3+1d Boundaries with Gravitational Anomaly of 4+1d Invertible Topological Order for Branch-Independent Bosonic Systems

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We study bosonic systems on a spacetime lattice (with imaginary time) defined by path integrals of commuting fields. We introduce a concept of branch-independent bosonic (BIB) systems, whose path integral is independent of the branch structure of the spacetime simplicial complex, even for a spacetime with boundaries. In contrast, a generic lattice bosonic (GLB) system's path integral may depend on the branch structure. We find the invertible topological order characterized by the Stiefel-Whitney cocycle (such as 4+1d \(w_3\)) to be nontrivial for branch-independent bosonic systems, but this topological order and a trivial gapped tensor product state belong to the same phase (via a smooth deformation without any phase transition) for generic lattice bosonic systems. This implies that the invertible topological orders in generic lattice bosonic systems on a spacetime lattice are not classified by the oriented cobordism. The branch dependence on a lattice may be related to the orthonormal frame of smooth manifolds and the framing anomaly of continuum field theories. In general, the branch structure on a discretized lattice may be related to a frame structure on a smooth manifold that trivializes any Stiefel-Whitney classes. We construct branch-independent bosonic systems to realize the \(w_3\) topological order, and its 3+1d gapped or gapless boundaries. One of the gapped boundaries is a 3+1d \(Z_2\) gauge theory with (1) fermionic \(Z_2\) gauge charge particle which trivializing \(w_2\) and (2) “fermionic” \(Z_2\) gauge flux line trivializing \(w_3\). In particular, if the flux loop’s worldsheet is unorientable, then the orientation-reversal 1d worldline must correspond to a fermion worldline that does not carry the \(Z_2\) gauge charge. We also explain why Spin and Spin\(^c\) structures trivialize the \(w_3\) nonperturbative global gravitational anomaly to zero (which helps to construct the anomalous 3+1d gapped \(Z_2\) and gapless all-fermion U(1) gauge theories), but the Spin\(^n\) and Spin\(^n\times Z_2\)Spin\((n \geq 3)\) structures modify the \(w_3\) into a nonperturbative global mixed gauge-gravitational anomaly, which helps to constrain Grand Unifications (e.g., \(n = 10, 18\)) or construct new models.

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I. INTRODUCTION

Gapped quantum states of matter (or more precisely, gapped quantum liquids [1, 2]) with no symmetry can be divided into two classes [3]:
(1) States with no topological order. All those states belong to the trivial phase represented by tensor product states, with no entanglement or short-range entanglement.
(2) States with topological order [4, 5] (i.e. gapped states with long-range entanglement [6]).

In the presence of global symmetry that is not spontaneously broken, the above two classes can be further divided into some subclasses:
(1a) States with no topological order and no symmetry-protected topological (SPT) order, or synonymously, symmetry-protected trivial (SPT) order, with no entanglement or short-range entanglement.
(1b) States with no topological order but with nontrivial SPT order [7–19], with short-range entanglement.
(2a) States with both topological order and symmetry. Those states are said to have symmetry enriched topological (SET) orders [20–31], with long-range entanglement. In this work, we aim to study those topological states of matter and their boundaries for bosonic systems. We realize that even bosonic systems without any symmetries can have many different types, such as bosonic systems on a spacetime lattice with imaginary time, bosonic systems in continuum spacetime with real or imaginary time, and bosonic systems defined via lattice Hamiltonian with real continuous time. Those different bosonic theories require different mathematical descriptions. In this work, we will only study bosonic systems on a spacetime lattice with imaginary time, that satisfy the reflection positivity. In fact, we will study a simpler problem – the so called invertible topological states of matter in the bulk, and their boundaries, with or without symmetry.

Stacking two topological states, $\mathcal{C}_1$ and $\mathcal{C}_2$, gives a third topological state $\mathcal{C}_3 = \mathcal{C}_1 \boxtimes \mathcal{C}_2$. The stacking operation $\boxtimes$ makes the set of topological states into a monoid. (A monoid is like a group except its elements may not have inverse.) If a topological state $\mathcal{C}$ has an inverse under the stacking operation $\boxtimes$, i.e. there exists a topological state $\mathcal{D}$ such that $\mathcal{C} \boxtimes \mathcal{D}$ is a trivial product state, then $\mathcal{C}$ will be called an invertible topological state. It turns out that all SPT states are invertible. Only a small fraction of topologically ordered states and SET states are invertible. For example, a fermionic integer quantum Hall state is an invertible topologically ordered state (a fermionic invertible SET with U(1) symmetry). An antiferromagnetic spin-1 Haldane chain is a SPT state protected by spin rotation SO(3) or time reversal $Z_2$ symmetry, which is always invertible. (In contrast, an antiferromagnetic spin-2 chain, another Haldane phase, has trivial SPT order.)

Invertible topological states of bosonic systems are characterized by a simple class of invertible topological invariants [16, 32–38]. In this article, we will derive the possible boundary theory from those topological invariants, especially the 4+1d invertible topological order characterized by the Stiefel-Whitney $w_3$ topological invariant in 5d. There are some earlier works in this direction [34, 39–49] which construct boundary theories of the $w_3$ invertible topological order. In this work, we will present a more complete and systematic derivation.

Branch-Independent Bosonic System vs Generic Lattice Bosonic System: Even bosonic systems on a spacetime lattice can have different types. We find that in order to discuss invertible topological order, we need to introduce a concept of branch-independent bosonic system (the L-type system studied in Ref. 50 happens to be a branch-independent bosonic system, L for Lagrangian formulated on the spacetime lattice), whose partition function computed from path integral is independent of the branch structures (or branching structures) of spacetime lattice (i.e. spacetime simplicial complex), even for spacetime with boundaries. Here a branch structure is a local ordering of the vertices for each simplex (see Fig. 1 and Appendix A). Generic lattice bosonic systems do not have this requirement, and their path integrals may depend on the branch structures of spacetime complex.

Branch-independence is a property of lattice bosonic systems. It appears that such a lattice property is related to frame-independence, a property of continuum
An orthonormal frame of a d-dimensional manifold is a set of d vector fields, which is orthonormal at every point respect to the metrics tensor of the manifold. We will abbreviate orthonormal frame as frame (which is also called the vielbein). After assigning a frame to a manifold, we can define an SO(d) connection to describe the curvature. Continuum field theory explicitly depends on the SO(d) connection and the frame. Thus the partition function of the continuum field theory may depend on the frame of the spacetime manifold. If the partition function does depend on the frame, we say that the theory has a framing anomaly [4]. Otherwise, the theory is free of framing anomaly.

In Ref. 51, 2+1d generic lattice bosonic systems are constructed to realize topological orders with any integral chiral central charge and the corresponding gravitational Chern-Simons term. Since the central charge is not 0 mod 24, those models contain framing anomaly and are not frame-independent. This implies that the partition function of generic lattice bosonic systems may depend on the frames of the spacetime manifold in the continuum limit. This example and the above discussions suggest a close relation between the branch structure on a lattice and the frame structure on a continuum manifold.1

We conjecture that the independence of the branch structure of spacetime complex for lattice models implies the independence of frame structure of spacetime manifold in the continuum limit. Under such a conjecture, the branch-independent bosonic systems on lattice only give rise to continuum effective field theories that are free of framing anomaly. As a result, a 2+1d branch-independent bosonic system can only realize topological orders with chiral central charge c = 0 mod 24, where the 2+1d invertible topological orders is generated by three copies of E8 quantum Hall states (say E8 topological order). Indeed, using SO(∞) non-linear σ-model, Ref. 50 constructed a branch-independent bosonic model that realize the E8 topological order with c = 24. For more details, see Appendix 1.

2 The 2+1d Abelian bosonic topological orders are classified (in a many-to-one fashion) by symmetric integral matrices with even diagonals[52], which are called K-matrices. Those topological orders are realized by K-matrix quantum Hall wavefunctions \( \Psi(z_i^I) = \prod_{i<j} e^{2 \pi i z_i^I z_j^J K_{IJ} e^{-\frac{1}{4} \sum |z_i|^2}} \) and are described by K-matrix U(1)-Chern-Simons theories \( K_{IJ} \sum_\alpha d_\alpha \). The K-matrices with \( |\det(K)| = 1 \) classify invertible topological orders. Here the E8 topological order is an invertible topological order described by a K-matrix given by the Cartan matrix of \( E_8 \), denoted as \( K_{E_8} \). The 1+1d boundary carries the chiral central charge \( c = 8 \) [53]. In contrast, the \( E_6^3 \) topological order is described by a K-matrix \( K = K_{E_8} \oplus K_{E_8} \oplus K_{E_8} \) and has its boundary carrying the chiral central charge \( c = 24 \). It was suggested that the \( \mathbb{Z} \) class of the oriented cobordism is generated by the \( E_8 \) topological order, see for example, Freed’s work [54] or Freed-Hopkins [16]. See Section 7 of [55] for an elaborated interpretation of the related cobordism invariants.

3 Perturbative local anomalies are detectable via infinitesimal gauge/diffeomorphism transformations continuously deformable from the identity, captured by perturbative Feynman diagrams [57]. Nonperturbative global anomalies are detectable via large gauge/diffeomorphism transformations that cannot be continuously deformed from the identity [58].
For example, it is known that the 4+1d invertible topological order has a gapped 3+1d boundary described by a \( Z_2 \) gauge theory with the \( w_2w_3 \) anomaly, where the \( Z_2 \) gauge charge is fermionic. However, the fermionic \( Z_2 \) gauge charge does not fully characterize the gravitational anomaly. In particular, there is also an anomaly-free 3+1d \( Z_2 \) gauge theory with a fermionic \( Z_2 \) gauge charge, \( i.e. \) there is a 3+1d lattice bosonic system that can realize a \( Z_2 \) gauge theory with a fermionic \( Z_2 \) gauge charge \([59]\). In this work, we show that the 2d worldsheet of the \( Z_2 \) gauge flux in the spacetime must carry a non-contractable 1d fermionic orientation-reversal worldline if the 2d worldsheet is unorientable. This 1d fermionic orientation-reversal worldline with neutral gauge charge is however distinct from the fermionic worldline of \( Z_2 \) gauge charge. This crucial property, together with the fermionic \( Z_2 \) gauge charge, characterizes the gravitational anomaly.

The above result about the fermion worldline on unorientable worldsheet of \( Z_2 \) flux line was first obtained in Section 3.3 of Ref. 44. In this paper we give a different derivation of the result using a path integral formulation on a spacetime lattice. This result was also obtained recently in Ref. 48 using Hamiltonian formulation on spatial lattice. The “fermionic” nature of the \( Z_2 \) flux line can also be characterized by the statistical hopping algebra for strings \([49]\), a generalization of the statistical hopping algebra for particles \([59]\).

### A. Notations and conventions

We denote the n’d for the spacetime dimensions to be \( n' = (n + 1) \) with an n dimensional space and a 1 dimensional time. Typically, in this article, the dimension always refer to the spacetime dimension altogether. We may simply call the 0d \( Z_2 \) gauge charge as \( Z_2 \) charge, whose spacetime trajectory is a 1d worldline. We may simply call the 1d \( Z_2 \) gauge flux loop as \( Z_2 \) flux, which can be a 1d loop which bounds a 2d surface enclosing gauge flux, whose spacetime trajectory is a 2d worldsheet.

In this work, we use a lot of formalisms of chain and cochain, as well as the associated derivative cup product, Steenrod square, etc. A brief coverage of those topic can found in Appendix A. The \( Z_n \) values are chosen to be \( \{0, 1, \cdots, n - 1\} \). In this article, we always use this set to extend \( Z_n \) values to \( Z \) values, and treat \( Z_n \)-valued quantities as \( Z \)-valued quantities. To help to express \( Z_n \)-valued relation using \( Z \)-valued quantities, we denote \( \overset{\equiv}{=} \) to mean equal up to a multiple of \( n \) (thus it is a mod \( n \) relation: two sides of the equality are equal mod \( n \)), and use \( \overset{\equiv}{=} \) to mean equal up to a coboundary \( df \) (\( i.e. \) with the coboundary operator \( d \)).

We denote the Lorentz group as SO (for bosonic systems) and Spin (for fermionic systems graded by the fermion parity \( Z_2 \)). In \( n + 1 \)d spacetime, the SO really means the \( SO(n + 1) \) for the Euclidean rotational symmetry group and the \( SO(n, 1) \) for the Lorentz rotational + boost symmetry group; the Spin really means the \( Spin(n + 1) \) for the Euclidean rotational symmetry group and the \( Spin(n, 1) \) for the Lorentz rotational + boost symmetry group. We denote \( \overset{\equiv}{n}d \) to mean equal up to a mod \( n \) relation and also equal up to a coboundary \( df \). We will use the group \( N \times_{e_2, \alpha} G \) to describe the extension of a group \( G \) by an abelian group \( N \) via

\[ 1 \rightarrow N \rightarrow N \times_{e_2, \alpha} G \rightarrow G \rightarrow 1, \]

which is characterized by \( e_2 \in H^2(G; N) \) of the second cohomology class, where \( N \) is an abelian group with a \( G \)-action via \( \alpha : G \rightarrow \text{Aut}[N] \). We will also use \( G_1 \times_N G_2 \equiv \frac{G_1 \times G_2}{N} \) to define as the product group of \( G_1 \) and \( G_2 \) modding out their common normal subgroup \( N \). Other mathematical conventions and definitions (such as Stiefel-Whitney class) can be found in Appendix B. We provide many appendices on the toolkits of cochain, cochain, cycle, cohomology, characteristic class, and cobordism.

**Dynamical gauge fields** are associated with the gauge connections of gauge bundles that are summed over in the path integral (or partition function). **Background gauge fields** are associated with the non-dynamical gauge connections of gauge bundles that are fixed, not summed over in the path integral. We will distinguish their gauge transformations: for dynamical fields as **dynamical gauge transformations**, for background fields as **background gauge transformations**, see Appendix J.

In this work, the **anomalous gauge theory** merely means its partition function alone is only **non-invariant** under **background gauge transformations** (but still **invariant** under **dynamical gauge transformations**) — namely, the anomalous gauge theory with only ’t Hooft anomaly of the global symmetry \([60]\) can still be well-defined on the boundary of one-higher dimensional invertible topological phase. The cancellation of **background gauge transformations** between the bulk and boundary theories are known as the anomaly inflow \([61]\).

### II. BRANCH-INDEPENDENT BOSONIC SYSTEM AND GENERIC LATTICE BOSONIC SYSTEM

In this work, we are going to use cochains on a spacetime simplicial complex as bosonic fields. In order to construct the action \( S \) in the path integral, using local Lagrangian term on each simplex, it is important to give the vertices of each simplex a local order. A local scheme to order the vertices is given by a branch structure \([10, 62, 63]\). A branch structure is a choice of orientation of each link in the complex so that there is no oriented loop on any triangle (see Fig. 1). Relative to a base branch structure, all other branch structure can be described by a \( Z \)-valued 1-cochain \( s \) (see Appendix A.6). After assigning a branch structure to the spacetime complex, we can define cup product \( \cup \) of cochains
that depend on the branch structure \( s \). For the base branch structure \( s = 0 \), we abbreviate \( \simeq \) by \( \sim \).

We find that for two cocycles, \( f \) and \( g \), \( f \sim g - f \sim g \) is coboundary, that depends on \( s, f, g \). Let us use \( dv(s, f, g) \) to denote such a coboundary (for details, see Appendix A.6):

\[
f \sim g + dv(s, f, g) = f \sim g. \tag{1}
\]

Using derivative and cup product of the cochains, we can construct a local action \( S \). So, in general, the action amplitude, \( e^{-S} \), may depend on the choices of the branch structures.

**A. Branch-Independent Bosonic (BIB) System**

Now we are ready to define the branch-independent bosonic system: A branch-independent bosonic system is defined by a path integral on a branch spacetime simplicial complex, such that the value of the path integral is independent of the choices of branch structures, even when the spacetime has boundaries.

Let us give an example of branch-independent bosonic system. The bosonic system has two fields: a \( \mathbb{Z}_2 \)-valued 1-cochain field \( a^{Z_2} \) and a \( \mathbb{Z}_2 \)-valued 2-cochain field \( b^{Z_2} \), which give rise to the following partition function

\[
Z = \sum_{a^{Z_2}, b^{Z_2}} e^{i\pi \int_{M^{5}} (w_2^e + da^{Z_2}) \cdot (w_2^{\nu} + db^{Z_2})} e^{i\pi \int_{\partial M^{5}} \nu_s (w_2^{e} + da^{Z_2}, w_2^{\nu} + db^{Z_2})}, \tag{2}
\]

where \( \sum_{a^{Z_2}, b^{Z_2}} \) sums over all the cochain fields. The summation \( \sum_{a^{Z_2}, b^{Z_2}} \) in the path integral is known as a summation of degrees of freedom. Here, the \( w_n^e \) is the \( n \)th Stiefel-Whitney cocycle computed from a simplicial complex \( M^5 \) with a branch structure \( s \), as described in Ref. 64 and in Appendix F. The cocycle \( w_n^e \) is a representation of the Stiefel-Whitney class \( w_n(TM) \) of the tangent bundle \( (TM) \) of the spacetime manifold \( M \). We omit the normalization factor here in the partition function (2).

Let \( w_n \) be the Stiefel-Whitney cocycle for the base branch structure: \( w_n \equiv w_n^0 = 0 \). In general, \( w_n^e \) depends on the branch structure \( s \) on \( M^5 \). However, \( w_n^e - w_n \) is a coboundary:

\[
w_n^e = w_n + dv_{n-1}(s). \tag{3}
\]

We can show that (2) is independent of branch structure \( s \). First, from the definition of \( \nu \), with a bulk \( M^5 \) but without any boundary \( \partial M^5 \), we have

\[
Z = \sum_{a^{Z_2}, b^{Z_2}} e^{i\pi \int_{M^{5}} (w_2^{e} + da^{Z_2}) \cdot (w_2^{\nu} + db^{Z_2})} = \sum_{a^{Z_2}, b^{Z_2}} e^{i\pi \int_{M^{5}} (w_2^{e} + da^{Z_2})(w_2^{\nu} + db^{Z_2})}. \tag{4}
\]

In this article, we abbreviate the cup product for the base branch structure, \( f \sim g \), as \( fg \). From the relation between \( w_n^e \) and \( w_n \), we have

\[
Z = \sum_{a^{Z_2}, b^{Z_2}} e^{i\pi \int_{M^{5}} (w_2 + dv_2(s))(w_3 + dv_2(s) + db^{Z_2})} = \sum_{a^{Z_2}, b^{Z_2}} e^{i\pi \int_{M^{5}} (w_2 + da^{Z_2})(w_3 + db^{Z_2})}. \tag{5}
\]

We see that the partition function (2) is indeed independent of branch structure.

The partition function for a generic bosonic system on a spacetime complex \( M \) in a quantum liquid phase has a form

\[
Z(M) = e^{-S^{\text{eff}}(M)} Z^{\text{top}}(M), \tag{6}
\]

in a thermodynamic limit, where \( S^{\text{eff}} = \int_M S^{\text{eff}} \) energy-density is the non-universal volume part, and \( Z^{\text{top}}(M) \) is the universal topological partition function (i.e. the topological invariant) that characterizes the topological order. (The topological partition function \( Z^{\text{top}} \) and its isolation are discussed in much more details in Ref. 35, 65, and 66.) For a branch-independent bosonic system, the topological invariant \( Z^{\text{top}}(M) \) does not depend on the branch structures on \( M \), and possibly, nor the framing of the spacetime manifold in continuum limit. This leads to the conjecture that the topological invariant \( Z^{\text{top}}(M) \) is a cobordism invariant \( 16, 34, 35 \) for invertible topological orders in the branch-independent bosonic systems.

The above example (2) is exactly solvable when \( M^5 \) has no boundary, since the partition function can be calculated exactly

\[
Z = 2^{N_l + N_f} e^{i\pi \int_{M^{5}} w_2 w_3} \tag{7}
\]

where \( N_l \) is the number of links (namely 1-simplices) that can be paired with \( a^{Z_2} \) and \( N_f \) is the number of faces (namely 2-simplices that can be paired with \( b^{Z_2} \)) in \( M^5 \). Thus the summation \( \sum_{a^{Z_2}, b^{Z_2}} \) gives a \( 2^{N_l + N_f} \) factor. After dropping the non-universal volume term

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4 Here we treat \( w_n \) as the Stiefel-Whitney cocycle. In contrast, the mathematical definition of Stiefel-Whitney class as cohomology class and characteristic class \( w_n \) is given in Appendix B. Because the cohomology class is also a cocycle, so we may also abuse the notation to write Stiefel-Whitney class \( w_n \) in terms of cocycle \( w_n^e \). A cohomology class is an equivalence class that has many representatives; any elements in the equivalence class are representatives. The \( w_n \) is Stiefel-Whitney class, also it can be referred to as Stiefel-Whitney cocycle when we choose a representative \( w_n \); the \( w_n \) is a Stiefel-Whitney number only when \( n \) is the top spacetime dimension.

The Stiefel-Whitney product cocycle \( w_2 w_3 \) is a representative of the cup product of two Stiefel-Whitney classes \( (w_2 \) and \( w_3) \). When the \( w_2 w_3 \) paired with the fundamental class, as \( \int_{M^{5}} w_2 w_3 \), is called the Stiefel-Whitney number.
\[ e^{-S_{\text{top}}} = 2^{N_i + N_f}, \] we see that the topological partition function is given by

\[ Z^{\text{top}}(M^5) = e^{i\pi \int_{M^5} w_2 w_3}. \quad (8) \]

The nontrivial topological invariant \( e^{i\pi \int_{M^5} w_2 w_3} \) suggests that the bosonic model (2) realizes a nontrivial topological order. The topological order is invertible since the topological invariant is a phase factor for any closed spacetime \( M^5 \). We will refer such an invertible topological order as the \( w_2 w_3 \)-topological order.

## B. Generic Lattice Bosonic (GLB) System

We also like to define a concept of generic lattice bosonic system, as a bosonic system on a spacetime simplicial complex, whose path integral may or may not depend on the branch structures. Indeed, when we define an generic lattice bosonic system, the local branch structure of spacetime complex is given, and the definition may depend on the choices of the local branch structure.

Let us give an example of a generic lattice bosonic system, that happens to have “no degrees of freedom.” Here “no degrees of freedom” means that there is only one term in the path integral, and also means that there is only one state \( \langle 0 \rangle_j \) per single site \( j \). By definition, the ground state of such a system is a trivial tensor product state. So such an action amplitude is only allowed by a system with no degrees of freedom, and thus corresponds to a nontrivial invertible topological order.

In other words, we can smoothly deform the topological order characterized by \( e^{i\pi \int_{M^5} w_2 w_3} \) into a trivial product state in the parameter space (or landscape) of generic lattice bosonic systems, but such a smooth deformation path does not exist in the parameter space (or landscape) of branch-independent bosonic systems. The smooth deformation is defined as the continuous and differentiable tuning of any coupling strength of Lagrangian/Hamiltonian terms in the parameter space, that do not cause any phase transitions.

