1. Introduction

Rossby waves and gravity waves, which are two important ingredients of meteorological disturbances, are ubiquitous in the atmosphere. As is well known, where there are vorticity (or potential vorticity) gradients, there are Rossby waves. In addition to the well-known horizontal Rossby waves associated with horizontal gradient of vertical vorticity (e.g., planetary Rossby waves), there exist vertical Rossby waves associated with vertical gradient of horizontal vorticity. As an example of a system with vertical Rossby waves, we consider a vertical-zonal two-dimensional flow having a downward gradient of basic horizontal vorticity at a level $z = z_2$ (see Fig. 1). Let a positive vorticity disturbance $q_2 > 0$ be initially present at $z = z_2$. The clockwise circulation induced by $q_2 > 0$ advects the basic vorticity $\mathcal{Q}$. Positive vorticity is advected by the circulation on the west side (i.e., left side) of $q_2$. Thus, the positive vorticity disturbance is displaced westward ($q_2 \rightarrow q_2'$). In the same way, the
Vorticity disturbance continues to propagate westward thereafter (from here on, propagation means propagation relative to the fluid). This is the Rossby wave. At the same time, the Rossby wave is advected eastward (i.e., to the right) by the basic flow $\bar{u}(z)$ there.

Similarly, where there are upward buoyancy gradients (i.e., potential temperature gradients), there exist gravity waves. As an example, we again consider a vertical-zonal two-dimensional flow having a basic upward buoyancy gradient at the level $z = z_1$ (Fig. 2). According to the buoyancy–vorticity wave interaction approach (hereafter called BV-thinking) of Harnik et al. (2008), we can interpret the propagation of gravity waves as follows. Let a positive vorticity disturbance $q_1 > 0$ and a positive buoyancy disturbance $b_1 > 0$ be initially present at $z = z_1$. The clockwise circulation induced by $q_1$ advects the basic buoyancy $b_1$. Larger buoyancy is advected by the circulation on the east side (i.e., right side) of $b_1$, and smaller buoyancy is advected by the circulation on the west side (i.e., left side) of $b_1$. Thus, the positive buoyancy disturbance is displaced eastward ($b_1 \to b'_1$).

Furthermore, the positive buoyancy disturbance $b_1 > 0$ implies an upward buoyancy force there, which generates a vorticity pair. Positive vorticity is generated by the buoyancy force on the east side of $b_1$, and negative vorticity is generated by the buoyancy force on the west side of $b_1$. Consequently, the positive vorticity disturbance is displaced eastward ($q_1 \to q'_1$). Combined, the initial buoyancy–vorticity disturbance is displaced eastward ($b_1 \cdot q_1 \to b'_1 \cdot q'_1$). In a similar manner, the buoyancy–vorticity disturbance continues to propagate eastward thereafter. This is the gravity wave. If the initial disturbance $b_1 > 0$, then the gravity wave propagates eastward. In contrast, if the initial disturbance $b_1 < 0$, then the gravity wave propagates westward.

Neither the Rossby wave nor the gravity wave alone can grow by itself. If an initially small wave does not grow and remains small, it does not participate in actual phenomena. Two mechanisms relevant to the growth of Rossby waves, namely barotropic instability (e.g., Heifetz et al. 1999) and baroclinic instability (e.g., Bretherton 1966), are well known.
In spite of their rather different appearances, the two instability mechanisms are of the same essential nature. That is, their growth is a result of resonant interaction between two phase-locked counter-propagating Rossby waves (e.g., Baines and Mitsudera 1994).

For example, the barotropic instability in a tropical cyclone is caused by two vortex Rossby waves at the inner and outer edges of the annular vortex ring (e.g., Montgomery and Kallenbach 1997; Ito and Kanehisa 2013). The inner vortex Rossby wave propagates cyclonically, and the outer vortex Rossby wave propagates anti-cyclonically. The inner cyclonic advection by the vortex flow is weaker, and the outer cyclonic advection by the vortex flow is stronger. These enable the two vortex Rossby waves to be phase-locked.

Moreover, for example, the synoptic-scale baroclinic instability in the mid-latitudes is caused by two synoptic-scale Rossby waves on the lower and upper levels. The lower Rossby wave manifests itself as an extratropical cyclone, and the upper Rossby wave manifests itself as an upper potential vorticity anomaly. The lower Rossby wave propagates eastward, and the upper Rossby wave propagates westward. The lower advection by the eastward wind is weaker, and the upper advection by the eastward wind is stronger. These enable the two Rossby waves to be phase-locked.

The growth of gravity waves is also caused by essentially the same mechanism as the above-mentioned mechanism for Rossby waves. That is, the growth is a result of the resonant interaction between two phase-locked counter-propagating gravity waves. The physical mechanism of the interaction between gravity and gravity waves is now clearly understood by the BV-thinking of Harnik et al. (2008).

In this paper, we consider the instability caused by the resonance between Rossby and gravity waves. Specifically, we examine the interaction between the above-mentioned Rossby and gravity waves in a vertical-zonal two-dimensional model that has a lower basic buoyancy gradient and an upper basic vorticity gradient (Fig. 3). The westward-propagating upper Rossby wave is advected eastward by basic zonal flow. If the westward propagation of the Rossby wave is so weak as to be phase-locked with the eastward-propagating lower gravity wave, then resonant interaction between them is possible, as is shown in Fig. 4. Because of the phase lock with phase difference $\delta \theta$ between the Rossby and gravity waves of $0 < \delta \theta < \pi$ ($\pi/2$ is optimal for the resonance), the circulation induced by the vorticity disturbance $q_1$ of the lower gravity wave enhances the vorticity disturbance $q_2$ of the upper Rossby wave. Thus, the circulation induced by the vorticity disturbance $q_2$ of the upper Rossby wave enhances the buoyancy disturbance $b_1$ of the lower gravity wave, which in turn enhances the vorticity disturbance $q_1$.

Baines and Mitsudera (1994) have already studied the model depicted in Fig. 3. They looked for disturbances in the form of normal modes and showed the existence of exponentially growing normal-mode solutions.

In this paper, we analytically solve the initial value problem for the disturbance introduced to the basic state depicted in Fig. 3. Following BV-thinking, we choose the buoyancy and vorticity disturbances, $b$ and $q$, as the fundamental variables, and derive the governing equation for them. By solving the equation we obtain an analytical solution. For the parameter values for exponential instability, the solution clearly shows the temporal evolution of disturbances from an initial value to the resonant instability configuration depicted in Fig. 4 in a manner consistent with BV-thinking. For the parameter values for marginal instability, the solution grows as a linear function of time and asymptotically approaches an in-phase or anti-phase configuration between the upper and lower disturbances. For the parameter values for stability, the upper and lower waves are not phase-locked but pass away from each other, just like free waves. Marginal linear growth, which is common to other instability problems (e.g., Kanehisa and Oda 2009; Oda and Kanehisa 2011), is never trivial in the present model. For small zonal wave numbers, the westward-propagating lower gravity wave is shown to play a role in the resonant interaction between the westward-propagating upper Rossby wave and the eastward-propagating lower gravity wave. The role can be interpreted in terms of BV-thinking in the same way as in the problem of the interaction between gravity and gravity waves (Rabinovich et al. 2011).

