Orbits and Hamilton bonds in a family of plane triangulations with vertices of degree three or six

Jan Florek

Institute of Mathematics, University of Economics,
53–345 Wrocław, ul. Komandorska 118/120, Poland

Abstract
Let $P$ be the family of all 2-connected plane triangulations with vertices of degree three or six. Grünbaum and Motzkin proved (in the dual terms) that every graph $P \in P$ is factorable into factors $P_0, P_1, P_2$ (indexed by elements of the cyclic group $Q = \{0, 1, 2\}$) such that every factor $P_q$ consists of two induced paths with the same length $M(q)$, and $K(q) - 1$ induced cycles with the same length $2M(q)$. For $q \in Q$, we define an integer $S^+(q)$ such that the vector $(K(q), M(q), S^+(q))$ determines the graph $P$ (if $P$ is simple) uniquely up to orientation-preserving isomorphism. We establish arithmetic equations that will allow calculate the vector $(K(q + 1), M(q + 1), S^+(q + 1))$ by the vector $(K(q), M(q), S^+(q))$, $q \in Q$. We present some applications of the equations. The set $\{(K(q), M(q), S^+(q)) : q \in Q\}$ is called the orbit of $P$. We characterize one point orbits of graphs in $P$. We prove that if $P$ is of order $4n + 2$, $n \in \mathbb{N}$, than it has a Hamilton bond such that the end-trees of the bond are equitable 2-colorable and have the same order. We prove that if $M(q)$ is odd and $K(q) \geq \frac{M(q)}{3}$, then there are two disjoint induced paths of the same order, which vertices together span all of $P$.

1. Introduction

Let $G_i$, $i = 1, 2$, be a plane graph with the vertex set $V(G_i)$, the edge set $E(G_i)$, and the face set $F(G_i)$. An isomorphism $\sigma$ between $G_1$ and $G_2$ is called combinatorial if it can be extended to a bijection

$$\sigma : V(G_1) \cup E(G_1) \cup F(G_1) \to V(G_2) \cup E(G_2) \cup F(G_2)$$

Email address: jan.florek@ue.wroc.pl (Jan Florek)

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that preserves incidence not only of vertices with edges but also of vertices and edges with faces (Diestel [2, p. 93]). Furthermore, we say that $G_1$ and $G_2$ are op-equivalent (equivalent up to orientation-preserving isomorphism) if $\sigma$ is a combinatorial isomorphism which preserves the counter clockwise orientation. (Formally: we require that $g_1, g_2, g_3$ are counter clockwise successive edges incident with a vertex $v$ if and only if $\sigma(g_1), \sigma(g_2), \sigma(g_3)$ are counter clockwise successive edges incident with $\sigma(v)$).

A factor of a graph is a subgraph whose vertex set is that of the whole graph. A graph $H$ is said to be factorable into factors $H_1, H_2, \ldots, H_t$ if these factors are pairwise edge-disjoint and $E(H) = E(H_1) \cup \ldots \cup E(H_t)$ (see Chartrand-Lesniak [1, p. 246]). An edge (respectively a subgraph) of $H$ is said to be of class $q$ if this edge (respectively any edge of this subgraph) belongs to the factor $H_q$.

Let $P$ be the family of all 2-connected plane triangulations all of whose vertices are of degree 3 or 6, and suppose that $P \in P$. Grünbaum and Motzkin [6, Lemma 2] proved (in the dual terms) that the graph $P$ is factorable into factors $P_0, P_1, P_2$ (indexed by elements of the cyclic group $Q = \{0, 1, 2\}$) satisfying the following condition:

\begin{align*}
\text{(**}) & \text{ for } q \in Q, \text{ there is a drawing of } P \text{ (called } q\text{-drawing) which is op-equivalent to } P. \text{ The } q\text{-drawing of } P \text{ consists of a maximal path of class } q \text{ with the length } M(q), \text{ and this path is surrounded by } K(q) - 1 \text{ disjoint cycles of class } q \text{ with the same length } 2M(q). \text{ Finally, there is another maximal path of class } q \text{ with the length } M(q) \text{ (called outer path) around the outside of the last cycle (see Example 1.1).}
\end{align*}

By (**

\begin{align*}
(1) & \text{ } 2K(q)M(q) + 2 \text{ is the order of } P.
\end{align*}

Notice that the outer path may be added at different positions. We define (Definition 2.2) the integer $0 \leq S^+(q) < M(q)$ (and also $0 < S^-(q) \leq M(q)$)
that determines the position. In Theorem 2.1 we show the following relation between $S^+(q)$ and $S^-(q)$:

$$(2) \quad S^-(q) - S^+(q) \equiv K(q) \pmod{M(q)}.$$  

The vector $(K(q), M(q), S^+(q))$, for $q \in Q$, is called a $q$-index-vector of $P$, and the set $\{(K(q), M(q), S^+(q)) : q \in Q\}$ is called the orbit of $P$. The purpose of this article is to establish arithmetic equations that will allow to calculate the $(q+1)$-index-vector by the $q$-index-vector of $P$. In Theorem 3.1 we prove the following equality:

$$(3) \quad K(q+1) = |S^+(q), M(q)|,$$

where $|s, m|$ is the greatest common divisor of integers $s \geq 0$ and $m \geq 1$ ($|0, m| = m$). Let $0 < b \leq M(q)/|S^+(q), M(q)|$ be an integer such that $bS^+(q) \equiv -|S^+(q), M(q)| \pmod{M(q)}$. In Theorem 3.2 we prove that

$$(4) \quad S^-(q+1) = bK(q).$$

Notice that every isomorphism between two 3-connected plane graphs is combinatorial (Diestel [2, p. 94]). Hence, if $P$ is simple, then it is determined by any of its index-vectors uniquely up to op-equivalence. Therefore, using equations (1)–(4) we can verify whether considered simple graphs in $P$ are op-equivalent.

**Example 1.1.** Let us consider the simple graph $S_0$ of Fig. 1, and simple graphs $S_1$, $S_2$ of Fig. 2. We assume that edges $g_0$, $g_1$, $g_2$ are of class 0, 1, 2, respectively, and they are incident with a common vertex of degree 3. The graph $S_0$ has the 0-index-vector $(1, 6, 3)$, $S_1$ has the 1-index-vector $(3, 2, 0)$ and $S_2$ has the 2-index-vector $(2, 3, 1)$. Using equations (1)–(4) we check that $\{(1, 6, 3), (3, 2, 0), (2, 3, 1)\}$ is their common orbit (see Example 3.1). Hence, the graph $S_0$ is the 0-drawing of the graph $S_1$ and $S_2$, $S_1$ is the 1-drawing of $S_0$ and $S_2$, and $S_2$ is the 2-drawing of $S_0$ and $S_1$. Therefore these graphs are op-equivalent.

We are going to present some applications of equations (1)–(4). From (**) follows that $X = \{(k, m, s) \in \mathbb{Z}^3 : 1 \leq k, 1 \leq m, 0 \leq s < m\}$ is the set of all index-vectors of graphs in $P$. In Theorem 4.1 we characterize one point orbits: $(k, m, s) \in X$ is a one point orbit of a graph in $P$ if and only if $m = kn$, $s = kx$, where $k$, $x$, $n$ are integers such that $k \geq 1$, $0 \leq x < n$ and $n$ is a divisor of $x^2 + x + 1$. By Schinzel and Sierpiński [8] the set of all integral solutions of the equation $x^2 + x + 1 = 3y^2$ is infinite. It follows that
Figure 1: The graph $S_0$ with the 0-index-vector $(1, 6, 3)$.

