On local description of two-dimensional geodesic flows with a polynomial first integral

Maxim V Pavlov¹,²,⁴,⁵ and Sergey P Tsarev³

¹ Sector of Mathematical Physics, Lebedev Physical Institute of Russian Academy of Sciences, Leninskij Prospekt 53, 119991 Moscow, Russia
² Department of Applied Mathematics, National Research Nuclear University MEPhI, Kashirskoe Shosse 31, 115409 Moscow, Russia
³ Siberian Federal University, Institute of Space and Information Technologies, 26 Kirenski str., ULK-311, Krasnoyarsk, 660074 Russia
⁴ Department of Mechanics and Mathematics, Novosibirsk State University, 2 Pirogova Street, 630090, Novosibirsk, Russia

E-mail: mpavlov@itp.ac.ru and sptsarev@mail.ru

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Abstract
In this paper we present a construction of multiparametric families of two-dimensional metrics with a polynomial first integral of arbitrary degree in momenta. Such integrable geodesic flows are described by solutions of some semi-Hamiltonian hydrodynamic-type system. We give a constructive algorithm for the solution of the derived hydrodynamic-type system, i.e. we found infinitely many conservation laws and commuting flows. Thus we were able to find infinitely many particular solutions of this hydrodynamic-type system by the generalized hodograph method. Therefore infinitely many particular two-dimensional metrics equipped with first integrals polynomial in momenta were constructed.

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1. Introduction

The problem of integration of geodesic flows on a two-dimensional surface appeared in classical mechanics very early and was extensively investigated in the 19th century focusing

* dedicated to the 55th birthday of our friend E V Ferapontov.
⁵ Author to whom any correspondence should be addressed.
mostly on local approaches. The 20th century was more centered on global behavior; the problems of local description of trajectories were not so popular. An overview of both aspects may be found in numerous sources, although we only give two references here [5, 6]. Partially, this lack of interest in the local problem was not only due to the importance of the global problems; it seems reasonable to ascribe this lack of interest to the absence of new ideas for the local integrability problem. The situation, in our opinion, has changed in the last decades after publications [1–4, 9], where the authors remarked that the equations for the coefficients of first integrals polynomial in momenta for two specific low-dimensional cases belong to a class of diagonalizable hydrodynamic-type systems integrable by differential-geometric means; the appropriate theory for such nonlinear systems of partial differential equations was developed at the very end of the 20th century (see [8, 25]). In [21], developing the preliminary results of [9], we demonstrated how to apply the techniques of integrable hydrodynamic-type systems to the case of so-called one-and-a-half-dimensional systems (one-dimensional mechanical systems with the potential depending on time). Below we investigate (using more sophisticated technologies) the problem of local description of two-dimensional Riemannian metrics with geodesic flows possessing a polynomial first integral of arbitrary high degree.

In [2] the \( N \) component hydrodynamic-type system

\[ a^0_i = a^1_i a^{N-1}_i, \quad a^k_i = a^{N-1}_i a^k_{i-1} + [(k + 1)a^{k+1}_i - (N + 1 - k)a^{k-1}_i] a^{N-1}_i, \tag{1} \]

where \( k = 1, \ldots, N - 1 \) and \( a^0 \equiv 1 \), was derived as the system for the coefficients \( a^k(t, x) \) of a (global smooth) polynomial first integral\(^6\)

\[ F(t, x, p_1, p_2) = \sum_{k=0}^{N} (-1)^k a^k \frac{g^{N-k}}{g_2} p_1^{N-k} p_2^k, \tag{2} \]

for metric in semi-geodesic coordinates

\[ ds^2 = g^2(t, x)dt^2 + dx^2 \tag{3} \]

on a two-dimensional torus, where \( g \equiv a^{N-1} \). As shown in [2], any smooth metric on a two-dimensional torus possessing a polynomial first integral can be reduced to such a form. The metric (3) corresponds to the Hamiltonian \( H(t, x, p_1, p_2) = \frac{1}{2} (p_1^2 + g^2(t, x) + p_2^2) \); the system (1) is equivalent to

\[ \{ F, H \} = \frac{\partial F}{\partial t} \frac{\partial H}{\partial p_1} - \frac{\partial F}{\partial p_1} \frac{\partial H}{\partial t} + \frac{\partial F}{\partial x} \frac{\partial H}{\partial p_2} - \frac{\partial F}{\partial p_2} \frac{\partial H}{\partial x} = 0. \]

The theorem proved in [2] states that the hydrodynamic-type system (1) is diagonalizable (i.e. possesses a complete set of Riemann invariants) and semi-Hamiltonian so may be integrated by the generalized hodograph method (see [25]). However, the authors of [2] did not give a constructive description of the necessary complete set of hydrodynamic conservation laws and commuting flows for (1).

In this paper we construct \( N \) infinite series of conservation laws and commuting flows. Thus one can construct a rich multiparametric family of particular solutions to (1) by the generalized hodograph method as described below in sections 2, 7 and 8. Our interest is focused on local properties of the system (1) and the respective coefficient \( g(t, x) \) in (3). Formal proof of local existence of metrics with (irreducible) polynomial integrals of high degree is not an issue (see for example [24]); Cauchy existence theorem for (1) and simple dimensionality considerations suffice for this. We should stress that the problem

\(^6\) The factor \((-1)^k\) was omitted in [2].
considered here is the problem of *local algorithmic construction of such metrics and their integrals* ‘in finite terms’. Global constructions are much more difficult. Many papers are devoted to the proof of non-existence of metrics with analytic integrals on surfaces of genus greater than 2 and investigation of the situation for the sphere and the torus (see for example [14]). If one is looking for a smooth \(C^\infty\) metric then the situation seems much less clear. We do not address this global problem here, referring to the previously given citations and [23].

The structure of the paper is as follows. In section 2 we discuss the semi-Hamiltonian property of the hydrodynamic-type system (1). We construct a generating function of conservation laws and find the equation of the associated Riemann surface playing the key role in our method. We rewrite the hydrodynamic-type system (1) in a diagonal form. In section 3 we remark that in the two-component case—corresponding to the case of first integrals quadratic in momenta—hydrodynamic-type system (1) is nothing but the simplest two-component linearly degenerate hydrodynamic-type system (which can be written in appropriate field variables in the form \(u_s = vu_s, v_t = uv_v\)), whose general solution can be presented in implicit form with explicit dependence on two arbitrary functions of a single variable. This result conforms to the well-known fact that geodesic flows with quadratic first integrals can be integrated by separation of variables. In section 4 we introduce new field variables \(b^k\) and show how the equation of a Riemann surface rewritten in these field variables \(b^k\) helps in constructing \(N\) infinite series of conservation law densities. In section 5 we introduce two hydrodynamic chains as infinite sets of equations compatible with the hydrodynamic-type system (1). This procedure allows us to construct infinitely many conservation laws (the so-called Kruskal series) in a compact form. In section 6 we present a way to construct infinitely many higher commuting flows and infinitely many associated conservation laws. In section 7 we adopted the generalized hodograph method for construction of a rich infinite parametric family of particular solutions for the case of the hydrodynamic-type system (1).

In section 8 we summarize our exposition in the form of an algorithm for construction of local two-dimensional metrics with local polynomial first integrals. For convenience of the reader we give here a succinct list of necessary steps:

1. Substitute the expansion (24) into (48) and obtain the quantities \(W^{(k)}_0(b)\) in the expansion (50).
2. Substitute the quantities \(W^{(k)}_0(b)\) into the generalized hodograph formula (51) with arbitrary parameters \(\sigma_k^p\). This gives the necessary system of \(N\) implicit equations for the symmetric hydrodynamic variables \(b^k(t, x), k = 1, \ldots, N\)—solutions of the system (23).
3. The metric (3) is found from the relation \(g(t, x) = a^{N-1}(t, x) = -\sum_{k=1}^{N} b^k(t, x)\). The coefficients \(a^i(t, x)\) of the first integral (2) are given by Vieta’s formulas from the equation (20).