To construct the above mentioned smooth deformation in the parameter space of generic lattice bosonic systems, let us consider the following generic lattice bosonic system

\[ Z = \sum_{a_{ij}, b_{ij}} e^{i\pi \int_{M^5} (w_2 + da_{ij}^2) (w_3 + db_{ij}^2)} e^{-U \int_{M^5} |a_{ij}|^2 + |b_{ij}|^2}, \quad (11) \]

where

\[ \int_{M^5} |a_{ij}|^2 = \sum_{(ij)} |a_{ij}|^2, \quad \int_{M^5} |b_{ij}|^2 = \sum_{(ijk)} |b_{ijk}|^2. \quad (12) \]

Changing \( U \) leads to the smooth deformation. When \( U = 0 \), the path integral (11) is (2). In the limit \( U \to \infty \), the path integral (11) becomes (10). The path integral (11) is still exactly solvable when \( M^5 \) has no boundaries

\[ Z = \sum_{a_{ij}, b_{ij}} e^{i\pi \int_{M^5} (w_2 + da_{ij}^2) (w_3 + db_{ij}^2)} e^{-U \int_{M^5} |a_{ij}|^2 + |b_{ij}|^2} = (1 + e^{-U})^{N_i + N_f} e^{i\pi \int_{M^5} w_2 w_3}. \quad (13) \]

The system is gapped, namely, with a short-range correlation for any values of \( U \). So the deformation from \( U = 0 \) to \( U = +\infty \) is a smooth deformation that does not cause any phase transition. This is why a \( w_2 w_3 \) topological order and a product state belong to the same phase for generic lattice bosonic systems.
From the above discussion, it is clear that invertible topological orders have different classifications for branch-independent bosonic systems and for generic lattice bosonic systems. However, the two kinds of invertible topological orders are still related. Let $\text{TP}$ denote a classification of “Topological Phases” following the notation of Freed-Hopkins [16]. Let $\text{TP}_d(\text{BIB})$ be the Abelian group that classifies the topological phases invertible topological orders for branch-independent bosonic (BIB) systems. Let $\text{TP}_d(\text{GLB})$ be the Abelian group that classifies invertible topological orders for generic lattice bosonic (GLB) systems. Since invertible topological orders for branch-independent bosonic systems can also be viewed as invertible topological orders for generic lattice bosonic systems, we have a group homomorphism

$$\text{TP}_d(\text{BIB}) \xrightarrow{p_d} \text{TP}_d(\text{GLB}). \quad (14)$$

Since invertible topological order characterized by Stiefel-Whitney classes all become trivial for generic lattice bosonic systems, the map $p_d$ sends all the Stiefel-Whitney class to zero in the topological invariant. The map $p_d$ is not injective. The map $p_d$ may also not be surjective. So it may only tell us a subset of invertible topological orders for generic lattice bosonic systems. In the rest of this article, we will mainly concentrate on branch-independent bosonic systems and its $w_2w_3$ invertible topological order.

C. Lattice Bosonic Systems with time reversal symmetry

The above discussion can be easily generalized to bosonic systems with time reversal symmetry, defined via path integrals on a spacetime lattice with a real action amplitude $e^{-S}$. In order words, we restrict ourselves to consider only the real action amplitudes $e^{-S}$, as a way to implement time reversal symmetry.

The simplest bosonic invertible topological order with time reversal symmetry (or time-reversal SPT order) appears in 2d and is characterized by the topological invariant $(-1)^{f_{M^2} w_1^2}$ on a closed spacetime $M^2$. A branch-independent bosonic model that realizes the SPT order is given by

$$Z = \sum_{g^{Z_2}} (-1)^{f_{M^2} (w_1 + d g^{Z_2})^2}, \quad (15)$$

where $\sum_{g^{Z_2}}$ sums over all $Z_2$-valued 0-cochains $g^{Z_2}$ and the spacetime $M^2$ can have boundaries. The branch-independence is ensured by the invariance of the action amplitude $(-1)^{f_{M^2} (w_1 + d g^{Z_2})^2}$ under the following transformation

$$w_1 \rightarrow w_1 + d v_0, \quad g^{Z_2} \rightarrow g^{Z_2} + v_0, \quad (16)$$

even when $M^2$ has boundaries.

We also have a generic lattice bosonic model given by

$$Z = \sum_{g^{Z_2}} (-1)^{f_{M^2} (w_1 + d g^{Z_2})^2} e^{-U f_{M^2} |g^{Z_2}|^2}. \quad (17)$$

The model has the time-reversal symmetry since the action amplitude is real. The model is exactly solvable when $M^2$ has no boundaries, and has the following partition function.

$$Z = (1 + e^{-U})^{N_v} (-1)^{f_{M^2} w_1^2} \quad (18)$$

where $N_v$ is the number of the vertices in the spacetime complex $M^2$. Since the partition function only depends on the area of spacetime $M^2$, but not depends on the shape of spacetime $M^2$, the model (17) is gapped (i.e. has short range correlations) for any values of $U$. There is no phase transition as we change $U$.

When $U = 0$, the model (17) becomes (15) and realize the time-reversal SPT order characterized by topological invariant $(-1)^{f_{M^2} w_1^2}$. When $U = \infty$, the model (17) becomes a model with no degrees of freedom which must correspond to a trivial tensor product state. We see that, for generic lattice bosonic models with time reversal symmetry, the time-reversal SPT order characterized by topological invariant $(-1)^{f_{M^2} w_1^2}$ and the trivial tensor product state belong to the same phase. This result suggests that the bosonic invertible topological orders with time-reversal symmetry (i.e. with real action amplitudes) for generic lattice bosonic systems are not classified by unoriented cobordism (i.e. manifolds with orthogonal group $O$ structures).

III. BOUNDARIES OF $w_2 w_3$ INVERTIBLE TOPOLOGICAL ORDER

A. The branch independence and background gauge invariance

In the last section, we studied a 5d branch-independent bosonic model on spacetime complex $M^5$ defined by the path integral (2). That path integral can be simplified by using the base branch structure to define the cup product:

$$Z = \sum_{a^{Z_2}, b^{Z_2}} e^{i \pi f_{M^5} (w_2^3 + da^{Z_2}) (w_1^3 + db^{Z_2})}. \quad (19)$$

When $M^5$ is a closed manifold with no boundary, the above path integral gives rise to a topological invariant (also known as a cobordism invariant) $e^{i \pi f_{M^5} w_2 w_3}$. In this section, we consider the case when $M^5$ has a boundary. We shall obtain the possible boundary theories for the $w_2w_3$-topological order.

One might guess that, when $M^5$ has a boundary, the partition function is still given by $Z(M^5) = e^{i \pi f_{M^5} w_2 w_3}$. If this was true, we could conclude that the boundary is gapped, since the partition function $Z(M^5) = e^{i \pi f_{M^5} w_2 w_3}$ does not depend on the metrics on the
boundary $B^4 \equiv \partial M^5$. (A gapless system must have a partition function that depends on the metrics (i.e. the shape and size) of the spacetime.)

Such a gapped boundary is possible for generic lattice bosonic systems, but it is impossible for the branch-independent bosonic model (19). This is because the partition function in (19) is independent of the branch structure on $M^5$ even when $M^5$ has boundaries, while our guess $Z(M^5) = e^{i \pi \int_{M^5} w_2^5 w_3^5}$ depends on the branch structure, since

$$w_2^5 = w_2 + dv_1(s), \quad w_3^5 = w_3 + dv_2(s). \quad (20)$$

Thus $e^{i \pi \int_{M^5} w_2^5 w_3^5}$ cannot be the partition function of (19) which must be independent of the branch structure.

Due to (20), we see that we can encode the branch structure independence, via the invariance under the following transformation

$$w_2 \rightarrow w_2 + dv_1, \quad w_3 \rightarrow w_3 + dv_2. \quad (21)$$

Thus the branch independence of (19) can be rephrased as the invariance of the following path integral

$$Z = \sum_{a^2, b^2} e^{i \pi \int_{M^5} (w_2 + da^2)(w_3 + db^2)}. \quad (22)$$

under the above transformation (21), even for spacetime $M^5$ with boundaries. We refer to (21) as a “background gauge transformation,” which is a change in the parameters in the Lagrangian rather than a change in the dynamical fields in the Lagrangian. See the comparison between “background gauge transformation” and “dynamical gauge transformation” in Appendix J.

In the following, we will use the background gauge invariance under (21) to ensure the independence of branch structures. For the branch-independent bosonic model (22), the boundary must be non-trivial. We may assume the boundary to be described by

$$Z(M^5) = \sum_{\phi} e^{i \pi \int_{M^5} w_2 w_3 - \int_{\partial M^5} L_{\text{boundary}}(\phi, w_2, w_3)}. \quad (23)$$

The boundary Lagrangian $L_{\text{boundary}}$ is not invariant under the background gauge transformation (21), which cancels the non-invariance of $w_2 w_3$ in $e^{i \pi \int_{M^5} w_2 w_3}$ when $M^5$ has boundaries. This cancellation of non-invariances of the bulk and boundary theories is actually the idea of anomaly inflow [61].

### B. 4d $Z_2$-gauge boundary of the $w_2 w_3$ topological order

In this section, we are going to explore the possibility that the boundary Lagrangian $L_{\text{boundary}}$ describes a $Z_2$ gauge theory, more precisely the dynamical Spin structure summed over in the path integral.

#### 1. Effective boundary theory

A 4d $Z_2$ gauge theory can be described by $Z_2$-valued 1-cochain $a^2 \equiv a^2$ and 2-cochain $b^2 \equiv b^2$ fields (for example, see [65, 68, 69]) with the following path integral

$$Z = \sum_{a^2, b^2} e^{i \pi \int_{M^5} a^2 \wedge db^2} = \sum_{a^2, b^2} e^{i \pi \int_{M^5} a^2 \wedge db^2}, \quad (24)$$

where $\sum w_{a^2, b^2}$ is a summation over $Z_2$-valued 1-cochains $a^2$ and 2-cochains $b^2$ on $B^4$. But such a theory is invariant under the background gauge transformation (21), and cannot cancel the non-invariant of $e^{i \pi \int_{M^5} w_2 w_3}$.

We can add coupling to $w_2$ and $w_3$ to fix this problem and obtain the following boundary theory (with the bulk topological invariant included)

$$Z(M^5, B^4 = \partial M^5) = \sum_{a^2, b^2 \text{ on } B^4} e^{i \pi \int_{M^5} w_2 w_3 + \int_{B^4} a^2 \wedge db^2 + a^2 \wedge w_2 b^2}. \quad (25)$$

Indeed, such a partition function is independent of the branch structure, since it is invariant under the background gauge transformation (21). To see this point, we note that the change in $w_2$ and $w_3$ can be absorbed by $a^2$ and $b^2$. In other words, the action amplitude

$$e^{i \pi \int_{M^5} w_2 w_3 + \int_{B^4} a^2 \wedge db^2 + a^2 \wedge w_2 b^2}$$

is invariant under the following generalized transformation

$$a^2 \rightarrow a^2 + v_1, \quad b^2 \rightarrow b^2 + v_2, \quad w_2 \rightarrow w_2 + dv_1, \quad w_3 \rightarrow w_3 + dv_2. \quad (26)$$

So (25) is a boundary theory of the branch-independent bosonic theory (22).

In fact we can obtain (25) directly from (22) by assuming $M^5$ to have boundaries and not adding anything on the boundaries:

$$Z = \sum_{a^2, b^2 \text{ on } M^5} e^{i \pi \int_{M^5} (w_2 + da^2)(w_3 + db^2)} \quad (27)$$

where $N_b^2$ and $N_f^b$ are the number of links and faces in $M^5$ that are not on the boundary $\partial M^5$.

Both (24) and (25) describe some kinds of $Z_2$ gauge theories. However, the two $Z_2$ gauge theory is very different. Eqn. (24) is anomaly-free and can be realized by a branch-independent bosonic model in 4d. In fact (24) itself is a branch-independent bosonic model in 4d that realize the anomaly-free $Z_2$ gauge theory. On the other hand, (25) has an invertible gravitational anomaly. It can only be realized as a boundary of 5d invertible topological order. In our example, (22) is a branch-independent
bosonic model that realizes the 5d invertible topological order, and (25) is a boundary theory of of the 5d model (22).

Due to different gravitational anomalies, the two $\mathbb{Z}_2$ gauge theories (24) and (25) have different properties. In the $\mathbb{Z}_2$ gauge theory (24), the $\mathbb{Z}_2$ gauge charge is a bosonic particle and the $\mathbb{Z}_2$ gauge flux line behave like a bosonic string. On the other hand, in the $\mathbb{Z}_2$ gauge theory (25), the $\mathbb{Z}_2$ gauge charge is a fermionic particle and the $\mathbb{Z}_2$ gauge flux line has certain fermionic nature.

To see the above result, we include the worldline of $\mathbb{Z}_2$ gauge charge and worldsheet of $\mathbb{Z}_2$ gauge flux into the boundary theory (25):

$$
\sum_{a^2z, b^2z} e^{i \pi f_{l_b} w_2 s_3} e^{i \pi f_{b^4} a^2z d b^2z + a^2z w_3 + w_2 b^2z} e^{i \pi f_{b^4} l_3 a^2z + s_2 b^2z}.
$$

where $l_3$ and $s_2$ are $\mathbb{Z}_2$-valued 3- and 2-cocycles which correspond to the Poincaré dual of the worldline and the worldsheet. But the above action with the worldlines and worldsheets is not invariant under the transformation (26), and thus is not a correct boundary theory for branch-independent bosonic systems. We may fix this problem by considering the following modified boundary theory

$$
\sum_{a^2z, b^2z} e^{i \pi f_{l_b} w_2 s_3} e^{i \pi f_{b^4} a^2z d b^2z + a^2z w_3 + b^2z w_2} e^{i \pi f_{b^4} l_3 a^2z + s_2 b^2z} e^{i \pi f_{l_b} l_3 w_2 + s_2 w_3}.
$$

In the above, we have assumed that $l_3$ and $s_2$ on the boundary $B^4$ can be extended to the bulk $M^5$ as cocycles. The invariance of the above path integral under transformation (26) can be checked directly.

But the above expression also has a problem: it depends on how we extend $l_3$ and $s_2$ on the boundary $B^4$ to the bulk $M^5$. To fix this problem, we propose the path integral

$$
Z = \sum_{a^2z, b^2z} e^{i \pi f_{l_b} w_2 s_3} e^{i \pi f_{b^4} a^2z d b^2z + a^2z w_3 + w_2 b^2z} e^{i \pi f_{b^4} l_3 a^2z + s_2 b^2z} e^{i \pi f_{l_b} l_3 w_2 + s_2 w_3},
$$

that is fully invariant under the transformations in (26).

Here $Sq$ is the Steenrod square (A20) and $\beta_2$ is the generalized Bockstein homomorphism (A9) that act on cocycles (see Appendix A for details).

Let us first show that the term $e^{i \pi f_{l_b} l_3 a^2z + l_3 w_2 + \beta_2 b^2z} e^{i \pi f_{l_b} n} Sq^n l_3 + l_3 w_2 + \beta_2 b^2z + s_2 w_3$ only depends on the fields on the boundary $B^4 = \partial M^5$, so that the above path integral is well-defined. To do so, we note that, according to Wu relation for a $\mathbb{Z}_2$-valued cocycle $x_{d-n}$ in the top $d$-dimension on the complex $M^d$:

$$
Sq^n x_{d-n} \equiv u_n x_{d-n},
$$

where $u_n$ is Wu class:

$$
\begin{align*}
  u_0 &\equiv 1, \quad u_1 \equiv w_1, \quad u_2 \equiv w_2 + w_2, \\
  u_3 &\equiv w_1 w_2, \quad u_4 \equiv w_1^2 + w_2 + w_1 w_3 + w_4, \\
  u_5 &\equiv w_1^2 w_2 + w_1^2 w_2 + w_2^2 w_3 + w_1 w_4, \\
  u_6 &\equiv w_1^2 w_2^2 + w_1^2 w_3 + w_1 w_2 w_3 + w_2^2 + w_1^2 w_4 + w_2 w_4, \\
  u_7 &\equiv w_1^2 w_2 w_3 + w_1 w_3^2 + w_1 w_2 w_4, \\
  u_8 &\equiv w_1^2 + w_2^2 + w_1^2 w_3^2 + w_1^2 w_2 w_4 + w_1 w_3 w_4 + w_4 + w_2^2 w_5 + w_3 w_5 + w_2^2 w_6 + w_2 w_6 + w_1 w_7 + w_8.
\end{align*}
$$

From (31) and (32), we can show that $Sq^2 l_3 + l_3 w_2$ is a coboundary on oriented $M^5$ with $w_1 \equiv 0$. Thus $e^{i \pi f_{l_b} l_3 + l_3 w_2}$ only depends on $l_3$ on the boundary $B^4 = \partial M^5$.

The term $e^{i \pi f_{l_b} l_3 + l_3 w_2}$ makes $l_3$ on the boundary to be a fermion worldline via a higher dimensional bosonization [17, 65, 70–72]. (For details, see Appendix C.) In other words, the anomalous $\mathbb{Z}_2$ gauge theory on the boundary has a special property that the $\mathbb{Z}_2$ gauge charge is a fermion.

There is another way to understand why the $\mathbb{Z}_2$ gauge charge is a fermion. The $w_2$-$w_3$-topological order, as a bosonic state, can live on any 5-dimensional orientable manifold with a $SO(5)$ connection for the tangent bundle. One way to obtain a gapped boundary of $w_2$-$w_3$-topological order is to trivialize $w_2$ for the $SO(5)$ connection on the boundary. Such a trivialization can be viewed as a group extension $Spin(5) = \mathbb{Z}_2 \times w_2 SO(5)$ via

$$
1 \rightarrow \mathbb{Z}_2 \rightarrow Spin(5) \rightarrow SO(5) \rightarrow 1.
$$

This is consistent with the fact that the Spin manifold is necessary and sufficient condition for its second Stiefel-Whitney class $w_2 = 0$ for its tangent bundle. Trivialize $w_2$ on the boundary can be thought as promoting the $SO(5)$ connection in the bulk to a $Spin(5)$ connection on the boundary, where $Spin(5)$ is a $\mathbb{Z}_2$ central extension of $SO(5)$. This implies that the boundary is described by a twisted $\mathbb{Z}_2$ gauge theory, where the $\mathbb{Z}_2$ connection 1-cochain $a^{Z_2}$ satisfies

$$
da^{Z_2} \equiv w_2,
$$

instead of $da^{Z_2} \equiv 0$. The above relation can be obtained from (25) by integrating out $b^{Z_2}$ first. In this case, the $\mathbb{Z}_2$ charge couples to the $Spin(5)$ connection on the boundary, instead of $\mathbb{Z}_2 \times SO(5)$ connection. So the $\mathbb{Z}_2$ charge

\[5\] Higher dimensional bosonization here especially in [71] means to use the purely bosonic commuting fields (i.e., cochains) with only Steenrod algebras (but without using Grassmann variables) to represent the fermionic systems with the fermionic parity symmetry $Z_2$. The fermionic system here may be regarded as a system with emergent fermions living on the boundary of bosonic topological order. See Appendix K for a summary.
carries a half-integer spin representation of the spacetime Spin group if we interpret the extended normal \(Z_2\) as the fermion parity \(Z_2^f\) in (33) as

\[ 1 \rightarrow Z_2^f \rightarrow \text{Spin}(5) \rightarrow \text{SO}(5) \rightarrow 1. \]  

(35)

The odd \(Z_2\) gauge charge in (33) is also the half-integer spin representation of Spin group, which is also odd under the fermion parity \(Z_2^f\) in (35). Then according to the usual lattice belief in terms of the spacetime-spin statistics relation, the half-integer spin representation of this \(Z_2\) gauge charge is also a fermion. The Spin structure, which contains the emergent fermion parity \(Z_2^f\) on the boundary, is also called the emergent dynamical Spin structure [44].

However, a 4d \(Z_2\) gauge theory with a fermionic \(Z_2\) gauge charge may not have gravitational anomaly, since such theory can be realized by a 4d bosonic model

\[ Z = \sum_{a^2, b^2} e^{i \pi f_{ab} b^2 z (w_2 + da^2 z)}. \]  

(36)

The action amplitude is invariant under the following transformation, even when \(B^4\) has boundaries

\[ a^2 z \rightarrow a^2 z + v_1, \quad b z \rightarrow b z, \quad w_2 \rightarrow w_2 + dv_1. \]  

(37)

After including the worldline of \(Z_2\) charge and the worldsheet of \(Z_2\) flux, the above becomes

\[ Z = \sum_{a^2, b^2 \text{ on } B^4} e^{i \pi f_{ab} a^2 d b^2 + b^2 w_2} e^{i \pi f_{ab} l_3 a^2 + s_2 b^2} e^{i \pi f_{ab} l_3 s_2}, \]  

(38)

which is the path integral description of the anomaly-free \(Z_2\) gauge theory with fermionic \(Z_2\) charge [59].

Therefore, the \(Z_2\)-flux line must also have some special properties as a realization of the anomaly in the \(Z_2\) gauge theory (25). Let us first show that \(S^2 \beta_2 s_2 + s_2 w_3\) is also a coboundary on \(M^5\). Using

\[ S^2 x = \beta_2 x, \]

\[ S^2 (w_j) = w_1 w_j + (j - 1) w_{j+1}, \]

we find that

\[ \beta_2 w_2 = S^2 w_2 = w_1 w_2 + w_3. \]  

(40)

Noticing \(w_1 = 0\), we find that

\[ s_2 w_3 = s_2 \beta_2 s_2 + w_2 \beta_2 s_2 + \beta_2 (w_2 s_2) = \beta_2 w_2 s_2 = 0, \]  

(41)

The first equality uses (40), and the second equality uses the chain rule. The third equality uses \(\beta_2 (w_2 s_2) = S^2 (w_2 s_2) = w_1 w_2 s_2 = 0\), where we have also used the Wu relation (31) and \(w_1 = 0\). Now, according to (41), we have \(w_2 \beta_2 s_2 + s_2 w_3 = 0\). Then using (31) and \(w_1 = 0\), we have \(S^2 \beta_2 s_2 + s_2 w_3 = 0\), and

\[ S^2 \beta_2 s_2 + w_2 \beta_2 s_2 = 0. \]  

Thus \(e^{i \pi f_{ab} S^2 \beta_2 s_2 + s_2 w_3}\) and \(e^{i \pi f_{ab} S^2 \beta_2 s_2 + w_2 \beta_2 s_2}\) only differ by a surface term on \(B^4 = \partial M^5\). Also, \(e^{i \pi f_{ab} S^2 \beta_2 s_2 + w_2 \beta_2 s_2}\) itself is a surface term on \(B^4 = \partial M^5\).

The term \(e^{i \pi f_{ab} S^2 \beta_2 s_2 + w_2 \beta_2 s_2}\) makes \(\beta_2 s_2\) on the boundary to be the Poincaré dual of a fermion worldline via a higher dimensional bosonization [17, 65, 71]. We note that if the 2d worldsheet for the \(Z_2\) flux loop is orientable, then its Poincaré dual \(s_2^2\) is a \(Z\)-valued 2-cocycle thus \(s_2 \in Z^2(M^2; Z)\). In this case \(\beta_2 s_2 = \frac{1}{2} d s_2 = 0\) because the cocycle condition imposes \(d s_2 = 0\).

