Controllable exact self-similar evolution of the Bose–Einstein condensate

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Abstract. We show analytically that the profile of the Bose–Einstein condensate can evolve exact self-similarly when the time-dependent interatomic interaction and harmonic potential satisfy a certain condition for one-, two- and three-dimensional geometries, respectively. Based on the exact scaling laws for the amplitude and width of the condensate profile, we propose experimentally feasible ways to improve the accuracy in measuring the stationary-state profile of the condensate when its spatial size is so small that the direct measurement is not available, and to squeeze the condensate to high local particle density to check the validity of the Gross–Pitaevskii equation via the controllable exact self-similar evolution of the condensate profile.
Contents

1. Introduction 2
2. The model equation 3
3. ESS evolution of BEC 4
   3.1. 1D geometry 4
   3.2. 2D geometry 6
   3.3. 3D geometry 7
4. Summary and discussion 9
Acknowledgments 9
References 10

1. Introduction

Since the trapped Bose–Einstein condensate (BEC) can create macroscopic quantum objects containing many atoms in the same quantum state, its static and dynamical evolution in the mean-field approximation can be described by the Gross–Pitaevskii equation (GPE). Moreover, the excellent agreement in the ground and excited states obtained by solving the GPE with most experiments (see [1] and references therein) confirms its validity. In general, the GPE is a three-dimensional (3D) nonlinear Schrödinger equation (NLSE), and under certain conditions, it can be reduced to 1D and 2D NLSE. It has been shown that the analytical soliton solution for a NLSE is only derived in 1D geometry when the interatomic interaction and harmonic potential satisfy certain conditions [2]–[4]. However, for 2D and 3D geometries, approximate methods such as the Thomas–Fermi approximation in the large particle number limit [5]–[7] and the variational technique using Gaussian ansatz [8] can be used to describe the static and dynamic properties of the BEC.

However, it should be noted that in experiments the produced BECs are generally in their stationary states instead of in the states described by soliton, inverted parabolic or Gaussian-type solutions, and moreover, various manipulations such as changing the interatomic interaction and external potential are performed on BECs to achieve different purposes. Thus, it is of interest to investigate how the initial stationary-state profiles evolve under the modulation of interatomic interaction and external potential. Specifically, the exact self-similar (ESS) evolution of the condensate profile is of considerable interest due to its unique character that the functional form of the condensate profile remains unchanged but the amplitude and width change according to some exact scaling laws. Therefore, as long as the initial stationary-state profile and the exact scaling laws are known, the condensate profile at the following time can be calculated or vice versa. Such a kind of ESS evolution of condensate profiles may have several experimental applications, as we shall demonstrate below.

Actually, the ESS evolution of the condensate profile has been achieved in 2D geometry by varying the frequency of the harmonic potential [9]. By using several coordinate transformations, the original GPE with time-dependent harmonic potential is converted to an analogous NLSE with time-independent harmonic potential, and the ESS solution is obtained when its initial condition takes the stationary solution of the converted NLSE; other self-similar solutions were also considered (see [9] and references therein); however, they cannot evolve exact self-similarly (ESSly). Another ESS solution is the analytical solitary wave solution of a
1D NLSE when the interatomic interaction and harmonic potential satisfy a certain condition, as mentioned above.

In this paper, we present a systemic method to investigate the ESS evolution of the condensate profile in 1D and 3D geometries. It is interesting that when the time-dependent interatomic interaction and harmonic potential satisfy certain conditions, the condensate profiles can evolve ESSly in 1D and 3D geometries, respectively. In particular, the analytical solitary wave solution [2]–[4] is recovered just as a special case of the ESS solution in 1D geometry. Furthermore, based on the exact scaling laws for the amplitude and width of the condensate profile, we discuss the potential experimental applications of ESS evolution of the condensate profile in 1D, 2D and 3D geometries, respectively.

