The theoretical investigation for quark stars has turned out to be a worldwide endeavor after the first prediction of the quark core at the center of the core collapsed neutron stars [1]. The concept of the quark star was brought to light by soviet physicists Ivanenko and Kurdgelaidze [1, 2], about five years after the Gell-Mann prediction of quarks [3]. Several recent observations of compact and comparatively cooler stars (SWIFTJ1749.4-2807 [4], RXJ185635-3754 and 3C58 [5]) also provide the evidence in favor of the existence of quark stars. Unlike other compact astrophysical objects and main sequence stars, the quark stars are self-bound by strong interaction rather than by gravity alone [6, 7]. Apart from this unique characteristic, the density of the quark stars are remarkably high (even higher than neutron star) and hence gains immense importance as natural laboratory for quark matter.

The quark star is considered to be the very end product of the stellar evolution. As the fuel of a star (neutron star) tends to get exhausted, the radiation pressure of the star can no longer balance the self-gravity. As a result, the core part of the star starts collapsing due to its own gravity. However, in certain cases, the density turns out to be substantially high and therefore a quark core is spontaneously generated (even without strangeness) at the center of the star. Although initially the collapse takes place in the core part, it grows over time and occupies the entire star eventually by capturing free neutrons from the vicinity of the surface in absence of the Coulomb barrier. Beside this process, a thin outer crust of comparatively lower density is also generated outside the quark core (∼ 1/1000 of the average density of the star), which does not contain any free neutrons [7]. The newly formed quark phase contains only 2 flavors of quarks (‘u’ and ‘d’ quark) as it is generated from the non-hyperonic baryons (mainly neutron). Later the strangeness can be developed in the 2 flavor quark state via weak interaction (ud → us), in excess of ‘d’ quark) by absorbing the energy of the quark star [8, 9] and thus transforms into a strange quark star (SQS). The SQS is basically composed of 3 flavors of quarks (contains strange quark in addition to the ‘u’ and ‘d’ quark) confined in a hypothetical large bag [10, 11], which is characterized by the Bag Constant. The strange quark matter can be treated as the perfect ground state of the strongly interacting matter as predicted by E. Witten [7]. Also, there are several alternative models [12–14] supporting the conjecture of the SQS (for a review, see [15]). But in another possibility, a quark star can be formed by clumping of ambient quark matters due to the gravitational interaction essentially like other normal stars. This is only possible if a significant percentage of strange quarks [7] present in the system. Such “pure” quark stars [16] might belong to a hypothetical quark galaxy or might result from the accumulation of strangelets [17–19]. Primordial strange star can be another possible example of the quark star. According to this conjecture, quark stars were formed due to the phase transition in the early Universe. Later those transform into strange stars in order to maintain the stability as assumed in the work of [7].

The key difference between the ordinary stars and the quark star is that, the mass of an ordinary star is almost entirely due to the baryons, whereas for the quark star, one has to define the effective mass for the ‘u’ and ‘d’ quarks, as those (quarks) are known to have very small masses. As the quark stars may be expected to be produced due to hadron quark transition in neutron stars, its limiting mass is expected to be very close to that of the neutron star. As a result, one can estimate the total mass of the star by solving the Tolman-Oppenheimer-Volkov (TOV) equations [20] numerically [21, 22] by adopting the conjecture, proposed by E. Witten [7]. In several recent works [23, 24], the limiting mass for spinning quark star is also studied using the same procedure as used for...
the case of static quark star. But there is no argument in
the literature that favors the existence of limiting mass
which, like ordinary compact objects \[25\], depends dom-
minantly on the fundamental constants. In the work of
Banerjee et. al. \[26\], an analytical study for the limiting
mass carried out for static quark stars, starting from the
energy balance relations as proposed by Landau \[27\], is
found to depend mainly on the fundamental constants
and the MIT bag constant \[28\]. Our aim in this work
is to extend the theoretical study of The Chandrasekhar
limit for quark star \[26\] for the case of rotating quark
stars.