Therefore a nontrivial \(\beta_2 s_2\) comes from an unorientable 2d worldsheet. On an unorientable worldsheet, we have a worldline that marks the reversal of the orientation, whose Poincaré dual is \(\beta_2 s_2\). So such an orientation-reversal worldline corresponds to a fermion worldline.

In other words, the anomalous \(Z_2\) gauge theory (25) on the boundary has a special property that an unorientable worldsheet of the \(Z_2\)-flux carries a non-contractable fermionic worldline. Such a fermionic worldline corresponds to the orientation reversal loop on the unorientable worldsheet.

2. Trivialization picture

We have used the trivialization picture to understand the half-integer spin and the Fermi statistic of the \(Z_2\) charge. Can we use the similar trivialization picture to understand the above “fermionic” properties of \(Z_2\)-flux lines?

In the above we have associated the Fermi statistic of the \(Z_2\) charge (described by \(l_3\) in spacetime \(B^4\)) with the topological invariant

\[ e^{i \pi f_{ab} l_3 a^2 z + i \pi f_{ab} S^2 l_3 l_1 w_2}, \]

expressed in one higher dimension \(M^5\). Similarly, we can associate the “Fermi statistic” of the \(Z_2\) flux line (described by \(s_2\) in spacetime \(B^4\)) with the topological invariant

\[ e^{i \pi f_{ab} S^2 b^2 z + i \pi f_{ab} S^2 S^4 s_2 w_3}, \]

(46)

Although we require the oriented and orientable special orthogonal SO symmetry for this 4d boundary and 5d bulk theory, we do allow unorientable worldsheets on 2d submanifolds. Earlier we wrote for oriented worldsheet \(s_2 \in Z^2(M^2; Z)\) with the topological term:

\[ e^{i \pi f_{ab} S^2 \beta_2 s_2 + w_2 \beta_2 s_2} = e^{i \pi f_{ab} S^2 \beta_2 s_2 + w_2 \beta_2 s_2}. \]  

(42)

However, for unorientable worldsheets \(s_2 \in Z^2(M^2; Z_2)\), we can use Steenrod square \(S^4\) to rewrite (42) as

\[ e^{i \pi f_{ab} S^2 S^4 s_2 + w_2 S^4 s_2} = e^{i \pi f_{ab} S^2 S^4 s_2 + w_2 S^4 s_2}. \]  

(43)

In Appendix H, we prove a generalized Wu relation

\[ S^4 S^4 x_{d-3} = (w_1 + w_2) x_{d-3} \]

(44)

on a manifold \(M^d\) where \(w_1\) is the Stiefel-Whitney class of \(M^d\).
To gain a better understanding of the “Fermi” statistics of $Z_2$ charged particle and $Z_2$ flux line. We like to express topological invariants in the same dimension rather than one-higher dimension.

We start with the path integral (25) describing the $Z_2$ boundary of the $w_2 w_3$ invertible bosonic topological order. Then we add the worldline for $Z_2$ charge and worldsheet for $Z_2$ flux line. But here we assume the worldline and worldsheet are boundaries. Thus their Poincaré dual’s, $l_3$ and $s_2$, are coboundaries

$$l_3 = d\tilde{l}_2, \quad s_2 = ds_1. \quad (47)$$

Adding the worldline for $Z_2$ charge and worksheet for $Z_2$ flux line to the boundary spacetime $B^4$, we obtain the following path integral

$$Z = \sum_{a^2, b^2} e^{i \pi f_{\mu \nu} w_2 w_3} e^{i \pi f_{\mu \nu} a^2 b^2 + a^2 w_2 + b^2 w_2} e^{i \pi f_{\mu \nu} (d\tilde{l}_2) a^2 + (d\tilde{s}_1) b^2}. \quad (48)$$

But the new term break the invariance under (26), which to ensure the branch independence. To fix this, we consider

$$Z = \sum_{a^2, b^2} e^{i \pi f_{\mu \nu} w_2 w_3} e^{i \pi f_{\mu \nu} a^2 b^2 + a^2 w_2 + b^2 w_2} e^{i \pi f_{\mu \nu} (d\tilde{l}_2) a^2 + (d\tilde{s}_1) b^2 + \tilde{s}_1 w_3}. \quad (49)$$

However, the fix causes another problem: $\tilde{l}_2$ and $\tilde{l}_2 + \tilde{d}_2$ described the same worldline if $\tilde{l}_2$ is a $Z_2$-valued cocycle; $\tilde{s}_1$ and $\tilde{s}_1 + \tilde{s}_1$ described the same worldsheet if $\tilde{s}_1$ is a $Z_2$-valued cocycle. Therefore, the path integral must be invariant under the following transformations

$$\begin{align*}
\tilde{l}_2 &\rightarrow \tilde{l}_2 + \tilde{d}_2, \quad d\tilde{l}_2 \overset{2}{=} 0; \\
\tilde{s}_1 &\rightarrow \tilde{s}_1 + \tilde{s}_1, \quad d\tilde{s}_1 \overset{2}{=} 0. \quad (50)
\end{align*}$$

To have such an invariance, we consider

$$Z = \sum_{a^2, b^2} e^{i \pi f_{\mu \nu} w_2 w_3} e^{i \pi f_{\mu \nu} a^2 b^2 + a^2 w_2 + b^2 w_2} e^{i \pi f_{\mu \nu} (d\tilde{l}_2) a^2 + (d\tilde{s}_1) b^2 + \tilde{s}_1 w_3 + \tilde{s}_1 \tilde{s}_1}. \quad (51)$$

where $Sq^n$ is the generalized Steenrod square introduced in Ref. 71, which acts on cochains and is defined in (A21). $Sq^n$ is equal to Pontryagin square mod 2 when acting on $n$-cochains. Thus the generalized Steenrod square $Sq^n$ is in general not the same as the convention Pontryagin square studied in [73, 74].

Using (A29), we find

$$(\tilde{l}_2 + \tilde{d}_2) w_2 + Sq^2(\tilde{l}_2 + \tilde{d}_2) \overset{2}{=} (\tilde{l}_2 + \tilde{d}_2) w_2 + Sq^2 \tilde{l}_2 + Sq^2 \tilde{l}_2 \overset{2}{=} (\tilde{l}_2 + \tilde{d}_2) w_2 + Sq^2 \tilde{l}_2 + w_2 \tilde{l}_2 \overset{2}{=} \tilde{l}_2 w_2 + Sq^2 \tilde{l}_2, \quad (52)$$

where we have used $Sq^2 \tilde{l}_2 \overset{2}{=} Sq^2 \tilde{l}_2 \overset{2}{=} (w_2 + w_2^2) \tilde{l}_2$ and $w_1 \overset{2}{=} 0$ for $B^4$. This allows us to show the invariance of the path integral (51) under $\tilde{l}_2 \rightarrow \tilde{l}_2 + \tilde{d}_2$.

Similarly, using (A29) and (A23), we find

$$(\tilde{s}_1 + \tilde{s}_1) w_3 + Sq^2 Sq^1(\tilde{s}_1 + \tilde{s}_1) \overset{2}{=} (\tilde{s}_1 + \tilde{s}_1) w_3 + Sq^2 Sq^1 \tilde{s}_1 + Sq^2 Sq^1 \tilde{s}_1 \overset{2}{=} \tilde{s}_1 w_3 + \tilde{s}_1 Sq^1 w_2 + Sq^2 Sq^1 \tilde{s}_1 + w_2 Sq^1 \tilde{s}_1 \quad (53)$$

where we have used $Sq^1 w_2 \overset{2}{=} w_3$ when $w_1 \overset{2}{=} 0$. Using (A33) and (A9), we have

$$\begin{align*}
\tilde{s}_1 Sq^1 w_2 + w_2 Sq^1 \tilde{s}_1 \overset{2}{=}& \tilde{s}_1 \beta_2 w_2 + w_2 \beta_2 \tilde{s}_1 \overset{2}{=}(\beta_2 w_2) \tilde{s}_1 + w_2 \beta_2 \tilde{s}_1 \overset{2}{=} \beta_2 (w_2 \tilde{s}_1) \overset{2}{=} 0 \quad (54)
\end{align*}$$

Therefore

$$(\tilde{s}_1 + \tilde{s}_1) w_3 + Sq^2 Sq^1(\tilde{s}_1 + \tilde{s}_1) \overset{2}{=} \tilde{s}_1 w_3 + Sq^2 Sq^1 \tilde{s}_1. \quad (55)$$

This allows us to show the invariance of the path integral (51) under $\tilde{s}_1 \rightarrow \tilde{s}_1 + \tilde{s}_1$. Thus (51) is a correct boundary theory for $w_2 w_3$ topological order.

Let us examine the $\tilde{l}_2$ terms in the theory. The term $e^{i \pi f_{\mu \nu} Sq^2 \tilde{l}_2}$, quadratic in $\tilde{l}_2$, gives the $Z_2$-charge (described by $l_3 = d\tilde{l}_2$) a Fermi statistics. The accompanying linear term $e^{i \pi f_{\mu \nu} w_3 \tilde{l}_2}$ gives the $Z_2$-charge a half integer spin, which is associated with the statement that fermion is related to the trivialization of $w_2$ (for details, see Appendix C).

Now let us examine the $\tilde{s}_1$ terms. Due to the very similar structure, we can say that the term $e^{i \pi f_{\mu \nu} Sq^2 \tilde{s}_1}$ gives the $Z_2$-flux line (described by $s_2 = d\tilde{s}_1$) an unusual statistics. We may also say that the accompanying linear term $e^{i \pi f_{\mu \nu} w_3 \tilde{s}_1}$ is associated with the trivialization of $w_3$. So the unusual statistics of the $Z_2$-flux line is associated with the trivialization of $w_3$, just like the Fermi statistics of a particle is associated with the trivialization of $w_2$.

One way to characterize the unusual statistics of the $Z_2$-flux line is to notice that $d(Sq^n \tilde{s}_1) = Sq^1 d\tilde{s}_1 = Sq^1 Sq^n \overset{2}{=} \beta_2 s_2$ describes the Poincaré dual of the orientation reversal line of the worldsheet of the $Z_2$-flux line. So the orientation reversal line of the worldsheet behaves like a fermion worldline due to the term $Sq^2(Sq^1 \tilde{s}_1)$.

From the relation $\beta_2 w_2 \overset{2}{=} w_3$, we see that the trivialization of $w_2$ on the boundary, also implies a trivialization of $w_3$. The worldsheet $s_2$ of the $Z_2$-flux line couples to a 2-cochain $b^{2,2}$ via $e^{i \pi f_{\mu \nu} b^{2,2}}$ (see (30)). For our anomalous $Z_2$ gauge theory, the $w_3$ is trivialized as a coboundary via the split into a lower-dimensional cochain $b^{2,2}$, namely

$$db^{2,2} \overset{2}{=} w_3. \quad (56)$$

Such a relation can be obtained from (25) by integrating out $a^{2,2}$ first. But (56) is not independent from (34), because $w_3 \overset{2}{=} Sq^1 w_2$.

To summarize, the 4+1D invertible topological order has a boundary described by
(1) A dynamical $Z_2$ gauge theory with gravitational anomaly.
(2) In the continuum limit, the gauge charge transforms as $Z_2 \times w_2$ SO($\infty$) = Spin($\infty$) under the $Z_2$ gauge transformation and spacetime rotation.
(3) Such a $Z_2$ gauge charge is a fermion in the spin-statistics.
(4) The orientation-reversal worldline on the unorientable worldsheet of $Z_2$-flux loop is a fermion worldline. But such a fermion worldline does not carry the $Z_2$ gauge charge.

3. A physical consequence of the fermion-carrying $Z_2$-flux worldsheet

In a usual anomaly-free $Z_2$-gauge theory, if we proliferate the $Z_2$-flux worldsheets in spacetime, we will get new gapped state, which corresponds to a confined phase $Z_2$-gauge theory with no topological order. Why proliferating the $Z_2$-flux worldsheet can give rise to a gapped state? This is because the path integral amplitude for the $Z_2$-flux worldsheets is positive. If the surface tension of the worldsheet is zero, the equal weight superposition $Z_2$ are not sure that the proliferation of orientable worldsheet give rise to a short-range correlated state. In this case, we are not sure that the proliferation the unorientable worldsheet. In this case, we will discuss another boundary — a $U(1)$-gauge boundary. Such a group extension to a total group Spin$^c$ is the U(1)$^f$ gauge theory with no topological order.

The above discussions also implies that the $w_2 w_3$ topological order cannot have a trivial confined phase.

If we only allow orientable worldsheet of the $Z_2$-flux lines, then the path integral amplitude for those orientable worldsheet can be all positive. We can have a phase where the orientable worldsheet proliferate. But the proliferation of orientable worldsheet give rise to a U(1) gauge theory, instead of $Z_2$ confined phase. Such a U(1)$^f$-gauge boundary of the $w_2 w_3$ topological order will be discussed in the next subsection.

C. 4d U(1)$^f$-gauge boundary of the $w_2 w_3$ topological order

The anomalous $Z_2$-gauge theory is only one possible boundaries of the $w_2 w_3$ topological order. In this section, we will discuss another boundary — a U(1)$^f$-gauge theory with gravitational anomaly. To obtain such an anomalous boundary U(1)$^f$-gauge theory, we first rewrite the topological invariant $w_2 w_3 \cong w_2 Sq^1 w_2$ as

$$Z^{top}(M^5) = e^{i \pi \int_{M^5} w_2 Sq^1 w_2} = e^{i \pi \int_{M^5} w_2 \beta_2 w_2} = e^{i \pi \int_{M^5} w_2 \frac{1}{2} dw_2},$$  \hspace{1cm} (57)

where we have used $Sq^1 w_2 \cong \beta_2 w_2$ (see (A33)) and $\beta_2 w_2 = \frac{1}{2} dw_2$ (see (A9)). In (57), both $(w_2)(\beta_2 w_2)$ and $(w_2)\frac{1}{2} dw_2$ pair between the $Z_2$-valued $w_2$ and the $Z$-valued $\beta_2 w_2$ or $(\frac{1}{2} dw_2)$, which altogether can be well-defined in the $Z_2$ value.

We find that if $M^5$ has a Spin$^c$ structure, then $Sq^1 w_2 = Sq^1 (c_1 \mod 2) \cong 0$, thus $\beta_2 w_2 = 0$ and $Z^{top}(M^5) = 1$ (See a proof in Appendix D 1's Remark 3). Thus, the $w_2 w_3$ is not a cobordism invariant for the Spin$^c$ structure. In this case, the SO($\infty$) connection on $M^5$ can be lifted into a U(1)$^f \times w_2$ SO($\infty$) connection. The U(1)$^f$ implies that

$$U(1)^f \supset Z_2^f$$

contains the fermion parity as a normal subgroup. Here $U(1)^f \times w_2$ SO($\infty$) is the U(1)$^f$ extension of SO($\infty$) characterized by $\frac{1}{2} w_2 \in H^2(BSO(\infty); \mathbb{R}/Z)$.

Such a group extension to a total group Spin$^c$ = Spin$\times_Z U(1)^f$ $U(1)^f$ implies that the $w_2 w_3$ in SO is trivialized in Spin$^c$. The Spin$^c$ structure, which contains the emergent U(1)$^f$ on the boundary, is also called the emergent dynamical Spin$^c$ structure [44].

The above discussions also implies that the $w_2 w_3$ topological order has another boundary described by a U(1)$^f$ gauge theory with gravitational anomaly. To write down such a U(1)$^f$ gauge theory, we start with a 5d branch-independent bosonic model that realize the $w_2 w_3$ invertible topological order:

$$Z = \sum_{a^{\infty/Z_2}, b^Z} e^{i \pi \int_{M^5} (w_2 + 2 d a^{\infty/Z_2} ) (\beta_2 w_2 + d b^Z)},$$ \hspace{1cm} (59)

The branching-independence is ensured by the invariance of the above path integral under the following dynamical gauge transformations $(\alpha_0)$ and background gauge transformations $(v_1$ and $v_2)$:

$$a^{\infty/Z_2} \rightarrow a^{\infty/Z_2} + d \alpha_0 + \frac{1}{2} v_1, \hspace{1cm} b^Z \rightarrow b^Z + v_2, \hspace{1cm} w_2 \rightarrow w_2 - dv_1 - 2v_2, \hspace{1cm} \beta_2 w_2 \rightarrow \beta_2 w_2 - \beta_2 dv_1 - dv_2 = \beta_2 w_2 - dv_2.$$ \hspace{1cm} (60)

The $v_1 \in C^1(M^5; \mathbb{Z})$ and $v_2 \in C^2(M^5; \mathbb{Z})$ are $\mathbb{Z}$-valued 1- and 2-cochains and $\alpha_0 \in C^0(B^4; \mathbb{R}/\mathbb{Z})$ is a $\mathbb{R}/\mathbb{Z}$-valued function (i.e. 0-cochain). Note that $\beta_2 dv_1 = 0$ because $\beta_2 = \frac{1}{2} d$, while $\frac{1}{2} v_1 \in C^1(M^5; \mathbb{Z})$, and $d d = d^2 = 0$ on $C^1(M^5; \mathbb{R}/\mathbb{Z})$. Here the gauge transformation of $a^{\infty/Z_2}$ is related to that of $\frac{1}{2} d b^Z$ where the gauge transformation of $a^{\infty/Z_2}$ is in (26).

When $M^5$ has boundaries and after integrating out $a^{\infty/Z_2}, b^Z$ in the bulk, we obtain the following boundary
theory

\[ Z = \sum \frac{e^{i\pi \int_{B^4} w_2 b_2}}{a^6/Z_2} e^{2\pi \int_{B^4} a^{R/Z_2} \, dB^2 + a^{R/Z_2} \, dB^2 + \frac{1}{2} w_2 b_2^2} \]

(61)

where \( a^{R/Z_2} \in C^1(B^4; \mathbb{R}/\mathbb{Z}) \) is a \( \mathbb{R}/\mathbb{Z} \)-valued 1-cochain and \( b^2 \in C^2(B^4; \mathbb{Z}) \) is a \( \mathbb{Z} \)-valued 2-cochain. We add a gauge invariant term \( e^{-\frac{i}{2} g \int_{B^4} [b + da^{R/Z_2} + \frac{1}{2} w_2 b_2]^2} \), which will produce the Maxwell term after we integrating out the \( b^2 \) field. We find that a small \( g \) leads to a semiclassical \( U(1)^J \) gauge theory. Thus, the above describes a \( U(1)^J \) gauge theory,\(^9\) with a gravitational anomaly of \( w_2 w_3 \).

\(^7\) Note that the isomorphism \( \mathbb{R}/\mathbb{Z} = U(1) \), however we use \( \mathbb{R}/\mathbb{Z} \) to emphasize the group operation is addition as in \( \mathbb{R}/\mathbb{Z} \), instead of multiplication in \( U(1) \). Also, we denote \( \mathbb{R}/\mathbb{Z} = U(1)/\mathbb{Z}_2 \) to include the fermion parity normal subgroup.

\(^8\) First let us clarify why \( b^2 \, da^{R/Z_2} + (2b_2 w_2) a^{R/Z_2} + b^2 (\frac{1}{2} w_2) \) is well-defined in \( \mathbb{R}/\mathbb{Z} \)-valued, for the cup product between a \( \mathbb{Z} \)-valued cohomology class (e.g., here \( b^2, b_2, w_2 \), etc.) and a \( \mathbb{R}/\mathbb{Z} \)-valued cohomology class (e.g., here \( da^{R/Z_2}, a^{R/Z_2}, \frac{1}{2} w_2 \), etc.). Generally, for \( z \in \mathbb{Z} \) and the equivalence class \( [x] \in \mathbb{R}/\mathbb{Z} \) where the representative \( x \in \mathbb{R} \), we have that the definition \( z(x) \equiv [x] \in \mathbb{R}/\mathbb{Z} \) is well-defined since if \( x = [y] \in \mathbb{R}/\mathbb{Z} \), then \( x - y \in \mathbb{Z} \) and \( z(x - y) \in \mathbb{Z} \). So \( [x] = [y] \in \mathbb{R}/\mathbb{Z} \), thus we prove that \( z(x) \in \mathbb{R}/\mathbb{Z} \) is well-defined. Thus, we prove that \( b^2 da^{R/Z_2} + (2b_2 w_2) a^{R/Z_2} + b^2 \) is well-defined in \( \mathbb{R}/\mathbb{Z} \)-valued.

\(^9\) We can motivate better about the 4d action \( \int_{B^4} b^2 da^{R/Z_2} \), schematically, without any other source or operator insertions in the path integral, we have the equation of motion \( db^2 = 0 \). Although the cocycle (closed) is not necessarily coboundary (exact), locally we can write \( b^2 = dv^{R/Z_2} \), since the \( b^2 \in C^2(B^4; \mathbb{Z}) \) has an integer quantized electric flux, then \( v^{R/Z_2} \in C^1(B^4; \mathbb{R}/\mathbb{Z}) \) is the dual gauge field (the ‘t Hooft magnetic gauge field). Since the pure abelian \( U(1) \) gauge theory has the action \( da^{R/Z_2} \wedge * da^{R/Z_2} = *dv^{R/Z_2} \wedge dv^{R/Z_2} \) with the \( U(1) \) gauge coupling suppressed, we can also regard \( b^2 \) as the integer quantized electric flux \( \int_{B^4} da^{R/Z_2} = \int_{B^4} *dv^{R/Z_2} \in \mathbb{Z} \) on a closed 2-cycle, while \( *b^2 \) as the integer quantized magnetic flux \( \int_{B^4} *dv^{R/Z_2} = \int_{B^4} da^{R/Z_2} \in \mathbb{Z} \) on a closed 2-cycle. Thus in this special case, we may also treat \( \int_{B^4} da^{R/Z_2} \wedge *da^{R/Z_2} = \int_{B^4} *dv^{R/Z_2} \wedge dv^{R/Z_2} \). By the Maxwell equations, \( ddv^{R/Z_2} = d * da^{R/Z_2} = 0 \), so \( da^{R/Z_2} \) is the harmonic form (closed and co-closed) and the Hodge dual \( *da^{R/Z_2} = dv^{R/Z_2} \) is also a harmonic form. By the Hodge theorem, each cohomology class has a unique harmonic representative. So Hodge star is an isomorphism from the harmonic representatives of \( H^{DR}_3(M) \) to the harmonic representatives of \( H^{DR}_3(M) \), here we can take \( M = B^4 \). Poincaré duality says that \( \alpha \wedge \beta = \int_M \alpha \wedge \beta \) is a perfect pairing between \( H^{DR}_k(M) \) and \( H^{DR}_{4-k}(M) \). So \( \int_M \alpha \wedge \alpha > 0 \) for any nonzero \( \alpha \) and we can define \( \int_M |\alpha|^2 \) as \( \int_M \alpha \wedge \alpha \). The sum of \( |\alpha|^2 \) on a 2-simplex over the spacetime simplex is also the action \( \int_M |\alpha|^2 \).