The organization of this paper is as follows. The governing equation is derived in Section 2. In Section 3, free (i.e., without interaction) upper Rossby and lower gravity waves are considered. In Section 4, the analytical solution of the initial value problem is derived. The oscillating and exponentially growing solutions are presented in Section 5. In Section 6, the marginal solution is derived and presented. In Sections 7, 8, and 9, the role played by the westward-propagating gravity wave is considered. Finally, Section 10 presents concluding remarks.
2. Governing equation

We begin with the vertical-zonal two-dimensional vorticity and buoyancy equations linearized about a steady basic flow:

\[
\begin{align*}
\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} q &= \frac{\partial}{\partial z} \frac{\partial \psi}{\partial x} + \frac{\partial}{\partial x} b = 0, \\
\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} b &= \frac{\partial}{\partial z} \frac{\partial \psi}{\partial x} = 0,
\end{align*}
\]

where \(x, z,\) and \(t\) are zonal, vertical, and temporal coordinates, respectively. The basic zonal flow and buoyancy are denoted by \(\bar{u}\) and \(\bar{b}\), respectively. The disturbance stream function and buoyancy are denoted by \(\psi\) and \(b\), respectively. The disturbance vorticity \(q\) is expressed in terms of \(\psi\) as \(q = \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x^2}\). The basic vorticity \(\bar{q} = \frac{\partial \psi}{\partial z}\) and buoyancy \(\bar{b}\) are assumed to be piecewise uniform in the vertical direction (Fig. 3).

\[
\begin{align*}
\bar{u} &= \Lambda(z - z_1) & \text{for} \ z < z_2, \\
\bar{b} &= B & \text{for} \ z < z_1,
\end{align*}
\]

and \(\bar{u} = \Lambda(z_2 - z_1) = \Lambda Z\) for \(z_2 < z\),

\[
\begin{align*}
\bar{q} &= \Lambda & \text{for} \ z < z_2, \\
\bar{b} &= B + \gamma & \text{for} \ z_1 < z,
\end{align*}
\]

where \(B, \gamma, \Lambda,\) and \(Z = z_2 - z_1\) are positive constants.
Because the coefficients in (1) for the disturbance are independent of the zonal coordinate $x$, we can perform Fourier transform on the equations with respect to $x$ with wave number $k$ as follows:

$$
\left( \frac{\partial}{\partial t} + i k \mathfrak{m} \right) \hat{q} - i k \frac{\partial}{\partial z} \hat{\psi} + i k \hat{b} = 0,
$$

$$
\left( \frac{\partial}{\partial t} + i k \mathfrak{m} \right) \hat{b} - i k \frac{\partial}{\partial z} \hat{\psi} = 0.
$$

(3)

In terms of the Green function

$$
G(z, z') = -\frac{1}{2|k|} e^{-|k| |z - z'|},
$$

(4)

which is the solution of

$$
\left( \frac{\partial^2}{\partial z^2} - k^2 \right) G(z, z') = \delta(z - z'),
$$

with $\lim_{z \to \pm \infty} G(z, z') = 0$, the stream function $\hat{\psi}$ can be expressed as

$$
\hat{\psi}(z, t) = \int_{-\infty}^{\infty} dz' G(z, z') \hat{q}(z', t).
$$

(5)

By the substitution of (5) into the equations (3), they can be rewritten as a system of equations for $\hat{q}$ and $\hat{b}$:

$$
\begin{align*}
\left( \frac{\partial}{\partial t} + i k \mathfrak{m} \right) \hat{q}(z, t) & - i k \frac{\partial}{\partial z} \int_{-\infty}^{\infty} dz' G(z, z') \hat{q}(z', t) + i k \hat{b}(z, t) = 0, \\
\left( \frac{\partial}{\partial t} + i k \mathfrak{m} \right) \hat{b}(z, t) & - i k \frac{\partial}{\partial z} \int_{-\infty}^{\infty} dz' G(z, z') \hat{q}(z', t) = 0.
\end{align*}
$$

(6)

Under the assumptions of (2), the vertical gradients of basic vorticity $\overline{\gamma}$ and buoyancy $\overline{f}$ are expressed in terms of the Dirac’s delta function as

$$
\frac{\partial \overline{\gamma}}{\partial z} = -\Lambda \delta(z - z_2), \quad \frac{\partial \overline{f}}{\partial z} = \gamma \delta(z - z_1).
$$

(7)

From equations (6) and (7), and the assuming the absence of an initial disturbance except at $z = z_1$, $z_2$, the disturbance can be also written in terms of the Dirac’s delta function as

$$
\hat{q}(z, t) = \sum_{n=1}^{2} \hat{q}_n(t) \delta(z - z_n),
$$

$$
\hat{b}(z, t) = \sum_{n=1}^{2} \hat{b}_n(t) \delta(z - z_n).
$$

(8)

By the substitution of (7) and (8) into the equations (6), they can be reduced to the following system of ordinary differential equations:

$$
\begin{align*}
\left( \frac{\partial}{\partial t} + i k \mathfrak{m} \right) \hat{q}_2(t) + i k \Lambda \sum_{n=1}^{2} G_{z2} \hat{q}_n(t) + i k \hat{b}_2(t) &= 0, \\
\left( \frac{\partial}{\partial t} + i k \mathfrak{m} \right) \hat{q}_1(t) + i k \hat{b}_1(t) &= 0, \\
\left( \frac{\partial}{\partial t} + i k \mathfrak{m} \right) \hat{b}_1(t) - i k \sum_{n=1}^{2} G_{z1} \hat{q}_n(t) &= 0,
\end{align*}
$$

(9)

where $\mathfrak{m}_n = \mathfrak{m}(z_n)$ and $G_{z1} = G(z_1, z_n)$. Because the third equation in (9) implies that $\hat{b}_2$ is simply advected by the basic zonal flow $\overline{u}$, we assume that $\hat{b}_2(0) = 0$ resulting in $\hat{b}_2(t) = 0$.

