A Graph Theoretical Analysis of Low-Power Coding Schemes for One-Hop Networks

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Abstract

Coding schemes with extremely low computational complexity are required for particular applications, such as wireless body area networks, in which case both very high data accuracy and very low power-consumption are required features. In this paper, coding schemes arising from incidence matrices of graphs are proposed. An analysis of the resilience of such codes to erasures is given using graph theoretical arguments; decodability of a graph is characterized in terms of the rank of its incidence matrix. Bounds are given on the number of decodable subgraphs of a graph and the number of edges that must be deleted in order to arrive at an undecodable subgraph. Algorithms to construct codes that are optimal with respect to these bounds are presented.

Index Terms

low-power coding, energy-efficiency, one-hop network, graph representation, incidence matrix, erasures, decodable/undecodable graph, decoding probability, decoding cut

I. INTRODUCTION

Network coding [3], [4] is by now a well established area offering demonstrated advantages over routing in communications networks such as increased throughput and reduced latency. Power consumption is a major concern for a number of applications. For example, wireless body area networks (WBANs) [5], [6], which are designed to give reliable unobtrusive support for the monitoring of person’s physiological data, favour ultra low-power coding schemes for communication between miniature sensors. At the same time, the number of re-transmissions requested due to errors should be minimized, since these are very costly from a power and/or memory perspective (see [7] and the references therein). Therefore, a major consideration in the design of any low-power coding scheme is that it be robust to packet loss, without incurring a large computational cost.

By a low-power coding scheme we mean one whose encoding functions have few arguments, and so in the linear case can be represented by a sparse matrix. In this paper, we discuss measures of robustness of coding schemes

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when only the summation of two packets is allowed. The analysis for the case that packets are vectors over $GF(2)$ has been considered in [1], [8]. These schemes have the practical advantage of very low complexity encoding and decoding [9]. Moreover, they can be identified with graphs and studied using that theory. In this paper, we extend these results to the more general case of coding schemes over arbitrary finite fields. The decoding criterion for a coding scheme over $GF(q)$ with $q$ odd is more general than for one over $GF(q)$ with $q$ even.

The rest of the paper is organized as follows. We present preliminaries in Section II and identify a graph as a code via its incidence matrix. In Section III we state a necessary and sufficient condition for a given coding scheme to be able to deliver all data at the terminal (in spite of packet erasures) in terms of its corresponding graph representation, in which case we call the graph decodable. In Section IV we give a definition of decoding probability $P_G(y, z)$ for a given coding scheme as the evaluation of transmission success at $y = p$, $z = 1 - p$, where $p$ is the reliability of a packet. In particular, we derive a recurrence relation on $P_G(y, z)$ for certain coding schemes, which is similar to that for the well-known chromatic polynomial in graph theory. We then discuss in Section V the minimal size $b_G$ of a decoding cut in order to find a coding scheme with high decoding probability, and in Section VI lower bounds on the number of undecodable graphs. Section VII proposes some algorithms to produce suitable coding schemes, and Section VIII shows comparisons between various coding schemes. Final remarks can be read in Section IX.

II. Preliminaries

For the framework we consider, a system consists of the sender $S$, a terminal $T$ and a set of relays. The sender $S$ wishes to send $n$ data packets $p_1, p_2, \ldots, p_n$ in $GF(q)^\ell$ to the terminal $T$, via the relays. Throughout the paper, we always assume that $n \geq 2$. At the relays, a total of $m$ encodings

$$f_1(p_1, p_2, \ldots, p_n), f_2(p_1, p_2, \ldots, p_n), \ldots, f_m(p_1, p_2, \ldots, p_n)$$

are computed and transmitted to $T$. The redundancy $r = m/n$ is the ratio of the number of packets sent to $T$ to the number of original packets. A (linear) coding scheme $C$ for the system is a collection of $GF(q)$-linear vectorial functions corresponding to packet encodings at relays; that is, $C = \{f_1, f_2, \ldots, f_m\}$. We assume for the remainder that each encoded packet $f_i(p_1, p_2, \ldots, p_n)$, $1 \leq i \leq m$, has the form $p_j$ or $p_j + p_k$ with $j \neq k$. This scheme offers both low encoding and decoding complexity.

We next present some preliminaries on graphs (see [10], [11] for further reading). Let $G = (V, E)$ be a finite (multi)-graph with vertex set $V = V(G)$ and edge multi-set $E = E(G) \subset V \times V$. An edge of $G$, $e = (x, y) \in V \times V$ is said to have initial vertex $x$ and end vertex $y$. Multi-edges may be distinguished by an edge labelling of $G$.

A walk starting at vertex $x$ and terminating at a vertex $y$ (or an $xy$-walk) is a sequence $\pi = x, e_1, v_1, e_2, v_2, \ldots, e_\ell, y$ such that each $e_k$ is an edge of $G$ with $e_1$ originating at $x$, $e_\ell$ ending at $y$ and such that the initial vertex $v_k$ of $e_{k+1}$ is the end vertex $v_k$ of $e_k$ for $k = 1, \ldots, \ell - 1$. If the vertices of walk $\pi$ are all distinct it is called a path. The number $\ell$ of edges appearing in a walk is referred to as its length. We write $x, v_1, \ldots, v_{\ell-1}, y$ to denote any $xy$-path of the form $x, e_1, v_1, e_2, v_2, \ldots, e_\ell, y$. If $G$ is simple (has no multiple edges) then $\pi$ is uniquely determined.
by the sequence of vertices $x, v_1, \ldots, v_{\ell-1}, y$. A walk or path is called a closed walk or cycle, respectively, if its initial vertex is the same as its terminal vertex. It is easy to see that an $xy$-path exists in $G$ if an $xy$-walk exists and that a closed walk of odd length implies the existence of a cycle of odd length. We say $x$ is connected to $y$ in $G$ when a walk from $x$ to $y$ exists in $G$. Graph $G$ is called connected if each pair of vertices are connected, otherwise $G$ is called disconnected. A connected component (or simply component) of $G$ is a maximal connected subgraph of $G$, where a subgraph $H$ of $G$ is a graph such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$ hold.

The adjacency matrix $A = A_G$ of $G$ is the $|V| \times |V|$ integer matrix whose $(i, j)$-entry is the number of edges with initial vertex $x$ and end vertex $y$. We write $\Delta(G)$ to denote the maximum multiplicity of any edge joining a pair of vertices of $G$; that is, the maximum entry in $A$. The incidence matrix $B = B_G$ of $G$ is the $|E| \times |V|$ 0-1 matrix whose $(i, j)$-entry is 1 if edge $e_i$ is incident with vertex $v_j$ with respect to some labelling of the vertices and edges of $G$. Observe that the $(i, j)$-entry of $A^t$ corresponds to the number of walks of length $\ell$ from $v_i$ to $v_j$.

We write $d_L(v)$ to denote the number of loops (length-1 cycles) incident with a given vertex $v$, and we let $\Delta_L(G) := \max\{d_L(v) : v \in V\}$. We denote by $L_G = \sum_{v \in V} d_L(v)$ the number of loops of $G$. We define the incidence degree of a vertex $v$, expressed $d_I(v)$, as the number of edges incident with $v$. Each loop at $v$ contributes a count of one to $d_I(v)$. We define $\delta_I(G) := \min\{d_I(v) : v \in V\}$. The sum of the incidence degrees $S_I(G)$ satisfies $S_I(G) := \sum_{v \in V} d_I(v) = 2|E| - L_G$.

Given a connected graph $G$, the edge-connectivity $\lambda(G)$ of $G$ is the smallest number of edges such that the resulting graph formed by deleting those edges is disconnected. Observe that $\lambda(G) \leq \delta_I(G)$ since deleting all edges attached to a vertex $v$ with incidence degree $d_I(v) = \delta_I(G)$ makes $v$ isolated.

