ON THE FOUNDATION OF ALGEBRAIC TOPOLOGY

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Abstract. In the 70:th, combinatorialists begun to systematically relate simplicial complexes and polynomial algebras, named Stanley-Reisner rings or face rings. This demanded an algebraization of the simplicial complexes, that turned the empty simplicial complex into a zero object w.r.t. to simplicial join, losing its former role as join-unit, a role taken over by a new (-1)-dimensional simplicial complex containing only the empty simplex.

There can be no realization functor targeting the classical category of topological spaces that turns the contemporary simplicial join into topological join unless a (-1)-dimensional space is introduced as a topological join-unit. This algebraization of general topology enables a homology theory that unifies the classical relative and reduced homology functors and allows a Künneth Theorem for simplicial resp. topological pair-joins.

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1991 Mathematics Subject Classification. Primary: 55Nxx, 55N10; Secondary: 57P05, 57Rxx.
Key words and phrases. Augmental Homology, join, manifolds, Stanley-Reisner rings.
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Part 1. Introduction

1. The Algebraic Topological Foundation is in Need of a Makeover

1.1. Simplicial complexes within Combinatorics. In Gräbe’s paper \[19\] §4.1 p. 171, the reduced homology groups, with field coefficients, of the simplicial join of two complexes were given as direct sums of tensor products, the relative analogue of which seemed non-existing, cf. p. \[3\]:

\[
\tilde{H}_{q+1}(\Sigma_1 \ast \Sigma_2; k) \cong \bigoplus_{i+j=q} \left( \tilde{H}_i(\Sigma_1; k) \otimes_k \tilde{H}_j(\Sigma_2; k) \right).
\]

\[19\] §4.2 p. 171 gives the boundary of the simplicial join of quasi-manifolds:

\[
\text{Bd}(\Sigma_1 \ast \Sigma_2) = ((\text{Bd}\Sigma_1) \ast \Sigma_2) \cup (\Sigma_1 \ast (\text{Bd}\Sigma_2)).
\]

This formula only works under Definition 1 below, partly due to the fact that manifolds now can have \(\emptyset\) as well as the “new” \((-1)\)-dimensional join unit \(\{\emptyset\}\) as its boundary.
Definition. 1) An augmented abstract simplicial complex $\Sigma$ on a vertex set $V_\Sigma$ is a collection of finite subsets $\sigma$, the simplices, of $V_\Sigma$ satisfying:

(a) If $v \in V_\Sigma$, then $\{v\} \in \Sigma$.
(b) If $\sigma \in \Sigma$ and $\tau \subset \sigma$, then $\tau \in \Sigma$.

Define the simplicial join $\Sigma_1 \ast \Sigma_2$, of two complexes $\Sigma_1$ and $\Sigma_2$ with $V_{\Sigma_1} \cap V_{\Sigma_2} = \emptyset$ to be:

$$\Sigma_1 \ast \Sigma_2 := \{\sigma_1 \cup \sigma_2 \mid \sigma_i \in \Sigma_i (i = 1, 2)\}.$$  

Definition 1 has three levels: a complex (level 1) is a set of simplices (level 2), which are finite sets (empty or non-empty) of vertices (level 3). The empty set, $\emptyset$, plays a dual role; it is both a $(-1)$-dimensional simplex and a $(-\infty)$-dimensional complex. See p. 3 for the definition of “dimension”.

Definition. 2) (\[45] p. 108-9) A classical (abstract) simplicial complex $\Sigma$ on a vertex set $V_\Sigma$ is a collection of non-empty finite subsets $\sigma$, the simplices, of $V_\Sigma$ satisfying:

(a) If $v \in V_\Sigma$, then $\{v\} \in \Sigma$.
(b) If $\sigma \in \Sigma$ and $\tau \subset \sigma$, then $\tau \in \Sigma$.

The classical simplicial join $\Sigma_1 \ast \Sigma_2$, of two disjoint classical complexes is defined to be:

$$\Sigma_1 \ast \Sigma_2 := \Sigma_1 \cup \Sigma_2 \cup \{\sigma_1 \cup \sigma_2 \mid \sigma_i \in \Sigma_i (i = 1, 2)\}.$$  

If $\text{card}(\sigma) = q + 1$ for a simplex $\sigma$ then $\dim \sigma := q$ and $\sigma$ is said to be a $q$-face or a $q$-simplex of $\Sigma$ and $\dim \Sigma := \sup\{\dim(\sigma) \mid \sigma \in \Sigma\}$. Writing $\emptyset$, when using $\emptyset$ as a simplex, we get $\dim(\emptyset) = -\infty$ and $\dim(\{\emptyset\}) = \dim(\emptyset, \emptyset) = -1$.

So, the dimension of any non-empty simplicial complex is a well-defined integer $\geq -1$, which for the join of two simplicial complexes with disjoint vertex sets results in the following dimension formula:

$$\dim(\Sigma_1 \ast \Sigma_2) = \dim \Sigma_1 + \dim \Sigma_2 + 1$$

implying that the join unit must be of dimension $-1$, which by definition is the dimension of the augmented abstract simplicial complex $\{\emptyset\}$.

The difference between Definition 1 and Definition 2, is that the former allows the empty set $\emptyset$ as a simplex that moreover is contained in every nonempty simplicial complex, implying that the simplicial complex generated by any vertex $v$, called a (combinatorial) 0-ball, is the set $\bullet := \{\emptyset, \{v\}\}$, resp. $\{\{v\}\}$. Similarly, the (combinatorial) 0-sphere is the set $\bullet \bullet := \{\emptyset, \{v\}, \{w\}\}$, resp. $\{\{v\}, \{w\}\}$. In dimension 1 and below, the boundary $\text{Bd}$ of any reasonably defined simplicial manifold of dimension $n$ is generated by the simplices of dimension $n - 1$ that is a face of only one simplex of dimension $n$. So, in particular, $\text{Bd}(\bullet) = \{\emptyset\} \neq \emptyset$ according to Definition 1 but $\text{Bd}(\bullet) = \emptyset$ according to Definition 2, while $\text{Bd}(\bullet \bullet) := \emptyset$ with respect to either of the two definitions.
The classical algebraic topological structure needed two separate homology functors, the relative and the ad-hoc invented reduced homology functor. The change of combinatorial foundation from Definition 2 to Definition 1 took place (≈1970) the minute combinatorialists began to use (“clashed into”, cf. [33]) commutative algebra to solve combinatorial problems. The only homology apparatus invented by the combinatorialists to handle this ingenious algebraization of the category of simplicial complexes were a jargon like - “We will use reduced homology with $\tilde{H}_{-1}(\{\emptyset\}) = \mathbb{Z}$”, which added even more inefficiency to algebraic topology than it already suffered from as a result of that unfortunate but necessary classical use of the two complementing homology functors just mentioned.

The category of topological spaces and continuous functions were left unattended. So, suddenly there were two vertex free simplicial complexes, $\emptyset$ and $\{\emptyset\}$, but still only one point-free topological space, i.e. $\emptyset$, which made it impossible to define, in any reasonable way, a faithful realization functor.

This unfortunate state of affairs is rectified in this article by performing an analogous algebraization of general topology by simply introducing a $(-1)$-dimensional topological join-unit (denoted “$\{\emptyset\}$”) that also is the realization of the $(-1)$-dimensional simplicial complex $\{\emptyset\}$. Now the topological join has the tensor product as its functorial counterpart within the tensor categories, which underlines its central role within algebraic topology.

The resulting homology theory, i.e. the augmental homology theory, is complete in the sense that no reduced homology functor is needed.

1.2. Looking elsewhere for “simplicial complexes”. In \[18\] Ch. 3 p. 110, Fritsch and Piccinini give the following definition and example.

**Definition 3** (Equivalent to our Definition 1.) A Simplicial Complex is a set $\Sigma$ of finite sets closed under the formation of subsets, i.e., any subset of a member of $\Sigma$ is also a member of $\Sigma$; more formally:

$$(\sigma \in \Sigma) \land (\tau \subset \sigma) \implies \tau \in \Sigma.$$  

**Example** (\[18\] p. 110) Let $\{U_{\lambda} : \lambda \in \Lambda\}$ be a family of arbitrary sets; then the set $K(\Lambda)$ of all finite subsets of $\Lambda$ such that $\bigcap_{\lambda \in x} U_{\lambda} \neq \emptyset$ is a simplicial complex. Note that in general the vertex set of this simplicial complex $K(\Lambda)$ is not the index set $\Lambda$ itself, but only its subsets consisting of the indices $\lambda$ with $U_{\lambda} \neq \emptyset$. Now, if $Z$ is a space and $\{U_{\lambda} : \lambda \in \Lambda\}$ is a covering of $Z$ (see Section A.3), then the simplicial complex $K(\Lambda)$ obtained in this way is called the nerve of the covering $\{U_{\lambda} : \lambda \in \Lambda\}$.

**Note.** $\bigcap_{\lambda \in \emptyset} U_{\lambda} = \text{Universe} \neq \emptyset \Rightarrow x = \emptyset \in K(\Lambda)$ (cf. [29], Kelley: General Topology, p. 256).

So, we note that any nerve $K(\Lambda)$ constructed on a nonempty index set $\Lambda$ within any nonempty universe, is nonempty, since it contains at least the simplex $\emptyset$.

\[18\] is an excellent book and due to its detailed excellence we have a chance to dwell somewhat on a difference between its chapter 3 and chapter 4.

In \[18\] Ch. 3 the empty simplex is allowed, as seen above, but the realization of the complex $\{\emptyset\}$ is nowhere explicit. This new simplicial complex $\{\emptyset\}$ has no natural realization within any Euclidian $n$-space within the old classical category of topological spaces and continuous maps.
In \([18]\) Ch. 4 however, Fritsch and Piccinini use the “topologist’s \(\Delta\)”, here denoted \(\Delta^\circ\), which they call the category of finite ordinals. This \(\Delta^\circ\) actually excludes the empty simplex and therefore also the complex \(\{\emptyset\}\), which makes the realization functor identical to the classical and therefore non-problematic.

Next definition, equivalent to our Definition 2, is given in p. 2 in \([27]\).

**Definition 4.** A Simplicial Complex \(K\) is a collection of non-empty, finite subsets (called simplices) of a given set \(V\) (of vertices) such that any non-empty subset of a simplex is a simplex.

Deleting the word “non-empty” makes it equivalent to our Definition 1, i.e.:

**Definition 5.** An (augmented abstract) Simplicial Complex \(K\) is a collection of finite subsets (called simplices) of a given set \(V\) (of vertices) such that any subset of a simplex is a simplex.

2. Remaking the Topological Foundation

2.1. Synchronizing Logic, Combinatorics, General Topology and Algebra.

The “\(\sigma_1 \cup \sigma_2\)”-operation in Definition 1 p. \([3]\) is strongly “ordinal addition”-related, see p. \([51]\), and we learn from Note iii p. \([34]\) that the Stanley-Reisner (St-Re) ring of a join of two simplicial complexes is the tensor product of their respective St-Re rings. This simple observation reveals a very strong relation between logics and algebra that also involves combinatorics and general topology. Still, the join operations is very much ignored in the literature, except of course for \([5], [12], \text{ and } [17]\).

A moment of reflexion on “realization functor”-candidates \(|\cdot|\), reveals the need for a real topological join-unit, \(\{\emptyset\} = |\{\emptyset\}| \neq |\emptyset| = \emptyset\), which also becomes our new \((-1)\)-dimensional geometrical standard simplex.

We are obviously dealing with a non-classical situation, which we must not try to squeeze into a classical framework. The category of topological spaces and continuous maps has been unchanged ever since the publication of Felix Hausdorff’s *Grundzüge der Mengenlehre* (1914), but now, modern algebra-based techniques makes the introduction of an \((-1)\)-dimensional topological space \(\{\emptyset\}\) unavoidable. The result of such a \(\{\emptyset\}\)-introduction is an algebraization of general and algebraic topology, i.e. a synchronization of these categories to that of the tensor categories within commutative algebra that is completely compatible with the switch from Definition 2 to Definition 1 within combinatorics in the 70th.

The pure “Def. 2”-based homology theory, i.e. classical homology theory, uses a combination of the relative homology theory and the ad-hoc invented reduced homology functor, the mere existence of which is a proof of the insufficiency of classical relative homology theory. These two functors are simply unified into one single homology functor \(\hat{H}\) through the external adjunction, to the classical category of topological spaces, of the non-empty \(-1\)-dimensional object \(\{\emptyset\}\) - unit element with respect to topological join. This is, through \(\{\emptyset\} := |\{\emptyset\}|\), fully compatible with the \(\{\emptyset\}\)-introduction in Definition 1 above: “Externally adjoined” means that \(\emptyset\) is assumed not to have been an element of any classical topological space.

The category of sets mustn’t be tempered with, but add, using topological sum, to each classical topological space \(X \in \mathcal{D}\) an external element \(\emptyset\), resulting in \(X_\emptyset := X + \{\emptyset\} \in \mathcal{D}_\emptyset\). Finally we add the universal initial object \(\emptyset\). This resembles a familiar routine from.
Homotopy Theory providing all free spaces with a common base point, cf. p. 103. Working with \( X_p \) instead of \( X \) makes it possible, e.g., to deduce the Künneth Formula for joins of topological pair-spaces, as in Theorem 4.3 p. 24.

The simplicial as well as the topological join-operation are (modulo realization, equivalent) cases of colimits, as being simplicial resp. topological attachments. Restricting to \( \omega \)-spaces, p. 157ff remains true also with our new (cf. p. 13) realization functor, i.e. geometric realization preserves finite limits and all colimits.

Let \( K \) be the classical category of simplicial complexes and simplicial maps resulting from Definition 2 p. 8 and let \( \mathcal{K}_o \) be the category of simplices resulting from Definition 1 p. 74 exercise. Define \( \mathcal{E}_o : \mathcal{K} \to \mathcal{K}_o \) to be the functor adjoining \( \emptyset \), as a simplex, to each classical simplicial complex. \( \mathcal{E}_o \) has an inverse \( \mathcal{E} : \mathcal{K}_o \to \mathcal{K} \) deleting \( \emptyset \). This idea is older than category theory itself. It was suggested by S. Eilenberg and S. MacLane in (1942), where they in page 820, when exploring the 0-dimensional homology of a simplicial complex \( \Sigma \) contrastar “the realization-functor” and defined by:

\[
\text{An alternative procedure would be to consider } K \text{ “augmented” by a single } (-1)\text{-cell } \sigma^{-1} \text{ such that } [\sigma^0_i : \sigma^{-1}] = 1 \text{ for all } \sigma^0_i.
\]

This approach is explored by Hilton and Wylie in p. 16 ff.

As for \( \mathcal{E}_o \) above, let \( \mathcal{F}_p : \mathcal{D} \to \mathcal{D}_p \) be the functor adjoining to each classical topological space a \textbf{new non-final element} \( \mathcal{D}_p \) denoted \( \wp \). Put \( X_p := X + \{\wp\} \), which makes \( X_p \) resemble Maunder’s modified \( K^+ \)-object defined in p. 317 and further explored in pp. 340-341. \( \mathcal{F}_p \) has an inverse \( \mathcal{F} : \mathcal{D}_p \to \mathcal{D} \) deleting \( \wp \).

Our Proposition 4.10 p. 23, partially quoted below, reveals a connection between the combinatorial and the topological local structures and is in itself a strong motivation for introducing a topological (−1)-dimensional object \( \{\wp\} \), imposing the following definition p. 13 of a “point-setminus” \( \setminus \), in \( \mathcal{D}_p \) using the convention \( x \leftrightarrow \{x, \wp\} \), as opposed to the classical \( x \leftrightarrow \{x\} \). “\setminus” denotes the classical “setminus”, also known in set theory as the \textbf{relative complement} while “\emptyset” denotes the empty set, \( \emptyset \), when regarded as a simplex and \textbf{not} as a simplicial complex.

We can not risk dropping out of our new category so for \( X_p \neq \emptyset \):

\[
X_p \setminus x := \begin{cases} \emptyset & \text{if } x = \wp \\ \mathcal{F}_p(X \setminus x) & \text{otherwise} \end{cases}
\]

In Part 3 we will find constant use for the \textbf{link-construction}, denoted \( Lk_\Sigma \sigma \), of a simplex \( \sigma \) with respect to a simplicial complex \( \Sigma \) and defined by:

\[
Lk_\Sigma \sigma := \{ \tau \in \Sigma \mid [\sigma \cap \tau = \emptyset] \wedge [\sigma \cup \tau \in \Sigma] \} \quad (\Rightarrow Lk_\Sigma \emptyset = \Sigma).
\]

We will also use:

\[
\text{"The contrastar of } \sigma \in \Sigma' = \text{cost}_\Sigma \sigma := \{ \tau \in \Sigma \mid \tau \not\supseteq \sigma \},
\]

implying that: \( \text{cost}_\Sigma \emptyset = \emptyset \) and \( \text{cost}_\Sigma \sigma = \Sigma \) iff \( \sigma \not\subseteq \Sigma \). In classical literature, the \textbf{contrastar}, or \textbf{costar}, of \( \Sigma \) w.r.t. \( \sigma \) is known as the \textbf{complement} of \( \Sigma \) w.r.t. \( \sigma \), e.g., see p. 74 exercise E. See also [37].

In p. 13 we define the \textbf{realization} \( |\Sigma| \) of the complex \( \Sigma \) by slightly modifying Spanier’s \textbf{realisation-functor}, which, relative to the classical definition, in the range space exhibits an additional \textbf{coordinate function} \( \alpha_0 \) with \( \alpha_0(v) \equiv 0 \forall v \in V_\Sigma \).
Proposition. Let $G$ be any module over a commutative ring $A$ with unit. With $\alpha \in \{ \alpha \in |\Sigma| \mid v \in \sigma \} \iff [\alpha(v) \neq 0]$ and $\alpha = \alpha_o$ if $\sigma = \emptyset$, the following $A$-module isomorphisms are all induced by chain equivalences:

$$\hat{H}_{i-\#}(Lk_{\Sigma}\sigma; G) \cong \hat{H}_{i}(\Sigma, \text{cost}_{\Sigma}\sigma; G) \cong \hat{H}_{i}(|\Sigma|, |\text{cost}_{\Sigma}\sigma|; G).$$

(The $\hat{H}$ in the first two $\hat{\text{Homology groups functor}}$ denotes the simplicial $\hat{\text{Homology groups functor}}$, while the two last represent the singular. "#" stands for "cardinality". The proof depends on [39] p. 279 Th. 46.2 quoted here in p. [23].)

Astro-Physical inspiration: Currently, astronomers hold it likely that every galaxy, including the Milky Way, possesses a single supermassive black hole active or inactive, here interpreted as $\{ \emptyset \}$, and so, eliminating it, would dispose off the whole galaxy, hinting at the above "$X_p \setminus \emptyset := \emptyset". Actually, the last proposition leaves us with no other option, since $Lk_{\Sigma}\emptyset := \Sigma$ by definition.

2.2. Exploring the boundary formula. The absence of a unit-space with respect to topological join cripples classical algebraic topology and this shortcoming can not be eliminated by the use of any technique from any mathematical field. The only remedy is to invent such a unit space $\{ \emptyset \}$. We use the 'Definition 1'-generated object $\{ \emptyset_o \}$ from p. [3] as a model-object and define $\{ \emptyset \}$ to be its realization, i.e. set: $\{ \emptyset \} := |\{ \emptyset_o \}|$.

Any link of a simplicial homology manifold is either a simplicial homology ball or a sphere. Its boundary is the set of simplices having ball-links, i.e.

$$\text{Bd} \Sigma := \{ \sigma \in \Sigma \mid \hat{H}_{i-\#}(Lk_{\Sigma}\sigma) = 0 \} \ (\approx \{ \sigma \in \Sigma \mid \hat{H}_{i}(Lk_{\Sigma}\sigma) = 0 \} \ \text{classically}).$$

Since, $\text{Lk}_{\Sigma}\emptyset := \Sigma$ ("the missing link"), the real projective plane $\mathbb{RP}^2$ has the boundary $\text{Bd}\mathbb{RP}^2 = \{ \emptyset_o \} \neq \emptyset$, which also holds for any one-point space $\bullet$, i.e. $\text{Bd}(\bullet) = \text{Bd}(\{ \emptyset_o, \{v\} \}) = \{ \emptyset_o \} = \text{the join-unit}". So, the boundary of a 0-ball is the -1- sphere. The one-point complex $\bullet := \{ \emptyset_o, \{v\} \}$ is the only finite orientable manifold having $\{ \emptyset_o \}$ as its boundary. For the 0-sphere $\text{Bd}(\bullet\bullet) = \emptyset$ - the join-zero with respect to Definition 1 p. [4]. The classical setting gives:

$$\text{Bd}(\{ \{v\} \}) = \emptyset = \text{Bd}(\{ \{v\}, \{w\} \}) = \emptyset = \text{the join-unit}" \text{ with respect to Definition 2, implying that the boundary formula below, for the join of two homology manifolds, can not hold classically but, as will be shown, will always hold in the ordinary $\emptyset$-augmented categories. Moreover, for any $n$-dimensional locally orientable simplicial manifold, the boundary is generated by those $(n-1)$-dimensional simplices whose link contains exactly one vertex.}

$$\mathcal{E}(\text{Bd}\Sigma) = \text{Bd}(\mathcal{E}(\Sigma)),$$

i.e., what remains after deleting $\emptyset_o$ is exactly the classical boundary - always! The new simplicial complex $\{ \emptyset_o \}$ will serve as the (abstract simplicial) $(-1)$-standard simplex.

We will see that the join-operation in the category of simplicial complexes and simplicial sets in their augmented form, will turn into (graded) tensor product when applied to either our homology functor $\hat{H}$ or the Stanley-Reisner ring functor p. [33], which is made explicit here in p. [37].
We give a few low-dimensional examples how the non-classical approach enriches algebraic topology through the following manifold boundary formula for topological as well as simplicial join, as given on p. 28 resp. p. 47:

\[ \text{Bd}(M_1 \ast M_2) = (\text{Bd}M_1) \ast M_2 \cup (M_1 \ast \text{Bd}M_2). \]

\[ \{\emptyset, \{v_1\}, \{v_2\}\{v_1, v_2\}\} = \bullet \ast \bullet \text{ is a figurative way of saying that the simplicial 1-ball, generated by a 1-simplex equals the join of two distinct simplicial 0-balls, each generated by a 0-simplex, so:} \]

\[ \text{Bd}(1\text{-ball}) = \text{Bd}(\bullet \ast \bullet) = (\{\emptyset\} \ast \bullet) \cup (\bullet \ast \{\emptyset\}) = \bullet \cup \bullet =: \bullet \bullet = 0\text{-sphere.} \]

\[ (\bullet \bullet) \ast (\bullet \bullet) = (\text{the simplicial} \ 1\text{-sphere}. \text{ So, since} \ \text{Bd}(\bullet \bullet) = \emptyset: \]

\[ \text{Bd}(1\text{-sphere}) = \text{Bd}((\bullet \bullet) \ast (\bullet \bullet)) = (\emptyset \ast (\bullet \bullet)) \cup ((\bullet \bullet) \ast \emptyset) = \emptyset \cup \emptyset = \emptyset. \]

The topological join might be thought of as the two factor-spaces along with arcs going from each point in one factor to each point in the other.

Our formula makes it trivial to calculate the boundaries of the following three low-dimensional joins, where \(\dim(X_1 \ast X_2) = \dim X_1 + \dim X_2 + 1\), which, by the way, implies that a unit space w.r.t. join has to be of dimension \(-1\).

1. \(\mathcal{M} \ast \bullet = \text{the cone of the Möbius band} \ \mathcal{M},\)
2. \(\mathbb{RP}^2 \ast \bullet = \text{the cone of the real projective plane} \ \mathbb{RP}^2,\)
3. \(S^2 \ast \bullet = \text{“the cone of the 2-sphere} \ S^2 = B^3, \text{ the solid 3-dimensional ball or 3-ball.} \)

The real projective plane \(\mathbb{RP}^2\) frequently appears in different mathematical fields. For example, within cobordism-theory it is known as the “first” manifold (with respect to dimension), that is not the boundary of any manifold, when the type of “manifold” is specified to mean “topological manifold”, which in dimension 2 is equivalent to both “homology manifold”, and to “quasi-manifold”. A simplicial complex is a quasi-manifold if and only if its links are all pseudo-manifolds, see p. 42 for definitions.

Our intuition easily gets hold of a useful picture of “the cone of an up-right standing cylinder” and also of a correct picture of its “boundary” if the cone vertex, in a realization, is placed in the point of inertia of the cylinder. Here “boundary” should be interpreted as those points where the local singular homology is trivial in the highest dimension, i.e. in dimension 3.

Now, make a straight cut in the cylinder, from the top to the bottom and glue these cut-ends back together again, after twisting one of them \(180^\circ\). The result is a Möbius band. With the cone-point in the point of inertia of the Möbius band we create \(\mathcal{M} \ast \bullet “\text{the cone of a Möbius band}”\). We engage the same set of points as in the cylinder case, but differently arranged, and there are reasons to believe that no sharp intuitive picture is revealing itself. \(\mathcal{M} \ast \bullet\) is 3-dimensional but it can not be inscribed in our ordinary room.

Regarding the factors as quasi-manifolds and using, say, the 3-element prime field \(\mathbb{Z}_3\) as the coefficient module for the homology groups, we get:

\( B^n \) is the \( n\)-dimensional disk or “\( n\)-ball” and \( S^n \) is the “\( n\)-sphere”.

1. \( \text{Bd}_{\mathbb{Z}_3}(\mathcal{M} \ast \bullet) = ((\text{Bd}_{\mathbb{Z}_3} \mathcal{M}) \ast \bullet) \cup (\mathcal{M} \ast \text{Bd}_{\mathbb{Z}_3} \bullet) = ((S^1) \ast \bullet) \cup (\mathcal{M} \ast \text{Bd}_{\mathbb{Z}_3} \bullet) = B^2 \cup (\mathcal{M} \ast \{\emptyset\}) = B^2 \cup \mathcal{M} = \mathbb{RP}^2: = \text{real projective plane} \)
(2) \( \text{Bd}_m(\mathbb{RP}^2 \ast \bullet) = ((\text{Bd}_m \mathbb{RP}^2) \ast \bullet) \cup (\mathbb{RP}^2 \ast \text{Bd}_m \bullet) = (\{0\} \ast \bullet) \cup (\mathbb{RP}^2 \ast \{0\}) = \bullet \cup \mathbb{RP}^2. \)

(3) \( \text{Bd}_m(S^2 \ast \bullet) = ((\text{Bd}_m S^2) \ast \bullet) \cup (S^2 \ast \text{Bd}_m \bullet) = (\emptyset \ast \bullet) \cup (S^2 \ast \{0\}) = S^2. \)

\( \mathbb{RP}^2 \ast \bullet \) is a homology \( p \)-3-manifold with boundary \( \bullet \cup \mathbb{RP}^2 \) if \( p \neq 2 \), by [20] p. 36 and Th. 7.9 p. 7, while \( M \ast \bullet \) fails to be one due to the cone-point. However, \( M \ast \bullet \) is a quasi-3-manifold, see p. [12], which by the boundary definition p. 7 has the real projective plane \( \mathbb{RP}^2 \) as its well-defined boundary, as seen above. The topological realisation, cf. p. 13 of \( \mathbb{RP}^2 \ast \bullet \) is not a topological manifold due to the properties at the cone-point. This is also true classically. Topological \( n \)-manifolds can only have \( n - 1 \)-, \( -1 \)- or \(-\infty\)-dimensional boundaries as homology manifolds, due to their local orientability. The global non-orientability of \( M \) and \( \mathbb{RP}^2 \) becomes local non-orientability at the cone-point in \( M \ast \bullet \), resp. \( \mathbb{RP}^2 \ast \bullet \). It is obviously rewarding to study combinatorial manifolds such as pseudo- and quasi-manifolds, also when the main object of study is topological manifolds. Our section 7 is therefore devoted to these generalized notions of simplicial manifolds.

2.3. Why classical search for “The Relative Künneth Formula for Joins” failed.

The short answer is that the objects in the classical categories simply do not contain any \((-1)\)-dimensional object that at the same time is a sub-object contained in every non-empty object and a unit element with respect to join. With respect to the category of simplicial complexes, Definition 1 p. 3 provide such a \((-1)\)-dimensional simplicial complex through \( \{0\} \), which is generated by the new empty simplex \( \emptyset \). Note that this new complex \( \{0\} \) also is the simplicial join-unit. Maybe this answer becomes more enlightening after a thorough check of the proof of the Eilenberg-Zilber theorem for simplicial join in p. 7. Our externally adjoined element \( \emptyset \) generates the join-unit \( \{0\} \) that allows an Eilenberg-Zilber theorem for topological join, resembling the classical one for products. The rest is not really category specific and is covered by homological algebra, as founded in the Cartan-Eilenberg masterpiece [4], for which the initial impetus was precisely the Künneth Formula.

That the join-operation is at the heart of algebraic topology, becomes apparent, for instance, by Milnor’s construction of the universal principal fiber bundle in [43], where he in Lemma 2.1 also reaches the limit for classical algebraic topology as he formulates a reduced version of the non-relative Künneth formula for joins of general topological spaces as (no classical relative version can exist due to the missing \((-1)\)-dimensional join-unit):

\[
\tilde{H}_i(X \ast Y) \cong \bigoplus_{i+j=q} \tilde{H}_i(X) \otimes \tilde{H}_j(Y) \oplus \text{Tor}^2(\tilde{H}_i(X), \tilde{H}_j(Y)),
\]

i.e. the \( X_2 = Y_2 = 0 \)-case in our Th. 7.21 p. 44. Milnor’s results, apparently, inspired G.W. Whitehead to introduce the augmented total chain complex \( \tilde{S}(\cdot) \) and augmented homology, \( \tilde{H}_*(\cdot) \), in [40]. \( \tilde{S}(\cdot) \) comprise an in-built dimension shift, which in our version becomes an application of the suspension operator on the underlying singular chain. G.W. Whitehead gives the empty space, \( \emptyset \), the status of \((-1)\)-dimensional standard simplex, but in his pair space theory, \( \tilde{H}_*(X_1, X_2 : G) \), he never took into account that the join-unit \( \emptyset \) then would have the identity map, \( \text{Id}_\emptyset \), as a generator for its \((-1)\)-dimensional singular augmental chain group, which, correctly interpreted, actually makes his pair space theory equivalent to the ordinary relative homology functor.
G.W. Whitehead states that $\tilde{S}(X \ast Y)$ and $\tilde{S}(X) \otimes \tilde{S}(Y)$ are chain equivalent, $\approx$, which, modulo the introduction of a $-1$-dimensional topological object, is confirmed by our Theorem [1.7] p. 23, and then in a footnote Whitehead points out, referring to [57] p. 431 Lemma 2.1, that;

This fact does not seem to be stated explicitly in the literature but is not difficult to deduce from Milnor’s proof of the “K"unneth theorem” for the homology groups of the join.

On the chain level it is indeed “not difficult” to see what is needed to achieve $\tilde{S}(X \ast Y) \approx \tilde{S}(X) \otimes \tilde{S}(Y)$, since the right hand side is well-known, but then to actually do it for the classical category of topological pair-spaces and within the frames of the Eilenberg-Steenrod axioms for relative homology is, unfortunately, impossible, since the need for a $(-1)$-dimensional standard simplex is indisputable and the initial object $\emptyset$ just won’t do, since it already plays another incompatible role within this axiomatization, cf. [1] p. 3-4. Indeed, the “convention” $H_i(\cdot) = H_i(\cdot, \emptyset; \mathbb{Z})$, cf. [1] pp. 3 + 273, is more than a mere “convention” in that it connects the single space and the pair space theories and thereby determine the chain-groups of $\emptyset$ to be trivial in all degrees. This was also observed in [34] p. 108, leading up to a refutation of Whitehead’s approach, sketched above.

Our Theorem [1.7] p. 23 answers G.W. Whitehead’s half a century old quest above for a proof. For arbitrary “Definition 1”-simplicial complexes, G.W. Whitehead’s “chain equivalent”-claim above (=the non-relative Eilenberg-Zilber theorem for joins), is easily proven as shown here in p. 57, where we quote the proof of this fact, taken from [16], which was the first T\TeX-version of the present article.

Our K"unneth Theorem for topological joins, i.e. Theorem [4.8] p. 24, holds for arbitrary topological spaces.

The generalization, i.e. Theorem [4.13] p. 28, of the "derivation"-like boundary formula p. 8 above, also holds for non-triangulable homology manifolds, which we have restricted to be locally compact Hausdorff, but weak Hausdorff $k$-spaces, cf. [18] p. 243ff, would also do, since we only need the "$T_1$-separation axiom" ($\iff$ “points are closed”).

Since classical Combinatorics and classical General Topology cannot separate the “join zero” from the “join-unit” or the boundary of the 0-ball from that of the 0-sphere, any theory constructed thereon — classical Algebraic Topology in particular — will need ad-hoc definitions/reasoning. The use of a relative and a separate reduced homology functor instead of one single homology functor, is only one example. These are probably the reasons for the ongoing marginalization of classical algebraic topology, as witnessed by M. Hovey at http://math.wesleyan.edu/~mhovey/problems/.

We quote Paul Dirac as he comments on the increasing abstraction within the realm of mathematics and physics. (P.A.M. Dirac, Proc. Roy. Soc. A 133, 60 (1931).) Quantised Singularities in the Electromagnetic Field

It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalization of the axioms at the base of the mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation.
Part 2. Augmental Homology Theory

3. Simplicial and singular augmental homology theory

3.1. Definition of underlying augmental categories. §3.1 presents a firm formal foundation for an overall augmented environment, suitable for (augmented) homology theories.

The combination of the complex-part of Definition 2 and the join-part of Definition 1 is not an option, since then the join-operation results in a simplicial complex (in any sense) only if one of the join-factors is the empty complex.

The typical morphisms in the classical category $K$ of simplicial complexes with vertices in $W$ are the simplicial maps as defined in [45] p. 109, implying in particular that:

$$
\begin{align*}
\text{Mor}_{K}(&\emptyset, \Sigma) = \{\emptyset, \Sigma\}, \\
\text{Mor}_{K}(\Sigma, \emptyset) = \emptyset, \text{ if } \Sigma \neq \emptyset, \\
\text{Mor}_{K}(\emptyset, \emptyset) = \{\emptyset, \emptyset\} = \{\text{id}_{\emptyset}\},
\end{align*}
$$

where $0_{\Sigma, \Sigma'}$ denotes $\emptyset$ = the empty function from $\Sigma$ to $\Sigma'$.

So; $0_{\Sigma, \Sigma'} \in \text{Mor}_{K}(\Sigma, \Sigma') \iff \Sigma = \emptyset$.

