}{12} F_{ij}(x) (D_i^2 + D_j^2) F_{ij}(x) + \ldots \right\}.
\] (10)

We see that the correct continuum limit is restored in the limit as \(a \to 0\). To cancel the order \(a^2\) error we introduce rectangular loops into the potential term. These are the next most complicated gauge invariant constructions on the lattice after plaquettes. We write the improved potential term as the linear combination

\[
V^{(1)} = \frac{2N a g^2}{N} \sum_{x,i<j} \left[ X P_{ij}(x) + \frac{Y}{2} (R_{ij}(x) + R_{ji}(x)) \right],
\] (11)

where \(R_{ij}(x)\) is the rectangle operator in the \((i, j)\) plane with the long side in the \(i\) direction, given by:

\[
R_{ij}(x) = 1 - \frac{1}{N} \text{Re} \text{Tr} \left\{ \begin{array}{c} i \\ \hline j \end{array} \right\}.
\] (12)

The constants \(X\) and \(Y\) are to be fixed so that the order \(a^2\) errors vanish.

The rectangle operator can be expanded in powers of \(a\) using Stokes’ theorem:

\[
R_{ij}(x) = \frac{4g^2 a^4}{2N} \text{Tr} \left\{ F_{ij}(x) F_{ij}(x) \right\} + \frac{4g^2 a^6}{24N} \text{Tr} \left\{ F_{ij}(x) (4D_i^2 + D_j^2) F_{ij}(x) \right\} + \ldots
\] (13)

Substituting eqns (8) and (12) into eqn (11) we see that cancelling the order \(a^2\) error in \(V^{(1)}\) requires \(X = 5/3\) and \(Y = -1/6\). This leads to the improved potential term

\[
V^{(1)} = \frac{2N a g^2}{N} \sum_{x,i<j} \left[ \frac{5}{3} P_{ij}(x) - \frac{1}{12} (R_{ij}(x) + R_{ji}(x)) \right].
\] (14)

In principle, the next lowest order classical errors could be corrected by including additional, more complicated terms in the potential term. This has not been done because many additional diagrams are required to cancel the large number of order \(a^4\) error terms. Since these errors are swamped by order \(a^2 g^2\) quantum errors, addressing quantum corrections in the Hamiltonian approach would seem to be of more immediate importance.
3.3 Improving the Kinetic Term

Constructing a kinetic Hamiltonian with a finite number of terms has proven to be a nontrivial exercise. Luo, Guo, Kröger and Schütter demonstrated an interesting trade off when using either the transfer matrix or Legendre transformation methods to derive an improved Hamiltonian [9]. Both techniques require the starting point to be an improved action. When one starts from an improved action incorporating rectangular terms the resulting Hamiltonian has infinitely many terms and couples links which are arbitrarily far apart. To produce a Hamiltonian which couples only nearest neighbour links, it was found necessary to start from a carefully constructed highly non-local improved action.

Here we demonstrate an alternative approach, similar in nature to the Symanzik improvement of the Wilson action. One only needs to include additional gauge invariant terms with appropriate continuum behaviour in the kinetic Hamiltonian. The coefficients of the additional terms are chosen so that the order $a^2$ errors vanish.

An important step in understanding the errors that arise in the kinetic Kogut-Susskind Hamiltonian involves making the distinction between lattice and continuum fields. In §2 we defined the lattice gluon field on a link to be the average of the continuum gluon field along the link:

$$A^L_{\mu}(x) = \frac{1}{a} \int_{\text{Link}} dx \cdot A = A_{\mu}(x) + \frac{a^2}{24} \partial^2_{\mu} A_{\mu}(x) + \frac{a^4}{1920} \partial^4_{\mu} A_{\mu}(x) + \ldots \quad (15)$$

From this we can build the following sequence of approximations to the lattice gluon field:

$$A^{(0)}_i(x) = A_i(x)$$
$$A^{(1)}_i(x) = A_i(x) + \frac{1}{24} a^2 \partial^2_i A_i(x)$$
$$A^{(2)}_i(x) = A_i(x) + \frac{1}{24} a^2 \partial^2_i A_i(x) + \frac{1}{1920} a^4 \partial^4_i A_i(x)$$

$$\vdots$$

Perhaps the most important property of the electric field is that it generate group transformations. Mathematically, this translates to the electric and gluon fields satisfying the commutation relations,

$$[E^\alpha_i(x), A^\beta_j(y)] = -\frac{i}{a^2} \delta_{xy} \delta_{ij} \delta_{\alpha\beta}. \quad (17)$$

