Alpha Decay in the Complex Energy Shell Model

R. Id Betan¹,²,³ and W. Nazarewicz¹,²,⁴

¹Department of Physics and Astronomy, University of Tennessee, Knoxville, Tennessee 37996, USA
²Physics Division, Oak Ridge National Laboratory, P.O. Box 2008, Oak Ridge, Tennessee 37831, USA
³Departamento de Química y Física, FCEIA(UNR) - Instituto de Física Rosario (CONICET), Av. Pellegrini 250, 2000 Rosario, Argentina
⁴Institute of Theoretical Physics, University of Warsaw, ul. Hoża 69, 00-681 Warsaw, Poland

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Background: Alpha emission from a nucleus is a fundamental decay process in which the alpha particle formed inside the nucleus tunnels out through the potential barrier.

Purpose: We describe alpha decay of ²¹²Po and ¹⁰⁴Te by means of the configuration interaction approach.

Method: To compute the preformation factor and penetrability, we use the complex-energy shell model with a separable T=1 interaction. The single-particle space is expanded in a Woods-Saxon basis that consists of bound and unbound resonant states. Special attention is paid to the treatment of the norm kernel appearing in the definition of the formation amplitude that guarantees the normalization of the channel function.

Results: Without explicitly considering the alpha-cluster component in the wave function of the parent nucleus, we reproduce the experimental alpha-decay width of ²¹²Po and predict an upper limit of $T_{1/2} = 5.5 \times 10^{-7}$ sec for the half-life of ¹⁰⁴Te.

Conclusions: The complex-energy shell model in a large valence configuration space is capable of providing a microscopic description of the alpha decay of heavy nuclei having two valence protons and two valence neutrons outside the doubly magic core. The inclusion of proton-neutron interaction between the valence nucleons is likely to shorten the predicted half-life of ¹⁰⁴Te.

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I. INTRODUCTION

According to Gamow theory of alpha decay [¹, ²], this fundamental radioactive decay can be considered as a two-step process [², ²]. In the first step, an alpha cluster is formed inside the parent nucleus. The resulting alpha particle resides in a metastable state of an average potential of the daughter system. In the second step, the particle tunnels through the potential barrier. Each step requires different theoretical treatment. To compute the preformation factor that describes the alpha formation probability, one needs to evaluate the overlap integral in-
The R-matrix expression for the width (11) depends on the channel radius $R$. This radius should be chosen large enough so that the alpha-daughter interaction in the external region is given by the Coulomb force alone [22]. The infinite range of the Coulomb force implies, however, that the asymptotic behavior of the R-matrix expression is reached only at large values of $R$, at which the asymptotic behavior of the shell-model s.p. basis (h.o. basis in most applications) used to calculate $\gamma_L^2(R)$, does matter. Due to the mismatch between the internal part of the s.p. wave function (well described in the h.o. basis) and the asymptotic part (poorly or not described in the h.o. basis), rather small changes in $R$ may produce appreciable variations in penetrability. Physically, the reason for this sensitivity is the fact that the alpha cluster is formed in the surface region of the nucleus in which the coupling to the alpha continuum that impacts the radial behavior of the formation amplitude is important [16]. Consequently, the absolute R-matrix width depends in general on the channel radius [5, 23], and this is an obvious drawback of the method [22].

Our renewed interest in the alpha-decay problem is stimulated by the recent experimental data above the doubly-magic $^{100}$Sn [24, 25] that demonstrate the presence of very fast alpha decays. Indeed, the observed enhancement of the reduced widths of $^{105,106}$Te relative to $^{212,213}$Po is two-to-three, thus confirming earlier expectations [20] of “superallowed” alpha decays in this region due to the large overlaps of valence s.p. shell model proton and neutron wave functions. Our long-term goal is to estimate alpha preformation factors in nuclei above $^{208}$Pb and $^{100}$Sn by using large valence s.p. spaces, including positive-energy Gamow states of a finite-depth WS potential [18, 19]. In this study, we focus on $^{212}$Po and $^{104}$Te nuclei having two valence protons and two valence neutrons outside doubly-magic cores.

Our paper is organized as follows. Section II briefly describes the alpha-decay formalism used in this work, with special emphasis on approximations used to describe wave functions of parent and daughter nuclei. Section III deals with the approximations employed and parameters used. In particular, we discuss the sensitivity of the calculated spectroscopic factor to the parameters defining the shifted Gaussian basis that is used to compute the normalization of the channel function. In Sec. IV we study the sensitivity of the reduced alpha width in $^{212}$Po on the choice of s.p. basis used. In Section V we discuss the absolute alpha-decay width of $^{212}$Po and in Sec. VI we compare it with the absolute width of the superallowed alpha emitter $^{104}$Te. Finally, the main conclusions of this work are summarized in Sec. VII.

II. FORMALISM

In this section, we discuss the R-matrix (11) and spectroscopic factor (2) expressions for the decay width. The connection between the two formulations is given in Ref. [8]. We also discuss the so-called delta-approximation for the formation amplitude.

A. R-matrix expression for the decay width

Within the R-matrix theory [8, 12], the absolute width is given by Eq. (11) with $P_L(R)$ being the barrier penetrability and $\gamma_L(R)$ — the reduced width amplitude [27]. While both quantities strongly depend on the value of the channel radius $R$, the absolute width should be $R$-independent.

For $P_L(R)$ we use the standard expression [6]:

$$P_L(R) = \frac{kR}{|H_L^0(\eta, kR)|^2}, \quad (3)$$

where $k$ is given by the alpha energy $E_\alpha = \frac{h^2 k^2}{2\mu}$, obtained from the experimental $Q_\alpha$ value by correcting for electron screening; $\mu = \frac{m_\alpha m_p}{m_\alpha + m_p}$ is the reduced mass of alpha particle with $m_\alpha$ being the mass of the daughter nucleus; $H_L^0(\eta, kR)$ is the outgoing spherical Coulomb-Hankel function; and $\eta = \frac{2Z_{\alpha}\mu e^2}{h k}$ is the Sommerfeld Coulomb parameter.

