Tachyonic perturbations in AdS\_5 orbifolds

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We show that scalar as well as vector and tensor metric perturbations in the Randall-Sundrum II braneworld allow normalizable tachyonic modes, i.e., possible instabilities. These instabilities require nonvanishing initial anisotropic stresses on the brane. We show with a specific example that within the Randall-Sundrum II model, even though the tachyonic modes are excited, no instability develops. We argue, however, that in the cosmological context instabilities might in principle be present. We conjecture that the tachyonic modes are due to the singularity of the orbifold construction. We illustrate this with a simple but explicit toy model.

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I. INTRODUCTION

Already at the beginning of the last century, the idea that our universe may have more than three spatial dimensions has been explored by Nordström [1], Kaluza [2] and Klein [3]. Since superstring theory, the most promising candidate for a theory of quantum gravity, is consistent only in ten space-time dimensions (11 dimensions for M-theory) these ideas have been revived in recent years [4–6]. It has also been found that string theories naturally predict lower dimensional “branes” to which fermions and gauge particles are confined, while gravitons (and the dilaton) propagate in the bulk [7–9].

Recently it has been emphasized that relatively large extra-dimensions (with typical length \( L \approx \mu m \)) can “solve” the hierarchy problem: The effective four-dimensional Newton constant given by \( G_4 \approx G/L^n \) can become very small even if the fundamental gravitational constant \( G \approx m_{\text{ew}}^{-(2+n)} \) is of the order of the electroweak scale [10–13]. Here \( n \) denotes the number of extra dimensions. It has also been shown that extra dimensions may even be infinite if the geometry contains a so-called “warp factor”. An especially attractive model of this type, where the bulk is a five-dimensional anti-de Sitter (AdS\_5) space has been developed by Randall and Sundrum [14]. This is the model which we discuss in this work; we shall call it RSII in what follows.

The size of the extra dimensions is constrained by the requirement of recovering usual four-dimensional Newton’s law on the brane, at least on scales tested by experiments [12–14].

Models with finite extra dimensions always have to invoke some nongravitational interaction in order to stabilize the graviscalar (which is equivalent to the radion) [15]. However, in the case of noncompact warped extra dimensions, it can happen that this mode is not normalizable and therefore cannot be excited. This is precisely what happens in the RSII model.

Therefore, there is justified hope that, for suitable parameters, this model can reproduce four-dimensional gravity without invoking ad hoc additional interactions. However, we show in this paper that the gravitational sector coupled to a brane with nonvanishing anisotropic stresses does have negative mass modes. We argue that, on the linearized level, these instabilities are not relevant for the Randall-Sundrum model, but they may be devastating in the cosmological context where the brane is moving.

The tachyonic modes are absent if there are no anisotropic stresses. Furthermore, if anisotropic stresses remain small, they cannot develop an instability. As we shall show, this is the case for the RSII model since there, to first order, anisotropic stresses evolve like in Minkowski space-time and hence remain small (if their Minkowski evolution is not already unstable). In the cosmological context, however, this is no longer true and large deviations from homogeneity and isotropy may in principle develop.

The outline of the paper is as follows: In the next section, the perturbation theory on RSII is briefly introduced and the relevant perturbation equations are given. We present the solutions to the bulk perturbation equations and the junction conditions for tensor, vector and scalar modes. We pay particular attention to the tachyonic modes which are new and represent a possible instability. In Section II we discuss the simple case of free-streaming, relativistic particles and show that they induce negative mass modes. We explicitly solve the equations for the RSII background and see that no instability is induced in this case. We then argue that, in principle, this behavior may change in a cosmological setting.

In Sec. IV, we present a simple 3 + 1 dimensional Minkowski orbifold who’s bulk modes exhibit the same instability as the AdS\_5 orbifold. We explicitly reconstruct the instability from the retarded Green’s function, showing that it is causal. In this toy model, instabilities develop due to nonlinear couplings. A final section is devoted to some conclusions.
II. PERTURBATIONS OF THE RSII MODEL

Our universe is considered to be a 3-brane embedded in five-dimensional anti-de Sitter space-time,

\[ ds^2 = g_{AB} dx^A dx^B = \frac{L^2}{y^2} [-dt^2 + \delta_{ij} dx^i dx^j + dy^2] \]  

(1)

Capital Latin indices \( A, B \) run from 0 to 4 and lower case Latin indices \( i, j \) from 1 to 3. Four-dimensional indices running from 0 to 3 will be denoted by lower case Greek letters. Anti-de Sitter space-time is a solution of Einstein’s equations with a negative cosmological constant \( \Lambda \),

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \]  

(2)

The curvature radius \( L \) is given by

\[ L^2 = -\frac{6}{\Lambda} \]  

(3)

In braneworld models one often uses Gaussian normal coordinates. For anti-de Sitter space time given by the transformation \( L^2/y^2 = \exp(-2\rho/L) \). The metric then takes the form

\[ ds^2 = g_{AB} dx^A dx^B = e^{-2\rho/L} \left(-dt^2 + \delta_{ij} dx^i dx^j + d\rho^2 \right) \]  

(4)

We now introduce one single brane at \( y = y_b = L \) (or equivalently \( \rho = 0 \)) and replace the “left hand side”, \( 0 < y < L \), of AdS by a second copy of the “right hand side”. We use the superscripts \( " > " \) and \( " < " \) for the bulk sides with \( y > y_b \) and \( y < y_b \), respectively. In terms of the coordinate \( y \), the value of \( y \) decreases continuously from \( \infty \) to \( L \) and then jumps to \( -L \) over the brane whereafter it continues to decrease. At the brane position, \( y_b^\nu = L, y_b^\nu = -L \), the metric function \( (L/y)^2 \) has a kink. The advantage of the coordinate \( \rho \) introduced in Eq. (4) is that the variable \( \rho \) does not jump, but the metric function in the presence of a brane becomes \( \exp(-2|\rho|/L) \).

