Derivative Formula and Harnack Inequality for SDEs Driven by Lévy Processes

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By using lower bound conditions of the Lévy measure, derivative formulae and Harnack inequalities are derived for linear stochastic differential equations driven by Lévy processes. As applications, explicit gradient estimates and heat kernel inequalities are presented. As byproduct, a new Girsanov theorem for Lévy processes is derived.

Keywords Derivative Formula; Gradient estimate; Harnack inequality; Lévy process.

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1. Introduction

The derivative formula enables one to derive explicit gradient estimates; while the Harnack inequality has been applied to the study of heat kernel estimates, contractivity properties, transportation-const inequalities and properties of the invariant probability measures, see, for example, [6, 25, 30] and references within (see Section 4.2 for some general results).

Recall that Bismut’s derivative formula of elliptic diffusion semigroups [4], also known as Bismut-Elworthy-Li formula due to [7], is a powerful tool for stochastic analysis on Riemannian manifolds and has been extended and applied to stochastic differential equations (SDEs) driven by noises with a non-trivial Gaussian parts, see e.g., [21] and references within for the study of diffusion-jump processes. But, up to our knowledge, explicit derivative formula relying only on the Lévy measure is not yet available.

On the other hand, by using couplings constructed through Girsanov transforms, the dimension-free Harnack inequality, first introduced by the author in [22] for diffusion semigroups on manifolds, has been established and applied to
various SDEs and SPDEs driven by Gaussian noises, see [2, 3, 6, 11, 13, 14, 18, 24, 25, 28–30, 33]. Since arguments used in these references essentially relies on special properties of the Brownian motion, they do not apply to the jump setting. Therefore, it is in particular interesting to built up a reasonable theory on derivative formula and Harnack inequality for pure jump processes.

In this article, we aim to establish derivative formula and Harnack inequality for the semigroup associated to SDEs driven by Lévy jump processes using lower bound conditions of the Lévy measure. As observed in a recent article [27], where the coupling property is confirmed for a class of linear SDEs driven by Lévy processes, the Mecke formula on the Poisson space will play an alternative role in the jump case to the Girsanov transform in the diffusion case. Indeed, with helps of this formula we will be able to establish explicit derivative formulae and Harnack inequalities for a class of jump processes (see Sections 3 and 4).

Before move on, let us introduce some recent results concerning regularity properties of the semigroup associated to the following linear SDE

$$dX_t - A_t X_t dt + \sigma_t dL_t,$$

(1.1)
on $\mathbb{R}^d$, where $A, \sigma : [0, \infty) \to \mathbb{R}^d \otimes \mathbb{R}^d$ are measurable such that $\sigma_t$ is invertible for $s \geq 0$ and $A, \sigma, \sigma^{-1}$ are locally bounded, $L_t$ is the Lévy process on $\mathbb{R}^d$ with Lévy measure $\nu$ (see, e.g., [1, 10]). Let $P_t$ be the semigroup associated to (1.1), that is,

$$P_t f(x) = \mathbb{E} f(X^x_t), \quad t \geq 0, \ x \in \mathbb{R}^d, \ f \in \mathcal{B}_b(\mathbb{R}^d),$$

where $\mathcal{B}_b(\mathbb{R}^d)$ is the set of all bounded measurable functions on $\mathbb{R}^d$, and $X^x_t$ is the solution with initial data $x$. To formulate the solution, for any $s \geq 0$ let $(T_{s,t})_{t \geq s}$ solve the equation on $\mathbb{R}^d \otimes \mathbb{R}^d$:

$$\frac{d}{dt} T_{s,t} = A_t T_{s,t}, \quad T_{s,s} = I.$$

Let $T_t = T_{0,t}$, $t \geq 0$. We have $T_t = T_{s,t} T_s$ for $t \geq s \geq 0$ and

$$X_t = T_t x + \int_0^t T_{s,t} \sigma_s dL_s, \quad t \geq 0.$$

(1.2)

By using lower bound conditions of the Lévy measure $\nu$, the coupling property and gradient estimates have been derived in [5, 19, 20, 26, 27]. Moreover, by using subordinations, the dimension-free Harnack inequality has been established in [8] for some jump processes in terms of known inequalities in the diffusion setting, where for the log-Harnack inequality the associated Bernstein function can be very broad but for the Harnack inequality with a power the function was assumed to have a growth stronger than $\sqrt{t}$. When $A_t$ and $\sigma_t$ are independent of $t$ and $\nu(\mathrm{d}z) \geq c|z|^{-(d+\alpha)} \mathrm{d}z$ for some constants $c > 0$ and $\alpha \in (0, 2)$, that is, the equation is time-homogenous with noise having an $\alpha$-stable part, a different version of Harnack inequality was presented in [31, Theorem 1.1 and Corollary 1.3]: for any $p \geq 1$ there exists a constant $C > 0$ such that

$$(P_t f(x+h))^p \leq C P_t f^p(x) \left(1 + \frac{|h|}{(t \wedge 1)^{\frac{1}{2}}} \right)^{p(d+\alpha)}, \quad t > 0, x, h \in \mathbb{R}^d$$

(1.3)
holds for positive \( f \in \mathcal{B}(\mathbb{R}^d) \). Since (1.3) allows \( p = 1 \) which is impossible even for the Brownian motion, this inequality is somehow stronger than general ones derived in the diffusion setting. On the other hand, however, this equality is not sharp for small distance since when \( h = 0 \) it is worse than the classical Jensen inequality. In particular, (1.3) does not imply the strong Feller property as the usual ones do (see Theorem 4.4(2) below).

The remainder of the article is organized as follows. In the next section, we present two lemmas, which will be used to establish derivative formulae and Harnack inequalities in Sections 3 and 4 respectively. In particular, a new Girsanov theorem is presented for Lévy processes using the Lévy measure is presented, which is interesting by itself. With concrete lower bounds of the Lévy measure, explicit gradient estimates and Harnack inequalities will also be addressed in Sections 3 and 4, which extend and improve the corresponding known results derived recently in [20, 26, 31], see Corollaries 3.3 and 4.3 for details.

2. Preliminary

Form now on, let \( t > 0 \) be fixed, and let

\[
W_t = \{ w : [0, t] \to \mathbb{R}^d \text{ is right-continuous with left limits} \}
\]

be the path space, which is a Polish space under the Skorokhod metric. Let \( L = (L_s)_{s \in [0, t]} \) be a Lévy process with Lévy measure \( \nu \), possibly with a Gaussian part and a drift part. Then its distribution \( \Lambda \) is a probability measure on \( W_t \). For any \( w \in W_t \) and \( s \in [0, t] \), let \( \Delta w_s = w_s - w_s^- = w_s - \lim_{s \downarrow t} w_s \). Then

\[
w(\cdot) := \sum_{x \in [0, t], \Delta w_x \neq 0} \delta_{(x, \cdot)}
\]

is a \( \mathbb{Z}_+ \cap [\infty] \)-valued measure on \( \mathbb{R}^d \times [0, t] \). For any nonnegative function \( g \) on \( \mathbb{R}^d \times [0, t] \), let

\[
w(g) = \int_{\mathbb{R}^d \times [0, t]} g(z, s)w(dz, ds) = \sum_{s \in [0, t], \Delta w_s \neq 0} g(z, s).
\]

Now, let \( \nu \geq \nu_0 \), where \( \nu_0 \) is another Lévy measure. We may write \( L = L^1 + L^0 \), where \( L^1 \) and \( L^0 \) are two independent Lévy processes with Lévy measure \( \nu - \nu_0 \) and \( \nu_0 \), respectively, and \( \Lambda^0 \) does not have Gaussian term. Let \( \Lambda^1 \) and \( \Lambda^0 \) be the distributions of \( L^1 \) and \( L^0 \) respectively. We have \( \Lambda = \Lambda^1 \ast \Lambda^0 \). In the sequel we will mainly use the \( L^0 \) part to establish derivative formulae of \( P_t \).

