HIGHER ORBITAL INTEGRALS, SHALIKA GERMS, AND THE HOCHSCHILD HOMOLOGY OF HECKE ALGEBRAS

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Abstract. We give a detailed calculation of the Hochschild and cyclic homology of the algebra \( C^\infty_c(G) \) of locally constant, compactly supported functions on a reductive \( p \)-adic group \( G \). We use these calculations to extend to arbitrary elements the definition the higher orbital integrals introduced by Blanc and Brylinski for regular semisimple elements. Then we extend to higher orbital integrals some results of Shalika. We also investigate the effect of the “induction morphism” on Hochschild homology.

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Introduction

Orbital integrals play an important role in the harmonic analysis of a reductive \( p \)-adic group \( G \); they are, for instance, one of the main ingredients in the Arthur–Selberg trace formula. Orbital integrals on unimodular groups are a particular case of invariant distribution, which have been used in \cite{2} to prove the irreducibility of certain induced representations of \( GL_n \) over a \( p \)-adic field.

By definition, an invariant distribution on a unimodular group \( G \) gives rise to a trace (i.e., a Hochschild cocycle of degree zero) on \( C^\infty_c(G) \), the Hecke algebra of compactly supported, locally constant, complex valued functions on \( G \). It is interesting then to try to analyse as completely as possible the Hochschild homology and cohomology groups of the algebra \( C^\infty_c(G) \) (denoted \( \text{HH}_*(C^\infty_c(G)) \) and \( \text{HH}^*(C^\infty_c(G)) \), respectively).

In this paper, \( G \) will be the set of \( F \)-rational points of a linear algebraic group \( \mathbb{G} \) defined over a finite extension \( F \) of the field \( \mathbb{Q}_p \) of \( p \)-adic numbers, \( p \) being a fixed prime number. The group \( \mathbb{G} \) does not have to be reductive, although this is

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1. We assume $G$ (or $G$, by abuse of language) to be reductive, we shall state this explicitly. For us, the most important topology to consider on $G$ will be the locally compact topology induced from an embedding of $G \subseteq GL_n(F)$. Nevertheless, the Zariski topology on $G$ will also play a role in our study.

To state the main result of this paper, we need to introduce first the concept of standard subgroup. For any set $A \subseteq G$, we shall denote

$$C(A) := \{ g \in G, ga = ag, \forall a \in A \}$$

and $Z(A) := A \cap C(A)$. This latter notation will be used only when $A$ is a subgroup of $G$. A commutative subgroup $S$ of $G$ is called standard if $S = Z(C(s))$ for some semi-simple element $s \in G$. Our results will be stated in terms of standard subgroups of $G$. We shall denote by $H_q$ the set of unipotent elements of a subgroup $H$. Sometimes, the set $C(S)_u$ is also denoted by $U_S$, in order to avoid having to many parentheses in our formulae. One of the main results of this paper identifies the groups $HH_q(C_c^\infty(G))$ in terms of the following data: the set $\Sigma$ of (conjugacy classes of) standard subgroups $S$ of $G$, the subset $S^{\text{reg}} \subseteq S$ of $S$-regular elements, the action of the Weyl group $W(S)$ of $S$ on $C_c^\infty(S)$, and the continuous cohomology of the $C(S)$-module

$$C_c^\infty(U_S)_\delta := C_c^\infty(C(S)_u) \otimes \Delta_{C(S)},$$

where $\Delta_{C(S)}$ denotes the modular function of the group $C(S)$. More precisely, if $G$ is a $p$-adic group defined over a field of characteristic zero, as before, then Theorem (1) states the existence of an isomorphism

$$\text{HH}_q(C_c^\infty(G)) \cong \bigoplus_{S \in \Sigma} C_c^\infty(S^{\text{reg}})^{W(S)} \otimes \text{HH}_q(C(S), C_c^\infty(U_S)_\delta),$$

which can be made natural by using a generalization of the Shalika germs.

It is important to relate this result with the periodic cyclic homology groups of $C_c^\infty(G)$. For the Hecke algebra $C_c^\infty(G)$, the periodic cyclic homology is related to Hochschild homology by

$$\text{HP}_q(C_c^\infty(G)) \cong \bigotimes_{k \in \mathbb{Z}} \text{HH}_{q+2k}(C_c^\infty(G))_{\text{comp}},$$

that is, $\text{HP}_*(C_c^\infty(G))$ is the localization of Hochschild homology to the $G$-invariant subset of compact elements of $G$. This relation is implicit in (1). Consequently, the results of this paper complement the results on the cyclic homology of $p$-adic groups in [10, 13]. It is interesting to remark that $\text{HP}_*(C_c^\infty(G))$ can also be described in terms of the admissible spectrum of $G$, see [12], and hence our results have significance for the representation theory of $p$-adic groups. See also [16] for similar results on (the groups of real points of) algebraic groups defined over $\mathbb{R}$. These periodic cyclic cohomology groups are isomorphic to $K_*(C_c^\infty(G))$, by combining results from [4], [13], and [14].

Assume for the moment that $G$ is reductive. Then, in order to better understand the role played by the groups $\text{HH}_*(C_c^\infty(G))$ and $\text{H}_*(G, C_c^\infty(G_u))$ in the representation theory of $G$, we relate $\text{H}_*(G, C_c^\infty(G_u))$ to the analogous cohomology groups, $\text{H}_*(P, C_c^\infty(P_u)_k)$ and $\text{H}_*(M, C_c^\infty(M_u)_k)$, associated to parabolic subgroups $P$ of $G$ and to their Levi components $M$. In particular, we define morphisms between these Hochschild homology groups that are analogous to the induction and inflation morphisms that play such a prominent role in the representation theory of $p$-adic groups. These morphisms are induced by morphisms of algebras.
In [4], Blanc and Brylinski have introduced higher orbital integrals associated to regular semisimple elements by proving first that

$$\text{HH}_q(C^\infty_c(G)) \simeq H_q(G, C^\infty_c(G)),$$

a result which they dubbed “the MacLane isomorphism.” (Actually, they did not have to twist with the modular function, because they worked only with unimodular groups $G$, see Lemma [1] for the slightly more general version needed in this paper), after that we rely more on filtrations of the $G$-module $C^\infty_c(G)$, rather than on localization. This allows us to define higher orbital integrals at arbitrary elements. Then, we study the properties of these orbital integrals and we obtain in particular a proof of the existence of abstract Shalika germs for the higher orbital integrals. Actually, the existence of Shalika germs turns out to be a consequence of some general homological properties of the ring of invariant, locally constant functions on the group $G$. We also use the techniques developed in [14] in the framework of real algebraic groups. It would be interesting to relate the results of this paper to those of [11].

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1. Homology of Hecke algebras

In this section we shall use several general results on Hochschild homology of algebras, on algebraic groups, and on the continuous cohomology of totally disconnected groups. Good references are [5, 6, 15], for the general theory, and [12] for questions related to Hochschild homology.

If $G$ is a group and $A \subset G$ is a subset, we denote by $C(A)$ the centralizer of $A$, that is, the set of elements of $G$ that commute with every element of $A$, and by $N(A)$ the normalizer of $A$, that is, the set of elements $g \in G$ such that $gAg^{-1} = A$. By $Z = Z(G) = C(G)$ we denote the center of $G$.

If $X$ is a totally disconnected, locally compact, non-discrete space $X$, we denote by $C^\infty_c(X)$ the space of compactly supported, locally constant, complex valued functions on $X$. Recall that, if $U \subset X$ is an open subset of $X$ as above, then restriction defines an isomorphism

$$C^\infty_c(U)/C^\infty_c(U) \simeq C^\infty_c(X \setminus U).$$

Let $G$ be a linear algebraic group defined over a totally disconnected locally compact field $F$. Thus $F$ is a finite algebraic extension of $\mathbb{Q}_p$, the field of $p$-adic numbers. The set $G(F)$ of $F$-rational points of $G$ is called a $p$-adic group and will be denoted simply by $G$. It is known [3] that $G = G(F)$ identifies with a closed subgroup of $GL_n(F)$, and hence it has a natural locally compact topology that makes it a totally disconnected space. We fix a Haar measure $dg$ on $G$.

Consider now the space $C^\infty_c(G)$ of compactly supported, locally constant functions on $G$. Fix a Haar measure $dh$ on $G$. Then the convolution product, denoted $\ast$, is defined by

$$f_1 \ast f_2(g) = \int_G f_1(h)f_2(h^{-1}g)dh,$$

makes $C^\infty_c(G)$ an algebra, the Hecke algebra of $G$. It is important in representation theory to determine the $(Ad_G)$-invariant linear functionals on $C^\infty_c(G)$. If $G$ is
unimodular, the space of invariant linear functionals on $C_c^\infty(G)$ coincides with the space of traces on $C_c^\infty(G)$; moreover, since the space of traces of $C_c^\infty(G)$ identifies with $\text{HH}^0(C_c^\infty(G))$, the first Hochschild cohomology group of $C_c^\infty(G)$, it is reasonable to ask what are the groups $\text{HH}^q(C_c^\infty(G))$, the Hochschild homology groups of $C_c^\infty(G)$, in general. Since $\text{HH}^q(C_c^\infty(G))$ is the algebraic dual of $\text{HH}_q(C_c^\infty(G))$, it is enough to concentrate on Hochschild homology. The computation of the groups $\text{HH}_q(C_c^\infty(G))$ is the main purpose of this paper.

Before stating and proving Lemma 1, we need to introduce some notation. First, the groups $\text{HH}_q(C_c^\infty(G))$, $q = 0, 1, \ldots$, be the usual (algebraic) tensor product of vector spaces. The Hochschild differential $b : C_c^\infty(G^q+2) \to C_c^\infty(G^{q+1})$ is given by

$$
(bf)(g_0, g_1, \ldots, g_q) = \sum_{j=0}^q (-1)^j \int_G f(g_0, \ldots, g_{j-1}, \gamma, \gamma^{-1} g_j, g_{j+1}, \ldots, g_q) dg.
$$

By definition, the $q$-th Hochschild homology group of $C_c^\infty(G)$, denoted $\text{HH}_q(C_c^\infty(G))$, is the $q$-th homology group of the complex $(C_c^\infty(G^{q+1}), b)$. Since $C_c^\infty(G)$ is an inductive limit of unital algebras, this definition coincides with the usual definition of Hochschild homology for non-unital algebras (using algebras with adjoint unit).

The group $G$ acts by conjugation on $C_c^\infty(G)$, and we denote by $C_c^\infty(G)_{\text{ad}}$ the $G$-module defined by this action. Also, let $\Delta_G$ denote the modular function of $G$, which we recall, is defined by the relation

$$
\Delta_G(h) \int_G f(gh) dg = \int_G f(g) dg.
$$

We shall be especially interested in the $G$-module $C_c^\infty(G)_\delta$ obtained from $C_c^\infty(G)_{\text{ad}}$ by twisting it with the modular function. More precisely, let $C_c^\infty(G)_{\delta} = C_c^\infty(G)$ as vector spaces, and let the action of $G$ on functions be given by the formula

$$
(\gamma \cdot f)(g) = \Delta_G(\gamma) f(\gamma^{-1} g \gamma), \quad f \in C_c^\infty(G)_{\delta}.
$$

The reason for this twisting is that, for $G$ non-unimodular, the traces of $C_c^\infty(G)$ are the $G$-invariant functionals on $C_c^\infty(G)_\delta$, not on $C_c^\infty(G)$ (this is an immediate consequence of Lemma 3). More generally, our approach to the Hochschild homology of $C_c^\infty(G)$ is based on Lemma 3.

Before stating and proving Lemma 3, we need to introduce some notation. First, if $M$ is an arbitrary $G$-module, we denote by $M \otimes \Delta_G$ the tensor product of the $G$-modules $M$ and $C$, where the action on $C$ is given by the multiplication with the modular function of $G$. (We sometimes write this as $C_c^\infty(G)_\delta = C_c^\infty(G) \otimes \Delta_G$.)

If $M$ is a right $G$-module and $M'$ is a left $G$-module, then $M \otimes_G M'$ is defined as the quotient of $M \otimes M'$ by the submodule generated by $mg \otimes m' - m \otimes gm'$. For example, if $H \subset G$ is a closed subgroup and if $X$ is a left $H$-space, then we have an isomorphism of $G$-spaces

$$
C_c^\infty(G) \otimes_H (C_c^\infty(X) \otimes \Delta_H) \simeq C_c^\infty(G \times_H X),
$$
where $G \times X$ is the quotient $(G \times X)/H$ for the action $h(g, x) = (gh, hx)$. This isomorphism is obtained by observing that the natural map
\[
t_X : C_c^\infty(G) \otimes C_c^\infty(X) = C_c^\infty(G \times X) \longrightarrow C_c^\infty(G \times X),
\]
\[
t_X(f)([g, x]) = \int_H f(gh, h^{-1}x)dh,
\]
passes to the quotient to give the desired isomorphism. Sometimes it will be convenient to regard a left $G$–module as a right $G$–module by replacing $g$ with $g^{-1}$.

