Erratum: Percolation on random Johnson–Mehl tessellations and related models

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The proof presented in [2] of the result that the critical probability for percolation on a random Johnson–Mehl tessellation is 1/2 contains a (glaring!) error; we are very grateful to Rob van den Berg for bringing this to our attention. Fortunately, the error is easy to correct; as is often the case when one applies sharp-threshold results such as those of Talagrand [6] or Friedgut and Kalai [1] (in both cases extending results of Kahn, Kalai and Katznelson [5] and Bourgain, Kahn, Kalai, Katznelson and Linial [3]) to ‘symmetric’ events, to obtain a sharp threshold one needs only enough symmetry to ensure that many variables are equivalent, rather than total symmetry.

Let $\mathbb{P}^n_{p_-,p_+}$ denote the probability measure on $\{-1,0,1\}^n$ in which each coordinate is independent, and is equal to +1 with probability $p_+$, and to −1 with probability $p_-$. An event $E \subset \{-1,0,1\}^n$ is increasing if whenever $x \in E$ and $x \leq x'$ holds coordinatewise, then $x' \in E$. We say that $E$ has symmetry of order $m$ if there is a group action on $[n] = \{1,2,\ldots,n\}$ in which each orbit has size at least $m$, such that the induced action on $\{-1,0,1\}^n$ preserves $E$. To correct the proof in [2] we need the following lemma.

**Lemma 1.** There is an absolute constant $c_3$ such that if $0 < q_- < p_- < 1/e$, $0 < p_+ < q_+ < 1/e$, $E \subset \{-1,0,1\}^n$ is increasing and has symmetry of order $m$, and $\mathbb{P}^n_{p_-,p_+}(E) > \eta$, then $\mathbb{P}^n_{q_-,q_+}(E) > 1 - \eta$ whenever

$$\min\{q_+ - p_+, p_- - q_-\} \geq c_3 \log(1/\eta)\max\log(1/\max)/\log m,$$

(1)

where $\max = \max\{q_+, p_-\}$.

Using this result in place of Theorem 2.2 of [1] (which is simply the special case when $m = n$, i.e., when $E$ is symmetric), the proof in [2] may be corrected with essentially no changes. Indeed, the event $E_3$ considered at the bottom of page 329, that some $3s/4$ by $s/12$ rectangle in $T(s)$ has a robustly black

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horizontal crossing, is symmetric under translations of the space $\mathbb{T}(s) \times [0, s]$ in which the Poisson point processes live through vectors of the form $(x, y, 0)$. Hence the corresponding discrete event $E^m$ considered on the next page has symmetry of order $m = (s/\delta)^2$. To deduce Theorem 6 of [2] from Lemma 1 one needs the inequality in the middle of page 330 of [2], but with $\log N$ replaced by $\log m$. Since $N = (s/\delta)^3$, this corresponds simply to a change in the constant, and all remaining calculations are unaffected.

To prove Lemma 1 one needs a suitable influence result. Such a result was proved for a product of 2-element spaces by Talagrand [6]; later, Friedgut and Kalai [4] used a different method to obtain slightly weaker results. One can adapt Talagrand’s proof to the 3-element setting, obtaining a slightly weaker form of Lemma 1 (see the remark at the end of this note), but it seems easier to follow the method of [4]. Unfortunately, even in the two element case, Friedgut and Kalai did not prove quite the result we need, although their method gives it.

Given a function $f$ on a product probability space $\Omega^n$, let $I_f(k)$ denote the influence of the $k$th coordinate with respect to $f$, i.e., the probability of the set of configurations $\omega$ with the property that there is some $\omega'$ differing from $\omega$ only in the $k$th coordinate for which $f(\omega') \neq f(\omega)$. For $A \subset \Omega^n$, let $I_A(k) = I_f(k)$ where $f$ is the characteristic function of $A$.

Following the notation of Friedgut and Kalai [4], let $V_n(p)$ denote the weighted cube, that is the $n$th power of the 2-element probability space in which $\mathbb{P}(0) = 1 - p$ and $\mathbb{P}(1) = p$. Bourgain, Kahn, Kalai, Katznelson and Linial [3] showed that if $f$ is any 0/1-valued function on the $n$th power of a probability space, then some influence $I_f(k)$ is at least a constant times $t \log n/n$, where $t = \min\{\mathbb{P}(f^{-1}(0)), \mathbb{P}(f^{-1}(1))\}$. Friedgut and Kalai [4] adapted their proof to prove two extensions (Theorems 3.1 and Theorem 3.4 in [4]). The following result combines these extensions. It is also implied by Corollary 1.2 of Talagrand [4]; see the remark below.

