Quantum Knizhnik-Zamolodchikov equation
associated with $U_q(A_2^{(2)})$ for $|q| = 1$

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Abstract

We present an integral representation to the quantum Knizhnik-Zamolodchikov equation associated with twisted affine symmetry $U_q(A_2^{(2)})$ for massless regime $|q| = 1$. Upon specialization, it leads to a conjectural formula for the correlation function of the Izergin-Korepin model in massless regime $|q| = 1$. In a limiting case $q \to -1$, our conjectural formula reproduce the correlation function for the Izergin-Korepin model \cite{1,2} at critical point $q = -1$.

1 Introduction

We shall consider the quantum Knizhnik-Zamolodchikov equation ($q$KZ equation) associated with twisted affine symmetry $U_q(A_2^{(2)})$ for massless regime $|q| = 1$. In the earlier work \cite{1,2}, the $q$KZ equation associated with twisted affine symmetry $U_q(A_2^{(2)})$ for massive regime $-1 < q < 0$, was studied within the framework of the representation theory of $U_q(A_2^{(2)})$, \cite{3,4}. Hou et al. gave free field realizations of the vertex operators, and realized integral representations of $q$KZ equation, as the trace of the vertex operators. Their integral representation gave the correlation function for the Izergin-Korepin model \cite{5} for massive regime $-1 < q < 0$. In this paper we present an integral representation for $q$KZ equation associated with twisted algebra $U_q(A_2^{(2)})$ for massless regime $|q| = 1$. Let $V = \mathbb{C}^3$, and consider the $R$-matrix $R(\beta) \in \text{End}(V \otimes V)$ associated with $U_q(A_2^{(2)})$ (see sec.2). The $q$KZ equation associated with $U_q(A_2^{(2)})$ for $|q| = 1$ is the following system of linear difference equations for an unknown function $G_{2N}(\beta_1, \cdots, \beta_{2N})$
taking value in the space $V^{\otimes 2N}$.

$$G_{2N}(\beta_1, \cdots, \beta_{j-1}, \beta_j - \lambda i, \beta_{j+1}, \cdots, \beta_{2N})$$

$$= R_{j,j+1}(\beta_j - \beta_{j+1} - \lambda i)^{-1} \cdots R_{j,2N}(\beta_j - \beta_{2N} - \lambda i)^{-1}$$

$$\times R_{1,j}(\beta_1 - \beta_j) \cdots R_{j-1,j}(\beta_{j-1} - \beta_j)G_{2N}(\beta_1, \cdots, \beta_{j-1}, \beta_j, \beta_{j+1}, \cdots, \beta_{2N}).$$

(1.1)

Here $R_{j,k}(\beta) \in \text{End}(V^{\otimes 2N})$ signifies the matrix acting as $R(\beta)$ on $(j,k)$-th tensor components and as identity elsewhere, and difference parameter $\lambda > 0$. Upon specialization of difference parameter $\lambda = 3\pi$ and spectral parameters $\beta_j = \beta + 3\pi i, \beta_{j+N} = \beta, (1 \leq j \leq N)$, our integral representation leads to a conjectural formula of the $N$-point correlation functions of the Izergin-Korepin model in massless regime $|q| = 1$.

$$G_{2N}(\beta + 3\pi i, \cdots, \beta + 3\pi i, \beta, \cdots, \beta).$$

(1.2)

Actually, in a limiting case $q \to -1$, our conjectural formula for the correlation function reproduce those of earlier work [1] at critical point $q = -1$.

In connection with “massless” $qKZ$, we should mention about the work [3, 4, 5], in which the authors presented an integral representation to $qKZ$ equation associated with $U_q(A^{(1)}_1)$ for $|q| = 1$. In pioneering work [3], F.Smirnov gave conjectural integral representations for the form factors of sine-Gordon model. M.Jimbo et al.’s integral representation [4] gave a conjectural formula of the correlation function for the massless XXZ chain. The higher spin (spin 1) generalization of work [4, 5] was achieved in [4]. The $U_q(A^{(1)}_{n-1})$ generalization of work [4, 5] was achieved in [11]. T.Miwa and Y.Takeyama [11] studied $qKZ$ equation associated with $U_q(A^{(1)}_{n-1})$ for $|q| = 1$, and presented hypergeometric pairing in terminology of V.Tarasov and A.Varchenko [12].

Now a few words about organization of this paper. In section 2 we formulate the system of difference equations. In section 3 we present an integral representation. In section 4 we show that our integral representation satisfies the system of difference equations. In section 5 we give a supporting argument of our conjectural formula.

## 2 Difference Equations

In this section, we formulate the system of equations we are going to study, including the $qKZ$ equation associated with $U_q(A^{(2)}_2)$ for $|q| = 1$. In this paper we parametrize the deformation parameter $q$ of $U_q(A^{(2)}_2)$ by $\xi$ as follows.

$$q = -e^{-\frac{\pi}{\sqrt{4\xi}}}, \quad \xi > 1.$$  

(2.1)
Consider the $R$-matrix $R(\beta) \in \text{End}(V \otimes V)$ of $U_q(A_2^{(2)})$ acting on the tensor product of $V = \mathbb{C}v_{-1} \oplus \mathbb{C}v_0 \oplus \mathbb{C}v_1$:

$$R(\beta)v_{j_1} \otimes v_{j_2} = \sum_{k_1, k_2 = \pm 1, 0} v_{k_1} \otimes v_{k_2} R(\beta)_{j_1 j_2}^{k_1 k_2},$$  \hspace{1cm} (2.2)

$$R(\beta) = \frac{1}{\kappa(\beta)} \overline{R}(\beta).$$  \hspace{1cm} (2.3)

The scalar function $\kappa(\beta)$ will be specified below. It is chosen to ensure that the $R$-matrix satisfies the unitarity (2.6) and the crossing symmetry (2.7). Nonzero entries of the $R$-matrix $\overline{R}(\beta)$ are given as follows.

$$\overline{R}(\beta)_{1,1}^{1,1} = \overline{R}(\beta)_{1,1}^{-1,-1} = 1,$$

$$\overline{R}(\beta)_{0,0}^{-1,0} = \overline{R}(\beta)_{0,1}^{0,1} = -e^{-\frac{\beta}{2\pi}} \text{sh} \left( \frac{\pi i}{\xi} \right) \frac{1}{\text{sh} \left( \frac{1}{2\pi} (\beta - 2\pi i) \right)},$$

$$\overline{R}(\beta)_{0,1}^{-0,-1} = \overline{R}(\beta)_{1,0}^{1,0} = \overline{R}(\beta)_{0,1}^{0,1} = \frac{-\text{sh} \left( \frac{\beta}{2\pi} \right)}{\text{sh} \left( \frac{1}{2\pi} (\beta - 2\pi i) \right)}$$

$$\overline{R}(\beta)_{1,1}^{-1,1} = \overline{R}(\beta)_{1,1}^{-1,-1} = \frac{\text{sh} \left( \frac{\beta}{2\pi} \right) \text{sh} \left( \frac{1}{2\pi} (\beta - \pi i) \right)}{\text{sh} \left( \frac{1}{2\pi} (\beta - 2\pi i) \right) \text{sh} \left( \frac{1}{2\pi} (\beta - 3\pi i) \right)},$$

$$\overline{R}(\beta)_{1,1}^{1,1} = \text{sh} \left( \frac{\pi i}{\xi} \right) \left( -e^{-\frac{\beta}{2\pi}} \text{ch} \left( \frac{\pi i}{\xi} \right) + e^{-\frac{3\beta}{2\pi}} \text{ch} \left( \frac{3\pi i}{\xi} \right) \right) \frac{1}{\text{sh} \left( \frac{1}{2\pi} (\beta - 2\pi i) \right) \text{sh} \left( \frac{1}{2\pi} (\beta - 3\pi i) \right)},$$

