Testing the Fair Sampling Assumption for EPR-Bell Experiments with Polarizing Beamsplitters

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(Dated: April 1, 2022)

In spite of many attempts, no local realistic model seems to be able to reproduce EPR-Bell type correlations, unless non ideal detection is allowed. The low efficiency of detectors in all experiments with photons makes the use of the fair sampling assumption unavoidable. However, since this very assumption is false in all existing local realistic models based on inefficient detection, we thus question its validity. We show that it is no more reasonable to assume fair sampling than it is impossible to test, and we actually propose an experimental test which would provides clear cut results in case of unfair sampling.

I. INTRODUCTION

The role of the efficiency of detectors in EPR-Bohm experiments [1, 2] with photons has been the subject of continuous investigation during the last thirty years, since allowing for non ideal detection is the only acknowledged way to get an apparent violation of CHSH-Bell Inequality [4, 5] with a local realistic model [24]. The main difference between existing experiments [25] and local realistic models thus based on inefficiency is the status of the fair sampling assumption. On one hand, in order to assess that a violation of Bell inequality has occurred, all EPR-Bell experiments with photons are interpreted assuming fair sampling, as it is taken to be both a reasonable and experimentally untestable assumption. On the other hand, this very assumption is precisely false in all existing local realistic models based on inefficiency. Yet, these models are supposed to be only ad hoc, designed to show that a violation of Bell inequality is possible in principle, but usually without any claim for relevance with physical reality.

The purpose of this article is to question this fair sampling assumption. We will show that a very straightforward model using contextual probabilities [5] can reproduce EPR correlations and that it is possible to overcome the usual understanding of the fair sampling issue with a generic test capable of disproving the fair sampling assumption.

II. A SIMPLE LOCAL HIDDEN-VARIABLE MODEL BASED ON THE DETECTION LOOPHOLE

The well-known scheme for an EPR-Bell experiment is represented on Fig. 1. A source sends pairs of photons, labelled 1 and 2, in an entangled singlet state $|\psi\rangle$, and polarization measurements are carried out on these photons using polarizing beam splitters (PBS) A and B oriented respectively along $\varphi_A$ and $\varphi_B$. The possible measurement results for each particle are labelled as +1 if the particle is detected in the ordinary channel, and as -1 if it is detected in the extraordinary channel.

The rates of the four possible coincident events are labelled $R_{++}$, $R_{+-}$, $R_{-+}$, and $R_{--}$. For instance $R_{+-}$ is the rate of coincident detection in the +1 channel of polarizing beamsplitter A and in the -1 channel of polarizing beamsplitter B. Let $R_d$ be the total rate of detected pairs, defined as:

$$R_d(\rho, \varphi_A, \varphi_B) = R_{++} + R_{+-} + R_{-+} + R_{--}.$$ (1)

It is then possible to define the correlation function as

$$E(\rho, \varphi_A, \varphi_B) = \frac{R_{++} - R_{+-} - R_{-+} + R_{--}}{R_d(\rho, \varphi_A, \varphi_B)}.$$ (2)

As is well known, experiments with pairs of photons [6, 7] show a correlation $E(\varphi_B - \varphi_A)$, in agreement with the predictions of quantum mechanics.

There are not many ways to get such an EPR-like correlation using a local realistic model based on the detection loophole. This point was already stressed by Clauser and Shimony [8], although they pointed this as a case against local hidden-variables models, whereas for us it grants the generality of our point. The model we will use here is indeed a simple variant of the one that has been used many times to show explicitly that a local realistic
model based on detection loophole can be in agreement with the experimental results [10, 11, 12, 13], and that these models need not even be “highly artificial” [8], but can in fact be sound consequences of known principles of physics [14, 15, 16]. In our model, each particle is provided with an internal parameter $\lambda$ — a polarization — that can take any value in the interval $[0, 2\pi]$, and the particles issued from one pair have the same polarization. The fate of a particle incoming into an analyzer orientated along the direction $\varphi$ is determined by the difference $\alpha = |\lambda - \varphi|$:

- If $\alpha$ is close to 0 modulo $\pi$, the particle goes into the channel labelled +1.
- If $\alpha$ is close to $\pi/2$ modulo $\pi$, the particle goes into the channel labelled −1.

