QUANTIZED NONPARAMETRIC ESTIMATION

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We present an extension to Pinsker’s theorem for nonparametric estimation over Sobolev ellipsoids when estimation is carried out under storage or communication constraints. Placing limits on the number of bits used to encode any estimator, we give tight lower and upper bounds on the excess risk due to quantization in terms of the number of bits, the signal size, and the noise level. This establishes the Pareto optimal minimax tradeoff between storage and risk under quantization constraints for Sobolev spaces. Our results and proof techniques combine elements of rate distortion theory and minimax analysis.

1. Introduction. In this paper we introduce a minimax framework for nonparametric estimation under storage constraints. In the classical statistical setting, the minimax risk for estimating a function \( f \) from a function class \( F \) using a sample of size \( n \) places no constraints on the estimator \( \hat{f}_n \), other than requiring it to be a measurable function of the data. However, if the estimator is to be constructed with restrictions on the computational resources used, it is of interest to understand how the error can degrade. Letting \( C(\hat{f}_n) \leq B_n \) indicate that the computational resources \( C(\hat{f}_n) \) used to construct \( \hat{f}_n \) are required to fall within a budget \( B_n \), the constrained minimax risk is

\[
R_n(F, B_n) = \inf_{\hat{f}_n: C(\hat{f}_n) \leq B_n} \sup_{f \in F} R(\hat{f}_n, f).
\]

Minimax lower bounds on the risk as a function of the computational budget thus determine a feasible region for computation constrained estimation, and a Pareto optimal tradeoff for risk versus computation as \( B_n \) varies.

In this paper we treat the case where the complexity \( C(\hat{f}_n) \) is measured by the storage or space used by the procedure. Specifically, we limit the number of bits used to represent the estimator \( \hat{f}_n \). We focus on the setting of nonparametric regression under standard smoothness assumptions, and study how the excess risk depends on the storage budget \( B_n \).

Our formulation of this problem is naturally motivated by certain applications. For instance, when data are collected and analyzed on board a remote

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satellite, the estimated values may need to be sent back to Earth for further analysis. To limit communication costs, the estimates can be quantized, and it becomes important to understand what, in principle, is lost in terms of statistical risk through quantization. A related scenario is a cloud computing environment where data are processed for many different statistical estimation problems, with the estimates then stored for future analysis. To limit the storage costs, which could dominate the compute costs in many scenarios, it is of interest to quantize the estimates, and the quantization-risk tradeoff again becomes an important concern. To impose energy constraints on computation, future processors may limit precision in arithmetic computations [9]; the cost of limited precision in terms of statistical risk must then also be accounted for.

With such applications as motivation, we address in this paper the problem of risk-storage tradeoffs in the normal means model of nonparametric estimation assuming the target function lies in a Sobolev space. As formulated in this paper, the problem is intimately related to classical rate distortion theory [10], and our results rely on a marriage of minimax theory and rate distortion ideas. We thus build on and refine the fundamental connection between function estimation and lossy source coding that was elucidated in Donoho’s 1998 Wald Lectures [7].

We work in the standard Gaussian white noise model
\[ dX(t) = f(t)dt + \varepsilon dW(t), \quad 0 \leq t \leq 1, \]
where \( W \) is a standard Wiener process on \([0, 1]\), \( \varepsilon \) is the standard deviation of the noise, and \( f \) lies in the periodic Sobolev space \( \tilde{W}(m, c) \) of order \( m \) and radius \( c \). In this classical setting, the minimax risk of estimation
\[ R_\varepsilon(m, c) = \inf_{\hat{f}_\varepsilon} \sup_{f \in \tilde{W}(m, c)} \mathbb{E} \| f - \hat{f}_\varepsilon \|^2 \]
is well known to satisfy
\[ \lim_{\varepsilon \to 0} \varepsilon^{-\frac{4m}{2m+1}} R_\varepsilon(m, c) = \left( \frac{c^2(2m+1)}{\pi^{2m}} \right)^{\frac{1}{2m+1}} \left( \frac{m}{m+1} \right)^{\frac{2m}{2m+1}} \triangleq P_{m,c} \]
where \( P_{m,c} \) is Pinsker’s constant. The constrained minimax risk for quantized estimation becomes
\[ R_\varepsilon(m, c, B_\varepsilon) = \inf_{\hat{f}_\varepsilon, C(\hat{f}_\varepsilon) \leq B_\varepsilon} \sup_{f \in \tilde{W}(m, c)} \mathbb{E} \| f - \hat{f}_\varepsilon \|^2 \]
where \( \hat{f}_\varepsilon \) is a quantized estimator that is required to use storage \( C(\hat{f}_\varepsilon) \) no greater than \( B_\varepsilon \) bits in total. Our main result identifies three separate quantization regimes.
• In the over-sufficient regime, the number of bits is very large, satisfying 
  \( B_\varepsilon \gg \varepsilon^{-\frac{2}{2m+1}} \) and the classical minimax rate of convergence 
  \( R_\varepsilon \asymp \varepsilon^{\frac{4m}{2m+1}} \) obtains. Moreover, the optimal constant is the Pinsker constant 
  \( P_{m,c} \).

• In the sufficient regime, the number of bits scales as 
  \( B_\varepsilon \asymp \varepsilon^{-\frac{2}{2m+1}} \). This level of quantization is just sufficient to preserve the classical 
  minimax rate of convergence, and thus in this regime \( R_\varepsilon(m,c,B_\varepsilon) \asymp \varepsilon^{\frac{4m}{2m+1}} \). However, the optimal constant degrades to a new constant 
  \( P_{m,c} + Q_{m,c,d} \), where \( Q_{m,c,d} \) is characterized in terms of the solution of 
  a certain variational problem, depending on \( d = \lim_{\varepsilon \to 0} B_\varepsilon \varepsilon^{\frac{2}{2m+1}} \).

• In the insufficient regime, the number of bits scales as 
  \( B_\varepsilon \ll \varepsilon^{-\frac{2}{2m+1}} \), with however \( B_\varepsilon \to \infty \). Under this scaling the number of bits is insufficient to preserve the unquantized minimax rate of convergence, and 
  the quantization error dominates the estimation error. We show that the quantized minimax risk in this case satisfies

  \[
  \lim_{\varepsilon \to 0} B_\varepsilon^{2m} R_\varepsilon(m,c,B_\varepsilon) = \frac{c^2 m^{2m}}{\pi^{2m}}.
  \]

  Thus, in the insufficient regime the quantized minimax rate of convergence is \( B_\varepsilon^{-2m} \), with optimal constant as shown above.

By using an upper bound for the family of constants \( Q_{m,c,d} \), the three regimes can be combined together to view the risk in terms of a decomposition into estimation error and quantization error. Specifically, we can write

\[
R_\varepsilon(m,c,B_\varepsilon) \approx P_{m,c} \varepsilon^{\frac{4m}{2m+1}} B_\varepsilon^{-2m} + \frac{c^2 m^{2m}}{\pi^{2m}} B_\varepsilon^{2m}.
\]

When \( B_\varepsilon \gg \varepsilon^{-\frac{2}{2m+1}} \), the estimation error dominates the quantization error, and the usual minimax rate and constant are obtained. In the insufficient case \( B_\varepsilon \ll \varepsilon^{-\frac{2}{2m+1}} \), only a slower rate of convergence is achievable. When \( B_\varepsilon \) and \( \varepsilon^{-\frac{2}{2m+1}} \) are comparable, the estimation error and quantization error are on the same order. The threshold \( \varepsilon^{-\frac{2}{2m+1}} \) should not be surprising, given that in classical unquantized estimation the optimal risk is achieved by keeping the first \( \varepsilon^{-\frac{2}{2m+1}} \) random variables and simply estimating the remaining coefficients to be zero. This corresponds to selecting a smoothing bandwidth that scales as \( h \asymp n^{-\frac{1}{2m+1}} \) with the sample size \( n \).
At a high level, our proof strategy integrates elements of minimax theory and source coding theory. In minimax analysis one computes lower bounds by thinking in Bayesian terms to look for least-favorable priors. In source coding analysis one constructs worst case distributions by setting up an optimization problem based on mutual information. Our quantized minimax analysis requires that these approaches be carefully combined to balance the estimation and quantization errors. To show achievability of the lower bounds we establish, we likewise need to construct an estimator and coding scheme together. Our approach is to quantize the blockwise James-Stein estimator, which achieves the classical Pinsker bound. However, our quantization scheme differs from the approach taken in classical rate distortion theory, where the generation of the codebook is determined once the source distribution is known. In our setting, we require the allocation of bits to be adaptive to the data, using more bits for blocks that have larger signal size. We therefore design a quantized estimation procedure that adaptively distributes the communication budget across the blocks. Assuming only a lower $m_0$ on the smoothness $m$ and an upper bound $c_0$ on the radius $c$ of the Sobolev space, our quantization-estimation procedure is adaptive to $m$ and $c$ in the standard statistical sense, and is also adaptive to the coding regime. In other words, given a storage budget $B_\epsilon$, the coding procedure achieves the optimal rate and constant for the unknown $m$ and $c$, operating in the corresponding regime for those parameters.

In the following section we formulate the general problem of quantized estimation, and present several examples. In Section 3 we state our main result, which establishes the quantized minimax lower bounds for the case of Sobolev ellipsoids, identifying the three distinct quantization regimes. We also given an outline of the proof of the lower bounds in this section. In Section 4 we show asymptotic achievability of these lower bounds, using a quantized estimation procedure based on adaptive James-Stein estimation and quantization in blocks. The detailed proofs of all the results are given in the supplementary material.

2. Quantized estimation and minimax risk. We now describe the general setting of quantized estimation and a framework for obtaining the quantized minimax risk. Suppose that $(X_1, \ldots, X_n) \in \mathcal{X}^n$ is a random vector drawn from a distribution $P_n$. We consider the problem of estimating a functional $\theta_n = \theta(P_n)$ of the distribution, assuming $\theta_n$ is restricted to lie in a parameter space $\Theta_n$. To unclutter some of the notation, we will suppress the subscript $n$ and write $\theta$ and $\Theta$ in the following, keeping in mind that nonparametric settings are allowed. The subscript $n$ will be maintained for
random variables. The minimax $\ell_2$ risk of estimating $\theta$ is then defined as

$$R_n(\Theta) = \inf_{\hat{\theta}_n, \theta \in \Theta} \sup_{\theta \in \Theta} \mathbb{E}_\theta \| \theta - \hat{\theta}_n \|^2$$

where the infimum is taken over all possible estimators $\hat{\theta}_n : \mathcal{X}^n \to \Theta$ that are measurable with respect to the data $X_1, \ldots, X_n$. We will abuse notation by using $\hat{\theta}_n$ to denote both the estimator and the estimate calculated based on an observed set of data. Among numerous approaches to obtaining the minimax risk, the Bayesian method is best aligned with quantized estimation. Consider a prior distribution $\pi(\theta)$ whose support is a subset of $\Theta$. Let $\delta(X_{1:n})$ be the posterior mean of $\theta$ given the data $X_1, \ldots, X_n$, which minimizes the integrated risk. Then for any estimator $\hat{\theta}_n$,

$$\sup_{\theta \in \Theta} \mathbb{E}_\theta \| \theta - \hat{\theta}_n \|^2 \geq \int_{\Theta} \mathbb{E}_\theta \| \theta - \hat{\theta}_n \|^2 d\pi(\theta) \geq \int_{\Theta} \mathbb{E}_\theta \| \theta - \delta(X_{1:n}) \|^2 d\pi(\theta).$$

Taking the infimum over $\hat{\theta}_n$ yields

$$\inf_{\hat{\theta}_n, \theta \in \Theta} \sup_{\theta \in \Theta} \mathbb{E}_\theta \| \theta - \hat{\theta}_n \|^2 \geq \int_{\Theta} \mathbb{E}_\theta \| \theta - \delta(X_{1:n}) \|^2 d\pi(\theta) \triangleq R_n(\Theta; \pi).$$

Thus, any prior distribution supported on $\Theta$ gives a lower bound on the minimax risk, and selecting the least-favorable prior leads to the largest lower bound provable by this approach.

Now consider the scenario where we have constraints on the storage or communication cost of our estimate. We restrict to the set of estimators that use no more than a total of $B_n$ bits; that is, the estimator takes at most $2^{B_n}$ different values. Such quantized estimators can be formulated by the following two-step procedure. First, an encoder maps the data $X_{1:n}$ to an index $\phi_n(X_{1:n})$, where

$$\phi_n : \mathcal{X}^n \to \{1, 2, \ldots, 2^{B_n}\}$$

is the encoding function. The decoder, after receiving or retrieving the index, represents the estimates based on a decoding function

$$\psi_n : \{1, 2, \ldots, 2^{B_n}\} \to \Theta,$$

mapping the index to a codebook of estimates. All that needs to be transmitted or stored is the $B_n$-bit-long index, and the quantized estimator $\hat{\theta}_n$ is simply $\psi_n \circ \phi_n$, the composition of the encoder and the decoder functions.
Denoting by $C(\hat{\theta}_n)$ the storage, in terms of the number of bits, required by an estimator $\hat{\theta}_n$, the minimax risk of quantized estimation is then defined as

$$R_n(\Theta, B_n) = \inf_{\hat{\theta}_n, C(\hat{\theta}_n) \leq B_n} \sup_{\theta \in \Theta} \mathbb{E}_\theta \| \theta - \hat{\theta}_n \|^2,$$

and we are interested in the effect of the constraint on the minimax risk. Once again, we consider a prior distribution $\pi(\theta)$ supported on $\Theta$ and let $\delta(X_1:n)$ be the posterior mean of $\theta$ given the data. The integrated risk can then be decomposed as

$$(2.1) \quad \int_\Theta \mathbb{E}_\theta \| \theta - \hat{\theta}_n \|^2 d\pi(\theta) = \mathbb{E} \| \theta - \delta(X_1:n) \|^2 + \mathbb{E} \| \delta(X_1:n) - \hat{\theta}_n \|^2$$

where the expectation is with respect to the joint distribution of $\theta \sim \pi(\theta)$ and $X_1:n | \theta \sim P_\theta$, and the second equality is due to

$$\mathbb{E} \langle \theta - \delta(X_1:n), \delta(X_1:n) - \hat{\theta}_n \rangle = \mathbb{E} \langle \mathbb{E}(\theta - \delta(X_1:n) | X_1:n), \mathbb{E} (\delta(X_1:n) - \hat{\theta}_n | X_1:n) \rangle = 0,$$

using the fact that $\theta \rightarrow X_1:n \rightarrow \hat{\theta}_n$ forms a Markov chain. The first term in the decomposition (2.1) is the Bayes risk $R_n(\Theta, \pi)$. The second term can be viewed as the excess risk due to quantization.

Let $T_n = T(X_1, \ldots, X_n)$ be a sufficient statistic for $\theta$. The posterior mean can be expressed in terms of $T_n$ and we will abuse notation and write it as $\delta(T_n)$. Since the quantized estimator $\hat{\theta}_n$ uses at most $B_n$ bits, we have

$$B_n \geq H(\hat{\theta}_n) \geq H(\hat{\theta}_n | \delta(T_n)) = I(\hat{\theta}_n; \delta(T_n)),$$

where $H$ and $I$ denote the Shannon entropy and mutual information, respectively. Now consider the optimization

$$\inf_{P(\cdot | \delta(T_n))} \mathbb{E} \| \delta(T_n) - \hat{\theta}_n \|^2$$

such that $I(\hat{\theta}_n; \delta(T_n)) \leq B_n$

where the infimum is over all conditional distributions $P(\hat{\theta}_n | \delta(T_n))$. This parallels the definition of the rate distortion function, minimizing the distortion under a constraint on mutual information [10]. Denoting the value
of this optimization by $Q_n(\Theta, B_n; \pi)$, we can lower bound the quantized minimax risk by

$$R_n(\Theta, B_n) \geq R_n(\Theta; \pi) + Q_n(\Theta, B_n; \pi).$$

Since each prior distribution $\pi(\theta)$ supported on $\Theta$ gives a lower bound, we have

$$R_n(\Theta, B_n) \geq \sup_{\pi} \{ R_n(\Theta; \pi) + Q_n(\Theta, B_n; \pi) \}$$

and the goal becomes to obtain a least favorable prior for the quantized risk.

Before turning to the case of quantized estimation over Sobolev spaces, we illustrate this technique on some simpler, more concrete examples.

