FINITE STATE MEAN FIELD GAMES WITH WRIGHT–FISHER COMMON NOISE

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ABSTRACT. We restore uniqueness in finite state mean field games by adding a Wright–Fisher common noise. We achieve this by analyzing the master equation of this game, which is a second-order partial differential equation whose stochastic characteristics are the stochastic forward-backward system that describes the fixed-point condition of the mean field game; see [8]. We show that this equation, which is a non-linear version of the Kimura type equation studied in [26], has a unique smooth solution. Among others, this requires a priori estimates of Hölder type for Kimura operators with merely continuous drift terms.

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1. INTRODUCTION

Motivated by restoration of uniqueness in the theory of mean field games (MFGs), a more complete account of which we provide below, we analyze here a system of parabolic partial

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differential equations (PDEs) with the main feature of being set on the space of probability measures on \([d] := \{1, \ldots, d\}\), the latter being referred to as the \((d-1)\)-dimensional simplex, for a fixed integer \(d \geq 1\). This system is indexed by the elements \(i\) of \([d]\) itself and has the following generic form:

\[
\begin{align*}
\partial_t U^i(t,p) &- \frac{1}{2} \sum_{j=1}^d (U^i - U^j)^2_+ + f^i(t,p) + \sum_{j=1}^d \varphi(p_j)[U^j - U^i] \\
+ &\sum_{1\leq j,k \leq d} p_k \left[\varphi(p_j) + (U^k - U^j)_+ \right] \left(\partial_{p_j} U^i - \partial_{p_k} U^i\right) \\
+ &\varepsilon^2 \sum_{j=1}^d p_j \left(\partial_{p_j} U^i - \partial_{p_j} U^i\right) + \frac{\varepsilon^2}{2} \sum_{1\leq j,k \leq d} (p_j \delta_{jk} - p_j p_k) \partial_{p_j p_k} U^i = 0,
\end{align*}
\]

(1.1)

for \(i \in [d]\), where \((t,p) \in [0,T] \times \mathcal{S}_{d-1}\) and \(\mathcal{S}_{d-1}\) is the \((d-1)\)-dimensional simplex. Whenever \(\varepsilon\) is equal to zero and \(\varphi\) is also identically equal to zero, this system is the so-called master equation that describes the values of the equilibria in a (finite state) MFG driven by a simple continuous time Markov decision process on \([d]\) and by the functions \((f^i)_{i \in [d]}\) and \((g^i)_{i \in [d]}\) as respective running and terminal costs. To wit, the first line in (1.1), which has a form similar to the concept of “restoration of uniqueness”. This terminology comes from the fact that equation (1.1) is associated with a new form of MFG, which we are going to describe later, in which equilibria are no longer deterministic but are subjected to a so-called common noise and are hence randomized. Under the action of the common noise, the master equation becomes a system of second order PDEs, the principal part of which is the second-order operator in the simplex (see [26, 42]). Accordingly, the master equation here reads as a system of non-linear parabolic PDEs of Kimura type. As for the additional function \(\varphi\) in the first two lines of (1.1), it should be understood as a forcing term in the dynamics of the equilibria that allow the latter to escape for free from the boundary of the simplex. In this context, one of our contributions (see Theorem 3.8) is to show that, when \(\varepsilon\) is strictly positive and the functions \((f^i)_{i \in [d]}\) and \((g^i)_{i \in [d]}\) satisfy some smoothness conditions, we can choose \(\varphi\) large enough in the neighborhood of the boundary of \(\mathcal{S}_{d-1}\) and null everywhere else in such a way that (1.1) has a unique smooth solution (in a so-called Wright–Fischer Hölder space of functions that are once differentiable in time and twice in space with a suitable behavior at the boundary of the simplex). Accordingly, our main result is that the corresponding MFG is uniquely solvable for a prescribed initial condition (see Theorem 2.9). Importantly, there are many examples for which the latter is false when \(\varepsilon = 0\), which explains why we refer quite often to the concept of “restoration of uniqueness”.

A general framework to analyze linear Kimura PDEs was introduced by Epstein and Mazzeo in [26] and this framework was extended subsequently in [27, 28, 52, 53]. Generally speaking, the analysis of Kimura PDEs suffers from two main difficulties: (1) the simplex boundary is not smooth, and (2) the PDE degenerates at the boundary. Despite these difficulties, the authors of [26] were able to prove the existence and uniqueness of smooth solutions to linear
Kimura PDEs under enough regularity of the coefficients. However, these results do not apply to \( (1.1) \) because the coefficients therein are time-dependent (Kimura operators are assumed to be time-homogeneous in \([26]\)) and, most of all, because the equation is non-linear. While the additional time dependence can be handled with relative ease (see Lemma 3.5), the non-linearity requires a sophisticated analysis, which, in fact, is the main technical part of this paper. In this respect, the main step in our study is Theorem 3.6 which provides an \( a \) \textit{priori} Hölder estimate to solutions of linear Kimura PDEs when driven by merely continuous drift terms that point inward the simplex in a sufficiently strong manner, whence our need for the additional \( \varphi \) in equation \( (1.1) \). The proof of this \( a \) \textit{priori} estimate uses a tailor-made \textit{coupling by reflection} argument inspired by earlier works on couplings for multidimensional processes (see e.g., \[18\]). However, the coupling by itself, as usually implemented in the literature for proving various types of smoothing effects for diffusion processes, is in fact not enough for our purpose. We indeed pay a price for the degeneracy of the equation at the boundary and, similar to other works on Kimura operators (see for instance \[1\]), we need to perform an induction over the dimension to handle the degeneracy properly; see Proposition 4.6 for the details of the induction property. Once we reach this point, the proof of existence of a solution to \( (1.1) \) is straightforward, provided that \( \varphi \) therein is chosen in a relevant way, and uses Schauder’s fixed point theorem on the proper Wright–Fisher space, as well as Schauder’s estimates derived in \[26\] for the linear equation and Lemma 3.5 mentioned earlier (see Theorem 3.8).

Let us now clarify our technical contribution into the context of MFGs. MFGs were introduced in the seminal works of Lasry and Lions \[45, 46, 47\], and Huang, Malhamé, and Caines \[40, 39\]. Merging intuition from statistical physics and classical game theory, this paradigm provides the asymptotic behavior of many weakly interacting strategic players who are in a Nash equilibrium. Formally, this asymptotic equilibrium is described as the fixed point of a best response map, which sends a given flow of measures to the distribution of a controlled state-dynamics. For recent theoretical developments and applications of this theory, we refer the reader to \[7, 4, 9, 10\] and the references therein. MFGs with (a fixed and) finite number of states were introduced by \[35, 36, 38\]; for a probabilistic approach to finite state MFGs we refer to \[11, 14\].

Typically, MFGs do not admit unique solutions. Two known instances of uniqueness are the small \( T \) case and the so-called monotonous case due to Lasry and Lions, see \[45\], Section 4 for the latter. The thrust of our paper is to establish uniqueness by adding a common noise \( \xi \) that emerges from the limiting behavior of Wright–Fisher population-genetics models. The special structure of the common noise we use leads to stochastic dynamics evolving inside the multidimensional simplex \( \mathcal{S}_{d-1} \) and eventually to the second order form of \( (1.1) \). Besides the restoration of uniqueness result, this is another interest of our work to incorporate population-genetics models into MFGs; to the best of our knowledge, this is a new feature in the field. In this regard, it is worth pointing out that, even though we say very few about it in this paper, we are in fact able to explain the common noise at the level of a particle system by a diffusion approximation. We just give a hint in Section 2.1.2 but we provide a more detailed explanation in the Appendix.

In fact, we must stress that the recent work \[6\] (to which we already alluded in the footnote \[2\]) also addresses a form of common noise for finite state MFGs. As explained therein, the key point in this direction is to force the finite-player system to have many simultaneous jumps at some random times prescribed by the common noise. Although we share a similar idea in our

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1 Most of the time, we just say Wright–Fisher space instead of Wright–Fisher Hölder space.

2 Recently, Bertucci, Lasry, and Lions \[6\] mentioned that “The addition of a common noise in the MFG setting remains one of the most important questions in the MFG theory.” We feel that our paper may be one step forward in this direction.
construction, our common noise structure is in the end different from \([6]\): While the simultaneous jumps in \([6]\) are governed by a deterministic transformation of the state space, they here obey a resampling procedure that is typical, as we have just said, of population-genetics models. Moreover, one of the questions in \([6]\) is to decide whether the solution preserves monotonicity of the coefficients; in this regard, restoration of uniqueness (outside the monotonous setting) is not discussed in \([6]\). In fact, restoration of uniqueness for MFGs was addressed in other works but in different settings. Recently, Delarue \([21]\) established a restoration of uniqueness result for a continuous state MFG obtained by forcing a deterministic (meaning that the players follow ordinary differential equations) MFG by means of a common noise. In this case, the common noise is infinite-dimensional and henceforth differs from the most frequent instance of common noise used in the literature, since the latter has very often a finite dimension, see e.g., \([10]\). At this point, it is worth mentioning that restoration of uniqueness is studied in \([56]\) under the action of a standard finite-dimensional common noise, but for a linear quadratic MFG. This is due to the fact that the equilibrium distribution in that paper is Gaussian and is parametrized by its mean and variance, which reduces the dimension of the problem. On a more prospective level, restoring uniqueness by common noise might enable a selection criterion by taking small noise limit for cases where the limiting problem does not have a unique equilibrium. This question was addressed in some specific cases in \([22]\) for a continuous state model and in \([12]\) for a finite state model (see also \([3]\)) and is the purpose of the forthcoming work \([13]\) in a more general setting (with a finite state space).

Once the master equation (1.1) has been solved, the equilibrium distribution of the mean field game, which becomes random under the action of the common noise, is provided by the setting (with a finite state space). The process \(P_t = (P_t^i)_{i \in \mathbb{D}}\) and is the purpose of the forthcoming work \([13]\) in a more general setting (with a finite state space).

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\begin{equation}
\begin{split}
dP_t^i &= \sum_{j=1}^d \left( P_t^i \left( \varphi(P_t^i) + (u_t^j - u_t^i)_+ \right) - P_t^j \left( \varphi(P_t^j) + (u_t^i - u_t^j)_+ \right) \right) dt \\
&\quad + \frac{\varepsilon}{\sqrt{2}} \sum_{j=1}^d \sqrt{P_t^i P_t^j} d\left( W_t^{i,j} - W_t^{j,i} \right), \\
du_t^i &= -\left( \sum_{j=1}^d \varphi(P_t^j) [u_t^j - u_t^i] - \frac{1}{2} \sum_{j=1}^d (u_t^i - u_t^j)_+^2 + f^i(P_t) \right) dt \\
&\quad - \frac{\varepsilon}{\sqrt{2}} \sum_{j=1}^d \frac{P_t^j}{P_t^i} \left( \varphi^i_{t;j,j} - \varphi^i_{t;j,i} \right) dt + \sum_{1 \leq j \neq k \leq d} \nu_{t,j,k}^i dW_{t,j}^k,
\end{split}
\end{equation}
by the following relationship.
\[ u_t^i = U^i(t, P_t), \quad \text{and} \quad v_t^{i,k} = V^{i,k}(t, P_t), \]

where
\[ V^{i,k}(t, p) = \frac{\varepsilon}{\sqrt{2}} (\partial_{p_j} U^i(t, p) - \partial_{p_k} U^i(t, p)) \sqrt{p_j p_k}. \]

In fact, this relationship is the cornerstone to prove uniqueness of the solution of [14] through a verification argument, see Theorem 3.3. This argument is inspired from the original four-step-scheme in [51]: in the framework of continuous state MFGs, it has already been used in [10, 8, 16]. On a more elaborated level, we should point out that the master equation has been a key tool (e.g., see [8] for continuous state mean field games and [2, 15] for finite state mean field games) to show convergence of the closed loop Nash-equilibrium of the N-player system to the MFG equilibrium. In both [2, 15], there is no common noise and, in all the latter three cases, the Lasry–Lions monotonicity condition is assumed to be force, which is obviously in stark contrast to the setting of the current paper. The convergence problem in our setup is thus an interesting question, which we leave for future work. The interested reader may find a hint in the Appendix.

The rest of the paper is organized as follows. In Section 2 we present the MFG model and provide the main result (Theorem 2.9) that states that the MFG with common noise admits a unique solution. In Section 3 we derive the master equation and connect it with the MFG forward-backward system (1.2). Theorem 3.3 states that if the master equation has a smooth solution then the MFG system admits a unique solution. The existence of a smooth solution to the master equation is established in Theorem 3.8. The proof relies on a priori Hölder regularity result given in Theorem 3.6, whose proof is the most demanding result of the paper and is given in Section 4. The main two ingredients in the proof are the coupling construction provided in Proposition 4.3 and the induction step in Proposition 4.6.

In the rest of this section we provide frequently used notation.

**Notation.** For \( a, b \in \mathbb{R} \), we let \( a \wedge b := \min\{a, b\} \). We use the notation \( M^\dagger \) to denote the transpose of a matrix \( M \). Moreover, we use the generic notation \( p = (p_i)_{i \in [d]} \) (with \( p \) in lower case and \( i \) in subscript) for elements of \( \mathbb{R}^d \), while processes are usually denoted by \( P = (P_t^i)_{i = 1, \ldots, d, 0 \leq t \leq T} \) (with \( P \) in upper case and \( i \) in superscript). For a subset \( A \) of a Euclidean space, we denote by \( \text{Int}(A) \) the interior of \( A \). Also, we recall the notation \( A_{d-1} := \{ (p_1, \ldots, p_d) \in (\mathbb{R}_+)^d : \sum_{i \in [d]} p_i = 1 \} \), where \( [d] := \{1, \ldots, d\} \). We can identify \( A_{d-1} \) with the convex polyhedron of \( \mathbb{R}^{d-1} \): \( \mathcal{S}_{d-1} := \{(x_1, \ldots, x_{d-1}) \in (\mathbb{R}_+)^{d-1} : \sum_{i \in [d-1]} x_i \leq 1\} \). In particular, we sometimes write “the interior” of \( \mathcal{S}_{d-1} \): in such a case, we implicitly consider the interior of \( \mathcal{S}_{d-1} \) as the \((d-1)\)-dimensional interior of \( \mathcal{S}_{d-1} \). Obviously, the interior of \( \mathcal{S}_{d-1} \), when regarded as a subset of \( \mathbb{R}^d \), is empty, which makes it of a little interest. To make it clear, for some \( p \in \mathcal{S}_{d-1} \), we sometimes write \( p \in \text{Int}(\mathcal{S}_{d-1}) \), meaning that \( p_i > 0 \) for any \( i \in [d] \). We use the same convention when speaking about the boundary of \( \mathcal{S}_{d-1} \): For some \( p \in \partial \mathcal{S}_{d-1} \), we may write \( p \in \partial \mathcal{S}_{d-1} \) to say that \( p_i = 0 \) for some \( i \in [d] \). For \( p \in \partial \mathcal{S}_{d-1} \), we write \( \sqrt{p} \) for the vector \((\sqrt{p_1}, \ldots, \sqrt{p_d})\). Finally, \( \delta_{i,j} \) is the Kronecker symbol and \( r_+ \) denotes the positive part of \( r \in \mathbb{R} \).

2. **Main results**

The purpose of this section is to introduce step by step the main results of the paper. As we already accounted for in the introduction, our general objective is to prove that a suitable form of common noise may restore uniqueness of equilibria to mean field games on a finite state space.
2.1. Preliminary version of the mean field game. The first point that we need to clarify is the form of the mean field game itself. Whilst it is absolutely standard when there is no common noise, the mean field game addressed below takes indeed a more intricate and less obvious form in the presence of common noise. In fact, the somewhat non-classical structure that we use throughout the paper is specifically designed in order to be in correspondence with the class of second order differential operators on the simplex, referred to as Kimura operators in the text, for which we can indeed prove the smoothing results announced in introduction, see Subsection 2.3 for a first account.

Clearly, the sharpest way to derive the form of mean field games that is used below would consist in going back to a game with a large but finite number $N$ of players and in justifying that, under the limit $N \to \infty$, this finite game converges in some sense to our form of mean field games. Although we could indeed do so, we feel better to hint about this approach here and to offer a more detailed overview of it in the Appendix. Accordingly, we directly write down our version of mean field game with a common noise on a finite state space. Our rationale for doing so is that it allows the reader to jump quickly into the article. If she or he is interested, she or he may have a look at the supplementary material available online.

2.1.1. Mean field game without a common noise. When there is no common noise, our form of mean field game is directly taken from the earlier work \cite{35,36}. In short, a given tagged player evolves according to the discrete Fokker–Planck (or Kolmogorov) equation:

$$\frac{d}{dt} P_{t,i} = \sum_{j \neq i} \beta_{t,i,j} P_{t,j}, \quad i \in [d],$$

(2.1)

For the initial statistical state $(P_{0,i})_{i \in [d]}$ being prescribed as an element of $\mathcal{S}_{d-1}$. In words, $P_{t,i}$ is the probability that the tagged player be in state $i$ at time $t$.

With the tagged player, we assign a cost functional depending on a deterministic time-measurable $\mathcal{S}_{d-1}$-valued path $(P_t)_{0 \leq t \leq T}$, referred to as an environment and starting from the same initial state as $(Q_t)_{0 \leq t \leq T}$, namely $Q_{0,i}^t = P_{0,i}^t$ for $i \in [d]$. Intuitively, $P_t$ is understood as the statistical state at time $t$ of all the other players in the continuum, which are basically assumed to be independent and identically distributed. Given $(P_t)_{0 \leq t \leq T}$, the cost to the tagged player is written in the form

$$\mathcal{J}((\beta_t)_{0 \leq t \leq T}, (P_t)_{0 \leq t \leq T}) := \sum_{i \in [d]} \left[ Q_T g(i, P_T) + \int_0^T Q_t \left( f(t,i, P_t) + \frac{1}{2} \sum_{j \neq i} |\beta_{t,i,j}|^2 \right) dt \right],$$

(2.3)

where $g$ is a function from $[d] \times \mathcal{S}_{d-1}$ into $\mathbb{R}$ and $f$ is a function from $[0,T] \times [d] \times \mathcal{S}_{d-1}$ into $\mathbb{R}$. To simplify the notations, we will sometimes write $\beta$ for $(\beta_t)_{0 \leq t \leq T}$ and $P$ for $(P_t)_{0 \leq t \leq T}$. Accordingly, we will write $\mathcal{J}(\beta,P)$ for the cost to the tagged player.

In this setting, a mean field equilibrium is a path $P = (P_t)_{0 \leq t \leq T}$ as before for which we can find an optimal control $(\beta^*_t)_{0 \leq t \leq T}$ to $\mathcal{J}(-,P)$ such that the corresponding solution to (2.2) is $(P_t)_{0 \leq t \leq T}$ itself. We stress the fact that here $P$ and $Q$ are deterministic paths.
2.1.2. Stochastic Fokker–Planck equation. We now introduce a special form of common noise in order to force the equilibria to satisfy a relevant form of diffusion processes with values in the simplex $\mathcal{S}_{d-1}$. To make it clear, our aim is to force equilibria to satisfy the following stochastic variant of equation (2.2):

$$
\mathrm{d}P_t^i = \sum_{j \in [d]} P_{t}^{i,j} \alpha_{t}^{i,j} \mathrm{d}t + \frac{\varepsilon}{\sqrt{2}} \sum_{j \in [d]} \sqrt{P_{t}^{i,j}} \sqrt{\sum_{l \in [d]} P_{t}^{l,j}} \mathrm{d}[W_{t}^{i,j} - W_{t}^{i,j}],
$$

(2.4)

for $t \in [0,T]$, where $((W_{t}^{i,j})_{0 \leq t \leq T})_{i,j \in [d]: i \neq j}$ is a collection of independent $1d$ Brownian motions, referred to as the common noise, and $\alpha = (\alpha_{t})_{0 \leq t \leq T}$ is a progressively measurable process (with respect to the augmented filtration $\mathbb{F}^{W} = (\mathcal{F}_{t}^{W})_{0 \leq t \leq T}$ generated by $W = ((W_{t}^{i,j})_{i,j \in [d]: i \neq j})_{0 \leq t \leq T}$) satisfying (2.1). Throughout, we use the convention $W^{i,i} = (W_{t}^{i,i})_{0 \leq t \leq T} \equiv 0$, for any $i \in [d]$. Above, the parameter $\varepsilon$ reads as the intensity of the common noise. Accordingly, the collection $((W_{t}^{i,j} := (W_{t}^{i,j} - W_{t}^{j,i})/\sqrt{2})_{0 \leq t \leq T})_{i,j \in [d]: i \neq j}$ forms an antisymmetric Brownian motion.

Although it looks rather unusual, the form of the stochastic integration in (2.4) is in fact directly inspired by stochastic models of population genetics. To wit, for $i, j \in [d]$, the $(i,j)$-bracket writes (with a somewhat abusive but quite useful notation in the first term in the right-hand side below)

$$
\frac{\mathrm{d}}{\mathrm{d}t} \langle P_{t}^{i}, P_{t}^{j} \rangle_{t} = \left\langle \frac{\varepsilon}{\sqrt{2}} \sum_{k \in [d]} \sqrt{P_{t}^{i,k}} \sqrt{P_{t}^{j,k}} \mathrm{d}[W_{t}^{i,k} - W_{t}^{j,k}], \frac{\varepsilon}{\sqrt{2}} \sum_{l \in [d]} \sqrt{P_{t}^{i,l}} \sqrt{P_{t}^{j,l}} \mathrm{d}[W_{t}^{i,l} - W_{t}^{j,l}] \right\rangle
$$

$$
= \varepsilon^{2} \sum_{k,l \in [d]} \sqrt{P_{t}^{i,k} P_{t}^{j,l}} \left( \delta_{i,j} \delta_{k,l} - \delta_{i,l} \delta_{k,j} \right) = \varepsilon^{2} \left( P_{t}^{i,j} - P_{t}^{j,i} \right).
$$

(2.5)

The last term on the right-hand side is known as being the diffusion matrix of the Wright–Fisher model, see for instance [31, 29, 54]. It is also the leading part of so-called Kimura operators, see Subsection 2.3.

Below, we will be specifically interested in cases when the equilibrium strategies are in feedback form, meaning that $\alpha_{t}^{i,j} = \alpha(t, i, P_{t})(j)$ for a function $\alpha : [0,T] \times [d] \times \mathcal{S}_{d-1} \times \mathbb{R}$ such that, for any $(t, p, j) \mapsto \alpha(t, i, p)(j) \in \mathbb{R}$ such that, for any $(t, p) \in [0,T] \times \mathcal{S}_{d-1}$ and any $i \in [d]$,

$$
\alpha(t, i, p)(j) \geq 0, \quad j \in [d] \setminus \{ i \}, \quad \alpha(t, i, p)(i) = -\sum_{j \neq i} \alpha(t, i, p)(j),
$$

(2.6)

in which case (2.4) becomes a stochastic differential equation, the well-posedness of which is addressed in the next section, at least in a setting that is relevant to us, see Proposition 2.1. The function $\alpha$ is said to be a feedback strategy. One of the key point in the latter statement is that the solution takes values in $\mathcal{S}_{d-1}$ itself. Another key point is that, whenever each $P_{0}^{i}$ is in $(0, +\infty)$ with probability 1 and each $\alpha(t, j, p)(i)$ remains away from zero for $p_{i}$ is in the right neighborhood of 0, the coordinates of the solution are shown to remain almost surely (strictly) positive, which plays a crucial role in the definition of our mean field game with common noise. Below, we ensure strict positivity of the rate transition from $j$ to $i$ for $p_{i}$ small enough by forcing accordingly the dynamics at the boundary of the simplex $\hat{\mathcal{S}}_{d-1}$ of the $(d-1)$-dimensional simplex (where $d$ is the cardinality of the state space) ; we make this point clear in (2.2.1). For the time being, we observe that the strict positivity of the solution (provided that we take it for granted)

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3 We recall the convention introduced in the very beginning of the paper according to which the boundary is here understood as the boundary of $\hat{\mathcal{S}}_{d-1}$ under the identification of $\mathcal{S}_{d-1}$ and $\hat{\mathcal{S}}_{d-1}$ (and similarly for the interior). We take this convention for granted in the rest of the paper.
permits to rewrite the equation \((2.7)\) in the form:

\[
dP^i_t = \sum_{j \in [d]} P^i_j \alpha(t, j, P_t)(i)dt + \varepsilon P^i_t \sum_{j \in [d]} \sqrt{\frac{P^i_j}{P^i_t}} dW^{i,j}_t, \quad t \in [0, T],
\]

where, for consistency, we have replaced \(\alpha^{i,j}_t\) by \(\alpha(t, j, P_t)(i)\).

Now that we have equation \((2.7)\), we can formulate our mean field game. As we already accounted for, the first observation is that the Brownian motions \(((W^{i,j}_t)_{0 \leq t \leq T})_{i,j \in [d], i \neq j}\) in \((2.7)\) should be regarded as common noises (or the whole collection should be regarded as a common noise). The second key point is that equation \((2.7)\) should be understood as the equation for an environment \((P_t)_{0 \leq t \leq T}\), candidate for being a solution of the mean field game. It thus remains to introduce the equation for a tagged player evolving within the environment \((P_t)_{0 \leq t \leq T}\). Our key idea in this respect is to linearize \((2.7)\) in order to describe the statistical marginal states of the tagged player, namely

\[
dQ^i_t = \sum_{j \in [d]} Q^i_j \beta(t, j, P_t)(i)dt + \varepsilon Q^i_t \sum_{j \in [d]} \sqrt{\frac{P^i_j}{P^i_t}} dW^{i,j}_t, \quad t \in [0, T],
\]

where \(\beta\) stands for the feedback function (hence satisfying \((2.6)\)) used by the tagged player to implement her own strategy in the form of a progressively-measurable (with respect to the filtration \(\mathbb{F}^{W^i}\)) process \(\beta = (\beta^{i,j}_t = \beta(t, i, P_t(j)))_{i,j \in [d], 0 \leq t \leq T}\). The main difficulty here is to interpret \((2.8)\) in a convenient manner. Notice in this regard that our choice to take here \((\beta_t)_{0 \leq t \leq T}\) in a closed feedback form (or semi-closed since it depends on the environment \((P_t)_{0 \leq t \leq T}\)) is only for consistency with \((2.7)\) and just plays a little role in our interpretation. Actually, one important fact in this respect is that the variable \(Q^i_t\) in \((2.8)\) should read as a conditional expected mass when the tagged player is in state \(i\) at time \(t\). Here, the reader must be aware of the terminology that we use: We say conditional expected mass instead of conditional probability because, although \(Q^i_t\) below to have non-negative entries, it may not be a probability measure, meaning that \(\sum_{i \in [d]} Q^i_t\) may differ from 1, which is the whole subtlety of our model.

In order to clarify the equation, we could also think of associating a Lagrangian or particle representation with \((2.8)\), but we are not able to do so directly. In fact, as we already mentioned, the most convincing way to give a meaning to \((2.8)\) is to make the connection with a finite-player game. For simplicity, we just give a hint about it below, but the interested reader may find more details in the Appendix. \(^4\) In short, the best we can say is that the pair \((2.7)–(2.8)\) arises as the approximation diffusion of the marginal states of a time discrete Markov chain \((X_n^{1,N}, \ldots, X_n^{N-1,N}, \mathcal{X}_n^{(N)}, Y_n^{(N)})_{n=0,\ldots,\lfloor NT \rfloor}\) with values in \([d]^{N-1} \times [d] \times \mathbb{R}_+\) after an appropriate hyperbolic time change \(t = n/N\), for \(n = 0, \ldots, \lfloor NT \rfloor\), and under the limit \(N \to \infty\). In short, \((X_n^{1,N}, \ldots, X_n^{N-1,N})_{n=0,\ldots,\lfloor NT \rfloor}\) should be understood as an \(N\)-discretization of what we have called so far the continuum formed by the other players. Accordingly, the solution to \((2.7)\) should read as the limit (in law), as \(N\) tends to \(\infty\), of the marginal empirical distributions

\[
\left(\mu_{t}^{N-1} := \frac{1}{N - 1} \sum_{i=1}^{N-1} \delta_{\mathcal{X}_{\lfloor nt \rfloor}^{i,N}}\right)_{0 \leq t \leq T}.
\]

In contrast, the state of the approximating tagged particle at time \(n \in \{0, \ldots, \lfloor NT \rfloor\}\) is represented by both \(\mathcal{X}_n^{(N)}\) and \(Y_n^{(N)}\): While \(\mathcal{X}_n^{(N)}\) obviously stands for the site occupied by the tagged

\(^4\)The reader may also skip ahead since the rest of the paper does not make any real use of this interpretation of the game.
particle in $[d]$ at time $n$, $Y_{n}^{(N)}$ carries an extra information in the form of a (non-negative) mass. There are two key features about this mass:

(i) The first one is that the masses of all the other players in the continuum are exactly 1 (which explains why those masses do not manifest in (2.7));

(ii) The second one is that the expectation of the mass (of the tagged player) with respect to all the noises supporting the discrete model is 1.

The second item is quite subtle since our model carries two sources of randomness. Obviously, a first source of randomness is the so-called common noise, which accounts for a noise that is felt by the whole system; we denote the corresponding expectation by $E$. In addition to the common noise, the tagged particle is also subjected to an idiosyncratic noise; to wit, equation (2.2) itself is implicitly associated with an idiosyncratic noise, which should be understood (in the case when there is no common noise) as the Poisson process governing the evolution of a $[d]$-valued Markov process with $(\beta_{t})_{0 \leq t \leq T}$ as transition rates; the expectation associated with the idiosyncratic noise is denoted by $E^0$. Hence, the second item right above may be reformulated in the form:

$$E E^0 [Y_{[N_t]}^{(N)}] = 1, \quad 0 \leq t \leq T.$$ 

The above says that the mass of the tagged particle is in fact a density on the entire probability space carrying both types of noise: Somehow, this density accounts for the way the tagged player perceives the world; when the tagged player is statistically equal to any of the other players of the continuum, the density becomes equal to 1 (this is a feature of our construction) and we recover the first of the two items right above. In this framework, the solution $(Q_{t})_{0 \leq t \leq T}$ to (2.8) should read as the limit (in law) of the conditional expected masses

$$\left(\left(Q_{N_t}^{(N)} := E E^0 [Y_{[N_t]}^{(N)} 1_{\{X_{[N_t]}^{(N)} = i\}}]\right)_{i \in [d]}\right)_{0 \leq t \leq T}.$$ 

We here recover the fact that the solution to (2.8) may not be normalized. At the same time, we also recover the fact (which is easily checked by taking expectations in (2.8)) that, for any $t \in [0, T]$,

$$E \left[ \sum_{i \in [d]} Q_{t}^{i} \right] = 1.$$ 

This prompts us to say that $Q_{t}^{i}$ stands for the non-normalized expected mass of state $i$ under the perception of the tagged particle, conditional on the realization of the common noise.

2.1.3. Cost functional and first formulation of the game. It now remains to associate a cost functional with the tagged player. Consistently with (2.3) we here let

$$J(\beta, (P_{t})_{0 \leq t \leq T}) := \sum_{i \in [d]} E \left[ Q_{T}^{i} g(i, P_{T}) + \int_{0}^{T} Q_{t}^{i} \left( f(t, i, P_{t}) + \frac{1}{2} \sum_{j \neq i} \left[ |\beta(t, i, P_{t})| (j) \right]^{2} \right) dt \right]. \quad (2.9)$$

In the above left-hand side, $\beta$ stands for the strategy used in the equation for $Q = (Q_{t})_{0 \leq t \leq T}$ in (2.3); also, $P = (P_{t})_{0 \leq t \leq T}$ in the left-hand side denotes the environment (as the cost functional does depend upon the environment), defined as the solution of (2.7).

Hence, for an initial condition $p_{0} = (p_{0,i})_{i \in [d]} \in S_{d-1}$ with positive entries (that is $p_{0,i} > 0$ for each $i \in [d]$), our definition of a mean field game solution comes in the following three steps:

1. Consider a feedback function $\alpha$ as in (2.6) such that (2.7), with $p_{0}$ as initial condition, has a unique solution $(P_{t})_{0 \leq t \leq T}$ (say on a probability space $(\Omega, \mathcal{A}, P)$) equipped with a collection of $1d$ Brownian motions $((W_{t}^{i,j})_{0 \leq t \leq T})_{i,j \in [d], i \neq j}$, with the same convention as
before that $W_{t,i} = (W_{t,i}^1)_{0 \leq t \leq T} \equiv 0$ for $i \in [d]$, which remains positive with probability 1; the process $(P_t)_{0 \leq t \leq T}$ is then called an environment;

(2) On the same space $(\Omega, \mathcal{A}, \mathbb{P})$, solve, for any bounded and measurable feedback function $\beta$, equation (2.8) for $(Q_t)_{0 \leq t \leq T}$ with $p_0$ as initial condition, and then find the optimal trajectories (if they exist) of the minimization problem

$$\inf_{\beta} J(\beta, P)$$

(2.10)

(3) Find an environment $(P_t)_{0 \leq t \leq T}$ such that $(P_t)_{0 \leq t \leq T}$ is an optimal trajectory of (2.10). Such a $(P_t)_{0 \leq t \leq T}$ is called an MFG equilibrium or a solution to the MFG.

The precise definition is given in the next section (Definition 2.5). It is worth noticing that, whenever $\varepsilon = 0$ in (2.7) and (2.8), the system (2.7)–(2.8) becomes a simpler system of two decoupled Fokker–Planck (or Kolmogorov) equations that are similar to (2.2). While existence of a mean field game solution (in the case $\varepsilon = 0$) is now well-understood, uniqueness remains a difficult issue. In fact, there are few generic conditions that ensure uniqueness. Generally speaking, the two known instances of uniqueness are (besides some specific examples that can be treated case by case) the short time horizon case (namely $T$ is small enough in comparison with the regularity properties of the underlying cost coefficients) and the so-called monotonous case due to Lasry and Lions [47, 46] (which does not require $T$ to be small enough). In short, the cost coefficients $f$ and $g$ are said to be monotonous (in the sense of Lasry and Lions) if, for any $p, p' \in S_{d-1}$ and for any $t \in [0, T]$,

$$\sum_{i \in [d]} (g(i, p) - g(i, p')) (p_i - p_i') \geq 0, \quad \sum_{i \in [d]} (f(t, i, p) - f(t, i, p')) (p_i - p_i') \geq 0. \quad (2.11)$$

The main goal of the rest of the paper is precisely to prove that, whenever $\varepsilon$ in (2.7)–(2.8) is strictly positive, uniqueness may hold true for our MFG under quite mild regularity conditions on the coefficients and in particular without requiring any monotonicity properties; in fact, the main constraint that we ask is that the coordinates of the solutions of (2.7) stay sufficiently far away from zero (provided that the coordinates of the initial condition themselves are not zero). We address this requirement in the next subsection: Basically, it will prompt us to introduce a new term in the dynamics of both $(P_t)_{0 \leq t \leq T}$ and $(Q_t)_{0 \leq t \leq T}$ to force the coordinates to stay positive.

2.2. New MFG and first meta-statement. As we already alluded to, an important observation is that, for any solution $((P_t^i)_{i \in [d]})_{0 \leq t \leq T}$ to (2.7), it holds

$$d \left( \sum_{i \in [d]} P_t^i \right) = 0, \quad (2.12)$$

which can be easily proved by summing over the coordinates in (2.7). In particular, since the initial condition is taken in $S_{d-1}$, the mass remains constant, equal to 1. Subsequently, if the coordinates of $(P_t)_{0 \leq t \leq T}$ remain non-negative (which we discuss right below), the process $(P_t)_{0 \leq t \leq T}$ lives in $S_{d-1}$, which is of a special interest for us. In fact, non-negativity of the coordinates may be easily seen by rewriting (2.7) in the form

$$dP_t^i = a_i(t, P_t)dt + \varepsilon \sqrt{P_t^i(1 - P_t^i)} dW_t^i, \quad (2.13)$$

for a new Brownian motion $(\tilde{W}_t^i)_{0 \leq t \leq T}$, where $a_i(t, p) := \sum_{j \in [d]} [p_j \alpha(t, j, p)(i) - p_i \alpha(t, i, p)(j)]$, for $i \in [d]$, and where the form of the stochastic integral follows from (2.5) with $i = j$ therein. (Notice that the form of $a_i$ differs from the writing used in (2.7), but both are obviously equivalent since $\sum_{j \in [d]} \alpha(t, i, p)(j) = 0$.) We have that $a_i(t, p) \geq -Cp_i$, for a constant $C > 0$, since
\( \alpha(t, j, p)(i) \geq 0 \) for \( j \neq i \) and we assume \( \alpha \) bounded. By stochastic comparison with Feller’s (1d) branching diffusion [23 Exercise 5.1], we easily deduce that the coordinates of \( (P_i)_{0 \leq t \leq T} \) should remain non-negative (the details are left to the reader and a rigorous statement, tailored to our framework, is given below).

2.2.1. Equations that are repelled from the boundary. In the sequel, we are interested in solutions to (2.7) that stay sufficiently far away from the boundary. As we already explained several times, the reason is that our restoration of uniqueness result is based upon the smoothing properties of the operator generated by (2.7). Since the latter degenerates at the boundary of the simplex, we want to keep the solutions to (2.7) as long as possible within the relative interior of \( S_{d-1} \). In this regard, it is worth observing from [23 Exercise 5.1] that the sole condition (2.6) is not enough to prevent solutions to (2.7) to touch the boundary of the simplex. To guarantee that no coordinate vanishes, more is needed. For instance, in Feller’s branching diffusions, the solution does not vanish if the drift is sufficiently positive in the neighborhood of 0. This prompts us to revisit the two equations (2.7) and (2.8) and to consider instead (notice that, in the two formulas below, the value of \( \alpha(t, i, P_t)(i) \) is in fact useless)

\[
d P_t^i = \sum_{j \in [d]} \left( P_t^j \left( \varphi(P_t^j) + \alpha(t, j, P_t)(i) \right) - P_t^i \left( \varphi(P_t^i) + \alpha(t, i, P_t)(j) \right) \right) dt + \varepsilon \sum_{j \in [d]} \sqrt{P_t^i P_t^j} d W_{t,j}^i,
\]

(2.14)

and

\[
d Q_t^i = \sum_{j \in [d]} \left( Q_t^j \left( \varphi(P_t^j) + \beta(t, j, P_t)(i) \right) - Q_t^i \left( \varphi(P_t^i) + \beta(t, i, P_t)(j) \right) \right) dt + \varepsilon Q_t^i \sum_{j \in [d]} \sqrt{P_t^i P_t^j} d W_{t,j}^i,
\]

(2.15)

for \( t \in [0, T] \), with the same deterministic initial condition \( P_0 = (P_0^i = p_{0,i})_{i \in [d]} \). Here the function \( \varphi \) is a non-increasing Lipschitz function from \([0, \infty)\) into itself such that

\[
\varphi(r) := \begin{cases} \kappa & r \leq \delta, \\ 0 & r > 2 \delta, \end{cases}
\]

(2.16)

\( \delta \) being a positive parameter whose value next is somewhat arbitrary. As for \( \kappa \), we clarify its role in the statement of Theorem 2.9 right below. In the two equations (2.14) and (2.15), \( \alpha \) and \( \beta \) are the same as in (2.7) and (2.8). Hence, the drift in the first equation now reads

\[
a_t^i(t, p) := \sum_{j \in [d]} \left( p_j \left( \varphi(p_j) + \alpha(t, j, p)(i) \right) - p_i \left( \varphi(p_i) + \alpha(t, i, p)(j) \right) \right).
\]

(2.17)

It still satisfies \( \sum_{i \in [d]} a_t^i(t, p) = 0 \). And, importantly, whenever \( p_i = 0 \) (with \( p = (p_1, \cdots, p_d) \in S_{d-1} \)), it satisfies \( a_t^i(t, p) \geq \kappa \). In this framework, we have the following three statements, the proofs of which are postponed to Subsection 2.4.

**Proposition 2.1.** Consider \( \varphi \) as in (2.16) with \( \delta \in (0, 1) \) and \( \kappa \geq \varepsilon^2 / 2, \) for \( \varepsilon > 0 \). Then, for a bounded (measurable) feedback function \( \alpha \) as in (2.6), the stochastic differential equation (2.14) has a unique (strong) solution whenever the random initial condition is prescribed and satisfies, with probability 1, \( p_{0,i} > 0 \) for each \( i \in [d] \) and \( \sum_{i \in [d]} p_{0,i} = 1 \). Moreover, the coordinates of the solution remain almost surely (strictly) positive and satisfy \( \sum_{i \in [d]} P_t^i = 1 \), for any time.

