Chiral gauge theories and non-abelian analogues of axions

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Abstract
The axion particle may or may not exist, but the axion field can be used, as shown here, in an explicitly local formulation of a chiral U(1) gauge theory with both classical and quantum gauge invariance. Non-abelian analogues of axion fields, which have recently been introduced, can be used, together with their special symmetries, in a similar construction of nonabelian chiral gauge theories. As in known cases, the gauge symmetry is broken and the gauge boson acquires a mass by swallowing the axions which are therefore not physical in this construction.

1 Introduction
It has been known for over half a century that the chiral symmetry which holds in classical Dirac theory with massless fermions interacting with gauge fields is broken upon quantization [1]. The transformation

$$\psi \rightarrow e^{i\alpha\gamma_5} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{i\alpha\gamma_5},$$

(1)
is a symmetry of the kinetic term $\bar{\psi}[i\not\partial] \psi$ and also of the interaction term $\bar{\psi}[iA] \psi$ with the gauge field $A_\mu$. But the axial current $\bar{\psi} \gamma_\mu \gamma_5 \psi$, which appears to be conserved from the equations of motion is found to violate this conservation when the fermion triangle diagram is carefully regularized and evaluated. This effect is the chiral anomaly.

In the early literature, only the one-loop diagram was considered, but subsequently it became clear that higher order loops do not cause any further damage. Instead of considering individual diagrams it is more natural to consider the action and regularize it by one of the many available methods. Calculations are done with the temporary regularized action, after which the realistic limit is taken. The divergence of the axial current is proportional to $F^{\mu\nu} F_{\mu\nu}$, $F$ being the field strength.

While global symmetries can usually be made local with the addition of gauge fields which have to be suitably transformed, this cannot be done in the

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case of chiral symmetries. This is because of the same anomaly. Gauge field
equations of motion require the current to be conserved, but the anomaly means
that the relevant current is not conserved. Therefore special efforts have to be
made to define a gauge theory with a local chiral symmetry. That anomalous
gauge theories can be made sense of in two dimensions was shown many years
ago [2]. Studies of the chiral Schwinger model showed that the photon of the
two-dimensional model acquires a mass and the gauge symmetry is broken. For
more realistic models, it is usual to follow the suggestion of [3] to reformulate
an anomalous model in a gauge invariant way by adding a group variable as an
extra field. The modified theory is not classically gauge invariant but becomes so
on quantization because of the anomaly! We show here how axions can be used
to reformulate chiral $U(1)$ gauge theories in four dimensions in an alternative
way which is manifestly gauge invariant both before and after quantization and
then proceed to generalize to nonabelian theories.

2 Peccei-Quinn symmetry and the axion

Chiral symmetry is classically broken by the mass term $m\bar{\psi}\psi$ in addition to the
quantum anomaly mentioned above. An artificial chiral symmetry for massive
fermions uses a new field $\varphi$ to absorb the chiral transformation. The mass term
is replaced by [4]

$$\bar{\psi}me^{i\varphi\gamma_5}\psi,$$

which is invariant if the field $\varphi$ transforms under

$$\psi \rightarrow e^{i\alpha\gamma_5}\psi, \quad \bar{\psi} \rightarrow \bar{\psi}e^{i\alpha\gamma_5}, \quad \varphi \rightarrow \varphi - 2\alpha.$$ (3)

This transformation leaves the action invariant provided the new field $\varphi$ is mass-
less. This is the Peccei-Quinn symmetry. The particle [5] corresponding to the
new field $\varphi$ [4] is called the axion. This construction was thought to solve the
so-called strong CP problem. The particle called the axion, which was duly
searched, has not been seen in any experiment [6]. An alternative solution of
the strong CP problem avoids the axion [7]. But axions are now being looked
upon as examples of the hiding dark matter and the search goes on.