The reduced width amplitude $\gamma_L(R)$ may be written in terms of the formation amplitude $g_L(R)$ [5, 14]:

$$\gamma_L = \sqrt{\frac{h^2 R}{2\mu}} g_L(R), \quad (4)$$

with

$$g_L(R) = \int d\Omega_R \int d\xi_D \int d\xi_a \int d\xi_D \Phi_{JM}^P A \left[ \phi_\alpha(\xi_a) \phi_\gamma(\xi_D) \phi_\gamma(\hat{R}) \right]^{*}_{JM}, \quad (5)$$

where $\phi_\alpha$ is the normalized wave function of the alpha particle with zero angular momentum, $Y_{LM}\ell$ is the angular part of the center-of-mass (c.o.m.) motion of the alpha particle, $\Phi_{JM}^\ell m$ is the wave function of the daughter nucleus, and $\Phi_{JM}^P$ is the wave function of the parent nucleus. The coordinates $\xi_a$ and $\xi_D$ are the intrinsic coordinates of the alpha particle and daughter nucleus, respectively. All wave functions are normalized in terms of the internal and c.o.m. coordinates [27]. By construction, the parent and daughter wave functions are antisymmetric. The antisymmetrization with respect to inter-fragment nucleons is done by means of the operator $\mathcal{A}$. Its action can be approximated by means of a factor $\left[ \frac{(N_{\ell} + 1)}{Z_{\ell}} \right]^{1/2}$ [14, 27, 28], with $N_{\ell}$ and $Z_{\ell}$ being, respectively, the numbers of valence neutrons and protons in the parent nucleus.

For the internal alpha-particle wave function we take...
the standard Gaussian ansatz \[\phi_\alpha(\rho_1, \rho_2, \alpha, \varphi_3) = \phi(\rho_1, \rho_2, \alpha, \varphi_3) \chi_\alpha(\varphi_3), \]

\[\chi_\alpha(\varphi_3) = |\varphi_3\rangle \langle \varphi_3|, \]

\[\phi(\rho_1, \rho_2, \alpha, \varphi_3) = \begin{cases} \frac{\beta^3}{\pi} \left( \frac{\rho_1}{\beta^2} + \frac{\rho_2}{\beta^2} \right), & \text{if } \rho_1, \rho_2 \leq \beta \varphi_3, \\ 0, & \text{otherwise}. \end{cases} \]

The parameter \(\beta = 0.057 \text{ fm}^{-2}\) depends on the root-mean-square alpha radius \(\sigma_\alpha = 1.57 \text{ fm}\). In Eq. \(12\) and in the following, the s.p. indices \(1, 2\) refer to neutrons while \(3, 4\) refer to protons. The coefficients \(b_{\nu_\nu, \nu_p}\) are the shell-model four-particle wave function amplitudes.

**B. Delta-function approximation**

In the calculation of alpha-decay rates based on h.o. wave functions, it was noticed [28] that the relative rates change little with the oscillator length \(b_{\text{h.o.}}\) of the basis. Using this argument, Mang proposed to take \(\beta \gg b_{\text{h.o.}}^2\). In this limit, the expression for the formation amplitude can be simplified (see also Ref. [31]). In the literature, this is known as delta-function approximation [32].

In practice, one assumes that the alpha particle wave function is constant inside a small volume of radius \(s_\alpha = 2.34 \text{ fm}\) [32] and zero outside. Within this approximation \(\rho_1 = 0\); hence, it immediately follows from Eqs. \(9\) that \(r_1 = r_2 = r_3 = r_4 = R\) [32, 33], and the formation amplitude reduces to

\[F_0(R) = \frac{\sqrt{8}}{16\pi^{3/2}} \left( \frac{4\pi s_\alpha^3}{3} \right)^{3/2} \left( \sum_{\nu_\nu} I^n_{\nu_\nu} \right) \left( \sum_{\nu_p} I^{np}_{\nu_p} \right), \]

with \(I^n_{\nu_\nu} = (-)^{\nu_\nu} b_{\nu_\nu, \nu_p}^* B_{\nu_\nu} u_{\nu_p}^2(R) / R^2\),

\[B_{\nu_\nu} = 1 - 0.013\nu_\nu (\nu_\nu + 1). \]

**C. Four-particle amplitudes**

For the g.s. alpha decay of \(^{212}\text{Po}\) and \(^{104}\text{Te}\), we are going to assume that the four valence nucleons move around the rigid, doubly-magic core. The parent-nucleus wave function is approximated by a product of two-neutron and two-proton seniority-zero states:

\[|\Phi^0_{2\nu_\nu, 2\nu_p} = |\Psi_{2\nu_\nu, 00}\rangle \otimes |\Psi_{2\nu_p, 00}\rangle, \]

where

\[|\Psi_{2\nu_p, 00}\rangle = \sum_{\nu_p} X^n_{\nu_p} |\nu_p, 00\rangle, \]

\(|\nu_\nu, 00\rangle = \frac{|\nu_\nu, 0\rangle}{\sqrt{2}} |\nu_\nu\rangle\), and \(|0\rangle = |0_0\rangle \otimes |0_p\rangle\) is the shell-model vacuum representing the \(^{208}\text{Pb}\) or \(^{100}\text{Sn}\) g.s. wave function. The four-particle amplitudes \(b_{\nu_\nu, \nu_p}\) in \((11)\) can thus be written in a separable form:

\[b_{\nu_\nu, \nu_p} = X^n_{\nu_\nu} X^\nu_{\nu_p}. \]
D. Alpha decay spectroscopic factor

Based on the general theoretical arguments [8–11], the absolute width can be expressed as a product of the alpha-particle spectroscopic factor and the single particle width, see Eq. (2). The spectroscopic factor $S_L$ contains information about the probability of forming an alpha cluster in the parent system. Since the alpha particle, when formed, occupies the resonant state, the s.p. width can be obtained from the so-called current expression [3, 44, 35]:

$$
\Gamma^{sp}_L = \frac{i \hbar^2 u^*_L(R) u_L(R) - u'_L(R) u^*_L(R)}{2 \mu \int |u_L(R)|^2 dR},
$$

(19)

where the Gamow function $u_L(R)$ is obtained as a solution of the Schrödinger equation with outgoing boundary condition. When the imaginary part of the complex energy eigenvalue $\epsilon_\alpha = \frac{k^2}{2 \mu}$ is small, which is always the case for the considered g.s. of alpha emitters, one can approximate [19] with [31]:

$$
\Gamma^{sp}_L = \frac{\hbar^2 \Re k(R)}{\mu} \frac{|u_L(R)|^2}{|H^L_k(\eta, kR)|^2}.
$$

(20)

The s.p. width obtained in this way should be identical to the value $-2 \text{Im}(\epsilon_\alpha)$ given by the imaginary part of the Gamow resonance energy, if the latter is computed with a sufficient precision.

