THEOREM OF EXISTENCE AND COMPLETENESS FOR HOLOMORPHIC POISSON STRUCTURES

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Abstract. In this paper, we define a concept of a family of compact holomorphic Poisson manifolds on the basis of Kodaira-Spencer’s deformation theory and deduce the integrability condition. We prove an analogue of their ‘Theorem of existence for complex analytic structures’ in the context of holomorphic Poisson deformations under some analytic assumption, and establish an analogue of their ‘Theorem of completeness for complex analytic structures’ for compact holomorphic Poisson surfaces.

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1. Introduction

In this paper, we study deformations of holomorphic Poisson structures in the framework of Kodaira and Spencer’s deformation theory of complex analytic structures ([KS58a], [KS60]). The main difference from Kodaira and Spencer’s deformation theory is that for deformations of a compact holomorphic Poisson manifold, we deform not only its complex structures, but also holomorphic Poisson structures. We will briefly review Kodaira-Spencer’s main idea and show how we can extend their idea in the context of deformations of holomorphic Poisson structures.

Kodaira and Spencer’s main idea of deformations of complex analytic structures is as follows [Kod05, p.182]. A n-dimensional compact complex manifold \( M \) is obtained by glueing domains \( U_1, ..., U_n \) in \( \mathbb{C}^n \): \( M = \bigcup_{j=1}^n U_j \) where \( \mathcal{U} = \{ U_j | j = 1, ..., n \} \) is a locally finite open covering of \( M \), and that each \( U_j \) is a polydisk:

\[ U_j = \{ z_j \in \mathbb{C}^n | |z_j^1| < 1, ..., |z_j^n| < 1 \} \]

and for \( p \in U_j \cap U_k \), the coordinate transformation

\[ f_{jk} : z_k \rightarrow z_j = (z_j^1, ..., z_j^n) = f_{jk}(z_k) \]

transforming the local coordinates \( z_k = (z_k^1, ..., z_k^n) = z_k(p) \) into the local coordinates \( z_j = (z_j^1, ..., z_j^n) = z_j(p) \) is biholomorphic. According to Kodaira,

“A deformation of \( M \) is considered to be the glueing of the same polydisks \( U_j \) via different identification. In other words, replacing \( f_{jk}^0(z_k) \) by the functions \( f_{jk}(z_k, t) = f_{jk}^0(z_k, t_1, ..., t_m), f_{jk}(z_k, 0) = f_{jk}^0(z_k) \) of \( z_k \), and the parameter \( t = (t_1, ..., t_m) \), we obtain deformations \( M_t \) of \( M = M_0 \) by glueing the polydisks \( U_1, ..., U_n \) by identifying \( z_k \in U_k \) with \( z_j = f_{jk}(z_k, t) \in U_j \)”

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We extend the main idea of Kodaira-Spencer in the context of deformations of holomorphic Poisson structures. A $n$-dimensional compact holomorphic Poisson manifold $M$ is a compact complex manifold such that the structure sheaf $\mathcal{O}_M$ is a sheaf of Poisson algebras (we refer to [LGPV13] for general information on Poisson geometry). The holomorphic Poisson structure is encoded in a holomorphic section (a holomorphic bivector field) $\Lambda \in H^0(M, \wedge^2 \Theta_M)$ with $[\Lambda, \Lambda] = 0$, where $\Theta_M$ is the sheaf of germs of holomorphic vector fields on $M$ and the bracket $[\cdot, \cdot]$ is the Schouten bracket on $M$. In the sequel a holomorphic Poisson manifold will be denoted by $(M, \Lambda)$. For deformations of a compact holomorphic Poisson manifold $(M, \Lambda)$, we extend the idea of Kodaira and Spencer. A $n$-dimensional compact holomorphic Poisson manifold is obtained by glueing the domains $U_1, \ldots, U_n$ in $\mathbb{C}^n$: $M = \bigcup_{j=1}^n U_j$ where $\Omega = \{U_j \mid j = 1, \ldots, n\}$ is a locally finite open covering of $M$ and each $U_j$ is a polydisk

$$U_j = \{z_j \in \mathbb{C}^n \mid |z_j|^2 < 1, \ldots, |z_n|^2 < 1\}$$

equipped with a holomorphic bivector field $\Lambda_j = \sum_{\alpha, \beta=1}^n g^{j}_{\alpha \beta}(z_j) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}$ such that $g^{j}_{\alpha \beta}(z_j) = -g^{j}_{\beta \alpha}(z_j)$ with $[\Lambda_j, \Lambda_j] = 0$ on $U_j$ and for $p \in U_j \cap U_k$, the coordinate transformation

$$f_{jk}: z_k \rightarrow z_j = (z_j^1, \ldots, z_j^n) = f_{jk}(z_k)$$
transforming the local coordinates $z_k = (z_k^1, \ldots, z_k^n) = z_k(p)$ into the local coordinates $z_j = (z_j^1, \ldots, z_j^n) = z_j(p)$ is a biholomorphic ‘Poisson’ map.

Deformations of a compact holomorphic Poisson manifold $(M, \Lambda)$ is the glueing of the Poisson polydisks $(U_j, \Lambda_j(t))$ parametrized by $t$. That is, replacing $f_{jk}^0(z_k)$ by $f_{jk}^t(z_k)$ of $z_k$), replacing $\Lambda_j = \sum_{\alpha, \beta=1}^n g^{j}_{\alpha \beta}(z_j) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}$ by $\Lambda_j(t) = \sum_{\alpha, \beta=1}^n g^{j}_{\alpha \beta}(z_j, t) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}$ with $[\Lambda_j(t), \Lambda_j(t)] = 0$ and $\Lambda_j(0) = \Lambda_j$, and the parameter $t = (t_1, \ldots, t_m)$, we obtain deformations $(M_t, \Lambda_t)$ by gluing the Poisson polydisks $(U_1, \Lambda_1(t)), \ldots, (U_n, \Lambda_n(t))$ by identifying $z_k \in U_k$ with $z_j = f_{jk}(z_k, t) \in U_j$. The work on deformations of holomorphic Poisson structures is based on this fundamental idea.

In section 2, we define a family of compact holomorphic Poisson manifolds, called a Poisson analytic family in the framework of Kodaira-Spencer’s deformation theory. In other words, when we ignore Poisson structures, a family of compact holomorphic Poisson manifolds is just a family of compact complex manifolds in the sense of Kodaira and Spencer. So deformations of compact holomorphic Poisson manifolds means that we deform complex structures as well as Poisson structures.

In section 3, we show that infinitesimal deformations of a holomorphic Poisson manifold $(M, \Lambda_0)$ in a Poisson analytic family are encoded in the second truncated holomorphic Poisson cohomology. More precisely, an infinitesimal deformation is realized as an element in the second hypercohomology group $HP^2(M, \Lambda_0)$ of a complex of sheaves $0 \rightarrow \Theta_M \rightarrow \wedge^2 \Theta_M \rightarrow \cdots \rightarrow \wedge^n \Theta_M \rightarrow 0$ induced by $[\Lambda_0, -]$. Analogously to deformations of complex structure, we define so called Poisson Kodaira-Spencer map where the Kodaira-Spencer map is realized as a component of the Poisson Kodaira-Spencer map.

In section 4, we study the integrability condition for a Poisson analytic family. Kodaira showed that given a family of deformations of a compact complex manifold $M$, locally the family is represented by a $C^\infty$ vector (0, 1)-form $\varphi(t) \in A^{0,1}(M, T_M)$ with $\varphi(0) = 0$ satisfying the integrability condition $\bar{\partial}\varphi(t) - \frac{1}{2}[\varphi(t), \varphi(t)] = 0$ (see [Kod05] §5.3). Here $T_M$ is the holomorphic tangent bundle of $M$ and we use the notation $A^{0,1}(M, T_M)$ instead of $\mathcal{L}^{0,1}(T_M)$ in [Kod05]. We show that given a family of deformations of a compact holomorphic Poisson manifold $(M, \Lambda_0)$, locally the family is represented by a $C^\infty$ vector (0, 1)-form $\varphi(t)$ with $\varphi(0) = 0$ and a $C^\infty$ bivector $\Lambda(t) \in A^{0,2}(M, \wedge^2 T_M)$ with $\Lambda(0) = \Lambda_0$ satisfying the integrability condition $[\Lambda(t), \Lambda(t)] = 0$, $\bar{\partial}\Lambda(t) - [\Lambda(t), \varphi(t)] = 0$, and $\bar{\partial}\varphi(t) - \frac{1}{2}[\varphi(t), \varphi(t)] = 0$. Replacing $\varphi(t)$ by $-\varphi(t)$ and putting $\Lambda'(t) := \Lambda(t) - \Lambda_0$ so that we have $\Lambda'(0) = 0$, the integrability condition is equivalent to $L(\varphi(t) + \Lambda'(t)) + \frac{1}{2}[\varphi(t) + \Lambda'(t), \varphi(t) + \Lambda'(t)] = 0$ where $L = \bar{\partial} + [\Lambda_0, -]$. Then $\varphi(t) + \Lambda'(t)$ is a solution of Maurer Cartan equation of the following differential graded Lie algebra

$$g = \bigoplus_{i \geq 0} g_i, g_i = \bigoplus_{p+q-1=i, p \geq 0, q \geq 1} A^{0, p}(M, \wedge^q T_M), L = \bar{\partial} + [\Lambda_0, -], [-, -])$$
where $[-, -]$ is the Schouten bracket on $M$, and $A^{0, p}(M, \wedge^q T_M)$ is the global section of $\mathcal{A}^{0, p}(\wedge^q T_M)$ the sheaf of germs of $C^\infty$-section of $\wedge^p T_M^* \otimes \wedge^q T_M$. Here $T_M^*$ is the dual bundle of antiholomorphic tangent bundle $T_M$ (see [Kod05] p.108). We remark that the integrability condition was proved in more general context in the language of generalized complex geometry (See [Gua11]). As $H^1(M, \Theta_M)$ is realized as a subspace of the second cohomology group of a compact complex manifold $M$ in the sense of generalized complex geometry, $HP^2(M, \Lambda_0)$ is realized as a subspace of the second cohomology group of a compact holomorphic Poisson manifold $(M, \Lambda_0)$ in the sense of generalized complex geometry. In this paper, we deduce the integrability condition by extending Kodaira-Spencer’s original approach, that is, by starting from a concept of a geometric family (a Poisson analytic family).

In section §4 under some analytic assumption, we establish an analogous theorem to the following theorem of Kodaira and Spencer ([KNS58], [Kod05] p.270).

**Theorem 1.0.2** (Theorem of existence for complex analytic structures). Let $M$ be a compact complex manifold and suppose $H^2(M, \Theta) = 0$. Then there exists a complex analytic family $(M, B, \omega)$ with $0 \in B \subset \mathbb{C}^m$ satisfying the following conditions:

1. $\omega^{-1}(0) = M$
2. The Kodaira-Spencer map $\rho_0 : \partial M \to (\frac{\partial}{\partial t})_{t=0}$ with $M_t = \omega^{-1}(t)$ is an isomorphism of $T_0(B)$ onto $H^1(M, \Theta_M) : T_0(B) \xrightarrow{\rho_0} H^1(M, \Theta_M)$.

Similarly, we prove ‘Theorem of existence for deformations of holomorphic Poisson structures’ (see Theorem 5.1.1).

**Theorem 1.0.3** (Theorem of existence for holomorphic Poisson structures). Let $(M, \Lambda_0)$ be a compact holomorphic Poisson manifold such that the associated Laplacian operator $\Box$ (induced from the operator $\partial + [\Lambda_0, -]$) is strongly elliptic and of diagonal type. Suppose that $HP^3(M, \Lambda_0) = 0$. Then there exists a Poisson analytic family $(M, \Lambda, B, \omega)$ with $0 \in B \subset \mathbb{C}^m$ satisfying the following conditions:

1. $\omega^{-1}(0) = (M, \Lambda_0)$
2. The Poisson Kodaira-Spencer map $\varphi_0 : \partial M \to (\frac{\partial(\Lambda_t M)}{\partial t})_{t=0}$ with $(M_t, \Lambda_t) = \omega^{-1}(t)$ is an isomorphism of $T_0(B)$ onto $HP^2(M, \Lambda_0) : T_0 B \xrightarrow{\varphi_0} HP^2(M, \Lambda_0)$.

The proof is rather formal. The proof follows from the Kuranishi’s method presented in [MK06]. The reason for the assumption on the associated Laplacian operator $\Box$ (induced from the operator $\partial + [\Lambda_0, -]$) is for applying the Kuranishi’s method in the holomorphic Poisson context.

In section §5 we establish an analogous theorem to the following theorem of Kodaira and Spencer ([KNS58], [Kod05] p.284).

**Theorem 1.0.4** (Theorem of completeness for complex analytic structures). Let $(M, B, \omega)$ be a complex analytic family of deformations of a compact complex manifold $M_0 = \omega^{-1}(0)$, $B$ a domain of $\mathbb{C}^m$ containing $0$. If the Kodaira-Spencer map $\rho_0 : T_0(B) \to H^1(M_0, \Theta_{M_0})$ is surjective, the complex analytic family $(M, B, \omega)$ is complete at $0 \in B$.

Similarly, we prove the following theorem which is an analogue of ‘Theorem of completeness’ by Kodaira-Spencer. However, we establish ‘Theorem of completeness for holomorphic Poisson structures’ only for dimension 2 (see Theorem 6.1.4 and Remark 6.1.5).

**Theorem 1.0.5** (Theorem of completeness for holomorphic Poisson structures). Let $(M, \Lambda_0, B, \omega)$ be a Poisson analytic family of deformations of a compact holomorphic Poisson surface $(M_0, \Lambda_0) = \omega^{-1}(0)$, $B$ a domain of $\mathbb{C}^m$ containing 0. If the Poisson Kodaira-Spencer map $\varphi_0 : T_0(B) \to HP^2(M, \Lambda_0)$ is surjective, the Poisson analytic family $(M, \Lambda_0, B, \omega)$ is complete at $0 \in B$.

2. Families of compact holomorphic Poisson manifolds

**Definition 2.0.6.** (compare [Kod05] p.59) Suppose that given a domain $B \subset \mathbb{C}^m$, there is a set $\{(M_t, \Lambda_t) | t \in B\}$ of $n$-dimensional compact holomorphic Poisson manifolds $(M_t, \Lambda_t)$, depending on $t = (t_1, ..., t_n) \in B$. We say that $\{(M_t, \Lambda_t) | t \in B\}$ is a family of compact holomorphic Poisson manifolds or a Poisson analytic family of compact holomorphic Poisson manifolds if there exists a
holomorphic Poisson manifold \((M,\Lambda)\) and a holomorphic map \(\omega : M \to B\) satisfying the following properties

1. \(\omega^{-1}(t)\) is a compact holomorphic Poisson submanifold of \((M,\Lambda)\) for each \(t \in B\).
2. \((M_t,\Lambda_t) = \omega^{-1}(t)(M_t has the induced Poisson holomorphic structure \Lambda_t from \Lambda)\).
3. The rank of Jacobian of \(\omega\) is equal to \(m\) at every point of \(M\).

We will denote a Poisson analytic family by \((M,\Lambda,B,\omega)\). We also call \((M,\Lambda,B,\omega)\) a Poisson analytic family of deformations of a compact holomorphic Poisson manifold \((M_0,\Lambda_0)\) for each fixed \(t_0 \in B\).

**Remark 2.0.7.** When we ignore Poisson structures, a Poisson analytic family \((M,\Lambda,B,\omega)\) is a complex analytic family \((M,B,\omega)\) in the sense of Kodaira-Spencer (see [Kod05] p.59).

**Remark 2.0.8.** Given a Poisson analytic family \((M,\Lambda,B,\omega)\) as in Definition 2.0.6 we can choose a locally finite open covering \(U = \{U_j\}\) of \(M\) such that \(U_j\) are coordinate polydisks with a system of local complex coordinates \(\{z_1,...,z_j,...\}\), where a local coordinate function \(z_j : p \to z_j(p)\) on \(U_j\) satisfies \(z_j(p) = (z_j^1(p),...,z_j^n(p))\) for \(1 \leq j \leq n\). Then for a fixed \(t_0 \in B\), \(U_j \cap U_k \neq \emptyset\) gives a system of local complex coordinates on \(M_{t_0}\). In terms of these coordinates, \(\omega\) is the projection given by \((z_j,t) \to (z_j^1,...,z_j^n,t)\) for \(j,k\) with \(U_j \cap U_k \neq \emptyset\), we denote the coordinate transformations from \(z_j\) to \(z_k\) by \(f_{jk} : (z_j^1,...,z_j^n,t) \to (z_k^1,...,z_k^n,t)\) for \(j,k\) with \(U_j \cap U_k \neq \emptyset\). For \(j,k\) with \(U_j \cap U_k \neq \emptyset\), we denote the coordinate transformations from \(z_j\) to \(z_k\) by \(f_{jk} : (z_j^1,...,z_j^n,t) \to (z_k^1,...,z_k^n,t)\) (for the detail, see [Kod05] p.60).

On the other hand, since \((M_t,\Lambda_t) \to (M,\Lambda)\) is a holomorphic Poisson submanifold for each \(t \in B\) and \(M = \bigcup M_t\), the holomorphic Poisson structure \(\Lambda\) on \(M\) can be expressed in terms of local coordinates as \(\Lambda = \sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}(z_j^1,...,z_j^n,t) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}\) on \(U_j\), where \(g_{\alpha\beta}(z_j,t) = g_{\alpha\beta}^j(z_j^1,...,z_j^n,t)\) is holomorphic with respect to \((z_j,t)\) with \(g_{\alpha\beta}(z_j,t) = \delta_{\alpha\beta}\). For a fixed \(t_0\), the holomorphic Poisson structure \(\Lambda_{t_0}\) on \(M_{t_0}\) is given by \(\sum_{\alpha,\beta=1}^{n} \delta_{\alpha\beta}(z_j^1,...,z_j^n,t_0) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}\) on \(U_j \cap M_{t_0}\).

**Remark 2.0.9.** Let \((M,\Lambda,B,\omega)\) be a Poisson analytic family. Let \(\Delta\) be an open set of \(B\). Then the restriction \((M_\Delta = \omega^{-1}(\Delta),\Lambda|_{M_\Delta},\Delta,\omega|_{M_\Delta})\) is also a Poisson analytic family. We will denote the family by \((M_\Delta,\Lambda_\Delta,\Delta,\omega)\).

**Example 2.0.10 (complex tori).** ([KS58a] p.408) Let \(S\) be the space of \(n \times n\) matrices \(s = (s_{ij})\) with \(\det(\text{Im}(s)) > 0\), where \(\alpha\) denotes the row index and \(\beta\) the column index, and \(\text{Im}(s)\) is the imaginary part of \(s\). For each matrix \(s \in S\) we define an \(n \times 2n\) matrix \(\omega(s) = (\omega_{ij}^s(s))\) by

\[
\begin{aligned}
\omega_{ij}^s(s) &= \begin{cases} 
\delta_{ij}, & \text{for } 1 \leq j \leq n \\
\delta_{ij}^s, & \text{for } j = n + \beta, 1 \leq \beta \leq n
\end{cases}
\end{aligned}
\]

Let \(G\) be the discontinuous abelian group of analytic automorphisms of \(\mathbb{C}^n \times S\) generated by \(g_j : (z,s) \to (z + \omega_j(s),s),\) \(j = 1,...,2n\), where \(\omega_j(s) = (\omega_1^j(s),...,\omega_n^j(s),...,\omega_n^j(s))\) is the \(j\)-th column vector of \(\omega(s)\). The quotient space \(M = \mathbb{C}^n \times S/G\) and \(\pi : M \to S\) induced from the canonical projection \(\mathbb{C}^n \times S \to S\) forms a complex analytic family of complex tori. We will put a holomorphic Poisson structure on \(M\) to make a Poisson analytic family. A holomorphic bivector field of the form \(\Lambda = \sum_{i,j=1}^{n} f_{ij}(s) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}\) on \(\mathbb{C}^n \times S\) where \(f_{ij}(s) = f_{ij}(z,s)\) are holomorphic functions on \(\mathbb{C}^n \times S\), independent of \(z\), is \(G\)-invariant bivector field on \(\mathbb{C}^n \times S\). So this induces a holomorphic bivector field on \(M\). Since \(f_{ij}(s)\) are independent of \(z\), we have \(\Lambda = 0\). So \((M,\Lambda,\Delta,\pi)\) is a Poisson analytic family.

**Example 2.0.11 (Hirzebruch-Nagata surface).** ([SU02] p.13) Take two \(\mathbb{C} \times \mathbb{P}^1_\mathbb{C} \times \mathbb{C}\) and write the coordinates as \((u, (\xi_0 : \xi_1), t), (v, (\eta_0 : \eta_1), t)\), respectively, where \(u, v, t\) are the coordinates of \(\mathbb{C}\) and \((\xi_0 : 1), (\eta_0 : 1)\) are the homogeneous coordinates of \(\mathbb{P}^1_\mathbb{C}\). By patching two \(\mathbb{C} \times \mathbb{P}^1_\mathbb{C} \times \mathbb{C}\) together by relation

\[
\begin{aligned}
u &= 1/v, \\
(\xi_0 : \xi_1) &= (\eta_0 : v^m \eta_1 + tv^k \eta_0), \quad m - 2 \leq 2k \leq m, \quad \text{where } m, k \text{ are natural numbers} \\
t &= t,
\end{aligned}
\]
we obtain a complex analytic family \( \pi : S \to \mathbb{C} \) which is induced from the natural projection \( \mathbb{C} \times \mathbb{P}^1_{\mathbb{C}} \times \mathbb{C} \to \mathbb{C} \) to the third component. We will put a holomorphic Poisson structure \( \Lambda \) on \( S \) so that \((S, \Lambda, \mathbb{C}, \pi) \) is a Poisson analytic family. \( S \) has four affine covers. For one \( \mathbb{C} \times \mathbb{P}^1_{\mathbb{C}} \times \mathbb{C} \) with coordinate \((u, \xi_0 : \xi_1), t)\), we have two affine covers, namely, \( \mathbb{C} \times \mathbb{C} \times \mathbb{C} \) and \( \mathbb{C} \times \mathbb{C} \times \mathbb{C} \). They are glued via \( \mathbb{C} \times (\mathbb{C} - \{0\}) \times \mathbb{C} \) and \( \mathbb{C} \times (\mathbb{C} - \{0\}) \times \mathbb{C} \) by \((u, x = \frac{\xi_1}{\xi_0}, t) \mapsto (u, \frac{1}{x}, t)\). Similarly for another \( \mathbb{C} \times \mathbb{P}^1_{\mathbb{C}} \times \mathbb{C} \), two affine covers are glued via \( \mathbb{C} \times (\mathbb{C} - \{0\}) \times \mathbb{C} \) and \( \mathbb{C} \times (\mathbb{C} - \{0\}) \times \mathbb{C} \) by \((v, w = \frac{n}{\eta_0}, t) \mapsto (v, z, t) = (v, \frac{1}{w} = \frac{m}{n_1}, t)\). We put holomorphic Poisson structures on each four affine covers which define a global bivector field \( \Lambda \) with \([\Lambda, \Lambda] = 0 \) on \( S \). On \((u, x, t)\) coordinate, we give \( g(t)x^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x}\), where \( g(t) \) is any holomorphic function, independent of \( t \). On \((u, y, t)\) coordinate, we give \(-g(t)\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial y}\). On \((v, w, t)\) coordinate, we give \(-g(t)v^{2k-m+2}(w^m-k+t)^2 \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial w}\). On \((v, z, t)\) coordinate, we give \( g(t)v^{2k-m+2}(v^{m-k}+tz)^2 \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial z}\). Then \((S, \Lambda, \mathbb{C}, \pi) \) is a Poisson analytic family.

**Example 2.0.12** (Hopf surfaces). We construct an one parameter Poisson analytic family of general Hopf surfaces. An automorphism of \( W \times \mathbb{C} \) given by \( g : (z_1, z_2, t) \mapsto (a z_1 + t z_2^n, b z_2, t)\) where \( 0 < |a| \leq |b| < 1 \) and \( b^m - a = 0 \) (i.e. \( a = b^m \)), generates an infinite cyclic group \( G \), which properly discontinuous and fixed point free. Hence \( M := W \times \mathbb{C} / G \) is a complex manifold. Since the projection of \( W \times \mathbb{C} \) to \( \mathbb{C} \) commutes with \( g \), it induces a holomorphic map \( \omega \) of \( M \) to \( \mathbb{C} \). So \((M, \mathbb{C}, \omega) \) is a complex analytic family. Since \( g^n \) is given by \( g^n : (z_1, z_2, t) \mapsto (z_1, z_2', t') = (a^n z_1 + n a^{n-1} t z_2^n, b^n z_2, t)\), we have

\[
\frac{\partial}{\partial z_1} = a^n \frac{\partial}{\partial z_1'} , \quad \frac{\partial}{\partial z_2} = m n a^{n-1} t z_1^{n-1} \frac{\partial}{\partial z_1'} + b^n \frac{\partial}{\partial z_2'} , \quad \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} = a^n b^n \frac{\partial}{\partial z_1'} \wedge \frac{\partial}{\partial z_2'}
\]

Then \( f(t)z_2^{m+1} \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \) where \( f(t) \) is any holomorphic function, independent of \( z \), is a \( G \)-invariant holomorphic bivector field on \( W \times \mathbb{C} \) and so define a holomorphic Poisson structure on \( M \). Hence \((M, f(t)z_2^{m+1} \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}, \mathbb{C}, \omega)\) is a Poisson analytic family of Poisson Hopf surfaces.

