On sums of subsets of Chen primes

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Abstract

In this paper we show that if \( A \) is a subset of Chen primes with positive relative density \( \alpha \), then \( A + A \) must have positive upper density at least \( c_0 e^{-C_3 \log(1/\alpha)^{2/3} (\log \log(1/\alpha))^{1/3}} \) in the natural numbers.

1. Introduction

In 1953, K. Roth [12] proved that any subset of positive integers of positive density contains non-trivial three-term arithmetic progressions. In recent years, Green [5] showed that any subset of primes of relative positive density also has this property. And later Roth’s theorem was extended to Chen primes in [6]. Moreover, a celebrated theorem was proved by Green and Tao [7], showing that the primes contain arbitrarily long arithmetic progressions.

The strategy developed by Green and Green-Tao is called “W-trick”. The primes are embedded to a set behaving more “pseudorandom”, meanwhile slight density-increment is gained. For various applications, one can see [2], [4], [9], [10], [11].... In [2], it is proved by Chipeniuk and Hamel that if \( A \) is a subset of primes, with positive relative density \( \alpha_0 \), then the set \( A + A \) has positive density at least

\[
C_1 \alpha_0 e^{-C_4 (\log(1/\alpha_0)^2/3 (\log \log(1/\alpha_0))^{1/3})}
\]

in the natural numbers. This result is not far from best possible due to some examples.

Let \( \mathcal{P}_c \) be the set of Chen primes, each of whom is a prime \( p \) for which \( p + 2 \) is either a prime or a product \( p_1 p_2 \) with \( p_1, p_2 > p^{3/11} \), according to Chen[1] and Iwaniec [8]. Chen’s famous theorem concludes that there are infinitely many such primes.

Theorem 0. (8) Let \( n \) be a large integer. Then the number of Chen primes less than \( n \) is at least \( c_1 n / \log^2 n \), for some absolute constant \( c_1 > 0 \).

In this paper, we extend the density result to subsets of Chen primes. For any set \( S \subseteq \mathbb{N} \), denote

\[
\overline{d}(S) = \limsup_{n \to \infty} \frac{|S \cap [1,n]|}{|1,n|}, \quad \overline{d}_{\mathcal{P}_c}(S) = \limsup_{n \to \infty} \frac{|S \cap \mathcal{P}_c \cap [1,n]|}{|\mathcal{P}_c \cap [1,n]|}.
\]

Theorem 1. Let \( A \subseteq \mathcal{P}_c \) with positive relative density \( \overline{d}_{\mathcal{P}_c}(A) = \alpha_0 \). Then

\[
\overline{d}(A + A) \geq C_3 \alpha_0 e^{-C_4 (\log(1/\alpha_0)^2/3 (\log \log(1/\alpha_0))^{1/3})}
\]

for some absolute positive constant \( C_3 \) and \( C_4 \).

Since \( \overline{d}_{\mathcal{P}_c}(A) = \alpha_0 \), there exist infinitely many \( n \) such that \( |A \cap [1,n]|/|\mathcal{P}_c \cap [1,n]| \geq \alpha_0/2 \). The previous theorem will follow from a finite version, with \( \alpha = \alpha_0/2 \).

Theorem 2. Suppose that \( n \) is a sufficiently large integer. Let \( A \subset \mathcal{P}_c \cap [1,n] \) with \( \frac{|A|}{|\mathcal{P}_c \cap [1,n]|} \geq \alpha \). Then

\[
|A + A| \geq C_5 \alpha e^{-C_6 \log(1/\alpha)^2/3 (\log \log(1/\alpha))^{1/3}} n.
\]

for some absolute positive constant \( C_5 \) and \( C_6 \).

We mainly follow arguments of Chipeniuk-Hamel[2] and combine the envelop sieve function of Green-Tao[6].

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2. Notations and Preliminary Lemmas

For a parameter $k$, we write $f \ll_k g$ or $f = O_k(g)$ to denote the estimate $f \leq C_k g$ for some positive constant $C_k$ depending only on $k$. For a set $S$, $|S|$ and $\#S$ both denote the cardinality of $S$ and the characteristic function $1_S(x)$ takes value $1$ for $x \in S$ and $0$ otherwise. The sum set $S + S' := \{ s + s' : s \in S, s' \in S' \}$. $c, c_0, c_1, c_2, \ldots$ are positive absolute constants. We write $\mathbb{Z}_N$ for the cyclic group $\mathbb{Z}/N\mathbb{Z}$ and $\mathbb{Z}_N^\ast$ for the multiplicative subgroup of integers modulo $N$. And $[1, N]$ denotes the set $\{1, 2, \ldots, N\}$.

Next we introduce Fourier analysis on $\mathbb{Z}_N$. If $f : \mathbb{Z}_N \to \mathbb{C}$ is a function and $S \subseteq \mathbb{Z}_N$, we define

$$E_{x \in S} f(x) := \frac{1}{|S|} \sum_{x \in S} f(x).$$

Let $e_N(x) := e^{2\pi i x/N}$. The Fourier transform and the Fourier inversion are the following

$$\hat{f}(\xi) = E_{x \in \mathbb{Z}_N} f(x) e_N(-\xi x), \quad f(x) = \sum_{\xi \in \mathbb{Z}_N} \hat{f}(\xi) e_N(\xi x).$$

The $L^q$, $L^\infty$, $l^q$, $l^\infty$-norms are defined to be

$$\|f\|_{L^q} = \left(\mathbb{E}_{x \in \mathbb{Z}_N} |f(x)|^q\right)^{1/q}, \quad \|f\|_{L^\infty} = \sup_{x \in \mathbb{Z}_N} |f(x)|.$$

$$\|\hat{f}\|_{l^q} = \left(\sum_{\xi \in \mathbb{Z}_N} |\hat{f}(\xi)|^q\right)^{1/q}, \quad \|\hat{f}\|_{l^\infty} = \sup_{\xi \in \mathbb{Z}_N} |\hat{f}(\xi)|.$$