A graph $G = (V, E)$ is called a bipartite graph or bi-colourable if $V$ can be partitioned into two sets $U$ and $W = V \setminus U$ such that every edge of $G$ is incident with exactly one member of $U$ and one member of $W$; that is, no two vertices in the same set are connected by an edge. (Note that a graph with loops is not bipartite, nor indeed colourable.) There are several characterizations of bipartite graphs. One that is relevant to this paper is König’s well-known result (see [10] Chapter 1)).

**Theorem II.1** A graph is bipartite if and only if it has no cycle of odd length.

We now relate identify a coding scheme with a multi-graph. Consider the case that the sender $S$ sends $n$ packets $p_1, p_2, \ldots, p_n \in GF(q)^\ell$ to the terminal $T$ with redundancy $r$. Given a coding scheme $C = \{f_1, f_2, \ldots, f_m\}$ we generate a multi-graph representation $G = G_C$ for $C$ as follows.

1) $G$ has as vertices $p_1, p_2, \ldots, p_n$.

2) For $j \neq j'$, $(p_j, p_{j'})$ is an edge of $G$ if $f_i(p_1, p_2, \ldots, p_n) = p_j + p_t$ for some $f_i$ (i.e., if $p_j + p_t$ is sent to the terminal $T$).

3) $G$ has a loop at $p_j$ if $f_i(p_1, p_2, \ldots, p_n) = p_j$ for some $f_i \in C$ (i.e., if $p_j$ itself is sent to $T$).

An example of a coding scheme $C$ and its corresponding graph representation is given in Table I and Figure I. The erasure of packets during a transmission is identified with deletions of corresponding edges in $G$. Also, for a
case with \( n \) packets and redundancy \( r \), any graph representation of a corresponding coding scheme must have \( n \) vertices and \( m = rn \) edges.

| Encoding at the sender \( S \) | \( p_1 \) | \( p_2 \) | \( p_3 + p_4 \) | \( p_4 + p_5 \) | \( p_5 + p_6 \) | \( p_6 + p_1 \) |
|-------------------------------|---------|---------|----------------|----------------|----------------|----------------|
| \( p_4 \)                     | \( p_7 \) | \( p_6 + p_7 \) | \( p_7 + p_8 \) | \( p_8 + p_9 \) | \( p_9 + p_4 \) |                       |
| \( p_7 \)                     | \( p_8 \) | \( p_9 + p_{10} \) | \( p_{10} + p_{11} \) | \( p_{11} + p_{12} \) | \( p_{12} + p_7 \) |                       |
| \( p_{10} \)                  | \( p_{11} \) | \( p_{12} + p_4 \) | \( p_4 + p_2 \) | \( p_2 + p_3 \) | \( p_3 + p_{10} \) |                       |

**TABLE I**  
AN EXAMPLE OF A CODING SCHEME \( C \)

Fig. 1. The graph representation \( G = G_C \) of coding scheme \( C \) in Table I.

**III. DECODABLE GRAPHS**

We provide a necessary and sufficient condition on the graph representation of a coding scheme for full packet retrieval at the terminal \( T \) extending [1, Theorem III.1]. For the remainder of this paper, when a case with \( n \) packets and redundancy \( r \) is given, we let \( C = \{ f_1, f_2, \ldots, f_r \} \) denote a coding scheme over \( GF(q) \) for the system, where each \( f_i \) is an encoding of packets \( p_1, p_2, \ldots, p_n \in GF(q)^l \), and we let \( G = G_C \) denote the graph representation of \( C \). \( H \) will denote a subgraph of \( G \) formed by deleting a corresponding edge of \( G \) for every erasure occurring in a given transmission.

**Definition III.1** \( H \) is called decodable if the linear system of equations corresponding to the edges of \( H \) can be solved for unique \( p_1, p_2, \ldots, p_n \). Otherwise \( H \) is called undecodable.

Throughout this paper, we denote by \( \mathcal{D}(n, m) \) the set of decodable graphs on \( n \) vertices and \( m \) edges. We always set \( m \geq n \) since it is impossible to uniquely retrieve all \( n \) packets from \( m \) received packets if \( m < n \). One criterion for decodability is given by the following.

**Lemma III.1** Suppose that the terminal \( T \) has retrieved the packet \( p_i \) for some \( i \in \{1, 2, \ldots, n\} \). Then for every \( p_j \) connected to \( p_i \) in \( H \), the packet \( p_j \) can be retrieved by \( T \).
Proof: Let $p_j$ be connected to $p_i$ in $H$, say by a path $\pi : p_i = p_{i_1}, p_{i_2}, \ldots, p_{i_h} = p_j$. This corresponds to the linear system of $h$ equations $g_1 = p_{i_1}, g_2 = p_{i_1} + p_{i_2}, \ldots, g_h = p_{i_{h-1}} + p_{i_h}$, which has rank $h$ and can be solved at $T$.

Lemma III.2 $H$ is decodable if and only if its incidence matrix $B_H$ has rank $n$, considered as a matrix over $GF(q)$.

Proof: Let $P$ be the $n \times \ell$ matrix whose rows are the vectors $p_1, p_2, \ldots, p_n$. The encoded packets received by the terminal decoder are the rows of the $|E(H)| \times \ell$ matrix $M = B_H P$. The equation $M = B_H P$ can be solved for unique $P$ if and only if $B_H$ has rank $n$ over $GF(q)$.

A criterion to determine whether or not the incidence matrix of a graph has full rank as a real matrix is already known. For example, in [12] it was shown that the incidence matrix of a loop-free graph has full $\mathbb{R}$-rank if and only if it has no bipartite connected components nor isolated vertices. It is not hard to see that, for odd $q$, the argument given there holds also for the $GF(q)$-rank of an incidence matrix (cf. [12]), with a minor modification to include the case of a graph with loops.

Theorem III.3 Let $\mathbb{F}$ be a field of characteristic not equal to 2. The rank over $\mathbb{F}$ of the incidence matrix of an undirected graph with $n$ vertices, $s$ bipartite components and $t$ isolated vertices is $n - s - t$.

Applying Lemma III.2 and Theorem III.3 for fields of odd characteristic we have the following characterization of decodable subgraphs.

Corollary III.4 Let $C$ be a coding scheme over $GF(q)$ for odd $q$. Then the subgraph $H$ of $G$ is decodable if and only if each connected component of $H$ is neither a bipartite graph nor an isolated vertex.

To summarize, we obtain the following theorem.

Theorem III.5 Let $C$ be a coding scheme over $GF(q)$, and $G$ be its graph representation. Then a subgraph $H$ of $G$ is decodable if and only if exactly one of the following holds:

1) $q$ is even and every connected component of $H$ has a loop,
2) $q$ is odd and every connected component of $H$ has an odd cycle.

Proof: Let $U$ be a connected component of $H$. If $U$ has a loop, then from Lemma III.1 each packet $p_i \in V(U)$ can be retrieved. Suppose that $q$ is even and that $U$ has no loop. Then every row of $B_U$ has Hamming weight exactly 2 so the sum of the columns of $B_U$ is zero over $GF(q)$ and the system has rank less than $|V(U)|$. If $q$ is odd, then from Corollary III.4 and Theorem III.3 $U$ has an odd cycle if and only if it is decodable.

The following is now immediate from Theorem III.5.

Corollary III.6 Let $C$ be a coding scheme over $GF(q)$ for odd $q$. Then a connected subgraph $H$ of $G$ is decodable if and only if the adjacency matrix $A = A_H$ of $H$ satisfies $\text{Tr} A^\ell > 0$ for some odd number $\ell \in \{1, 2, \ldots, n\}$.
Proof: The \((i,j)\)-entry of \(A^\ell\) is the number of walks of length \(\ell\) from \(v_i\) to \(v_j\). Hence, \(\text{Tr}A^\ell > 0\) holds for some odd number \(\ell\) if and only if there exists a closed walk of odd length \(\ell\) (and hence, if and only if there exists an odd cycle). The result now follows from Theorem III.5.

The robustness of a coding scheme against packet loss can be measured as a function of the number of decodable subgraphs found upon deleting some edges. We will next define and discuss such a function.

