Families of Vector Fields which Generate the Group of Diffeomorphisms

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Abstract

Given a compact manifold $M$, we prove that any bracket generating and invariant under multiplication on smooth functions family of vector fields on $M$ generates the connected component of unit of the group $\text{Diff} M$.

Let $M$ be a smooth $n$-dimensional compact manifold, $\text{Vec} M$ the space of smooth vector fields on $M$ and $\text{Diff}_0 M$ the group of isotopic to the identity diffeomorphisms of $M$.

Given $f \in \text{Vec} M$, we denote by $t \mapsto e^{tf}$, $t \in \mathbb{R}$, the flow on $M$ generated by $f$; then $e^{tf}$, $t \in \mathbb{R}$, is a one-parametric subgroup of $\text{Diff}_0 M$. Let $\mathcal{F} \subset \text{Vec} M$; the subgroup of $\text{Diff}_0 M$ generated by $e^{tf}$, $f \in \mathcal{F}$, $t \in \mathbb{R}$, is denoted by $\text{Gr}\mathcal{F}$.

**Theorem.** Let $\mathcal{F} \subset \text{Vec} M$; if $\text{Gr}\mathcal{F}$ acts transitively on $M$, then

$$\text{Gr}\{af : a \in C^\infty(M), f \in \mathcal{F}\} = \text{Diff}_0 M.$$ 

**Corollary 1.** Let $\Delta \subset TM$ be a completely nonholonomic vector distribution. Then any isotopic to the identity diffeomorphism of $M$ has a form $e^{f_1} \circ \cdots \circ e^{f_k}$, where $f_1, \ldots, f_k$ are sections of $\Delta$.

**Remark.** Recall that $\text{Gr}\{f_1, f_2\}$ acts transitively on $M$ for a generic pair of smooth vector fields $f_1, f_2$.

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We start the proof of the theorem with an auxiliary lemma that is actually the main part of the proof. Let $B \subset \mathbb{R}^n$ be diffeomorphic to a cube, $0 \in B$; we set $C_0^\infty(B) = \{ a \in C^\infty(B) : a(0) = 0 \}$ and assume that $C_0^\infty(B)$ is endowed with the standard $C^\infty$-topology.

**Lemma 1 (Main Lemma).** Let $X_i \in \text{Vec}\mathbb{R}^n$, $a_i \in C^\infty(\mathbb{R}^n)$, $i = 1, \ldots, n$, and the following conditions hold:
- $\text{span}\{X_1(0), \ldots, X_n(0)\} = \mathbb{R}^n$,
- $a_i(0) = 0$, $\langle d_0 a_i, X_i(0) \rangle < 0$, $i = 1, \ldots, n$;

then there exist $\epsilon, \varepsilon > 0$ and a neighborhood $\mathcal{O}$ of $(\epsilon a_1, \ldots, \epsilon a_n)|_{B_\varepsilon}$ in $C_0^\infty(B_\varepsilon)^n$ such that the mapping

$$
\Phi : (b_1, \ldots, b_n) \mapsto (e^{b_1 X_1} \circ \cdots \circ e^{b_n X_n})|_{B_\varepsilon}
$$

(1)
is an open map from $\mathcal{O}$ into $C_0^\infty(B_\varepsilon)^n$, where

$$
B_\varepsilon = \{e^{s_1 X_1} \circ \cdots \circ e^{s_n X_n}(0) : |s_i| \leq \varepsilon, i = 1, \ldots, n\}.
$$

**Sketch of proof.** Openness of the map (1) is derived from the Hamilton’s version of the Nash–Moser inverse function theorem [2]. Set $\bar{a} = (\epsilon a_1, \ldots, \epsilon a_n)$. In order to apply the Nash–Moser theorem we have to invert the differential of $\Phi$ at $\bar{a}$ and show that inverse is “tame” with respect to $\bar{a}$. Here we make computations only for fixed $\bar{a}$ and leave the boring check of the tame dependence on $\bar{a}$ for the detailed paper.

