REDUCIBILITY OF SCHRÖDINGER EQUATION ON A ZOLL MANIFOLD WITH UNBOUNDED POTENTIAL

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ABSTRACT. In this article we prove a reducibility result for the linear Schrödinger equation on a Zoll manifold with quasi-periodic in time pseudo-differential perturbation of order less or equal than $1/2$. As far as we know, this is the first reducibility results for an unbounded perturbation of a linear system which is not integrable.

CONTENTS

1. Introduction .................................................................................................................. 1
2. Functional setting ......................................................................................................... 4
3. Regularization procedure ............................................................................................. 14
4. KAM reducibility ......................................................................................................... 20
5. Proof of Theorem 1.1 .................................................................................................... 30
Appendix A. Technical lemmata ..................................................................................... 31
References ......................................................................................................................... 35

1. INTRODUCTION

In this article we are interested in the problem of reducibility for the linear Schrödinger equation on a Zoll manifold with quasi-periodic in time pseudo-differential perturbation of order less or equal than $1/2$. We first recall that a Zoll manifold of dimension $n \in \mathbb{N}$ is a compact Riemannian manifold $(\mathbb{M}^n, g)$ such that all the geodesic curves have all the same period $T$. In this paper we assume $T := 2\pi$. For example the $n$-dimensional sphere $\mathbb{S}^n$ is a Zoll manifold. We denote by $\Delta_g$ the positive Laplace-Beltrami operator on $(\mathbb{M}^n, g)$ and we define $H^s(\mathbb{M}^n) := \text{dom}(\sqrt{1 + \Delta_g})^s$ with $s \in \mathbb{R}$ the usual scale of Sobolev spaces. We denote by $S^m_{cl}(\mathbb{M}^n)$ the space of classical real valued symbols of order $m \in \mathbb{R}$ on the cotangent bundle $T^* (\mathbb{M}^n)$ and we define $\mathcal{A}_m$ the associated class of pseudo-differential operators (see for instance Hörmander [Hor85] for a definition of pseudo-differential operators on a manifold see also [BGMRT19] in the case of a Zoll manifold).

We consider the following linear Schrödinger

$$i \partial_t u = \Delta_g u + \varepsilon W(\omega t)u, \quad u = u(t, x), \quad t \in \mathbb{R}, \quad x \in \mathbb{M}^n, \quad (LS)$$

where $\varepsilon > 0$ is a small parameter and $W(\omega t)$ is a time dependent unbounded operator from $H^s(\mathbb{M}^n) \to H^{s-\delta}(\mathbb{M}^n)$ for some $\delta \leq 1/2$. More precisely we assume that $W \in C^\infty(\mathbb{T}^d, \mathcal{A}_d)$ with $\delta \leq 1/2$, $d \geq 1$. So the potential $t \mapsto W(\omega t)$ depends on time quasi-periodically with frequency vector $\omega \in \mathbb{R}^d$ and for any...
\( \varphi \in T^d := (\mathbb{R}/2\pi \mathbb{Z})^d \), the linear operator \( W(\varphi) \) is a pseudo-differential operator of order \( \delta \), i.e., belongs to \( \mathcal{A}_{\delta} \).

The purpose of this article is to construct a change of variables that transforms the non-autonomous equation \( (LS) \) into an autonomous equation. Our main result is the following.

**Theorem 1.1.** Let \( 0 < \alpha < 1 \) and \( \delta \in \mathbb{R}, \delta \leq 1/2 \). Assume that the map \( \varphi \mapsto W(\varphi, \cdot) \in \mathcal{A}_{\delta} \) is \( C^\infty \) in \( \varphi \in T^d \). Then for any \( s \in \mathbb{R}, s > n/2 \) there exists \( \varepsilon_0 > 0 \) and \( C > 0 \) such that, for any \( 0 < \varepsilon \leq \varepsilon_0 \) there is a set \( \mathcal{G}_{\varepsilon} \subset [1/2, 3/2]^d \subset \mathbb{R}^d \) with

\[
\text{meas}(\mathbb{R}^d \setminus \mathcal{G}_{\varepsilon}) \leq C \varepsilon^\alpha
\]

such that the following holds. For any \( \omega \in \mathcal{O}_{\varepsilon} \) there exists a family of linear isomorphisms \( \Psi(\varphi) \in \mathcal{L}(H^s(\mathbb{M}^n; \mathbb{C})) \) and a Hermitian operator \( Z \in \mathcal{A}_{\delta} \) commuting with the Laplacian \( \Delta \) and satisfying

\[
\|Z\|_{\mathcal{L}(H^{s-n}(\mathbb{M}^n))} \leq C \varepsilon.
\]

