Levy Processes, Generators

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Abstract. For a broad class of the Levy processes the new form (convolution type) of the infinitesimal generators is introduced. It leads to the new notions: a truncated generator, a quasi-potential. The probability of the Levy process remaining within the given domain is estimated.

1 Main notions

Let us consider the Levy processes $X_t$ on $R$. If $P(X_0 = 0) = 1$ then Levy-Khinchine formula gives [1],[6]

$$
\mu(x, t) = E\{\exp[izX_t]\} = \exp[-t\lambda(z)], \quad t \geq 0,
$$

(1.1)

where

$$
\lambda(z) = \frac{1}{2} Az^2 - i\gamma z - \int_{-\infty}^{\infty} (e^{ixz} - 1 - ixz1_{D(x)})d\nu(x).
$$

(1.2)

Here $A \geq 0, \quad \gamma = \tau$, and $D = \{x : |x| \leq 1\}$ is closed unit ball, $\nu(x)$ is monotonically increasing function satisfying the conditions

$$
\int_{-\infty}^{\infty} \frac{x^2}{1 + x^2}d\nu(x) < \infty.
$$

(1.3)

By $P_t(x_0, B)$ we denote the probability $P(X_t \in B)$ when $P(X_0 = x_0) = 1$ and $B \in R$. The transition operator is defined by the formula

$$
P_t f(x) = \int_{-\infty}^{\infty} P_t(x, dy)f(y).
$$

(1.4)

Let $C_0$ be the Banakh space of continuous functions $f(x)$ $(-\infty < x < \infty)$ satisfying the condition $\lim_{x \to \infty} f(x) = 0$, $\lim_{x \to -\infty} f(x) = 0$ with the norm $||f|| = \sup_{x \in R}|f(x)|$. We denote by $C_0^n$ the set of $f(x) \in C_0$ such that $f^k(x) \in C_0$, $1 \leq k \leq n$. It is known that [6]

$$
P_t f \in C_0
$$

(1.5)
if \( f(x) \in C_0 \).
Now we formulate the following important result (see [2],[6]) .

**Theorem 1.1.** The family of the operators \( P_t \) \((t \geq 0)\) defined by the Levy process \( X_t \) is a strong continuous semigroup on \( C_0 \) with norm \( \|P_t\| = 1 \). Let \( L \) be its infinitesimal generator. Then

\[
L f = \frac{1}{2} A^2 f + \gamma f + \int_{\infty}^{\infty} \left( f(x + y) - f(x) - y \frac{df}{dx} 1_{D(x)} \right) d\nu(x),
\]

where \( f \in C^2_0 \).

### 2 Convolution type form of infinitesimal generator

1. In this section we prove that under some conditions the infinitesimal generator \( L \) can be represented in the special convolution type form

\[
Lf = \frac{d}{dx} S \frac{df}{dx},
\]

where the operator \( S \) is defined by the relation

\[
S f = \frac{1}{2} A f + \int_{-\infty}^{\infty} k(y-x) f(y) dy,
\]

and for arbitrary \( M(0 < M < \infty) \) we have

\[
\int_{-M}^{M} |k(t)| dt < \infty.
\]

The representation \( L \) in form (2.1) is convenient as the operator \( L \) is expressed with the help of the classic differential and convolution operators.

By \( C_c \) we denote the set of functions \( f(x) \in C_0 \) with compact support.

**Lemma 2.1.** Let the following conditions be fulfilled.

1. The function \( \nu(x) \) is monotonically increasing, has the derivative when \( x \neq 0 \) and

\[
\int_{-\infty}^{\infty} \frac{x^2}{1 + x^2} d\nu(x) = \int_{-\infty}^{\infty} \frac{x^2}{1 + x^2} \nu'(x) dx < \infty,
\]

\[
\nu(x) \to 0, \quad x \to \infty.
\]
2. For arbitrary $M(0 < M < \infty)$ we have
\[
\int_{-M}^{M} |\nu(x)| \, dx < \infty, \quad \int_{-M}^{M} |x| \nu'(x) \, dx < \infty.
\] (2.6)

