The Fourier coefficients of metaplectic theta series on GL(2) over rational function fields

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To Jeff Hoffstein, to mark his 60th birthday.

Abstract

The idea of the metaplectic theta function was introduced by Tomio Kubota in the 1960s. These theta functions are constructed as residues of Eisenstein series and are only known completely in the case of double covers and, up to the ambiguity of a constant, for triple covers. In 1992 Jeff Hoffstein published formulæ by which these can be computed in certain cases over a rational function field. The author gave an alternative approach in 2007. Both of these methods give the coefficients in a closed form. The rational function field is unusual in that it has a large automorphism group. In this paper we show that this group has an operation on the coefficients. This operation is not visible from the explicit formulæ.

1 Introduction

The purpose of this paper is to take up the investigation begin in the paper [9]. The main result of this paper is Theorem 1 which describes the action of the group of automorphisms of the rational function field on the coefficients of metaplectic theta series.

We fix an integer $n$ and an odd prime power $q$, $q \equiv 1 \pmod{n}$. The coefficients of the theta functions are quantities denoted by $\rho_0(r, \varepsilon, i)$ where

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is an integer $0 \leq i < n$ and $r$ is a non–zero element of $\mathbb{F}_q[x]$. The quantities describe the asymptotic distribution of Gauss sums of order $n$ over $\mathbb{F}_q[x]$. A crucial property of $\rho_0(r, \varepsilon, i)$ is that it depends only on $r$ modulo $n^{th}$ powers. Although the $\rho_0(r, \varepsilon, i)$ were constructed only for polynomial $r$ and they can be extended to arbitrary rational functions $r$ by this property.

The automorphism group of $\mathbb{F}_q(x)$ over $\mathbb{F}_q$ is $\text{PGL}_2(\mathbb{F}_q)$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a non-degenerate matrix over $\mathbb{F}_q$. We shall show that $\rho_0(r_1, \varepsilon, i) = \rho_0(r_2, \varepsilon, i)$ where $r_2(x) = r_1((ax + b)/(cx + d))((ad - bc)/(cx + d)^2)^{1-i}$. It follows that there are a large number of relationships between the $\rho_0(r, \varepsilon, i)$. In particular this very much simplifies the task of computing tables of this function for given $n$ and $q$.

Section 2 is dedicated to summarizing and completing the results of [9] and in Section 3 we give the proof of Theorem 1. It should be noted that such a statement should hold follows from the general discussion of the formalism of the $\rho_0$ given in [7] and [8].

In Section 4 we shall discuss the consequences of Theorem 1 when there are at most three irreducible factors of $r$ and these are of degree 1. Although these are very special they exhibit a number of interesting features which are suggestive. In particular these coefficients are, as was already noted in [9], related to Selberg sums and some generalizations of these. Selberg sums were evaluated by G.W. Anderson, [11], R.J. Evans, [3] and P.B. van Wamelen, [12], and the theory of the metaplectic group throws a new light on the general class of these sums.

2 Summary of notations and previous results

We shall here both establish the notations we shall need and recall those results from [9] which we shall make use of here. The notations will, as the result of experience differ a little from those in the previous paper.

We let $q$ and $n$ be as above and let $k = \mathbb{F}_q(x)$ and $R = \mathbb{F}_q[x]$. We define the map $\chi : \mathbb{F}_q^* \rightarrow \mu_n(\mathbb{F}_q); x \mapsto x^{(q-1)/n}$ where for any field $k \mu_n(k)$ is the set of $n^{th}$ roots of 1 in $k$. Let $\varepsilon$ be an embedding of $\mu_n(\mathbb{F}_q)$ in $\mathbb{C}^\times$. Let $e_\varepsilon : F_q \rightarrow \mathbb{C}^\times$ be a additive character. It is convenient, and involves no loss of generality for our purposes, to assume that $e_\varepsilon$ maps $a$ to $e^{2\pi i j/p}$ where the residue class $j \pmod{p}$ represents $\text{Tr}_{\mathbb{F}_q/F_p}(a)$. We define an additive character $\varepsilon$ on $k$ by $\varepsilon(f) = e_\varepsilon(\sum_v \text{Res}_v(f dx)) = e_\varepsilon(-\text{Res}_\infty(f dx))$ where the sum over $v$ is over all finite places of $k$. Note that in terms of the uniformizer $x_\infty = x^{-1}$ at $\infty$ the latter expression is $e_\varepsilon(\text{Res}_{\infty}(fx^{-2}dx_\infty))$. We define the Gauss sums over $R$ to be $g(r, \varepsilon, c) = \sum_{\xi \pmod{c}} \varepsilon((\xi/c)) e_\varepsilon(r\xi/c)$. Here $r$ and $c$ are non-zero...
elements of $R$. For a character $\omega$ of $F_q^\times$ we define the finite field Gauss sum $\tau(\omega) = \sum_{j \in F_q} \omega(j)e_\omega(j)$.