Since, the quark stars are self-bound by strong inter-
action \[29\] (which is far stronger than the gravitational
field), it can withstand high rotational frequency with-
out having notable shape deformation. Moreover, in
most of the cases, such core collapse stars exhibit ex-
treme magnetic field, which significantly prevents the
rotational contribution in shape deformation (ellipticity
$\geq 10^{-4} - 10^{-7}$) \[30, 31\] even in the case for millisecond
(time period) stars. So, the modeled quark stars are con-
sidered to be rigid and spherical in our entire calculation.
The effect of the bag constant and rotational frequency
in the limiting mass and radius have also been studied in
theoretical regime.

The paper is organized as follow, in the section \[\Pi\] and
\[\Pi\] we give an account of the bag constant and the fermi
energy respectively. In the section \[\IV\] the effective mass
per particle of the quark star is calculated in the context
of particle physics. Section \[\IV\] deals with the total energy
calculation of the star and the results for the limiting
case. Finally in the section \[\VI\] the concluding remarks
are given.

\section*{II. BAG CONSTANT}

MIT Bag model has turned out to be a successful
model of hadronic structure and achieved immense suc-
scesses in hadron spectroscopy \[32–34\]. According to this
phenomenological model, massless point-like quarks are
confined in a hypothetical bag and the state is character-
ized by a parameter bag constant $B$.

The bag constant $B$ depicts the difference between the
vacuum energy densities of the non-perturbative and the
perturbative ground states of the quarks \[28\] and depends
on density (and temperature, in general). In the case of
quark star, the entire star is assumed to be such a bag,
which contains all the constituent quarks.

The primary objective of this paper is to obtain an-
alytical results of a rotating star that can be compared
relative to a static star. In order to achieve this, we have
made use of only the essential features of this model, to
the extent of providing a dynamic mass to the constituent
light quarks. This approach has also been adopted in
several other recent works \[11, 12, 35\] which address the
issue of the static mass limit of similar compact stars.

\section*{III. FERMI ENERGY}

In the current work, the limiting mass for the quark
star is estimated by adopting the simple energy balance
picture, as proposed by Landau \[27\]. According to this
approach, the total energy per fermion ($e$) attains the
minima at the limiting case of the star, given by,

$$e = e_f + e_G + e_{rot},$$

where, $e_f$, $e_G$ and $e_{rot}$ represent the contributions of the
fermi energy, gravitational energy and the kinetic energy
due to the rotation of the star respectively. The Fermi
energy density of the non-interacting fermions measures
the maximum occupied energy by the fermions per unit volume, at the ground state of the system \[36\]. In the case
of the quark star, the fermi energy occupies a significant
fraction of the total energy of the star, mainly depending
on the fermi number density ($n$) and radius of the star ($R$). The fermi energy density can be expressed in terms of
chemical potential ($\mu$), given by \[26\],

$$\mathcal{E}_f = \frac{g}{8\pi^2} \mu^4,$$

where, $g$ is the statistical degeneracy factor of the system.
The chemical potential ($\mu$) for a star having $N$ number
of fermions is described as,

$$\mu = \left(\frac{9\pi}{2g}\right)^{\frac{1}{2}} \frac{N^{\frac{1}{2}}}{R}. \quad (3)$$

So, the expression for the Fermi energy per particle takes
the form,

$$e_f = \frac{\mathcal{E}_f}{n} = \frac{3}{4} \left(\frac{9\pi}{2g}\right)^{\frac{1}{2}} \frac{N^{\frac{1}{2}}}{R}. \quad (4)$$

\section*{IV. EFFECTIVE MASS PER PARTICLE}

The effective mass of the quarks is an essential quan-
tity, in order to estimate the total mass of the star as well
as the gravitational potential and the rotational kinetic
energy. According to our assumption (rigid and spherical
star), the effective mass of the entire star ($M$) is given by,

$$M = e_f N + \frac{4}{3} \pi R^3 B. \quad (5)$$

Extremizing the above expression of total mass $M$ with
respect to $R$, and further simplifying, the expression for
bag constant ($B$) is reduced.

