CANONICALLY CODABLE POINTS AND IRREDICUBLE CODINGS

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Abstract. Let \( f : M \to M \) be a \( C^{1+\beta} \) diffeomorphism, where \( \beta > 0 \) and \( M \) is a compact Riemannian manifold without boundary. In [Sar13], for all \( \chi > 0 \), for every small enough \( \epsilon > 0 \), Sarig had first constructed a coding \( \pi : \hat{\Sigma} \to M \) which covers the set of all Lyapunov regular \( \chi \)-hyperbolic points when \( \dim M = 2 \), where \( \hat{\Sigma} \) is a topological Markov shift (TMS) over a locally-finite and countable directed graph. \( \pi \) is Hölder continuous, and is finite-to-one on \( \hat{\Sigma}^\# := \{ u \in \hat{\Sigma} : \exists v, w \text{ s.t. } \#\{ i \geq 0 : u_i = v \} = \infty, \#\{ i \leq 0 : u_i = w \} = \infty \} \); and \( \hat{\pi}[\hat{\Sigma}^\#] \supseteq \{ \text{Lyapunov regular and temperable } \chi \text{-hyperbolic points} \} \). We later extended Sarig’s result for the case \( \dim M \geq 2 \) in [BO18]. In this work, we offer an improved construction for [BO18] such that (\( \forall \epsilon > 0 \) small enough) we could identify canonically the set \( \hat{\pi}[\hat{\Sigma}^\#] \). We introduce the notions of \( \chi \)-summable, and \( \epsilon \)-weakly temperable points.

In [BCS], in the case where \( \dim M = 2 \), the authors show that for each homoclinic class of a periodic hyperbolic point \( p \), there exists a maximal irreducible component \( \tilde{\Sigma} \subseteq \hat{\Sigma} \) s.t. all invariant ergodic probability \( \chi \)-hyperbolic measures which are carried by the homoclinic class of \( p \) can be lifted to \( \tilde{\Sigma} \). We use their construction in the context of ergodic homoclinic classes, to show the stronger claim, for all \( \dim M \geq 2 \), \( \hat{\pi}[\tilde{\Sigma} \cap \hat{\Sigma}^\#] = H(p) \) modulo conservative measures; where \( H(p) \) is the ergodic homoclinic class of \( p \), as defined in [RHRHTU11], with the (canonically identified) recurrently-codable points replacing the Lyapunov regular points in the definition in [RHRHTU11].

Contents

1. Motivation and Introduction
2. Definitions and Basic Properties
   2.1. A Set of Hyperbolic and Weakly Temperable Points
3. Symbolic Dynamics
4. Ergodic Homoclinic Classes and Maximal Irreducible Components
References

1. Motivation and Introduction

Symbolic dynamics are a powerful tool which allows us to derive many strong conclusions on smooth dynamical systems which admit them. For example, construction of Gibbs and SRB measures for uniformly hyperbolic systems [Bow08, BM77, BR75], classification of toral automorphisms (in the monumental work of Adler and Weiss) [AW70], counting periodic points [Sar13], counting of measures of maximal entropy [BCS], and many others. In the general setup of a non-uniformly hyperbolic diffeomorphism of a compact boundaryless Riemannian manifold, in [Sar13], Sarig had constructed a Markov partition when the dimension of the manifold is 2, and we later extended his result to any dimension in [BO18]. Generally, diffeomorphisms might have many points which are not hyperbolic (elliptic islands, homoclinic tangency, etc.); thus the codings of [BO18, Sar13] usually code a set \( \subseteq M \). However, not every point which demonstrates hyperbolic behavior is necessarily covered by these Markov partitions. The authors in both [Sar13, BO18] construct a set of Lyapunov regular points which are covered by the Markov partition; but this comes with two clear disadvantages: 1) Lyapunov regularity restricts us to the study of probability measures, which are known to be carried by the Lyapunov regular points, but do not include the rich ergodic theory of infinite conservative invariant measures (see Definition [2.1] for the definition of a conservative measure in our context). 2) There could be many other points which demonstrate the relatively-easy-to-study behavior of the Markov structure, which are not Lyapunov regular, and which are being ignored- thus not using the strength of the coding to
its fullest. One could simply choose to work with the image of the coding, or the set covered by the Markov partition, but this prevents us from making any new definitions and extending objects, since defining them based on a specific construction seems less natural. We therefore wish to find a Markov partition (or coding) and a set of hyperbolic points with the following three properties: 1) The Markov partition covers that set of hyperbolic points; 2) Every point which is covered by the Markov partition belongs to that set of hyperbolic points; 3) That set of hyperbolic points is defined canonically, and not by the specific coding (i.e. based on the quality of hyperbolicity of a point along its orbit, and not on a specific choice or construction). We offer such a Markov partition and such a set of hyperbolic points, by presenting an improved way to carry-out the construction in [BO18, and its analysis. This paper treats diffeomorphisms. The case of flows brings in new difficulties, because of issues related to singularities in the Poincaré section. The paper [LS19] codes a smaller set than the set of Lyapunov regular points. The author has been informed by Y. Lima that together with J. Buzzi and S. Crovisier he now has a coding which captures a larger set than the set of Lyapunov regular orbits (work in progress). It would be interesting to know if the set of coded points can be characterized completely as we do in this paper for discrete time systems.

In [Sma67], Smale introduced the notion of the homoclinic relation between two orbits of periodic hyperbolic points by, \( O \sim O' \iff \text{the global stable leaf of a point in } O \text{ intersects transversely with full codimension the global unstable leaf of a point in } O' \), and vice-versa. In [New72], Newhouse showed that this relation is in fact an equivalence relation, and so the notion of a homoclinic class of a periodic hyperbolic point rose naturally. The closure of \( \{O': O \sim O'\} \) is a closed and transitive set, and as such it is used often in the studying of transitive hyperbolic dynamics. In [RHRHTU11], Rodriguez-Hertz, Rodriguez-Hertz, Tahzibi and Ures have introduced a new notion to consider hyperbolic points which are associated with the orbit of a hyperbolic periodic point, which is called an ergodic homoclinic class. They have shown that ergodic homoclinic classes hold the property of admitting at most one SRB measure. We consider this object with the larger set of hyperbolic points (as mentioned in the paragraph above) replacing the Lyapunov regular points. This allows us to not restrict ourselves to probability measures, while having a canonical way of studying conservative (possibly infinite) invariant measures with the powerful tool of symbolic dynamics. We show in addition that an ergodic homoclinic class, admits a point with a forward orbit which is dense in a set which carries all conservative measures (possibly infinite) on it. More generally, an ergodic homoclinic class has a subset which carries all conservative measures on it, and which can be coded by an irreducible component.

2. Definitions and Basic Properties

This work uses tools which were previously developed in [BO18, Sar13]. In the following subsection, we introduce two notions of hyperbolic points, in order to have a canonical characterization for a set of points which our symbolic extension codes (see [BO18, Sar13]).

2.1. A Set of Hyperbolic and Weakly Temperable Points. Let \( M \) be a compact Riemannian manifold without boundary, of dimension \( d \geq 2 \). Let \( f \in \text{Diff}^{1+\beta}(M), \beta > 0 \) (i.e. \( f \) is invertible, \( f,f^{-1} \) are differentiable, and both \( d,f,d,f^{-1} \) are \( \beta \)-Hölder continuous). \( \forall x \in M, \langle \cdot, \cdot \rangle_x : T_x M \times T_x M \to \mathbb{R} \) is the inner product on the tangent space of \( x \) given by the Riemannian metric. \( |\cdot|_x : T_x M \to \mathbb{R} \) is the norm induced by the inner product, \( |\xi|^2_x := \langle \xi, \xi \rangle_x, \forall \xi \in T_x M \). We often omit the \( x \) subscript of the inner product and of the norm, when the tangent space in domain is clear by their argument.

Notations:

1. For every \( a, b \in \mathbb{R}, c \in \mathbb{R}^+, a = e^{\pm c} \cdot b \) means \( e^{-c} \cdot b \leq a \leq e^{c} \cdot b \).
2. \( \forall a, b \in \mathbb{R}, a \wedge b := \min\{a, b\} \).
3. For every topological Markov shift \( \Sigma \) induced by a graph \( \mathcal{G} := (\mathcal{V}, \mathcal{E}) \) (e.g. Theorem 2.4), \( \forall v \in \mathcal{V}, [v] := \{u \in \Sigma : u_0 = v\} \).

1This does not harm the uniqueness of SRB measures, as every invariant probability measure is carried by the Lyapunov regular points regardless.

2In this context, “canonical” means definitions which do not rely on a specific construction of symbolic dynamics, but which depend only on the quality of hyperbolicity of the orbit of the point.
Definition 2.1. Let \((X,\mathcal{B},\mu,T)\) be an invertible (perhaps infinite) measure preserving transformation. \(\mu\) is said to be conservative, if it satisfies the statement of the Poincaré recurrence theorem. I.e.
\[
\forall A \in \mathcal{B}, \mu(A \setminus \{x \in A : \exists n_k, m_k \uparrow \infty \text{ s.t. } f^{n_k}(x), f^{-m_k}(x) \in A, \forall k \geq 0\}) = 0.
\]

Definition 2.2.

(1) \(\chi\)-summ := \(\{x \in M : \exists a \text{ splitting } T_xM = H^s(x) \oplus H^u(x) \text{ s.t.} \\]
\[
\sup_{\xi_s \in H^s(x), |\xi_s|=1} \sum_{m=0}^{\infty} |d_x f^m \xi_s|^2 e^{2\chi m} < \infty, \sup_{\xi_u \in H^u(x), |\xi_u|=1} \sum_{m=0}^{\infty} |d_x f^{-m} \xi_u|^2 e^{2\chi m} < \infty.\]

(2) \(\chi\)-hyp := \(\{x \in M : \exists a \text{ splitting } T_xM = H^s(x) \oplus H^u(x) \text{ s.t.} \forall \xi_s \in H^s(x) \setminus \{0\}, \xi_u \in H^u(x) \setminus \{0\},\]
\[
\lim_{n \to \infty} \frac{1}{n} \log |d_x f^n \xi_s|, \lim_{n \to \infty} \frac{1}{n} \log |d_x f^{-n} \xi_u| < -\chi.\]

(3) We define for each \(x \in \chi\)-hyp,
\[
\chi(x) := -\max\{ \sup_{\xi_s \in H^s(x)} \lim_{n \to \infty} \frac{1}{n} \log |d_x f^n \xi_s|, \sup_{\xi_u \in H^u(x)} \lim_{n \to \infty} \frac{1}{n} \log |d_x f^{-n} \xi_u| \} > \chi.
\]

(4) \(\forall x \in \chi\)-summ, \(\forall \xi, \eta \in T_xM, \text{ write } \xi = \xi_s + \xi_u, \eta = \eta_s + \eta_u \text{ with } \xi_s, \eta_s \in H^s(x), \xi_u, \eta_u \in H^u(x),\]
\[
\langle \xi_s, \eta_s \rangle'_{x,s} := 2 \sum_{m=0}^{\infty} \langle d_x f^m \xi_s, d_x f^m \eta_s \rangle_x e^{2\chi m},
\]
\[
\langle \xi_u, \eta_u \rangle'_{x,u} := 2 \sum_{m=0}^{\infty} \langle d_x f^{-m} \xi_u, d_x f^{-m} \eta_u \rangle_x e^{2\chi m},
\]
\[
\langle \xi, \eta \rangle'_x := \langle \xi_s, \eta_s \rangle'_{x,s} + \langle \xi_u, \eta_u \rangle'_{x,u}.
\]

Notice that \(\chi\)-hyp \(\subseteq\) \(\chi\)-summ. For each \(x \in \chi\)-summ, write \(s(x) := \dim(H^s(x)), u(x) := \dim(H^u(x)).\)

The following theorem is a version of the Pesin-Oseledets reduction theorem, which we prove in [BO18 Theorem 2.4, Definition. 2.5]).