The conventional alpha spectroscopic factor as introduced in Ref. [8] is defined by

$$
S_L = |\langle A \left[ \phi_\alpha(\xi_\alpha) \Psi^D_j(\xi_D) \right] \psi_L(R) \rangle_{JM}|^2.
$$

(21)

where $\psi_{LM}(R) = \frac{u_L(R)}{R} Y_{LM}(R)$ represents the relative motion alpha particle with respect to the daughter. In terms of the formation amplitude, $S_L$ reads [3, 10, 37]:

$$
S_L = \int_0^\infty g^2_L(R) R^2 dR.
$$

(22)

E. Modified spectroscopic factor

Since the overlap of the parent wave function with the daughter-alpha product state, one would be tempted to associate it with the probability amplitude that in the parent wave function $\Phi^D_{JM}$ an alpha particle $\phi_\alpha$ and a daughter nucleus $\Psi^D_{jJM}$ are at a distance $R$. The value of $S_L$ would then be associated with the total probability of formation of an alpha particle. However, the fundamental problem with this interpretation is that the channel function $\langle A \left[ \phi_\alpha(\xi_\alpha) \Psi^D_j(\xi_D) \right] \psi_L(R) \rangle_{JM}$ is not properly normalized [3, 10, 11, 38, 41].

The properly defined spectroscopic factor (sometimes referred to as “the amount of clustering”) [21, 38, 42, 45] is given by

$$
S_L = \int_0^\infty G^2_L(R) R^2 dR,
$$

(23)

where

$$
G_L(R) = \int N_L^{-1/2}(R, R') g_L(R') R^2 dR'
$$

(24)

is the modified formation amplitude. The norm kernel $N_L$ appearing in Eq. (23) is

$$
N_L(R, R') = \langle A \left[ \delta(R_\alpha - R) \right] \phi_\alpha \left[ Y_L \Psi^D_j(\xi_D) \right] \rangle
\frac{A \left[ \delta(R_\alpha - R') \right]}{R'^2} \phi_\alpha \left[ Y_L \Psi^D_j(\xi_D) \right] \rangle.
$$

(25)

The presence of the norm kernel $N_L$ effectively enhances the spectroscopic factor by one-to-two orders of magnitude [21, 43, 44, 43, 40].

To compute $N_L^{-1/2}(R, R')$, we expand the eigenfunctions of the norm kernel in an orthonormalized shifted Gaussian basis (SGB) [43],

$$
F_L(R, R_k) = \sum_{k'} \left( N_L^{-1/2} \right)_{kk'} F_{L, kk'}(R, R_{k'}),
$$

(26)

with $R_k$ equidistant mesh points in the interval $(0, R_{max})$ and $k = 1, \ldots, M$, where $M$ is the dimension of the basis. The SGB is given by

$$
F_L(R, R_k) = 4\pi \left( \frac{8\beta' \pi}{\beta'} \right)^{3/4} e^{-4\beta'(R^2 + R_k^2)} I_{L, jL} \left( -i 8\beta' R R_k \right),
$$

(27)

while the SGB overlap $\langle N_F \rangle_{kk'}$ is given by

$$
\langle N_F \rangle_{kk'} = \int F^*_L(R, R_k) F_L(R, R_{k'}) R^2 dR
\quad = 4\pi e^{-2\beta'(R_k^2 + R_{k'}^2)} I_{L, jL} \left( -i 4\beta' R_k R_{k'} \right).
$$

(28)

Using the SGB overlaps, the eigenvalue equation for the norm matrix can be expressed in the form:

$$
\sum_{k'} N^F_{kk'} c_{k'} = n, c_{k'}^n,
$$

(29)

where

$$
N^F_{kk'} = \sum_{nn'} \left( N_F^{-1/2} \right)_{kn} N^F_{nn'} \left( N_F^{-1/2} \right)_{n'n'},
$$

(30)

For $\beta' = 4\beta$, the core-projected norm $N^F_{kk'}$ in Eq. (30) reduces to a simple expression [21, 43, 47]:

$$
N^F_{kk'} = \langle \psi_k(\mu, L) | \psi_{k'}(\mu, L) \rangle^2 \langle \psi_k(\pi, L) | \psi_{k'}(\pi, L) \rangle^2
$$

(31)

where

$$
\langle \psi_k(\mu, L) | \psi_{k'}(\mu, L) \rangle = \langle \phi^L_k | \phi^L_{k'} \rangle - \sum_{nlj \in \text{core}} \delta_{ll} \langle \phi^L_k | R_{nlj} \rangle \langle R_{nlj} | \phi^L_{k'} \rangle.
$$

(32)

with $\phi^L_k(R) = F_L(R, R_k)(\beta' \rightarrow \beta)$ and $R_{nlj}(R) = u_{nlj}/R$ are the radial s.p. wave functions of the core.
In terms of eigenstates $c_k^L$ of (29), the spectral representation of the norm kernel can be written as:

$$N_L^{-1/2}(R, R') = \sum_{\nu}(n_\nu > n_{\text{min}}) n_\nu^{-1/2} u_\nu^L(R) u_\nu^L(R'),$$

(33)

where the eigenfunctions $u_\nu^L(R)$ of the norm kernel are

$$u_\nu^L(R) = \sum_k c_k^L \tilde{F}_L(R, R_k),$$

(34)

and $n_{\text{min}}$ represents the usual cutoff on the eigenvalue of the norm kernel. The final expression for the modified formation amplitude in the normalized SGB becomes [43]:

$$G_L(R) = \sum_{\nu}(n_\nu > n_{\text{min}}) n_\nu^{-1/2} u_\nu^L(R) g_\nu^L,$$

(35)

with

$$g_\nu^L = \int u_\nu^L(R) g_L(R) R^2 dR.$$  

(36)

