The spectrum of the Dirac operator near zero virtuality

for $N_c = 2$

Jacobus Verbaarschot
Department of Physics
SUNY, Stony Brook, New York 11794

Abstract
We study the spectrum of the QCD Dirac operator near zero virtuality for $N_c = 2$. According to a universality argument, it can be described by a random matrix theory with the chiral structure of QCD, but with real matrix elements.

Using results derived by Mehta and Mahoux and Nagao and Wadati, we are able to obtain an analytical result for the microscopic spectral density that in turn is the generating function for Leutwyler-Smilga type spectral sum rules.
1. Introduction

Quantum Chromodynamics (QCD) for two colors ($N_c = 2$) shares many common features with QCD for $N_c = 3$, generally believed to be the correct theory of strong interactions. However, there are important differences. In particular, the low-energy excitations do not only involve mesons but also baryons, which consist out of two quarks and are bosons. The corresponding effective theory \[1\] has a much richer structure with many Goldstone particles than for $N_c = 3$. For example, for $N_f = 2$ we have nine Goldstone bosons \[1\] for the $SU(4) \to SO(4)$ symmetry breaking scheme, instead of the usual 3 (see \[2\], \[3\], \[4\] for a detailed discussion of the different chiral symmetry breaking schemes for $N_c = 2$).

As we have learnt from the work of Leutwyler and Smilga \[5\], the existence of a low-energy effective theory imposes severe restrictions on the spectrum of the Dirac operator in the form of sum rules for the inverse powers of its eigenvalues. Recently, we have shown \[6\], \[7\] that the spectral sum rules can also be obtained from a random matrix theory with the chiral symmetry of the Dirac operator. This theory enabled us to derive the joint spectral density that generates all sum rules obtained by Leutwyler and Smilga. In fact, arguments from random matrix theory imply that the spectral correlations near zero virtuality are uniquely determined by the symmetries of the system \[8\], \[9\]. In other words they are universal (see \[8\] for a systematic study of this issue). This point has been known for quite some time in the study of spectra of classically chaotic systems \[8\], \[9\], \[10\] and the theory of $S$-matrix fluctuations \[12\]. For example, both Ericson fluctuations \[13\] and universal conductance fluctuations \[14\], \[15\] can be unified within the latter context (see \[16\] for a discussion of this similarity). In particular, this means that the microscopic spectral density, defined as the $V_4 \to \infty$ limit of the spectral density while the eigenvalues are rescaled $\sim V_4$, is a universal function.

The random matrix theory that corresponds to the standard scheme of chiral symmetry breaking, has complex matrix elements. For this reason we have called it the chiral unitary ensemble (chGUE). In the framework of this model, new sum rules \[17\] can be derived with the help of the Selberg \[18\], \[19\] integral formula.

Because the effective theory for $N_c = 2$ is different from the generic case, we expect
a different microscopic spectral density and different sum rules. This raises the question what is the correct random matrix theory for $N_c = 2$. The answer becomes clear if one considers the matrix elements of the Dirac operator. In this case it is possible to choose a basis in which they are real, as opposed to three and more colors where they are complex. This reminds us of the three universality classes in random matrix theory \[20\], the Gaussian Orthogonal Ensemble (GOE), the Gaussian Unitary Ensemble (GUE) and the Gaussian Symplectic Ensemble (GSE). The first correspond the real symmetric matrices, the second to Hermitean complex matrices and the third to quaternion matrices. It is clear that the correct random matrix ensemble should not only embody the chiral symmetry of the Dirac operator but also satisfy the additional constraint that the matrix elements are real. From now on we will call this matrix ensemble the chiral orthogonal ensemble (chGOE). The third possibility is realized for fermions in the adjoint representation \[21\]. In this case it is possible to regroup the matrix elements of the Dirac operator in terms of quaternions. The corresponding ensemble will called the chiral symplectic ensemble (chGSE).

In a separate publication \[22\] we will show that the chGOE describes the spectrum of the Dirac operator in a liquid of instantons \[23\], whereas it is not given by the chGUE. In this model the Dirac operator is diagonalized in the space of fermionic zero modes. Indeed, the matrix elements of the Dirac operator are real. Also the corresponding sum rules will be given elsewhere \[17\].

In this work we report on the calculation of the microscopic spectral density (defined in section 2) of the Dirac operator for $N_c = 2$. The argument that the matrix elements of the Dirac operator are real is presented in section 3. The corresponding random matrix theory and the joint eigenvalue distribution is derived in section 4. The exact spectral density for any finite size random matrix is calculated in section 5. The microscopic spectral density is obtained in section 6 and concluding remarks are made in section 7. Some technical details are worked out in three appendices.