The Davenport-Hasse theorem implies that for $r$ coprime to $c$ one has

$$g(r, \varepsilon, c) = \mu(c)\varepsilon\left(\frac{r}{c}\right)_n \left(\frac{c'}{c}\right)_n (-\tau(\varepsilon\chi))^{\deg(c)}$$

where $\mu$ denotes the Möbius function in $R$. The case where $c$ and $r$ are no longer assumed to be coprime can be reduced to this case. We shall come back to this later.

The functions which concern us here are

$$\psi(r, \varepsilon, \eta, s) = (1 - q^{n-ns})^{-1} \sum_{c \in R, c \sim \eta} g(r, \varepsilon, c)q^{-\deg(c)s}$$

where $s \in \mathbb{C}, \Re(s) > 3/2$ and $\eta \in k_{\infty}^\times/k_{\infty}^\times$ where $k_{\infty}$ denotes the completion of $k$ at the infinite place. The condition $c \sim \eta$ means that $\eta/c \in R^\times k_{\infty}^\times$.

The function $\psi(r, \varepsilon, \eta, s)$ has at most one pole modulo $2\pi \sqrt{-1}n\log q$ in $\Re(s) > 1$ located at $s = 1 + 1/n$. If it exists it is simple and we denote by $\rho(r, \varepsilon, \eta)$ the residue of $\psi(r, \varepsilon, \eta, s)$ at $s = 1 + 1/n$.

The general theory of Eisenstein series over function fields (see [4]) also shows that there exists a polynomial $\Psi(r, \varepsilon, i, T)$ so that

$$\psi(r, \varepsilon, \pi_i^{-1}, s) = q^{-is}(1 - q^{n+1-ns})^{-1}\Psi(r, \varepsilon, i, q^{-ns}).$$

The function $\Psi(r, \varepsilon, i, T)$ depends only on $i \pmod{n}$. This leads to

$$\rho(r, \varepsilon, \pi_i^{-1}) = c_1q^{-i(n+1)/n}\Psi(r, \varepsilon, i, q^{-n-1})$$

where $c_1 = 1/(n \log q)$. The notation in [9] is different; there the residue was replaced by the value of $(1 - q^{1+\frac{1}{n}-s})\psi(r, \varepsilon, \pi_i^{-1}, s)$ at $s = 1 + 1/n$. This is inessential for our purposes. We shall write

$$\rho_0(r, \varepsilon, i) = \Psi(r, \varepsilon, i, q^{-n-1})$$

which is a much more convenient function to use.

We have also for $i$ with $0 \leq i < n$

$$(1 - q^{n-ns})\psi(r, \varepsilon, \pi_i^{-1}, s) = \frac{q - 1}{n} \sum_{i' \geq i \pmod{n}} C(r, \varepsilon, i')q^{-i's}$$

where

$$C(r, \varepsilon, i) = \sum_{\deg(c) = i, \text{monic}} g(r, \varepsilon, c).$$
If we let
\[ C^*(r, \varepsilon, i) = \sum_{\deg(c) = i, c \text{ monic, } \gcd(r, c) = 1} g(r, \varepsilon, c) \]
then
\[ C(r, \varepsilon, i) = \sum_{r^*} g(r, \varepsilon, r^*) \varepsilon(\chi(-1))^{(i-1)\deg(r^*)} C^*(rr^*n^{-2}, \varepsilon, i - \deg(r^*)) \]
where \( r^* \) runs through the set of integers (modulo units) all of whose prime factors divide \( r \) and where \( g(r, \varepsilon, r^*) \neq 0 \). If we assume, as we shall do in this paper, that no non-trivial \((n-1)^{st}\) power divides \( r \) then we can describe the set of \( r^* \) as follows. Let \( \Sigma \) be the set of primes dividing \( r \), each of which is to be represented by the corresponding monic polynomial. Denote, for \( \pi \in S \) the exponent of \( \pi \) dividing \( r \) by \( e(\pi) \). Then the set of \( r^* \) can be parametrized by the subsets \( S \subset \Sigma \) via \( r^*S = \prod_{\pi \in S} \pi^{e(\pi)+1} \) with the restriction \( \sum_{\pi \in S} (e(\pi) + 1) \deg(\pi) \leq i \). One can reduce the \( g(r, \varepsilon, r^*) \) to a product of Legendre symbols, Gauss sums of the type \( \tau(\varepsilon \chi^j) \) and powers of \( q \). The general formula is neither illuminating nor computationally helpful so that we shall not discuss it here. If we are involved in calculating \( C(r, \varepsilon, i) \) and \( i \leq e(\pi) \deg(\pi) \) for all \( \pi | r \) then the only \( r^* \) in the sum is 1.