III. THE MODEL

A. Single-particle space

The s.p. space is spanned on resonant states of a WS+Coulomb average potential. The parameters of the s.p. Hamiltonian, namely the WS potential depth $V_0$, spin-orbit potential depth $V_{so}$, diffuseness $a (= a_{so})$, radius $r_0 (= r_{0,so})$, and the radius of the uniform charge distribution $r_c$ defining the Coulomb potential are listed in Table II. The resulting neutron and proton s.p. ener-

| Orbit | Neutrons | Protons |
|-------|----------|---------|
| $1g_{9/2}$ | $-3.396$ | $-3.784$ |
| $0h_{9/2}$ | $-3.542$ | $-3.784$ |
| $1j_{15/2}$ | $1f_7/2$ | $-2.673$ |
| $0i_{13/2}$ | $1f_5/2$ | $-2.673$ |
| $2d_{3/2}$ | $0i_{11/2}$ | $0.491 - i0.200 \times 10^{-11}$ |
| $3f_{3/2}$ | $0i_{11/2}$ | $4.028 - i0.130 \times 10^{-7}$ |
| $3g_{9/2}$ | $0i_{11/2}$ | $5.434 - i0.990 \times 10^{-8}$ |
| $1h_{11/2}$ | $0j_{15/2}$ | $5.960 - i0.115 \times 10^{-7}$ |
| $0o_{13/2}$ | $2d_{5/2}$ | $6.748 - i0.184 \times 10^{-2}$ |
| $1j_{7/2}$ | $3s_{1/2}$ | $7.843 - i0.367 \times 10^{-1}$ |
| $1g_{7/2}$ | $1j_{7/2}$ | $8.087 - i0.898 \times 10^{-3}$ |
| $2d_{5/2}$ | $2f_{7/2}$ | $8.530 - i0.284 \times 10^{-1}$ |
| $1h_{11/2}$ | $0o_{13/2}$ | $11.390 - i0.215 \times 10^{-1}$ |
| $0o_{13/2}$ | $1h_{9/2}$ | $15.086 - i0.493 \times 10^{-2}$ |
| $1h_{9/2}$ | $1h_{9/2}$ | $15.964 - i0.393$ |

B. Two-particle interaction

The correlated two particle wave functions $|\Psi_{2\tau,0}\rangle$ [17] have been obtained using a separable two-body $T = 1$ pair interaction [51]:

$$\langle \nu\nu', 00|V|\nu'\nu', 00\rangle = -G f(\nu, \tau)f(\nu', \tau),$$

(37)

where

$$f(\nu, \tau) = \frac{(-)^\nu}{\sqrt{2}} (j_\nu||Y_0||j_\nu) I(\nu, \tau).$$

(38)

In Eq. (38) we used the Condon-Shortley phase convention for $(j_\nu||Y_0||j_\nu)$ and

$$I(\nu, \tau) = \int u_\nu^2(r) f_\tau(r) dr.$$  

(39)
For the radial form factor \( f_r(r) \) we took the derivative of the WS potential multiplied by \( r \):

\[
f_r(r) = \frac{r}{a_{\text{WS}}} \frac{e^{-r/a_{\text{WS}}}}{1 + e^{-r/a_{\text{WS}}}}^2. \tag{40}
\]

In the case of \( ^{212}\text{Po} \) and \( ^{104}\text{Te} \) the two-particle amplitudes of Eq. (17) were obtained exactly in the Tamm-Dancoff approximation [22, 58]:

\[
X_\nu^\tau = N_0 \frac{f(\nu, \tau)}{2E_\nu^\tau - E_0}. \tag{41}
\]

where \( E_\nu^\tau \) are s.p. energies, \( E_0^\tau \) is the correlated two-particle energy and \( N_0 \) is the normalization constant fixed by the condition \( \sum\nu (X_\nu^\tau)^2 = 1 \).

The parameters \( a_{\text{WS}} \) and \( a_{\text{WS}} \) defining the radial form factor (40) for \( ^{210}\text{Pb} \) and \( ^{210}\text{Po} \) were chosen to reproduce the wave functions used by Harada [14]. Since such data are not available for \( ^{102}\text{Sn} \) and \( ^{102}\text{Te} \), in this case we adopted the values of the WS potential for \( ^{100}\text{Sn} \) shown in Table I. The pairing strength \( G_\tau \) was adjusted to fit the experimental two-nucleon separation energies \( S_{2\nu} \) through the dispersion relation

\[
\frac{1}{G_\tau} = \sum\nu \frac{f^2(\nu, \tau)}{2E_\nu^\tau - E_0^\tau}. \tag{42}
\]

Since the proton-unbound nucleus \( ^{102}\text{Te} \) is not known experimentally, for this system we adopted the value of \( S_{2\nu} = -2.14 \text{MeV} \) obtained by extrapolating down from the heavier Te isotopes [54]. This value is in reasonable agreement with recent phenomenological estimates [48]. Table IV lists the parameters of the residual interaction used in our study.

TABLE IV. Parameters \( a_{\text{WS}}^\tau \) and \( a_{\text{WS}}^\tau \) of the residual interaction (47). The last column lists the value of \( S_{2\nu} \) that has been used to constrain the pairing strength \( G_\tau \) for various configuration spaces considered.

| nucleus    | \( a_{\text{WS}}^\tau \) (fm) | \( a_{\text{WS}}^\tau \) (fm) | \( S_{2\nu^\tau} \) (MeV) |
|------------|-----------------|-----------------|-----------------|
| \( ^{210}\text{Pb} \) | 7.255            | 0.70             | 9.123            |
| \( ^{210}\text{Po} \) | 5.451            | 0.75             | 8.783            |
| \( ^{102}\text{Sn} \) | 5.895            | 0.70             | 24.3             |
| \( ^{102}\text{Te} \) | 5.895            | 0.70             | -2.14            |

C. Configuration space

To study the dependence of the formation amplitude on the size of valence space, and to compare with previous work, we considered several model spaces. Those used in the description of the alpha decay of \( ^{212}\text{Po} \) are given in Table V. The model space M0 contains only one valence shell. The space M1 contains one major shell, including the unusual-parity intruder orbit. The model space M2 is that used by Harada [14]. The model space M3 is that of Glendenning and Harada [55]. Finally, M4 is the extended shell model space employed by Tonozuka and Arima. The model spaces used to describe \( ^{104}\text{Te} \) alpha decay are shown in Table VI. M1 consists of one major shell, including the unusual-parity intruder orbit, while M4 consists of states with width less than 1 MeV.

