A VERAGE DECAY OF THE FOURIER TRANSFORM OF MEASURES WITH APPLICATIONS

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Abstract. We consider spherical averages of the Fourier transform of fractal measures and improve the lower bound on the rate of decay. Maximal estimates with respect to fractal measures are deduced for the Schrödinger and wave equations. This refines the almost everywhere convergence of the solution to its initial datum as time tends to zero. A consequence is that the solution to the wave equation cannot diverge on a \((d-1)\)-dimensional manifold if the data belongs to the energy space \(H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)\).

1. Introduction

Consider the Schrödinger equation, \(i \partial_t u + \Delta u = 0\), on \(\mathbb{R}^d = \mathbb{R}^{n+1}\), with initial data \(u(\cdot, 0) = u_0\) in \(H^s\) defined by

\[ H^s := \{ G_s * f : f \in L^2(\mathbb{R}^n) \}. \]

Here \(G_s\) is the Bessel kernel defined as usual by \(\hat{G}_s := (1 + |\cdot|^2)^{-s/2}\), where \(\hat{}\) is the Fourier transform. In [14], Carleson considered the problem of identifying the exponents \(s > 0\) for which

\[ \lim_{t \to 0} u(x, t) = u_0(x), \quad \text{a.e. } x \in \mathbb{R}^n, \quad \forall \ u_0 \in H^s, \quad (1.1) \]

and proved that this is true as long as \(s \geq 1/4\) in the one-dimensional case. Dahlberg and Kenig [18] then showed that (1.1) does not hold if \(s < 1/4\). The higher dimensional case has since been studied by many authors; see for example [17, 12, 33, 32, 30, 37, 36]. The best known positive result to date, that (1.1) holds if \(s > 1/2 - 1/(4n)\), is due to Lee [25] when \(n = 2\) and Bourgain [8] when \(n \geq 3\). Bourgain also showed that \(s \geq 1/2 - 1/n\) is necessary for (1.1) to hold.

A natural refinement of the problem is to bound the size of the divergence sets

\[ D(u_0) := \left\{ x \in \mathbb{R}^n : \lim_{t \to 0} u(x, t) \neq u_0(x) \right\}, \]

and in particular we consider

\[ \alpha_n(s) := \sup_{u_0 \in H^s} \dim_H (D(u_0)), \]

where \(\dim_H\) denotes the Hausdorff dimension. A completely satisfactory theory has already been developed in the one-dimensional case; see [1, 5, or 15]. Indeed

\[ \alpha_n(s) \leq n - 2s, \quad \text{if } \frac{n}{2} \leq s \leq \frac{n}{2}, \]

and this bound is sharp in the sense that initial data in \(H^s\) can be singular on \(\alpha\)-dimensional sets when \(\alpha < n - 2s\); see [12]. On the other hand, the solution is continuous (and so \(\alpha_n(s) = 0\)) when \(s > n/2\), and the example of Dahlberg and Kenig tells us that \(\alpha_n(s) = n\) when \(s < 1/4\). Noting that altogether, when \(n = 1\), we have covered the whole range, we see that \(\alpha_1\) is known and it is discontinuous.

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at \( s = 1/4 \). These results and a more gentle introduction to the problem can also be found in [28] Chapter 17.

Here we improve the best known upper bounds for \( \alpha_n(s) \) in the remaining range of interest, when \( s < n/4 \), in higher dimensions. In particular, we prove the following theorem that refines the almost everywhere convergence due to Bourgain and Lee. At the same time, we improve the bounds \( \alpha_n(s) \leq n + 1 - 2s \) due to Sjörgen and Sjölin [32] and \( \alpha_n(s) \leq \frac{n+1}{n+1}(n-2s) \) due to Barceló, Bennett, Carbery and the second author [1].

**Theorem 1.1.** Let \( n \geq 2 \). Then

\[
\alpha_n(s) \leq \begin{cases} 
  n + 1 - \left(2 + \frac{2}{2n-1}\right)s, & \frac{1}{2} - \frac{1}{4n} < s \leq 1 - \frac{3}{2(n+1)}, \\
  n + 1 - \frac{1}{n+1} - 2s, & 1 - \frac{3}{2(n+1)} \leq s < \frac{n}{2}.
\end{cases}
\]

This will be a consequence of a maximal estimate (see Theorem 6.2) that holds uniformly with respect to fractal measures in the following class. To avoid repetition, we include positivity and a support condition inside the definition of ‘\( \alpha \)-dimensional’.

**Definition 1.2.** Let \( 0 < \alpha \leq d \). We say that \( \mu \) is (at least) \( \alpha \)-dimensional if it is a positive Borel measure, supported in the unit ball \( B(0,1) \), that satisfies

\[
c_\alpha(\mu) := \sup_{x \in \mu} \frac{\mu(B(x,r))}{r^\alpha} < \infty.
\]

The Fourier transform of such a measure need not decay (for example the Fourier transform of a piece of the surface measure on a hyperplane does not decay in the normal direction), however it must decay on average. As the class contains measures that are supported on \( \alpha \)-dimensional sets, the uncertainty principle suggests that there should be less decay for smaller values of \( \alpha \). Let \( \beta_\alpha(\alpha) \) denote the supremum of the numbers \( \beta \) for which

\[
\|\hat{\mu}(R \cdot)\|_{L^2(G^{d-1})}^2 \lesssim c_\alpha(\mu)\|\mu\|R^{-\beta} \tag{1.2}
\]

whenever \( R > 1 \) and \( \mu \) is \( \alpha \)-dimensional. The problem of identifying the precise value of \( \beta_\alpha(\alpha) \) was proposed by Mattila; see for example [28, pp. 42] or [29] Chapter 15. In two dimensions, the sharp decay rates are now known;

\[
\beta_2(\alpha) = \begin{cases} 
  \alpha, & \alpha \in (0, 1/2], \\
  1/2, & \alpha \in [1/2, 1], \\
  \alpha/2, & \alpha \in [1, 2],
\end{cases} \quad \text{(Mattila [27])}
\]

The work of Wolff, later simplified by Erdoğan [19], improved upon a lower bound due to Bourgain [27] who was the first to bring Fourier restriction theory to bear on the problem. In higher dimensions, the best known lower bounds are

\[
\beta_d(\alpha) \geq \begin{cases} 
  \alpha, & \alpha \in (0, \frac{d-1}{2}], \\
  \frac{d-1}{2}, & \alpha \in [\frac{d-1}{2}, \frac{d-1}{2}], \\
  \frac{d-1}{2} + \frac{d}{4}, & \alpha \in [\frac{d-1}{2}, \frac{d+2}{2}], \quad \text{(Erdoğan [20] [21])} \\
  \alpha - 1, & \alpha \in [\frac{d+2}{2}, d], \quad \text{(Sjölin [34])}.
\end{cases}
\]

\footnote{We write \( A \leq B \) if \( A \leq CB \) for some constant \( C > 0 \) that only depends on the dimension \( d \) and/or a small parameter \( \varepsilon \), in this case \( \varepsilon = \beta_d(\alpha) - \beta \). If the constant depends on anything else, say a power of \( N \), we write \( A \lesssim_N B \). We also write \( A \simeq B \) if \( A \lesssim B \) and \( B \lesssim A \).}
On the other hand, by considering limits of very simple measures supported on small sets (the so called ‘Knapp examples’, see for example [29, Chapter 15.2]), it is easy to show that

\[
\beta_d(\alpha) \leq \begin{cases} 
\alpha, & \alpha \in (0, d-2], \\
\frac{d+\alpha-2}{2}, & \alpha \in [d-2, d].
\end{cases}
\]

We see that the difference between the best known upper and lower bounds is never more than one and the bounds coincide when \(\alpha < \frac{d-1}{2}\) or \(\alpha = d\). Worse counterexamples have been constructed for signed measures by Iosevich and Rudnev [24], or when the averages are taken over a piece of paraboloid rather than the sphere by Barceló, Bennett, Carbery, Ruiz and Vilela [2]. Indeed, there is an extensive literature regarding averages over different manifolds and other generalisations; see for example [10, 11, 22, 23, 35] and the references therein.

We will prove the following theorem.

**Theorem 1.3.** Let \(d \geq 3\). Then

\[
\beta_d(\alpha) \geq \alpha - 1 + \frac{(d - \alpha)^2}{(d - 1)(2d - \alpha - 1)}.
\]

This improves the estimate of Sjölin for all \(\alpha < d\) and the estimate of Erdoğan for \(\alpha \geq d/2 + 2/3 + 1/d\). This is not enough to improve the state-of-the-art for the Falconer distance set conjecture (the argument of Mattila [27] combined with (1.3) implies that distance sets associated to \(\alpha\)-dimensional sets have positive Lebesgue measure whenever \(\alpha > d/2 + 5/12\)). On the other hand, the difference between the best known upper and lower bounds is now strictly less than one (never more than 1

\[
1 - \frac{4}{(d-1)(d+1)},
\]

from which we can deduce new information regarding the pointwise convergence of solutions to the wave equation.

Considering \(D_t v = \Delta v\) on \(\mathbb{R}^{d+1}\), with \(v(\cdot, 0) = v_0\) and \(D_t v(\cdot, 0) = v_1\), we take the initial data in the homogeneous space \(\dot{H}^s \times \dot{H}^{s-1}\), where

\[
\dot{H}^s := \left\{ I_s * f : f \in L^2(\mathbb{R}^d) \right\}.
\]

Here \(I_s\) is the Riesz kernel defined by \(\hat{I}_s := |\cdot|^{-s}\). The almost everywhere convergence question was first considered by Cowling [17], who proved

\[
\lim_{t \to 0} v(x, t) = v_0(x), \quad \text{a.e.} \quad x \in \mathbb{R}^d, \quad \forall (v_0, v_1) \in \dot{H}^s \times \dot{H}^{s-1}
\]
as long as \(s > 1/2\). Walther [40] then proved that this is not true when \(s \leq 1/2\), and so the Lebesgue measure question is completely solved for the wave equation. As before we write

\[
\mathcal{D}(v_0, v_1) := \left\{ x \in \mathbb{R}^d : \lim_{t \to 0} v(x, t) \neq v_0(x) \right\},
\]

and consider the refined problem of providing upper bounds for

\[
\gamma_d(s) := \sup_{(v_0, v_1) \in \dot{H}^s \times \dot{H}^{s-1}} \dim_H (\mathcal{D}(v_0, v_1)).
\]

Sharp estimates were proven in the two-dimensional case in [1], using the following proposition which forms the link with the decay estimate (1.2).

**Proposition 1.4.** Let \(d \geq 2\) and \(0 < s < d/2\). Then \(\beta_d(\alpha) > d - 2s \Rightarrow \gamma_d(s) \leq \alpha\).

Estimates for the inhomogeneous spaces \(H^s(\mathbb{R}^d)\) were proven in [1], which puts unnecessary restrictions on the data \(v_1\), but we will see that the implication also holds in this slightly more general context. Using Sjölin’s bound \(\beta_d(\alpha) \geq \alpha - 1\) they deduced that \(\gamma_d(s) \leq d + 1 - 2s\), so a consequence of Theorem 1.3 is that

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\(^2\)in fact in a very slightly larger range.
γ_d(1) < d − 1, ruling out divergence on spheres if the initial data belongs to the energy space $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$.

The exponent $\beta_d(\alpha)$ is also connected to dimension estimates for orthogonal projections; see for example the recent work of Oberlin–Oberlin [31]. For a related problem regarding Fourier convergence at the points where the function is zero, see [13] or [10] and the references therein.

Although Theorem 1.3 yields new bounds for the Schrödinger equation, via an appropriate version of Proposition 1.4 those presented in Theorem 1.1 follow by a more direct use of the techniques developed to prove Theorem 1.3. Compared to the cone, the paraboloid has an extra nonzero principal curvature, and so it is not always efficient to use Proposition 1.4 in that case. For this reason we have presented the results for the Schrödinger equation in $\mathbb{R}^{n+1}$, where $d = n + 1$, and this convention will be maintained throughout.

The key ingredient will be the multilinear extension estimate due to Bennett, Carbery and Tao [4], which was first successfully employed to prove linear estimates by Bourgain and Guth [9]. We present the multilinear estimates in the following section and a decomposition due to Bourgain and Guth in Section 3. In Section 4 we prove Theorem 1.3 and in Section 6 we prove Theorem 1.1. In Section 5 we present the simple proof of Proposition 1.4, via polar coordinates.

2. Multilinear extension estimates

Here we present the multilinear extension estimates due to Bennett, Carbery and Tao [4]. The extension operator, defined below, is also the adjoint of the operator that restricts the Fourier transform to a surface, and so they are also referred to as restriction estimates. We consider the surfaces

$S := \{(\xi, \phi(\xi)) \in \mathbb{R}^d : |\xi| \leq 1/2\}$

with $\phi(\xi) = -|\xi|^2$ or $\phi(\xi) = \sqrt{1 - |\xi|^2}$. For a cap $\tau = \{ (\xi, \phi(\xi)) : \xi \in Q \} \subset S$ associated to a cube $Q$, we define the extension operator $T_\tau$ by

$T_\tau g(x, t) := \int_Q g(\xi) e^{ix \cdot \xi + i t \phi(\xi)} d\xi,$

Letting $Y(\xi) \in S^{d-1}$ be the outward unit normal vector at a point $(\xi, \phi(\xi)) \in S$, we say that the caps $\tau_1, \ldots, \tau_m$ are $m$-transversal with constant $\theta > 0$ if

$|Y(\xi_1) \wedge \cdots \wedge Y(\xi_m)| > \theta,$

for all $\xi_1 \in Q_1, \ldots, \xi_m \in Q_m$. In the following theorem, and throughout, $B_R$ denotes a ball of radius $R$ with arbitrary centre.

