Noncommutativity from Canonical and Noncanonical Structures

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Abstract. Using arbitrary symplectic structures and parametrization invariant actions, we develop a formalism, based on Dirac’s quantization procedure, that allows us to consider theories with both space-space as well as space-time noncommutativity. Because the formalism has as a starting point an action, the procedure admits quantizing the theory either by obtaining the quantum evolution equations or by using the path integral techniques. For both approaches we only need to select a complete basis of commutative observables. We show that for certain choices of the potentials that generate a given symplectic structure, the phase of the quantum transition function between the admissible bases corresponds to a linear canonical transformation, by means of which the actions associated to each of these bases may be related and hence lead to equivalent quantizations. There are however other potentials that result in actions which can not be related to the previous ones by canonical transformations, and for which the fixed end-points, in terms of the admissible bases, can only be realized by means of a Darboux map. In such cases the original arbitrary symplectic structure is reduced to its canonical form and therefore each of these actions results in a different quantum theory. One interesting feature of the formalism here discussed is that it can be introduced both at the levels of particle systems as well as of field theory.

1. Introduction

In recent years, space-time noncommutativity has become the subject of increasing interest. In field theory stimulated by some results in low energy string theory, and in quantum mechanics because it is in the context of this formalism that space-time noncommutativity is more naturally understood in terms of space and time operators acting on a Hilbert space and also, because quantum mechanics viewed as a minisuperspace reduction of field theory, could reasonably be expected to provided further insight into how quantum mechanical noncommutativity reflects...
itself in field theory. Some of the more relevant work related to the approach here considered may be found in [1], [2], [3], [4], [5], [6], [7].

An interesting idea that allows us to consider in a full setting the space-time noncommutativity in the context of particle mechanics, is to use the concept of parametrization invariance [5], [7]. In this way the time is taken as an extra canonical variable of the system and it is then easy to introduce a non-canonical structure in this extended phase-space. The usual way to study the parametrization invariance of a system is by using the Dirac method of canonical analysis. Because not all the momenta are independent due to the invariance under parametrizations, this approach requires that a constraint on the system be introduced. For a parametrized particle, this constraint is at the classical level the Hamilton-Jacobi equation and at the quantum level the Schrödinger equation. So the Dirac method associates to the symmetry of parametrizations the classical or quantum evolution equations [8].

Here we want to generalize the above mentioned procedure in order to be able to consider noncommutative theories at the quantum level resulting both from canonical and non-canonical structures. The noncommutativity will then appear as a consequence of the existence of second class constraints, and the implementation of these constraints in terms of Dirac brackets. The interesting point of the procedure is that on the one hand we get the classical and quantum evolution equations for the noncommutative systems and on the other hand we also obtain a classical action that can be quantized using the path integral formalism. Furthermore, the analysis is not restricted to noncommutative theories with constant deformation parameters, since the procedure naturally incorporates arbitrary canonical potentials. Another interesting property of the method is that it can be naturally extended to field theory.

Our starting point is to consider a parametrization invariant system. This means that if the system is not naturally invariant under parametrizations we promote the original parameters of the theory, for example the time in the case of particle dynamics, to the level of canonical variables. The second step is to perform the canonical analysis of this theory. One point that we must be careful with is that, since we add new variables to the system, we have to introduce constraints associated to the parametrization invariance symmetry of the theory in order that the number of degrees of freedom are preserved. The third step is to introduce an arbitrary canonical potential that allows us to realize the required noncommutativity. The next step is to show that under the Dirac brackets the first class constraint (or constraints) generate the symmetry. This means that we will probably need to modify the constraints. At this point, if we have several constraints, we need to check that the algebra of these first class constraints closes. Once we finish this procedure we obtain the quantum evolution equations for our system. Alternatively, we can introduce the canonical potential in the action and select an appropriate basis in order to quantize the system using the path integral formalism. For certain choices of the potentials that generate a given symplectic structure, the phase of the quantum transition function between the admissible bases corresponds to a linear canonical transformation, by means of which the actions associated to each of these bases may be related and hence lead to equivalent quantizations. We must stress that in contradistinction to the case when time plays the role of a parameter, the canonical transformation here is implemented in an extended phase space, where the time and its conjugate momentum are included.
With the purpose of examining all the above mentioned facets of the space-time noncommutativity, our presentation has been structured as follows: In Section 2 we consider the canonical formalism of parametrization invariant systems. In Section 3 we introduce an arbitrary symplectic structure in the action, and after the canonical analysis we construct the Dirac brackets associated to the theory and also obtain the action for the reduced system. In Section 4, we quantize the theory using different bases, and using both path integral methods and the quantum evolution equations. We conclude the paper with some remarks and possible extensions.

2. Parametrization invariant systems

We begin here by reviewing the essentials of the canonical analysis of parametrized systems following the approach in [8]. To this end, consider the action for a particle in a $N$-dimensional configuration space, in an arbitrary potential:

$$ S = \frac{1}{2} m \left( \frac{d q^i}{dt} \right)^2 - V(q^i, t), $$

where $i = 1, \ldots, N$. In this action the time $t$ plays the role of a parameter in the theory. To study the non-commutativity of the space and time it is more convenient to consider the time as another coordinate of our theory, i.e. we extend our configuration space with one extra dimension $t = q^0$. To do this, we parametrize the action by introducing a new parameter $\tau$ and assume that the coordinates $q^i(t)$ are scalars under this parametrization, i.e.,

$$ t \rightarrow \tau $$
$$ q^i(t) \rightarrow q^i(\tau) $$

The action (2.1) takes the form

$$ S = \int^{\tau_2}_{\tau_1} d\tau \left( \frac{1}{2} m \left( \frac{d q^i}{d\tau} \right)^2 \left( \frac{d\tau}{dt} \right) - V(q^i, \tau) \left( \frac{d\tau}{dt} \right) \right), $$

where $t = q^0$ now plays the role of a new coordinate in the theory. Making the identifications $\dot{q}^i \equiv \left( \frac{dq^i}{d\tau} \right)$ and $\dot{q}^0 \equiv \frac{dt}{d\tau}$, we can rewrite (2.3) in the form

$$ S = \int^{\tau_2}_{\tau_1} d\tau \left( \frac{1}{2} m \frac{(\dot{q}^i)^2}{\dot{q}^0} - V(q^i, q^0)\dot{q}^0 \right). $$

In Hamiltonian form the action (2.4) reads

$$ S = \int^{\tau_2}_{\tau_1} d\tau \left( p_0 \dot{q}^0 + p_i \dot{q}^i - \lambda \varphi \right), $$

where $\varphi = p_0 + H \approx 0$ is the first class primary constraint associated to the symmetry under parametrizations (which needs to be included in (2.5) in order to account for the fact that by introducing a new variable in the theory, restrictions must be added to the physical evolution of the system that indicate that the $N+1$ new coordinates are not all independent), $H$ is the canonical Hamiltonian of the action (2.4), and $\lambda(\tau)$ is a Lagrange multiplier. The action (2.5) is invariant up to
a total derivative under the transformations generated by the constraint \( \varphi \), given by
\[
\delta q_0 = \{ q_0, \varepsilon \varphi \}, \quad \delta p_0 = \{ p_0, \varepsilon \varphi \}, \quad \delta p_i = \{ p_i, \varepsilon \varphi \}, \quad \delta q^i = \{ q^i, \varepsilon \varphi \} \quad \delta \lambda = \dot{\varepsilon},
\]
where the variation of the Lagrange multiplier is imposed in such a way that when varying the action it should vanish up to a boundary term.

