Interacting Fermi liquid  
in two dimensions at finite temperature  
Part II: Renormalization

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Abstract
This is a companion paper to [DR1]. Using the method of continuous renormalization group around the Fermi surface and the results of [DR1], we achieve the proof that a two-dimensional jellium system of interacting Fermions at low temperature $T$ is a Fermi liquid above the BCS temperature. Following [S1], this means proving analyticity in the coupling constant $\lambda$ for $|\lambda||\log T| \leq K$ where $K$ is some numerical constant, and some uniform bounds on the derivatives of the self-energy.

I Introduction

For general introduction we refer to the [DR1] paper. We assume all its results and notations. In [DR1] the “convergent contributions” to the vertex functions of a two-dimensional weakly interacting Fermi liquid were controlled hence the results of [FMRT] were essentially reproduced but with a continuous renormalization group analysis, as advocated in [S1]. In this paper we consider the complete sum of all graphs, perform renormalization of the two point subgraphs and obtain our main theorem. This is not a trivial extension of the methods of [FMRT] and [FT1-2], since renormalization has to be performed in phase space, not momentum space. This raises a delicate point: since angular sector decomposition has to be anisotropic [FMRT], it is not obvious that one gains anything by renormalizing in phase space, if the sector directions of the spanning tree used for spatial integration do
not match the sector directions of the external legs. This non-trivial problem is solved here by a somewhat delicate one-particle irreducibility analysis for two point subgraphs that must respect the determinant structure of the Fermionic loop variables and Gram’s inequality.

Here we go.

II Renormalization

We consider now the sum over all (not necessarily convergent) attributions. By [DR1], eq.(IV.51-53) the four point and two point subgraphs are convergent at finite temperature, but diverge logarithmically and linearly respectively when \( T \to 0 \). Remark that, as we keep \( T \geq T_c > 0 \), we could avoid performing renormalization at all, but in this case the estimation of the convergence radius would be bad. Actually, we would have to bound a sum such as

\[
\sum_{n=1}^{\infty} \sum_{n_4+n_2 \leq n} |\lambda|^{n_2} K_2^{n_4} \log w_T |^{n_4} w_T^{-\frac{n_2}{2}}
\]

where \( n_4 \) and \( n_2 \) are the number of four point and two point subgraphs respectively. Since \( n_2 + n_4 \leq n \) it is easy to check that the convergence radius of this sum is defined by the upper bound on the critical temperature

\[
T_{\text{upper}} = \max \left[ T_{c(4)}^{(4)}, T_{c(2)}^{(2)} \right] = \frac{1}{\pi\sqrt{2}} \max \left[ e^{-\frac{1}{|\lambda|^{2}K_2}}, \left( |\lambda|^{2K_2} \right) \right] = \frac{|\lambda|^{2K_2}}{\pi\sqrt{2}}.
\]

Actually one can do slightly better and find a bound in \( |\lambda|^2 \), because tadpoles vanish, so that one has effectively \( n_2 \leq n/2 \). But we see that without renormalization of the two point subgraphs, we cannot get an upper bound on the critical temperature of the non-perturbative form predicted by the theory of superconductivity\(^1\), namely:

\[
T_{c}^{\text{true}} \simeq C_1 e^{-\frac{1}{|\lambda|^{2}K_2}}.
\]

\(^1\)We recall that in dimension \( d = 2 \) by the Mermin-Wagner theorem there is no continuous symmetry breaking at finite temperature, but there ought to be a critical temperature associated to a Kosterlitz-Thouless phase. At zero temperature, there are three non-compact dimensions (space plus imaginary time) and there should be a continuous symmetry breaking with an associated Goldstone boson.
where \( C_1 \) and \( C_2 \) are two constants related to the physical parameters of the model such as the Debye frequency, the electron mass, the interatomic distance, and the particular crystalline lattice structure.

Our goal in this paper is to prove an upper bound on \( T_c \) i.e. give a value of \( T_{c \text{ upper}} \) which is non-perturbative like (II.3) but with different constants \( K_1 \) and \( K_2 \). To obtain this behavior we need to perform renormalization, but only for two-point subgraphs, which amounts to a computation of the flow of the chemical potential only.

Hence in this paper we will use the interacting action

\[
S_V = \frac{\lambda}{2} \int_V d^3x \left( \sum_a \bar{\psi}_a \psi_a \right)^2 + \delta \mu_\Lambda \int_V d^3x \left( \sum_a \bar{\psi}_a \psi_a \right)
\]

(II.4)

where \( \lambda \) is the bare coupling constant and \( \delta \mu_\Lambda \) is the bare chemical potential counterterm, which is function of the ultraviolet cut-off \( \Lambda_0 = 1 \) and the infrared cut-off \( \Lambda \). The free covariance is as usual

\[
\hat{C}_{ab}(k) = \delta_{ab} \frac{1}{ik_0 - (\vec{k}^2 - \mu)}
\]

(II.5)

where \( \mu = 1 \) is the renormalized chemical potential and we have taken \( 2m = 1 \). The BPHZ condition states

\[
\delta \mu_{\text{ren}}(\Lambda) = \delta \mu_\Lambda = \hat{\Sigma}^\Lambda(k_F) = \int d^3x \ e^{-ik_fx} \Sigma^\Lambda(0, x) = 0
\]

(II.6)

where \( \Sigma^\Lambda \) is the two point vertex function \( \Gamma_{\Lambda_0,\Lambda}^\Lambda(x_1, x_2) \) (\( \Lambda_0 = 1 \)), \( k_f \) is some vector as near as possible to the Fermi surface, (the Fermi surface cannot be reached at finite temperature because of the antiperiodicity of Fermions) hence with \( |k_{F0}| = \pi T \) and \( |\vec{k}_F| = 1 \). This function actually coincides with the 1PI one, as the Gevrey cut-off on internal lines fixes the 1PR contributions to zero. By rotation invariance, this condition does not depend on the angular part of \( \vec{k}_F \). On the other hand, to conserve the parity in the imaginary time direction we should take the mean value \( 1/2[\Sigma^\Lambda(\pi T, \vec{k}_F) + \Sigma^\Lambda(-\pi T, \vec{k}_F)] \), but in our computations this is not necessary. The main result of our paper is

\footnote{To find the exact constant \( K_1 = C_1 \) in our Theorem 3 is trivial, but to find a bound with the exact constant \( K_2 = C_2 \) requires to compute the flows of the coupling constant also. This is almost certainly also doable within the methods of this paper, but introduces some painful complications, since there are really infinitely many running coupling constants [FT2].}
Theorem 1 The limit $\Lambda \to 0$ of $\Gamma_{2p}^{\Lambda\Lambda_0}(\phi_1, \ldots, \phi_{2p})$ is analytic in the bare coupling constant $\lambda$, for all values of $\lambda \in \mathbb{C}$ such that $|\lambda| \leq c$, with $c$ given by the equivalent relations

$$T = K_1 e^{-\frac{1}{K_2}} \quad ; \quad c = \frac{1}{K_2 |\log T/K_1|}$$

for some constants $K_1$ and $K_2$ (this relation being limited to the interesting low temperature regime $T/K_1 < 1$).

This theorem is in a sense a generalization of [DR1], Theorem 1, and the remaining part of this section is devoted to its proof.

With the new action (II.4) the expression to bound becomes:

$$\Gamma_{2p}^{\Lambda\Lambda_0}(\phi_1, \ldots, \phi_{2p}) = \sum_{n \geq 1} \frac{\lambda^n}{n!} \frac{(\delta \mu_\Lambda)^{n'}}{n'!} \sum_{a-T} \sum_{E} \sum_{\Omega} \varepsilon(T, \Omega) \int d^3x_1 \ldots d^3x_\bar{n}$$

where $n$ is the number of four point vertices (with coupling constant $\lambda$), $n'$ is the number of two point vertices (with coupling constant $\delta \mu_\Lambda$) and we defined $\bar{n} = n + n'$. Now, we can insert band attributions and classes exactly as in [DR1].

II.1 Extracting loop lines.

Before introducing sectors, we must perform an additional expansion of the loop determinant. This is necessary for two reasons:

- to select the two-point subgraphs that really need renormalization;
- to optimize sector counting by reducing the number of possible sector choices, in order to perform renormalization.

We introduce some notations. For any class $C$ we define $D_C$ as the set of “potentially dangerous” two-point subgraphs $g_i$. They are defined by the following property: by cutting a single tree line on the path joining the two external vertices of $g_i$ we cannot separate $g_i$ into two disconnected subgraphs.
Figure 1: examples of subgraphs not belonging to $D_C$; tree lines are solid and loop fields are wavy.

Figure 2: a subgraph $g$ and the reduced correspondent subgraph $g^r$; $g_1$, $g_2$ and $g_3$ belong to $A(g)$

$g_j(C)$ and $g_{j'}(C)$, one of them, say $g_j(C)$, being a two point subgraph. This property is similar but not equal to 1PI (one particle irreducibility). In Fig.1 there are some examples of subgraphs not belonging to $D_C$.

By the relation of partial order in the CTS, $D_C$ has a forest structure (see [R]). This means that for any pair $g$ and $g' \in D_C$ we have $g \cap g' = \emptyset$ or $g \subseteq g'$ or $g' \subseteq g$. Now, for any $g \in D_C$, we define the set $A(g)$ of maximal subgraphs $g' \in D_C$, $g' \subset g$. The loop determinant is then factorized on the product of several terms: one for each set $il_j(C)$, $g_j \in A(g)$, one containing the remaining internal loop fields in $g_i$, and a last term containing all the other loop fields. Then, the good object to study is not $g$, but the reduced graph $g^r := g/D_C$, where each $g_j \in A(g)$ has been reduced to a single point (see Fig.2). For each $g_i^r$ we denote the set of internal loop half-lines by $il_i^r$ and the set of vertices by $V_i^r$. 
For each \( g_i \), \( g_i \in D_C \), we call \( h_i^{(1)} \) the external half-line \( h_i^{\text{root}} \) and \( h_i^{(2)} \) the other external half-line. In the same way we define \( v_i^{(1)} \) and \( v_i^{(2)} \). With these definitions, we introduce the chain \( C_r^i \) which joins the dot vertex \( v_{h_i^{(2)}} \) to the cross vertex just above the cross \( t(i) \) (see Fig.3).

On this chain we define the set \( J_i \) of crosses (and eventually one dot) indices \( j \) corresponding to four-point subgraphs \( |e_{g_j}(C)| = 4 \). We order them starting from the lowest index \( j_1 \) and going up to the highest \( j_{|J_i|} \). Remark that, by definition of \( D_C \), there is no index \( j \) on the chain with \( |e_{g_j}(C)| = 2 \).

Again we introduce the reduced subgraphs \( g_j^r(C) := g_j(C) / g_{j+1}(C) \) (see Fig.4), the set of internal loop half-lines of \( g_j^r \), \( il_{j}^r \), and that of internal vertices, \( V_{j}^r \). Then the corresponding loop determinant is factorized

\[
\det(il_{j}^r) = \det(il_{j+1}^r) \det(il_{j+2}^r). \tag{II.8}
\]

For the first step of the induction we define \( j_0 = i, g_{j_0} := g_i^r \) (it is a two point subgraph!) and \( g_{j_0}^r := g_{j_0}/g_{j_1} \).

For each \( g_j^r, q = 1, \ldots, |J_i| \) we call \( h_j^{(1)} \) the external tree half-line \( h_j^{\text{root}} \), \( l_j^{\text{root}} \) the corresponding tree line, \( h_j^{(2)} = h_j^{(2)} \) (remark that \( h_j^{\text{root}} \) can never coincide with \( h_i^{(2)} \) by construction), and \( h_j^{(3)}, h_j^{(4)} \) the remaining two external half-lines. The line \( l_j^{\text{root}} \) cuts the tree \( t_i \) into two connected components. We call \( T_{j_1}(i) \) the component that contains the vertex \( x_i^{(1)} \), and \( T_{j_2}(i) \) the other component, that contains the vertex \( x_i^{(2)} \). Remark that all vertices in \( g_j^r \),

\[
\text{Figure 3: example of } C_r^i: \text{ the dashed lines belong to the chain}
\]
belong to $\mathcal{T}_{j_q}^R(i)$.

For each $q = 1, \ldots, |J_i|$ (starting from the lowest and going up) we test if there is some loop line $l_{j_q}^\ell$ with $f, g \in \set{l_i^\ell}$ connecting $\mathcal{T}_{j_q}^L(i)$ with $\mathcal{T}_{j_q}^R(i)$. If for some $j_q \in J_i$ there is no loop line $g_i$ is actually 1PR (one particle reducible) and, by momentum conservation, it does not need to be renormalized (as it is shown below). On the other hand, if $\forall j \in J_i$ we can find a loop line, then $g_i$ is 1PI and it must be renormalized.

We perform this test inductively. At each subgraph $g_{j_q}$ we define

$$
\begin{align*}
L^R_{j_q}(i) &:= \{ a \in i{l_{j_q-1}^r} \mid a \in \mathcal{T}_{j_q}^R(i), m(a, C) \leq i(l_{j_q}^{\text{root}}) \leq \mathcal{A}(j_q) \} \\
L^L_{j_q}(i) &:= \{ a \in i{l_{j_q-1}^r} \mid a \in \mathcal{T}_{j_q}^L(i), m(a, C) \leq i(l_{j_q}^{\text{root}}) \leq \mathcal{A}(j_q) \},
\end{align*}
$$

(II.9)

(where we recall that $\mathcal{A}(j_q)$ (defined in [DR1]) is the index of the highest external tree line of $g_{j_q}^r$). Actually, $L^R_{j_q}(i)$ is the set of internal loop half-lines of $g_{j_q-1}^r$ which are hooked to $\mathcal{T}_{j_q}^R(i)$ and may connect somewhere in $\mathcal{T}_{j_q}^L(i)$. By construction, no internal loop half-line of $g_{j_q}^r$ and no external loop half-line of $g_{j_q-1}^r$ belongs to $L^L_{j_q}(i) \cup L^R_{j_q}(i)$. This is the main reason for which this expansion does not develop any new factorial.

We distinguish three situations:

1. $h_{j_q}^{(3)}$ and $h_{j_q}^{(4)} \in L$ (see Fig[4]). Then $l_{j_q}^{\text{root}} = l_{\mathcal{A}(j_q)}$, $L^R_{j_q}(i)$ is reduced to two elements and we develop the determinant to chose where they contract, applying two times the following formula:

$$
\det \mathcal{M'} = \sum_a \mathcal{M'}_{h_{j_q}^{(3)}, a} \varepsilon(h_{j_q}^{(3)}, a) \det \mathcal{M'}_{\text{red}},
$$

(II.10)
If $g_i$ is 1PR. If not, we have $|L^L_{j_q}(i)|^2$ choices to contract them. Remark that if $h^{(3)}_{j_q}$ or $h^{(4)}_{j_q}$, or both are external lines at some $g_{j_{q'}}$, with $q' < q$, then they have already been extracted from the determinant and we do not touch them.

2. $h^{(3)}_{j_q} \in L$ and $h^{(4)}_{j_q} \in t_i$ (see Fig.8).

If $h^{(3)}_{j_q}$ has not been already contracted at some lower scale, we develop the determinant as before to choose where $h^{(3)}_{j_q}$ contracts. If $h^{(3)}_{j_q}$ has already been contracted at some lower scale we do not touch it.

In any case, if $h^{(3)}_{j_q}$ contracts with some element of $L^R_{j_q}(i)$, then 1PI is assured and we go to the step $q+1$. If not (Fig.9a), we test the loop determinant in the following way:

$$
\text{det } \mathcal{M}'(C) = \text{det } \mathcal{M}'(C)(0) + \int_0^1 ds_{j_q} \frac{d}{ds_{j_q}} \text{det } \mathcal{M}'(s_{j_q}). \quad \text{(II.11)}
$$

where we defined

$$
\mathcal{M}'_{x_f,x_g}(C)(s_{j_q}) = s_{j_q} \mathcal{M}'_{x_f,x_g}(C) \quad s_{j_q} \in [0,1] \quad \text{(II.12)}
$$

if $(f,g)$ or $(g,f)$ belong to $L^R_{j_q}(i) \times L^L_{j_q}(i)$ and

$$
\mathcal{M}'_{x_f,x_g}(C)(s_{j_q}) = \mathcal{M}'_{x_f,x_g}(C) \quad \text{(II.13)}
$$

otherwise. The term $s \neq 0$ extracts from the determinant the loop line we wanted (see Fig.9a). The term $s = 0$ means that $g_i$ is 1PR. The number of choices is bounded by $|L^R_{j_q}(i)|^2|L^L_{j_q}(i)|$.

3. $h^{(3)}_{j_q}$ and $h^{(4)}_{j_q} \in t_i$. Then we apply directly the interpolation formulas (II.11)-(II.12). Again we distinguish the case $s = 0$, that corresponds to $g_i$ 1PR, and the case $s \neq 0$ that corresponds to $g_i$ 1PI and has at most $|L^L_{j_q}(i)||L^R_{j_q}(i)|$ terms (see Fig.10a).

Repeating the same procedure for all $j \in J_i$ we extract from the loop determinant at most $2|J_i|$ internal loop line propagators. For each class $C$, the process $J^0_i$ specifies the set of $j_q \in J_i$ for which one or two loop lines have been extracted simply developing the determinant, $J^1_i$ specifies the set of $j_q \in J_i$ for which one loop line has been extracted applying (II.12). In the
same way the process $P_0$ and $P_1$ specifies which loop fields are contracted in $J^0_i$ and $J^1_i$ for all $i$. Then

$$\det \mathcal{M}'(C) = \sum_J \sum_P g_i \prod_{l_{fg} \in D_C} \left[ \left( \prod_{l_{fg} \in J^0_i} \mathcal{M}_{j_{fg}}(C) \right) \prod_{l_{fg} \in J^1_i} \int_0^1 ds_{j_{fg}} \det \mathcal{M}'(\{s_{j_{fg}}\}) \right]. \tag{II.14}$$

where $J$ defines the sets $J^0_i$ and $J^1_i$ for all $i$. For each loop line $l_{fg}$ extracted, the set of band indices accessible for both $f$ and $g$ is reduced to

$$M'(f, C) = M'(g, C) = \min[M(f, C), M(g, C)]$$

$$m'(f, C) = m'(g, C) = \max[m(f, C), m(g, C)] \tag{II.15}$$

We have to verify that the new matrix $\mathcal{M}'(C)(\{s_{j_{fg}}\})$ still satisfies a Gram’s inequality and that the sum over processes does not develop a factorial. This is done in the following two lemmas. Remark that the sum over $J$ is not dangerous. Actually at each $j_{fg}$ we have two choices, hence $|J| \leq 2^n$.