**Theorem 2.1.** [4] Let $d \geq 2$, $\epsilon > 0$ and let $\tau_1, \ldots, \tau_d \subset S$ be $d$-transversal caps with constant $\theta > 0$. Then, for all $R > 1$,

$$\left\| \prod_{k=1}^d T_{\tau_k} g \right\|_{L^{\frac{2d}{d-1}}(B_R)} \lesssim c(\theta) R^\epsilon \prod_{k=1}^d \| g \|_{L^2(Q_k)}.$$

The exact dependence of $c$ on $\theta$ is an interesting open question. The following version is lower dimensional and it has also been discretised as in [9, pp. 1250]. This is the version we will require in the following section.

**Proposition 2.2.** [9] Let $0 < \epsilon < \frac{1}{m^2}$ and let $\tau_1, \ldots, \tau_m \subset \tau$ be $m$-transversal caps with constant $\theta$, where $2 \leq m \leq d-1$. Let $V_m$ be an $m$-dimensional subspace of $\mathbb{R}^d$.
and let $Q_{j_k} \subset Q_k$ be disjoint cubes of side length $1/K$ such that $\text{dist}(Y(\xi), \mathbb{V}_m) \leq 1/K$ for some $\xi \in Q_{j_k}$. Then, for all $K > 1$,

$$\int_{B_K} \prod_{k=1}^m \left| \sum_{j_k} T_{\tau_{j_k}} g \right|^{2/d} \lesssim (\int_{B_K} \prod_{k=1}^m \left( \sum_{j_k} |T_{\tau_{j_k}} g|^2 \right)^{\frac{d}{2}})^{\frac{1}{d}}. \quad (2.3)$$

In fact, due to rescaling arguments we will require these estimates for slightly more general phases $\phi$. Note first that, as we are only interested in the modulus of the extension operator, we are free to add and subtract constants to the phase $\phi$ and so we work instead with $\phi(\xi) = \sqrt{1-|\xi|^2} - 1$ in the spherical case so that it looks very similar to the parabolic case. Then, for $\xi_0 \in \{\xi \in \mathbb{R}^{d-1}: |\xi| \leq 1/2 - \delta/2\}$ and $0 < \delta < 1$, we define the scaling map $S_{\xi_0, \delta}$ by

$$S_{\xi_0, \delta}(\xi) = \delta^{-2} \left( \phi(\xi + \delta \xi) - \delta \nabla \phi(\xi_0) \cdot \xi - \phi(\xi_0) \right).$$

Note that the paraboloid is unchanged by this operation, and the sphere is changed only very mildly. The estimates of this section hold uniformly for all the extension operators defined with a phase obtained by applying the scaling map a finite number of times to $\phi$.

Finally we present a globalised-in-space version of Theorem 2.1 that we will need in the final sections. It follows by a standard localisation argument.

**Proposition 2.3.** Let $\varepsilon > 0$, $p = \frac{2d}{d+1}$ and let $\tau_1, \ldots, \tau_d \subset S$ be $d$-transversal caps with constant $\theta > 0$. Let $\{\Omega\}$ be a partition of $\mathbb{R}^{d-1}$ into cubes of side length $R$. Then, for all $R > 1$,

$$\left\| \prod_{k=1}^d |T_{\tau_k} g|^{2/d} \right\|_{L^p(\mathbb{R}^{d-1} \times (-R, R))} \lesssim \sum_{\Omega} \left\| \prod_{k=1}^d |T_{\tau_k} g|^{2/d} \right\|_{L^p(\Omega \times (-R, R))} \lesssim \varepsilon(\theta) R^\varepsilon \|g\|_2^2. \quad (2.4)$$

**Proof.** Noting that the first inequality is nothing more than the inclusion $\ell^2 \subset \ell^p$, it remains to prove the second which we rewrite as

$$\sum_{\Omega} \left\| \prod_{k=1}^d |T_{\tau_k} g|^{2/d} \right\|_{L^p(\Omega \times (-R, R))} \lesssim \varepsilon(\theta) R^\varepsilon \|g\|_2^2.$$

For this we write $g_\Omega = (\chi_{Q_\xi}^{\varepsilon})^\vee$ and $g_\Omega^c = g \chi_{Q_\xi} - g_\Omega$, where $\chi_{Q_\xi}$ is a Schwartz function adapted to the cube $\Omega^c$, with same centre as $\Omega$, but with side length $10 \sup_{|\xi| \leq 1/2} |1 + \nabla \phi(\xi)| R$. (2.1)

Now that we have taken the support restriction inside the definition of the functions, we will consider the operator $T$ defined by

$$T g(x, t) := \int_{\mathbb{R}^{d-1}} \psi(\xi) g(\xi) e^{i t \cdot \xi + i t \phi(\xi)} d\xi,$$

where $\psi$ is a Schwartz function supported in the unit ball and equal to one on $|\xi| \leq 1/2$. By applications of the triangle inequality it would then suffice to bound the main term as

$$\sum_{\Omega} \left\| \prod_{k=1}^d T g_\Omega \right\|_{L^p(\Omega \times (-R, R))} \lesssim \varepsilon(\theta) R^\varepsilon \|g\|_2^2,$$

and prove other mixed inequalities, like for example

$$\sum_{\Omega^c} \left\| T g_\Omega^c \prod_{k=2}^d T g_\Omega \right\|_{L^p(\Omega^c \times (-R, R))} \lesssim \|g\|_2^2. \quad (2.3)$$

\[\text{Figure 2.1: Diagram of the argument.} \]
The main term is bounded directly using Theorem 2.1 and the finite overlapping of the frequency supports. For the second estimate we first note that by Hölder’s inequality, followed by Bernstein’s inequality (or Young’s inequality given the compact frequency support and the reproducing formula that it yields, see below) and Plancherel’s identity in the $x$-variable, the left-hand side of (2.3) is bounded by

$$\sum_{\Omega} \left( \|T g_{\Omega}\|_{L^2(\Omega)} \|g_{\Omega} e^{it\phi(\cdot)}\|_{L^{d-1}_{\Omega}} \right)^{2/d}.$$ 

Then by Hölder’s inequality in the time integral, we see that this is bounded by

$$R^{d/(d-2)} \sum_{\Omega} \left( \|T g_{\Omega}\|_{L^2(\Omega \times (-R, R))} \|g_{\Omega}\|_{L^2(\Omega \times (-R, R))} \right)^{1/2}.$$ 

A final application of Hölder’s inequality in the sum, and the finite overlapping of the frequency supports, shows that this is bounded by

$$R^{d/(d-2)} \|g\|_2 \left( \sum_{\Omega} \|T g_{\Omega}\|_{L^2(\Omega \times (-R, R))} \right)^{1/2}.$$ 

Thus in order to complete the proof of (2.3), we need only prove that

$$\sum_{\Omega} \|T g_{\Omega}\|_{L^2(\Omega \times (-R, R))}^2 \lesssim R^{-N} \|g\|_2^2 \tag{2.4}$$

for large enough $N \in \mathbb{N}$.

For this we write the operator as a convolution,

$$T g_{\Omega}(x, t) = \int_{\mathbb{R}^{d-1}} \psi(\xi) g_{\Omega}(\xi) e^{ix \xi + it\phi(\xi)} d\xi = \int_{z \in \mathbb{R}^{d-1}} T[1](z, t) (g \chi_{Q_1})^\vee(z - x) dz.$$ 

Recalling the definitions (2.1) and (2.2), we have that $|z| \geq 2|\nabla \phi(\xi)|$ when $(x, t) \in \Omega \times (-R, R)$, so by repeated integration by parts we see that

$$|T[1](z, t)| \lesssim C_N (1 + |z|)^{-N-d-1},$$

so that, for $(x, t) \in \Omega \times (-R, R)$, we have

$$|T g_{\Omega}(x, t)| \lesssim R^{-N-1} \int_{\mathbb{R}^{d-1}} (1 + |z|)^{-d} |(g \chi_{Q_1})^\vee(z - x)| dz.$$ 

Plugging this into (2.4), and integrating in time, we see that

$$\sum_{\Omega} \|T g_{\Omega}\|_{L^2(\Omega \times (-R, R))}^2 \lesssim R^{-N} \int_{\mathbb{R}^{d-1}} \left( \int_{1 + |z|} (1 + |z|)^{-d} |(g \chi_{Q_1})^\vee(z - x)| dz \right)^2 dx$$

$$\lesssim R^{-N} \|g\|_2^2,$$

where the final inequality is by Young’s inequality and the Plancherel identity. This completes the proof of (2.3) and thus (2.3), and the other mixed terms are bounded in an analogous manner. $\square$

3. The Bourgain–Guth decomposition

In order to take advantage of the multilinear estimates, we must first decompose the operator in such a way that transversality presents itself. In order to take advantage of bilinear estimates, this can be done by employing something like a Whitney decomposition. A triumph of the work of Bourgain and Guth [9] was to achieve something similar in the multilinear setting. In fact they use the lower dimensional multilinear estimates of Proposition 2.2 in order to create the ‘decomposition’ (really it is an inequality) and in the coming sections we will need pointwise control this. Indeed we will make essential use of the fact that the right-hand side of the inequality is almost constant at certain scales. As they point out, this only holds after mollifications, and the final decomposition is obtained by an iteration. In
this section, we keep track of some of the details that they omitted so as to check that these approximations, as well as the lack of control of the constant $c$ from the previous section, do not feedback in an uncontrolled way.

Let $Q \subset \{ \xi \in \mathbb{R}^{d-1} : |\xi| \leq 1/2 \}$ be a box of side length $\delta$ and let $\tau$ denote the associated cap. Take $0 < \varepsilon < \frac{1}{4}$, and $R > 1$ and introduce $d$ different scales

$$R^{1/\varepsilon} < K_{2} < \cdots < K_{d+1} < R^\varepsilon$$

that satisfy $K_{m}^\varepsilon \varepsilon(K_{m}^{-m}) \leq K_{m+1}^{-1}$, where $\varepsilon(\varepsilon) \geq 1/\varepsilon^{2d}$ dominates the constant from the previous section. As long as it does not blow up at zero in a very unexpectedly fast way, it would suffice to take $K_{m} \simeq R^{2(d+2-m)}$. One can calculate that we also have $R^{\varepsilon} K_{m} \leq K_{m+1}$.

Take a partition $\{Q_{m,j}\}$ of $Q$ made of pairwise disjoint cubes of side length $\delta/K_{2}$ and centered in $\xi_{j}$. Then, for all $m = 3, \ldots, d$, define recursively a sub-partition $\{Q_{m,j}\}$ made by pairwise disjoint cubes of side length $\delta/K_{m}$ and centered at $\xi_{j}$ in such a way that for every $Q_{m,j}$ there exists an $Q_{m-1,j}$ that contains it. For this we need to suppose that $R > 2^{\varepsilon(\varepsilon)}$ in order to have room to choose the scales appropriately, and so this is assumed from now on.

We say that the caps $\tau_{m,j}$ associated to $Q_{m,j}$ are at scale $\delta/K_{m}$. Recalling that

$$T_{\tau_{m,j}} g(x,t) = \int_{Q_{m,j}} g(\xi) e^{ix \cdot \xi + it \cdot \phi(\xi)} d\xi,$$

for each $m = 2, \ldots, d$, we have

$$T_{\tau} g = \sum_{j} T_{\tau_{m,j}} g.$$  

We will also need a restricted version of $T_{\tau_{m,j}}$. Let $\mathbb{V}_{m}$ be an $m$-dimensional subspace of $\mathbb{R}^{d}$ and define

$$T_{\tau_{m,j}}^{\mathbb{V}_{m}} g := \sum_{\nu \subset \tau_{m,j} \in \tau} T_{\nu} g,$$

where $V_{\tau_{m,j}} := \{ \tau_{m+1,j} \subset \tau : \text{dist}(Y(\nu), \mathbb{V}_{m}) \leq \delta/K_{m+1} \text{ for some } \nu \in \mathbb{V}_{\tau_{m,j}} \}$.

The following pointwise estimate [9] pp. 1256] will be a key ingredient:

$$|T_{\tau} g(x,t)| \lesssim K_{d}^{2d} \max_{m=2} \prod_{k=1}^{d} |T_{\tau_{m,j}} g(x,t)|^{1/2}$$

$$+ \sum_{m=2}^{d-1} K_{m}^{2m} \max_{\nu \subset \tau_{m,j}} \prod_{k=1}^{m} |T_{\nu} g(x,t)|^{1/2} + \sum_{m=2}^{d} \max_{\tau_{m,j} \subset \tau} |T_{\tau_{m,j}} g(x,t)|.$$  

Here the caps $\tau_{1}, \ldots, \tau_{m}$ in the first two maxima are $m$-transversal at scale $\delta/K_{m}$, and the final maximum is over caps $\tau_{m}$ at scale $\delta/K_{m}$. This is proved by iterating the following dichotomy: either the operator is bounded by a product of $m+1$ operators associated to transversal caps, or it is not, in which case, given $m$ caps where the operator is large and the hyperplane $\mathbb{V}_{m}$ that their normals lie on outside of $\mathbb{V}_{m}$ must be small.

The uncertainty principle tells us that the terms should be essentially constant at different scales $\delta/K$. This can be formalised by replacing them with suitable majorant functions. Indeed, define the dual set $\tau'$ to be the $d$-dimensional cuboid with dimensions $\delta^{-1} \times \ldots \times \delta^{-1} \times \delta^{-2}$ centred at the origin, and with long side normal to $\tau$ (pointing in the direction of the normal $Y_{\tau}$ to the centre of a cap $\tau$). The scaled version $K^{\tau'}$ denotes the similar set but with dimensions $K^{d-1} \times \ldots \times K^{d-1} \times K^{d-2}$. Let $\psi = \psi_{o} * \psi_{o}$ be a smooth radially symmetric cut-off function, supported on
Definition 3.1. The subspace which the operator is large. These caps are at scale $K_m^3$. Remark $K$.

