The general solution of the matrix equation
\[ w_t + \sum_{k=1}^{n} w_{xk} \rho^{(k)}(w) = \rho(w) + [w, T\tilde{\rho}(w)] \]

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Abstract

We construct the general solution of the equation
\[ w_t + \sum_{k=1}^{n} w_{xk} \rho^{(k)}(w) = \rho(w) + [w, T\tilde{\rho}(w)], \]
for the \(N \times N\) matrix \(w\), where \(T\) is any constant diagonal matrix, \(n, N \in \mathbb{N}_+\) and \(\rho^{(k)}, \rho, \tilde{\rho} : \mathbb{R} \rightarrow \mathbb{R}\) are arbitrary analytic functions. Such a solution is based on the observation that, as \(w\) evolves according to the above equation, the evolution of its spectrum decouples, and it is ruled by the scalar analogue of the above equation. Therefore the eigenvalues of \(w\) and suitably normalized eigenvectors are the \(N^2\) Riemann invariants. We also obtain, in the case \(\rho = \tilde{\rho} = 0\), a system of \(N^2\) non-differential equations characterizing such a general solution. We finally discuss reductions of the above matrix equation to systems of \(N\) equations admitting, as Riemann invariants, the eigenvalues of \(w\). The simplest example of such reductions is a particular case of the gas dynamics equations.

1 Introduction

It is well-known that the general solution of the scalar first order quasi-linear Partial Differential Equation (PDE):
\[ u_t + \sum_{k=1}^{n} u_{xk} \rho^{(k)}(u) = \rho(u), \quad u : \mathbb{R}^{n+1} \rightarrow \mathbb{R}, \quad (1) \]
can be constructed by the method of characteristics \([1]\), converting (1) into the system of ODEs:
\[ \frac{du}{dt} = \rho(u), \quad \frac{dx_k}{dt} = \rho^{(k)}(u) , \quad k = 1, \ldots, n, \quad (2) \]
defined on the characteristic curves described by equations (2b). If, in particular, \(\rho = 0\), then the characteristic curves become straight lines and equations (2) are integrated in the form:
\[ u = f(\eta_1, \ldots, \eta_N), \quad x_k = \rho^{(k)}(u)t + \eta_k, \quad k = 1, \ldots, n, \quad (3) \]
where \(\eta_k, \quad k = 1, \ldots, N\) are arbitrary constants, \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) is an arbitrary scalar function that can be identified with the initial condition: \(f(x_1, \ldots, x_n) = u(x_1, \ldots, x_n, 0)\) of the Cauchy problem.
in \( \mathbb{R}^n \), and \( u \) is constant along the characteristic straight lines. From equations (3) it also follows that the general solution \( u(x, t) \) of (1) for \( \rho = 0 \) is defined implicitly by the non-differential equation

\[
   u = f \left( x_1 - \rho^{(1)}(u)t, ..., x_n - \rho^{(n)}(u)t \right).
\]

(4)

Vector generalizations of equation (1) have been considered in several papers [2, 3, 4, 5, 6, 7].

The main result of our work is the construction, using elementary spectral means, of the general solution of the following matrix generalization of equation (1): 

\[
   w_t + \sum_{k=1}^{n} w_{x_k} \rho^{(k)}(w) = \rho(w) + [w, T \tilde{\rho}(w)],
\]

(5)

where \( w \) is the unknown \( N \times N \) matrix function of the \( n+1 \) independent variables \( (x_1, ..., x_n, t) \in \mathbb{R}^{n+1}, T \) is any constant diagonal matrix, \([\cdot, \cdot]\) is the usual commutator between matrices, and \( \rho^{(k)}, \rho, \tilde{\rho} : \mathbb{R} \to \mathbb{R}, k = 1, ..., n \) are \( n+2 \) arbitrary scalar functions representable by positive power series, so that the quantities \( \rho^{(k)}(w), k = 1, ..., n, \rho(w) \) and \( \tilde{\rho}(w) \) are well-defined functions of the matrix \( w \).

In \( \S 2 \) we show that, as \( w \) evolves according to (5), the evolution of its spectrum decouples, and it is ruled by the scalar analogue of the above equation. Therefore the eigenvalues and suitably normalized eigenvectors of \( w \) are the \( N^2 \) Riemann invariants. In addition, the number of characteristic curves is only \( N \) (and not \( N^2 \), as it would be for a more generic vector system of \( N^2 \) equations).