Theorem 2.3. \(\forall x \in \chi\)-summ, \(\exists C_\chi(x) : \mathbb{R}^d \to T_xM\text{ a linear invertible map, such that } \forall u, v \in \mathbb{R}^d,\)
\[
\langle C_\chi^{-1}(x)u, C_\chi^{-1}(x)v \rangle_x = \langle u, v \rangle_2, \text{ where } \langle \cdot, \cdot \rangle_2 \text{ is the Euclidean inner product on } \mathbb{R}^d.\]

In addition,
\[
C_\chi^{-1}(f(x)) \circ d_x f \circ C_\chi(x) = \begin{pmatrix} D_s(x) \\ D_u(x) \end{pmatrix},
\]
where \(D_s(x), D_u(x)\) are square matrices of dimensions \(s(x), u(x)\) respectively, and \(\|D_s(x)\|, \|D_u^{-1}(x)\| \leq e^{-\chi}, \|D_u(1)\|, \|D_u(1)\| \leq \kappa \text{ for some constant } \kappa = \kappa(f, \chi) > 1.\)

Claim 2.4. \(\forall x \in \chi\)-summ, \(C_\chi(x)\) is a contraction.

See [BO18 Lemma 2.9], [Sar13 Lemma 2.5].

Definition 2.5. Let \(x \in \chi\)-summ with \(T_xM = H^s(x) \oplus H^u(x),\)

(1) \(c_\chi(x) := \sup_{\xi_s \in H^s(x), \xi_u \in H^u(x), |\xi_s + \xi_u|=1} \sqrt{2 \sum_{m=0}^{\infty} |d_x f^m \xi_s|^2 e^{2\chi m} + 2 \sum_{m=0}^{\infty} |d_x f^{-m} \xi_u|^2 e^{2\chi m}}.\)

(2) \(\forall \xi_s \in H^s(x), \xi_u \in H^u(x),\)
\[
S^2(x, \xi_s) := 2 \sum_{m=0}^{\infty} |d_x f^m \xi_s|^2 e^{2\chi m}, \quad U^2(x, \xi_u) := 2 \sum_{m=0}^{\infty} |d_x f^{-m} \xi_u|^2 e^{2\chi m}.
\]

\(c_\chi(x)\) is a measurement of the hyperbolicity of \(x\) w.r.t decomposition \(T_xM = H^s(x) \oplus H^u(x)\). The greater it is, the worse the hyperbolicity (i.e. slow contraction/expansion on stable/unstable tangent spaces, or small angle between the stable and unstable tangent spaces).
Claim 2.6. For $x \in \chi-$summ, $\|C_\chi^{-1}(x)\| = c_\chi(x)$.

This fact follows from the definition of the maps $C_\chi(\cdot)$, and can be seen in [BO13 Theorem 2.4] where they are defined. It is important to note that the expression for the norm of $C_\chi^{-1}$ depends only on the decomposition $H^s(x) \oplus H^a(x)$, independently of the choice of $C_\chi(\cdot)$.

Theorem 2.7. $\forall \chi > 0$ s.t. $\exists \rho \in \chi$-hyp a periodic hyperbolic point, $\exists \epsilon_\chi > 0$ (which depends on $M, f, \beta, \chi$) s.t. $\forall 0 < \epsilon \leq \epsilon_\chi$ a countable and locally-finite directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}) = (\chi(e), \mathcal{E}(e))$ which induces a topological Markov shift $\Sigma = \Sigma(\chi, \epsilon) := \{ \mu \in \mathcal{V} : (u_i, u_{i-1}) \in \mathcal{E}, \forall i \in \mathbb{Z} \}$. $\Sigma$ admits a factor map $\pi : \Sigma \rightarrow M$ with the following properties:

1. $\sigma : \Sigma \rightarrow \Sigma$, $(\sigma \mu)_i := u_{i+1}, i \in \mathbb{Z}$ (the left-shift); $\pi \circ \sigma = f \circ \pi$.
2. $\pi$ is a H"older continuous map w.r.t to the metric $d(\mu, \nu) := \exp(-\min \{ i \geq 0 : u_i \neq v_i \text{ or } u_{i-1} \neq v_{i-1} \})$.
3. $\Sigma^\# := \{ \mu \in \Sigma : \exists n_k, m_k \uparrow \infty \text{ s.t. } u_{n_k} = u_{m_0}, u_{-n_k} = u_{-m_0}, \forall k \geq 0 \}$, $\pi[\Sigma^\#]$ carries all $f$-invariant, $\chi$-hyperbolic probability measures.

This theorem is the content of [BO13 Theorem 3.13] (and similarly, the content of [Sar13, Theorem 4.16] when $d = 2$). $\mathcal{V}$ is a collection of double Pesin-charts (see Definition 2.9), which is discrete.

Definition 2.8. Let $\epsilon > 0$ and $x \in \chi$-summ,

$$Q_\epsilon(x) := \max\{ Q \in \{ e^{-\epsilon t} \}_{t \in \mathbb{N}} : Q \leq \frac{1}{3^\alpha} (c_\chi(x))^{-\frac{\epsilon}{\beta}} \}.$$

Definition 2.9. (Pesin-charts) Since $M$ is compact, $\exists r(M) > 0$ s.t. the exponential map $\exp_x : \{ v \in T_xM : |v| \leq r \} \rightarrow M$ is well defined and smooth. When $\epsilon \leq r$, the following is well defined since $C_\chi(\cdot)$ is a contraction (see Claim 2.4):

1. $\psi_x^\# := \exp_x \circ C_\chi(x) : \{ v \in T_xM : |v| \leq \eta \} \rightarrow M, \eta \in (0, Q_\epsilon(x)]$, is called a Pesin-chart.
2. A double Pesin-chart is an ordered couple $\psi_x^\#: (\psi_x^\#, \psi_x^\#)$, where $\psi_x^\#$ and $\psi_x^\#$ are Pesin-charts.

Definition 2.10. A point $x \in \chi$-summ is called $\epsilon$-weakly temperable if $\exists q : \{ f^n(x) \}_{n \in \mathbb{Z}} \rightarrow (0, \epsilon) \cap \{ e^{-\epsilon t} \}_{t \in \mathbb{N}}$ s.t.

1. $\frac{\sum q_{k \uparrow}}{q} = e^{\pm \epsilon t}$.
2. $\limsup_{n \rightarrow \pm \infty} q(f^n(x)) > 0$.
3. $\forall n \in \mathbb{Z}$, $q(f^n(x)) \leq Q_\epsilon(f^n(x))$.

The set of all $\epsilon$-weakly temperable points is denoted by $\epsilon$-w.t.

Remark:

1. Notice that item (3) could have been replaced by "$\exists a > 0$ s.t. $\forall n \in \mathbb{Z}$, $q(f^n(x)) \leq \frac{a}{(c_\chi(f^n(x)))^{\frac{\epsilon}{\beta}}}$, since $\forall \epsilon > 0 \exists \alpha \epsilon > 0$ s.t. $Q_\epsilon(\cdot) = a_\epsilon^{\pm 1} \frac{1}{(c_\chi(\cdot))^{\frac{\epsilon}{\beta}}}$, and $q$ can be rescaled. The $\frac{\epsilon}{\beta}$ exponent is not an intrinsic property- every fixed, sufficiently large, power of $c_\chi(\cdot)$ in the definition of $Q_\epsilon(\cdot)$ would have sufficed; altering the power alters both the set of hyperbolic points, and the coding, via the $\epsilon$-overlap condition, [BO13 Definition 2.18].

2. In [BO13 Claim 2.11], we show that $\exists \epsilon > 0$, almost every point is $\epsilon$-weakly temperable w.r.t. every invariant probability measure carried by $\chi$-summ, (by the fact that for almost every $x \in \chi$-summ, $\lim_{n \rightarrow \pm \infty} n \log \|C_\chi^{-1}(f^n(x))\| = 0$, and the construction of a Pesin’s tempered kernel $q_\epsilon(x) := \frac{1}{\Sigma_{n \in \mathbb{Z}} q_\epsilon(f^n(x)) e^{-\frac{\epsilon}{\beta} n}}$; and finally $\limsup_{n \rightarrow \pm \infty} q_\epsilon(f^n(x)) > 0$ for almost every point by Poincaré’s recurrence theorem).

Definition 2.11. $\text{HWT}^\chi_\epsilon := \chi$-summ $\cap \epsilon$-w.t, are the hyperbolic and weakly temperable points, with parameters $\chi, \epsilon > 0$. 

31.i.e. hyperbolic measures with Lyapunov exponents greater than $\chi$ in absolute value.

3Every $v \in \mathcal{V}$ is a double Pesin-chart of the form $v = \psi_x^{p_u} \cdot p^u$ with $0 < p^s, p^u \leq Q_\epsilon(x)$; and discreteness means that $\forall \eta > 0 : \# \{ v \in \mathcal{V} : v = \psi_x^{p_u} \cdot p^u \text{ and } p^u > \eta \} < \infty$. 

4
Remark: $Q_\epsilon(\cdot)$ depends only on $\epsilon$ and the norm of $C^{-1}_\epsilon(\cdot)$ (a Lyapunov norm on the tangent space of a point), which is given by Claim 2.6 and depends only on the decomposition $T.M = H^s(\cdot) \oplus H^u(\cdot)$. By equation (1), if $x \in \chi-$summ is also $\epsilon$-weakly tamperable (and $\epsilon$ is small w.r.t $\chi, \beta$, as imposed by the assumption $\epsilon \leq \epsilon_\beta$ from Theorem 2.7), then the decomposition $T.M = H^s(x) \oplus H^u(x)$ must be unique. Therefore, $Q_\epsilon(\cdot)$ is defined canonically on $\epsilon$-w.t, and does not depend on the choice of $C_\chi(\cdot)$. Thus, $\forall \epsilon \in (0, \epsilon_\chi]$, HWT$^\epsilon_\chi$ is defined canonically.

Remark: HWT$^\epsilon_\chi$ is of full measure w.r.t. every invariant probability measure carried by $\chi-$summ. In [BO18], and in [Sar13], the authors construct a Markov partition and show that it covers a smaller set than HWT$^\epsilon_\chi$. Theorem 3.3 shows that the same construction in [BO18] can be done to cover all of HWT$^\epsilon_\chi$ from Definition 2.8.

Claim 2.12. $\forall \epsilon > 0$ and $\epsilon' \geq \frac{2}{3}\epsilon$, HWT$^\epsilon_\chi \subseteq$ HWT$^{\epsilon'}_\chi$.