**Theorem 2.** The obtained metrics (3) have first integrals (2) of the given degree \(N\) obtained in step 3 of the algorithm.

Finally in the conclusion (section 9) we discuss the problem of completeness of the constructed series of conservation laws and briefly expose further perspectives of integrability of two-dimensional geodesic flows.
2. Semi-Hamiltonian systems and their integration

The origins of our method of construction of metrics (3) with polynomial integrals lie in the theory of diagonalizable semi-Hamiltonian hydrodynamic-type systems (see [25] where the detailed exposition of the respective differential-geometric theory was given).

As we show in this section (see also [2]), the system (1) is diagonalizable, i.e. it possesses $N$ Riemann invariants $\mathbf{r} = (r^1(\mathbf{a}), \ldots, r^N(\mathbf{a}))$ which reduce (1) to the diagonal form

$$r^i_t = v_i(\mathbf{r}) r^i_x, \quad i = 1, \ldots, N,$$

and has another remarkable property:

**Definition 1.** A diagonal system (4) is called semi-Hamiltonian if the characteristic velocities $v_i(\mathbf{r})$ of the system satisfy the following identity:

$$\frac{\partial_i}{v_k - v_i} \frac{\partial_j v_i}{v_j - v_i} = \frac{\partial_j v_i}{v_j - v_i} \frac{\partial_i}{\partial_j r^i}, \quad i \neq j \neq k. \quad (5)$$

Such systems have infinitely many hydrodynamic conservation laws and locally any solution $r^i(t, x)$ can be constructed implicitly as the solution of the following algebraic system (the so-called generalized hodograph method):

$$w_i(r^1, \ldots, r^N) = v_i(r^1, \ldots, r^N) \cdot t + x, \quad i = 1, \ldots, N, \quad (6)$$

where $w_i(\mathbf{r})$ are the coefficients of one of the infinitely many flows commuting with (1)\(^7\)

$$r^i_t = w_i(\mathbf{r}) r^i_x, \quad i = 1, \ldots, N, \quad (7)$$

(no summation over repeated indices is assumed everywhere in this paper). The characteristic velocities $v_i(\mathbf{r})$ are solutions of the following linear system with variable coefficients

$$\partial_k w^i = \frac{\partial_k v_i}{v_k - v_i} (w^k - w^i), \quad i \neq k \quad (8)$$

(again $\partial_i \equiv \partial/\partial r^i$). The overdetermined system (8) (with given $v_i(\mathbf{r})$) has infinitely many solutions parameterized by $N$ functions of one variable\(^8\). In addition, semi-Hamiltonian systems possess an infinite set of conservation laws. The reader should consult [25] for motivations and proofs.

In this section we present the equation of a Riemann surface $\lambda(q, \mathbf{a})$ associated with the hydrodynamic-type system (1). Its branch points $r^i = \lambda_{q_i = a_i}$, where $q_i$ are solutions of the algebraic equation $\lambda_{q_i = a_i} = 0$, are the Riemann invariants of (1). When an associated Riemann surface is known for a semi-Hamiltonian system, it can be used for effective construction of commuting flows (7) (see [16]). The details of this approach are given in the subsequent sections.

**Theorem [2]:** Hydrodynamic-type system (1) is semi-Hamiltonian.

In this section we give an alternative proof of this theorem obtaining the associated Riemann surface playing the key role in our construction of the explicit formulas for the conservation laws and commuting flows for (1).

\(^7\) The new variable $\tau$ means that $r_i$ now also depends on it, $r^i = r^i(\tau, x)$, and the compatibility conditions ($r^i_{\tau}$) = ($r^i$)\(_x\) equivalent to (8) hold for the systems (4) and (7).

\(^8\) The compatibility conditions for (8) which guarantee this functional freedom are nothing but (5).
Proof. According to [2], hydrodynamic-type system (1) can be equally derived for the coefficients \(a^k(t, x)\) of a polynomial first integral\(^9\)

\[
\tilde{F}(t, x, p) = \sum_{k=0}^{N} (-1)^k a^k(t, x) p^k (1 - p^2)^{N-k}/2
\]  

(9)

of Hamilton’s equations\(^10\)

\[
x' = \frac{\partial \tilde{H}}{\partial p}, \quad p' = -\frac{\partial \tilde{H}}{\partial x},
\]

(10)

where the effective Hamiltonian function is\(^11\)

\[
\tilde{H}(t, x, p) = -a^{N-1}(t, x)\sqrt{1 - p^2}.
\]

This means that in fact we have a one-dimensional mechanical system with Hamiltonian depending explicitly on the ‘time’ \(t\). Such systems were called ‘1.5-dimensional’ in [13]. An integrable subclass of 1.5-dimensional systems was studied in [21] where the general technology of hydrodynamic reductions of integrable nonlinear hydrodynamic chains was used for explicit description of various integrable 1.5-dimensional cases. In the present paper we develop a more general approach for the case studied.

In this 1.5-dimensional setting the corresponding Liouville equation\(^12\)

\[
f_{\lambda} = -\{f, \tilde{H}\} = \frac{\partial f}{\partial p} \frac{\partial \tilde{H}}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial \tilde{H}}{\partial p},
\]

takes the form

\[
f_{\lambda} = a^{N-1}q_{\lambda} + (1 + q^2)f_{\lambda} q_{\lambda}^{N-1},
\]

(11)

where instead of the variable \(p\) we introduce the auxiliary variable \(q\) connected with \(p\) via the point transformation

\[
q = -\frac{p}{\sqrt{1 - p^2}}, \quad p = -\frac{q}{\sqrt{1 + q^2}}.
\]

(12)

Substitution of a general ansatz \(f(t, x, p) = \lambda(q, a(t, x))\) with a set of some (formally unspecified) field variables \(a \equiv (a^0(t, x), \ldots, a^{N-1}(t, x))\) into (11) leads to an overdetermined compatible systems of equations on \(\partial \lambda/\partial q, \partial \lambda/\partial a\) if \(a^k(t, x)\) are solutions of hydrodynamic-type system (1). Now \(\lambda(q, a(t, x))\) can be found explicitly:

\[
\lambda(q, a) = (1 + q^2)^{-N/2} \left[ q^N + \sum_{k=0}^{N-1} q^k a^k \right].
\]

(13)

In fact (13) can be obtained by a direct substitution of (12) into (9). Thus a Riemann surface with parameters \((\lambda, q)\) used below for the explicit constructions of the conservations laws and commuting flows is defined. And vice versa, substitution of (13) into (11) and splitting with respect to \(q\) leads to (1). In a generic case the algebraic equation \(\lambda_q = 0\) for \(\lambda\) of the form

\(^9\) The corresponding formula in [2] contains misprints.

\(^10\) This means that geodesic flows can be written either as stationary Hamilton’s equations with two degrees of freedom or alternatively as non-stationary Hamilton’s equations with one-and-a-half degrees of freedom. Here the ‘prime’ means the derivative with respect to \(t\).

\(^11\) We remind the reader that the coefficient \(g\) of the metric \(ds^2 = g^2(t, x)dt^2 + dx^2\) is equal to \(a^{N-1}\).

\(^12\) The Poisson bracket is standard \(\{f, H\} = f_{\lambda} H_{\lambda} - f_{\lambda} H_{\lambda}\). We remind that \(f(t, x, p)\) is a first integral for the given Hamilton’s equations, for example \(f = \tilde{F}\).
There is a remarkable class of conservation laws usually called the Kruskal series. We describe its construction in more detail below in section 5.