**Lemma 2.** Let $0 < p \leq 1/2$ and let $f : V_n(p) \to \{0, 1\}$ with $\mathbb{P}(f^{-1}(1)) = t$. If $I_f(k) \leq \delta$ for every $k$ then $I_f = \sum_{k=1}^n I_f(k)$ satisfies the inequality

$$I_f \geq c \cdot \frac{t(1-t)}{p \log(1/p)} \log \left(\frac{ct(1-t)}{\delta^{1/2} I_f} \right),$$

where $c > 0$ is an absolute constant. In particular, if for some $a \leq 1/2$ we have $I_f(k) \leq a p^2 \log(1/p)^2$ for every $k$, then

$$I_f \geq c' \cdot \frac{t(1-t)}{p \log(1/p)} \log(1/a),$$

where $c' > 0$ is an absolute constant.

In the related results in [4], it is assumed that $t \leq 1/2$, in which case the factor $(1 - t)$ can be dropped. Of course this makes no difference; indeed, $t(1 - t)$ can be replaced by $\min\{t, (1 - t)\}$ above, changing the constants appropriately.
The condition \( p \leq 1/2 \) can be replaced by \( p \leq 1 - \varepsilon \) for any constant \( \varepsilon > 0 \), but in any case the main interest is when \( p \) is small. (Such a condition is assumed implicitly in [4].)

Although Lemma 2 is not given in [4], it might as well have been: to prove it one simply combines the two modifications to the Bourgain, Kahn, Kalai, Katznelson and Linial [3] proof that Friedgut and Kalai [4] gave; these modifications can be applied simultaneously without any problems.

**Proof.** As in [4], we phrase the proof in terms of modifications to that in [3]; what follows is not intended to be read on its own.

As in [3], the first step is to replace each factor \( V_1(p) \) in the product space \( V_n(p) \) by the probability space \( Y = \{0, 1\}^m \) with uniform measure; one can assume that \( p \) is a dyadic rational, choose \( m \) so that \( 2^m p \) is an integer, and take the first \( (1 - p)2^m \) points of \( Y \) (in the binary order) to correspond to \( 0 \in V_1(p) \) and the last \( p2^m \) to \( 1 \in V_1(p) \). Then, as noted in [4], for any function \( f : V_1(p) \to \{0, 1\} \), the sum \( w(f) \) of the influences of the corresponding function on \( Y \) satisfies

\[
w(f) \leq c_1 p \log(1/p)
\]

for some absolute constant \( c_1 \). Using this in place of the bound \( w(f) \leq 2 \) one can replace relation (14) of [3] by

\[
\|W_k\|_2^2 \leq c_1 p \log(1/p) I_k
\]

Writing \( \delta_k \) for \( I_f(k) \) and using the first part of (15) in [3], it then follows that more than half the weight of the sum

\[
t(1 - t) = \|f - \mathbb{E} f\|_2^2 = \sum_{S_1 \subseteq [m], \ldots, S_n \subseteq [m]: |S_1| + \cdots + |S_n| > 0} \hat{f}^2(S_1, \ldots, S_n)
\]

is concentrated on terms \( \hat{f}^2(S_1, \ldots, S_n) \) with

\[
0 < \sum |S_i| \leq 3c_1(t(1 - t))^{-1} p \log(1/p) \sum_k \delta_k.
\]

[We have slightly modified the argument in [4] to exclude the term with all \( S_i \) empty. This seems to be needed later to correct a trivial error in [4].]