IV. THE DECODING PROBABILITY

We first recall definitions provided in [1]. Given a graph \(G \in \mathcal{D}(n,m)\), we denote by \(c^G_x\) the number of decodable subgraphs of \(G\) formed by deleting \(x\) edges of \(G\), and we write \(u^G_x = \binom{m}{x} - c^G_x\) to denote the number of undecodable subgraphs of \(G\) found by deleting some \(x\)-set of its edges. (Here, we assume that edges in \(G\) are labeled distinctly in order to distinguish them. Thus, deleting different edges produces different subgraphs at all times.) By convention, we set \(c^G_x = 0\) if \(x > m\) or \(x < 0\). We define the decoding probability of \(G\) by

\[
P_G(y, z) := \sum_{x=0}^{m} c^G_x y^{m-x} z^x, \tag{1}
\]

and \(p\) is the probability that an edge is not deleted; that is, the probability that a packet is successfully transmitted to the terminal. In other words, \(1 - p\) is equal to the packet loss rate for each packet. It is clear that the decoding probability is the probability that all packets are retrieved at the terminal even though some packet loss occurs during transmission, and indeed, measures the robustness of a coding scheme against packet loss.

In this section, we will first make an approach for analyzing the decoding probability of a graph \(G\), using pre-computed decoding probabilities of graphs \(G - e^j\) and \(G \cdot e^k\) obtainable by the deletion and the contraction of edges between \(u\) and \(v\), respectively, as denoted below.

For a graph \(G\) and distinct vertices \(u, v\) in \(G\), suppose that there are \(k \geq 1\) edges \(e_1, e_2, \ldots, e_k\) connecting \(u\) and \(v\). Set \(e^j = e^j(u,v) = \{e_1, e_2, \ldots, e_j\}\) for \(1 \leq j \leq k\). We define \(G - e^j\) \((1 \leq j \leq k)\) to be the graph obtained by deleting \(j\) edges from those \(k\) edges, and \(G \cdot e^k\) to be the graph obtained by identifying the ends \(u, v\) as a single vertex (i.e., by applying the contraction of edges between \(u\) and \(v\)). The following lemma states a relationship between \(G, G - e^j\) and \(G \cdot e^k\).

**Lemma IV.1** Let \(G \in \mathcal{D}(n,m)\) be a graph representation of a coding scheme over \(GF(q)\), and suppose there are \(k\) edges \(e_1, e_2, \ldots, e_k\) connecting distinct vertices \(u\) and \(v\) in \(G\). Then for the case when

1) If \(q\) is even; or

2) If \(q\) is odd and no edge in \(e^k(= e^k(u,v))\) can be included in a cycle in \(G\),

we have

\[
c^G_x = \sum_{j=1}^{k} \binom{k}{j} c^{G - e^j}_{x-j} + c^G_x e^k.
\]

**Proof:** Observe that there are \(k + 1\) mutually exclusive cases when deleting \(x\) edges from \(G\);
(A) the $x$ edges include $j$ edges ($1 \leq j \leq k$) from $e_k$; or

(B) the $x$ edges do not include any edge in $e_k$.

It is trivial that the number of decodable graphs in case (A) is given by $\binom{k}{j} c_{x}^{G-e^j}$, here $\binom{k}{j}$ comes from the number of choices deleting $j$ edges from $e_k$ and observe that deletion of any such $j$ edges results in an isomorphic graph (a graph with the same topology). So we will show that the number of decodable graphs counted in case (B) is equal to $c_{x}^{G\cdot e^k}$. To do so, we define $X$ and $Y$ to be the sets of decodable graphs counted in case (B) and in $c_{x}^{G\cdot e^k}$, respectively, and then show the existence of a bijection $f : X \rightarrow Y$.

Let $H$ be a subgraph of $G$ such that $e_k \subset E(H)$. Then, it is straightforward to check that $H$ and $H \cdot e_k$ have the same number of components and loops at each component. Furthermore, for $GF(q)$ with $q$ odd, from the assumption on $e \in e_k$, we have that $H$ has an odd cycle if and only if $H \cdot e_k$ has an odd cycle. Hence, for any $q$, we can conclude that $H$ is decodable if and only if $H \cdot e_k$ is decodable. Therefore, set $f : X \rightarrow Y$ to be $f(H) = H \cdot e_k$, and then $f$ is obviously bijective, which completes the proof.

From Lemma IV.1, we can obtain the following corollary which indicates the relationship of $P_G$, $P_{G-e^j}$ and $P_{G\cdot e^k}$. A similar result holds for the chromatic polynomial in graph theory.

**Corollary IV.2** Let $G \in D(n, m)$ be a graph representation of a coding scheme over $GF(q)$, and suppose there are $k$ edges $e_1, e_2, \ldots, e_k$ connecting distinct vertices $u$ and $v$ in $G$. If assumption 1) or 2) in the statement of Lemma IV.1 holds, then we have

$$P_G(y, z) = \sum_{j=1}^{k} \binom{k}{j} z^j P_{G-e^j}(y, z) + y^k P_{G\cdot e^k}(y, z).$$

**Proof:**

$$P_G(y, z) = \sum_{x=0}^{m} c_{x}^{G} y^{m-x} z^x$$

$$= \sum_{x=0}^{m} \left( \sum_{j=1}^{k} \binom{k}{j} c_{x}^{G-e^j} + c_{x}^{G\cdot e^k} \right) y^{m-x} z^x \quad (2)$$

$$= \sum_{j=1}^{k} \binom{k}{j} \sum_{x=0}^{m} c_{x}^{G-e^j} y^{m-x} z^x + \sum_{x=0}^{m} c_{x}^{G\cdot e^k} y^{m-x} z^x$$

$$= \sum_{j=1}^{k} \binom{k}{j} z^j \sum_{x=0}^{m-j} c_{x}^{G-e^j} y^{m-x} z^x \quad (3)$$

$$+ y^k \sum_{x=0}^{m-k} c_{x}^{G\cdot e^k} y^{m-x} z^x$$

$$= \sum_{j=1}^{k} \binom{k}{j} z^j P_{G-e^j}(y, z) + y^k P_{G\cdot e^k}(y, z), \quad (4)$$
where (2) is from Lemma [IV.1] (3) is obtainable from

\[
\sum_{x=0}^{m} e_x^{G-e^j} y^{m-x} z^x = \sum_{x=-j}^{m-j} e_x^{G-e^j} y^{m-x} z^{x+j} \\
= \sum_{x=0}^{m-j} e_x^{G-e^j} y^{m-x} z^x
\]

observing that \(e_x^{G-e^j} = 0\) when \(x < 0\), and (4) is from the fact that \(e_x^{G-e^j} = 0\) for \(x > m-k\) as \(G \cdot e^k\) has \(m-k\) edges.

Corollary [IV.2] implies that for \(G \in \mathcal{D}(n,m)\) if we have computed decoding probabilities of graphs in \(\mathcal{D}(n',m')\) with \(n' \leq n\) and \(m' < m\) in advance, the decoding probability of \(G\) can be obtained recursively. Since \(p > 1 - p\) in general, if \(P_{G,e^b}(p,1-p)\) is large enough, then \(P_{G}(p,1-p)\) could also have large value.

V. Decoding Cuts

Our interest is to find a graph \(G \in \mathcal{D}(n,m)\) (for some fixed \(m\) and \(n\)) which has the maximum decoding probability amongst all graphs in \(\mathcal{D}(n,m)\). To this end, we define a decoding cut \(L\) of \(G\), as follows. Given \(G \in \mathcal{D}(n,m)\), a decoding cut (D-cut) \(L\) is a subset of \(E\) such that graph \(\tilde{G} = (V,E \setminus L)\) is undecodable. We denote by \(b_G\) the smallest cardinality of any D-cut of \(G\). In other words, \(b_G\) is the smallest \(x\) such that \(u_x^G > 0\) (that is, \(c_x^G < (m)\)). Clearly a high value of \(b_G\) is desirable for a good coding scheme.

