THE FREE LOCALLY CONVEX SPACE \( L(s) \) OVER A CONVERGENT SEQUENCE \( s \) IS NOT A MACKEY SPACE

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Abstract. We show that the free locally convex space \( L(s) \) over a convergent sequence \( s \) is not a Mackey space. Consequently \( L(s) \) is not a Mackey group that answers negatively a question posed in [4].

1. Introduction

Let \((E, \tau)\) be a locally convex space. A locally convex vector topology \(\nu\) on \(E\) is called compatible with \(\tau\) if the spaces \((E, \tau)\) and \((E, \nu)\) have the same topological dual space. The classical Mackey–Arens theorem states that for every locally convex space \((E, \tau)\) there exists the finest locally convex vector space topology \(\mu\) on \(E\) compatible with \(\tau\). The topology \(\mu\) is called the Mackey topology on \(E\) associated with \(\tau\), and if \(\mu = \tau\), the space \(E\) is called a Mackey space.

For an abelian topological group \((G, \tau)\) we denote by \(\widehat{G}\) the group of all continuous characters of \((G, \tau)\). Two topologies \(\mu\) and \(\nu\) on an abelian group \(G\) are said to be compatible if \((\widehat{G}, \mu) = (\widehat{G}, \nu)\). Being motivated by the Mackey–Arens Theorem and following [1], a locally quasi-convex abelian group \((G, \mu)\) is called a Mackey group if for every locally quasi-convex group topology \(\nu\) on \(G\) compatible with \(\tau\) it follows that \(\nu \leq \mu\). It follows from Proposition 2.5 of [4] that if a locally convex space \(E\) is not Mackey, then \(E\) also is not a Mackey group.

In [4, Question 4.3], we posed the following question: Is the free locally convex space \(L(s)\) over a convergent sequence \(s\) a Mackey group? In this note we answer this question in the negative in a stronger form:

**Theorem 1.1.** The locally convex space \(L(s)\) is not a Mackey space. Consequently \(L(s)\) is not a Mackey group.

We prove Theorem 1.1 in the next section.

2. Proof of Theorem 1.1

Set \(\mathbb{N} := \{1, 2, \ldots\}\) and \(\omega = \{0\} \cup \mathbb{N}\). Denote by \(S\) the unit circle group and set \(S_+ := \{z \in S : \text{Re}(z) \geq 0\}\). Let \(G\) be an abelian topological group. If \(\chi \in \widehat{G}\), it is considered as a homomorphism from \(G\) into \(S\). A subset \(A\) of \(G\) is called quasi-convex if for every \(g \in G \setminus A\) there exists \(\chi \in \widehat{G}\) such that \(\chi(g) \notin S_+\) and \(\chi(A) \subseteq S_+\). An abelian topological group \(G\) is called locally quasi-convex if it admits a neighborhood base at the neutral element 0 consisting of quasi-convex sets. It is well known that the class of locally quasi-convex abelian groups is closed.

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under taking products and subgroups. The dual group $\hat{G}$ of $G$ endowed with the compact-open topology is denoted by $G^\wedge$.

Let $G$ be an abelian topological group and let $\mathcal{N}(G)$ be the filter of all open neighborhoods at zero of $G$. Denote by $G^N$ the group of all sequences $(x_n)_{n \in \mathbb{N}}$. It is easy to check that the collection $\{V^N : V \in \mathcal{N}(G)\}$ forms a base at 0 for a group topology in $G^N$. This topology is called the uniform topology and is denoted by $u$. Following [2], denote by $c_0(G)$ the following subgroup of $G^N$

$$c_0(G) := \left\{(x_n)_{n \in \mathbb{N}} \in G^N : \lim_{n \to \infty} x_n = 0\right\}.$$ 

The uniform group topology on $c_0(G)$ induced from $(G^N, u)$ we denote by $u_0$ and set $\mathfrak{F}_0(G) := (c_0(G), u_0)$. Clearly, the group $\mathfrak{F}_0(\mathbb{R})$ coincides with the classical Banach space $c_0$.