Proof. Let $x \in$ HWT$^\epsilon_\chi$, and let $q : \{f^n(x)\}_{n \in \mathbb{Z}} \to (0, \epsilon) \cap \{e^{-\frac{2}{3}\epsilon}\}_{\ell \geq 0}$ be given by the $\epsilon$-weak temperability of $x$. Define $\tilde{q}(f^n(x)) := \max\{t \in \{e^{-\frac{2}{3}\epsilon}\}_{\ell \geq 0} : t \leq q(f^n(x))\}$. It follows that $\tilde{q}(f^n(x)) = e\pm(\epsilon + \frac{2}{3}\epsilon) = e\pm\epsilon'$. It follows from Definition 2.8 that $\exists \tilde{b}(\epsilon, \epsilon') > 0$ s.t. $\forall n \in \mathbb{Z}$, $\tilde{b}(\epsilon, \epsilon') \cdot Q_{\epsilon'}(f^n(x)) \leq Q_{\epsilon'}(f^n(x))$. Let $b(\epsilon, \epsilon') := \max\{t \in \{e^{-\frac{2}{3}\epsilon}\}_{\ell \geq 0} : t \leq \tilde{b}(\epsilon, \epsilon')\}$, and define $q'(f^n(x)) := b(\epsilon, \epsilon') \cdot \tilde{q}(f^n(x))$, $n \in \mathbb{Z}$. Since $\{e^{-\frac{2}{3}\epsilon}\}_{\ell \geq 0}$ is closed under multiplication, it follows that $q'$ satisfies the assumptions of $\epsilon'$-weak temperability for $x$, and so $x \in$ HWT$^{\epsilon'}_\chi$. □

3. Symbolic Dynamics

We now present some changes to the construction of $\epsilon_\chi$, $\chi$, and in Theorem 3.3, we will show that this does not affect the statements of [BO18]. On the other hand, these changes will allow us to construct the symbolic dynamics in such a way that we could characterize the image of $\Sigma^u_\chi$ (for every $\epsilon > 0$ small enough).

Assume that there exists a $\chi$-hyperbolic periodic point $p$, so that $\exists \epsilon_\chi > 0$ as in Theorem 2.7.

Definition 3.1.

(1) $\epsilon_\chi := \min\{\epsilon_2, \epsilon_\chi\} > 0$.
(2) $\check{\mathcal{V}}_\chi, \epsilon \in (0, \epsilon_\chi]$ is a collection of double Pesin-charts in the set $\check{A}_\epsilon$, where $\check{A}_\epsilon$ is constructed in the Coarse Graining process for $\chi-$summ $\cap$ e.w.t (instead of Pesin-charts with centers in NUH$^*_\chi$, see [BO18 Proposition 2.22, Sar13 Proposition 3.5]).
(3) A vertex $v \in \check{\mathcal{V}}_\epsilon'$ is called $\epsilon'$-relevant, if $\exists u \in \check{\Sigma}_\epsilon \cap [v]$ s.t. $\pi(u) \in$ HWT$^{\epsilon'}_\chi$ (instead of the previous definition, [BO18 Definition 3.14]).
(4) $\check{E}_\epsilon \subseteq \check{\mathcal{V}}_\epsilon \times \check{\mathcal{V}}_\epsilon'$ is a set of edges, characterized by the same $\epsilon$-overlap condition with no change (see [BO18 § 3.0.2,Definition 2.23,Definition 2.18]).
(5) $\check{G}_\epsilon = (\check{\mathcal{V}}_\epsilon, \check{E}_\epsilon)$ is a countable locally-finite directed graph (the local-finiteness of $\check{G}_\epsilon$ follows from the discreteness of $\check{\mathcal{V}}_\epsilon$, see footnote 3).
(6) $\check{\Sigma}_\epsilon^z := \{u \in \check{\mathcal{V}}_\epsilon^z : (u_i,u_{i+1}) \in \check{E}_\epsilon, \forall i \in \mathbb{Z}\}$ is the topological Markov shift induced by $\check{G}_\epsilon$.
(7) $\check{\Sigma}_\epsilon^x := \{u \in \check{\Sigma}_\epsilon : \exists n_k,m_k \uparrow \infty$ s.t. $u_{n_k} = u_{m_k} = u_{-m_k} = u_{-n_k}, \forall k \geq 0\}$.

Without loss of generality, we dismiss all vertex in $\check{\mathcal{V}}_\epsilon'$ which are not $\epsilon'$-relevant, and assume that every $v \in \check{\mathcal{V}}_\epsilon$ is $\epsilon'$-relevant.

Lemma 3.2. Let $\epsilon \in (0, \epsilon_\chi]$, and let $\bar{u} \in \check{\Sigma}^x_\epsilon$ be an admissible chain of double Pesin-charts. Then $p := \pi(\bar{u}) \in \chi-$summ.
Proof. Write $u_i = \psi_{x,i}^r, i \in \mathbb{Z}$. For each $i \in \mathbb{Z}, x_i \in \chi-$summm, whence, in particular, $x_i \in r-$hyp \( \forall r \in [\frac{1}{2}, \chi) \). By the definition of $\chi$, since $\epsilon \leq \epsilon_* \chi$, $Q_r(x_i)$ is small enough for the Graph Transform with the Lyapunov change of coordinate of $C_r(\epsilon)$, \( \forall r \in [\frac{1}{2}, \chi) \) (see [BO18 Theorem 3.6 § 2.1.2]). We prove that \( \sup_{\xi \in \mathbb{T} \nu_r^*} \sum_{|\xi_i| = 1} \sum_{m=0}^{\infty} |d_{f}f^m\xi_i|^2 \chi^m < \infty \), the case for $H^u(p)$ is similar. W.l.o.g., $u_{n_k} = u_0$ for $n_k \uparrow \infty$.

Step 1: By the relevance of $u_0$, take some chain $\mathbb{w} \in \Sigma \cap [u_0]$ s.t. $\zeta := \pi(\mathbb{w}) \in \text{HWT}_r^c = \chi-$summm \( \cap \epsilon-$w.t., whence in $r-$hyp \( \forall r \in [\frac{1}{2}, \chi) \). In the proof of [BO18 Lemma 4.5], temperability can be replaced by $\epsilon$-weak temperability (for $0 < \epsilon \leq \epsilon_*$), a more relaxed assumption. It follows then, that \( \forall r \in [\frac{1}{2}, \chi) \) \( \exists C = C(\zeta, \frac{1}{2}, r) \), s.t. \( \forall y \in V^s(\mathbb{w}) \), $\sup_{\xi \in \mathbb{T} \nu_r^*} \sum_{|\xi_i| = 1} \sum_{m=0}^{\infty} |d_{f}f^m\xi_i|^2 \chi^m \leq C < \infty$. Now, since \( r - \frac{\chi - r}{2} \geq \frac{1}{2} \), we are free to use [BO18 Lemma 4.6] and claim 2 in [BO18 Lemma 4.7], and get

\[
\sup_{\xi \in \mathbb{T} \nu_r^*} \sum_{|\xi_i| = 1} \sum_{m=0}^{\infty} |d_{f}f^m\xi_i|^2 \chi^m \leq \epsilon^{\frac{1}{m}} \quad \text{and} \quad \sup_{\xi \in \mathbb{T} \nu_r^*} \sum_{|\xi_i| = 1} \sum_{m=0}^{\infty} |d_{x_0}f^m\xi_i|^2 \chi^m \leq \frac{5}{4} \left| C_{\chi}^{-1}(x_0) \right|^2.
\]

The last inequality is true since $t \mapsto \left| C^{-1}_t(x) \right|$ is monotonous for every $x \in \chi-$summm. Thus, $p \in r-$hyp \( \forall r' < \chi \).

Step 2: Fix $\xi \in \mathbb{T} \nu_r^*(\mathbb{w}), |\xi| = 1$. If $\sum_{m=0}^{\infty} |d_{f}f^m\xi_i|^2 \chi^m \geq \sqrt{2} \left| C_{\chi}^{-1}(x_0) \right|^2$, then choose $N > 0$ s.t. \( \sum_{m=0}^{N} |d_{f}f^m\xi_i|^2 \chi^m \geq \frac{1}{4} \left| C_{\chi}^{-1}(x_0) \right|^2 \). Choose $r' < \chi$ s.t. $\sum_{m=0}^{\infty} |d_{f}f^m\xi_i|^2 \chi^m \geq \frac{1}{4} \left| C_{\chi}^{-1}(x_0) \right|^2$, whence \( \sum_{m=0}^{\infty} |d_{f}f^m\xi_i|^2 \chi^m \leq \frac{1}{8} \left| C_{\chi}^{-1}(x_0) \right|^2 \), a contradiction to equation [2] from step 1! Therefore,

\[
\forall \xi \in \mathbb{T} \nu_r^*(\mathbb{w}), |\xi| = 1, \quad \sum_{m=0}^{\infty} |d_{f}f^m\xi_i|^2 \chi^m \leq \sqrt{2} \left| C_{\chi}^{-1}(x_0) \right|^2,
\]

and similarly with $\mathbb{T} \nu_r^*(\mathbb{w})$; and so $p \in \chi-$summm. \( \square \)

**Theorem 3.3.** \( \forall \epsilon \in (0, \epsilon_*], \pi[\Sigma^*_\epsilon] = \text{HWT}_r^c \).

In [BO18 Definition 2.17] (and similarly in [Sar13 § 2.5] when $d = 2$) the authors offer a set which is covered by the constructed Markov partition $\mathcal{R}$, and denote this set by $NUH^*_\epsilon$. The definition of $NUH^*_\epsilon$ involves Lyapunov regularity, and we claim that this was unnecessary, when wishing to work with conservative (perhaps infinite) measures. In Definition 2.11 we introduce a definition for a more inculsive set $\text{HWT}_r^c$, which does not depend on Lyapunov regularity. We show in the proof below that $\text{HWT}_r^c$ still admits the property of being coded by a Markov partition; but in fact it carries additional natural properties which we will see later (e.g. Corollary 3.3 Proposition 3.10).

**Proof.** Let $\epsilon \in (0, \epsilon_*]$. The following steps are done both in [BO18], and in [Sar13] when $d = 2$. We follow each step and give references to [BO18] for the case $d \geq 2$, although these references are analogues to the preceding work in [Sar13].

Step 1: In [BO18 Definition 2.10], the authors introduce $NUH^*_\epsilon$, a set of Lyapunov regular $\chi$-hyperbolic points, such that $\forall x \in NUH^*_\epsilon, \lim_{n \to \pm \infty} \frac{1}{n} \log \left| \mathcal{C}^{-1}_\epsilon(f^n(x)) \right| = 0$ (BO18 Claim 2.11). This allows us to define Pesin’s tempered kernel $\forall x \in NUH^*_\epsilon, q_\epsilon(x) := \frac{1}{\sum_{x \in \mathcal{A}_\epsilon} q_\epsilon(x)}$. This allows us to define $NUH^*_\epsilon := \{ x \in NUH^*_\epsilon : \lim_{n \to \pm \infty} q_\epsilon(f^n(x)) = 0 \}$ as well ([BO18 Definition 2.17]).

Step 2: By [BO18 Proposition 2.22], there exists a discrete and sufficient collection of Pesin-charts with centers in $NUH^*_\epsilon, \mathcal{A}$, as in Definition 3.11. $\mathcal{A}$ is constructed similarly, over a collection of Pesin-charts with centers in $\text{HWT}_r^c\chi$.