3. Infinitesimal deformations

3.1. Infinitesimal deformations and truncated holomorphic Poisson cohomology.

In this subsection, we show that given a Poisson analytic family \((M, \Lambda, B, \omega)\), an infinitesimal deformation of a compact holomorphic Poisson manifold \( \omega^{-1}(t) = (M_t, \Lambda_t) \) with dimension \( n \) is captured by an element in the second hypercohomology group of the complex of sheaves \( 0 \to \Theta_{M_t} \to \wedge^2 \Theta_{M_t} \to \cdots \to \wedge^n \Theta_{M_t} \to 0 \) induced by \([\Lambda_t, -] \) analogously to how an infinitesimal deformation of a compact complex manifold \( M_t \) is captured by an element in the first cohomology group \( H^1(M_t, \Theta_t) \).

Let \((M, \Lambda_0)\) be a compact holomorphic Poisson manifold and consider the complex of sheaves

\[
(3.1.1) \quad 0 \to \Theta_M \xrightarrow{[\Lambda_0, -]} \wedge^2 \Theta_M \xrightarrow{[\Lambda_0, -]} \cdots \xrightarrow{[\Lambda_0, -]} \wedge^n \Theta_M \to 0
\]

where \( \Theta_M \) is the sheaf of germs of holomorphic vector fields on \( M \). Let \( U = \{U_j\} \) be sufficiently fine open covering of \( M \) such that \( U_j \) are coordinate polydisks of \( M \), that is, \( U_j = \{(z_1^j, \ldots, z_n^j) \in \mathbb{C}^n \mid |z_\alpha^j| < r_\alpha^j, \alpha = 1, \ldots, n\} \) where \( z_\alpha = (z_1^\alpha, \ldots, z_n^\alpha) \) is a local coordinate on \( U_j \) and \( r_\alpha^j > 0 \) is a constant. Then we can compute the hypercohomology group of the complex of sheaves \((3.1.1)\) by the following Čech resolution (see \[EV92\] Appendix). Here \( \delta \) is the Čech map.
Poisson structure $\Lambda$ is expressed in terms of local complex coordinate system on $z$ polydisks of $M \in \mathfrak{t}$, with notational consistency with (3.1.6).

Remark 3.1.3. In [ELW99], the holomorphic Poisson cohomology for a holomorphic Poisson manifold $(M, \Lambda_0)$ is defined by the $i$-th hypercohomology group of complex of sheaves $\mathcal{O}_M \to \Theta_M \to \wedge^2 \Theta_M \to \cdots \to \wedge^n \Theta_M \to 0$ induced by $[\Lambda_0, -]$. Since there is no role of the structure sheaf $\mathcal{O}_M$ in deformations of compact holomorphic Poisson manifolds, we truncate the complex of sheaves. For the expression $H^i(M, \Lambda_0)$, we adopted the notation from [Nam08] by which this present work was inspired. By general philosophy of deformation theory, it might be natural to shift the grading after truncation so that the $0$-th cohomology group corresponds to infinitesimal Poisson automorphisms, the first cohomology group corresponds to infinitesimal Poisson deformations and the third cohomology group corresponds to obstructions. However, we do not shift the grading to maintain notational consistency with [Nam08]. That’s why we put $0 \to 0 \to 0 \cdots$ on the bottom of the Čech resolution.

We will relate the $2n$-th truncated holomorphic Poisson cohomology group $H^2(M_t, \Lambda_t)$ to infinitesimal deformations of $\omega^{-1}(t) = (M_t, \Lambda_t)$ in a Poisson analytic family $(M, \Lambda, B, \omega)$ for each $t \in B$. As in Remark 2.0.8, let $\mathcal{U} = \{\mathcal{U}_t\}$ be an open covering of $\mathcal{M}$ such that $\mathcal{U}_t$ are coordinate polydisks of $\mathcal{M}$, $\{(z_j, t)\} = \{(z_1^n, t_1, \ldots, t_m)\}$ is a local complex coordinate system on $\mathcal{U}_t$, and $z_\alpha^j = f_{jk}^\alpha(z_k^r, \ldots, z_\beta^n, t_1, \ldots, t_m)$, $\alpha = 1, \ldots, n$ is a holomorphic transition function from $z_k$ to $z_j$. The Poisson structure $\Lambda$ is expressed in terms of local complex coordinate system on $\mathcal{U}_t$ as

$$\Lambda = \Lambda_j = \sum_{\alpha, \beta=1}^n g_{\alpha\beta}^j(z_j, t) \frac{\partial}{\partial z_\alpha^j} \wedge \frac{\partial}{\partial z_\beta^j}$$

where $g_{\alpha\beta}^j(z_j, t)$ is a holomorphic function on $\mathcal{U}_t$ with $g_{\alpha\beta}^j(z_j, t) = -g_{\beta\alpha}^j(z_j, t)$ and we have

$$[\Lambda, \Lambda] = \sum_{\alpha, \beta=1}^n g_{\alpha\beta}^j(z_j, t) \frac{\partial}{\partial z_\alpha^j} \wedge \frac{\partial}{\partial z_\beta^j} \sum_{\alpha, \beta=1}^n g_{\alpha\beta}^j(z_j, t) \frac{\partial}{\partial z_\alpha^j} \wedge \frac{\partial}{\partial z_\beta^j} = 0$$

Since $f_{jk}(z_k, t) = (f_{jk}^1(z_k, t), \ldots, f_{jk}^n(z_k, t), t_1, \ldots, t_m)$ is a Poisson map, we have

$$g_{\alpha\beta}(f_{jk}^1(z_k, t), \ldots, f_{jk}^n(z_k, t)) = \sum_{r, s=1}^m g_{rs}(z_k, t) \frac{\partial f_{jk}^r}{\partial z_k^s} \frac{\partial f_{jk}^s}{\partial z_k^r}$$

on $\mathcal{U}_t \cap \mathcal{U}_k$. Set $\mathcal{U}_t^t := \mathcal{U}_t \cap M_t$. Then for each $t \in B$, $\mathcal{U}_t^t := \{\mathcal{U}_t^t\}$ is an open covering of $M_t$. Recall that $\Lambda_t$ is the Poisson structure on $M_t$ induced from $(M, \Lambda)$. Let $\frac{\partial}{\partial t} = \sum_{\lambda=1}^m c_\lambda \frac{\partial}{\partial z_\lambda}$, $c_\lambda \in \mathbb{C}$ be a tangent vector of $B$. Then we have

Proposition 3.1.7.

$$(\{\lambda_j(t) = \sum_{\alpha, \beta=1}^n \frac{\partial g_{\alpha\beta}^j(z_j, t)}{\partial t} \frac{\partial}{\partial z_\alpha^j} \wedge \frac{\partial}{\partial z_\beta^j}, \{\theta_{jk}(t) = \sum_{\alpha=1}^n \frac{\partial f_{jk}^\alpha(z_k, t)}{\partial t} \frac{\partial}{\partial z_\alpha^j}\} \in C^0(\mathcal{U}_t, \wedge^2 \Theta_{M_t}) \oplus C^1(\mathcal{U}_t, \Theta_{M_t})$$
define a 2-cocycle and call its cohomology class in $HP^2(M_t, Λ_t)$ the infinitesimal (Poisson) deformation along $\frac{δ}{δt}$. This expression is independent of the choice of system of local coordinates.

Proof. First we note that $δ(\{θ_{jk}(t)\}) = 0$ (See [Kod05] p.201). Second, by taking the derivative of (3.1.5) with respect to $t$, we have $\sum_{α, β = 1}^n g^j_{αβ}(z_j, t) \frac{δ}{δz_j} + \sum_{α, β = 1}^n g^j_{αβ}(z_j, t) \frac{δ}{δz_j} = 0$. It remains to show that $δ(\{λ_j(t)\}) + [Λ_t, \{θ_{jk}(t)\}] = 0$. More precisely, on $U_j ∩ U_k ≠ ∅$, we show that $λ_k(t) - λ_j(t) + [Λ_t, θ_{jk}(t)] = 0$. In other words,

$$\sum_{r,s} \frac{∂g^j_{rs}(z_j, t)}{∂t} \frac{δ}{δz_k} - \sum_{α, β = 1}^n g^j_{αβ}(z_j, t) \frac{δ}{δz_j} + \frac{∂g^j_{rs}(z_j, t)}{∂t} \frac{δ}{δz_j} + \sum_{c=1}^n \frac{∂f^j_{rk}(z_k, t)}{∂t} \frac{δ}{δz_j} = 0$$

(3.1.8)

Since $z^α = f^α_{jk}(z^1_k, ..., z^n_k, t_1, ..., t_m)$ for $α = 1, ..., n$, we have $\frac{δ}{δz_k} = ∑_{r,s = 1}^n g^j_{rs} \frac{δ}{δz_k} \frac{∂}{∂z_j}$ for $r = 1, ..., n$. Hence the first term of (3.1.8) is

$$\sum_{r,s} \frac{∂g^j_{rs}(z_j, t)}{∂t} \frac{δ}{δz_k} = \sum_{r,s,a,b = 1}^n \frac{∂g^j_{rs}(z_j, t)}{∂t} \frac{δ}{δz_k} \frac{∂}{∂z_j} + \frac{∂g^j_{rs}(z_j, t)}{∂t} \frac{δ}{δz_j} \frac{∂}{∂z_j}$$

We compute the third term of (3.1.8):

$$\sum_{r,s,c}^n \frac{∂g^j_{rs}(z_j, t)}{∂t} \frac{δ}{δz_j} \frac{∂}{∂z_j} = \sum_{r,s,c}^n \frac{∂f^j_{rk}(z_k, t)}{∂t} \frac{δ}{δz_j} \frac{∂}{∂z_j} + \sum_{c=1}^n \frac{∂f^j_{sk}(z_k, t)}{∂t} \frac{δ}{δz_j} \frac{∂}{∂z_j}$$

(3.1.9)

By considering the coefficients of $\frac{∂}{∂z_j} \frac{∂}{∂z_j}$, (3.1.8) is equivalent to

$$\sum_{r,s}^n \frac{∂g^j_{rs}(z_j, t)}{∂t} \frac{δ}{δz_k} \frac{∂}{∂z_j} = \sum_{r,s,a,b = 1}^n \frac{∂g^j_{rs}(z_j, t)}{∂t} \frac{δ}{δz_k} \frac{∂}{∂z_j} + \sum_{c=1}^n \frac{∂g^j_{rs}(z_j, t)}{∂t} \frac{δ}{δz_j} \frac{∂}{∂z_j}$$

On the other hand, from (3.1.6), we have

$$g^j_{ab}(f^1_{jk}(z, t), ..., f^m_{jk}(z, t), t_1, ..., t_m) = ∑_{r,s = 1}^n g^j_{rs} \frac{δ}{δz_k} \frac{∂}{∂z_j}$$

(3.1.10)

By taking the derivative of (3.1.10) with respect to $t$, we have

$$\sum_{c=1}^n \frac{∂g^j_{ab}(f^c_{jk}(z, t))}{∂t} \frac{∂}{∂z_j} = \sum_{r,s = 1}^n \frac{∂g^j_{rs}(z_j, t)}{∂t} \frac{∂}{∂z_k}$$

(3.1.11)

Indeed, the left hand side and right hand side of (3.1.11) coincide: from (3.1.6)

$$\sum_{c=1}^n (g^j_{ab} \frac{∂}{∂z_j} (\frac{∂}{∂z_j} (\frac{∂}{∂z_j} (\frac{∂}{∂z_j} (\frac{∂}{∂z_j} (\frac{∂}{∂z_j})))) = \sum_{r,s = 1}^n g^j_{rs} (\frac{δ}{δz_k} \frac{∂}{∂z_j} + \frac{δ}{δz_j} \frac{∂}{δz_j})$$

This proves the first claim. It remains to show that $\{λ_j(t)\}, \{θ_{jk}(t)\}$ is independent of the choice of systems of local coordinates. We can show that the infinitesimal deformation does not change under the refinement of the open covering (See [Kod05] p.190). Since we can choose a common refinement for two system of local coordinates, it is sufficient to show that given two local coordinates $x_j =$
$$(z_j, t)$$ and $${u_j = (w_j, t)}$$ on each $${U_j}$$, the infinitesimal Poisson deformation $${\{(\pi_j(t)), \{\eta_{jk}(t)\}\}}$$ with respect to $${\{u_j\}}$$ coincides with $${\{(\lambda_j(t)), \{\theta_{jk}(t)\}\}}$$ with respect to $${\{x_j\}}$$. Let the Poisson structure $${\Lambda}$$ in (3.1.12) be expressed in terms of local coordinates $${u_j}$$ as $${\Lambda = \Pi_j = \sum_{\alpha, \beta = 1}^n \Pi_{\alpha \beta}^j (w_j, t) \frac{\partial}{\partial w_j^\alpha} \wedge \frac{\partial}{\partial w_j^\beta}}$$. Let $${(w_k, t) \to (w_j, t) = (e_{jk}(w_k, t), t)}$$ be the coordinate transformation of $${\{u_j\}}$$ on $${U_j \cap U_k \neq \emptyset}$$. Now we set

$$\eta_{jk}(t) = \sum_{\alpha = 1}^n \frac{\partial \eta_{jk}^\alpha (w_k, t)}{\partial t} \frac{\partial}{\partial w_j^\alpha}, \; w_k = e_{kj}(w_j, t), \; \pi_j(t) = \sum_{\alpha, \beta = 1}^n \frac{\partial \Pi_{\alpha \beta}^j (w_j, t)}{\partial t} \frac{\partial}{\partial w_j^\alpha} \wedge \frac{\partial}{\partial w_j^\beta}$$

We show that $${\{(\lambda_j(t)), \{\theta_{jk}(t)\}\}}$$ is cohomologous to $${\{(\pi_j(t)), \{\eta_{jk}(t)\}\}}$$. Let $${w_0^\alpha = h_0^\alpha (z_1^j, ..., z_n^j, t)}$$, $${\alpha = 1, ..., n}$$, define the coordinate transformation from $${x_j = (z_j, t) \to u_j = (w_j, t)}$$ which is a Poisson map. So we have $${\frac{\partial}{\partial z_j^\alpha} = \sum_{r=1}^n \frac{\partial h_r^\alpha}{\partial z_j^\alpha} \frac{\partial}{\partial z_j^\beta}}$$ and the following relation holds

$$(3.1.12) \quad \Pi_{\alpha \beta}^j (h_1^j(z_j, t), ..., h_n^j(z_j, t), t) = \sum_{r,s=1}^n g_{r,s}^j (z_j, t) \frac{\partial h_r^\alpha}{\partial z_j^\alpha} \frac{\partial h_s^\beta}{\partial z_j^\beta}.$$ 

Set $${\theta_j(t) = \sum_{s=1}^n \frac{\partial h^s_r (z_j, t)}{\partial t} \frac{\partial}{\partial z_j^s}}$$, $${w_0^j = h_0^j (z_j, t)}$$. Then we claim that $${(\lambda_j(t), \theta_{jk}(t)) - (\pi_j(t), \eta_{jk}(t)) = \theta_k(t) - \theta_j(t) - [\Lambda_t, \theta_{jk}(t)] = -\delta (\theta_j(t))}$$, which means $${\{(\lambda_j(t)), \{\theta_{jk}(t)\}\}}$$ is cohomologous to $${\{(\pi_j(t)), \{\eta_{jk}(t)\}\}}$$. Since $${\delta (\{\theta_j(t)\}) = \{\theta_{jk}(t)\} - \{\eta_{jk}(t)\}}$$ for the detail, see [Kod05] p.191-192), we only need to see $${\lambda_j(t) - \pi_j(t) + \Lambda_t = 0}$$, equivalently, $${\sum_{r,s=1}^n \frac{\partial g_{r,s}^j (z_j, t)}{\partial t} \frac{\partial}{\partial z_j^r} \wedge \frac{\partial}{\partial z_j^s} - \sum_{\alpha, \beta = 1}^n \frac{\partial \Pi_{\alpha \beta}^j (w_j, t)}{\partial t} \frac{\partial}{\partial w_j^\alpha} \wedge \frac{\partial}{\partial w_j^\beta} + \sum_{\alpha, \beta = 1}^n \Pi_{\alpha \beta}^j (w_j, t) \frac{\partial}{\partial w_j^\alpha} \wedge \frac{\partial}{\partial w_j^\beta} = 0}$$

which follows from taking the derivative (3.1.12) with respect to $${t}$$ as in the proof of the first claim.

**Definition 3.1.13** (holomorphic Poisson Kodaira-Spencer map). Let $${(M, \Lambda, B, \omega)}$$ be a Poisson analytic family, where $${B}$$ is a domain of $${\mathbb{C}^n}$$, and $${\{z_j, t\}}$$ a local complex coordinate system on $${U_j}$$, $${(z_j, t) \in U_j}$$. The Poisson structure $${\Lambda}$$ is expressed as $${\sum_{\alpha, \beta = 1}^n g_{\alpha \beta}^j (z_j, t) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}}$$ on $${U_j}$$ where $${g_{\alpha \beta}^j (z_j, t)}$$ is a holomorphic function with $${g_{\alpha \beta}^j (z_j, t) = -g_{\beta \alpha}^j (z_j, t)}$$. For a tangent vector $${\frac{\partial}{\partial t} = \sum_{\lambda=1}^m c_\lambda \frac{\partial}{\partial x_\lambda}}$$, $${c_\lambda \in \mathbb{C}}$$, of $${B}$$, we put $${\frac{\partial \Lambda_t}{\partial t} = \sum_{\alpha, \beta = 1}^n \left[ \sum_{\lambda=1}^m c_\lambda \frac{\partial}{\partial z_j^\alpha} \frac{\partial}{\partial \lambda} \right] \frac{\partial}{\partial z_j^\beta} \wedge \frac{\partial}{\partial z_j^\beta}}$$. The (holomorphic) Poisson Kodaira-Spencer map is defined to be a $${\mathbb{C}}$$-linear map

$$\varphi_t : T_t(B) \to HP^2 (M_t, \Lambda_t)$$

$$\frac{\partial}{\partial t} \mapsto \left[ p_t \left( \frac{\partial}{\partial t} \right) \left( \frac{\partial M_t}{\partial t} \right), \frac{\partial \Lambda_t}{\partial t} \right] = \frac{\partial (M_t, \Lambda_t)}{\partial t}$$

where $${p_t : T_t(B) \to H^1 (M_t, \Theta_t)}$$ is the Kodaira-Spencer map of the complex analytic family $${(M, B, \omega)}$$ (see [Kod05] p.201).

### 4. Integrability condition

In a complex analytic family $${(M, B, \omega)}$$ of deformations of a complex manifold $${M = \omega^{-1}(0)}$$, the deformations near $${M}$$ are represented by $${C^\infty}$$ vector (1, 0)-forms $${\varphi(t) \in A^{0,1}(M, TM)}$$ on $${M}$$ satisfying $${\varphi(0) = 0}$$ and the integrability condition $${\partial \varphi(t) - \frac{\partial}{\partial t} \varphi(t) = 0}$$ where $${\varphi(t) \in \Delta}$$ a sufficiently small polydisk in $${B}$$ (see [Kod05] section §5.3). In this section, we show that in a Poisson analytic family $${(M, B, \Lambda, \omega)}$$ of deformations of a compact holomorphic Poisson manifold $${(M, \Lambda_0) = \omega^{-1}(0)}$$, the deformations near $${(M, \Lambda_0)}$$ are represented by $${C^\infty}$$ vector (0, 1)-forms $${\varphi(t) \in A^{0,1}(M, TM)}$$ and $${C^\infty}$$ bivectors $${\Lambda(t) \in A^{0,0}(M, \wedge^2 TM)}$$ satisfying $${\varphi(0) = 0}$$, $${\Lambda(0) = \Lambda_0}$$ and the integrability condition

$$\partial (\varphi(t) + \Lambda(t)) + \frac{1}{2} [\varphi(t) + \Lambda(t), \varphi(t) + \Lambda(t)] = 0.$$ 

To deduce the integrability condition, we extend Kodaira’s approach ([Kod05] section §5.3) in the context of a Poisson analytic family.
4.1. Preliminaries.

We extend the argument of [Kod05] p.259-261 (to which we refer for the detail) in the context of a Poisson analytic family. We tried to maintain notational consistency with [Kod05].

Let $(M, \Lambda, B, \omega)$ be a Poisson analytic family of compact Poisson holomorphic manifolds, where $B$ is a domain of $\mathbb{C}^m$ containing the origin 0. Define $|t| = \max_\lambda |t_\lambda|$ for $t = (t_1, ..., t_m) \in \mathbb{C}^m$, and let $\Delta = \Delta_r = \{ t \in \mathbb{C}^m||t| < r \}$ the polydisk of radius $r > 0$. If we take a sufficiently small $\Delta \subset B$, then $(M_\Delta, \Lambda_\Delta) = \omega^{-1}(\Delta)$ is represented in the form

$$(M_\Delta, \Lambda_\Delta) = \bigcup_j (U_j \times \Delta, \Lambda_{U_j \times \Delta})$$

We denote a point of $U_j$ by $\xi_j = (\xi_j^1, ..., \xi_j^m)$ and its holomorphic Poisson structure $\Lambda_{|U_j \times \Delta}$ by

$$g_{\alpha\beta}(\xi_j, t) \frac{\partial}{\partial \xi_j^\alpha} \wedge \frac{\partial}{\partial \xi_j^\beta}$$
on the $U_j \times \Delta$ with $g_{\alpha\beta}(\xi_j, t) = -g_{\beta\alpha}(\xi_j, t)$. For simplicity, we assume that $U_j \subset \mathbb{C}^m ||\xi_j|| < 1$ where $\xi = \max_\alpha |\xi_j^\alpha|$. $(\xi_j, t) \in U_j \times \Delta$ and $(\xi_k, t) \in U_k \times \Delta$ are the same point on $M_\Delta$ if $\xi_j^\alpha = f_{jk}^\alpha(\xi_k, t), \alpha = 1, ..., n$ where $f_{jk}(\xi_k, t)$ is a Poisson holomorphic map of $\xi_k^1, ..., \xi_k^n, t_1, ..., t_m$, defined on $U_k \times \Delta \cap U_j \times \Delta$, and so we have the following relation

$$(4.1.1) g_{\alpha\beta}^j(f_{jk}^1(\xi_k, t), ..., f_{jk}^n(\xi_k, t)) = \sum_{r,s=1}^n g_{rs}(\xi_k, t) \frac{\partial f_{jk}^r}{\partial \xi_k^r} \frac{\partial f_{jk}^s}{\partial \xi_k^s}$$

We note that $\omega^{-1}(t_0) = (M_{t_0}, \Lambda_{t_0}) = \bigcup_j (U_j, \sum_{\alpha, \beta=1}^n g_{\alpha\beta}^{\xi_j}(\xi_j, t_0) \frac{\partial}{\partial \xi_j^\alpha} \wedge \frac{\partial}{\partial \xi_j^\beta})$ for $t_0 \in \Delta$.

By [Kod05] Theorem 2.3, when we ignore complex structures and Poisson structures, $M_t$ is diffeomorphic to $M_0 = \omega^{-1}(0)$ as differentiable manifolds for each $t \in \Delta$. We put $M := M_0$. By [Kod05] Theorem 2.5, if we take a sufficiently small $\Delta$, there is a diffeomorphism $\Psi$ of $M \times \Delta$ onto $M_\Delta$ as differentiable manifolds such that $\omega \circ \Psi$ is the projection $M \times \Delta \to \Delta$. Let $z = (z_1, ..., z_n)$ be local complex coordinates of $M = M_0$. Then we have $\omega \circ \Psi(z, t) = t, \quad t \in \Delta$. For $\Psi(z, t) \in U_j \times \Delta$, put

$$(4.1.2) \Psi(z, t) = (\xi_j^1(z, t), ..., \xi_j^n(z, t), t_1, ..., t_m).$$

Then each component $\xi_j^\alpha = \xi_j^\alpha(z, t), \alpha = 1, ..., n$ is a $C^\infty$ function. If we identify $M_\Delta = \Psi(M \times \Delta)$ with $M \times \Delta$ via $\Psi$, $(M_\Delta, \Lambda_\Delta)$ is considered as a holomorphic Poisson manifold with the complex structure defined on the $C^\infty$ manifold $M \times \Delta$ by the system of local coordinates on $U_j \times \Delta$

$$\{(\xi_j, t) | j = 1, 2, 3, ... \}, \quad (\xi_j, t) = (\xi_j^1(z, t), ..., \xi_j^n(z, t), t_1, ..., t_m).$$

and the holomorphic Poisson structure given by on $U_j \times \Delta$

$$(4.1.3) \sum_{\alpha, \beta=1}^n g_{\alpha\beta}^{\xi_j}(\xi_j(z, t), t) \frac{\partial}{\partial \xi_j^\alpha} \wedge \frac{\partial}{\partial \xi_j^\beta} | j = 1, 2, 3, ... \}$$

We note that since $(z_1, ..., z_n)$ and $(\xi_j^1(z, 0), ..., \xi_j^n(z, 0))$ are local complex coordinates on $M = M_0$,

$$(4.1.4) \xi_j^\alpha(z, 0) \text{ are holomorphic functions of } z_1, ..., z_n, \alpha = 1, ..., n$$

We also note that if we take $\Delta$ sufficiently small, we have

$$(4.1.5) \det \left( \frac{\partial \xi_j^\alpha(z, t)}{\partial z_\lambda} \right)_{\alpha, \lambda = 1, ..., n} \neq 0$$

for any $t \in \Delta$.

With this preparation, we identify the holomorphic Poisson deformations near $(M, \Lambda_0)$ in the Poisson analytic family $(M, \Lambda, B, \omega)$ with $\varphi(t) + \Lambda(t)$ where $\varphi(t)$ is a $C^\infty$ vector $(0, 1)$-form and $\Lambda(t)$ is a $C^\infty$ bivector on $M$ for $t \in \Delta$. 
4.2. Identification of the deformations of complex structures with $\varphi(t) \in A^{0,1}(M, TM)$.