The Einstein equations at the brane position are simply

\[ K_{\mu\nu}^\gamma - K_{\mu\nu}^\gamma = \kappa_5 \left(S_{\mu\nu} - \frac{1}{3} S q_{\mu\nu} \right) \equiv \kappa_5 \tilde{S}_{\mu\nu} \]  

(5)

where \( S_{\mu\nu} \) is the energy-momentum tensor on the brane with trace \( S \), and

\[ \kappa_5 = 6\pi^2 G_5 = \frac{1}{M_5^3} \]  

(6)

\( M_5 \) and \( G_5 \) are the five-dimensional (fundamental) reduced Planck mass and Newton constant, respectively. \( K_{\mu\nu} \) is the extrinsic curvature of the brane and \( q_{\mu\nu} \) is the induced metric on the brane. Equation (5) is usually referred to as the second junction condition. The first junction condition simply states that the induced metric, the first fundamental form,

\[ q_{\mu\nu} = e_\mu^A e_\nu^B g_{AB} \]  

(7)

be continuous across the brane. Here the vectors \( e_\nu \) are tangent to the brane. In more detail, if we parametrize the brane by coordinates \( (z^\nu) \) and its position in the bulk is given by functions \( X^A_B(z^\nu) \), the vectors \( e_\nu \) are defined by

\[ e_\mu^\nu = \partial_\mu X^A_B(z) \]  

(8)

Denoting the brane normal by \( n \), we have \( g_{AB} e_\mu^A e_\nu^B = 0 \). The extrinsic curvature can be expressed purely in terms of the internal brane coordinates \( \vec{y} \), \( K = K_{\mu\nu} dz^\nu dz^\mu \), with

\[ K_{\mu\nu} = -\frac{1}{2} \left[g_{AB} e_\mu^A e_\nu^B n^A + e_\nu^A e_\mu^i n^i + e_\mu^A e_\nu^B n^c g_{AB,c} \right] \]  

(9)

In the case we are interested in, the background space-time consists of two copies of the part of AdS, with \( |y| \geq |y_b| = L \). We actually let the coordinate \( y \) jump from \( y = L \) to \( y = -L \) across the brane. Since the metric is symmetric in \( y \), the first junction condition is trivially fulfilled. The second fundamental form is proportional to the induced metric, \( K_{\mu\nu} = \pm L^{-1} q_{\mu\nu} \), hence the energy-momentum tensor on the brane is a pure brane tension \( T \), \( S_{\mu\nu} = -T q_{\mu\nu} \). With

\[ K_{\mu\nu}^{\gamma} - K_{\mu\nu}^{\gamma} = [K_{\mu\nu}] = 2K_{\mu\nu}^{\gamma} \]  

(10)

the second junction condition becomes

\[ [K_{\mu0}]|_{\nu = 0} = \frac{-2}{L} = -\frac{1}{3}\kappa_5 \tilde{T} \]  

(11)

\[ [K_{\nu\nu}]|_{\nu = 0} = \frac{2}{L} = \frac{1}{3}\kappa_5 \tilde{T} \]  

(12)

This leads to the well-known RS-fine-tuning condition,

\[ -\Lambda = \frac{6}{L^2} = \frac{1}{6\kappa_5^2 \tilde{T}^2} \]  

(13)

The most general perturbation of the AdS metric \( g_{AB} \) is of the form

\[ ds^2 = g_{AB} dx^A dx^B \]

\[ = \frac{L^2}{y^2} \left[-(1 + 2\Psi)dt^2 - 4\Sigma_i dt dx^i - 4B dt dy + (1 + 2\Psi)dx^i dx^j + 4\Xi_i dx^i dy + (1 + 2\Psi)dy^2 \right] \]  

(14)

Here \( \Sigma_i \) and \( \Xi_i \) are divergenceless vectors and \( H_{ij} \) is a divergenceless, traceless tensor. It is easy to show that there exists one fully specified gauge in which the perturbation variables take this form; vectors have no “scalar component” and tensors have neither a vector nor
We call this the generalized longitudinal gauge (see also [24, 27]). We shall use it in the following. Within linear perturbation theory, the variables with different spin, the tensor $H_{ij}$, the vectors $\Sigma_i$, and $\Xi_i$, and the scalars $\Psi, \Phi, B, C$ do not couple. We can therefore study the perturbations of each type separately. We shall do so in the next subsections. There we write down the perturbed Einstein equations for a fixed Fourier-mode $k$ for which we have $k \cdot \Sigma = k \cdot \Xi = k^i H_{ij} = 0$. We do not perform a Fourier decomposition in time.

We want to study the perturbations in an empty bulk with possible perturbations on the brane. The five-dimensional Einstein equation implies the perturbation equations in the bulk,

$$\delta G_{AB} = -\Lambda \delta g_{AB} ,$$  \hspace{1cm} (15)

and the junction conditions at the brane,

$$2 \delta K_{\mu\nu} = \kappa_5 \delta \tilde{S}_{\mu\nu} .$$  \hspace{1cm} (16)

We first discuss tensor perturbations. As we shall see, the homogeneous equations reduce to the same Bessel equations for all three types of perturbations (see also [24]).

A. Tensor perturbations

In this paragraph we first discuss the simplest case, the tensor perturbation equations. We write them down for a fixed Fourier-mode $k$ and determine their solutions. We consider only $H_{ij} \neq 0$. For this case, Eq. (15) reduces to

$$\left( \partial_t^2 + k^2 - \partial_y^2 + \frac{3}{y} \partial_y \right) H_{ij} = 0 .$$  \hspace{1cm} (17)

For a given polarization, $H_\star = H_\perp$ or $H_\star = H_\times$, we make the ansatz $H_\star = f(t)g(y)$ leading to

$$\frac{\partial_t^2 f}{f} + k^2 = \frac{\left( \partial_y^2 - \frac{3}{y} \partial_y \right) g}{g} = Z ,$$  \hspace{1cm} (18)

where $Z$ is an arbitrary separation constant. The behavior of the solutions to these equations depends strongly on the sign of $Z$. If $Z = -m^2$ is negative, we obtain

$$f = \exp(\pm i t \sqrt{m^2 + k^2}) \equiv \exp(\pm i \omega t) ,$$  \hspace{1cm} (19)

$$g = N(my)^2 \times \left\{ \begin{array}{l} J_2(my) , \\
Y_2(my) . \end{array} \right.$$  \hspace{1cm} (20)

Here $J_\nu$ and $Y_\nu$ denote the Bessel functions of order $\nu$. They are oscillating and decaying. They are “δ-function normalizable” perturbations like harmonic waves in flat space, in the sense that $H_m = f g$ satisfies [28, 29]

$$\int_0^\infty H_m H_m' \frac{dy}{m^2 y^3} \propto m \delta(m - m') .$$  \hspace{1cm} (21)

These are just the ordinary gravity modes of four-dimensional mass $m$ without a mass gap which are discussed in the original RS paper [14]. However, if $Z = -m^2$ is positive, the solutions take the form

$$f = \exp(\pm t \sqrt{Z - k^2}) = \exp(\pm \omega t) ,$$  \hspace{1cm} (22)

$$g = N(|m|y)^2 \times \left\{ \begin{array}{l} K_2(|m|y) , \\
I_2(|m|y) . \end{array} \right.$$  \hspace{1cm} (23)

Here $K_\nu$ and $I_\nu$ are the modified Bessel functions of order $\nu$. The second case $g \propto I_2$ grows exponentially in $y$. This is not normalizable and therefore cannot represent a physical, small perturbation. However, the mode $K_2$ decays exponentially and is normalizable and small for sufficiently small initial amplitudes. However, even with arbitrary small initial data this mode grows exponentially in time for sufficiently small wave numbers, $k^2 < -m^2$; it is a tachyonic instability.