Also, recall that a $G$–module $M$ is smooth if, and only if, the stabilizer of each element of $M$ is open in $G$. Then one can define the continuous homology groups of $G$ with coefficients in the smooth module $M$, denoted $H_k(G, M)$, using tensor products as follows. Let $B_q(G) = C_c^\infty(G^{q+1})$, $q = 0, 1, \ldots$, be the Bar complex of the group $G$, with differential
\[
(df)(g_0, g_1, \ldots, g_q) = \sum_{j=0}^{q+1} (-1)^j \int_G f(g_0, \ldots, g_{j-1}, \gamma, g_j, \ldots, g_q)d\gamma.
\]

Then the complex $(B_q, d)$ gives a resolution of $\mathbb{C}$ with projective $C_c^\infty(G)$–modules, and the complex
\[
(8) \quad B_q(G) \otimes_G M
\]
computes $H_q(G, M)$. See [3, 6].

We shall need the following extension of a result from [1]:

**Lemma 1.** Let $C_c^\infty(G) = C_c^\infty(G) \otimes \Delta_G$ be the $G$–module obtained by twisting the adjoint action of $G$ on $C_c^\infty(G)$ by the modular function. Then we have a natural isomorphism
\[
(9) \quad HH_q(C_c^\infty(G)) \simeq H_q(G, C_c^\infty(G)) = C_c^\infty(G^{q+1}) \longrightarrow C_c^\infty(G) \otimes_G C_c^\infty(G).
\]

**Proof.** Consider the complex (8), which computes the continuous cohomology of $M = C_c^\infty(G)$, and let $h_G : B_q(G) \otimes C_c^\infty(G) \simeq C_c^\infty(G) \otimes C_c^\infty(G^{q+1}) = C_c^\infty(G^{q+2}) \rightarrow C_c^\infty(G) \otimes C_c^\infty(G)$ be the map
\[
h_G(f)(g_0, g_1, \ldots, g_q) = \int_G f(g^{-1}h, g^{-1}g_0, \ldots, g^{-1}g_0 g_1, \ldots, g^{-1}g_0 g_1 \ldots g_q)dg.
\]

As in equation (7), the map $h_G$ descends to the quotient to induce an isomorphism
\[
\hat{h}_G : B_q(G) \otimes C_c^\infty(G) \simeq C_c^\infty(G^{q+2}) \otimes C_c^\infty(G) \simeq C_c^\infty(G) \otimes C_c^\infty(G)
\]
of complexes, that is $\hat{h}_G \circ (d \otimes_G 1) = b \circ \hat{h}_G$, which establishes the isomorphism $H_q(G, C_c^\infty(G)) \simeq HH_q(C_c^\infty(G))$, as desired.

To better justify the twisting of the module $C_c^\infty(G)$ by the modular function in the above lemma, note that the trivial representation of $G$ gives rise to an obvious morphism $\pi_0 : C_c^\infty(G) \rightarrow \mathbb{C}$, by $\pi_0(f) = \int_G f(g)dg$, which hence defines a trace on $C_c^\infty(G)$. However $\pi_0$ is not $G$–invariant for the usual action of $G$, but is invariant if we twist the adjoint action of $G$ by the modular function, as indicated.

We proceed now to a detailed study of the $G$-module $C_c^\infty(G)$, called the standard stratification of $G$. The standard stratification of $G$ is constructed by considering the action of $G$ on the space $V = \mathbb{C}$, via the trivial representation, and then passing to the quotient $G \times V / \Delta_G$ to obtain a $G$–module $M$.

We denote the pullback of the trivial representation $\pi_0$ by $\pi_0 : C_c^\infty(G) \rightarrow \mathbb{C}$, and the pullback of the modular function $\delta : C_c^\infty(G) \rightarrow \mathbb{C}$ by $\delta : C_c^\infty(G) \rightarrow \mathbb{C}$. Then we have the following result:

**Theorem 1.** Let $\mathbb{C}$ be the $G$–module obtained by twisting the adjoint action of $G$ on $\mathbb{C}$ by the modular function. Then we have a natural isomorphism
\[
\hat{H}_q(G, \mathbb{C}) \simeq HH_q(C_c^\infty(G)),
\]
which establishes the isomorphism $H_q(G, \mathbb{C}) \simeq HH_q(C_c^\infty(G))$, as desired.
Let \( g \) be the Lie algebra of \( G \) in the sense of linear algebraic groups. Denote by \( a_i(g) \) the coefficients of the polynomial \( \det(t + 1 - Ad_g) \),
\[
\det(t + 1 - Ad_g) = \sum_{i=0}^{m} a_i(g)t^i \in \mathbb{F}[t].
\]

Let \( a_r \) be the first non-zero coefficient \( a_i \), and define
\[
V_k = \{ a_r = a_{r+1} = \ldots = a_{r+k-1} = 0 \}.
\]
Thus \( V_0 = G \), by convention, and \( G \setminus V_1 = G' \), the set of regular elements of \( G \) if \( G \) is reductive. Also, \( V_{m+1} = \emptyset \) because \( a_m = 1 \). We observe that the functions \( a_i(g) \) are \( G \)-invariant polynomial functions on \( G \), and that they depend only on the semisimple part of \( g \).

The description of the Hochschild homology of \( C^\infty_c(G) \) that we shall obtain is formulated in terms of certain commutative subgroups of \( G \), called \emph{standard}, that we now define.

**Definition 1.** A commutative subgroup \( S \subset G \) is called \emph{standard} if, and only if, there exists a semisimple element \( s_0 \in G \) such that \( S \) is the center of \( C(s_0) \), the centralizer of \( s_0 \) in \( G \).

If \( s_0 \) and \( S \) are as in the definition above, then \( S \) is the set of \( \mathbb{F} \)-rational points of a subgroup \( \mathcal{S} \) of \( G \) and \( C(S) = C(s_0) \). The element \( s_0 \) in the above definition is not unique in general, and, for a standard subgroup \( S \subset G \), we denote by \( S^{\text{reg}} \subset S \) the set of semisimple elements \( s \in S \) such that \( C(S) = C(s) \). An element \( s \in S^{\text{reg}} \) will be called \emph{\( S \)-regular}. This set is not empty by the definition of a standard subgroup. Note that \( S^{\text{reg}} \) depends on \( G \) also, and not only on \( S \).

Standard subgroups exist. Indeed, if \( \gamma \in G \) is semisimple, then \( S(\gamma) := Z(C(\gamma)) \), the center of the centralizer of \( \gamma \), is a standard subgroup of \( G \). In particular, every semisimple element of \( G \) belongs to \( S^{\text{reg}} \), for some standard subgroup \( S \) of \( G \). For all standard subgroups \( S \), the set \( S^{\text{reg}} \) is an open subset of \( S \) in the Zariski topology.

For any \( p \)-adic group \( H \), we denote by \( H_u \) the set of unipotent elements of \( H \), and call it the \emph{unipotent variety} of \( H \). In the particular case of \( H = C(S) \), where \( S \subset G \) is a standard subgroup, we also denote \( C(S) = U_S \).

In order to proceed further, recall that the Jordan decomposition of an element \( g \in G \) is \( g = g_s g_u \), where \( g_s \) is semisimple, \( g_u \) is unipotent, and \( g_s g_u = g_u g_s \). This decomposition is unique. If \( g = g_s g_u \) is the Jordan decomposition of \( g \in G \) and if \( g_u \in S^{\text{reg}} \), then \( g_u \in U_S \), by definition, and hence \( g \in S^{\text{reg}} U_S \).

Consider now a standard subgroup \( S \subset G \) and let
\[
F_S = \text{Ad}_G(S^{\text{reg}}), \quad F_S^S = \text{Ad}_G(S^{\text{reg}} U_S)
\]
be the set of semisimple elements of \( G \) conjugated to an element of \( S^{\text{reg}} \) and, respectively, the set of elements \( g \in G \) conjugated to an element of \( S^{\text{reg}} U_S \) (i.e., such that the semisimple part of \( g \) is in \( F_S \)). Also, let \( N(S) \) be the normalizer of \( S \) and \( W(S) = N(S)/C(S) \). Since \( N(S) \) leaves \( S^{\text{reg}} \) invariant and is actually the normalizer of this set, it follows that the quotient \( W(S) \) can be identified with a set of automorphisms of \( S^{\text{reg}} \), the subgroup of semisimple elements of \( S \). Since \( N(S) \) is the set of \( \mathbb{F} \)-rational points of an algebraic group, the rigidity of tori (5, page 117) shows that \( W(S) \) is finite.
The natural map \((g, s) \rightarrow gsg^{-1}\) descends to a map

\[ \phi_S : (G \times S^{reg})/N(S) \ni (g, s) \rightarrow gsg^{-1} \in F_S. \]

Similarly, we obtain a map

\[ \phi^u_S : (G \times S^{reg}U_S)/N(S) \ni (g, su) \rightarrow gsg^{-1} \in F^u_S. \]

In the following proposition we consider the locally compact (and Hausdorff) topology of \(G\), and not the Zariski topology. Denote by \(G_{ss}\) the set of semisimple elements of \(G\).

**Proposition 1.** Let \(S\) be a standard subgroup of \(G\). Using the above notation, we have:

(i) The set \(F_S\) is an analytic submanifold of \(G\), and the maps \(\phi_S\) and \(\phi^u_S\) are homeomorphisms.

(ii) For each \(k\), the set \(V_k \setminus V_{k+1}\) is a disjoint union of sets of the form \(F^u_k\), and each \(F^u_k \subset V_k \setminus V_{k+1}\) is an open subset of \(V_k\). Moreover, the set \(G_{ss} \cap (V_k \setminus V_{k+1})\) is a disjoint union of sets of the form \(F_S\), and each \(F_S \subset V_k \setminus V_{k+1}\) is an open subset of \(G_{ss} \cap V_k\).

**Proof.** (i) First we check that \(\phi_S\) and \(\phi^u_S\) are injective. Indeed, assume that \(g_1s_1g_1^{-1} = g_2s_2g_2^{-1}\), for some \(s_1, s_2 \in S^{reg}\). Then, if \(g = g_2^{-1}g_1\), we have

\[ gC(s_1)g^{-1} = C(s_2) \Rightarrow gC(S)g^{-1} = C(S) \Rightarrow gSg^{-1} = S, \]

and hence \(g \in N(S)\). Consequently, we have \((g_1, s_1) = (g_2, g^{-1}s_2g) = g^{-1}(g_2, s_2)\), with \(g \in N(S)\), as desired. The same argument shows that if \(F_S\) and \(F_{S'}\) have a point in common, then the standard subgroups \(S\) and \(S'\) are conjugated in \(G\).

The injectivity of \(\phi^u_S\) follows from the injectivity of \(\phi_S\), indeed, if \(g_1(s_1u_1)g_1^{-1} = g_2(s_2u_2)g_2^{-1}\), let \(g = g_2^{-1}g_1\) as above, and conclude that \(g_1s_1g_1^{-1} = s_2\), by the uniqueness of the Jordan decomposition. As above, this implies that \(g \in N(S)\).

Since the differential \(d\phi_S\) is a linear isomorphism onto its image (i.e. it is injective) and \(\phi_S\) is injective, it follows that \(\phi_S\) is a local homeomorphism onto its image (for the locally compact topologies), and that its image is an analytic submanifold ([23], p. 38, Theorem (2.3)). The set \(G_{ss} \cap (V_k \setminus V_{k+1})\) is an algebraic variety on which \(G\) acts with orbits of the same dimension, and hence \(\phi_S\) is proper. This proves that \(\phi_S\) is a homeomorphism. Using an inverse for \(\phi_S\), we obtain that \(\phi^u_S\) is also a homeomorphism.

To prove now (ii), consider a standard subgroup \(S \subset G\), and let \(d\) be the dimension of \(C(S)\). Then \(a_0 = a_1 = \ldots = a_{d-1} = 0\) on \(S\), and \(S \cap \{ad \neq 0\}\) contains \(S^{reg}\) as an open component. It follows that, if \(s \in G_{ss} \cap (V_k \setminus V_{k+1}) \cap S^{reg}\), then \(F_S \subset V_k \setminus V_{k+1}\). This shows that \(G_{ss} \cap (V_k \setminus V_{k+1})\) is a union of sets of the form \(F_S\). This must then be a disjoint union because the sets \(F_S\) are either equal or disjoint, as proved above.

Now, if \(g \in V_k \setminus V_{k+1}\) has semisimple part \(s\), then \(s \in F_S \subset G_s \cap (V_k \setminus V_{k+1})\), for some standard subgroup \(S\), and hence \(g \in F^u_S \subset V_k \setminus V_{k+1}\). The sets \(F^u_S\) are open in the induced topology because the map \(V_k \setminus V_{k+1} \rightarrow G_s \cap (V_k \setminus V_{k+1})\) is continuous.

See also [24].