On a different front, it has been noted recently that the Peccei-Quinn symme-
try is not anomalous [5], though it is spontaneously broken. The basic technique
is to introduce a regularization, say Pauli-Villars. With a careful construction
of the regulator mass term to include a coupling to the axion, one can preserve
the Peccei-Quinn symmetry. The Lagrangian density

$$\mathcal{L}_{reg} = \bar{\psi}[i\slashed{D} - me^{i\varphi\gamma_5}]\psi + \bar{\chi}[i\slashed{D} - M]\chi$$ (4)

including a Pauli-Villars regulator field $\chi$ is invariant under (3) but not under
the chiral transformation of the regulator, namely

$$\chi \rightarrow e^{i\alpha\gamma_5/2}\chi, \quad \bar{\chi} \rightarrow \bar{\chi}e^{i\alpha\gamma_5/2}.$$ (5)
That is why it was believed that the Peccei-Quinn symmetry does not survive quantization. However, the regulator field must behave in the same way as the physical field. Thus the field $\chi$ should be coupled to the gauge field $A_\mu$ and also the axion:

$$L_{\text{reg,corr}} = \bar{\psi}[i\not\!D - m e^{i\varphi \gamma_5}]\psi + \bar{\chi}[i\not\!D - M e^{i\varphi \gamma_5}]\chi.$$  

(6)

With this correction, the action, including the kinetic term of the axion, is invariant under the full Peccei-Quinn transformation

$$\psi \to e^{i\alpha \gamma_5} \psi, \quad \bar{\psi} \to \bar{\psi} e^{i\alpha \gamma_5},$$

$$\chi \to e^{i\alpha \gamma_5} \chi, \quad \bar{\chi} \to \bar{\chi} e^{i\alpha \gamma_5},$$

$$\varphi \to \varphi - 2\alpha.$$  

(7)

Thus this Pauli-Villars regularization respects the Peccei-Quinn symmetry, which accordingly is not anomalous but survives quantization [8]. Note that the action is not invariant under just the chiral transformations of the fermion and regulator fields: the axion field also has to transform. That is why there is a chiral anomaly but no Peccei-Quinn anomaly.

### 3 Chiral $U(1)$ theory

We note now that the global symmetry can be made local by having an appropriate gauge field $B_\mu$ for the purpose:

$$L_B = \bar{\psi}[i\not\!D + B\gamma_5 - m e^{i\varphi \gamma_5}]\psi + \frac{1}{2} F^2 (\partial^\mu \varphi + 2 B_\mu)(\partial^\mu \varphi + 2 B^\mu),$$  

(8)

where

$$B_\mu \to B_\mu + \partial_\mu \alpha$$  

(9)

under a local Peccei-Quinn transformation [3]. Note that it has to couple to both $\psi$ and $\varphi$. Here $F$ is a constant of mass dimension such that the axion kinetic term is $\frac{1}{2} F^2 \partial_\mu \varphi \partial^\mu \varphi$ and there is an additional kinetic term involving the gauge field strength $G_{\mu\nu}$. The argument for the absence of an anomaly in the Peccei-Quinn symmetry continues to hold because the regulator mass term can again be made invariant.

A mass term is generated for the gauge field $B$ because the symmetry is spontaneously broken. Spontaneous breaking is clear because the transformation of $\varphi$ by an additive term cannot be consistent with any vacuum expectation value. The axion is the Goldstone field which vanishes in the unitary gauge and may be considered to be swallowed by the gauge field $B$ which becomes massive, as may be seen by considering a local Peccei-Quinn transformation with $\alpha = -\frac{1}{2} \varphi$. This is in fact the usual Higgs phenomenon.