If we use the Davenport-Hasse theorem to evaluate the Gauss sums in the definition of \( C^*(r, \varepsilon, i) \) then we see that
\[ C^*(r, \varepsilon, i) = (-1)^i \tau(\varepsilon \chi^i) \sum_c \mu(c) \varepsilon(\left(\frac{c}{r^*n}\right)) \varepsilon(\left(\frac{c'}{r^*n}\right)) \]
where the sum over \( c \) is as before. The polynomial \( \Psi(r, \varepsilon, i, T) \) satisfies a functional equation which is given implicitly in [11, top of p. 251]. One can make this much more useful by a simple observation. Let \( \sigma = \deg(r) + 1 \) and let \( i \) be so that \( 0 \leq i < n \). Let \( R = [\sigma - i)/n] \). Let \( i' \) be the least non-negative residue modulo \( n \) of \( \sigma - i \). We note first that \( [\sigma - i')/n] = R \). To see this we observe that we obtain an equivalent statement if we replace \( \sigma \) by \( \sigma_1 \) where \( \sigma \equiv \sigma_1 \mod n \). We can then assume that \( i \leq \sigma_1 < i + n \) so that \( R = 0 \). Then \( i' = \sigma_1 - i \) and \( \sigma_1 - i' = i \). As \( 0 \leq i < n \) we see that \( [\sigma_1 - i')/n] = 0 \) as required.

Next let \( R_1 = [\sigma - 2i)/n] \) and \( R_2 = [\sigma - 2i')/n] \). Assume that \( i \neq i' \). It is clear that \( R - 1 \leq R_1, R_2 \leq R \). Assume that \( \sigma_1 \) is as above; then \( \sigma_1 - 2i' = 2i - \sigma_1 \) and so \( [\sigma_1 - 2i)/n] + [\sigma_1 - 2i')/n] = -1 \). In the general case we therefore have \( R_1 + R_2 = 2R - 1 \). Thus if \( i < i' \) we have \( R_1 = R \) and \( R_2 = R - 1 \).
With these observations we can now formulate the functional equation for $\Psi(r, \varepsilon, i, \ast)$ in two equivalent forms:

$$
\Psi(r, \varepsilon, i, T) = \left( T^{p_0} \right)^{R + 1} \left( 1 - \frac{1}{T} \right)^{1-p-1} \Psi(r, \varepsilon, i, (p^{2n} T)^{-1}) \\
+ \varepsilon \chi(-1)^{\sigma} \tau(\varepsilon \chi^{i-i'}) (p^n T)^R \Psi(r, \varepsilon, i', (p^{2n} T)^{-1})
$$

and

$$
\Psi(r, \varepsilon, i', T) = \left( T^{q_0} \right)^{R + 1} \left( 1 - \frac{1}{T} \right)^{1-q-1} \Psi(r, \varepsilon, i', (q^{2n} T)^{-1}) \\
+ \varepsilon \chi(-1)^{\sigma} \tau(\varepsilon \chi^{i'-i}) (q^n T)^R \Psi(r, \varepsilon, i, (q^{2n} T)^{-1}).
$$

Here $\sigma$ and $R$ are as above and we assume here and henceforth that $i < i'$. With these notations the “Hecke relations at infinity” can be formulated as follows:

$$
\rho_0(r, \varepsilon, i) = \varepsilon \chi(-1)^{i} \tau(\varepsilon \chi^{i-i'}) q^{-1} \rho_0(r, \varepsilon, i')
$$

and

$$
\rho_0(r, \varepsilon, i') = \varepsilon \chi(-1)^{i'} \tau(\varepsilon \chi^{i'-i}) \rho_0(r, \varepsilon, i).
$$

If $i = i'$ then $\rho_0(r, \varepsilon, i) = 0$ which is the reason for excluding this case in the functional equations.

The corresponding “Hecke relation” for a finite prime $\pi$ takes the form for $r_0$ coprime to $\pi$:

$$
\rho_0(r_0 \pi^j, \varepsilon, i) = \varepsilon \chi(-1)^{(j+1) \deg(\pi)} q^{-\left( (j+1) \deg(\pi) \right)} g(r_0, \varepsilon^{j+1}, \pi) \\
\rho_0(r_0 \pi^{n-2-j}, \varepsilon, i - (j + 1) \deg(\pi))
$$

for $0 \leq j < n - 1$ and

$$
\rho_0(r_0 \pi^{n-1}, \varepsilon, i) = 0.
$$

In [9] the relation (7) refers not to $\rho(r, \varepsilon, i)$ as used here but to the corresponding residue of $\psi(r, \varepsilon, i)$ so that the power of $N(\pi)$ has had to be modified correspondingly. It is convenient to avoid fractional powers of $q$.

We retain the assumption that $i < i'$. Because of the functional equations for the $\Psi(r, \varepsilon, i, T)$ we can determine the polynomial from the coefficients of $\Psi(r, \varepsilon, i, T)$ up to that of $T^{R/2}$ together with those of $\Psi(r, \varepsilon, i', T)$ up to $T^{R/2-1}$ if $R$ is even; if $R$ is odd these are both to be replaced by $T^{(R-1)/2}$. This means that we need only evaluate the corresponding $C(r, \varepsilon, j)$ We shall give more details of the cases in Section 4.

