Applications of non-perturbative renormalization

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Abstract. A short survey of the renormalization problem in QCD and its non-perturbative solution by means of numerical simulations on the lattice is given. Most emphasis is on scale dependent renormalizations, which can be reliably addressed via a recursive finite-size scaling procedure employing a suitable intermediate renormalization scheme. To illustrate these concepts we discuss some — partly recent — computations of phenomenologically relevant quantities: the running QCD gauge coupling, renormalization group invariant quark masses and the renormalization of the static-light axial current.

1 Introduction

Apart from its well established rôle as a non-perturbative framework to calculate relations between Standard Model parameters and experimental quantities from first principles [1], Lattice Field Theory is particularly designed to solve various renormalization problems in QCD [2, 3]. Since renormalized perturbation theory as analytical tool is limited to high energy processes, where the QCD coupling is sufficiently small, but inadequate for bound states and momentum transfers of the order of typical hadronic scales, \( \mu \simeq 1 \text{ GeV}/c \), a genuinely non-perturbative solution of the theory is generally required. This is achieved by numerical Monte Carlo simulations of the Euclidean QCD path integral on a space-time lattice. Though renormalization is an ultraviolet phenomenon (relevant scales \( \mu^{-1} \sim a \)) and QCD asymptotically free, tolerable simulation costs prevent the lattice spacing \( a \) from becoming much smaller than the extent of physical observables so that a truncation of the lattice perturbative series is often not justified. Therefore, it is far more safe to perform renormalizations non-perturbatively.

In addition, Lattice QCD has a large potential to address the computation of fundamental parameters of the theory, which escape a direct determination by experiments. The most prominent ones among them are the QCD coupling constant itself and the quark masses, whose running with the energy scale is desirable to be understood on a quantitative level beyond perturbation theory — the central subject of the next sections. The knowledge of these quantities (e.g. at some common reference point, \( \alpha_s(M_Z) \) or \( \overline{m}(2 \text{ GeV}) \)) might then also provide essential input to theoretical analyses of observables of phenomenological interest. For instance, the mixing ratio \( \epsilon'/\epsilon \) in the neutral kaon system incorporates the strange quark mass value.

The renormalization properties of many other quantities have been investigated with lattice methods, e.g. (bilinear) quark composite operators, \( \Delta S = 2 \) matrix elements and, as presented at this conference too, structure functions [4]. Later we will briefly discuss the non-perturbative renormalization of the static axial current as a further example.

During the last few years the lattice community has seen much theoretical and numerical advances [1, 5]. Here it is worth to mention at least the issue of \( O(a) \) discretization effects inherent in the Wilson fermion action. In case of the quenched approximation to QCD, where all dynamics due to virtual quark loops is ignored, they have been systematically eliminated through a non-perturbative realization of Symanzik’s improvement programme [6, 3, 7]. Hence, lattice artifacts can be extrapolated away linearly in \( a^2 \), which allows to precisely extract many physical quantities in the continuum limit, \( a \to 0 \).

2 Intermediate schemes

As a representative example for a non-perturbative renormalization problem we may consider the calculation of quark masses through the PCAC relation,

\[
F_K m_K^2 = (\overline{m_u} + \overline{m_s}) \langle 0|\overline{q}_s \gamma_5 q_u|K\rangle \quad (1)
\]
\[ L_{\text{max}} = C/F_\pi \cdot O(1/2 \text{fm}) \quad \text{hadronic scheme} \leftrightarrow \text{SF} \quad \rightarrow \quad \alpha_{\text{SF}} (\mu = 1/L_{\text{max}}) \]

\[ \alpha_{\text{SF}} (\mu = 2/L_{\text{max}}) \downarrow \]

\[ \alpha_{\text{SF}} (\mu = 2^n/L_{\text{max}}) \downarrow \cdots \cdots \downarrow \]

\[ \alpha_{\text{SF}} (\mu = 2^n/L_{\text{max}}) \downarrow \text{PT} \downarrow \lambda_{\text{SF} L_{\text{max}}} \]

