CONTINUITY OF RADIAL AND TWO-SIDED RADIAL \( SLE_\kappa \) AT THE TERMINAL POINT

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Abstract. We prove that radial \( SLE_\kappa \) and two-sided radial \( SLE_\kappa \) are continuous at their terminal point.

1. Introduction

We answer a question posed by Dapeng Zhan about radial Schramm-Loewner evolution (\( SLE_\kappa \)) and discuss a similar question about two-sided \( SLE_\kappa \) that arose in work of the author with Brent Werness [4]. Radial \( SLE_\kappa \) was invented by Oded Schramm [8] and is a one-parameter family of random curves \( \gamma : [0, \infty) \to \mathbb{D} \), \( \gamma(0) \in \partial \mathbb{D} \), where \( \mathbb{D} \) denotes the unit disk. The definition implies that \( \gamma(t) \neq 0 \) for every \( t \) and

\[
\liminf_{t \to \infty} |\gamma(t)| = 0.
\]

Zhan asked for a proof that with probability one

\[
(1) \quad \lim_{t \to \infty} \gamma(t) = 0.
\]

For \( \kappa > 4 \), for which the \( SLE \) paths intersect themselves, this is not difficult to prove because the path makes closed loops about the origin. The harder case is \( \kappa \leq 4 \). Here we establish (1) for \( \kappa \leq 4 \) by proving a stronger result.

To state the result, let

\[
\mathbb{D}_n = e^{-n} \mathbb{D} = \{ z \in \mathbb{C} : |z| < e^{-n} \},
\]

\[
\rho_n = \inf \{ t : |\gamma(t)| = e^{-n} \},
\]

and let \( \mathcal{G}_n \) denote the \( \sigma \)-algebra generated by \( \{ \gamma(s) : 0 \leq s \leq \rho_n \} \). We fix

\[
\alpha = \frac{8}{\kappa} - 1,
\]

which is positive for \( \kappa < 8 \).

Theorem 1. For every \( 0 < \kappa < 8 \), there exists \( c > 0 \) such that if \( \gamma \) is radial \( SLE_\kappa \) from 1 to 0 in \( \mathbb{D} \) and \( j, k, n \) are positive integers, then

\[
(2) \quad \mathbb{P} \{ \gamma[\rho_{n+k}, \infty) \subset \mathbb{D}_j \mid \mathcal{G}_{n+k} \} \geq [1 - c e^{-\alpha n/2}] 1\{ \gamma[\rho_k, \rho_{n+k}] \subset \mathbb{D}_j \}.
\]

Moreover, if \( 0 < \kappa \leq 4 \), then

\[
(3) \quad \mathbb{P} \{ \gamma[\rho_{n+k}, \infty) \subset \mathbb{D}_k \mid \mathcal{G}_{n+k} \} \geq 1 - c e^{-\alpha n/2}.
\]

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There is another version of $SLE$, sometimes called two-sided radial $SLE_\kappa$ which corresponds to chordal $SLE_\kappa$ conditioned to go through an interior point. We consider the case of chordal $SLE_\kappa$ in $D$ from 1 to $-1$ conditioned to go through the origin stopped when it reaches the origin (see Section 3.3 for precise definitions).

**Theorem 2.** For every $0 < \kappa < 8$, there exists $c > 0$ such that if $\gamma$ is two-sided radial $SLE_\kappa$ from 1 to $-1$ through 0 in $\mathbb{D}$ and $j, k, n$ are positive integers, then

$$P\{\gamma[\rho_{n+k}, \infty] \subset \mathbb{D}_j \ | \ G_{n+k}\} \geq \left[1 - ce^{-n\alpha/2}\right] 1\{\gamma[\rho_k, \rho_{n+k}] \subset \mathbb{D}_j\}.$$  

Using these theorems, we are able to obtain the following corollary. Unfortunately, we are not able to estimate the exponent $u$ that appears.

**Theorem 3.** For every $0 < \kappa < 8$, there exist $c < \infty, u > 0$ such that the following holds. Suppose $\gamma$ is either radial $SLE_\kappa$ from 1 to 0 in $\mathbb{D}$ or two-sided radial $SLE_\kappa$ from 1 to $-1$ through 0 stopped when it reaches the origin. Then, for all nonnegative integers $k, n$,

$$P\{\gamma[\rho_{n+k}, \infty] \cap \partial \mathbb{D}_k \neq \emptyset \ | \ G_k\} \leq ce^{-un},$$

and hence

$$P\{\gamma[\rho_{n+k}, \infty] \cap \partial \mathbb{D}_k = \emptyset\} \leq ce^{-un}.$$ In particular, if $\gamma$ has the radial parametrization, then with probability one,

$$\lim_{t \to \infty} \gamma(t) = 0.$$

Note that (3) is not as strong a result as (2). At the moment, we do not have uniform bounds for

$$P\{\gamma[\rho_{n+k}, \infty] \cap \partial \mathbb{D}_k \neq \emptyset \ | \ G_{n+k}\}$$

for radial $SLE_\kappa$ with $4 < \kappa < 8$ or two-sided radial $SLE_\kappa$ for $0 < \kappa < 8$.

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**1.1. Outline of the paper.** When studying $SLE$, one uses many kinds of estimates: results for all conformal maps; results that hold for solutions of the (deterministic) Loewner differential equation; results about stochastic differential equations (SDE), often simple equations of one variable; and finally results that combine them all. We have separated the non-$SLE$ results into a “preliminary” section with subsections emphasizing the different aspects.

We discuss three kinds of $SLE_\kappa$: radial, chordal, and two-sided radial. They are probability measures on curves (modulo reparametrization) in simply connected domains connecting, respectively: boundary point to interior point, two distinct boundary points, and two distinct boundary points conditioned to go through an interior point. In all three cases, the measures are conformally invariant and hence we can choose any convenient domain. For the radial equation, the unit disk $\mathbb{D}$ is most convenient and for this one gets the Loewner equation as originally studied by Loewner. For this equation a radial parametrization is used which depends on the interior point. For the chordal case, Schramm [3] showed that the half-plane with boundary points 0 and $\infty$ was most convenient, and the corresponding
continuity of radial and two-sided radial SLE

Loewner equation is probably the easiest for studying fine properties. Here a **chordal parametrization** depending on the target boundary point (infinity) is most convenient. The two-sided radial, which was introduced in [7, 3] and can be considered as a type of SLE(κ, ρ) process as defined in [6], has both an interior point and a boundary point. If one is studying this path up to the time it reaches the interior point, which is all that we do in this paper, then one can use either the radial or the chordal parametrization.

The three kinds of SLEκ, considered as measures on curves modulo reparametrization, are locally absolutely continuous with respect to each other. To make this precise, it is easiest if one studies them simultaneously in a single domain with a single choice of parametrization. We do this here choosing the radial parametrization in the unit disk D. We review the radial Loewner equation in Section 2.1. We write the equation slightly differently than in [8]. First, we add a parameter a that gives a linear time change. We also write a point on the unit circle as e^{-2iθ} rather than e^{iθ}; this makes the SDEs slightly easier and also shows the relationship between this quantity and the argument of a point in the chordal case. Indeed, if F is a conformal transformation of the unit disk to the upper half plane with F(1) = 0 and F(e^{2iθ}) = ∞, then sin[arg F(0)] = sin θ.

The radial Loewner equation describes the evolution of a curve γ from 1 to 0 in D. More precisely, if D_t denotes the connected component of D \ γ(0, t] containing the origin, and g_t : D_t → D is the conformal transformation with g_t(0) = 0, g'_t(0) > 0, then the equation describes the evolution of g_t. At time t, the relevant information is g_t(γ(t)) which we write as e^{2iU_t}. To compare radial SLEκ to chordal or two-sided radial SLEκ with target boundary point w = e^{2iθ}, we also need to keep track of g_t(w) which we write as e^{2iθ_t}.

Radial SLEκ is obtained by solving the Loewner equation with a = 2/κ and U_t = -B_t a standard Brownian motion. If X_t = θ_t - U_t, then X_t satisfies

\[ dX_t = \beta \cot X_t \, dt + dB_t, \]

with \( \beta = a \). Much of the study of SLEκ in the radial parametrization can be done by considering the SDE above. In fact, the three versions of SLEκ can be obtained by choosing different \( \beta \). In Section 2.2 we discuss the properties of this SDE that we will need. We use the Girsanov theorem to estimate the Radon-Nikodym derivative of the measures on paths for different values of \( \beta \).

Section 2.3 gives estimates for conformal maps that will be needed. The first two subsections discuss crosscuts and the argument of a point. If D is a simply connected subdomain of D containing the origin, then the intersection of D with the circle ∂D_k can contain many components. We discuss such crosscuts in Section 2.3.1 and state a simple topological fact, Lemma 2.3, that is used in the proofs of (2) and (4).

A classical conformally invariant measure of distance between boundary arcs is extremal distance or extremal length. We will only need to consider distance between arcs in a conformal rectangle for which it is useful to estimate harmonic measure, that is, hitting probabilities for Brownian motion. We discuss the general strategy for proving such estimates in Section 2.3.3. The following subsections give specific estimates that will be needed for radial and two-sided radial. The results in this section do not depend much at all on the Loewner equation — one fact that is used is that we are stopping a curve at the first time it reaches ∂D_n for some n.
The Beurling estimate (see [5, Section 3.8]) is the major tool for getting uniform estimates.

The main results of this paper can be found in Section 3. The first three subsections define the three types of $SLE_\kappa$, radial, chordal, two-sided radial, in terms of radial. (To be more precise, it defines these processes up to the time the path separates the origin from the boundary point $w$). Section 4.1 contains the hardest new result in this paper. It is an analogue for to radial $SLE_\kappa$ of a known estimate for chordal $SLE_\kappa$ on the probability of hitting a set near the boundary. This is the main technical estimate for Theorem 1. A different estimate is proved in Section 4.2 for two-sided radial. The final section finishes the proof Theorem 3 by using a known technique to show exponential rates of convergence.