To have a complete solution to the perturbation equations we need to discuss the boundary conditions at the brane, i.e., the junction conditions.

A short computation shows that the nonvanishing components of the extrinsic curvature tensor perturbations are in our case

$$\delta K_{ij} \big|_{y_0} = \left( \frac{2}{L} H_{ij} - \partial_y H_{ij} \right) \big|_{y_0} , \quad \text{hence}$$  \hspace{1cm} (24)

$$-2(\partial_y H_{ij}) \big|_{y_0} = \kappa_5 \Pi_{ij}^{(T)} ,$$  \hspace{1cm} (25)

where $\Pi^{(T)}$ are tensor-type anisotropic stresses on the brane.

Let us first consider the homogeneous case $\Pi^{(T)} = 0$. For $m^2 > 0$, the solutions are of the form

$$H = \exp(\pm i \omega t)(my)^2 \left[ AJ_2(my) + BY_2(my) \right] .$$  \hspace{1cm} (25)

The junction condition (24) then requires

$$B = -A \frac{J_1(ml)}{Y_1(ml)} \approx \frac{\pi}{4}(ml)^2 A ,$$  \hspace{1cm} (26)

where the last expression is a good approximation for $mL \ll 1$. This is precisely the result of Randall and Sundrum [14]. This is not modified even if we allow for the negative mass modes, $Z = -m^2 > 0$, because a physical solution has to be of the form

$$H = C \exp(\pm t \sqrt{Z - k^2})(|m|y)^2 K_2(|m|y) ,$$  \hspace{1cm} (27)

and since $K_1$ has no zero, the junction condition (24) requires $C = 0$.

But in a realistic brane universe, $\Pi^{(T)}$ is not exactly zero. In cosmology, it is just typically a factor of 10 smaller than other perturbations of the energy-momentum tensor on the brane. We therefore can not set $C \equiv 0$. However, as long as $\Pi^{(T)}$ remains small, we do not expect the unstable modes to be present; hence we expect $C(k, m) = 0$ for $k^2 < -m^2$. In the next section,
we shall show in a specific example that this is indeed the case in RSII, where the brane is Minkowski space and to linear order the anisotropic stress follows its background equation of motion. In the cosmological context, however, the evolution of $\Pi^{(T)}$ contains $H$ which then can in principle feed the instability.

One may ask, whether the unstable $K$ mode is a consequence of the thin brane limit. However, it is clear that in a thick brane, the kink of the $K$ mode at the brane position will simply be replaced by a rapid but gradual contraction and not the limit of an infinitely thin brane. However, due to its negative mass square, the bulk, but moves along the brane. However, due to its instability once non–linear effects are taken into account. In that sense it is not clear that the RSII model is safe from this problem of vector perturbations has been considered in Ref. [30].

The constraint equation Eq. (30) fixes the relative amplitudes of $\Sigma$ and $\Xi$, showing that there is only one independent vector perturbation in the bulk (the “graviphoton”). One can check that these equations are consistent, e.g., with the master function approach of Ref. [31].

As in the tensor case, one can solve the equations with a separation ansatz. For a negative separation constant, $Z = -m^2 < 0$, one obtains the expected oscillatory modes,

$$\Sigma = \exp(\pm i\omega t)(my)^2 [AJ_2(my) + BY_2(my)] ,$$

$$\Xi = \pm \frac{i\omega}{m} \exp(\pm i\omega t)(my)^2 [AJ_1(my) + BY_1(my)] ,$$

where $\omega = \sqrt{m^2 + k^2}$. These solutions have been found in Ref. [32]. For a positive separation constant $Z = -m^2 > 0$, we obtain again tachyonic solutions. Like in the tensor case, the solution containing the modified Bessel function $I_0$ cannot be accepted as it is exponentially growing and thus represents a non-normalizable perturbation. However, the $K_z$-solution is exponentially decaying and therefore perfectly acceptable. For tachyonic vector perturbations with $\omega = \sqrt{Z - k^2}$ we have

$$\Sigma = C \exp(\pm i\omega t)(|m|y)^2 K_2(|m|y) ,$$

$$\Xi = \pm \omega |m| C \exp(\pm i\omega t)(|m|y)^2 K_1(|m|y) .$$

For large enough scales, $-m^2 > k^2$, these solutions can grow exponentially.

Again, the boundary conditions at the brane relate the perturbations to the brane energy-momentum tensor. For the energy-momentum tensor on the brane, the vector degrees of freedom are defined according to

$$S_{\mu\nu} = \left( \begin{array}{cc} 0 & 2V_i \\ 2V_i & 4\Pi_{ij}^{(V)} \end{array} \right) - \mathcal{T} q_{\mu\nu} ,$$

where $V_i$ and $\Pi_{ij}^{(V)}$ are divergence-free vectors fields and $\Pi_{ij}^{(V)} \equiv \frac{1}{2}(\partial_i \Pi_j^{(V)} + \partial_j \Pi_i^{(V)})$. The first junction condition simply requires that $\Sigma$ be continuous at the brane, which it is since the (modified) Bessel functions of even index are even functions. The second junction condition results in (for a detailed derivation, see [33])

$$\left. (\partial_t \Xi + \partial_y \Sigma) \right|_{y=b} = \kappa_v V ,$$

$$\Xi \right|_{y=b} = \kappa_v \Pi^{(V)} ,$$

$$\partial_t V = -k^2 \Pi^{(V)} .$$

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1 Since the new “K modes” are not free, they have to vanish if anisotropic stresses are absent, they are not modes in the usual sense of the term: free solutions to some linear hyperbolic equation. We shall nevertheless use this term here, committing a slight abuse of language.
The last equation follows from (38) and (17) and the bulk equations of motion (28)–(30). It represents momentum conservation on the brane, which is guaranteed as long we have vanishing energy flux off the brane and $Z_2$-symmetry.

Like for tensor perturbations, we first consider homogeneous solutions, setting $\Pi^{(V)} \equiv V \equiv 0$. This requires $\Xi((mL) = 0$, hence
\begin{align*}
B &= -A \frac{J_1(mL)}{Y_1(mL)} \quad \text{for } m^2 > 0, \quad (39) \\
C &\equiv 0 \quad \text{for } m^2 < 0. \quad (40)
\end{align*}
Equation (39) is then identically satisfied.