### 2. Formulation of the problem
The distribution of the eigenvalues of the Dirac operator are determined by the fluctuations of the gauge field which are subject to the Euclidean QCD partition function

\[ Z = \sum_{\nu} e^{i\mu \theta} \prod_{f=1}^{N_f} \prod_{\lambda_n > 0} \left( \lambda_n^2 + m_f^2 \right) \langle \nu \rangle_{S_\nu(A)}, \tag{2.1} \]

where the average \( \langle \cdots \rangle_{S_\nu(A)} \) is over gauge field configurations with topological quantum number \( \nu \) weighted by the gauge field action \( S_\nu(A) \). The product is over all eigenvalues of the Dirac operator, and relevant observables are obtained by differentiation with respect to the masses \( m_f \). The number of flavors is denoted by \( N_f \). The factor \( \exp i\theta \nu \) represents the topological term in the action.

The condensate can be expressed as a derivative of the partition function

\[ \langle \bar{q} q \rangle = \lim_{m_f \to 0} \lim_{V \to \infty} \frac{i}{V} \frac{d}{dm_f} \log Z(m_f). \tag{2.2} \]

According to the Banks-Casher [24] formula we have

\[ \langle \bar{q} q \rangle = i\pi \frac{\langle \rho(0) \rangle}{V^4}, \tag{2.3} \]

where the spectral density \( \rho(\lambda) \) is defined as

\[ \rho(\lambda) = \sum_{\lambda_n} \delta(\lambda - \lambda_n). \tag{2.4} \]

It is now clear that, in order to obtain a nonzero value of \( \langle \bar{q} q \rangle \), we should have

\[ \langle \rho(0) \rangle \sim V^4, \tag{2.5} \]

or, put differently, the spacing between the eigenvalues near zero virtuality is \( \sim 1/V^4 \) (Note that for a non-interacting system the spacing between the eigenvalues is \( \sim 1/V^{1/4} \)).

As was observed by Leutwyler and Smilga [8], (2.5) implies the existence of a family of new sum rules. The simplest one involves the sum

\[ \frac{1}{V^2} \sum_{\lambda_n > 0} \left\langle \frac{1}{\lambda_n^2} \right\rangle_{\nu} \tag{2.6} \]

which should converge to a finite limit for \( V \to \infty \).

The above mentioned sum-rules can be expressed in the microscopic spectral density defined by

\[ \rho_S(x) = \lim_{V_4 \to \infty} \frac{1}{V_4} \left\langle \rho(x) \right\rangle_{\nu}, \tag{2.7} \]
in the sector of topological charge \( \nu \). For the sum (2.6) we find

\[
\int dx \frac{\rho_S(x)}{x^2}.
\]  

(2.8)

In this paper we will obtain an analytical expression for \( \rho_S(x) \) for \( N_f \) flavors and arbitrary topological charge \( \nu \).

3. Symmetries of the Dirac operator for \( N_c = 2 \)

In this paper we study the Euclidean Dirac operator

\[
D \equiv i\gamma \partial + \gamma A,
\]  

(3.1)

where \( A \) is an \( SU(2) \) valued gauge field. The spectrum of \( D \) is defined by the eigenvalue equation

\[
D \phi_\lambda = \lambda \phi_\lambda.
\]  

(3.2)

The Dirac operator for \( SU(2) \) has two symmetries. First, the chiral symmetry, which is present for any \( SU(N_c) \)-valued gauge group,

\[
\{ \gamma_5, D \} = 0,
\]  

(3.3)

and second, a symmetry specific to \( SU(2) \):

\[
[C^{-1} \tau_2 K, D] = 0,
\]  

(3.4)

where \( C \) is the charge conjugation operator (\( \gamma^*_\mu = -C \gamma_\mu C^{-1} \) and \( CC^* = -1 \)), and \( K \) denotes the complex conjugation operator. Because of the first symmetry, the eigenvalues occur in pairs \( \pm \lambda \). The second symmetry operator has the property that

\[
(C^{-1} \tau_2 K)^2 = 1.
\]  

(3.5)

As is well known from the analysis of the of the time-reversal operator in random matrix theory (see for example refs. [25, 19]), this property allows us to choose a basis in which
the Dirac operator is real. In a chiral basis the Dirac operator therefore has the following
general structure

\[
\begin{pmatrix}
0 & T \\
\tilde{T} & 0
\end{pmatrix},
\]

(3.6)
where \( T \) is a real matrix. This is the only information of the Dirac operator that will be
injected into the random matrix model to be defined in section 4.