The following statement provides a stronger version.
Proposition 2.2. Under the assumptions and notation of Proposition 2.1, for \( \kappa \) as in (2.16) and for \( \lambda > 0 \), let \( \gamma := \kappa - (1 + \lambda) \varepsilon^2 / 2 \). Then, the solution to (2.14) satisfies
\[
\mathbb{E} \left[ \exp \left( \lambda \gamma \int_0^T \frac{1}{P_i^t} \, ds \right) \right] \leq C P_{0,i}^{-\lambda}, \quad \text{for each } i \in [d],
\]
(2.18)

and for each \( i \in [d] \),
\[
\sup_{0 \leq t \leq T} \mathbb{E} \left[ (P_i^t)^{-\lambda} \right] \leq C P_{0,i}^{-\lambda},
\]
(2.19)

for a constant \( C \) that only depends on \( \delta, \kappa, \lambda, T \) and on the supremum norm of \( \alpha \).

Proposition 2.3. Under the assumptions and notation of Proposition 2.1, assume that \( \beta \) is bounded and measurable and that \( \kappa \) in (2.16) satisfies \( \kappa \geq 61 \varepsilon^2 \), then (2.15), for a (deterministic) initial condition \( p_0 = (p_{0,i})_{i \in [d]} \in \mathcal{S}_{d-1} \) with positive entries (that is \( p_{0,i} > 0 \) for each \( i \in [d] \)), has a unique pathwise solution \((Q_i^t)_{0 \leq t \leq T} \) for each \( i \in [d] \). The coordinates of the solution are (strictly) positive and satisfy
\[
\mathbb{E} \left[ \sum_{i \in [d]} Q_i^T \right] = 1.
\]
(2.20)

2.2.2. Reformulation of the game. We now have most of the needed ingredients to formulate the setting to which our main result applies. Roughly speaking, our result addresses the mean field game associated with the pair (2.14)–(2.15) instead of (2.7)–(2.8) and with the cost functional (2.9); so this is the same MFG as the one described in Subsection 2.1 except for the fact that we included the forcing \( \varphi \) in the state equations and that, in the equation (2.15), we will allow for a more general (random) rate function instead of the feedback function \( \beta \). So, in lieu of (2.15), we will consider (in the mean field game)
\[
dQ_i^t = \sum_{j \in [d]} \left( Q_i^t (\varphi(P_i^t) + \beta_t^{i,j}) - Q_i^t (\varphi(P_j^t) + \beta_t^{i,j}) \right) \, dt + \varepsilon Q_i^t \sum_{j \in [d]} \sqrt{P_i^t P_j^t} \, dW_t^{i,j}, \quad t \in [0,T].
\]
(2.21)

Here, \((\beta_t^{i,j})_{0 \leq t \leq T} \) is a collection of bounded \( F^W \)-progressively-measurable processes that are required to satisfy
\[
\beta_t^{i,j} \geq 0, \quad i, j \in [d], \quad i \neq j.
\]
(2.22)

(Notice that the diagonal terms \((\beta_t^{i,i})_{0 \leq t \leq T} \) play no role.) Such processes are called admissible open-loop strategies. Somewhere, this is to say that we can work (at least for (2.15)) with strategies that may depend upon the whole past of the environment \( P = (P_t)_{0 \leq t \leq T} \), which is in contrast to strategies of the form \((B(t, i, P_t(j)))_{0 \leq t \leq T} \) in (2.15) which depend, at time \( t \), on the environment through its current state only. Latter strategies are said to be semi-closed. We explain below how such semi-closed strategies manifest through the master equation.

Remark 2.4. We let the reader check that Propositions 2.1 and 2.2, given below, see Subsection 2.2.4 for their proofs, can be also extended to the case where the process \((\alpha_t(i, P_t(j)))_{i \in [d]} \) in (2.14) is replaced by a more general bounded progressively measurable process \((\alpha_t^{i,j})_{i, j \in [d]} \) satisfying the analogue of (2.22). The proof of the solvability of (2.14) in the statement of Proposition 2.1 is even simpler since \((\alpha_t^{i,j})_{i, j \in [d]} \) is then in open-loop form.

Importantly, we regard the two cost coefficients \( f \) and \( g \) in (2.9) as being defined on \([0, T] \times [d] \times \mathcal{S}_{d-1} \) and \([d] \times \mathcal{S}_{d-1} \) respectively. To make it simpler, we write \( g(t, p) \) for \( g(i, p) \) and \( f(i, t, p) \)
for \( f(t, i, p) \). Accordingly, for a progressively-measurable strategy \( \beta = ((\beta_t^{i,j})_{i,j \in [d]})_{0 \leq t \leq T} \), the cost functional becomes

\[
\mathcal{J}(\beta, P) := \sum_{i \in [d]} \mathbb{E} \left[ Q_t^i g^i(P_T) + \int_0^T Q_t^i \left( f^i(t, P_t) + \frac{1}{2} \sum_{j \neq i} |\beta_t^{i,j}|^2 \right) dt \right],
\]

where \((Q_t)_{0 \leq t \leq T}\) solves (2.21), with \( Q_0 = P_0 \) (the latter being equal to some deterministic \( p_0 \in \mathcal{S}_{d-1} \)).

**Definition 2.5.** Given a deterministic initial condition \( p_0 \), a solution of the mean field game (with common noise) is a pair \((P, \alpha)\) such that

1. \( P = (P_t)_{0 \leq t \leq T} \) is an \( \mathcal{S}_{d-1} \)-valued process, progressively measurable with respect to \( \mathbb{F}^W \), with \( p_0 \) as initial condition, and \( \alpha : [0, T] \times [d] \times \mathcal{S}_{d-1} \times [d] \to \mathbb{R} \) is a bounded feedback strategy;
2. \( P \) and \( \alpha \) satisfy Equation (2.14) in the strong sense;
3. \( \mathcal{J}(\alpha, P) \leq \mathcal{J}(\beta, P) \) for any admissible open-loop strategy \( \beta \).

We say that the solution \((P, \alpha)\) is unique if given another solution \((\tilde{P}, \tilde{\alpha})\), we have \( P_t = \tilde{P}_t \) for any \( t \in [0, T] \), \( \mathbb{P}\)-a.s., and \( \alpha(t, i, P_t)(j) = \tilde{\alpha}(t, i, P_t)(j) \) \( dt \otimes \mathbb{P}\)-a.e., for each \( i, j \in [d] \).

We recall that the probability space and the Brownian motion \( W \) are fixed and then Equations (2.14) and (2.21) have unique strong solutions. The above hence defines strong mean field game solutions, in the sense that \( P \) is adapted to \( \mathbb{F}^W \). For a comparison between strong and weak MFG solutions, in the diffusion case, we refer to [10, Chapter 2].

Here is now a meta-form of our main statement.

**Meta-Theorem 2.6.** Assume that the coefficients \( f \) and \( g \) are sufficiently regular. Then, for any \( \varepsilon \in (0, 1) \), there exists a threshold \( \kappa_0 > 0 \), only depending on \( \varepsilon \), \( \|f\|_\infty \), \( \|g\|_\infty \) and \( T \), such that, for any \( \kappa \geq \kappa_0 \) and \( \delta \in (0, 1/(4\sqrt{d})) \), and for any (deterministic) initial condition \((p_{0,i})_{i \in [d]} \in \mathcal{S}_{d-1} \) with positive entries, the mean field game has a unique solution as defined by Definition 2.5.

The statement is said to be in meta-form since the assumptions on \( f \) and \( g \) are not clear. The definitive version is given in Theorem 2.9 below.

**Remark 2.7.** At this stage, it is worth mentioning that our notion of solution, as defined in Definition 2.5, could be relaxed: Instead of requiring the strategies to be in feedback form (namely, in the form \(((\alpha(t, i, P_t)(j))_{i,j \in [d]})_{0 \leq t \leq T} \)), we could allow them to be in open-loop form (namely, to be given by more general bounded progressively-measurable processes \(((\alpha_t^{i,j})_{i,j \in [d]})_{0 \leq t \leq T} \)). Our claim is that Meta-Theorem 2.6, and in fact Theorem 2.7 as well, extend to this case: The solution given by Meta-Theorem 2.6 and Theorem 2.4 remains unique within the larger class of open-loop solutions. The proof is exactly the same. Actually, our choice to use feedback strategies is for convenience only since we feel better to keep, in our main statements, the same framework as the one used in the exposition of the problem.

Moreover, we are confident that our result also extends to random initial conditions, but the proof would certainly require an additional effort since the initial conditions should then satisfy suitable integrability properties. To wit, an expectation must be added to the right-hand side of both (2.18) and (2.19) when \((p_{0,i})_{i \in [d]} \) becomes random: The resulting expectations might be infinite unless some integrability properties are indeed satisfied.

### 2.3. From Kimura diffusions to the main statement

We now elucidate the choice of the functional spaces for \( f \) and \( g \) in Meta-Theorem 2.6. Basically, we take those spaces from a recent Schauder like theory due to Epstein and Mazzeo [26] for what we called Kimura operators, the
latter being operators of the very same structure as the second order generator of \((L_t)_{0 \leq t \leq T}\), which we will denote by \((L_t)_{0 \leq t \leq T}\).

2.3.1. Wright–Fisher model. Following \((2.5)\), we get (at least informally) that, for any twice differentiable real-valued function \(h\) on \(\mathbb{R}^d\),

\[
L_t h(p) = \sum_{i \in [d]} a_i(t, p) \partial_i h(p) + \frac{\varepsilon^2}{2} \sum_{i,j \in [d]} (p_i \delta_{i,j} - p_i p_j) \partial_{p_i p_j}^2 h(p).
\]  

(2.24)

Pay attention that the above writing is rather abusive since \(a_i(t, p)\) only makes sense at points \(p \in \mathbb{R}^d\) that belong to the simplex \(\hat{S}_{d-1}\). Of course, this here makes sense since, as we already alluded to, the set \(\hat{S}_{d-1}\) is invariant under the dynamics \((2.7)\). In this regard, it is worth mentioning that the second-order term in \(L_t\) is degenerate, which follows from the obvious fact that the matrix \((p_i \delta_{i,j} - p_i p_j)_{i,j \in [d]}\) has \((1, \ldots, 1)\) in its kernel whenever \((p_1, \ldots, p_d)\) is in \(\hat{S}_{d-1}\). Somehow, the degeneracy of \(L_t\) is the price to pay for forcing the solutions to solutions to \((2.7)\) to stay within \(\hat{S}_{d-1}\).

In case when the drift \(a\) in \((2.24)\) is zero, \(L_t\) becomes time independent and coincides with the generator of the \(d\)-dimensional Wright–Fisher model. In case when \(a\) is non-zero but is time-independent and satisfies \(a_i(p) \geq 0\) if \(p_i = 0\),

(2.25)

(which means that \(a\) points inward at points \(p\) that belong to the boundary of \(\hat{S}_{d-1}\)), the operator \(L_t\) itself becomes a time-homogeneous Kimura diffusion operator. Below, we make an intense use of the recent monograph of Epstein and Mazzeo \([26]\) on those types of operators, see also \([17, 42, 55]\) for earlier results. The key feature is that, under the identification of \(\hat{S}_{d-1}\) with \(\hat{S}_{d-1} = \{(x_1, \ldots, x_{d-1}) \in (\mathbb{R}_+)^{d-1} : \sum_{i \in [d-1]} x_i \leq 1\}\) (see the introduction for the notation), we may regard the simplex as a \(d-1\) dimensional manifold with corners, the corners being obtained by intersecting at most \(d-1\) of the hyperplanes \(\{x \in \mathbb{R}^{d-1} : x_1 = 0\}, \ldots, \{x \in \mathbb{R}^{d-1} : x_{d-1} = 0\}, \{x \in \mathbb{R}^{d-1} : x_1 + \cdots + x_{d-1} = 1\}\) with \(\hat{S}_{d-1}\) (we then call the codimension of the corner the number of hyperplanes showing up in the intersection). Accordingly, we can rewrite \((2.24)\) as an operator acting on functions from \(\hat{S}_{d-1}\) to \(\mathbb{R}\), the resulting operator being then a Kimura diffusion operator on \(\hat{S}_{d-1}\). In words, we can reformulate \((2.24)\) in terms of the sole \(d - 1\) first coordinates \((p_1, \ldots, p_{d-1})\) or, more generally, in terms of \((p_i)_{i \in [d]\backslash\{l\}}\) for any given coordinate \(l \in [d]\). Somehow, choosing the coordinate \(l\) amounts to choosing a system of local coordinates and, as we explain below, the choice of \(l\) is mostly dictated by the position of \((p_1, \ldots, p_d)\) inside the simplex. For instance, whenever all the entries of \(p = (p_1, \ldots, p_d)\) are positive, meaning that \((p_1, \ldots, p_{d-1})\) belongs to the interior of \(\hat{S}_{d-1}\), the choice of \(l\) does not really matter and we may work, for convenience, with \(l = d\). We then rewrite the generator \(L_t\), as given by \((2.24)\), in the form

\[
\hat{L}_t \hat{h}(x) = \sum_{i=1}^{d-1} \hat{a}_i(t, x) \partial_{x_i} \hat{h}(x) + \frac{\varepsilon^2}{2} \sum_{i,j=1}^{d-1} (x_i \delta_{i,j} - x_i x_j) \partial_{x_i x_j}^2 \hat{h}(x),
\]

(2.26)

where now \(x \in \hat{S}_{d-1}\), \(\hat{h}\) is a smooth function on \(\mathbb{R}^{d-1}\) and \(\hat{a}_i(t, x) = a_i(t, \bar{x})\), with \(\bar{x} = (x_1, \ldots, x_{d-1}, 1 - x_1 - \cdots - x_{d-1})\). The formal connection between \((2.24)\) and \((2.26)\) is that \(\hat{L}_t \hat{h}(x) = L_t \{\hat{h}(\bar{x})\}\), whenever \(\hat{h}\) is defined from \(h\) through the identity \(h(x) = \hat{h}(\bar{x})\). Once again, the intuitive argument behind this formal connection is that expanding \((h(P^1_t, \ldots, P^d_t))_{0 \leq t \leq T}\) (by Itô’s formula) should be the same as expanding \((h(P^1_t, 1<P^d_t, \ldots, P^d_t))_{0 \leq t \leq T}\) = \(h(P^1_t, \ldots, P^d_t))_{0 \leq t \leq T}\).
Importantly, we then notice that, in the interior of $\hat{\mathcal{S}}_{d-1}$, $\hat{\mathcal{L}}_t$ is elliptic. Indeed, for any $(x_1, \ldots, x_{d-1}) \in \hat{\mathcal{S}}_{d-1}$ and $(\xi_1, \ldots, \xi_{d-1}) \in \mathbb{R}^{d-1}$,

$$\sum_{i,j=1}^{d-1} \xi_i(x_i \delta_{i,j} - x_i x_j) \xi_j = \sum_{i=1}^{d-1} \xi_i^2 x_i - \left( \sum_{i=1}^{d-1} \xi_i x_i \right)^2 \geq \sum_{j=1}^{d-1} \xi_j^2 x_j - \sum_{j=1}^{d-1} x_j \sum_{i=1}^{d-1} \xi_i^2 x_i$$

$$= \sum_{i=1}^{d-1} \xi_i^2 x_i \left(1 - \sum_{j=1}^{d-1} x_j\right),$$

which suffices to prove ellipticity whenever $x_1, \ldots, x_{d-1} > 0$ and $\sum_{j=1}^{d-1} x_j < 1$.

Take now a corner of codimension $l \in \{1, \ldots, d-1\}$. If the $l$ hyperplanes entering the definition of the corner are of the form $\mathcal{H}_{i_1} = \{x \in \mathbb{R}^{d-1} : x_{i_1} = 0\}$, $\ldots$, $\mathcal{H}_{i_l} = \{x \in \mathbb{R}^{d-1} : x_{i_l} = 0\}$, for $1 \leq i_1 < i_2 < \cdots < i_l \leq d - 1$, then we can rewrite (2.26) in the form

$$\hat{\mathcal{L}}_t \hat{h}(x) = \frac{\varepsilon^2}{2} \sum_{i \in I} (1 - x_i) x_i \partial^2_{x_i} \hat{h}(x) + \frac{\varepsilon^2}{2} \sum_{j,k \notin I} x_j (\delta_{j,k} - x_k) \partial^2_{x_j x_k} \hat{h}(x) + \sum_{i \in I} \hat{a}_i(t,x) \partial_{x_i} \hat{h}(x)$$

$$- \frac{\varepsilon^2}{2} \sum_{i,j \in I : i \neq j} x_i x_j \partial^2_{x_i x_j} \hat{h}(x) - \varepsilon^2 \sum_{i \in I, j \notin I} x_i x_j \partial^2_{x_i x_j} \hat{h}(x) + \sum_{i \in I} \hat{a}_i(t,x) \partial_{x_i} \hat{h}(x),$$

with $I = \{i_1, \ldots, i_l\}$. Here, we observe from (2.25) that $\hat{a}_i(t,x) \geq 0$ if $i \in I$ and $x \in \cap_{j \notin I} \mathcal{H}_j$. Also, as long as $x$ belongs to $\cap_{j \in I} \mathcal{H}_j$ but $(x_j)_{j \notin I}$ and $1 - \sum_{j \notin I} x_j$ remain positive (which is necessarily true in the relative interior of $\cap_{j \in I} \mathcal{H}_j \cap \hat{\mathcal{S}}_{d-1}$), then, by the same argument as in (2.27), the matrix $(x_j (\delta_{j,k} - x_k))_{j,k \notin I}$ is non degenerate. Hence, up to the intensity factor $\varepsilon$, the above decomposition fits (2.4) in [26, Definition 2.2.1], which is of crucial interest for us.

Assume now that the corner of codimension $l$ is given by the intersection of the hyperplanes $\{x \in \mathbb{R}^{d-1} : x_{i_1} = 0\}, \ldots, \{x \in \mathbb{R}^{d-1} : x_{i_{l-1}} = 0\}$ and $\{x \in \mathbb{R}^{d-1} : x_1 + \cdots + x_{d-1} = 1\}$. In order to recover the same form as in [26, (2.4)] (or equivalently in (2.28)), we perform the following change of variables: consider $(y_1, \ldots, y_{d-1}) := (p_1, \ldots, p_{i_1-1}, p_{i_1+1}, \ldots, p_d)$ as a new system of local coordinates, for some given index $i_1 \in \{i_1+1, \ldots, d-1\}$ (which is indeed possible if $i_1+1 < d-1$; if not, $i_1$ must be chosen as the largest index that is different from $i_1, \ldots, i_{l-1}, d$ and hence is lower than $i_{l-1}$, which asks for reordering the coordinates in the change of variables). For a test function $h$ as before, we then expand $L_t[h(p_1, \ldots, p_{i_1-1}, 1 - \sum_{j \neq i_1} p_j, p_{i_1+1}, \ldots, p_d)]$ as before and check that we recover the same structure as in [26, (2.4)], but in the new coordinates. This demonstrates how to check the setting of [26, Definition 2.2.1].

2.3.2. Wright–Fisher spaces. The rationale for double-checking [26, Definition 2.2.1] is that we want to use next the Schauder theory developed in [26] for Kimura operators. This prompts us to introduce here the functional spaces that underpin the corresponding Schauder estimates.

For a point $x^0 \in \hat{\mathcal{S}}_{d-1}$ in the relative interior of a corner $C$ of $\hat{\mathcal{S}}_{d-1}$ of codimension $l \in \{0, \ldots, d-1\}$ (if $l = 0$, then $x^0$ is in the interior of $\hat{\mathcal{S}}_{d-1}$), we may consider a new system of coordinates $(y_1, \ldots, y_{d-1})$ (of the same type as in the previous paragraph) such that $C = \{y \in \hat{\mathcal{S}}_{d-1} : y_i = \cdots = y_i = 0\}$, for $1 \leq i_1 < \cdots < i_l$. Letting $I := \{i_1, \ldots, i_l\}$ and denoting by

$^5$The reader may also notice that, in [26], the operator $L_t$ is passed in a normal form, see Proposition 2.2.3 therein which guarantees that such a normal form indeed exists. Here the change of variable to get the normal form may be easily elucidated by adapting the 1d case accordingly, see [26]. It suffices to change $x_i$ into the new coordinate $\arcsin^2(\sqrt{x_i})$. In fact, the normal form in [26] plays a key role in the definition of the Wright-Fisher Hölder spaces that we recall in the next paragraph.
(y_1^0, \cdots, y_{d-1}^0) the coordinates of x^0 in the new system (for sure y_i^0 = 0 for j = 1, \cdots, l), we may find a \delta^0 > 0 such that:

1. the closure \( \overline{U}(\delta^0, x^0) \) of \( U(\delta^0, x^0) := \{ y \in (\mathbb{R}_+)^{d-1} : \sup_{i=1, \cdots, d-1} |y_i - y_i^0| < \delta^0 \} \) is included in \( \mathcal{S}_{d-1} \),
2. for \( y \) in \( U(\delta^0, x^0) \), for \( j \not\in I, y_j > 0 
3. for \( y \) in \( \overline{U}(\delta^0, x^0) \), for \( y_1 + \cdots + y_{d-1} < 1 - \delta^0 \).

A function \( \hat{h} \) defined on \( \overline{U}(\delta^0, x^0) \) is then said to belong to \( \mathcal{E}^{\gamma}_{WF}(\overline{U}(\delta^0, x^0)) \), for some \( \gamma \in (0, 1) \), if, in the new system of coordinates, \( \hat{h} \) is Hölder continuous on \( \overline{U}(\delta^0, x^0) \) with respect to the distance

\[
d(y, y') := \sum_{i=1}^{d-1} |\sqrt{y_i} - \sqrt{y_i'}|.
\]

We then let

\[
\| \hat{h} \|_{\gamma; \overline{U}(\delta^0, x^0)} := \sup_{y \in \overline{U}(\delta^0, x^0)} |\hat{h}(y)| + \sup_{y, y' \in \overline{U}(\delta^0, x^0)} \frac{|\hat{h}(y) - \hat{h}(y')|}{d(y, y')^\gamma}.
\]

Following [26, Lemma 5.2.5 and Definition 10.1.1], we say that a function \( \hat{h} \) defined on \( U(\delta^0, x^0) \) belongs to \( \mathcal{E}^{\gamma}_{WF}(U(\delta^0, x^0)) \) if, in the new system of coordinates,

1. \( \hat{h} \) is continuously differentiable on \( U(\delta^0, x^0) \) and \( \hat{h} \) and its derivatives extend continuously to \( \overline{U}(\delta^0, x^0) \) and the resulting extensions belong to \( \mathcal{E}^{\gamma}_{WF}(\overline{U}(\delta^0, x^0)) \);
2. The function \( \hat{h} \) is twice continuously differentiable on \( U_+(\delta^0, x^0) = U(\delta^0, x^0) \cap \{ (y_1, \cdots, y_{d-1}) \in (\mathbb{R}_+)^{d-1} : \forall i \in I, y_i > 0 \} \). Moreover, for any \( i, j \in I \) and any \( k, l \not\in I \),

\[
\lim_{\min(y_i, y_j) \to 0^+} \sqrt{y_i y_j} \partial^2_{y_i y_j} \hat{h}(y) = 0, \quad \lim_{y_i \to 0^+} \sqrt{y_i} \partial^2_{y_i y_k} \hat{h}(y) = 0,
\]

and the functions \( y \mapsto \sqrt{y_i y_j} \partial^2_{y_i y_j} \hat{h}(y), y \mapsto \sqrt{y_i} \partial^2_{y_i y_k} \hat{h}(y) \) and \( y \mapsto \partial^2_{y_i y_k} \hat{h}(y) \) belong to \( \mathcal{E}^{\gamma}_{WF}(\overline{U}(\delta^0, x^0)) \) (meaning in particular that they can be extended by continuity to \( \overline{U}(\delta^0, x^0) \)).

Then we call

\[
\| \hat{h} \|_{2+; \overline{U}(\delta^0, x^0)} := \| \hat{h} \|_{\gamma; \overline{U}(\delta^0, x^0)} + \sum_{i=1}^{d-1} \| \partial_{y_i} \hat{h} \|_{\gamma; \overline{U}(\delta^0, x^0)} + \sum_{i,j \in I} \| \sqrt{y_i y_j} \partial^2_{y_i y_j} \hat{h} \|_{\gamma; \overline{U}(\delta^0, x^0)} + \sum_{k,l \not\in I} \| \sqrt{y_i} \partial^2_{y_i y_k} \hat{h} \|_{\gamma; \overline{U}(\delta^0, x^0)}.
\]

---

6 There is a subtlety here: In fact, the distance used in [26] (1.32), (5.42) is defined in terms of the coordinates that are used in the normal form of the operator, see footnote 5. So rigorously, we should not use the variables \((y_1, \cdots, y_{d-1})\) but the variables \((\arcsin(\sqrt{y_1}), \cdots, \arcsin(\sqrt{y_{d-1}}))\) in the definition of the distance. Fortunately, since we have the condition \( y_1 + \cdots + y_{d-1} < 1 - \delta^0 \), the change of variable \( y_i \mapsto \arcsin(\sqrt{y_i}) \) is a smooth diffeomorphism, from which we deduce that the distance (2.29) is equivalent to the same distance but with the new variables. And, in fact, once we have made the change of variables, there is another subtlety: The reader may indeed notice that the distance defined in (2.29) does not match the distance defined in [26] (1.32), (5.42) since, for \( j \not\in I \), we should consider \( |y_j - y_j'| \) instead of \( |\sqrt{y_j} - \sqrt{y_j'}| \). Anyhow, since \( y_j \) and \( y_j' \) are here required to be away from 0, the distance used in (2.29) is equivalent to ours.

7 Similar to footnote 5 the reader should observe that, in [26] Lemma 5.2.5 and Definition 10.1.1, the two limits in (2.30) are in fact regarded in terms of the coordinates used in the normal form of the operator. To make it clear, we should here require \( \lim_{\min(z_i, z_j) \to 0^+} \sqrt{z_i z_j} \partial^2_{z_i z_j} \phi(z) = 0 \) with \( \phi(z_1, \cdots, z_{d-1}) = \hat{h}(\arcsin(\sqrt{z_1}), \cdots, \arcsin(\sqrt{z_{d-1}})) \) (and similarly for the second limit). It is an easy exercise to check that (2.30) would then follow.
Following [26, Lemma 5.2.7], we say that a function \( \hat{h} \) is continuously differentiable on \( [0, T] \) if \( \hat{h} \) belongs to each \( C^{2+\gamma}_{WF}(\mathcal{U}(\delta^0, x^0)) \). We then let

\[
\| \hat{h} \|_{2+\gamma} := \sum_{i=1}^{K} \| \hat{h} \|_{2+\gamma, \mathcal{U}(\delta^0, x^0)}.
\]

We refer to [26, Chapter 10] for more details.

A similar definition holds for the space \( C^{1+\gamma}_{WF}(0, T \times \mathcal{S}_{d-1}) \) of functions that are once continuously differentiable in time and twice continuously differentiable in space, with derivatives that are locally \( \gamma \)-Hölder continuous with respect to the time-space distance (in the local system of coordinates)

\[
D((t, y), (t', y')) := |t - t'|^{1/2} + d(y, y').
\]

To make it clear, a function \( \hat{h} \) defined on \([0, T] \times \mathcal{U}(\delta^0, x^0)\) is then said to belong to \( C^{\gamma/2}_{WF}(0, T \times \mathcal{U}(\delta^0, x^0)) \), for some \( \gamma \in (0, 1) \), if, in the new system of coordinates, \( \hat{h} \) is Hölder continuous on \([0, T] \times \mathcal{U}(\delta^0, x^0)\) with respect to the distance \( D \). We then let

\[
\| \hat{h} \|_{\gamma/2, \gamma; [0, T] \times \mathcal{U}(\delta^0, x^0)} := \sup_{(t, y) \in [0, T] \times \mathcal{U}(\delta^0, x^0)} |\hat{h}(t, y)| + \sup_{t, t' \in [0, T], \ y, y' \in \mathcal{U}(\delta^0, x^0)} \frac{|\hat{h}(t, y) - \hat{h}(t', y')|}{D((t, y), (t', y'))}.
\]

Following [26, Lemma 5.2.7], we say that a function \( \hat{h} \) defined on \( \mathcal{U}(\delta^0, x^0) \) belongs to the space \( C^{1+\gamma/2, 2+\gamma}_{WF}([0, T] \times \mathcal{U}(\delta^0, x^0)) \). We then let

\[
\| \hat{h} \|_{1+\gamma/2, 2+\gamma} := \| \hat{h} \|_{\gamma/2, \gamma, \mathcal{U}(\delta^0, x^0)} + \| \partial_t \hat{h} \|_{\gamma/2, \gamma, \mathcal{U}(\delta^0, x^0)} + \sum_{i=1}^{d-1} \| \partial_{y_i} \hat{h} \|_{\gamma/2, \gamma, \mathcal{U}(\delta^0, x^0)}
\]

\[
+ \sum_{i, j \in I} \| \sqrt{g_{ij}} \partial_{y_i} \partial_{y_j} \hat{h} \|_{\gamma/2, \gamma, \mathcal{U}(\delta^0, x^0)} + \sum_{k, l \in I} \| \partial_{y_k y_l} \hat{h} \|_{\gamma/2, \gamma, \mathcal{U}(\delta^0, x^0)}
\]

\[
+ \sum_{i=1}^{d-1} \sum_{k \notin I} \| \sqrt{g_{ik}} \partial_{y_i} \hat{h} \|_{\gamma/2, \gamma, \mathcal{U}(\delta^0, x^0)}.
\]

For a given finite covering \( \bigcup_{i=1}^{K} \mathcal{U}(\delta^0, x^0) \) of \( \mathcal{S}_{d-1} \), a function \( \hat{h} \) (or equivalently the associated function \( h \) defined on \([0, T] \times \mathcal{S}_{d-1}\)) is said to be in \( C^{1+\gamma/2, 2+\gamma}_{WF}([0, T] \times \mathcal{S}_{d-1}) \) if \( \hat{h} \) belongs to each \( C^{\gamma/2}_{WF}([0, T] \times \mathcal{S}_{d-1}) \). We then let

\[
\| \hat{h} \|_{1+\gamma/2, 2+\gamma} := \sum_{i=1}^{K} \| \hat{h} \|_{1+\gamma/2, 2+\gamma, \mathcal{U}(\delta^0, x^0)}.
\]
We stress the fact that the finite covering that we use in the sequel is fixed once for all. There is no need to change it.

**Remark 2.8.** Importantly, for \( y, y' \in \mathcal{U}(\delta^0, x^0) \), and for a constant \( c \geq 1 \), \( c \) depending on \( \delta_0 \),
\[
\left| \left( 1 - \sum_{i=1}^{d-1} y_i \right)^{1/2} - \left( 1 - \sum_{i=1}^{d-1} y'_i \right)^{1/2} \right| \leq \frac{c-1}{2} \sum_{i=1}^{d-1} (y_i - y'_i) \leq (c-1) \sum_{i=1}^{d-1} \sqrt{y_i - y'_i}.
\]
Recalling that the vector \( y = (y_1, \cdots, y_{d-1}) \) (resp. \( y' = (y'_1, \cdots, y'_{d-1}) \)) stand for the new coordinates of an element \( x \in \mathcal{S}_{d-1} \) (resp. \( x' \)) and that \( x \) (resp. \( x' \)) itself is canonically associated with \( \bar{x} = (x_1, \cdots, x_{d-1}, 1 - x_1 - \cdots - x_{d-1}) \in \mathcal{S}_{d-1} \) (resp. \( \bar{x}' \)), we deduce that
\[
c^{-1} \sum_{i=1}^{d} \left| \sqrt{x_i} - \sqrt{x'_i} \right| \leq d(y, y') \leq c \sum_{i=1}^{d} \left| \sqrt{x_i} - \sqrt{x'_i} \right|,
\]
which permits to reformulate the modulus of continuity showing up in the Hölder condition of the Wright–Fisher space in an intrinsic manner. Since the number of neighborhoods of the form \( \mathcal{U}(\delta^0, x^0) \) used to cover \( \mathcal{S}_{d-1} \) is finite, we can choose the same \( c \) for all those neighborhoods.

2.3.3. **Complete version of the main statement.** We now have all the ingredients to clarify Meta-Theorem 2.6 and to formulate our main statement in a rigorous manner.

**Theorem 2.9.** Assume that, for some \( \gamma > 0 \), each \( f^i \), for \( i \in [d] \), belongs to \( \mathcal{C}^{\gamma/2+\gamma}([0, T] \times \mathcal{S}_{d-1}) \), and each \( g^i \), for \( i \in [d] \), belongs to \( \mathcal{C}^{2+\gamma}(\mathcal{S}_{d-1}) \). Then, for any \( \varepsilon \in (0, 1) \), there exists a threshold \( \kappa_0 > 0 \), only depending on \( \varepsilon, \|f\|_{\infty}, \|g\|_{\infty} \) and \( T \), such that, for any \( \kappa \geq \kappa_0 \) and \( \delta \in (0, 1/(4\sqrt{d})) \), and for any (deterministic) initial condition \( (p_{0,i})_{i \in [d]} \in \mathcal{S}_{d-1} \) with positive entries, the mean field game has a unique solution, in the sense of Definition 2.7.

**Remark 2.10.** As already emphasized in Remark 2.7, we could certainly extend the uniqueness result to the larger class of open-loop strategies and also to the case when the initial condition is random.

As for the assumptions on the coefficients, the key fact is that there is no need for any monotonicity condition in the statement. Still, it would be interesting to see whether the result remains true under lower regularity conditions on the function \( g \). Assuming \( g \) to have two Hölder continuous derivatives (as we do here) is quite convenient since it allows to find a solution to the master equation (see the next section) that remains smooth up to the boundary at time \( T \). More effort would be needed to allow for more general (and hence less regular) terminal costs; accordingly, it would require to address with care the rate at which the derivatives of the solution to the master equation would blow up at terminal time. We leave this problem for future work.

2.4. **Proofs of auxiliary results.** We now prove some of the results stated right above.

2.4.1. **Proof of Proposition 2.7.** The proof holds in two steps. We give a sketch of it only.

**First Step.** By (2.12), it suffices to solve the equation for \( (P^1_t, \cdots, P^{d-1}_t)_{0 \leq t \leq T} \). As long as the latter stays in the interior of \( \mathcal{S}_{d-1} \), the equation satisfied by the process is non-degenerate, see (2.27). Moreover, the diffusivity matrix is Lipschitz away from the boundary of the simplex and the drift is bounded. Therefore, by a standard localization argument, we can easily adapt the strong existence and uniqueness result of Veretennikov [57] (see Remark 3 therein for the case when the initial condition is random and Remark 4 therein for the case when the state variable and the underpinning Brownian motion do not have the same dimension) and then deduce that, up until one coordinate (including the \( d \)th coordinate, as given by \( P^d_t = 1 - (P^1_t + \cdots + P^{d-1}_t) \)) reaches a given positive threshold \( \varepsilon \), a (strong) solution exists and is pathwise
unique. By letting $\varepsilon$ tend to 0, we deduce that there exists a solution up to the first time it reaches the boundary of the simplex (or, equivalently, one of the coordinates vanishes) and this solution is pathwise unique (once again, up to the first time it reaches the boundary).

Second Step. The second step is to prove that, for $\kappa \geq \varepsilon^2/2$, the solution of (2.14) up until it reaches the boundary of $\hat{S}_{d-1}$ stays in fact away from the boundary of $\hat{S}_{d-1}$ and hence is a solution of (2.14) on the entire $[0, T]$. To do so, we come back to (2.13), namely, we write the dynamics of the $i$th coordinate (for $i = 1, \cdots, d - 1$) in the form

$$dP_t^i = a_i(t, (P_t^1, \cdots, P_t^d))dt + \varepsilon \sqrt{P_t^i(1 - P_t^i)}d\tilde{W}_t^i,$$

with $a_i$ as in (2.17). The above holds true up until the first time $\tau := \inf\{t \in [0, T] : (P_t^1, \cdots, P_t^{d-1}) \in \partial\hat{S}_{d-1}\} \wedge T$. Then, using the fact that $\alpha$ is bounded and $\alpha(t, j, x)(i) \geq 0$ for $i \neq j$ in (2.17), we can easily compare the process $(P_t^i)_{0 \leq t \leq \tau}$ with the solution of the equation

$$d\bar{P}_t^i = (\varphi(\bar{P}_t^i) - C\bar{P}_t^i)dt + \varepsilon \sqrt{(\bar{P}_t^i)_+ + (1 - \bar{P}_t^i)_+}d\tilde{W}_t^i,$$

with $\bar{P}_0^i = P_0^i$ as initial condition, for a constant $C \geq 0$. Above, $(\cdot)_+$ stands for the positive part. By [11] Chapter 5, Proposition 2.13, the above equation has a unique strong solution. Letting $\tau^i := \inf\{t \in [0, T] : \bar{P}_t^i \in \{0, 1\}\} \wedge T$ and choosing $C$ large enough, we have $\bar{P}_t^i \leq P_t^i$, for all $t \leq \tau \wedge \tau^i$. We then apply Feller’s test (see [11] Chapter 5, Proposition 5.22) to $(\bar{P}_t^i)_{0 \leq t \leq \tau}$ (the reader may notice that the fact that the initial condition is random is not a hindrance since it belongs to $(0, 1)$ with probability 1). The natural scale (see [11] Chapter 5, (5.42))) is here given by

$$\Phi(r) := \int_\delta^r \exp\left(-2 \int_\delta^s \frac{\varphi(u) - Cu}{\varepsilon^2u(1 - u)}du\right)ds, \quad r \in (0, 1).$$

For $r \in (0, \delta)$,

$$-\Phi(r) \geq \int_r^\delta \exp\left(\frac{2\kappa}{\varepsilon^2} \ln\left(\frac{\delta}{s}\right) - \frac{2C\delta}{\varepsilon(1 - \delta)}\right)ds,$$

from which we get that $\Phi(0+) = -\infty$ if $2\kappa/\varepsilon^2 \geq 1$. We deduce that, if the latter is true, $(\bar{P}_t^i)_{0 \leq t \leq \tau}$ does not touch 0. By comparison, we deduce that $(P_t^i)_{0 \leq t \leq \tau}$ cannot touch 0 before it touches 1, that is $P_\tau^i = 1$ if the set $\{t \in [0, \tau] : P_t^i \in \{0, 1\}\}$ is not empty. This holds true for $i = 1, \cdots, d - 1$, but by choosing another system of coordinates, we get the same result for the coordinate $i = d$. Assume now that we can find some coordinate $i \in [d]$ such that $P_\tau^i \in \{0, 1\}$, in which case our analysis says that $P_\tau^i = 1$. Since $\sum_{j=1}^d P_\tau^j = 1$, we deduce that $P_\tau^j = 0$ for all $j \in [d] \setminus \{i\}$, which is a contradiction with our analysis. So, the conclusion is that, at time $\tau$, we must have $P_\tau^i \in (0, 1)$ for all $i \in [d]$. That is, $\tau = T$ and the process $(P_t = (P_t^1, \cdots, P_t^d))_{0 \leq t \leq T}$ remains in the $(d - 1)$-dimensional interior of $S_{d-1}$. $\square$

2.4.2. Proof of Proposition 2.2. As in the proof of Proposition 2.1 we write the equation for $(P_t^i)_{0 \leq t \leq \tau}$ (for a given $i \in [d]$, $i$ being possibly equal to $d$) in the form

$$dP_t^i = a_i(t, P_t)dt + \varepsilon \sqrt{P_t^i(1 - P_t^i)}d\tilde{W}_t^i,$$
with $a_i$ as in (2.17). Then, we get, by Itô’s formula (recall that the left-hand side below is well-defined since $(P^i_t)_{0 \leq t \leq T}$ does not vanish),

$$
\begin{align*}
&\frac{d}{dt} \ln P^i_t = \sum_{j \in [d]} \left[ \frac{P^j_i}{P^i_t} \left( \varphi(P^i_t) + \alpha(t, j, P_t)(i) \right) - \left( \varphi(P^j_i) + \alpha(t, i, P_t)(j) \right) \right] dt - \frac{\varepsilon^2}{2} \frac{1 - P^i_t}{P^i_t} dt \\
&\quad + \varepsilon \sqrt{1 - \frac{P^i_t}{P^i_t}} \, d\tilde{W}^i_t.
\end{align*}
$$

(2.33)

For a constant $C$ depending on the same parameters as those quoted in the statement, we can lower bound the drift in (2.33) as follows (using the definition of $\varphi$ in (2.16) together with the fact that $\alpha(t, j, P_t)(i)$ is non-negative if $j \neq i$)

$$
\begin{align*}
&\sum_{j \in [d]} \left[ \frac{P^j_i}{P^i_t} \left( \varphi(P^i_t) + \alpha(t, j, P_t)(i) \right) - \left( \varphi(P^j_i) + \alpha(t, i, P_t)(j) \right) \right] - \frac{\varepsilon^2}{2} \frac{1 - P^i_t}{P^i_t} \\
&\quad \geq \frac{1 - P^i_t}{P^i_t} \kappa 1_{\{p^i_t \leq \delta\}} - \frac{\varepsilon^2}{2} \frac{1 - P^i_t}{P^i_t} - C.
\end{align*}
$$

Allowing the value of $C$ to change from line to line and recalling that $\kappa \geq \varepsilon^2/2$, we get

$$
\begin{align*}
&\sum_{j \in [d]} \left[ \frac{P^j_i}{P^i_t} \left( \varphi(P^i_t) + \alpha(t, j, P_t)(i) \right) - \left( \varphi(P^j_i) + \alpha(t, i, P_t)(j) \right) \right] - \frac{\varepsilon^2}{2} \frac{1 - P^i_t}{P^i_t} \\
&\quad \geq \frac{1 - P^i_t}{P^i_t} \left( \kappa - \frac{\varepsilon^2}{2} \right) - C.
\end{align*}
$$

Hence, integrating (2.33) from 0 to some stopping time $\tau$ (with values in $[0, T]$), adding and subtracting the compensator $(\lambda \varepsilon^2/2) \int_0^\tau (1 - P^i_t)/P^i_t dt$, multiplying by $\lambda$ and then taking the exponential, we get

$$
\begin{align*}
&\left( P^i_{\tau} \right)^{\lambda} \exp \left( -\lambda \varepsilon \int_0^\tau \sqrt{1 - \frac{P^i_t}{P^i_t}} \, d\tilde{W}^i_t - \frac{\lambda^2 \varepsilon^2}{2} \int_0^\tau \frac{1 - P^i_t}{P^i_t} dt \right) \\
&\quad \geq \left( p_{0, i} \right)^{\lambda} \exp \left( \lambda \left[ \kappa - \frac{\varepsilon^2}{2} (1 + \lambda) \right] \int_0^\tau \frac{1}{P^i_t} dt - C \right).
\end{align*}
$$

(2.34)

Choosing $\tau = \inf \{ t \in [0, T] : P^i_t \leq \varepsilon \} \wedge T$, for $\varepsilon > 0$ as small as needed, the left-hand side has conditional expectation less than 1. So, taking expectation and letting $\varepsilon$ tend to 0, we deduce that

$$
E \left[ \exp \left( \lambda \left[ \kappa - \frac{\varepsilon^2}{2} (1 + \lambda) \right] \int_0^T \frac{1}{P^i_t} dt \right) \right] \leq C \left( p_{0, i} \right)^{-\lambda}.
$$

The bound (2.18) easily follows.