We should stress here that although the $\rho(r, \varepsilon, i)$ only depend on $r$ modulo $n^{th}$ powers the same is not true of $\Psi(r, \varepsilon, i, T)$. We saw above that the global Gauss sums for $k$ could be expressed explicitly in terms of the Gauss sums for the field of constants and certain other
quantities. First of all the connection between the Legendre symbol and the resultant shows that
\[
\left(\frac{c'}{c}\right)_n = \begin{cases} 
\chi(D(c)) & \text{if } \deg(c) \equiv 0, 1 \pmod{4} \\
\chi(-D(c)) & \text{if } \deg(c) \equiv 2, 3 \pmod{4}
\end{cases}
\]
where \(D(c)\) is the discriminant of the polynomial \(c\) (assumed monic).

The evaluation of the quadratic Gauss sum is also valid in \(k\); details are given in, for example, [13, XIII, §12]. This yields a relation, known as Pellet’s formula,
\[
\omega(D(c)) = \mu(c)(-1)^{\deg(c)},
\]
valid of monic polynomials \(c\) where \(\omega\) denotes the quadratic character of \(F_q^\times\).

Several proofs are available - see, for example, [10]. It is this formula that establishes the connection of the \(C^*(r, \varepsilon, i)\) in the special case \(r = x^{e_0}(x-1)^{e_1}\) with a Selberg sum.

3 The main theorem

In [9, §3] we explained how an analogue of \(\psi(r, \varepsilon, \eta, s)\) can be defined over a ring obtained from \(R\) through the inversion of an arbitrary finite set of primes and the relationship of these new functions with the original ones. The formulæ given there need a little further explanation. The variable “\(\eta\)” in \(\psi_S(r, \varepsilon, \eta, s)\) and \(\psi_{S,\cup\{\pi\}}(r, \varepsilon, \eta, s)\) should strictly speaking be considered as elements of \(k_S^\times\) and \(k_S^\times \times k^\times\pi\) respectively. What was written as \(\psi_{S,\cup\{\pi\}}(r, \varepsilon, \eta, s)\) should be \(\sum_{\theta \in r\pi/\pi \times n} \psi_{S,\cup\{\pi\}}(r, \varepsilon, \eta \times \theta, s)\); here \(r\pi\) denotes the ring of integers of \(k\pi\), the completion of \(k\) at \(\pi\).

Now we can move to the following theorem:

**Theorem 1** Let \(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(F_q)\) and set \(\Delta = \det(g)\). Then if \(r \in k\) and \(r_i^\theta\) is defined by \(r_i^\theta(x) = r(\frac{ax+b}{cx+d})\left(\frac{\Delta}{(cx+d)^2}\right)^{1-i}\) then
\[
\rho(r, \varepsilon, i) = \rho(r_i^\theta, \varepsilon, i)
\]

**Proof:** The case where \(c = 0\) is elementary so that we can concentrate on the case \(c \neq 0\). Let \(\pi_1 = x + d/c\) and \(\pi_2 = x - a/c\). Let \(R_1\) (resp. \(R_2\)) be the ring obtained from \(R\) by inverting \(\pi_1\) (resp. \(\pi_2\)). Consider the map \(g : k \to k; f(x) \mapsto f^g(x) = f((ax+b)/(cx+d))\). Set \(\pi_\infty = x^{-1}\). Then \(\pi_\pi^\theta = -c^2\pi_1/\Delta + \mathcal{O}(\pi_1^2)\) (in \(k_\pi\)) and \(\pi_\pi^\theta = -(\Delta/c^2)\pi_\infty + \mathcal{O}(\pi_\infty^2)\) (in \(k_\pi\)). This means that \(g\) maps \(R_2\) to \(R_1\). We also have \(d(x^\theta) = \Delta dx/(cx+d)^2\). We can conclude that, for \(\theta \in \mathbb{F}_q^\times\),
\[
\psi_{\{\infty, a/c\}}(r, \varepsilon, \pi_\infty^{m_1} \times \theta \pi_\pi^{m_2}, s)
\]
is equal to
\[ \psi_{-d/c,\infty}(r^g(x)(\Delta/(cx+d)^2), \varepsilon, (\delta^{-1}\pi_i)^{m_1} \times \theta(\delta \pi_\infty)^{m_2}, s) \]

