States and Quantum Effects in the Collective Field Theory of a Deformed Matrix Model

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We derive an equation which gives the tree-level scattering amplitudes for tachyons in the black hole background using the exact states of the collective field hamiltonian corresponding to a deformed matrix model recently proposed by Jevicki and Yoneya. Using directly the symmetry algebra we obtain explicit expression for a class of amplitudes in the tree approximation. We also study the quantum effects in the corresponding collective field theory. In particular, we compute the ground state energy and the free energy at finite temperature up to two loops, and the first quantum correction to the two-point function.

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1. Introduction

Matrix models have been a very powerful tool for studying two-dimensional quantum gravity and low-dimensional string theories. Although there are still some puzzles in establishing the exact relationship between the matrix model and the standard continuum formulation of string theory, it is generally believed that the one-dimensional hermitian matrix model in the double scaling limit describes the linear dilaton solution with tachyon condensation of the two-dimensional critical string theory in a flat space-time. The main reason for this belief is an exact agreement of all known results for the correlation functions and the existence of the $W_\infty$ symmetry in both approaches (for reviews see e.g. [1,2]).

If, however, the matrix model is to be taken as a (nonperturbative) definition of string theory it must in some way encompass all solutions of string theory, in particular the black hole background [3–5]. The importance of a full quantum mechanical (and possibly nonperturbative) understanding of black hole physics can hardly be overemphasized. Since the matrix model, at the present time, provides the most elegant framework for explicit computations it is very important to find a matrix model formulation of the black hole background. Despite numerous attempts [6] the problem remains unsolved. An important step in this direction has recently been taken by Jevicki and Yoneya [7].

Their proposal is based on several lessons we have learned from the standard $c = 1$ model. First, the massless scalar collective field [8] is related to the string theory tachyon through a non-local redefinition [9] which transforms the Klein-Gordon equation of the collective field theory into the Virasoro condition in the string theory. It is remarkable that the wave function renormalization induced by this transformation precisely accounts for the “external leg factors” in the tachyon scattering amplitudes which have the following factorized form [10]:

$$A\text{ (tachyons)} = (\text{external leg factors}) \times A\text{ (collective field theory)}.$$ 

Although in Minkowski space external leg factors are pure phases and therefore have no physical effect they still contain physical information about the background. Namely, after analytic continuation to Euclidean space they have poles at special discrete values of momenta for which the incoming tachyon wave is in resonance with the “wall” condensate. The on-shell collective field amplitudes, on the other hand, are simple polynomials in certain combinations of the (absolute values) of momenta and have no poles [11].
Another crucial point in the proposal is related to the identification of the string coupling constant $g_{st}$. In the standard $c = 1$ case it is given by $g_{st} \sim 1/\mu$, where $\mu$ is the (negative) Fermi energy which can be thought of as a constant deformation of the inverted harmonic potential in the corresponding matrix model.

In the black hole background, the non-local field redefinition of the collective field which relates it to the tachyon field was discussed in. It is given a new interpretation in where it is also argued that the existence of such transformation is necessary, but not sufficient for the matrix model to describe the black hole background. In addition one must have the correct relationship between the string coupling constant and the black hole mass, $g_{st} \sim 1/\sqrt{M}$, which can be achieved if one deforms the inverted harmonic oscillator potential by a singular term $M/x^2$ and sets the Fermi energy to zero. This is precisely the deformation of the inverted harmonic potential found by Avan and Jevicki in a search for potentials with algebraic structures similar to the one found in the standard $c = 1$ case.

It is also shown by Jevicki and Yoneya that the factorization assumption is consistent only if the amplitudes for odd number of tachyons vanish. This imposes strong constraints on the form of the collective field theory, and they demonstrated that the deformed potential indeed gives scattering amplitudes which satisfy this constraint (at least in the tree approximation). Therefore, the model proposed in is the first matrix model consistent with minimal requirements to make connection with the black hole background and deserves a careful analysis. In this article we study further the properties of the collective field formulation of the deformed matrix model concentrating on states, symmetries, quantum and finite temperature effects.

This paper is organized as follows: In sect. 2 we review the collective field formulation of a general matrix model including normal ordering and apply it to the deformed matrix model. In sect. 3 we summarize some results on the exact eigenstates and symmetries of the collective hamiltonian which are needed for our discussion. Following ref. we then derive an equation which describes the tree-level scattering amplitudes using these exact states. In sect. 4 we give another derivation of scattering amplitudes using directly the symmetry algebra. In sect. 5 we present some examples of loop calculations. In particular, we compute the ground state energy and the finite temperature free energy up to two loops and the two-point function up to one loop. We find that the free energy is not invariant under the duality transformation $2\pi T \rightarrow 1/(2\pi T)$. Sect. 6 is reserved for discussion and
conclusions. In appendix A we derive some of the results of sect. 5 directly in the fermionic picture of the matrix model in order to confirm some of our results and to provide further evidence of the completeness (at least perturbatively) of the normal ordered collective hamiltonian.

2. Collective field theory of a deformed matrix model

Let us consider a general one-dimensional hermitian matrix model defined by

$$H = \text{Tr} \left( \frac{1}{2} M^2(t) + V(M) \right).$$

(2.1)

The singlet sector of this hamiltonian describes the dynamics of $N$ decoupled nonrelativistic fermions in an external potential with the single particle hamiltonian given by

$$h(p, x) = \frac{1}{2} p^2 + v(x) - \mu_F,$$

(2.2)

where $\mu_F$ is the Fermi energy. The collective field theory is most easily derived in the Fermi liquid picture first discussed by Polchinski with the result

$$H = \int_{-\infty}^{\infty} \frac{dx}{2\pi} \int_{\alpha_+(x)}^{\alpha_-} d\alpha \ h(\alpha, x).$$

(2.3)

As noted in one could equally take the collective hamiltonian to be

$$H = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \int_{\beta_+(p)}^{\beta_-} d\beta \ h(p, \beta),$$

(2.4)

which would lead to a differently looking theory but which should give the same physical results. In the following we will discuss only the first representation.

Integrating (2.3) one finds

$$H = \int_{-\infty}^{\infty} \frac{dx}{2\pi} \left[ \frac{1}{6} \left( \alpha_+^3 - \alpha_-^3 \right) - (\mu_F - v(x)) \left( \alpha_+ - \alpha_- \right) \right].$$

(2.5)

The fields $\alpha_{\pm}(x, t) \equiv \partial_x \Pi \pm \pi \phi$ represent linear combinations of the canonical field and momentum, and can be identified with the two branches of the profile function of the Fermi surface in the classical phase space. They obey Poisson brackets $\{\alpha_{\pm}(x), \alpha_{\pm}(y)\} = \ldots$
The classical equation of motion $\partial_t \alpha_{\pm} = \{\alpha_{\pm}, H\}$ has a static solution which is easily found to be

$$\pi \phi_0(x) = \sqrt{2[\mu_F - v(x)]} \theta[\mu_F - v(x)] .$$  \hfill (2.6)