Following Dirac [11], we propose that at the quantum level the physical states of the theory are invariant under the above transformations, i.e.,
\[
e^{i\varepsilon \hat{\varphi}} |\psi_P\rangle = |\psi_P\rangle.
\]
So in infinitesimal form we get
\[
\hat{\varphi} |\psi_P\rangle = 0.
\]
We thus see that the constraint leads to a supplementary condition on the physical states, and is another way to reduce the quantum theory to its physical sector without imposing a gauge condition.

Now if we consider the configuration representation with basis \( |q^0, q^i\rangle \), equation (2.8) yields,
\[
\hat{\varphi} |\psi_P\rangle = 0 \Rightarrow \left( -i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2m} \nabla^2 + V(q^i, t) \right) \psi(q^i, t) = 0,
\]
where we have identified \( t = q^0 \). We therefore obtain the Schrödinger equation as a result of imposing at the quantum level the classical invariance under parametrizations of the theory.

In the following section we shall apply the same procedure to the case of arbitrary symplectic structures.

### 3. Non-commutativity and Dirac Brackets

Let \( z^a = (q^0, q^i, p_0, p_i) \), with \( a = 1, ..., 2N + 2 \), denote the \( 2N + 2 \) phase-space variables of a parametrized system in the Hamiltonian formulation. In this case we don’t have a second order action to begin with as in (2.1). We can however consider a general first order action, equivalent to (2.5), given by
\[
S = \int_{\tau_1}^{\tau_2} d\tau \left( A_a(z) z^a - \lambda \varphi(z) \right),
\]
where \( A_a(z) \) is a vector potential which we shall use to generate an arbitrary symplectic structure associated to the Poisson brackets in the Hamiltonian formulation.

Applying the Dirac’s method for constrained systems, we have from (3.1) that the corresponding canonical Hamiltonian is given by
\[
H_c = \lambda \varphi(z),
\]
and the canonical momenta lead to the set of primary constraints,
\[
\chi_a = p_{za} - A_a(z).
\]
Consequently, the total Hamiltonian for this theory is
\[
H_T = \lambda \varphi + \mu^a \chi_a.
\]
Moreover, from the evolution of the constraints we obtain the following consistency conditions

\( \dot{\chi}_a = \{ p_{za} - A_a (z), H_T \} = -\lambda \frac{\partial \phi}{\partial z^a} + \mu^b \omega_{ab} \approx 0, \)

where

\( \omega_{ab} := \partial_a A_b - \partial_b A_a = \{ \chi_a, \chi_b \}. \)

This antisymmetric matrix will play the role of the symplectic structure of the theory. Assuming further that \( \omega_{ab} \) is invertible so all the Lagrange’s multipliers \( \mu^a \) in (3.5) can be determined, it then follows from (3.6) that the constraints \( \chi_a \) are second class. Note that in the case where the symplectic structure is degenerate, at least one of the \( \chi_a \)’s will be first class, but in this case the number of degrees of freedom of the generalized theory will not correspond to the degrees of freedom of the original theory. Hence in what follows we will assume that all the constraints \( \chi_a \) are second class.

Now, in order to impose these constraints as strong conditions when quantizing, we construct the associated Dirac brackets which are given by

\( \{ A, B \}^* = \{ A, B \} - \{ A, \chi_a \} \omega^{ab} \{ \chi_b, B \}, \)

where \( \omega^{ab} \) is the inverse matrix of \( \omega_{ab} \). Computing the Dirac’s brackets of the coordinates with the above expression we obtain

\( \{ z^a, z^b \}^* = \omega^{ab}. \)

Thus, quantizing a theory constrained by symmetries under parametrization results in the noncommutativity of the quantum operators corresponding to the phase space coordinates:

\( [\hat{z}^a, \hat{z}^b] = i\hbar \omega^{ab}. \)

The simplest case corresponds to the usual Heisenberg algebra of ordinary Quantum Mechanics, for which the inverse matrix of the canonical symplectic structure takes the form

\( J^{ab} := \omega^{ab}_{|\theta=0} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \)

4. Non-commutative Quantum Mechanics

In the previous section we have considered a general procedure for quantizing a theory with an arbitrary symplectic structure. One interesting feature of this formalism is that by including time as a canonical variable allows us to consider also noncommutativity between the time and the spatial coordinates. Now, given such a symplectic structure we can quantize either by using the Dirac’s procedure where the first class constraints act as operators on the physical states, imposing supplementary conditions on them, and the Dirac brackets of the second class constraints are replaced by commutators, or, alternatively, we can also quantize by first evaluating the generating potentials of the symplectic structure and then applying path integral methods in order to derive the Feynman propagators.

It should be noted, however, that for a given symplectic structure the solution for the potentials \( A_a \) is not unique, although all the possible resulting actions and
resulting classical theories are related by canonical transformations. Furthermore, in the Dirac quantization the commutators of the generators of the extended Heisenberg algebra define the possible complete sets of commuting observables of the theory and the correlative admissible bases (labeled by the eigenvalues of these sets). For each of these admissible bases, we obtain a realization of the Heisenberg algebra and of the subsidiary condition and, correspondingly in the path integral formalism, the Feynman propagators derived from the transition functions in each of these bases. This means that in the path integral calculation of a transition function, the only admissible actions are those for which the fixed end-points in a variational principle are the same as the dynamical variables labeling the basis used for the evaluation of the transition function.