Put $U_j = \Psi^{-1}(U_j \times \Delta)$. Then $U_j \subset M \times \Delta$ is the domain of $\xi_j(z, t)$. From Equation (4.1.5), we can define a $(0, 1)$-form $\varphi^\lambda_j(z, t) = \sum_{v=1}^n \varphi^\lambda_{jv}(z, t) d\bar{z}_v$ in the following way:

$$
\left( \varphi_1^\lambda(z, t), \ldots, \varphi_n^\lambda(z, t) \right) := \left( \frac{\partial \xi^1_1}{\partial z_1}, \ldots, \frac{\partial \xi^1_n}{\partial z_n} \right)^{-1} \left( \bar{\partial} \xi^1_j \right)
$$

Then the coefficients $\varphi^\alpha_{jv}(z, t)$ are $C^\infty$ functions on $U_j$ and $\bar{\partial} \xi^\alpha_j(z, t) = \sum_{\lambda=1}^n \varphi^\lambda_j(z, t) \frac{\partial \xi^\alpha_j}{\partial z_\lambda}$, $\alpha = 1, \ldots, n$. So we have

$$
\frac{\partial \xi^\alpha_j}{\partial z_v} = \sum_{\lambda=1}^n \varphi^\lambda_{jv}(z, t) \frac{\partial \xi^\alpha_j}{\partial z_\lambda}
$$

Lemma 4.2.2. On $U_j \cap U_k$, we have

$$
\sum_{\lambda=1}^n \varphi^\lambda_j(z, t) \frac{\partial}{\partial z_\lambda} = \sum_{\lambda=1}^n \varphi^\lambda_k(z, t) \frac{\partial}{\partial z_\lambda}
$$

Proof. See [Kod05] p.262.

If for $(z, t) \in U_j$, we define

$$
\varphi(z, t) := \sum_{\lambda=1}^n \varphi^\lambda_j(z, t) \frac{\partial}{\partial z_\lambda} = \sum_{\lambda=1}^n \varphi^\lambda_j(z, t) \frac{\partial}{\partial z_\lambda} = \sum_{\lambda=1}^n \varphi^\lambda_j(z, t) d\bar{z}_v \frac{\partial}{\partial z_\lambda}
$$

By Lemma 4.2.2, $\varphi(t) = \varphi(z, t) \in A^{0,1}(M, TM)$ is a $C^\infty$ vector $(0, 1)$-form on $M$ for every $t \in \Delta$ and we have

$$
\varphi(0) = 0, \quad \bar{\partial} \varphi(t) - \frac{1}{2} [\varphi(t), \varphi(t)] = 0
$$

(see [Kod05] p.263,p.265). We also point out that

Theorem 4.2.5. If we take a sufficiently small polydisk $\Delta$ as in subsection 4.1, then for $t \in \Delta$, a local $C^\infty$ function $f$ on $M$ is holomorphic with respect to the complex structure $M_t$ if and only if $f$ satisfies the equation

$$
(\bar{\partial} - \varphi(t)) f = 0
$$

Proof. See [Kod05] Theorem 5.3 p.263.

4.3. Identification of the deformations of Poisson structures with $\Lambda(t) \in A^{0,0}(M, \wedge^2 TM)$.

For the holomorphic Poisson structure $\sum_{\alpha, \beta=1}^n g^\alpha_{\beta j}(\xi_j(z, t), t) \frac{\partial}{\partial \xi^\alpha_j} \wedge \frac{\partial}{\partial \xi^\beta_j}$ on each $U_j \times \Delta$ from Equation (4.1.3), there exists the unique bivector field $\Lambda_j(z, t) := \sum_{r,s=1}^n h^j_{rs}(z, t) \frac{\partial}{\partial z_r} \wedge \frac{\partial}{\partial z_s}$ on $U_j = \Psi^{-1}(U_j \times \Delta)$ such that

$$
\sum_{r,s=1}^n h^j_{rs}(z, t) \frac{\partial \xi^\alpha_j}{\partial z_r} \frac{\partial \xi^\beta_j}{\partial z_s} = g^\alpha_{\beta j}(\xi_j(z, t), t).
$$

Indeed, from (4.1.5), we set

$$
\left( h^1_{rj}(z, t) \ldots h^n_{rj}(z, t) \right) := \left( \frac{\partial \xi^1_1}{\partial z_1}, \ldots, \frac{\partial \xi^1_n}{\partial z_n} \right)^{-1} \left( g^1_{rj}(\xi_j(z, t), t) \ldots g^n_{rj}(\xi_j(z, t), t) \right)
$$

We note that since $g^\alpha_{\beta j}(\xi_j(z, t)) = -g^\beta_{\alpha j}(\xi_j(z, t))$, we have $h^j_{rs}(z, t) = -h^j_{sr}(z, t)$.

Lemma 4.3.2. On $U_j \cap U_k$, we have $h^j_{rs}(z, t) = h^k_{rs}(z, t)$. 

Proof. From (4.3.1), (4.1.1) and $\frac{\partial \xi_{\alpha}}{\partial z_{\beta}} = \sum_{p=1}^{n} \frac{\partial \xi_{\alpha}}{\partial \bar{\xi}_{\beta}}$, we have

\[
\sum_{r,s=1}^{n} h_{rs}^{j}(z, t) \frac{\partial \xi_{\alpha}}{\partial z_{r}} \frac{\partial \xi_{\beta}}{\partial z_{s}} = g_{\alpha\beta}^{j}(\xi_{j}(z, t), t) = \sum_{p,q=1}^{n} g_{pq}^{k}(\xi_{k}(z, t), t) \frac{\partial \xi_{\alpha}}{\partial \bar{\xi}_{p}} \frac{\partial \xi_{\beta}}{\partial \bar{\xi}_{q}} = \sum_{p,q,r,s=1}^{n} h_{rs}^{k}(z, t) \frac{\partial \xi_{\alpha}}{\partial z_{r}} \frac{\partial \xi_{\beta}}{\partial z_{s}}.
\]

From (4.1.5), we have $h_{rs}^{j}(z, t) = h_{rs}^{k}(z, t)$.

If for $(z, t) \in U_{j}$, we define

(4.3.3) $\Lambda(z, t) := \sum_{r,s=1}^{n} h_{rs}^{j}(z, t) \frac{\partial}{\partial z_{r}} \wedge \frac{\partial}{\partial z_{s}}$ on $U_{j}$ satisfying

By Lemma 4.3.2, $\Lambda(t) := \Lambda(z, t) \in A^{0,0}(M, \wedge^{2} T_{M})$ is a $C^{\infty}$ bivector field on $M$ for every $t \in \Delta$ with $\Lambda(0) = \Lambda_{0}$.

Theorem 4.3.4. If we take a sufficiently small polydisk $\Delta$ as in subsection 4.1 then for the Poisson structure $\sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}^{j}(\xi_{j}, t) \frac{\partial}{\partial \xi_{\alpha}} \wedge \frac{\partial}{\partial \xi_{\beta}}$ on $U_{j} \times \Delta$ for each $j$, there exists the unique bivector field $\Lambda_{j}(t) = \sum_{r,s=1}^{n} h_{rs}^{j}(z, t) \frac{\partial}{\partial z_{r}} \wedge \frac{\partial}{\partial z_{s}}$ on $U_{j}$ satisfying

1. $\sum_{r,s=1}^{n} h_{rs}^{j}(z, t) \frac{\partial \xi_{\alpha}}{\partial z_{r}} \frac{\partial \xi_{\beta}}{\partial z_{s}} = g_{\alpha\beta}^{j}(\xi_{j}(z, t), t)$
2. $\Lambda_{j}(t)$ are glued together to define a $C^{\infty}$ bivector field $\Lambda(t)$ on $M \times \Delta$
3. for each $j$, $[\Lambda_{j}(t), \Lambda_{j}(t)] = 0$. Hence we have $[\Lambda(t), \Lambda(t)] = 0$

We will use the following lemma to prove the theorem.

Lemma 4.3.5. If $\sigma = \sum_{\alpha,\beta=1}^{n} \sigma_{\alpha\beta} \frac{\partial}{\partial z_{\alpha}} \wedge \frac{\partial}{\partial z_{\beta}}$ with $\sigma_{\alpha\beta} = -\sigma_{\beta\alpha}$, then $[\sigma, \sigma] = 0$ is equivalent to

(4.3.6) $\sum_{i=1}^{n} (\sigma_{ik} \frac{\partial \sigma_{ij}}{\partial z_{l}} + \sigma_{il} \frac{\partial \sigma_{jk}}{\partial z_{l}} + \sigma_{ij} \frac{\partial \sigma_{ki}}{\partial z_{l}}) = 0$

for each $1 \leq i, j, k \leq n$.

Proof of Theorem 4.3.4. We have already showed (1) and (2). It remains to show (3). We note that

(4.3.7) $\frac{\partial}{\partial \xi_{\alpha}} \left( \sum_{a,b=1}^{n} h_{ab}^{j}(z, t) \frac{\partial \xi_{\alpha}}{\partial z_{a}} \frac{\partial \xi_{\beta}}{\partial z_{b}} \right) = \sum_{a,b=1}^{n} \frac{\partial}{\partial \xi_{\alpha}} \left( h_{ab}^{j}(z, t) \frac{\partial \xi_{\alpha}}{\partial z_{a}} \frac{\partial \xi_{\beta}}{\partial z_{b}} \right) = 0$.

Since $g_{\alpha\beta}^{j}(\xi_{j}(z, t), t) = \sum_{a,b=1}^{n} h_{ab}^{j}(z, t) \frac{\partial \xi_{\alpha}}{\partial z_{a}} \frac{\partial \xi_{\beta}}{\partial z_{b}}$ is holomorphic with respect to $\xi_{j} = (\xi_{\alpha}^{j})$, $\alpha = 1, ..., n$, we have

In the following, for simplicity, we denote $\xi_{\alpha}^{j}(z, t)$ by $\xi_{\alpha}$ and $h_{ab}^{j}(z, t)$ by $h_{ab}$. By (4.3.6), Lemma 4.3.5 and (4.3.7), and by the property $h_{ab} = -h_{ba}$ and $\frac{\partial}{\partial z_{a}} = \sum_{l=1}^{n} \frac{\partial}{\partial z_{l}} \frac{\partial}{\partial \xi_{\alpha}} + \sum_{l=1}^{n} \frac{\partial}{\partial z_{l}} \frac{\partial}{\partial \xi_{\beta}}$, we have
0 = \sum_{a,b,c,d,l=1}^{n} (h_{ab} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_k}{\partial z_b} \frac{\partial \xi_j}{\partial \xi_l} + h_{ac} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_k}{\partial z_c} \frac{\partial \xi_j}{\partial \xi_l} + h_{ad} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_k}{\partial z_d} \frac{\partial \xi_j}{\partial \xi_l} + h_{cd} \frac{\partial \xi_l}{\partial z_c} \frac{\partial \xi_k}{\partial z_d} \frac{\partial \xi_j}{\partial \xi_l}) + \sum_{a,b,c,d,l=1}^{n} (h_{ab} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_k}{\partial z_b} \frac{\partial \xi_j}{\partial \xi_l} + h_{ac} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_k}{\partial z_c} \frac{\partial \xi_j}{\partial \xi_l} + h_{ad} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_k}{\partial z_d} \frac{\partial \xi_j}{\partial \xi_l} + h_{cd} \frac{\partial \xi_l}{\partial z_c} \frac{\partial \xi_k}{\partial z_d} \frac{\partial \xi_j}{\partial \xi_l})

= \sum_{a,b,c,d,l=1}^{n} (h_{ab} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_k}{\partial z_b} + h_{ac} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_k}{\partial z_c} + h_{ad} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_k}{\partial z_d} + h_{cd} \frac{\partial \xi_l}{\partial z_c} \frac{\partial \xi_k}{\partial z_d})

From (4.1.5), we have \( \sum_{a=1}^{n} h_{ab} \frac{\partial h_{cd}}{\partial z_a} + h_{ac} \frac{\partial h_{bd}}{\partial z_a} + h_{ad} \frac{\partial h_{bc}}{\partial z_a} = 0 \) for each \( b, c, d \). So by Lemma 4.3.3, \( \Lambda_j(t), \Lambda_j(t) = 0 \).

**Remark 4.3.8.** For the compact holomorphic Poisson manifold \((M_t, \Lambda_t)\) for each \( t \in \Delta \) in the Poisson analytic family \((M, \Lambda, B, \omega)\), we showed that there exists a bivector field \( \Lambda(t) \) on \( M = M_0 \) with \([\Lambda(t), \Lambda(t)] = 0\) for \( t \in \Delta \) by Theorem 4.3.1. Let \( J_t : T_\mathbb{R}M \to T_\mathbb{R}M \) with \( J_t^2 = -id \) be the almost complex structure associated to the complex structure \( M_t \) (induced by \( \varphi(t) \)) where \( T_\mathbb{R}M \) is a real tangent bundle of the underlying differentiable manifold \( M \). Then \( J_t \) induces a type decomposition of complexified tangent bundle \( T_\mathbb{C}M = T^0_{\mathbb{C}}M \oplus T^1_{\mathbb{C}}M \) (see [KN96] Chapter IX section 2) so that we have \( \Lambda^{2_{\mathbb{C}}}T_\mathbb{C}M = \Lambda^{2_{\mathbb{C}}}T^0_{\mathbb{C}}M \oplus \Lambda^{2_{\mathbb{C}}}T^1_{\mathbb{C}}M \). If \( \Lambda \) is a \( C^\infty \) section of \( \Lambda^{2_{\mathbb{C}}}T_\mathbb{C}M \) on \( M \), then we denote by \( \Lambda^{2,0} \) the component of \( \Lambda^{2_{\mathbb{C}}}T^1_{\mathbb{C}}M \), by \( \Lambda^{1,1} \) the component of \( T^0_{\mathbb{C}}M \otimes T^1_{\mathbb{C}}M \), and by \( \Lambda^{0,2} \) the component of \( \Lambda^{2_{\mathbb{C}}}T^0_{\mathbb{C}}M \). So we have \( \Lambda = \Lambda^{2,0} + \Lambda^{1,1} + \Lambda^{0,2} \). We call \( \Lambda^{2,0} \) the type \((2,0)\)-part of \( \Lambda \). With this notation, the type \((2,0)\)-part of \( \Psi \Lambda(t) \) is \( \Lambda_t \) for \( t \in \Delta \), where \( \Psi \Lambda(t) \) is the bivector field induced from \( \Lambda(t) \) via diffeomorphism \( \Psi \) in (4.1.2). So we can say that \( \Lambda(t)^{2,0} = \Lambda_t \).
Remark 4.3.9. Let $\Lambda$ be a $C^\infty$-section of $\wedge^k T^* M$. From $\wedge^k T^* M = \bigoplus_{p+q=k} \wedge^p T^* T^* M \otimes \wedge^q T^* M$, we can define the type $(p, q)$ part $\Lambda^{p,q}$ of $\Lambda$ in an obvious way as in Remark 4.3.8.

Next we discuss the condition when a given $C^\infty$ bivector field $\Lambda \in A^{0,0}(M, \wedge^2 T M)$ on $M$ with $[\Lambda, \Lambda] = 0$ gives a holomorphic bivector field $\Lambda^{2,0} \in A^{0,0}(M, \wedge^2 T M)$ with respect to the complex structure $M_t$ induced by $\varphi(t)$. Before proceeding our discussion, we recall the Schouten bracket $[-, -]$ on $\bigoplus_{i \geq 0} \bigoplus_{p+q-i = 0, q \geq 1} A^{0,p}(M, \wedge^q T M)$ (see (4.1.1)) which we need for the computation of the integrability condition (4.3.5). The Schouten bracket $[-, -]$ is defined in the following way:

$$[-, -] : A^{0,p}(M, \wedge^q T M) \times A^{0,q'}(M, \wedge^{q'} T M) \to A^{0,p+q'}(M, \wedge^{q+q'-1} T M)$$

In local coordinates it is given by

$$(4.3.10) \quad [f dz_i \frac{\partial}{\partial z_j}, g dz_K \frac{\partial}{\partial z_L}] = (-1)^{|I||J|+1} d\bar{z}_I \wedge d\bar{z}_J [f \frac{\partial}{\partial z_I}, g \frac{\partial}{\partial z_J}]$$

where $f, g$ are $C^\infty$ functions on $M$ and $d\bar{z}_I = d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_I}$, $\frac{\partial}{\partial z_I}$ = $\frac{\partial}{\partial z_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial z_{i_I}}$ (similarly for $d\bar{z}_J$, $\frac{\partial}{\partial z_J}$).

Then

$$(4.3.11) \quad g = (\bigoplus_{i \geq 0} g_i, g_i = \bigoplus_{p+q-i = 0, q \geq 1} A^{0,p}(M, \wedge^q T M), L = \bar{\partial} + [\Lambda, -], [-, -]),$$

is a differential graded Lie algebra. So we have the following properties: for $a \in A^{0,p}(M, \wedge^q T M)$, $b \in A^{0,q'}(M, \wedge^{q'} T M)$, and $c \in A^{0,q''}(M, \wedge^{q''} T M)$

1. $[a, b] = (-1)^{(p+q+1)(p'+q''+1)}[a, b]
2. [a, [b, c]] = [[a, b], c] + (-1)^{(p+q+1)(p'+q''+1)}[b, [a, c]]
3. $\bar{\partial}[a, b] = [\bar{\partial}a, b] + (-1)^{(p+q+1)}[a, \bar{\partial}b]$)

Theorem 4.3.12. If we take a sufficiently small polydisk $\Delta$ as in subsection 4.1, then for $t \in \Delta$, a type $(2, 0)$-part $\Lambda^{2,0}$ of a $C^\infty$ bivector field $\Lambda = \sum_{r,s=1}^n h_{rs}(z) \frac{\partial}{\partial z_r} \wedge \frac{\partial}{\partial z_s}$ on $M$ is holomorphic with respect to the complex structure $M_t$ induced by $\varphi(t)$ if and only if it satisfies the equation

$$\bar{\partial} \Lambda - [\Lambda, \varphi(t)] = 0$$

Moreover, if $[\Lambda, \Lambda] = 0$, then $[\Lambda^{2,0}, \Lambda^{2,0}] = 0$.

Proof. We note that the type $(2, 0)$-part of $\Lambda = \sum_{r,s=1}^n h_{rs}(z) \frac{\partial}{\partial z_r} \wedge \frac{\partial}{\partial z_s}$ with respect to complex structure $M_t$ is $\sum_{r,s,a,\beta=1}^n h_{rs}(z) \frac{\partial}{\partial z_r} \frac{\partial}{\partial z_s} \frac{\partial}{\partial z_a} \wedge \frac{\partial}{\partial z_\beta}$. Hence by Theorem 4.2.5 it suffices to show that

$$\bar{\partial} \Lambda - [\Lambda, \varphi(t)] = 0$$

First we note that from (4.2.3) and (4.3.10), we have

$$\bar{\partial} \Lambda - [\Lambda, \varphi(t)]$$

(4.3.13)

For each $\alpha, \beta$, ($\bar{\partial} - \varphi(t)$)($\sum_{r,s=1}^n h_{rs} \frac{\partial}{\partial z_r} \frac{\partial}{\partial z_s}$) = 0 if and only if $\bar{\partial} \Lambda - [\Lambda, \varphi(t)] = 0$

First we note that from (4.2.3) and (4.3.10), we have

$$\bar{\partial} \Lambda - [\Lambda, \varphi(t)]$$

(4.3.14)

By considering the coefficients of $\frac{\partial}{\partial z_e} \frac{\partial}{\partial z_r} \wedge \frac{\partial}{\partial z_s}$ in (4.3.14), $\bar{\partial} \Lambda - [\Lambda, \varphi(t)] = 0$ is equivalent to

$$\bar{\partial} \Lambda - [\Lambda, \varphi(t)] = 0$$

for each $r, s, v.$
On the other hand, from (4.2.0), \((\bar{\partial} - \varphi(t))(\sum_{r,s=1}^{n} h_{rs} \frac{\partial \xi^a_j}{\partial z_r} \frac{\partial \xi^b_j}{\partial z_s}) = 0\) for each \(\alpha, \beta\) is equivalent to

\[(4.3.16)\]
\[
\sum_{r,s=1}^{n} \left( \frac{\partial h_{rs}}{\partial z_v} \frac{\partial \xi^a_j}{\partial z_r} \frac{\partial \xi^b_j}{\partial z_s} + h_{rs} \frac{\partial}{\partial z_r} \left( \frac{\partial \xi^a_j}{\partial z_v} \frac{\partial \xi^b_j}{\partial z_s} + h_{rs} \frac{\partial \xi^a_j}{\partial z_r} \frac{\partial \xi^b_j}{\partial z_s} \right) \right)
- \sum_{r,s,c=1}^{n} \varphi_v^c \left( \frac{\partial h_{rs}}{\partial z_c} \frac{\partial \xi^a_j}{\partial z_r} \frac{\partial \xi^b_j}{\partial z_c} + h_{rs} \frac{\partial^2 \xi^a_j}{\partial z_r^2} \frac{\partial \xi^b_j}{\partial z_s} + h_{rs} \frac{\partial \xi^a_j}{\partial z_c} \frac{\partial^2 \xi^b_j}{\partial z_c \partial z_r} \right) = 0 \quad \text{for each } \alpha, \beta, v.
\]

From (4.2.0), we have \(\frac{\partial \xi^a_j}{\partial z_c} = \sum_{c=1}^{n} \xi_{ij}^a \varphi^c_v\) and \(\frac{\partial \xi^b_j}{\partial z_c} = \sum_{c=1}^{n} \xi_{ij}^b \varphi^c_v\). So (4.3.16) is equivalent to

\[(4.3.17)\]
\[
\sum_{r,s=1}^{n} \left( \frac{\partial h_{rs}}{\partial z_v} \frac{\partial \xi^a_j}{\partial z_r} \frac{\partial \xi^b_j}{\partial z_s} + h_{rs} \frac{\partial}{\partial z_r} \left( \frac{\partial \xi^a_j}{\partial z_v} \frac{\partial \xi^b_j}{\partial z_s} + h_{rs} \frac{\partial \xi^a_j}{\partial z_r} \frac{\partial \xi^b_j}{\partial z_s} \right) \right)
- \sum_{r,s,c=1}^{n} \varphi_v^c \left( \frac{\partial h_{rs}}{\partial z_c} \frac{\partial \xi^a_j}{\partial z_r} \frac{\partial \xi^b_j}{\partial z_c} + h_{rs} \frac{\partial^2 \xi^a_j}{\partial z_r^2} \frac{\partial \xi^b_j}{\partial z_s} + h_{rs} \frac{\partial \xi^a_j}{\partial z_c} \frac{\partial^2 \xi^b_j}{\partial z_c \partial z_r} \right) = 0
\]

So (4.3.17) is equivalent to

\[(4.3.18)\]
\[
\sum_{r,s=1}^{n} \left( \frac{\partial h_{rs}}{\partial z_v} + \sum_{c=1}^{n} (h_{cs} \frac{\partial \varphi^r_v}{\partial z_c} \varphi^s_v \frac{\partial h_{rs}}{\partial z_c} + h_{rc} \frac{\partial \varphi^s_v}{\partial z_c} \varphi^r_v \frac{\partial h_{rs}}{\partial z_c}) \right) \frac{\partial \xi^a_j}{\partial z_r} \frac{\partial \xi^b_j}{\partial z_s} = 0 \quad \text{for each } \alpha, \beta, v.
\]

From (4.1.5), the equation (4.3.18) is equivalent to

\[(4.3.19)\]
\[
\frac{\partial h_{rs}}{\partial z_v} + \sum_{c=1}^{n} (h_{cs} \frac{\partial \varphi^r_v}{\partial z_c} \varphi^s_v \frac{\partial h_{rs}}{\partial z_c} + h_{rc} \frac{\partial \varphi^s_v}{\partial z_c} \varphi^r_v \frac{\partial h_{rs}}{\partial z_c}) = 0 \quad \text{for each } r, s, v.
\]

Note that (4.3.15) is same to (4.3.19), which proves (4.3.13).

For the second statement of Theorem 4.3.12, we note that

\[
\Lambda = \sum_{r,s=1}^{n} h_{rs} \frac{\partial}{\partial z_r} \wedge \frac{\partial}{\partial z_s} = \sum_{r,s,i,j=1}^{n} \left( h_{rs} \frac{\partial \xi^a_j}{\partial z_r} \frac{\partial \xi^b_j}{\partial z_s} \frac{\partial}{\partial \xi^a_j} \wedge \frac{\partial}{\partial \xi^b_j} + 2 h_{rs} \frac{\partial \xi^a_j}{\partial z_r} \frac{\partial \xi^b_j}{\partial z_s} \frac{\partial}{\partial \xi^a_j} \wedge \frac{\partial}{\partial \xi^b_j} \wedge \frac{\partial}{\partial \xi^a_j} \wedge \frac{\partial}{\partial \xi^b_j} + h_{rs} \frac{\partial \xi^a_j}{\partial z_r} \frac{\partial \xi^b_j}{\partial z_s} \frac{\partial}{\partial \xi^a_j} \wedge \frac{\partial}{\partial \xi^b_j} \wedge \frac{\partial}{\partial \xi^a_j} \wedge \frac{\partial}{\partial \xi^b_j} \right)
\]
\[
= \Lambda^{2,0} + \Lambda^{1,1} + \Lambda^{2,0}.
\]

Since \([\Lambda, \Lambda] = 0\), the type \((3, 0)\) part \([\Lambda, \Lambda]^{3,0} = [\Lambda^{2,0}, \Lambda^{2,0}] + [\Lambda^{2,0}, \Lambda^{1,1}]^{3,0} = 0\) (see Remark 4.3.9). Since \(\Lambda^{2,0}\) is holomorphic with respect to the complex structure induced by \(\varphi(t)\), we have \([\Lambda^{2,0}, \Lambda^{1,1}]^{3,0} = 0\). Hence \([\Lambda^{2,0}, \Lambda^{2,0}] = 0\).

**Remark 4.3.20.** A $C^\infty$ complex bivector field \(\Lambda \in A^{0,0}(M, \Lambda^2T_M)\) on \(M\) with \([\Lambda, \Lambda] = 0\) gives a Poisson bracket \(\{-, -\}\) on \(C^\infty\) complex valued functions on \(M\). We point out that when we restrict the Poisson bracket \(\{-, -\}\) to holomorphic functions with respect to the complex structure \(M_t\) induced by \(\varphi(t)\), this is exactly the (holomorphic) Poisson bracket induced from \(\Lambda^{2,0}\) when \(\bar{\partial} \Lambda - [\Lambda, \varphi(t)] = 0\).

