Partial Transpose As A General Criterion for the Separability of Quantum States

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The property of quantum entanglement or the inseparability of quantum states lies at the heart of several fields like quantum information theory, quantum teleportation, quantum cryptography etc. However for a given a quantum state it is a non trivial question to find out whether a given state is entangled or not. So several criterion’s such as Peres Horodecki Positivity of Partial Transpose (PPT) criterion[1], Von Neumann entropy, Schmidt decomposition etc. have already been proposed to find out whether a given state is entangled or not. However, the entanglement of indistinguishable particles are not well studied since such wave function of such particles are symmetrised or antisymmetrised product states which may not be separable in the usual sense. But these inseparability does not imply entanglement as it need not lead to any useful correlations. So the above mentioned separability criterion may not work in the case of indistinguishable particles. Von Neumann entropy[5] is one generalized criterion which works for distinguishable particles and indistinguishable particles. But the entropy does change

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for distinguishable particles, fermions and bosons. Slater decomposition or slater rank is another one criterion proposed for fermions.

In this paper, partial transpose operation is proposed as one such general separability criterion for distinguishable particles and indistinguishable fermions and bosons for bipartite systems. In this paper we propose partial transposition criterion in which if we can write the partially transposed density matrix as product of two density matrices (not necessarily the density matrix of one particle states) the state is separable.

**DISTINGUISHABLE PARTICLES**

For distinguishable particles, a bipartite pure quantum state can be represented as

$$|\psi\rangle = \sum_{ij} \Omega_{ij} a_i^\dagger b_j^\dagger |0\rangle$$

where $a_i^\dagger$ and $b_j^\dagger$ are creation operators acting on the vacuum state $|0\rangle$ to create the particles. Then by using the Schmidt decomposition theorem, the above state can be written in the Schmidt basis as

$$|\psi\rangle = \sum_i \omega_i a_i^\dagger b_i^\dagger |0\rangle.$$ 

Then the density matrix for the system in the Schmidt basis can be written as

$$\rho = \sum_{ij} \omega_i \omega_j a_i^\dagger b_j^\dagger |0\rangle \langle 0| b_j a_j$$

The partial transpose operation is done on the density matrix by exchanging the indices for one of the subsystems. The partially transposed density matrix for the given state is

$$\rho^{PT} = \sum_{ij} \omega_i \omega_j a_i^\dagger b_j^\dagger |0\rangle \langle 0| b_i a_j$$

For separable system

$$\rho = \rho_1 \otimes \rho_2$$

Since $\rho_1 = \rho_1^T$ we can write

$$\rho = \rho_1^T \otimes \rho_2 = \rho^{PT}$$
Then for the separability we may expect $\rho = \rho^{PT}$. From (3) and (4) $\rho \neq \rho^{PT}$ and then the state is entangled.

Consider a system consisting of two distinguishable particles with Schmidt number equal to 1. Then the state is

$$\ket{\psi} = \omega_i a_i^\dagger b_i^\dagger \ket{0}$$

$$\rho = \omega_i^2 a_i^\dagger b_i^\dagger a_i \bra{0} + \omega_j^2 a_j^\dagger b_j^\dagger a_j \bra{0} = \omega_i a_i^\dagger b_i^\dagger a_i \bra{0} + \omega_j a_j^\dagger b_j^\dagger a_j \bra{0}$$

(5)

Evidently $\rho = \rho^{PT} = \rho_1 \otimes \rho_2$ and the state is separable.

But for a system with Schmidt rank greater than 1, say 2, the state can be written as

$$\ket{\psi} = \omega_i a_i^\dagger b_i^\dagger \ket{0} + \omega_j a_j^\dagger b_j^\dagger \ket{0}$$

(6)

Then the density matrix for the above state can be written as

$$\rho = \left( \omega_i a_i^\dagger b_i^\dagger \ket{0} + \omega_j a_j^\dagger b_j^\dagger \ket{0} \right) \left( \omega_i \bra{0} a_i + \omega_j \bra{0} a_j \right)$$

$$= \omega_i^2 a_i^\dagger b_i^\dagger \ket{0} \bra{0} + \omega_j^2 a_j^\dagger b_j^\dagger \ket{0} \bra{0} = \omega_i a_i^\dagger b_i^\dagger a_i \bra{0} + \omega_j a_j^\dagger b_j^\dagger a_j \bra{0}$$

(7)

On exchanging $i$ and $j$ indices of operator $b$ we get

$$\rho^{PT} = \left( \omega_i a_i^\dagger b_i^\dagger \ket{0} + \omega_j a_j^\dagger b_j^\dagger \ket{0} \right) \left( \omega_i \bra{0} a_i + \omega_j \bra{0} a_j \right)$$

$$= \omega_i a_i^\dagger b_i^\dagger a_i \bra{0} + \omega_j a_j^\dagger b_j^\dagger a_j \bra{0}$$