However, it seems more realistic to allow for a small but nonvanishing anisotropic stress contribution $\Pi^{(V)}$ and corresponding vorticity $V$. In this case, again, solutions with $C \neq 0$ may exist, and small initial data could lead to exponential growth like for tensor perturbations.

Using the normalization condition (21) for the $m = 0$ mode of the variable $\Xi \propto y$ (this is the one which enters as dynamical variable in the perturbed action; see (23)), one finds that contrary to the tensor case, the vector zero-mode is not normalizable. Therefore, on the brane there is only the ordinary massless spin-2 graviton, but there are an infinity of massive spin-2 and spin-1 particles (the modes discussed here, with $m \neq 0$).

\[ \frac{ds^2}{y^2} = -(1 + 2\Phi)dt^2 - 4\mathcal{B}dt\,dy + (1 - 2\Phi)\delta_{ij}dx^i\,dx^j + (1 + 2\mathcal{C})dy^2. \] (41)

The bulk Einstein perturbation equations for the mode $k$ become, after some manipulations and introducing the combination $\Gamma \equiv \Phi + \Psi$,
\begin{align*}
\Phi - \Psi &= \mathcal{C}, \quad (42) \\
\left( \partial_y^2 - \frac{3}{y} \partial_y \right) \Gamma &= \left( \partial_y^2 + k^2 \right) \Gamma, \quad (43) \\
\partial_y \Phi + \left( \partial_y - \frac{3}{y} \right) \mathcal{C} &= -\partial_y \mathcal{B}, \quad (45) \\
\frac{3}{y} \left( \partial_y - \frac{2}{y} \right) \mathcal{C} &= 3\partial_y^2 \Phi + k^2(\Phi + \mathcal{C}), \quad (46) \\
3\partial_y \left( \partial_y \Phi - \frac{C}{y} \right) &= k^2 \mathcal{B}, \quad (47) \\
\partial_y (2\Phi - \mathcal{C}) &= \left( \partial_y - \frac{3}{y} \right) \mathcal{B}. \quad (48)
\end{align*}

Clearly these equations are not all independent; Eqs. (17) and (18) are identically satisfied if Eqs. (13)–(14) are. The solutions are obtained as for tensor and vector perturbations. For a negative separation constant $Z = -m^2 < 0$, we obtain ($\omega = \sqrt{m^2 + k^2}$)
\begin{align*}
\Gamma &= \exp(\pm i\omega t)(my)^2 [A J_2(my) + B Y_2(my)], \quad (49) \\
\Phi &= \frac{1}{2} \exp(\pm i\omega t)(my)^2 [A J_2(my) + B Y_2(my)] \\
&\quad + AJ_0(my) + BY_0(my), \quad (50) \\
\Psi &= \frac{1}{2} \exp(\pm i\omega t)(my)^2 [A J_2(my) + B Y_2(my)] \\
&\quad - AJ_0(my) - BY_0(my), \quad (51) \\
\mathcal{B} &= \frac{\pm i m^3 y^2}{2\omega} \exp(\pm i\omega t) \left[ (A' - 3A) J_1(my) + (B' - 3B) Y_1(my) \right], \quad (52) \\
A' &= 3A - \frac{m^2}{m^2 + 2\omega^2}, \quad B' = 3B - \frac{m^2}{m^2 + 2\omega^2}. \quad (54)
\end{align*}

For a positive separation constant, $Z = -m^2 > 0$, we find ($\omega = \sqrt{Z - k^2}$)
\begin{align*}
\Gamma &= \exp(\pm i\omega t)(|m|y)^2 C' K_2(|m|y), \quad (55) \\
\Phi &= \frac{1}{2} \exp(\pm i\omega t)(|m|y)^2 C' K_2(|m|y) + C K_0(|m|y), \quad (56) \\
\Psi &= \frac{1}{2} \exp(\pm i\omega t)(|m|y)^2 [C' K_2(|m|y) - C K_0(|m|y)], \quad (57) \\
\mathcal{B} &= \frac{\pm |m|^3 y^2}{2\omega} \exp(\pm i\omega t) [C' + 3C'] K_1(|m|y), \quad (58) \\
C' &= -3C' - \frac{|m|^2}{|m|^2 + 2\omega^2}, \quad (60)
\end{align*}
where we have already used that the $I$ mode is not normalizable and therefore cannot contribute. Like for vector and tensor perturbations, we find again tachyonic solutions with $m^2 < 0$ which represent an exponential instability for sufficiently small wave number $k$ (large scales).

Determining the boundary conditions via the first and second junction conditions now requires a bit more care. Since we have already fully specified our coordinate system by the adopted choice of perturbation variables, we must allow for brane bending. We cannot fix the brane at $y_b = L$, but we must allow for $y_b' = L + \mathcal{E}$ and $y_b'' = -L - \mathcal{E}$, respectively. The antisymmetry $y_b'' = -y_b'$ is an expression of $Z_2$ symmetry. The introduction of the new perturbation variable $\mathcal{E}(z^\mu)$ describing brane bending affects the first and second fundamental forms. From Eq. (4), we obtain $g_{\mu\nu} = g_{\mu\nu}$ to first order, which implies that $\Phi$ and $\Psi$, hence $\mathcal{C}$, have to be continuous. At the brane position, the perturbed components of the extrin-
The second junction condition reads
\[ \delta K_{ij} = \left[ \frac{1}{L} \left( \Phi - 3\Psi + 2\xi \right) + \partial_x \Psi - 2\partial_b B + \partial_y \xi \right] \delta_{ij} + \partial_y \partial_x \mathcal{E} . \tag{63} \]

For the energy-momentum tensor on the brane, we parametrize the 4 degrees of freedom according to
\[ S_{\mu\nu} = \left( \frac{\rho}{v_i} P \delta_{ij} + \Pi^{(S)}_{ij} \right) \delta_{\mu\nu} - \mathcal{T} q_{\mu\nu} , \tag{64} \]
where \( v_i = \partial_i v \) and \( \Pi^{(S)}_{ij} \equiv \left( \partial_i \partial_j - \frac{1}{3} \delta_{ij} \right) \Pi^{(S)} \). With Eqs. (61)–(63), the second junction condition reads
\[ \frac{1}{T} \left( 2\rho + 3P \right) = \left[ \Phi - \Psi + L \partial_t \left( \partial_t \mathcal{E} - 2B \right) + L \partial_y \xi \right] v_i , \tag{65} \]
\[ \frac{3}{T} \mathcal{L} v = \left[ \partial_t \mathcal{E} - B \right] v_i , \tag{66} \]
\[ \frac{3}{T} \mathcal{L} \Pi^{(S)} = \mathcal{E} , \tag{67} \]
\[ \frac{1}{T} \left[ \rho - \Delta \Pi^{(S)} \right] = \left[ \Psi - \Phi + L \partial_y \xi \right] v_i . \tag{68} \]
Combining the time derivative of Eq. (64) with Eqs. (65), (66) and (68), we obtain momentum conservation on the brane,
\[ \partial_t \rho = \frac{2}{3} \Delta \Pi^{(S)} + P . \tag{69} \]