Let \(R^\infty := R^\infty(G)\) be the ring of locally constant \(Ad_G\)-invariant functions on \(G\) with the pointwise product, which we regard as a subset of the set of endomorphisms
of the \(G\)-module \(C^\infty_c(G) = C^\infty_c(G) \otimes \Delta_G\). For each \(k \geq 1\), denote by \(I_k \subset R^\infty\) the ideal generated by functions \(f : G \to \mathbb{C}\) of the form
\[
f = \phi(a_r, a_{r+1}, \ldots, a_{r+k-1}),
\]
where \(\phi\) is a locally constant function \(\phi : \mathbb{F}^k \to \mathbb{C}\) such that \(\phi(0,0,\ldots,0) = 0\). (Recall that each of the polynomials \(a_0,\ldots,a_{r-1}\) is the 0 polynomial.) By convention, we set \(I_0 = (0)\); also, it follows that \(I_{m+1} = R^\infty\).

Fix now \(k\), and let \(p_n \in I_k\) be the function \(\phi_n(a_r, a_{r+1}, \ldots, a_{r+k-1})\), where \(\phi_n : \mathbb{F}^k \to \mathbb{C}\) is a locally constant function \(\phi : \mathbb{F}^k \to \mathbb{C}\) such that \(\phi(0,0,\ldots,0) = 0\). Then, for each \(q \in I_k\), we have an isomorphism
\[
\{\xi = (\xi_0, \ldots, \xi_{k-1}) \in \mathbb{F}^k, \max |\xi_i| \geq q^{-n}\}
\]
and vanishes outside this set. (Here \(q\) is the number of elements of the residual field of \(\mathbb{F}\), and the non-archimedean norm \(||\cdot||\) is normalized such that its range is \(\{0\} \cup \{q^n, n \in \mathbb{Z}\}\).) Then \(p_n = p_n^2 = p_np_{n+1}\) and \(I_k = \cup p_n R^\infty\). For further reference, we state as a lemma a basic property of the constructions we have introduced.

**Lemma 2.** If \(M\) is a \(R^\infty\)-module, then \(I_k M = \cup p_n M\).

As a consequence of this above lemma, we obtain the following result.

**Corollary 1.** Consider the \(G\)-module
\[
M_k = (I_{k+1}/I_k) \otimes_{R^\infty} C^\infty_c(G) = I_{k+1}C^\infty_c(G)/I_k C^\infty_c(G) = I_{k+1}C^\infty_c(G)/I_k C^\infty_c(G).
\]
Then, for each \(q \geq 0\), we have an isomorphism
\[
H_q(G, C^\infty_c(G)) = \bigoplus_{k=0}^m H_q(G, M_k)
\]
of vector spaces.

**Proof.** There exists a (not natural) isomorphism
\[
H_q(G, C^\infty_c(G)) \simeq \bigoplus_{k=0}^m H_q(G, M_k) = \bigoplus_{k=0}^m H_q(G, I_mC^\infty_c(G))/I_k H_q(G, C^\infty_c(G)),
\]
of vector spaces.

By the above lemma, the inclusion of \(I_k C^\infty_c(G) \subset C^\infty_c(G)\) of \(G\)-modules induces natural isomorphisms
\[
H_q(G, I_k C^\infty_c(G)) \simeq H_q(G, \lim_{\to} p_n C^\infty_c(G)) \simeq \lim_{\to} p_n H_q(G, C^\infty_c(G)) \simeq I_k H_q(G, C^\infty_c(G)),
\]
because the functor \(H_q\) is compatible with inductive limits and with direct sums.

The naturality of these isomorphisms and the Five Lemma show that
\[
H_q(G, I_{k+1}C^\infty_c(G))/I_k C^\infty_c(G) \simeq I_{k+1} H_q(G, C^\infty_c(G))/I_k H_q(G, C^\infty_c(G)),
\]
This is enough to complete the proof. \(\Box\)

We now study the homology of the subquotients \(M_k = I_{k+1}C^\infty_c(G)/I_k C^\infty_c(G)\) by identifying them with induced modules. Let \(\Sigma_k\) be a set of representative of conjugacy classes of standard subgroups \(S\) such that \(F_S \subset V_k \setminus V_{k+1}\) (or, equivalently, \(F'_S \subset V_k \setminus V_{k+1}\)).
Lemma 3. Using the above notation, we have $I_k C^\infty_c(G) = C^\infty_c(G \setminus V_k)$ and 

$$I_{k+1} C^\infty_c(G)/I_k C^\infty_c(G) \cong \bigoplus_{S \in \Sigma_k} C^\infty_c(F^w_S).$$

Proof. It follows from the definition of $I_k$ that, if $f \in I_k C^\infty_c(G)$, then $f$ vanishes in a neighborhood of $V_k$. Conversely, if $f$ is in $C^\infty_c(G \setminus V_k)$, then we can find some polynomial $a_i$, with $i \leq r + k - 1$, such that $|a_i|$ is bounded from below on the support of $f$ by, say, $q^{-n}$, then $p_n f = f$. The second isomorphism follows from the first isomorphism using (3) and Lemma 2. \[ \square \]

If $H \subset G$ is a closed subgroup and $M$ is a smooth (left) $H$–module (that is, the stabilizer of each $m \in M$ is an open subgroup of $M$), we denote

$$\text{ind}_H^G(M) = C^\infty_c(G) \otimes_H M = C^\infty_c(G) \otimes M/\{fh \otimes m - f \otimes hm, h \in H\}.$$ 

Where the right $H$–module structure on $C^\infty_c(G)$ is $(fh)(g) = f(gh^{-1})$. Then Shapiro’s lemma, see [7], states that

$$H_k(G, \text{ind}_H^G(M) \otimes \Delta_G) \cong H_k(H, M).$$

(A proof of Shapiro’s lemma in our setting is contained in the proof of Theorem 1.)

The basic examples of induced modules are obtained from $H$–spaces. If $X$ is an $H$–space (we agree that $H$ acts on $X$ from the left), then

$$C^\infty_c(G \times X)/H \cong \text{ind}_H^G(C^\infty_c(X) \otimes \Delta_H) \cong \text{ind}_H^G(C^\infty_c(X)_{\delta})$$

as $G$–modules, where $H$ acts on $G \times X$ by $h(g, x) = (gh^{-1}, hx)$. For example, Proposition 2 identifies $C^\infty_c(F^w_S)$ with an induced module:

$$C^\infty_c(F^w_S) \cong \text{ind}_{N(S)}^G(C^\infty_c(S_{\text{reg}} U_S) \otimes \Delta_{N(S)}) \cong \text{ind}_{N(S)}^G(C^\infty_c(S_{\text{reg}} U_S)_{\delta}).$$

Shapiro’s lemma is an easy consequence of the Serre–Hochschild spectral sequence, see (4), which states the following. Let $M$ be a smooth $G$–module and $H \subset G$ be a normal subgroup. Then the action of $G$ on $H_k(H, M)$ descends to an action of $G/H$, and there exists a spectral sequence with $E^2_{p,q} = H_p(G/H, H_q(H, M))$, convergent to $H_{p+q}(G, M)$.

Let $M_k = I_{k+1} C^\infty_c(G)_{\delta}/I_k C^\infty_c(G)_{\delta}$, as before.

Proposition 2. Using the above notation, we have a natural

$$H_q(G, M_k) \cong \bigoplus_{(S) \in \Sigma_k} C^\infty_c(S_{\text{reg}})^W(S) \otimes H_q(C(S), C^\infty_c(U_S)_{\delta})$$

isomorphism of $R^\infty$–modules.

Proof. Let $S$ be a standard subgroup of $G$. Recall first that $W(S) = N(S)/C(S)$ is a finite group that acts freely on $S_{\text{reg}}$, which gives a $N(S)$–equivariant isomorphism

$$C^\infty_c(S_{\text{reg}} U_S)_{\delta} \cong C^\infty_c(U_S)_{\delta} \otimes C^\infty_c(S_{\text{reg}}).$$

Let $M$ be a smooth $N(S)$–module. The Hochschild–Serre spectral sequence applied to the module $M$ and the normal subgroup $C(S) \subset N(S)$ gives natural isomorphisms

$$H_q(N(S), M) \cong H_0(W(S), H_q(C(S), M)) \cong H_q(C(S), M)^{W(S)}.$$
Combining these two isomorphisms, we obtain
\[ H_k(G, C^\infty_c(F^\delta_S)) \simeq H_k(G, \text{ind}_N^G(C^\infty_c(S^{\text{reg}}U_S)\delta)) \simeq H_k(N(S), C^\infty_c(S^{\text{reg}}U_S)\delta) \]
\[ \simeq (H_k(C(S), C^\infty_c(U_S)\delta) \otimes C^\infty_c(S^{\text{reg}})) \simeq C^\infty_c(S^{\text{reg}})^W(S) \otimes H_k(C(S), C^\infty_c(U_S)\delta). \]

The result then follows from Lemma 3, which implies directly that
\[ M_k \simeq \bigoplus_{S \in \Sigma} C^\infty_c(F^\delta_S). \]

The proof is now complete. \qed

Combining this proposition with Corollary [1], we obtain the main result of this section. Recall that a $p$-adic group $G = \mathbb{G}(F)$ is the set of $F$-rational points of a linear algebraic group $\mathbb{G}$ defined over a non-archimedean, non-discrete, locally compact field $F$ of characteristic zero. Also, recall that $U_S$ is the set of unipotent elements commuting with the standard subgroup $S$, and that the action of $C(S)$ on $C^\infty_c(U_S)$ is twisted by the modular function of $C(S)$, yielding the module $C^\infty_c(U_S)\delta = C^\infty_c(U_S) \otimes \Delta_{C(S)}$.

**Theorem 1.** Let $G$ be a $p$-adic group. Let $\Sigma$ be a set of representative of conjugacy classes of standard subgroups of $S \subset G$ and $W(S) = N(S)/C(S)$, then we have an isomorphism
\[
\text{HH}_q(C^\infty_c(G)) \simeq \bigoplus_{S \in \Sigma} C^\infty_c(S^{\text{reg}})^W(S) \otimes H_q(C(S), C^\infty_c(U_S)\delta).
\]

**Remark.** The isomorphism of the above theorem is not natural. A more natural description of $\text{HH}_q(C^\infty_c(G))$ will be obtained in one of the following sections by considering higher orbital integrals and their Shalika germs

2. Higher orbital integrals and their Shalika germs

Proposition 3 of the previous section allows us to determine the structure of the localized cohomology groups $\text{HH}_q(C^\infty_c(G))_m$, where $m$ is a maximal ideal of $R^\infty(G)$. This will lead to an extension of the higher orbital integrals introduced by Blanc and Brylinski in [1], and to a generalization of some results of Shalika [22] to higher orbital integrals. In this way, we shall also obtain a more natural description of $\text{HH}_q(C^\infty_c(G))$.

First recall the following result.

**Proposition 3.** Let $G$ be a reductive $p$-adic group over a field of characteristic $0$, $S \subset G$ be a standard subgroup, and $\gamma \in S^{\text{reg}}$ (that is, $\gamma$ is a semisimple element such that $C(S) = C(\gamma)$). Then there exists a $N(S)$–invariant closed and open neighborhood of $\gamma$ in $C(S)$ such that

\[ G \times U \subset (g, h) \rightarrow ghg^{-1} \in G \]

defines a homeomorphism of $(G \times U)/N(S)$ onto a $G$–invariant, closed and open subset $V \subset G$ containing $\gamma$.

**Proof.** The result follows from Luna’s Lemma. For $p$-adic groups, Luna’s Lemma is proved in [17], page 109, Properties “C” and “D.” \qed

From this proposition we obtain the following consequences for the ring $R^\infty(G)$.
Corollary 2. Let $U$ and $V$ be as in Proposition 3 above.

(i) The ring $R^\infty(G)$ decomposes as the direct sum $C^\infty(V)^G \oplus C^\infty(V^c)^G$, and $C^\infty(V)^G \simeq C^\infty(U)^{N(S)} \subset C^\infty(C(S))^{N(S)} = R^\infty(C(S))^{W(S)}$. (Here $V^c$ is the complement of $V$ in $G$.)

(ii) For any two semisimple elements $\gamma, \gamma' \in G$, if $\phi(\gamma) = \phi(\gamma')$ for all $\phi \in R^\infty(G)$, then $\gamma$ and $\gamma'$ are conjugated in $G$.

(iii) Let $m \in R^\infty(G)$ be the maximal ideal consisting of functions that vanish at a semisimple element $\gamma \in G$. Then $m$ is generated by an increasing sequence of projections, and $M_m \simeq M/mM$, for any $R^\infty(G)$–module $M$.

Proof. (i) is an immediate consequence of Proposition 3. (ii) follows from [17], Proposition 2.5.

To prove (iii), observe that the maximal ideal $m$ is generated by a sequence of projections $p_n$, that is, $m = \cup p_n R^\infty(G)$, with $p_n^2 = p_n$.