$$\overline{R}(\beta)_{1,1}^{-1,1} = \text{sh} \left( \frac{\pi i}{\xi} \right) \left( e^{\frac{\pi i}{2\pi}} \text{ch} \left( \frac{n i}{\xi} \right) - e^{-\frac{3\beta}{2\pi}} \text{ch} \left( \frac{3\pi i}{\xi} \right) \right) \frac{1}{\text{sh} \left( \frac{1}{2\pi} (\beta - 2\pi i) \right) \text{sh} \left( \frac{1}{2\pi} (\beta - 3\pi i) \right)}.$$

$$\overline{R}(\beta)_{0,0}^{-1,0} = \overline{R}(\beta)_{1,1}^{0,0} = -ie^{-\frac{\beta}{2\pi}} \text{sh} \left( \frac{\pi i}{\xi} \right) \frac{1}{\text{sh} \left( \frac{1}{2\pi} (\beta - 2\pi i) \right) \text{sh} \left( \frac{1}{2\pi} (\beta - 3\pi i) \right)},$$

$$\overline{R}(\beta)_{0,0}^{-0,-1} = -ie^{-\frac{3\beta}{2\pi}} \text{sh} \left( \frac{\pi i}{\xi} \right) \frac{1}{\text{sh} \left( \frac{1}{2\pi} (\beta - 2\pi i) \right) \text{sh} \left( \frac{1}{2\pi} (\beta - 3\pi i) \right)},$$

$$\overline{R}(\beta)_{0,0}^{0,0} = \frac{-\text{sh} \left( \frac{\beta}{2\pi} \right)}{\text{sh} \left( \frac{1}{2\pi} (\beta - 2\pi i) \right)} - \frac{\text{sh} \left( \frac{\pi i}{\xi} \right) \text{sh} \left( \frac{3\pi i}{\xi} \right)}{\text{sh} \left( \frac{1}{2\pi} (\beta - 2\pi i) \right) \text{sh} \left( \frac{1}{2\pi} (\beta - 3\pi i) \right)}.$$  \hspace{1cm} (2.4)
Our $R$-matrix $R(\beta)$ gives the Boltzmann weight of the Izergin-Korepin model \cite{4}. The $R$-matrix $R(\beta)$ satisfies the Yang-Baxter equation.

$$R_{12}(\beta_1 - \beta_2)R_{13}(\beta_1 - \beta_3)R_{23}(\beta_2 - \beta_3) = R_{23}(\beta_2 - \beta_3)R_{13}(\beta_1 - \beta_3)R_{12}(\beta_1 - \beta_2), \quad (2.5)$$

the unitarity condition,

$$R_{12}(\beta)R_{21}(-\beta) = id, \quad (2.6)$$

and the crossing symmetry,

$$R \left( \beta + \frac{3\pi i}{2} \right)_{j_1,j_2}^{k_1,k_2} = (ie^{-\frac{\pi i}{2}})^{k_2-j_2} \times R \left( -\beta + \frac{3\pi i}{2} \right)_{-j_2,k_1}^{-j_2,k_1}. \quad (2.7)$$

Let $N$ be a non-negative integer. Consider a $V \otimes 2^N$-valued function $G_{2N}(\beta_1, \cdots, \beta_{2N})$, depending on the 'spectral parameter' $\beta_1, \cdots, \beta_{2N}$.

$$G_{2N}(\beta_1, \cdots, \beta_{2N}) = \sum_{j_1, \cdots, j_{2N} = \pm 1, 0} v_{j_1} \otimes \cdots \otimes v_{j_{2N}} G_{2N}(\beta_1, \cdots, \beta_{2N})_{j_1, \cdots, j_{2N}}. \quad (2.8)$$

We study the following system of difference equations for $G_{2N}(\beta_1, \cdots, \beta_{2N})_{j_1, \cdots, j_{2N}}$ involving parameter $\lambda$. Hereafter we assume that

$$2(\xi + 1)\pi > \lambda > 0, \quad \xi > 1. \quad (2.9)$$

**$R$-matrix Symmetry**

$$G_{2N}(\beta_1, \cdots, \beta_{s+1}, \beta_s, \cdots, \beta_{2N})_{j_1, \cdots, j_{s+1}, j_s, \cdots, j_{2N}} = \sum_{j_s, j_{s+1} = \pm 1, 0} R(\beta_s - \beta_{s+1})_{j_s, j_{s+1}}^{j_s', j_{s+1}'} G_{2N}(\beta_1, \cdots, \beta_s, \beta_{s+1}, \cdots, \beta_{2N})_{j_1, \cdots, j_s', j_{s+1}', \cdots, j_{2N}}. \quad (2.10)$$

**Cyclicity Condition**

$$G_{2N}(\beta_1, \cdots, \beta_{2N-1}, \beta_{2N} - i\lambda)_{j_1, \cdots, j_{2N}} = G_{2N}(\beta_2, \cdots, \beta_{2N-1})_{j_2N, j_1, \cdots, j_{2N-1}}. \quad (2.11)$$

**Recursion Relation**

$$G_{2N}(\beta_1, \cdots, \beta_{s-1}, \beta_s, \beta + 3\pi i, \beta_{s+1}, \cdots, \beta_{2N})_{j_1, \cdots, j_{s-1}, j_s, j_{s+1}, \cdots, j_{2N}} = C_j \cdot G_{2N-2}(\beta_1, \cdots, \beta_{s-1}, \beta_{s+1}, \cdots, \beta_{2N})_{j_1, \cdots, j_{s-2}, j_{s-1}, j_{s+2}, \cdots, j_{2N}}. \quad (2.12)$$

Here $C_j$ are specified below. See \cite{4, 27, 128} and \cite{29}. Note that the above system of equations involve only the functions $G_{2N}(\beta_1, \cdots, \beta_{2N})_{j_1, \cdots, j_{2N}}$ with fixed value of 'spin' $j_1 + \cdots + j_{2N} = 0$. 


3 Integral Representation

The purpose of this section is to present an integral representation. In what follows we use the Multiple-Gamma functions $\Gamma_r(x|\omega_1 \cdots \omega_r)$ and the Multiple-Trigonometric functions $S_r(x|\omega_1 \cdots \omega_r)$ introduced in [7] as follows.

\[
\log \Gamma_r(x|\omega_1 \cdots \omega_r) = \gamma \frac{(-1)^r}{r!} B_{r,r}(x|\omega_1, \cdots, \omega_r) + \int_{C'} \frac{e^{-xt} \log(-t)}{\prod_{j=1}^{r}(1 - e^{-\omega_j t})} \frac{dt}{2\pi it}, \quad (\text{Re} x > 0), \tag{3.1}
\]

where $\gamma =$ Euler’s constant, the integral contour $C'$ is shown in below Figure (Contour $C'$), and Multiple-Bernoulli polynomials $B_{r,r}(x|\omega_1 \cdots \omega_r)$ are given by

\[
\frac{t^r e^{xt}}{\prod_{j=1}^{r}(e^{\omega_j t} - 1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_{r,n}(x|\omega_1, \cdots, \omega_r). \tag{3.2}
\]

The Multiple-Gamma function $\Gamma_r(x|\omega_1 \cdots \omega_r)$ is an entire function of $x$. $\Gamma_r(x|\omega_1 \cdots \omega_r)$ is meromorphic function with poles at $x = n_1 \omega_1 + \cdots + n_r \omega_r, (n_1, \cdots, n_r \leq 0)$.