Figure 2: The graph $S_1$ with the 1-index-vector $(3, 2, 0)$, and $S_2$ with the 2-index-vector $(2, 3, 1)$. 
there is an infinite family of graphs in \( \mathcal{P} \) with one point orbit.

If \( P \) has an index-vector \((K(q), M(q), S^+(q))\), then its mirror reflection has the index-vector \((K(q), M(q), M(q) - S^-(q))\). We say that \( P \) is double mirror symmetric if there exists \( q_1, q_2 \in Q \) such that \( S^+(q_i) = M(q_i) - S^-(q_i) \), for \( i = 1, 2 \). In Theorem 4.2 we show that \( P \) is double mirror symmetric if and only if \( P \) has a one point orbit of the form \( \{(k, k, 0)\} \) or \( \{(k, 3k, k)\} \) for some \( k \in \mathbb{N} \).

A bond of a connected plane graph \( G \) is a minimal non-empty edge cut (Diestel [2, p. 25]). A Hamilton bond of \( G \) is a bond \( B \) such that both components of \( G \setminus B \) are trees, and the trees are called end-trees of the bond. It is known that a Hamilton bond in \( G \) is the algebraic dual of a Hamilton cycle in the dual graph (see Stein [10]). Goodey [4] showed that every 2-connected cubic plane graph whose faces are only triangles or hexagons has a Hamilton cycle. Hence, every graph in \( \mathcal{P} \) has a Hamilton bond. In Theorem 5.1 we prove that for every Hamilton bond of a graph in \( \mathcal{P} \) the end-trees of the bond have the same order. In Theorem 5.2 we prove a similar result for the family \( \mathcal{H} \) of all 2-connected plane triangulations all of whose vertices are of degree at most 6. Namely, for every Hamilton bond of a graph in \( \mathcal{H} \) the orders of end-trees of the bond differ by at most 3.

In [7] Meyer introduced the following notation of equitable colorability. A graph \( G \) is equitable \( k \)-colorable if there exists a proper \( k \)-coloring of \( G \) such that the size of any two color classes differ by at most one. It is easy to see, by condition (**), that every graph in \( \mathcal{P} \) is equitable 4-colorable. We know that every graph in \( \mathcal{P} \) has a Hamilton bond and the end-trees of the bond have the same order. One may guess that every graph in \( \mathcal{P} \) has a Hamilton bond such that end-trees are equitable 2-colorable. In fact, in Theorem 6.1 we prove (using the equations (1)–(3)) that this is the case if \( P \) is of order \( 4n + 2 \), \( n \in \mathbb{N} \). We also prove that if \( P \) has an index-vector \((K(q), M(q), S^+(q))\) such that \( M(q) \) is odd and \( K(q) \geq \frac{M(q)}{4} \), then there are two disjoint induced paths which vertices together span all of \( P \) (see Theorem 6.2).

2. Index-vector

Let \( \mathcal{P} \) be the family of all 2-connected plane triangulations all of whose vertices are of degree 3 or 6. Fix \( P \in \mathcal{P} \). Let \( P \) be factorable into factors \( P_0, P_1, P_2 \) (indexed by elements of the cyclic group \( Q = \{0, 1, 2\} \)) satisfying the condition (*). We recall that a subgraph of \( P \) is said to be of class \( q \in Q \) if any edge of the subgraph belongs to the factor \( P_q \). Let \( M(q) \) be the length of
a maximal path of class $q$, and $K(q)$ the distance between the two maximal paths of this class in $P$.

**Definition 2.1.** Let $A$ be a vertex of degree 3 in the graph $P$, and suppose that $[A, q]$ is a maximal path of class $q$ with a fixed orientation $v_0v_1 \ldots v_{M(q)}$ such that $A = v_0$ is its initial and $A_q = v_{M(q)}$ is its terminal vertex. An edge $e$ adjacent to the path $[A, q]$ is called a left branch of the path if it is branching off from $[A, q]$ to the left (more precisely, if $v_jv_{j+1}, e$, $0 \leq j < M(q)$, or $e, v_{j-1}v_j$, $0 < j \leq M(q)$, are counter clockwise successive edges incident with the vertex $v_j$). Otherwise, it is called a right branch of the path. We put

$$[A, q](e) = \begin{cases} j & \text{if } e \text{ is a left branch of } [A, q], \\ 2M(q) - j & \text{if } e \text{ is a right branch of } [A, q]. \end{cases}$$

**Remark 2.1.** Notice that $[A, q] = v_0v_1 \ldots v_{M(q)}$ if and only if $[A_q, q] = v_{M(q)}v_{M(q)-1} \ldots v_0$. An edge $e$ is a left branch of the path $[A, q]$ if and only if it is a right branch of the path $[A_q, q]$. Moreover, we have

$$|[A_q, q](e) - [A, q](e)| = M(q).$$

**Lemma 2.1.** Let $A, C$ be ends of two different maximal paths of class $q$.

(1) If $e, \hat{e}$ and $f, \hat{f}$ are pairs of end-edges of two minimal paths of class $q + 1$ so that $e, f$ are adjacent to the path $[A, q]$ and $\hat{e}, \hat{f}$ are adjacent to the path $[C, q]$, then

$$[A, q](f) - [A, q](e) \equiv [C, q](\hat{e}) - [C, q](\hat{f}) \pmod{2M(q)}.$$

(2) Moreover, if the edge $e$ is incident with $A$, and the edge $\hat{f}$ is incident with $C$, then

$$[A, q](f) = [C, q](\hat{e}).$$

**Proof.** It is more clear when we consider the $q$-drawing of $P$. Notice that

$$[A', q](f) - [A', q](e) = [C', q](\hat{e}) - [C', q](\hat{f})$$

for some $A' \in \{A, A_q\}$ and for some $C' \in \{C, C_q\}$. Hence, by Remark 2.1 we obtain (1). By ($*$), $e$ is a left branch of the path $[A, q]$ and $\hat{f}$ is a left branch of the path $[C, q]$. Hence, $[A, q](e) = 0$ and $[C, q](\hat{f}) = 0$, which yields (2). □
Definition 2.2. Let $A, C$ be ends of two different maximal paths of class $q$ in the graph $P$, and suppose $f$ (or $g$) is the first edge of the path $[C, q+1]$ (or $[C, q-1]$, respectively) which is adjacent to the path $[A, q]$. Let

$$S^+(q) = \begin{cases} [A, q](f), & \text{if } f \text{ is a left branch of } [A, q], \\ [A, q](f) - M(q), & \text{if } f \text{ is a right branch of } [A, q], \end{cases}$$

$$S^-(q) = \begin{cases} [A, q](g), & \text{if } g \text{ is a left branch of } [A, q], \\ [A, q](g) - M(q), & \text{if } g \text{ is a right branch of } [A, q], \end{cases}$$

Notice that by Remark 2.1 and Lemma 2.1(2) the definition of $S^+(q)$ and $S^-(q)$ do not depend on the choice of ends of two different maximal paths of class $q$. The following theorem shows that $S^+(q)$ is determined by $S^-(q)$ and vice versa.