2. The model equation

We consider a Bose–Einstein gas of $^{87}\text{Rb}$ atoms within the $|f, m_f\rangle = |1, 1\rangle$ hyperfine state and confined in a harmonic potential. When the particle density and temperature of the condensate are sufficiently low, the dynamics of the BEC in the mean-field approximation can be described by the GPE [10]

$$\dot{\Psi} - \frac{i\hbar}{2m} \nabla^2 \Psi + V \Psi + U(t) |\Psi|^2 \Psi = 0$$

Here, $\Psi(t, \vec{r})$ is the macroscopic wavefunction and satisfies the normalization condition $\int |\Psi|^2 d^3 \vec{r} = N$, where $N$ is the number of atoms. The strength of the interatomic interaction is governed by $U(t) = 4\pi \hbar^2 a_s(t)/m$, where $m$ is the atomic mass and $a_s(t)$ is the s-wave scattering length controlled by the magnetic Feshbach resonance technique as follows: $a_s(t) = a_{s, \infty}[1 - \Delta/(B(t) - B_0)]$ with $a_{s, \infty}$, $\Delta$, $B(t)$ and $B_0$ being the asymptotic value of the scattering length far from resonance, the width of the resonance, the time-dependent external magnetic field and the resonant value of the magnetic field, respectively. For $^{87}\text{Rb}$ atoms within the $|f, m_f\rangle = |1, 1\rangle$ hyperfine state, the parameters can be given by $\Delta = 0.2 \text{G}$, $B_0 = 1007.4 \text{G}$ and $a_{s, \infty} = 100a_0$ with $a_0$ being the Bohr radius [11]. The external harmonic potential is chosen as $V = \frac{1}{2}m[\omega_x^2 x^2 + \omega_z^2 (y^2 + z^2)]$ with $\omega_x$ and $\omega_z$ being the frequencies in the longitudinal and transversal directions, respectively, and can be controlled continuously by manipulating the currents in Ioffe–Pritchard coils [12].

When the transversal frequency is much larger than the longitudinal frequency, namely, $\omega_z \gg \omega_x$, the BEC is cigar-shaped, and its dynamics can be self-consistently reduced to a 1D NLSE in the dimensionless form [13]–[15]

$$\partial_t u + u_{xx} - 2T g(t) |u|^2 u - k_1(t)x^2 u = 0$$

with the normalized wave function $u$ satisfying the following equation: $\Psi = \langle \epsilon_1/\sqrt{2\pi \langle a_s(0) |a_\perp \rangle} \rangle \exp(-i\omega_x t - (y^2 + z^2)/2a_\perp^2) u$, where $x$ and $t$ are measured in units of $a_\perp / \epsilon_1$ and $2/\epsilon_1^2 \omega_\perp$, with $a_\perp = \sqrt{\hbar/m \omega_\perp}$ and $\epsilon_1 = N \langle a_s(0) / a_\perp \rangle \ll 1$, respectively. Here, we assume that the time-dependent nonlinearity parameter $-2T g(t)$ with $\sigma = \text{sgn}(a_s)$ and $g(t) = a_s(t)/a_s(0)$ does not change its sign during the manipulation of magnetic field, whereas the time-dependent harmonic potential parameter $k_1(t) = \omega_x(t)^2 / \omega_z^2 \epsilon_1^4 \ll 1$ depends only on the time-dependent frequency $\omega_x$.

In contrast, when the longitudinal frequency is much larger than the transversal frequency, namely, $\omega_x \gg \omega_z$, the BEC is pancake-shaped, and its dynamical equation becomes the...
following 2D NLSE:

\[ iu_t + u_{yy} + u_{zz} - \sqrt{2}g(t)|u|^2u - k_2(t)(y^2 + z^2)u = 0. \] (3)

Here, the wavefunction \( u \) is normalized via the equation \( \Psi = (\epsilon_2/\sqrt{2\pi}|a_s(0)|a_s) \times \exp(-i\omega_s t/2 - x^2/2a_s^2)u \), where \( x, y \) and \( t \) are measured in units of \( a_s/\epsilon_2 \) and \( 2/\epsilon_2^2 \omega_s \) with \( a_s = \sqrt{\hbar/m\omega_s} \) and \( \epsilon_2 = N|a_s(0)|/a_s \ll 1 \), respectively. And now the time-dependent harmonic potential parameter \( k_2(t) = \omega_\perp(t)^2/\omega^2_x \epsilon_s^2 \ll 1 \) depends only on the time-dependent frequency \( \omega_\perp \).