For a cohomology class \( \alpha \), the integral \( \int_M |\alpha|^2 \) is defined to be \( \int_M |\alpha|^2 \) for any cocycle representative \( \alpha \) of the cohomology class.

In the presence of the 1d worldline of \( U(1)^J \)-electric charge (i.e. Wilson line) and the worldline of \( U(1)^J \)-magnetic monopole (i.e. ‘t Hooft line) on the boundary, the boundary theory becomes

\[ Z = \sum \frac{e^{i\pi \int_{M^5} w_2 b_2}}{a^6/Z_2} e^{2\pi \int_{M^5} a^{R/Z_2} \, dB^2 + a^{R/Z_2} \, dB^2 + \frac{1}{2} w_2 b_2^2} \]

(62)

where \( l_3 \) is a \( \mathbb{Z} \)-valued 3-cocycle – the Poincaré dual of the Wilson line \( e^{i \frac{1}{2} \eta_3^{R/Z_2}} \) (the worldline of the \( U(1)^J \) electric charge), and \( \eta_2^{R/Z_2} \) is a \( \mathbb{R}/\mathbb{Z} \)-valued 2-cocycle, i.e. it satisfies

\[ d\eta_2^{R/Z_2} = \frac{1}{l_3} Z. \]

(63)

Here \( \frac{1}{l_3} Z \) is a \( \mathbb{Z} \)-valued cocycle, which is the Poincaré dual of the boundary of the \( \eta_2^{R/Z_2} / U(1)^J \) (the worldline of the \( U(1)^J \) magnetic monopole) written in terms of the dual gauge field.\(^{10}\) We note that \( \frac{1}{l_3} Z \) (of \( \mathbb{R}/\mathbb{Z} \) or \( U(1) \) value) in (62) is related to \( \frac{1}{2} s_2 \) where \( s_2 \) (of \( \tilde{Z}_2 \) or \( Z \) value) appears in (30).\(^{11}\)

The theory (62) is invariant under the gauge transformation (60). Also, the term

\[ e^{i\pi \int_{M^5} \eta_2^{R/Z_2} w_2 + \frac{1}{2} \eta_2^{R/Z_2} b_2} \]

only depends on the fields on the boundary \( B^4 = \partial M^5 \).

Based on the relation (63), this term can be also read

\[ a. \]

In particular, we can choose the harmonic representative. For a harmonic form \( \alpha \), we have \( \int_M |\alpha|^2 = \int_M \alpha \wedge * \alpha \) since \( \alpha \wedge * \alpha = |\alpha|^2 \).

\(^{10}\) Note that \( e^{i \frac{1}{2} \eta_3^{R/Z_2} \eta_2^{R/Z_2} b_2} \) in (62) can be schematically rewritten as

\[ e^{i \frac{1}{2} \eta_3^{R/Z_2} \eta_2^{R/Z_2} b_2} = e^{i \frac{1}{2} \eta_3^{R/Z_2} \eta_2^{R/Z_2} b_2 + \eta_2^{R/Z_2} \eta_2^{R/Z_2}} \]

(64)

via \( b_2 = dv^{R/Z_2} \wedge *dv^{R/Z_2} \). Hence we can identify the Poincaré dual of the Wilson line \( (\eta_3^{R/Z_2}) \) as \( l_3 \) \( Z \), while the Poincaré dual of the \( \text{‘} \text{t} \text{Hooft line} \) \( (\eta_2^{R/Z_2}) \) as \( \frac{1}{l_3} Z \).

\(^{11}\) If we map \( \eta_2^{R/Z_2} \mapsto \frac{1}{2} s_2 \), both are \( \mathbb{R}/\mathbb{Z} \)-valued, then we can map between the expressions in (62) and (30):

\[ \text{Sq}^2 d\eta_2^{R/Z_2} + \eta_2^{R/Z_2} \text{dw}_2 \mapsto \text{Sq}^2 \frac{1}{2} s_2 + \frac{1}{2} s_2 \text{dw}_2 \]

(65)
schematically as
\[
e^{i\pi \int_{M^5} (\text{Sq}^2 + w_2)(l_5^1 Z + l_5^3 Z)}.
\]

This term makes the U(1)\(^f\) electric charge (Wilson line \(e^{i\phi_{\beta}} a^e_{/Z} f\) as Poincaré dual of \(l_5^1 Z\)) and the U(1)\(^f\) magnetic monopole ('t Hooft line \(e^{i\phi_{\beta}} a^e_{/Z} f\) as Poincaré dual of \(l_5^3 Z\)) to be fermions, via a higher dimensional bosonization [17, 65, 71]. Thus, we show that the boundary of 5d \(w_2\) topological order has the U(1)\(^f\) electric charge has the fermionic statistics, the U(1)\(^f\) magnetic monopole has the fermionic statistics, and their bound object U(1)\(^f\) dyon also has the fermionic statistics. This is known as the all fermion quantum electrodynamics (QED\(_4\)).

Before ending this section, we like to write down two bosonic theories with no gravitational anomaly. The first one is given by the following path integral

\[
Z = \sum_{a^e_{/Zf}, b^z} e^{i2\pi \int_{M^4} a^e_{/Zf} \, da^e_{/Zf} - \frac{1}{2} g \int_{M^4} b^z \wedge b^z}
\]

which is invariant under the following gauge transformation

\[
a^e_{/Zf} \rightarrow a^e_{/Zf} + d\alpha_0, \quad b^z \rightarrow b^z.
\]

When \(g\) is small, the above bosonic model realizes a U(1) gauge theory at low energies, where both electric charge described by \(l_5^1 Z\) and magnetic charge described by \(d\theta_2^m \, \text{R}^{\text{Z}}\) are bosons.

The second bosonic model is given by

\[
Z = \sum_{a^e_{/Zf}, b^z} e^{i2\pi \int_{M^4} a^e_{/Zf} \, da^e_{/Zf} - \frac{1}{2} g \int_{M^4} b^z \wedge b^z}
\]

\[
\left. e^{i2\pi \int_{M^4} l_5^1 Z \, a^e_{/Zf} + \eta_2^e_{/Zf} b^z} e^{i\pi \int_{M^5} \text{Sq}^2 l_5^1 Z + l_5^3 Z w_2} \right)
\]

which is invariant under the following gauge transformation

\[
a^e_{/Zf} \rightarrow a^e_{/Zf} + d\alpha_0 + \frac{1}{2} \psi_0, \quad b^z \rightarrow b^z,
\]

\[
w_2 \rightarrow w_2 - d\psi_0.
\]

When \(g\) is small, the above bosonic model realizes a U(1) gauge theory at low energies, where the electric charge described by \(l_5^1 Z\) is a fermion, and the magnetic charge described by \(d\theta_2^m \, \text{R}^{\text{Z}}\) is a boson.

### IV. 4D Boundary of SU(2) or Other Spin(\(n\)) Internal Symmetric Theories

So far we have formulated two kinds of 4d boundary theories of 5d \(w_2\)\(_3\): the 4d \(Z_2\) gauge theories (where the local \(Z_2\) gauge field is more precisely the dynamical Spin structure summed over in the path integral) and the 4d U(1) gauge theories (where the local U(1) gauge field is more precisely the dynamical U(1) gauge connection of Spin\(^c\) structure summed over in the path integral). We also have provided these boundary gauge theory constructions as the trivialization of \(w_2\)\(_3\) of SO structure via its pullback \(p\) to the \(p^* w_2\)\(_3\) = 0 in Spin or Spin\(^c\) structures (Namely, 1 \(\rightarrow\) \(Z_2\) \(\rightarrow\) Spin \(\bigotimes_2^2\) SO \(\rightarrow\) 1 and 1 \(\rightarrow\) U(1) \(\rightarrow\) Spin\(^c\) \(\bigotimes_2^2\) SO \(\rightarrow\) 1 see further details in Appendix D). The Spin\(^c\) = Spin \(\times\) \(Z_2\) U(1) structure constrains that the fermions carry odd U(1) charge while the bosons carry even U(1) charge. In this section, we consider the Spin\(^h\) = Spin \(\times\) \(Z_2\) SU(2) structure such that the fermions are in even dimensional representation (e.g., \(2, 4, \ldots\); or isospin \(\frac{1}{2}, \frac{3}{2}, \ldots\)) of SU(2) while the bosons are in odd dimensional representation (e.g., \(1, 3, \ldots\); or isospin 0, 1, \ldots) of SU(2).

More generally, we can consider the Spin \(\times\) \(Z_2\) Spin\((n)\) structure. For example, for \(n = 10\), fermions are in the spinor representation of Spin(10) (e.g., \(16, \ldots\)) while bosons are in other tensor representation of Spin(10) (e.g., \(10, \ldots\)). We list down the cobordism invariants from TP\(_5\)(Spin \(\times\) \(Z_2\) Spin\((n)\)) in Appendix D. In this section, we summarize and enumerate other 4d boundary theories of 5d \(w_2\)\(_3\), based on the Spin \(\times\) \(Z_2\) Spin\((n)\) structure construction, in particular \(n = 3\) and 10.

1. Boundary with the SU(2) and Spin\(^h\) = Spin \(\times\) \(Z_2\) SU(2) = Spin \(\times\) \(Z_2\) Spin(3) structures:

   (a) When the SU(2) is a global symmetry, 
   
   - An odd number of the fundamental \(2\)-dimensional representation (Rep) of SU(2) of the spacetime Weyl spinor \(\psi\) cannot be gapped by quadratic mass term while preserving the Lorentz and SU(2) symmetries. This is due to that the only quadratic mass term \(e^{i\alpha \beta} \psi_{\alpha i} \psi_{\beta j}\) = 0 (where \(\alpha, \beta\) are the Lorentz indices and \(i, j \in \{1, 2\}\) are SU(2) indices; we take both the singlet \(1\) out of \(2 \otimes 2 = 1 \oplus 3\) vanish due to the fermi statistics. This suggests a possible anomaly — another hint is that the fermion spectrum under the SU(2) gauge bundle over \(S^4\) with an instanton number 1 background gives an odd number of fermion zero mode. More generally, an odd number of \(4\mathbf{r} + 2\) Rep of SU(2) Weyl spinor has the same anomaly known

\[
a^e_{/Zf} \rightarrow a^e_{/Zf} + d\alpha_0, \quad b^z \rightarrow b^z.
\]

\[
(66)
\]

\[
(67)
\]

\[
(68)
\]

\[
(69)
\]

\[
(70)
\]
as Witten anomaly as a ’t Hooft anomaly of the Spin$^h$-symmetry (See Table I). But these 4d theories live on the boundary of another 5d cobordism invariant, known as $\partial PD(c_2(V_{SU(2)})).$ These 4d theories do not live on the boundary of the 5d $w_2 w_3$.

- An odd number of the 4-dimensional representation (Rep) of $SU(2)$ of the spacetime Weyl spinor $\Psi$ also cannot be gapped by quadratic mass term while preserving the Lorentz and $SU(2)$ symmetries. The singlet of both Lorentz and $SU(2)$ symmetry requires any quadratic mass term vanishes: $\epsilon^{\alpha\beta} C^{IJ} \Psi_{\alpha I} \Psi_{\beta J} = 0.$ Ref. 44 shows that on a 4d non-spin manifold, the complex projective space $\mathbb{CP}^2$, with an appropriate large diffeomorphism by complex conjugation the $\mathbb{CP}^2$ coordinates $z_j \rightarrow \bar{z}_j$ and an appropriate $SU(2)$ large gauge transformation, we can construct a mapping torus 5d Dold manifold $\mathbb{CP}^2 \times S^1$ such that the large gauge-diffeomorphism is transformed along the fifth dimension. Moreover, together with the $SU(2)$ bundle, the whole theory is compatible with the Spin$^h$ structure. But the path integral gets a $(-1)$ sign under this large gauge-diffeomorphism transformation. This odd $(-1)$ non-invariance shows the new $SU(2)$ anomaly. More generally, an odd number of $8r + 4$ Rep of $SU(2)$ Weyl spinor has the same anomaly known as the new $SU(2)$ anomaly as a ’t Hooft anomaly of the Spin$^h$-symmetry (See Table I).

| $SU(2)$ isospin | $0 \mod 4$ | $1 \mod 4$ | $2 \mod 4$ | $3 \mod 4$ | $2r + \frac{1}{2} \mod 4$ | $4r + \frac{1}{2} \mod 4$ |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $SU(2)$ Rep $R$ (dim) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | mod 8 | $4r + 2$ | $8r + 4$ | mod 8 |
| Witten $SU(2)$ anomaly | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| New $SU(2)$ anomaly | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ |

TABLE I. For a single spacetime Weyl spinor $2_L$ in 4d, it has a Witten $SU(2)$ anomaly if the spacetime Weyl spinor is also the 4$r + 2$-dimensional representation (Rep) of $SU(2)$ (or isospin $2r + \frac{1}{2}$), for some nonnegative integer $r$. It has a new $SU(2)$ anomaly if the spacetime Weyl spinor is also the 8$r + 4$-dimensional Rep of $SU(2)$ (or isospin $4r + \frac{3}{2}$), for some nonnegative integer $r$. The checkmark $\checkmark$ means the fermion theory has the corresponding anomaly. These $SU(2)$ anomalies can be interpreted as either a ’t Hooft anomaly of global symmetry (if the $SU(2)$ is global symmetry not gauged), or dynamical gauge anomaly (if the $SU(2)$ is dynamically gauged).

(b) When the $SU(2)$ is dynamically gauged,

- Witten anomaly gives a dynamical gauge anomaly constraint such that an odd number of $4r + 2$ Rep of $SU(2)$ Weyl spinor coupled to dynamical $SU(2)$ gauge fields are ill-defined. It is not physically sensible to study its gauge dynamics.

- The gauge theories with new $SU(2)$ anomalies are still well-defined theories with well-defined gauge dynamics on Spin manifolds, because $w_2 = 0$ means no $w_2 w_3$ anomaly on the Spin manifolds. However, their gauge dynamics become ill-defined in 4d on non-Spin manifolds.

(c) Let us discuss further about the $SU(2)$ theory with a 4 Rep of $SU(2)$ Weyl spinor.

- **Explicit symmetry breaking**: If we are allowed to break this $SU(2)$ theory with the new $SU(2)$ anomaly, for example by choosing the quadratic fermion mass term via the $\epsilon^{\alpha\beta} C^{IJ} \Psi_{\alpha I} \Psi_{\beta J}$ such that the pairing $C^{IJ} \Psi_{\alpha I} \Psi_{\beta J}$ selects the 3-dimensional Rep of $SU(2)$ (which is also the vector Rep of SO(3)), and the isospin-1 Rep of $SU(2)$ and SO(3)), then $C^{IJ}$ is symmetric under $I \leftrightarrow J$, and such a mass term does not vanish under fermi statistics. This mass term explicitly break the $SU(2)$ down to U(1) symmetry.

- **Spontaneous symmetry breaking**: Instead we can consider the spontaneous symmetry breaking by introducing the Yukawa-Higgs term $\epsilon^{\alpha\beta} (C^{IJ} \Psi_{\alpha I} \Psi_{\beta J}) \Phi$ such that not merely $(C^{IJ} \Psi_{\alpha I} \Psi_{\beta J})$ is the 3 of $SU(2)$ but also the Higgs scalar $\Phi$ is also the 3 of $SU(2)$, while we again select the $SU(2)$ singlet $1$ out of

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12 The $\eta$ is a mod 2 index of 1d Dirac operator from $TP_1(\text{Spin}) = \mathbb{Z}_2$ or $\Omega^\text{Spin} = \mathbb{Z}_2$. A 1d manifold generator for the cobordism invariant $\eta$ is a 1d $S^1$ for fermions with periodic boundary condition, so called the Ramond circle. A 4d manifold generator for the $c_2(V_{SU(2)})$ is the nontrivial $SU(2)$ bundle over the $S^4$, such that the instanton number is 1. The PD is Poincaré dual.

13 Here $I, J \in \{1, 2, 3, 4\}$ are $SU(2)$ indices which correspond to isospin $\frac{1}{2}$ indices $\{\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\}$. We take the singlet 1 out of the tensor product of 4 of $SU(2)$: $4 \otimes 4 = 1 \oplus 3 \oplus 5 \oplus 7$. Based on the Clebsch-Gordan coefficients $\frac{1}{2}(\{\frac{1}{2}, \frac{1}{2}\}, \{\frac{1}{2}, \frac{1}{2}\}) - \{\frac{1}{2}, -\frac{1}{2}\}, \{\frac{1}{2}, 1\}) = (0, 0)$, we have $C^{ij} \Psi_{\alpha i} \Psi_{\beta j} = \frac{1}{2} (\Psi_{\alpha i} \Psi_{\beta j} - \Psi_{\alpha j} \Psi_{\beta i} - (\Psi_{\alpha i} \Psi_{\beta j} = (0, 0)$.

14 Here we take the 3 out of the tensor product of 4 of $SU(2)$: $4 \otimes 4 = 1 \oplus 3 \oplus 5 \oplus 7$. Based on the Clebsch-Gordan coefficients, we may choose $(\sqrt{\frac{2}{20}} (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})) = (1, 0)$, we have $C^{ij} \Psi_{\alpha i} \Psi_{\beta j} = (\sqrt{\frac{2}{20}} (\Psi_{\alpha i} \Psi_{\beta j} + \Psi_{\alpha j} \Psi_{\beta i} - (\Psi_{\alpha i} \Psi_{\beta j} = (0, 0)$. Since $\epsilon^{\alpha\beta}$ and $C^{ij}$ are both anti-symmetric, while all $\Psi$ are Grassmannian variables due to fermi statistics, thus $\epsilon^{\alpha\beta} C^{ij} \Psi_{\alpha i} \Psi_{\beta j} = 0$.
their tensor product pairing $3 \otimes 3 = 1 \oplus 3 \oplus 5$. Then the Higgs condensation $\langle \bar{\Phi} \rangle \neq 0$ only breaks the SU(2) \textit{spontaneously} down to U(1).

- **SU(2) to all-fermion U(1) gauge theories**: Ref. 44 shows that the spherical rotationally symmetric 't Hooft-Polyakov monopole (up to SU(2) gauge transformation) in the presence of $4$ Rep of SU(2) Weyl spinor traps fermionic zero modes in the spectrum. Ref. 44 shows that the 't Hooft-Polyakov monopole becomes fermion. This means under the breaking from SU(2) to U(1), such that the isospin $\frac{3}{2}$ Weyl fermion of SU(2) becomes the fermionic electric charge 1 of U(1).

- **SU(2) to all-fermion U(1) gauge theories**: Together with the fermionic electric charge of U(1), the dyon of U(1) is also a fermion. Ref. 44 suggests that a possible renormalization group (RG) flow such that the high energy theory is an asymptotic free SU(2) gauge theory while the low energy theory is the all-fermion U(1) gauge theory. The subtlety is that this 4d SU(2) gauge theory is ill-defined on a non-spin manifold and cannot be put on the boundary of 5d $w_2w_3$. But once we break SU(2) down to U(1), the 4d U(1) gauge theory can be put on a non-spin manifold on the boundary of 5d $w_2w_3$.

- **All-fermion U(1) to $Z_2$ gauge theories**: If we introduce another Higgs also the 3 of SU(2), with a different vacuum expectation value, we can further Higgs down the U(1) down to $Z_2$, such that the fermionic electric charge 1 of U(1) becomes the fermionic electric charge 1 of $Z_2$.

- **All-fermion U(1) to $Z_2$ gauge theories**: The fermionic monopole's 1d 't Hooft line of U(1) becomes the orientation-reversal 1d fermionic worldline on an unorientable 2d worldsheet of $Z_2$ gauge theory.

2. Boundary with the Spin(10) and Spin $\times Z_2$ Spin(10) structures:

(a) The standard $so(10)$ Grand Unification [75, 76] with Spin(10) internal symmetry group and with Weyl fermions in the 16 of Spin(10) does not have the $w_2w_3$ anomaly (more precisely, it is the $w_2w_3 = w'_2w'_3$ mixed gauge-gravitational anomaly, which is also a nonperturbative global anomaly, with $w_j = w_j(TM)$ and $w'_j = w_j(VSO(n=10))$). This is the only 5d cobordism invariant from TP$_3$(Spin $\times Z_2$ Spin(10)), thus the only 4d global anomaly for Spin $\times Z_2$ Spin(10) structure. The absence of $w_2w_3 = w'_2w'_3$ anomaly means that the standard $so(10)$ Grand Unification is free from all perturbative local and nonperturbative global anomalies within Spin $\times Z_2$ Spin(10) structure [44, 55, 77].

(b) However, it is possible to construct a modified $so(10)$ Grand Unification with Spin(10) internal symmetry group, also with Weyl fermions in the 16 of Spin(10), but with additional discrete torsion class of Wess-Zumino-Witten like term written on the 4d boundary and 5d bulk coupled system [45–47]. Here we summarize the results in [45–47]:

- **When Spin(10) internal symmetry group is treated as a global symmetry**, this modified $so(10)$ Grand Unification can live on the boundary of 5d $w_2w_3 = w'_2w'_3$ invertible topological order. The $w_2w_3 = w'_2w'_3$ anomaly is saturated by the discrete torsion class of Wess-Zumino-Witten like term alone. The discrete torsion class of Wess-Zumino-Witten like term gives rise to various possible gauge theory realizations of low energy dynamics in 4d. The various possible gauge theory realizations are the emergent gauge theories (similar to the emergent dynamical Spin structure of the $Z_2$ gauge theory and emergent dynamical Spin$^c$ structure of the all-fermion U(1) gauge theory that we studied earlier).

- **When Spin(10) internal symmetry group is dynamically gauged**, the Spin(10) gauge field in the 5d bulk $w_2w_3 = w'_2w'_3$ is also gauged. The 5d bulk is no longer a gapped invertible topological order; the 5d bulk becomes gapless and further long-range entangled. Thus the Spin(10) gauge fields can live only on the 4d boundary, but also propagate into the 5d bulk.

V. ACKNOWLEDGEMENTS

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Appendix A: Spacetime Complex with Branch Structure, Cochains, Higher Cup Product

1. Spacetime complex and branch structure

In this article, we consider models defined on a spacetime lattice. A spacetime lattice is a triangulation of the $d$-dimensional spacetime, which is denoted by $M^d$. We will also call the triangulation $M^d$ as a spacetime complex, which is formed by simplices – the vertices, links, triangles, etc. We will use $i,j,\cdots$ to label vertices of the spacetime complex. The links of the complex (the 1-simplices) will be labeled by $(i,j), (j,k), \cdots$. Similarly, the triangles of the complex (the 2-simplices) will be labeled by $(i,j,k), (j,k,l), \cdots$.

In order to define a generic lattice theory on the spacetime complex $M^d$ using local Lagrangian term on each simplex, it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branch structure.[10, 62, 63] A branch structure is a choice of orientation of each link in the $d$-dimensional complex so that there is no oriented loop on any triangle (see Fig. 2).