I would like to thank Dapeng Zhan for bringing up the fact that this result is not in the literature and Joan Lind and Steffen Rohde for useful conversations.

1.2. Notation. We let
\[ \mathbb{D} = \{ |z| < 1 \}, \quad \mathbb{D}_n = e^{-n} \mathbb{D} = \{ |z| < e^{-n} \}. \]
If $\gamma$ is a curve, then
\[ \rho_n = \inf \{ t : \gamma(t) \in \partial \mathbb{D}_n \}. \]
If $\gamma$ is random, then $\mathcal{F}_t$ denotes the $\sigma$-algebra generated by $\{ \gamma(s) : s \leq t \}$ and $\mathcal{G}_n = \mathcal{F}_{\rho_n}$ is the $\sigma$-algebra generated by $\{ \gamma(t) : t \leq \rho_n \}$. Let $D_t$ be the connected component of $\mathbb{D} \setminus \gamma(0,t]$ containing the origin and
\[ H_n = D_{\rho_n}. \]
If $D$ is a domain, $z \in D$, $V \subset \partial D$, we let $h_D(z,V)$ denote the harmonic measure starting at $z$, that is, the probability that a Brownian motion starting at $z$ exits $D$ at $V$.

When discussing $SLE_\kappa$ we will fix $\kappa$ and assume that $0 < \kappa < 8$. We let
\[ a = \frac{2}{\kappa}, \quad \alpha = \frac{8}{\kappa} - 1 = 4a - 1 > 0. \]

2. Preliminaries

2.1. Radial Loewner equation. Here we review the radial Loewner differential equation; see [5] for more details. The radial Loewner equation describes the evolution of a curve from 1 to 0 in the unit disk $\mathbb{D}$. Let $a > 0$, and let $U_t : [0, \infty) \to \mathbb{R}$ be a continuous function with $U_0 = 0$. Let $g_t$ be the solution to the initial value problem
\[ \partial_t g_t(z) = 2a g_t(z) \frac{e^{2iU_t} + g_t(z)}{e^{2iU_t} - g_t(z)}, \quad g_0(z) = z. \]
For each $z \in \overline{\mathbb{D}} \setminus \{ 1 \}$, the solution of this equation exists up to a time $T_z \in (0, \infty]$. Note that $T_0 = \infty$ and $g_t(0) = 0$ for all $t$. For each $t \geq 0$, $D_t$, as defined above, equals $\{ z \in \mathbb{D} : T_z > t \}$, and $g_t$ is the unique conformal transformation of $D_t$ onto $\mathbb{D}$ with $g_t(0) = 0, g_t'(0) > 0$. By differentiating (7) with respect to $z$, we see that $\partial_z g_t'(0) = 2ag_t'(0)$ which implies that $g'_t(0) = e^{2at}$.

If we define $h_t(z)$ to be the continuous function of $t$ such that
\[ g_t(e^{2iz}) = \exp \{ 2ih_t(z) \}, \quad h_0(z) = z, \]
the Loewner equation becomes
\begin{equation}
\partial_t h_t(z) = a \cot(h_t(z) - U_t), \quad h_0(z) = z.
\end{equation}
We will consider this primarily for real \( z = x \in (0, \pi) \). Note that if \( x \in (0, \pi) \) and \( D_t \) agrees with \( \mathbb{D} \) in a neighborhood of \( e^{2ix} \), then
\begin{equation}
|g_t'(e^{2ix})| = h_t'(x).
\end{equation}

The radial equation can also be used to study curves whose “target” point is a boundary point \( w = e^{2i\theta_0}, 0 < \theta_0 < \pi \). If we let \( \theta_t = h_t(\theta_0) \), then (8) becomes
\[ \partial_t \theta_t = a \cot(\theta_t - U_t), \]
which is valid for \( t < T_w \). Using (9), we get
\[ |g_t'(w)| = h_t'(\theta_0) = \exp \left\{-a \int_0^t \frac{ds}{\sin^2(\theta_s - U_s)} \right\}. \]

\textbf{Definition}

- A curve arising from the Loewner equation will be called a \textit{Loewner curve}.
  Two such curves are equivalent if one is obtained from the other by increasing reparametrization.
- A Loewner curve has the a-radial parametrization if \( g_t'(0) = e^{2at} \).

Recall that \( \rho_n = \inf\{t : |\gamma(t)| = e^{-n}\} \). A simple consequence of the Koebe 1/4-theorem is the existence of \( c < \infty \) such that for all \( n \)
\begin{equation}
\rho_{n+1} \leq \rho_n + c.
\end{equation}

\textbf{2.2. Radial Bessel equation.} Analysis of radial SLE leads to studying a simple one-dimensional SDE (12) that we call the \textit{radial Bessel equation}. This equation can be obtained using the Girsanov theorem by “weighting” or “tilting” a standard Brownian motion as we now describe. Suppose \( X_t \) is a standard one-dimensional Brownian motion defined on a probability space \((\Omega, \mathbb{P})\) with \( 0 < X_0 < \pi \) and let \( \tau = \inf\{t : \sin X_t = 0\} \). Roughly speaking, the radial Bessel equation with parameter \( \beta \) (up to time \( \tau \)) is obtained by weighting the Brownian motion locally by \( (\sin X_t)^\beta \). Since \( (\sin X_t)^\beta \) is not a local martingale, we need to compensate it by a \( C^1 \) (in time) process \( e^{\beta X_t} \) such that \( e^{-\beta t} (\sin X_t)^\beta \) is a local martingale. The appropriate compensator is found easily using Itô’s formula; indeed,
\[ M_t = M_{t,\beta} = (\sin X_t)^\beta e^{\beta^2 t/2} \exp \left\{ \frac{(1-\beta)\beta}{2} \int_0^t \frac{ds}{\sin^2 X_s} \right\}, \quad 0 \leq t < \tau, \]
is a local martingale satisfying
\begin{equation}
    dM_t = \beta M_t \cot X_t \, dX_t.
\end{equation}

In particular, for every \( \epsilon > 0 \) and \( t_0 < \infty \), there exists \( C = C(\beta, \epsilon, t_0) < \infty \) such that if \( \tau_\epsilon = \inf \{ t : \sin X_t \leq \epsilon \} \), then
\[
    C^{-1} \leq M_t \leq C, \quad 0 \leq t \leq t_0 \wedge \tau_\epsilon.
\]

Let \( \mathbb{P}_\beta \) denote the probability measure on paths \( X_t, 0 \leq t < \tau \) such that for each \( \epsilon > 0, t_0 < \infty \), the measure \( \mathbb{P}_\beta \) on paths \( X_t, 0 \leq t \leq t_0 \wedge \tau_\epsilon \) is given by
\[
    d\mathbb{P}_\beta = \frac{M_{t_0 \wedge \tau_\epsilon} M_0}{M_0} \, d\mathbb{P}.
\]

The Girsanov theorem states that
\[
    B_t = B_{t, \beta} := X_t - \beta \int_0^t \cot X_s \, ds, \quad 0 \leq t < \tau
\]
is a standard Brownian motion with respect to the measure \( \mathbb{P}_\beta \). In other words,
\[
    dX_t = \beta \cot X_t \, dt + dB_t, \quad 0 \leq t < \tau.
\]

We call this the radial Bessel equation (with parameter \( \beta \)). By comparison with the usual Bessel equation, we can see that
\[
    \mathbb{P}_\beta \{ \tau = \infty \} = 1 \text{ if and only if } \beta \geq \frac{1}{2}.
\]

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2.2.1. **An estimate.** Here we establish an estimate \( \mathbb{1} \) for the radial Bessel equation which we will use in the proof of continuity of two-sided radial SLE. Suppose that \( X_t \) satisfies
\begin{equation}
    dX_t = \beta \cot X_t \, dt + dB_t, \quad 0 \leq t < \tau,
\end{equation}
where \( \beta \in \mathbb{R} \), \( B_t \) is a standard Brownian motion, and \( \tau = \inf \{ t : \sin X_t = 0 \} \). Let
\[
    F(x) = F_\beta(x) = \int_x^{\pi/2} (\sin t)^{-2\beta} \, dt, \quad 0 < x < \pi,
\]
which satisfies
\begin{equation}
    F''(x) + 2\beta \left( \cot x \right) F'(x) = 0.
\end{equation}

**Lemma 2.1.** For every \( \beta > 1/2 \), there exists \( c_\beta < \infty \) such that if \( 0 < \epsilon < x \leq \pi/2 \), \( X_t \) satisfies \( \mathbb{1} \) with \( X_0 = x \), and
\[
    \tau_\epsilon = \inf \{ t \geq 0 : X_t = \epsilon \text{ or } \pi/2 \},
\]
then
\[
    \mathbb{P}\{ X_{\tau_\epsilon} = \epsilon \} \leq c_\beta \left( \epsilon/x \right)^{2\beta-1}.
\]
Proof: Itô’s formula and (13) show that \( F(X_{t\wedge \tau}) \) is a bounded martingale, and hence the optional sampling theorem implies that

\[
F(x) = \mathbb{P}\{X_{\tau} = \epsilon\} F(\epsilon) + \mathbb{P}\{X_{\tau} = \pi/2\} F(\pi/2) = \mathbb{P}\{X_{\tau} = \epsilon\} F(\epsilon).
\]

Therefore,

\[
\mathbb{P}\{X_{\tau} = \epsilon\} = \frac{F(x)}{F(\epsilon)}.
\]

If \( \beta > 1/2 \), then

\[
F(\epsilon) \sim \frac{1}{2\beta - 1} \epsilon^{1-2\beta}, \quad \epsilon \to 0+,
\]

from which the lemma follows. \(\square\)