3. \quad x \nu(x) \to 0, \quad x \to 0. \quad \text{(2.7)}

Then the equality
\[
J = \int_{-\infty}^{\infty} \left[ f(y - x) - f(x) \right] \nu'(y) \, dy,
\] (2.8)

is true, where $f(x) \in C_0^2$, \quad $k(x) = \int_0^x \nu(y) \, dy$ and
\[
J = \frac{d}{dx} \int_{-\infty}^{\infty} f'(y) k(y - x) \, dy. \quad \text{(2.9)}
\]

Proof. For every $f(x) \in C_0$ there exists such $M(0 < M < \infty)$ that
\[
f(x) = 0, \quad x \notin [-M, M]. \quad \text{(2.10)}
\]

Let us introduce the following notations
\[
J_1 = \frac{d}{dx} \int_{-\infty}^{x} f'(y) k(y - x) \, dy, \quad \text{(2.11)}
\]
\[
J_2 = \frac{d}{dx} \int_{x}^{\infty} f'(y) k(y - x) \, dy. \quad \text{(2.12)}
\]

Using (2.11) we have
\[
J_1 = -\frac{d}{dx} \int_{-M}^{x} \left[ f(y) - f(x) + f(x) \right] k'(y - x) \, dy. \quad \text{(2.13)}
\]

From (2.11) and (2.13) we deduce the relation
\[
J_1 = f(x) k'(x - M) + \int_{-M}^{x} [f(y) - f(x)] k''(y - x) \, dy. \quad \text{(2.14)}
\]

When $M \to \infty$ we obtain the equality
\[
J_1 = \int_{-\infty}^{0} [f(y + x) - f(x)] k''(y) \, dy. \quad \text{(2.15)}
\]
In the same way we deduce the relation

\[ J_2 = \int_0^\infty [f(y + x) - f(x)]k''(y)dy. \] (2.16)

The relation (2.9) follows directly from (2.15), (2.16) and equality \( J = J_1 + J_2 \). The lemma is proved.

**Lemma 2.2.** Let the following conditions be fulfilled.

1. The function \( \nu(x) \) satisfies conditions of Lemma 2.1.
2. For arbitrary \( M \) \((0 < M < \infty)\) we have

\[ \int_{-M}^{M} |k(x)|dx < \infty, \quad \int_{-M}^{M} |x\nu(x)|dx < \infty, \] (2.17)

where

\[ k'(x) = \nu(x), \quad x \neq 0. \] (2.18)

3. \( xk(x) \to 0, \quad x \to 0; \quad x^2\nu(x) \to 0, \quad x \to 0. \) (2.19)

Then the equality

\[ J = \int_{-\infty}^{\infty} [f(y + x) - f(x) - y\frac{df(x)}{dx}1_{D(y)}]\nu'(y)dy + \Gamma f'(x), \] (2.20)

is true, where \( \Gamma = \Gamma \) and \( f(x) \in \mathcal{C}_c \).

**Proof.** From (2.11) we obtain the relation

\[ J_1 = f'(x)\gamma_1 - \int_{x-1}^{x} [f'(y) - f'(x)]k'(y-x)dy - \int_{-M}^{x-1} f'(y)k'(y-x)dy, \] (2.21)

where \( \gamma_1 = k(-1) \). We introduce the notations

\[ P_1(x, y) = f(y) - f(x) - (y - x)f'(x), \quad P_2(x, y) = f(y) - f(x). \] (2.22)

Integrating by parts (2.21) and passing to limit when \( M \to \infty \) we deduce that

\[ J_1 = f'(x)\gamma_2 + \int_{x-1}^{x} P_1(x, y)k''(y-x)dy + \int_{-M}^{x-1} P_2(x, y)k''(y-x)dy, \] (2.23)

where \( \gamma_2 = k(-1) - k'(-1) \). It follows from (2.22) and (2.23) that

\[ J_1 = \int_{-\infty}^{\infty} [f(y + x) - f(x) - y\frac{df(x)}{dx}1_{D(y)}]\nu'(y)dy + \gamma_2 f'(x). \] (2.24)
In the same way can be proved the relation

\[ J_2 = \int_{x}^{\infty} [f(y + x) - f(x) - y \frac{df(x)}{dx} 1_{D(y)}] \nu'(y) dy + \gamma_3 f'(x), \tag{2.25} \]

where \( \gamma_3 = -k(1) + k'(1) \). The relation (2.20) follows directly from (2.24) and (2.25). Here \( \Gamma = \gamma_2 + \gamma_3 \). The lemma is proved.