Step 3: $\text{HWT}_r^c \subseteq \epsilon-$w.t., and so an analogue to Pesin’s tempered kernel exists by definition $\forall x \in \text{HWT}_r^c\chi$. Consequently, as in [BO18 Proposition 2.30], $\pi[\Sigma^*_\epsilon] \supseteq \text{HWT}_r^c \chi$. In addition, for [BO18 Theorem 3.13(3), Proposition 3.16], $\pi[\Sigma^*_\epsilon] \supseteq \text{HWT}_r^c \chi$.

Step 4: Let $\underline{\mathbb{w}} \in \Sigma^*_\epsilon$, and write $\zeta := \pi(\underline{\mathbb{w}})$. W.l.o.g., write $u_{n_k} = u_0, \forall k \geq 0$, and $u_n \uparrow \infty$. Consider the finite periodic word $\underline{\mathbb{w}} = (u_0, u_1, \ldots, u_{n_1-1}, u_0)$, and associate it with its periodic extension to a chain.
Consider the chains $u^{(l)} \in \hat{\Sigma}^e$, $l \geq 0$, where $u^{(l)}_i = \{ u_i, u_{i-n_l} \}$ for $i \leq n_l$, $u^{(l)}_i = \{ u_{i-n_l}, u_i \}$ for $i \geq n_l$. Write $z_l := \pi(u^{(l)})$. In [BO18 Lemma 4.7], the author uses [BO18 Lemma 4.5] in order to show that $\sup_{t \geq 0} \sup_{x \in T_t V^u(u)} |\xi| |z(t, \xi) < \infty$; equation 23 in Lemma 3.2 takes the place of [BO18 Lemma 4.5] in this argument. It follows that [BO18 Lemma 4.6,Lemma 4.7(Claim 2)] can be carried out verbatim, and so $\exists$ a linear invertible map $\pi_{x_0}^s : T_1 V^s(u) \to H^s(x_0)$ s.t. $\|\pi_{x_0}^s, \| (\pi_{x_0}^s)^{-1} \| \leq e^{2q}, x_0 \|^{\frac{1}{2}}$, and

$$\forall \xi \in T_1 V^u(u), |\xi| = 1, S(z, \xi) = e^{\pm \sqrt{\tau}} S(x_0, \pi_{x_0}^s \xi).$$

A similar statement holds for $\pi_{x_0}^u : T_1 V^u(u) \to H^u(x_0)$. $\pi_{x_0}^s$ and $\pi_{x_0}^u$ extend to the invertible linear map $\pi_{x_0}^s : T_1 M \to T_{x_0} M$ by $\pi_{x_0}^s |T_1 V^u(u) = \pi_{x_0}^s$ and $\pi_{x_0}^u |T_1 V^u(u) = \pi_{x_0}^u$. In particular, $z \in \chi$-sum. It then follows from the proof of [BO18 Proposition 4.8] (though not specified) that $\|C^{-1}(e)\| = e^{\pm(4\sqrt{\tau} + \epsilon)} = e^{\pm \sqrt{\tau}}$.

Step 5: By Lemma 3.2 $\forall u \in \hat{\Sigma}^e$, $\pi(u) \in \chi$-sum. In addition, $q(f^n(u)) = b_n \cdot p_n^u \cdot p_n^s$, where $b_n := \max\{t \in \{ e^{-\frac{1}{\sqrt{\tau}}} \} \} \leq e \cdot 200 \cdot \pi$ and $u_n = \psi^{p_n^u \cdot p_n^s}$. By the definition of $\hat{A}_n$ (and so $\hat{V}_e$), $p_n^u \cdot p_n^s \in \{ e^{-\frac{1}{\sqrt{\tau}}} \} \leq 0$, $\forall n \in Z$; thus, since $\{ e^{-\frac{1}{\sqrt{\tau}}} \} \leq 0$ is closed under multiplication, $q : \{ f^n(u) \}_{n \in Z} \to (0, e) \cap \{ e^{-\frac{1}{\sqrt{\tau}}} \}$. $q$ satisfies the assumptions of $e$-weak temperability:

1. By the $e$-overlap condition, [BO18 Definition 2.18], $q \cdot f^l(q) = e^{\pm \epsilon}$.
2. Since $u \in \hat{\Sigma}^e$, $\lim_{n \to 0 \pm \infty} q \cdot f^l(x) > 0$.
3. By the definition of double Pesin-charts, $q(f^l(u)) \leq Q_e(x_l), \forall i \in Z$. Thus, $\forall n \in Z$.

By step 4 (recall Definition 2.8 for definition of $Q_e$),

$$p_n^u \cdot p_n^s \leq Q_e(x_n) \leq e^{\frac{200}{3}} \cdot \frac{\|C^{-1}(x_n)\|^{\frac{1}{2}}}{\|C^{-1}(f^n(u))\|^{\frac{1}{2}}} \leq e^{\pm \sqrt{\tau} \cdot 48} \cdot Q_e(f^n(u)) \cdot e^{\pm \epsilon} \leq e^{\frac{300 \cdot \pi}{\sqrt{\tau}}} \cdot Q_e(f^n(u)).$$

Therefore $q(f^n(u)) \leq Q_e(f^n(u))$.

It follows that $\pi(u) \in \chi$-sum $\cap e$-w.t. $W = \text{HWT}_e[\chi]$. Thus, $\pi(\hat{\Sigma}^e) \subseteq \text{HWT}_e[\chi]$, and together with step 3, $\pi(\hat{\Sigma}^e) = \text{HWT}_e[\chi]$.

$\square$

**Corollary 3.4.** $\forall \epsilon \in (0, \epsilon), \forall u \in \hat{V}_e$, $\# \{ v \in \hat{V}_e : \hat{Z}_e(u) \cap \hat{Z}_e(v) \subset \emptyset \} < \infty$, where $\hat{Z}_e(v) := \pi(\hat{\Sigma}_e \cap [v])$.

This is the content of [BO18 Theorem 5.2] (and similarly [Sar13 Theorem 10.2] when $d = 2$), where (as in step 4 in Theorem 3.3 Lemma 3.2 replaces [BO18 Lemma 4.5] in the proof [BO18 Lemma 4.7]; and the rest of the Inverse Problem ([BO18 §4]) can be carried out verbatim with $\hat{\Sigma}_e$ replacing $\Sigma$.

Assume that there exists a periodic point $p \in \chi$-hyp, and let $\epsilon \in (0, \epsilon)$. Let $\epsilon \in (0, \epsilon)$.

**Definition 3.5.**

1. $\forall u \in \hat{V}_e$, $\hat{Z}_e(u) = \{ [u] \cap \hat{\Sigma}_e \}$, $\hat{Z}_e := \{ \hat{Z}_e(u) : u \in \hat{V}_e \}$.
2. $\hat{R}_e$ is a countable partition of $\bigcup_{v \in \hat{V}_e} \hat{Z}_e(v) = \pi(\hat{\Sigma}_e)$, s.t.
   a. $\hat{R}_e$ is a refinement of $\hat{Z}_e$; $\forall Z \in \hat{Z}_e, \hat{R} \subseteq \hat{Z}_e(v) \subset \hat{Z}_e$.
   b. $\forall v \in \hat{V}_e$, $\# \{ R \in \hat{R}_e : R \subseteq \hat{Z}_e(v) \} < \infty$ (Sar13 §11).
   c. The Markov property: $\forall R \in \hat{R}_e \forall x, y \in R \exists z := [x,y] \in R \cap [y,z], \forall i \geq 0, R(f^i(z)) = R(f^i(y)), R(f^{-i}(z)) = R(f^{-i}(x))$, where $R(t)$ is the unique partition member of $\hat{R}_e$ containing $t$, for $t \in \pi(\hat{\Sigma}_e)$.
3. $\forall R, S \in \hat{R}_e$, we say $R \rightarrow S$ if $R \cap f^{-1}[S] \neq \emptyset$, i.e. $\hat{R}_e = \{ (R,S) \in \hat{R}_e^2 : f^{-1}[S] \cap R \neq \emptyset \}$.
4. $\hat{\Sigma}_e := \{ R \in \hat{R}_e^Z : R_l \rightarrow R_{l+1}, \forall i \in Z \}$. 

7
Given $Z$, such a refining partition as $\mathcal{R}$ exists by the Bowen-Sinai refinement, see [Sar13 § 11.1].

Definition 3.6.

(1) $\hat{\Sigma}^e := \{R \in \hat{\Sigma} : \exists n_k, m_k \uparrow \infty \text{ s.t. } R_{n_k} = R_{m_k}, R_{-m_k} = R_{-n_k}, \forall k \geq 0\}.$

(2) Every two partition members $R, S \in \mathcal{R}_e$ are said to be $e$-affiliated if $\exists u, v \in \hat{V}$ s.t. $R \subseteq \hat{Z}(u), S \subseteq \hat{Z}(v)$ and $\hat{Z}(u) \cap \hat{Z}(v) \neq \emptyset$ (this is due to O. Sarig, [Sar13 § 12.3]).

Remark: By Corollary 3.4 and Definition 3.5(2)(b), it follows that every partition member of $\mathcal{R}_e$ has only a finite number of partition members $e$-affiliated to it.

Theorem 3.7. Given $\hat{\Sigma}$ from Definition 3.5, there exists a factor map $\hat{\pi} : \hat{\Sigma} \to M$ s.t.

(1) $\hat{\pi}$ is Hölder continuous w.r.t the metric $d(R, S) = \exp(-\min\{i \geq 0 : R_i \neq S_i \text{ or } R_{-i} \neq S_{-i}\}).$

(2) $f \circ \hat{\pi} = \hat{\pi} \circ \sigma$, where $\sigma$ denotes the left-shift on $\hat{\Sigma}$.

(3) $\hat{\pi}|_{\hat{\Sigma}^e}$ is finite-to-one.

(4) $\forall R \in \hat{\Sigma}_e, \hat{\pi}(R) \in \mathcal{R}_0$.

(5) $\hat{\pi}|_{\hat{\Sigma}^e}$ carries all $\chi$-hyperbolic invariant probability measures.

This theorem is the content of the main theorem of [BO18], Theorem 1.1 (and similarly the content of [Sar13 Theorem 1.3] when $d = 2$).

Proposition 3.8. $\forall \epsilon \in (0, \epsilon_X],$

$$\hat{\pi}[\hat{\Sigma}^e] = \hat{\pi}[\hat{\Sigma}^e] = \bigcup \mathcal{R}_e.$$  

Proof. $\pi[\hat{\Sigma}^e] = \bigcup \mathcal{R}_e$ by definition, since $\mathcal{R}_e$ is a partition of $\pi[\hat{\Sigma}^e]$. We need to show $\hat{\pi}[\hat{\Sigma}^e] = \hat{\pi}[\hat{\Sigma}^e]$.

$\supseteq$: Let $u \in \hat{\Sigma}^e$, $\pi(u) \in \mathcal{R}_e$. Write $R_i := \text{unique element of } \mathcal{R}_e$ which contains $f^i(\pi(u)), i \in \mathbb{Z}$. It follows that $R = (R_i)_{i \in \mathbb{Z}} \in \hat{\Sigma}_e$, and that $\hat{\pi}(R) = \pi(u)$ by the uniqueness of a shadowed orbit. Then by definition, $\forall i \in \mathbb{Z}, R_i \subseteq \hat{Z}(u_i)$, and so by the pigeonhole principle, $R \in \hat{\Sigma}^e$ (see Definition 3.5(b)).

