The quantum pigeonhole principle as a violation of the principle of bivalence

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Abstract

In the paper, it is argued that the phenomenon known as the quantum pigeonhole principle (namely, three quantum particles are put in two boxes, yet no two particles are in the same box) can be explained not as a violation of Dirichlet’s box principle in the case of quantum particles but as a nonvalidness of a bivalent logic for describing not-yet verified propositions relating to quantum mechanical experiments.

Keywords: Quantum mechanics; Truth values; Bivalence; Many-valued logics; Pigeonhole principle.

1 Introduction

Do quantum systems always possess intrinsic properties? In accordance with a realist interpretation of quantum mechanics [1], it is appropriate to say that an individual system possesses values of its physical quantities even before these values can be measured. In this context, “appropriate” means that propositions asserting possession of the physical quantities can be handled using the standard propositional logic obeying the principle of bivalence (saying that “A proposition cannot be neither true nor false” and “A proposition cannot be both true and false” [2]).

However, such an assumption brings about a violation of an abstract principle of combinatorial analysis, namely, Dirichlet’s box principle also known as the pigeonhole principle [3].

Indeed, let us consider three quantum particles and suppose that each particle has either the property $x$ or the property $\neg x$. Let $X_j$ denote the proposition asserting that the particle $j \in \{1, 2, 3\}$ possesses the property $x$ and, correspondingly, let $\neg X_j$ denote the proposition that this particle possesses the alternative property $\neg x$.

Using the language of the paper [4], one may say that the particle $j$ is in the box “$x$” if the particle possesses the property $x$ and analogously the particle $j$ is in the box “$\neg x$” if it has the property $\neg x$.

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not $-x$.

Let $[\diamond]_v$, where the symbol $\diamond$ can be replaced by any proposition (compound or simple), refer to a valuation, that is, an assignment of a truth-value $v$ to a proposition $\diamond$, explicitly,

$$[\diamond]_v = v \in \{v\},$$

where $\{v\}$ is the set of the truth-values ranging from the value 0 (denoting the falsity) to the value 1 (denoting the truth). Furthermore, let the truth value of the negation $[-X_j]_v$ be defined by the following axiom

$$[-X_j]_v = 1 - [X_j]_v.$$  \hspace{1cm} (2)

According to the classical distributive law, for any two particles $j$ and $k$, where $j < k \leq 3$, the equality must hold

$$(X_j \lor -X_j) \land (X_k \lor -X_k) = \text{Same}_{jk} \lor \text{Diff}_{jk}.$$  \hspace{1cm} (3)

where

$$\text{Same}_{jk} \equiv (X_j \land X_k) \lor (-X_j \land -X_k),$$  \hspace{1cm} (4)

$$\text{Diff}_{jk} \equiv (X_j \land -X_k) \lor (-X_j \land X_k).$$  \hspace{1cm} (5)

At this point, let us consider the case of a bivalent logic with the set of the truth-values $\{v\} = \{0, 1\}$. In such a case, among three propositions $X_j$ there are at least two having the same truth-value, 0 or 1. This means that ahead of the verification of the propositions $X_j$, a pair of the particles is always in the same box – either “$x$” or “not–$x$”, which can be presented in the form of the pigeonhole principle

$$[\text{Same}_{12} \lor \text{Same}_{13} \lor \text{Same}_{23}]_v = 1.$$  \hspace{1cm} (6)

By contrast, let us assume that the cardinality of the set of the truth-values concerning unperformed quantum mechanical experiments is not 2 but, say 3, specifically, $\{v\} = \{0, \frac{1}{2}, 1\}$ where the additional truth value $\frac{1}{2}$ is interpreted as “neither true nor false” (and 1 is the only designated truth value).\footnote{This can be Kleene’s (strong) logic $K_3$ or the 3-valued Lukasiewicz system \cite{5, 6}.}
Using the corresponding three-valued truth-table, it is straightforward to show that in the given case there are instances in which no two propositions \( X_j \) and \( X_k \) have the same truth-value.