**Example 2.1 (Normal means in a hypercube).** Let $X_i \sim \mathcal{N}(\theta, \sigma^2 I_d)$ for $i = 1, 2, \ldots, n$. Suppose that $\sigma^2$ is known and $\theta \in [-\tau, \tau]^d$ is to be estimated. We choose the prior $\pi(\theta)$ on $\theta$ to be a product distribution with density

$$p(\theta) = \prod_{j=1}^{d} \frac{3}{2\tau^3} \left( \tau - |\theta_j| \right)^2_+.$$

Then it can be shown [13] that

$$R_n(\Theta; \pi) \geq \frac{\sigma^2 d}{n} \frac{\tau^2}{\tau^2 + 12\sigma^2/n} \geq c_1 \frac{\sigma^2 d}{n}$$

for some constant $c_1$ independent of $n$ and $d$. Turning to $Q_n(\Theta, B_n; \pi)$, there exist a constant $c_2$ independent of $n$, $d$ and $B_n$ such that

$$Q_n(\Theta, B_n; \pi) \geq c_2 d 2^{-\frac{2B_n}{d}}.$$

Combining the two terms, we obtain that for estimating means in a $d$-dimensional hypercube,

$$R_n(\Theta, B_n) \geq c_1 \frac{\sigma^2 d}{n} + c_2 d 2^{-\frac{2B_n}{d}}.$$

This lower bound intuitively shows the risk is regulated by two factors, the estimation error and the quantization error; whichever is larger dominates the risk. The scaling behavior of this lower bound (ignoring constants) can be achieved by first quantizing each of the $d$ intervals $[-\tau, \tau]$ using $B_n/d$ bits each, and then mapping the MLE to its closest codeword.
Example 2.2 (Binomial). Let $X_i \sim \text{Bern}(\theta)$ be independent samples from a Bernoulli distribution, for $i = 1, 2, \ldots, n$, and take $\pi(\theta) = 1$ to be the uniform prior on $[0, 1]$. Then $T_n = \bar{X}_n$ and
\[
\delta(T_n) = \frac{1}{n + 2} \bar{X}_n + \frac{2}{n + 1} \cdot \frac{1}{2}
\]
with $I(\tilde{\theta}_n, \delta(T_n)) = I(\tilde{\theta}_n, \bar{X}_n)$. In this case it can be shown that
\[
R_n(\Theta, B_n) \geq c_1 n + c_2 H^{-1} \left( 1 - \frac{B_n}{n} \right)
\]
for constants $c_1$ and $c_2$, where $H^{-1}$ is the inverse of the binary entropy function on $[0, \frac{1}{2}]$.

Example 2.3 (Gaussian sequences in Euclidean balls). In the example shown above, the lower bound is tight only in terms of the scaling of the key parameters. In some instances, we are able to find an asymptotically tight lower bound for which we can show achievability of both the rate and the constants. Estimating the mean vector of a Gaussian sequence with an $\ell_2$ norm constraint on the mean is one of such case, as we showed in previous work [23].

Specifically, let $X_i \sim \mathcal{N}(\theta_i, \sigma_i^2)$ for $i = 1, 2, \ldots, n$, where $\sigma_i^2 = \sigma^2/n$. Suppose that the parameter $\theta = (\theta_1, \ldots, \theta_n)$ lies in the Euclidean ball $\Theta_n(c) = \{ \theta : \sum_{i=1}^{n} \theta_i^2 \leq c^2 \}$. Furthermore, suppose that $B_n = nB$. Then using the prior $\theta_i \sim \mathcal{N}(0, c^2)$ it can be shown that
\[
\liminf_{n \to \infty} R_n(\Theta_n(c), B_n) \geq \frac{\sigma^2 c^2}{\sigma^2 + c^2} + \frac{c^4 2^{-2B}}{\sigma^2 + c^2}.
\]
The asymptotic estimation error $\sigma^2 c^2/(\sigma^2 + c^2)$ is the well-known Pinsker bound for the Euclidean ball case. As shown in [23], an explicit quantization scheme can be constructed that asymptotically achieves this lower bound, realizing the smallest possible quantization error $c^4 2^{-2B}/(\sigma^2 + c^2)$ for a budget of $B_n = nB$ bits.

The Euclidean ball case is clearly relevant to the Sobolev ellipsoid case. In classical adaptive estimation over Sobolev ellipsoids, the sequence is blocked, and the James-Stein estimator is applied within each block [18]. However, the quantized estimation framework just described does not suffice for the Sobolev case; new coding strategies and proof techniques are required. In particular, as will be made clear in the sequel, an adaptive allocation of bits is required across blocks, using more bits for blocks that have larger signal
Moreover, determination of the optimal constants requires a detailed analysis of the worst case prior distributions and the solution of a series of variational problems.

3. Quantized estimation over Sobolev spaces. We now present the first of our main results, a lower bound on the asymptotic minimax risk for quantized estimation over Sobolev spaces. In Section 4 we show that this bound is tight, by demonstrating an explicit quantization scheme that asymptotically achieves it. We begin by recalling the standard definitions of the required function spaces and ellipsoids.

The Gaussian white noise model is based on the diffusion equation

$$dX(t) = f(t)dt + \varepsilon dW(t), \quad 0 \leq t \leq 1,$$

where $f$ is an unknown function on $[0, 1]$, $W$ is a standard Wiener process on $[0, 1]$ and $\varepsilon$ is the standard deviation parameter. The statistician observes a random sample path $X(t)$ of this diffusion, with the goal of estimating the unknown function $f$. If the function $f$ is allowed to be chosen without any constraint, this estimation problem can be arbitrarily hard. We shall place a standard smoothness assumption on $f$.

Recall that the Sobolev space of order $m$ and radius $c$ is defined by

$$W(m, c) = \{ f \in [0, 1] \to \mathbb{R} : f^{(m-1)} \text{ is absolutely continuous and}$$

$$\int_0^1 (f^{(m)}(x))^2 dx \leq c^2 \}.$$

The periodic Sobolev space is defined by

$$\tilde{W}(m, c) = \{ f \in W(m, c) : f^{(j)}(0) = f^{(j)}(1), \ j = 0, 1, \ldots, m - 1 \}.$$

Pinsker [15] derived the precise asymptotic rate of the minimax risk

$$R_\varepsilon(m, c) = \inf_{\tilde{f}_\varepsilon} \sup_{f \in \tilde{W}(m, c)} \mathbb{E} \| \tilde{f}_\varepsilon - f \|_2^2$$

for this function space, showing that

$$\lim_{\varepsilon \to 0} \varepsilon^{-\frac{4m}{2m+1}} R_\varepsilon(m, c) = \left( \frac{c^2 (2m + 1)}{\pi^{2m+1}} \right)^{\frac{1}{2m+1}} \left( \frac{m}{m+1} \right)^{\frac{2m}{2m+1}} \triangleq P_{m,c}$$

where the quantity $P_{m,c}$ is now called Pinsker’s constant. The white noise model is equivalent to making $n$ equally space observations along the sample
path, \( Y_i = f(i/n) + \varepsilon_i \), where \( \varepsilon_i \sim \mathcal{N}(0, 1) \) [4]. In this formulation, the noise level scales as \( \varepsilon_i^2 = \sigma^2 / n \), and the rate of convergence takes the familiar form \( n^{-2m/(2m+1)} \) where \( n \) is the number of observations.

To carry out quantized estimation we require an encoder
\[
\phi_\varepsilon : \mathbb{R}^{[0,1]} \to \{1, 2, \ldots, 2^{B_\varepsilon}\}
\]
which is a function applied to the sample path \( X(t) \). The decoding function then takes the form
\[
\psi_\varepsilon : \{1, 2, \ldots, 2^{B_\varepsilon}\} \to \mathbb{R}^{[0,1]}
\]
and maps the index to a function estimate. As in the previous section, we write the composition of the encoder and the decoder as \( \tilde{f}_\varepsilon = \psi_\varepsilon \circ \phi_\varepsilon \), which we call the quantized estimator. In this case, the communication or storage \( C(\tilde{f}_\varepsilon) \) required by this quantized estimator is no more than \( B_\varepsilon \) bits. We define the minimax risk for quantized estimation as
\[
R_\varepsilon(m, c, B_\varepsilon) = \inf_{\tilde{f}_\varepsilon, C(\tilde{f}_\varepsilon) \leq B_\varepsilon} \sup_{f \in \widetilde{W}(m, c)} \mathbb{E} \| f - \tilde{f}_\varepsilon \|_2^2.
\]
As in Pinsker’s theorem, we are interested in the asymptotic risk as \( \varepsilon \to 0 \).

It will be convenient to recast the quantized estimation in terms of an infinite sequence model. Specifically, let \( (\varphi_j)_{j=1}^\infty \) be the trigonometric basis, and let
\[
\theta_j = \int_0^1 \varphi_j(t)f(t)dt, \quad j = 1, 2, \ldots,
\]
be the Fourier coefficients. It is well known [18] that \( f = \sum_{j=1}^\infty \theta_j \varphi_j \) belongs to \( \widetilde{W}(m, c) \) if and only if the Fourier coefficients \( \theta \) belong to the Sobolev ellipsoid defined as
\[
\Theta(m, c) = \left\{ \theta \in \ell_2 : \sum_{j=1}^\infty a_j^2 \theta_j^2 \leq \frac{c^2}{\pi^{2m}} \right\}
\]
where
\[
a_j = \begin{cases} j^m, & \text{for even } j, \\ (j-1)^m, & \text{for odd } j. \end{cases}
\]
Although this is the standard definition of a Sobolev ellipsoid, for the rest of the paper we will set \( a_j = j^m, j = 1, 2, \ldots \) for convenience of analysis. All the results in fact hold for both definitions of \( a_j \). Also note that (3.3) actually gives a more general definition, since \( m \) is no longer assumed to be...
an integer, as it is in (3.1). Expanding with respect to the same orthonormal basis, the observed path $X(t)$ is converted into an infinite Gaussian sequence

$$Y_j = \int_0^1 \varphi_j(t) dX(t), \quad j = 1, 2, \ldots,$$

with $Y_j \sim N(\theta_j, \varepsilon^2)$. For an estimator $(\hat{\theta}_j)_{j=1}^\infty$ of $(Y_j)_{j=1}^\infty$, an estimate of $f$ is obtained by

$$\hat{f}(x) = \sum_{j=1}^\infty \hat{\theta}_j \varphi_j(x)$$

with squared error $\|\hat{f} - f\|_2^2 = \|\hat{\theta} - \theta\|_2^2$. In terms of this standard reduction, the quantized minimax risk is thus reformulated as

$$(3.4) \quad R_\varepsilon(m, c, B_\varepsilon) = \inf_{\hat{\theta}_\varepsilon, C(\hat{\theta}_\varepsilon) \leq B_\varepsilon} \sup_{\theta \in \Theta(m, c)} E_\theta \|\theta - \hat{\theta}_\varepsilon\|_2^2.$$

Finally, to state our result, we need to define the value of the following variational problem:

$$(3.5) \quad V_{m, c, d} \triangleq \max_{(\sigma^2, x_0) \in F(m, c, d)} \int_0^{x_0} \frac{\sigma^2(x)}{\sigma^2(x) + 1} dx + x_0 \exp \left( \frac{1}{x_0} \int_0^{x_0} \log \frac{\sigma^4(x)}{\sigma^2(x) + 1} dx - \frac{2d}{x_0} \right)$$

where the feasible set $F(m, c, d)$ is the collection of increasing functions $\sigma^2(x)$ and values $x_0$ satisfying

$$\int_0^{x_0} x^{2m} \sigma^2(x) dx \leq c^2$$

$$\frac{\sigma^4(x)}{\sigma^2(x) + 1} \geq \exp \left( \frac{1}{x_0} \int_0^{x_0} \log \frac{\sigma^4(x)}{\sigma^2(x) + 1} dx - \frac{2d}{x_0} \right)$$

for all $x \leq x_0$.

The following result establishes the rate of convergence and explicit constants for three distinct quantization regimes. A key role is played by the above variational problem. The significance and interpretation of the variational problem will become apparent as we outline the proof of this result.

**Theorem 3.1.** Let $R_\varepsilon(m, c, B_\varepsilon)$ be defined as in (3.4), for $m > 0$ and $c > 0$. 

(i) If $B_{\varepsilon}\varepsilon^{-\frac{2}{2m+1}} \to \infty$ as $\varepsilon \to 0$, then

$$\liminf_{\varepsilon \to 0} \varepsilon^{-\frac{4m}{2m+1}} R_{\varepsilon}(m, c, B_{\varepsilon}) \geq P_{m,c}$$

where $P_{m,c}$ is Pinker's constant defined in (3.2).

(ii) If $B_{\varepsilon}\varepsilon^{-\frac{2}{2m+1}} \to d$ for some constant $d$ as $\varepsilon \to 0$, then

$$\liminf_{\varepsilon \to 0} \varepsilon^{-\frac{4m}{2m+1}} R_{\varepsilon}(m, c, B_{\varepsilon}) \geq P_{m,c} + Q_{m,c,d} = \mathcal{V}_{m,c,d}$$

where $\mathcal{V}_{m,c,d}$ is the value of the variational problem (3.5).

(iii) If $B_{\varepsilon}\varepsilon^{-\frac{2}{2m+1}} \to 0$ and $B_{\varepsilon} \to \infty$ as $\varepsilon \to 0$, then

$$\liminf_{\varepsilon \to 0} B_{\varepsilon}^{2m} R_{\varepsilon}(m, c, B_{\varepsilon}) \geq \frac{2^2 m^{2m}}{\pi^{2m}}.$$

Although the above theorem gives only lower bounds on the minimax risks for quantized estimation, it will be shown in the following section that the lower bounds are achievable. As described in the theorem, there are three different asymptotic regimes based on how the communication budget grows as a function of the noise level. It turns out that the critical threshold is $\varepsilon^{-\frac{2}{2m+1}}$. This value should not be surprising. Indeed, in classical unquantized estimation, in order to achieve the optimal risk, only the first $\varepsilon^{-\frac{2}{2m+1}}$ random variables are kept and the rest are simply estimated to be zero. In the finite sample setting this corresponds to truncating the infinite sequence at $O\left(n^{\frac{1}{2m+1}}\right)$ basis functions.

In the first regime where the number of bits $B_{\varepsilon}$ is much greater than $\varepsilon^{-\frac{2}{2m+1}}$, we recover the same convergence result as in Pinsker’s theorem, in terms of both convergence rate and leading constant. The proof of the lower bound for this regime can directly follow the proof of Pinsker’s theorem, since the set of estimators considered in our minimax framework is a subset of all possible estimators.

In the second regime where we have “just enough” bits to preserve the rate, we suffer a loss in terms of the leading constant. In this “Goldilocks” regime, the optimal rate $\varepsilon^{-\frac{4m}{2m+1}}$ is achieved but the constant in front of the rate is Pinsker’s constant $P_{m,c}$ plus a positive quantity $Q_{m,c,d}$ determined by the variational problem. While the solution to this variational problem does not appear to have an explicit form, it can be computed numerically. We
Fig 1. The constants $P_{m,c} + Q_{m,c,d}$ as a function of quantization level $d$ in the sufficient regime, together with the upper bound (3.8) and two lower bounds. The lower bounds correspond to specific choices of prior parameters $\sigma_j^2$. The first lower bound is based on the same prior used for unquantized estimation, and is accurate for large $d$; the second uses the least favorable prior for the insufficient regime, and is accurate for small $d$. These lower bounds are defined explicitly in (A.6) and (A.7). The classical Pinsker constant is shown as a horizontal line at 1. The inset shows the point where the two lower bounds cross over.

discuss this term at length in the sequel, where we explain the origin of the variational problem, compute the constant numerically and approximate it from above and below. The constants $P_{m,c}$ and $Q_{m,c,d}$ are shown graphically in Figure 1.

In the third regime where the communication budget is insufficient for the estimator to achieve the optimal rate, we obtain a sub-optimal rate which no longer depends on the noise level $\varepsilon$ of the model. In this regime, quantization error dominates, and the risk decays at a rate of $B^{-\frac{m}{m+1}}$ no matter how fast $\varepsilon$ approaches zero, as long as $B \ll \varepsilon^{-\frac{2}{2m+1}}$. Here the analogue of Pinsker’s constant takes a very simple form.

The constant for the third regime yields an upper bound for $Q_{m,c,d}$, and we can thus informally write

$$R_\varepsilon(m, c, B_\varepsilon) \approx P_{m,c} \varepsilon^{\frac{4m}{2m+1}} + \frac{c^2 m^{2m}}{\pi^{2m}} B_\varepsilon^{-2m}$$
which (approximately) decomposes the risk into an estimation error term and a quantization error term, combining together the three regimes.