Note finally that there are also actions originating from solutions of for which no fixed end-points, corresponding to one of the admissible bases in the Dirac quantization exits. However, can be defined using a Darboux map. This map, involves introducing new dynamical variables in terms of linear combinations of the original ones and, consequently implies a change in the initial symplectic structure to a canonical one. Compatible, although non-equivalent, path integral and Dirac quantizations result from promoting to the rank of operators these new variables, which will satisfy the Heisenberg algebra of ordinary quantum mechanics. So in these cases the deformation of the symplectic structure at the classical level is reflected at the quantum level in a deformed Hamiltonian while the standard Heisenberg algebra of the usual quantum mechanics is preserved.

To further illustrate the above observations, we next consider some examples of quantum noncommutativity schemes in the context of both the Dirac and path integral formalisms. For analytical simplicity we assume a 1+1 space-time, generalization to higher order dimensions is fairly straightforward.

4.1. Space-time noncommutativity. Let us consider first the case where the Dirac brackets determine a symplectic structure of the form

\[
\omega^{ab} = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \omega_{ab} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & \theta \\ 0 & 1 & -\theta & 0 \end{pmatrix}.
\]

Quantizing according to Dirac’s prescription by using leads to the commutators

\[
[\hat{t}, \hat{x}] = i\hbar \theta, \quad [\hat{x}, \hat{p}_x] = i\hbar, \quad [\hat{t}, \hat{p}_t] = i\hbar, \quad [\hat{p}_t, \hat{p}_x] = 0,
\]

and, using to the supplementary condition

\[
\hat{\phi} | \psi \rangle = 0,
\]

where \( \hat{\phi} \) is given by

\[
\hat{\phi} = \hat{p}_t + H(\hat{t}, \hat{x}, \hat{p}_x).
\]

It is obvious from that for a mechanical Hamiltonian the sets of complete commuting observables in this case are \( \{ \hat{x}, \hat{p}_t \}, \{ \hat{t}, \hat{p}_x \} \) and \( \{ \hat{p}_t, \hat{p}_x \} \). The admissible bases in Hilbert space are then \( \{|x, p_t\rangle \}, \{|t, p_x\rangle \} \) and \( \{|p_t, p_x\rangle \} \), respectively.
4.1.1. Basis \(|x, p_t\)|. For the basis \(\{\hat{x}, \hat{p}_t\}\) the algebra (4.2) is realized by
\[
\hat{p}_x \psi(x, p_t) = -i\hbar \partial_x \psi(x, p_t), \quad \hat{x} \psi(x, p_t) = \psi(x, p_t),
\]
while the remaining generators of the extended Heisenberg algebra are just multiplicative quantities. Also projecting on (4.3) with \((x, p_t)\) and substituting (4.5) into (4.4), with a Hamiltonian of the form
\[
H = \frac{k^2}{2m} + V(x, t),
\]
we need to compute the transition function
\[
\psi(x, p_t).
\]
One interesting feature of the Dirac quantization resulting from the use of this basis is that for a \(t\) independent potential, equation (4.6) becomes
\[
\left( p_t - \frac{k^2}{2m} \partial_x^2 + V(x) \right) \psi(x, p_t) = 0.
\]
For such a time independent Hamiltonian, (4.7) may be interpreted as an eigenvalue equation, with \(-p_t\) the energy eigenvalues of the system and \(\psi(x, p_t)\) the corresponding eigenvectors. Note that the energy spectrum of the resulting theory does not have any corrections from the noncommutativity of the space-time. A similar result was obtained by Balachandran, et al [4] by means of a very different approach.

Now, in order to obtain the equivalent quantization by means of path integrals, we need to compute the transition function \(\langle x(\tau_2), p_t(\tau_2)|x(\tau_1), p_t(\tau_1)\rangle\). For this purpose we need first to derive the appropriate action function, which according to our previous observations has to have as fixed end-points the variables \(x, p_t\). This, as implied by (4.10), requires in turn deriving the proper generating potentials \(A_a(z)\) for the symplectic structure \(4.13\) by solving the equations,
\[
\frac{\partial A_1}{\partial p_t} - \frac{\partial A_3}{\partial t} = 1, \quad \frac{\partial A_2}{\partial p_x} - \frac{\partial A_4}{\partial x} = 1, \quad \frac{\partial A_4}{\partial p_t} - \frac{\partial A_3}{\partial p_x} = \theta.
\]

It is not difficult to verify that the needed solution is
\[
A_1 = 0, \quad A_2 = p_x, \quad A_3 = -(t + \theta p_x), \quad A_4 = 0.
\]
In fact, inserting (4.9) in the action (3.11) results in
\[
S_1 = \int_{\tau_1}^{\tau_2} dt \left( p_x \dot{x} - \theta p_x \dot{p}_t - t \dot{p}_t - \lambda \left( p_t + H(t, x, p_x) \right) \right),
\]
which indeed has the appropriate variational fixed end-points \(x, p_t\). With (4.10) we can now compute the propagator
\[
\langle x(\tau_2), p_t(\tau_2)|x(\tau_1), p_t(\tau_1)\rangle = \int Dt Dp_t Dx Dp_x \delta(\chi) \delta(\varphi) \{\varphi, \chi\}^* \exp \left( \frac{i}{\hbar} S_1 \right),
\]
where we have introduced a canonical gauge fixing condition \(\chi = \chi(\tau, t, p_t, x, p_x)\). This gauge must first be a good canonical gauge in the Dirac’s sense, i.e. the Dirac bracket \(\{\varphi, \chi\}^*\) must be invertible and second the gauge must be consistent with
the boundary conditions. Because, we are fixing at the end points \((x, p_t)\), it is not possible to use the usual gauge \(t = f(\tau)\), we will use instead the gauge condition

\[
\chi = x - f(\tau) \approx 0.
\]

The Dirac’s bracket between this gauge condition and the constraint is given by

\[
\{ \varphi, \chi \}^* = -\frac{p_x}{m} + \theta \frac{\partial V}{\partial t}
\]

This gauge is a good canonical gauge for \(p_x \neq 0\), in which case the path integral has two different branches, one corresponding to \(p_x > 0\) and the other for negative \(p_x\). It can also be seen that this term leads to corrections of first order in \(\theta\) which are, however, proportional to the time dependence of the potential. Consequently, if we assume that the potential is time independent, this corrections cancel and we can then integrate \((4.11)\) over \(t\) to obtain

\[
\langle x(\tau_2), p_t(\tau_2) | x(\tau_1), p_t(\tau_1) \rangle = \int \mathcal{D}x \mathcal{D}p_t \mathcal{D}p_x \delta(x - f) \delta(\dot{p}_t) \left( -\frac{p_x}{m} \right) \times \left( \exp \left( \frac{i}{\hbar} \int_{\tau_1}^{\tau_2} d\tau \left( p_x (\dot{x} - \theta \dot{p}_t) - \lambda \varphi(p_t, p_x, x) \right) \right) \right).
\]

Note now that the only dependence on \(\theta\) in the above expression appears multiplying \(\dot{p}_t\), but taking into account that this term is zero due to the delta functional in the path integral we do not get noncommutative corrections to the propagator. This is in agreement with our previous results derived by using the Dirac’s quantization.

