Cut-free LK Quasi-Polynomially Simulates Resolution

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Abstract

In this paper, the relative efficiency of two propositional systems is studied: resolution and cut-free LK in DAG. We give an upper bound for translation of resolution refutation to cut-free LK proofs. The best upper bound known was $2^n$ and we improve it to $n^{2+\log n}$.

1 Introduction

The interest in the lengths of proofs of resolution and cut-free LK is threefold. In [3], Cook and Reckhow showed that the problem of whether there exists a propositional proof system in which every tautology has a polynomial size proof is equivalent to the problem of $\text{NP}=\text{co-NP}$. In the same paper, they introduced a notion of polynomial simulation ($\text{p-simulation}$) to compare efficiencies of propositional systems so that we can order them in terms of complexity. In the early researches, propositional systems were separated into the equivalence classes of $\text{p-simulation}$, and the hierarchy of propositional calculi is obtained. Hakken showed a lower bound for resolution by proving that resolution does not polynomially prove the pigeonhole principle [4]. Then, his technique was applied to cut-free LK expressed as directed acyclic graphs (DAG) to prove its lower bound [9]. In recent researches, it was revealed that the hierarchy of propositional systems corresponds to that of circuit classes. Thus, separating two propositional systems is equivalent to or strongly related to separating corresponding circuit classes. Most of questions of relative strength have been settled for systems weaker than LK, with the exception of the relative efficiency of resolution and cut-free LK [10].

The second source of interest is classical proof theory. Most of the important theorems in classical propositional calculus, such as interpolation theorem and Beth’s theorem, are obtained as consequences of Gentzen’s cut elimination

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theorem. Unfortunately, cut-elimination for propositional calculus requires the function $2^n$ [6]. Even worse, reducing cuts of degree $k$ to $k - 1$ requires $2^n$ [1]. Consequently, theorems mentioned above remain open when we restrict our interest in polynomial-time algorithm [5]. For example, whether there exists a polynomial-time algorithm to find an interpolant for a given end-sequent is an open problem.

Resolution is a system which is almost equivalent to cut-free LK plus cuts of atomic formulas. Finding a polynomial-time or quasi-polynomial-time algorithm to eliminate atomic cuts may help us to answer these open problems.

The third source of interest is theory of automated theorem proving. Cut-free LK and resolution are among the most frequently mentioned propositional calculi in the field of automated reasoning. Many proof-search programs are benefited by the simplicity of cut-free LK and resolution: In cut-free LK, the proof of a given tautology is completely determined by the structure of the tautology itself by virtue of its subformula property, and resolution has only one inference and no axiom. Thus, the question whether there exists a feasible function which translates resolution proofs to cut-free LK proofs is counted as one of the major open problems in the theory of automated theorem proving. In [3] and [1], it was shown that any translation of tree resolution to cut-free tree LK requires $2^n$. The best known translation of resolution to cut-free DAG LK also requires $2^n$ and it was widely believed to be its lower bound.

In this paper, we show that $n^{2+3\log n}$ is enough to translate resolution into cut-free DAG LK.

## 2 Resolution and GCNF

A proof system is defined to be a polynomial time computable function $f$ from $\{0,1\}^*$ onto the set of tautologies. When a proof $P$ of a proof system is given, we measure its size by the total number of symbols appearing in $P$, and its length by the total number of lines (inferences) of $P$. The size of $P$ is denoted by $\text{size}(P)$ and the length by $\text{len}(P)$.

Let $S_1$ and $S_2$ be proof systems for propositional calculus. $S_1$ simulates $S_2$ if and only if there exists a polynomial function $p$ such that for any formula $A$ and any proof $P_2$ of $A$ in $S_2$, there exists a $S_1$-proof $P_1$ of $A$ (translated into $S_1$ language) so that

$$\text{size}(P_1) \leq p(\text{size}(P_2))$$

In other words, a system $S_1$ simulates $S_2$ if $S_1$ is not less efficient than $S_2$ as a proof system. In particular, we say that $S_1$ polynomially simulates ($p$-simulates) $S_2$ if there is a polynomial-time algorithm which, given an $S_2$-proof of a formula $A$, produces an $S_1$-proof of $A$.

We begin with reviewing two propositional systems, resolution and GCNF [8] [2]. Both resolution and GCNF are refuting systems: we show that a given
formula is a tautology by showing its negation, put into conjunctive normal form, is unsatisfiable.