Proof. We now outline the proof Theorem 3.1; the details are deferred to Section A. Consider a Gaussian prior distribution on $\theta = (\theta_j)_{j=1}^{\infty}$ with $\theta_j \sim \mathcal{N}(0, \sigma_j^2)$ for $j = 1, 2, \ldots$, in terms of parameters $\sigma^2 = (\sigma_j^2)_{j=1}^{\infty}$ to be specified later. One requirement for the variances is

$$\sum_{j=1}^{\infty} \sigma_j^2 \sigma_j^2 \leq \frac{c^2}{\pi^2 m}.$$ 

We denote this prior distribution by $\pi(\theta; \sigma^2)$, and show in Section A that it is asymptotically concentrated on the ellipsoid $\Theta(m, c)$. Under this prior the model is

$$\theta_j \sim \mathcal{N}(0, \sigma_j^2),$$

$$\forall_j : \theta_j \sim \mathcal{N}(\theta_j, \varepsilon^2), \quad j = 1, 2, \ldots$$

and the marginal distribution of $Y_j$ is thus $\mathcal{N}(0, \sigma_j^2 + \varepsilon^2$. Following the strategy outlined in Section 2, let $\delta$ denote the posterior mean of $\theta$ given $Y$ under this prior, and consider the optimization

$$\inf \mathbb{E}\|\delta - \tilde{\theta}\|^2$$

such that $I(\delta; \tilde{\theta}) \leq B\varepsilon$

where the infimum is over all distributions on $\tilde{\theta}$ such that $\theta \rightarrow Y \rightarrow \tilde{\theta}$ forms a Markov chain. Now, the posterior mean satisfies $\delta_j = \gamma_j Y_j$ where $\gamma_j = \sigma_j^2 / (\sigma_j^2 + \varepsilon^2)$. Note that the Bayes risk under this prior is

$$\mathbb{E}\|\theta - \delta\|^2 = \sum_{j=1}^{\infty} \frac{\sigma_j^2 \varepsilon^2}{\sigma_j^2 + \varepsilon^2}.$$ 

Define

$$\mu_j^2 \triangleq \mathbb{E}(\delta_j - \tilde{\theta}_j)^2.$$ 

Then the classical rate distortion argument [6] gives that

\[
I(\delta; \bar{\theta}) \geq \sum_{j=1}^{\infty} I(\gamma_j Y_j; \bar{\theta}_j)
\]

\[
\geq \sum_{j=1}^{\infty} \frac{1}{2} \log_+ \left( \frac{\gamma_j^2 (\sigma_j^2 + \epsilon^2)}{\mu_j^2} \right)
\]

\[
= \sum_{j=1}^{\infty} \frac{1}{2} \log_+ \left( \frac{\sigma_j^4}{\mu_j^2 (\sigma_j^2 + \epsilon^2)} \right)
\]

where \( \log_+ (x) = \max(\log x, 0) \). Therefore, the quantized minimax risk is lower bounded by

\[
R_{\epsilon}(m, c, B_{\epsilon}) = \inf_{\hat{\theta}_{\epsilon}, C(\hat{\theta}_{\epsilon}) \leq B_{\epsilon}} \sup_{\theta \in \Theta(m, c)} \mathbb{E} \| \theta - \hat{\theta}_{\epsilon} \|^2 \geq V_{\epsilon}(B_{\epsilon}, m, c)(1 + o(1))
\]

where \( V_{\epsilon}(B_{\epsilon}, m, c) \) is the value of the optimization

\[
(P_1)
\]

\[
\max_{\sigma^2} \min_{\mu^2} \sum_{j=1}^{\infty} \mu_j^2 + \sum_{j=1}^{\infty} \frac{\sigma_j^2 \epsilon^2}{\sigma_j^2 + \epsilon^2}
\]

such that \( \sum_{j=1}^{\infty} \frac{1}{2} \log_+ \left( \frac{\sigma_j^4}{\mu_j^2 (\sigma_j^2 + \epsilon^2)} \right) \leq B_{\epsilon} \)

\[
\sum_{j=1}^{\infty} a_j^2 \sigma_j^2 \leq \frac{c^2}{\pi^2 m}
\]

and the \((1 + o(1))\) deviation term is analyzed in the supplementary material.

Observe that the quantity \( V_{\epsilon}(B_{\epsilon}, m, c) \) can be upper and lower bounded by

(3.6)

\[
\max \left\{ R_{\epsilon}(m, c), Q_{\epsilon}(m, c, B_{\epsilon}) \right\} \leq V_{\epsilon}(m, c, B_{\epsilon}) \leq R_{\epsilon}(m, c) + Q_{\epsilon}(m, c, B_{\epsilon})
\]

where the estimation error term \( R_{\epsilon}(m, c) \) is the value of the optimization

\[
(R_1)
\]

\[
\max_{\sigma^2} \sum_{j=1}^{\infty} \frac{\sigma_j^2 \epsilon^2}{\sigma_j^2 + \epsilon^2}
\]

such that \( \sum_{j=1}^{\infty} a_j^2 \sigma_j^2 \leq \frac{c^2}{\pi^2 m} \)
and the quantization error term $Q_\epsilon(m, c, B_\epsilon)$ is the value of the optimization

\[
\max_{\sigma^2} \min_{\mu^2} \sum_{j=1}^\infty \mu_j^2
\]

such that

\[
\sum_{j=1}^\infty \frac{1}{2} \log_+ \left( \frac{\sigma_j^4}{\mu_j^2(\sigma_j^2 + \epsilon^2)} \right) \leq B_\epsilon
\]

\[
\sum_{j=1}^\infty a_j^2 \sigma_j^2 \leq \frac{c^2}{\pi 2m}.
\]

The following results specify the leading order asymptotics of these quantities.

**Lemma 3.2.** As $\epsilon \to 0$,

\[
R_\epsilon(m, c) = P_{m, c, \epsilon} \frac{2m}{\pi^{m+1}} (1 + o(1)).
\]

**Lemma 3.3.** As $\epsilon \to 0$,

\[
Q_\epsilon(m, c, B_\epsilon) \leq \frac{c^2 m^{2m}}{\pi^{2m}} B^{-2m}_\epsilon (1 + o(1)).
\]

Moreover, if $B_\epsilon \epsilon^{2m+1} \to 0$ and $B_\epsilon \to \infty$,

\[
Q_\epsilon(m, c, B_\epsilon) = \frac{c^2 m^{2m}}{\pi^{2m}} B^{-2m}_\epsilon (1 + o(1)).
\]

This yields the following closed form upper bound.

**Corollary 3.4.** Suppose that $B_\epsilon \to \infty$ and $\epsilon \to 0$. Then

\[
V_\epsilon(m, c, B_\epsilon) \leq \left( P_{m, c} \epsilon^{2m+1} + \frac{c^2 m^{2m}}{\pi^{2m}} B^{-2m}_\epsilon \right) (1 + o(1)).
\]

In the insufficient regime $B_\epsilon \epsilon^{2m+1} \to 0$ and $B_\epsilon \to \infty$ as $\epsilon \to 0$, equation (3.6) and Lemma 3.3 show that

\[
V_\epsilon(m, c, B_\epsilon) = \frac{c^2 m^{2m}}{\pi^{2m}} B^{-2m}_\epsilon (1 + o(1)).
\]

Similarly, in the over-sufficient regime $B_\epsilon \epsilon^{2m+1} \to \infty$ as $\epsilon \to 0$, we conclude that

\[
V_\epsilon(m, c, B_\epsilon) = P_{m, c} \epsilon^{2m+1} (1 + o(1)).
\]
We now turn to the sufficient regime $B_\varepsilon \varepsilon^{\frac{2}{2m+1}} \to d$. We begin by making three observations about the solution to the optimization $\mathcal{P}_1$. First, we note that the series $(\sigma_j^2)_{j=1}^\infty$ that solves $\mathcal{P}_1$ can be assumed to be decreasing. If $(\sigma_j^2)$ were not in decreasing order, we could rearrange it to be decreasing, and correspondingly rearrange $(\mu_j^2)$, without violating the constraints or changing the value of the optimization. Second, we note that given $(\sigma_j^2)$, the optimal $(\mu_j^2)$ is obtained by the “reverse water-filling” scheme [6]. Specifically, there exists $\eta > 0$ such that

$$
\mu_j^2 = \begin{cases} 
\eta & \text{if } \frac{\sigma_j^4}{\sigma_j^2 + \varepsilon^2} \geq \eta \\
\frac{\sigma_j^4}{\sigma_j^2 + \varepsilon^2} & \text{otherwise,}
\end{cases}
$$

where $\eta$ is chosen so that

$$
\frac{1}{2} \sum_{j=1}^\infty \log_+ \left( \frac{\sigma_j^4}{\mu_j^2(\sigma_j^2 + \varepsilon^2)} \right) \leq B_\varepsilon.
$$

Third, there exists an integer $J > 0$ such that the optimal series $(\sigma_j^2)$ satisfies

$$
\frac{\sigma_j^4}{\sigma_j^2 + \varepsilon^2} \geq \eta, \text{ for } j = 1, \ldots, J \quad \text{and} \quad \sigma_j^2 = 0, \text{ for } j > J,
$$

where $\eta$ is the “water-filling level” for $(\mu_j^2)$. Using these three observations, the optimization $\mathcal{P}_1$ can be reformulated as

$$
\max_{\sigma^2, J} \quad J\eta + \sum_{j=1}^J \frac{\sigma_j^2 \varepsilon^2}{\sigma_j^2 + \varepsilon^2}
$$

such that

$$
\frac{1}{2} \sum_{j=1}^J \log_+ \left( \frac{\sigma_j^4}{\eta(\sigma_j^2 + \varepsilon^2)} \right) = B_\varepsilon
$$

$$
\sum_{j=1}^J a_j^2 \sigma_j^2 \leq \frac{c^2}{\pi^{2m}}
$$

$(\sigma_j^2)$ is decreasing and $\frac{\sigma_j^4}{\sigma_j^2 + \varepsilon^2} \geq \eta$.

To derive the solution to $\mathcal{P}_2$, we use a continuous approximation of $\sigma^2$, writing

$$
\sigma_j^2 = \sigma^2(jh)h^{2m+1}
$$
where \( h \) is the bandwidth to be specified and \( \sigma^2(\cdot) \) is a function defined on \((0, \infty)\). The constraint that \( \sum_{j=1}^{\infty} a_j^2 \sigma_j^2 \leq \frac{c^2}{\pi^2m} \) becomes the integral constraint

\[
\int_0^\infty x^{2m} \sigma^2(x) dx \leq \frac{c^2}{\pi^2m}.
\]

We now set the bandwidth so that \( h^2m + 1 = \varepsilon^2 \). This choice of bandwidth will balance the two terms in the objective function, and thus gives the hardest prior distribution. Applying the above three observations under this continuous approximation, we transform problem \((P_2)\) to the following optimization:

\[
(P_3) \quad \max_{\sigma^2, x_0} x_0 \eta + \int_0^{x_0} \frac{\sigma^2(x)}{\sigma^2(x) + 1} dx \quad \text{such that} \quad \int_0^{x_0} \frac{1}{2} \log \left( \frac{\sigma^4(x)}{\eta(\sigma^2(x) + 1)} \right) = d
\]

\[
\int_0^{x_0} x^{2m} \sigma^2(x) dx \leq \frac{c^2}{\pi^2m}
\]

\( \sigma^2(x) \) is decreasing and \( \frac{\sigma^4(x)}{\sigma^2(x) + 1} \geq \eta \) for all \( x \leq x_0 \).

Note that here we omit the convergence rate \( h^{2m} = \varepsilon \frac{4m}{m+1} \) in the objective function. Solving the first constraint for \( \eta \) yields

\[
(P_4) \quad \max_{\sigma^2, x_0} \int_0^{x_0} \frac{\sigma^2(x)}{\sigma^2(x) + 1} dx + x_0 \exp \left( \frac{1}{x_0} \int_0^{x_0} \log \frac{\sigma^4(x)}{\sigma^2(x) + 1} dx - \frac{2d}{x_0} \right)
\]

\( \sigma^2(x) \) is decreasing

\[
\frac{\sigma^4(x)}{\sigma^2(x) + 1} \geq \exp \left( \frac{1}{x_0} \int_0^{x_0} \log \frac{\sigma^4(x)}{\sigma^2(x) + 1} dx - \frac{2d}{x_0} \right)
\]

for all \( x \leq x_0 \).

The following is proved using a variational argument in the supplementary material.

**Lemma 3.5.** The solution to \((P_4)\) satisfies

\[
\frac{1}{(\sigma^2(x) + 1)^2} + \exp \left( \frac{1}{x_0} \int_0^{x_0} \log \frac{\sigma^4(x)}{\sigma^2(x) + 1} dx - \frac{2d}{x_0} \right) \frac{\sigma^2(x) + 2}{\sigma^2(x)(\sigma^2(x) + 1)} = \lambda x^{2m}
\]

for some \( \lambda > 0 \).
Fixing $x_0$, the lemma shows that by setting

$$\alpha = \exp \left( \frac{1}{x_0} \int_0^{x_0} \log \frac{\sigma^4(x)}{\sigma^2(x) + 1} \, dx - \frac{2d}{x_0} \right)$$

we can express $\sigma^2(x)$ implicitly as the unique positive root of a third-order polynomial in $y$,

$$\lambda x^{2m} y^3 + (2\lambda x^{2m} - \alpha) y^2 + (\lambda x^{2m} - 3\alpha - 1)y - 2\alpha.$$

This allows us to compute the values of the optimization numerically.

To summarize, in the regime $B_{\varepsilon} \varepsilon^{\frac{1}{2m+1}} \to d$ as $\varepsilon \to 0$, we obtain

$$V_\varepsilon(m, c, B_{\varepsilon}) = (P_{m,c} + Q_{m,c,d}) \varepsilon^{\frac{4m}{2m+1}} (1 + o(1)),$$

where we denote by $P_{m,c} + Q_{m,c,d}$ the values of the optimization $(P_4)$. This result is shown graphically in Figure 1.

4. Achievability. In this section, we show that the lower bounds in Theorem 3.1 are achievable by a quantized estimator using a random coding scheme. The basic idea of our quantized estimation procedure is to conduct blockwise estimation and quantization together, using a quantized form of the Stein estimator.

We begin by defining the block system to be used, which is usually referred to as the weakly geometric system of blocks [18]. Let $N_\varepsilon = \lfloor 1/\varepsilon^2 \rfloor$ and $\rho_\varepsilon = (\log(1/\varepsilon))^{-1}$. Let $J_1, \ldots, J_K$ be a partition of the set $\{1, \ldots, N_\varepsilon\}$ such that

$$\bigcup_{k=1}^K J_k = \{1, \ldots, N_\varepsilon\}, \quad J_{k_1} \cap J_{k_2} = \emptyset \text{ for } k_1 \neq k_2,$$

and $\min\{j : j \in J_k\} > \max\{j : j \in J_{k-1}\}$.

Let $T_k$ be the cardinality of the $k$th block and suppose that $T_1, \ldots, T_k$ satisfy

$$T_1 = \lfloor \rho_\varepsilon^{-1} \rfloor = \lfloor \log(1/\varepsilon) \rfloor,$$

$$T_2 = \lfloor T_1 (1 + \rho_\varepsilon) \rfloor,$$

$$\vdots$$

$$T_{K-1} = \lfloor T_1 (1 + \rho_\varepsilon)^{K-2} \rfloor,$$

$$T_K = N_\varepsilon - \sum_{k=1}^{K-1} T_k.$$
For an infinite sequence \( x \in \ell_2 \), denote by \( x_{(k)} \) the vector \( (x_j)_{j \in J_k} \in \mathbb{R}^{T_k} \). We also write \( j_k = \sum_{l=1}^{k-1} T_l + 1 \), which is the smallest index in block \( J_k \).

We are now ready to describe the quantized estimation scheme. In contrast to rate distortion theory, where the codebook and allocation of the bits are determined once the source distribution is known, here the codebook and allocation of bits are adaptive to the data—more bits are used for blocks having larger signal size. Thus, in the quantized estimation procedure, the first step is to allocate the communication budget \( B_\epsilon \equiv B \) based on the data, and then quantize each block using a variant of the scheme proposed in [23], separately coding the directions and magnitudes of each block vector. The quantized estimation scheme is detailed below.

**Step 1. Base code generation.**

1.1. Generate codebook \( S_k = \{ \sqrt{T_k\epsilon^2} + i\epsilon^2 : i = 0, 1, \ldots, s_k \} \) where \( s_k = \lceil \epsilon^{-2}c(j_k\pi)^{-m} \rceil \) for \( k = 1, \ldots, K \).

1.2. Generate base code \( Z \), a \( 2^B \times T_K \) matrix with i.i.d. \( N(0,1) \) entries. \((S_k)\) and \( Z\) are shared between the encoder and the decoder, before seeing any data.

**Step 2. Encoding.**

2.1. Encoding block radius. For \( k = 1, \ldots, K \), encode \( \tilde{S}_k = \arg \min \{ |s - S_k| : s \in S_k \} \) where

\[
S_k = \begin{cases} \sqrt{T_k\epsilon^2} & \text{if } \|Y_{(k)}\| < \sqrt{T_k\epsilon^2} \\ \sqrt{T_k\epsilon^2} + c(j_k\pi)^{-m} & \text{if } \|Y_{(k)}\| > \sqrt{T_k\epsilon^2} + c(j_k\pi)^{-m} \\ \|Y_{(k)}\| & \text{otherwise.} \end{cases}
\]

2.2. Bits allocation. Let \( (\tilde{b}_k)_{k=1}^K \) be the solution to the optimization

\[
\min_{\tilde{b}} \sum_{k=1}^K \frac{(\tilde{S}_k^2 - T_k\epsilon^2)^2}{S_k^2} \cdot 2^{-2\tilde{b}_k} \quad (4.2)
\]

such that \( \sum_{k=1}^K T_k\tilde{b}_k \leq B, \tilde{b}_k \geq 0 \).