The branch structure induces a local order of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming links, and the second vertex is the vertex with only one incoming link, etc. So the simplex in Fig. 2a has the following vertex ordering: $0, 1, 2, 3$. We always use ordered vertices to label a simplex. So the simplex in Fig. 2a and 2b are labeled as $[0, 1, 2, 3]$.

The branch structure also gives the simplex (and its sub-simplices) a canonical orientation. Fig. 2 illustrates two 3-simplices with opposite canonical orientations compared with the 3-dimensional space in which they are embedded. The blue arrows indicate the canonical orientations of the 2-simplices. The black arrows indicate the canonical orientations of the 1-simplices.

2. Chain, cochain, cycle, cocycle

Given an Abelian group $(\mathbb{M}, +)$, $\mathbb{M}$ can also be viewed as a $\mathbb{Z}$-module (i.e., a vector space with integer coefficient) that also allows scaling by an integer:

$$x + y \in \mathbb{M}, \quad x \ast y \in \mathbb{M}, \quad mx = x + x + \cdots + x \in M, \quad \text{(m terms)}$$

$$x, y \in \mathbb{M}, \quad m \in \mathbb{Z}. \quad \text{(A1)}$$

The direct sum of two modules $\mathbb{M}_1 \oplus \mathbb{M}_2$ (as vector spaces) is equal to the direct product of the two modules (as sets):

$$\mathbb{M}_1 \oplus \mathbb{M}_2 \text{ as set} = \mathbb{M}_1 \times \mathbb{M}_2 \quad \text{(A2)}$$

An $n$-cochain $\alpha_n$ in $M^d$ is a formal combination of $n$-simplices in $M^d$ with coefficients in $\mathbb{M}$:

$$\alpha_n = \sum_{[i,j,\cdots,k]} \alpha_{n,i,j,\cdots,k} \in \mathbb{M}, \quad \text{(A3)}$$

where $\sum_{[i,j,\cdots,k]}$ sums over all simplexes in $M^d$. The collection of all such $n$-chains is denoted as $C_n(M^d; \mathbb{M})$. For example, a 2-chain can be a 2-simplex: $[i,j,k]$, a sum of two 2-simplices: $[i,j,k] + [j,k,l]$, a more general composition of 2-simplices: $[i,j,k] - 2[j,k,l]$, etc.

An $n$-cochain $f_n$ in $M^d$ is an assignment of values in $\mathbb{M}$ to each $n$-simplex $M^d$. For example a value $f_n(i,j,\cdots,k) \in \mathbb{M}$ is assigned to $n$-simplex $(i,j,\cdots,k)$. So a cochain $f_n$ can be viewed as a bosonic field, $f_n([i,j,\cdots,k])$, on the spacetime lattice $M^d$. To be more precise $f_n$ is a linear map $f_n : n$-chain $\rightarrow \mathbb{M}$. We can denote the linear map as $f_n(n$-chain), or

$$f_n(\alpha_n) = \sum_{[i,j,\cdots,k]} \alpha_{n,i,j,\cdots,k} f_n([i,j,\cdots,k]) \in \mathbb{M}. \quad \text{(A4)}$$

where $\sum_{[i,j,\cdots,k]}$ sums over all simplexes in $M^d$.

We will use $C^n(M^d; \mathbb{M})$ to denote the set of all $n$-cochains on $M^d$. $C^n(M^d; \mathbb{M})$ can also be viewed as a set all $\mathbb{M}$-valued fields (or paths) on $M^d$. Note that $C^n(M^d; \mathbb{M})$ is an Abelian group under the $+$-operation.

The total spacetime lattice $M^d$ correspond to a $d$-...
so called cohomology group denoted by \( f \),

\[
B_n(M^d;\mathcal{M}) \text{ is also an Abelian group.}
\]

For the \( \mathbb{Z}_N \)-valued cochain \( x_n \), we lift \( \mathbb{Z}_N \) to \( \mathbb{Z} \), via \( \{0, 1, \cdots, N - 1\} \subset \mathbb{Z}_N \) to \( \{0, 1, \cdots, N - 1\} \subset \mathbb{Z} \), and define

\[
\beta_N x_n = \frac{1}{N} dx_n. 
\tag{A9}
\]

When \( x_n \) is a cocycle, we have \( dx_n \overset{\mathbb{N}}{=} 0 \). In this case, \( \beta_N x_n \) is a \( \mathbb{Z} \)-valued cocycle, and \( \beta_N \) is Bockstein homomorphism.

4. Cup product and higher cup product

From two cochains \( f_m \) and \( h_n \), we can construct a third cochain \( p_{m+n} \) via the cup product (see Fig. 4):

\[
p_{m+n} = f_m \smile h_n, 
\tag{A10}
\]

\[
p_{m+n}([0 \rightarrow m + n]) = f_m([0 \rightarrow m])h_n([m \rightarrow m + n]),
\]

where \( i \rightarrow j \) is the consecutive sequence from \( i \) to \( j \):

\[
i \rightarrow j \equiv i, i + 1, \cdots, j - 1, j. 
\tag{A11}
\]

Note that the order of vertices in a simplex \( (0 \rightarrow m) \) and the notion of consecutive sequence are determined by the branch structure. Thus the cup product (and the higher cup product below) on a simplicial complex can be defined only after we assign a branch structure to the simplicial complex. The value of the cup product depends on the branch structure.

The cup product has the following property

\[
d(h_n \smile f_m) = (dh_n) \smile f_m + (-)^n h_n \smile (df_m) 
\tag{A12}
\]

for cochains with global or local values. We see that \( h_n \smile f_m \) is a cocycle if both \( f_m \) and \( h_n \) are cocycles. If both \( f_m \) and \( h_n \) are cocycles, then \( f_m \smile h_n \) is a coboundary if one of \( f_m \) and \( h_n \) is a coboundary. So the cup product is also an operation on cohomology groups \( \smile : H^n(M^d;\mathcal{M}) \times H^n(M^d;\mathcal{M}) \rightarrow H^{n+m}(M^d;\mathcal{M}) \). The cup product of two cochains has the following property (see Fig. 4)

\[
f_m \smile h_n = (-)^{mn} h_n \smile f_m + \text{coboundary} \quad \quad \quad \tag{A13}
\]

We can also define higher cup product \( f_m \smile_k h_n \) which gives rise to a \( (m-n) \)-cochain [78]:

\[
(f_m \smile_k h_n)([0,1,\cdots,m+n-k]) = \sum_{0 \leq i_0 < \cdots < i_k \leq n+m-k} (-)^p f_m([0 \rightarrow i_0, i_1 \rightarrow i_2, \cdots, i_k]) \times h_n([i_0 \rightarrow i_1, i_2 \rightarrow i_3, \cdots, i_k]), 
\tag{A14}
\]

and \( f_m \smile_k h_n = 0 \) for \( k < 0 \) or for \( k > m \) or \( n \). Here \( i \rightarrow j \) is the sequence \( i, i + 1, \cdots, j - 1, j \), and \( p \) is the number of permutations to bring the sequence

\[
0 \rightarrow i_0, i_1 \rightarrow i_2, \cdots; i_0 + 1 \rightarrow i_1 - 1, i_2 + 1 \rightarrow i_3 - 1, \cdots 
\tag{A15}
\]
to the sequence
\[ 0 \to m + n - k. \]  
(A16)

For example
\[
(f_m \sim h_n)([0 \to m + n - 1]) = \sum_{i=0}^{m-1} (-1)^{(m-i)(n+1)} \times
f_m([0 \to i, i + n \to m + n - 1])h_n([i \to i + n]). \quad \text{(A17)}
\]

We can see that \( \sim = \sim. \) Unlike cup product at \( k = 0, \)
the higher cup product \( \sim \) of two cocycles may not be a cocycle. For cocycles \( f_m, h_n, \) we have
\[
d(f_m \sim h_n) = df_m \sim h_n + (-)^m f_m \sim dh_n + (\sim)^{m+n-k} f_m \sim \quad \text{(A18)}
\]

Let \( f_m \) and \( h_n \) be cocycles and \( c_l \) be a cocochain, from (A18) we can obtain
\[
d(f_m \sim h_n) = (-)^{m+n-k} f_m \sim h_n + (-)^{m+n+k} h_n \sim f_m,
\]
\[
d(f_m \sim f_m) = [(-)^{k} + (-)^{m}] f_m \sim f_m,
\]
\[
d(c_l \sim c_l + c_l \sim dc_l) = dc_l \sim dc_l
\]
\[
- [(-)^{k} - (-)^{l}] (c_l \sim c_l + c_l \sim dc_l). \quad \text{(A19)}
\]

5. Steenrod square and generalized Steenrod square

From (A19), we see that, for \( Z_2 \)-valued cocycles \( z_n, \)
\[
\text{Sq}^{n-k}(z_n) \equiv z_n \sim z_n \quad \text{(A20)}
\]
is always a cocycle. Here \( \text{Sq} \) is called the Steenrod square. More generally \( h_n \sim h_n \) is a cocycle if \( n+k = \text{odd} \) and \( h_n \) is a cocycle. Usually, the Steenrod square is defined only for \( Z_2 \)-valued cocycles or cohomology classes. Here, we like to define a generalized Steenrod square for \( M \)-valued cocycles \( c_n: \)
\[
\text{Sq}^{n-k}c_n \equiv c_n \sim c_n + c_n \sim dc_n. \quad \text{(A21)}
\]

From (A19), we see that
\[
d\text{Sq}^k c_n = d(c_n \sim c_n + c_n \sim dc_n) \quad \text{(A22)}
\]
\[
= \text{Sq}^k dc_n + (-)^n \begin{cases} 
0, & k = \text{odd} \\
2\text{Sq}^{k+1}c_n, & k = \text{even}.
\end{cases}
\]

In particular, when \( c_n \) is a \( Z_2 \)-valued cocochain, we have
\[
d\text{Sq}^k c_n \equiv \text{Sq}^k dc_n. \quad \text{(A23)}
\]

Next, let us consider the action of \( \text{Sq}^k \) on the sum of two \( M \)-valued cocycles \( c_n \) and \( c'_n: \)
\[
\text{Sq}^k(c_n + c_n') = \text{Sq}^k c_n + \text{Sq}^k c_n' +
\begin{align*}
&c_n \sim c_n' + c_n \sim c_n + c_n \sim dc_n' + c_n \sim dc_n' \sim dc_n \\
&= \text{Sq}^k c_n + \text{Sq}^k c_n' + [1 + (-)^k]c_n \sim c_n' \\
&- (-)^{n-k}[(-)^{n-k}c_n \sim c_n + (-)^n c_n \sim c_n'] \\
&+ c_n \sim dc_n' + c_n' \sim dc_n'.
\end{align*}
\quad \text{(A24)}
\]

Notice that (see (A18))
\[
- (-)^{n-k}c_n \sim c_n + (-)^n c_n \sim c_n' 
\quad \text{(A25)}
\]
\[
d(c_n \sim c_n) - dc_n' \sim c_n - (-)^n dc_n' \sim dc_n',
\]

we see that
\[
\begin{align*}
\text{Sq}^k(c_n + c_n') &= \text{Sq}^k c_n + \text{Sq}^k c_n' + [1 + (-)^k]c_n \sim c_n' \\
&+ (-)^{n-k}[dc_n' \sim c_n + (-)^n c_n \sim dc_n] \\
&- (-)^{n-k}d(c_n' \sim c_n + c_n \sim dc_n' + c_n \sim dc_n') \\
&= \text{Sq}^k c_n + \text{Sq}^k c_n' + [1 + (-)^k]c_n \sim c_n' \\
&+ [1 + (-)^k]c_n \sim dc_n - (-)^{n-k}d(c_n' \sim c_n) \\
&- [(-)^{n-k+1}dc_n' \sim c_n - c_n \sim dc_n'].
\end{align*}
\quad \text{(A26)}
\]

Notice that (see (A18))
\[
(-)^{n-k+1}dc_n' \sim c_n - c_n \sim dc_n' = d(dc_n' \sim c_n) + (-)^n dc_n' \sim dc_n',
\quad \text{(A27)}
\]

we find
\[
\begin{align*}
\text{Sq}^k(c_n + c_n') &= \text{Sq}^k c_n + \text{Sq}^k c_n' + [1 + (-)^k]c_n \sim c_n' \\
&+ [1 + (-)^k]c_n \sim dc_n - (-)^{n-k}d(c_n' \sim c_n) \\
&- d(dc_n' \sim c_n + c_n \sim dc_n') - (-)^n dc_n' \sim dc_n \\
&= \text{Sq}^k c_n + \text{Sq}^k c_n' - (-)^n dc_n' \sim dc_n \\
&+ [1 + (-)^k][c_n \sim c_n' + c_n' \sim dc_n] \\
&- (-)^{n-k}d(c_n' \sim c_n - dc_n') - (-)^n dc_n' \sim dc_n).
\end{align*}
\quad \text{(A28)}
\]

We see that, if one of the \( c_n \) and \( c'_n \) is a cocycle,
\[
\text{Sq}^k(c_n + c_n') \equiv \text{Sq}^k c_n + \text{Sq}^k c_n'.
\quad \text{(A29)}
\]
We also see that
\[ \text{Sq}^k(c_n + df_{n-1}) = \text{Sq}^k c_n + \text{Sq}^k df_{n-1} + [1 + (-)^k] df_{n-1} \sim c_n \]
\[ - (-)^{n-k} d(c_{n-k+1} \sim df_{n-1}) - d(df_{n-1}) \]
\[ = \text{Sq}^k c_n + [1 + (-)^k] [df_{n-1} \sim c_n + (-)^n \text{Sq}^{k+1} f_{n-1}] \]
\[ + d[\text{Sq}^k f_{n-1} - (-)^{n-k} c_{n-k+1} \sim df_{n-1} - dc_{n-k+2} \sim df_{n-1}]. \]

Using (A28), we can also obtain the following result if \( dc_n = 0 \)
\[ \text{Sq}^k(c_n + 2c_n') = \text{Sq}^k c_n + 2d(c_n \sim c_n') + 2dc_n \sim c_n' \]
\[ = \text{Sq}^k c_n + 2d(c_n \sim c_n') \quad (A31) \]

As another application, we note that, for a \( \mathbb{Q} \)-valued cochain \( m_d \) and using (A18),
\[ \text{Sq}^1(m_d) = m_d \sim m_d + m_d \sim dm_d \]
\[ = \frac{1}{2} (-)^d [d(m_d \sim m_d) - dm_d \sim m_d] + \frac{1}{2} m_d \sim dm_d \]
\[ = (-)^d \beta_2(m_d \sim m_d) - (-)^d \beta_2 m_d \sim m_d + m_d \sim \beta_2 m_d \]
\[ = (-)^d \beta_2 \text{Sq}^0 m_d - 2(-)^d \beta_2 m_d \sim \beta_2 m_d \]
\[ = (-)^d \beta_2 \text{Sq}^0 m_d - 2(-)^d \text{Sq}^0 \beta_2 m_d \quad (A32) \]
This way, we obtain a relation between Steenrod square and Bockstein homomorphism, when \( m_d \) is a \( \mathbb{Z}_2 \)-valued cocycle
\[ \text{Sq}^1(m_d) \overset{d}{=} \beta_2 m_d \quad (A33) \]
where we have used \( \text{Sq}^0 m_d = m_d \) when the value of the cochain is only 0,1.

For a \( k \)-cochain \( a_k \), \( k \) odd, we find that
\[ \text{Sq}^k a_k = a_k a_k + a_k \sim da_k \]
\[ = \frac{1}{2} [da_k \sim a_k - a_k \sim da_k - d(a_k \sim a_k)] + a_k \sim da_k \]
\[ = \frac{1}{2} [da_k \sim da_k - d(da_k \sim a_k)] - \frac{1}{2} d(a_k \sim a_k) \]
\[ = \frac{1}{4} d(da_k \sim da_k) - \frac{1}{2} d(a_k \sim a_k + da_k \sim a_k) \]
Thus \( \text{Sq}^k a_k \) is always a \( \mathbb{Q} \)-valued coboundary, when \( k \) is odd.

6. Branch structure dependence

Note that the concepts of chain and cochain do not depend on the branch structure. Although the definition of the derivative operator \( d \) formally depends on the branch structure, in fact, it is independent of the branch structure as a map between cochains.

However, the cup product and higher cup product do depend on the branch structure, as maps from two cochains to one cochain. To stress this dependence, we write a higher cup product as \( B \), where \( B \) denotes the branch structure. In this section, we like to study this branch structure dependence. First we need to find a quantitative way to describe a change of branch structures.

Let us compare two branch structures \( B_0 \) and \( B \). We can use a \( \mathbb{Z}_2 \)-valued 1-cochain \( s \) to describe the difference between \( B_0 \) and \( B \): \( s_{ij} = 1 \) if the arrow on the link \((ij)\) is different for \( B_0 \) and \( B \), and \( s_{ij} = 0 \) otherwise. However, not every 1-cochain \( s \) corresponds to the difference between two branch structures. We find that \( s \) describes the difference between two branch structures if and only if (iff) on every triangle \((ijk)\), \( i < j < k \), \( s \) has a form
\[ s_{ij} = 1, \ s_{jk} = 0, \ s_{ik} = 0; \]
or \[ s_{ij} = 0, \ s_{jk} = 1, \ s_{ik} = 0; \]
or \[ s_{ij} = 0, \ s_{jk} = 1, \ s_{ik} = 1; \]
or \[ s_{ij} = 1, \ s_{jk} = 0, \ s_{ik} = 1. \quad (A35) \]

We believe that, for cocycles \( f, g, f \sim g \sim g \) is a coboundary. Thus
\[ f \sim g + d\nu(s, f, g) = f \sim g. \quad (A36) \]
If \( f \) and \( g \) are 1-cocycles, then we find that
\[ f \sim g - f \sim g \]
\[ = d(s \sim f \sim g) + 2(s \sim f) \sim g + 2f \sim (s \sim g) \]
\[ - 2(s \sim f) \sim (s \sim g) - 2(s \sim g) \sim (s \sim f) \]
\[ - 2(s \sim f) \sim (g \sim s) + 2(s \sim g) \sim (s \sim (s \sim f)) \]
\[ + 2(s \sim f) \sim ((s \sim g) \sim s) \quad (A37) \]
holds on a triangle \((ijk)\) for all the 5 choices of \( s \) in (A35). We prove it as follows. The value of the right hand side of (A37) on a triangle \((ijk)\) with the branch structure \( B_0 \) is
\[ s_{ij}f_{ij}g_{ij} + s_{jk}f_{jk}g_{jk} - s_{ik}f_{ik}g_{ik} \]
\[ + 2s_{ij}f_{ij}g_{jk} + 2f_{ij}s_{jk}g_{jk} - 2s_{ij}f_{ij}s_{jk}g_{jk} \]
\[ + 2s_{ik}g_{ik}s_{ij}f_{ik} + 2s_{ik}f_{ik}s_{ij}g_{jk} \]
\[ - 2(g_{ik}f_{jk} + g_{ij}f_{ik})s_{ij}g_{jk}s_{ik} \quad (A38) \]
where we have used (A17).

- If \( s_{ij} = 1, s_{jk} = 0, s_{ik} = 0 \), then the value of the left hand side of (A37) on the triangle \((ijk)\) is 
  \[ f_{ij} g_{jk} - f_{ik} g_{kj}, \]
  while the value of (A38) is 
  \[ f_{ij} g_{jk} + 2 f_{ij} g_{ik} \]
  since \( g_{ij} + g_{jk} - g_{ik} = 0 \) where we have used the cocycle condition for \( g \).

- If \( s_{ij} = 0, s_{jk} = 1, s_{ik} = 0 \), then the value of the left hand side of (A37) on the triangle \((ijk)\) is 
  \[ f_{ij} g_{jk} - f_{ik} g_{kj}, \]
  while the value of (A38) is 
  \[ f_{ij} g_{jk} + 2 f_{ij} g_{ik} \]
  since \( g_{ij} + g_{jk} - g_{ik} = 0 \) where we have used the cocycle condition for \( f \) and \( g \).

- If \( s_{ij} = 0, s_{jk} = 1, s_{ik} = 1 \), then the value of the left hand side of (A37) on the triangle \((ijk)\) is 
  \[ f_{ij} g_{jk} - f_{ik} g_{ji}, \]
  since \( f_{ij} + f_{jk} - f_{ik} = 0 \) and \( g_{ij} + g_{jk} - g_{ik} = 0 \) where we have used the cocycle condition for \( f \) and \( g \).

- If \( s_{ij} = 1, s_{jk} = 0, s_{ik} = 1 \), then the value of the left hand side of (A37) on the triangle \((ijk)\) is 
  \[ f_{ij} g_{jk} - f_{kj} g_{ik}, \]
  since \( f_{ij} + f_{jk} - f_{ki} = 0 \) and \( g_{ij} + g_{jk} - g_{ki} = 0 \) where we have used the cocycle condition for \( f \) and \( g \).

Thus we have proved that (A37) holds on a triangle \((ijk)\) for all the 5 choices of \( s \) in (A35). So \( f \sim g \sim f \) is a coboundary modulo 2 if \( f \) and \( g \) are 1-cocycles.

### 7. Poincaré dual and pseudo-inverse of Poincaré dual

The Poincaré dual of a cochain \( f \in C^n(K; \mathbb{Z}_2) \) is defined to be the cap product \( [K] \hookrightarrow f \in C_{m-n}(K; \mathbb{Z}_2) \) where \([K]\) is the fundamental class of \( K \) (the sum modulo 2 of all \( m \)-simplices of \( K \)). The cap product \( \sigma \hookrightarrow f \) for an \( m \)-simplex \( \sigma = [v_0, v_1, ..., v_n] \) and \( f \in C^n(K; \mathbb{Z}_2) \) is an \((m-n)\)-chain, which is defined as:

\[
\sigma \hookrightarrow f := f([v_0, ..., v_n]) [v_1, ..., v_m].
\]  
(A42)

So the Poincaré dual \( PD(f) = [K] \hookrightarrow f \) is

\[
PD(f) = \sum_{[v_0, ..., v_n]} f([v_0, ..., v_n]) [v_n, ..., v_m].
\]  
(A43)

where \( \sum_{[v_0, ..., v_n]} \) is the sum of all \( m \)-simplices of \( K \).

Since the Poincaré dual is an isomorphism between cohomology and homology, it has a pseudo-inverse (defined on cycles and cocycles and up to a boundary or coboundary). The pseudo-inverse Poincaré dual of a cycle \( \psi \in C_{m-n}(K; \mathbb{Z}_2) \) is a cocycle \( PD(\psi) \in C^n(K; \mathbb{Z}_2) \), which is defined via its values on all the \( n \)-simplices \([v_0, ..., v_n] \): first assume that no summand of \( \psi \) is of the form \([v, ..., v] \) where \( v \) is any one of the first \( n \) vertices according to the order given by the branch structure. We can first determine the value of \( PD(\psi) \) on the “minimal” subset of all the \( n \)-simplices \([v_0, ..., v_n] \):

\[
PD(\psi)([v_0, ..., v_n]) = \psi_{v_1 ..., v_m}
\]  
(A44)

for \( \psi = \sum_{[v_n, ..., v_m]} \psi_{v_n ..., v_m} [v_n, ..., v_m] \)

where \( \psi_{v_1 ..., v_m} = 0,1 \). Here by minimal we mean: since \( PD(\psi) \) has to be a cocycle, its value on any boundary is zero, if we have determined the values of \( PD(\psi) \) on the \( n \)-simplices consist of vertices that are prior according to the order given by the branch structure, then we can determine the values of \( PD(\psi) \) on other \( n \)-simplices \([v_0, ..., v_n] \) and \( PD(\psi) \) is defined up to a coboundary.