Under the potential substitution (14) for the hydrodynamic-type system (1), Liouville equation (11) leads to the hydrodynamic-type system (1), written in the diagonal form

\[ r^i_a = a^{N-1}q_i r^i_a, \]

where the Riemann invariants \( r^i = \lambda_{q_i} \) are determined by (13), i.e.

\[ r^i(a) = \lambda(q_i, a) = (1 + (q_i)^2)^{-N/2} \left( (q_i)^N + \sum_{m=0}^{N-1} (q_i)^m a^m \right). \]

It is easy to see that the constructed \( r^i(a) \) are functionally independent since the characteristic velocities in (15) are distinct. (Functional independence of the velocities \( v_i = a^{N-1}q_i \) will be shown below in section 3.)

Under the functional inversion transformation \( \lambda = \lambda(q_i, a) \rightarrow q = q(\lambda, a) \) the linear equation (11) transforms to

\[ q_i = a^{N-1}q q_i - (1 + q_i^2) a^{N-1}, \]

which is equivalent to the equation (13)

\[ p_i = (a^{N-1} \sqrt{1 - p_i^2}) \lambda \]

for the so-called generating function of conservation law densities \( p = p(\lambda, a^0, a^1, \ldots, a^{N-1}) \) for the hydrodynamic-type system (1) (here \( \lambda \) is a free parameter). Infinitely many conservation laws can be found directly from the equation of the Riemann surface (13) expanding the inverse function \( q = q(\lambda, a) \) (as \( q \rightarrow \infty \), while \( \lambda \rightarrow 1 \)) with respect to the powers of \( (\lambda - 1) \) and substituting into the second relationship of (12) and then into (18)\(^{14}\). Since coefficients of the resulting expansion of \( p(\lambda, a) \) are conservation law densities (see (18)), and the hydrodynamic-type system (1) is diagonalizable (see (15)), we conclude that (1) is semi-Hamiltonian. The theorem is proved.

The next important step in the integration of the semi-Hamiltonian system (1) is the construction of the necessary complete set of \( w_i(\mathbf{r}) \) (depending on \( N \) functions of one variable, see [25]) in order to be able to construct any solution of (1) locally using the generalized hodograph formula (6). This problem will be discussed in sections 6 and 7. In the following three sections we consider different forms of the system (1) and the associated Riemann surface (13) in more detail in order to expose the necessary techniques.

3. Triviality of the quadratic case (\( N = 2 \))

The case \( N = 2 \) deserves a special treatment. Geometrically it corresponds to the case when the two-dimensional metric (3) has a first integral (2) which is a quadratic polynomial in momenta \( p_1 \). In this case the algebraic relations (14), (16) between the original variables \( a_i \), branch points \( q_i \), Riemann invariants \( r^i \) and velocities \( v_i = a^{N-1}q_i \) (see (15)) can be easily

\(^{13}\) Under the potential substitution \( p = S_1 \) this equation reduces to the Hamilton–Jacobi equation \( S_1 = a^{N-1} \sqrt{1 - (S_1)^2} \) for the mechanical system with the specified Hamiltonian \( H(t, x, p) \).

\(^{14}\) This remarkable class of conservation laws is usually called the Kruskal series. We describe its construction in more detail below in section 5.
solved producing a very simple expression of velocities in terms of the Riemann invariants:

\[ v_2 = 2 - 2r^1, \quad v_1 = 2 - 2r^2. \]

Thus, hydrodynamic-type system (1) in the two-component case

\[ a^0_i = a^1_i, \quad a^1_i = a^0_i + 2(1 - a^0)a^1_i, \]

where \( a^0 = r^1 + r^2 - 1 \), \( (a^1)^2 = -4(r^1 - 1)(r^2 - 1) \), is linearly degenerate, that is \( \partial v_1 / \partial r^1 = 0 \) and \( \partial v_2 / \partial r^2 = 0 \). In order to keep the metric positively definite we set \( r^1 < 1 \), \( r^2 > 1 \). Its general solution \( r^1(t, x), r^2(t, x) \) depending on two arbitrary functions \( \beta(u) \) and \( \gamma(v) \) of one variable is well known (see for example [22]) and may be presented in implicit form:

\[ t = \beta'(u) + \gamma'(v), \quad x = \beta(u) - u\beta'(u) + \gamma(v) - v\gamma'(v), \tag{19} \]

where (for simplicity) we denoted \( u = 2(1 - r^1) > 0 \), \( v = 2(1 - r^2) < 0 \).

The fact of complete integrability in the case of quadratic first integrals is also well known; its exposition can be found for example in [5] and [6 ch 11]. Actually the standard exposition of this case (for the isothermic form \( ds^2 = (f(u) + g(v))(du^2 + dv^2) \) of the metric) in the references given above is equivalent to our result for the metric in semi-geodesic coordinates (3). Indeed, substituting the above expressions into (3), one can obtain

\[
d s^2 = (a^1)^2 dr^2 + dr^2 = [u\beta''(u)du + v\gamma''(v)dv]^2 - uv[\beta''(u)du + \gamma''(v)dv]^2
\]

\[
= (u - v)u[d\beta'(u)]^2 - v(u - v)[d\gamma'(v)]^2. \]

Introducing new coordinates \( U, V \) such that \( dU = u^{1/2}d\beta'(u) \) and \( dV = (v)^{1/2}d\gamma'(u) \), finally we obtain

\[
d s^2 = [u(U) - v(V)](dU^2 + dV^2), \]

where the dependencies \( u(U) \) and \( v(V) \) can be found by inversion from

\[
U = \int u^{1/2}d\beta'(u), \quad V = \int (v)^{1/2}d\gamma'(u). \]

Recomputation of the quadratic first integral is based on the introduction of new momenta \( \tilde{p}_1, \tilde{p}_2 \) by virtue of the identity

\[
p_1dt + p_2dx = \tilde{p}_1dU + \tilde{p}_2dV. \]

Indeed, substituting (19) as well as \( dU = u^{1/2}d\beta'(u) \) and \( dV = (v)^{1/2}d\gamma'(v) \) into the above relationship, one can immediately obtain

\[
\tilde{p}_1 = p_1u^{-1/2} - p_2u^{1/2}, \quad \tilde{p}_2 = p_1(-v)^{-1/2} + p_2(-v)^{1/2}. \]

Then the first integral (2) takes the form

\[
F(t, x, p_1, p_2) = \frac{a^0}{g^2}p_1^2 - p_1p_2 + p_2^2 = \tilde{F}(U, V, \tilde{p}_1, \tilde{p}_2)
\]

\[
= \frac{(2 - u(U))\tilde{p}_1^2 + (2 - v(V))\tilde{p}_2^2}{2(u(U) - v(V))}, \]

which coincides with the formulas in [5 theorem 5], if we introduce their notations \( f = u - 2 \), \( g = 2 - v \).
4. Symmetric field variables and $N$ principal series of conservation laws

Let us introduce the roots $b^k$ of the polynomial

$$q^N + \sum_{k=0}^{N-1} q^k a_k = \prod_{k=1}^{N} (q - b^k),$$

(20)

so all field variables $a^k$ become elementary symmetric polynomials of new field variables $b^m$. For instance,

$$a^{N-1} = -\sum_{k=1}^{N} b^k.$$  

(21)

Thus, the equation of the Riemann surface (13) takes the form

$$\lambda = (1 + q^2)^{-N/2} \prod_{k=1}^{N} (q - b^k).$$

(22)