With this function, the decomposition (3.3) can be rewritten as

$$B(0, d) \subset \mathbb{R}^d$$ and equal to one on $B(0, \sqrt{d}) \subset \mathbb{R}^d$ and let $\psi_{K\tau'}$ denote the scaled version of $\psi$ adapted to $K\tau'$. By this we mean that

$$\psi_{K\tau'}(x, t) := \frac{\delta^{d+1}}{K^d} \psi\left(\frac{dx'}{\delta}, \frac{\delta^2 t'}{K}\right), \quad (x', t') = \Lambda_t(x, t), \quad (3.2)$$

where $\Lambda_t \in SO(d)$ and $\Lambda_t(Y_t) = (0, \ldots, 0, 1)$. By the modulated reproducing formula,

$$|T_r g| \leq |T_r g| * |\psi_r|,$$

and one can also calculate (see Lemma 6.7 of the appendix) that

$$|T_r g| \lesssim \left(|T_r g| \frac{m}{K}\right)^m,$$

for any $m \geq 1$. This yields

$$|T_r g| \lesssim |T_r g| \frac{m}{K} * \zeta_r, \quad \zeta(x, t) := (1 + |x|^2 + |t|^2)^{-c(x)},$$

and as $\zeta_r$ is essentially constant on translates of $r'$, which is a property that is preserved under convolution, we have majorised by an essentially constant function. By elementary trigonometry one sees that $\frac{1}{2} K_m r' \subset v'$ whenever $v \subset r$ is at scale $\delta/K_m$, so that the dual of the latter is contained in the former and so

$$|T_{V_m} g| \frac{m}{K} \lesssim |T_{V_m} g| \frac{m}{K} * \zeta_{K_m r}.$$

Using these observations, (3.1) can be rewritten as

$$|T_r g| \lesssim K^2 \max_{\tau_1, \ldots, \tau_d} \prod_{k=1}^d |T_{\tau_k} g| \frac{m}{K} * \zeta_{\tau_k}$$

$$+ \sum_{m=2}^{d-1} K^m \max_{\tau_1, \ldots, \tau_m} \prod_{k=1}^m |T_{V_k} g| \frac{m}{K} * \zeta_{K_m r} + \sum_{m=1}^{d-1} \max_{\tau_{m+1}} |T_{\tau_{m+1}} g| * \zeta_{\tau_{m+1}},$$

where as before $\tau_1, \ldots, \tau_m \subset r$ are $m$-transversal caps at scale $\delta/K_m$ and the maximum in the last term is taken over caps of size $\delta/K_{m+1}$.

Remark 3.1. The maximum over $\tau_1, \ldots, \tau_m, V_m$ depends on the value of $(x, t)$, but we can now choose the same $\tau_1, \ldots, \tau_m, V_m$ for all $(x, t)$ in a translate of $K_m r'$. In fact, given the dichotomy with which the initial decomposition is obtained, $V_m$ can be chosen to be the same in any translate of $K_m r'$. This is because we only need to consider this lower dimensional case in the absence of $m+1$ transversal caps for which the operator is large. These caps are at scale $K_{m+1}$ and so the definition of the subspace $V_m$ can be taken uniformly at that scale.

Definition 3.1. Set $\Phi_{r, V_m, r_2} = 1$ and, for $m = 2, \ldots, d-1$, define

$$\Phi_{r, V_m, r_{m+1}} := \frac{K^{2m}}{m} \max_{\tau_1, \ldots, \tau_m \subset r} \prod_{k=1}^m |T_{V_k} g| \frac{m}{K} * \zeta_{K_m r} + |T_{\tau_{m+1}} g| * \zeta_{\tau_{m+1}},$$

$$\left(\sum_{v \in V_m \cup \{\tau_{m+1}\}} (|T_v g| * \zeta_v)^2 \right)^{1/2} + R^{-1/2} ||g||_{L^2}.$$
where the remainder term $R_\tau(g)$ is defined by

$$R_\tau(g) := R^{-1/\epsilon} \sum_{m=1}^{d-1} \max_{V_m, \tau_m+1} \Phi_{V_m, \tau_m+1} \|g\|_{L^2}^2.$$  

Although $\Phi_{V_m, \tau_m+1}$ looks complicated, we will no longer care about its explicit form, and focus instead on its properties. These properties, one of which we prove now using the multilinear extension estimate, hold uniformly for all hyperplanes $\mathcal{V}_m$ and caps $\tau_m+1$ at scale $\delta/K_m+1$.

**Lemma 3.2.** Let $0 < \epsilon < \frac{1}{m}$ and $0 < \delta \leq 1$. Let $\tau_1, \ldots, \tau_m$ be $m$-transversal caps at scale $\delta/K_m$ and let $\tau$ be a cap at scale $\delta$ that contains them. Then, for all $\mathcal{V}_m \subset \mathbb{R}^d$ and $a \in \mathbb{R}^d$,

$$\int_{a+K_m+1} \left( \prod_{k=1}^{m} |T_{r_k} g|^{\frac{1}{q}} \cdot \zeta_{K_m, \tau} \right)^q \lesssim c(K_m^{-m})^{K_m+1+1} \left( \sum_{v \in \mathcal{V}_m} \left( |T_v g| \cdot \zeta_{v'} \right)^2 (a) \right)^{q/2} \left( \int_{a+K_m+1} \|g\|_{L^2}^q \right)^{\frac{q}{2}}.$$  

Proof. Denoting $q := \frac{2m}{m+1}$, by the trivial bound $\|T_{r_m+1} g\|_{L^\infty} \leq K_m^{-n/2} \|g\|_{L^2}$, and the definition of the $K_m$, this would follow from the slightly stronger estimate

$$\int_{a+K_m+1} \left( \prod_{k=1}^{m} |T_{r_k} g|^{\frac{1}{q}} \cdot \zeta_{K_m, \tau} \right)^q \lesssim c(K_m^{-m})^{K_m+1+1} \left( \sum_{v \in \mathcal{V}_m} \left( |T_v g| \cdot \zeta_{v'} \right)^2 (a) \right)^{q/2} \left( \frac{K_m}{K_m+1} \right)^{\frac{q}{2}} K_m^{-m} \max_{v \in \mathcal{V}_m} \|T_v g\|_{L^\infty}^q.$$  

By scaling as in the proof of the forthcoming Lemma 4.3, it will be enough to prove this with $\delta = 1$, so we can replace $a+K_m+1 \tau'$ by $B_{K_m+1}$ centred at $a$. By Hölder’s inequality and Fubini’s theorem, we see that

$$\int_{B_{K_m+1}} \left( \prod_{k=1}^{m} |T_{r_k} g((x, t) - y_k)|^{\frac{1}{q}} \cdot \zeta_{K_m, \tau} \cdot \zeta_{K_m, \tau'}^{-(1/q)} (y_k) \right) dy_k dx dt \lesssim \int_{B_{K_m+1}} \left( \prod_{k=1}^{m} |T_{r_k} g((x, t) - y_k)|^{\frac{2}{m}} dx dt \right) w(y) dy,$$

where $\prod_{k=1}^{m} \zeta_{K_m, \tau} (y_k) dy_1 \ldots dy_m =: w(y) dy$. Then by Proposition 2.2 (with $K = K_{m+1}$ and $\theta = K_{m-1}$), Hölder’s inequality and Fubini, this is bounded by a constant multiple of

$$c(K_m^{-m})^{K_m+1} \left( \int_{B_{K_m+1}} \left( \sum_{v \in \mathcal{V}_m} |T_v g((x, t) - y_k)|^2 \right) \frac{dx dt}{w(y) dy} \right)^q \lesssim c(K_m^{-m})^{K_m+1} \int_{B_{K_m+1}} \left( \sum_{v \in \mathcal{V}_m} |T_v g((x, t) - y_k)|^2 \right) \frac{dx dt}{w(y) dy}. $$
By Hölder’s inequality again and the reproducing formula, we can bound this as
\[
\leq c(K_m^{-m})K_{m+1}^\varepsilon \int_{B_{K_{m+1}}} \prod_{k=1}^m \left( \sum_{v \in V_{r,m}} \left( |T_v g| * \zeta_{\tau'} \right)^2 \right)^{q/2} \frac{dxdt}{K_{m+1}^{n/2}} \left| \zeta_{K_{m+1}}(x,t) \right| \psi \leq c(K_m^{-m})K_{m+1}^\varepsilon \int_{B_{K_{m+1}}} \left( \sum_{v \in V_{r,m}} \left( |T_v g| * \zeta_{\tau'} \right)^2 \right)^{q/2} \frac{dxdt}{K_{m+1}^{n/2}} \left| \zeta_{K_{m+1}}(x,t) \right| \psi.
\]
Finally we can apply Lemma 6.3 of the appendix, with \( K = K_{m+1} \) and \( K' = K_m \), to conclude that this is bounded by a constant multiple of
\[
c(K_m^{-m})K_{m+1}^\varepsilon \left( \sum_{v \in V_{r,m}} \left( |T_v g| * \zeta_{\tau'} \right)^2 \right)^{q/2} + \left( \frac{K_m}{K_{m+1}} \right)^{n/2} \max_{v} ||T_v g||_{L^\infty}^q.
\]
The chain of inequalities yields (3.5) and hence the result. □

**Property 3.1.** It is clear that \( \Phi_{\tau, \tau', \tau^m_{m+1}} \) is essentially constant on translates of \( K_m \tau' \). Given that by definition \( K_{m+1} \leq K_m^\varepsilon \), Lemma 3.2 yields
\[
\int_{a+K_{m+1}\tau'} \Phi_{\tau, \tau', \tau^m_{m+1}} \lesssim K_{m+1}^{2\varepsilon},
\]
where \( m = 2, \ldots, d-1 \). By Hölder’s inequality this also implies that
\[
\int_{a+K_{m+1}\tau'} \Phi_{\tau, \tau', \tau^m_{m+1}} \lesssim K_{m+1}^{2\varepsilon},
\]
uniformly over all \( a \in \mathbb{R}^d, \tau_m \subset \mathbb{R}^d \) and \( \tau_{m+1} \subset \tau \) at scale \( \delta/K_{m+1} \).

We could have convolved both sides of (3.3) with \( \zeta_{\tau'} \), before introducing the function \( \Phi_{\tau, \tau', \tau^m_{m+1}} \). In order to then replace the double convolutions on the right-hand side by single convolutions we again use Lemma 6.3 of the appendix. Introducing \( \Phi_{\tau, \tau, \tau_{m+1}} \) after this process, we can also write
\[
|T_v g| * \zeta_{\tau'} \lesssim K_d^2 \max_{\tau_{m+1} \subset \tau} \prod_{k=1}^d |T_{\tau_k} g| \psi \zeta_{\tau_k} + \phi_{\tau, \tau, \tau_{m+1}} \left( \sum_{v \in V_{r,m} \cup \{\tau_{m+1}\}} \left( |T_v g| * \zeta_{\tau'} \right)^2 \right)^{1/2} + \mathcal{R}_{\tau} \tag{3.6}
\]
As the terms on the right-hand side have the same form as the left-hand side at a different scale, we can iterate this inequality to obtain the following theorem. From now on we write \( \tau \sim \delta/K \) if \( \tau \) is a cap at scale \( \delta/K \).

**Definition 3.3.** Define \( \Psi \psi \) recursively by
\[
\Psi_1 := 1, \quad \Psi_v := \Psi_{\tau, \tau_{m+1}} \Phi_{\tau, \tau_{m+1}} \Psi_1 \quad v \sim 1, \quad v \subset \tau, \quad \psi \sim \delta/K_{m+1}, \quad \psi \sim \delta.
\]
We keep track of the maximal number of caps in the following sets \( E_5 \) as this information is used when proving linear restriction estimates. However the cardinality will have no consequence in this article - it will only be important that the caps of these sets are disjoint.
Proposition 3.4. Let $0 < \varepsilon < \frac{1}{4d}$ and let $S = \{((\xi, \phi(\xi)) : |\xi| \leq 1/2\}$. Then, for all $N \in \mathbb{N}$,

$$|T_S g| \lesssim_N K_2^{2d} \sum_{K_2^{-N} \leq \delta \leq 1} \max_{E_\delta} \left( \sum_{\tau \in E_\delta} \Psi_\tau^2 \left( \max_{\tau_1, \ldots, \tau_d \subset \tau} \prod_{k=1}^d |T_{\tau_k} g|^{\frac{1}{2}} \ast \zeta_{\tau_k} \right)^2 \right)^{1/2}$$

$$+ \sum_{K_2^{-N} \leq \delta \leq 1} \max_{E_\delta} \left( \sum_{\tau \in E_\delta} \Psi_\tau^2 \left( \max_{\tau_1, \ldots, \tau_d \subset \tau} (|T_{\tau_1} g| \ast \zeta_{\tau_1})^2 \right)^{1/2} \right)^{1/2}$$

$$+ \sum_{K_2^{-N} \leq \delta \leq 1} \max_{E_\delta} \left( \sum_{\tau \in E_\delta} \Psi_\tau^2 |T_{\tau} g| \ast \zeta_{\tau} \ast \zeta_{\tau} \right)^{1/2} R^{-1/\varepsilon} \|g\|_{L^2},$$

(3.7)

provided $\text{supp} \ g \subset \{\xi \in \mathbb{R}^{d-1} : |\xi| \leq 1/2\}$. Here $\delta$ is restricted to taking values of the form $K_2^{-\gamma_1} \ldots K_2^{-\gamma_d}$ with $\gamma_1, \ldots, \gamma_d \in \mathbb{N} \cup \{0\}$ and $\tau_1, \ldots, \tau_d$ are $d$-transversal caps at scale $\delta/K_d$. The sets $E_\delta$ consist of at most $4N \delta^{2-d}$ disjoint caps at scale $\delta$.