For example, when \([X_1]_v = 1, [X_2]_v = \frac{1}{2}\) and \([X_3]_v = 0\), one can say that before the verification, the particle 1 is in the box “\( x \)”, the particle 3 is in the box “not – \( x \)”, while the particle 2 is neither in the box “\( x \)” nor in the box “not – \( x \)”. In other words, in this instance, no two particles are in the same box.

In the said instance, as the conjunction \( X_1 \land X_2 \) along with the conjunction \( \neg X_2 \land \neg X_3 \) cannot be evaluated to the truth \([\text{Same}_{12}]_v \) and \([\text{Same}_{23}]_v \) are not equal to 1. Thus, in case of a non-bivalent logic one must get
\[
[ \text{Same}_{12} \lor \text{Same}_{13} \lor \text{Same}_{23} ]_v \neq 1 \quad . \tag{7}
\]

As follows, a violation of the pigeonhole principle described in the paper \([4]\) (namely, three quantum particles are put in two boxes, yet no two particles are in the same box) can be viewed not as a failure of Dirichlet’s box principle in the case of quantum particles but as a nonvalidness of a bivalent semantics for treating not-yet verified propositions about properties of quantum mechanical systems.

Let us develop this line of argument further in this paper.

### 2 Preliminaries

Following the setup introduced in the paper \([4]\) let us consider the complex Hilbert space \( \mathcal{H} \) of finite dimension 4, i.e., \( \mathcal{H} \equiv \mathbb{C}^4 \), related to the two-qubit system with each qubit (called a “particle”) prepared in the superposition
\[
|\Psi_j^z\rangle = \frac{1}{\sqrt{2}} (|\Psi_j^x\rangle + |\Psi_j^y\rangle) \quad , \tag{8}
\]
where \(|\cdot\rangle\) are the normalized eigenvectors of the Pauli spin matrices.

The projection operator \( \hat{P}_{jk}^{z+} \)
\[
\hat{P}_{jk}^{z+} \equiv |\Psi_j^z\rangle\langle \Psi_j^z| \otimes |\Psi_k^z\rangle\langle \Psi_k^z| = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{9}
\]

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\( ^2 \)As long as the conjunction of two propositions is the weakest proposition among the two.
corresponds to the proposition $Z_{jk}^+$ asserting that both particles of the bipartite composite system have the same spin angular momentum value $+\hbar/2$ along the axis $z$. Together with this, the projection operators $\hat{P}_{\text{Same}}$ and $\hat{P}_{\text{Diff}}$ corresponding to the propositions Same$_{jk}$ and Diff$_{jk}$ introduced in (4) and (5) are given explicitly by

$$\hat{P}_{\text{Same}} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (10)$$

$$\hat{P}_{\text{Diff}} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}. \quad (11)$$

As it follows from here,

$$\hat{P}_{\text{Same}} \hat{P}_{\text{Diff}} = \hat{P}_{\text{Diff}} \hat{P}_{\text{Same}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \equiv \hat{0}, \quad (12)$$

$$\hat{P}_{\text{Same}} + \hat{P}_{\text{Diff}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv \hat{1}, \quad (13)$$

where $\hat{0}$ is the zero matrix and $\hat{1}$ is the identity matrix (the operator of the identity mapping) on $\mathbb{C}^4$.

Let us consider a lattice $L(\mathbb{C}^4)$ of the subspaces of $\mathbb{C}^4$ in which the partial order $\leq$ is set inclusion $\subseteq$, the meet $\cap$ is set intersection $\cap$ and the join $\sqcup$ is the internal direct sum of any pairwise disjoint sequence of the subspaces of $\mathbb{C}^4$. The lattice $L(\mathbb{C}^4)$ is bounded, with the trivial space $\{0\}$ equal to the range (column space) of the zero matrix, $\text{ran}(\hat{0}) = \{0\}$, as the bottom and the whole space $\mathbb{C}^4$ equal to the range of the identity matrix, $\text{ran}(\hat{1}) = \mathbb{C}^4$, as the top.