\[ \left( \pi \gamma_5 s \right)_{\text{MS}} = Z_P (g_0, a\mu) \left( \pi \gamma_5 s \right)_{\text{lattice}}, \quad (2) \]

in which the scale and scheme dependent renormalization constant \( Z_P \) relates the lattice results to the \( \overline{\text{MS}} \) scheme and is computable in lattice perturbation theory. But since this expansion introduces errors which are difficult to control, a non-perturbative determination of the renormalization factor is needed. A non-perturbative renormalization condition between the two schemes can, however, not be formulated, because \( \overline{\text{MS}} \) is only defined perturbatively.

The idea to overcome this problem is the introduction of an intermediate renormalization scheme: the lattice observable is first matched at some fixed scale \( \mu_0 \) to the corresponding one in the intermediate scheme, and afterwards it is evolved from \( \mu_0 \) up to high energies, where perturbation theory (PT) is expected to work well. Nonetheless, as in a simulation one then has to cover many scales (the box size \( L \), \( \mu \approx 0.2 \text{ GeV} - 10 \text{ GeV} \) and the lattice cutoff \( a^{-1} \)) simultaneously, the task to reliably match the low energy regime with the high energy one, i.e., the applicability domain of perturbation theory, gets quite complicated. In the present context two implementations of such schemes are available, the regularization independent approach \( \text{A} \) and the QCD Schrödinger functional (SF) \( \text{B} \). Whereas the former may suffer from the scale hierarchy problem in practice, the basic strategy of the SF approach is to recourse to an intermediate finite-volume renormalization scheme, where one identifies two of the before-mentioned scales, \( \mu = 1/L \), and takes low energy data as input in order to use the non-perturbative renormalization group to scale up to high energies \( \text{A} \) \( \text{B} \).

A schematic view of a non-perturbative computation of short distance parameters on the lattice along these lines, here in case of the running QCD coupling \( \alpha (\mu) \), is given in the diagram above; the same can also be set up for the running quark masses. It is important to note that all relations ‘\( \rightarrow \)’ are accessible in the continuum limit and in this sense universal by construction.

### 3 \( \Lambda_{\text{QCD}} \) and \( M_{\text{quark, RGI}} \) via the SF

The Schrödinger functional is the QCD partition function with certain Dirichlet boundary conditions in time imposed on the quark and gluon fields, for which a renormalized coupling constant can be defined as the response to an infinitesimal variation of the boundary conditions \( \text{B} \). By help of the so-called step scaling function, being a measure for the change in the coupling when changing the box size \( L \) (and thus having the meaning of a discrete \( \beta \)-function), one is now able in the SF scheme to make contact with the high-energy regime of perturbative scaling:

\[ \lambda \equiv \lim_{\mu \rightarrow \infty} \left\{ \mu (b_0 g^2 (\mu))^{-b_1/2 b_2} e^{-1/2 b_0 g^2} \right\} \]

\[ b_0 = 11/(4 \pi^2), \quad b_1 = 102/(4 \pi^4). \quad (3) \]

Every step during the non-perturbative evolution towards the perturbative regime has been extrapolated to the continuum limit in the quenched approximation \( \text{A} \), and upon conversion to the \( \overline{\text{MS}} \) scheme this results in a value for the \( \Lambda \)-parameter:

\[ \Lambda_{\text{MS}}^{(0)} = 238(19) \text{ MeV}. \quad (4) \]

An extension of this investigation to the situation with two dynamical quarks is already in progress by the ALPHA Collaboration.