**Theorem 2.1.**

$$S^-(q) - S^+(q) \equiv K(q) \pmod{M(q)}.$$

**Proof.** It is more clear if one considers the $q$-drawing of $P$. Let $A, C$ be ends of two different maximal paths of class $q$ in $P$, and suppose $f$ (or $g$) is the first edge of the path $[C, q+1]$ (or $[C, q-1]$) which is adjacent to the path $[A, q]$, say in a vertex $E$ (or $F$, respectively). If $V$ is the last common vertex of the path $[C, q+1]$ with a segment $CF$ of $[C, q-1]$, then we have

$$[A, q](g) - [A, q](f) \equiv |VE| \equiv K(q) \pmod{2M(q)}.$$

Hence,

$$S^-(q) - S^+(q) \equiv [A, q](g) - [A, q](f) \equiv K(q) \pmod{M(q)},$$

which completes the proof. \qed

3. Billiards and structure of graphs in $\mathcal{P}$

Let $\mathcal{P}$ be the family of all 2-connected plane triangulations all whose vertices are of degree 3 or 6. Fix $P \in \mathcal{P}$ and $q \in Q$ (where $Q = \{0, 1, 2\}$ is the cyclic group). Let $(K(q), M(q), S^+(q))$ be the $q$-index-vector of $P$. 

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If $0 < \theta < 1$, then a $\theta$-billiard sequence is a sequence $F(j) \in [0, 1)$, $j \in \mathbb{N}$, which satisfies the following conditions (see [3]): $F(1) = 0$ and

\[
F(j) + F(j + 1) = \begin{cases} 
\theta & \text{or } 1 + \theta, \quad \text{for an odd } j, \\
0 & \text{or } 1, \quad \text{for an even } j.
\end{cases}
\]

We consider a billiard table rectangle with perimeter of length 1 with the bottom left vertex labeled $v_0$, and the others, in a clockwise direction $v_1, v_2$ and $v_3$. The distance from $v_0$ to $v_1$ is $\theta/2$. We describe the position of points on the perimeter by their distance along the perimeter measured in the clockwise direction from $v_0$, so that $v_1$ is at position $\theta/2$, $v_2$ at $1/2$ and $v_3$ at $(\theta + 1)/2$. If a billiard ball is pushed from position $F(1) = 0$ at the angle of $\pi/4$, then it will rebound against the sides of the rectangle consecutively at points $F(2), F(3), \ldots$.

The following Lemma 3.1 comes from [3, Theorem 3.3(2) and Example 3.1].

**Lemma 3.1.** If $0 < s/m < 1$ is a fraction, $d = |s, m|$ and $F(j), j \in \mathbb{N}$, is the $s/m$-billiard sequence, then,

1. $\{2mF(1), 2mF(2), \ldots, 2mF(m/d)\} = \{0, 2d, 4d, \ldots, 2m - 2d\}$.
2. $2mF(m/d) = \begin{cases} 
s, & \text{for } s/d \text{ even,} \\
m, & \text{for } m/d \text{ even,} \\
s + m, & \text{for } s/d \text{ and } m/d \text{ both odd,}
\end{cases}$

and $2mF(j) \notin \{s, m, s + m\}$, for $1 \leq j < m/d$.

3. If $a, b$ are natural numbers, $am - bs = d$ and $b \leq m/d$, then,

\[
2mF(b) = \begin{cases} 
2s + d, & \text{for } a \text{ even,} \\
2m - d, & \text{for } b \text{ even,} \\
2s + m + d, & \text{for } a \text{ and } b \neq 1 \text{ both odd,} \\
0, & \text{for } a = b = 1.
\end{cases}
\]

**Remark 3.1.** The sequence of all reduced fractions of the interval $[0, 1]$ with denominators not exceeding $n$, listed in order of their size, is called the Farey sequence of order $n$ ($0/1$ is the smallest and $1/1$ the biggest fraction of any Farey sequence). Let $0 \leq s/m < 1$ be a fraction, and suppose that $s'/m' = s/m$ is a fraction in lowest terms. Then $s'/m' < a/b$ are consecutive fractions
in the Farey sequence of order $m'$ if and only if $am - bs = |s, m|$ and $b \leq m'$ (see Schmidt [4]).

The following theorem shows that the structure of the graph $P$ is closely related to the $S^+(q)/M(q)$-billiard sequences.

**Theorem 3.1.** Let $A$ be a vertex of degree 3 in $P$, and suppose that $e_1, e_2, \ldots, e_n$ is a sequence of all consecutive edges of the path $[A, q + 1]$ which are adjacent to the path $[A, q]$.

1. If $n > 1$, then

$$[A, q](e_j) = 2M(q)F(j), \text{ for } 1 \leq j \leq n,$$

where $F(j)$, $j \in \mathbb{N}$, is the $S^+(q)/M(q)$-billiard sequence,

2. $n = M(q)/|S^+(q), M(q)|$,
3. $K(q + 1) = |S^+(q), M(q)|$,
4. $K(q + 1)M(q + 1) = K(q)M(q)$.

**Proof.** Since $2K(q)M(q) + 2$ is the order of $P$, condition (4) holds.

If $n = 1$, then $S^+(q) = 0$, $M(q + 1) = K(q)$ and, by (4), $K(q + 1) = M(q)$. Hence, conditions (2) and (3) are satisfied.

Let $n > 1$. Let $C$ be a vertex of degree 3, $C \neq A$, $C \neq A_q$, and suppose that $f$ is the first edge of the path $[C, q + 1]$ which is adjacent to the path $[A, q]$. Without loss of generality we can assume, by Remark 2.1, that $f$ is a left branch of $[A, q]$. Hence, $[A, q](f) = S^+(q)$. Suppose that $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n$ is a sequence of all consecutive edges of the path $[A, q + 1]$ which are adjacent to the path $[C, q]$. Note that the edges $\hat{e}_{2j-1}$, $\hat{e}_{2j}$ are incident with the same vertex of the path $[C, q]$ and that they are on the opposite sides of this path. Hence, we get

$$[C, q](\hat{e}_{2j-1}) + [C, q](\hat{e}_{2j}) = 2M(q).$$

By Lemma 2.1(1), we have

$$[A, q](e_j) + [C, q](\hat{e}_j) \equiv [A, q](f) \equiv S^+(q) \pmod{2M(q)}, \text{ for } 1 \leq j \leq n.$$

Hence, we obtain

$$[A, q](e_{2j-1}) + [A, q](e_{2j}) \equiv 2S^+(q) \pmod{2M(q)}, \text{ for } 2 \leq 2j \leq n.$$
Since \(0 \leq [A, q](e_{2j-1}) + [A, q](e_{2j}) < 4M(q)\) and \(0 < S^+(q) < M(q)\) we get

(i) \([A, q](e_{2j-1}) + [A, q](e_{2j}) = 2S^+(q)\) or \(2M(q) + 2S^+(q), \ 2 \leq 2j \leq n.\)

By analogy, the edges \(e_{2j}, e_{2j+1}\) are incident with the same vertex of the path \([A, q]\) and, therefore, they are on the opposite sides of this path. Hence, we have

(ii) \([A, q](e_{2j}) + [A, q](e_{2j+1}) = 2M(q), \ 2 \leq 2j \leq n - 1.\)

By (i) and (ii) we obtain (1).