Plancherel’s equality tells that $\|f\|_{L^2} = \|\hat{f}\|_{l^2}$. We also write

$$f * g(x) = E_{y \in \mathbb{Z}_N} f(x - y) g(y)$$

for convolution. A basic identity for convolution is $\hat{f} * \hat{g} = \hat{f} \cdot \hat{g}$. For non-negative valued function $f$ and $g$, it obeys that

$$\|f * g\|_{L^1} = \|f\|_{L^1} \cdot \|g\|_{L^1}. \leqno(1)$$

Hausdroff-Young inequality shows that

$$\|\hat{f}\|_{l^{q'}} \leq \|f\|_{L^q},$$

where $1 \leq q \leq 2$ and $1/q + 1/q' = 1$. Hölder’s inequality says that

$$\|fg\|_{L^r} \leq \|f\|_{L^q} \|g\|_{L^q'}$$

whenever $0 < q, q', r \leq \infty$ are such that $1/q + 1/q' = 1/r$. And Young inequality tells that

$$\|f * g\|_{L^r} \leq \|f\|_{L^q} \|g\|_{L^{q'}}$$

for $1 \leq q, q', r \leq \infty$ and $1/q + 1/q' = 1/r + 1$. (See [13] for details.)

Throughout this paper, $A \subseteq P_c \cap [1, n]$ with $|A|/|P_c \cap [1, n]| \geq \alpha$. And $p$ always denotes a prime. Let $\lambda, \varepsilon, \delta$ be small parameters and $t \gg 1$ be a very large real number to be specified later.

Write $W := \prod_{3 \leq p \leq t} p$. For $b$ with $(b, W) = (b + 2, W) = 1$, write $F(b)(x) = (Wx + b)(Wx + b + 2)$, and let $R = \lfloor N^{1/20} \rfloor$. In [6], Green and Tao constructed an enveloping sieve function $\beta^{(b)}_R$ with the property (See [6] Proposition 3.1)

$$\beta^{(b)}_R(x) \gg \mathcal{S}^{-1}_{F(b)} \log^2 R \cdot 1_{X^{(b)}_{R!}}(x) \leqno(2)$$

(One can check that the constant does not depend on $b$.) with

$$\mathcal{S}_{F(b)} = \prod_p \frac{\gamma^{(b)}(p)}{1 - \frac{1}{p}}, \quad \gamma^{(b)}(p) = \frac{1}{p} \{ n \in \mathbb{Z}/p\mathbb{Z} : (p, F^{(b)}(n)) = 1 \},$$

and

$$X^{(b)}_{R!} = \{ n \in \mathbb{Z} : (d, F^{(b)}(n)) = 1 \text{ for all } 1 \leq d \leq R \}.$$
By computation (or See \[5\] (6.3)),
\[
\log^2 t \ll \mathcal{G}_{F(b)} \ll \log^2 t.
\]

Restrict \(\beta_R^{(b)}\) to the set \([1, N]\), which we identify with \(\mathbb{Z}_N\), and let \(\nu^{(b)} : \mathbb{Z}_N \to \mathbb{R}^+\) be the resulting function. We have the following

**Lemma 1.** \([5]\), Lemma 6.1 For \(\xi \in \mathbb{Z}_N\), we have
\[
\widehat{\nu^{(b)}}(\xi) = \delta_{\xi,0} + O(t^{-1/2}),
\]
where \(\delta_{\xi,0}\) is the Kronecker delta function. Also the constant does not depend on \(b\).

**Lemma 2.** \([5]\), Prop 4.2, Lemma 3R.1 Let \(q > 2\) be a real number and \(\{a_x\}\) be an arbitrary sequence of complex numbers. Suppose that \(1 \ll R \ll N^{1/10}\) and \(F^{(b)}, \beta^{(b)}_R\) are defined above. Then we have
\[
\left(\sum_{\xi \in \mathbb{Z}_N} \left| \mathbb{E}_{1 \leq x \leq N} a_x \beta^{(b)}_R(x) e_N(-\xi x) \right|^q \right)^{1/q} \ll q \left( \mathbb{E}_{1 \leq x \leq N} |a_x|^2 \beta^{(b)}_R(x) \right)^{1/2}
\]
and
\[
\mathbb{E}_{1 \leq x \leq N} \beta^{(b)}_R(x) \ll 1.
\]

3. **W-trick to Chen Primes**

Denote \(A_n := A \cap (\sqrt{n}, n]\). And let \(\Phi_W := \{b \in \mathbb{Z}_W : (b, W) = (b + 2, W) = 1\}\). Observe that
\[
\varphi_W := |\Phi_W| = W \prod_{3 \leq p \leq t} \left( 1 - \frac{2}{p} \right) \ll \frac{W}{\log^2 t}.
\]

Choose a prime \(N \in \left(\frac{2n}{W}, \frac{4n}{W}\right]\). For \(b \in \Phi_W\), let
\[
A_N^{(b)} := \{x \in A_n : x \equiv b(\mod W)\}, \quad A_N^{(b)} := \{x \leq N : Wx + b \in A_N^{(b)}\}.
\]

The choice of \(N\) ensures that \(A_N^{(b)} + A_N^{(b)}\) in \(\mathbb{Z}\) can be identified with \(A_N^{(b)} + A_N^{(b)}\) in \(\mathbb{Z}_N\). Noting that \(A_N^{(b)} \subseteq X_R^{(b)}\). Combining \([2]\) and \([3]\), we can define \(f^{(b)} : \mathbb{Z}_N \to \mathbb{R}^+\) by
\[
f^{(b)}(x) = c \frac{\log^2 N}{\log^2 t} A_N^{(b)}(x),
\]
with \(c > 0\) chosen sufficiently small such that \(f^{(b)}(x) \leq 1\). (Noting that the choice of \(c\) does not depend on \(b\).)