Furthermore

- \( \Psi(\varphi) \) is unitary on \( L^2(\mathbb{M}^n) \);  
- for any \( \frac{n}{2} < s' \leq s \) and any \( \omega \in \mathcal{O}_{\varepsilon} \)

\[
\|\Psi(\varphi) - \text{Id}\|_{\mathcal{L}(H^{s-n}(\mathbb{M}^n), H^{s-\delta}(\mathbb{M}^n))} + \|\Psi(\varphi)^{-1} - \text{Id}\|_{\mathcal{L}(H^{s-n}(\mathbb{M}^n), H^{s-\delta}(\mathbb{M}^n))} \leq C \varepsilon^{1-\alpha},
\]

(1.3)

- for any \( \frac{n}{2} < s' \leq s \) and any \( \omega \in \mathcal{O}_{\varepsilon} \)

\[
\|\Psi(\varphi)\|_{\mathcal{L}(H^{s-n}(\mathbb{M}^n))} + \|\Psi(\varphi)^{-1}\|_{\mathcal{L}(H^{s-n}(\mathbb{M}^n))} \leq 1 + C \varepsilon^{1-\alpha},
\]

(1.4)

As a consequence of our reducibility result, we get a control of the flow generated by the \( (LS) \) equation in the scale of Sobolev spaces:

**Corollary 1.2.** Let \( W \in C^\infty(T^d, \mathcal{A}_{\delta}) \) with \( \delta \leq 1/2 \). Then for any \( s \in \mathbb{R}, s > n/2 \) there exists \( \varepsilon_0 > 0 \) and \( C > 0 \) such that, for any \( 0 < \varepsilon \leq \varepsilon_0 \) there is a set \( \mathcal{G}_{\varepsilon} \subset [1/2, 3/2]^d \subset \mathbb{R}^d \) satisfying (1.1) such that for any \( \omega \in \mathcal{O}_{\varepsilon} \) the flow generated by the \( (LS) \) equation is bounded in \( H^s(\mathbb{M}^n; \mathbb{C}) \).

More precisely if \( u_0 \in H^s(\mathbb{S}^2; \mathbb{C}) \) then there exists a unique solution \( u \in C^1(\mathbb{R}; H^s) \) of \( (LS) \) such that \( u(0) = u_0 \). Moreover, \( u \) is almost-periodic in time and satisfies

\[
(1 - \varepsilon C)\|u_0\|_{H^s} \leq \|u(t)\|_{H^s} \leq (1 + \varepsilon C)\|u_0\|_{H^s}, \quad \forall t \in \mathbb{R},
\]

(1.5)

for some \( C = C(s) > 0 \).

Following the pioneering work \cite{BBM14} we prove Theorem 1.1 in two steps:

- **The regularization step** where we use the pseudo-differential calculus (and in particular the technics developed in \cite{BGMR19}) to transform equation \( (LS) \) in a system with a smoothing perturbation, still depending on time;
- **The KAM step** where we use a KAM procedure (going back to \cite{Kuk93} but using recent development in \cite{BBHM18}) on infinite dimensional matrices to eliminate the time in the new system.

\(^1\)Actually \( Z \) and \( \Delta_\delta \) can be diagonalized in the same basis of \( L^2(\mathbb{M}^n) \).
The same strategy was recently successfully applied in [BLM18] to prove the reducibility of non-resonant transport equation on the torus $\mathbb{T}^d$. Our main contribution consists in merging these two recent technics in the context of linear Schrödinger equation on Zoll manifold which, in contrast to the transport equation on the torus, is not an integrable system.