### 3. ESS evolution of BEC

In this section, we aim to find the relation between the nonlinearity parameter and the harmonic potential parameter for the ESS evolution of condensate profiles and consider their potential experimental applications.

#### 3.1. 1D geometry

In order to make the condensate wavefunction evolve ESSly in 1D geometry, the expansion velocity of the condensate profile must be proportional to \( \sqrt{\epsilon_2} \). Fortunately, the lens-type transformation

\[ u(t, x) = \sqrt{g(t)}\Phi(X, T) \exp[i f_1(t)x^2] \] (4)

with \( X = g(t)x, T = \int_0^t g(t')^2 \, dt' \) and \( f_1(t) = -g_1/4g \) just satisfies these conditions. When \( g(t) \) and the harmonic potential parameter \( k_1(t) \) satisfy the following equation with a constant \( K_1 \):

\[ k_1 = K_1 g^4 + \frac{g\mu g^2 - 2g_1^2}{4g^2}, \] (5)

we can obtain the following equation:

\[ i\Phi_T + \Phi_{XX} - 2\sigma |\Phi|^2\Phi - K_1 X^2 \Phi = 0. \] (6)

It should be noted that equation (6) with time-independent nonlinearity and harmonic potential is analogous to equation (2). This similarity enables us to investigate the evolution of the normalized wavefunction \( u(t, x) \) in terms of equation (6). In particular, when \( K_1 = 0 \), equation (6) turns into the standard homogeneous NLSE which processes analytical soliton solutions, and the ESS dynamics of the BEC bright solitary wave of equation (2) has been discussed [4].

In the present paper, we are more interested in the case that \( K_1 \) is a positive constant. In this case, the stationary-state solutions of equation (6) can be written as \( \Phi(T, X) = G(X)e^{-i\mu T} \), where \( \mu \) is the chemical potential and the corresponding normalized wavefunction \( u(t, x) \) is given by \( u(t, x) = \sqrt{g(t)} \exp[i f_1(t)x^2 - i\mu T]G[g(t)x] \). This normalized wavefunction can describe the controllable ESS evolution of the condensate profile when \( g(t) \) and \( k_1(t) \) are changed according to the Feshbach resonance and the ESS condition (5), respectively. The exact scaling laws for the amplitude and width of the condensate profile are that the amplitude is proportional to \( \sqrt{g(t)} \), while the spatial width \( W = \sqrt{\int_{-\infty}^{\infty} x^2 |u|^2 \, dx} \) is proportional to \( g(t)^{-1} \).
Besides, we find that the coefficient of quadratic phase $f_1(t)$ is proportional to $W_1/W$, which is consistent with the result obtained by the variational technique [8]. The ESS evolution of condensate profiles are confirmed by the direct numerical simulations of equation (2) using the split-step Fourier method when the initial condition is chosen as the ground-state solutions of equation (6), which are obtained numerically by an imaginary-time algorithm [17]. From figures 1 and 2, we can see that the initial ground-state profiles expand ESSly when $g(t)$ is decreased. Further numerical simulations show that, when the initial conditions of equation (2) are chosen as the first excited stationary-state solutions of equation (6), the condensate profiles can also evolve ESSly. 