**Lemma 1** $\mathcal{M}'(C)(\{s_{j_{fg}}\})$ satisfies the same Gram inequality as $\mathcal{M}(C)$ in [DR1], (IV.4), which does not depend on the parameters $s_{j_{fg}}$.

**Proof** The proof is identical to that of [DR1], Lemma 4. The only difference is that now $W^k_{v,a,v',a'}$ contains an additional $s$ dependent factor $S^k_{v,a,v',a'}$. By (II.8) or (II.17) below, we recall that the determinant for the set $L^{R}_{j_q}(i) \cup L^{L}_{j_q}(i)$ of fields and antifields which may be concerned by the $s_{j_q}$ interpolation step factorize in the big loop determinant, so we need only to consider a single such factor $S^{k,j_q}_{v,a,v',a'}$, and prove that it is still a positive matrix. This is obvious if we reason on the index space for the vertices $v$ and $v'$ to which the fields and antifields hook (and not on the fields or antifields indices $a$ and $a'$ themselves). Indeed $S^{k,j_q}_{v,a,v',a'}$ is $\chi^v_a, \chi^v_{a'}$ (the positive matrix which is 1 if $a$ hooks to $v$ and $a'$ hooks to $v'$, and 0 otherwise) times the combination with positive coefficients $s_{j_q} M_{v,v'} + (1 - s_{j_q}) N_{v,v'}$ of the positive matrix $M$ which has each coefficient equal to 1 and the positive block matrix $N$ which has $N_{v,v'} = 1$ if $v$ and $v'$ belong both to $T^{R}_{j_q}$ or both to $T^{L}_{j_q}$ and $N_{v,v'} = 0$ otherwise. Therefore the matrix $S^{k,j_q}_{v,a,v',a'}$ is positive in the big tensor space.
spanned by pairs of indices \( v, a \), it has a diagonal bounded by 1, and we can complete the proof as in [DR1], Lemma 4. The conclusion is that the additional interpolation parameters \( s_{jq} \) do not change the Gram estimate and the norms of \( F_f \) and \( G_g \) given in [DR1].

**Lemma 2** The cardinal of \( P \) is bounded by \( K^n \) for some constant \( K \).

**Proof** The loop determinant is factorized on determinants restricted to each reduced two-point subgraph in \( D_c \):

\[
\det M' = \prod_{g^r_i \in D_c/A(g_i)} \det M'(iI^r_i). \tag{II.16}
\]

Each determinant \( \det M'(iI(g^r_i)) \) is in turn factorized on determinants restricted to internal loop fields for each reduced subgraph \( g^r_{jq}, q = 0, \ldots, |J_i| \):

\[
\det M'(iI(g^r_i)) = \prod_{q=0}^{|J_i|} \det M'(iI^r_{jq}). \tag{II.17}
\]

We have seen that for each \( g_{jq} \) the number of terms in \( P \) is bounded by \( |L^R_{jq}(i)|^2 |L^L_{jq}(i)| \). Then

\[
|P| \leq \prod_{g^r_i \in D_c/A(g_i)} \prod_{q=0}^{|J_i|} (|L^L_{jq}(i)| |L^R_{jq}(i)|^2)
\]

\[
\leq 2^n e \sum_{g^r_i \in D_c/A(g_i)} \sum_{q=0}^{|J_i|} (|L^R_{jq}(i)| + |L^L_{jq}(i)|)
\]

\[
\leq 2^n e 4 \sum_{g^r_i \in D_c/A(g_i)} |V^r_i| \leq K^n, \tag{II.18}
\]

where we applied

\[
\sum_{q=0}^{|J_i|} (|L^R_{jq}(i)| + |L^L_{jq}(i)|) \leq 4 |V^r_i| ; \quad \sum_{g^r_i \in D_c/A(g_i)} |V^r_i| \leq \bar{n}. \tag{II.19}
\]

This completes the proof.

Now we can insert sector decouplings exactly as we did in [DR1], but with a few additional operations.
II.2 Sector refinement.

For each $g_i \in D_c$ and 1PI we introduce one more sector decomposition on $h_i^{(2)}$, in order to optimize the bounds from renormalization (Sec.II.7). Actually, the finest sector of size $\Lambda^{\frac{1}{2}}(w_{i(h_i^{(2)})})$ is further decomposed in a smaller sector of size

$$\Lambda^{\frac{1}{2}}(w_{j(h_j^{(2)})}) := \Lambda^{\frac{1}{2}}(w_{i(h_i^{(2)})}) \ z_i$$

where $i(h_i^{(2)}) \leq A(i)$ is the band index of $h_i^{(2)}$ and $0 < z_i \leq 1$ is a factor to be chosen. This sector is introduced applying the identity [DR1](III.11) with $\alpha_s = \alpha_{j_{h,1}}$ defined by $\alpha_{j_{h,1}}^{-\frac{4}{3}} := \Lambda^{\frac{1}{2}}(w_{j_{h,1}})$. All the other larger sectors are introduced through the identity [DR1](III.13). The effect of last refinement is an additional factor $1/z_i$ from sector counting and spatial integration (this only if the refined line is a tree one), and a factor $z_i$ from the volume in impulsion space. Then, we have are left with a global factor $1/z_i$. The optimal value is $z_i = \Lambda^{\frac{1}{2}}(w_{i(i)})$, as will be explained at the end of Sec. II.7.

The expression to bound is then similar to [DR1] (III.16):

$$\Gamma_{\Delta 0}(\phi_1^{\Delta T}, \ldots, \phi_{2p}^{\Delta T}) = \sum_{n \geq 1} \frac{\lambda^n (\delta \mu_A)_{n'}}{n!}$$

$$\sum_{CST} \sum_{u-T} \sum_{E} \sum_{E} \sum_{L} \sum_{C} \sum_{J} \sum_{P} \varepsilon(T, \Omega) \int_{w_T \leq w_A(i)} \frac{\prod_{q=1}^{n-1} dw_q}{d\theta h,n_h} \left[ \prod_{h \in E T \cup E} \left[ \int_{\Omega_{j,h,n_h}} d\theta_h,n_h \right] \right]$$

$$\prod_{h_{j,h,n_h}} \left[ \prod_{r=2}^{n_h} \chi_{\alpha_{j_{h},r}}(\theta_{h,1}) \right]$$

$$\prod_{v \in V \cup V^T} \Upsilon(\theta_{r_{out}}, \theta_{h,v})$$

$$\prod_{g_i} \varepsilon_{r_{out}}(\theta_{h,v})$$

$$\int d^3x_1 \ldots d^3x_h \ \phi_1^{\Delta T}(x_{1, \theta_{e_{1,1}}}) \ldots \phi_{2p}^{\Delta T}(x_{j, \theta_{e_{2p,1}}}) \left[ \prod_{q=1}^{n-1} C_{\mu q}(x_q, \bar{q}, \theta_{h,1}) \right]$$

$$\prod_{l_{f,g} \in P} \mathcal{M}_{f,g}(C, E, \{\theta_{a,1}\}) \prod_{j_{q} \in P} \int_{s_{j}}^{1} ds_{j} \det \mathcal{M}(C, E, \{\theta_{a,1}\}, \{s_{j}\})$$

$$1$$
where we defined $V$ and $V'$ as the set of four point and two point vertex respectively. To perform renormalization we apply to the amplitude of each two point subgraph $g$ the operator $(1 - \tau_g) + \tau_g$, where $\tau_g$ selects the linearly divergent term in $g$ giving a local counterterm for $\delta \mu$ that depends on the scale of the external lines of $g$. We start the renormalization from the leaves of the $CTS$ (hence from the smallest subgraphs at highest scale) and go down.

II.3 Momentum space

The Taylor expansion of $\hat{g}(k)$ around a vector $k_F$ near the Fermi surface gives two possible sources of counterterms. The term of order 0 in the Taylor expansion is linearly divergent and gives rise to a chemical potential counterterm; the term of order 1 is logarithmic and would give rise to wave function counterterms (in fact proportional to $k_0$ and $\vec{k}^2$), that we do not need to consider for our upper bound, Theorem 1. As we said, for this kind of bound we need only to perform the linearly divergent renormalization. Therefore we define the localization operator acting on a two-point function as:

$$\delta(k_1 + k_2) \tau_g \hat{g}(k_2) = \delta(k_1 + k_2) \hat{g}(k_F). \quad (\text{II.22})$$

Remark that by rotational invariance there is no ambiguity in the choice of the spatial component of $k_F$. For the temporal component we choice $k_{F0} = \pi T$, to simplify computations. This choice breaks parity in the imaginary time direction, but in our context this is not essential.

II.3.1 Not Dangerous subgraphs.

We do not need to renormalize all two point subgraphs but only the subset

$$D(C, P) := \{g_i \mid |e_{g_i}(C)| = 2, \ g_i \ 1\text{PR}\} \quad (\text{II.23})$$

in the sense explained in the section II.1. By momentum conservation it is easy to see that, if $g_i(C)$ is 1PR and $g_j(C)$ is the two-point subgraph we obtain cutting one tree line of $g_i$

$$\tau_{g_i}(C) \left(1 - \tau_{g_j}(C)\right) = \left(1 - \tau_{g_i}(C)\right) \tau_{g_j}(C) = 0 \quad (\text{II.24})$$
hence the renormalization of $g_i(C)$ is ensured by that of $g_j(C)$. Remark that, by the relation of partial order in the CTS, $D(C, P)$ has a forest structure (see [R] and [DR2]).

We denote by $ND(C, P)$ (not-dangerous...) the set of two point subgraphs which are 1PR, hence are not renormalized. It is the union of the set of two point subgraphs not in $D_C$, for which we knew one particle reducibility from start, and the set $D_C \setminus D(C, P)$, for which we learnt it after the loop extraction process. For any $g_i \in ND(C, P)$ one internal line $l_j$ must have the same momentum as the external line $l_{A(i)}$. Then the internal and external scales of $g_i$ cannot be far; this imposes a constraint on the integral over the parameter $w_i$ that allows to avoid renormalizing these subgraphs.

II.4 Real space

The formulation of renormalization in momentum space is the one of [FT2] and is sufficient for perturbative results. In this formulation the localization operator is rotation invariant. However for constructive bounds we need a phase space analysis, hence a direct space “dual version” of this operator [R].

In the space of positions, the dual localization operator, which acts on the external lines of the subgraph, is never unique. In relativistic euclidean field theory it depends on the choice of an arbitrary localization point (see [R]), a convenient choice being the position of one of the external vertices. Here, in condensed matter, this dual operator depends on an additional choice, namely a direction on the Fermi surface. A convenient choice is found thanks to the sector decomposition. Actually, before performing the sum over sector attributions, the two external propagators of a graph $g_i$ belong to well defined sectors $\Sigma(\alpha_{j(1)}, \theta_1)$ and $\Sigma(\alpha_{j(2)}, \theta_2)$ with sector center on the vectors $(0, \vec{r}_k)$, $k = 1, 2$, where $j(1), j(2) \leq A(i)$. Therefore we define the operator $\tau_g$ as a first order Taylor expansion around the momentum $k_2 = -r_2 = (-\pi T, -\vec{r}_2)$ (the minus sign corresponding to integration by parts). The dual $x$-space operator $\tau_g^*$ acts on the product of external propagators $C_{\theta_1}(x_1, y_1)C_{\theta_2}(x_2, y_2)$ by

$$\tau_g^* C_{\theta_1}(x_1, y_1)C_{\theta_2}(x_2, y_2) = e^{ir_2(x_2-x_1)} C_{\theta_1}(x_1, y_1)C_{\theta_2}(x_1, y_2). \quad (I1.25)$$

This formula does not coincide with the usual one (see [R]) and can be justified observing that $C_{\theta_s}(x, y)$ is not a slowly varying function with $x$, but
has a spatial momentum of order 1, hence oscillates wildly. The good slowly varying function to move is $C'_{\theta_s}(x, y)$ defined by:

$$C_{\theta_s}(x, y) = \frac{e^{ir_s(x-y)}}{(2\pi)^2} \int d^3k \, e^{i(k-r_s)(x-y)} C_{\theta_s}(k) := e^{ir_s(x-y)} C'_{\theta_s}(x, y).$$

(II.26)

The expression (II.25) can also be obtained defining

$$\tau_g^* C'_{\theta_1}(x_1, y_1) C'_{\theta_2}(x_2, y_2) = C'_{\theta_1}(x_1, y_1) C'_{\theta_2}(x_1, y_2).$$

(II.27)

**Choice of the reference vertex.** The choice of $x_1$ as fixed vertex instead of $x_2$ is arbitrary. In this paper we use the rule that most simplifies notations and calculations (not exactly the same as in [DR2]). For each $g_i \in D(C, P)$ we chose as reference vertex the one hooked to the half-line $h_i^{(1)} = h_i^{root}, v_i^{(1)}$ with position $x_i^{(1)}$. The moved vertex is then $x_i^{(2)}$. This rule implies that tree lines have never both ends moved, and that the root vertex $x_1$, which is essential in spatial integration, is always fixed.

In the following we will denote by $D_t(C, P), D_l(C, P), D_e(C, P)$ the subgraphs in $D(C, P)$ for which the moved line is tree, loop or external respectively.

**II.5 Effective Constants**

At each vertex $v$ we can now resum the series of all counterterms obtained applying $\tau_g$ to all $g \in D(C, P)$ (for different classes $C$, processes $P$ and perturbation orders $\bar{n}$) that have the same set of external lines as $v$ itself. In this way we obtain an effective coupling constant which depends on the scale $\Lambda(w_{v_i})$ of the highest tree line hooked to the vertex $v$. This is automatically true for a two point vertex (and in fact would also be true as in [DR2] for a four point vertex because tadpoles are zero by [DR1], Lemma 2). Each counterterm is now a function

$$F_{\theta_1, \theta_2}(y_1, y_2) = \int d^3x_1 \, C_{\theta_1}(x_1, y_1) C_{\theta_2}(x_1, y_2) \left[ \int d^3x_2 \, g(x_1, x_2) e^{ir_2(x_2-x_1)} \right]$$

$$= \int d^3x_1 \, C_{\theta_1}(x_1, y_1) C_{\theta_2}(x_1, y_2) \hat{g}(-r_2)$$

(II.28)
where we applied the translational invariance of $g$. Now remark that $\hat{g}(k)$ is invariant under rotations of the spatial component $\vec{k}$ of $k$ as the free propagator depends only on the absolute value of $\vec{k}$. Therefore

$$\hat{g}(-r_2) = \hat{g}(-\pi T, |r_2|) = \hat{g}(-\pi T, 1)$$

(II.29)

is independent from $\theta_1$ and $\theta_2$.

**Theorem 2** If we apply to each two point subgraph $g \in D(\mathcal{C}, P)$, for any class $\mathcal{C}$ and process $P$, the operator $(1 - \tau_g) + \tau_g = R_g + \tau_g$, (II.21) can be written as

$$\Gamma^{\Lambda_0}_{2p}(\phi_1^{\Lambda_T}, \ldots, \phi_{2p}^{\Lambda_T}) = \sum_{n \geq 1} \frac{\lambda^n}{n! n'}$$

(II.30)

$$\sum_{C \subseteq S} \sum_{u - T} \mathcal{L} \sum_{\mathcal{E} \Omega} \sum_{\mathcal{C} \ p} \sum_{\mathcal{E} \Omega} \mathcal{E} (T, \Omega) \int_{w_T \leq w_{A(i)} \leq w_1} \prod_{q=1}^{n-1} d w_q$$

$$\prod_{h \in L \cup T \cup E} \left\{ \left[ \frac{1}{2} \Lambda^{-\frac{1}{2}} (w_{j,h,nh}) \right] \int_{0}^{2\pi} d \theta_{h,nh} \left[ -\frac{1}{2} \Lambda^{-\frac{1}{2}} (w_{j,h,nh-1}) \right] \prod_{r=2}^{n_h} \theta_{h,nh-1} \right\}$$

$$\prod_{g_i \in C} \prod_{h \in e_g^i \cup e_g^i} \left. \prod_{v \in V \cup V'} \mathcal{Y} \left( \theta_{i,v}, \{ \theta_{h,T(i)} \}_{h \in e_g^i} \right) \prod_{v \in V \cup V'} \mathcal{Y} \left( \theta_{i,v}, \{ \theta_{h,nh} \}_{h \in e_g^i} \right) \right\}$$

$$\int d^3 x_1 \ldots d^3 x_h \phi_1^{\Lambda_T} (x_i, \theta_{1,i}) \ldots \phi_{2p}^{\Lambda_T} (x_j, \theta_{2p,j}) \left[ \prod_{v \in V'} \delta \mu_\Lambda^w (\lambda) \right]$$

$$\prod_{g_i \in D(\mathcal{C}, P)} \left\{ \left[ \prod_{q=1}^{n-1} C_{w} (x_q, \bar{x}_q, \theta_{1,h}) \right] \prod_{l \in g \in P} \left[ \mathcal{M}_{f,g}(\mathcal{C}, E, \{ \theta_{a,1} \}) \right] \left[ \prod_{j \in J^1} \int_{0}^{1} d s_{j} \right] \det \mathcal{M}^i(\mathcal{C}, E, \{ \theta_{a,1} \}, \{ s_{j} \}) \right\},$$

where $\delta \mu_\Lambda^w (\lambda)$, the effective constant defined by:

$$\delta \mu_\Lambda^w (\lambda) = \hat{\Sigma}^w (-r_2) = \int d^3 x_2 \Sigma^L(w)(0, x_2) e^{-i x_2 x_2},$$

(II.31)
is independent of the choice of the angular component of $\vec{r}_2$. This effective constant is the vertex function $\Gamma^{(w)}_2 \Lambda^{(w)}(\Lambda)$ for an effective theory with IR parameter $\Lambda(w)$, and bare counterterm $\delta \mu^1_\Lambda$. Furthermore $\delta \mu^{(w)}_\Lambda(\lambda)$ is analytic in $\lambda$ and is bounded by

$$\left| \delta \mu^{(w)}_\Lambda(\lambda) \right| \leq K |\lambda| (\Lambda(w) - \Lambda)$$

(II.32)

for some constant $K$. The renormalized $\delta \mu_{\text{ren}}(\Lambda)$ is then the vertex function for an effective theory with IR parameter $\Lambda(0) = \Lambda$

$$\delta \mu_{\text{ren}}(\Lambda) = \delta \mu^1_\Lambda(\lambda) = 0.$$  

(II.33)

Finally the first and second derivatives of the self-energy $\hat{\Sigma}(k)$ are uniformly bounded:

$$\left| \frac{\partial}{\partial k_i} \hat{\Sigma}_{|k_0=\frac{\pi}{\beta},e(k)=0} \right| \leq K|\lambda|^2 ; \left\| \frac{\partial^2}{\partial k_i \partial k_j} \hat{\Sigma}(k) \right\|_{\infty} \leq K$$

where $i$ and $j$ take values $0, 1, 2$, and $K$ is some constant. These bounds are proved in Appendix B.