It is desirable for this hold on the lattice for any degree of approximation. Let us consider what happens to these commutation relations on the lattice for the approximation labelled by the superscript (1) in eqns [2]

$$[E^{(1)\alpha}_i(x), A^{(1)\beta}_j(y)] = [E^{(1)\alpha}_i(x), A^{\beta}_j(y) + \frac{a^2}{24} \partial^2_j A^{\beta}_j(y)]. \quad (18)$$

We observe that if the lattice electric field is taken to be the continuum electric field, order $a^2$ errors arise in the commutation relations. To cancel this error we set

$$E^{(1)\alpha}_i(x) = E^\alpha_i(x) - \frac{a^2}{24} \partial^2_i E^\alpha_i(x). \quad (19)$$
We can take this to order $a^4$ by setting
\[ E_i^{(2)\alpha}(x) = E_i^\alpha(x) - \frac{a^2}{24} \partial_i^2 E_i^\alpha(x) + \frac{7a^4}{5760} \partial_i^4 E_i^\alpha(x). \]  
\( \text{(20)} \)

In this way a sequence of approximations to the lattice electric field $E^L$ can be constructed.

Making use of these approximations we can analyse the classical errors arising in the kinetic Hamiltonian. To cancel these errors we take the approach of adding new terms and fixing their coefficients in order to cancel the order $a^2$ error. We have a great deal of freedom in choosing additional terms. They are restricted only by gauge invariance and the need for an appropriate continuum limit.

To understand the construction of gauge invariant kinetic terms it is important to recall that the electric field and link operator transform as follows under a local gauge transformation $\Lambda(x)$:
\[ E_i(x) \rightarrow \Lambda(x) E_i(x) \Lambda^\dagger(x) \]  
\( \text{(21)} \)
\[ U_i(x) \rightarrow \Lambda(x) U_i(x) \Lambda^\dagger(x + a). \]  
\( \text{(22)} \)

Consequently, the next most complicated gauge invariant term we can construct (after $\text{Tr} E^L E^L$) couples nearest neighbour electric fields:
\[ \text{Tr} \left\{ E_i(x) U_i(x) E_i(x + a) U_i^\dagger(x) \right\}. \]  
\( \text{(23)} \)

More complicated gauge invariant terms are easily constructed. One only needs to couple electric fields on different links anywhere around a closed loop. Consequently, generating Hamiltonians with higher degrees of improvement would be seem to be more readily achieved within this approach.

Incorporating nearest neighbour interactions leads to the simplest improved kinetic Hamiltonian:
\[ K^{(1)} = \frac{a^3}{2} \sum_{x,i} \text{Tr} \left\{ X E_i^L(x) E_i^L(x) + Y E_i^L(x) U_i(x) E_i^L(x + a) U_i^\dagger(x) \right\}. \]  
\( \text{(24)} \)

We fix the coefficients $X$ and $Y$ to cancel the order $a^2$ error. To do this we expand the second term in a Taylor series in $a$. Ignoring $O(g^2a^2)$ errors, we then substitute $E^L \approx E^{(1)}$ from eqn (13). To cancel the order $a^2$ error we must set $X = 5/6$ and $Y = 1/6$. This results in the order $a^2$ improved kinetic Hamiltonian
\[ K^{(1)} = \frac{a^3}{2} \sum_{x,i} \text{Tr} \left\{ \frac{5}{6} E_i^L(x) E_i^L(x) + \frac{1}{6} E_i^L(x) U_i(x) E_i^L(x + a) U_i^\dagger(x) \right\}. \]  
\( \text{(25)} \)

This is the result of Luo, Guo, Kröger and Schütter [9]. We can take the this to order $a^4$ by including next nearest neighbour interactions. A similar calculation using $E^L \approx E^{(2)}$ from eqn (20) gives
\[ K^{(2)} = \frac{a^3}{2} \sum_{x,i} \text{Tr} \left\{ \frac{97}{120} E_i^L(x) E_i^L(x) + \frac{1}{5} E_i^L(x) U_i(x) E_i^L(x + a) U_i^\dagger(x) \right. \]
\[ \left. - \frac{1}{120} E_i^L(x) U_i(x) U_i(x + a) E_i^L(x + 2a) U_i^\dagger(x + a) U_i^\dagger(x) \right\}. \]  
\( \text{(26)} \)
4 Tadpole Improvement

Tadpole improvement, developed by Lepage and Mackenzie [12], is an important step in removing errors from lattice gauge theory. It is necessary for close agreement between lattice perturbation theory and Monte Carlo calculations on coarse lattices.