We know from [17], Proposition 2.5, that $R^\infty(G)$ is isomorphic to $C^\infty$, for some locally compact, totally disconnected topological space $X$. Moreover, if $M$ is a $C^\infty(X)$–module and $m$ is the maximal ideal of functions vanishing at $x_0$, for some fixed point $x_0 \in X$, then $C^\infty(X)_m \simeq C^\infty(X)/mC^\infty(X)$, and hence

$$M_m = M \otimes_{C^\infty(X)} C^\infty(X)_m \simeq M \otimes_{C^\infty(X)} C^\infty(X)/mC^\infty(X) \simeq M/mM.$$ 

Since $X$ is metrizable, we can choose a basis $V_n$ of compact open neighborhoods of $x_0$ in $X$. Then, if we let $p_n$ to be the characteristic function of $V_n^c$, then $p_n$ are projections generating $m$. By choosing $V_n$ to be decreasing, we obtain a decreasing sequence $p_n$.

We now consider for each maximal ideal $m \subset R^\infty = R^\infty(G)$ the localization $\text{HH}_q(C^\infty_c(G))_m$.

Proposition 4. Let $m$ be a maximal ideal of $R^\infty(G)$. If $m$ consists of the functions that vanish at the semisimple element $\gamma \in G$ and $S \subset G$ is a standard subgroup such that $\gamma \in S^{\text{reg}}$, then

$$\text{HH}_q(C^\infty_c(G))_m \simeq \text{HH}_q(C(S), C^\infty_c(U_S)),$$

For all other maximal ideals $m \subset R^\infty(G)$, we have $\text{HH}_q(C^\infty_c(G))_m = 0$.

Note that $\text{HH}_q(C(S), C^\infty_c(U_S)) \simeq \text{HH}_q(C(\gamma), C^\infty_c(C(\gamma)^{U_S}))$.

Proof. The vanishing of $\text{HH}_q(C^\infty_c(G))_m$ in the second half of the proposition follows because $C^\infty_c(G)_m = 0$ in that case.

Assume now that $m$ consists of functions vanishing at $\gamma$, a semisimple element of $G$. The localization functor $V \rightarrow V_m$ is exact by standard homological algebra. The sequence of ideals $(I_k)_m$ is an increasing sequence satisfying $(I_0)_m \simeq 0$ and $(I_m+1)_m \simeq \mathbb{C}$. Choose $k$ such that $(I_k)_m \simeq 0$ and $(I_{k+1})_m \simeq \mathbb{C}$. (This happens if, and only if, $\gamma \in V_k \setminus V_{k+1}$.) It follows that

$$\text{HH}_q(G, C^\infty_c(G))_m \simeq \text{HH}_q(G, I_{k+1}C^\infty_c(G)/I_kC^\infty_c(G))_m.$$

Since all the isomorphisms of Proposition 4 are compatible with this the localization functor, we obtain that

$$\text{HH}_q(C^\infty_c(G))_m \simeq \text{HH}_q(G, C^\infty_c(G))_m,$$

$$\simeq \bigoplus_{(S) \in \Sigma_k} \text{HH}_q(C(S), C^\infty_c(U_S)) \otimes (C^\infty_c(S^{\text{reg}}))^{W(S)}/mC^\infty_c(S^{\text{reg}})^{W(S)}.$$
The only quotient $\mathcal{C}_c^\infty(S^\text{reg})^{W(S)}/\mathfrak{m}\mathcal{C}_c^\infty(S^\text{reg})^{W(S)}$ that does not vanish is the one containing (a conjugate of) $\gamma$, and then it is isomorphic to $\mathbb{C}$. This completes the proof.

An alternative proof can be obtained by writing

$$H_q(G, C_c^\infty(G)_\delta)_m \simeq H_q(G, C_c^\infty(G)_\delta m) \simeq H_q(G, C_c^\infty(G)_\delta)/\mathfrak{m}C_c^\infty(G)_\delta,$$

and then observing that $C_c^\infty(G)_\delta/\mathfrak{m}C_c^\infty(G)_\delta \simeq C_c^\infty(U_S)$, by Corollary 2 (iii). However the above proof is more convenient when dealing with orbital integrals. See also [16], first circulated in 1990 as a preprint of the Mathematical Institute of the Romanian Academy (INCREST) Nr. 18, March 1990, and where the localization techniques were first introduced.

We now extend the definition of higher orbital integrals introduced by Blanc and Brylinski to cover non–regular semisimple elements also. Fix a standard subgroup $S \subset G$, and let $k$ be such that $S^\text{reg} \subset V_k \setminus V_{k+1}$. As in the above proof, Proposition 3 gives a natural $R^\infty(G)$–linear, degree preserving, surjective morphism

$$H_*(G, I_{k+1}C_c^\infty(G)_\delta/I_kC_c^\infty(G)_\delta) \longrightarrow C_c^\infty(S^\text{reg})^{W(S)} \otimes H_*(C(S), C_c^\infty(U_S)_\delta),$$

and hence a linear map

$$I_{k+1} HH_*(C_c^\infty(G)) = H_*(G, I_{k+1}C_c^\infty(G)_\delta) \longrightarrow C_c^\infty(S^\text{reg})^{W(S)} \otimes H_*(C(S), C_c^\infty(U_S)_\delta).$$

Fix $c \in H^q(C(S), C_c^\infty(U_S)_\delta)$ and $\gamma \in S^\text{reg}$, and let

$$O_{\gamma,c} = O^{S}_{\gamma,c} : I_{k+1} HH_q(C_c^\infty(G)) \longrightarrow \mathbb{C}$$

be the evaluation of the map at $\gamma$ and $c$ in (13). We obtain, in particular, that for any $f \in I_{k+1} HH_q(C_c^\infty(G))$, the function $\gamma \to O_{\gamma,c}(f)$ is a locally constant, compactly supported function on $S^\text{reg}$. The function $O_{\gamma,c}$ can then be extended to the whole group $HH_q(C_c^\infty(G))$ using a simple observation. For any $\gamma \in S^\text{reg}$ there exists a locally constant function $\phi \in I_{k+1}$ such that $\phi(\gamma) = 1$. Then let

$$O_{\gamma,c}(f) := O_{\gamma,c}(\phi f),$$

which is independent of $\phi$. It follows from definition of $O_{\gamma,c}$ that, for any $f \in HH_q(C_c^\infty(G))$, the function $\gamma \to O_{\gamma,c}(f)$, obtained as above, is a locally constant function on $S^\text{reg}$, but not necessarily compactly supported. We thus obtain the following result.

**Proposition 5.** Let $S \subset G$ be a standard subgroup. Then there exists a degree-preserving, $R^\infty(G)$–linear map

$$O^S : HH_*(C_c^\infty(G)) \longrightarrow C_c^\infty(S^\text{reg})^{W(S)} \otimes H_*(C(S), C_c^\infty(U_S)_\delta),$$

which is an isomorphism when localized at each maximal ideal $m \subset R^\infty(G)$ consisting of functions vanishing at an element $\gamma \in S^\text{reg}$.

We call the maps $O^S$ and $O_{\gamma,c} = O^{S}_{\gamma,c}$ “higher orbital integrals” because they generalize the usual notion of orbital integral. (If $c$ is a cocycle of dimension $q$, we call $O_{\gamma,c}$ a $q$–higher orbital integral.) Indeed, assume that $G$ and $C(S)$ are unimodular, following thorough the identifications in the previous section, we obtain,
for $c_0 = 1 \in H^0(C(S), C_c \infty(U_S)_{S})$ the evaluation at the identity element $e \in G$, and
\[ f \in C_c \infty(G) = HH_b(C_c \infty(G)), \]
that
\[ O_{\gamma,c_0}(f) = O_{\gamma,1}(f) = \int_{G/C(S)} f(gg^{-1})d\mathcal{G}, \]
where $d\mathcal{G}$ is the induced measure on $G/C(S)$.

If $\gamma \in G$ is a semisimple element and $S$ is a standard subgroup of $G$ such that
$C(\gamma) = C(S)$, (i.e., $\gamma \in S_{\text{reg}}$), then restriction at $\gamma$ defines a map
\[ O_{\gamma} = O_{\gamma}^S : HH_q(C_c \infty(G)) \to H_q(C(S), C_c \infty(U_S)_{S}) \]
such that $c(O_{\gamma}(f)) = O_{\gamma,c}(f)$, for all $c \in H^0(C(S), C_c \infty(U_S)_{S})$.

A word on notation, whenever we write $O_{\gamma,\cdot}$ or $O_{\cdot,\cdot}^S$, we assume that $\gamma \in S_{\text{reg}}$,
which actually determines $S$. This means that we can omit $S$ from notation. However,
if we want to write that $O_{\gamma,\cdot} = O_{\gamma,\cdot}^S$ is obtained by specializing
\[ O^S : HH_q(C_c \infty(G)) \to C_c \infty(S_{\text{reg}}^1) \otimes H_q(C(S), C_c \infty(U_S)_{S}) \]
to a point $\gamma \in S_{\text{reg}}$ and then by evaluating at $c$, that is that
\[ O^S_{\gamma,c}(f) = (O^S(f)(\gamma), c), \]
then it is obviously more convenient to include $S$ in the notation.

Let $\gamma \in G$ be a semisimple element. We want now to investigate the behavior of the orbital integrals $O_{\gamma,\cdot}$ with $g$ in a small neighborhood of $\gamma$. Fix a standard
subgroup $S \subset G$ such that $\gamma$ is in the closure of $Ad_G(S_{\text{reg}})$, but is not in $Ad_G(S_{\text{reg}})$,
and a class $c \in H^0(C(S), C_c \infty(U_S))$. More precisely, we want to study the germ of the function $g \mapsto O_{g,c}(f)$ at an element $\gamma$, where $f \in HH_q(C_c \infty(G))$ is arbitrary.
The germ of a function $h$ at $\gamma$ will be denoted $h_\gamma$.

The following theorem extends one of the basic properties of Shalika germs from usual orbital integrals to higher orbital integrals.

**Theorem 2.** Let $S \subset G$ be a standard subgroup $\gamma \in S$ an element in the closure of $S_{\text{reg}}$, but $\gamma \notin S_{\text{reg}}$. Then there exists a degree preserving linear map
\[ \sigma^S_\gamma : H_*(C(\gamma), C_c \infty(C(\gamma)\cdot_U S)_{S}) \to C_c \infty(S_{\text{reg}}^1) \otimes H_*(C(S), C_c \infty(U_S)_{S}), \]
such that
\[ O^S(f)_\gamma = \sigma^S_\gamma(O(f)), \]
for all $f \in HH_q(C_c \infty(G))$.

Note that, in the notation for the maps $\sigma^S_\gamma$, the standard subgroup $S$ is no longer
determined by $\gamma$.

**Proof.** By the definition of the localization of a module, the map
\[ O^S : HH_q(C_c \infty(G)) \to C_c \infty(S_{\text{reg}}^1) \otimes H_*(C(S), C_c \infty(U_S)_{S}) \]
factors through a map
\[ F : HH_q(C_c \infty(G))_\gamma \to C_c \infty(S_{\text{reg}}^1) \otimes H_*(C(S), C_c \infty(U_S)_{S}). \]
Since $O_\gamma : HH_q(C_c \infty(G))_\gamma \to H_*(C(\gamma), C_c \infty(C(\gamma)\cdot_U S)_{S})$ is an isomorphism by Proposition 3, we may define
\[ \sigma^S_\gamma = F \circ O^{-1}_\gamma, \]
and all desired properties for $\sigma^S_\gamma$ will be satisfied. \qed
Let $\gamma \in S \setminus S_{\text{reg}}$ be such that $\gamma$ is in the closure of $S_{\text{reg}}$, as above, and also let $c \in H^0(C(S), C_{c,\infty}^\infty(U_S)_{\delta})$. Then a consequence of the above theorem, Theorem 2, is that the germ at $\gamma$ of the higher orbital integrals $O_{g,c}^S$ depends only on $O_{\gamma}$. More precisely, if $g \in S_{\text{reg}}$, $f \in HH_0(C_{c,\infty}^\infty(G))$, and we regard $O_{g,c}^S(f)$ as a function of $g$, then its germ at $\gamma$, denoted $O_{g,c}^S(f)_{\gamma}$, is given by

$$O_{g,c}^S(f)_{\gamma} = \langle \sigma_{\gamma}^S(O_{\gamma}(f)), c \rangle.$$ 

This observation allows us to relate Theorem 2 with results of Shalika [22] and Vigneras [23]. So assume now that $G$ is reductive and let $\xi_i \in H_0(C(\gamma), C_{c,\infty}^\infty(C(\gamma)_{u})_{\delta})$ be the basis dual to the basis of $H^0(C(\gamma), C_{c,\infty}^\infty(C(\gamma)_{u})_{\delta})$ given by the orbital integrals over the orbits of $\gamma u$, for $u$ nilpotent in $C(\gamma)$. If we let $F_i = \sigma_{\gamma}(\xi_i)$, then we recover the usual definition of Shalika germs. Due to this fact, we shall call the maps $\sigma_{\gamma}^S$ introduced in Theorem 2 the higher Shalika germs.