$\subseteq$: Let $R, S \in \mathcal{R}_e$ s.t. $\exists x \in R \cap f^{-1}[S]$. Let $u \in \hat{V}$ s.t. $R \subseteq \hat{Z}(u)$. Then $\exists u \in \hat{\Sigma}^e \cap [u]$ s.t. $\pi(u) = x$, and so $S \subseteq \hat{Z}(u)$, by the uniqueness of a shadowed orbit, $\pi(u) = \hat{\pi}(R)$. By Corollary 3.4 and the pigeonhole principle, $u \in \hat{\Sigma}^e$.

Corollary 3.9. Let $p$ be a $\chi$-hyperbolic periodic point, such that $\text{HWT}_\chi^r \neq \emptyset$. Then $\forall \epsilon \in (0, \epsilon_X],$

$$\hat{\pi}[\hat{\Sigma}^e] = \hat{\pi}[\hat{\Sigma}^e] = \text{HWT}_\chi^r.$$  

Proof. In Theorem 3.7 we saw $\pi[\hat{\Sigma}^e] = \text{HWT}_\chi^r$. In Proposition 3.8 we showed the quick argument for the equality $\hat{\pi}[\hat{\Sigma}^e] = \hat{\pi}[\hat{\Sigma}^e]$. Therefore we are done.

Definition 3.10. $\text{HWT}_\chi^r := \text{HWT}_\chi^r$ is called the recurrently-codable points.

Notice, by Claim 2.12 $\bigcup_{0 < \epsilon \leq \epsilon_X} \text{HWT}_\chi^r \subseteq \text{HWT}_\chi^r$.

4. Ergodic Homoclinic Classes and Maximal Irreducible Components

In this section $\epsilon$ is fixed and equals $\epsilon_X$. The $\epsilon$ subscript of $\hat{\Sigma}_e, \hat{\Sigma}^e, \mathcal{R}_e, \Sigma_e, \Sigma^e$ will be omitted to ease notation.

Let $p$ be a periodic point in $\chi-$summ. Since $p$ is periodic, $||C^{-1}()||$ is bounded along the orbit of $p$, and therefore $p \in \text{HWT}_\chi^r$. Every point $x \in \text{HWT}_\chi^r$ is (recurrently-)codable, and so has a local stable manifold $V^s(x)$ (e.g. $V^s(u), u \in \pi^{-1}([x]) \cap \Sigma^e$), and a global stable manifold $W^s(x) := \bigcup_{n \geq 0} f^{-n}[V^s(f^n(x))]$ (similarly for a global unstable manifold).
Definition 4.1. The ergodic homoclinic class of $p$ is

$$H(p) := \{x \in \text{HWT}_\chi^* : W^u(x) \cap W^s(o(p)) \neq \emptyset, W^s(x) \cap W^u(o(p)) \neq \emptyset\},$$

where $\cap$ denotes transverse intersections of full codimension, $o(p)$ is the (finite) orbit of $p$, and $W^s(\cdot), W^u(\cdot)$ are the global stable and unstable manifolds of a point (or points in an orbit), respectively.

This notion was introduced in [RHRHTU11], with a set of Lyapunov regular points replacing HWT$^*_\chi$. Every ergodic conservative $\chi$-hyperbolic measure, is carried by an ergodic homoclinic class of some periodic hyperbolic point.

Definition 4.2. (1) Define $\sim \subseteq \mathcal{R} \times \mathcal{R}$ by $R \sim S \iff \exists n_{RS}, n_{SR} \in \mathbb{N}$ s.t. $R \xrightarrow{n_{RS}} S, S \xrightarrow{n_{SR}} R$, i.e. there is a path of length $n_{RS}$ connecting $R$ to $S$, and a path of length $n_{SR}$ connecting $S$ to $R$. The relation $\sim$ is transitive and symmetric. When restricted to $\{R \in \mathcal{R} : R \sim R\}$, it is also reflexive, and thus an equivalence relation. Denote the corresponding equivalence class of some representative $R \in \mathcal{R}$, $R \sim R$ by $(R)$.

(2) A maximal irreducible component in $\hat{\Sigma}$, corresponding to $R \in \mathcal{R}$ s.t. $R \sim R$, is $\{R \in \hat{\Sigma} : R \in (R)\}$. 

Lemma 4.3. Let $p \in \chi$–summ $\mathbb{S}$ t. $\exists \in \mathbb{N}$ s.t. $f^l(p) = p$, then $p \in \chi$–hyp.

Proof. We prove an exponential contraction strictly stronger than $e^{-\chi}$ on $H^s(p)$. The case for $H^u(p)$ is similar. First assume that $f(p) = p$. Since $p$ is $\chi$-summable, $\forall \xi \in H^s(p)$ with $|\xi| = 1$, $\sum_{m=0}^{\infty} |d_pf^m\xi|^2 e^{2\chi m} < \infty$. Let $\{\xi_i^{(m)}\}_{i=1}^{s(m)}$ be an orthonormal basis for $H^s(p)$ (w.r.t $\langle \cdot, \cdot \rangle_p$, the Riemannian form at $T_pM$). For all $m \geq 0$, $\exists \xi^{(m)} \in H^s(p)$ with $|\xi^{(m)}| = 1$ s.t. $|d_pf^m\xi^{(m)}| = ||d_pf^m||_{H^s(p)}\|$. Whence, for $a_i^{(m)} := (\xi^{(m)}, \xi)_p$,

$$\sum_{m=0}^{\infty} ||d_pf^m||_{H^s(p)}^2 e^{2\chi m} = \sum_{m=0}^{\infty} |d_pf^m\xi^{(m)}|^2 e^{2\chi m} \leq \sum_{m=0}^{\infty} \left( \sum_{i=1}^{s(p)} |a_i^{(m)}| \cdot |d_pf^m\xi_i^{(m)}| \right)^2 e^{2\chi m} \leq \sum_{i=1}^{s(p)} |d_pf^m\xi_i^{(m)}|^2 e^{2\chi m},$$

where the third line is by the Cauchy-Schwarz inequality, and the second line is since $|a_i^{(m)}| \leq 1$, $i \leq s(p)$ (by the Cauchy-Schwarz inequality as well). $d_pf : H^s(p) \to H^s(p)$ is a linear map, then by working in coordinates, it is sufficient to assume w.l.o.g. that $d_pf$ is of the form of a Jordan block $J_{s(p)}(\lambda)$ for some $\lambda \in \mathbb{R} \setminus \{0\}$ (since $f$ is a diffeomorphism). Let $N$ denote the nilpotent matrix whose superdiagonal (entries right above the diagonal) is all ones, and all other entries are zero, and $I$ denotes the identity matrix. Then $N^{s(p)} = 0$, and trivially $N$ and $\lambda I$ commute. Then $J_{s(p)}(\lambda) = \lambda I + N$, and by the binomial theorem (w.l.o.g. $m \geq s(p))$,

$$J_{s(p)}(\lambda)^m = (\lambda I + N)^m = \sum_{k=0}^{s(p)} \binom{m}{k} \lambda^{m-k} N^k.$$ 

(5)

Since $J_{s(p)}(\lambda)$ is a Jordan block, it admits an eigenvector with an eigenvalue $\lambda$, and so $||J_{s(p)}(\lambda)|| \geq |\lambda|$, and $||d_pf^m||_{H^s(p)} \| \geq |\lambda|^m$. Hence, by equation (4), $\sum_{m=0}^{\infty} |\lambda|^m e^{2\chi m} \leq \sum_{m=0}^{\infty} \|d_pf^m\|_{H^s(p)}^2 e^{2\chi m} < \infty$, and so $0 < |\lambda| < e^{-\chi}$. On the other hand, by equation (5), $\|d_pf^m\|_{H^s(p)} \| \leq C \cdot m^{s(p)} \cdot |\lambda|^m$ for some $C = C(p) > 0$. Whence, $\|d_pf^m\|_{H^s(p)} \| \leq C \cdot m^{s(p)} \cdot e^{-\chi m}$, where $\chi' := \log|\lambda| < -\chi$. Thus, $\limsup_{m \to \infty} \frac{1}{m} \log \|d_pf^m\|_{H^s(p)} \| < -\chi$. This concludes the proof for the case $f(p) = p$. 

9
In the case where the period of $p$ is $l > 1$, write $\forall \xi \in \mathcal{H}^s(p)$,

$$
\sum_{m=0}^{\infty} |d_p f^m \xi|^2 e^{2\gamma m} = \sum_{i=0}^{l-1} e^{2\gamma} \sum_{m=0}^{\infty} |d_p f^m (f^i \xi)|^2 e^{2(\gamma_m)m} < \infty.
$$

Then by the first part of this proof, $\exists \lambda_i$, $i = 0, \ldots, l-1$ s.t. $0 < |\lambda| < e^{-\gamma}$ and $C_i = C_i(p) > 0$ s.t. $\forall 0 \leq i \leq l-1, l \geq 0, \|d_p f^{m+i}|_{\mathcal{H}^s(p)}\| \leq \max \{C_i\} \max \{|\lambda|^m\} \cdot m^a(p)$. As in the case of $f(p) = p$, this is sufficient.

The following two definitions are due to Sarig in [Sar13, § 4.2, Definition 4.8] (the version here corresponds to the case $d \geq 2$ from [BOT8, Definition 3.1, Definition 3.2]).

**Definition 4.4.** Let $x \in \text{HWT}^s_\chi$, a $u$–manifold in $\psi_x$ is a manifold $V^u \subset M$ of the form

$$
V^u = \psi_x[\{(F^u_1(t_s(x)+1, \ldots, t_d), \ldots, F^u_s(t_s(x)+1, \ldots, t_d), t_s(x)+1, \ldots t_d) : |t_i| \leq q\}],
$$

where $0 < q \leq Q_\epsilon(x)$, and $\widehat{F}^u$ is a $C^{1+\beta/3}$ function s.t. $\max_{\mathcal{R}_q(0)} |\widehat{F}^u|_\infty \leq Q_\epsilon(x)$.

Similarly we define an $s$–manifold in $\psi_x$:

$$
V^s = \psi_x[\{(t_1, \ldots, t_s), F^s_{r_s}(t_1, \ldots, t_s), \ldots, F^s_{r_s}(t_1, \ldots, t_s) : |t_i| \leq q\}],
$$

with the same requirements for $\widehat{F}^s$ and $q$. We will use the superscript “$u/s$” in statements which apply to both the $u$ case and the $s$ case. The function $\widehat{F} = \widehat{F}^{u/s}$ is called the representing function of $V^{u/s}$ at $\psi_x$.

The parameters of a $u/s$ manifold in $\psi_x$ are:

- **$\sigma$–parameter:** $\sigma(V^{u/s}) := \|d_p \widehat{F}^u\|_{\beta/3} := \max_{\mathcal{R}_q(0)} \|d_p \widehat{F}^u\| + \text{H"{o}l}_{\beta/3}(d_p \widehat{F}^u),$

  where $\text{H"{o}l}_{\beta/3}(d_p \widehat{F}^u) := \max_{t_i,t_j \in \mathcal{R}_q(0)} \|d_p \widehat{F}^u - d_p \widehat{F}^u\| / |t_i-t_j|^{\beta/3}$ and $\|A\| := \sup_{v \neq 0} |A_v|_\infty$.