**Proof** The first part of the theorem actually consists in a reshuffling of perturbation theory, and can be proved by standard combinatorial arguments as in [R]. The only difficulty that is not in [R] is to prove that the parameter $w$ of the effective constant always corresponds to the highest tree line of the vertex: as we said above this is obvious for two point vertices. The second part of the theorem, that is the analyticity of $\delta \mu$ and the bound (II.32), corresponds in statistical mechanics to the problem of fixing the bare mass in such a way that the renormalized mass is zero. This is a standard problem, now well understood. For instance, a proof in the case of the critical $\phi^4_4$ model, can be found in [FMRS] and [GK]. For completeness we recall the arguments of the proof in Appendix A. Finally the bound of the first and second derivatives of the self-energy allows a Taylor expansion around the Fermi surface which proves Fermi liquid behavior [S1]; they would be false in $d = 1$, were Luttinger liquid behavior is known to occur [BGPS]-[BM].

\[\text{Recall that the self-energy is the sum of all non-trivial one-particle-irreducible two point subgraphs.}\]
II.6 Convergence of the Effective Expansion

**Theorem 3** Let $\varepsilon > 0$ and $\Lambda_0 = 1$ be fixed. The series (II.30) is absolutely convergent for $|\lambda| \leq c$ and

$$c \leq \frac{1}{K_2|\log(T/K_1)|}$$  \hspace{1cm} (II.34)

for some constants $K_1$, $K_2$. This convergence is uniform in $\Lambda$, then the IR limits of the vertex functions $\Gamma_{2p}^{\Lambda_0} = \lim_{\Lambda \to 0} \Gamma_{2p}^{\Lambda_0}$ exist, they are analytic in $\lambda$ in a disk of radius $c$, and they obey the bounds

$$|\Gamma_{2p>4}^{\Lambda_0}(\phi_1^{\Lambda T}, ..., \phi_{2p}^{\Lambda T})| \leq K_0 \|\phi_1\| \prod_{i=2}^{2p} \|\hat{\phi}_i\|_{\infty,\delta} T^{\frac{p}{2} - \frac{1}{2p}} [K_1(\varepsilon)]^p (pl)^2 K(c, T) e^{-(1-\varepsilon)\Lambda_1^\frac{1}{2} d_T^\frac{1}{2} (\Omega_1, ..., \Omega_4)}$$  \hspace{1cm} (II.35)

$$|\Gamma_{4}^{\Lambda_0}(\phi_1^{\Lambda T}, ..., \phi_4^{\Lambda T})| \leq K_0' \|\phi_1\| \prod_{i=2}^{4} \|\hat{\phi}_i\|_{\infty,\delta} T^{\frac{4}{2} - \frac{1}{2}} K(c, T) e^{-(1-\varepsilon)\Lambda_1^\frac{1}{2} d_T^\frac{1}{2} (\Omega_1, ..., \Omega_4)}$$  \hspace{1cm} (II.36)

$$|\Gamma_{2}^{\Lambda_0}(\phi_1^{\Lambda T}, \phi_2^{\Lambda T})| \leq K_0''(\varepsilon) \|\phi_1\| \|\phi_2\|_{\infty,\delta} T^2 K(c, T) e^{-(1-\varepsilon)\Lambda_1^\frac{1}{2} d_T^\frac{1}{2} (\Omega_1, \Omega_2)}$$  \hspace{1cm} (II.37)

where $\Omega_i$ is the compact support of $\phi_i$, $K_1(\varepsilon)$, $K_0'(\varepsilon)$ and $K_0''(\varepsilon)$ are functions of $\varepsilon$ only, $d_T(\Omega_1, ..., \Omega_2p)$ is defined as in [DR1], Theorem 2, $K(c, T)$ is a function which tends to 0 when $c \to 0$, and

$$\|\hat{\phi}_i\|_{\infty,2} := \left(\|\hat{\phi}_i\|_{\infty} + \|\hat{\phi}_i'\|_{\infty} + \|\hat{\phi}_i''\|_{\infty}\right). \hspace{1cm} (II.38)$$

This Theorem (that is a generalization of [DR1], Theorem 2) means that one can build in a constructive sense the infrared limit of the Fermi liquid at a finite temperature higher than some exponentially small function of the coupling constant simply by summing up perturbation theory.

The rest of this paper is devoted to the proof of that theorem.

II.7 Lines interpolation

Before performing any bound we must study the action of $\prod_{g \in D_C} R_g^*$. For each $g_i \in D_C$ the action of $R_g^*$ on the external lines of $g_i$ is

$$R_{g_i}^* C_{\theta_1} (x^{(1)}, y^{(1)}) C_{\theta_2} (x^{(2)}, y^{(2)})$$
where we applied a first order development on $C_{\theta_2}(x(2), y(2)) e^{-ir_2 x(2)}$ and $x(2)(t)$ is any differentiable path with $x(2)(0) = x(1)$ and $x(2)(1) = x(2)$. The external line hooked to $x(2)$ has then been hooked to the point $x(t)$ (see Fig.5) and has now propagator:

$$C_{\mu \theta_2}(x(2)(t), y(2)) := e^{ir_2 x(2)(t)} \frac{d}{dt} \left[ C_{\theta_2}(x(2)(t), y(2)) e^{-ir_2 x(2)(t)} \right].$$

(II.40)

The easiest choice for the path is a linear interpolation between $x(1)$ and $x(2)$:

$$x(2)(t) = x(1) + t(x(2) - x(1)).$$

(II.41)

This is actually the kind of path we will take if the moved line is a loop or an external one. The interpolated line can then be written as

$$C_{\mu \theta_2}(x(2)(t), y(2)) = e^{ir_2 x(2)(t) - x(2)(t)} (x(2) - x(1))^\mu$$

(II.42)

$$\left[ -ir_2 + \frac{\partial}{\partial x(2)(t)} \right] \left[ C_{\theta_2}(x(2)(t), y(2)) \right]$$

$$= e^{ir_2 x(2) - x(2)(t)} \int d^3 k e^{ik(x(2)(t) - y(2))} \left[ i(x(2) - x(1))(k - r_2) \right] C_{\theta_2}(k).$$

When applied to a tree line, this interpolation does not “follow the tree” as the point $x(t)$ in general no longer hooks to some point on a segment corresponding to a tree line. This leads to some difficulties when integrating over
spatial positions. To avoid this we take \( x(t) \) as the path in the tree joining \( x^{(2)} \) to \( x^{(1)} \), as in [DR2]. This path has in general \( q \) lines with vertices \( x_0, \ldots, x_q \) with the conditions \( x_0 = x^{(1)} \) and \( x_q = x^{(2)} \). Remark that, with this rule, the renormalization at higher scales modifies the tree used for renormalization at lower scales. We will define below the modified tree by an induction process. The interpolated line can then be written as

\[
e^{ir_2 x^{(2)}} \left[ e^{-ir_2 x^{(2)}} C_{\theta_2}(x^{(2)}, y^{(2)}) - e^{-ir_2 x^{(1)}} C_{\theta_2}(x^{(1)}, y^{(2)}) \right]
= \sum_{j=1}^{q} \int_{0}^{1} dt \ C_{\theta_2}^{m}(x_j(t), y^{(2)}) , \tag{II.43}
\]

where we defined

\[
C_{\theta_2}^{m}(x_j(t), y^{(2)}) = e^{ir_2 x^{(2)}} \frac{d}{dt} \left[ C_{\theta_2}(x_j(t), y^{(2)}) e^{-ir_2 x_j(t)} \right] \tag{II.44}
\]

\[
e^{ir_2 (x^{(2)} - x_j(t))} \left( x_j - x_{j-1} \right)^\mu \left[ -ir_2 + \frac{\partial}{\partial x_j(t)} \right]_{\mu} \left[ C_{\theta_2}(x_j(t), y^{(2)}) \right]
= \frac{e^{ir_2 (x^{(2)} - x_j(t))}}{(2\pi)^2} \int d^3 k \ e^{ik(x_j(t) - y^{(2)})} \left[ i(x_j - x_{j-1})(k - r_2) \right] C_{\theta_2}(k)
\]

and

\[
x_j(t) = x_{j-1} + t(x_j - x_{j-1}) . \tag{II.45}
\]

### II.7.1 Second order expansion

The renormalizing factor is \((k - r_2)(x_j - x_{j-1})\), or \((k - r_2)(x^{(2)} - x^{(1)})\). The size of \((k - r_2)\) is fixed by the cut-off of the propagator \( C_{\theta_2} \):

\[
(k - r_2)_0 \simeq \Lambda(w_{i_2}) \leq \Lambda(w_{A(i)}) \leq \Lambda(w_{A(i)})
(k - r_2)_r(r_2) \simeq \Lambda(w_{i_2}) \leq \Lambda(w_{A(i)})
(k - r_2)_t(r_2) \simeq \Lambda_{\frac{1}{2}}(w_{i_2}) z_i \leq \Lambda_{\frac{1}{2}}(w_{A(i)}) z_i \tag{II.46}
\]

where \((k - r_2)_r(r_2)\) is the spatial component on the direction \( \vec{r}_2 \) and \((k - r_2)_t(r_2)\) is the spatial component on the direction orthogonal to \( \vec{r}_2 \). Remark that the size of the tangential component \((k - r_2)_t(r_2)\) is the size of the finest sector of the propagator \( C_{\theta_2} \); as we have said in the precedent subsection we have cut
its finest sector scale $\Lambda^\frac{r}{z}(w_{i_2})$ in a smaller sector, to improve the renormalizing factor.

On the other hand $(x_j - x_{j-1})$ is bounded using a fraction of the exponential decay of tree line propagators and give the scale factors (we will perform the detailed calculation in the following):

$$(x_j - x_{j-1})_0 \simeq \Lambda^{-1}(w_{l(i)})$$  \hspace{1cm} (II.47)

$$(x_j - x_{j-1})_{r(r_j)} \simeq \Lambda^{-1}(w_{l(i)})$$

$$(x_j - x_{j-1})_{t(r_j)} \simeq \Lambda^{-\frac{r}{z}}(w_{l(i)}) .$$

$(x^{(2)} - x^{(1)})$ give the same factors, as it can be written as $\sum_j (x_j - x_{j-1})$. One sees immediately that the components $(k - r_2)_0(x_j - x_{j-1})$ and $(k - r_2)_{r(r_2)}(x_j - x_{j-1})_{r(r_2)}$ give the factor $\Lambda(w_{A(i)}) \Lambda^{-1}(w_{l(i)})$ that we need to renormalize, but $(k - r_2)_{t(r_2)}(x_j - x_{j-1})_{t(r_2)}$ gives only $\Lambda^\frac{r}{z}(w_{A(i)}) z_i \Lambda^{-1}(w_{l(i)})$ that is not sufficient. This is the main difficulty, announced in the Introduction, that we met in this paper: when trying to renormalize in phase space with anisotropic sectors, the internal decay of the tree does not necessarily match the external sector scales. To solve this problem we expand to second order, by $\int_0^1 dt F''(t) = F''(0) + \int_0^1 dt (1 - t) F'''(t)$, and we prove that the first order term which gives the bad power counting factor is actually zero. Then we optimize the bound obtained with respect to $z_i$. Indeed this second order Taylor formula gives for loop and external lines

$$\int_0^1 dt \frac{d}{dt} \left[ C_{\theta_2}(x^{(2)}(t), y^{(2)}) e^{-ir_2x^{(2)}(t)} \right] =$$

$$(x^{(2)} - x^{(1)})^\mu \frac{\partial}{\partial x^{(1)}\mu} \left[ C_{\theta_2}(x^{(1)}, y^{(2)}) e^{-ir_2x^{(1)}} \right] + (x^{(2)} - x^{(1)})^\mu (x^{(2)} - x^{(1)})^\nu \int_0^1 dt (1 - t) \frac{\partial}{\partial x^{(2)\mu}(t)} \frac{\partial}{\partial x^{(2)\nu}(t)} \left[ C_{\theta_2}(x^{(2)}(t), y^{(2)}) e^{-ir_2x^{(2)}(t)} \right]$$  \hspace{1cm} (II.48)

where we applied $x^{(2)}(0) = x^{(1)}$. For tree lines we have

$$\sum_{j=1}^q \int_0^1 dt \left[ C_{\theta_2}(x_j(t), y^{(2)}) e^{-ir_2x_j(t)} \right] = \sum_{j=1}^q \left[ \frac{d}{dt} C_{\theta_2}(x_j(t), y^{(2)}) e^{-ir_2x_j(t)} \right] \bigg|_{t=0}$$

$$+ \sum_{j=1}^q \int_0^1 dt (1 - t) \frac{d^2}{dt^2} \left[ C_{\theta_2}(x_j(t), y_j) e^{-ir_2x_j(t)} \right].$$  \hspace{1cm} (II.49)
The last sum on the right hand of the equation is a second order term:

\[
\sum_{j=1}^{q} (x_j - x_{j-1})^\mu (x_j - x_{j-1})^\nu \int_0^1 dt (1-t) \frac{\partial}{\partial x_j^\mu (t)} \frac{\partial}{\partial x_j^\nu (t)} [C_\theta_2(x_j(t), y^{(2)})e^{-ir_2 x_j(t)}].
\]

(II.50)

The first sum on the right hand of the equation contains a first order and a second order term:

\[
\sum_{j=1}^{q} (x_j - x_{j-1})^\mu \frac{\partial}{\partial x_j^\mu - 1} \left[ C_\theta_2(x_j, y^{(2)})e^{-ir_2 x_j} \right] = (II.51)
\]

\[
(x^{(2)} - x^{(1)})^\mu \frac{\partial}{\partial x_1^\mu} \left[ C_\theta_2(x^{(1)}, y^{(2)})e^{-ir_2 x^{(1)}} \right] + \sum_{j=1}^{q} (x_j - x_{j-1})^\mu \sum_{k=1}^{j-1} \left[ \frac{\partial}{\partial x_k^\mu} \left[ C_\theta_2(x_k, y^{(2)})e^{-ir_2 x_k} \right] - \frac{\partial}{\partial x_{k-1}^\mu} \left[ C_\theta_2(x_{k-1}, y^{(2)})e^{-ir_2 x_{k-1}} \right] \right]
\]

where we applied \( x_j(0) = x_{j-1} \). The first term is the same first order term we obtain for loop or external lines, while the second term can be written as:

\[
\sum_{j=1}^{q} (x_j - x_{j-1})^\mu \sum_{k=1}^{j-1} \int_0^1 dt \frac{d}{dt} \left[ C_\theta_2(x_k(t), y^{(2)})e^{-ir_2 x_k(t)} \right] = (II.52)
\]

\[
\sum_{j=1}^{q} \sum_{k=1}^{j-1} (x_j - x_{j-1})^\mu (x_k - x_{k-1})^\nu \int_0^1 dt \frac{\partial}{\partial x_k^\mu (t)} \frac{\partial}{\partial x_k^\nu (t)} [C_\theta_2(x_k(t), y^{(2)})e^{-ir_2 x_k(t)}]
\]

and gives a second order term that adds to (II.50).