A literal is a propositional variable or its conjugate. A clause is a finite set of literals which is often expressed as a product of literals. The meaning of a clause is the disjunction of the literals in the clause. For example $l_1 \cdots l_n$ means $l_1 \lor \cdots \lor l_n$. (A clause consisting of a single literal is also called a literal if there is no danger of confusion.) Let $H$ be a finite set of clauses. A resolution of $H$ is a directed acyclic graph such that any leaf of the graph is labeled by a clause in $H$, the root by an empty clause, and any inner node associated with its two upper nodes by the resolution rule

\[
\frac{C_1 \cup \{x\} \quad C_2 \cup \{\overline{x}\}}{C_1 \cup C_2}
\]

where neither $C_1$ nor $C_2$ contains the literals $x$ or $\overline{x}$.

A cedent is a sequence of clauses which is often expressed by a capital Greek letter. A sequent is a string of the form $\Gamma \Rightarrow \Delta$, where $\Gamma$ is called the antecedent and $\Delta$ the succedent. LK is a Gentzen style sequent calculus \cite{7}. We restrict LK to obtain a propositional calculus, GCNF, on clauses rather than ordinary formulas. GCNF is equipped with an axiom scheme,

\[
\text{axioms} \quad p, \overline{p} \Rightarrow
\]

and the following two inference rules.

\[
\begin{align*}
\text{structural inference} & \quad \frac{\Gamma \Rightarrow}{\Gamma^* \Rightarrow} \\
\text{logical inference} & \quad \frac{l_1, \Gamma_1 \Rightarrow \cdots \quad l_n, \Gamma_n \Rightarrow}{l_1 \cdots l_n, \Gamma_1, \ldots, \Gamma_n \Rightarrow}.
\end{align*}
\]

where $l_i$ ($i = 1, \ldots, n$) are literals. In the logical inference, we infer a single sequent $l_1 \cdots l_n, \Gamma_1, \ldots, \Gamma_n \Rightarrow$ from $n$ ($\geq 2$) sequents, $l_j, \Gamma_j \Rightarrow$ ($1 \leq j \leq n$). The indicated occurrence of the clause $l_1 \cdots l_n$ in the lower sequent is called the principal clause, and the indicated occurrences of the clauses $l_1, \ldots, l_n$ in the upper sequents are called the auxiliary clauses of this inference. The principal clause is the successor of auxiliary clauses. The successors of clauses in $\Gamma_j$'s are defined as those in the structural inference.

\text{Remark}: A logical inference in GCNF is an abbreviation of several $[\lor, \text{right}]$ inferences in LK.

We introduce another inference rule to GCNF, called the atomic cut.

\[
\begin{align*}
\text{atomic cut} & \quad \frac{p, \Gamma \Rightarrow \overline{p}, \Gamma \Rightarrow}{\Gamma \Rightarrow}.
\end{align*}
\]
where \( p \) is a propositional variable. We call \( p \) the *cut-formula* of this inference. The occurrence of \( p \) (resp. \( \overline{p} \)) appearing in the left (resp. right) upper sequent is called the *left* (resp. *right*) *cut-formula*. Cut-formulas have no *successor*.

Let \( P \) be a GCNF or GCNF + Atomic cut proof, and \( C, D \) occurrences of clauses in \( P \). \( C \) is an *ancestor* of \( D \) either if \( D \) is a successor of \( C \) or if the successor of \( C \) is an ancestor of \( D \). A sequent \( S \) is a *left-ancestor* (resp. *right-ancestor*) of another sequent \( T \) if the lower sequent of \( S \) is a left-ancestor (resp. right-ancestor) of \( T \). \( S \) is an *ancestor* of \( T \) if \( S \) is either a left- or a right-ancestor of \( T \). Successents in a GCNF or GCNF + Atomic cut proof are always empty. Hence, we omit →'s for the sake of simplicity. When a propositional variable \( p \) occurs in a sequent \( \Gamma \) as a clause, we say \( p \) *occurs positively* in \( \Gamma \). When a conjugate of a propositional variable \( \overline{p} \) occurs in \( \Gamma \) as a clause, we say \( p \) *occurs negatively*.

Resolution can be also defined as a subsystem of GCNF + Atomic cut.

**Definition 1** A GCNF + Atomic cut proof \( P \) is *resolution-like* if and only if \( P \) satisfies the following condition.