Finally, substituting $\mathfrak{m}_1 = 0$, $\mathfrak{m}_2 = \Lambda z$, and the Green’s function (4) into (9), and defining the non-dimensional parameters and variables $\kappa = kZ$, $\tau = tA$, $R = \frac{\gamma}{\Lambda \kappa}$, $\hat{q}_n(\tau) = \hat{q}_n(t) / (\Lambda Z)^2$, and $\hat{b}(\tau) = \hat{b}(t) / (\Lambda Z)^2$, we obtain the following governing equation:

$$
\frac{\partial}{\partial \tau} |\hat{q}(\tau)| + i \kappa M |\hat{q}(\tau)| = 0,
$$

(10)

where

$$
M = \begin{bmatrix}
1 & -\frac{1}{2|\kappa|} & -\frac{1}{2|\kappa|} e^{-|\kappa|} & 0 \\
0 & 0 & 0 & 0 \\
R & 0 & 0 & 0 \\
R & 2/|\kappa| & 0 & 0 \\
\end{bmatrix}.
$$

(11)

3. Free Rossby and gravity waves

If the interaction between the lower and upper disturbances is neglected, then the governing equation (10) is reduced to the upper and lower equations without interaction:

$$
\frac{\partial}{\partial \tau} |\hat{q}_2(\tau)| + i \kappa M_2 |\hat{q}_2(\tau)| = 0,
$$

(12)

$$
\frac{\partial}{\partial \tau} |\hat{q}_1(\tau)| + i \kappa M_1 |\hat{q}_1(\tau)| = 0,
$$

(13)

where $|\hat{q}_1| = |\hat{q}_1| / |\hat{b}_1|$ and $M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$. The solution to (12) is the free Rossby wave on the upper level $z = z_2$ due to the vertical gradient of basic vorticity there:
\( \dot{q}_2(\tau) = e^{-i\kappa C_R \tau} \dot{q}_2(0). \)  

(14)

The phase velocity \( C_R \) in (14) is given by

\[ C_R = 1 - \frac{1}{2|\kappa|^2} \]  

(15)

The first term of \( C_R \) in (15) is the eastward advection velocity by the basic flow, and the second term is the westward propagation velocity relative to the basic flow. If \( |\kappa| > \frac{1}{2} \), then the Rossby wave moves eastward. If \( |\kappa| < \frac{1}{2} \), then the Rossby wave moves westward.

Meanwhile, the solution to (13) is the free gravity wave on the lower level \( z = z_1 \) due to the vertical gradient of basic buoyancy there. This is given by

\[ \dot{q}_1(\tau) = e^{-i\kappa C_g \tau} \dot{q}_1(0) \]  

(16)

where \( \lambda_\pm, |r_{y2}^\pm| \) and \( \langle l_{y2}^\pm | \) are the eigenvalues and the right and left eigenvectors of the coefficient matrix \( M_g \) in (13), respectively:

\[ \lambda_\pm = \pm C_g, \quad |r_{y2}^\pm| = \begin{pmatrix} 1 \\ \pm C_g \end{pmatrix}, \quad \langle l_{y2}^\pm | = |\pm C_g \rangle. \]  

(17)

The phase velocity \( \lambda_\pm = C_g \) in (17) are given by

\[ \pm C_g = \pm \sqrt{R \over 2|\kappa|}. \]  

(18)

In (16), the scalar product \( \langle l_{y2}^\pm | r_{y2}^\pm \rangle \) is a scalar, and the dyadic product \( r_{y2}^\pm \langle l_{y2}^\pm | \) is a \( 2 \times 2 \) matrix as follows:

\[ \langle l_{y2}^\pm | r_{y2}^\pm \rangle = \begin{pmatrix} 1 & 1 \\ \pm C_g & \pm C_g \end{pmatrix} = \pm 2C_g, \]  

\[ |r_{y2}^\pm \rangle \langle l_{y2}^\pm | = \begin{pmatrix} 1 & 1 \\ \pm C_g & \pm C_g \end{pmatrix} = \begin{pmatrix} \pm C_g & 1 \\ C_g^2 & \pm C_g \end{pmatrix}. \]  

(19)

Taking note of the eigenvalue decomposition \( M_g = \sum_\pm \lambda_\pm |r_{y2}^\pm \rangle \langle l_{y2}^\pm | \) and the completeness relation \( \sum_\pm |r_{y2}^\pm \rangle \langle l_{y2}^\pm | = 1 \), we can easily see that (16) is indeed the solution of (13) with the initial value \( |\dot{q}_1(0)\rangle \). Because the basic flow is zero on the lower level \( z = z_1 \), the phase velocities (18) are equal to the propagation velocities relative to the basic flow.

The first and second terms on the RHS of (16) represent the eastward- and westward-propagating waves, respectively. Because \( \ddot{b}_l = +C_g \dot{q}_1(\dot{b}_l = -C_g \dot{q}_1) \) for the eastward-(westward-) propagating wave from the right eigenvectors \( |r_{y2}^\pm\rangle \) in (17), the vorticity disturbance \( \ddot{q}_1 \) and the buoyancy disturbance \( \dot{b}_l \) are in phase(antiphased) for the eastward(westward)-propagating wave.

4. Solution with interaction

The solution to (10) with an initial value \( |\ddot{q}(0)\rangle \) is given by the following formula:

\[ |\ddot{q}(\tau)\rangle = \sum_{n=0}^{2} e^{-i\kappa \lambda_n \tau} \frac{\langle r_n |}{\langle l_n |} |\ddot{q}(0)\rangle, \]  

(20)

Where \( \lambda_n, |r_n\rangle, \) and \( \langle l_n \rangle, \ n = 0, 1, 2, \) are the eigenvalues and right and left eigenvectors of the matrix \( M \) in (11), respectively. Taking note of the eigenvalue decomposition \( M = \sum_{n=0}^{2} \lambda_n |r_n\rangle \langle l_n | \) and the completeness relation \( \sum_{n=0}^{2} \langle r_n | = 1 \), we can easily see that (20) is indeed the solution of (10) with the initial value \( |\ddot{q}(0)\rangle \). The eigenvalues are given by

\[ \lambda_n = \begin{pmatrix} \lambda_n^2 & -\frac{e^{-|\kappa|}}{2|\kappa|} \lambda_n & -\frac{e^{-|\kappa|}}{2|\kappa|} \end{pmatrix}, \]  

(21)

Where \( C_g = \sqrt{R \over 2|\kappa|} \) is the propagation speed of the free gravity wave in (18). In (20), the scalar product \( \langle l_n | r_n \rangle \) is a scalar

\[ \langle l_n | r_n \rangle = \begin{pmatrix} \lambda_n^2 - C_g^2 \\ -\frac{e^{-|\kappa|}}{2|\kappa|} \lambda_n \\ -\frac{e^{-|\kappa|}}{2|\kappa|} \end{pmatrix} \]  