D. Wave functions

For the alpha formation amplitude in \( ^{212}\text{Po} \) discussed in Sec. IV we considered the model spaces M2, M3 and M4. The wave function amplitudes in M2 were taken from Refs. [14, 56]. For calculations in M3, we took the \( T=1 \) seniority-zero amplitudes of Ref. [55] and renormalized them accordingly. For calculations in the extended space M4, we used the renormalized amplitudes of Ref. [15]; here we retained only configurations having width smaller than 1 MeV. The comparison between \( ^{212}\text{Po} \) and \( ^{104}\text{Te} \) discussed in Sec. VI was carried out in the model spaces M1 and M4. The corresponding wave functions were calculated in the two-particle approximation described in Sec. III B except for \( ^{212}\text{Po} \) in the M4 model space, where Ref. [55] was used instead.

E. Penetration factor

The s.p. alpha width \( \Gamma_{0}^{\text{sp}} \) has been obtained from the current expression [20]. The alpha-core potential was
assumed to be of a WS+Coulomb form with the parameters of Ref. 57: \( r_0 = R_c = 1.315 \text{ fm}, a = 0.65 \text{ fm}. \) The strength of the WS potential has been adjusted to reproduce the measured \( Q_\alpha \) value corrected by the electron screening term \[3, 13, 58, 60]:

\[
Q_\alpha = E_\alpha \frac{A_p}{A_d} + \Delta E_{sc}
\]

where

\[
\Delta E_{sc} = 65.3Z_p^{-4} - 80Z_p^{0.4} \text{ (eV)}.
\]

For \(^{212}\text{Po}\), \( E_\alpha = 8.785 \text{ MeV} \) and \( \Delta E_{sc} = 31.8 \text{ keV} \); hence, \( Q_\alpha = 8.986 \text{ MeV}. \) The g.s. alpha decay of \(^{104}\text{Te}\) has not been observed. For that reason, we took the value \( Q_{104}^{\text{BE}} = 5.135 \text{ MeV} \) extrapolated down from the binding energy differences in \(^{108}\text{Te}\) (3.445 MeV) and \(^{106}\text{Te}\) (4.290 MeV) \[61\]. By adding the screening correction \( \Delta E_{sc} = 16.1 \text{ keV} \), we arrived at \( Q_\alpha = 5.151 \text{ MeV} \). The resulting WS potential strength is \( V_0 = 143.49 \text{ MeV} \) for \(^{212}\text{Po}\) and 149.64 MeV for \(^{104}\text{Te}\).

The Gamow wave functions were obtained by means of the code ANTI \[50\]. The complex energy of the metastable alpha state is \( E_\alpha = (8.986 - i0.632 \times 10^{-13}) \text{ MeV} \) for \(^{212}\text{Po}\) and \( E_\alpha = (5.151 - i0.814 \times 10^{-13}) \text{ MeV} \) for \(^{104}\text{Te}\). The outgoing spherical Coulomb-Hankel function \( H^+ \) was calculated using the code \[62\].

**F. Calculation of the spectroscopic factor**

The radial integration in the expressions for the spectroscopic factor \[28\] and the formation amplitude in the normalized SGB \[36\] have been carried out using 200 Gauss-Legendre mesh-points with the maximum radius of 20 fm.

The s.p. core wave functions entering Eq. \(32\) are those of the s.p. Hamiltonian of Table I. The radial mesh \( R_k \) defining the normalized SGB \[36\] was taken at equidistant points \( R_k = k \Delta R \). In order to determine the step \( \Delta R \) we expanded the s.p. core states \( u(r) \) in the normalized SGB: \( \tilde{u}(r) \equiv \sum_{k=1}^{M} a_k [rF_0(r, R_k)] \). Under the condition that \( u_{\text{diff}}(r) = |u(r) - \tilde{u}(r)| < 0.005 \text{ fm}^{-1/2} \) we found that 0.44 fm \( \lesssim \Delta R \lesssim 0.57 \text{ fm} \) and \( R_{\text{max}} \gtrsim 14 \text{ fm} \). For this range of \( \Delta R \) and \( R_{\text{max}} \) the normalized SGB is orthonormal with an accuracy better than \( 10^{-9} \). To illustrate the quality of the resulting expansion, Fig. 1 shows \( u_{\text{diff}}(r) \) for the neutron core states in \(^{208}\text{Pb}\).

To calculate the modified formation amplitude \( G(R) \), one needs to determine the eigenvalue cutoff \( n_{\text{min}} \). To this end, we show in Figs. 2 and 3 typical distribution of the eigenvalues \( n_\nu \) of the norm kernel \[37\] for \(^{212}\text{Po}\) and \(^{104}\text{Te}\), respectively, for different values of \( \Delta R \). One may observe that a significant fraction of them accumulate at zero \[53, 63\]. To eliminate these spurious eigenvectors, we define the cutoff at the value where the eigenvalue distribution changes slope. For \(^{212}\text{Po}\) and \(^{104}\text{Te}\) this happens at \( n_\nu \) around \( 10^{-3} \). Consequently, in our calculations, we adopt the cutoff value of \( n_{\text{min}} = 0.001 \).

The eigenfunctions \( u_\nu^L(R) \) of the norm kernel \[37\] are orthonormal with an accuracy of \( 10^{-10} \) for all eigenvalues. The eigenfunctions with \( n_\nu < n_{\text{min}} \) oscillate inside the nuclear volume and vanish outside the surface region. To further check the quality of \( u_\nu^L(R) \) we compute expression \[54\] by assuming \( n_{\text{min}} = 0 \) and \( n_\nu = 1 \) for all \( \nu \). In this case, Eq. \[54\] formally reduces to \( g(R) \). Figure 4 shows \( g(R) \) for \(^{212}\text{Po}\) calculated in this way. The agreement with the original formation amplitude is excellent, except for a small deviation close to \( R = 0 \) and a small oscillation around and beyond the nuclear surface, which is not visible in the scale of Fig. 4.