By definition of \(A_q, C\) and \(C_q\), we have

\[
A_{q+1} = A_q \text{ and } j = n \iff [A, q](e_j) = M(q),
\]
\[
A_{q+1} = C \text{ and } j = n \iff [C, q](e_j) = 0 \iff [A, q](e_j) = S^+(q),
\]
\[
A_{q+1} = C_q \text{ and } j = n \iff [C, q](e_j) = M(q) \iff [A, q](e_j) = M(q) + S^+(q).
\]

Accordingly,

(iii) \([A, q](e_n) \in \{M(q), S^+(q), M(q) + S^+(q)\}\), and

\([A, q](e_j) \notin \{M(q), S^+(q), M(q) + S^+(q)\}\) for \(j < n.\)

By (1) and Lemma 3.1(2), condition (iii) leads to \(n = M(q)/|S^+(q), M(q)|.\)

Since \(n = M(q)/|S^+(q), M(q)|\) condition (4) shows that

\[
M(q + 1) = nK(q) = \frac{M(q)K(q)}{|S^+(q), M(q)|} = \frac{M(q + 1)K(q + 1)}{|S^+(q), M(q)|}.
\]

Thus \(K(q + 1) = |S^+(q), M(q)|\) and condition (3) holds. \(\square\)

By analogy, we obtain the following corollary:

**Corollary 3.1.** Let \(A\) be a vertex of degree 3 in \(P\), and suppose that \(e_1, e_2, \ldots, e_n\) is a sequence of all consecutive edges of the path \([A, q - 1]\) which are adjacent to the path \([A, q]\).

(1) If \(n > 1\), then

\([A, q](e_j) = 2M(q)F(j), \ \text{for } 1 \leq j \leq n,\)

where \(F(j), j \in \mathbb{N},\) is the \(S^-(q)/M(q)\)-billiard sequence,

(2) \(n = M(q)/|S^-(q), M(q)|,\)

(3) \(K(q - 1) = |S^-(q), M(q)|.\)
Theorem 3.2. Let \( A \) be a vertex of degree 3 in \( P \), and suppose that \( a, b \) are natural numbers such that \( aM(q) - bS^+(q) = d \) and \( b \leq M(q)/d \), where \( d = |S^+(q), M(q)| \). Then we have:

1. \( S^-(q + 1) = bK(q) \),
2. \( S^+(q + 1) \equiv bK(q) + K(q + 1) \pmod{M(q + 1)} \).

**Proof.** Suppose that \( e_1, e_2, \ldots, e_n \) is a sequence of all consecutive edges of the path \([A, q + 1]\) which are adjacent to the path \([A, q]\) in vertices \( A = E_1, E_2, \ldots, E_n \), respectively.

If \( n = 1 \), then \( A \) is the only common vertex of paths \([A, q + 1]\) and \([A, q]\). Hence, \( S^+(q) = 0 \) and \( S^-(q + 1) = M(q + 1) = K(q) \). Then \( a = b = 1 \), and condition (1) holds.

Let \( n > 1 \). Let \( C \) be a vertex of degree 3, \( C \neq A, C \neq A_q \), and suppose that \( f \) is the first edge of the path \([C, q + 1]\) which is adjacent to the path \([A, q]\). Without loss of generality we can assume, by Remark 2.1, that \( f \) is a left branch of \([A, q]\). Hence, \([A, q](f) = S^+(q) \). Suppose that \( \hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n \) is a sequence of all consecutive edges of the path \([A, q + 1]\) which are adjacent to the path \([C, q]\) in vertices \( \hat{E}_1, \hat{E}_2, \ldots, \hat{E}_n \), respectively. Note that \( E_j = \hat{E}_{j+1} \) for \( j \) even, \( \hat{E}_j = \hat{E}_{j+1} \) for \( j \) odd, and a segment \( E_j \hat{E}_j \) of the path \([A, q + 1]\) has the length \( |E_j \hat{E}_j| = K(q) \). Hence, segments \( AE_b \) and \( A\hat{E}_b \) of the path \([A, q + 1]\) have the lengths:

\[
(i) \begin{cases} |AE_b| = bK(q), & \text{for } b \text{ even,} \\ |A\hat{E}_b| = bK(q), & \text{for } b \text{ odd.} \end{cases}
\]

By Remark 2.1 and Lemma 2.1(1), we have

\[
[A_q, q](e_j) - [A_q, q](e_i) \equiv [A, q](e_j) - [A, q](e_i) \equiv [C, q](\hat{e}_i) - [C, q](\hat{e}_j) \\
\equiv [C_q, q](\hat{e}_i) - [C_q, q](\hat{e}_j) \pmod{2M(q)}, \text{ for } 1 \leq i, j \leq n.
\]

From Theorem 3.1(2) it follows that \( n = M(q)/d \). Hence, by Lemma 3.1(1), we obtain

\[
(ii) \begin{cases} [A_q, q](e_j) - [A_q, q](e_i) \equiv [C, q](\hat{e}_i) - [C, q](\hat{e}_j) \\
\equiv [C_q, q](\hat{e}_i) - [C_q, q](\hat{e}_j) \equiv 0 \pmod{2d}, & \text{for } 1 \leq i, j \leq n. \end{cases}
\]

By Lemma 3.1(3) we get
\[ [A, q](e_b) = \begin{cases} 
S^+(q) + d, & \text{for } a \text{ even,} \\
M(q) - d, & \text{for } b \text{ even,} \\
S^+(q) + M(q) + d, & \text{for } a \text{ and } b \neq 1 \text{ both odd.} 
\end{cases} \]

Since \([A, q](f) = S^+(q)\), Lemma 2.1(1) shows that

\[ [C, q](\hat{e}_b) \equiv [A, q](f) - [A, q](e_b) \equiv S^+(q) - [A, q](e_b) \pmod{2M(q)}. \]

Accordingly, by Remark 2.1, we obtain

\[ \begin{aligned}
& (iii) \quad [C, q](\hat{e}_b) = 2M(q) - d, & & \text{for } a \text{ even,} \\
& [A, q](e_b) = 2M(q) - d, & & \text{for } b \text{ even,} \\
& [C_q, q](\hat{e}_b) = 2M(q) - d, & & \text{for } a \text{ and } b \neq 1 \text{ both odd.} 
\end{aligned} \]

Hence, for \(b \text{ even } (b \neq 1 \text{ odd})\) \(e_b\) (\(\hat{e}_b\), respectively) is a right branch of the path \([A_q, q]\) ([\(C, q]\), or \([C_q, q]\), respectively). For \(b \text{ even } (b \neq 1 \text{ odd})\), suppose that \(g\) is the first edge of the directed path \([A_q, q]\) ([\(C, q]\), or \([C_q, q]\)) which is adjacent to the directed path \([A, q+1]\). By (ii)–(iii), \(E_b\) (\(\hat{E}_b\), respectively) is the common head of the arcs \(g\) and \(e_b\) (\(\hat{e}_b\), respectively). Hence, \(g\) is a left branch of the path \([A, q+1]\). Thus, by (i), \(S^-(q+1) = [A, q+1](g) = bK(q)\), and condition (1) holds.