Note that the summand of the Poincaré dual of any cochain \( f \in C^n(K; \mathbb{Z}_2) \) can not be of the form \([v, ..., v] \) where \( v \) is any one of the first \( n \) vertices according to the order given by the branch structure. If \( \psi \in C_{m-n}(K; \mathbb{Z}_2) \) is a cycle, we can modify \( \psi \) by a boundary such that no summand of \( \psi \) is of the form \([v, ..., v] \) where \( v \) is any one of the first \( n \) vertices according to the order given by the branch structure. So the definition of the pseudo-inverse Poincaré dual is complete.

For example, let \( K \) be the surface of a tetrahedron and \([K] = S^2 \). Given the branch structure on \( K \) so that the 4 vertices of \( K \) are ordered as \( v_0, v_1, v_2, \) and \( v_3 \), see Figure 5. If \( \psi = [v_0, v_1] + [v_0, v_3] + [v_2, v_1] + [v_3, v_2] \), then modify \( \psi \) by a boundary \([v_0, v_1] + [v_0, v_3] + [v_1, v_2] \) such that no summand of \( \psi \) is of the form \([v_0, ..., v_0] \), we get \( \psi' = [v_1, v_2] + [v_1, v_3] + [v_2, v_3] \). By (A44), \( PD(\psi') \) takes value 1 on \([v_0, v_1] \), so the sum of the values of \( PD(\psi') \) on \([v_0, v_2] \) and \([v_1, v_2] \) is 1. The values of
PD(ψ') on other 1-simplices can also be determined and PD(ψ') is determined up to a coboundary. By (A43), PD(PD(ψ')) = [v_1, v_2] + [v_1, v_3] + [v_2, v_3]. So PD(PD(ψ')) and ψ' are equal.

Since v_0, ..., v_n, ..., v_m are ordered according to the branch structure, the Poincaré dual of a cochain and the pseudo-inverse Poincaré dual of a chain depends on the branch structure, i.e. the same cochain can have different Poincaré duals and the same chain can have different pseudo-inverse Poincaré duals for different branch structures.

Appendix B: Comparison with Standard Mathematical Conventions and Stiefel-Whitney Class

In the main text of this article, we use the Stiefel-Whitney coycle w_n and the Steenrod algebra for cochains, for example, summarized in Sec. A. In this section, instead, we make the comparison with the standard mathematical conventions and Stiefel-Whitney characteristic class w_n. Some of the math notations/conventions can be found also in the summary of [79].

Let us define mathematically carefully about Stiefel-Whitney class. The Stiefel-Whitney classes of a real vector bundle ξ : ℝ^n → ξ → B(ξ) (here ξ → B(ξ) is the total space of ξ and B(ξ) is the base of ξ) are the cohomology classes w_j(ξ) ∈ H^j(B(ξ); Z_2) (j = 0, 1, 2, ...) satisfying the following axioms:

A1: w_0(ξ) = 1 ∈ H^0(B(ξ); Z_2), w_j(ξ) = 0, for νj > n.
A2: Naturality — For f : B(ξ) → B(η) covered by a bundle map (so that ξ = f*η), w_j(ξ) = f^*w_j(η).
A3: Whitney sum formula — If ξ and η are vector bundles over the same base B, then w_k(ξ ⊕ η) = Σ_w_j(ξ) ⊵ w_k−j(η).
A4: For a canonical line bundle γ^1 over ℝ^1, w_1(γ^1) ≠ 0 (i.e. The γ^1 is the M"{o}bius strip. γ^½ is the canonical line bundle over ℝ^½).

The Steenrod square is Sq^{n−k}c_n ≡ c_n ⊵ c_n for a Z_2-valued n-cohomology class c_n ∈ H^n(−, Z_2). The first Steenrod square Sq^1 : H^n(−, Z_2) → H^{n+1}(−, Z_2) is the Bockstein homomorphism associated with the group extension Z_2 → Z_4 → Z_2. The β_2 : H^n(−, Z_2) → H^{n+1}(−, Z) is the Bockstein homomorphism associated with the group extension Z → Z → Z_2. Poincaré dual means PD(B) = B ∼ [M] where ∼ is the cap product, the PD maps a cohomology class B to a homology class, and [M] is the fundamental class of the manifold. So PD is the cap product between a cohomology class and the fundamental class of the manifold. The cup product ⋅ is a product between a cochain and another cochain. We shall make the cup product ⋅ implicit whenever the product is clear written between cochains.

Appendix C: Emergence of Half-Integer Spin and Fermi Statistics

In the following, we like to explain more carefully why Z_2 gauge charge current l_3 is a fermion current, or why the Z_2 gauge charge is a fermion. We like to show that for a twisted Z_2-gauge theory satisfying da^{Z_2} = q^{Z_2} w_2, the corresponding Z_2 gauge charge is a fermion. This is because da^{Z_2} = q^{Z_2} w_2 implies that under a combined Z_2-gauge and SO(∞) spacetime rotation transformation, the Z_2 gauge charge transforms as Z_2 × w_2 SO(∞). In other words, the Z_2 gauge charge couple to a Z_2 × w_2 SO(∞) connection in spacetime. Let us use

\( (a_1^{Z_2}, γ_1), (a_2^{Z_2}, γ_2) \) in link \((ij)\) to describe a Z_2 × w_2 SO(∞). Here we use a pair

\( (a^{Z_2}, γ), a^{Z_2} ∈ Z_2, γ ∈ SO(∞) \)

on link \((ij)\) to label an element in Z_2 × w_2 SO(∞). The group multiplication in Z_2 × w_2 SO(∞) is given by

\[ (a_1^{Z_2}, γ_1) (a_2^{Z_2}, γ_2) = (a_1^{Z_2} + a_2^{Z_2} + w_2(γ_1, γ_2), γ_1γ_2) \]

where w_2(γ_1, γ_2) ∈ H^2(BSO(∞); Z_2).

For a nearly flat connection \((a_1^{Z_2}, γ_1)\) on a triangle \((ijk)\), we have

\[ (a_1^{Z_2}, γ_1) (a_2^{Z_2}, γ_2) = (a_1^{Z_2} + a_2^{Z_2} + w_2(γ_1, γ_2), γ_1γ_2) \approx (a_1^{Z_2}, γ_1γ_2) \]

We see that

\[ w_2(γ_1, γ_2) \approx a_4^{Z_2} \]

which is w_2 \(= da^{Z_2}\). This way we show that the twisted Z_2-gauge theory has a Z_2 gauge charge that transforms as Z_2 × w_2 SO(∞) = Spin(∞) simply denoted as Spin. In other words, the Z_2 gauge charge carries a half-integer spin, and is a fermion using the spin-statistics theorem.

We may also compute the statistics of the Z_2 gauge charge directly (which is phrased as the high dimensional bosonization in Ref. 17 and 65). Let us assume the worldline of the Z_2 gauge charge is a boundary. In this case, the Poincaré dual of the worldline is a coboundary

\[ l_3 \overset{Z_2}{=} dB_c. \]

Now we can rewrite

\[ e^{iπ f_{M^*} Sq^2 i_1 + i_1 w_2} = e^{iπ f_{M^*} Sq^2 db^{Z_2} + db^{Z_2} w_2} \]

\[ = e^{iπ f_{M^*} Sq^2 db^{Z_2} + db^{Z_2} w_2} = e^{iπ f_{M^*} dSq^2 b^{Z_2} + db^{Z_2} w_2} \]

\[ = e^{iπ f_{M^*} Sq^2 b^{Z_2} + db^{Z_2} w_2}. \]
Here $\mathcal{S}_q$ is a generalized Steenrod square that acts on a cochain $c_n$
\[
\mathcal{S}_q^{n-k}c_n \equiv c_n \smile c_n + c_{n+k} \smile d c_n,
\] (C8)
where $\smile$ is the higher cup product. It has properties
\[
d\mathcal{S}_q^k c_n \equiv \mathcal{S}_q^k d c_n,
\]
\[
\mathcal{S}_q^k (c_n + c_n) \equiv \mathcal{S}_q^k c_n + \mathcal{S}_q^k c_n \smile d c_n
\]
\[
+ d(c_n \smile d c_n) + d(d c_n \smile c_n) \quad (C9)
\]
Now the term in the path integral that contain $l_3$ becomes
\[
e^{i\pi \int_B l_3 aZ^2 + \mathcal{S}_q^2 bZ^2 + bZ^2 w_2},
\] (C10)
where $bZ^2$ is given by $l_3$ via $d bZ^2 \equiv l_3$. The above phase factor has a gauge invariance
\[
w_2 \rightarrow w_2 + d u_1, \quad aZ^2 \rightarrow aZ^2 + u_1.
\] (C11)
We note that $bZ^2$ is determined up to a cocycle $bZ^2$. Since
\[
e^{i\pi \int_B l_3 aZ^2 + \mathcal{S}_q^2 (bZ^2 + bZ^2) + (bZ^2 + bZ^2) w_2}
\]
\[
e^{i\pi \int_B l_3 aZ^2 + \mathcal{S}_q^2 bZ^2 + \mathcal{S}_q^2 bZ^2 + (bZ^2 + bZ^2) w_2}
\]
\[
e^{i\pi \int_B l_3 aZ^2 + \mathcal{S}_q^2 bZ^2 + bZ^2 w_2}
\] (C12)
therefore the phase factor (C10) does not depend on this $bZ^2$ ambiguity. In the above, we have used $\mathcal{S}_q^2 bZ^2 + bZ^2 w_2 \equiv 0$ since $w_1 \equiv 0$.

The linear $l_3$-term in the phase factor
\[
e^{i\pi \int_B l_3 aZ^2 + bZ^2 w_2},
\] (C13)

describes the coupling to the background $Z_2 \times w_3 SO(\infty)$-connection, which indicates that the $Z_2$ gauge charge carry half-integer spin. The quadratic $l_3$-term
\[
e^{i\pi \int_B \mathcal{S}_q^2 bZ^2} = e^{i\pi \int_B (bZ^2)^2 + l_3 \frac{1}{2} bZ^2}
\] (C14)
describes the Fermi statistics of the $Z_2$ gauge charge. The absence of $bZ^2$ cocycle ambiguity requires the linear $l_3$-term and the quadratic $l_3$-term to appear together as a combination (C12). Similarly, the WZW-like phase factor $e^{i\pi \int_B \mathcal{S}_q^2 l_3 w_2}$ will not depend on how we extend from $B^4$ to $M^5$ only when the linear $l_3$-term $w_2 l_3$ and the quadratic $l_3$-term $\mathcal{S}_q^2 l_3$ to appear together. This corresponds to the spin statistical theorem.

To summarize, adding a phase factor $e^{i\pi \int_B \mathcal{S}_q^2 l_3 w_2}$ will make the current $l_3$ on the boundary $B^4 = \partial M^5$ to become a fermion current where the fermions carry a half-integer spin. Similarly, adding a phase factor $e^{i\pi \int_B \mathcal{S}_q^2 \beta_{x_2} + w_2 \beta_{x_2}}$ will make $\beta_{x_2}$ on the boundary to become a fermion current as well.

**Appendix D: Cobordism Group Data and Anomaly Classification**

Let us systematically enumerate the pertinent cobordism group $TP_d (G)$ with some spacetime-internal $G$ symmetry in Table II.

| $d$ | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|-----|---|---|---|---|---|---|-----|
| $TP_d (SO)$ | 0 | 0 | $Z_2$ | 0 | $Z_2$ | 0 | $Z_2$ | ... |
| $TP_d (Spin)$ | $Z_2$ | $Z_2$ | 0 | 0 | 0 | 0 | ... |
| $TP_d (Spin \times U(1))$ | $Z \times Z_2$ | $Z_2$ | $Z^2$ | 0 | $Z^2$ | 0 | $Z^2$ | ... |
| $TP_d (Spin \times SU(2))$ | $Z_2$ | $Z_2$ | 0 | 0 | 0 | 0 | ... |
| $TP_d (Spin \times Z_2 SU(2))$ | 0 | 0 | $Z^2$ | 0 | $Z^2$ | 0 | $Z^2$ | ... |
| $TP_d (Spin \times SO(3))$ | $Z_2$ | $Z_2$ | 0 | 0 | 0 | 0 | ... |
| $TP_d (Spin \times Spin(n \geq 7))$ | $Z_2$ | $Z_2$ | 0 | 0 | 0 | 0 | ... |
| $TP_d (Spin \times Spin(10))$ | $Z_2$ | $Z_2$ | 0 | 0 | 0 | 0 | ... |
| $TP_d (Spin \times Spin(n \geq 7))$ | $Z_2$ | $Z_2$ | 0 | 0 | 0 | 0 | ... |
| $TP_d (Spin \times Spin \times Spin(n \geq 7))$ | $Z_2$ | $Z_2$ | 0 | 0 | 0 | 0 | ... |
| $TP_d (Spin \times Spin \times Spin(10))$ | $Z_2$ | $Z_2$ | 0 | 0 | 0 | 0 | ... |

**TABLE II** The cobordism group $TP_d (G)$ classifies the invertible topological phases or invertible topological field theories (including both the $G$-SPT state and the invertible topological order with $G$-symmetry) of spacetime-internal symmetry $G$ in the $d$-dimensional spacetime. See the cobordism computations in [55, 77, 79].

In particular, we focus on $TP_5 (G)$ which classifies the invertible topological phases of spacetime-internal symmetry $G$ in the 5-dimensional spacetime. We would like to comment why the invertible topological order characterized by $w_2 w_3$ is present or absent in the given $G$ symmetry.

Here we denote the $w_j = w_j (TM) = w_j (\alpha^{SO})$ as the $j$th Stiefel-Whitney class of the spacetime tangent bundle $(TM)$ of the spacetime manifold $M$, while $w_j^{\prime} = w_j (V_{SO(n)})$ is the $j$th Stiefel-Whitney class of the associated vector bundle of the principal gauge bundle of $SO(n) = Spin(n) / Z_2$.

Below let us explain why the cobordism invariant $w_2 w_3$ vanishes in some $G$ symmetry (e.g., Spin and Spin$^C$ $\equiv Spin \times \mathbb{Z}_2 U(1)$), but why the $w_2 w_3$ persists in other symmetry (e.g., SO and Spin$^h = Spin \times \mathbb{Z}_2 SU(2)$).

1. **Spacetime and gauge bundle constraint**

1. There is no particular constraint on $w_2$ or $w_3$ for the SO structure and SO manifold, thus the cobordism invariant $w_2 w_3$ derived in the cobordism group of an SO structure still survives.

2. The constraint for the Spin structure and Spin manifold is $w_2 = 0$, thus $w_3 = Sq^1 w_2 = 0$, and $w_2 w_3 = 0$. We had also used the symmetry extension to trivialize the $w_2 w_3$ term in an SO structure.
via a pullback to 0 in a Spin structure under the
group extension $1 \to \mathbb{Z}_2 \to \text{Spin} \to \text{SO} \to 1$.

3. The constraint for the Spin$^c \equiv \text{Spin} \times \mathbb{Z}_2$ $U(1)$
structure is $w_2 = c_1 \mod 2$ also $w_3 = 0$, so $w_2w_3 = 0$. To derive this, we use the $w_3 = 
\text{Sq}^1w_2 = \text{Sq}^1(c_1 \mod 2) = 0$, since $\text{Sq}^1\rho = 0$ where $\rho$ is the mod 2 map. The $\text{Sq}^1 = \rho \beta_2$ where $\beta_2$ is the Bockstein associated with the short exact
sequence $1 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 1$, which
induces the fiber sequence in their classifying spaces as $\cdots \to B^2\mathbb{Z} \to B^2\mathbb{Z}_2 \to B^2\mathbb{Z} \to \cdots$. The $\beta_2$ sends the $\rho c_1 = c_1 \mod 2 \in H^2(M; \mathbb{Z}_2)$ to
$\beta_2\rho c_1 = \beta_2(c_1 \mod 2) \in \mathbb{Z}$ in $H^2(M; \mathbb{Z})$. Moreover, the group of homotopy classes of the maps from $M$
to the higher classifying space $B^n G$ is the cohomology
structure still requires a Spin manifold ($w_2 \neq 0$) with a tensor product structure of spacetime tangent bundle
and the principal $U(1)$ gauge bundle, thus $w_2w_3 = 0$.

4. The constraint for the Spin$^h \equiv \text{Spin} \times \mathbb{Z}_2$ $SU(2)$
structure includes $w_2 = w'_2$, where we denote $w'_2 = w_j(V_{SO(3)})$. Thus $\text{Sq}^1w_2 = \text{Sq}^1w'_2 \Rightarrow w_3 = w'_3$, so $w_2w_3 = w'_2w'_3$ can be non-zero.

The constraint for the Spin$\times SU(2)$ or Spin$\times SO(3)$
structure still requires a Spin manifold ($w_2 = 0$) with a tensor product structure of spacetime
tangent bundle and the principal $SU(2)$ or $SO(3)$ gauge
bundle, thus $w_2w_3 = 0$.

5. Now we discuss Spin $\times \text{Spin}(n \geq 7)$, Spin $\times \mathbb{Z}_2$
Spin ($n \geq 7$) and Spin $\times SO(n \geq 7)$, especially when
$n = 10$ or 18 suitable for Grand Unified Theories
[76, 80, 81].

The constraint for the Spin $\times \text{Spin}(n \geq 7)$ and
Spin $\times SO(n \geq 7)$ structure still requires a Spin
manifold ($w_2 = 0$) with a tensor product structure of spacetime tangent bundle and the principal
$SU(2)$ or $SO(3)$ gauge bundle, thus $w_2w_3 = 0$.

The constraint for the Spin $\times \mathbb{Z}_2$ Spin ($n \geq 7$) structure
includes $w_2 = w'_2$, where we denote $w'_2 = w_j(V_{SO(n)})$. Thus $\text{Sq}^1w_2 = \text{Sq}^1w'_2 \Rightarrow w_3 = w'_3$, so $w_2w_3 = w'_2w'_3$ can be non-zero.

2. Cobordism invariants

To summarize their 5d cobordism invariants:

1. $TP_5(\text{SO}) = \mathbb{Z}_2$ is generated by the cobordism invariant $w_2w_3$. The manifold generator is a non-
Spin manifold such as a Wu manifold

$$\text{SU}(3)/\text{SO}(3)$$

or a Dold manifold

$$\mathbb{C}P^2 \times S^1$$

(which identifies the complex conjugation of coordinates in $z \in \mathbb{C}P^2$ with the antipodal inversion of
$x \in S^1$, so $(z, x) \sim (\bar{z}, -x)$).

2. $TP_5(\text{Spin}) = 0$ trivializes the cobordism invariant

$w_2w_3$ to none via a pullback from SO to Spin.

3. $TP_5(\text{Spin} \times U(1)) = \mathbb{Z}^2$ classes are generated by
two 5d cobordism invariants $ac_1^2$ and $\mu(\text{PD}(c_1))$.
The 5d $ac_1$ corresponds to the perturbative local anomaly captured by Feynman diagram of $U(1)$-
$U(1)-U(1)$ fields acting on the three vertices of the triangle diagram. The 5d $\mu(\text{PD}(c_1))$ corresponds to
the perturbative local anomaly captured by Feynman
diagram of $U(1)$-gravity-gravity fields acting
on the three vertices of the triangle diagram. Here the $a$ is the $U(1)$ 1-form gauge connection. Here
the first Chern class $c_1 = c_1(V_{U(1)})$ is written as the associated vector bundle of $U(1)$. The $\mu(\text{PD}(c_1))$ is the 3d Rokhlin invariant of $\text{PD}(c_1)$, where $\text{PD}(c_1)$ is the submanifold of a Spin 5-
manifold which represents the Poincaré dual of $c_1$. In general, the Poincaré dual means $\text{PD}(B) = B \cap [M]$ where $\cap$ is the cap product, $\text{PD}$ maps a cobordism class $B$ to a homology class, and $[M]$ is the fundamental class of the manifold.

The $TP_5(\text{Spin} \times U(1)) = \mathbb{Z}^2$ are also descended from the two 6d topological invariants of the bordism
$\Omega^6_0(\text{Spin}^c)$. $c_1^2$ and $\frac{1}{8}(\sigma(\text{PD}(c_1)) - F \cdot F)$ from the free part of the bordism group
$\Omega^6_0(\text{SU}(2))$. The $\text{PD}(c_1)$ is the submanifold of a Spin 6-manifold which represents the Poincaré dual of $c_1$. The signature of the 4-manifold $\text{PD}(c_1)$. The $F$ is a 2d characteristic surface of the 4-manifold $\text{PD}(c_1)$, where $F$ represents $\text{PD}(B)$ where $B \in H^2(\text{PD}(c_1); \mathbb{Z})$. The $F$-$F$ is the intersection form of the 4-manifold $\text{PD}(c_1)$. The intersection form $F \cdot F = (B \cap B, [\text{PD}(c_1)])$ is computed via the pairing between a cobordism class with a homology class, where $[\text{PD}(c_1)]$ is the fundamental class of $\text{PD}(c_1)$. By Rokhlin’s theorem, $\sigma(\text{PD}(c_1)) - F \cdot F$ is a multiple of 8 and $\frac{1}{8}(\sigma(\text{PD}(c_1)) - F \cdot F) = \text{Arf}(\text{PD}(c_1))$ mod 2. The $\text{Arf}(\text{PD}(c_1), F)$ is the Arf invariant of a quadratic form $\tilde{q} : H_1(F; \mathbb{Z}_2) \to \mathbb{Z}_2$, it is $\mathbb{Z}_2$-valued, the LHS is $\mathbb{Z}$-valued and equals to the RHS modulo 2. The $F \cdot h = h \cdot h$ mod 2 is true for all $h \in H_2(\text{PD}(c_1); \mathbb{Z})$.

4. $TP_5(\text{Spin} \times \mathbb{Z}_2.U(1)) = TP_5(\text{Spin}^c) = \mathbb{Z}^2$ classes are generated by two 5d cobordism invariants $\frac{1}{8}ac_1^2$.
and \( \frac{1}{32} c_1 \mathrm{CS}_3(TM) \). The 5d \( \frac{1}{2} \sigma^2 \) corresponds to the perturbative local anomaly captured by Feynman diagram of \( U(1)_{-1}-U(1)_1 \) fields acting on the three vertices of the triangle diagram. The 5d \( \frac{1}{32} c_1 \mathrm{CS}_3(TM) \) corresponds to the perturbative local anomaly captured by Feynman diagram of \( U(1) \)-gravity-gravity fields acting on the three vertices of the triangle diagram.