Similar manipulations imply energy conservation on the brane,
\[ \partial_t \rho = \Delta v . \tag{70} \]

Like for tensor and vector perturbations, we first look for solutions with vanishing brane matter. Setting \( \Pi^{(S)} \equiv \rho \equiv P \equiv v \equiv 0 \) forbids brane bending, \( \mathcal{E} = 0 \). Then Eq. (64) implies \( B(mL) = 0 \), thus
\[ B' - 3B = -(A' - 3A) \frac{J_1(mL)}{Y_1(mL)} \quad \text{for } m^2 > 0 , \tag{71} \]
\[ C' + 3C = 0 \quad \text{for } m^2 < 0 . \tag{72} \]
The other equations are satisfied if we require separately
\[ \frac{B}{A} = \frac{B'}{A'} = \frac{J_1(mL)}{Y_1(mL)} \quad \text{for } m^2 > 0 , \tag{73} \]
\[ C \equiv 0 \quad \text{for } m^2 < 0 . \tag{74} \]
Equations (61) and (62) are of course equivalent.

As for vector perturbations, the \( m = 0 \) scalar mode is not normalize. Like for tensor and vector perturbations, we have found “scalar gravitons” which appear on the brane as massive particles. If the brane matter is unperturbed, only oscillating \( m^2 > 0 \) solutions are possible. However, if we allow for nonvanishing matter perturbations on the brane, we can have \( C \neq 0 \) and the tachyonic modes \( m^2 < 0 \) can appear exactly like in the tensor and vector sectors.

It is not surprising that the same instability appears in the scalar, vector and tensor sectors, because all modes describe the same bulk particle, the five-dimensional graviton.

\section{III. TACHYONIC MODES FROM FREELY PROPAGATING RELATIVISTIC PARTICLES}

\subsection{A. The Liouville equation in 4 dimensions}
As long as \( Z_2 \) symmetry is satisfied, we expect free (collisionless) particles on the brane to move along brane geodesics. Their one-particle distribution function therefore obeys the Liouville equation. Here we sketch a derivation of perturbation to the 4D Liouville equation. Many more details can be found, e.g., in \cite{13} or \cite{15}.

An ensemble of freely propagating particles on the brane is described by the Liouville equation,
\[ \left[ p^{\mu} \partial_{\mu} - \Gamma^i_{\alpha \beta} p^\alpha p^\beta \frac{\partial}{\partial p^i} \right] f = L X_s f = 0 . \tag{75} \]
Here \( f \) is the one-particle distribution function defined on the mass shell,
\[ P_M \equiv \left\{ (x^\mu, p^\nu) \mid g_{\mu\nu}(x) p^\mu p^\nu = -M^2 \right\} , \tag{76} \]
which we parametrize by the \( 7 \) coordinates \( (x^\mu, p^\nu) \). The energy \( p^0 \) is then determined via the mass-shell condition. The energy-momentum tensor of the particles is given by
\[ T^{\mu\nu} = \int d^3 p \sqrt{g} \frac{p^\mu p^\nu}{p^0} f . \tag{77} \]
To maximize anisotropic stresses, we consider relativistic particles, \( M^2 = 0 \). In this case we obtain the simple relation
\[ \rho = -T^0_0 = 3P = T^i_i , \tag{78} \]
where \( \rho \) is the energy density and \( P \) the “pressure” of the collisionless particles. With respect to an orthonormal frame we parametrize the particle momentum \( p^i = pm^i \) and define the “brightness”
\[ M = \frac{4\pi a^4}{3} \int dp p^3 f . \tag{79} \]
Here \( a \) is the cosmic scale factor which we can simply set to \( a = 1 \) in the case of a nonexpanding background. The components of the energy-momentum tensor are then
\[ T^{\mu\nu} = \int d^3 p \sqrt{g} \frac{p^\mu p^\nu}{p^0} f . \tag{77} \]
given by integrals over the momentum directions \( \mathbf{n} \). The anisotropic stress is

\[
\Pi_{ij} = \frac{1}{4\pi a^2} \int d\Omega \left( n_i n_j - \frac{1}{3} \delta_{ij} \right) \mathcal{M} . \tag{80}
\]

Let us first consider tensor perturbations of the metric. Then, the Liouville equation \((75)\) for the spatial Fourier transform of \( \mathcal{M} \), becomes, to first order in the metric perturbations, \[34\]

\[
\dot{\mathcal{M}}^{(T)} + i k \mu \mathcal{M}^{(T)} = -4a^4 P n^i n^j \dot{H}_{ij} , \tag{81}
\]

where \( \mu = \mathbf{k} \cdot \mathbf{n} \) is the direction cosine between \( \mathbf{k} \) and \( \mathbf{n} \). The overdot denotes the derivative with respect to conformal time on the brane, \( \eta \). We now choose the coordinate system so that \( \mathbf{k} \) is in 3-direction and \((\theta, \varphi)\) denote the usual polar angles with \( \mu = \cos(\theta) \). In order to represent a pure tensor perturbation, \( \mathcal{M}^{(T)} \) must be of the form

\[
\mathcal{M}^{(T)}(\mathbf{k}, \mathbf{n}, \eta) = (1 - \mu^2) \left[ \mathcal{M}_0(k, \mu, \eta) \cos(2\varphi) + \mathcal{M}_x(k, \mu, \eta) \sin(2\varphi) \right] . \tag{82}
\]

Using this ansatz, the two modes of \( H_{ij} \) decouple and Eq. \((81)\) reduces to

\[
\dot{\mathcal{M}}_* + i k \mu \mathcal{M}_* = -4a^4 P n^i n^j \dot{H}_{ij} , \tag{83}
\]

where “\( \cdot \)” stands for “\( \times \)” or “\( + \)”. Decomposing the tensor-type anisotropic stress into the two standard helicities \( + \) and \( \times \), we obtain

\[
\Pi^{(T)}_* = \frac{3}{8a^4} \int_{-1}^{1} d\mu \left( 1 - \mu^2 \right)^2 \mathcal{M}_*(k, \mu, \eta) . \tag{84}
\]