Note that this is a construction of a chiral $U(1)$ gauge theory using an axion. The presence of the axion may raise eyebrows, but this is similar to
the reformulation used earlier [3, 9, 10, 11, 12]. That way of going about a chiral gauge theory with an anomaly is to make it non-anomalous by adding a Wess-Zumino term [3, 9] which includes a gauge group element as a new field. Instead of adding a non-invariant Wess-Zumino term to nullify the anomaly in the quantum theory, the present approach includes the axion to increase the classical symmetry, but this Peccei-Quinn symmetry turns out to be free from anomaly, leading also to quantum invariance. The axion does not occur in the physical spectrum, just like the gauge group element of the Wess-Zumino term: both become trivial in the unitary gauge. In the Wess-Zumino term approach, there need not be a kinetic term for the gauge group field, so that a gauge boson mass need not arise when one goes to the unitary gauge. In the axion construction the mass necessarily arises and this is as in the chiral Schwinger model [2, 13]. The mass of the gauge boson is not determinate in the sense that $F$ is not fixed. In the original studies on the chiral Schwinger model too it was undetermined, because it was related to a regularization ambiguity of the fermion determinant. In the present approach, no determinant is calculated and the discussion holds even in four dimensions. The photon acquires a mass.

Note that it is also possible to make an anomalous gauge theory non-anomalous without introducing new fields [14]. There have been many studies on different aspects of chiral gauge theories. Some recent work can be found, for example, in [15, 16].

4 Nonabelian chiral symmetry and nonabelian axions

Next we consider the nonabelian generalization of this construction. First we need a generalized axion. This is not related to its alleged connection with the strong CP problem, but it could be relevant to dark matter. The above axion couples to the flavour-singlet meson. As it has not been detected so far, it is reasonable to think of non-singlet particles which could be present in the form of dark matter [17]. Even if such particles do not exist, we may consider generalizing axion fields for the sake of chiral gauge theory and see what consequences arise. The abovementioned chiral symmetry is under a transformation of the fermion in spinor space. If the fermion is an $SU(N)$ multiplet, there exist nonabelian chiral symmetries. The kinetic piece

$$\bar{\psi}i\partial_t\psi = \bar{\psi}_L i\partial_t\psi_L + \bar{\psi}_R i\partial_t\psi_R$$

is invariant under the chiral transformations

$$\psi_L \rightarrow U_L \psi_L, \quad \psi_R \rightarrow U_R \psi_R,$$

where $U_L, U_R$ are spacetime independent $SU(N)$ matrices acting on the two chiral projections of $\psi$. The gauge interactions will also be invariant under these provided the matrix $A_\mu$ commutes with $U_L, U_R$: if the $SU(N)$ is a flavour group, the colour $SU(3)$ could be the gauge group.
The usual mass term \( m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) \) is not invariant under (11) unless \( U_L = U_R \), in which case of course the transformation is not really a chiral transformation. The analogue of the Peccei-Quinn term \( \bar{\psi}'\psi \exp(i\varphi\gamma_5) = m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) \), with \( W \) an \( SU(N) \) matrix field. It is analogous to the exponentiated pion field used in chiral models. But this is different from chiral models because these are not related to quarks and pions but are new fields. Considering that the original \( U(1) \) axion has not yet been detected, one may have doubts about the existence of such new objects, but they could be looked for in the search for dark matter. Whether they exist or not, the idea of these fields will be theoretically useful for nonabelian chiral gauge theories as we shall see.

The above term can be made invariant under (11) by transforming \( W \) as [17]
\[
W \to U_L W U_R^\dagger.
\]
(12)

As an \( SU(N) \) matrix it involves \( N^2 - 1 \) parameters which become fields. The kinetic term for this matrix field has to be of the form \( \frac{1}{2} F^2 Tr[\partial_\mu W\partial_\nu W^\dagger] \) familiar from chiral models. This is invariant under (12). Here \( F \) is a constant of mass dimension 1. Thus the full action is invariant under the generalized Peccei-Quinn symmetry
\[
\psi_L \to U_L \psi_L, \quad \psi_R \to U_R \psi_R, \quad W \to U_L W U_R^\dagger.
\]
(13)

