NOTE

A note on time-symmetric hypersurfaces in the Schwarzschild geometry

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Abstract
In this note, we show that any inextensible time-symmetric space-like hypersurface of differentiability class $C^k$, $k \geq 2$, isometrically embedded in the maximal Schwarzschild geometry must intersect the bifurcation sphere.

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1. Introduction

The Schwarzschild solution is one of the most relevant vacuum solution of the Einstein field equations and its geometric properties have been widely studied. Among these we quote the analysis of space-like embedded hypersurfaces (submanifolds of co-dimension 1) fulfilling certain geometric requirements. These requirements have been mostly inspired by the analysis of the constraint equations which arise from the orthogonal splitting of the Einstein field equations. These equations adopt a simpler form if one makes assumptions on the extrinsic curvature and it is therefore necessary to ensure that the corresponding assumptions can be made. The most common assumptions are to impose that the extrinsic curvature vanishes or has zero trace and this naturally leads to the question about the existence of embedded hypersurfaces possessing these properties. When the extrinsic curvature vanishes then the hypersurface is totally geodesic. The set of points which are invariant under a time-reflection isometry with respect to a given time function forms a space-like hypersurface with vanishing extrinsic curvature and for that reason one uses the nomenclature ‘time-symmetric’ for these hypersurfaces.

In [11] all the foliations of the Schwarzschild–Kruskal spacetime, whose leaves are spherically symmetric, maximal hypersurfaces were found. This work was later generalized to the case in which the extrinsic curvature trace is a constant [4]. Examples of these foliations can be found in [1, 9]. A maximal slicing of a subset of the Schwarzschild–Kruskal spacetime containing a Cauchy hypersurface was constructed in [2] and the implications in numerical simulations were addressed. A different kind of hypersurfaces were considered in [8] where spherically symmetric hypersurfaces having constant scalar curvature were constructed. Yet
another kind of foliation is that composed by hypersurfaces whose leaves have vanishing Riemann tensor (flat foliations). The uniqueness of these foliations in static spherically symmetric spacetimes was addressed in [3] and explicit examples were constructed in the Schwarzschild and Reissner Nordström solutions [10] and in the Kottler–Schwarzschild–de Sitter spacetime [12].

In this note we study embedded, time-symmetric hypersurfaces in the maximal extended Schwarzschild–Kruskal spacetime. The most trivial example of such a hypersurface comprises any integral hypersurface of the static Killing vector but no other instance of such a hypersurface is known. A natural question is whether these hypersurfaces are indeed the only embedded time-symmetric hypersurfaces which exist in the Schwarzschild–Kruskal spacetime. We believe that the answer to this question is affirmative but so far a complete proof of this issue is lacking. However, one can show other interesting properties of embedded time-symmetric hypersurfaces in the Schwarzschild–Kruskal spacetime. It is the aim of this short note to show that any embedded time-symmetric hypersurface in the Schwarzschild–Kruskal spacetime cannot intersect the black hole region. To the best of our knowledge, a rigorous proof of this statement is not available in the literature. A consequence of this result is that in the Schwarzschild geometry, all inextensible time-symmetric hypersurfaces embedded in the Schwarzschild–Kruskal spacetime should intersect the bifurcation sphere (see corollary 1). A similar property is fulfilled in any static (resp. stationary axisymmetric) spacetime admitting a bifurcate Killing horizon by the static (resp. stationary axisymmetric) hypersurfaces. See [13] for further details.

The outline of this work is as follows: in section 2 we summarize some known results about the Schwarzschild solution which are needed in this work. Section 3 contains the main results of this work which are the non-existence of embedded time-symmetric hypersurfaces in the non-static regions of the Schwarzschild spacetime (proposition 1) and its corollary. Finally, we discuss some applications in section 4.

2. Preliminaries

The maximal extension of the Schwarzschild geometry is the well-known Schwarzschild–Kruskal spacetime [7] and this is what will be understood in this work by the Schwarzschild–Kruskal spacetime. This is a globally hyperbolic Lorentzian manifold, which we shall denote by $M$ with the Lorentzian metric $g_{ab}$ (small Latin letters $a, b, c, \ldots$ will be used to denote indices for tensors fields on any tensor bundle built from $T(M)$ and its dual $T^*(M)$). The global causal structure of the Kruskal maximal extension is well known and it is displayed in figure 1 where the different regions of this spacetime are represented.