(22)

and the dyadic product \( |r_n\rangle \langle l_n | \) is a \( 3 \times 3 \) matrix. The eigenvalues \( \lambda_n \) are the roots of the cubic algebraic equation \( (\lambda - C_g)(\lambda^2 - C_g^2) + \frac{e^{-2|\kappa|}}{2|\kappa|} C_g^2 = 0 \), where
\[ C_R = 1 - \frac{1}{2|\kappa|} \] is the phase velocity of the free Rossby wave in (15). The roots are given by the following expression:

\[
\lambda_n = \frac{1}{3} C_R + \left( -A + \sqrt{A^2 - B^3} \right)^{\frac{1}{3}} e^{\frac{2 n \pi i}{3}} + \left( -A - \sqrt{A^2 - B^3} \right)^{\frac{1}{3}} e^{\frac{2 n \pi i}{3}},
\]

where

\[
A = -\frac{1}{27} C_R^3 + \frac{1}{3} C_R C_g^2 + \frac{e^{-2|\kappa|}}{4|\kappa|} C_g^2,
\]

\[
B = \frac{1}{9} C_R^2 + \frac{1}{3} C_g^2.
\]

From (23), the solution in (20) exponentially grows if \( A^2 - B^3 > 0 \) (see equation (31)), and oscillates if \( A^2 - B^3 < 0 \) (see equation (29). The discriminant \( A^2 - B^3 \) is rewritten as

\[
PQ = C \left( \frac{9}{2} \frac{e^{-2|\kappa|}}{|\kappa|} C_R + \frac{27}{16} \frac{e^{-4|\kappa|}}{|\kappa|^2} \right),
\]

For the instability, the quadratic expression of \( C_g^2 \) inside the bracket in (25) must be negative. This implies

\[
P^2 - 4Q = \frac{16}{|\kappa|} \left( C_R + \frac{9}{16} \frac{e^{-2|\kappa|}}{|\kappa|^2} \right)^3 > 0,
\]

\[
\frac{1}{2} \left( P - \sqrt{P^2 - 4Q} \right) < C_g^2 < \frac{1}{2} \left( P + \sqrt{P^2 - 4Q} \right).
\]

Because \( C_R + \frac{9}{16} \frac{e^{-2|\kappa|}}{|\kappa|^2} = 1 - \frac{1}{2|\kappa|} + \frac{9}{16} \frac{e^{-2|\kappa|}}{|\kappa|^2} \) is positive definite, the condition (26) is automatically satisfied. Thus, the instability criterion is reduced to (27). That is,

\[
|\kappa| \left( P - \sqrt{P^2 - 4Q} \right) < C_g^2 < \frac{1}{2} \left( P + \sqrt{P^2 - 4Q} \right)
\]

for instability.

The disturbance grows in the region between two marginal curves \( R = |\kappa| \left( P \mp \sqrt{P^2 - 4Q} \right) \) in the \((R, |\kappa|)\) parameter space (Fig. 5).

![Fig. 5. The unstable region in the \((R, |\kappa|)\) space.](image)

Between the left curve \( R = |\kappa| \left( P + \sqrt{P^2 - 4Q} \right) \) and right curve \( R = |\kappa| \left( P - \sqrt{P^2 - 4Q} \right) \), the disturbance grows exponentially. The gray curve \( C_R = C_g \) is included in the unstable region.

The exponential growth is caused by the interaction between the westward-propagating upper Rossby wave and the eastward-propagating lower gravity wave (Rayleigh’s condition). Furthermore, the phase lock between the two waves is required for such growth (Fjøerstoft’s condition). Therefore, on the curve where the phase velocities of free Rossby and eastward-propagating gravity waves are equal, i.e., \( C_R = C_g \), in the \((R, |\kappa|)\) parameter space, exponential growth is expected. Indeed, the curve \( C_R = C_g \), i.e.,

\[
1 - \frac{1}{2|\kappa|} = \sqrt{\frac{R}{2|\kappa|}},
\]

is included in the unstable region in (28).

5. Oscillating and exponentially growing solutions

Outside of the unstable region (28), the discriminant in (25) is negative, \( A^2 - B^3 < 0 \), and all the eigenvalues \( \lambda_n, n = 0, 1, 2 \), in (23) are real:

\[
\lambda_n = \frac{1}{3} C_R - 2B^2 \cos \left( \frac{\Theta}{3} - \frac{2n\pi}{3} \right),
\]

where \( \Theta = \tan^{-1} \frac{\sqrt{B^3 - A^2}}{A} \). The reality of eigenvalues implies that all of the eigenvectors \( |n_n\rangle \) and \( |l_n| \), \( n = 0, 1, 2 \) in (21) are also real. Assuming that the initial
value \( |\tilde{q}(0)| \) is real, multiplying (20) by \( e^{i\kappa \xi} \) and taking the real part gives the following oscillating solution in the physical space \((\xi, \tau)\):

\[
|q(\xi, \tau)| = \sum_{n=0}^{2} \cos(\kappa - \lambda_n \tau) \left| \frac{r_n}{l_n} \right| |\tilde{q}(0)|,
\]

where \( \xi = \frac{x}{Z} \) is the nondimensional zonal coordinate. An example of the oscillating solution (30) with \((\mathcal{R}, |\kappa|) = (1, 2)\) is shown in Fig. 6. Because \( C_R > C_g \) for the parameter values \((\mathcal{R}, |\kappa|) = (1, 2)\), it is expected that the upper Rossby wave moves eastward faster than the lower gravity wave, implying no phase lock, and therefore implying no growth. As is shown in Fig. 7, this is indeed the case.

Inside of the unstable region (28), the discriminant in (25) is positive, \( A^2 - B^3 > 0 \), and the first and second eigenvalues \( \lambda_{1,2} \) in (23) become complex conjugates, although the 0th eigenvalue \( \lambda_0 \) remains real:

\[
\lambda_0 = \frac{1}{3} C_R - \left( A - \sqrt{A^2 - B^3} \right)^{\frac{1}{3}} - \left( A + \sqrt{A^2 - B^3} \right)^{\frac{1}{3}},
\]

\[
\lambda_{1,2} = \frac{1}{3} C_R + \frac{1}{2} \left[ \left( A + \sqrt{A^2 - B^3} \right)^{\frac{1}{3}} + \left( A - \sqrt{A^2 - B^3} \right)^{\frac{1}{3}} \right] \pm \frac{i\sqrt{3}}{2} \left[ \left( A + \sqrt{A^2 - B^3} \right)^{\frac{1}{3}} - \left( A - \sqrt{A^2 - B^3} \right)^{\frac{1}{3}} \right],
\]

\[
= \lambda_R \pm i\lambda_I.
\]

Fig. 6. An example of the oscillating solution with \((\mathcal{R}, |\kappa|) = (1, 2)\). The initial values are set to zero except for \( \tilde{q}_2 = (0) \neq 0 \). The black sinusoidal lines represent the vorticity disturbances. The gray sinusoidal lines represent the buoyancy disturbance. Small squares are added to indicate the movement of disturbances. Time passes from (a) to (d).
The reality of $\lambda_0$ implies that the 0th eigenvectors $\langle r_0 |$ and $| l_0 \rangle$ in (21) are real, and the complexity of $\lambda_{1,2}$ implies that the 1st and 2nd eigenvectors $\langle r_{1,2} |$ and $| l_{1,2} \rangle$ in (21) are complex conjugates:

$$\frac{| r_{1,2} \rangle \langle l_{1,2} |}{\langle l_{1,2} | r_{1,2} \rangle} = \pm i \frac{| r_{1,2} \rangle \langle l_{1,2} |}{\langle l_{1,2} | r_{1,2} \rangle}.$$  \hfill (32)