**Remark 4.3.21.** By Theorem 4.3.12 \(\varphi(t)\) in (4.2.0) and \(\Lambda(t)\) in (4.3.3) satisfy

\[(4.3.22)\]
\[
\bar{\partial} \Lambda(t) - [\Lambda(t), \varphi(t)] = 0 \quad \text{for each } t
\]

and \(\Lambda(t)^{2,0} = \Lambda_t\) for each \(t\) (see Remark 4.3.9).
4.4. Expression of infinitesimal deformations in terms of $\varphi(t)$ and $\Lambda(t)$.

In this subsection, we study how an infinitesimal deformation of $(M, \Lambda_0) = \omega^{-1}(0)$ in the Poisson analytic family $(M, \Lambda, B, \omega)$ (in subsection 4.1) is represented in terms of $\varphi(t)$ (4.2.3) and $\Lambda(t)$ (4.3.3). Recall that an infinitesimal deformation at $(M, \Lambda_0)$ is captured by an element $(\partial(M, \Lambda_0))/\partial t \in HP^2(M, \Lambda_0)$ of the complex of sheaves (3.1.1) by using the following Čech resolution associated with the open covering $U^0 = \{U^0_j := U_j \times \mathbb{R}\}$ (see Proposition 3.1.7 and Definition 3.1.13).

We can also compute the hypercohomology group of the complex of sheaves (3.1.1) by using the following Dolbeault resolution.

We describe how a 2-cocycle in the Čech resolution looks like in the Dolbeault resolution. In the picture below, we connect two resolutions. We only depict a part of resolutions that we need in the following diagram. Recall that $\mathcal{A}^{0,p}(\Lambda^qT_M)$ is the sheaf of germs of $C^\infty$-section of $\Lambda^pT_M^* \otimes \Lambda^qT_M$ (see (1.0.1)).
Note that each horizontal complex is exact except for edges of the “real wall”.

Now we explicitly construct the isomorphism of the second hypercohomology group from Čech resolution to the second hypercohomology group from Dolbeault resolution, namely

\[
(4.4.1) \quad \frac{\ker(C^0(\mathcal{U}^0, \wedge^2 \Theta_M) \oplus C^1(\mathcal{U}^0, \Theta_M) \rightarrow C^0(\mathcal{U}^0, \wedge^3 \Theta_M) \oplus C^1(\mathcal{U}^0, \wedge^2 \Theta_M) \oplus C^2(\mathcal{U}^0, \Theta_M))}{C^0(\mathcal{U}^0, \Theta_M) \rightarrow C^0(\mathcal{U}^0, \wedge^2 \Theta_M) \oplus C^1(\mathcal{U}^0, \Theta_M)}
\]

\[
\cong \frac{\ker(A^{0,0}(M, \wedge^2 T_M) \oplus A^{0,1}(M, T_M) \rightarrow A^{0,0}(M, \wedge^3 T_M) \oplus A^{0,1}(M, \wedge^2 T_M) \oplus A^{0,2}(M, T_M))}{\text{im}(A^{0,0}(M, T_M) \rightarrow A^{0,0}(M, \wedge^2 T_M) \oplus A^{0,1}(M, T_M))}
\]

\[(b, a) \mapsto ([\Lambda_0, c] - b, \bar{\partial}c)\]

We define the map in the following way: let \((b, a) \in C^0(\mathcal{U}^0, \wedge^2 \Theta_M) \oplus C^1(\mathcal{U}^0, \Theta_M)\) be a cohomology class of Čech resolution. Since \(\bar{\partial}a = 0\), there exists a \(c \in C^0(\mathcal{U}^0, \wedge^0(\Theta_M))\) such that \(-\bar{\partial}c = a\). Since \(a\) is holomorphic (\(\bar{\partial}a = 0\)), by the commutativity \(\bar{\partial}c \in A^{0,1}(M, T_M)\). We claim that 

\[\hat{\delta}(\Lambda_0, c) - b \in A^{0,0}(M, \wedge^2 T_M)\]

Indeed, \(\hat{\delta}(\Lambda_0, c) - b = -\Lambda_0 - \bar{\partial}c - \bar{\partial}b = -\Lambda_0 - \bar{\partial}b = 0\). We show that \((\bar{\partial}c, \Lambda_0, c) - b) is a cohomology class from Dolbeault resolution. Indeed, \(\hat{\partial}(\bar{\partial}c) = 0\).

\[\hat{\delta}([\Lambda_0, c] - b) + [\Lambda_0, \bar{\partial}c] = 0\]

We define the map by

\[(b, a) \mapsto ([\Lambda_0, c] - b, \bar{\partial}c)\]

We show that this map is well defined.

1. (Independence of choice of \(c\)) Let \(c' \in A^{0,0}(M, T_M)\). Then \([\Lambda_0, c] - b, \bar{\partial}c - [\Lambda_0, c'] - b, \bar{\partial}c') = ([\Lambda_0, c'] - b, \bar{\partial}c') = \hat{\delta}d + [\Lambda_0, d]\)

2. (Independence of choice of \((b, a)\)) Let \((b, a)\) and \((b', a')\) are in the same cohomology class.

We show that \((b' - b, a - a')\) is mapped to \((0, 0) \in A^{0,0}(\mathcal{U}^0, \wedge^2 T_M) \oplus A^{0,1}(\mathcal{U}^0, M, T_M)\). Indeed, there exists \(e \in C^0(\mathcal{U}^0, \Theta_M)\) such that \(-\bar{\partial}e = a - a', [\Lambda_0, e] = b - b'\). We can use \(e\) as \(c\).

Then \((b' - b, a - a')\) is mapped to \([\Lambda_0, e] - (b - b'), \bar{\partial}e) = (0, 0)\).

For the inverse map, let \((\beta, \alpha) \in A^{0,0}(\mathcal{U}^0, \wedge^2 T_M) \oplus A^{0,1}(\mathcal{U}^0, M, T_M)\) be a cohomology class from Dolbeault resolution. Then there exists \(a \in C^0(\mathcal{U}^0, \wedge^0(\Theta_M))\) such that \(\hat{\partial}c = a\). We define the inverse map \((\beta, \alpha) \mapsto ([\Lambda_0, e] - \bar{\partial}c)\).

**Theorem 4.4.2.** \((-\frac{\partial f(t)}{\partial \tau}|_{t=0}, \frac{\partial f(t)}{\partial \tau}|_{t=0}) \in A^{0,0}(\mathcal{U}^0, \wedge^2 T_M) \oplus A^{0,1}(\mathcal{U}^0, M, T_M)\) satisfies

\[\Lambda_0 - \hat{\delta}\Lambda(t) = 0, \hat{\delta}\Lambda(t) - [\Lambda(t), \varphi(t)] = 0\]

and \(\hat{\delta}f(t) - \frac{1}{2}[\varphi(t), \varphi(t)] = 0\) with \(\varphi(0) = 0, \Lambda(0) = \Lambda_0\). By taking the derivative of these equations with respect to \(t\) and plugging 0, we get the first claim. Next we show the second claim. Put

\[\theta_{jk} = \frac{\partial f_{jk}(\xi_k, t)}{\partial \tau}|_{t=0}, \sigma_j = \frac{\partial g_{rs}(\xi_1, t)}{\partial \tau}|_{t=0}\]

The infinitesimal deformation \((-\frac{\partial f_{jk}(\xi_k, t)}{\partial \tau}|_{t=0}, \frac{\partial f_{jk}(\xi_k, t)}{\partial \tau}|_{t=0}) \in HP^2(M, \Lambda_0)\) is the cohomology class of the \([\{\sigma_j\}, \{\theta_{jk}\}, \hat{\delta}f(t)] \in C^0(\mathcal{U}^0, \wedge^2 \Theta_M) \oplus C^1(\mathcal{U}^0, \Theta_M)\) (see Proposition \[3.1.7\] and Definition \[3.1.13\]). We fix a tangent vector \(\frac{\partial f}{\partial \tau} \in T_0(\Delta)\), denote \((-\frac{\partial f(t)}{\partial \tau}|_{t=0})\) by \(f\) for a \(C^\infty\) function \(f(t), t \in \Delta\). With this notation, we put

\[\xi_j = \sum_{\alpha=1}^n \hat{\xi}_j^\alpha \frac{\partial}{\partial \xi_j^\alpha}, \text{ where } \hat{\xi}_j^\alpha = \left(\frac{\partial \xi_j^\alpha(z, t)}{\partial \tau}|_{t=0}\right)
\]

for each \(j\). Then we have

\[1.1.3) \delta(\xi_j) = -\theta_j, \text{ and } \bar{\partial}\xi_j = \sum_{\alpha=1}^n \frac{\partial f_{jk}(\xi_k, t)}{\partial \tau}|_{t=0} \frac{\partial}{\partial \xi_j^\alpha} = \sum_{\alpha=1}^n \varphi^\alpha \frac{\partial}{\partial \xi_j^\alpha} = \varphi\]

(for the detail, see \[Kod05\] Theorem 5.4 p.266). On the other hand,

**Lemma 4.4.4.** We have \(\hat{\Lambda} - \sigma_j + [\Lambda_0, \xi_j] = 0\). More precisely,
\[
\sum_{r,s=1}^{n} \left( \frac{\partial h_{rs}(z,t)}{\partial t} \right)_{t=0} = \sum_{r,s=1}^{n} \left( \frac{\partial g_{a\beta}^{j}(\xi_j,t)}{\partial t} \right)_{t=0} + \sum_{r,s=1}^{n} \left( \frac{\partial g_{r,s}^{j}(\xi_j,0)}{\partial \xi_j} \right) + \sum_{r,s=1}^{n} \left( \frac{\partial \xi_j}{\partial \xi_j} \right)
\]
equivalently (with the notation above),
\[
(4.4.5) \quad \sum_{r,s=1}^{n} h_{rs}^{j} \frac{\partial}{\partial z_{r}} \wedge \frac{\partial}{\partial z_{s}} - \sum_{\alpha,\beta=1}^{n} g_{a\beta}^{j} \frac{\partial}{\partial \xi_{j}} \wedge \frac{\partial}{\partial \xi_{j}} + \sum_{r,s=1}^{n} g_{r,s}^{j}(\xi_j,0) \frac{\partial}{\partial \xi_j} \wedge \frac{\partial}{\partial \xi_j} + \sum_{c=1}^{n} \xi_j^{c} \frac{\partial}{\partial \xi_j} = 0
\]
Proof. From (4.4.4), the first term of (4.4.5) is
\[
\sum_{r,s=1}^{n} h_{rs}^{j} \frac{\partial}{\partial z_{r}} \wedge \frac{\partial}{\partial z_{s}} = \sum_{r,s,a,b=1}^{n} h_{rs}^{j} \frac{\partial \xi_{j}^{a}(z,0)}{\partial \xi_{j}} \frac{\partial \xi_{j}^{b}(z,0)}{\partial \xi_{j}} \frac{\partial}{\partial \xi_{j}} \wedge \frac{\partial}{\partial \xi_{j}}
\]
Let’s compute the third term of (4.4.5):
\[
\sum_{r,s,c=1}^{n} \left[ g_{r,s}^{j}(\xi_j,0) \frac{\partial}{\partial \xi_{j}} \wedge \frac{\partial}{\partial \xi_{j}} \right] = \sum_{r,s,c=1}^{n} \left[ g_{r,s}^{j}(\xi_j,0) \frac{\partial}{\partial \xi_{j}} \wedge \frac{\partial}{\partial \xi_{j}} - g_{r,s}^{j}(\xi_j,0) \frac{\partial}{\partial \xi_{j}} \wedge \frac{\partial}{\partial \xi_{j}} \right]
\]
By considering the coefficients of \( \frac{\partial}{\partial \xi_{j}} \wedge \frac{\partial}{\partial \xi_{j}} \), (4.4.5) is equivalent to
\[
(4.4.6) \quad \sum_{r,s=1}^{n} h_{rs}^{j} \frac{\partial \xi_{j}^{a}(z,0)}{\partial \xi_{j}} \frac{\partial \xi_{j}^{b}(z,0)}{\partial \xi_{j}} \frac{\partial}{\partial \xi_{j}} - \sum_{c=1}^{n} \xi_j^{c} \frac{\partial g_{ab}(\xi_j,0)}{\partial \xi_j} + \sum_{c=1}^{n} (g_{c}^{j}(\xi_j,0) \frac{\partial \xi_{j}^{a}}{\partial \xi_{j}} + g_{ac}(\xi_j,0) \frac{\partial \xi_{j}^{b}}{\partial \xi_{j}}) = 0
\]
On the other hand, from (4.3.1), we have
\[
(4.4.7) \quad g_{ab}^{j}(\xi_j(z,t),\ldots,\xi_j(z,t),t_1,\ldots,t_m) = \sum_{r,s=1}^{n} h_{rs}(z,t) \frac{\partial \xi_{j}^{a}(z,t)}{\partial z_{r}} \frac{\partial \xi_{j}^{b}(z,t)}{\partial z_{s}}
\]
By taking the derivative of (4.4.7) with respect to \( t \) and putting \( t = 0 \), we have
\[
\sum_{c=1}^{n} \frac{\partial g_{ab}^{j}(\xi_j,0)}{\partial \xi_j} \xi_j^{c} + \sum_{c=1}^{n} \xi_j^{c} \frac{\partial g_{ab}(\xi_j,0)}{\partial \xi_j} + \sum_{c=1}^{n} (g_{c}^{j}(\xi_j,0) \frac{\partial \xi_{j}^{a}}{\partial \xi_{j}} + g_{ac}(\xi_j,0) \frac{\partial \xi_{j}^{b}}{\partial \xi_{j}}) = 0
\]
Hence (4.4.6) is equivalent to
\[
(4.4.8) \quad \sum_{c=1}^{n} g_{ab}^{j}(\xi_j,0) \frac{\partial \xi_{j}^{a}}{\partial \xi_j} + g_{ac}(\xi_j,0) \frac{\partial \xi_{j}^{b}}{\partial \xi_j} = \sum_{r,s=1}^{n} (h_{rs}(z,0) \frac{\partial \xi_{j}^{a}}{\partial z_{r}} \frac{\partial \xi_{j}^{b}}{\partial z_{s}} + h_{rs}(z,0) \frac{\partial \xi_{j}^{a}}{\partial z_{r}} \frac{\partial \xi_{j}^{b}}{\partial z_{s}})
\]
Indeed, the left hand side and right hand side of (4.4.8) coincide: from (4.4.7) and (4.4.3),
\[
\sum_{c=1}^{n} g_{ab}^{j}(\xi_j,0) \frac{\partial \xi_{j}^{a}}{\partial \xi_j} + g_{ac}(\xi_j,0) \frac{\partial \xi_{j}^{b}}{\partial \xi_j} = \sum_{r,s=1}^{n} (h_{rs}(z,0) \frac{\partial \xi_{j}^{a}}{\partial z_{r}} \frac{\partial \xi_{j}^{b}}{\partial z_{s}} + h_{rs}(z,0) \frac{\partial \xi_{j}^{a}}{\partial z_{r}} \frac{\partial \xi_{j}^{b}}{\partial z_{s}})
\]
This completes Lemma 4.4.4

Going back to the proof of Theorem 4.4.2, we defined the isomorphism (4.4.1): \( (b,a) \mapsto [(\Lambda_0,c) - b, c] \) where \( -\delta c = a \). We take \((b,a) = (\{\sigma_j\}, \{\delta_j\})\) and \( c = \{\xi_j\} \). Note \( -\delta \{\xi_j\} = \{\delta_j\} \) by (4.4.3). Then by the isomorphism (4.4.1), \( \{\sigma_j\}, \{\delta_j\} \) is mapped to \( [(\Lambda_0, \{\xi_j\}) - \{\sigma_j\}, \delta \{\xi_j\}] \) which is \( (-\Lambda, \psi) \) by Lemma 4.4.4 and (4.4.3). This completes the proof of Theorem 4.4.2
4.5. Integrability condition.

We have showed that given a Poisson analytic family \((\mathcal{M}, \Lambda, B, \omega)\), the deformations \((M_t, \Lambda_t)\) of \(M = M_0\) near \((M_0, \Lambda_0)\) is represented by the \(C^\infty\) vector \((0,1)\)-form \(\varphi(t)\) \((4.2.3)\) and the \(C^\infty\) bivector field \(\Lambda(t)\) of type \((2,0)\) \((4.3.3)\) on \(M\) with \(\varphi(0) = 0\) and \(\Lambda(0) = \Lambda_0\) satisfying the conditions:

\[
(1) |\Lambda(t), \Lambda(t)| = 0, (2) \partial \Lambda(t) - [\Lambda(t), \varphi(t)] = 0 \quad \text{and} \quad (3) \partial \varphi(t) - \frac{1}{2}[\varphi(t), \varphi(t)] = 0 \quad \text{for each } t \in \Delta \text{ by Theorem } 4.3.12 (3), (4.3.22) \text{ and } (4.2.4).
\]

Conversely, we will show that on a compact holomorphic Poisson manifold \((M, \Lambda_0)\), a \(C^\infty\) vector \((0,1)\)-form \(\varphi \in \mathfrak{A}^{0,1}(M, T_M)\) and a \(C^\infty\) type \((2,0)\) bivector field \(\Lambda \in \mathfrak{A}^{0,0}(M, \wedge^2 T_M)\) on \(M\) such that \(\varphi\) and \(\Lambda_0 + \Lambda\) satisfying the integrability condition (1),(2),(3) define another holomorphic Poisson structure on the underlying differentiable manifold \(M\). Indeed, let \(\varphi = \sum_{\alpha=1}^n \varphi^\alpha(z) \partial z^\alpha\) be a \(C^\infty\) vector \((0,1)\)-form and \(\Lambda = \sum_{\alpha,\beta=1}^n h_{\alpha\beta}(z) \partial z^\alpha \wedge \partial z^\beta\) be a \(C^\infty\) bivector field of type \((2,0)\) on a compact holomorphic Poisson manifold \((M, \Lambda_0)\). Suppose det\((\delta_\varphi - \sum_{\mu=1}^n \varphi^\mu(z) \partial \varphi^\mu(z))\lambda,\mu=1,\ldots,n \neq 0\), and \(\varphi, \Lambda\) satisfy the integrability condition:

\[
\begin{align*}
(4.5.1) & \quad [\Lambda_0 + \Lambda, \Lambda_0 + \Lambda] = 0 \\
(4.5.2) & \quad \partial(\Lambda_0 + \Lambda) - [\Lambda_0 + \Lambda, \varphi] = 0 \\
(4.5.3) & \quad \partial \varphi - \frac{1}{2}[\varphi, \varphi] = 0
\end{align*}
\]

Then by the Newlander-Nirenberg theorem(\cite{NN57, Kod05}), the condition \((4.5.3)\) gives a finite open covering \(\{U_j\}\) of \(M\) and \(C^\infty\)-functions \(\xi^\alpha_j = \xi^\alpha_j(z), \alpha = 1, \ldots, n\) on each \(U_j\) such that \(\xi_j : z \mapsto \xi_j(z) = (\xi^1_j(z), \ldots, \xi^n_j(z))\) gives complex coordinates on \(U_j\), and \(\{\xi_1, \ldots, \xi_j, \ldots\}\) defines another complex structure on \(M\), which we denote by \(M_\psi\). By Theorem \(4.3.12\) the conditions \((4.5.1)\) and \((4.5.2)\) gives a holomorphic Poisson structure \((\Lambda_0 + \Lambda)^2.0\) on \(M_\psi\). Recall that \((\Lambda_0 + \Lambda)^2.0\) means the type \((2,0)\)-part of \(\Lambda_0 + \Lambda\) with respect to the complex structure induced by \(\varphi\) (see Remark \(4.3.8\)).

**Remark 4.5.4.** If we replace \(\varphi\) by \(-\varphi\), then \((4.5.1), (4.5.2), \text{ and } (4.5.3)\) are equivalent to

\[
(4.5.5) \quad L(\Lambda + \varphi) + \frac{1}{2}[\Lambda + \varphi, \Lambda + \varphi] = 0 \quad \text{where} \quad L = \partial + [\Lambda_0, -]
\]

which is a solution of the Maurer-Cartan equation of a differential graded Lie algebra \((\mathfrak{g} = \bigoplus_{i \geq 0} g^i = \bigoplus_{p+q-1=i, q \geq 1} A^{p,q}(M, \wedge^q T_M), L, [-,-]\). This differential graded Lie algebra controls deformations of compact holomorphic Poisson manifolds in the language of functor of Artin rings (see the second part of the author’s Ph.D. thesis \cite{Kim14}).

**Example 4.5.6** (Hitchin-Goto Poisson analytic family). Let \((M, \sigma)\) be a compact holomorphic Poisson manifold which satisfies the \(\partial \bar{\partial}\)-lemma. Then any class \(\sigma([\omega]) \in H^1(M, \Theta_M^g)\) for \([\omega] \in H^1(M, \Theta_M^g)\) is tangent to a deformation of complex structure induced by \(\phi(t) = \sigma(\alpha)\) where \(\alpha = t \omega + \partial(\beta_2 + t^3 \beta_3 + \cdots)\) for \((0,1)\)-forms \(\beta_i\) with respect to the original complex structure (see \cite{Hit12} Theorem 1). Suppose that \(\phi(t) = \sigma(\alpha)\) converges for \(t \in \Delta \subset \mathbb{C}\). We can consider \(\psi = \psi(t) := -\phi(t)\) as a \(C^\infty\) vector \((0,1)\)-form on \(M \times \Delta\), and \(\sigma\) as a \(C^\infty\) type \((2,0)\) bivector on \(M \times \Delta\). We note that \((\psi(t), \sigma)\) satisfies \((\sigma, \sigma) = 0, \partial \sigma - [\sigma, \psi(t)] = 0 \quad \text{and} \quad \partial \psi(t) - \frac{1}{2}[\psi(t), \psi(t)] = 0\). Then by Newlander-Nirenberg theorem(\cite{NN57, Kod05} p.268), we can give a holomorphic coordinate on \(M \times \Delta\) induced by \(\psi\). Let’s denote the complex manifold induced by \(\psi\) by \(M\). On the other hand, the type \((2,0)\) part \(\sigma^{2.0}\) of \(\sigma\) with respect to the complex structure \(M\) defines a holomorphic Poisson structure on \(M\). Then the natural projection \(\pi : (M, \sigma^{2.0}) \rightarrow \Delta\) defines a Poisson analytic family of deformations of \((M, \sigma)\). Since \(\sigma\) does not depend on \(t\), we have \(0\) in the Poisson direction under the Poisson Kodaira-Spencer map \(\varphi_0 : T_0 \Delta \rightarrow H^2(M, \sigma)\) by Theorem \(4.4.2\). More precisely, we have \(\varphi_0(\frac{\partial}{\partial t}) = (0, -\sigma([\omega]))\).

5. THEOREM OF EXISTENCE FOR HOLOMORPHIC POISSON STRUCTURES

In this section, we prove ‘Theorem of existence for holomorphic Poisson structures’ as an analogue of ‘Theorem of existence for complex analytic structures’ by Kodaira-Spencer under the assumption that the associated Laplacian operator \(\Box\) (induced from the operator \(\partial + [\Lambda_0, -]\)) is strongly elliptic and of diagonal type.
5.1. Statement of Theorem of existence for holomorphic Poisson structures.

**Theorem 5.1.1** (Theorem of existence for holomorphic Poisson structures). Let \((M, \Lambda_0)\) be a compact holomorphic Poisson manifold such that the associated Laplacian operator \(\Box\) (induced from the operator \(\overline{\partial} + [\Lambda_0, -]\)) is strongly elliptic and of diagonal type. Suppose that \(\text{HP}^3(M, \Lambda_0) = 0\). Then there exists a Poisson analytic family \((M, \Lambda, B, \omega)\) with \(0 \in B \subset \mathbb{C}^n\) satisfying the following conditions:

1. \(\omega^{-1}(0) = (M, \Lambda_0)\)
2. The Poisson Kodaira-Spencer map \(\varphi_0 : \frac{\partial}{\partial t} \to \left(\frac{\partial(M_t, \Lambda_t)}{\partial t}\right)\)
   such that \((M_t, \Lambda_t) = \omega^{-1}(t)\) is an isomorphism of \(T_0(B)\) onto \(\text{HP}^2(M, \Lambda_0) : T_0B \xrightarrow{\varphi_0} \text{HP}^2(M, \Lambda_0)\).