Condition (2) follows from (1) and Theorem 2.1.

\[ \square \]

**Example 3.1.** Let \(\{a_j\}\) be the Fibonacci sequence:

\[ a_1 = a_2 = 1 \quad \text{and} \quad a_{j+2} = a_j + a_{j+1} \quad \text{for } j \in \mathbb{N}. \]

We will check that

\[ \{(1, a_{2n+1}a_{2n+2}, a_{2n}a_{2n+2}), (a_{2n+2}, a_{2n+1}, 0), (a_{2n+1}, a_{2n+2}, a_{2n})\} \]

is the orbit of a graph in \(P\). Notice that for \(n = 1\) we obtain the orbit

\[ \{(1, 6, 3), (3, 2, 0), (2, 3, 1)\}. \]

**Proof.** Since \(a_j/a_{j+1}\) is the \(j\)-th convergence to \((\sqrt{5} - 1)/2, j \in \mathbb{N}\), we have the following conditions (see Schmidt [8, Lemma 3C, 3D]):

\begin{enumerate}
  \item \(a_{j+1}^2 - a_ja_{j+2} = (-1)^j\),
  \item \(a_{j+3}a_j - a_{j+2}a_{j+1} = (-1)^{j+1}\).
\end{enumerate}
If \((K(1), M(1), S^+(1)) = (1, a_{2n+1}a_{2n+2}, a_{2n}a_{2n+2})\) then, by (1),
\[ a_{2n-1}M(1) - a_{2n}S^+(1) = a_{2n+2}. \]
Hence, by Theorem 3.1(3-4) we have
\[ K(2) = a_{2n+2}, \quad M(2) = a_{2n+1}, \]
and, by Theorem 3.2(2),
\[ S^+(2) \equiv a_{2n}K(1) - K(2) = a_{2n} - a_{2n+2} = -a_{2n+1} \equiv 0 \pmod{a_{2n+1}}. \]
If \((K(2), M(2), S^+(2)) = (a_{2n+2}, a_{2n+1}, 0)\), then \(M(2) - S^+(2) = a_{2n+1}\).
Hence, by Theorem 3.1(3-4) we obtain
\[ K(3) = a_{2n+1}, \quad M(3) = a_{2n+2}, \]
and, by Theorem 3.2(2),
\[ S^+(3) \equiv K(2) - K(3) = a_{2n+2} - a_{2n+1} = a_{2n} \pmod{a_{2n+2}}. \]
If \((K(3), M(3), S^+(3)) = (a_{2n+1}, a_{2n+2}, a_{2n})\), then, by (2),
\[ a_{2n-1}M(3) = a_{2n+1}S^+(3) = 1. \]
Hence, by Theorem 3.1(3-4) we have
\[ K(1) = 1, \quad M(1) = a_{2n+1}a_{2n+2}, \]
and, by Theorem 3.2(2) and (1),
\[ S^+(1) \equiv a_{2n+1}K(3) - K(1) = a^2_{2n+1} - 1 = a_{2n}a_{2n+2} \pmod{a_{2n+1}a_{2n+2}}. \]
\[ \square \]

4. One point orbits of graphs in \(\mathcal{P}\)

We recall that \(X = \{(k, m, s) \in \mathbb{Z}^3 : 1 \leq k, 1 \leq m, 0 \leq s < m\}\) is the set of all index-vectors of graphs in \(\mathcal{P}\). In the following theorem we characterize one point orbits.
Theorem 4.1. \{(k, m, s)\} ∈ X is a one point orbit of a graph in \(P\) if and only if \(m = kn, s = kx\), where \(0 ≤ x < n\) are integers such that \(n\) is a divisor of \(x^2 + x + 1\).

**Proof.** Let \((k, m, s)\) be an index-vector of a graph in \(P\). It is easy to prove that the following conditions are equivalent (equivalence (ii)–(iii) follows from Theorem 3.1(3–4) and Theorem 3.2(2)):

(i) \(\{(k, m, s)\}\) is a one point orbit,

(ii) \((k, m, s) = (K(q), M(q), S^+(q)) = (K(q + 1), M(q + 1), S^+(q + 1))\),

(iii) \(k = |s, m|\) and \(s = bk - k\), where \(b\) is an integer such that \(0 < b ≤ m/k\) and \(bs ≡ -k \pmod{m}\),

(iv) \(m = kn, s = kx = bk - k\), where \(n ≥ 1, x ≥ 0\) and \(0 < b ≤ n\) are integers such that \(bx ≡ -1 \pmod{n}\),

(v) \(m = kn, s = kx\), where \(0 ≤ x < n\) are integers such that \(n\) is a divisor of \(x^2 + x + 1\). □

**Remark 4.1.** Notice that if \(\{(k, m, s)\}\) is a one point orbit of a graph \(G \in P\), then, by Theorem 2.1, \(\{(k, m, m - s - k)\}\) is the one point orbit of the mirror reflection of \(G\). Hence, by Theorem 4.1, \(n\) is a divisor of \(x^2 + x + 1\) if and only if \(n\) is a divisor of \((n - x - 1)^2 + (n - x - 1) + 1\), which is confirmed by the following equivalence

\[ x^2 + x + 1 = an ⇔ (n - x - 1)^2 + (n - x - 1) + 1 = (n - 2x - 1 + a)n. \]

**Example 4.1.** Notice that \((a, n, x) = (1, 1, 0), (1, 3, 1), (1, 7, 2)\) and \((1, 13, 3)\) are all integral solutions of the diophantine equation

\[ x^2 + x + 1 = a \cdot n, \quad \text{for} \quad 0 ≤ x ≤ 3 \quad \text{and} \quad x < n. \]

Hence, by Theorem 4.1, \(\{(k, k, 0)\}\), for \(k \in \mathbb{N}\), \(\{(1, 3, 1)\}, \{(1, 7, 2)\}\) and \(\{(1, 13, 3)\}\) are all one point orbits with \(s ≤ 3\). Notice that \(K^4\) (tetrahedron) has the one point orbit \(\{(1, 1, 0)\}\). Let \(G_0, G_1, G_2\) and \(G_3\) be graphs in \(P\) with one point orbits

\(\{(4, 4, 0)\}, \{(1, 3, 1)\}, \{(1, 7, 2)\}\) and \(\{(1, 13, 3)\}\),

respectively. Let us consider a solid regular tetrahedron with closed 3-faces \(f_1, f_2, f_4, f_4\). It is easy to check that \(G_j, j = 0, 1, 2, 3\), can be embedded on
the sphere of the solid regular tetrahedron in such a way that all four induced plane graphs $G_j[V_j \cap f_1], \ldots, G_j[V_j \cap f_4]$ are op-equivalent to the plane graph $Q_j$ shown in Fig. 3.

We conjecture that each graph $G \in \mathcal{P}$ with one point orbit, and the vertex set $V$, can be embedded on the sphere of the solid regular tetrahedron in such a way that all four induced plane graphs $G[V \cap f_1], \ldots, G[V \cap f_4]$ are op-equivalent.

**Theorem 4.2.** $G \in \mathcal{P}$ is double mirror symmetric if and only if $G$ has a one point orbit of the form $\{(k, k, 0)\}$ or $\{(k, 3k, k)\}$ for some $k \in \mathbb{N}$.