1. No logical inference appears below any cut.
2. Upper sequents of logical inferences are initial sequents.
3. No atomic formula appears more than once in any sequent.
4. The cut-formula does not appear in the lower sequent of a cut.
5. Non-initial sequents do not contain both a propositional variable and its negation.
6. Formulas in the end-sequent does not appear as a cut-formula of any cut.

Let \( s \) be an occurrence of a sequent. We say that \( s \) occurs in the *end-part* of \( P \) if there exists no logical inference below \( s \).

**Lemma 1** Let \( H = \{l, C_1 \cup l, \ldots, C_n \cup l, D_1 \cup \overline{l}, \ldots, D_m \cup \overline{l}, \Gamma\} \) be a set of clauses such that the occurrence of \( l \) and \( \overline{l} \) are fully indicated, and none of \( D_i \) (1 ≤ \( i \) ≤ \( m \)) is empty. Let \( H^* \) be a set of clauses, \{\( D_1, \ldots, D_m, \Gamma \}\}. For any resolution, \( P \), of \( H \), there exists a resolution \( Q \) of \( H^* \) such that size(\( Q \)) ≤ size(\( P \)) and \( \operatorname{len}(Q) \leq \operatorname{len}(P) \).

(Proof) First, change every node \( C_i \cup l \) for \( l \) in \( Q \). Secondly, omit every occurrence of \( l \) and \( \overline{l} \) in \( Q \). After a minor modification, we obtain a resolution proof of \( H^* \) without increasing its size and length. \( \square \)

By lemma 1, we can assume that no clause in the hypothesis consists of a single literal.

4
Theorem 1 Let $P$ be a resolution of $C_1, \ldots, C_n$. Then, there exists a resolution-like proof $P^*$ of $C_1, \ldots, C_n$ such that $\text{size}(P^*) = O(\text{size}(P))$ and $\text{len}(P^*) = O(\text{len}(P))$.

Proof. First, change every node labeled by $C_i$ ($1 \leq i \leq n$) in $P$ for the subproof

$$
\overline{t_{1i}, \ldots, t_{ki}} \overline{C_i, t_{1i}, \ldots, t_{ki}, C_i}
$$

where $\overline{t_1 \cdots t_k} = C_i$.

When nodes $\{x\} \cup D_1$ and $\{\overline{x}\} \cup D_2$ in $P$ are replaced by the sequents $\overline{x}, \Gamma_1$ and $x, \Gamma_2$ respectively, change the resolution of the form

$$
\begin{array}{c}
\overline{x} \cup D_1 \\
\Gamma_1 \\
\overline{x} \cup D_2
\end{array}
$$

for

$$
\begin{array}{c}
\overline{\overline{x}, \Gamma_1} \\
\overline{x, \Gamma_1 \cup \Gamma_2} \\
\overline{x, \Gamma_1 \cup \Gamma_2}
\end{array}
$$

After minor modification (inserting some structural inferences), we obtain a GCNF + Atomic cut proof $P^*$. We show that $P^*$ satisfies the conditions (1)-(7). (1), (2) and (3) are obvious from the construction. (4)-(6) follows from the nature of resolution. (7) follows from the remark given above. □

3 Cut-elimination

In this section, we give a non-deterministic algorithm to eliminate all the atomic cuts from a given resolution-like proof to obtain a DAG GCNF proof of which size is bounded by quasi-polynomial of the size of the original proof. Note that the number of nodes in a DAG is equal to the number of different nodes in a tree. For simplicity, we express GCNF proofs as trees and measure their complexity by the number of different sequents contained in them.

Definition 2 Let $P$ be a tree GCNF proof. We define its DAG-length, denoted by $\text{dlen}(P)$, by the number of the different lines in $P$. Similarly, we define its DAG-size, $\text{dsize}(P)$, by the total number of symbols contained in different lines in $P$.

(Cut-elimination Algorithm)

Let $\vdash$ a cut occurring in a resolution-like (tree) proof of the form

$$
\begin{array}{c}
\vdash Q_1 \\
\vdash Q_2 \\
\Gamma
\end{array}
$$

\begin{enumerate}
\item \text{Drop the cut.}
\end{enumerate}
We define two cut-elimination algorithms, called $L$-reduction and $R$-reduction as follows.