### 4.1.2. Basis \(|t, p_x\rangle\).

Let us next consider the basis \(\{\hat{t}, \hat{p}_x\}\) in which the operators \(\hat{x}\) and \(\hat{p}_t\) are realized by

\[
\hat{x} \psi(t, p_x) = i\hbar (\partial_{p_x} - \theta \partial_t) \psi(t, p_x), \quad \hat{p}_t \psi(t, p_x) = -i\hbar \partial_t \psi(t, p_x).
\]

In the Dirac quantization we have that a realization of the supplementary condition \((2.8)\) in this basis results from projecting with \(|t, p_x\rangle\) and substituting \((4.15)\) into the first class constraint \((4.4)\), we thus get

\[
\left( -i\hbar \partial_t + \frac{p_x^2}{2m} + V(t, i\hbar (\partial_{p_x} - \theta \partial_t)) \right) \psi(t, p_x) = 0.
\]

Note that contrary to what we had in the case of the basis \(|x, p_t\rangle\) where the supplementary condition was independent of time, here we have a time evolution equation. However, because of the time derivative in the potential in \((4.16)\) we may lose the usual probability amplitude interpretation for \(\psi(t, p_x)\) for time derivatives of order higher than one, regardless of whether or not the potential has an explicit dependence on time. It is conceivable, nonetheless, that for certain forms of the potential a probabilistic interpretation may be recovered by modifying the product in the algebra of the wave functions or by redefining hermicity, in analogy to what occurs in Feshbach-Villars formulation of the Klein-Gordon equation.
It is natural to ask how (4.10) related to (4.7) for a time independent potential. For this purpose note that
\[ (t, p_x|V(\dot{x})|\psi) = V(i\hbar(\partial_{p_x} - \theta \partial_t)) \psi(t, p_x) = \int dx \, dp_x V(i\hbar(\partial_{p_x} - \theta \partial_t)) \langle t, p_x|x, p_t\rangle \psi(x, p_t). \]

But
\[ (t, p_x|x, p_t) = x(t, p_x|x, p_t) = i\hbar(\partial_{p_x} - \theta \partial_t)\langle t, p_x|x, p_t\rangle. \]
So
\[ V(i\hbar(\partial_{p_x} - \theta \partial_t)) \langle t, p_x|x, p_t\rangle = V(x)\langle t, p_x|x, p_t\rangle, \]
and using
\[ (t, p_x|x, p_t) = (2\pi\hbar)^{-1}e^{-\frac{i}{\hbar}(xp_x - \theta p_tp_x - tp_t)}, \]
(see e.g. (4.13) for details of a procedure used to derive a similar transition function), we get
\[ (t, p_x|V(\dot{x})|\psi) = (2\pi\hbar)^{-1}\int dx \, dp_t V(x) e^{-\frac{i}{\hbar}(xp_x - \theta p_tp_x - tp_t)} \psi(x, p_t). \]
Finally, substituting this result in (4.16) we get the integro-differential equation
\[ \left( -i\hbar \partial_t + \frac{p_x^2}{2m} \right) \psi(t, p_x) + (2\pi\hbar)^{-2}\int dx \, dp_t \, dt' \, dp'_x V(x) e^{-\frac{i}{\hbar}[xp'_x - \theta p.tp_x - \theta p_tp'_x - (t' - t)p_t]} \psi(t', p'_x) = 0. \]

On the other hand, if \( \psi(x, p_t) \) is a solution of (4.7) then
\[ \psi(t, p_x) = \int dx dp_t \langle t, p_x|x, p_t\rangle \psi(x, p_t) = (2\pi\hbar)^{-1}\int dx dp_t e^{-\frac{i}{\hbar}(xp_x - \theta p_tp_x - tp_t)} \psi(x, p_t), \]
is a solution of (4.16). Indeed acting with \( \left( -i\hbar \partial_t + \frac{p_x^2}{2m} + V(t, i\hbar(\partial_{p_x} - \theta \partial_t)) \right) \) on (4.16) and making use of (4.17) and (4.21) we get
\[ \left( -i\hbar \partial_t + \frac{p_x^2}{2m} + V(t, i\hbar(\partial_{p_x} - \theta \partial_t)) \right) \psi(t, p_x) = (2\pi\hbar)^{-1}\int dx \, dp_t \, e^{-\frac{i}{\hbar}(xp_x - \theta p_tp_x - tp_t)} [p_t - \frac{\hbar^2}{2m} \partial_x + V(x)] \psi(x, p_t). \]
Now, if \( \psi(x, p_t) \) satisfies (4.7) the right side of (4.24) is zero, hence \( \psi(t, p_x) \) as given by (4.24) satisfies (4.16). Q.E.D.

Let us now turn to the path integral quantization for this case and the calculation of the propagator \( \langle t(t_2), p_x(t_2)|t(t_1), p_x(t_1)\rangle \). The appropriate solution to the equations (4.3) for which \( t, p_x \) are the fixed end points of the action are
\[ A_1 = p_t, \, A_2 = 0, \, A_3 = 0, \, A_4 = \theta p_t - x. \]
Inserting this solution into the action \(3.1\) we then obtain,

\[(4.26)\]

\[S_2 = \int_{\tau_1}^{\tau_2} d\tau \left( p_t \dot{t} + \theta p_t \dot{p}_x - x \dot{p}_x - \lambda \varphi \right),\]

Observe that the action \(4.26\) and the action \(4.10\) are indeed related by a linear canonical transformation generated by \(F_1 = p_x x - \theta p_t p_x - p_t t\).

The propagator for the admissible basis \(|t, p_x\rangle\) is then

\[(4.27)\]

\[\langle t(\tau_2), p_x(\tau_2)|t(\tau_1), p_x(\tau_1)\rangle = \int D\tau Dp_x \delta(\chi) \exp\left(\frac{i}{\hbar}S_2\right),\]

and for the boundary conditions that we are considering, the usual gauge

\[(4.28)\]

\[t = f(\tau),\]

is a good gauge condition.