Because any subspace of $\mathbb{C}^4$ is the range of some unique projection operator on $\mathbb{C}^4$, there is a one-to-one correspondence between the subspaces of $\mathbb{C}^4$ and the corresponding projection operators. Thus, one can take the projection operators to be the elements of $L(\mathbb{C}^4)$.

Specifically, as

$$\text{ran}(\hat{P}_{\text{Same}}) \subseteq \text{ker}(\hat{P}_{\text{Diff}}) = \text{ran}(\hat{1} - \hat{P}_{\text{Diff}}), \quad (14)$$

one can define the partial order $\hat{P}_{\text{Same}} \leq (\hat{1} - \hat{P}_{\text{Diff}})$ by setting $\hat{P}_{\text{Same}} \cap (\hat{1} - \hat{P}_{\text{Diff}}) = \hat{P}_{\text{Same}}$ which means that the meet of $\hat{P}_{\text{Same}}$ and $\hat{P}_{\text{Diff}}$ in $L(\mathbb{C}^4)$ can be defined by
\[ \hat{P}_{\text{Same}} \cap \hat{P}_{\text{Diff}} = \hat{P}_{\text{Same}} \hat{P}_{\text{Diff}} = \hat{0} \quad . \]  

(15)

Since the subspaces \( \text{ran}(\hat{P}_{\text{Same}}) \) and \( \text{ran}(\hat{P}_{\text{Diff}}) \) are disjoint, namely

\[ \text{ran}(\hat{P}_{\text{Same}}) \cap \text{ran}(\hat{P}_{\text{Diff}}) = \text{ran}(\hat{P}_{\text{Same}} \hat{P}_{\text{Diff}}) = \text{ran}(\hat{0}) = \{0\} \quad , \]

(16)

the join of \( \hat{P}_{\text{Same}} \) and \( \hat{P}_{\text{Diff}} \) in \( L(\mathbb{C}^4) \) can be defined as their sum, i.e.,

\[ \hat{P}_{\text{Same}} \sqcup \hat{P}_{\text{Diff}} = \hat{P}_{\text{Same}} + \hat{P}_{\text{Diff}} = 1 \quad . \]

(17)

As an immediate consequence of such definitions, it follows that \( \hat{P}_{z^+} \cap \hat{P}_{\text{Same}} \) and \( \hat{P}_{z^+} \sqcup \hat{P}_{\text{Same}} \) are not defined in \( L(\mathbb{C}^4) \) because \( \hat{P}_{z^+} \hat{P}_{\text{Same}} \neq \hat{P}_{\text{Same}} \hat{P}_{z^+} \) and therefore neither \( \hat{P}_{z^+} \hat{P}_{\text{Same}} \) nor \( \hat{P}_{\text{Same}} \hat{P}_{z^+} \) is the projection operator on \( \mathbb{C}^4 \) (the same concerns \( \hat{P}_{z^+} \hat{P}_{\text{Diff}} \) and \( \hat{P}_{\text{Diff}} \hat{P}_{z^+} \)).

Given that the projection operator \( \hat{1} \) leaves invariant any vector lying in the space \( \mathbb{C}^4 \), the range of \( \hat{1} \), a proposition represented by \( \hat{1} \) must be true in any state of the system, i.e., such a proposition must be a tautology \( \top \). Also, as the projection operator \( \hat{0} \) annihilates any vector in \( \mathbb{C}^4 \), the null space of \( \hat{0} \), a proposition represented by \( \hat{0} \) must be false in any state of the system, i.e., this proposition must be a contradiction \( \bot \).

This can be written as

\[ |\Psi\rangle \in \text{ran}(\hat{1}) \quad \implies \quad v(\hat{1}) = [\top]_v = 1 \quad , \]

(18)

\[ |\Psi\rangle \in \ker(\hat{0}) \quad \implies \quad v(\hat{0}) = [\bot]_v = 0 \quad , \]

(19)

where the symbol \( \implies \) means “implies” or “if \ldots then”, \( v \) denotes the truth-function that maps a given projection operator to the truth value of the corresponding proposition.