It then remains to prove (2.19). To do so, we come back to (2.34). Using the fact that $\gamma$ is positive and choosing $\tau = t \in [0, T]$, we rewrite it in the form

$$
\left( P^i_t \right)^{-\lambda} \leq \left( p_{0, i} \right)^{-\lambda} \exp \left( C - \lambda \varepsilon \int_0^t \sqrt{1 - \frac{P^i_s}{P^i_t}} \, d\tilde{W}^i_s - \frac{\lambda^2 \varepsilon^2}{2} \int_0^t \frac{1 - P^i_s}{P^i_t} ds \right).
$$

Taking expectations on both sides, we easily complete the proof of (2.19). □
2.4.3. Proof of Proposition 2.3. For each \( i \in [\ell] \), we call \((\mathcal{E}_t^i)_{0 \leq t \leq T}\) the Doléans–Dade exponential
\[
\mathcal{E}_t^i := \exp \left( \varepsilon \sum_{j \in [\ell]} \int_0^t \sqrt{P_{ij}^j} dW_{ij}^s - \frac{\varepsilon^2}{2} \int_0^t \frac{1 - P_{ij}^j}{P_{ij}^i} ds \right), \quad t \in [0, T].
\]
Then, \((Q_t^i)_{0 \leq t \leq T}\) is a solution to (2.15) if and only if
\[
d\left[ (\mathcal{E}_t^i)^{-1} Q_t^i \right] = \sum_{j \in [\ell]} (\mathcal{E}_t^i)^{-1} \left( Q_t^i (\varphi(P_t^j) + \beta(t, j, P_t)(i)) - Q_t^i (\varphi(P_t^j) + \beta(t, i, P_t)(j)) \right) dt,
\]
which may be rewritten in the form
\[
d\tilde{Q}_t^i = \sum_{j \in [\ell]} \left( (\mathcal{E}_t^i)^{-1} \tilde{Q}_t^j (\varphi(P_t^j) + \beta(t, j, P_t)(i)) - \tilde{Q}_t^j (\varphi(P_t^j) + \beta(t, i, P_t)(j)) \right) dt, \tag{2.35}
\]
for \( t \in [0, T] \), with \((\tilde{Q}_t^i = p_{0,i})_{i \in [\ell]} \in \mathcal{S}_{d-1}\) as initial condition and under the change of variable
\[
\tilde{Q}_t^i := (\mathcal{E}_t^i)^{-1} Q_t^i, \quad t \in [0, T].
\tag{2.36}
\]
Obviously, (2.35) has a unique pathwise solution. It is continuous and adapted to the filtration \( \mathbb{F}^W \). Since \( \beta(t, j, P_t)(i) \geq 0 \) for \( j \neq i \), it is pretty easy to check that all the coordinates remain (strictly) positive.

Given the solution to (2.35), we may reconstruct \((Q_t^i)_{0 \leq t \leq T}\) from the change of variable (2.36). Then, taking the power \( l \) in (2.15), for an exponent \( l \geq 1 \), we get
\[
d(Q_t^i)^l = l \sum_{j \in [\ell]} (Q_t^j)^{l-1} \left( Q_t^j (\varphi(P_t^j) + \beta(t, j, P_t)(i)) - Q_t^j (\varphi(P_t^j) + \beta(t, i, P_t)(j)) \right) dt
\]
\[
+ \frac{l(l-1)}{2} \sum_{j \in [\ell]} \left( Q_t^j \right)^{1-l} \frac{1 - P_{ij}^j}{P_{ij}^i} dt + \varepsilon l (Q_t^i)^l \sum_{j \in [\ell]} P_{ij}^j d\tilde{W}_t^j,
\tag{2.37}
\]
for \( t \in [0, T] \). As a result, we can find a constant \( C \), only depending on \( l, \kappa \) and on the supremum norm of \( \beta \), such that
\[
d \left[ \sum_{i \in [\ell]} (Q_t^i)^l \right] \leq \left[ C + \varepsilon^2 \frac{l(l-1)}{2} \sum_{j \in [\ell]} \frac{1 - P_{ij}^j}{P_{ij}^i} \right] \left[ \sum_{i \in [\ell]} (Q_t^i)^l \right] dt + dm_t,
\]
where \((m_t)_{0 \leq t \leq T}\) is a local martingale.\footnote{Here and throughout, the notation \( dX_t^i \geq dX_t^j, \quad t \in [0, T] \), for two stochastic processes \((X_t^i)_{0 \leq t \leq T}\) is understood as \((X_t^i - X_t^j)_{0 \leq t \leq T}\) is a non-decreasing process.} By a standard localization argument, we end up with
\[
\sup_{0 \leq t \leq T} \mathbb{E} \left[ \left( \sum_{i \in [\ell]} (Q_t^i)^l \right) \exp \left( -\varepsilon^2 \frac{l(l-1)}{2} \int_0^T \sum_{j \in [\ell]} \frac{1 - P_{ij}^j}{P_{ij}^i} ds \right) \right] \leq C,
\]
for a new value of \( C \). And then, applying Cauchy-Schwarz inequality and invoking the above inequality with \( 2l \) instead of \( l \), we get
\[
\sup_{0 \leq t \leq T} \mathbb{E} \left[ \sum_{i \in [\ell]} (Q_t^i)^l \right] \leq C \sup_{0 \leq t \leq T} \mathbb{E} \left[ \exp \left( \varepsilon^2 l(2l-1) \int_0^T \sum_{j \in [\ell]} \frac{1 - P_{ij}^j}{P_{ij}^i} dt \right) \right]^{1/2}.
\]
Take \( l = 8 \) and choose \( \gamma = \varepsilon^2 \) and \( \lambda = 120 \) in the statement of Proposition 2.2. Then, the right-hand side is upper bounded. Returning to (2.37), invoking Burkholder–Davis–Gundy inequalities, (2.19) and the bound \( \sup_{0 \leq t \leq T} \mathbb{E}[(Q_t^i)^4/P_t^i] \leq \sup_{0 \leq t \leq T} \mathbb{E}[(Q_t^i)^8]^{1/2} \sup_{0 \leq t \leq T} \mathbb{E}(P_t^i)^{-2} \),
for \( i \in [d] \), we deduce that \( \sup_{0 \leq t \leq T} |Q_t^i| \) has a finite fourth moment for each \( i \in [d] \). Equality (2.20) is easily proved by summing over \( i \in [d] \) in (2.15).

3. From the MFG system to the master equation

The proof of Theorem 2.9 is highly based upon the so-called master equation associated with the mean field game in hand. We refer to [7, 8, 10, 49] for foundations of the topics for mean field games set on \( \mathbb{R}^d \) and to [2, 13, 56] for related issues for mean field games with a finite state space. Generally speaking, the master equation here takes the form of a system of nonlinear parabolic equations driven by a Kimura operator. Throughout the section, we assume that the condition \( \kappa \geq (61 + d)\varepsilon^2 \) is in force.

3.1. MFG system. With the optimization problem driven by the cost functional (2.9) and the state equation (2.21) (within the environment (2.14)), we may associate a value function. Obviously, we may expect this value function to solve a stochastic (because of the common noise) variant of the usual Hamilton–Jacobi–Bellman equation for a stochastic optimal control problem. In any case, it is a continuous distribution \( Q \) therein is regarded as a stochastic environment. Typically, it is the solution of an equation of the form (2.14). In any case, it is a continuous parabolic equations driven by a Kimura operator. Throughout the section, we assume that the condition \( \kappa \geq (61 + d)\varepsilon^2 \) is in force.

3.1.1. Formulation of the system. In order to proceed, we recall (2.23). Importantly, \( (P_t)_{0 \leq t \leq T} \) therein is regarded as a stochastic environment. Typically, it is the solution of an equation of the form (2.14). In any case, it is a continuous \( \mathcal{S}_{d-1} \)-valued process that is progressively-measurable with respect to the filtration \( \mathcal{F}_t^W \) and that satisfies the conclusion of Proposition 2.2 see (2.18) and (2.19). In particular, it remains away from the boundary of the simplex.

The related value function at time \( t \in [0, T] \) is defined as

\[
\begin{align*}
u^l(t, (P_s)_{t \leq s \leq T}) &:= \text{ess inf}_{(\beta_s)_{t \leq s \leq T}} J^l(t, (\beta_s)_{t \leq s \leq T}, (P_s)_{t \leq s \leq T}), \quad l \in [d], \\
J^l(t, (\beta_s)_{t \leq s \leq T}, (P_s)_{t \leq s \leq T}) &:= \sum_{i \in [d]} \mathbb{E} \left[ Q^i_t[t, l] g(P_T) + \int_t^T (Q^i_s[t, l] f^i(s, P_s) + \frac{1}{2} \sum_{j \neq i} \beta^{i,j}_s) \right] ds \bigg| \mathcal{F}_t^W, \quad l \in [d],
\end{align*}
\]

whereas \( (Q^i_s[t, l])_{t \leq s \leq T} \) is the solution to (2.21) when the initial time is \( t \in [0, T) \) and the initial distribution is \( Q^i_0[t, l] = \delta_{i,i} \), for \( i \in [d] \). Importantly, the value function is random: Stochasticity accounts for the fact that the cost functionals \( f \) and \( g \) in the optimal control problem depend upon the environment \( (P_s)_{0 \leq s \leq T} \), which is random itself. Hence the corresponding HJB equation is a backward stochastic HJB equation (SHJB) that takes the form of a system of backward SDEs indexed by \( i \in [d] \):

\[
\begin{align*}
du^i_t &= -\left( \sum_{j \in [d]} \varphi(P_t^j)[u^j_t - u^i_t] + H^i(u_t) + f^i(t, P_t) \right) dt - \frac{\varepsilon}{\sqrt{2}} \sum_{j \in [d], j \neq i} \sqrt{\frac{P_t^j}{P_t^i}} (\nu^i_{t,i,j} - \nu^j_{t,i,j}) dt \\
&\quad + \sum_{j,k \in [d], j \neq k} \nu^i_{t,j,k} dW_t^{j,k}, \\
u_T^i &= g^i(P_T),
\end{align*}
\]

(3.2)
where $H^i$ is the Hamiltonian

$$H^i(y) := -\frac{1}{2} \sum_{j \in [d]} (y_i - y_j)^2, \quad y = (y_j)_{j \in [d]}.$$  

It is worth emphasizing that, in the equation (3.2), the unknown is the larger family of processes $((u^i_t)_{i \in [d]}), (\nu^{i,j,k}_t)_{i,j,k \in [d]; j \neq k})_{0 \leq t \leq T}$, which are required to be progressively measurable with respect to $\mathcal{F}^W$. This is a standard fact in the theory of backward SDEs and the role of the processes $((\nu^{i,j,k}_t)_{i,j,k \in [d]; j \neq k})_{0 \leq t \leq T}$ is precisely to force the solution of the stochastic HJB equation to be non-anticipative. The reason why we here choose indices $(i, j, k)$ with $j \neq k$ is quite clear: there are no noises of the form $((W^{i,j}_t)_{0 \leq t \leq T})_{j \in [d]}$ in the forward equation.

3.1.2. Verification argument. Interestingly, the following verification argument clarifies the connection between (3.1) and (3.2).

**Lemma 3.1.** For an environment $P = (P_t)_{0 \leq t \leq T}$ as before (satisfying in particular (2.18) and (2.19)), assume that there exists a solution $((u^i_t)_{i \in [d]}), (\nu^{i,j,k}_t)_{i,j,k \in [d]; j \neq k})_{0 \leq t \leq T}$ to (3.2) such that

$$\sum_{i,j,k \in [d]; j \neq k} \sum_{t=0}^{T} \mathbb{E} \left[ \int_0^T |\nu^{i,j,k}_t|^2 dt \right] < \infty.$$  

For $t \in [0, T]$, let $(\beta^s)_{i,j} := (u^i_s - u^j_s)_+$ for $i \neq j$ and $s \in [t, T]$. Then, $J^s_l(t, (\beta^s)_t \leq s \leq T; (P_s)_{t \leq s \leq T}) = u^i_t$ and, for any other (bounded) strategy, say $(\beta^*)_{t \leq s \leq T}$, such that

$$\sum_{i,j \in [d]; i \neq j} \int_t^T \mathbb{P} (\beta^s_{t, j} \neq (\beta^*)_{i,j}) ds > 0,$$

the cost $J^s_l(t, (\beta^s)_{t \leq s \leq T}; (P_s)_{t \leq s \leq T})$ is strictly higher than $u^i_t$.

In words, the above says that $((\beta^s)^{i,j} = (u^i - u^j)_+)_{i,j \in [d]; i \neq j})_{t \leq s \leq T}$ is the unique optimal control. In fact, the solvability of the equation (3.2) is guaranteed by the following lemma.

**Lemma 3.2.** For $P = (P_t)_{0 \leq t \leq T}$ as before (satisfying (2.18) and (2.19)), (3.2) has a unique (progressively-measurable) solution $((u^i_t)_{i \in [d]}), (\nu^{i,j,k}_t)_{i,j,k \in [d]; j \neq k})_{0 \leq t \leq T}$ such that $((u^i_t)_{i \in [d]})_{0 \leq t \leq T}$ is almost surely bounded by a deterministic constant and $((\nu^{i,j,k}_t)_{i,j,k \in [d]; j \neq k})_{0 \leq t \leq T}$ satisfies

$$\sum_{i,j,k \in [d]; j \neq k} \mathbb{E} \left[ \int_0^T \exp \left( \varepsilon^2 \sum_{l \in [d]} \int_0^t \frac{1}{P^l_s} ds \right) |\nu^{i,j,k}_t|^2 dt \right] < \infty.$$  

Abusively, such a solution is said to be bounded.

The two lemmas will be proved in Subsection 3.1.3 below. For the time being, we observe, by combining the two of them, that, for a given $P = (P_t)_{0 \leq t \leq T}$ satisfying (2.18) and (2.19), the solution to the optimal control problem (2.23) is entirely described by the SHJB equation (3.2), as it suffices to solve the forward equation (2.21) with $((\beta^i_t = (u^i - u^j)_+)_{i,j \in [d]; i \neq j})_{0 \leq t \leq T}$ therein. Now, Definition 2.5 implies that an environment $P$ is a solution to the MFG in hand.
if and only if it solves the forward equation in the forward-backward system:

\[
\begin{align*}
\frac{dP^i_t}{dt} &= \sum_{j \in [d]} \left( P^i_t (\varphi(P^i_t) + (u^i_t - u^j_t)_+) - P^j_t (\varphi(P^j_t) + (u^j_t - u^i_t)_+) \right) dt \\
&\quad + \varepsilon \sum_{j \in [d]} \sqrt{P^i_t P^j_t} dW^{i,j}_t, \\
\frac{du^i_t}{dt} &= -\left( \sum_{j \in [d]} \varphi(P^j_t) [u^j_t - u^i_t] + H^i(u_t) + f^i(t, P_t) \right) dt \\
&\quad - \frac{\varepsilon}{\sqrt{2}} \sum_{j \in [d]: j \neq i} \sqrt{P^i_t P^j_t} (\nu^i_{t, i, j} - \nu^j_{t, j, i}) dt + \sum_{j, k \in [d]: j \neq k} \nu^i_{t, j, k} dW^{j, k}_t, 
\end{align*}
\]

with \((P^0_t = p_{0,i})_{i \in [d]} \in S_{d-1}\) as deterministic initial condition for the forward equation and \((u^{i}_t = g^i(P^i_t))_{i \in [d]}\) as terminal boundary condition for the backward equation. System (3.3) is the (stochastic) MFG system that characterizes the solutions of the MFG described in Definition 2.3. Hence, proving Theorem 2.9 is here the same as proving that (3.3) is uniquely solvable (within the space of processes that satisfy the conditions described in Proposition 2.3 and Lemma 3.2).

3.1.3. Proofs of Lemmas 3.1 and 3.2

Proof of Lemma 3.1: Call \(((Q^i_s)_{i \in [d]})_{t \leq s \leq T} \text{ the solution to (2.21)}\) with \(Q^i_t = \delta_{i,t}\) for some \(l \in [d]\) and expand

\[
\begin{align*}
\frac{d}{dt} \left( \sum_{i \in [d]} Q^i_s u^i_s + \int_t^s \sum_{i \in [d]} Q^i_r (f^i(r, P_r) + \frac{1}{2} \sum_{j \neq i} |\beta^i_{r, j}|^2) dr \right) \\
&= -\sum_{i \in [d]} \sum_{j \in [d]} Q^i_s (\varphi(P^j_s) [u^j_s - u^i_s] + H^i(u_s)) ds - \varepsilon \sum_{i \in [d]} \sum_{j \neq i} Q^i_s \sqrt{P^i_s P^j_s} (\nu^i_{s, i, j} - \nu^j_{s, j, i}) ds \\
&\quad + \sum_{i \in [d]} u^i_s \sum_{j \in [d]} \left( Q^i_s (\varphi(P^j_s) + \beta^i_{s, j}) - Q^j_s (\varphi(P^i_s) + \beta^j_{s, i}) \right) ds + \frac{1}{2} \sum_{i \in [d]} \sum_{j \neq i} Q^i_s |\beta^i_{s, j}|^2 ds \\
&\quad + \frac{\varepsilon}{\sqrt{2}} \sum_{i \in [d]} \sum_{j \in [d]} Q^i_s \sqrt{P^i_s} d[W^i_s - W^j_s] \cdot \sum_{j, k \in [d]: j \neq k} \nu^i_{s, j, k} dW^{j, k}_s + dm_s,
\end{align*}
\]

where \((m_s)_{t \leq s \leq T}\) is a uniformly integrable martingale. On the last line, the dot in the first term is used to compute the underlying bracket. On the second line,

\[
\sum_{i \in [d]} \sum_{j \in [d]} Q^i_s \varphi(P^j_s) = \sum_{i \in [d]} \sum_{j \in [d]} Q^j_s \varphi(P^i_s) = \sum_{i \in [d]} Q^i_s \varphi(P^i_s)(u^i_s - u^j_s),
\]
which cancels out with the first term on the first line. Moreover,

\[
- \sum_{i \in [d]} Q_i^s H^i(u_s) + \sum_{i, j \in [d]} u_i^s(Q_i^s \beta_{i,j}^s - Q_j^s \beta_{j,i}^s) + \frac{1}{2} \sum_{i \in [d]} Q_i^s \sum_{j \neq i} |\beta_{i,j}^s|^2
\]

\[
= \frac{1}{2} \sum_{i \in [d]} Q_i^s \sum_{j \neq i} (u_i^s - u_j^s)^2 + \sum_{i \in [d]} \beta_{i,j}^s (u_i^s - u_j^s) + \frac{1}{2} \sum_{i \in [d]} Q_i^s \sum_{j \neq i} |\beta_{i,j}^s|^2
\]

\[
\geq \frac{1}{2} \sum_{i \in [d]} Q_i^s \sum_{j \neq i} (u_i^s - u_j^s)^2 - \sum_{i \in [d]} \beta_{i,j}^s (u_i^s - u_j^s) + \frac{1}{2} \sum_{i \in [d]} Q_i^s \sum_{j \neq i} |\beta_{i,j}^s|^2
\]

\[
= \frac{1}{2} \sum_{i \in [d]} Q_i^s \sum_{j \neq i} |\beta_{i,j}^s - (u_i^s - u_j^s)|^2,
\]

the inequality being in fact an equality if \( \beta \equiv \beta^* \).

It remains to compute the bracket on the last line of (3.4). We get

\[
\frac{\varepsilon}{\sqrt{2}} Q_s^i \sum_{j \in [d]} \sqrt{P_t^j} \left[ W_t^{i,j} - W_t^{j,i} \right] \cdot \sum_{j, k \in [d], j \neq k} \nu_{s, i,j,k}^j dW_t^{j,k} = \frac{\varepsilon}{\sqrt{2}} Q_s^i \sum_{j \in [d], j \neq k} \sqrt{P_t^j} \left( \nu_{s, i,j,k}^j - \nu_{s, j,k,i}^j \right) ds,
\]

which cancels out with the last term on the first line of (3.4).

Integrating from \( t \) to \( T \) and taking conditional expectation in (3.4), we then deduce that

\[
\sum_{i \in [d]} Q_t^i u_t^i + \frac{1}{2} \mathbb{E} \left[ \sum_{i \in [d]} \int_t^T Q_s^i \sum_{j \neq i} |\beta_{i,j}^s - (u_i^s - u_j^s)|^2 ds \mid \mathcal{F}_t^W \right]
\]

\[
\leq \mathbb{E} \left[ \sum_{i \in [d]} Q_T^i u_T^i + \int_t^T \sum_{i \in [d]} Q_s^i \left( f^i(s, P_s) + \frac{1}{2} \sum_{j \neq i} |\beta_{i,j}^s|^2 \right) ds \mid \mathcal{F}_t^W \right],
\]

the inequality being an equality if \( \beta \equiv \beta^* \). Recalling that \( Q_t^i = \delta_{i,1} \), this is what we want. □

**Proof of Lemma 3.2 First Step.** The first step of the proof is to consider a truncated version of (3.2). Hence, for a given constant \( c > 0 \), we consider the equation

\[
du_t^i = - \left( \sum_{j \in [d]} \varphi \left( P_t^j \right) [u_t^j - u_t^i] + H_t^i(u_t) + f^i(t, P_t) \right) dt - \frac{\varepsilon}{\sqrt{2}} \sum_{j \neq k} \sqrt{P_t^j} \left( \nu_{i,j,k}^j - \nu_{i,k,j}^j \right) dt
\]

\[
+ \sum_{j \neq k} \nu_{i,j,k}^j dW_t^{j,k},
\]

\[
u_T^i = g^i(P_T),
\]

where \( H_t^i \) stands for the truncated Hamiltonian

\[
H_t^i(y) := - \frac{1}{2} \sum_{j \in [d]} \left[ \left( y_i - y_j \right)^2 \mathbf{1}_{\{y_i - y_j \leq c\}} + \left( 2c(y_i - y_j) - c^2 \right) \mathbf{1}_{\{y_i - y_j > c\}} \right], \quad y = (y_j)_{j \in [d]}.
\]

Then, (3.6) is a backward equation with a time dependent driver that is Lipschitz continuous with respect to the entries \((u_t^i)_{i \in [d]}\) and \((\nu_t^{i,j,k})_{i,j,k \in [d], j \neq k}\), the Lipschitz constant with respect to the entries \((u_t^i)_{i \in [d]}\) being bounded by a deterministic constant \( C \) (possibly depending on \( c \)) and the Lipschitz constant with respect to \((\nu_t^{i,j,k})_{i,j,k \in [d], j \neq k}\) being bounded by

\[
c_t := \frac{\varepsilon}{\sqrt{2}} \left[ \sum_{i \in [d]} \frac{1}{P_t^i} \right]^{1/2},
\]
in the sense that (using the fact that the driver is linear in \((\nu_t^{i,j,k})_{i,j,k \in [d]:j \neq k}\))
\[
\frac{\varepsilon}{\sqrt{2}} \left( \sum_{i \in [d]} \sum_{j \neq i} \sqrt{\frac{p_{ij}^t}{p_{ii}^t}} (\nu_t^{i,i,j} - \nu_t^{i,j,i}) \right)^{1/2} \leq \frac{\varepsilon}{\sqrt{2}} \left( \sum_{i \in [d]} \frac{1}{p_{ii}^t} \sum_{j \neq i} (\nu_t^{i,i,j} - \nu_t^{i,j,i})^2 \right)^{1/2}
\]
\[
\leq \frac{\varepsilon}{\sqrt{2}} \left( \sum_{i \in [d]} \frac{1}{p_{ii}^t} \right)^{1/2} \left( \sum_{i,j \in [d]:j \neq i} (\nu_t^{i,i,j} - \nu_t^{i,j,i})^2 \right)^{1/2}
\]
\[
= c_t \left( \sum_{i,j \in [d]:j \neq i} (\nu_t^{i,i,j} - \nu_t^{i,j,i})^2 \right)^{1/2}.
\]

By Proposition \[2.2\] (with \(\lambda = 2d - 1\) and \(\gamma = 60 \varepsilon^2\)) and from the condition \(\kappa \geq (61 + d)\varepsilon^2\) together with Hölder’s inequality, we notice that \(\mathbb{E}[\exp(2\varepsilon^2 \sum_{l \in [d]} \int_0^T (1/p_t^l) ds)] < \infty\). Then, by \[32\] Theorem 2.1 (i) (with, using the notations therein, \(\gamma \equiv 1\), \(\beta_1\) a positive constant, \(c_1\) a non-negative constant, \(\beta_2 = 2\) and \(c_0(t) = c_t\)), there exists a unique solution to \[3.6\] satisfying
\[
\sum_{i \in [d]} \mathbb{E}\left[ \sup_{0 \leq t \leq T} \left( \exp\left( \varepsilon^2 \sum_{l \in [d]} \int_0^t \frac{1}{p^l_s} ds \right) |u_t^i|^2 \right) \right] < \infty,
\]
\[
\sum_{i,j,k \in [d]:j \neq k} \mathbb{E}\left[ \int_0^T \exp\left( \varepsilon^2 \sum_{l \in [d]} \int_0^t \frac{1}{p^l_s} ds \right) \left| \nu_t^{i,j,k} \right|^2 + \left( 1 + \sum_{l \in [d]} \int_0^t \frac{1}{p^l_s} ds \right) |u_t^i|^2 \right] dt < \infty.
\]

Second Step. We now prove that we can find a bound for the solution that is independent of \(c\). To do so, we follow the proof of Lemma \[3.1\] noticing that the Hamiltonian \(H_t\) introduced in the first step is associated with the same cost functional \(\mathcal{J}\) as in \[2.23\] except that the processes \((\beta_t^{i,j})_{0 \leq t \leq T})_{i,j \in [d]:i \neq j}\) therein are required to be bounded by \(c\), and similarly for \(\mathcal{J}\) in \[3.1\]. In particular, \(u_t^l\) defined in the first step satisfies \(u_t^l = \text{ess inf}_{\beta, \beta^{i,j} \leq c} \mathcal{J}^l(t, (\beta_s)_{t \leq s \leq T}, (P_s)_{t \leq s \leq T})\). Call now \((Q_t^i)_{i \in [d]}\) the solution to \[2.21\] with \(Q_t^i = \delta_{i,t}\) for some \(l \in [d]\) and \(\beta \equiv 0\). Then, by \[3.3\] (but with the solution \((u_t^l)_{i \in [d]}\) in \[2.21\]) and \[3.6\] and so with the new Hamiltonian
\[
u_t^l \leq \mathbb{E}\left[ \sum_{i \in [d]} Q_t^i u_t^i + \int_0^T \sum_{i \in [d]} Q_t^i (f^i(s, P_s) + \frac{1}{2} \sum_{j \neq i} |\beta_t^{i,j}|^2) ds \right| F_t^W].
\]

Here, \(\beta \equiv 0\) and \(u_t^l = g^i(P_T)\), which provides an upper bound for \((u_t^l)_{i \in [d]}\) in \([0, T]\), by using \[2.20\] and the \(L^\infty\) bounds on \(f\) and \(g\). Importantly, the upper bound is independent of \(c\). In order to obtain a lower bound, we call \((Q_t^i)_{i \in [d]}\) the solution to \[2.21\] with \(Q_t^i = \delta_{i,t}\) for some \(l \in [d]\), given an open-loop strategy \(\beta\) whose coordinates are bounded by \(c\). Using again the bounds on \(f\) and \(g\), we get
\[
\mathcal{J}^l(t, (\beta_s)_{t \leq s \leq T}, (P_s)_{t \leq s \leq T}) = \mathbb{E}\left[ \sum_{i \in [d]} Q_t^i u_t^i + \int_0^T \sum_{i \in [d]} Q_t^i (f^i(s, P_s) + \frac{1}{2} \sum_{j \neq i} |\beta_t^{i,j}|^2) ds \right| F_t^W]
\]
\[
\geq \mathbb{E}\left[ - \sum_{i \in [d]} Q_t^i |g^i|_{L^\infty} - \int_0^T \sum_{i \in [d]} Q_t^i |f^i|_{L^\infty} ds \right| F_t^W] \geq -C_0.
\]

by \[2.20\], for a constant \(C_0\) independent of \(c\) and \(\beta\); so \(u_t^l \geq -C_0\).

In the end, we may find a constant \(C_0\) such that, whatever the value of \(c\) in \[3.6\], the solution is bounded by \(C_0\). We deduce that, whenever \(c \geq 2C_0\), the solution of \[3.6\] is also a solution of the backward equation \[3.2\]. This proves the existence of a bounded solution to \[3.2\]. As for uniqueness, it suffices to notice that a bounded solution to \[3.2\] is also a solution to \[3.4\], but
for a large enough \( c \) inside. Hence, we get that any bounded solution to (3.2) is bounded by \( C_0 \), which shows uniqueness. \( \square \)

3.2. **Master equation.** The system (3.3) is what we call a forward-backward stochastic differential equation. But, differently from most of the cases that have been addressed so far in the literature (see for instance [9] Chapter 3), solutions to the forward equation are here regarded as processes with values in \( S_{d-1} \). This requires a special treatment.

A standard strategy for solving forward-backward stochastic differential equations (at least in the so-called Markovian case, see for instance [51, 20]) is to regard the system formed by the derivatives that appear in the forward and backward equations as the characteristics of a system of parabolic second order PDEs. In our framework, this system of PDEs is precisely what we call the master equation.

In words, the master equation of the mean field game is here a system of second order PDEs stated on the \((d - 1)\)-dimensional simplex: A key fact is that it features some degeneracy at the boundary of the simplex. Formally, the system writes

\[
\partial_t U^i(t, p) + H^i\left((U^j(t, p))_{j \in [d]}\right) + f^i(t, p) + \sum_{j \in [d]} \varphi(p_j)\left[U^j(t, p) - U^i(t, p)\right] + \sum_{j, k \in [d]} p_k\left[\varphi(p_j) + (U^k(t, p) - U^j(t, p))\right] \left(\partial_{p_j} U^i(t, p) - \partial_{p_k} U^i(t, p)\right) + \varepsilon^2 \sum_{j \in [d]} \left(p_j \partial_{p_j} U^i(t, p) - \partial_{p_j} U^i(t, p)\right) + \frac{\varepsilon^2}{2} \sum_{j, k \in [d]} (p_j \delta_{jk} - p_j p_k) \partial^2_{p_j p_k} U^i(t, p) = 0,
\]

\[U^i(T, p) = g^i(p),\]

for \( i \in [d] \), where \((t, p) \in [0, T] \times S_{d-1}\). In the above, the unknown is the tuple of functions \((U^i)_{i \in [d]}\), each \( U^i \) standing for a real valued function defined on \([0, T] \times S_{d-1}\). Equation (3.7) is formally obtained by imposing \( u^i = U^i(t, P_t) \) and expanding \( U \) using Itô formula; this is basically what we do later for proving Theorem 3.3. Although the above formulation looks quite appealing, it remains rather abusive since \( S_{d-1} \) has empty interior in \( \mathbb{R}^d \): In other words, except if \( U^i \) is in fact defined on a neighborhood of the simplex, the derivatives that appear in the above equation remain quite obscure at this stage of the paper.

3.2.1. **Derivatives on the simplex.** This prompts us to revisit first derivatives on the simplex, whenever the latter is seen as a \((d - 1)\)-dimensional manifold. Obviously, the basic definition relies upon the same local coordinates as in (2.3.1). Indeed, for a real-valued function \( h \) defined on \( S_{d-1} \), we may define the function

\[
\hat{h}^i(p^{-i}) := h(p) = h\left(p_1, \cdots, p_{i-1}, 1 - \sum_{k \neq i} p_k, p_{i+1}, \cdots, p_d\right),
\]

with \( p^{-i} = \left(p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_d\right) \), for \( p \in S_{d-1} \) and hence \( p^{-i} \in S_{d-1} \), and for \( i \in [d] \). We then say that \( h \) is differentiable on \( S_{d-1} \) if \( \hat{h}^i \) is differentiable on \( S_{d-1} \) for some (and hence for any\footnote{In order to prove the differentiability of \( \hat{h}^i \) for any \( j \in [d] \setminus \{i\} \), it suffices to see that (say that \( j > i \) to simplify) \( \hat{h}^j(p^{-j}) = h(p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{j-1}, 1 - \sum_{k \neq j} p_k, p_{j+1}, \cdots) \).} i \in [d] \). In particular, any function \( h \in C^{2,+,2}_{WF}(S_{d-1}) \) fits this definition. In case when \( h \) is defined on a \((d\text{-dimensional})\) neighborhood of \( S_{d-1} \), we then have \( \partial_{p_j} \hat{h}^i(p^{-i}) = \partial_{p_j} h(p) - \partial_{p_i} h(p) \), for \( j \neq i \). We then end up with (writing \( V \) as a function of \((p_1, \cdots, p_d) \in S_{d-1}\))

\[
\partial_{p_j} \hat{h}^i(p^{-i}) - \partial_{p_i} \hat{h}^i(p^{-i}) = \partial_{p_j} h(p) - \partial_{p_i} h(p),
\]

\[\partial_{p_j} \hat{h}^i(p^{-i}) = \hat{h}^i(p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_{j-1}, 1 - \sum_{k \neq j} p_k, p_{j+1}, \cdots).\]
for any \( j, k \in [d] \setminus \{i\} \) and \( p \in S_{d-1} \). Similarly, the second order derivative may be written as
\[
\frac{\partial^2}{\partial p_j \partial p_k} \hat{h}^i(p^-) = \frac{\partial^2}{\partial p_j \partial p_k} h(p) - \frac{\partial^2}{\partial p_j \partial p_k} h(p) - \frac{\partial^2}{\partial p_i \partial p_k} h(p) + \frac{\partial^2}{\partial p_i \partial p_j} h(p),
\]
and the second order term in (3.7) with \( h = U^i \) (the reader should not make any confusion between \( U^i \) and \( \hat{h}^i \); \( U^i \) is the \( i \)th coordinate of the solution to the master equation whilst \( \hat{h}^i \) is the projection of \( h \), whenever the latter is real-valued, onto a real-valued function on \( \mathcal{S}_{d-1} \)) becomes
\[
\frac{\varepsilon^2}{2} \sum_{j,k \in [d]} (p_j \delta_{jk} - p_j p_k) \frac{\partial^2}{\partial p_j \partial p_k} h(p) = \frac{\varepsilon^2}{2} \sum_{j,k \neq i} (p_j \delta_{jk} - p_j p_k) \frac{\partial^2}{\partial p_j \partial p_k} \hat{h}^i(p^-),
\]
for \( p \in S_{d-1} \). In fact, according to the definition of the Wright–Fisher spaces in (2.3.2) we will consider functions that are just twice-differentiable on the interior \( \text{Int}(\mathcal{S}_{d-1}) \) of the simplex and for which the second-order derivatives may not extend to the boundary of the simplex.

Equivalently, we may formulate the derivatives in (3.7) in terms of the intrinsic gradient on \( S_{d-1} \), regarded as a \((d-1)\)-dimensional manifold. Indeed, whenever \( h \) is defined on a neighborhood of \( S_{d-1} \), we may denote by \( \nabla h = (\partial_{p_1} h, \ldots, \partial_{p_d} h) \) the standard gradient in \( \mathbb{R}^d \). Identifying the tangent space \( \mathcal{T}_{d-1} \) to the simplex at a given point \( p \in S_{d-1} \) with the orthogonal space to the \( d \)-dimensional vector \( \mathbf{1} = (1, \ldots, 1) \), the intrinsic gradient of \( h \), seen as a function defined on the simplex, at \( p \) identifies with the orthogonal projection of \( \nabla h \) on \( \mathcal{T}_{d-1} \). We denote it by \( \mathcal{D} h := (\partial_1 h, \ldots, \partial_d h) \), that is
\[
\mathcal{D} h := \nabla h - \frac{1}{d} (\mathbf{1} \cdot \nabla h) \mathbf{1}; \quad \partial_p h := \partial_{p_i} h - \frac{1}{d} \sum_{j \in [d]} \partial_{p_j} h, \quad i \in [d],
\]
or, equivalently (which allows to define \( \mathcal{D} h \) when \( h \) is just defined on \( S_{d-1} \)),
\[
\partial_{p_i} h(p) = -\frac{1}{d} \sum_{j \neq i} \partial_{p_j} \hat{h}^i(p^-), \quad p \in S_{d-1}.
\]
Of course we have \( \sum_j \partial_{p_j} h = \mathbf{1} \cdot \mathcal{D} h = 0 \) by construction. And the following holds true
\[
\partial_{p_i} h(p) - \partial_{p_j} h(p) = \partial_{p_i} h(p) - \partial_{p_j} h(p),
\]
for \( i, j \in [d] \) and \( p \in S_{d-1} \). As for the second order derivatives, we have
\[
\frac{\partial^2}{\partial p_i \partial p_j} h(p) = \frac{\partial^2}{\partial p_i \partial p_j} h(p) - \frac{1}{d} \sum_{k \in [d]} (\partial^2_{p_k p_k} h(p) + \partial^2_{p_k p_k} h(p)) + \frac{1}{d^2} \sum_{k,l \in [d]} \partial^2_{p_k p_l} h(p) \]
\[
= \frac{1}{d^2} \sum_{k,l \neq i} \partial^2_{p_k p_l} \hat{h}^i(p^-) - \frac{1}{d} \sum_{k \neq i} \partial^2_{p_k p_i} \hat{h}^i(p^-),
\]
(3.12)

---

\(^{10}\)When \( h \) is just defined on the simplex, the proof is slightly less obvious, but it may be achieved by slightly checking that \( \partial_{p_i} \hat{h}^i(p^-) = -\partial_{p_i} \hat{h}^i(p^-) \), for \( i \neq j \). And then, \( \sum_{i \in [d]} \sum_{j \neq i} \partial_{p_j} \hat{h}^i(p^-) = -\sum_{i \in [d]} \sum_{j \neq i} \partial_{p_i} \hat{h}^i(p^-) \). By Fubini’s theorem, the latter is also equal to \(-\sum_{j \in [d]} \sum_{i \neq j} \partial_{p_j} \hat{h}^i(p^-) \). By this, we deduce that it is indeed equal to \( 0 \).