It is well known that \( \Lambda^0 \) can be represented by using the Poisson measure \( \Pi \), with intensity

\[
\mu_t(dz, ds) = 1_{[0, t]}(s)\nu_0(dz) \times ds,
\]

which is a probability measure on the configuration space

\[
\Gamma_t := \left\{ \gamma := \sum_{i=1}^n \delta_{(z_i, s_i)} : n \in \mathbb{Z}_+ \cup [\infty], z_i \in \mathbb{R}^d, s_i \in [0, t], \gamma([|z| \geq \varepsilon] \times [0, t]) < \infty \text{ for } \varepsilon > 0 \right\}
\]
equipped with the vague topology, where \( \mathbb{R}^d = \mathbb{R}^d \setminus \{0\} \). More precisely (see, e.g., [32, (4.2)]),

\[
\Lambda^0 = \Pi_t \circ \phi^{-1}
\]  

(2.2)

holds for

\[
\phi(\gamma) := Bt + \int_{[0, t] \times \{|\cdot| \geq 1\}} z 1_{[\cdot, 1]}(\gamma)(ds, dz) + \int_{[0, t] \times \{|\cdot| < 1\}} z 1_{[\cdot, 1]}(\gamma - \mu_t)(ds, dz), \quad \gamma \in \Gamma,
\]

where \( B \in \mathbb{R}^d \) is a constant. Since \( \mu_t \) is a Lévy measure on \([0, t] \times \mathbb{R}^d\), \( \phi(\gamma) \in W_t \) is well-defined for \( \Pi_t \)-a.s. \( \gamma \). It is easy to see that

\[
\phi(\gamma - \delta(z, s)) = \phi(\gamma) - z 1_{[\cdot, 1]}, \quad \text{for } \delta(z, s) \leq \gamma,
\]  

(2.3)

and by (2.1),

\[
\gamma(dz, ds) = \phi(\gamma)(dz, ds).
\]  

(2.4)

This and the Mecke formula for the Poisson measure imply the following lemma, which is crucial for our study.

**Lemma 2.1.** For any \( h \in L^1(W_t \times \mathbb{R}^d \times [0, t]; \Lambda^0 \times v_0 \times ds) \),

\[
\int_{W_t \times \mathbb{R}^d \times [0, t]} h(w, z, s) \Lambda^0(dw) \mu_t(dz, ds)
= \int_{W_t} \Lambda^0(dw) \int_{\mathbb{R}^d \times [0, t]} h(w - z 1_{[\cdot, 1]}, z, s) w(dz, ds).
\]  

(2.5)

Consequently, for \( X_t^i \) solving (1.1) with initial data \( x \),

\[
\mathbb{E} \int_{\mathbb{R}^d \times [0, t]} f(X_t^i + T_z \sigma z) h(L^0, z, s) \mu_t(dz, ds)
= \mathbb{E} \int_{\mathbb{R}^d \times [0, t]} f(X_t^i) h(L^0 - z 1_{[\cdot, 1]}, z, s) L^0(dz, ds).
\]  

(2.6)

**Proof.** By the Mecke formula [12] (see [16, Lemma 6.7]), for any \( F \in L^1(\Pi_t \times \mu_t) \) we have

\[
\int_{\Gamma_t} \Pi_t(dy) \int_{\mathbb{R}^d \times [0, t]} F(\gamma, z, s) \mu_t(dz, ds) = \int_{\Gamma_t} \Pi_t(dy) \int_{\mathbb{R}^d \times [0, t]} F(\gamma - \delta(z, s), z, s) \gamma(dz, ds).
\]

Combining this with (2.2), (2.3), and (2.4), we obtain

\[
\int_{W_t \times \mathbb{R}^d \times [0, t]} h(w, z, s) \Lambda^0(dw) \mu_t(dz, ds)
= \int_{\Gamma_t} \Pi_t(dy) \int_{\mathbb{R}^d \times [0, t]} h(\phi(\gamma), z, s) \mu_t(dz, ds)
= \int_{\Gamma_t} \Pi_t(dy) \int_{\mathbb{R}^d \times [0, t]} h(\phi(\gamma - \delta(z, s)), z, s) \gamma(dz, ds)
\]
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\[
= \int_{t_1} \Pi_t(\mathcal{D}_t) \int_{\mathbb{R}^d \times [0, t]} h(\phi(\gamma) - z_{[\xi, \tau]}, z, s) \phi(\gamma)(dz, ds)
\]

\[
= \int_{W_t} \Lambda^0(\mathcal{D}_t) \int_{\mathbb{R}^d \times [0, t]} h(w - z_{[\xi, \tau]}, z, s)w(dz, ds).
\]

Hence, (2.5) holds.

Next, let

\[
\psi(w) = T_x + \int_0^t T_x, \sigma_s dw_s,
\]

where the integral w.r.t. \( dw_s \) is the Itô integral which is \( \Lambda \)-a.s. defined on \( W_t \). By (1.2) we have

\[
f(X_s^t) = f \circ \psi(L^1 + L^0), \quad f(X_s^t + T_x, \sigma_s, z) = f \circ \psi(L^1 + L^0 + z_{[\xi, \tau]}).
\]

Combining this with (2.5) and noting that \( L^1 \) and \( L^0 \) are independent with distributions \( \Lambda^1 \) and \( \Lambda^0 \), respectively, we obtain

\[
\mathbb{E} \int_{\mathbb{R}^d \times [0, t]} f(X_s^t + T_x, \sigma_s, z)h(L^0, z, s)\mu_t(dz, ds)
\]

\[
= \int_{W_t} \Lambda^1(\mathcal{D}_t) \int_{\mathbb{R}^d \times [0, t]} f \circ \psi(w^1 + w^0 + z_{[\xi, \tau]}h(w^0, z, s)\Lambda^0(\mathcal{D}_t)\mu_t(dz, ds)
\]

\[
= \int_{W_t \times W_t} \Lambda^1(\mathcal{D}_t) \Lambda^0(\mathcal{D}_t) \int_{\mathbb{R}^d \times [0, t]} f \circ \psi(w^1 + w^0)h(w^0 - z_{[\xi, \tau]}, z, s)w^0(dz, ds)
\]

\[
= \mathbb{E} \int_{\mathbb{R}^d \times [0, t]} f(X_s^t)h(L^0 - z_{[\xi, \tau]}, z, s)L^0(dz, ds).
\]

\[\square\]

As an application of Lemma 2.1, we have the following Girsanov theorem, which might be interesting by itself.

**Theorem 2.2.** Let \( G \geq 0 \) be a measurable function on \( W_t \times \mathbb{R}^d \times [0, t] \) such that \( (\Lambda^1 \times \mu_t)(G) = 1 \). Let \( (\xi, \tau) \) be a random variable on \( \mathbb{R}^d \times [0, t] \) such that the distribution of \( (L^0, \xi, \tau) \) is \( G(w, z, s)\Lambda^0(\mathcal{D}_t)\mu_t(dz) \). Let

\[
g(z, s) = \int_{W_t} G(w, z, s)\Lambda^0(\mathcal{D}_t),
\]

which is the distribution density of \( (\xi, \tau) \) w.r.t. \( \mu_t \). If \( \mu_t(g > 0) = \infty \) and

\[
1_{\{G(w, z, s) > 0\}}g(z, s) = g(z, s), \quad (\Lambda^0 \times \mu_t)-a.e.
\]

Then the process

\[
L^0 + \xi 1_{[\xi, \tau]} := (L^0 + \xi 1_{[\xi, \tau]}(s))_{s \in [0, t]}
\]

has distribution \( \Lambda^0 \) under the probability measure \( Q := R \mathbb{P} \), where

\[
R = \frac{g(\xi, \tau)}{G(L^0, \xi, \tau)(L^0(g) + g(\xi, \tau))}.
\]
Proof. Since $G$ is the distribution density of $(L^0, \xi, \tau)$ w.r.t. $\Lambda^0 \times \mu$, we have $G(L^0, \xi, \tau) > 0$ a.s. Similarly, $g(\xi, \tau) > 0$ a.s. as well. Moreover, it is easy to see that $\mathbb{E}L^0(g) = 1$ so that $w(g) < \infty$. Therefore, $R \in (0, \infty)$ a.s.