Now let $\Sigma \subset M$ be a $C^k$, $k > 2$, embedded space-like submanifold and let $\vec{N}$ be the unit time-like normal vector to $\Sigma$ (our signature convention is such that $g(\vec{N}, \vec{N}) = -1$). Unless otherwise stated, all hypersurfaces considered in this work will be space-like. The first fundamental form $h_{\alpha\beta}$ and the extrinsic curvature $K_{\alpha\beta}$ are defined by the standard formulae

$$h_{\alpha\beta} \equiv g(\vec{e}_\alpha, \vec{e}_\beta), \quad K_{\alpha\beta} \equiv g(\vec{e}_\beta, \nabla_{\vec{e}_\alpha}\vec{N}),$$  

where $\{\vec{e}_\alpha\}, \alpha = 1, 2, 3$ is an arbitrary frame on $\Sigma$ and $\nabla$ represents the Levi-Civita connection on $M$. As is well known, the first fundamental form and the extrinsic curvature can be regarded as symmetric tensor fields on $\Sigma$ (indices for tensor fields on any tensor bundle with $\Sigma$ as the base manifold will be denoted with Greek characters $\alpha, \beta, \ldots$). The pair $(\Sigma, h_{\alpha\beta})$ is a Riemmannian manifold and therefore one can introduce the Levi-Civita connection $D_\gamma$ of the Riemannian metric $h_{\alpha\beta}$. An embedded hypersurface $\Sigma$ is said to be inextensible if it is not a proper subset
of another embedded hypersurface in $\Sigma$. When the extrinsic curvature is zero on every point of $\Sigma$ then we say that it is a time-symmetric hypersurface.

The Schwarzschild spacetime can be characterized locally in terms of scalar and tensor concomitants of the Weyl tensor $W_{abcd}$ [5]. Since some of the conditions arising from this characterization are required later on, we review next its main aspects. First of all let us define the tensor

$$G_{abcd} \equiv g_{ac} g_{bd} - g_{ad} g_{bc}. \quad (2)$$

The tensor $G_{abcd}$ has algebraic properties similar to those of the Riemann tensor. Next we introduce the Weyl tensor scalar concomitants

$$\rho \equiv \frac{1}{2} \left( \frac{W_{cd} W_{pq} W_{ab} W_{pq}}{12} \right)^{1/3}, \quad \alpha \equiv 2 \rho + \frac{\nabla_a \rho \nabla_a \rho}{9 \rho^2}, \quad (3)$$

and tensorial concomitants

$$S_{abcd} \equiv \frac{1}{3 \rho} (W_{abcd} + \rho G_{abcd}), \quad Q_{ab} \equiv S_{apbq} \nabla^p \rho \nabla^q \rho, \quad P_{ab} \equiv W^*_{apbq} \nabla^p \rho \nabla^q \rho, \quad (4)$$

where $W^*_{apbq} \equiv \eta_{abcd} W_{ap} / 2$ and $\eta_{abcd}$ is the volume element associated with the metric $g_{ab}$.

One now has the following result proven in [5].

**Theorem 1** (Ferrando and Sáez [5]). A spacetime $(\mathcal{M}, g_{ab})$ is locally isometric to the Schwarzschild solution if and only if the following conditions are fulfilled:

$$-S_{abpq} + \frac{1}{2} S_{ab} S_{cd} S_{dcpq} = 0, \quad P_{ab} = 0, \quad \alpha > 0, \quad Q_{ab} Q^b_c - Q_{ac} \nabla^b \rho \nabla^b \rho = 0, \quad \rho \neq 0. \quad (5)$$

3. Main results

**Proposition 1.** Let $\Sigma \subset \mathcal{M}$ be an embedded $C^k$, $k > 2$, hypersurface and assume further that $\Sigma$ is time-symmetric. Then $\Sigma$ does not intersect the interior of the black hole region of the Kruskal maximal extension.
Proof. Define a one-parameter family of embedded $C^k$ space-like hypersurfaces $\{\Sigma_t\}_{t \in I}$, where $I$ is an open real interval, $\Sigma_t \cap \Sigma_{t'} = \emptyset$ if $t_1 \neq t_2$ and $\Sigma_0 = \Sigma$. The family $\{\Sigma_t\}$ can then be regarded as a foliation of the set $\mathcal{U} = \bigcup_{t \in I} \Sigma_t$. The unit time-like normal to each leaf $\Sigma_t$ is a $C^k$ vector field on $\text{int}(\mathcal{U})$ and we shall denote this vector field by $n_t$. We now take conditions (5) and compute their orthogonal splitting with respect to the vector field $n_t$. Such a computation was carried out in [6] and we shall only quote here the result which we need, referring the reader to the aforementioned reference for a complete derivation. The needed result is the orthogonal splitting of the gradient of the scalar $\rho$ which reads