Next we study the sensitivity of \( S \) to the choice of \( R_{\text{max}}, \Delta R, \) and \( n_{\text{min}} \). For this analysis we relax the condition for \( u_{\text{diff}}(r) \) in order to access a wider range of \( \Delta R \). First, we study the sensitivity of \( S \) as a function of \( R_{\text{max}} \) for various values of \( \Delta R \). Figure 5 shows the result for \( 0.33 \text{ fm} \leq \Delta R \leq 0.59 \text{ fm} \) for \(^{212}\text{Po}\) in the model

![FIG. 1.](image1.png) | ![FIG. 2.](image2.png)
FIG. 3. Similar as in Fig. 2 except for $^{104}$Te and $R_{\text{max}} = 10$ fm.

FIG. 4. Formation amplitude $g(R)$ for $^{212}$Po in the M4 model space expanded in eigenfunctions of the norm kernel for $\Delta R = 0.500$ fm and $R_{\text{max}} = 13$ fm.

space M4 and $n_{\text{min}} = 0.001$. Except for a small value of $\Delta R = 0.53$ fm, which does not produce stable results, a plateau in $R_{\text{max}}$ is reached around 14 fm.

The dependence of $S$ on $\Delta R$ displayed in Fig. 5 reflects the fact that for too small values of the step the basis functions become numerically linearly dependent, while for too large $\Delta R$'s the basis cannot capture high Fourier components [43, 53, 63]. Figure 6 shows $S$ for $^{212}$Po in the model space M4 and $n_{\text{min}} = 0.001$ as a function of $\Delta R$. In general, appreciable oscillations of $S$ can be seen except for the “safe” region $0.54 \text{ fm} \leq \Delta R \leq 0.59$ fm, where results weakly depend on $R_{\text{max}}$.

Finally, Fig. 7 shows the behavior of $S$ as a function of the eigenvalue cutoff $n_{\text{min}}$ for $\Delta R = 0.57$ fm. The cutoff used in Figs. 5 and 6 corresponds to $n_{\text{min}}^{-1/2} = (0.001)^{-1/2} \approx 31.5$.

G. Integral over intrinsic coordinates

The multidimensional integral (12) depends on the nucleonic coordinates, which are parametrized in terms of intrinsic variables through Eqs. (9) and (10). The integration over $\tilde{\varphi}_i$ can easily be done analytically. Since the coordinates of particles 1 and 2 depend only on the relative coordinates 1 and 3, and the particle coordinates 3 and 4 depend only on the relative coordinates 2 and 3, one can greatly simplify the remaining six-dimensional integral by making first the integration over the relative coordinates 1 and 2 and then the integration over the coordinate 3:

$$
\int d\rho_3 \left[ \left( \int \ldots d\rho_1 \right) \left( \int \ldots d\rho_2 \right) \right].
$$

The integration has been carried out using the Gauss-Legendre quadrature using 10 points for the radial integrals and 8 points for the angular coordinates. This
IV. REDUCED WIDTH FOR $^{212}$Po

A. Single-$j$ configuration

Following Rasmussen [32], it is instructive to compute relative reduced widths assuming a pure single-$j$ shell model orbital assignment for the neutron pair, while the proton pair fills the 0$\hbar_9/2$ shell. For simplicity, the results are expressed relative to the $^{210}$Po reference (a neutron pair in 2$p_{1/2}$).

In the delta-function approximation of Sec. II B, the ratio $r_\delta$ of the reduced widths is given by a simple expression:

$$r_\delta = \frac{\gamma_j^2}{\gamma_{2p_{1/2}}^2} = \frac{2j_n + 1}{2} \left( \frac{u_{j_n}(R)}{u_{2p_{1/2}}(R)} \right)^4. \quad (46)$$

In a more general case expressed by Eq. (11), the ratio $r$ depends on the proton wave function:

$$r = \frac{\gamma_j^2}{\gamma_{2p_{1/2}}^2,0\hbar_9/2} = \frac{2j_n + 1}{2} \left( \frac{I_{j_n,0\hbar_9/2}(R)}{I_{2p_{1/2},0\hbar_9/2}(R)} \right)^2. \quad (47)$$

Table VII compares the ratio $r_\delta$ given by Eq. (46) using the WS wave functions with that of Table I of Rasmussen [32] based on the rounded square well potential of Blomqvist and Wahlborn [64] for several neutron configurations at $R = 9.5$ fm. We find excellent agreement between these two calculations, and we checked that this agreement also holds for $R = 9.0$ fm. This is not surprising as both calculations employ finite-depth potentials. The fourth column of Table VII displays the ratio $r$ given by Eq. (47) using the WS wave functions; they are compared with the h.o. values of Ref. [65] (last column). It is seen that h.o. calculations underestimate WS values for high-$j$ orbits by a factor two-to-three.

It has been early recognized [28, 32] that the delta-function approximation overestimates the contributions of high-$j$ orbitals. One can see it clearly by comparing the values of $r_\delta$ of Eq. (46) with those of $r$ (47), i.e., the third and fourth columns of Table VII. To cure this deficiency, a correction factor $B_\nu$ [15] was introduced [32] in Eq. (14) that depends on the relative angular momentum.

B. Enhancement due to configuration mixing

As was first shown by Harada [14] and later by Rasmussen et al. [32], the reduced width at the surface region is strongly enhanced by the configuration mixing because contributions from various shell model orbits add coherently. To assess the effect of collective enhancement due to the configuration mixing, we carried out calculations in the M2 space. For $R = 8$ fm, our WS calculations yield the enhancement factor of $\zeta = 8.5$ with respect to the valence-shell configuration M0. This is to be compared with $\zeta = 11$ obtained in the delta-function approximation; $\zeta = 10$ obtained by Rasmussen [32]; and $\zeta = 5.5$ of Harada [14] using h.o. wave functions.