4. The chiral random matrix model

As has been shown in previous section, the matrix elements of the Dirac operator can
be chosen real. The corresponding random matrix theory with the chiral structure of
QCD is defined by the partition function

\[
Z_{\nu} = \int DTP(T) \prod_{f} \det \left( \begin{array}{cc}
m_f & iT \\
i\tilde{T} & m_f
\end{array} \right).
\]

(4.1)
This model is defined for \( N_f \) flavors with masses \( m_f \) in the chiral limit \((m_f \to 0)\) and in
the sector with topological charge \( \nu \). The latter property is implemented by choosing \( T \)
an \( m \times n \) matrix (for definiteness, \( m \geq n \)) with \(|m - n| = \nu \). With this choice the matrix
of which the determinant is calculated has exactly \( \nu \) zero eigenvalues for \( m_f = 0 \). The
integral is over all matrix elements of \( T \), i.e., \( DT \) is the Haar measure. As follows form
the the maximum entropy principle \[26\] the distribution function of the overlap matrix
elements \( P(T) \) is chosen Gaussian

\[
P(T) = \exp\left(-\frac{n\beta}{2\sigma^2} \text{Tr}TT\right).
\]

(4.2)
In the present case (\( T \) real) the value of \( \beta = 1 \), whereas for complex matrix elements
\( \beta = 2 \). With this convention, a mean-field argument shows that the average level density
for the chGOE is the same as for the chGUE (see appendix A). One can easily show that
the eigenvalues of a matrix with this block structure occur in pairs \( \pm \lambda \), a property that
is well-known for the nonzero eigenvalues of the massless Dirac operator. The density of
modes \( N/V_4 \) \((N = m + n)\) is taken equal to one, which allows us to identify \( N \) with the
volume of space time \( V_4 \), which we will do from now on. In agreement with general QCD
relations we will always assume that \( \nu \ll N \).
The matrix $T$ can be diagonalized by an $n \times n$ orthogonal matrix $O_1$ and an $m \times m$ orthogonal matrix $O_2$:

$$T = O_1 \Lambda O_2. \quad (4.3)$$

Here, $\Lambda$ is an $n \times m$ diagonal matrix with diagonal matrix elements $\lambda_k$. The joint eigenvalue distribution is obtained immediately by using $\Lambda$, $O_1$ and $O_2$ as new integration variables. Because the integrand only depends on $\Lambda$, the integration over the orthogonal matrices can be performed trivially. The Jacobian corresponding to the transformation (4.3) is given by (see appendix B)

$$J(\Lambda) = \prod_{k<l} |\lambda_k^2 - \lambda_l^2| \prod_{k=1}^n \lambda_k^n. \quad (4.4)$$

The joint probability density of the nonzero eigenvalues is therefore given by

$$\rho(\lambda_1, \ldots, \lambda_n) = \prod_{k<l} |\lambda_k^2 - \lambda_l^2| \prod_{k=1}^n \lambda_k^{2N_f+\nu} \exp\left(-\frac{n \beta}{2 \sigma^2} \sum \lambda_k^2\right). \quad (4.5)$$

Since this distribution is symmetric in all eigenvalues, the spectral density is simply obtained by integrating it over all eigenvalues except for one:

$$\rho(\lambda_1) = \frac{\int d\lambda_2 \cdots d\lambda_n \rho(\lambda_1, \ldots, \lambda_n)}{\int d\lambda_1 \cdots d\lambda_n \rho(\lambda_1, \ldots, \lambda_n)}. \quad (4.6)$$

The normalization integral in the denominator will be denoted by $Z(n)$.

5. Calculation of the spectral density

To make contact with what is also called the orthogonal generalized Laguerre ensemble, we introduce new integration variables by

$$x_k = \frac{n \lambda_k^2}{2 \sigma^2}. \quad (5.1)$$

If we absorb the constants in a redefinition of the normalization constant $Z$, the spectral density is given by

$$\rho(\lambda_1) d\lambda_1 = \rho_L(x_1) dx_1 = \frac{dx_1}{Z} \int_0^\infty dx_2 \cdots dx_n \prod_{k<l} |x_k - x_l| \prod_{k=1}^n x_k^n \exp\left(-\sum_{k=1}^n x_k\right). \quad (5.2)$$
For convenience we have introduced the notation
\[ a = N_f - \frac{1}{2} + \nu. \]  

(5.3)

General expressions for spectral densities given by these types of integrals have been derived by Mahoux and Mehta [27]. The result is expressed in skew-orthogonal polynomials \( R_k \) defined by
\[
\langle R_{2k}, R_{2l+1} \rangle = r_k \delta_{kl},
\]
\[
\langle R_{2k}, R_{2l} \rangle = \langle R_{2k+1}, R_{2l+1} \rangle = 0,
\]

(5.4)

with the scalar product given by
\[
\langle f, g \rangle = \int_0^\infty dx x^a e^{-x} \int_0^\infty dy y^a e^{-y} \epsilon(x - y) f(x) g(y),
\]

(5.5)

where \( \epsilon(x - y) = \frac{1}{2} \text{sign}(x - y) \). For this weight function the normalization constants \( r_k \) are known explicitly
\[
r_k = h_{2m}^{2a+1} = \frac{\Gamma(k+1)\Gamma(2a+k+2)}{2^{2a+2k+2}}.
\]

(5.6)

where \( h_{2m}^{2a+1} \) are the normalization constants of the monic generalized Laguerre polynomials with index \( 2a + 1 \). They are fixed by the normalization integrals \( Z(n) \) (see below (4.6)).