where \( \delta = -\Delta/c^2 \). Using the transformation properties of \( \psi_{(\infty,a/c)} \) (resp. \( \psi_{(-d/c,\infty)} \)) under units of \( R_2 \) (resp. \( R_1 \)) we deduce that
\[ \psi_{(\infty,a/c)}(r, \varepsilon, \pi_\infty^{m_1} \pi_2^{-m_2} \times \theta, s) \varepsilon \chi(\theta)^{-m_2} \varepsilon \chi(-1)^{m_2+m_1m_2} \]
is equal to
\[ \psi_{(-d/c,\infty)}(r^g(x)(\Delta/(cx+d)^2), \varepsilon, \theta^{-1} \times (\delta \pi_\infty)^{m_1+m_2}, s) \varepsilon \chi(\theta)^{2m_1+2m_2} \varepsilon \chi(-1)^{m_1+m_2} \varepsilon \chi(\delta)^{(m_1+m_2)^2}, \]
consequently
\[ \psi_{(\infty,a/c)}(r, \varepsilon, \pi_\infty^{m_1} \pi_2^{-m_2} \times \theta, s) \]
is equal to
\[ \psi_{(-d/c,\infty)}(r^g(x)(\Delta/(cx+d)^2), \varepsilon, \theta^{-1} \times (\delta \pi_\infty)^{m_1+m_2}, s) \varepsilon \chi(\theta)^{2m_1+2m_2} \varepsilon \chi(-1)^{m_1+m_2} \varepsilon \chi(\delta)^{(m_1+m_2)^2}. \]

Again the behaviour of \( \psi_{(-d/c,\infty)} \) in the first variable under multiplication by a unit (in our case this will be \( (\Delta/(cx+d)^2) \) yields that
\[ \psi_{(-d/c,\infty)}(((\Delta/(cx+d)^2) \varepsilon, \theta^{-1} \times (\delta \pi_\infty)^{m_1+m_2}, s) \varepsilon \chi(\theta)^{2m_1+2m_2} \varepsilon \chi(-1)^{m_1+m_2} \varepsilon \chi(\delta)^{(m_1+m_2)^2}. \]
Together these show that
\[ \psi_{(\infty,a/c)}(r, \varepsilon, \pi_\infty^{m_1} \pi_2^{-m_2} \times \theta, s) \]
is equal to
\[ \psi_{(-d/c,\infty)}(((\Delta/(cx+d)^2) \varepsilon, \theta^{-1} \times (\delta \pi_\infty)^{m_1+m_2}, s). \]
We take the residue at \( s = 1 + 1/n \) and sum over \( \theta \in \mathbb{F}_q^\times \). We now get
\[ \rho_{(\infty)}(r, \varepsilon, \pi_\infty^{m_1} \pi_2^{-m_2}) = \rho_{(\infty)}(((\Delta/(cx+d)^2) \varepsilon, \theta^{-1} \times (\delta \pi_\infty)^{m_1+m_2}. \]

The behaviour under units shows that both sides depend on \( \delta \) in the same way. We therefore obtain the formula of the theorem if we take residues and write \( i \) for \( -m_1 - m_2 \).
4 Consequences

We noted in [9] the elementary statement
\[ \rho_0(r, \varepsilon, i) \in \tau(\varepsilon \chi^i) \times q^{-((n+1)/2)\{(1+\deg(r)-i)/n\}} \mathbb{Z}^{1/n} \]
where the estimate for the denominator can probably be improved. The real challenge is to find the factor lying in \( \mathbb{Z}^{1/n} \). It follows from the results of Section 2 that \( \rho_0(r, \varepsilon, i) = 0 \) if \( R = [(1 + \deg(r) - i)/n] < 0 \). Let \( i \) and \( i' \) be as in §2 with \( i < i' \) and \( 0 \leq i, i' < 1 \). Recall that we have \( R = [(1 + \deg(r) - i')/n] \). The pair \( \rho_0(r, \varepsilon, i) \) and \( \rho_0(r, \varepsilon, i') \) are connected with one another by the Hecke relation at infinity. In this connection we recall that \( \tau(\varepsilon \chi^a)^\varepsilon \) is \((-1)^{1-i}\tau(\varepsilon \chi^a)\) times an (integral) Jacobi sum.

We shall in this section consider the cases where \( r \) has at most three prime factors and that these are of degree 1. By means of a linear transformation we can assume then that \( r \) has the form \( x^{e_0}(x-1)^{e_1}(x-\lambda)^{e_3} \) where \( \lambda \in \mathbb{F}_q \) and \( \lambda \neq 0, 1 \). By the Hecke relations at \( x, x-1 \) and \( x-\lambda \) we can take \( e_j \leq [n/2] - 1 \) for \( j = 0, 1, \lambda \).

If just one prime divides \( r \) we can take \( r \) to be \( x^{e_0} \), the degree of which is \( e_0 \). It follows that that \( \rho_0(x^{e_0}, \varepsilon, i) = 0 \) if \( i > e_0 + 1 \). By means of the Hecke relations at infinity we only need to determine these coefficients when \( i < (e_0 + 1)/2 \). This means that \( C(x^{e_0}, \varepsilon, i) = C^*(x^{e_0}, \varepsilon, i) \). This sum is a degenerate Selberg sum which we could evaluate by means of the results in [12].