By the substituting (31) and (32), the solution (20) becomes

$$| \bar{q}(\tau) \rangle = e^{-i\lambda_0 \tau} \frac{| n_0 \rangle \langle l_0 |}{\langle l_0 | n_0 \rangle} | \tilde{q}(0) \rangle$$

$$+ e^{i\lambda_{1,2} \tau} e^{-i\lambda_{1,2} \tau} \left[ \frac{| r_{1,2} \rangle \langle l_{1,2} |}{\langle l_{1,2} | r_{1,2} \rangle} + i \frac{| r_{1,2} \rangle \langle l_{1,2} |}{\langle l_{1,2} | r_{1,2} \rangle} \right] | \tilde{q}(0) \rangle$$

$$+ e^{-i\lambda_{1,2} \tau} e^{i\lambda_{1,2} \tau} \left[ \frac{| r_{1,2} \rangle \langle l_{1,2} |}{\langle l_{1,2} | r_{1,2} \rangle} - i \frac{| r_{1,2} \rangle \langle l_{1,2} |}{\langle l_{1,2} | r_{1,2} \rangle} \right] | \tilde{q}(0) \rangle.$$  \hfill (33)

Assuming that the initial value $| \tilde{q}(0) \rangle$ is real, multiplying (33) by $e^{i\lambda_0 \tau}$ and taking the real part gives the following exponentially growing solution in the physical space $(\xi, \tau)$:

$$| q(\xi, \tau) \rangle = \cos \kappa (\xi - \lambda_0 \tau) \left[ \frac{| n_0 \rangle \langle l_0 |}{\langle l_0 | n_0 \rangle} | \tilde{q}(0) \rangle \right.$$  

$$+ 2 \cosh \kappa \lambda_{1,2} \tau \cos \kappa (\xi - \lambda_{1,2} \tau) \frac{| r_{1,2} \rangle \langle l_{1,2} |}{\langle l_{1,2} | r_{1,2} \rangle} | \tilde{q}(0) \rangle$$

$$- 2 \sinh \kappa \lambda_{1,2} \tau \sin \kappa (\xi - \lambda_{1,2} \tau) \frac{| r_{1,2} \rangle \langle l_{1,2} |}{\langle l_{1,2} | r_{1,2} \rangle} | \tilde{q}(0) \rangle.$$  \hfill (34)

An example of the exponentially growing solution in (34) with $(R, | \kappa \rangle) = (1.35, 0.3)$ is shown in Fig. 8. Because $C_R = C_g$ for the parameter values $(R, | \kappa \rangle) = \left( 1, \frac{3 + \sqrt{5}}{2} \right)$, it is expected that the upper Rossby wave and the lower gravity wave eventually take the optimal phase-locked configuration with $\delta \theta = \frac{\pi}{2}$ for the strong interaction. Here, $\delta \theta$ is the phase difference between the upper Rossby wave and the lower gravity wave. As is shown in Fig. 8, this is indeed the case. Another example of the exponentially growing solution
(34) with \(R_\kappa = (1, 1)\) is shown in Fig. 9. Because \(C_g > C_R\) for the parameter values \(R_\kappa = (1, 1)\), it is expected that the lower gravity wave shifts eastward from the optimal configuration of \(\delta \theta = \frac{\pi}{2}\) although the phase lock takes place. Consequently, the phase difference \(\delta \theta\) eventually becomes greater than \(\frac{\pi}{2}\) (and less than \(\pi\) in order to grow), i.e., \(\frac{\pi}{2} < \delta \theta < \pi\). As is shown in Fig. 9, this is indeed the case.

6. Marginal solution

On the boundary between the stable and unstable regions in the parameter space \((R, |\kappa|)\), where \(A^2 - B^3 = 0\), the 1st and 2nd eigenvalues \(\lambda_{1,2}\) in (29) degenerate into one as follows:

\[
\lim_{\Theta \to 0} \lambda_{1,2} = \lim_{\Theta \to 0} \left[ \frac{1}{3} C_R - \frac{1}{2} B^2 \cos \left( \frac{2\pi}{3} + \frac{\Theta}{3} \right) \right] = \frac{1}{3} C_R + \frac{1}{2} B^2 = \lambda_c.
\]  

(35)

In the case of degeneracy of eigenvalues, the scalar products of the eigenvectors vanish:

\[
\lim_{\Theta \to 0} \langle l_{1,2} | n_{1,2} \rangle = \langle l_c | n_c \rangle = 0.
\]

Outside of the unstable region in (28), the solution is oscillating and written as (30). By the use of the completeness relation of eigenvectors \(\sum_{n=0}^{\infty} |l_n \rangle \langle n| = 1\), the solution (30) is rewritten as

\[
|q(\xi, \tau)\rangle = \cos \kappa (\xi - \lambda_3 \tau) |\tilde{q}(0)\rangle \\
+ \cos \kappa (\xi - \lambda_0 \tau) \\
- \cos \kappa (\xi - \lambda_2 \tau) |\tilde{l}_0 \rangle \langle \tilde{l}_0 | n_0 \rangle |\tilde{q}(0)\rangle \\
+ \cos \kappa (\xi - \lambda_1 \tau) \\
- \cos \kappa (\xi - \lambda_2 \tau) |\tilde{l}_1 \rangle \langle \tilde{l}_1 | n_1 \rangle |\tilde{q}(0)\rangle.
\]

(37)

In the limit of \(\Theta \to 0\), i.e., in the limit of \((B^3 - A^2) \to 0\), the denominator \(\langle l_1 | n_1 \rangle\) of the third term on the
RHS of (37) vanishes because of (36). At the same time, \( \{\cos(\xi - \lambda_\tau) - \cos(\xi - \lambda_\tau)\} \) also vanishes because of (35). By applying L'Hopital's rule, the limit of \( \Theta \to 0 \) of the third term is rewritten as

\[
\lim_{\Theta \to 0} \frac{\partial}{\partial \Theta} \{\cos(\xi - \lambda_\tau) - \cos(\xi - \lambda_\tau)\} \frac{\partial}{\partial \Theta} \langle h \mid \eta \rangle.
\]

(38)

Let \( |\kappa| \to \kappa_c \) and \( R \to R_c \) as \( \Theta \to 0 \). In the calculation of (38), we first take the limit of \( |\kappa| \to \kappa_c \) and \( R \to R_c \) except for \( \Theta \) of \( \lambda_{1,2} \) in (35), and thereafter we take the limit of \( \Theta \to 0 \) of \( \lambda_{1,2} \) in (35). First,

\[
\lim_{\Theta \to 0} \frac{\partial}{\partial \Theta} \{\cos(\xi - \lambda_\tau) - \cos(\xi - \lambda_\tau)\} \frac{\partial}{\partial \Theta} \langle h \mid \eta \rangle = \kappa_c \tau \sin(\xi - \lambda_\tau) \frac{\partial}{\partial \kappa} \langle h \mid \eta \rangle,
\]