When the $L$-reduction is applied to $\alpha$, we eliminate the cut $\iota$ to obtain a new proof of $\Gamma, \Gamma$ as follows. Suppose that $s$ is a non-initial sequent in $Q_1$ of the form $\Pi_1, p, \Pi_2$. We have a new sequent $s'$ ($\equiv \Pi_1, \Pi_2, \Gamma$) replace $s$ if the indicated occurrence of $p$ in $s$ is an ancestor of the left cut-formula of $\iota$. The end-sequent of $Q_1$ turns into $\Gamma, \Gamma$ after the transformation. We say that $s$ produces $s'$ and that $\iota$ changes $s$. For initial sequents of the form, $p, \overline{p}$, we delete these sequents and paste the subproof $Q_2$ instead if the indicated occurrences of $p$ are ancestors of the left cut-formula of $\iota$. More than one copies of $t$ are possibly produced by the reduction. We say that $t$ produces the copies of $t$ in the new proof.

When the $R$-reduction is applied to $\alpha$, we obtain a proof of $\Gamma, \Gamma$ symmetrically to the $L$-reduction. Suppose that $s$ is a non-initial sequent in $Q_2$ of the form $\Lambda_1, \overline{p}, \Lambda_2$. We have a new sequent $s'$ ($\equiv \Lambda_1, \Lambda_2, \Gamma$) replace $s$ if the indicated occurrence of $\overline{p}$ is an ancestor of the right cut-formula of $\iota$. We say that $s$ produces $s'$ and that $\iota$ changes $s$. For the initial sequents of the form, $p, \overline{p}$, we delete these sequents and paste the subproof $Q_1$ instead if the indicated occurrences of $\overline{p}$ are ancestors of the right cut-formula of $\iota$. We say that an occurrence of a sequent $t$ in $Q_1$ produces the copies of $t$ in the new proof.

Let $P$ be a resolution-like (tree) proof of $\Gamma, \Gamma$ symmetrically to the $L$-reduction. Suppose that $s$ is a non-initial sequent in $Q_2$ of the form $\Lambda_1, \overline{p}, \Lambda_2$. We have a new sequent $s'$ ($\equiv \Lambda_1, \Lambda_2, \Gamma$) replace $s$ if the indicated occurrence of $\overline{p}$ is an ancestor of the right cut-formula of $\iota$. We say that $s$ produces $s'$ and that $\iota$ changes $s$. For the initial sequents of the form, $p, \overline{p}$, we delete these sequents and paste the subproof $Q_1$ instead if the indicated occurrences of $\overline{p}$ are ancestors of the right cut-formula of $\iota$. We say that an occurrence of a sequent $t$ in $Q_1$ produces the copies of $t$ in the new proof.

Let $P$ be a resolution-like (tree) proof of $\Gamma, \Gamma$ and $t_1, \ldots, t_n$ be the list of cuts occurring in $P$. A string of $R$ and $L$ of length $n$ is called a strategy (on $P$). We denote the set of strategies (on $P$) by $A$. Note that $\text{card}(A) = 2^n$.

For each strategy $\alpha \in A$, we denote the $i^{th}$ ($1 \leq i \leq n$) element of $\alpha$ by $a_i$.

Now we define algorithms to eliminate all the cuts. Given a strategy $\alpha$, we eliminate the cuts in $P$ bottom-up following $\alpha$: for each $i$ ($1 \leq i \leq n$), we apply $\alpha_i$-reduction to eliminate all the cuts produced from $t_i$.

Let $\alpha$ be a strategy and $s$ an occurrence of a sequent. We define the influence of $\alpha$ on $s$ by the number of cuts which change $s$. Formally we define it as follows.

**Definition 3** Let $\alpha$ be a strategy on $P$ and $s$ an occurrence of sequent in the end-part of $P$. We define the influence of $\alpha$ on $s$ by the number of literals $l$ in $s$ satisfying the either one of the following;

1. There exists $1 \leq i \leq n$ such that $a_i = L$, and $l$ is an ancestor of the left cut-formula of $t_i$.
2. There exists $1 \leq i \leq n$ such that $a_i = R$, and $l$ is an ancestor of the right cut-formula of $t_i$.

We define the influence of $\alpha$ on $P$ by the maximum influence of $\alpha$ on all the occurrences of sequents in $P$. Finally we define the influence (of $A$) on $P$ by
the minimum of influences of all the strategies in $\mathcal{A}$ on $P$. When $\alpha \in \mathcal{A}$ has the minimum influence on $P$, we call $\alpha$ a winning strategy.