If in a category $\varphi_{i} \in \text{Mor}(R_{i}, S_{i}), \ i = 1, 2$, we put:

$$
\varphi_{1} \sqcup \varphi_{2} : R_{1} \sqcup R_{2} \to S_{1} \sqcup S_{2} : r \mapsto \begin{cases} 
\varphi_{1}(r) & \text{if } r \in R_{1} \\
\varphi_{2}(r) & \text{if } r \in R_{2}
\end{cases}
$$

where $\sqcup$ := “disjoint union”.

**Definition.** (of the objects in $K_{o}$) An (abstract) simplicial complex $\Sigma$ on a vertex set $V_{\Sigma}$ is a collection (empty or non-empty) of finite (empty or non-empty) subsets $\sigma$ of $V_{\Sigma}$ satisfying;

(a) If $v \in V_{\Sigma}$, then $\{v\} \in \Sigma$.

(b) If $\sigma \in \Sigma$ and $\tau \subset \sigma$ then $\tau \in \Sigma$.

So, $\{\emptyset\}$ is allowed as an object in $K_{o}$. We will write “concept_{o}” or “concept_{p}” when we want to stress that a concept is to be related to our modified categories.

If $\text{card}(\sigma) = q+1$ then $\text{dim}(\sigma) := q$ and $\sigma$ is said to be a $q$-face_{o} or a $q$-simplex_{o} of $\Sigma_{o}$ and $\text{dim}(\Sigma) := \sup\{\text{dim}(\sigma) | \sigma \in \Sigma_{o}\}$. Writing $\emptyset_{o}$ when using $\emptyset$ as a simplex, we get $\text{dim}(\emptyset) = -\infty$ and $\text{dim}(\emptyset_{o}) = -1$.

**Note.** Any simplicial complex $\Sigma_{o} \neq \emptyset$ in $K_{o}$ includes $\{\emptyset\}$ as a subcomplex.

So, a typical object_{o} in $K_{o}$ is $\Sigma \sqcup \{\emptyset\}$ or $\emptyset$, where $\Sigma \in K$ and $\chi$ is a morphism in $K_{o}$ if either:

a) $\chi = \varphi \sqcup \text{id}_{\emptyset_{o}}$ for some $\varphi \in \text{Mor}_{K}(\Sigma, \Sigma')$ or

b) $\chi = 0_{\emptyset, \Sigma_{o}}$.

In particular, $\text{Mor}_{K_{o}}(\Sigma_{o}, \emptyset_{o}) = \emptyset$ if and only if $\Sigma_{o} \neq \emptyset_{o}$, $\emptyset$.

A functor $E : K_{o} \to K$:

Set $E(\Sigma_{o}) = \Sigma_{o} \setminus \{\emptyset_{o}\} \in \text{Obj}(K)$ and given a morphism $\chi : \Sigma_{o} \to \Sigma'_{o}$ we define:

$$
E(\chi) = \begin{cases} 
\varphi & \text{if } \chi \text{ fulfills a) above and} \\
0_{\emptyset, E(\Sigma_{o})} & \text{if } \chi \text{ fulfills b) above.}
\end{cases}
$$
A functor $\mathcal{E}_o: \mathcal{K} \rightarrow \mathcal{K}_o$:

Set $\mathcal{E}_o(\Sigma) = \Sigma \sqcup \{0_o\} \in \text{Obj}(\mathcal{K}_o)$ and given $\varphi: \Sigma \rightarrow \Sigma'$, we put $\chi := \varphi \sqcup \text{id}_{\{0_o\}}$, which gives; $\mathcal{E} \mathcal{E}_o = \text{id}_\chi$; $\text{im} \mathcal{E}_o := \text{Obj}(\mathcal{K}_o) \setminus \{0_o\}$ and $\mathcal{E}_o \mathcal{E} = \text{id}_\chi$ except for $\mathcal{E}_o(\emptyset) = \{\emptyset\}$.

Similarly, let $\mathcal{C}$ be the category of topological spaces and continuous maps. Consider the category $\mathcal{D}_o$ with objects: $\emptyset$ together with $X_\varphi := X + \{\varphi\}$ for all $X \in \text{Obj}(\mathcal{C})$, i.e., the set $X_\varphi := X \sqcup \{\varphi\}$ with the weak topology, $\tau_{\varphi}$, with respect to $X$ and $\{\varphi\}$.

$f_\varphi \in \text{Mor}_{\mathcal{D}_o}(X_\varphi, Y)$ if:

$$f_\varphi = \begin{cases} 
  a) & f + \text{id}_{\{\varphi\}} := f \sqcup \text{id}_{\{\varphi\}} \text{ with } f \in \text{Mor}(X, Y) \text{ and } f \text{ is on } X \text{ to } Y, \text{i.e the domain of } f \text{ is the whole of } X \text{ and } X_\varphi = X + \{\varphi\}, Y_\varphi = Y + \{\varphi\} \text{ or } \\
  b) & 0_{\emptyset, X_\varphi} := \emptyset = \emptyset \text{ is the empty function from } \emptyset \text{ to } Y_\varphi.
\end{cases}$$

There are functors:

$$\mathcal{F}_o: \mathcal{C} \rightarrow \mathcal{D}_o \quad \text{and} \quad \mathcal{F}: \mathcal{X} \rightarrow \mathcal{X} + \{\varphi\} \rightarrow \mathcal{X} \text{ resembling } \mathcal{E}_o, \text{ resp. } \mathcal{E}.$$

Note. The “$\mathcal{F}_o$-lift topologies”,

$$\tau_{\varphi} := \mathcal{F}_o(\tau_X) \cup \{\emptyset\} = \{O_o \cup \{\varphi\} \mid O \in \tau_X \cup \{\emptyset\}$$

and

$$\tau_X := \tau_X \cup \{X\} = \{O \mid O = X \setminus (\mathcal{N} \cup \{\varphi\}) \text{; } \mathcal{N} \text{ closed in } X \} \cup \{X\}$$

would also give $\mathcal{D}_o$ due to the domain restriction in a), disallowing “partial” functions. This makes $\mathcal{D}_o$ a link between the two constructions of partial maps treated in [pp. 184-6].

No extra morphisms of has been allowed into $\mathcal{D}_o (\mathcal{K}_o)$ in the sense that the morphisms $\varphi$ are all targets under $\mathcal{F}_o (\mathcal{E}_o)$ except $0_{\emptyset, X_\varphi}$ defined in b), re-establishing $\emptyset$ as the unique initial object.

The underlying principle for our definitions is that a concept in $\mathcal{C}$ ($\mathcal{K}$) is carried over to $\mathcal{D}_o$ ($\mathcal{K}_o$) by $\mathcal{F}_o$ ($\mathcal{E}_o$) with addition of definitions of the concept, for cases which are not proper images under $\mathcal{F}_o$ ($\mathcal{E}_o$). The definitions of the product/join operations “$\times_o$”, “$*$”, “$\hat{*}$” in page [184] as well as “$/$”, “$+_o$” and “$\backslash_o$” below certainly comply with this principle.

Definition. $X_{\varphi_1}/oX_{\varphi_2} := \begin{cases} 
  \emptyset & \text{if } X_{\varphi_1} = \emptyset \\
  \mathcal{F}_o(\mathcal{F}(X_{\varphi_1})/\mathcal{F}(X_{\varphi_2})) & \text{if } X_{\varphi_2} \neq \emptyset \text{ in } \mathcal{D}_o \text{ for } X_{\varphi_2} \supset X_{\varphi_1}.
\end{cases}$

“$/$” is the classical “quotient” except that $\mathcal{F}(X_{\varphi_1})/\emptyset := \mathcal{F}(X_{\varphi_1})$, cf. [184] p. 102.

So, for example: $(X_{\varphi_1}/oX_{\varphi_2}, X_{\varphi_2}/oX_{\varphi_2}) := \begin{cases} 
  (X_{\varphi_1}, \emptyset) & \text{if } X_{\varphi_2} = \emptyset, \\
  (X_{\varphi_1}, \{\varphi\}) & \text{if } X_{\varphi_2} = \{\varphi\}, \\
  (X_{\varphi_1} \times_o X_{\varphi_2}, \bullet) & \text{if } X_{\varphi_2} \neq \emptyset \neq X_{\varphi_1}.
\end{cases}$

Definition. Let “$+$” denote the classical “topological sum”, then:

$$X_{\varphi_1} +_o X_{\varphi_2} := \begin{cases} 
  X_{\varphi_1} & \text{if } X_{\varphi_2} = \emptyset \\
  X_{\varphi_2} & \text{if } X_{\varphi_1} = \emptyset \\
  \mathcal{F}_o(\mathcal{F}(X_{\varphi_1}) + \mathcal{F}(X_{\varphi_2})) & \text{if } X_{\varphi_1} \neq \emptyset \neq X_{\varphi_2}.
\end{cases}$$
To avoid dropping out of the category $\mathcal{D}_o$ we introduce a setminus "\(\setminus\)" in $\mathcal{D}_o$, giving Proposition [110], p. 29 its compact form. Obs, that "\(\setminus\)" might be non-associative if \(\{\varnothing\}\) is involved. ("\(\setminus\)" := classical "setminus").

**Definition.** \(X_o \setminus X'_o := \{\emptyset\}\) if \(X_o = \emptyset\), \(X_o \not\subseteq X'_o\) or \(X'_o = \{\varnothing\}\); else.

### 3.2. Realization of a simplicial complex \(o\).

A "point" in a classical topological space is simply a subspace containing a single element. In a simplicial complex \(o\) a "point" is a subcomplex \(o\) containing only one vertex. The simplicial complexes \(o\) \(\emptyset\), \(\{\emptyset\}\) are both pointless and non-final. Any functorial definition of the realization of a simplicial complex \(o\) forces their realizations \(|\emptyset|\) and \(|\{\emptyset\}|\) resp. to be separate pointless objects, which implies that no realization functor could target any classical Euclidian space of any dimension, since each of these spaces only possess one pointless subspace - namely the empty space \(\emptyset\).

So, to be able to create something similar to the classical realization functor one need to (externally) add a non-final object \(\{\varnothing\} = \{|\emptyset|\}\) into the classical category of topological spaces as join-unit and (-1)-dimensional standard simplex. If \(X_o \neq \emptyset\), \(\{\varnothing\}\), then \((X_o, \tau_{X_o})\) is a non-connected space. We therefore define \(X_o \neq \emptyset\), \(\{\varnothing\}\) to have a certain topological "property" if \(\mathcal{F}(X_o)\) has the "property", e.g., \(X_o\) is connected, iff \(X\) is connected.

We will use Spanier’s definition of the *function space realization* \(|\Sigma_o|\) as given in [15], p. 110, unaltered, except for the "\(\setminus\)"'s and the underlined addition below, where \(\varnothing := \alpha_0\) and \(\alpha_0(v) \equiv 0\) for all \(v \in V_o\). We now define a covariant functor from the category of simplicial complexes, and simplicial maps, to the category of topological spaces, and continuous maps. Given a nonempty simplicial complex \(\Sigma_o\), let \(|\Sigma_o|\) be the set of all functions \(\alpha\) from the set of vertices of \(\Sigma_o\) to \(I := [0, 1]\) such that:

(a) For any \(\alpha\), \(\{v \in V_{\Sigma_o} \mid \alpha(v) \neq 0\}\) is a simplex \(o\) of \(\Sigma_o\).

(in particular, \(\alpha(v) \neq 0\) for only a finite set of vertices).

(b) For any \(\alpha \neq \alpha_0\), \(\sum_{v \in V_{\Sigma_o}} \alpha(v) = 1\).

If \(\Sigma_o = \emptyset\), we define \(|\Sigma_o| = \emptyset\).

The *barycentric coordinates* \(o\) define a metric
\[
d(\alpha, \beta) = \sqrt{\sum_{v \in V_{\Sigma_o}} \left[\alpha(v) - \beta(v)\right]^2}
\]
on \(|\Sigma_o|\) providing it with the *metric topology* and thereafter denoting it \(|\Sigma_o|_d\).

We will equip \(|\Sigma_o|\) with another, more commonly used topology and for this purpose we define the *closed geometric simplex* \(|\sigma_o|\) of \(\sigma_o \in \Sigma_o\) to be:
\[
|\sigma_o| := \{\alpha \in |\Sigma_o| \mid [\alpha(v) \neq 0] \implies [v \in \sigma_o]\}.
\]

**Definition.** For \(\Sigma_o \neq \emptyset\), \(|\Sigma_o|\) is topologized by setting \(|\Sigma_o| := |\mathcal{E}(\Sigma_o)| + \{\alpha_0\}\). This is equivalent to giving \(|\Sigma_o|\) the weak topology with respect to the \(|\sigma_o|\)'s naturally imbedded in \(\mathbb{R}^n + \{\varnothing\}\), and we define \(\Sigma_o\) to be connected if \(|\Sigma_o|\) is, i.e., if \(\mathcal{F}(|\Sigma_o|) \simeq |\mathcal{E}(\Sigma_o)|\) is.
Proposition.
\[ |\Sigma| \text{ is always homotopy equivalent to } |\Sigma|_d \quad (\text{[18] pp. 115, 226}.). \]
\[ |\Sigma| \text{ is homeomorphic to } |\Sigma|_d \quad \text{iff } \Sigma_o \text{ is countable and locally finite } (\text{[18] p. 119 Th. 8}). \]

Theorem. (\text{[18] p. 120}) If \( \Sigma_o \) has a realization in \( \mathbb{R}^n + \{ \varnothing \} \), then \( \Sigma_o \) is countable and locally finite, and \( \dim \Sigma_o \leq n \). Conversely, if \( \Sigma_o \) is countable and locally finite, and \( \dim \Sigma_o \leq n \), then \( \Sigma_o \) has a realization as a closed subset in \( \mathbb{R}^{2n+1} + \{ \varnothing \} \).

3.3. Topological properties of realizations. \text{[14] p. 352 ff., [14] p. 54 and [20] p. 171 emphasize the importance of triangulable spaces, i.e., spaces homeomorphic to the realization, (p. [13]) of a simplicial complex. Non-triangulable topological (i.e., 0-differentiable) manifolds have been constructed, cf. [14], but since all continuously \( n \)-differentiable (\( n > 0 \)) manifolds are triangulable, cf. [38] p. 103 Th. 10.6, it is clear that essentially all spaces in practical use, in particular those within mathematical physics, are triangulable and therefore CW-complexes.

A topological space has the homotopy type (or, of a polytope for short) if and only if it has the type of an absolute neighborhood retraction (ANR), which it has iff it has the type of a CW-complex, cf. [18] p. 226 Th. 5.2.1.

Our CW-complexes will have a \((-1)\)-cell \( \{ \varnothing \} \), as the “relative CW-complexes” defined in [18] p. 26, but their topology is such that they belong to the category \( D_o \), as defined in p. [14].

CW-complexes are compactly generated, perfectly normal spaces, cf. [18], pp. 22, 112, 242, that are locally contractible, in a strong sense and (hereditarily) paracompact, cf. [18], pp. 28-9 Th. 1.3.2 + Th. 1.3.5 (Ex. 1, p. 33).

{\varnothing} is called an “ideal cell” in [1], p. 122.

Notations. We have used \( \tau_X := \) the topology of \( X \). We will also use; \( \text{PID} := \) Principal Ideal Domain and “an \( \text{R-PID} \)-module \( \mathcal{G} := \mathcal{G} \) is a module over a \( \text{PID} \) \( \text{R} \). \( \text{LHS} := \) Long Homology Sequence, \( \text{M-Vs} := \) Mayer-Vietoris sequence. “\( \simeq \)” denotes “homeomorphism” or “chain isomorphism”.

Let, here in Chapter 3 \( \Delta = \{ \Delta^n, \partial \} \) be the classical singular chain complex.

Note. In Definition 1 p. [3] we had \( \{ \varnothing \} = \{ \emptyset_o \} \) and in the realization definition we had \( \{ \varnothing \} = \{ \alpha_0 \} \). \( \theta \) is unquestionably the universal initial object. Our \( \{ \varnothing \} \) could be regarded as a local initial object and \( \varnothing \) as a member identifier, since every object with \( \varnothing \) as its unique “\(-1\)-dimensional” element, has therefore a “natural” membership tag, that fully complies with Grothendieck’s assumption that, for every set \( \mathcal{S} \), there is a “universe” \( U \) with \( \mathcal{S} \in U \), cf. [32] p. 24. Note also that \( d(\alpha_0, \alpha) \equiv 1 \) for any \( \alpha \), i.e., \( \alpha_0 \) is an isolated point.

3.4. Simplicial augmental homology theory. \( \hat{H} \) denotes the simplicial as well as the singular augmental (co)homology functor. We begin with ten (10) lines that constitute a firm formal foundation for the simplicial augmental pair-(co)homology functor.

Two vertex-orderings are equivalent if an even permutation makes them equal. Choose ordered \( q \)-simplices to generate the simplicial chain complex \( C_q(\Sigma_o; G) := \{ C_q(\Sigma_o; G) \} \) where the coefficient module \( G \) is a unital \( \leftrightarrow 1 \alpha \circ q = q \) module over any commutative ring \( A \) with unit. We use ordered simplices instead of \( \text{oriented} \) since the former is more natural in relation to our definition p. [28] of the ordered simplicial cartesian product.
Now, with 0 as the additive unit element,

- \( C^o(\emptyset; G) \equiv 0 \) in all dimensions.
- \( C^p(\{\emptyset\}; G) \equiv 0 \) in all dimensions except for \( C^0_2(\{\emptyset\}; G) \cong G \).
- \( C^o(\Sigma_o; G) \equiv \tilde{C}(E(\Sigma_o); G) \equiv \) the classical \( \{\emptyset\} \)-augmented chain.

Totally analogous to the classical introduction of simplicial homology and by just hanging on to the \( \{\emptyset\} \)-augmented chains, also when defining relative chains, we get the relative simplicial augmental homology functor for \( K_\sigma \) pairs, denoted \( \tilde{H}_\sigma \). The graded quotient \( C(\Sigma_{10}, \Sigma_{20}; G) := C(\Sigma_{10}; G)/C^0(\Sigma_{20}; G) := \{C^o_q(\Sigma_{10}; G)/C^0_q(\Sigma_{20}; G)\}_q \) induces the “relative chains for the simplicial pair \((\Sigma_{10}, \Sigma_{20})\)”.

\[
\tilde{H}_p(\Sigma_{o1}, \Sigma_{o2}; G) = \begin{cases} 
H_1(E(\Sigma_{o1}), E(\Sigma_{o2}); G) & \text{if } \Sigma_{o2} \neq \emptyset \\
\tilde{H}_1(E(\Sigma_{o1}), G) & \text{if } \Sigma_{o1} \neq \{\emptyset\}, \emptyset \text{ and } \Sigma_{o2} = \emptyset \\
\cong G & \text{if } i = -1 \text{ and } \Sigma_{o1} = \{\emptyset\} \text{ and } \Sigma_{o2} = \emptyset \\
0 & \text{if } i \neq -1 \\
& \text{for all } i \text{ when } \Sigma_{o1} = \Sigma_{o2} = \emptyset.
\end{cases}
\]

\( H_\sigma(\tilde{H}_\sigma) \) denotes the classical (reduced) homology functor.

3.5. Singular augmental homology theory. The \(|q|\), defined in p. 13, imbedded in \( \mathbb{R}^n + \{\varphi\} \) generates a satisfying set of “standard simplices,” and “singular simplices,”. This implies, in particular, that the “\( p \)-standard simplices,”, denoted \( \Delta^p \), are defined by \( \Delta^p := \Delta + \{\varphi\} \) where \( \Delta \) denotes the usual \( p \)-dimensional standard simplex and + is the topological sum, i.e. \( \Delta^p := \Delta \sqcup \{\varphi\} \), and \( \{\varphi\} \). Now, and most important: \( \Delta^p := \{\varphi\} \).

Let \( T^p \) denote an arbitrary classical singular \( p \)-simplex \( (p \geq 0) \). The “singular \( p \)-simplex,” denoted \( \sigma^p \), now stands for a function of the following kind, where the bar, \( \bar{\ } \), in \( \sigma^p \) indicates “restriction”:

\[
\sigma^p : \Delta^p = \Delta + \{\varphi\} \longrightarrow X + \{\varphi\} \quad \text{where } \sigma^p(\varphi) = \varphi \quad \text{and} \quad \sigma^p = T^p \quad \text{for some classical } p \text{-dimensional singular simplex } T^p \text{ for all } p \geq 0.
\]

In particular;

\[
\sigma^{p(-0)} : \{\varphi\} \longrightarrow X_p = X + \{\varphi\} : \varphi \mapsto \varphi.
\]

The boundary function \( \partial_p \) is defined by \( \partial_p(\sigma^p) := \mathcal{F}(\partial_p(T^p)) \) where \( \partial_p \) is the classical singular boundary function, and \( \partial_0(\sigma^0) \equiv \sigma^{p(-0)} \) for every singular \( 0 \)-simplex \( \sigma^0 \). Let \( \Delta = \{\Delta^p, \partial_p\} \) denote the singular augmental chain complex. Observe that:

\[
|\Sigma| \neq \emptyset \Longrightarrow |\Sigma| = \mathcal{F}_0(|E(\Sigma)|) \in \mathcal{D}_{>0}.
\]

It is now obvious that e.g. \( \hat{H}_i(|\Sigma_{o1}|, |\Sigma_{o2}|; G) = \hat{H}_i(\Sigma_{o1}, \Sigma_{o2}; G) \). By the strong analogy to classical singular homology, we omit the proof of the next lemma.
**Lemma.** (Equivalently for coHomology – with the subindex $i$ as an index.)

$$
\hat{H}_i(X_{\varphi_1}, X_{\varphi_2}; G) = \begin{cases} 
H_i(F(X_{\varphi_1}), F(X_{\varphi_2}); G) & \text{if } X_{\varphi_2} \neq \emptyset \\
\hat{H}_i(F(X_{\varphi_1}); G) & \text{if } X_{\varphi_1} \neq \{\varphi\}, \emptyset \text{ and } X_{\varphi_2} = \emptyset \\
\{0\} & \text{if } i = -1 \\
\{\emptyset\} & \text{if } i \neq -1 \\
0 & \text{for all } i \text{ when } X_{\varphi_1} = X_{\varphi_2} = \emptyset.
\end{cases}
$$

**Note.**

i) $\Delta(X_1, X_2; G) \cong \Delta(F_{\varphi}(X_1), F_{\varphi}(X_2); G)$ always. So,

$$
\hat{H}_{-1}(F(X_1), F(X_2); G) \equiv 0, \text{ for any pair } (X_1, X_2) \text{ of classical spaces.}
$$

ii) $\Delta^e(X_{\varphi_1}, X_{\varphi_2}) \cong \Delta(F(X_{\varphi_1}), F(X_{\varphi_2}))$

except if, and only if, $X_{\varphi_1} \neq X_{\varphi_2} = \emptyset$, when non-isomorphisms occur only in $\text{deg} = -1$, i.e.,

$$
\Delta^e(X_{\varphi_1}, \emptyset) \cong \mathbb{Z} \not\cong 0 \cong \Delta^e(X_{\varphi_1}, \emptyset)
$$

which, if $X_{\varphi_1} \neq \emptyset$, gives,

$$
\hat{H}_i(X_{\varphi_1}, \emptyset) \oplus \mathbb{Z} \cong H_0(F(X_{\varphi_1}), \emptyset).
$$

Remember that: $\text{Mor}_{\text{coframe}}((X_{\varphi_1}, X_{\varphi_2}), (X_{\varphi_1}, \emptyset)) \neq \emptyset$ if, and only if, $X_{\varphi_1} \neq \emptyset$.

iii) $\mathcal{C}(\Sigma_{\varphi_1}, \Sigma_{\varphi_2}; G) \cong \Delta(\Sigma_{\varphi_1}; \Sigma_{\varphi_2}; G)$ connects the simplicial and singular functor, where “$\cong$” stands for “chain equivalence”.

iv) $\hat{H}_0(X_{\varphi_1} Y_{\varphi_2}, \{\varphi\}; G) = \hat{H}_0(X_{\varphi_1}, \{\varphi\}; G) \oplus \hat{H}_0(Y_{\varphi_2}, \{\varphi\}; G)$ but

$$
\hat{H}_0(X_{\varphi_1} Y_{\varphi_2}, \emptyset; G) = \hat{H}_0(X_{\varphi_1}, \emptyset; G) \oplus \hat{H}_0(Y_{\varphi_2}, \{\varphi\}; G) = \hat{H}_0(X_{\varphi_1}, \{\varphi\}; G) \oplus \hat{H}_0(Y_{\varphi_2}, \emptyset; G).
$$

**Definition.** The $p$th singular augmental homology group of $X_{\varphi}$ with respect to $G$ is $\hat{H}_p(X_{\varphi}; G) := \hat{H}_p(X_{\varphi}, \emptyset; G)$. The coefficient group $:\hat{H}_{-1}(\emptyset, \emptyset; G) \cong G.

Using $F_{\varphi}(\mathcal{E}_o)$, we “lift” the foundational concepts of homotopy, excision and point in $\mathcal{C}(\mathcal{K})$ into $D_{\varphi}$-concepts ($\mathcal{K}_o$-concepts) homotopy, excision and point, ($\bullet$), respectively.

So, $f_o, g_o \in D_{\varphi}$ are homotopic if and only if $f_o = g_o = 0_{\emptyset Y_{\varphi}}$ or there are homotopic maps $f_1, g_1 \in \mathcal{C}$ such that $f_o = f_1 + \text{Id}_{\{\varphi\}}, g_o = g_1 + \text{Id}_{\{\varphi\}}$.

An inclusion

$$(i_o, i_o A_o) : (X_o \setminus U_o, A_o \setminus U_o) \longrightarrow (X_o, A_o), \; U_o \neq \{\varphi\},$$

is an excision if and only if there is an excision

$$$(i, i_A) : (X \setminus U, A \setminus U) \longrightarrow (X, A)$$

such that $i_o = i + \text{Id}_{\{\varphi\}}$ and $i_{o A_o} = i_A + \text{Id}_{\{\varphi\}}$.

$\{P, \varphi\} \in D_{\varphi}$ is a point if and only if $\{P\} + \{\varphi\} = F_{\varphi}(\{P\})$ and $\{P\} \in \mathcal{C}$ is a point. So, $\{\varphi\}$ is not a point. 


Conclusions: \( \hat{\mathcal{H}}, \partial \), abbreviated \( \hat{\mathcal{H}} \), is a homology theory on the \( h \)-category of pairs from \( \mathcal{D}_\varphi (\mathcal{K}_\varphi) \), c.f. \[14\] p. 117, i.e. \( \hat{\mathcal{H}} \) fulfills the \( h \)-category analogues, given in \[14\] \( \S \S 8-9 \) pp. 114-118, of the classical "Eilenberg-Steenrod axioms" from \[14\] \( \S 3 \) pp. 10-13. The necessary verifications are either equivalent to the classical or completely trivial. E.g. the dimension axiom is fulfilled since \( \{\varphi\} \) is not a point, and the "excision"-concept doesn’t apply to a construction like \((X_\varphi \setminus \varphi, A_\varphi \setminus \varphi)\).

Since the exactness of the relative Mayer-Vietoris sequence of a proper triad follows from the axioms, cf. \[14\] p. 43, we will use this without further motivation, paying proper attention to Note iv above.

\( \hat{\mathcal{H}}(X) = \hat{\mathcal{H}}(\mathcal{F}_\varphi(X), \emptyset) \) explains all the ad-hoc reasoning surrounding the classical reduced homology functor \( \check{\mathcal{H}} \).

4. Augmental homology modules for products and joins

4.1. Background to the product- and join definitions. What is usually called simply [the product topology] with respect to products including infinitely many factors is actually the Tychonoff product topology which became the dominant product topology when the Tychonoff product theorem for compact spaces was introduced. J.L. Kelley (1950) (J. Lo"{s} and C. Ryll-Nardzewski (1954)) has proven the [Tychonoff product theorem] (Tychonoff product theorem for Hausdorff spaces) to be equivalent to the [axiom of choice] (the [Boolean prime ideal theorem], which in its equivalent dual formulation is known as the [ultra filter lemma]). Compare [4].

Our "realization" of simplicial complexes p. [13] turns out to be a contravariant functor.

As was stated in p. [4]: The simplicial as well as the topological join-operation are (modulo realization, equivalent) cases of [colimits] as being simplicial resp. topological attachments. Restricting to topological k-spaces, [18] p. 157ff remains true also with our new realization functor, i.e. geometric realization preserves finite limits and all colimits.

For a background on the join-definitions, see [12] and [17] and for the topological join-definitions, see the introductions in [15] and [51], the definitions of which are quoted below.

Since \( \emptyset \) is a simplicial zero-element with respect to simplicial join and \( \{\emptyset\} \) is the join-unit, it is absolutely impossible to functorially realize the (augmented abstract) simplicial complexes into the classical topological spaces including the classical Euclidian spaces.

That there should be a join-unit also in the category of classical topological spaces and continuous maps has been taken for granted. The join-unit in classical literature has been introduce with a jargon like: "We use \( \emptyset \) as join-unit", or for example as in [11] p. 135: "...(it is a convention that \( \emptyset \ast Y = Y, X \ast \emptyset = X \)". The "new" topological space \( \{\varphi\} \) now provide general topology with a natural join-unit.

4.2. Classical definitions of topological product and join.

Definition 4.1. (E.H. Spanier [15] p. 4) The topological product of an indexed collection of topological spaces \( \{X\}_{j \in J} \) is the cartesian product \( \times_{j \in J} X_j \) given the topology induced by the projection maps \( p : \times_{j \in J} X_j \rightarrow X_j \) for \( j \in J \).

Definition 4.2. ([15] p. 234) Given topological pairs \((X, A)\) and \((Y, B)\), we define their product \((X, A) \times (Y, B)\) to be the pair \( X \times Y, (X \times B) \cup (A \times Y) \).
Definition 4.3. (G.W. Whitehead [34] p. 56) Let $X$ and $Y$ be spaces, which we assume to
be disjoint from each other and from $X \times Y \times I$, where $I$ is the unit interval $\{t \mid 0 \leq t \leq 1\}$. Let $W = X \cup (X \times Y \times I) \cup Y$; we topologize $W$ by defining a subset to be open if and only if its intersection with each of the spaces $X$, $(X \times Y \times I)$ and $Y$ is open. The join of $X$ and $Y$ is the identification space $X * Y$ obtained from $W$ by identifying each $x \in X$ with all of the points $(x, y, 0)$ and each $y \in Y$ with all of the points of $(x, y, 1)$. The identification map sends $X$ and $Y$ homeomorphically into $X * Y$; hence we may consider $X$ and $Y$ as subspaces of $X * Y$. Let $(t - 1)x \oplus ty$ be the image of $(x, y, t)$ in $X * Y$.

The join operation is easily seen to be commutative (up to a natural homeomorphism). The join of $X$ with the empty set $\emptyset$ is $X$.

Definition 4.4. (J.W. Milnor, [35] p. 430.) The join $A_1 \circ \cdots \circ A_n$ of $n$ topological spaces $A_1, \ldots, A_n$ can be defined as follows. A point of the join is specified by

(1) $n$ real numbers $t_1, \ldots, t_n$ satisfying $t_i \geq 0$, $t_1 + \cdots + t_n = 1$, and

(2) a point $a_i \in A_i$ for each $i$ such that $t_i \neq 0$. Such a point in $A_1 \circ \cdots \circ A_n$ will be denoted by the symbol $t_1 a_1 \oplus \cdots \oplus t_n a_n$, where the element $a_i$ may be chosen arbitrarily or omitted whenever the corresponding $t_i$ vanishes.

By the strong topology in $A_1 \circ \cdots \circ A_n$ we mean the strongest topology such that the coordinate functions $t_i : A_1 \circ \cdots \circ A_n \rightarrow [0, 1]$ and $a_i : t_i : [0, 1] \rightarrow A_i$ are continuous. Thus a sub-basis for the open sets is given by the sets of the following two types

(1) the set of all $t_1 a_1 \oplus \cdots \oplus t_n a_n$ such that $\alpha < t_i < \beta$,

(2) the set of all $t_1 a_1 \oplus \cdots \oplus t_n a_n$ such that $t_i \neq 0$ and $a_i \in U$, where $U$ is an arbitrarily open subset of $A_i$.

The join of infinitely many topological spaces in the strong topology can be defined in exactly the same manner, with the restriction that all but a finite number of the $t_i$ should vanish. It is clear from the definition that the formation of finite or infinite joins in the strong topology is an associative, commutative operation.

The strong topology is not the same as the more conventional weak topology, in which $A_1 \circ \cdots \circ A_n$ is considered as an identification space of the product of $A_1 \times \cdots \times A_n$ with an $(n - 1)$-simplex.

Let $A_1 \circ \cdots \circ A_n$ denote $A_1 \circ \cdots \circ A_n$ above equipped with the strong topology.

4.3. Augmentals versions of products and joins.

Let $\vee$ denote the above classical:

- topological product $\times$ from [36] p. 4 or
- topological join $*$ from [37] p. 56 or
- topological join $\$ from [35] p. 430.

Definition 2 p. 34 implies that the join-unit in the category of classical (abstract) simplicial complexes is $\emptyset$, the classical realization of which, $\emptyset$, is assumed to be the join-unit in the category of classical topological spaces, i.e.;

$$X \$ \emptyset = X * \emptyset = X = \emptyset * X = \emptyset \$ X.$$

Definition.

$$X_{\vee 1} \$ X_{\vee 2} := \begin{cases} \emptyset & \text{if } X_{\vee 1} = \emptyset \text{ or } X_{\vee 2} = \emptyset \\ \mathcal{F}_{\vee} (\mathcal{F}(X_{\vee 1}) \vee \mathcal{F}(X_{\vee 2})) & \text{if } X_{\vee 1} \neq \emptyset \neq X_{\vee 2}. \end{cases}$$
From now on we will delete the $\varnothing/o$-indices. So, e.g. $\times,*,$ $\bowtie$ now means $\times_0,*_0,$ $\bowtie_0,$ respectively, while “$X$ connected” means “$\mathcal{F}(X)$ connected”.

**Equivalent Join Definition.** Put

$$\emptyset \sqcup X = X \sqcup \emptyset := \emptyset$$

and

$$\{\varnothing\} \sqcup X = \{\varnothing\}, \; X \sqcup \{\varnothing\} = X, \; \text{if} \; X \neq \emptyset.$$  

For $X,Y \neq \emptyset$ or $\varnothing$, let $X \sqcup Y$ be “$X \times Y \times (0,1]$ pasted to $X$” along the function

$$\varphi_1 : X \times Y \times \{1\} \rightarrow X; \; (x,y,1) \mapsto x,$$

i.e. the quotient set of $(X \times Y \times (0,1]) \sqcup X$, under the equivalence relation $(x,y,1) \sim x$ and let $p_1 : (X \times Y \times (0,1]) \sqcup X \rightarrow X \sqcup Y$ be the quotient function.