Let \(\{(\pi_1, \eta_1), \ldots, (\pi_m, \eta_m)\}\) be a basis of \(\text{HP}^2(M, \Lambda_0)\) where \((\pi_\lambda, \eta_\lambda) \in A^{0,0}(M, \wedge^2 T_M) \oplus A^{0,1}(M, T_M)\) for \(\lambda = 1, \ldots, m\). Let \(\Delta_\epsilon = \{t \in \mathbb{C}^m\|t\| < \epsilon\}\) for some \(\epsilon > 0\). Assume that there is a family \(\{(\varphi(t), \Lambda(t))|t \in \Delta_\epsilon\}\) of \(C^\infty\) vector \((0,1)\)-forms \(\varphi(t) = \sum_{\lambda=1}^n \sum_{\mu=1}^n \varphi_\mu^\lambda(z,t) \frac{\partial}{\partial z_\mu} \in A^{0,1}(M, T_M)\) and \(C^\infty\) type \((2,0)\) bivectors \(\Lambda(t) = \sum_{\alpha, \beta=1}^n \Lambda_{\alpha\beta}(z,t) \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_\beta} \in A^{0,0}(M, \wedge^2 T_M)\) on \(M\), which satisfy

1. \([\Lambda(t), \Lambda(t)] = 0\)
2. \(\overline{\partial}\Lambda(t) - [\Lambda(t), \varphi(t)] = 0\)
3. and the initial conditions \(\varphi(0) = 0, \Lambda(0) = \Lambda_0, (\frac{\partial \Lambda(t)}{\partial \lambda})_{t=0}, (\frac{\partial \varphi(t)}{\partial \lambda})_{t=0}\) = \((\pi_\lambda, \eta_\lambda), \lambda = 1, \ldots, m,\)

Since \(\varphi(0) = 0\), we may assume \(\det(\delta^\lambda_v - \sum_{\mu=1}^n \varphi_\mu^\lambda(z,t) \varphi_\mu^\lambda(z,t))_{\lambda, \mu = 1, \ldots, n} \neq 0\) if \(\Delta_\epsilon\) is sufficiently small. Therefore, as in subsection 4.5 by the Newlander-Nirenberg theorem (NN57, [Kod55] p.268), each \(\varphi(t)\) determines a complex structure \(M_{\varphi(t)}\) on \(M\). The conditions (2) and (3) imply that \((2,0)\)-part \(\Lambda(t)^{2,0}\) of \(\Lambda(t)\) with respect to the complex structure induced from \(\varphi(t)\) is a holomorphic Poisson structure on \(M_{\varphi(t)}\). If the family \(\{(M_{\varphi(t)}, \Lambda(t)^{2,0})|t \in \Delta_\epsilon\}\) is a Poisson analytic family, it satisfies the conditions (1) and (2) in Theorem 5.1.1 by Theorem 4.4.2. We will construct such a family \(\{(\varphi(t), \Lambda(t))|t \in \Delta_\epsilon\}\) under the assumption \(\text{HP}^3(M, \Lambda_0) = 0\) and then show that \(\{(M_{\varphi(t)}, \Lambda(t)^{2,0})|t \in \Delta_\epsilon\}\) is a Poisson analytic family in the subsection 5.3 which completes the proof of Theorem 5.1.1.

**Remark 5.1.2.** By replacing \(\varphi(t)\) by \(-\varphi(t)\), it is sufficient to construct \(\varphi(t)\) and \(\Lambda(t)\) satisfying

1. \([\Lambda(t), \Lambda(t)] = 0\)
2. \(\overline{\partial}\Lambda(t) + [\Lambda(t), \varphi(t)] = 0\)
3. and the initial conditions \(\varphi(0) = 0, \Lambda(0) = \Lambda_0, (\frac{\partial \Lambda(t)}{\partial \lambda})_{t=0}, (\frac{\partial \varphi(t)}{\partial \lambda})_{t=0}\) = \((\pi_\lambda, \eta_\lambda), \lambda = 1, \ldots, m,\)

We note that (1), (2), (3) are equivalent to

(5.1.3) \(\overline{\partial}(\varphi(t) + \Lambda(t)) + \frac{1}{2}[\varphi(t) + \Lambda(t), \varphi(t) + \Lambda(t)] = 0\)

We construct such \(\alpha(t) := \varphi(t) + \Lambda(t)\) in the following subsection.

5.2. Construction of \(\alpha(t) = \varphi(t) + \Lambda(t)\).

We use the Kuranishi’s method presented in [MK06] to construct \(\alpha(t)\). First we note the following: let \(A^p = A^{0,p-1}(M, T_M) \oplus \cdots \oplus A^{0,0}(M, \wedge^p T_M)\) and \(L = \overline{\partial} + [\Lambda_0, -]\). Then the sequence

\[0 \to A^1 \xrightarrow{L} \cdots \xrightarrow{L} A^n \xrightarrow{L} A^{n+1} \to 0\]
is an elliptic complex. So we have the adjoint operator $L^*$, Green's operator $G$, Laplacian operator
\( \Box := LL^* + L^*L \) and $H$ where $H$ is the orthogonal projection to the $\Box$-harmonic subspace $\mathbb{H}$ of $\mathbb{H}$. In particular we have $H : A^p \to \mathbb{H}^p \cong H^p(M, \Lambda_0)$. For the detail, we refer to [We08].

We introduce the Hölder norms in the spaces $A^p = A^{0,p-1}(M, T_M) \oplus \cdots \oplus A^{0,0}(M, \Lambda^p T_M)$ in the following way: we fix a finite open covering \( \{ U_j \} \) of $M$ such that $z_j = (z_j^1, ..., z_j^n)$ are local coordinates on $U_j$. Let $\phi \in A^p$ which is locally expressed on $U_j$ as
\[
\phi = \sum_{r+s=p, a \geq 1} \phi_{j, a_1, ..., a_r, b_1, ..., b_s} (z) dz_{j}^{a_1} \wedge \cdots \wedge dz_{j}^{a_r} \wedge \frac{\partial}{\partial z_{j}^{b_1}} \wedge \cdots \wedge \frac{\partial}{\partial z_{j}^{b_s}}
\]
Let $k \in \mathbb{Z}, k \geq 0, \theta \in \mathbb{R}, 0 < \theta < 1$. Let $h = (h_1, ..., h_{2n}), h_i \geq 0, |h| := \sum_{i=1}^{2n} h_i$ where $n = \dim M$. Then denote
\[
D_h^\theta = \left( \frac{\partial}{\partial x_{j}^{a_1}} \right)^{h_1} \cdots \left( \frac{\partial}{\partial x_{j}^{a_{2n}}} \right)^{h_{2n}}, \quad z_j^{a_1} = x_j^{2\alpha - 1} + ix_j^{2\beta}
\]
Then the Hölder norm $||\phi||_{k+\theta}$ is defined as follows:
\[
||\phi||_{k+\theta} = \max_j \left( \frac{\sup_{t, |h| \leq k} |D_h^\theta \phi_{j, a_1, ..., a_s, b_1, ..., b_s} (z)|}{\sup_{y, z \in U_j, |h| = k} \left| \frac{D_h^\theta \phi_{j, a_1, ..., a_s, b_1, ..., b_s} (y) - D_h^\theta \phi_{j, a_1, ..., a_s, b_1, ..., b_s} (z)}{|y - z|^{\theta}} \right|} \right),
\]
where the sup is over all $\alpha_1, ..., \alpha_r, \beta_1, ..., \beta_s$.

Now suppose that the associated Laplacian operator $\Box$ induced from $L = \bar{\partial} + [\Lambda_0, -]$ is a strongly elliptic operator whose principal part is of diagonal type. Then by [Kod05] Appendix Theorem 4.3 page 436, we have a priori estimate
\[
||\phi||_{k+\theta} \leq C(\||\Box \phi||_{k-2+\theta} + ||\phi||_0)
\]
where $k \geq 2$, $C$ is a constant which is independent of $\varphi$ and
\[
||\phi||_0 = \max_{j, a_1, ..., a_s} \sup_{z \in U_j} |\phi_{j, a_1, ..., a_s, b_1, ..., b_s} (z)|.
\]

We will use the following two lemmas.

**Lemma 5.2.2.** For $\phi, \psi \in A^2$, we have $||[\phi, \psi]||_{k+\theta} \leq C||\phi||_{k+1+\theta}||\psi||_{k+1+\theta}$, where $C$ is independent of $\phi$ and $\psi$.

**Lemma 5.2.3.** For $\phi \in A^2$, we have $||G\phi||_{k+\theta} \leq C||\phi||_{k-2+\theta}, k \geq 2$, where $C$ depends only on $k$ and $\theta$, not on $\phi$.

**Proof.** This follows from (5.2.1). See [MK06] p.160 Proposition 2.3. \( \square \)

With this preparation, we construct $\alpha(t) := \varphi(t) + \Lambda(t) = \Lambda_0 + \sum_{\mu=1}^{\infty} (\varphi_{\mu}(t) + \Lambda_{\mu}(t))$, where
\[
\varphi_{\mu}(t) + \Lambda_{\mu}(t) = \sum_{v_1 + \cdots + v_m = \mu} (\varphi_{v_1, \cdots, v_m} + \Lambda_{v_1, \cdots, v_m}) t_1^{v_1} \cdots t_m^{v_m}
\]
with $\varphi_{v_1, \cdots, v_m} + \Lambda_{v_1, \cdots, v_m} \in A^{0,1}(M, T_M) \oplus A^{0,0}(M, \Lambda^2 T_M)$ such that
\[
\bar{\partial} \alpha(t) + \frac{1}{2} (\alpha(t), \alpha(t)) = 0, \quad \alpha_1(t) = \varphi_1(t) + \Lambda_1(t) = \sum_{v \in \Lambda_0} \eta_v t_v + \pi_v t_v,
\]
where $\{\eta_v + \pi_v\}$ is a basis for $\mathbb{H}$. Let $\beta(t) := \alpha(t) - \Lambda_0 = \sum_{\mu=1}^{\infty} (\varphi_{\mu}(t) + \Lambda_{\mu}(t))$. Then (5.2.4) is equivalent to
\[
L \beta(t) + \frac{1}{2} [\beta(t), \beta(t)] = 0, \quad \beta_1(t) = \alpha_1(t)
\]
Construing $\alpha(t)$ is equivalent to constructing $\beta(t)$. We will construct $\beta(t)$ satisfying (5.2.5). Consider the equation
\[
\beta(t) = \beta_1(t) - \frac{1}{2} L^* G[\beta(t), \beta(t)],
\]
where $\beta_1(t) = \alpha_1(t)$. Then (5.2.6) has a unique formal power series solution $\beta(t) = \sum_{\mu=1}^{\infty} \beta_{\mu}(t)$, and there exists a $\epsilon > 0$ such that for $t \in \Delta_\epsilon = \{ t \in \mathbb{C}^n \mid |t| < \epsilon \}$, $\beta(t) = \sum_{\mu=1}^{\infty} \beta_{\mu}(t)$ converges in
the norm $|| \cdot ||_{k+\theta}$ (for the detail, see [MK06] p.162 Proposition 2.4. By virtue of the integrability condition [5.2.5], we can formally apply their argument.

**Proposition 5.2.7.** $\beta(t)$ satisfies $L \beta(t) + \frac{1}{2} [\beta(t), \beta(t)] = 0$ if and only if $H[\beta(t), \beta(t)] = 0$, where $H : A^3 = A^{0,2}(M, T_M) \oplus A^{0,1}(M, \Lambda^2 T_M) \oplus A^{0,0}(M, \Lambda^3 T_M) \to \mathbb{H}^3 \cong HP^3(M, \Lambda_0)$ is the orthogonal projection to the harmonic subspace of $A^3$.

**Proof.** We simply note that $(\bigoplus_{i \geq 0} g_{\alpha}, g_{\alpha} = \bigoplus_{p+q-1=i, p \geq 0, q \geq 1} A^{0,p}(M, \wedge^q T_M), L = \partial + [\Lambda_0, -], [-, -])$ is a differential graded Lie algebra and so the argument in the proof of [MK06] p.163 Proposition 2.5 is formally applied to our case by Lemma [5.2.2] and Lemma [5.2.3].

Now suppose that $HP^3(M, \Lambda_0) = 0$. Then by Proposition [5.2.7], $\beta(t)$ satisfies (5.2.5) for $t \in \Delta_\epsilon$. Hence $\alpha(t) = \beta(t) + \Lambda_0 = \varphi(t) + \Lambda(t)$ is the desired one satisfying (5.2.3). We note that $\alpha(t)$ has the following property which we need in the construction of a Poisson analytic family in the next subsection.

**Proposition 5.2.8.** $\alpha(t) = \beta(t) + \Lambda_0 = \varphi(t) + \Lambda(t)$ is $C^\infty$ in $(z, t)$ and holomorphic in $t$.

**Proof.** We note that $\square$ is a strongly elliptic differential operator whose principal part is of diagonal type by our assumption. So we can formally apply the argument of [MK06] p.163 Proposition 2.6. See also [Kod05] Appendix p.452 §8.

5.3. Construction of a Poisson analytic family.

In the previous subsection, we have constructed a family $\{(\varphi(t), \Lambda(t)) | t \in \Delta_\epsilon\}$ of $C^\infty$ vector $(0, 1)$-forms $\varphi(t)$ and $C^\infty$ type $(2, 0)$ bivectors $\Lambda(t)$

$$\varphi(t) = \sum_{\lambda=1}^{n} \sum_{v=1}^{n} \varphi^\lambda_v(z, t) d\bar{z}_v \frac{\partial}{\partial z_\lambda}, \quad \Lambda(t) = \sum_{\alpha, \beta=1}^{n} \Lambda_{\alpha \beta}(z, t) \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_\beta}$$

satisfying the integrability condition $[\Lambda(t), \Lambda(t)] = 0, \bar{\partial} \Lambda(t) = [\Lambda(t), \varphi(t)], \bar{\partial} \varphi(t) = \frac{1}{2} [\varphi(t), \varphi(t)]$ and the initial conditions $\varphi(0) = 0, \Lambda(0) = \Lambda_0(\{- \frac{\partial \Lambda(t)}{\partial t_0}\}_{t=0}^{t_0} = \{- \pi_\lambda, - \eta_\lambda\}, \lambda = 1, ..., m$, where $\varphi^\lambda_v(z, t)$ and $\Lambda_{\alpha \beta}(z, t)$ are $C^\infty$ functions of $z_1, ..., z^n, t_1, ..., t_m$ and holomorphic in $t_1, ..., t_m$.

$(\varphi(t), \Lambda(t))$ determines a Poisson Poisson structure $\mathcal{P}_F(t), \Lambda(t)^{2,0})$ on $M$ for each $t \in \Delta_\epsilon$. In order to show that $\{(\mathcal{P}_F(t), \Lambda(t)^{2,0}) | t \in \Delta_\epsilon\}$ is a Poisson analytic family, we consider $\varphi := \varphi(t)$ as a vector $(0, 1)$-form on the complex manifold $M \times \Delta_\epsilon$, and $\Lambda := \Lambda(t)$ as a $(2, 0)$ bivector on $M \times \Delta_\epsilon$.

Then since $\varphi^\lambda_v = \varphi^\lambda_v(z, t)$ are holomorphic in $t_1, ..., t_m$ (Proposition 5.2.8), we have $\frac{\partial \varphi^\lambda_v}{\partial t_\mu} = 0$ in

$$\bar{\partial} \varphi = \sum_{\lambda, \mu=1}^{n} \left( \sum_{\beta=1}^{n} \frac{\partial \varphi^\lambda_v}{\partial z_\beta} \bar{d}z_\beta + \sum_{\mu=1}^{m} \frac{\partial \varphi^\lambda_v}{\partial t_\mu} \bar{d}t_\mu \right) \wedge \frac{\partial}{\partial z_\lambda}$$

Similarly since $\Lambda_{\alpha \beta}(z, t)$ is holomorphic in $t_1, ..., t_m$ (Proposition 5.2.8), we have $\frac{\partial \Lambda_{\alpha \beta}}{\partial t_\mu} = 0$ in

$$\bar{\partial} \Lambda = \sum_{\alpha, \beta} \left( \sum_{\lambda=1}^{n} \frac{\partial \Lambda_{\alpha \beta}}{\partial z_\lambda} \bar{d}z_\lambda + \sum_{\mu=1}^{m} \frac{\partial \Lambda_{\alpha \beta}}{\partial t_\mu} \bar{d}t_\mu \right) \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_\beta}$$

Hence $\varphi$ and $\Lambda$ satisfy $\bar{\partial} \varphi = \frac{1}{2} [\varphi, \varphi], \bar{\partial} \Lambda = [\Lambda, \varphi], \text{ and } [\Lambda, \Lambda] = 0$. Then by the Newlander-Nirenberg theorem ([NN57], [Kod05] p.268), $\varphi$ defines a complex structure $\mathcal{M}$ on $M \times \Delta_\epsilon$ and $(2, 0)$-part $\Lambda^{2,0}$ of $\Lambda$ defines a holomorphic Poisson structure on $\mathcal{M}$. Let $\omega : \mathcal{M} \to \Delta_\epsilon$ be the natural projection. Then $\{(\mathcal{P}_F(t), \Lambda(t)^{2,0}) | t \in \Delta_\epsilon\}$ forms a Poisson analytic family $(\mathcal{M}, \Lambda^{2,0}, \Delta_\epsilon, \omega)$ (for the detail, see [Kod05] p.282). This completes the proof of Theorem 5.1.1.

6. Theorem of completeness for compact holomorphic Poisson structures

6.1. Statement of Theorem of completeness for holomorphic Poisson structures.
6.1.1. Change of parameters. (compare [Kod05] p.205)
Consider a Poisson analytic family $\{(M_t,\Lambda_t)|(M_t,\Lambda_t) = \omega^{-1}(t), t \in B\} = (M,\Lambda,B,\omega)$ of compact holomorphic Poisson manifolds, where $B$ is a domain of $\mathbb{C}^m$. Let $D$ be a domain of $\mathbb{C}^r$ and $h : s \to t = h(s)$, $s \in D$, a holomorphic map of $D$ into $B$. Then by changing the parameter from $t$ to $s$, we will construct a Poisson analytic family $\{(M_{h(t)},\Lambda_{h(t)})|s \in D\}$ on the parameter space $D$ in the following.

Let $M \times_B D := \{(p,s) \in M \times B|\omega(p) = h(s)\}$. Then we have the following commutative diagram

\[
\begin{array}{ccc}
M \times_B D & \xrightarrow{\pi} & M \\
\downarrow & & \downarrow \\
D & \xrightarrow{h} & B \\
\end{array}
\]

such that $(M \times_B D, D, \pi)$ is a complex analytic family in the sense of Kodaira-Spencer and $\pi^{-1}(s) = M_{h(s)}$. We show that $(M \times_B D, D, \pi)$ is naturally a Poisson analytic family such that $\pi^{-1}(s) = (M_{h(s)},\Lambda_{h(s)})$ and $p$ is a Poisson map. Note that the bivector field $\Lambda$ on $M$ can be considered as a bivector field on $M \times D$ which gives a holomorphic Poisson structure on $M \times D$. So $(M \times D,\Lambda)$ is a holomorphic Poisson manifold. We show that $M \times_B D$ is a holomorphic Poisson submanifold of $(M \times D,\Lambda)$ and defines a Poisson analytic family. Let $(p_0,s_0) \in M \times_B D$. Taking a sufficiently small coordinate polydisk $\Delta$ with $h(s_0) \in \Delta$, we represent $(M_{\Delta},\Lambda_{\Delta}) = \omega^{-1}(\Delta)$ in the form of

\[
(M_{\Delta},\Lambda_{\Delta}) = \left( \bigcup_{j=1}^l U_j \times \Delta, \sum_{\alpha,\beta=1}^n g_{\alpha\beta}(z_j,t) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta} \right)
\]

where each $U_j$ is a polydisk independent of $t$, and $(z_j,t) \in U_j \times \Delta$ and $(z_k,t) \in U_k \times \Delta$ are the same point on $M_\Delta$ if $z_j = f_{jk}(z_k,t)$, $t = \alpha = 1,\ldots,n$. Let $E$ be a sufficiently small polydisk of $D$ such that $s_0 \in E$ and $h(E) \subset \Delta$. Then we can represent $(M \times D,\Lambda)$ around $(p_0,s_0)$ in the form of

\[
(M_\Delta \times E, \Lambda|_{M_\Delta \times E}) = \left( \bigcup_{j=1}^l U_j \times \Delta \times E, \sum_{\alpha,\beta=1}^n g_{\alpha\beta}(z_j,t) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta} \right)
\]

where $(z_j,t,s) \in U_j \times \Delta \times E$ and $(z_k,t,s) \in U_k \times \Delta \times E$ are the same point on $M_\Delta \times E$ if $z_j = f_{jk}(z_k,t)$. Then we can represent $M \times_B D$ around $(p_0,s_0)$ in the form of $\bigcup_{j=1}^l U_j \times G_E$, where $G_E = \{(h(s),s)|s \in E\} \subset \Delta \times E$, and $(z_j,h(s),s) \in U_j \times G_E$ and $(z_k,h(s),s) \in U_k \times G_E$ are the same point if $z_j = f_{jk}(z_k,h(s))$. We note that at $(p_0,s_0) \in M \times_B D \subset M \times D$, we have $\Lambda_{(p_0,s_0)} = \sum_{\alpha,\beta=1}^n \frac{\partial}{\partial z_j^\alpha} (p_0,h(s_0)) \frac{\partial}{\partial z_j^\beta} \Lambda_{p_0} \wedge \frac{\partial}{\partial z_j^\beta} |_{p_0} \in \Lambda^2 T_{M \times_B D}$. Hence $M \times_B D$ is a holomorphic Poisson submanifold of $(M \times D,\Lambda)$, and $p : (M \times_B D,\Lambda|_{M \times_B D}) \to (M,\Lambda)$ is a Poisson map.

Since $G_E$ is biholomorphic to $E$. The holomorphic Poisson manifold $(M \times_B D,\Lambda|_{M \times_B D})$ is represented locally by the form

\[
\left( \bigcup_{j=1}^l U_j \times E, \sum_{\alpha,\beta=1}^n g_{\alpha\beta}(z_j,h(s)) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta} \right)
\]

where $(z_k,s) \in U_k \times E$ and $(z_j,s) \in U_j \times E$ are the same point if $z_j = f_{jk}(z_k,h(s))$, which shows that $(M \times_B D,\Lambda|_{M \times_B D},\pi)$ is a Poisson analytic family and $\pi^{-1}(s) = (M_{h(s)},\Lambda_{h(s)})$.

**Definition 6.1.1.** The Poisson analytic family $(M \times_B D,\Lambda|_{M \times_B D})$ is called the Poisson analytic family induced from $(M,B,\Lambda,\omega)$ by the holomorphic map $h : D \to B$.

We point out that change of variable formula holds for infinitesimal Poisson deformations as in infinitesimal deformations of complex structures ([Kod05] Theorem 4.7 p.207).

**Theorem 6.1.2.** For any tangent vector $\frac{\partial}{\partial s} = c_1 \frac{\partial}{\partial t_1} + \cdots + c_r \frac{\partial}{\partial t_r} \in T_s(D)$, the infinitesimal Poisson deformation of $(M_{h(s)},\Lambda_{h(s)})$ along $\frac{\partial}{\partial s}$ is given by

\[
\frac{\partial(M_{h(s)},\Lambda_{h(s)})}{\partial s} = \left( \sum_{\lambda=1}^m \frac{\partial t_\lambda}{\partial s} \frac{\partial M_\lambda}{\partial t_\lambda}, \sum_{\lambda=1}^m \frac{\partial t_\lambda}{\partial s} \frac{\partial \Lambda_\lambda}{\partial t_\lambda} \right)
\]

With this preparation, we discuss a concept of completeness and ‘Theorem of completeness’ in the context of deformations of compact holomorphic Poisson manifolds in the next subsection.
6.1.2. Statement of ‘Theorem of completeness for holomorphic Poisson structures’

**Definition 6.1.3.** Let \((\mathcal{M}, \Lambda_{\mathcal{M}}, B, \omega)\) be a Poisson analytic family of compact holomorphic Poisson manifolds, and \(\theta \in B\). Then \((\mathcal{M}, \Lambda_{\mathcal{M}}, B, \omega)\) is called complete at \(\theta \in B\) if for any Poisson analytic family \((\mathcal{N}, \Lambda_{\mathcal{N}}, D, \pi)\) such that \(D\) is a domain of \(\mathbb{C}^l\) containing 0 and that \(\pi^{-1}(0) = \omega^{-1}(\theta)\), there is a sufficiently small domain \(\Delta\) with \(0 \in \Delta \subset D\), and a holomorphic map \(h : s \rightarrow t = h(s)\) with \(h(0) = \theta\) such that \((\mathcal{N}_{\Delta}, \Lambda_{\mathcal{N}_{\Delta}}, \Delta, \pi)\) is the Poisson analytic family induced from \((\mathcal{M}, \Lambda_{\mathcal{M}}, B, \omega)\) by \(h\) where \((\mathcal{N}_{\Delta}, \Lambda_{\mathcal{N}_{\Delta}}, \Delta, \pi)\) is the restriction of \((\mathcal{N}, \Lambda_{\mathcal{N}}, D, \pi)\) to \(\Delta\) (see Remark 6.1.9).

We will prove the following theorem which is an analogue of ‘Theorem of completeness’ by Kodaira-Spencer (see Theorem 6.1.7). However, we establish ‘Theorem of completeness for holomorphic Poisson structures’ only for dimension 2.

**Theorem 6.1.4 (Theorem of completeness for holomorphic Poisson structures).** Let \((\mathcal{M}, \Lambda_{\mathcal{M}}, B, \omega)\) be a Poisson analytic family of deformations of a compact holomorphic Poisson surface \((M_0, \Lambda_0) = \omega^{-1}(0), B\) a domain of \(\mathbb{C}^m\) containing 0. If the Poisson Kodaira-Spencer map \(\varphi_0 : T_0(B) \rightarrow H\mathcal{P}^2(M_0, \Lambda_0)\) is surjective, the Poisson analytic family \((\mathcal{M}, \Lambda_{\mathcal{M}}, B, \omega)\) is complete at \(0 \in B\).

**Remark 6.1.5.** Even though we prove Theorem 6.1.4 only for dimension 2, we present the proof for \(n\) dimensional compact holomorphic Poisson manifold in general. We use the assumption of \(\dim_{\mathbb{C}} M_0 = 2\) only in the inductive step in the Poisson direction (see the equation (6.3.27)). The author believes that this inductive step holds for general compact holomorphic Poisson manifolds and thus ‘Theorem of completeness’ holds for any dimension \(n\). However, the author could not prove this inductive step (6.3.27) for dimension \(> 2\).