Assuming now that the Hamiltonian is independent of \(t\), we can easily integrate \(4.27\) over the variables \(t\) and \(p_t\), using the gauge condition \(4.28\) and the constraint, we get

\[(4.29)\]

\[\langle f(\tau_2), p_x(\tau_2)|f(\tau_1), p_x(\tau_1)\rangle = \int Dx Dp_x (-1 - \theta \partial_x V) \exp\left(\frac{-i}{\hbar} \int_{f_1}^{f_2} df \left( (\theta H + x) \frac{dp_x}{df} + H \right) \right),\]

where the parametrization in the action has been eliminated.

Note that in the limit \(\theta = 0\) both \(4.16\) and \(4.29\) reduce to the usual Quantum Mechanics. The same is true for a free particle, as it is immediately evident from \(4.16\), and it also follows for \(4.29\) since in this case the Hamiltonian is independent of \(x\), so by integrating over this variable the term with \(\theta = 0\) disappears.

4.1.3. Basis \(|p_t, p_x\rangle\). To conclude our analysis of the Dirac and path integral quantization realized on the three admissible bases for the extended Heisenberg algebra \(4.2\) that we are studying in this section, consider now the representation of the operators \((\hat{t}, \hat{x})\) in \(|p_t, p_x\rangle\). For this basis we have

\[(4.30)\]

\[\hat{t}\psi(p_t, p_x) = (i\hbar \partial_{p_t} + a \theta p_x)\psi(p_t, p_x), \quad \hat{x}\psi(p_t, p_x) = (i\hbar \partial_{p_x} + (1 + a) \theta p_t)\psi(p_t, p_x).\]

It is interesting to note that in this representation we have introduced an extra parameter \(a\), that can translate the noncommutativity from the coordinate operator to the time operator. (Observe that this characteristic is also present when we impose noncommutativity of the space so we can also translate the noncommutativity parameter from one coordinate to the another). For this representation the constraint equation \(4.3\) takes the form

\[(4.31)\]

\[\left( p_t + \frac{p_x^2}{2m} + V (i\hbar \partial_{p_t} + a \theta p_x, i\hbar \partial_{p_x} + (1 + a) \theta p_t) \right) \psi(p_t, p_x) = 0.\]

Note that in this case, when the potential is time independent so that \(4.31\) reduces to

\[(4.32)\]

\[\left( p_t + \frac{p_x^2}{2m} + V (i\hbar \partial_{p_x} + (1 + a) \theta p_t) \right) \psi(p_t, p_x) = 0,\]
we do have noncommutative corrections except when we choose the parameter $a = 0$, or for the case of a free particle.

For the path integral formulation in this basis, an appropriate action (having $p_t, p_x$ as fixed end-points) is given by

\[ S_3 = \int_{\tau_1}^{\tau_2} d\tau \left( -t\dot{p}_t + a\theta p_t \dot{p}_x - (1 - a)\theta p_x \dot{p}_t - x\dot{p}_x - \lambda \varphi \right), \]

from which we can obtain results equivalent to those derived from the analysis of the constraint equation (4.31).

Contrary to the actions $S_1$ and $S_2$ which are unique solutions of (3.6) for their corresponding fixed end-points, there are several canonically equivalent admissible actions with fixed points $p_t, p_x$. Thus, for example, $S_4 = \int_{\tau_1}^{\tau_2} d\tau (-t\dot{p}_t - \theta \dot{p}_x \dot{p}_t - \dot{p}_x)$ can be obtained from $S_3$ by subtracting the total derivative of $F_2 = a\theta p_t p_x$ from the integrand in $S_3$. Other canonically equivalent actions follow from $S_3$ and $S_4$ by means of the generator $F_3 = \theta p_t p_x$.

4.1.4. Noncanonical related actions. Up to this point we have considered path integral quantizations based on actions which are compatible with the extended Heisenberg algebra (4.2), derived by means of the Dirac quantization procedure. There are, however, other solutions to the equations (4.8) which, although indistinguishable at the classical level from the ones considered so far, they are not canonically related to them, in the sense that there is no generating function for mapping canonically the actions resulting from these solutions to the ones previously considered. We shall see that in these cases the transformations needed for fixing the end-points required for a path integral quantization are actually transformations which map the original phase-space variables with symplectic structure (4.2) to another set of variables related to the canonical symplectic structure (3.10).

Classically, as it is well known from the Darboux theorem [12], this map is always possible (at least locally). To each of these Darboux maps corresponds, however, a different quantum mechanics, generated by what in some works in the literature has been called the equivalent of the Seiberg-Witten map for “noncommutative quantum mechanics”.

To exhibit in more detail the above considerations, let us begin with the solutions:

\[ A_1 = p_t, \ A_2 = p_x, \ A_3 = 0, \ A_4 = \theta p_t, \]

\[ A_1 = p_t, \ A_2 = p_x, \ A_3 = -\frac{\theta}{2} p_x, \ A_4 = \frac{\theta}{2} p_t. \]

With the first set of equations in (4.34), the canonical action takes the form

\[ S_5 = \int_{\tau_1}^{\tau_2} d\tau \left( p_t (t + \theta p_x) + p_x \dot{x} - \lambda (p_t + H(t, x, p_x)) \right). \]

We therefore see from (4.35) that from the original phase-space variables of the theory we do not have a set of fixed end-points for the action from which a quantization can be developed. Nonetheless a natural pair $(\tilde{t}, x)$ can be constructed by
making the change of variables
\begin{equation}
\tilde{t} := t + \theta p_x, \quad \tilde{x} = x,
\end{equation}
where \(\tilde{t}\) is a new canonical variable associated to the time. In terms of this new pair of variables, the symplectic structure is reduced to (3.10), and introducing this new time in the action (4.35), results in
\begin{equation}
S_5 = \int_{\tau_1}^{\tau_2} d\tau \left( p_t \dot{\tilde{t}} + p_x \dot{\tilde{x}} - \lambda \left( p_t + H(\tilde{t} - \theta p_x, x, p_x) \right) \right).
\end{equation}

Note that if the original Hamiltonian was time-dependent, the modified one introduces a new kind of interaction that is proportional to the parameter \(\theta\) of noncommutativity and to the momenta in the spatial direction. Also note that in terms of the modified symplectic structure (3.10) the Dirac brackets (3.8) lead, upon quantization, to the commutators
\begin{equation}
\left[ \hat{\tilde{t}}, \hat{p}_t \right] = i\hbar, \quad \left[ \hat{\tilde{t}}, \hat{x} \right] = 0, \quad \left[ \hat{x}, \hat{p}_x \right] = i\hbar, \quad \left[ \hat{x}, \hat{p}_t \right] = 0, \quad \left[ \hat{p}_x, \hat{p}_t \right] = 0.
\end{equation}

From these commutators we clearly see that a new complete set of commuting observables is \(\hat{\tilde{t}}, \hat{\tilde{x}}\), which label the admissible associated basis of coordinate states \(\{ | \tilde{t}, x \rangle \}\). The Dirac’s supplementary condition in this basis is now,
\begin{equation}
\left( -i\hbar \frac{\partial}{\partial \tilde{t}} + \dot{\tilde{t}} + i\hbar \theta \partial_{\tilde{x}} - i\hbar \partial_{\tilde{x}} \right) \psi(\tilde{t}, \tilde{x}) = 0,
\end{equation}
and we note that in the case that the Hamiltonian does not depend explicitly on the time the Schrödinger equation is not modified by the noncommutativity.