For $X \bowtie Y = \emptyset$ or $\{\varnothing\}$ put $X \sqcup Y := Y \sqcup X$ and else $X \sqcup Y := X \times Y \times [0,1)$ pasted to the set $Y$ along the function

$$\varphi_2 : X \times Y \times \{0\} \rightarrow Y; \; (x,y,0) \mapsto y,$$

and let

$$p_2 : (X \times Y \times [0,1)) \sqcup Y \rightarrow X \sqcup Y$$

be the quotient function. Put

$$X \circ Y := (X \sqcup Y) \sqcup (X \sqcup Y).$$

To any $(x,y,t) \in X \times Y \times [0,1]$ there corresponds precisely one point $(x,y,t) \in X \sqcup Y \cap X \sqcup Y$, if $0 < t < 1$ and the “equivalence class containing $x$” if $t = 1$ ($y$ if $t = 0$), which is denoted $(x,1)$ ($(y,0)$). This allows one to introduce “coordinate functions”

$$\xi : X \circ Y \rightarrow [0,1],$$

$$\eta_1 : X \sqcup Y \rightarrow X,$$

$$\eta_2 : X \sqcup Y \rightarrow Y$$

extendable to $X \circ Y$, by setting

$$\eta_1(y,0) := x_0 \in X,$$

resp.

$$\eta_2(x,1) := y_0 \in Y.$$

Combining $p_1$ and $p_2$ we also have a projection $p$;

$$p : X \sqcup (X \times Y \times [0,1]) \sqcup Y \rightarrow X \circ Y.$$  

Let $X \bowtie Y$ denote $X \circ Y$ equipped with the smallest topology making $\xi, \eta_1, \eta_2$ continuous and let $X \bowtie Y$ be $X \circ Y$ with the quotient topology with respect to $p$, i.e. the largest topology making $p$ continuous ($\Rightarrow \tau_{X \bowtie Y} \subset \tau_{X \circ Y}$).

**Pair-definitions.** Let

$$(X_1, X_2) \bowtie (Y_1, Y_2) := (X_1 \bowtie Y_1, (X_1 \bowtie Y_2) \sqcup (X_2 \bowtie Y_1)),$$

where $\sqcup$ stand for “$\cup$” or “$\sqcup$” and if either $X_2$ or $Y_2$ is not closed (open),

$$(X_1 \bowtie Y_1, (X_1 \bowtie Y_2) \sqcup (X_2 \bowtie Y_1)) := (X_1 \bowtie Y_1, (X_1 \bowtie Y_2) \sqcup (X_2 \bowtie Y_1)),$$
where $\ast$ indicates that the subspace topology is to be used, i.e. the pair

$$(X_1 \ast Y_1, (X_1 \circ Y_1) \ominus (X_1 \circ Y_1))$$

with the subspace topology in the 2:nd component. (We will use similar pair-definitions for simplicial complexes with, "$\times$" ("$\ast$") from [13] p. 67 Def. 8.8 (app. p. 109 Ex. 7) - of course, without any topologizing considerations.)

**Note.**

i. With $t \in [0, 1]$, the "Equivalent Join Definition" above, suggest the notation

$$(X \ast Y)_{t>0} = X \cup Y \text{ resp. } (X \ast Y)_{t<1} = X \ominus Y \text{ and } (X \ast Y)_{t=1} = X \text{ resp. } (X \ast Y)_{t=0} = Y.$$  

Now, for any $s, t \in (0, 1)$ we see that $(X \ast Y)_{t\geq s}$ is homeomorphic to the mapping cylinder $C(q_1)$ with respect to the coordinate map $q_1 : X \times Y \rightarrow X$ and that $(X \ast Y)_{t\leq s}$ is homeomorphic to the mapping cylinder $C(q_2)$ with respect to the coordinate map $q_2 : X \times Y \rightarrow Y$.

ii. $X_1 \ast Y_1$ is a subspace of $X_1 \hat{\times} Y_1$ by [3] 5.7.3 p. 163. $X_2 \ast Y_2$ is a subspace of $X_1 \ast Y_1$ if $X_2, Y_2$ are both closed (open), cf. [1] Th. 2.1(1) p. 122.

iii. $(X_1 \circ Y_1) \cap X_1 \circ Y_1 = X_1 \circ Y_1$ and with $\vee := \vee_1$ for short in pair operations; $(X_1, \{\emptyset\}) \times (Y_1, Y_2) = (X_1, \emptyset) \times (Y_1, Y_2)$ if $Y_2 \neq \emptyset$.

iv. $\hat{\times}$ and $\ast$ are both commutative but, while $\hat{\times}$ is associative by [3] p. 161, $\ast$ is not in general, cf. p. 28.

v. "$\chi_\ast$" is (still, cf. [1] p. 15,) the categorical product on pairs from $\mathcal{D}_0$.

### 4.4. Augmental homology for products and joins.

The following theorem and comments comes from [13], page 235 and it is the classical Künneth formula for products.

![Cross product](image)

**Theorem** (10). [The classical relative Künneth formula for products (from [13] p. 235).]

If $\{X \times B, A \times Y\}$ is an excisive couple in $X \times Y$ and $G$ and $G'$ are modules over a principal ideal domain $R$ such that $\text{Tor}_1^R(G, G') = 0$, there is a functorial short exact sequence;

$$0 \rightarrow [H(X, A; G) \otimes H(Y, B; G')]_q \rightarrow H_q((X, A) \times (Y, B); G \otimes G') \rightarrow [\text{Tor}_1^R(H(X, A; G), H(Y, B; G'))]_{q-1} \rightarrow 0$$

and this sequence is split.

In particular, if the right hand side vanishes (which always happens if $R$ is a field) then the cross product $\mu'$ is an isomorphism.

By using Lemma + Note ii p. [14] we convert the classical Künneth formula above, mimicking what Milnor did at the end of his proof of Lemma 2.1 [3] p. 431, where $H_0(X) = \overline{H}_0(X) \oplus \mathbb{Z}$ and $H_r(X) = \overline{H}_r(X)$ for $r \neq 0$, is used to reach the first line in Theorem 4.3 below. Recall that $\overline{H}_r(X) = \overline{H}_r(X, \emptyset)$ and that a pair $\{X, Y\}$ is an excisive couple of subsets if the inclusion chain map $\Delta^p(X) + \Delta^p(Y) \subset \Delta^p(X \cup Y)$ induces an isomorphism in homology. The "new" object $\{\emptyset\}$ gives additional strength to the Künneth formula w.r.t. the classical Künneth formula (≡4:th line in Th. [14]), but much of the classical beauty is lost – a loss which is regained in the join version i.e. in Theorem 4.8 p. 24 – but still, this is now how the Künneth formula for products looks in any augmented environment. Th. 4.8 is, via Note 5.2 iii p. 34 at the heart of modern algebraic topology.
Theorem 4.5. [The relative Künneth formula for products.]
If \( \{X_1 \times Y_2, X_2 \times Y_1\} \) is an excisive couple of subsets (Def. [15] p. 188), \( q \geq 0 \), \( R \) a PID, and assuming \( \text{Tor}_i^R(G, G') = 0 \) for \( R \)-modules \( G \) and \( G' \), then;

\[
\text{H}_q((X_1, X_2) \times (Y_1, Y_2); G \otimes_R G') \cong \begin{cases} 
[\hat{H}_i(X_1; G) \otimes \hat{H}_j(Y_1; G')]_q \oplus (\hat{H}_i(X_1; G) \otimes G') \oplus (G \otimes \hat{H}_j(Y_1; G')) \oplus T_1 & \text{if } C_1 \\
[\hat{H}_i(X_1; G) \otimes \hat{H}_j(Y_1, Y_2; G')]_q \oplus (G \otimes \hat{H}_j(Y_1, Y_2; G')) \oplus T_2 & \text{if } C_2 \\
[\hat{H}_i(X_1, X_2; G) \otimes \hat{H}_j(Y_1; G')]_q \oplus (\hat{H}_j(X_1, X_2; G) \otimes G') \oplus T_3 & \text{if } C_3, \\
[\hat{H}_i(X_1, X_2; G) \otimes \hat{H}_j(Y_1, Y_2; G')]_q \oplus T_4 & \text{if } C_4,
\end{cases}
\]

where the torsion terms, i.e. the \( T \)-terms, split as those ahead of them, e.g.,

\[
T_i = [\text{Tor}_i^R(\hat{H}_i(X_1; G), \hat{H}_j(Y_1; G'))]_{q-1} \oplus \text{Tor}_1^R(\hat{H}_i(X_1; G), G') \otimes \text{Tor}_i^R(G, \hat{H}_j(Y_1; G')),
\]

and

\[
T_4 = [\text{Tor}_1^R(\hat{H}_i(X_1, X_2; G), \hat{H}_j(Y_1, Y_2; G'))]_{q-1}.
\]

\( C_i, (i = 1 \to 4) \), are “conditions” and should be interpreted as follows, resp.,

\[
\begin{cases} 
C_1 := "X_1 \times Y_1 \neq \emptyset, \ \{0\} \text{ and } X_2 = \emptyset = Y_2", \\
C_2 := "X_1 \times Y_1 \neq \emptyset, \ \{0\} \text{ and } X_2 \neq 0 \neq Y_2", \\
C_3 := "X_1 \times Y_1 \neq \emptyset, \ \{0\} \text{ and } X_2 \neq 0 = Y_2", \\
C_4 := "X_1 \times Y_1 = \emptyset, \ \{0\} \text{ or } X_2 \neq 0 \neq Y_2".
\end{cases}
\]

Let \([\ldots]_q\), as in [15] p. 235 Th. 10, be interpreted as \( \bigoplus_{i+j=q \& i,j \geq 0} \ldots \).

Lemma. For a relative homeomorphism \( f : (X, A) \to (Y, B) \) (i.e. \( f : X \to Y \) is continuous and \( f : X \setminus A \to Y \setminus B \) is a homeomorphism), let \( F : N \times I \to N \) be a strong neighborhood deformation retraction of \( N \) onto \( A \). If \( B \) and \( f(N) \) are closed in \( N' := f(N \setminus A) \cup B \), then \( B \) is a strong neighborhood deformation retraction of \( N' \) by means of,

\[
F' : N' \times I \to N' ; \begin{cases} 
(y,t) \mapsto (y) & \text{if } y \in B, t \in I \\
(y,t) \mapsto f \circ F(f^{-1}(y), t) & \text{if } y \in f(N \setminus A), \ t \in I.
\end{cases}
\]

Proof. (cf. p. [34]) \( F' \) is continuous by [15] p. 34; 2.5.12 as being so when restricted to any one of the closed subspaces \( f(N) \times I \), resp. \( B \times I \), where \( N' \times I = (f(N) \times I) \cup (B \times I) \).

Note. \( X \) (resp. \( Y \)) is a strong deformation retract of “the mapping cylinder with respect to its product projection”, which is homeomorphic to \( (X \ast Y)^{t \geq s} \) (resp. \( (X \ast Y)^{t \leq 1-s} \)), with \( s, t \in [0, 1] \). Equivalently for “\( \ast \)”-join, by the Lemma. So,

\[
r : X \ast Y \to X \ast Y; \begin{cases} 
(x, y, t) \mapsto x \text{ if } t \geq 0.9, \\
(x, y, t) \mapsto y \text{ if } t \leq 0.1, \\
(x, y, t) \mapsto (x, y, 0.5(t - 0.5) + 0.5) \text{ otherwise,}
\end{cases}
\]

is a homotopy inverse of the identity, i.e.,

\( X \ast Y \) and \( X \ast Y \) are homotopy equivalent.

This homotopy is a well-known fact and it allows us to substitute “\( \ast \)” for any occurrence of “\( \ast \)”, and vice versa, in any discussion of homology groups.
Theorem 4.6. (Analogously for \( \adj \) by the last Note (mutatis mutandis).)
If \((X_1, X_2) \neq (\{\varnothing\}, \emptyset) \neq (Y_1, Y_2)\) and \(G\) is an \(A\)-module, then

\[
\hat{H}_q((X_1, X_2) \times (Y_1, Y_2); G) \cong \hat{H}_{q+1}((X_1, X_2) * (Y_1, Y_2); G) \oplus \hat{H}_q((X_1, X_2) * (Y_1, Y_2); G) \]
\[
\cong \begin{cases} 
\hat{H}_{q+1}(X_1 * Y_1; G) \oplus \hat{H}_q(X_1; G) \oplus \hat{H}_q(Y_1; G) & \text{if } C_1 \\
\hat{H}_{q+1}(X_1, Y_1) * (Y_1, Y_2); G) \oplus \hat{H}_q(Y_1, Y_2; G) & \text{if } C_2 \\
\hat{H}_{q+1}(X_1, X_2) * (Y_1, Y_2); G) \oplus \hat{H}_q(X_1, X_2; G) & \text{if } C_3 \\
\hat{H}_{q+1}(X_1, X_2) * (Y_1, Y_2); G) & \text{if } C_4 
\end{cases}
\]

where

\[
\begin{cases} 
C_1 := "X_1 \times Y_1 \neq \emptyset, \ \{\varnothing\} \ and \ X_2 = \emptyset = Y_2", \\
C_2 := "X_1 \times Y_1 \neq \emptyset, \ \{\varnothing\} \ and \ X_2 = \emptyset \neq Y_2", \\
C_3 := "X_1 \times Y_1 \neq \emptyset, \ \{\varnothing\} \ and \ X_2 \neq \emptyset = Y_2", \\
C_4 := "X_1 \times Y_1 = \emptyset, \ \{\varnothing\} \ or \ X_2 \neq \emptyset \neq Y_2."
\end{cases}
\]

Proof. Splitting \(X \ast Y\) at \(t = 0.5\) gives \((X \ast Y)^{\leq 0.5}\) and \((X \ast Y)^{\geq 0.5}\). Now, \((X \ast Y)^{\geq 0.5}\) is homeomorphic to the mapping cylinder \(C(p_X)\) of the product projection \(p_X : X \times Y \rightarrow X; \ (x, y) \mapsto x\) of which \(X\) is a strong deformation retract. Equivalently, \((X \ast Y)^{\leq 0.5}\) is homeomorphic to \(C(p_Y)\) of the projection \(p_Y\).

The relative \(M\)-Vs with respect to the excisive couple of pairs

\[\{((X_1, X_2) \ast (Y_1, Y_2))^{\geq 0.5}, \ (X_1, X_2) \ast (Y_1, Y_2))^{\leq 0.5}\}\]
splits since the inclusion of their topological sum into \((X_1, X_2) \ast (Y_1, Y_2)\) is pair null-homotopic, cf. [39] p. 141 Ex. 6c, and [24] p. 32 Prop. 1.6.8. Since the 1:st (2:nd) pair is acyclic if \(Y_2 (X_1) \neq \emptyset\), we get Theorem 1.6.

This proven homomorphism remains true also when \(\adj\) is substituted for \(\ast\), by the homotopy equivalence in the last Note. \(\square\)

Milnor finished his proof of [35] Lemma 2.1 p. 431 by simply comparing the r.h.s. of the \(C_1\)-case in Eq. 1 with that of Eq. 2. Since we are aiming at a stronger result of “natural chain equivalence” in Theorem 1.7 this is not strong enough. We will therefore use the following three auxiliary results to prove our next two theorems. We hereby avoid explicit use of “proof by acyclic models”.

5.7.4. (from [3] p. 164.) \((E^0 := \{e, \varnothing\} = \bullet\) denotes a point, i.e. a 0-disc.)
There is a homeomorphism:

\[\nu : X \ast Y \ast E^0 \rightarrow (X \ast E^0) \times (Y \ast E^0)\]

which restricts to a homeomorphism:

\[X \ast Y \rightarrow ((X \ast E^0) \times Y) \cup (X \times (Y \ast E^0)). \] \(\square\)

Corollary (5.7.9). (from [24] p. 210.) (“\(\approx\)” stands for “chain equivalence”.)
If \(\phi : C \approx E\) with inverse \(\chi\) and \(\phi' : C' \approx E'\) with inverse \(\chi'\), then

\[\phi \otimes \phi' : C \otimes C' \approx E \otimes E'\] with inverse \(\chi \otimes \chi'\). \(\square\)
The $\otimes$-operation in the \textbf{monoidal category} (= tensor category) in this corollary can be substituted for any monoid-inducing operation in any category, cf. \textbf{III} Ex. 2 p. 168.

\textbf{Theorem} (46.2). (from \textbf{III} p. 279)

Let $\mathcal{C}$ and $\mathcal{D}$ be free chain complexes that vanish below a certain dimension; let $\lambda : \mathcal{C} \to \mathcal{D}$ be a chain map. If $\lambda$ induces homology isomorphisms in all dimensions, then $\lambda$ is a chain equivalence.

\textbf{Theorem 4.7.} (The relative Eilenberg-Zilber theorem for topological join.) For an excisive couple $\{X \hat{\otimes} Y_2, X_2 \hat{\otimes} Y\}$ from the category of ordered pairs $((X, X_2), (Y, Y_2))$ of topological pairs $\tilde{s}$,

$$s(\Delta(X, X_2) \otimes \Delta(Y, Y_2)) \text{ is naturally chain equivalent to } \Delta((X, X_2) \hat{\otimes} (Y, Y_2)).$$

(“$s$” stands for suspension i.e. the suspended chain equals the original one except that the dimension $i$ in the original chain becomes $i + 1$ in the suspended chain. Th. 4.7 is the join version of the classical Th. 9 in \textbf{I} p. 234, with 1) “$\Delta$”, 2) “$\Delta$”, resp. $s$ $\Delta$, 3) “Theorem 6” in the original $\times$-proof in \textbf{I} p. 228 together with the complete classical $\times$-proof in \textbf{I} p. 232 Theorem 6.)

Proof. The second isomorphism is the key one and is induced by the pair homeomorphism in \textbf{I} 5.7.4 p. 164. For the 2:nd last isomorphism we use \textbf{II} p. 210 Corollary 5.7.9 and that $\hat{\text{LHS}}$-homomorphisms are “chain map”-induced. The last follows from the $\hat{\text{LHS}}$ since cones are null-homotopic. Note that the second component in the third module is an excisive union.

$$\hat{H}_i(X \hat{\otimes} Y) \overset{\cong}{=} \hat{H}_i(X \hat{\otimes} (Y \hat{\otimes} \{v, \varphi\}), X \hat{\otimes} Y) \overset{\cong}{=}$$

$$\hat{H}_i(\{(X \hat{\otimes} \{u, \varphi\}) \times (Y \hat{\otimes} \{v, \varphi\}), (X \hat{\otimes} \{u, \varphi\}) \times Y) \cup (X \times (Y \hat{\otimes} \{v, \varphi\}))\} =$$

$$= \hat{H}_i((X \hat{\otimes} \{u, \varphi\}, X) \times (Y \hat{\otimes} \{v, \varphi\}, Y)) \overset{\cong}{=}$$

$$\overset{\cong}{=} \hat{H}_i(\Delta(X \hat{\otimes} \{u, \varphi\}, X) \hat{\otimes} \Delta(Y \hat{\otimes} \{v, \varphi\}, Y)) \overset{\cong}{=}$$

$$\overset{\cong}{=} \hat{H}_i(\hat{s}(\Delta(X) \hat{\otimes} \Delta(Y))) \overset{\cong}{=} \hat{H}_i(s(\Delta(X) \otimes \Delta(Y))).$$

So, $\Delta(X \hat{\otimes} Y)$ is naturally chain equivalent to $s(\Delta(X) \otimes \Delta(Y))$ by \textbf{III} p. 279 Th. 46.2 quoted above, proving the non-relative Eilenberg-Zilber Theorem for joins.

(The \textbf{boundary map} for the $\otimes$-complex is given in \textbf{II} p. 228 together with the complete classical $\times$-proof in \textbf{II} p. 232 Theorem 6.)

Substituting 1) “$\times$”, 2) “$\Delta$”, 3) “Theorem 6” in the original $\times$-proof for pairs given in \textbf{II} p. 234, with 1) “$\hat{\otimes}$”, 2) “$\hat{\Delta}$” resp. $s\hat{\Delta}$, 3) “Theorem I.7. 1st part” respectively, will do since; (\textbf{II} Th. 35 p. 184 gives the algebraic motivation.)

$$s(\Delta(X_j) \otimes \Delta(Y_i)) / (s(\Delta(X_j) \otimes \Delta(Y_2)) + s(\Delta(X_2) \otimes \Delta(Y_j))) =$$

$$= s\{\Delta(X_j) \otimes \Delta(Y_j) / ((\Delta(X_j) \otimes \Delta(Y_2) + (\Delta(X_2) \otimes \Delta(Y_j)))\} =$$

$$= s\{(\Delta(X_j) / \Delta(Y_j) \otimes (\Delta(Y_j) / \Delta(Y_2))\}.$$

\qed
Corollary 4.9. (from [E] p. 231.) Given torsion-free chain complexes $C$ and $C'$ and modules $G$ and $G'$ such that $\text{Tor}_1(G, G') = 0$, there is a functorial short exact sequence $0 \to [H(C; G) \otimes H(C'; G')]_q \otimes H_q(C \otimes C'; G \otimes G') \to [\text{Tor}_1(H(C; G), H(C'; G'))]_q \to 0$ and this sequence is split exact.

Corollary 4 [E] p. 231 now gives Theorem 1.8, since $\check{H}(\cdot) \cong s\check{H}(\cdot) \cong \check{H}(s(\cdot))$ and $\Delta((X, X)_* (Y, Y)) \cong \Delta((X, X) \hat{\otimes} (Y, Y))$ by Th. 4.3 and 3.4 p. 279 Th. 46.2, quoted above.

The couple $\{X_1 \times Y_2, X_2 \times Y_1\}$ is excisive iff $\{X_1 \times Y_2, X_2 \times Y_1\}$ is excisive, which is seen through the $M$-$V$s-stuffed 9-Lemma p. 234ff and Theorem 1.0 (line 4).

Theorem 4.8. (The Relative Künneth Formula for Topological Joins; cf. [E] p. 235.) If $\{X_1 \# Y_2, X_2 \# Y_1\}$ is an excisive couple in $X_1 \# Y_1$, $G$ a PID, $G$ and $G'$ $R$-modules and $\text{Tor}_1(G, G') = 0$, then the functorial sequences below are (non-naturally) split exact:

$$0 \to \bigoplus_{i+j=q} \left[ \check{H}_i(X_1, X_2; G) \otimes_R \check{H}_j(Y_1, Y_2; G') \right] \to \check{H}_{q+i}(X_1, X_2 \hat{\otimes} (Y_1, Y_2); G \otimes_R G') \to \bigoplus_{i+j=-q} \text{Tor}_1^R(\check{H}_i(X_1, X_2; G), \check{H}_j(Y_1, Y_2; G')) \to 0 \quad \square$$

Analogously for the $\star$-join.

[E] p. 247 Th. 11 gives the co$\check{H}$omology-analog of Theorem 4.8. $(X_1, X_2) = (\emptyset, \emptyset)$ in Theorem 4.8 immediately gives our next theorem.

Theorem 4.9. [The Universal Coefficient Theorem for (co)$\check{H}$omology]

$$\check{H}_i(Y_1, Y_2; G) \cong [E] p. 214 \cong \check{H}_i(Y_1, Y_2; R \otimes_R G) \cong (\check{H}_i(Y_1, Y_2; R) \otimes_R G) \oplus \text{Tor}_1^R(\hat{\check{H}}_{i+1}(Y_1, Y_2; R), G),$$

for any $R$-PID module $G$.

If all $\check{H}_i(Y_1, Y_2; R)$ are of finite type or $G$ is finitely generated, then

$$\hat{\check{H}}_i(Y_1, Y_2; G) \cong \hat{\check{H}}_i(Y_1, Y_2; R \otimes_R G) \cong (\hat{\check{H}}_i(Y_1, Y_2; R) \otimes_R G) \oplus \text{Tor}_1^R(\hat{\check{H}}_{i+1}(Y_1, Y_2; R), G). \quad \square$$

N.B. There can be no classical relative Künneth Formula for Topological Joins, cf. p. 4.

4.5. Local homology groups for products and joins. Proposition 1.10 below motivates in itself the introduction of a topological $(-1)$-object, which imposed the definition p. 13 of a “setminus”, “\$\setminus\$”, in $D_{\varphi}$, revealing the shortcomings of the classical boundary definitions with respect to manifolds, cf. p. 7 and p. 27. Somewhat specialized, Proposition 1.11 below is found in [H] p. 162 and partially also in [H] p. 116 Lemma 3.3. “$X \setminus \{x\}$” usually stands for “$X \setminus \{x\}$” and we will write $x$ for $\{x, \varphi\}$ as a notational convention. Recall the definition of $\alpha_{\varphi}$ p. 13 and that dim $Lk_{\sigma} = \dim \Sigma - \# \sigma$, where $\# \sigma$ stands for cardinality.

The contrastar of $\sigma \in \Sigma = \text{cost}_p : = \{ \tau \in \Sigma \mid \tau \nsubseteq \sigma \}$. So, $\text{cost}_p 0 = \emptyset$ and $\text{cost}_p \Sigma = \Sigma$ if $\sigma \not\subseteq \Sigma$. In the proof we use the following simplicial complexes, $\bar{\sigma} := \{ \tau \mid \tau \subseteq \sigma \}$ and $\sigma := \{ \tau \mid \tau \nsubseteq \sigma \}$. So, $\bar{\varnothing} = \{ \emptyset \}$ and $\emptyset = \emptyset$. Int$(\sigma) := \{ \alpha \in [\Sigma] \mid [v \in \sigma] \iff [\alpha(v) \neq 0] \}$.
Proposition 4.10. Let $G$ be any module over a commutative ring $A$ with unit. With
$\alpha \in \text{Int}\sigma$ and $\alpha = \alpha_0$ iff $\sigma = \emptyset$, the following module isomorphisms are all induced by chain
equivalences (cf. [23] p. 279 Th. 46.2 quoted here in p. [23])
\[
\hat{H}_{i-\sigma} (\text{Lk}_2; \sigma; G) \cong \hat{H}_i(\Sigma, \text{cost}_\sigma; G) \cong \hat{H}_i(\Sigma, |\text{cost}_\sigma|; G) \cong \hat{H}_i(|\Sigma|, |\Sigma| \setminus \alpha; G),
\]
\[
\hat{H}^i-\sigma (\text{Lk}_2; \sigma; G) \cong \hat{H}^i(\Sigma, \text{cost}_\sigma; G) \cong \hat{H}^i(\Sigma, |\text{cost}_\sigma|; G) \cong \hat{H}^i(|\Sigma|, |\Sigma| \setminus \alpha; G).
\]

Proof. (Cf. definitions p. [24].) The \"$\alpha$\"-definition p. [13] and [23] Th. 46.2 p. 279 + pp. 194-199 Lemma 35.1-35.2 + Lemma 63.1 p. 374 gives the two last isomorphisms since $|\text{cost}_\sigma|$ is a deformation retract of $|\Sigma| \setminus \alpha$, while already on the chain level we have
\[
C^\alpha_*(\Sigma, \text{cost}_\sigma) = C^\alpha_*(\Sigma, \sigma \circ \text{Lk}_2; \sigma) = C^\alpha_*(\sigma \circ \text{Lk}_2; \sigma) \simeq C^\alpha_*(\text{Lk}_2, \sigma).
\]

\[
\square
\]

Lemma. If $x$ (y) is a closed point in $X$ ($Y$), then both $\{X \times (Y \setminus y), (X \setminus x) \times Y\}$ and
$\{X \times (Y \setminus y), (X \setminus x) \times Y\}$ are excisive pairs.

Proof. [13] p. 188 Th. 3, since $X \times (Y \setminus y)$ is open in $(X \times (Y \setminus y)) \cup ((X \setminus x) \times Y)$
(resp. $(X \times (Y \setminus y)) \cup ((X \setminus x) \times Y)$, which proves the excisivity.
\[
\square
\]

Theorem 4.11. If $(t_1, \tilde{x} \times y, t_2) := \{(x, y, t) \mid 0 < t_1 < t < t_2 < 1\}$ for closed points $x \in X$ and
$y \in Y$ we have
\[
\hat{H}_{t_1} (\hat{X} \setminus Y, X \setminus Y \setminus \sigma(x), y, t) ; G) \cong \hat{H}_{t_1} (\hat{X} \setminus Y, X \setminus Y \setminus \sigma(x), y, t) ; G) \cong \hat{H}_{t_1} (\hat{X} \setminus Y, X \setminus Y \setminus \sigma(x), y, t) ; G) \cong \hat{H}_{t_1} (\hat{X} \setminus Y, X \setminus Y \setminus \sigma(x), y, t) ; G)
\]

\[
\text{Motivation: A simple calculation.}
\]

and equivalently for the $(x, 1)$-points.

All isomorphisms are induced by chain (homotopy) equivalences (cf. [23] p. 279 Th. 46.2 quoted here in p. [23]).

Analogously for $\text{co}$Homology and for \"$\ast$\” substituted for \"$\hat{\ast}$\”.

Proof. i. \[
\{ A := X \setminus Y \setminus \sigma \{x_0, y_0, t_0 \mid t_1 \leq t < 1\} \}
\]
\[
\{ B := X \setminus Y \setminus \sigma \{x_0, y_0, t_0 \mid 0 < t \leq t_2\} \}
\]

\[
\Rightarrow \quad \{ A \cup B = X \setminus Y \setminus \sigma(t_1, \tilde{x} \times y, t_2) \}
\]

\[
\Rightarrow \quad \{ A \cap B = X \setminus Y \times (0, 1) \setminus \sigma(x_0) \times \sigma(y) \times (0, 1) \}
\]

with $x_0 \times y_0 \times (0, 1) := \{x_0\} \times \{y_0\} \times \{t \mid t \in (0, 1)\}$ and $(x_0, y_0, t_0) := \{(x_0, y_0, t_0) \}$.

Now, using the null-homotopy in the relative $\text{M}$-Vs with respect to $\{(X \setminus Y, A), (X \setminus Y, B)\}$ and
the resulting splitting of it, see Note i p. [24] and the involved pair deformation retractions as in the proof of Th. [1.6] we get
\[
\hat{H}_{t_1} (\hat{X} \setminus Y, X \setminus Y \setminus \sigma(t_1, \tilde{x} \times y, t_2)) \cong \hat{H}_q ((X \setminus Y \times (0, 1) \setminus \sigma(x_0) \times \{y\} \times (0, 1) )) \cong \hat{H}_q ((X \setminus Y \times (0, 1) \setminus \sigma(x_0) \times \{y\} \times (0, 1) ))
\]
\[ \hat{\mathcal{H}}(X \times Y \times \{b, \emptyset\}, X \times Y \times \{b, \emptyset\} \setminus \{(x_0, y_0, b)\}) \triangleq \hat{\mathcal{H}}(X \times Y, X \times Y \setminus \{(x_0, y_0)\}) = \hat{\mathcal{H}}(X \times Y, X \times Y \setminus \{(x_0, y_0)\}) \].

\[ \hat{\mathcal{H}}(X \times Y, (X \times (Y \setminus y_0)) \cup ((X \setminus x_0) \times Y)) = \hat{\mathcal{H}}((X, X \setminus y_0) \times (Y, Y \setminus y_0)). \]

\[ A := X \sqcup Y, \quad B := X \sqcup Y \times \{y_0\} \times [0, 1] \implies A \cup B = X \hat{\otimes} Y \setminus y_0, 0 \quad A \cap B = X \times (Y \setminus y_0) \times (0, 1) \]

\[ (x_0, y_0, t) \in X \times \{y_0\} \times [0, 1] \text{ is independent of } x_0 \text{ and } (x_0, y_0, t) := \{(x_0, y_0, t)\}. \]

Now use Th. [1.5 p. 21] line 2 and that the r.h.s. is a pair deformation retract of the l.h.s.;

\[ (X \times Y \times (0, 1), X \times (Y \setminus y_0) \times (0, 1)) \sim (X \times Y, X \times (Y \setminus y_0)) = (X, \emptyset) \times (Y, Y \setminus y_0). \]

**Definition.** Put \( \text{Hip}_X = \{ \text{The homologically strongly unstable points in } X \} := \{ x \in X \mid \hat{\mathcal{H}}(X, X \setminus x; G) = 0 \text{ for all } i \in \mathbb{Z} \}. \)

So, for products and joins we get

\[ \text{Hip}_{X_1 \vee X_2} \supset (X_1 \vee \text{Hip}_{X_2}) \cup ((\text{Hip}_{X_1}) \vee X_2). \]

**4.6. Homology manifolds under product and join.**

**Definition.** \( \emptyset \) is said to be a weak homology \( \text{whm}_G \)-\((-\infty)\)-manifold. Any nonempty \( \text{A}-\text{space} \) (\( \Leftrightarrow \) all points are closed) \( X \in \mathcal{D}_G \) is a weak homology \( n \)-manifold \( n \text{-whm}_G \) if, for some \( \text{A-module } \mathcal{R}; \)

\[ \hat{\mathcal{H}}(X, X \setminus x; G) = 0 \text{ if } i \neq n \text{ for all } \emptyset \neq x \in X, \quad (4.1) \]

\[ \hat{\mathcal{H}}(X, X \setminus x; G) \cong G \oplus \mathcal{R} \text{ for some } \emptyset \neq x \in X \text{ if } X \neq \{\emptyset\}. \quad (4.1') \]

An \( n \text{-whm}_G \) \( X \) is joinable \( (n \text{-jwhm}_G) \) if \( (4.1) \) holds also for \( x = \emptyset \).

An \( n \text{-jwhm}_G \) \( X \) is a weak homology \( n \text{-sphere}_G \) \( (n \text{-whsp}) \) if \( \hat{\mathcal{H}}(X \setminus x; G) = 0 \forall x \in X. \)

**Definition.** \( X \) is acyclic \( G \) if \( \hat{\mathcal{H}}(X, \emptyset; G) = 0 \forall i \in \mathbb{Z}. \) So, \( \{\emptyset\} = \{\{\emptyset\}\} \) isn’t acyclic \( G. \)

An arbitrary \( X \) is ordinary \( G \) if \( (4.1') \) implies that \( \hat{\mathcal{H}}(X \setminus x; G) = 0, \forall i \geq n \text{ and } \forall x \in X. \)

\( X \) is locally weakly direct \( G \) if \( \hat{\mathcal{H}}(X, X \setminus x; G) \cong G \oplus \mathcal{Q} \) for some \( i, \text{ some } \text{A-module } \mathcal{Q} \) and some \( \emptyset \neq x \in X. \) (This definition was introduced to avoid troublesome tensor annihilations in case of general \( G. \) )

**Note.** 1. \( \text{Lk}_G \sigma = \{\emptyset\} \) for any maxidimensional simplex \( \sigma \) in any any simplicial complex \( \Sigma. \) So, for such a \( \sigma \) and if \( \alpha \in \text{Int } \Sigma \) we get, \( \hat{\mathcal{H}}(\text{dim}\Sigma, \Sigma \setminus \partial \alpha; G) \cong G \) by Proposition [1.10] p. 22 and Lemma p. 14. Quasi-manifolds are ordinary by Note 1 p. 14.