**Remark 6.1.6.** In order to prove Theorem 6.1.4 as in [Kod05] Lemma 6.1 p.284, it suffices to show that for any given Poisson analytic family \((\mathcal{N}, \Lambda_{\mathcal{N}}, D, \pi)\) with \(\pi^{-1}(0) = (M_0, \Lambda_0)\), if we take a sufficiently small domain \(\Delta\) with \(0 \in \Delta \subset D\), we can construct a holomorphic map \(h : s \rightarrow t = h(s)\), \(h(0) = 0\), of \(\Delta\) into \(B\), and a Poisson holomorphic map \(g\) of \((\mathcal{N}_{\Delta}, \Lambda_{\mathcal{N}_{\Delta}}) = \pi^{-1}(\Delta)\) into \((\mathcal{M}, \Lambda_{\mathcal{M}})\) satisfying the following condition: \(g\) is a Poisson holomorphic map extending the identity \(g_0 : \pi^{-1}(0) = (M_0, \Lambda_0) \rightarrow (M_0, \Lambda_0)\), and \(g\) maps each \((N_s, \Lambda_{N_s})\) \(= \pi^{-1}(s)\) Poisson biholomorphically onto \((M_{h(s)}, \Lambda_{M_{h(s)}})\). We will construct such \(h\) and \(g\) by extending Kodaira’s elementary method (see [Kod05] Chapter 6).

6.2. Preliminaries.

We extend the argument of [Kod05] p.285-286 (to which we refer for the detail) in the context of a Poisson analytic family. We try to keep notational consistency with [Kod05].

Since the problem is local with respect to \(B\), we may assume that \(B = \{t \in \mathbb{C}^m||t| < 1\}\), and \((\mathcal{M}, \Lambda_{\mathcal{M}}, B, \omega)\) is written in the following form

\[(\mathcal{M}, \Lambda_{\mathcal{M}}) = \bigcup_j (\mathcal{U}_j, \Lambda_{\mathcal{M}_j}), \quad \mathcal{U}_j = \{(\xi_j, t) \in \mathbb{C}^n \times B||\xi_j|| < 1\}\]

where the Poisson structure \(\Lambda_{\mathcal{M}_j}\) is given by \(\Lambda_{\mathcal{M}_j} = \sum_{r,s=1}^{n} \Lambda_{r,s}^r(\xi_j, t)\frac{\partial}{\partial \xi_j^r} \wedge \frac{\partial}{\partial \xi_j^s}\) on \(\mathcal{U}_j\) with \(\Lambda_{r,s}^r(\xi_j, t) = -\Lambda_{s,r}^r(\xi_j, t)\), and \(\omega(\xi_j, t) = t\). For \(\mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset\), \((\xi_j, t)\) and \((\xi_k, t)\) are the same point of \(\mathcal{M}\) if

\[(6.2.1) \quad \xi_j = g_{jk}(\xi_k, t) = (g_{jk}^1(\xi_k, t), ..., g_{jk}^n(\xi_k, t))\]

where \(g_{jk}^\alpha(\xi_k, t), \alpha = 1, ..., n\), are holomorphic functions on \(\mathcal{U}_j \cap \mathcal{U}_k\), and we have the following relations

\[(6.2.2) \quad \Lambda_{r,s}^r_{M_j}(g_{jk}(\xi_k, t), t) = \sum_{p,q=1}^{n} \Lambda_{p,q}^{p,s}_{M_k}(\xi_k, t) \frac{\partial g_{jk}^p}{\partial \xi_k^p} \frac{\partial g_{jk}^q}{\partial \xi_k^q}\]

Similarly we assume that \(D = \{s \in \mathbb{C}^l||s| < 1\}\), and \((\mathcal{N}, \Lambda_{\mathcal{N}}, D, \pi)\) is written in the following form

\[\mathcal{N}, \Lambda_{\mathcal{N}}) = \bigcup_j (\mathcal{W}_j, \Lambda_{\mathcal{N}_j}), \quad \mathcal{W}_j = \{(z_j, s) \in \mathbb{C}^n \times D||z_j|| < 1\}\]
where the Poisson structure $\Lambda_N$ is given by $\Lambda_{N_j} = \sum_{\alpha, \beta=1}^n \Lambda_{N_j}^{\alpha \beta}(z_j, t) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}$ on $W_j$ with $\Lambda_{N_j}^{\alpha \beta}(z_j, t) = -\Lambda_{N_j}^{\beta \alpha}(z_j, t)$, and $\pi(z_j, s) = s$. For $W_j \cap W_k \neq \emptyset, (z_j, s)$ and $(z_k, s)$ are the same point of $N$ if

$$z_j = f_{jk}(z_k, s) = (f_{jk}^1(z_k, s), ..., f_{jk}^n(z_k, s)).$$

and we have

$$\Lambda_{N_j}^{\alpha \beta}(f_{jk}(z_k, s), s) = \sum_{a,b=1}^n \Lambda_{N_k}^{ab}(z_k, s) \frac{\partial f_{jk}^a}{\partial z_k^a} \frac{\partial f_{jk}^b}{\partial z_k^b}.$$ 

Since $(N_0, \Lambda_{N_0}) = \pi^{-1}(0) = (M_0, \Lambda_0) = \omega^{-1}(0) = (M_0, \Lambda_{M_0})$, we may assume $(W_j \cap N_0, \Lambda_{N_0}) = (U_j \cap M_0, \Lambda_{M_0})$ where $\Lambda_{N_0} := \sum_{\alpha, \beta=1}^n \Lambda_{N_j}^{\alpha \beta}(z_j, 0) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}$ and $\Lambda_{M_0} := \sum_{s=1}^n \Lambda_{M_j}^{s}(\xi_j, 0) \frac{\partial}{\partial \xi_j^s} \wedge \frac{\partial}{\partial \xi_j^s}$, and assume that the local coordinates $(\xi_j, 0)$ and $(z_j, 0)$ coincide on $W_j \cap N_0 = U_j \cap M_0$. In other words, if $\xi_j^1 = z_j^1, ..., \xi_j^n = z_j^n$, $(\xi_j, 0)$ and $(z_j, 0)$ are the same point of $W_j \cap N_0 = U_j \cap M_0$, and we have $\Lambda_{N_j}^{\alpha \beta}(z_j, 0) = \Lambda_{M_j}^{\alpha \beta}(\xi_j, 0)$. Putting

$$b_{jk}(\xi_k) := g_{jk}(\xi_k, 0), \quad \Lambda_{M_0}^{\alpha \beta}(\xi_j, 0) := \Lambda_{M_j}^{\alpha \beta}(\xi_j, 0)$$

Then from (6.2.1) and (6.2.3), we have

$$b_{jk}(z_k) = f_{jk}(z_k, 0), \quad \Lambda_{M_0}^{\alpha \beta}(z_j) = \Lambda_{M_j}^{\alpha \beta}(z_j, 0) = \Lambda_{N_j}^{\alpha \beta}(z_j, 0).$$

In conclusion, we have

$$(N_0, \Lambda_{N_0}) = (M_0, \Lambda_0 = \Lambda_0) = \bigcup_j (U_j, \Lambda_{M_0}), \quad U_j = W_j \cap N_0 = U_j \cap M_0,$$

such that $\{z_j\}, z_j = (z_j^1, ..., z_j^n)$, is a system of local complex coordinates of the complex manifold $N_0 = M_0$ with respect to $U_j$, and the Poisson structure is given by $\Lambda_{M_0} = \sum_{\alpha, \beta=1}^n \Lambda_{M_0}^{\alpha \beta}(z_j) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}$ on $U_j$ with $\Lambda_{M_0}^{\alpha \beta}(z_j) = -\Lambda_{M_0}^{\beta \alpha}(z_j)$. The coordinate transformation on $U_j \cap U_k$ is given by $z_j^\alpha = b_{jk}^\alpha(z_k), \alpha = 1, ..., n$, and we have

$$\Lambda_{M_0}^{\alpha \beta}(b_{jk}(z_k)) = \sum_{a,b=1}^n \Lambda_{M_0}^{ab}(z_k) \frac{\partial b_{jk}^a}{\partial z_k^a} \frac{\partial b_{jk}^b}{\partial z_k^b}.$$ 

### 6.3. Construction of Formal Power Series.

As in Remark 6.1.3, we have to define a holomorphic map $h : s \to t = h(s)$ with $h(0) = 0$ of $\Delta = \{s \in D \mid |s| < \epsilon\}$ into $B$ for a sufficiently small $\epsilon > 0$, and to extend the identity $g_0 : (N_0, \Lambda_0) \to (M_0 = N_0, \Lambda_0)$ to a Poisson holomorphic map $g : \pi^{-1}(\Delta) = (N_\Delta, \Lambda_{N_\Delta}) \to (M, \Lambda)$ such that $\omega \circ g = h \circ \pi$.

We begin with constructing formal power series $h(s) = \sum_{i=0}^\infty h_i(s)$ of $s_1, ..., s_l$ where $h_i(s)$ is a homogeneous polynomial of degree $v$ of $s_1, ..., s_l$, and formal power series $g_{jk}(z_j, s) = z_j + \sum_{i=1}^\infty g_{jk}^i(z_j, s)$ in terms of $s_1, ..., s_l$ for each $U_j$ in (6.2.7), whose coefficients are vector valued holomorphic functions on $U_j$ where $g_{jk}(z_j, s) = \sum_{v_1, ..., v_l=0}^\infty \sum_{s_1, ..., s_l} g_{jk}^{v_1, ..., v_l}(z_j) s_1^{v_1} \cdots s_l^{v_l}$ is a homogeneous polynomial of degree $v$ of $s_1, ..., s_l$, and each component $g_{jk}^{v_1, ..., v_l}(z_j), \alpha = 1, ..., n$ of the coefficient $g_{jk}^{v_1, ..., v_l}(z_j) = (g_{jk1, ..., v_l}(z_j), ..., g_{jkn, ..., v_l}(z_j))$ is a holomorphic function of $z_j^1, ..., z_j^n$ defined on $U_j$. The formal power series $h(s)$ and $g_{jk}(z_j, s)$ will satisfy

$$g_j(f_{jk}(z_k, s), s) = g_{jk}(g_k(z_k, s), h(s)) \quad \text{on } U_j \cap U_k \neq \emptyset$$

and

$$\Lambda_{M_j}^{s}(g_j(z_j, s), h(s)) = \sum_{\alpha, \beta=1}^n \Lambda_{N_j}^{s}(z_j, s) \frac{\partial g_j^\alpha}{\partial z_j^\alpha} \frac{\partial g_j^\beta}{\partial z_j^\beta} \quad \text{on } U_j.$$ 

For the meaning of (6.3.1), we refer to [Kor05] p.286-288. (6.3.1) is a crucial condition for the proof of ‘Theorem of completeness for complex analytic structures’ (Theorem 1.0.4). However, in order to prove ‘Theorem of completeness for holomorphic Poisson structures’ (Theorem 6.1.4), we need to impose additional condition (6.3.2) which means that $g_j(z_j, s)$ is a Poisson map.
We will write
\[ h^v(s) := h_1(s) + \cdots + h_v(s). \]
\[ g_j^v(z_j, s) := z_j + g_{j1}(z_j, s) + \cdots g_{jv}(z_j, s). \]

The equalities (6.3.1) and (6.3.2) are equivalent to the following system of the infinitely many congruences:

\[(6.3.3)\]
\[ g_j^v(f_{jk}(z_k, s), s) \equiv v g_{jk}(g_j^v(z_k, s), h^v(s)) \]
\[(6.3.4)\]
\[ \Lambda_{M_j}^{r,s}(g_j^v(z_j, s), h^v(s)) \equiv_v \sum_{\alpha, \beta = 1}^n \Lambda_{N_j}^{\alpha,\beta}(z_j, s) \frac{\partial g_j^{r,v}}{\partial z_j^\alpha} \frac{\partial g_j^{s,v}}{\partial z_j^\beta} \]

for \( v = 0, 1, 2, 3, \ldots \) where we indicate by \( \equiv_v \) that the power series expansions with respect to \( s \) of both sides of (6.3.3) and (6.3.4) coincide up to the term of degree \( v \).

We will construct \( h^v(s), g_j^v(z_j, s) \) satisfying (6.3.3), (6.3.4) inductively on \( v \). Then the resulting formal power series \( h(s) \) and \( g_j(z, s) \) will satisfy (6.3.1) and (6.3.2). For \( v = 0 \), since \( h^0(s) = 0 \) and \( g_j^0(z_j, s) = z_j \), (6.3.3)0 and (6.3.4)0 hold by (6.2.5), (6.2.6). Now suppose that \( h^{v-1}(s) \) and \( g_j^{v-1}(z_j, s) \) are already constructed in such a manner that, for each \( U_j \cap U_k \neq \emptyset \),

\[(6.3.5)\]
\[ g_j^{v-1}(f_{jk}(z_k, s), s) \equiv_{v-1} g_{jk}(g_k^{v-1}(z_k, s), h^{v-1}(s)) \]

and for each \( U_j \),

\[(6.3.6)\]
\[ \Lambda_{M_j}^{r,s}(g_j^{v-1}(z_j, s), h^{v-1}(s)) \equiv_{v-1} \sum_{\alpha, \beta = 1}^n \Lambda_{N_j}^{\alpha,\beta}(z_j, s) \frac{\partial g_j^{r,v-1}}{\partial z_j^\alpha} \frac{\partial g_j^{s,v-1}}{\partial z_j^\beta} \]

hold. We will find \( h_v(s) \) and \( g_j|_v(z_j, s) \) such that \( h^v(s) = h^{v-1}(s) + h_v(s) \), and \( g_j^v(z_j, s) = g_j^{v-1}(z_j, s) + g_j|_v(z_j, s) \) satisfy (6.3.3), on each \( U_j \cap U_k \) and (6.3.4), on each \( U_j \).

For this purpose, we start from finding the equivalent conditions to (6.3.3), (6.3.4), and then interpret them cohomologically by using Čech resolution of the complex of sheaves (3.1.1) with respect to the open covering (6.2.7) of \( M_0 = N_0 \) (see Lemma 6.3.22 below).

For the equivalent condition to (6.3.3), we briefly summarize Kodaira’s result in the following: if we let \( \Gamma_{jk|v} \) denote the sum of the terms of degree \( v \) of \( g_j^{v-1}(f_{jk}(z_k, s), s) - g_{jk}(g_k^{v-1}(z_k, s), h^{v-1}(s)) \):

\[(6.3.7)\]
\[ \Gamma_{jk}(z_j, s) \equiv_v g_j^{v-1}(f_{jk}(z_k, s), s) - g_{jk}(g_k^{v-1}(z_k, s), h^{v-1}(s)), \]

then (6.3.3) is equivalent to the following:

\[(6.3.8)\]
\[ \Gamma_{jk|v}(z_j, s) = \sum_{\beta = 1}^m \frac{\partial z_j^\beta}{\partial z_k^\beta} g_j^v(z_k, s) - g_j|_v(z_j, s) + \sum_{u=1}^m \left( \frac{\partial g_{jk}(z_k, t)}{\partial t_u} \right)_{t=0} h_u|_v(s) \]

where \( z_k \) and \( z_j = b_{jk}(z_k) \) are the local coordinates of the same point of \( N_0 = M_0 \) (for the detail, see [Kod05] p.289-290).

On the other hand, let’s find the equivalent condition to (6.3.4). We note that

\[(6.3.9)\]
\[ \Lambda_{M_j}^{r,s}(g_j^v(z_j, s), h^v(s)) = \Lambda_{M_j}^{r,s}(g_j^{v-1}(z_j, s) + g_j|_v(z_j, s), h^{v-1}(s) + h_v(s)) \]

By expanding \( \Lambda_{M_j}^{r,s}(\xi_j + \xi, t + \omega) \) into power series of \( \xi^1, \ldots, \xi^n, \omega_1, \ldots, \omega_m \), we obtain

\[(6.3.10)\]
\[ \Lambda_{M_j}^{r,s}(\xi_j + \xi, t + \omega) = \Lambda_{N_j}^{r,s}(\xi_j, t) + \sum_{\beta = 1}^m \frac{\partial \Lambda_{M_j}^{r,s}(\xi_j, t)}{\partial \xi_j^\beta} (\xi_j, t) \xi^\beta + \sum_{u=1}^m \frac{\partial \Lambda_{M_j}^{r,s}(\xi_j, t)}{\partial t_u} (\xi_j, t) \omega_u + \cdots \]
where \cdots denotes the terms of degree \( \geq 2 \) in \( \xi^1, \ldots, \xi^n, \omega_1, \ldots, \omega_m \). Let’s consider the left hand side of (6.3.4)\( _v \). Then from (6.3.9), (6.3.10), and (6.2.6), we have

\[
(6.3.11) \quad \Lambda_{M_j}^{r,s}(g^v_j(z_j, s), h^v(s)) - \Lambda_{M_j}^{r,s}(g^{v-1}_j(z_j, s), h^{v-1}(s)) = v \sum_{\beta=1}^n \frac{\partial \Lambda_{M_j}^{r,s}}{\partial \beta}(g^{v-1}_j(z_j, s), h^{v-1}(s)) g^\beta_j(z_j, s) + \sum_{u=1}^m \frac{\partial \Lambda_{M_j}^{r,s}}{\partial t_u}(g^{v-1}_j(z_j, s), h^{v-1}(s)) h_{u|v}(s)
\]

\[
= v \sum_{\beta=1}^n \frac{\partial \Lambda_{M uuid}(r,s)}{\partial \beta}(g^{v-1}_j(z_j, s), h^{v-1}(0)) g^\beta_j(z_j, s) + \sum_{u=1}^m \frac{\partial \Lambda_{M uuid}(r,s)}{\partial t_u}(g^{v-1}_j(z_j, s), h^{v-1}(0)) h_{u|v}(s)
\]

\[
= v \sum_{\beta=1}^n \frac{\partial \Lambda_{M uuid}(r,s)}{\partial \beta}(g^{v-1}_j(z_j, s), h^{v-1}(0)) g^\beta_j(z_j, s) + \sum_{u=1}^m \left( \frac{\partial \Lambda_{M uuid}(r,s)}{\partial t_u}(z_j, t) \right) h_{u|v}(s)
\]

On the other hand, let’s consider the right hand side of (6.3.4)\( _v \). Then from (6.2.6), we have

\[
(6.3.12) \quad \sum_{\alpha,\beta=1}^n \Lambda_{N_j}^{\alpha,\beta}(z_j, s) \frac{\partial g^{r,v}}{\partial \alpha}(z_j, s) \frac{\partial g^{s,v}}{\partial \beta}(z_j, s) = \sum_{\alpha,\beta=1}^n \Lambda_{N_j}^{\alpha,\beta}(z_j, s) \frac{\partial (g^{r,v}_j + g^{s,v}_j)}{\partial \alpha}(z_j, s) \frac{\partial (g^{r,v}_j + g^{s,v}_j)}{\partial \beta}(z_j, s)
\]

\[
= v \sum_{\alpha,\beta=1}^n \Lambda_{N_j}^{\alpha,\beta}(z_j, s) \frac{\partial g^{r,v}_j}{\partial \alpha}(z_j, s) \frac{\partial g^{s,v}_j}{\partial \beta}(z_j, s) + \sum_{\alpha,\beta=1}^n \Lambda_{N_j}^{\alpha,\beta}(z_j, s) \frac{\partial g^{r,v}_j}{\partial \alpha}(z_j, s) \frac{\partial g^{s,v}_j}{\partial \beta}(z_j, s) + \sum_{\alpha,\beta=1}^n \Lambda_{N_j}^{\alpha,\beta}(z_j, s) \frac{\partial g^{r,v}_j}{\partial \alpha}(z_j, s) \frac{\partial g^{s,v}_j}{\partial \beta}(z_j, s)
\]

Then from (6.3.11) and (6.3.12), the congruence (6.3.4)\( _v \) is equivalent to the following:

\[
(6.3.13) \quad -\Lambda_{M_j}^{r,s}(g^{v-1}_j(z_j, s), h^{v-1}(s)) + \sum_{\alpha,\beta=1}^n \Lambda_{N_j}^{\alpha,\beta}(z_j, s) \frac{\partial g^{r,v}_j}{\partial \alpha}(z_j, s) \frac{\partial g^{s,v}_j}{\partial \beta}(z_j, s)
\]

\[
= v \sum_{\alpha,\beta=1}^n \Lambda_{M uuid(j)}^{r,s}(z_j, s) \frac{\partial g^{r,v}_j}{\partial \alpha}(z_j, s) \frac{\partial g^{s,v}_j}{\partial \beta}(z_j, s) + \sum_{u=1}^m \left( \frac{\partial \Lambda_{M uuid(j)}^{r,s}}{\partial t_u}(z_j, t) \right) h_{u|v}(s) - \sum_{\alpha,\beta=1}^n \Lambda_{M uuid(j)}^{\alpha,\beta}(z_j) \frac{\partial g^{r,v}_j}{\partial \alpha}(z_j, s) \frac{\partial g^{s,v}_j}{\partial \beta}(z_j, s) - \sum_{\alpha,\beta=1}^n \Lambda_{M uuid(j)}^{\alpha,\beta}(z_j) \frac{\partial g^{r,v}_j}{\partial \alpha}(z_j, s) \frac{\partial g^{s,v}_j}{\partial \beta}(z_j, s)
\]

By induction hypothesis (6.3.6), the left hand side of (6.3.13) \( \equiv 0 \). Hence if we let \( \lambda_{j|v}^{r,s} \) denote the terms of degree \( v \) of the left hand side of (6.3.13), we have

\[
(6.3.14) \quad \lambda_{j|v}^{r,s}(z_j, s) \equiv v - \Lambda_{M_j}^{r,s}(g^{v-1}_j(z_j, s), h^{v-1}(s)) + \sum_{\alpha,\beta=1}^n \Lambda_{N_j}^{\alpha,\beta}(z_j, s) \frac{\partial g^{r,v}_j}{\partial \alpha}(z_j, s) \frac{\partial g^{s,v}_j}{\partial \beta}(z_j, s)
\]

Hence from (6.3.13) and (6.3.14), the congruence (6.3.4)\( _v \) is equivalent to the following:

\[
(6.3.15) \quad \lambda_{j|v}^{r,s}(z_j, s) = \sum_{\beta=1}^n \frac{\partial \Lambda_{M uuid(j)}^{r,s}}{\partial \beta}(z_j, s) g^{\beta}_j(z_j, s) + \sum_{u=1}^m \left( \frac{\partial \Lambda_{M uuid(j)}^{r,s}}{\partial \alpha}(z_j, t) \right) h_{u|v}(s) - \sum_{\beta=1}^n \Lambda_{M uuid(j)}^{\alpha,\beta}(z_j) \frac{\partial g^{r,v}_j}{\partial \alpha}(z_j, s) \frac{\partial g^{s,v}_j}{\partial \beta}(z_j, s) - \sum_{\alpha=1}^n \Lambda_{M uuid(j)}^{\alpha,\beta}(z_j) \frac{\partial g^{r,v}_j}{\partial \alpha}(z_j, s) \frac{\partial g^{s,v}_j}{\partial \beta}(z_j, s)
\]

where \( z_k \) and \( z_j = b_{jk}(z_k) \) are the local coordinates of the same point of \( N_0 = M_0 \). We note that \( \lambda_{j|v}^{r,s}(z_j, s) = -\lambda_{j|v}^{r,s}(z_j, s) \).

As in [Kod03] p.291, to interpret the meaning of (6.3.8)\( _v \), and (6.3.15)\( _v \) in terms of Čech resolution of the complex of sheaves (3.1.1) with respect to the open covering (6.2.7) of \( M_0 = N_0 \), we introduce
Here Proof. First, we have $\theta_{ujk} = \sum_{\alpha=1}^{n} \theta_{ajk} (z_j) \frac{\partial}{\partial z_j} = \sum_{\alpha=1}^{n} \left( \left. \frac{\partial \theta_{ajk} (z_j, \tau)}{\partial \tau} \right|_{\tau=0} \right) \frac{\partial}{\partial z_j}$, $z_k = b_{jk} (z_j)$

(6.3.17) \[ \Lambda'_{uj} = \sum_{r,s} \Lambda'_{r,s} (z_j) \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j} := \sum_{r,s=1}^{n} \left( \frac{\partial \Lambda'_{r,s} (z_j, \tau)}{\partial \tau} \right) \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j} \]

(6.3.18) \[ \Gamma_{jk|v}(s) = \sum_{\alpha=1}^{n} \Gamma_{jk|v} (z_j, s) \frac{\partial}{\partial z_j} \]

(6.3.19) \[ g_{k|v}(s) = \sum_{\beta=1}^{n} g_{k|v} (z_k, s) \frac{\partial}{\partial z_k} \]

(6.3.20) \[ \lambda_{j|v}(s) = \sum_{r,s=1}^{n} \lambda_{j|v} (z_j, s) \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j} \]

By (6.2.7), $\mathcal{U} := \{ U_j \}$ is a finite open covering of $M_0 = N_0$. We assume that $\lambda_j = z_j$, $\alpha = 1, ..., n$ in subsection 6.2, the 2-cocycle $(\{ \Lambda'_{uj} \}, \{ \theta_{ujk} \}) \in C^0 (\mathcal{U}, \wedge^2 \Theta_{M_0}) \oplus C^1 (\mathcal{U}, \Theta_{M_0})$ in (6.3.16) and (6.3.17) represents the infinitesimal Poisson deformation $(\Lambda'_{uj}, \theta_{ujk}) = \varphi_0 (\frac{\partial}{\partial \tau}) \in HP^2 (M_0, \Lambda_0)$ where $\varphi_0$ is the Poisson Kodaira-Spencer map of the Poisson analytic family $(M, M, B, \omega)$ (see Proposition 3.1.7 and Definition 3.1.13). Since the coefficients $\Gamma_{jk|v}(s)$ of the homogeneous polynomial $\Gamma_{jk|v}(s) = \sum_{v_1 + ... + v_l = 0} \Gamma_{j|v} (z_j) \cdot s_1^{v_1} \cdot ... \cdot s_l^{v_l}$ are holomorphic vector fields on $U_j \cap U_k$, $\Gamma_{jk|v}(s)$ is a holomorphic polynomial of degree $v$ whose coefficients are $\{ \Gamma_{jk|v} \} \in C^1 (\mathcal{U}, \Theta_{M_0})$. Similarly, $\{ g_{j|v}(s) \} = \sum_{v_1 + ... + v_l = 0} \{ g_{j|v} (z_j) \cdot s_1^{v_1} \cdot ... \cdot s_l^{v_l} \}$ is a holomorphic polynomial of degree $v$ whose coefficients are $\{ g_{j|v} \} \in C^0 (\mathcal{U}, \Lambda_0)$. We claim that

Lemma 6.3.21. The following equation holds

(6.3.22) \[ \{ \lambda_{j|v}(s) \}, \{ \Gamma_{jk|v}(s) \} = \sum_{u=1}^{m} h_{u|v}(s) (\{ \Lambda'_{uj} \}, \{ \theta_{ujk} \}) - \delta_{HP} (\{ g_{j|v}(s) \}) \]

where $\delta_{HP} (\{ g_{j|v}(s) \}) := (-\delta (\{ g_{j|v} \}) = \{ g_{j|v} (s) - g_{k|v}(s) \} \cdot \{ \sum_{r,s=1}^{n} \Lambda'_{r,s} (z_j) \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j} \}, \{ g_{j|v}(s) \})$. Here $\delta$ is the Čech map.