For vector perturbations one finds, correspondingly

\[
\dot{\mathcal{M}}^{(V)} + i k \mu \mathcal{M}^{(V)} = -4a^4 P i k \mu (\mathbf{n} \cdot \Sigma) , \tag{85}
\]

where \( \Sigma_i \) is the perturbation of \( q_0_i \) [corresponding to our vector perturbation in Eq. \((83)\) at fixed \( y \)]. Using the coordinate system where \( \mathbf{k} \) points in the third axis, \( \mathcal{M}^{(V)} \) must be of the following form to represent a pure vector perturbation,

\[
\mathcal{M}^{(V)}(\mathbf{k}, \mathbf{n}, \eta) = \sqrt{1 - \mu^2} \left[ \mathcal{M}_0(k, \mu, \eta) \cos(\varphi) + \mathcal{M}_x(k, \mu, \eta) \sin(\varphi) \right] . \tag{86}
\]

Again, with this ansatz the equations for the two helicities decouple into

\[
\dot{\mathcal{M}}_* + i k \mu \mathcal{M}_* = -4a^4 P i k \mu \Sigma_*, \tag{87}
\]

and the components of the anisotropic stress potential \( \Pi^{(V)}_j \) are given by

\[
\Pi^{(V)}_j = \frac{-i}{2a^4} \int_{-1}^{1} d\mu \left( 1 - \mu^2 \right) \mu \mathcal{M}_*(k, \mu, \eta) . \tag{88}
\]

Finally, for scalar perturbations of the metric, which are given by the Bardeen potentials, \( \Psi_b \) and \( \Phi_b \), which are the longitudinal perturbations of the induced metric \( g_{\mu \nu} \) on the brane, one finds

\[
\dot{\mathcal{M}}^{(S)} + i k \mu \mathcal{M}^{(S)} = -4a^4 P i k \mu (\Psi_b + \Phi_b) . \tag{89}
\]

To represent a pure scalar mode, \( \mathcal{M}^{(S)} \) must be independent of the polar angle \( \varphi \). The anisotropic stress potential \( \Pi^{(S)} \) is thus given by

\[
\Pi^{(S)} = \frac{1}{4a^4} \int_{-1}^{1} d\mu \left( 1 - \mu^2 \right) \mathcal{M}^{(S)}(k, \mu, \eta) . \tag{90}
\]

### B. Solution in the RSII model

In the RSII model, the brane is Minkowski space-time. We therefore can simply fix \( a = 1 \) and conformal time is identical to physical time, \( \eta = t \). Furthermore, there is no matter in the background so that the functions \( \mathcal{M} \) and the pressure \( P \) are both of first order. Therefore, the right-hand sides of the perturbation Eqs. \((81), (85)\) and \((89)\) are of second order and have to be dropped in a consistent first order treatment. The solution is therefore of the same form in all cases,

\[
\mathcal{M} = F(k, \mu, \varphi) \exp(-i k \mu t) . \tag{91}
\]

Before we determine the prefactor \( F \), and the resulting anisotropic stress \( \Pi_{ij} \), we Fourier transform \( \mathcal{M} \) with respect to time. This yields

\[
\mathcal{M}(k, \mu, \varphi, \omega) = F(k, \mu, \varphi) \delta(\omega - k \mu) . \tag{92}
\]

For tensor perturbations,

\[
F(k, \mu, \varphi) = (1 - \mu^2) \left[ \mathcal{M}_0(k, \mu, \varphi) \cos(2\varphi) + \mathcal{M}_x(k, \mu, \varphi) \sin(2\varphi) \right] , \tag{93}
\]

we find

\[
\Pi^{(T)}_* = \begin{cases} \frac{3}{8} \mathcal{M}_*(\omega/k, k) \left( 1 - \frac{\omega^2}{k^2} \right)^2 & \text{for } \omega^2 < k^2 , \\ 0 & \text{else} . \end{cases} \tag{94}
\]

Comparing this with Eq. \((84)\) and using \( \omega^2 = k^2 + m^2 \), we find that only modes with \(-k^2 \leq m^2 < 0\) are excited and

\[
C_*(m^2, k) = \frac{-3\kappa_5}{16} \mathcal{F} \left( \frac{\sqrt{k^2 + m^2}}{k} \right) \frac{|m|^2}{L^2 k^4 K_1(|m|L)} \text{ if } -k^2 \leq m^2 < 0 , \tag{95}
\]

\[
C_*(m^2, k) = 0 \text{ else} .
\]

Nevertheless, \( \omega = \sqrt{k^2 + m^2} \in \mathbb{R} \), so that only oscillating and no growing modes are excited.

Physically this result is not surprising. The distribution function of relativistic particles cannot change
faster than with the speed of light and hence modes with frequencies \( \omega > k \) cannot be excited. Therefore, only the tachyonic modes are relevant. However, since \( \Pi \) is not unstable, there is no instability. Only modes with \( k^2 \geq -m^2 \), and hence \( \omega \in \mathbb{R} \) are excited. We expect this to hold true whenever the temporal change in the perturbations is due to the motion of particles.

For vector and scalar perturbations, one obtains similar results. For vector perturbations, we have

\[
\Pi_{(V)} = \begin{cases} \frac{\omega}{k^2} \mathcal{M}_v(\omega/k, k) \left( 1 - \frac{\omega^2}{k^2} \right) & \text{for } \omega^2 < k^2, \\ 0 & \text{else.} \end{cases}
\]

and for scalar perturbations

\[
\Pi_{(S)} = \begin{cases} \frac{1}{2} \mathcal{M}_s(\omega/k, k) \left( \frac{1}{2} - \frac{\omega^2}{k^2} \right) & \text{for } \omega^2 < k^2, \\ 0 & \text{else.} \end{cases}
\]

The detailed junction conditions are somewhat involved, but they evidently imply again that only the modes with \( 0 \leq \omega^2 < k^2 \) are excited,

\[
C_v \neq 0 \quad \text{only if } \quad -k^2 < m^2 \leq 0.
\]

Hence only tachyonic modes, \( m^2 < 0 \) with \( \omega \in \mathbb{R} \) are present. But, since the distribution function \( \mathcal{M} \) does not grow exponentially in time, truly unstable modes with \( \omega^2 = k^2 + m^2 < 0 \) are not allowed.

This example is interesting in the sense that the new tachyonic modes play an important role, but no instability develops.