Now follow \([5]\) and \([6]\), we decompose \(f^{(b)}\) into one anti-uniform component and one uniform component. Denote
\[
R^{(b)} = \{\xi \in \mathbb{Z}_N : |\hat{f}^{(b)}(\xi)| \geq \delta\}, \quad B^{(b)} = \{x \in \mathbb{Z}_N : \sup_{\xi \in R^{(b)}} |1 - e_N(\xi x)| \leq \varepsilon\}.
\]

Let \(\beta^{(b)}(x) = \frac{N}{|B^{(b)}|} 1_{B^{(b)}}(x)\) and
\[
f^{(b)}_1 = f^{(b)} * \beta^{(b)} * \beta^{(b)}, \quad f^{(b)}_2 = f^{(b)} - f^{(b)}_1.
\]

**Lemma 3.** The functions \(f^{(b)}, f^{(b)}_1, f^{(b)}_2\) defined above have the following properties:

(i) \(\|f^{(b)}_1\|_{L^1} = \|f^{(b)}\|_{L^1} \leq 1 + O(t^{-1/2})\).

(ii) \(\left\|f^{(b)}_2\right\|_{L^\infty} \ll \delta + \varepsilon \cdot (1 + O(t^{-1/2}))\).

(iii) \(\left\|f^{(b)}_1\right\|_{L^{2+\lambda}}, \left\|f^{(b)}_2\right\|_{L^{2+\lambda}} \ll \left\|f^{(b)}\right\|_{L^{2+\lambda}} \ll \lambda\).

(iv) \(\left\|f^{(b)}_1\right\|_{L^\infty} \leq 1 + O(t^{-1/2}) + O_{\lambda}(\varepsilon^{-c\lambda\delta^{-2(2+\lambda)} \cdot t^{-1/2}})\)

Proof: The proofs follow as in \([5]\). We reiterate them here in order to specify \(\varepsilon\) and \(\delta\).
(i) One can deduce that $\|\beta^{(b)}\|_{L^1} = 1$. Then by (i) and Lemma 1, we have

$$\|f_1^{(b)}\|_{L^1} = \|f^{(b)}\|_{L^1} \leq \|\nu^{(b)}\|_{L^1} = |\nu^{(b)}(0)| = 1 + O(t^{-1/2}).$$

(ii) Noting that

$$f_2^{(b)}(\xi) = f^{(b)}(\xi) - f_1^{(b)}(\xi) = f^{(b)}(\xi)(1 - \beta^{(b)}(\xi)^2).$$

For $\xi \in R^{(b)}$, we have

$$|1 - \beta^{(b)}(\xi)| = |E_x \in B^{(b)}(1 - e_N(-\xi x))| \leq E_x \in B^{(b)}|1 - e_N(-\xi x)| \leq \varepsilon.$$  

By triangle inequality, $|1 - \beta^{(b)}(\xi)^2| \leq 2\varepsilon$. Also we have

$$\|f^{(b)}\|_{l^\infty} \leq \|f^{(b)}\|_{L^1} \leq \|\nu^{(b)}\|_{L^1} = 1 + O(t^{-1/2})$$

by Hausdorff-Young inequality and Lemma 1. Hence

$$|f_2^{(b)}(\xi)| \ll \varepsilon \cdot (1 + O(t^{-1/2})).$$

And for $\xi \notin R^{(b)}$, we have

$$|f_2^{(b)}(\xi)| \ll |f^{(b)}(\xi)| \ll \delta.$$  

(iii) For arbitrary $\lambda > 0$, taking $a_x = f^{(b)}(x)/\beta^{(b)}_R(x)$ and $q = 2 + \lambda$, Lemma 2 tells

$$\|f^{(b)}\|_{l^{2+\lambda}} \ll_\lambda 1.$$  

Since $\|\beta^{(b)}\|_{l^\infty} \leq \|\beta^{(b)}\|_{L^1} = 1$, we have

$$\left| f_1^{(b)}(\xi) \right| = \left| f^{(b)}(\xi) \right| \beta^{(b)}(\xi)^2 \leq \left| f^{(b)}(\xi) \right|,$$

and

$$\left| f_2^{(b)}(\xi) \right| \leq \left| f^{(b)}(\xi) \right| + \left| f_1^{(b)}(\xi) \right| \ll \left| f^{(b)}(\xi) \right|.$$

Then (iii) follows.

(iv)

$$|f_1^{(b)}(x)| = |f^{(b)} * \beta^{(b)} * \beta^{(b)}(x)|$$

$$\leq |\nu^{(b)} * \beta^{(b)} * \beta^{(b)}(x)|$$

$$= \| \sum_{\xi \in Z_N} \nu^{(b)}(\xi) \beta^{(b)}(\xi)^2 e_N(\xi x) \|$$

$$\leq \| \nu^{(b)}(0) \| \beta^{(b)}(0) \|^2 + \sup_{\xi \neq 0} \| \nu^{(b)}(\xi) \| \sum_{\xi} |\beta^{(b)}(\xi)|^2$$

$$\leq (1 + O(t^{-1/2})) + O(t^{-1/2}) \cdot \frac{N}{|B^{(b)}|}.$$  

In the last inequality we have used Plancherel’s equality to get

$$\sum_{\xi} |\beta^{(b)}(\xi)|^2 = \mathbb{E}_{x \in Z_N} \left| \beta^{(b)}(x) \right|^2 = \frac{N}{|B^{(b)}|}.$$  