We may ask whether this generalization of nonabelian chiral symmetry survives quantization. The naive way of regularization would be to introduce a Pauli-Villars \( SU(N) \) multiplet with the kinetic term \( \bar{\chi} i D \chi \) and a mass term \( M(\bar{\chi}_L\chi_R + \bar{\chi}_R\chi_L) \). This second term will not be invariant under chiral transformations of \( \chi \). But Pauli-Villars requires a coupling of \( \chi \) to \( W \):
\[
M(\bar{\chi}_L W \chi_R + \bar{\chi}_R \chi_L W^\dagger) \chi_L.
\]
Adding this interaction with the \( W \) field ensures invariance of the regularized action. For the full Pauli-Villars regularization [18] one has
\[
\mathcal{L}_{reg} = \bar{\psi}[i D - m(W P_R + W^\dagger P_L)] \psi + \sum_j |c_j| \sum_k \bar{\chi}_{jk}[i D - M_j(W P_R + W^\dagger P_L)] \chi_{jk},
\]
(14)

where \( P_L, P_R \) are the projection operators for left and right chirality respectively and \( c_j \) are integers whose signs are related to the statistics assigned to the spinor fields \( \chi_{jk} \). They are positive for Fermi statistics and negative for Bose statistics. They have to satisfy some conditions to ensure regularization of the divergences:
\[
1 + \sum_j c_j = 0, \quad m^2 + \sum_j c_j M_j^2 = 0.
\]
(15)

This regularized action is invariant under the transformations
\[
\psi_L \to U_L \psi_L, \quad \psi_R \to U_R \psi_R, \quad \chi_L \to U_L \chi_L, \quad \chi_R \to U_R \chi_R, \quad W \to U_L W U_R^\dagger.
\]
(16)

Hence like the abelian Pecccei-Quinn symmetry [8], this nonabelian classical symmetry too survives quantization and is not anomalous.
Instead of a Pauli-Villars regularization, one may use other regularizations. Some are discussed in [17], and a lattice regularization [19] is briefly discussed in the Appendix.

5 Gauging the nonabelian Peccei-Quinn symmetry

We shall now go on to our gauge theory. As before, gauge fields can be used to extend the global Peccei-Quinn-like symmetries to local ones. For example, for the left handed symmetry under \( U_L \), one needs an \( SU(N) \) gauge field matrix \( B_\mu \):

\[
\mathcal{L}_B = \bar{\psi} \left[ i \partial_\mu + B_\mu - m \left( WP_R + W^\dagger P_L \right) \right] \psi + \frac{1}{2} F^2 Tr \left[ \left( \partial_\mu - i B_\mu \right) W \left( \partial_\mu W^\dagger + i W^\dagger B_\mu \right) \right] - \frac{1}{4} Tr \left[ G_{\mu\nu} G^{\mu\nu} \right] \quad (17)
\]

Here the transformation of the new gauge field under \( (13) \) is given by

\[
\left( \partial_\mu - i B_\mu \right) \rightarrow U_L \left( \partial_\mu - i B_\mu \right) U_L^{-1}. \quad (18)
\]

The action is invariant under local gauge transformations \( U_L \) affecting \( \psi_L, W, B \). The right handed chiral symmetry too can be gauged if desired in a similar way. The quantity \( F \) has mass dimension one, as in the abelian case.

The Pauli-Villars regularization now reads

\[
\mathcal{L}_{B,reg} = \bar{\psi} \left[ i \partial_\mu + B_\mu - m \left( WP_R + W^\dagger P_L \right) \right] \psi + \frac{1}{2} F^2 Tr \left[ \left( \partial_\mu - i B_\mu \right) W \left( \partial_\mu W^\dagger + i W^\dagger B_\mu \right) \right] - \frac{1}{4} Tr \left[ G_{\mu\nu} G^{\mu\nu} \right] + \sum_j \sum_k \tilde{\chi}_{jk} \left[ i \partial_\mu + B_\mu - M_j \left( WP_R + W^\dagger P_L \right) \right] \chi_{jk}. \quad (19)
\]

This continues to be invariant under local \( U_L \) transformations. Thus there is no anomaly in the Peccei-Quinn type \( U_L \) symmetry even now although there is the usual nonabelian chiral anomaly when \( W \) is not transformed. The result of the construction is that there is formally a gauge theory having a local chiral nonabelian Peccei-Quinn symmetry including axions \( W \). The axions again play the rôle of gauge group valued fields in the Wess-Zumino action [3, 9].