(39)

Then, the calculation of (38) is performed as follows:

\[
\lim_{\Theta \to 0} \left[ \text{the third term} \right] = \kappa_c \tau \sin(\xi - \lambda_\tau) \frac{\partial}{\partial \kappa} \langle h \mid \eta \rangle.
\]

(40)

Fig. 9. Same as in Fig. 6, but for the exponentially growing solution with \( (R, |\kappa|) = (1, 1) \).
\[ \frac{R_c}{2 \kappa_c} \] from (22). In the second line of (40), we used 
\[ \frac{\partial \lambda_2}{\partial \Theta} = - \frac{\partial \lambda_1}{\partial \Theta}, \] which is derived from (35). Taking the limit of \( \Theta \to 0 \) in (37) and substituting (40), we obtain the marginal solution, which grows linearly in time:

\[
|q(\xi, \tau)| = \cos \kappa_c (\xi - \lambda_c) |q(0)| 
+ \{ \cos \kappa_c (\xi - \lambda_c) \} \frac{\langle r_0 \rangle}{\langle \hat{r}_0 \rangle} |q(0)| 
- \cos \kappa_c (\xi - \lambda_c) \frac{\langle r_0 \rangle}{\langle \hat{r}_0 \rangle} |q(0)| 
+ 2 \kappa_c \tau \sin \kappa_c (\xi - \lambda_c) \frac{\langle r_0 \rangle}{\langle \hat{r}_0 \rangle} |q(0)|.
\]

(41)

An example of the marginal solution in (41) with 
\( (R, |\kappa|) = (1, 0.5766) \), which is on the left curve 
\( R = |\kappa| (P + \sqrt{P^2 - 4Q}) \) in Fig. 5, is shown in Fig. 10. Because \( C_g > C_R \) for the parameter values 
\( (R, |\kappa|) = (1, 0.5766) \), it is expected that the lower gravity wave shifts eastward from the configuration of \( \delta \theta = \frac{\pi}{2} \). Consequently, the phase difference becomes greater than \( \frac{\pi}{2} \) (but less than \( \pi \) in order to grow), i.e., \( \frac{\pi}{2} < \delta \theta < \pi \). Because there is no phase lock, the phase difference \( \delta \theta \) asymptotically approaches \( \pi \) as time goes on. Another example of the marginal solution in (41) with 
\( (R, |\kappa|) = (1, 1.5776) \), which is on the right curve 
\( R = |\kappa| (P + \sqrt{P^2 - 4Q}) \) in Fig. 5, is shown in Fig. 11. Because \( C_R > C_g \) for the parameter values 
\( (R, |\kappa|) = (1, 1.5776) \), it is expected that the upper Rossby wave shifts eastward from the configuration of \( \delta \theta = \frac{\pi}{2} \). Consequently, the phase difference becomes smaller than \( \frac{\pi}{2} \) (but greater than 0 in order to grow), i.e., \( 0 < \delta \theta < \frac{\pi}{2} \). Because there is no phase lock, the phase difference \( \delta \theta \) asymptotically approaches 0 as time goes on.

7. Free mode expansion

The exponential growth of disturbance is caused by the interaction between the westward-propagating upper Rossby wave and the eastward-propagating lower gravity wave. To examine the role played by the
westward-propagating lower gravity wave, we rewrite the governing equation in the form of the interaction between the Rossby and gravity modes.

The disturbance $|\tilde{q}(\tau)\rangle = |q(\tau)\rangle$ can be expanded in terms of the free Rossby, eastward-propagating gravity, and westward-propagating gravity modes in Section 3 as follows:

$$|\tilde{q}(\tau)\rangle = Q_R(\tau) |r_R\rangle + Q_G^+ (\tau) |r_G^+\rangle + Q_G^- (\tau) |r_G^-\rangle,$$

where

$$|r_R\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |r_G^+\rangle = \begin{bmatrix} 0 \\ 1 \\ +C_g \end{bmatrix}, \quad |r_G^-\rangle = \begin{bmatrix} 0 \\ 1 \\ -C_g \end{bmatrix}$$

are the right eigenvectors of the coefficient matrix $M$ in (11) without the interaction. The corresponding left eigenvectors are given by

$$\langle l_R | = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \langle l_G^+ | = \begin{bmatrix} 0 & +C_g & 1 \end{bmatrix},$$

$$\langle l_G^- | = \begin{bmatrix} 0 & -C_g & 1 \end{bmatrix}.$$  \hfill (44)

By the construction of the right eigenvectors in (43), the mode amplitudes, $Q_R$, $Q_G^+$, and $Q_G^-$, represent the amplitudes of vorticity of the mode. Substituting the expansion equation (42) into the governing equation (10), and taking the scalar products with the left eigenvectors in (44), we obtain the following evolution equations for the mode amplitudes, $Q_R$, $Q_G^+$, and $Q_G^-$:

$$\frac{\partial Q_R}{\partial \tau} = -i\kappa C_R Q_R + ik\frac{e^{-|\kappa|}}{2|\kappa|} (Q_G^+ + Q_G^-),$$

$$\frac{\partial Q_G^+}{\partial \tau} = -i\kappa C_g Q_G^+ - ik\frac{C_g^2}{2} e^{-|\kappa|} Q_R,$$

$$\frac{\partial Q_G^-}{\partial \tau} = +i\kappa C_g Q_G^- + ik\frac{C_g^2}{2} e^{-|\kappa|} Q_R.$$

These equations are combined into a vector equation

![Fig. 11. Same as in Fig. 6, but for the marginal solution with $(R,|\kappa|) = (1, 1.5776)$.](image-url)
The eigenvalues of the coefficient matrix in (46) are the roots of the following cubic algebraic equation:

\[(\lambda - C_R)(\lambda^2 - C_g^2) + \frac{C_g^2}{2|\kappa|}e^{-2|\kappa|} = 0.\] (47)

This is of course the same as that of the coefficient matrix \(M\) in the governing equation (10).