The result for the spectral density is given by
\[
\rho_L(x) = \sum_{m=0}^{(n/2)-1} \frac{1}{r_m} \left( \phi_{2m}(x) \phi'_{2m+1}(x) - \phi_{2m+1}(x) \phi'_{2m}(x) \right),
\]

(5.7)

where
\[
\phi_k(x) = \int_0^\infty dy y^a e^{-y} \epsilon(x - y) R_k(y).
\]

(5.8)

From the definition of the skew-symmetric scalar product one finds for the normalization
\[
\int_0^\infty \rho_L(x) dx = n.
\]

The polynomials \( R_k \) can be obtained most conveniently by expanding them in the monic generalized Laguerre polynomials \( C_n^{2a}(x) \equiv n! L_n^{2a}(2x)/(-2)^n \):
\[
R_{2m}(x) = \sum_{k=0}^{2m} a_{mk} C_{2m-k}^{2a}(x),
\]
\[
R_{2m+1}(x) = \sum_{k=0}^{2m+1} b_{mk} C_{2m+1-k}^{2a}(x).
\]

(5.9)
Starting from the identity
\[ \langle \frac{a}{x} - 1 \rangle f, g \rangle - \langle f', g \rangle = -\int_0^\infty x^{2a} e^{-2x} f(x) g(x) dx \] (5.10)

Nagao and Wadati \cite{29} were able to derive recursion relations for the expansion coefficients
\begin{align*}
a_{m,n+1} &= -\frac{1}{2}(2m - n)a_{m,n}, \quad n \geq 0, \\
b_{m,n+1} &= -\frac{1}{2}(2m - n + 1)b_{m,n}, \quad n \geq 2, \\
b_{m,2} &= -\frac{m}{2}(2b_{m,1} + 2a + 2m + 1), \quad (5.11)
\end{align*}

where \( b_{m,1} \) is not determined by the recursion relation, and in fact, does not contribute the spectral correlation functions. The initial conditions are fixed by
\[ a_{m,0} = b_{m,0} = 1. \] (5.12)

The solution of these recursion relations is straightforward:
\begin{align*}
a_{m,n} &= \frac{1}{(2)^n} \binom{2m}{n} n!, \quad (5.13) \\
b_{m,n} &= \frac{2a + 2m + 1}{(-2)^{n-1}} \binom{2m}{n-1} (n-1)!, \quad n \geq 2 \quad (5.14)
\end{align*}

where we made the choice (does not satisfy (5.14) for \( n = 1 \))
\[ b_{m,1} = \frac{1}{2}(2a + 2m + 1). \] (5.15)

Remarkably, using the identity \( \sum_{k=0}^{m} L_{m-k}^{a} = L_{m}^{a+1} \) (see \cite{33}) the sums in eq. (5.9) can be performed exactly. The result is
\begin{align*}
R_{2m}(x) &= C_{2m}^{2a+1}(x), \\
R_{2m+1}(x) &= C_{2m+1}^{2a}(x) + \frac{2a + 2m + 1}{2}(C_{2m}^{2a+1}(x) - mC_{2m-1}^{2a+1}(x)) \quad (5.16)
\end{align*}

The spectral density (5.7) can be written as the sum of two terms
\begin{align*}
\rho_L(x) &= x^a e^{-x} \int dy y^a e^{-y} \epsilon(x - y) \left( \sum_{m=0}^{(n/2)-1} \frac{1}{r_m} (C_{2m+1}^{2a+1}(x)C_{2m}^{2a+1}(y) - C_{2m}^{2a+1}(x)C_{2m+1}^{2a+1}(y)) \right) \\
&\quad - \sum_{m=0}^{(n/2)-1} \frac{m(2a + 2m + 1)}{2r_m} (C_{2m-1}^{2a+1}(x)C_{2m}^{2a+1}(y) - C_{2m}^{2a+1}(x)C_{2m-1}^{2a+1}(y)) \right), \quad (5.17)
\end{align*}
The two sums can be combined into a single sum. This expression can be simplified further by applying the inverse Christoffel-Darboux formula (see [33]) and collecting all terms. We find
\[
\rho_L(x) = x^a e^{-x} \int_0^\infty y^a e^{-y} \frac{|x-y|}{2} \sum_{m=0}^{n-2} \frac{n-(m+1)}{h_{2m+1}} C_{m+1}^{2a+1}(x) C_{m+1}^{2a+1}(y)
\]
(5.18)