Changing variables from $x$ to the “time-of-flight” variable $\tau$

$$d\tau = \frac{dx}{\pi \phi_0(x)} ,$$  \hfill (2.7)

and shifting the fields $\alpha_{\pm}(x, t)$ by the classical background $\pi \phi_0$:

$$\alpha_{\pm}(x, t) = \pm \pi \phi_0 + \frac{1}{\pi \phi_0} \tilde{\alpha}_{\pm}(\tau, t) ,$$  \hfill (2.8)

one obtains the following hamiltonian

$$H = \frac{1}{2} \int_{0}^{\infty} \frac{d\tau}{2\pi} (\tilde{\alpha}_{+}^2 + \tilde{\alpha}_{-}^2) + \frac{1}{6} \int_{0}^{\infty} \frac{d\tau}{2\pi} \frac{1}{(\pi \phi_0)^2} (\tilde{\alpha}_{+}^3 - \tilde{\alpha}_{-}^3) + E^{(0)} ,$$  \hfill (2.9)

where the constant $E^{(0)}$ is the classical energy of the background field

$$E^{(0)} = -\frac{1}{3\pi} \int dx \ [2(\mu_F - v(x))]^{3/2} .$$  \hfill (2.10)

In the quadratic part of the hamiltonian one easily recognizes a massless two-dimensional scalar theory since $\tilde{\alpha}_{\pm}(\tau) = -\Pi \zeta \pm \pi \partial_\tau \zeta$. This is true for any matrix model provided there is a static classical solution. The coupling constant of the cubic interaction is space dependent and encodes the specific form of the matrix model potential through $\pi \phi_0$.

At the linearized level, $\tilde{\alpha}_{\pm}$ are simply the right and left moving massless modes which satisfy $(\partial_t \pm \partial_\tau) \tilde{\alpha}_{\pm}(\tau, t) = 0$, and can be expanded as

$$\tilde{\alpha}_{\pm}(\tau, t) = \pm \int_{-\infty}^{\infty} dk \ \alpha_{k}^{\pm} e^{-ik(t\mp\tau)} , \quad [\alpha_{k}^{\pm}, \alpha_{k'}^{\pm}] = k\delta(k + k') , \quad [\alpha_{k}^{\pm}, \alpha_{k'}^{\mp}] = 0 .$$  \hfill (2.11)

Due to the Dirichlet boundary conditions for the shifted field, the “plus” and “minus” sectors of the theory (2.9) are not independent. It was argued in ref. [14] that the definition of the $\tilde{\alpha}_{\pm}$ fields can be extended to the whole real line by requiring $\tilde{\alpha}_{\pm}(-\tau) = -\tilde{\alpha}_{\pm}(\tau)$, which is equivalent to

$$\alpha_{k}^{\mp} = -\alpha_{k}^{\pm} .$$  \hfill (2.12)

In physical terms this means that creation of a right-mover must be associated with the creation of a left-mover but with opposite amplitude (and similarly for the annihilation
processes). In the standard perturbative expansion of the collective field theory, this condition is explicitly built into the field expansions, and if one chooses $\tilde{\alpha}_+$ as the independent field (as we will choose to do in the following), one finds that all contributions from the “minus” sector of the theory always extend the range of $\tau$ integration of the “plus” sector to the whole real line. Thus the Hamiltonian (2.9) can be written as

$$H = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \alpha^2(\tau) + \frac{1}{6} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \frac{1}{(\pi \phi_0)^2} \alpha^3(\tau) .$$

To simplify notation we have dropped the subscript $+$ and the tilde. We will see later that condition (2.12), when imposed on exact states expressed in terms of asymptotic variable expansions, determines all tree-level scattering amplitudes.

The full quantum Hamiltonian is obtained by normal ordering [11], by taking into account the fact that the Hamiltonian (2.13) was obtained following a reparametrization from the $x$ to the $\tau$ coordinate. Denoting normal ordering in $x$-space by

$$N_x(\alpha^2) = \alpha^2 - \langle \alpha \alpha \rangle_x ,$$

and in $\tau$-space by

$$N_\tau(\alpha^2) = \alpha^2 - \langle \alpha \alpha \rangle_\tau ,$$

we have

$$N_\tau(\alpha^2) = N_x(\alpha^2) + \Delta(\tau) .$$

$\Delta(\tau)$ is simply the (finite) difference between the two propagators and it can be expressed solely in terms of classical solution $\phi_0$ and its derivatives as follows:

$$\Delta(\tau) = \frac{1}{12} \left[ 2 \left( \frac{\phi''_0}{\phi_0} \right) - 3 \left( \frac{\phi'_0}{\phi_0} \right)^2 \right] .$$

For an arbitrary polynomial in $\alpha$ one has the following relation between normal ordering in the two spaces

$$N_x[P(\alpha)] = N_\tau \left( e^{\frac{1}{2} \Delta(\tau) \frac{\delta^2}{\delta \alpha^2} P(\alpha)} \right) .$$

The normal ordered (in $x$-space) Hamiltonian (2.13) reads

$$H = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} : \alpha^2 : + \frac{1}{6} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \frac{1}{(\pi \phi_0)^2} : \alpha^3 : + \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \frac{\Delta(\tau)}{(\pi \phi_0)^2} : \alpha : + E^{(1)} ,$$

(2.16)
where double dots denote the normal ordering in $\tau$-space. The normal ordering introduces a finite linear tadpole, and the last term in (2.16) which is independent of $\alpha$ is the first quantum correction to the ground state energy

$$E^{(1)} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \Delta(\tau).$$

(2.17)

For practical purposes it is convenient to express the hamiltonian in momentum space. One finds

$$H = \frac{1}{2} \int_{-\infty}^{\infty} dk \ : \alpha_k \alpha_{-k} : + \frac{1}{6} \int_{-\infty}^{\infty} d^3k \ f(k_1 + k_2 + k_3) : \alpha_{k_1} \alpha_{k_2} \alpha_{k_3} : + \frac{1}{2} \int_{-\infty}^{\infty} dk \ g(k) \alpha_k$$

where

$$f(k) = \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} f(\tau) e^{ik\tau}, \quad f(\tau) = \frac{1}{(\pi \phi_0)^2},$$

$$g(k) = \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} g(\tau) e^{ik\tau}, \quad g(\tau) = \frac{\Delta(\tau)}{(\pi \phi_0)^2}.$$ (2.19)

$\alpha_k$ is creation (annihilation) operator for $k < 0 \ (k > 0)$.