After substitution of this expression into (11) the hydrodynamic-type system (1) reduces to another simple symmetric form

$$b^k = (1 + (b^k)^2)\sum_{m=1}^{N} b^m - \left(\sum_{n=1}^{N} b^n\right) b^k b^k.$$  

(23)

Indeed, the hydrodynamic-type system (23) can be derived in three steps:

1. Compute the partial derivatives of $\ln \lambda$ with respect to the independent variables $t, x, q$:

$$\ln \lambda_t = -\sum_{m=1}^{N} \frac{b_m}{q - b^m},$$

$$\ln \lambda_x = -\sum_{m=1}^{N} \frac{b^m}{q - b^m},$$

$$\ln \lambda_q = -\frac{Nq}{1 + q^2} + \sum_{m=1}^{N} \frac{1}{q - b^m}.$$  

2. Substitution of these derivatives into (11) leads to

$$\sum_{m=1}^{N} \frac{b^m}{q - b^m} = a^{N-1} q \sum_{m=1}^{N} \frac{b^m}{q - b^m} + \left[\sum_{m=1}^{N} \frac{q^2 + 1}{q - b^m}\right] a^{N-1}$$

or

$$\sum_{m=1}^{N} \frac{b^m}{q - b^m} = a^{N-1} \sum_{m=1}^{N} b^m + a^{N-1} \sum_{m=1}^{N} \frac{b^m b^m}{q - b^m}$$

$$= \left[\sum_{m=1}^{N} b^m + \sum_{m=1}^{N} \frac{(b^m)^2 + 1}{q - b^m}\right] a^{N-1},$$

which simplifies to

$$\sum_{m=1}^{N} \frac{b^m}{q - b^m} + \left[(b^m)^2 + 1\right] a^{N-1} - a^{N-1} b^m b^m = a^{N-1} \sum_{m=1}^{N} b^m - \sum_{m=1}^{N} b^m a^{N-1}.$$
The variables \( b^k \) are very convenient for explicit computation of commuting flows and conservation laws for the system (1). Since we will need in fact not only the conservation law densities (for example \( p \) appearing in the left-hand sides of equations like (18)) but also the fluxes—expressions inside the \((\ldots)\) on the right-hand sides of such equations—we will call conservation laws the equalities (18) themselves.

Now we are ready to explain our method of deriving explicit formulas for conservation laws of (1) using the Riemann surface associated with this integrable hydrodynamic-type system. One should note that the \( N \) component semi-Hamiltonian hydrodynamic-type system has in infinitely many conservation laws and commuting flows—both families are parameterized by \( N \) arbitrary functions of a single variable (see details in [25]). In many interesting cases this functional dependence cannot be presented in explicit form. Nevertheless, \( N \) infinite series of conservation laws and commuting flows can be constructed, for instance, if the corresponding equation of the associated Riemann surface is known. As one can prove (see for example [20]) for a vast class of semi-Hamiltonian systems such series form the complete basis for the (infinite-dimensional) linear space of all conservation laws (or commuting flows). Another technique for proving completeness can be found in [21].

\( N \) infinite series of conservation laws for (23) can be found in three steps:

1. Let us expand\(^{15}\) \( q \) with respect to the local parameter \( \lambda \) at the vicinity of each root \( b^k \):

\[
q^{(k)}(\lambda) = b^k + \lambda q_1^{(k)} + \lambda^2 q_2^{(k)} + \lambda^3 q_3^{(k)} + \ldots, \quad \lambda \to 0.
\] (24)

All coefficients \( q_n^{(k)} \) can be found recursively, for instance

\[
q_1^{(k)} = \frac{(1 + (b^k)^2)^{N/2}}{\prod_{m \neq k} (b^k - b^m)}.
\]

2. Substitute the series \( q^{(k)}(\lambda) \) into (12), obtaining expansion

\[
p^{(k)}(\lambda) = -\frac{q^{(k)}(\lambda)}{\sqrt{1 + (q^{(k)}(\lambda))^2}} = h^k + \lambda p_1^{(k)} + \lambda^2 p_2^{(k)} + \lambda^3 p_3^{(k)} + \ldots,
\] (25)

where

\[
h^k = -\frac{b^k}{\sqrt{1 + (b^k)^2}}.
\] (26)

Then again all conservation law densities \( p_m^{(k)} \) can be found recursively, for instance

\[
p_1^{(k)} = -\frac{(1 + (b^k)^2)^{(N-3)/2}}{\prod_{m \neq k} (b^k - b^m)}.
\] (27)

3. Substitute \( p^{(k)}(\lambda) \) into the generating function \( p \) of conservation laws (18)

\[
(p^{(k)}(\lambda))_t + \left(\sqrt{1 - (p^{(k)}(\lambda))^2} \sum_{a=1}^{N} b^a \right)_x = 0
\]

\(^{15}\) This expansion is given by the so-called Lagrange–Bürmann inversion (see [21] for an example of its use).
and expand both sides with respect to the parameter $\lambda \to 0$. Matching the coefficients of the same powers of $\lambda$ one obtains $N$ infinite series of conservation laws, namely

\[
(h^k)_t + \left(1 - (h^k)^2 \sum_{n=1}^{N} b^n \right)_x = 0, \\
(p^{(k)}_1)_t = \frac{h^k p^{(k)}_1}{\sqrt{1 - (h^k)^2} \sum_{n=1}^{N} b^n}, \ldots.
\]  

(28)

Remark. As a by-product we obtain the hydrodynamic-type system (1), (23) in a symmetric conservative form

\[
(h^k)_t = \left(1 - (h^k)^2 \sum_{n=1}^{N} \frac{h^n}{\sqrt{1 - (h^m)^2}} \right)_x, \quad k = 1, \ldots, N,
\]

where we utilized the inverse point transformation (see (12), (26))

\[
b^k = - \frac{h^k}{\sqrt{1 - (h^k)^2}}.
\]

(30)

Its first $N$ conservation laws are (28) expressed in variables $h^k$ using (26).

Alongside with the constructed $N$ series of conservation laws one can obtain another (incomplete) set of conservation laws usually called the Kruskal series. Its form is much simpler and is symmetric with respect to the variables $b^i$. We give Kruskal series below in section 5.

5. Hydrodynamic chains and Kruskal series of conservation laws

If we introduce new variables (called moments)

\[
B^k = \frac{1}{k+1} \sum_{m=1}^{N} (h^m)^{k+1}, \quad k = 0, 1, \ldots,
\]

(31)

then the hydrodynamic-type system (23) implies infinitely many equations

\[
B^0_t = (N + 2B^1)_t B^0_\tau - B^0 B^1, \\
B^k_t = (kB^{k-1} + (k+2)b^{k+1})B^{k-1}_\tau - B^0 B^{k+1}_\tau, \quad k = 1, 2, \ldots;
\]

(32)

the first $N$ of them are independent since, due to (31), all higher moments $B^N_b, B^{N+1}_b, \ldots$ are polynomial expressions of the first $N$ moments; these expressions can be found from (31) using the standard combinatorial results on symmetric polynomials.

Nevertheless, one can consider infinitely many equations (32) without the above restrictions. Similar infinite chains of quasilinear first-order equations (called hydrodynamic chains) are very useful for integration of various semi-Hamiltonian systems appearing in applications. An overview of this approach can be found in [18, 21].