\(^{11}\)The second line follows from the identity \( \partial_{p_i} \hat{h}^i(p^-) = \partial_{p_i} \hat{h}^i(p^-) - \partial_{p_i} \hat{h}^i(p^-) \) if \( i \neq l \) and \( \partial_{p_l} \hat{h}^i(p^-) = -\partial_{p_l} \hat{h}^i(p^-) \) if \( l = i \), which, in turn, implies
\[
\partial_{p_i} (\partial_{p_j} h) = -\frac{1}{d} \sum_{k \neq i} \partial_{p_k} \partial_{p_j} \hat{h}^i(p^-) = -\frac{1}{d} \sum_{k \neq i} \partial_{p_k} \left( -\frac{1}{d} \sum_{l \neq i,j} \partial_{p_l} \hat{h}^i(p^-) - \partial_{p_j} \hat{h}^i(p^-) + \frac{1}{d} \partial_{p_j} \hat{h}^i(p^-) \right) \]
\[
= -\frac{1}{d} \sum_{k \neq i} \partial_{p_k} \left( -\frac{1}{d} \sum_{l \neq i,j} \partial_{p_l} \hat{h}^i(p^-) + \partial_{p_j} \hat{h}^i(p^-) \right).
\]
and then the second order term in (3.7) (with \( h = U^i \), see (3.11)) becomes\(^{12}\)

\[
\sum_{j,k \in [d]} (p_j \delta_{jk} - p_j p_k) \partial_{p_j p_k} \hat{h}(p) = \sum_{j,k \in [d]} (p_j \delta_{jk} - p_j p_k) \partial_{p_j p_k}^2 \hat{h},
\]

since \( \sum_{k \in [d]} (p_j \delta_{jk} - p_j p_k) = 0 \) for any \( j \in [d] \).

In the end, the master equation (3.7) may be written in two equivalent forms. The first one may be written in terms of the derivatives in (3.9)–(3.10):

\[
\partial_t U^i(t, p) + H^i \left( \left( U^j(t, p) \right)_{j \in [d]} \right) + f^i(t, p) + \sum_{j \in [d]} \varphi(p_j) [U^j(t, p) - U^i(t, p)]
\]

\[
+ \sum_{j \neq i} \left( p_k \left[ \varphi(p_j) + (U^k(t, p) - U^j(t, p)) \right] - p_j \left[ \varphi(p_k) + (U^j(t, p) - U^k(t, p)) \right] \right) \partial_{p_i} (U^i(t, p)) = 0,
\]

\[
U^i(T, p) = g^i(p),
\]

for \((t, p) \in [0, T] \times \text{Int}(\hat{S}_{d-1})\). Above, the function \( \left( U^i \right) \) is defined on \( \hat{S}_{d-1} \) in the same way as before, namely \( \left( U^i \right) = U^i(t, p_1, \cdots, p_{i-1}, 1 - \sum_{j \neq i} p_j, p_{i+1}, \cdots, p_d) \), for \( p \in S_{d-1} \). For sure, we could rewrite the equation for \( U^i \) in terms of the variable \( p^{-i} \) (instead of \( p^{-j} \)) for another index \( j \neq i \), but this would be of little interest for us. In fact, we will make greater use of a second form of (3.7) that may be written in terms of the intrinsic derivative:

\[
\partial_t U^i(t, p) + H^i \left( \left( U^j(t, p) \right)_{j \in [d]} \right) + f^i(t, p) + \sum_{j \in [d]} \varphi(p_j) [U^j(t, p) - U^i(t, p)]
\]

\[
+ \left( p_k \left[ \varphi(p_j) + (U^k(t, p) - U^j(t, p)) \right] - p_j \left[ \varphi(p_k) + (U^j(t, p) - U^k(t, p)) \right] \right) \partial_{p_i} (U^i(t, p))
\]

\[
+ \varepsilon^2 \sum_{j \neq i} p_j (\partial_{p_i} U^i(t, p) - \partial_{p_j} U^j(t, p)) + \varepsilon^2 \sum_{j, k \in [d]} (p_j \delta_{jk} - p_j p_k) \partial_{p_j p_k}^2 U^i(t, p) = 0,
\]

\[
U^i(T, p) = g^i(p),
\]

for \((t, p) \in [0, T] \times \text{Int}(\hat{S}_{d-1})\).

\(^{12}\)The result may be also proved by combining (3.11) and (3.12), hence bypassing the derivatives of \( h \) themselves. Indeed,

\[
\sum_{j, k \in [d]} (p_j \delta_{jk} - p_j p_k) \partial_{p_j p_k}^2 \hat{h} = -\frac{1}{d} \sum_{j, k \in [d]} (p_j \delta_{jk} - p_j p_k) \sum_{l \neq k} \partial_{p_j p_k}^2 \hat{h}(p^{-k})
\]

\[
= -\frac{1}{d} \sum_{j, k \in [d]} (p_j \delta_{jk} - p_j p_k) \left( \sum_{l \neq k, i} \partial_{p_j p_k}^2 \hat{h}^i(p^{-i}) - \partial_{p_j p_k}^2 \hat{h}^i(p^{-i}) \right)
\]

\[
= -\frac{1}{d} \sum_{j, k \in [d]} (p_j \delta_{jk} - p_j p_k) \left( \sum_{l \neq k} \partial_{p_j p_k}^2 \hat{h}^i(p^{-i}) - d \cdot \partial_{p_j p_k}^2 \hat{h}^i(p^{-i}) \right)
\]

\[
= \sum_{j, k \in [d]} (p_j \delta_{jk} - p_j p_k) \partial_{p_j p_k}^2 \hat{h}^i(p^{-i}).
\]
Actually, we must point out that, in order to apply Theorem 10.0.2 in [26], as we do later, the master equation should be satisfied also in boundary points, under the appropriate local chart; see (10.1) therein. This would require to formulate (3.13) for each local chart used in the construction of the Wright–Fisher spaces, see [2.25] that is, for any projection \( p^{-1} \) with \( i \in [d] \), since we verified in (2.25) that those changes of variables make the second order operator fit the setup of [26]. In this respect, we observe that there is no hindrance for us to restate (3.13) in the right system of coordinates. Also, we already noticed that, as long as we look for solutions \( U^1, \ldots, U^d \) in the space \( C_{WF}^{1+\gamma/2,2+\gamma}(0, T) \times S_{d-1} \) for some \( \gamma \in (0, 1) \), the first order derivatives always extend by continuity up to the boundary. Still, the second order derivatives are defined only in the interior of the simplex and are allowed to blow-up at the boundary. Fortunately, the rate of explosion of those second order derivatives, as prescribed by (2.32), combine well with the degeneracy property of the operator on the boundary. Hence, by a standard continuity argument, it is enough to require that (3.14), which is written in terms of the intrinsic derivative, holds in \( \text{Int}(\hat{S}_{d-1}) \).

3.2.2. Connection between the MFG system and the master equation. The connection between the master equation (3.13)–(3.14) and the MFG system (3.3) is given by the following statement.

**Theorem 3.3.** Assume that there exists a \( d \)-tuple \((U^1, \ldots, U^d)\) of real valued functions defined on \([0, T] \times S_{d-1}\) such that, for any \( i \in [d] \), \( U^i \) belongs to the Wright–Fisher space \( C_{WF}^{1+\gamma'/2,2+\gamma'}([0, T] \times S_{d-1}) \) for some \( \gamma' > 0 \) (see (2.2.3) for the definition), and for any \((t, p) \in [0, T] \times \text{Int}(\hat{S}_{d-1})\), equation (3.13) holds at \((t, p)\). Then, for any (deterministic) initial condition \((p_{0,i})_{i \in [d]} \in S_{d-1}\), with \( p_{0,i} > 0 \) for any \( i \in [d] \), the MFG system (3.3) has a unique solution \((P_t = (P_t^i)_{i \in [d]}, U_t = (u_t^i)_{i \in [d]}, \nu_t = (\nu_t^{i,j,k})_{i,j,k \in [d]: j \neq k})_{0 \leq t \leq T}\) in the class of \( \mathbb{P}^W \)-progressively-measurable processes \((\tilde{P}_t = (\tilde{P}_t^i)_{i \in [d]}, \tilde{u}_t = (\tilde{u}_t^i)_{i \in [d]}, \tilde{\nu}_t = (\tilde{\nu}_t^{i,j,k})_{i,j,k \in [d]: j \neq k})_{0 \leq t \leq T}\) such that \((\tilde{P}_t)_{0 \leq t \leq T}\) is continuous and takes values in \( S_{d-1}\), \((\tilde{u}_t)_{0 \leq t \leq T}\) is continuous and is bounded by a deterministic constant and \((\tilde{\nu}_t)_{0 \leq t \leq T}\) satisfies \( \mathbb{E}[\int_0^T \exp(\varepsilon^2 \sum_{i \in [d]} \int_0^t (1/\tilde{P}^i_s)ds)|\tilde{\nu}_t|^2 dt] < \infty\).

The solution satisfies, \( \mathbb{dP}\) almost surely, for all \( t \in [0, T] \) and all \( i \in [d] \), \( u_t^i = U^i(t, P_t)\), and, \( \mathbb{dP} \otimes dt \) almost everywhere, for all \( i, j, k \in [d] \), with \( j \neq k \), \( \nu_t^{i,j,k} = V^{i,j,k}(t, P_t)\), where

\[
V^{i,j,k}(t, p) := \frac{\varepsilon}{\sqrt{2}}(\mathcal{O}_{p_j}U^i(t, p) - \mathcal{O}_{p_k}U^j(t, p)) \sqrt{p_j p_k}.
\]

We remark that the main result, Theorem 2.9, follows from the above theorem, thanks to what we discussed in (3.1.2). Before we perform the proof of Theorem 3.3, we state the following corollary.

**Corollary 3.4.** The master equation (3.13)–(3.14) is at most uniquely solvable in the classical sense. In words, it has at most one solution \((U^1, \ldots, U^d)\) such that, for any \( i \in [d] \), \( U^i \) belongs to the Wright–Fisher space \( C_{WF}^{1+\gamma'/2,2+\gamma'}([0, T] \times S_{d-1}) \) for some \( \gamma' > 0 \), and for any \((t, p) \in [0, T] \times \text{Int}(\hat{S}_{d-1})\), (3.13)–(3.14) hold at \((t, p)\).

**Proof of Corollary 3.4.** By Theorem 3.3, we know that, for any initial condition \( p_0 = (p_{0,i})_{i \in [d]} \in \text{Int}(\hat{S}_{d-1})\), the system (3.3) has a unique solution. Hence, for any two solutions \( U \) and \( U' \) to the master equation, one has \( U(0, p_0) = U'(0, p_0)\). Since \( p_0 \) is arbitrary, we get that \( U(0, \cdot) \) and \( U'(0, \cdot) \) coincide on \( \text{Int}(\hat{S}_{d-1})\). By continuity, they coincide up to the boundary. Here, the initial time is arbitrary and we can replace the initial time 0 by any other initial time \( t \in (0, T)\). \( \square \)

3.2.3. Proof of Theorem 3.3. The proof is inspired by [51], but the fact that the master equation is set on the simplex makes it more difficult. Also, we recall that \( \kappa \geq (61 + d)^2 \).
**First Step.** In order to prove the existence of a solution, one may first solve the SDE

\[
dP_t^i = \sum_{j \in [d]} \left( P_t^j [\varphi(P_t^i) + (U^j(t, P_t) - U^i(t, P_t))_+] - P_t^i [\varphi(P_t^j) + (U^i(t, P_t) - U^j(t, P_t))_+] \right) dt + \varepsilon \sum_{j \in [d]: j \neq i} \sqrt{P_t^i P_t^j} dW^{i,j}_t, \quad t \in [0, T],
\]

with \( p_0 = (p_{0,i})_{i \in [d]} \) as initial condition. Solvability is a mere consequence of Proposition 2.1.

Then, it suffices to let \( u_t^i := U^i(t, P_t) \) and \( \nu_t^{i,j,k} := V^{i,j,k}(t, P_t) \), for \( t \in [0, T] \) and \( i, j, k \in [d] \), with \( j \neq k \). By Itô’s formula (see the next step if needed for the details), we may easily expand \( (u_t^i)_{0 \leq t \leq T} \) and check that it solves the backward equation in (3.3). (The fact that the second-order derivatives are just defined on the interior of simplex is not a hindrance since \((P_t)_{0 \leq t \leq T}\) does not touch the boundary, see Proposition 2.1.) Obviously, the processes \((u_t^i)_{0 \leq t \leq T}, i \in [d]\) and \((\nu_t^{i,j,k})_{0 \leq t \leq T}, i, j, k \in [d], j \neq k\) are bounded and hence satisfy the required growth conditions.

**Second Step.** Consider now another solution, say

\[
\left( ((\tilde{P}_t^i)_{0 \leq t \leq T})_{i \in [d]}, ((\tilde{u}_t^i)_{0 \leq t \leq T})_{i \in [d]}, (\tilde{\nu}_t^{i,j,k})_{0 \leq t \leq T}, i, j, k \in [d], j \neq k \right)
\]

to (3.3). We denote \( \tilde{U}_t^i := U^i(t, \tilde{P}_t), \partial_{p_j} \tilde{U}_t^i := \partial_{p_j} U^i(t, \tilde{P}_t), \partial_{p_j p_k}^2 \tilde{U}_t^i := \partial_{p_j p_k}^2 U^i(t, \tilde{P}_t) \), for \( t \in [0, T] \) and \( i, j, k \in [d], j \neq k \). Thanks to the fact that \( U^i \in C^{1,2}([0, T] \times Int(\hat{S}_{d-1})) \), we can apply Itô’s formula to \((\tilde{U}_t^i)_{0 \leq t \leq T}\) (which obviously coincides with \((\tilde{U}_t^i)_{0 \leq t \leq T}\)). We get (the computations of the various intrinsic derivatives that appear in the expansion are similar to those in (3.14))

\[
d\tilde{U}_t^i = \left\{ \partial_t \tilde{U}_t^i + \frac{\varepsilon^2}{2} \sum_{j,k \in [d]} (\tilde{P}_t^j \delta_{jk} - \tilde{P}_t^j \tilde{P}_t^k) \partial_{p_j p_k}^2 \tilde{U}_t^i \right. \\
+ \sum_{j,k \in [d]} \tilde{P}_t^k \left[ \varphi(\tilde{P}_t^i, \tilde{U}_t^i - \tilde{U}_t^j) \right] \left( \partial_{p_j} \tilde{U}_t^i - \partial_{p_k} \tilde{U}_t^i \right) \right\} dt \\
+ \varepsilon \sqrt{2} \sum_{j,k \in [d]: j \neq k} \sqrt{\tilde{P}_t^i \tilde{P}_t^k} \left( \partial_{p_j} \tilde{U}_t^i - \partial_{p_k} \tilde{U}_t^i \right) dW_t^{j,k} \\
= -\left\{ H^i(\tilde{U}_t) + f^i(t, \tilde{P}_t) + \sum_{j \in [d]} \varphi(\tilde{P}_t^j) [\tilde{U}_t^j - \tilde{U}_t^i] \right. \\
+ \varepsilon^2 \sum_{j \in [d]} \tilde{P}_t^j \left( \partial_{p_j} \tilde{U}_t^i - \partial_{p_j} \tilde{U}_t^i \right) \right\} dt + \frac{\varepsilon}{\sqrt{2}} \sum_{j,k \in [d]: j \neq k} \sqrt{\tilde{P}_t^i \tilde{P}_t^k} \left( \partial_{p_j} \tilde{U}_t^i - \partial_{p_k} \tilde{U}_t^i \right) dW_t^{j,k},
\]

where in the last equality we used the equation (3.14) satisfied by \( U \). This prompts us to let

\[
\tilde{V}_t^{i,j,k} = \frac{\varepsilon}{\sqrt{2}} \sqrt{\tilde{P}_t^i \tilde{P}_t^k} \left( \partial_{p_j} \tilde{U}_t^i - \partial_{p_k} \tilde{U}_t^i \right), \quad t \in [0, T], \quad i, j, k \in [d], \quad j \neq k.
\]
Subtracting the equation satisfied by \((\widetilde{u}_i^*)_{i \in [d]}\), we get
\[
d(\widetilde{U}_t^i - \widetilde{u}_i^*) = - \left\{ H^i(\widetilde{U}_t) - H^i(\widetilde{u}_t) + \sum_{j \in [d]} \varphi(\widetilde{P}_t^j) \left[ \widetilde{U}_t^j - \widetilde{u}_t^j - (\widetilde{U}_t^i - \widetilde{u}_t^i) \right] \right\} dt + \frac{\varepsilon}{\sqrt{2}} \sum_{j \in [d]} \sqrt{\frac{P_t^j}{P_t}} \left( \widetilde{V}_t^{i,i,j} - \widetilde{v}_t^{i,i,j} - (\widetilde{V}_t^{i,j,i} - \widetilde{v}_t^{i,j,i}) \right) dt + \sum_{j,k \in [d], j \neq k} (\widetilde{V}_t^{i,j,k} - \widetilde{v}_t^{i,j,k}) dW_t^{j,k}, \quad t \in [0, T], \quad i \in [d].
\]

Consider now
\[
e_t := \exp \left( \varepsilon^2 \int_0^t \frac{1}{P_s} ds \right), \quad t \in [0, T].
\]

Then, by Itô’s formula, we obtain, for any \(t \in [0, T]\),
\[
e_t |\widetilde{U}_t^i - \widetilde{u}_t^i|^2 + \int_t^T \varepsilon^2 e_s |\widetilde{U}_s^i - \widetilde{u}_s^i|^2 \left( \sum_{j \in [d]} \frac{1}{P_s^j} \right) ds + \int_t^T e_s \sum_{j,k \in [d], j \neq k} |\widetilde{V}_s^{i,j,k} - \widetilde{v}_s^{i,j,k}|^2 ds
\]
\[
= 2 \int_t^T e_s (\widetilde{U}_s^i - \widetilde{u}_s^i) \left\{ H^i(\widetilde{U}_s) - H^i(\widetilde{u}_s) + \sum_{j \in [d]} \varphi(\widetilde{P}_s^j) \left[ \widetilde{U}_s^j - \widetilde{u}_s^j - (\widetilde{U}_s^i - \widetilde{u}_s^i) \right] \right\}
\]
\[
+ \frac{\varepsilon}{\sqrt{2}} \sum_{j \in [d]} \sqrt{\frac{P_t^j}{P_t}} \left( \widetilde{V}_s^{i,i,j} - \widetilde{v}_s^{i,i,j} - (\widetilde{V}_s^{i,j,i} - \widetilde{v}_s^{i,j,i}) \right) ds + \sum_{j,k \in [d], j \neq k} (\widetilde{V}_s^{i,j,k} - \widetilde{v}_s^{i,j,k}) dW_s^{j,k}.
\]

By Proposition 2.2 (together with Remark 2.4), with \(\lambda = 2d - 1, \gamma = 60 \varepsilon^2\), and \(\kappa \geq (61 + d) \varepsilon^2\), therein, and by Hólder’s inequality, \(E[e_t^2]\) is finite. Since \((\widetilde{U}_t^i)_{i \in [d]}\) and \((\widetilde{u}_t^i)_{i \in [d]}\) are bounded (by deterministic constants) and \((\widetilde{V}_t^{i,j,k})_{i,j,k \in [d], j \neq k}\) are square-integrable, all the terms in the right-hand side have integrable sup norm (over \(t \in [0, T]\)); as for the last term in the right-hand side, the latter follows from Burkholder–Davis–Gundy inequalities. Also, we can treat the difference \(H^i(\widetilde{U}_s) - H^i(\widetilde{u}_s)\) as a Lipschitz difference, since \(U\) and \(\bar{u}\) are bounded. Hence, taking expectations and applying Young’s inequality, we can find a constant \(C\) such that
\[
E \left[ e_t |\widetilde{U}_t^i - \widetilde{u}_t^i|^2 + \int_t^T \varepsilon^2 e_s |\widetilde{U}_s^i - \widetilde{u}_s^i|^2 \left( \sum_{j \in [d]} \frac{1}{P_s^j} \right) ds + \int_t^T e_s \sum_{j,k \in [d], j \neq k} |\widetilde{V}_s^{i,j,k} - \widetilde{v}_s^{i,j,k}|^2 ds \right]
\]
\[
\leq C \sum_{j \in [d]} E \left[ \int_t^T e_s |\widetilde{U}_s^j - \widetilde{u}_s^j|^2 ds \right] + \varepsilon^2 E \left[ \int_t^T e_s |\widetilde{U}_s^i - \widetilde{u}_s^i|^2 ds \right]
\]
\[
+ E \left[ \int_t^T e_s \sum_{j,k \in [d], j \neq k} |\widetilde{V}_s^{i,j,k} - \widetilde{v}_s^{i,j,k}|^2 ds \right].
\]

We obtain,
\[
\sum_{j \in [d]} E \left[ e_t |\widetilde{U}_t^j - \widetilde{u}_t^j|^2 \right] \leq C \sum_{j \in [d]} \int_t^T E \left[ e_s |\widetilde{U}_s^j - \widetilde{u}_s^j|^2 \right] ds,
\]
and thus Gronwall’s lemma yields, for any $i \in [d]$ and any $t \in [0, T]$,  
$$\mathbb{P} \left( \tilde{u}_i^t = \tilde{U}_i^t = U^i(t, \tilde{P}_i) \right) = 1.$$  
This permits to identify $(\tilde{P}_i)_{0 \leq t \leq T}$ with the solution of (3.15). It is then pretty straightforward to show that $(\tilde{u}_i^t)_{0 \leq t \leq T}$ coincides with $(u_i^t)_{0 \leq t \leq T}$, for each $i \in [d]$, and then that $(\tilde{r}_i^{t,j,k})_{0 \leq t \leq T}$ coincides with $(r_i^{t,j,k})_{0 \leq t \leq T}$, for each $i, j, k \in [d], j \neq k$.  

### 3.3. Solvability of the master equation.

Solvability of the master equation (3.13)–(3.14) is the main issue. The first point is to observe that it may be rewritten in a somewhat generic form. Indeed, for a given coordinate $i \in [d]$, we may let

$$B^i_j(t,p,y) := \varphi(p_j) + \sum_{k \in [d]} p_k (y_k - y_j)_+ - p_j \sum_{k \in [d]} [\varphi(p_k) + (y_j - y_k)_+] + \varepsilon^2 (\delta_{i,j} - p_j),$$

$$F^i(t,p,y) := H^i(y) + f^i(t,p) + \sum_{k \in [d]} \varphi(p_j)(y_j - y_i),$$

where $t \in [0, T], p \in S_{d-1}$ and $y = (y_k)_{k \in [d]} \in \mathbb{R}^d$. Then, we may rewrite (3.14) in the form

$$\partial_t U^i(t,p) + F^i(t,p,U(t,p)) + \sum_{j \in [d]} B^i_j(t,p,U(t,p)) \partial_{p_j} U^i(t,p)$$

$$+ \frac{\varepsilon^2}{2} \sum_{j,k \in [d]} (p_j \delta_{j,k} - p_j p_k) \partial^2_{p_j p_k} U^i(t,p) = 0,$$

for $t \in [0, T]$ and $p \in \text{Int}(S_{d-1})$, with the shorten notation $U(t,p) = (U^i(t,p))_{i \in [d]}$. For sure, we could write (3.13) in a similar form. In fact, what really matters is that

$$\sum_{j \in [d]} B^i_j(t,p,y) = 0,$$

for any $i \in [d], t \in [0, T], p \in \text{Int}(S_{d-1})$ and $y = (y_i)_{i \in [d]} \in \mathbb{R}^d$, and that $B^i_j(t,p,y) > 0$ whenever $p_j = 0$.

In the sequel, solvability of (3.18) is addressed in several steps. The first one is to address the solvability of the linear version of (3.18) obtained by freezing the nonlinear component $U$ in $B^i$ and $F^i$; as we make it clear below, this mostly follows from the earlier results of [26]. The second one is to prove a priori estimates for the solutions to the latter linear version independently of the nonlinear component $U$ that is frozen in the coefficients $B^i$ and $F^i$; actually, this is the core of our paper. The last step is to deduce the existence of a classical solution to the master equation by means of Schauder’s fixed point theorem.

#### 3.3.1. Linear version.

The linear analogue of the equation (3.18) for $U^i$, for one given $i \in [d]$, may be written in the generic form

$$\partial_t u(t,p) + \sum_{j \in [d]} \left( \varphi(p_j) + b_j(t,p) + p_j b^\circ_j(t,p) \right) \partial_{p_j} u(t,p)$$

$$+ \frac{\varepsilon^2}{2} \sum_{j,k \in [d]} (p_j \delta_{j,k} - p_j p_k) \partial^2_{p_j p_k} u(t,p) + h(t,p) = 0,$$

$$u(T,p) = \ell(p),$$

for $t \in [0, T], p \in \text{Int}(S_{d-1})$ and $u = (u_i)_{i \in [d]} \in \mathbb{R}^d$.
for \((t, p) \in [0, T] \times \text{Int}(\tilde{S}_{d-1})\), \(b = (b_j)_{j \in [d]} : [0, T] \times S_{d-1} \rightarrow \mathbb{R}^d\), \(b^\circ = (b_j^\circ)_{j \in [d]} : [0, T] \times S_{d-1} \rightarrow \mathbb{R}^d\), \(h : [0, T] \times S_{d-1} \rightarrow \mathbb{R}\) and \(\ell : S_{d-1} \rightarrow \mathbb{R}\) are bounded and satisfy
\[
\sum_{j \in [d]} \left( \varphi(p_j) + b_j(t, p) + p_j b_j^\circ(t, p) \right) = 0, \quad t \in [0, T], \quad p \in S_{d-1},
\]
the function \(\varphi\) being as in \([2.16]\) and the unknown \(u\) in \((3.19)\) being here real-valued (in words, it is an equation and not a system of equations). Our first lemma is

**Lemma 3.5.** Assume that the functions \((b_j)_{j \in [d]}\) and \((b_j^\circ)_{j \in [d]}\) are in \(\mathcal{C}^{\eta/2, \eta}([0, T] \times S_{d-1})\), that \(h\) is in \(\mathcal{C}^{2+\eta/2}_{\text{WF}}([0, T] \times S_{d-1})\) and \(\ell\) is in \(\mathcal{C}^{2+\eta/2}_{\text{WF}}(S_{d-1})\), for some \(\eta \in (0, 1)\). Then, equation \((3.19)\) has a unique classical solution in the space \(\mathcal{C}^{1+\eta/4, 2+\eta/2}_{\text{WF}}([0, T] \times S_{d-1})\).

**Proof.** For a fixed \(t_0 \in [0, T]\), we rewrite \((3.19)\) in the form
\[
\partial_t u(t, p) + \sum_{j \in [d]} \left( \varphi(p_j) + b_j(t_0, p) + p_j b_j^\circ(t_0, p) \right) \partial_{p_j} u(t, p) + \frac{\varepsilon^2}{2} \sum_{j, k \in [d]} \left(p_j \delta_{j k} - p_j p_k\right) \partial_{p_j}^2 u(t, p)
\]
\[+ \sum_{j \in [d]} \left((b_j(t, p) + p_j b_j^\circ(t, p)) - (b_j(t_0, p) + p_j b_j^\circ(t_0, p))\right) \partial_{p_j} u(t, p) + h(t, p) = 0,
\]
\(u(T, p) = \ell(p),\)
for \((t, p) \in [0, T] \times \text{Int}(\tilde{S}_{d-1})\). Our first goal is to solve the equation on \([t_0, T]\) \times \text{Int}(\tilde{S}_{d-1})\) provided \(t_0\) is chosen close enough to \(T\).

In order to solve the above equation, we define the following mapping. For a vector-valued function \(w = (w_j)_{j \in [d]} : [0, T] \times S_{d-1} \rightarrow \mathbb{R}^d\) whose components are in \(\mathcal{C}^{\eta/4, \eta/2}_{\text{WF}}([0, T] \times S_{d-1})\), we call \(v\) the solution of the equation
\[
\partial_t v(t, p) + \sum_{j \in [d]} \left( \varphi(p_j) + b_j(t_0, p) + p_j b_j^\circ(t_0, p) \right) \partial_{p_j} v(t, p) + \frac{\varepsilon^2}{2} \sum_{j, k \in [d]} \left(p_j \delta_{j k} - p_j p_k\right) \partial_{p_j}^2 v(t, p)
\]
\[+ \sum_{j \in [d]} \left((b_j(t, p) + p_j b_j^\circ(t, p)) - (b_j(t_0, p) + p_j b_j^\circ(t_0, p))\right) w_j(t, p) + h(t, p) = 0,
\]
\(v(T, p) = \ell(p),\)
for \((t, p) \in [t_0, T]\) \times \text{Int}(\tilde{S}_{d-1})\), the solution being known, by \([26]\) Theorem 10.0.2, to exist and to satisfy
\[
\|v\|_{1+\eta/4, 2+\eta/2; [t_0, T]} \leq C \left(\|\varphi\|_{2+\eta/2} + \|W\|_{\eta/4, \eta/2; [t_0, T]} + \|h\|_{\eta/4, \eta/2; [t_0, T]}\right),
\]
where we added the notation \([t_0, T]\) in the Wright–Fisher norm in order to emphasize the fact that the underlying domain is \([t_0, T]\) \times \(S_{d-1}\) and not \([0, T]\) \times \(S_{d-1}\), and with
\[
W(t, p) := \sum_{j \in [d]} \left((b_j(t, p) + p_j b_j^\circ(t, p)) - (b_j(t_0, p) + p_j b_j^\circ(t_0, p))\right) w_j(t, p).
\]
Clearly, we can find a universal constant \(c > 0\) such that
\[
\|W\|_{\eta/4, \eta/2; [t_0, T]} \leq c \sum_{j \in [d]} \|B_j^\circ\|_{\eta/4, \eta/2; [t_0, T]} \|w_j\|_{\eta/4, \eta/2; [t_0, T]},
\]
with
\[
B_j^\circ(t, p) := b_j(t, p) + p_j b_j^\circ(t, p) - \left(b_j(t_0, p) + p_j b_j^\circ(t_0, p)\right), \quad (t, p) \in [t_0, T] \times S_{d-1}.
\]
Now, we can find a constant $C$, only depending on the Wright–Fisher norm of $b = (b_j)_{j \in [d]}$ such that, for any $s, t \in [0, T]$ and any $p, q \in S_{d-1}$,
\[
|b(t, p) - b(t, p) - (b(s, q) - b(t, q))| \\
= |b(t, p) - b(t, p) - (b(s, q) - b(t, q))|^{1/2} |b(t, p) - b(t, p) - (b(s, q) - b(t, q))|^{1/2} \\
\leq C \left( |b(t, p) - b(t, p)| + |b(s, q) - b(t, q)| \right)^{1/2} \left( |b(t, p) - b(s, q)| + |b(t, p) - b(t, q)| \right)^{1/2} \\
\leq C (T - t_0)^{n/4} \left( |t - s|^{n/4} + |\sqrt{p} - \sqrt{q}|^{n/2} \right),
\]
which shows that the Wright–Fisher Hölder norm (of exponents $(\eta/4, \eta/2)$) of $b - b(t_0, \cdot)$ is small with $T - t_0$ (it is easy to see that sup norm is small with $T - t_0$). Proceeding in a similar way with the other functions entering the definition of $B^\circ$, we deduce that the Wright–Fisher Hölder norm (of exponent $\eta/2$) of $B^\circ$ is small with $T - t_0$.

Therefore, Schauder’s estimates \((3.22)\) imply that, for $T - t_0$ small enough
\[
\|\partial_p v\|_{n/4,n/2;[t_0,T]} \leq C \left( \|\ell\|_{2+n/2;[t_0,T]} + \|h\|_{n/4,n/2;[t_0,T]} \right) + \frac{1}{2} \|w\|_{n/4,n/2;[t_0,T]},
\]
for a constant $C$ which is independent of $w$ and $t_0$. This shows in particular that $\|\partial_p v\|_{n/4,n/2;[t_0,T]} \leq 2C (\|\ell\|_{2+n/2;[t_0,T]} + \|h\|_{n/4,n/2;[t_0,T]})$ whenever $\|w\|_{n/4,n/2;[t_0,T]} \leq 2C (\|\ell\|_{2+n/2;[t_0,T]} + \|h\|_{n/4,n/2;[t_0,T]}).

In particular, the map $w \mapsto v$ preserves a closed ball of $[C_{WF}^{n/4,n/2}([0, T] \times S_{d-1})]^d$. By linearity, the map $w \mapsto v$ is obviously continuous from $[C_{WF}^{n/4,n/2}([0, T] \times S_{d-1})]^d$ into itself, for any $n' \in (0, n]$. By Schauder’s theorem (regarding any closed ball of $[C_{WF}^{n/4,n/2}([0, T] \times S_{d-1})]^d$ as a compact subset of $[C_{WF}^{1+n/4,n/2}([0, T] \times S_{d-1})]^d$, for $n' \in (0, n)$, we deduce that there exists a solution $v$ to \((3.21)\) (and hence to \((3.19)\)) in $C_{WF}^{1+n/4,2+n/2}([0, T] \times S_{d-1})$ (so on $[0, T] \times S_{d-1}$).

By iterating in time, we deduce that there exists a solution to \((3.19)\) on the entire $[0, T] \times S_{d-1}$ in the space $C_{WF}^{1+n/4,2+n/2}([0, T] \times S_{d-1})$.

Uniqueness follows from a straightforward application of Kolmogorov representation formula, see Proposition \((4.1)\) if needed.

The main technical result of the paper may be formulated as follows:

**Theorem 3.6.** Assume that $(b_j)_{j \in [d]}$, $(b_j^2)_{j \in [d]}$ and $\ell$ is Lipschitz continuous (we let $\|\ell\|_{1,\infty} = \|\ell\|_{1,\infty} + \sup_{p \neq q} |\ell(p) - \ell(q)|/|p - q|$). Then, there exists an exponent $\eta \in (0, 1)$ such that, for any given $\delta \in (0, 1/(4\sqrt{d})$ and $\varepsilon \in (0, 1)$, we can find a threshold $\kappa_0 > 0$, depending on $\varepsilon$ and $(\|b_j\|_{1,\infty})_{j \in [d]}$, such that, for any $\kappa \geq \kappa_0$, we can find another constant $C$, only depending on $\delta$, $\varepsilon$, $\kappa$, $(\|b_j\|_{1,\infty})_{j \in [d]}$, $(\|b_j^2\|_{1,\infty})_{j \in [d]}$, $\|h\|_{1,\infty}$, $\|\ell\|_{1,\infty}$ and $T$, such that any solution $u \in C^{1,2}([0, T] \times \text{Int}(S_{d-1}), \mathbb{R}) \cap C^0([0, T] \times S_{d-1}, \mathbb{R})$ of \((3.19)\) satisfies\(^3\)
\[
|u(t, p) - u(s, q)| \leq C (|t - s|^{n/2} + |p - q|^n), \quad (s, t) \in [0, T], \quad (p, q) \in S_{d-1}.
\]
Moreover, $\|u\|_{\infty}$ is less than $\|\ell\|_{\infty} + T\|h\|_{\infty}$.

**Remark 3.7.** We stress that we are not aware of any similar a priori Hölder estimate in the literature. There are some papers about the Hölder regularity of elliptic equations with degeneracies near the boundary, but they do not fit our framework (besides the obvious fact that the underlying equations are elliptic while ours is parabolic): We refer for instance to \(^{30}\) for a case with a specific instance of drift that does not cover our needs. We also emphasize that the

\(^3\)The notation $C^{1,2}$ is here understood in the usual sense: $u$ is required to be once continuously differentiable in $t$ and twice continuously differentiable in $p$ on the interior of the simplex, the notion of derivative being the same as in \(^{21}\). As for the notation $C^0$, it refers to functions that are continuous in $(t, p)$. 
\textbf{Hölder estimate in Theorem 3.6} does not depend on the modulus of continuity of the coefficients \((b_j)_{j \in [d]}, (b^2_j)_{j \in [d]}\) and \(h\). In fact, we here assume the latter to be continuous for convenience only as it suffices for our own purposes. We strongly believe that the result would remain true if \((b_j)_{j \in [d]}, (b^2_j)_{j \in [d]}\) and \(h\) were merely bounded and measurable, but this would certainly ask for an additional effort.

On another matter, it is worth noticing that we may trace back explicitly the dependence of \(\kappa_0\) over \(\varepsilon\). The key point in the proof is inequality (4.35), which shows that \(\kappa_0\) may be taken of the form \(\kappa_0 = \varepsilon^\gamma \kappa_00\), for \(\kappa_00\) only depending on \((\|b_j\|_\infty)_{j \in [d]}\). The parameter \(\eta\) therein is a free parameter that is eventually chosen as \(1/2\), see the discussion after Proposition 4.6.

The proof relies on a coupling argument, which is addressed in the next section.

### 3.3.2. Fixed point argument via Schauder’s theorem

Here is now the last step of our proof. To make it clear, Theorem 3.8 below together with the previous Theorem 3.3 imply the main Theorem 2.9.

\textbf{Theorem 3.8.} Assume that, for some \(\gamma \in (0, 1/2]\), each \(f_i\), for \(i \in [d]\), belongs to \(\mathcal{C}^{\gamma/2, \gamma}_{\operatorname{WF}}([0, T] \times S_{d-1})\), and each \(g_i\), for \(i \in [d]\), belongs to \(\mathcal{C}^{2+\gamma}_{\operatorname{WF}}(S_{d-1})\). Then, for any \(\varepsilon \in (0, 1)\), there exist a universal exponent \(\eta \in (0, 1)\) (hence independent of \(\varepsilon\)) and a threshold \(\kappa_0 > 0\), only depending on \(\varepsilon\), \(\|f\|_\infty\), \(\|g\|_\infty\), and \(T\), such that, for any \(\kappa \geq \kappa_0\) and \(\delta \in (0, 1/(4\sqrt{d})\), the master equation (3.13) has a solution in \([\mathcal{C}^{1+\gamma/2, 2+\gamma}_{\operatorname{WF}}([0, T] \times S_{d-1})]^d\), for \(\gamma' = \min(\gamma, \eta)/2\).

\textbf{Proof.} The proof holds in two steps.

**First Step.** We first consider the following nonlinear variant of (3.19):

\begin{equation}
\begin{aligned}
\partial_t U^i(t, p) + \sum_{j \in [d]} \left( \varphi(p_j) + b_j^1(t, p, U(t, p)) + p_j b_j^2(t, p, U(t, p)) \right) \partial_{p_j} U^i(t, p) \\
+ \frac{\varepsilon^2}{2} \sum_{j, k \in [d]} \left( p_j \delta_{jk} - p_j p_k \right) b_j^2(t, p, U(t, p)) + h^i(t, p, U(t, p)) = 0,
\end{aligned}
\end{equation}

where for any \(i \in [d]\), \(b_i = (b_j^1)_{j \in [d]} : [0, T] \times S_{d-1} \times \mathbb{R}^d \to (\mathbb{R}_+)^d\), \(b^o = (b^o_j)_{j \in [d]} : [0, T] \times S_{d-1} \times \mathbb{R}^d \to \mathbb{R}^d\) and \(h^i : [0, T] \times S_{d-1} \times \mathbb{R}^d \to \mathbb{R}\). We are going to prove the existence of a solution \(U = (U^1, \cdots, U^d)\) to (3.23) whenever, for some constant \(C_0 \geq 0\), the functions \((b^o_i)_{i \in [d]}\), \(b^o\) and \((h^i)_{i \in [d]}\) are bounded by \(C_0\) and satisfy the following regularity properties

\begin{equation}
\begin{aligned}
&|b^i(t, p, y) - b^i(s, q, z)| + |b^o(t, p, y) - b^o(s, q, z)| + |h^i(t, p, y) - h^i(s, q, z)| \\
&\leq C_0(|t - s|^{\gamma/2} + |p - q|\,|y - z|),
\end{aligned}
\end{equation}

for \(i \in [d], s, t \in [0, T], p, q, y, z \in \mathbb{R}^d\). Without any loss of generality, we can assume that \(\max_{i \in [d]} \|g_i\|_{1, \infty} \leq C_0\). Existence of a classical solution to (3.23) is then proved by a new application of Schauder’s fixed point theorem. To do so, we call \(\eta\) and \(C\) the exponent and the constant from Theorem 3.6 when \(\|b\|_\infty\), \(\|b^o\|_\infty\), \(\|h\|_\infty\) and \(\|\ell\|_{1, \infty}\) are less than \(C_0\). We then take an input function \(V = (V^1, \cdots, V^d) \in [\mathcal{C}^{\eta/2, \eta}_{\operatorname{WF}}([0, T] \times S_{d-1})]^d\) such that, for each \(i \in [d]\), \(\|V^i\|_\eta/2, \eta \leq C\). By Lemma 3.3, with \(\eta\) therein being replaced by \(\min(\eta, \gamma)\), we can solve (3.23) for each \(i \in [d]\) when, in the nonlinear terms, \(U\) is replaced by \(V\). We call \(U = (U^1, \cdots, U^d)\) the solution. It belongs to \([\mathcal{C}^{1+\gamma/2, 2+\gamma}_{\operatorname{WF}}([0, T] \times S_{d-1})]^d\). By Theorem 3.6 it also satisfies \(\|U^i\|_\eta/2, \eta \leq C\), for each \(i \in [d]\). Revisiting if needed the proof of Lemma 3.3, there is no
difficulty in proving that the resulting map \( V \mapsto U \) is continuous from \([0,T] \times \mathcal{S}_{d-1}\)^d into itself, for any \( \eta' \in (0,\eta) \). This permits to apply Schauder’s theorem.