By pigeonhole principle (or see [2], Lemma 3) we have $|B^{(b)}| \gg (\varepsilon/2\pi)^{|R^{(b)}|} N$. And from (iii), there exists some positive integer $c_\lambda$ such that

$$c_\lambda \geq \|f^{(b)}\|_{l^{2+\lambda}} \geq \sum_{\xi \in R} \left| \tilde{f}(\xi) \right|^{2+\lambda} \geq |R^{(b)}| \cdot \delta^{2+\lambda},$$

which implies $|R^{(b)}| \leq c_\lambda \delta^{-(2+\lambda)}$. Hence we get

$$0 \leq f_1^{(b)}(x) \leq 1 + O(t^{-1/2}) + O_\lambda(\varepsilon^{-c_\lambda \delta^{-(2+\lambda)} t^{-1/2}}).$$
Now for \( b \in \Phi_W \), define
\[
\delta_b := \frac{|A_n(b)|}{\log^2 n \cdot \varphi_W}.
\]
By (i) of Lemma 3, the definition of \( f(b) \) and recalling that \( N \in (\frac{2n}{W}, \frac{4n}{W}] \), we can get
\[
\delta_b \ll \|f(b)\|_{L^1} \ll \delta_b \ll 1, \tag{4}
\]
when \( t \gg 1 \) is sufficient large. However, \( \delta_b \ll 1 \) is equivalent to
\[
|A_n(b)| \ll \frac{n}{\log^2 n \cdot \varphi_W}. \tag{5}
\]

**Lemma 4.** The convolution of functions \( f_1(b) \) and \( f_2(b) \) defined above have following properties:

(i) \( \delta_b \delta_{b_2} \ll \|f_1^{(b_1)} \ast f_2^{(b_2)}\|_{L^1} \ll \delta_b \delta_{b_2} \),

(ii) \( \|f_1^{(b_1)} \ast f_2^{(b_2)}\|_{L^2} \ll (\delta + \varepsilon \cdot (1 + O(t^{-1/2}))^{1-\lambda/2}, \quad \|f_2^{(b_1)} \ast f_2^{(b_2)}\|_{L^2} \ll (\delta + \varepsilon \cdot (1 + O(t^{-1/2}))^{1-\lambda/2}. \)

(iii) \( \|f_1^{(b_1)} \ast f_2^{(b_2)}\|_{L^\infty} \ll \min\{\delta_b, \delta_{b_2}\}(1 + O(t^{-1/2}) + O(\varepsilon^{-c \lambda \delta^{-2+\lambda} t^{-1/2}})). \)

**Proof:**

(i) By (1) and (4), we have
\[
\|f_1^{(b_1)} \ast f_2^{(b_2)}\|_{L^1} = \|f_1^{(b_1)}\|_{L^1} \cdot \|f_2^{(b_2)}\|_{L^1} = \|f_1^{(b_1)}\|_{L^1} \cdot \|f_2^{(b_2)}\|_{L^1} \gg \delta_b \delta_{b_2}.
\]

(ii) Using Plancherel’s equality, Hölder’s inequality, and together with Lemma 3, we can obtain
\[
\|f_1^{(b_1)} \ast f_2^{(b_2)}\|_{L^2}^2 = \left\| \frac{\hat{f}_1^{(b_1)}}{\lambda} \cdot \frac{\hat{f}_2^{(b_2)}}{\lambda} \right\|_{L^1}^2 = \left\| \frac{\hat{f}_1^{(b_1)}}{\lambda} \cdot \frac{\hat{f}_2^{(b_2)}}{\lambda} \right\|_{L^1}^2 \leq \left\| \frac{\hat{f}_1^{(b_1)}}{\lambda} \right\|_{L^\infty} \cdot \left\| \frac{\hat{f}_2^{(b_2)}}{\lambda} \right\|_{L^1} \leq \left\| \hat{f}_1^{(b_1)} \right\|_{L^\infty} \cdot \left\| \hat{f}_2^{(b_2)} \right\|_{L^1} \leq \left\| \hat{f}_1^{(b_1)} \right\|_{L^\infty} \cdot \left\| \hat{f}_2^{(b_2)} \right\|_{L^1} \ll (\delta + \varepsilon \cdot (1 + O(t^{-1/2}))^{2-\lambda}.
\]

(iii) With the Young inequality and Lemma 3, it follows that
\[
\|f_1^{(b_1)} \ast f_2^{(b_2)}\|_{L^\infty} \leq \|f_1^{(b_1)}\|_{L^1} \cdot \|f_2^{(b_2)}\|_{L^\infty} \ll \delta_b(1 + O(t^{-1/2}) + O(\varepsilon^{-c \lambda \delta^{-2+\lambda} t^{-1/2}})).
\]

Similarly, we have
\[
\|f_1^{(b_1)} \ast f_2^{(b_2)}\|_{L^\infty} \ll \delta_{b_2}(1 + O(t^{-1/2}) + O(\varepsilon^{-c \lambda \delta^{-2+\lambda} t^{-1/2}})).
\]

For any \( b_1, b_2 \in \Phi_W \), define \( T^{(b_1, b_2)} = \{x \in \mathbb{Z}_N : f^{(b_1)} \ast f^{(b_2)}(x) > 0\} \).

**Proposition 1.** For any \( b_1, b_2 \in \Phi_W \), we have
\[
|T^{(b_1, b_2)}| \gg (\delta_{b_1} + \delta_{b_2})N,
\]
provided that \( \varepsilon, \delta \ll \min\{\delta_{b_1}, \delta_{b_2}\} \frac{\lambda}{c_2} \) and \( \lambda \gg (\frac{1}{\delta})^{2+\lambda} \log \frac{1}{\varepsilon} \).