For a subset $A$ of a topological vector space $E$, we denote the convex hull of $A$ by $\text{conv}(A)$, so

$$\text{conv}(A) := \left\{ \lambda_1 a_1 + \cdots + \lambda_n a_n : \lambda_1, \ldots, \lambda_n \geq 0, \sum_{i=1}^n \lambda_i = 1, a_1, \ldots, a_n \in A, n \in \mathbb{N} \right\}.$$ 

If $n \in \mathbb{N}$, we set $(1)A := A$ and $(n + 1)A := (n)A + A$.

Let $X$ be a Tychonoff space and $e \in X$ a distinguished point. Recall that the free locally convex space $L(X)$ over $X$ is a pair consisting of a locally convex space $L(X)$ and a continuous map $i : X \to L(X)$ such that every continuous map $f$ from $X$ to a locally convex space (lcs) $E$ gives rise to a unique continuous linear operator $\tilde{f} : L(X) \to E$ with $f = \tilde{f} \circ i$. For every Tychonoff space $X$, the free locally convex space $L(X)$ exists, is unique up to isomorphism of locally convex spaces, and $L(X)$ is algebraically the free vector space on $X$.

It is more convenient for our purpose to consider Graev free locally convex spaces instead of free locally convex spaces. Recall that the Graev free locally convex space $L_G(X)$ over $X$ is a pair consisting of a locally convex space $L_G(X)$ and a continuous map $i : X \to L_G(X)$ such that $i(e) = 0$ and every continuous map $f$ from $X$ to a locally convex space $E$ gives rise to a unique continuous linear operator $\tilde{f} : L_G(X) \to E$ with $f = \tilde{f} \circ i$. Also the Graev free locally convex space $L_G(X)$ exists, is unique up to isomorphism of locally convex spaces, and is independent of the choice of $e$ in $X$. Further, $L_G(X)$ is algebraically the free vector space on $X \setminus \{e\}$. We shall use the following fact which can be obtained exactly as in [7] or Proposition 2.10 in [4]: For every Tychonoff space $X$, one holds

$$L(X) = \mathbb{R} \oplus L_G(X).$$

Therefore $L(X)$ is a Mackey space if and only if $L_G(X)$ is a Mackey space, see Corollary 8.7.5 and Theorem 8.8.5 of [10].

Let $s = \left\{ \frac{1}{n} \right\}_{n \in \mathbb{N}} \cup \{0\}$ be a convergent sequence with the usual topology induced from $\mathbb{R}$. Set $s^* := s \setminus \{1\}$, so $s$ is the disjoint union of $\{1\}$ and $s^*$. Then

$$L_G(s) = L_G(\{1\}) \oplus L_G(s^*) = \mathbb{R} \oplus L_G(s^*) = \mathbb{R} \oplus L_G(s) = L(s).$$

Dealing with the concrete case $X = s$ it is convenient and useful to describe the space $L_G(s)$ and its topology $\nu^s$ explicitly that we do below. We shall use this description also in forthcoming papers. The free vector space $\mathbb{V}_s$ over $s$ is just the direct sum $\mathbb{R}^{(\mathbb{N})}$ of $\aleph_0$-many copies of the reals $\mathbb{R}$. Denote by $\pi_n$ the projection of $\mathbb{R}^{(\mathbb{N})}$ onto the $n$th coordinate, and we denote by $0$ the zero vector.
of $\mathbb{V}_s$. For every $n \in \mathbb{N}$, set $e_n = (0, \ldots, 0, 1, 0, \ldots) \in \mathbb{V}_s$, where 1 is placed in position $n$. Below we also find the maximal vector topology $\mu_s^G$ and the maximal locally convex vector topology $\nu_s^G$ on $\mathbb{V}_s$ in which the sequence $s$ converges to 0.

Then (see [6]) the topological vector space $\mathbb{V}_G(s) := (\mathbb{V}_s, \mu_s^G)$ is called the Graev free topological vector space over $s$, and the space $L_G(s) := (\mathbb{V}_s, \nu_s^G)$ is the Graev free locally convex space on $s$.

Below we describe the topologies $\mu_s^G$ and $\nu_s^G$. Let $\mathcal{A}$ be the family of all sequences $a = (a_n)_{n \in \mathbb{N}}$ such that $a_n > 0$ for all $n \in \mathbb{N}$ and $\lim_n a_n = \infty$. Denote by $\mathcal{S}_s$ the family of all subsets of $\mathbb{V}_s$ of the form

$$S(a) := \bigcup_{n \in \mathbb{N}} \left(-a_n, a_n\right) \times \prod_{j \in \mathbb{N}, j \neq n} \{0\},$$

where $a \in \mathcal{A}$.