**Lemma 2.2.** For every \( \beta > 1/2, t_0 < \infty \), there exists \( c = c_{\beta,t_0} < \infty \) such that if \( X_t \) satisfies (12) with \( X_0 \in (0, \pi) \), then

\[ (14) \quad \mathbb{P}\left\{ \min_{0 \leq t \leq t_0} \sin X_t \leq \epsilon \sin X_0 \right\} \leq c \epsilon^{2\beta-1}. \]

Proof: We allow constants to depend on \( \beta, t_0 \). Let \( r = \sin X_0 \). It suffices to prove the result when \( r \leq 1/2 \). Let \( \sigma = \inf\{t : \sin X_t = 1 \text{ or } \epsilon r\} \) and let \( \rho = \inf\{t > \sigma : \sin X_t = r\} \). Using the previous lemma we see that

\[
\mathbb{P}\{\sin X_\sigma = \epsilon r\} \leq c \epsilon^{2\beta-1}.
\]

Since \( r \leq 1/2 \) and there is positive probability that the process started at \( \pi/2 \) stays in \([\pi/4, 3\pi/4]\) up to time \( t_0 \), we can see that

\[
\mathbb{P}\{\rho > t_0 \mid \sin X_\sigma = 1\} \geq c_1.
\]

Hence, if \( q \) denotes the probability on the left-hand side of (14), we get

\[
q \leq c \epsilon^{2\beta-1} + (1 - c_1) q.
\]

\(\square\)

### 2.3. Deterministic lemmas.

#### 2.3.1. Crosscuts in \( \partial \mathbb{D}_k \).

**Definition** A crosscut of a domain \( D \) is the image of a simple curve \( \eta : (0, 1) \to D \) with \( \eta(0+) , \eta(1-) \) in \( \partial D \).

Recall that \( H_n \) is the connected component of \( \mathbb{D} \setminus \gamma(0, \rho_n) \) containing the origin. Let

\[
\partial^0_n = \partial H_n \setminus \gamma(0, \rho_n),
\]

which is either empty or is an open subarc of \( \partial \mathbb{D} \).

For each \( 0 < k < n \), let \( V_{n,k} \) denote the connected component of \( H_n \cap \mathbb{D}_k \) that contains the origin, and let \( \partial_{n,k} = \partial V_{n,k} \cap H_n \). The connected components of \( \partial_{n,k} \) comprise a collection \( \mathcal{A}_{n,k} \) of open subarcs of \( \partial \mathbb{D}_k \). Each arc \( l \in \mathcal{A}_{n,k} \) is a crosscut of \( H_n \) such that \( H_n \setminus l \) has two connected components. Let \( V_{n,k,l} \) denote the component of \( H_n \setminus l \) that does not contain the origin; note that these components are disjoint for distinct \( l \in \mathcal{A}_{n,k} \). If \( \partial^0_n \neq \emptyset \), there is a unique arc \( l^* = l^*_{n,k} \in \mathcal{A}_{n,k} \) such that \( \partial^0_n \subset \partial V_{n,k,l^*} \).

\(\blacktriangleleft \) Note that each \( l \in \mathcal{A}_{n,k} \) is a connected component of \( \partial \mathbb{D}_k \cap H_n \); however, there may be components of \( \partial \mathbb{D}_k \cap H_n \) that are not in \( \mathcal{A}_{n,k} \). In particular, it is possible that \( V_{n,k,l} \cap \mathbb{D}_k \neq \emptyset \). The arc \( l^* \) is the unique arc in \( \mathcal{A}_{n,k} \) such that each path from 0 to \( \partial^0_n \) in \( H_n \) must pass through
$l^*$. One can construct examples where there are other components $l$ of $\partial B_k \cap H_n$ with the property that every path from 0 to $\partial B_k$ in $H_n$ must pass through $l$. However, these components are not in $A_{n,k}$.

If $k < n$ and $\gamma([\rho_n, \infty)) \cap \partial B_k \neq \emptyset$, then the first visit to $\partial B_k$ after time $\rho_n$ must be to the closure of the one of the crosscuts in $A_{n,k}$. In this paper we will estimate the probability of hitting a given crosscut. Since there can be many crosscuts, it is not immediate how to use this estimate to bound the probability of hitting any crosscut. This is the technical issue that prevents us from extending (3) to all $\kappa < 8$. The next lemma, however, shows that if the curve has not returned to $\partial B_j$ after time $\rho_k$, then there is only one crosscut in $A_{n,k}$ from which one can access $\partial B_j$.

**Lemma 2.3.** Suppose $j < k < n$ and $\gamma$ is a Loewner curve in $\mathbb{D}$ starting at 1 with $\rho_n < \infty$, $H_n \not\subset D_j$, and $\gamma[\rho_k, \rho_n] \subset D_j$. Then there exists a unique crosscut $l \in A_{n,k}$ such that if $\eta : [0,1) \to H_n \cap D_j$ is a simple curve with $\eta(0) = 0, \eta(1-) \in \partial B_j$ and $\eta$ is a Loewner curve, then there is only one crosscut in $A_{n,k}$.

**Proof.** Call $l \in A_{n,k}$ good if there exists a curve $\eta$ as above with $\eta(s_0) \in l$. Since $H_n \not\subset \partial B_j$, there exists at least one good $l$. Also, if $-1 \in \partial B_n$, then $l_{n,k}^* \subset A_{n,k}$. Suppose $\eta^1, \eta^2$ are two such curves with times $s_0^1, s_0^2$ and let $z_j = \eta(s_j^0)$. We need to show that $z_1$ and $z_2$ are in the same crosscut in $A_{n,k}$. If $z_1 = z_2$ this is trivial, so assume $z_1 \neq z_2$. Let $l^1, l^2$ denote the two subarcs of $\partial B_k$ obtained by removing $z_1, z_2$ (these are not crosscuts in $A_{n,k}$). Let $l^1$ denote the arc that contains $\gamma(\rho_k)$ and let $U$ denote the connected component of $(H_k \cap D_j) \setminus (\eta^1 \cup \eta^2)$ that contains $\gamma(\rho_k)$. Our assumptions imply that $\gamma[\rho_k, \rho_n] \subset U$. In particular, $l^2 \cap \gamma[\rho_k, \rho_n] = \emptyset$. Therefore $l^2, z_1, z_2$ lie in the same component of $H_n$ and hence in the same crosscut of $A_{n,k}$. \[ \Box \]

2.3.2. **Argument.**

**Definition** If $\gamma$ is a Loewner curve in $\mathbb{D}$ starting at 1, $w \in \partial B \setminus \{1\}$, and $t < T_w$, then

$$S_t = S_{t,0,w} = \sin \arg F_t(0),$$

where $F_t : D_0 \to \mathbb{H}$ is a conformal transformation with $F_t(\gamma(t)) = 0, F_t(w) = \infty$.

If $z \in \mathbb{H}$, let $h_+(z) = h_{\mathbb{H}}(z, (0, \infty))$ denote the probability that a Brownian motion starting at $z$ leaves $\mathbb{H}$ at $(0, \infty)$ and let $h_-(z) = 1 - h_+(z)$ be the probability of leaving at $(-\infty, 0)$. Using the explicit form of the Poisson kernel in $\mathbb{H}$, one can see that $h_-(z) = \arg(z)/\pi$. Using this, we can see that

$$S_0 = \sin \theta_0$$

and

$$\sin \arg(z) \asymp \min \{h_+(z), h_-(z)\},$$

where $\asymp$ means each side is bounded by an absolute constant times the other side.

If $t < T_w$, we can write $\partial D_t = \{\gamma(t)\} \cup \{w\} \cup \partial^+ \cup \partial^-$ where $\partial^+ (\partial^-)$ is the part of $\partial D_t$ that is sent to the positive (resp., negative) real axis by $F_t$. Using conformal invariance and (15), we see that

$$S_t \asymp \min \{h_{D_t}(0, \partial^+), h_{D_t}(0, \partial^-)\}.$$
2.3.3. \textit{Extremal length}. The proofs of our deterministic lemmas will use estimates of extremal length. These can be obtained by considering appropriate estimates for Brownian motion which are contained in the next lemma. Let $R_L$ denote the open region bounded by a rectangle,

$$R_L = \{x+iy \in \mathbb{C} : 0 < x < L, 0 < y < \pi\}.$$ 

We write $\partial R_L = \partial_0 \cup \partial_l \cup \partial^+_L \cup \partial^-_L$ where

$$\partial_0 = [0, i\pi], \quad \partial_l = [L, L+i\pi], \quad \partial^+_L = (i\pi, L+i\pi), \quad \partial^-_L = (0, L).$$

If $D$ is a simply connected domain and $A_1, A_2$ are disjoint arcs on $\partial D$, then the $\pi$-extremal distance (\pi times the usual extremal distance or length) is the unique $L$ such that there is a conformal transformation of $D$ onto $R_L$ mapping $A_1, A_2$ onto $\partial_0$ and $\partial_L$, respectively. Estimates for the Poisson kernel in $R_L$ are standard, see, for example, [5, Sections 2.3 and 5.2]. The next two lemmas which we state without proof give the estimates that we need.

\textbf{Lemma 2.4.} There exist $0 < c_1 < c_2 < \infty$ such that the following holds. Suppose $L \geq 2$, and $V$ is the closed disk of radius $1/4$ about $1 + (\pi/2)i$.

- If $z \in V$,

\begin{equation}
(17) \quad c_1 \leq h_{R_L}(z, \partial_0), h_{R_L}(z, \partial^+_L), h_{R_L}(z, \partial^-_L) \leq c_2,
\end{equation}

- If $z \in V$ and $A \subset \partial_L$, then

\begin{equation}
(18) \quad h_{R_L}(z, A) \leq c_2 e^{-L} |A|,
\end{equation}

where $| \cdot |$ denotes length.