The study of the reducibility problem for Schrödinger equations with quasi-periodic in time perturbation has been intensively studied in recent years. The first results adapting the KAM technics were due to Kuksin [Kuk93] followed by many results in one dimensional context (see in particular [BG01, LY10, GT11]). More recently the technics were adapted to the higher dimensional case [EK09, FGK16, GP16]. To consider unbounded perturbations, a new strategy has been developed in [BBM14, BBM15] using the pseudo-differential calculus. Without trying to be exhaustive we quote also [FI14, BM19, BBHM18, FGP18] regarding KAM theory for quasi-linear PDEs in one space dimension. This technics were successfully applied for reducibility problems in various case. For one dimensional linear equations with unbounded potential we quote [Bam17, Bam18, BM18, FGP19]. In higher space dimensions we refer to [EK09, GP16] for bounded potential, and to [BGMR18, Mon19, FGMP19, BLM18] for the unbounded cases.

**Scheme of the proof**

As said above the proof consists in a regularization step (section 3) and a KAM step (section 4). In section 5 we merge the two procedure to prove Theorem 1.1.

In the regularization step we prove that we can transform (by using a symplectic map: $u = \Phi(v)$) the original Schrödinger equation (LS) in a new one

$$i\partial_t v = \Delta_g v + \varepsilon(Z + R(\omega t))v,$$

where $Z$ is a pseudo-differential operator of order $\delta$ independent on time and commuting with $\Delta_g$ and $R$ is a $\rho$-regularizing operator in $L(H^s, H^{s+\rho})$ with $\rho$ arbitrary large. It is based on a normal form procedure developed in [BGMR19]. The central idea consists in averaging the Schrödinger operator by the flow of $K_0$ where $K_0 = \sqrt{\Lambda_g + Q}$ with $Q$ is a pseudo of order $-1$ is chosen (following [CdV79]) such that the spectrum of $K_0$ is included in $\mathbb{N} + \lambda$ for some constant $\lambda \in \mathbb{R}^+$. This crucial property makes the $K_0$ flow periodic and motivates us to use it to average the original operator; if $A$ is a pseudo-differential operator of order $\delta$ then its average with respect to the flow of $K_0$ (see (3.11)) is independent of $t$. In addition the homological equation (3.23) has a solution $S$ of order $\delta$ and thus $M \mapsto ad_{\lambda S}M$ maps a pseudo of order $m$ to a pseudo of order $m + \delta - 1 < m$ (see Lemma 3.2). This idea was already used in a pioneering work of Weinstein [Wein77]. In [BGMR19] such a procedure was iterated to obtain an equivalent equation like (1.6) but with $Z$ still depending on time (typically $Z = \langle A \rangle$, see (3.15)). In this paper we alternate the averaging procedure with a time elimination procedure based on the use of the operator (3.32) which solves the homological equation (3.33) and thus the Lie transform $\Phi_T = e^{iT}$ will kill the dependence on time in $Z = \langle A \rangle$ (see Lemma 3.4). This time elimination procedure requires a non resonance hypothesis on the frequency vector $\omega$ (see (3.2)).

Throughout section 3 we work at the pseudo-differential level and the main difficulty is to precisely control the flow generated by pseudo-differential operator of positive order (see Appendix A.3 and in particular hypothesis (A.13)). We notice that all this section holds true upon the hypothesis $\delta < 1$.

In the KAM step we kill the remainder term $R$ in (1.6) which still depends on time but is now a regularizing operator. As in [BBHM18] (see also [Mon19] and [BLM18]) we use a reducibility scheme where the regularizing property of the perturbation compensates the bad non resonance estimates satisfied by the eigenvalues of $\Delta_g + \varepsilon Z$ (see (4.13)). The condition $\delta \leq 1/2$ is used to ensure that condition (4.13) is preserved during the KAM iteration as long as a small part of the parameters $\omega$ are excised (see Lemma 4.2 where $\kappa = 1 - 2\delta$). This constraint in the KAM procedure was not necessary in [BLM18] (they
obtain the reducibility for perturbation of order $1 - e$ for any $e > 0$ when the transport operator is of order 1 essentially because the unperturbed system is integrable. In the context of the transport equation, the integrability allows Bambusi-Langela-Montalto to prove that the perturbed eigenvalues have the form, $\lambda_j = \lambda_j^{(0)} + z(j) + \text{remainder}$, where $z$ is the symbol of $Z$ (see formula (4.43) in [BLM18]). In our case we just know that $Z$ commutes with $\Delta_g$ and thus we can just prove that the spectrum of $\Delta_g + V$ preserves the cluster structure inherited from $\Delta_g$ on a Zoll manifold. That means that, once written in the laplacian diagonalization basis, the matrix of $Z$ is block-diagonal. By the way throughout section 4 we work at the matrix level.