$$\nabla_a \rho = P n_a + P_a, \quad (6)$$

where

$$P \equiv -\frac{1}{2} E_{a \beta} K_{a \beta} - \rho K_{\alpha \beta} - \frac{1}{6} \epsilon_{\alpha \beta} (E_{a \gamma} D_b B_{bd} + B_{a \gamma} D_b E_{bd}), \quad (7a)$$

$$P_a \equiv \frac{1}{6 \rho} (-B_{a \beta} D_b B_{bd} + E_{a \beta} D_b E_{bd}) = D_b \rho \quad (7b)$$

and

$$h_{a b} \equiv g_{a b} + n_a n_b, \quad K_{a \beta} \equiv -\frac{1}{2} \mathcal{L}_{n} h_{a \beta}, \quad \epsilon_{a b c} \equiv \eta_{d a b c} n^d, \quad (8)$$

$$E_{a \beta} \equiv W_{a c b d} n^c n^d, \quad B_{a \beta} \equiv W_{a c b d} n^c n^d. \quad (9)$$

The operator $D_b$ appearing in some of the previous equations is defined for any tensor field $T_{b_1 \cdots b_q}$ by

$$D_b T_{b_1 \cdots b_q} \equiv h_{d_1}^{b_1} \cdots h_{d_p}^{b_p} h_{b_1}^{a_1} \cdots h_{b_q}^{a_q} h^c_{d_1} \cdots h^c_{d_p} \nabla_c T_{a_1 \cdots a_q}. \quad (10)$$

From (6) we easily deduce

$$\nabla_a \rho \nabla^a \rho = -P^2 + P_a P^a. \quad (11)$$

We realize that $P$ and $P_a$ are given in terms of spatial quantities (quantities invariant under the action of the spatial projection $h_{b a}$) and that makes it easy to compute the pull-back of them to any of the hypersurfaces $\Sigma_t$ under the inclusion embedding $i : \Sigma_t \subseteq \mathcal{M}$. In particular the pull-back of $h_{a b}$ ($K_{a b}$) under this embedding is the first fundamental form (the extrinsic curvature) of $\Sigma_t$ which is denoted by $h^{(t)}_{\alpha \beta}$ ($K^{(t)}_{\alpha \beta}$). Also the pull-backs of $E_{a \beta}$, $B_{a \beta}$ can be rendered in the form (see again [6])

$$E_{a \beta}^{(t)} \equiv r_{a \beta}^{(t)} + K_{a \gamma}^{(t) \gamma} K_{\gamma \beta}^{(t)} - K_{a \gamma}^{(t) \gamma} K_{\beta \gamma}^{(t)} B_{a \beta}^{(t)} \equiv \epsilon_{a \beta}^{(t)} \gamma^\delta D_{\gamma}^{(t)} K_{\delta \beta}^{(t)}, \quad (12)$$

where $r_{a \beta}^{(t)}$ is the Ricci tensor computed from the curvature arising from the covariant derivative $D_{\gamma}^{(t)}$ (we follow the convention of adding the superscript $t$ to those quantities intrinsic to the Riemannian manifold $(\Sigma_t, h_{a b}^{(t)})$). Now, by assumption, we have $K_{a \beta}^{(0)} = 0$ which entails, via (12), $B_{a \beta}^{(0)} = 0$. Hence, from (7a) we get $P|_{|_\Sigma} = 0$ and thus

$$\nabla_a \rho \nabla^a \rho|_{|_\Sigma} \geq 0. \quad (13)$$

We can now compute the explicit value of the scalar $\nabla_a \rho \nabla^a \rho$ at any point of the Schwarzschild spacetime if we choose a coordinate representation. For example, we may choose the standard Schwarzschild coordinates

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad 0 < \theta < \pi, \quad 0 < \phi < 2\pi, \quad (14)$$

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where $M$ is the Schwarzschild mass. As is very well known when $r > 2M$ the Schwarzschild coordinates cover either of the two asymptotically flat exterior regions $D_1$ and $D_2$ whereas for $0 < r < 2M$ the coordinates cover either of the two interior regions $B^+$ or $B^-$. In the Schwarzschild coordinates one has

$$\nabla_a \rho \nabla^a \rho = \frac{-9M^2(2M - r)}{r^9}. \tag{15}$$

Note that since the scalar $\nabla_a \rho \nabla^a \rho$ is an invariant quantity, the relation (15) gives its value at any point of the maximal extension $M$, including the region not covered by the Schwarzschild coordinates (this region comprises the points such that $r = 2M$ which correspond to the region $\mathcal{H}^+ \cup \mathcal{H}^-$). Now, from (15), it is straightforward to see that in the interior of any of the black hole regions the inequality $\nabla_a \rho \nabla^a \rho < 0$ holds and combining this property with (13) the required result follows.

\[ \square \]

Corollary 1. Any inextensible connected time-symmetric $C^k$, $k > 2$, hypersurface $\Sigma$ embedded in the Schwarzschild spacetime must intersect the bifurcation sphere.