As such, the model is incapable of providing anything better than the saw tooth, however we define “close” and whatever the polarization distribution of the pairs of particles (a consequence of Bell’s Theorem). However, if there exists a third channel, labelled 0, corresponding to a non detection, measurements which tend to reduce the curve to a saw tooth can be discarded, thus yielding to a $S$ greater than 2, up to 4. The particles that must remain undetected for this purpose are the ones for which $\alpha \simeq \pi/4$ modulo $\pi/2$, while all other sampling of undetected particles would reduce the correlation curve to a saw tooth, resulting in no apparent violation of Bell Inequalities [27]. We may refer conveniently thereafter to the particles for which $\alpha \simeq \pi/4$ modulo $\pi/2$ as shaky particles. By this terminology, we emphasized that a small perturbation of the internal polarization $\lambda$ of a shaky particle would induce a change in the channel the particle would choose if it was to be detected [28].

This sampling process presents the crucial feature of being unfair. A sampling process is thus said to be unfair if the probability $P_{\text{unfair}}$ for a particle to be rejected depends on its hidden parameters (i.e., the internal polarization $\lambda$ in our model) and on the measurement settings (i.e., the orientation $\varphi$ of the PBS). One can write this dependence explicitly by stating that the probability of non detection is of the form:

$$P_{\text{unfair}} = P_{\text{unfair}}(\lambda, \varphi).$$

A direct consequence of this local unfair sampling is that for each pair of measurement settings $\varphi_1$ and $\varphi_2$ the ensemble of detected pairs $S_{\varphi_1\varphi_2}$ belongs to a specific probability spaces $\Omega_{\varphi_1\varphi_2}$ which does depend on the measurement settings $\varphi_1$ and $\varphi_2$. For each $\varphi_1$ and $\varphi_2$, some specific regions of the Kolmogorov space are simply never recorded: the probability space has become contextual. In other words, a straightforward consequence of unfair sampling is the *contextuality* of the associated probabilities [3]. Bell’s theorem, which is derived by using one fixed space [17], is therefore no longer valid here, and multi-context framework generalizations of Bell and CHSH inequalities are required [17, 18, 19].

We performed a first numerical simulation [29] in the case where both particles from one pair do share exactly the same polarization $\lambda$. The correlation obtained in this case shows unrealistic sharp edges (see Fig. 4). This unrealistic correlation function can be smoothed very near a cosinus simply by slightly breaking the statistical alignment of the polarization of particles in accordance with a gaussian distribution centered on perfect alignment [31]. The correlation then shows a good agreement with the predictions of Quantum Mechanics (see Fig. 3). Note that this was obtained here with unsharp polarization correlations of particles [31] and sharp boundaries for the channels in the analyzer. It should however be possible to proceed the other way round and get the same result.

### III. Comparing Stern-Gerlach Devices with Polarizing Beamsplitters

The question then arising is, where this rejection occurs and why. The purpose of this article is not to discuss in details the likeness of some possible explanations, but to focus on the observable consequences of either fair or unfair sampling. We would like nevertheless to make few
By order of magnitude, the most important places for this rejection are the detectors, since for the experiments performed with photons, their efficiency was at best 10%. The important aspect is however not the magnitude of rejections, but the selectiveness of these rejections. For this purpose, the analyzers providing two-channel measurement are much better candidates \[14\]. Indeed the experiments carried out so far were not performed with Stern-Gerlach devices and atoms, but with polarizing beamsplitters and photons. While Stern-Gerlach devices can in principle perform an ideal two-channel measurement on all emitted atoms, as the separation of the beam of atoms in two channels occurs in a vacuum, the separation of the photons into two channels by a polarizing beamsplitter occurs in a solid, i.e. two birefringent prisms assembled together. Therefore, assuming that a polarizing beamsplitter is an ideal analyzer similar to a Stern-Gerlach device is a risky assumption. It would mean that the polarizing beamsplitter treats all impinging photons equally, providing a clear +1 or -1 as a result, while a 0 would occur independently of the internal polarization of the photon. Even though this problem was acknowledged on some rare occasions \[20\], it was obviously not considered serious enough to prevent the possibility of an experimental test using these polarizing beamsplitters. Nevertheless, in our view this assumption of fair sampling cannot a priori be accepted as reasonable for two-channel polarizing beamsplitters. In other words, our hypothesis is that on the contrary a polarizing beamsplitter is unfair and that this behavior is actually responsible for the observed EPR-Bell correlations \[32\].

### IV. TESTING THE FAIR SAMPLING ASSUMPTION FOR LOCAL HIDDEN VARIABLE MODELS

Fair sampling is not only reputed to be a reasonable assumption, which as we have seen above can be criticized \[33\], but it is also reputed impossible to test experimentally. Yet, it is rather difficult to find in the literature any clear justification for this line of thought. At most some statements can be found relative to the one-channel type experiment \[4\], but to the best of our knowledge, there is however no such a justification for the two-channel type experiment. We will actually show hereafter how fair sampling can in principle perform an ideal two-channel measurement. For a particle to be rejected according to the measurement settings (i.e., the orientation \(\varphi\) of the PBS).