Now, for any nonnegative measurable function $F$ on $W_t$, applying (2.5) to

$$h(w, z, s) = \frac{F(w, z + 1_{\{L^0\}})}{g(z, s) + w(g)} 1_{\{G > 0\}}(w, z, s),$$

which is finite since $\mu_t(g > 0) = \infty$ implies that $w(g) > 0$ holds $\Lambda^0$-a.e., and using $1_{\{G(\xi, \tau) > 0\}}$ holds $\Lambda^0$-a.e., and using

$$\mathbb{E}_q F(L^0 + 1_{\{L^0\}}) = \mathbb{E}_q \frac{F(L^0 + 1_{\{L^0\}})g(\xi, \tau)}{G(L^0, \xi, \tau)\{g(\xi, \tau) + L^0(g)\}} = \int_{G(\xi, \tau) > 0} \frac{F(w + 1_{\{L^0\}})G(w, z, s)g(z, s)}{G(w, z, s)\{g(z, s) + w(g)\}} \Lambda^0(dw)\mu_t\{dz, ds\} = \int_{W_t \times \mathbb{R}^d \times [0, t]} \frac{F(w + 1_{\{L^0\}})g(z, s)}{g(z, s) + w(g)} \Lambda^0(dw)\mu_t\{dz, ds\} = \int_{W_t} \Lambda^0(dw) \int_{\mathbb{R}^d \times [0, t]} \frac{F(w)g(z, s)}{w(g)} w(dz, ds) = \int_{W_t} F(w)\Lambda^0(dw),$$

This completes the proof. \qed

A simple choice of $G$ in the above Theorem is that $G(w, z, s) = g(z, s)$, that is, $(\xi, \tau)$ is independent of $L^0$. To derive gradient estimate from Theorem 3.1 below, we need the following $\Gamma$-function:

$$\Gamma(r) = \int_0^\infty s^{r-1}e^{-s}ds, \quad r > 0.$$

Lemma 2.3. Let $\Lambda^0$ be the distribution of a Lévy process with Lévy measure $\nu_0$ which is not necessarily absolutely continuous w.r.t. the Lebesgue measure. Let $\mu_t(dz, ds) = \nu_0(dz) \times ds$ on $\mathbb{R}^d \times [0, t]$. Then for any nonnegative measurable function $g$ on $\mathbb{R}^d \times [0, t]$.

$$\int_{W_t} \frac{\Lambda^0(dw)}{w(g)^{\theta}} = \frac{1}{\Gamma(\theta)} \int_0^\infty r^{\theta-1} \exp[-\mu_t(1 - e^{-r})]dr, \quad \theta > 0.$$

Proof. Noting that

$$\frac{1}{s^\theta} = \frac{1}{\Gamma(\theta)} \int_0^\infty r^{\theta-1}e^{-r}sdr, \quad s > 0,$$

it follows from (2.2) that

$$\int_{W_t} \frac{\Lambda^0(dw)}{w(g)^{\theta}} = \int_{\Gamma_t} \frac{\Pi_t(d\gamma)}{\gamma(g)^{\theta}} = \frac{1}{\Gamma(\theta)} \int_0^\infty r^{\theta-1}dr \int_{\Gamma_t} e^{-r(\gamma)}\Pi_t(d\gamma) = \frac{1}{\Gamma(\theta)} \int_0^\infty r^{\theta-1} \exp[-\mu_t(1 - e^{-r})]dr. \quad \square$$
3. Derivative Formula and Gradient Estimates

To establish a derivative formula for $P_t$, we need an absolutely continuous lower bound of $v$. Let

$$v(dz) \geq v_0(dz) := \rho_0(z)dz$$

such that $v_0(\mathbb{R}^d) = \infty$. Recall that the infinity of $v$ is essential to ensure the strong Feller property of $P_t$, which is necessary for the differentiability of the semigroup (see [15] and references within for criteria on the strong Feller property). Thus, the assumption $v_0(\mathbb{R}^d) = \infty$ is reasonable in order to establish a derivative formula of $P_t$.

Let $L^0 = (L^0_t)_{t \in [0, \epsilon]}$ be the Lévy process with Lévy measure $v_0$, and let $L^1 = (L^1_s)_{s \in [0, \epsilon]}$ be the Lévy processes with Lévy measures $v_1 := v - v_0$ independent of $L^0$, so that $L := L^0 + L^1$ is the Lévy process with Lévy measure $v$ introduced above. Let $\hat{\mathbb{R}}^d = \mathbb{R}^d \setminus \{0\}$ and let $v_0(g)$ be the integral of $g$ w.r.t. $v_0$.

**Theorem 3.1.** Let $v(dz) \geq \rho_0(z)dz$ for some nonnegative $\rho_0 \in W^{1,1}_{loc}(\hat{\mathbb{R}}^d)$ such that $v_0(dz) := \rho_0(z)dz$ is an infinite measure. If there exists a nonnegative measurable function $g$ on $\mathbb{R}^d \times [0, \epsilon]$ differentiable in $z \in \mathbb{R}^d$ such that

$$\int_0^\infty \exp[-\mu_t(1 - e^{-\mu t})]dt + \int_{\mathbb{R}^d \times [0, \epsilon]} \left\{ \rho_0 g + \rho_0 |\nabla g| + g |\nabla \rho_0| \right\}(z, s)dzds < \infty, \quad (3.1)$$

where $\nabla$ is the gradient in $z \in \mathbb{R}^d$, then for any $f \in \mathcal{B}_b(\mathbb{R}^d)$,

$$\nabla P_t f(x) = -\mathbb{E} \int_{\mathbb{R}^d \times [0, \epsilon]} f(X_t^y + T_{t,s}, \sigma) \left( \frac{(\sigma^{-1} T_s)^* \{ L_t^0(g) \nabla(\rho_0 g) + g \nabla \rho_0 \} (z, s) dz ds}{L_t^0(g) + g} \right)$$

$$= \mathbb{E} \left[ f(X_t) \int_{\mathbb{R}^d \times [0, \epsilon]} \left( \frac{(\sigma^{-1} T_s)^* \{ \rho_0 g \nabla g - L_t^0(g) \nabla(\rho_0 g) \} (z, s) L_t^0(dz, ds) }{L_t^0(g) + g} \right) \right], \quad (3.2)$$

where

$$L_t^0(g) := \int_{\mathbb{R}^d \times [0, \epsilon]} g dL_t^0 = \sum_{s \in [0, \epsilon], \Delta L_t^s \neq 0} g(\Delta L_t^s, s).$$

**Proof.** Noting that the second equality in (3.2) follows from the first and (2.6), we only need to prove the first formula.

(a) We first prove for the case where $g$ has a compact support $K$. Let $\Lambda = \Lambda^0 \star \Lambda^1$ be the distribution of $L$. For $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $\epsilon \in (0, 1)$, let

$$h_\epsilon(w) = f \left( T_\epsilon(x + \epsilon h) + \int_0^\epsilon T_{\epsilon,s} \sigma dw_s \right).$$

By (1.2) and noting that $L^0$ and $L^1$ are independent with distributions $\Lambda^0$ and $\Lambda^1$ respectively, we have $f(X_t^{\epsilon + \epsilon h}) = h_\epsilon(L_t^0 + L_t^1)$ and

$$P_t f(x + \epsilon h) = \int_{\mathbb{R}^d \times \mathbb{R}^d} h_\epsilon(u^1 + u^0) \Lambda^0(du^0) \Lambda^1(du^1).$$
Since $T_i = T_{s,i}T_s$ for $s \in \mathbb{R}$, and since due to (3.1) and Lemma 2.3 $w(g) > 0$ holds for $\Lambda^0$-a.s. $w \in W_T$, this implies

\begin{align*}
P_s f(x + \varepsilon h) &= \int_{W \times W} \Lambda^1(dw^1) \Lambda^0(dw^0) \int_{\mathbb{R}^{d} \times [0,1]} h_0(w^1 + w^0 + z + \varepsilon \sigma^{-1}_s T_s h 1_{[s,\tau]})(g(s), s) w^0(dz, ds) \\
&= \int_{W \times W} \Lambda^1(dw^1) \Lambda^0(dw^0) \\
&\times \int_{\mathbb{R}^{d} \times [0,1]} h_0(w^1 + w^0 + \varepsilon \sigma^{-1}_s T_s h 1_{[s,\tau]})(g(s), s) w^0(dz, ds).
\end{align*}