**Proof:** Without loss of generality, we suppose that \( \delta_{b_1} \leq \delta_{b_2} \). And suppose \( \|f_1^{(b_1)} \ast f_2^{(b_2)}\|_{L^1} = c_2 \delta_{b_1} \delta_{b_2} \) with \( 1 \ll c_2 \ll 1 \) (Recalling Lemma 4(i)). Let
\[
T_{1,1} = \{x \in \mathbb{Z}_N : f_1^{(b_1)} \ast f_1^{(b_2)}(x) > c_2 \delta_{b_1} \delta_{b_2}/2\},
\]
and
\[
T_{i,j} = \{x \in \mathbb{Z}_N : |f_i^{(b_1)} \ast f_j^{(b_2)}(x)| > c_2 \delta_{b_1} \delta_{b_2}/20\}.
\]
for \((i, j) = (1, 2), (2, 1)\) or \((2, 2)\). Then
\[
T^{(b_1, b_2)} \supseteq T_{1,1} \cap (T_{1,2} \cup T_{2,1} \cup T_{2,2})^c.
\]
So
\[
|T^{(b_1, b_2)}| \geq |T_{1,1}| - |T_{1,2}| - |T_{2,1}| - |T_{2,2}|.
\]
Combining Lemma 4, we can get
\[
c_2 \delta_b \delta_{b_2} = \|f_{i}^{(b_1)} * f_{j}^{(b_2)}\|_{L^1} = \|f_{i}^{(b_1)} * f_{1}^{(b_2)}(x)\|
= \frac{1}{N} \left(|T_{1,1}| \cdot \delta_{b}(1 + O(t^{-1/2}) + O(\varepsilon^{-c_2 \delta_{b}^{-2-\lambda}} t^{-1/2})) + N \cdot c_2 \delta_b \delta_{b_2}/2\right).
\]
We conclude that
\[
|T_{1,1}| \geq \delta_2 (1 + O(t^{-1/2}) + O(\varepsilon^{-c_2 \delta_{b}^{-2-\lambda}} t^{-1/2}))^{-1} N.
\]
For \((i, j) = (1, 2), (2, 1)\) or \((2, 2)\), it appears that
\[
|T_{i,j}| \ll (\delta + \varepsilon \cdot (1 + O(t^{-1/2}))) ^{-2-\lambda} \|f_{i}^{(b_1)} * f_{j}^{(b_2)}\|_{L^2} \geq \frac{1}{N} |T_{i,j}| \cdot (c_2 \delta_b \delta_{b_2}/20)^2.
\]
Choose \(\varepsilon = \delta \ll \min\{\delta_b, \delta_{b_2}\} \frac{\theta_b}{\sqrt{n}}, t \gg 1\) and \(\log t \gg \lambda (\frac{1}{2})^{-2-\lambda} \log \frac{1}{\varepsilon}\) such that
\[
|T_{1,1}| > 4 \max\{|T_{1,2}|, |T_{2,1}|, |T_{2,2}|\},
\]
then Proposition 1 follows from (6). (7) and (8).

\[
\square
\]

4. Sums of Subsets of Chen Primes

Noting that \(A \subseteq \mathcal{P}_c \cap [1, n]\) with \(|A| \geq \alpha |\mathcal{P}_c \cap [1, n]|\) and \(A_n = A \cap (\sqrt{n}, n]\). By Theorem 0, we can assert that
\[
|A_n| \geq \frac{\alpha c_1 n}{2 \log^2 n}.
\]
To pick out the \(A_n^{(b)}\)'s with ‘many’ elements, define
\[
G := \left\{ b \in \Phi_W : \delta_{b} \geq \frac{\alpha}{4} \right\}.
\]
Recall (5), i.e.
\[
|A_n^{(b)}| \ll \frac{n}{\log^2 n \cdot \varphi_W}.
\]
Since
\[
\frac{\alpha c_1 n}{2 \log^2 n} \leq \sum_{b \in \Phi_W} |A_n^{(b)}| \leq |G| O(1) \frac{n}{\log^2 n \cdot \varphi_W} + \varphi_W \cdot \frac{\alpha}{4} \cdot \frac{c_1 n}{\log^2 n \cdot \varphi_W},
\]
we conclude that
\[
|G| \gg \alpha \varphi_W.
\]

Proposition 1 tells that
\[
|A_n^{(b_1)} + A_n^{(b_2)}| = |A_N^{(b_1)} + A_N^{(b_2)}| \geq |T^{(b_1, b_2)}| \gg (\delta_{b_1} + \delta_{b_2}) N \gg (\delta_{b_1} + \delta_{b_2}) n/W,
\]
provided that we set \(\varepsilon = \delta = c_3 \alpha \frac{\hat{\varphi}_b}{\sqrt{n}}\) and \(\log t = c_4 c_3 \alpha^{-\frac{5(2+\lambda)}{2\lambda}} \log \alpha^{-1}\).

Denote \(\Delta_x = \max_{(b_1, b_2) \in G \times G} (\delta_{b_1} + \delta_{b_2})\). we have
\[
|A_n + A_n| \geq \sum_{x \in G + G} \max_{(b_1, b_2) \in G \times G} |A_n^{(b_1)} + A_n^{(b_2)}| \gg \sum_{x \in G + G} \Delta_x \cdot n/W.
\]
By (9),
\[
\sum_{b \in \Phi_W} \delta_{b} \geq \alpha \varphi_W/2.
\]
Then we have
\[ \sum_{b \in G} \delta_b \geq \alpha \varphi_W / 4. \]

\[ \sum_{(b_1, b_2) \in G \times G} (\delta_{b_1} + \delta_{b_2}) \gg \alpha \varphi_W |G|. \]

For \( B \subseteq \mathbb{Z}_W \) and \( x \in \mathbb{Z}_W \), denote \( r_B(x) = \# \{(b_1, b_2) \in G \times G : b_1 + b_2 = x \} \).

**Lemma 5.** Suppose \( W \in \mathbb{Z}^+ \) is a sufficiently large squarefree integer. Let \( \alpha > 0 \) and \( k \) sufficiently large. And let \( B \subseteq \Phi_W \) satisfy \( |B| \geq \alpha \varphi_W W^{k-1} \).