Let us assume that we have at disposal a stable source of entangled photons, so that the rate of particles emitted by the source is a constant of time. Let \(R\) be the total rate of pairs entering in the coincidence circuitry. We assume that \(R\) is independent of the polarization distribution of the source and of the measurement parameters \(\varphi_A\) and \(\varphi_B\), since the source of entangled photons is rotationally invariant. Let \(R_{\text{unfair}}\) be the rate of pairs rejected according to the unfair sampling process. This rate depends on the polarization distribution \(\rho\) of the source and on the orientations \(\varphi_A\) and \(\varphi_B\) of the two-channel devices, that is, \(R_{\text{unfair}} = R_{\text{unfair}}(\rho, \varphi_A, \varphi_B)\) \[34\]. Finally, let \(R_{\text{fair}}\) be the rate of pairs rejected according to the fair sampling process, which is on the contrary completely independent of these same variables.

The experimentally available total rate of detected pairs in case of unfair sampling can then be written as:

\[
R_d(\rho, \varphi_A, \varphi_B) = R - R_{\text{unfair}}(\rho, \varphi_A, \varphi_B) - R_{\text{fair}}. \tag{4}
\]

The fair sampling assumption is then written as \(R_{\text{unfair}}(\rho, \varphi_A, \varphi_B) = 0\), so that the total rate of detected pairs \(R_d\) is no more dependent on \(\rho\), \(\varphi_A\), and \(\varphi_B\) than are \(R\) and \(R_{\text{fair}}\):

\[
R_d(\rho, \varphi_A, \varphi_B) = R - R_{\text{fair}}, \quad \forall \rho, \varphi_A, \varphi_B. \tag{5}
\]

Testing the fair sampling assumption therefore means to test which one among equations \(\text{4}\) and \(\text{5}\) holds experimentally. Unlike with the one-channel experiment, this can be decided experimentally by making the settings \(\rho\), \(\varphi_A\), and \(\varphi_B\) vary \[35\]. In case of unfair sampling, the measured total rate of detected pairs should depend on these settings, according to Eq. \(\text{4}\), whereas it should remain independent of them in case of fair sampling, according to Eq. \(\text{5}\).

#### A. A passive fair sampling test

A first possible way to test the fair sampling assumption is to vary the angle between the polarizing beamsplitters \(\Delta \varphi = |\varphi_B - \varphi_A|\), and check whether a variation of \(R_d\) is observed.

We made a numerical simulation using the model described above with settings exhibiting sharp correlations as in Fig. \(\text{2}\) and observed that the size of \(R_d\) is not constant (see Fig. \(\text{1}\)) when \(\Delta \varphi\) is varied from 0 to \(\pi\). This is due to the fact that the errors are dependent: if both analyzers are oriented in the same direction modulo \(\pi/2\), then the chances that only one particle remains undetected are smaller than for other relative angles, and since the total number of undetected particles is a constant, the total number of detected pairs is larger. This effect might however be difficult to observe experimentally. We have carried out a numerical simulation with the same loose correlations that gave us a close fit with the predictions
Unfair sampling and dependent errors induce a sharp variation of $R_d$.

FIG. 4: Total coincidence rate for the unrealistic violation of CHSH-Bell inequality observed in Fig. 2. Unfair sampling of Quantum Mechanics, as in Fig. 1 and found that the error dependence is more difficult to observe (see Fig. 5), not to mention that in our model no dark rates or no mistakes of any kind are implemented, which would undoubtedly make this dependence even more difficult to observe, so that checking fair sampling with this test can hardly be conclusive [36].

B. An active fair sampling test

Nevertheless, another approach to testing fair sampling is possible by varying not only $\varphi_A$ and $\varphi_B$ like in the previous section, but also the polarization distribution $\rho$ of the source, on which also depends $R_{unfair}$. Instead of the rotationally invariant polarization distribution $\rho^*$ of the source, we propose to use a source state with probability distribution $\rho_\theta$ centered on a particular polarization angle $\theta$. We will see that varying $\rho_\theta$, allows to exhibit clear cut discrepancies depending on whether or not the fair sampling assumption holds [37].

In order to control the source experimentally, our proposal is therefore to insert two aligned polarizing beam-splitters, both oriented along the same angle $\theta$, in the coincidence circuitry right after the rotationally invariant source $\rho^*$ of entangled photons and before the two-channel measurement devices [38].