A. Upper bounds on \(b_G\) for \(GF(q)\) with \(q\) even

We first discuss some upper bounds on \(b_G\) for \(GF(q)\) with \(q\) even. Recall from Theorem [III.5] that a graph \(H\) is decodable if and only if each component of \(H\) has a loop.

Remark 1 For \(b_G\) of a graph \(G\), we note the following.

1) \(b_G \leq \min(L_G, \delta I(G))\) since deleting all loops in \(G\) or deleting all edges attached to a vertex \(v\) with incidence degree \(d_I(v) = \delta I(G)\) yields an undecodable graph.

2) If \(L_G \geq \lambda(G)\), then \(\lambda(G) \leq b_G\) since a resulting graph \(\tilde{G}\) of \(G\) after deletion of some edges cannot be undecodable, if \(L_G \neq 0\), unless \(\tilde{G}\) is disconnected.

The following lemma can be easily obtained using elementary graph theory.

Lemma V.1 Let \(G \in \mathcal{D}(n,m)\). Then, \(\delta I(G) \leq 2m/n-1\) and \(b_G \leq 2m/(n+1)\). In particular, \(b_G \leq \min([2m/n-1],[2m/(n+1)])\)

Proof: Since \(G\) is decodable, \(L_G \geq 1\), and we have \(n\delta I(G) \leq S_I(G) = 2m - L_G < 2m\). Furthermore, since \(b_G \leq L_G\), we have \(nb_G \leq n\delta I(G) \leq S_I(G) = 2m - L_G < 2m - b_G\).

From Lemma [V.1] it follows that for a system sending \(n\) packets with redundancy \(r\), any graph representation \(G\) of a coding scheme satisfies \(b_G \leq \min(2r - 1, \lfloor 2m/n+1 \rfloor) = \lfloor 2m/n+1 \rfloor\), which is simply \(2r - 1\) whenever \(r \leq \frac{n+1}{2}\).
B. Upper bounds on $b_G$ for $GF(q)$ with $q$ odd

We next discuss some upper bounds on $b_G$ for $GF(q)$ with $q$ odd, where the notion of cuts in graph theory plays an important role.

Recall that a cut of a graph $G = (V, E)$ is a partition of the vertex set $V$ into a pair of disjoint subsets $V_1$ and $V_2$. The cut-set of the partition is the set of edges of $E$ that have one endpoint in $V_1$ and the other in $V_2$. A cut is called a maximal cut if no other cut has a larger cut-set. A maximal cut of a graph can be identified with a maximal bipartite subgraph as follows. A subgraph $(V, E')$ of $G$ is bipartite if and only if its edge set $E'$ form a cut-set of a partition of the vertices of $V$. Then a bipartite subgraph has a maximum number of edges when its edges correspond to a maximal cut of $G$. We denote this number by $\Gamma(G)$.

Lemma V.2 The size of the minimum D-cut $b_G$ of $G = (V, E)$ is upper-bounded by the following:

(i) $\delta_I(G) = \min \{d_I(v) : v \in V\}$,
(ii) $|E| - \Gamma(G)$.

Proof: The first item is clear since we can make an isolated vertex by deleting $\delta_I(G)$ attached edges from a vertex $v$ with incidence degree $d_I(v) = \delta_I(G)$, and the isolated vertex is unretrievable.

To see that (ii) holds, note that to produce a bipartite subgraph $(V, E \setminus L)$ of $G$ by deleting edges $L$ from $E$ we must have $|L| \geq |E| - \Gamma(G)$ and we have equality exactly when $(V, E \setminus L)$ is a maximal bipartite subgraph of $G$.

There are several bounds on the size of a maximal bipartite subgraph. We will use the following results of Edwards (cf. [13], [14] and [15]).

Theorem V.3 (Edwards) Let $G = (V, E)$ have $n$ vertices and $m$ edges. Then

1) $\Gamma(G) \geq \frac{m}{2} + \frac{1}{8}(\sqrt{8m + 1} - 1)$
2) $\Gamma(G) \geq \frac{m}{2} + \frac{n-1}{4}$ if $G$ is connected.

Applying Theorem V.3 to (ii) of Lemma V.2 immediately gives the following bounds on $b_G$.

Corollary V.4 Let $G = (V, E)$ have $n$ vertices and $m$ edges. Then

1) $b_G \leq \frac{m}{2} - \frac{1}{8}(\sqrt{8m + 1} - 1)$, and
2) $b_G \leq \frac{m}{2} - \frac{n-1}{4}$ if $G$ is connected.

The following lemma gives us an upper bound on $b_G$ from the perspective of the minimum incidence degree $\delta_I(G)$.

Lemma V.5 For $G \in \mathcal{D}(n, m)$, $\delta_I(G) \leq 2m/n$, and therefore, $b_G \leq 2m/n$. 
Proof: Recall that the sum of the incidence degrees $S_I(G)$ in $G$ is given by $S_I(G) = 2|E(G)| - L_G \leq 2|E(G)|$. Therefore, $n\delta_I(G) \leq S_I(G) \leq 2m$, which implies $\delta_I(G) \leq 2m/n.$

VI. LOWER BOUNDS ON THE NUMBER OF UNDECODABLE GRAPHS

We have discussed upper bounds of $b_G$, which counts the number of edges that must be deleted to produce a undecodable subgraph. In this section, we give lower bounds on the number of undecodable subgraphs $u^G_x$ of $G$ (or equivalently, upper bounds on the number of decodable subgraphs $c^G_x$ of $G$) for $x \geq b_G$.

A. For $GF(q)$ with $q$ even

We first present the following lemma which describes a sharp lower bound on $u^G_{b_G}$.

Lemma VI.1 Let $G$ be a decodable graph with $n$ vertices and $m$ edges. Then for any $G \in D(n,m)$, we have

$$u^G_{b_G} \geq b_G(n+1) + n - 2m + 1.$$ 

Proof: First recall $b_G \leq \min(L_G, \delta_I(G))$. Let $\alpha$ be the number of vertices with incidence degree $b_G$ and let $\beta$ be the number of vertices with incidence degree at least $b_G + 2$. Then by considering the sum of incidence degrees, we have

$$b_G\alpha + (b_G + 1)(n - \alpha - \beta) + (b_G + 2)\beta \leq 2m - L_G,$$

which implies that

$$\alpha \geq L_G + nb_G + n - 2m + \beta \geq b_G(n+1) + n - 2m,$$

since $L_G \geq b_G$ and $\beta \geq 0$. Clearly $u^G_{b_G} \geq \alpha$, since deleting any $b_G$ edges incident with a vertex of incidence degree $b_G$ results in an undecodable graph. Since $G$ is decodable, no vertex of incidence degree $b_G$ is incident with all loops of $G$. Therefore, if $\alpha = b_G(n+1) + n - 2m$, then $L_G = b_G$ and so $u^G_{b_G} \geq \alpha + 1$ since we also have to count the case of deleting all $L_G$ loops from $G$. If $\alpha > b_G(n+1) + n - 2m$, the result follows trivially.

We can also show a tight lower bound of $u^G_{b_G+y}$ when $y$ is small. We first note the following lemma which can be used in the latter lemma.

Lemma VI.2 Let $G \in D(n,m)$ satisfy $u^G_{b_G} = b_G(n+1) + n - 2m + 1$. Then, with the same notation as in Lemma VI.1, $\beta = 0$ and either

1) $\alpha = b_G(n+1) + n - 2m$ and $L_G = b_G$, or
2) $\alpha = b_G(n+1) + n - 2m + 1$ and $L_G = b_G + 1$.

Proof: Let $\theta = b_G(n+1) + n - 2m$. Recall that, as in the proof of Lemma VI.1

$$\alpha \geq L_G + nb_G + n - 2m + \beta \geq b_G(n+1) + n - 2m = \theta.$$ (5)
Therefore,
\[ \theta + 1 = u^G_{bg} \geq \alpha \geq \theta, \]
so either \( \alpha = \theta \), in which case \( u^G_{bg} = \alpha + 1 \), or \( \alpha = \theta + 1 \) and \( u^G_{bg} = \alpha \).