Proof. When $N = 1$, there is only one term in the sum over $K_2^{-N} \leq \delta \leq 1$ and the inequality follows from (3.4) at scale one. So we proceed by induction on $N$.

Suppose the inequality is true for $N$. Note that if it were not for the upper bound on $\delta$ in the second sum on the right-hand side, the inequality with $N + 1$ would immediately follow from the $N$th version. Thus it remains to bound the part of the sum that appears in the $N$th version that does not appear in the version with $N + 1$;

$$\sum_{K_2^{-(N+1)} \leq \delta \leq K_2^{-N}} \max_{E_\delta} \left( \sum_{\tau \in E_\delta} \Psi_\tau^2 \left( |T_{\tau} g| \ast \zeta_{\tau} \right)^2 \right)^{1/2}.$$ 

Applying (3.6) to the summands, this is bounded by a constant multiple of

$$\sum_{K_2^{-(N+1)} \leq \delta \leq K_2^{-N}} \max_{E_\delta} \left( \sum_{\tau \in E_\delta} \Psi_\tau^2 \left( K_2^{2d} \max_{\tau_1, \ldots, \tau_d \subset \tau} \prod_{k=1}^d |T_{\tau_k} g|^{\frac{1}{2}} \ast \zeta_{\tau_k} \right)^2 \right)^{1/2}$$

$$+ \sum_{K_2^{-(N+1)} \leq \delta \leq K_2^{-N}} \max_{E_\delta} \left( \sum_{\tau \in E_\delta} \Psi_\tau^2 \left( \sum_{m=1}^{d-1} \sum_{\tau_m, \tau_{m+1}} \max_{\tau \in V_{\tau_m, \tau_{m+1}}} \sum_{\xi \in \mathcal{V}_{\tau_m, \tau_{m+1}}} (|T_{\xi} g| \ast \zeta_{\tau})^2 \right)^{1/2} \right)^{1/2}$$

$$+ \sum_{K_2^{-(N+1)} \leq \delta \leq K_2^{-N}} \max_{E_\delta} \left( \sum_{\tau \in E_\delta} \Psi_\tau^2 R_{\tau}^2 (g) \right)^{1/2}.$$ 

Here $\tau_1, \ldots, \tau_d$ are $d$-transversal caps of size $\delta/K_d$ and $V_{\tau_m}$ is the set of all the caps $\nu \subset \tau$ of size $\delta/K_{m+1}$ and such that $\text{dist}(Y(\xi), V_m) \leq \delta/K_{m+1}$ for some $\xi$ in the orthogonal projection of $\nu$. The first term is clearly acceptable and, by the definitions of $\Psi_\tau$ and $R_{\tau}$, we can bound the other two as

$$\sum_{K_2^{-(N+1)} \leq \delta \leq K_2^{-N}} \max_{E_\delta} \left( \sum \sum_{m=2}^{d-1} \sum_{\tau_1, \ldots, \tau_d \subset \tau} \Psi_\tau^2 (|T_{\tau_1} g| \ast \zeta_{\tau_1})^2 \right)^{1/2}$$

$$+ \sum_{K_2^{-(N+1)} \leq \delta \leq K_2^{-N}} \max_{E_\delta} \left( \sum_{\tau \in E_\delta, \nu \subset \tau} \Psi_\tau^2 R_{\tau}^{-2/\varepsilon} \|g\|_{L^2}^2 \right)^{1/2}.$$ 

Using the induction hypothesis again, there are at most

$$4^N \delta^{2-d} (\delta/(\delta/K_m))^{d-2} = 4^{N+1} (\delta/K_m)^{2-d}$$
We can of course do this in such a way that a constant multiple of the averaged property. Then we have
\[ \max_{E_d} \left( \sum_{v \in E_d} \Psi^2_v (|T_v g| \ast \zeta_v)^2 \right)^{1/2} \]
\[ + \sum_{K_2^{-N+1} < \delta \leq K_2^{-N+1}} \max_{E_d} \left( \sum_{v \in E_d} \Psi^2_v \right)^{1/2} R^{-1/2} \|g\|_{L^2}. \]

This is also acceptable and so the proof is complete. \( \square \)

**Definition 3.5.** If \( \tau \) is a cap at scale \( K_2^{-\gamma_2} \cdots K_d^{-\gamma_d} \) we write \( l(\tau) := \sum_{j=2}^d \gamma_j \).

The functions \( \Psi_\tau \) also have good essentially constant properties, that we record in the following proposition.

**Proposition 3.6.** Let \( 0 < \varepsilon < \frac{1}{16} \). Then the functions \( \Psi_\tau \) are essentially constant at scale one. Moreover, for all \( a \in \mathbb{R}^d \),
\[ \int_{a + \tau'} |x| dx dt \lesssim l(\tau) |\tau'|^\varepsilon. \]

**Proof.** The essentially constant property is an immediate consequence of the definition and the corresponding property for \( \Phi_v \) with \( v \subset \tau \), so it remains to prove the averaged property.

If \( \tau \sim 1 \), then \( \Psi_\tau \) := 1 and the estimate is trivially satisfied. If \( \tau \sim 1/K_{m+1} \), then \( \Psi_v := \max_{\tau_v, v, \tau_{m+1}} \Phi_{\tau_v, v, \tau_{m+1}} \) where \( v \subset \tau \sim 1 \) and we can cover \( a + v' \) with a family of translates of \( K_{m+1}^{\tau'} \) which are essentially balls \( B_j \) of diameter \( K_{m+1} \).

We can of course do this in such a way that
\[ \bigcup_j B_j \subset a + 4v'. \]

Then we have
\[ \int_{a + v'} \Psi^q_v \lesssim \sum_j \int_{B_j} \max_{\tau_v, v, \tau_{m+1}} \Phi^q_{\tau_v, v, \tau_{m+1}}. \]
\[ \lesssim \sum_j |B_j| \left( \int_{B_j} \max_{\tau_v, v, \tau_{m+1}} \Phi^q_{\tau_v, v, \tau_{m+1}} \right). \]

Recalling Remark 3.3 in fact we have the same \( \tau_v \) for all \( (x, t) \in B_j \). Similarly as \( \Phi_{\tau_v, v, \tau_{m+1}} \) is essentially constant on \( B_j \) we can suppose that the maximum is attained on the same \( \tau_{m+1} \) for a given \( B_j \). Thus, taking \( q = \frac{2(d-1)}{d-2} \), by Property 3.3 we obtain
\[ \int_{a + v'} \Psi^q_v \lesssim \sum_{B_j} |B_j| K_{m+1}^{2\varepsilon} \lesssim |v'| K_{m+1}^{2\varepsilon} \leq |v'|^{1+\varepsilon} \]
as claimed.

We have proved the proposition for \( \tau \) such that \( l(\tau) = 0 \) or 1. Thus we can proceed by induction on this quantity. Supposing that we have the estimate for \( \tau \) such that \( l(\tau) = N \), it will suffice to prove the estimate for \( v \) such that \( l(v) = N+1 \). That is we suppose that
\[ \int_{a + \tau'} \Psi^q_v \lesssim |\tau'|^\varepsilon, \quad a \in \mathbb{R}^d, \quad (3.8) \]
and attempt to prove the same for \( v \) at scale \( \delta/K_{m+1} \) such that \( v \subset \tau \) at scale \( \delta \). We cover \( a + v' \) with a family \( \{ T_\ell \} \) of pairwise disjoint translates of \( \tau' \) with centres at \((x_\ell, t_\ell)\). We can do this in such a way that
\[
\bigcup_\ell T_\ell \subset a + 2v'.
\]

As \( \Phi^q_{\tau,Y_m,T_{m+1}} \) is essentially constant on \( T_\ell \), we have
\[
\int_{a + v'} \Psi^q_v \leq \sum_\ell \int_{T_\ell} \Phi^q_{\tau,Y_m,T_{m+1}} \max_{v \in T_{m+1}} (x_\ell, t_\ell) |T_\ell| \int_{T_\ell} \Psi^q_v.
\]

Then, by the induction hypothesis (3.8), we see that
\[
\int_{a + v'} \Psi^q_v \leq |\tau'| |x_\ell| \sum_\ell \max_{v \in T_{m+1}} (x_\ell, t_\ell) |T_\ell| \leq |\tau'| \sum_\ell \max_{v \in T_{m+1}} \Phi^q_{\tau,Y_m,T_{m+1}}.
\]

We are now in a similar position as in the case \( l(\tau) = 1 \). We cover \( \bigcup_j T_j \) with a family \( \{ T_j \} \) of disjoint translates of \( K_{m+1} \tau' \). As the angle between \( Y_\nu \) and \( Y_\tau \) is bounded by \( \delta \), elementary trigonometry tells us that we can do this so that
\[
\bigcup_j T_j \subset a + 4v'.
\]

Thus, by Remark 3.3 and Property 3.1,
\[
\int_{a + v'} \Psi^q_v \leq |\tau'| \sum_j |T_j| \max_{v \in T_{m+1}} \int_{T_j} \Phi^q_{\tau,Y_m,T_{m+1}} \leq |\tau'| K_{m+1}^{2\epsilon} \sum_j |T_j| \leq |\tau'| K_{m+1}^{2\epsilon} |v'| \leq |v'|^{1+\epsilon}
\]
where in the final inequality we used that \( |v'|^{1+\epsilon} = |\tau'|^{\epsilon} K_{m+1}^{d(\epsilon+1)} |u'| \), and so the proof is complete.

Returning to the decomposition (3.7), we stop the iteration at the biggest value of \( N \) such that \( K_N^N K_d < R^\lambda \), where \( \lambda > 0 \), so that
\[
|T_S g| \lesssim R^\epsilon \sum_{R^{-\lambda} < \delta \leq 1} \max_{E_\delta} \left( \sum_{\tau \in E_\delta} \left( \max_{1 \leq r \leq d} \prod_{k=1}^d |T_k g|^{1/2} \right)^2 \right)^{1/2}
\]
\[
+ \sum_{R^{-\lambda} < \delta \leq R^{-\lambda+\epsilon}} \max_{E_\delta} \left( \sum_{\tau \in E_\delta} \left( \max_{1 \leq r \leq d} \prod_{k=1}^d |T_k g|^{1/2} \right)^2 \right)^{1/2}
\]
\[
+ \sum_{R^{-\lambda} < \delta \leq 1} \max_{E_\delta} \left( \sum_{\tau \in E_\delta} \left( \max_{1 \leq r \leq d} \prod_{k=1}^d |T_k g|^{1/2} \right)^2 \right)^{1/2} R^{-1/\epsilon} \|g\|_{L^2}.
\]  
This is what we call the Bourgain–Guth decomposition [9] pp. 1259]. Note that as \( |\tau'| < R^{d+1} \), we have
\[
\int \Psi^q_{\tau^{d-1}} \lesssim R^{(d+1)\lambda}, \quad a \in \mathbb{R}^d,
\]

Later we will dispose of the sets \( E_d \) and take the inner sums in \( \tau \) over the full partition of \( S \). The outer sum (over the scales at which the partition is taken) has
less than $\lambda(\varepsilon)$ terms in it, where $\varepsilon$ is the constant from the Bennett–Carbery–Tao extension estimate. The inequality recalls the way in which the Whitney decomposition can be used to take advantage of bilinear estimates, stopping at a scale for which easy estimates are available. The big difference between this and the Whitney decomposition are the functions $\Psi_{\tau}$, which have reasonably nice properties, but will prove to be something of a hindrance. Indeed, the easy estimates for the linear terms are no longer so good that we can ignore them completely. Our final bounds are obtained by compromising between the scale $\lambda$ that is good for the multilinear term and that which is good for the linear term.