The procedure to simulate our active fair sampling test is the following:

1. In all model based on the detection loophole, a detection pattern for the couple PBS+Detector is given. It processes an input, dispatching each particle in the proper channel in order to obtain the EPR-Bell statistics when a specific rotationally invariant source $\rho^*$ impinges onto it. The model usually says nothing as to the output polarization distribution of each channel, and that is what we need since our test uses the output of a polarizer to control the polarization distribution $\rho_\theta$ of the pairs.

2. This output polarization distribution must be set consistently when impinging on the PBS+Detector pattern with the known behavior of two successive polarizer, i.e., consistently with the Malus law, and also consistently with the projection postulate of Quantum Mechanics (that is, a polarizer does not only filter a polarized beam, it also rotates its main polarization direction, as can be seen by inserting a polarizer between two crossed polarizers).

3. As described above, our active fair sampling test consists in inserting two aligned polarizing beam-splitters oriented along an angle $\theta$ in the coincidence circuitry before the two-channel measurement devices, both oriented along the same angle $\varphi$.

Note that this procedure is quite general and need not apply solely to our model. Although local realist models based on the detection loophole are always concerned only with reproducing the EPR correlations (Step. 1), it is straightforward in most cases to complete the model so as to get the Malus law (Step. 2), and thereof perform our fair sampling test (Step. 3). We have actually followed this procedure both with our model and the model that seemed to be the least alike ours, that is Larsson’s model, and found that the output necessary to reproduce Step. 2 is the same, and that the result of our test
FIG. 7: Total coincidence rate for a source controlled with aligned polarizing beamsplitters at angle 0 and for realistic settings (as in Fig. 3 and Fig. 5). The variation of $R_d$ due to unfair sampling appears sharply.

in Step. 3 shows exactly the same behavior, as exhibited in Fig. 5.

To be more specific, we gave a gaussian distribution to the output polarization of the photons in the considered channel (+1 in this case). In other words, the output state of each particle dispatched in channel +1 was modified randomly according to the context of the encountered polarizing beamsplitter so that this output has statistically the form of a gaussian distribution centered on the main polarization direction of the polarizer $\frac{\pi}{2}$. In a quantum formulation, this modification of the state is nothing but the collapsing of the initial vector state, from the rotationally invariant singlet state to $|\uparrow\uparrow\rangle_\theta$. By doing so we have obtained the Malus law behavior without in any way modifying our detection pattern for the EPR-Bell experiment. The numerical simulation with the same settings as for the realistic violation of Bell inequality of Fig. 4 and $\theta = 0$, shows a much clearer characteristic of unfair sampling (see Fig. 7), with very clear oscillations: the contrast is roughly five times better than for the passive fair sampling test of Fig. 5 (1/3 against 1/15). This behavior can be explained in the following, all pairs coming from the controlled source are such that $\lambda \simeq \theta$ modulo $\pi$, so that if both analyzers are oriented in the same direction $\varphi_A = \varphi_B = \varphi$, the sum of all four coincidence rates $R_d$ drops when $\varphi = \theta + \pi/4$ modulo $\pi/2$, because then almost all particles are shaky.

Thanks to its higher contrast, this behavior should be possible to observe even with some noise and not so bright and accurate source of entangled photons. It must be noted that these oscillations cannot be made arbitrarily small without hindering the observed EPR-Bell correlations: the higher the violation of Bell Inequalities, the higher the oscillations of the total coincidence rate in our fair sampling test.

Our test can therefore rule out fair sampling if these oscillations are observed experimentally [40], since no fair sampling process could account for such contrasty oscillations thus connected to an apparent violation of Bell inequalities.

V. CONCLUSION

The detection loophole still remains the most stringent of all loopholes, so that the available EPR-Bell experiments would be far more convincing if fair sampling was thoroughly investigated, instead of assumed as being reasonable. The new experiment we have proposed here should be simple to implement, and should allow to check whether the fairness of the sampling in an EPR-Bell experiments is a reasonable assumption.

Acknowledgments

We are most grateful to Jan-Åke Larsson for fruitful and rather critical discussions on the issues raised in this article. We are also indebted to Gregg Jaeger, Martin Salomon, Emilio Santos, Afshin Shafiee and Johan Summhammer for their valuable comments. This work was supported by the EU-Network “Quantum Probability and Applications”.

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These rates depends on the experimental settings \(\varphi_A\) and \(\varphi_B\) and on the polarization distribution \(\rho\) of the source. This dependence is left here implicit for simplicity.

The rejection process of the particles is a completely local process. It mimics nonlocality only because of the necessary coincidence circuitry: each time one particle remains undetected at one location, the whole pair is logically rejected as providing no correlation measurement.