2.3. Encoding block direction. Form the data-dependent codebook as follows.

Divide the rows of \( Z \) into blocks of sizes \( 2^{[T_1\tilde{b}_1]}, \ldots, 2^{[T_K\tilde{b}_K]} \). Based on the \( k \)th block of rows, construct the data-dependent
codebook $\tilde{Z}_k$ by keeping only the first $T_k$ entries and normalizing each truncated row; specifically, the $j$th row of $\tilde{Z}_k$ is given by

$$\tilde{Z}_{k,j} = \frac{Z_{i,1:T_k}}{\|Z_{i,1:T_k}\|} \in S_{T_k-1}$$

where $i$ is the appropriate row of the base code $Z$ and $Z_{i,1:t}$ denotes the first $t$ entries of the row vector. A graphical illustration is shown below in Figure 2.

With this data-dependent codebook, encode

$$Z_{(k)} = \arg \min \{ \langle z, Y_{(k)} \rangle : z \in \tilde{Z}_k \}$$

for $k = 1, \ldots, K$.

**Fig 2.** An illustration of the data-dependent codebook. The big matrix represents the base code $Z$, and the shaded areas are $(\tilde{Z}_k)$, sub-matrices of size $T_k \times 2^{|T_k|}$ with rows normalized.

**Step 3. Transmission.** Transmit or store $(\tilde{S}_k)_{k=1}^K$ and $(\tilde{Z}_{(k)})_{k=1}^K$ by their corresponding indices.

**Step 4. Decoding & Estimation.**

1. Recover $(\tilde{S}_k)$ based on the transmitted or stored indices and the common codebook $(S_k)$.
2. Solve (4.2) and get $(\tilde{b}_k)$. Reconstruct $(\tilde{Z}_k)$ using $Z$ and $(\tilde{b}_k)$.
3. Recover $(\tilde{Z}_{(k)})$ based on the transmitted or stored indices and the reconstructed codebook $(\tilde{Z}_k)$. 
4.4. Estimate $\theta(k)$ by

$$\hat{\theta}(k) = \frac{S_k^2 - T_k\varepsilon^2}{S_k} \sqrt{1 - 2^{-2b_k}} \cdot \tilde{Z}(k).$$

4.5. Estimate the entire vector $\theta$ by concatenating the $\hat{\theta}(k)$ vectors and padding with zeros; thus,

$$\hat{\theta} = (\hat{\theta}(1), \ldots, \hat{\theta}(K), 0, 0, \ldots).$$

The following theorem establishes the asymptotic optimality of this quantized estimator.

**Theorem 4.1.** Let $\hat{\theta}$ be the quantized estimator defined above.

(i) If $B \varepsilon^{2m+1} \to \infty$, then

$$\lim_{\varepsilon \to 0} \varepsilon^{-4m} \sup_{\theta \in \Theta(m,c)} \mathbb{E}\|\theta - \hat{\theta}\|^2 = P_{m,c}.$$  

(ii) If $B \varepsilon^{2m+1} \to d$ for some constant $d$ as $\varepsilon \to 0$, then

$$\lim_{\varepsilon \to 0} \varepsilon^{-4m} \sup_{\theta \in \Theta(m,c)} \mathbb{E}\|\theta - \hat{\theta}\|^2 = P_{m,c} + Q_{d,m,c}.$$  

(iii) If $B \varepsilon^{2m+1} \to 0$ and $B(\log(1/\varepsilon))^{-3} \to \infty$, then

$$\lim_{\varepsilon \to 0} B^{2m} \sup_{\theta \in \Theta(m,c)} \mathbb{E}\|\theta - \hat{\theta}\|^2 = \frac{c^2m^{2m}}{\pi^{2m}}.$$  

The expectations are with respect to the random quantized estimation scheme $Q$ and the distribution of the data.

We pause to make several remarks on this result before outlining the proof.

**Remark 1.** The total number of bits used by this quantized estimation scheme is

$$\sum_{k=1}^{K} T_k \beta_k + \sum_{k=1}^{K} \log[\varepsilon^{-2}c(j_k\pi)^{-m}] \leq \sum_{k=1}^{K} T_k \beta_k + \sum_{k=1}^{K} \log[\varepsilon^{-2}c]$$

$$\leq B + K + 2K \rho_{\varepsilon}^{-1} + K \log[c]$$

$$= B + O((\log(1/\varepsilon))^3).$$

Therefore, as long as $B(\log(1/\varepsilon))^{-3} \to \infty$, the total number of bits used is asymptotically no more than $B$, the given communication budget.
Remark 2. The quantized estimation scheme does not make essential use of the parameters of the Sobolev space, namely the smoothness $m$ and the radius $c$. The only exception is that in Step 1.1 the size of the codebook $S_k$ depends on $m$ and $c$. However, suppose that we know a lower bound on the smoothness $m$, say $m \geq m_0$, and an upper bound on the radius $c$, say $c \leq c_0$. By replacing $m$ and $c$ by $m_0$ and $c_0$ respectively, we make the codebook independent of the parameters. We shall assume $m_0 > 1/2$, which leads to continuous functions. This modification does not, however, significantly increase the number of bits; in fact, the total number of bits is still $B + O(\rho^{-3}_\varepsilon)$. Thus, we can easily make this quantized estimator minimax adaptive to the class of Sobolev ellipsoids $\{\Theta(m,c) : m \geq m_0, c \leq c_0\}$, as long as $B$ grows faster than $(\log(1/\varepsilon))^3$. More formally, we have

Corollary 4.2. Suppose that $B_\varepsilon$ satisfies $B_\varepsilon((\log(1/\varepsilon))^{-3} \to \infty$. Let $\hat{\theta}'$ be the quantized estimator with the modification described above, which does not assume knowledge of $m$ and $c$. Then for $m \geq m_0$ and $c \leq c_0$,

$$\lim_{\varepsilon \to 0} \quad \sup_{\theta \in \Theta(m,c)} \mathbb{E}\|\theta - \hat{\theta}'\|^2 \inf_{\hat{\theta}, \hat{C}(\hat{\theta}) \leq B} \sup_{\theta \in \Theta(m,c)} \mathbb{E}\|\theta - \hat{\theta}\|^2 = 1,$$

where the expectation in the numerator is with respect to the data and the randomized coding scheme, while the expectation in the denominator is only with respect to the data.

Remark 3. When $B$ grows at a rate comparable to or slower than $(\log(1/\varepsilon))^3$, the lower bound is still achievable, just no longer by the quantized estimator we described above. The main reason is that when $B$ does not grow faster than $\log(1/\varepsilon)^3$, the block size $T_1 = \lceil \log(1/\varepsilon) \rceil$ is too large. The blocking needs to be modified to get achieveability in this case.

Remark 4. In classical rate distortion [6, 10], the probabilistic method applied to a randomized coding scheme shows the existence of a code achieving the rate distortion bounds. According to Theorem 3.1, the expected risk, averaged over the randomness in the codebook, similarly achieves the quantized minimax lower bound. However, note that the average over the codebook is inside the supremum over the Sobolev space, implying that the code achieving the bound may vary over the ellipsoid. In other words, while the coding scheme generates a codebook that is used for different $\theta$, it is not known whether there is one code generated by this randomized scheme that is “universal,” and achieves the risk lower bound with high probability over the ellipsoid. The existence or non-existence of such “universal codes” is an interesting direction for further study.
Proof of Theorem 4.1. We now sketch the proof of Theorem 4.1, deferring the full details to Section A. To provide only an informal outline of the proof, we shall write $A_1 \approx A_2$ as a shorthand for $A_1 = A_2(1 + o(1))$, and $A_1 \lesssim A_2$ for $A_1 \leq A_2(1 + o(1))$, without specifying here what these $o(1)$ terms are.

To upper bound the risk $\mathbb{E}\|\hat{\theta} - \theta\|^2$, we adopt the following sequence of approximations and inequalities. First, we discard the components whose index is greater than $N$ and show that

$$\mathbb{E}\|\hat{\theta} - \theta\|^2 \approx \sum_{k=1}^{K} \|\hat{\theta}(k) - \theta(k)\|^2.$$ 

Since $\hat{S}_k$ is close enough to $S_k$, we can then safely replace $\hat{\theta}(k)$ by $\tilde{\theta}(k) = \frac{S_k^2 - T_k \varepsilon^2}{S_k^2} \sqrt{1 - 2^{-2\hat{b}_k}} \cdot \hat{Z}_k$ and obtain

$$\approx \sum_{k=1}^{K} \|\tilde{\theta}(k) - \theta(k)\|^2.$$ 

Writing $\lambda_k = \frac{S_k^2 - T_k \varepsilon^2}{S_k^2}$, we further decompose the risk into

$$= \sum_{k=1}^{K} \left( \|\tilde{\theta}(k) - \lambda_k Y(k)\|^2 + \|\lambda_k Y(k) - \theta(k)\|^2 \\
+ 2 \langle \tilde{\theta}(k) - \lambda_k Y(k), \lambda_k Y(k) - \theta(k) \rangle \right).$$

Conditioning on the data $Y$ and taking the expectation with respect to the random codebook yields

$$\lesssim \sum_{k=1}^{K} \left( \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} 2^{-2\hat{b}_k} + \|\lambda_k Y(k) - \theta(k)\|^2 \right).$$

By two oracle inequalities upper bounding the expectations with respect to the data, and the fact that $\hat{b}$ is the solution to (4.2),

$$\lesssim \min_{b \in \Pi_{blk}(B)} \sum_{k=1}^{K} \left( \frac{\|\theta(k)\|^4}{\|\theta(k)\|^2 + T_k \varepsilon^2} 2^{-2\hat{b}_k} + \frac{\|\theta(k)\|^2 T_k \varepsilon^2}{\|\theta(k)\|^2 + T_k \varepsilon^2} \right).$$
Showing that the blockwise constant oracles are almost as good as the mono-
tone oracle, we get for some $B' \approx B$

$$\leq \min_{b \in \Pi_{\text{mon}}(B')}, \omega \in \Omega_{\text{mon}} \sum_{j=1}^{N} \left( \frac{\theta_j^4}{\theta_j^2 + \varepsilon^2} 2^{-2b_{j}} + (1 - \omega_{j})^2 \theta_j^2 + \omega_{j}^2 \varepsilon^2 \right),$$

where $\Pi_{\text{blk}}(B)$, $\Pi_{\text{mon}}(B)$ are the classes of blockwise constant and mono-
tone allocations of the bits defined in (A.8), (A.9), and $\Omega_{\text{mon}}$ is the class of
monotone weights defined in (A.11). The proof is then completed by Lemma
A.8 showing that the last quantity is equal to $V_e(m, c, B)$.

5. Related work and future directions. Concepts related to quan-
tized nonparametric estimation appear in multiple communities. As men-
tioned in the introduction, Donoho’s 1997 Wald Lectures (on the eve of the
50th anniversary of Shannon’s 1948 paper), drew sharp parallels between
rate distortion, metric entropy and minimax rates, focusing on the same
Sobolev function spaces we treat here. One view of the present work is that
we take this correspondence further by studying how the risk continuously
degrades with the level of quantization. We have analyzed the precise leading
order asymptotics for quantized regression over the Sobolev spaces, showing
that these rates and constants are realized with coding schemes that are
adaptive to the smoothness $m$ and radius $c$ of the ellipsoid, achieving auto-
matically the optimal rate for the regime corresponding to those parameters
given the specified communication budget. Our detailed analysis is possible
due to what Nussbaum [14] calls the “Pinsker phenomenon,” refering to the
fact that linear filters attain the minimax rate in the over-sufficient regime.
It will be interesting to study quantized nonparametric estimation in cases
where the Pinsker phenomenon does not hold, for example over Besov bodies
and different $L_p$ spaces.

Many problems of rate distortion type are similar to quantized regression.
The standard “reverse water filling” construction to quantize a Gaussian
source with varying noise levels plays a key role in our analysis, as shown
in Section 3. In our case the Sobolev ellipsoid is an infinite Gaussian se-
quence model, requiring truncation of the sequence at the appropriate level
depending on the targeted quantization and estimation error. In the case
of Euclidean balls, Draper and Wornell [8] study rate distortion problems
motivated by communication in sensor networks; this is closely related to
the problem of quantized minimax estimation over Euclidean balls that we
analyzed in [23]. The essential difference between rate distortion and our
quantized minimax framework is that in rate distortion the quantization is
carried out for a random source, while in quantized estimation we quantize
our estimate of the deterministic and unknown basis coefficients. Since linear
estimators are asymptotically minimax for Sobolev spaces under squared er-
error (the “Pinanker phenomenon”), this naturally leads to an alternative view
of quantizing the observations, or said differently, of compressing the data
before estimation.

Statistical estimation from compressed data has appeared previously in
different communities. In [22] a procedure is analyzed that compresses data
by random linear transformations in the setting of sparse linear regression.
Zhang and Berger [21] study estimation problems when the data are com-
municated from multiple sources; Ahlswede and Csizár [2] consider testing
problems under communication constraints; the use of side information is
studied by Ahlswede and Burnashev [1]; other formulations in terms of mul-
titerminal information theory are given by Han and Amari [12]; nonpara-
metric problems are considered by Raginsky in [16]. In a distributed setting
the data may be divided across different compute nodes, with distributed
estimates then aggregated or pooled by communicating with a central node.
The general “CEO problem” of distributed estimation was introduced by
Berger, Zhang and Viswanathan [3], and has been recently studied in para-
metric settings in [20, 11]. These papers take the view that the data are
communicated to the statistician at a certain rate, which may introduce
distortion, and the goal is to study the degradation of the estimation error.
In contrast, in our setting we can view the unquantized data as being fully
available to the statistician at the time of estimation, with communication
constraints being imposed when communicating the estimated model to a
remote location.

Finally, our quantized minimax analysis shows achievability using random
coding schemes, which are not computationally efficient. A natural problem
is to develop practical coding schemes that come close to the minimax lower
bounds. In our view, the most promising approach currently is to exploit
source coding schemes based on greedy sparse regression [19], applying such
techniques blockwise according to the procedure we developed in Section 4.

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APPENDIX A: PROOFS OF TECHNICAL RESULTS

In this section, we provide proofs for Theorems 3.1 and 4.1.

A.1. Proof of Theorem 3.1. We first show

Lemma A.1. The quantized minimax risk is lower bounded by $V_\varepsilon(m, c, B_\varepsilon)$, the value of the optimization $(P_1)$.