In a very similar way, in terms of the current quark mass renormalization factor \( Z_P \) of eq. \( \text{B} \) replacing the SF coupling to build up another step scaling function, the scale and scheme independent renormalization group invariant (RGI) quark masses

\[ M \equiv \lim_{\mu \rightarrow \infty} \left\{ (2 b_0 g^2 (\mu))^{-d_0/2 b_0} \overline{m} (\mu) \right\} \]

\[ b_0 = 11/(4 \pi^2), \quad d_0 = 8/(4 \pi^2) \quad (5) \]

were obtained in the same reference. Both evolutions are displayed in Fig. \( \text{B} \), and at the scale \( \mu_0 \) (leftmost point in Fig. \( \text{B} \)) the matching between the lattice regularization and \( \overline{\text{MS}} \) via the SF is completed:

\[ \frac{M}{m_{\text{SF}} (\mu_0)} = 1.157(12), \quad \mu_0 \simeq 275 \text{ MeV}. \quad (6) \]
can also be summarized as renormalization scheme as utilized here, these results as known from chiral perturbation theory \((\chi PT)\)\(^\text{[11]}\). Their ratios are independent renormalization schemes too, one arrives in the case of quenched QCD \([\text{14}]\). The remaining task is now to calculate \(M_{\text{ref}}\) from Lattice QCD.

As the foregoing discussion holds true in mass independent renormalization schemes too, one arrives by virtue of the PCAC relation applied to the vacuum-to-pseudoscalar matrix elements at the central relation

\[
2r_0 M_{\text{ref}} = Z_M R \left| m_{PS}^2 r_0^2 = 1.5736 \right. 
\]

where \(Z_M\) is the flavour independent renormalization factor of the previous section, which directly leads to the RGI quark masses, being pure numbers and not depending on the scheme. By means of numerical simulations of the SF in large volumes of size \((1.5 \text{fm})^3 \times 3 \text{fm}\), the ratio \(R/a\) and the meson mass \(m_{PS}^2\) can be computed accurately as a function of the bare quark mass and the bare coupling by evaluating suitable correlation functions \([\text{13} \text{[14]}]\). With the only using Lattice QCD. They concern the applicability of \(\chi PT\) in general, i.e. in how far the lowest orders dominate the full result, and the problem that the parameters in the chiral Lagrangian (at a given order in the expansion) can not be inferred with great precision from experimental data alone. This statement holds in particular for the overall scale of the quark masses, which is only defined once the connection with the fundamental theory, QCD, is made. Since the parameters in the chiral Lagrangian (the so-called low energy constants) are independent of the quark masses, it is important to realize that these problems can be dealt with by working with unphysical — of course not too large — quark masses, where it is essential or at least of significant advantage to explore a certain range of quark masses. While a determination of some low energy constants based on these ideas has been recently tested in \([\text{15}]\), we focus in the following on the computation of the renormalization-group invariant mass of the strange quark by combining \(\chi PT\) with lattice techniques.

To this end, and in the spirit of the considerations before, we define a reference quark mass \(M_{\text{ref}}\) implicitly through

\[
m^2_{PS}(M_{\text{ref}}) r_0^2 = (m_{K^0})^2 = 1.5736 \text{.} \tag{9}\]

Here \(m^2_{PS}(M)\) is the pseudoscalar meson mass as a function of the quark mass for mass-degenerate quarks, and \(r_0 = 0.5 \text{ fm}\) and \(\frac{1}{2} (m_{K^+}^2 + m_{K^0}^2) \text{ pure QCD} = (495 \text{ MeV})^2\) enter the r.h.s. of eq. \((8)\). \(\chi PT\) in full QCD relates \(M_{\text{ref}}\) to the other light quark masses viz.

\[
2 M_{\text{ref}} \simeq M_s + \tilde{M}, \tag{10}\]

which has been substantiated also numerically in the case of quenched QCD \([\text{14}]\). The remaining task is now to calculate \(M_{\text{ref}}\) from Lattice QCD.

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\]

Figure 1: Non-perturbative scale evolution of \(\alpha_{\text{SF}}\) and \(m^\text{SF}/M\) computed from simulations of the SF in the quenched approximation. The lines represent perturbative predictions involving the 2– and 3–loop \(\beta\)-function (a) and 1/2–, 2/2– and 2/3–loop expressions for the \(\tau\– and \(\beta\–functions, respectively (b).