Then
\[ \sum_{x \in G + G} r_B(x)^k \leq e^{\tilde{C} k^3 \log k} |G|^k \frac{\varphi_W^k}{W^{k-1}}. \]

for some absolute constant \( \tilde{C} > 0 \).

This lemma is an analog of Proposition 14 of Chipeniuk and Hamel\[2\]. It can be extended to any subsets of ‘sieve-type’ without much modification. We put the long proof in the appendix.

By Lemma 5, we conclude
\[ \sum_{x \in G + G} r_B(x)^k \leq e^{O(k^3 \log k)} |G|^k \frac{\varphi_W^k}{W^{k-1}}. \]

By Hölder’s inequality,
\[ \alpha \varphi_W |G| \ll \sum_{x \in G + G} r_G(x) \left( \sum_{x \in G + G} (\delta_{b_1} + \delta_{b_2}) \right) \]
\[ \leq \sum_{x \in G + G} r_G(x) \cdot \Delta_x \]
\[ \leq \left( \sum_{x \in G + G} r_G(x)^k \right)^{1/k} \left( \sum_{x \in G + G} \Delta_x^{k/(k-1)} \right)^{(k-1)/k} \]
\[ \ll \frac{e^{O(k^3 \log k)} |G|^k \varphi_W^k}{\alpha^{2/k} W^{(k-1)/k}} \left( \sum_{x \in G + G} \Delta_x^{k/(k-1)} \right)^{(k-1)/k}. \]

Noting that \( \Delta_x \leq 2 \). Calculation shows that
\[ \sum_{x \in G + G} \Delta_x \gg \sum_{x \in G + G} \Delta_x^{k/(k-1)} \]
\[ \gg \alpha^{1 + \frac{3}{k-1}} e^{-O(\frac{k^3 \log k}{k-1})} W. \]

(11)

Combining (10) and (11), yields
\[ |A_n + A_n| \gg \alpha e^{-O(\frac{k^3 \log k}{k-1}) + \frac{3 \log \alpha}{k-1} n}. \]

If \( \alpha \) is small enough, it can be deduced that
\[ |A + A| \geq |A_n + A_n| \gg \alpha e^{-O((\log \alpha^{-1})^{2/3} (\log \log \alpha^{-1})^{1/3}) n} \]

by taking \( k = \lceil (\log \alpha^{-1})^{-1} \rceil \). For \( \alpha \) is not small, Theorem 2 can also follow by partition \( B \) into the union of smaller subsets such that the above argument can be applied. (See \[2\])

\[ \square \]
5. Appendix: Addition in $\Phi_W$

Proof of Lemma 5: Let

$$R(x) := \# \{ (b, r) \in B \times \Phi_W : b + r = x \}$$

$$= \# \{ b \in B : (b - x)(b - x + 2) \not\equiv 0 \pmod{p} \text{ for all } p | W \}.$$ 

Put

$$X_d := \{ x \in [0, W - 1] : (x, W) = d \}$$

$$= \{ x \in [0, W - 1] : x = dl \text{ for some } l \in [0, W/d - 1] \text{ with } (l, W/d) = 1 \}.$$ 

We have

$$S = \sum_{x \in \mathbb{Z}_W} r_B(x)^k$$

$$\leq \sum_{x \in \mathbb{Z}_W} R(x)^k$$

$$= \sum_{d | W} \sum_{x \in X_d} R(x)^k$$

$$= \sum_{d | W} \sum_{x \in X_d} \# \{ b \in B : (b - x)(b - x + 2) \not\equiv 0 \pmod{p} \text{ for all } p | W/d \}^k$$

$$\leq \sum_{d | W} \sum_{b_1, \ldots, b_k \in B} \# \{ l \in [0, W/d - 1] : (l, W/d) = 1, \}$$

$$(b_i d^{-1} - l)(b_i d^{-1} - l + 2d^{-1}) \not\equiv 0 \pmod{p} \text{ for all } p | W/d \text{ and } 1 \leq i \leq k \}$$

$$\leq \sum_{d | W} \sum_{b_1, \ldots, b_k \in B} \# \{ l \in [0, W/d - 1] : l \not\equiv 0 \text{ for all } p | W/d, \}$$

$$(b_i d^{-1} - l)(b_i d^{-1} - l + 2d^{-1}) \not\equiv 0 \pmod{p} \text{ for all } p | W/d \text{ and } 1 \leq i \leq k \}$$

$$= \sum_{d | W} \sum_{b_1, \ldots, b_k \in B} \prod_{p | W/d} (p - r_p(b_1, \ldots, b_k) - 1)$$

$$= \sum_{d | W} \sum_{b_1, \ldots, b_k \in B} \frac{W}{d} \prod_{p | W/d} \left( 1 - \frac{r_p(b_1, \ldots, b_k) + 1}{p} \right),$$

where

$$r_p(b_1, \ldots, b_k) = \# \{ s \in [0, p - 1] : (b_i d^{-1} - s)(b_i d^{-1} - s + 2d^{-1}) \equiv 0 \pmod{p} \text{ for some } 1 \leq i \leq k \}.$$ 

Now we fix a $d | W$.