Let \( \hat{P}_A \) and \( \hat{P}_B \) denote the projection operators representing the propositions \( A \) and \( B \). Then, to decide the truth values of disjunction, conjunction and negation of these propositions, let the following valuational axioms hold

\[ v(\hat{P}_A \sqcup \hat{P}_B) = [A \lor B]_v \quad , \]

(20)

\[ v(\hat{P}_A \sqcap \hat{P}_B) = [A \land B]_v \quad , \]

(21)

\[ v(\hat{1} - \hat{P}_A) = [\neg A]_v \quad . \]

(22)
In this manner, disjunction and conjunction on the propositions \( Z_{jk}^+ \) and \( \text{Same}_{jk} \) are undefined since \( \hat{P}_{jk}^{z_+} \cap \hat{P}_{jk}^{\text{Same}} \) and \( \hat{P}_{jk}^{z_+} \cup \hat{P}_{jk}^{\text{Same}} \) are not defined in \( L(\mathbb{C}^4) \).

According to such valuations, one gets

\[
v(\hat{P}_{jk}^{\text{Same}} \sqcup \hat{P}_{jk}^{\text{Diff}}) = \left[ [\text{Same}_{jk} \lor \text{Diff}_{jk}] \right]_v = 1 ,
\]

\[
v(\hat{P}_{jk}^{\text{Same}} \sqcap \hat{P}_{jk}^{\text{Diff}}) = \left[ [\text{Same}_{jk} \land \text{Diff}_{jk}] \right]_v = 0 ,
\]

which means that the statement “Either two particles are in the same box or it is not the case that two particles are in the same box” is always true.

3 The intermediate truth-value of the proposition \( \text{Same}_{jk} \)

After the preparation, the two-qubit system’s spin state is preselected in the \( z^+ \) direction, i.e., in the state \( |\Psi_{jk}^{z_+}\rangle \)

\[
|\Psi_{jk}^{z_+}\rangle \equiv |\Psi_{j}^{z_+}\rangle \otimes |\Psi_{k}^{z_+}\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

lying in the range of the projection operator \( \hat{P}_{jk}^{z_+} \):

\[
\text{ran}(\hat{P}_{jk}^{z_+}) = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid a \in \mathbb{R} \right\} , \quad \text{ker}(\hat{P}_{jk}^{z_+}) = \left\{ \begin{bmatrix} 0 \\ b \\ c \\ d \end{bmatrix} \mid b, c, d \in \mathbb{R} \right\} .
\]

In this state, the proposition \( Z_{jk}^+ \) is definite and has the truth-value 1. Hence, one can say that in the preselected state, the two-qubit system possesses an intrinsic property, specifically, both qubits have the same spin value \(+\hbar/2\) along the \( z \)-axis.

Given that each projection operator leaves invariant any vector lying in its range and annihilates any vector lying in its null space, the definiteness of the proposition \( Z_{jk}^+ \) can be written down as its bivalence, i.e.,

\[
|\Psi_{jk}^{z_+}\rangle \in \text{ran}(\hat{P}_{jk}^{z_+}) \implies v(\hat{P}_{jk}^{z_+}) = [Z_{jk}^+]_v = 1 ,
\]

\[
|\Psi_{jk}^{z_+}\rangle \notin \text{ker}(\hat{P}_{jk}^{z_+}) \implies v(\hat{P}_{jk}^{z_+}) = [Z_{jk}^+]_v \neq 0 .
\]
Now, consider the range and the null space of the projection operator $\hat{P}^{\text{Same}}_{jk}$:

\[
\text{ran}(\hat{P}^{\text{Same}}_{jk}) = \left\{ \begin{bmatrix} a \\ b \\ b \\ a \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}, \quad \text{ker}(\hat{P}^{\text{Same}}_{jk}) = \left\{ \begin{bmatrix} -d \\ -c \\ c \\ d \end{bmatrix} \middle| c, d \in \mathbb{R} \right\}.
\]