Just as in the abelian case the axion could be transformed away to zero by a local Peccei-Quinn transformation, here too \( W \) can be transformed away to the identity by a local transformation \( U_L = W^{-1} \), whereby

\[
W \rightarrow U_L W = 1. \quad (20)
\]

The Lagrangian density can then be written as

\[
\mathcal{L}_{red} = \bar{\psi} \left[ i \partial_\mu + B_\mu - m \right] \psi + \frac{1}{2} F^2 Tr \left[ B_\mu B_\mu \right] - \frac{1}{4} Tr \left[ G_{\mu\nu} G^{\mu\nu} \right]. \quad (21)
\]
Pauli-Villars fields \( D \) decouples, as indicated by the local symmetry. The measure is gauge invariant in the absence of gauge fixing. This confirms that \( [7] \). Hence in the final step the original measure is restored. The gauge field \( B \) which means that the integration measures, shown with primes, include primes on the other fields are dropped in the next step, as a change of variable, clearly, gauge invariance under \( U_L \) is lost in this reduced form and the gauge field \( B \) has a mass whose value is undetermined because \( F \) is not known. Global chiral symmetry is completely lost in the absence of \( W \). The nonabelian axions have in effect been swallowed by the gauge field. These axions are thus physical only if the Peccei-Quinn symmetry is not gauged.

In the functional integral formulation, for each \( W \) configuration, the above \( U_L \) transformation may be used to transform \( W \) away to unity. If we suppress the indices in \( \chi \) and \( M \) for convenience,

\[
Z \equiv \int \int \int \int DWD\psi D\bar{\psi} D\chi D\bar{\chi} DB \\
\exp i \int d^4x \left( \bar{\psi}[i\partial + \mathcal{B}\mathcal{P}_L - m(W\mathcal{P}_R + W^\dagger\mathcal{P}_L)]\psi + \bar{\chi}[i\partial + \mathcal{B}\mathcal{P}_L - M(W\mathcal{P}_R + W^\dagger\mathcal{P}_L)]\chi \\
+ \frac{1}{2}F^2 Tr[(\partial_{\mu} - iB_{\mu})W(\partial^{\mu}W^\dagger + iW^\dagger B^{\mu})] - \frac{1}{4}Tr[G_{\mu\nu}G^{\mu\nu}] \right) \\
= \int \int \int \int DWD\psi D\bar{\psi} D\chi D\bar{\chi} DB \\
\exp i \int d^4x \left( \bar{\psi}[i\partial + \mathcal{B}\mathcal{P}_L - m(W'\mathcal{P}_R + W'^\dagger\mathcal{P}_L)]\psi' + \bar{\chi}'[i\partial + \mathcal{B}'\mathcal{P}_L - M(W'\mathcal{P}_R + W'^\dagger\mathcal{P}_L)]\chi' \\
+ \frac{1}{2}F^2 Tr[(\partial_{\mu} - iB'_{\mu})W'(\partial^{\mu}W'^\dagger + iW'^\dagger B'^{\mu})] - \frac{1}{4}Tr[G_{\mu\nu}G^{\mu\nu}] \right) \\
= \int \int \int \int DWD\psi D\bar{\psi} D\chi D\bar{\chi} DB \\
\exp i \int d^4x \left( \bar{\psi}[i\partial + \mathcal{B}\mathcal{P}_L - m]\psi + \bar{\chi}[i\partial + \mathcal{B}\mathcal{P}_L - M]\chi \\
+ \frac{1}{2}F^2 Tr[B_{\mu}B^{\mu}] - \frac{1}{4}Tr[G_{\mu\nu}G^{\mu\nu}] \right) \\
= \int \int \int \int DWD\psi D\bar{\psi} D\chi D\bar{\chi} DB \\
\exp i \int d^4x \left( \bar{\psi}[i\partial + \mathcal{B}\mathcal{P}_L - m]\psi + \bar{\chi}[i\partial + \mathcal{B}\mathcal{P}_L - M]\chi \\
+ \frac{1}{2}F^2 Tr[B_{\mu}B^{\mu}] - \frac{1}{4}Tr[G_{\mu\nu}G^{\mu\nu}] \right) \\
\tag{22}
\]