Let us set Multiple-Trigonometric function by

\[
S_r(x|\omega_1 \cdots \omega_r) = \frac{\Gamma_r(x+\omega_r|\omega_1, \cdots, \omega_r) - 1}{\Gamma_r(x|\omega_1, \cdots, \omega_r)}, \tag{3.3}
\]

Properties of these functions are listed in [7]. For examples they enjoy

\[
\frac{\Gamma_r(x + \omega_r|\omega_1, \cdots, \omega_r)}{\Gamma_r(x|\omega_1, \cdots, \omega_r)} = \frac{1}{\Gamma_{r-1}(x|\omega_1, \cdots, \omega_{r-1})}, \tag{3.4}
\]

\[
\frac{S_r(x + \omega_r|\omega_1, \cdots, \omega_r)}{S_r(x|\omega_1, \cdots, \omega_r)} = \frac{1}{S_{r-1}(x|\omega_1, \cdots, \omega_{r-1})}, \tag{3.5}
\]

and

\[
\Gamma_1(x|\omega) = \omega^{\frac{x}{\omega}} \frac{1}{2} \Gamma(x/\omega), \quad S_1(x|\omega) = 2\sin\left(\frac{\pi x}{\omega}\right). \tag{3.6}
\]

Here $\Gamma(x)$ is Gamma function. As $x \to \infty$ $(\pm \text{Im} x > 0)$, the function $S_2(x|\omega_1 \omega_2)$ behaves as follows.

\[
\log S_2(x|\omega_1 \omega_2) = \pm i \left( \frac{x^2}{2\omega_1 \omega_2} - \frac{\omega_1 + \omega_2}{2\omega_1 \omega_2} x - \frac{1}{12} \left( \frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} + 3 \right) \right) + o(1). \tag{3.7}
\]
3.1 Auxiliary functions

The integral formula involves certain special functions $\kappa(\beta), \rho(\beta), \varphi(\beta), \psi(\beta)$.

- $\kappa(\beta)$: Let us set
  \[
  \kappa(\beta) = \frac{S_2(-i\beta + 2\pi)S_2(-i\beta + 3\pi)S_2(i\beta + 5\pi)S_2(i\beta + 6\pi)}{S_2(i\beta + 2\pi)S_2(i\beta + 3\pi)S_2(-i\beta + 5\pi)S_2(-i\beta + 6\pi)}, \quad (S_2(x) = S_2(x|2\pi \xi, 6\pi)).
  \]
  The function $\kappa(\alpha)$ satisfies the following difference equations, which ensure the unitarity and the crossing symmetry of $R$-matrix.
  \[
  \kappa(\beta)\kappa(-\beta) = 1, \quad (3.9)
  \]
  \[
  \kappa(\beta)\kappa(\beta - 3\pi i) = \frac{\text{sh} \left( \frac{1}{2\pi}(\beta - \pi i) \right) \text{sh} \left( \frac{\beta}{2\pi} \right)}{\text{sh} \left( \frac{1}{2\pi}(\beta - 2\pi i) \right) \text{sh} \left( \frac{1}{2\pi}(\beta - 3\pi i) \right)}. \quad (3.10)
  \]

- $\rho(\beta)$: Let us set
  \[
  \rho(\beta) = \frac{S_3(-i\beta + 2\pi)S_3(-i\beta + 3\pi)S_3(i\beta + 3\pi + \lambda)S_3(i\beta + 2\pi + \lambda)}{S_3(-i\beta + 2\pi \xi)S_3(-i\beta + \pi + 2\pi \xi)S_3(i\beta + \pi + 2\pi \xi + \lambda)S_3(i\beta + 2\pi \xi + \lambda)}, \quad (S_3(x) = S_3(x|2\pi \xi, \lambda, 6\pi)).
  \]
  The function $\rho(\alpha)$ satisfies the following difference equations.
  \[
  \frac{\rho(\beta)}{\rho(-\beta)} = \kappa(\beta), \quad \rho(i\lambda - \beta) = \rho(\beta). \quad (3.12)
  \]

- $\varphi(\alpha)$: Let us set
  \[
  \varphi(\alpha) = \frac{1}{S_2(i\alpha + \pi |\lambda, 2\pi \xi)S_2(-i\alpha + \pi |\lambda, 2\pi \xi)}. \quad (3.13)
  \]
  The function $\varphi(\alpha)$ satisfies following difference equations.
  \[
  \varphi(\alpha) = \varphi(-\alpha), \quad (3.14)
  \]
  \[
  \frac{\varphi(\alpha - i\lambda)}{\varphi(\alpha)} = -\frac{\text{sh} \left( \frac{1}{2\pi}(\alpha - \pi i) \right)}{\text{sh} \left( \frac{1}{2\pi}(\alpha + \pi i - \lambda) \right)}, \quad (3.15)
  \]
  \[
  \frac{\varphi(\alpha \pm 2\pi \xi i)}{\varphi(\alpha)} = -\frac{\text{sh} \left( \frac{\pi}{\lambda}(\alpha \pm \pi i) \right)}{\text{sh} \left( \frac{\pi}{\lambda}(\alpha \mp \pi i \pm 2\pi \xi i) \right)}. \quad (3.16)
  \]
  The function $\varphi(\alpha)$ has poles at
  \[
  \varphi(\alpha) = \pm i \left( n_1 \lambda + n_2 2\pi \xi + \pi \right), \quad n_1, n_2 \geq 0. \quad (3.17)
  \]
  The function $\varphi(\alpha)$ is evaluated as follows.
  \[
  \left| \varphi(\alpha) \right| \leq \text{Const.} \exp \left( \frac{2\pi - (\lambda + 2\pi \xi)}{2\xi \lambda} |\alpha| \right), \quad |\alpha| \to \infty, \quad (3.18)
  \]
  \[
  \varphi(\alpha) = \sqrt{\frac{\lambda}{2\pi \xi}} \times \frac{1}{iS_2(2\pi |\lambda, 2\pi \xi)} \times \frac{1}{\alpha - \pi i} + \cdots, \quad \alpha \to \pi i. \quad (3.19)
  \]
• $\psi(\alpha)$: Let us set

$$
\psi(\alpha) = \frac{1}{S_2(i\alpha - 2\pi|\lambda, 2\pi|\xi)S_2(-i\alpha - 2\pi|\lambda, 2\pi|\xi)}.
$$

(3.20)

The function $\psi(\alpha)$ satisfies the following difference equations.

$$
\psi(\alpha) = \psi(-\alpha),
$$

(3.21)

$$
\psi(\alpha - i\lambda) = -\frac{\sh\left(\frac{1}{2\xi}(\alpha + 2\pi i)\right)}{\sh\left(\frac{1}{2\xi}(\alpha - \lambda i - 2\pi i)\right)},
$$

(3.22)

$$
\psi(\alpha \pm 2\pi\xi i) = -\frac{\sh\left(\frac{\pi}{\lambda}(\alpha \mp 2\pi i)\right)}{\sh\left(\frac{\pi}{\lambda}(\alpha \pm 2\pi i \pm 2\pi\xi i)\right)},
$$

(3.23)

and

$$
\varphi(\alpha + \pi i)\psi(\alpha)\varphi(\alpha - \pi i)
= 2^{-6} \left\{ \sh\left(\frac{1}{2\xi}(\alpha + 2\pi i)\right) \sh\left(\frac{\alpha}{2\xi}\right) \sh\left(\frac{1}{2\xi}(\alpha - 2\pi i)\right) \right\}^{-1} \times \left\{ \sh\left(\frac{\pi}{\lambda}(\alpha - 2\pi i)\right) \sh\left(\frac{\pi}{\lambda}(\alpha + 2\pi i)\right) \right\}^{-1}.
$$

(3.24)

(3.25)

The function $\psi(\alpha)$ has poles at

$$
\alpha = \pm i(n_1\lambda + n_22\pi\xi - 2\pi), \quad n_1, n_2 \geq 0.
$$

(3.26)

The function $\psi(\alpha)$ is evaluated as follows.

$$
|\psi(\alpha)| \leq \text{Const. exp}\left(-\frac{4\pi + \lambda + 2\pi\xi}{2\xi\lambda}|\alpha|\right), \quad |\alpha| \to \infty.
$$

(3.27)

### 3.2 Integral Representation

In this section we present the integral representation for $G_{2N}(\beta_1, \ldots, \beta_{2N})_{j_1, \ldots, j_{2N}}$, which is our main result of this paper. At first we demonstrate integral representations for $N = 1$ case, $G_{2}(\beta_1, \beta_2)_{0,0}, G_{2}(\beta_1, \beta_2)_{1,-1}$, and $G_{2}(\beta_1, \beta_2)_{-1,1}$.