Note that although the explicit form of the function $G(X)$ cannot be exactly solved when $K_1 \neq 0$, it can be directly related to the stationary state of the experimentally produced BEC. With such a relation and exact scaling laws for the amplitude and width of the condensate profile, we can design an experiment to improve the accuracy in measuring the BEC ground-state profiles when the direct measurements are not available. We recall that when the size of the condensate profile is smaller than the spatial resolution of measurement, the usual method is to switch off the harmonic potential suddenly and let the condensate expand freely; then, by measuring the expanded condensate profile, one gets the initial ground-state profile by the variational technique using a trial function such as the Gaussian ansatz [16]. However, this method may not be of high accuracy since the interatomic interaction may change the condensate profile significantly with respect to the Gaussian. Now, this drawback can be eliminated by the ESS expansion of the condensate profile: after the BEC is produced in its...
ground state, we decrease $g(t)$ and change $k_1(t)$ accordingly to let the condensate expand ESSly; then the initial ground-state profile can be calculated by measuring the expanded profile according to the exact scaling laws for the amplitude and width of the condensate profile.

It should be pointed out that the quadratic phase $f_1 x^2$ of the initial ground state is zero, because if it is not equal to zero, the derivative of the condensate width with respect to time is not zero, thus the condensate width will change, which violates the fact that the BEC is in its stationary state. Therefore, the variation of nonlinearity parameter $g(t)$ should satisfy $g(t)|_{t=0} = 0$; this means that $B_1|_{t=0} = 0$. Moreover, the stationary-state solution of equation (6) is consistent with the experimentally produced BEC stationary state; therefore, $k_1(0)$ should be equal to $K_1$, that is, $g_{tt}|_{t=0}$ must be zero according to the ESS condition (5). There are many choices for the functional form of the external magnetic field $B(t)$. In this paper, we take the simplest polynomial form of the external magnetic field $B(t) = B(0) + \sum_n c_n t^n$ with $n > 2$ to fulfill the above requirements. However, it is remarkable that, with such a kind of external magnetic field, time $t$ is sometimes confined in a small region to assure that the variation of the nonlinearity parameter does not cross the zero point.

3.2. 2D geometry

The lens-type transformation can be improved to study the ESS evolution of the condensate in 2D geometry. Although the ESS condition was obtained [9], we present the lens-type
transformation for 2D NLSE (3) for consistency and comparability:

\[ u(t, y, z) = \ell(t)e^{i\int_0^t \ell(\tau)y^2 + z^2 d\tau} \Phi(T, Y, Z), \]

where \( \ell(0) = 1 \), \( (Y, Z) = (y, z)\ell \), \( T = \int_0^t \ell(\tau)^2 d\tau \) and \( f_2(t) = -\ell(t)/4\ell \). When the harmonic potential parameter satisfies the following equation with a positive constant \( K_2 \):

\[ k_2 = K_2 \ell^4 + \frac{\ell t_\ell - 2\ell^2}{4\ell^2}, \]

we obtain the following equation:

\[ i\Phi_T + \Phi_{YY} + \Phi_{ZZ} - \sqrt{2}\sigma g|\phi|^2\Phi - K_2(Y^2 + Z^2)\Phi = 0, \]

which is analogous to equation (3) but with time-independent harmonic potential, and processes stationary-state solutions when \( g(t) \) is time-independent. Similar to the above 1D cases, when the initial conditions of equation (3) are chosen as the stationary-state solutions of equation (9), the initial profiles will evolve ESSly if \( k_2(t) \) satisfies the ESS condition (8): the amplitudes and widths of condensate profiles are proportional to \( \ell \) and \( \ell^{-1} \), respectively\(^4\). The stationary-state solutions of equation (9) are related directly to the experimentally produced BEC when the first and second derivatives of \( \ell \) with respect to time vanish. Therefore, the ESS expansion of the 2D condensate profile can be used to improve the accuracy in measuring the stationary-state profile when its size is so small that the direct measurement is not available. It is interesting that the ESS conditions for 1D and 2D NLSEs are equivalent if we replace \( \ell(t) \) by \( g(t) \). However, theoretically, \( \ell(t) \) can be of arbitrary functional form, whereas \( g(t) \) is restricted to the Feshbach resonance technique in 1D geometry.