**Theorem 4.12.** For compact triangulable spaces \( X_1, X_2 \ (x_1 \neq \emptyset, \{\emptyset\}); \)

i. \( X_1 \times X_2 \ (n_1 + n_2) \text{-whm}_k \iff X_1, X_2 \text{ both whm}_k. \)

ii. If \( n_i > n_i, i = 1, 2, \text{ then } \]

\[ [X_1 \times X_2(n_1 + n_2) \text{-jwhm}_k] \iff [X_1, X_2 \text{ both } n_i \text{-jwhm}_k \text{ and acyclic}_k]. \]

(Since by Eq. 2 p. 22: \( \hat{\mathcal{H}}(X_1 \times X_2; k) = 0 \text{ for } i \neq n_1 + n_2 ] \iff [X_1, X_2 \text{ both acyclic}_k ] \iff [X_1 \times X_2, \text{ acyclic}_k]. \) So \( X_1 \times X_2 \) is never a whsp.)
iii. \( X_1 \ast X_2 \ (n_1 + n_2 + 1)\)-whsp \( k \) \iff \( X_1, X_2 \) both \( n_i\)-jwhm \( k \) \iff \( X_1 \ast X_2 \) jwhm \( k \).

iv. \( X_1, X_2 \) are both whsp \( k \) if \( X_1 \ast X_2 \) is a whsp.

**Proof.** [Originating from \( \mathbb{E} \).] (i-iii) Use Th. \( \mathbb{E} \). \( \mathbb{E} \mathbb{E} \) and the weak directness \( k \) to transpose non-zeros from one side to the other, using Th. \( \mathbb{E} \mathbb{E} \) only for joins, i.e., in particular, with \( \epsilon = 0 \) or 1 depending on whether \( \vee = \times \) or \(* \) respectively, use

\[
(\hat{H}_{p+\epsilon}(X_1 \ast X_2, X_1 \ast X_2 \setminus o(x_i, x_j); k) \cong \hat{H}_{p}(X_1, X_1 \setminus o(x_i); k) \boxplus \hat{H}_{i}(X_1, X_1 \setminus o(x_i); k)) \cong \hat{H}_{p}(X_1, X_1 \setminus o(x_i); k)
\]

\( \delta \)

iv. Use, by the Five Lemma, the chain equivalence of the second component in the first and the last item of Theorem \( \mathbb{E} \) and the M-Vs with respect to \( (X \ast (Y \setminus o_0)), ((X \setminus o_0) \ast Y) \).

**Note.** 2. With \( \epsilon = 0 \) or 1 depending on whether \( \vee = \times \) or \(* \) respectively, Th. \( \mathbb{E} \mathbb{E} \) will handle any case encountered in practice, since e.g., for any module \( G \):

\[
(\hat{H}_{p+\epsilon}(X_1 \ast X_2, X_1 \ast X_2 \setminus o(x_i, x_j); G) \cong \hat{H}_{p}(X_1, X_1 \setminus o(x_i); G) \boxplus \hat{H}_{i}(X_1, X_1 \setminus o(x_i); G)) \cong \hat{H}_{p}(X_1, X_1 \setminus o(x_i); G)
\]

\( \cong \)

\( \boxplus \)

\( \mathbb{E} \mathbb{E} \mathbb{E} \)

**Note.** 3. If all the involved homology groups are of finite type with \( Z \) as coefficient module, then the Universal Coefficient Theorem implies that the above field-versions of Th. \( \mathbb{E} \mathbb{E} \) and Th. \( \mathbb{E} \mathbb{E} \mathbb{E} \) are true also with \( Z \) as coefficient module, cp. \( \mathbb{E} \mathbb{E} \mathbb{E} \mathbb{E} \) Proposition 2.4 p. 181-182.

**Definition.** \( \emptyset \) is said to be a homology \(-\infty\)-manifold and \( X = \bullet \) is a homology \( \emptyset \) manifold. Any other (than \( \emptyset \) or \( \bullet \)) connected, locally compact Hausdorff space \( X \in \mathbb{D} \) is a \( \emptyset \) manifold \( n\)-manifold \( (n\hsp) \) if

\[
\hat{H}_{i}(X, X \setminus o; G) = 0 \quad \text{if} \quad i \neq n \quad \forall \emptyset \neq x \in X,
\]

\( \emptyset \neq x \in X \)

\( (4.\text{i}) \)

\[
\hat{H}_{n}(X, X \setminus o; G) \cong 0 \quad \text{or} \quad G \quad \forall \emptyset \neq x \in X \text{and} = G \quad \text{for some} \ x \in X.
\]

\( (4.\text{ii}) \)

The boundary is defined to be: \( \text{Bd}_{G} X := \{ x \in X \mid \hat{H}_{n}(X, X \setminus o; G) = 0 \} \). If \( \text{Bd}_{G} X \neq \emptyset \), \( X \) is said to be a homology \( n\hsp \) manifold with boundary.

A compact \( n\hsp \) manifold \( S \) is orientable \( G \) if \( \hat{H}_{n}(S, \text{Bd} S; G) \cong G \). An \( n\hsp \) manifold is orientable \( G \) if all its compact \( n\hsp \) submanifolds are orientable; otherwise it is non-orientable \( G \). Orientability of \( \emptyset \) is not defined.

An \( n\hsp \) \( X \) is joinable if (4) holds also for \( x = \emptyset \).

An \( n\hsp \) \( X \neq \emptyset \) is a homology \( n\hsp \) sphere \( (n\hsp \text{sph}) \) if, for all \( x \in X \), \( \hat{H}_{i}(X, X \setminus o; G) = G \) if \( i = n \) and 0 otherwise. So, a triangulable \( n\hsp \) \( \text{sph} \) is a compact space.

When \( \vee \) in Theorem \( \mathbb{E} \mathbb{E} \) is interpreted throughout as \( \times \), the symbol “\( \text{hm} \)" on the r.h.s. of Th. \( \mathbb{E} \mathbb{E} \) temporarily excludes \( \emptyset \), \( \{ \emptyset \} \) and \( \bullet \), and we assume \( \epsilon := 0 \). When \( \vee \) is interpreted throughout as \( \ast \), put \( \epsilon := 1 \), and let the symbol “\( \text{hm} \)" on the right hand side of Th. \( \mathbb{E} \mathbb{E} \) be limited to “any compact joinable homology \( \hsp \) \( n\hsp \) manifold”.

Theorem 4.13. For compact triangulable spaces \(X_1, X_2\) and any field \(k\):
1. \(X_1 \circ X_2\) is a homology \(k\)-manifold if \(n_1 + n_2 + \epsilon\) and \(X_1\) is a \(n_i\)-homology manifold for \(i = 1, 2\).
2. \(\text{Bd}_k(\bullet \times X) = \bullet \times (\text{Bd}_k(X)).\) Else \(\text{Bd}_k((X_1 \circ X_2) = ((\text{Bd}_kX_1) \circ X_2) \cup (X_1 \circ (\text{Bd}_kX_2)).\)
3. \(X_1 \circ X_2\) is orientable if both \(X_1, X_2\) are orientable.

Proof. Th. 4.13 is trivially true for \(X \sim \bullet\) and \(X_1 \ast \{\emptyset\}\). Otherwise, exactly as in the proof for Theorem 4.12 above, adding for 4.13.1 that for Hausdorff-like spaces (all compact subsets are locally compact), in particular for Hausdorff spaces, \(X_1 \ast X_2\) is locally compact (Hausdorff) if and only if both \(X_1, X_2\) are compact (Hausdorff), cf. [3] p. 224.

Note. 4. \((X_1 \vee X_2, \text{Bd}_k(X_1 \vee X_2)) = \text{[Eq. 3 p. 8]} = (X_1, \text{Bd}_kX_1)(X_2, \text{Bd}_kX_2).

Test case. Using join and quotient to calculate the \(\tilde{H}\) homology groups of the \(\varnothing\)-figure. \((\tilde{H}_i(\varnothing, G) = 0\) if \(i \neq 1\)) Prop. iii p. 342 gives the second equality, the join-definition the two next, while Eq. 3 p. 24 and Lemma p. 13 take care of the rest in the following calculation:

\[
\tilde{H}_i(\varnothing, G) = \tilde{H}_i(\{\emptyset\}, \emptyset ; G) = \tilde{H}_i(\{\emptyset\}, \emptyset ; G) = [\text{Def. p. 13}] = \tilde{H}_i(\{\emptyset\}, \emptyset ; G) = [\text{Eq. 3 p. 24}] = \bigoplus_{i+j=0} \tilde{H}_i(\emptyset, \emptyset ; R) \otimes \tilde{H}_j(R) \otimes \tilde{H}_j(R) \otimes \emptyset ; G) = \tilde{H}_i(\emptyset, \emptyset ; R) \otimes \tilde{H}_j(R) \otimes \emptyset ; G) = [\text{Lemma p. 13}] = R \otimes (G \oplus G) = G \oplus G.
\]

Part 3. Merging Combinatorics, Topology & Commutative Algebra

5. RELATING GENERAL TOPOLOGY TO COMBINATORICS

5.1. Realizations for simplicial products and joins. The \(k\)-ification \(k(X)\) or \(X_k\) of \(X\) is \(X\) with its topology enlarged to the weak topology with respect to all continuous maps with Hausdorff domain. Generally, this topology differs from the \(k\)-ification \(\text{generated}\) enlargement \(X_C\), but they coincide for Hausdorff spaces. For any two simplicial complexes \(\Sigma_1, \Sigma_2\), \(|\Sigma_1| \times |\Sigma_2|, |\Sigma_1| \ast |\Sigma_2|\) and \(|\Sigma_1| \ast |\Sigma_2|\) are all Hausdorff. Put

\[X \times Y := k(X \times Y)\]

For CW-complexes, this is a proper topology-enlargement only if none of the two underlying complexes is locally finite and at least one is uncountable. Let \(X \ast Y\) be the quotient space with respect to \(p : (X \times Y) \times I \to X \circ Y\) from p. 17. Related subtleties are examined in [17] and [31]. Simplicial \(\times\) and \(\ast\) both “commute” with realization by turning into \(\tilde{\times}, \tilde{\ast}\), respectively. Unlike \(\ast\), Def. p. 17 \(\times\) is associative for arbitrary topological spaces, cf. [44] §3.

Definition 5.1. (cf. [14] Def. 8.8 p. 67.) Given ordered simplicial complexes \(\Delta\) and \(\Delta'\) i.e. the vertex sets \(V_\Delta\) and \(V_{\Delta'}\) are partially ordered so that each simplex becomes linearly ordered resp. The Ordered Simplicial Cartesian Product \(\Delta \times \Delta'\) of \(\Delta\) and \(\Delta'\) (triangulates \(|\Delta| \times |\Delta'|\) and is defined as

\[V_{\Delta \times \Delta'} := \{(v_i, v'_j)\} = V_\Delta \times V_{\Delta'}\]

Put \(w_{i,j} := (v_i, v'_j)\). Simplices in \(\Delta \times \Delta'\) are sets \(\{w_{i_0,j_0}, w_{i_1,j_1}, ..., w_{i_k,j_k}\}\), with \(w_{i_0,j_0} \neq w_{i_s,j_s+1}\) and \(v_{i_0} \leq v_{i_1} \leq ... \leq v_{i_k}\) \(v'_{j_0} \leq v'_{j_1} \leq ... \leq v'_{j_k}\), where \(v_{i_0}, v'_{j_0}, v_{i_1}, v'_{j_1}, ..., v_{i_k}, v'_{j_k}\) (resp. \(v'_{j_0}, v'_{j_1}, ..., v'_{j_k}\)) is a sequence of vertices, with repetitions possible, constituting a simplex in \(\Delta'\) (resp. \(\Delta''\)).
Lemma 5.2. (cf. [4] p. 68.) If \( p_i : \Sigma_i \times \Sigma_i \rightarrow \Sigma_i \) is the simplicial projection, then 
\[
\eta := (|p_1|, |p_2|), \quad |\Sigma_1 \times \Sigma_2| \hookrightarrow |\Sigma_1| \times |\Sigma_2|
\]
triangulates \( |\Sigma_1| \times |\Sigma_2| \).

If \( L_1 \) and \( L_2 \) are subcomplexes of \( \Sigma_1 \) and \( \Sigma_2 \), then \( \eta \) carries \( |L_1 \times L_2| \) onto \( |L_1| \times |L_2| \).

This triangulation has the property that, for each vertex \( B \) of \( \Sigma_2 \), say, the correspondence
\[
x \rightarrow (x, B)
\]
is a simplicial map of \( \Sigma_1 \) into \( \Sigma_1 \times \Sigma_2 \).

Similarly for joins, with \( \eta : |\Sigma_1 \ast \Sigma_2| \hookrightarrow |\Sigma_1| \ast |\Sigma_2| \) defined in the proof.

Proof. \((\ast)\). The simplicial projections \( p_i : \Sigma_i \times \Sigma_i \rightarrow \Sigma_i \) gives realized continuous maps \( |p_i| \), for \( i = 1, 2 \). The map \( \eta := (|p_1|, |p_2|): |\Sigma_1 \times \Sigma_2| \rightarrow |\Sigma_1| \times |\Sigma_2| \) is bijective and continuous, cf. [4] 2.5.6 p. 32 + Ex. 12, 14 p. 106-7. The topology \( \tau_{p_1 \times p_2} \) (resp. \( \tau_{p_1 \times p_2} \)) is the weak topology with respect to the compact subspaces \( \{|\Sigma_1 \times \Sigma_2|\}_{\gamma \subseteq \Sigma_i} \) (resp. \( \{\mid \Sigma_1 \times \Sigma_2\mid\}_{\gamma \subseteq \Sigma_i} \), cf. [8] p. 246 Prop. A.2.1. Since,
\[
(|p_1|, |p_2|)(|\Sigma_1 \times \Sigma_2| \cap A) = (|\Sigma_1| \times |\Sigma_2|) \cap (|p_1|, |p_2|)(A),
\]
the inverse \((|p_1|, |p_2|)^{-1} \) is continuous, i.e., \((|p_1|, |p_2|) \) is a homeomorphism. \(\star\) This follows from [13] p. 99, using the map \( \eta \) below. With \( \Sigma_i \ast \Sigma_i := \{\sigma \cup \sigma \mid \sigma \in \Sigma_i, \text{ for } i = 1, 2\} \), and if \( \sigma = \{v_1', \ldots, v_i', v_i'' \ldots, v_i'\} \in V_{\Sigma_1 \ast \Sigma_2} \), then put
\[
\{t_{i_1} v_1', \ldots, t_{i_j} v_j', t_{i_j+i_1} v_j', \ldots, t_{i_j+i_1} v_j'' \} := \alpha_{\sigma} : V_\Sigma \rightarrow [0, 1];
\]
\[
\alpha_{\sigma}(v) = \begin{cases}
    t_v & \text{if } v \in \sigma, \\
    0 & \text{otherwise}.
\end{cases}
\]

Set \( t' := \sum_{i \leq q} t_i \) and \( t'' := 1 - t' = \sum_{q+1 \leq j \leq q+r} t_j \). Now:
\[
\eta : |\Sigma_i \ast \Sigma_i| \hookrightarrow |\Sigma_1| \ast |\Sigma_2| ; \quad \{v_1', \ldots, v_i', v_i'' \ldots, v_i'\} \mapsto
\]
\[
\mapsto (t', \{t_{i_1} v_1', \ldots, t_{i_j} v_j', t_{i_j+i_1} v_j', \ldots, t_{i_j+i_1} v_j'' \}, t''(\frac{t_{i_1} v_1''}{t'}, \ldots, \frac{t_{i_j} v_j''}{t'})) \}
\]
\[
\text{where } (tx, (1-t)y) := (x, y, t).
\]

(50) (3.3) p. 59 is useful, when deling moore explicitly with the compact subspaces of \( |\Sigma_1| \ast |\Sigma_2| \).
with the join of simplicial sets, \( \ast \), as well as the motivations taken from [17].

The Milnor realization \( \hat{\Xi} \) of any simplicial set \( \Xi \) is triangulable by \([18]\) p. 209 Cor. 4.6.12. E.g., the augmental singular complex \( \Delta(X) \) with respect to any topological space \( X \), is a simplicial set and, cf. \([19]\) p. 362 Th. 4, the map \( j : \hat{\Delta(X)} \to X \) is a weak homotopy equivalence i.e. induces isomorphisms in homotopy groups, and \( j \) is a true homotopy equivalence if \( X \) is of CW-type, cf. \([18]\) pp. 76–77, 170, 189ff, 221–2. Also the identity map \( k(X) \equiv X \) is at least a weak homotopy equivalence.

**5.2. Local homology for simplicial products and joins.** Lemma \([5, 2] \) p. 25 and \([11]\) Th. 12.4 p. 89, also implies that

\[
\eta : (|\Sigma| \ast |\Delta|) / o(\alpha_1, \alpha_2) \xrightarrow{\sim} (|\Sigma| \ast |\Delta|) \setminus o(\alpha_1, \alpha_2)
\]

is a homeomorphism if \( \eta(\alpha_1, \alpha_2) = (\alpha_1, \alpha_2) \).

We conclude, cf. \([11]\) p. 99, that the following four topological spaces

\[
k(|\Sigma| \ast |\Delta|) = |\Sigma| \ast |\Delta| = |\Sigma \ast \Delta| = k(|\Sigma| \ast |\Delta|),
\]

are homeomorphic, since the underlying spaces are all Hausdorff and their respective compact subspaces are homeomorphic, i.e., they are homeomorphic compactly generated Hausdorff spaces.

Moreover, if \( \Delta \subset \Sigma' \), \( \Delta' \subset \Sigma'' \) then, \( |\Sigma'| \ast |\Delta'| \) is a subspace of \( |\Sigma| \ast |\Delta''| \) and \( \dim(\Sigma \ast |\Delta|) = \dim \Sigma + \dim |\Delta| \) while \( \dim(\Sigma \ast |\Delta|) = \dim \Sigma + \dim |\Delta| + 1 \).

With \( \alpha_i \in \text{Int}\sigma_i \subset |\Sigma_i| \) and

\[
(\sigma_1, \sigma_2) := \eta^{-1}(\alpha_1, \alpha_2) \in \text{Int}\sigma \subset |\Sigma_1 \times \Sigma_2|, \quad c_\sigma := \dim \sigma_1 + \dim \sigma_2 - \dim \sigma,
\]

we conclude that, \( c_\sigma \geq 0 \) and \( c_\sigma = 0 \) if and only if \( \sigma \) is a maximal simplex in \( \sigma_1 \times \sigma_2 \subset \Sigma_1 \times \Sigma_2 \).

**Corollary 5.3.** (to \([18, 14, 17]\). Let \( G, G' \) be arbitrary modules over a PID \( R \) such that \( \text{Tor}_1^R(G, G') = 0 \), then, for any \( \emptyset \neq \sigma \in \Sigma_1 \times \Sigma_2 \) with \( \eta(\text{Int}(\sigma)) \subset \text{Int}(\sigma_1) \times \text{Int}(\sigma_2) \),

\[
\hat{\text{H}}_{i+\sigma_1}(\text{Lk}_{\Sigma_1 \times \Sigma_2}(\sigma; G \otimes_R G')) \cong \\
\hat{\text{H}}_i(\text{Lk}_{\Sigma_1 \times \Sigma_2}(\sigma_1; G)) \otimes_R \hat{\text{H}}_i(\text{Lk}_{\Sigma_1 \times \Sigma_2}(\sigma_2; G')) \cong \\
\bigoplus_{p+q=1}^{p+q \leq -1} \text{Tor}_1^R(\hat{\text{H}}_p(\text{Lk}_{\Sigma_1}(\sigma_1; G) \otimes_R \hat{\text{H}}_q(\text{Lk}_{\Sigma_2}(\sigma_2; G)) \cong \\
\hat{\text{H}}_{i+1}(\text{Lk}_{\Sigma_1 \times \Sigma_2}(\sigma_1 \cup \sigma_2; G) \otimes_R G')).
\]

So, if \( \emptyset \neq \sigma \) and \( c_\sigma = 0 \) then

\[
\hat{\text{H}}_i(\text{Lk}_{\Sigma_1}(\sigma_1; G) \cong \hat{\text{H}}_i(\text{Lk}_{\Sigma_1}(\sigma_1 \cup \sigma_2; G))
\]

and, since \( \text{Tor}_1^R(G, G') = 0 \),

\[
\hat{\text{H}}_0(\text{Lk}_{\Sigma_1 \times \Sigma_2}(\sigma; G \otimes G')) \cong \\
\hat{\text{H}}_0(\text{Lk}_{\Sigma_1}(\sigma_1; G) \otimes \hat{\text{H}}_1(\text{Lk}_{\Sigma_2}(\sigma_2; G')) \oplus \hat{\text{H}}_1(\text{Lk}_{\Sigma_1}(\sigma_1; G) \otimes \hat{\text{H}}_0(\text{Lk}_{\Sigma_2}(\sigma_2; G')).
\]
Proof. Note that \( \sigma \neq \emptyset_0 \Rightarrow \sigma \neq \emptyset_o \), \( j = 1, 2 \). The isomorphisms of the underlined modules are, by Proposition \( 1.11 \) p. 24 just a simplicial version of Theorem \( 4.11 \) p. 24 and holds even without the PID-assumption. Prop. \( 4.11 \) p. 24, and Theorem \( 4.3 \) p. 24, gives the second isomorphism even for \( \sigma_1 = \emptyset_o \) and/or \( \sigma_2 = \emptyset_o \). □

The above module homomorphisms concerns only simplicial homology, so, it should be possible to prove them purely in terms of simplicial homology. This is, however, a rather cumbersome task, mainly due to the fact that \( \Sigma_1 \times \Sigma_2 \) is not a subcomplex of \( \Sigma_1 \ast \Sigma_2 \).

A somewhat more explicit proof is provided in p. 24.

Lemmas \( 5.3 \) and \( 5.6 \) below are related to the defining properties for quasi-manifolds, resp. pseudomanifolds, cf. p. 172. We write out all seven items mainly just to be able to see the details once and for all.

In the \( * \)-case, lemma \( 5.4 \) below is trivially true if any \( \Sigma_i = \{ \emptyset \} \). “codim\( \sigma \geq 2 \)” (\( \Rightarrow \dim \text{Lk}_O \sigma \geq 1 \)) means that a maximal simplex, say \( \tau \), containing \( \sigma \) fulfills \( \dim \tau \geq \dim \sigma + 2 \).

**Lemma 5.4.** Read \( \vee \) below as “\( \times \)” or throughout as “\( * \)”. When \( \vee \) is \( * \)-substituted, \( \Sigma_i \) is assumed to be connected or 0-dimensional. \( G_1, G_2 \) are \( A \)-modules such that \( \text{Tor}_1^A(G_1, G_2) = 0 \). Now, if \( \dim \Sigma_i \geq 0 \) and \( v_i := \dim \sigma_i (i = 1, 2) \) then \( D_1 - D \) are all equivalent;

( \( \text{Int}(\sigma) := \{ \alpha \in |\Sigma| \mid [v \in \sigma] \leftrightarrow [\alpha(v) \neq 0]\} \) and \( \bar{\sigma} := \{ \tau \mid \tau \subseteq \sigma \} \).

\( D_1 \) \( \hat{H}_0(\text{Lk}_{\Sigma_i \vee \Sigma_2})G_1 \otimes G_2) = 0 \) for \( \emptyset_o \neq \sigma \in \Sigma_1 \vee \Sigma_2 \), whenever \( \text{codim}\sigma \geq 2 \).

\( D_1' \) \( \hat{H}_0(\text{Lk}_{\Sigma_i}G_i) = 0 \) for \( \emptyset_o \neq \sigma \in \Sigma_i \), whenever \( \text{codim}\sigma_i \geq 2 \) (\( i = 1, 2 \)).

\( D_2 \) \( \hat{H}_{i+1}(\Sigma_1 \vee \Sigma_2, \text{cost}_{\Sigma_1 \vee \Sigma_2}\sigma; G_1 \otimes G_2) = 0 \) for \( \emptyset_o \neq \sigma \in \Sigma_1 \vee \Sigma_2 \), if \( \text{codim}\sigma \geq 2 \).

\( D_2' \) \( \hat{H}_{i+1}(\Sigma_i, \text{cost}_{\Sigma_i}\sigma; G_i) = 0 \) for \( \emptyset_o \neq \sigma \in \Sigma_i \), whenever \( \text{codim}\sigma_i \geq 2 \) (\( i = 1, 2 \)).

\( D_3 \) \( \hat{H}_{i+1}(|\Sigma_1 \vee \Sigma_2|, |\Sigma_1 \vee \Sigma_2| \setminus \alpha; G_1 \otimes G_2) = 0 \) for all \( \alpha_0 \neq \alpha \in \text{Int}(\sigma) \) if \( \text{codim}\sigma \geq 2 \).

\( D_3' \) \( \hat{H}_{i+1}(|\Sigma_i|, |\Sigma_i| \setminus \alpha_i; G_i) = 0 \) for all \( \alpha_0 \neq \alpha_i \in \text{Int}(\sigma_i) \), if \( \text{codim}\sigma_i \geq 2 \) (\( i = 1, 2 \)).

\( D \) \( \hat{H}_{i+1}(|\Sigma_1 \vee \Sigma_2|, |\Sigma_1 \vee \Sigma_2| \setminus (\alpha_1, \alpha_2); G_1 \otimes G_2) = 0 \) for all \( \alpha_0 \neq (\alpha_1, \alpha_2) \in \text{Int}(\sigma) \) \( \subset |\Sigma_1 \vee \Sigma_2| \) if \( \text{codim}\sigma \geq 2 \), where \( \eta: |\Sigma_1 \vee \Sigma_2| \xrightarrow{\sim} |\Sigma_i \vee \Sigma_2| \) and \( \eta(\alpha_1, \alpha_2) = (\alpha_1, \alpha_2) \) (k-ifications never effect the homology modules).

Proof. By the homogeneity of the interior of \( |\bar{\sigma} \times \bar{\sigma}| \) we only need to deal with simplices \( \sigma \) fulfilling \( c = 0 \). By Prop. 1 p. 25 and Eq. 3 above, all non-primed, resp. primed, items are equivalent among themselves. \( D_1 \Leftrightarrow D_1' \) by Corollary \( 5.3 \) above, since \( \text{Tor}_1^A(G_1, G_2) = 0 \). For joins of finite complexes, this is done explicitly in p. 172. □
5.3. Connectedness for simplicial complexes.

Any $\Sigma$ is representable as $\Sigma = \bigcup_{\sigma^m \in \Sigma} \sigma^m$, where $\sigma^m$ denotes the simplicial complex generated by the maximal simplex $\sigma^m$.

**Definition 5.5.** Two maximal faces $\sigma, \tau \in \Sigma$ are strongly connected if they can be connected by a finite sequence $\sigma = \delta_1, \ldots, \delta_i = \tau$ of maximal faces with $\#(\delta_i \cap \delta_{i+1}) = \max_{\sigma_j \subseteq \delta_i} \# \delta_j - 1$ for consecutives. Strong connectedness imposes an equivalence relation among the maximal faces, the equivalence classes of which defines the maximal strongly connected components of $\Sigma$, cf. [3] p. 419ff. A complex $\Sigma$ is said to be strongly connected if each pair of its maximal simplices are strongly connected. A submaximal face has exactly one vertex less then some maximal face containing it.

Note that strongly connected complexes are pure, i.e., $\sigma \in \Sigma$ maximal $\Rightarrow$ $\dim \sigma = \dim \Sigma$.

**Lemma 5.6.** (This lemma concerns the defining properties for pseudomanifolds. [3] p. 81 gives a proof, valid for finite-dimensional complexes.)

A) If $d_i := \dim \Sigma_i \geq 0$ then $\Sigma_1 \times \Sigma_2$ is pure $\iff$ $\Sigma_1$ and $\Sigma_2$ are both pure.

B) If $\dim \sigma^m_i \geq 1$ for each maximal simplex $\sigma^m_i \in \Sigma_i$ then

Any submaximal face in $\Sigma_1 \times \Sigma_2$ lies in at most (exactly) two maximal faces $\iff$ Any submaximal face in $\Sigma_i$ lies in at most (exactly) two maximal faces of $\Sigma_i$, where $i = 1, 2$.

C) For $d_i > 0$: $\Sigma_1 \times \Sigma_2$ strongly connected $\iff$ $\Sigma_1, \Sigma_2$ both strongly connected.

**Note 5.7.** Lemma 5.6 is true also for $*$ with exactly the same reading but now with no other restriction than that $\Sigma \neq \emptyset$ and this includes, in particular, heading B.

**Definition 5.8.** $\Delta \setminus \Delta := \{ \delta \in \Delta \mid \delta \not\in \Delta \}$ is connected as a poset (partially ordered set) with respect to simplex inclusion if, for every pair $\sigma, \tau \in \Delta \setminus \Delta$, there is a chain $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_\ell = \tau$, where $\sigma_i \in \Delta \setminus \Delta$ and $\sigma_i \subset \sigma_{i+1}$ or $\sigma_i \supset \sigma_{i+1}$.

**Note 5.9.** (cf. [3] p. 162.) Let $\Delta \supseteq \Delta \supseteq \{0\}$. $\Delta \setminus \Delta$ is connected as a poset iff $|\Delta| \setminus \{0\}$ is pathwise connected. When $\Delta = \{0\}$, then the notion of connectedness as a poset is equivalent to the usual one for $\Delta$. Let $\Delta^\emptyset := \{ \sigma \in \Delta \mid \# \sigma \leq p + 1 \}$; then $|\Delta|$ is connected iff $|\Delta^\emptyset|$ is connected.

**Lemma 5.10.** ([3] p. 163) $\Delta \setminus \Delta$ is connected as a poset iff for each pair of maximal simplices $\sigma, \tau \in \Delta \setminus \Delta$, there is a chain, $\sigma = \sigma_0 \supseteq \sigma_1 \supseteq \sigma_2 \supseteq \ldots \supseteq \sigma_m = \tau$ in $\Delta \setminus \Delta$, where the $\sigma_i$s are maximal faces and $\sigma_{2i} \setminus \sigma_{2i+1}$ and $\sigma_{2i+2} \setminus \sigma_{2i+1}$ are situated in different components of $\text{Lk}_\Sigma \sigma_{2i+1}$ ($i = 0, 1, \ldots, m-1$).

**Lemma 5.11.** (A. Björner 1995.) (A direct consequence of Lemma 5.10.) Let $\Sigma$ be a finite-dimensional simplicial complex, and assume that $\text{Lk}_\Sigma \sigma$ is connected for all $\sigma \in \Sigma$, including $\emptyset \in \Sigma$, such that $\dim \text{Lk}_\Sigma \sigma \geq 1$. Then $\Sigma$ is pure and strongly connected.
6. Relating Combinatorics to Commutative Algebra

6.1. Definition of Stanley-Reisner rings.

---***--- What we are aiming at. ---***---

§6.2. Through the Stanley-Reisner Functor below, attributes like Buchsbaum, Cohen-Macaulay and Gorenstein on (graded) rings and algebras (13H10 in MSC2000), became relevant also within Combinatorics as the classical definition (Def. 2 p. 3) of simplicial complexes was altered to Def. 1 p. 3. Now this extends to General/Algebraic Topology, cf. p. 34. Theorem 6.10 p. 46 indicates that simplicial homology manifolds can be inductively generated.

§7.2. Corollary 7.1 p. 48, tells us that there is no n-manifold (≠ •) with an (n − 2)-dimensional boundary. The examples on p. 44 implicitly raise the arithmetic-geometrical question: Which boundary-dimensions are accessible for quasi-n-manifolds with respect to different coefficient-modules?

§7.4. Corollary 7.10 p. 72 confirms Bredon’s conjecture in 3 p. 384, that homology manifolds with Z as coefficient-module are locally orientable, and so, these manifolds have either an empty, a (−1)-dimensional or an (n − 1)-dimensional boundary.

---***--- ---***---

Definition 6.1. A subset \( s \subset W \supset V_\Delta \) is said to be a non-simplex (with respect to \( W \)) of a simplicial complex \( \Delta \), denoted \( s \preceq \Delta \), if \( s \not\preceq \Delta \) but \( s = (\bar{s})^{(\text{dim } s)−1} \subset \Delta \) (i.e., the \((\text{dim } s − 1)\)-dimensional skeleton of \( \bar{s} \), consisting of all proper subsets of \( s \), is a subcomplex of \( \Delta \)). For a simplex \( \bar{s} = \{v_i, \ldots, v_k\} \) we define \( m_{\bar{s}} \) to be the square-free monic monomial \( m_{\bar{s}} := 1_A v_i \cdots v_k \in A[W] \) where \( A[W] \) is the graded polynomial algebra on the variable set \( W \) over the commutative ring \( A \) with unit \( 1_A \). So, \( m_{\bar{s}} = 1_A \).

Let \( A(\Delta) := A[W]/I_\Delta \) where \( I_\Delta \) is the ideal generated by \( \{m_{\bar{s}} \mid \delta \preceq \Delta \} \). \( A(\Delta) \) is the face ring or Stanley-Reisner (St-Re) ring of \( \Delta \) over \( A \). Frequently \( A = k \), a field, e.g. a prime field, the real numbers \( \mathbb{R} \) or the rational numbers \( \mathbb{Q} \).

Stanley-Reisner (St-Re) ring theory is a basic tool within combinatorics, where it supports the use of commutative algebra. The generating set for the ideal \( I_\Delta \) above is minimal, while in the traditional definition \( I_\Delta \) is generated by all squarefree monomials \( p_{\bar{s}} = v_i \cdots v_k \in A[W] \) such that \( \{v_i, \ldots, v_k\} \not\preceq \Delta \), i.e. \( I_\Delta = \{p_{\bar{s}} \mid \delta \not\preceq \Delta \} \). This latter definition of \( p_{\bar{s}} \) in \( I_\Delta = \{p_{\bar{s}} \mid \delta \not\preceq \Delta \} \) leaves the questions “\( A(\{0\}) \equiv \)?” and “\( A(\emptyset) = \)?” open, while Note 6.2. ii below, shows that: \( A(\{0\}) \equiv A \neq 0 = \text{“The trivial ring”} = A(\emptyset) \).

Note 6.2. i. Let \( \mathcal{P} \) be the set of all finite subsets of a set \( \mathcal{P} \), like the \( \bar{s} \) above. Now, \( A(\mathcal{P}) = A[\mathcal{P}] \). \( \mathcal{P} \) is known as the “the full simplicial complex on \( \mathcal{P} \)”. The set of natural numbers \( \mathbb{N} \) gives \( \mathcal{N} \), namely the “infinite simplex”.

Another useful concept is the costar, defined by \( \text{cost}_{\Sigma} \sigma := \{\tau \in \Sigma \mid \tau \not\supset \sigma \} \). We have \( \Delta = \bigcap_{\Sigma \in A} \text{cost}_\Sigma \), for any universe \( W \) containing the vertex set \( V_\Delta \).