**Second Step.** The goal now is to choose \((b^i)_{i \in [d]}, b^o\) and \((h^i)_{i \in [d]}\) (and hence \(C_0\) as well) accordingly so that the solution to (3.23) is in fact a solution to the master equation (3.18). In order to proceed, we follow the same idea as in the proof of Lemma 3.2 and recall the truncated Hamiltonian

\[
H^i_c(y) = \frac{1}{2} \sum_{j \in [d]} ((y_i - y_j)^2 + 1_{\{y_i - y_j \leq c\}} (2c(y_i - y_j) - c^2) 1_{\{y_i - y_j > c\}}), \quad y = (y_j)_{j \in [d]},
\]

for a constant \( c \) to be fixed later. Also, for another constant \( \Gamma \), the value of which will be also fixed later on, we call \( \psi_{T} \) the function

\[
\psi_{T}(r) := \begin{cases} r, & \text{if } |r| \leq \Gamma, \\ \Gamma \text{sign}(r), & \text{if } |r| \geq \Gamma, \\ r \in \mathbb{R}. \end{cases}
\]

Given these notations, we let (compare with (3.17))

\[
b_i^o(t,p,y) := \sum_{k \in [d]} p_k \min (c, (y_k - y_j)_+) + \varepsilon^2 \delta_{i,j}, \quad i, j \in [d],
\]

\[
b_j^c(t,p,y) := -\sum_{k \in [d]} \left[ \varphi(p_k) + \min (c, (y_k - y_j)_+) \right] - \varepsilon^2, \quad j \in [d],
\]

\[
h_i^o(t,p,y) := \psi_{T} \left( H^i_c(y) + f^o(t,p) + \sum_{j \in [d]} \varphi(p_j) \psi_{T}(y_j - y_i) \right), \quad i \in [d],
\]

for \((t, p, y) \in [0,T] \times \mathcal{S}_{d-1} \times \mathbb{R}^d\). For a given value of \( c \), we can choose \( \Gamma \) (hence depending on \( c \)) such that the above coefficients are bounded by \( \Gamma \). Moreover, the coefficients satisfy (3.24) for a suitable choice of \( C_0 \) therein (notice in this regard that this is the specific interest of the second occurrence of \( \psi_{T} \) to force the whole term to be jointly Lipschitz in \((p,y)\)). By the first step, there exists a classical solution, say \( U = (U^1, \ldots, U^d) \), to (3.23). As in the proof of Theorem 3.3, we can represent \( U \) through a forward-backward stochastic differential equation. Following (3.3), the backward equation writes

\[
du^i_t = -h^i(t,P_t,u_t)dt - \frac{\varepsilon}{\sqrt{2}} \sum_{j \in [d]} \sqrt{P^i_t} \left( \nu^i_{t,j} - \nu^j_{t,i} \right) dt + \sum_{j,k \in [d]} \nu^i_{t,j,k} dW^j_{t,k},
\]

with \( u^i_T = g^i(P_T) \) as terminal boundary condition, where \((P_t)_{0 \leq t \leq T}\) is the solution to the corresponding forward equation (but there is no need to write it down). The key point here is to observe that \( h^i \) is at most of linear growth in \( u \), uniformly in \((t,p)\), the constant in the linear growth depending on \( c \) but not on \( \Gamma \). By considering the drifted Brownian motions \(((W^i_{t,j}) = W^i_{t,j} - \int_0^t (\varepsilon \sqrt{P^i_s})/(\sqrt{2P^i_s} ds)_{0 \leq s \leq T})_{j \in [d]: j \neq i}\) and \(((W^j_{t,i}) = W^j_{t,i} + \int_0^t (\varepsilon \sqrt{P^j_s})/(\sqrt{2P^j_s} ds)_{0 \leq s \leq T})_{j \in [d]: j \neq i}\), for a given value of \( i \in [d] \), we can apply Girsanov theorem to get rid of the second term in the equation for \((u_t)_{0 \leq t \leq T}\) in (3.24), the application of Girsanov theorem being here made licit by Proposition 2.2. We easily deduce that, with \( U \) as in (3.26), \( \|U(t,\cdot)\|_{\infty} \leq C(1 + \int_0^T \|U(s,\cdot)\|_{\infty} ds) \) and then deduce that \( U \) and hence \((u_t)_{0 \leq t \leq T}\) are bounded by a constant \( C \) that depends on \( c \) but not on \( \Gamma \). In particular, it makes sense to choose \( \Gamma \) large enough such that, for \( y = (y_j)_{j \in [d]} \) with \(|y| \leq C \), \( h^i(t,p,y) \) in (3.25) is also equal to

\[
h^i(t,p,y) = H^i_c(y) + f^i(t,p) + \sum_{j \in [d]} \varphi(p_j) (y_j - y_i).
\]
It says that the backward equation (3.26) identifies with the backward equation (3.6) in the proof of Lemma 3.2. But the point in the second step of the proof of Lemma 3.2 is precisely to show that the solution to (3.26) can be bounded independently of $c$. In words, we can find a constant $C_1$, independent of $c$ (and of course of $\Gamma$) such that $\bar{U} = (U^i)_{i \in [d]}$ is bounded by $C_1$. Then, choosing $c \geq 2C_1$ (and $\Gamma$ large enough as before), we have that, for any $y = (y_j)_{j \in [d]}$ with $|y| \leq C_1$, and any $(t, p) \in [0, T] \times S_{d-1}$ and $i, j \in [d]$,

$$B^i_j(t, p, y) = \varphi(p_j) + b^i_j(t, p, y) + b^j_i(t, p, y)p_j, \quad F^i(t, p, y) = h^i(t, p, y),$$

with $B^i$ and $F^i$ as in (3.17). This shows that $U = (U^1, \ldots, U^d)$ solves the master equation (3.18). □

4. Proof of the a priori Hölder estimate

The proof of Theorem 3.6 is the core of the paper. The main ingredient is a coupling estimate for the diffusion process associated with the linear equation (3.19), see the statement of Proposition 4.3. Whilst this approach is mostly inspired by earlier coupling arguments used to prove regularity of various classes of harmonic functions (see for instance [18, 19, 24, 50]), we here need a tailored version that fits the specificities of Kimura operators. In short, the coupling estimate we obtain below does not suffice to conclude directly in full generality. In fact, it just permits to derive the required Hölder estimate in the case $d = 2$. In the higher dimensional setting, we need an additional argument that uses induction on the dimension of the state space to pass from the coupling estimate to the Hölder bound; see Remark 4.4 for a first account and Subsection 4.2 for more details. In short, the rationale for this additional induction argument is that the coupling estimate obtained in Proposition 4.3 blows up near the boundary, except when $d = 2$. As for the induction argument itself, it is based on a conditioning property that is proper to Kimura type operators: Roughly speaking, the last $d - m$ coordinates of the diffusion process associated with the linear equation (3.19) behave, conditional on the first $m$ coordinates, as a diffusion associated with a linear equation of the same type as (3.19) but in dimension $d - m$ instead of $d$, see Proposition 4.2 for the complete statement. It is worth mentioning that our induction argument is inspired by the work [1]. Therein, the authors prove a gradient estimate for simpler and more regular forms of drifts by iterating on the dimension of the state space. Differently from ours, their approach is purely deterministic: As a result, the conditioning principle exposed in Proposition 4.2 manifests implicitly in [1] through the form of the underlying Kimura operators.

Throughout the section, we are given coefficients $b = (b_i)_{i \in [d]}$, $b^0 = (b^0_i)_{i \in [d]}$, $h$ and $\ell$ as in the statement of Theorem 3.6. Then, the aforementioned diffusion process associated with (3.19) is given by the following statement.

**Proposition 4.1.** Consider $\varphi$ as in (2.16) with $\delta \in (0, 1)$ and $\kappa \geq \varepsilon^2/2$, for $\varepsilon > 0$. Then, the stochastic differential equation

$$dP^i_s = \left(\varphi(P^i_s) + b_i(s, P_s) + P^i_s b^0_i(s, P_s)\right)dt + \varepsilon \sqrt{P^i_s} \sum_{j \in [d]} \sqrt{P^j_s} dW^i_j, \quad s \in [t, T], \quad i \in [d], \quad (4.1)$$

is uniquely solvable for any initial time in $t \in [0, T]$ and any (possibly random) initial condition in $\text{Int}(S_{d-1})$. Moreover, the coordinates of the solution remain almost surely strictly positive. In particular, for any $(t, p) \in [0, T] \times \text{Int}(S_{d-1})$, for any $[t, T]$-valued stopping time $\tau$ (with respect to the filtration $\mathbb{F}^W$), any function $u$ as in the statement of Theorem 3.6 admits the representation

$$u(t, p) = \mathbb{E}\left[u(\tau, P^i_s) + \int_t^\tau h(s, P^i_s)ds\right], \quad (4.2)$$
where $P^{t,p}$ is the $d$-dimensional process whose dynamics are given by (4.1) and starts from $p$ at time $t$. In particular, the $L^\infty$ bound in Theorem 2.4 holds true.

Proof. Strong existence and uniqueness may be proven in Proposition 2.1 (Equation (4.1) is slightly more general than the equation handled in Proposition 2.1 but the proof works in the same way). Representation of $u$ is a straightforward consequence of Itô’s formula (as in the proof of Theorem 3.3).

It is worth observing that, taking $\tau = T$ in (4.2), $u$ has the (standard) representation:

$$u(t,p) = \mathbb{E} \left[ \ell(P_T^{t,p}) + \int_t^T h(s, P_s^{t,p}) ds \right], \quad (t,p) \in [0,T] \times \text{Int}(\hat{S}_{d-1}),$$

(4.3)

which is of course very useful to us. Indeed, using a standard mollification argument (taking benefit of the fact that the coefficients $(b_i)_{i \in [d]}$ and $(b_i^0)_{i \in [d]}$ are continuous), we can easily approximate the coefficients $(b_i)_{i \in [d]}$ and $(b_i^0)_{i \in [d]}$ for the sup norm by sequences of coefficients $((b_i^n)_{i \in [d]})_{n \geq 1}$ and $((b_i^{0,n})_{i \in [d]})_{n \geq 1}$ that are time-space continuous and Lipschitz continuous in the space variable (uniformly in the time variable). Hence, if we prove that $u$ in (4.3) satisfies the Hölder estimate stated in Theorem 3.6 for coefficients $(b_i)_{i \in [d]}$ and $(b_i^0)_{i \in [d]}$ that are Lipschitz continuous in space (uniformly in time), we can deduce that the same holds when $(b_i)_{i \in [d]}$ and $(b_i^0)_{i \in [d]}$ are merely continuous by passing to the limit along the aforementioned mollification.

In other words, we may assume for our purpose that $(b_i)_{i \in [d]}$ and $(b_i^0)_{i \in [d]}$ are Lipschitz continuous in space, uniformly in time, provided that we prove that the resulting Hölder estimate does not depend on the Lipschitz constants of $(b_i)_{i \in [d]}$ and $(b_i^0)_{i \in [d]}$.

Throughout the section, we assume that, as in the statement of Theorem 3.6, $\varepsilon$ belongs to $(0,1)$.

4.1. Preliminary results on coupling and conditioning.

4.1.1. Conditioning on the $m$ first coordinates. The core of the analysis is based upon the probabilistic representation (4.2) and in turn on the properties of the process $P = (P_t^1, \ldots, P_t^d)_{0 \leq t \leq T}$ solving equation (4.1).

As we already alluded to a few lines before, our general strategy relies on an induction argument based upon the dimension of the state variable. This is precisely the goal of this paragraph to clarify the way we may reduce dimension inductively. General speaking, the arguments is based on a conditioning argument.

In order to make it clear, we rewrite (4.1), but using (at least for the sole purpose of the statement of Proposition 1.2 right below) the letter $X_t$ instead of $P_t$ for the unknown:

$$dX^i_t = \left( \varphi(X^i_t) + b_i(t, X_t) + X^i_t b^0_i(t, X_t) \right) dt + \varepsilon \sqrt{X^i_t} \sum_{j \in [d]} \sqrt{X^j_t} dW^j_t, \quad t \in [t_0, T], \; i \in [d],$$

(4.4)

for a given initial time $t_0$. Our rationale to change $P_t$ into $X_t$ is motivated by the fact that we feel better to keep the letter $P_t$ for the new state variable once the dimension has been reduced. The objective is then to write the law of $(X_t)_{t_0 \leq t \leq T}$ in the form

$$(X_t)_{t_0 \leq t \leq T} \overset{\text{law}}{=} (P_t^{o_1}, \ldots, P_t^{o_m}, \varsigma^2(P_t^{o_1}P_t^1, \ldots, \varsigma^2(P_t^{o_1}P_t^{d-m}), (5.4))$$

where $P^o = (P_t^{o_1}, \ldots, P_t^{o_m})_{t_0 \leq t \leq T}$ and $P = (P_t^1, \ldots, P_t^{d-m})_{t_0 \leq t \leq T}$ are new stochastic processes taking respectively values within the set $\hat{S}_m = \{ (p_1, \ldots, p_m) \in (\mathbb{R}_+)^m : \sum_{i=1}^m p_i^o \leq 1 \}$ and $S_{d-m-1} = \{ (p_1, \ldots, p_{d-m}) \in (\mathbb{R}_+)^{d-m} : \sum_{i=1}^{d-m} p_i = 1 \}$. Above, $\varsigma$ is given by

$$\varsigma(p^o) := \sqrt{1 - \left( \sum_{i=1}^{m} p_i^o \right)^2}, \quad (p_1^o, \ldots, p_m^o) \in \hat{S}_m.$$
Proposition 4.2. Given coefficients \((b_i)_{i\in [d]}\) and \((b_i^\circ)_{i\in [d]}\) as in the statement of Theorem 3.6, there exist new coefficients

- \((\bar{b}_i)_{i\in [d-m]}\), with values in \((\mathbb{R}_+)^d\), that are bounded by a constant that only depends on \(\\|b_i\|_{\infty}\)\(\in [d]\),
- \((\bar{b}_i^\circ)_{i\in [d-m]}\), with values in \(\mathbb{R}^{d-m}\), that are bounded by a constant that only depends on \(\\|b_i\|_{\infty}\)\(\in [d]\) and \(\\|b_i^\circ\|_{\infty}\)\(\in [d]\),

and that are Lipschitz continuous in space uniformly in time such that

- whenever \(\delta \in (0, 1)\) and \(\kappa \geq \varepsilon^2/2\),
- for any family of antisymmetric Brownian motions \(\mathbf{W}^\circ = (\mathbf{W}^0_t = (W^0_{t,i,j})_{i,j\in [d]: i\neq j})_{0 \leq t \leq T}\) of dimension \(d(d-1)/2\) that is independent of \(\mathbf{W} = (\mathbf{W}_t = (W_{t,i,j})_{i,j\in [d]: i\neq j})_{0 \leq t \leq T}\),
- for any given initial condition \((t_0, p^0, p) \in [0, T] \times \hat{S}_m \times S_{d-m-1}\),

the system

\[
dP_t^i = \kappa^{-2}(P_t^0) (\varphi(\kappa^2(P_t^0) P_t^i) + \bar{b}_i(t, P_t^0, P_t) + P_t^i \tilde{b}^\circ(t, P_t^0, P_t)) \, dt
\]

\[
+ \varepsilon^{-1}(P_t^0) \sum_{j\in [d-m]: j\neq i} \sqrt{P_t^i P_t^j} dW_t^{i,j}, \quad i \in [d-m],
\]

\[
dP_t^{\circ,i} = \left( \varphi(P_t^{\circ,i}) + b_i(t, (P_t^0, \kappa^2(P_t^0) P_t)) + P_t^{\circ,i} \tilde{b}^\circ(t, P_t^0, \kappa^2(P_t^0) P_t) \right) \, dt
\]

\[
+ \varepsilon \sum_{j\in [m]: j\neq i} \sqrt{P_t^{\circ,i} P_t^{\circ,j}} dW_t^{\circ,i,j} + \varepsilon \kappa \left( P_t^{\circ,i} \right) \sum_{j\in [d-m]} \sqrt{P_t^j} dW_t^{\circ,i,m+j}, \quad i \in [m],
\]

for \(t \in [t_0, T]\), with \((P_{t_0}^0, P_{t_0}) = (p^0, p)\), has a unique strong solution, which satisfies the identity in law \((1.5)\) whenever \((1.4)\) is initialized from \((p^0, \kappa^2(p^0)p)\) at time \(t_0\).

The proof of Proposition 4.2 is deferred to Subsection 4.4. Throughout, we denote by \(\mathbb{F}^{\mathbf{W}^\circ, \mathbf{W}} = (\mathcal{F}_{t}^{\mathbf{W}^0, \mathbf{W}})_{0 \leq t \leq T}\) the augmented filtration generated by \(\mathbf{W}^\circ = ((W^{\circ,i,j})_{0 \leq t \leq T})_{i,j\in [d]: i\neq j}\) and \(\mathbf{W} = ((W_{t,i,j})_{0 \leq t \leq T})_{i,j\in [d]: i\neq j}\), the latter two being implicitly understood as two independent collections of Brownian motions such that \(\mathbf{W}^{\circ,i,j} = (W^{\circ,i,j} - W^{\circ,j,i})/\sqrt{2}\) and \(\mathbf{W}^{i,j} = (W^{i,j} - W^{j,i})/\sqrt{2}\) when \(\mathbf{W}^\circ\) and \(\mathbf{W}\) are as in the statement of Proposition 4.2.

4.1.2. Main coupling estimate.

Proposition 4.3. Assume that \(\delta\) in \((2.10)\) belongs to \((0, 1/(4\sqrt{d}))\) Moreover, take two initial conditions \((t_0, p^0, p)\) and \((t_0, q^0, q)\) in \([0, T] \times \hat{S}_m \times S_{d-m-1}\), with \(m \in \{1, \ldots, d-1\}\), such that \(\|p^0\|_1 := p^0 + \cdots + p^0_m \leq 1/2, \|q^0\|_1 \leq 1/2\) and \(|p - q| < \delta^2/(64\sqrt{d})\). On the (filtered) probability space carrying \(\mathbb{F}^{\mathbf{W}^\circ, \mathbf{W}}\) and \(\mathbb{F}^{\mathbf{W}^0, \mathbf{W}}\), call \((P_t^0 = (P_t^{0,1}, \ldots, P_t^{0,m}), P_t = (P_t^{1,\ldots, P_t^{d-m}}))_{t_0 \leq t \leq T}\) the solution to \((1.6)\) with \((t_0, p^0, p)\) as initial condition.

Then, for any \(\eta \in (0, 1)\), there exists a threshold \(\kappa_0 \geq 2\), only depending on \(\eta, \varepsilon\) and \((\|b_i\|_{\infty})_{i\in [d]}\) (but not on \(\delta\)) such that, for any \(\kappa \geq \kappa_0\), we can find a constant \(C\), depending on \(\delta, \varepsilon, \kappa, \eta, (\|b_i\|_{\infty})_{i\in [d]}, (\|b_i^\circ\|_{\infty})_{i\in [d]}\) and \(T\) such that, provided that \(|p - q|^{1/3} \leq T - t_0\), there exists an adapted process \((Q_t^0 = (Q_t^{0,1}, \ldots, Q_t^{0,m}), Q_t = (Q_t^1, \ldots, Q_t^{d-m}))_{t_0 \leq t \leq T}\) that has the same law as the solution to \((1.6)\) with \((t_0, q^0, q)\) as initial condition and for which the following property holds true.

If we call

\[
\tilde{P}_t := \left( \sqrt{P_t^1}, \ldots, \sqrt{P_t^{d-m}} \right), \quad \tilde{Q}_t := \left( \sqrt{Q_t^1}, \ldots, \sqrt{Q_t^{d-m}} \right), \quad t \in [t_0, T],
\]
and
\[
\varrho := \inf \{ s \geq t_0 : |P^s_s - Q^s_s| > |\tilde{P} - \tilde{Q}| \}, \quad \rho := \inf \{ s \geq t_0 : |P^s_s|_1 \geq 3/4 \},
\]
\[
\sigma := \inf \{ s \geq t_0 : |\tilde{P}_s - \tilde{Q}_s| \geq \delta/4 \}, \quad \tau := \inf \{ s \geq t_0 : |\tilde{P}_s - \tilde{Q}_s| = 0 \},
\]
then
\[
\mathbb{P}(\{ \varpi_S < \tau \wedge \varrho \wedge \rho \}) \leq C \frac{|p - q|^{1/12}}{\min_{i \in [d-m]} (\max(p_i, q_i))^\eta},
\]
where \( \varpi_S := \varpi \wedge S \), with \( \varpi := \rho \wedge \sigma \wedge \tau \) and \( S := t_0 + |p - q|^{1/3} \).

The proof of Proposition 4.3 is deferred to Subsection 4.3.

**Remark 4.4.** We now explain the difficulty when the dimension \( d \) is greater than or equal to 3 and the reason why we need an induction argument to derive the required Hölder estimate.

A naive way to proceed is indeed to choose \( m = 0 \) in the above statement. In such a case, the process \((P^1, \ldots, P^{d-m})\) in (4.6) coincides with the solution \((P^1, \ldots, P^d)\) in (4.1). In other words, Proposition 4.3 with \( m = 0 \) reads as a coupling estimate for the solution to (4.1).

Let us now see what the right hand side of (4.8) becomes when \( m = 0 \). Up to an obvious change of coordinates, we then may assume that \( \min_{i \in [d]} (\max(p_i, q_i)) = \max(p_1, q_1) \), in which case the right-hand side of (4.8) writes \( |p - q|^{1/12}/\max(p_1, q_1)^\eta \), both \( p = (p_1, \cdots, p_d) \) and \( q = (q_1, \cdots, q_d) \) being now regarded as \( d \)-dimensional vectors. The point is then to upper bound \( |p - q|^{1/12}/\max(p_1, q_1)^\eta \). When \( d = 2 \), this is pretty easy because
\[
|p - q| = \sqrt{|p_1 - q_1|^2 + |p_2 - q_2|^2} = \sqrt{|p_1 - q_1|^2 + |(1 - p_1) - (1 - q_1)|^2} \leq \sqrt{2}|p_1 - q_1|,
\]
and then we get
\[
\frac{|p - q|^{1/12}}{\max(p_1, q_1)^\eta} \leq \sqrt{2}|p - q|^{1/12-\eta}.
\]
Unfortunately, this argument no longer works when \( d \geq 3 \) since, in that case, one of the entries \( |p_i - q_i| \) of \( p, q \) may be much larger than \( |p_1 - q_1| \).

### 4.2. Derivation of the Hölder estimate and proof of Theorem 3.6

We now explain how to derive the Hölder estimate in Theorem 3.6 from Proposition 4.3. As we already alluded to, it relies on an additional iteration on the dimension, which is in turn inspired by earlier PDE results on Kimura diffusions, see for instance [1]. The induction assumption takes the following form.

Take \( h, \ell \) and \( u \) as in (3.19). For a given \( m \in [d] \), call \( \mathcal{P}_m \) the following property: For any \( \varepsilon, \eta \in (0, 1) \) and \( \delta \in (0, 1/(4 \sqrt{d})) \), there exist a threshold \( \kappa_0 \), only depending on \( \varepsilon, \eta, m \) and \( (\|b_i\|_\infty)_{i \in [d]} \), and an exponent \( \alpha \in (0, 1) \), only depending on \( \eta \) and \( m \), such that, for any \( \kappa \geq \kappa_0 \), we can find a constant \( C_\varepsilon \), depending on \( \varepsilon, \delta, \kappa, \eta, (\|b_i\|_\infty)_{i \in [d]}, (\|b_i^\gamma\|_\infty)_{i \in [d]}, \|h\|_\infty, \|\ell\|_{1, \infty} \) and \( T \) such that, for any \( p = (p_1, \cdots, p_d) \) and \( q = (q_1, \cdots, q_d) \) in \( \mathcal{S}_{d-1} \),
\[
|u(t, p_1, \cdots, p_d) - u(t, q_1, \cdots, q_d)| \leq C_\varepsilon \frac{|p - q|^{\alpha}}{\max(p, q)^{\alpha}},
\]
where \( \max(p, q) \) denotes the \( m \)th element in the increasing reordering of \( \max(p, q) = (\max(p_1, q_1), \cdots, \max(p_d, q_d)) \).

We then have the following two propositions:

**Proposition 4.5.** Within the framework of Theorem 3.6, \( \mathcal{P}_1 \) holds true.

**Proposition 4.6.** Within the framework of Theorem 3.6, assume that there exists an integer \( m \in \{1, \cdots, d - 1\} \) such that \( \mathcal{P}_m \) holds true. Then, \( \mathcal{P}_{m+1} \) holds true.
Notice that Proposition 4.6 implies Theorem 3.6. It suffices to choose \( \eta = 1/2 \) in \( \mathcal{P}_d \), noticing that \( \max(p, q)_{(d)} \) is necessarily greater than \( 1/d \). Below, we directly prove Proposition 4.6. The proof of Proposition 4.5 is completely similar (somehow, everything works as if we had a property \( \mathcal{P}_0 \)).

**Proof.** For some \( m \in \{1, \ldots, d-1\} \) and some \( \eta \in (0, 1/4) \), we consider \( \kappa_0 \) as being the maximum of \( \kappa_0 \) given by Proposition 4.5 with \( \eta \) replaced by \( \eta/12 \) therein and of \( \kappa_0 \) given by \( \mathcal{P}_m \) with \( \eta \) therein. Also, we consider \( \alpha \) given by \( \mathcal{P}_m \) and \( \kappa \geq \kappa_0 \). We then assume that \( \mathcal{P}_m \) holds true and we take \( p, q \in \mathcal{S}_{d-1} \) together with \( t_0 \in [0, T] \). Without any loss of generality, we may assume that

\[
\max(p_1, q_1) \leq \max(p_2, q_2) \leq \cdots \leq \max(p_m, q_m) \leq \frac{1}{2md}.
\]

(4.10)

Observe that if the last inequality is not satisfied, the bound (4.9) at rank \( m+1 \) (at \( t_0 \) instead of \( t \)) is a straightforward consequence of the bound at rank \( m \) with \( (2md)^\eta C \) instead of \( C \) as constant. For sure, we may also assume that

\[
\max(p_1, q_1) \leq \max(p_2, q_2) \leq \cdots \leq \max(p_d, q_d),
\]

(4.11)

in which case \( \max(p_d, q_d) \) is the largest element in the sequence \( \max(p_i, q_i) \). In particular, at least \( p_d \) or \( q_d \) is above \( 1/d \) (since one of the two elements dominates all the other elements in the family \( (p_1, \ldots, p_d, q_1, \ldots, q_d) \)). Hence, we may assume that \( \min(p_d, q_d) \geq 1/(2d) \). Again, the proof is over if not since \( |p - q| \) is then necessarily larger than \( 1/(2d) \): Tuning \( C \) accordingly, (4.9) follows from the fact that \( u \) is bounded, see Proposition 4.1. By the same argument, we may assume that \( |p - q| < \delta^2/(128d^3/2) \).

**First Step.** Clearly,

\[
|u(t_0, p_1, \ldots, p_d) - u(t_0, q_1, \ldots, q_d)|
\leq \sum_{i \in [m]} \left( |u(t_0, q_1, \ldots, q_{i-1}, p_i, \ldots, p_{d-1}, 1 - q_1 - \cdots - q_{i-1} - p_i - \cdots - p_{d-1})
\quad - u(t_0, q_1, \ldots, q_i, p_{i+1}, \ldots, 1 - q_1 - \cdots - q_i - p_{i+1} - \cdots - p_{d-1})|ight)
\quad + |u(t_0, q_1, \ldots, q_m, p_{m+1}, \ldots, p_d, 1 - q_1 - \cdots - q_m - p_{m+1} - \cdots - p_d)
\quad - u(t_0, q_1, \ldots, q_m, q_{m+1}, \ldots)|, \tag{4.12}
\]

with the obvious convention that \((q_1, \ldots, q_{i-1}, p_i, \ldots, p_{d-1}, 1 - q_1 - \cdots - q_{i-1} - p_i - \cdots - p_{d-1}) = (p_1, \ldots, p_d)\) when \( i = 1 \). Notice from (4.10) and from the bound \( \min(p_d, q_d) \geq 1/(2d) \) that, for \( i \in [m], q_1 + \cdots + q_i \leq 1/(2d) \) and \( p_1 + \cdots + p_{d-1} \leq 1 - 1/(2d) \), which fully justifies the fact that all the entries above are non-negative. Obviously, by the induction assumption, for any \( i \in [m], \)

\[
\sum_{i \in [m]} \left( |u(t_0, q_1, \ldots, q_{i-1}, p_i, \ldots, p_{d-1}, 1 - q_1 - \cdots - q_{i-1} - p_i - \cdots - p_{d-1})
\quad - u(t_0, q_1, \ldots, q_{i+1}, p_{i+1}, \ldots, 1 - q_1 - \cdots - q_{i+1} - p_{i+1} - \cdots - p_{d-1})| \right)
\leq C \sum_{i \in [m]} \frac{|p_i - q_i|_\alpha}{\max(p, q)_{(m)}} \leq C |p - q|^{(1 - \eta)\alpha},
\]

where we used the fact that \( \max(p, q)_{(m)} = \max(p_m, q_m) \) and where we modified the value of \( C \) in the last term. The conclusion is that, in (4.12), we can focus on the last term. Equivalently, we can assume that \( p_i = q_i \), for \( i = 1, \cdots, m \), provided we replace (4.11) by (which is weaker, but
which is the right assumption here since there is no way to compare properly the last coordinates in the last term of (4.12)

\[
\max_{i \in [m]} p_i = \max_{i \in [m]} q_i \leq \max(p_{m+1}, q_{m+1}) \leq \cdots \leq \max(p_{d-1}, q_{d-1}),
\]

with \(q_d \geq 1/(2d)\). We now invoke Proposition 4.1 to represent \(u(t, p_1, \ldots, p_d)\) and \(u(t, q_1, \ldots, q_d)\) through the respective solutions to (4.1) together with Proposition 4.2 above which provides another representation for the process used in the Kolmogorov formula (4.2). In particular, we can find \((P_0^p, \ldots, P_{d-m}^p, P_1^p, \ldots, P_{d-m}^p)\) and \(u(t, P_1^p, \ldots, P_{d-m}^p)\) such that the tuple \((P_0^p, \ldots, P_{d-m}^p, \pi_2(P^p_0), \ldots, \pi_2(P^p_{d-m}))\) has the same law as the solution to (4.1) when starting from \(p\) at time \(t_0\), and, in a similar manner, \((Q_0^q, \ldots, Q_{d-m}^q, \pi_2(Q^q_0), \ldots, \pi_2(Q^q_{d-m}))\) such that the tuple \((Q_0^q, \ldots, Q_{d-m}^q, \pi_2(Q^q_0), \ldots, \pi_2(Q^q_{d-m}))\) has the same law as the solution to (4.1) when starting from \(q\) at time \(t_0\). In particular, we have \((P_0^t, \ldots, P_{d-m}^t) = (Q_0^t, \ldots, Q_{d-m}^t) = p^\circ\), \(P_1^t, \ldots, P_{d-m}^t = \pi_2(p^\circ)\) \((p_{m+1}, \ldots, p_d)\) and \((Q_1^t, \ldots, Q_{d-m}^t) = \pi_2(p^\circ)\) \((q_{m+1}, \ldots, q_d)\).

Then, for any deterministic time \(S \in [t_0, T]\), using the same notation as in the statement of Proposition 4.3,

\[
u(t_0, p_1, \ldots, p_d) = \mathbb{E}\left[u\left(\varpi_S, P_1^p, \ldots, P_{d-m}^p, \pi_2(P^p_0), \ldots, \pi_2(P^p_{d-m})\right)\right] + O(S - t_0),
\]

\[
u(t_0, q_1, \ldots, q_d) = \mathbb{E}\left[u\left(\varpi_S, Q_1^q, \ldots, Q_{d-m}^q, \pi_2(Q^q_0), \ldots, \pi_2(Q^q_{d-m})\right)\right] + O(S - t_0),
\]

where \(|O(r)| \leq \|h\|_{\infty} r\). To make it simpler, we also let (the notation \((X_t)_{t_0 \leq t \leq T}\) below is rather abusive since \((X_t)_{t_0 \leq t \leq T}\) also denotes the solution to (4.1), but, in fact, Proposition 4.2 says both \((X_t)_{t_0 \leq t \leq T}\)'s have the same law)

\[X_t = \left(\Pi_{t,0}^0, \ldots, \Pi_{t,m}^0, \pi_2(P_0^p), \ldots, \pi_2(P_{d-m}^p)\right), \]

\[Y_t = \left(\Pi_{t,0}^1, \ldots, \Pi_{t,m}^1, \pi_2(Q_0^q), \ldots, \pi_2(Q_{d-m}^q)\right), \quad t \in [t_0, T].\]

We then denote by \((\max(X_t, Y_t)_{(1)}, \ldots, \max(X_t, Y_t)_{(d)})\) the order statistic of the \(d\)-dimensional tuple \((\max(X_t, Y_t)_{(1)}, \ldots, \max(X_t, Y_t)_{(d)})\).

\textbf{Second Step.} We first assume that \(S := t_0 + |p - q|^{1/3} \leq T\). The strategy is to split into four events the set \(\Omega\) over which the expectations appearing in (4.14) are computed.

\textbf{1st event.} On the event \(E_1 := \{\varpi_S = \tau\} \subset \{(P_1^p, \ldots, P_{d-m}^p) = (Q_1^q, \ldots, Q_{d-m}^q)\}\), we have, by the induction assumption and from the Lipschitz property of \(\pi_2\),

\[
|u(\varpi_S, X_{\varpi_S}) - u(\varpi_S, Y_{\varpi_S})| = |u(\varpi_S, P_1^p, \ldots, P_{d-m}^p, \pi_2(P_0^p), \ldots, \pi_2(P_{d-m}^p)) - u(\varpi_S, Q_1^q, \ldots, Q_{d-m}^q, \pi_2(Q_0^q), \ldots, \pi_2(Q_{d-m}^q))| \\ \leq \frac{C}{\max(X_{\varpi_S}, Y_{\varpi_S})_{(m)}^{\alpha}}|P_{\varpi_S}^p - Q_{\varpi_S}^q|^\alpha,
\]

the constant \(C\) being allowed to change from line to line provided that it only depends on the parameters listed in the induction assumption.

Assume that \(\max(X_{\varpi_S}, Y_{\varpi_S})_{(l)} < \max(X_{\varpi_S}, Y_{\varpi_S})_{(l)}\) for any \(l = m\) \(+ \cdots, d\), then necessarily \(\max(X_{\varpi_S}, Y_{\varpi_S})_{(m)} \geq \max(X_{\varpi_S}, Y_{\varpi_S})_{(i)}\) for any \(i = m\), \(\ldots, m\). Then we obtain \(C|P_{\varpi_S}^p - Q_{\varpi_S}^q|^{(1-\eta)\alpha}\) as upper bound for the right-hand side of (4.15). Therefore, we can focus on the complementary
event when \( \max(X_{\bar{w}_S}, Y_{\bar{w}_S})_{(m)} \geq \max(X_{\bar{w}_S}, Y_{\bar{w}_S})_{l} \) for some \( l = m + 1, \ldots, d \). We obtain
\[
|u(\bar{w}_S, X_{\bar{w}_S}) - u(\bar{w}_S, Y_{\bar{w}_S})| \leq C|P^o_{\bar{w}_S} - Q^o_{\bar{w}_S}|^{(1-\eta)^\alpha} + \sum_{l=m+1}^d \frac{C}{\max(X^l_{\bar{w}_S}, Y^l_{\bar{w}_S})^{\eta \alpha}} |P^o_{\bar{w}_S} - Q^o_{\bar{w}_S}|^{\alpha},
\]
which we rewrite into (recalling that \( p_1 = q_1, \ldots, p_m = q_m \))
\[
|u(\bar{w}_S, X_{\bar{w}_S}) - u(\bar{w}_S, Y_{\bar{w}_S})| \leq C|P^o_{\bar{w}_S} - P^o_{t_0} - (Q^o_{\bar{w}_S} - Q^o_{t_0})|^{(1-\eta)^\alpha}
+ \sum_{l=m+1}^d \frac{C}{\max(X^l_{\bar{w}_S}, Y^l_{\bar{w}_S})^{\eta \alpha}} |P^o_{\bar{w}_S} - P^o_{t_0} - (Q^o_{\bar{w}_S} - Q^o_{t_0})|^{\alpha}. \tag{4.16}
\]

In order to upper bound \( C/(\max(X^l_{\bar{w}_S}, Y^l_{\bar{w}_S})^{\eta \alpha}) \), we expand \( ((X^l_t)^{-2\eta \alpha})_{t_0 \leq t \leq \bar{w}_S} \) by Itô’s formula, for \( l = m + 1, \ldots, d \). To do so, we recall that \( (X^1, \ldots, X^d) \) has the same law as the solution of (4.13). It is an Itô process with bounded coefficients (in terms of \( (\|b_i\|_{\infty})_{i=1, \ldots, d} \) and \( (\|b^2_i\|_{\infty})_{i=1, \ldots, d} \)). Importantly, the drift of the \( l \)th coordinate is lower bounded by \( \kappa - \|b^0_i\|_{\infty}p_l \) when \( p_l \leq \delta \). Recalling that \( \kappa_0 \geq 2\varepsilon^2 \), we easily deduce that, for a new value of \( C \) (say \( C \geq 1 \)) whose value is allowed to change from line to line (see also footnote [N] for the meaning of the inequality right below),
\[
d\left((X^l_t)^{-2\eta \alpha} (t-t_0)^{\alpha/2}\right)
\leq \left(C(t-t_0)^{\alpha/2 - 1} (X^l_t)^{-2\eta \alpha} - \eta \alpha (2 \kappa - \varepsilon^2 (1 + 2 \eta \alpha)) (t-t_0)^{\alpha/2} (X^l_t)^{-1+2\eta \alpha}\right) dt + dm_t
\leq \left(C(t-t_0)^{\alpha/2 - 1} (X^l_t)^{-2\eta \alpha} - C^{-1} (t-t_0)^{\alpha/2} (X^l_t)^{-1+2\eta \alpha}\right) dt + dm_t
= (t-t_0)^{\alpha/2} (X^l_t)^{-2\eta \alpha} \left(C(t-t_0)^{-1} - C^{-1} (X^l_t)^{-1}\right) dt + dm_t
\leq (t-t_0)^{\alpha/2} (X^l_t)^{-2\eta \alpha} \left(C(t-t_0)^{-1} - C^{-1} (X^l_t)^{-1}\right) 1_{\{X^l_t \geq (t-t_0)/C^2\}} dt + dm_t
\leq C(t-t_0)^{\alpha/2 - 1-2\eta \alpha} dt + dm_t,
\]
where \( (m_t)_{t \geq 0} \) is a local martingale. Recall that \( \eta < 1/4 \) (see the remark at the very beginning of the proof), we get by a standard localization argument:
\[
\mathbb{E}\left[|X^l_{\bar{w}_S} - 2\eta \alpha (\bar{w}_S-t_0)^{\alpha/2}\right] \leq C.
\]

Using the fact that the coefficients in the second equation of (4.16) are bounded, we deduce that, from Kolmogorov–Centsov theorem, \( P^o - Q^o \) has a version that is \( 1/3 \)-Hölder continuous and the Hölder constant \( \Lambda \) has a finite fourth moment, which we may assume to be bounded by \( C \). Hence,
\[
\mathbb{E}\left[|P^o_{\bar{w}_S} - Q^o_{\bar{w}_S} - (P^o_{t_0} - Q^o_{t_0})|^{2\eta \alpha} \right]/(\bar{w}_S-t_0)^{\alpha/2} \leq \mathbb{E}[\Lambda^{2\eta \alpha} (\bar{w}_S-t_0)^{\alpha/6}] \leq CE[(\bar{w}_S-t_0)^{\alpha/3}]^{1/2}.
\]

Similarly,
\[
\mathbb{E}\left[|P^o_{\bar{w}_S} - Q^o_{\bar{w}_S} - (P^o_{t_0} - Q^o_{t_0})|^{1-\eta \alpha} \right] \leq \mathbb{E}[\Lambda^{(1-\eta)\alpha} (\bar{w}_S-t_0)^{(1-\eta)\alpha/3}] \leq CE[(\bar{w}_S-t_0)^{2(1-\eta)\alpha/3}]^{1/2}.
\]

From (4.16), we get (use the condition \( \eta < 1/4 \) to get the last bound)
\[
\mathbb{E}\left[|u(\bar{w}_S, X_{\bar{w}_S}) - u(\bar{w}_S, Y_{\bar{w}_S})|_{E_1}\right] \leq CE[(\bar{w}_S-t_0)^{2(1-\eta)\alpha/3}]^{1/2} + CE[(\bar{w}_S-t_0)^{\alpha/3}]^{1/4}
\leq C|p-q|^{\alpha/36} + C|p-q|^{(1-\eta)\alpha/9} \leq C|p-q|^{\alpha/36}. \tag{4.17}
\]
2nd event. On the event \( E_2 := \{ |P_{\omega_S}^\circ - Q_{\omega_S}^\circ| \geq |\bar{P}_{\omega_S} - \bar{Q}_{\omega_S}| \} \supset \{ \omega_S = \rho \} \) (with the same notation as in the statement of Proposition 4.13), we have

\[
|X_{\omega_S} - Y_{\omega_S}|^2 = \sum_{i \in [m]} |P_{\omega_S}^{\circ,i} - Q_{\omega_S}^{\circ,i}|^2 + \sum_{i \in [d-m]} |P_i^{\circ} - Q_i^{\circ}|^2
= \sum_{i \in [m]} |P_{\omega_S}^{\circ,i} - Q_{\omega_S}^{\circ,i}|^2 + \sum_{i \in [d-m]} |\bar{P}_i^{\circ} + \bar{Q}_i^{\circ}|^2 |\bar{P}_i^{\circ} - \bar{Q}_i^{\circ}|^2 \leq 5|P_{\omega_S}^\circ - Q_{\omega_S}^\circ|^2.
\]

Modifying the value of the constant \( C \) in (4.19), we deduce that

\[
|u(\omega_S, X_{\omega_S}) - u(\omega_S, Y_{\omega_S})| = \left| u(\omega_S, P_{\omega_S}^{\circ,1}, \ldots, P_{\omega_S}^{\circ,m}, \bar{P}_1^{\circ}, \ldots, \bar{P}_{d-m}^{\circ}, \bar{Q}_1^{\circ}, \ldots, \bar{Q}_{d-m}^{\circ}) \right|
- u(\omega_S, Q_{\omega_S}^{\circ,1}, \ldots, Q_{\omega_S}^{\circ,m}, \bar{Q}_1^{\circ}, \ldots, \bar{Q}_{d-m}^{\circ}) \right| \leq \frac{C}{\max(X_{\omega_S}, Y_{\omega_S})} |P_{\omega_S}^\circ - Q_{\omega_S}^\circ|^\alpha,
\]

which is the same as (4.15). Therefore, we get the same conclusion as in the first step, see (4.17):

\[
\mathbb{E} \left[ \left| u(\omega_S, X_{\omega_S}) - u(\omega_S, Y_{\omega_S}) \right| 1_{E_2} \right] \leq C |p - q|^{\alpha/36}.
\]

3rd event. We now consider the event \( E_3 := \{ \sup_{t \in [a,t]} |P_t^\circ| \geq \frac{3}{4} \} \supset \{ \omega_S = \rho \} \), where we recall from (4.10) that \( |P_t^\circ| = \sum_{i=1}^m p_i \leq \frac{1}{2d} \leq \frac{3}{4} \). Obviously (since the SDE for \( P^\circ \) has bounded coefficients), there exists a constant \( c \) such that \( \mathbb{E} \sup_{t \in [a,t]} |P_t^\circ - P_0^\circ|^2 \leq c(S - t_0) = c|p - q|^{1/3} \).