On the other hand, as noted by Friedgut and Kalai [4], with \( \varepsilon = 1/\sqrt{3} \) relations (18) and (19) in [3] give

\[
\sum_{k=1}^n \|T_k R_k\|_2^2 \leq \sum_{k=1}^n \|R_k\|_{4/3}^2 \leq \sum_{k=1}^n (3\delta_k)^{3/2},
\]

and it follows from [3, (20)] that more than half the weight of \( \|f - \mathbb{E} f\|_2^2 \) is on terms with

\[
(|S_1| + \cdots + |S_n|) \varepsilon^2 |S_1| + \cdots + 2|S_n| \leq c_2^{-1} (t(1 - t))^{-1} \sum_{k=1}^n \delta_k^{3/2},
\]

for some absolute constant \( c_2 > 0 \). Hence, some weight of \( \|f - \mathbb{E} f\|_2^2 \) sits where both [3] and [4] hold, so these inequalities can hold simultaneously.
Let $s$ denote a value of $|S_1| + \cdots + |S_n|$ for which both (3) and (4) hold. Since $s \geq 1$ we have $s^2 s = s^{3-s} \geq 3^{-s}$, so from (4)

$$3^{-s} \leq c_2^{-1} (t(1-t))^{-1} \sum_{k=1}^n \delta_k^{3/2},$$

i.e.,

$$s \geq \log \left( \frac{c_2 t(1-t)}{\sum \delta_k^{3/2}} \right) / \log 3.$$

Combined with (3), this gives

$$3c_1 (t(1-t))^{-1} p \log(1/p) \sum_k \delta_k \geq \log \left( \frac{c_2 t(1-t)}{\sum \delta_k^{3/2}} \right) / \log 3,$$

i.e.,

$$I_f \geq c_3 \frac{t(1-t)}{p \log(1/p)} \log \left( \frac{c_2 t(1-t)}{\sum \delta_k^{3/2}} \right)$$

for some absolute $c_3 > 0$, where $I_f = \sum \delta_k$ is the sum of the influences.

Note that (5) is valid for any 0/1-valued function on $V_n(p)$. Assuming now that $\delta_k \leq \delta$ for all $k$, we have $\sum \delta_k^{3/2} \leq \delta^{3/2} \sum \delta_k = \delta^{1/2} I_f$, so

$$I_f \geq c_3 \frac{t(1-t)}{p \log(1/p)} \log \left( \frac{c_2 t(1-t)}{ \delta^{1/2} I_f} \right),$$

proving the first part of the result.

Define $a$ by $\delta = a(p \log(1/p))^2$, and set $x = t(1-t)/(p \log(1/p))$. Suppose that $I_f \leq bx$. Then we have

$$I_f \geq c_3 x \log \left( \frac{c_2 t(1-t)}{a^{1/2} p \log(1/p) b t(1-t)(p \log(1/p))^{-1}} \right) = c_3 x \log \left( \frac{c_2}{a^{1/2} b} \right).$$

Since $I_f = bx$ it follows that $b \geq c_3 \log(c_2/(a^{1/2} b))$. Assuming that $a \leq 1/2$, say, it follows that

$$I_f \geq 2c_4 \log(a^{-1/2}) x = c_4 \frac{t(1-t)}{p \log(1/p)} \log(1/a)$$

for some absolute constant $c_4 > 0$, as claimed. \qed

As noted in [1], because of the form of the proof, the extension to the probability space $W_n^{p_-, p_+}$, i.e., $\{-1, 0, 1\}^n$ with the product measure $P_n^{p_-, p_+}$, is immediate. We state only the second part.

**Corollary 3.** For every $0 < p_-, p_+ \leq 1/e$ and every function $f : W_n^{p_-, p_+} \to \{0, 1\}$ with $P(f(1)) = t$, if $a \leq 1/2$ and $I_f(k) \leq a p_{\max}^2 \log(1/p_{\max})^2$ for every $k$, then

$$I_f \geq c \frac{t(1-t)}{p_{\max} \log(1/p_{\max})} \log(1/a),$$

where $p_{\max} = \max\{p_-, p_+\}$ and $c > 0$ is an absolute constant.
Proof. The proof is almost identical to that of Lemma 2; the first step is to replace each factor $W_{p_-,p_+}$ by $Y = \{0,1\}^n$, noting that this time one has

$$w(f) \leq cp_+ \log(1/p_+) + cp_- \log(1/p_-) \leq 2cp_{\max} \log(1/p_{\max})$$

in place of (2). From this point on the original probability space is irrelevant. \\[
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Using standard methods, it is easy to deduce Lemma 1 from Corollary 3.