C. Comments about cosmology

In a cosmological setup, we assume the brane to move in \( \text{AdS}_5 \). The scale factor is then given by \( a = L/y_b(\eta) \), where \( y \) is conformal time on the brane. The cosmological equations for the unperturbed case are well known (see e.g., [38]),

\[
\left( \frac{\dot{a}}{a} \right)^2 = \mathcal{H}^2 = \frac{\kappa_4}{3} a^2 \left( 1 + \frac{\rho}{2T} \right),
\]

\[
\dot{\rho} + 3\mathcal{H}(\rho + P) = 0,
\]

with \( \kappa_5 T = 6/L \) and \( \kappa_4 = \kappa_5/L \). At low energy, \( \rho \ll T \) or \( \mathcal{H} \ll a/L \), we recover the standard Friedmann equations.

Brane motion significantly alters the junction condition. For instance, the perturbed junction condition for tensor perturbations [34] becomes

\[
L \mathcal{H} \partial_t H_* - a \left[ 1 + \left( \frac{L \mathcal{H}}{a} \right)^2 \right]^{1/2} \partial_y H_* = \frac{\kappa_5}{2} a^2 \Pi_0,
\]

where \( t \) and \( y \) denote bulk coordinates. Clearly, \( H_*(t, y) \) is no longer separable and we have to rely on numerical simulations for its determination (see [36, 37] for recent works on the homogeneous case, \( \Pi_0 = 0 \)).

In a cosmology dominated by relativistic collisionless particles, metric perturbations enter in the first order perturbation Eq. (83), since \( P = \rho/3 \) is now a background quantity. Hence \( \Pi_0 \) and \( H_* \) must be determined simultaneously via Eqs. (83), (84) and (101). It is not yet clear to us whether an instability may develop in the presence of anisotropic stresses. A detailed study is in preparation [38].

IV. A SIMPLE ORBIFOLD MODEL

A. The toy model

In Sec. II we have found exponentially growing perturbations in the scalar, vector and tensor sectors of a RSII background. These can be generated from small, everywhere regular initial data. To complete the discussion, we have to specify a brane equation of motion for the anisotropic stresses and solve the full system. In the previous section we have shown that in the RSII model linear perturbations are still stable, but in a cosmological context, the coupling to metric perturbations might induce an instability. Here we present, instead, a simple toy model, to show that nonlinearities can lead to instabilities already if the brane is flat Minkowski space.

We want to show the following: The instability which we have found is not due to an instability in the equation of motion of the anisotropic stresses on the brane nor to an instability of the bulk in absence of a brane. It is also not coming from the choice of the wrong boundary conditions (incoming wave, advanced instead of retarded solution, etc.), but it is due entirely to the singular orbifold construction used in RSII.

Only the fact that we have two copies of \( y \geq y_b \) renders the \( K \) mode normalizable, which would otherwise diverge exponentially for \( y \rightarrow 0 \). In the same way one obtains an instability if one glues together twice the same half of a simple Minkowski space. To see this we consider four-dimensional Minkowski space-time, with orbifoldlike spatial sections which can be identified with two copies of \( z \geq 0 \). The \( 2 + 1 \) dimensional “brane” is represented by the plane \( z = 0 \) and the “bulk” by two copies of \( z > 0 \).

We consider a bulk field \( \phi(t, x_\parallel, z) \), which satisfies the ordinary hyperbolic wave equation,

\[
\Box \phi = -(\partial_t^2 - \Delta_\parallel - \partial_z^2) \phi = 0.
\]

Here \( x_\parallel = (x, y) \) are the coordinates parallel to the “brane” and \( \Delta_\parallel = \partial_x^2 + \partial_y^2 \). Like for the metric in RSII, we require \( \phi \) to be continuous across the brane (\( z = 0 \)), but it may have a kink, i.e., its derivative may jump.

Confining on the brane is a field \( \Pi \) which is given by

\[
\lim_{z \to 0^+} \partial_z \phi = \Pi.
\]
Finally, we specify the equation of motion for $\Pi$ on the brane (the matter equation):

$$
(\partial_t^2 - \Delta)\Pi = -\frac{\partial_{\mu}\phi \partial^{\mu}\Pi}{\phi},
$$

(104)

where $(\partial_{\mu}) = (\partial_t, \partial_x, \partial_y)$ is the gradient on the brane.

This resembles a “covariant derivative” with $\partial_{\mu}\phi/\phi$ playing the role of the Christoffel symbols.

**B. Stable and unstable modes**

We first show that, if we do not allow for any “singularity”, i.e., all fields and their first and second derivatives have to be continuous everywhere, this system has no instability.

In this case, the fundamental solutions are simply

$$
\phi = \phi_0 \exp \left[ \pm i(\omega t - p \cdot x) \right], \quad \omega^2 - p^2 = 0,
$$

(105)

with $x = (x, y, z) = (x_{||}, z)$ and $p = (p_x, p_y, p_z) = (p_{||}, p_{\perp})$. From Eq. (103) we obtain

$$
\Pi = \mp ip_{\perp} \phi_0 \exp \left[ \pm i(\omega t - p_{||} \cdot x_{||}) \right].
$$

(106)

It is easy to verify that this is compatible with the equation of motion (104) for $\Pi$. Clearly, these solutions do not exhibit any instability. They do not grow, but oscillate in time. For an observer confined to the brane, $\phi$ appears as a field with mass $m^2 = p_{||}^2 \geq 0$. For $p_{||}^2 = m^2$ fixed, this system can be seen as a system of two fields $\Pi, \phi_m$ on the brane, satisfying

$$
(\partial_t^2 - \Delta_{||})\phi_m = m^2 \phi_m,
$$

(107)

and Eq. (104). This corresponds to a stable system of scalar fields on the brane.

Now we proceed to the orbifold construction. We require $Z_2$ symmetry, $\phi(-z) = \phi(z)$, but allow $\partial_z \phi$ to jump at $z = 0$. All the equations of motion remain the same. Now, the previous solutions for $\phi$ have to be combined into $Z_2$-symmetric linear combinations,

$$
\phi = \phi_0 \exp \left[ \pm i(\omega t - p \cdot x) \right] + \phi_0 \exp \left[ \pm i(\omega t - p \cdot x) \right],
$$

(108)

where $p = (p_{||}, -p_{\perp})$ if $p = (p_{||}, p_{\perp})$. Such solutions are still regular (continuous derivatives) at $z = 0$. However, there now appears a new set of solutions:

$$
\phi = \phi_0 \exp \left[ \omega t - ip_{||} \cdot x_{||} - k|z| \right],
$$

(109)

$$
\Pi = -k \phi_0 \exp \left[ \omega t - ip_{||} \cdot x_{||} \right],
$$

(110)

with $\omega^2 = k^2 - p_{||}^2$, $k > 0$.