**Lemma 3** The contribution coming from the component orthogonal to \( \vec{r}_2 \) of the first order term

\[
\int d^3 x^{(1)} d^3 x^{(2)} g(x^{(1)}, x^{(2)}) C_\theta_1(x^{(1)}, y^{(1)}) C_\theta_2(x^{(1)}, y^{(2)})
\]

\[
e^{ir_2 (x^{(2)} - x^{(1)})} (x^{(2)} - x^{(1)})_{t(r_2)} \frac{-ir_2 + \frac{\partial}{\partial x^{(1)}_{t(r_2)}}}{C_\theta_2(x^{(1)}, y^{(2)})}
\]

is zero.
The complete first order term is

\[
\int d^3x(1) d^3x(2) \ g(x(1), x(2)) \ C_{\theta_1}(x(1), y(1)) \\
eir_2(x(2) - x(1)) \left(x(2) - x(1)^)^\mu \left(-r_2 + \frac{1}{i} \frac{\partial}{\partial x(1)}\right)_\mu C_{\theta_2}(x(1), y(2))
\]

\[
= \left[ \int d^3x(1) C_{\theta_1}(x(1), y_1) \left(-r_2 + \frac{1}{i} \frac{\partial}{\partial x(1)}\right)_\mu C_{\theta_2}(x(1), y(2)) \right] \\
\left[ i \int d^3x(2) g(0, x(2)) e^{ir_2x(2)x(2)^\mu} \right] \tag{II.54}
\]

where we applied the translational invariance of \( g(x(1), x(2)) \). Now

\[
i \int d^3x_2 \ g(0, x(2)) e^{ir_2x(2)x(2)^\mu} = \frac{\partial}{\partial r_2\mu} \int d^3x(2) \ g(0, x(2)) e^{ir_2x(2)}
\]

\[
= - \left[ \frac{\partial}{\partial k_\mu} \hat{g}(k) \right]_{|k=-r_2^2} \tag{II.55}
\]

To compute the expression we take the two spatial axes on the directions parallel and orthogonal to \(-r_2^2\). Then \(k = -r_2^2\) means \(k_1 = 1\), \(k_2 = 0\) or in radial coordinates \(\rho = 1\) and \(\theta = 0\). As we said before, \(\hat{g}(k)\) depends only on the zero component \(k_0\) and on the module of the spatial vector \(\rho\):

\[
\frac{\partial}{\partial \theta} \hat{g}(k) = 0 \ \ \forall \theta . \tag{II.56}
\]

Now, applying

\[
\frac{\partial \hat{g}(k)}{\partial k_i} = \frac{\partial \hat{g}(k)}{\partial \rho} \frac{\partial \rho}{\partial k_i} + \frac{\partial \hat{g}(k)}{\partial \theta} \frac{\partial \theta}{\partial k_i} \tag{II.57}
\]

for \(i = 1, 2\) and the relations:

\[
\frac{\partial}{\partial k_1} \rho = \frac{k_1}{\rho} , \quad \frac{\partial}{\partial k_2} \rho = \frac{k_2}{\rho} \tag{II.58}
\]

we obtain

\[
\left[ \frac{\partial \hat{g}(k)}{\partial k_0} \right]_{|k=-r_2^2} \neq 0 , \left[ \frac{\partial \hat{g}(k)}{\partial k_{r(r_2)}} \right]_{|k=-r_2^2} \neq 0 , \left[ \frac{\partial \hat{g}(k)}{\partial k_{i(r_2)}} \right]_{|k=-r_2^2} = 0 . \tag{II.59}
\]

This ends the proof.
Choice of $z_i$  After putting the dangerous first order term to zero we are left with the bound

$$
\Lambda_1^{\frac{1}{2}} \left( w_{t(i)} \right) \frac{1}{z_i} \left[ \Lambda (w_{A(i)}) \Lambda (w_{A(i)}) + \frac{\Lambda^2 (w_{A(i)})}{\Lambda^2 (w_{t(i)})} + \frac{\Lambda (w_{A(i)}) z_i^2}{\Lambda^2 (w_{t(i)})} + \frac{\Lambda \left( w_{A(i)} \right) z_i}{\Lambda^2 (w_{t(i)})} \right] \leq \Lambda_1^{\frac{1}{2}} \left( w_{t(i)} \right) \frac{\Lambda (w_{A(i)})}{\Lambda (w_{t(i)})} \left[ \frac{2}{z_i} + \frac{z_i}{\Lambda (w_{t(i)})} + \frac{\Lambda \left( w_{A(i)} \right)}{\Lambda \left( w_{t(i)} \right)} \right]
$$

where $t(i)$ is the band index of the lowest tree line in the path joining the two external vertices of $g_i$, the global factor $\Lambda_1^{\frac{1}{2}} (w_{t(i)})$ comes from a gain in sector sum, as explained in Sec. III.4 and in the second line we have extracted the renormalization factor and bounded $\left[ 1 + \frac{\Lambda (w_{t(i)})}{\Lambda (w_{A(i)})} \right] \leq 2$. To optimize the bound we study the function

$$
f(z_i) = \left[ \frac{2}{z_i} + \frac{z_i}{b} + c \right]
$$

This function has a minimum at $z_i = \sqrt{2b} = \sqrt{2} \Lambda_1^{\frac{1}{2}} (w_{t(i)})$ whose value is

$$
\left[ \frac{2\sqrt{2}}{\Lambda_1^{\frac{1}{2}} (w_{t(i)})} + \frac{\Lambda \left( w_{A(i)} \right)}{\Lambda \left( w_{t(i)} \right)} \right] \leq \frac{1}{\Lambda_1^{\frac{1}{2}} (w_{t(i)})} \left[ 2\sqrt{2} + \frac{\Lambda \left( w_{A(i)} \right)}{\Lambda \left( w_{t(i)} \right)} \right] \leq \frac{K}{\Lambda_1^{\frac{1}{2}} (w_{t(i)})}
$$

This bad factor is compensated by the gain on the sector sum.

### III Main bound

Now we have all we need to perform the bound. We introduce absolute values inside the sums and integrals. As in [DR1], tree line propagators are used to perform spatial integrals and the loop propagator is bounded through a Gram inequality. The difference is that now some propagators (tree, loop or external) have been moved, and bear one or two derivatives, hence giving a different scaling factor. Furthermore some loop propagators have been taken out of the determinant, and there are some additional distance factors to bound, coming from the renormalization factors.
III.1 Loop lines

For each $g_i \in D_l(C,P)$ the interpolation (III.4) applies to the determinant, or to a matrix element that has been extracted. The distance factors and the integral over $t$ are taken out of the determinant by multi-linearity. Then we apply Gram inequality as in section IV.1 of [DR1]. Loop lines in $P$ are bounded by a Schwartz inequality

$$| < F_f(x_f), G_g(x_g) > | \leq ||F_f|| \cdot ||G_g||.$$  \hspace{1cm} (III.1)

The interpolated half-line functions $F_f$ or $G_g$ will have some factors $(k - r_f)^u$ or $(k - r_g)^u$ (actually the two ends of a matrix element could be both interpolated), that modify the estimation of their norms $||F_f||, ||G_g||$. For $f \in L$ being the interpolated line for the subgraph $g_i$, each $(k - r_f)_0$ and $(k - r_f)_{t(r_f)}$ adds a factor $(\alpha - \frac{1}{2})^2$ in the integral [DR1](IV.14), while each $(k - r_f)_{t(r_f)}$ adds a factor $\left[ \Lambda^\frac{1}{4}(w_{M(f,C)})\Lambda^\frac{1}{2}(w_{t(i)}) \right]^2$ as we are integrating $|F_f|^2$. Hence, for each $g_i \in D_l(C,P)$ the contribution to the bound at the first order is

$$\left( |x_i^{(2)} - x_i^{(1)}|_0 + |x_i^{(2)} - x_i^{(1)}|_{r(r_2)} \right) \Lambda^\frac{1}{4}(w_{M(h_i^{(2)},C)}) \left[ \Lambda^3_{M(h_i^{(2)},C)} - \Lambda^3_{A(m(h_i^{(2)},C))} \right]^{\frac{1}{2}}.$$  \hspace{1cm} (III.2)

At the second order it is given by three terms:

$$\Lambda^\frac{1}{4}(w_{M(h_i^{(2)},C)}) \left[ \Lambda^5_{M(h_i^{(2)},C)} - \Lambda^5_{A(m(h_i^{(2)},C))} \right]^{\frac{1}{2}}$$  \hspace{1cm} (III.3)

for the distance factor

$$\left( |x_i^{(2)} - x_i^{(1)}|_0 + |x_i^{(2)} - x_i^{(1)}|_{r(r_2)} \right)^2,$$  \hspace{1cm} (III.4)

$$\Lambda^\frac{1}{4}(w_{M(h_i^{(2)},C)}) \left[ \Lambda^3_{M(h_i^{(2)},C)} - \Lambda^3_{A(m(h_i^{(2)},C))} \right]^{\frac{1}{2}} \Lambda^\frac{1}{4}(w_{M(h_i^{(2)},C)}) \Lambda^\frac{1}{4}(w_{t(i)})$$  \hspace{1cm} (III.5)

for the distance factor

$$\left( |x_i^{(2)} - x_i^{(1)}|_0 + |x_i^{(2)} - x_i^{(1)}|_{r(r_2)} \right) \left| x_i^{(2)} - x_i^{(1)} \right|_{t(r_2)},$$  \hspace{1cm} (III.6)

and

$$\Lambda^\frac{1}{4}(w_{M(h_i^{(2)},C)}) \left[ \Lambda_{M(a,C)} - \Lambda_{A(m(a,C))} \right]^{\frac{1}{2}} \Lambda(w_{M(h_i^{(2)},C)}) \Lambda(w_{t(i)})$$  \hspace{1cm} (III.7)
for

$$|x_i^{(2)} - x_i^{(1)}|_{t(r_2)}^2.$$  \hspace{1cm} (III.8)

Then, the loop determinant times the product of extracted loop propagators is bounded by the usual term

$$\prod_{a \in L} \Lambda_{M(a,C)}^3 \left[ 1 - \frac{\Lambda_{\mathcal{A}(m(a,C))}}{\Lambda_{M(a,C)}} \right]^{\frac{1}{2}}$$

(where we applied the relations $\sqrt{1 - \frac{2x}{1-x}} \leq \sqrt{3}$ and $\sqrt{1 - \frac{x^2}{1-x}} \leq \sqrt{5}$ for $x \leq 1$) times the terms coming from renormalization:

$$\prod_{g_i \in D_1(C,P)} \left\{ \left( |x_i^{(2)} - x_i^{(1)}|_0 + |x_i^{(2)} - x_i^{(1)}|_{t(r_2)} \right) \Lambda(w_{\mathcal{A}(i)}) \right\}$$

$$+ \left( |x_i^{(2)} - x_i^{(1)}|_0 + |x_i^{(2)} - x_i^{(1)}|_{t(r_2)} \right)^2 \Lambda^2(w_{\mathcal{A}(i)})$$

$$+ \left( |x_i^{(2)} - x_i^{(1)}|_0 + |x_i^{(2)} - x_i^{(1)}|_{r(r_2)} \right) \left| x_i^{(2)} - x_i^{(1)} \right|_{t(t_2)} \Lambda^2(w_{\mathcal{A}(i)}) \Lambda^2(w_{t(i)})$$

$$\leq \prod_{g_i \in D_1(C,P)} \left\{ \left( |x_i^{(2)} - x_i^{(1)}| \right) \Lambda(w_{\mathcal{A}(i)}) + \left| x_i^{(2)} - x_i^{(1)} \right|^2 \Lambda(w_{\mathcal{A}(i)}) \Lambda(w_{t(i)}) \right\}$$

\textbf{III.2 External lines}

It is easy to see that, when some external test function is moved, the bound obtained in [DR1], section IV.2, becomes

$$||\phi_i||_1 \left[ \prod_{i=2}^{2p} ||\hat{\phi}_i||_{\infty,2} \right] \left( \Lambda_{\mathcal{F}}^5 \right)^{(2p-1)}$$

multiplied by the factor

$$\prod_{g_i \in D_{c}(C,P)} \left\{ \left( |x_i^{(2)} - x_i^{(1)}|_0 + |x_i^{(2)} - x_i^{(1)}|_{r(r_2)} \right) \Lambda(w_{\mathcal{A}(i)}) \right\}$$

$$+ \left( |x_i^{(2)} - x_i^{(1)}|_0 + |x_i^{(2)} - x_i^{(1)}|_{r(r_2)} \right)^2 \Lambda^2(w_{\mathcal{A}(i)})$$

$$+ \left( |x_i^{(2)} - x_i^{(1)}|_0 + |x_i^{(2)} - x_i^{(1)}|_{r(r_2)} \right) \left| x_i^{(2)} - x_i^{(1)} \right|_{t(t_2)} \Lambda^2(w_{\mathcal{A}(i)}) \Lambda^2(w_{t(i)})$$

$$\leq \left\{ \left( |x_i^{(2)} - x_i^{(1)}| \right) \Lambda(w_{\mathcal{A}(i)}) + \left| x_i^{(2)} - x_i^{(1)} \right|^2 \Lambda(w_{\mathcal{A}(i)}) \Lambda(w_{t(i)}) \right\}.$$  \hspace{1cm} (III.11)
where $||\hat{\phi}_i||_{\infty,2}$ has been defined in (II.38).

### III.3 Tree lines.

As we said, interpolated tree lines are moved along the connection between the external vertices of any graph provided by the tree. But, as the tree itself is modified by renormalization, this process has to be inductive, starting from the smallest graph and going down towards the biggest. We take for this construction the same rules as in [DR2], with some simplifications as we do not treat four point subgraphs. Remark that only the renormalization of subgraphs in $D_t(\mathcal{C},P)$ can modify the tree. Our induction creates progressively a new tree $\mathcal{T}(\mathcal{J})$. To describe it, we number the subgraphs in $D_t(\mathcal{C},P)$ in the order we meet then $g_1, \ldots g_r$. At the stage $1 \leq p \leq r$, before renormalization of $g_p$, the tree is called $\mathcal{T}(\mathcal{J}_{p-1})$. Then we interpolate the external line of $g_p$ following the unique path in $\mathcal{T}(\mathcal{J}_{p-1})$ connecting the two external vertices of $g_p$. Then we update $\mathcal{J}$ and $\mathcal{T}$. We define $\mathcal{J}_p = \mathcal{J}_{p-1}$ for the first order term, as the propagator hooks to the reference vertex, $\mathcal{J}_p = \mathcal{J}_{p-1} \cup \{j\} \cup \{k\}$ for the second order term, where $j$ and $k$ are the indices of the lines of $\mathcal{T}(\mathcal{J}_{p-1})$ chosen by renormalization. Finally we update the tree according to Fig.6.

In the following we will call $D^1_t(\mathcal{C},P)$ the set of subgraphs with the interpolated line fixed by $\mathcal{J}$ on the reference vertex (hence giving a first order term), and $D^2_t(\mathcal{C},P)$ the set of subgraphs with the interpolated line fixed by $\mathcal{J}$ on some tree line $x_k - x_{k-1}$, $k \leq j$ (hence giving a second order term).
### III.3.1 Spatial decay

It is easy to see that the interpolated line for the subgraph $g_i$ has the same spatial decay as the non interpolated one [DR1](IV.24), times a factor

$$\left(\left|\bar{x}_i^{(2)} - \bar{x}_i^{(1)}\right|_0 + \left|\bar{x}_i^{(2)} - \bar{x}_i^{(1)}\right|_r\right)\Lambda(w_{A(i)}) \quad \text{(III.12)}$$

if $g_i \in D^1_1(C, P)$. If $g_i \in D^2_1(C, P)$ we have three multiplying factors depending on the components of the scaling factors

$$|x_j - x_{j-1}|_\mu |x_k - x_{k-1}|_\nu. \quad \text{(III.13)}$$

If $\mu$ and $\nu \in (0, r(r_2))$ we have the multiplying factor $\Lambda^2(w_{A(i)})$. If $\mu$, or $\nu$ is $t(r_2)$ and the other belongs to $(0, r(r_2))$ we have the factor $\Lambda^2(w_{A(i)})\Lambda^2(w_{l(i)})$. Finally, if $\mu$ and $\nu = t(r_2)$ we have the factor $\Lambda(w_{A(i)})\Lambda(w_{l(i)})$.

Before going on we take a fraction $(1 - \varepsilon)$ of the exponential decay to ensure the decay between the test function supports of Theorem 3 as in [DR1](IV.25). Of the remaining decay a fraction $\frac{\varepsilon}{2}$ will be used to bound the distance factors and the other to perform spatial integrals.

### III.3.2 Bounding distance factors

For each renormalized subgraph $g_i$, we have to bound one or two distance factors, depending if it belongs to $D^1(C, P)$ or $D^2(C, P)$, which are the subsets of subgraphs that give a first order or a second order term respectively. These sets can be cut in turn into $D^m_1(C, P), D^m_e(C, P)$ and $D^m_t(C, P), m = 1, 2$, for loop, external and tree lines respectively moved. Then we have to bound the quantity

$$A(x, J, T) = \prod_{g_i \in D^1_1(C, P)} |x_i^{(2)} - x_i^{(1)}| \prod_{g_i \in D^2_1(C, P)} |x_j - x_{j-1}| |x_k - x_{k-1}|$$

$$\prod_{g_i \in D^2_1(C, P) \cup D^2_2(C, P)} |x_i^{(2)} - x_i^{(1)}|^2 \prod_{l \in T(J)} e^{-\frac{\varepsilon}{2}(|\bar{x}_l^{(1)} - \bar{x}_l^{(0)}|\Lambda(w_{l})) \frac{1}{2}}$$

where we have taken the same spatial decay (actually the worst) for all directions. For each loop or external line the difference $|x_i^{(2)} - x_i^{(1)}|$ can be bounded, applying several triangular inequalities, by the sum over the tree lines on the unique path in $T(J)$ connecting $x_i^{(2)}$ to $x_i^{(1)}$. 27
We observe that the same tree line $l_j$ can appear in several paths connecting different pairs of points $x_i^{(2)}$, $x_i^{(1)}$. Using the same fraction of its exponential decay many times might generate some unwanted factorials as $\sup_x x^n \exp(-x) = (n/e)^n$. To avoid this problem we define $D_j$ as the set of subgraphs $g_i \in D(C, P)$ that use the tree distance $|\bar{x}_{lj} - x_{lj}|$ and we apply the relation

$$e^{-a \frac{d}{|\bar{x}_{lj} - x_{lj}|}} \leq \left(\frac{n}{e}\right)^n$$  \hspace{1cm} (III.14)

With this expression a different decay factor is used for each subgraph. Now applying this result and the inequality $xe^{-(x)^{1/s}} \leq s!$ we prove the bound:

$$\sup_x |A(x, J, T)| \leq K(s) \prod_{g_i \in D^1(C, P)} \Lambda(w_{l(i)})^{-1} \left[1 - \left(\frac{\Lambda(w_{A(i)})}{\Lambda(w_{l(i)})}\right)^{\frac{1}{s}}\right]^{-s} \prod_{g_i \in D^2(C, P)} \Lambda(w_{l(i)})^{-2} \left[1 - \left(\frac{\Lambda(w_{A(i)})}{\Lambda(w_{l(i)})}\right)^{\frac{1}{s}}\right]^{-2s}$$  \hspace{1cm} (III.15)

where $K(s)$ is some function of $s$. The remaining differences are dangerous as they appear with a negative exponent. This happens because in this continuous formalism one has to perform renormalization even when the differences between internal and external scales of subgraphs are arbitrary small. The solution of this problem is given by loop lines factors. Indeed any renormalized subgraph has necessarily internal loop lines, which give small factors when the differences between internal and external scales of subgraphs become arbitrarily small. By Lemma 9 in [DR2] we know that, for each $g_i \in D(C, P)$ there are at least two loop lines internal to $g_i$ which satisfy $\Lambda(w_{M(a, C)}) \leq \Lambda(w_{l(i)})$ and $\Lambda(w_{A(m(a, C))}) \geq \Lambda(w_{A(i)})$. Then for each $g_i \in D^1(C, P)$ we have to bound

$$f_1(x) = \frac{[1-x]^2}{[1-(x)^{\frac{1}{2}}]^s}$$  \hspace{1cm} (III.16)

and for each $g_i \in D^2(C, P)$

$$f_2(x) = \frac{[1-x]^2}{[1-(x)^{\frac{1}{2}}]^2}\hspace{1cm},$$  \hspace{1cm} (III.17)

28
where we defined $x = \Lambda(w_{A(i)})/\Lambda(w_{t(i)})$. Remark that $f_1(x) \simeq (1 - x)^{2-s}$ for $x \to 1$ while $f_2(x) \simeq (1 - x)^{-2(s-1)}$. Therefore choosing $1 < s < 3/2$, $f_1$ is bounded near $x = 1$, and $f_2$ is integrable. We bound

$$f_1(x) \leq \sup_{x \in [0,1]} f_1(x)$$

(III.18)

and we keep $f_2$ to be bounded when the integration over the parameters $w$ will be performed. Finally the factors $\left[1 - \frac{\Lambda(w_{A(m(a,c)})}{\Lambda(w_{M(a,c)})}\right]^{1/2}$ that are not used are bounded by 1.