For the case \( \alpha = \theta \), from (5), we must have \( L_G = b_G \) and \( \beta = 0 \). For the case \( \alpha = \theta + 1 \), we have
\[ \alpha = \theta + 1 \geq L_G + nb_G + n - 2m + \beta = \theta - b_G + L_G + \beta, \]
which gives \( b_G + 1 \geq L_G + \beta \geq b_G + \beta \). Therefore, either \( \beta = 1 \) and \( L_G = b_G \) or \( \beta = 0 \) and \( L_G = b_G + 1 \). Since for \( \alpha = \theta + 1 \) we have \( u^G_{bg} = \alpha \), every undecodable subgraph of \( G \) found by deleting \( b_G \) edges is constructed by deleting the \( b_G \) edges that meet a vertex of incidence degree \( b_G \). If \( L_G = b_G \), then \( G \) has a vertex of incidence degree \( b_G \) that is incident with every loop of \( G \), contradicting the decodability of \( G \). We deduce that \( \beta = 0 \) and \( L_G = b_G + 1 \).

Using Lemma VI.2, we obtain the following lemma giving a lower bound on \( u^G_{bg+y} \) for small values of \( y \). Recall from Section II that \( \Omega(G) \) denotes the maximum multiplicity of any edge joining a pair of vertices of \( G \) and \( \Delta_L(G) := \max\{d_L(v) : v \in V\} \), where \( d_L(v) \) is the number of loops incident with vertex \( v \).

**Lemma VI.3** Let \( G \) be a graph satisfying the hypothesis of Lemma VI.2. Let \( \theta = b_G(n+1) + n - 2m \). Then
\[ u^G_{bg+y} \geq (\theta + 1)\left(\frac{m-b_G}{y}\right) + (n-\theta)\left(\frac{m-b_G-1}{y-1}\right), \]
for any \( y \) satisfying \( 1 \leq y \leq b_G - \max(\Omega(G), \Delta_L(G)) - 1 \).

**Proof:** Let \( y \in \{1, 2, \ldots, b_G - \mu\} \), where \( \mu = \max(\Omega(G) + 1, \Delta_L(G) + 1) \). Consider the following operations, each of which results in an undecodable subgraph of \( G \) with \( m - b_G - y \) edges.

1) Delete \( b_G \) edges incident with a vertex of incidence degree \( b_G \) and delete a further \( y \) edges arbitrarily.

2) Delete \( b_G + 1 \) edges incident with a vertex of incidence degree \( b_G + 1 \) and delete a further \( y - 1 \) edges arbitrarily.

3) Delete all \( L_G \) loops of \( G \), and then delete a further \( b_G + y - L_G \) edges arbitrarily.

Observe first there are exactly \( \alpha\left(\frac{m-b_G}{y}\right) \) (respectively \( (n-\alpha)\left(\frac{m-b_G-1}{y-1}\right) \)) ways to produce an undecodable subgraph by the operation 1) (respectively, by the operation 2)).

The operations 1) and 2) are mutually exclusive, since in 1) at most \( y \leq b_G - 1 \) edges are deleted from a vertex of incidence degree \( b_G + 1 \). Moreover, the operations 2) and 3) are exclusive to each other, since when \( b_G - y \) edges are deleted so that a vertex \( v \) of incidence degree \( b_G + 1 \) is isolated, at most \( d_L(v) + y - 1 \) loops can be deleted and
\[
\begin{align*}
d_L(v) + y - 1 &\leq d_L(v) + (b_G - \Delta_L(G) - 1) - 1 \\
&= b_G - (\Delta_L(G) - d_L(v)) - 2 \\
&\leq b_G - 2 < L_G.
\end{align*}
\]
Similarly, 1) and 3) are exclusive, since when a vertex \( v \) of incidence degree \( b_G \) is isolated, at most \( d_L(v) - y \) loops can be deleted and \( d_L(v) - y \leq b_G - 1 < L_G \).

It follows that

\[
G_{v_G + y} \geq \alpha \left( \frac{m - b_G}{y} \right) + (n - \alpha) \left( \frac{m - b_G - 1}{y - 1} \right) + \left( \frac{m - L_G}{b_G + y - L_G} \right),
\]

which yields for any \( G \in D(n,m) \),

\[
G_{v_G + y} \geq (\theta + 1) \left( \frac{m - b_G}{y} \right) + (n - \theta) \left( \frac{m - b_G - 1}{y - 1} \right).
\]

For given \( G_x \), we can compute a lower bound on \( G_{x+z} \) for \( z \geq 0 \) by using the following easy result.

**Lemma VI.4** Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Then

\[
G_{x+z} \geq G_x \left( \frac{m - x}{z} \right) / \left( \frac{x + z}{z} \right)
\]

for any \( z \geq 0 \).

**Proof:** From a undecodable graph of \( m - x \) edges, we can generate \( \left( \frac{m - x}{z} \right) \) undecodable subgraphs of \( m - x - z \) edges. On the other hand, given a undecodable graph \( K \) of \( m - x - z \) edges, there are at most \( \left( \frac{x + z}{z} \right) \) undecodable graphs of \( m - x \) edges which have \( K \) as a subgraph. Hence, there are at least \( u_x G_{x+z} \) undecodable graphs of \( m - x - z \) edges.

The following corollary is now immediate.

**Corollary VI.5** Let \( G \in D(n,m) \) satisfy the hypothesis of Lemma **VI.2**. Let \( \mu = \max(\Omega(G) + 1, \Delta_t(G) + 1) \) Then for each \( z \geq 0 \), we have

\[
G_{2b_G - \mu + z} \geq G_{2b_G - \mu} \left( \frac{m - 2b_G + \mu}{z} \right) / \left( \frac{2b_G - \mu + z}{z} \right).
\]

**B. For GF(q) with q odd**

Finding good lower bounds on \( G_x \) can be a hard task since not only loops but also odd cycles must be taken into consideration for determining whether a given graph is decodable or not. We have, up to this moment, the following sharp lower bound on \( G_x \) for some special class of graphs.

**Lemma VI.6** Let \( G \) be a graph in \( D(n,rn) \), where \( r \geq 2 \) and \( n \geq 4 \), with the minimum incidence degree \( \delta_I(G) = 2r \). Then we have

\[
G_{2r} \geq n.
\]
Proof: First observe that if $\delta_1(G) = 2r$, then each vertex has incidence degree $2r$ (i.e. $G$ is a $2r$-regular graph). Indeed, if there exists a vertex with incidence degree strictly greater than $2r$, then we have $n\delta_1(G) < S_1(G)$, which is a contradiction since $2rn = n\delta_1(G) \leq S_1(G) \leq 2|E(G)| = 2rn$.

If $b_G = \delta_1(G) = 2r$, then it is straightforward to check that $u_{2r}^G \geq n$ since for each vertex $v$ in $G$, deleting $2r$ edges attached to $v$ in $G$ generates an undecodable graph. Furthermore, if $b_G < \delta_1(G) = 2r$, then we can construct at least $(r^n-b_G)$ undecodable graphs with $rn-2r$ edges by first generating an undecodable graph $K$ consisting of $rn-b_G$ edges, and then deleting another $2r-b_G$ edges arbitrarily from the edges in $K$. Since

$$\begin{align*}
(rn-b_G, 2r-b_G) & \geq rn-b_G > rn-2r \geq n
\end{align*}$$

holds by assumption on $n$ and $r$, we have $u_{2r}^G \geq n$.

\section{Encoding Schemes}

In this section, we introduce two algorithms that produce graphs meeting the bounds derived in the previous sections.