Now, if we consider the second set in (4.34) of solutions to (4.8) the resulting action is given by
\begin{equation}
S_6 = \int_{\tau_1}^{\tau_2} d\tau \left( p_t (t + \frac{\theta}{2} p_x) + p_x (x - \frac{\theta}{2} p_t) - \lambda \left( p_t + H(t, x, p_x) \right) \right).
\end{equation}

Following the same logic as in the previous case, it is natural to introduce in this equation the new set \(\tilde{t} = t + \frac{\theta}{2} p_x, \tilde{x} = x - \frac{\theta}{2} p_t\) of time and spatial coordinate. Here then the action (4.40) is reduced to
\begin{equation}
S_6 = \int_{\tau_1}^{\tau_2} d\tau \left( p_t \dot{\tilde{t}} + p_x \dot{\tilde{x}} - \lambda \left( p_t + H(t + \frac{\theta}{2} p_x, \tilde{x} + \frac{\theta}{2} p_t, p_x) \right) \right),
\end{equation}
and, upon Dirac quantization, the corresponding new set of dynamical observables satisfies the following commutation relations,
\begin{equation}
\left[ \hat{\tilde{t}}, \hat{\tilde{x}} \right] = 0, \quad \left[ \hat{\tilde{t}}, \hat{p}_t \right] = i\hbar, \quad \left[ \hat{\tilde{x}}, \hat{p}_x \right] = i\hbar, \quad \left[ \hat{p}_t, \hat{p}_x \right] = 0.
\end{equation}

Using as a complete set of commuting observables the variables \(\hat{\tilde{t}}, \hat{\tilde{x}}\), the new supplementary Dirac condition is
\begin{equation}
\left( -i\hbar \frac{\partial}{\partial \tilde{t}} + \dot{\tilde{t}} + i\hbar \frac{\theta}{2} \partial_{\tilde{x}} - i\hbar \partial_{\tilde{x}} \right) \psi(\tilde{t}, \tilde{x}) = 0.
\end{equation}
For this Schrödinger equation we see that including the case when the Hamiltonian does not depend explicitly on time we do have modifications originated by the
noncommutativity. Furthermore we see that the new theory could be non-unitary, since partials with respect to $\dot{t}$ appear to an order that depends on the kind of interaction. This type of quantization can been formulated directly by using the Moyal product:

$$H(\dot{t} + \frac{i\hbar\theta}{2}\partial_{\dot{x}}, \dot{x} - \frac{i\hbar\theta}{2}\partial_{\dot{t}}, -i\hbar\partial_{\dot{x}})\psi(\dot{t}, \dot{x}) = H(\dot{t}, \dot{x}, -i\hbar\partial_{\dot{x}}) *_{\theta} \psi(\dot{t}, \dot{x}),$$

where

$$*_{\theta} = \exp \left[ \frac{i\hbar\theta}{2} \left( \partial_{\dot{t}} \partial_{\dot{x}} - \partial_{\dot{x}} \partial_{\dot{t}} \right) \right].$$

So for this selection of symplectic potentials the theory is not unitary and this result is equivalent to the obtained in Ref. [15] in the context of noncommutative field theory.

To quantize these two cases by means of the path integral method we make use of the basis $\{\dot{t}, \dot{x}\}$ and the respective actions (4.37) and (4.41) to compute the propagator

$$\langle \dot{t}_2, x_2 | \dot{t}_1, x_1 \rangle.$$

Following the normal procedure to quantize a theory with first class constraints [8], we have only two extra points to consider. First we have to impose a gauge condition, which in this case can be the normal canonical gauge $\dot{t} = f(\tau)$, since in difference with the approach used in [5] and [7] we are imposing the noncommutativity at the level of the action, using the symplectic structure, and not at the level of the gauge condition. The second point that we need to take into account is the extra appearance in the Hamiltonian of the $\theta p_x$ shifted term when we have a $t$ dependent theory, this can imply that it may not be possible to compute the path integral over the momenta. These are however the usual problems that one finds when computing path integrals with actions in terms of variables with powers larger than two.

One additional point to notice is that for both types of solutions of the equations (4.8) considered in this section, the Dirac constraint is not modified, since in both cases the new time is canonical conjugated to the original $p_t$ and then the constraint generates the parametrization invariance. It is not difficult to see that this is not the case when the above analysis is extended to the more general case of symplectic structures that upon quantization result in an extended Heisenberg algebra that includes noncommutativity of the momenta.

For such a generalization one would have to consider a symplectic structure of the form

$$\omega^{ab} = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & \beta \\ 0 & -1 & -\beta & 0 \end{pmatrix}, \quad \omega^{ab} = \frac{1}{\gamma} \begin{pmatrix} 0 & \beta & -1 & 0 \\ -\beta & 0 & 0 & -1 \\ 1 & 0 & 0 & \theta \\ 0 & 1 & -\theta & 0 \end{pmatrix},$$

where

$$\gamma = 1 - \beta \theta.$$
Here the quantization of the Dirac brackets would then result in the extended Heisenberg algebra

\[ [\hat{t}, \hat{x}] = i\hbar \theta, \quad [\hat{x}, \hat{p}_x] = i\hbar, \quad [\hat{t}, \hat{p}_t] = i\hbar \beta, \]

and, in contradistinction to what occurred for the previously considered symplectic structure, we would only have two complete sets of commuting fundamental observables: \((\hat{x}, \hat{p}_t)\) and \((\hat{t}, \hat{p}_x)\), with their respective admissible bases: \(\{|x, p_t\}\) and \(\{|t, p_x\}\).

Except for some differences such as the ones mentioned above, the analysis of the Dirac and path integral quantizations relative to these bases, as well as others resulting from considering canonical transformations of their respective associated actions followed by Darboux maps, is qualitatively similar (see [14]) to what we have already done, so for the sake of brevity we shall omit the details here.

Rather, and in preparation for a future investigation of how our analysis of space-time noncommutativity in the discrete realm of quantum mechanics can be extended to the continuum of relativistic field theory, we turn next our consideration to the case of a relativistic particle.