Note that $e^{\epsilon a_j X_j}$ are closed to identity diffeomorphisms, hence $\frac{\partial \Phi}{\partial b_i}|_{\bar{a}}$ is obtained from $\frac{\partial}{\partial b_i} e^{b_i X_i}|_{\epsilon a_i}$ by a closed to identity change of variables. We have

$$
\left( \frac{\partial}{\partial a} e^{a X} \right) : b \mapsto e^{a X} \left( \int_0^1 e^{\int_0^t \langle da, X \rangle} e^{\tau a X} d\tau \big|_b \circ e^{t a X} dt X \right) \circ e^{a X}.
$$

This equality follows from the standard “variations formula” (see [1]) and the relation:

$$
(e^{t a X})_* : X \mapsto \left( e^{\int_0^t \langle da, X \rangle e^{-\tau a X} d\tau} \right) X.
$$

Let us define an operator $A(a, X) : C_0^\infty(B_\varepsilon) \to C_0^\infty(B_\varepsilon)$ by the formula

$$
A(a, X)b = \int_0^1 e^{\int_0^t \langle da, X \rangle e^{\tau a X} d\tau} b \circ e^{t a X} dt,
$$
where \( \hat{B}_\epsilon = \{ e^{sX}(x) : |s| \leq \epsilon, x \in \Pi^{n-1} \} \) and \( \Pi^{n-1} \) is a transversal to \( X \) small \((n-1)\)-dimensional box. We see that invertibility of \( A(\varepsilon a_i, X_i), i = 1, \ldots, n, \) implies invertibility of \( D_\Phi \).

Now set \( \mathcal{X} = \{ bX : b \in C^\infty(M) \} \subset \text{Vec} M \). The map
\[
(bX) \mapsto (A(a, x)b) X
\]
has a clear intrinsic meaning as a linear operator on the space \( \mathcal{X} \); moreover, this operator depends only on the vector field \( aX \in \mathcal{X} \). Indeed,
\[
(A(a, X)b) X = e^{-aX} \left( (D(aX)Exp|_X)(bX) \right) \circ e^{-aX},
\]
where \( D_Y Exp \) is the differential at the point \( Y \in \text{Vec} M \) of the map \( Y \mapsto e^Y, Y \in \text{Vec} M \).

Recall that \( a(0) = 0, \langle d_0 a, X(0) \rangle < 0 \). In particular, \( X \) is transversal to the hypersurface \( a^{-1}(0) \). We may rectify the field \( X \) in such a way that, in new coordinates, \( X = \frac{\partial}{\partial x_1}, a(0, x_2, \ldots, x_n) = 0 \). Now the field \( aX \) can be treated as a depending on \( y = (x_2, \ldots, x_n) \) family of 1-dimensional vector fields \( a(x_1, y) \frac{\partial}{\partial x_1} \). Moreover, \( a(0, y) = 0, \frac{\partial a}{\partial x_1}(0, y) = \alpha(y) < 0 \).

A hyperbolic 1-dimensional field \( a(x_1, y) \frac{\partial}{\partial x_1} \) can be linearized by a smooth change of variable and this smooth change of variable smoothly depends on \( y \). Hence we may assume that \( aX = \alpha(y)x_1 \frac{\partial}{\partial x_1} \). Then \( b \circ e^{taX}(x_1, y) = b(e^{\alpha(y)t}x_1, y) \).