The above description of the collective field theory is perfectly general and it applies to any matrix model potential. In what follows we consider the model with zero Fermi energy ($\mu_F = 0$) and with a deformed inverted harmonic potential

$$v(x) = -\frac{1}{2} x^2 + \frac{M}{2x^2},$$

(2.20)

suggested in [7] as a candidate for describing the black hole background. In this case the classical solution (2.6) reads

$$\pi \phi_0(x) = \sqrt{x^2 - \frac{M}{x^2}}.$$ (2.21)

Integrating eq. (2.7) and then inverting $\tau(x)$ one finds

$$x(\tau) = M^{1/4} \sqrt{\cosh 2\tau},$$

(2.22)

which then gives

$$\pi \phi_0(\tau) = M^{1/4} \frac{\sinh 2\tau}{\sqrt{\cosh 2\tau}}.$$ (2.23)
From this one simply finds

\[ f(\tau) = \frac{1}{\sqrt{M}} \frac{\cosh 2\tau}{\sinh^2 2\tau}, \]

\[ \Delta(\tau) = -\frac{1}{12} \left( 1 + \frac{3}{\cosh^2 2\tau} + \frac{12}{\sinh^2 2\tau} \right), \]  

(2.24)

\[ g(\tau) = \frac{1}{\sqrt{M}} \left( \frac{1}{4 \cosh 2\tau} + \frac{2 \cosh 2\tau}{3 \sinh^2 2\tau} - \frac{\cosh^3 2\tau}{\sinh^4 2\tau} \right). \]

After a Fourier transform one gets

\[ f(k) = \frac{-k}{8\sqrt{M}} \tanh\left(\frac{\pi}{4} k\right), \]

\[ g(k) = \frac{1}{16\sqrt{M}} \left[ \frac{1}{\cosh\left(\frac{\pi}{4} k\right)} + \frac{1}{3} \left( k - \frac{k^3}{4} \right) \tanh\left(\frac{\pi}{4} k\right) \right]. \]  

(2.25)

In performing Fourier transforms, any poles on the contour of integration have been handled by means of a principal part prescription, as it was physically justified in ref. [11].

The cubic vertex \( f(k) \) was already derived in [7]. It has an interesting property that \( f(0) = 0 \) which is crucial for the tree-level three- and five-point amplitudes to vanish on-shell. We note that both the cubic vertex \( f(k) \) and the linear tadpole \( g(k) \) have poles at discrete imaginary values of momenta \( ik = 2(2n + 1) \), i.e. when the momentum is twice an odd integer. For comparison, in the standard \( c = 1 \) case with potential \( v(x) = -x^2/2 \) vertex functions are proportional to \( \coth(\pi k/2) \), and therefore have poles when the (imaginary) momentum equals an even integer, i.e. \( ik = 2n \).

The cubic vertex \( f(k) \) and the linear tadpole \( g(k) \) provide a basis for systematic perturbative computations. Before presenting some examples of perturbative calculations using the above hamiltonian we consider in the next section states and symmetries of the classical theory and derive the tree-level scattering equation.

### 3. Exact states and symmetries

The symmetry structure of the collective hamiltonian with the potential (2.20) was described by Avan and Jevicki [12] in terms of three-index operators:

\[ O_{j,m}^a \equiv \int \frac{dx}{2\pi} \int_{\alpha_-}^{\alpha_+} d\alpha \left( \alpha^2 - x^2 + \frac{M}{x^2} \right)^a \left( (\alpha + x)^2 + \frac{M}{x^2} \right)^{\frac{j+m}{2}} \left( (\alpha - x)^2 + \frac{M}{x^2} \right)^{\frac{j-m}{2}} \]  

(3.1)
which satisfy the following algebra:

\[
[O^{a_1}_{j_1,m_1}, O^{a_2}_{j_2,m_2}] = -4i(j_1m_2 - m_1j_2) O^{a_1+a_2+1}_{j_1+j_2-2,m_1+m_2} - 4i(a_1m_2 - m_1a_2) O^{a_1+a_2-1}_{j_1+j_2,m_1+m_2}. 
\]

(3.2)

This algebra can be reduced to a two-index algebra by means of the identity

\[
O^{a+2}_{j,m} = O^{a}_{j+2,m} - 4MO^a_{j,m}. 
\]

(3.3)

The operators (3.1) satisfy the following commutation relation with the collective hamiltonian \(H\):

\[
[H, O^a_{j,m}] = -i2mO^a_{j,m}. 
\]

(3.4)

In the above, \(a\) and \(j\) are integers and \(m = -j, -j+2, ..., j-2, j\). This choice has been made to ensure that the exponents in (3.1) are always integers which then guarantees that the operators \(O^a_{j,m}\) are polynomial eigenstates (i.e., they have finite expansions) in the sense of refs. [12,13]. It then follows from (3.4) that the energies of the discrete states of the theory are (imaginary) \(even\) numbers, in agreement with other independent considerations [14]. A consequence of this choice is that in the limit \(M \to 0\) one recovers only “half” the \(W_{\infty}\) algebra of the \(c = 1\) model. This is very precisely related to the choice of physically acceptable solutions of the Schrödinger equation associated with the potential (2.20) as it will be discussed in appendix A.

Special cases of the operators (3.1) which will be useful in our future discussion are:

\[
O^a_{0,j} = \int \frac{dx}{2\pi} \int_{\alpha_-}^{\alpha_+} d\alpha \left[ (\alpha + x)^2 + \frac{M}{x^2} \right]^j \equiv T^{(+)}_{2j}, 
\]

\[
O^a_{-j} = \int \frac{dx}{2\pi} \int_{\alpha_-}^{\alpha_+} d\alpha \left[ (\alpha - x)^2 + \frac{M}{x^2} \right]^j \equiv T^{(-)}_{2j}, 
\]

(3.5)

\[
O^a_{0,0} = \int \frac{dx}{2\pi} \int_{\alpha_-}^{\alpha_+} d\alpha \left[ \alpha^2 - x^2 + \frac{M}{x^2} \right] \equiv 2H. 
\]

In order to discuss tachyon amplitudes one needs to make an analytic continuation of the operators (3.1) and the algebra (3.2). This procedure was justified for the standard \(c = 1\) model in ref. [14]. Here we will argue that the same is true in the case of a deformed matrix model. By letting \(j \to \pm ik/2\) one obtains from (3.5) operators \(T^{(-)}_{-ik}\) (\(T^{(+)}_{ik}\)) which
when acting on vacuum create an in (out) state, as we now demonstrate. Consider for example the operator

\[ T^{(\pm)}_{ik} = \int \frac{dx}{2\pi} \int_{\alpha_-}^{\alpha_+} d\alpha \left[ (\alpha + x)^2 + \frac{M}{x^2} \right]^{ik/2}. \]  

(3.6)

Shifting \( \alpha(x) \) as in (2.8) and expanding it up to quadratic order in \( \tilde{\alpha} \), one finds

\[ T^{(\pm)}_{ik} = c + \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \left( \Lambda_{\pm} + \frac{M}{x^2} \right)^{ik/2} \tilde{\alpha} + \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} (ik)^{\frac{ik}{2} - 1} \frac{\tilde{\alpha}^2}{2} + O(\tilde{\alpha}^3) \]

where we have introduced

\[ \Lambda_{\pm} \equiv \pm \pi \phi_0 + x = \pm M^{1/4} \frac{e^{\pm 2\tau}}{\sqrt{\cosh 2\tau}}. \]  

(3.7)

Now using the fact that

\[ \Lambda_{\pm} + \frac{M}{x^2} = 2 M^{1/2} e^{\pm 2\tau}, \]

and

\[ \frac{\Lambda_{\pm}}{\pi \phi_0} = \pm \frac{e^{\pm 2\tau}}{\sinh 2\tau}, \]

the above expansion becomes

\[ T^{(\pm)}_{ik} = (2\sqrt{M})^{ik/2} \left[ c + \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} e^{ik\tau} \tilde{\alpha} + \frac{i k}{4\sqrt{M}} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} e^{ik\tau} \tilde{\alpha}^2 + O(\tilde{\alpha}^3) \right]. \]  