The associated Riemann surface (22) can be expanded at infinity as $q \to \infty$ and $\lambda \to 1$. For convenience we replace $\lambda$ below by $\mu = -\ln \lambda$, so $\mu \to 0$ and

\[
\mu = \frac{N}{2} \ln(1 + q^2) - \sum_{k=1}^{N} \ln(q - b^k).
\]
Then (see (31)) asymptotically
\[
\mu = \frac{N}{2} \ln \left( 1 + \frac{1}{q^2} \right) + \sum_{k=0}^{\infty} \frac{B_k}{q^{k+1}} = \sum_{k=0}^{\infty} \frac{C_k}{q^{k+1}}, \quad q \to \infty, \tag{33}
\]
and introducing new field variables \( C_k \) using their relation to \( B_k \) in (33) we obtain another remarkable hydrodynamic chain
\[
C_k = (kC^{k-1} + (k + 2)C^{k+1})C_k^0 = C_k^0C^{k+1}_k, \quad k = 0, 1, 2, \ldots, \tag{34}
\]
where \( C^{2k} = B^{2k} \) and \( C^{2k-1} = B^{2k-1} - (-1)^k \frac{N}{2k} \).

Embedding the hydrodynamic-type system (23) into the hydrodynamic chain (34) for arbitrary \( N \) allows us to find Kruskal conservation laws\(^\text{16}\) in a compact form. First we (using the Lagrange–Bürmann inversion) get from (33) the asymptotic decomposition of \( q \) and \( p \) as \( \mu \to 0 \):
\[
q(\mu) = \frac{C_0}{\mu} + \frac{C_1}{C_0} + \mu \left( \frac{C_2}{(C_0)^2} - \frac{(C_1)^2}{(C_0)^3} \right) + \mu^2 \left( \frac{C_3}{(C_0)^3} - \frac{3C_2^2}{2(C_0)^4} + \frac{2(C_1)^3}{(C_0)^5} \right) + \ldots, \tag{35}
\]
\[
p(\mu) = -1 + \frac{\mu^2}{2(C_0)^2} - \frac{\mu^3}{3(C_0)^3} + \mu^4 \left( \frac{C_2^2}{(C_0)^4} + \frac{5(C_1)^2}{2(C_0)^6} - \frac{3}{8(C_0)^4} \right) + \ldots. \tag{36}
\]

Substituting the last expansion into (18), we note that \( a^{N-1} = -C_0 \) and equate the coefficients at equal powers of \( \mu \) obtaining the Kruskal series of conservation laws. The first few of them are:
\[
((C_0)^{-2})_x = \left[ 2C_1(C_0)^{-2} \right]_x, \quad (C_1(C_0)^{-4})_x = \left( \frac{2(C_1)^2}{(C_0)^3} - \frac{C_2}{(C_0)^3} - \frac{1}{2(C_0)^2} \right)_x, \tag{37}
\]
\[
\left( \frac{C_2}{(C_0)^3} - \frac{5(C_1)^2}{2(C_0)^5} + \frac{3}{8(C_0)^4} \right)_x = \left( \frac{3C_1}{2(C_0)^4} + \frac{5C_2C_1}{(C_0)^5} - \frac{5(C_1)^3}{(C_0)^6} - \frac{C_3}{(C_0)^4} \right)_x.
\]

6. Commuting flows

The most important part of the integration procedure for a semi-Hamiltonian system (as described briefly in section 2) is construction (preferably in an explicit form) of sufficiently many commuting flows, either in a diagonal form (7) or in non-diagonal representation in terms of the variables \( a^k \) or \( b^k \). We choose the latter possibility in this section; the next section is devoted to a suitable modification of the generalized hodograph formula (6) to the non-diagonal form of the commuting flows.

In the case considered in this paper the conservation laws for the generating function have the form (see [16])
\[
p_x = (S_x(p, b^1, b^2, \ldots, b^N))_x
\]
where the flux \( S(p, b^1, b^2, \ldots, b^N) \) can be represented in a much simpler form \( S(p, B^0) \) (see [17]), where \( B^0(b) \) is the ‘zeroth’ moment (of the corresponding hydrodynamic chain (32)). In this paper we consider the case \( S = S_1(p, B^0) = -B^0\sqrt{1 - p^2} \). The situation with higher

\(^\text{16}\) We call this asymptotic expansion Kruskal, because M. Kruskal was the first to introduce a similar construction for the KdV equation.
commuting flows for the chain (32) is as follows: the first commuting flow has the flux (corresponding to the same generating function \( p(b, \lambda) \)) of the form \( S_2(p, B^0, B^1) \), the second commuting flow has the flux \( S_3(p, B^0, B^1, B^2) \), etc. Assume that all these functions \( S_i(p, B^0) \), \( S_2(p, B^0, B^1) \), \( S_3(p, B^0, B^1, B^2) \) ... are the coefficients of an expansion of some more complicated function with respect to an extra parameter \( \zeta \), we come to the ansatz that such a function should be written also in a compact form \( G(p(\lambda), p(\zeta)) \) (see a more general exposition in [18]).

The conservation laws for the generating function of all higher commuting flows were found in [18]. For this particular case they can be represented formally as

\[
\partial_{\tau^{(q)}} p(\mu) = \partial_\tau G(p(\mu), p(\eta)),
\]

where \( \mu = -\ln \lambda, p(\eta) \) is obtained by formally replacing the parameter \( \mu \) by the parameter \( \eta \) and

\[
G(p(\mu), p(\eta)) = \frac{\sqrt{1 - p^2(\mu)}}{\sqrt{1 - p^2(\eta)}} + \frac{1}{2} p(\mu) \ln \frac{p(\eta) + 1}{p(\eta) - 1} + \ln \frac{p(\mu) - p(\eta)}{\sqrt{1 - p^2(\mu)} + \sqrt{1 - p^2(\eta)}}.
\]

The so-called ‘vertex’ operator \( \partial_{\tau^{(q)}} \) is not yet determined and should be specified separately for different cases below. We do not give a detailed account of the construction of the formulas (38) and (39) here; the only fact we really need is compatibility of (38) with \( G \) given by (39) and the generator (18) of conservation laws for the system under study. One can easily verify this compatibility using any computer algebra system and assuming \( \tau \) for the moment a new ‘time’ (as \( \tau \) in (7)), both \( p(\mu) \) and \( p(\lambda) \) should satisfy (18); in order to find \( \partial_{\tau^{(q)}} a^{N-1} \) needed to complete the compatibility check one should substitute the expansion (36) into (38) and keep the terms with \( \mu^2 \) (we recall that \( a^{N-1} = -C^0 = g \)).

6.1. Kruskal series of commuting flows

For this series infinitely many respective fluxes for the generating function of conservation laws of the higher commuting flows can be found substituting the following asymptotic expansions for \( \eta \to 0 \) (see (35) and (36))

\[
q(\eta) = \frac{C^0}{\eta} + \frac{C^1}{C^0} + \eta \left( \frac{C^2}{(C^0)^2} - \frac{(C^1)^2}{(C^0)^3} \right) + \eta^2 \left( \frac{C^3}{(C^0)^3} - \frac{3C^2}{(C^0)^4} + \frac{2(C^1)^3}{(C^0)^5} \right) + ..., \quad (40)
\]

\[
p(\eta) = -1 + \frac{\eta^2}{2(C^0)^2} - \eta^3 \frac{C^1}{(C^0)^4} + \frac{\eta^4}{2} \left( \frac{C^2}{(C^0)^5} + \frac{5(C^1)^2}{5(C^0)^6} - \frac{3}{8(C^0)^7} \right) + ... \quad (41)
\]

into (39) and (38) and specifying the expansion of the vertex operator

\[
\partial_{\tau^{(q)}} = \ln \eta \partial_\rho + \frac{1}{\eta} \partial_\tau + \partial_\tau^{(q)} + \eta \partial_\tau^{(q)} + \eta^2 \partial_\tau^{(q)} + ...
\]
to match with the expansion
\[
G(p(\mu), p(\eta)) = p(\mu) \ln \eta + \frac{C_0 \sqrt{1 - p^2(\mu)}}{\eta} \\
+ \left[ C^1 \sqrt{1 - p^2(\mu)} - p(\mu) \ln C^0 + \frac{1}{2} \ln \frac{p(\mu) + 1}{p(\mu) - 1} - p(\mu) \ln 2 \right] \\
+ \eta \left[ \frac{C^2}{(C^0)^2} - \frac{(C^0)^2}{(C^0)^3} + \frac{1}{2C^0} \right] \sqrt{1 - p^2(\mu)} - \frac{C^1}{(C^0)^2} p(\mu) - \frac{1}{C^0 \sqrt{1 - p^2(\mu)}} \right] + \ldots
\]