Proof. As will be clear to the reader, $V_\varepsilon(m, c, B_\varepsilon)$ is achieved by some $\sigma^2$ that is non-increasing and finitely supported. Let $\sigma^2$ be such that

$$\sigma_1^2 \geq \cdots \geq \sigma_n^2 > 0 = \sigma_{n+1} = \cdots, \sum_{j=1}^n \sigma_j^2 = \frac{c^2}{\pi^2 m},$$

and let

$$\Theta_n(m, c) = \{\theta \in \ell_2 : \sum_{j=1}^n \theta_j^2 \leq \frac{c^2}{\pi^2 m}, \theta_j = 0 \text{ for } j \geq n+1\} \subset \Theta(m, c).$$

For $\tau \in (0, 1)$, write $s_j^2 = (1-\tau)\sigma_j^2$ and let $\pi_\tau(\theta; \sigma^2)$ be a the prior distribution on $\theta$ such that

$$\theta_j \sim N(0, s_j^2), \quad j = 1, \ldots, n,$$

$$P(\theta_j = 0) = 1, \quad j \geq n+1.$$

We observe that

$$R_\varepsilon(m, c, B_\varepsilon) \geq \inf_{\hat{\theta}, C(\hat{\theta}) \leq B_\varepsilon} \sup_{\theta \in \Theta_n(m, c)} \mathbb{E}\|\theta - \hat{\theta}\|^2$$

$$\geq \inf_{\hat{\theta}, C(\hat{\theta}) \leq B_\varepsilon} \int_{\Theta_n(m, c)} \mathbb{E}\|\theta - \hat{\theta}\|^2 d\pi_\tau(\theta; \sigma^2)$$

$$\geq I - r$$

where $I$ is the integrated risk of the optimal quantized estimator

$$I = \inf_{\hat{\theta}, C(\hat{\theta}) \leq B_\varepsilon} \int_{\mathbb{R}^n \otimes \{0\}^\infty} \mathbb{E}\|\theta - \hat{\theta}\|^2 d\pi_\tau(\theta; \sigma^2)$$
and $r$ is the residual

$$r = \sup_{\hat{\theta} \in \Theta(m,c)} \int_{\Theta(m,c)} \mathbb{E} \|\theta - \hat{\theta}\|^2 d\pi_\tau(\theta; \sigma^2)$$

where $\Theta(m,c) = (\mathbb{R}^n \otimes \{0\}^\infty) \setminus \Theta_n(m,c)$. As shown in Section 3, $\lim_{\tau \to 0} I$ is lower bounded by the value of the optimization

$$\min_{\mu^2} \sum_{j=1}^\infty \mu_j^2 + \sum_{j=1}^\infty \frac{\sigma_j^4 \varepsilon^2}{\sigma_j^2 + \varepsilon^2}$$

such that

$$\sum_{j=1}^\infty \frac{1}{2} \log_+ \left( \frac{\sigma_j^4}{\mu_j^2(\sigma_j^2 + \varepsilon^2)} \right) \leq B_\varepsilon.$$

It then suffices to show that $r = o(I)$ as $\varepsilon \to 0$. Let $d_n = \sup_{\theta \in \Theta_n(m,c)} \|\theta\|$. We have

$$r = \sup_{\hat{\theta} \in \Theta(m,c)} \int_{\Theta_n(m,c)} \mathbb{E} \|\theta - \hat{\theta}\|^2 d\pi_\tau(\theta; \sigma^2)$$

$$\leq 2 \int_{\Theta_n(m,c)} (d_n^4 + \mathbb{E}\|\theta\|^2) d\pi_\tau(\theta; \sigma^2)$$

$$= 2 \left( d_n^2 \mathbb{P}(\theta \notin \Theta_n(m,c)) + (\mathbb{P}(\theta \notin \Theta_n(m,c))\mathbb{E}\|\theta\|^4)^{1/2} \right)$$

where we use the Cauchy-Schwarz inequality. Noticing that

$$\mathbb{E}\|\theta\|^4 = \mathbb{E}\left( \left( \sum_{j=1}^n \theta_j^2 \right)^2 \right)$$

$$= \sum_{j_1 \neq j_2} \mathbb{E}(\theta_{j_1}^2)\mathbb{E}(\theta_{j_2}^2) + \sum_{j=1}^n \mathbb{E}(\theta_j^4)$$

$$\leq \sum_{j_1 \neq j_2} s_{j_1}^2 s_{j_2}^2 + 3 \sum_{j=1}^n s_j^4$$

$$\leq 3 \left( \sum_{j=1}^n s_j^2 \right)^2 \leq 3d_n^4,$$

we obtain

$$r \leq 2d_n^2 \left( \mathbb{P}(\theta \notin \Theta_n(m,c)) + \sqrt{3\mathbb{P}(\theta \notin \Theta_n(m,c))} \right)$$

$$\leq 6d_n^2 \sqrt{\mathbb{P}(\theta \notin \Theta_n(m,c))}.$$
Thus, we only need to show that $\sqrt{\mathbb{P}(\theta \notin \Theta_n(m, c))} = o(I)$. In fact,

$$
\mathbb{P}(\theta \notin \Theta_n(m, c))
\begin{align*}
&= \mathbb{P}\left( \sum_{j=1}^{n} a_j^2 \theta_j^2 > \frac{c^2}{2m} \right) \\
&= \mathbb{P}\left( \sum_{j=1}^{n} a_j^2 (\theta_j^2 - \mathbb{E}(\theta_j^2)) > \frac{c^2}{2m} - (1 - \tau) \sum_{j=1}^{n} a_j^2 \sigma_j^2 \right) \\
&= \mathbb{P}\left( \sum_{j=1}^{n} a_j^2 (\theta_j^2 - \mathbb{E}(\theta_j^2)) > \frac{\tau c^2}{2m} \right) \\
&= \mathbb{P}\left( \sum_{j=1}^{n} a_j^2 s_j^2 (Z_j^2 - 1) > \frac{\tau}{1 - \tau} \sum_{j=1}^{n} a_j^2 s_j^2 \right)
\end{align*}
$$

where $Z_j \sim \mathcal{N}(0, 1)$. By Lemma A.2, we get

$$
\mathbb{P}(\theta \notin \Theta_n(m, c)) \leq \exp \left( -\frac{\tau^2}{8(1 - \tau)^2} \frac{\sum_{j=1}^{n} a_j^2 s_j^2}{\max_{1 \leq j \leq n} a_j^2 s_j^2} \right).
$$

For the $\sigma^2$ that achieves $V_\varepsilon(m, c, B_\varepsilon)$, we have that $\sqrt{\mathbb{P}(\theta \notin \Theta_n(m, c))} = o(I)$. □

**Lemma A.2 (Lemma 3.5 in [18]).** Suppose that $X_1, \ldots, X_n$ are i.i.d. $\mathcal{N}(0, 1)$. For $t \in (0, 1)$ and $\omega_j > 0$, $j = 1, \ldots, n$, we have

$$
\mathbb{P}\left( \sum_{j=1}^{n} \omega_j (X_j^2 - 1) > t \sum_{j=1}^{n} X_j \right) \leq \exp \left( -\frac{t^2}{8 \max_{1 \leq j \leq n} \omega_j} \right).
$$

**Proof of Lemma 3.2.** This is in fact Pinsker’s theorem, which gives the exact asymptotic minimax risk of estimation of normal means in the Sobolev ellipsoid. The proof can be found in [14] and [18]. □

**Proof of Lemma 3.3.** As argued in Section 3 for the lower bound in
the sufficient regime, optimization problem \( (Q_1) \) can be reformulated as

\[
\max_{\sigma^2, J} \ J \eta \\
\text{such that } \frac{1}{2} \sum_{j=1}^{J} \log_{+} \left( \frac{\sigma_j^4}{\eta (\sigma_j^2 + \epsilon^2)} \right) \leq B \epsilon \\
\sum_{j=1}^{J} a_j^2 \sigma_j^2 \leq \frac{c^2}{\pi^2 m} \\
(\sigma_j^2) \text{ is decreasing and } \frac{\sigma_j^4}{\sigma_j^2 + \epsilon^2} \geq \eta.
\]

\((Q_2)\)

Now suppose that we have a series \((\sigma_j^2)\) which satisfies the last constraint and is supported on \(\{1, \ldots, J\}\). By the first constraint, we have that

\[
J \eta = J \exp \left( -\frac{2B \epsilon}{J} \right) \left( \prod_{j=1}^{J} \frac{\sigma_j^4}{\sigma_j^2 + \epsilon^2} \right)^{\frac{1}{J}} \\
\leq J \exp \left( -\frac{2B \epsilon}{J} \right) \left( \prod_{j=1}^{J} \sigma_j^2 \right)^{\frac{1}{J}} \\
= J \exp \left( -\frac{2B \epsilon}{J} \right) \left( \prod_{j=1}^{J} a_j^2 \sigma_j^2 \right)^{\frac{1}{J}} \left( \prod_{j=1}^{J} a_j^{-2} \right)^{\frac{1}{J}} \\
\leq \exp \left( -\frac{2B \epsilon}{J} \right) \left( \sum_{j=1}^{J} a_j^2 \sigma_j^2 \right)^{\frac{1}{J}} \left( \prod_{j=1}^{J} a_j^{-2} \right)^{\frac{1}{J}} \\
\leq \frac{c^2}{\pi^2 m} \exp \left( -\frac{2B \epsilon}{J} \right) \left( \prod_{j=1}^{J} a_j^{-2} \right)^{\frac{1}{J}} \\
= \frac{c^2}{\pi^2 m} \left( \exp \left( \frac{B \epsilon}{m} \right) J! \right)^{-\frac{2m}{J}}.
\]

\((A.1)\)

This provides a series of upper bounds for \(Q_\epsilon(m, c, B \epsilon)\) parameterized by \(J\). Minimizing \((A.1)\) over \(J\), we obtain that the optimal \(J\) satisfies

\[
\frac{J^J}{J!} < \exp \left( \frac{B \epsilon}{m} \right) \leq \frac{(J + 1)^{J+1}}{(J + 1)!}.
\]

\((A.2)\)
Denote this optimal \( J \) by \( J_\varepsilon \). By Stirling’s approximation, we have
\[
\lim_{\varepsilon \to 0} \frac{B_\varepsilon}{J_\varepsilon} = 1,
\]
and plugging this asymptote into (A.1), we get as \( \varepsilon \to 0 \)
\[
\frac{c^2}{\pi^{2m}} \left( \exp \left( \frac{B_\varepsilon}{m} \right) J_\varepsilon! \right)^{\frac{2m}{J_\varepsilon}} \sim \frac{c^2}{\pi^{2m}} J_\varepsilon^{-2m} \sim \frac{c^2 m^{2m}}{\pi^{2m}} B^{-2m}_\varepsilon.
\]
This gives the desired upper bound (3.7).

Next we show that the upper bound (3.7) is asymptotically achievable
when \( B_\varepsilon \varepsilon^2 \rightarrow 0 \) and \( B_\varepsilon \rightarrow \infty \). It suffices to find a feasible solution that
attains (3.7). Let
\[
\tilde{\sigma}_j^2 = \frac{c^2}{\pi^{2m}} J_\varepsilon a_j^2, \quad j = 1, \ldots, J_\varepsilon.
\]
Note that the entire sequence of \( (\tilde{\sigma}_j^2)_{j=1}^{J_\varepsilon} \) does not qualify for a feasible solution, since the first constraint in (Q2) won’t be satisfied for any \( \eta \leq \frac{\tilde{\sigma}_j^2}{a_j^2} \).
We keep only the first \( J_\varepsilon' \) terms of \( (\tilde{\sigma}_j^2) \), where \( J_\varepsilon' \) is the largest \( j \) such that
\[
(A.3) \quad \frac{\tilde{\sigma}_j^2}{J_\varepsilon a_j^2} \geq \tilde{\sigma}_j^2.
\]
Thus,
\[
\sum_{j=1}^{J_\varepsilon'} \frac{1}{2} \log_+ \left( \frac{\tilde{\sigma}_j^2}{\tilde{\sigma}_j^2 + \varepsilon} \right) \leq \sum_{j=1}^{J_\varepsilon} \frac{1}{2} \log_+ \left( \frac{\tilde{\sigma}_j^2}{\tilde{\sigma}_j^2} \right) \leq \sum_{j=1}^{J_\varepsilon} \frac{1}{2} \log_+ \left( \frac{\tilde{\sigma}_j^2}{\tilde{\sigma}_j^2} \right) \leq B_\varepsilon,
\]
where the last inequality is due to (A.2). This tells us that setting \( \eta = \tilde{\sigma}_j^2 \)
leads to a feasible solution to (Q2). As a result,
\[
(A.4) \quad Q_\varepsilon(m, c, B_\varepsilon) \geq J_\varepsilon' \tilde{\sigma}_j^2.
\]
If we can show that \( J_\varepsilon' \sim J_\varepsilon \), then
\[
(J_\varepsilon' \tilde{\sigma}_j^2) \sim J_\varepsilon \tilde{\sigma}_j^2 \sim \frac{c^2 m^{2m}}{\pi^{2m}} B^{-2m}_\varepsilon.
\]
To show that \( J_\varepsilon' \sim J_\varepsilon \), it suffices to show that \( a_{J_\varepsilon'} \sim a_{J_\varepsilon} \). Plugging the formula of \( \tilde{\sigma}_j^2 \) into (A.3) and solving for \( a_{J_\varepsilon'}^2 \), we get
\[
a_{J_\varepsilon'}^2 \sim \frac{-c^2}{\pi^{2m} J_\varepsilon} + \sqrt{\left( \frac{c^2}{\pi^{2m} J_\varepsilon} \right)^2 + \frac{4 c^2}{\pi^{2m} J_\varepsilon} \varepsilon^2 a_{J_\varepsilon}^2} \sim a_{J_\varepsilon}^2.
\]
where the last equivalence is due to the assumption $B_{\varepsilon_{m+1}} \to 0$ and L'Hopital's rule.

**Proof of Lemma 3.5.** Suppose that $\sigma^2(x)$ with $x_0$ solves ($P_4$). Consider function $\sigma^2(x) + \xi v(x)$ such that it is still feasible for ($P_4$), and thus we have

$$\int_0^{x_0} x^{2m} v(x) dx \leq 0.$$  

Now plugging $\sigma^2(x) + \xi v(x)$ for $\sigma^2(x)$ in the objective function of ($P_4$), taking derivative with respect to $\xi$, and letting $\xi \to 0$, we must have

$$\int_0^{x_0} \frac{v(x)}{(\sigma^2(x) + 1)^2} dx + x_0 \exp \left( \frac{1}{x_0} \int_0^{x_0} \log \frac{\sigma^4(x)}{\sigma^2(x) + 1} dx - \frac{2d}{x_0} \right) \frac{1}{x_0} \int_0^{x_0} \frac{2v(x)}{\sigma^2(x)} - \frac{v(x)}{\sigma^2(x) + 1} dx \leq 0,$$

which, after some calculation and rearrangement of terms, yields

$$\int_0^{x_0} v(x) \left( \frac{1}{(\sigma^2(x) + 1)^2} + \exp \left( \frac{1}{x_0} \int_0^{x_0} \log \frac{\sigma^4(x)}{\sigma^2(x) + 1} dx - \frac{2d}{x_0} \right) \frac{\sigma^2(x) + 2}{\sigma^2(x)(\sigma^2(x) + 1)} \right) dx \leq 0.$$  

Thus, we obtain that, for some $\lambda$

$$\frac{1}{(\sigma^2(x) + 1)^2} + \exp \left( \frac{1}{x_0} \int_0^{x_0} \log \frac{\sigma^4(x)}{\sigma^2(x) + 1} dx - \frac{2d}{x_0} \right) \frac{\sigma^2(x) + 2}{\sigma^2(x)(\sigma^2(x) + 1)} = \lambda x^{2m}.$$  

**Two lower bounds.** Here we give two lower bounds on the constant $P_{m,c} + Q_{m,c,d}$, i.e., the value of the optimization ($P_3$). Our approach to finding the lower bounds is to pick some particular feasible function and compute the corresponding objective function. Our first lower bound is based on the choice

$$\sigma^2(x) = ((\kappa x)^{-m} - 1) _ +$$

where

$$\kappa = \left( \frac{m}{m+1} \left( \frac{\pi^2 m}{2m+1} \right) \right).$$

This corresponds to the least favorable prior for the over-sufficient regime. The associated lower bound is

$$(A.6) \quad V_{m,c,d} = V(d^{-1}(d))$$

where $V$ is the function

$$V(x) = P_{m,c} + \frac{1}{\kappa} \left( \frac{m}{1-m}(\kappa x)^{1-m} + \frac{m}{m+1}(\kappa x)^{m+1} - \frac{2m^2}{m^2-1} \right).$$
and $d^{-1}$ is the inverse function of
\[ d(x_0) = m \left( \int_0^{x_0} \frac{1}{1 - (\kappa y)^{m}} dy - \frac{1}{2} x_0 \right). \]

The second lower bound is based on the following choice of $\sigma^2(x)$
\[ \sigma^2(x; x_0) = \frac{c^2}{x_0 \pi^{2m}} x^{-2m} \]
for some $x_0 > 0$. This corresponds to using the least favorable prior for the insufficient regime. The associated lower bound is given by
\[(A.7) \quad V_{m,c,d} = V(d^{-1}(d))\]
where now $V$ is the function
\[ V(x) = \frac{(\frac{c^2}{\pi^{2m}})^2}{\frac{c^2}{\pi^{2m}} x^{2m} + x^{4m}} + \int_0^x \frac{\frac{c^2}{\pi^{2m}}}{\frac{c^2}{\pi^{2m}} + xy^{2m}} dy \]
and $d^{-1}$ is the inverse function of
\[ d(x_0) = 2mx_0 - m \int_0^{x_0} \frac{\frac{c^2}{\pi^{2m}}}{\frac{c^2}{\pi^{2m}} + xy^{2m}} dy. \]

As shown in Figure 1, the first of these lower bounds is fairly tight for $d$ larger than about 3, and the second is fairly tight for smaller $d$. These bounds are simpler to compute than the exact solution to the variational problem.

**A.2. Proof of Theorem 4.1.** Now we give the details of the proof of Theorem 4.1. For the purpose of our analysis, we define two allocations of bits, the monotone allocation and the blockwise constant allocation,
\[(A.8) \quad \Pi_{\text{blk}}(B) = \left\{ (b_j)_{j=1}^\infty : \sum_{j=1}^\infty b_j \leq B, \ b_j = \bar{b}_k \text{ for } j \in J_k, \ 0 \leq b_j \leq b_{\max} \right\}, \]
\[(A.9) \quad \Pi_{\text{mon}}(B) = \left\{ (b_j)_{j=1}^\infty : \sum_{j=1}^\infty b_j \leq B, \ b_{j-1} \geq b_j, \ 0 \leq b_j \leq b_{\max} \right\}, \]
where \( b_{\text{max}} = 2 \log(1/\varepsilon) \). We also define two classes of weights, the monotonic weights and the blockwise constant weights,

\[
\Omega_{\text{blk}} = \{(\omega_j)_{j=1}^{\infty} : \omega_j = \bar{\omega}_k \text{ for } j \in J_k, \ 0 \leq \omega_j \leq 1\}, \\
\Omega_{\text{mon}} = \{(\omega_j)_{j=1}^{\infty} : \omega_{j-1} \geq \omega_j, \ 0 \leq \omega_j \leq 1\}.
\]

We will also need the following results from [18] regarding the weakly geometric system of blocks.