- If $B_t$ is a standard Brownian motion, $\tau_L = \inf\{t : B_t \notin R_L\}$, $\sigma = \inf\{t : \text{Re}(B_t) = 1\}$, then if $0 < x < 1/2$ and $0 < y < \pi$,

\begin{equation}
(19) \quad \mathbb{P}^{x+iy}\{B_{\sigma} \in V \mid \sigma < \tau_L\} \geq c_1.
\end{equation}

\textbf{Lemma 2.5.} For every $\delta > 0$, there exists $c > 0$ such that if $L \geq \delta$ and $z \in R_L$ with $\text{Re}(z) < 1/2$,

$$h_{R_L}(z, \partial_L) \leq c e^{-L} \min\{h_{R_L}(z, \partial^+_L), h_{R_L}(z, \partial^-_L)\}. $$

We explain the basic idea on how we will use these estimates. Suppose $D$ is a domain and $l$ is a crosscut of $D$ that divides $D$ into two components $D_1, D_2$. Suppose $D_2$ is simply connected and $A$ is a closed subarc of $\partial D_2$ with $\partial D_2 \cap l = \emptyset$. Let $\partial^+, \partial^-$ denote the connected components of $\partial D_2 \setminus \{l, A\}$. We consider $A, \partial^+, \partial^-$ as arcs of $\partial D$ in the sense of prime ends. Let $F : D_2 \to R_L$ be a conformal transformation sending $l$ to $\partial_0$ and $A$ to $\partial_L$ and suppose that $L \geq 2$. Let $l_1 = F^{-1}(1 + i(0, \pi))$. Let $\tau = \inf\{t : B_t \notin D\}, \sigma = \inf\{t : B_t \in l_1\}$. Then if $z \in D_1$ and $A_1 \subset A$,

$$h_D(z, \partial^+), h_D(z, \partial^-) \geq c \mathbb{P}^{z}\{\sigma < \tau\}. $$

$$h_D(z, A_1) \leq c \mathbb{P}^{z}\{\sigma < \tau\} e^{-L} |F(A_1)|. $$

In particular, there exists $c < \infty$ such that for $z \in D_1, A_1 \subset A$,

$$h_D(z, A_1) \leq c e^{-L} |F(A_1)| \min\{h_D(z, \partial^+), h_D(z, \partial^-)\}. $$
2.3.4. **Radial case.** We will need some lemmas that hold for all curves \( \gamma \) stopped at the first time they reach the sphere of a given radius or the first time they reach a given vertical line. If \( D \) is a domain and \( \eta : (0, 1) \to D \) is a crosscut, we write \( \eta \) for the image \( \eta(0, 1) \) and \( \overline{\eta} = \eta[0, 1] \).

- The next lemma is a lemma about Loewner curves, that is, curves modulo reparametrization. To make the statement nicer, we choose a parametrization such that \( \rho_{n+k} = 1 \). Although the parametrization is not important, it is important that we are stopping the curve at the first time it reaches \( \partial\mathbb{D}_{n+k} \).

**Lemma 2.6.** There exists \( c < \infty \) such that the following is true. Suppose \( k > 0, n \geq 4 \) and \( \gamma : [0, 1] \to \mathbb{D} \) is a Loewner curve with \( \gamma(0) = 1; |\gamma(1)| = e^{-n-k}; \) and \( e^{-n-k} < |\gamma(t)| < 1 \) for \( 0 < t < 1 \). Let \( D \) be the connected component of \( \mathbb{D} \setminus \gamma(0,1) \) containing the origin, and let

\[
\eta = \{ e^{-k+i\theta} : \theta_1 < \theta < \theta_2 \} \in A_{n+k,k}
\]

be a crosscut of \( D \) contained in \( \partial\mathbb{D}_k \).

Let \( F : D \to \mathbb{D} \) be the unique conformal transformation with \( F(0) = 0, F(\gamma(1)) = 1 \). Suppose that we write \( \partial\mathbb{D} \) as a disjoint union

\[
\partial\mathbb{D} = \{ 1 \} \cup V_1 \cup V_2 \cup V_3,
\]

where \( V_3 \) is the closed interval of \( \partial\mathbb{D} \) not containing 1 whose endpoints are the images under \( F \) of \( \eta(0+), \eta(1-) \) and \( V_1, V_2 \) are connected, open intervals. Then

\[
\text{diam}(F(\eta)) \leq c e^{-n/2}(\theta_2 - \theta_1) \min\{|V_1|, |V_2|\},
\]

where \( |\cdot| \) denotes length.

- It is important for our purposes to show not only that \( F(\eta) \) is small, but also that it is smaller than both \( V_1 \) and \( V_2 \). When we apply the proposition, one of the intervals \( V_1, V_2 \) may be very small.

**Proof.** Let \( U \) denote the connected component of \( D \setminus \eta \) that contains the origin and note that \( U \) is simply connected. Let

\[
U^* = U \cap \{ |z| > e^{-n-k} \}.
\]

Since \( \gamma(0,1) \subset \{ |z| > e^{-n-k} \} \) and \( |\gamma(1)| = e^{-n-k}, \) we can see that \( U^* \) is simply connected with \( \eta \cup \partial\mathbb{D}_{n+k} \subset \partial U^* \). Let

\[
g : \mathcal{R}_L \to U^*
\]

be a conformal transformation mapping \( \partial_0 \) onto \( \partial\mathbb{D}_{n+k} \) and \( \partial_L \) onto \( \overline{\eta} \). Such a transformation exists for only one value of \( L \), the \( \pi \)-extremal distance between \( \partial\mathbb{D}_{n+k} \) and \( \overline{\eta} \) in \( U^* \). Since \( \eta \cap \mathbb{D}_k = \emptyset \), and the complement of \( U^* \) contains a curve connecting \( \partial\mathbb{D}_k \) and \( \partial\mathbb{D}_{n+k} \), see that \( L \geq n/2 \geq 2 \) (this can be done by comparison with an annulus, see. e.g., [5 Example 3.72]). We write

\[
\partial U^* = \partial\mathbb{D}_{n+k} \cup \overline{\eta} \cup \partial_\gamma \cup \partial_\gamma^+
\]

where \( \partial_\gamma \) (resp. \( \partial_\gamma^+ \)) is the image of \( \partial\mathbb{D}_L \) (resp. \( \partial\mathbb{D}_L^+ \)) under \( g \). Here we are considering boundaries in terms of prime ends, e.g., if \( \gamma \) is simple then each point on \( \gamma(0,1) \)
corresponds to two points in \( \partial D \). Note that \( \{F(\partial_{-}), F(\partial_{+})\} = \{V_{1}, V_{2}\} \), so we can rewrite the conclusion of the lemma as
\[
\begin{aligned}
&
\end{aligned}
\]
(20) 
\[
\begin{aligned}
&
\end{aligned}
\]

Let \( \ell = g(1 + i(0, \pi)) \) which separates \( \partial \mathbb{D}_{n+k} \) from \( \eta \), and hence also separates the origin from \( \eta \) in \( U \). Let \( B_{\ell} \) be a Brownian motion starting at the origin and let \( \sigma = \inf\{t : B_{t} \in \ell\}, \quad \tau = \inf\{t : B_{t} \not\in U\} \).

Using conformal invariance and (17), we can see that if \( z \) exits \( U^{*} \) at \( \partial \mathbb{D}_{n+k} \) at \( \eta \) is contained in \( \mathbb{D}_{n+k} \). We claim that there exists \( c \) such that the probability that a Brownian motion starting at \( z \in \mathbb{D}_{n+k} \) exits \( U^{*} \) at \( \eta \) is bounded above by \( c e^{-n/2} (\theta_{2} - \theta_{1}) \). Indeed, the Beurling estimate implies that the probability to reach \( \partial \mathbb{D}_{k+1} \) without leaving \( U^{*} \) is \( O(e^{-n/2}) \), and using the Poisson kernel in the disk we know that the probability that a Brownian motion starting on \( \partial \mathbb{D}_{k+1} \) exits \( \mathbb{D}_{k} \) at \( \eta \) is bounded above by \( c (\theta_{2} - \theta_{1}) \).

Using conformal invariance and (17), we can see that if \( z \in g(V) \), the probability that a Brownian motion starting at \( z \) exits \( U^{*} \) at \( \partial \mathbb{D}_{n+k} \) is at least \( c \).

Hence from Lemma 2.5 and (16) we see that there exists \( c \) such that the probability that \( \min \{\{B_{t} \in \partial_{-} \mid \sigma < \tau\}, \{B_{t} \in \partial_{+} \mid \sigma < \tau\}\} \geq c \).

Combining this with (17) we see that
\[
\begin{aligned}
&
\end{aligned}
\]
(21) 
\[
\begin{aligned}
&
\end{aligned}
\]

Using (19) and conformal invariance, we can see that
\[
\begin{aligned}
&
\end{aligned}
\]
and combining this with (17) we see that
\[
\begin{aligned}
&
\end{aligned}
\]
\[
\begin{aligned}
&
\end{aligned}
\]
In particular, there exists \( c \) such that
\[
\begin{aligned}
&
\end{aligned}
\]
Combining this with (21), we get (20). \( \square \)

2.3.5. An estimate for two-sided radial. Recall that \( \psi_{n,k} \) is the first time after \( \rho_{n} \) that the curve \( \gamma \) intersects \( l_{n,k}^{*} \), the crosscut defining \( V_{n,k}^{*} \).