We can now characterize the range of the higher orbital integrals. Combining all higher orbital integrals for $S \subset G$ ranging through a set $\Sigma$ of representatives of standard subgroups of $G$, we obtain a map

$$O : HH_*(C_{c,\infty}^\infty(G)) \to \bigoplus_{S \in \Sigma} C^\infty(S_{\text{reg}})W(S) \otimes H_*(C(S), C_{c,\infty}^\infty(U_S)).$$

**Theorem 3.** Let $\Sigma$ be a set of representatives of standard subgroups of $G$ and $\sigma_{\gamma}^S$ be the maps introduced in Theorem 2 for $\gamma \in S_{\text{reg}} \setminus S_{\text{reg}}$. Also, let

$$F \subset \bigoplus_{S \in \Sigma} C^\infty(S_{\text{reg}}) \otimes H_*(C(S), C_{c,\infty}^\infty(U_S)_{\delta}),$$

be the space of sections $\xi$ satisfying $\xi_{\gamma} = \sigma_{\gamma}^S(\xi(\gamma))$. Then $O$ establishes a $R^\infty(G)$–linear isomorphism

$$O : HH_*(C_{c,\infty}^\infty(G)) \longrightarrow F.$$

**Proof.** Note first that the map $O$ is well defined, that is, that its range is contained in $F$, by Theorem 2.

To prove that $O$ is an isomorphism, filter both $HH_*(C_{c,\infty}^\infty(G))$ and $F$ by the subgroups $I_k HH_*(C_{c,\infty}^\infty(G))$ and, respectively, by $I_k F$, using the ideals $I_k$ introduced in Section 3. Since $O$ is $R^\infty(G)$–linear, it preserves this filtration and induces maps

$$I_{k+1} HH_*(C_{c,\infty}^\infty(G))/I_k HH_*(C_{c,\infty}^\infty(G)) \to I_{k+1} F/I_k F.$$

These maps are, by construction, exactly the isomorphisms of Proposition 2. Standard homological algebra then implies that $O$ itself is an isomorphism, as desired.

A consequence of the above result the following “density” corollary.

**Corollary 3.** Let $a \in HH_4(C_{c,\infty}^\infty(G))$. If all $q$-higher orbital integrals of $a$ vanish, then $a = 0$.

We shall also need certain specific cocycles below. Let $\tau_0$ be the trace $\tau_0(f) = f(e)$ on $C_{c,\infty}^\infty(G)$, $G$ unimodular, obtained by evaluating $f$ at the identity $e$ of $G$. Let $G_0$ be the kernel of all unramified characters of $G$. Then $G/G_0 \simeq Z^r$, where $r$ is the rank of a split component of $G$. Let $p_j : G \to Z$ be the morphisms obtained by considering the $j$th component of $Z^r$. Then

$$\delta_j(f)(g) = p_j(g)f(g)$$
defines a derivation of $C_c^{\infty}(G)$. Moreover, we can identify $H^*(G)$ with $\Lambda^*C^r$, the exterior algebra with generators $\delta_1, \ldots, \delta_r$. Fix $c \in H^1(G)$. We can assume that $c = \delta_1 \wedge \ldots \wedge \delta_q$, and then we define the map $D_c : C_c^{\infty}(G)^{\otimes q+1} \to C_c^{\infty}(G)$ by the formula

$$ D_c(f_0, \ldots, f_q) = (q!)^{-1} \sum_{\sigma \in S_q} \epsilon(\sigma)f_0\delta_{\sigma(1)}(f_1)\delta_{\sigma(2)}(f_2)\ldots \delta_{\sigma(q)}(f_q) $$

Then $\tau_c = \tau_0 \circ D_c(f_0, \ldots, f_q)$ defines a Hochschild $q$ cocycle on $C_c^{\infty}(G)$, and

$$ \tau_c = \mathcal{O}_{c,c}, $$

if we naturally identify $c$ with an element of the cohomology group $H^*(G, C_c^{\infty}(G_u))$.

### 3. The Cohomology of the Unipotent Variety

It follows from the main result of the first section, Theorem 3, that, in order to obtain a more precise description of the Hochschild homology of $C_c^{\infty}(G)$, we need to understand the continuous cohomology of the $H$–module $C_c^{\infty}(H_u)_\delta$, where $H$ ranges through the set of centralizers of standard subgroups of $G$ and $H_u$ is the variety of unipotent elements in $H$. (We call the variety $H_u$ the unipotent variety of $H$.) Since the cohomology groups $H_k(H, C_c^{\infty}(H_u)_\delta)$ depend only on $H$, it is enough to consider the case $H = G$. In this section we gather some results on the groups $H_k(G, C_c^{\infty}(G_u)_\delta)$.

We first need to recall the computation of the groups $H_k(G) = H_k(G, \mathbb{C})$, [6]. More generally, we also need to compute $H_k(G, \mathbb{C}_\chi)$, where $\chi : G \to \mathbb{C}^*$ is a character of $G$ and $\mathbb{C}_\chi = \mathbb{C}$ as a vector space, but with $G$–action given by the character $\chi$.

Assume first that $G = S$ is a commutative $p$–adic group, and let $S_0$ be the union of all compact-open subgroups of $S$. Then $S_0$ is a subgroup of $S$ and $S/S_0$ is a free abelian subgroup, whose rank we denote by $\text{rk}(S)$. For this group we then have

$$ H_k(S) \simeq H_k(S/S_0, \mathbb{C}) \simeq \Lambda^k\mathbb{C}^{\text{rk}(S)}. $$

For an arbitrary $p$–adic group $G$, we may identify the cohomology groups $H_k(G)$ with those of a commutative $p$–adic group. Indeed, if $G^0$ is the connected component of $G$ (in the sense of algebraic groups) then $G/G^0$ is finite, and hence $H_k(G) \simeq H_k(G^0)$, by the Hochschild–Serre spectral sequence. This tells us that we may assume $G$ to be connected as an algebraic group. Choose then a Levi decomposition $G = MN$, where $N$ is the unipotent radical of $G$, $M$ is a reductive subgroup, uniquely determined up to conjugation, and the product $MN$ is a semidirect product. Since $H_q(N) = 0$ for $q > 0$, it follows that $H_q(G) \simeq H_q(M)$.

Let $M_1 \subset M$ be the commutator subgroup of $M$, which is also a $p$–adic group, see [5]. The cohomology groups $H_q(M_1)$ were computed in [4], Proposition 6.1, page 316, and [5] and they also vanish for $q > 0$ (because the fundamental domain of the building of $G$ is a simplex). All in all, we obtain that

$$ H_q(G) \simeq H_q(M) \simeq H_q(M^{ab}), $$

where $M^{ab} = M/M_1$ is the abelianization of $M$.

We summarize the above discussion in the following well known statement.

**Lemma 4.** Let $G$ be a $p$–adic group, not necessarily reductive, and let $r$ be the rank of a split component of the reductive quotient of $G$. Then

$$ H_q(G) = H_q(G, \mathbb{C}) \simeq \Lambda^q\mathbb{C}^r, $$
and $H_q(G, C_\chi) = 0$, if $\chi$ is a nontrivial character of $G$.

We continue with a few elementary remarks on $H_k(G, C_c^\infty(G_u)_{\delta})$.

Remark 1. If $G_1 \to G$ is a surjective morphism with finite kernel $F$, then there exists a natural homeomorphism $G_{1u} \simeq G_u$ of the unipotent varieties of the two groups. Since the kernel $F$ acts trivially on $G_{1u}$, using the Hochschild-Serre spectral sequence we obtain an isomorphism

$$H_k(G_1, C_c^\infty(G_{1u})_{\delta}) \simeq H_k(G_1, C_c^\infty(G_u)) \simeq H_k(G, C_c^\infty(G_u)_{\delta}).$$

Remark 2. If $G \subset G_1$ is a normal $p$–adic subgroup with $F \simeq G_1/G$ finite, then we again have a natural homeomorphism $G_{1u} = G_u$. This gives

$$H_k(G_1, C_c^\infty(G_{1u})_{\delta}) \simeq H_k(G_1, C_c^\infty(G_u)_{\delta}) \simeq H_k(G, C_c^\infty(G_u)_{\delta})^F,$$

using once again the Hochschild-Serre spectral sequence. In particular, if the characteristic morphism $F \to Aut(G)/Inn(G)$ is trivial, then we get a natural isomorphism $H_k(G, C_c^\infty(G_u)_{\delta}) \simeq H_k(G_1, C_c^\infty(G_{1u})_{\delta})$.

Remark 3. If $G = G' \times G''$, then $G_u = G'_u \times G''_u$ naturally, and hence $C_c^\infty(G_u) \simeq C_c^\infty(G'_u) \otimes C_c^\infty(G''_u)$. This gives

$$H_k(G, C_c^\infty(G_u)_{\delta}) \simeq \bigoplus_{i+j=k} H_i(G', C_c^\infty(G'_u)) \otimes H_j(G'', C_c^\infty(G''_u)).$$

Remark 4. If $Z$ is a commutative $p$–adic group of split rank $r$, then

$$H_k(Z, C_c^\infty(Z_u)_{\delta}) \simeq C_c^\infty(Z_u) \otimes \Lambda^k C^r.$$

Remark 5. The above isomorphisms reduce the computation of $H_k(G, C_c^\infty(G_u)_{\delta})$ for $G$ reductive, to the computation of the cohomology groups corresponding to its semisimple quotient $H := G/Z(G)$:

$$H_k(G, C_c^\infty(G_u)_{\delta}) = H_k(G, C_c^\infty(G_u)) \simeq \bigoplus_{i+j=k} H_i(H, C_c^\infty(H_u)_{\delta}) \otimes \Lambda^j C^r,$$

where $r$ is the rank of a split component of $G$.

Let $\tau_0$ be the trace obtained by evaluating at the identity. Using $\tau_0$, we obtain an injection $H^1(G) \ni c \mapsto \tau_0 \otimes c \in H^1(G, C_c^\infty(G_u))$.

In order to obtain more precise results on $H_*(G, C_c^\infty(G_u)_{\delta})$, we need to take a closer look at the structure of $C_c^\infty(G_u)$ as a $G$–module. For a $G$–space $X$, we denote by $\langle X \rangle$ the quotient space $X/G$ with the induced topology, which may be non-Hausdorff. Thus $\langle G_u \rangle$ is the set of unipotent conjugacy classes of $G$.

Assume now that $\langle G_u \rangle$ is a finite set. (This happens for example if $G$ is reductive, because the ground field $\mathbb{F}$ has characteristic zero.) Then the space $G_u$ can be written as an increasing union of open $G$–invariant sets $U_1 \subset G_u$, $U_1 = \emptyset$, such that each difference set $U_1 \setminus U_{1-1}$ is a disjoint union of open and closed $G$–orbits,

$$U_1 \setminus U_{1-1} = \bigcup X_{l,j}.$$ A filtration $U_l$ with these properties will be called “nice.” There may be several nice filtrations of $G_u$.

A nice filtration of $G_u$, as above, gives rise, by standard arguments, to a spectral sequence converging to $H_k(G, C_c^\infty(G_u)_{\delta})$, as follows. First, let $\langle g \rangle \in \langle G_u \rangle$ be the
orbit through an element $g \in G_u$. Also, let $C(g)$ denote the centralizer of $g \in G_u$ and $r_g$ denote the rank of a split component of $C(g)$ if $C(g)$ is unimodular, $r_g = 0$ otherwise. This definition of $r_g$ is justified by $H_k(C(g), \Delta_{C(g)}) \simeq \Lambda^k C^r_u$.

**Proposition 6.** Let $G$ be a $p$–adic group with finitely many unipotent orbits (i.e., $\langle G_u \rangle$ is finite). Then, for any nice filtration $(U_i)$ of $G_u$ by open $G$–invariant subsets, there exists a natural spectral sequence with

$$E^2_{p,q} = \bigoplus_{(u) \in (U_p \setminus U_{p-1})} \Lambda^{p+q} C^r_u,$$

convergent to $H_{p+q}(G, C^\infty_c(G_u)_\delta)$.

**Proof.** The argument is standard and goes as follows. Recall first that any filtration $0 = F_0 \subset F_1 \subset \ldots \subset F_N = C^\infty_c(G_u)_\delta$ by $G$–submodules gives rise to a spectral sequence with $E^1_{p,q} = H_{p+q}(G, F_p/F_{p-1})$, convergent to $H_{p+q}(G, C^\infty_c(G_u)_\delta)$.