$$B = \frac{e_f N}{3V}, \quad (6)$$

where $V$ is the volume of the star. Applying the above
expression of bag constant ($B$) in Equ. \[5\] the simplified
form of the total mass \( M \) of the star is obtained, given by

\[
M = 4V B = \frac{16}{3} \pi R^3 B,
\]

(7)

and thus the effective mass \( m \) of each quark inside the star,

\[
\frac{M}{N} = 4V B.
\]

(8)

Being fermions, the effective mass of the quark coincides with the quark chemical potential. As a result, from the limit of vanishing quark density \([26, 37, 38]\) we get,

\[
\gamma = \frac{\mu}{B}.
\]

(9)

Now applying Eq. (3) and (9) in Eq. (8), the desired expression for effective mass per quark particle \( m \) can be obtained in terms of fundamental constants and bag constants.

\[
\mu = B n, \\
N = \mu.
\]

V. ENERGY

The gravitational potential and rotational kinetic energy per particle at the point \((R, \theta, \phi)\) (in spherical polar coordinate system) with respect to the center of the star, can be written in terms of effective mass and other parameters as,

\[
e_G = -G \frac{M m}{R}, \quad \text{and} \quad e_{\text{rot}} = m(\gamma - 1),
\]

(10)

where \( \gamma = \sqrt{1 - \left(\frac{2\mu c R \cos \theta}{c}\right)^2} \). From the above expression it can be observed that, unlike the gravitational and fermi energy, the kinetic energy per particle changes over \( R \) for any fixed value of \( \theta \). So, in order to overcome the \( \theta \)-dependence in further calculations, a \( \theta \)-averaged kinetic energy term \( \langle e_{\text{rot}} \rangle \) is taken into account. Now the total effective energy per particle takes the form

\[
e = e_f + e_G + \langle e_{\text{rot}} \rangle.
\]

(11)

According to the Landau’s energy balance picture, the limiting radius \((R_{\text{max}})\) and the corresponding mass \((M_{\text{max}})\) of the star can be obtained by extremizing the total energy per particle \( e \) with respect to the total number of particles \( N \).

The limiting mass and radius, which are calculated by extremizing the total energy \( e \), is independent of the degeneracy factor \( g \). Consequently, we address a general solution for the both types of quark stars (2-flavored normal quark star and 3-flavored strange quark star). In the current work, we also studied the dependence of the limiting mass, radius and the total number of particles \( N \) on the bag constant as well as the rotational frequency. The rotational effect of the star is parameterized by the rotational frequency \( \omega \), however, due to the extreme compactness and mass of the star, the relativistic effects emerge prominently in the observed rotational frequency \( \nu \) (as observed by a far away observer) \([39, 40]\). As a result, a far away observer would observe a red shifted form of the actual frequency \( \omega \). Consequently, a significant deviation from \( \omega \) is obtained in the observed frequency \( \nu \) in the case of fast spinning stars as shown in (see Fig. 1). The variation of the limiting mass \( M_{\text{max}} \) with observed red-shifted frequency \( \nu \) and actual frequency \( \omega \) is described in Fig. 1 for a chosen value of bag constant \( B = (145 \text{ MeV})^4 \).

From Fig. 2b, it is evident that, for each chosen value of bag constant, the limiting mass remains almost unchanged in the lower frequency range. But as the frequency \( \nu \) reaches 300 Hz, the kinetic energy becomes sufficiently high to be taken into consideration for the evaluation of the effective mass. As a result \( M_{\text{max}} \) starts increasing gradually with frequency, however it suffers a rapid increase above the frequency of 600 Hz. Beyond a certain limit of \( M_{\text{max}} \) for a given bag constant \( B \) the star become so massive that observed frequency \( \nu \) starts falling with increasing \( \omega \) due to gravitational effect. This provides a limiting value of \( \nu \) (i.e. \( \nu_{\text{max}} \)) (see Fig. 2b) for each value of bag constant. Fig. 2a and Fig. 2c describe the variation of limiting radius \( R_{\text{max}} \) and corresponding number of particle containing the star \((M_{\text{max}})\) respectively. In both the case, the same limiting frequencies \((\nu_{\text{max}})\) are observed for individual values of bag constants. The limiting frequencies \((\nu_{\text{max}})\) are found to be different for different values of bag constant. The corresponding variation of \( \nu_{\text{max}} \) is plotted in Fig. 3 showing that, the \( \nu_{\text{max}} \) increases almost linearly with \( B^{1/4} \). According to the work of \([32, 41, 42]\) the value of the bag constant for stable quark matter is \( \sim (145 \text{ MeV})^4 - (162 \text{ MeV})^4 \). The above mentioned range