Combining this with (2.5) for $\Lambda^0$ in place of $\Lambda$ and

\[ h(w, z, s) := \frac{h_0(w^1 + w + (z + \varepsilon \sigma^{-1}_s T_s h) 1_{[s,\tau]})g(s, z)}{(w^0 + z 1_{[s,\tau]})(g)} \quad \text{for} \quad w \in W_T, \]

we arrive at

\begin{align*}
P_s f(x + \varepsilon h) &= \int_{W} \Lambda^1(dw^1) \int_{\mathbb{R}^{d} \times [0,1]} h_0(w^1 + w^0 + (z + \varepsilon \sigma^{-1}_s T_s h) 1_{[s,\tau]})(g(s), s) w^0(g) + g(z, s) \\
&\times \Lambda^0(dw^0) \mu_i(dz, ds) \\
&= \int_{W} \Lambda^1(dw^1) \int_{\mathbb{R}^{d} \times [0,1]} h_0(w^1 + w^0 + (z + \varepsilon \sigma^{-1}_s T_s h) 1_{[s,\tau]})(\rho_0 g)(z, s) w^0(g) + g(z, s) \\
&\times \Lambda^0(dw^0) dz ds.
\end{align*}

Using the integral transform $z \mapsto z - \varepsilon \sigma^{-1}_s T_s h$, it follows that

\begin{align*}
P_s f(x + \varepsilon h) &= \int_{W} \Lambda^1(dw^1) \int_{\mathbb{R}^{d} \times [0,1]} h_0(w^1 + w^0 + z 1_{[s,\tau]})(\rho_0 g)(z - \varepsilon \sigma^{-1}_s T_s h, s) w^0(g) + g(z - \varepsilon \sigma^{-1}_s T_s h, s) \\
&\times \Lambda^0(dw^0) dz ds.
\end{align*}

Therefore,

\begin{align*}
\frac{P_s f(x + \varepsilon h) - P_s f(x)}{\varepsilon} &= \int_{W} \Lambda^1(dw^1) \int_{\mathbb{R}^{d} \times [0,1]} h_0(w^1 + w^0 + z 1_{[s,\tau]}) \Phi_\varepsilon(w^0, z, s) \Lambda^0(dw^0) dz ds \\
&= \int_{W} \Lambda^1(dw^1) \int_{\mathbb{R}^{d} \times [0,1]} h_0(w^1 + w^0 + z 1_{[s,\tau]}) \Phi_\varepsilon(w^0, z, s) \Lambda^0(dw^0) dz ds
\end{align*}

holds for

\[ \Phi_\varepsilon(w^0, z, s) := \frac{1}{\varepsilon} \left( \frac{\rho_0 g}{w^0(g) + g} (z - \varepsilon \sigma^{-1}_s T_s h, s) - \frac{\rho_0 g}{w^0(g) + g} (z, s) \right). \]

Since

\[ \lim_{\varepsilon \to 0} \Phi_\varepsilon(w, z, s) = -\left( \nabla \frac{\rho_0 g}{w^0(g) + g} (z, s), \sigma^{-1}_s T_s h \right) \\
= -\left( \sigma^{-1}_s T_s \right)^* \nabla \frac{\rho_0 g}{w^0(g)^2} (z, s) h, \]

we have

\[ \Phi_\varepsilon(w^0, z, s) \]
to derive the desired derivative formula by letting $\varepsilon \to 0$, we need to make use of the dominated convergence theorem. Since $g$ has a compact support and $\sup_{x \in [0,1]} \sigma^{-1}_j T_j h < \infty$, there is a compact set $K \subset \mathbb{R}^d$ such that $\text{supp } \Phi_{\varepsilon} \subset W \times K \times [0,t]$ holds for all $\varepsilon \in (0, 1)$. Since $\Lambda^0(dw) \times dz \times ds$ is finite on $W \times K \times [0,t]$, it suffices to show that $\{\Phi_{\varepsilon}\}_{\varepsilon \in (0,1)}$ is uniformly integrable w.r.t. this measure. Noting that

$$\left| \nabla \frac{\rho_0 g}{w(g) + g} \right| \leq \frac{\left| \nabla (\rho_0 g) \right|}{w(g) + g} + \frac{\rho_0 g |\nabla g|}{(w(g) + g)^2} \leq \frac{2(\rho_0 |\nabla g| + g |\rho_0|)}{w(g)},$$

there exists a constant $c > 0$ such that we have

$$|\Phi_{\varepsilon}(u^0, z, s)| = \frac{1}{\varepsilon} \int_0^\varepsilon \left\{ \frac{d}{dr} \left( \frac{\rho_0 g}{w^0(g) + g} \right) (z - r \sigma^{-1}_j T_j h, s) \right\} dr \leq \frac{c}{\varepsilon w(g)} \int_0^\varepsilon \left( \rho_0 |\nabla g| + g |\nabla \rho_0| \right) (z - r \sigma^{-1}_j T_j h, s) dr.$$

By (3.1) and Lemma 2.3, we see that $\int_{W} \frac{1}{w(g)} \Lambda^0(dw) < \infty$. So, it suffices to show that

$$\Psi_{\varepsilon}(z, s) := \frac{1}{\varepsilon} \int_0^\varepsilon \left( \rho_0 |\nabla g| + g |\nabla \rho_0| - R \right) (z - r \sigma^{-1}_j T_j h, s) dr, \quad \varepsilon \in (0, 1)$$

is uniformly integrable w.r.t. $dz \times ds$ on $K \times [0,t]$. Since the function $s \mapsto (s-R)^+$ is convex, by the Jensen inequality we have

$$(\Psi_{\varepsilon} - R)^+(z, s) \leq \int_0^\varepsilon \left( \rho_0 |\nabla g| + g |\nabla \rho_0| - R \right)^+(z - r \sigma^{-1}_j T_j h, s) dr.$$

So,

$$\int_{K \times [0,t]} (\Psi_{\varepsilon}(z, s) - R)^+ dz ds \leq \frac{1}{\varepsilon} \int_{\mathbb{R}^d \times [0,t] \times [0,\varepsilon]} \left( \rho_0 |\nabla g| + g |\nabla \rho_0| - R \right)^+(z - r \sigma^{-1}_j T_j h, s) dz ds dr \leq \int_{\mathbb{R}^d \times [0,t]} \left( \rho_0 |\nabla g| + g |\nabla \rho_0| - R \right)^+(z, s) dz ds, \quad \varepsilon \in (0, 1),$$

where the last step is due to the integral transform $z \mapsto z + re^{-\sigma h}$ for the integral w.r.t. $dz$. Combining this with (3.1) we see that

$$\lim_{\varepsilon \to 0} \sup_{z \in (0,1)} \int_{K \times [0,t]} (\Psi_{\varepsilon}(z, s) - R)^+ dz ds = 0,$$

that is, $\{\Psi_{\varepsilon}\}_{\varepsilon \in (0,1)}$ is uniform integrable w.r.t. $dz \times ds$ on $K \times [0,t]$.