8. Reduced 2-mode system

In the case where the disturbances grow, the westward-propagating Rossby wave and the eastward-propagating gravity wave strongly interact (Rayleigh’s condition). Therefore, to first approximation, \(Q_g\) can be neglected and the three-component equation (46) is reduced to the following two-component one:

\[
\frac{\partial}{\partial \tau} \begin{bmatrix} Q_R \\ Q_g^+ \end{bmatrix} = -i\kappa \begin{bmatrix} C_R & -\frac{1}{2|\kappa|}e^{-|\kappa|} \\ \frac{C_g}{2}e^{-|\kappa|} & C_g \end{bmatrix} \begin{bmatrix} Q_R \\ Q_g^+ \end{bmatrix}. \tag{48}
\]

The eigenvalues and the eigenvectors of the coefficient matrix in (48) are

\[
\lambda_{\pm} = \frac{1}{2}\left((C_R + C_g) \pm \sqrt{D}\right) \quad \text{with} \quad D = (C_R - C_g)^2 - \frac{e^{-2|\kappa|}}{|\kappa|}C_g, \tag{49}
\]

\[
|R_{\pm}| = \left|\frac{2e^{\kappa}}{C_g}(\lambda_{\pm} - C_g)\right| = \left|\frac{e^{\kappa}}{C_g}(C_R - C_g \pm \sqrt{D})\right|, \tag{50}
\]

\[
\langle L_{\pm}\rangle = \left|1 \frac{2e^{\kappa}}{C_g}(\lambda_{\pm} - C_R)\right| = \left|1 \frac{e^{\kappa}}{C_g}(C_R - C_g \pm \sqrt{D})\right|, \tag{51}
\]

The scalar products of the right and left eigenvectors are

\[
\langle L_{\pm}\mid R_{\pm}\rangle = \frac{2e^{\kappa}}{C_g}(2\lambda_{\pm} - C_R - C_g) = \pm \frac{2}{C_g}\sqrt{D}. \tag{52}
\]

The solution to (48) is given in terms of the eigenvalues and eigenvectors by

\[
\begin{bmatrix} Q_R(\tau) \\ Q_g^+(\tau) \end{bmatrix} = e^{-i\omega_{\pm}\tau} \begin{bmatrix} |R_{\pm}| \langle L_{\pm}\mid R_{\pm}\rangle \end{bmatrix} \begin{bmatrix} Q_R(0) \\ Q_g^+(0) \end{bmatrix} + e^{-i\omega_{\pm}\tau} \begin{bmatrix} R_{\pm} \langle L_{\pm}\mid R_{\pm}\rangle \end{bmatrix} \begin{bmatrix} Q_R(0) \\ Q_g^+(0) \end{bmatrix}. \tag{53}
\]

If the discriminant \(D\) in (49) is negative, then the eigenvalues \(\lambda_{\pm}\) in (49) are complex conjugates, i.e., \(\lambda_{\pm} = \lambda_R \pm i\lambda_I = \frac{C_R + C_g}{2} \pm i\lambda_I\) with \(\lambda_I > 0\), and then the disturbance in (53) grows exponentially. For the growing disturbance, the first term on the RHS of (53) eventually dominates over the second. Then, from the right eigenvector \(Q_R\) in (50), the ratio \(Q_R/Q_g^+\) is estimated as

\[
\frac{Q_R}{Q_g^+} = \frac{2e^{\kappa}}{C_g}(\lambda_{+} - C_g) = \frac{2e^{\kappa}}{C_g}\sqrt{(C_R - C_g)^2 - \frac{e^{-2|\kappa|}}{|\kappa|}C_g} + \lambda_I^2 e^{i(\theta_R - \theta_g^+)}, \tag{54}
\]

where the phase difference is given by \(\theta_R - \theta_g^+ = \tan^{-1}\frac{2\lambda_I}{C_R - C_g}\). Because \(\lambda_I > 0\) by definition, the phase difference satisfies \(0 < \theta_R - \theta_g^+ < \pi\). The phase difference between \(Q_R\) and \(Q_g^+\) enables the mutual amplification.

The unstable region \(D = (C_R - C_g)^2 - \frac{e^{-2|\kappa|}}{|\kappa|}C_g < 0\) in the parameter space \((R, |\kappa|)\) is shown in Fig. 12. It is a little narrower than the original unstable region depicted in Fig. 5. The narrowness implies that the westward-propagating gravity wave \(Q_g^+\), which is neglected in the reduced system in (48), plays a role for the growth of \(Q_R\) and \(Q_g^+\).

9. 3-mode system

The solution to equation (46) is given by

\[
|\tilde{\Omega}(\tau)| = \sum_{n=0}^{2} e^{-i\omega_n \tau} \left|\frac{R_n}{\langle L_n \mid R_n\rangle}\right| \langle L_n \mid \tilde{\Omega}(0)\rangle, \tag{55}
\]
Fig. 12. The unstable region in \((R, |\kappa|)\) space in the reduced 2-mode system. The unstable region 
\[ D = (C_R - C_g)^2 - \frac{e^{-|\kappa|}}{|\kappa|} C_g < 0 \]
is between the left and right marginal curves.

where \( |\tilde{Q}| = \frac{|Q_R|}{Q_g^+} \), and \( |R_n| \) and \( |L_n| \) are the right and left eigenvectors of the coefficient matrix in (46), respectively:

\[
|R_n| = \begin{bmatrix}
\lambda_n^2 - C_g^2 \\
\frac{1}{2} e^{-|\kappa|} C_g (\lambda_n + C_g) \\
-\frac{1}{2} e^{-|\kappa|} C_g (\lambda_n - C_g)
\end{bmatrix},
\]

(56)

\[
|L_n| = \begin{bmatrix}
\lambda_n^2 - C_g^2 \\
\frac{1}{2} e^{-|\kappa|} C_g (\lambda_n + C_g) \\
-\frac{1}{2} e^{-|\kappa|} C_g (\lambda_n - C_g)
\end{bmatrix}.
\]

(57)

The eigenvalues \( \lambda_n, n = 0, 1, 2 \), are given by (23). In the unstable region of (28) between two marginal curves \( R = |\kappa| \left( \sqrt{P^2 - 4Q} \right) \) in the parameter space \((R, |\kappa|)\), the 1st and 2nd eigenvalues are complex conjugates, i.e., \( \lambda_{1,2} = \lambda_R \pm \phi_1 \) with \( \phi_1 > 0 \).