Now, associated to the open sets $U_i$ of a nice filtration, there exists an increasing filtration $F_i = C^\infty_c(U_i)_\delta \subset C^\infty_c(G_u)_\delta$ by $G$–submodules such that

$$C^\infty_c(U_i)_\delta/C^\infty_c(U_{i-1})_\delta \simeq \bigoplus_j C^\infty_c(X_{i,j})_\delta,$$

where each $X_{i,j}$ is the orbit of a unipotent element, and $U_i \setminus U_{i-1}$ has the topology given by the disjoint union of the orbits $X_{i,j}$. Fix $l$ and $j$, and let $u$ be a unipotent element in $X_{i,j}$ (so that then $X_{i,j}$ is the orbit through $u$), which implies that $C^\infty_c(X_{i,j}) \simeq \text{ind}_{C(u)}^G(\Delta_{C(u)})$. Finally, from Shapiro’s lemma we obtain that

$$H_k(G, C^\infty_c(X_{i,j})_\delta) \simeq H_k(C(u), \Delta_{C(u)}) \simeq \Lambda^k C^r_u,$$

and this completes the proof. □

We expect this spectral sequence to converge for $G$ reductive. This is the case, for example for $G = GL_n(\mathbb{F})$ and for $SL_n(\mathbb{F})$. See Section 3. The convergence of the spectral sequence implies, in particular, the convergence of the orbital integrals of unipotent elements in reductive groups (which is a well known fact due to Deligne and Rao [3]). In general, the convergence of the spectral sequence of the above proposition can be interpreted to represent the convergence of “higher orbital integrals.”

4. Induction and the unipotent variety

We assume from now on that $G$ is reductive, and we fix a parabolic subgroup $P \subset G$, $P \neq G$, and a Levi subgroup $M \subset P$, so that $P = MN$, where $N$ is the unipotent radical of $P$, and the product is a semidirect product. In this section, we relate $H_*(G, C^\infty_c(G_u)_\delta)$ to the groups, $H_*(P, C^\infty_c(P_u)_\delta)$ and $H_*(M, C^\infty_c(M_u)_\delta)$. He have considered non-unimodular subgroups in the previous sections in order to be able to handle subgroups like $P$.

Let $K$ be a “good” maximal compact subgroup of $G$ (see 3, Theorem 5), so that $G = KP$. This decomposition shows that the map

$$K \times P \ni (k, p) \rightarrow kpk^{-1} \in G$$

is proper, and hence the map $G \times_P P := (G \times P)/P \ni (g, p) \rightarrow gpg^{-1} \in G$ is also proper. This gives a map

$$C^\infty_c(G)_\delta \rightarrow C^\infty_c(G \times P) \simeq \text{ind}_P^G(C^\infty_c(P) \otimes \Delta_P) = \text{ind}_P^G(C^\infty_c(P)_\delta)$$

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of $G$-modules. This map of $G$–modules and the standard identification of Hochschild homology with continuous cohomology, equation (21), then give a morphism
\[ \text{ind}_P^G : \text{HH}_*(C_c^\infty(G)) \rightarrow \text{HH}_*(C_c^\infty(P)), \]
defined as the composition of the following sequence of morphisms
\[ \text{HH}_*(C_c^\infty(G)) \simeq \text{H}_*(G, C_c^\infty(G)_\delta) \rightarrow \text{H}_*(G, \text{ind}_P^G(C_c^\infty(P)_\delta) \otimes \Delta_G) \simeq \text{H}_k(P, C_c^\infty(P)_\delta) \simeq \text{HH}_*(C_c^\infty(P)) \]
of Hochschild homology groups. The main result of this section states that $\text{ind}_P^G$ is induced by a morphism of algebras, which we now proceed to define.

Let $dk$ be the normalized Haar measure on the maximal compact subgroup $K$, normalized such that $K$ has volume 1. The composition of kernels
\[ T_1T_2(k_1, k_2) = \int_{G/P} T_1(k_1, k)T_2(k, k_2)dk \]
defines on $C^\infty(K \times K)$ an algebra structure. Let
\[ \phi_P^G : C_c^\infty(G) \rightarrow C^\infty(K \times K) \otimes C_c^\infty(P), \]
be defined by $\phi_P^G(f)(k_1, k_2, p) = f(k_1pk_2^{-1})$.

Recall that the push-forward of the product $dpdk$ of Haar measure on $P \times K$, via the multiplication map $P \times K \ni (p, k) \mapsto pk \in G$, is a left invariant measure on $G$, and hence a multiple $\lambda dg$ of the Haar measure $dg$ on $G$. Suppose that the measure $dk$ of $K$ is the restriction of $dg$ to $K$, and has total mass 1. Then the Haar measures on $G$ and $P$ will be called compatible if $\lambda = 1$. We shall need the following result of Harish Chandra (implicitly stated in (23)):

**Lemma 5.** Suppose the Haar measures on $G$ and $P$ are compatible. Then the linear map $\phi_P^G$, defined above in equation (23), is a morphism of algebras.

**Proof.** The product on $C^\infty(K \times K) \otimes C_c^\infty(P) = C_c^\infty(K \times K \times P)$ is given by the formula
\[ (h_1h_2)(k_1, k_2, p) = \int_K \int_P h_1(k_1, k, q)h_2(k, k_2, q^{-1}p)dqdk. \]
Let $*$ denote the multiplication (i.e., convolution product) on $C_c^\infty(G)$. Thus, we need to prove that
\[ f_1 * f_2(k_1pk_2^{-1}) = \int_K \int_P f_1(k_1qk^{-1})f_2(\overline{qk^{-1}pk_2^{-1}})dqdk, \]
for all $f_1, f_2 \in C_c^\infty(G)$. Consider the map $P \times K \ni (q, k) \mapsto g := qk^{-1} \in G$, and let $d\mu$ be the push-forward of the measure $dqdk$. Then the right-hand side of the above formula becomes
\[ \int_K \int_P f_1(k_1qk^{-1})f_2(\overline{qk^{-1}pk_2^{-1}})dqdk = \int_G f_1(k_1g)f_2(\overline{g^{-1}pk_2^{-1}})d\mu(g). \]
We know that $d\mu = dg$, by assumptions (see the discussion before the statement of this lemma), and then
\[ \int_G f_1(k_1g)f_2(\overline{g^{-1}pk_2^{-1}})d\mu(g) = \int_G f_1(g)f_2(\overline{g^{-1}k_1pk_2^{-1}})d\mu(g) = f_1 * f_2(k_1pk_2^{-1}), \]
by the invariance of the Haar measure. The lemma is proved. \qed
The trace $C^\infty(K \times K) \to \mathbb{C}$ induces an isomorphism
$$\tau : \text{HH}_* (C^\infty(K \times K) \otimes C^\infty_c(P)) \simeq \text{HH}_* (C^\infty_c(P)).$$
Explicitly, this isomorphism is given at the level of chains by
$$\tau (f_0 \otimes f_1 \otimes \ldots \otimes f_q)(p_0, p_1, \ldots, p_q)$$
$$:= \int_{K^{q+1}} f_0(k_0, k_1, p_0)f_1(k_1, k_2, p_1)\ldots f_q(k_q, k_0, p_q)dk_0 \ldots dk_q.$$
This isomorphism combines with $\phi^G_P$ to give a morphism
$$\tag{24} (\phi^G_P)_* : \text{HH}_* (C^\infty_c(G)) \to \text{HH}_* (C^\infty_c(P)).$$

**Theorem 4.** Let $P$ be a parabolic subgroup of a reductive $p$–adic group $G$. Consider the morphisms $(\phi^G_P)_*$ and $\text{ind}^G_P : \text{HH}_* (C^\infty_c(G)) \to \text{HH}_* (C^\infty_c(P))$, defined above (Equations (24) and (22)). Then $(\phi^G_P)_* = \text{ind}^G_P$.

**Proof.** Let $M_1$ and $M_2$ be two left $G$–modules. We can regard $M_1$ as a right module, and then the tensor product $M_1 \otimes_G M_2$ is the quotient of $M_1 \otimes M_2$ by the group generated by the elements $gm_1 \otimes gm_2 - m_1 \otimes m_2$, as before. Alternatively, we can think of $M_1 \otimes_G M_2$ as $(M_1 \otimes M_2) \otimes_G \mathbb{C}$. This justifies the notation $f \otimes_G 1$ for a morphism $f : M_1 \otimes_G M_2 \to M'_1 \otimes_G M'_2$ induced by a morphism
$$f = f_1 \otimes f_2 : M_1 \otimes M_2 \to M'_1 \otimes M'_2.$$

We shall prove the theorem by an explicit computation. To this end, we shall use the results and notation $(h_G$ and $\bar{h}_G = h_G \otimes 1)$ of Lemma 1.

By direct computation, we see that the morphism
$$\tau \circ \phi^G_P : C^\infty_c(G)^\otimes q+1 \to C^\infty_c(P)^\otimes q+1$$
between Hochschild complexes, is given by the formula
$$\tag{25} \tau \circ \phi^G_P(f)(p_0, p_1, \ldots, p_q) = \int_{K^{q+1}} f(k_0k_1k_2^{-1}, k_1p_1k_2^{-1}, \ldots, k_qp_qk_0^{-1})dk_0dk_1\ldots dk_q.$$

We now want to realize the map $\text{ind}^G_P : \text{HH}_* (C^\infty_c(G)) \to \text{HH}_* (C^\infty_c(P))$, at the level of complexes. In the process, it will be convenient to identify the smooth $G$–module $C^\infty_c((G \times P)/P)$ with a subspace of the space of functions on $G \times P$, using the projection $G \times P \to (G \times P)/P$.

Consider the $G$–morphism
$$l : B_q(G) \otimes C^\infty_c(G) \to B_q(G) \otimes \text{ind}^G_P(C^\infty_c(P)_\delta)$$
induced by the morphism
$$C^\infty_c(G) \to \text{ind}^G_P(C^\infty_c(P)_\delta) \subset C^\infty_c(G \times P).$$
Explicitly,
$$l(f) (g_0, g_1, \ldots, g_q, g, p) = f(g_0, g_1, \ldots, g_q, gp^{-1}).$$
Then the resulting morphism
$$l \otimes_G 1 : H_q(G, C^\infty_c(G)_\delta) = H_q(G, C^\infty_c(G)) \to H_q(G, \text{ind}^G_P(C^\infty_c(P)_\delta))$$
is the morphism $H_q(G, C^\infty_c(G)_\delta) \to H_q(G, \text{ind}^G_P(C^\infty_c(P)_\delta))$ on homology corresponding to the $G$–morphism $C^\infty_c(G) \to \text{ind}^G_P(C^\infty_c(P)_\delta)$. 

\[\text{HOMOLOGY OF } p\text{-ADIC GROUPS } 19\]
The $G$–morphism

$$r : \mathcal{B}_q(G) \otimes \text{ind}_P^G(\mathcal{C}_\infty^G(P)_{\delta}) \to \text{ind}_P^G(\mathcal{B}_q(P) \otimes \mathcal{C}_\infty^G(P)_{\delta})$$

$$= \mathcal{C}_\infty^G(G) \otimes _P (\mathcal{B}_q(P) \otimes \mathcal{C}_\infty^G(P)_{\delta}) \subset \mathcal{C}_\infty^G(G \times P^{q+2}),$$

given by the formula

$$r(f)(g, p_0, p_1, \ldots, p_q, p) = \int_{K \times K'^{+1}} f(g, p_0k_1^{-1}, g, p_1k_2^{-1}, \ldots, g, p_nk_0^{-1}, g, p)dkdk',$n

$(dk = dk_0 \ldots dk_q)$ is well defined and induces an isomorphism in homology, because the only nonzero homology groups are in dimension 0, and they are both isomorphic to $\text{ind}_P^G(\mathcal{C}_\infty^G(P)_{\delta})$. We have an isomorphism

$$\chi : \text{ind}_P^G(\mathcal{B}_q(P) \otimes \mathcal{C}_\infty^G(P)_{\delta}) \otimes_G \mathbb{C} \to (\mathcal{B}_q(P) \otimes \mathcal{C}_\infty^G(P)_{\delta}) \otimes_P \mathbb{C},$$
of complexes. This shows that the homology of the second complex in (26) is obtained from $h$ by the morphism of complexes $\tilde{h}$ of complexes satisfying

$$dk \cdot \chi = \int_{K \times K'^{+1}} f(g, p_0k_1^{-1}, g, p_1k_2^{-1}, \ldots, g, p_nk_0^{-1}, g, p)dkdk',$n

where $(dk = dk_0 \ldots dk_q)$ is well defined and induces an isomorphism in homology, because the only nonzero homology groups are in dimension 0, and they are both isomorphic to $\text{ind}_P^G(\mathcal{C}_\infty^G(P)_{\delta})$. We have an isomorphism

$$\chi : \text{ind}_P^G(\mathcal{B}_q(P) \otimes \mathcal{C}_\infty^G(P)_{\delta}) \otimes_G \mathbb{C} \to (\mathcal{B}_q(P) \otimes \mathcal{C}_\infty^G(P)_{\delta}) \otimes_P \mathbb{C},$$
of complexes. This shows that the homology of the second complex in (26) is isomorphic to $H_q(P, C_\infty^G(P)_{\delta})$, and that the map induced on homology, that is

$$\chi(r \otimes_G 1) : H_q(G, \text{ind}_P^G(\mathcal{C}_\infty^G(P)_{\delta})) \to H_q(P, C_\infty^G(P)_{\delta}),$$
is the Shapiro isomorphism.