In \( \S 3 \) we construct, in the case \( \rho = \tilde{\rho} = 0 \), the matrix generalization of the non-differential equation (4), given by a nonlinear system of \( N^2 \) non-differential equations for the components of \( w \).

In \( \S 4 \) we introduce a distinguished reduction of equation (5) to a system of \( N \) equations exhibiting, as Riemann invariants, the eigenvalues of \( w \).

We end this introduction with the obvious remark that solving equation (5) is equivalent to solving the equation transposed to (5)

\[
   \tilde{w}_t + \sum_{k=1}^{n} \rho^{(k)}(\tilde{w}) \tilde{w}_{x_k} = \rho(\tilde{w}) + [\tilde{\rho}(\tilde{w})T, \tilde{w}],
\]

(6)

whose solution is just the transposed of \( w \): \( \tilde{w} = w^T \).

## 2 The spectral solution of (5)

The main characterization of the matrix equation (5) is that, although the \( N^2 \) components \( w_{ij}, i, j = 1, ..., N \) of \( w \) are coupled by the evolution, such a coupling disappears in spectral space, and the eigenvalues of \( w \) evolve according to the scalar analogue (1) of (5).

**Proposition 1** Let \( \{e^{(1)}, ..., e^{(N)}\} \) be the eigenvalues of matrix \( w \), let \( \{v^{(1)}, ..., v^{(N)}\} \) be the corresponding right eigenvectors, conveniently normalized to satisfy the \( N \) conditions \( v^{(j)}_j = 1, j = 1, ..., N \), where \( v^{(j)}_i \) is the \( i \)-component of vector \( v^{(j)} \). Then, if \( w \) evolves according to the matrix PDE (5), the dynamics in spectral space is decoupled in the following way.

i) Each eigenvalue evolves separately according to the scalar analogue of equation (5):

\[
   e^{(j)}_t + \sum_{k=1}^{n} e^{(j)}_{x_k} \rho^{(k)}(e^{(j)}) = \rho(e^{(j)}), \quad j = 1, ..., N.
\]

(7)
ii) Known \( e^{(j)} \) from equation \( 7 \), the corresponding normalized eigenvector \( v^{(j)} \) evolves according to the linear equation:

\[
\left( v_i^{(j)} \right)_t + \sum_{k=1}^{n} \left( v_i^{(j)} \right)_{x_k} \rho^{(k)}(e^{(j)}) = (T_j - T_i)v_i^{(j)} \tilde{\rho}(e^{(j)}), \quad i, j = 1, \ldots, N.
\]

(8)

Proof. The proof is by direct calculation. We substitute the spectral decomposition of \( w \) into \( 5 \), where \( E = \text{diag}(e^{(1)}, \ldots, e^{(N)}) \) and \( v \) is the matrix of the normalized eigenvectors: \( v_{ij} = v_i^{(j)} \). After multiplying this equation from the left and from the right respectively by \( v^{-1} \) and \( v \), we obtain, after some manipulation:

\[
E_t + \sum_{k=1}^{n} E_{x_k} \rho^{(k)}(E) - \rho(E) + \left[ v^{-1} \left( v_t + \sum_{k=1}^{n} v_{x_k} \rho^{(k)}(E) + T v \tilde{\rho}(E) \right) \right] E = 0.
\]

(10)

Separating the diagonal and off-diagonal parts of this equation, we obtain:

\[
E_t + \sum_{k=1}^{n} E_{x_k} \rho^{(k)}(E) = \rho(E),
\]

\[
v_t + \sum_{k=1}^{n} v_{x_k} \rho^{(k)}(E) + T v \tilde{\rho}(E) = v h, \quad h \text{ diagonal}
\]

(11)

The diagonal matrix \( h \) is fixed by the normalization of matrix \( v \). In our case, the diagonal part of \( v \) is the identity matrix; therefore, taking the diagonal part of \( 11b \), one infers that \( h = T \tilde{\rho}(E) \). Equations \( 11a \) and \( 11b \) with \( h = T \tilde{\rho}(E) \) are just equations \( 7 \) and \( 8 \) respectively. □