Algorithm $\text{ICHEMES}$ yields an optimal coding scheme for $GF(q)$ with $q$ even. Note that the subscripts $i$ of the packets $p_i$ are computed modulo $n$ in what follows, unless explicitly stated otherwise.

\begin{algorithm}
\caption{A coding scheme for a system with $n$ packets $p_1, p_2, \ldots, p_n$ and redundancy $r$.}
\begin{algorithmic}[1]
\Require Let $k$ be a proper divisor of $n$; so $n = sk$ for some integer $s \geq 1$.
\State Prepare a $k \times rs$ list such that its $(a, b)$ entry is $f_{b+(a-1)rs}$ with the column-wise order \(h_c(a,b) = a + (b-1)k\).
\ForAll{$(a, b)$ with $1 \leq a \leq k, 1 \leq b \leq rs$}
\If{\$1 \leq h_c(a,b) \leq L_G$
\State set $f_{b+(a-1)rs} := p_{b+(a-1)s}$
\ElseIf{$b \leq s-1$
\State set $f_{b+(a-1)rs} := p_{b+(a-1)s} + p_{b+as}$
\ElseIf{$s \leq b \leq rs-1$
\State set $f_{b+(a-1)rs} := p_{b+(a-1)s} + p_{b+(a-1)s+1}$
\ElseIf{$b = rs$
\State set $f_{b+(a-1)rs} := p_{b+(a-1)s} + p_{1+(a-1)s}$
\EndIf
\EndIf
\EndFor
\State \Return $C = \{f_1, f_2, \ldots, f_{rn}\}$ as a coding scheme.
\end{algorithmic}
\end{algorithm}

\begin{example}
Consider the case of 9 packets, redundancy 2. Suppose that $k = 3$ and $L_G = 4$ in Algorithm $\text{ICHEMES}$

First prepare a $3 \times 6$ list (together with the column-wise orders in brackets) as shown below.

\begin{itemize}
\item \textbf{Step 1:} Since $L_G = 4$, for all $(a, b)$ satisfying $1 \leq h_c(a,b) \leq 4$ which are $(1,1), (2,1), (3,1), (1,2)$, set $f_1 = p_1, f_7 = p_4, f_{13} = p_7, f_2 = p_2$.
\item \textbf{Step 2:} Else,
\end{itemize}
\end{example}
is indeed possible to generate a graph $G$ for which $b_G = 2r - 1$ holds based on Algorithm 1.

Proposition VII.1 Let $C$ be the coding scheme for a system sending $n$ packets defined as in Algorithm 1 where the integers $k$ and $r$ satisfy $k, r \geq 2$. Let $s = n/k$ and let the graph representation $G$ of $C$ satisfy $L_G \geq 2r - 1$. If $k \leq L_G \leq (s - 1)k$ then it holds that $b_G = \delta_1(G) = 2r - 1$. 

Recall from Subsection V-A that $b_G \leq 2r - 1$ for $GF(q)$ with $q$ even. The following proposition shows that it is indeed possible to generate a graph $G$ for which $b_G = 2r - 1$ holds based on Algorithm 1.

Algorithm 2 produces optimal schemes over $GF(q)$ for $q$ odd. Table III shows the coding scheme under Algorithm 2 for 9 packets and redundancy 2.

**Algorithm 2**: A coding scheme for a system with $n$ packets $p_1, p_2, \ldots, p_n$ and redundancy $r$.

1. For all $1 \leq i \leq nr$ do
   
   - Set $f_i = p_i + r_i + \lceil \frac{r_i}{k} \rceil$.

2. Return $C = \{f_1, f_2, \ldots, f_{rn}\}$ as a coding scheme.

| Encodings at the relays |
|-------------------------|
| $p_1 + p_2$ | $p_2 + p_3$ | $p_3 + p_4$ | $p_4 + p_5$ | $p_5 + p_6$ | $p_6 + p_7$ |
| $p_7 + p_8$ | $p_8 + p_9$ | $p_9 + p_{10}$ |
| $p_1 + p_2$ | $p_2 + p_3$ | $p_3 + p_4$ | $p_4 + p_5$ | $p_5 + p_6$ | $p_6 + p_7$ |
| $p_7 + p_8$ | $p_8 + p_9$ | $p_9 + p_{10}$ |

**Table II**

The coding scheme under Algorithm 1 for 9 packets and redundancy 2 when $k = 3$ and $L_G = 4$

Table II shows the resulting coding scheme $C$ under Algorithm 7
Proof: Let \( t = sr \). Observe that the graph representation \( G \) of \( \mathcal{C} \) under Algorithm \[ \] satisfies the following properties.

1) Any \((a, 1)\) with \( 1 \leq a \leq k \) satisfies \( h_e(a, 1) = a \leq k \leq L_G \). Since \( n = sk \) by assumption, each vertex \( p_i \) with \( i \equiv 1 \pmod{s} \) has a loop.

2) Any \((a, b)\) such that \( b \geq s \) satisfies \( h_e(a, b) \geq a + (s - 1)k > L_G \), and therefore, lines 7-8 in the algorithm are applied to all pairs \((a, b)\) with \((s - 1)k \leq b \leq rs - 1\). Since there are in total \( sk(r - 1) = n(r - 1) \) of such pairs, for each \( 1 \leq i \leq n \), the number of edges between vertices \( p_i \) and \( p_{i+1} \) is \( r - 1 \).

3) From 2), \( G \) has a connected subgraph consisting of multi-edges \((p_i, p_{i+1})\), \( 1 \leq i \leq n \), and hence, \( G \) itself is connected. Furthermore, \( G \) cannot be disconnected without deleting the multi-edges \((p_i, p_{i+1})\) and \((p_j, p_{j+1})\) for some pair \((i, j)\) with \( i \neq j \).

4) If a vertex \( p_i \) does not have a loop, then
   - it is adjacent to vertex \( p_{i+s} \) when \( i \not\equiv 0 \pmod{s} \) from lines 5-6 in the algorithm.
   - it is adjacent to vertex \( p_{i-t+1} \) otherwise from lines 9-10 in the algorithm.

From the statements above, it is straightforward to see that \( G \) is connected and \( \delta_f(G) = 2r - 1 \) holds, which automatically implies \( b_G \geq 2r - 1 \) from 1) in Remark \[ ] So we focus in the rest of the proof that \( b_G \geq 2r - 1 \) also holds.

Now suppose that for some pair \((i, j)\) with \( i \neq j \), the multi-edges \((p_i, p_{i+1})\) and \((p_j, p_{j+1})\) are deleted from \( G \) (so \( 2(r - 1) \) edges are deleted in total), and call the resulting graph \( \widehat{G} \). Denote by \( H_i \) the subgraph of \( \widehat{G} \) induced by the vertices \( p_{i+1}, p_{i+2}, \ldots, p_{j} \), and by \( H_j \) the one induced by the vertices \( p_{j+1}, p_{j+2}, \ldots, p_{j} \). If both \( H_i \) and \( H_j \) contain loops, then we can conclude that \( b_G \geq 2r - 1 \). Furthermore, it cannot happen that neither \( H_i \) nor \( H_j \) have loops since \( L_G = L_{\widehat{G}} \geq 1 \). Therefore, we need only to consider the case for which \( H_i \) contains a loop but \( H_j \) does not. In this case, we will focus on \( H_i \) and \( H_j \) as subgraphs of \( \widehat{G} \), and show the existence of an edge in \( E(\widehat{G}) \) joining them, which implies that \( b_G \geq 2r - 1 \).

As \( H_j \) does not contain loops, \( |V(H_j)| < s \) since otherwise, at least one of the vertices \( p_{t} \) in \( H_j \) satisfies \( \ell \equiv 1 \pmod{s} \), and therefore, \( H_j \) contains a loop from Property 1).

If \( H_j \) contains a vertex \( p_{t} \) with \( \ell \equiv 0 \pmod{s} \), \( p_{t} \) is adjacent to the vertex \( p_{t-t+1} \), where \( \ell - t + 1 \equiv 1 \pmod{s} \) as \( t = sr \). Since \( p_{\ell-t+1} \) has a loop from Property 1), it is in \( H_i \). If each vertex \( p_{\ell} \) in \( H_j \) satisfies \( \ell \not\equiv 0 \pmod{s} \), then \( p_{\ell} \) is adjacent to \( p_{\ell+s} \). Since \( |V(H_j)| < s \) and \( |V(H_i)| > n - s = (k - 1)s \geq s \), \( p_{\ell+s} \) is in \( H_i \). In each case, there exists an edge in \( E(\widehat{G}) \) joining \( H_i \) and \( H_j \) as required.