$$
G_{2}(\beta_1, \beta_2)_{0,0} = 2^9 e^{-\frac{\pi i}{\xi}} \sh\left(\frac{\pi i}{\xi}\right) \rho(\beta_1 - \beta_2) \int_C \frac{d\alpha_1}{2\pi i} \int_C \frac{d\alpha_2}{2\pi i} \times \left\{ \varphi(\alpha_1 - \alpha_2)\psi(\alpha_1 - \alpha_2) \prod_{s,t=1,2} \varphi(\alpha_s - \beta_t) \right\} e^{\frac{3\pi i}{\xi}(\alpha_1 + \alpha_2 - \beta_1 - \beta_2)} h_\lambda(\alpha_1 - \alpha_2) \times \sh\left(\frac{1}{2\xi}(\alpha_1 - \alpha_2 + 2\pi i)\right) \sh\left(\frac{1}{2\xi}(-\alpha_1 + \alpha_2 + \pi i)\right) \sh\left(\frac{1}{2\xi}(\alpha_2 - \beta_1 + \pi i)\right) \sh\left(\frac{1}{2\xi}(-\alpha_1 + \beta_2 + \pi i)\right).
$$

(3.28)
The index set \( A \) of this paper. Given a set of indices \( \{j_1, \cdots, j_{2N}\} \subset \{1, 2, \cdots, 2N\} \). Let us set \( A_j \) by
\[
A_j = \{1 \leq s \leq 2N| j_s \geq j \}.
\] (3.32)
The index set \( A_{-1} = \{1, \cdots, 2N\} \). Let us set \( \alpha_{-1,s} = \beta_s, (1 \leq s \leq 2N) \). Let us define the kernel function \( \Psi(\{\alpha\}|\{\beta\}) \) by
\[
\Psi(\{\alpha\}|\{\beta\}) = \prod_{s=1}^{2N} \prod_{j=0,1} \prod_{t \in A_j} \varphi(\beta_s - \alpha_{j,t}) \prod_{s \in A_j, t \in A_1} \varphi(\alpha_{0,s} - \alpha_{1,t}) \varphi(\alpha_{0,s} - \alpha_{1,t}) \times \prod_{j=0,1} \prod_{s < t \in A_j} \varphi(\alpha_{j,s} - \alpha_{j,t}) \psi(\alpha_{j,s} - \alpha_{j,t}).
\] (3.33)
Here we have use the function $\varphi(\alpha), \psi(\alpha)$ in (3.13), (3.20). Let us define the auxiliary functions $g_{k}({\{\alpha_{j}\}_{j=-1}^{k}}), (k = \pm 1, 0)$ by

$$g_{-1}(\{\alpha_{-1}\}) = 1,$$  \hspace{1cm} (3.34)

$$g_{0}(\{\alpha_{-1}, \alpha_{0}\}) = \frac{e^{\frac{1}{\xi}(\alpha_{0} - \alpha_{-1})} \sh(\frac{\pi i}{\xi})}{\sh(\frac{1}{\xi}(\alpha_{0} - \alpha_{-1} + \pi i))},$$  \hspace{1cm} (3.35)

$$g_{1}(\{\alpha_{-1}, \alpha_{0}, \alpha_{1}\}) = \frac{ie^{-\frac{1}{\xi}\sh(\frac{\pi i}{\xi})}}{\sh(\frac{1}{\xi}(\alpha_{0} - \alpha_{-1} + \pi i)) \sh(\frac{1}{\xi}(\alpha_{1} - \alpha_{-1} + \pi i))} \times \left( e^{\frac{1}{2\xi}\alpha_{0}\sh\left(\frac{1}{2\xi}(\alpha_{1} - \alpha_{-1} + \pi i)\right)} - e^{\frac{1}{2\xi}\alpha_{1}\sh\left(\frac{1}{2\xi}(\alpha_{1} - \alpha_{0} + \pi i)\right)} \right).$$  \hspace{1cm} (3.36)

Let us set the integral representation $G_{2N}$ by

$$G_{2N}(\beta_{1}, \cdots, \beta_{2N})_{j_{1}, \cdots, j_{2N}}$$

$$= 2^{N(14N-5)} \prod_{1 \leq s < t \leq 2N} \rho(\beta_{s} - \beta_{t}) \prod_{s=1}^{2N} \prod_{j=0}^{j_{s}} \frac{d\alpha_{j,s}}{2\pi i} \Psi(\{\alpha\}, \{\beta\})$$

$$\times \prod_{j=0, 1} \prod_{s,t \in A_{j}} h_{\lambda}(\alpha_{j,s} - \alpha_{j,t}) \prod_{s \in A_{0}, t \in A_{1}} h_{\lambda}(\alpha_{0,s} - \alpha_{1,t}) \prod_{s \in A_{1}, t \in A_{0}} h_{\lambda}(\alpha_{1,t} - \alpha_{0,s})$$

$$\times \prod_{s=1}^{2N} g_{j_{s}}(\{\alpha_{j,s}\}_{j=-1}^{j_{s}}) \prod_{j=0, 1} \prod_{s,t \leq t} \sh(\frac{1}{2\xi}(\alpha_{j,s} - \alpha_{j,t} + 2\pi i)) \sh(\frac{1}{2\xi}(-\alpha_{j,s} + \alpha_{j,t} + \pi i))$$

$$\times \prod_{s \in A_{0}, t \in A_{1}} \sh(\frac{1}{2\xi}(\alpha_{0,s} - \alpha_{1,t} + 2\pi i)) \sh(\frac{1}{2\xi}(-\alpha_{0,s} + \alpha_{1,t} + \pi i))$$

$$\times \prod_{s \in A_{1}, t \geq t} \sh(\frac{1}{2\xi}(\alpha_{1,t} - \alpha_{0,s} + 2\pi i)) \sh(\frac{1}{2\xi}(-\alpha_{1,t} + \alpha_{0,s} + \pi i))$$

$$\times \prod_{s=1}^{2N} \prod_{j=0, 1} \prod_{t \leq t} \sh(\frac{1}{2\xi}(-\beta_{s} + \alpha_{j,t} + \pi i)) \prod_{t \leq t} \sh(\frac{1}{2\xi}(\alpha_{j,t} + \beta_{s} + \pi i)).$$  \hspace{1cm} (3.37)

Here we have used $\rho(\beta)$ in (3.11), $\Psi(\{\alpha\}, \{\beta\})$ in (3.33), $h_{\lambda}(\alpha)$ in (3.31), $g_{j_{s}}(\{\alpha_{j,s}\}_{j=-1}^{j_{s}})$ in (3.34), (3.35), (3.36). Here the integral contour $C$ is taken along a path going from $-\infty$ to $\infty$ in such a way that $-\pi < \text{Im}(\alpha_{k} - \beta_{j}) < \pi$ for all $k, j$. We understand $G_{0} = 1$.