We can however proceed in a different way which is, in some respects, illuminating. We apply Theorem 1 with \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and we see that \( \rho_0(x^{e_0}, \varepsilon, i) = \rho_0(x^{-e_0+2i-2}, \varepsilon, i) \). We can now apply the Hecke relation at \( x \) and we see that this latter function is a multiple of \( \rho_0(x^{e_0-2i}, \varepsilon, e_0+1-i) \); the multiplier is made up of a power of \( q \), a power of \( \varepsilon \chi(-1) \) and \( \tau(\varepsilon \chi^{2i-e_0-1}) \). We note that \( i = 0 \) then \( \rho_0(x^{e_0}, \varepsilon, i) = 1 \). If \( i > 0 \) then \( e_0 - 2i > -1 \) and so \( 0 \leq e_0 - 2i < e_0 \). This means that we can compute \( \rho_0(x^{e_0}, \varepsilon, i) \) recursively. We deduce that \( \rho_0(x^{e_0}, \varepsilon, i) \) is \( \tau(\varepsilon \chi^i) \) times a Jacobi sum, that is, a product of elementary Jacobi sums. This is also what the theory of Selberg sums yields.

We now move on to the case where there are two prime factors, thus \( r \) is of the form \( x^{e_0}(x-1)^{e_1} \) where \( 1 \leq e_0, e_1 \leq [n/2] - 1 \). Now \( R = [(e_0 + e_1 + 1 - i)/n] \) and we see that \( \rho_0(x^{e_0}(x-1)^{e_1}, \varepsilon, i) = 0 \) if \( i > e_0 + e_1 + 1 \). Again the Hecke relation at infinity means that we can restrict our attention to the case \( i \leq [(e_0 + e_1 - 1)/2] \). It is clear that we can exchange the roles of \( e_0 \) and \( e_1 \) without changing the function. We may assume also that \( e_0 \leq e_1 \). As we have already studied the case where one of the exponents is 0 we assume that \( e_0 \geq 1 \).
With these restrictions we have \( R = 0 \) in the notation used above. This means that, as we shall see below, \( \rho_0(x^{e_0}(x-1)^{e_1}, \varepsilon, i) \) is equal to \( c_1 \mathcal{C}(x^{e_0}(x-1)^{e_1}, \varepsilon, i) \) where now \( i \) is to be understood not as a residue class but as the least non-negative element of that class. There are two cases to be distinguished. If \( e_0 + 1 > i \) then \( \mathcal{C}(x^{e_0}(x-1)^{e_1}, \varepsilon, i) = \mathcal{C}(x^{e_0}(x-1)^{e_1}, \varepsilon, i) \) and this means that \( \rho_0(x^{e_0}(x-1)^{e_1}, \varepsilon, i) \) is equal to \( \tau(\varepsilon \chi^i) \) times a Selberg sum which itself is a Jacobi sum.

If \( e_0 + 1 \leq i \) then there are precisely two elements \( r^* \) in the terminology of Section 2, 1 and \( x^{e_0+1} \). We can now apply Theorem 1 again but now inverting \( x-1 \) we find that \( \rho_0(x^{e_0}(x-1)^{e_1}, \varepsilon, i) = \rho_0((x-1)^{2n-2}x^{e_0}, \varepsilon, i) \). If we now make use of the Hecke relation at \( (x-1) \) we obtain a multiple of \( \rho_0((x-1)^{e_0+e_1-2i}x^{e_0}, \varepsilon, e_0 + e_1 - i + 1) \). The case \( i = 0 \) does not arise and so \( e_0 + e_1 - 2i + 1 < e_0 + e_1 - i + 1 \) whence it follows that \( \rho_0((x-1)^{e_0+e_1-2i}x^{e_0}, \varepsilon, e_0 + e_1 - i + 1) = 0 \); indeed the corresponding \( \Psi \) function vanishes. It therefore follows that if \( e_0 + 1 \leq i \)

\[
\rho_0(x^{e_0}(x-1)^{e_1}, \varepsilon, i) = 0
\]

and in the other case, namely \( e_0 + 1 > i \)

\[
\rho_0(x^{e_0}(x-1)^{e_1}, \varepsilon, i)
\]

is a power of \( q \) times \( c_1 \tau(\varepsilon \chi^i) \) times a specific Jacobi sum. We shall come back to the details in a later publication. What is rather remarkable is that there is, in these cases, an “explicit formula” for \( \rho_0(x^{e_0}(x-1)^{e_1}, \varepsilon, i) \) and that this is in terms of Jacobi sums. I have not found a method of demonstrating this without the use of the theorem of Anderson, Evans and v. Wamel

This is not the rule. If we next consider \( r \) of the form \( x^{e_0}(x-1)^{e_1}(x-\lambda)^{e_\lambda} \), now with \( 0 < e_0, e_1, e_\lambda \leq [n/2] - 1 \), then we find a large number of relationships between various \( \rho_0(x^{e_0}(x-1)^{e_1}(x-\lambda)^{e_\lambda}, \varepsilon, i) \). The structure of this set of relations is that which one knows from the theory of the hypergeometric function (see [14]). If we have \( \text{Min}(e_0, e_1, e_\lambda) + 1 > i \) and \( e_0 + e_1 + e_2 \leq n \) then we see, as before, that \( \rho_0(x^{e_0}(x-1)^{e_1}(x-\lambda)^{e_\lambda}, \varepsilon, i) = \mathcal{C}(x^{e_0}(x-1)^{e_1}(x-\lambda)^{e_\lambda}, \varepsilon, i) \). The latter sum is an analogue of the hypergeometric function in the same sense that the standard Selberg sum is an analogue of the beta function.