Claim. For $(b_1, \ldots, b_k) \in B^k$, let

$$f(b_1, \ldots, b_k) = \sum_{r_p(b_1, \ldots, b_k) \leq 2^{k-1}} \frac{1}{p},$$

and

$$K = \{ (b_1, \ldots, b_k) \in B^k : f(b_1, \ldots, b_k) \geq \beta \}.$$ 

Then there exists an absolute constant $c > 0$ such that

$$|K| \leq k^2 2^{-\exp(\beta/c^2)} |B|^{k-2} \varphi_2^{-\beta},$$

holds uniformly for every $\beta > 0.$
Proof: Given \((b_1, \ldots, b_k) \in K\), we have

\[
\beta \leq \sum_{p | W, r_p(b_1, \ldots, b_k) \leq 2k-1} \frac{1}{p}
\]

\[
\leq \sum_{p | W, p| (b_i - b_j)(b_i - b_j - 2) \text{ for some } i \neq j} \frac{1}{p}
\]

\[
\leq \sum_{\{i, j\} \subseteq \{1, \ldots, k\}, i \neq j} \frac{1}{p | (b_i - b_j)(b_i - b_j - 2)}
\]

By the pigeon hole principle, there exists some pair \(\{i, j\}\) with \(i \neq j\) such that

\[
\frac{\beta}{k(k-1)/2} \leq \sum_{p | W, p| (b_i-b_j)(b_i-b_j-2)} \frac{1}{p}.
\]

Since each \((b_1, \ldots, b_k) \in K\) contribute at least one such pair \(\{i, j\}\) and a given \(\{b_i, b_j\}\) comes from at most \(k^2 |B|^{k-2} k\)-tuples, hence we can conclude

\[
\frac{|K|}{k^2 |B|^{k-2} 2^l \beta^l} \leq \left( \sum_{\{b, c\} \subseteq B, b \neq c} \left( \sum_{p | W, p| (b-c)(b-c-2)} \frac{1}{p} \right)^l \right)^{-1}.
\]

Furthermore, we have

\[
\sum_{\{b, c\} \subseteq B, b \neq c} \left( \sum_{p | W, p| (b-c)(b-c-2)} \frac{1}{p} \right)^l
\]

\[
\leq \sum_{\{b, c\} \in B^2} \left( \sum_{p | W, p| (b-c)(b-c-2)} \frac{1}{p} \right)^l
\]

\[
= \sum_{p_1, \ldots, p_l | W} \frac{1}{p_1 \cdots p_l} \sum_{\{b, c\} \in B^2, (b-c)(b-c-2) \equiv 0 (\text{mod} \ lcm[p_1, \ldots, p_l])} 1
\]

Write \(p_0 = \max_{1 \leq i \leq l} p_i\) for fixed \(p_1, \ldots, p_l\), one can deduce that

\[
\sum_{\{b, c\} \in B^2, (b-c)(b-c-2) \equiv 0 (\text{mod} \ lcm[p_1, \ldots, p_l])} 1 \leq \sum_{b \in \Phi_W} 1 \leq \sum_{c \in \Phi_W, (b-c)(b-c-2) \equiv 0 (\text{mod} \ p_0)} 1
\]

\[
\leq 2 \sum_{b \in \Phi_W} \max_{a \in Z_W} \sum_{c \in \Phi_W, c \equiv a (\text{mod} \ p_0)} 1 \leq 2 \varphi_W \max_{a \in Z_W} \sum_{c \equiv a (\text{mod} \ p_0), c \neq 0, -2 (\text{mod} \ p)} 1
\]

\[
\leq 2 \varphi_W \varphi_W / p_0 - 2 \leq \frac{2 \varphi_W^2}{p_0 - 2} \leq \frac{2 \varphi_W^2}{p_0 - 2} = \frac{2 \varphi_W^2}{p_0 - 2}.
\]
Then

\[
\sum_{\{b, c\} \subseteq B \atop b \neq c} \left( \sum_{p \mid W} \frac{1}{p} \right)^l \leq 2 \varphi_W^2 \sum_{p_1, \ldots, p_l \mid W} \frac{1}{\prod_{1 \leq i \leq l} p_i (p_i - 2)^{1/l}}
\]

\[
= 2 \varphi_W^2 \left( \sum_{p \mid W} \frac{1}{p (p - 2)^{1/l}} \right)^l
\]

\[
\leq 2 \varphi_W^2 \left( \sum_{p \leq 5k} \frac{1}{p} + \sum_{n \geq 5k} \frac{1}{n (n - 2)^{1/l}} \right)^l
\]

\[
\leq \varphi_W^2 (c \log l)^l
\]

for some absolute constant \( c > 0 \). Combining (12) and the above formula, gives

\[
|K| \leq 2^{-l} \beta^{-l} k^{l+2} (k - 1)^l l^l (\log l)^l |B|^{k-2} \varphi_W^2.
\]

The Claim follows by taking \( l = \exp(\beta/ck^2) \).

\[
\square
\]

Writing

\[
W_1 = \prod_{p \mid W \atop p \leq 5k} p, W_2 = \prod_{p \mid W \atop p > 5k} p.
\]

\[
d_1 = \prod_{p \mid d \atop p \leq 5k} p, d_2 = \prod_{p \mid d \atop p > 5k} p.
\]

The estimate below will be useful later.

\[
\log \left( \prod_{p \mid W_2/d_2 \atop r_p(b_1, \ldots, b_k) \leq 2k-1} \left( 1 - \frac{2k+1}{p} \right)^{-1} \right)
\]

\[
= - \sum_{p \mid W_2/d_2 \atop r_p(b_1, \ldots, b_k) \leq 2k-1} \log \left( 1 - \frac{2k+1}{p} \right)
\]

\[
= \sum_{p \mid W_2/d_2 \atop r_p(b_1, \ldots, b_k) \leq 2k-1} \frac{2k+1}{p} \sum_{t=0}^\infty \frac{1}{t+1} \left( \frac{2k+1}{p} \right)^t
\]

\[
\leq 2(2k+1) \sum_{p \mid W_2/d_2 \atop r_p(b_1, \ldots, b_k) \leq 2k-1} \frac{1}{p}
\]

\[
\leq 2(2k+1) f(b_1, \ldots, b_k).
\]
The last step is resulted from the fact that \( \frac{1}{t+1} \left( \frac{2k+1}{p} \right)^t \leq \frac{1}{2^t} \) for \( t \geq 1 \) and \( p > 5k \). Noting that \( r_p(b_1, \ldots, b_k) \geq 1 \). We continue to estimate \( S \).