(3.8)

After an integration by parts, the above expression in momentum space reads:

\[ T^{(\pm)}_{ik} = (2\sqrt{M})^{ik/2} \left[ c + \alpha_{-k} + k \int dp_1 \int dp_2 \frac{f(k + p_1 + p_2)}{k + p_1 + p_2 - i\epsilon} \alpha_{p_1} \alpha_{p_2} + O(\alpha^3) \right]. \]  

(3.9)

When acting on the vacuum, this gives precisely the result one finds for the connected contribution to an out state in the lowest order of perturbation theory. To obtain (3.9) use has been made of the fact that \( f(0) = 0. \) (Strictly speaking, this also implies that the sign of the \( i\epsilon \) prescription is not fixed to this order. However, in the \( c = 1 \) case, where \( f(0) \neq 0, \) a similar analysis establishes the above operator as an out-state.) A similar analysis can be carried out for \( T^{(-)}_{-ik}. \)

We will now provide further evidence that our interpretation of the analytically continued operators \( T^{(\pm)}_{-ik} \) \((T^{(\pm)}_{ik})\) is correct by deriving the tree-level scattering equation of ref.[7] following the procedure developed in [14]. We first write \( T^{(+)}_{ik} \) as

\[ T^{(+)}_{ik} = T^{(+)}_{ik,+} - T^{(+)}_{ik,-} \]
where
\[ T_{ik,+}^{(+)} = \int \frac{dx}{2\pi} \int_{0}^{\alpha+} d\alpha \left[ (\alpha + x)^2 + \frac{M}{x^2} \right]^{ik/2}, \]  
(3.10)
and
\[ T_{ik,-}^{(+)} = \int \frac{dx}{2\pi} \int_{0}^{\alpha-} d\alpha \left[ (\alpha + x)^2 + \frac{M}{x^2} \right]^{ik/2}. \]  
(3.11)
After a change of variables to the “time-of-flight” variable \( \tau \), we will extend the limits of integration to the full line in both the left and right sector. Dirichlet boundary conditions then require
\[ T_{ik,+}^{(+)} = -T_{ik,-}^{(+)}. \]  
(3.12)
Following Polchinski[15], we consider the asymptotic expansion
\[ \alpha_{\pm}(x, t) = \pm x + \frac{1}{x} \hat{\alpha}_{\pm}(\tau, t) + \mathcal{O}(\frac{1}{x^2}). \]  
(3.13)
For large \( \tau \), we have
\[ \pi \phi_0(x) \rightarrow x, \quad x \rightarrow \frac{M^{1/4}}{\sqrt{2}} e^{\tau}, \]
and therefore
\[ T_{ik,-}^{(+)} = \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \int_{-\infty}^{\infty} d\hat{\alpha} \left( \hat{\alpha}^{2} + \frac{M}{x^2} \right)^{ik/2} \]  
\[ = c + \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \frac{M^{ik/2}}{x^{ik}} \sum_{p=0}^{\infty} \frac{1}{M^p} \frac{\Gamma(1 + ik/2)}{p!(2p + 1) \Gamma(1 + ik/2 - p)} \hat{\alpha}_{-}^{2p+1}. \]  
(3.14)
On the other hand
\[ T_{ik,+}^{(+)} = \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \int_{-\infty}^{\infty} d\hat{\alpha} \left( 2M^{1/2} e^{2\tau} \right)^{ik/2} \]  
\[ = c + \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} (2M^{1/2})^{ik/2} e^{ik\tau} \hat{\alpha}_{+}. \]  
(3.15)
Equation (3.12) now implies
\[ -\hat{\alpha}_{+}(\tau) = \sum_{p=0}^{\infty} \frac{1}{M^p} \frac{\Gamma(1 - \partial/2)}{p!(2p + 1) \Gamma(1 - \partial/2 - p)} \hat{\alpha}_{-}^{2p+1}(-\tau), \]  
(3.16)
in agreement with the solution of the scattering problem obtained by Jevicki and Yoneya[7]. Expanding
\[ \hat{\alpha}_{\pm}(\tau) = \int_{-\infty}^{\infty} dk \hat{c}^{\pm}_{k} e^{\pm ik\tau} \]  
(3.17)
the scattering equation (3.16) in momentum space becomes

$$-\hat{c}_k^+ = \sum_{p=0}^{\infty} \frac{1}{M^p} \frac{\Gamma(1 - ik/2)}{p! (2p + 1) \Gamma(1 - ik/2 - p)} \prod_{i=1}^{2p+1} \int_{-\infty}^{\infty} dk_i \hat{c}_{k_1}^- \hat{c}_{k_2}^- \cdots \hat{c}_{k_{2p+1}}^- \delta(k - \sum k_i).$$

This equation contains the complete information about tree-level scattering amplitudes and can be used to obtain explicit expressions for arbitrary $N \to M$ amplitudes. An analogous equation was derived for the standard $c = 1$ model in ref. [16]. In the next section we shall describe an alternative method to obtain scattering amplitudes using directly the symmetry algebra. This method was developed for the $c = 1$ problem in [17].

4. Scattering amplitudes

We now proceed to obtain $1 \to N$ scattering amplitudes from the analytically continued algebra (3.2). The previous section established, at least perturbatively, the existence of a vacuum state with respect to which

$$T_{-ik}^{(-)} |0\rangle = |k; \text{in}\rangle, \quad T_{ik}^{(+)} |0\rangle = |k; \text{out}\rangle, \quad T_{-ik}^{(+)} |0\rangle = 0.$$  (4.1)

In order to compute a $1 \to N$ amplitude, it is therefore sufficient to obtain the ground state expectation value of the nested commutator

$$\langle 0 | [T_{-ik_N}^{(-)}, [T_{-ik_3}^{(-)}, [T_{-ik_2}^{(-)}, [T_{-ik_1}^{(-)}]] \ldots]]] |0\rangle_c,$$  (4.2)

where the subscript $c$ refers to connected diagrams. Since the interaction part of the collective hamiltonian consists of a cubic plus a tadpole term it is easy to see that any connected diagram in a $1 \to N$ tree amplitude will involve exclusively the cubic interaction (the tadpole term will contribute only to disconnected diagrams). Because the tadpole term is a quantum-mechanical effect resulting from the normal ordering of the cubic interaction (see sect. 2), for tree-level amplitudes one can use the classical algebra (3.2). We will first evaluate

$$[T_{2n_N}^{(+)} \ldots, [T_{2n_3}^{(+)}, [T_{2n_2}^{(+)}], T_{2n_1}^{(-)}]] \ldots$$  (4.3)

and then analytically continue the result and compute (4.2).
From the algebra (3.2) and the relationship (3.3) one easily obtains the following useful results:

\[
\begin{align*}
T^{(+)}_{2n_N} \cdot O^{a=0}_{N-1-j,N-1-2n_1} & = 4i n_N (2n_1 - j) O^{a=1}_{N-j-2,N-2n_1}, \\
T^{(+)}_{2n_N} \cdot O^{a=1}_{N-1-j,N-1-2n_1} & = 4 i n_N \left( (2n_1 + 1 - j) O^{a=0}_{N-j,N-2n_1} - 4M(2n_1 - j) O^{a=0}_{N-j-2,N-2n_1} \right),
\end{align*}
\]