Furthermore, we can also prove that when \( 2r - 1 \leq L_G \leq 2r \) the graph \( G \) in Proposition \[ \] satisfies \( u_{b_G}^G = u_{2r-1}^G = 2r \), which is the lower bound of \( u_{b_G}^G \) obtained in Lemma \[ \] as stated below.

**Proposition VII.2** Let \( G \) be the graph satisfying the conditions described in Proposition \[ \] If \( k \geq 3, r \geq 2 \) and \( L_G \) is either \( 2r - 1 \) or \( 2r \), then we have \( u_{b_G}^G = u_{2r-1}^G = 2r \).

**Proof:** It is straightforward to check from the construction of \( G \) that the number of vertices with incidence
degree $2r-1$ (resp. $2r$) is $L_G$ (resp. $n-L_G$). So when $L_G = 2r-1$ or $L_G = 2r$, $2r$ undecodable subgraphs can be found by deleting $2r-1$ edges from $G$ by

(1) making a vertex $v$ with $d_I(v) = 2r-1$ isolated;
(2) deleting all $L_G$ loops from $G$ (when $L_G = 2r-1$)

We will show that we cannot construct other undecodable subgraphs by deleting $2r-1$ edges from $G$ (that is, other undecodable subgraphs of $G$ with exactly $nr-2r+1$ edges).

Assume by contradiction that there exists an undecodable subgraph $H$ of $G$ with $nr-2r+1$ edges constructed by either (1) nor (2) above. Then $H$ must be disconnected since otherwise, we have to delete all $L_G = 2r-1$ loops to make $H$ undecodable. Therefore, from 3) in the proof of Proposition VII.1, $2r-2$ edges $(p_i, p_{i+1})$ and $(p_j, p_{j+1})$, $i \neq j$, must be deleted from $G$ to generate $H$.

As before, let $\hat{G}$ be the subgraph of $G$ found by deleting the multi-edges $(p_i, p_{i+1})$ and $(p_j, p_{j+1})$ for some $i \neq j$. Again, let $H_i$ and $H_j$ be the subgraphs of $\hat{G}$ induced by the vertices $p_{i+1}, p_{i+2}, \ldots, p_j$, and $p_{j+1}, p_{j+2}, \ldots, p_i$, respectively.

First suppose $r \geq 3$, so that neither $H_i$ nor $H_j$ can be disconnected by deleting another a single edge. If $|V(H_i)| = 1$, say $V(H_i) = \{v\}$ for some $v \in V$, then $d_I(v) = 2r$ in $G$ since the case of $d_I(v) = 2r-1$ has been counted in (1). As $d_I(v) = 2r$ and $2r-2$ edges amongst these $2r$ edges have been deleted already, there are two remaining edges attached to $v$. Since a non-loop edge attached to $v$ (if it exists) is joined with a vertex in $H_j$, we can conclude that we need to delete two or more edges from $\hat{G}$ to make $H$, which shows that $|E(H)| < nr-2r+1$.

So now assume that $|V(H_i)|, |V(H_j)| \geq 2$. If both $H_i$ and $H_j$ contain at least two loops, it is trivial that at least two edges must be deleted to make $H$ from $\hat{G}$. If $H_j$ contains no loops, then we can use the same argument in the proof of Proposition VII.1 to show the existence of at least $|V(H_i)| \geq 2$ edges joining $|V(H_i)|$ and $|V(H_j)|$. Furthermore, if $H_j$ contains only one loop (in which case $H_i$ has at least two loops since $L_G \geq 2r-1 \geq 3$), then we have $2 \leq |V(H_i)| \leq 2s-1$, since $H_j$ contains at least two loops whenever $|V(H_j)| > 2s-1$. Thus, $|V(H_i)| \geq n-(2s-1) = s(k-2)+1 > s$ as $k \geq 3$ by assumption. Hence, from 4) in the proof of Proposition VII.1, we have there exists an edge joining $H_i$ and $H_j$. For each case, we can conclude that $|E(H)| < nr-2r+1$, and hence, it is impossible to make an undecodable graph with $nr-2r+1$ edges except for deleting edges according to (1) and (2), as required.

We next suppose that $r = 2$. If neither $H_i$ nor $H_j$ can be disconnected by deleting another a single edge, then use the same argument above. So suppose that $H_j$ can be disconnected by deleting a single edge.

As before we can assume that $|V(H_i)|, |V(H_j)| \geq 2$. Now, assume that an edge $(a, a+1)$ is deleted from $H_j$ and the resulting graph is disconnected. There are two cases to consider. In the first case we suppose that $H_i$ contains a loop. If $|V(H_j)| < s$, each vertex in $H_j$ with no loops is adjacent to some vertex in $H_j$. If $|V(H_j)| \geq s$, at least one vertex in $H_j$ has a loop, which implies that at least one of two components in $H_j$ has a loop. If each of the connected components of $H_j$ contains loop, then we are done, as $H_j$ is decodable. If there exists a component with no loops, then it has at most $s-1$ vertices and some of these are adjacent to vertices of $H_j$ or the other component with loops. For each case, we can conclude that the resulting graph is decodable.
Now suppose that $H_i$ has no loops (so all loops are in $H_j$). Then $|V(H_i)| < s$. Also each vertex in $H_i$ is adjacent to some vertex in $H_j$. If both of two components in $H_j$ contain loops, or if $H_i$ contains two vertices such that one vertex is joining to one component in $H_j$ and the other vertex is joining to the other component in $H_j$, then we are done. If each vertex in $H_i$ is adjacent to vertices in a component in $H_j$ with no loops, then the size of the component is also bounded by $s - 1$. Thus, $n$ vertices are partitioned into 3 parts in the resulting graph, each of which consists of consecutive integers, so that 2 parts contain $s - 1$ or fewer vertices all of which are not equivalent to 1 modulo $s$, yielding a contradiction.

We have the following proposition, for Algorithm 2

**Proposition VII.3** Let $C$ be the coding scheme for a system sending $n$ packets and redundancy $r$, where $n > 3$, defined as in Algorithm 2. Suppose that $n/2 < 4r < n$. Then the representation $G = G_C$ of $C$ satisfies $b_G = \delta_I(G) = 2r$, and furthermore, that $u^2_{b_G} = u^2_{2r} = n$.

**Proof:** Observe that each vertex $p_i$ of $G$ is adjacent to $2r$ vertices $p_{i \pm k}$, $1 \leq k \leq r$, and hence, $d_I(p_i) = 2r$ for each $i$. We claim that except for the case of deleting $2r$ edges incident to a vertex, deleting $2r$ or fewer edges results in a decodable subgraph.

We first show that deleting $2r$ or fewer edges rather than $2r$ incident edges to a vertex always gives us a connected graph. Consider a cut of vertex set $V$ of $G$; so suppose that $V$ is partitioned into two sets $V_1$ and $V_2$. Let $|V_1| = t > 1$ and assume without loss of generality that $t \leq n/2$ since the case of $t > n/2$ implies that $|V_2| = n - t \leq n/2$ and the argument below follows by replacing $V_1$ with $V_2$.

Since there are no multi-edges in $G$, at most $\binom{t}{2}$ edges are used to connect vertices in $V_1$, and therefore, the size of the cut-set is at least $2rt - \binom{t}{2}$. From the following inequality

$$2rt - \binom{t}{2} > 2r$$

$$\Leftrightarrow (t - 1)(4r - t) > 0$$

we have that the size of the cut-set is bigger than $2r$ as long as $1 < t < 4r$. Therefore, we can conclude that the resulting graph is connected whenever $n/2 < 4r$.