The cone points in a simplicial complex \( \Delta \) are those vertices \( v \in V_\Delta \) that are contained in every maximal simplex of \( \Delta \). So, \( \hat{\Delta} := \{v \in V_\Delta \mid v \text{ cone point} \} \in \Delta \). The set of cone points in \( \Delta \) is also characterized as the intersection of all maximal simplices in \( \Delta \), and thus resembling the \textit{radical} in algebra.

Many more basic simplicial constructions are collected in pp. 44ff.
ii. \( A(\Delta) \cong \frac{A[\mathcal{V}]}{\{m_\delta \in A[\mathcal{V}] \mid \delta \vartriangleleft \Delta\}} \) if \( \Delta \neq \emptyset, \{\emptyset\} \). So, the choice of the universe \( W \) isn’t all that critical.

If \( \Delta = \emptyset \), then the set of non-simplices equals \( \{\emptyset\} \), since \( \emptyset \vartriangleleft \emptyset \), and \( \emptyset_{(\dim \emptyset - 1)} = \{\emptyset\}^{(-2)} = \emptyset \subset \emptyset \), implying \( A(\emptyset) = 0 = \text{“The trivial ring” since } m_\emptyset = 1_A \).

Since \( \emptyset \in \Delta \) for every simplicial complex \( \Delta \neq \emptyset \), \( \{v\} \) is a non-simplex of \( \Delta \) for every \( v \in W \setminus V_\Delta \), i.e.,

\[ [v \not\in V_\Delta \neq \emptyset] \iff [\{v\} \vartriangleleft \Delta \neq \emptyset]. \]

So \( A(\{\emptyset\}) = A \) since \( \{\delta \mid \delta \vartriangleleft \Delta\} = W \).

iii. \( k(\Delta_1 \vartriangleleft \Delta_2) \cong k(\Delta_1) \otimes k(\Delta_2) \) (R. Fröberg, 1988) with \( I_{\Delta_1 \vartriangleleft \Delta_2} = (\{m_\delta \mid [\delta \vartriangleleft \Delta_1 \vartriangleleft \Delta_2] \land [\delta \not\in \Delta_1 \vartriangleleft \Delta_2]\}) \) in \( A[W] \).

The Stanley-Reisner ring assignment functor defines a monomorphism on distributive lattices \( (\Sigma^0_W, \cup, \cap, \{\emptyset\}) \rightarrow (F[W], \cap, +, I) \), which is an isomorphism for finite \( W \).

6.2. Buchsbaum complexes are weak manifolds. Combinatorialists call a finite simplicial complex a \textit{Buchsbaum} (Bbm) \( \text{(Cohen-Macaulay} (CM) \text{, Gorenstein} (Gor)) \) complex if its Stanley-Reisner ring is a Buchsbaum (Cohen-Macaulay, Gorenstein) ring. The following three propositions, found in \([16]\) together with proof-references, can be used as "definitions" when restricted to finite simplicial complexes. The proof of Proposition \([6.4]\) iv is given by A. Björner as an appendix in \([17]\).

Proposition 6.4. (Schenzel) \([14]\) Th. 8.1 \( \Sigma \) be a finite simplicial complex and let \( k \) be a field. Then the following are equivalent:

(i) \( \Sigma \) is Buchsbaum over \( k \).

(ii) \( \Sigma \) is pure, and \( k(\Sigma)_p \) is Cohen-Macaulay for all prime ideals \( p \) different from the unique homogeneous maximal ideal i.e., the irrelevant ideal.

(iii) For all \( \sigma \in \Sigma, \sigma \neq \emptyset \) and \( i < \dim(Lk_{\Sigma}\sigma), H_i(Lk_{\Sigma}\sigma; k) = 0 \).

(iv) For all \( \alpha \in |\Sigma|, \alpha \neq \alpha_0 \) and \( i < \dim(\Sigma, H_i(|\Sigma|, |\Sigma| \setminus \alpha; k) = 0. \)
Proposition 6.5. (10 Prop. 4.3, Cor. 4.2) Let $\Sigma$ be a finite simplicial complex and let $k$ be a field. Then the following are equivalent:

(i) $\Sigma$ is Cohen-Macaulay over $k$.

(ii) (Reisner) For all $\sigma \in \Sigma$ and $i < \dim(Lk_\sigma \Sigma; k) = 0$.

(iii) (Munkres) For all $\alpha \in \vert \Sigma \vert$ and $i < \dim \Sigma, \check{H}_i(\Sigma, \vert \Sigma \vert \setminus \alpha; k) = 0$. $\square$

Proposition 6.6. (10 Th. 5.1) Let $\Sigma$ be a finite simplicial complex, $k$ a field or $\mathbb{Z}$ and $\Gamma := \text{core}\Sigma := Lk_\Sigma \delta_\Sigma$. Then the following are equivalent:

(i) $\Sigma$ is Gorenstein over $k$.

(ii) For all $\sigma \in \Gamma$, $\check{H}_i(Lk_\sigma \Sigma; k) \cong \begin{cases} k & \text{if } i = \dim(Lk_\sigma \Sigma), \\
0 & \text{if } i < \dim(Lk_\sigma \Sigma) \end{cases}$

(iii) For all $\alpha \in \vert \Gamma \vert$, $\check{H}_i(\vert \Gamma \vert, \vert \Gamma \vert \setminus \alpha; k) = \begin{cases} k & \text{if } i = \dim \Gamma, \\
0 & \text{if } i < \dim \Gamma. \end{cases}$

(iv) (A. Björner) $\Sigma$ is Gorenstein $k \iff \Sigma$ is C-M$_k$, and $\Gamma$ is an orientable pseudomanifold without boundary.

(v) Either (1) $\Sigma = \{\emptyset\}, \bullet, \bullet\bullet$, or (2) $\Sigma$ is Cohen-Macaulay over $k$, $\dim \Sigma \geq 1$, and the link of every $(\dim(\Sigma) - 2)$-face is either a circle or a line with two or three vertices and $\check{\chi}(\Gamma) = (-1)^{\dim \Gamma}$. $\square$

Through Prop. 4.10 p. 23, the following definitions for arbitrary modules and topological spaces are consistent with the original.

Definition 6.7. $X$ is “Bbm$_{\mathbb{G}}$” (“CM$_{\mathbb{G}}$”, 2-“CM$_{\mathbb{G}}$”) if $X$ is an $n$-whm$_{\mathbb{G}}$ ($n$-jwhm$_{\mathbb{G}}$, $n$-whsp$_{\mathbb{G}}$).

A simplicial complex $\Sigma$ is defined to be “Bbm$_{\mathbb{G}}$”, “CM$_{\mathbb{G}}$” resp. 2-“CM$_{\mathbb{G}}$” if $|\Sigma|$ is. In particular, $\Delta$ is 2-“CM$_{\mathbb{G}}$” iff $\Delta$ is “CM$_{\mathbb{G}}$” and $\check{H}_i(\text{cost}_\delta; \mathbb{G}) = 0, \forall \delta \in \Delta$, cf. 10 Prop. 3.7 p. 94.

(The $n$ in “n-Bbm$_{\mathbb{G}}$” (“n-CM$_{\mathbb{G}}$” resp. 2-“n-CM$_{\mathbb{G}}$”) is deleted since any interior point $\alpha$ of a realization of a maxi-dimensional simplex gives $\check{H}_i(\text{cost}_\delta; \mathbb{G}) = 0, \forall \delta \in \Delta$, cf. 10 Prop. 3.7 p. 94.)

Recall that $\sigma \in \Sigma$ is said to be maxi-dimensional if $\dim \sigma = \dim \Sigma$. The $|\{\emptyset\}|$-extended “$\setminus$” is defined in p. 13.

So we are simply renaming $n$-whm$_{\mathbb{G}}$, $n$-jwhm$_{\mathbb{G}}$ and $n$-whsp$_{\mathbb{G}}$ to “Bbm$_{\mathbb{G}}$”, “CM$_{\mathbb{G}}$” resp. 2-“CM$_{\mathbb{G}}$”, where the quotation marks indicate that we are not limited to compact spaces nor to just $\mathbb{Z}$ or $k$ as coefficient modules.

N.B. The definition p. 30 of $|\Delta(X)|$ provides each topological space $X$ with a Stanley-Reisner ring, which, with respect to “Bbm$_{\mathbb{G}}$”, “CM$_{\mathbb{G}}$”- and 2-“CM$_{\mathbb{G}}$”-ness, is triangulation invariant.

Proposition 6.8. The following conditions are equivalent:

a. $\Delta$ is “Bbm$_{\mathbb{G}}$”.

b. (Schenzel, 27 p. 96) $\Delta$ is pure and $Lk_\delta \Delta$ is “CM$_{\mathbb{G}}$” for all $\emptyset \neq \delta \in \Delta$.

c. (Reisner, 8 5.3.16.(b) p. 229) $\Delta$ is pure and $Lk_v \Delta$ is “CM$_{\mathbb{G}}$” for all $v \in V_\Delta$.

Proof. Use Prop. 4.10 p. 23, and Lemma 5.11 p. 32, and then use Eq. 1 p. 54. $\square$

Example. When limited to compact polytopes and a field $k$ as coefficient module, we add, from 10 p. 73, the following Buchsbaum-equivalence using local cohomology;
\( \Delta \) is Buchsbaum iff \( \dim \mathbb{H}^i_{k(\Delta)}(k(\Delta)) \leq \infty \) if \( 0 \leq i < \dim k(\Delta) \), in which case \( \mathbb{H}^i_{k(\Delta)}(k(\Delta)) \cong \mathbb{H}^i_{k(\Delta)}(\Delta; k) \); for proof cf. [17] p. 144. Here, “dim” is the Krull dimension, which for Stanley-Reisner rings is simply “1 + the simplicial dimension”.

For \( \Gamma_1, \Gamma_2 \) finite and \( CM_k \), a K"unneth formula for ring theoretical local cohomology follows; (“+” indicates the unique homogeneous maximal ideal i.e., the irrelevant ideal of “-“.)

1. \( \mathbb{H}^i_{(k(\Gamma_1) \otimes k(\Gamma_2))} \cong \mathbb{H}^i_{k(\Gamma_1)}(k(\Gamma_2)) \)

Motivation: \( k(\Gamma) \cong k(\Gamma \setminus 0) \) by Note 3.3 in p. [4] and [17] p. 144 and Theorem 4.12 in p. [4].

\[ \cong \bigoplus_{i+j=q} \mathbb{H}^i_{k(\Gamma_1)}(k(\Gamma_2)). \]

2. Put:
\[ \beta_{i}(X) := \inf \{ j \mid \exists x \in X \land \mathbb{H}_j(X, X \setminus x; G) \neq 0 \}. \]

For a finite \( \Delta, \beta_{i}(\Delta) \) is related to the concepts “depth of the ring \( k(\Delta) \)” and “C-M-ness of \( k(\Delta) \)” through \( \beta_{i}(\Delta) = \text{depth}(k(\Delta)) - 1 \), in [3] Ex. 5.1.23 p. 214 and [10] p. 142 Ex. 34. See also [21] and [14].

Proposition 6.9. a) \( [\Delta \text{ is } CM_k] \Leftrightarrow [\mathbb{H}_i(\Delta, \text{cost}_{\delta}; G) = 0 \ \forall \delta \in \Delta \text{ and } i \leq n - 1] \).

b) \( [\Delta \text{ is } 2-CM_k] \Leftrightarrow [\mathbb{H}_i(\text{cost}_{\delta}; G) = 0 \ \forall \delta \in \Delta \text{ and } i \leq n - 1] \).

Proof. Use the LHS (long homology sequence) with respect to \( (\Delta, \text{cost}_{\delta}) \), Prop. 4.10 p. 239 and the fact that \( \text{cost}_{\Delta_0} = 0 \) resp. \( \text{cost}_{\delta} = \Delta \) if \( \delta \notin \Delta \).

Put
\[ \Delta^{(0)} := \{ \delta \in \Delta \mid \# \delta \leq p + 1 \}, \quad \Delta^{(p)} := \Delta^{(p)} \setminus \Delta^{(p - 1)} \quad \Delta^{(n - 1)} := \Delta^{(n - 1)}, \quad n := \dim \Delta. \]

So, in particular \( \Delta^{(0)} = \Delta \).

Theorem 6.10.

a) \( \Delta \text{ is } CM_k \iff \Delta^{(0)} = 2 - CM_k \text{ and } \mathbb{H}_i(\Delta, \text{cost}_{\delta}; G) = 0 \text{ for all } \delta \in \Delta. \)

b) \( \Delta \text{ is } CM_k \iff \Delta^{(0)} = 2 - CM_k \text{ and } \mathbb{H}_i(\text{cost}_{\delta}; G) = 0 \text{ for all } \delta \in \Delta. \)

Proof. Proposition 3.9 above together with the fact that adding or deleting \( n \)-simplices does not effect homology groups of degree \( \leq n - 2 \). See Proposition 3.3 e p. [3].

Theorem 6.10 is partially deducible, using commutative algebra, from [22] pp. 358-360.

Our next corollary was originally, for \( G = k \), ring theoretically proven by T. Hibi. We will essentially keep Hibi’s formulation, though using:
\[ \Delta := \Delta \setminus \{ \tau \in \Delta \mid \tau \supset \delta_i \text{ for some } i \in I \} = \bigcap_{i \in I} \text{cost}_{\delta_i}. \]

Corollary 6.11. (27) Corollary p. 95-96) Let \( \Delta \) be a pure simplicial complex of dimension \( n \) and \( \{ \delta_i \}_{i \in I} \), a finite set of faces in \( \Delta \) satisfying \( \delta_i \cap \delta_j \notin \Delta \) for all \( i \neq j \). Set, \( \Delta := \bigcap_{i \in I} \text{cost}_{\delta_i}. \)

a) If \( \Delta \text{ is } CM_k \) and \( \dim \Delta < n \), then \( \dim \Delta = n - 1 \) and \( \Delta \text{ is } CM_k. \)

b) If \( \Delta \) is \( CM_k \) for all \( i \in I \) and \( \Delta \) is \( CM_k \) of dimension \( n \), then \( \Delta \text{ is } CM_k. \)
Proof. a) \[ \delta_i \cup \delta_j \notin \Delta \text{ for all } i \neq j \in I \] \[ \iff (\text{cost}_\Delta \cup (\bigcap_{j \neq i} \text{cost}_{\delta_j})) = \Delta \iff (\text{cost}_\Delta \cup (\bigcap_{j \neq i} \text{cost}_{\delta_j})) \cap \Delta = \Delta \iff \dim \Delta = n-1 \iff \Delta = \Delta \cap \Delta = (\bigcap_{i \in I} \text{cost}_{\delta_i}) \cap \Delta = \bigcap_{i \in I} \text{cost}_{\delta_i}. \]

By Th. \[ \delta \text{10} \] we know that \( \Delta \) is 2-"CM\( _\text{G} \)" implying that \text{cost}_\Delta \text{ is } "CM\( _\text{G} \)" for all \( i \in I \).

Induction using the M-Vs with respect to \( (\text{cost}_\Delta, \bigcap_{j \neq i} \text{cost}_{\delta_j}) \) gives \( \hat{H}(\Delta; G) = 0 \) for all \( i < n-1. \)

For links, use Prop. \[ \delta \text{2} p. \delta \text{3}. \] E.g.,

\[ \text{Lk}_\Delta \delta = \text{Lk}_\Delta \delta = \text{Lk}_\Delta \delta = [\text{Prop. } \delta \text{2} a p. \delta \text{3}] = \bigcap_{i \in I} \text{Lk}_{\text{cost}_{\delta_i}} \]

where \text{cost}_\Delta \text{ and so } \text{Lk}_\Delta \text{ is } "CM\( _\text{G} \), for all } i \in I. \quad \blacksquare

b) \( \Delta = \bigcup_{i \in I} \text{cost}_{\delta_i} \) and

\[ \text{cost}_{\Delta} \cap \Delta = \left[ \text{Motivation: Eq. } \text{H}+\text{HII}, \text{ p. } \delta \text{3} \text{ plus that } \delta_i \cup \delta_j \notin \Delta \right] = \text{cost}_{\Delta} \cap \text{cost}_{\alpha} = \delta \ast \text{Lk}_{\Delta}. \]

So, by Th. \[ \delta \text{12} i, p. \delta \text{2} \] \( \text{cost}_{\Delta} \cap \Delta \) is "CM\( _\text{G} \)" iff \( \text{cost}_{\Delta} \) is. Induction, using the M-Vs with respect to \( (\text{cost}_{\Delta}, \bigcup_{i \in I} \text{cost}_{\delta_i}) \) gives \( \hat{H}(\Delta; G) = 0 \) for all \( i < n-1. \) End as in a. \( \square \)

Now, we will show that our generalized definition of 2-"CM"-ness p. \[ \delta \text{3} \] is consistent with K. Baclawski’s original definition in \[ \text{I} \] p. 295.

Lemma 6.12. \( \Delta \) "CM\( _\text{G} \)" \( \Rightarrow \)

\[ \begin{align*}
\text{(a)} & \quad \hat{H}(\text{cost}_{\delta_i}; G) = 0 \text{ for all } \delta_i \in \Delta \text{ if } i \leq n-2, \\
\text{(b)} & \quad \hat{H}(\text{cost}_{\delta_i}; G) = 0 \text{ for all } \delta_i, \delta_2 \in \Delta \text{ if } i \leq n-3.
\end{align*} \]

Proof. a) Use Proposition \[ \delta \text{1} \text{ p. } \delta \text{3}, \] the definition of "CM"-ness and the LHS with respect to \( (\Delta, \text{cost}_{\delta_i}) \), which reads;

\[ \begin{align*}
\beta & \Rightarrow \hat{H}_{\alpha}(\Delta, \text{cost}_{\delta_i}; G) \overset{\beta_i}{\Rightarrow} \hat{H}_{\alpha}(\text{cost}_{\delta_i}; G) \overset{\alpha_1}{\Rightarrow} \hat{H}_{\alpha}(\Delta; G) \overset{\beta_1}{\Rightarrow} \hat{H}_{\alpha}(\Delta, \text{cost}_{\delta_i}; G) \overset{\delta_1}{\Rightarrow} \hat{H}_{\alpha}(\text{cost}_{\delta_i}; G) \overset{\alpha_1}{\Rightarrow} \hat{H}_{\alpha}(\Delta, \text{cost}_{\delta_i}; G). \quad \blacksquare
\end{align*} \]

b) Apply Prop. \[ \delta \text{3} p. \delta \text{3} a+b \] to the M-Vs with respect to \( (\text{cost}_{\delta_i}, \text{cost}_{\delta_2}) \), then use a), i.e.,

\[ \begin{align*}
\cdots & \Rightarrow \hat{H}_{\alpha}(\text{cost}_{\delta_1} \cup \delta_2; G) \overset{\beta_i}{\Rightarrow} \hat{H}_{\alpha}(\text{cost}_{\delta_1}; G) \overset{\alpha_1}{\Rightarrow} \hat{H}_{\alpha}(\text{cost}_{\delta_1}; G) \oplus \hat{H}_{\alpha}(\text{cost}_{\delta_2}; G) \overset{\beta_i}{\Rightarrow} \\
& \Rightarrow \hat{H}_{\alpha}(\text{cost}_{\delta_1}; G) \overset{\alpha_1}{\Rightarrow} \hat{H}_{\alpha}(\text{cost}_{\delta_1}; G) \oplus \hat{H}_{\alpha}(\text{cost}_{\delta_2}; G) \overset{\beta_i}{\Rightarrow} \hat{H}_{\alpha}(\text{cost}_{\delta_1}; G) \oplus \hat{H}_{\alpha}(\text{cost}_{\delta_2}; G) \overset{\beta_i}{\Rightarrow} \cdots \square
\end{align*} \]

Observation. To turn the implication in Lemma \[ \delta \text{12} \int \] into an equivalence we just have to add \( \hat{H}(\alpha; G) = 0 \) for \( i \leq n-1 \), giving us the equivalence in;

\[ \Delta \text{ "CM\( _\text{G} \)" } \iff \begin{cases}
\text{(i) } & \hat{H}(\Delta; G) = 0 \text{ for } i \leq n-1 \\
\text{(ii) } & \hat{H}(\text{cost}_{\delta_i}; G) = 0 \text{ for } i \leq n-2 \\
\text{(iii) } & \hat{H}(\text{cost}_{\delta_i}; G) = 0 \text{ for } i \leq n-3
\end{cases} \] \( \text{(1)} \)

The property \( \hat{H}_{\alpha}(\text{cost}_{\delta_i}; G) = 0 \) for all \( \delta_i \in \Delta \) allows one more step in the proof of Lemma 6.12b, i.e.,
[\Delta \text{ is } \text{"CM"}_\sigma \text{ and } \hat{H}_i((\text{cost}_\delta; G) = 0 \text{ for all } \delta \in \Delta] \iff \left\{ \begin{array}{l}
i \hat{H}_i(\Delta; G) = 0 \text{ for } i \leq n-1, \\
ii \hat{H}_i((\text{cost}_\delta; G) = 0 \text{ for all } \delta \in \Delta \text{ for } i \leq n-1, \\
iii \hat{H}_i((\text{cost}_\delta; G) = 0 \text{ for all } \delta_1, \delta_2 \in \Delta \text{ for } i \leq n-2. \end{array} \right. \tag{2}

Note. a) Item i) in Eq. 2 follows from ii) and iii) by the M-Vs above.
b) The l.h.s. in Eq. 2 is, by definition, equivalent to \Delta being 2-"CM".

Since,

\[ [\Delta \text{"CM"}_\sigma] \iff [\hat{H}_i(\Delta, \text{cost}_\delta; G) = 0 \text{ for all } \delta \in \Delta \text{ and } i \leq n-1], \]

it is obvious that item iii) in Eq. 1 above, by the LHS with respect to (\Delta, \text{cost}_\delta), is totally superfluous as far as the equivalence is concerned but nevertheless it becomes quite useful when substituting cost_\delta for every occurrence of \Delta and, using that cost_\delta_1 = cost_\delta for \delta \notin \text{cost}_\delta, we get:

\[ \text{cost}_\delta \text{"CM"}_\sigma \text{ for all } \delta \in \Delta \iff \left\{ \begin{array}{l}
i \hat{H}_i((\text{cost}_\delta; G) = 0 \text{ for all } \delta \in \Delta \text{ for } i \leq n-1, \\
ii \hat{H}_i((\text{cost}_\delta_1; G) = 0 \text{ for all } \delta, \delta_1 \in \Delta. \end{array} \right. \tag{3} \]

Remarks 1) From the LHS with respect to (\Delta, \text{cost}_\delta), Theorem 1.12 ii p. 26 and Note 1 p. 44 we conclude that: If \Delta is 2-"CM" then Note 1 p. 26 plus Proposition 4.10 p. 25 implies that \( \hat{H}_{n-1}(\Delta; G) \neq 0. \)

2) It is always true that

\[ n_{\sigma} - 1 \leq n_{\tau} \leq n_{\sigma} \leq n_{\Delta} \text{ if } \tau \subset \sigma, \]

where \( n_{\sigma} := \dim \text{cost}_\sigma \) and \( n_{\Delta} := \dim \Delta. \)

Any \( \Delta \) is representable as \( \Delta = \bigcup_{\sigma} \sigma^m \), where \( \sigma^m \) denotes the simplicial complex generated by the maximal simplex \( \sigma^m \in \Delta. \)

The cone points in a simplicial complex \( \Delta \) are those vertices \( \textbf{v} \in V_\Delta \) that are contained in every maximal simplex of \( \Delta. \) Set \( \hat{\Delta} := \{ \textbf{v} \in V_\Delta \mid \textbf{v} \text{ cone point} \} \in \Delta. \) So, \( \hat{\Delta} = \bigcap_{\sigma \in \Delta} \sigma^m. \)

Let the subindex \( m \) in \( \sigma_m \) denote that the simplex is maxi-dimensional, i.e., \( \dim \sigma_m = n_{\Delta}. \)

Set \( \in \sigma := \{0, \{v_1\}, \{v_2\}, \{v_3\}, \{v_2, v_3\}\}. \) Now,

\[ \left\{ \begin{array}{l}
\text{[all } \textbf{v} \text{ in } V_\Delta \text{ pure and } n_{\sigma} = n_{\Delta} \text{ and } \text{cost}_\sigma \text{ pure } } \forall \emptyset \neq \sigma \in \Delta \Rightarrow \\
\text{[all } \textbf{v} \text{ in } V_\Delta \text{ is a cone point] } \iff [n_{\sigma} = n_{\Delta} - 1] \iff [\text{all } \textbf{v} \in \sigma \in \Delta \text{ if } \dim \sigma_m = n_{\Delta}]. \end{array} \right. \]

Since, \( n_{\varphi} := \dim \text{cost}_\sigma = n_{\Delta} - 1 \iff \emptyset \neq \varphi \subset \sigma_m \in \Delta \) for all \( \sigma_m \in \Delta \) with \( \dim \sigma_m = n_{\Delta}, \) we conclude that; if \( \Delta \) is pure, then \( \varphi \) contains only cone points.

So,

\[ [\Delta \text{ pure and } n_{\sigma} = \dim \Delta \text{ for all } \sigma \in \Delta] \iff [\Delta \text{ pure and has no cone points}]. \]

We also note that

\[ [n_{\sigma} = n_{\Delta} - 1 \forall \textbf{v} \in V_\Delta] \iff [V_\Delta \text{ is finite and } \Delta = \overline{V_\Delta}], \]

where \( \overline{V_\Delta} := \{ \sigma \subset V_\Delta \mid \sigma \text{ finite} \}, \) i.e., the full complex with respect to \( V_\Delta. \)
Theorem 6.13. Each one of the following two double conditions are equivalent to “Δ is 2-“CM₁””:

a) \[
\begin{cases}
\text{i.} \; \text{cost}_\Delta \delta \text{ is } \text{“CM}_1^\ast, \; \forall \delta \in \Delta \\
\text{ii.} \; n_i := \dim \text{cost}_\Delta \delta = \dim \Delta =: n_\delta \; \forall \delta \notin \Delta.
\end{cases}
\]

b) \[
\begin{cases}
\text{i.} \; \text{cost}_\Delta \mathbf{v} \text{ is } \text{“CM}_1^\ast, \; \forall \mathbf{v} \in V_\Delta \\
\text{ii.} \; \{\bullet \bullet\} \neq \Delta \text{ has no cone points.}
\end{cases}
\]

Proof. With no dimension collapse in Eq. 3 above, this equation is equivalent to Eq. 2. \(\square\)

Definition. \(\Delta \setminus \{\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}\} := \{\delta \in \Delta | \delta \cap \{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \emptyset\}. \quad (\Delta \setminus \{\mathbf{v}\} = \text{cost}_\mathbf{v})\)

Permutations and partitions within \(\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}\) does not effect the result:

Lemma 6.14. \(\Delta \setminus \{\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}\} = (\Delta \setminus \{\{\mathbf{v}_1', \ldots, \mathbf{v}_p'\}\}) \setminus \{\{\mathbf{v}_1'', \ldots, \mathbf{v}_q''\}\} \quad \text{and}
\]
\[
\Delta \setminus \{\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}\} = \bigcap_{i=1}^p \text{cost}_{\Delta \setminus \{\mathbf{v}_i\}} \mathbf{v}_i = \text{cost}_{\bigcap_{i=1}^p \Delta \setminus \{\mathbf{v}_i\}} \mathbf{v}_i \quad \square
\]

Definition. (From \(\boxed{}\) p. 295.) For \(k \in \mathbb{N}\), \(\Delta\) is \(k-“CM_1^\ast”\) if for every subset \(T \subset V_\Delta\) such that \(#T = k - 1\), we have:
\[
\begin{cases}
\text{i. } \Delta \setminus \{T\} \text{ is } \text{“CM}_1^\ast, \\
\text{ii. } \dim \Delta \setminus \{T\} = \dim \Delta =: n_\delta =: n.
\end{cases}
\]

Changing “\(#T = k - 1\)” to “\(#T < k\)” does not alter the extension of the last definition (Iterate in Th. 6.13 mutatis mutandis.). So, for \(G = k\), a field, our definition of \(k-“CM_1^\ast”\) is equivalent to Kenneth Baclawski’s original definition in \(\boxed{}\) p. 295.

6.3. Stanley-Reisner rings for simplicial products are Segre products.

Definition. The \textit{Segre product} of the graded \(A\)-algebras \(R_1\) and \(R_2\), denoted \(R = \sigma_A (R_1, R_2)\) or \(R = \sigma (R_1, R_2)\), is defined through:
\[
[R]_p := [R_1]_p \otimes_A [R_2]_p \; \forall p \in \mathbb{N}.
\]

Example. 1) The trivial Segre product, \(R_1 \sigma 0 \otimes R_2\), is equipped with the trivial product, i.e., every product of elements, both of which lacks ring term, equals 0.

2) The “canonical” Segre product, \(R_1 \otimes_\sigma R_2\), is equipped with a product induced by extending (linearly and distributively) the componentwise multiplication on simple homogeneous elements: If \(m_1' \otimes m_1'' \in [R_1 \otimes R_2]_\alpha\) and \(m_2' \otimes m_2'' \in [R_1 \otimes R_2]_\beta\) then
\[
(m_1' \otimes m_1'')(m_2' \otimes m_2'') := m_1' m_2' \otimes m_1'' m_2'' \in [R_1 \otimes R_2]_{\alpha + \beta}.
\]

3) The “canonical” generator-order sensitive Segre product, \(R_1 \otimes_\sigma R_2\), of two graded \(k\)-standard algebras \(R_1\) and \(R_2\) presupposes the existence of a uniquely defined partially ordered minimal set of generators for \(R_1\) (\(R_2\)) in \([R_1]_1\) (\([R_2]_1\)) and is equipped with a product induced by extending (distributively and linearly) the following operation defined on simple homogeneous elements, each of which now are presumed to be written, in product form, as an increasing chain of the specified linearly ordered generators:
If \((m_{11} \otimes m_{21}) \in \left[R_1 \otimes R_2 \right]_\alpha\) and \((m_{12} \otimes m_{22}) \in \left[R_1 \otimes R_2 \right]_\beta\) then

\[
(m_{11} \otimes m_{21})(m_{12} \otimes m_{22}) := ((m_{11}m_{12} \otimes m_{21}m_{22}) \in \left[R_1 \otimes R_2 \right]_{\alpha+\beta}
\]

if by “pairwise” permutations, \((m_{11}m_{12}, m_{21}m_{22})\) can be made into a chain in the product ordering, and 0 otherwise. Here, \((x, y)\) is a pair in \((m_{11}m_{12}, m_{21}m_{22})\) if \(x\) occupy the same position as \(y\) counting from left to right in \(m_{11}m_{12}\) and \(m_{21}m_{22}\) respectively.

**Note.** 1. (\[17\] p. 39-40) Every Segre product of \(R_1\) and \(R_2\) is module-isomorphic by definition and so, they all have the same Hilbert series. The Hilbert series of a graded \(k\)-algebra \(R= \bigoplus_{i \geq 0} R_i\) is

\[
\text{Hilb}_R(t) := \sum_{i \geq 0} (H(R, i))t^i := \sum_{i \geq 0} (\dim_k R_i)t^i
\]

where \(\dim\) is Krull dimension and \(H\) stands for the Hilbert function.

2. If \(R_1, R_2\) are graded algebras finitely generated (over \(k\)) by \(x_1, \ldots, x_n \in \left[R_1 \right]_1, y_1, \ldots, y_m \in \left[R_2 \right]_1\), resp., then \(R_1 \otimes R_2\) and \(R_1 \otimes R_2\) are generated by \((x_1 \otimes y_1), \ldots, (x_n \otimes y_m)\), and

\[
\dim R_1 \otimes R_2 = \dim R_1 \otimes R_2 = \dim R_1 + \dim R_2 - 1.
\]

3. The generator-order sensitive case covers all cases above. In the theory of Hodge Algebras and in particularly in its specialization to Algebras with Straightening Laws (ASLs), the generator-order is the main issue, cf. \[4\] §7.1 p. 289ff. and \[23\] p. 123ff.

\[13\] p. 72 Lemma gives a reduced (Gröbner) basis, \(C' \cup D\), for “\(T\)” in

\[
k(\Delta_1 \times \Delta_2) \cong k[V_{\Delta_1} \times V_{\Delta_2}] / I
\]

with

\[
C' := \{w_{\lambda, \nu} | \lambda < \nu \wedge \mu > \xi\}, w_{\lambda, \nu} := (v_\lambda, v_\nu) , v_\lambda \in V_{\Delta_1}, v_\nu \in V_{\Delta_2}
\]

where the subindices reflect the assumed linear ordering on the factor simplices and with \(\overline{\pi_i}\) as the projection down onto the \(i\):th factor;

\[
D := \{ w = w_{\lambda_{\mu_1}} \cdots w_{\lambda_{\mu_k}} | [[{\overline{\pi_1}(w)} \times \Delta_1] \wedge [{\overline{\pi_2}(w)} \times \Delta_2] \wedge [{\lambda_{\mu_1} < \cdots < \lambda_{\mu_k}}] \wedge [{\overline{\pi_1}(w)} \times \Delta_1] \wedge [{\overline{\pi_2}(w)} \times \Delta_2] \wedge [{\lambda_{\mu_1} < \cdots < \lambda_{\mu_k}}] \}
\]

Now, \(C' \cup D = \{m_\delta | \delta \propto \Delta_1 \times \Delta_2\}\) and the identification \((v_\lambda, v_\mu) \leftrightarrow v_\lambda \otimes v_\mu\) gives, see \[13\] Theorem 1 p. 71, the following graded \(k\)-algebra isomorphism of degree zero; \(k(\Delta_1 \times \Delta_2) \cong k(\Delta_1) \otimes k(\Delta_2)\), which, in the Hodge Algebra terminology, is the discrete algebra with the same data as \(k(\Delta_1) \otimes k(\Delta_2)\) cf. \[4\] §7.1. If the discrete algebra is “C-M” or Gorenstein (defined below), so is the original by \[4\] Corollary 7.1.6. Any finitely generated graded \(k\)-algebra has a Hodge Algebra structure, see \[23\] p. 145.
6.4. Gorenstein complexes without cone points are homology spheres.

Definition 6.15. The cone points in $\Delta$ are those vertices $v \in V_\Delta$ that are contained in every maximal simplex $\sigma^m$. Set $\Delta := \{v \in V_\Delta \mid v$ cone point $\} = \bigcap_{\sigma^m} \sigma^m$.

Definition 6.16. (from [19] Th. 5.1 p. 65) Let $\Sigma$ be an arbitrary (finite) complex and put; $\Gamma := \text{core} \Sigma := \{\sigma \in \Sigma \mid \sigma$ contains no cone points $\}$.