In order to take the limit \( n \to \infty \) the \( n \)-dependence has to be made more explicit. To achieve this, we apply the Christoffel-Darboux formula once again to the term proportional to \( n \), and, after using the identity,
\[
C_{m+1}^{2a+1} = \frac{1}{m+1} \frac{d}{dx} C_{m+1}^{2a},
\]
(5.19)
to the term proportional to \((m+1)\). The term proportional to \( n \) cancels against one of the terms obtained from the differentiation with respect to \( x \) and \( y \). We finally obtain
\[
\rho_L(x) = \frac{2^{2a} n!}{\Gamma(n+2a)} x^a e^{-x} \int_0^\infty dy y^a e^{-y} \epsilon(x-y) \frac{L_n^{2a}(2x) L_n^{2a-1}(2y) - L_n^{2a-1}(2x) L_n^{2a}(2y)}{(x-y)^2}
\]
\[
+ L_n^{2a-1}(2x) L_n^{2a+1}(2y) + L_n^{2a+1}(2x) L_n^{2a-1}(2y) - 2 L_n^{2a}(2x) L_n^{2a}(2y)
\]
(5.20

This sum can be written more compactly as an integral over the unitary kernel \( K(2x, 2y) \)
\[
\rho_L(x) = x^a e^{-x} \int_0^\infty dy y^a e^{-y} \epsilon(x-y) \left( \frac{d}{dy} - \frac{d}{dx} \right) K(2x, 2y),
\]
(5.21)

with the kernel \( K(2x, 2y) \) defined by
\[
K(2x, 2y) = \frac{2^{2a} n!}{\Gamma(n+2a)} \frac{L_n^{2a}(2x) L_n^{2a}(2y) - L_n^{2a-1}(2x) L_n^{2a}(2y)}{2x-2y}.
\]
(5.22)

This kernel was first considered by Fox and Kahn [30], and was studied in great detail by Bronk [31]. The relation (5.21) shows an intimate and, an as yet not well understood, relation between the unitary kernel and the orthogonal spectral density.

6. The microscopic limit

In this section we derive the microscopic limit of the spectral density. First, we express the parameter \( \sigma \) in the mean level density. An expression suitable for the analysis of the
spectral density many level spacings away from the origin but yet far from the edge of
the semi-circle is obtained by commuting the $\epsilon$-function through the derivative operators.
The result can then be written as the sum of the chGUE spectral density plus a remaining
oscillatory term:

$$\rho_L(x) = 2x^2a e^{-2x} K(2x, 2x) + x^a e^{-x} \int_0^\infty dy y^a e^{-y} (\frac{d}{dy} - \frac{d}{dx})\epsilon(x - y) K(2x, 2y). \tag{6.1}$$

Because

$$2 \int_0^\infty dx x^2a e^{-2x} K(2x, 2x) = n, \tag{6.2}$$

the second term does not contribute to the total number of levels. As is well known for
the chGUE the large $n$-limit of the first term is a semicircle. In the normalization (4.2) of
the distribution of the matrix elements, the average level density does not depend on $\beta$
(see appendix A). The second term in (6.1) therefore does not contribute to the average
level density. From the asymptotic formula for the Laguerre polynomials (see eq. (C.2)),
it follows that in the thermodynamic limit ($n \gg 1$)

$$\rho_L(x) \sim \frac{2}{\pi} \frac{n}{\sqrt{2nx}}, \tag{6.3}$$

for $x \to 0$. The large-$n$ limit of the spectral density (4.6) at $\lambda = 0$ is then given by

$$\rho(\lambda = 0) = \rho_L(x) \frac{dx}{d\lambda} = \frac{N}{\pi \sigma}. \tag{6.4}$$

According to Banks-Casher formula \[24\], we can identify the parameter $\sigma$ as

$$\sigma = \frac{1}{\Sigma}, \tag{6.5}$$

where $\Sigma$ is the chiral condensate.

The microscopic limit of the spectral density is given by

$$\rho_S(z) = \lim_{N \to \infty} \rho(\lambda = \frac{z}{N}) \frac{d\lambda}{dz} = \lim_{N \to \infty} \rho_L(x = \frac{z^2}{8n\sigma^2}) \frac{dx}{dz}. \tag{6.6}$$

In order to evaluate this limit we start from expression (5.20) for the spectral density and
use the microscopic variables $z$ and $w$ defined by

$$x = \frac{z^2}{8n\sigma^2}, \quad \text{and} \quad y = \frac{w^2}{8n\sigma^2}, \tag{6.7}$$
where \( z \) is related to the original eigenvalues by \( z = \lambda N \).