As usual the homological equation (4.16) is solved blockwise and it is well known that the increasing size of the blocks may generate loss of regularity. In [1] Eliasson-Kuksin used geometrical arguments (related to a Bourgain’s Lemma, see Lemma 8.1 in [Bou99]) to control the size of the blocks, in [GP16] or [FG19] authors used a different argument introduced by Delort-Szeftel in [DS04] (see Lemma 4.3 in [GP16]). In this paper, as a consequence of the regularization step, we can solve the homological equation with loss of regularity and thus this step is simplified.

On the other hand the KAM procedure of [BBHM18] requires a tame property to deal with product of matrices. This motivates the definition of the space $\mathcal{M}_s$ of matrices with $s$-decay norm (see Definition 2.8) which was first introduced in [BCP15] (see also [BP11]). The tame property for the $s$-decay norm is stated in Lemma 2.11. It is crucial to obtain (4.38) and (4.39) which express the control of the new remainder $R_L$ after one KAM step in two different norms, a low $s$-decay norm and a high $s + \beta$-decay norm. The parameter $N$ measures the truncature in the Fourier variable associate to the angle $\varphi = \omega t$ and in the off-diagonal distance in the matrix (see (4.20)). When iterating the procedure, this special form of estimates (4.38)-(4.39) allows to obtain a convergent scheme for the sequence of remainders $R_k$ when choosing conveniently the sequence of truncature parameter $N_k$.

Section 3 and section 4 are independent and in fact are at different levels: while all section 3 takes place in the context of pseudo-differential operators, all section 4 takes place at matrix level. In section 5 we merge the two sections and for that we need the Lemma 2.14 which makes the link between $\rho$-smoothing operators and $\beta$- regularizing matrices.

**Notation.** We shall use the notation $A \preceq B$ to denote $A \leq CB$ where $C$ is a positive constant depending on parameters fixed once for all: $d$, $n$, $\delta$. We shall use the notation $A \leq_s B$ to denote $A \leq C(s)B$ where $C(s) > 0$ is a constant depending also on $s$.

2. **Functional setting**

In this section we introduce the space of functions, sequences, linear operators and pseudo differential operators we shall use along the paper.

2.1. **Spectral decomposition.** Following Theorem 1 of Colin de Verdière [CdV79], we introduce $Q$ the pseudo-differential operator of order $-1$, commuting with $\Delta_g$ such that, setting

$$K_0 := \sqrt{\Delta_g} + Q,$$

we have $\text{spec}(K_0) \subset \mathbb{N} + \lambda$ for some constant $\lambda \in \mathbb{R}^+$. We notice that our original Schrödinger operator $H(t) := \Delta_g + \varepsilon V(\omega t)$ reads

$$H(t) = \Delta_g + \varepsilon V(\omega t) = K_0^2 + Q_0 + \varepsilon V(\omega t)$$