**Remark 2.1.** The operator \( L_0 f = \frac{df}{dx} \) can be represented in form (2.1), where

\[ S_0 f = \int_{-\infty}^{\infty} p_0(y - x) f(y) dy, \tag{2.26} \]

\[ p_0(x) = \frac{1}{2} \text{sign}(x). \tag{2.27} \]

From Lemmas 2.1, 2.2 and Remark 2.1 we deduce the following assertion.

**Theorem 2.1.** Let the conditions of Lemma 2.1 or Lemma 2.2 be fulfilled. Then the corresponding operator \( L \) has form (2.1),(2.2).

**Proposition 2.1.** The generator \( L \) of the Levy process \( X_t \) admits the convolution representation (2.1), (2.2) if there exist such \( C > 0 \) and \( 0 < \alpha < 2, \alpha \neq 1 \) that

\[ \nu'(y) \leq C |y|^{-\alpha - 1}, \tag{2.28} \]

**Proof.** The function \( \nu(y) \) has the form

\[ \nu(y) = \int_{-\infty}^{y} \nu'(t) dt 1_{y < 0} - \int_{y}^{\infty} \nu'(t) dt 1_{y > 0}. \tag{2.29} \]

We suppose that \( 1 < \alpha < 2 \) and introduce the function

\[ k_0(y) = \int_{-\infty}^{y} (y - t) \nu'(t) dt 1_{y < 0} - \int_{y}^{\infty} (t - y) \nu'(t) dt 1_{y > 0}. \tag{2.30} \]

We obtain the relation

\[ k(y) = k_0(y) + (\gamma - \Gamma) p_0(y), \tag{2.31} \]

where \( k_0(y) \) and \( p_0(y) \) are defined by (2.27) and (2.30) respectively. The constant \( \Gamma \) is defined by relations:

\[ \Gamma = k_0(-1) - k_0'(-1) - k_0(-1) + k_0'(1), \quad 1 \leq \alpha < 2, \tag{2.32} \]
It follows from (2.28)-(2.30) that the conditions of Theorem 2.1 are fulfilled. Hence the proposition is true when \(1 < \alpha < 2\). Let us consider the case when \(0 < \alpha < 1\). In this case we have

\[
k_0(y) = \int_{-\infty}^{y} \nu'(t) dt + \int_{y}^{0} \nu'(t) dt, \quad y < 0,
\]

\[
k_0(y) = -\int_{y}^{\infty} \nu'(t) dt - \int_{0}^{y} \nu'(t) dt, \quad y > 0,
\]

and

\[
k(y) = k_0(y) + \gamma p_0(y) \quad 0 < \alpha < 1.
\]

In view of (2.28) and (2.33),(2.34) the conditions of Theorem 2.1 are fulfilled. Hence the proposition is proved.

**Corollary 2.1.** If condition (2.28) is fulfilled then

\[
k_0(y) \geq 0, \quad -\infty < y < \infty, \quad 1 < \alpha < 2,
\]

\[
k_0(y) \leq 0, \quad -\infty < y < \infty, \quad 0 < \alpha < 1.
\]

**Example 2.1.** The stable processes.

For the stable processes we have \(A = 0\), \(\gamma = \overline{\gamma}\) and

\[
\nu'(y) = |y|^{-\alpha-1}(C_11_{y<0} + C_21_{y>0}),
\]

where \(C_1 > 0\) \(C_2 > 0\). Hence the function \(\nu(y)\) has the form

\[
\nu(y) = \frac{1}{\alpha} |y|^{-\alpha}(C_11_{y<0} - C_21_{y>0}).
\]