For the model space M3 of Glendenning and Harada [59], obtained by adding the intruder neutron state 0$j_{15/2}$ to M2, we obtain $\zeta = 21$. This should be compared with $\zeta = 24$ obtained in the delta-function approximation and $\zeta = 30$ obtained in Ref. [55] (also within the delta-function approximation) using a fairly rich wave function that also includes proton-neutron correlations and $J > 0$ two-particle couplings. It is worth noting that our enhancement is around 80% of that by Glendenning and Harada, and that the seniority-zero component in their wave function is also 80%.

C. Extended shell model space

Due to the strong collective enhancement of the reduced width due to configuration mixing, it is important to consider extended shell-model space by taking into account higher-lying orbitals [16]. For finite-depth shell-model potentials, such as the WS potential used in this study, this necessitates a proper treatment of the particle
continuum. An appropriate representation to deal with the continuum space is the complex Berggren ensemble representing bound and unbound s.p. states \[66, 67\].

Here we consider the large configuration space M4 of Tonozuka and Arima \[15\], i.e., all s.p. orbits up to \( N = 7 \) harmonic oscillator shell except for broad resonances with widths greater than 1 MeV. The shell-model amplitudes were taken from Ref. \[15\] and renormalized to the reduced model space. For the sake of comparison with Ref. \[15\], we consider the relative reduced width \( \Gamma(R) \) in which the absolute width varies weakly around the nuclear surface. Here, we find a fairly weak variation of \( \Gamma(R) \) between 7 fm and 11 fm.

As seen in Fig. 9 and discussed in Sec. \[15, 69, 74\], the width strongly increases with the size of the shell-model space. Indeed, in the surface region, \( \Gamma(R) \) obtained in M3 shows an enhancement \( \sim 15 \) with respect to M0, and in the extended space M4 the enhancement is \( \sim 260 \). Compared to experimental value, however, the width obtained in M4 is still 600 times smaller than the experimental value \( \Gamma_{\exp} = 0.153 \times 10^{-14} \) MeV \[54\].

A further enhancement in the reduced width is due to
the antisymmetrization and normalization of the channel decay \[^{212}\text{Po}\]. This is achieved by replacing the standard formation amplitude \(g(R)\) with the modified formation amplitude \(G(R)\) of Eq. (24). Figure 10 shows \(G(R)\) calculated in the M4 model space with \(\Delta R = 0.56\) fm, \(R_{\text{max}} = 11.76\) fm \((M = 21)\) and \(n_{\text{min}} = 0.001\). A small oscillation at the tail of \(G(R)\) can be seen. The amplitude of this oscillation, around the asymptotic behavior given by \(H_0^+(\eta, kR)\), varies very little with \(R_{\text{max}}\) for this value of \(\Delta R\). As discussed in, e.g., [3, 42, 45], the behavior of \(g(R)\) and \(G(R)\) is generally very different. This can be seen by comparing Figs. 9 and 10.

The absolute alpha-decay width obtained by using the R-matrix expression (1) with the formation amplitude \(G(R)\) of Fig. 10 is shown in Fig. 11. There appears a small plateau in the region of nuclear surface that corresponds to \(\Gamma \approx 0.0042 \times 10^{-14}\) MeV. This value is \(~36\) times smaller than \(\Gamma_{\text{exp}}\). At larger distances \(R > 9\) fm, the result obtained by assuming \(G(R) \propto H_0^+(\eta, kR)\) is marked by a dotted line.

The absolute width can also be obtained from expression (2), which involves the alpha-particle spectroscopic factor \(S\) and the s.p. decay width. Figure 12 shows the result of the current expression (20) for \(\Gamma^{\text{sp}}\) as a function of the channel radius. As discussed in Ref. [36], \(\Gamma^{\text{sp}}\) calculated this way should be independent of \(R\) if \(R\) is large enough. This is precisely what is seen in Fig. 12: the s.p. width converges beyond the range of the WS potential to \(\Gamma^{\text{sp}} = 0.1247 \times 10^{-12}\) MeV, which is indeed very close to the value of \(2\Im(\xi_s) = 0.1265 \times 10^{-12}\) MeV given by the imaginary part of the Gamow resonance.

Using the modified formation amplitude \(G(R)\) of Fig. 10 we compute the spectroscopic factor \(S = 0.011\), which – combined with the value of \(\Gamma^{\text{sp}}\) above – yields \(\Gamma = 0.14 \times 10^{-14}\) MeV. Using \(\Delta R = 0.55\) fm we obtain \(S = 0.0080\) and \(\Gamma = 0.10 \times 10^{-14}\) MeV. Both these values are close to \(\Gamma_{\text{exp}} = 0.153 \times 10^{-14}\) MeV.

VI. COMPARISON BETWEEN GROUND-STATE ALPHA DECAY OF \(^{212}\text{Po}\) AND \(^{104}\text{Te}\)

To compare absolute widths of \(^{212}\text{Po}\) and \(^{104}\text{Te}\) in a consistent way, we consider similar M1 and M4 model spaces for both nuclei. The norm kernel eigenvalues \(n_{\nu}\) do not depend on the model space in which \(g(R)\) is calculated, so we take the cutoff \(n_{\text{min}} = 0.001\).

Let us begin with \(^{212}\text{Po}\) by making a convergence analysis of \(S\) in the M1 model space as a function of \(\Delta R\) and \(R_{\text{max}}\) (as in Fig. 9). For \(\Delta R = 0.53, 0.54, 0.55, \) and
of the rapid oscillation of the eigenfunctions inside the surface region – the overlap with those eigenfunctions is too small, as expected from Fig. 8. This deficiency is related to the poor quality of the interaction used to describe $^{212}$Po in M1. To better understand this fact, let us take a look of the spectroscopic factor in terms of the spectral representation of the norm kernel,

$$S = \sum_{\nu} \frac{g_\nu^2}{n_\nu},$$

(49)

where the sum is truncated by the condition $n_\nu > n_{\text{min}}$. The summation range and eigenvalues $n_\nu$ are the same for M1 and M4; the only difference comes from $g_\nu$. Because of the rapid oscillation of the eigenfunctions inside the nucleus, only the eigenfunctions which are peaked at and beyond the nuclear surface will contribute significantly to the sum. But – because $g(R)$ in M1 is small in the surface region – the overlap with those eigenfunctions is small, and this gives rise to a very reduced value of $S$.