Let us check the convergence of the integral of $G_{2N}$. Let us set

$$I_{\lambda}(\alpha) = \varphi(\alpha)\psi(\alpha)h_{\lambda}(\alpha)\sh\left(\frac{1}{2\xi}(\alpha + 2\pi i)\right)\sh\left(\frac{1}{2\xi}(-\alpha + \pi i)\right),$$  \hspace{1cm} (3.38)

$$J_{\lambda}(\alpha) = \varphi(\alpha)\sh\left(\frac{1}{2\xi}(-\alpha + \pi i)\right).$$  \hspace{1cm} (3.39)

9
Let us consider the evaluation of $|\alpha_{k,t}| \to \infty$. In our integral representation, factors involving the variable $\alpha_{k,t}$ consist of following three parts.

\[
g_{j_t}(\{\alpha_{j,t}\}_{j=t}^{j=-1}) \times \prod_{s \leq t} J_\lambda(-\beta_s + \alpha_{k,t}) \prod_{t < s} J_\lambda(-\alpha_{k,t} + \beta_s) \prod_{j=0,1} \prod_{t < s} I_\lambda(\alpha_{k,t} - \alpha_{j,s}) \prod_{j \neq k} I_\lambda(\text{sgn}(k - j)(\alpha_{j,t} - \alpha_{k,t})).
\]

(3.40)

For $|\alpha| \to \infty$, the factors $I_\lambda(\alpha), J_\lambda(\alpha)$ are evaluated as follows.

\[
I_\lambda(\alpha) = \text{Const.} \exp \left( \frac{\pi}{\xi \lambda} (\xi - 1)|\alpha| \right),
\]

(3.41)

\[
J_\lambda(\alpha) = \text{Const.} \exp \left( \frac{\pi}{\xi \lambda} (1 - \xi)|\alpha| \right).
\]

(3.42)

We have $g_{j_t}(\{\alpha_{j,t}\}_{j=t}^{j=-1}) \to 1, \ (\alpha_{k,t} \to \infty)$. Let us consider the evaluation for variable $|\alpha_{k,t}| \to \infty$ in the integrand of our integral formula. Because of spin condition, $j_1 + \cdots + j_{2N} = 0$, there exist $2N$ factors of type (3.39), $(2N - 1)$ factors of type (3.38), and one $g_{j_t}(\{\alpha_{j,t}\}_{j=t}^{j=-1})$, in the integrand. Using above estimates (3.41) (3.42), we know convergence of the integral representation upon condition $\xi > 1$.

4 Proof of Difference Equations

In this section we prove the integral formula (3.37) satisfies the system of difference equations (2.10), (2.11), and (2.12).

4.1 $R$-matrix Symmetry

In this section we prove $R$-matrix symmetry (2.10). Let us set

\[
\overline{G}_{2N}(\beta_1, \cdots, \beta_{2N})_{j_1, \cdots, j_{2N}} = \prod_{1 \leq s < t \leq 2N} \rho(\beta_s - \beta_t)^{-1} G_{2N}(\beta_1, \cdots, \beta_{2N})_{j_1, \cdots, j_{2N}}.
\]

(4.1)

Because of the properties of $\frac{\rho(\beta)}{\rho(-\beta)} = \kappa(\beta), \rho(i \lambda - \beta) = \rho(\beta)$, the equation (2.10) is reduced to the same equation for $\overline{G}_{2N}$ where $R$ is replaced to $\overline{R}$.

\[
\overline{G}_{2N}(\beta_1, \cdots, \beta_{s+1}, \beta_s, \cdots, \beta_{2N})_{j_1, \cdots, j_{s+1}, j_s, \cdots, j_{2N}} = \sum_{j_s' j_{s+1}' = \pm 1, 0} \overline{R}(\beta_s - \beta_{s+1})^{j_s' j_{s+1}'} \overline{G}_{2N}(\beta_1, \cdots, \beta_{s+1}, \cdots, \beta_{2N})_{j_1, \cdots, j_{s+1}', j_s', \cdots, j_{2N}}.
\]

(4.2)
• $N = 1$ Case.

First we demonstrate how to prove (2.10) in simple cases, $N = 1$. There exist 3 cases to consider. We prove (2.10) by checking every cases. Let us consider the case,

$$G_2(\beta_2, \beta_1)_{-1,1},$$

(4.3)

$$= R(\beta_1 - \beta_2)_{1, -1} G_2(\beta_1, \beta_2)_{1, -1} + R(\beta_1 - \beta_2)_{0, 0} G_2(\beta_1, \beta_2)_{0, 0} + R(\beta_1 - \beta_2)_{-1,1} G_2(\beta_1, \beta_2)_{-1,1}.$$

Considering the integrand of (LHS)-(RHS), we get

$$2^9 e^{-\frac{\pi i}{\xi}} \sh \left( \frac{\pi i}{\xi} \right) \int_C \int_C \frac{d\alpha_1}{2\pi i} \frac{d\alpha_2}{2\pi i} \left\{ \psi(\alpha_1 - \alpha_2) \varphi(\alpha_1 - \alpha_2) \prod_{s,t=1,2} \varphi(\alpha_s - \beta_t) \right\} \times h_\lambda(\alpha_1 - \alpha_2) \text{Int}(\alpha_1 \alpha_2 | \beta_1 \beta_2)_{-1,1},$$

(4.4)

where we set

$$\text{Int}(\alpha_1 \alpha_2 | \beta_1 \beta_2)_{-1,1}$$

(4.5)

$$= \sh \left( \frac{1}{2\xi}(\alpha_1 - \alpha_2 + 2\pi i) \right) \sh \left( \frac{1}{2\xi}(-\alpha_1 + \alpha_2 + \pi i) \right)$$

$$\times \left\{ \text{ish} \left( \frac{1}{2\xi}(-\alpha_1 + \beta_1 + \pi i) \right) \sh \left( \frac{1}{2\xi}(-\alpha_2 + \beta_1 + \pi i) \right) e^{-\frac{1}{2\xi}\beta_2} \right\}$$

$$\times \left\{ e^{\frac{1}{2\xi} \alpha_1} \sh \left( \frac{1}{2\xi}(\alpha_2 - \beta_2 + \pi i) \right) - e^{\frac{1}{2\xi} \alpha_2} \sh \left( \frac{1}{2\xi}(\beta_2 - \alpha_1 + \pi i) \right) \right\}$$

$$- R(\beta_1 - \beta_2)_{1, -1} \text{ish} \left( \frac{1}{2\xi}(-\alpha_1 + \beta_2 + \pi i) \right) \sh \left( \frac{1}{2\xi}(-\alpha_2 + \beta_2 + \pi i) \right) e^{-\frac{1}{2\xi}\beta_1}$$

$$\times \left\{ e^{\frac{1}{2\xi} \alpha_1} \sh \left( \frac{1}{2\xi}(\alpha_2 - \beta_1 + \pi i) \right) - e^{\frac{1}{2\xi} \alpha_2} \sh \left( \frac{1}{2\xi}(\beta_1 - \alpha_1 + \pi i) \right) \right\}$$

$$- R(\beta_1 - \beta_2)_{0, 0} \text{ish} \left( \frac{1}{2\xi}(\alpha_2 - \beta_1 + \pi i) \right) \sh \left( \frac{1}{2\xi}(-\alpha_1 + \beta_2 + \pi i) \right) e^{\frac{1}{2\xi}(\alpha_1 + \alpha_2 - \beta_1 - \beta_2)}$$

$$- R(\beta_1 - \beta_2)_{-1,1} \text{ish} \left( \frac{1}{2\xi}(\alpha_1 - \beta_1 + \pi i) \right) \sh \left( \frac{1}{2\xi}(\alpha_2 - \beta_1 + \pi i) \right) e^{-\frac{1}{2\xi}\beta_2}$$

$$\times \left\{ e^{\frac{1}{2\xi} \alpha_1} \sh \left( \frac{1}{2\xi}(\alpha_2 - \beta_2 + \pi i) \right) - e^{\frac{1}{2\xi} \alpha_2} \sh \left( \frac{1}{2\xi}(\beta_2 - \alpha_1 + \pi i) \right) \right\} \right\}.$$

(4.6)

Because of antisymmetric relation $h_\lambda(\alpha) = -h_\lambda(-\alpha)$, we get the equation (4.3) from the following relation of trigonometric function.

$$\text{Int}(\alpha_1 \alpha_2 | \beta_1 \beta_2)_{-1,1} - \text{Int}(\alpha_2 \alpha_1 | \beta_1 \beta_2)_{-1,1} = 0.$$ 

(4.7)

Other cases are shown as the same manner as the above case. We compare integrands directly.