Recall now that the isomorphism $H_q(G, C_\infty^G(G)_{\delta}) \simeq \text{HH}_q(C_\infty^G(G))$ is induced by the morphism of complexes $\tilde{h}$ defined in Lemma [3], Equation (6). From the definition of the morphism $\text{ind}_P^G : \text{HH}_1(C_\infty^G(G)) \to \text{HH}_1(C_\infty^G(P))$ and the above discussion, we obtain the equality of the morphisms $H_q(G, C_\infty^G(G)_{\delta}) \to H_q(P, C_\infty^G(P)_{\delta})$ induced by $\chi(rl \otimes_G 1)$ and $\tilde{h}_P \circ \text{ind}_P^G \circ \tilde{h}_G$. Thus, in order to complete the proof, it would be enough to check that $\tilde{h}_P \circ \chi(rl \otimes_G 1) = \tau \circ \phi_P^G \circ \tilde{h}_G$ at the level of complexes. Let

$$t : \mathcal{B}_q(G) \otimes \text{ind}_P^G(\mathcal{C}_\infty^G(P)_{\delta}) \to \mathcal{B}_q(G) \otimes_G \text{ind}_P^G(\mathcal{C}_\infty^G(P)_{\delta})$$

be the projection. Since $h_G$ is surjective, it is also enough to check that $\tilde{h}_P \chi(r \otimes_G 1)t = \tau \phi_P^G h_G$.

Let

$$r'(f)(p_0, p_1, \ldots, p_q, p) = \int_{K \times K'^{+1}} f(k'p_0k_1^{-1}, k'p_1k_2^{-1}, \ldots, k'p_nk_0^{-1}, \quad , p)dkdk',$n

where $(dk = dk_0 \ldots dk_q)$, as before. Then $r'$ induces a morphism

$$r' : \mathcal{B}_q(G) \otimes \text{ind}_P^G(\mathcal{C}_\infty^G(P)_{\delta}) \to \mathcal{B}_q(P) \otimes \mathcal{C}_\infty^G(P)_{\delta}$$
of complexes satisfying $h_P \circ r' = \tilde{h}_P \chi(r \otimes_G 1)t$. Directly from the definitions we obtain then that $h_P \circ r' \circ l = \tau \circ \phi_P^G \circ h_G$. This completes the proof. $$\square$$

For simplicity, we have stated and proved the above result only for $G$ reductive, however, it extends to general $G$ and $P$ such that $G/P$ is compact, by including the modular function of $G$, where appropriate.

In order to better understand the effect of the morphism

$$\text{ind}_P^G = (\phi_P^G)_* : \text{HH}_*(C_\infty^G(G)) \longrightarrow \text{HH}_*(C_\infty^G(P)),$$

it is sometimes useful to look at its action on the geometric fibers of the group $\text{HH}_*(C_\infty^G(G))$. This is especially useful because the action on the geometric fibers also recovers the classical results on the characters of induced representations.
First we observe that restriction defines a morphism \( \rho^G_P : R^\infty(G) \to R^\infty(P) \). In case the group \( G \) is reductive and \( M \) is a Levi component of the parabolic subgroup \( P \), we also have \( R^\infty(P) \simeq R^\infty(M) \).

**Lemma 6.** Let \( P \) be a parabolic subgroup of a reductive \( p \)-adic group \( G \), and let \( \rho^G_P : R^\infty(G) \to R^\infty(P) \) be the morphism induced by restriction, used to define a \( R^\infty(G) \)-module structure on \( HH_q(C^\infty_c(P)) \). Then

\[
\text{ind}^G_P : HH_q(C^\infty_c(G)) \to HH_q(C^\infty_c(P))
\]

is \( R^\infty(G) \)-linear, in the sense that \( \text{ind}^G_P(f \xi) = \rho^G_P(f) \text{ind}^G_P(\xi) \), for all \( f \in R^\infty(G) \) and all \( \xi \in HH_q(C^\infty_c(G)) \).

**Proof.** The result of the lemma follows from the fact that the map

\[
C^\infty_c(G) \to \text{ind}^G_P(C^\infty_c(P))
\]

is \( R^\infty(G) \)-linear and the isomorphism of Shapiro’s Lemma,

\[
H_q(G, \text{ind}^G_P(C^\infty_c(P))) \simeq H_q(P, C^\infty(P)),
\]

is natural.

Alternatively, one can use the explicit formula of equation [27].

If \( m = m_\gamma \subset R^\infty(G) \) is the maximal ideal of functions vanishing at a semisimple element \( \gamma \in G \), then its image \( (\rho^G_P)_\gamma(m) := \rho^G_P(m)R^\infty(P) \subset R^\infty(P) = R^\infty(M) \) is the ideal of functions vanishing at all \( g \in M \) that are conjugate to \( \gamma \) in \( G \). If \( \gamma \) is elliptic, then \( m = R^\infty(P) \). If \( \gamma \in M \), then \( m \) need not, in general, be maximal. Nevertheless, we obtain a morphism

\[
(\rho^G_P)_\gamma : \mathbb{C} \simeq R^\infty(G)/\gamma = R^\infty(G)/m \to R^\infty(M)/m \simeq \mathbb{C}^{#(\gamma)},
\]

where \( #(\gamma) \) = \( l \) is the set of conjugacy classes in \( M \) that consist of elements that are conjugated to \( \gamma \) in the bigger group \( G \). Let \( \gamma_1, \gamma_2, \ldots, \gamma_l \in M \) be representatives of the conjugacy classes of elements in \( M \) that are conjugated to \( \gamma \) in \( G \).

We are ready now to study the morphisms

\[
(\text{ind}^G_P)_\gamma : HH_q(C^\infty_c(G)) \to HH_q(C^\infty_c(P))/m HH_q(C^\infty_c(G))
\]

\[
\longrightarrow HH_q(C^\infty_c(P))/((\rho^G_P)_\gamma(m) \simeq \bigoplus_{j=1}^l HH_q(C^\infty_c(P))_{\gamma_j}.
\]

Let \( C_P(\gamma_j) \) be the centralizer of \( \gamma_j \) in \( P \) and \( C_G(\gamma_j) \simeq C_G(\gamma_j) \) be the centralizer of \( \gamma_j \) in \( G \). Then \( C_P(\gamma_j) \) identifies with a subspace of \( C_G(\gamma_j) \), which gives rise to a continuous proper map \( C_G(\gamma_j) \times C_P(\gamma_j) \to C_G(\gamma_j) \), and hence to a morphism

\[
C^\infty_c(C_G(\gamma_j)) \to \text{ind}^{C_G(\gamma_j)}_{C_P(\gamma_j)}(C^\infty_c(C_P(\gamma_j)_{\delta}))
\]

of \( C_G(\gamma) \)-modules. Passing to cohomology, we obtain using Shapiro’s Lemma a morphism

\[
\iota^G_j : H_q(C_G(\gamma_j), C^\infty_c(C_G(\gamma_j))) \to H_q(C_P(\gamma_j), C^\infty_c(C_P(\gamma_j)_{\delta})�.
\]

Recall that Proposition [4] gives isomorphisms

\[
HH_q(C^\infty_c(G)) \simeq H_q(C_G(\gamma), C^\infty_c(C_G(\gamma))),
\]

and

\[
HH_q(C^\infty_c(P))_{\gamma_j} \simeq H_q(C_P(\gamma_j), C^\infty_c(C_P(\gamma_j)_{\delta}).
\]
Proposition 7. Let $\gamma \in G$ be a semisimple element and $M \subset P$ be as above. If the conjugacy class of $\gamma$ does not intersect $M$, then $\text{HH}_*(\mathcal{C}^\infty_c(P))_{\gamma} = 0$, and hence $(\text{ind}^G_P)_\gamma = 0$. Otherwise, using the notation above, we have

$$(\text{ind}^G_P)_\gamma = \oplus_{j=1}^l \iota_{\gamma_j} : \text{HH}_*(\mathcal{C}^\infty_c(G))_{\gamma} \to \text{HH}_*(\mathcal{C}^\infty_c(P))_{\gamma}.$$ 

Proof. This follows from definitions if we observe that in the sequence of maps

$$G \times_P P \times_{C_P(\gamma_i)} (\gamma_i C_P(\gamma_i)u) \simeq G \times_{C_G(\gamma_i)} C_G(\gamma_i) \times_{C_P(\gamma_i)} (\gamma_i C_P(\gamma_i)u)
\to G \times_{C_G(\gamma_i)} (\gamma_i C_G(\gamma_i)u)$$

the second map is induced by $C_G(\gamma_i) \times_{C_P(\gamma_i)} C_P(\gamma_i)u \to C_G(\gamma_i)u$ and their composition induces on homology the direct summand $\iota_{\gamma_j}$ of the map $(\text{ind}^G_P)_\gamma$. 

Another morphism that is likely to play an important role is the “inflation,” which we now define. Let $N \subset P$ be the unipotent radical of an algebraic $p$-adic group, and let $M = P/N$ be its reductive quotient. Then integration over $N$ defines an algebra morphism

$$\psi^N_M : \mathcal{C}^\infty_c(P) \to \mathcal{C}^\infty_c(M), \quad \psi^N_M(f)(m) = \int_N f(mn)dn.$$ 

Integration over $N$ also defines a $G$-morphism $\mathcal{C}^\infty_c(P)_\delta \to \mathcal{C}^\infty_c(M)_\delta$, and, since $N$ is a union of compact subgroups, we finally obtain morphisms

$$\text{H}_k(P, \mathcal{C}^\infty_c(P)_\delta) \to \text{H}_k(P, \mathcal{C}^\infty_c(M)_\delta) \simeq \text{H}_k(M, \mathcal{C}^\infty_c(M)_\delta),$$

whose composition we denote $\text{inf}^P_M$.

Theorem 5. If $M$ is a Levi component of a $p$-adic group $P$ over a field of characteristic zero, as above. Then we have

$$(\psi^P_M)_* = \text{inf}^P_M : \text{HH}_*(\mathcal{C}^\infty_c(P)) \to \text{HH}_*(\mathcal{C}^\infty_c(M)).$$

Proof. Integration over $N$ defines a morphism

$$f : \mathcal{B}(P) \otimes \mathcal{C}^\infty_c(P)_\delta \to \mathcal{B}(M) \otimes \mathcal{C}^\infty_c(M),$$

which commutes with the action of $P$. Then $f \otimes_P 1$ coincides with the morphism of complexes induced by $\psi^P_M$.

Consider now the maps $h_D$ defined in the proof of Lemma §. Then $\psi^P_M \circ h_P = h_M \circ f$, and hence $\psi^P_M \circ \tilde{h}_P = \tilde{h}_M \circ (f \otimes_P 1)$, from which the result follows. 

We now want to proceed by analogy and establish the explicit form of the action of $\text{inf}^P_M$ on the geometric fibers. Fix $\gamma \in M$. Again, integration over the nilpotent radical of $C_P(\gamma)$, the centralizer of $\gamma$ in $P$, induces a morphism

$$\mathcal{C}^\infty_c(C_P(\gamma)u)_\delta = \mathcal{C}^\infty_c(C_P(\gamma)u) \otimes \Delta_{C_P(\gamma)} \to \mathcal{C}^\infty_c(C_M(\gamma)u),$$

of $P$-modules. Let

$$j_\gamma : \text{HH}_*(\mathcal{C}^\infty_c(P))_{\gamma} = \text{H}_*(C_P(\gamma), \mathcal{C}^\infty_c(C_P(\gamma)u)_\delta)$$

$$\to \text{H}_*(C_M(\gamma), \mathcal{C}^\infty_c(C_M(\gamma)u)) = \text{HH}_*(\mathcal{C}^\infty_c(M))_{\gamma}$$

be the induced morphism.
Proposition 8. Let $\gamma$ be a semisimple element of a Levi component $M$ of the group $P$. Let $d(\gamma)$ be the determinant of $Ad_{\gamma}^{-1} - 1$ acting on $\text{Lie}(N)/\ker(Ad_{\gamma} - 1)$. Then, using localization at the maximal ideal defined by $\gamma$ in $R^\infty(G) = R^\infty(P)$ and the above notation,

$$(\inf_M^P)_{\gamma} = |d(\gamma)|^{-1} j_{\gamma} : \text{HH}_*(C_c^\infty(P)) \to \text{HH}_*(C_c^\infty(M))_{\gamma}.$$ 

Proof. Fix $\gamma \in G$, not necessarily semisimple and let $N_\gamma$ be the subgroup of elements of $N$ commuting with $\gamma$. We choose a complement $V_\gamma$ of $\text{Lie}(N_\gamma)$ in $\text{Lie}(N)$ and we use the exponential map to identify $V_\gamma$ with a subset of $N$. Then the Jacobian of the map

$$V_\gamma \times N_\gamma \ni (n, n') \to \gamma^{-1} n \gamma n^{-1} n' \in N = V_\gamma N_\gamma$$

is $d(\gamma)$, and from this the result follows.

This result is compatible with the results of van Dijk on characters of induced representations, see [8].

5. Examples

Our results can be used to obtain some very explicit results in certain particular cases.