Then; $\emptyset$ is not Gorenstein$_G$, and $\emptyset \neq \Sigma$ is Gorenstein$_G$ (Gor$_G$) if

$$\overline{H}_i(|\Gamma|, |\Gamma| \setminus \alpha; G) = \begin{cases} 0 & \text{if } i \neq \dim \Gamma \\ G & \text{if } i = \dim \Gamma \end{cases} \forall \alpha \in |\Gamma|.$$ 

Note. $\Sigma$ Gorenstein$_G$ $\iff$ $\Sigma$ finite and $|\Gamma|$ is a homology sphere as defined in p. [27]. Now; $v$ is a cone point iff $\overline{st}_v v = \Sigma$ and so,

$$\overline{st}_v \Delta := \Sigma = (\text{core} \Sigma) \ast \Delta$$

and core$\Sigma = \text{Lk}_{\Delta \ast} \Delta := \{\tau \in \Sigma \mid [\Delta \ast \tau = \emptyset] \wedge [\Delta \ast \tau \cup \tau \in \Sigma]\}$.

Proposition 6.17. ([13] p. 77.) If $G$ is a field $k$ or the integers $\mathbb{Z}$ then,

$$\Sigma_1 \ast \Sigma_2 \text{ Gorenstein}_G \iff \Sigma_1, \Sigma_2 \text{ both Gorenstein}_G.$$ 

Gorensteinness is, unlike “Bbmg”, “CMG”, and 2-“CMG”-ness, triangulation sensitive and in particular, the Gorensteinness for products is sensitive to the partial orders, assumed in the definition, given to the vertex sets of the factors; for $\{m_{\delta} \mid \delta \propto \Delta_1 \times \Delta_2\}$, see p. [10].

In [13] p. 80, this product is represented in the form of matrices, one for each pair $(\Delta_1, \Delta_2)$ of maximal simplices $\Delta_i \in \Delta_i, \ i = 1, 2$. It is then easily seen that a cone point must occupy the upper left corner in each matrix or the lower right corner in each matrix. So a product $(\dim \Delta_i \geq 1)$ can never have more than two cone points. For Gorensteinness to be preserved under product the factors must have at least one cone point to preserve even “CMG”-ness, by Theorem [12] ii p. [20].

Bd(core$(\Delta_1 \times \Delta_2)$) = $\emptyset$ forces each $\Delta_i$ to have as many cone points as $\Delta_1 \times \Delta_2$.

Proposition 6.18. (For proof see [13] p. 83ff.) Let $G$ be a field $k$ or the integers $\mathbb{Z}$ and let $\Delta_1, \Delta_2$ be two arbitrary finite simplicial complexes with $\dim \Delta_i \geq 1$ $(i = 1, 2)$ and a linear order defined on their vertex sets $V_{\Delta_1}, V_{\Delta_2}$, respectively, then

$$\Delta_1 \times \Delta_2 \text{ Gorenstein}_G \iff \Delta_1, \Delta_2 \text{ both Gorenstein}_G$$

and either condition I or II holds:

(I) $\Delta_1, \Delta_2$ has one cone point each, either both minimal or both maximal.

(II) $\Delta_1, \Delta_2$ has two cone points each, one minimal and the other maximal. $\square$

Example. Gorensteinness is character sensitive!

Let $\Gamma := \text{core} \Sigma = \Sigma$ be a 3-dimensional Gorenstein$_G$ complex, where $k$ is the prime field $\mathbb{Z}_p$ of characteristic $p$. This implies, in particular, that $\Gamma$ is a homology 3-manifold. Put $\mathcal{H}_i := \overline{H}_i(\Gamma; \mathbb{Z})$, then $\mathcal{H}_0 = 0, \mathcal{H}_3 = \mathbb{Z}$ and $\mathcal{H}_2$ has no torsion by Lemma [17] p. [43]. Poincare’ duality and [19] p. 244 Corollary 4 gives $\mathcal{H}_1 = \mathcal{H}_2 \oplus T \mathcal{H}_1$, where $T \circ :=$ the torsion-submodule of $\circ$. So $\Sigma = \Gamma$ with a pure torsion $\mathcal{H}_1 = \mathbb{Z}_p$, say, gives us an example of a presumptive character sensitive Gorenstein complex. Examples of such orientable compact
combinatorial manifolds without boundary are given by the 3-dimensional projective space \( \mathbb{R}P^3 \) and the lens space \( L(n, k) \) where \( \tilde{H}_1(\mathbb{R}P^3; \mathbb{Z}) = \mathbb{Z}_2 \) and \( \tilde{H}_1(L(n, k); \mathbb{Z}) = \mathbb{Z}_n \).

So, \( \mathbb{R}P^3 \ast (L(n, k) \ast \bullet) \) is \( \text{Gor}_k \) for \( \text{char} \, k \neq 2 \) (\( \text{char} \, k \neq n \)) while it is not even Buchsbaum for \( \text{char} \, k = 2 \) (resp. \( \text{char} \, k = n \)), cf. [45] p. 231-243 for details on \( \mathbb{R}P^3 \) and \( L(n, k) \) and [40] Prop. 5.1 p. 65 or [5] p. 75 for Gorenstein equivalences. A Gorenstein \( \Delta \) is not in general shellable, since if it were, \( \Delta \) would have been CM but \( L(n, k) \) and \( \mathbb{R}P^3 \) is not. Indeed, in 1958 M.E. Rudin publish a paper [43]: An unshellable triangulation of a tetrahedron.

Other examples are given by Jeff Weeks’ computer program “SnapPea” hosted at http://geometrygames.org/SnapPea/. E.g. \( \tilde{H}_1(\Sigma(5, 1); \mathbb{Z}) = \mathbb{Z}_5 \) for the old tutorial example of SnapPea, here denoted \( \Sigma \), i.e. the Dehn surgery filling with respect to a figure eight complement with diffeomorphism kernel generated by \((5, 1)\). \( \Sigma(5, 1) \ast \bullet \) is \( \text{Gor}_k \) if \( \text{char} \, k \neq 5 \) but not even \( \text{Bbm}_k \) if \( \text{char} \, k = 5 \).

7. Simplicial manifolds

**Definitions.** We will make extensive use of Proposition [14] p. 225 without explicit notification. A topological space will be called a *n-pseudomanifold* or a *quasi-n-manifold* if it can be triangulated into a simplicial complex that is a *n-pseudomanifold* resp. a *quasi-n-manifold*.

**Definition 1.** An *n*-dimensional pseudomanifold is a locally finite \( n \)-complex \( \Sigma \) such that;

(α) \( \Sigma \) is pure, i.e. the maximal simplices in \( \Sigma \) are all \( n \)-dimensional.

(β) Every \((n - 1)\)-simplex of \( \Sigma \) is the face of at most two \( n \)-simplices.

(γ) If \( s \) and \( s' \) are \( n \)-simplices in \( \Sigma \), there is a finite sequence \( s = s_0, s_1, \ldots, s_m = s' \) of \( n \)-simplices in \( \Sigma \) such that \( s_i \cap s_{i+1} \) is an \((n - 1)\)-simplex for \( 0 \leq i < m \).

The *boundary*, \( \text{Bd}\Sigma \), of an *n*-dimensional pseudomanifold \( \Sigma \), is the subcomplex generated by those \((n - 1)\)-simplices which are faces of exactly one \( n \)-simplex in \( \Sigma \).

**Definition 2.** \( \Sigma \) is a *quasi-n-manifold* if it is an \( n \)-dimensional, locally finite complex fulfilling;

(α) \( \Sigma \) is pure. (α is redundant since it follows from γ by Lemma [51] p. 32.)

(β) Every \((n - 1)\)-simplex of \( \Sigma \) is the face of at most two \( n \)-simplices.

(γ) \( \text{Lk}_s \sigma \) is connected i.e. \( \tilde{H}_0(\text{Lk}_s \sigma; G) = 0 \) for all \( \sigma \in \Sigma \), s.a. \( \text{dim } \sigma < n - 1 \) (\( \equiv \text{dim } \text{Lk}_s \sigma \geq 1 \)).

The *boundary with respect to* \( G \), denoted \( \text{Bd}\_G \Sigma \), of a quasi-*n*-manifold \( \Sigma \), is the set of simplices \( \text{Bd}\_G \Sigma := \{ \sigma \in \Sigma \mid \tilde{H}_n(\Sigma, \text{cost}_s \sigma; G) = 0 \} \), where \( G \) is a unital module over a commutative ring \( A \). (According to β in Definition 1 and 2, there are no other 0-manifolds than \( \bullet \) and \( \bullet \), due to the presens of the \((−1)\)-dimensional simplex \( \emptyset \).)

**Note.** \( \Sigma \) is (locally) finite \( \iff |\Sigma| \) is (locally) compact. If \( X = |\Sigma| \) is a homology \( n \)-manifold \((n\text{-hm})\) we will call \( \Sigma \) a *n-hm* \( \kappa \). Now, by Theorem [14] p. 224: \( \Sigma \) is a *n-hm* \( \kappa \) for any \( R\text{-PID} \) module \( G \).

[43] p. 207-8 + p. 277-8 treats the case \( R = G = \mathbb{Z} := \{ \text{the integers} \} \). A simplicial complex \( \Sigma \) is called a \( \text{hm} \kappa \) if \( |\Sigma| \) is. The \( n \) in \( n\text{-hm} \kappa \) is deleted, since now \( n = \text{dim } \Sigma \). From a purely technical point of view we really don’t need the “locally finiteness”-assumption, as is seen from Theorem [112] p. 28.
Definition 3. Let “manifold” stand for pseudo-, quasi- or homology manifold. A compact \( n \)-manifold, \( S \), is \textit{orientable} if \( \hat{H}_n(S, \text{Bd}S; G) \cong G \). An \( n \)-manifold is \textit{orientable} if all its compact \( n \)-submanifolds are orientable – else, \textit{non-orientable}. Orientability is left undefined for \( \emptyset \).

Note 2. For a classical homology manifold \( X \neq \emptyset \) s.a. \( \text{Bd}X = \emptyset \); \( \text{Bd}_2F_\emptyset(X) = \emptyset \) if \( X \) is compact and orientable and \( \text{Bd}_2F_\emptyset(X) = \{ \emptyset \} \) else. \* is the only compact orientable manifold with boundary \( = \{ \emptyset \} \).

Definition 4. \( \{ B_{G, j}^\Sigma \}_{j \in I} \) is the set of strongly connected boundary components of \( \Sigma \) if \( \{ B_{G, j}^\Sigma \}_{j \in I} \) is the maximal strongly connected components of \( \text{Bd}G \Sigma \) from Definition 5.5 p. 32 \( \Rightarrow B_{G, j}^\Sigma \) is pure and if \( \sigma \) is a maximal simplex in \( B_j \), then \( \text{Lk}_{\text{Bd}G} \Sigma \sigma = \text{Lk}_{B_j} \sigma = \{ \emptyset \} \).

Note 3. \( \emptyset \), \( \{ \emptyset \} \), and 0-dimensional complexes with either one, \( \bullet \), or two, \( \bullet \), vertices are the only manifolds in dimensions \( \leq 0 \), and the \( |1\)-manifolds| are finite/infinitesimal 1-circles and (half)lines, while \( \Sigma \) is a quasi-2-manifold \( \Longleftrightarrow \Sigma \) is a homology 2-manifold. Def. 1. \( \gamma \) is paraphrased by “\( \Sigma \) is strongly connected” and \( \bullet \bullet \)-complexes, though strongly connected, are the only non-connected manifolds. Note also that \( S^{-1} \) is the boundary of the 0-ball, \( \bullet \), the double of which is the 0-sphere, \( \bullet \bullet \). Both the \( (-1) \)-sphere \( \{ \emptyset \} \) and the 0-sphere \( \bullet \bullet \) has, as preferred, empty boundary.

7.2. Auxiliaries.

Lemma 7.1. For a finite \( n \)-pseudomanifold \( \Sigma \);

i. (cf. [14] p. 206 Ex. E2.)

\[ \hat{H}_n(\Sigma, \text{Bd} \Sigma; Z) = Z \] and \( \hat{H}_{n-1}(\Sigma, \text{Bd} \Sigma; Z) \) has no torsion, or

\[ \hat{H}_n(\Sigma, \text{Bd} \Sigma; Z) = 0 \] and the torsion submodule of \( \hat{H}_{n-1}(\Sigma, \text{Bd} \Sigma; Z) \) is isomorphic to \( Z_2 \).

ii. \( \Sigma \) orientable \( \Longleftrightarrow \Sigma \) orientable \( \bullet \) or \( \text{Tor}_{1}^{2}(Z_2, G) = G \).

So, any complex \( \Sigma \) is orientable with respect to \( Z_2 \).

iii. \( \hat{H}_n(\Sigma, \text{Bd} \Sigma; Z) = Z (Z_2) \) when \( \Sigma \) is (non-)orientable.

Proof. By conditions \( \alpha \), \( \beta \) in Def. 1 p. [14], a possible relative \( n \)-cycle in \( C^n(\Sigma, \text{Bd} \Sigma; Z_m) \) must include all oriented \( n \)-simplices all of which with coefficients of one and the same value. When the boundary function is applied to such a possible relative \( n \)-cycle the result is an \( (n - 1) \)-chain that includes all oriented \( (n - 1) \)-simplices, not supported by the boundary, all of which with coefficients 0 or \( \pm 2c \in Z_m \).

So, \( \hat{H}_n(\Sigma, \text{Bd} \Sigma; Z_2) = Z_2 \) – always, and

\[ \hat{H}_n(\Sigma, \text{Bd} \Sigma; Z) = Z \] \( (0) \) \( \Longleftrightarrow \hat{H}_n(\Sigma, \text{Bd} \Sigma; Z_m) = Z_m \) \( (0) \) if \( m \neq 2 \).

The Universal Coefficient Theorem \( (= \text{Th.}[E9] \text{p. 23}) \) now gives;

\[ Z_2 = \hat{H}_n(\Sigma, \text{Bd} \Sigma; Z_2) \cong \hat{H}_n(\Sigma, \text{Bd} \Sigma; Z \otimes Z_2) \cong \]

\[ \cong \hat{H}_n(\Sigma, \text{Bd} \Sigma; Z) \otimes Z_2 \cong \text{Tor}_{1}^{2}(\hat{H}_{n-1}(\Sigma, \text{Bd} \Sigma; Z), Z_2) \] and,

\[ \hat{H}_n(\Sigma, \text{Bd} \Sigma; Z_m) \cong \hat{H}_n(\Sigma, \text{Bd} \Sigma; Z \otimes Z_m) \cong \]

\[ \cong \hat{H}_n(\Sigma, \text{Bd} \Sigma; Z) \otimes Z_m \cong \text{Tor}_{1}^{2}(\hat{H}_{n-1}(\Sigma, \text{Bd} \Sigma; Z), Z_m). \]
where the last homology module in each formula, by \[4.9\] p. 225 Cor. 11, can be substituted by its torsion submodule. Since \(\Sigma\) is finite, \(\hat{H}_{n-1}(\Sigma; \text{Bd}\Sigma; \mathbb{Z}) = C_1 \oplus C_2 \oplus \ldots \oplus C_S\) by The Structure Theorem for Finitely Generated Modules over \(\text{PIDs}\), cf. \[4.9\] p. 9. Now, a simple check, using \[4.9\] p. 221 Example 4, gives \(i\), which gives \(\text{iii} \) by \[4.9\] p. 244 Corollary. Theorem \[4.4\] p. \(\text{p.}\) and \(i\) implies \(\text{ii}\).

Proposition \[6.8\] p. \(\text{p.}\) together with Proposition \[4.10\] p. \(\text{p.}\) gives the next Lemma.

**Lemma 7.2.**

(i) \(\Sigma \) is a \(n\)-hm\(\alpha\) \(\Leftrightarrow\) \(\Sigma \) is a quasi-manifold \(\Leftrightarrow\) \(\hat{\Sigma}\) is an \(n\)-pseudomanifold.

(ii) \(\Sigma \) is a \(n\)-hm\(\alpha\) \(\Leftrightarrow\) it is a “Bbm\(\alpha\)" pseudomanifold and \(\hat{H}_{n}(\Sigma; G) = 0\) or \(G\) \(\forall\) \(\alpha \neq 0\).

**Def.** \[6.3\] p. \(\text{p.}\) makes perfect sense even for non-simplicial posets like \(\Gamma \setminus \Delta\) (Def. \[6.5\] p. \(\text{p.}\)) which allow us to say that \(\Gamma \setminus \Delta\) is or is not strongly connected (as a poset) depending on whether \(\Gamma \setminus \Delta\) fulfills Def. \[6.3\] p. \(\text{p.}\) or not. Now, for quasi-manifolds \(\Gamma \setminus \Delta\) connected as a poset (Def. \[6.3\] p. \(\text{p.}\)) is equivalent to \(\Gamma \setminus \Delta\) strongly connected which is a simple consequence of Lemma \[6.10\] p. \(\text{p.}\) and the definition of quasi-manifolds, cf. \[13\] p. 165 Lemma 4 & 5. \(\Sigma \setminus \text{cost}\(\alpha\)\(\Sigma\) is a connected as a poset for any simplicial complex \(\Sigma\) and any \(\sigma \neq 0\) \(\Sigma \setminus \text{Bd}\Sigma\) is strongly connected for any pseudomanifold \(\Sigma\). \(\Delta^p\) is defined at the end of p. \[34\].

If \(\Delta \subset \Sigma \subset \Sigma\) the \(\text{LHS}\) with respect to any simplicial complex \(\Sigma\), with \(\dim \Sigma = n\) reads:

\[\ldots 0 \rightarrow \hat{H}_{n}(\Sigma, \Delta; G) \rightarrow \hat{H}_{n}(\Sigma, \Delta; G) \rightarrow \hat{H}_{n}(\Sigma, \Delta; G) \rightarrow \ldots \]

(1)

With \(\sigma \subset \tau\) then \(\Delta := \text{cost}\(\sigma\)\(\Sigma\) \(\subset\) \(\Gamma := \text{cost}\(\tau\)\(\Sigma\) \(\subset\) \(\Sigma\) we get \(a\) in the next Corollary and (1) becomes:

\[\ldots 0 \rightarrow \hat{H}_{n}(\text{cost}\(\sigma\)\(\Sigma\), \text{cost}\(\tau\)\(\Sigma\), G) \rightarrow \hat{H}_{n}(\Sigma, \Delta; G) \rightarrow \hat{H}_{n}(\Sigma, \Delta; G) \rightarrow \ldots \]

(2)

Next theorem is given in \[13\] p. 166 (3.2 Haupplemma) with the integers \(\mathbb{Z}\) as coefficients.

**Theorem 7.3.** (For proof see \[19\] p. 166 plus Th. \[13\] p. \(\text{p.}\)) Let \(\text{Tor}_1^{\mathbb{Z}}(\hat{H}_{n-1}(\Sigma, \Delta; G)) = 0\). For a finite quasi-n-manifold \(\Sigma\) with subcomplexes \(\Delta \subset \Sigma \subset \Sigma\) and \(\dim \Sigma = n\), then \(\hat{H}_{n}(\Sigma, \Delta; G) = 0\) if \(\text{Sim}\(\Sigma\) \(\Delta\) is strongly connected, which obviously is equivalent to \(\hat{H}_{n}(\Sigma, \Delta; G) \rightarrow \hat{H}_{n}(\Sigma, \Delta; G)\) is injective in the relative \(\text{LHS}\) with respect to \(\Sigma, \Delta, \Sigma\).

With \(\Gamma = \text{cost}\(\sigma\)\(\Sigma\) and \(\Delta = \text{cost}\(\tau\)\(\Sigma\) \(\subset\) \(\Sigma\) \(\subset\) \(\Sigma\) resp. \(\text{cost}\(\tau\)\(\Sigma\) \(\subset\) \(\Sigma\) \(\subset\) \(\Sigma\) \(\subset\) \(\Sigma\) we get \(b\) in the next Corollary.

**Corollary 7.4.** (\(G\) as in Th. \[7.3\]) If \(\Sigma\) is a finite quasi-n-manifold and \(\sigma \subset \tau\) \(\in\) \(\Sigma\), then:

a. \(\hat{H}_{n}(\Sigma, \tau; G) \rightarrow \hat{H}_{n}(\Sigma, \tau; G)\) is injective if \(\text{Sim}\(\Sigma\) \(\Delta\) is strongly connected.

b. \(\hat{H}_{n}(\Sigma, \tau; G) \rightarrow \hat{H}_{n}(\Sigma, \tau; G)\) is injective if \(\text{Sim}\(\Sigma\) \(\Delta\) is strongly connected.

(Cf. the proof of Proposition \[1.10\] p. \(\text{p.}\).)

**Note.** \(a\) above, implies that the boundary of a quasi-n-manifold is a subcomplex and \(b\) that simplicial quasi-manifolds are “ordinary”, i.e. \(\hat{H}_{n}(\Sigma; G) = 0\) as defined in p. \(\text{p.}\) and so, for quasi-manifolds, \(\text{Bd}\(\Sigma\) \(\neq\) 0 \(\Leftrightarrow\) \(\hat{H}_{n}(\Sigma; G) = 0\), since \(\hat{H}_{n}(\Sigma; G) = \hat{H}_{n}(\Sigma; \text{cost}\(\delta\)\(\Sigma\); G) = 0 \(\iff\) \(\delta \in \text{Bd}\(\Sigma\) \(\text{by the LHS.}\) Also recall Proposition \[1.10\] p. \(\text{p.}\).
Corollary 7.5. i. If $\Sigma$ is a finite quasi-$n$-manifold with $\#I \geq 2$ then; 
$\tilde{H}_i(\Sigma; \bigcup_{j \neq i} B_j; G) = 0$ and $\tilde{H}_i(\Sigma, B_i; G) = 0$, (with $B_j := B_{E_j}$ from Definition 4 p. [13]). 

ii. Both $\tilde{H}_i(\Sigma; \mathbb{Z})$ and $\tilde{H}_i(\Sigma, Bd\Sigma; \mathbb{Z})$ equals 0 or $\mathbb{Z}$.

Proof. i. $\bigcup_{j \neq i} B_j \subseteq \text{cost}_n \sigma$ for some $B_i$-maxidimensional $\sigma \in B_i$ and vice versa. 

ii. $[\dim \tau = n] \Rightarrow [[\tilde{H}_i(\Sigma, \text{cost}_n \tau; G) = G] \land [\text{Bd} \Sigma \subset \text{cost}_n \tau]]$ and $\Sigma \setminus \text{Bd} \Sigma$ strongly connected. Theorem 7.3 and the LHS gives the injections $\tilde{H}_i(\Sigma; G) \rightarrow \tilde{H}_i(\Sigma, \text{Bd} \Sigma; G) \rightarrow G$. □

Note. 2. $\Sigma_1 \ast \Sigma_2 \not\subseteq \Sigma_i$ locally finite $\iff$ $\Sigma_1, \Sigma_2$ both finite, and; $\Sigma$ Gorenstein $\iff$ $\Sigma$ finite.

Theorem 7.6. i.a (cf. [19] p. 168, [20] p. 32.) Let $G$ denote a field $k$ or $\mathbb{Z}$. $\Sigma$ is a quasi-manifold iff $\Sigma = \bullet \bullet$ or $\Sigma$ is connected and $Lk_\sigma$ is a finite quasi-manifold for all $\emptyset \neq \sigma \in \Sigma$.

i.b. $\Sigma$ is a homology manifold iff $\Sigma = \bullet \bullet$ or $\tilde{H}_i(\Sigma; G) = 0$ and $Lk_\sigma$ is a finite CM$_\text{L}$-homology manifold $\forall \emptyset \neq \sigma \in \Sigma$.

ii. $\Sigma$ is a quasi-manifold $\iff$ $\text{Bd}_G(Lk_\sigma) = Lk_\sigma$ if $\sigma \in \text{Bd}_G \Sigma$ and $\text{Bd}_G(Lk_\sigma) \equiv \emptyset$ else.

iii. $\Sigma$ is a quasi-manifold $\iff$ $Lk_\sigma$ is a pseudomanifold $\forall \sigma \in \Sigma$, including $\sigma = \emptyset$.

Proof. i. A simple check confirms all our claims for dim $\Sigma \leq 1$, cf. Note 3 p. [13]. So, assume dim $\Sigma \geq 2$ and note that $\sigma \in \text{Bd}_G \Sigma$ if $\text{Bd}_G(Lk_\sigma) \neq \emptyset$.

i.a. ($\Leftarrow$) That $Lk_\sigma$, with dim $Lk_\sigma = 0$, is a quasi-$0$-manifold implies definition condition 2. Theorem 7.2 and since the other “links” are all connected condition 2 follow.

($\Rightarrow$) Definition condition 2 p. [42] implies that $0$-dimensional links are $\bullet \bullet$ or $\bullet \circ$ while Eq. I p. [42] gives the necessary connectedness of ‘links of links’, cf. Lemma 5.1 p. [52].

i.b. Lemma 7.2 above plus Proposition 7.8 p. [47] and Eq. I p. [54].

ii. Pureness is a local property, i.e. $\Sigma$ pure $\iff$ $Lk_\sigma$ pure. Put $n := \dim \Sigma$.

Now; $\varepsilon \in \text{Bd}_G(Lk_\sigma) \iff 0 = \tilde{H}_{n-\#(Lk_\sigma)}(\Sigma; G) = \left[\tilde{H}_{n-\#(Lk_\sigma)}(\Sigma; G) = \tilde{H}_{n-\#(Lk_\sigma)}(\Sigma; G) \cup \tilde{H}_n(Lk_\sigma \cup \varepsilon) \right] & \varepsilon \in Lk_\sigma$.

So; $\varepsilon \in \text{Bd}_G(Lk_\sigma) \iff [\sigma \cup \varepsilon \in \text{Bd}_G \Sigma \and \varepsilon \in Lk_\sigma] \iff [\sigma \cup \varepsilon \in \text{Bd}_G \Sigma \and \sigma \cap \varepsilon = \emptyset] \iff [\varepsilon \in \text{Bd}_G(Lk_\sigma)]$.

iii. ($\Rightarrow$) Lemma 7.2 i p. [44] and Th. 7.6 i.a.

($\Leftarrow$) All links are connected, except $\bullet$.

Note. 3. For a finite $n$-manifold $\Sigma$ and a $n$-submanifold $\Delta$, put $U := \Sigma \setminus \Delta$, implying that $\text{Bd}_G \Delta \cup U$ is the polytope of a subcomplex, $\Gamma$, of $\Sigma$ i.e. $[\Gamma] = \text{Bd}_G \Delta \cup U$, and $\text{Bd}_G \Sigma \subset \Gamma$, cf. [39] p. 427-429. Consistency of Definition 3 p. [13] follows by excision in simplicial homology since; $\tilde{H}_i(\Sigma, \text{Bd}_G \Sigma; G) \rightarrow \tilde{H}_i(\Sigma, \Gamma; G) \cong \tilde{H}_i(\Sigma \setminus U, \Gamma \setminus U; G) = \tilde{H}_i(\Delta, \text{Bd}_G \Delta; G)$. E.g., $\overline{\delta} \times Lk_\sigma$, is an orientable quasi-manifold if $\Sigma_i$ is, as $Lk_\sigma$ by Theorem 7.4 and Theorem 7.3 below. Moreover, $\delta \subset \sigma \Rightarrow \overline{\delta} \times \sigma$ non-orientable if $\overline{\delta} \times \sigma$ is.

Corollary 7.7. For any quasi-$n$-manifold $\Sigma$ except infinite 1-circles; 
$\dim B_j \geq n-2 \Rightarrow \dim B_j = n-1$. 
Proof. Check $n \leq 1$ and then assume that $n \geq 2$. If $\dim \sigma = \dim B_j = n - 2$ and $\sigma \in B_j$ then;

\[ L_k \sigma = L_k \sigma = \emptyset. \]

Now, by Th. 7.8 ii,

\[ L_k \sigma = \text{Bd} \sigma (\text{Lk} \sigma, \sigma) = \begin{cases} L_k \sigma \text{ is, by Th. 7.8 i, a finite quasi-1-manifold i.e. (a circle or) a line.} & = (\emptyset \text{ or } \bullet \bullet. \\
\end{cases} \]

Contradiction! □

Denote $\Sigma$ by $\Sigma_i \Sigma_j$ and $\Sigma_n$ when it is assumed to be a pseudo-, quasi- resp. a homology manifold. Note also that; $\sigma \in \text{Bd} \Sigma_i \iff \text{Bd} \sigma (\text{Lk} \sigma, \sigma) = L_k \sigma \neq \emptyset$ by Theorem 7.8 ii above.

Corollary 7.8. i. If $\Sigma$ is finite quasi-n-manifold with $\text{Bd} \Sigma_i = \bigcup B_j$ and $-1 \leq \dim B_i < \dim \Sigma - 1$ for some $i \in I$ then, $H_n(\Sigma, \text{Bd} \Sigma; G) = 0$. In particular, $\Sigma$ is non-orientable.$G$

ii. If $\text{Bd} \Sigma_i = \bigcup B_j$ with $\dim B_i := \dim \text{Bd} \Sigma_i$, then $B_i$ is a pseudomanifold.

iii. $H_n(\Sigma, \text{Bd} \Sigma; G') = H_n(\Sigma, \text{Bd} \Sigma; G) = H_n(\Sigma, \text{Bd} \Sigma; G)$, $i = 0, 1$ even if $G \neq G'$. Orientability is independent of both $G$ and $G'$, as long as $\text{Tor}_2(Z_2, G') \neq G'$ (Lemma 7.8 ii p. 43). Moreover, $Bd \Sigma_i = Bd \Sigma_h$ always, while $(\text{Bd} \Sigma_i)^* = (\text{Bd} \Sigma_j)^*$ and $(\text{Bd} \Sigma_i)^* = (\text{Bd} \Sigma_j)^*$ except for infinite 1-circles in which case $(\text{Bd} \Sigma_i)^* \neq \emptyset \neq \{0\}$; $Bd \Sigma_i^*$.

iv. For any finite orientable$G$ quasi-n-manifold $\Sigma$, each boundary component $B_i := B_i \Sigma \notin \{0\}$ in Def. 4 p. 43, is an orientable $(n-1)$-pseudomanifold without boundary.

v. $\text{Tor}_2(Z_2, G) = 0 \iff \text{Bd} \Sigma_i = B_{d \Sigma_i}^*.$

vi. $\text{Bd} \Sigma_i = \text{Bd} \Sigma_2 \leq \text{Bd} \Sigma_3 \leq \text{Bd} \Sigma_4$ with equality if $\text{Bd} \Sigma_0 = \emptyset$ or $\dim B_i = n - 1 \forall j \in I$, except if $\Sigma$ is infinite and $\text{Bd} \Sigma_0 = \emptyset$, $\neq \emptyset$ (by Lemma 1 i ii p. 43 plus Th. 1.9 p. 22) since $\emptyset \neq \text{Bd} \Sigma_1 \neq 0$ if $\Sigma$ is infinite. If $\Sigma$ Gorenstein, then; $Bd \Sigma_i = Bd \Sigma_i$.

Proof. i. $\dim B < \dim \Sigma - 2$ from Cor. 1 above, gives the 2:nd equality and Cor. 1.a p. 44 gives the injection-arrow in;

\[ H_n(\Sigma, B_d \Sigma; G) = H_n(\Sigma, B_i \cup (\bigcup B_j); G) = \]

\[ \Rightarrow H_n(\Sigma, \text{cost} \sigma; G) = 0 \iff \sigma \in \text{Bd} \Sigma_0 \cup B_i. \]

\[ \text{ii. The claim is true if } \dim \Sigma \leq 1 \text{ and assume it is true for dimensions } \leq n - 1. \alpha \text{ and } \gamma \text{ are true by definition of } B_i \text{ so only } \beta \text{ remains. If } \sigma \in B_i \text{ with } \dim \sigma = \dim B_i - 1, \text{ then } \sigma \notin \cup B_j \text{ and so, } \text{Bd} \sigma = L_k \sigma = L_k \sigma, \text{ implying, by the induction assumption, that the r.h.s. is a 0-pseudomanifold i.e. } \sigma = \bullet. \]

\[ \text{iii. } \text{Use Note 1.a p. 44 plus Corollary 7.7 above and the fact that; } [\sigma \in \Sigma^* \cap \text{Bd} \Sigma \iff \text{Lk} \sigma = \emptyset]. \]

\[ \text{iv. i and ii implies that only orientability and } \text{Bd}(B_i) \sigma = \emptyset \text{ remains. } \text{Bd}(B_i) \sigma \neq \emptyset \iff \text{H}_{n-1}(B_i \sigma) = 0 \text{ by Note 1 p. 44 and so, } \text{Bd} \sigma = \emptyset \text{ by iii. From Def. 4 p. 43 we get, } \text{dim } B_i = \text{dim } B_j = n - 1, \text{ s, t } \in I \implies \text{dim } B_i \cap B_j \leq n - 3 \text{ if } s \neq t, \text{ which, since } (\text{Bd} \Sigma_i)^* = (\text{Bd} \Sigma_j)^* \iff \text{H}_n(\Sigma, B \Sigma; G') \leq 0 \text{ for dimensional reasons. } \]

\[ \text{by Cor. 7.8 p. 45 = G by assumption if } G' = G. \]
\[ \tilde{H}_{\ast}(\text{Bd} \Sigma \cup \text{Bd}_i; G') = [\text{For dimensional reasons.}] = \tilde{H}_{\ast}(\text{Bd}_i; G). \]

In the above truncated relative LHS with respect to \((\Sigma, \text{Bd} \Sigma \cup \text{Bd}_i)\), the choice of \(L\) is irrelevant by iii. Lemma \(7.1\) ii p. \(\text{l}\) gives the orientability \(\sigma\). If \(G = \mathbb{Z}\) in the LHS, the injection gives our claim. Otherwise \(\text{Tor}^2(\mathbb{Z}, G) = G\) and we are done.

**v.** \(\text{Bd} \Sigma_{n_i} \not\ni \sigma \in \text{Bd} \Sigma_{n_i} \iff \tilde{H}(\langle \text{Lk} \sigma \rangle; G) \neq 0 \iff \tilde{H}(\langle \text{Lk} \sigma \rangle; \mathbb{Z}) \iff \text{Bd} \Sigma_{n_i} \neq 0 \iff \text{Bd} \Sigma_{n_i} (\mathbb{Z}, G) = 0\). Contradiction!

\((- \text{Tor}^2(\mathbb{Z}, G) = 0\) and the torsion module of \(\tilde{H}(\langle \text{Lk} \sigma \rangle; \text{Bd} \Sigma_{n_i} ; \mathbb{Z})\) is either 0 or homomorphic to \(\mathbb{Z}\); by Lemma \(7.1\) i p. \(\text{l}\) give the contradiction, since only the torsion submodules matter in the torsion product by \([19\) Corollary 11 p. 225.\)]

**vi.** iv and that \(\text{Bd} \Sigma_{n_i} \subseteq \text{Bd} \Sigma_{n_i}\), by Corollary \(7.3\) p. \(\text{l}\) plus Theorem \(4.9\) p. \(\text{m}^\circ\).

### 7.3. Products and joins of simplicial manifolds.

Let in the next theorem, when \(\ast\), all through, is interpreted as \(\times\), the word “manifold(s)” in \(7.3.1\) temporarily excludes \(\emptyset\), \(\{\emptyset\}\) and \(\ast\).