![Graph](image_url)
FIG. 2. Variation of (a) limiting radius \((R_{\text{max}})\), (b) limiting mass \((M_{\text{max}})\) and (c) \(N_{\text{max}}\) with \(\nu\) for three different values of bag constant.

of the bag constant corresponds to the frequency range \(1043.5 \sim 1301\) Hz, (see Fig. 3) which addresses the possible upper-bound of the observed frequency of quark stars (millisecond order) \([43]\). The maximum admissible mass of a static quark star having bag constant \(B = (145 \text{ MeV})^4\) is \(\sim 1.54 \text{ M}_\odot\) \([20]\), it may reach up to \(6.24 \text{ M}_\odot\) for rotating quark star, as described in Fig. 3. The feasibility of the newly discovered extremely massive compact star PSR J0740+6620 (mass \(\sim 2.14 \text{ M}_\odot\) \([44]\)) is also shown in the Fig. 4 where the value of bag constant is assumed according to the work of \([41]\).

In the current work we have looked at an significant quantity given by the radius to the Schwarzschild radius ratio \(\left(\frac{R_{\text{max}}}{R_{\text{sch}}}\right)\), inverse of compactness) of the star. The variation of the ratio \(\left(\frac{R_{\text{max}}}{R_{\text{sch}}}\right)\) with frequency \(\nu\) are traced out for different values of bag constants (see Fig. 5). In the low frequency range \((\omega < 200\) Hz), the numerical value of the ratio is found to be independent of the bag constant and remains almost unchanged \((\sim 8/3\) in geometric unit, as obtained from the work of \([26]\)) with frequency (Fig. 5). But it starts falling gradually with increasing frequency as the rotational frequency reaches 300 Hz. In this range of frequencies, a minute dependence of the bag constant is observable. However, at higher frequencies, all the \(\frac{R_{\text{max}}}{R_{\text{sch}}}\) curves for the different bag constants suffer a quick fall toward unity. From the above study, it is evident that, the compactness of the quark star is extremely
FIG. 5. Radius mass ratio vs $\nu$ graph

high, but it cannot become a black hole even in the limiting case. Thus they (strange stars) and black hole could co-exist as candidates of cold dark matter. While their phenomenological signatures may not be sufficient to distinguish between them, their different signatures in the gravitational wave scenario may be interesting to study [45].

VI. CONCLUSION

In this work we have studied the limiting mass for the rotating quark star, which is introduced here as the “Chandrasekhar limit for rotating quark stars”. The limit mostly depends on the universal constants and the Bag parameter as well as the angular velocity of the star. A relation between the limiting radius ($R_{\text{max}}$) and corresponding Schwarzschild radius ($R_{\text{sch}}$) for a range of rotational frequencies has also been resolved in our current work. The numerical value of the quantity $R_{\text{max}}/R_{\text{sch}}$ lie at $8/3$ for a static star (as obtained from the work of [26]), however it tends toward unity for an extreme case of the rotation ($\omega \rightarrow R_{\text{max}}$). Although the Schwarzschild radius can describe the event horizon of the static stars (or slowly rotating stars), it cannot address the same for the fast spinning stars [46]. So, in order to obtain the event horizon for such spinning bodies, the Kerr space time has to be taken into account, which provides comparatively smaller horizon than that of the Schwarzschild metric system [46]. Consequently, one can conclude that a quark star can never behave as black hole.

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