In the action formulation tadpole improvement is handled by dividing all link operators by the mean link $u_0$. In the Hamiltonian approach two conflicting implementations have been suggested. The earliest starts from a tadpole improved action and carries factors of $u_0$ into the Hamiltonian [9]. More recently it was suggested that no tadpole improvement was necessary in the kinetic term of the improved Hamiltonian [13]. Here we present our own views on the correct implementation.

In the Hamiltonian approach the question of whether the electric field should be scaled arises. This question is easily answered by considering the commutation relations between the link operator and electric field:

$$[E^a_i(x), U_j(y)] = \frac{g}{a^2} \delta_{ij} \delta_{xy} \lambda^a U_i(x).$$  \hspace{1cm} (27)

We see that if we divide all link operators by $u_0$ we have

$$[E^a_i(x), \frac{1}{u_0} U_j(y)] = \frac{g}{a^2} \delta_{ij} \delta_{xy} \lambda^a \frac{1}{u_0} U_i(x).$$  \hspace{1cm} (28)

We observe that the electric field cannot be rescaled and still maintain the correct commutation relations. Thus under tadpole improvement the electric field cannot change. We must, however, divide the second of the kinetic terms by a factor of $u_0^2$. Tadpoles arise in this term because the electric and gluon fields do not commute.

Including tadpole improvement in eqns 14 and 25 leads to the order $a^2$ tadpole improved Hamiltonian:

$$H^{(1)} = K^{(1)} + V^{(1)}$$

$$= \frac{a^3}{2} \sum_{x,i} \text{Tr} \left\{ \frac{5}{6} E^L_i(x) E^L_i(x) + \frac{1}{6u_0^2} E^L_i(x) U_i(x) E^L_i(x + ai) U^\dagger_i(x) \right\}$$

$$- \frac{2N}{ag^2} \sum_{x,i<j} \left[ \frac{5}{3u_0^4} P_{ij}(x) - \frac{1}{12u_0^6} (R_{ij}(x) + R_{ji}(x)) \right].$$  \hspace{1cm} (29)

5 Conclusion

We have demonstrated that direct improvement of the Kogut-Susskind Hamiltonian by demanding the correct continuum limit is possible. The advantage of our direct approach is that it is easily extended to more complicated Hamiltonians. One simply needs to construct suitable gauge invariant terms to add to the kinetic Hamiltonian and fix the coefficients so that higher order errors are cancelled.

Our next step is to perform variational and coupled cluster SU(3) calculations to determine precisely the level of improvement achieved by the improved Hamiltonians. Other groups
have made progress with these calculations for U(1) \cite{13} and SU(2) \cite{14} with promising results.

In the near future we intend to extend the direct approach to the cancellation of quantum errors which have not yet been examined in Hamiltonian lattice gauge theory.

References

[1] K. G. Wilson, Phys. Rev. D 10, 2445 (1974).
[2] J. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975).
[3] M. Creutz, *Quarks, Gluons and Lattices* (Cambridge University Press, Cambridge, 1985).
[4] J. B. Kogut, Rev. Mod. Phys. 55, 775 (1983).
[5] G. P. Lepage, hep-lat/9607076.
[6] K. Symanzik, Nucl. Phys. B 226, 187 (1983).
[7] B. H. J. McKellar, C. R. Leonard and L. C. Hollenberg, Int. J. Mod. Phys. B 14, 2023 (2000).
[8] E. B. Gregory, S. H. Guo, H. Kroger and X. Q. Luo, Phys. Rev. D 62, 054508 (2000) hep-lat/9912054.
[9] X. Q. Luo, S. H. Guo, H. Kroger and D. Schutte, Phys. Rev. D 59, 034503 (1999) hep-lat/9804029.
[10] M. Lüscher and P. Weisz, Phys. Lett. B 158, 250 (1985).
[11] B. H. McKellar and J. Carlsson, Nucl. Phys. Proc. Suppl. 90, 185 (2000).
[12] G. P. Lepage and P. B. Mackenzie, Phys. Rev. D 48, 2250 (1993) hep-lat/9209022.
[13] X. Fang, S. Guo and J. Liu, Mod. Phys. Lett. A 15, 737 (2000).
[14] J. Li, S. Guo and X. Luo, Commun. Theor. Phys. 34, 301 (2000).