For every sequence $S = \{S(a_k)\}_{k \in \omega} \subset \mathcal{S}_s$, we put

$$U(S) = \sum_{k \in \omega} S(a_k) := \bigcup_{k \in \omega} (S(a_0) + S(a_1) + \cdots + S(a_k)).$$

If $a_k = (a_{n,k})_{n \in \mathbb{N}}$ for every $k \in \omega$, then $v \in \text{conv}(U(S))$ if and only if there are $m \in \mathbb{N}$ and $r \in \omega$ such that $v$ has the form

$$v = \sum_{i=1}^{m} \lambda_i \left(\sum_{j=0}^{r} t_{j,i} e_{l(j,i)} \right),$$

where

(a) $\lambda_1, \ldots, \lambda_m \in (0, 1]$ and $\lambda_1 + \cdots + \lambda_m = 1$;

(b) $l(j,i) \in \mathbb{N}$ for every $0 \leq j \leq r$ and $1 \leq i \leq m$;

(c) $|t_{j,i}| < a_{l(j,i),j}$ for every $0 \leq j \leq r$ and $1 \leq i \leq m$.

Denote by $\mathcal{N}_s$ the family of all subsets of $\mathbb{V}_s$ of the form $U(S)$.

**Proposition 2.1.** The family $\mathcal{N}_s$ forms a base at 0 for $\mu_s^G$. Thus the family

$$\hat{\mathcal{N}}_s := \{\text{conv}(V) : V \in \mathcal{N}_s\}$$

is a base at 0 for $\nu_s^G$. In particular, for every neighborhood $U$ of zero in $\nu_s^G$ and for every $t \in \mathbb{N}$ and each $a > 0$, there exists $q \in \mathbb{N}$ such that every vector of the form

$$v = \lambda_1 ae_{m_1} + \cdots + \lambda_t ae_{m_t},$$

where $q < m_1 < \cdots < m_t$ and $\lambda_1, \ldots, \lambda_t \in [-1, 1]$, belongs to $U$.

**Proof.** First we check that the family $\mathcal{N}_s$ satisfies the following conditions:

(a) for each $U(S) \in \mathcal{N}_s$ there is a $U(S') \in \mathcal{N}_s$ with $U(S') + U(S') \subseteq U(S)$;

(b) for each $U(S) \in \mathcal{N}_s$ there is a $U(S') \in \mathcal{N}_s$ such that $\lambda U(S') \subseteq U(S)$ for all $|\lambda| \leq 1$;

(c) for each $U(S) \in \mathcal{N}_s$ and every $x \in \mathbb{V}_s$, there is $n \in \mathbb{N}$ such that $x \in nU(S)$;

(d) $\bigcap \mathcal{N}_s = \{0\}$.

Indeed, let $S = \{S(a_k)\}_{k \in \omega}$, where $a_k = (a_{n,k})_{n \in \mathbb{N}} \subseteq (0, \infty)$. Set

$$S' = \{S(a'_k)\}_{k \in \omega},$$

where

$$a'_k := \left(\frac{1}{2} \min\{a_{n,0}, \ldots, a_{n,2k+1}\}\right)_{n \in \mathbb{N}}.$$

Then

$(a)$

$$S(a'_0) + \cdots + S(a'_k) \subseteq S(a_0) + \cdots + S(a_{2k+2})$$

and $U(S') + U(S') \subseteq U(S)$,

$(b)$

$$S(a'_0) + \cdots + S(a'_k) \subseteq S(a_0) + \cdots + S(a_{2k+2})$$

for every $i \in \omega$. Therefore

$(2)$

$$S(a'_0) + \cdots + S(a'_k) \subseteq S(a_0) + \cdots + S(a_{2k+2})$$

and $U(S') + U(S') \subseteq U(S)$,
that proves (a). Items (b)-(d) are trivial. So, by [11] \([15, 2(1)])\), \(\mathcal{N}_s\) forms a base for some vector topology \(\tau\) on \(V_s\). Moreover, \(e_n \to 0\) in \(\tau\), since \(e_n \in S(\mathcal{a}_0)\) for all sufficiently large \(n\).