We thus have to invert the operator
\[
\hat{A} : b(x_1, y) \mapsto \int_0^1 e^{-t\alpha(y)}b \left( e^{\alpha(y)t}x_1, y \right) dt
\]
acting in the space of smooth functions on a box. We can write
\[
b(x_1, y) = b_0(y) + x_1b_1(y) + x_1^2u(x_1, y),
\]
where \( u \) is a smooth function. Then \( \hat{A}b_0 = \frac{1}{\alpha}(1 - e^{-\alpha})b_0, \hat{A}(x_1b_1) = x_1b_1 \) and
\[
\hat{A} \left( x_1^2u(x_1, y) \right) = x_1^2 \int_0^1 e^{\alpha(y)t}u \left( e^{\alpha(y)t}x_1, y \right) dt = -\frac{x_1^2}{\alpha(y)} \int_0^1 u(\tau x_1, y) d\tau.
\]
What remains is to invert the operator
\[ B : u(x_1, y) \mapsto \int_{e^{\alpha(y)}} u(\tau x_1, y) \, d\tau. \]
We set \( v(x_1, y) = \frac{1}{x_1} \int_0^{x_1} u(s, y) \, ds; \) then
\[
(Bu)(x_1, y) = \left( v(x_1, y) - e^{\alpha(y)}v(e^{\alpha(y)}x_1, y) \right). \tag{2}
\]
We introduce one more operator:
\[ R : v(x_1, y) \mapsto e^{\alpha(y)}v(e^{\alpha(y)}x_1, y). \]
Let \( \|v\|_{C^k,0} = \sup_{1 \leq i \leq k} \|\frac{\partial^i v}{\partial x_1^i}\|_{C^0}. \) Obviously, \( \|R\|_{C^k,0} \leq e^{\sup \alpha} < 1, \forall k. \) Hence \((I - R)^{-1}\) transforms a smooth on the box function \( \psi \) in the function \( \varphi = (I - R)^{-1}\psi \) that is smooth with respect to \( x_1. \) As usually, the chain rule for the differentiation allows to demonstrate that function \( \varphi \) is also smooth on the box and to compute its derivatives:
\[
\frac{\partial \varphi}{\partial y_i} = (I - R)^{-1} \left( \frac{\partial \psi}{\partial y_i} - e^{\alpha} \frac{\partial \alpha}{\partial y_i} \varphi - e^{2\alpha} \frac{\partial \alpha}{\partial y_i} \frac{\partial \varphi}{\partial x_1} \right), \text{ e.t.c.}
\]
Coming back to equation (2), we obtain: \( v = (I - R)^{-1}Bu. \) Finally,
\[ B^{-1} : w \mapsto \frac{\partial}{\partial x_1} (x_1(I - R)^{-1}w). \]

Now set
\[ \mathcal{P} = \text{Gr} \{ af : a \in C^\infty(M), f \in \mathcal{F} \}, \quad \mathcal{P}_q = \{ P \in \mathcal{P} : P(q) = q \}, \quad q \in M. \]

**Lemma 2.** Any \( q \in M \) possesses a neighborhood \( U_q \subset M \) such that the set
\[
\left\{ P|_{U_q} : P \in \mathcal{P}_q \right\} \tag{3}
\]
has a nonempty interior in \( C^\infty_q(U_q, M) \), where \( C^\infty_q(U_q, M) \) is the Fréchet manifold of smooth maps \( F : U_q \rightarrow M \) such that \( F(q) = q. \)
Proof. According to the Orbit Theorem of Sussmann [4] (see also the textbook [1]), transitivity of the action of $\text{Gr}\mathcal{F}$ on $M$ implies that 

$$T_qM = \text{span}\{P_*f(q) : p \in \text{Gr}\mathcal{F}, f \in \mathcal{F}\}.$$ 

Take $X_i = P_i^*f_i$, $i = 1, \ldots, n$, such that $P_i \in \text{Gr}\mathcal{F}$, $f_i \in \mathcal{F}$, and $X_1(q), \ldots, X_n(q)$ form a basis of $T_qM$. Then for any vanished at $q$ smooth functions $a_1, \ldots, a_n$, the diffeomorphism 

$$e^{a_1X_1} \circ \cdots \circ e^{a_nX_n} = P_1 \circ e^{(a_1P_1)f_1} \circ P_1^{-1} \circ \cdots \circ P_n \circ e^{(a_nP_n)f_n} \circ P_n^{-1}$$ 

belongs to the group $\mathcal{P}_q$. The desired result now follows from Main Lemma.