Let us introduce the functions

\[
k_0(y) = \frac{1}{\alpha(\alpha - 1)} |y|^{1-\alpha}(C_11_{y<0} + C_21_{y>0}),
\]

where \(0 < \alpha < 2\), \(\alpha \neq 1\). When \(\alpha = 1\) we have

\[
k_0(y) = -\log |y| (C_11_{y<0} + C_21_{y>0}).
\]

It means that the conditions of proposition 2.1 are fulfilled. Hence the generator \(L\) for the stable processes admits the convolution type representation (2.1),(2.2).
**Proposition 2.2.** The kernel \( k(y) \) of the operator \( S \) in representation (2.1) for the stable processes has form (2.31), when \( 1 \leq \alpha < 2 \), and has form (2.35) when \( 0 < \alpha < 1 \).

**Example 2.2.** The variance damped Levy processes [7].

For the variance damped Levy processes we have \( A = 0 \), \( \gamma = \gamma \) and

\[
\nu'(y) = C_1 e^{-\lambda_1 |y| |y|^{-\alpha-1}} 1_{y < 0} + C_2 e^{-\lambda_2 |y| |y|^{-\alpha-1}} 1_{y > 0},
\]

where \( C_1 > 0 \), \( C_2 > 0 \), \( \lambda_1 > 0 \), \( \lambda_2 > 0 \), \( 0 < \alpha < 2 \), \( \alpha \neq 1 \). It follows from (2.42) that the conditions of Proposition 2.1 are fulfilled. Hence the generator \( L \) for the variance damped Levy processes admits the convolution type representation (2.1), (2.2) and the kernel \( k(y) \) is defined by formulas (2.30), (2.31), when \( 1 \leq \alpha < 2 \), and by formula (2.35) when \( 1 \leq \alpha < 2 \).

**Example 2.3.** The variance Gamma process [7].

For the variance Gamma process we have \( A = 0 \), \( \gamma = \gamma \) and

\[
\nu'(y) = C_1 e^{-G |y| |y|^{-1}} 1_{y < 0} + C_2 e^{-M |y| |y|^{-1}} 1_{y > 0},
\]

where \( C_1 > 0 \), \( C_2 > 0 \), \( G > 0 \), \( M > 0 \). It follows from (2.43) that the conditions of Proposition 2.1 are fulfilled and the generator \( L \) of variance Gamma process admits the convolution type representation (2.1), (2.2). The kernel \( k(y) \) is defined by relations (2.33) and (2.34).

**Example 2.4.** The normal inverse Gaussian process [7].

In the case of the normal inverse Gaussian process we have \( A = 0 \), \( \gamma = \gamma \) and

\[
\nu'(y) = C e^{\beta y} K_1(|y| |y|^{-1}), \quad C > 0, \quad -1 \leq \beta \leq 1,
\]

where \( K_\lambda(x) \) denotes the modified Bessel function of the third kind with index \( \lambda \). Using equalities

\[
|K_1(|x|)| \leq M e^{-|x|/|x|}, \quad M > 0, \quad 0 < x_0 \leq |x|,
\]

\[
|K_1(|x||x|) \leq M, \quad 0 \leq |x| \leq x_0
\]

we see that the conditions of Proposition 2.1 are fulfilled. Hence the corresponding generator \( L \) admits the convolution type representation (2.1), (2.2) and the kernel \( k(y) \) is defined by relations (2.33) and (2.34).

**Example 2.5.** The Meixner process [7].

For the Meixner process we have

\[
\nu'(y) = C \frac{e^{\beta x}}{x \sinh \pi x},
\]
where $C > 0$, $-\pi < \beta < \pi$. The conditions of Proposition 2.1 are fulfilled. Hence the corresponding generator $L$ admits the convolution type representation (2.1),(2.2) and the kernel $k(y)$ is defined by relations (2.33),(2.34).