4. Proof of Theorem 1.3

Letting $\sigma$ denote the surface measure on the unit sphere, by duality, the desired estimate (1.2) is equivalent to

$$
\| (f d\sigma)^\vee (R \cdot) \|_{L^2(d\mu)} \lesssim R^{-\beta/2} \sqrt{c_\alpha(\mu)} \| f \|_{L^2(\mathbb{R}^{d-1})}.
$$

Thus, by Hölder’s inequality, it will suffice to prove

$$
\| (f d\sigma)^\vee (R \cdot) \|_{L^2(d\mu)} \lesssim R^{-\beta/2} \sqrt{c_\alpha(\mu)} \| f \|_{L^2(\mathbb{R}^{d-1})}
$$

with

$$
\beta > \alpha - 1 + \frac{(d - \alpha)^2}{(d - 1)(2d - \alpha - 1)}.
$$

Defining the measure $\mu_R$ by $d\mu_R(x) = R^{\alpha} d\mu(x/R)$, it is easy to check that $c_\alpha(\mu_R) = c_\alpha(\mu)$. Then (1.3) is equivalent to

$$
\| (f d\sigma)^\vee \|_{L^2(d\mu_R)} \lesssim R^{-\beta/2} \sqrt{c_\alpha(\mu)} \| f \|_{L^2(\mathbb{R}^{d-1})}.
$$

By a finite splitting, the triangle inequality and the rotational invariance of the inequality (which holds uniformly for all $\alpha$-dimensional measures $\mu_R$) we can suppose that $\sigma$ is supported on $S = \{(\xi, \phi(\xi)) : |\xi| \leq 1/2\}$, where $\phi(\xi) = \sqrt{1 - |\xi|^2}$. Defining

$$
g(\xi) := \frac{f(\xi, \phi(\xi))}{\sqrt{1 - |\xi|^2}},
$$

we can write

$$
(f d\sigma)^\vee(x, t) = \int_{|\xi| \leq 1/2} g(\xi) e^{ix \cdot \xi + it \phi(\xi)} d\xi,
$$

so we see that (4.2) is equivalent to

$$
\| T_s g \|_{L^2(d\mu_R)} \lesssim R^{-\frac{\alpha - d}{2d - \alpha - 1}} \sqrt{c_\alpha(\mu)} \| g \|_{L^2(\mathbb{R}^{d-1})}.
$$

For this we will use the Bourgain–Guth decomposition with $\lambda = \frac{d - \alpha}{2d - \alpha - 1}$;

$$
|T_s g| \lesssim \sum_{R^{-\lambda} \leq \delta \leq R^{-\lambda + \varepsilon}} \left( \sum_{\tau \sim \delta} \left( \max_{\tau_1, \ldots, \tau_d \subset \tau} \prod_{k=1}^d [T_{\tau_k} g] \ast \zeta_{\tau_k} \right)^2 \right)^{1/2}
$$

$$
+ \sum_{R^{-\lambda} \leq \delta \leq R^{-\lambda + \varepsilon}} \left( \sum_{\tau \sim \delta} (\Psi_{\tau} |T_{\tau} g| \ast \zeta_{\tau})^2 \right)^{1/2}
$$

$$
+ \sum_{R^{-\lambda} \leq \delta \leq 1} \left( \sum_{\tau \sim \delta} \Psi_{\tau}^2 \right)^{1/2} R^{-1/2} \| g \|_{L^2(\mathbb{R}^{d-1})},
$$

which follows from estimate (3.9) by summing in $\tau$ over the full partition of $S$ at scale $\delta$ instead of over the restricted subsets $E_\delta$. 
Recalling that there are less that \( \lambda(\varepsilon) < \varepsilon \) terms in each of the \( \delta \)-sums, by the triangle inequality, we need only prove estimates which are uniform in \( \delta \). Writing \( g_\tau := g_\chi_\tau \), if we could prove

\[
\left\| \Psi_\tau[T_\tau g] \ast \zeta_\tau \right\|_{L^2(\mathcal{M}_n)} \lesssim \sqrt{c_0(\mu)} R^{\frac{d}{2} - \frac{\alpha}{4} - \frac{d\alpha}{2(2\alpha + 1)} + \varepsilon} \|g_\tau\|_2, \tag{4.5}
\]

uniformly for \( \tau \) at scale \( \delta \) with \( R^{-\lambda} \leq \delta \leq R^{-\lambda + \varepsilon} \), then using orthogonality, we could bound the middle term on the right-hand side of (4.4). Similarly, replacing the \( \max_{\tau_1, \ldots, \tau_d} \) with an \( \ell^2 \)-norm, and using the fact that there are no more than \( R^\varepsilon \) choices in such a sum, in order to treat the first term it will suffice to prove

\[
\left\| \Psi_\tau \prod_{k=1}^d |T_{\tau_k} g|^{\frac{1}{d}} \ast \zeta_{\tau_k} \right\|_{L^2(\mathcal{M}_n)} \lesssim \sqrt{c_0(\mu)} R^{\frac{d}{2} - \frac{\alpha}{4} - \frac{d\alpha}{2(2\alpha + 1)} + \varepsilon} \|g_\tau\|_2, \tag{4.6}
\]

uniformly for \( \tau \) at scale \( \delta \) with \( R^{-\lambda} \leq \delta \leq 1 \) and uniformly for choices of transversal caps \( \tau_1, \ldots, \tau_d \subset \tau \). In fact we will only prove this for \( \alpha > 1 \) however we can safely ignore the other cases as Mattila already proved the sharp bound for \( \beta_d \) in low dimensions [27]. Finally, in order to deal with the remainder term, by taking \( \varepsilon \) sufficiently small, it will suffice to prove that

\[
\|\Psi_\tau\|_{L^2(\mathcal{M}_n)} \lesssim \sqrt{c_0(\mu)} R^{d/2 + \lambda}, \tag{4.7}
\]

uniformly for \( \tau \) at scale \( \delta \) with \( R^{-\lambda} \leq \delta \leq 1 \). Taking for granted the proofs of (4.5), (4.6) and (4.7), which we will present in the forthcoming lemmas, starting with the easier (4.7), this completes the proof of Theorem 1.3.

**Lemma 4.1.** Let \( 0 < \varepsilon < \frac{1}{d} \). Then, for all caps \( \tau \sim \delta \) with \( R^{-\lambda} \leq \delta \leq 1 \),

\[
\|\Psi_\tau\|_{L^2(\mathcal{M}_n)} \lesssim \sqrt{c_0(\mu)} R^{d/2 + \lambda}.
\]

**Proof.** Writing \( q = \frac{2(d-1)}{d-2} \), we prepare to use the property [38.10]. First of all, as \( \Psi_\tau \) is essentially constant at scale one, we can bound

\[
\|\Psi_\tau\|_{L^2(\mathcal{M}_n)} \lesssim \mu_{R(B_R)} \frac{1}{R^{\frac{d}{2}}} \|\Psi_\tau\|_{L^2(\mathcal{M}_n)} \lesssim c_0(\mu) R^{\frac{d}{2} - \frac{\alpha}{4} - \frac{d\alpha}{2(2\alpha + 1)}} \|\Psi_\tau\|_{L^2(\mathcal{M}_n)} = \sqrt{c_0(\mu)} R^{\frac{d}{2} - \frac{\alpha}{4}} \|\Psi_\tau\|_{L^2(\mathcal{M}_n)}.
\]

Covering \( B_R \) with a family \( \{T_j\} \) of translates of \( \tau' \) with disjoint interiors, cuboids of dimension \( \delta^{-1} \times \cdots \times \delta^{-1} \times \delta^{-2} \), we can then bound this as

\[
\|\Psi_\tau\|_{L^2(\mathcal{M}_n)} \lesssim \sqrt{c_0(\mu)} R^{\frac{d}{2} - \frac{\alpha}{4}} \left( \sum_j \|\Psi_\tau\|_{L^2(T_j)}^q \right)^{1/q} \lesssim \sqrt{c_0(\mu)} R^{\frac{d}{2} - \frac{\alpha}{4}} \left( \sum_j |T_j|^{\frac{d}{2} - \frac{\alpha}{4}} \right)^{1/q} \lesssim \sqrt{c_0(\mu)} R^{\frac{d}{2} - \frac{\alpha}{4}} \delta^{-\frac{(d+1)\alpha}{4}},
\]

where the second inequality is by Proposition 3.6. For the range of \( \delta \) under consideration, this is easily enough to give the stated bound. \( \square \)

**Lemma 4.2.** Let \( 0 < \varepsilon < \frac{1}{2d} \). Then, for all caps \( \tau \sim \delta \) with \( R^{-\lambda} \leq \delta \leq R^{-\lambda + \varepsilon} \),

\[
\left\| \Psi_\tau[T_\tau g] \ast \zeta_\tau \right\|_{L^2(\mathcal{M}_n)} \lesssim \sqrt{c_0(\mu)} R^{\frac{d}{2} - \frac{d\alpha}{2(2\alpha + 1)} + \varepsilon} \|g_\tau\|_2. \tag{4.8}
\]
Proof. Again we cover $B_R$ by a family $\{T_j\}$ of translations of $\tau'$ with disjoint interiors. Setting $G_\tau := |T_\tau g| \ast \zeta_{\tau'}$, and denoting the measure $d\mu_R$ restricted to $T_j$ by $d\mu_R^j$, we can write
\[
\| \Psi \tau G_\tau \|_{L^2(d\mu_R^j)} = \left( \sum_j \| \Psi \tau G_\tau \|_{L^2(d\mu_R^j)}^2 \right)^{1/2}. \tag{4.9}
\]
As in the previous lemma, we use that $\Psi \tau$ is essentially constant at scale one, so
\[
\| \Psi \tau \|_{L^2(d\mu_R^j)} \lesssim \mu_R(T_j)^{\frac{1}{2} - \frac{d}{4}} \| \Psi \tau \|_{L^2(T_j)} \lesssim \mu_R(T_j)^{\frac{1}{2} - \frac{d}{4}} \| \Psi \tau \|_{L^2(T_j)} \lesssim c_\alpha(\mu)^{\delta} \mu_R(T_j)^{\frac{1}{2} - \frac{d}{4}}|T_j|^{\frac{1}{2}},
\]
where the final inequality is by the property (3.10). Using this and the fact that $G_\tau$ is essentially constant on $T_j$,
\[
\| \Psi \tau G_\tau \|_{L^2(d\mu_R^j)} \lesssim c_\alpha(\mu)^{\delta} \mu_R(T_j)^{\frac{1}{2} - \frac{d}{4}}|T_j|^{\frac{1}{2}} \left( \sum_j \| G_\tau \|_{L^2(T_j)}^2 \right)^{1/2}
\]
Plugging into (4.9), we obtain
\[
\| \Psi \tau G_\tau \|_{L^2(d\mu_R^j)} \lesssim c_\alpha(\mu)^{\delta} \mu_R(T_j)^{\frac{1}{2} - \frac{d}{4}}|T_j|^{\frac{1}{2}} \left( \sum_j \| G_\tau \|_{L^2(T_j)}^2 \right)^{1/2}
\]
where in the second inequality we use $\mu_R(T_j) \lesssim c_\alpha(\mu)\delta^{-(\alpha+1)}$ which follows by covering the $T_j$ by $\delta^{-1}$ balls of radius $\delta^{-1}$.

On the other hand, by Minkowski’s integral inequality, we can bound
\[
\| G_\tau \|_{L^2(B_R^j)} = \| T_\tau g \ast \zeta_{\tau'} \|_{L^2(B_R^j)} \leq \int \| T_\tau g(\cdot - y) \|_{L^2(B_R^j)} \zeta_{\tau'}(y) \, dy
\]
\[
\leq \left\| \left\| g_y \right\|_{L^2(\mathbb{R}^{d-1})} \right\|_{L^2(|t| \leq R)} \zeta_{\tau'}(y) \, dy,
\]
where $g_y(\xi) := g(\xi) \chi_{\tau}(\xi)e^{-i\pi(y)\xi + i(t-t_y)\phi(\xi)}$, $t_y := y - \pi(y)$.

Here $\pi$ is the orthogonal projection onto $\mathbb{R}^{d-1}$. Then by Plancherel’s theorem, the fact that $\|g_y\|_2 = \|g_r\|_2$, and the fact that the integral of $\zeta_{\tau'}$ is bounded, we obtain
\[
\| G_\tau \|_{L^2(B_R^j)} \lesssim R^{1/2}\|g_r\|_2.
\]

Plugging this into (4.10), we see that
\[
\| \Psi \tau |T_\tau g| \ast \zeta_{\tau'} \|_{L^2(d\mu_R^j)} \lesssim c_\alpha(\mu)R^{\frac{1}{2} - \frac{d}{4} - \frac{1}{2}}\delta^{\frac{d-\alpha}{2}} \|g_r\|_2, \tag{4.11}
\]
which, with $R^{-\lambda} \leq \delta \leq R^{-\lambda+\epsilon}$, yields the desired uniform estimate. □

Lemma 4.3. Let $0 < \epsilon < \frac{1}{2}$ and $\alpha > 1$ and $\lambda = \frac{d-\alpha}{2d-2}$. Then, for all caps $\tau \sim \delta$ with $R^{-\lambda} \leq \delta \leq 1$ and all $d$-transversal caps $\tau_1, \ldots, \tau_d \sim \delta/K_d$ contained in $\tau$,
\[
\| \Psi \tau \prod_{k=1}^d |T_{\tau_k} g|^{\frac{1}{2}} \ast \zeta_{\tau_k} \|_{L^2(d\mu_R)} \lesssim c_\alpha(\mu)R^{\frac{1}{2} - \frac{d(\alpha-\epsilon)}{2d-2} + \frac{d-\alpha}{2}} \|g_r\|_2. \tag{4.12}
\]
Proof. Setting $G_\tau := \prod_{k=1}^d |T_{\tau_k} g|^{\frac{1}{d}} \ast \zeta_{\tau_*}^\flat$, we will prove that

$$\| \Psi_\tau G_\tau \|_{L^2(\mu_R)} \lesssim \sqrt{c_0(\mu)} R^{\frac{d}{2} \delta} R_{\tau} \delta^{\frac{d}{2} \delta} \| g_\tau \|_2,$$  \hspace{1cm} (4.13)

which on can calculate gives the required bound for $\delta \geq R^{-\frac{1}{d} \frac{d}{2} \delta}$. By Hölder’s inequality with $p = \frac{d}{2} \delta$, we first note that

$$\| \Psi_\tau G_\tau \|_{L^2(\mu_R)} \lesssim c_0(\mu)^{\frac{1}{2} \frac{d}{2} \delta} R^{\frac{d}{2} \delta} \| \Psi_\tau G_\tau \|_{L^p(\mu_R)}$$

$$= c_0(\mu)^{\frac{1}{2} \frac{d}{2} \delta} R^{\frac{d}{2} \delta} \left( \sum_j \| \Psi_\tau G_\tau \|_{L^p(\mu_R)} \right)^{1/p},$$

where $d\mu_R^j$ denotes the measure $d\mu_R$ a member of the cover of $B_R$ by translates of $\tau'$. Using that $\Psi_\tau$ is essentially constant at scale one,

$$\| \Psi_\tau \|_{L^p(\mu_R)} \lesssim \mu(T_j)^{\frac{1}{2} \frac{d}{2} \delta} \| \Psi_\tau \|_{L^\infty(\mu_R)}$$

$$\lesssim \mu(T_j)^{\frac{1}{2} \frac{d}{2} \delta} c_0(\mu)^{\frac{1}{2} \frac{d}{2} \delta} \| \Psi_\tau \|_{L^\infty(T_j)}$$

$$\lesssim \mu(T_j)^{\frac{1}{2} \frac{d}{2} \delta} c_0(\mu)^{\frac{1}{2} \frac{d}{2} \delta} |T_j|^{\frac{1}{2} \frac{d}{2} \delta} R^{\frac{d}{2} \delta}.$$ 

where the final inequality is by the property (3.10). As $\tau' \subset \tau'_k$ we still have that that $G_\tau$ is essentially constant on $T_j$, so that

$$\| \Psi_\tau G_\tau \|_{L^p(\mu_R)} \lesssim \mu(T_j)^{\frac{1}{2} \frac{d}{2} \delta} c_0(\mu)^{\frac{1}{2} \frac{d}{2} \delta} |T_j|^{\frac{1}{2} \frac{d}{2} \delta} R^{\frac{d}{2} \delta} \| G_\tau \|_{L^\infty(T_j)}$$

$$\lesssim \mu(T_j)^{\frac{1}{2} \frac{d}{2} \delta} c_0(\mu)^{\frac{1}{2} \frac{d}{2} \delta} \tau d \delta^{\frac{d}{2} \delta} \| G_\tau \|_{L^\infty(T_j)}$$

$$\lesssim c_0(\mu)^{\frac{1}{2} \frac{d}{2} \delta} \tau d \delta^{\frac{d}{2} \delta} \| G_\tau \|_{L^\infty(T_j)}$$