(b) Let $g$ satisfy (3.1). For any $n \geq 1$, let $g_n(z, s) = g(z, s) [1 \wedge (1 + n - |z|)^+]$, which has a compact support. By (a) we have

$$\nabla P_{\varepsilon} f(x) = -\varepsilon \int_{\mathbb{R}^d \times [0,t]} f(X_j^\varepsilon + e^{(s-t)\sigma_j} z) \times \left( \sigma^{-1}_j T_j \right)^* \left\{ L^0(g_n) \nabla (g_n \rho_0) + g_n^2 \nabla \rho_0 \right\} (z, s) dz ds$$

(3.5)
Let \( c = e^{[\mathcal{A}]}. \) It is easy to see that

\[
\left| \left( \sigma_s^{-1} T_s \right)^* \left[ L^0(g_s) \nabla (g_s \rho_0) + g_s^2 \nabla \rho_0 \right] \right| \leq \frac{c \left( \rho_0 \| \nabla g \| + g | \nabla \rho_0 | + \rho_0 \right)}{L^0(g_s)} , \quad n \geq 1
\]

holds for some constant \( c > 0. \) Then, according to (3.1), the desired formula follows from the dominated convergence theorem by letting \( n \to \infty \) in (3.5), provided

\[
\int_{w_i} \frac{\Lambda^0(dw)}{w(g_i)} < \infty . \tag{3.6}
\]

By Lemma 2.3 and (3.1), we have

\[
\int_{w_i} \frac{\Lambda^0(dw)}{w(g_i)} = \frac{1}{\Gamma(1)} \int_0^\infty \exp[-\mu_i(1 - e^{-r_i})]dr
\]

\[
\leq \frac{1}{\Gamma(1)} \int_0^\infty \exp[tv_0(|z| > 1) - \mu_i(1 - e^{-r_i})]dr < \infty
\]

since \( v_0(|z| \geq 1) \leq v(|z| \geq 1) < \infty \) as \( v \) is a Lévy measure. Therefore, the proof is finished.

**Corollary 3.2.** Let \( \rho_0 \in W^{1,1}_m(\mathbb{R}^d) \) be nonnegative such that \( v(dz) \geq \rho_0(z)dz, \) and let \( g \) be a nonnegative measurable function on \( \mathbb{R}^d \times [0, t] \) differentiable in the first variable such that \( \mu_i(g) < \infty. \) Then for any \( p \in (1, \infty) \) and \( f \in B_0(\mathbb{R}^d), \)

\[
|\nabla f| \leq (P_i[f]^p)^\frac{1}{p} \left( \frac{1}{\Gamma(1)} \int_0^\infty \exp[-\mu_i(1 - e^{-r_i})]dr \right)^\frac{1}{p'}
\]

\[
\times \left( \int_{\mathbb{R}^d \times [0, t]} \| \sigma_i^{-1} T_s \| (g | \nabla \log \rho_0 | + | \nabla \log (g \rho_0) |) \right)^\frac{1}{p'} g(z, s) \mu_i(dz, ds) \right)^\frac{1}{p'} .
\]

**Proof.** Assume that the desired upper bound is finite. Then (3.1) holds. On the other hand, according to (2.6), for any \( f \in B_0(\mathbb{R}^d) \) we have

\[
P_i f(x) = \mathbb{E} \int_{W_i \times \mathbb{R}^d \times [0, t]} f(X_s + T_s, \sigma_s z) \frac{(g \rho_0)(z, s)}{L^0(g) + g(z, s)} dz ds . \tag{3.7}
\]

Combining this with the first formula in (3.2) and using the Hölder inequality, we obtain

\[
|\nabla f| \leq \mathbb{E} \int_{\mathbb{R}^d \times [0, t]} |f(X_s + T_s, \sigma_s z) | \| \sigma_i^{-1} T_s \|
\]

\[
\times \left\{ (g | \nabla \log \rho_0 | + | \nabla \log (g \rho_0) |) \frac{(g \rho_0)}{L^0(g) + g} \right\} (z, s) dz ds
\]

\[
\leq (P_i[f]^p)^\frac{1}{p} \left( \mathbb{E} \int_{\mathbb{R}^d \times [0, t]} \| \sigma_i^{-1} T_s \| (g | \nabla \log \rho_0 | + | \nabla \log (g \rho_0) |) \right)^\frac{1}{p'} \frac{g}{L^0(g) + g} d\mu_i \right)^\frac{1}{p'} .
\]
Then the desired gradient estimate follows from Lemma 2.3 by noting that \( \frac{1}{L_d^{0}(g)} \leq \frac{g}{L_d^{0}(g)} + \int W_{\frac{1}{L_d^{0}(g)}} A_{\theta}(dw) \).

To illustrate Corollary 3.2, we present explicit conditions on the lower bound of \( \nu \) for the gradient estimate and Harnack inequality. Comparing with results in [20, 26] where the uniform gradient estimates are derived, in the following result \( \rho_0 \) is not necessary corresponding to a Bernstein function and more general \( L^p \) gradient estimates are also provided. For a \( d \times d \) matrix \( M \) and a constant \( \lambda \in \mathbb{R} \), we write \( M \leq \lambda I \) provided \( \langle M, a \rangle \leq |a|^2 \) holds for all \( a \in \mathbb{R}^d \).

**Corollary 3.3.** Let \( A_i \leq \lambda I \) and \( \| \sigma_i^{-1} \| \leq \lambda \) for some constants \( \lambda \in \mathbb{R} \) and \( \lambda > 0 \). Let \( v(dz) \geq |z|^{-d} S(|z|^{-2}) 1_{|z| \leq r_0} \) for some constant \( r_0 > 0 \) and positive function \( S \in C^1([0, \infty)) \) such that

\[
\limsup_{r \to \infty} \frac{|S'(r)|r}{S(r)} < \infty. \tag{3.8}
\]

For \( p > 1 \) and \( k > 2 + \frac{d}{p-1} \), let

\[
\psi_k(r) = \left( 1 - \frac{1}{e} \right) \frac{k}{2k} \int_{2k^{-1/4}}^{1} \frac{S(s^{-2})}{s} \, ds, \quad r > 0,
\]

where \( \kappa(d) \) is the area of the unit sphere in \( \mathbb{R}^d \). If \( \int_0^\infty e^{-\psi_k(r)} \, dr < \infty \), then there exists a constant \( c > 0 \) such that

\[
|\nabla P_t f| \leq c \left( \frac{e^{\psi_k} - 1}{2} \left( P_t |f|^p \right)^{\frac{1}{p}} \left( \int_0^\infty e^{-\psi_k(r)} \, dr \right)^{\frac{1}{p}} \right)
\]

holds for \( f \in B_0(\mathbb{R}^d) \). In particular, if \( S(r) = c_0 \log^s (1 + r) \) for some \( c_0, \varepsilon > 0 \), then for any \( p > 1 \) there exists a constant \( c > 0 \) such that

\[
|\nabla P_t f| \leq (P_t |f|^p)^{\frac{1}{p}} \exp[c(1 + t)^{-\frac{1}{s}}], \quad t > 0
\]

holds for all positive \( f \in B_0(\mathbb{R}^d) \).

**Proof.** Let \( \rho_0(z) = |z|^{-d} S(|z|^{-2})(1 - \frac{|z|^2}{r_0^2})^{\frac{1}{s}} \) and \( g(z, s) = g(z) = |z|^s \). Obviously, \( \int_0^\infty e^{-\psi_k(r)} \, dr < \infty \) implies that \( \int_{\mathbb{R}^d} \rho_0(z) \, dz = \infty \). Since \( \| S' \|_\infty < \infty \) implies that \( S(s^{-2}) \leq cs^{-2} \) for some constant \( c > 0 \) and all \( s \leq r_0 \), we have

\[
(g\rho_0)(z) \leq c |z|^{k-d-2} 1_{|z| \leq r_0},
\]

so that \( \mu_t(g) = t \int_{\mathbb{R}^d} (g\rho_0)(z) \, dz < \infty \). Next, it is easy to see from (3.8) that

\[
\{ |g| |\nabla \log \rho_0| + |\nabla \log (g\rho_0)| \}(z) \leq c \frac{c}{|z|(|r_0 - |z|)^{\frac{1}{s}}}, \quad |z| < r_0
\]

holds for some constant \( c > 0 \). Thus, there exists a constant \( c > 0 \) such that