The PDEs \( 7 \) and \( 8 \) are converted into the system of ODEs:

\[
\frac{d e^{(j)}}{d t} = \rho(e^{(j)}), \quad j = 1, \ldots, N
\]

\[
\frac{d v^{(j)}}{d t} = (T_j - T_i)v_i^{(j)} \tilde{\rho}(e^{(j)}), \quad i, j = 1, \ldots, N
\]

\[
\frac{d x_k}{d t} = \rho^{(k)}(e^{(j)}), \quad k = 1, \ldots, n,
\]

(12)

defined on the characteristic curves described by equations \( 12a \). On such characteristic curves, the eigenvalues are obtained by the quadratures associated with \( 12b \); once the eigenvalues are constructed, the eigenvectors are obtained solving the linear equations \( 12b \).

If \( \rho = \tilde{\rho} = 0 \), the characteristic curves become the straight lines

\[
x_k = \rho^{(k)}(e^{(j)}) t + \eta_k, \quad k = 1, \ldots, n,
\]

(13)

where \( \eta_k \) are arbitrary integration constants, and each eigenvalue \( e^{(j)}(x_1, \ldots, x_n, t) \) is defined by the implicit equation

\[
e^{(j)}( \eta_1, \ldots, \eta_n ) = e^{(j)} \left( x_1 - \rho^{(1)}(e^{(j)}) t, \ldots, x_n - \rho^{(n)}(e^{(j)}) t \right),
\]

(14)

where \( e^{(j)} : \mathbb{R}^n \to \mathbb{R} \) is an arbitrary function. For the normalized eigenvector \( v^{(j)} \), constant along the characteristic curve \( 13 \), the general solution is explicit in terms of \( e^{(j)} \):

\[
v^{(j)}(x_1, \ldots, x_n, t) = \mathbf{v}^{(j)} \left( x_1 - \rho^{(1)}(e^{(j)}) t, \ldots, x_n - \rho^{(n)}(e^{(j)}) t \right),
\]

(15)
where $v^{(j)} : \mathbb{R}^n \to \mathbb{R}^N$ is an arbitrary vector function. Then the general solution of the matrix system (5) is achieved through the spectral formula (9). Such a solution depends, as it has to be, on the $N^2$ arbitrary scalar functions $\epsilon^{(j)}$, $j = 1, \ldots, N$ and $v_i^{(j)}$, $i, j = 1, \ldots, N$, $i \neq j$. We remark that, although the evolution in spectral space is decoupled, the components of $w$ evolve along all the $N$ characteristics, through the algebraic coupling given by (9).

If, in particular, one is interested in solving the Cauchy problem for the matrix system (5) on $\mathbb{R}^n$, for $\rho = \tilde{\rho} = 0$, one first remarks that, in this case, $\epsilon^{(j)}$, $j = 1, \ldots, N$ and $v_i^{(j)}$, $i, j = 1, \ldots, N$, $i \neq j$ are identified with the initial conditions for the eigenvalues and eigenvectors:

$$\begin{align*}
\epsilon^{(j)}(x_1, \ldots, x_n) &= \epsilon^{(j)}(x_1, \ldots, x_n, 0), \\
v^{(j)}(x_1, \ldots, x_n) &= v^{(j)}(x_1, \ldots, x_n, 0),
\end{align*}$$

Therefore: i) given the initial condition

$$f(x_1, \ldots, x_n) = w(x_1, \ldots, x_n, 0),$$

one uniquely constructs the initial conditions $\epsilon^{(j)}(x_1, \ldots, x_n)$ and $v^{(j)}(x_1, \ldots, x_n)$ for the eigenvalues and eigenvectors solving the associated eigenvalue problem:

$$f v^{(j)} = \epsilon^{(j)} v^{(j)}, \quad j = 1, \ldots, N$$

with the above normalization for the eigenvectors. ii) Given $\epsilon^{(j)}$ and $v^{(j)}$, one obtains the decoupled evolution of the spectrum through the equations (14) and (15). iii) The reconstruction of $w(x_1, \ldots, x_n, t)$ is finally achieved through the formula (9).

The elementary spectral solution of the nonlinear PDE (5) presented in this section is, to the best of our knowledge, new.