Proof of Lemma 7. Since the left-hand side of (1) is at most $p_{\max}$, taking $c_3$ large we may assume that $m \geq 100 \log(1/p_{\max})$, say, i.e., that $m \geq p_{\max}^{100}$.

For $0 \leq h \leq m_{\max} = \min\{q_+ - p_+, p_- - q_\}$, let $r_+ = p_+ + h$ and $r_- = p_- - h$, let $g(h) = \mathbb{P}_{r_-,r_+}(E)$, and let $\tilde{g}(h) = \log(g(h)/(1-g(h)))$. Note that $g(0) = \mathbb{P}_{p_-,p_+}(E) \geq \eta$, so $\tilde{g}(0) \geq -\log(1/\eta)$, while $\mathbb{P}_{q_-,q_+}(E) \geq g(h_{\max})$. We claim that

$$\frac{d\tilde{g}}{dh} \geq \frac{2 \log m}{c_3 p_{\max} \log (1/p_{\max})}$$

for any $0 \leq h \leq m_{\max}$. Assuming this, using the lower bound (1) on $m_{\max}$, we then have $\tilde{g}(h_{\max}) \geq \tilde{g}(0) + 2 \log (1/\eta) \geq \log (1/\eta)$, giving $g(h_{\max}) > 1 - \eta$, and hence $\mathbb{P}_{q_-,q_+}(E) > 1 - \eta$, as required.

To prove (6), note that $d\tilde{g}/dh = (g(1-g))^{-1}dg/dh$, and that, by a form of the Margulis–Russo formula, the derivative of $g(h)$ is at least $I_f = \sum_k I_f(k)$, where $f$ is the characteristic function of $E$ and we evaluate the influences in the product space $\mathbb{P}_{r_-,r_+}^n$. Hence (6) follows if we can show that

$$I_f \geq 2c_3^{-1} \frac{t(1-t)}{p_{\max} \log (1/p_{\max})} \log m,$$

where $t = g(h) = \mathbb{P}_{r_-,r_+}^n(E)$.

Suppose first that some influence $I_f(k)$ is at least $m^{-1/2}$, say. Then, from the symmetry assumption, at least $m$ influences are at least this large, and $I_f \geq m^{1/2} \geq m^{1/3}p_{\max}^{-2}$. Taking $c_3$ large enough, this is much larger than the bound in (7). (The factor $t(1-t)$ works in our favour.) On the other hand, if $I_f(k) \leq m^{-1/2}$ for all $k$, then $a = \max I_f(k)p_{\max}^{-2} \log (1/p_{\max})^{-2} \leq m^{-1/3}$, say, and Corollary 3 gives (7). \\[
\]

Let us remark briefly on the relationship of the results above to those of Talagrand [6]. Note first that given an increasing subset $A$ of the weighted cube $V_n(p)$, the quantity $\mu_p(A)$ in (6) is exactly $pI_A(i)$, where $I_A(i) = I_f(i)$ with $f$ the characteristic function of $A$. Theorem 1.1 of [6] thus states in the notation above that for some universal constant $c > 0$ and any $A \subset V_n(p)$ with $\mathbb{P}(A) = t$,

$$\sum_{k=1}^n \frac{p(1-p)I_A(k)}{\log[1/(p(1-p)I_A(k))]} \geq c \frac{t(1-t)}{\log[2/(p(1-p))]}.$$

and Corollary 1.2 in [6] gives

$$I_A \geq \frac{ct(1-t)}{p(1-p) \log[2/(p(1-p))]} \log(1/\epsilon)$$

(8)
whenever \(p(1-p)I_A(i) \leq \varepsilon\) for all \(i\). (In fact, we have reinserted an irrelevant factor \((1-p)\) omitted in \([6]\). Also, Talagrand states his results for monotone sub-
sets, but this condition is not used.) These results immediately imply Lemma 2
indeed, Theorem 1.1 of \([6]\) is stronger. The corollary \([3]\) is superficially stronger
than Lemma 2 but in practice most likely exactly equivalent, as in the applications
one always assumes that \(\varepsilon\) is smaller than some large power of \(p\), and
then the apparent differences are irrelevant up to changing the constants.