For the \(O(a)\) improved theory and a massless renormalization scheme as utilized here, these results can also be summarized as

\[
M = Z_M (g_0) \times m(g_0) + O(a^2) \\
Z_M (g_0) = \frac{M}{\langle m(\mu) \rangle} \times \frac{m(\mu)}{m(g_0)} , \tag{7}\]

where \(m\) is the bare current quark mass, and the flavour independent total renormalization factor \(Z_M\), non-perturbatively known for a range of bare couplings \(g_0\) in the quenched approximation \([\text{14}]\), is composed of an universal part, \(M/\langle m \rangle\), and of \(\langle m \rangle/m = Z_A/Z_F\) depending on the lattice regularization.

4 The strange quark’s mass

In order to illustrate the non-perturbative quark mass renormalization just explained in a concrete numerical application, we first sketch our strategy for the computation of light quark masses \([\text{14}]\). Their ratios are known from chiral perturbation theory \((\chi PT)\)\([\text{11}]\). Their ratios are known from chiral perturbation theory \((\chi PT)\)\([\text{11}]\). Their ratios are known from chiral perturbation theory \((\chi PT)\)\([\text{11}]\). Their ratios are independent renormalization schemes too, one arrives in the case of quenched QCD \([\text{14}]\). The remaining task is now to calculate \(M_{\text{ref}}\) from Lattice QCD.

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\[
R \equiv \frac{F_{PS}}{G_{PS}} \tag{11}\]

with \(\tilde{M} = \frac{1}{2} (M_u + M_d) \tag{3}\). Nevertheless there are still questions, which might be answered decisively
values for the scale \( r_0 / a \) from \(^{10}\), a mild extrapolation yields \( R / a \) at the point \( m_{\text{ref}}^2 r_0^2 = 1.5736 \). Then the quantity \( 2 r_0 M_{\text{ref}} \) is extrapolated to the continuum limit. Both fits are shown in Fig. 2. In view of the still significant slope in the latter, we discard the point furthest away from the continuum in this extrapolation as a safeguard against higher order lattice spacing effects. Moreover, the analysis was repeated for \( M_{\text{ref}} \) in units of the kaon decay constant, which amounts to substitute eq. (11) by

\[
2 M_{\text{ref}} = \frac{M}{m} \frac{1}{Z_P r_0^2 G_{\text{PS}}} 1.5736. \tag{12}
\]

Here we observe a weaker lattice spacing dependence. The final results of these analyses

\[
2 r_0 M_{\text{ref}} = 0.36(1) \quad r_0 = 0.5 \text{ fm} \quad 2 M_{\text{ref}} = 143(5) \text{ MeV} \\
2 M_{\text{ref}} = 0.87(3) \quad (F_K)^R = 160 \text{ MeV} \quad 2 M_{\text{ref}} = 140(5) \text{ MeV}
\]

are completely consistent with each other. But, as also pointed out in that reference, the assignment of physical units is intrinsically ambiguous in the quenched approximation. Consulting e.g. the recent results of the CP-PACS Collaboration \(^{17}\), roughly 10% larger numbers would be obtained, if the scale \( r_0 \) were replaced by one of the masses of the stable light hadrons. \( \overline{\text{MS}} \) masses for finite renormalization scales \( \mu \) can be obtained through perturbative conversion factors known up to 4–loop precision. A typical result is

\[
m_{\pi}^{\overline{\text{MS}}} (2 \text{ GeV}) = 97(4) \text{ MeV}, \tag{13}
\]

where the uncertainty in \( \Lambda_{\overline{\text{MS}}}^{(0)} \), eq. (1), entering the relation of the running quark masses in the \( \overline{\text{MS}} \) scheme to the RGI masses, eq. (3), and the quark mass ratios from full QCD chiral perturbation theory, eq. (5), were taken into account \(^{14}\).