Lemma 2.7. There exists \( c < \infty \) such that if \( 0 < k < n \) and \( \psi = \psi_{n,k}^{*} < \infty \), then
\[
\begin{aligned}
&
\end{aligned}
\]
\[
\begin{aligned}
&
\end{aligned}
\]
Proof. Let \( \eta = l_{n,k}^{*} \) and let \( U^{*} \) be as in the proof of Lemma 2.6. Since \( \eta \) disconnects \( -1 \) from 0, we can see that when we write
\[
\begin{aligned}
&
\end{aligned}
\]
\[
\begin{aligned}
&
\end{aligned}
\]
then \( \partial_{-} \subset \partial_{-}U, \partial_{+} \subset \partial_{+}U \) (or the other way around). We also have a universal lower bound on \( h_{H_{n} \setminus \eta}(0, \eta) \). Hence from Lemma 2.5 and (16) we see that
\[
\begin{aligned}
&
\end{aligned}
\]
There is a crosscut \( l \) of \( D_{\psi} \) that is contained in \( l^{*} \), has one of its endpoints equal to \( \gamma(\psi) \), and such that 0 is disconnected from \(-1 \) in \( D_{\psi} \) by \( l \). If \( V \) denotes the
connected component of $D_{\psi_n} \setminus l$ containing the origin, then $\partial V \cap \partial D_\psi$ (considered as prime ends) is contained in either $\partial_+ D_\psi$ or $\partial_- D_\psi$. Therefore,

$$S_{D_\psi}(0) \leq c h_{D_\psi \setminus l}(0, l) \leq c h_{H_n \setminus \eta}(0, \eta) \leq c e^{(k-n)/2} S_{H_n}(0).$$

\[ \square \]

\[ \clubsuit \] This proof uses strongly the fact that $l^*$ separates $-1$ from $0$ in $H_n$. The reader may wish to draw some pictures to see that for other crosscuts $l \in A_{n,k}$, $S_{D_\psi}(0)$ need not be small.

3. SCHRAMM-LOEWNER EVOLUTION (SLE)

Suppose $D$ is a simply connected domain with two distinct boundary points $w_1, w_2$ and one interior point $z$. There are three closely related versions of $SLE_\kappa$ in $D$: chordal $SLE_\kappa$ from $w_1$ to $w_2$; radial from $w_1$ to $z$; and two-sided radial from $w_1$ to $w_2$ going through $z$. The last of these can be thought of as chordal $SLE_\kappa$ from $w_1$ to $w_2$ conditioned to go through $z$. All of these processes are conformally invariant and are defined only up to increasing reparametrizations. Usually chordal $SLE_\kappa$ is parametrized using a “half-plane” or “chordal” capacity with respect to $w_2$ and radial and two-sided radial $SLE_\kappa$ are defined with a radial parametrization with respect to $z$, but this is only a convenience. If the same parametrization is used for all three processes, then they are mutually absolutely continuous with each other if one stops the process at a time before which that paths separate $z$ and $w_2$ in the domain.

We now give precise definitions. For ease we will choose $D = \mathbb{D}$, $z = 0$, $w_1 = 1$ and $w_2 = w = e^{2i\theta_0}$ with $0 < \theta_0 < \pi$. We will use a radial parametrization. We first define radial $SLE_\kappa$ (for which the point $w$ plays no role in the definition) and then define chordal $SLE_\kappa$ (for which the point $0$ is irrelevant when one considers processes up to reparametrization but is important here since our parametrization depends on this point) and two-sided $SLE_\kappa$ in terms of radial. The definition using the Girsanov transformation is really just one example of a general process of producing “$SLE(\kappa, \rho)$ processes”.

Let $h_t(x)$ be the solution of (8) with $h_0(x) = \theta_0$ and let

$$X_t = h_t(w) - U_t, \quad S_t = \sin X_t.$$

Note that $S_t$ is the same as defined in Section 2.3.2 and

$$h_t'(w) = \exp \left\{ -a \int_0^t \frac{ds}{S_s^2} \right\}.$$

Let

$$\tau_\epsilon = \tau_\epsilon(w) = \inf\{t \geq 0 : S_t = \epsilon\},$$

$$\tau = \tau_0(w) = \inf\{t \geq 0 : S_t = 0\} = \inf \{ t : \text{dist}(w, \mathbb{D} \setminus D_t) = 0 \}.$$
3.1. **Radial SLE**. If \( \kappa > 0 \), then radial SLE \( \kappa \) (parametrized so that \( g_t'(0) = e^{a t / \kappa} \)) is the solution of the Loewner equation \( \partial_t g_t(z) = \frac{a}{g_t(z) - U_t} \) with \( a = 2 / \kappa \) and \( U_t = -B_t \) where \( B_t \) is a standard Brownian motion. This definition does not reference the point \( w \).

However, if we define \( X_t \) by \( \text{(22)} \), we have

\[
dX_t = a \cot X_t \, dt + dB_t.
\]

Suppose that \( (\Omega, \mathcal{F}, \mathbb{P}_0) \) is a probability space under which \( X_t \) is a Brownian motion. Then, see Section 2.2, for each \( \beta \in \mathbb{R} \) there is a probability \( \mathbb{P}_\beta \) such that

\[
B_t,\beta = X_t - \beta \int_0^t \cot X_s \, ds, \quad 0 \leq s < \tau,
\]

is a standard Brownian motion. In other words,

\[
dX_t = \beta \cot X_t \, dt + dB_t,\beta.
\]

In particular, \( B_t = B_t,0 \). We call this radial SLE \( \kappa \) weighted locally by \( S_{\beta-t} \), where \( S_t = \sin X_t \). Radial SLE \( \kappa \) is obtained by choosing \( \beta = a \). Using \( \text{(23)} \) we can write the local martingale in \( \text{(11)} \) as

\[
M_{t,\beta} = S_{\beta-t} e^{t^2 / 2 \beta} h_t'(w)^{\beta(\beta-1)} / 2a.
\]

We summarize the discussion in Section 2.2 as follows. If \( \sigma \) is a stopping time, let \( \mathcal{F}_\sigma \) denote the \( \sigma \)-algebra generated by \( \{X_s : 0 \leq s < \infty\} \).

**Lemma 3.1.** Suppose \( \sigma \) is a stopping time with \( \sigma \leq \tau_\varepsilon \) for some \( \varepsilon > 0 \). Then the measures \( \mathbb{P}_\alpha \) and \( \mathbb{P}_\beta \) are mutually absolutely continuous on \( \Omega, \mathcal{F}_\sigma \).

More precisely, if \( t_0 < \infty \), there exists \( c = c(\varepsilon, t_0, \alpha, \beta) < \infty \) such that if \( \sigma \leq \tau_\varepsilon \land t_0 \),

\[
\frac{1}{c} \leq \frac{d\mathbb{P}_\alpha}{d\mathbb{P}_\beta} \leq c.
\]

Clearly we can give more precise estimates for the Radon-Nikodym derivative, but this is all we will need in this paper.

Different values of \( \beta \) give different processes; chordal and two-sided radial SLE \( \kappa \) correspond to particular values.

3.2. **Chordal SLE** \( \kappa \): \( \beta = 1 - 2a \). Chordal SLE \( \kappa \) (from \( 1 \) to \( w \) in \( \mathbb{D} \) in the radial parametrization stopped at time \( T_w \)) is obtained from radial SLE \( \kappa \) by weighting locally by \( S_{t-3a}^1 \). In other words,

\[
dX_t = (1 - 2a) \cot X_t \, dt + dB_{t,1-2a},
\]

where \( B_{t,1-2a} \) is a Brownian motion with respect to \( \mathbb{P}_{1-2a} \).

This is not the usual way chordal SLE \( \kappa \) is defined so let us relate this to the usual definition. SLE \( \kappa \) from \( 0 \) to \( \infty \) in \( \mathbb{H} \) is defined by considering the Loewner equation

\[
\partial_t g_t(z) = \frac{a}{g_t(z) - U_t},
\]

where \( U_t = -B_t \) is a standard Brownian motion. There is a random curve \( \gamma : [0, \infty) \to \mathbb{H} \) such that the domain of \( g_t \) is the unbounded component of \( \mathbb{H} \setminus \gamma(0,t) \). SLE \( \kappa \) connected boundary points of other simply connected domains is defined (modulo time change) by conformal transformation. One can use Itô’s formula to check that our definition agrees (up to time change) with the usual definition.
If $D$ is a simply connected domain and $w_1, w_2$ are boundary points at which $\partial D$ is locally smooth, the chordal $SLE_\kappa$ partition function is defined (up to an unimportant multiplicative constant) by
\[
H_H(x_1, x_2) = |x_2 - x_1|^{-2b},
\]
and the scaling rule
\[
H_D(w_1, w_2) = |f'(w_1)|^b |f'(w_2)|^b H_{f(D)}(f(w_1), f(w_2)),
\]
where $b = (3a - 1)/2$ is the boundary scaling exponent. To obtain $SLE_\kappa$ from 0 to $x$ in $\mathbb{H}$ one can take $SLE_\kappa$ from 0 to $\infty$ and then weight locally by the value of the partition function between $g_t(x)$ and $U_t$, i.e., by $|g_t(x) - U_t|^{-2b}$. A simple computation shows that
\[
H_H(e^{2i\theta_1}, e^{2i\theta_2}) = |\sin(\theta_1 - \theta_2)|^{-2b} = |\sin(\theta_1 - \theta_2)|^{1-3a}.
\]
Hence we see that chordal $SLE_\kappa$ in $D$ is obtained from radial $SLE_\kappa$ by weighting locally by the chordal partition function.

### 3.3. Two-sided radial $SLE_\kappa$: $\beta = 2a$

If $\kappa < 8$, Two-sided radial $SLE_\kappa$ (from 1 to $w$ in $\mathbb{H}$ going through 0 stopped when it reaches 0) is obtained by weighting chordal $SLE_\kappa$ locally by $(\sin X_t)^{(4a-1)}$. Equivalently, we can think of this as weighing radial $SLE_\kappa$ locally by $(\sin X_t)^a$. It should be considered as chordal $SLE_\kappa$ from 1 to $w$ conditioned to go through 0.

If $\kappa \geq 8$, $SLE_\kappa$ paths are plane-filling and hence conditioning the path to go through a point is a trivial conditioning. For this reason, the discussion of two-sided radial is restricted to $\kappa < 8$.