The nonlinear PDEs (5) have been recently identified by the authors because they arise as the simplest examples of a class of nonlinear systems of PDEs in arbitrary dimensions generated by a novel dressing procedure based on a homogeneous integral equation with nontrivial kernel [8].

### 3 The analogue of the implicit equation (4)

Starting with the spectral solution of (5) obtained in the previous section, it is possible to construct, if $\rho = \tilde{\rho} = 0$, the matrix analogue of the implicit non-differential equation (4).

**Proposition 2** Let $f_{ij} : \mathbb{R}^n \to \mathbb{R}$, $i, j = 1, \ldots, N$ be $N^2$ arbitrary scalar functions admitting formal positive power expansions, so that $f_{ij}(M_1, \ldots, M_n)$ are well-defined $N \times N$ matrices, where $M_1, \ldots, M_n$ are arbitrary matrices $N \times N$. Denote by $(f_{ij}(M_1, \ldots, M_n))_{kl}$ the $(kl)$-component of the matrix $f_{ij}(M_1, \ldots, M_n)$. Then the general solution $w$ of equation (5) is characterized implicitly by the following system of $N^2$ non-differential equation:

$$w_{ij} = \sum_{k=1}^{N} \left( f_{ik} \left( x_1 I - \rho^{(1)}(w)t, \ldots, x_n I - \rho^{(n)}(w)t \right) \right)_{kj}, \quad i, j = 1, \ldots, N,$$

where $w_{ij}$ is the $(ij)$-component of matrix $w$ and $I$ is the $N \times N$ identity matrix.
Proof. Using equations (9), (7) and (8), we have that:

\[
w_{ij} = \sum_{k=1}^{N} v_{ik} \epsilon^{(k)}(v^{-1})_{kj} = \sum_{i=1}^{N} v_{ik} (x_1 - \rho^{(1)}(\epsilon^{(k)})t, \ldots, x_n - \rho^{(n)}(\epsilon^{(k)})t) \times \\
\epsilon^{(k)}(x_1 - \rho^{(1)}(\epsilon^{(k)})t, \ldots, x_n - \rho^{(n)}(\epsilon^{(k)})t) (v^{-1})_{kj}
\] (20)

Let \( f \) be the matrix having \( \epsilon^{(l)} \), \( l = 1, \ldots, N \) as eigenvalues and \( v \) as matrix of eigenvectors, i.e.: \( f_{ij} = \sum_{l=1}^{N} v_{il} \epsilon^{(l)} v^{-1}_{lj} \), then equation (20) becomes:

\[
w_{ij} = \sum_{k,l=1}^{N} v_{ik} (x_1 - \rho^{(1)}(\epsilon^{(k)})t, \ldots, x_n - \rho^{(n)}(\epsilon^{(k)})t) f_{il} (x_1 - \rho^{(1)}(\epsilon^{(k)})t, \ldots, x_n - \rho^{(n)}(\epsilon^{(k)})t) (v^{-1})_{kj}.
\] (21)

Replacing now the definition of function of a matrix:

\[
f_{il} (x_1 - \rho^{(1)}(\epsilon^{(k)})t, \ldots, x_n - \rho^{(n)}(\epsilon^{(k)})t) = \\
\sum_{m,r=1}^{N} (v^{-1})_{km} (f_{il} (x_1 I - \rho^{(1)}(\epsilon^{(k)})t, \ldots, x_n I - \rho^{(n)}(\epsilon^{(k)})t))_{mr} v_{rs},
\] (22)

into equation (21), we obtain the result. \( \Box \)

We end this section remarking that, as a byproduct of Propositions 1 and 2, the complicated system of \( N^2 \) non-differential equations (19) defining the solution \( w \) is reduced to the solution of the single non-differential equation (14) for the eigenvalues of \( w \).

4 A distinguished reduction

In this section we briefly describe a reduction of the matrix equation (5) exhibiting Riemann invariants coinciding with the eigenvalues \( e^{(j)} \), \( j = 1, \ldots, N \).