4 Main theorem

In the previous section, we define an algorithm to eliminate atomic cuts from GCNF (tree) proofs. Given a strategy, the DAG lengths of the result proof may grow exponentially to that of the original proof. In this section, however, we show that there must exist a strategy which keeps the DAG-length quasi-polynomial to that of the original proof. Our proof is sketched as follows. Suppose that the influence on a given GCNF (tree) proof, $P$, is $k$. We first show that the DAG length of the result proof is bounded by $n^{O(k)}$. Then, we show a combinatorial theorem which guarantees that $k$ must be bounded by $\log(n)$.

Theorem 2 Let $P$ be a resolution-like (tree) proof of $\Gamma$, and $t_1, \ldots, t_n$ be the list of cuts occurring in $P$. Suppose that $\alpha$ is a winning strategy on $P$. Then, $\text{dlen}(P; \alpha) \leq \text{dlen}(P)^{1+2k}$. The size of each sequent in $(P; \alpha)$ is bounded by $k \cdot \text{dsize}(P)$

(Proof.) Let $S$ be a sequent, and $s$ an occurrence of $S$ in $P$. Without loss of generality, we can assume that $S$ is of the form

$$l_1, \ldots, l_m, \Gamma.$$  

Note that the final sequents in $(P; \alpha)$ produced from $s$ are in the same form.

Now we want to compute how many different sequents can be produced from occurrences of $S$ in $P$. We can choose $k'$ (\$k\$) literals among $l_1, \ldots, l_m$ which satisfy either one of the condition given in definition 3. For such $l_i$, there may exist many (\$\leq \text{dlen}(P)$) cuts of which cut-formula is $l_i$. Hence, we have at most

$$\binom{m}{k'} \cdot \text{dlen}(P)^{k'} \leq \text{dlen}(P)^{2k}$$

different sequents produced from $S$. □

In principle, we try both R-reduction and L-reduction to every cut in $P$ and adopt the strategy of which influence is minimum. However, there are cases such that it is apparent which reduction is suitable to adopt. Consider the following case; suppose that $t_1$ is the lowermost cut of the form

$$\Gamma \vdash Q_1, \Gamma \vdash Q_2$$

$$\frac{\bar{p}, \Gamma}{\overline{p}, \Gamma}{t_1}$$
where the depth of $Q_1$ is $100$ and that of $Q_2$ is $2$ and $k = 15$. Define the subset $A^*$ of $A$ by $\{\alpha|\alpha_1 = R\}$. Then, it is obvious that for any strategy $\alpha$ and sequents in $Q_2$, the influence of $\alpha$ on $s$ is less than $k$. The influence of $A^*$ on $P$ cannot exceed $k$. We can find a winning strategy in $A^*$.

We define $S$ by the set of occurrences of sequents in $P$;

$$S = \{s|\exists \alpha \in A \text{(the influence of } \alpha \text{ on } s \text{ is greater than or equal to } k)\}.$$ 

$S \neq \emptyset$ by the definition.

**Lemma 2** Suppose that $t_i$ is a cut in $P$ of the form

$$\begin{array}{c}
\vdash \overset{\Sigma}{Q_1} \\
\vdash \overset{\Sigma}{Q_2} \\
\vdash \overset{\Sigma}{t_i}
\end{array}$$

Let $s$ be an occurrence of a sequent in $Q_1$ (resp. $Q_2$), and $\alpha$ a strategy. Define $\alpha^*$ in $A$ by

$$\alpha_j^* = \begin{cases} R \text{ (resp. L)} & \text{if } j = i \\ \alpha_j & \text{otherwise} \end{cases}$$

Then, the influence of $\alpha^*$ on $s$ is less than or equal to that of $\alpha$ on $s$.

**(Short-cut algorithm)**

Suppose that $t_i$ is a lowermost cut in $P$ such that it is of the form

$$\begin{array}{c}
\vdash \overset{\Sigma}{Q_1} \\
\vdash \overset{\Sigma}{Q_2} \\
\vdash \overset{\Sigma}{t_i}
\end{array}$$

and that there is no occurrence of a sequent in $S$ in $Q_1$ (resp. $Q_2$). Then, we determine $\alpha_i = L$ (resp. $\alpha_i = R$).

Define the new $A$ by $\{\alpha|\alpha_i = L\}$ (resp. $\{\alpha|\alpha_i = R\}$). Then, the influence of new $A$ on $P$ is also $k$. New $S$ (determined by new $A$) is obtained, and it may be different from the old one. (Note that new $S$ is a subset of old $S$.) See if there is any cut which satisfies the condition above for new $S$. Continue the algorithm until there is no such cut left.