The definition comes from the Green's function for chordal $SLE_\kappa$. If $\gamma$ is a chordal $SLE_\kappa$ curve from 0 to $\infty$ and $z \in \mathbb{H}$, let $R_t$ denote the conformal radius of the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$ with respect to $z$, and let $R = \lim_{t \to \infty} R_t$. The Green’s function $G(z) = G_H(z; 0, \infty)$ can be defined (up to multiplicative constant) by the relation
\[
P\{R_t \leq \epsilon\} \sim c \epsilon^{2-d} G(z), \quad z \to \infty,
\]
where $d = \max\{1+\frac{\kappa}{2}, 2\}$ is the Hausdorff dimension of the paths. Roughly speaking, the probability that a chordal $SLE_\kappa$ in $\mathbb{H}$ from 0 to $\infty$ gets within distance $\epsilon$ of $z$ looks like $c G(z) \epsilon^{2-d}$. For other simply connected domains, the Green’s function is obtained by conformal covariance
\[
G_D(z; w_1, w_2) = |f'(z)|^{2-d} G_{f(D)}(f(z); f(w_1), f(w_2)),
\]
assuming smoothness at the boundary. In particular, one can show that (up to an unimportant multiplicative constant)
\[
G_D(0; 1, e^{i\theta}) = (\sin \theta)^{4a-1}, \quad \kappa < 8.
\]
4. Proofs of main results

4.1. Continuity of radial SLE. The key step to proving continuity of radial \( \text{SLE}_\kappa \) is an extension of an estimate for chordal \( \text{SLE}_\kappa \) to radial \( \text{SLE}_\kappa \). The next lemma gives the analogous estimate for chordal \( \text{SLE}_\kappa \); a proof can be found in [1]. Recall that \( \alpha = (8/\kappa - 1) \).

**Lemma 4.1.** For every \( 0 < \kappa < 8 \), there exists \( c < \infty \) such that if \( \eta \) is a crosscut in \( \mathbb{H} \) and \( \gamma \) is a chordal \( \text{SLE}_\kappa \) curve from \( 0 \) to \( \infty \) in \( \mathbb{H} \), then

\[
P\{\gamma(0, \infty) \cap \eta \neq \emptyset\} \leq c \left( \frac{\text{diam}(\eta)}{\text{dist}(0, \eta)} \right)^\alpha.
\]

We will prove the corresponding result for radial \( \text{SLE}_\kappa \). We start by establishing the estimate up to a fixed time (this is the hardest estimate), and then extending the result to infinite time.

**Lemma 4.2.** For every \( t < \infty \), there exists \( C_t < \infty \) such that the following holds. Suppose \( \eta \) is a crosscut of \( \mathbb{D} \) and \( \gamma \) is a radial \( \text{SLE}_\kappa \) curve from \( 1 \) to \( 0 \) in \( \mathbb{D} \). Then

\[
P\{\gamma(0, t] \cap \eta \neq \emptyset\} \leq C_t \left( \frac{\text{diam}(\eta)}{\text{dist}(1, \eta)} \right)^\alpha.
\]

**Proof.** Fix a positive integer \( n \) sufficiently large so that \( \gamma(0, t] \cap \mathbb{D}_n = \emptyset \). All constants in this proof may depend on \( n \) (and hence on \( t \)).

Since \( \text{dist}(1, \eta) \leq 2 \), it suffices to prove the lemma for crosscuts satisfying \( \text{diam}(\eta) < 1/100 \) and \( \text{dist}(1, \eta) > 100 \text{diam}(\eta) \). Such crosscuts do not disconnect 1 from 0 in \( \mathbb{D} \).

Let \( V = V_0 \) denote the connected component of \( \mathbb{D} \setminus \eta \) containing the origin, and let \( F = F_\gamma \) be a conformal transformation of \( V \) onto \( \mathbb{D} \) with \( F(0) = 0 \). We write \( \partial V \) as a disjoint union:

\[
\partial V = \{1\} \cup \eta[0, 1] \cup \partial_1 \cup \partial_2,
\]

where \( \partial_1, \partial_2 \) are open connected subarcs of \( \partial \mathbb{D} \). Let

\[
L(\eta) = \frac{1}{2\pi} |F(\eta)| = h_{V_0}(0, \eta),
\]

\[
L^*(\eta) = \frac{1}{2\pi} \min \{|F(\partial_1)|, |F(\partial_2)|\} = \min \{h_{V_0}(0, \partial_1), h_{V_0}(0, \partial_2)\},
\]

where \( | \cdot | \) denotes length. Note that

\[
\text{diam}(\eta) \asymp L(\eta), \quad \text{dist}(1, \eta) \asymp L^*(\eta),
\]

and hence we can write the conclusion of the lemma as

\[
P\{\gamma(0, t] \cap \eta \neq \emptyset\} \leq C \left( \frac{L(\eta)}{L^*(\eta)} \right)^\alpha,
\]

which is what we will prove.

Let \( \gamma \) be a radial \( \text{SLE}_\kappa \) curve. If \( \gamma(0, t] \cap \eta = \emptyset \) and \( \eta(0, 1) \subset D_t \), let \( V_t \) be the connected component of \( D_t \setminus \eta \) containing the origin with corresponding maps \( F_t \). We write

\[
\partial V_t = \{\gamma(t)\} \cup \eta[0, 1] \cup \partial_{1,t} \cup \partial_{2,t},
\]

where the boundaries are considered in terms of prime ends. Let

\[
L_t(\eta) = \frac{1}{2\pi} |F_t(\eta)| = h_{V_t}(0, \eta),
\]

\[ L_t^*(\eta) = \frac{1}{2\pi} \min \{ |F_t(\partial_1,t)|, |F_t(\partial_2,t)| \} = \min \{ h_{V_t}(0,\partial_1,t), h_{V_t}(0,\partial_2,t) \}. \]

Note that \( L_t(\eta) \) decreases with \( t \) but \( L_t^*(\eta) \) is not monotone in \( t \).

As before, let \( \rho = \rho_n \) be the first time \( s \) that \( |\gamma(s)| \leq e^{-n} \); our assumption on \( n \) implies that \( \rho \geq t \). Let \( \sigma = \sigma_n \) be the first time \( s \) that \( \text{Re}\gamma(s) \leq e^{-2n} \). Our proof will include a series of claims each of which will be proved after their statement.

- **Claim 1.** There exists \( u > 0 \) (depending on \( n \)), such that

\[ \mathbb{P}\{\sigma \wedge \rho \geq t\} \geq u. \]  

Deterministic estimates using the Loewner equation show that if \( U_t \) stays sufficiently close to 0, then \( \rho < \sigma \). Therefore, since \( \rho \geq t \),

\[ \mathbb{P}\{\sigma \wedge \rho > t\} \geq \mathbb{P}\{\rho < \sigma\} > 0. \]

- **Claim 2.** There exists \( c < \infty \) such that

\[ \mathbb{P}\{\gamma(0,\sigma) \cap \eta \neq \emptyset\} \leq c \left( \frac{\text{diam}(\eta)}{\text{dist}(0,\eta)} \right)^\alpha. \]

To show this we compare radial \( \text{SLE}_\kappa \) from 1 to 0 with chordal \( \text{SLE}_\kappa \) from 1 to \(-1\). Note by (10) that \( \sigma \) is uniformly bounded. Straightforward geometric arguments show that there exists \( c \) (recall that constants may depend on \( n \)) such that \( c^{-1} \leq h_\kappa'(-1) \leq c \) and \( \sin X_\kappa \geq c^{-1} \). By (24) the Radon-Nikodym derivative of radial \( \text{SLE}_\kappa \) with respect to chordal \( \text{SLE}_\kappa \) is uniformly bounded away from 0 and \( \infty \) and therefore if \( \tilde{\gamma} \) denotes a chordal \( \text{SLE}_\kappa \) path from 1 to \(-1\),

\[ \mathbb{P}\{\gamma(0,\sigma) \cap \eta = \emptyset\} \leq \mathbb{P}\{\tilde{\gamma}(0,\sigma) \cap \eta = \emptyset\}. \]

Hence (29) follows from (26).

- **Claim 3.** There exists \( \delta > 0 \) such that if \( \Lambda(\eta), L^*(\eta) \leq \delta \), then on the event

\[ \{\gamma(0,\sigma) \cap \eta = \emptyset\}, \]

we have

\[ \frac{L_\sigma(\eta)}{L^*_\sigma(\eta)} \leq \frac{L(\eta)}{L^*(\eta)}. \]

It suffices to consider \( \eta \) with \( \Lambda(\eta), L^*(\eta) \leq 1/10 \), and without loss of generality we assume that \( \eta \) is “above” 1 in the sense that its endpoints are \( e^{i\theta_1}, e^{i\theta_2} \) with \( 0 < \theta_1 < \theta_2 < 1/4 \). Let \( \gamma(\sigma) = e^{-2n} + iy \), and let \( V_\sigma = V_{\gamma,\sigma} \) be the connected component of \( V \setminus \gamma(0,\sigma) \) containing the origin. Suppose \( \gamma(0,\sigma) \cap \eta = \emptyset \) and

\[ \eta \subset V_\sigma, \]

(If \( \eta \not\subset V_\sigma \), then \( L_\sigma(\eta) = 0 \).) As before we write

\[ \partial V_\sigma = \{\gamma(\sigma)\} \cup \eta[0,1] \cup \partial_1, \sigma \cup \partial_2, \sigma, \]

where we write \( \partial_1, \sigma \) for the component of the boundary that includes \(-1\). Note that (30) and (32) imply that \( \partial_1, \sigma \) in fact contains \( \{e^{i\theta} : \theta_2 < \theta < 3\pi/2\} \). Let \( \ell \) denote the crosscut of \( V_\sigma \) given by the vertical line segment whose lowest point is \( \gamma(\sigma) \) and whose highest point is on \( \{e^{i\theta} : 0 < \theta < \pi/2\} \). Note that \( V_\sigma \setminus \ell \) has two connected components, one containing the origin and the other, which we denote by \( V^* = V_{\sigma,\eta}^* \), with \( \eta \subset \partial V^* \). Let \( \epsilon \) denote the length of \( \ell \) and for the moment assume that \( \epsilon < 1/4 \). Topological considerations using (30) and (32) imply that all
the points in $\partial V^* \cap \{ z \in \mathbb{D} : e^{-2n} < \text{Re}(z) < e^{-1} \}$ (considered as prime ends) are in $\partial_{2,\sigma}$.