### III.3.3 Sum over $\mathcal{J}$

We bound the sum over $\mathcal{J}$ by taking the $\sup_{\mathcal{J}}$ times the cardinal of $\mathcal{J}$. In [DR2], Lemma 7, it is proved that $|\mathcal{J}| \leq K^n$ for some constant $K$.

### III.3.4 Spatial integration

To perform spatial integration we use the remaining tree line decay

$$\prod_{l \in T(\mathcal{J})} e^{-a \frac{1}{2} \left[ \left| (\delta x_l)_0 \Lambda(w_l) \right|^\frac{1}{2} + \left| (\delta x_l)_1 \Lambda(w_l) \right|^\frac{1}{2} + \left| (\delta x_l)_2 \Lambda(w_l) \right|^\frac{1}{2} \right]}.$$  

(III.19)

These lines depend in general from the interpolation parameters $t$. In [DR2] it is proved that spatial integration performed with interpolated tree lines does not depend from the interpolating factor $t$ and give the same result as integration with the starting tree $\mathcal{T}$.

Summarizing the results, tree lines are used for several purposes: extracting the exponential decay between the test functions supports, bounding distance factors and performing spatial integration. The resulting bound is:

$$e^{-a \left(1 - \varepsilon\right) \Lambda^\frac{1}{2} \frac{1}{2} \left(\Omega_1, \ldots, \Omega_{2p}\right) \prod_{q=1}^{n} \Lambda(\Lambda(w_q)) \prod_{g_i \in D^1(C,P)} \Lambda(\Lambda(w_{t(i)}))^{-1} \prod_{g_i \in D^2(C,P)} \Lambda(\Lambda(w_{l(i)}))^{-2} \left[1 - \left(\frac{\Lambda(w_{A(i)})}{\Lambda(w_{l(i)})}\right)^2\right]^{-2s}}.$$  

(III.20)
III.4 Sector sum

We still have to perform the sum over sector choices corresponding to [DR1], (IV.28). We do it in the same way as in section IV.3 of [DR1]. The only difference is that, for a two-point subgraph $g_i$, by momentum conservation, there is no sector choice at all:

\[
\left[ \frac{4 \Lambda^{-\frac{1}{2}}(w_i(2))}{\Lambda^2(w)} \right] \int_{\Sigma_{h_i(2),r(i)+1}} d\theta_{h_i(2), r(i)} \left( \theta_{\text{root}}, \theta_{h_i(2), r(i)} \right) \leq K \ , \quad \text{(III.21)}
\]

and, for each $g_i \in D(C, P)$ we have to count the number of choices for the additional refinement for the half-line $h_i(2)$ from a sector of size $\Lambda^{\frac{1}{2}}(w_i(2))$ into a sector of size $\Lambda^{\frac{1}{2}}(w_{h_i(2)})$. This costs a factor $\Lambda^{-\frac{1}{2}}(w_{h_i(2)})$. This term is dangerous as it is on the denominator. To compensate it we extract a factor $\Lambda^{\frac{1}{2}}(w_{h_i(2)})$ from the subgraphs $g_j$ of $g_i$ defined above. This factor is extracted inductively for $j \in C_i^r$. For each subgraph $g_j$ we distinguish two situations:

- if $|eg_j(C)| > 4$ we insert the identity $1 = \frac{\Lambda^{\frac{1}{2}}(w_{A(j)})}{\Lambda^{\frac{1}{2}}(w_j)} \frac{\Lambda^{\frac{1}{2}}(w_j)}{\Lambda^{\frac{1}{2}}(w_{A(j)})}$, where the second factor will be compensated by the convergent power counting of the subgraph $g_j$;

- if $|eg_j(C)| = 4$ we observe (see Lemma 4 below) that we have counted one unnecessary sum over sector choices and we gain again a factor $\Lambda^{\frac{1}{2}}(w_{A(j)})/\Lambda^{\frac{1}{2}}(w_j)$.

Putting together all these terms we obtain the factor we want, namely $\Lambda^{\frac{1}{2}}(w_{t(i)})$, times a factor

\[
\prod_{g_j, j \in C_i^r, |eg_j(C)| > 4} \frac{\Lambda^{\frac{1}{2}}(w_j)}{\Lambda^{\frac{1}{2}}(w_{A(j)})} . \quad \text{(III.22)}
\]

**Lemma 4** Let the two point subgraph $g_i \in D(C, P)$, and the four point subgraph $g_j, j \in J_i$, be fixed. Then the number of sector choices predicted by [DR1], Lemma 6, (IV.31) must be modified:

\[
\prod_{m=2}^{4} \left[ \frac{4 \Lambda^{-\frac{1}{2}}(w_{A(j)})}{\Lambda^2(w)} \right] \int_{\Sigma_{h_j(m),r(j)+1}} d\theta_{h_j(m), r(j)} \left( \theta_{\text{root}}, \theta_{h_j(m), r(j)} \right)_{m=2,3,4} \leq K \]

\[
\quad \text{(III.23)}
\]
Figure 7: Tree lines are solid, loop lines are wavy; the arrows show the direction of the sector sum

for some constant $K$.

Proof We observe that $\theta_{h_{j}}^{(2)}$ actually is fixed by the momentum conservation for the two external lines of $g_{i}$ on an interval of size $\Lambda^{\frac{1}{2}}(w_{A(i)})$.

In the following we write explicitly the $q$ dependence: $j = j_{q}$. We distinguish then three possible situations.

1. $h_{j_{q}}^{(3)}$ and $h_{j_{q}}^{(4)}$ are both loop half-lines (see Fig.7). Then they are contracted to some half-lines $h_{j_{q}}^{''(3)}$ and $h_{j_{q}}^{''(4)}$, that belong to some sector of size $\Lambda^{\frac{1}{2}}(w_{M_{r}}(h_{j_{q}}^{(m)},c)) \leq \Lambda^{\frac{1}{2}}(w_{A(j_{q})}), m = 3, 4$. Therefore by momentum conservation $\theta_{h_{j_{q}}^{(m)}}$ is restricted on the sector of $h_{j_{q}}^{''(m)}$, for $m = 3, 4$.

2. $h_{j_{q}}^{(3)} \in L$ and $h_{j_{q}}^{(4)} \in t_{i}$. Then we have two situations.

When $h_{j_{q}}^{(3)}$ contracts with some element of $L_{j_{q}}^{L}(i)$ (see Fig.8), repeating the argument above, $\theta_{h_{j_{q}}^{(3)}}$ is restricted to an interval of width $\Lambda^{\frac{1}{2}}(w_{A(j_{q})})$, and, by momentum conservation ([DR1], Appendix B) $\theta_{h_{j_{q}}^{(4)}}$ is restricted to an interval of the same size.

When $h_{j_{q}}^{(3)}$ contracts with some element of $L_{j_{q}}^{R}(i)$, there is a loop line $a$ connecting $L_{j_{q}}^{L}(i)$ with $L_{j_{q}}^{R}(i)$. This line is external line of some subgraph in $T_{j_{q}}^{R}(i)$, say $g_{j'}$ (see Fig.8a). Then we chose as root half-line for $g_{j'}$ the
Figure 8: $h_j^{(3)}$ contracts with some element of $L^R_j(i)$

loop half-line $a$ instead of the tree half-line $h^\text{root}_j$ and, for all tree lines on the unique path connecting $v^\text{root}_{j'}$ to $v^{(4)}_j$, we can exchange $h^L$ and $h^R$ (see Fig.9b; the new arrows show the direction towards this new root). Then $\theta_{h^{(4)}_j}$ is fixed in an interval of size $\Lambda^{\uparrow}(w_{A(j_q)})$.

3. $h^{(3)}_{j_q}$ and $h^{(4)}_{j_q} \in t_i$. Remark that $T^R_{j_q}(i)$ is separated into two subtrees, $T^R_{j_q}(3)(i)$ which is connected to $g_{j_q}$ through $l^{(3)}_j$ and $T^R_{j_q}(4)(i)$ which is connected to $g_{j_q}$ through $l^{(4)}_j$. There is a loop half-line $a$ hooked to $T^R_{j_q}(3)(i)$ or to $T^R_{j_q}(4)(i)$, contracting to some loop half-line in $L^L_{j_q}(i)$. Let’s say $a$ is hooked in $T^R_{j_q}(3)(i)$ (see Fig.10a). Then, repeating the same argument above (see Fig.10b), $\theta_{h^{(3)}_j}$ and $\theta_{h^{(4)}_j}$ are fixed in an interval of size $\Lambda^{\uparrow}(w_{A(j_q)})$. This ends the proof. ■

III.5 Integration over the parameters $w_i$

Putting everything together, we can bound the sum (II.30):

$$|\Gamma^{\Lambda_0}_{2p}| \leq e^{-a(1-\varepsilon)} \Lambda^{\uparrow}_T d_T^{\uparrow}(\Omega_1,...,\Omega_{2p}) ||\phi_1||_1 \prod_{i=2}^{2p} ||\hat{\phi}_i||_{\infty,2}$$

(III.24)
Figure 9: $h_{j_q}^{(3)}$ contracts with some element of $L_j^L(i)$

Figure 10: $h_{j_q}^{(3)}$ and $h_{j_q}^{(3)}$ are tree half-lines, and there is a loop line connecting $T_{j_q}^{L(3)}(i)$ with $T_{j_q}^{R}(i)$
\[
\left[ \Lambda^2 \left( w_T \right) \right]^{(2p-1)} \prod_{i=1}^{\frac{n-1}{2}} \left( \frac{1}{\Lambda^2 \left( w_i \right)} \right) \prod_{a \in L} \left( \frac{\Lambda^2 \left( w_M(a,c) \right)}{\Lambda^2 \left( w_{i_a} \right)} \right) \prod_{v \in V'} \Lambda \left( w_{i_v} \right)
\]

\[
= \prod_{g_i} \prod_{h \in \{0,\frac{1}{2},1\}} \left( \frac{\Lambda^2 \left( w_{j_{h,i}} \right)}{\Lambda^2 \left( w_{j_{h-1,i}} \right)} \right) \prod_{h \in I(i)} \left( \frac{\Lambda^2 \left( w_{j_{h,r(i)+1}} \right)}{\Lambda^2 \left( w_{j_{h,r(i)}} \right)} \right)
\]

where we bounded \( \delta \mu_{\Lambda}^{w_{i_v}}(\lambda) \leq K |\lambda| \Lambda(\lambda) \) and \( |\lambda| \leq c \). Remark that sector counting for vertex gives a factor depending from \( V \) only, as for two point vertex no sum has to be paid. Now we can send \( \Lambda \) to zero, hence \( \Lambda(w) = \sqrt{w} \) as \( \Lambda_0 = 1 \). The bound becomes

\[
|\Gamma_{2p}^{\Lambda_{0}}| \leq K_0 e^{-a \left( 1 - e \right) \Lambda^2 \left( \Omega_1, \ldots, \Omega_{2p} \right)} |\phi_1| \prod_{i=2}^{2p} |\phi_{i}^{2p}| \sum_{n \geq 1} \frac{c^n}{n! n!} K^n \quad \text{(III.25)}
\]

\[
\sum_{CTS} u \cdots \sum_{L \in C} \sum_{C} \sum_{J} \sum_{P} \int_{w_T \leq w_{A(i)} \leq w_1} \prod_{v \in V'} \left[ w_{i_v}^{\frac{1}{2}} \prod_{g_i} \prod_{h \in \{0,\frac{1}{2},1\}} \left( \frac{\Lambda^2 \left( w_{j_{h,i}} \right)}{\Lambda^2 \left( w_{j_{h-1,i}} \right)} \right) \prod_{h \in I(i)} \left( \frac{\Lambda^2 \left( w_{j_{h,r(i)+1}} \right)}{\Lambda^2 \left( w_{j_{h,r(i)}} \right)} \right) \right]
\]

Now we can introduce the variables \( \beta_i \) exactly as in section IV.4 of \([DR1]\) (see (IV.38-40)) and obtain in these new coordinates:
\[ |\Gamma_{2p}^\Lambda| \leq K_0 \|\phi_1\|_1 \prod_{i=2}^{2p} \|\phi_i\|_{\infty,2} e^{-a (1-\varepsilon) \Lambda T \frac{1}{2} \sum_{i=1}^{2} \frac{T_i}{2} (\Omega_1, \ldots, \Omega_{2p})} \sum_{\bar{n} \geq 1} \frac{c^\bar{n}}{\bar{n}! \bar{n}'!} |K|^\bar{n} \quad (III.26) \]

\[
\sum_{\text{CTs}} \sum_{\text{L}} \sum_{\text{E}} \sum_{\text{C}} \sum_{\text{JP}} \sum_{\text{w}_T} \sum_{i=1}^{\bar{n}-1} d_{\beta_i} \prod_{j=1}^{n-1} \prod_{i=1}^{n-1} \left[ \left( \prod_{j \in C_i} \beta_j^{\frac{3}{2}} \right) w_T^{-\frac{3}{2}} \right] \prod_{a \in L} \left[ \left( \prod_{j \in C_{h(i,c)}} \beta_j^{\frac{3}{2}} \right) w_T^\frac{3}{2} \right] \prod_{v \in V'} \left[ \left( \prod_{j \in C_{v(i)}} \beta_j^{\frac{3}{2}} \right) w_T^\frac{3}{2} \right]
\]

where \( \bar{n}_i = n_i + n_i' \), and \( n_i, n_i' \) are respectively the number of four points and two points vertex in \( g_i \). Now we compute power counting as in [DR1], section IV.4, and we obtain the same expressions, substituting \( n \) by \( \bar{n} \). The only different expressions are

\[
\prod_{v \in V} \left( \prod_{j \in C_{v(i)}} \beta_j \right)^{-\frac{1}{2}} = \prod_{i=1}^{\bar{n}-1} \beta_i^{-\frac{n_i}{2}}
\]

\[
\prod_{v \in V'} \left( \prod_{j \in C_{v(i)}} \beta_j \right)^{-\frac{1}{2}} = \prod_{i=1}^{\bar{n}-1} \beta_i^{-\frac{n_i'}{2}} \quad (III.27)
\]

Then we obtain

\[ |\Gamma_{2p}^\Lambda| \leq K_0 \|\phi_1\|_1 \prod_{i=2}^{2p} \|\phi_i\|_{\infty,2} e^{-a (1-\varepsilon) \Lambda T \frac{1}{2} \sum_{i=1}^{2} \frac{T_i}{2} (\Omega_1, \ldots, \Omega_{2p})} \sum_{\bar{n} \geq 1} \frac{c^\bar{n}}{\bar{n}! \bar{n}'!} |K|^\bar{n} \]

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To compute $x_i$ we apply the following relation

$$|il_i(C)| = 2n_i + 2 - |eg_i(C)|.$$ (III.29)

For all $g_i$ with $|eg_i(C)| > 4$ such that there exists some $g_{i'} \in D(C, P)$ with $i \in C_i \setminus J_{i'}$, the factor $x_i$ is given by:

$$x_i = \frac{1}{2}(\overline{n}_i - 1) - \frac{3}{8}|il_i(C)| + \frac{1}{4}n_i - \frac{1}{2}n_i' - \frac{1}{4}(|eg_i(C)| - 3) - \frac{1}{4}$$

$$= \frac{1}{8}(|eg_i(C)| - 6) \quad \text{if} \quad 4 < |eg_i(C)| \leq 10$$

$$x_i \geq \frac{1}{2}(\overline{n}_i - 1) - \frac{3}{8}|il_i(C)| + \frac{1}{4}n_i - \frac{1}{2}n_i' - \frac{1}{4}(|eg_i(C)| - 1) - \frac{1}{4}$$

$$= \frac{1}{8}(|eg_i(C)| - 10) \quad \text{if} \quad |eg_i(C)| > 10 ,$$ (III.30)

where the last term $-1/4$ corresponds to the factor extracted to perform sector sum in section III.4. For the remaining $g_i$ with $|eg_i(C)| \geq 4$ we have the usual power counting

$$x_i = \frac{1}{2}(\overline{n}_i - 1) - \frac{3}{8}|il_i(C)| + \frac{1}{4}n_i - \frac{1}{2}n_i' - \frac{1}{4}(|eg_i(C)| - 3)$$

$$= \frac{1}{8}(|eg_i(C)| - 4) \quad \text{if} \quad 4 \leq |eg_i(C)| \leq 10$$

$$x_i \geq \frac{1}{2}(\overline{n}_i - 1) - \frac{3}{8}|il_i(C)| + \frac{1}{4}n_i - \frac{1}{2}n_i' - \frac{1}{4}(|eg_i(C)| - 1)$$

$$= \frac{1}{8}(|eg_i(C)| - 8) \quad \text{if} \quad |eg_i(C)| > 10 .$$ (III.31)

Remark that in the first situation six-points subgraphs become logarithmic divergent, while the other ones still have $x_i > 0$; this is a price to pay for our anisotropic analysis. However by Lemma 3 $x_i$ is still proportional to the number of tree external lines $|et_i|$, which is crucial to perform the sum over partial orders. This is the reason why, when introducing classes in [DR1], we have selected up to 11 external lines per subgraph.
Finally we consider two-point subgraphs. For all $g_i \in D(C, P)$ we have

$$x_i = \frac{1}{2}(n_i - 1) - \frac{3}{8}|\mathcal{I}_i(C)| + \frac{1}{4}n_i - \frac{1}{2}n_i' + \frac{1}{2} = \frac{1}{2} - \frac{1}{2} = 0 \quad (III.32)$$

where the term $1/2$ comes from renormalization. The corresponding power counting is logarithmic in $T$. We have still to consider the IPR two-point subgraphs $g_i \in ND(C, P)$, that have $x_i = -1/2$. Their external momentum at scale $A(i)$ is equal to that of some internal line $l_j$. Since our Gevrey cutoffs have compact support, this forces a relation between external and internal scales, namely $\Lambda(w_i) \leq \sqrt{2} \Lambda(w_{A(i)})$. This means $w_i \leq 2w_{A(i)}$, or equivalently $\beta_i \geq 1/2$. The corresponding integral is then bounded by a constant:

$$\int_{\frac{1}{2}}^{1} d\beta_i \beta_i^{-1-\xi} = 2(\sqrt{2} - 1).$$

Now Lemma 8 in [DR1] can be generalized:

**Lemma 5** For any subgraph $g_i \ (i \neq r)$ with $|eg_i| > 6$ we have

$$x_i \geq \frac{|et_i|}{88}. \quad (III.33)$$

**Proof** For $g_i$ with $|eg_i| > 4$ and such that there is no $g_i' \in D(C, P)$ with $i \in C_i' \setminus J_i'$, Lemma 8 of [DR1] applies directly. For $g_i$ with $|eg_i| > 6$ and such that there is some $g_i' \in D(C, P)$ with $|eg_i| > 6$, $i \in C_i' \setminus J_i'$, we have to bound $\frac{1}{8}(|eg_i(C)| - 10)$ for $|eg_i| > 10$ or $\frac{1}{8}(|eg_i(C)| - 6)$ for $|eg_i| \leq 10$. Then we apply the same reasonings as in Lemma 8 of [DR1].