Therefore, using the fact that \( u \) is bounded together with Markov’s inequality, we deduce that

\[
\mathbb{E} \left[ \left| u(\omega \land S, X_{\omega \land S}) - u(\omega \land S, Y_{\omega \land S}) \right| 1_{E_3} \right] \leq C |p - q|^{1/3}.
\]

4th event. Lastly, we let \( E_4 := \{ \omega_S < \tau \land \varrho \land \rho \} \). Since \( \zeta^2(p^\circ) = \zeta^2(q^\circ) \leq 2d \) and \( |p - q| < \delta^2/(128d^{3/2}) \), we have \( |\zeta^2(p^\circ)(p_{m+1}, \ldots, p_d) - \zeta^2(q^\circ)(q_{m+1}, \ldots, q_d)| < \delta^2/(6\sqrt{d}) \).

Hence, by Proposition 4.13 (with \( p \) therein being given by \( \zeta^2(p^\circ)(p_{m+1}, \ldots, p_d) \) and similarly for \( q \)) and with \( \eta \) therein being replaced by \( \eta/12 \), we know that, for a new value of \( C \),

\[
P(E_4) \leq C \frac{|p - q|^{1/12}}{\min_{i=m+1, \ldots, d}(\max(p_i, q_i))^{\eta/12}} \leq C \frac{|p - q|^{1/12}}{\max(p_{m+1}, q_{m+1})^{\eta/12}} = C \frac{|p - q|^{1/12}}{\max(p, q^{\frac{\eta}{12}})}
\]

where the derivation of the last two terms follows from (4.13) and from the condition \( \max(p_d, q_d) \geq 1/(2d) \). Since the left-hand side is less than \( 1/2 \), we deduce that, for any exponent \( \beta \in (0, 1) \),

\[
P(E_4) \leq P(E_4)^\beta \leq C^{\beta} \frac{|p - q|^{\beta/12}}{\max(p, q^{\frac{\beta\eta}{12}})}
\]

and then, for a new value of \( C \) possibly depending on \( \beta \),

\[
\mathbb{E} \left[ \left| u(\omega \land S, X_{\omega \land S}) - u(\omega \land S, Y_{\omega \land S}) \right| 1_{E_4} \right] \leq C \frac{|p - q|^{\beta/12}}{\max(p, q^{\frac{\beta\eta}{12}})}.
\]

Conclusion. Here is now the conclusion of the second step. For the same \( \eta \) and \( \alpha \) as before, choose the largest \( \beta \in (0, 1) \) such that \( \beta/12 \leq \alpha/36 \). Finally, let \( \alpha' = \beta/12 \). Deduce that, for a
possibly new value of the constant $C$ therein, all the terms in (4.17), (4.18), (4.19) and (4.20) are bounded by $C|p - q|^\alpha'/\max(p, q)^\nu/(m+1)$. Since $E_1 \cup E_2 \cup E_3 \cup E_4 = \Omega$, we deduce that

$$
\mathbb{E} \left[ \left| u(\varpi \wedge S, X_{\varpi \wedge S}) - u(\varpi \wedge S, Y_{\varpi \wedge S}) \right| \right] \leq C \frac{|p - q|^\alpha'}{\max(p, q)^\alpha/\eta},
$$

which together with (4.14), is $\mathcal{P}_{m+1}$, at least for initial conditions $(t_0, p)$ and $(t_0, q)$ such that $T - t_0 \geq |p - q|^{1/3}$, and $\eta < 1/4$. As for the requirement $\eta < 1/4$, this is not a hindrance: Since the denominator in (4.9) is less than 1, the exponent can be increased for free. As for the case $T - t_0 < |p - q|^{1/3}$, it is discussed in the next step.

**Third Step.** It remains to handle the case that $t_0 + |p - q|^{1/3} \geq T$. We then rewrite (4.14) in the form

$$
\begin{align*}
&u(t_0, p_1, \cdots, p_d) = \mathbb{E} \left[ \ell \left( P_{T_0}^1, \cdots, P_{T_0}^m, \beta^2(P_{T_0}^i)P_{T_0}^1, \cdots, \beta^2(P_{T_0}^i)P_{T_0}^d - m \right) \right] + O(T - t_0), \\
&u(t_0, q_1, \cdots, q_d) = \mathbb{E} \left[ \ell \left( Q_{T_0}^1, \cdots, Q_{T_0}^m, \beta^2(Q_{T_0}^i)Q_{T_0}^1, \cdots, \beta^2(Q_{T_0}^i)Q_{T_0}^d - m \right) \right] + O(T - t_0).
\end{align*}
$$

By (4.5), each expectation may be (directly) rewritten by means of the solution to the stochastic differential equation (4.1). Since the latter has bounded coefficients and since $\ell$ is Lipschitz continuous, we deduce that

$$
|u(t_0, p) - u(t_0, q)| \leq |\ell(p) - \ell(q)| + |u(t_0, p) - \ell(p)| + |u(t_0, q) - \ell(q)| \leq C|p - q| + C(T - t_0)^{1/2},
$$

for a constant $C$ that only depends on $\kappa$, $\|\ell\|_{1, \infty}$, $\|b_i\|_{\infty}$, $\|\beta_i\|_{\infty}$, $\|\nu_i\|_{\infty}$, and $T$. Since $T - t_0 \leq |p - q|^{1/3}$, this completes the proof of $\mathcal{P}_{m+1}$ in the remaining case when $T - t_0 \leq |p - q|^{1/3}$. \hfill $\square$

### 4.3. Proof of the coupling property.

We now prove Proposition 4.3 by means of a reflection coupling, inspired by [18, 48]. Throughout, we use the notation $Z_t := \tilde{P}_t - \tilde{Q}_t$, for $t \in [0, T]$.

#### 4.3.1. Preliminary result.

**Proposition 4.7.** Under the same assumption and notation as in the statement of Proposition 4.3, there exists a constant $C$, only depending on $\delta$, $\kappa$, $\|b_i\|_{\infty}$, $\|\beta_i\|_{\infty}$, $\|\nu_i\|_{\infty}$ and $T$ such that (see footnote 8 for the meaning of the inequality right below),

$$
\begin{align*}
d|Z_t| &\leq Cd t + \sum_{i \in [d - m]} \frac{Z_i^j}{|Z_t|} \frac{\beta_i^j}{\max(P_i^j, Q_i^j)} dt + \varepsilon \frac{s^{-1}(P_i^j)}{2} + \frac{s^{-1}(Q_i^j)}{2} \sum_{i \in [d - m], i \neq j} \frac{Z_i^j}{|Z_t|} dW_t^{i,j} \tilde{P}_i^j,
\end{align*}
$$

for $t \in [t_0, \varpi \wedge T)$, where for each $i \in [d - m]$, $(\beta_i^j)_{0 \leq t \leq T}$ is a progressively measurable process that is dominated by $4\|\tilde{b}_i\|_{\infty}$.

**Proof.** Throughout the proof, we use the convention $\overline{W}_t^{i,j} := 0$, for $t \in [0, T]$ and $i \in [d - m]$.

**First Step.** The first step is to perform a change of variable in equation (4.16), letting therein

$$
\tilde{P}_i^j := \sqrt{P_i^j}, \quad t \in [t_0, T], \quad i \in [d - m].
$$

By Itô’s formula, we get

$$
\begin{align*}
d\tilde{P}_i^j &= \varepsilon^{-2}(P_i^j) \left( \varphi'(s^2(P_i^j) P_i^j) + \tilde{b}_i(t, P_i^j, P_i^j) + (\tilde{P}_i^j)^2 \tilde{b}_i(t, P_i^j, P_i^j) - \frac{\varepsilon^2}{8} \frac{P_i^j}{P_i^j} + \frac{\varepsilon^2}{8} \tilde{P}_i^j \right) dt \\
&\quad + \varepsilon \frac{s^{-1}(P_i^j)}{2} \sum_{j \in [d - m]} \tilde{P}_i^j d\overline{W}_t^{i,j}, \tag{4.22}
\end{align*}
$$
for \( t \in [t_0, T] \), which prompts us to let

\[
\bar{B}_t(t, r^\circ, r) := \frac{\varphi(r^\circ r)}{2\sqrt{r_t}} + \bar{b}_t(t, r^\circ, r) + r_t \bar{b}_t(t, r^\circ, r) - \frac{\epsilon^2}{8\sqrt{r_t}} + \frac{\epsilon^2}{8\sqrt{r_t}},
\]

for \( r^\circ \in \bar{S}_m \) and \( r \in \mathbb{S}_{d-m-1} \). Denoting by \( B^\circ \) and \( \Sigma^\circ \) the drift and diffusion coefficients in the dynamics of \( P^\circ \) in equation (4.6), we then look at the solutions of the coupled SDEs:

\[
\begin{align*}
    d\bar{P}_t^i &= \sigma^{-2}(P_t^i)\bar{B}_t(t, P_t^i, P_t)dt + \epsilon \sigma^{-1}(P_t^i) \sum_{j \in [d-m]} \bar{P}_t^j dW_t^{i,j} \\
    dP_t^i &= B^\circ(t, P_t^i, P_t)dt + \Sigma^\circ(t, P_t^i, P_t) dW_t^\circ \\
    d\bar{Q}_t^i &= \sigma^{-2}(Q_t^i)\bar{B}_t(t, Q_t^i, Q_t)dt + \epsilon \sigma^{-1}(Q_t^i) \sum_{j \in [d-m]} \bar{Q}_t^j (R_t dW_t^i R_t)^{i,j}, \\
    dQ_t^i &= B^\circ(t, Q_t^i, Q_t)dt + \Sigma^\circ(t, Q_t^i, Q_t) dW_t^\circ,
\end{align*}
\]

(4.23)

where \( Q_t^i := (\bar{Q}_t)^2 \), for \( t \in [t_0, T] \) and \( i \in [d-m] \), and where \( R_t \) denotes the reflection matrix:

\[
R_t := I_{d-m} - 2\frac{(\bar{P}_t - \bar{Q}_t)(\bar{P}_t - \bar{Q}_t)^\dagger}{|\bar{P}_t - \bar{Q}_t|^2}1_{t < \tau}, \quad t \in [t_0, T],
\]

(4.24)

with \( \tau := \inf \{ t \geq t_0 : \bar{P}_t = \bar{Q}_t \} \), \( I_{d-m} \) standing for the identity matrix of dimension \( d - m \). The initial conditions are \( \bar{P}_{t_0} = \bar{p} := \sqrt{p} \), \( P_{t_0}^i = p^i \), \( \bar{Q}_{t_0} = \bar{q} := \sqrt{q} \) and \( Q_{t_0}^i = q^i \), for the same \( p, p^i, q \) and \( q^i \) as in the statement of Proposition 4.3.

We claim that (4.23) is uniquely solvable in the strong sense, see Lemma 4.9. Importantly, we prove in Lemma 4.8 below that \( \int_{t_0}^T R_t dW_s R_s \) is an antisymmetric Brownian motion of dimension \((d-m)(d-m-1)/2\). Since (4.6) is uniquely solvable, this proves that the law of \((Q_t^i, \bar{Q}_t)_{t_0 \leq t \leq T}\) coincides with the law of the solution to the first two equations in (4.23) when the latter are initiated from \((q^i, \sqrt{q})\) at time \( t_0 \). In other words, the law of \((Q_t^i, \bar{Q}_t)_{t_0 \leq t \leq T}\) coincides with the law of the solution to (4.5) when the latter is initiated from \((q^i, q)\) at time \( t_0 \). We then define the stopping times \( \rho, \vartheta \) and \( \sigma \) as in the statement of Proposition 4.3.

**Second Step.** We now have a look at \((Z_t = \bar{P}_t - \bar{Q}_t)_{t_0 \leq t \leq T}\). Using the fact that \( R_t \bar{Q}_t = \bar{P}_t \), for \( t \in [t_0, \tau \wedge T] \), we get

\[
\begin{align*}
    d\bar{W}_t \bar{P}_t - R_t d\bar{W}_t R_t \bar{Q}_t &= d\bar{W}_t \bar{P}_t - R_t d\bar{W}_t \bar{P}_t = 2\frac{Z_t^i Z_t^i}{|Z_t|^2} d\bar{W}_t \bar{P}_t.
\end{align*}
\]

We deduce the following expression

\[
\frac{1}{dt} \left( \frac{Z_t^i}{|Z_t|^2} d\bar{W}_t \bar{P}_t \right) = \sum_{i,j \in [d-m]} \sum_{i', j' \in [d-m]} \frac{Z_t^i Z_t^{i'}}{|Z_t|^2} (\delta_{i,i'} \delta_{j,j'} - \delta_{i,j} \delta_{i',j'}) \bar{P}_t^i \bar{P}_t^{i'}.
\]

(4.25)

Now, we may compute the bracket of the above right-hand side. We get

\[
\begin{align*}
    \frac{1}{dt} \left( \frac{Z_t^i}{|Z_t|^2} d\bar{W}_t \bar{P}_t \right)^2 &= 1 - \left( \frac{\bar{Q}_t}{\tilde{Q}_t} \right)^2 = 1 - \frac{(1 - \bar{Q}_t)}{2} (\tilde{Q}_t, \tilde{P}_t) = 1 - \frac{1}{2} (1 - \bar{Q}_t, \tilde{P}_t) = 1 - \frac{1}{4} |Z_t|^2,
\end{align*}
\]
which holds true for $t < T \wedge \tau$. More generally, we need to compute the bracket of $(\int_{t_0}^{t \wedge \tau} d\bar{W}_s \bar{P}_s - R_s d\bar{W}_s R_\tau \bar{Q}_s)_{t_0 \leq t \leq T}$. For $i, j \in [d - m]$ and $t \in (t_0, \tau)$, we have

$$
\langle (d\bar{W}_t \bar{P}_t - R_d d\bar{W}_d R_\tau \bar{Q}_\tau)^i, (d\bar{W}_t \bar{P}_t - R_d d\bar{W}_d R_\tau \bar{Q}_\tau)^j \rangle = 4 \langle \left( \frac{Z_t Z_i^\dagger}{|Z_t|^2} d\bar{W}_t \bar{P}_t \right)^i, \left( \frac{Z_t Z_i^\dagger}{|Z_t|^2} d\bar{W}_t \bar{P}_t \right)^j \rangle
$$

Here,

$$
\langle \left( \frac{Z_t Z_i^\dagger}{|Z_t|^2} d\bar{W}_t \bar{P}_t \right)^i, \left( \frac{Z_t Z_i^\dagger}{|Z_t|^2} d\bar{W}_t \bar{P}_t \right)^j \rangle = \sum_{k, l \in [d - m]} \frac{Z_k^i Z_l^j Z_k^l Z_l^i}{|Z_t|^4} \langle (d\bar{W}_t \bar{P}_t)^k, (d\bar{W}_t \bar{P}_t)^l \rangle.
$$

Now,

$$
\frac{1}{dt} \langle (d\bar{W}_t \bar{P}_t)^i, (d\bar{W}_t \bar{P}_t)^j \rangle = \sum_{k, l \in [d - m]} \frac{1}{dt} \langle d\bar{W}_t^{i,k} \bar{P}_t^k, d\bar{W}_t^{j,l} \bar{P}_t^l \rangle
$$

$$
= \sum_{k, l \in [d - m]} \bar{P}_t^k \bar{P}_t^l (\delta_{i,j} \delta_{k,l} - \delta_{i,l} \delta_{j,k}) = \delta_{i,j} - \bar{P}_t^i \bar{P}_t^j.
$$

So,

$$
\frac{1}{dt} \langle \left( \frac{Z_t Z_i^\dagger}{|Z_t|^2} d\bar{W}_t \bar{P}_t \right)^i, \left( \frac{Z_t Z_i^\dagger}{|Z_t|^2} d\bar{W}_t \bar{P}_t \right)^j \rangle = \sum_{k, l \in [d - m]} \frac{Z_k^i Z_l^j Z_k^l Z_l^i}{|Z_t|^4} (\delta_{k,l} - \bar{P}_t^k \bar{P}_t^l)
$$

$$
= \frac{Z_k^i Z_l^j}{|Z_t|^2} - \frac{Z_k^i Z_l^j}{|Z_t|^2} \langle Z_t, \bar{P}_t \rangle^2.
$$

And then,

$$
\frac{1}{dt} \langle (d\bar{W}_t \bar{P}_t - R_d d\bar{W}_d R_\tau \bar{Q}_\tau)^i, (d\bar{W}_t \bar{P}_t - R_d d\bar{W}_d R_\tau \bar{Q}_\tau)^j \rangle = 4 \left( \frac{Z_k^i Z_l^j}{|Z_t|^2} - \frac{Z_k^i Z_l^j}{|Z_t|^2} \langle Z_t, \bar{P}_t \rangle^2 \right).
$$

(4.26)

**Third Step.** Now, we return to the equation satisfied by $(Z_t)_{t_0 \leq t \leq T}$:

$$
dZ_t = \left[ \kappa^{-2}(P_t^0) \bar{B}(t, P_t^0, P_t) - \kappa^{-2}(Q_t^0) \bar{B}(t, Q_t^0, Q_t) \right] dt
$$

$$
+ \varepsilon \frac{\kappa^{-1}(P_t^0)}{2} d\bar{W}_t \bar{P}_t - \varepsilon \frac{\kappa^{-1}(Q_t^0)}{2} R_t d\bar{W}_t R_t \bar{Q}_t,
$$

for $t \in [0, T]$. The point is to apply Itô’s formula to $|Z|$. Using the relationship $R_t \bar{Q}_t = \bar{P}_t$ together with the fact that $Z_t^i R_t = (R_t Z_t)^i = -Z_t^i$, we get, for $t < \varpi \wedge T$,

$$
d|Z_t| = \frac{Z_t^i}{|Z_t|} \left[ \kappa^{-2}(P_t^0) \bar{B}(t, P_t^0, P_t) - \kappa^{-2}(Q_t^0) \bar{B}(t, Q_t^0, Q_t) \right] dt + \varepsilon \frac{\kappa^{-1}(P_t^0) + \kappa^{-1}(Q_t^0)}{2} \frac{Z_t^i}{|Z_t|} d\bar{W}_t \bar{P}_t
$$

$$
+ \varepsilon^2 \frac{1}{8} \sum_{i,j \in [d - m]} \left[ \left( \kappa^{-1}(P_t^0) d\bar{W}_t \bar{P}_t - \kappa^{-1}(Q_t^0) R_t d\bar{W}_t R_t \bar{Q}_t \right)^i \right.
$$

$$
\left. \left( \kappa^{-1}(P_t^0) d\bar{W}_t \bar{P}_t - \kappa^{-1}(Q_t^0) R_t d\bar{W}_t R_t \bar{Q}_t \right)^j \right\} \times \frac{1}{|Z_t|} \left( \delta_{i,j} - \frac{Z_t^i Z_t^j}{|Z_t|^2} \right).
At this stage of the proof, we have a special look at the brackets in the last two lines in the above expression. By Kunita–Watanabe inequality, we get, for $t < \varpi \wedge T$,

$$
\left\langle \left( \varsigma^{-1}(P^o_t) d\bar{W}_t \tilde{P}_t - \varsigma^{-1}(Q^o_t) R_t d\bar{W}_t R_t \tilde{Q}_t \right)^i, \left( \varsigma^{-1}(P^o_t) d\bar{W}_t \tilde{P}_t - \varsigma^{-1}(Q^o_t) R_t d\bar{W}_t R_t \tilde{Q}_t \right)^j \right\rangle
$$

$$
= \varsigma^{-2}(P^o_t) \left\langle \left( d\bar{W}_t \tilde{P}_t - R_t d\bar{W}_t R_t \tilde{Q}_t \right)^i, \left( d\bar{W}_t \tilde{P}_t - R_t d\bar{W}_t R_t \tilde{Q}_t \right)^j \right\rangle + O \left( \left| \varsigma^{-1}(P^o_t) - \varsigma^{-1}(Q^o_t) \right| \right) dt,
$$

where $(O(\varsigma^{-1}(P^o_t) - \varsigma^{-1}(Q^o_t)))_{0 \leq t < \varpi \wedge T}$ stands for a progressively measurable process that is dominated by $C((\varsigma^{-1}(P^o_t) - \varsigma^{-1}(Q^o_t)))_{0 \leq t < \varpi \wedge T}$ for a universal constant $C$. Invoking (4.26), we get

$$
\left\langle \left( \varsigma^{-1}(P^o_t) d\bar{W}_t \tilde{P}_t - \varsigma^{-1}(Q^o_t) R_t d\bar{W}_t R_t \tilde{Q}_t \right)^i, \left( \varsigma^{-1}(P^o_t) d\bar{W}_t \tilde{P}_t - \varsigma^{-1}(Q^o_t) R_t d\bar{W}_t R_t \tilde{Q}_t \right)^j \right\rangle
$$

$$
= 4\varsigma^{-2}(P^o_t) \left( \frac{Z_i Z_j}{|Z_t|^2} - \frac{Z_i^i Z_j}{|Z_t|^4} \langle Z_t, \tilde{P}_t \rangle^2 \right) dt + O \left( |P^o_t - Q^o_t| \right) dt
$$

$$
= 4\varsigma^{-2}(P^o_t) \left( \frac{Z_i Z_j}{|Z_t|^2} - \frac{Z_i^i Z_j}{|Z_t|^4} \langle Z_t, \tilde{P}_t \rangle^2 \right) dt + O(|Z_t|) dt,
$$

where we used the fact that $t < \varpi$ to derive the last line of the statement. As before, the process $(O(|Z_t|))_{0 \leq t < \varpi \wedge T}$ is a progressively measurable process that is dominated by $(C|Z_t|)_{0 \leq t < \varpi \wedge T}$ for a universal constant $C$. Returning to the expression for $d|Z_t|$, we then get that

$$
\frac{1}{8} \sum_{i,j \in [d-m]} \left\langle \left( \varsigma^{-1}(P^o_t) d\bar{W}_t \tilde{P}_t - \varsigma^{-1}(Q^o_t) R_t d\bar{W}_t R_t \tilde{Q}_t \right)^i, \left( \varsigma^{-1}(P^o_t) d\bar{W}_t \tilde{P}_t - \varsigma^{-1}(Q^o_t) R_t d\bar{W}_t R_t \tilde{Q}_t \right)^j \right\rangle \times \frac{1}{|Z_t|} \left( \delta_{i,j} - \frac{Z_i Z_j^i}{|Z_t|^2} \right)
$$

$$
= \frac{1}{2|Z_t|} \varsigma^{-2}(P^o_t) \sum_{i,j \in [d-m]} \left( \frac{Z_i Z_j^i}{|Z_t|^2} - \frac{Z_i Z_j}{|Z_t|^4} \langle Z_t, \tilde{P}_t \rangle^2 \right) (\delta_{i,j} - \frac{Z_i Z_j}{|Z_t|^2}) \ dt + O(1) dt,
$$

where $(O(1))_{0 \leq t < \varpi \wedge T}$ stands for a progressively measurable process that is dominated by $C$ for a universal constant $C$. The key fact here is that

$$
\sum_{i,j \in [d-m]} \frac{Z_i Z_j^i}{|Z_t|^2} (\delta_{i,j} - \frac{Z_i Z_j}{|Z_t|^2}) = \sum_{i \in [d-m]} \frac{(Z_i)^2}{|Z_t|^2} - \left( \sum_{i \in [d-m]} \frac{(Z_i)^2}{|Z_t|^2} \right)^2 = 1 - 1 = 0.
$$

We deduce that the last line in (4.27) reduces to $O(1) dt$. We end up with

$$
d|Z_t| = \frac{Z_t^t}{|Z_t|} \left[ \varsigma^{-2}(P^o_t) \tilde{B}(t, P_t^o, \tilde{P}_t) - \varsigma^{-2}(Q^o_t) \tilde{B}(t, Q_t^o, \tilde{Q}_t) \right] dt + O(1) dt, \quad t < \varpi \wedge T.
$$

**Fourth Step.** We now have a look at the drift in a more precise way. Recalling that

$$
\tilde{B}_t(t, P_t^o, \tilde{P}_t) = \frac{\varphi(\varsigma^2(P_t^o)P_t^i) + \tilde{b}_t(t, P_t^o, \tilde{P}_t) + (\tilde{P}_t)^2 \tilde{b}^i_0(t, P_t^o, \tilde{P}_t)}{2 P_t^i} - \frac{\varphi^2}{8 P_t^i} + \frac{\varphi^2}{8} \tilde{P}_t^i,
$$
we write

\[
d|Z_t| = \sum_{i \in [d-m]} \frac{Z_t^i}{|Z_t|} \left[ \frac{\varsigma^2(P_0^i)}{2P_t^i} \varphi(\varsigma^2(P_0^i)P_t^i) + \bar{b}_i(t, P_0^i, P_t) - \epsilon^2/4 - \frac{\varphi(\varsigma^2(Q_0^i)Q_t^i)}{2Q_t^i} + \bar{b}_i(t, Q_0^i, Q_t) - \epsilon^2/4 \right] dt + O(1) dt
\]

where the constant dominating \(O(1)\) is now allowed to depend on \((\|b_i^0\|_\infty)_{i \in [d-m]}\). Fix now an index \(i \in [d-m]\). Then, on the event \(P_t^i \leq Q_t^i\), we have, for \(t < \varpi \wedge T\),

\[
\frac{Z_t^i}{|Z_t|} |\varsigma^2(P_0^i) - \varsigma^2(Q_0^i)| \leq C \frac{|Z_t^i|}{|Z_t|} |P_t^i - Q_t^i| \leq C |Z_t^i| \leq C(|\tilde{P}_t^i + \tilde{Q}_t^i|) \leq 2C \tilde{Q}_t^i,
\]

for a universal constant \(C\). Proceeding similarly whenever \(Q_t^i \leq P_t^i\), we can find a collection of non-negative bounded processes \((\tilde{Q}_t^i)_{0 \leq t \leq T})_{i \in [d-m]}\) (bounded by 4 since \(\varsigma^2(P_0^i)\) and \(\varsigma^2(Q_0^i)\) are bounded by 4 for \(t < \varpi \wedge T\)) such that

\[
d|Z_t| = \sum_{i=1}^{d-m} \frac{Z_t^i}{|Z_t|} \left[ \frac{\varsigma^2(P_0^i)}{2P_t^i} \varphi(\varsigma^2(P_0^i)P_t^i) + \bar{b}_i(t, P_0^i, P_t) - \epsilon^2/4 - \frac{\varphi(\varsigma^2(Q_0^i)Q_t^i)}{2Q_t^i} + \bar{b}_i(t, Q_0^i, Q_t) - \epsilon^2/4 \right] dt
\]

\[+ O(1) dt + \epsilon \frac{\varsigma^{-1}(P_0^i) + \varsigma^{-1}(Q_0^i)}{2} \frac{Z_t^i}{|Z_t|} d\tilde{W}_t\tilde{P}_t, \quad t < \varpi \wedge T.\]

Notice that, on the event \(\{\max(P_t^i, Q_t^i) > \delta\}\), we have \(\min(P_t^i, Q_t^i) > \delta/2\), for \(t < \varpi \wedge T\), since \(|P_t^i - Q_t^i| \leq 2|\tilde{P}_t^i - \tilde{Q}_t^i| \leq \delta/2\). Allowing \(O(1)\) to depend on \(\delta, \kappa (\|b_i^0\|_\infty)_{i \in [d-m]}\) and \((\|b_i^0\|_\infty)_{i \in [d-m]}\), we get

\[
d|Z_t| \leq \sum_{i=1}^{d-m} \frac{Z_t^i}{|Z_t|} \left[ \frac{\varsigma^2(P_0^i)}{2P_t^i} \varphi(\varsigma^2(P_0^i)P_t^i) + \bar{b}_i(t, P_0^i, P_t) - \epsilon^2/4 - \frac{\varphi(\varsigma^2(Q_0^i)Q_t^i)}{2Q_t^i} + \bar{b}_i(t, Q_0^i, Q_t) - \epsilon^2/4 \right] \mathbf{1}_{\{\max(P_t^i, Q_t^i) \leq \delta\}} dt
\]

\[+ O(1) dt + \epsilon \frac{\varsigma^{-1}(P_0^i) + \varsigma^{-1}(Q_0^i)}{2} \frac{Z_t^i}{|Z_t|} d\tilde{W}_t\tilde{P}_t, \quad t < \varpi \wedge T.\]
Now, it remains to see that
\[
\frac{\varphi(\varepsilon^2(P^i_t) P^i_t) + \tilde{b}_i(t, P^i_t, P_t) - \varepsilon^2/4}{2P^i_t} - \frac{\varphi(\varepsilon^2(Q^i_t) Q^i_t) + \tilde{b}_i(t, Q^i_t, Q_t) - \varepsilon^2/4}{2Q^i_t}
\]
\[
= - \frac{\varphi(\varepsilon^2(Q^i_t) Q^i_t) + \tilde{b}_i(t, Q^i_t, Q_t) - \varepsilon^2/4}{2} \left( \frac{1}{Q^i_t} - \frac{1}{P^i_t} \right)
\]
\[
+ \frac{\varphi(\varepsilon^2(P^i_t) P^i_t) + \tilde{b}_i(t, P^i_t, P_t) - [\varphi(\varepsilon^2(Q^i_t) Q^i_t) + \tilde{b}_i(t, Q^i_t, Q_t)]}{2P^i_t}
\]
\[
= - \frac{\varphi(\varepsilon^2(Q^i_t) Q^i_t) + \tilde{b}_i(t, Q^i_t, Q_t) - \varepsilon^2/4}{2} \left( \frac{Z^i_t}{P^i_t Q^i_t} \right)
\]
\[
+ \frac{\varphi(\varepsilon^2(P^i_t) P^i_t) + \tilde{b}_i(t, P^i_t, P_t) - [\varphi(\varepsilon^2(Q^i_t) Q^i_t) + \tilde{b}_i(t, Q^i_t, Q_t)]}{2P^i_t}
\]

Recalling that \( \tilde{b}_i \), as given by Proposition \ref{prop:4.2}, for each \( i \in \{1, \cdots, d - m\} \), has non-negative values, that \( \kappa \geq \kappa_0 \geq 2 \), see Proposition \ref{prop:4.3}, and that \( \varepsilon \in (0, 1) \), we deduce that, on the event \( \{ \max(P^i_t, Q^i_t) \leq \delta \} \), \( \varphi(\varepsilon^2(Q^i_t) Q^i_t) + \tilde{b}_i(t, Q^i_t, Q_t) - \varepsilon^2/4 \geq 0 \). Therefore, whenever \( P^i_t \geq Q^i_t \) and \( P^i_t \leq \delta \),
\[
\frac{\varphi(\varepsilon^2(P^i_t) P^i_t) + \tilde{b}_i(t, P^i_t, P_t) - \varepsilon^2/4}{2P^i_t} - \frac{\varphi(\varepsilon^2(Q^i_t) Q^i_t) + \tilde{b}_i(t, Q^i_t, Q_t) - \varepsilon^2/4}{2Q^i_t}
\]
\[
\leq \frac{\varphi(\varepsilon^2(P^i_t) P^i_t) + \tilde{b}_i(t, P^i_t, P_t) - [\varphi(\varepsilon^2(Q^i_t) Q^i_t) + \tilde{b}_i(t, Q^i_t, Q_t)]}{2P^i_t}
\]
\[
= \frac{\kappa + \tilde{b}_i(t, P^i_t, P_t) - [\kappa + \tilde{b}_i(t, Q^i_t, Q_t)]}{2P^i_t}
\]
\[
\leq \frac{\|\tilde{b}_i\|_{\infty}}{\max(P^i_t, Q^i_t)}.
\]

Proceeding similarly when \( Q^i_t \geq P^i_t \) and \( Q^i_t \leq \delta \) and then letting
\[
\tilde{\beta}^i_t := \zeta^i_t \|\tilde{b}_i\|_{\infty} \mathbf{1}_{\{\max(P^i_t, Q^i_t) \leq \delta\}},
\]
we get
\[
d|Z_t| \leq \sum_{i=1}^{d-m} \frac{\tilde{\beta}^i_t}{|Z_t|_{\max(P^i_t, Q^i_t)}} dt + O(1) dt + \varepsilon \frac{\zeta^{-1}(P^o_t) + \zeta^{-1}(Q^o_t)}{2} \frac{Z^i_t}{|Z_t|} d\tilde{W}^i_t \tilde{P}_t, \quad t < \tau \land T,
\]
which completes the proof. \( \square \)

**Lemma 4.8.** Take \( (\tilde{P}_t)_{0 \leq t \leq T} \) and \( (\tilde{Q}_t)_{0 \leq t \leq T} \) two continuous \( \mathbb{R}^{W^o, W} \)-adapted processes with values in the intersection of the orthant \( (\mathbb{R}_+)^d \) and of the sphere of dimension \( d \). Then, letting
\[
R_t := I_d - 2 (\tilde{P}_t - \tilde{Q}_t)(\tilde{P}_t - \tilde{Q}_t)^t \mathbf{1}_{t < \tau}, \quad t \in [0, T],
\]
with \( \tau = \inf \{ t \geq t_0 : \tilde{P}_t = \tilde{Q}_t \} \), \( I_d \) standing for the identity matrix of dimension \( d \), the process \( (\int_0^t R_s d\tilde{W}_s \tilde{R}_s)_{0 \leq t \leq T} \), with the convention \( \tilde{W}_t^{i,i} := 0 \) for \( t \in [0, T] \) and \( i \in [d] \), is an antisymmetric Brownian motion of dimension \( d(d - 1)/2 \) independent of \( W^o \).

**Proof.** We first extend the family \( W \) into a new family \( \tilde{W} \) of independent Brownian motions, by letting \( \tilde{W}^{i,j} = W^{i,j} \) for \( i, j \in [d] \) with \( i \neq j \) and by assuming that the family \( (W^{i,i})_{i \in [d]} \).
is a collection of Brownian motions that is independent of $W$. We then observe that, for any $i, j \in [d]$ with $i \neq j$,

$$
\left( \int_0^t R_s d\tilde{W}_s R_s \right)_{i,j} = \frac{1}{\sqrt{2}} \left[ \left( \int_0^t R_s d\tilde{W}_s R_s \right)_{i,i} - \left( \int_0^t R_s d\tilde{W}_s R_s \right)_{j,j} \right], \quad t \in [0,T].
$$

In order to complete the proof, it suffices to show that the family $((\int_0^t R_s d\tilde{W}_s R_s)_{0 \leq t \leq T})_{i,j} \in [d] ; i \neq j$ forms a collection of independent Brownian motions that is independent of $W^\circ$. Independence between $((\int_0^t R_s d\tilde{W}_s R_s)_{0 \leq t \leq T})_{i,j} \in [d]$ and $W^\circ$ is obvious. It thus remains to compute the brackets of the family $((\int_0^t R_s d\tilde{W}_s R_s)_{0 \leq t \leq T})_{i,j} \in [d]$ to conclude. Using the fact that $R_s^i = R_s$ and $R_s R_s = I_d$, we have

$$
\sum_{k,l \in [d]} R_s^{i,k} d\tilde{W}_s^{k,l} R_s^{l,j} \cdot \sum_{k',l' \in [d]} R_s^{i,k'} d\tilde{W}_s^{k',l'} R_s^{l',j'} = \sum_{k,l \in [d]} \sum_{k',l' \in [d]} \left( R_s^{i,k} R_s^{i,k'} R_s^{l,j} R_s^{l',j'} \delta_{k,k'} \delta_{l,l'} \right) = \sum_{k,l \in [d]} \left( R_s^{i,k} R_s^{i,k'} R_s^{l,j} R_s^{l',j'} \right) = \delta_{i,i} \delta_{j,j},
$$

which completes the proof.

\[ \square \]

**Lemma 4.9.** Under the assumption and notations of Propositions 4.3 and 4.7, Equation (4.23) is uniquely solvable (in the strong sense).

**Proof.** We first observe that, at any time $t \in [0,T]$, the coefficients $B(t, \cdot)$ and $B^\circ(t, \cdot)$ are a priori defined as functions of the space variable $(r^\circ, r) \in \hat{S}_m \times \mathbb{S}_{d-m-1}$. By projecting $\mathbb{R}^m$ onto $\hat{S}_m$ (which is convex) and then $\mathbb{R}^{d-m}$ onto $\mathbb{S}_{d-m-1}$ (which is also convex), we may easily extend them to the entire $\mathbb{R}^d$. We then observe from Proposition 4.2 that the full-fledged drift coefficient in the system (4.23) remains Lipschitz continuous in the four entries $(P_0^\circ, P_t, Q_0^\circ, Q_t)$ as long the coordinates of the latter remain away from zero. Similarly, the diffusion coefficient remains Lipschitz continuous in the same four entries as long as the coordinates of the latter remain strictly positive and the distance between $P_t$ and $Q_t$ remains also strictly positive.

Therefore, we deduce that, for any small $a > 0$, the system (4.23) is uniquely solvable up to the first time $\tau^a$ when one of the coordinates of the vector $(P_{\tau^a}^\circ, P_{\tau^a}, Q_{\tau^a}^\circ, Q_{\tau^a})$ is less than $a$ or the distance between $P_{\tau^a}$ and $Q_{\tau^a}$ becomes less than $a$. Letting $a$ tend to 0, we deduce that (4.23) is uniquely solvable up to $\tau = \lim_{a \searrow 0} \tau^a \wedge T$.