We consider another crosscut $\ell'$ defined as follows. Let $x = e^{-2n} + \epsilon$. Start at $x + i \sqrt{1 - x^2} \in \partial \mathbb{D}$ and take a vertical segment downward of length $2\epsilon$ to the point $z' = x + i (\sqrt{1 - x^2} - 2\epsilon)$. From $z'$ take a horizontal segment to the left ending at $\{ \text{Re}(z) = e^{-2n} \}$. This curve, which is a concatenation of two line segments, must intersect the path $\gamma(0, \sigma)$ at some point; let $\ell'$ be the crosscut obtained by stopping this curve at the first such intersection. Let $V'$ be the connected component of $V^* \setminus \ell'$ that contains $\ell$ on its boundary. The key observations are: $\text{dist}(\ell, \ell') \geq c\epsilon$, $\text{diam}(\ell) = O(\epsilon)$, $\text{diam}(\ell') = O(\epsilon)$ and $\text{area}(V') = O(\epsilon^2)$. In particular (see, e.g., [5, Lemma 3.74]) the $\pi$-extremal distance between $\ell$ and $\ell'$ is bounded below by a positive constant $c_1$ independent of $\epsilon$.

Let $B_t$ be a standard complex Brownian motion starting at the origin and let

$$\tau = \inf\{ t : B_t \notin V_\sigma \}, \quad \xi = \inf\{ t : B_t \in \ell' \}.$$ 

Then using Lemma [2, 4] we see that

$$\min\{ \mathbb{P}\{ B_\tau \in \partial_{1,\sigma} \}, \mathbb{P}\{ B_\tau \in \partial_{2,\sigma} \} \} \geq c_2 \mathbb{P}\{ \xi < \tau \}. \tag{33}$$

Also, we claim that

$$\mathbb{P}\{ B_\tau \in \eta | \xi < \tau \} \leq c \epsilon L(\eta).$$

To justify this last estimate, note that $\text{dist}(B_\xi, \partial \mathbb{D}) = O(\epsilon)$. It suffices to consider the probability that a Brownian motion starting at $B_\xi$ hits $\eta$ before leaving $\mathbb{D}$. The gambler’s ruin estimate implies that the probability that a Brownian motion starting at $B_\xi$ reaches $\{ \text{Re}(z) = 1/2 \}$ before leaving $\mathbb{D}$ is $O(\epsilon)$. Given that we reach $\{ \text{Re}(z) = 1/2 \}$, the probability to hit $\eta$ before leaving $\mathbb{D}$ is $O(L(\eta))$. Therefore,

$$\mathbb{P}\{ B_\tau \in \eta \} \leq c \epsilon L(\xi) \mathbb{P}\{ \xi < \tau \}. \tag{34}$$

By combining (33) and (34), we see that we can choose $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$,

$$\mathbb{P}\{ B_\tau \in \eta \} \leq L(\eta) \min\{ \mathbb{P}\{ B_\tau \in \partial_{1,\sigma} \}, \mathbb{P}\{ B_\tau \in \partial_{2,\sigma} \} \},$$

and hence

$$L_\sigma(\eta) \leq L(\eta) L^*_\sigma(\eta) \quad \text{if} \quad \epsilon \leq \epsilon_0. \tag{35}$$

One we have fixed $\epsilon_0$, we note there exists $\epsilon = c(\epsilon_0) > 0$ such that if $\epsilon \geq \epsilon_0$,

$$L^*_\sigma(\eta) = \min\{ \mathbb{P}\{ B_\tau \in \partial_{1,\sigma} \}, \mathbb{P}\{ B_\tau \in \partial_{2,\sigma} \} \} \geq c_2.$$

Indeed, to bound $\mathbb{P}\{ B_\tau \in \partial_{1,\sigma} \}$ from below we consider Brownian paths starting at the origin that leave $\partial \mathbb{D}$ before reaching $\{ z : \text{Re}(z) \geq e^{-2n} \}$. To bound $\mathbb{P}\{ B_\tau \in \partial_{2,\sigma} \}$ consider Brownian paths in the disk that start at the origin, go through the crosscut $l$ (defined using $\epsilon = \epsilon_0$), and then make a clockwise loop about $\gamma(\sigma)$ before leaving $\mathbb{D}$ and before reaching $\{ \text{Re}(z) \geq 1/2 \}$. Topological considerations show that these paths exit $V_\sigma$ at $\partial_{2,\sigma}$. Combining this with (33) and the estimate $L_\sigma(\eta) \leq L(\eta)$, we see that there exists $c_1 > 0$ such that for all $\eta$,

$$L_\sigma(\eta) \leq c_1^{-1} L(\eta) L^*_\sigma(\eta).$$

In particular,

$$\frac{L_\sigma(\eta)}{L^*_\sigma(\eta)} \leq \frac{L(\eta)}{L^*(\eta)} \quad \text{if} \quad L^*(\eta) \leq c_1.$$

From this we conclude (33).
Fix \( \delta \) such that (31) holds, and let \( \phi(r) \) be the supremum of
\[
P\{\gamma(0,t] \cap \eta \neq \emptyset\}
\]
where the supremum is over all \( \eta \) with
\[
L(\eta) \leq r \min\{\delta, L^*(\eta)\}.
\]

**Claim 4.** If \( r < \delta \), then \( \phi(r) \) equals \( \tilde{\phi}(r) \) which is defined to be the supremum of
\[
P\{\gamma(0,t] \cap \eta \neq \emptyset\}
\]
where the supremum is over all \( \eta \) with
\[
L(\eta) \leq \min\{\delta, r L^*(\eta)\}.
\]
To see this, suppose \( \eta \) is a curve with \( L(\eta) \leq \delta, L^*(\eta) > \delta \). Let \( S \) be the first time \( s \) such that \( L_s^*(\eta) = \delta \). Note that \( S < \inf\{s : \gamma(s) \in \eta\} \). Since \( L(\eta) \leq r \delta \), we see that
\[
P\{\gamma(0,t] \cap \eta \neq \emptyset\} \leq P\{\gamma(0,t] \cap \eta \neq \emptyset | S < \infty\} \leq \tilde{\phi}(r).
\]
This establishes the claim.

To finish the proof of the lemma, suppose \( r < \delta \). Since \( \tilde{\phi}(r) = \phi(r) \), we can find a crosscut \( \eta \) with \( L(\eta) \leq r L^*(\eta) \) and \( L^*(\eta) \leq \delta \) such that
\[
\phi(r) = P\{\gamma(0,t] \cap \eta \neq \emptyset\}.
\]
(For notational ease we are assuming the supremum is obtained. We do not need to assume this, but could rather take a sequence of crosscuts \( \eta_j \) with \( P\{\gamma(0,t] \cap \eta_j \neq \emptyset\} \to \phi(r) \).) Using Claim 3, we see that if \( \gamma(0,\sigma] \cap \eta = \emptyset \), then
\[
L(\sigma) \leq L(\eta) \leq r, \quad L^*(\sigma) \leq L^*(\eta).
\]
Therefore,
\[
P\{\gamma(0,t] \cap \eta \neq \emptyset | \gamma(0,\sigma \wedge \rho] \cap \eta = \emptyset\} \leq \phi(r).
\]
Hence, using (28),
\[
\phi(r) = P\{\gamma(0,t] \cap \eta \neq \emptyset\} \leq P\{\gamma(0,\sigma \wedge \rho] \cap \eta \neq \emptyset\} + (1 - u) \phi(r),
\]
which implies
\[
\phi(r) \leq \frac{1}{1 - u} P\{\gamma(0,\sigma \wedge \rho] \cap \eta \neq \emptyset\}.
\]
Combining this with (29), we get
\[
\phi(r) \leq \frac{c}{1 - u} \left[ \frac{L(\eta)}{L^*(\eta)} \right]^{\alpha} \leq c' r^\alpha.
\]

**Proposition 4.3.** If \( 0 < \kappa < 8 \), there exists \( c < \infty \) such that the following holds. Suppose \( \eta \) is a crosscut of \( \mathbb{D} \) and \( \gamma \) is a radial \( \text{SLE}_\kappa \) curve from 1 to 0 in \( \mathbb{D} \). Then
\[
P\{\gamma(0,\infty) \cap \eta \neq \emptyset\} \leq c \left[ \frac{\text{diam}(\eta)}{\text{dist}(1,\eta)} \right]^\alpha.
\]
Proof. We may assume that $\eta \cap \mathbb{D}_1 = \emptyset$. By Lemma 2.6 for $n \geq 5$, conditioned on $\gamma[0, \rho_n] = \emptyset$, we know that

$$L_{\rho_n} \leq c L[\eta] e^{-n/2} L^*_{\rho_n}.$$ 

Since $\rho_{n+1} - \rho_n$ is uniformly bounded in $n$, we can use Lemma 4.2 to conclude that

$$\mathbb{P}\{\gamma[0, \rho_5] \cap \eta \neq \emptyset\} \leq c \left[ \frac{L[\eta]}{L^*[\eta]} \right]^\alpha,$$

and for $n \geq 5$,

$$\mathbb{P}\{\gamma[0, \rho_{n+1}] \cap \eta \neq \emptyset \mid \gamma[0, \rho_n] \cap \eta = \emptyset\} \leq c L[\eta]^\alpha e^{-\alpha / 2} \leq c e^{-\alpha / 2} \left[ \frac{L[\eta]}{L^*[\eta]} \right]^\alpha.$$ 

By summing over $n$ we get the proposition.