(4.4)

where \( \Sigma_L \equiv \sum_{i=1}^{L} n_i \). It is then straightforward to evaluate the commutator (4.3). Denoting by \( h_i \equiv (2n_1 - i) \) the result can be written as:

\[
\begin{align*}
N = 2 & : \quad 2 (4i) \left( \prod n_i \right) O^1_{N-2,N-2n_1}, \\
N = 3 & : \quad 2 (4i)^2 \left( \prod n_i \right) \left[ h_1 O^0_{N-2,N-2n_1} - 4M h_2 O^0_{N-4,N-2n_1} \right], \\
N = 4 & : \quad 2 (4i)^3 \left( \prod n_i \right) \left[ h_1 h_2 O^1_{N-4,N-2n_1} - 4M h_4 O^0_{N-6,N-2n_1} \right], \\
N = 5 & : \quad 2 (4i)^4 \left( \prod n_i \right) \left[ h_1 h_2 h_3 O^0_{N-4,N-2n_1} - 4M h_4 (h_1 + h_5) O^0_{N-6,N-2n_1} \\
& \quad + (4M)^2 h_4 h_6 O^0_{N-8,N-2n_1} \right], \\
N = 6 & : \quad 2 (4i)^5 \left( \prod n_i \right) \left[ h_1 h_2 h_3 h_4 O^1_{N-6,N-2n_1} \\
& \quad - 4M h_4 h_6 (h_1 + h_5) O^1_{N-8,N-2n_1} \\
& \quad + (4M)^2 h_4 h_6 h_8 O^1_{N-10,N-2n_1} \right], \\
N = 7 & : \quad 2 (4i)^6 \left( \prod n_i \right) \left[ h_1 h_2 h_3 h_4 h_5 O^0_{N-6,N-2n_1} \\
& \quad - 4M h_4 h_6 (h_1 h_3 + h_1 h_7 + h_5 h_7) O^0_{N-8,N-2n_1} \\
& \quad + (4M)^2 h_4 h_6 h_8 (h_1 + h_5 + h_9) O^0_{N-10,N-2n_1} \right], \\
N = 8 & : \quad 2 (4i)^7 \left( \prod n_i \right) \left[ h_1 h_2 h_3 h_4 h_5 h_6 O^1_{N-8,N-2n_1} \\
& \quad - 4M h_4 h_6 h_8 (h_1 h_3 + h_1 h_7 + h_5 h_7) O^1_{N-10,N-2n_1} \\
& \quad + (4M)^2 h_4 h_6 h_8 h_{10} (h_1 + h_5 + h_9) O^1_{N-12,N-2n_1} \right], \\
& \quad - (4M)^3 h_4 h_6 h_8 h_{10} h_{12} O^1_{N-14,N-2n_1} \right].
\end{align*}
\]

(4.5)
We have also calculated nine- and ten-point commutators. The structure is now clear: for even (odd) $N$, we need to calculate the $c$-number contribution to the ground state expectation value of (analytically continued) operators $O^{a=0}$ ($O^{a=1}$). Due to energy conservation the commutators depend only on the momentum of the single incoming particle, as is the case in the standard $c = 1$ model.

In order to calculate the $c$-number contribution to $O^{a=1}$ it is sufficient to obtain, say, $T_{2n}^\pm$ to linear order in $\tilde{\alpha}$. We find

\begin{align*}
T_{2n}^{(+)} &= c + 2^n M^{n/2} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \ e^{2n\tau} \ \tilde{\alpha}_+^2(\tau) + O(\tilde{\alpha}_+^2) , \\
T_{2n}^{(-)} &= c + 2^n M^{n/2} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \ e^{-2n\tau} \ \tilde{\alpha}_+^2(\tau) + O(\tilde{\alpha}_+^2) ,
\end{align*}

and therefore

\[ [T_{2n_2}^{(+)}, T_{2n_1}^{(-)}] = 2in_1 2^{n_1+n_2} M^{(n_1+n_2)/2} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \ e^{2(n_2-n_1)\tau} + O(\tilde{\alpha}_+) . \]

Comparing this expression with what we expect from the algebra (3.2), we obtain

\[ O^{a=1}_{2j,m} \bigg|_{c-number} = \frac{(4M)^{j+1}}{4(j+1)} \delta_{m,0} , \tag{4.6} \]

written in a form which anticipates the energy conservation delta-function which is obtained once the analytical continuation $n \rightarrow -ik/2$ is performed.

Tree-level amplitudes with an odd number of particles always correspond to the $c$-number contribution to a commutator of the type

\[ [T_{2n}^{(+)}, O^{a=1}_{j,m}] . \tag{4.7} \]

We will now show that the operator $O^{a=1}_{j,m}$ has no term linear in $\tilde{\alpha}_+$. We recall that

\[ O^{a=1}_{j,m} \equiv \int \frac{dx}{2\pi} \int_{\alpha_-}^{\alpha_+} d\alpha \ \left( \alpha^2 - x^2 + \frac{M}{x^2} \right) \left( (\alpha + x)^2 + \frac{M}{x^2} \right)^{\frac{i+m}{2}} \left( (\alpha - x)^2 + \frac{M}{x^2} \right)^{\frac{i-m}{2}} . \]

After shifting by $\pi \phi_0$ the first term in brackets becomes

\[ \frac{\tilde{\alpha}^2}{(\pi \phi_0)^2} + 2\tilde{\alpha} + (\pi \phi_0)^2 - x^2 + \frac{M}{x^2} . \]
The $\tilde{\alpha}$-independent term vanishes because of (2.21) and, apart from a $c$-number, $O_{j,m}^{\alpha=1}$ starts quadratically in $\tilde{\alpha}_+$. Therefore, the commutator (4.7) has no $c$-number, and all tree-level amplitudes for an odd number of particles vanish. This is clearly related to the fact that the Fermi energy of the problem is zero, since otherwise the $\tilde{\alpha}$-independent term above would be $2\mu_F$, giving a nonvanishing $c$-number contribution to (4.7).