- **$\gamma$–parameter:** $\gamma(V^{u/s}) := \|d_p \widehat{F}^u\|$

- **$\varphi$–parameter:** $\varphi(V^{u/s}) := |\widehat{F}^u(0)|_\infty$

- **$q$–parameter:** $q(V^{u/s}) := q$

A $(u/s, \sigma, \gamma, \varphi, q)$–manifold in $\psi_x$ is a $u/s$ manifold $V^{u/s}$ in $\psi_x$ whose parameters satisfy $\sigma(V^{u/s}) \leq \sigma, \gamma(V^{u/s}) \leq \gamma, \varphi(V^{u/s}) \leq \varphi, q(V^{u/s}) \leq q$.

Notice that the dimensions of an $s$ or a $u$ manifold in $\psi_x$ depend on $x$. Their sum is $d$.

**Definition 4.5.** Suppose $x \in \text{HWT}^s_\chi$ and $0 < p^u, p^s \leq Q_\epsilon(x)$ (i.e. $\psi_x^{p^u,p^s}$ is a double Pesin-chart). A $u/s$-admissible manifold in $\psi_x^{p^u,p^s}$ is a $(u/s, \sigma, \gamma, \varphi, q)$–manifold in $\psi_x$ s.t.

$$
\sigma \leq \frac{1}{2}, \gamma \leq \frac{1}{2}(p^s \wedge p^u)^{\beta/3}, \varphi \leq 10^{-3}(p^s \wedge p^u), \text{ and } q = \left\{ \begin{array}{ll} p^u, & \text{u – manifolds} \\ p^s, & \text{s – manifolds} \end{array} \right. .
$$

Recall: $\forall \mathcal{U} \in \Sigma$ there exists a local stable manifold for $\pi(\mathcal{U}), V^s(\mathcal{U}) = V^s((u_i)_{i \geq 0})$, and a local unstable manifold for $\pi(\mathcal{U}), V^u(\mathcal{U}) = V^u((u_i)_{i \leq 0})$ (see [BOT8, Proposition 3.12, Proposition 4.4]; or [Sar13, Proposition 4.15, Proposition 6.3] in the case $d = 2$). $V^s(\mathcal{U}), V^u(\mathcal{U})$ are admissible manifolds in $u_0 = \psi_x^{p_0^{u},p_0^{s}}$.

**Definition 4.6.** (This definition was introduced in [Sar13, Lemma 4.6], and is due to F. Ledrappier) $\forall x \in \text{HWT}^s_\chi$,

$$
p^u_n(x) := \max\{t \in \{e^{-\delta N}\}_{\ell \geq 0} : e^{-\epsilon N}t \leq Q_\epsilon(f^{-N}(x)), \forall N \geq 0\},
$$

$$
p^s_n(x) := \max\{t \in \{e^{-\delta N}\}_{\ell \geq 0} : e^{-\epsilon N}t \leq Q_\epsilon(f^{N}(x)), \forall N \geq 0\}.
$$

Note: the chain $\{\psi_{p_0^{u,n}(x)}, p_0^{s,n}(x))_{n \in \mathbb{Z}}$ is admissible (see [BOT8] Definition 2.23 for the conditions for an edge between two double Pesin-charts).
Lemma 4.7. Let \( p \) be a \( \chi \)-hyperbolic periodic point, and let \( u \) be the admissible (periodic) chain \( \{\psi_{f^n(p)}^p\}_{n \in \mathbb{Z}} \). Let \( x \in V^u(u) \cap \text{HWT}_x^\chi \) s.t. \( \dim H^s(x) = \dim H^s(p) \). Then

\[
\limsup_{n \to \infty} \sup_{\xi_n \in H^s(f^{-n}(x))} |S(f^{-n}(x), \xi_n)| \leq \max \{ \sup_{\eta_i \in H^s(f^i(p)), |\eta_i| = 1} S(f^i(p), \eta_i) \} < \infty.
\]

A similar claim holds for \( x \in V^s(u) \cap \text{HWT}_x^\chi \).

Proof. Fix \( \delta \in (0, 1) \). Assume w.l.o.g. that \( f(p) = p \). By Definition \[\text{[\text{\text{BOS}}2]}\], if \( f(p) = p \), then \( \sigma u = u \). By the Inclination Lemma (\[\text{[\text{\text{BOS}}2]}\]), we may assume w.l.o.g. that \( d_C(V^s(f^{-i}(x)), V^s(u)) \leq \delta \) \( \forall i \geq 0 \), where \( V^s(f^{-i}(x)) \) is the part of \( W^s(f^{-i}(x)) \) which is close in \( C^1 \)-norm to \( V^s(u) \), and the \( C^1 \)-distance is calculated in the chart \( \psi_p^{Q^i(p)} \). In particular, since \( V^s(u) \) is an admissible manifold in \( \psi_p^{Q^i(p)} \), \( V^s(f^{-i}(x)) \) is the graph of a \( C^1 \)-smooth function. Denote the function representing the graph of \( V^s(u) \) by \( F \), and the function representing the graph of \( V^s(f^{-i}(x)) \) by \( G_i, i \geq 0 \). Let \( P_\delta : \mathbb{R}^d \to \mathbb{R}^{s(x)} \) be the projection to the \( s(x) \) first coordinates, and let \( \xi \in H^s(x) \) s.t. \( |d_{\xi}(f^{-i}\xi)| = 1 \). \( \xi = d_{\psi_p^{-i}(\psi_p^{-1}(\psi_p(x)) \psi_p(u, d_{\psi_p^{-i}(\psi_p(x))} G_0 u)} \) for some \( u \in \mathbb{R}^{s(p)} \).

Define \( \eta = \eta(\xi) := d_0 \psi_p(u, d_0 Fu) \in H^s(p) \). Write again \( d_x(f^{-1})\xi = d_{\psi_p^{-i}(\psi_p(x)) \psi_p(v, d_{\psi_p^{-i}(\psi_p(x))} G_1 v)} \) for some \( v \in \mathbb{R}^{s(x)} \), and define \( \zeta = \zeta(\xi) := d_0 \psi_p(v, d_0 Fv) \in H^s(p) \). Notice, \( \eta : T_x V^s(x) \to H^s(p) \) is a linear map, and in fact the definition extends naturally to \( \eta : T_{f^{-i}(x)} V^s(f^{-i}(x)) \to H^s(p), i \geq 0 \). Thus, by the Inclination Lemma, assume w.l.o.g. that \( \left| d_x(f^{-1})\xi \right|^2 - |d_p(f^{-1})\xi|^2 \right| \leq \delta \). Define \( \rho := \max \{ \frac{S(x, \xi)}{S(p, \eta)}, \frac{S(p, \eta)}{S(x, \xi)} \} \).

Step 1:

\[
S^2(f^{-1}(x), d_x(f^{-1})\xi) = 2 \sum_{m=0}^\infty |d_{f^{-1}(x)} f^m d_x(f^{-1})\xi|^2 e^{2\chi m} = S^2(x, \xi)e^{2\chi} + 2|d_x(f^{-1})\xi|^2 \\
\leq \rho^2 e^{2\chi} S^2(p, \eta) + 2|d_x(f^{-1})\xi|^2.
\]

Then,

\[
\frac{S^2(f^{-1}(x), d_x(f^{-1})\xi)}{S^2(p, d_p(f^{-1})\eta)} \leq \frac{\rho^2 e^{2\chi} S^2(p, \eta) + 2|d_x(f^{-1})\xi|^2}{S^2(p, d_p(f^{-1})\eta)} \\
= \rho^2 - \frac{2(\rho^2 - 1)|d_p(f^{-1})\eta|^2 + 2(|d_p(f^{-1})\eta|^2)^2 - |d_x(f^{-1})\xi|^2)}{S^2(p, d_p(f^{-1})\eta)}.
\]

Now, if \( \rho \geq e^{\sqrt{\delta}} \), then \( \rho^2 - 1 \geq 2\sqrt{\delta} \), and so \( (\rho^2 - 1)|d_p(f^{-1})\eta|^2 + (|d_p(f^{-1})\eta|^2)^2 - |d_x(f^{-1})\xi|^2 \geq (\rho^2 - 1)(1 - \delta) - \delta \geq (\rho^2 - 1)(1 - \delta) - (\rho^2 - 1)\frac{\delta}{2\sqrt{\delta}} \geq (\rho^2 - 1)(1 - \frac{\sqrt{\delta}}{2}) \geq (\rho^2 - 1)e^{-\sqrt{\delta}} \), for small \( \delta \in (0, 1) \). We then get that all together,

\[
\frac{S^2(f^{-1}(x), d_x(f^{-1})\xi)}{S^2(p, d_p(f^{-1})\eta)} \leq \rho^2 - \frac{2(\rho^2 - 1)e^{-2\sqrt{\delta}}}{S^2(p, d_p(f^{-1})\eta)} \leq \rho^2 - \frac{2(\rho^2 - 1)e^{-2\sqrt{\delta}}}{\|C_\chi^{-1}(p)\| \cdot |d_p(f^{-1})\eta|} \leq \rho^2 - \frac{2(\rho^2 - 1)e^{-2\sqrt{\delta}}}{\|C_\chi^{-1}(p)\| \cdot (1 + \delta)} \\
\leq \rho^2 \left( 1 - \frac{2(1 - \frac{1}{\rho^2})e^{-2\sqrt{\delta}}}{\|C_\chi^{-1}(p)\| e^{\delta}} \right) \leq \rho^2 \left( 1 - \frac{2(1 - \frac{1}{\rho^2})e^{-2\sqrt{\delta}}}{\|C_\chi^{-1}(p)\| e^{\delta}} \right) \leq \rho^2 \left( 1 - \frac{2\sqrt{\delta}e^{-2\sqrt{\delta}}}{\|C_\chi^{-1}(p)\| e^{\delta}} \right).\]

For \( \delta \in (0, 1) \) small enough so \( \frac{\sqrt{\delta}}{2} + \frac{1}{\rho^2} \geq 1 \),

\[
\frac{S^2(f^{-1}(x), d_x(f^{-1})\xi)}{S^2(p, d_p(f^{-1})\eta)} \leq \rho^2(1 - \delta) \leq \rho^2 e^{-2\delta}.
\]

Since both \( \zeta \) and \( \eta \) depend continuously on \( \xi \), and can be made arbitrarily close with \( \delta > 0 \) small enough, and since \( S^2(p, \cdot) \) is continuous (see \[\text{[\text{\text{BOS}}2]}\]), we may assume w.l.o.g. that \( \frac{S^2(p, \cdot)}{S^2(p, d_p(f^{-1})\eta)} \in [e^{-\delta}, e^{\delta}] \). Thus we get in total,

\[
\frac{S^2(f^{-1}(x), d_x(f^{-1})\xi)}{S^2(p, \xi)} \leq \rho^2 e^{-\delta}.
\]
Similarly one obtains \( \frac{S^2(f^{-n}(x), d_x(f^{-n}(x))}{S^2(p, \eta(d_x(f^{-n}(x))) \geq \rho^{-2e^\delta}. \)