By Lemma 4.8, we know that, up to time $\tau$, we may see $(P_t^\circ, P_t)_{t_0 \leq t \leq \tau}$ and $(Q_t^\circ, Q_t)_{t_0 \leq t \leq \tau}$ as solutions of an SDE of the same type as (4.0). Hence, by identity in law (4.5) and by Proposition 4.1 (or equivalently by Proposition 2.1 recalling that $\kappa_0 \geq 2$), we deduce that, both processes take values in $\hat{S}_m \times \mathbb{S}_{d-m}$ and that

$$
P\left( \inf_{t \in [m]} t_0 \leq t \leq \tau P_{t}^i > 0, \inf_{t \in [m]} t_0 \leq t \leq \tau Q_{t}^i > 0, \inf_{t \in [d-m]} t_0 \leq t \leq \tau P_{t}^j > 0, \inf_{t \in [d-m]} t_0 \leq t \leq \tau Q_{t}^j > 0 \right) = 1.
$$

This shows in particular that the drift in (4.23) remains bounded up to time $\tau$ and that it makes sense to extend (by continuity) the process $(P, P^\circ, Q, Q^\circ)$ to the closed interval $[0, \tau]$. Moreover, we must have

$$
P(\tau = \tau) = 1,
$$

where we recall that $\tau$ denotes the first time when the two processes $(P_t)_{0 \leq t \leq \tau}$ and $(Q_t)_{0 \leq t \leq \tau}$ meet. This proves the unique strong solvability on $[0, \tau]$. Unique solvability from $\tau$ to $T$ is addressed in a similar manner noting that the diffusion coefficient then becomes simpler, see (4.24). \[ \square \]
4.3.2. Proof of Proposition 4.3.2 We recall that $|p - q| < \delta^2/(64\sqrt{d})$.

**First Step.** The proof mostly relies on a Girsanov argument. Using the same notations as in the statement and in the proof of Proposition 4.1, we let (see (4.22) and (4.23))

$$W^{i,j}_t := W^{i,j}_t + \frac{1}{\varepsilon} \int_t^T \Psi^i_j ds, \quad \Psi^i_j := \frac{2\sqrt{2}P^j_t}{\langle s^{-1}(P^j_t) + s^{-1}(Q^j_t) \rangle \max(P^j_t, Q^j_t)} 1_{t < \infty},$$

for $t \in [t_0, T]$ and $i, j \in [d - m]$ with $i \neq j$, and $W^{i,i}_t = 0$ for $i \in [d - m]$. Then, for all $t \in [t_0, \varpi \wedge T)$,

$$\varepsilon \frac{Z^\dagger_t}{Z_t} d\tilde{P}_t = \varepsilon \frac{Z^\dagger_t}{Z_t} d\tilde{P}_t - \frac{Z^\dagger_t}{\sqrt{2}|Z_t|} (\Psi_t - \Psi^i_t) \tilde{P}_tdt,$$

with

$$\frac{Z^\dagger_t}{\sqrt{2}|Z_t|} (\Psi_t - \Psi^i_t) \tilde{P}_t = \frac{2}{s^{-1}(P^i_t) + s^{-1}(Q^i_t)} \sum_{i,j \in [d - m]} Z^i_t \left( \frac{\tilde{P}^j_t}{\max(P^j_t, Q^j_t)} - \frac{\tilde{P}^i_t}{\max(P^i_t, Q^i_t)} \right) \tilde{P}^j_t = \frac{2}{s^{-1}(P^i_t) + s^{-1}(Q^i_t)} \sum_{i,j \in [d - m]} Z^i_t \left( \frac{\tilde{P}^j_t}{\max(P^j_t, Q^j_t)} - \frac{\tilde{P}^i_t}{\max(P^i_t, Q^i_t)} \right),$$

where we used the identity $\sum_{j \in [d - m]} (\tilde{P}^j_t)^2 = 1$. Plugging the above identity into (4.21), we get

$$d|Z_t| \leq C dt + \varepsilon s^{-1}(P^i_t) + s^{-1}(Q^i_t) \frac{Z^\dagger_t}{|Z_t|} d\tilde{P}_t, \quad t \in [t_0, \varpi \wedge T),$$

where $C$ is a constant only depending on $\delta, \kappa, (\|b_i\|_{\infty})_{i=1,\ldots,d}$, $(\|b^0_i\|_{\infty})_{i=1,\ldots,d}$ and $T$.

**Second Step.** We now introduce the probability measure:

$$\frac{dQ}{dP} = \exp \left( -\frac{1}{\varepsilon} \sum_{i,j \in [d - m]: i \neq j} \int_0^{\varpi \wedge T} \Psi^i_j dW^i_j - \frac{1}{2\varepsilon^2} \sum_{i,j \in [d - m]: i \neq j} \int_0^{\varpi \wedge T} |\Psi^i_j|^2 dt \right).$$

Under $Q$, the processes $(W^{i,j}_{t_0 \leq t \leq T})_{i,j \in [d - m]: i \neq j}$ are independent Brownian motions (the fact that we can apply Girsanov’s theorem is fully justified in the third step of the proof). By (1.25), the bracket of the martingale part in (4.29) is given by (up to the leading multiplicative factor)

$$\frac{1}{dt} \left( \frac{Z^\dagger_t}{|Z_t|} dW^i_t \right) = 1 - \frac{1}{4}|Z_t|^2.$$

In particular, there exists a Brownian motion $(B_t)_{t_0 \leq t \leq T}$ under $Q$ such that

$$d|Z_t| \leq C dt + \varepsilon \frac{s^{-1}(P^0_t) + s^{-1}(Q^0_t)}{2} \sqrt{1 - \frac{1}{4}|Z_t|^2} dB_t, \quad t \in [t_0, \varpi \wedge T).$$

Let now

$$d\Theta_t = C dt + \varepsilon \theta_t dB_t, \quad \theta_t := \min \left( \frac{s^{-1}(P^0_t) + s^{-1}(Q^0_t)}{2}, c' \right) \sqrt{1 - \frac{1}{4}\min(|Z_t|^2, \delta^2/4)},$$

with $|\Theta_t| = |Z_t| = |\tilde{p} - \tilde{q}|$, with $\tilde{p} = (\sqrt{P_1}, \ldots, \sqrt{P_{d-m}})$ and similarly for $\tilde{q}$, and where $c' = \min\{s^{-1}(p^\circ), |p^\circ|_1 \geq 3/4 + \delta/\sqrt{d} \}$ and similarly for $q^\circ$, and where $c = \min\{s^{-1}(p^\circ), |p^\circ|_1 \geq 3/4 + \delta/\sqrt{d} \}$. (Note that, for $t \in [t_0, \varpi \wedge T)$, $|P^0_t| \leq 3/4$ and $|Q^0_t| \leq 3/4 + |P^0_t - Q^0_t| \leq 3/4 + \sqrt{m} |P^0_t - Q^0_t| < 3/4 + \delta / \sqrt{d} < 3/4 + 1/16 = 13/16.$) Obviously, $|Z_t| \leq \Theta_t$ for all $t \in [t_0, \varpi \wedge T]$ (because, up to time $\varpi \wedge T$, $\theta_t$ coincides with the
integrating in the stochastic integral appearing in the right-hand side of (4.31). Since \((\theta_t)_{t_0 \leq t \leq T}\) stays in a (universal) deterministic compact subset of \((0, +\infty)\), we deduce from a new application of Girsanov’s theorem that there exists a new probability measure \(Q'\) under which

\[ d\Theta_t = \varepsilon\theta_t dB'_t, \quad t \in [t_0, T], \]

\((B'_t)_{t_0 \leq t \leq T}\) being a Brownian motion under \(Q'\). By expanding the Girsanov transformation, we can check that \(E^{Q'}[(dQ'/dQ')^2] = E^{Q}[dQ'/dQ] \leq \gamma^2\), that is \(Q(A) = E^{Q'}[(dQ'/dQ')1_A] \leq \gamma Q'(A)^{1/2}\) for any event \(A \in \mathcal{F}_t\).

Clearly, \((B'_t)_{t_0 \leq t \leq T}\) can be extended into a Brownian motion (under \(Q'\)) on the entire \([t_0, +\infty)\) and, similarly, \((\theta_t)_{t_0 \leq t \leq T}\) can be also extended to the entire \([t_0, +\infty)\) by letting \(\theta_t = c'\) for \(t > T\). The process \((\Theta_t)_{t_0 \leq t \leq T}\) can be extended accordingly to the entire \([t_0, +\infty)\). Representing \((\Theta_t)_{t \geq t_0}\) in the form a time-changed Brownian motion, there exists a new Brownian motion \((\hat{B}_t)_{t \geq 0}\) under \(Q'\) (with respect to a time-changed filtration) such that

\[ \Theta_t = |\tilde{p} - \tilde{q}| + \varepsilon\hat{B}_t, \quad I_t := \int_{t_0}^t \theta_s^2 ds, \quad t \geq t_0. \]

Obviously, there exists a universal constant \(\Gamma \geq 1\) such that, with probability 1 under \(Q'\),

\[ \Gamma^{-1}(t - t_0) \leq I_t \leq \Gamma(t - t_0), \quad t \geq t_0. \]

We now call

\[ \sigma(\Theta) := \inf\{s \geq t_0 : |\Theta_s| \geq \frac{\delta}{4}\}, \quad \tau(\Theta) := \inf\{s \geq t_0 : \Theta_s = 0\}. \]

Then, recalling that \(|p - q| < \delta^2/(64\sqrt{d})\) and observing that

\[ |\tilde{p} - \tilde{q}|^2 = \sum_{i \in [d-m]} |\sqrt{p_i} - \sqrt{q_i}|^2 \leq \sum_{i \in [d-m]} |p_i - q_i| \leq \sqrt{d}|p - q|. \] (4.32)

we get \(|\tilde{p} - \tilde{q}| < \delta/8\) and then, for \(t \in [t_0, T]\),

\[ \begin{align*}
Q'(\tau(\Theta) < \sigma(\Theta), \tau(\Theta) \leq t) & \geq Q'(\varepsilon \inf_{0 \leq s \leq I_t} \hat{B}_s \leq |\tilde{p} - \tilde{q}|, \varepsilon \sup_{0 \leq s \leq I_t} \hat{B}_s \leq \frac{\delta}{8}) \\
& \geq 1 - Q'(\varepsilon \sup_{0 \leq s \leq I_t} \hat{B}_s \leq |\tilde{p} - \tilde{q}|) - Q'(\varepsilon \sup_{0 \leq s \leq I_t} \hat{B}_s \geq \frac{\delta}{8}) \\
& \geq 1 - Q'(\varepsilon \sup_{0 \leq s \leq (t - t_0)/\Gamma} \hat{B}_s \leq |\tilde{p} - \tilde{q}|) - Q'(\varepsilon \sup_{0 \leq s \leq (t - t_0)/\Gamma} \hat{B}_s \geq \frac{\delta}{8}).
\end{align*} \]

We deduce that

\[ \begin{align*}
Q'(\tau(\Theta) < \sigma(\Theta), \tau(\Theta) \leq t) & \geq 1 - C' \frac{|\tilde{p} - \tilde{q}|}{\varepsilon \sqrt{t - t_0}} - C' \exp(-\frac{1}{C'\varepsilon^2(t - t_0)}),
\end{align*} \]

for a new constant \(C'\) that is independent of \(\varepsilon\). Therefore, by (4.32), up to a new value of \(C'\),

\[ \begin{align*}
Q'(\tau(\Theta) < \sigma(\Theta), \tau(\Theta) \leq t) & \geq 1 - C' \frac{\sqrt{|p - q|}}{\varepsilon \sqrt{t - t_0}} - C' \exp(-\frac{1}{C'\varepsilon^2(t - t_0)}).
\end{align*} \]

In particular, choosing \(t - t_0 = |p - q|^{1/3}/2\), which is possible since \(|p - q|^{1/3} \leq T - t_0\), we deduce that (with \(S := t_0 + |p - q|^{1/3}\))

\[ Q'(\tau(\Theta) < \sigma(\Theta), \tau(\Theta) < S) \geq 1 - \frac{C'}{\varepsilon}|p - q|^{1/3}, \]

for a new constant \(C'\).
or, equivalently,
\[ Q' \left( \left\{ \tau(\Theta) < \sigma(\Theta), \tau(\Theta) \leq S \right\}^C \right) \leq C|p - q|^{1/3}, \]
where we recall that \( C \) is allowed to depend on \( \varepsilon \). Then, returning to \( Q \), we get, for a new value of \( C \),
\[ Q(\tau(\Theta) \geq \sigma(\Theta) \land S) = Q \left( \left\{ \tau(\Theta) < \sigma(\Theta), \tau(\Theta) \leq S \right\}^C \right) \leq C|p - q|^{1/6}. \]
We then notice from the inequality \(|Z_t| \leq \Theta_t\) for \( t \in [t_0, \infty \land T] \), that \( t \leq \infty \land T \) implies \( t \leq \tau(\Theta) \) and that \( \sigma \leq \infty \land T \) implies \( \sigma(\Theta) \leq \sigma \). Therefore, on the event \( \{ \infty \land S \} \), we have \( S \leq \tau(\Theta) \).

Moreover, on the event \( \{ \tau \land \sigma \leq \infty \land T \} \),
\[ \{ \sigma(\Theta) \leq \tau(\Theta) \} \cup \{ \sigma \leq \tau \} = \{ \tau < \sigma \}^C. \]

Hence,
\[ Q(\infty \land S < \tau \land \infty \land \rho) \leq Q(\{ S \leq \infty \} \cup \{ \sigma \leq \tau, \tau \land \sigma \leq \infty \land T \}) \leq Q(\{ S \leq \tau(\Theta) \} \cup \{ \sigma(\Theta) \leq \tau(\Theta) \}) = Q(\tau(\Theta) \geq \sigma(\Theta) \land S) \leq C|p - q|^{1/6}. \]

**Third Step.** In order to complete the proof, it remains to prove that, for a new value of the constant \( C \), for any event \( A \in \mathcal{F}_T \), \( \mathbb{P}(A) \leq C\mathbb{Q}(A)^{1/2} \), provided that \( \kappa \) is chosen large enough. As for the comparison of \( Q \) and \( Q' \) in the previous step, it suffices to prove that \( \mathbb{E}[d\mathbb{P}/d\mathbb{Q}]^{1/2} \leq C \) (since \( \mathbb{P}(A) = \mathbb{E}^\mathbb{Q}[(d\mathbb{P}/d\mathbb{Q})\mathbb{1}_A] \leq \mathbb{E}[d\mathbb{P}/d\mathbb{Q}]^{1/2}\mathbb{Q}(A)^{1/2} \)). Here (compare with (3.30)),
\[ \frac{d\mathbb{P}}{d\mathbb{Q}} = \exp\left( \frac{1}{\varepsilon} \sum_{i,j \in [d-m]: i \neq j} \int_{t_0}^{\infty \land T} \Psi_{t}^{i,j} dW_{t}^{i,j} + \frac{1}{2\varepsilon^2} \sum_{i,j \in [d-m]: i \neq j} \int_{t_0}^{\infty \land T} |\Psi_{t}^{i,j}|^2 dt \right). \]

Letting
\[ (M_t := \frac{1}{\varepsilon} \sum_{i,j \in [d-m]: i \neq j} \int_{t_0}^{t} \Psi_{s}^{i,j} dW_{s}^{i,j})_{t_0 \leq t \leq T}, \]
this may be rewritten as
\[ \frac{d\mathbb{P}}{d\mathbb{Q}} = \exp\left( M_{\infty \land T} + \frac{1}{2} \langle M \rangle_{\infty \land T} \right), \]
and then
\[ \mathbb{E}\left[ \frac{d\mathbb{P}}{d\mathbb{Q}} \right] = \mathbb{E}\left[ \exp\left( M_{\infty \land T} + \frac{1}{2} \langle M \rangle_{\infty \land T} \right) \right] \leq \mathbb{E}\left[ \exp\left( M_{\infty \land T} - \langle M \rangle_{\infty \land T} + \frac{1}{2} \langle M \rangle_{\infty \land T} \right) \right] \leq \mathbb{E}\left[ \exp\left( \frac{3}{2} \langle M \rangle_{\infty \land T} \right) \right]^{1/2}, \]
where to get the last line, we used the fact that \( \mathbb{E}[\exp(2M_{\infty \land T} - 2\langle M \rangle_{\infty \land T})] \leq 1 \). Returning to the definition of \( \Psi \) in (4.28) and recalling that \( \varepsilon^{-1} \) is lower bounded by \( 1 \), the point is to prove that
\[ \mathbb{E}\left[ \exp\left( \frac{3}{\varepsilon^2} \sum_{i,j \in [d-m]: i \neq j} \int_{t_0}^{\infty \land T} |\Psi_{t}^{i,j}|^2 dt \right) \right] \leq \mathbb{E}\left[ \exp\left( \frac{6}{\varepsilon^2} \sum_{i \in [d-m]} \int_{t_0}^{\infty \land T} \frac{1}{\max(P_{t}^{i}, Q_{t}^{i})} |\frac{\beta_{t}^{i}}{\varepsilon^2}|^2 dt \right) \right] \]
is finite, provided that \( \kappa \) is chosen large enough and then to find a tractable bound. The proof is similar to that of Proposition 2.2, but we feel better to expand it as it plays a key role in the determination of the constant \( \kappa \). Recalling the bound for \( (\beta_{t}^{i})_{0 \leq t \leq T} \) in the statement of
Proposition 4.7 using the fact that \((\tilde{P}_t^i)^2 = P_t^i\) and invoking Hölder’s inequality, it suffices to upper bound
\[
\sup_{i \in \mathbb{I}_{[d-m]}} \mathbb{E} \left[ \exp \left( \frac{6d}{\varepsilon^2} \int_{t_0}^{\omega \wedge T} \frac{(4\|\tilde{b}_i\|_\infty^2)^2}{\max(P_t^i, Q_t^i)} dt \right) \right]^{1/d}.
\] (4.33)

Here, we recall from Proposition 4.2 that the coefficients \((\tilde{b}_j)_{j \in [d-m]}\) are bounded by a constant that only depends on \((\|b_j\|_\infty)_{j \in [d]}\). Moreover, we recall from (4.10) that
\[
dP_t^i = \varepsilon^{-2}(P_0^i) \left( \varphi (\varepsilon^2 (P_t^i) P_t^i) + \tilde{b}_i(t, P_t^\circ, P_t^i) + P_t^\circ \tilde{b}_i(t, P_t^\circ, P_t^i) \right) dt + \varepsilon \varsigma^{-1}(P_0^i) \sum_{j=1}^{d-m} \sqrt{P_t^i} P_t^j dW_t^{ij},
\]
for \(i \in \mathbb{I}_{[d-m]}\) and \(t \in [t_0, T]\). Using again the fact that \((1/dt) \sum_{j=1}^{d-m} \sqrt{P_t^i} P_t^j dW_t^{ij} = P_t^i (1 - P_0^i)\), we get, by Itô’s formula,
\[
d[\ln P_t^i] = \varepsilon^{-2}(P_0^i) \left( \frac{\varphi (\varepsilon^2 (P_t^i) P_t^i) + \tilde{b}_i(t, P_t^\circ, P_t^i) + \tilde{b}_i^\circ(t, P_t^\circ, P_t^i)}{P_t^i} \right) dt
+ \varepsilon \varsigma^{-1}(P_0^i) \frac{1}{\sqrt{P_t^i}} \sum_{j=1}^{d-m} \sqrt{P_t^j} dW_t^{ij} - \frac{\varepsilon^2}{2} \varsigma^{-2}(P_0^i) \frac{1}{P_t^i} \frac{dP_t^i}{P_t^i}, \quad t \in [t_0, T].
\] (4.34)

Recalling that \(\tilde{b}_i\) takes non-negative values and denoting by \((O(1))_{0 \leq t \leq T}\) a progressively measurable process that is dominated by a constant \(C\) that may depend on \(\delta, \kappa, (\|b_j\|_\infty)_{j \in [d]}\), \((\|b_j\|_\infty)_{j \in [d]}\) and \(T\), recalling that \(\varphi \equiv \kappa\) on \([0, \delta]\) and \(\varsigma^{-2}(P_t^i) \leq 4\) for \(t \leq \omega\), and choosing \(\kappa\) as large as needed (in terms of the sole \((\|b_j\|_\infty)_{j \in [d]}\), we get, for \(t \leq \omega \wedge T\),
\[
\varsigma^{-2}(P_t^i) \left( \frac{\varphi (\varepsilon^2 (P_t^i) P_t^i) + \tilde{b}_i(t, P_t^\circ, P_t^i) + \tilde{b}_i^\circ(t, P_t^\circ, P_t^i)}{P_t^i} \right)
\geq \frac{\varsigma^{-2}(P_0^i)}{P_t^i} (\kappa - \frac{1}{2}) \frac{dP_t^i}{P_t^i} - O(1) \geq \frac{\kappa \varsigma^{-2}(P_0^i)}{2} - O(1).
\]

Hence, integrating (4.34), multiplying by some \(\eta > 0\) and then taking exponential,
\[
(P_{\omega \wedge T}^i)^{\eta} \exp \left( -\eta \varepsilon \int_{t_0}^{\omega \wedge T} \varsigma^{-1}(P_t^i) \frac{1}{\sqrt{P_t^i}} \sum_{j=1}^{d-m} \sqrt{P_t^j} dW_t^{ij} - \frac{\eta \varepsilon^2}{2} \int_{t_0}^{\omega \wedge T} \varsigma^{-2}(P_t^i) \frac{1}{P_t^i} \frac{dP_t^i}{P_t^i} dt \right)
\geq (P_0^i)^{\eta} \exp \left( \frac{(\eta \kappa - \eta^2 \varepsilon^2)}{2} \int_{t_0}^{\omega \wedge T} \varsigma^{-2}(P_t^i) \frac{1}{P_t^i} \frac{dP_t^i}{P_t^i} dt - C \right),
\]
where \(C\) is a constant as before. For any given \(\eta \in (0, 1)\), we can choose \(\kappa\) as large as needed (\(\kappa\) now depending on \(\varepsilon, \eta\) and \((\|b_j\|_\infty)_{j \in [d]}\) such that
\[
\frac{\eta \kappa - \eta^2 \varepsilon^2}{2} \geq \frac{6d}{\varepsilon^2} \left( \max_{j \in [d-m]} \|\tilde{b}_j\|_\infty^2 \right),
\] (4.35)
and then (compare with (4.33))
\[
\mathbb{E} \left[ \exp \left( \frac{6d}{\varepsilon^2} \int_{t_0}^{\omega \wedge T} \varsigma^{-2}(P_t^i) \frac{(4 \max_{j \in [d-m]} \|\tilde{b}_j\|_\infty^2)^2}{P_t^i} dt \right) \right] \leq C \frac{p_0^i}{p_0^i},
\]
where \(C\) is independent of \(p_0\) but depends on \(\delta, \varepsilon, (\|b_j\|_\infty)_{j \in [d]}\) and \((\|b_j\|_\infty)_{j \in [d]}\) and \(T\). Since \(\varsigma^{-1}\) is above 1,
\[
\mathbb{E} \left[ \exp \left( \frac{6d}{\varepsilon^2} \int_{t_0}^{\omega \wedge T} (4 \max_{j \in [d-m]} \|\tilde{b}_j\|_\infty^2)^2 dt \right) \right] \leq C \frac{p_0^i}{p_0^i},
\]
Similarly, we have the same inequality, but replacing $P_t^i$ by $Q_t^i$ in the left-hand side and $p_i$ by $q_i$ in the right-hand side. Hence, we can upper bound (4.33) by $C/\max(p_i,q_i)^{1/2}$.

**Conclusion.** We deduce from the conclusions of the second and third steps that

$$ \mathbb{P}(\omega_S < \tau \wedge q \wedge p) \leq C \frac{|p - q|^{1/2}}{\min_{i \in [d-m]}(\max(p_i,q_i))^{\eta/2}}, $$

where $C$ depends on $\delta$, $\varepsilon$, $\kappa$, $\eta$, $(\|b_i\|_{\infty})_{i=1,\ldots,d}$, $(\|b_i\|_{\infty})_{i=1,\ldots,d}$ and $T$. Since the value of $\eta$ is arbitrary (provided that it belongs to $(0,1)$), we can easily apply the above inequality with $2\eta/d$ instead of $\eta$ (observe that, whenever $\eta \in (0,1)$, $2\eta/d$ also belongs to $(0,1)$, since $d \geq 2$).

4.4. **Proof of Proposition 4.2.**

**Proof. First Step.** We introduce some useful notations. Having in mind the shape of the coefficients in equation (4.1), we let, for $i \in [d]$ and for $p \in \mathcal{S}_{d-1}$,

$$ b_i(t,p) := \varphi(p_i) + b_i(t,p) + p_i b_i^* (t,p). $$

Importantly, we recall from (3.20) that, for any $(\cdot,\cdot)$ in the right-hand side. Hence, we can upper bound (4.33) by $C/\max(p_i,q_i)^{1/2}$, we deduce from the conclusions of the second and third steps that

$$ \mathbb{P}(\omega_S < \tau \wedge q \wedge p) \leq C \frac{|p - q|^{1/2}}{\min_{i \in [d-m]}(\max(p_i,q_i))^{\eta/2}}, $$

where $C$ depends on $\delta$, $\varepsilon$, $\kappa$, $\eta$, $(\|b_i\|_{\infty})_{i=1,\ldots,d}$, $(\|b_i\|_{\infty})_{i=1,\ldots,d}$ and $T$. Since the value of $\eta$ is arbitrary (provided that it belongs to $(0,1)$), we can easily apply the above inequality with $2\eta/d$ instead of $\eta$ (observe that, whenever $\eta \in (0,1)$, $2\eta/d$ also belongs to $(0,1)$, since $d \geq 2$).

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where $C$ depends on $\delta$, $\varepsilon$, $\kappa$, $\eta$, $(\|b_i\|_{\infty})_{i=1,\ldots,d}$, $(\|b_i\|_{\infty})_{i=1,\ldots,d}$ and $T$. Since the value of $\eta$ is arbitrary (provided that it belongs to $(0,1)$), we can easily apply the above inequality with $2\eta/d$ instead of $\eta$ (observe that, whenever $\eta \in (0,1)$, $2\eta/d$ also belongs to $(0,1)$, since $d \geq 2$).

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$$ b_i(t,p) := \varphi(p_i) + b_i(t,p) + p_i b_i^* (t,p). $$

Importantly, we recall from (3.20) that, for any $(t,p) \in [0,T] \times \mathcal{S}_{d-1}$, $\sum_{i \in [d]} b_i(t,p) = 0$. In fact, we can easily extend $b_i$, for each $i \in [d]$, to the entire $[0,T] \times \mathbb{R}^d$ by composing $b_i$ with the orthogonal projection from $\mathbb{R}^d$ into $\mathcal{S}_{d-1}$. This allows us to define the drift $B^\circ$ entering the dynamics of the second equation in (4.6). For a given coordinate $i \in [m]$, we indeed let

$$ B_i^\circ (t,r^o,r) := b_i \left( t, (r^o, \varsigma^2 (r^o) r) \right), \quad t \in [0,T], $$

for $r^o \in \mathbb{R}^m$ and $r \in \mathbb{R}^{d-m}$. Notice that the definition is especially interesting for our purpose whenever $r^o \in \mathcal{S}_m$ and $r \in \mathcal{S}_{d-m-1}$, but it is well defined in any case (with the obvious convention that $\varsigma^2 (r^o) = 1 - (r^o_1 + \cdots + r^o_m)$ even if it is negative). Similarly, for $i \in [d-m]$, we let

$$ B_i (t,r^o,r) := b_{m+i} \left( t, (r^o, \varsigma^2 (r^o) r) \right), \quad t \in [0,T]. $$

For a new collection of antisymmetric Brownian motions $\mathbf{W}^o = (\mathbf{W}^o_t)_{t \in [d]; \exists j \in [d]}$ of dimension $d(d-1)/2$ (with the convention that $\mathbf{W}^o_{\tau^0,i;i} \equiv 0$ for $i \in [d]$), we consider the system

$$ \begin{align*}
    dP_t^i &= \varsigma^2 (P_t^o) \left( B_i (t,P_t^o,P_t) - P_t^i \sum_{j \in [d-m]} B_j (t,P_t^o,P_t) \right) dt \\
    &\quad + \varepsilon \varsigma^{-1} (P_t^o) \sum_{j \in [d-m]} \sqrt{P_t^j P_t^j} d\mathbf{W}_t^{i+m,j+m}, \quad i \in [d-m], \\
    d(P_t^o)^i &= B_i^\circ (t,P_t^o,P_t) dt + \varepsilon \sum_{j \in [m]} \sqrt{(P_t^o)^j (P_t^o)^j} d\mathbf{W}_t^{i,j} \\
    &\quad + \varepsilon (P_t^o) \sum_{j \in [d-m]} \sqrt{(P_t^o)^j (P_t^o)^j} d\mathbf{W}_t^{i,m+j}, \quad i \in [m],
\end{align*} $$

(4.36)

for $t \in [t_0,T]$. The unique solvability of (4.36) is addressed in the next two steps.

**Second Step.** Observing that $b$ is Lipschitz continuous, we deduce that the coefficients of (4.36) are Lipschitz continuous in the entries $(P^o,P)$ as long the coordinates of the latter remain bounded and away from zero and as long as the sum of the coordinates of $P^o$ remains away below 1, we deduce that, for any small $a > 0$, the system (4.36) is uniquely solvable up to the first $\tau^o$ when one of the coordinates of $P^o_{\tau^o} \lor P_{\tau^o}$ becomes lower than $a$ or when the sum of the coordinates of $P^o_{\tau^o} \lor P_{\tau^o}$ becomes greater than $1 - a$. Letting $a$ tend to 0, we deduce that (4.36) is uniquely solvable up to time $\tau = \lim_{a \searrow 0} \tau^o \wedge T$. 

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Hence, unique solvability follows if we can prove that

\[(\inf_{i \in [m]} \inf_{t_0 \leq t < \tau} P_{t_0}^{i,i} > 0, \sup_{t_0 \leq t < \tau} \sum_{i \in [m]} P_{t_0}^{i,i} < 1, \forall t \in [t_0, \tau), \sum_{i \in [d-m]} P_{t}^{i} = 1) = 1, \] 

(4.37)
since the latter implies that \( \tau = T \).

In order to check (4.37), we first observe that, for \( t \in [t_0, \tau) \), \( d(\sum_{i \in [d-m]} P_{t}^{i}) = 0 \). Hence,

\[ \sum_{i \in [d-m]} P_{t}^{i} = 1, \quad t \in [t_0, \tau]. \]

(Notice that the time interval is closed: Observing that the coefficients in (4.36) are bounded, we may indeed easily extend the solution in hand at time \( \tau \) itself.) This prompts us to let

\[ \tilde{X}_{t}^{i} = (P_{t}^{i})^{i}, \quad i \in [m] ; \quad \tilde{X}_{t}^{i} = \varsigma^{2}(P_{t}^{i}) P_{t}^{i-m}, \quad i = m + 1, \ldots, d, \]

for \( t \in [t_0, \tau] \). Observe in particular that \( \sum_{i \in [d]} \tilde{X}_{t}^{i} = 1 \), for all \( t \in [t_0, \tau] \). If we prove that \( (\tilde{X}_{t}^{1}, \ldots, \tilde{X}_{t}^{d})_{t_0 \leq t \leq \tau} \) satisfies the SDE (4.1) but for a new choice of the noise, then we are done: Not only we then deduce from Proposition 4.1 (or, equivalently, Proposition 2.1) that (4.37) indeed holds true, but we also obtain the required identity in law, see (4.5).

**Third Step.** In order to prove that \( (\tilde{X}_{t}^{1}, \ldots, \tilde{X}_{t}^{d})_{t_0 \leq t \leq \tau} \) satisfies (4.1) (for a new choice of noise), we proceed as follows. First, we notice that, for \( i \in [m] \),

\[ d\tilde{X}_{t}^{i} = b_{i}(t, \tilde{X}_{t})dt + \varepsilon \sum_{j \in [d]} \sqrt{\tilde{X}_{t}^{i}} \sqrt{\tilde{X}_{t}^{j}} dW_{t}^{i,j}, \quad t \in [t_0, \tau]. \]

And, for \( i = m + 1, \ldots, d \),

\[ d\tilde{X}_{t}^{i} = \left( B_{i-m}(t, P_{t}^{0}, P_{t}) - P_{t}^{i-m} \sum_{j \in [d-m]} B_{j}(t, P_{t}^{0}, P_{t}) \right) dt + \varepsilon(\varsigma P_{t}^{i}) \sum_{j \in [d-m]} \sqrt{P_{t}^{i-m} P_{t}^{j} dW_{t}^{i,j+m}} - P_{t}^{i-m} \sum_{j \in [m]} B_{j}^{0}(t, P_{t}^{0}, P_{t}) dt - \sum_{j \in [d-m]} d(\varsigma P_{t}^{i}) \sum_{j \in [d-m]} \varsigma(\varsigma P_{t}^{i}) \sum_{j \in [m]} \sum_{l \in [d-m]} (P_{t}^{0})^{l} dW_{t}^{i,j+l} \] 

\[ (4.38) \]

Obviously, the bracket on the second line is zero since the underlying noises are independent. Hence, using the fact that \( \sum_{i \in [d]} b_{i}(t, p) = 0 \) for any \( (t, p) \in [0, T] \times \mathbb{R}^{d} \), the drift reads

\[ B_{i-m}(t, P_{t}^{0}, P_{t}) - P_{t}^{i-m} \sum_{j \in [d-m]} B_{j}(t, P_{t}^{0}, P_{t}) - P_{t}^{i-m} \sum_{j \in [m]} B_{j}^{0}(t, P_{t}^{0}, P_{t}) = b_{i}(t, \tilde{X}_{t}). \]

Therefore, in order to prove that \( (\tilde{X}_{t})_{t_0 \leq t \leq \tau} \) satisfies (4.1) (for a new choice of noise), it suffices to identify the martingale structure in (4.38). To do so, we rewrite the three martingale increments
in the above expansion for $i = m + 1, \ldots, d$ in the form

$$
\varepsilon \kappa (P_t^o) \sum_{j \in [d-m]} \sqrt{P_t^{i-m} P_t^j} dW_t^{j+m} - \varepsilon P_t^{i-m} \sum_{j,l \in [m]} \sqrt{(P_t^o)^j (P_t^o)^l} dW_t^{j,l} - \varepsilon P_t^{i-m} \sum_{j \in [m]} \sum_{l \in [d-m]} \sqrt{(P_t^o)^j (P_t^o)^l} dW_t^{j,m+l}
$$

$$
= \varepsilon \kappa^{-1} (P_t^o) \sum_{j=m+1}^d \sqrt{X_t^i X_t^j} dW_t^{i,j} - \varepsilon \kappa^{-2} (P_t^o) \sum_{j \in [m]} \sum_{l \in [d]} \sqrt{X_t^j X_t^l} dW_t^{i,j}. \tag{\ref{eq:expansion}}
$$

Hence, in order to complete the analysis, it remains to compute the various brackets $d \langle \bar{X}_t^i, \bar{X}_t^j \rangle$ for $i, j \in [d]$. Obviously, whenever $i, j \in [m]$, \[ d \langle \bar{X}_t^i, \bar{X}_t^j \rangle = \varepsilon^2 \kappa^{-2} (P_t^o) \sqrt{X_t^i X_t^j} \sum_{l=0}^d \sqrt{X_t^i X_t^j} (\delta_{i,l} \delta_{i,l'} - \delta_{i,l} \delta_{j,l'}) dt
\]

$$
+ \varepsilon^2 \kappa^{-4} (P_t^o) \sum_{l=0}^d \sum_{k,l' \in [m]} \sqrt{X_t^k X_t^l} (\delta_{k,l'} \delta_{l,l'} - \delta_{k,l'} \delta_{l,l'}) dt
$$

$$
= \varepsilon^2 \kappa^{-2} (P_t^o) \delta_{i,j} \sum_{l=0}^d \sqrt{X_t^l X_t^j} dt
$$

$$
+ \varepsilon^2 \kappa^{-4} (P_t^o) \sum_{l=0}^d \sum_{k,l' \in [m]} \sqrt{X_t^k X_t^l} (\delta_{k,l'} \delta_{l,l'} - \delta_{k,l'} \delta_{l,l'}) dt
$$

$$
= \varepsilon^2 \kappa^{-2} (P_t^o) \delta_{i,j} \sum_{l=0}^d \sqrt{X_t^l X_t^j} dt. \tag{\ref{eq:brackets}}
$$

Now, the key point is to observe that $\sum_{i=0}^d X_t^i = 1 - \sum_{i=1}^m X_t^i = \kappa^2 (P_t^o)$. Therefore, \[ d \langle \bar{X}_t^i, \bar{X}_t^j \rangle = \left( \varepsilon^2 \delta_{i,j} \sum_{l=0}^d \sqrt{X_t^i X_t^j} \sum_{k,l' \in [m]} \sqrt{X_t^k X_t^l} (\delta_{k,l'} \delta_{l,l'} - \delta_{k,l'} \delta_{l,l'}) dt \right) dt
\]

$$
= \varepsilon^2 \delta_{i,j} \sum_{l=0}^d \sqrt{X_t^i X_t^j} dt. \tag{\ref{eq:finite}}
$$

Now, for $i = 1, \ldots, m$ and for $j = m + 1, \ldots, d$, \[ d \langle \bar{X}_t^i, \bar{X}_t^j \rangle = -\varepsilon^2 \kappa^{-2} (P_t^o) \sum_{l'=1}^d \sum_{k=1}^m \sqrt{X_t^i X_t^j} \sqrt{X_t^k X_t^{l'}} (\delta_{i,k} \delta_{l,l'} - \delta_{i,k} \delta_{l,l'}) dt
\]

$$
= \left( -\varepsilon^2 \kappa^{-2} (P_t^o) \sum_{l'=1}^d \sqrt{X_t^i X_t^j} \sum_{k=1}^m \sqrt{X_t^k X_t^{l'}} dt \right) dt = -\varepsilon^2 \bar{X}_t^i \bar{X}_t^j dt,
$$

which completes the proof. \hfill \blacksquare

**APPENDIX A. MFGs with a finite state space in discrete time**

The purpose of this section is to introduce the finite player version of the game in discrete time and discuss convergence.
A.1. Discrete time model. As the structure of the common noise that we use is taken from earlier discrete time models from population genetics, we feel that it is fair to define the finite player game in discrete time as well.

A.1.1. A system of controlled Markov chains. We first recall the form of the $N$-player game when there is no common noise, see for instance [35]. The time horizon is denoted by $M$ and the time indices are labelled by $m = 0, 1, \cdots, M$. The state of the system at time $m$ is described by a tuple $(X^1_m, \ldots, X^N_m)$, $N$ denoting the total number of players in the finite player game and $X^\ell_m$ the state of player $\ell$ at time $m$. The tuple $(X^1_m, \ldots, X^N_m)$ takes values in $[d]^N$.

Player $\ell$ is assumed to choose a strategy $\alpha^{\ell,N} : \{0, \cdots, M\} \times [d]^N \rightarrow S_{d-1}$ (so differently from the core of the paper, $\alpha^{\ell,N}$ is here a probability measure and not a transition rate). Given the strategies $\alpha^{1,N}, \ldots, \alpha^{N,N}$, the process $(X^1, \ldots, X^N) := (X^1_m, \ldots, X^N_m)_{m=0,\cdots,M}$ may be defined as a time-inhomogeneous Markov chain with transition matrix:

$$
P \left( (X^1_{m+1}, \ldots, X^N_{m+1}) = j \mid (X^1_m, \ldots, X^N_m) = i \right) = \prod_{l=1}^N [\bar{\alpha}^{l,N}(m, \bar{i})](j_l), \quad (A.1)$$

where both $\bar{i} = (i_1, \cdots, i_N)$ and $j = (j_1, \cdots, j_N)$ are in $[d]^N$. In words, given the state of $(X^1, \ldots, X^N)$ at time $m$, the above definition says that players jump independently from one state to another according to the transitions prescribed by the strategies they have chosen.

Say once again that each $\bar{\alpha}^{l,N}(m, \bar{i})(\cdot)$ is required to be in $S_{d-1}$, that is $\bar{\alpha}^{l,N}(m, \bar{i})(j) \geq 0$ for all $j \in [d]$ and $\sum_{j \in [d]} \bar{\alpha}^{l,N}(m, \bar{i})(j) = 1$.

Importantly, we will assume below that, whenever $j_l \neq i_l$, $\bar{\alpha}^{l,N}(m, \bar{i})(j_l)$ is (at most) of order $1/N$, see Definition A.1 below. The latter implies in particular that the probability that a particle jumps at time $m$ is also of order $1/N$ and hence that the total number of jumps in the system at the same time $m$ is stochastically dominated by a Poisson random variable with an intensity of order 1.