Consider the subspace of \( N \times N \) matrices spanned by the basis \( \{\omega_0, \ldots, \omega_{N-1}\} \) given by:

\[
\omega_0 = I, \quad (\omega_1)_{ij} = \delta_{i+1,j}, \quad \ldots, \quad (\omega_k)_{ij} = \delta_{i+k,j}, \quad \ldots, \quad (\omega_{N-1})_{ij} = \delta_{i+N-1,j}, \quad \text{mod}N.
\] (23)

This subspace is left invariant under matrix multiplication, since:

\[
\omega_j \omega_k = \omega_k \omega_j = \omega_{j+k}, \quad \text{mod}N,
\] (24)

therefore it defines a reduction of (5) from the \( N^2 \) components of \( w \) to the \( N \) scalar coefficients \( \nu_k, \ k = 1, \ldots, N \) of the expansion:

\[
w = \sum_{k=1}^{N} \nu_k \omega_{k-1}.
\] (25)

We remark that the mapping between the \( N \) dependent variables \( \nu_k, \ k = 1, \ldots, N \) of the reduced system and the eigenvalues \( e^{(j)} \), \( j = 1, \ldots, N \) allows one to decouple completely the above dynamics; therefore the eigenvalues \( e^{(j)} \), \( j = 1, \ldots, N \) are the Riemann invariants for this reduced class of equations.

In the remaining part of this section we show some examples of reduced systems.

A system of 2 interacting fields in 2 + 1 dimensions is obtained, for instance, choosing \( N = 2, \ n = 2, \ \rho^{(1)}(x) = x, \ \rho^{(2)}(x) = ax^2, \ \rho = \tilde{\rho} = 0; \)

\[
\nu_1 t + \nu_1 \nu_{1x_1} + \nu_2 \nu_{2x_1} + a(\nu_1^2 + \nu_2^2)\nu_{1x_2} + 2a\nu_1 \nu_2 \nu_{2x_2} = 0, \\
\nu_2 t + \nu_2 \nu_{1x_1} + \nu_1 \nu_{2x_1} + 2a\nu_1 \nu_2 \nu_{1x_2} + a(\nu_1^2 + \nu_2^2)\nu_{2x_2} = 0.
\] (26)
If $a = 0$, equation (26) reduces to a particular case of the well-known gas dynamics equations [1]:

\[
\begin{align*}
\nu_1 t + \nu_1 \nu_1 x_1 + \nu_2 \nu_2 x_1 &= 0, \\
\nu_2 t + \nu_2 \nu_1 x_1 + \nu_1 \nu_2 x_1 &= 0.
\end{align*}
\] (27)

The eigenvalues of $w$:

\[
\begin{align*}
e^{(1)} &= \nu_1 + \nu_2, \\
e^{(2)} &= \nu_1 - \nu_2,
\end{align*}
\] (28)

evolve decoupled according to equation (7) and are the Riemann invariants of the above two systems (26) and (27).

The simplest example of a system of 3 interacting fields arises choosing $N = 3$, $n = 1$, $\rho^{(1)}(x) = x$, $\rho = \tilde{\rho} = 0$:

\[
\begin{align*}
\nu_1 t + \nu_1 \nu_1 x_1 + \nu_3 \nu_2 x_1 + \nu_2 \nu_3 x_1 &= 0, \\
\nu_2 t + \nu_2 \nu_1 x_1 + \nu_1 \nu_2 x_1 + \nu_3 \nu_3 x_1 &= 0, \\
\nu_3 t + \nu_3 \nu_1 x_1 + \nu_2 \nu_2 x_1 + \nu_1 \nu_3 x_1 &= 0.
\end{align*}
\] (29)

The three eigenvalues

\[
\begin{align*}
e^{(1)} &= \nu_1 + \nu_2 + \nu_3, \\
e^{(2)} &= \frac{1}{2} \left[ 2\nu_1 - \nu_2 - \nu_3 + i\sqrt{3}|\nu_2 - \nu_3| \right], \\
e^{(3)} &= \frac{1}{2} \left[ 2\nu_1 - \nu_2 - \nu_3 - i\sqrt{3}|\nu_2 - \nu_3| \right]
\end{align*}
\] (30)

evolve decoupled according to equation (7). But only $e^{(1)}$ is real, therefore the system is not hyperbolic and the eigenvalues $e^{(2)}$ and $e^{(3)}$ can be called Riemann invariants only in an extended sense.

The systematic study of this reduced class and of other reductions of equation (5), in the search for integrable and applicative equations, is postponed to a subsequent paper.

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