**Remark 2.1.** Examples 2.1-2.5 are used in the finance problems [7].

**Example 2.6.** Compound Poisson process.

We consider the case when $A = 0$, $\gamma = 0$ and

$$M = \int_{-\infty}^{\infty} \nu'(y)dy < \infty. \tag{2.48}$$

Using formulas (2.1) and (2.2) we deduce that the corresponding generator $L$ has the following form

$$Lf = -Mf(x) + \int_{-\infty}^{\infty} \nu'(y-x)f(y)dy. \tag{2.49}$$

3 Potential

The operator

$$Qf = \int_{0}^{\infty} (P_t f)dt \tag{3.1}$$

is called *potential* of the semigroup $P_t$. The generator $L$ and the potential $Q$ are (in general) unbounded operators. Therefore the operators $L$ and $Q$ are defined not in the whole space $L^2(-\infty, \infty)$ but only in the subsets $D_L$ and $D_Q$ respectively. We use the following property of potential $Q$ (see[6]).

**Proposition 3.1.** If $f = Qg$, $(g \in D_q)$ then $f \in D_L$ and

$$-Lf = g. \tag{3.2}$$

**Example 3.1.** Compound Poisson process.

Let the generator $L$ has form (2.49) where

$$M = \int_{-\infty}^{\infty} \nu'(x)dx < \infty, \quad \int_{-\infty}^{\infty} [\nu'(x)]^2dx < \infty. \tag{3.3}$$

We introduce the functions

$$K(u) = -\frac{1}{M \sqrt{2\pi}} \int_{-\infty}^{\infty} \nu'(x)e^{-iu x}dx, \tag{3.4}$$
$$N(u) = \frac{K(u)}{1 - \sqrt{2\pi}K(u)}. \quad (3.5)$$

Let us note that
$$|K(u)| < \frac{1}{\sqrt{2\pi}}, \quad u \neq 0; \quad K(0) = -\frac{1}{\sqrt{2\pi}}. \quad (3.6)$$

It means that $N(u) \in L^2(-\infty, \infty)$. Hence the function
$$n(x) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} N(u)e^{-ixu}du \quad (3.7)$$

belongs to $L^2(-\infty, \infty)$ as well. It follows from (2.49), (3.2) and (3.7) that the corresponding potential $Q$ has form (see [8], Ch.11)
$$Qf = \frac{1}{M}[f(x) + \int_{-\infty}^{\infty} f(y)n(x-y)dy]. \quad (3.8)$$

**Proposition 3.2.** Let conditions (3.3) be fulfilled. Then the operators $L$ and $Q$ are bounded in the space $L^2(-\infty, \infty)$.

Now we shall give an example when the kernel $n(x)$ can be written in an explicit form.

**Example 3.2.** We consider the case when
$$\nu'(x) = e^{-|x|}, \quad -\infty < x < \infty. \quad (3.9)$$

In this case $M = 2$ and the operator $L$ takes the form
$$Lf = -2f(x) + \int_{-\infty}^{\infty} f(y)e^{-|x-y|}dy. \quad (3.10)$$

Formulas (3.4)-(3.7) imply that
$$Qf = \frac{1}{2}f(x) - \frac{1}{4\sqrt{2}} \int_{-\infty}^{\infty} f(y)e^{-|x-y|\sqrt{2}}dy. \quad (3.11)$$

### 4 Truncated generators and quasi-potentials

Let us denote by $\Delta$ the set of segments $[a_k, b_k]$ such that
$a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n, \quad 1 \leq k \leq n$. By $C_\Delta$ we denote the set of functions $g(x)$ on $L^2(\Delta)$ such that
$$g(a_k) = g(b_k) = g'(a_k) = g'(b_k), \quad 1 \leq k \leq n, \quad g''(x) \in L^2(\Delta). \quad (4.1)$$
We introduce the operator $P_{\Delta}$ by relation $P_{\Delta}f(x) = f(x)$ if $x \in \Delta$ and $P_{\Delta}f(x) = 0$ if $x \notin \Delta$.

**Definition 4.1.** The operator

$$L_{\Delta} = P_{\Delta}LP_{\Delta}$$

is called a truncated generator.