This shakiness is not a property of a particle in itself since it depends on the relative angle \(\alpha\); it is but a quick way to label a class of particles in the context of a given analyzer.

The initial number of pairs is 10000 for each point. All simulations described in this paper are available upon request to G. Adenier.

As a matter of detail, we set the standard deviation of the gaussian distribution to \(\pi/16.80\), and the size of each of the four shaky regions was \(\pi/13.39\).

Contrary to the usual understanding of entanglement as being a stronger correlation than would allow classical physics, the use of the detection loophole allows for loosely correlated particles to reproduce the EPR-Bohm statistics. In our model, they actually do the job better than strongly correlated particles.

We can think of few possible explanations for such such a behavior. For instance, the shaky photon might have a greater probability than other photons to be absorbed inside the polarizing cube. Another possibility is that the state of the impinging shaky photon is modified in such a way that it has a greater probability of remaining undetected than others. For instance, if a photon behaves in the polarizing beamsplitter more like an electromagnetic wave than like a particle \(\Box\), it would split into two waves with reduces intensities, and if the probability to generate a detectable signal depends on the amplitude of the wave, then shaky photons would yield the least detection probability.

Not to mention that attempting to justify fair sampling on the basis of the symmetry of the experiment scheme \(\Box\) is not a convincing argument either, since our local realistic model conforms exactly to the same requirements although it exhibits unfair sampling.

The fact that \(R_{\text{unfair}}\) depends on \(\rho\) doesn’t mean in any way that the sampling process itself is modified by the source that is sent onto it. The sampling process characterizing the detection pattern in Eq. \(\Box\) is independent of the polarization distribution of the source \(\rho\) that is sent onto it: each particle is following the defined detection pattern of Eq. \(\Box\) —in our case deterministically— independently of the polarization distribution of the source it belongs to (i.e., there is no memory loophole). In other words, the fairness or unfairness of the sampling concerns the measurement setup, not the statistical properties of the source that is sent onto it, and that precisely what makes it possible to test experimentally.

The trouble with the one-channel experiment that makes checking of the fair sampling assumption impossible is that the only measurement result that can be recorded is the +1 result (channels 1 and 0 cannot be distinguished as they correspond both to a non detection) so that the only available rate is \(R_{+}\), which always depends on \(\rho, \varphi_A\), and \(\varphi_B\), whether or not the fair sampling assumption holds.

Note that an ad-hoc local realistic model with independent error can be built \cite{12}, by defining two distinct detector patterns for each particle in such a way that the rate of detected particles remains independent of \(\Delta\rho\) for the specific polarization distribution \(\rho^*\) of the source reproducing the EPR-Bohm statistics, so that this passive test can only check that fair sampling is reasonable, but cannot logically rule out unfair sampling.

It is perhaps necessary to stress that we are not in any way suggesting that the source of entangled photons itself is not rotationally invariant and that we want to put this to a test. We do assume that the experimenter has checked before doing our test that the source of entangled state is indeed rotationally invariant, as it is clearly a crucial feature of the singlet state, necessary to demonstrate any violation of a Bell inequality. Our test is about breaking this rotationally invariant on purpose to see how the measurement setup react to a source with a preferred polarization.

We would like to stress that the measurement setup itself is left unchanged with respect to the ordinary EPR-Bell setup of Fig. \(\Box\). In our fair sampling test, represented in Fig. \(\Box\) the one and only modification concerns the source, and nothing else.

The gaussian distribution yielding a good agreement with the Malus law has here a standard deviation of \(\pi/9\). The choice of a gaussian distribution is quite arbitrary, as another distribution might give us a slightly better fit to the Malus law, but whatever the output state it would just the same have to be centered on a specific polarization \(\theta\) with the same qualitative result under our test.

Note that it is possible to introduce an additional hidden-variable that tells each photon whether the PBS should treat it according to a fair sampling process or to an unfair one, depending on whether the photon comes from the output of another polarizer of from the output of a source of entangled photons. This would allow for the model to reproduce both Malus law and EPR-Bell correlation, yet with a constant \(R_d\) under our test—we are grateful to Jan-Ake Larsson for pointing out this idea. Although this sounds definitely too ad hoc for our taste, we must acknowledge that it means that our test cannot logically rule out unfair sampling. Since our test has never been performed, it is nevertheless premature for now to try to thus hide a behavior that might well be observed experimentally. We believe in any case that nature itself cannot be that conspiratory in hiding unfair sampling, so that our test can nevertheless tell whether or not fair sampling is a reasonable assumption.