Step 2: In step 1, we fixed \( \delta > 0 \), and made some assumptions without losing generality, that hold for all \( f^{-n}(x) \) for \( n \geq 0 \) large enough, and hence don’t affect the limit. Denote by \( n_0 \geq 0 \) the smallest backward-iteration of \( x \) to satisfy this way the choice of \( \delta \) (thus step 1 treats all \( f^{-n}(x) \) \( \forall n \geq n_0 \). That means that the ratio in equation (7) is either in \( [e^{-\sqrt{7}}, e^{\sqrt{7}}] \), or it improves by a factor of at least \( e^{\delta} \), with each iteration of \( f^{-1} \) (starting from \( f^{-n_0}(x) \)). On the other hand, when \( 1 \leq \rho \leq e^{\sqrt{7}} \), by equation (8), the bound may deteriorate by a factor of at most \( e^{2\delta} \). Thus,

\[
\limsup_{n \to \infty} \frac{S^2(f^{-n}(x), d_x(f^{-n}(x)))}{S^2(p, \eta(d_x(f^{-n}(x)))} \leq e^{\sqrt{7}+2\delta}, \text{ and so } \liminf_{n \to \infty} \frac{S^2(f^{-n}(x), d_x(f^{-n}(x)))}{S^2(p, \eta(d_x(f^{-n}(x)))} \geq e^{-\sqrt{7}-2\delta}.
\]

Now, since \( \delta > 0 \) was arbitrary, we get \( \forall \xi \in H^s(x) \setminus \{0\}, \)

\[
\lim_{n \to \infty} \frac{S^2(f^{-n}(x), d_x(f^{-n}(x)))}{S^2(p, \eta(d_x(f^{-n}(x)))} = 1, \text{ and so } \lim_{n \to \infty} \frac{S^2(f^{-n}(x), d_x(f^{-n}(x)))}{S^2(p, \eta(d_x(f^{-n}(x)))} = 1.
\]

From that, since \( \lim_{n \to \infty} |\eta(d_x(f^{-n}(x)))| = 1 \) (by the Inclination Lemma and the definition of \( \eta : T_{f^{-n}(x)}V^s(f^{-n}(x)) \to H^s(p) \)), the lemma follows. \( \square \)

The reason that we got a better result here, than [BOTS Lemma 4.6] and [Sar13 Lemma 7.2], is that here the centers of charts are \( f^{-i}(p) \), and there is no distortion as a consequence of non-full overlap between \( f^{-1}(x) \) and \( x_{i-1} \).

**Definition 4.8.** \( \forall x \in \text{HWT}_{\chi}^* = \bigcup \mathcal{R}, \) the itinerary of \( x \) is \( R(x) := (R(f^i(x)))_{i \in \mathbb{Z}}, \) \( \)where \( R(f^i(x)) := \text{unique element of } \mathcal{R} \text{ which contains } f^i(x). \)

Notice that \( \hat{R}() : \text{HWT}_{\chi}^* \to \hat{\Sigma} \) is a bijection onto its image and is a measurable map s.t. \( R \circ f = \sigma \circ \hat{R} \)

and \( \hat{\pi} \circ \hat{R} = \text{Id} \) (in particular \( \forall x \in \text{HWT}_{\chi}^* \), \( R(x) \) is an admissible chain in \( \hat{\Sigma}^\# \), as demonstrated in the proof of Proposition 3.8).

**Proposition 4.9.** For every \( \chi \)-hyperbolic periodic point \( p, \exists \hat{\Sigma} \subseteq \hat{\Sigma} \) a maximal irreducible component, s.t. \( \hat{\pi}([\hat{\Sigma}]) \geq H(p) \mod \text{ulo all conservative (possibly infinite) measures which are carried by } H(p). \)

This is an adaptation of the proof by Buzzi, Crovisier and Sarig in [BCS Lemma 3.11, Lemma 3.12].

**Proof.** Let \( \mu \) be a conservative measure carried by \( H(p) \). By Corollary 3.3 and Proposition 3.8 \( H(p) \subseteq \hat{\pi}([\hat{\Sigma}]) \), and so \( \hat{\mu} := \mu \circ \hat{R}^{-1} \) is a well defined invariant and conservative measure on \( \hat{\Sigma}^\# \), and \( \hat{\mu} \circ \hat{\pi}^{-1} = \hat{\mu} \). Whence \( \hat{\mu} \) is carried by

\[ \hat{\Sigma}^\# := \{R \in \hat{\Sigma}^# : \exists a \in \mathcal{R} \text{ s.t. } R_i = a \text{ for infinitely many } i \geq 0 \text{ and for infinitely many } i \leq 0 \}. \]

Therefore, \( \hat{\pi}([\hat{\Sigma}^\#]) \cap H(p) \) carries all conservative measures which are carried by \( H(p) \). Each chain in \( \hat{\Sigma}^\# \) has the following form:

\[ \ldots a, \tilde{p}_{-i}, a, \tilde{p}_{-i+1}, a, \ldots, \tilde{p}_1, a, \ldots, a, \tilde{p}_{-1}, a, \ldots a, \tilde{p}_i, a, \ldots \]

where \( \tilde{p}_i \) denotes a finite word which connects \( a \) to \( a \). For each word \( w_n := a, \tilde{p}_{-n}, a, \tilde{p}_{-n+1}, a, \ldots, a, \tilde{p}_{-n}, a, \ldots \), let \( x_{R^n} \) be the image of the periodic extension of \( w_n \).

**Step 0:** For every chain \( x_{R^n} \) as in equation (6), let \( x_{R^{(n)}} := \text{the admissible concatenation of } w_n \text{ to itself}. \) Then \( x_{R^{(n)}} \to x_{R^n} \). As demonstrated in the proof of Proposition 3.8, \( x_{R^{(n)}, R^n} \in \hat{\Sigma}^\# \) s.t. \( x_{R^{(n)}} \to x_{R^n} \) and \( \pi(x_{R^{(n)}}) = \hat{\pi}(x_{R^n}) \). It follows that \( V^{u}(x_{R^{(n)}}) \to V^{u}(x_{R^n}), V^{s}(x_{R^{(n)}}) \to V^{s}(x_{R^n}) \) as admissible manifolds in \( u_0 \) (i.e. the representing functions converge in \( || \cdot ||_{\text{\infty}} \)-norm). Since \( \pi(x_{R^n}) = \hat{\pi}(x_{R^n}) \in H(p), \) \( \exists N = N_{R^n} \) s.t. \( f^N[V^{u}(\sigma^{-N}x_{R^{(n)}})] \cap W^{s}(o(p)) \neq \emptyset, f^{-N}[V^{s}(\sigma^Nx_{R^{(n)}})] \cap W^{s}(o(p)) \neq \emptyset. \) Whence, \( \exists N_{R^n} \) s.t. \( \forall n \geq N_{R^n}, f^N[V^{u}(\sigma^{-N}x_{R^{(n)}})] \cap W^{s}(o(p)) \neq \emptyset, f^{-N}[V^{s}(\sigma^Nx_{R^{(n)}})] \cap W^{s}(o(p)) \neq \emptyset. \) Let \( P_{R^n} := \{x_{R^{(n)}}\}_{|n| > n_{R^n}}. \) Then \( \forall n \geq n_{R^n}, \hat{\pi}(x_{R^{(n)}}) \in H(p), \) and \( W^{s}(\hat{\pi}(x_{R^{(n)}})) = W^{s}(x_{R^{(n)}}), W^{u}(\hat{\pi}(x_{R^{(n)}})) = W^{u}(x_{R^{(n)}}); \) therefore \( P_{R^n} \subseteq H(p). \)
Consider the countable collection of all periodic points generated in this manner \( \{p_i\}_{i \geq 0} = \bigcup \{P_R : R \in \hat{\Sigma}^\# \cap \hat{\tau}^{-1}\{\{x\}\}, x \in H(p)\} \subseteq HWT_\chi^* \), Theorem 3.3). Then by the transitivity of the homoclinic relation ([New89 Proposition 2.1]), \( \forall i, j \geq 0, p_i \in H(p_j) \). Assume w.l.o.g. that \( \exists N_i \uparrow \infty \) s.t. \( \forall l \in \mathbb{N}, \forall i \leq N_l, o(p_i) \subseteq \{p_j\}_{j \leq N_l} \). Fix \( N \in \{N_l\}_{l \geq 0} \).

**Step 1:** \( \forall 0 \leq i, j \leq N \exists i_j \in \left(W^u(p_i) \cap W^s(p_j)\right) \cap H(p) \), and \( t_{ij} \) has a uniformly hyperbolic orbit, and its coding involves finitely many symbols.

**Proof:** Since \( p_i \in H(p_j) \), \( \exists t_{ij} \in W^u(p_i) \cap W^s(p_j) \). Showing that \( t_{ij} \) has a uniformly hyperbolic orbit would yield that \( t_{ij} \in HWT_\chi^* \), and so, since \( p_i, p_j \in H(p) \), also \( t_{ij} \in H(p) \). By Lemma 3.3, \( p_i, p_j \in \chi \)-hyp.; whence, by [BO18 Lemma 4.5], \( t_{ij} \in \chi \)-hyp. By the Inclination Lemma ([BS02 Theorem 5.7.2]), the angle between \( W^s(t_{ij}) \) and \( W^u(t_{ij}) \) is bounded away from zero along the orbit of \( t_{ij} \). Therefore, by [BO18 Lemma 4.5] and by Lemma 4.7 \( \{\|C^{-1}_\chi\{f^k(t_{ij})\}\}_{k \in \mathbb{Z}} \) is bounded along the orbit of \( t_{ij} \). Thus, \( t_{ij} \in HWT_\chi^* \) and can be coded with finitely many symbols.

**Step 2:** Let \( \{t_{ij}\}_{i,j \leq N} \) be as defined in step 1, and choose some \( \zeta_{ij} \in \hat{\tau}^{-1}\{\{t_{ij}\}\} \) (which involves only finitely many symbols). As defined before step 0, each point \( p \in \{p_i\}_{i \geq 0} \) is the image of a periodic extension of a finite word \( w(p) = a_i \bar{p} a \); in the following definition \( w(p_i) \) is the periodic extension of \( w(p_i) \) which induces \( p_i, 0 \leq i \leq N \).

\[ \hat{L} := \{w(p_i) : 0 \leq i \leq N\} \bigcup \{\zeta_{ij} \}_{i,j \leq N} \]

\( \hat{L} \) is finite and involves only finitely many symbols.

**Step 3:**

Define \( L := \bigcup_{y \in \hat{L}} \{\sigma^j \mu \}_{j \in \mathbb{Z}} \).

By the Hölder-continuity of \( \hat{\tau} \), it follows that \( \forall y \in \hat{\tau}[L], \exists j(y), j(y) \leq N \) s.t. \( \lim_{n \to \infty} d(f^{-n}(y), f^{-n}(p_j(y))) = 0 \), \( \lim_{n \to \infty} d(f^n(y), f^n(p_j(y))) = 0 \). Therefore, \( \hat{\tau}[L] \) is compact, \( f \)-invariant, and \( \chi' \)-uniformly hyperbolic for some \( \chi' > \chi \) (by the proof of step 1).

**Step 4:** We now follow the argument in [BCS, Lemma 3.12]: By the Shadowing Lemma, there are \( \epsilon' > 0, \delta > 0 \) s.t.