\[
\{ \rho_0 |g| |\nabla \log \rho_0|^{\frac{1}{p}} + |\nabla \log (g\rho_0)|^{\frac{1}{p}} \}(z) \leq c |z|^{k-d-2, \frac{1}{s}} (r_0 - |z|)^{1-k+\frac{1}{s}}
\]
holds for some constant $c > 0$. Since $\|\sigma_s^{-1} T_s \| \leq \lambda e^{\alpha t}$ and $k > 2 + \frac{p-1}{p}$, this implies
\[ \int_{\mathbb{R}^d \times [0,1]} \{ \|\sigma_s^{-1} T_s \| (g|\nabla \log \rho_0| + |\nabla \log (\rho_0 g)|)(z, s) \} \frac{p}{p-1} g(z, s) \mu_s(\,dz, \,ds) \leq \frac{c(e^{\alpha t} - 1)}{\alpha} \int_{\mathbb{R}^d} \{ \rho_0 g(|g|\nabla \log \rho_0|^{\frac{p}{p-1}} + |\nabla \log (\rho_0 g)|^{\frac{p}{p-1}})\}(z) \,dz \leq \frac{c(e^{\alpha t} - 1)}{\alpha} \] (3.9)
for some constant $c > 0$. Next, for $r \geq (2/r_0)^k$ we have
\[ v_0(1 - e^{-r^p}) \geq \frac{1}{2^k} \int_{\{ |z|^2 \leq r^2 \}} |\nabla S(\frac{1}{2})| \,dz \]
\[ = \frac{k(d)}{2^k} \int_0^{r_0^2} s^{-1} S(s^{-2})(1 - e^{-r^p}) \,ds \]
\[ \geq \frac{k(d)}{2^k} \int_{\frac{r_0^2}{2}}^{r_0^2} s^{-1} \frac{S(s^{-2})(1 - e^{-1})}{s} \,ds = \psi_k(r). \]
Combining this with (3.9), we prove the first assertion by Corollary 3.2.

Next, let $S(r) = c_0 \log^*(1 + r)$. By the semigroup property and the Jensen inequality, it suffices to prove the desired gradient estimate for $t \in (0, 1]$. It is easy to see that
\[ t \psi_k(r) \geq c_1 t \log^{1+\epsilon}(1 + r) - c_2 t \geq 2 \log(1 + r) - c_3 t^{-1/\epsilon} \]
holds for some constants $c_1, c_2, c_3 > 0$. Then the desired gradient estimate follows from the first part of this Corollary.

Note that the second estimate in Corollary 3.3 improves and extends [26, Example 1.3] to $L^p$ gradient estimate with better short time behavior. On the other hand, however, Corollary 3.3 does not provide sharp estimate for the $x$-stable case. In general, to drive sharper gradient estimates, it might be necessary to take $g$ depending also on $s$.

4. Harnack Inequality and Applications

We first investigate the Harnack inequality with a power in the sense of [22] and the log-Harnack inequality introduced in [18, 25], then present some applications of these inequalities in an abstract framework. Recently, these type of inequalities have been established in [8] for some jump processes with using subordinations from diffusion processes and in [31] using heat kernel bounds of the $x$-stable process.

4.1. Harnack Inequality

For positive measurable functions $\rho_0, g$ on $\hat{\mathbb{R}}^d$, let $v_0(\,dz) = \rho_0(\,dz) \,dz$ and
\[ \gamma_{\rho_0,s}(\theta, t) = \frac{1}{\Gamma(\theta)} \int_0^\infty r^{\theta-1} \exp[-r v_0(1 - e^{-s})] \,dr, \quad \theta, t > 0. \]
Theorem 4.1. Let $\lambda \in \mathbb{R}$ and $\lambda \geq 0$ be such that $A_s \in \mathbb{R}$ and $\|\sigma_s^{-1}\|_s \leq \lambda$ for $s \in [0, 1]$. Let $\rho_0 \in W^{1,1}_c(\mathbb{R}^d)$ and $g \in W^{1,1}_c(\mathbb{R}^d)$ be positive such that $v(z) \geq \rho_0(z)dz$ and $v_0(g > 0) = \infty$. Then for any $p > 1$ and positive $f \in \mathcal{B}_b(\mathbb{R}^d)$,

$$\frac{(P_s f)(x+h)}{P_s f(x)} \leq \left\{ \int_{W_d \times [0, 1]} \left( \frac{(\rho_0 g)(z)}{w(g) + g(z)} \right)^{1/p} \frac{d^2}{w(g) + g(z)} \Lambda^0(dz) \right\}^{p-1}$$

holds for $x, h \in \mathbb{R}^d$ and $t > 0$.

Proof. Since $v_0(g > 0) = \infty$ implies $w(g) > 0$ for $\Lambda^0$-a.e. $w$, the right-hand side of the first inequality makes sense (could be infinite). Let $g(z, s) = g(z)$. By (3.3) and the Hölder inequality, we obtain

$$(P_s f(x+h))^p \leq \left\{ \int_{W_d \times [0, 1]} \left( \frac{(\rho_0 g)(z)}{w(g) + g(z)} \right)^{1/p} \frac{d^2}{w(g) + g(z)} \Lambda^0(dz) \right\}^{p-1}$$

where in the last step we have used the transform $z \mapsto z + \sigma_s^{-1}T_s h$ for the integral w.r.t. $dz$. This proves the first inequality.

Next, due to the semigroup property and the Jensen inequality, for the second inequality it suffices to consider $t \in (0, 1)$. Then

$$(P_s f(x+h))^p \leq \left( \int_{W_d \times [0, 1]} \frac{g(z)}{w(g) + g(z)} (B_1B_2)^{1/p} \Lambda^0(dz)v_0(dz) \right)^{p-1}$$

holds for

$$B_1 = B_1(w, z, s) := \frac{w(g) + g(z + \sigma_s^{-1}T_s h)}{w(g) + g(z)}, \quad B_2 = B_2(w, z, s) := \frac{(\rho_0 g)(z)}{(\rho_0 g)(z + \sigma_s^{-1}T_s h)}.$$

Since $t \in (0, 1]$ and $\|\sigma_s^{-1}T_s \| \leq \lambda e^s$ for $s \in (0, 1]$, we have

$$B_1 \leq 1 + \frac{|g(z + \sigma_s^{-1}T_s h) - g(z)|}{w(g) + g(z)} \leq 1 + \frac{\lambda \|\nabla g\|_{\infty} e^s |h|}{w(g) + g(z)},$$

$$B_2 = \exp[\|\nabla \log(\rho_0 g)\|_{\infty} \lambda e^s |h|], \quad s \in [0, t].$$
Moreover, due to Lemma 2.1

\[
\int_{W_t \times \mathbb{R}^d \times [0, t]} \left( 1 + \frac{c}{w(g) + g(z)} \right)^{\frac{1}{\lambda_t}} \frac{g(z)}{w(g) + g(z)} \Lambda^0(du)\nu_0(dz) \, ds = \int_{W_t} \left( 1 + \frac{c}{w(g)} \right)^{\frac{1}{\lambda_t}} \Lambda^0(du) \int_{\mathbb{R}^d \times [0, t]} \frac{g(z)}{w(g)} \nu_0(dz) \, ds
\]

\[
= \int_{W_t} \left( 1 + \frac{c}{w(g)} \right)^{\frac{1}{\lambda_t}} \Lambda^0(du) = \int_{W_t} \left( 1 + \frac{c}{w(g)} \right)^{\frac{1}{\lambda_t}} \Lambda^0(du) \tag{4.2}
\]

holds for \( c \geq 0 \). So, it follows from (4.1) that

\[
\left( \frac{(P_t f(x + h))^p}{P_t f(x)} \right)^{\rho} \leq \exp\left[ \| \nabla \log(\rho_0 g) \|_\infty \lambda e^s |h| \right] \left( \int_{W_t} \left( 1 + \frac{\| \nabla g \|_\infty e^s |h|}{w(g)} \right)^{\frac{1}{\lambda_t}} \Lambda^0(du) \right)^{p-1}.
\]

This implies the second inequality since

\[
\left( \int_{W_t} \left( 1 + \frac{c}{w(g)} \right)^{\frac{1}{\lambda_t}} \Lambda^0(du) \right)^{p-1} \leq 1 + c^{\frac{\rho - 1}{\rho - 1}} \gamma_{\rho_d, g} \left( \frac{1}{p - 1}, t \right)^{(p-1)\lambda_t} \left( p-1 \right)^{(p-1)\lambda_t}
\]

holds for \( c \geq 0 \) according to Lemma 2.3 and the triangle inequality for the norm

\[
\|F\|^{\frac{1}{\lambda_t}} := \left( \int_{W_t} |F| \Lambda^0(du) \right)^{\frac{1}{\lambda_t}} (p-1)\lambda_t, \quad r > 0.
\]

Next, we consider the log-Harnack inequality.