The term in the integral proportional to \( L_n^{2a+1}L_n^{2a-1} \) does not satisfy the conditions necessary to interchange the limit and the integration. However, if we add and subtract the term

\[
\frac{2^{2a-1}n!}{\Gamma(n+2a)} x^a e^{-x} \int_0^\infty dy y^{a-1} e^{-y} L_n^{2a+1}(2x)L_n^{2a-1}(2y),
\]

(6.8)
to the integral, it can be proved by dominated convergence that in the subtracted integral the limit \( n \to \infty \) can be taken before integration. The integral (6.8) has to be performed exactly first. The result is (see appendix C)

\[
\lim_{n \to \infty} n^{-a+1} \int_0^\infty dy y^{a-1} e^{-y} L_n^{2a+1}(2y) = 2^{-a+1}.
\]

(6.9)

From the asymptotic formula for the Laguerre polynomials (C.3) it then follows that the microscopic limit of the spectral density is given by

\[
\rho_S(z) = \frac{\Sigma}{4} J_{2a+1}(z\Sigma) + \frac{\Sigma}{2} \int_0^\infty dw (zw)^{2a+1} \epsilon(z-w) \left( \frac{1}{w} \frac{d}{dw} - \frac{1}{z} \frac{d}{dz} \right) \times \frac{w J_{2a}(z\Sigma) J_{2a-1}(w\Sigma) - z J_{2a-1}(z\Sigma) J_{2a}(w\Sigma)}{(zw)^{2a}(z^2-w^2)}. \]

(6.10)

Note that half of the subtracted term is again reabsorbed into the integral. To achieve this we have used the identity

\[
\frac{\Sigma}{4} J_{2a+1}(z\Sigma) = -\frac{\Sigma}{4} \int_0^\infty dw J_{2a-1}(w\Sigma) z^{2a} \frac{d}{dz}(\epsilon z^{-2a} J_{2a}(\Sigma z)).
\]

(6.11)

In eq (6.10) we have expressed the microscopic spectral density in terms of an integral over the Bessel kernel. This kernel was studied extensively by Widom and Tracy [32].

The leading term in the small \( z \)-expansion is obtained by replacing \( z^2 - w^2 \to -w^2 \) and \( \epsilon(z-w) \to -\frac{1}{2} \). Then all integrals can be performed exactly, and after a cancellation only the integral (6.11) and the first term in (6.10) contribute to leading order. This results in

\[
\rho(z) \sim \frac{\Sigma}{2\Gamma(2a+2)} \left( \frac{z\Sigma}{2} \right)^{2a+1}.
\]

(6.12)

The simplest spectral sum rule is given by

\[
\int_0^\infty dz \frac{\rho_S(z)}{z^2} = \frac{\Sigma^2}{8(N_f + \frac{\Sigma}{2})}.
\]

(6.13)
This sum rule can also be derived from the partition function without reference to the spectral density \[17\].

In the case of zero flavors \((N_f = 0)\) and zero topological charge \((\nu = 0)\) a direct numerical simulation of the random matrix model (4.1) is particularly simple. To convince the reader that (6.10) is correct we show in Fig. 1 a histogram obtained from the diagonalization of 10,000 \(128 \times 128\) matrices (dashed curve) and the exact microscopic limit (6.10) for \(a = 0\) (full line). Perfect agreement is no exaggeration in this case. In Fig. 2 we show the results of the microscopic spectral density for \(N_f = 1, 2\) and 3. We observe much less oscillations than in the chGUE case. This agrees with general property known from the classical random matrix ensembles that spectra of real matrices are much less rigid than spectra of complex matrices. Therefore, the variation of each level about its average position, called level motion, is much larger resulting in the (almost) absence of oscillations.

7. Conclusions

In this work we have studied the spectrum of the QCD Dirac operator for an \(SU(2)\)-valued gauge field in the fundamental representation. This case differs from any other nonabelian gauge group in the fundamental representation by the reality constraints of the eigenfunctions: it is possible to choose a basis in which the matrix elements of the Dirac operator are real. As we have argued before for the chGUE case, the correlations between the eigenvalues of the QCD Dirac operator near zero virtuality are insensitive to the detailed dynamics of the system and can be described by a random matrix model with as only its symmetries as input. There is no reason to believe that the present case is different. However, the appropriate random matrix ensemble not only has the chiral structure of QCD but also has real matrix elements. For this reason, it will be called the chiral orthogonal ensemble, abbreviated by chGOE.

Because of the spontaneous breaking of chiral symmetry, the spectral density near zero is \(\sim V_4\). This property allows us to define a limit \(V_4 \rightarrow \infty\) of the spectral density in which the eigenvalues are at the same time rescaled by a factor \(V_4\). The resulting spectral
density, called the microscopic spectral density \( \rho_S(z) \), is insensitive to the dynamics of the system. It is a universal function that is entirely determined by the symmetries of the Dirac operator.