1. Every \( \epsilon' \)-pseudo-orbit in \( L^Z \) is \( \delta \)-shadowed by at least one real orbit [KH95 Theorem 18.1.2].
2. Every \( \epsilon' \)-pseudo-orbit in \( L^Z \) is \( 2\delta \)-shadowed by at most one orbit by expansivity, see [KH95 Theorem 18.1.3] (in particular every orbit as in the first item is unique).

Since \( \hat{L} \) is finite, there is some \( m \geq 0 \) large enough so \( d(f^m(y), f^m(p_j(y))) < \frac{\epsilon'}{2}, d(f^{-m}(y), f^{-m}(p_j(y))) < \frac{\epsilon'}{2} \), \( \forall y \in \hat{L} \). Let \( L_m := \bigcup_{j=-m}^{m} \{f^j(y) : y \in \hat{\tau}[\hat{L}] \} \), which is also finite. Let \( K := \{x \in M : \text{the orbit of } x \text{ is } \delta \text{-shadowed by an } \epsilon' \text{-pseudo-orbit in } L^Z_m \} \).

This set contains \( \hat{L} \), and since \( L_m \) is finite, it is also closed. It is also invariant and uniformly \( \chi' \)-hyperbolic for some \( \chi' > \chi \) (whence \( \subseteq HWT_\chi^* \)). We construct a point in \( K \) with a dense forward-orbit in \( K \) in the following way: take a list of all admissible finite words \( \{\omega_i\}_{i \geq 0} \) with letters in \( L_m \). Each \( y \in L_m \) connects by such admissible word of length at most \( m \), to some periodic point \( p_j \in L_m \), and has some periodic point \( p_j^{-}(y) \) connecting to it by an admissible word of length at most \( m \). For each two periodic points \( p_i, p_j \in L_m \), \( p_i \) connects to \( p_j \) by an admissible word of length at most \( 2m \) through \( t_{ij} \). Therefore, every two admissible finite words \( \omega, \omega' \) of letters in \( L_m \) can be concatenated by some admissible finite word of letters in \( L_m \). Concatenate this way all words in \( \{\omega_i\}_{i \geq 0} \), and take any admissible continuation to the past. This yields an
\(\varepsilon\)-pseudo-orbit, and the unique orbit in \(K\) it \(\delta\)-shadows must be dense in \(K\) by expansivity. Denote this orbit by \(o(x)\).

Step 5: The orbit of \(x\) lies in \(K\), which is an invariant \(\chi\)'-uniformly hyperbolic set for some \(\chi' > \chi\) (whence \(|C_{\chi'}^{-1}(\cdot)|\) is uniformly bounded on \(K\)), and by the same argument as in footnote \(3\), \(x\) has a pre-image in \(\hat{\Sigma}^\#\) which involves only finitely many symbols. Choose one pre-image as such, and denote it by \(\hat{v}\). There exists at least one symbol \(v'\) such that the forward-orbit of \(\hat{v}\) visits its cylinder \([v']\) infinitely often. Let \(\hat{v}^+ := \sigma^{\min\{i \geq 0 : \sigma^i [v'] \}} \hat{v}\) and let \(\hat{v}^−\) be some periodic chain in \([v']\). Define \(v'_i := \begin{cases} v^+_i & \text{if } i \geq 0 \\ v^−_i & \text{if } i \leq 0 \end{cases}\), and write \(x' := \tilde{\pi}(v')\). Then the forward orbit of \(x'\) is dense in \(K\), and \(v' \in \hat{\Sigma}_\infty := \{ \hat{v} \in \hat{\Sigma} : \hat{v} \in [v']^\# \}\) (recall Definition \(4.2\)). \(\hat{\Sigma}_\infty\) is a maximal irreducible component of \(\hat{\Sigma}\) containing a compact set which contains the orbit of \(v'\). For each \(y \in \hat{\Sigma} \subseteq K\), the orbit of \(x'\) has a subsequence converging to it. Since \(v'\) is made of finitely-many letters, the orbit of \(v'\) belongs to a compact subset of \(\hat{\Sigma}_\infty\). Therefore, the subsequence \(\{ f^n(x') \}_{k \geq 0}\) which converges to \(y\), has a subsubsequence \(\{ \sigma^{n_k}y' \}_{k \geq 0}\) converges as well. By the continuity of \(\tilde{\pi}\), that limit must code \(y\). Therefore \(\tilde{\pi}[\hat{\Sigma}_\infty] \supseteq \hat{L}\), and moreover, each term in \(L\) has a pre-image in \(\hat{\Sigma}_\infty\) made of finitely-many symbols.

Step 6: For each \(N \in \{N_i\}_{i \geq 0}\), \(p_0 \in \tilde{\pi}[\hat{\Sigma}_\infty]\). Since \(\tilde{\pi}|_{\hat{\Sigma}^\#}\) is finite-to-one, there could be only finitely many maximal irreducible components in \(\hat{\Sigma}\), which can code \(p_0\). Therefore, there is some subsequence of the maximal irreducible components from step 5, \(\hat{\Sigma}_\infty, l_j \uparrow \infty\) which is constant. Denote this component by \(\hat{\Sigma}\). For each fixed \(N \in \{N_i\}_{i \geq 0}\), we constructed a set \(\hat{L}\), and all such sets must be covered by \(\tilde{\pi}[\hat{\Sigma}]\).

Step 7: Given a point \(z \in H(p) \cap \tilde{\pi}[\hat{\Sigma}^\#]\), consider its coding as in equation \(\text{(2)}\). This coding has a sequence of finitely-lettered chains which converge to it in \(\hat{L}\). These chains all have their images coded by a chain in \(\hat{\Sigma}\), and they all belong to the same cylinder of their zeroth symbol, denoted by \(b\). The corresponding coding chains in \(\hat{\Sigma}\) (which have been shown in step 5 to be made of finitely-many letters, and as such lie in \(\hat{\Sigma}^\#\)) all belong to a cylinder from \(\{ [S] \in \hat{\Sigma} : b \cap \tilde{\pi}([S] \cap \hat{\Sigma}^\#) \neq \emptyset \}\), which is a finite collection. Thus they have a converging subsequence with a limit in \(\hat{\Sigma}\) (since it is a closed set). By the continuity of \(\tilde{\pi}\), that limit must code \(z\), and so \(\tilde{\pi}[\hat{\Sigma}] \supseteq H(p) \cap \tilde{\pi}[\hat{\Sigma}^\#]\) = \(H(p)\) modolu all conservative measures which are carried by \(H(p)\).

In [BCS], Lemma 3.13, the authors offer a technique of lifting invariant probability measures which are carried by \(H(p)\) to an irreducible coding. Their technique involves restriction to points which are generic w.r.t Birkhoff’s ergodic theorem, which we may not always be able to do if the measure we wish to lift is infinite. The lifting is being done by the formula \(\tilde{v} := \int \sum_{x \in N} \delta_{\tilde{\pi}(x)} \mu(x)\), where \(N := \tilde{\pi}^{-1}([x]) \cap \hat{\Sigma}^\#\) and \(\hat{\Sigma}^\#: = \{ \hat{v} \in \hat{\Sigma} : \exists w, v \text{ s.t. } \#\{i > 0 : u_i = v\} = \infty, \#\{i < 0 : u_i = w\} = \infty \}\). When \(\mu\) is carried by \(\tilde{\pi}[\hat{\Sigma}^\#]\), this lifting is well defined since \(\tilde{\pi}\) is finite-to-one on \(\hat{\Sigma}^\#\). So, we are required to find a different way which does not depend on generic points to show that all conservative measures which are carried by \(H(p)\) are also carried by \(\tilde{\pi}[\hat{\Sigma}^\#]\). Proposition \(4.9\) was written in a way to guarantee that we can do that. In addition, [BCS], Lemma 3.13 is done for \(HO(p) := \{ q \in H(p) : q \text{ is periodic} \}\) as the homoclinic class which is being lifted to an irreducible component (modolu all invariant ergodic probability measures carried by it). Both Proposition \(4.9\) and Theorem \(4.10\) are done in the context of ergodic homoclinic classes, which is relevant to specific objects (e.g. SRB measures on ergodic homoclinic classes, see [RHRHTU11]).

**Theorem 4.10.** Let \(p\) be a \(\chi\)-hyperbolic periodic point. Let \(\hat{\Sigma}\) be the irreducible TMS which we construct in Proposition \(4.9\) to cover \(H(p)\). Then, \(\tilde{\pi}[\hat{\Sigma}^\#]\) = \(H(p)\) modolu all conservative measures which are carried by \(H(p)\).

**Proof.** \(p\) is clearly in \(H(p) \cap \tilde{\pi}[\hat{\Sigma}^\#]\), and so it is has a coding in \(\hat{\Sigma}\). \(\forall \hat{v} \in \hat{\Sigma}^\#\), \(\tilde{\pi}(\hat{v}) \in H(p)\) by the irreducibility of \(\hat{\Sigma}\) (and since \(\tilde{\pi}[\hat{\Sigma}^\#]\) = \(HWT^*\)), thus the inclusion \(\subseteq\) is evident. We are therefore left to show only the inclusion \(\subseteq\). Given each \(x \in H(p) \cap \tilde{\pi}[\hat{\Sigma}^\#]\)(= \(H(p)\) modolu all conservative measures on \(H(p)\)), \(x\) has

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\(\hat{v}\) is a continuous map (the topology on the space of pseudo-orbits is the metric topology generated by cylinders); and in addition \(\tau \circ \sigma = f \circ \tau\), where \(\sigma\) denotes the left-shift on \(\varepsilon\)-pseudo-orbits. Thus, shadowing longer intervals of the orbit of \(x\) forces being closer to it.
a coding \( v \in \hat{\Sigma}^{\#} \) as in equation \([1]\). In step 2 of Proposition \([13]\) we describe a corresponding sequence of periodic chains which converge to \( v \). Call these chains \( \{v^{(i)}\}_{i \geq 0} \). By step 5 of Proposition \([13]\) \( \{\hat{\pi}(v^{(i)})\}_{i \geq 0} \) all have codings made of finitely many letters in \( \hat{\Sigma} \), denoted by \( \{u^{(i)}\}_{i \geq 0} \), and by step 7 of Proposition \([13]\) we can assume w.l.o.g. that \( \{u^{(i)}\}_{i \geq 0} \) converge to \( v \). Let \( \{v_{ji}\}_{i \geq 0} \) be a subsequence of the symbols of \( v \) which is constant (it exists since \( v \in \hat{\Sigma}^{\#} \)); denote by \( w \) the symbol which satisfies \( v_{ji} = w, \forall i \geq 0 \). For all \( i \geq 0 \) \( v^{(i)}_{ji} = v_{ji} = w \). If \( i \) is big enough so \( d(u^{(i)}_{ji}, u_{ji}) \leq e^{-j} \), then \( u^{(i)}_{ji}, u^{(i)}_{ji} \) both code the same point and lie in \( \hat{\Sigma}^{\#} \), therefore \( u^{(i)}_{ji} \) is affiliated to \( v^{(i)}_{ji} \). Whence, \( w \) is affiliated to \( u_{ji} \). Since \( \#\{v' : v' \text{ is affiliated to } w\} < \infty \), we get by the pigeonhole principle that some symbol must repeat in \( u \) for infinitely many positive indices. Similarly it follows that some symbol must repeat in \( u \) for infinitely many negative indices. Therefore, \( u \in \hat{\Sigma}^{\#} \), and \( \hat{\pi}(u) = x \). \( \Box \)

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