In the first step, the fields are transformed by \( U_L \) and the transformed versions are indicated by primes. With the choice of \( U_L \) mentioned above, \( W' = 1 \). The primes on the other fields are dropped in the next step, as a change of variable, which means that the integration measures, shown with primes, include \( U_L \): \( \mathcal{D}'\psi = \mathcal{D}(U_L^{-1}\psi_L)\mathcal{D}\psi_R \) etc. But the measure for fermions together with the Pauli-Villars fields \( \mathcal{D}\psi\mathcal{D}\psi\mathcal{D}\chi\mathcal{D}\bar{\chi} \) is invariant under the chiral \( U_1 \) transformation, as in the regularized theory any anomaly arises from the regulator mass term \( [7] \). Hence in the final step the original measure is restored. The gauge field measure is gauge invariant in the absence of gauge fixing. This confirms that \( W \) decouples, as indicated by the local symmetry.
6 Discussion

To sum up, we have used axions and their nonabelian analogues which are associated with a nonabelian generalization of the Peccei-Quinn symmetry. The Peccei-Quinn symmetry is not just a classical symmetry but is non-anomalous, as shown [8, 17] by constructing a regularization involving the axions consistent with it. The abelian or nonabelian chiral Peccei-Quinn symmetry can be made local by introducing a new gauge field. This symmetry is spontaneously broken and the corresponding gauge field becomes massive by swallowing the axion. Thus there emerges a new gauge invariant axion formulation of abelian or nonabelian chiral gauge theory even if the axion does not exist physically as a particle. The old results of the chiral Schwinger model hold: the gauge symmetry is broken and the gauge boson becomes massive. Similar results follow in four dimensions and for nonabelian cases. The advantages over the Wess-Zumino reformulation are that the action is explicit, manifestly local, possesses gauge invariance even at the classical level and necessarily yields a mass term for the gauge bosons.

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Appendix

The lattice regularization, which may be regarded as a kind of point-splitting regularization, also preserves Peccei-Quinn-like symmetries of the axion actions discussed above using Pauli-Villars regularization.

The discrete Lagrangian density for a Wilson fermion takes the form

\[ L_{\text{lat}} = \bar{\psi} \left[ \frac{1}{2} (D^L + D^{L*}) - a D^L D^{L*} \right] \psi. \]  

(23)

Here \( a \) is the lattice spacing, \( D^L_\mu \) is the covariant forward difference (in the \( \mu \)-direction) divided by \( a \) and \( D^{L*}_\mu \) the covariant backward difference, again divided by \( a \). While the kinetic term is chirally invariant as in the continuum, there is the double derivative term which breaks this invariance on the lattice [19]. In the axion case, the axion coupling has to be introduced at two places in the action because there are two terms which are chirally non-invariant:

\[ L_1 = \bar{\psi} \left[ \frac{1}{2} (D^L + D^{L*}) - a e^{i\varphi/2} (D^L D^L) e^{i\varphi/2} - m e^{i\varphi} \right] \psi. \]  

(24)

In the SU(\( N \)) case, one needs

\[ L_2 = \bar{\psi} \left[ \frac{1}{2} (D^L + D^{L*}) - a (D^L D^L) W P_R - a W^\dagger P_L (D^L D^L) - m (W P_R + W^\dagger P_L) \right] \psi. \]  

(25)

These lattice actions clearly preserve the Peccei-Quinn symmetry of the continuum action when \( \varphi, W \) are transformed correctly.