**Proof.** Proposition 1 implies that $\Sigma$ must be included in the complementary set of the interior of the black hole regions, which is the set $D_1 \cup D_2 \cup \mathcal{H}^+ \cup \mathcal{H}^-$. Moreover, since $\Sigma$ is space-like and inextensible, the intersection of $\Sigma$ and the set $(\mathcal{H}^+ \cup \mathcal{H}^-) \backslash \Omega$ must be the empty set from which we conclude that the intersection of $\Sigma$ and $\Omega$ is never empty.

\[ \square \]

Corollary 2. A point in the event horizon, but not in the bifurcation sphere, never belongs to the boundary of a connected, time-symmetric, analytic hypersurface $\Sigma$ embedded in any of the exterior regions of the Schwarzschild spacetime.

**Proof.** Suppose the contrary and let $\Sigma \cap (\mathcal{H} \backslash \Omega)$ be non-empty, where $\mathcal{H} \equiv \mathcal{H}^+ \cup \mathcal{H}^-$. Since $\Sigma$ is analytic, we may construct an analytic extension $\tilde{\Sigma}$ of $\Sigma$ through the set $\Sigma \cap (\mathcal{H} \backslash \Omega)$ such that $\Sigma \subset \tilde{\Sigma}$ and $\tilde{\Sigma} \cap \text{int}(B) \neq \emptyset$, where $B \equiv B^+ \cup B^-$. We should distinguish now two possibilities: either $\tilde{\Sigma}$ is space-like at every point or it contains a subset in which it is null. In the first case one can define a symmetric tensor field $K_{\alpha\beta}$ on $\tilde{\Sigma}$ through equation (1) which by construction is analytic and vanishes in a proper open subset of $\tilde{\Sigma}$ which is impossible. In the second alternative we can still use equation (1) to define an analytic symmetric tensor field $\tilde{K}_{\alpha\beta}$ at any point of $\tilde{\Sigma}$ if we remove the normalization condition on the vector $\tilde{N}$. On those points belonging to $\Sigma$ it is easy to check that the relation $\tilde{K}_{\alpha\beta} = \Psi K_{\alpha\beta}$ holds, where $K_{\alpha\beta}$ is the tensor obtained with the normalization condition on $\tilde{N}$ and $\Psi$ is a certain scalar function. Since $K_{\alpha\beta} = 0$ we get $\tilde{K}_{\alpha\beta} = 0$ on $\Sigma$, which again is impossible for an analytic tensor on $\tilde{\Sigma}$.

\[ \square \]

4. Applications and conclusions

In this work we have shown that inextensible time-symmetric hypersurfaces embedded in the Schwarzschild–Kruskal spacetime never intersect the black hole region and always intersect the bifurcation sphere. An interesting issue not solved in this work (nor, as far as we know, in any other place of the literature) deals with the uniqueness of embedded time-symmetric hypersurfaces in the Schwarzschild–Kruskal spacetime. It is tempting to conjecture that the integral hypersurfaces of the static Killing vector are the only time-symmetric hypersurfaces and one expects that our techniques should contribute to settle this question. If this conjecture
were true then one could seek a uniqueness result of the Schwarzschild spacetime formulated in terms of the uniqueness in the existence of embedded time-symmetric hypersurfaces.

As mentioned in section 2, a time-symmetric hypersurface has been regarded as space-like by definition. However, the second fundamental form can be defined for any kind of hypersurface by means of equation (1) (the vector $\vec{N}$ need not be normalized) and therefore a logical question would be: What would happen when arbitrary embedded hypersurfaces with vanishing second fundamental form are considered (this condition is independent from the normalization for $\vec{N}$ chosen). When we deal with a time-like hypersurface, then equations (6) and (7a)–(7b) adopt a similar form and the quantity $P$ does vanish if the second fundamental form is zero but $P_a$ could still be time-like or space-like. Thus, the sign of the right-hand side of (11) is not determined in general and hence the main argument used in the proof of proposition 1 is no longer valid. Therefore, we would have to consider the possibility in which the hypersurface has null regions and space-like regions. In the null regions one does not have a well-defined orthogonal splitting with respect to the null vector orthogonal to the hypersurface and hence it is not possible to decompose $\nabla_a \rho$ on these regions in the manner shown in equation (6).

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