**Proof.** Let $G \in \mathcal{P}$ and suppose that $\{(K(q), M(q), S^+(q)) : q \in Q\}$ is the orbit of $G$. First we prove that if $S^+(q) + S^-(q) = M(q)$, for $q = 1, 2$, then $G$ has a one point orbit of the form $\{(k, 2s + k, s)\}$. If $S^+(q) + S^-(q) = M(q)$, for $q = 1, 2$, then by Theorem 3.1(3) and Corollary 3.1(3) we conclude that $K(0) = K(1) = K(2) = k$. Hence, $M(0) = M(1) = M(2) = m$, by Theorem 3.1(4). Suppose that $a_q, b_q$, for $q \in Q$, are integers such that $a_q m - b_q S^+(q) = k$ and $1 \leq b_q \leq m/k$. By Theorem 3.2(1–2), we deduce
that $S^+(q+1) = b_qk - k$ and $S^-(q+1) = b_qk$. Since $S^+(q+1) + S^-(q+1) = m$ for $q = 0, 1$, we see that $b_0 = b_1$, $S^+(1) = S^+(2) = s$, and $s + (s + k) = m$. Since $(K(1), M(q), S^+(1)) = (K(2), M(2), S^+(2)) = (k, 2s + k, s)$, we have $(K(0), M(0), S^+(0)) = (k, 2s + k, s)$. This completes the proof of the implication. The opposite implication follows from Theorem 2.1.

It is easy to see that the following conditions are equivalent (the equivalence (i)–(ii) follows from Theorem 4.1):

(i) $\{(k, 2s + k, s)\}$ is a one point orbit of $G$,

(ii) $m = 2s + k = kn$, $s = kx$, where $0 \leq x < n$ are integers such that $n$ is a divisor of $x^2 + x + 1$,

(iii) $m = k(2x + 1)$, $s = kx$, where integers $x \geq 0$ and $a > 0$ are solutions of the equation $x^2 + x + 1 = a(2x + 1)$.

Let $D$ be the determinant of the quadratic equation $x^2 + x(1 - 2a) + 1 - a = 0$. Since $D = 4a^2 - 3$ is a square of an integer, it follows that $a = 1$. Hence, $x = 0$ or $x = 1$, which completes the proof. \[\Box\]

5. Hamilton bonds with the end-trees of the same order

Let $G$ be a 2-connected plane triangulation, and suppose that $S$ and $T$ are the end-trees of a Hamilton bond in $G$. Let us denote by $f'_i$ ($f''_i$) the number of vertices of degree $i$ contained in $S$ ($T$, respectively). Tutte [11] proved the following identity, which is the dual version of the well-known Grinberg’s theorem [5]:

$$\sum_i (i - 2)f'_i = \sum_i (i - 2)f''_i. \quad (1)$$

Let us denote by $f_i$ the number of vertices of degree $i$ of the graph $G$. Euler’s equation becomes:

$$\sum_i (6 - i)f_i = 12. \quad (2)$$

Recall that $\mathcal{P}$ ($\mathcal{H}$) is the family of all 2-connected plane triangulations all of whose vertices are of degree 3 or 6 (at most 6, respectively).

Theorem 5.1. If $G \in \mathcal{P}$, then for every Hamilton bond, the end-trees of the bond have the same number of vertices of degree 6, and the same number of vertices of degree 3 in $G$. 

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Proof. Let $S$ and $T$ be the end-trees of a Hamilton bond in $G$. By equality (1) we have $4f'_6 + f'_3 = 4f''_6 + f''_3$. Hence, $f'_3 \equiv f''_3 \pmod{4}$. Because of $f'_3 + f''_3 = 4$ we have two cases: $f'_3 = 4$ or $f'_3 = 2 = f''_3$. In the first case we have $4f'_6 + 4 = 4f''_6$. Accordingly, the number $f_6$ is odd. Hence, we have a contradiction, because the order of $G$ is even. In the second case we have $f'_3 = 2 = f''_3$ and we obtain $f'_6 = f''_6$.

Theorem 5.2. If $G \in \mathcal{H}$, then for every Hamilton bond in $G$, the difference in the orders of the end-trees of the bond is not greater than 3.

Proof. Let $S$ and $T$ be the end-trees of a Hamilton bond in $G$. By equality (1) and (2) we obtain

$$\left|\sum_{i=3}^{6} f''_i - \sum_{i=3}^{6} f'_i\right| = \left|\sum_{i=3}^{5} (f''_i - f'_i) - \sum_{i=3}^{5} \frac{i-2}{4} (f''_i - f'_i)\right|$$

$$= \left|\frac{1}{4} \sum_{i=3}^{5} (6-i)(f''_i - f'_i)\right| \leq \frac{1}{4} \sum_{i=3}^{5} (6-i)f_i = 3,$$

which completes the proof.

6. Induced Caterpillars

Let $\mathcal{P}$ be the family of all 2-connected plane triangulations all of whose vertices are of degree 3 or 6. We recall that a graph $P \in \mathcal{P}$ is factorable into factors $P_0, P_1, P_2$ (indexed by elements of the cyclic group $Q = \{0, 1, 2\}$) satisfying the condition (*). Notice that $P$ has a Hamilton bond if and only if there exists a partitioning of the vertex set of $P$ into two subsets so that each induces a tree. We show an example that such trees are not always equitable 2-colorable.

Example 6.1. Let $G \in \mathcal{P}$ be the graph of Figure 4. Notice that $G$ contains two disjoint and induced trees whose vertices together span all of $G$. However, the induced trees are not equitable 2-colorable.

A $k$-caterpillar, $k \geq 1$, is a tree $T$ which contains a path $T_0$ such that $T - V(T_0)$ is a family of independent paths of the same order $k$. The path $T_0$ is referred to as the spine of $T$ (see Chartrand and Lesniak [1]). Paths
and $k$-caterpillars, for $k$ even, are called even caterpillars. Notice that even caterpillars are equitable 2-colorable.

Let $H \in \mathcal{P}$ with $K(q) = 2$, for some $q \in Q$, where $(K(q), M(q), S^+(q))$ is a $q$-index-vector of the graph $H$. Goodey [4] constructed a Hamiltonian cycle in every 2-connected cubic plane graph whose faces are only triangles or hexagons. In Lemma 6.1 we use a dual version of the Goodey’s construction to partition the vertex set of $H$ into two subsets so that each induces an even caterpillar.

**Lemma 6.1.** The graph $H$ contains two disjoint and induced even caterpillars $T$ and $S$ ($T$ is a $(2d-2)$-caterpillar, where $d = |S^+(q) + 1, M(q)|$, and $S$ is a path) whose vertices together span all of $H$. Moreover, if $\gamma_1$ is the cycle of class $q$ in $H$, then

1. $T \cap \gamma_1$ is a family of independent paths in $H$ with the same order $2d - 1$, and $S \cap \gamma_1$ is an independent set of vertices.
PROOF. Let $\gamma = v_0v_1\ldots v_{M(q)}$ and $\gamma'$ be two maximal paths of class $q$, and suppose that $\gamma_1 = t_0t_1\ldots t_{2M(q)−1}$ is the clock-wise oriented cycle of class $q$ in $H$. Without loss of generality we can assume that vertices $t_0$, $t_1$ are adjacent to $v_1$ (see Fig. 6). If $S^+(q) = M(q) − 1$, then there exists a vertex $u \neq v_0$ of degree 3 which is adjacent to $t_{2M(q)−1}$ and $t_0$. Then the set $W = \{u,v_0,t_0,t_1,\ldots,t_{2M(q)−2}\}$ ($V(H) − W$) induces a $(2M(q)−2)$-caterpillar $T$ (a path $S$, respectively) satisfying condition (1).

