Construction of a Penrose diagram for an accreting black hole

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Abstract
A Penrose diagram is constructed for a spatially coherent black hole that accretes at stepwise steady rates as measured by a distant observer from an initial state described by a metric of Minkowski form. Coordinate lines are computationally derived, and radial lightlike trajectories verify the viability of the diagram. Coordinate dependences of significant features, such as the horizon and radial mass scale, are clearly demonstrated on the diagram. The onset of a singularity at the origin is shown to open a new region in space time that contains the interior of the black hole.

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1. Introduction

Astrophysical black holes are expected to evolve, i.e. accrete or evaporate, over time. One expects that the initiation and final evaporation processes of black hole dynamics should involve spatial coherence on scales comparable to those describing the medium-scale structure of the space time, since the geometries during those periods are defined by micro-physics. It is of particular interest to examine the correspondence of low-curvature space time with a dynamic black hole geometry.

The non-orthogonal temporal coordinate associated with the river model of black holes [1, 2] has been shown to provide a convenient parameter for describing the evolution of a black hole without physical singularities in the vicinity of the horizon. In a previous paper, we examined an example dynamic black hole undergoing a steady rate of evaporation [3]. To complement evaporation, in this paper we will examine the global causal structure of a black hole undergoing periods of stepwise steady accretion. This is accomplished via the construction of a computationally derived Penrose diagram. The viability of this diagram can be verified by computing and plotting radial lightlike trajectories as curves of slope $\pm$ unity.
The complete evolutionary cycle of a black hole should involve a merger of the techniques involved in these companion papers [4].

2. Form of the metric and conformal coordinates

2.1. Form of the metric

The dynamic space time metric will be assumed to take the form
\[
\text{d}s^2 = -\left(1 - \frac{R_M(\text{ct})}{r}\right) (\text{dct})^2 + 2\sqrt{\frac{R_M(\text{ct})}{r}} \text{dct} \, \text{dr} + r^2 (\text{d}\theta^2 + \sin^2 \theta \, \text{d}\phi^2).
\]

In this equation, \(R_M(\text{ct}) \equiv 2G_N M(\text{ct})/c^2\) is a time-dependent form of the Schwarzschild radius that we refer to as the radial mass scale. This metric was examined as a dynamic extension of the river model of (static) black holes discussed in the literature [1, 5]. The metric takes the form of a Minkowski space time both asymptotically \(\, r \rightarrow \infty\) as well as when the radial mass scale vanishes \((R_M(\text{ct}) \rightarrow 0)\). Therefore, the temporal and radial coordinates are those of an observer far from the black hole. All curvature components generated by this metric are non-singular near the lightlike surface defining the horizon, \(\dot{R}_H = 1 - \sqrt{R_M/R_H}\), as well as the surface defined by the radial mass scale, \(R_M(\text{ct})\). This simplifies descriptions of the physics in the vicinity of \(R_H\) and \(R_M\).

2.2. Form of conformal coordinates

The form of the conformal temporal and radial coordinates that are used for the construction of the Penrose diagram have been developed in companion papers [3, 5]:
\[
\begin{align*}
\text{c}_t^* &= \frac{r}{2} \left( \exp \left[ \int \frac{\text{d}\zeta'}{\zeta' (1 + \sqrt{\zeta'}) + R_M} \right] \frac{(1 + \sqrt{\zeta'}) \, \text{d}\zeta'}{\zeta' (1 + \sqrt{\zeta'}) + R_M} - \exp \left[ \int \frac{\text{d}\zeta'}{\zeta' (1 - \sqrt{\zeta'}) - R_M} \right] \frac{(1 - \sqrt{\zeta'}) \, \text{d}\zeta'}{\zeta' (1 - \sqrt{\zeta'}) - R_M} \right], \\
\text{r}_* &= \frac{r}{2} \left( \exp \left[ \int \frac{\text{d}\zeta'}{\zeta' (1 + \sqrt{\zeta'}) + R_M} \right] \frac{(1 + \sqrt{\zeta'}) \, \text{d}\zeta'}{\zeta' (1 + \sqrt{\zeta'}) + R_M} + \exp \left[ \int \frac{\text{d}\zeta'}{\zeta' (1 - \sqrt{\zeta'}) - R_M} \right] \frac{(1 - \sqrt{\zeta'}) \, \text{d}\zeta'}{\zeta' (1 - \sqrt{\zeta'}) - R_M} \right).
\end{align*}
\]

The transformations are valid for constant accretion rates \(\dot{R}_M = 0\). These equations relate the space-time coordinates of an asymptotic observer \((\text{ct}, r)\) with the conformal coordinates \((\text{c}_t^*, \text{r}_*)\). In subsequent calculations, the constants associated with the integrations are chosen so that the conformal space-time parameters \((\text{ct}_*, \text{r}_*)\) asymptotically (i.e., as \(\frac{\text{R}_M(\text{ct})}{\text{R}_H} \rightarrow 0\)) correspond to Minkowski space-time parameters \((\text{ct}, r)\), which are likewise conformal.