So now we can identify \( t^0 = x \) and \( t^1 = t \), so
\[
(p(\mu))_{t^0} = (p(\mu))_x, \quad (p(\mu))_{t^1} = -(C^0 \sqrt{1 - p^2(\mu)})_x,
\]
while higher conservation laws are (see (18) and (21))
\[
(p(\mu))_{t^1} = \left[ \frac{C^1}{C^0} \sqrt{1 - p^2(\mu)} - p(\mu) \ln C^0 + \frac{1}{2} \ln \frac{p(\mu) + 1}{p(\mu) - 1} - p(\mu) \ln 2 \right]_x,
\]
\[
(p(\mu))_{t^2} = \left[ \left( \frac{C^2}{(C^0)^2} - \frac{(C^0)^2}{(C^0)^3} + \frac{1}{2C^0} \right) \sqrt{1 - p^2(\mu)} - \frac{C^1}{(C^0)^2} p(\mu) - \frac{1}{C^0 \sqrt{1 - p^2(\mu)}} \right]_x, \ldots
\]

Corresponding higher Liouville equations are associated with higher commuting hydrodynamic chains. For instance, the first higher Liouville equation
\[
f_{t^2} = (q^2 + \frac{C^1}{C^0} a - \ln C^0 + 1 - \ln 2)_{x} + (q^2 + 1)f_{t^1} (q \ln C^0 + \frac{C^1}{C^0})_{x}
\]
is associated with the first higher hydrodynamic chain commuting with (34):
\[
C^k_{t^2} = C^k_{x} + \frac{C^1}{C^0} C^k_{x} - (\ln C^0 - 1 + \ln 2) C^k_{x} - [(k + 2) C^k_{x} + k C^{k+1}] \frac{C^1}{C^0} C^k_{x}
\]
\[
+ [C^1 (k C^{k-1} + (k + 2) C^{k+1}) - C^0 ((k + 1) C^k + (k + 3) C^{k+2})] \frac{C^0}{(C^0)^2} C^k_{x}.
\]

6.2. N principal series of commuting flows

As we remarked above, any \( N \) component semi-Hamiltonian hydrodynamic-type system has infinitely many conservation laws and commuting flows parameterized by \( N \) arbitrary functions of a single variable (see details in [25]), but in many interesting cases only \( N \) infinite series of conservation laws and commuting flows can be constructed (see [20, 21, 25]); they usually form a complete basis in the respective linear spaces. The completeness of \( N \) series constructed here is discussed in the conclusion.

In this case we again (as in the previous subsection) start from (38) and (39).

The fluxes for the generating function of conservation laws of corresponding higher commuting flows can be found from the equations written in a conservative form
\[
(h_t)_{t^0(\zeta)} = \partial_x G(h_t, p(\zeta)).
\]
This is obtained by substituting (25) into (38) and keeping only the first term. Also, in order to check the correctness of (42) independently, one can check its compatibility with (29) using (18), (21) and (30).

Substitution (see (25)) of the expansion of \( p(\zeta) \) as \( \zeta \to 0 \) leads to the construction of higher commuting flows. However, this derivation is not so trivial. Indeed, the corresponding expansion of the ‘vertex’ operator \( \partial_{\tau(\zeta)} \) matching the expansion of the right-hand side of (42) obtained below is simple:

\[
\partial_{\tau(\zeta)} = \partial_{\ell^1} + \zeta \partial_{\ell^2} + \zeta^2 \partial_{\ell^3} + \zeta^3 \partial_{\ell^4} + \ldots,
\]

where the index \( k = 1, \ldots, N \) in \( \tau^{(k)}(\zeta) \) means the \( k \)-th branch of the Riemann surface (22).

However, direct substitution of (25) and (43) yields the desirable infinite set of equations

\[
(h^k)_{\ell^1} = \partial_{x} G(h^k, h^k), \quad (h^k)_{\ell^2} = \partial_{x} G_{1}(h^k, h^k, p^{(k)}_1), \quad (h^k)_{\ell^3} = \partial_{x} G_{2}(h^k, h^k, p^{(k)}_1, p^{(k)}_2), \quad \ldots
\]

only for distinct indices \( i \) and \( k \), because the function \( G(x, y) \) has a singularity (see (39)) for \( x = y \):

\[
G(x, y) = Q(x, y) + \ln(x - y),
\]

with the non-singular part

\[
Q(x, y) = \frac{1 - x^2}{\sqrt{1 - y^2}} + \frac{1}{2} x \ln \frac{y + 1}{y - 1} - \ln \left(\sqrt{1 - x^2} + \sqrt{1 - y^2}\right).
\]

To complete the construction for \( i = k \), one should observe that substitution of (25) into (44) leads to the asymptotic expansion \( \zeta \to 0 \)

\[
G(h^k, p^{(k)}(\zeta)) = Q(h^k, h^k + \zeta p^{(k)}_1 + \zeta^2 p^{(k)}_2 + \zeta^3 p^{(k)}_3 + \ldots) + \ln(p^{(k)}_1 + \zeta p^{(k)}_2 + \zeta^2 p^{(k)}_3 + \ldots) + \ln \zeta.
\]

Since the leading term \( \ln \zeta \) disappears in (42) after differentiation, we obtain the necessary matching expansions for \( G \) and \( \partial_{\tau(\zeta)} \) for \( i = k \). For instance,

\[
(h^k)_{\ell^1} = \left[ Q(h^k, h^k) + \ln p^{(k)}_1 \right] e^{x}, \quad (h^k)_{\ell^2} = \left( p^{(k)}_1 \frac{\partial Q(h^k, p)}{\partial p} \right)_{p = h^k} + p^{(k)}_1, \quad \ldots
\]

Conservation law densities \( p^{(k)}_m \) were described in section 4.

7. Generalized hodograph method

In this section we finish our procedure for the integration of the hydrodynamic-type system (1), also written in equivalent forms (23), (29).

According to the generalized hodograph method (see details in [25]), any generic solution \( r'(t, x) \) of a semi-Hamiltonian diagonal hydrodynamic-type system (4) in a neighborhood of a generic point is given in an implicit form by the algebraic system (6) for the unknowns \( r'(t, x) \), where \( w'(r) \) are the velocities of a generic commuting flow (7). In arbitrary hydrodynamic variables \( u'(r) \) one can easily rewrite the algebraic system (6) (see [25]) as

\[
x \delta_k^j - n'_j(u) = w'_j(u),
\]

(45)
where the hydrodynamic-type system (4) has the form
\[ u^i_t = \sum_j v^i_j(u)u^j_t, \quad i, j = 1, \ldots, N, \]
while commuting hydrodynamic-type systems (7) have the form
\[ u^i_t = \sum_j w^i_j(u)u^j_t, \quad i, j = 1, \ldots, N. \]

In this section we will modify (6), (45) further in order to get the simplest form suitable for the case studied.