In this paper we have evaluated \( \rho_S(z) \) for the case of \( SU(2) \) in which the Dirac operator is real. This turned out to be much more difficult than for a complex Dirac operator, a well known property of random matrix theory. The spectral density in the present case differs from the chGUE case by the absence of strong oscillations. As is also the case for the classical random matrix ensembles, the spectrum of a complex matrix is much more rigid than the spectrum of a real matrix. This can be made more quantitative in terms of the so called level motion which turns out to be much larger for the chGOE than for the chGUE.

The microscopic spectral density is generating function for the Leutwyler-Smilga sum rules. The sum rules for for \( N_c = 2 \) have not been obtained before. In view of the fact that the corresponding static effective field theory involves both baryons (which are bosons in this case) and mesons [1], it is not surprising that the results are different from for other nonabelian gauge groups in the fundamental representation. It would be interesting to derive the sum rules from the static limit of the effective field theory also in this case. In the case of one flavor the effective theory for \( N_c = 2 \) and other nonabelian gauge groups coincides. Although the spectral density is different, all spectral sum rules for \( N_f = 1 \) turn out to be the same for the chGOE and chGUE. Further work to clarify this issue is in progress [17].

Appendix A

The joint eigenvalue density valid for both the chGOE (\( \beta = 1 \)) and the chGUE (\( \beta = 2 \)) is given by

\[
\rho_\beta(\lambda_1, \ldots, \lambda_n) = C_{\beta,n} \prod_{k,l} |\lambda_k^2 - \lambda_l^2|^\beta \prod_k \lambda_k^\alpha \exp\left(-\frac{n\beta \Sigma^2}{2} \sum_k \lambda_k^2\right).
\]

(A.1)

where \( C_{n,\beta} \) are normalization constants and \( \alpha = (2N_f + \beta \nu + \beta - 1) \).
The normalization integral \( \int d\lambda_1 \cdots d\lambda_n \rho_\beta(\lambda_1, \cdots, \lambda_n) \) can be approximated by

\[
\exp \left[ \beta \int d\lambda d\lambda' \log |\lambda^2 - \lambda'^2| \bar{\rho}(\lambda) \bar{\rho}(\lambda') + \int d\lambda \bar{\rho}(\lambda) \left( -\frac{n\beta}{2\sigma^2} \lambda^2 + \alpha \log \lambda + \mu \bar{\rho}(\lambda) \right) \right],
\]

(A.2)

where the average level density \( \bar{\rho}(\lambda) \) satisfies a 'mean-field' equation obtained by minimizing the exponent:

\[
2\beta \int d\lambda' \log |\lambda^2 - \lambda'^2| \bar{\rho}(\lambda') - \frac{n\beta}{2\sigma^2} \lambda^2 + \alpha \log \lambda + \mu = 0.
\]

(A.3)

The normalization of the level density is introduced via a Lagrange multiplier. By differentiation with respect to \( \lambda^2 \) we obtain the principal value equation

\[
2\beta P \int \frac{d\lambda'}{\lambda^2 - \lambda'^2} \bar{\rho}(\lambda') - \frac{n\beta}{2\sigma^2} + \frac{\alpha}{2\lambda^2} = 0.
\]

(A.4)

Since \( \int \bar{\rho}(\lambda) d\lambda = n \) the third term is subleading for \( n \to \infty \) (in agreement with general properties of topological fluctuations we have \( \nu \ll n \)). Consequently, the 'mean-field' equation for \( \bar{\rho} \) does not depend on \( \beta \). A more elaborate discussion of this argument for the classical random matrix ensembles can be found in the book by Mehta [19].

**Appendix B**

In this appendix we calculate the Jacobian of the transformation of the matrix valued variables \( T \) into its eigenvalues and eigenangles using techniques developed in [34]. For an arbitrary real \( n \times m \) matrix we have

\[
T = O_1 \Lambda O_2,
\]

(B.1)

where \( \Lambda \) is a positive \( n \times m \) diagonal matrix and, the \( n \times n \) matrix \( O_1 \) and the \( m \times m \) matrix \( O_2 \) are orthogonal (for definiteness \( m > n \)). By differentiation of (B.1) it can be shown that that the variation \( dT \) can be expressed in variations \( \delta O_i \) of \( O_i \) near the identity

\[
O_1^{-1} dT O_2 = \delta O_1 \Lambda + d\Lambda - \Lambda \delta O_2, \quad (B.2)
\]

\[
O_2^{-1} dT O_1 = -\Lambda \delta O_1 + d\Lambda + \delta O_2 \Lambda. \quad (B.3)
\]
where we have introduced $\delta \mathcal{O}_i = \mathcal{O}_i^{-1}d\mathcal{O}_i$. Note that the matrices $\delta \mathcal{O}_i$ are anti-symmetric, and, in particular the diagonal matrix elements are zero. For the invariant distance we find