To complete the bound we must factorize the integrals over the $\beta$ parameters as in [DR1]. Some sets of $\beta_j$ are not independent yet:

$$\prod_{g_i \in D^2(C, P)} \left[ \prod_{j \in C_{l(i)} \setminus C_{A(i)}} \int_{w_T}^{1} d\beta_j \beta_j^{-1+x_j} \left[ \frac{1 - \prod_{j \in C_{l(i)} \setminus C_{A(i)}} \beta_j}{1 - \left( \prod_{j \in C_{l(i)} \setminus C_{A(i)}} \beta_j \right)^{1+2s}} \right] \right]. \quad (III.34)$$

The mixed term has an integrable singularity at the point $\beta_j = 1$, $\forall j \in C_{l(i)} \setminus C_{A(i)}$. We decompose the integration domain of each $\beta_j$ into two subsets $[w_T, 1] = I^1 \cup I^2$ where $I^1 = [w_T, 1/2]$ and $I^2 = [1/2, 1]$. The integral above,
for a fixed \( g_i \in D^2(\mathcal{C}, P) \) is written as

\[
\prod_{j \in C_{t(i)} \setminus C_{A(i)}} \sum_{m_j=1,2} \int_{m_j}^{1} d\beta_j \beta_j^{-1+x_j} \frac{[1 - \prod_{j \in C_{t(i)} \setminus C_{A(i)}} \beta_j]^2}{[1 - \left( \prod_{j \in C_{t(i)} \setminus C_{A(i)}} \beta_j \right)^{\frac{1}{2s}}]}^{2s} \tag{III.35}
\]

We distinguish two situations.

1. If \( m_j = 1 \) for some \( j \), then some \( \beta_j \leq 1/2 \), and the mixed term can be bounded by \( 1/[1 - (1/2)^{\frac{1}{2s}}]^{2s} \) and taken out of the integral. The integrals in (III.35) are then factorized.

2. If \( m_j = 2 \ \forall j \), we have to compute

\[
\prod_{j \in C_{t(i)} \setminus C_i} \int_{\frac{1}{2}}^{1} d\beta_j \beta_j^{-1+x_j} \int_{\frac{1}{2}}^{1} d\beta_i \beta_i^{-1} \frac{[1 - \prod_{j \in C_{t(i)} \setminus C_{A(i)}} \beta_j]^2}{[1 - \left( \prod_{j \in C_{t(i)} \setminus C_{A(i)}} \beta_j \right)^{\frac{1}{2s}}]}^{2s} , \tag{III.36}
\]

where \( \beta_i \) appears with exponent \(-1\) because \( g_i \) is a two point renormalized subgraph. Then \( x_i = 0 \). We perform the change of variable on \( \beta_i \):

\[
z := \beta_i c_i , \quad c_i := \left( \prod_{j \in C_{t(i)} \setminus C_i} \beta_j \right) , \tag{III.37}
\]

and the integral becomes:

\[
\prod_{j \in C_{t(i)} \setminus C_i} \int_{\frac{1}{2}}^{1} d\beta_j \beta_j^{-1+x_j} \int_{\frac{1}{2}}^{c_i} dz \ z^{-1} \frac{[1 - z]^{2s}}{[1 - z^{\frac{1}{2s}}]^{2s}} . \tag{III.38}
\]

We observe that \( c_i \) varies on the interval \([2^{-|c_i|}, 1]\), where we defined \(|c_i|\) as the number of \( \beta_j \) in \( C_{t(i)} \setminus C_i \). To bound the integral over \( z \) and verify this bound does not depend from \( c_i \) we distinguish two cases.

\textbf{a:} \( c_i \geq \frac{1}{2} \) then

\[
\int_{\frac{1}{2}}^{c_i} dz \ z^{-1} \frac{[1 - z]^{2s}}{[1 - z^{\frac{1}{2s}}]^{2s}} \leq \int_{\frac{1}{2}}^{\frac{1}{2}} dz \ z^{-1} + \int_{\frac{1}{2}}^{c_i} dz \ z^{-1} \frac{[1 - z]^{2s}}{[1 - z^{\frac{1}{2s}}]^{2s}} \leq K ; \tag{III.39}
\]
\[ \int_{\frac{1}{2}}^{c_i} dz \, z^{-1} \frac{|1 - z|^2}{|1 - z^s|^2} \leq K \int_{\frac{1}{2}}^{c_i} dz \, z^{-1} = K \log 2, \quad (III.40) \]

where \( K \) is a constant. In both cases the bound does not depend on \( c_i \) and the integrals in (III.33) are factorized.

Finally we can bound the integrals over the parameters \( \beta_i \):

\[
\prod_{g_i \notin D(C,P)} \int_{w_T}^{1} d\beta_i \beta_i^{1+x_i} \leq \prod_{g_i \in D(C,P)} | \log w_T | \prod_{g_i \mid |e_g| = 4} | \log w_T | \prod_{g_i \mid |e_g| > 6} \frac{1}{x_i}. \quad (III.41)
\]

Now, like in [DR1], Lemma 8, we can bound the vertex functions by

\[
|\Gamma_{2p>4}^{\Lambda_0}| \leq K_0 \| \phi \|_1 \prod_{i=2}^{2p} \| \phi_i \|_{\infty,2} e^{-a (1-\epsilon) \Lambda_T^{' \frac{1}{2}} d_T^{1/2} (\Omega_1, \ldots, \Omega_{2p})} \frac{w_T^{2p}}{2p-4} \sum_{n \geq 1} \frac{c^n}{n!n'} K^n \quad (III.42)
\]

where the sum over \( I \) gives the choices of the integration domain of \( \beta_i \) between \( I^1 \) and \( I^2 \). The set \( I \) has then cardinal proportional to \( 2^n \). If \( 2p = 4 \) or \( 2p = 2 \) we substitute an additional factor \( | \log w_T | \) to the global factor \( 1/(2p-4) \) in front of (III.42). Now we bound all the sums exactly as in [DR1] (the only difference being that we are working with \( n \) instead of \( n \)). Finally we obtain

\[
|\Gamma_{2p>4}^{\Lambda_0}| \leq K_0 \| \phi \|_1 \prod_{i=2}^{2p} \| \phi_i \|_{\infty,2} e^{-a (1-\epsilon) \Lambda_T^{' \frac{1}{2}} d_T^{1/2} (\Omega_1, \ldots, \Omega_{2p})} \frac{w_T^{2p}}{2p-4} K_1^p (pl)^2 \sum_{n \geq 1} \frac{1}{n!n'} (cK_2 | \log w_T |)^n, \quad (III.43)
\]

\[
|\Gamma_{2p\leq4}^{\Lambda_0}| \leq K_0 \| \phi \|_1 \prod_{i=2}^{2p} \| \phi_i \|_{\infty,2} e^{-a (1-\epsilon) \Lambda_T^{' \frac{1}{2}} d_T^{1/2} (\Omega_1, \ldots, \Omega_{2p})} \frac{w_T^{2p}}{2p-4} \sum_{n \geq 1} \frac{1}{n!n'} (cK_2 | \log w_T |)^n. \quad (III.44)
\]
These sums are convergent for $cK_2|\log w_T| < 1$ which achieves the proof of Theorem 3.

Appendix A: Flow of $\delta \mu$

To study the flow of the chemical potential counterterm we introduce some definitions. We define $\Sigma[\delta \mu, C]$ as the two point vertex function $\Gamma_2(\phi_1^0, \phi_2^0)$ 1PI and with at least one internal line, for a theory with bare chemical potential counterterm $\delta \mu$ and propagator $C$. The test functions are $\delta \phi^0(x) = \delta(x)$ and $\phi^0_2(x) = e^{ik_F x}$, hence the external impulsion is fixed to $k_F$, as near a possible to the Fermi surface. Remark that, as the external impulsion is fixed near the Fermi surface, we do not introduce any Gevrey cut-off on the test functions, and, when performing sector sum, the factor for the root sector $\Lambda^{-\frac{1}{2}}$ does not appear. The two fundamental equations are then

$$
\begin{align*}
\delta \mu_{\Lambda}^0(\lambda) &= \Sigma[\delta \mu_{\Lambda}^1(\lambda), C_{\Lambda}^1] \\
\delta \mu_{\Lambda}^{\Lambda'}(\lambda) &= \delta \mu_{\Lambda}^1(\lambda) - \Sigma[\delta \mu_{\Lambda}^1(\lambda), C_{\Lambda}^1] & \Lambda \leq \Lambda' \leq 1
\end{align*}
$$

These equations are consistent with the BPHZ condition $\delta \mu_{\Lambda}^1(\lambda) = 0$. To study the flow we write $\Sigma[\delta \mu_{\Lambda}^1(\lambda), C_{\Lambda}^1]$ as an expression where the dependence from $\Lambda$ and $\Lambda'$ is explicit:

$$
\Sigma[\delta \mu_{\Lambda}^1(\lambda), C_{\Lambda}^1] = \sum_{n,n'=2}^{\Lambda^2} \frac{\Lambda^n (\delta \mu_{\Lambda}^1)^{n'}}{n! n'}
$$

$$
\sum_{C,T,S} \sum_{u-T} \sum_{E \Omega} \sum_{c} \sum_{J,P} \sum_{\varepsilon(T, \Omega)} (2)^{n-1} \int_{\Lambda_T \leq \Lambda_{A(u)} \leq \Lambda_s \leq 1} \prod_{q=1}^{n-1} \Lambda_q d\Lambda_q
$$

$$
\prod_{h \in H_T \cup \Lambda_T \cup E} \left\{ \left[ \frac{4}{\Lambda} - \frac{1}{2} (w_{j_h,n_h}) \right] \right\}

\int_0^{2\pi} d\theta_h n_h \left[ \frac{4}{\Lambda} - \frac{1}{2} (w_{j_h,n_h-1}) \right] \int_{\Sigma_{j_h,n_h}} d\theta_{h,n_h-1}
$$

... $\frac{4}{\Lambda} - \frac{1}{2} (w_{j_h,1})$ \int_{\Sigma_{j_h,2}} d\theta_{h,1} \prod_{r=2}^{n_h} \chi_{\alpha_{j_h,r}}(\theta_{h,1})

$$
\prod_{q_j \leq 1 \text{ or } |q_j(C)| \leq 8} \prod_{v \in V \cup V'} \gamma \left( \theta_{v,\text{root}}, \{ \theta_{h,n_h} \}_{h \in H^*(v)} \right)
$$

$$
\int d^3 x_1 ... d^3 x_{\bar{h}} \phi_1(x_{i_1}, \theta_{e_1,1}) \phi_2(x_{j_1}, \theta_{e_2,1}) \left[ \prod_{q=1}^{n-1} \frac{1}{\Lambda_q} C_{\alpha=q} C_{\gamma^2} (x_q, \bar{x}_q, \theta_{h,1}) \right]
$$

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where we performed the change of variable \( w_i = (\Lambda_i^2 - \Lambda_{i}^\prime)^2/(1 - \Lambda_{i}^\prime)^2 \), and 
\( \Lambda_{i}^\prime = \Lambda' \) for \( \Lambda' \geq \sqrt{2}\pi T \). Remark that the \( \Lambda_{i}^\prime \) dependent factor \( 1/(1 - \Lambda_{i}^\prime) \) coming from the Jacobian is cancelled by the corresponding factor \( (1 - \Lambda_{i}^\prime) \) coming from tree line propagators. Then \( \Lambda' \) appears only in the \( \Lambda_r \) integration and in some loop line propagators (those of the loop fields with \( m(a, C) = \Lambda' \)). The power counting is performed as usual, passing to the variables \( \beta_i \) defined by \( \Lambda_i^2 = \Lambda_i^2 A_i(\Lambda) / \beta_i. \)

We want to prove, by induction, that the property \( H(\Lambda) \), defined by

\[
\left| \delta \mu_{\Lambda}^{\Lambda'}(\lambda) \right| \leq K_1 |\lambda| |(\Lambda' - \Lambda)| \quad \forall \Lambda' \geq \Lambda \quad (A.3)
\]
is true \( \forall \ 0 \leq \Lambda \leq 1 \). We suppose \( H(\Lambda) \) is true for a certain \( \Lambda \), then we prove Lemma 6 and 7, about the existence and the bound satisfied by the derivative. These lemmas ensure that \( H(\Lambda) \) is true for all \( \Lambda \). Indeed otherwise there exists \( \Lambda_m > 0 \) defined as

\[
\Lambda_m = \inf_{\Lambda \in [0,1]} \{ \Lambda | H(\Lambda) \text{ is true} \} \quad (A.4)
\]

Then by lemma 7 we can write a Taylor expansion at first order

\[
\left| \delta \mu_{\Lambda-\varepsilon}^{\Lambda'}(\lambda) \right| \leq \left| \delta \mu_{\Lambda}^{\Lambda'}(\lambda) \right| + \left| \frac{d}{d\Lambda} \delta \mu_{\Lambda}^{\Lambda'}(\lambda) \right| \varepsilon + o(\varepsilon) \quad (A.5)
\]

\[
\leq K_1 |\lambda| |(\Lambda' - \Lambda)| + K_3 |\lambda| \varepsilon + o(\varepsilon) \leq K_1 |\lambda| |(\Lambda' - (\Lambda - \varepsilon))|
\]

for all \( \Lambda' \geq \Lambda \). The same bound in the case \( \Lambda - \varepsilon \leq \Lambda' \leq \Lambda \) is proven in Lemma 8. This result contradicts with the definition of \( \Lambda_m \), therefore \( \Lambda_m = 0 \).

**Lemma 6** If \( H(\Lambda) \) is true then the derivative \( \frac{d}{d\Lambda} \delta \mu_{\Lambda}^{1}(\lambda), \Lambda \leq \Lambda' \leq 1 \) exists and satisfies the bounds:

\[
\left| \frac{d}{d\Lambda} \delta \mu_{\Lambda}^{1}(\lambda) \right| \leq K_2 |\lambda|. \quad (A.6)
\]
Figure 11: Two possible schema for the lowest band in a two point 1PI graph

**Proof** If the derivative exists, it satisfies the formal equation:

$$\frac{d}{d\Lambda} \delta \mu_\Lambda^1 = A + \frac{d}{d\Lambda} \delta \mu_\Lambda^1 \cdot B \tag{A.7}$$

where $A$ is the expression for $\delta \mu_\Lambda^1$ with the derivative $d/d\Lambda$ applied to one propagator, and $B$ is the same expression for $\delta \mu_\Lambda^1$, but with one special two point vertex with value 1 instead of $\delta \mu_\Lambda^1$ (as the derivative of the corresponding factor has been taken out of the sum in (A.7)). Then, formally, the solution for (A.7) is

$$\frac{d}{d\Lambda} \delta \mu_\Lambda^1 = \frac{A}{1 - B} \tag{A.8}$$

Now, if we can prove that $A \leq K' \lambda$ and $B \leq 1/2$, we obtain (A.6) with $K_2 = 2K'$.

**Bound on $A$.** Remark that, as shown in Fig.11 a,b, there is at least one loop line in the first band, obtained in the first case by 1PI, in the second case by parity of the number of external lines for any subgraph.

Therefore the derivative $d/d\Lambda$ may apply only to a loop line propagator. Indeed, if it applies to the $\Lambda_r$ integral, the first band width and the corresponding loop line amplitude are reduced to zero.

The action of the derivative on the loop propagator is given by

$$\frac{d}{d\Lambda} C_{\Lambda_r}^{\Lambda M} = -\frac{2}{\Lambda^3} C_{\alpha=\Lambda^{-2}} \tag{A.9}$$
Indeed, this is the reason for which we can extract only one coupling constant
where the integral limit $\Lambda^2$ cancels with the global factor $\Lambda_T$. The power counting of
terms, that appear in eq. III.28 and come from renormalized two point
subgraphs, as their power counting is not modified at all. The factor 1
where we the factor $\Lambda_T^{1/2}$ has disappeared as the external impulsion
(hence the sector too) is fixed. Remark that the set of renormalized subgraphs
$D(C, P)$ does not contain the global graph $g_r$. Passing to the variables $\beta_i = \frac{\Lambda^2_{A(i)}}{\Lambda_T^2}$, we obtain

$$|\tilde{\Sigma}| \leq K_0 \sum_{\tilde{n} \geq 2} \frac{\epsilon^n}{n!n'^!} K^n \sum_{\text{CTS}} \sum_{u-T} \sum_{E \subset C} \sum_{J \subset P} \Lambda_T^{\tilde{n}-1} \prod_{i=1}^{\tilde{n}-1} \int_{A(i)} \frac{1}{A_T} \prod_{g_i \in C} \frac{\Lambda_{A(i)}}{\Lambda_{A(i)'}} \left( \frac{\Lambda(w_{A(i)})}{\Lambda(w_{A(i)'})} \right) \right] \prod_{g_i \in D^2(C, P)} \left[ 1 - \left( \frac{\Lambda(w_{A(i)1})}{\Lambda(w_{A(i)'})} \right)^2 \right]^{2^n} \right],$$

where the integral limit $\Lambda^2_{A(i)} \geq \Lambda_T$. We have not written the non factorized
terms, that appear in eq. III.28 and come from renormalized two point
subgraphs, as their power counting is not modified at all. The factor $1/\Lambda_T$ cancels with the global factor $\Lambda_T$, giving a constant independent from $\Lambda_T$. The power counting of $\beta_i$ becomes logarithmic, instead of linearly divergent; this is the reason for which we can extract only one coupling constant $\lambda$. Indeed:

$$A \leq |\lambda| \sum_{\tilde{n} \geq 2} \left( |\lambda| |\ln \Lambda| \right)^{\tilde{n}-1} \leq |\lambda| \frac{|\lambda| |\ln \Lambda|}{1 - |\lambda| |\ln \Lambda|} \leq K_2 |\lambda|. \quad (A.12)$$
**Bound on B.** The estimation for $B$ is performed as that for $\delta\mu_1^\Lambda$. The only difference is that, when a two point subgraph contains the special insertion, it is not renormalized, as the power counting is logarithmic instead of linearly divergent. This happens because there is one two point insertion (the special one) that is not compensated by the corresponding $\delta\mu$ scaling factor, then we have

$$\int_\Lambda^1 d\beta_r \beta_r^{-1-\frac{1}{2}+\frac{1}{2}} \leq \log|\Lambda|$$  \hspace{1cm} (A.13)

Of course, the $\beta_r$ power counting becomes logarithmic too, as $g_r$ always contains the special insertion. The global factor $\Lambda$ is then cancelled by the global factor coming from the special insertion. Then

$$B \leq |\lambda| \sum_{n \geq 2} (|\lambda \ln \Lambda|)^{n-1} \leq |\lambda| |\lambda \ln \Lambda| \leq K_2 |\lambda| \leq \frac{1}{2}$$  \hspace{1cm} (A.14)

for $\lambda$ small enough.