Substituting eq. (4.6) into eqs. (4.5) we obtain:

\begin{equation}
N = 2 : (4i) \left( \prod n_i \right) (4M)^{n_1} \left( \frac{1}{2n_1} \right) \delta_{\Sigma N - 2n_1, 0} ,
\end{equation}

\begin{equation}
N = 4 : (4i)^3 \left( \prod n_i \right) (4M)^{n_1} \left( \frac{1}{4M} \right) \delta_{\Sigma N - 2n_1, 0} ,
\end{equation}

\begin{equation}
N = 6 : (4i)^5 \left( \prod n_i \right) (4M)^{n_1} \left[ \frac{3 (2n_1 - 2)}{(4M)^2} \right] \delta_{\Sigma N - 2n_1, 0} ,
\end{equation}

\begin{equation}
N = 8 : (4i)^7 \left( \prod n_i \right) (4M)^{n_1} \left[ \frac{3 \cdot 5 (2n_1 - 2) (2n_1 - 4)}{(4M)^3} \right] \delta_{\Sigma N - 2n_1, 0} ,
\end{equation}

\begin{equation}
N = 10 : (4i)^9 \left( \prod n_i \right) (4M)^{n_1} \left[ \frac{3 \cdot 5 \cdot 7 (2n_1 - 2) (2n_1 - 4) (2n_1 - 6)}{(4M)^4} \right] \delta_{\Sigma N - 2n_1, 0} .
\end{equation}

The structure of the amplitudes is now clear. Rescaling the exact operators so that the coefficient of the linear term $\alpha_k$ equals one (see e.g. eq. (3.8)), and analytically continuing $n \rightarrow -ik/2$, we finally obtain the general expression for $1 \rightarrow 2N - 1$ amplitudes ($N \geq 1$):

\begin{equation}
A(1 \rightarrow 2N - 1) = \left( \prod_{i=1}^{2N} k_i \right) \frac{(-1)^{N-1}}{M^{N-1}} \frac{1}{i^{k_{2N}}} \left[ (2N - 3)!! \left( \prod_{p=0}^{N-2} (ik_{2N} + 2p) \right) \right] \delta \left( \sum_{i=1}^{2N-1} k_i - k_{2N} \right)
\end{equation}

with all $1 \rightarrow 2N$ amplitudes vanishing. To conform with usual practice we have denoted by $k_{2N}$ the momentum of the single incoming particle. We have confirmed this result by computing $2N - 1 \rightarrow 1$ amplitudes using the asymptotic in-out relationhsip (3.18) derived in sect. 3.

We close this section with a remark on scattering of special states. If one assumes the existence of a vacuum state with respect to which $T_{2n}^{(-)} |0\rangle = 0$, $T_{2n}^{(+)} |0\rangle = 0$ (see, for instance, [18]), it then follows from our analysis that the structure of $2N - 1 \rightarrow 1$ amplitudes is identical to the one derived above.
5. Quantum effects

In this section we present some examples of loop computations in quantum collective field theory. The motivation for this is twofold: First, we want to demonstrate the completeness at the quantum level of the normal ordered collective hamiltonian by comparing our perturbative results with the corresponding results obtained in the fermionic picture (and summarized in appendix A). We believe that the same procedure of normal ordering can then be applied to the classical exact states discussed in sect. 3 and will give exact quantum mechanical scattering amplitudes. Second, there is certainly great interest in obtaining quantum corrections to some of the results established in [7] and in the previous sections. These, in any case, still remain to be derived in the conformal field theory approach.

We begin with the computation of the ground state energy. The classical energy of the background field is obtained from (2.10) and reads

\[ E^{(0)} = -\frac{1}{3\pi} \int dx \left[-2v(x)\right]^{3/2} = -\frac{1}{3\pi} \int_{M^{1/4}}^{\infty} dx \left(x^2 - \frac{M}{x^2}\right)^{3/2} = -\frac{1}{8\pi} M \ln M. \quad (5.1) \]

The one-loop contribution is given by (2.17)

\[ E^{(1)} = \frac{1}{2} \int \frac{d\tau}{2\pi} \Delta(\tau) = -\frac{L}{24\pi} + \text{regular terms}, \quad (5.2) \]

where \( L \) is the extent of the \( \tau \) coordinate

\[ L = \int d\tau = \int \frac{dx}{\pi \phi_0} = -\frac{1}{4} \ln M, \]

giving

\[ E^{(1)} = \frac{1}{96\pi} \ln M. \quad (5.3) \]

![Fig. 1: Ground state energy – two-loop diagrams.](image)
The two-loop contribution is obtained from the second order perturbation theory. There are two diagrams shown in fig. 1 contributing:

\[
E^{(2)}_a = -\frac{1}{6} \int_0^\infty d^3k k_1k_2k_3 \frac{f^2(k_1 + k_2 + k_3)}{k_1 + k_2 + k_3} = \frac{31}{3780\pi} \frac{1}{M},
\]

(5.4)

\[
E^{(2)}_b = -\frac{1}{4} \int_0^\infty dk g^2(k) = \frac{1943}{241920\pi} \frac{1}{M},
\]

which altogether gives

\[
E^{(2)} = \frac{187}{11520\pi} \frac{1}{M}.
\]

(5.5)

In evaluating integrals we have used the \(\zeta\)-function regularization as discussed in ref. [11] and the following results [19]:

\[
\int_0^\infty dx \frac{x^{2m}}{\cosh^2 ax} = \frac{2^{2m} - 2}{a} \left(\frac{\pi}{2a}\right)^{2m} |B_{2m}|, \quad m \geq 1,
\]

(5.6)

\[
\int_0^\infty dx \frac{x^{2m+1}}{\cosh^2 ax} = \frac{2m + 1}{a} \left(\frac{\pi}{2a}\right)^{2m} |E_{2m}|, \quad m \geq 0,
\]

\[
\int_0^\infty \frac{dx}{\cosh^2 ax} = \frac{1}{a}.
\]

\(B_{2m}\) and \(E_{2m}\) are Bernoulli and Euler numbers, respectively. Results (5.4), (5.3) and (5.5) are in precise agreement with the first three terms in the weak coupling expansion of the exact result given in appendix A.

\(\begin{minipage}{0.5\textwidth}
\begin{align*}
\text{Fig. 2:} & \quad \text{First quantum correction to the two-point function.}
\end{align*}
\end{minipage}\)

Next we evaluate the first quantum correction to the two-point function using

\[
S(k; k') = 1 - 2\pi i \delta(k - k') T(k; k').
\]

(5.7)
Again, there are two diagrams shown in fig. 2. Their contributions are:

\begin{align}
T^{(2)}_{2,a}(k) &= -\frac{k}{12} \int_{-\infty}^{\infty} dp \, p^3 \frac{f^2(p + k)}{p + k - i\epsilon \text{sign} p} = \frac{1}{M} \frac{1}{360\pi} (28k + 15k^3), \\
T^{(2)}_{2,b}(k) &= -k \int_{0}^{\infty} dp \, f(p) \, g(p) = \frac{1}{M} \frac{49k}{720\pi},
\end{align}

which together give

\begin{equation}
T^{(2)}_2(k) = \frac{1}{M} \frac{1}{48\pi} (7k + 2k^3).
\end{equation}

Finally, we discuss the finite temperature free energy. The one-loop contribution to the free energy is simply

\begin{equation}
F^{(1)} = E^{(1)} + T \frac{L}{\pi} \int_0^{\infty} dk \, \ln(1 - e^{-k/T}) = \frac{1}{96\pi} (2\pi T) \left[\frac{1}{2\pi T} + 2\pi T\right] \ln M.
\end{equation}

The temperature dependent part above is just the free energy of a massles boson on a half-line. This is precisely the same answer (when expressed in terms of the length \(L\) of the spatial coordinate) as in the standard \(c = 1\) case and simply means that the two models cannot be distinguished at the quadratic level. However, the order \(g^2\) result for the ground state energy is different for the two models. With this in mind we next compute the two-loop result for the free energy which contains physical information about degrees of freedom.