Let $S^+(q) < M(q) − 1$. In the graph $H − V(\gamma)$ we identify successive vertices and edges of the path $t_0t_1\ldots t_{M(q)}$ with successive vertices and edges of the path $t_0t_{2M(q)−1}t_{2M(q)−2}\ldots t_{M(q)}$. After the identification we obtain a path $\delta = w_0w_1\ldots w_{M(q)}$ and a graph $H_\gamma \in \mathcal{P}$ (see Fig. 5). Notice that $\delta$ and $\gamma'$ are two maximal paths of the same class (say $q$) in $H_\gamma$. Hence, $(K_\gamma(q), M_\gamma(q), S^+_\gamma(q)) = (1, M(q), S^+(q))$ is the $q$-index-vector of the graph $H_\gamma$. Let $e_1,e_2,\ldots,e_n$ be a sequence of all consecutive edges of the path $[w_0,q − 1]$ which are adjacent to the path $\delta$ (see Fig. 5). Since $S^-_\gamma(q) = S^+_\gamma(q) + 1 = S^+(q) + 1 < M(q)$, we have $n > 1$. By Lemma 3.1(1) and Corollary 3.1(2), we obtain

\begin{equation}
(2) \quad \{[w_0,q](e_1),[w_0,q](e_2),\ldots,[w_0,q](e_n)\} = \{0,2d,4d,\ldots,2M(q)−2d\},
\end{equation}

where $d = |S^+(q) + 1, M(q)|$. Let $I = \{0 \leq i \leq M(q) : w_i \in V([w_0,q − 1])\}$. We can consider $V_0 = V([w_0,q − 1]) \cap V(\gamma')$ as a set of vertices in $H$. It is not difficult to see that the following set

\[ V_1 = V_0 \cup \bigcup_{i \in I} \{v_i\} \cup \bigcup_{i \in I} \{t_i\} \cup \bigcup_{i \in I \setminus \{0,M(q)\}} \{t_{2M(q)−i}\} \]

induces a path $T_0$ in $H$ (see Fig. 6). Accordingly, by (2), the following set

\[ V_2 = V_1 \cup \bigcup_{i \in I} \{t_{i+1},t_{i+2},\ldots,t_{i+2d−2}\} \]

\[ \cup \bigcup_{i \in I \setminus \{0,M(q)\}} \{t_{2M(q)−i+1},t_{2M(q)−i+2},\ldots,t_{2M(q)−i+2d−2}\} \]

induces a $(2d − 2)$-caterpillar $T$ in $H$ with the spine $T_0$ (see Fig. 6). Notice that $V(H) − V_2$ induces a path $S$ in $H$, and condition (1) is satisfied. \[\square\]

In Theorem 6.1 we prove that if $P \in \mathcal{P}$ has the order $2n$ and $n$ is odd, then there is possible to partition the vertex set of $P$ into two subsets so that each
induces an even caterpillar. Hence, by Theorem 5.1, the two even caterpillars are equitable 2-colorable and have the same order. In Theorem 6.2 we prove that if \( P \) has an index-vector \( (K(q), M(q), S^+(q)) \) such that \( M(q) \) is odd and \( K(q) \geq \frac{M(q)}{3} \), then there is a partitioning of the vertex set of \( P \) into two subsets so that each induces a path.

**Theorem 6.1.** Let \( P \in P \). If \( P \) has the order \( 2n \) and \( n \) is odd, then \( P \) contains two disjoint and induced even caterpillars which vertices together span all of \( P \).

**Proof.** Let \( P \in P \) have the order \( 2n \), and suppose that \( n \) is odd. Let \( (K(q), M(q), S^+(q)) \) be the \( q \)-index-vector of \( P, q \in Q \). First we prove that \( K(q) \) is even for some \( q \in Q \). We know that \( 2K(q)M(q) + 2 = 2n \) for every \( q \in Q \). Hence, if \( K(q) \) is odd, then \( M(q) \) is even, because \( K(q)M(q) = n - 1 \) is even. By Theorem 2.1, \( S^-(q) - S^+(q) \equiv K(q) \pmod{M(q)} \), whence \( S^+(q) \) or \( S^-(q) \) is even. By Theorem 3.1(3) and Corollary 3.1(3), \( K(q \pm 1) = |S^\pm(q), M(q)| \), whence \( K(q + 1) \) or \( K(q - 1) \) is even.

Let now \( K(q) = k \) be even, and suppose that \( \gamma_0, \gamma' \) are maximal paths of class \( q \), and \( \gamma_1, \gamma_2, \ldots, \gamma_{k-1} \) are clock-wise oriented cycles of class \( q \) such that vertices of \( \gamma_j \) are adjacent to vertices of \( \gamma_{j-1} \), \( 1 \leq j < k \). We will prove that \( P \) contains two disjoint and induced even caterpillars \( T_k \) and \( S_k \) which vertices together span all of \( P \), and the following condition is satisfied.
Figure 7: An even caterpillar \( T \) (in bold) in the graph \( P \)

\[
\{T_k \cap \gamma_j : j \text{ odd}, 1 \leq j < k\} \cup \{S_k \cap \gamma_j : j \text{ even}, 1 \leq j < k\}
\]
is a family of independent paths in \( P \) with the same odd order, and
\[
\{T_k \cap \gamma_j : j \text{ even}, 1 \leq j < k\} \cup \{S_k \cap \gamma_j : j \text{ odd}, 1 \leq j < k\}
\]
is an independent set of vertices in \( P \).

We proceed by induction on the even number \( K(q) = k \). By Lemma 6.1, we can assume that \( k \geq 4 \). Let \( \gamma_{k-3} = x_0x_1 \ldots x_{M(q)-1} \), \( \gamma_{k-2} = y_0y_1 \ldots y_{2M(q)-1} \), \( \gamma_{k-1} = z_0z_1 \ldots z_{2M(q)-1} \). Without loss of generality we can assume that \( y_0, y_1 \) are adjacent to \( x_1 \), and \( z_0, z_1 \) are adjacent to \( y_0 \) (see Fig. 7). In the graph \( P - V(\gamma_{k-2}) \) we identify successive vertices and edges of the cycle \( \gamma_{k-1} \) with successive vertices and edges of the cycle \( \gamma_{k-3} \). After the identification we obtain a cycle \( \delta = t_0t_1 \cdots t_{2M(q)-1} \) and a graph \( H \in \mathcal{P} \) (see Fig. 6). Notice that \( \gamma_0, \gamma' \) (or \( \gamma_j \) for \( 1 \leq j \leq k-3 \)) are maximal paths (or cycles, respectively) of the same class (say \( q \)) in \( H \). By induction \( H \) contains two disjoint and induced even caterpillars \( T_{k-2} \) and \( S_{k-2} \) which vertices together span all of \( H \), and condition (3) holds (for \( k \) replaced with \( k - 2 \), and \( P \) replaced with
Let $I = \{0 \leq i < 2M(q) : t_i \in V(T_{k-2})\}$ and $J = \{0 \leq i < 2M(q) : t_i \in V(S_{k-2})\}$. We can consider $V_T = V(T_{k-2}) \setminus V(\delta)$ and $V_S = V(S_{k-2}) \setminus V(\delta)$ as sets of vertices in the graph $P$. Hence, the following sets

$$V_T \cup \{x_i : i \in I\} \cup \{z_i : i \in I\} \cup \{y_i : i \in J\},$$

$$V_S \cup \{x_i : i \in J\} \cup \{z_i : i \in J\} \cup \{y_i : i \in I\}$$

induce (respectively) two disjoint even-caterpillars $T_k$ and $S_k$ which vertices together span all of $P$, and condition (3) holds.