**Existence of the derivative** We still have to prove that the derivative exists. For that we apply the definition

$$\frac{d}{d\Lambda} \delta\mu_1^\Lambda = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \Sigma[\delta\mu_1^{\Lambda-\varepsilon}, C_\Lambda^{1\Lambda-\varepsilon}] - \Sigma[\delta\mu_1^\Lambda, C_\Lambda^1] \right) = \lim_{\varepsilon \to 0} \frac{\Delta_{1,\varepsilon}}{\varepsilon}$$  \hspace{1cm} (A.15)

The difference $\Delta_{1,\varepsilon}$ can be written as

$$\Delta_{1,\varepsilon} = \left( \Sigma[\delta\mu_1^{\Lambda-\varepsilon}, C_\Lambda^{1\Lambda-\varepsilon}] - \Sigma[\delta\mu_1^\Lambda, C_\Lambda^1] \right) + \left( \Sigma[\delta\mu_1^{\Lambda-\varepsilon}, C_\Lambda^{1\Lambda-\varepsilon}] - \Sigma[\delta\mu_1^\Lambda, C_\Lambda^1] \right) = \sum_{p=1}^\infty \left( \Delta_{1,\varepsilon}^p \right) F_p + A_1 + A_2$$  \hspace{1cm} (A.16)

where $A_1$ is the expression for $\Sigma[\delta\mu_1^\Lambda, C_\Lambda^{1\Lambda-\varepsilon}]$ with one loop propagator $C_\Lambda^{1\Lambda-\varepsilon}$, and $A_2$ is the same expression, but this time with the first band $\Lambda_r \leq \Lambda$. Finally $F_p$ is the expression for $\Sigma[\delta\mu_1^\Lambda, C_\Lambda^{1\Lambda-\varepsilon}]$ with $p$ insertions of special two point vertex, obtained by substituting the coefficient by 1. With the same kind of argument as before we can prove that $|F_p| \leq \varepsilon |\lambda|/\Lambda^{p-1}$, $|A_1| \leq \varepsilon |\lambda|$ and $|A_2| \leq \varepsilon^2 |\lambda|$. Then we can prove that $\Delta_{1,\varepsilon}$ exists and the derivative takes the form (A.7).

**Lemma 7** If $H(\Lambda)$ is true then the derivative $\frac{d}{d\Lambda} \delta\mu_1^N(\lambda)$ exists and satisfies the bound:

$$\left| \frac{d}{d\Lambda} \delta\mu_1^N(\lambda) \right| \leq K_3 |\lambda|.$$  \hspace{1cm} (A.17)
Proof The proof is a direct consequence of Lemma 6. By the definition of \( \delta \mu_{\Lambda}'(\lambda) \) the derivative is given by

\[
\frac{d}{d\Lambda} \delta \mu_{\Lambda}' = \frac{d}{d\Lambda} \delta \mu_{\Lambda}(1 - F) \tag{A.18}
\]

where \( F \) is the expression for \( \Sigma[\delta \mu_{\Lambda}', C_{\Lambda}] \) with one special insertion, that means one two point vertex factor substituted by 1. As for \( B \) in Lemma 6, we can prove that

\[
|F| \leq |\lambda| \tag{A.19}
\]

then

\[
\left| \frac{d}{d\Lambda} \delta \mu_{\Lambda}' \right| \leq K_2 |\lambda| (1 + |\lambda|) \leq K_3 |\lambda|. \tag{A.20}
\]

The existence of this derivative is a consequence of Lemma 6, as

\[
\Delta_{\Lambda,\epsilon}^N = \Delta_{\Lambda,\epsilon}^1 - \sum_{p=1}^{\infty} \left( \Delta_{\Lambda,\epsilon}^1 \right)^p F_p' \tag{A.21}
\]

where we can prove that \( |F'_p| \leq |\lambda|/\Lambda^{p-1} \).

Lemma 8 If the bound

\[
|\delta \mu_{\Lambda}'| \leq K_1 |\lambda| (\Lambda' - \Lambda) \tag{A.22}
\]

is true for all \( \Lambda' \geq \Lambda'_0 = \Lambda + \epsilon \) then it is true for \( \Lambda'_0 - \epsilon' \) for \( \epsilon' < \epsilon \) small enough.

Proof For all \( \Lambda' \geq \Lambda'_0 \) we can prove (by the same arguments as before) that

\[
\left| \frac{d}{d\Lambda'} \delta \mu_{\Lambda}' \right| \leq K_2 |\lambda| \tag{A.23}
\]

Then we can perform a first order Taylor expansion

\[
\left| \delta \mu_{\Lambda,\epsilon}^{N'_0 - \epsilon'} \right| \leq \left| \delta \mu_{\Lambda,\epsilon}^{N'_0} \right| + \left| \frac{d}{d\Lambda} \delta \mu_{\Lambda}' \right| \epsilon' + o(\epsilon') \\
\leq K_2 |\lambda| (\epsilon + |\epsilon'|) + o(\epsilon) + o(\epsilon') \leq K_1 |\lambda| (\epsilon - |\epsilon'|) \tag{A.24}
\]

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for $\varepsilon'$ small enough. Remark that we used the inequality
\[
\left| \delta \mu_{\Lambda}^{N_0} \right| = \left| \delta \mu_{\Lambda}^{\Lambda + \varepsilon} \right| \leq \left| \delta \mu_{\Lambda + \varepsilon}^{\Lambda + \varepsilon} \right|
\]
\[
+ \left| \frac{d}{d\Lambda^\prime} \delta \mu_{\Lambda^\prime}^{N_0} \right|_{\Lambda^\prime = \Lambda^\prime = \Lambda + \varepsilon} \varepsilon + o(\varepsilon)
\]
\[
\leq K_2 |\lambda| \varepsilon + o(\varepsilon)
\]
(A.25)

Remark that the differential RG equations (A.8) is simpler than its discretized counterpart (A.16). This is an advantage of the differential version of the RG.

Appendix B: Study of the selfenergy.

B1: A loop expansion for extracting the self energy $\Sigma$

(in collaboration with D. Iagolnitzer and J. Magnen)

Our tree formula selects connected graphs. But the self energy $\Sigma$ is the sum over all non trivial two point connected subgraphs which are furthermore 1PI-irreducible (with respect to the single channel between the two external points). In this appendix we apply an (unpublished) formula, due to D. Iagolnitzer and J. Magnen, which proves that this additional information of 1PI in a single channel can be extracted by expanding some loops out of the determinant without generating any factorials in the bounds.

The arch expansion We consider a two point connected graph with $n$ vertices, equipped with its spanning tree $T$. If the two (amputated) external lines are hooked to the same vertex, we have a “generalized tadpole” which is automatically 1PI, hence belongs to the self energy. In that case no additional expansion is performed. Otherwise there is a unique non-empty linear path $P_{\infty, \varepsilon}$ made of $p-1 \leq n-1$ lines in the tree $T$ joining the two external vertices $x^{(1)} = x_1$ and $x^{(2)} = x_p$ through the intermediate vertices $x_2, \ldots, x_{p-1}$. The set $V$ of vertices of $G$ is then the disjoint union of the sets $V_j$, $j = 1, \ldots, p$,
Figure 12: Example of tree, with $p = 5$ vertices on $\mathcal{P}_{1,2}$. Loop fields are dashed, tree lines are solid, and the amputated external lines are darker.

where a vertex belongs to $V_j$ if and only if the unique path in $\mathcal{T}$ joining it to the root $x_1$ passes through $x_j$ but not through $x_{j+1}$ (Fig. 12).

We call $F(\{C_{i,j}\})$ the loop determinant of the remaining fields (it depends on the weakening $w$ parameters, but this is completely irrelevant in what follows). Expanding completely $F$ would cost $n!$. But we just want to know if the graph is 1PI, which means 1PI with respect to the $p - 1$ lines of $\mathcal{P}_{1,2}$, since by parity there cannot be 1 particle reducibility in a 0-2 channel. We perform an auxiliary expansion à la Brydges-Battle-Federbush, and we call it the “arch expansion”. This means that we first test if some vertex of $V_1$ is linked to some vertex of $V_{k_1}$, with $k_1 > 1$. This is done by introducing the interpolation parameter $0 \leq s_1 \leq 1$ and defining

$$C_{ij}(s_1) := \begin{cases} s_1 C_{ij} & \text{if } i \in V_1, j \not\in V_1 \\ C_{ij} & \text{otherwise} \end{cases} \quad (B.1)$$

Then we can write

$$F(\{C_{i,j}\}) = F(\{C_{i,j}(s_1)\}) \big|_{s_1=1} = F(\{C_{i,j}(s_1)\}) \big|_{s_1=0} + \int_0^1 ds_1 \frac{d}{ds_1} F(s_1) \quad (B.2)$$

The first term $s_1 = 0$ means that the graph is 1PR (by cutting the first line of $\mathcal{P}_{1,2}$ as no loop line connects $V_1$ to its complement). Otherwise we
Figure 13: Extraction of one loop line from $V_1$. The loop line is dashed.

derive an explicit loop line out of the determinant, which connects a vertex of
$V_1$ to a vertex of $V_{k_1}$, for some $k_1 > 1$ (see Fig. 13 where we have $k_1 = 3$). If
$k_1 = p$ we are done since the full graph is $1$PI. Otherwise we repeat the same
procedure, but bewteen $\cup_{l=1}^{k_1} V_l$ and its non empty complement, introducing
a second interpolation parameter $0 \leq s_2 \leq 1$:

\[
C_{ij}(s_1, s_2) := \begin{cases} s_2 C_{ij}(s_1) & \text{if } i \in \cup_{l=1}^{k_1} V_l, j \notin \cup_{l=1}^{k_1} V_l \\ C_{ij}(s_1) & \text{otherwise} \end{cases}
\]  

(B.3)

Then we can write

\[
F_1(\{C_{ij}(s_1)\}) = F_1(\{C_{i,j}(s_1, s_2)\})_{s_2=1} = \\
= F(\{C_{i,j}(s_1, s_2)\})_{s_2=0} + \int_0^1 ds_2 \frac{d}{ds_2} F_1(s_1, s_2) \quad \text{(B.4)}
\]

Once again the first term at $s_2 = 0$ means that the block $\cup_{l=1}^{k_1} V_l$ is not linked
to its complement by any loop line, and the graph is $1$PR across the line
number $k_1$ of $P_{1,2}$. The second term corresponds to extract a new loop line
(see Fig. 14) and can be written again as

\[
\int_0^1 ds_2 \frac{d}{ds_2} F_1(s_1, s_2) = \sum_{i_2 \in \cup_{l=1}^{k_1} V_l} \int_0^1 ds_2 \frac{\partial}{\partial s_2} C_{i_2j_2}(s_1, s_2) \frac{\partial}{\partial C_{i_2j_2}} F_1(s_1, s_2)
\]  

(B.5)
Remark that
\[
\frac{\partial}{\partial s_2} C_{i_2j_2}(s_1, s_2) = C_{i_2j_2} \quad \text{if} \quad i_2 \in \bigcup_{l=2}^{k_1} V_l \\
= s_1 C_{i_2j_2} \quad \text{if} \quad i_2 \in V_1
\] (B.6)

We repeat this procedure until we reach $V_p$ with a loop line. Then, if the process stops at the $q$-th step we have the expression

\[
A(q, n) = \sum_{1<k_1<...<k_q=p} \sum_{i_1 \in V_1} ... \sum_{i_q \in \bigcup_{l=1}^{k_q-1} V_l \setminus V_{l-1}} \int_0^1 ds_1 ... \int_0^1 ds_q \frac{\partial}{\partial s_1} C_{i_1j_1}(s_1) \frac{\partial}{\partial s_2} C_{i_2j_2}(s_1, s_2) ... \frac{\partial^q F_1(s_1, s_2, ..., s_q)}{\partial C_{i_1j_1} \partial C_{i_2j_2} ... \partial C_{i_qj_q}}
\] (B.7)

To extract the exact expression for the derived propagators we introduce some notations. We call $l(V)$ the number of loop fields hooked to the vertices in $V$, $W_i$ the set of vertices from where the line $i$ may start from, and $m_i$ the number of loop fields where $i$ may contract without crossing more than one arch:

\[
W_i = V_1 \cup V_2 \cup ... \cup V_{k_i-1} \\
m_i = l(W_i \setminus W_{i-1})
\] (B.8)
Remark that \( m_1 = l(V_1) \) and \( m_2 = l(V_1 \cup V_2 \cup \ldots \cup V_{k_1} \setminus V_1) \). Now we observe that the interpolated propagator is given by

\[
C_{irjr}(s_1, \ldots, s_r) = s_{r+1}s_{r+2}\ldots s_r C_{irjr}, \tag{B.9}
\]

if \( i_r \in m_{r'} \). Then the derivative just takes away the factor \( s_r \). Remark that the remaining determinant satisfies all properties of the initial one, in particular, as the interpolation respects positivity, a Gram inequality can be applied.

Hence the functional \( F \) has been developed as follows

\[
F(\{C_{ij}\}) = \sum_{\mathcal{LF}} \prod_{I \in \mathcal{LF}} A_I \tag{B.10}
\]

where \( \mathcal{LF} \) is a set of subsets \( I \) of the path \( \{x_1, \ldots, x_n\} \) which form 1PI clusters and

\[
A_I = \sum_{q=1}^{n_i/2} A(q, n_i) \tag{B.11}
\]

where \( n_i \leq n \) is the number of vertices belonging to \( I \) and \( q \) is the number of loop lines ensuring 1PI. Now we want to prove that \( A_I \leq K \). As the functional and the propagators can always be bounded by a constant, the problem is to prove the following lemma:

**Lemma 9** The sum over all possible arch systems that connect \( p \) points in such a way to obtain a 1PI block does not develop a factorial, in other words:

\[
\sum_{q=1}^{p} \sum_{1<k_1<\ldots<k_q=p} \sum_{j_{r} \in V_{r}} \int_{0}^{1} ds_1 \ldots \int_{0}^{1} ds_q \sum_{i_{r} \in W_{r}} \sum_{r=1,\ldots,q} a(s_1, \ldots, s_q, i_1, \ldots, i_q) \leq K^n \tag{B.12}
\]

where \( a \) is the function we obtain after bounding the determinant and the propagators by a constant.

**Proof.** We start observing that

\[
\sum_{i_{r} \in W_{r}} \sum_{r=1,\ldots,q} a(s_1, \ldots, s_q, i_1, \ldots, i_q) \leq \prod_{r=1}^{q} a_r(s_1, \ldots, s_{r-1}) \tag{B.13}
\]
where \( a_r \) is defined inductively by \( a_1 = m_1 \) and \( a_r(s_1, \ldots, s_{r-1}) = \sum s_{r-1}a_{r-1}(s_1, \ldots, s_{r-2}) \). To see this we remark that we have \( m_1 \) choices to choose \( i_1 \). In the same way, we have \( m_2 \) choices to choose \( i_2 \) if it does not hook to \( V_1 \). If it does hook to \( V_1 \), we have \( m_1 = a_1 \) choices, but we also have a factor \( s_1 \). Remark that this is an overestimate, as, once fixed \( i_1 \) we have only \( m_1 - 1 \) choices for \( i_2 \). Hence we have

\[
\int_0^1 \prod_{s_1}^{q} ds_r \prod_{r=1}^{q} a_r(s_1, \ldots, s_{r-1}) \leq e^{\sum_{r=1}^{q} m_r}.
\]

(B.14)

This is indeed obvious if we use inductively the fact that for \( a > 0, b > 0 \),

\[
\int_0^1 (as + b) ds \leq (1/a)e^{a+b}.
\]

Now, as \( m_r = l(W_r \setminus W_{r-1}) \), we have

\[
\sum_{r=1}^{q} m_r \leq \sum_{i=1}^{p} l(V_i) < 4n
\]

(B.15)

Finally we prove that

\[
\sum_{q=1}^{p/2} \sum_{1 < k_1 < \ldots < k_q = p} \sum_{j_r \in V_r}^{r=1\ldots q} 1 \leq K^n
\]

(B.16)

Actually

\[
\sum_{j_r \in V_r}^{r=1\ldots q} 1 = \sum_{r=1}^{q} l(V_r) < 4n
\]

(B.17)

and \( \sum_{1 < k_1 < \ldots < k_q = p} 1 \) corresponds to the number of partitions of \( \{1, \ldots, p\} \) into \( q \) intervals, hence is bounded by \( 2^p \leq 2^n \). This ends the proof.