**Theorem 4.2.** Let \( z, \lambda \) and \( \rho_0, g \) be in Theorem 4.1. For any positive \( f \in \mathcal{B}_h(\mathbb{R}^d) \),

\[
P_t \lambda \log f(x + h) \leq \log P_t f(x) + \lambda e^s |h| (\| \nabla \log(\rho_0 g) \|_\infty + \| \nabla g \|_\infty \gamma_{\rho_d, g} (1, t \wedge 1))
\]

holds for \( t > 0 \) and \( x, h \in \mathbb{R}^d \).

**Proof.** Again, due to the semigroup property and the Jensen inequality, it suffices to prove for \( t \in (0, 1] \). Let

\[
\Omega(du^1, du^0, dz, ds) = \frac{g(z)}{w(g) + g(z)} \Lambda^1(du^1) \Lambda^0(du^0) \nu_0(dz) \, ds,
\]

which is a probability measure on \( W_t \times W_t \times \mathbb{R}^d \times [0, t] \) according to (3.3) for \( g(z, s) = g(z), f = 1 \) (hence, \( h_e = 1 \) and \( e = 0 \). Let

\[
G(w, z, s) = \frac{g(z)}{w(g) + g(z)} \left( \frac{(w(g) + g(z))(\rho_0 g)(z - \sigma^{-1}_e T_s h)}{(w(g) + g(z - \sigma^{-1}_e T_s h)) \rho_0 g(z)} \right).
\]
which is a probability density w.r.t. $\Omega$ by the same reason. Moreover, using $\log f$ to replace $f$ in (3.3) with $\epsilon = 1$, we have

$$P_t \log f(x + h) = \int_{W_t \times W_t \times \mathbb{R}^d \times [0, t]} (\log h_0)(w^1 + w^0 + z_1)\Omega(dw^1, dw^0, dz, ds).$$

So, by the Young inequality (see [3, Lemma 2.4]) and (3.3) with $\epsilon = 0$, we obtain

$$P_t \log f(x + h) \leq \log \int_{W_t \times W_t \times \mathbb{R}^d \times [0, t]} h_0(w^1 + w^0 + z_1)\Omega(dw^1, dw^0, dz, ds) + \Omega(G \log G) = \log P_t f(x) + \int_{W_t \times \mathbb{R}^d \times [0, t]} \left\{ \frac{(\rho_0 g)(z - \sigma^{-1}_s T_h)}{w(g) + g(z - \sigma^{-1}_s T_h)} \log G(w, z, s) \right\} \Lambda^0(dw)dzds,$$

where $\Omega(G \log G)$ is the integral of $G \log G$ w.r.t. the probability measure $\Omega$. Since for $t \in (0, 1]$ one has

$$G(w, z, s) \leq \exp \left[ \left( \frac{\|\nabla g\|_\infty}{w(g)} + \|\nabla \log(\rho_0 g)\|_\infty \right) \delta \epsilon_t |h| \right],$$

and since (4.2) and the integral transform $z \mapsto z + \sigma^{-1}_s T_h$ imply that

$$\int_{W_t \times \mathbb{R}^d \times [0, t]} \frac{(\rho_0 g)(z - \sigma^{-1}_s T_h)}{w(g) + g(z - \sigma^{-1}_s T_h)} \Lambda^0(dw)dzds = \int_{W_t \times \mathbb{R}^d \times [0, t]} \frac{g(z)}{w(g) + g(z)} \Lambda^0(dw) = 1,$$

we conclude that

$$P_t \log f(x + h) \leq \log P_t f(x) + \delta \epsilon_t \left( \|\nabla \log(\rho_0 g)\|_\infty + \|\nabla g\|_\infty \int_{W_t} \frac{1}{w(g)} \Lambda^0(dw) \right).$$

This completes the proof according to Lemma 2.3. \qed

Finally, we consider a specific situation for $\nu$ having an $\alpha$-stable like lower bound. Comparing with the Harnack inequality (1.3) derived recently in [31], our result (4.4) is better for small time and small $|h|$, and we only need the specific lower bound in a neighborhood of 0.

**Corollary 4.3.** Let $A_\alpha$ and $\|\sigma^{-1}_s\|$ be bounded above, and let $\nu(dz) \geq h(|z|)dz$ for some positive decreasing function $h \in C^1((0, \infty))$ such that

$$\sup_{r < 0} \frac{|h'(r)|}{h(r) + h(r)^2} < \infty. \quad (4.3)$$

Then for any $p > 1$ there exist two constants $c_1, c_2 > 0$ such that for any positive $f \in \mathcal{B}_h(\mathbb{R}^d)$,

$$\left( P_t f(x + h) \right)^p \leq P_t f^p(x) e^{c_2 |h|} \left( 1 + c_1 |h| \int_0^\infty \frac{1}{r^{p-1}} e^{-c_1(t+1)r^{(p-1)/p}} \, dr \right)^{(p-1) \vee 1} \quad (4.4).$$
holds for $t > 0, x, h \in \mathbb{R}^d$. Moreover, there exist constants $c_1, c_2 > 0$ such that

$$P_t \log f(x + h) \leq \log P_t f(x) + c_2 |h| \int_0^\infty e^{-c_1(t^{-1})r(h^{-1}(r))^d} dr, \quad x, h \in \mathbb{R}^d, \quad t > 0 \quad (4.5)$$

holds for positive $f \in \mathcal{B}_b(\mathbb{R}^d)$. 

**Proof.** Obviously, it suffices to prove for $t \in (0, 1]$. Let $\rho_0(z) = h(|z|)$ and $g(z) = \frac{1}{\mathbb{V} \rho_0(z)} = \frac{1}{\mathbb{V} h(|z|)}$. Then it is easy to see from (4.3) that $\|\nabla \log(\rho_0 g)\|_{\infty}, \|\nabla g\|_{\infty} < \infty$. Moreover, since

$$(\rho_0 g)(z) = h(|z|) \wedge 1 = 1, \quad \text{if } g(z) \leq 1,$$

for $r \geq 1$ we have

$$v_0(1 - e^{-\nu r}) \geq \frac{r}{2} v_0(g \mathbf{1}_{(s \leq r^{-1})}) \geq \frac{\kappa(d) r}{2} \int_0^{h^{-1}(r)} s^{d-1} ds \geq c_1(h^{-1}(r))^d$$

for some constants $c_1 > 0$. Thus, for any $\theta > 1$, there exists constants $c_2 > 0$ such that

$$v_{\rho_0 g}(\theta, t) \leq c_2 \int_0^\infty r^{d-1} \exp[-tc_1 r(h^{-1}(r))^d] dr$$

holds for $\theta = \frac{r}{\rho^{-1}}$. Therefore, (4.4) and (4.5) follow from Theorems 4.1 and 4.2, respectively. 