When \(\ast\), all through, is interpreted as \(\ast\) let the word “manifold(s)” in \(7.3.1\) stand for finite “pseudo-manifold(s)” (”quasi-manifold(s)”, cf. \([19\) 4.2 pp. 171-2]. We conclude, with respect to joins, that Th. \(7.3\) is trivial if either \(\Sigma_1\) or \(\Sigma_2\) equals \(\{\emptyset\}\) and that \(\Sigma_1\) and \(\Sigma_2\) must be finite since otherwise, their join \(\Sigma_1 \ast \Sigma_2\) won’t be locally finite. Moreover, set \(\epsilon = 0\) if \(\ast = \times\) and \(\epsilon = 1\) if \(\ast = \ast\). Only quasi-manifold boundaries and orientability depend on the choice of coefficient module in the next theorem. Also disposing off all torsion terms by using a field \(k\) as coefficient module, we are essentially back in well known cases proven for instance as Theorem 10 in \(13\) p. 82 (pseudomanifolds) and in \(19\) p. 171 as “Satz” (joins of quasi-manifolds).

**Theorem 7.9.** If \(V_{\Sigma_{n_i}} \neq \emptyset\) then;

1. \(\Sigma_1 \ast \Sigma_2\) is a \((n_1 + n_2 + \epsilon)\) -manifold \(\iff \Sigma_i\) is a \(n_i\)-manifold.
2. \(\text{Bd}(\ast \times \Sigma) = \ast \times (\text{Bd} \Sigma)\). Else; \(\text{Bd}(\Sigma_1 \ast \Sigma_2) = ((\text{Bd} \Sigma_1) \ast \Sigma_2) \cup (\Sigma_1 \ast ((\text{Bd} \Sigma_2)))\).
3. If any side of \(7.3.1\) holds; \(\Sigma_1 \ast \Sigma_2\) is orientable \(_k\) \(\iff \Sigma_1, \Sigma_2\) are both orientable \(_k\).

**Proof.** \(7.3.1\) [Pseudomanifolds.] Lemma \(5.6\) p. \(32\)

Proposition \(4.11\) p. \(28\) and Theorem \(4.13\) p. \(28\) gives the rest, but we still add a semi-combinatorial proof - as follows.

\(- [\text{Quasi-manifolds; }\times]\) Put \(n := \dim \Sigma_1 \times \Sigma_2 = \dim \Sigma_1 + \dim \Sigma_2 = n_1 + n_2\). The invariance of local Homology within \(\text{Int} \Sigma_1 \times \text{Int} \Sigma_2\) implies, through Prop. 1 p. \(23\) that, without loss of generality, we will only need to study simplices with \(c_\ast = 0\) (Def. p. \(31\)). Put \(v := \dim \sigma = \dim \sigma_1 + \dim \sigma_2 =: v_1 + v_2\). Recalling from Theorem \(7.0.1\) a p. \(15\) that now,
all links are finite quasi-manifolds, “σ ∈ BdΣ₁ × Σ₂ ⇐⇒ σ₁ ∈ BdΣ₁ or σ₂ ∈ BdΣ₂” follows from Corollary 5.3. p. 30 which, after deletion of a trivial torsion term through Note 1 p. 26, gives;

\[ \tilde{H}_{n-1}(Lkσ; k)^{k} \cong \tilde{H}_{n-1}(Lkσ; k \otimes k) \cong \tilde{H}_{n-1}(Lkσ₁; k) \otimes k \tilde{H}_{n-1}(Lkσ₂; k) \]

[Pseudomanifolds; ×] Reason as for Quasi-manifolds but restrict to submaximal simplices only, the links of which are 0-pseudomanifolds. ▷

[Pseudo- and Quasi-manifolds; *] Use Corollary 5.3. p. 30 as for products, cf. [13] 4.2 pp. 171-2.

\[(7.9.3) \quad (Σ₁ ∨ Σ₂, Bd⁺(Σ₁ ∨ Σ₂)) = \]

\[= \left[ \text{Motivation: According to the pair-definition p. 13} \right] = \]

\[= (Σ₁, BdΣ₁) ∨ (Σ₂, BdΣ₂). \]

Use Eq. 1 p. 24 (resp. Eq. 3 p. 27) and Note 3 p. 15. □

**Note.** In general, the equivalence “σ ∈ BdΣ₁ × Σ₂ ⇐⇒ σ₁ ∈ BdΣ₁ or σ₂ ∈ BdΣ₂” for quasi-manifolds (and homology manifolds) can be investigated through the general case of Corollary 5.3. p. 30 which, after deletion of a trivial torsion term through Note 1 p. 26, gives;

\[ \tilde{H}_{n-1}(Lkσ; G) \overset{Z}{\cong} \tilde{H}_{n-1}(Lkσ₁; Z) \otimes \tilde{H}_{n-1}(Lkσ₂; G) \oplus\]

\[\oplus \text{Tor}^{1}_{Z}(\tilde{H}_{n-1}(Lkσ₁; Z), \tilde{H}_{n-1}(Lkσ₂; G)) .\]

The coefficient module in the last theorem was restricted to a field k, but it is possible to work with arbitrary coefficient modules and infinite manifolds with respect to orientation in Th. 7.8.3 as follows. By Cor. 7.8 iii p. 10 one can without loss of generality confine the study to pseudomanifolds, and then choose the coefficient module to be, say, a field k (\text{char} k \neq 2). Since any finite max-dimensional submanifold, i.e. a submanifold of maximal dimension, in Σ₁ × Σ₂ (Σ₁ * Σ₂, cf. [50] (3.3) p. 59) can be embedded in the product (join) of two finite max-dimensional submanifolds, and vice versa, one might confine, without loss of generality, the attention to finite max-dimensional submanifolds \(S₁, S₂\) of \(Σ\), resp. \(Σ₂\).

Now, use Eq. 1 p. 24 (Eq. 3 p. 27) and Note 3 p. 15.

**Example. 1.** For a triangulation \(C\) of a two-dimensional cylinder \(\text{Bd} \mathcal{C} = \text{two circles}\).

\(\text{Bd} \cdot = \{\emptyset\}\). By Th. 7.8.3 \(\text{Bd} (C * \cdot)_{q} = \mathcal{C} \cup \{\text{two circles}\} * \cdot\). So, \(\text{Bd}_2(C * \cdot)_{q}, \mathbb{R}^3\)-realizable as a pinched torus (or inwards as in p. 8), is a 2-pseudomanifold but not a quasi-manifold.

**2.** Cut, twist and glue along the cut to turn the cylinder \(C\) above into a Möbius band \(\mathcal{M}\).

“The boundary (with respect to \(Z\)) of the cone of \(\mathcal{M}\)” =

\(\text{Bd}_2(\mathcal{M} * \cdot)_{q} = (\mathcal{M} * \{\emptyset\}) \cup \{\text{a circle}\} * \cdot = \mathcal{M} \cup \{\text{a 2-disk}\}\)

which is a well-known representation of the real projective plane \(\mathbb{RP}^2\), i.e. a homology₂p 2-manifold with boundary \(\{\emptyset\} \neq 0\) if \(p \neq 2\), cf. [24] p. 36.

\(\mathbb{Z}_p := \text{The prime-number field modulo } p\), i.e. of characteristic \(p\).
\( \mathbb{RP}^2 \# S^2 = \mathbb{RP}^2 = \mathcal{M} \cup_{\text{bd}} \{ \text{a 2-disk} \} \) confirms the obvious — that the \( n \)-sphere is the unit element with respect to the connected sum of two \( n \)-manifolds, cf. \( p. 38ff + \text{Ex. 3 p. 366} \). Leaving the \( S^2 \)-hole empty turns \( \mathbb{RP}^2 \) into \( \mathcal{M} \).

Let “\( \cup \)” denote “union through identification of boundaries”.

\[ \text{Bd}_n(\mathcal{M} \ast S^1)_q = (\mathcal{M} \ast \emptyset) \cup (S^1 \ast S^1) = S^3. \]

So,

\[ \text{Bd}_n(\mathcal{M}_1 \ast \mathcal{M}_2)_q = (S^1 \ast \mathcal{M}_2)_q \cup (\mathcal{M}_1 \ast S^1)_q; \]

is \( \text{Bd}_n(\mathcal{M}_1 \ast \mathcal{M}_2)_q \) orientable or is \( \text{Bd}_n(\text{Bd}_n(\mathcal{M}_1 \ast \mathcal{M}_2)_q)_q \)?

3. Let \( \mathbb{RP}^2_2, \mathbb{RP}^4 \) be triangulations of the 2-, resp. 4-dimensional real projective plane/space, then \( \text{Bd}_n(\mathbb{RP}^2_2) = \text{Bd}_n(\mathbb{RP}^4_2) = \{ \emptyset \} \). So, by Theorem \( \text{p. 17} \),

\[
\begin{cases}
\text{Bd}_n(\mathbb{RP}^2_2 \ast \mathbb{RP}^4_2)_q = \mathbb{RP}^4_2 \cup \mathbb{RP}^2_2, & \text{if } p \neq 2,
\text{Bd}_n(\mathbb{RP}^2_2 \ast \mathbb{RP}^4_2)_q = \emptyset, & \text{if } p = 2.
\end{cases}
\]

Note that \( \dim \text{Bd}_n(\mathbb{RP}^2_2 \ast \mathbb{RP}^4_2)_2 = 4 \) while \( \dim(\mathbb{RP}^2_2 \ast \mathbb{RP}^4_2)_2 = 7 \), cf. Corollary \( \text{p. 45} \), \( \text{Bd}_n(\mathbb{RP}^2_2 \ast \ast)_q = \ast \ast \) and \( \text{Bd}_n(\mathbb{RP}^2_2 \ast \ast)_q = \mathbb{RP}^2 \cup \ast \).

4. (cp. \( \text{p. 34} \) p. 198 Ex. 16 (Surgery)) If \( \mathcal{E}^m := \text{“the } m \text{-unit ball” } \) and \( n := p + q, p, q \geq 0 \), then,

\[ \mathcal{S}^n = \text{Bd}(\mathcal{E}^{n+1}) \approx \text{Bd}(\mathcal{E}^p \ast \mathcal{E}^q) \approx \mathcal{E}^p \ast \mathcal{S}^{q-1} \cup \mathcal{S}^p \ast \mathcal{E}^q \approx \text{Bd}(\mathcal{E}^{p+1} \times \mathcal{E}^q) \approx \mathcal{E}^{p+1} \times \mathcal{S}^{q-1} \cup \mathcal{S}^p \times \mathcal{E}^q, \]

by Theorem \( \text{p. 3} \) and Lemma \( \text{p. 17} \), \( \mathcal{S}^{n+1} \approx \mathcal{S}^p \ast \mathcal{S}^q \) and \( \mathcal{E}^{n+1} \approx \mathcal{E}^p \ast \mathcal{S}^q \) also hold.

See also \( \text{p. 376} \) for some non-intuitive manifold examples. Also \( \text{pp. 123-131} \) gives insights on different aspects of different kinds of simplicial manifolds.

**Proposition 7.10.** If \( \Sigma \) is finite and \( -1 \leq \dim \mathcal{B}_n \delta < \dim \Sigma - 1 \) then \( \text{Lk}_\delta \Sigma \) is non-orientable for all \( \delta \in \mathcal{B}_{n-1} \) (Note that Cor. 2.i p. \( \text{p. 16} \) is the special case \( \text{Lk}_0 \emptyset = \Sigma \).)

**Proof.** If \( \sigma \in \text{Bd}_n(\text{Lk}_{\delta})_q \) then \( \dim(\text{Bd}_n(\text{Lk}_{\delta})_q \leq n - \# \sigma - 3 \) in \( \text{rk}_{\Sigma} \sigma \), cf. Note 3 p. \( \text{p. 16} \) \( \square \)

### 7.4. Boundaries of simplicial homology manifolds.

In this section we will work mainly with finite simplicial complexes.

The coefficient module plays, through the St-R ring functor, a more delicate role in commutative ring theory than it does here in our Homology theory. So, having a CM-complex, its St-R ring may not be CM if the coefficient module is changed into one that isn’t a CM-ring.

**Lemma 7.11.** i. \( \Sigma \) is a homology manifold if and only if; \( \{ \Sigma = \bullet \bullet \text{ or } [ \Sigma \text{ is connected and} \text{Lk} \beta \text{ is a finite CM}_m \text{ pseudo manifold for all vertices } \beta \in \mathcal{V}_m \} \text{ and } \{ \text{Bd}_n(\Sigma) = \text{Bd}_n(\Sigma) \text{ or else; } [ \text{Bd}_n(\Sigma) = \emptyset \neq \emptyset = \emptyset \} = \text{Bd}_n(\Sigma) \text{ or } [ \exists \emptyset \neq \emptyset \in \mathcal{S}^{n-3} \text{ and } \text{Tor}^2(\mathbb{Z}_2, \mathbb{G}) = \mathbb{G} \} \} \} \}

ii. For a CM homology manifold \( \Delta \), \( \text{Bd}_n(\Delta) = \text{Bd}_n(\Delta) \) for any module \( \mathbb{G} \). So, for any homology manifold \( \Sigma \), \( \text{Bd}_n(\Sigma) = \emptyset, \{ \emptyset \} \) or \( \dim \mathcal{B}_n = (n-1) \) for boundary components.

**Proof.** i. Prop. \( \text{p. 34} \), Lemma \( \text{p. 17} \) and the proof of Corollary \( \text{p. 17} \) p. \( \text{p. 17} \)

ii. Use Theorem \( \text{p. 17} \) and i along with \( \mathbb{G} = \mathbb{Z}_2 \) in Corollary \( \text{p. 17} \).
Lemma 7.12. For a finite $\Sigma$, the boundary homomorphism

$$\delta_{q}: \hat{H}_{q}(\Sigma, \text{Bd}_q \Sigma; G) \to \hat{H}_{q-1}(\Sigma, \bigcup_{j \neq i} (\cup B_j); G)$$

in the relative $M$-Vs, with respect to $\{(\Sigma, B_{i}), (\Sigma, \bigcup_{j \neq i} B_{j})\}$ is injective if $\# I \geq 2$ in Def. 4 p. 43.

So,

$$[\hat{H}_{q-1}(\Sigma, \{0\}; G) = 0] \implies [\hat{H}_{q}(\Sigma, \text{Bd}_q \Sigma; G) = 0 \text{ or } \text{Bd}_q \Sigma \text{ is strongly connected},$$

as is the case e.g. if $\Sigma \neq \emptyset$, $\bullet \bullet$ is a $CM_{\Sigma}$ quasi-$n$-manifold.

If $\Sigma$ is a finite $CM_{\Sigma}$ quasi-$n$-manifold then $\text{Bd}_q \Sigma = \emptyset$ or it is strongly connected and $\dim(\text{Bd}_q \Sigma) = n_{\Sigma} - 1$ (by Lemma 7.11).

Proof. Use the relative $M$-Vs with respect to $\{(\Sigma, B_{i}), (\Sigma, \bigcup_{j \neq i} B_{j})\}$, $\dim(B_i \cap \bigcup_{j \neq i} B_{j}) \leq n_{\Sigma} - 3$ and Corollary 7.5 i p. 15.

\[ \square \]

Theorem 7.13. i. If $\Sigma$ is a finite orientable $G$ $CM_{\Sigma}$ homology $n$-manifold with boundary then, $\text{Bd}_q \Sigma$ is an orientable homology $(n-1)$-manifold without boundary.

ii. Moreover; $\text{Bd}_q \Sigma$ is Gorenstein $G$.

(This theorem is a special case of a theorem, attributed to M. Hochster, given in [10] p. 70 Th. 7.3, where it is proven through the use of the canonical module of the St-R ring of $\Sigma$)

Proof. i. Induction over the dimension, using Theorem 7.1 i-ii, once the connectedness of the boundary is established through Lemma 7.12, while orientability, resp. $\text{Bd}_q(\text{Bd}_q \Sigma) = \emptyset$ follows from Corollary 7.8 iii-iv p. 43.

ii. $\Sigma \ast (\bullet \bullet)$ is a finite orientable $G$ $CM_{\Sigma}$ homology $(n+1)$-manifold with boundary by Th. 7.3 + Th. 7.12 iii p. 21.

$$\text{Bd}_q(\Sigma \ast \bullet \bullet) = \frac{[\text{Th. VII}]}{[\text{Th. VII}]} \bigcup (\Sigma \ast (\text{Bd}_q(\bullet \bullet))) \bigcup ((\text{Bd}_q \Sigma) \ast (\bullet \bullet)) =$$

$$\Sigma \ast \emptyset \bigcup (\text{Bd}_q \Sigma) \ast (\bullet \bullet) = (\text{Bd}_q \Sigma) \ast (\bullet \bullet)$$

where the l.h.s. is an orientable homology $n$-manifold without boundary by the first part.

So, $\text{Bd}_q \Sigma$ is an orientable $G$ $CM_{\Sigma}$ homology $(n-1)$-manifold without boundary by Th. 7.3 i.e. it is Gorenstein $G$.

\[ \square \]

Note 1. $\emptyset \neq \Delta$ is a 2-$CM_{\Sigma}$ $\text{hm}_{\Sigma}$ $\iff \Delta = \text{core} \Delta$ is Gorenstein $G$ $\iff \Delta$ is a homology $G$ sphere.

Corollary 7.14. (Cf. [10] p. 190.) If $\Sigma$ is a finite orientable $G$ homology $n$-manifold with boundary, so is $2 \Sigma$ except that $\text{Bd}_q(2 \Sigma) = \emptyset$. $2 \Sigma = \text{"the double of } \Sigma := \Sigma \cup \hat{\Sigma}$ where $\hat{\Sigma}$ is a disjoint copy of $\Sigma$ and $\text{"} \cup \text{"}$ is "the union through identification of the boundary vertices". If $\Sigma$ is $CM_{\Sigma}$ then $2 \Sigma$ is $2$-$CM_{\Sigma}$.

Proof. Use [23] p. 57 (23.6) Lemma, i.e., apply the non-relative augmental $M$-Vs to the pair $(\text{Lk}_\Sigma \text{v}, \text{Lk}_\Sigma \text{v})$ using Prop. 6.2 a p. 53 and then to $(\Sigma, \hat{\Sigma})$ for the $CM_{\Sigma}$ case.

\[ \square \]
Theorem 7.15. If $\Sigma$ is a finite $CM_{\mathbb{Z}}$-homology $\mathbb{Z}n$-manifold, then $\Sigma$ is orientable.

Proof. $\Sigma$ finite $CM_{\mathbb{Z}}$ $\iff$ $\Sigma$ CM $\mathbb{Z}_{p}$ for all prime fields $\mathbb{Z}_{p}$, i.e. for characteristic $p > 0$, (M.A. Reisner, 1976, cf. [23] p. 181-2.) by induction over dim $\Sigma$, Theorem 4.9 p. 24 and the Structure Theorem for Finitely Generated Modules over PIDs. So, $\Sigma$ is a finite $CM_{\mathbb{Z}_{p}}$ homology $\mathbb{Z}_{p}n$-manifold for any prime number $p$ by Lemma 7.11, since $Bd_{\mathbb{Z}}\Sigma = Bd_{\mathbb{Z}_{p}}\Sigma$ by Lemma 7.11 ii above. In particular, $Bd_{\mathbb{Z}_{2}}\Sigma$ is Gorenstein, by Lemma 7.1 ii p. 43 and Theorem 7.13 above. If $Bd_{\mathbb{Z}_{2}}\Sigma = \emptyset$ then $\Sigma$ is orientable. Now, if $Bd_{\mathbb{Z}_{2}}\Sigma \neq \emptyset$ then dim $Bd_{\mathbb{Z}_{2}}\Sigma = n - 1$ by Cor. 7.8 i+iv p. 46, and, in particular, $Bd_{\mathbb{Z}_{2}}\Sigma = Bd_{\mathbb{Z}}\Sigma$ is a quasi-$(n-1)$-manifold.

$Bd_{\mathbb{Z}_{2}}(Bd_{\mathbb{Z}_{2}}\Sigma) = \emptyset$ since $Bd_{\mathbb{Z}_{2}}\Sigma$ is Gorenstein and so, dim $Bd_{\mathbb{Z}_{2}}(Bd_{\mathbb{Z}_{2}}\Sigma) \leq n - 4$ by Cor. 7.7 p. 45. So if $Bd_{\mathbb{Z}_{2}}(Bd_{\mathbb{Z}_{2}}\Sigma) \neq \emptyset$ then, by Cor. 7.8 ii p. 46, $Bd_{\mathbb{Z}_{2}}\Sigma$ is nonorientable i.e.

$$\hat{H}_{n-1}(Bd_{\mathbb{Z}_{2}}\Sigma, Bd_{\mathbb{Z}_{2}}(Bd_{\mathbb{Z}_{2}}\Sigma); \mathbb{Z}) = \hat{H}_{n-1}(Bd_{\mathbb{Z}_{2}}\Sigma; \mathbb{Z}) = 0$$

and the torsion submodule of

$$\hat{H}_{n-2}(Bd_{\mathbb{Z}_{2}}\Sigma, Bd_{\mathbb{Z}_{2}}(Bd_{\mathbb{Z}_{2}}\Sigma); \mathbb{Z}) = \left[\text{For dimensional reasons.}\right] = \hat{H}_{n-2}(Bd_{\mathbb{Z}}\Sigma; \mathbb{Z})$$

is isomorphic to $\mathbb{Z}_{2}$ by Lemma 7.1 i p. 13. In particular $\hat{H}_{n-2}(Bd_{\mathbb{Z}_{2}}\Sigma; \mathbb{Z}) \otimes \mathbb{Z}_{2} \neq 0$ implying, by Th. 4.9 p. 24, that

$$\hat{H}_{n-2}(Bd_{\mathbb{Z}_{2}}\Sigma; \mathbb{Z}) = \hat{H}_{n-2}(Bd_{\mathbb{Z}_{2}}\Sigma; \mathbb{Z}_{2}) = \hat{H}_{n-2}(Bd_{\mathbb{Z}}\Sigma; \mathbb{Z}) \otimes \mathbb{Z}_{2} \oplus \text{Tor}^{\mathbb{Z}}_{1}(\hat{H}_{n-3}(Bd_{\mathbb{Z}}\Sigma; \mathbb{Z}), \mathbb{Z}_{2}) \neq 0$$

contradicting the Gorenstein of $Bd_{\mathbb{Z}_{2}}\Sigma$. \hfill $\Box$

Corollary 7.16. Each simplicial homology $\mathbb{Z}_{n}$ manifold (hm$_{\mathbb{Z}_{2}}$) is locally orientable.

Proof. Proposition 6.8 p. 35. \hfill $\Box$

Note. 2. Corollary 7.16 confirms G.E. Bredon's conjecture stated in [3] Remark p. 384 just after the definition of Borel-Moore cohomology manifolds with boundary, the reading of which is aiming at Poincaré duality and therefore only allows $BdX = \emptyset$ or dim $BdX = n - 1$, cf. our Ex. 3 p. 19. Weak homology manifolds over PIDs are defined in [3] p. 329 and again we would like to draw the attention to their connection to Buchsbaum rings, cf. our p. 33 and that, for polytopes, “join” becomes “tensor product” under the St-R ring functor as mentioned in our Note iii p. 34.
Part 4. Appendices and Addenda

8. APPENDICES

8.1. The $3 \times 3$-lemma (also called “the 9-lemma”). In this appendix we will make use of the relative Mayer-Vietoris sequence (M-Vs) to form a 9-Lemma-grid clarifying some relations between Products and Joins, e.g. that $\{X_1 \Join Y_2, \ X_2 \Join Y_1\}$ is excisive if $\{X_1 \times Y_2, X_2 \times Y_1\}$ is.

12 Ex. 6.16 p. 175-6 ($3 \times 3$ Lemma) (J.J. Rotman: An Intr. to Homological Algebra.) Consider the commutative diagram of modules (on the right): If the columns are exact and if the bottom (or the top two) rows are exact, then the top row (or the bottom row) is exact. (Hint: Either use the Snake lemma or proceed as follows: first show that $\alpha \alpha^\prime = 0$, then regard each row as a complex and the diagram as a short exact sequence of complexes, and apply theorem 6.3.)

12 Ex. 5, p. 208. (S. MacLane: Categories for the Working Mathematician.) A $3 \times 3$ diagram is one of the form (to the right) (bordered by zeros).

(a) Give a direct proof of the $3 \times 3$ lemma: If a $3 \times 3$ diagram is commutative and all three columns and the last (first) two rows are short exact sequences, then so is the last (first) row.

(b) Show that this lemma also follows from the ker-coker sequence.

(c) Prove the middle $3 \times 3$ lemma: If a $3 \times 3$ diagram is commutative, and all three columns and the first and third rows are short exact sequences, then so is the middle row.

12 Ex. 16 p. 227 (P.J. Hilton and S. Wylie: Homology Theory.) Let (the diagram on the right) be a commutative diagram in which each row and each column is an exact sequence of differential groups (Def. p. 99). Then there are defined (see 5.5.1 (p. 196)) homomorphism $\nu^\flat: H(C_i) \to H(C_j)$, $\nu^\sharp: H(C_j) \to H(A_i)$, $\nu^\#$: $H(C_j) \to H(A_i)$. Prove that $\nu^\flat \nu^\sharp = -\nu^\sharp \nu^\flat$. (Reversed notation.)

12 Ex. 2 p. 53 (A. Dold: Lectures on Algebraic Topology.) If (the diagram on the right) is a commutative diagram of chain maps with exact rows and columns then the homology sequence of these rows and columns constitute a 2-dimensional lattice of group homomorphisms. It is commutative except for the $(q,q)$-squares which anticommute.

Put

\[
\begin{align*}
\hat{C}X := & X \Join \{(x_0, 1)\} \text{ where } x_0 \in X_2 \text{ and } (x_0, 1) := (\overline{x_0}, y_0, 1) \in X \Join Y \\
\hat{C}Y := & Y \Join \{(y_0, 0)\} \text{ where } y_0 \in Y_2 \text{ and } (y_0, 0) := (x_0, y_0, 0) \in X \Join Y \quad \text{(}\#\text{-def. p. 13)}.
\end{align*}
\]

Put

\[
\begin{align*}
A := & \hat{C}X \times Y_2 \\
B := & \hat{C}X_2 \times Y_1 \\
C := & X_2 \times \hat{C}Y_1 \\
D := & X_2 \times \hat{C}Y_1
\end{align*}
\]

then

\[
\begin{align*}
\{A \cap B \cap C \cap D = \{A \cap (B \cup C)\} & = X_2 \times Y_2 \\
\{A \cap (B \cup C) \cap D = \{A \cup (B \cap C)\} & = X_2 \Join Y_2 \\
\{A \cup (B \cap C) \cap D = \{A \cup (B \cup C)\} & = X_2 \times Y_2 \cup X_2 \times Y_2
\end{align*}
\]

All operations are assumed to take place within $X_1 \Join Y_1$.

The last commutative square provide us with a “9-lemma”-grid as that above, constituted by two horizontal and two vertical relative Mayer-Vietoris sequences, using the naturality of the M-Vs as in Munkres p. 186-7.
The entries consist of three levels where the upper (lower) concerns the vertical (horizontal) M-Vs. $X_2$, $Y_2$ are assumed to be non-empty.

\[ \delta_3 \downarrow \delta_1 \downarrow \delta_2 \quad (#) \quad \delta_1 \downarrow \delta_2 \quad (#) \quad \delta_2 \downarrow \quad (#) \]

The groups within curly braces in the last row doesn’t play any part in the deduction of the excisivity equivalence but fits the underlying pattern.
8.2. Simplicial calculus. The complex of all subsets of a simplex \( \sigma \) is denoted \( \bar{\sigma} \), while the boundary of \( \sigma \), \( \partial \), is the set of all proper subsets.

So, \( \bar{\sigma} := \{ \tau \mid \tau \nsubseteq \sigma \} = \bar{\sigma} \setminus \{ \sigma \} \), \( \emptyset \) and \( \emptyset \).

The closed star of \( \sigma \) with respect to \( \Sigma \) = \( \mathbb{S}_{\Sigma} \sigma := \{ \tau \in \Sigma \mid \sigma \cup \tau \in \Sigma \} \).

The open (realized) star of \( \sigma \) with respect to \( \Sigma = \mathcal{S}_{\Sigma} \sigma := \{ \alpha \in |\Sigma| \mid [v \in \sigma] \implies [\alpha(v) \neq 0] \} \).

So, \( \mathcal{S}_{\Sigma} \sigma \) = \( \{ \alpha \in |\Sigma| \mid \alpha \in \bigcap_{v \in \mathcal{S}_{\Sigma} \sigma} [v] \} \), \( \alpha \neq \mathcal{S}_{\Sigma} \sigma \) except for \( \mathcal{S}_{\Sigma} \emptyset = |\Sigma| \).

The closed geometrical simplex \( |\sigma| \) with respect to \( \Sigma \) = “The (realized) closure of \( \sigma \) with respect to \( \Sigma \) = \( \{ \alpha \in |\Sigma| \mid [\alpha(v) \neq 0] \implies [v \in \sigma] \} \). So, \( |\emptyset| = \{ \alpha_0 \} \).

The interior of \( \sigma \) with respect to \( \Sigma \) = \( \text{Int}(\sigma) := \{ \alpha \in |\Sigma| \mid [\alpha(v) \neq 0] \implies [v \in \sigma] \} \). So, \( \text{Int}(\emptyset) := \{ \alpha_0 \} \).

The barycenter \( \hat{\sigma} \) of \( \sigma \neq \emptyset \) is \( \alpha \in \text{Int}(\sigma) \) fulfilling \( v \in \sigma \implies \alpha(v) = \frac{1}{|\sigma|} \) while \( \emptyset \).

The link of \( \sigma \) with respect to \( \Sigma \) = \( \text{Lk}_\Sigma \sigma := \{ \tau \in \Sigma \mid [\sigma \cap \tau = \emptyset] \land [\sigma \cup \tau \in \Sigma] \} \). So, \( \text{Lk}_\Sigma \emptyset = \Sigma \), \( \sigma \in \text{Lk}_\Sigma \tau \iff \tau \in \text{Lk}_\Sigma \sigma \) while \( \text{Lk}_\Sigma \emptyset = \{ \emptyset \} \) if \( \tau \in \Sigma \) is maximal.

The join \( \Sigma_1 \star \Sigma_2 := \{ \sigma_1 \cup \sigma_2 \mid \sigma_i \in \Sigma_i \ (i = 1, 2) \} \). In particular \( \Sigma \star \emptyset = \emptyset \).

**Proposition 8.1.** If \( \Sigma_1 \) and \( \Sigma_2 \) has mutually exclusive vertex sets, \( V_{\Sigma_1} \cap V_{\Sigma_2} = \emptyset \) then,

a) \( \tau \in \Sigma_1 \star \Sigma_2 \iff \exists! \sigma_i \in \Sigma_i \) so that \( \tau = \sigma_1 \cup \sigma_2 \),

and:

b) \( \text{Lk}_\Sigma (\sigma_1 \cup \sigma_2) = (\text{Lk}_\Sigma \sigma_1) \star (\text{Lk}_\Sigma \sigma_2) \).

**Proof.** a) follows immediately from the definition.

b) Suppose that \( V_{\Sigma_1} \cap V_{\Sigma_2} = \emptyset \) and that \( \sigma_1 \cup \sigma_2 \in \Sigma_1 \star \Sigma_2 \) where \( \sigma_i \in \Sigma_i \ (i = 1, 2) \).

\[
\text{Lk}_{\Sigma_1 \star \Sigma_2} (\sigma_1 \cup \sigma_2) \overset{\text{Def.}}{=} \{ \mu_0 \in \Sigma_1 \star \Sigma_2 \mid \mu_0 \cap (\sigma_1 \cup \sigma_2) = \emptyset \} \text{ and } \mu_0 \cup (\sigma_1 \cup \sigma_2) \in \Sigma_1 \star \Sigma_2 \}
\]

This equality follows from the first part i.e. from a):

\[
\mu_0 = \mu_1 \cup \mu_2 | [\mu_i \in \Sigma_i \ (i = 1, 2)] \text{ and } [\mu_1 \cup \mu_2 \cap (\sigma_1 \cup \sigma_2) = \emptyset] \}
\]

\[
[(\mu_1 \cup \mu_2) \cup (\sigma_1 \cup \sigma_2) \in \Sigma_1 \star \Sigma_2] = \{ \text{[Here we use that } V_{\Sigma_1} \cap V_{\Sigma_2} = \emptyset] \}
\]

\[
\{ \mu_1 \cup \mu_2 | [\mu_i \in \Sigma_i \ (i = 1, 2)] \text{ and } [\mu_1 \cup \sigma_1 = \emptyset] \} \land [\mu_2 \cup \sigma_2 = \emptyset] \}
\]

\[
[(\mu_1 \cup \sigma_1) \cup (\mu_2 \cup \sigma_2) \in \Sigma_1 \star \Sigma_2] = \{ \text{[Uniqueness from a]} \}
\]

\[
= \{ \mu_1 \cup \mu_2 | [\mu_i \in \Sigma_i \ (i = 1, 2)] \text{ and } [\mu_1 \cup \sigma_1 = \emptyset] \} \land [\mu_2 \cup \sigma_2 = \emptyset] \}
\]

\[
= \{ \text{[By the definition of “Lk”]} \}
\]

\[
= (\text{Lk}_{\Sigma_1} \sigma_1) \star (\text{Lk}_{\Sigma_2} \sigma_2)
\]

With the convention \( \text{Lk}_\emptyset \sigma := \emptyset \), any link is an iterated link of vertices and

\[
\text{Lk}_{\emptyset} \sigma = \emptyset \star \text{Lk}_{\emptyset} \sigma = \text{Lk}_{\emptyset} \sigma = \text{Lk}_{\emptyset} (\sigma \cup \tau) = \text{Lk}_{\emptyset} (\sigma \cup \tau) = \text{Lk}_{\emptyset} (\sigma \cup \tau) = \text{Lk}_{\emptyset} (\sigma \cup \tau)
\]

So, \( \tau \neq \text{Lk}_{\emptyset} \sigma \Rightarrow \text{Lk}_{\emptyset} \tau = \emptyset \) while:

\[
\tau \in \text{Lk}_{\emptyset} \sigma \Rightarrow \text{Lk}_{\emptyset} \tau = \text{Lk}_{\emptyset} (\sigma \cup \tau) = \text{Lk}_{\emptyset} (\sigma \cup \tau) \subset \text{Lk}_{\emptyset} \sigma \cap \text{Lk}_{\emptyset} \tau).
\]
With $\dim \Sigma = 2$, the **edge contraction** of $\{v, w\} \in \Sigma$ in $|\Sigma|$ is topology preserving if and only if $Lk_{\Sigma}^{|v|} = Lk_{\Sigma}^{|v|} \cap Lk_{\Sigma}^{|v|}$.

$$\tau \in Lk_{\Sigma}^{|\Sigma|} \implies \overline{Lk_{\Sigma}^{|\Sigma|}} = \overline{\tau} \ast Lk_{\Sigma}^{|\Sigma|} = [\text{Eq. I}] = \overline{\tau} \ast Lk_{\Sigma}^{|\Sigma|} \ast \tau = \overline{Lk_{\Sigma}^{|\Sigma|}} = Lk_{\Sigma}^{|\Sigma|}$$

The $p$-**skeleton** is defined by, $\Delta^{(p)} := \{ \delta \in \Delta \mid \# \delta \leq p + 1 \}$. So, $\Delta^{(n)} = \Delta$, if $n := \dim \Delta$.