Selfenergy

Now, we can apply the arch formula to the two point vertex function and extract the following expression for the selfenergy.

\[
\Sigma^\Lambda(\phi_1, \phi_2) = \sum_{n \geq 1} \frac{\lambda^n}{n!} \frac{(\delta \mu_1 \Lambda)^{n'}}{n'} \sum_{CT} \sum_{S} \sum_{u \setminus T} \sum_{E} \sum_{\Omega} \sum_{\mathcal{L}} \sum_{J,P} \sum_{L_c} \sum \frac{\lambda^n}{n!}
\]

\[
\varepsilon(T, \Omega) \int_{w_T \leq w_{A(i)} \leq w_i} dw_q \int_{0}^{\frac{|L_c|}{q}} ds_q \prod_{h \in L \cup T \cup E} \left\{ \frac{4\Lambda^\frac{1}{2}(w_{j,h,n_h})}{\pi} \int_{0}^{2\pi} d\theta_{h,n_h} \frac{1}{\pi} \right\} \int_{\Sigma_{j,h,n_h}} d\theta_{h,n_h} - 1
\]

\[
\leq \sum_{r=1}^{q} l(V_r) < 4n
\]

(B.18)
where we took $\phi_1(x) = \delta(x)$ and $\phi_2(x) = e^{-ixk}$, to obtain $\hat{\Sigma}(k)$. $L_e$ is the set of loop lines extracted to ensure 1PI, $s_q$ is the set of interpolation parameters used to extract them while $P$ is the set of loop lines extracted in Sec.II.1. With this expression we can perform the same bound as for the vertex function $\Gamma$, as the additional sums do not generate any factorial. The only difference is that, when performing sector counting, the real external impulsion is not always near the Fermi surface. This does not change the counting lemmas, as this impulsion is fixed with a precision $T$.

**B2: First Derivative of the Selfenergy at the Fermi Surface**

The bound on the first derivative below already proves that our system is not a Luttinger liquid [S1][BGPS][BM].

We want to prove that the first order derivative of the selfenergy computed at the impulsion $k_F$ is bounded by

$$\left| \partial_{k \alpha} \hat{\Sigma}(k) \right|_{k_F} \leq |\lambda|^2 M_1$$

for $\alpha = 0, 1, 2$ and for all $\lambda$ and $T$ satisfying $|\lambda \ln T| \leq M_0$, where $M_0$ and $M_1$ are some constants. The derivative actually corresponds to the multiplication by a factor $x - y$ in position space

$$\partial_{k \alpha} \hat{\Sigma}(k) \big|_{k_F} = \int d^3x \ e^{i(x-y)} \delta(x) (x - y)_\alpha \Sigma(x, y)$$
Then we can perform power counting as usual, the only difference being an additional factor $1/\Lambda(w(t(r)))$, where $t(r)$ is the band index of the lowest tree line in the path joining the two external points $x$ and $y$. Nevertheless, as the two point function $\Sigma$ itself is not renormalized, the factor $\Lambda^{\frac{1}{2}}(w(t(r)))$ coming from loop contractions is not consumed. Then we are left with the factor

$$\frac{1}{\Lambda^{\frac{1}{2}}(w(t(r)))} = \left( \prod_{j \in C(t(r))} \beta_j^{\frac{1}{2}} \right) \frac{1}{\Lambda_T^{\frac{1}{2}}} \quad \text{(B.21)}$$

We remark that, by 1PI, all $j \in C(t(r))$, except for the last one $j = r$, correspond to a subgraph with at least four external legs. Then a factor $\beta_j^{\frac{1}{2}}$ just makes their power counting even more convergent. The last subgraph gives

$$\Lambda_T \int_{\Lambda_T^2}^1 d\beta_r \beta_r^{-1-\frac{1}{2}} \beta_r^{\frac{1}{2}} \frac{1}{\Lambda_T^{\frac{3}{2}}} \leq K \quad \text{(B.22)}$$

Hence the derivative is bounded by

$$\left| \partial_{k_a} \hat{\Sigma}(k) \right|_{k_F} \leq \sum_{n=2}^{\infty} (K|\lambda|^n \ln T)^{n-2} \leq |\lambda|^2 M_1 \quad \text{(B.23)}$$

for $|\lambda| \ln T \leq 1/K = M_0$. The extraction of two coupling constants from the sum does not affect the convergence as there are at most $n - 2$ subgraphs logarithmic divergent. Actually, there are $n - 1$ subgraphs, and one of them, $g_r$, does not give a logarithm, as shown in the equation above. This ends the proof.

B3: Second Derivative of the Selfenergy

The bound on the second derivative is the one which proves really “Fermi liquid behavior” [S1].

We want to prove that the second order derivative of the selfenergy computed at any impulsion $k$ is bounded by

$$\left| \partial_{k_a} \partial_{k_b} \hat{\Sigma}(k) \right| \leq M_3 \quad \text{(B.24)}$$
for $\alpha, \beta = 0, 1, 2$ and for all $\lambda$ and $T$ satisfying $|\lambda \ln T| \leq M_0$, where $M_0$ and $M_3$ are some constants. Applying a double derivative in impulsion space corresponds to multiply by a factor $|x - y|^2$ in position space

$$
\partial_{k_\alpha} \partial_{k_\beta} \hat{\Sigma}(k) = \int d^3 x \ e^{ik(x-y)} \delta(x)(x-y)_{\alpha}(x-y)_{\beta} \Sigma(x,y) \quad \text{(B.25)}
$$

This time we have the bad factor $\Lambda^{-2}(w_{t(r)})$, then the factor $\Lambda^{\frac{3}{2}}(w_{t(r)})$ extracted from sector sum is not enough to assure the bound. Actually we need to extract a second factor $\Lambda^{\frac{3}{2}}(w_{t(r)})$. It turns out that this is almost possible but not quite. One can only extract $\Lambda^{\frac{3}{2}}(w_{t(r)}) |\ln \Lambda(w_{t(r)})|$, using the so-called “volume effect”. This explains the absence of any $\lambda$ in the final bound (B.30). Indeed the second $\lambda$ is also consumend since $g_r$ becomes logarithmic.

**Extracting a second factor $\Lambda^{\frac{3}{2}}(w_{t(r)})$.** When extracting loop lines in Sec.II.1, we introduced the chain $C_{r}^{i}$ ($i = r$ in this case), joining the dot vertex $v_{h^{(2)}}$ to the cross vertex just above $t(r)$ (see Fig.3), where $h^{(1)}$ is the root external half-line of the self-energy and $h^{(2)}$ is the other one. Now we introduce the equivalent chain $C^{nr}$ for $h^{(1)}$, joining the dot vertex $v_{h^{(1)}}$ to the cross vertex just above $t(r)$ (see Fig.15).

Remark that, for all $j \in C^{nr}$ we must have $|eg_j(C)| \geq 4$, by 1PI. As for $C^r$ we call $J_r'$ the set of $g_j$ in the chain $C^{nr}$ with $|eg_j(C)| = 4$, ordered from the lowest and going up. For each $g_j$, with $j \in J_r'$, we call $h^{(1)}_j$ the real external half-line $h^{(1)}$, $h^{(2)}_j$ the tree half-line going towards the external vertex $v_{h^{(2)}}$, $h^{(3)}_j$ and $h^{(4)}_j$ the two remaining external half-lines. The tree line
\( l_j^{(2)} \) (corresponding to \( h_j^{(2)} \)) cuts the tree in two connected components. By analogy with the definitions in Sec.II.1, we call \( T_j^L \) the component containing the vertex \( v_{h(j)} \), and \( T_j^R \) the component containing the vertex \( v_{h(2)} \). Remark that \( g_j \) belongs to \( T_j^L \). Finally we call \( L_j^L \) the set of internal loop half-lines of \( g_{j-1} \), hooked to \( T_j^L \), that may connect somewhere in \( T_j^R \), and have not been already contracted. In the same way we introduce \( L_j^R \).

Now we extract the factor we need inductively for \( j \in C''_r \). For each subgraph \( g_j \) we distinguish two situations:

- if \(|e_{g_j}(C)| > 4\) we extract the factor \( \frac{\Lambda^{\pm}(w_{A(j)})}{\Lambda^{\pm}(w_j)} \) from the convergent power counting of \( g_j \), and we pass to the cross above in the chain;

- if \(|e_{g_j}(C)| = 4\) we test the number of loop lines connecting \( T_j^L \) to \( T_j^R \). Remark that this number must be always even, and cannot be zero by 1PI. If there are two loop lines, we know, by Lemma 9, that a finer estimation of the sector volumes gives an additional factor \( \Lambda^{\pm}(w_{A(j)}) \ln \Lambda(w_{A(j)}) \). Then we stop the induction. If there are four or more, we observe (Lemma 10) that we have counted one unnecessary sum on sector choices and we gain the factor \( \frac{\Lambda^{\pm}(w_{A(j)})}{\Lambda^{\pm}(w_j)} \). Then we pass to the following cross in the chain.

Putting together all these terms we obtain the factor we wanted times a logarithm.

\[
\Lambda^{\pm}(w_{t(r)}) \ln \Lambda(w_{t(r)})
\]

(B.26)

**Extracting loop lines.** We consider the four point subgraph \( g_j \) on the chain. We distinguish three situations.

1. If \( h_j^{(3)} \) and \( h_j^{(4)} \) are both loop half-lines we contract them developing the determinant (see eq.(II.10)). As in Sec.II.1, the number of choices is bounded by \(|L_j^R|^2\).
2. If \( h_j^{(3)} \) is a tree half-line and \( h_j^{(4)} \) is a loop one, we contract \( h_j^{(4)} \) by developing the determinant. If it contracts to \( T_j^R \), then we have to extract, applying several times the formula eq.\((11.11-11.12)\), one or three loop lines joining \( T_j^L \) with \( T_j^R \) (depending if there are two or more loop lines joining \( T_j^L \) with \( T_j^R \)). If \( h_j^{(4)} \) contracts to \( T_j^L \), then we have to extract two or four additional loop lines (depending if there are two or more loop lines joining \( T_j^L \) with \( T_j^R \)). In any case the number of choices is bounded by \(|L_j^R|^4|L_j^L|^5|\).

3. If \( h_j^{(3)} \) and \( h_j^{(4)} \) are both tree half-lines, then we call \( T_j^{(3)} \) the subtree connected to \( g_j \) through \( h_j^{(3)} \), and \( T_j^{(4)} \) the one connected to \( g_j \) through \( h_j^{(4)} \) (see Fig.14). In the same way we define \( L_j^{(3)} \) and \( L_j^{(4)} \) \((L_j^L = L_j^{(3)} \cup L_j^{(4)})\). Then we apply eq.\((11.11-11.12)\) several times, until we extract two or four loop lines joining \( T_j^L \) with \( T_j^R \). Finally, if there are four loop lines we perform an additional analysis. If four or two loop lines extracted are hooked to \( T_j^{(3)} \), then we apply eq.\((11.11)\) once more to extract a loop line joining \( T_j^{(4)} \) to \( T_j^{(3)} \) or to \( T_j^{(3)} \) (there must be one by 1PI, and by the parity of the number of external lines). In any case the number of choices is bounded by \(|L_j^{(3)}|^5|L_j^{(4)}|^5|L_j^L|^5|\).

**Number of choices.** Applying Lemma 1 and 2 we see that the remaining determinant still satisfies a Gram inequality, and the number of choices to extract the loop lines is bounded by \( K^n \).

Now we prove the following lemmas.

**Lemma 10** Let \( g_j \) be a four point subgraph on the chain \( C^\alpha \). If there are only two loop lines \( l_1^{(1)} \) and \( l_2^{(2)} \), connecting \( T_j^R \) to \( T_j^L \), then the power counting has an additional volume factor \( \Lambda^\frac{5}{2}(w_{A(j)}) \left| \ln \Lambda(w_{A(j)}) \right| \).

**Proof** As there are only two loop lines, then \( T_j^L \) actually is a four point subgraph \( G_j \) (but not necessarily quasi-local) with external lines \( l_1(G_j) = h_j^{(1)} \), \( l_2(G_j) = l_j^{(2)} \), \( l_3(G_j) = l_1^{(1)} \) and \( l_4(G_j) = l_2^{(2)} \). For an example see Fig.17.

Now we refine the sectors of \( l_i(G_j), i = 1, ..., 4 \) (that may be of different sizes \( \Lambda^\frac{5}{2}(l(G_j)) \leq \Lambda^\frac{5}{2}(w_{A(j)}) \)) in smaller sectors multiplying their size by \( \Lambda^\frac{5}{2}(w_{A(j)}) \). Remark that, when \( \Lambda^\frac{5}{2}(l_i(G_j)) = \Lambda^\frac{5}{2}(w_{A(j)}) \forall i \), this means we
Figure 16: example of $T_{j}^{(3)}$ and $T_{j}^{(4)}$

Figure 17: Example of a non quasi-local four point subgraph $G_j$. 

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are passing to isotropic sectors. By Lemma 2, Sec.II, in [FMRT] the new sector sum costs
\[
\frac{1}{\Lambda^{\frac{1}{2}}(w_{A(j)})} \left| \ln \Lambda(w_{A(j)}) \right| \tag{B.27}
\]
Remark that the bad factor \(\Lambda^{-\frac{1}{2}}(w_{A(j)})\) coming from the worse spatial decay of the tree line is compensated by the good factor \(\Lambda^{\frac{1}{2}}(w_{A(j)})\) from the smaller volume in impulsion space. Loop lines are not used for spatial integration, therefore their smaller volume factor in impulsion space is not consumed. Each loop line gives therefore a net bonus \(\Lambda^{\frac{1}{2}}(w_{A(j)})\). Finally we are left with the factor
\[
\Lambda(w_{A(q)}) \Lambda^{-\frac{1}{2}}(w_{A(q)}) \left| \ln \Lambda(w_{A(q)}) \right| = \Lambda^{\frac{1}{2}}(w_{A(q)}) \left| \ln \Lambda(w_{A(q)}) \right| \tag{B.28}
\]
When there are at least four loop lines joining \(\mathcal{T}_{j}^{L}\) with \(\mathcal{T}_{j}^{R}\), the following lemma proves that we have paid one unnecessary sector refinement.

**Lemma 11** Let the four point subgraph \(g_{j}\) on the chain \(C^{\alpha}\) and the four loop lines \(l_{1,j}, \ldots, l_{4,j}\), joining \(\mathcal{T}_{j}^{L}\) with \(\mathcal{T}_{j}^{R}\), be fixed. Then the number of sector choices predicted by [DR1], Lemma 6, (IV.31) must be modified:
\[
\prod_{m=2}^{4} \left[ \frac{1}{2} \Lambda^{-\frac{1}{2}}(w_{A(j)}) \right] \int_{\Sigma_{h_{j}^{(m)}, r(j)+1}} d\theta_{h_{j}^{(m)}, \{\theta^{(\text{root})}_{j}, \{\theta^{(m)}, r(j)\}_{m=2,4}} \leq K \tag{B.29}
\]
for some constant \(K\).

**Proof** The proof is quite similar to that of Lemma 4, but slightly more complicated, as this time the sector of the external tree line on the path joining \(x^{(1)}\) with \(x^{(2)}\) must be summed. We distinguish two cases.

1. If \(h_{j}^{(3)}\) is a loop half-line and \(h_{j}^{(4)}\) is a tree one, we know there are at least three loop lines (different from \(h_{j}^{(3)}\)), called \(l_{1,j}, l_{2,j}, l_{3,j}\), joining \(\mathcal{T}_{j}^{R}\) to \(\mathcal{T}_{j}^{L}\) (see Fig.18a). One of these lines, say \(l_{1,j}\), may have been used to gain a sector sum on the other chain \(C^{\alpha}\) (in Lemma 4). At least one of the two remaining loop lines, say \(l_{2,j}\), has been paid in refining sectors. This was not necessary, as its sector is fixed by impulsion conservation along the loop line. Then this
Figure 18: a. shows the usual sector counting and b. shows the new sector counting taking \(l^2_j\) as root.

Figure line can be chosen as a new root to perform sector counting (see Fig. 18b). This permits to fix the sector of \(h^{(4)}_j\). The sector of \(h^{(3)}_j\) is fixed by impulsion conservation along the loop line.

2. If \(h^{(3)}_j\) and \(h^{(4)}_j\) are both tree half-lines, then we know there are four loop lines \(l^i_j, i = 1, \ldots, 4\), connecting \(T^L_j\) with \(T^R_j\). Now we have three situations, shown on Fig. 20.

- \(l^1_j, l^2_j\), and \(l^3_j\) are hooked to \(T^{(3)}_j\) and only \(l^4_j\) is hooked to \(T^{(4)}_j\) (see Fig. 20a). One of the first three lines, say \(l^1_j\), may have been used to gain a sector sum on the other chain \(C^{x}\), then, among \(l^2_j\) and \(l^3_j\), we choose as new root for sector counting in \(T^{(3)}_j\), the one that has been summed in usual sector refinement. In \(T^{(4)}_j\), we choose as new root the unique loop line \(l^4_j\).

- \(l^1_j\) and \(l^2_j\) are hooked to \(T^{(3)}_j\) while \(l^3_j\) and \(l^4_j\) are hooked to \(T^{(4)}_j\) (see Fig. 20a). Then there is a fifth loop line \(l^5_j\) hooked to \(T^{(4)}_j\), and we repeat the same argument above, exchanging \(T^{(4)}_j\) with \(T^{(3)}_j\).

- All the four loop lines are hooked to \(T^{(3)}_j\) (see Fig. 20b). Then there is a fifth loop line \(l^5_j\) hooked to \(T^{(4)}_j\), and we repeat the same argument above.
Figure 19: Three loop lines hook to $T_j^{(3)}$ and only one to $T_j^{(4)}$.

Figure 20: a.: two loop lines hook to $T_j^{(3)}$ and two to $T_j^{(4)}$. b.: four loop lines hook to $T_j^{(3)}$ and none to $T_j^{(4)}$. 

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In any case, the two sectors, for \( h_j^{(3)} \) and \( h_j^{(4)} \), are fixed. Then, as the sector of \( h^{(1)} \) is always fixed, three sectors are known, hence the fourth one too and there is no sector refinement to pay. Then we gain the factor \( \frac{\Lambda^{1/2}(w_{A(j)})}{\Lambda^{1/2}(w_j)} \) and iterate the process.

**Final bound.** Inserting all these results, and performing power counting we find the bound

\[
\left| \partial_{k_a} \partial_{k_b} \tilde{\Sigma}(k) \right| \leq \sum_{n=2}^{\infty} (|\lambda \ln T|)^n \leq M_3 \tag{B.30}
\]

for \( |\lambda \ln T| \leq 1/K = M_3 \). Remark that there is no factor \( \lambda^2 \) as in eq.(B.23), as there are two additional logarithms, coming one from the power counting of the subgraph \( g_r \), and the other from the bound eq.(B.24) on sector counting.

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