**Definition 4.2.** The operator $B$ with dense in $L^2(\Delta)$ definition domain $L_B$ is called a quasi-potential if the functions $f = Bg, \quad (g \in L_B)$ belong to $C_{\Delta}$ and

$$-L_{\Delta}f = g. \quad (4.3)$$

It follows from (4.3) that

$$-P_{\Delta}Lf = g, \quad (f = Bg). \quad (4.4)$$

**Remark 4.1.** In a number of cases (see the next section) we need relation (4.4). In these cases we can use the quasi-potential $B$, which is often simpler than the corresponding potential $Q$.

**Remark 4.2.** The operators of type (4.2) are investigated in book ([5],Ch.2).

**Definition 4.3.** We call the operator $B$ a regular one if the following conditions are fulfilled.

1). The operator $B$ has the form

$$Bf = \int_{\Delta} \Phi(x,y)f(y)dy, \quad f(y) \in L^2(\Delta), \quad (4.5)$$

where the function $\Phi(x,y)$ can have a discontinuity only when $x = y$ and

$$|\Phi(x,y)| \leq M|x - y|^{-\beta}, \quad 0 \leq \beta < 1. \quad (4.6)$$

2). The range of $B$ is dense in $L^2(\Delta)$.

It follows from condition 2) and relation (4.4) that the range of $L$ is dense in $L^2(\Delta)$. Further we assume that the quasi-potential $B$ is regular.

5 **The Probability of the Levy process remaining within the given domain**

In many theoretical and applied problems it is important to estimate the quantity

$$p(t,\Delta) = P\{X_{\tau} \in \Delta\}, \quad 0 \leq \tau \leq t, \quad (5.1)$$
i.e. the probability that a sample of the process $X_\tau$ remains inside $\Delta$ for $0 \leq \tau \leq t$.

To derive the integro-differential equations corresponding to Levy processes we use the argumentation by Kac [3] and our own argumentation (see [5]). Now we get rid of the requirement for the process to be stable.

We suppose that

$$M(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mu(x,t)| dx < \infty, \quad t > 0$$

(5.2)

and

$$\int_{0}^{1} M(t) dt < \infty.$$  \hspace{1cm} (5.3)

According to (1.2) we have the inequality

$$\text{Re}[\lambda(x)] \geq 0.$$  \hspace{1cm} (5.4)

**Remark 5.1.** It follows from (5.2) and (5.4) that

$$\int_{-\infty}^{\infty} |\mu(x,t)|^p dx < \infty, \quad t > 0, \quad p \geq 1.$$  \hspace{1cm} (5.5)

**Proposition 5.1.** (see [5], Ch.5.) Let condition (5.2) be fulfilled. Then the corresponding Levy process has the continuous density

$$\rho(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixz} \mu(z,t) dz, \quad t > 0$$

(5.6)

and

$$\rho(x,t) \leq M(t).$$  \hspace{1cm} (5.7)

Now we introduce the sequence of functions

$$Q_{n+1}(x,t) = \int_{0}^{t} \int_{-\infty}^{\infty} Q_0(x - \xi, t - \tau) V(\xi) Q_n(\xi, \tau) d\xi d\tau,$$

(5.8)

where

$$Q_0(x,t) = \rho(x,t), \quad V(x) = 1 - 1_\Delta.$$  \hspace{1cm} (5.9)

For Levy processes the following relation

$$Q_0(x,t) = \int_{-\infty}^{\infty} Q_0(x - \xi, t - \tau) Q_0(\xi, \tau) d\xi$$

(5.10)
is true. Using (5.8) and (5.10) we have
\[ 0 \leq Q_n(x, t) \leq t^n Q_0(x, t)/n!. \quad (5.11) \]