• $N \geq 2$, General Case
Next we prove general case. There exists the following decomposition of the integral representation $G_{2N}(\beta_1 \cdots \beta_{2N})_{j_1 \cdots j_{2N}}$.

$$G_{2N}(\beta_1, \cdots, j_s, j_{s+1}, \cdots, \beta_{2N})_{j_1 \cdots j_{2N}}$$

$$= \cdots \int_C \frac{d\alpha_1'}{2\pi i} \cdots \int_C \frac{d\alpha_{s+j_{s+1}+2}'}{2\pi i} J(\alpha_1' \cdots \alpha_{s+j_{s+1}+2}'|\beta_s, \beta_{s+1})_{j_s j_{s+1}}$$

$$\times \text{Sym}(\cdots, \alpha_1' \cdots \alpha_{s+j_{s+1}+2}'|\beta_s, \beta_{s+1}, \cdots).$$

Here $J(\alpha_1' \cdots \alpha_{s+j_{s+1}+2}'|\beta_s, \beta_{s+1})_{j_s j_{s+1}}$ is the integrand of $G_{2N}(\beta_s, \beta_{s+1})_{j_s j_{s+1}}$.

$$G_{2N}(\beta_s, \beta_{s+1})_{j_s j_{s+1}} = \int_C \frac{d\alpha_1'}{2\pi i} \cdots \int_C \frac{d\alpha_{s+j_{s+1}+2}'}{2\pi i} J(\alpha_1' \cdots \alpha_{s+j_{s+1}+2}'|\beta_s, \beta_{s+1})_{j_s j_{s+1}}. $$

Here $G_{2N}(\beta_1, \beta_2)_{j_s j_{s+1}}$ is formally introduced by using the equation (3.37) for not only $j_s + j_{s+1} = 0$ but also $j_s + j_{s+1} = \pm 1, \pm 2$. The function $\text{Sym}(\cdots, \alpha_1' \cdots \alpha_{s+j_{s+1}+2}'|\cdots, \beta_s, \beta_{s+1}, \cdots)$ is symmetric with respect to $\alpha' \leftrightarrow \alpha$ and $\beta \leftrightarrow \beta_+$. Therefore $R$-matrix symmetry (4.2) is reduced to the following relations of trigonometric functions.

$$\sum_{\sigma \in S_{(j_s+j_{s+1}+2)}} J(\alpha_1'_{\sigma(1)}, \cdots, \alpha'_{\sigma(j_s+j_{s+1}+2)}|\beta_{s+1}, \beta_s)_{j_s j_{s+1}}$$

$$= \sum_{\sigma \in S_{(j_s+j_{s+1}+2)}} \sum_{j_s', j_{s+1}' = \pm 1, 0} \prod (\beta_s - \beta_{s+1})_{j_s j_{s+1}} J(\alpha_1'_{\sigma(1)}, \cdots, \alpha'_{\sigma(j_s+j_{s+1}+2)}|\beta_{s+1}, \beta_s)_{j_s' j_{s+1}'}. $$

The above equations for $j_s + j_{s+1} = 0$, have appeared in $N = 1$ case. For $j_s + j_{s+1} = \pm 1, \pm 2$, we have to do tedious checking.

### 4.2 Cyclicity Condition

Because of the relation $\rho(\beta) = \rho(i\lambda - \beta)$, cyclicity condition (2.11) is equivalent to

$$G_{2N}(\beta_1, \cdots, \beta_{2N-1}, \beta_{2N} - i\lambda)_{j_1 \cdots j_{2N}} = G_{2N}(\beta_{2N}, \beta_1, \cdots, \beta_{2N-1})_{j_{2N} j_1 \cdots j_{2N-1}}. $$

- $N = 1$ Case

First we demonstrate how to prove (2.11) in simple case, $N = 1$. Let us consider the case,

$$G_2(\beta_1, \beta_2 - i\lambda)_{-1,1} = G_2(\beta_2, \beta_1)_{1,1}. $$

In the RHS we change the variable $\alpha_1 \rightarrow \alpha_1 - i\lambda, \alpha_2 \rightarrow \alpha_2 - i\lambda$. The integrands of the LHS and the RHS coincides because of the relation,

$$\varphi(\alpha - i\lambda) \varphi(\alpha) = \frac{\text{sh} \left( \frac{i}{\pi}(\alpha - \pi i) \right)}{\text{sh} \left( \frac{i}{\pi}(\alpha + \pi i - \lambda i) \right)}. $$
We have to check that the contours for the LHS and the RHS are the same. Consider the LHS. The points \( \alpha_j = \beta_1 + \pi i \) which appear in poles of the kernel \( \varphi(\alpha_j - \beta_1) \) are actually not poles, because there exist zeros from the factor \( \frac{1}{\pi i}(-\alpha_1 + \beta_1 + \pi i) \). Therefore the integral \( \int_{C+i\lambda} d\alpha_1 \int_{C+i\lambda} d\alpha_2 \) can be deformed to \( \int_{C} d\alpha_1 \int_{C} d\alpha_2 \). The parameter condition,

\[
\lambda < 2\pi (\xi + 1),
\]

ensures that there is no poles in deforming strips of the integral. Therefore there exists no difference in integral contours between the LHS and the RHS. Other cases are proved as the same manner as this case.

- **General \( N \geq 2 \) Case**
  Let us consider general case (4.11). For simplicity we consider the case \( j_{2N} = +1 \). We make the following change of integration variables :

\[
\alpha_{0,2N} \rightarrow \alpha_{0,2N} - i\lambda, \quad \alpha_{1,2N} \rightarrow \alpha_{1,2N} - i\lambda, \quad \text{in the LHS},
\]

and

\[
\{\alpha_{j,1}\}_{j=0}^{j_{2N}} \rightarrow \{\alpha_{j,2N}\}_{j=0}^{j_{2N}}, \quad \{\alpha_{j,2}\}_{j=0}^{j_{1}} \rightarrow \{\alpha_{j,1}\}_{j=0}^{j_{1}}, \cdots, \quad \{\alpha_{j,2N-1}\}_{j=0}^{j_{2N-1}} \rightarrow \{\alpha_{j,2N-1}\}_{j=0}^{j_{2N-1}},
\]

in the RHS.

After changing variables, the difference between the LHS and the RHS appears only in the factors involving variables \( \beta_{2N}, \alpha_{0,2N}, \alpha_{1,2N} \). The factor of LHS, which involves the variables \( \beta_{2N}, \alpha_{0,2N}, \alpha_{1,2N} \) is

\[
g_{j_{2N}}(\{\alpha_{j,2N}\}_{j=\pm 1,0}) \prod_{s=1}^{2N-1} \prod_{j=0}^{j_s} J_\lambda(\alpha_{j,s} - \beta_{2N} + i\lambda) \prod_{j=0,1} J_\lambda(\beta_{2N} - \alpha_{j,2N}) I_\lambda(\alpha_{0,2N} - \alpha_{1,2N})
\]

\[
\times \prod_{s=1}^{2N-1} \prod_{j=0,1} J_\lambda(\beta_{s} - \alpha_{j,2N} + i\lambda) \prod_{s=1}^{2N-1} \prod_{j=0,1} I_\lambda(\alpha_{j,s} - \alpha_{j',2N} + i\lambda),
\]