Now we assume that we already applied the procedure given above to a given resolution-like proof; to some cuts, either L- or R-reduction are pre-assigned on some cuts and others remain unassigned. Those cuts to which neither reductions are pre-assigned are called free cuts.

In the remainder of the argument, we fix $P$ as a resolution-like (tree) proof, $t_1, \ldots, t_n$ the list of free cuts in $P$, $A$ the set of strategies over $t_1, \ldots, t_n$, $k$ the
influence of $A$ on $P$, and $S$ as defined above. We define $\overline{S}$ by the set of the lowermost occurrences of sequents in $S$.

**Definition 4** Let $\alpha$ be a strategy on $P$. The path determined by $\alpha$ is a sequence of sequents $s_1, \ldots, s_h$ satisfying the following conditions.

1. The end-sequent is $s_1$.
2. For all $i$, $s_{i+1}$ is an upper sequent of $s_i$.
   (a) If $s_i$ is the lower sequent of a structural inference, $s_{i+1}$ is the upper sequent of $s_i$.
   (b) If $s_i$ is the lower sequent of a non-free cut, and if L-reduction is assigned to the cut, then $s_{i+1}$ is the right upper sequent of $s_i$.
   (c) If $s_i$ is the lower sequent of a non-free cut, and if R-reduction is assigned to the cut, then $s_{i+1}$ is the left upper sequent of $s_i$.
   (d) If $s_i$ is the lower sequent of a free cut, $s_j$, and $\alpha_j = R$, then $s_{i+1}$ is the right upper sequent of $s_i$.
   (e) If $s_i$ is the lower sequent of a free cut, $s_j$, and $\alpha_j = L$, then $s_{i+1}$ is the left upper sequent of $s_i$.
3. In the end-part of $P$, $s_h$ is one of the uppermost sequents.

**Lemma 3** Let $\alpha$ be a strategy on $P$. Then, there exists an occurrence of a sequent in $\overline{S}$ on the path determined by $\alpha$.

(Proof.) Obvious from the reduction defined by the short-cut algorithm.

Converse is also true: for any $s \in \overline{S}$, there exists a strategy such that $s$ is on the path determined by the strategy, and the influence of $\alpha$ on $s$ is equal to $k$.

**Lemma 4** Let $s \in \overline{S}$ and $\alpha \in A$. Suppose that $s$ is on the path determined by $\alpha$. Then, the influence of $\alpha$ on $s$ is equal to $k$.

(Proof.) It suffices to show that for any $\beta \in A$, the influence of $\beta$ on $s$ is less than or equal to that of $\alpha$ on $s$, but it is obvious from the definition of influence. □

**Definition 5** Let $s$ be an occurrence of a sequent and $p$ a variable. We say $s$ is on the left (resp. right) tree of $p$ if there exists an occurrence of a cut $\iota$ such that $\iota$ is the uppermost cut below $s$ of which cut-formula is $p$ and $s$ is a left-ancestor (right-ancestor) of the lower sequent of $\iota$.

**Lemma 5** Let $S$ be a sequent, $s$ occurrences of $S$ in $P$. Suppose that $s$ is in the left (resp. right) tree of some variable $p$. Then, $p$ cannot occur negatively (resp. positively) in $S$. 

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(Proof.) Without loss of generality, suppose that $s$ is in the left tree of $p$. Suppose that $p$ occurs negatively in $S$. By the definition of resolution-like proofs and lemma 1, the clause $\bar{p}$ in $s$ must be eliminated by the uppermost cut of which cut-formula is $p$ below $s$. It contradicts with the hypothesis that $s$ is in the left tree of $p$. □

Lemma 5 tells us that if $s$ is an occurrence of $S$, and if $s$ is in the left (resp. right) tree of $p$, either the following conditions hold:

1. $p$ occurs neither positively nor negatively in $S$,

2. for any occurrence of $t$ of $S$, $t$ must also occur in the left (resp. right) tree of $p$.

Lemma 6 Let $S$ be a sequent, $s,t$ occurrences of $S$, and $p$ a variable. Suppose that $s$ occurs in the left tree of $p$ and $t$ in the right tree of $p$. Then, neither $s$ nor $t$ is influenced by any cut of which cut-formula is $p$.

(Proof.) Straightforward from lemma 5 and the definition of influence. □

Next we define a new notion, an effect, which is a similar concept to an influence.