In the matrix form we can write it as

$$\rho^{PT} = \begin{pmatrix}
\omega_i^2 & \omega_i \omega_j & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \omega_i \omega_j & \omega_j^2
\end{pmatrix}$$

(7)

Here $\rho \neq \rho^{PT}$. For a separable state its Schmidt number is equal to 1 and it is greater than 1 for an entangled state. We have seen that for a state with Schmidt number one, the partially transposed density matrix is same as the original density matrix while it is not the same as the original
for a state with Schmidt number greater than 1. So to conclude, we can say that for a bipartite system consisting of distinguishable particles, the state is separable if the partially transposed density matrix is same as the original density matrix as we expected. Later when we discuss bosons, we will show that for the separability $\rho = \rho^{PT}$ is not as necessary condition. There we will show that states with $\rho^{PT} = \tilde{\rho}_1 \otimes \tilde{\rho}_2$, where $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are one particle mixed states are also separable. $\rho^{PT}$ in equation (7) cannot be written as tensor product of density matrices and hence the state is separable.

**INDISTINGUISHABLE PARTICLES**

Hilbert space of two distinguishable particles, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Where $\mathcal{H}_1$ belongs to particle one and $\mathcal{H}_2$ belongs to particles two. For distinguishable particles entanglement is attributed to states which cannot be written in product form. But a system of indistinguishable particle in general, cannot be written in product form because both the particles share the same Hilbert space. Therefore many of the criteria that is discussed for distinguishable particles is not applicable. For example, particles with VonNeuman entropy of the reduced density matrix equal to one can be a separable states for Bosons and Fermions. But this inseparability does not mean entanglement. So special care is to be taken when entanglement of indistinguishable particles are studied. In this paper we propose partial transposition criterion in which if we can write the partially transposed density matrix as product of two density matrices (not necessarily the density matrix of one particle states) the state is separable.

**BOSONS**

Bosons can be represented in the Schmidt basis as

$$|\psi\rangle = \sum_i d_i a_i^\dagger a_i^\dagger |0\rangle$$

And hence

$$\rho = \sum_{ij} d_i d_j a_i^\dagger a_j^\dagger |0\rangle \langle 0| a_j a_j$$

(8)

$$\rho^{PT} = \sum_{ij} d_i d_j a_i^\dagger a_j^\dagger |0\rangle \langle 0| a_i a_j$$
In general $\rho \neq \rho^{PT}$.

For bosons, the states are separable when the Schmidt number is 1 or 2 and it is entangled when the Schmidt number is greater than 2 \[5\]. However when Schmidt’s number is one,

$$\rho = d_i^2 a_i^+ a_i^+ |0\rangle \langle 0| a_i a_i$$

Then

$$\rho = \rho^{PT}$$

When Schmidt’s number is 2,

$$|\psi\rangle = d_i a_i^+ a_i^+ |0\rangle + d_j a_j^+ a_j^+ |0\rangle$$

$$\rho = \left( d_i a_i^+ a_i^+ |0\rangle + d_j a_j^+ a_j^+ |0\rangle \right) \left( \langle 0| a_i a_i d_i + \langle 0| a_j a_j d_j \right)$$

$$= d_i^2 a_i^+ a_i^+ |0\rangle \langle 0| a_i a_i + d_j^2 a_j^+ a_j^+ |0\rangle \langle 0| a_j a_j$$

$$\rho^{PT} =$$

$$d_i d_j a_i^+ a_j^+ |0\rangle \langle 0| a_i a_j + d_i d_j a_j^+ a_i^+ |0\rangle \langle 0| a_j a_i$$

For bosons for all values of $i$ and $j$

$$[a_i^+, a_j^+] = 0$$

(11)

By using (11) we can write,

$$\rho^{PT} =$$

$$\left\{ d_i^2 a_i^+ a_i^+ |0\rangle \langle 0| a_i a_i + d_j^2 a_j^+ a_j^+ |0\rangle \langle 0| a_j a_j \right\}$$

$$= \left( \begin{array}{ccc} d_i^2 & 0 & 0 \\ 0 & d_i d_j & 0 \\ 0 & 0 & d_j^2 \end{array} \right) = \left( \begin{array}{cc} d_i & 0 \\ 0 & d_j \end{array} \right) \otimes \left( \begin{array}{cc} d_i & 0 \\ 0 & d_j \end{array} \right) = \tilde{\rho}_1 \otimes \tilde{\rho}_2$$

(12)
That is it is possible to write it as product of density matrices. From (9) and (10) it is evident that $\rho \neq \rho^{PT}$. Even then the state is separable\cite{5}. Therefore to be in accordance with the result of Gherardi\cite{5} we propose that the state is separable if the partially transposed density matrix is a tensor product of density matrices. Above result is not trivial because $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are not one particle density matrices. If we consider a state with Schmidt’s number 3, it is not possible to write the density matrix as product of density matrices and hence the state is entangled.