Observe that, in order to complete the description of the game, we should assign a cost functional to each player. Although the reader may easily guess that the form of such a cost functional should be directly inspired from (2.9), we feel better to introduce it later on and to focus, for the time being, on variants of (A.1) with a common noise.

A.1.2. Particle system with a pure common noise. As explained in [6], we may introduce a common noise in a system of $N$ players by allowing the system to have a massive number of jumps at the same time. Obviously, the latter feature is in contrast to (A.1).

So, we forget for a while (A.1) and we introduce a simple particle system without control in which many players may indeed jump simultaneously. A typical instance for such a particle system is the Wright–Fisher model used in population genetics: at time $m+1$, the states of the players (also called children in genetic models) are sampled independently from the empirical distribution of the system at $m$ (or, equivalently, from the empirical distribution of the parents). Put it differently, the Wright–Fisher version of (A.1) is

$$
P \left( (X^1_{m+1}, \ldots, X^N_{m+1}) = j \mid (X^1_m, \ldots, X^N_m) = i \right) = \prod_{l=1}^N \bar{\mu}^N_m(j_l), \quad (A.2)$$

where $\bar{\mu}^N_m$ is the empirical distribution of $(X^1_m, \ldots, X^N_m)$, namely $\bar{\mu}^N_m = \mu(X^N_m)$, $X^N_m$ being an abbreviated notation\footnote{Pay attention that we here use boldfaces for $N$-tuples. We thus use boldfaces for both processes and for $N$-tuples. We guess that there is no possible confusion for the reader.} for the $N$-tuple $(X^1_m, \ldots, X^N_m)$ and, for any $\bar{i} = (i_1, \cdots, i_N) \in [d]^N$,
\(\mu(i)\), given by
\[
\mu(i) = \frac{1}{N} \sum_{k=1}^{N} \delta_{i_k} = \frac{1}{N} \sum_{e \in [d]} \#\{k \in \{1, \cdots, N\} : i_k = e\} \delta_e,
\]
is the uniform distribution on the set \(\{i_1, \cdots, i_N\}\) (pay attention that two \(i_k\)'s may be the same in which case the masses are obviously added).

To make (A.2) clear, we may have a look at the sole expression of \(\bar{\mu}_{m+1}^N\). Indeed, we may represent \(\bar{\mu}_{m+1}^N\) in the form
\[
\bar{\mu}_{m+1}^N(i) = \frac{1}{N} S_m(i) = \frac{\bar{\mu}_{m}^N(i)}{N\mu_m^{N}(i)}, \tag{A.3}
\]
where, conditional on the state of the population at time \(m\), \(S_m = (S_m(1), \cdots, S_m(d))\) is a random variable with multinomial distribution of parameters \(N\) and \((\bar{\mu}_m^N(1), \cdots, \bar{\mu}_m^N(d))\). The above formulation says that, in (A.2), the empirical distribution of the system at point \(i\) is multiplied by \(S_{m+1}(i)/(N\mu_m^N(i))\) between times \(m\) and \(m+1\).

A.1.3. A controlled system with common noise. Below, we address a mixture of (A.1) and (A.2), meaning that we deal with players that may control their own dynamics (as it is the case in (A.1)) but that are also subjected to a common noise. Whilst the common noise we choose below is mainly inspired by (A.3), it is worth mentioning that, for reasons that will be clear in the text, we work with a variant of it. Basically, this variant has the same macroscopic behavior, in the sense that the empirical distribution satisfies an equation close to (A.3), but it allows for different microscopic behaviors, meaning that the given player \(l\) can evolve really differently. In short, the issue with the Wright–Fisher model is that, at each time, players are resampled from the empirical measure of the system. Hence, if, at any time, we toss a coin with a fixed parameter (as we do below) to decide whether we follow (A.1) or (A.2), then resampling occurs quite often (meaning at a frequency of order 1) and, consequently, the empirical measure becomes strongly attractive; in turn, the latter precludes any interesting deviating phenomenon. This is in contrast to the model we introduce below: Therein, players may really deviate (in law) from the empirical measure of the whole population, which is more consistent with the basics of the MFG theory.

The main idea of our model is that we now attach a mass to each particle. While the usual MFG theory is constructed on the idea that players have all the same masses (namely 1), we here allow the masses to be non-uniform\(^{15}\). The whole point is then to imitate (A.3) in order to define the evolution of those masses. To wit, we assume that the state of a player \(l \in \{1, \cdots, N\}\) comprises of an element of \([d]\) as well as an element of \(\mathbb{Q}_+\), the set of the nonnegative rational numbers. We thus encode the state of player \(l\) at time \(m\) in the form \(Z_m^l = (X_m^l, Y_m^l) \in [d] \times \mathbb{Q}_+\). While the first of the two coordinates describes the location of the player \(l\) at time \(m\), the second one must be understood as the mass of \(l\). We also fix a parameter \(\varepsilon \in (0, 1)\), which stands for the intensity of the common noise.

We first provide an informal description of the model. At time \(m\), conditional on the event \(\{(X_m^1, \cdots, X_m^N) = i, (Y_m^1, \cdots, Y_m^N) = g\}\), for \(i \in [d]^N\) and \(g \in \mathbb{Q}_+^N\), each player \(l \in \{1, \cdots, N\}\) chooses a probability weight \([\bar{\alpha}^l\cdot N(m, i, g)](\cdot) \in S_{d-1}\); we then flip a coin \(T_{m+1}\) of parameter \(\varepsilon\) and we sample independently a random variable \(S_{m+1}\) from multinomial distribution on \([d]\).

\(^{15}\)Observe that mean field models with non-uniform weights (but without any game inside) were already addressed in earlier works, see for instance [13, 44].
with parameters $N$ and $(\mu(1), \cdots, \mu(d))$, where $\mu$ is here understood as
\[
\mu := \left(\sum_{\ell=1}^{N} q_\ell\right)^{-1} \sum_{\ell=1}^{N} q_\ell \delta_{\ell i}.
\]

(Importantly, $\mu$ is a probability measure on $[d]$.) We then have the following two scenarios:

1. If the coin is unsuccessful, i.e. $T_{m+1} = 0$, then
   (a) the locations of the players change independently, conditional on the states at time $m$ and the realization of the coin, according to the aforementioned transitions $([\tilde{\alpha}^{l, N}(m, i, \varrho)](\cdot))_{l=1, \ldots, N}$; this is consistent with (A.1);
   (b) the masses of the players remain constant.

2. If the coin is successful, i.e. $T_{m+1} = 1$, then
   (a) the locations of the players remain constant;
   (b) moreover, conditional on the states at time $m$, the realization of the coin and the realization of $S_m$, the masses are multiplied, between times $m$ and $m+1$, by the factors $(S_{m+1}(i_l)/\mu(i_l) \sum_{l'=1}^{N} \varrho_{l'})_{l=1, \ldots, N}$; this is consistent with (A.3).

To make it clear, we have, for $i = (i_1, \cdots, i_N)$ and $j = (j_1, \cdots, j_N)$ in $[d]^N$, for $\varrho = (\varrho_1, \cdots, \varrho_N) \in \mathbb{Q}_+^N$, and for $k \in \{0, \cdots, N\}^d$, with $k_1 + \cdots + k_d = N$,
\[
\mathbb{P}\left(Z_{m+1} = (j, \varrho) \mid Z_m = (i, \varrho), T_{m+1} = 0\right) = \prod_{l=1}^{N} [\tilde{\alpha}^{l, N}(m, i, \varrho)](j_l),
\]
\[
\mathbb{P}\left(Z_{m+1} = (i, \sigma[\varrho, k]) \mid Z_m = (i, \varrho), T_{m+1} = 1, S_{m+1} = k\right) = 1,
\]
where we let $Z_m = (X_m, Y_m)$, with $X_m = (X^1_m, \cdots, X^N_m)$ and $Y_m = (Y^1_m, \cdots, Y^N_m)$, and
\[
\sigma[\varrho, k]_l := \frac{q_l k_{i_l}}{\mu(i_l) \sum_{l'=1}^{N} \varrho_{l'}} = \frac{q_l k_{i_l}}{\sum_{l': i_{l'} = i_l} \varrho_{l'}}.
\]
for $l \in \{1, \cdots, N\}$ and $\mu$ as in (A.4). Importantly, it must be observed that
\[
\sum_{l=1}^{N} \sigma[\varrho, k]_l = \sum_{l=1}^{N} \frac{q_l k_{i_l}}{\sum_{l': i_{l'} = i_l} \varrho_{l'}} = \sum_{i \in [d]} \sum_{l: i_l = i_l} \sum_{l': i_{l'} = i_l} \frac{q_l k_{i_l}}{\varrho_{l'}} = \sum_{i \in [d]} k_i = N,
\]
which says that, whatever the two cases $T_{m+1} = 0$ or $T_{m+1} = 1$, the global mass remains equal to $N$ if it is initialized from $N$ (that is $\sum_{l=1}^{N} Y^l_m = N$ for any $m \in \{0, \cdots, M\}$ if $\sum_{l=1}^{N} Y^l_0 = N$). In the sequel, we always choose $Y^l_0 = 1$, for any $l \in \{1, \cdots, N\}$. In particular, (A.4) becomes
\[
\mu = N^{-1} \sum_{l=1}^{N} q_l \delta_{i_l}.
\]
Equivalently, letting
\[
\bar{\mu}^N = \frac{1}{N} \sum_{l=1}^{N} \chi^l_m \delta_{X^l_m}, \quad m \in \{0, \cdots, M\},
\]
we get that, for any $m \in \{0, \cdots, M\}$, $\bar{\mu}^N$ is a random probability measure on $[d]$, which stands, from the modeling point of view, for the collective state of the population.
A.1.4. Mean field strategy. Take as a typical instance
\[
\left[\alpha^{l,N}(m, i, \varrho)\right](j) = \left[\alpha^{l,N}(m, i_l, \mu)\right](j), \quad \text{if } j \neq i_l,
\]  
for \(l \in \{1, \ldots, N\}, i \in [d]^N, \varrho \in \mathbb{Q}_+^N\) and with \(\mu\) given by (A.5). Notice that, at some point, we shall take \(\alpha^{l,N}\) independent of \(l\). In the latter case, the strategy is said to be mean field: The transition probabilities only depend upon the private state of the player (here \(i_l\) since the label of the player is \(l\)) and upon the global state of the population (here \(\mu\), which is in \(S_{d-1}\)); in particular, mean field strategies do not depend on the own labels of the particles.

In order to define the model rigorously in this framework, we proceed as follows. First, for any \(l \in \{1, \ldots, N\}, i \in [d]^N\) and \(\mu \in S_{d-1}\), we call \(q^{l,N}(m, i, \mu) : [0,1] \ni u \mapsto q^{l,N}(m, i, \mu)(u)\) a quantile function of \((\alpha^{l,N}(m, i, \mu)(j))_{j \in [d]}\), meaning that, for a random variable \(U\) with uniform law on \([0,1]\) and for \(j \in [d]\),
\[
P(q^{l,N}(m, i, \mu)(U) = j) = \alpha^{l,N}(m, i, \mu)(j).
\]
Accordingly, we introduce a collection \(((U^l_m))_{l \in \{1, \ldots, N\}}\) of independent random variables with uniform law on \([0,1]\). Then, we choose the sequence \((T_m)_{m \geq 0}\) as a sequence of independent Bernoulli random variables with parameter \(\varepsilon\). Lastly, we represent the multinomial random variables in a canonical way. Indeed, we may be given a collection of random variables \(((S_{m}(\mu))_{\mu \in S_{d-1}})_{m \geq 0}\) such that the \(\sigma\)-algebras \((\sigma(S_{m}(\mu), \mu \in S_{d-1}))_{m \geq 0}\) are independent and, for each \(m \geq 0\) and \(\mu \in S_{d-1}\), \(S_{m}(\mu)\) has a multinomial distribution on \([d]^N\) with parameters \(N\) and \((\mu(i))_{i \in [d]}\).

For instance, we may choose
\[
S_{m}(\mu)(i) = \sum_{k=1}^{N} \mathbf{1}_{\mu(1) + \cdots + \mu(i-1) \leq V^m_k < \mu(1) + \cdots + \mu(i)}, \quad i \in [d],
\]
where \(((V^k_m)_{k \in \{1, \ldots, N\}})_{m \geq 0}\) is a new collection of independent random variables with uniform law on \([0,1]\). We then assume that the three collections \(((U^l_m))_{l \in \{1, \ldots, N\}})_{m \geq 0}, \,(T_m)_{m \geq 0}\) and \(((V^k_m))_{k \in \{1, \ldots, N\}})_{m \geq 0}\) are independent.

With these notations in hand, we then have the following representation of the dynamics of the model\footnote{Notice that, differently from \([6]\), the map that sends the state of the system at time \(m\) onto the new state at time \(m+1\) is intrinsically nonlinear.}
\[
Y^l_{m+1}(X^l_{m+1}) = \sum_{j \in [d]} 1_{\{T_{m+1}=0\}} Y^l_{m} 1_{\{X^l_{m}=j\}} 1_{\{q^{l,N}(m,j,\bar{\mu}^N_m) (U^l_{m+1}) = i\}} + 1_{\{T_{m+1}=1\}} 1_{\{X^l_{m}=i\}} Y^l_{m} \frac{S_{m+1}(\bar{\mu}^N_m)(i)}{N \bar{\mu}^N_m(i)},
\]
for \(l \in \{1, \ldots, N\}, m \in \{0, \ldots, M-1\}\) and \(i \in [d]\). (In the above, the ratio \(S_{m+1}(\bar{\mu}^N_m)(i)/\bar{\mu}^N_m(i)\) is treated as 0 if the denominator \(\bar{\mu}^N_m(i)\) is also 0.) This writing makes clear the fact that there are indeed two types of noise in the model:

1. As the agent \(l\) only sees the collection \(((U^r_m))_{r \in \{1, \ldots, N\}})_{m \geq 0}\) through the sub-collection \((U^l_m)_{m \geq 0}\), the former should be regarded as an idiosyncratic noise;
2. This is in contrast to the two families \((T_m)_{m \geq 0}\) and \(((V^k_m))_{k \in \{1, \ldots, N\}})_{m \geq 0}\), which are common to all the agents and hence should be regarded as common noises.

This prompts to use two probability spaces instead of one in order to identify the types of the various noises. Hence we assume below that: \((\Omega, \mathcal{A}, \mathbb{P})\) is given as the tensorial product of two probability spaces \((\Omega^0, \mathcal{A}^0, \mathbb{P}^0)\) and \((\Omega^1, \mathcal{A}^1, \mathbb{P}^1)\):
A.1.5. Marginal empirical distribution versus conditional marginal masses. Recalling (A.6) and taking the mean over \( l \) in (A.8), we get

\[
\tilde{\mu}_{m+1}^N(i) = \frac{1}{N} \sum_{j \in \llbracket d \rrbracket} 1_{\{T_{m+1} = 0\}} \sum_{l=1}^N Y^l_m 1_{\{X^l_m = j\}} 1_{\{Y^l_N(m,j,\tilde{\mu}_m^N(U^l_{m+1}) = i\}} \\
+ 1_{\{T_{m+1} = 1\}} \frac{S_{m+1}(\tilde{\mu}_m^N)(i)}{N},
\]

for \( m \in \{0, \ldots, M - 1\} \) and \( i \in \llbracket d \rrbracket \). Obviously, (A.9) is the equation for the state of the population as defined by (A.6). At this stage, it is not closed (meaning that the first term in the right-hand side depends on other unknowns than \( \tilde{\mu}_m^N \) itself). Fortunately, whenever \( Y^l_m \) is independent of \( l \) (which is the case when working with a mean field strategy, see (A.1.4) and the first term in the right-hand side is replaced by the conditional expectation given the past up until time \( m \), it becomes closed. The latter is at the roots of the diffusion approximation result discussed below.

Another quantity of interest is what we called in (2.8) the conditional expected mass of player \( l \) at time \( m \). We define it as the vector

\[
\left( Q^{l,N}_{m+1} := \mathbb{E}^0[Y^l_m(\mathbb{1}_{X^l_m = i})] \right)_{i \in \llbracket d \rrbracket},
\]

Our terminology is somewhat an abuse of notation since the above vector does not define a probability measure, but it is well-understood: In the standard case where the mass \( Y^l_m \) is deterministic and is equal to 1, the above matches

\[
\left( \mathbb{P}^0(\mathbb{1}_{X^l_m = i}) \right)_{i \in \llbracket d \rrbracket}.
\]

(Observe that the latter is in fact independent of \( l \) since we assumed the initial conditions to be independent and identically distributed.) Taking \( \mathbb{E}^0 \) in (A.8) (and using the independence of \( U^l_{m+1} \) with the past before \( m \)), we get

\[
Q^{l,N}_{m+1}(i) = \sum_{j \in \llbracket d \rrbracket} 1_{\{T_{m+1} = 0\}} \mathbb{E}^0\left[Y^l_m 1_{\{X^l_m = j\}} \alpha^{l,N}(m,j,\tilde{\mu}_m^N(i))\right] \\
+ 1_{\{T_{m+1} = 1\}} \mathbb{E}^0\left[1_{\{X^l_m = i\}} Y^l_m \frac{S_{m+1}(\tilde{\mu}_m^N)(i)}{N\tilde{\mu}_m^N(i)}\right].
\]

Obviously, (A.11) is no more closed than (A.9) and, in fact, it even looks more complicated. However, the good point is that, as usual with mean field models (even those featuring a common noise, see for instance [10 Chapter 2]), we may expect for some form of propagation of chaos as \( N \) tends to \( +\infty \). To make it clear, we may expect \( \tilde{\mu}_N^N \) to become independent of the idiosyncratic noises under the limit \( N \to +\infty \) (at least under suitable assumptions and in particular for quantiles in (A.9) that are mean field, namely that are independent of the indices of the players). For sure, this is a guess only and this asks for a rigorous proof, the analysis of which is deferred.
to future works, but, in the end, we claim that it looks reasonable to approximate (A.11) by the following simpler equation, at least in some suitable cases:

$$Q_{m+1}^{l,i}(i) = \sum_{j \in [d]} 1_{\{T_{m+1} = 0\}} Q_m^{l,i}(j) [\alpha^{l,i}(m, j, \bar{\mu}_m)](i) + 1_{\{T_{m+1} = 1\}} Q_m^{l,i}(i) \frac{S_{m+1}(\bar{\mu}_N)(i)}{N \bar{\mu}_m(i)}, \quad (A.12)$$

for \( l \in \{1, \ldots, N\}, i \in [d] \) and \( m \in \{0, \ldots, M - 1\} \).

### A.2. Diffusion approximation

The passage from the discrete to the continuous models mostly relies on diffusion approximation techniques that are quite standard in the theory of stochastic processes and that permit, among others, to pass from the discrete Wright–Fisher model introduced in (A.4) to the time continuous diffusive version given in Section 2.

#### A.2.1. Associating a game with the particle system

In fact, it would be more relevant, in our framework, to associate first a game with the discrete model and to pass to the limit, in a suitable sense (think of the vast literature that has been published on this matter in the MFG theory, see for instance the notes and complements in [10, Chapter 6]), toward the MFG defined in Definition 2.3. Although this would indeed make perfect sense, this would demand however a longer discussion, which would go beyond the scope of this appendix. Hence, our choice is quite clear: Here, we just explain intuitively how the discrete model introduced right above can be rescaled into a time continuous diffusive model whenever the number of particles \( N \) tends to \( \infty \). This will suffice to motivate the form of mean field games addressed in the paper. And we will leave the rigorous asymptotic analysis of the finite game for future work.

For the reader who knows very few of mean field games, say that the derivation of a mean field game from the discrete model defined above consists in three steps. The first one is to associate with any particle (or player) \( l \in \{1, \ldots, N\} \) a cost functional \( J_l^N \) depending explicitly on the private states \( X_m^l \) of the player, for \( m \) in the time window \( \{0, \ldots, M\} \), on the collective states \( \bar{\mu}_m^N \) of the population and on the strategy \( \alpha^{l,N}(m, \cdot, \cdot) \) chosen by \( l \), for the same values of \( m \). A typical instance (obviously inspired from (2.3)) is

$$J_l^N = \mathbb{E} \left[ g(X_M^l, \bar{\mu}_M^N) + \sum_{m=0}^{M-1} f(m, X_m^l, \bar{\mu}_m^N) + \frac{1}{2} \sum_{m=0}^{M-1} \sum_{j \neq X_m^l} |[\alpha^{l,N}(m, X_m^l, \bar{\mu}_m^N)](j)|^2 \right],$$

$$(A.13)$$

with \( \alpha^{l,N} \) as in (A.7). For \( m \geq 0, i \in [d] \) and \( \mu \in \mathcal{S}_{d-1} \), \( f(m, i, \mu) \) and \( g(i, \mu) \) are interaction costs (or potential energies) between \( i \) and \( \mu \) (in fact, \( g \) here coincides with the terminal cost in (2.3) and, right below, we connect \( f \) with the running cost in (2.3)). As for the last term, it may be interpreted as a form of kinetic energy: The higher the transition probabilities from the current state to a new state, the higher the cost. It is worth pointing out that, implicitly, \( J_l^N \) depends upon all the strategies \( \alpha^{l',N} \), with \( l' \in \{1, \ldots, N\} \), even though the latter ones do not show up explicitly in the formula. So, we may write \( J_l^N \) in the form \( J_l^N(\alpha^{1,N}, \ldots, \alpha^{N,N}) \).

In fact, it is more appropriate, for our purpose, to consider a variant of (A.13) that is closer to (2.9). Indeed, we insert the mass process \((Y_m^l)_{m=0,\ldots,M}\) into the cost functionals and, for
reasons that will become clear in the next paragraph, we renormalize the two sums, letting

\[ \mathcal{J}^N_i = \sum_{i \in [d]} \mathbb{E} \left[ Y_M 1_{\{M = i\}} g(i, \bar{\mu}^N_M) \right] + \frac{1}{N} \sum_{m=0}^{M-1} Y_M 1_{\{M = i\}} f\left( \frac{m}{N}, i, \bar{\mu}_m^N \right) + \frac{N}{2} \sum_{m=0}^{M-1} \sum_{j \neq i} Y_M 1_{\{M = i\}} \left| \left[ \alpha^{N}(m, i, \bar{\mu}_m^N) \right] (j) \right|^2, \]

(A.14)

\( f \) being now defined on \([0, \infty) \times E \times \mathcal{S}_{d-1}\) (this \( f \) now coincides with the running cost in \((2.3)\) and \((2.9)\)).

Now that we have a collection of cost functionals, as given by \((A.14)\), the combination of \((A.8)\) together with \((A.14)\) forms a game. Hence, the second step towards an MFG is to look for Nash equilibria within this game. In short, a Nash equilibrium is a collection of strategies (say to simplify of the same form \((\alpha^{N})_{l=1,...,N} \) as in \((A.7)\)) from which no player can get better by deviating unilaterally, see for instance \([9, \text{Chapter 2}]\) for the details. In the folklore of MFG theory, Nash equilibria for the game with \(N\) players are typically expected to be (or at least to be “almost”) in mean field form according to the definition postulated in \((A.14)\) meaning that \(\bar{\alpha} \) in \((A.7)\) is independent of \(l\). To formulate the generic form of such a Nash equilibrium when \(N\) tends to \(+\infty\), we tackle first the diffusion approximation of \((A.9)\) when \(\alpha\) therein is precisely a mean field strategy and hence \(q\) is independent of \(l\).

The last step in the derivation of an MFG is to address the asymptotic dynamics of a player that deviates unilaterally from a Nash equilibrium. Here it may be worth clarifying the meaning of “unilateral deviation”: A Nash equilibrium is said to be computed over closed loop strategies if, whenever one player deviates (namely uses another strategy), the others still use the same feedback function \(\alpha\) (in which case the realizations of the strategies may change since the arguments plugged in entry of the feedback functions may change). Differently, the Nash equilibrium is said to be computed over open loop strategies if, whenever one player deviates, the others keep the same realizations of the strategies (meaning that, \(\omega\) per \(\omega\) in \(\Omega\), they use the same \(\omega\)-realizations of their strategies). We feel it to be useless to say more about the difference between the two: It is also part of the folklore of MFGs that the two notions lead to the same game. Another, and maybe more intuitive way, to catch the asymptotic dynamics of a deviating player is to proceed as follows: Take one given player, allow it to change its own strategy, but freeze (at the same time) the realization of the empirical measure of the system. This perfectly makes sense from a computational point of view since the strategy only depends on the other players through the empirical distribution and, more generally, all the other quantities in \((A.8)\) (except the own state of player \(l\) itself) only depend on the global state of the population. Of course, this would require a careful mathematical analysis, but, as we already explained, we feel better to leave the details of the asymptotic analysis for a future work. In particular, right below, we take for granted the fact that, asymptotically, \((A.11)\) can be replaced by \((A.12)\). Hence, in the end, we claim that, in order to recover the game addressed in Definition 2.5 it suffices to provide a time continuous diffusive approximation of \((A.12)\) (which means that we only care of the statistical behavior of a given player conditional on the realization of the common noise) whenever \(\alpha^{N}(m, j, \bar{\mu}_m^N(t))\) therein is replaced by an arbitrary \(\beta(m, j, \bar{\mu}_m^N(t))\) (standing for the deviation of the given player) and \(\bar{\mu}^N\) is exactly given by \((A.9)\) (in other words, the empirical distribution is kept frozen).

A.2.2. Hyperbolic scaling. The aforementioned diffusive regime appears in fact under a proper hyperbolic scaling. In short, for a given time horizon \(T > 0\), one chooses \(M = \lfloor NT \rfloor\) and then one considers \((\mu_{[NT]}^N)_{0 \leq t \leq T}\) in the equation \((A.9)\) for the empirical measure and, similarly,
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\[ \mathbb{P}^{0}[Y_{[N t]}^{l} I_{\{X_{[N t]}^{l} = i\}}]_{0 \leq t \leq T} \] in the equation (A.12) for the conditional marginal mass (of a given player \( l \)). In words, time is accelerated, which prompts us to force the jump rates to decay with \( N \) (as otherwise the dynamics would blow-up in finite time). Accordingly, we let

**Definition A.1.** We call an admissible mean field strategy \((\check{\alpha}^{1,N}, \ldots, \check{\alpha}^{N,N})\) a strategy for which there exists a bounded and continuous function \( \alpha : [0, +\infty) \times [d] \times \mathcal{S}_{d-1} \rightarrow [0, +\infty)^{d} \) such that, for any \( m \in \mathbb{N} \), any \( i \in [d] \), any \( \varrho \in \mathbb{Q}_{+}^{N} \) with \( \sum_{l=1}^{N} \varrho_{l} = N \), and any \( l \in \{1, \ldots, N\} \),

\[
\check{\alpha}^{l,N}(m, i, \varrho)(j) = \frac{1}{N} \alpha\left(\frac{m}{N}, i, \mu\right)(j), \quad j \neq i, \tag{A.15}
\]

where \( \mu \) is given by (A.4). The function \( \alpha \) is called a mean field rate function.

Of course, whenever \( \check{\alpha}^{l,N} \) is directly given in the form (A.7), the above rewrites

\[
\alpha^{l,N}(m, i, \mu)(j) = \frac{1}{N} \alpha\left(\frac{m}{N}, i, \mu\right)(j), \quad j \neq i, \tag{A.16}
\]

for \( m \in \mathbb{N} \), \( i, j \in [d] \) and \( \mu \in \mathcal{S}_{d-1} \).

The reader may easily check that (A.15) fits (A.7). Hence, (A.16) may seem a bit redundant, but, since we use it many times below, we feel it useful to write it explicitly. In fact the key point here is that the function \( \alpha \) that appears in the above right-hand side will be kept fixed as \( N \) tend to \(+\infty\). As \( N \) gets bigger and bigger, it is more and more difficult to jump from one state to another, which is consistent with the fact that the underlying physical time scale has to be thought as \( 1/N \) (because of the hyperbolic scaling).

**A.2.3. Derivation of the diffusive approximation.** We here provide a sketch of the diffusion approximation. We stress the fact that it is not a rigorous proof, but we think it to be useful for the reader to have some intuition on the way we pass from the aforementioned model to the mean field game that was addressed in the paper. We also refer to [30, Chapter 7, Theorem 4.1] for general results on diffusion approximation.

Generally speaking, the strategy is to expand the increments in the underlying dynamics in the form of the sum of a previsible term and of a martingale part and to show that both the previsible increment and the bracket of the martingale increment are of the same order as the time step (to ensure that the weak limits of both are absolutely continuous with respect to the Lebesgue measure). Provided that the weak limit of the dynamics has no jumps, it may be written as a Brownian semi-martingale: The drift derives from the previsible part in the aforementioned expansion and the bracket of the Brownian stochastic integral derives from the bracket of the martingale part (also in the aforementioned expansion).

Back to (A.9), we assume that \( q^{l} \) therein is independent of \( l \) (which is licit since the strategy is mean field, see A.1.4) and that the corresponding transition is admissible in the sense of
Definition [A.1] with $\alpha$ as mean field transition rate. We then write

$$
\tilde{\mu}^N_{m+1}(i) = \mu^N_m(i) + \frac{1}{N} \mathbb{1}_{\{T_{m+1}=0\}} \sum_{l=1}^{N} Y_{l}^f \mathbb{1}_{\{X^m_l=i\}} \left(1_{q(m,i,\tilde{\mu}^N_m)(U^l_{m+1})=i} - 1\right) \\
+ \frac{1}{N} \mathbb{1}_{\{T_{m+1}=0\}} \sum_{l=1}^{N} \sum_{j \neq i} Y_{l}^f \mathbb{1}_{\{X^m_l=j\}} 1_{q(m,j,\tilde{\mu}^N_m)(U^l_{m+1})=i} \\
+ \mathbb{1}_{\{T_{m+1}=1\}} \left(\frac{S_{m+1}(\tilde{\mu}^N_m)(i)}{N} - \tilde{\mu}^N_m(i)\right),
$$

(A.17)

for $m \in \{0, \ldots, M-1\}$ and $i \in [d]$. Using the same notations as in (A.8), we then call $\mathcal{F}^N_m$ the $\sigma$-field generated by the variables $(\{U^l_n\}_{l \in \{1, \ldots, N\}})_{0 \leq n \leq m}$, $(T_n)_{0 \leq n \leq m}$ and $(\{V^k_n\}_{k \in \{1, \ldots, N\}})_{0 \leq n \leq m}$. Taking the conditional expectation with respect to $\mathcal{F}^N_m$ under the probability $\mathbb{P}$ and using (A.16), we get

$$
\mathbb{E}[\tilde{\mu}^N_{m+1}(i) - \tilde{\mu}^N_m(i) \mid \mathcal{F}^N_m] = \frac{1 - \varepsilon}{N} \sum_{j \in [d]} \left(\mu^N_m(j) \alpha(N, j, \tilde{\mu}^N_m)(i) - \tilde{\mu}^N_m(i) \alpha(N, i, \tilde{\mu}^N_m)(j)\right),
$$

which may be rewritten in the form

$$
\mathbb{E}[\tilde{\mu}^N_{m+1}(i) - \tilde{\mu}^N_m(i) \mid \mathcal{F}^N_m] = \frac{1 - \varepsilon}{N} a_i(N, \tilde{\mu}^N_m),
$$

with $a_i(t, \mu) = \sum_{j \in [d]} \left(\mu(j) \alpha(t, j, \mu)(i) - \mu(i) \alpha(t, i, \mu)(j)\right),

(A.18)

where $t \geq 0$, $i \in [d]$ and $\mu \in S_{d-1}$ in the last line. (The reader may compare the notation $a$ with the one used in (2.13).)

Now, we may proceed similarly to compute the conditional covariance matrix of the vector $(\tilde{\mu}^N_{m+1}(i) - \tilde{\mu}^N_m(i))_{i \in [d]}$. The main idea is that the leading term is given by the covariance matrix of the last term in the right-hand side of (A.17). The latter is nothing but (up to the indicator function of $\{T_{m+1} = 1\}$) the covariance matrix of a multinomial distribution. So, we have, for any two $i, j \in [d],$

$$
\mathbb{E}\left[\mathbb{1}_{\{T_{m+1}=1\}} \left(\frac{S_{m+1}(\tilde{\mu}^N_m)(i)}{N} - \tilde{\mu}^N_m(i)\right) \left(\frac{S_{m+1}(\tilde{\mu}^N_m)(j)}{N} - \tilde{\mu}^N_m(j)\right) \mid \mathcal{F}^N_m\right] = \frac{\varepsilon}{N} \Xi_{i,j}(\tilde{\mu}^N_m),
$$

(A.19)

This may be rewritten in the form

$$
\mathbb{E}\left[\mathbb{1}_{\{T_{m+1}=1\}} \left(\frac{S_{m+1}(\tilde{\mu}^N_m)(i)}{N} - \tilde{\mu}^N_m(i)\right) \left(\frac{S_{m+1}(\tilde{\mu}^N_m)(j)}{N} - \tilde{\mu}^N_m(j)\right) \mid \mathcal{F}^N_m\right] = \frac{\varepsilon}{N} \Xi_{i,j}(\tilde{\mu}^N_m),
$$

with $\Xi_{i,j}(\mu) := \mu(i) \delta_{i,j} - \mu(i) \mu(j),

for $i, j \in [d]$ and $\mu \in S_{d-1}$. As for the conditional covariances of the second and third terms in the right-hand side of (A.17), they may be treated in the same way. Focus for instance on the first of those two ones. Since the variables $(U^l_{m+1})_{1 \leq l \leq N}$ are independent, we get as upper
Following (A.17), we may indeed rewrite (A.12) in the form

\[
\mathbb{E}
\left[
\left(
\frac{1}{N}
\sum_{l=1}^{N} Y_m^l 1_{\{X_m^l = i\}} \left(1_{\{T_{m+1} = 0\}} \left[1_{\{q(m,i,\bar{\mu}_m^N)(U_m^l) = i\}} - 1\right] - (1 - \varepsilon) \left[\alpha(m, i, \bar{\mu}_m^N)(i) - 1\right]\right) \right)^2 \bigg| \mathcal{F}_m^N
\right]
\]

\[
\leq \frac{2}{N^2} \sum_{l=1}^{N} (Y_m^l 1_{\{X_m^l = i\}})^2 \mathbb{E}
\left[
\left(1_{\{q(m,i,\bar{\mu}_m^N)(U_m^l) = i\}} - \alpha(m, i, \bar{\mu}_m^N)(i)\right)^2 \bigg| \mathcal{F}_m^N
\right]
\]

\[
+ 2 \mathbb{E}
\left[
\left(\frac{1}{N} \sum_{l=1}^{N} Y_m^l 1_{\{X_m^l = i\}} \left(1_{\{T_{m+1} = 0\}} - (1 - \varepsilon) \left[\alpha(m, i, \bar{\mu}_m^N)(i) - 1\right]\right)\right)^2 \bigg| \mathcal{F}_m^N
\right]
\]

\[
\leq \frac{C}{N^3} \sum_{l=1}^{N} (Y_m^l 1_{\{X_m^l = i\}})^2 + \frac{C}{N^2},
\]

where we used \(\alpha(m, i, \bar{\mu}_m^N)(i) = \mathbb{E}\left[1_{\{q(m,i,\bar{\mu}_m^N)(U_m^l) = i\}} \bigg| \mathcal{F}_m^N\right]\) in the first line, and (A.16) together with the equality \(\sum_{l=1}^{N} Y_m^l = N\) to derive the last line for a constant \(C\) that is independent of \(N\).

Part of the difficulty in the proof is to show that the first term in the last line of (A.20) is of order \(o(1/N)\). Taking it for granted, the combination of (A.17), (A.18) and (A.19) suggests that the dynamics of \((\bar{\mu}_m^{N,N})_{0\leq l \leq T}\) (see (A.17)) might be approximated (in law) by a diffusion process with values in the set \(S_{d-1}\) with \((1 - \varepsilon)\alpha\) (see (A.18)) as drift and \(\Xi\) (see (A.19)) as diffusion matrix. Up to factor \((1 - \varepsilon)\) (which we removed for aesthetic reason in the main core of the paper but which can obviously be absorbed in the drift if needed), we recover (2.7).

Once the limit of the (marginal) empirical measures has been understood, it becomes easier to address the asymptotic behavior of the conditional marginal masses as given by (A.12). Following (A.17), we may indeed rewrite (A.12) in the form

\[
Q_{m+1}^{l,N}(i) = Q_m^{l,N}(i) + 1_{\{T_{m+1} = 0\}} Q_m^{l,N}(i) \left[\alpha^{l,N}(m, i, \bar{\mu}_m^N)\right] (i) - 1
\]

\[
+ \sum_{j \neq i} 1_{\{T_{m+1} = 0\}} Q_m^{l,N}(j) \left[\alpha^{l,N}(m, j, \bar{\mu}_m^N)\right] (i) + 1_{\{T_{m+1} = 1\}} Q_m^{l,N}(i) \left(S_{m+1}(\bar{\mu}_m^N)(i)/N \bar{\mu}_m^N(i) - 1\right),
\]

(A.21)

for \(l \in \{1, \cdots, N\}\), \(i \in [d]\) and \(m \in \{0, \cdots, M - 1\}\). Differently from (A.17), \(l\) here is fixed and the fact that \(\alpha\) depends on \(l\) does not matter. As we already explained, we may easily remove the superscript \(l\) in \(\alpha^{l,N}\) and, in order to distinguish the latter from the transition \(\alpha\) used in (A.17) via the quantile, we denote it by \(\beta\) and, then, we assume that there exists a mean field rate function \(\beta\) as in Definition (A.1), namely

\[
\beta(m,i,\mu)(j) = \frac{1}{N} \beta\left(\frac{m}{N}, i, \mu\right)(j), \quad j \neq i; \quad \beta(m,i,\mu)(i) = 1 - \frac{1}{N} \sum_{j \neq i} \beta\left(\frac{m}{N}, i, \mu\right)(j).
\]
Following (A.18), we can compute the local conditional mean of the increments $Q_{m+1}^{l,N} - Q_{m}^{l,N}$, for an integer $m \in \{0, \ldots , M - 1\}$,

$$\mathbb{E} \left[ Q_{m+1}^{l,N}(i) - Q_{m}^{l,N}(i) \mid \mathcal{F}_{m}^{N} \right] = \frac{1 - \varepsilon}{N} b_{i}(\frac{m}{N}, Q_{m}^{l,N}, \bar{\mu}_{m}^{N}),$$

with $b_{i}(t,q,\mu) = \sum_{j \in [d]} \left( q(j)\beta(t,j,\mu)(i) - q(i)\beta(t,i,\mu)(j) \right)$, (A.22)

for $i \in [d]$, $t \geq 0$, $q \in (\mathbb{R}_{+})^{d}$ and $\mu \in S_{d-1}$, which provides the asymptotic form of the drift in the dynamics of $(Q_{l,N}^{m})_{0 \leq t \leq T}$ when $N$ tends to $\infty$. We then proceed similarly to compute the conditional covariance matrix of the increments of $(Q_{l,N}^{m})_{m=0,...,M}$. Similar to the conditional covariance matrix of the increments of $(\bar{\mu}_{m}^{N})_{m=0,...,M}$, only the last term in (A.21) really matters. In fact, this last term may be rewritten in the form

$$1_{\{T_{m+1}=1\}}Q_{m}^{l,N}(i) \left( \frac{S_{m+1}(\bar{\mu}_{m}^{N})(i)}{N\bar{\mu}_{m}^{N}(i)} - 1 \right) = \frac{Q_{m}^{l,N}(i)}{\bar{\mu}_{m}^{N}(i)} 1_{\{T_{m+1}=1\}} \left( \frac{S_{m+1}(\bar{\mu}_{m}^{N})(i)}{N} - \bar{\mu}_{m}^{N}(i) \right),$$

and, then, we recognize, up to the leading factor $Q_{m}^{l,N}(i)/\bar{\mu}_{m}^{N}(i)$ the leading martingale term in the expansion of (A.17).

The whole suggests that, as $N$ tends to $\infty$, the dynamics of $(Q_{l,N}^{m})_{0 \leq t \leq T}$ might be approximated (in law) by an Itô process $(Q_{t})_{0 \leq t \leq T}$ with values in $(\mathbb{R}_{+})^{d}$, constructed on the same space as the limit in law $(\mu_{t})_{0 \leq t \leq T}$ of $(\bar{\mu}_{l,N}^{N})_{0 \leq t \leq T}$ and expanding in the form

$$dQ_{t}(i) = (1 - \varepsilon)b_{i}(t,Q_{t},\mu_{t})dt + \frac{Q_{t}(i)}{\mu_{t}(i)}dm_{t}(i),$$

where $(m_{t})_{0 \leq t \leq T}$ stands for the martingale part of the diffusive process $(\mu_{t})_{t \geq 0}$. Up to factor $(1 - \varepsilon)$ in the drift, we recover (2.8).

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