Those of RHS is

\[
g_{j_{2N}}(\{\alpha_{j,2N}\}_{j=\pm 1,0}) \prod_{s=1}^{2N-1} \prod_{j=0}^{j_s} J_\lambda(\beta_{2N} - \alpha_{j,s}) \prod_{j=0,1} J_\lambda(\beta_{2N} - \alpha_{j,2N}) I_\lambda(\alpha_{0,2N} - \alpha_{1,2N})
\]

\[
\times \prod_{s=1}^{2N-1} \prod_{j=0,1} J_\lambda(\alpha_{j,2N} - \beta_{s}) \prod_{s=1}^{2N-1} \prod_{j=0,1} I_\lambda(\alpha_{j',2N} - \alpha_{j,s}).
\]
Here we have used $I_\lambda(\alpha), J_\lambda(\alpha)$ in (3.38), (3.39). Using the following property,

$$I_\lambda(\alpha) = I_\lambda(i\lambda - \alpha), \quad J_\lambda(\alpha) = J_\lambda(i\lambda - \alpha), \quad (4.19)$$

we know that (4.17) and (4.18) are the same. We have shown that the integrands of the LHS and the RHS are the same. We also have to check that the contours for the LHS and the RHS are the same. Consider the LHS. The points $\alpha_{j,2N} = \beta_s + \pi i, (j = 0, +1; s = 1, \ldots, 2N - 1)$ which appear in poles of the kernel $\varphi(\alpha_{j,2N} - \beta_s)$ are actually not poles, because there exist zeros from the factor $\prod_{s=1}^{2N-1} \prod_{j=0,+1} \text{sh} \left( \frac{1}{2\xi}(-\alpha_{j,2N} + \beta_s + \pi i) \right)$. Therefore the integral $\int_{C+i\lambda} da_{0,2N} \int_{C+i\lambda} da_{1,2N}$ can be deformed to $\int_{C} da_{0,2N} \int_{C} da_{1,2N}$. The parameter condition $\lambda < 2\pi(\xi + 1)$ ensures that there is no poles in deforming strips of the integral. We have proved the equation (4.11) for $j_{2N} = +1$. For $j_{2N} = 0, -1$ case we can show the relation (4.11) as the same manner as the case $j_{2N} = +1$.

### 4.3 Recursion Relation

It can be shown that Recursion relation (2.12) are consequence of $R$-matrix symmetry (2.10), unitarity (2.6), crossing symmetry (2.7), and

$$G_{2N}(\beta_1, \ldots, \beta_{2N-2}, \beta, \beta + 3\pi i)_{j_1, \ldots, j_{2N-2}, \beta, -j} = C_jG_{2N-2}(\beta_1, \ldots, \beta_{2N-2})_{j_1, \ldots, j_{2N-2}}. \quad (4.20)$$

We shall prove (4.20). Consider the integral formula (3.37). The factor $\rho(\beta_{2N-1} - \beta_{2N})$ has zero at $\beta_{2N} = \beta_{2N-1} + 3\pi i$. We see that

$$\rho(\beta_{2N-1} - \beta_{2N}) = \rho(3\pi i) \frac{i}{4\xi} \left( \cos \left( \frac{\pi}{2\xi} \right) \sin \left( \frac{3\pi}{2\xi} \right) \right)^{-1} (\beta_{2N} - \beta_{2N-1} - 3\pi i) + \cdots. \quad (4.21)$$

The integral may have a pole at $\beta_{2N} = \beta_{2N-1} + 3\pi i$ because the integral contour $(-\infty, \infty)$ is pinched by the pole of the kernel function $\Psi(\{\alpha\}|\{\beta\})$. We explain this procedure using simple example. For regular function $f(\alpha_1, \alpha_2)$, the following estimate holds.

$$\int_{C_{1}} \frac{d\alpha_1}{2\pi i} \int_{C_{2}} \frac{d\alpha_2}{2\pi i} \frac{f(\alpha_1, \alpha_2)}{(\beta_1 - \alpha_1 + \pi i)(\alpha_1 - \alpha_2 + \pi i)(\alpha_2 - \beta_2 + \pi i)} \frac{1}{(\beta_1 + \pi i, \beta_1 + 2\pi i)(\beta_1 - \beta_2 + 3\pi i)} + \text{Regular}, \quad (\beta_2 \rightarrow \beta_1 + 3\pi i). \quad (4.22)$$

Here the contour $C_1$ is taken along a path from $-\infty$ to $\infty$ in such a way that $-\pi < \text{Im}(\alpha_2 - \beta_2)$, $\text{Im}(\alpha_1 - \alpha_2) < \pi$, and the contour $C_2$ is taken along a path from $-\infty$ to $\infty$ in such a way that $-\pi < \text{Im}(\alpha_1 - \alpha_2), \text{Im}(\beta_1 - \alpha_1) < \pi$. We will check this procedure for our considering case, and calculate the residue of $\overline{G}_{2N}$.
• \( N = 1 \) Case

First we demonstrate simple case, \( N = 1 \).

\[
G_2(\beta_1, \beta_2)_{j,-j}, \quad \beta_2 \rightarrow \beta_1 + 3\pi i.
\]  

(4.23)

The integral \( \int_C d\alpha_1 \) is pinched by the pole of \( \varphi(\alpha_1 - \beta_1) \), and the integral \( \int_C d\alpha_2 \) is pinched by the pole of \( \varphi(\alpha_2 - \alpha_1) \). Because \( S_2(\omega_1|\omega_1\omega_2) = \sqrt{\varphi(\alpha_1 - \alpha_2)\psi(\alpha_1 - \alpha_2)} \prod_{s,t=1,2} \varphi(\alpha_s - \beta_t) \) behaves for \( \beta_1 \rightarrow \alpha_1 + \pi i, \alpha_1 \rightarrow \alpha_2 + 2\pi i, \alpha_2 \rightarrow \beta_2 + \pi i \).

\[
\varphi(2\pi i)^2\psi(\pi i) \left( \frac{\sqrt{\frac{\lambda}{\pi}}}{iS_2(2\pi|\lambda,2\pi\xi)} \right)^3 \frac{-1}{(\beta_1 - \alpha_1 + \pi i)(\alpha_1 - \alpha_2 + \pi i)(\alpha_2 - \beta_2 + \pi i)} + \cdots. \tag{4.24}
\]

Note that

\[
\varphi(2\pi i)\psi(\pi i) h_{\lambda}(\pi i) = \left( 2^6 \varphi(0) \sh{\pi i \over 2\xi} \sh{3\pi i \over 2\xi} \right)^{-1}. \tag{4.25}
\]

We know the following is independent of the spectral parameter \( \beta \).

\[
G_2(\beta, \beta + 3\pi i)_{j,-j} = C_j, \quad (j = \pm 1, 0), \tag{4.26}
\]

where

\[
C_1 = \frac{\sqrt{2e^{-\pi i}}}{\xi} \sqrt{\frac{\lambda \xi^3}{\pi}} \sh{\pi i \over \xi} \rho(3\pi i) \varphi(2\pi i) \tag{4.27}
\]

\[
C_0 = \frac{\sqrt{2e^{-\pi i}}}{i\xi} \sqrt{\frac{\lambda \xi^3}{\pi}} \sh{\pi i \over 2\xi} \rho(3\pi i) \varphi(2\pi i) \tag{4.28}
\]

\[
C_{-1} = \frac{\sqrt{2e^{-3\pi i}}}{\xi} \sqrt{\frac{\lambda \xi^3}{\pi}} \sh{\pi i \over \xi} \rho(3\pi i) \varphi(2\pi i) \tag{4.29}
\]

• General \( N \geq 2 \) Case

We show that general \( N \geq 2 \) case is reduced to \( N = 1 \) case. Consider the integrand of the following integral formula.

\[
G_{2N}(\beta_1, \cdots, \beta_{2N-2}, \beta_1', \beta_2')_{j_1 \cdots j_{2N-2}, j,-j}. \tag{4.30}
\]

Let us set integral variables,

\[
\{\alpha_1', \alpha_2'\} = \{\{\alpha_k,2N-1\}_{k=0}^j, \{\alpha_k,2N\}_{k=0}^{-j}\}.
\]

The factor interacting both \( \alpha_{j,s},(s = 1, \cdots, 2N - 2; j = \pm 1, 0) \) and \( \beta_1', \beta_2', \alpha_1', \alpha_2' \) is given by

\[
\prod_{s=1}^{2N-2} \rho(\beta_s - \beta_1')\rho(\beta_s - \beta_2')J_{\lambda}(\beta_s - \alpha_1')J_{\lambda}(\beta_s - \alpha_2')
\]

\[
\times \prod_{s=1}^{2N-2} \prod_{j_s} J_{\lambda}(\alpha_{j,s} - \beta_1')J_{\lambda}(\alpha_{j,s} - \beta_2')I_{\lambda}(\alpha_{j,s} - \alpha_1')I_{\lambda}(\alpha_{j,s} - \alpha_2'). \tag{4.31}
\]
The above factor (4.31) simplifies for \( \beta' \to \alpha' + \pi i, \alpha' \to \alpha' + \pi i, \alpha' \to \beta' + \pi i, \)

\[(4.31) \to 2^{-28(N-1)}, \tag{4.32}\]

because of the following relations.

\[
\rho(\beta) \rho(\beta - 3\pi i) J_\lambda(\beta - \pi i) J_\lambda(\beta - 2\pi i) = 2^{-2}, \tag{4.33}
\]

\[
J_\lambda(\beta) J_\lambda(\beta - 3\pi i) I_\lambda(\beta - \pi i) I_\lambda(\beta - 2\pi i) = 2^{-12}. \tag{4.34}
\]

This simplification teaches us that general \( N \geq 2 \) case is reduced to the simplest \( N = 1 \) case.