Why then did we start from (a form of) the Friedgut–Kalai result instead of
Talagrand’s? The answer is that the extension to a power of a 3-element space is
 clearer, at least to us. One approach is as follows. First, consider any function
\(f\) defined on the weighted cube \(V_n(p_1, \ldots, p_n)\), i.e., the product of the probability
spaces \(V_1(p_1), \ldots, V_1(p_n)\), with state space \(\{0,1\}^n\). The proof of Lemma 2 gives
the following result; note that assuming \(p_i \leq 1/2\) loses no generality, as one can
replace \(p_i\) by \(1-p_i\).

**Lemma 4.** Let \(0 < p_1, \ldots, p_n \leq 1/2\) and let \(f : V_n(p_1, \ldots, p_n) \to \{0,1\}\) with
\(\mathbb{P}(f^{-1}(1)) = t\). If \(I_f(k) \leq \delta\) for every \(k\) then \(I_f = \sum_{k=1}^n I_f(k)\) satisfies the
inequality
\[
I_f \geq c \frac{t(1-t)}{\max_i p_i \log(1/p_i)} \log \left( \frac{ct(1-t)}{\delta^2 I_f} \right),
\]
where \(c > 0\) is an absolute constant.

Turning to Talagrand’s version, for \(x \in \{0,1\}^n\), write \(U_i(x)\) for the point
obtained by changing the \(i\)th coordinate of \(x\), and, adapting Talagrand’s defi-
nition in the obvious way, set \(\Delta_i f(x) = (1-p_i)(f(x) - f(U_i(x)))\) if \(x_i = 1\) and
\(\Delta_i f(x) = p_i (f(x) - f(U_i(x)))\) if \(x_i = 0\). The proof of Theorem 1.5 in \([6]\) goes
through *mutatis mutandis* to give the following result.

**Theorem 5.** There is an absolute constant \(K\) such that, for any function \(f\) on
any space \(V_n(p_1, \ldots, p_n)\) with \(\mathbb{E} f = 0\), we have
\[
\|f\|_2^2 \leq K \log \left( \frac{2}{\min_i p_i(1-p_i)} \right) \sum_i \frac{\|\Delta_i f\|_2^2}{\log(e\|\Delta_i f\|_2/\|\Delta_i f\|_1)}
\]
where the expectation \(\mathbb{E}\) and norms \(\| \cdot \|_q\) are calculated with respect to the
probability measure on \(V_n(p_1, \ldots, p_n)\).

Since the modifications to Talagrand’s proof are essentially trivial, we omit
the details. The key point is that, much of the time, one coordinate is considered
at a time, and one should simply replace \(p\) by the corresponding \(p_i\) wherever it
appears (for example, in the definition of \(r_i(x)\)). The minimum over \(i\) comes in
when the upper bound \(|r_i(x) - r_i(y)| \leq \theta\) on page 1580 of \([6]\) is used; this now
requires \(\theta = \max_i 1/\sqrt{p_i(1-p_i)}\).

Translating back to influences in the case where \(f\) is the characteristic function
of \(A\) with the constant \(\mathbb{P}(A)\) subtracted, one has
\[
\|\Delta_i f\|_q^2 = I_A(i) (p_i(1-p_i)^q + (1-p_i)p_i^q),
\]

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giving $||\Delta f||_2 = \sqrt{I_A(i)p_i(1-p_i)}$ and $||\Delta f||_1 = 2I_A(i)p_i(1-p_i)$. With $\varepsilon = \max_i p_i(1-p_i)I_A(i) \leq \max I_A(i)$, the equivalent of Corollary 1.2 in [6] one obtains is thus that

$$I_A \geq \frac{ct(1-t)}{(\max_i p_i(1-p_i)) \log[2/(\min_i p_i(1-p_i))] \log(1/\varepsilon)}.$$

If the $p_i$ vary wildly, this inequality seems to be weaker then Lemma 4 although the difference may not matter. If the maximum and minimum are close, it is stronger than Lemma 4 although most likely equivalent for almost all applications.

To apply this result to [2], roughly speaking one can replace each copy of $\{-1,0,1\}$ by $\{0,1\}^2$ with an appropriate measure, with $p_1$ close to $1 - p_-$ and $p_2$ close to $p_+$. Since in this case $p_-$ and $p_+$ are close, while we do not quite obtain Lemma 4 we obtain a result that is good enough for our application. Since we have given a different proof above, we omit the details.

References

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