The two-loop contribution to the free energy is easily computed once we know the second order two- and four-point functions, as was done in [11] using the result of ref.[20]:

\begin{equation}
F^{(2)} = E^{(2)} + \int_0^{\infty} dk \, n(k) \, \text{Re} T^{(2)}_2(k; k) + \int_0^{\infty} dk_1 \int_0^{\infty} dk_2 \, n(k_1) n(k_2) \, \text{Re} T^{(2)}_4(k_1, k_2; k_1, k_2)
\end{equation}

where \(n(k) = 1/(e^{k/T} - 1)\). Using the result (5.9) for \(T^{(2)}_2(k)\) and the expression for the four-point amplitude derived in [7]:

\begin{equation}
T^{(2)}_4(k_1, k_2; k_1, k_2) = \frac{1}{M} \frac{1}{4\pi} k_1 k_2,
\end{equation}

we get

\begin{equation}
F^{(2)} = \frac{1}{M} \frac{1}{11520\pi} (2\pi T)^2 \left[\frac{187}{(2\pi T)^2} + 70 + 7(2\pi T)^2\right].
\end{equation}

As a check, equation (5.13) is derived in the fermionic description of the model at the end of appendix A.
The beauty of the formula (5.11) is that it allows one to easily see which states in the spectrum contribute to the free energy at this order. Namely, one simply finds from the integral representation of scattering amplitudes which states in the intermediate channel give rise to the real part of the amplitude. In our case the contribution from the exchange of continuous mode (which would be imaginary) vanishes since it is proportional to $f(0)$, and the total answers for the second order two- and four-point amplitudes come entirely from an exchange of states at discrete values of momenta discussed at the end of sect. 2. There is, however, a puzzle concerning result (5.13) in that it is not symmetric under the duality transformation

$$2\pi T \rightarrow \frac{1}{2\pi T}. \quad (5.14)$$

If it were, the expression in the brackets in (5.13) would be symmetric under (5.14). We do not yet have a complete understanding of this. However, we believe this to be related to the fact that the collective field amplitudes used in (5.11) are only a part of the total tachyon amplitudes. If the factorization assumption is correct, these amplitudes have to be multiplied by external leg factors as discussed in the introduction. Since one expects the string theory free energy to be symmetric under the duality transformation, our result would imply that the external leg factors (in Minkowski space) must have such a specific momentum dependence that when the full amplitudes are integrated according to formula (5.11) the obtained answer is symmetric under (5.14). This observation may be useful in finding the correct completion of the collective field amplitudes to the full answer. We note that in the standard $c = 1$ case collective field theory gave the full (dual) answer for the free energy up to second order[11], which could be related to the fact that external leg factors in this case are pure phases.

6. Discussion and conclusions

We have investigated further the collective field theory description of a deformed matrix model with emphasis on states, symmetries and higher-loop corrections. We have identified the exact operators which create in and out tachyon states. By imposing an appropriate boundary condition we showed that these states encode the complete information on the tree-level scattering amplitudes which can be written in a form of a scattering equation[4]. We then proceeded to derive amplitudes using directly (analytic continuation of) the symmetry algebra. In both computations it was clear that vanishing of the
“odd-point” on-shell amplitudes is closely related to the choice of zero Fermi energy. In the collective hamiltonian this choice is seen as the vanishing of the three-string vertex at zero momentum.

We computed loop corrections to the ground state energy, finite temperature free energy and the two-point function, and found the interesting result that the free energy is not invariant under the duality transformation typical of compactified string theory. This result may be helpful for a better understanding of the full tachyon amplitudes in the black hole background.

We hope we have further clarified some issues and pointed out to some problems in the matrix model approach to the black hole solution of two-dimensional string theory. In particular, we believe that our discussion of classical exact states can be extended to the quantum level by normal ordering as explained in sect. 2. This would then give the full quantum collective field amplitudes and allow one to check the consistency of the factorization assumption at the loop-level. The exact collective field amplitudes can probably be obtained in the fermionic picture of matrix model following [21], and work in this direction is in progress. Finally, any explicit result, either for amplitudes or for the partition function [22] obtained in the continuum approach would greatly help to improve our understanding of a possible matrix model formulation of the black hole problem. It is precisely this interplay between the two approaches which has lead to remarkable progress in our understanding of the standard $c = 1$ problem, and we certainly hope this to continue.

**Note added**

During the completion of this manuscript we have learned of the work of Danielsson (ref.[23]) who derived some of the results presented in sect. 5 using different methods from ours.

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Appendix A.

In this appendix we derive an expansion for the ground state energy directly in the fermionic picture of the matrix model. It is given by

\[ E = \int_{\mu_F}^{\mu_F} \rho(e) \, de \]  \hspace{1cm} (A.1)

where \( \rho(e) \) is the density of states, and in our case \( \mu_F = 0 \). In the standard \( c = 1 \) model one had \( \mu_F = -\mu \neq 0 \) and the problem could be solved by knowing only the density of states at the Fermi level\[24\]. A similar trick can be used here, but now one needs to know the density of states as a function of energy, and only at the end can one put \( e = 0 \).

The density of states is given by

\[ \rho(e) = \frac{1}{\pi} \text{Im} \sum_{n=0}^{\infty} \frac{1}{\epsilon_n - e - i\epsilon} \]  \hspace{1cm} (A.2)

where \( \epsilon_n \) are the single particle energy levels which can be obtained by solving the Schrödinger equation with potential (2.20). In ref.\[25\] the eigenvalue problem for a “deformed harmonic potential”

\[ v(x) = \frac{1}{2} \omega^2 x^2 + \frac{\lambda}{x^2} \]  \hspace{1cm} (A.3)

was solved with the result

\[ \epsilon_n = (2n + 1 + a) \omega, \quad a = \frac{1}{2} \sqrt{1 + 8\lambda} \]  \hspace{1cm} (A.4)

Strictly speaking, there is a second set of wave functions with energies \( \epsilon_n = (2n + 1 - a) \omega \), which although singular at the origin, are also normalizable on the half-line. Both branches can be extended to the full line, the first being chosen to be odd and the second even under parity\[25\]. As \( \lambda \to 0 \), these wave functions and corresponding eigenvalues map smoothly into the odd (respectively even) solutions of the harmonic oscillator problem. However, in the first two references of\[25\] it is argued that the second set of wave functions is not physically acceptable because, for instance, the laplacian operator would not be hermitian. For us, not to consider this set is a reasonable choice for at least two reasons: first, we only consider the right branch of the potential and the first set of wave functions indeed vanishes at the origin; second, as \( \lambda \to 0 \), we only recover half the energies of the harmonic oscillator. This is a desirable feature following our previous observation that for the inverted case,
the full algebra of the deformed matrix model reduces to “half” the discrete state $W_{\infty}$ algebra of the $M \to 0$ case.