In order to construct solutions of (23) we first need to prove the following result suitable for our particular case of system (1) (see [20, 21]):

**Lemma.** Hydrodynamic-type system (23) together with commuting flows (42) has the common conservation law
\[ dz = \frac{1}{(C^0)^2}dx + \frac{2C^4}{(C^0)^2}dr + \left( \frac{1}{2(C^0)^2} \ln \frac{p(\eta) + 1}{p(\eta) - 1} + \frac{p(\eta)}{(C^0)^2(1 - p^2(\eta))} \right) d\tau(\eta). \]

**Proof.** If \( \mu \to 0, \)
\[ G(p(\mu), p(\eta)) = \mu^2 \left( \frac{1}{4(C^0)^2} \ln \frac{p(\eta) + 1}{p(\eta) - 1} + \frac{p(\eta)}{2(C^0)^2(1 - p^2(\eta))} \right) + \mu^3(...) + ... \]
Taking into account (38) and (41), one can obtain
\[ \left( \frac{1}{(C^0)^2} \right) \tau(\eta) = \left( \frac{1}{2(C^0)^2} \ln \frac{p(\eta) + 1}{p(\eta) - 1} + \frac{p(\eta)}{(C^0)^2(1 - p^2(\eta))} \right). \]
This conservation law together with (37) can be written in the above potential form, where \( z \) is a potential function such that \( z_t = (C^0)^{-2}. \) The Lemma is proved.

Next we modify the initial generalized hodograph formula (6). It should be reduced (following the idea in [21]) to a form suitable for our non-diagonal systems (6) and (23).

Algebraic system (6) (or (45) in arbitrary variables) can be written in the form (here \( \partial_i = \partial/\partial r^i \))
\[ x + \frac{\partial_i G_1}{\partial_i H_0} = \frac{\partial_i G_\infty}{\partial_i H_0}, \quad (46) \]
where we denoted \( H_0 = (C^0)^{-2}, \quad G_1 = 2C^4(C^0)^{-2} \)
and
\[ G_\infty = \frac{1}{2(C^0)^2} \ln \frac{p(\lambda) + 1}{p(\lambda) - 1} + \frac{p(\lambda)}{(C^0)^2(1 - p^2(\lambda))}. \]
Here we returned to the original parameter \( \lambda \) to indicate that now we are working not with any particular asymptotic expansion (see (25), (41)), but with the original algebraic surface as a whole. Indeed, the hydrodynamic-type system (4) has a conservation law \( \partial_i H_0 = \partial_i G_1, \)
while the commuting hydrodynamic system has the conservation law \( \partial_i H_0 = \partial_i G_\infty. \) This means that \( \sum \partial_i H_0 \cdot r_i^t = \sum \partial_i G_1 \cdot r_i^t \) and \( \sum \partial_i H_0 \cdot r_i^t = \sum \partial_i G_\infty \cdot r_i^t. \)
Taking into account (4), (7), (6) and splitting with respect to \( r_i^t, \) one obtains (46). Multiplying (46) by \( \partial_i H_0 \cdot dr^i \) and summing up, one arrives at
Now we rewrite this equation after the invertible point transformation \((r) \rightarrow (b)\) as
\[ x \partial H_0 / \partial b^i + t \partial G_1 / \partial b^i = \partial G_{\infty} / \partial b^i. \]
Taking into account \(\partial_t H_0 = -2(C_0)^{-3}, \partial_t G_1 = 2(C_0)^{-2}b^i - 4C_1(C_0)^{-3}\) (see (31), here \(\partial_t = \partial / \partial t(b^i)\), we obtain the algebraic system
\[ x + t(2C_1 - C_0b^i) = \ln(\sqrt{1 + q^2} - q) - q\sqrt{1 + q^2} + C_0\frac{1 + q^2}{q - b^i} \left( \sum_{m=1}^{N} \frac{1}{q - b^m} - \frac{Nq}{1 + q^2} \right)^{-1}, \tag{47} \]
where \(q(b, \lambda)\) is the (multivalued) inverse function to the function \(\lambda(b, q)\) (22). These \(N\) equations are nothing but the diagonal part of the matrix algebraic system (45). All off-diagonal equations are compatible with the diagonal part (25).

So we proved:

**Theorem 1.** A hydrodynamic-type system (23) has infinitely many particular solutions \(b^i(t, x)\) in the implicit form given by (47) with a free parameter \(\lambda\).

The algebraic system (47) determines the one-parametric family of solutions \(b^i(t, x, \lambda)\) in implicit form and simultaneously \(g(t, x, \lambda) = -C_0b(t, x, \lambda)\). Thus we found one parametric family of Hamilton’s equations (10), which are Liouville integrable.

In fact, using the generalized hodograph method and the nonlinear superposition principle implied by this method (see below) we easily obtain multiparametric families of integrable metrics. Namely expanding the generating function \(p(b, \lambda)\) at different points on the Riemann surface \(p = p(b, \lambda)\) with the parameters \((p, \lambda)\) (for example when \(p \rightarrow -1\) or \(p \rightarrow b^i\)), one can construct infinite multiparametric series of new solutions \(g(b(t, x))\). Now we demonstrate this idea. To avoid large repeated expressions, we introduce functions
\[ W_i(b, \lambda) = \ln(\sqrt{1 + q^2} - q) - q\sqrt{1 + q^2} + C_0\frac{1 + q^2}{q - b^i} \left( \sum_{m=1}^{N} \frac{1}{q - b^m} - \frac{Nq}{1 + q^2} \right)^{-1}. \tag{48} \]

1. *Kruskal series.* Substitution of asymptotic expansion (35) into (48) leads to
\[ W_i(b, \mu) = \ln \mu + \frac{1}{\mu} W_i^{(-1)}(b) + W_i^{(0)}(b) + \mu W_i^{(1)}(b) + \mu^2 W_i^{(2)}(b) + \ldots, \tag{49} \]
where, for instance,
\[ W_i^{(-1)}(b) = C_0b^i - 2C_1. \]

\[ W_i^{(0)}(b) = (b^i)^2 - \frac{C_1}{C_0b^i} + \frac{2(C_1)^2 - 3C_0C_2}{(C_0)^2} - \log C_0 - \log 2. \]
2. $N$ principal series. Substitution of asymptotic series (24) into (48) leads to

$$W^{(k)}_i(b, \lambda) = W^{(k)}_{0i}(b) + \lambda W^{(k)}_{1i}(b) + \lambda^2 W^{(k)}_{2i}(b) + ..., \quad (50)$$

where, for instance,

$$W^{(k)}_{0i}(b) = \ln(\sqrt{1 + (b^k)^2} - b^k) + (C^0 b^k - b^k)\sqrt{1 + (b^k)^2}.$$

Once we found all these coefficients $W^{(k)}_{mi}(b)$ and the Kruskal series $W^{(k)}_j(b)$, we can construct infinitely many particular solutions parameterized by an arbitrary number of constants $\sigma^m_k$:

$$x + t(2C^0 - C^0 b^k) = \sum_{k=1}^{N} \sum_{m=0}^{\infty} \sigma^m_k W^{(k)}_{mi}(b), \quad i = 1, ..., N, \quad (51)$$

or a functional parameter $\varphi(\lambda)$:

$$x + t(2C^0 - C^0 b^k) = \oint \varphi(\lambda) W_j(b, \lambda) d\lambda, \quad i = 1, ..., N, \quad (52)$$

where $\varphi(\lambda)$ and the contour can be chosen in many special forms. Formulae (51) and (52) present the nonlinear superposition principle implied by the generalized hodograph method.

8. Complete algorithm for construction of two-dimensional metrics with polynomial integrals

In this section we put together the steps necessary for obtaining multiparametric families of metrics

$$dx^2 = g^2(t, x)dt^2 + dx^2 \quad (53)$$

possessing first integrals

$$F(t, x, p_1, p_2) = \sum_{k=0}^{N} \frac{(-1)^k a^k}{g^{N-k}} p_1^{N-k} p_2^k \quad (54)$$

of a given degree $N$.