Proof of Theorem 1. We start by proving the stronger result for $\kappa \leq 4$. Note that $\partial \mathbb{D}_k \cap H_n$ is a disjoint union of crosscuts $\eta = \{ e^{-k+\theta} : \theta_1, \eta < \theta < \theta_2, \eta \}$. For each $\eta$, we use Lemma 2.6 and Proposition 4.3 to see that

$$\mathbb{P}\{\gamma[\rho_{n+k}, \infty) \cap \eta \neq \emptyset \mid \mathcal{F}_{\rho_n}\} \leq c e^{-\alpha / 2} (\theta_2 - \theta_1)^\alpha.$$ 

However, since $\alpha \geq 1$ (here we use the fact that $\alpha \leq 4$),

(37) \[ \sum_{\eta} (\theta_2, \eta - \theta_1, \eta)^\alpha \leq \left[ \sum_{\eta} (\theta_2, \eta - \theta_1, \eta) \right]^\alpha \leq (2\pi)^\alpha. \]

We will now prove (2), assuming only $\kappa < 8$. Let $E = E_{j, k, n}$ denote the event $\gamma[\rho_k, \rho_{n+k}] \subset \mathbb{D}_j$. Lemma 2.5 implies that on the event $E$, there is a unique crosscut $l \in A_{n+k, k}$ such that every curve from the origin to $\partial \mathbb{D}_j$ in $H_{n+k}$ intersects $l$. Hence, on $E$

$$\mathbb{P}\{\gamma[\rho_{n+k}, \infty) \not\subset \mathbb{D}_j \mid \mathcal{G}_{n+k}\}$$

is bounded above by the supremum of

$$\mathbb{P}\{\gamma[\rho_{n+k}, \infty) \cap l \neq \emptyset \mid \mathcal{G}_{n+k}\},$$

where the supremum is over all $l \in A_{n+k, k}$. For each such crosscut $l$, we use Lemma 2.6 and Proposition 4.2 to see that

$$\mathbb{P}\{\gamma[\rho_{n+k}, \infty) \cap l \neq \emptyset \mid \mathcal{G}_n\} \leq c e^{-\alpha / 2}.$$ 

4.2. Two-sided radial $SLE_\kappa$. In order to prove that two-sided radial $SLE_\kappa$ is continuous at the origin, we will prove the following estimate. It is the analogue of Proposition 4.1 restricted to the crosscut that separates the origin from $-1$.

Proposition 4.4. If $\kappa < 8$ there exist $c'$ such if $\gamma$ is two-sided radial from 1 to $-1$ through 0 in $\mathbb{D}$, then for all $k$, $n > 0$, if $l = l^*_{n+k, k}$,

(38) \[ \mathbb{P}\{\gamma[\rho_{n+k}, \infty) \cap \bar{l} \neq \emptyset \mid \mathcal{G}_{n+k}\} \leq c' e^{-\alpha / 2}. \]

Proof. Let $\rho = \rho_{n+k}$ and as in Lemma 2.7 let $\psi = \psi_{n+k, k}$ be the first time $t \geq \rho$ that $\gamma(t) \in \bar{l}$. It suffices to show that

$$\mathbb{P}\{\psi < \rho_{n+k+1} \mid \mathcal{G}_{n+k}\} \leq c e^{-\alpha / 2},$$
for then we can iterate and sum over $n$. By Lemma 2.7 we know that
\begin{equation}
S_{\psi}(0) \leq c e^{-n/2} S_{\rho}(0).
\end{equation}
Also, (10) gives $\rho_{n+k+1} - \rho \leq c_1$ for some uniform $c_1 < \infty$. Recalling that two-sided $SLE_\kappa$ corresponds to the radial Bessel equation (11) with $\beta = 2a$, we see from Lemma 2.2 that
\begin{equation}
\mathbb{P}\left\{ \min_{0 \leq t \leq \rho + c_1} S_{t}(0) \leq c S_{\rho}(0) \mid G_{n+k} \right\} \leq e^{c_1 - 1/2} = c e^{\alpha}.
\end{equation}
Combining this with (39) gives the first inequality. □

Proof of Theorem 2. To prove (4), we recall Lemma 2.3 which tells us that if $\gamma[\rho_k, \rho_{n+k}] \subset D_j$, then in order for $\gamma[\rho_{n+k}, \infty)$ to intersect $D_j$ is is necessary for it to intersect $I$.

4.3. Proof of Theorem 3. Here we finish the proof of Theorem 3. We have already proved the main estimates (2) and (4). The proof is essentially the same for radial and two-sided radial; we will do the two-sided radial case. We will use the following lemma which has been used by a number of authors to prove exponential rates of convergence, see, e.g., [2]. Since it is not very long, we give the proof. An important thing to note about the proof is that it does not give a good estimate for the exponent $u$.

Lemma 4.5. Let $\epsilon_j$ be a decreasing sequence of numbers in $[0,1)$ such that
\begin{equation}
\limsup_{n \to \infty} \epsilon_n^{1/n} < 1.
\end{equation}
Then there exist $c, u$ such that the following holds. Let $X_n$ be a discrete time Markov chain on state space $\{0,1,2\ldots\}$ with transition probabilities
\begin{equation}
p(j,0) = 1 - p(j,j+1) \leq \epsilon_j.
\end{equation}
Then,
\begin{equation}
\mathbb{P}\{X_n < n/2 \mid X_0 = 0\} \leq c e^{-nu}.
\end{equation}

♣ The assumption that $\epsilon_j$ decrease is not needed since one can always consider $\delta_j = \min\{\epsilon_1, \ldots, \epsilon_j\}$ but it makes the coupling argument described below easier.

Proof. We will assume that $p(j,0) = \epsilon_j$. The more general result can be obtained by a simple coupling argument defining $(Y_n, X_n)$ on the same space where
\begin{equation}
\mathbb{P}\{Y_{j+1} = 0 \mid Y_j = n\} = 1 - \mathbb{P}\{Y_{j+1} = n + 1 \mid Y_j = n\} = \epsilon_n,
\end{equation}
in a way such that $Y_n \leq X_n$ for all $n$. Let $p_n = \mathbb{P}\{X_n = 0 \mid X_0 = 0\}$, with corresponding generating function
\begin{equation}
G(\xi) = \sum_{n=0}^{\infty} p_n \xi^n.
\end{equation}
Let
\begin{equation}
\delta = \mathbb{P}\{X_n \neq 0 \text{ for all } n \geq 1 \mid X_0 = 0\} = \prod_{n=0}^{\infty} [1 - p(n,n+1)] = \prod_{n=0}^{\infty} [1 - \epsilon_n] > 0.
\end{equation}
For $n \geq 1$, let
\[ P\{X_n = 0; X_j \neq 0, 1 \leq j \leq n - 1 \mid X_0 = 0\} \]
with generating function
\[ F(\xi) = \sum_{n=1}^{\infty} q_n \xi^n. \]
Note that
\[ q_n = p(0,1) \cdot p(1,2) \cdots \cdot p(n-2,n-1) \cdot p(n-1,0) \leq \epsilon_{n-1}. \]
Therefore, (40) implies that the radius of convergence of $F$ is strictly greater than 1. Since $F(1) = 1 - \delta < 1$, we can find $t > 1$ with $F(t) < 1$, and hence
\[ G(t) = [1 - F(t)]^{-1} < \infty, \]
In particular, if $e^{2u} < t$, then there exists $c < \infty$ such that for all $n$,
\[ p_n \leq c e^{-2un}. \]
Let $A_n$ be the event that $X_m = 0$ for some $m \geq n/2$. Then,
\[ P(A_n) \leq \sum_{j \geq n/2} p_j \leq c' e^{-un}. \]
But on the complement of $A_n$, we can see that $X_n \geq n/2$.

Proof of Theorem 3. The proof is the same for radial or two-sided $SLE_\kappa$. Let us assume the latter. The important observation is that for every $0 < k < m < \infty$, we can find $\epsilon > 0$ such that for all $n$,
\[ P\{\gamma[\rho_{n+k}, \rho_{n+5+m+k}] \subset D_{n+5} \mid G_n\} \geq \epsilon. \]
(This can be shown by considering the event that the driving function stays almost constant for a long interval of time after $\rho_n$. We omit the details.) By combining this with Proposition 4.4, we can see that there exists $m, \epsilon$ such that
\[ P\{\gamma[\rho_n + m, \infty) \subset D_{n+5} \mid G_n\} \geq \epsilon. \]
To finish the argument, let us fix $k$. Let $c', u = \alpha/4$ be the constants from (41) and let $m$ be sufficiently large so that $c' e^{-nu} \leq 1/2$ for $n \geq m$. For positive integer $n$ define $L_n$ to be the largest integer $j$ such that
\[ \gamma[\rho_{n+k-j}, \rho_{n+k}] \subset D_k. \]
The integer $j$ exists but could equal zero. From (41), we know that
\[ P\{L_{n+k+1} = L_{n+k} + 1 \mid G_{n+k}\} \geq 1 - c' e^{-nL_{n+k} u}, \]
and if $L_{n+k} \geq m$, the right-hand side is greater than 1/2.
We see that the distribution of $L_{n+k}$ is stochastically bounded below by that of a Markov chain $X_n$ of the type in Lemma 4.3 Using this we see that there exists $C', \delta$ such that
\[ P\{L_{n+k} \leq n/2 \mid G_k\} \leq C' e^{-\delta n}. \]
On the event $P\{L_{n+k} \geq n/2\}$, we can use (55) to conclude that the conditional probability of returning to $\partial D_k$ after time $\rho_{n+k}$ given $L_{n+k} \geq n/2$ is $O(e^{-n\alpha/4})$. This completes the proof with $u = \min\{\delta, \alpha/4\}$. \qed
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