Having justified our choice of energies, we proceed with evaluation of the density of states (A.2). For our problem with an inverted harmonic potential we have to analytically continue $\omega \to -i\omega$ (and set $\omega = 1$) which gives

$$\rho(e) = -\frac{1}{2\pi} \text{Re} \sum_{n=0}^{\infty} \frac{1}{n + \frac{1}{2} + \frac{a-ie}{2}}, \quad a = \sqrt{M + 1/4}. \quad (A.5)$$

Expanding this as in ref.[24] one finds

$$\rho(e) = -\frac{1}{2\pi} \text{Re} \left[ \ln(a-ie) + \sum_{m=1}^{\infty} \frac{(2^{2m-1} - 1) B_{2m}}{m} \frac{1}{(a-ie)^{2m}} \right]. \quad (A.6)$$

The first few terms in this expansion read

$$\rho(e) = -\frac{1}{4\pi} \ln(a^2 + e^2) - \frac{1}{2\pi} \left[ \frac{1}{6} \frac{a^2 - e^2}{(a^2 + e^2)^2} - \frac{7}{60} \frac{a^4 - 6a^2e^2 + e^4}{(a^2 + e^2)^4} + \cdots \right]. \quad (A.7)$$

One could now simply evaluate integral in (A.1) using (A.7) term by term. However, one can avoid this procedure by noting that the energy given by (A.1) corresponds to the solution of the equation

$$\rho(e) = -\frac{\partial^2 E(e)}{\partial e^2} \quad (A.8)$$

for $e = 0$. One easily finds that (A.8) is solved by

$$E(e) = \frac{1}{2\pi} \text{Re} \left[ -\frac{1}{2} (a-ie)^2 \ln(a-ie) + \frac{1}{6} \ln(a-ie) - \sum_{m=2}^{\infty} \frac{(2^{2m-1} - 1) B_{2m}}{m(2m-2)(2m-1)} \frac{1}{(a-ie)^{2m-2}} \right] \quad (A.9)$$

from which it follows that

$$E = -\frac{1}{2\pi} \left[ \frac{1}{2} a^2 \ln a - \frac{1}{6} \ln a + \sum_{m=1}^{\infty} \frac{(2^{2m+1} - 1) B_{2m+2}}{2m(m+1)(2m+1)} \frac{1}{a^{2m}} \right]. \quad (A.10)$$

However, we are interested in an expansion in $g_{st}^2 = 1/M$, and therefore one still needs to reexpand (A.10) in powers of $1/M$ using that $a = \sqrt{M + 1/4}$. This gives

$$E = -\frac{1}{8\pi} M \ln M + \frac{1}{96\pi} \ln M + \sum_{n=1}^{\infty} \frac{a_n}{M^n}, \quad (A.11)$$
with the coefficients $a_n$ given by

$$a_n = \frac{1}{2\pi} \frac{(-1)^{n+1}}{4^n} \left[ \frac{4n+1}{48n(n+1)} + \sum_{m=1}^{n} \frac{(-1)^m 4^m (2^{2m+1} - 1) B_{2m+2}}{2m(m+1)(2m+1)} \frac{(n-1)!}{(m-1)!(n-m)!} \right].$$  \hspace{1cm} \text{(A.12)}$$

The first few terms in this expansion are:

$$E = -\frac{1}{8\pi} M \ln M + \frac{1}{96\pi} \ln M + \frac{187}{11520\pi} \frac{1}{M} - \frac{3083}{322560\pi} \frac{1}{M^2} + \frac{4719}{286720\pi} \frac{1}{M^3} - \cdots \hspace{1cm} \text{(A.13)}$$

We have also checked that the analytic continuation we have done is justified by computing the first few terms in the expansion of $\rho(e)$ using the Gelfand-Dikii formula\[26,24\]

$$\rho(e) = \frac{2}{\pi} \int dy \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{R_{\ell}[2(v(x) - v(y))]}{[2(e - v(y))]^{\ell+\frac{1}{2}}} \bigg|_{x=y}$$

where $R_{\ell}[u]$ are Gelfand-Dikii polynomials and the integral goes over the interval where $e - v(y) \geq 0$. We have found precise agreement with (A.7). This is to be expected since the expressions for the energy (5.4), when expressed as $x$-space integrals, are in agreement with the Gelfand-Dikii expansion for the density of states for an arbitrary potential, once use has been made of eq. (A.8), as it was pointed out in the first of refs.[11].

Finally, we summarize the computation of the finite temperature free energy in the fermionic picture. It can be written as\[27\]:

$$F = \int de E(e) \frac{\partial}{\partial \mu_F} \left( \frac{1}{1 + e^{(e-\mu)/T}} \right) = \int de E(e) \frac{1}{4T \cosh^2 \left( \frac{e-\mu}{2T} \right)}, \hspace{1cm} \text{(A.14)}$$

where $E(e)$ is defined in (A.1). Integrating by parts we obtain

$$F = -\int de \rho(e) \frac{\partial^2}{\partial e^2} \frac{1}{4T \cosh^2 \left( \frac{e-\mu}{2T} \right)} = -\frac{\partial^2}{\partial \mu_F^2} \int de \frac{\rho(e)}{4T \cosh^2 \left( \frac{e-\mu}{2T} \right)} \equiv -\frac{\partial^2}{\partial \mu_F^2} \tilde{\rho}(\mu_F).$$

It is straightforward to check that for the standard $c = 1$ model the above equation is in agreement with the relationship between $F$ and $\tilde{\rho}$ derived in\[27\]. From eqs. (A.14) and (A.9), we obtain (after setting $\mu_F = 0$):

$$F = -\frac{1}{2\pi} \int de \text{Re} \left[ \frac{1}{2} (a - ie)^2 \ln(a - ie) - B_2 \ln(a - ie) \right. \left. + \sum_{m=2}^{\infty} \frac{(2^{2m-1} - 1) B_{2m}}{m(2m - 1)(2m - 1)} \frac{1}{(a - ie)^{2m-2}} \right] \frac{1}{4T \cosh^2 \left( \frac{e}{2T} \right)}. \hspace{1cm} \text{(A.15)}$$
Expanding, the first few terms in powers of $1/M$ are:

\[
F = -\frac{1}{2\pi} \left\{ \frac{1}{4} I_0 M \ln M + \left[\left(\frac{1}{16} - \frac{B_2}{2}\right) I_0 - \frac{1}{4} I_2 \right] \ln M \\
+ \left[\left(\frac{1}{128} - \frac{B_2}{8} + \frac{7B_4}{12}\right) I_0 - \left(\frac{1}{16} + \frac{B_2}{2}\right) I_2 - \frac{1}{24} I_4 \right] \frac{1}{M} + \ldots \right\},
\]

where

\[
I_{2n} = \int_{-\infty}^{\infty} dx \frac{x^{2n}}{4T \cosh^2 \left(\frac{e}{2T}\right)} = (1 - 2^{1-2n}) |B_{2n}| (2\pi T)^{2n}, \quad n = 1, 2, 3, \ldots
\]

and $I_0 = 1$. This finally gives

\[
F = -\frac{1}{8\pi} M \ln M + \frac{1}{96\pi} \left[(1 + (2\pi T)^2) \ln M + \frac{1}{M} \frac{1}{11520\pi} \left[187 + 70(2\pi T)^2 + 7(2\pi T)^4\right] \right],
\]

in precise agreement with our results of sect. 5.
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