The \( \mathrm{TP}_5(\mathrm{Spin}^c) = \mathbb{Z}^2 \) are also descended from the two 6d topological invariants of the bordism group \( \Omega^5_{\mathrm{Spin}^c} = \frac{1}{2} c^3_1 \) and \( \frac{1}{16} \sigma(\mathrm{PD}(c_1)) \) from the free part of the bordism group \( \Omega^5_{\mathrm{Spin}^c} \). The \( \mathrm{PD}(c_1) \) is a Spin submanifold of the Spin\(^c\) 6-manifold which represents the Poincaré dual of \( c_1 \).

5. \( \mathrm{TP}_5(\mathrm{Spin} \times \mathrm{SU}(2)) = \mathbb{Z}_2 \) class is generated by a 5d cobordism invariant \( \tilde{\eta} \mathrm{PD}(c_2(V_{\mathrm{SU}(2)}) \), where the \( \tilde{\eta} \) is a mod 2 index of 1d Dirac operator from \( \mathrm{TP}_1(\mathrm{Spin}) = \mathbb{Z}_2 \) or \( \Omega^1_{\mathrm{Spin}^c} = \mathbb{Z}_2 \). A 1d manifold generator for the cobordism invariant \( \tilde{\eta} \) is a 1d \( S^1 \) for fermions with periodic boundary condition, so called the Ramond circle. A 4d manifold generator for the \( c_2(V_{\mathrm{SU}(2)}) \) is the nontrivial \( SU(2) \) bundle over the \( S^4 \), such that the instanton number is 1. So the 5d manifold generator for the cobordism invariant \( \tilde{\eta} \mathrm{PD}(c_2(V_{\mathrm{SU}(2)}) \) is the \( S^1 \times S^4 \) with the fermionic periodic boundary condition on \( S^1 \) and the SU(2) bundle over \( S^4 \) with an instanton number 1. The 4d boundary for a 5d \( \tilde{\eta} \mathrm{PD}(c_2(V_{\mathrm{SU}(2)}) \) captures the Witten SU(2) anomaly [82].

6. \( \mathrm{TP}_5(\mathrm{Spin} \times \mathbb{Z}_2 \mathrm{SU}(2)) = \mathrm{TP}_5(\mathrm{Spin}^h) = \mathbb{Z}_2^2 \) classes are generated by two 5d cobordism invariants. One is the similar cobordism invariant as that of the \( \mathrm{TP}_5(\mathrm{Spin} \times \mathrm{SU}(2)) = \mathbb{Z}_2 \) whose 4d boundary has the Witten SU(2) anomaly [82]. The other is the \( w_2 w_3 = w_2' w_3' \) with \( w_j = w_j(V_5(\mathrm{SU}(3))) \).

7. \( \mathrm{TP}_5(\mathrm{Spin} \times \mathbb{Z}_2 \mathrm{Spin}(n)) = \mathbb{Z}_2 \) has a \( \mathbb{Z}_2 \) class generated by \( w_2 w_3 = w_2' w_3' \) with \( w_j = w_j(V_5(\mathrm{SO}(n))) \).

a. **Anomalies in SU(2) vs SO(3): cobordism vs homotopy group**

We note that the 4d nonperturbative global anomalies of an internal SU(2) symmetric theory on non-spin manifolds are classified by

\[
\mathrm{TP}_5(\mathrm{Spin} \times \mathbb{Z}_2 \mathrm{SU}(2)) = \mathbb{Z}_2^2,
\]

whose generator is the Witten SU(2) anomaly. The global anomalies of an internal SO(3) symmetric theory on non-spin manifolds are classified by

\[
\mathrm{TP}_5(\mathrm{SO} \times \mathrm{SO}(3)) = \mathbb{Z}_2^2,
\]

where generators are the \( w_2 w_3 \) gravitational anomaly from the \( w_j(TM) \) of spacetime tangent bundle \( TM \) and the \( w_2' w_3' \) gauge anomaly from the \( w_j' (V_{\mathrm{SO}(3)}) \) of internal gauge bundle; while on spin manifolds are classified by

\[
\mathrm{TP}_5(\mathrm{Spin} \times \mathrm{SO}(3)) = 0.
\]

We shall compare the cobordism group classification of global anomalies with the traditional homotopy group analysis [82] of global anomalies. We will find that the homotopy group analysis is insufficient, such that the homotopy group sometimes leads to incomplete or misleading results.

For example, in Table III, we learn that the homotopy group \( \pi_4(\mathrm{SU}(2)) = \mathbb{Z}_2 \) only gives the Witten SU(2) anomaly [82] but misses the new SU(2) anomaly [44]. We also learn that the homotopy group \( \pi_4(\mathrm{SO}(3)) = \mathbb{Z}_2 \) gives a possible global anomaly but in fact the topological invariant in the homotopy theory does not correspond to any 5d cobordism invariant on Spin manifolds (with \( \mathrm{Spin} \times \mathrm{SO}(3) \) structures). Thus there is no corresponding Witten SU(2) anomaly in an internal SO(3) symmetric theory.

| d | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|-----|
| \( \pi_4(\mathrm{SU}(2)) \) | 0 | 0 | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | ... |
| \( \pi_4(\mathrm{SO}(3)) \) | 0 | 0 | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | ... |

**TABLE III.** The homotopy group \( \pi_4(G) \) sometimes leads to incomplete or misleading results of global anomalies. The lesson is that we should use the dth cobordism group \( \mathrm{TP}_d(G) \) to classify perturbative local and nonperturbative global anomalies in the \((d-1)d\) spacetime, such as the cobordism group data in Table II. See the cobordism computations in [55, 79].

### 3. Trivialization via group extension

1. **Trivialization via the pullback** \( p^* \) in

\[
1 \to \mathbb{Z}_2 \to \mathrm{Spin}(5) \overset{p}{\to} \mathrm{SO}(5) \to 1,
\]

following (33) and the symmetry extension method [43], gauging the normal subgroup \( \mathbb{Z}_2 \) provides a boundary \( \mathbb{Z}_2 \) gauge theory construction of the 5d bulk \( w_2 w_3 \).

- The 5d cobordism invariant \( w_2 w_3 \) for the 5d spacetime with the SO symmetry becomes trivialized to 0 for the spacetime with the Spin symmetry, because the Spin structure requires \( w_2 = 0 \) and \( w_3 = \mathrm{Sq}^1 w_2 = 0 \) on the Spin manifold.

- The \( w_2 w_3 \) of \( \mathrm{TP}_5(\mathrm{SO}) \) is trivialized as \( p^*(w_2 w_3) = 0 \) in \( \mathrm{TP}_5(\mathrm{Spin}) \).
3. The group extension $p$ external SU(2) spinor, in the representation of ($w_2 w_3$) in 4d with the boundary anomaly of 5d $w_2 w_3$.

- This means the 4d boundary anomaly of 5d $w_2 w_3$ of SO is also vanished in Spin. However, dynamically gauging the normal $Z_2$ subgroup provides a boundary $Z_2$ gauge theory that preserves the SO symmetry but with the 't Hooft anomaly of $w_2 w_3$.

2. Trivialization via the pullback $p^*$ in

$$1 \to U(1) \to Spin^c(5) \xrightarrow{p} SO(5) \to 1,$$

following the symmetry extension method [43], gauging the normal subgroup U(1) provides a boundary U(1) gauge theory construction of the 5d bulk $w_2 w_3$.

- The 5d cobordism invariant $w_2 w_3$ for the 5d spacetime with the SO symmetry becomes trivialized to 0 for the spacetime with the Spin$^c$ symmetry, because the Spin$^c$ structure requires $w_2 = c_1 \mod 2$ and $w_3 = Sq^1 w_2 = Sq^1 c_1 \mod 2 = 0$ on the Spin$^c$ manifold.

- The $w_2 w_3$ of TP$_5$(SO) is trivialized as $p^*(w_2 w_3) = 0$ in TP$_5$(Spin$^c$).

- So there is no Spin$^c$ symmetric theory with some internal global U(1) symmetry in 4d with the boundary anomaly of 5d $w_2 w_3$.

- This means the 4d boundary anomaly of 5d $w_2 w_3$ of SO is also vanished in Spin$^c$. However, dynamically gauging the normal U(1) subgroup provides a boundary U(1) gauge theory that preserves the SO symmetry but with the 't Hooft anomaly of $w_2 w_3$.

3. The group extension

$$1 \to SU(2) \to Spin^h(5) \xrightarrow{p} SO(5) \to 1$$

however does not provide the trivialization of $w_2 w_3$.

- The 5d cobordism invariant $w_2 w_3$ for the 5d spacetime with the SO symmetry becomes $w_2 w_3 = w_2 w_3'$ for the spacetime with the Spin$^h$ symmetry, because the Spin$^h$ structure requires $w_2 = w_2'$ and $w_3 = Sq^1 w_2 = Sq^1 w_3 = w_3'$ on the Spin$^h$ manifold. This also means the 4d gravitational anomaly on the boundary of 5d $w_2 w_3$ term becomes the 4d mixed gauge-gravitational anomaly on the boundary of 5d $w_2 w_3'$ term.

- The $w_2 w_3$ of TP$_5$(SO) is not trivialized but becomes $p^*(w_2 w_3) = w_2 w_3 = w_2 w_3'$ in TP$_5$(Spin$^h$).

- So there indeed exists certain Spin$^h$ symmetric theory with some internal global SU(2) symmetry in 4d with the boundary 't Hooft anomaly of 5d $w_2 w_3'$ = $w_2 w_3$. In fact, the Weyl fermion as a 2-component spacetime spinor and a 4-component internal SU(2) spinor, in the representation of ($2_L, 4$) of Spin$\times Z_2$ SU(2) $\equiv$ Spin$^h$ has this precise so-called new SU(2) anomaly [44] of 5d $w_2 w_3'=w_2 w_3$. However, we can ask whether it is sensible to dynamically gauge the normal SU(2) subgroup in this Spin$^h$ symmetric Weyl fermion theory with the new SU(2) anomaly [44].

- If we only restrict to the Spin manifold with $w_2 = w_2' = 0$ thus also $w_3 = w_3' = 0$, then, yes, we can obtain a well-defined SU(2) gauge theory on a Spin manifold (such as a flat Euclidean or Minkowski spacetime) and we can study its dynamics [44].

- If we construct this SU(2) gauge theory on a generic non-Spin manifold with a Spin$^h$ structure, then we have $w_2 = w_2' \neq 0$ thus also $w_3 = w_3' \neq 0$. Then, no, we obtain an ill-defined SU(2) gauge theory by summing over the SU(2) bundle with the SU(2) connections on a generic non-Spin manifold. We cannot study the dynamics of an ill-defined SU(2) gauge theory with dynamical gauge-gravitational anomaly uncanceled [44].

4. The group extension

$$1 \to Spin(n \geq 3) \to Spin(5) \times Z_2 Spin(n \geq 3) \xrightarrow{p} SO(5) \to 1$$

however also does not provide the trivialization of $w_2 w_3$, but modifies the $w_2 w_3$ to $w_2 w_3 = w_2' w_3'$. The situation for $n \geq 3$ is similar to our previous remark on Spin(3) $\equiv$ SU(2).

- In comparison to the Spin(2) = U(1) case, there exists an all fermion QED$_4$ as a U(1) gauge theory definable on a generic non-Spin manifold with a pure 4d gravitational anomaly as a 't Hooft anomaly of the spacetime diffeomorphism SO symmetry from the 5d $w_2 w_3$.

- But for Spin($n \geq 3$), we do not have a Spin($n \geq 3$) gauge theory — such a 4d gauge theory is not definable on a generic non-Spin manifold — with a pure 4d gravitational anomaly as a 't Hooft anomaly of the spacetime diffeomorphism SO symmetry from the 5d $w_2 w_3$.

- For Spin($n \geq 3$), we do have a Spin($n$)-symmetric theory definable on a generic non-Spin manifold with a 4d mixed gauge-gravitational anomaly as a 't Hooft anomaly of the gauge-diffeomorphism SO symmetry from the 5d $w_2 w_3$.

- Dynamically gauging the Spin($n$) in 4d alone makes sense only on a Spin manifold, which results in a 4d Spin($n$) gauge theory with a well-defined dynamics on a 4d Spin manifold.

- Dynamically gauging the Spin($n$) on a non-Spin manifold is ill-defined. But dynamically gauging the Spin($n$) on a non-Spin manifold can result in a well-defined 4d-5d coupled fully...
gauged system. This 4d-5d coupled system for Spin($n = 10$) is studied in [45–47].

Appendix E: Oriented Bordism Groups and Manifold Generators

In Thom’s famous 1954 article [83], he showed that the oriented bordism ring is isomorphic to stable homotopy groups of the Thom spectrum $MSO$: $\Omega^*_{SO} = \pi_*(MSO)$. All of the homotopy groups are a direct sum $\mathbb{Z}^{r} \oplus \mathbb{Z}_{2}^{r}$. Bordism classes of oriented manifolds are completely determined by their Pontryagin and Stiefel-Whitney numbers. The mod 2 cohomology of $BSO$, a polynomial ring on the Stiefel-Whitney classes $w_2, w_3, \ldots$ whose Poincaré series is

$$\prod_{i \geq 2} \frac{1}{1 - t^i}.$$  

Rationally, the oriented bordism ring is a polynomial algebra $\mathbb{Q}[x_4, x_8, x_{12}, \ldots]$ on generators in degrees that are a multiple of 4. This tells us the rank $r$ of each group. The Poincaré series for the free part of $\Omega^*_{SO}$ is thus

$$p_{\text{free}}(t) = \prod_{i \geq 1} \frac{1}{1 - t^{4i}}.$$  

2-locally, the Thom spectrum $MSO$ is a wedge sum of suspensions of Eilenberg-Mac Lane spectra $H\mathbb{Z}_2$ and $H\mathbb{Z}$. This allows us to write

$$H^*(MSO; \mathbb{Z}_2) \cong \bigoplus_{\text{free summands}} H^*(HZ; \mathbb{Z}_2) \bigoplus \bigoplus_{\text{torsion summands}} H^*(HZ_2; \mathbb{Z}_2).$$  

(E1)

Let the Poincaré series for $\text{H}^*(HZ_2; \mathbb{Z}_2)$ and $\text{H}^*(HZ; \mathbb{Z}_2)$ be $p_{HZ_2}(t)$ and $p_{HZ}(t)$ respectively, then by [84], we have

$$p_{HZ_2}(t) = \prod_{k \geq 2^{-1}} \frac{1}{1 - t^k}$$  

and

$$p_{HZ}(t) = \frac{1}{1 + t} p_{HZ_2}(t).$$

Since the Poincaré series $p_{\text{tors}}(t)$ of the torsion part in $\Omega^*_{SO}$ satisfies

$$p_{\text{tors}}(t) \cdot p_{HZ_2}(t) + p_{\text{free}}(t) \cdot p_{HZ}(t) = \prod_{k \geq 2} \frac{1}{1 - t^k},$$

we can solve

$$p_{\text{tors}}(t) = \left[ (1-t) \prod_{k \geq 2, k \neq 2^{-1}} \left( \frac{1}{1 - t^k} \right) \right] - \left[ \frac{1}{1 + t} \prod_{k \geq 1} \left( \frac{1}{1 - t^{4k}} \right) \right].$$

In particular, we have Table IV [85, page 203].

| $d$ | $\Omega^*_{SO}$ manifold generators |
|-----|----------------------------------|
| 0   | $\mathbb{Z}$                     |
| 1   | $\mathbb{Z}$                     |
| 2   | $\mathbb{Z}$                     |
| 3   | $\mathbb{Z}$                     |
| 4   | $\mathbb{Z}$                     |
| 5   | $\mathbb{Z}_2$, $\text{CP}^2$    |
| 6   | $\mathbb{Z}$                     |
| 7   | $\mathbb{Z}$                     |
| 8   | $\mathbb{Z}_2$, $\text{CP}^2$, $\mathbb{CP}^2 \times \mathbb{CP}^2$ |
| 9   | $\mathbb{Z}_2$, $\text{CP}^2 \times \mathbb{CP}^2$ |
| 10  | $\mathbb{Z}_2$, $\text{CP}^2 \times \mathbb{CP}^2$ |
| 11  | $\mathbb{Z}_2$, $\mathbb{CP}^2$ |

TABLE IV. Oriented bordism groups and manifold generators. As manifold $Y^5$ (respectively $Y^9, Y^{11}$) we may take the nonsingular hypersurface of degree $(1,1)$ in the product $\mathbb{R}P^2 \times \mathbb{R}P^4$ (respectively $\mathbb{R}P^2 \times \mathbb{R}P^4 \times \mathbb{R}P^8$) of real projective spaces. These manifolds are called real Milnor manifolds. The 5d Wu manifold is $SU(3)/SO(3)$. The 5d Dold manifold is $S^1 \times \mathbb{CP}^2$ were the involution $\tau$ sends $(x, [y])$ to $(-x, [\bar{y}])$.

Appendix F: Combinatorial Formula for Stiefel-Whitney Classes

In 1940, Whitney obtained an explicit combinatorial formula for the Stiefel-Whitney classes [86]. The formula is as follows. Let $K$ be an $m$-dimensional combinatorial manifold and $K'$ the first barycentric subdivision of $K$. Let $C_n$ be the sum modulo 2 of all $(m-n)$-dimensional simplices of $K'$. Then the chain $C_n$ is a cycle modulo 2 and represents the homology class $W_n$ Poincaré dual to the $n$-th Stiefel-Whitney class of $K$.

In [64], the authors obtained a formula for the Stiefel-Whitney homology classes in the original triangulation without passing to the first barycentric subdivision. Their formula is as follows. A branch structure on a triangulation is an orientation of the links with no closed loops which in turn provides an order to the vertices of simplices. Given a branch structure on $K$ so that any representation of a simplex in $K$ is written with its vertices in increasing order. Let $s$ be an $(m-n)$-simplex in $K$, say $s = [v_0, v_1, \ldots, v_{m-n}]$. Let $t$ be another simplex which has $s$ as a face, i.e. $s \subset t$ (s may be equal to $t$). Let $B_{s-t} = $ set of vertices of $t$ less than $v_0$, $B_0 = $ set of vertices of $t$ strictly between $v_0$ and $v_1$, $B_k = $ set of vertices of $t$ strictly between $v_k$ and $v_{k+1}$, $B_{m-n} = $ set of vertices of $t$ greater than $v_{m-n}$. We say that $s$ is regular in $t$, if $B_k$ is empty for every odd $k$. Let $\partial_{m-n}(t)$ denote the mod 2 chain which consists of all $(m-n)$-simplices $s$ in $t$ so that $s$ is regular in $t$. Then $C_n = \sum_{\dim t = m-n} \partial_{m-n}(t)$ is a chain which represents the homology class $W_n$ Poincaré dual to the $n$-th Stiefel-Whitney class of $K$.

For example, let $K$ be the surface of a tetrahedron and $|K| = S^2$. Then $C_1 = \sum_{\dim t = 1} \partial_1(t)$. Given the branch structure on $K$ so that the 4 vertices of $K$ are ordered as $v_0, v_1, v_2$, and $v_3$, see Figure 5. For $\dim t \geq 1$, $t$ can
be chosen as \([v_0, v_1], \ldots, [v_{t-1}, v_t]\) if \(t = 1\) and \([v_0, v_1], [v_0, v_2], v_3, [v_0, v_2], v_3, [v_1, v_2], v_3\) if \(t = 2\). If \(t = 1\) and \(s \in \partial_1(t)\), then \(s = t\), if \(t = 2\) and \(s \in \partial_1(t)\), then \(s\) is the 1-simplex whose two vertices are the smallest and the greatest vertices of \(t\). Therefore, \(C'_t = [v_0, v_1] + [v_0, v_2] + [v_1, v_2] + [v_2, v_3]\). If another branch structure is given on \(K\) so that the order between \(v_0\) and \(v_1\) is reversed while other orders remain the same, see Figure 6, then \(C'_t\) changes to \([v_0, v_1] + [v_0, v_2] + [v_1, v_3] + [v_2, v_3] + [v_1, v_2] + [v_2, v_3]\) and the difference with the original \(C'_t\) is \([v_0, v_3] + [v_0, v_2] + [v_1, v_2] + [v_2, v_3]\) which is a boundary. So \(C'_t\) depends on the branch structure and different choices of branch structures can only change \(C'_t\) by a coboundary.

The Poincaré dual (see Appendix A 7) also depends on the branch structure. Thus \(w_n\) depends on the branch structure and different choices of branch structures can only change \(w_n\) by a coboundary.

**Appendix G: Compute \(w_2w_3\) on Real Milnor, Wu, and Dold Manifolds**

The 5d real Milnor manifold \([87]\) \(Y^5 = H(2, 4)\) is the submanifold of \(\mathbb{RP}^2 \times \mathbb{RP}^4\) given by

\[
H(2, 4) = \{(x_0, x_1, x_2); [y_0, \ldots, y_4] \in \mathbb{RP}^2 \times \mathbb{RP}^4 : \sum_{i=0}^{2} x_i y_i = 0\}. \quad (G1)
\]

In fact, \(H(2, 4)\) is the submanifold of \(\mathbb{RP}^2 \times \mathbb{RP}^4\) Poincaré dual to \((a + b)\) where \(a\) and \(b\) are the generators of \(H^*(\mathbb{RP}^2; \mathbb{Z}_2)\) and \(H^*(\mathbb{RP}^4; \mathbb{Z}_2)\) respectively. Note that \(a^5 = 0\) and \(b^5 = 0\). The total Stiefel-Whitney class \(w(H(2, 4))\) of \(H(2, 4)\) is given by the restriction to \(H(2, 4)\) of the expression

\[
(1 + a)^3(1 + b)^5/(1 + a + b).
\]

By direct computation, we find that \(w_2 = a^2 + ab\) and \(w_3 = ab^2 + a^2b\). So \(w_2w_3 = a^2b^3\) and the Stiefel-Whitney number \(\langle w_2w_3, [Y^5] \rangle = \langle (a + b)w_2w_3, [\mathbb{RP}^2 \times \mathbb{RP}^4] \rangle = 1\).

The Wu manifold \(W := SU(3)/SO(3)\) has cohomology ring \(H^*(W; \mathbb{Z}_2) = \mathbb{Z}_2[z_2, z_3]/(z_2^2, z_3^2)\) with the total Stiefel-Whitney class \(w(W) = 1 + z_2 + z_3\), \(Sq(z_2) = z_2 + z_3\), and \(Sq(z_3) = z_3 + z_2z_3\) where \(Sq := Sq^0 + Sq^1 + Sq^2 + \cdots\) is the total Steenrod square. So the Stiefel-Whitney number \(\langle w_2w_3, [W] \rangle\) is 1.