We have proved recursion relation (2.12).

5 Supporting Argument

The purpose of this section is to give a supporting argument on physical meaning of our integral representation. Let us consider the integral formula for \( \lambda = 3\pi \). In this case following simplifications occur.

\[
\rho(\beta) = \frac{S_2(-i\beta + 2\pi|6\pi, 2\pi \xi)S_2(-i\beta + 3\pi|6\pi, 2\pi \xi)}{S_2(-i\beta + 2\pi \xi|6\pi, 2\pi \xi)S_2(-i\beta + 2\pi \xi + \pi|6\pi, 2\pi \xi)}, \tag{5.1}
\]

and

\[
\psi(\alpha) = \varphi(\alpha) \times \frac{1}{4 \text{sh} \left( \frac{1}{2\pi}(-\alpha - 2\pi i) \right) \text{sh} \left( \frac{1}{2\pi}(\alpha - 2\pi i) \right)}. \tag{5.2}
\]

For the case \( \lambda = 3\pi \), our integral formula (3.37) gives a conjectural formula for the Izergin-Korepin model [1]. Hou et al. [2] considered the correlation functions for the Izergin-Korepin model for massive regime \(-1 < q < 0\), within the framework of representation theory of \( U_q(A_2^{(2)}) \). They gave free field realizations of the vertex operators, and realized integral representation of the correlation functions, as the trace of the vertex operators. In the limiting case, our integral representation reproduce the correlation function for the Izergin-Korepin model at critical point \( q = -1 \), which was derived by Hou et al. [1, 2]. This gives a supporting argument that our integral representation (3.37) gives a conjectural formula of the correlation function of the Izergin-Korepin model at massless regime \( |q| = 1 \).

In the special case where \( \lambda = 3\pi \) and \( \xi \to \infty \), the auxiliary functions \( g_k(\{\alpha_j\}_{j=-1}^k), (k = \pm 1, 0) \) tend to \( \tilde{g}_k(\{\alpha_j\}_{j=-1}^k), (k = \pm 1, 0) \) given by

\[
\tilde{g}_{-1}(\{\alpha_{-1}\}) = 1, \tag{5.3}
\]

\[
\tilde{g}_0(\{\alpha_{-1}, \alpha_0\}) = \frac{2\pi i}{\alpha_0 - \alpha_{-1} + \pi i}, \tag{5.4}
\]

\[
\tilde{g}_1(\{\alpha_{-1}, \alpha_0, \alpha_1\}) = \frac{2\pi(2\alpha_{-1} - \alpha_0 - \alpha_1)}{(\alpha_0 - \alpha_{-1} + \pi i)(\alpha_1 - \alpha_{-1} + \pi i)}. \tag{5.5}
\]
The integral representation $G_{2N}$, (3.37) becomes as follows.

\[
\prod_{1 \leq s < t \leq 2N} \Gamma \left( \frac{i(\beta_s - \beta_t)}{6\pi} + 1 \right) \Gamma \left( \frac{i(\beta_t - \beta_t)}{6\pi} + \frac{5}{3} \right) \Gamma \left( \frac{i(-\beta_s + \beta_t)}{6\pi} + \frac{1}{2} \right) \Gamma \left( \frac{i(-\beta_s + \beta_t)}{6\pi} + \frac{1}{2} \right) \times 2N \prod_{j,s=1}^{2N} \int_{-\infty}^{\infty} \frac{d\alpha_{j,s}}{2\pi i} \Phi(\{\alpha\}|\{\beta\}) \prod_{s=1}^{2N} \tilde{g}_{j,s}(\{\alpha_{j,s}\}_{j=-1})
\]

\[
\times \prod_{s=1}^{2N} \prod_{j=0,1}^{N} \prod_{t \in A_j, s \leq t} (-\beta_s + \alpha_{j,t} + \pi i) \prod_{t \in A_j, t \leq s} (-\alpha_{j,t} + \beta_s + \pi i)
\]

\[
\times \prod_{j=0,1}^{N} \prod_{s \in A_j, t \in A_j, s \leq t} \frac{\alpha_{j,s} - \alpha_{j,t} - \pi i}{\alpha_{j,s} - \alpha_{j,t} - 2\pi i} \prod_{s \in A_0, t \in A_1, s \leq t} \frac{\alpha_{0,s} - \alpha_{1,t} - \pi i}{\alpha_{0,s} - \alpha_{1,t} - 2\pi i} \prod_{t \in A_0, s \in A_1, t \leq s} \frac{\alpha_{1,t} - \alpha_{0,s} - \pi i}{\alpha_{1,t} - \alpha_{0,s} - 2\pi i}
\]

\[
\times \prod_{j=0,1}^{N} \prod_{s \in A_j, t \in A_j, s \leq t} \text{sh} \left( \frac{1}{3}(\alpha_{j,s} - \alpha_{j,t}) \right) \prod_{s \in A_0, t \in A_1, s \leq t} \text{sh} \left( \frac{1}{3}(\alpha_{0,s} - \alpha_{1,t}) \right) \prod_{t \in A_0, s \in A_1, t \leq s} \text{sh} \left( \frac{1}{3}(\alpha_{1,t} - \alpha_{0,s}) \right).
\]

(5.6)

Here we have set the kernel function $\tilde{\Phi}(\{\alpha\}|\{\beta\})$ by

\[
\tilde{\Phi}(\{\alpha\}|\{\beta\}) = \prod_{s=1}^{2N} \prod_{j=0,1}^{N} \prod_{t \in A_j, s \leq t} \tilde{\varphi}(\beta_s - \alpha_{j,t}) \prod_{s \in A_0, t \in A_1, s \leq t} \tilde{\varphi}(\alpha_{0,s} - \alpha_{1,t}) \tilde{\varphi}(\alpha_{0,s} - \alpha_{1,t})
\]

\[
\times \prod_{j=0,1}^{N} \prod_{s \in A_j, t \in A_j, s \leq t} \tilde{\varphi}(\alpha_{j,s} - \alpha_{j,t}) \tilde{\varphi}(\alpha_{j,s} - \alpha_{j,t}),
\]

(5.7)

where

\[
\varphi(\alpha) = \Gamma \left( \frac{i\alpha}{3\pi} + \frac{1}{3} \right) \Gamma \left( -\frac{i\alpha}{3\pi} + \frac{1}{3} \right),
\]

\[
\tilde{\varphi}(\alpha) = \frac{1}{\Gamma \left( \frac{i\alpha}{3\pi} + \frac{2}{3} \right) \Gamma \left( -\frac{i\alpha}{3\pi} + \frac{2}{3} \right)}.
\]

(5.8)

In the limiting case, our integral representation reproduce those of the correlation function at critical point $q = -1$ \cite{1}.

Recently Kitanine, Maillet, Terras \cite{13} derived integral representation of the correlation function for the six-vertex model ($A_1^{(1)}$ symmetry) by means of the quantum inverse scattering method. Their method is available for both massive and massless regime. To construct $A_2^{(2)}$ analogue of the paper \cite{13} and reproduce the integral representation (3.37) is interesting problem.

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