Plugging this into (4.11), we obtain

$$\| \Psi_\tau G_\tau \|_{L^2(\mu_R)} \lesssim \sqrt{c_0(\mu)} R^{\frac{d}{2} \frac{d}{2} \delta} R_{\tau} \delta^{\frac{d}{2} \delta} \| G_\tau \|_{L^p(\mu_R)}.$$

In order to bound $\| G_\tau \|_{L^p(B_R)}$, we write

$$G_\tau(x,t) = \prod_{k=1}^d \int |T_{\tau_k} g|^{\frac{1}{d}} ((x,t) - y_k) \zeta_{\tau_*}^\flat(y_k) \, dy_k$$

$$= \prod_{k=1}^d \int |T_{\tau_k} g|^{\frac{1}{d}} (x,t) \zeta_{\tau_*}^\flat(y_k) \, dy_k,$$

where this time

$$g_{y_k} := g \chi_{\tau_k} e^{-i\pi(y_k) \cdot \xi - u_k \cdot \phi(\xi)}, \hspace{1cm} t_k := y_k - \pi(y_k).$$

Then, by Minkowski’s integral inequality, it will suffice to bound

$$\int \left\| \prod_{k=1}^d |T_{\tau_k} g|^{\frac{1}{d}} \right\|_{L^p(B_R)} \prod_{k=1}^d \zeta_{\tau_*}^\flat(y_k) \, dy_1 \ldots dy_d.$$

Again $\|g_{y_k}\|_2 = \|g_{\tau_*}\|_2$, and so it remains to prove the multilinear extension estimate

$$\left\| \prod_{k=1}^d |T_{\tau_k} g|^{\frac{1}{d}} \right\|_{L^p(B_R)} \lesssim R^d \delta^{-\frac{d}{2} \delta} \| g_\tau \|_2.$$

(4.15)

We recall that $\tau_k$ are traversal caps at scale $\delta/K_d$ and so a direct application of Theorem 2.1 would give us the inequality with the constant $c(\delta^d K_d^{-d})$. We do not know how large this is, however we have chosen the scales so that at least we know
that $c(K_{d}^{-d}) \leq R^{-2}$. Thus, using the fact the caps $\tau_{k}$ are contained in $\tau$ at scale $\delta$, we can first modulate and scale the inequality in order to get into this situation.

Denoting by $\xi_{0}$ the center of $\pi(\tau) = Q$ we let $\tilde{Q}_{k}$ be the scaled versions of $Q_{k}$ which are first translated by $-\xi_{0}$. Indeed, introducing new variables, 

$$(x', t') = (\delta x, \delta^{2} t), \quad \xi - \xi_{0} = \delta \xi',$$

and writing 

$$f(\xi') := \delta^{-d+1} g(\xi_{0} + \delta \xi'),$$

so that $\|f\|_{2} = \|g\|_{2}$, it is trivial to calculate that 

$$|T_{\tau} g(x, t)| = \delta^{-d+1} \left| \tilde{T}_{\tau} f(x' + \delta^{-1} \nabla \phi(\xi_{0}) t', t') \right|,$$

where 

$$\tilde{T}_{\tau} f(x, t) := \int_{\tilde{Q}_{k}} e^{i(x \cdot \xi + it S_{\xi_{0}, \delta \phi(\xi_{0})}) f(\xi)} d\xi$$

and the scaled phase is given by 

$$S_{\xi_{0}, \delta \phi(\xi')} = \delta^{-2} \left( \phi(\xi_{0} + \delta \xi') - \delta \nabla \phi(\xi_{0}) \cdot \xi' - \phi(\xi_{0}) \right).$$

The $d$-transversal caps $\tilde{\tau}_{k}$ satisfying $\pi(\tilde{\tau}_{k}) = \tilde{Q}_{k}$ are now at scale $1/K_{d}$. Writing 

$$\prod_{k=1}^{d} |T_{\tau} g|^{\frac{1}{p}}(x, t) = \delta^{-d+1} \prod_{k=1}^{d} \left| \tilde{T}_{\tau} f \right|^{\frac{1}{p}}(x' + \delta^{-1} \nabla \phi(\xi_{0}) t', t'),$$

we see that the left-hand side of $\ref{eq:14.15}$ is bounded by 

$$\delta^{-\frac{d}{2} - \frac{d-1}{p}} \left( \int_{|t'| \leq \delta R} \int_{|x'| \leq \delta R} \left| \prod_{k=1}^{d} \left| \tilde{T}_{\tau} f \right|^{\frac{1}{p}}(x' + \delta^{-1} \nabla \phi(\xi_{0}) t', t') \right|^{p} dx' dt' \right)^{1/p}$$

$$\leq \delta^{-\frac{d}{2} - \frac{d-1}{p}} \left\| \prod_{k=1}^{d} \left| \tilde{T}_{\tau} f \right|^{\frac{1}{p}} \right\|_{L^{p}[0, \delta^{2} R \times B_{3 \delta R}]}.$$

Here, we change variables $x = x' + \delta^{-1} \nabla \phi(\xi_{0}) t'$ and use that $\delta^{-1} \nabla \phi(\xi_{0}) t'$ is bounded above by $\delta R$ so that the oblique tube can be covered by the fatter cylinder. Now, by Proposition \ref{prop:2.8}, 

$$\left\| \prod_{k=1}^{d} \left| \tilde{T}_{\tau} f \right|^{\frac{1}{p}} \right\|_{L^{p}[0, \delta^{2} R \times B_{3 \delta R}]} \leq c(\varepsilon)(\delta^{2} R)^{\varepsilon} \|f\|_{2},$$

and so altogether we get $\ref{eq:14.15}$, which completes the proof. \hfill \Box

The conjectured $m$-linear extension estimates [7, Conjecture 4], with $m \leq d - 1$, combined with the arguments of this section, would yield 

$$\beta_{d}(\alpha) \geq \min \left\{ \alpha - 1 + \frac{(d - \alpha)(d + m - 2\alpha)}{2(m-1)(d+\alpha-1)}, \alpha - \frac{2\alpha}{d+m} \right\},$$

whenever $3 \leq m \leq d - 1$. Comparing the second term in the minimum with the bound of Theorem \ref{thm:1.3} it is clear that this is not an improvement for larger $\alpha$. However, by taking $m = d/2 + 1$ (assuming that $d$ is even), this would improve our bound and Erd\'os’s in a neighbourhood of $\alpha = d/2 + 3$. It would not be sufficient to improve the state-of-the-art for Falconer’s conjecture however. Using the partial results for $m$-linear restriction already proven in [4, formula (40)], by the same argument one obtains 

$$\beta_{d}(\alpha) \geq \min \left\{ \alpha - 1 + \frac{(d - \alpha)(m - \alpha)}{(m-1)(2m-\alpha-1)}, \alpha - \frac{\alpha}{m} \right\}.$$
gained by using these. The reason that they can be effective is that the decomposition of Bourgain and Guth improves if we take the initial dichotomy at a lower level of multilinearity. The improvement manifests itself in the fact that the functions $Ψ_{τ}$ have better integrability properties and so we pay less while removing them. This kind of thing was first observed by Temur in the context of the linear restriction problem \[38\]. Here, the reduced integrability in the estimates leads to both \[4.13\] and \[4.16\] having a worse dependency on $R$ (this produces the second term in the minimum), however the improved properties of $Ψ_{τ}$ lead to both \[4.13\] and \[4.16\] having a better dependency on $δ$, and together they would yield \[4.17\] after choosing the limiting scale $λ$ in an optimal fashion.

5. Proof of Proposition 1.4

In order to avoid repetition in the following section, we consider $m \geq 1$, however it will suffice to consider $m = 1$ here. If $v_0$ and $v_1$ are in the Schwartz class then the solution $v$ to the wave equation with this initial data can be written as

$$v(\cdot, t) = \cos(t\sqrt{-\Delta})v_0 + \sin(t\sqrt{-\Delta})v_1 = e^{it\sqrt{-\Delta}}f_+ + e^{-it\sqrt{-\Delta}}f_-.$$ 

Here $f_+ = \frac{1}{2}(v_0 - iI_1 \ast v_1)$ and $f_- = \frac{1}{2}(v_0 + iI_1 \ast v_1)$, where $I_1$ is the Riesz kernel, and

$$e^{it(-\Delta)^{m/2}}f(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix \cdot \xi + it|\xi|^m} d\xi.$$ 

For data in $\dot{H}^s \times \dot{H}^{s-1}$, both $f_+$ and $f_-$ belong to $\dot{H}^s$, however this integral does not necessarily exist in the sense of Lebesgue for $s \leq n/2$. Instead we define $v(x, t)$ to be the pointwise limit

$$v(x, t) := \lim_{N \to \infty} S^{N, 1}_t f_+(x) + S^{N, 1}_t f_-(x),$$

whenever the limit exists, where

$$S^{N, m}_t f := \int_{\mathbb{R}^d} \psi \left( \frac{|\xi|}{N} \right) \hat{f}(\xi) e^{ix \cdot \xi + it|\xi|^m} d\xi$$

and $\psi$ is a positive Schwartz function that equals $(2\pi)^{-d/2}$ at the origin. This coincides almost everywhere with the classical solution defined via the $L^2$-limit.

Writing $\|I_s \ast f\|_{H^s} := \|f\|_2$, we know that $f_+, f_-$ and the limit \[5.1\] are well-defined with respect to fractal measures provided that $\alpha > d - 2s$ due to the inequalities

$$\|I_s \ast f\|_{L^1(\mu)} \lesssim e^{c_0(\mu)}\|\mu\| \|f\|_2,$$

$$\sup_{N \geq 1} \|S^{N, m}_s I_s \ast f\|_{L^1(\mu)} \lesssim e^{c_0(\mu)}\|\mu\| \|f\|_2;$$

see for example \[1\], \[3\] or \[29\] Chapter 17. Then by standard arguments (see for example Appendix B of \[5\]) and an application of Frostman’s lemma (see for example \[29\] Theorem 2.7)), the implication

$$\beta_d(\alpha) > d - 2s \implies \gamma_d(s) \leq \alpha$$

can be deduced from from the following lemma.

**Lemma 5.1.** Let $m \geq 1$, $d \geq 2$ and $0 < s < d/2$. Then

$$\sup_{t \in \mathbb{R}} \sup_{N \geq 1} \|S^{N, m}_s I_s \ast f\|_{L^1(\mu)} \lesssim e^{c_0(\mu)}\|\mu\| \|f\|_2$$

whenever $f \in L^2(\mathbb{R}^d)$, $\mu$ is an $\alpha$-dimensional measure and $s > \frac{d - \beta_d(\alpha)}{2}$. 
Proof. First of all we remark that the maximal function is Borel measurable by comparing with the maximum function with time restricted to the rationals; see [29, Lemma 17.7]. Then, using polar coordinates we write

\[ |S_{t}^{N,m} I_{*} f(x)| = \left| \int_{\mathbb{R}^{d}} \psi(N^{-1}|\xi|) |\xi|^{-s} \hat{f}(\xi) e^{i(x \cdot \xi + t |\xi|^{m})} d\xi \right| \]

\[ = \left| \int_{0}^{\infty} \psi(N^{-1} R) R^{d-1-s} e^{it R^{\alpha}} \int_{S^{d-1}} \hat{f}(R \omega) e^{i R x \cdot \omega} d\sigma(\omega) dR \right| \]

\[ \lesssim \int_{0}^{\infty} R^{d-1-s} \left| \int_{S^{d-1}} \hat{f}(R \omega) e^{i R x \cdot \omega} d\sigma(\omega) \right| dR, \]

so that, by Fubini’s theorem,

\[ \left\| \sup_{t \geq 0} \left| S_{t}^{N} I_{*} f \right| \right\|_{L^{1}(d\mu)} \lesssim \int_{0}^{\infty} R^{d-1-s} \left\| (\hat{f}(R \cdot) d\sigma)^{\vee} (R \cdot) \right\|_{L^{1}(d\mu)} dR. \] (5.2)