**Lemma A.3.** Let \( \{J_k\} \) be a weakly geometric block system defined by (4.1). Then there exists \( 0 < \varepsilon_0 < 1 \) and \( C > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \),

\[
K \leq C \log^2(1/\varepsilon), \\
\max_{1 \leq k \leq K-1} \frac{T_{k+1}}{T_k} \leq 1 + 3\rho_\varepsilon.
\]

We divide the proof into four steps.

**Step 1. Truncation and replacement.** The loss of the quantized estimator \( \hat{\theta} \) can be decomposed into

\[
\|\hat{\theta} - \theta\|^2 = \sum_{k=1}^{K} \|\hat{\theta}(k) - \theta(k)\|^2 + \sum_{j=N+1}^{\infty} \theta_j^2,
\]

where the remainder term satisfies

\[
\sum_{j=N+1}^{\infty} \theta_j^2 \leq N^{-2m} \sum_{j=N+1}^{\infty} a_j^2 \theta_j^2 = O(N^{-2m}).
\]

If we assume that \( m > 1/2 \), which corresponds to classes of continuous functions, the remainder term is then \( o(\varepsilon^2) \). If \( m \leq 1/2 \), the remainder term is on the order of \( O(\varepsilon^{4m}) \), which is still negligible compared to the order of the lower bound \( \varepsilon^{4m+1} \). To ease the notation, we will assume that \( m > 1/2 \), and write the remainder term as \( o(\varepsilon^2) \), but need to bear in mind that the proof works for all \( m > 0 \). We can thus discard the remainder term in our analysis. Recall that the quantized estimate for each block is given by

\[
\hat{\theta}(k) = \frac{\tilde{S}_k^2 - T_k \varepsilon^2}{\tilde{S}_k} \sqrt{1 - 2^{-2b_k} \tilde{Z}(k)},
\]

and consider the following estimate with \( \tilde{S}_k \) replaced by \( S_k \)

\[
\hat{\theta}(k) = \frac{S_k^2 - T_k \varepsilon^2}{S_k} \sqrt{1 - 2^{-2b_k} \tilde{Z}(k)}.
\]
Notice that
\[ \| \hat{\theta}(k) - \theta(k) \| = \left| \frac{\tilde{S}_k^2 - T_k \varepsilon^2}{\tilde{S}_k} - \frac{S_k^2 - T_k \varepsilon^2}{S_k} \right| \sqrt{1 - 2 - 2b_k \| Z(k) \|} \]
\[ \leq \left| \frac{\tilde{S}_k S_k + T_k \varepsilon^2}{S_k S_k} \right| | \tilde{S}_k - S_k | \]
\[ \leq 2 \varepsilon^2 \]

where the last inequality is because \( \tilde{S}_k S_k \geq T_k \varepsilon^2 \) and \( | \tilde{S}_k - S_k | \leq \varepsilon^2 \). Thus we can safely replace \( \hat{\theta}(k) \) by \( \tilde{\theta}(k) \) because
\[ \| \tilde{\theta}(k) - \theta(k) \|^2 = \| \tilde{\theta}(k) - \hat{\theta}(k) + \hat{\theta}(k) - \theta(k) \|^2 \]
\[ \leq \| \tilde{\theta}(k) - \hat{\theta}(k) \|^2 + \| \hat{\theta}(k) - \theta(k) \|^2 + 2\| \tilde{\theta}(k) - \hat{\theta}(k) \| \| \hat{\theta}(k) - \theta(k) \| \]
\[ = \| \tilde{\theta}(k) - \theta(k) \|^2 + O(\varepsilon^2). \]

Therefore, we have
\[ \mathbb{E} \| \tilde{\theta} - \theta \|^2 = \mathbb{E} \sum_{k=1}^{K} \| \tilde{\theta}(k) - \theta(k) \|^2 + O(K \varepsilon^2). \]

**Step 2. Expectation over codebooks.** Now conditioning on the data \( Y \), we work under the probability measure introduced by the random codebook. Write
\[ \lambda_k = \frac{S_k^2 - T_k \varepsilon^2}{S_k^2} \quad \text{and} \quad Z(k) = \frac{Y(k)}{\| Y(k) \|}. \]

We decompose and examine the following term
\[ A_k = \| \tilde{\theta}(k) - \theta(k) \|^2 \]
\[ = \| \tilde{\theta}(k) - \lambda_k S_k Z(k) + \lambda_k S_k Z(k) - \theta(k) \|^2 \]
\[ = \left\| \begin{array}{c} \tilde{\theta}(k) - \lambda_k S_k Z(k) \mid_{A_{k,1}} \\ \lambda_k S_k Z(k) - \theta(k) \mid_{A_{k,2}} \\ + 2 \langle \tilde{\theta}(k) - \lambda_k S_k Z(k), \lambda_k S_k Z(k) - \theta(k) \rangle \mid_{A_{k,3}} \end{array} \right\| \]

To bound the expectation of the first term \( A_{k,1} \), we need the following lemma, which bounds the probability of the distortion of a codeword exceeding the desired value.
Lemma A.4. Suppose that $Z_1, \ldots, Z_n$ are independent and each follows the uniform distribution on the $t$-dimensional unit sphere $\mathbb{S}^{t-1}$. Let $y \in \mathbb{S}^{t-1}$ be a fixed vector, and

$$Z^* = \arg \min_{z \in Z_{1:n}} \left\| \sqrt{1 - 2^{-2b}} z - y \right\|^2.$$

If $n = 2^{q_t}$, then

$$E \left\| \sqrt{1 - 2^{-2q}} Z^* - y \right\|^2 \leq 2^{-2q} (1 + \nu) + 2e^{-2t}$$

where

$$\nu = \frac{3 \log t + 5}{t - 3 \log t - 6}.$$

Observe that

$$A_{k,1} = \left\| \hat{\theta}(k) - \lambda_k S_k Z(k) \right\|^2 = \left\| \lambda_k S_k \sqrt{1 - 2^{-2b_k}} \tilde{Z}(k) - \lambda_k S_k Z(k) \right\|^2 = \lambda_k^2 S_k^2 \left\| \sqrt{1 - 2^{-2b_k}} \tilde{Z}(k) - Z(k) \right\|^2.$$

Then, it follows as a result of Lemma A.4 that

$$E \left( A_{k,1} | Y(k) \right) \leq \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} \left( 2^{-2b_k} (1 + \nu_\varepsilon) + 2e^{-2T_k} \right)$$

$$\leq \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} \left( 2^{-2b_k} (1 + \nu_\varepsilon) + 2e^{-2T_1} \right)$$

$$\leq \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} 2^{-2b_k} (1 + \nu_\varepsilon) + \frac{2e^2}{(jk \pi)^{2m} \varepsilon^2},$$

where $\nu_\varepsilon = \frac{3 \log T_1 - 5}{T_1 - 3 \log T_1 - 6}$. Since $A_{k,2}$ only depends on $Y(k)$, $E \left( A_{k,2} | Y(k) \right) = A_{k,2}$. Next we consider the cross term $A_{k,3}$. Write $\gamma_k = \frac{(\hat{\theta}(k), Y(k))}{\|Y(k)\|^2}$ and

$$A_{k,3} = 2 \left\langle \hat{\theta}(k) - \lambda_k S_k Z(k), \lambda_k S_k Z(k) - \theta(k) \right\rangle$$

$$= 2 \left\langle \hat{\theta}(k) - \lambda_k S_k Z(k), \lambda_k S_k Z(k) - \theta(k) \right\rangle$$

$$= \underbrace{2 \left\langle \hat{\theta}(k) - \lambda_k S_k Z(k), \lambda_k S_k Z(k) - \gamma_k Y(k) \right\rangle}_A_{k,3a}$$

$$+ \underbrace{2 \left\langle \hat{\theta}(k) - \lambda_k S_k Z(k), \lambda_k S_k Z(k) - \gamma_k Y(k) \right\rangle}_{A_{k,3b}}.$$
The quantity $\gamma_k$ is chosen such that $\langle Y_{(k)}, \gamma_k Y_{(k)} - \theta_{(k)} \rangle = 0$ and therefore

$$A_{k,3a} = 2 \left\langle \hat{\theta}_{(k)} - \lambda_k S_k Z_{(k)}, \gamma_k Y_{(k)} - \theta_{(k)} \right\rangle$$

$$= 2 \left\langle \Pi_{Y_{(k)}^\perp} (\hat{\theta}_{(k)} - \lambda_k S_k Z_{(k)}), \gamma_k Y_{(k)} - \theta_{(k)} \right\rangle$$

where $\Pi_{Y_{(k)}^\perp}$ denotes the projection onto the orthogonal complement of $Y_{(k)}$. Due to the choice of $\tilde{Z}_{(k)}$, the projection $\Pi_{Y_{(k)}^\perp} (\hat{\theta}_{(k)} - \lambda_k S_k Z_{(k)})$ is rotation symmetric and hence $\mathbb{E} \left( A_{k,3a} \mid Y_{(k)} \right) = 0$. Finally, for $A_{k,3b}$ we have

$$\mathbb{E} \left( A_{k,3b} \mid Y_{(k)} \right)$$

$$\leq 2\|\lambda_k S_k Z_{(k)} - \gamma_k Y_{(k)}\| \mathbb{E} \left( \|\hat{\theta}_{(k)} - \lambda_k S_k Z_{(k)}\| \mid Y_{(k)} \right)$$

$$\leq 2\|\lambda_k S_k Z_{(k)} - \gamma_k Y_{(k)}\| \sqrt{\mathbb{E} \left( \|\hat{\theta}_{(k)} - \lambda_k S_k Z_{(k)}\|^2 \mid Y_{(k)} \right)}$$

$$\leq 2\|\lambda_k S_k Z_{(k)} - \gamma_k Y_{(k)}\| \sqrt{(S_k^2 - T_k \varepsilon^2)^2 S_k^2 - 2 - 2\tilde{b}_k (1 + \nu) + \frac{2c^2}{(j_k \pi)^2 m} \varepsilon^2}.$$

Combining all the analyses above, we have

$$\mathbb{E} \left( A_k \mid Y_{(k)} \right)$$

$$\leq \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} 2^{-2\tilde{b}_k (1 + \nu)} + \frac{2c^2}{(j_k \pi)^2 m} \varepsilon^2 + \|\lambda_k S_k Z_{(k)} - \theta_{(k)}\|^2$$

$$+ 2\|\lambda_k S_k Z_{(k)} - \gamma_k Y_{(k)}\| \sqrt{\frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} 2^{-2\tilde{b}_k (1 + \nu)} + \frac{2c^2}{(j_k \pi)^2 m} \varepsilon^2}.$$

and summing over $k$ we get

$$\mathbb{E} \left( \|\hat{\theta} - \theta\|^2 \mid Y \right)$$

$$\leq \sum_{k=1}^K \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} 2^{-2\tilde{b}_k (1 + \nu)} + \sum_{k=1}^K \|\lambda_k S_k Z_{(k)} - \theta_{(k)}\|^2$$

$$+ 2 \sum_{k=1}^K \|\lambda_k S_k Z_{(k)} - \gamma_k Y_{(k)}\| \sqrt{\frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} 2^{-2\tilde{b}_k (1 + \nu)} + O(\varepsilon^2) + O(K \varepsilon^2)}.$$

**Step 3. Expectation over data.** First we will state three lemmas, which bound the deviation of the expectation of some particular functions of the norm of a Gaussian vector to the desired quantities. The proofs are given in Section A.2.
Lemma A.5. Suppose that $X_i \sim \mathcal{N}(\theta, \sigma^2)$ independently for $i = 1, \ldots, n$, where $\|\theta\|^2 \leq c^2$. Let $S$ be given by

$$S = \begin{cases} \sqrt{n\sigma^2} & \text{if } \|X\| < \sqrt{n\sigma^2} \\ \sqrt{n\sigma^2} + c & \text{if } \|X\| > \sqrt{n\sigma^2} + c \\ \|X\| & \text{otherwise.} \end{cases}$$

Then there exists some absolute constant $C_0$ such that

$$\mathbb{E}\left( \frac{S^2 - n\sigma^2}{S} - \frac{\langle \theta, X \rangle}{\|X\|} \right)^2 \leq C_0\sigma^2.$$

Lemma A.6. Let $X$ and $S$ be the same as defined in Lemma A.5. Then for $n > 4$

$$\mathbb{E}\left( \frac{(S^2 - n\sigma^2)^2}{S^2} \right) \leq \frac{\|\theta\|^4}{\|\theta\|^2 + n\sigma^2} + \frac{4n}{n - 4}\sigma^2.$$

Lemma A.7. Let $X$ and $S$ be the same as defined in Lemma A.5. Define

$$\hat{\theta}_+ = \left( \frac{\|X\|^2 - n\sigma^2}{\|X\|^2} \right) + X, \quad \hat{\theta}_1 = \left( \frac{S^2 - n\sigma^2}{S\|X\|} \right) X.$$

Then

$$\mathbb{E}\|\hat{\theta}_1 - \theta\|^2 \leq \mathbb{E}\|\hat{\theta}_+ - \theta\|^2 \leq \frac{n\sigma^2\|\theta\|^2}{\|\theta\|^2 + n\sigma^2} + 4\sigma^2.$$

We now take the expectation with respect to the data on both sides of (A.2). First, by the Cauchy-Schwarz inequality

$$\mathbb{E}\left( \|\lambda_k S_k Z_k - \gamma_k Y_k\| \sqrt{\frac{(S_k^2 - T_k\varepsilon^2)^2}{S_k^2} - 2^{-2T_k} (1 + \nu_\varepsilon) + O(\varepsilon^2)} \right) \leq \sqrt{\mathbb{E}\|\lambda_k S_k Z_k - \gamma_k Y_k\|^2} \sqrt{\mathbb{E}\left( \frac{(S_k^2 - T_k\varepsilon^2)^2}{S_k^2} - 2^{-2T_k} (1 + \nu_\varepsilon) + O(\varepsilon^2) \right)}.$$

We then calculate

$$\sqrt{\mathbb{E}\|\lambda_k S_k Z_k - \gamma_k Y_k\|^2} = \sqrt{\mathbb{E}\left( \frac{S_k^2 - T_k\varepsilon^2}{S_k} \frac{Y_k}{\|Y_k\|} - \frac{\langle \gamma_k, Y_k \rangle}{\|Y_k\|} \right)^2} \leq C_0\varepsilon,$$
where the last inequality is due to Lemma A.5, and $C_0$ is the constant therein. Plugging this in (A.2) and summing over $k$, we get

$$
\sum_{k=1}^{K} \mathbb{E} \left( \| \lambda_k S_k Z_{(k)} - \gamma_k Y_{(k)} \| \sqrt{\frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} - 2^{-2b_k} (1 + \nu \varepsilon) + O(\varepsilon^2)} \right)
\leq C_0 \varepsilon \sum_{k=1}^{K} \mathbb{E} \left( \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} - 2^{-2b_k} (1 + \nu \varepsilon) + O(\varepsilon^2) \right)
\leq C_0 \sqrt{K} \varepsilon \sum_{k=1}^{K} \left( \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} - 2^{-2b_k} (1 + \nu \varepsilon) + O(K \varepsilon^2) \right).
$$

Therefore,

$$
\mathbb{E} \| \hat{\theta} - \theta \|^2
\leq \mathbb{E} \sum_{k=1}^{K} \left( \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} - 2^{-2b_k} (1 + \nu \varepsilon) + \mathbb{E} \sum_{k=1}^{K} \| \lambda_k S_k Z_{(k)} - \theta_{(k)} \|^2 \right)
+ \sum_{k=1}^{K} \left( \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} - 2^{-2b_k} (1 + \nu \varepsilon) + O(K \varepsilon^2) \right)
+ O(K \varepsilon^2).
$$

Now we deal with the term $B_1$. Recall that the sequence $\tilde{b}$ solves problem (4.2), so for any sequence $\tilde{b} \in \Pi_{\text{blk}}$

$$
\sum_{k=1}^{K} \left( \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} - 2^{-2b_k} \right)
\leq \sum_{k=1}^{K} \left( \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} - 2^{-2\tilde{b}_k} \right).
$$

Notice that

$$
\left| \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} - \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} \right| = \left| S_k^2 - S_k^2 \right| \left| \frac{S_k^2}{S_k^2} - \frac{T_k \varepsilon^2}{S_k^2} \right| = O(\varepsilon^2)
$$

and thus,

$$
\sum_{k=1}^{K} \left( \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} - 2^{-2\tilde{b}_k} \right)
\leq \sum_{k=1}^{K} \left( \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} - 2^{-2b_k} + O(K \varepsilon^2) \right).
$$
Taking the expectation, we get
\[
\mathbb{E} \sum_{k=1}^{K} \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} 2^{-2\delta_k} \leq \sum_{k=1}^{K} \frac{\mathbb{E}(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} 2^{-2\delta_k} + O(K \varepsilon^2).
\]