Definition 6 Let $t \in 3$ of the form $T$ and $t$ a free cut below $t$. We say that $t$ effects $t$ if and only if $T$ does not have occurrences in $P$ such that one is in the left tree of $p$ and another in the right tree of $p$.

We define the effect on $t$ by the number of cuts which effect $t$, and the effect on $P$ by the minimum of effects on all the occurrences of sequents in $3$.

Lemma 7 Let $s \in 3$, and $\alpha \in \mathcal{A}$. Suppose that $s$ is on the path determined by $\alpha$. Then, the effect on $s$ is greater than or equal to the influence of $\alpha$ on $s$.

(Proof.) By lemma 6, $t$ effects $s$, if it influences $s$. □

Lemma 7 guarantees that the effect on $P$ is greater than or equal to the influence on $P$, which is $k$.

Now we show the following.

Theorem 3 When the influence on $P$ is $k$, $k \leq \log(dlen(P))$.

We prove the theorem by showing that the number of different sequents in $3$ must be greater than or equal to $2^k$. 

1/0
Before we start the proof, we extract “critical informations” from $P$ and construct a backbone of $P$. For example, suppose that $P$ is of the form

$$
\begin{align*}
&s_1 = p_1 p_2, p_1 \bar{p}_2, p_1 \bar{p}_2 \\
&s_2 = p_1, p_2, p_1 \bar{p}_2 , p_1 \bar{p}_2 \\
&t_1 = p_1, p_2, p_1 \bar{p}_2 , p_1 \bar{p}_2 \\
&t_2 = p_1, p_2, p_1 \bar{p}_2 , p_1 \bar{p}_2 \\
&t_3 = p_1, p_2, p_1 \bar{p}_2 , p_1 \bar{p}_2
\end{align*}
$$

where $t_1, t_2, t_3$ are the list of free cuts, $k = 2$, and $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$. Then, we extract the information consists of follows to get the backbone of $P$;

1. sequents in $\mathcal{S}$
2. free cuts and their lower sequents, where the lower sequents of cuts are labeled by the cut-formulas.

In this particular example, the backbone of $P$ is illustrated as Figure 1

![Figure 1](image)

Figure 1:

Notice that there are 4 ($= 2^2$) different sequents in $\mathcal{S}$ while the influence on $P$ is equal to 2, which happens to be equal to the effect on $P$. We need technical definitions to count the number of different sequents in $\mathcal{S}$.

**Definition 7** A tree evolved from $P$ is a binary labeled tree $T$ satisfying the following conditions.

1. The number of edges leading out from a node is either 2 or 0.
2. Every leaf is labeled by a sequent in $\mathcal{S}$.
Every inner nodes are labeled by a variable.

The terminology for proofs, such as ancestors and DAG-length, are also used on trees evolved from $P$, and they are defined in a similar manner. However, we need to be careful about the definition of left and right tree.

**Definition 8** Let $Q$ be a tree evolved from $P$. A leaf $s$ of $Q$ is in the left tree (resp. right tree) of a variable $p$ if there exists a node $t$ such that $t$ is the uppermost node labeled by $p$ below $s$ and $s$ is a left-ancestor (resp. right-ancestor) of $t$.

We define effects on the occurrences of sequents exactly as we did for resolution-like proofs. The backbone of $P$ is obviously a tree evolved from $P$. We construct a sequence of trees evolved from $P$, $T_0, T_1, \ldots$, so that there exists $l \geq o$ such that leaves of $T_l$ are all different and the effect on $T_l$ is equal to that on $T_0$.

Define $T_0$ by the backbone of $P$. Let $k^*$ be the effect on $T_0$. We define $m^*$ by the number of different variables labeled on inner nodes in $T_0$.

For $i \geq 0$, we construct $T_{i+1}$ from $T_i$ as follows. Suppose that the root of $T_i$ is labeled by $p$. Define $L_1$ by the set of sequents labeled to the leaves in the left tree of $p$, and $L_2$ by those in the right tree of $p$. Note that any leaf labeled by a sequent in $L_1 \cap L_2$ is not effected by $p$.

**(Case 1)** Suppose that $\text{card}(L_1) \geq \text{card}(L_2)$. Let $f$ be a matching between $L_1/L_2$ and $L_2/L_1$. (A matching between sets $D$ and $E$ is a set of mutually disjoint unordered pairs $\{i, j\}$, where $i \in D$ and $j \in E$.) For every node $t$ labeled by $p$, we remove $t$ and all the branches leading out from its left upper node. As a result, we obtain a tree, $T_i^*$, which is also evolved from $P$, but the effect may decrease by 1. We construct a branch, called $B_s$, as Figure 2.