In the 5d Dold manifold \([88]\) \(P(1, 2)\) is the quotient \(S^1 \times \tau\) \(\mathbb{CP}^2\) where the involution \(\tau\) sends \((x, y)\) to \((-x, [y])\). The ring structure of \(H^*(P(1, 2); \mathbb{Z}_2)\) is

\[
H^*(P(1, 2); \mathbb{Z}_2) = [\mathbb{Z}_2[c]/(c^2 = 0)] \otimes [\mathbb{Z}_2[d]/(d^5 = 0)],
\]

and the total Stiefel-Whitney class of \(P(1, 2)\) is

\[
w(P(1, 2)) = (1 + c)(1 + c + d)^3,
\]

where \(c \in H^1(P(1, 2); \mathbb{Z}_2)\) and \(d \in H^2(P(1, 2); \mathbb{Z}_2)\). The Steenrod squares act by

\[
Sq^0 = id, \quad Sq^1(c) = 0, \quad Sq^1(d) = cd, \quad Sq^2(d) = d^2,
\]

and all other Steenrod squares act trivially on \(c\) and \(d\). By direct computation, we find that \(w_2 = d\) and \(w_3 = cd\). So the Stiefel-Whitney number \(\langle w_2w_3, [P(1, 2)] \rangle = 1\).

**Appendix H: Generalized Wu Relation**

The classical Wu relation (31) expresses the action of a single Steenrod square \(Sq^n\) on a \(\mathbb{Z}_2\)-valued cocycle \(x_{d-n}\) in the top \(d\)-dimension on a manifold \(M^d\) as the cup product \(u_nx_{d-n}\) where \(u_n\) is the Wu class (32). In this section, we generalize this Wu relation to other elements in the mod 2 Steenrod algebra \(A_2\).
By Adem relation, \( \text{Sq}^1 \text{Sq}^1 = 0 \) and \( \text{Sq}^1 \text{Sq}^2 = \text{Sq}^3 \). So the simplest element in \( A_2 \) which is not a single Steenrod square is \( \text{Sq}^2 \text{Sq}^1 \). We claim that \( \text{Sq}^2 \text{Sq}^1 x_{d-3} = (w_1^2 + w_2^2) x_{d-3} \) on a manifold \( M^d \) where \( w_i \) is the Stiefel-Whitney class of \( M^d \). In fact, 
\[
\begin{align*}
\text{Sq}^2 \text{Sq}^1 x_{d-3} &= (w_1^2 + w_2^2) (\text{Sq}^1 x_{d-3}) \\
&= \text{Sq}^1 (w_1^2 x_{d-3}) + (\text{Sq}^1 w_2) x_{d-3} + \text{Sq}^1 (w_1 w_2 x_{d-3}) \\
&= w_1^2 x_{d-3} + (w_1 w_2 + w_3) x_{d-3} + w_1 w_2 x_{d-3} \\
&= (w_1^2 + w_2^2) x_{d-3}.
\end{align*}
\] 
In the first equality, we used the Wu relation (31) for \( \text{Sq}^2 \). In the second equality, we used the product formula for Steenrod square \( \text{Sq}^k (x \sim y) = \sum_{i+j=k} \text{Sq}^i x \sim \text{Sq}^j y \) and \( \text{Sq}^1 (w_1^2) = 0 \). In the third equality, we used the Wu relation (31) for \( \text{Sq}^1 \) and \( \text{Sq}^1 w_2 = w_1 w_2 + w_3 \). This (H1) is a new generalized Wu relation, which is mentioned in (44).

**Appendix I: Pullback Construction of Branch-Independent Bosonic Models**

In this section, we are going to present a general systematic construction of branch-independent bosonic models. We will first construct a model with a finite \( G \) symmetry, realizing a \( G \)-SPT order. The degrees of freedom in our model are described by \( g_i \in G \) on each vertex-\( i \). The model on space time \( M^d \) is defined by the path integral 
\[
Z(M^d) = \sum_{g_i} e^{-S(g_i)}.
\] 
We can rewrite that model as 
\[
Z(M^d) = \sum_{g_i} e^{-S(a_{ij}^G)} a_{ij}^G = g_i b_j^{-1}.
\] 
We can add a background flat \( G \)-gauge field \( A_{ij}^G \in G \)
\[
A_{ij}^G a_{jk}^G = A_{ik}^G
\] 

to describe the symmetry twist, and consider the following gauged model 
\[
Z(M^d, A^G) = \sum_{g_i} e^{-S(a_{ij}^G)} a_{ij}^G = g_i A_{ij}^G b_j^{-1}.
\] 
Note that \( a_{ij}^G \) satisfy a flat condition 
\[
a_{ij} a_{jk}^G = a_{ik}^G.
\] 
When \( A_{ij}^G = 1 \), the partition function in (14) automatically has the \( G \) symmetry 
\[
g_i \rightarrow g_i h, \quad h \in G,
\] 
even for spacetime \( M^d \) with boundaries. This implies that the \( G \) symmetry is anomaly-free.

We can choose a proper \( S(a_{ij}^G) \) so that the bosonic model (14) realized a bosonic SPT order. To do so, let us consider the classifying space of the group \( G \), and choose an one-vertex triangulation of the classifying space. We denote the resulting simplicial complex as \( B_G \) For the details about the one-vertex triangulation, see Ref. 89. \( B_G \) has the properties that \( \pi_1(B_G) = G \) and \( \pi_{n \neq 1}(B_G) = 0 \). Since \( B_G \) has only one vertex, the links in \( B_G \) are labeled by \( a_{ik}^G \in G \), that satisfy the condition 
\[
a_{ij}^G a_{jk}^G = a_{ik}^G
\] 
for every triangles \( (ijk) \) in \( B_G \).

\( a_{ij}^G \) on the links of spacetime simplicial complex \( M^d \) defines a homomorphism of simplicial complex 
\[
M^d \xrightarrow{\phi(a_{ij}^G)} B_G.
\] 
\( \phi(a_{ij}^G) \) maps the vertices in \( M^d \) to the only vertex in \( B_G \). \( \phi(a_{ij}^G) \) maps the link in \( M^d \) with value \( a_{ij}^G \) to the link in \( B_G \) labeled by \( a_{ik}^G \). The two flat conditions (15) and (17) ensure that \( \phi(a_{ij}^G) \) maps the triangles in \( M^d \) to triangles in \( B_G \), etc. Thus \( \phi(a_{ij}^G) \) is a homomorphism of simplicial complex, and \( a_{ij}^G \) is the pullback of \( a_{ij}^G \) by \( \phi(a_{ij}^G) \):
\[
a_{ij}^G = \phi^*(a_{ij}^G) a_{ij}^G.
\] 
Let \( \bar{\omega}_d \) be a \( \mathbb{R}/\mathbb{Z} \)-valued cocycle in \( B_G \):
\[
\bar{\omega}_d \in H^d(B_G; \mathbb{R}/\mathbb{Z}).
\] 
Now we can construct a bosonic model using \( \bar{\omega}_d \):
\[
Z(M^d, A^G) = \sum_{g_i} e^{i 2 \pi \int_M d a_{ij}^G \phi^*(a_{ij}^G) \bar{\omega}_d}, \quad a_{ij}^G = g_i A_{ij}^G b_j^{-1}.
\] 
We refer to this construction as pullback construction. Clearly, the resulting bosonic model is branch-independent, since during the whole construction, the branch structure is not even specified. We also note that the action amplitude \( e^{i 2 \pi \int_M d a_{ij}^G \phi^*(a_{ij}^G) \bar{\omega}_d} \) does not depend on the Stiefel-Whitney cocycle \( w_n \).

The branch-independent bosonic model (11) realizes the SPT order labeled by \( \bar{\omega}_d \in H^d(B_G; \mathbb{R}/\mathbb{Z}) \), and described by the SPT invariant [32, 33] 
\[
Z^{\text{top}}(M^d, A^G) = e^{i 2 \pi \int_M \phi^*(A_{ij}^G) \bar{\omega}_d}.
\] 
This is consistent with the group cohomology classification of SPT order [10].

The above construction of branch-independent bosonic models can be generalized to compact continuous group \( G \) with less rigor, where the branch-independent bosonic model is given by the following path integral 
\[
Z(M^d, A^G) = \sum_{\phi} e^{i 2 \pi \int_M d a_{ij}^G \phi^*(a_{ij}^G) \bar{\omega}_d} e^{-\int_M L(\phi)}.
\]
Here we give the classifying space of $G$ a triangulation and denote the resulting simplicial complex as $BG$. The triangulation temporarily breaks the $G$ symmetry. $\phi$ is a homomorphism from spacetime simplicial complex $M^d$ to classifying space simplicial complex $BG$:

$$M^d \xrightarrow{\phi} BG.$$  \hspace{1cm} (114)

The path integral is a sum over the homomorphisms $\phi$. This is similar to the definition of non-linear $\sigma$-model on a continuum spacetime, where the path integral is a sum over the continuous maps from spacetime manifold to target space. The term $e^{-\int_{M^d} \mathcal{L}(\phi)}$ is chosen such that the model (I13) describes a disordered state. In the limit of fine triangulation the model has a $G$ symmetry, and the disordered state is $G$ symmetric.

Also, $\tilde{\omega}_d$ in the path integral (I13) is a $\mathbb{R}/\mathbb{Z}$-valued cocycle in $BG$:

$$\tilde{\omega}_d \in H^d(BG; \mathbb{R}/\mathbb{Z}).$$  \hspace{1cm} (115)

If $H^n(BG; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_n \oplus \cdots$, then $H^n(BG; \mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z} \oplus \mathbb{Z}_n \oplus \cdots$ according to the universal coefficient theorem:

$$H^d(X; \mathbb{R}) \simeq \mathbb{R} \otimes_{\mathbb{Z}} H^d(X; \mathbb{Z}) \oplus \text{Tor}(\mathbb{R}, H^{d+1}(X; \mathbb{Z})).$$  \hspace{1cm} (116)

We see that the torsion part of $H^d(BG; \mathbb{R}/\mathbb{Z})$ and $H^{d+1}(BG; \mathbb{R}/\mathbb{Z})$ coincide:

$$\text{Tor}(H^d(BG; \mathbb{R}/\mathbb{Z})) = \text{Tor}(H^{d+1}(BG; \mathbb{R}/\mathbb{Z})).$$  \hspace{1cm} (117)

The cocycle that generates $\mathbb{R}/\mathbb{Z}$ part of $H^n(BG; \mathbb{R}/\mathbb{Z})$ does not have a quantized coefficient and can be continuously changed to zero. So the $\mathbb{R}/\mathbb{Z}$ part of $H^d(BG; \mathbb{R}/\mathbb{Z})$ does not characterize a topological phase. Only $\tilde{\omega}_d \in \text{Tor}(H^d(BG; \mathbb{R}/\mathbb{Z})) = \text{Tor}(\mathbb{R}/\mathbb{Z}, H^{d+1}(BG; \mathbb{Z}))$ gives rise to distinct topological phase via the model (I13), with is a $G$-SPT phase.

When $G$ is continuous, some $G$-SPT order can belong to $\mathbb{Z}$ class which is not a torsion. To construct branch-dependent bosonic model to realize this kind of SPT order, we need to generalize the above model (I13) to the following form

$$Z(M^d, A^G) = \sum_{\phi} e^{-\int_{M^d} \mathcal{L}(\phi)} e^{i2\pi [\int_{M^d} \phi \cdot \tilde{\omega}_d + \int_{M^{d+1}} \phi_N \tilde{\nu}_{d+1}]}.$$  \hspace{1cm} (118)

The term $e^{i2\pi [\int_{M^{d+1}} \phi_N \tilde{\nu}_{d+1}]}$, living in one higher dimension, is a Wess-Zumino-Witten like term, and $\phi_N$ is a homomorphism of simplicial complex

$$N^{d+1} \xrightarrow{\phi_N} BG$$  \hspace{1cm} (119)

such that at the boundary $M^d = \partial N^{d+1}$, $\phi_N = \phi$. $\tilde{\nu}_{d+1}$ is an $\mathbb{R}$-valued cocycle that satisfies a quantization condition

$$\int_{N^{d+1}} \tilde{\nu}_{d+1} \in \mathbb{Z},$$  \hspace{1cm} (120)

for all closed $N^{d+1}$ in $BG$. In other words, the $\mathbb{R}$-valued cocycle $\tilde{\nu}_{d+1}$ represents a cohomology class in the free part of $H^{d+1}(BG; \mathbb{Z})$:

$$\tilde{\nu}_{d+1} \in \text{Free}(H^{d+1}(BG; \mathbb{Z})).$$  \hspace{1cm} (121)

In the disordered phase, (I18) realizes a $G$-SPT order characterized by $(\tilde{\omega}_d, \tilde{\nu}_{d+1})$ in $\text{Tor}(H^d(BG; \mathbb{R}/\mathbb{Z})) = \text{Tor}(H^{d+1}(BG; \mathbb{Z}))$ and $\text{Free}(H^{d+1}(BG; \mathbb{Z}))$. In other words, the $G$-SPT order characterized is characterized by the elements in $H^{d+1}(BG; \mathbb{Z})$, which agree with the group cohomology theory of SPT order for symmetries described by compact groups.

When $G = \text{SO}(\infty)$, the term $e^{i2\pi [\int_{M^{d+1}} \phi_N \tilde{\nu}_{d+1}]}$ gives rise to the $\text{SO}(\infty)$ Chern-Simons term on $M^d = \partial N^{d+1}$, whose generator is the Pontryagin class (for $d + 1 = 0 \mod 4$). The pullback of different maps $\phi$ gives rise to different $\text{SO}(\infty)$ bundle over $M^d$. If we restrict $\sum_{\phi}$ in (I18) to a subset of maps $\phi$, such that the $\text{SO}(\infty)$ bundle over $M^d$ is the same as the stabilized tangent bundle of $M^d$, the model (I18) may realize a bosonic invertible topological order. The $\mathbb{Z}$-class of invertible topological orders are described by gravitational Chern-Simons term, which is also $\text{SO}(\infty)$ Chern-Simons term. We see that the model (I18) can only realize gravitational Chern-Simons terms generated by Pontryagin classes, which have no framing anomaly. Thus in 3d, the model (I18) can only realize invertible topological orders generated by $\text{E}_8$ topological order. The $\text{E}_8$ topological orders is characterized by a gravitational Chern-Simons term that corresponds to $\frac{1}{3}$ of the first Pontryagin class, which has a framing anomaly.

### Appendix J: Background vs Dynamical Gauge Transformations

In Sec. III B 1, we describe the invariance or non-invariance of path integral in terms of the change of coboundaries and branch structures. Here we fill in some additional terminology more accessible for quantum field theorists: in terms of background gauge transformations vs dynamical gauge transformations.

For 4d $\mathbb{Z}_2$ gauge theory (24) described by $\mathbb{Z}_2$-valued 1-cochain and 2-cochain dynamical fields $a^{Z_2}$ and $b^{Z_2}$

$$Z = \sum_{a^{Z_2}, b^{Z_2}} e^{i\pi [\int_{a^{Z_2}} a^{Z_2} \cdot b^{Z_2}]};$$

where $\sum_{a^{Z_2}, b^{Z_2}}$ is a summation over $\mathbb{Z}_2$-valued 1- and 2-cochains. The above theory has dynamical gauge transformations for dynamical fields:

$$a^{Z_2} \rightarrow a^{Z_2} + du_0, \quad b^{Z_2} \rightarrow b^{Z_2} + du_1,$$  \hspace{1cm} (J1)

where $u_0 \in C^0(B^4; \mathbb{Z}_2)$ and $u_1 \in C^1(B^4; \mathbb{Z}_2)$ are $\mathbb{Z}_2$-valued 0- and 1-cochain fields.
Now let us discuss another interpretation of \((25)\). The Stiefel-Whitney classes \(w_2 \in H^2(M^5;\mathbb{Z}_2)\) and \(w_3 \in H^3(M^5;\mathbb{Z}_2)\) are special cohomology classes satisfying the extra axioms A1–A4 listed earlier, with the base manifold \(M^5\) for the real vector bundle. Since \(M^5\) is orientable, we have \(w_1 \cong 0\) and \(w_3 \cong \text{Sq}^1 w_2\).

When \(M^5\) has a boundary, the partition function \((25)\) depends on the choice of the coboundaries in \(w_2\) and \(w_3\). i.e. under the following background gauge transformation for non-dynamical fields:

\[
w_2 \rightarrow w_2 + d\nu_1, \\
w_3 \rightarrow w_3 + \text{Sq}^1 \nu_1 + d\nu_2 \rightarrow w_3 + d\nu_2.
\]

Although the Stiefel-Whitney classes have the relation \(\text{Sq}^1 w_2 = w_3\) so that the transformation \(d\nu_1\) can be related to \(\text{Sq}^1 \nu_1\), but they can differ by a coboundary \(d\nu_2\) which thus absorbs \(\text{Sq}^1 \nu_1\).

An anomalous \(\mathbb{Z}_2\)-gauge theory (that has ’t Hooft anomaly of spacetime SO diffeomorphism) on the boundary \(B^4 = \partial M^5\) of the topological state is described by \((25)\) which has not only the dynamical gauge transformations (involving \(u_0\) and \(u_1\) in \((J1)\)) but also additional background gauge transformations (involving \(v_1\) in \((J2)\)):

\[
a^{Z_2} \rightarrow a^{Z_2} + du_0 + v_1, \\
b^{Z_2} \rightarrow b^{Z_2} + du_1 + v_2, \\
w_2 \rightarrow w_2 + d\nu_1, \\
w_3 \rightarrow w_3 + d\nu_2.
\]

It turns out that the background gauge transformations at the lattice scale of the simplicial complex are important to ensure the anomaly inflow or anomaly cancellation between the bulk and boundary for the ’t Hooft anomaly of global symmetries. In contrast, the dynamical gauge transformations at the lattice scale of the simplicial complex are not so crucial or fundamental — the dynamical gauge invariance at the lattice scale, even if we break them locally, the dynamical gauge invariance can re-emerge at a larger length scale. So only the emergent dynamical gauge invariance is crucial.

**Appendix K: \(\mathbb{Z}_2\) topological order with emergent fermion and higher dimensional bosonization**

In this section, we review and summarize the higher dimensional bosonization following Ref. 71. In \(d + 1\)-dimensional spacetime, a bosonic model that realizes a \(\mathbb{Z}_2\) topological order is described by the following path integral

\[
Z(M^{d+1}) = \sum_{d\text{z}_2 \cong 0} 1.
\]

where \(\sum_{d\text{z}_2 \cong 0}\) sums over all \(\mathbb{Z}_2\) valued 1-cocycles \(a^{Z_2}\).

The low energy effective theory of the \(\mathbb{Z}_2\) topological order is a \(\mathbb{Z}_2\) gauge theory where the point-like \(\mathbb{Z}_2\) charge is a boson. Such a \(\mathbb{Z}_2\) topological order has another re-alization in terms of \(\mathbb{Z}_2\)-valued \(d-1\) cocycles \(f^{Z_2}\)

\[
Z(M^{d+1}) = \sum_{d\text{z}_2 \cong 0} 1.
\]

There is a twisted \(\mathbb{Z}_2\) topological order \([59]\), whose low energy effective theory is a twisted \(\mathbb{Z}_2\) gauge theory where the point-like \(\mathbb{Z}_2\) charge is a fermion. Such a twisted \(\mathbb{Z}_2\) topological order is realized by the following bosonic model

\[
Z(M^{d+1}) = \sum_{d\text{z}_2 \cong 0} e^{i\pi f_{M^{d+1}} \text{Sq}^1 f_{M^{d+1}}},
\]

The above path integral does not contain a \(\mathbb{Z}_2\) charge. To include a \(\mathbb{Z}_2\) charge, we note that the worldline of a particle can be described by its Poincaré dual \(d\), which is a \(\mathbb{Z}_2\)-valued \(d\)-coboundary. The path integral including such a worldline is given by

\[
Z(M^{d+1}) = \sum_{d\text{z}_2 \cong 0} e^{i\pi f_{M^{d+1}} \text{Sq}^1 f_{M^{d+1}}},
\]

The term \(e^{i\pi f_{M^{d+1}} \text{Sq}^1 f_{M^{d+1}}}\) gives the worldline (described by \(f_{M^{d+1}}\)) a Fermi statistics.

The \((d + 1)d\) twisted \(\mathbb{Z}_2\) topological order has a \(d\)-dimensional boundary, formed by condensing the \(\mathbb{Z}_2\) flux. Such a boundary contains only point-like topological excitations, which are fermions, coming from the bulk fermionic \(\mathbb{Z}_2\) charge. Such a boundary is the canonical boundary of the path integral \((K4)\), which is described by the following path integral

\[
Z(M^{d+1}) = \sum_{d\text{z}_2 \cong 0} e^{i\pi f_{M^{d+1}} \text{Sq}^1 f_{M^{d+1}}},
\]

\(B^d = \partial M^{d+1}\).

The cocycle \(f_{M^{d+1}}\) on the boundary \(B^d\) can be viewed as the Poincaré dual of the worldline of boundary fermions.

Now, let us try to view the above path integral \((K4)\) as a path integral on the boundary \(B^d\) for the field \(l_{d-1}\), and view the term \(e^{i\pi f_{M^{d+1}} \text{Sq}^1 f_{M^{d+1}}}\) as a Wess-Zumino-Witten like term defined in one higher dimension. But such a viewpoint is not quite correct, since the \(e^{i\pi f_{M^{d+1}} \text{Sq}^1 f_{M^{d+1}}}\) only depends on \(l_{d-1}^2\) on the boundary \(B^d = \partial M^{d+1}\), but also depends on \(M^{d+1}\), i.e. how we extend \(B^d\). However, when \(M^{d+1}\) is oriented and spin, \(w_1 = 0\) and \(w_2 = 0\). In this case, \(e^{i\pi f_{M^{d+1}} \text{Sq}^1 f_{M^{d+1}}}\) only depends on \(l_{d-1}^2\) on the boundary \(B^d = \partial M^{d+1}\), and can indeed be viewed as Wess-Zumino-Witten like term.

We can modify the path integral \((K5)\) to relax the requirement for \(M^{d+1}\) to be spin:

\[
Z(M^{d+1}) = \sum_{d\text{z}_2 \cong 0} e^{i\pi f_{M^{d+1}} l_{d-1}^{2d} + f_{M^{d+1}} \text{Sq}^1 f_{M^{d+1}} + w_2 l_{d-1}^{2d}},
\]

\(B^d = \partial M^{d+1}, \ w_2 = dA Z^2\) on \(B^d\).
In the above, \( M^{d+1} \) is orientable but may not be spin, and \( B^d \) is spin such that \( w_2 \cong dA^{2d} \) on \( B^d \). In this case, \( e^{i\pi f_{d+1}} \sigma_1w_2^2 \) is a Wess-Zumino-Witten like term, that only depend on \( f_{d+1} \) and \( w_2 \) on \( B^d \). Also, \((K6)\) is invariant under the “gauge” transformation
\[
w_2 \rightarrow w_2 + d\tau^2, \quad A^{2d} \rightarrow A^{2d} + v^2, \quad (K7)
\]
so that it is branch independent.

\( A^{2d} \) is a \( \mathbb{Z}_2 \)-valued 1-cochain, which corresponds to the spin structure on \( B^d \). The relation \( w_2 \cong dA^{2d} \) tells us that the worldline \( f_{d+1}^2 \) couples to the \( SO(n) \) tangent bundle of \( B^d \) in such a way that the worldline corresponds to an half-integer spin.

\[\begin{align*}
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