$$
\text{Tr} \tilde{T} d\tilde{T} = \sum_k (d\Lambda_k)^2 + \sum_{k<l} \frac{1}{2} (\delta \mathcal{O}_1 - \delta \mathcal{O}_2)_{kl}^2 (\lambda_k + \lambda_l)^2 + \sum_{k<l} \frac{1}{2} (\delta \mathcal{O}_1 + \delta \mathcal{O}_2)_{kl}^2 (\lambda_k - \lambda_l)^2 + \sum_{k=1}^n \sum_{l=n+1}^m (\delta \mathcal{O}_1)_{kl}^2 \lambda_k^2. \quad (B.4)
$$

This allows us to read off the Jacobian of the transformation to the variables $d\Lambda_k$ and $(\delta \mathcal{O}_1 \pm \delta \mathcal{O}_2)/\sqrt{2}$ from the Lamé-coefficients:

$$
J = \prod_{k<l} (\lambda_k - \lambda_l)^2 \prod_k \lambda_n^{m-n}. \quad (B.5)
$$

Note that the total powers of lambda can be verified by a dimensional argument.

**Appendix C**

In this appendix we consider the large $n$ limit of integrals of the type

$$
I_n^{a,k} = n^{k+1-a} \int_0^\infty L_{n}^{2a-k}(2y)y^a e^{-y} dy, \quad (C.1)
$$

where $k$ is an integer, and $n$ is even. From the asymptotic expansion for $n \to \infty$ of the generalized Laguerre polynomials (see \[33\] for this and other properties of the generalized Laguerre polynomials used below), among others,

$$
L_\alpha^\alpha(x) \sim \frac{1}{\sqrt{\pi} x^{3/4} (\alpha - 1/2)^n} e^{2x} \cos(2\sqrt{nx} - \frac{\alpha \pi}{4} - \frac{\pi}{4}), \quad (C.2)
$$

it is clear that for $k < -3/2$ and $a > -1$ the absolute value of the integrand can be majorized by an integrable function. In that case we can, after introducing a new integration variable by $y = w^2/2n$, interchange the limit $n \to \infty$ and the integration over $w$. Using the asymptotic result

$$
\lim_{n \to \infty} L_\alpha^\alpha(\frac{w^2}{n}) \sim n^{-a} w^{-a} J_\alpha(2w), \quad (C.3)
$$

with $J_\alpha$ a Bessel function, we find that

$$
\lim_{n \to \infty} I_n^{a,k} = 2^{-a} \int dw w^{k+1} J_{2a-k}(2w). \quad (C.4)
$$
This integral can be evaluated analytically, resulting in

$$\lim_{n \to \infty} I_{n}^{a,k} = 2^{-a-1} \frac{\Gamma(a+1)}{\Gamma(a-k)}.$$  \hfill (C.5)

For $-\frac{3}{2} < k < -\frac{1}{2}$ (since $k$ is an integer: $k = -1$) and $a > -1$ the integral still converges but no longer satisfies the conditions necessary to interchange the limit and the integral. In this case there is an important contribution from the region around the largest zero of $L_{n}^{\alpha}$. However, for $k = -1$, it is particularly simple to evaluate the integral for any finite value of $n$. From the recursion relation

$$L_{n}^{2\alpha+1}(2y) = \sum_{m=0}^{n} L_{m}^{\alpha}(y)L_{n-m}^{\alpha}(y)$$  \hfill (C.6)

we can reduce the integral to a normalization integral for the Laguerre polynomials resulting in

$$I_{n}^{\alpha,-1} = n^{-a} \frac{\Gamma(a + \frac{n}{2} + 1)}{\Gamma(\frac{\alpha}{2} + 1)}.$$  \hfill (C.7)

The asymptotic limit follows immediately from Stirling’s formula

$$\lim_{n \to \infty} I_{n}^{\alpha,-1} = 2^{-a},$$  \hfill (C.8)

which is a factor 2 bigger than the result given in (C.5). We conclude that half of the contribution to this integral is from the region near the largest zero of the Laguerre polynomial.
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Figure Captions

Fig. 1. The microscopic spectral density $\rho_S(z)$ versus $z$ for $N_f = 0, \nu = 0$ and $\Sigma = 2$. The full line represents the exact analytical result and the cashed curve shows data obtained by diagonalizing 10,000 random matrices distributed according to (4.1).

Fig. 2. The microscopic spectral density $\rho_S(z)$ versus $z$ for $N_f = 1$ (dotted curve), $N_f = 2$ (dashed curve) and $N_f = 3$ (full curve) all for $\nu = 0$ and $\Sigma = 2$. 