By making a similar analysis for $^{104}$Te in M1, we found $S = 0.067, 0.024, 0.0066,$ and $0.00046$ for $\Delta R = 0.53, 0.54, 0.55,$ and $0.56$ fm, respectively. In the model space M4 we found $S = 0.21, 0.088, 0.032,$ and $0.0051$ for the same values of $\Delta R$. Clearly, the convergence in $S$ has not been achieved for $^{104}$Te. We would like to attribute this to the impact of the proton continuum on $g_\nu$, which results in increased oscillations of $G(R)$ in the surface area. Table IX compares the values of $S$ and the corresponding absolute widths for $^{212}$Po and $^{104}$Te at $\Delta R = 0.56$ fm. (The single particle width for Te is $\Gamma^{sp} = 0.162 \times 10^{-12}$ MeV.)

It is interesting to compare our current results for $^{104}$Te with the estimates of phenomenological alpha-decay models based on semi-classical approximation [72]. The assumed large value of $Q_\alpha = 6.12$ MeV in Ref. 72 results in a very short half-life of $7 \times 10^{-11}$ sec. The alpha-decay energies of $5.05$ MeV [76] and $5.42 \pm 0.07$ MeV [77] result in $T_{1/2} \sim 10^{-7}$ sec and $\sim 5 \times 10^{-9}$ sec, respectively, and these estimates are not inconsistent with our value (M4 model space) $T_{1/2} = 5.5 \times 10^{-7}$ sec ($Q_\alpha = 5.151$ MeV). As the value of $Q_\alpha$ in $^{104}$Te is very uncertain, we show in Fig. 13 the absolute width and half-life $T_{1/2}$ as a function of $Q_\alpha$ for the model space M4.

![FIG. 12. Single particle width of $^{212}$Po from current expression (20).](image)

![FIG. 13. Ground-state alpha-decay width (left scale) and half-life (right scale) in $^{104}$Te as functions of the decay energy.](image)

| Model Space | $^{212}$Po $S$ | $^{104}$Te $S$ | $\Gamma \times 10^{14}$ MeV $^{212}$Po | $\Gamma \times 10^{14}$ MeV $^{104}$Te |
|-------------|----------------|----------------|----------------------------------|----------------------------------|
| M1          | 0.00032        | 0.00046        | 0.0040                          | 0.0075                          |
| M4          | 0.011          | 0.0051         | 0.14                            | 0.083                           |

TABLE IX. Alpha decay spectroscopic factor and absolute width for $^{212}$Po and $^{104}$Te computed in the configuration spaces M1 and M4, with $n_{\text{min}} = 0.001, \Delta R = 0.56$ and $R_{\text{max}} = 11.76$ fm ($M = 21$).

Our predicted spectroscopic factors in M4 for $^{104}$Te and $^{212}$Po are about 0.5% and 1%, respectively. As mentioned above, a fairly small value of $S$ in $^{104}$Te could be a consequence of the proximity of the proton continuum. Indeed, all the valence proton shells are resonances. The small value of $S$ in $^{104}$Te could also be attributed to the poor quality of the valence interaction assumed, and the neglect of the $T = 0$ force. The effect of the proton-neutron interaction was examined in, e.g., Refs. 14, 78 for $^{212}$Po and was found to be minor due to the fact that neutrons and protons in $^{212}$Po occupy different shells. This is no longer true in the $N = Z$ nucleus $^{104}$Te, in which the major enhancement of $S$ is expected due to $T = 0$ correlations. Therefore, our predictions for $S$ and $\Gamma$ in $^{104}$Te given in Table IX should be considered as a very conservative lower limit.

VII. CONCLUSIONS

The g.s. alpha decay of $^{212}$Po has been studied within the complex-energy shell model framework with the Berggren ensemble of the average Woods-Saxon potential. We applied the pole approximation by consider-
ing s.p. resonant states only. The overlap integral involving alpha-cluster nucleons was computed exactly, without resorting to the delta-function approximation. We considered the large valence space of Tonozuka and Arima that is necessary to produce the collective enhancement of the formation amplitude.

The absolute alpha-decay width was computed using the reduced width obtained in the framework of the R-matrix theory and also from the alpha spectroscopic factor. The latter approach yielded results consistent with experimental value, but only after considering the antisymmetrization and normalization of the decay channel wave function. The R-matrix estimate underestimates the experimental width by a factor of \( \sim 36 \). The R-matrix expression depends on the asymptotic value of the formation amplitude that is very sensitive to the size of the configuration space. On the other hand, the reaction-theory expression involves the spectroscopic factor – an integral quantity that depends less on the size of the basis used. It is very encouraging to see that a reasonable agreement with the experimental width of \(^{212}\text{Po}\) has been obtained without explicitly considering the alpha-cluster component in the wave function of the parent nucleus. In this context, we believe that the improved treatment of the particle continuum has been essential.

We have also provided an estimate of the alpha-decay rate in \(^{104}\text{Te}\). Unfortunately, due to the fact that the valence proton shells in this nucleus lie in the continuum, no fully convergent result has been achieved. We hope to improve the situation in the future by inclusion of the non-resonant continuum space that will remove some of the undesired oscillations in \( G(R) \) at large distances. In addition, since the residual interaction employed in our work neglects the proton-neutron components, and the wave function has a seniority-zero character based on \( T = 1 \) nucleonic pairs, the predicted alpha width in this \( N = Z \) nucleus should be viewed as a conservative low limit. Indeed, the inclusion of \( T = 0 \) correlations is expected to increase the value of \( \Gamma \) significantly.

The calculations presented in this study should be considered as an important step towards an improved microscopic understanding of the alpha-decay process. Still, as this work demonstrates, further improvements are needed. The neglect of the non-resonant continuum, i.e., complex-energy scattering states in the Berggren ensemble, slightly violates the completeness relation at a one-body level. This results in small imaginary contributions to spectroscopic factors and reduced widths, and most importantly – can affect the behavior of formation amplitudes at very large distances. The second crucial development will be the use of large-scale shell model calculations, including realistic \( T = 0 \) and \( T = 1 \) interactions, to compute wave function amplitudes. This will enable us to provide a more meaningful estimate of \(^{104}\text{Te}\) alpha decay rate. The work in both directions is underway.

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