Comparing (25) with (29) makes it evident that the preselected vector $|\Psi_{jk}^+\rangle$ does not lie in the range of the projection operator $\hat{P}^{\text{Same}}_{jk}$ or in its null space. For that reason, one can assert that in the state $|\Psi_{jk}^+\rangle$ the truth-value of the proposition Same$_{jk}$ cannot be 1 or 0, that is, Same$_{jk}$ does not obey the principle of bivalence, explicitly,

\[
|\Psi_{jk}^+\rangle \notin \text{ran}(\hat{P}^{\text{Same}}_{jk}) \implies v(\hat{P}^{\text{Same}}_{jk}) = [\text{Same}_{jk}]_v \neq 1,
\]

\[
|\Psi_{jk}^+\rangle \notin \ker(\hat{P}^{\text{Same}}_{jk}) \implies v(\hat{P}^{\text{Same}}_{jk}) = [\text{Same}_{jk}]_v \neq 0.
\]

Expressed differently, in the intermediate state that exists after the preparation but before the (strong and simultaneous) measurement of particles’ spins along the $x$-axis (that is to say, particles’ presence in the boxes “spin $x+$” and “spin $x-$”) the statement “Two particles are in the same box” is neither true nor false.

Next, consider the disjunction Same$_{12} \lor$ Same$_{13}$: As stated by the valuational axiom (20), its intermediate truth-value is determined by the join of the projection operators $\hat{P}^{\text{Same}}_{12}$ and $\hat{P}^{\text{Same}}_{13}$ in $L(C^4)$

\[
v(\hat{P}^{\text{Same}}_{12} \sqcup \hat{P}^{\text{Same}}_{13}) = [\text{Same}_{12} \lor \text{Same}_{13}]_v,
\]

which is given by $\hat{P}^{\text{Same}}_{12} \sqcup \hat{P}^{\text{Same}}_{13} = \hat{P}^{\text{Same}}_{13}$ consistent with the set inclusion ran($\hat{P}^{\text{Same}}_{12}$) $\subseteq$ ran($\hat{P}^{\text{Same}}_{13}$).

Thus, it must be

\[
v(\hat{P}^{\text{Same}}_{jk}) = [\text{Same}_{12} \lor \text{Same}_{13} \lor \text{Same}_{23}]_v
\]

since

\[
(\hat{P}^{\text{Same}}_{12} \sqcup \hat{P}^{\text{Same}}_{13}) \sqcup \hat{P}^{\text{Same}}_{23} = \hat{P}^{\text{Same}}_{13} \sqcup \hat{P}^{\text{Same}}_{23} = \hat{P}^{\text{Same}}_{23}
\]

in agreement with

\[
\text{ran} \left( \hat{P}^{\text{Same}}_{12} \sqcup \hat{P}^{\text{Same}}_{13} \right) \subseteq \text{ran} \left( \hat{P}^{\text{Same}}_{23} \right).
\]
From (33) it immediately follows that the pigeonhole principle does not hold in the intermediate state of the quantum particles, that is,

$$\left[ \text{Same}_{12} \lor \text{Same}_{13} \lor \text{Same}_{23} \right]_v \neq \{0, 1\} . \quad (36)$$

4 Concluding remarks

In logical terms, the pigeonhole principle boils down to the statement that among three propositions $X_1$, $X_2$ and $X_3$ there exist at least two that have the same bivalent truth-value, i.e., 0 or 1. Consequently, the disjunction of the set of three logical connectives $\text{Same}_{jk} \stackrel{\text{def}}{=} X_j \iff X_k$ (where the symbol $\iff$ denotes “equivalent” and $j \neq k$) must always have the value 1.

In the paper [4] it is suggested that a quantum violation of the pigeonhole principle is an indication that Dirichlet’s box principle (which “encapsulates abstract mathematical notions that go to the core of what numbers and counting are, so it underlies, implicitly or explicitly, virtually the whole of mathematics”) does not hold in the case of quantum particles.

But, as it has been just demonstrated in the presented paper, the quantum violation of the pigeonhole principle may have another, “less dramatic”, so to speak, explanation: It can be a sign that a logic defined as the relations between projection operators associated with quantum particles does not obey the principle of bivalence.

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