(111)

For $k^2 > p_{||}^2$, we have $\omega^2 > 0$ and can thus choose $\omega > 0$, so that these solutions grow exponentially in time. Evidently, $\phi$ obeys the bulk wave Eq. (102) and $\Pi$ satisfies its equation of motion (104).

From the brane point of view, $\phi$ is now a tachyonic field with $m^2 = -\omega^2 - p_{||}^2 < 0$. In the pure brane model (without a bulk) this solution would therefore not appear. It is clearly only due to the orbifold construction. Since $\phi$ decays exponentially away from the brane, this solution also does not describe an ingoing wave.

**C. Green’s function approach**

We finally want to address the question whether this solution may contain “a-causal” contributions; or, more formally, whether it can be constructed from the retarded Green’s function alone.

We choose as Green’s function for $\phi$ the one with

$$
x = (t, x, y, z), \quad \square G(x; x') = \delta^{(4)}(x - x'),
$$

(112)

$$
\partial_\nu G(x, x')|_{z = 0} = 0.
$$

(113)

Then, Green’s formula gives (see, e.g., [39])

$$
\phi(x) = \int_{z' = 0} dt' dx_{||}' \left[ \partial_\nu G(x, x') \phi(x') - G(x, x') \partial_\nu \phi(x') \right].
$$

(114)

Using the boundary conditions at $z = 0$ for $G$ and $\phi$, this yields

$$
\phi(x) = -\int_{z' = 0} dt' dx_{||}' G(x, x') \Pi(t', x_{||}').
$$

(115)

The analogous expression for the RSII model is given in [10].

We now construct the retarded Green’s function obeying the boundary condition (113). We perform a Fourier transform in $t'$ and $x_{||}'$ so that (112) becomes

$$
\left( \omega^2 - p_{||}^2 + \partial_{z'}^2 \right) \tilde{G}(z; z') = \delta(z - z').
$$

(116)

The boundary condition (113) is satisfied by the modes

$$
u_q(z) = \frac{1}{\sqrt{2}} \left( e^{i qz} + e^{-i qz} \right), \quad q^2 = \omega^2 - p_{||}^2.
$$

(117)

With the correct normalization (see, e.g., [10]), we obtain the retarded Green’s function

$$
G(x, x')|_{z = 0} = \frac{1}{\pi} \int_0^\infty dq \left( e^{iqz} + e^{-iqz} \right) \int \frac{d\omega d^2 p_{||}}{(2\pi)^3}
$$

$$
\times \frac{e^{-i\omega(t - t') + ip_{||} \cdot (x_{||} - x_{||}')}}{q^2 + p_{||}^2 - (\omega + i\epsilon)^2}
$$

$$
= 2 \int \frac{dt d^2 p_{||}}{(2\pi)^3} \frac{e^{i p_{||} \cdot (x - x') \cdot t}}{p_{||}^2 - (\omega + i\epsilon)^2}
$$

$$
= \frac{1}{(2\pi)^3} \frac{1}{\sqrt{z^2 + |x_{||} - x_{||}'|^2}}.
$$

(118)
where $\theta$ is the Heaviside function, 
\[
\theta(x) = \begin{cases} 
1 & \text{if } x \geq 0, \\
0 & \text{if } x < 0.
\end{cases}
\]

Up to a factor 2 which is due to $Z_2$ symmetry, this is just the standard retarded Green’s function of the four-dimensional wave equation.

In order to show that our unstable solution (109) does not invoke any a-causalities, i.e., contributions from an advanced Green’s function, we now show that it can be obtained from the retarded Green’s function (118) by means of Eq. (115). We perform the calculations for $z > 0$ and then invoke $Z_2$ symmetry for $z < 0$. Equations (110) and (115) imply
\[
\phi(t, \mathbf{x}_\parallel, z) = k \phi_0 \int \frac{dt' d^2 \mathbf{x}'_\parallel}{2\pi} \frac{e^{i\omega (t-t') - i\mathbf{p}_\parallel \cdot \mathbf{x}'_\parallel}}{\sqrt{\omega^2 + \mathbf{p}_\parallel^2}} \theta(t - t') 
\times \delta(t - t' - \sqrt{z^2 + \mathbf{r}^2}) ,
\]
where $\mathbf{r} = \mathbf{x}_\parallel - \mathbf{x}'_\parallel$. With $d^2 \mathbf{x}_\parallel = d^2 \mathbf{r} = r dr d\varphi$, integration over $t'$ and $\varphi$ finally gives
\[
\phi(t, \mathbf{x}_\parallel, z) = k \phi_0 e^{i\omega t - i\mathbf{p}_\parallel \cdot \mathbf{x}_\parallel} \int_0^\infty dr \frac{e^{-\sqrt{\omega^2 + r^2}}}{\sqrt{\omega^2 + r^2}} J_0(|\mathbf{p}_\parallel|r)
= \phi_0 e^{i\omega t - i\mathbf{p}_\parallel \cdot \mathbf{x}_\parallel - k z}.
\]
Here $J_0$ is the Bessel function of order 0. For the last equal sign we used the integral No. 6.645.2 in [41]. This proves that $\phi$ from Eq. (109) is a purely retarded solution and hence represents a true physical instability of the system.

V. CONCLUSIONS

We have shown that the perturbation equations of the RSII model allow for tachyonic modes which can be exponentially unstable. We have then argued that, within first order perturbation theory, the RSII model remains stable. Nevertheless, we have given an example where the new tachyonic modes play a crucial role. In the cosmological context, the situation is significantly different and tachyonic modes could in principle lead to instabilities. Since metric perturbations are no longer separable, we have to rely on numerical simulations. We do not yet know whether an instability may develop in the presence of anisotropic stresses. A more detailed study will be presented elsewhere [38].

Within a toy model, we have shown that the exponentially growing bulk modes do not come from an instability of the brane equations of motion or from the use of an inadequate Green’s function, but are due to the singular orbifold construction. As we show explicitly in our toy model, the unstable modes can be obtained using the retarded Green’s function. The toy model also shows that, taking into account nonlinearities, even a flat Minkowski-space brane can become unstable.

Starting with small regular initial data on some hypersurface of constant time, an exponential instability can build up in an AdS$_5$ orbifold. It remains to be examined whether this instability stays exponential also in the cosmological context of a brane moving through AdS$_5$, or if it disappears in an expanding universe like the Jeans instability of Newtonian gravity. This question is of utmost importance. If exponentially unstable modes are generated either on the first or second order, this will inhibit the realization of cosmology in terms of a RSII braneworld.

One may then go even further and ask whether such orbifold constructions may not lead to instabilities in a much broader sense than what has been discussed here.

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