Noting that, even when \( R \) is small, we have

\[ \| \tilde{\mu}(R \cdot) \|_{L^{2}(\mathbb{S}^{d-1})} \lesssim \| \mu \|^{2} \lesssim c_{\alpha}(\mu) \| \mu \|, \]

the inequality (5.2) implies by duality that

\[ \left\| (\hat{f}(R \cdot) d\sigma)^{\vee} (R \cdot) \right\|_{L^{1}(d\mu)} \lesssim \sqrt{c_{\alpha}(\mu)} \| \mu \| (1 + R)^{-\beta/2} \| \hat{f}(R \cdot) \|_{L^{2}(\mathbb{S}^{d-1})}, \]

for all \( \beta < \beta_{d}(\alpha) \), so that (5.2) is bounded by

\[ \lesssim \sqrt{c_{\alpha}(\mu)} \| \mu \| \int_{0}^{\infty} \frac{R^{d-1-s}}{(1 + R)^{\beta/2}} \| \hat{f}(R \cdot) \|_{L^{2}(\mathbb{S}^{d-1})} dR. \]

Finally, by an application of the Cauchy–Schwarz inequality, we can continue to estimate as

\[ \lesssim \sqrt{c_{\alpha}(\mu)} \| \mu \| \left( \int_{0}^{\infty} \frac{R^{d-1-2\beta}}{(1 + R)^{\beta} dR} \right)^{1/2} \left( \int_{0}^{\infty} \left\| \hat{f}(R \cdot) \right\|_{L^{2}(\mathbb{S}^{d-1})}^{2} R^{d-1} dR \right)^{1/2} \]

\[ \lesssim \sqrt{c_{\alpha}(\mu)} \| \mu \| \| f \|_{L^{2}(\mathbb{S}^{d})}, \]

where in the final inequality we choose \( \beta \) so that \( \beta_{d}(\alpha) > \beta > d - 2s \) as we may. \( \square \)

6. Proof of the Theorem

As in the previous section, if \( i \partial_{t} u + \Delta u = 0 \) and the initial data \( u_{0} \) is in the Schwartz class, we can write

\[ u(x, t) = e^{i t \Delta} u_{0}(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n}} \hat{u}_{0}(\xi) e^{i x \cdot \xi - i t |\xi|^{2}} d\xi, \]

however for data in \( H^{\alpha} \) we define

\[ u(x, t) := \lim_{N \to \infty} S_{t}^{N,2} u_{0}(x) \] (6.1)

whenever the limit exists. This coincides almost everywhere with the classical solution defined via the \( L^{2} \)-limit. Then, by standard arguments, an upper bound for \( \alpha_{n}(s) \) can be obtained from appropriate maximal inequalities with respect to fractal measures. We summarise this in the following lemma.

Lemma 6.1. \( \square \) Let \( \alpha > \alpha_{0} \geq n - 2s \) and suppose that

\[ \left\| \sup_{0 \leq t \leq 1} |e^{i t \Delta} u_{0}| \right\|_{L^{1}(d\mu)} \lesssim \sqrt{c_{\alpha}(\mu)} \| \mu \| \| u_{0} \|_{H^{\alpha}(\mathbb{R}^{n})} \]

whenever \( u_{0} \) is in the Schwartz class and \( \mu \) is an \( \alpha \)-dimensional. Then \( \alpha_{n}(s) \leq \alpha_{0} \).
Let $n \geq 1$ and
\[
\sup_{0 < t < 1} \| e^{itf} \|_{L^2(B_1)} \lesssim \| f \|_{H^s(\mathbb{R}^n)}
\]
whenever $f$ is Schwartz and $\mu$ is $\alpha$-dimensional.

Proof of Theorem 6.3. Let $n \geq 1$ and $s > \frac{1}{2} + \frac{\alpha}{4n}$. Then
\[
\sup_{0 < t < 1} \| e^{itf} \|_{L^2(B_1)} \lesssim \| f \|_{H^s(\mathbb{R}^n)}.
\]

Writing $\hat{f}_R = R^{-\alpha} \hat{f}(R^{-1})$ and scaling, we see that
\[
\sup_{0 < t < 1/R} \| e^{itf} \|_{L^2(\mathbb{R}^n)} = R^{-\alpha/2} \left( \sup_{0 < t < R} \| e^{itf} \|_{L^2(\mathbb{R}^n)} \right)^{1/2}
\]

provided $\hat{f} \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2^{\ell(c)} \}$, by a dyadic decomposition in frequency, the inequality (6.2) would follow from
\[
\sup_{0 < t < 1} \| e^{itf} \|_{L^2(\mu)} \lesssim \sqrt{c_\alpha(\mu)} R^{n+\varepsilon} \| f \|_2
\]
so that, by writing $d\mu_R(x) := R^\alpha d\mu(x/R)$, this is equivalent to
\[
\left\| \sup_{0 < t < R} |e^{it\Delta} f| \right\|_{L^2(d\mu_R)} \lesssim c_\alpha(\mu) R^{\frac{n+1}{2} + \delta + \varepsilon} \|f\|_2, 
\]
provided $\text{supp} \hat{f} \subset \{ \xi : 1/8 \leq |\xi| \leq 1/2 \}$. It is easy to check that $c_\alpha(\mu R) = c_\alpha(\mu)$.

Now by taking $\lambda = 1/2$ in (3.9) we have the pointwise bound
\[
|e^{it\Delta} f| \lesssim R^\varepsilon \sum_{R^{-1/2} \leq \delta \leq 1} \left( \sum_{\tau \sim \delta} \max_{\tau_1, \ldots, \tau_{n+1} \subset \tau} \prod_{k=1}^{n+1} |T_{\tau_k} \hat{f} \ast \zeta_{\tau_k'}|^2 \right)^{1/2} 
+ \sum_{R^{-1/3} \leq \delta \leq R^{-1/2} + \varepsilon} \left( \sum_{\tau \sim \delta} \left( \Psi_{\tau} |T_{\tau} \hat{f} \ast \zeta_{\tau'}|^2 \right)^{1/2} \right) 
+ \sum_{R^{-1/2} \leq \delta \leq 1} \left( \sum_{\tau \sim \delta} \Psi_{\tau}^2 \right)^{1/2} R^{-1/\varepsilon} \|\hat{f}\|_2. 
\]

Recalling that there are a finite number, independent of $R$, of terms in each of the $\delta$-sums, by the triangle inequality, we need only prove estimates which are uniform in $\delta$. Writing $g_{\tau} := \hat{f}_{\lambda_{\tau}}$, if we could prove
\[
\left\| \sup_{0 < t < R} \Psi_{\tau} T_{\tau} g \ast \zeta_{\tau'} \right\|_{L^2(d\mu_R)} \lesssim c_\alpha(\mu) R^{\frac{n+1}{2} + \delta + \varepsilon} \|g_{\tau}\|_2, 
\]
uniformly for $\tau$ at scale $\delta$ with $R^{-1/2} \leq \delta \leq R^{-1/2 + \varepsilon}$, then using orthogonality, we could bound the middle term on the right-hand side of (6.2). Similarly, replacing the $\max_{\tau_1, \ldots, \tau_{n+1} \subset \tau}$ with a $L^2$-norm, and using the fact that there are no more than $R^\varepsilon$ choices in such a sum, in order to treat the first term it will suffice to prove
\[
\left\| \sup_{0 < t < R} \prod_{k=1}^{n+1} T_{\tau_k} g \ast \zeta_{\tau_k'} \right\|_{L^2(d\mu_R)} \lesssim c_\alpha(\mu) R^{\frac{n+1}{2} + \delta + \varepsilon} \|g_{\tau}\|_2, 
\]
uniformly for $\tau$ at scale $\delta$ with $R^{-1/2} \leq \delta \leq 1$ and uniformly for choices of transversal caps $\tau_1, \ldots, \tau_{n+1} \subset \tau$. Finally, in order to deal with the remainder term, by taking $\varepsilon$ sufficiently small, it will suffice to prove that
\[
\| \sup_{0 < t < R} \Psi_{\tau} \|_{L^2(d\mu_R)} \lesssim c_\alpha(\mu) R^{n+1}, 
\]
uniformly for $\tau$ at scale $\delta$ with $R^{-1/2} \leq \delta \leq 1$. Taking for granted the proofs of (6.2), (6.3) and (6.4), which we will present in the forthcoming lemmas, this completes the proof of Theorem 6.2. \hfill \Box

From now on, for nested norms, we write $\|f\|_{XY} := \|\|f\|_X\|_Y$.\hfill 

**Lemma 6.4.** Let $0 < \varepsilon < \frac{1}{n}$. Then, for all caps $\tau \sim \delta$ with $R^{-1/2} \leq \delta \leq 1$,
\[
\|\Psi_{\tau}\|_{L^2(d\mu_R)L^\infty(0,R)} \lesssim \sqrt{c_\alpha(\mu)} R^{n+1}. 
\]

**Proof.** Writing $q = \frac{2n}{n+1}$, we prepare to apply Proposition 6.9. First of all, as $\Psi_{\tau}$ is essentially constant at scale one, we can bound
\[
\|\Psi_{\tau}\|_{L^2(d\mu_R)L^\infty(0,R)} \lesssim \|\Psi_{\tau}\|_{L^2(d\mu_R)L^q(0,R)} 
\lesssim \sqrt{c_\alpha(\mu)} \|\Psi_{\tau}\|_{L^2(B_R)L^q(0,R)} 
\lesssim \sqrt{c_\alpha(\mu)} R^{n+1} \|\Psi_{\tau}\|_{L^q(B_R \times (0,R))}. 
\]
Noting that \( n(\frac{1}{2} - \frac{1}{q}) = \frac{1}{2} \), and covering \( B_R \times (0, R) \) with a family \( \{T_j\} \) of translates of \( \tau' \) with disjoint interiors, we can bound this as
\[
\|\Psi_\tau\|_{L^2(d\mu_R) L^q(0, R)} \lesssim \sqrt{c_\alpha(\mu)} R^{\delta} \left( \sum_j \|\Psi_\tau\|_{L^q(T_j)}^q \right)^{1/q}
\]
\[
\lesssim \sqrt{c_\alpha(\mu)} R^{\delta} \left( \sum_j |T_j| \|\tau'\|^\epsilon \right)^{1/q}
\]
\[
\lesssim \sqrt{c_\alpha(\mu)} R^{\frac{n+\delta}{2}} \delta^{-\frac{(n+2\epsilon)}{8}},
\]
where the second inequality is by Proposition 4.10. For the range of \( \delta \) under consideration, this is more than enough to give the desired bound.

**Lemma 6.5.** Let \( 0 < \varepsilon < \frac{1}{8n} \). Then, for all caps \( \tau \sim \delta \) with \( R^{-1/2} \leq \delta \leq R^{-1/2+\varepsilon} \),
\[
\|\Psi_\tau T_\tau g \ast \zeta_{\tau'}\|_{L^2(d\mu_R) L^\infty(0, R)} \lesssim \sqrt{c_\alpha(\mu)} R^{\frac{1}{2} - \frac{1}{2n} - \frac{n+\epsilon}{8n}} \|g_\tau\|_2.
\]

**Proof.** We cover \( B_R \times (0, R) \) by a family \( \{T_{jk}\} \) of translates of \( \tau' \) with disjoint interiors. Denote by \( I_j \) the projection orthogonal to time of \( T_{jk} \) onto \( \mathbb{R}^n \). Recall that the sets \( T_{jk} \) have dimensions \( \delta^{-1} \times \cdots \times \delta^{-1} \times \delta^{-2} \) and, as our functions are frequency supported in the the unit annulus, the sets \( \tau' \) make an angle greater than \( \pi/8 \) with the time axis. Thus the projections \( I_j \) also have a long side of length a constant multiple of \( \delta^{-2} \).

Set \( G_T := [T_T g] \ast \zeta_{\tau'} \). Denoting by \( d\mu_R^T \) the measure \( d\mu_R \) restricted to \( I_j \), by Hölder’s inequality
\[
\|\Psi_\tau G_T\|_{L^2(d\mu_R) L^\infty(0, R)} \left( \sum_j \|\Psi_\tau G_T\|_{L^2(d\mu_R^T) L^\infty(0, R)}^2 \right)^{1/2}
\]
\[
\leq \mu R(I_j)^{\frac{1}{2} - \frac{1}{p}} \left( \sum_j \|\Psi_\tau G_T\|_{L^p(d\mu_R) L^\infty(0, R)}^p \right)^{1/p},
\]
Denoting \( T_{jk}^x = \{(y, t) \in T_{jk} : y = x\} \), on the other hand we have
\[
\sup_{0 < t < R} |\Psi_\tau G_T|(x, t) \leq \left( \sum_k \|\Psi_\tau G_T\|_{L^p(T_{jk}^x)}^p \right)^{1/p},
\]
for all \( x \in I_j \), so that
\[
\|\Psi_\tau G_T\|_{L^2(d\mu_R) L^\infty(0, R)} \lesssim \left( \sum_j \left( \sum_k \|\Psi_\tau G_T\|_{L^p(T_{jk}^x, d\mu_R dt)}^p \right)^{2/p} \right)^{1/2}.
\]