In certain special cases, when \( n = 4 \), or when \( n = 6 \) and \( e_0, e_1, e_\lambda \) are all 2 conjectures of Eckhardt and Patterson and of Chinta, Friedburg and Hoffstein respectively suggest that these sums take on a special form and this is what one finds. It seems as if the corresponding statements can now be proved by a new method due to S. Friedberg and D. Ginzburg [4] but there is still work to be done to complete the proof. In other cases, as one
would expect, the values are irregular. These evaluations go beyond the range considered in [5] and make it clear that with increasing complexity of the $r$ the arithmetical nature of the $\rho_0(r, \varepsilon, i)$ also becomes more complex. Furthermore a complete evaluation does not seem to reasonable expectation and one will have to be satisfied with less specific questions. One can, for example, make estimates for the $\rho_0(r, \varepsilon, i)$ in different metrics. One would suspect that one can do better than relatively elementary convexity bounds.

We now return to the expression of $\rho_0(r, \varepsilon, i)$ in terms of $C(r, \varepsilon, i)$ or of $C^*(r, \varepsilon, i)$ for $R = 0, 1, 2$ where, as before, $R = [(1 + \deg(r) - i)/n]$. Let $0 \leq i < n$ and $i' = (1 + \deg(r) - i)_n$. We assume, as before that $i < i'$. The condition $R \leq 3$ means that we cover all cases with $\deg(r) < 3n$. This is much more than we need for the purposes of this paper.

We recall that by construction

$$\Psi(r, \varepsilon, i, T) = \frac{1 - q^{n+1}T}{1 - q^n} \frac{q - 1}{n} \sum_{i' \geq i \equiv i \pmod{n}} C(r, \varepsilon, i') T^{(i'-i)/n}.$$ 

We shall assume that $0 \leq i < n$.

If $R = 0$ then $\Psi(r, \varepsilon, i, T) = C(r, \varepsilon, i)$ and there is nothing left to be said.

If $R = 1$ then a direct application of the definition yields

$$\Psi(r, \varepsilon, i, T) = C(r, \varepsilon, i) + (C(r, \varepsilon, i + n) - (q - 1)q^n C(r, \varepsilon, i)) T.$$

If $R = 2$ then

$$\Psi(r, \varepsilon, i, T) = C(r, \varepsilon, i) + (C(r, \varepsilon, i + n) - (q - 1)q^n C(r, \varepsilon, i)) T + (C(r, \varepsilon, i + 2n) - (q - 1)q^n C(r, \varepsilon, i + n) - (q - 1)q^{2n} C(r, \varepsilon, i)) T^2$$

It follows that in the three cases we have

$$\Psi(r, \varepsilon, i, q^{-n}) = C(r, \varepsilon, i),$$

$$\Psi(r, \varepsilon, i, q^{-n-1}) = q^{-1}C(r, \varepsilon, i) + q^{-n}C(r, \varepsilon, i + n)$$

and

$$\Psi(r, \varepsilon, i, q^{-n}) = C(r, \varepsilon, i)q^{-2} + C(r, \varepsilon, i + n)q^{-n-2} + C(r, \varepsilon, i + 2n)q^{-n-2}.$$ 

respectively. These are perfectly usable expressions but are not as practical as they might be as the number of summands in $C(r, \varepsilon, i + n)$ and $C(r, \varepsilon, i + 2n)$ is large. It is advantageous to exploit the functional equation when $R = 1$ and $R = 2$; the method is a simple version of the “approximate function equation”.
To carry this out we write \( C_j = C(r, \varepsilon, i + jn) \) and \( C_j' = C(r, \varepsilon, i' + jn) \). We shall use \( X \) with \( X = q^nT \) rather than \( T \). Let \( F(X) = \Psi(r, \varepsilon, i, q^{-n}X) \) and \( G(X) = \Psi(r, \varepsilon, i', q^{-n}X) \). Then \( F(X) = \sum_{0 \leq j \leq R} D_j X^j \) and \( G(X) = \sum_{0 \leq j \leq R} D'_j X^j \) where \( D_j = C_j q^{-nj} - (q-1) \sum_{0 \leq t < j} C_t q^{-nt} \) and \( D'_j = C'_j q^{-nj} - (q-1) \sum_{0 \leq t < j} C'_t q^{-nt} \). Let \( \eta = \varepsilon \chi(-1)^{i \sigma}(\varepsilon \chi^{i'-i}) \) and \( \eta' = \varepsilon \chi(-1)^{i \sigma}(\varepsilon \chi^{i'-i}) \). Note that \( \eta \eta' = q^{-1} \). Then the functional equation takes on the two equivalent forms

\[
F(X) = x^{R+1} \left( \frac{1 - q^{-1}}{1 - q^{-1}X} F\left(\frac{1}{X}\right) + \eta \frac{1 - X}{1 - q^{-1}X} G\left(\frac{1}{X}\right) \right)
\]

and

\[
G(X) = x^R \left( \frac{1 - q^{-1}}{1 - q^{-1}X} G\left(\frac{1}{X}\right) + \eta' \frac{1 - X}{1 - q^{-1}X} F\left(\frac{1}{X}\right) \right)
\]