A compilation of lattice results on the strange quark mass in the quenched approximation can be found e.g. in \(^{18}\). Most of these differ in the Ward identity used and in whether non-perturbative renormalization and a continuum extrapolation has been performed or not; also systematic errors often are not estimated uniformly either. Our result (13) includes all errors except quenching. Finally it is interesting to note that, as reported by the CP-PACS Collaboration in their comprehensive study about simulations with two dynamical flavours \(^{19}\), dynamical quark effect appear to decrease the estimates for the strange quark mass by \( \sim 20\% \) or less.

![Figure 2: Extrapolations of the ratio \( R \) to the kaon mass scale and, in units of \( r_0 \), to the continuum limit.](image)

\section{5 The static-light axial current}

Let us turn to another example, where a scale and scheme dependent renormalization is encountered, i.e. the matrix element \( \langle 0 | (A_R)_{\mu} | B(p) \rangle = i p_{\mu} F_B \) describing leptonic B–decays in the theory with heavy quarks. It involves the renormalized axial current, \( (A_R)_{\mu} = Z_{\mu} \vec{D}_{\mu} \gamma_\tau d \), and the decay constant \( F_B \), which is by its own an interesting quantity for a first principles computation on the lattice. Since \( m_b \sim 4 \text{ GeV} \gg \Lambda_{\text{QCD}} \) implies large discretization errors of \( O((am_b)^2) \), a direct treatment assuming a relativistic b quark is difficult on the lattice. Therefore, in the first place one may restrict to an effective theory, one possibility being the static approximation, where the b quark is taken to be infinitely heavy.

As at the end we want to relate the physical matrix element \( \Phi \) at a scale \( \mu = m_b \),

\[
F_B \sqrt{m_B} \equiv \Phi(\mu) + O(\frac{\Lambda_{\text{QCD}}}{m_b}), \tag{14}
\]

to the one determined on the lattice at some matching scale \( \mu_0 \), a crucial ingredient in its (scale and scheme independent) renormalization group invariant counterpart

\[
\Phi_{\text{RGI}} = \lim_{\mu \to \infty} \left\{ (2b_0 \vec{F}(\mu))^{-\gamma_0/2b_0} \Phi(\mu) \right\}
\]

\[
b_0 = 11/(4\pi^2), \quad \gamma_0 = 3 - 1/4\pi^2 \tag{15}
\]

to be passed into the factorization

\[
\Phi(\mu) = \frac{\Phi(\mu)}{\Phi_{\text{RGI}}} \frac{\Phi_{\text{SF}(\mu_0)}}{\Phi_{\text{SF}(\mu_0)}}, \tag{16}
\]
As already anticipated in the notation, the further strategy is basically analogous to that explained when considering the coupling and the quark masses: we again adopt the SF framework and invoke an appropriate step scaling function, while everything is meant in the static approximation now.

The definition of the renormalized static axial current and the step scaling function, together with the 2–loop anomalous dimension, has recently been worked out perturbatively \cite{20}. The preliminary status of the outcome of the corresponding non-perturbative investigation by numerical simulations of the SF in the quenched approximation is depicted in Fig. 3. The leftmost factor in eq. (16) is then supposed to be inferable in that region, where perturbation theory is feasible again.

6 Conclusions

Numerical simulations on the lattice can be applied to renormalization problems in QCD. In particular, the Schrödinger functional scheme offers a clean and flexible approach to deal with the accompanying scale differences. As a consequence of good control over statistical, discretization and systematic errors, non-perturbative coupling and quark mass renormalization can be performed with confidence, and solid results for $\lambda^{(0)}_{\overline{MS}}$ and $m^{\overline{MS}}_{\overline{MS}}$ with high precision of the order of a few % were reached in the quenched approximation. Similar ideas are now carried over to the heavy quark sector of QCD, where first steps towards a computation of renormalization group invariant matrix elements in the static approximation are underway.

The presented concepts will be valuable also for full QCD. Despite more powerful (super-)computers continuously being developed, a quantitative understanding of dynamical sea quark effects is a great challenge which, albeit in sight, still demands for much effort on the theoretical as well as on the technical/implementational side of Lattice QCD.

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