Applying Lemma A.6, we get for \(T_k > 4\)
\[
\mathbb{E} \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} \leq \frac{\|\theta(k)\|^4}{\|\theta(k)\|^2 + T_k \varepsilon^2} + \frac{4T_k}{T_k - 4} \varepsilon^2
\]
and it follows that
\[
\mathbb{E} \sum_{k=1}^{K} \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} 2^{-2\delta_k} \leq \sum_{k=1}^{K} \frac{\|\theta(k)\|^4}{\|\theta(k)\|^2 + T_k \varepsilon^2} 2^{-2\delta_k} + O(K \varepsilon^2).
\]

Since \(b \in \Pi_{\text{blk}}\) is arbitrary,
\[
\mathbb{E} \sum_{k=1}^{K} \frac{(S_k^2 - T_k \varepsilon^2)^2}{S_k^2} 2^{-2\delta_k} \leq \min_{b \in \Pi_{\text{blk}}} \sum_{k=1}^{K} \frac{\|\theta(k)\|^4}{\|\theta(k)\|^2 + T_k \varepsilon^2} 2^{-2\delta_k} + O(K \varepsilon^2).
\]

Turning to the term \(B_2\), as a result of Lemma A.7 we have
\[
\|\lambda_k S_k Z(k) - \theta(k)\|^2 \leq \frac{\|\theta(k)\|^2 T_k \varepsilon^2}{\|\theta(k)\|^2 + T_k \varepsilon^2} + 4 \varepsilon^2.
\]

Combining the above results, we have shown that
\[
(A.12) \quad \mathbb{E}\|\hat{\theta} - \theta\|^2 \leq M + O(K \varepsilon^2) + C_0 \sqrt{K} \varepsilon \sqrt{M + O(K \varepsilon^2)}
\]
where
\[
M = (1 + \nu \varepsilon) \min_{b \in \Pi_{\text{blk}}(B)} \sum_{k=1}^{K} \frac{\|\theta(k)\|^4}{\|\theta(k)\|^2 + T_k \varepsilon^2} 2^{-2\delta_k} + \sum_{k=1}^{K} \frac{\|\theta(k)\|^2 T_k \varepsilon^2}{\|\theta(k)\|^2 + T_k \varepsilon^2} = (1 + \nu \varepsilon) \min_{b \in \Pi_{\text{blk}}(B)} \sum_{k=1}^{K} \frac{\|\theta(k)\|^4}{\|\theta(k)\|^2 + T_k \varepsilon^2} 2^{-2\delta_k}
\]
\[
+ \min_{\omega \in \Pi_{\text{blk}}} \sum_{k=1}^{K} \left( (1 - \tilde{\omega}) \|\theta(k)\|^2 + \tilde{\omega} T_k \varepsilon^2 \right).
\]
Step 4. Blockwise constant is almost optimal. We now show that in terms of both bit allocation and weight assignment, blockwise constant is almost optimal. Let’s first consider bit allocation. Let $B' = \frac{1}{1 + 3\rho \varepsilon} (B - T_1 b_{\max})$. We are going to show that

$$\min_{b \in \Pi_{blk}(B)} \sum_{k=1}^{K} \frac{\|\theta_{(k)}\|^4}{\theta_{(k)}^2 + T_k \varepsilon^2} 2^{-2b_k} \leq \min_{b \in \Pi_{mon}(B')} \sum_{j=1}^{N} \frac{\theta_j^4}{\theta_j^2 + \varepsilon^2} 2^{-2b_j}.$$

In fact, suppose that $b^* \in \Pi_{mon}(B')$ achieves the minimum on the right hand side, and define $b^*$ by

$$b^*_j = \begin{cases} \max_{i \in B_k} b^*_i & j \in B_k \\ 0 & j \geq N \end{cases}.$$

The sum of the elements in $b^*$ then satisfies

$$\sum_{j=1}^{\infty} b^*_j = \sum_{k=1}^{K-1} T_{k+1} \max_{j \in B_{k+1}} b^*_j$$

$$= T_1 b^*_1 + \sum_{k=1}^{K-1} T_{k+1} \max_{j \in B_{k+1}} b^*_j$$

$$\leq T_1 b_{\max} + \sum_{k=1}^{K-1} \frac{T_{k+1}}{T_k} \sum_{j \in B_k} b^*_j$$

$$\leq T_1 b_{\max} + (1 + 3\rho \varepsilon) \sum_{k=1}^{K-1} \sum_{j \in B_k} b^*_j$$

$$\leq T_1 b_{\max} + (1 + 3\rho \varepsilon) B'$$

$$= B,$$
which means that \( b^* \in \Pi_{\text{blk}}(B) \). It then follows that

\[
\min_{b \in \Pi_{\text{blk}}(B)} \sum_{k=1}^{K} \frac{\|\theta(k)\|_4^4}{\|\theta(k)\|_2^2 + T_k \varepsilon^2} 2^{-2b_k^*} \leq \sum_{k=1}^{K} \frac{\|\theta(k)\|_4^4}{\|\theta(k)\|_2^2 + T_k \varepsilon^2} 2^{-2b_k^*} \leq \sum_{k=1}^{K} \sum_{j \in B_k} \frac{\theta_j^4}{\theta_j^2 + \varepsilon^2} 2^{-2b_j^*} \leq \sum_{j=1}^{N} \frac{\theta_j^4}{\theta_j^2 + \varepsilon^2} 2^{-2b_j^*} = \min_{b \in \Pi_{\text{mon}}(B')} \sum_{j=1}^{N} \frac{\theta_j^4}{\theta_j^2 + \varepsilon^2} 2^{-2b_j},
\]

where (A.13) is due to Jensen’s inequality on the convex function \( \frac{x^2}{x^2 + \varepsilon^2} \)

\[
\left( \frac{1}{T_k} \|\theta(k)\|_2^2 + \varepsilon^2 \right)^2 \leq \frac{1}{T_k} \sum_{j \in B_k} \frac{\theta_j^4}{\theta_j^2 + \varepsilon^2}.
\]

Next, for the weights assignment, by Lemma 3.11 in [18], we have

\[
\min_{\omega \in \Pi_{\text{blk}}} \sum_{k=1}^{K} \left( (1 - \bar{\omega}_k)^2 \|\theta(k)\|_2^2 + \bar{\omega}_k^2 T_k \varepsilon^2 \right) \leq (1 + 3 \rho \varepsilon) \left( \min_{\omega \in \Pi_{\text{mon}}} \sum_{k=1}^{K} \left( (1 - \omega_j)^2 \theta_j^2 + \omega_j^2 \varepsilon^2 \right) \right) + T_1 \varepsilon^2.
\]
Combining (A.2) and (A.14), we get

$$M = (1 + \nu_\varepsilon) \min_{b \in \Pi_{lk}(B)} \sum_{k=1}^{K} \frac{\|\theta(k)\|^4}{\|\theta(k)\|^2 + T_k \varepsilon^2} 2^{-2b_k}$$

$$+ \min_{\omega \in \Omega_{lk}} \sum_{k=1}^{K} ((1 - \tilde{\omega}_k)^2\|\theta(k)\|^2 + \tilde{\omega}_k^2 T_k \varepsilon^2)$$

$$\leq (1 + \nu_\varepsilon) \min_{b \in \Pi_{lk}(B)} \sum_{k=1}^{K} \frac{\|\theta(k)\|^4}{\|\theta(k)\|^2 + T_k \varepsilon^2} 2^{-2b_k}$$

$$+ (1 + 3\rho_\varepsilon) \min_{\omega \in \Omega_{lk}} \sum_{k=1}^{K} ((1 - \tilde{\omega}_k)^2\|\theta(k)\|^2 + \tilde{\omega}_k^2 T_k \varepsilon^2) + T_1 \varepsilon^2$$

$$\leq (1 + \nu_\varepsilon) \left( \min_{b \in \Pi_{mon}(B')} \sum_{j=1}^{N} \theta_j^4 \frac{\theta_j^2}{\theta_j^2 + \varepsilon^2} 2^{-2b_j} \right) + \min_{\omega \in \Omega_{mon}} \sum_{k=1}^{K} ((1 - \omega_j)^2\theta_j^2 + \omega_j^2 \varepsilon^2) + T_1 \varepsilon^2.$$

Then by Lemma A.8,

$$M \leq (1 + \nu_\varepsilon) V_\varepsilon(m, c, B') + T_1 \varepsilon^2.$$ which, plugged into (A.12), gives us

$$\mathbb{E}\|\hat{\theta} - \theta\|^2 \leq (1 + \nu_\varepsilon) V_\varepsilon(m, c, B') + O(K \varepsilon^2)$$

$$+ C_0 \sqrt{K} \varepsilon \sqrt{(1 + \nu_\varepsilon) V_\varepsilon(m, c, B') + O(K \varepsilon^2)}.$$}

Recall that

$$\nu_\varepsilon = O\left( \frac{\log \log(1/\varepsilon)}{\log(1/\varepsilon)} \right), \quad K = O(\log^2(1/\varepsilon)),$$

and that

$$\lim_{\varepsilon \to 0} \frac{B'}{B} = \lim_{\varepsilon \to 0} \frac{1}{1 + 3\rho_\varepsilon} \left( 1 - \frac{T_1 b_{\text{max}}}{B} \right) = 1.$$ Thus,

$$\lim_{\varepsilon \to 0} \frac{V_\varepsilon(m, c, B')}{V_\varepsilon(m, c, B)} = 1.$$
Also notice that no matter how $B$ grows as $\varepsilon \to 0$, $V_{\varepsilon}(m, c, B) = O(\varepsilon^{\frac{4m}{2m+1}})$. Therefore,

\[
\lim_{\varepsilon \to 0} \frac{\mathbb{E}\|\hat{\theta} - \theta\|^2}{V_{\varepsilon}(B, m, c)} \leq \lim_{\varepsilon \to 0} \left( (1 + \nu_\varepsilon) \frac{V_{\varepsilon}(B', m, c)}{V_{\varepsilon}(B, m, c)} + \frac{O(K\varepsilon^2)}{V_{\varepsilon}(B, m, c)} \right) + C_0 \sqrt{\frac{K\varepsilon}{V_{\varepsilon}(B, m, c)}} \frac{V_{\varepsilon}(B, m, c)}{V_{\varepsilon}(B, m, c)} + C_0 \varepsilon^2 = 1
\]

which concludes the proof.

**Lemma A.8.** Let $V_1$ be the value of the optimization

\[
(A_1) \quad \max_{\theta} \min_{b} \sum_{j=1}^{N} \left( \frac{\theta_j^4}{\theta_j^2 + \varepsilon^2} 2^{-2b_j} + \frac{\theta_j^2 \varepsilon^2}{\theta_j^2 + \varepsilon^2} \right)
\]

such that $\sum_{j=1}^{N} b_j \leq B$, $b_j \geq 0$, $\sum_{j=1}^{J} a_j^2 \theta_j^2 \leq \frac{c^2}{\pi 2m}$,

and let $V_2$ be the value of the optimization

\[
(A_2) \quad \max_{\theta} \min_{b, \omega} \sum_{j=1}^{N} \left( \frac{\theta_j^4}{\theta_j^2 + \varepsilon^2} 2^{-2b_j} + (1 - \omega_j)^2 \theta_j^2 + \omega_j^2 \varepsilon^2 \right)
\]

such that $\sum_{j=1}^{N} b_j \leq B$, $b_j \geq b_j$, $0 \leq b_j \leq b_{\max}$, $\omega_{j-1} \geq \omega_j$,

$\sum_{j=1}^{J} a_j^2 \theta_j^2 \leq \frac{c^2}{\pi 2m}$.

Then $V_1 = V_2$.

**Proof of Lemmas.**

**Proof of Lemma A.4.** Let $\zeta(t)$ be a positive function of $t$ to be specified later. Let

\[
p_0 = \mathbb{P} \left( \|\sqrt{1 - 2^{-2b_j}Z_j - y}\| \leq 2^{-q}(1 + \zeta(t)^{-1}) \right).
\]
Using a result from [17], \( p_0 \) can be bounded by

\[
p_0 \geq \frac{\Gamma\left(\frac{t}{2} + 1\right)}{\pi t \Gamma\left(\frac{t+1}{2}\right)} 2^{-q(t-1)}(1 + \zeta(t)^{-1})^{t-1}.
\]

We obtain that

\[
\mathbb{E} \left\| \sqrt{1 - 2^{-2b}} Z^* - y \right\|^2 \\
\leq 2^{-2q}(1 + \zeta(t)^{-1})^2 + 2 \mathbb{P} \left( \left\| \sqrt{1 - 2^{-2b}} Z^* - y \right\| > 2^{-q}(1 + \zeta(t)^{-1}) \right) \\
\leq 2^{-2q}(1 + 2\zeta(t)^{-1}) + 2(1 - p_0)^n.
\]

To upper bound \((1 - p_0)^n\), we consider

\[
\log ((1 - p_0)^n) = n \log(1 - p_0) \leq -np_0 \\
\leq -2^{qt} \frac{\Gamma\left(\frac{t}{2} + 1\right)}{\pi t \Gamma\left(\frac{t+1}{2}\right)} 2^{-q(t-1)}(1 + \zeta(t)^{-1})^{t-1} \\
\leq -2^{qt} \frac{\Gamma\left(\frac{t}{2} + 1\right)}{\pi t \Gamma\left(\frac{t+1}{2}\right)} (1 + \zeta(t)^{-1})^{(\zeta(t)+1)^{t-1} + 1} \\
\leq - \sqrt{2\pi} \left( \frac{t}{2} \right)^{\frac{1}{2} + \frac{1}{2} e^{-\frac{t}{2}}} e^{\frac{t-1}{\zeta(t)+1}} \\
= - \frac{1}{\sqrt{\pi} e^{\frac{t}{2}}} t^{-\frac{1}{2}} \left( \frac{t}{t-1} \right)^{\frac{1}{2}} e^{\frac{t-1}{\zeta(t)+1}} \\
\leq - \frac{1}{\sqrt{\pi} e^{\frac{t}{2}}} t^{-\frac{1}{2}} e^{\frac{t-1}{\zeta(t)+1}}
\]

where we have used the Stirling’s approximation

\[
\sqrt{2\pi} z^{z+1/2} e^{-z} \leq \Gamma(z+1) \leq ez^{z+1/2} e^{-z}.
\]

In order for \((1 - p_0)^n \leq e^{-2t}\) to hold, we need

\[
-2t = - \frac{1}{\sqrt{\pi} e^{\frac{t}{2}}} t^{-\frac{1}{2}} e^{\frac{t-1}{\zeta(t)+1}},
\]

which leads to the choice of \(\zeta(t)\)

\[
\zeta(t) = \frac{t-1}{\log(2\sqrt{\pi} e^{\frac{t}{2}})} - 1.
\]