As explained in Section 5, the exponentially growing solution in the physical space \((\xi, \tau)\) is given by

\[
|\tilde{Q}(\xi, \tau)\rangle = \cos \kappa (\xi - \lambda_0 \tau) \left|\frac{R_0}{L_0} |Q_0\rangle \right| \langle \tilde{Q}(0) \right| + 2 \cosh \kappa \lambda_I \tau \cos \kappa (\xi - \lambda_R \tau) \left|\frac{|R|}{\langle L | R \rangle} \right| \langle \tilde{Q}(0) \right| (58)
\]

\[-2 \sinh \kappa \lambda_I \tau \sin \kappa (\xi - \lambda_R \tau) \left|\frac{|R|}{\langle L | R \rangle} \right| \langle \tilde{Q}(0) \right|.
\]

For the growing disturbance, the term with \( \lambda_1 = \lambda_R + i\phi_1 \) in (55) eventually dominates the others. Then, from the right eigenvector \( |R_i\rangle \) in (56), the ratio \( \frac{Q_R}{Q_g} \) is estimated as

\[
\frac{Q_R}{Q_g} = \frac{2 e^{i|\kappa|}}{C_g} (\lambda_i - C_g)
\]

\[= \frac{2 e^{i|\kappa|}}{C_g} \sqrt{(\lambda_R - C_g)^2 + \lambda_I^2 e^{i(\theta_R - \theta_g)}} ,
\]

(59)

where the phase difference is given by \( \theta_R - \theta_g = \tan^{-1} \left( \frac{-\lambda_I}{\lambda_R - C_g} \right) \). Because \( \lambda_I > 0 \) by definition, the phase difference satisfies \( 0 < \theta_R - \theta_g < \pi \), which implies the mutual amplification of \( Q_R \) and \( Q_g \). Next, from the right eigenvector \( |R_i\rangle \) in (56), the ratio \( \frac{Q_R}{Q_g} \) is estimated as

\[
\frac{Q_R}{Q_g} = -\frac{2 e^{i|\kappa|}}{C_g} (\lambda_i + C_g)
\]

\[= \frac{2 e^{i|\kappa|}}{C_g} \sqrt{(\lambda_R + C_g)^2 + \lambda_I^2 e^{i(\theta_R - \theta_g)}} ,
\]

(60)

where the phase difference is given by \( \theta_R - \theta_g = \tan^{-1} \left( \frac{-\lambda_I}{-(\lambda_R + C_g)} \right) \). Because \( \lambda_R > 0 \) for instability (growing wave moves eastward), the phase difference \( \theta_R - \theta_g \) is estimated as

\[\pi < \theta_R - \theta_g < \frac{3\pi}{2} .
\]

(61)

An example of the exponentially growing solution in (58) with the parameter values \((R, |\kappa|) = (0.5, 0.3)\), which is stable in the reduced 2-mode system, is shown in Fig. 13. The eventual evolution in Fig. 13 indeed shows \( 0 < \theta_R - \theta_g < \pi \) and \( \pi < \theta_R - \theta_g < \frac{3\pi}{2} \).

In the reduced 2-mode system, the disturbance grows exponentially in the region between the right- and left-marginal lines on the \((R, |\kappa|)\) plane in Fig. 12. In the region to the right of the right-marginal line, where \( |\kappa| \) is relatively large, both the free westward
propagation speed $\frac{1}{2|\kappa|}$ of the upper Rossby wave $Q_R$ and the free eastward propagation speed $C_g = \sqrt{\frac{R}{2|\kappa|}}$ of the lower gravity wave $Q_g^+$ are too small for the two waves to be phase-locked with each other against the upper eastward advection speed $\bar{u}_2 = 1$ (or $U = \Lambda Z$ in the dimensional form). This is because both propagation speeds are decreasing functions of $|\kappa|$. The presence of the westward-propagating gravity wave $Q_g^-$ does not alter the situation, because $Q_g^-$ is negligible there because of (60).

In the region to the left of the left-marginal line, where $|\kappa|$ is relatively small, both the free westward propagation speed of the upper Rossby wave and the free eastward propagation speed of the lower gravity wave are too large for the two waves to be phase-locked with each other. This is because both propagation speeds are increasing functions of $\frac{1}{|\kappa|}$. In the absence of phase-lock, the disturbance cannot grow exponentially. However, in the presence of $Q_g^-$, because of (61), the circulation induced by $Q_g^-$ acts to reduce the westward propagation speed of the upper Rossby wave $Q_R$ (Fig. 14). The reduction of propagation speed helps $Q_R$ and $Q_g^+$ to be phase-locked with each other. Consequently, the unstable region is enlarged by the westward-propagating gravity wave $Q_g^-$. The difference between Figs. 5 and 12 represents the effect of $Q_g^-$.  

10. Concluding remarks

In this paper, we analytically investigated the interaction between Rossby and gravity waves in a simple vertical-zonal two-dimensional model, having an upper vorticity gradient and a lower buoyancy gradient. The upper vorticity gradient supports a west-
ward-propagating Rossby wave. The lower buoyancy gradient supports eastward-propagating and westward-propagating gravity waves. By propagation, we mean propagation relative to the fluid. The analytical solution of the initial value problem for the interaction between the Rossby and gravity waves was obtained. From an initial value, the solution temporally evolves in a manner consistent with the BV-thinking of Harnik et al. (2008).

Under the condition that $C_R$ and $C_g$ are not very different, the westward-propagating upper Rossby wave and the eastward-propagating lower gravity wave can interact resonantly, resulting in exponential growth. Here, $C_R$ and $C_p$, which depend on the Richardson number $R$ and the zonal wave number $|\kappa|$, are the phase velocities of the free (i.e., without interaction) Rossby and gravity waves, respectively. As in other instability problems, to amplify each other (Rayleigh’s condition), two interacting waves must be counter-propagating. Furthermore, to be phase-locked with each other (Fjoeoft’s condition), their free phase velocities must not be so different.

When $C_R$ and $C_g$ are sufficiently different such that the two counter-propagating waves cannot be phase-locked with each other, the solution shows stable oscillation instead of unstable exponential growth. As in other instability problems, between the stable and unstable regions in the parameter space $(R,|\kappa|)$, the marginal solution grows as a linear function of time.

Although the resonant interaction occurs between the westward-propagating upper Rossby wave and the eastward-propagating lower gravity wave, the westward-propagating lower gravity wave also plays a role in the resonance; this is the same as in the problem of the resonant interaction between gravity and gravity waves (Rabinovich et al. 2011). The westward-propagating gravity wave regulates the westward propagation velocity of the Rossby wave, helping the Rossby and eastward-propagating gravity waves to become phase-locked. By the presence of the westward-propagating gravity wave, the unstable region in the parameter space is enlarged.

In this paper, the Rossby wave, which is supported by the vertical gradient of the horizontal vorticity, is a wave on the vertical plane, not on a horizontal plane. The interaction between the horizontal Rossby wave, which is supported by the horizontal gradient of vertical vorticity (or potential vorticity), and the gravity wave is also to be examined.

An example is the interaction between the vortex Rossby and gravity waves in a tropical cyclone (e.g., Schecter and Montgomery 2004). The instability caused by the interaction is called Rossby-inertia-buoyancy (RIB) mode instability (Hodyss and Nolan 2008). Although RIB instability is now well known, to our knowledge, the basic mechanism is not yet as clear as that of the barotropic or baroclinic instability based on PV-thinking (Hoskins et al. 1985).
Similar to the barotropic or baroclinic instability, RIB instability may be caused by resonant interaction between two phase-locked counter-propagating waves. One is an anti-cyclonically propagating vortex Rossby wave in the vortex region, and the other is a cyclonically propagating gravity wave in the outer region.

Clarifying the RIB instability based on the BV-thinking is a subject of our future study.

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