where $Q_0 = -2Q \sqrt{\Delta_g} - Q^2$ is a pseudo differential operators of order 0.
Let us denote by \( \lambda_k \) the eigenvalue of \( K_0 \) and by \( E_k \) be the eigenspace associated to \( \lambda_k \). We have
\[
\lambda_k \sim k
\]
\[
\dim E_k := d_k \leq k^{n-1}.
\] (2.3)
We denote by
\[
\Phi_{[k]}(x) := \{\Phi_{k,m}(x), m = 1, \ldots, d_k\}
\] (2.4)
an orthonormal basis of \( E_k \). By formula (2.1) we also deduce that
\[
\Delta_g := K_0^2 + Q_0,
\] (2.5)
where \( Q_0 \) is a pseudo differential operator commuting both with the Laplacian \( \Delta_g \) and \( K_0 \). For this reason \( K_0 \) and \( \Delta_g \) diagonalizes simultaneously, hence
\[
\Delta_g \Phi_{k,j} = \Lambda_{k,j} \Phi_{k,j}, \quad k \in \mathbb{N}, \quad j = 1, \ldots, d_k,
\] (2.6)
with
\[
\Lambda_{k,j} = \lambda_k^2 + \eta_{k,j}, \quad |\eta_{k,j}| \lesssim 1.
\]
In particular there exists \( c_0 > 0 \) such that
\[
\Lambda_{k,j} \geq c_0 k^2, \quad |\Lambda_{k,j} - \Lambda_{k',j'}| \geq c_0 (k + k'), \quad \forall k \neq k',
\] (2.7)
and for any \( j = 1, \ldots, d_k, j' = 1, \ldots, d_{k'} \).

2.2. Space of functions and sequences. Using the spectral decomposition of the space \( L^2(M; \mathbb{C}) = \oplus_{k \in \mathbb{N}} E_k \), any function \( u \in L^2(M; \mathbb{C}) \) can be written as
\[
\Pi E_k u \Phi_{k,m}(x) = \sum_{k \in \mathbb{N}} \sum_{m=1}^{d_k} z[k] \Phi_{[k]}(x),
\]
\[
z[k] = (z_{k,1}, \ldots, z_{k,d_k}) \in \mathbb{C}^{d_k},
\]
where \( , \cdot \) denotes the usual scalar product in \( \mathbb{R}^{d_k} \). We denote by \( \Pi E_k \) the \( L^2 \)-projector on the eigenspace \( E_k \), i.e., for any \( k \in \mathbb{N} \),
\[
(\Pi E_k u)(x) = z[k] \cdot \Phi_{[k]}(x) \quad \Rightarrow \quad (\sqrt{-\Delta} + Q) \Pi E_k u = \lambda_k \Pi E_k u.
\] (2.9)
For \( s \geq 0 \), we define the (Sobolev) scale of Hilbert sequence spaces
\[
h_s := \left\{ z = \{z[k]\}_{k \in \mathbb{N}}, z[k] \in \mathbb{C}^{d_k} : \|z\|_{h_s}^2 := \sum_{k \in \mathbb{N}} \langle k \rangle^{2s} \|z[k]\|^2 < +\infty \right\},
\] (2.10)
where \( \langle k \rangle := \sqrt{1 + |k|^2} \) and \( \| \cdot \| \) denotes the \( L^2(\mathbb{C}^{d_k}) \)-norm. By a slight abuse of notation we define the operator \( \Pi E_k \) on sequences as \( \Pi E_k z = z[k] \) for any \( z \in h^s \) and \( k \in \mathbb{N} \).

We notice that the weight \( \langle k \rangle \) we use in the norm in (2.10) is related to the eigenvalues of \( K_0 \), indeed
\[
c \langle k \rangle \leq \lambda_k \leq C \langle k \rangle
\] (2.11)
for some suitable constants \( 0 < c \leq C \).

As a consequence the space
\[
H^s = H^s(\mathcal{M}^n, \mathbb{C}) := \{ u(x) = \sum_{k \in \mathbb{N}} z[k] \cdot \Phi_{[k]}(x) \mid z \in h^s \}
\]
is the usual Sobolev space \( H^s = \text{dom}(\{(K_0)^s\}) = \text{dom}(\sqrt{1 + K_0^2})^s \) and \( \|u\|_{H^s} := \|z\|_{h^s} \) is equivalent to the standard Sobolev norm \( \|u\|_{H^s} \sim \|K_0^s u\|_{L^2(\mathcal{M}^n)} \).
In the paper we shall also deal with quasi periodic in time functions $\mathbb{R} \times M \ni (t, x) \mapsto u(\omega t, x)$ where $\omega \in \mathbb{R}^d$ is a frequency vector and $u$ is periodic in its first variable. To this end we introduce the space $H^r(\mathbb{T}^d; H^s(\mathbb{M}^n; \mathbb{C}))$ defined as the set of functions $u : \mathbb{T}^d \ni \varphi \mapsto H^s(\mathbb{M}^n; \mathbb{C})$ which are Sobolev in $\varphi \in \mathbb{T}^d$ with values in $H^s(\mathbb{M}^n; \mathbb{C})$.