### 4.2. Space-time Noncommutativity for a Relativistic particle

Our starting point is the action for the free relativistic particle

\[ S = \int_{\tau_1}^{\tau_2} d\tau \left( -m\sqrt{-\eta_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta} \right), \quad \alpha = 1, \ldots, n. \]

In Hamiltonian form we have

\[ S = \int_{\tau_1}^{\tau_2} d\tau \left( p_\alpha \dot{x}^\alpha - \lambda \left( p^2 + m^2 \right) \right), \]

where now the first class primary constraint \(\varphi\) is given by

\[ \varphi = p^2 + m^2 \approx 0. \]

As discussed above in Sec. 3, for an arbitrary symplectic structure the action has the form

\[ S = \int_{\tau_1}^{\tau_2} d\tau \left( A_a(z) \dot{z}^a - \lambda (\varphi(z)) \right), \quad z^a = (x^\alpha, p_\alpha), \quad a = 1, \ldots, 2n. \]

Again, arising from the definition of the momenta, we have the primary constraints

\[ \chi_a = p_{z_a} - A_a(z). \]

These constraints are second class and the corresponding Dirac brackets are identical in form to those in the non-relativistic case, given by Eq. (3.8).

Let us consider now a symplectic structure which is determined by the following Dirac brackets involving the space-time and momentum variables:

\[ \{x^\alpha, x^\beta\}^* = g^{\alpha\beta}, \quad \{x^\alpha, p_\beta\}^* = \delta^\alpha_\beta, \]
where $\theta^{\alpha\beta}$ is a constant antisymmetric tensor. Then, the symplectic structure takes the explicit form:

\[
\omega^{ab} = \begin{pmatrix} \theta^{\alpha\beta} & \text{I} \\ -\text{I} & 0 \end{pmatrix}, \quad \omega_{ab} = \begin{pmatrix} 0 & -\text{I} \\ \text{I} & \theta^{\alpha\beta} \end{pmatrix}.
\]

Note, as it was the case before, that the solutions of

\[
\omega^{ab} = \partial_a A_b - \partial_b A_a,
\]

for the generating potentials $A_a$, are not unique, but they are all related by canonical transformations. One possible covariant solution is

\[
A_\alpha = 0, \quad A_{n+\alpha} = -x_\alpha - \frac{\theta^{\alpha\beta}}{2} \beta, \quad \alpha = 1 \ldots n.
\]

Introducing this symplectic potential in the action (4.53), we obtain

\[
S = \int_{\tau_1}^{\tau_2} d\tau \left( -x^\alpha \dot{p}_\alpha + \frac{\theta^{\alpha\beta}}{2} p_\alpha \dot{p}_\beta - \lambda \left( p^2 + m^2 \right) \right).
\]

So the variables with fixed end points in the action are the momenta $p_\alpha$. Dirac quantization in this case results in the commutators

\[
[\hat{x}^\alpha, \hat{p}^\beta] = i\hbar \theta^{\alpha\beta}, \quad [\hat{x}^\alpha, \hat{p}_\beta] = i\hbar \delta^\alpha_\beta, \quad [\hat{p}_\alpha, \hat{p}_\beta] = 0,
\]

and the supplementary condition

\[
\hat{\psi}(\hat{x}, \hat{p})\psi(p) = (p_\alpha p^\alpha + m^2)\psi(p) = 0,
\]

referred to the admissible basis $\{|p\}$. As is to be expected, this merely states that

\[
(p_\alpha p^\alpha + m^2) = 0.
\]

To compute the propagator for the theory (4.59) using path integrals, the more convenient technique is to use a non-canonical gauge and the BFV-BRST path integral procedure. The full action, after introducing the gauge fixing term and ghost terms, is

\[
S = \int_{\tau_1}^{\tau_2} d\tau \left( -x^\alpha \dot{p}_\alpha + \frac{\theta^{\alpha\beta}}{2} p_\alpha \dot{p}_\beta - \lambda \left( p^2 + m^2 \right) - \pi \dot{C} + \bar{C} \dot{\pi} - i\bar{P} \dot{P} - \lambda \left( p^2 + m^2 \right) \right).
\]

Here the boundary conditions on the ghost, the momenta conjugate to the coordinates $p_\alpha$ and the Lagrange multiplier $\pi$ are

\[
\pi(\tau_1) = \pi(\tau_2) = \bar{C}(\tau_1) = \bar{C}(\tau_2) = C(\tau_1) = C(\tau_2) = 0,
\]

\[
p_\alpha(\tau_1) = p_{\alpha 1}, \quad p_\alpha(\tau_2) = p_{\alpha 2}.
\]

From the path integral over the ghosts we get a multiplicative factor of $(\tau_1 - \tau_2)$, this term is very useful since it allows to eliminate the dependence of the propagator on the parameter $\tau$. Using the path integral over $x^\alpha$, we obtain delta functions that we then use to integrate over the momenta $p_\alpha$. As a result these integrals cancel the $\theta^{\alpha\beta}$ correction term, since this term is multiplied by $\dot{p}_\alpha$. So, finally we get the usual propagator in the basis where the momenta are fixed at the end points

\[
\langle p_\beta(\tau_2) | p_\alpha(\tau_1) \rangle = \frac{-i\eta^{\alpha\beta}(p_{\alpha 2} - p_{\alpha 1})}{p^2 + m^2}.
\]

This result is fully consistent with the previous result (4.61).
Other admissible bases compatible with the Heisenberg algebra (4.60) are obtained from (4.59) by a canonical transformation generated by $F_p^\alpha = p_\alpha x^\alpha$, for $\alpha$ fixed. These sets of admissible bases are

$$\{\{x^\alpha, p_\beta, p_\gamma, p_\lambda\}; \alpha \neq \beta \neq \gamma \neq \lambda\}.$$  

Refer to them, the Dirac subsidiary condition results in

$$(4.66) \quad \hat{\varphi}(\hat{x}, \hat{\hat{p}})\psi(x_\alpha, p_\beta, p_\gamma, p_\lambda) = (\hbar^2 (\partial_{x_\alpha})^2 + (p_\beta)^2 + (p_\gamma)^2 + (p_\lambda)^2) \psi(x_\alpha, p_\beta, p_\gamma, p_\lambda) = 0,$$

where indices here are not summed over.

So, even though the deformation parameter $\theta$ does not appear in these constraint equations the space-time noncommutativity is reflected in their violation of Lorentz invariance.