We can rewrite the first as

\[
G(X) = \eta^{-1} \left( X^R F\left(\frac{1}{X}\right) + (1 - q^{-1}) \frac{F(X) - X^R F\left(\frac{1}{X}\right)}{1 - X} \right)
\]

and the second as

\[
F(X) = \eta'^{-1} \left( X^R G\left(\frac{1}{X}\right) + (1 - q^{-1}) \frac{X G(X) - X^R G\left(\frac{1}{X}\right)}{1 - X} \right)
\]

These can be written as linear expression for the \( D_* \) in terms of the \( D'_* \) or conversely. Explicitly one has

\[
D_k = \eta'^{-1} (D'_{R-k} - (1 - q^{-1}) D_k' + (1 - q^{-1}) \sum_{j < \min(k,R-k)} D'_j - D'_{R-j-1})
\]

and

\[
D'_k = \eta^{-1} (D_{R-k} - (1 - q^{-1}) \sum_{j < \min(k+1,R-k)} D_j - D_{R-j-1}).
\]

These are not precisely what we need for an “approximate functional equation”. What is need to do is to express the \( D_* \) and \( D'_* \) in terms of \( D_0, \ldots, D_{[R/2]} \) and \( D'_0, \ldots, D'_{[R/2]} \). This is easy to do for any given value of \( R \). Thus we find if \( R = 0 \)

\[
D_0 = \eta D'_0,
\]

if \( R = 1 \)

\[
D_1 = \eta'^{-1} D'_0 - (q - 1) D_0
\]

and if \( R = 2 \)

\[
D_2 = \eta'^{-1} D'_0 - (q - 1) D_0.
\]

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It follows that in the three cases we have \( F(X) = D_0 \), \( F(X) = D_0(1 - (q - 1)X) + \eta^{-1}D_0'X \) and \( F(X) = D_0(1 - (q - 1)X^2) + D_1X + \eta^{-1}D_0'X^2 \) respectively. We obtain the values of \( F(1/q) \) in terms of \( D_0 \), of \( D_0 \) and \( D_0' \) and of \( D_0, D_1 \) and \( D_0' \) respectively.

It should be noted that whereas on the one hand the theory of Selberg sums allows us to obtain closed expressions for the coefficients of metaplectic sums on the other hand the theory of metaplectic forms, and, in particular Theorem 1, leads to a number of new relations between Selberg sums which do not seem to be accessible by elementary methods or those of Anderson. This will be the subject of a future paper.

5 Outlook - Curves of higher genus

The case discussed above is very straightforward as the structure of the rational curve is explicit. For other curves there is, in general, no “natural” ring of integers, especially if we demand that it be a principal ideal ring. It seems therefore a considerable challenge to gather numerical evidence in such a case. It is unclear as to whether the nature of the \( \rho_0(r, \varepsilon, i) \) are typical of what happens in the case of curves of genus \( \geq 1 \). There seems to be no reason why not but nevertheless one is able to exploit so much in the rational case that one is cautious about making any too large extrapolations.

One case that is probably deserving of study is that of elliptic curves. To gain some idea of what is needed we consider the function field of an elliptic curve over a field of characteristic \( \neq 2, 3 \). We can then represent it in Weierstrass form and as usual we make the point at infinity the identity element of the group. Hasse’s estimate shows that there is at least one further rational point on the curve and one knows that the group of points over a finite field is either a (non-trivial) cyclic group or a product of two cyclic groups; see \cite{2} Prop. 7.1.9. Let \( P_1 \) (resp. \( P_1 \) and \( P_2 \)) be generators of the group of rational points in these two cases. Then the ring \( R \) of functions integral outside \( \{ \infty, P_1 \} \) (resp. \( \{ \infty, P_1, P_2 \} \)) is a principal ideal domain. The methods of the theory of algebraic curves allow us to represent the ring \( R \) as a quotient of a polynomial ring. This comes down to working with a model of the curve in 3- or 4-dimensional projective space. The determination of the elements of this ring and especially of the prime elements is possible but it is no longer as easy as in the case of the rational function field. Also the description of the relation \( \sim \) in \( k_\mathbb{C}^\times \) is considerably more intricate than before. It seems at the moment that the computational effort needed would be considerable and that it would only be justified it there were good reason to expect that the behaviour of the \( \psi(r, \varepsilon, \eta, s) \) and \( \rho_0(r, \varepsilon, \eta) \) would show
features that do not appear in the case of rational function fields. Whether this is so is an open question at present.

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