Observing that

\[
2\zeta(t)^{-1} \leq \frac{3 \log t + 5}{t - 3 \log t - 6}
\]

completes the proof.
Proof of Lemma A.5. We first claim that
\[
E\left( \frac{S^2 - n\sigma^2}{S} - \langle \theta, X \rangle \right)^2 \leq E\left( \frac{\|X\|^2 - n\sigma^2}{\|X\|} - \langle \theta, X \rangle \right)^2.
\]
In fact, writing \( E_r(\cdot) \) for the conditional expectation \( E(\cdot | \|X\| = r) \), it suffices to show that for \( r < \sqrt{n\sigma^2} \) and \( r > \sqrt{n\sigma^2 + c} \)
\[
E_r\left( \frac{S^2 - n\sigma^2}{S} - \langle \theta, X \rangle \right)^2 \leq E_r\left( \frac{\|X\|^2 - n\sigma^2}{\|X\|} - \langle \theta, X \rangle \right)^2.
\]
When \( r < \sqrt{n\sigma^2} \), it is equivalent to
\[
E_r\left( \frac{\|X\|^2 - n\sigma^2}{\|X\|} - \langle \theta, X \rangle \right)^2 \leq E_r\left( \|X\|^2 - n\sigma^2 \right) - \frac{\langle \theta, X \rangle}{\|X\|} \geq 0.
\]
This can be obtained by following a similar argument as in Lemma A.6 in [18]. When \( r > \sqrt{n\sigma^2 + c} \), we need to show that
\[
E_r\left( \frac{(\sqrt{n\sigma^2 + c})^2 - n\sigma^2}{\sqrt{n\sigma^2 + c}} - \langle \theta, X \rangle \right)^2 \leq E_r\left( \|X\|^2 - n\sigma^2 \right) - \frac{\langle \theta, X \rangle}{\|X\|} \geq 0.
\]
which, after some algebra, boils down to
\[
\frac{(\sqrt{n\sigma^2 + c})^2 - n\sigma^2}{\sqrt{n\sigma^2 + c}} + \frac{r^2 - n\sigma^2}{r} \geq 2\frac{E_r(\theta, X)}{r}.
\]
This holds because
\[
r \left( \frac{(\sqrt{n\sigma^2 + c})^2 - n\sigma^2}{\sqrt{n\sigma^2 + c}} + \frac{r^2 - n\sigma^2}{r} - \frac{2}{r} E_r(\theta, X) \right)
\geq \|\theta\|^2 + r^2 - n\sigma^2 - 2E_r(\theta, X)
\geq E_r\|X - \theta\|^2 - n\sigma^2
\geq 0
\]
where we have used the assumption that \( r > \sqrt{n\sigma^2 + c} \), \( \|\theta\| \leq c \) and that
\[
E_r\|X - \theta\| \geq E_r\|X\| - \|\theta\| \geq \sqrt{n\sigma^2}.
\]
Now that we have shown (A.2) and noting that
\[
E\left( \frac{\|X\|^2 - n\sigma^2}{\|X\|} - \langle \theta, X \rangle \right)^2 \geq \sigma^2 E\left( \frac{\|X/\sigma\|^2 - n}{\|X/\sigma\|} - \frac{\langle \theta/\sigma, X/\sigma \rangle}{\|X/\sigma\|} \right)^2,
\]
we can assume that $X \sim N(\theta, I_n)$ and equivalently show that there exists a universal constant $C_0$ such that

$$\mathbb{E} \left( \frac{\|X\|^2 - n \langle \theta, X \rangle}{\|X\|} \right)^2 \leq C_0$$

holds for any $n$ and $\theta$. Letting $Z = X - \theta$ and writing $\|\theta\|^2 = \xi$, we have

$$\mathbb{E} \left( \frac{\|X\|^2 - n \langle \theta, X \rangle}{\|X\|} \right)^2 = \mathbb{E} \left( \frac{\|Z + \theta\|^2 - n - \xi}{\|Z + \theta\|} \right)^2 \leq 2\mathbb{E} \left( \frac{\|Z + \theta\|^2 - n - \xi}{\|Z + \theta\|} \right)^2 + 2\mathbb{E} \left( \frac{\langle \theta, Z \rangle}{\|Z + \theta\|} \right)^2 \leq 2\mathbb{E} \|Z + \theta\|^2 - 4(n + \xi) + 2\mathbb{E} \left( \frac{\langle \theta, Z \rangle}{\|Z + \theta\|} \right)^2 = \frac{8(n + \xi)}{n + \xi - 4} + 2\mathbb{E} \left( \frac{\langle \theta, Z \rangle}{\|Z + \theta\|} \right)^2,$$

where the last inequality is due to Lemma A.9. To bound the last term, we apply the Cauchy-Schwarz inequality and get

$$\mathbb{E} \left( \frac{\langle \theta, Z \rangle}{\|Z + \theta\|} \right)^2 \leq \sqrt{\mathbb{E} \|Z + \theta\|^4 \mathbb{E} \langle \theta, Z \rangle^4} \leq \sqrt{\frac{3(n - 4)\xi^2}{(n - 6)(n + \xi - 4)(n + \xi - 6)}}$$

where the last inequality is again due to Lemma A.9. Thus we just need to take $C_0$ to be

$$\sup_{n \geq 7, \xi \geq 0} \frac{8(n + \xi)}{n + \xi - 4} + 2\sqrt{\frac{3(n - 4)\xi^2}{(n - 6)(n + \xi - 4)(n + \xi - 6)}},$$

which is apparently a finite quantity.

Proof of Lemma A.6. Since the function $(x^2 - n\sigma^2)^2/x^2$ is decreasing on $(0, \sqrt{n\sigma^2})$ and increasing on $(\sqrt{n\sigma^2}, \infty)$, we have

$$\frac{(S^2 - n\sigma^2)^2}{S^2} \leq \frac{\langle \|X\|^2 - n\sigma^2 \rangle^2}{\|X\|^2},$$
and it follows that if $n > 4$

\begin{equation}
E \frac{(S^2 - n\sigma^2)^2}{S^2} \leq E \frac{\|X\|^2 - n\sigma^2}{\|X\|^2}
\end{equation}

\begin{equation}
= E\|X\|^2 - 2n\sigma^2 + n^2\sigma^4E \left( \frac{1}{\|X\|^2} \right)
\end{equation}

\begin{equation}
\leq \|\theta\|^2 - n\sigma^2 + \frac{n^2\sigma^4}{\|\theta\|^2 + n\sigma^2 - 4\sigma^2}
\end{equation}

\begin{equation}
\leq \|\theta\|^2 + n\sigma^2 + \frac{4n}{n - 4}\sigma^2
\end{equation}

where (A.17) is due to Lemma A.9, and (A.18) is obtained by

\[
\|\theta\|^2 - n\sigma^2 + \frac{n^2\sigma^4}{\|\theta\|^2 + n\sigma^2 - 4\sigma^2}
= \|\theta\|^4 + 4\sigma^2(n\sigma^2 - \|\theta\|^2) - \|\theta\|^4
\]

\[
= \frac{4n^2\sigma^6}{4n\sigma^2 - \|\theta\|^2 + n\sigma^2}
\]

\[
\leq \frac{4n}{n - 4}\sigma^2.
\]

Proof of Lemma A.7. First, the second inequality

\[
E\|\hat{\theta}_+ - \theta\|^2 \leq \frac{n\sigma^2\|\theta\|^2}{\|\theta\|^2 + n\sigma^2} + 4\sigma^2
\]

is given by Lemma 3.10 from [18]. We thus focus on the first inequality. For convenience we write

\[
g_+(x) = \left( \frac{\|x\|^2 - n\sigma^2}{\|x\|^2} \right)_+, \quad g_\| (x) = \frac{s(x)^2 - na^2}{s(x)\|x\|}
\]

with

\[
s(x) = \begin{cases} \sqrt{n\sigma^2} & \text{if } \|x\| < \sqrt{n\sigma^2} \\ \sqrt{n\sigma^2} + c & \text{if } \|x\| > \sqrt{n\sigma^2} + c \\ \|x\| & \text{otherwise} \end{cases}
\]

Notice that $g_+(x) = g_\| (x)$ when $\|x\| \leq \sqrt{n\sigma^2} + c$ and $g_+(x) > g_\| (x)$ when $\|x\| > \sqrt{n\sigma^2} + c$. Since $g_\|$ and $g_+$ both only depend on $\|x\|$, we sometimes
will also write \( g_1(\|x\|) \) for \( g_1(x) \) and \( g_\pm(\|x\|) \) for \( g_\pm(x) \). Setting \( \mathbb{E}_r(\cdot) \) to denote the conditional expectation \( \mathbb{E}(\cdot|\|X\| = r) \) for brevity, it suffices to show that for \( r \geq \sqrt{n\sigma^2 + c} \)

\[
\mathbb{E}_r(\|g_1(X)X - \theta\|^2) \leq \mathbb{E}_r(\|g_\pm(X)X - \theta\|^2)
\]

\[
\iff g_1(r)^2 r^2 - 2g_1(r)\mathbb{E}_r(X, \theta) \leq g_\pm(r)^2 r^2 - 2g_\pm(r)\mathbb{E}_r(X, \theta)
\]

\[
\iff (g_1(r)^2 - g_\pm(r)^2) r^2 \geq 2(g_1(r) - g_\pm(r))\mathbb{E}_r(X, \theta)
\]

\[
\iff (g_1(r) + g_\pm(r))^2 \geq 2\mathbb{E}_r(X, \theta).
\]  

(A.19)

On the other hand, we have

\[
(g_1(r) + g_\pm(r))^2 \geq \left( \frac{\|\theta\|^2}{r^2} + \frac{r^2 - n\sigma^2}{r^2} \right) r^2
\]

\[
= \|\theta\|^2 + r^2 - n\sigma^2
\]

\[
= \|\theta\|^2 + r^2 - 2\mathbb{E}_r(X, \theta) - n\sigma^2 + 2\mathbb{E}_r(X, \theta)
\]

\[
= \mathbb{E}_r(\|X - \theta\|^2 - n\sigma^2 + 2\mathbb{E}_r(X, \theta)
\]

\[
\geq 2\mathbb{E}_r(X, \theta)
\]

where the last inequality is because

\[
\|X - \theta\|^2 \geq (\|X\| - \|\theta\|)^2 \geq n\sigma^2.
\]

Thus, (A.19) holds and hence \( \mathbb{E}\|\hat{\theta}_1 - \theta\|^2 \leq \mathbb{E}\|\hat{\theta}_\pm - \theta\|^2 \). \(\square\)

**Proof of Lemma A.8.** It is easy to see that \( V_1 \leq V_2 \), because for any \( \theta \) the inside minimum is smaller for \((A_1)\) than for \((A_2)\). Next, we will show \( V_1 \geq V_2 \).

Suppose that \( \theta^* \) achieves the value \( V_2 \), with corresponding \( b^* \) and \( \omega^* \). We claim that \( \theta^* \) is non-increasing. In fact, if \( \theta^* \) is not non-increasing, then there must exist an index \( j \) such that \( \theta^*_j < \theta^*_{j+1} \) and for simplicity let’s assume that \( \theta^*_j < \theta^*_2 \). We are going to show that this leads to \( b^*_1 = b^*_2 \) and \( \omega^*_1 = \omega^*_2 \). Write

\[
s_1 = \frac{\theta^*_1}{\theta^*_1 + \varepsilon^2}, \quad s_2 = \frac{\theta^*_2}{\theta^*_2 + \varepsilon^2}.
\]
We have $s_1 < s_2$. Let $\tilde{b}^* = \frac{b_1^* + b_2^*}{2}$ and observe that $b_1^* \geq \tilde{b}^* \geq b_2^*$. Notice that

\[
\left(s_1 2^{-2b_1^*} + s_2 2^{-2b_2^*}\right) - \left(s_1 2^{-2\tilde{b}^*} + s_2 2^{-2\tilde{b}^*}\right) = s_1 \left(2^{-2b_1^*} - 2^{-2\tilde{b}^*}\right) + s_2 \left(2^{-2b_2^*} - 2^{-2\tilde{b}^*}\right) \geq s_2 \left(2^{-2b_2^*} - 2^{-2\tilde{b}^*}\right) \geq 0,
\]

where equality holds if and only if $b_1^* = b_2^*$, since $s_2 > s_1 \geq 0$. Hence, $b_1^*$ and $b_2^*$ have to be equal, or otherwise it would contradict with the assumption that $b^*$ achieves the inside minimum of $(A_2)$. Now turn to $\omega^*$. Write $\bar{\omega}^* = \frac{\omega_1^* + \omega_2^*}{2}$ and note that $\omega_1^* \geq \bar{\omega}^* \geq \omega_2^*$. Consider

\[
\left((1 - \omega_1^*)^2 \theta_1^2 + \omega_1^2 \varepsilon^2\right) + \left((1 - \omega_2^*)^2 \theta_2^2 + \omega_2^2 \varepsilon^2\right) - \left((1 - \bar{\omega}^*)^2 (\theta_1^2 + \theta_2^2) + 2\bar{\omega}^2 \varepsilon^2\right) = \left((1 - \omega_1^*)^2 (1 - \bar{\omega}^*)^2 \theta_1^2 + (1 - \omega_2^*)^2 (1 - \bar{\omega}^*)^2 \theta_2^2 + (\omega_1^2 + \omega_2^2 - 2\bar{\omega}^2) \varepsilon^2\right) \geq 0,
\]

where the equality holds if and only if $\omega_1^* = \omega_2^*$. Therefore, $\omega_1^*$ and $\omega_2^*$ must be equal. Now, with $b_1^* = b_2^*$ and $\omega_1^* = \omega_2^*$, we can switch $\theta_1^*$ and $\theta_2^*$ without increasing the objective function and violating the constraints. Thus, our claim that $\theta^*$ is non-increasing is justified.

Next, we will show that $b_1^* < b_{\text{max}}$. If $b_1^* = b_{\text{max}}$, then for $j = 1, \ldots, N$

\[
\frac{\theta_j^* \varepsilon^2}{\theta_j^2 + \varepsilon^2} 2^{-2b_j^*} \leq \frac{\theta_j^* \varepsilon^2}{\theta_j^2 + \varepsilon^2} 2^{-2b_{\text{max}}} \leq c^2 2^{-4 \log(1/\varepsilon)} = c^2 \varepsilon^4,
\]

and therefore

\[
\sum_{j=1}^N \frac{\theta_j^* \varepsilon^2}{\theta_j^2 + \varepsilon^2} 2^{-2b_j^*} \leq N c^2 \varepsilon^4 = o\left(\varepsilon^{\frac{4m}{m+1}}\right).
\]

Now we have shown that the solution triplet $(\theta^*, b^*, \omega^*)$ to $(A_2)$ satisfy that $\theta^*$ is non-decreasing and $b_1^* < b_{\text{max}}$. In order to prove $V_1 \geq V_2$, it then suffices to show that if we take $\theta = \theta^*$ in $(A_1)$, the minimizer $b^*$ is non-increasing and $b_1^* \leq b_{\text{max}}$. In fact, if so, we will have $b^* = b^*$ as well as $\omega^* = \frac{\theta_j^* \varepsilon^2}{\theta_j^2 + \varepsilon^2}$ and then

\[
V_1 \geq \min_{b : \sum_{j=1}^N b_j \leq B} \sum_{j=1}^N \left(\frac{\theta_j^* \varepsilon^2}{\theta_j^2 + \varepsilon^2} 2^{-2b_j^*} + \frac{\theta_j^* \varepsilon^2}{\theta_j^2 + \varepsilon^2}\right) \geq V_2;
\]
Lemma A.9. Suppose that $W_{n,\xi}$ follows a non-central chi-square distribution with $n$ degrees of freedom and non-centrality parameter $\xi$. We have for $n \geq 5$

$$E\left(W_{n,\xi}^{-1}\right) \leq \frac{1}{n + \xi - 4},$$

and for $n \geq 7$

$$E\left(W_{n,\xi}^{-2}\right) \leq \frac{n - 4}{(n - 6)(n + \xi - 4)(n + \xi - 6)}.$$

Proof. It is well known that the non-central chi-square random variable $W_{n,\xi}$ can be written as a Poisson-weighted mixture of central chi-square distributions, i.e., $W_{n,\xi} \sim \chi^2_{n+2K}$ with $K \sim \text{Poisson}(\xi/2)$. Then

$$E\left(W_{n,\xi}^{-1}\right) = E\left(E(W_{n,\xi}^{-1} \mid K)\right) = E\left(\frac{1}{n + 2K - 2}\right) \geq \frac{1}{n + 2EK - 2} = \frac{1}{n + \xi - 2},$$

where we have used the fact that $E(1/\chi^2_{n}) = n - 2$ and Jensen’s inequality. Similarly, we have

$$E\left(W_{n,\xi}^{-2}\right) = E\left(E(W_{n,\xi}^{-2} \mid K)\right) = E\left(\frac{1}{(n + 2K - 2)(n + 2K - 4)}\right) \geq \frac{1}{(n + 2EK - 2)(n + 2EK - 4)} = \frac{1}{(n + \xi - 2)(n + \xi - 4)}.$$

Using the Poisson-weighted mixture representation, the following recurrence relation can be derived [5]

(A.20) \hspace{1cm} 1 = \xi E\left(W_{n+4,\xi}^{-1}\right) + n E\left(W_{n+2,\xi}^{-1}\right),

(A.21) \hspace{1cm} E\left(W_{n,\xi}^{-1}\right) = \xi E\left(W_{n+4,\xi}^{-2}\right) + n E\left(W_{n+2,\xi}^{-2}\right),

for $n \geq 3$. Thus,

$$E\left(W_{n+4,\xi}^{-1}\right) = \frac{1}{\xi} - \frac{n}{\xi} E\left(W_{n+2,\xi}^{-1}\right) \leq \frac{1}{\xi} - \frac{n}{\xi} \frac{1}{n + \xi} = \frac{1}{n + \xi}.$$
Replacing $n$ by $n - 4$ proves (A.9). On the other hand, rearranging (A.20), we get

$$
\mathbb{E}(W_{n+2,\xi}^{-1}) = \frac{1}{n} - \frac{\xi}{n} \mathbb{E}(W_{n+4,\xi}^{-1}) \\
\leq \frac{1}{n} - \frac{\xi}{n(n + \xi + 2)} \\
= \frac{n + 2}{n(n + \xi + 2)}.
$$

Now using (A.21), we have

$$
\mathbb{E}(W_{n+4,\xi}^{-2}) = \frac{1}{\xi} \mathbb{E}(W_{n,\xi}^{-1}) - \frac{n}{\xi} \mathbb{E}(W_{n+2,\xi}^{-2}) \\
\leq \frac{n}{\xi(n - 2)(n + \xi)} - \frac{n}{\xi(n + \xi)(n + \xi - 2)} \\
= \frac{(n - 2)(n + \xi)(n + \xi - 2)}{(n - 2)(n + \xi)(n + \xi - 2)}.
$$

Replacing $n$ by $n - 4$ proves (A.9). \qed