3. Penrose diagram of the accreting black hole

3.1. Procedure for constructing the Penrose diagram

The diagram that is developed uses hyperbolic tangents of a scaled multiple of the conformal coordinates in equation (2.2) to map the infinite domain of those conformal coordinates onto a finite region. More specifically, the vertical coordinate \(Y_{t*}\) takes the form \(\text{tanh}\left(\left(\int \text{d}t^* + \text{tanh} \left(\frac{\text{ct}^* - \text{r}^*}{\text{scale}}\right)\right)/2\right)\) and the horizontal coordinate \(Y_{r*}\) takes the form \(\text{tanh}\left(\left(\int \text{d}r^* - \text{tanh} \left(\frac{\text{ct}^* - \text{r}^*}{\text{scale}}\right)\right)/2\right)\) in the exterior region. In the interior region \(r < R_H(\text{ct})\), the timelike and spacelike coordinates \((Y_{t*} \, \text{and} \, Y_{r*})\) are interchanged (because of a sign change in the coefficient of \([-d\text{ct}^*]^2 + [dr^*]^2\] in the conformal metric as one crosses the horizon) and shifted in a manner that preserves the coordinates of the horizon. The overall transformation
assures that the slopes of any outgoing/ingoing lightlike radial trajectories on the diagram will be $\pm 1$, and that the diagram has its domain and range bounded by $\pm 1$.

The initial metric will be assumed to be of the form of Minkowski space time for $t \leq 0$. The accretion begins at $t = 0$ from an initial radial mass scale $R_M(0) = 0$, which smoothly transitions the dynamic metric in equation (2.1) from a Minkowski metric form. The radial coordinates must smoothly match across the volume $t = 0$, since $4\pi r^2$ measures the area of any sphere of radial coordinate $r$ in both metric forms. Since the conformal coordinates developed require a steady rate of accretion, changes in that rate (if desired) can be modeled using stepwise steady accretions, and matching the coordinates of the radial mass scale $R_M$ (a physical parameter directly related to the mass of the black hole) across differing rates with instantaneously matching metrics.

3.2. Features of the Penrose diagram

The Penrose diagram in figure 1 demonstrates the expected global structure of this spherically symmetric black hole that accretes at a steady rate of change in the radial mass scale $R_M(ct)$ with respect to the distant observer’s time coordinate $ct$. In the diagram, the red curves that are timelike in the right-hand regions represent curves of constant $r$, originally graded from $r = 0$ in hundredths, tenths, then in multiples, and decades of the chosen scale. The curves of constant radial coordinate $r$ all originate at the bottom corner of the diagram representing $t = -\infty$, and terminate at the uppermost corner representing $t = +\infty$. The green curves that are spacelike in the right-hand regions represent curves of constant $ct$ graded in multiples of
the given scale. All constant $ct$ curves originate on the curve $r = 0$ and terminate at the far right corner of the diagram representing $r = \infty$. The various $ct = \text{constant}$ and $r = \text{constant}$ curves each intersect at only one point on the diagram. The lightlike bounding curves $r = \infty$ on the right are those of a Minkowski space time.