**Corollary 2.** Interior of the set (3) contains the identical map.

**Proof.** Let $O$ be an open subset of $C^\infty_q(U_q, M)$ that is contained in (3) and $P_0|_{U_q} \in O$. Then $P_0^{-1} \circ O$ is a contained in (3) neighborhood of the identity.

**Definition 1.** Given $P \in \text{Diff}M$, we set $\text{supp} P = \{x \in M : P(x) \neq x\}$.

**Lemma 3.** Let $O$ be a neighborhood of the identity in $\text{Diff}M$. Then for any $q \in M$ and any neighborhood $U_q \subset M$ of $q$, we have:

$$q \in \text{int}\{P(q) : P \in O \cap \mathcal{P}, \text{supp} P \subset U_q\}.$$ 

**Proof.** Let vector fields $X_1, \ldots, X_n$ be as in the proof of Lemma 2 and $b \in C^\infty(M)$ a cut-off function such that $\text{supp} b \subset U_q$ and $q \in \text{int} b^{-1}(1)$. Then the diffeomorphism 

$$Q(s_1, \ldots, s_n) = e^{s_1bX_1} \circ \cdots \circ e^{s_nbX_n}$$ 

belongs to $O \cap \mathcal{P}$ for all sufficiently close to 0 real numbers $s_1, \ldots, s_n$ and $\text{supp} Q(s_1, \ldots, s_n) \subset U_q$. On the other hand, the map 

$$(s_1, \ldots, s_n) \mapsto Q(s_1, \ldots, s_n)(q)$$ 

is a local diffeomorphism in a neighborhood of 0.

**Lemma 4.** Let $\bigcup_j U_j = M$ be a covering of $M$ by open subsets and $O$ be a neighborhood of identity in $\text{Diff}M$. Then the group $\text{Diff}_0M$ is generated by the subset 

$$\{P \in O : \exists j \text{ such that } \text{supp} P \subset U_j\}.$$
Proof. The group $\text{Diff}_0 M$ is obviously generated by any neighborhood of the identity. We may assume that the covering of $M$ is finite and any $U_j$ is contained in a coordinate neighborhood. Moreover, taking a finer covering and a smaller neighborhood $\mathcal{O}$ if necessary, we may assume that for any $P \in \mathcal{O}$ and any $U_j$, the coordinate representation of $P|_{U_j}$ has a form $P : x \mapsto x + \varphi_P(x)$, where $\varphi$ is a $C^1$-small smooth vector function.

Now consider a refined covering $\bigcup O_i = M$, so that $\overline{O_i} \subset U_{j_i}$ for some $j_i$ and cut-off functions $a_i$ such that $a_i|_{O_i} = 1$, $\text{supp} a_i \subset U_{j_i}$. Given $P \in \mathcal{O}$, we set

$$P_i(x) = x + a_i(x)\varphi_P(x), \forall x \in U_{j_i} \text{ and } P_i(q) = q, \forall q \in M \setminus U_{j_i}.$$ 

Then $\text{supp} (P_i^{-1} \circ P) \subset \text{supp} P \setminus O_i$. Now, by the induction with respect to $i$, we step by step arrive to a diffeomorphism with empty support. In other words, we present $P$ as a composition of diffeomorphisms whose supports are contained in $U_j$.

Proof of the Theorem. According to Lemma 4, it is sufficient to prove that there exist a neighborhood $U_q \subset M$ and a neighborhood of the identity $\mathcal{O} \subset \text{Diff} M$ such that any diffeomorphism $P \in \mathcal{O}$ whose support is contained in $U_q$ belongs to $\mathcal{P}$. Moreover, Lemma 3 allows to assume that $P(q) = q$. Finally, the corollary to Lemma 2 completes the job.

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References

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