\[
S \leq \sum_{d|W} \sum_{b_1, \ldots, b_k \in B} \frac{W}{d} \prod_{p|W/d} \left( 1 - \frac{r_p(b_1, \ldots, b_k) + 1}{p} \right)
\]

\[
\leq \sum_{d|W} \sum_{b_1, \ldots, b_k \in B} \frac{W_1}{d_1} \prod_{p|W_1/d_1} \left( 1 - \frac{2}{p} \right) \frac{W_2}{d_2} \prod_{p|W_2/d_2} \left( 1 - \frac{r_p(b_1, \ldots, b_k) + 1}{p} \right)
\]

\[
= \sum_{d|W} \varphi_{W_1/d_1} \prod_{b_1, \ldots, b_k \in B} \frac{W_2}{d_2} \prod_{p|W_2/d_2} \left( 1 - \frac{r_p(b_1, \ldots, b_k) + 1}{p} \right)
\]

\[
\leq \sum_{d|W} \varphi_{W_1/d_1} \prod_{p|W_2/d_2} \left( 1 - \frac{2k+1}{p} \right) \sum_{j=-\infty}^{\infty} \sum_{b_1, \ldots, b_k \in B, f(b_1, \ldots, b_k) \in \{2^j, 2^{j+1}\}} e^{(2k+1)2^{j+2}}
\]

\[
\leq \sum_{d|W} k^2 |B|^{k-2} \varphi_W \varphi_{W_1/d_1} \prod_{p|W_2/d_2} \left( 1 - \frac{2k+1}{p} \right) \sum_{j=0}^{\infty} 2^{-\exp(2j/c^2)} e^{(2k+1)2^{j+2}}
\]

\[
= C_k k^2 |B|^{k-2} \varphi_W \varphi_{W_1/d_1} \prod_{p|W_2/d_2} \left( 1 - \frac{2k+1}{p} \right),
\]

where

\[
C_k = \sum_{j=0}^{\infty} 2^{-\exp(2j/c^2)} e^{(2k+1)2^{j+2}}
\]

\[
\leq \sum_{j=0}^{\infty} e^{(2k+1)2^{j+2} - \exp(2j/c^2) \log 2}
\]

\[
\leq \sum_{j=0}^{\infty} e^{(2k+1)2^{j+2} - \log 2 \left( \frac{2^j}{c^2} + \frac{2^j}{c^2} \right)}.
\]

The range of summation in \( j \) over \((-\infty, 0)\) can be omitted since \( f(b_1, \ldots, b_k) \geq \sum_{p \leq 2k} \gg \log \log k > 1 \) for sufficiently large \( k \). Now for \( j \geq j_1 := \frac{\log(8c^2k^4(2k+1)/\log 2)}{\log 2} \), we have \( \frac{2^j \log 2}{c^2} \geq (2k + 1)2^{j+2} \) and \( \frac{2^j}{c^2} \geq 2^{j+2} \) \( \geq j \). Then

\[
\sum_{j \geq j_1} e^{(2k+1)2^{j+2} - \log 2 \left( \frac{2^j}{c^2} + \frac{2^j}{c^2} \right)} \leq \sum_{j \geq j_1} e^{-\log 2 \frac{2^j}{c^2}} \leq \sum_{j \geq j_1} 2^{-j} \leq 1.
\]

For \( j \leq j_1 \), the exponent \( (2k + 1)2^{j+2} - \exp(2j/c^2) \log 2 \) is maximized when

\[
2^j = c^2k^2 \log(4(2k + 1)c^2/\log 2)
\]

and

\[
\sum_{j=0}^{j_1} e^{(2k+1)2^{j+2} - \exp(2j/c^2) \log 2}
\]

\[
\leq \left( \frac{\log(8c^2k^4(2k+1)/\log 2)}{\log 2} \right) \cdot e^{2(2k+1)c^2 \log(2\log(2(2k+1)c^2)/\log 2)}
\]

Now we get

\[
C_k \leq e^{c_5 k^3 \log k}
\]

for some absolute constant \( c_5 > 0 \).
Now we turn back to $S$. The number of divisors $d$ of $W$ which gives the same $d_2$ is smaller than $\sum_{t=0}^{5k} \binom{5k}{t} = 2^{5k}$. And recall $|B| = \alpha \varphi_W$. Hence

$$S \leq C_k k^2 |B|^{-\alpha - 2} \varphi_W \sum_{d_1 | W} \varphi_W / d_1 \prod_{p | W} \left( 1 - \frac{2k+1}{p} \right) \prod_{d_2 | W} \left( 1 - \frac{2k+1}{p} \right)$$

$$\leq C_k^2 \alpha^{-2} |B|^{-2} \varphi_W W \sum_{d_2 | W} 1/d_2 \prod_{p | W} \left( 1 - \frac{2k+1}{p} \right)$$

$$\leq C_k^2 \alpha^{-2} |B|^{-2} \varphi_W W \sum_{d_2 | W} 1/d_2 \prod_{p | W} \left( 1 - \frac{2k+1}{p} \right)^{-1}$$

$$\leq C_k^2 \alpha^{-2} |B|^{-2} \varphi_W W \prod_{p | W} \left( 1 - \frac{2k+1}{p} \right) \left( 1 + \frac{1}{p - 2k - 1} \right)$$

$$\leq C_k^2 \alpha^{-2} |B|^{-2} \varphi_W W \prod_{p | W} \left( 1 - \frac{2}{p} \right)^{-2}$$

$$= C_k^2 \alpha^{-2} |B|^{-2} \varphi_W W \prod_{p | W} \left( 1 - \frac{2}{p} \right)^{-k}$$

$$\leq C_k^3 \alpha^{-2} |B|^{-2} \varphi_W W \prod_{p | W} \left( 1 - \frac{2}{p} \right)^{-k}$$

$$\leq \frac{e^{c_k k \log k} |B|^{k \varphi_W k}}{\alpha^2 W^{k-1}}.$$

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