**Step 1.** First we should choose some set (possibly infinite) of indices $S_{ij} = \{(i, j_1), j_2, k_2, \ldots\}$ and/or set $S_{ij} = \{(i, j_1), j_2, k_2, \ldots\}, \quad 1 \leq i, j \leq N, \quad 1 \leq k, \leq N, \quad 0 \leq j \leq \infty$. For each element pair $(i, j)$ of $S_{ij}$ take the appropriate quantities $W_{ij}^{(0)}(b)$ obtained in (49) and for each element triple $(i, j, k)$ of $S_{ijk}$ take the appropriate quantities $W_{ijk}^{(0)}(b)$ obtained in (50). We recall that the necessary quantities $W_{ijk}^{(0)}$ are obtained by substitution of the expansion (35) into (48), and (24) into (48). The expansions (24) and (35) are obtained from equation (22) of the Riemann surface (with $\mu = -\ln \lambda$), and the quantities $C^k$ are related to the symmetric field variables $b^k$ by the formulas of section 5.

**Step 2.** Substitute the quantities $W_{ijk}^{(0)}$ into the generalized hodograph formula (51) with arbitrary parameters $\sigma^m_k$ (if $S_{ij}$ was given in step 1, add a linear combination of $W_{ijk}^{(0)}(b)$ from the Kruskal series). This gives the necessary system of $N$ implicit equations for the symmetric hydrodynamic variables $b^k(t, x), \quad k = 1, ..., N$—solutions of the system (23).

**Step 3.** The metric (53) is found from the relation $g(t, x) = a^{N-1}(t, x) = -\sum_{k=1}^{N} b^k(t, x)$. The coefficients $a^i(t, x)$ of the first integral (54) are given by Vieta’s formulas from equation (20).
Theorem 2. The local metrics (53) obtained by the method described in this section have first integrals (54) of the given degree \( N \) obtained in step 3.

Note that the background on conservation laws, etc., exposed in sections 4 and 5 was necessary to find the algorithm exposed above and corroborate its correctness (so providing a proof of theorem 2) although formally the content of these sections does not appear in the algorithm itself.

As one can notice, equation (51) of the generalized hodograph method suggests to change the variables \((t, x)\) in (53) to some pair of independent hydrodynamic variables \( b^k \). Namely, if we express \( t = T(b) \) and \( x = X(b) \) from any pair of equation (51) and substitute these expressions for \( t \) and \( x \) into the other \( N - 2 \) equations we obtain a system of \( N - 2 \) equations for the variables \( b^1, \ldots, b^N \). Suppose for example that \( b^1, b^2 \) give an independent set of variables for these \( N - 2 \) equations, then \( b^3 = \beta_3(b^1, b^2), \ldots, b^N = \beta_N(b^1, b^2) \). Substituting \( t = T(b), x = X(b) \) and the expressions for \( b^3, \ldots, b^N \) into (53) (we recall that \( g(t, x) = e^{N-1}(t, x) = -\sum_{k=1}^{N} b^k(t, x) \)) we obtain the metric in another non-diagonal form:

\[
\frac{ds^2}{dx^2} = \tilde{g}_{11}(b^1, b^2)(db^1)^2 + 2\tilde{g}_{12}(b^1, b^2)db^1db^2 + \tilde{g}_{22}(b^1, b^2)(db^2)^2. \tag{55}
\]

The corresponding first integral for this metric can be easily found substituting the expressions of \( a^i \) in terms of \( b^k \) into (54) and changing \( p_i \) to \( \tilde{p}_i \) using

\[
p_1 dt + p_2 dx = p_1 dT(b^1, b^2) + p_2 dX(b^1, b^2) = \tilde{p}_1 db^1 + \tilde{p}_2 db^2.
\]

Another (although formally less rich) one-parametric family of metrics with polynomial integrals of a given degree \( N \) can be obtained from (47) as stated in theorem 1. In fact (as we explained in the previous section) the multiparametric families of metrics obtained by the algorithm in this section are nonlinear superpositions of the one-parametric family of theorem 1.

An important issue is irreducibility of first integrals obtained by this algorithm. The solutions of the system (1) describe all polynomial first integrals of a given degree \( N \), so we can easily check by counting functional freedom of its solutions that a generic solution of it for any particular \( N \) cannot be a power of a lower-order integral and/or power or a product of the Hamiltonian \( \mathcal{H}(t, x, p_1, p_2) = \frac{1}{2} (p_1^2 + p_2^2) \) with lower-order integrals. In many similar cases ([19–21, 25]) one can prove completeness of the constructed series of commuting flows and conservation laws. We discuss completeness of the construction given in this paper in the conclusion. Even if the constructed series of commuting flows do not form a complete basis in the linear space of all commuting flows, the parametric freedom still seems to be sufficient to prove irreducibility of the constructed first integrals for a generic choice of the coefficients \( a_k^m \) in step 2.

9. Conclusion

In this paper we considered the integrability of the semi-Hamiltonian hydrodynamic-type system (1). We found and presented \( N \) infinite (principal) series of conservation laws and commuting flows. This means that one can extract infinitely many corresponding solutions by the generalized hodograph method; section 8 gives the details of our algorithm for construction of local metrics with polynomial integrals. Polynomial integrals was a very important direction in the theory of integrable geodesic flows. Plenty of explicit examples were found (see, for example, [10]). Our claim is that we presented a method which allows
(local) computation of infinitely many metrics (equipped with polynomial integrals) parameterised by an arbitrary number of constants.

Here we did not investigate the completeness of conservation law densities (and corresponding commuting flows). This problem should be investigated elsewhere. Here we just mention that (see (27)) for $N = 2n + 1$, $n > 0$

$$
\sum_{k=1}^{2n+1} p^{(k)}_1 = -2n+1 \frac{1 + (b^k)^2}{\prod_{m \neq k} (b^k - b^m)} = 0.
$$

This means that corresponding conservation law densities $p^{(k)}_1$ are not linearly independent for odd $N$. Thus one should probably construct additional conservation law densities to get the complete set.

The semi-Hamiltonian hydrodynamic-type system (1) holds for the coefficients $a^k(t, x)$ of a first (polynomial) integral (2). In this paper we found a more appropriate set of field variables $b^k$. This natural choice of unknown functions follows also from factorization of the polynomial expression for the first integral with respect to the ratio of both momenta\(^17\):

$$
F = \sum_{k=0}^{N} \frac{(-1)^k a^k}{8^{N-k}} p^{N-k}_1 p^k_2 = \left( \frac{p^1}{8} \right)^N \prod_{k=1}^{N} (q - b^k),
$$

where

$$
q = \frac{B^0 p_2}{p_1}.
$$

This first integral can also be written in two other equivalent forms (see (31) and (33)):

$$
F = (p_2)^N \exp \left( -\sum_{k=0}^{\infty} \frac{B^k}{q^{k+1}} \right) = \left( p_2 \right)^2 + \left( \frac{p^1}{B^0} \right)^2 \lambda(q, B).
$$

These two equivalent representations for the first integral are of great interest: if one finds another finite-dimensional parameterization of the moments $B^k(b)$ compatible with (32), the first integral associated with this solution will no longer be polynomial (see, for instance, the discussion on the existence of rational first integrals in the two-dimensional case in [15], in the multi-dimensional case in [7], and several examples of non-polynomial integrals in [12]; see also [11]). This more general problem should be considered in a separate publication.

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\(^{17}\) We remind the reader that $a^N = 1$ and $a^{N-1} = g$. 
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