To illustrate the Corollary 4.3, we consider $v(\text{d}z) \geq bc_0 |z|^{-(d+\alpha)}$ for some $c_0 > 0$ and $\alpha \in (0, 2)$. Letting $h(r) = c_0 r^{-(d+\alpha)}$ we have

$$\int_0^\infty r^{\frac{\alpha+d}{d+\alpha}} e^{-c_1 r(h^{-1}(r))^d} dr \leq c' t^{-\frac{\alpha + \alpha}{d + \alpha}}$$

for some constant $c' > 0$ and all $t \in (0, 1]$. Therefore,

$$(P_tf(x + h))^p \leq P_tf^p(x) e^{c|h|} (t \wedge 1)^{-\frac{d + \alpha}{d + \alpha}}$$

and

$$P_t \log f(x + h) \leq \log P_t f(x) + \frac{c|h|}{(t \wedge 1)^{\frac{\alpha}{d + \alpha}}}$$

hold for all $t > 0, x, h \in \mathbb{R}^d$ and positive $f \in \mathcal{B}_b(\mathbb{R}^d)$. One may also derive explicit Harnack and log-Harnack inequalities for the case that $v(\text{d}z) \geq c_0 |z|^{-d} \log^\prime (1 + |z|^{-1})$ for some $c_0, \varepsilon > 0$.

**4.2. Applications**

For applications of our results derived in this section, we introduce some applications of Harnack inequalities which are essentially organized or generalized from [25, 29, 30]. As most results presented below are not yet well known, we include brief proofs for readers’ convenience.
Let $E$ be a topological space with Borel $\sigma$-field $\mathcal{B}$, let $\mathcal{B}(E)$ (resp. $\mathcal{B}_b(E)$, $\mathcal{B}_b^+(E)$) denote the set of all measurable (resp. bounded measurable, bounded nonnegative measurable) functions on $E$, and let $C(E)$ (resp. $C_b(E)$, $C_b^+(E)$) stands for the set of continuous (resp. bounded continuous, bounded nonnegative continuous) functions on $E$. We recall some notions which will be considered in this subsection.

**Definition 4.1.** Let $\mu$ be a probability measure on $(E, \mathcal{B})$, and let $P$ be a bounded linear operator on $\mathcal{B}_b(E)$.

(i) $\mu$ is called quasi-invariant of $P$, if $\mu P$ is absolutely continuous w.r.t. $\mu$, where $(\mu P)(A) := \mu(P1_A)$, $A \in \mathcal{F}$. If $\mu P = \mu$ then $\mu$ is called an invariant probability measure of $P$.

(ii) A measurable function $p$ on $E^2$ is called the kernel of $P$ w.r.t. $\mu$, if

$$Pf = \int_E p(\cdot, y)f(y)\mu(dy), \quad f \in \mathcal{B}_b(E).$$

(iii) $P$ is called a Feller operator, if $PC_b(E) \subset C_b(E)$, while it is called a strong Feller operator if $P\mathcal{B}_b(E) \subset C_b(E)$.

From now on, we let $P$ be a Markov operator given by

$$Pf(x) = \int_E f(y)P(x, dy), \quad f \in \mathcal{B}_b(E), \quad x \in E$$

for a transition probability measure $P(x, dy)$. We will consider the following general version of Harnack type inequality for $P$:

$$\Phi(Pf(x)) \leq \{P\Phi(f)(y)|e^{q(x,y)}\}, \quad x, y \in E, \quad f \in \mathcal{B}_b^+(E), \quad (4.6)$$

where $\Phi$ is a nonnegative function on $[0, \infty)$ and $\Psi$ is a measurable nonnegative function on $E^2$. In particular, the log-Harnack inequality and Harnack inequality with a power $p > 1$ addressed above refer to $\Phi(r) = e^r$ and $\Phi(r) = r^p$, respectively.

**Theorem 4.4.** Let $\mu$ be a quasi-invariant probability measure of $P$. Let $\Phi \in C^1([0, \infty))$ be an increasing function with $\Phi(1) > 0$ and $\Phi(\infty) := \lim_{r \to \infty} \Phi(r) = \infty$, such that (4.6) holds.

1. For any $x, y \in E$, $P(x, \cdot)$ and $P(y, \cdot)$ are equivalent.
2. If $\lim_{r \to \infty} [\Psi(x, y) + \Psi(y, x)] = 0$ holds for all $x \in E$, then P is strong Feller.
3. $P$ has a kernel $p$ w.r.t. $\mu$, so that any invariant probability measure of $P$ is absolutely continuous w.r.t. $\mu$.
4. $P$ has at most one invariant probability measure and if it has, the kernel of $P$ w.r.t. the invariant probability measure is strictly positive.
5. The kernel $p$ of $P$ w.r.t. $\mu$ satisfies

$$\int_E p(x, \cdot)\Phi^{-1}\left(\frac{p(x, \cdot)}{p(y, \cdot)}\right)\,d\mu \leq \Phi^{-1}(e^{q(x,y)}), \quad x, y \in E,$$

where $\Phi^{-1}(\infty) := \infty$ by convention.
(6) If $r\Phi^{-1}(r)$ is convex for $r \geq 0$, then the kernel $p$ of $P$ w.r.t. $\mu$ satisfies

$$\int_E p(x, \cdot) p(y, \cdot) d\mu \geq e^{-\Phi(x, y)}, \quad x, y \in E.$$ 

Proof. We only need to prove (1) since other assertions are included in [30, Proposition 3.1]. Let $A \in \mathcal{B}$ be such that $P(y, A) = 0$. Applying (4.6) to $n1_A$, we obtain

$$\Phi(nP1_A(x)) \leq e^{\Phi(x, y)} P\Phi(n1_A)(y) = e^{\Phi(x, y)} \Phi(0).$$

Since $\Phi(r) \to \infty$ as $r \to \infty$, letting $n \to \infty$ we conclude that $P(x, A) = P1_A(x) = 0$. That is, $P(x, \cdot)$ is absolutely continuous w.r.t. $P(y, \cdot)$ and vice versa. \qed

By Theorem 4.4(1), if (4.6) holds then

$$p_{x,y}(z) := \frac{P(x, dz)}{P(y, dz)}$$

exists. We aim to describe this function using the Harnack inequality. For simplicity, we only consider the Harnack inequality with a power $p > 1$

$$(Pf(x))^p \leq (Pf^p(y)) e^{\Phi(x, y)}, \quad x, y \in E, \quad f \in \mathcal{B}_b(E)$$

and the log-Harnack inequality

$$P(\log f)(x) \leq \log Pf(y) + \Psi(x, y), \quad x, y \in E, \quad f \geq 1, \quad f \in \mathcal{B}_b(E).$$

The following result is organized from [25, Section 2].

**Proposition 4.5.** (4.7) holds if and only if $p_{x,y}$ exists and satisfies

$$P[p_{x,y}^{1/(p-1)}](x) \leq e^{\Phi(x, y)/(p-1)}, \quad x, y \in E;$$

while (4.8) holds if and only if $p_{x,y}$ exists and satisfies

$$P[\log p_{x,y}](x) \leq \Psi(x, y), \quad x, y \in E.$$

Finally, we consider the hyperbounded property and the entropy-cost inequality implied by (4.7) and (4.8). Let $P$ have an invariant probability measure $\mu$. Then $\| \cdot \|_{p=q}$ stands for the operator norm from $L^p(\mu)$ to $L^q(\mu)$. Moreover, for a nonnegative measurable function $\Psi$ on $E \times E$, and for $\mathcal{C}(v, \mu)$ the class of all couplings of $\mu$ and $v$, let

$$W_\Psi(\mu, v) = \inf_{\pi \in \mathcal{C}(v, \mu)} \int_{E \times E} \Psi(x, y) \pi(dx, dy)$$

be the transportation-cost from $v$ to $\mu$ induced by the cost-function $\Psi$. The following result can be deduced as in the proof of [18, Corollary 1.2] and [23, Section 2].

**Proposition 4.6.** Let $P$ have an invariant probability measure $\mu$. 

(1) (4.7) implies
\[
\|P\|_{p \to p} \leq \int_E \frac{\mu(dx)}{\int_E \exp[-\Psi(x, y)\mu(dy)]\delta}, \quad \delta > 1.
\]

(2) Let \(P^*\) be the adjoint operator of \(P\) in \(L^2(\mu)\). Then (4.8) implies
\[
\int_E (P^*f) \log P^*f d\mu \leq W_\Psi(f\mu, \mu), \quad f \geq 0, \quad \int_E f d\mu = 1.
\]

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