FERMIONS

A generic state for a two fermion system can be represented as

$$|\psi\rangle = \sum_{ij} \Omega_{ij} f_i^\dagger f_j^\dagger |0\rangle$$

(13)

where $f^\dagger$ is the fermionic creation operating on the vacuum state $|0\rangle$ to create the fermions and $\Omega_{ij} = -\Omega_{ji}$ is an antisymmetric matrix. Similar to the Schmidt decomposition used before, there exist a decomposition\cite{7,6} for antisymmetric matrices known as slater decomposition by which the above state can be written as

$$|\psi\rangle = \sum_l z_l f_{1l}^\dagger f_{2l}^\dagger |0\rangle$$

(14)

where the number of non vanishing coefficients $z_l$ gives the Slater rank. For fermions, the separable states have a slater rank of one and it is greater than one for entangled states. For slater rank 1, the density matrix is given by

$$\rho = z^2 f_{11}^\dagger f_{22}^\dagger |0\rangle \langle 0| f_{22} f_{11} = \rho^{PT} = \rho_1 \otimes \rho_2 .$$

(15)

The density matrix has only one element and hence it corresponds to a separable state which is expected for a system with slater rank 1.

But for slater rank 2, the state of the system is given by

$$|\psi\rangle = z_i f_{1i}^\dagger f_{2i}^\dagger |0\rangle + z_j f_{1j}^\dagger f_{2j}^\dagger |0\rangle$$

(16)

and the density matrix is

$$\rho = \left\{ 
\begin{array}{l}
  z_i^2 f_{1i}^\dagger f_{2i}^\dagger |0\rangle \langle 0| f_{2i} f_{1i} + z_j^2 f_{1j}^\dagger f_{2j}^\dagger |0\rangle \langle 0| f_{2j} f_{1j}, \\
  z_i z_j^* f_{1i}^\dagger f_{2j}^\dagger |0\rangle \langle 0| f_{2j} f_{1i} + z_j z_i^* f_{1j}^\dagger f_{2i}^\dagger |0\rangle \langle 0| f_{2i} f_{1j}, \\
\end{array}
\right\}$$
\[
\begin{pmatrix}
  z_i^2 & 0 & 0 & z_i z_j^* \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  z_j z_i^* & 0 & 0 & z_j^2
\end{pmatrix}
\]

(17)

\[
\rho^{PT} = \begin{cases}
  z_i^2 f_1^1 f_2^1 |0\rangle \langle 0| f_2, f_1, + z_j^2 f_1^1 f_2^1 |0\rangle \langle 0| f_2, f_1, \\
  z_i z_j^* f_1^1 f_2^1 |0\rangle \langle 0| f_2, f_1, + z_j z_i^* f_1^1 f_2^1 |0\rangle \langle 0| f_2, f_1,
\end{cases}
\]

By using \(\{b_i^\dagger, b_j^\dagger\} = 0\),

\[
\begin{pmatrix}
  z_i^2 & 0 & 0 & 0 \\
  0 & -z_i z_j^* & 0 & 0 \\
  0 & 0 & -z_j z_i^* & 0 \\
  0 & 0 & 0 & z_j^2
\end{pmatrix}
\]

(18)

According to our criteria, for separable states, the partially transposed density matrix should be written as the product of two density matrices. \(\rho\) in equation (18) cannot be written as tensor product of matrices and hence fermions with slater rank 2 are not separable.

**CONCLUSION**

In summary, we have presented a general criteria for the separability of quantum states for a bipartite system based on partial transposition operation for both distinguishable and indistinguishable particles. For distinguishable particles and bosons, the partial transpose is taken on the Schmidt basis, while it is done using slater decomposition for fermions. The partially transposed matrix is identical to the original density matrix for the distinguishable particles and any change in the matrix is only due to entanglement. In the case of fermions and bosons, we check whether the partially transposed matrix can be written as the product of two density matrices. If it is possible, then the states are separable, otherwise entangled.

**References**

[1] Peres A, Phys. Rev. Lett. 77 1413(1996)
[2] K. Eckert, J. Schliemann, D. Bru, M. Lewenstein, Ann. Phys. 299 (2002) 88.

[3] ZHAO Xin, WU Hua, LI Yan-Song, LONG Gui-Lu, CHIN. PHYS. LETT. Vol. 26, No. 6 (2009)

[4] GianCarlo Ghirardi, Luca Marinatto, Fortschr. Phys. 51, 379-387 (2003)

[5] GianCarlo Ghirardi, Luca Marinatto, Phys. Rev. A. 70, 012109 (2004)

[6] John Schliemann, J. Ignacio Cirac, Marek Kus, Maciej Lewenstein, Daniel Loss, Phys. Rev. A, 64, 022303 (2001)

[7] M. L. Mehta, Elements of Matrix Theory, Hindustan Publishing Corporation, Delhi (1977)