We next show that any resulting graph after deletion of edges has an odd cycle. Consider a subgraph $\tilde{G}$ of $G$ consisting of $n$ triangles (cycles of length 3) $\tau_i : p_i, p_{i+1}, p_{i-1}, p_i$ with $1 \leq i \leq n$. It is trivial that $\tau_i$ and $\tau_{i'}$ are edge-disjoint (i.e., they do not share the same edges) if $i$ and $i'$ are both even, and hence, there exist $\lfloor n/2 \rfloor$ edge-disjoint triangles in $\tilde{G}$. Thus, at least $\lfloor n/2 \rfloor$ edges should be deleted from $\tilde{G}$ to make a decodable graph.

If $n$ is even, then $n/2 = \lfloor n/2 \rfloor = \lceil n/2 \rceil$. If $n$ is odd, then $\lfloor n/2 \rfloor = (n - 1)/2$. However, since each edge is used within at most 2 triangles, at least another edge must be deleted to remove all $n$ triangles, which results in a total of $(n - 1)/2 + 1 = (n + 1)/2 = \lceil n/2 \rceil$ edges.

It follows that for any $n$, at least $\lceil n/2 \rceil$ edges must be deleted if the resulting graph is undecodable. Then as long as $n/2 > 2r$, the result holds.
VIII. COMPARISON OF VARIOUS CODING SCHEMES

In this section, we will analyze robustness against packet loss for the coding scheme given by Algorithm 1 and Algorithm 2. We compare each coding scheme over $GF(q)$ with $q$ even and with $q$ odd by computing its decoding probability.

Throughout this section, set the number of packets $n = 9$ and redundancy $r = 2$. We consider the following 12 graphs for comparison.

- $G_0, G_1, \ldots, G_9$ are the graph representations of coding schemes obtained by Algorithm 1 with divisor $k = 3$, where we set the number of uncoded packets (i.e. the number of loops) to be 0, 1, $\ldots, 9$, respectively.
- $G'$ is the graph representation of a coding scheme obtained by Algorithm 2.
- $G$ be the graph corresponding to transmitting packets without coding (i.e., each packet is sent twice to the terminal without coding).

Table IV presents $b_H$ for each graph $H$ for $GF(q)$ with $q$ even and with $q$ odd.

| Graph $H$ | $b_H$ with $q$ even | $b_H$ with $q$ odd |
|-----------|---------------------|---------------------|
| $G_0$     | 0                   | 3                   |
| $G_1$     | 1                   | 3                   |
| $G_2$     | 2                   | 3                   |
| $G_3$     | 3                   | 3                   |
| $G_4$     | 3                   | 3                   |
| $G_5$     | 3                   | 3                   |
| $G_6$     | 3                   | 3                   |
| $G_7$     | 3                   | 3                   |
| $G_8$     | 3                   | 3                   |
| $G_9$     | 3                   | 3                   |
| $G'$      | 0                   | 4                   |
| $G$       | 2                   | 2                   |

TABLE IV
$b_H$ FOR EACH GRAPH $H$

Tables V, VI and VII provide the decoding probabilities $P_H(p, 1-p)$ for each graph $H$ when $p = 0.6$, $p = 0.7$, and $p = 0.8$, respectively. Recall that $p$ is the probability that each packet is successfully transmitted, and graphs with higher decoding probabilities are preferred.

Observe that the decoding probability of a coding scheme over $GF(q)$ with $q$ odd is higher that over $GF(q)$ with $q$ even; for coding schemes over fields of even characteristic decoding relies solely on the presence of loops in each connected component, whilst over fields of odd characteristic any odd cycle in each component will suffice. Furthermore, we can also confirm that the decoding probability of $H$ increases as $b_H$ gets larger.

More importantly, $G_3$ (a graph with 3 loops) and $G'$ give the highest decoding probability amongst those 12 coding schemes over $GF(q)$ with $q$ even and with $q$ odd, respectively, which supports the results provided in this paper. Figures 2 and 3 show the decoding probabilities $P_H(p, 1-p)$ for $G_3$, $G'$ and $G$ over $GF(q)$ with $q$ even with $q$ odd, respectively, for comparison. From these figures, we can see that decoding probability can be increased by up to 0.42 (at $p = 0.65$) for $q$ even and 0.51 (at $p = 0.64$) for $q$ odd, by using the proposed low-power coding schemes.
TABLE V
THE DECODING PROBABILITIES WHEN $p = 0.6$

| Graph | Decoding Prob. with $q$ even | Decoding Prob. with $q$ odd |
|-------|-----------------------------|-----------------------------|
| $G_0$ | 0.405309                    | 0.639395                    |
| $G_1$ | 0.554439                    | 0.646345                    |
| $G_2$ | 0.607826                    | 0.644499                    |
| $G_3$ | 0.685513                    | 0.617078                    |
| $G_4$ | 0.584724                    | 0.592104                    |
| $G_5$ | 0.567454                    | 0.569576                    |
| $G_6$ | 0.541135                    | 0.542086                    |
| $G_7$ | 0.515860                    | 0.516201                    |
| $G_8$ | 0.491854                    | 0.491856                    |
| $G_9$ | 0.703957                    | 0.703957                    |
| $G'$  | 0.208216                    | 0.208216                    |

TABLE VI
THE DECODING PROBABILITIES WHEN $p = 0.7$

| Graph | Decoding Prob. with $q$ even | Decoding Prob. with $q$ odd |
|-------|-----------------------------|-----------------------------|
| $G_0$ | 0.608349                    | 0.856136                    |
| $G_1$ | 0.784283                    | 0.859695                    |
| $G_2$ | 0.812120                    | 0.859901                    |
| $G_3$ | 0.812605                    | 0.838849                    |
| $G_4$ | 0.818934                    | 0.821020                    |
| $G_5$ | 0.803590                    | 0.804415                    |
| $G_6$ | 0.784735                    | 0.784735                    |
| $G_7$ | 0.765603                    | 0.765757                    |
| $G_8$ | 0.747395                    | 0.747396                    |
| $G'$  | 0.427930                    | 0.427930                    |
| $G$   | 0.692534                    | 0.692534                    |

TABLE VII
THE DECODING PROBABILITIES WHEN $p = 0.8$

| Graph | Decoding Prob. with $q$ even | Decoding Prob. with $q$ odd |
|-------|-----------------------------|-----------------------------|
| $G_0$ | 0.847225                    | 0.961702                    |
| $G_1$ | 0.772801                    | 0.964009                    |
| $G_2$ | 0.925741                    | 0.964775                    |
| $G_3$ | 0.955110                    | 0.963999                    |
| $G_4$ | 0.955182                    | 0.956759                    |
| $G_5$ | 0.949392                    | 0.949776                    |
| $G_6$ | 0.934297                    | 0.934350                    |
| $G_7$ | 0.935511                    | 0.935368                    |
| $G_8$ | 0.927755                    | 0.927783                    |
| $G_9$ | 0.920293                    | 0.920293                    |
| $G'$  | 0.932510                    | 0.932510                    |
| $G$   | 0.692534                    | 0.692534                    |

IX. Final Remarks

In this paper, we theoretically analyzed the robustness of coding schemes against packet loss, using only additions of two packets over $GF(q)$. Such coding schemes are well-suited to systems for which data reliability and low computational complexity are strongly preferred (e.g. WBANs since they require little energy for encoding and decoding). We introduced some criteria for a coding scheme to have high decoding probability using graph theory.
Fig. 2. The decoding probabilities of $G_3, G'$ and $G$ over $GF(q)$ with $q$ even.

Fig. 3. The decoding probabilities of $G_3, G'$ and $G$ over $GF(q)$ with $q$ odd.

We also compared decoding probabilities of different schemes. Our results suggest that coding schemes defined over $GF(q)$ with $q$ odd may outperform their counterparts defined over $GF(q)$ with $q$ even.

The results here are related to problems such as the number of bipartite subgraphs of a graph and the size of a maximal cut of a graph. We remark that for linear functions $f_i(p_1, p_2, \ldots, p_n) = \sum_{j \in J} p_{ij}$ with $J \subset \{1, 2, \ldots, n\}$ of size greater than 2, a hypergraph representation can apply. Then as in Lemma III.2 the scheme is decodable if and only if its incidence matrix has full rank $n$.

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