To show that \(\tau = \mu^G_s\), fix an arbitrary vector topology \(\tau'\) on \(V_s\) in which \(e_n \to 0\) and let \(V \in \tau'\). Choose inductively a sequence of symmetric open circled neighborhoods \(\{V_k\}_{k \in \omega}\) of \(0\) in \(\tau'\) such that \(V_0 + V_0 \subseteq V\) and \(V_{k+1} + V_{k+1} \subseteq V_k\) for each \(k \in \omega\), so \(\sum_{k \in \omega} V_k \subseteq V\). For every \(k \in \omega\) and each \(n \in \mathbb{N}\), set

\[
a_{nk} := \text{sup} \left\{ \lambda \in (0, n] : \text{if } |x| < \lambda, \text{ then } x \in V_k \cap \left( \mathbb{R}_n \times \prod_{j \in \mathbb{N}, j \neq n} \{0\} \right) \right\}.
\]

We show that \(\lim_{n} a_{nk} = \infty\) for every \(k \in \omega\). Indeed, if \(a_{n,k} < A\) for some sequence \(n_1 < n_2 < \ldots\) and a positive number \(A\), then \(e_{n_i} \notin \frac{1}{k} V_k\) for every \(i \in \mathbb{N}\) (we recall that \(V_k\) is circled). Hence \(e_n \notin 0\) in \(\tau'\), a contradiction. For every \(k \in \omega\), we set \(a_k = (a_{nk})_{n \in \mathbb{N}}\) and let \(S = \{S(\mathcal{a}_k)\}\), so \(S \subseteq \mathcal{S}_s\). Then, by construction, \(U(S) \subseteq V\). Thus \(\tau' \subseteq \tau\) and \(\tau = \mu^G_s\).

The fact that the family \(\mathcal{N}_s\) is a base for \(\nu^G_s\) follows from the definitions of \(\mu^G_s\) and \(\nu^G_s\) (cf. Proposition 5.1 of [11]). Let us prove the last assertion.

We assume that the neighborhood \(U\) is absolutely convex and contains \(S(\mathcal{a})\) for some \(\mathcal{a} = (a_n)_{n \in \mathbb{N}} \in \mathcal{A}\). Choose \(q \in \mathbb{N}\) such that for every \(n \geq q\) it follows that \(ta < a_n\). Hence \(\lambda(\mathcal{a})/e_n \in S(\mathcal{a})\) for every \(\lambda \in [-1, 1]\). Now if \(q < m_1 < \ldots < m_t\) and \(\lambda_1, \ldots, \lambda_t \in [-1, 1]\), then

\[
v = \frac{1}{t} \cdot \lambda_1(\mathcal{a})e_{m_1} + \cdots + \frac{1}{t} \cdot \lambda_t(\mathcal{a})e_{m_t}
\]

belongs to \(U\).

Below we describe the dual space of \(L_G(s)\).

**Proposition 2.2.** The dual space \(E\) of \(L_G(s)\) is linearly isomorphic to the space \(c_0\) of all real-valued sequences converging to zero.

**Proof.** From the description of the base \(\mathcal{N}_s\) of the topology \(\nu^G_s\) of \(L_G(s)\) given in Proposition 2.1, it immediately follows that \(\nu^G_s\) is strictly weaker than the box topology \(\mathcal{T}_s\) on \(\mathbb{R}^{(s)}\). It is well known that the dual space of \((\mathbb{R}^{(s)}, \mathcal{T}_s)\) is linearly isomorphic to the direct product \(\mathbb{R}^{\mathbb{N}}\). Now the continuity of the identity map \((\mathbb{R}^{(s)}, \mathcal{T}_s) \to L_G(s)\) implies that \(E \subseteq \mathbb{R}^{\mathbb{N}}\). Thus, to prove the proposition we have to show that \(\chi = (y_n) \in E\) if and only if \(y_n \to 0\).