Prior to the formation of the black hole, the space time is taken to be that described by Minkowski, represented by the lower portion of the diagram. At $t = 0$, the singularity develops via an ingoing lightlike transition ($ct = 0, r = 0$), during which a new region in the space time opens up. This new region, which contains the interior of the black hole, is represented by the upper-left quadrant in figure 1. The curve $r = 0$ is initially a timelike trajectory bounding the Minkowski space time on the left. It then undergoes the lightlike transition to become a spacelike trajectory bounding the upper interior region of the black hole from above. The Penrose coordinates of the singularity $r = 0$ somewhat matches the behavior in the vertical Penrose coordinate of a static Schwarzschild singularity $Y_{\text{Schwarzschild}}^{\text{singularity}} = +1$. For an accreting black hole, the radial mass scale $R_M(ct)$ (indicated by the solid black spacelike curve between the singularity and the horizon $R_H$) can be shown generally to lie within the horizon [5], and it is seen to mirror the behavior of the singularity $r = 0$. Radial coordinate curves are seen to transition from timelike to spacelike as they cross the radial mass scale. This occurs because the outgoing lightlike trajectories, which on this geometry satisfy $\dot{r} = 1 - \sqrt{R_M/r}$, are temporarily stationary in the radial coordinate $\dot{r} = 0$ at the radial mass scale.

The mass scale and horizon of the black hole are both dynamic. The horizon is represented by the dashed diagonal blue line indicated by $R_H$ in figure 1, and it lies completely within the new region of space time opened up by the formation of the black hole singularity. Unlike as is the case for a static Schwarzschild black hole, the horizon $R_H(ct)$ is not a $t = \infty$ surface. Both radial and temporal coordinate curves are seen to cross the horizon and radial mass scale on the diagram. Thus, the distant observer’s time is qualitatively different from that of a distant Schwarzschild observer, even for a very slowly accreting black hole.

Finally, there is a region of significant coordinate distortion along the lightlike curve labeled $R_2$, which corresponds to one of the singular curves of the conformal metric [4]. The coefficient of the factor $-(ct)^2 + (dr)^2$ in the metric becomes singular on the outgoing lightlike curve $R_2(ct)$ and vanishes on the horizon $R_H(ct)$. However, this anomalous coordinate behavior near $R_2$ does not correspond to any physical singularity on the geometry (for example, the Ricci scalar of the geometry, which is of the form $\mathcal{R} = \frac{1}{2c^{2}} \sqrt{\frac{1}{R_H}}$, is non-singular near $R_2$). For the steadily accreting black hole, in the distant future the area of the horizon of the black hole is unbounded ($R_H \to \infty$ as $t \to \infty$). However, as was the case for the steadily excreting black hole [3], it remains possible to stay external to the black hole, but not with a fixed radial coordinate $r$. A radial coordinate that is a fixed multiple of that of the horizon remains external if that multiple is greater than unity.

4. Conclusions and discussion

The large-scale geometry of a steadily accreting black hole revealed several aspects of interest. The radial mass scale was seen to have a trajectory internal to the horizon of the black hole as expected from analytical examinations. The diagrammatic representation of the speed of light formation of the singularity was unexpected, but necessary, since low curvature space times have non-negative conformal radial coordinates. This lightlike formation of the singularity concurrently expands the large-scale structure of the space time. Both the radial mass scale and the horizon were seen to lie completely within the region of space time opened during the formation of the singularity.
The observation that the radial mass scale does not coincide with the horizon demonstrates that the dynamics described using our temporal coordinate $t$ is qualitatively different from what would result from a mass scale dependence upon a Schwarzschild observer's temporal coordinate, no matter how small the rate of accretion. Parametric descriptions of both the radial mass scale and the horizon are non-singular in the coordinates $(ct, r)$. In addition, all physical curvatures are well defined near these trajectories.

The asymptotic correspondence of the black hole description was seen to fix the scales of the past and future lightlike infinities of the black hole to be the same as those of Minkowski space time. Using the coordinates employed for the spatially coherent accreting black hole, when the accretion initiates there is continuity in the metric with that of static Minkowski space, but a discontinuity in curvature. The primary concerns of the present paper has been the construction of the Penrose diagram rather than the development of a model for the micro-physics describing the initial stage of accretion, which likely occurs on time scales small compared to the resolution of the diagram. Since the metrics precisely match at the transition, it is felt that the large-scale structure has accurately been portrayed.

A Penrose diagram representing the complete life cycle of a model black hole can likewise be constructed using stepwise steady rates of accretion/evaporation, and matching metric forms across the transitions. A diagram demonstrating the initiation of accretion from a low curvature space time, an end of accretion and beginning of evaporation, and an end of evaporation as the black hole mass vanishes, is presently being examined by the authors, and its form will be presented in a future manuscript. We ultimately want to examine the micro-physics that allows the initial accretion and final evaporation processes of a black hole. This will involve exploring general dynamics in the mass scale $R_M(ct)$ in a geometry with rates which can smoothly transition from and to low curvature space times.

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