Hence the series
\[ Q(x, t, u) = \sum_{n=0}^{\infty} (-1)^n u^n Q_n(x, t) \quad (5.12) \]
converges. The probabilistic meaning of \( Q(x, t, u) \) is defined by the relation (see [4], Ch. 4)
\[ E\{\exp[-u \int_0^t V(X(\tau)) d\tau], c_1 < X(t) < c_2\} = \int_{c_1}^{c_2} Q(x, t, u) dx. \quad (5.13) \]
The inequality \( V(x) \geq 0 \) and relation (5.13) imply that the function \( Q(x, t, u) \) monotonically decreases with respect to the variable "u" and the formulas
\[ 0 \leq Q(x, t, u) \leq Q(x, t, 0) = Q_0(x, t) = \rho(x, t) \quad (5.14) \]
are true. In view of (5.6), (5.7) and (5.14) the Laplace transform
\[ \psi(x, s, u) = \int_0^{\infty} e^{-st} Q(x, t, u) dt, \quad s > 0. \quad (5.15) \]
has the meaning. According to (5.8) the function \( Q(x, t, u) \) is the solution of the equation
\[ Q(x, t, u) + u \int_0^t \int_{-\infty}^{\infty} \rho(x - \xi, t - \tau) V(\xi) Q(\xi, \tau, u) d\xi d\tau = \rho(x, t) \quad (5.16) \]
Taking from both parts of (5.16) the Laplace transform and bearing in mind (5.15) we obtain
\[ \psi(x, s, u) + u \int_{-\infty}^{\infty} V(\xi) R(x - \xi, s) \psi(\xi, s, u) d\xi = R(x, s), \quad (5.17) \]
where
\[ R(x, s) = \int_0^{\infty} e^{-st} \rho(x, t) dt. \quad (5.18) \]
Multiplying both parts (5.17) by \( \exp(ixp) \) and integrating them with respect to \( x \) \(( -\infty < x < \infty \) we have
\[ \int_{-\infty}^{\infty} \psi(x, s, u) e^{ixp}[s + \lambda(p) + uV(x)] dx = 1. \quad (5.19) \]
Here we use relations (1.1), (5.6) and (5.18). Now we introduce the function

\[ h(p) = \frac{1}{2\pi} \int_{\Delta} e^{-ixp} f(x) dx, \quad (5.20) \]

where the function \( f(x) \) belongs to \( C_{\Delta} \). Multiplying both parts of (5.19) by \( h(p) \) and integrating them with respect to \( p(-\infty < p < \infty) \) we deduce the equality

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x, s, u) e^{ixp} [s + \lambda(p)] h(p) dx dp = f(0). \quad (5.21) \]

We have used the relations

\[ V(x)f(x) = 0, \quad -\infty < x < \infty, \quad (5.22) \]

\[ \frac{1}{2\pi} \lim_{N \to \infty} \int_{-N}^{N} \int_{\Delta} e^{-ixp} f(x) dx dp = f(0), \quad (5.23) \]

Since the function \( Q(x, t, u) \) monotonically decreases with respect to "\( u \)" then by (5.15) this is true for the function \( \psi(x, s, u) \) also. Hence there exists the limit

\[ \psi(x, s) = \lim_{u \to \infty} \psi(x, s, u), \quad (5.24) \]

where

\[ \psi(x, s) = 0, \quad x \notin \Delta. \quad (5.25) \]

The probabilistic meaning of \( \psi(x, s) \) follows from the equality

\[ \int_{-\infty}^{\infty} e^{-st} p(t, \Delta) dt = \int_{\Delta} \psi(x, s) dx. \quad (5.26) \]

Using the properties of the Fourier transform and conditions (5.24) , (5.25) we deduce from (5.21) the following assertion.

**Proposition 5.2.** Let conditions (5.2) and (5.3) be fulfilled. Then the relation

\[ ((sI - L_{\Delta}) f, \psi(x, s))_{\Delta} = f(0) \quad (5.27) \]

is true.

The important function \( \psi(x, s) \) can be expressed with the help of the quasi-potential \( B \).

**Theorem 5.1.** Let conditions (5.2), (5.3) be fulfilled and let the quasi-potential be regular. Then in the space \( L(\Delta) \) there is one and only one function

\[ \psi(x, s) = (I + sB^*)^{-1} \Phi(0, x) \quad (5.28) \]
which satisfies relation (5.27).