**Claim 1.** If \(\chi = (y_n) \in E\), then \(y_n \to 0\). Indeed, suppose for a contradiction that there is a sequence \(0 < r_1 < r_2 < \ldots\) of indices such that \(|y_{r_i}| > a\) for some \(a > 0\). We assume that all \(y_{r_i}\) are positive. Let \(U\) be a neighborhood of zero in \(L_G(s)\) such that \(\chi(U) \subseteq [-1, 1]\) and take \(t \in \mathbb{N}\) such that \(ta > 1\). By Proposition 2.1, there is \(q \in \mathbb{N}\) such that every vector of the form

\[
v = e_{m_1} + \cdots + e_{m_t}, \quad \text{where } q < m_1 < \ldots < m_t,
\]

belongs to \(U\). Take \(i \in \mathbb{N}\) such that \(r_i > q\) and set \(w := e_{r_{i+1}} + \cdots + e_{r_{i+t}}\). Then \(w \in U\) and

\[
\chi(w) = y_{r_{i+1}} + \cdots + y_{r_{i+t}} > t \cdot a > 1,
\]

a contradiction. Thus \(y_n \to 0\).
Claim 2. If \( y_n \to 0 \), then \( \chi \in E \). Indeed, since \( \chi \) is linear, by Proposition 2.1 it is sufficient to show that there is a sequence \( S = \{ S(a_k) \}_k \subseteq S_n \) such that \( \chi(U(S)) \subseteq [-1, 1] \). To this end, for every \( i \in \omega \), we shall find \( a_i \in A \) such that

\[
\chi(S(a_i)) \subseteq \left[ -\frac{1}{2^{i+1}}, \frac{1}{2^{i+1}} \right].
\]

Set \( I_1 := \{ n \in \mathbb{N} : y_n = 0 \} \) and \( I_2 := \mathbb{N} \setminus I_1 \). If \( n \in I_1 \), set \( a_i(n) := n \), and set

\[
a_i(n) := \frac{1}{2y_n \cdot 2^{i+1}}, \quad \text{for every } n \in I_2.
\]

It is clear that \( a_i(n) \to \infty \) at \( n \to \infty \), so \( a_i \in A \). The inclusion \( \chi(2.2) \) holds trivially by the construction of \( a_i \) and \( S(a_i) \). Now, if \( w \in U(S) \) has a decomposition \( w = v_0 + \cdots + v_k \) with \( v_i \in S(a_i) \), we obtain

\[
|\chi(w)| \leq \sum_{i=0}^k |\chi(v_i)| \leq \sum_{i=0}^k \frac{1}{2^{i+1}} < 1.
\]

Thus \( \chi \in E \). \( \square \)

The next important result is proved in [9, 12, see also 8, 23.32].

Fact 2.3. Let \( E \) be a locally convex space. Then the mapping \( p : E' \to \hat{E} \), defined by the equality

\[
p(f) = \exp\{2\pi if\}, \quad \text{for all } f \in E',
\]

is a group isomorphism between \( E' \) and \( \hat{E} \).

Now Theorem 1.1 immediately follows from (2.1) and the following result.

Theorem 2.4. The space \( L_C(s) \) is not a Mackey space. Consequently \( L_G(s) \) is not a Mackey group.

Proof. First we describe a method for constructing of compatible group topologies on \( L_G(s) \). Let \( h \) be a continuous homomorphism from \( \mathbb{R} \) to a locally quasi-convex abelian group \( H \). Define an algebraic monomorphism \( T_H : \mathbb{R}^{(N)} \to L_G(s) \times \mathfrak{F}_0(H) \) by

\[
T_H((x_k)) := \left( (x_k), (h(x_k)) \right), \quad \forall (x_k) \in \mathbb{R}^{(N)},
\]

and let \( T_H \) be the locally quasi-convex group topology on \( \mathbb{R}^{(N)} \) induced from \( L_G(s) \times \mathfrak{F}_0(H) \).