Functions in $H^r(\mathbb{T}^d; H^s(\mathbb{M}^n; \mathbb{C}))$ can be expanded, using the standard Fourier theory, as

$$u(\varphi, x) = \sum_{\ell \in \mathbb{Z}^d, k \in \mathbb{N}} z[k](\ell) \cdot \Phi[k](x) e^{i \varphi \ell}, \quad z[k](\ell) \in \mathbb{C}^{d_k} \quad (2.12)$$

where $e^{i \varphi \ell} \Phi_{k,m}(x), l \in \mathbb{Z}^d, k \in \mathbb{N}, m = 1, \ldots, d_k$ is an orthogonal basis of $L^2(\mathbb{T}^d \times \mathbb{M}^n; \mathbb{C})$. We define the space of sequence (recall (2.10))

$$h_{s,r} := \{ z = \{ z[k](\ell) \}_{\ell \in \mathbb{Z}^d, k \in \mathbb{N}} : z[k] \in \mathbb{C}^{d_k} : \| z \|_{h_{s,r}}^2 := \sum_{\ell \in \mathbb{Z}^d} \langle l, k \rangle^{2r} \| z(l) \|_{H^s}^2 < +\infty \}. \quad (2.13)$$

Along the paper we shall also consider the space, for $p \in \mathbb{N}$ with $p > \frac{d+n}{2}$,

$$\ell_p := \bigcap_{r+d/2, s+n/2} h_{s,r}. \quad (2.14)$$

We endow the space $\ell_p$ with the norm

$$\| z \|_{\ell_p}^2 := \sum_{l \in \mathbb{Z}^d, k \in \mathbb{N}} \langle l, k \rangle^{2p} \| z[k](\ell) \|^2. \quad (2.15)$$

**Lipschitz norm.** Consider a compact subset $\mathcal{O}$ of $\mathbb{R}^d$, $d \geq 1$. For functions $f : \mathcal{O} \rightarrow E$, with $(E, \| \cdot \|_E)$ some Banach space, we define the sup norm and the lipschitz semi-norm as

$$\| f \|_{E}^{\sup} := \| f \|_{E}^{\sup, \mathcal{O}} := \sup_{\omega \in \mathcal{O}} \| f(\omega) \|_E,$$

$$\| f \|_{E}^{lip} := \| f \|_{E}^{lip, \mathcal{O}} := \sup_{\omega_1, \omega_2 \in \mathcal{O}, \omega_1 \neq \omega_2} \| \frac{f(\omega_1) - f(\omega_2)}{|\omega_1 - \omega_2|} \|_E. \quad (2.16)$$

For any $\gamma > 0$ we introduce the weighted Lipschitz norms

$$\| f \|_{E}^{\gamma, \mathcal{O}} := \| f \|_{E}^{\sup, \mathcal{O}} + \gamma \| f \|_{E}^{lip, \mathcal{O}}. \quad (2.17)$$

We finally define the space of sequences

$$h_{s,r}^{\gamma, \mathcal{O}} := \{ \mathcal{O} \ni \omega \mapsto z(\omega) \in h_{s,r} : \| z \|_{h_{s,r}} < +\infty \}, \quad (2.18)$$

and consequently the space (recall (2.14))

$$\ell_p^{\gamma, \mathcal{O}} := \bigcap_{s+r=p} h_{s,r}^{\gamma, \mathcal{O}}, \quad (2.19)$$

endowed with the norm

$$\| z \|_{\ell_p^{\gamma, \mathcal{O}}} := \| z \|_{\ell_p^{\sup, \mathcal{O}}} + \gamma \| z \|_{\ell_p^{lip, \mathcal{O}}}. \quad (2.20)$$
2.3. Pseudo-differential operators. In this paper we consider operators which are pseudo-differential. Here we recall some fundamental properties of operators in $A_m$ which are collected in \cite{BGMRT19}. First $A_m$ is a Fréchet space for a family of filtering semi-norms $\{N_{m,p}\}_{p \geq 1}$ such that the embedding $A_m \hookrightarrow \bigcap_{s \in \mathbb{R}} L(H^s, H^{s-m})$ is continuous. We can also chose the semi-norms in an increasing way, i.e. $N_{m,p}(A) \leq N_{m,p+1}(A)$ for $p \geq 1$ and $A \in A_m$. To state the other properties we need to introduce the following definition.