Example 1. Let $Z$ be a commutative $p$-adic group of split rank $r$ (so that $H_q(Z) \simeq \Lambda^q \mathbb{C}^r$, for all $q \geq 0$). Then

$$(27) \quad \text{HH}_q(C_c^\infty(Z)) \simeq C_c^\infty(Z) \otimes \Lambda^q \mathbb{C}^r.$$ 

Example 2. Let $P$ be the (parabolic) subgroup of upper triangular matrices in $SL_2(F)$, and $A \subset P$ be the subgroup of diagonal matrices. Then inflation defines a morphism

$$\inf_A^P : \text{HH}_*(C_c^\infty(P)) \to \text{HH}_*(C_c^\infty(A)) = C_c^\infty(A) \otimes \Lambda^* \mathbb{C}$$ 

whose range is $C_c^\infty(A) \oplus C_c^\infty(A \setminus \{\pm I\})$, with $I$ the identity matrix of $SL_2(F)$. (We see this by localizing at each $\gamma$.) To describe the kernel of $\inf_A^P$, let

$$(28) \quad u_b = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}.$$ 

Then, if we choose $b$ to range through $\Sigma_u$, a set of representative of $F^*/F^{*2}$, the set of elements $u_b$ forms a set of representatives of the set of nontrivial conjugacy classes of unipotent elements of $P$. Recall that $F$ has characteristic zero, so $\Sigma_u$ is a discrete set. Let $O_{b,+}$ be the orbital integral associated to $u_b$, and let $O_{b,-}$ be the orbital integral associated to $-u_b$, then

$$F_\pm = \oplus_{b} O_{b,\pm} : C_c^\infty(G) \to C^\Sigma_u$$

identify the kernel of $\inf_A^P$ as follows. The map $F_+ \oplus F_- : \ker(\inf_A^P) \to C^{\pm \Sigma_u}$ is injective, and the range of each of $F_\pm$ is the set of elements with zero sum.

All in all, let us consider the map $\Phi = \inf_A^P \oplus F_+ \oplus F_-,$

$$\Phi : \text{HH}_*(C_c^\infty(P)) \to (C_c^\infty(A) \oplus C^{\pm \Sigma_u}(0)) \oplus (C_c^\infty(A \setminus \{\pm I\})),$$

where the lower index $(i)$ represents the degree. Then $\Phi$ is surjective in degree 1, and, in degree 0, it consists of $(f, \lambda_b, \epsilon)$, $f \in C_c^\infty(A)$, $\lambda \in \mathbb{C}$, for $\epsilon \in \{\pm 1\}$ and $b \in \Sigma_u \simeq F^*/F^{*2}$, such that $\sum_b \lambda_b \epsilon = f(\epsilon I)$, for $\epsilon = \pm 1$. 

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Note that evaluation at \( \pm I \) does not define a trace on \( \mathcal{C}_c^\infty(P) \). Actually, in the spectral sequence of Proposition 8, the obstruction to extend the evaluation at \( I \) to a trace is responsible for “killing” the 1 cohomology supported at \( I \) (which explains our claim on the range of \( \inf_A^P \) above).

**Example 3.** Consider now the group \( G = SL_2(\mathbb{F}) \), where \( \mathbb{F} \) is a \( p \)-adic field of characteristic zero such that the characteristic of the residual field is not 2, for simplicity. Let \( F_q \) be the residual field of \( \mathbb{F} \) (thus \( q = p^n \) for some \( n \in \mathbb{N} \) and some prime \( p \), denotes the number of elements of \( F_q \). We choose \( c \) in the valuation ring of \( \mathbb{F} \), whose image in \( F_q \) is not a square. Also, let \( \tau \) be a generator of the (unique) maximal ideal of the valuation ring of \( \mathbb{F} \). We shall use the notation of [20] and thus let \( \theta \) range through the set \( \{ \epsilon, \tau, \epsilon \tau \} \) and let \( T_0, T_\theta \) be the elliptic tori defined there. (Recall that \( T_0 = \{ [a_{ij}], a_{11} = a_{22}, a_{21} = \theta a_{12} \} \) and \( T_\theta = \{ [a_{ij}], a_{11} = a_{22}, a_{21} = \theta^2 a_{12} \} \), where \( \theta^2 = \theta \) for some \( a \in \mathbb{F}^2 \) not in the image of the norm map \( N : \mathbb{F}[\theta^*] \to \mathbb{F}^* \).) We distinguish two cases, first the case where \( -1 \) is a square and then the case where it is not a square. If \( -1 \) is a square, then the Weyl group of each of the tori \( T = T_0, T_\theta \) has order 2. Otherwise \( W(T) = \{ 1 \} \), for each \( T = T_0 \) or \( T = T_\theta \), but \( T_0 \) and \( T_\theta \) are conjugate for each fixed \( \theta \).

Let \( X = \bigcup T_0/S_2 \cup \bigcup T_\theta/S_2 \), if \( -1 \) is a square, and \( X = \bigcup T_0 \) otherwise, with the induced topology. Then \( X \setminus \{ \pm I \} \) is the set of all elliptic conjugacy classes of \( SL_2(\mathbb{F}) \).

Denote by \( A \subset SL_2(\mathbb{F}) \) the set of diagonal matrices in \( SL_2(\mathbb{F}) \). Let \( W(A) = S_2 \) act on \( \mathcal{C}_c^\infty(A) \otimes \Lambda^* \mathbb{C} \) by conjugation on \( \mathcal{C}_c^\infty(A) \) and act by the nontrivial character on \( \mathbb{C} \). Then we have the following. Recall that there are 10 conjugacy classes of unipotent elements in \( SL_2(\mathbb{F}) \), if \( p \neq 2 \).

**Proposition 9.** The composition
\[
\phi := \inf_A^P \circ \text{ind}_G^C : \text{HH}_*(\mathcal{C}_c^\infty(SL_2(\mathbb{F}))) \to \text{HH}_*(A) = \mathcal{C}_C^\infty(A) \otimes \Lambda^* \mathbb{C}
\]
has range consisting of \( W(A) \) invariant elements, and the kernel of \( \phi \) is isomorphic to \( \mathcal{C}_C^\infty(X \setminus \{ \pm I \}) \otimes \mathbb{C}^{10} \), via orbital integrals with respect to elliptic and unipotent elements.

**Proof.** First of all, it is clear that the composition \( \phi = \inf_A^P \circ \text{ind}_G^C \) is invariant with respect to the Weyl group \( W(A) \), and hence its range consists of \( W(A) \)-invariant elements.

The localization of \( \phi \) at a regular, diagonal conjugacy class \( \gamma \) is onto by Proposition 8. Next, we know that every orbital integral extends to \( \mathcal{C}_C^\infty(SL_2(\mathbb{F})) \), and this implies directly that the spectral sequence of Proposition 8 collapses at the \( E^2 \) term. This proves that the localization of \( \phi \) at \( \gamma = 1 \) is also onto, and hence \( \phi \) is onto. The rest of the proposition follows also from Proposition 8 by localization.

This example is also discussed in [20], but from a different perspective.

**Example 4.** We end this section with a description of the ingredients entering in the formula (8) for the of the Hochschild homology of \( \mathcal{C}_C^\infty(G) \), if \( G = GL_n(\mathbb{F}) \). Let \( \gamma \in G \) be a semisimple element. The minimal polynomial \( Q_\gamma \) of \( \gamma \) decomposes as \( Q_\gamma = p_{1}p_{2} \cdots p_{r} \) into irreducible polynomials with coefficients in \( \mathbb{F} \). (We assume, for simplicity, that each polynomial \( p_j \) is a monic polynomial.) Also, let \( P_\gamma = p_{1}^{l_{1}}p_{2}^{l_{2}} \cdots p_{r}^{l_{r}} \) be the characteristic polynomial of \( \gamma \). Then the algebra generated by
\[ F[\gamma] \simeq K_1 \oplus \ldots \oplus K_r, \]
where \( K_i = \mathbb{F}[t]/(p_i(t)) \) are not necessarily distinct fields. The commutant \( \{ \gamma \}' \) of \( \gamma \) in \( M_n(F) \) is the commutant of this algebra, and hence

\[ \{ \gamma \}' \simeq M_{l_1}(K_1) \oplus M_{l_2}(K_2) \oplus \cdots \oplus M_{l_r}(K_r), \]

\[ C(\gamma) \simeq \prod_{i=1}^r GL_{l_i}(K_i), \quad S := Z(C(\gamma)) \simeq \prod_{i=1}^r K_i^*, \]
and

\[ S^{\text{reg}} = \{(x_i) \in S, x_i \text{ generates } K_i \text{ and the minimal polynomials of } x_i \text{ are distinct}\}. \]

By the Skolem–Noether theorem, the Weyl group \( W(S) = N(S)/C(S) \) coincides with the group of algebra automorphisms of \( \{ \gamma \}' \). This group has as quotient a group isomorphic to the subgroup \( \Pi \subset N(S) \) which permutes the algebras \( M_{l_i}(K_i) \).

Then \( \Pi \cong S_{m_1} \times \ldots \times S_{m_r} \), that is, \( \Pi \) is a product of symmetric groups. We denote the kernel of this morphism by \( W_0(S) \). It is isomorphic to \( \prod_{i=1}^r Aut_s(K_i) \) (again by the Skolem–Noether theorem).

The group \( W(S) \) is then the semidirect product of \( W_0(S) \) by \( \Pi \). We hence obtain exact sequences

\[ 1 \rightarrow N_0(S) \rightarrow N(S) \rightarrow \Pi \rightarrow 1 \]
and

\[ 1 \rightarrow C(S) \rightarrow N_0(S) \rightarrow W_0(S) \rightarrow 1. \]

According to (3), the only other ingredients necessary to compute \( H^*(C_n^\infty(G)) \) are the groups \( H_*(C(S), C_n^\infty(U_n)) \).

Now, the unipotent variety of \( C(S) \) is the product of the unipotent varieties of \( GL_{l_i}(K_i) \), \( i = 1, r \), and the subgroup \( C(S) \) preserves this product decomposition.

We see then that in order to prove that the spectral sequence of Proposition 3 collapses (for any choice of open subsets \( U_i \)), it is enough to check this for the spectral sequence converging to the cohomology of \( C_n^\infty(GL_n(K)) \), for an arbitrary characteristic zero \( p \)-adic field \( \mathbb{K} \).

Fix a unipotent element \( \gamma \in GL_n(\mathbb{K}) \). Define then \( V_0 = 0 \), \( V_i = \ker(\gamma - 1)^l \subset \mathbb{K}^n \), if \( l > 0 \), also, choose \( W_i \) such that \( V_i = V_{i-1} \oplus W_i \), and define

\[ P = \{ \gamma \in GL_n(\mathbb{K}), \gamma V_i \subset V_i \}, \quad \text{and} \]
\[ M = \{ \gamma \in GL_n(\mathbb{K}), \gamma W_i = W_i \}. \]

Then \( P \) is a parabolic subgroup with unipotent radical

\[ N = \{ \gamma \in GL_n(\mathbb{K}), (\gamma - 1)V_i \subset V_{i-1} \}, \]
and \( M \) is a Levi component of \( P \). It is easy to check, from definition, that the \( P \)-orbit of \( u \) in \( N \) is dense. The centralizer of \( u \) is then contained in \( P \) and has split rank \( \leq \) the split rank of \( P \). Fix a maximal split torus \( A \) in the centralizer of \( u \). We can assume that this split torus is contained in \( M \). From the definition and by direct inspection, the map \( H_*(A) \rightarrow H_*(M) \) is injective, and hence the map

\[ H^*(M) \rightarrow H^*(A) = H^*(C(u)) \]

is surjective.

Fix now a cohomology class \( c_0 \in H^q(C(\gamma)) \simeq H^q(A) \) and choose a cohomology class \( c \in H^q(M) \) that maps to \( c_0 \) under the above restriction map. Also, let \( \tau \) be
the trace on $\tau_0(f) = f(e)$ on $C^\infty_c(M)$ (obtained by evaluation at the identity $e$). Then the formula
\begin{equation}
\phi_0(f_0, \ldots, f_q) = \tau_0(D_c(f_0, \ldots, f_q))
\end{equation}
defines a Hochschild cyclic cocycle on $C^\infty_c(M)$. Consequently,
\begin{equation}
\phi = \phi_0 \circ \inf_{A}^P \circ \circ \text{ind}_G^P
\end{equation}
defines a Hochschild cocycle on $C^\infty_c(G)$. For any filtration $U_i$ of $G$ by open, invariant open sets, such that each $U_i \setminus U_{i-1}$ consists of a single orbit. Suppose that the orbit $U_i \setminus U_{i-1}$ is the orbit of $\gamma \in GL_n(F)$ considered above. Then the cocycle $\phi$ will vanish on $C^\infty_c(U_i)$ and represent the cohomology class $c \in H^q(C(\gamma)) \cong H^q(G, C^\infty_c(U_i \setminus U_{i-1})))$.

From this it follows that the spectral sequence of Proposition \[\] degenerates at $E^2$.

It is very likely that the above argument extends to arbitrary reductive $G$ by choosing $M$ and $P$ as in [18].

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