Now that the condensate profile can expand ESSly, it can also be compressed ESSly. Since the GPE works well at low particle density, it is of importance to compress the matter wave to a higher density to check at what particle density the GPE is valid. In [2], it was proposed that the matter wave can be compressed to very high local matter density in the form of a bright soliton by increasing the absolute value of the atomic scattering length in 1D geometry when interatomic interaction is attractive. Here, we are able to compress the matter wave to higher densities ESSly in 1D and 2D geometries both for attractive and repulsive interatomic interactions, by increasing \( g \) in 1D geometry (or \( \ell \) in 2D geometry) and changing \( k_{1,2} \) accordingly. However, the self-consistent condition in the reduction of the 3D GPE to 1D and 2D NLSEs requires that \( \epsilon_{1,2} \ll 1 \) and \( k_{1,2}(t) \lesssim 1 \) [14], which prohibits the compression of the matter wave to a very high density within the ESS process: from the ESS conditions (5) and (8), we know that \( k_1 \sim K_1 g^4 \) (or \( k_2 \sim K_2 \ell^4 \)), which means that \( k_1 \) (or \( k_2 \)) will quickly exceed 1 as \( g \) (or \( \ell \)) increases. Thus one cannot assure that the invalidation of GPE is totally caused by a high particle density. Therefore, in order to determine at what particle density the GPE is invalid, we should adopt 3D geometry.

3.3. 3D geometry

In 3D geometry, when the harmonic trap is isotropic with frequency \( \omega_\perp \), we get the following 3D NLSE [13]:

\[ iu_t + \nabla^2 u - g_3(t)|u|^2u - k_3(t)r^2u = 0, \]

\(^4\) Note that the ESS condition (8) for the 2D NLSE is equivalent to equation (9) in [9] if \( \ell(t) \) is replaced by \( b(t)^{-1} \).
where the wavefunction $u$ is normalized through the equation $u = \sqrt{a_0^3/N}\Psi$, while temporal and spatial variables are measured in units of $2/\omega_\perp(0)$ and $a_0 = \sqrt{\hbar/m\omega_\perp(0)}$, respectively. It is assumed that the nonlinearity parameter $g_3(t) = 8\pi a_0(t)N/a_0$ does not change its sign during manipulation of the external magnetic field, and the harmonic potential parameter is given by $k_3(t) = \omega_\perp(t)^2/\omega_\perp(0)^2$.

Similarly, when the nonlinearity parameter $g_3(t)$ and the harmonic potential parameter $k_3(t)$ satisfy the following equation with a positive constant $K_3$:

$$k_3 = K_3\frac{g_3(0)^4}{g_3^3} - \frac{g_3\ell_t}{4g_3}, \quad (11)$$

we can obtain an equation which is analogous to equation (10) but with time-independent nonlinearity and harmonic potential

$$i\Phi_T + \nabla^2\Phi - g_3(0)|\Phi|^2\Phi - K_3R^2\Phi = 0 \quad (12)$$

through the lens-type transformation

$$u(t, r) = \left[\frac{g_3(0)}{g_3(t)}\right]^{3/2} e^{i\int_0^t g_3(0)^2/g_3(\tau)^2 d\tau} \Phi(T, R), \quad (13)$$

where $R = r g_3(0)/g_3$, $T = \int_0^t g_3(0)^2/g_3(\tau)^2 d\tau$ and $f_3(t) = g_3\ell_t/4g_3$.