![Figure 2](image)

Now we construct a new tree $T_{i+1}$ from $T_i^*$ as follows. For every leaf in $T_i^*$, which are labeled with $f(s)$, we graft the branch $B_s$ onto the left-hand side of the edge adjacent to the leaf (Figure 3).
Figure 3:

$T_{i+1}$ is a tree evolved from $P$ as is $T_i$. Note that $p$ effects leaves labeled by $s$ and $f(s)$ since it is labeled to the uppermost node below them. $f(s)$; the leaves labeled by $f(s)$ are all effected by $p$ in $T_{i+1}$. The effect on $T_{i+1}$ is equal to $k^*$ assuming that the effect on $T_i$ is also $k^*$. No repeating occurrence of $p$ in any path in $T_{i+1}$.

(Case 2)
Suppose that $\text{card}(L_1) < \text{card}(L_2)$. We define an algorithm to transform $T_i$ in a similar manner to the algorithm given in (case 1), excepting that we interchange left and right throughout. As a result, we obtain a tree $T_{i+1}$ evolved from $P$, on which effect is $k^*$ assuming that the effect on $T_i$ is also $k^*$.

**Lemma 8** Let $p$ be a variable. There is no repeating occurrence of $p$ in any path in $T_{m^*}$.

**Lemma 9** There exists $l \geq m^*$ such that no two leaves in $T_l$ have the same label.

(Proof.) Let $s \in \mathcal{S}$ and $i \geq 0$. Define overlap of $s$ in $T_i$, denoted by $\text{overlap}(s, T_i)$, the number of leaves labeled by $s$ in $T_i$. The overlap of $T_i$ is defined by the maximum of overlaps of all the sequents in $\mathcal{S}$. We also define the distance of $s$ in $T_i$, denoted by $d(s, T_i)$, by the smallest number of edges connecting two nodes labeled by $s$. The distance of $T_i$, $d(T_i)$, is defined by the minimum of distances of all the sequents hit the maximum overlap. For any $i \geq m^*$, $d(T_i) \leq 2m^*$ and overlap$(T_i) \geq 1$. We show for each $i \geq 0$, one of the following condition holds.

1. Labels of leaves in $T_i$ are all different,

2. overlap$(T_{i+1}) < \text{overlap}(T_i)$, or
3. $d(T_{i+1}) > d(T_i)$. 

Suppose that neither 1 nor 2 holds. Then, there exists $s$ such that $s$ is hitting the maximum overlap in $T_{i+1}$, and $\text{overlap}(s, T_i) \geq \text{overlap}(s, T_{i+1})$. Let $p$ denote the label of the root of $T_i$. Since $\text{overlap}(s, T_i) \geq \text{overlap}(s, T_{i+1})$, $s$ is occurring either in the left or right tree but not both. Suppose that $s$ is relocated when we transform $T_i$ into $T_{i+1}$; $s$ is labeled on leaves in $B_s$, and grafted onto the edge adjacent to $f(s)$. Then, $\text{overlap}(s, T_i) = \text{overlap}(f(s), T_{i+1})$. At the same time, $d(f(s), T_{i+1}) > d(f(s), T_i)$. Suppose that $s$ is not relocated. Then, it is apparent that $d(s, T_{i+1}) > d(s, T_i)$. In either case, we have $d(T_{i+1}) > d(T_i)$. $\Box$

Now we resume the proof of theorem 3. 

(Proof of theorem.) For any leaf $s$ in $T_i$, the number of edges connecting the root of $T_i$ and $s$ must be at least $k^*$ since the effect on $T_i$ is $k^*$. That means $T_i$ has more than $2k^*$ leaves. By lemma 4, there are at least $2k^*$ different sequents in $\mathcal{G}$. Since $k^* \geq k$, we have $k \leq \log(dlen(P))$. $\Box$

As a result of theorem 1, 2 and 3, we obtain the following result; cut-free Gentzen system (LK) quasi-polynomially simulates resolution.

**Theorem 4** For any resolution refutation $P$ of a set of clauses $C_1,\ldots,C_k$ of length $n$ and size $m$, there exists a cut-free LK proof $P^*$ of $C_1,\ldots,C_k \rightarrow$ of length $n^{1+\frac{1}{2}\log n}$ and size $m^{2+3\log n}$.

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