Proof. First, we have $\{ \Gamma_{jk|v}(s) \} = \sum_{u=1}^{m} h_{u|v}(s) \{ \theta_{rjk} \} + \delta (\{ g_{j|v}(s) \}$ (see Kod5 p.291).

It remains to show that $\{ \lambda_{j|v}(s) \} = \sum_{u=1}^{m} h_{u|v}(s) \{ \Lambda'_{uj} \} - \{ \sum_{r,s} \Lambda_{r,s} (z_j) \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j} \}, \{ g_{j|v}(s) \})$. Indeed,

\[
\begin{align*}
\sum_{u=1}^{m} h_{u|v}(s) \Lambda'_{uj} - \sum_{r,s=1}^{n} \Lambda_{r,s} (z_j, s) \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j} = & \sum_{u=1}^{m} h_{u|v}(s) \Lambda'_{uj} - \sum_{r,s=1}^{n} \Lambda_{r,s} (z_j) \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j} \\
= & \sum_{u=1}^{m} h_{u|v}(s) \Lambda'_{uj} - \sum_{r,s=1}^{n} \Lambda_{r,s} (z_j) \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j} = \lambda_{j|v}(s)
\end{align*}
\]

by (6.3.15), (6.3.17) and (6.3.20).
Lemma 6.3.26. We will prove that this is possible when the Poisson Kodaira-Spencer map \( \varphi_0 : T_0(B) \to HP^2(M_0, \Lambda_0) \) is surjective and \( M_0 \) is a surface.

If solutions \( h_{u|v}(s), u = 1, \ldots, m, \{g_{j|v}(s)\} \) exist, from (6.3.22), we have

\[
(6.3.23) \quad \begin{cases}
\sum_{r,s=1}^{n} \Lambda_{M_0}^{r,s}(z_j) \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j}, \lambda_{j|v}(s) = 0 \\
\lambda_{k|v}(s) - \lambda_{j|v}(s) + \sum_{r,s=1}^{n} \Lambda_{M_0}^{r,s}(z_j) \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j}, \Gamma_{jk|v}(s) = 0 \\
\Gamma_{jk|v}(s) - \Gamma_{ik|v}(s) + \Gamma_{ij|v}(s) = 0
\end{cases}
\]

Conversely,

Lemma 6.3.24. If \( \{(\lambda_{j|v}(s)), (\Gamma_{jk|v}(s))\} \) satisfies (6.3.23), then

\[
(6.3.25) \quad (\{(\lambda_{j|v}(s)), (\Gamma_{jk|v}(s))\}) = \sum_{u=1}^{m} h_{u|v}(s)(\{\Lambda'_{uj}, \theta_{ujk}\} - \delta_{HP}(\{g_{j|v}(s)\})
\]

has solutions \( h_{u|v}(s), u = 1, \ldots, m, \{g_{j|v}(s)\} \) when the Poisson Kodaira-Spencer map \( \varphi_0 : T_0(B) \to HP^2(M_0, \Lambda_0) \) is surjective.

Proof. Let \( h_{u|v}(s) = \sum_{v_1, \ldots, v_n = \nu} h_{uv_1 \cdots v_n} s_1^{v_1} \cdots s_n^{v_n} \). Then by considering the coefficients of \( s_1^{v_1} \cdots s_n^{v_n} \),

(6.3.25) can be written as

\[
(\{(\lambda_{j|v_1 \cdots v_n}(s)), (\Gamma_{jkv_1 \cdots v_n}(s))\}) = \sum_{u=1}^{m} h_{uv_1 \cdots v_n}(\{\Lambda'_{uj}, \theta_{ujk}\} - \delta_{HP}(\{g_{j|v_1 \cdots v_n}(s)\}).
\]

Thus it suffices to prove that any 2-cycle \( (\{(\lambda_j), (\Gamma_{jk})\}) \in C^0(\mathcal{U}, \wedge^2 \Theta_0) \oplus C^1(\mathcal{U}, \Theta_0) \) such that \([\Lambda_0, \lambda_j] = 0, \lambda_k - \lambda_j + [\Lambda_0, \Gamma_{jk}] = 0, \Gamma_{jk} - \Gamma_{ik} + \Gamma_{ij} = 0\) can be written in the form

\[
(\{(\lambda_j), (\Gamma_{jk})\}) = \sum_{u=1}^{m} h_u(\{\Lambda'_{uj}, \theta_{ujk}\} - \delta_{HP}(\{g_{j}(s)\}), \text{ for some } h_u \in \mathbb{C}, \{g_{j}\} \in C^0(\mathcal{U}, \Theta_0)
\]

Let \( (\eta, \gamma) \in HP^2(M_0, \Lambda_0) \) be the cohomology class of \( (\{(\lambda_j), (\Gamma_{jk})\}) \). Since \( \varphi_0 : T_0(B) \to HP^2(M_0, \Lambda_0) \) is surjective, \( (\eta, \gamma) \) is written in the form of a linear combination of the \( (\Lambda'_{uj}, \theta_{ujk}) \) (i.e., the cohomology class of \( (\{\Lambda'_{uj}, \theta_{ujk}\}), u = 1, \ldots, m \) in (6.3.16), (6.3.17)) as

\[
(\eta, \gamma) = \sum_{u=1}^{m} h_u(\Lambda'_{uj}, \theta_{ujk}), \quad h_u \in \mathbb{C}
\]

So \( \sum_{u=1}^{m} h_u(\{\Lambda'_{uj}, \theta_{ujk}\}) \) is cohomologous to \( (\{(\lambda_j), (\Gamma_{jk})\}) \). Therefore there exists \( \{g_{j}\} \in C^0(\mathcal{U}, \Theta_0) \) such that \( \delta_{HP}(\{g_{j}\}) = \sum_{u=1}^{m} h_u(\{\Lambda'_{uj}, \theta_{ujk}\} - (\{(\lambda_j), (\Gamma_{jk})\}) \)

Next we will prove that

Lemma 6.3.26. \( (\{(\lambda_{j|v}(s)), (\Gamma_{jk|v}(s))\}) \) satisfies (6.3.23) when \( (M_0, \Lambda_0) \) is a compact holomorphic Poisson surface.

Proof. First, we have \( \Gamma_{jk|v}(s) - \Gamma_{ik|v}(s) + \Gamma_{ij|v}(s) = 0 \) (see [Kod05] p.292). Second, we want to show that

\[
(6.3.27) \quad \sum_{r,s=1}^{n} \Lambda_{M_0}^{r,s}(z_j) \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j}, \lambda_{j|v}(s) = 0
\]

When \( M_0 \) is a surface, the equation (6.3.27) automatically holds. This is the reason why we assume that \( (M_0, \Lambda_0) \) is a Poisson surface in the statement of ‘Theorem of completeness for holomorphic Poisson structures’ (see Theorem 6.1.4). The author could not prove this inductive step (6.3.27) for general \( n \)-dimensional compact holomorphic Poisson manifolds.

Next we will show that

\[
(6.3.28) \quad \lambda_{k|v}(s) - \lambda_{j|v}(s) + \sum_{r,s=1}^{n} \Lambda_{M_0}^{r,s}(z_j) \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_j}, \Gamma_{jk|v}(s) = 0.
\]
First we compute the third term of (6.3.28)

\[ (6.3.29) \quad \sum_{r,s=1}^{n} \Lambda_{M_{0j}}^{r,s}(z_{j}) \frac{\partial}{\partial z_{j}} \wedge \frac{\partial}{\partial z_{j}} \Gamma_{jk|v}(s) = \sum_{r,s,\beta=1}^{n} \Lambda_{M_{0j}}^{r,s}(z_{j}) \frac{\partial}{\partial z_{j}} \wedge \frac{\partial}{\partial z_{j}} \Gamma_{jk|v}(s) \]

\[ = \sum_{r,s,\beta=1}^{n} \left( \Lambda_{M_{0j}}^{r,s} \frac{\partial \Gamma_{jk}^{\beta}}{\partial z_{j}} \frac{\partial}{\partial z_{j}} \wedge \frac{\partial}{\partial z_{j}} - \Gamma_{jk|v}(s) \frac{\partial \Lambda_{M_{0j}}^{r,s}}{\partial z_{j}} \frac{\partial}{\partial z_{j}} \wedge \frac{\partial}{\partial z_{j}} + \Lambda_{M_{0j}}^{r,s} \frac{\partial \Gamma_{jk|v}(s)}{\partial z_{j}} \frac{\partial}{\partial z_{j}} \wedge \frac{\partial}{\partial z_{j}} \right) \]

We consider the first term \( \lambda_{jk|v}(s) \) of (6.3.28). From (6.3.14) and (6.2.6), we have

\[ (6.3.30) \quad \lambda_{jk|v}(s) \equiv v \sum_{p,q=1}^{n} \left( -\Lambda_{M_{k}}^{p,q}(g_{k}^{p-1}(z_{k}, s), h^{v-1}(s)) + \sum_{a,b=1}^{n} \Lambda_{N_{k}}^{a,b}(z_{k}, s) \frac{\partial g_{k}^{a}}{\partial z_{k}^{a}} \frac{\partial g_{k}^{b}}{\partial z_{k}^{b}} \right) \frac{\partial}{\partial z_{k}^{j}} \wedge \frac{\partial}{\partial z_{k}^{j}} \]

We consider the second term \(-\lambda_{jk|v}(s)\) of (6.3.28). We note that since \( z_{j} = b_{jk}(z_{k}) = f_{jk}(z_{k}, 0) \) from (6.2.6), we have \( \lambda_{jk|v}(z_{j}, s) \equiv v \lambda_{jk|v}(f_{jk}(z_{k}, s), s) \). Then from (6.3.14) and by induction hypothesis (6.3.6), we have

\[ (6.3.31) \quad -\lambda_{jk|v}(z_{j}, s) \equiv v \sum_{r,s=1}^{n} \left( \Lambda_{M_{k}}^{r,s}(g_{k}^{r-1}(f_{jk}(z_{k}, s), h^{v-1}(s)) - \sum_{a,b=1}^{n} \Lambda_{N_{k}}^{a,b}(z_{k}, s) \frac{\partial g_{k}^{a}}{\partial z_{k}^{a}} \frac{\partial g_{k}^{b}}{\partial z_{k}^{b}} \right) \frac{\partial}{\partial z_{k}^{j}} \wedge \frac{\partial}{\partial z_{k}^{j}} \]

We consider the first term of (6.3.31). From (6.3.7) and (6.2.2), we have

\[ (6.3.32) \quad \Lambda_{M_{k}}^{r,s}(g_{k}^{r-1}(f_{jk}(z_{k}, s), h^{v-1}(s)) \equiv v \Lambda_{M_{k}}^{r,s}(g_{k}^{r-1}(z_{k}, s), h^{v-1}(s)) + \Gamma_{jk|v}(z_{j}, s), h^{v-1}(s)) \]

\[ \equiv v \sum_{r,s=1}^{n} \left( \Lambda_{M_{k}}^{r,s}(g_{k}^{r-1}(z_{k}, s), h^{v-1}(s)) + \sum_{\beta=1}^{n} \frac{\partial \Lambda_{M_{0j}}^{r,s}}{\partial z_{j}^{\beta}} \Gamma_{jk|v}(z_{j}, s) \right) \frac{\partial}{\partial z_{k}^{j}} \wedge \frac{\partial}{\partial z_{k}^{j}} \]
On the other hand, we consider the second term of \((6.3.31)\). We note that from \((6.2.14)\), \((6.3.7)\) and \((6.2.8)\), we have

\[(6.3.33)\]

\[
\sum_{a,b=1}^{n} \Lambda_{Nk}^{ab}(f_{jk}(z_{k}, s), s) \frac{\partial g^{a}_{jv}(f_{jk}(z_{k}, s), s)}{\partial z^{a}_{j}} (f_{jk}(z_{k}, s), s) \frac{\partial g^{b}_{v}(f_{jk}(z_{k}, s), s)}{\partial z^{b}_{j}} (f_{jk}(z_{k}, s), s) = \sum_{a,b=1}^{n} \Lambda_{Nk}^{ab}(z_{k}, s) \frac{\partial g^{a}_{jv}(f_{jk}(z_{k}, s), s)}{\partial z^{a}_{k}} (f_{jk}(z_{k}, s), s) \frac{\partial g^{b}_{v}(f_{jk}(z_{k}, s), s)}{\partial z^{b}_{k}} (f_{jk}(z_{k}, s), s)
\]

\[
\equiv_{v} \sum_{a,b=1}^{n} \Lambda_{Nk}^{ab}(z_{k}, s) \frac{\partial g^{a}_{jv}(g_{k}^{v-1}(z_{k}, s), h^{v-1}(s)) + \Gamma_{jk|v}(z_{j}, s)}{\partial z^{a}_{k}} + \sum_{a,b=1}^{n} \Lambda_{Mk}^{ab}(z_{k}) \frac{\partial g^{b}_{v}(g_{k}^{v-1}(z_{k}, s), h^{v-1}(s)) + \Gamma_{jk|v}(z_{j}, s)}{\partial z^{b}_{k}}
\]

where we mean \(\frac{\partial g^{a}_{jv}}{\partial z^{a}_{k}}\) and \(\frac{\partial g^{b}_{v}}{\partial z^{b}_{k}}\) by

\[(6.3.34)\]

\[
\frac{\partial g^{a}_{jv}}{\partial z^{a}_{k}} := \frac{\partial g^{a}_{jv}}{\partial z^{a}_{k}} (g_{k}^{v-1}(z_{k}, s), h^{v-1}(s)), \quad \frac{\partial g^{b}_{v}}{\partial z^{b}_{k}} := \frac{\partial g^{b}_{v}}{\partial z^{b}_{k}} (g_{k}^{v-1}(z_{k}, s), h^{v-1}(s))
\]

Hence from \((6.3.31)\), \((6.3.32)\), and \((6.3.33)\), we have

\[(6.3.35)\]

\[
- \Lambda_{j|v}(s) \equiv_{v} \sum_{a,b=1}^{n} \left( \sum_{p,q=1}^{n} \Lambda_{Mk}^{ab}(g_{k}^{v-1}(z_{k}, s), h^{v-1}(s)) \frac{\partial g^{a}_{jv}}{\partial z^{a}_{k}} \frac{\partial g^{b}_{v}}{\partial z^{b}_{k}} + \sum_{a,b=1}^{n} \Lambda_{Mk}^{ab}(z_{k}) \frac{\partial g^{a}_{jv}}{\partial z^{a}_{k}} + \sum_{a,b=1}^{n} \Lambda_{Mk}^{ab}(z_{k}) \frac{\partial g^{b}_{v}}{\partial z^{b}_{k}} \right) \frac{\partial}{\partial z^{a}_{j}} + \frac{\partial}{\partial z^{b}_{j}}
\]

From \((6.3.29)\), \((6.3.30)\) and \((6.3.35)\), to show \((6.3.28)\) is equivalent to show that for each \(r, s\),

\[(6.3.36)\]

\[
\sum_{p,q=1}^{n} \left( -\Lambda_{Mk}^{pq}(g_{k}^{v-1}(z_{k}, s), h^{v-1}(s)) + \sum_{a,b=1}^{n} \Lambda_{Nk}^{ab}(z_{k}, s) \frac{\partial g^{a}_{jv}}{\partial z^{a}_{k}} \frac{\partial g^{b}_{v}}{\partial z^{b}_{k}} \right) \frac{\partial g^{p}_{jv}}{\partial z^{p}_{k}} + \sum_{a,b=1}^{n} \Lambda_{Nk}^{ab}(z_{k}, s) \frac{\partial g^{a}_{jv}}{\partial z^{a}_{k}} \frac{\partial g^{b}_{v}}{\partial z^{b}_{k}} \equiv_{v} 0
\]

\[(6.3.36)\] is equivalent to

\[(6.3.37)\]

\[
\sum_{p,q=1}^{n} \left( -\Lambda_{Mk}^{pq}(g_{k}^{v-1}(z_{k}, s), h^{v-1}(s)) + \sum_{a,b=1}^{n} \Lambda_{Nk}^{ab}(z_{k}, s) \frac{\partial g^{a}_{jv}}{\partial z^{a}_{k}} \frac{\partial g^{b}_{v}}{\partial z^{b}_{k}} \right) \left( \frac{\partial g^{p}_{jv}}{\partial z^{p}_{k}} + \frac{\partial g^{q}_{jv}}{\partial z^{q}_{k}} \right) \equiv_{v} 0
\]
By induction hypothesis (6.3.6), we have \( \Lambda_{M_k}^{\alpha}(g_k^{-1}(z_k, s), h^r(s)) \equiv_{v-1} - \sum_{a,b=1}^n \Lambda_{N_k}^{a,b}(z_k, s) \frac{\partial y_a^{v-1}}{\partial z_k} \frac{\partial y_b^{v-1}}{\partial z_k} \), and we have \( \left( \frac{\partial y_a^{v}}{\partial z_k}, \frac{\partial y_b^{v}}{\partial z_k} \right) \equiv_{v} 0 \) since from (6.3.31), we have \( \frac{\partial y_a^{v}}{\partial z_k}(g_k^{-1}(z_k, 0), h^r(1)) = \frac{\partial y_a^{v}}{\partial z_k}(z_k, 0) = \frac{\partial y_a^{v}}{\partial z_k}(z_k, 0) \) and similarly for \( \frac{\partial y_b^{v}}{\partial z_k} \). Hence we have (6.3.37). This completes Lemma 6.3.26. □

### 6.4. Proof of Convergence.

By Lemma 6.3.24 and Lemma 6.3.26, we can find \( h_{u|v}, u = 1, \ldots, m, \) and \( \{g_{j|v}(z_j, s)\} \) inductively on \( v \) such that \( h^v(s) = h^{v-1}(s) + h_{u|v}(s) \) and \( g_j^v(z_j, s) = g_j^{v-1}(z_j, s) + g_{j|v}(z_j, s) \) satisfy (6.3.3) and (6.3.4), so that we have formal power series \( h(s) \) and \( g_j(z_j, s) \) satisfying (6.3.1) and (6.3.2). In this subsection, we will prove that we can choose appropriate solutions \( h_{u|v}(s) \) and \( \{g_{j|v}(z_j, s)\} \) in each inductive step so that \( h(s) \) and \( g_j(z_j, s) \) converge absolutely in \( |s| < \epsilon \) if \( \epsilon > 0 \) is sufficiently small. As in [Kod05] p.294-302, our approach is to estimate \( \Gamma_{jk}(z_j, s), \lambda_j(z_j) \) and use Lemma 6.4.10 below concerning the “magnitude” of the solutions \( h_{u|v}(s), u = 1, \ldots, m, \{g_{j|v}(z_j, s)\} \) of the equation (6.3.22).

#### Definition 6.4.1. Let \( \mathcal{U} := \{U_j\} \) be a open covering of \( M_0 \) in (6.2.7). We denote a 2-cocycle \( \{(\lambda_j), \{\Gamma_{jk}\}\} \in C^0(\mathcal{U}, \Lambda^2 \Theta_{M_0}) + C^1(\mathcal{U}, \Theta_{M_0}) \) by \( (\lambda, \Gamma) \) in the Čech resolution of the complex of sheaves (5.1.1), and define its norm by

\[
| (\lambda, \Gamma) | := \max_{j, z_j \in U_j} | \lambda_j(z_j) | + \max_{j, k, z_j \in U_j \cap U_k} | \Gamma_{jk}(z_j) |
\]

#### Remark 6.4.3. We explain the meaning of \( | \lambda_j(z_j) | \) and \( | \Gamma_{jk}(z_j) | \) in (6.4.2). We regard holomorphic vector field \( \Gamma_{jk}(z_j) = \sum_{a=1}^n \Gamma_{jk}^a(z_j) \frac{\partial}{\partial z_j^a} \) as a vector-valued holomorphic function \( (\Gamma_{jk}^a(z_j), \ldots, \Gamma_{jk}^n(z_j)) \) and regard holomorphic bivector field \( \lambda_j(z_j) = \sum_{r,s=1}^n \lambda_{jk}^{r,s}(z_j) \frac{\partial}{\partial z_j^r} \wedge \frac{\partial}{\partial z_j^s} \) as a holomorphic vector valued function \( (\lambda_{jk}^{1,1}(z_j), \ldots, \lambda_{jk}^{n,n}(z_j)) \). For \( j \in U_j \cap U_k \), we define \( | \Gamma_{jk}(z_j) | := \max_{\alpha, \beta} | \Gamma_{jk}^\alpha(z_j) | \). On the other hand, for \( z_j \in U_j \), we define \( | \lambda_j(z_j) | := \max_{r,s} | \lambda_{jk}^{r,s}(z_j) | \).

#### Remark 6.4.4. Since each \( U_j = \{z_j \in \mathbb{C}^n \mid |z_j| < 1\} \) in (6.2.7) is a coordinate polydisk, we may assume that the coordinate function \( z_j \) is defined on a domain of \( M_0 \) containing \( \overline{U_j} \) (the closure of \( U_j \)). Hence there exists a constant \( L_1 > 0 \) such that for all \( \alpha, \beta = 1, \ldots, n \), and for all \( U_k \cap U_j \neq \emptyset \),

\[
| \frac{\partial z_j^\alpha}{\partial z_k^\beta} (z_k) | = \left| \frac{\partial \lambda_{jk}^\alpha}{\partial \lambda_{jk}^\beta} (z_k) \right| < L_1, \quad z_k \in U_k \cap U_j,
\]

and there exists constants \( C, C' > 0 \) such that for all \( r, s, \beta = 1, \ldots, n \) and for all \( U_j \),

\[
| \Lambda_{M_0}^{r,s}(z_j) | < C, \quad \left| \frac{\partial \Lambda_{M_0}^{r,s}}{\partial z_j^\beta} (z_j) \right| < C', \quad z_j \in U_j.
\]

We define the norm of the matrix \( B_{jk}(z_k) := \left( \frac{\partial \lambda_{jk}^\alpha}{\partial \lambda_{jk}^\beta} (z_k) \right)_{\alpha, \beta = 1, \ldots, n} \) by \( | B_{jk}(z_k) | = \max_{\alpha} \sum_{\beta} \left| \frac{\partial \lambda_{jk}^\alpha}{\partial \lambda_{jk}^\beta} (z_k) \right| \).

Then there exists a constant \( K_1 > 1 \) such that for all \( U_j \cap U_k \neq \emptyset \),

\[
| B_{jk}(z_k) | < K_1, \quad z_k \in U_k \cap U_j
\]

Since \( \theta_{u|jk}(z_k) = \left( \frac{\partial \lambda_{jk}^\alpha}{\partial \lambda_{jk}^\beta} (z_k, t) \right)_{t=0} \) in (6.3.16) are bounded on \( U_j \cap U_k \neq \emptyset \), there exists a constant \( K_2 \) such that

\[
| \theta_{u|jk}(z_k) | = \left| \sum_{\alpha=1}^n \theta_{u|jk}^\alpha (z_k) \frac{\partial}{\partial z_k^\alpha} \right| := \max_{\alpha} | \theta_{u|jk}^\alpha (z_k) | < K_2
\]

Since \( \Lambda_{M_0}^{r,s}(z_j, t) = \left( \frac{\partial \Lambda_{M_0}^{r,s}(z_j, t)}{\partial u} \right)_{t=0} \) in (6.3.17) are bounded on \( U_j \), there exists a constant \( L_2 \) such that

\[
| \Lambda_{M_0}^{r,s}(z_j) | < \left| \sum_{r,s=1}^n \Lambda_{u|j}^{r,s} (z_j) \frac{\partial}{\partial z_j^r} \wedge \frac{\partial}{\partial z_j^s} \right| := \max_{r,s} | \Lambda_{u|j}^{r,s}(z_j) | < L_2
\]
Lemma 6.4.10 (compare [Kod05] Lemma 6.2 p.295). There exist solutions \( h_u, u = 1, \ldots, m \), and \( \{g_j(z_j) = \sum_{\alpha=1}^n g^\alpha_j(z_j) \frac{\partial}{\partial z^\alpha_j} \} \) of the equation

\[
(6.4.11) \quad (\{\lambda_j\}, \{\Gamma_{jk}\}) = \sum_{u=1}^m h_u(\{\Lambda'_{uj}\}, \{\theta_{u,jk}\}) - \delta_{HP} \{g_j(z_j)\}
\]

which satisfy

\[
|h_u| \leq M|\langle \lambda, \Gamma \rangle|, \quad |g_j(z_j)| := \max_{\alpha} |g^\alpha_j(z_j)| \leq M|\langle \lambda, \Gamma \rangle| \quad \text{for } z_j \in U_j
\]

where \( M \) is a constant independent of a 2-cocycle \( (\lambda, \Gamma) = (\{\lambda_j\}, \{\Gamma_{jk}\}) \).