**Proof.** In view of (4.4) we have

\[-BL_\Delta f = f, \quad f \in C_\Delta.\] (5.29)

Relations (5.28) and (5.29) imply that

\[((sI - L_\Delta)f, \psi(x, s))_\Delta = -((I + sB)L_\Delta f, \psi)_\Delta = -(L_\Delta f, \Phi(0, x))_\Delta.\] (5.30)

Since \(\Phi(0, x) = B^*\delta(x), (\delta(x) is the Dirac function) then according to (5.24) and (5.30) relation (5.27) is true. Let us suppose that in \(L(\Delta)\) there is another function \(\psi_1(x, s)\) satisfying (5.27). Then the equality

\[((sI - L_\Delta)f, \phi(x, s))_\Delta = 0, \quad \phi = \psi - \psi_1\] (5.31)

is valid. We write relation (5.31) in the form

\((L_\Delta f, (I + sB^*)\phi)_\Delta = 0.\) (5.32)

The range of \(L_\Delta\) is dense in \(L^\infty(\Delta)\). Hence in view of (5.32) we have \(\phi = 0\). This proves the theorem.

### 6 The kernel of the quasi-potential

In this section we shall investigate the properties of the kernel \(\Phi(x, y)\) of the quasi-potential \(B\).

**Proposition 6.1.** Let conditions (5.2), (5.3) be fulfilled and let the quasi-potential \(B\) be regular. Then the corresponding kernel \(\Phi(x, y)\) is non-negative i.e.

\(\Phi(x, y) \geq 0.\) (6.1)

**Proof.** In view of (5.14) and (5.15) we have \(\psi(x, s, u) \geq 0\). Relation (5.24) implies that \(\psi(x, s) \geq 0\). Now it follows from (5.28) that

\(\Phi(0, x) = \psi(0, x) \geq 0.\) (6.2)

Let us consider the domains \(\Delta_1\) and \(\Delta_2\) which are connected by relation \(\Delta_2 = \Delta_1 + \delta\). We denote the corresponding truncated generators, quasi-poten
tials kernels by \(L_k, B_k\) and \(\Phi_k(x, y), (k=1,2)\). We introduce the unitary operator

\(Uf = f(x - \delta),\) (6.3)
which maps the space $L^2(\Delta_2)$ onto $L^2(\Delta_1)$. At the beginning we suppose that the conditions of Theorem 2.1 are fulfilled. Using formulas (2.1) and (2.2) we deduce that

$$L_2 = U^{-1}L_1U.$$  \hfill (6.4)

Hence the equality

$$L_2 = U^{-1}L_1U.$$  \hfill (6.5)

Hence the equality

$$B_2 = U^{-1}B_1U$$  \hfill (6.6)

is valid. The last equality can be written in the terms of the kernels

$$\Phi_2(x, y) = \Phi_1(x + \delta, y + \delta).$$  \hfill (6.7)

According to (6.2) and (6.6) we have

$$\Phi_1(x + \delta, y + \delta) \geq 0.$$  \hfill (6.8)

As $\delta$ is an arbitrary real number relation (6.1) follows directly from (6.7).

We remark that an arbitrary generator operator $L$ can be approximated by the operators of form (2.1) (see [], Ch.). Hence the proposition is proved.

Proposition 6.2. Let the quasi-potential $B$ be regular. Then the equalities

$$\Phi(a_k, y) = \Phi(b_k, y) = 0 \quad 1 \leq k \leq n$$  \hfill (6.9)

are valid.

7 Sectorial operators

We introduce the following notions.

Definition 7.1. The bounded operator $B$ in the space $L^2(\Delta)$ is called sectorial if

$$(Bf, f) \neq 0, \quad f \neq 0$$  \hfill (7.1)

and

$$-\frac{\pi}{2} \beta \leq \arg(Bf, f) \leq \frac{\pi}{2} \beta, \quad 0 \leq \beta \leq 1.$$  \hfill (7.2)

Definition 7.2. The sectorial operator $B$ is called strong sectorial if $\beta < 1$.

It is easy to see that the following assertion is true.

Proposition 7.1. Let the operator $B$ be sectorial. Then the operator $(I + sB)^{-1}$ is bounded when $s \geq 0$. 

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