As in the previous lemma, we use that \( \Psi_\tau \) is essentially constant at scale one, so that
\[
\|\Psi_\tau\|_{L^p L^\infty(T_{jk}, d\mu_R dt)} \lesssim \|\Psi_\tau\|_{L^p L^\infty(T_{jk}, d\mu_R dt)} \lesssim \mu R(I_j)^{\frac{1}{2} - \frac{1}{p}} \|\tau'\|_{L^p(T_{jk}, dx)} \lesssim c_\alpha(\mu)^{\frac{1}{2}} \mu R(I_j)^{\frac{1}{2} - \frac{1}{p}} R^{\frac{n+2\epsilon}{8}} |T_{jk}|^{\frac{1}{2}},
\]
where the final inequality is by 4.10. Using this and the fact that \( G_T \) is essentially constant on \( T_{jk} \),
\[
\|\Psi_\tau G_T\|_{L^p L^\infty(T_{jk}, d\mu_R dt)} \lesssim c_\alpha(\mu)^{\frac{1}{2}} \mu R(I_j)^{\frac{1}{2} - \frac{1}{p}} R^{\frac{n+2\epsilon}{8}} |T_{jk}|^{\frac{1}{2}} \|G_T\|_{L^\infty(L^\infty(T_{jk}))} \lesssim c_\alpha(\mu)^{\frac{1}{2}} \mu R(I_j)^{\frac{1}{2} - \frac{1}{p}} R^{\frac{n+2\epsilon}{8}} |T_{jk}|^{\frac{1}{2}} \delta^{-\frac{n+1}{8}} \delta^{\frac{1}{8}} \|G_T\|_{L^2 L^p(T_{jk})}.
\]
Plugging this into (5.5), we obtain

\[
\|\Psi \Gamma g\|_{L^2(d\mu_R)L^\infty(0,R)} \lesssim c_\alpha(\mu)^{\frac{1}{q} + \epsilon} R^{\frac{n\alpha}{n} + \frac{1}{2} - \frac{1}{q}} \left( \sum_j \left( \sum_k \|G_j\|_{L^2(T_{jk})}^p \right)^{2/p} \right)^{1/2},
\]

where in the second inequality we use \(\mu_R(I_j) \lesssim c_\alpha(\mu)\delta^{-(\alpha+1)}\) which follows by covering the \(I_j\) by \(\delta^{-1}\) balls of radius \(\delta^{-1}\). Finally, using that \(\Gamma\) is essentially constant on \(T_{jk}\) and \(\frac{1}{q} - \frac{1}{2} = \frac{-2\alpha}{n}\), we can sum up to obtain

\[
\|\Psi \Gamma g\|_{L^2(d\mu_R)L^\infty(0,R)} \lesssim \sqrt{c_\alpha(\mu) R^{\frac{n\alpha}{n} + \frac{1}{2} - \frac{1}{q}}} \|G\|_{L^2(B_R)L^p(0,R)}.
\]

In fact we have only performed this argument for general \(p\) to facilitate the proof of the following lemma. Here we set \(p = 2\) and so it remains to bound

\[
\left| \int T_{\tau} g \right| \leq \int \left| T_{\tau} g \right| \leq \int |g_{\tau}| \leq \|g_{\tau}\| L^2(\mathbb{R}^n) \leq \|g\| L^2(\mathbb{R}^n),
\]

by Fubini, Minkowski’s integral inequality and Plancherel. Plugging this into the previous estimate, we see that

\[
\|\Psi \Gamma g\|_{L^2(d\mu_R)L^\infty(0,R)} \lesssim \sqrt{c_\alpha(\mu) R^{\frac{n\alpha}{n} + \frac{1}{2} - \frac{1}{q}}} \|g\| L^2(\mathbb{R}^n),
\]

which, with \(R^{-1/2} \leq \delta \leq R^{-1/2+\epsilon}\), yields the desired uniform estimate. □

**Lemma 6.6.** Let \(0 < \epsilon < \frac{1}{8n}\). Then, for all caps \(\tau \sim \delta\) with \(R^{-1/2} \leq \delta \leq 1\) and all \((n+1)\)-transversal caps \(\tau_1, \ldots, \tau_{n+1} \sim \delta/K_{n+1}\) contained in \(\tau\),

\[
\left\| \Psi_\tau \prod_{k=1}^{n+1} |T_{\tau_k} g|^{\frac{1}{n+1}} \right\|_{L^2(d\mu_R)L^\infty(0,R)} \lesssim \sqrt{c_\alpha(\mu) R^{\frac{n\alpha}{n} + \frac{1}{2} - \frac{1}{q}}} \|g\| L^2(\mathbb{R}^n),
\]

**Proof.** As before we set \(G_\tau := \prod_{k=1}^{n+1} |T_{\tau_k} g|^{\frac{1}{n+1}} \right\|_{L^2(d\mu_R)L^\infty(0,R)} \lesssim \sqrt{c_\alpha(\mu) R^{\frac{n\alpha}{n} + \frac{1}{2} - \frac{1}{q}}} \|g\| L^2(\mathbb{R}^n),
\]

which yields the desired estimate uniform in the range \(R^{-1/2} \leq \delta \leq 1\). Covering \(B_R \times (0,R)\) by translations of \(\tau'\), as \(\tau' \subset \tau_k\) we still have that \(G_\tau\) is essentially constant at this scale. Repeating the previous argument, this time with \(p := \frac{2(\alpha+1)}{n}\), by (5.6) we have

\[
\|\Psi \Gamma g\|_{L^2(B_R)L^p(0,R)} \lesssim \sqrt{c_\alpha(\mu) R^{\frac{n\alpha}{n} + \frac{1}{2} - \frac{1}{q}}} \|G\|_{L^2(B_R)L^p(0,R)},
\]

and so it remains to bound \(\|G\|_{L^2(B_R)L^p(0,R)}\). By Minkowski’s integral inequality, it will suffice to treat

\[
\int \left\| \prod_{k=1}^{n+1} |T_{\tau_k} g_{y_k}|^{\frac{1}{n+1}} \right\|_{L^2(B_R)L^p(0,R)} \prod_{k=1}^{n+1} \zeta_{\tau_k} (y_k) dy_1 \ldots dy_{n+1},
\]

where

\[
g_{y_k} := g(x_{\tau_k} e^{-i\pi(y_k)} \xi + t_k) e^{y_k^2}, \quad t_k := y_k - \pi(y_k).
\]
Noting that $\frac{1}{p} - \frac{1}{q} = \frac{1}{2m} - \frac{1}{2(n+1)}$ and $\|g_{\tau_n}\|_2 = \|g_{\tau}\|_2$, it remains to prove

$$\left\| \prod_{k=1}^{n+1} [T_{\tau_k} g](\tau) \right\|_{L^2(B_{\tau_n})L^p(0,R)} \lesssim R^{\frac{n-1}{2m + 1} + \epsilon} \delta^{- \frac{1}{2m + 1} + \epsilon} \|g\|_2.$$  

By scaling as in the proof of Lemma 1.3 (see (4.16) for the definition), this would follow from

$$\left\| \prod_{k=1}^{n+1} [\tilde{T}_{\tau_k} f](x' - 2\delta^{-1} \xi_0 t', t') \right\|_{L^2(B_{\tau_n})L^p(0,\delta^2 R)} \lesssim R^{\frac{n-1}{2m + 1} + \epsilon} \delta^{- \frac{1}{2m + 1} + \epsilon} \|f\|_2.$$  

By a rotation we can suppose that $\xi_0$ is parallel to $x_n$, so by an application of Hölder’s inequality, and making the change of variables $x = x' - 2\delta^{-1} \xi_0 t'$, it would suffice to prove

$$\left\| \prod_{k=1}^{n+1} [\tilde{T}_{\tau_k} f](\tau) \right\|_{L^2(B_{\tau_n})L^p(0,\delta^2 R)} \lesssim R^{\frac{n-1}{2m + 1} + \epsilon} \delta^{- \frac{1}{2m + 1} + \epsilon} \|f\|_2.$$  

Now partitioning $\mathbb{R}^{n-1}$ into cubes $\Omega$ of side length $\delta^2 R$, and applying Hölder’s inequality, the left-hand side is bounded by

$$\left( \delta^2 R \right)^{(n-1)(\frac{1}{p} - \frac{1}{2})} \left( \sum_{\Omega} \left\| \prod_{k=1}^{n+1} [\tilde{T}_{\tau_k} f](\tau) \right\|_{L^p(\Omega) L^p(0,\delta^2 R)} \right)^{1/2}.$$  

Noting that

$$2(n - 1) \left( \frac{1}{2} - \frac{1}{p} \right) = \frac{n - 1}{n + 1} = \frac{3}{p} - \frac{1}{2(n + 1)},$$

the proof is completed by an application of Proposition 2.3. \hfill \square

**Appendix**

The following lemma is well-known; see for example [38, pp. 1024].

**Lemma 6.7.** Let $\tilde{\psi} = \psi_0 \ast \psi_o$ be a smooth radially symmetric cut-off function supported in $B(0,d) \subset \mathbb{R}^d$ and equal to one on $B(0,\sqrt{d})$ and consider the scaled version $\phi_{\tau'}$ adapted to $\tau'$. Then, for all $m \geq 1$,

$$|F(x, t)| \lesssim \left( |F|^{m'} \ast |\psi_{\tau'}|^{m'}(x, t) \right)^m,$$  

(6.7)

provided supp $\tilde{F} \subset \tau \subset \mathbb{R}^d$.

**Proof.** As usual we set $m' := m/(m - 1)$. Letting

$$\eta(x, t) := \psi_{\tau'}(x, t)e^{ix \cdot \xi_0 + it \phi(\xi_o)},$$  

(6.8)

where $(\xi_0, \phi(\xi_o))$ is the centre of $\tau$, we note that

$$|\eta|^{m'} = |\tau'|^{m'} |\psi_{\tau'}|^{m'}.$$  

(6.9)

By the self reproducing formula $F = F \ast \eta$

$$|F(x, t)| \leq \int |F((x, t) - y)\eta(y)| \; dy \leq \|F((x, t) - \cdot)\eta\|^{m'}_{L^{m'}} \int |F((x, t) - y)\eta(y)|^{m'} \; dy,$$

(6.10)

$$\leq \|F((x, t) - \cdot)\eta\|^{m'}_{L^{m'}} \int |F((x, t) - y)\eta(y)|^{m'} \; dy \lesssim |\tau'|^{-m'} \|F((x, t) - \cdot)\eta\|^{m'}_{L^{m'}} \int |F((x, t) - y)\eta(y)|^{m'} \; dy,$$
where in the last inequality we have used Bernstein’s inequality. Hence by dividing by \(\|F((x, t) - \cdot)\eta\|_{L^1_T}^\frac{\delta}{2}\), we see that
\[
\left( \int |F((x, t) - y)\eta(y)| \, dy \right)^\frac{\delta}{2} \lesssim |\tau'|^{-\frac{\delta}{2}} \left( \int |F((x, t) - y)\eta(y)|^\frac{\delta}{2} \, dy \right)
\]
\[
= \left( \int |F((x, t) - y)|^\frac{\delta}{2} |\psi_{\tau'}(y)|^\frac{\delta}{2} \, dy \right), \tag{6.11}
\]
where in the final identity we have used (6.9). Then (6.7) follows using (6.10). \(\square\)

**Lemma 6.8.** Let \(0 < \delta \leq 1\) and let \(K > (K')^2 > 1\). Let \(\Lambda_1, \Lambda_2 \in SO(d)\) be such that \(\Lambda_1\Lambda_2^{-1}\) is a rotation by an angle less than \(\delta\). Then if \(F : \mathbb{R}^d \to \mathbb{R}^+\) is essentially constant on translates of \(\Lambda_1^{-1}(T)\) where
\[
T := \left[ -\frac{K}{\delta}, \frac{K}{\delta} \right] \times \cdots \times \left[ -\frac{K}{\delta}, \frac{K}{\delta} \right] \times \left[ -\frac{K}{\delta^2}, \frac{K}{\delta^2} \right],
\]
and
\[
\zeta(x, t) \lesssim \frac{\delta^{d+1}}{(K')^d} \left( 1 + \left| \frac{\delta x}{K'} \right|^2 + \left| \frac{\delta t}{K'} \right|^2 \right)^{-N},
\]
or
\[
\zeta(x, t) \lesssim \frac{\delta^{d+1}}{(K')^d} \left( 1 + \left| \frac{\delta x}{K'} \right|^2 + \left| \frac{\delta t}{(K')^2} \right|^2 \right)^{-N},
\]
for some \(N \geq d\), then
\[
F \ast \zeta(\Lambda_2(-))(x_1, t_1) \lesssim_L F(x_2, t_2) + \|F\|_{L^\infty} \left( \frac{K'}{K} \right)^N
\]
whenever \((x_1, t_1) - (x_2, t_2) \in \Lambda_1^{-1}(T)\).

**Proof.** If \(\zeta\) takes the second form, then by a change of variables,
\[
\int_{\mathbb{R}^d/T} \zeta(x, t) \, dx dt \lesssim \int_{\mathbb{R}^d/[-K/K', K/K']^{d-1} \times [-K/(K')^2, K/(K')^2]} |(x, t)|^{-2N} \, dx dt
\]
\[
\lesssim \int_{K/K'}^\infty \rho^{-2N+d-1} \, d\rho \lesssim N \left( \frac{K'}{K} \right)^{2N - d},
\]
and the same is true if \(\zeta\) takes the first form. Then note that
\[
\int_T F((x_1, t_1) - y)\zeta(\Lambda_2(y)) \, dy = \int_T F((x_1, t_1) - \Lambda_2^{-1}y)\zeta(y) \, dy
\]
\[
= \int_T F((x_1, t_1) - \Lambda_2^{-1}y)\zeta(y) \, dy + \int_{\mathbb{R}^d/T} F((x_1, t_1) - \Lambda_2^{-1}y)\zeta(y) \, dy =: I + II.
\]
By trigonometry and the essentially constant assumption, we have
\[
F((x_1, t_1) - \Lambda_2^{-1}y) \bigg|_{y \in \Lambda_1^{-1}T} \lesssim F(x_2, t_2),
\]
whenever \((x_1, t_1) - (x_2, t_2) \in \Lambda_1^{-1}T\) so that \(I \lesssim F(x_2, t_2)\). On the other hand, we have that
\[
II \lesssim \|F\|_{L^\infty} \int_{\mathbb{R}^d/T} \zeta(y) \, dy \lesssim \|F\|_{L^\infty} \left( \frac{K'}{K} \right)^{2N - d},
\]
from before, and so the desired estimate follows by adding the two bounds. \(\square\)
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