Proof. The proof is similar to [Kod05] Lemma 6.2 p.295 to which we refer for the detail. For a 2-cocycle \( (\lambda, \Gamma) = (\{\lambda_j\}, \{\Gamma_{jk}\}) \) with \( |\langle \lambda, \Gamma \rangle| < \infty \), we define \( \iota(\lambda, \Gamma) \) by

\[
\iota(\lambda, \Gamma) = \inf \{ |h_u|, \sup_{u,j} |g_j(z_j)| \}
\]

where \( \inf \) is taken with respect to all the solutions \( h_u, u = 1, \ldots, m \), and \( g_j(z_j) \) of \( (6.4.11) \). We will show that there exists a constant \( M \) such that for all 2-cocycles \( (\lambda, \Gamma) \in C^0(\mathcal{U}, \Lambda^2 \Theta_{M_0}) \cup C^1(\mathcal{U}, \Theta_{M_0}) \), we have

\[
\iota(\lambda, \Gamma) \leq M|\langle \lambda, \Gamma \rangle|
\]

Suppose there is no such constant \( M \). Then we can find a sequence of 2-cocycles \( (\lambda^{(v)}, \Gamma^{(v)}) = (\{\lambda_j^{(v)}\}, \{\Gamma_{jk}^{(v)}\}) \in C^0(\mathcal{U}, \Lambda^2 \Theta_{M_0}) \cup C^1(\mathcal{U}, \Theta_{M_0}), v = 1, 2, 3 \cdots \), and their solutions \( g_j^{(v)}(z_j), h_u^{(v)}, u = 1, \ldots, m \) such that

\[
(6.4.12) \quad \iota(\lambda^{(v)}, \Gamma^{(v)}) = 1, \quad |\langle \lambda^{(v)}, \Gamma^{(v)} \rangle| < \frac{1}{v^2},
\]

and the sequence \( \{h_u^{(v)}\}, u = 1, \ldots, m \) converge and \( \{g_j^{(v)}(z_j)\} \) converges uniformly on \( U_j \).

Put \( h_u = \lim_{v \to \infty} h_u^{(v)} \), and \( g_j(z_j) = \lim_{v \to \infty} g_j^{(v)}(z_j) \) and note that

\[
\sum_{u=1}^m h_u \theta_{u,jk}(z_j) + B_{jk}(z_k) g_k^{(v)}(z_k) - g_j^{(v)}(z_j), \quad \sum_{u=1}^m h_u \Lambda'_{uj}(z_j) - \left[ \sum_{r,s=1}^n \Lambda^r_{M_0}(z_j) \frac{\partial}{\partial z_j^r} \wedge \frac{\partial}{\partial z_j^s} \sum_{\alpha=1}^n g^\alpha_j(z_j) \frac{\partial}{\partial z^\alpha_j} \right]
\]

Since \( |\Gamma_{jk}^{(v)}(z_j)| \leq |\langle \lambda^{(v)}, \Gamma^{(v)} \rangle| \to 0 \) and \( |\lambda_j^{(v)}(z_j)| \leq |\langle \lambda^{(v)}, \Gamma^{(v)} \rangle| \to 0 \) as \( v \to \infty \), we have

\[
0 = \sum_{u=1}^m h_u \theta_{u,jk}(z_j) + B_{jk}(z_k) g_k(z_k) - g_j(z_j), \quad 0 = \sum_{u=1}^m h_u \Lambda'_{uj}(z_j) - \left[ \sum_{r,s=1}^n \Lambda^r_{M_0}(z_j) \frac{\partial}{\partial z_j^r} \wedge \frac{\partial}{\partial z_j^s} \sum_{\alpha=1}^n g^\alpha_j(z_j) \frac{\partial}{\partial z^\alpha_j} \right]
\]

By putting \( \tilde{h}_u^{(v)} = h_u^{(v)} - h_u \), and \( \tilde{g}^{(v)}_j(z_j) = \tilde{g}_j^{(v)}(z_j) - g_j(z_j) \), we obtain

\[
\Gamma_{jk}^{(v)}(z_j) = \sum_{u=1}^m \tilde{h}_u^{(v)} \theta_{u,jk}(z_j) + B_{jk}(z_k) \tilde{g}_k^{(v)}(z_k) - \tilde{g}_j^{(v)}(z_j), \quad \lambda_j^{(v)}(z_j) = \sum_{u=1}^m \tilde{h}_u^{(v)} \Lambda'_{uj}(z_j) - \left[ \sum_{r,s=1}^n \Lambda^r_{M_0}(z_j) \frac{\partial}{\partial z_j^r} \wedge \frac{\partial}{\partial z_j^s} \sum_{\alpha=1}^n \tilde{g}_j^{(v)}(z_j) \frac{\partial}{\partial z^\alpha_j} \right]
\]

Hence \( \tilde{h}_u^{(v)}, u = 1, \ldots, m \), and \( \{\tilde{g}_j^{(v)}(z_j)\} \) satisfy the equation \( (6.4.11) \) for \( (\lambda, \Gamma) = (\{\lambda^{(v)}\}, \{\Gamma^{(v)}\}) \). This is a contradiction to \( \iota(\lambda, \Gamma) = 1 \) \((6.4.12)\) since we have \( \tilde{h}_u^{(v)} \to 0 \) and \( \sup_{z_j \in U_j} |\tilde{g}_j^{(v)}(z_j)| \to 0 \). \( \square \)
Next we will prove that we can choose appropriate solutions $h_{|v}(s), u = 1, \ldots, m$ and $\{g_{j|v}(z_j, s)\}$ in each inductive step by estimating $\Gamma_{jk|v}(z_j, s), \lambda_{jk}(z_j, s)$ and using Lemma 6.4.10 so that the formal power series $h(s)$ and $g_j(z_j, s)$ converge absolutely in $|s| < \epsilon$ if $\epsilon > 0$ is sufficiently small. Before the proof, we remark the following.

**Remark 6.4.13.**

1. For two power series of $s_1, \ldots, s_t$, 
\[
P(s) = \sum_{v_1, \ldots, v_t = 0}^\infty P_{v_1, \ldots, v_t} s_1^{v_1} \cdots s_t^{v_t}, \ P_{v_1, \ldots, v_t} \in \mathbb{C}^n,\]
\[
a(s) = \sum_{v_1, \ldots, v_t = 0}^\infty a_{v_1, \ldots, v_t} s_1^{v_1} \cdots s_t^{v_t}, \ a_{v_1, \ldots, v_t} \geq 0,
\]
we write $P(s) \ll a(s)$ if $|P_{v_1, \ldots, v_t}| \leq a_{v_1, \ldots, v_t}, \ v_1, \ldots, v_t = 0, 1, 2, \ldots$.

2. For a power series $P(s)$, we denote by $[P(s)]_v$ the term of homogeneous part of degree $v$ with respect to $s$.

3. For $A(s) = \frac{b_0}{16c_0} \sum_{v=1}^\infty \frac{c_v(s_1 + \cdots + s_t)^v}{v^2}, b > 0, c > 0$, we have $A(s)^v \ll \left(\frac{b}{c}\right)^{v-1} A(s), v = 2, 3, \ldots$

4. For each $U_j = \{z_j \in \mathbb{C}^n \mid z_j < 1\}$, set $U^\delta_j = \{z_j \in U_j \mid |z_j| < 1 - \delta\}$ for a given $\delta$. Then $M_0 = \bigcup_j U^\delta_j$ for a sufficiently small $\delta$.

To prove the convergence of $h(s)$ and $g_j(z_j, s)$, we will show the estimates $h(s) \ll A(s), \ g_j(z_j, s) - z_j \ll A(s)$ for suitable constants $b$ and $c$ in Remark 6.4.13 (3), equivalently
\[
h^v(s) \ll A(s), \ g_j^v(z_j, s) - z_j \ll A(s)\]for $v = 1, 2, 3, \ldots$. We will prove this by induction on $v = 1, 2, 3, \ldots$. For $v = 1$, since the linear term of $A(s)$ is $\frac{b_0}{16c_0} (s_1 + \cdots + s_t)$, the estimate holds if $b$ is sufficiently large. Let $v \geq 2$ and assume that the induction holds for $v - 1$. In other words,
\[
h^{v-1}(s) \ll A(s), \ g_j^{v-1}(z_j, s) - z_j \ll A(s)\]
We will prove that (6.4.14) holds. For this, we estimate $\Gamma_{jk|v}(z_j, s)$ and $\lambda_{jk|v}(z_j, s)$. For the estimation of $\Gamma_{jk|v}(z_j, s)$, we briefly summarize Kodaira's estimation presented in [Kod05] p.298-302 in the following: since $f_{jk}(z_j) = b_{jk}(z_k) + \sum_{v=1}^\infty f_{jk|v}(z_j, s)$ are given vector-valued holomorphic functions, we may assume that
\[
f_{jk}(z_k) - b_{jk}(z_k) \ll A_0(s), \ A_0(s) = \frac{b_0}{16c_0} \sum_{v=1}^\infty \frac{c_v(s_1 + \cdots + s_t)^v}{v^2}\]
holds for $z_k \in U_k \cap U_j$ with $b_0 > 0$ and $c_0 > 0$ such that $\frac{b_0}{c_0} < \frac{1}{2}$, where $\delta$ from Remark 6.4.13 (4).

\[
\Gamma_{jk|v}(z_j, s) \ll 2K_1 K^* A(s), \ z_j \in U_j \cap U_k.\]
where $K^* = \frac{2^{n+1}b_0}{c_0} + \frac{2b_0(4n+4)^2}{c}$ and $K_1$ from (6.4.7) (for the detail, see page [Kod05] 298-302).

Next we estimate $\lambda_{jk|v}(z_j, s)$ (see (6.3.20)). To estimate it, we estimate $\lambda_{jk|v}^r(s, z_j, s)$ for each pair $(r, s)$ where $r, s = 1, \ldots, n$. We note that from (6.3.14), we have
\[
\lambda_{jk|v}^r(s, z_j) = [-\lambda_{M_j}^r(g_j^{v-1}(z_j, s), h^{v-1}(s))]_v + \sum_{p, q=1}^n \Lambda_{M_j}^{p,q}(z_j, s) \frac{\partial g_j^{v-1}}{\partial z_p} \frac{\partial g_j^{v-1}}{\partial z_q} \bigg|_v
\]
First we estimate $[\lambda_{M_j}^r(g_j^{v-1}(z_j, s), h^{v-1}(s))]_v$ in (6.4.17). We expand $\lambda_{M_j}^r(z_j, s)$ into power series in $x_1, \ldots, x_n, t_1, \ldots, t_m$, and let $L(x, t)$ be its linear term. Since $\lambda_{M_j}^r(z_j, 0) = \lambda_{M_0}^r(z_j)$ from (6.2.6), we may assume that for all the pairs $(r, s)$,
\[
\lambda_{M_j}^r(z_j, s) - \lambda_{M_0}^r(z_j) - L(x, t) \ll \sum_{\mu=2}^\infty d_0^\mu (x_1 + \cdots + x_n + t_1 + \cdots + t_m)\]for some constant $d_0 > 0$. 

Set \( \xi = g_j^{v-1}(z_j, s) - z_j \), and \( t = h^{v-1}(s) \). Since \( \xi \ll A(s) \) and \( t \ll A(s) \) by induction hypothesis (6.4.15), we have from Remark 6.4.13 (3) for some constants \( (6.4.22) \)

\[
\Lambda_{M, j}^{r, s}(g_j^{v-1}(z_j, s), h^{v-1}(s)) - \Lambda_{M_0, j}^{r, s}(z_j) - L(g_j^{v-1}(z_j, s)) - z_j, h^{v-1}(s) \\
\ll \sum_{\mu=2}^{\infty} d_0^n(n + m)^\mu A(s)^\mu \ll \sum_{\mu=2}^{\infty} d_0^n(m + n)^\mu \left( \frac{b}{c} \right)^{\mu-1} A(s) = \frac{b d_0^2(m + n)^2}{c} \sum_{\mu=0}^{\infty} \left( \frac{b d_0(m + n)}{c} \right)^{\mu-2} A(s)
\]

Hence we have

\[
[\Lambda_{M, j}^{r, s}(g_j^{v-1}(z_j, s), h^{v-1}(s))]_v \ll \frac{b d_0^2(m + n)^2}{c} \sum_{\mu=0}^{\infty} \left( \frac{b d_0(m + n)}{c} \right)^{\mu} A(s)
\]

Choose a constant \( c \) such that \( \frac{b d_0(m + n)}{c} < \frac{1}{2} \). Then we have

\[
(6.4.18) \quad [\Lambda_{M, j}^{r, s}(g_j^{v-1}(z_j, s), h^{v-1}(s))]_v \ll \frac{2 b d_0^2(m + n)^2}{c} A(s), \quad z_j \in U_j
\]

Next we estimate \( \sum_{p, q=1}^{n} \Lambda_{N_j}^{p, q}(z_j, s) \partial g_j^{v-1}(z_j, s) \partial g_j^{v-1}(z_j, s) \partial z_j \) in (6.4.17). By induction hypothesis (6.4.15), set

\[
(6.4.19) \quad \alpha_j^{v-1}(z_j, s) := g_j^{v-1}(z_j, s) - z_j^v \ll A(s) \quad \text{for all} \quad r = 1, ..., n
\]

Since \( \Lambda_{N_j}^{p, q}(z_j, s) \) is holomorphic, and \( \Lambda_{N_j}^{p, q}(z_j, 0) = \Lambda_{M_0, j}^{p, q}(z_j) \) from (6.2.6), we may assume that for all the pairs \((p, q)\),

\[
(6.4.20) \quad \Pi_j^{p, q}(z_j, s) := \Lambda_{N_j}^{p, q}(z_j, s) - \Lambda_{M_0, j}^{p, q}(z_j) \ll A_1(s) = \frac{b_1}{16c_1} \sum_{v=1}^{\infty} c_v^1(s_1 + \cdots + s_l)^v
\]

for some constants \( b_1, c_1 > 0 \). If we choose \( b > b_1 \) and \( c > c_1 \), then we have

\[
(6.4.21) \quad \Pi_j^{p, q}(z_j, s) \ll \frac{b_1}{b} A(s).
\]

Now assume that \( z_j = (z_j^1, ..., z_j^n) \in U_j^{\delta} \) from Remark 6.4.13 (4). Then by Cauchy’s integral formula, and (6.4.19), we have, for \( p = 1, ..., n, \)

\[
\frac{\partial \alpha_j^{v-1}(z_j, s)}{\partial z_j^p} = \frac{1}{2\pi i} \int_{|\xi - \xi_j^p| = \delta} \frac{\alpha_j^{v-1}(z_j^1, ..., \xi^p, ..., z_j^n, s)}{(|\xi - \xi_j^p|^2)} d\xi
\]

Hence we have, for \( p = 1, ..., n, \)

\[
(6.4.22) \quad \frac{\partial \alpha_j^{v-1}(z_j, s)}{\partial z_j^p} \ll \frac{A(s)}{\delta}
\]
Then from (6.4.19), (6.4.20), (6.4.21), (6.4.22) and (6.4.23), we have

\begin{equation}
[\sum_{p,q=1}^{n} \Lambda_{p,q}^{n}(z_j, s) \frac{\partial g_{j}^{n-1}(z_j, s)}{\partial z_{j}^{p}} \frac{\partial g_{j}^{n-1}(z_j, s)}{\partial z_{j}^{q}}]_{v}
= \sum_{p,q=1}^{n} \left( \Pi_{p,q}^{n}(z_j, s) + \Lambda_{p,q}^{n}(z_j) \right) \frac{\partial (\alpha_{j}^{n-1}(z_j, s) + z_{j}^{p})}{\partial z_{j}^{p}} \frac{\partial (\alpha_{j}^{n-1}(z_j, s) + z_{j}^{q})}{\partial z_{j}^{q}}]_{v}
= \left[ \sum_{p,q=1}^{n} \Pi_{p,q}^{n}(z_j, s) \frac{\partial \alpha_{j}^{n-1}(z_j, s)}{\partial z_{j}^{p}} \frac{\partial \alpha_{j}^{n-1}(z_j, s)}{\partial z_{j}^{q}} \right]_{v} + \left[ \sum_{p,q=1}^{n} \Pi_{p,q}^{n}(z_j, s) \frac{\partial z_{j}^{p}}{\partial z_{j}^{p}} \frac{\partial z_{j}^{q}}{\partial z_{j}^{q}} \right]_{v} + \left[ \sum_{p,q=1}^{n} \Lambda_{p,q}^{n}(z_j) \frac{\partial \alpha_{j}^{n-1}(z_j, s)}{\partial z_{j}^{p}} \frac{\partial \alpha_{j}^{n-1}(z_j, s)}{\partial z_{j}^{q}} \right]_{v}
\end{equation}

\[\leq \frac{n^2b_{1}}{b^2c^2}A(s)^3 + \frac{2nb_{1}}{b^2}A(s)^2 + \frac{b_{1}}{b}A(s) + Cn^2 \delta A(s)^2 \text{ for } C > 0 \text{ from (6.4.24) which does not depend on } b, c.
\]

Hence from (6.4.17), (6.4.18), (6.4.23), we have

\begin{equation}
\lambda_{r,s}^{j}(z_j, s) \ll LA(s), \quad z_j \in U^\delta_j
\end{equation}

where \(L = \frac{2bd_2(n+1)}{c} + \frac{n^2b_{1}}{c^2d^2} + \frac{2nb_{1}}{c^2d^2} + \frac{b_{1}}{c} + Cn^2 \delta \). Now we estimate \(\lambda_{r,s}^{j}(z_j, s)\) for arbitrary \(z_j \in U_j\). Since \(M_0 = \bigcup_{j} U^\delta_j\), if \(z_j \in U_j\) and \(z_j \notin U^\delta_j\), then \(z_j\) is contained in some \(U^\delta_k(j \neq k) : z_j = b_{jk}(z_k), z_k \in U^\delta_k\). Since \(\lambda_{r,s}^{j}(z_j, s) - \lambda_{r,s}^{j}(s) + \sum_{r,s=1}^{n} \Lambda_{r,s}^{j}(z_j) \frac{\partial}{\partial z_{j}^{r}} \wedge \frac{\partial}{\partial z_{j}^{s}}, \Gamma_{r,s}^{j}(z_j, s) = 0\) by Lemma 6.3.26, we have

\begin{equation}
\lambda_{r,s}^{j}(z_j, s) = \sum_{p,q=1}^{n} \Lambda_{p,q}^{j}(z_k, s) \frac{\partial b_{jk}^{p}}{\partial z_{j}^{p}} \frac{\partial b_{jk}^{q}}{\partial z_{j}^{q}} + \sum_{\beta=1}^{n} \Lambda_{\beta,s}^{j}(z_j) \frac{\partial \Gamma_{jk}^{r,s}^{j}(z_j)}{\partial z_{j}^{\beta}} - \sum_{\beta=1}^{n} \Gamma_{jk}^{r,s}^{j}(z_j, s) \frac{\partial \Lambda_{s}^{j}}{\partial z_{j}^{\beta}} + \sum_{\alpha,\beta=1}^{n} \Lambda_{r,s}^{j}(z_j) \frac{\partial b_{jk}^{\alpha}}{\partial z_{j}^{\alpha}} \frac{\partial \Gamma_{jk}^{r,s}^{j}(z_j)}{\partial z_{j}^{\beta}}
\end{equation}

We note that \(\Gamma_{jk}^{r,s}^{j}(z_j, s) = \Gamma_{jk}^{r,s}^{j}(b_{jk}(z_k), s)\) and so by Cauchy’s integral formula and (6.4.16), we have, for \(n = 1, \ldots, n\),

\begin{equation}
\frac{\partial \Gamma_{jk}^{r,s}^{j}(z_j, s)}{\partial z_{j}^{\alpha}} = \frac{\partial \Gamma_{jk}^{r,s}^{j}(b_{jk}(z_k), s)}{\partial z_{j}^{\alpha}} = \frac{1}{2\pi i} \int_{|z_k - z_j| = \delta} \frac{\Gamma_{jk}^{r,s}^{j}(b_{jk}(z_k), \ldots, \xi, \ldots, z_k^\alpha)}{(\xi - z_j^\alpha)^2} \, d\xi \ll 2K_1 K^* A(s) / \delta
\end{equation}

Hence from (6.4.25), (6.4.24) (when \(j\) is replaced by \(k\)), (6.4.5), (6.4.6), (6.4.26), and (6.4.16),

\begin{equation}
\lambda_{r,s}^{j}(z_j, s) \ll T_1 LA(s) + \frac{T_2 K^*}{\delta} A(s) + T_3 K^* A(s)
\end{equation}

for \(z_j \in U_j\), and some constants \(T_1, T_2, T_3 > 0\) which does not depend on \(b, c\). Since \(|\lambda_{r,s}^{j}(z_j, s)| = \max_{r,s} |\lambda_{r,s}^{j}(z_j, s)|\), from (6.4.27) we have

\begin{equation}
\lambda_{r,s}^{j}(z_j, s) \ll T_1 LA(s) + \frac{T_2 K^*}{\delta} A(s) + T_3 K^* A(s)
\end{equation}

Then by Lemma 6.4.10, 6.4.16, and 6.4.28, we can choose solutions \(h_{u,j}(s), u = 1, \ldots, m, \{g_{j}^{j}(s)\}\) such that

\begin{equation}
h_{u,j}(s) \ll NA(s), \quad g_{j}^{j}(s) \ll NA(s), \quad \text{where } N = M \left( 2K_1 K^* + T_1 L + \frac{T_2 K^*}{\delta} + T_3 K^* \right)
\end{equation}
Note that \( N \) is independent of \( v \) and \( K^* = 2n+1 \nu_0 + \nu_0 + 2b_0(n+n)^2 \), \( L = 2b_0(n+n)^2 + w_0^2b_0 + 2b_0 + \frac{b_0}{b_0} + \frac{c_n+b_0}{c_n+b_0} \). If we first choose a sufficiently large \( b \), and then choose \( c \) so that \( \frac{c_n+b_0}{c_n+b_0} \) is sufficiently large (so that \( \frac{b_0}{b_0} \) is sufficiently small), then we obtain \( N \leq 1 \). Note that \( b \) and \( c \) satisfy \( b > \max \{b_0, b_1\}, \; c > \max \{c_0, c_1\}, \; \frac{b_0(n+n)^2}{c_n+b_0} < \frac{1}{2} \) and \( \frac{b_0(n+n)^2}{c_n+b_0} < \frac{1}{2} \).

Hence the above solution \( h_{u|v}(s) \), \( u = 1, ..., m \), \( \{g_{j|v}(s)\} \) satisfy the inequalities

\[
h_{u}(s) \ll A(s), \quad g_{j|v}(s) \ll A(s)\]

Since \( h^n(s) = h^{n-1}(s) + g_{u}(s) \), \( g_{j}^n(z_j, s) = g_{j}^{n-1}(z_j, s) + g_{j|v}(z_j, s) \), we have \( h^n(s) \ll A(s) \), and \( g_{j}^n(z_j, s) \ll A(s) \). This completes the induction, and so we have \( h(s) \ll A(s) \) and \( g_{j}(z_j, s) - z_j \ll A(s) \). These inequalities imply that, if \( |s| < \frac{1}{10} \), \( h(s) \) converges absolutely, and \( g_{j}(z_j, s) \) converges absolutely and uniformly for \( z_j \in U_j \).

### 6.5. Proof of Theorem 6.1.4

By the same argument presented in page 303-304, we can glue together each \( g_{j} \) on \( U_j \times \Delta \) to construct a Poisson holomorphic map \( g : \pi^{-1}(\Delta) = (U_j \times \Delta, \lambda_N|_{\Delta_{n}}) \to (\mathcal{M}, \lambda_{\mathcal{M}}) \) which extends the identity map \( g_0 : (N_0, \lambda_0) \to (M_0 = N_0, \lambda_0) \) (see [Kod05] page 303-304 for the details and notations). This completes the proof of Theorem 6.1.4.

**Example 6.5.1.** Let \( U_i = \{[z_0, z_1, z_2] | z_i \neq 0 \} \) \( i = 0, 1, 2 \) be an open cover of complex projective plane \( \mathbb{P}_\mathbb{C}^2 \). Let \( x = \frac{a_0}{z_0} \) and \( w = \frac{a_0}{z_0} \) be coordinates on \( U_0 \). Then the holomorphic Poisson structures on \( U_0 \) are parametrized by \( t = (t_1, ..., t_{10}) \in \mathbb{C}^{10} \)

\[
(t_1 + t_2x + t_3w + t_4x^2 + t_5xw + t_6w^2 + t_7x^3 + t_8x^2w + t_9xw^2 + t_{10}w^3) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial w}
\]

This parametrizes the whole holomorphic Poisson structures on \( \mathbb{P}_\mathbb{C}^2 \) (see [HX11] Proposition 2.2).

Let \( \Lambda_0 = x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial w} \) be the holomorphic Poisson structure on \( \mathbb{P}_\mathbb{C}^2 \). Then \( HP^2(\mathbb{P}_\mathbb{C}^2, \Lambda_0) = 5 \), \( HP^3(\mathbb{P}_\mathbb{C}^2, \Lambda_0) = 0 \) (see [HX11] Example 3.5 ). \( w^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial w}, x^3 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial w}, x^2w \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial w}, xw^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial w} \) and \( w^3 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial w} \) are the representatives of the cohomology classes consisting of the basis of \( HP^2(\mathbb{P}_\mathbb{C}^2, \Lambda_0) \).

Let \( t = (t_1, t_2, t_3, t_4, t_5) \in \mathbb{C}^{5} \). Let \( \Lambda(t) = (t_1w^2 + x + t_2x^3 + t_3x^2w + t_4xw^2 + t_5w^3) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial w} \) be the holomorphic Poisson structure on \( \mathbb{P}_\mathbb{C}^2 \times \mathbb{C}^5 \). Then \( (\mathbb{P}_\mathbb{C}^2 \times \mathbb{C}^5, \Lambda(t), \mathbb{C}^5, \omega) \), where \( \omega \) is the natural projection, is a Poisson analytic family with \( \omega^{-1}(0) = (\mathbb{P}_\mathbb{C}^2, \Lambda_0) \). Since the complex structure does not change in the family, the Poisson Kodaira-Spencer map is an isomorphism. Hence the Poisson analytic family is complete at 0.

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