Step 1. The topology \( T_H \) is weaker than the box topology \( T_b \) on \( \mathbb{R}^{(N)} \). As we explained in the beginning of the proof of Proposition 2.2 the topology \( \nu_H \) is weaker than \( T_b \). Therefore we have to show only that the map

\[
p : (\mathbb{R}^{(N)}, T_b) \to \mathfrak{F}_0(H), \quad p((x_k)) := (h(x_k)),
\]

is continuous at zero. Fix a neighborhood \( U \) of the identity in \( H \). Choose a neighborhood \( V \) of zero in \( \mathbb{R} \) such that \( h(V) \subseteq U \). Set \( W := \mathbb{R}^{(N)} \cap V^N \). Then \( W \) is a neighborhood of zero in \( (\mathbb{R}^{(N)}, T_b) \) such that for every \( (x_k) \in W \) (recall that \( (x_k) \) has finite support) we have

\[
p((x_k)) = (h(x_k)) \in c_0(H) \cap U^N.
\]

Hence \( p \) is continuous. Thus \( T_H \) is weaker than \( T_b \).
Step 2. We claim that the topology $T_H$ is compatible with $\nu_a^G$ such that $\nu_a^G \leq T_H$. Indeed, set $G := (\mathbb{R}^{(0)}, T_H)$. Step 1 implies that
\[(L_G(s))^\wedge \subseteq G^\wedge \subseteq (\mathbb{R}^{(0)}, T_0)^\wedge.\]
Taking into account Proposition 2.2 and Fact 2.3 we can identify $G^\wedge$ with a subgroup $Y$ of $(\mathbb{R}^{(0)}, T_0)' = \mathbb{R}^N$ containing $c_0$. Thus, to prove the claim we have to show that $Y = c_0$.

Fix arbitrarily $\chi = (y_n) \in Y$. We have to show that $y_n$ converges to zero. Suppose for a contradiction that $y_n \not\to 0$. So there is a sequence $0 < m_1 < m_2 < \ldots$ of indices and $A \in \mathbb{N}$ such that $|A \cdot y_{m_j}| > 1$ for every $i \in \mathbb{N}$. We assume that all $y_{m_j}$ are positive. Since $\chi$ is $T_H$-continuous, there exists a standard neighborhood $W = T_H^{-1}(U \times V^N)$ of zero in $G$, where $U$ is a neighborhood of zero in $L_G(s)$ and $V$ is a neighborhood of the identity in $H$, such that $\chi(W) \subseteq \mathbb{S}_+$. Observe that, by (2.3), $(x_k) \in W$ if and only if
\[(2.4) \quad (x_k) \in U \text{ and } h(x_k) \in V \text{ for every } k \in \mathbb{N}.\]
Choose $l \in \mathbb{N}$ such that $h(\lbrack -1/2l, 1/2l \rbrack) \subseteq V$. Set
\[t := lA, \quad a := \frac{1}{2l} \quad \text{and} \quad \lambda_j := \frac{1}{A \cdot y_{m+j}} \text{ for } j = 1, \ldots, t.\]
Now Proposition 2.1 implies that there exists $q \in \mathbb{N}$ such that the vector
\[v := \lambda_1 \cdot \frac{1}{2l}e_{m_1} + \cdots + \lambda_t \cdot \frac{1}{2l}e_{m_t}\]
belongs to $U$. Since $0 < \lambda_1 < 1$, the choice of $l$ implies that $h(\lambda_j/2l) \in V$ for every $j = 1, \ldots, t$. Hence, by (2.4), we obtain $v \in W$. Therefore
\[\chi(v) = \exp \left\{ 2\pi i \sum_{j=1}^{t} y_{m+j} \cdot \frac{1}{2lA \cdot y_{m+j}} \right\} = \exp \left\{ 2\pi i \frac{t}{2lA} \right\} = e^{\pi i} = -1 \in \mathbb{S}_+,\]
a contradiction. Thus $y_n \to 0$ and $Y = c_0$. The inequality $\nu_a^G \leq T_H$ holds trivially.

Step 3. $L_G(s)$ is not a Mackey space. Indeed, let $H = \mathbb{R}$ and $h : \mathbb{R} \to H$ be the identity map. Then $\mathbb{S}_0(\mathbb{R})$ is the classical Banach space $c_0$. Therefore the compatible topology $T_H$ is a locally convex vector topology such that $\nu_a^G \leq T_H$. We show that $\nu_a^G \neq T_H$. Indeed, by the definition of $\nu_a^G$, we have $e_n \to 0$ in $\nu_a^G$. However, $e_n \not\to 0$ in the Banach space $c_0$, and hence $e_n \not\to 0$ in $T_H$.

The space $L_G(s)$ is not a Mackey group by (i) of Proposition 2.5 of [1].

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