Similarly, when the first- and second-order derivatives of $g_3(t)$ with respect to time vanish, equations (10) and (12) are the same at $t = 0$. Therefore, if the initial condition of equation (10) is chosen as the stationary-state solution of equation (12), the condensate will evolve ESSly when $g_3(t)$ and $k_3(t)$ satisfy the ESS condition (11). Thus, when we increase the absolute value of $g_3(t)$ and change $k_3(t)$ accordingly, the condensate expands ESSly, which can be used to improve the accuracy in measuring the stationary-state profile when the direct measuring is not available, similarly to the discussion in the 1D case when decreasing $g_3(t)$ and in the 2D case when decreasing $\ell(t)$. On the contrary, if we decrease $g_3(t)$ and change $k_3(t)$ accordingly, we can compress the condensate to very high local matter density to check the validity of GPE. The experiment can be designed as follows: (i) produce a BEC with the fixed parameters $g_3(0)$ and $k_3(0)$, and measure the density profile using the phase-contract imaging which is nondestructive. (ii) Increase the condensate density according to the ESS condition (10). Considering the fact that measuring the density profile may become more and more difficult or even unfeasible when the density is increased, we choose the external magnetic field as $B(t) = B(0) + c t^3 + d t^4$, where $cd < 0$. With this kind of magnetic field, we can compress the condensate into an assumed maximum peak density firstly and then expand the condensate ESSly (see figure 3). (iii) Measure the expanded density profile, and select another set of parameters $c$ and $d$ to increase (decrease) the maximum peak condensate density if the experimentally obtained data (condensate amplitude and width) satisfy (do not satisfy) the exact scaling laws\(^5\). On repeating steps (i)–(iii) several times, we can obtain the maximum particle density for the validity of GPE. The above analysis can be extended to anisotropic harmonic potential by demanding that $k_{3i} = K_{3i}g_3(0)^4/g_3^3 - g_{3i}/4g_3$, where $k_{3i}$ is related to the confining frequency of the harmonic potential in the $i$ direction with $i = x$, $y$ and $z$ and $K_{3i}$ being positive constants.

\(^5\) The dissatisfication with the exact scaling properties is caused by mechanisms not included in the Gross–Pitaevskii theory such as inelastic two- and three-body collisions that cause a loss of atoms.
Figure 3. Peak matter wave density (in units of initial peak density) for (a) attractive and (b) repulsive interatomic interactions. The parameters are \( K = 1 \), (a) \( B(0) - B_0 = 0.15 \), \( c = 0.00001 \), and from top to bottom, \( d = -c/32, -c/31, -c/30 \), (b) \( B(0) - B_0 = -0.05 \), \( c = -0.0001 \), and from top to bottom, \( d = -c/35, -c/30, -c/25 \).

4. Summary and discussion

In summary, with the help of lens-type transformation, we have found the exact scaling laws for the amplitude and width of the condensate profile when the time-dependent interatomic interaction and harmonic trapping potential satisfy the ESS condition for 1D, 2D and 3D GPEs, respectively. Based on these controllable exact scaling laws, we have designed two experiments to expand the condensate to improve the accuracy in measuring the initial stationary-state profile and to squeeze the matter wave to very high local matter density to check the validity of the GPE by ESS expansion and compression of the condensate, respectively.

It is interesting that when the matter wave is compressed to high density, other mechanisms than the two-body elastic scattering, such as two-body inelastic collision and three-body recombination, begin to dominate and cause the loss of atoms. When taking these effects into account, the governing equation of BEC will be the original GP equation with an additional phenomenological nonlinear loss term. So far, however, we cannot find the self-similar solution for such an equation. But for the GP equation with a linear loss term, the self-similar solution can be easily found by lens-type transformation, and such a model equation is very popular in nonlinear optical systems \([18, 19]\), but not in BECs. We believe that the lens-type transformation presented in this paper can be extended to nonlinear optical systems. And since the analytical bright solitary wave solution of the 1D NLSE when the interatomic interaction and harmonic potential satisfy a certain condition is just a special solution of the ESS solution that we have just found, we predict that other types of solutions than those solitary wave solutions should be found to evolve ESSly in nonlinear optical systems.

Acknowledgments

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