The following definitions simplifies notations: $\Delta' := \Delta^{(n-1)}$, $\Delta' := \Delta^{(p)} \setminus \Delta^{(p-1)}$.

$\Gamma \subset \Sigma$ is said so be **full** in $\Sigma$ if for all $\sigma \in \Sigma$; $\sigma \subset V_{\Gamma} \implies \sigma \in \Gamma$.

**Proposition 8.2.**

1. $Lk_{\delta} = Lk_{\delta} \cup Lk_{\delta}$.
2. $Lk_{\delta} = Lk_{\delta} \cap Lk_{\delta}$.
3. $\big( \Delta \ast \Delta \big) = \big( \Delta \ast \Delta \big) \cup \big( \Delta \ast \Delta \big)$.
4. $\Delta \ast \big( \bigcup_{i=1}^{n} \Delta \big) = \bigcup_{i=1}^{n} \Delta \ast \Delta$, and $\Delta \ast \big( \bigcup_{i=1}^{n} \Delta \big) = \bigcup_{i=1}^{n} \Delta \ast \Delta$.
5. (iv) also holds for arbitrary topological spaces under the $\hat{\text{I}}$-join.

**Proof.**

1. Associativity and distributivity of the logical connectives.

2. $\Delta$ pure $\iff Lk_{\delta} = Lk_{\delta} \cap \delta \forall \delta \in \Delta$.

3. $[\Gamma \cap Lk_{\delta} = Lk_{\delta} \cap \delta \in \Gamma] \iff [\Gamma \text{ is full in } \Delta] \iff [\Gamma \cap st_{\delta} = \overline{st}_{\delta} \forall \delta \in \Gamma]$.

**Proof.**

1. $\Delta$ pure $\implies \Delta$ pure, so; $Lk_{\delta} = \{ \tau \in \Delta \mid \tau \cap \delta = \emptyset \wedge \tau \cup \delta \in \Delta \} = \{ \tau \in \Delta \mid \tau \cap \delta = \emptyset \wedge \tau \cup \delta \in \Delta \wedge \#(\tau \cup \delta) \leq n \} = \{ \tau \in Lk_{\delta} \mid \#(\tau \cup \delta) \leq n \} = \{ \tau \in Lk_{\delta} \mid \#(\tau \cup \delta) \leq \dim Lk_{\delta} \} = \big( Lk_{\delta} \big)'$.

2. $\Delta$ non-pure, then $\exists \delta_{m} \in \Delta'$ maximal in both $\Delta'$ and $\Delta$ i.e.,

$$\left( Lk_{\delta_{m}} \right)' = \left( \{ \emptyset \} \right)' = \emptyset \neq \{ \emptyset \} = Lk_{\delta_{m}}.$$ 

3. $Lk_{\sigma} = \{ \tau \mid \tau \cap \sigma = \emptyset \wedge \tau \cup \sigma \in \Gamma \} = \left[ \text{True \ for \ all } \sigma \in \Gamma \iff \Gamma \text{ is full} \right] = \{ \tau \in \Gamma \mid \tau \cap \sigma = \emptyset \wedge \tau \cup \sigma \in \Delta \} = \Gamma \cap Lk_{\sigma}$.

$$\Gamma \cap \overline{st_{\delta}} = \Gamma \cap \{ \tau \in \Gamma \mid \tau \cup \sigma \in \Delta \} \cup \{ \tau \in \Delta \mid \tau \notin \Gamma \wedge \tau \cup \sigma \in \Delta \} = \{ \tau \in \Gamma \mid \tau \cup \sigma \in \Delta \} = \left[ \text{True \ for \ all } \sigma \in \Gamma \iff \Gamma \text{ is a full subcomplex.} \right] = \{ \tau \in \Gamma \mid \tau \cup \sigma \in \Gamma \} = \overline{st_{\delta}}.$$ 

$\square$
The contrastwr of $\sigma$ with respect to $\Sigma = \text{cost,}\sigma := \{\tau \in \Sigma \mid \tau \not\subset \sigma\}$.
So, $\text{cost,}\emptyset = \emptyset$ and $\text{cost,}\sigma = \Sigma$ iff $\sigma \not\subset \Sigma$.

**Proposition 8.3.** Whether $\delta_1, \delta_2 \in \Delta$ or not, the following holds;

a. $\text{cost,}\delta_1 \cup \text{cost,}\delta_2 = \text{cost,}\delta_1 \cup \text{cost,}\delta_2$ and $\delta = \{v_1, \ldots, v_p\}$ \implies $\text{cost,}\delta = \bigcup_{i=1, p} \text{cost,}\delta_i$.

b. $\text{cost,}\delta_1 \cap \text{cost,}\delta_2 = \text{cost,}\delta_1 \cap \text{cost,}\delta_2$ and $\bigcap_{i=1, q} \text{cost,}\delta_i = \text{cost,}\delta_{\delta q}$

c. \begin{align*}
i. \ & \delta \not\in \Delta \iff [\text{Lk,}\delta = \text{cost,}\varnothing = \emptyset]. \\
ii. \ & \delta \in \Delta \iff [v \not\in \delta \iff \delta \in \text{cost,}\varnothing \iff \text{Lk,}\delta = \text{cost,}\varnothing \supset \{\varnothing\} (\not= \emptyset)]. \\
& (\delta \in \Delta \iff [v \in \delta \iff \delta \notin \text{cost,}\varnothing \iff \text{Lk,}\delta = \emptyset = \{\varnothing\} \subset \text{Lk,}\delta = \text{cost,}\varnothing].)
\end{align*}

d. $\delta, \tau \in \Delta \iff \{\delta \in \text{cost,}\tau \iff \text{Lk,}\delta \not\in \text{cost,}\tau \iff \text{Lk,}\delta \not\in \text{cost,}\tau \not\in \Delta \} \text{ or equivalently, } \text{iff } \tau \not\in \text{cost,}\tau \not\in \Delta.$

e. If $\delta \not\in \emptyset$ then;
\begin{enumerate}
\item $[\text{cost,}\delta] = \text{cost,}\delta \iff [n_\delta = n_\Delta]$
\item $[\text{cost,}\delta] = \text{cost,}\delta \iff [n_\delta = n_\Delta - 1]$
\end{enumerate}

 where $n_\Delta := \dim \text{cost,}\varnothing$ and $\Delta := \Delta^{(n_\Delta - 1)}$, with $n_\Delta := \dim \Delta < \infty$.

(\text{It is always true that;} \tau \subset \delta \implies n_\delta - 1 \leq n_\tau \leq n_\delta$, if $\emptyset \not\subset \tau, \delta$.)

f. $\text{st,}\delta \cap \text{cost,}\delta = \text{cost,}\delta \cap \text{st,}\delta \cap \text{Lk,}\delta$. (So, $\text{cost,}\delta$ is a quasi-/homology manifold if $\Delta$ is, by Th 7/2 p. 14, Th 7/9 p. 17 and 6/8 p. 35.)

\textbf{Proof.} If $\Gamma \subset \Delta$ then $\text{cost,}\gamma = \text{cost,}\gamma \cap \Gamma \forall \gamma$ and $\text{cost,}\delta_1 \cup \text{cost,}\delta_2 = \{\tau \in \Delta \mid \neg[\delta_1 \cup \delta_2 \subset \tau]\} = \{\tau \in \Delta \mid \neg[\delta_1 \subset \tau] \vee \neg[\delta_2 \subset \tau]\} = \text{cost,}\delta_1 \cup \text{cost,}\delta_2$, giving $a$ and $b$.

c. A "brute force"-check gives $c$, which is the "$\tau = \{v\}$"-case of $d$.

d. $\tau \in \text{Lk,}\delta \iff \{v\} \in \text{Lk,}\delta \forall \{v\} \in \Delta \iff \delta \in \text{Lk,}\varnothing \forall \varnothing \in \Delta$ gives $d$ from $a, c$ and Proposition 8.2 a.i above.

e. $\text{cost,}\delta = [\text{iff } n_\delta = n_\Delta - 1] = (\text{cost,}\delta) \cap \Delta = \text{cost,}\delta \supset (\text{cost,}\delta)' = [\text{iff } n_\delta = n_\Delta] = (\text{cost,}\delta) \cap \Delta'$.

f. $\text{st,}\delta = \text{st,}\delta \cap \text{Lk,}\delta = \text{st,}\delta \cap \{\tau \in \Delta \mid \delta \subset \tau\}$. \hfill \Box

With $\delta, \tau \in \Sigma; \delta \cup \tau \notin \Sigma \iff \text{st,}\delta \cap \text{st,}\tau = \{\alpha_\delta\} \iff \text{st,}\delta \cap \text{st,}\tau = \{\alpha_\tau\}$ and $|\text{st,}\delta| \setminus \text{st,}\tau = |\text{st,}\delta \cap \text{cost,}\tau| = |\text{st,}\delta \cap \text{cost,}\tau|$, by 372, 62.6 and Proposition 8.3 f above.

$$|\text{st,}\delta| = \{\tau \in \Sigma \mid \sigma \cup \tau \in \Sigma\} = \sigma \ast \text{Lk,}\sigma.$$  \hfill (III)

Identifying $|\text{cost,}\sigma|$ with its homeomorphic image in $|\Sigma|$ we get; $|\text{cost,}\sigma| \simeq |\Sigma| \setminus \text{st,}\sigma$ and $\text{st,}\sigma = |\Sigma| \setminus |\text{cost,}\sigma|$. 

8.3. Homology groups for arbitrary simplicial joins. As said in p. 10, it is easily seen what Whitehead’s $\tilde{S}(X \ast Y)$ need to fulfill to make $\tilde{S}(X \ast Y) \cong S(X) \otimes S(Y)$ true, since the right hand side is well known as soon as $\tilde{S}(X)$ and $\tilde{S}(Y)$ are known. But, a priori, we don’t know what $\tilde{S}((X, X) \ast (Y, Y))$ actually looks like. Now, Theorem 4.7 p. 23 gives the following complete answer:

$$\Delta((X, X) \ast (Y, Y)) \cong S(\Delta(X, X) \otimes \Delta(Y, Y)).$$

Below, we will express $H_q((\Gamma_1, \Delta_1) \ast (\Gamma_2, \Delta_2); G \otimes G')$ in terms of $H_i(\Gamma_1, \Delta_1; G)$ for arbitrary simplicial complexes. By Prop. 5.4, every simplex $\gamma \in \Gamma_i \ast \Gamma_2$ splits uniquely into $\gamma_1 \cup \gamma_2$ where $\gamma_i \in \Gamma_i \ast \Gamma_2$ “after” the vertices in $\Gamma_i$. Let $[\gamma_1] \cup [\gamma_2] := [\gamma]$ stand for the chosen generator representing $\gamma$ in $C^o_{v_1,v_2}(\Gamma_1 \ast \Gamma_2)$ and, extend the “$\ast$” distributively and linearly, c.f. 13 p. 228,

$$C^o_{v_1,v_2}(\Gamma_1 \ast \Gamma_2) = \bigoplus_{p+q=q_{v_1,v_2}}(C^p_{\Gamma_1} \ast C^q_{\Gamma_2}).$$

The following construction is taken from 14 p. 14.

The boundary function operates on generators as;

$$\delta^i_{v_1,v_2}([\gamma]) = \delta^i_{\dim \gamma_1 + \dim \gamma_2}([\gamma]) = \big((\delta^i_{\dim \gamma_1}([\gamma_1]) \ast [\gamma_2]) + ([\gamma_1] \ast (-1)^{i+1}\delta^i_{\dim \gamma_2}([\gamma_2])\big).$$

Define a graded morphism $f^\ast$ through its effect on the generators of $C^o_{v_1}(\Gamma_1 \ast \Gamma_2) = C^o_{v_1,v_2}((\Gamma_1 \ast \Gamma_2)$ by

$$f_+(\gamma) = f_{v+1}([\gamma_1] \ast [\gamma_2]) := [\gamma_1] \ast (-1)^{v+1}[\gamma_2],$$

$$[C^o(\Gamma_1) \otimes C^o(\Gamma_2)]_{v+1} = \bigoplus_{p+q=p_{v+1}+q_{v+1}}(C^p_{\Gamma_1} \otimes C^q_{\Gamma_2}).$$

The boundary function is given through its effect on generators:

$$\delta^o_{\Gamma_1} \otimes \delta^o_{\Gamma_2}([\gamma]) = \delta^o_{\dim \gamma_1 + \dim \gamma_2}([\gamma_1] \otimes [\gamma_2]) = \big((\delta^o_{\dim \gamma_1}([\gamma_1]) \otimes [\gamma_2]) + ([\gamma_1] \otimes (-1)^{i+1}\delta^o_{\dim \gamma_2}([\gamma_2])\big).$$

$f^\ast$ is obviously a chain isomorphism of degree -1, and we can conclude that

$$C^o(\Gamma_1 \ast \Gamma_2)$$

and $s(C^o(\Gamma_1) \otimes C^o(\Gamma_2))$ are isomorphic as chains, where the “$\ast$” stands for “suspension” meaning that the suspended chain equals the original except that the dimension $i$ in the original is dimension $i + 1$ in the suspended chain.

The argument now motivating the formula for relative simplicial homology is the same as that for the “relative singular homology”-case in p. 23.

Lemma. (The Eilenberg-Zilber theorem for simplicial join.) On the category of ordered pairs of arbitrary simplicial pairs $(\Gamma_i, \Delta_i)$ and $(\Gamma_j, \Delta_j)$ there is a natural chain equivalence of with

$$C^o(\Gamma_1 \ast \Gamma_2) / C^o((\Gamma_1 \ast \Delta_2) \cup (\Delta_1 \ast \Gamma_2))$$

$$s\left([C^o(\Gamma_1) / C^o(\Delta_1)] \otimes [C^o(\Gamma_2) / C^o(\Delta_2)]\right)$$

Proposition. (The Künneth Theorem for Arbitrary Simplicial Joins.) With $R$ a PID, $G$ and $G'$ $R$-modules and $\text{Tor}^R_1(G, G') = 0$, 13 p. 231 Cor. 4 gives;

$$\hat{H}_{q+1}(\Gamma_1, \Delta_1 \ast \Gamma_2, \Delta_2; G \otimes G') \cong \bigoplus_{i+j=q}(\hat{H}_i(\Gamma_1, \Delta_1; G) \otimes \hat{H}_j(\Gamma_2, \Delta_2; G')) \oplus \bigoplus_{i+j=q-1}(\hat{H}_i(\Gamma_1, \Delta_1; G) \otimes \hat{H}_j(\Gamma_2, \Delta_2; G')).$$

8.4. Local homology for products and joins of arbitrary simplicial complexes. 

**Corollary 5.3** (from p. 21) Let \( G \) and \( G' \) be arbitrary modules over a principal ideal domain \( R \) such that \( \text{Tor}_1^R(G, G') = 0 \), then, for any \( \emptyset \neq \sigma \in \Sigma_1 \times \Sigma_2 \) with \( \eta(\text{Int}(\sigma)) \subset \text{Int}(\sigma_1) \times \text{Int}(\sigma_2) \) and \( c_\sigma := \text{dim} \sigma_1 + \text{dim} \sigma_2 - \text{dim} \sigma \),

\[
\hat{H}_{i+\alpha+1}(\text{Lk}_{\Sigma_1 \times \Sigma_2} \sigma; G \otimes_R G') \cong \bigoplus_{p+q \geq 1} \text{Tor}_1^R(\hat{H}_p(\text{Lk}_{\Sigma_1} \sigma_1; G) \otimes_R \hat{H}_q(\text{Lk}_{\Sigma_2} \sigma_2; G')) \oplus \bigoplus_{p+q \geq 1} \text{Tor}_1^R(\hat{H}_p(\text{Lk}_{\Sigma_1} \sigma_1; G) \otimes R \hat{H}_q(\text{Lk}_{\Sigma_2} \sigma_2; G')).
\]

If \( \text{dim} \sigma = \text{dim} \sigma_1 + \text{dim} \sigma_2 \) then \( c_\sigma = 0 \) and so:

\[
\hat{H}_i(\text{Lk}_{\Sigma_1 \times \Sigma_2} \sigma; G) \cong \hat{H}_i(\text{Lk}_{\Sigma_1 \times \Sigma_2} (\sigma_1 \cup \sigma_2); G) \oplus \hat{H}_i(\text{Lk}_{\Sigma_1 \times \Sigma_2} \sigma; G) \otimes_R G'.
\]

**Proof.** The one-vertex complex \( \circ := \{\{v\}, \emptyset\} \) is a kind of unit-element with respect to “\( \times \)”, so, our claims are true if either \( \Sigma_1 \) or \( \Sigma_2 \) equals \( \circ \). Therefore, suppose that \( \Sigma_1 \neq \circ \neq \Sigma_2 \).

Lemma 5.2 p. 21 and [11] Th. 12.4 p. 89 gives (*),

\[
(*): \eta \circ \{\{\Sigma_1 \setminus \Sigma_j', |\Sigma_1 \setminus \Sigma_j| / (\alpha_1, \alpha_2)\} \sim \{\{\Sigma_1 \setminus \Sigma_j', |\Sigma_1 \setminus \Sigma_j| / (\alpha_1, \alpha_2)\}, \text{with} \eta(\alpha_1, \alpha_2) = (\alpha_1, \alpha_2).
\]

Note that \( \sigma = \emptyset \Rightarrow \sigma_j = \emptyset, j = 1, 2 \). Put \( v_j := \text{dim} \sigma_j, j = 1, 2 \) and \( v := v_1 + v_2 \).

\[
\hat{H}_{-i-\alpha-1}(\text{Lk}_{\Sigma_1 \times \Sigma_2} \sigma; G \otimes G') \cong [\text{Prop. 4.10 p. 22}] \cong \hat{H}_s(|\Sigma_1 \times \Sigma_2|, |\Sigma_1 \times \Sigma_2| \setminus \emptyset (\alpha_1, \alpha_2); G \otimes G') \cong \hat{H}_s(|\Sigma_1 \times \Sigma_2|, |\Sigma_1 \times \Sigma_2| \setminus \emptyset (\alpha_1, \alpha_2); G \otimes G').
\]

\[
\hat{H}_s(|\Sigma_1|, |\Sigma_1| \setminus \emptyset \alpha_1) \times (|\Sigma_2|, |\Sigma_2| \setminus \emptyset \alpha_2); G \otimes G') \cong \hat{H}_{p_1}(\Sigma_1, |\Sigma_1| \setminus \emptyset \alpha_1; G) \otimes R \hat{H}_{p_2}(\Sigma_2, |\Sigma_2| \setminus \emptyset \alpha_2; G').
\]

\[
\text{Eq. (3)p. 22} \cong \hat{H}_{p_1}(\Sigma_1, |\Sigma_1| \setminus \emptyset \alpha_1; G), \hat{H}_{p_2}(\Sigma_2, |\Sigma_2| \setminus \emptyset \alpha_2; G'))]
\]

Hence, Proposition 8.1 p. 54 and Theorem 4.8 p. 24 gives the second isomorphism even for \( \sigma_1 = \emptyset \), and/or \( \sigma_2 = \emptyset \).
8.5. On Relative Homeomorphisms and Quotients. An excision map \((X \setminus P, Y \setminus P) \rightarrow (X, Y)\) induces an isomorphism in singular homology if \(P \subseteq \text{Int}(Y)\) while “P open in X” is not needed, cf. \([12]\) p. 180-1. This will be used in the next proposition. Note that \(\tilde{H}(X; \bullet; G) \cong \tilde{H}(X; G)\).

**Proposition.** Let \(f: (X, A) \to (Y, B)\) be a relative homeomorphism, i.e., \(f: X \to Y\) is continuous, and \(f: X \setminus A \to Y \setminus B\) is a homeomorphism. If \(A \neq \emptyset\), \(\{\emptyset\}\) is a strong deformation retract of a neighborhood \(N\) and \(B\) and \(f(N)\) are closed subsets of \(N':=f(N \setminus A) \cup B = f(N) \cup B\), since \(f(A) \subset B\), then:

i. (from p. \([2]\)) \(B\) is a strong deformation retract of \(N'\).

If also \(A \subseteq \text{Int}(X), \tilde{B} \subseteq \text{Int}(Y')\) and \(\hat{f}: (X \setminus A, N \setminus A) \to (Y \setminus B, N' \setminus B)\) is a pair homeomorphism (which it is if \(X, Y\) is Hausdorff and \(N\) compact) then;

ii. (cp. \([7]\) p. 66 Lemma 7.3) \(\hat{H}(X, A; G) \cong \hat{H}(X, Y; B; G)\) for \(i \in \mathbb{Z}\) and \(\Delta(X, A) \cong \Delta(Y, B)\). \(G\) is an \(A\)-module where \(A\) is a commutative ring with unit.

iii. A closed \(\Rightarrow \hat{H}(X, A; G) \cong \hat{H}(X/A; \bullet; G) \cong \hat{H}(X; G)\). \(\Delta(X, A) \cong \Delta(X/A, \bullet)\).

**Proof.** (cp. \([4]\) p. 202 Th. 9, proof) Let \(F: N \times I \to N\) be the postulated strong neighborhood deformation retraction of \(N\) down onto \(A\) and define;

\[F': N' \times I \to N'\ : \begin{cases} (y, t) \mapsto y & \text{if } y \in B, t \in I \\ (y, t) \mapsto f \circ F(f^{-1}(y), t) & \text{if } y \in f(N) \setminus A, t \in I \end{cases}\]

i. \(F'\) is well-defined and it is continuous as being so when restricted to any one of the closed subspaces \(f(N) \times I\) resp. \(B \times I\) that together cover \(N' \times I\), cf. \([15]\) 4 p. 5. (N.B. Changing the domain \(f(N) \setminus B\) to \(f(N)\) doesn’t alter \(F'\).)

ii. The following standard diagram gives ii.

\[
\begin{array}{ccc}
\hat{H}(X(A); G) & \xrightarrow{\sim} & \hat{H}(X(N); G) \cong \hat{H}(X; A, N \setminus A; G) \\
\hat{H}(Y; B; G) & \xrightarrow{\sim} & \hat{H}(Y; N; G) \cong \hat{H}(X; B, N' \setminus B; G) \\
\end{array}
\]

iii. (\([11]\) p. 125 Ex. 1) \(f(\cdot) := \text{(restriction of) projection. Use as second diagram row; }\)

\[
\hat{H}(X(A); G) \cong \hat{H}(X(A); N; G) \cong \hat{H}(X(A); N; A; G) \cong \hat{H}(X(A); N \setminus A; G) \cong \hat{H}(X(A); N \setminus A; G)
\]

\([30]\) Th. 46.2 p. 279, see p. \([23]\) gives the chain equivalences in ii and iii.

The quotient maps \(p_{\ast}, p_{\ast}^\ast\) w.r.t. \(*\)-join and suspension, makes \(p_{\ast} \circ p_{\ast}^\ast\) continuous and \((p_{\ast} \circ p_{\ast}^\ast)\) a homeomorphism, cf. \([11]\) p. 134 Ex. 5.

\(v_1, v_2\) are the suspension points.

**Lemma.** (\([10]\) p. 225) \(f: Z \to W\) is null-homotopic iff \(W\) is a retract of the mapping cone \((CZ) \cup W\). \((CZ) := Z \ast \bullet\).

\(Z \subseteq W, i := \text{id}_Z\) and excising the cone-top followed by a retraction gives;

\[
\hat{H}((CZ) \cup W) \cong \hat{H}(CZ \cup W, CZ) \cong \hat{H}(W, Z)
\]

So, if \(i\) is null-homotopic, \(i\) is trivial and \(l\) splits, i.e:

\[
\hat{H}(W, Z) \cong \hat{H}(W, Z) \oplus \hat{H}(Z)
\]

Now, \(i + i_{\ast}: X + Y \to X \ast Y\) is null-homotopic, so:

\[
\hat{H}(X \ast Y, X + Y; G) \cong \hat{H}(X \ast Y; G) \oplus \hat{H}(X + Y; G) \cong \hat{H}(X; G) \oplus \hat{H}(X; G)
\]

The l.h.s. equals \(\hat{H}(X \ast Y, \{\emptyset\}; G)\). Deleting \(\{\emptyset\}\) gives Eq. 2 p. \([22]\) first line.
9. Addenda

9.1. The importance of simplicial complexes. In [14] p. 54 Eilenberg and Steenrod wrote:

Although triangulable spaces appear to form a rather narrow class, a major portion of the spaces occurring in applications within topology, geometry and analysis are of this type. Furthermore, it is shown in Chapter X that any compact space can be expressed as a limit of triangulable spaces in a reasonable sense. In this sense, triangulable spaces are dense in the family of compact spaces.

Moreover, not only are the major portion of the spaces occurring in applications triangulable, but the triangulable spaces plays an even more significant theoretical role than that, when it comes to determining the homology and homotopy groups of arbitrary topological spaces, or as Sze-Tsen Hu, in his Homotopy Theory ([26] 1959) p. 171, states, after giving a description of the Milnor realization of the Singular Complex $S(X)$ of an arbitrary space $X$:

The significance of this is that, in computing the homotopy groups of a space $X$, we may assume without loss of generality that $X$ is triangulable and hence locally contractible. In fact, we may replace $X$ by $S(X)$.

Recall that the Milnor realization of $S(X)$ of any space $X$ is weakly homotopy equivalent to $X$ and triangulable, as is any Milnor realization of any simplicial set, cf. [18] p. 209 Cor. 4.6.12. Triangulable spaces are frequently encountered and their singular homology is often most easily determined by calculating the simplicial homology of a triangulation of the space. Well, the usefulness also goes the other way: E.G. Sklyarenko: Homology and Cohomology Theories of General Spaces ([14] p. 132):

The singular theory is irreplaceable in problems of homotopic topology, in the study of spaces of continuous maps and in the theory of fibrations. Its importance, however, is not limited to the fact that it is applicable beyond the realm of the category of polyhedra. The singular theory is necessary for the in depth understanding of the homology and cohomology theory of polyhedra themselves. . . . In particular it is used in the problems of homotopic classification of continuous maps of polyhedra and in the description of homology and cohomology of cell complexes. The fact that the simplicial homology and cohomology share the same properties is proved by showing that they are isomorphic to the singular one.

9.2. The simplicial category. The simplicial category is a somewhat controversial concept among algebraic topologists. Indeed, this becomes apparent as Fritsch and Piccinini are quite explicit about it when defining their version of the simplicial category, which they call the finite ordinals and we quote from [18] p. 220-1:

A short but systematic treatment of the category of finite ordinals can be found in MacLane (1971) under the name “the simplicial category”. Our category of finite ordinals is not exactly the same as MacLane’s; however, it is isomorphic to the subcategory obtained from MacLane’s category by removing its initial object.
The last sentence pinpoints the disguised "initial simplex"-deletion described in the following quotation from Mac Lane [32] p. 178 (=2:nd ed. of "Mac Lane (1971)"):

By $\Delta^+$ we denote the full subcategory of $\Delta$ with objects all the positive ordinals $\{1, 2, 3, \ldots\}$ (omit only zero). Topologists use this category, call it $\Delta$, and rewrite its objects (using the geometric dimension) as $\{0, 1, 2, \ldots\}$. Here we stick to our $\Delta$, which contains the real 0, an object which is necessary if all face and degeneracy operations are to be expressed as in (3), in terms of binary product $\mu$ and unit $\eta$.

This mentioned Eq. (3) is part of MacLane's description in [32] p. 175ff, of the Simplicial Category with objects being the generic ordered abstractions of the simplices in Definition 1 p. 3, and with the empty set $\emptyset$, denoted 0 by MacLane, as an initial object which induces a unit operation $\eta$ in a "universal monoid"-structure imposed by ordinal addition.

"Ordinal addition" is very much equivalent of the union $\sigma_1 \cup \sigma_2$ involved in the join-definition in Definition 1 p. 3, once a compatible strict vertex order is imposed on each maximal simplex in the join-factors together with the agreement that all the vertices of the left factor is strictly less than all those of the right one. Since this vertex-ordering is needed also in the definition p. 28 of the binary simplex-operation called ordered simplicial cartesian product, we have used these ordered simplicial complexes all through this paper. In particular we used it when we defined the action of the boundary function related to the chain complexes in our homology definitions.

The empty simplex plays a central role in modern mathematical physics, as hinted at in [30] p. 188 where J. Kock describe the classical use of the “topologist $\Delta$” within category theory, there named “topologist’s delta” and denoted $\Delta$, in the following words:

In this way, the topologist’s delta is a sort of bridge between category theory and topology. In these contexts the empty ordinal (empty simplex) is not used, but in our context it is important to keep it, because without it we could not have a monoidal structure.

In his series This Week’s Finds in Mathematical Physics, John Baez gives a compact but very lucid and accessible presentation (Week 115ff) of the modern category theory based homotopy theory, which, on his suggestion, could be called homotopical algebra. He ends section B in Week 115 with the following, after a description of The Category of Simplices, which he calls Delta, the objects of which are the nonempty simplices, “1” consisting of a single vertex, “the simplex with two vertices” := an intervall, “the simplex with three vertices” := a triangle, and so on:

We can be slicker if we are willing to work with a category *equivalent* to Delta (in the technical sense described in “week76”), namely, the category of *all* nonempty totally ordered sets, with order-preserving maps as morphisms. This has a lot more objects than just $\{0\}$, $\{0, 1\}$, $\{0, 1, 2\}$, etc., but all of its objects are isomorphic to one of these. In category theory, equivalent categories are the same for all practical purposes - so we brazenly call this category Delta, too. If we do so, we have following *incredibly* slick description of the category of simplices: it’s just the category of finite nonempty totally ordered sets!

If you are a true mathematician, you will wonder “why not use the empty set, too?” Generally it’s bad to leave out the empty set. It may seem like
"nothing", but "nothing" is usually very important. Here it corresponds to the "empty simplex", with no vertices! Topologists often leave this one out, but sometimes regret it later and put it back in (the buzzword is "augmentation"). True category theorists, like Mac Lane, never leave it out. They define Delta to be the category of *all* totally ordered finite sets. For a beautiful introduction to this approach, try: Saunders Mac Lane, Categories for the Working Mathematician, Springer, Berlin, 1988.

9.3. Simplicial sets. A simplicial set, is a contravariant functor from the simplicial category to the category of sets. The natural transformations constitute the morphisms.

So, the classical simplicial sets are objects in the following functor category:

\[(\text{Category of sets})^{\text{Simplicial Category}}\]^

A simplicial set is usually perceived through a representation in its target as a dimension-indexed collection of its non-degenerate simplices, which itself usually is envisioned through its realization, i.e. as a geometric simplicial complex, cf. p. [31] middle and Theorem [3.2] p. [4]. The morphisms in the category of simplicial sets, i.e. the natural transformations, now materializes as realisations of simplicial maps. By mainly just well-ordering the vertices in each simplex of a simplicial complex – in an overall consistent way, cf. [18] p. 152 Ex. 4, each “classical simplicial complex” defined as in Definition 2 p. [3] can be regarded as a “classical simplicial set” in the functor category of classical Simplicial Sets.

The augmented simplicial complexes, resulting from Definition 1 p. [3], can, equivalently, be associated with an “augmented simplicial set” in the following functor category:

\[(\text{Category of sets})^{\text{Augmented Simplicial Category}}\]^

as S. Mac Lane does in [32] p. 179. There Mac Lane, in p. 178, chooses the empty topological space $\emptyset$ as the standard $(-1)$-dimensional affine simplex, implying that the empty topological space $\emptyset$ would have a cyclic chain generator in degree $-1$, which isn’t compatible with the “Eilenberg-Steenrod”-formalism for pair-spaces, where $\emptyset$ has a fixt privileged role, including nothing but trivial homology groups in all dimensions.

In “Week 116 E” J. Baez gives an alternative specification of a simplicial set as: -“... a presheaf on the category Delta” and then he also pays attention to the realization functor.

In “Week 117 J” John Baez describe the Nerv of a Category, as a functor from the Category of Small Categories (Cat) to the Category of Simplicial Sets, and ends the paragraph with the following:

I should also point out that topologists usually do this stuff with the topologist’s version of Delta, which does not include the “empty simplex”.

This exclusion of the “empty simplex” within general topology is probably a consequence of the fact that there is no “natural” $(-1)$-dimensional topological space in classical general topology that can serve as the realisation of the simplicial complex $\{\emptyset\}$. 
9.4. A category theoretical realization functor. Our plea for the use of an augmented topological category rests on the otherwise lost realization functor. In [28], D.M. Kan introduced the realization functor and the simplicial singular functor as an example of an adjoint pair of functors. In [18] p. 303, with its suggestive notations, Kan’s example is given a somewhat generalized category theoretical formulation as follows:

Let $D$ be a small category and let $D\text{Sets}$ denote the category of contravariant functors $D \rightarrow \text{Sets}$. One might view the objects of $D\text{Sets}$ as sets graded by the objects of $D$ with the morphisms of $D$ operating on the right. Let $\Phi : D \rightarrow \text{Sets}$ be an arbitrarily given covariant functor; associated to it, construct a pair of adjoint functors

$\Gamma_{\Phi} : \text{DSets} \rightarrow \text{Sets}$,

$S_{\Phi} : \text{Sets} \rightarrow \text{DSets}$,

as follows:

(1) The left adjoint functor $\Gamma_{\Phi}$ - called realization functor associates to each object $X$ of $D\text{Sets}$ the set of equivalence classes of pairs $(x, t) \sqcup X(d) \times \Phi(d)$ modulo the relation $(x, \Phi(\alpha)(t))$. Given a morphism $f : Y \rightarrow X$, i.e., a natural transformation, one has the function $\Gamma_{\Phi}f : \Gamma_{\Phi}Y \rightarrow \Gamma_{\Phi}X$, $[x, t] \mapsto [f(x), t]$ where $[x, t]$ denotes the equivalence class represented by the pair $(x, t)$.

(2) The right adjoint functor $S_{\Phi}$ - called singular functor - associates to each set $Z$ the object of $D\text{Sets}$ given by $(S_{\Phi}Z)(d) = Z^{\Phi(d)}$, for each object $d$ of $D$, and $\alpha^{*} : Z^{\Phi(d)} \rightarrow Z^{\Phi(d')}$, $x \mapsto x \circ \Phi(\alpha)$, for each morphism $\alpha : d' \rightarrow d$ of $D$.

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