**Lemma 6.2.** If $m \geq 3$, $0 \leq a < b \leq m$ are integers, and $b - a \geq \frac{m}{3} - 1$, then interval $[a, b]$ contains an integer $2^k$ or $m - 2^k$ for some integer $k$.

**Proof.** Let $k$ be integer such that $2^k \leq \frac{m}{3} < 2^{k+1}$. Since $2^{k+1} - 2^k \leq \frac{m}{3}$ and $(m - 2^{k+1}) - 2^{k+1} < \frac{m}{3}$, the interval $[a, b]$ of the length at least $\frac{m}{3} - 1$ contains one of the integers:

$$1, 2^k, 2^{k+1}, m - 2^{k+1}, m - 2^k, m - 1,$$

which completes the proof.

**Theorem 6.2.** Let $P \in \mathcal{P}$. If $M(q)$ is odd and $K(q) \geq \frac{M(q)}{3}$ for some $q \in Q$, then $P$ has two disjoint and induced paths which together span $P$.

**Proof.** Let $(K(q), M(q), S^+(q))$ be a $q$-index-vector of $P$ such that $M(q)$ is odd and $K(q) \geq \frac{M(q)}{3}$. If $M(q) = 1$ ($K(q) = 1$), then the union of two maximal paths of class $q + 1$ (respectively) is a spanning subgraph of $P$. Hence we assume that $M(q) \geq 3$ and $K(q) \geq 2$. By Lemma 5.2, there exists integer $s$ such that $\max(S^+(q), 1) \leq s < \min(S^+(q) + K(q), M(q))$ and $|s, M(q)| = 1$. If $S^+(q) = s$, then, by Theorem 3.1(3), $K(q + 1) = 1$ and our theorem holds.

Let $l = s - S^+(q) > 0$. Assume that $\gamma_0 = v_0^0v_1^0 \ldots v_{M(q)}^0$, $\gamma'$ are maximal paths of class $q$, and $\gamma_k = v_0^kv_1^k \ldots v_{2M(q)-1}^k$, $1 \leq k < K(q)$, are clock-wise oriented cycles of class $q$ in $P$. Without loss of generality we can assume that vertices $v_1^k$, $v_2^k$ are adjacent to $v_1^{k-1}$, $1 \leq k < K(q)$. In $P - \bigcup_{0 \leq k < l} V(\gamma_k)$ we identify successive vertices and edges of the path $v_0^l v_1^l \ldots v_{M(q)}^l$ with successive vertices and edges of the path $v_0^l v_{2M(q)-1}^l v_{2M(q)-2}^l \ldots v_{M(q)}^l$. After the identification we obtain a path $\delta = w_0w_1 \ldots w_{M(q)}$ and a graph $H \in \mathcal{P}$. Notice that
\[ \delta \text{ and } \gamma' (\gamma_k, \text{ for } l < k < K(q)) \text{ are two maximal paths } (K(q) - l - 1 \text{ cycles, respectively}) \text{ of the same class (say } q) \text{ in } H. \text{ Hence, } (K_H(q), M_H(q), S_H^+(q)) = (K(q) - l, M(q), S^+_H(q)) \text{ is the } q\text{-index-vector of } H. \text{ Let us consider the segment } v_{S^+(q)}^0 v_{S^+(q)+1}^1 \cdots v_{s}^l v_{s+1}^{l+1} \cdots v_{S^+(q)+K(q)-1}^{K(q)-1} v \text{ of class } q+1 \text{ in } P, \text{ where the vertex } v \text{ belongs to } \gamma'. \text{ By the definition of } S^+(q), \deg_{P}(v) = 3. \text{ Thus, the segment } w_s v_{s+1}^{l+1} \cdots v_{S^+(q)+K(q)-1}^{K(q)-1} v \text{ is of class } q+1 \text{ in } H \text{ and } \deg_{H}(v) = 3, \text{ whence } S_H^+(q) = s. \text{ Therefore, by Theorem 3.1(3), } K_H(q + 1) = |s, M(q)| = 1. \text{ Hence, } H \text{ has two maximal paths } \alpha \text{ and } \beta \text{ of class } q+1 \text{ whose vertices together span all of } H. \text{ Let } I = \{0 \leq i \leq M(q) : w_i \in V(\alpha)\} \text{ and } J = \{0 \leq i \leq M(q) : w_i \in V(\beta)\}. \text{ We can consider } V_\alpha = V(\alpha) \setminus V(\delta) \text{ and } V_\beta = V(\beta) \setminus V(\delta) \text{ as sets of vertices in the graph } P. \text{ Notice that } v_i^l v_{i-1}^{l} \cdots v_i^0, 0 \leq i \leq M(q), v_i^0 v_{2M(q) - i}^0 \cdots v_{2M(q) - i}^0, 0 < i < M(q), \text{ are segments of class } q-1 \text{ in } P. \text{ Since a vertex } w_i \text{ of the path } \delta \text{ is obtained by the identification of the vertices } v_i^l \text{ and } v_{2M(q) - i}^l \text{ in } P, \text{ the following sets }

V_\alpha \cup \bigcup_{i \in I \setminus \{0,M(q)\}} \{v_i^l, v_i^{l-1}, \ldots, v_i^0\} \cup \bigcup_{i \in I \setminus \{0,M(q)\}} \{v_i^l \cdots v_i^0 v_{2M(q) - i}^0 \cdots v_{2M(q) - i}^l\},

and

V_\beta \cup \bigcup_{i \in J \setminus \{0,M(q)\}} \{v_i^l, v_i^{l-1}, \ldots, v_i^0\} \cup \bigcup_{i \in J \setminus \{0,M(q)\}} \{v_i^l \cdots v_i^0 v_{2M(q) - i}^0 \cdots v_{2M(q) - i}^l\}

induce paths in } P \text{ whose vertices together span all of } P. \square

7. Orbits of non simple graphs in } P

In the following theorem we characterize orbits of plane triangulations in } P \text{ which are not simple.

**Theorem 7.1.** } G \in \mathcal{P} \text{ is not simple if and only if } G \text{ has the orbit of the form }

\{(n,1,0),(1,n,n-1),(1,n,0)\}, \text{ for some integer } n > 1.

**Proof.** Let } G \in \mathcal{P}. \text{ It is easy to prove that the following conditions are equivalent (the last equivalence follows from Theorem 3.1(3–4) and 3.2(2)):

(i) } G \text{ is not simple,

(ii) } G \text{ has a cycle of class } q \text{ with the length } 2, \text{ for some } q \in Q,
(iii) $G \neq K_4$ and it has two edges of class $q$ with ends of degree 3, for some $q \in Q$,
(iv) $G$ has an index-vector of the form $(n, 1, 0)$, for some $n > 1$,
(v) $G$ has an orbit of the form $\{(n, 1, 0), (1, n, n-1), (1, n, 0)\}$, for some $n > 1$.
This completes the proof. $\square$

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