On the other hand, canonically transforming (4.59) with $F_p^\alpha = p_\alpha x^\alpha$, where now we sum over $\alpha$, we get, after regrouping terms,

$$\begin{align*}
(4.67) \quad S &= \int_\tau^\tau d\tau \left( p_\alpha (x_\alpha + \frac{\theta_{\alpha\beta}}{2} p_\beta)^* - \lambda (p^2 + m^2) \right).
\end{align*}$$

Here we see that it is natural to define as fixed end-point variables of the action the new set of coordinates given by

$$(4.68) \quad \tilde{x}_\alpha = x_\alpha + \frac{\theta_{\alpha\beta}}{2} p_\beta.$$  

The Dirac bracket between these new coordinates vanishes and, in consequence, so does their commutator:

$$(4.69) \quad [\tilde{x}_\alpha, \tilde{x}_\beta] = 0,$$

while

$$(4.70) \quad [\tilde{x}_\alpha, p_\beta] = i\hbar \delta_{\alpha\beta}.$$  

Note, however, that (4.68) is a Darboux map and not a canonical transformation of the action (4.59). Consequently this is a different Dirac quantization, related to the canonical symplectic form and not to the original one given by (4.56). The Dirac supplementary condition in this case is

$$(4.71) \quad \hat{\varphi}(\hat{\tilde{x}}, \hat{\hat{p}})\psi(\tilde{x}) = (\partial_{\tilde{x}_\alpha} \tilde{x}^\alpha + m^2) \psi(\tilde{x}) = 0.$$  

So, quantizing the theory in this way we obtain that a relativistic particle satisfies the Klein-Gordon equation, and thus arrive at the well known result that for a free particle we do not obtain any deformation of the theory. However, if we consider that the particle lives in a given background, we will get the deformation produced by the new choice of coordinates.

To further illustrate this point, consider the interaction of the relativistic particle with a constant external field. Here the constraint will be of the form

$$(4.72) \quad \left( \Pi_\mu - \frac{1}{2} F_{\mu\nu} x^\nu \right) \left( \Pi^\mu - \frac{1}{2} F^{\mu\sigma} x_\sigma \right) + m^2 \approx 0.$$  

Using the $\tilde{x}^\alpha$ coordinates, which will have the same form as in (4.68), except for the substitution $p_\beta \to \Pi_\beta$, the Dirac supplementary condition in the basis $\{\{\tilde{x}^\alpha\}\}$
is of the form
\[
\left[ \left( -i\hbar \partial_\mu - \frac{1}{2} F_{\mu\nu} \left( \tilde{x}^\nu + i\hbar \frac{1}{2} \theta^{\nu\rho} \partial_\rho \right) \right) \times \right.
\left. \left( -i\hbar \partial_\mu - \frac{1}{2} F^{\mu\sigma} \left( \tilde{x}_\sigma + i\hbar \frac{1}{2} \theta_{\sigma\rho} \partial^\rho \right) \right) + m^2 \right] \psi(\tilde{x}) = 0,
\]\n(4.73)
which indeed shows corrections containing the deformation parameter \(\theta\).

5. Concluding remarks

We have seen that according to the Dirac quantization scheme for constrained systems, it is the first class constraints and the symplectic structure resulting from the Dirac brackets that uniquely define a particular quantum theory, irrespectively of the fact that there are many possible solutions for the potentials \(A_a\) corresponding to the same symplectic structure \(\omega\). On the other hand, if we use these solutions as the starting point for evaluating the action in the path integral formulation, then depending on the type of solutions that we propose for the equations (3.6), we could get different quantizations. We have seen moreover, that if there is a linear canonical transformation relating these actions, as is the case for the actions \(S_1, S_2\) and \(S_3\) considered in subsections 4.1.1-4.1.3, then the corresponding quantizations are actually equivalent to each other and differ only by the fact that they are referred to the three admissible bases compatible with the extended Heisenberg algebra (4.2). Indeed, the phases of the quantum mechanical transition functions corresponding to changes between these bases (cf. e.g. Eq. (4.20)) are nothing other than the classical generating functions of the linear canonical transformations among the three actions, and the associated symplectic transformation leaving invariant their common symplectic structure \(\omega\) is, for each of these three cases, the identity element of the group.

Alternatively, for the type of solutions to (3.6) leading to the actions considered in subsection 4.1.4, the situation is actually quite different because there is no generating function that permits to canonically transform such actions to the ones previously considered, and because at the classical level fixing the end-points of these actions involves a change of variables in extended phase-space which results in a Darboux map from the original symplectic structure to the canonical one given by (3.10). Quantizing in these cases via either the Dirac or path integral formalisms is then tantamount to applying standard quantum mechanics with a Hamiltonian modified with the new variables, which are formally promoted to the rank of operators satisfying the commutation relations (4.42). But in axiomatic quantum mechanics the operators acting on vectors in Hilbert space are observables, i.e. operators functions of the basic dynamical variables of the theory, with eigenvalues given by quantities measurable by experiment. For the systems we have been considering and the construction followed in subsection 4.1.4, this would imply that the new time and coordinate variables are the observables of the theory and, since they obey the commutation relations (1.43), the new time and coordinate operators commute. Physically this would then mean that experiments could be designed to measure simultaneously the eigenvalues of these space-time operators. This, however, begs the question of what is then the true physical interpretation for the \(\theta\) parameter that
appears in the modified quantum expressions of the theory, such as the Hamiltonian? We could try to further argue that both the old and new space-time operators are observables and that $\theta$ reflects the noncommutativity of the old observables. This, however, brings in a somewhat Bohmian flavor of hidden variables to the new quantization which is, to say the least, subject to questioning (for additional arguments regarding this issue see [16]). Thus, from our point of view, it would seem preferable to conclude that in the case of the quantizations discussed in subsection 4.1.4, the term “space-time noncommutativity” is a misnomer. Nonetheless, since the different quantizations here discussed lead to different (at least conceptually) experimental predictions, it is experiment then that will determine which, if any, of these theories can be closer related to reality. The same can be said regarding the different cases discussed in Section 4.2 for the relativistic particle.

Of course it could also be contended that the use of the Dirac and path integral quantizations, which have been so successful in extending classical mechanics and field theory to a certain range of the quantum realm, is not justified a priori when dealing with distances of the order of the Planck length where quantum gravity becomes relevant. This could very well be so and it may involve having to drop the very concept of manifold, which underlies the mathematics of all of our present day physical constructions, in favor of new geometrical paradigms in which quantization is built in ab initio, such as the noncommutative geometry proposed by Connes [17] a few years ago. Be it as it may, we believe that the analysis presented here, the more axiomatic one presented in [13] and references within, as well as many other related works that have appeared in the literature, could provide some guidance for further work in that ultimate direction.

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