**Definition 2.1.** Let $S \in \mathcal{L}(H)$. We say that $S$ is $\rho$-smoothing, and we will write $S \in R_\rho$, if $S$ can be extended to an operator in $\mathcal{L}(H^s, H^{s+\rho})$ for any $s \in \mathbb{R}$. When this is true for every $\rho \geq 0$, we say that $S$ is a smoothing operator.

Then we have the following properties concerning the class $A_m$ equipped with the semi-norms $\{N_{m,p}\}_{p \geq 1}$:

(i) let $A \in A_m$, for any $s \in \mathbb{R}$ there exist constants $C = C(m, s) > 0$, $p = p(m, s) \geq 1$ which is an increasing function of $s$ such that

$$
\|A\|_{L(H^s, H^{s-m})} \leq CN_{m,p}(A).
$$

(ii) Let $A \in A_m$, $B \in A_n$ then $AB \in A_{m+n}$. Furthermore for any $\rho \geq 0$ there exists $S$ a $\rho$-smoothing operator such that for any $p \geq 1$ for any $s \in \mathbb{R}$ there are constants $C = C(m, n, p, s, \rho) > 0$, $q = q(m, n, p, s, \rho) \geq p$ such that

$$
N_{m+n,p}(AB - S) \leq CN_{m,q}(A)N_{n,q}(B),
$$

$$
\|S\|_{L(H^s, H^{s+\rho})} \leq CN_{m,q}(A)N_{n,q}(B).
$$

(iii) Let $A \in A_m$, $B \in A_n$ then $[A, B] \in A_{m+n-1}$. Furthermore for any $\rho \geq 0$ there exists $S$ a $\rho$-smoothing operator such that for any $p \geq 1$ for any $s \in \mathbb{R}$ there are constants $C = C(m, n, p, s, \rho) > 0$, $q = q(m, n, p, s, \rho) \geq p$ such that

$$
N_{m+n-1,p}([A, B] - S) \leq CN_{m,q}(A)N_{n,q}(B),
$$

$$
\|S\|_{L(H^s, H^{s+\rho})} \leq CN_{m,q}(A)N_{n,q}(B).
$$

(iv) The map $\tau \rightarrow A(\tau) := e^{-itK_0}Ae^{itK_0} \in C^0_0(\mathcal{R}, A_m)$. Furthermore for any $\rho \geq 0$ there exists $S$ a $\rho$-smoothing operator such that for any $p \geq 1$ for any $s \in \mathbb{R}$ there are constants $C = C(m, n, p, s, \rho) > 0$, $q = q(m, n, p, s, \rho) \geq p$ such that

$$
N_{m+n-1,p}(e^{-itK_0}Ae^{itK_0} - S) \leq CN_{m,q}(A),
$$

$$
\|S\|_{L(H^s, H^{s+\rho})} \leq CN_{m,q}(A).
$$

**Remark 2.2.** In (ii), (iii) and (iv) the smoothing correction doesn't play an important role since it can be chosen as regularizing as one want. In the KAM scheme the level of regularization will be fix once for all. Thus, by a slight abuse of notation, we will often omit in the following the smoothing correction and will just write

$$
N_{m+n,p}(AB) \leq CN_{m,q}(A)N_{n,q}(B),
$$

$$
N_{m+n-1,p}([A, B]) \leq CN_{m,q}(A)N_{n,q}(B).
$$

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2This fact is quite evident in the case of pseudo-differential operators on $\mathbb{R}^n$ and thus extends to pseudo-differential operators on $\mathbb{M}^n$ by passing to local charts.
We shall also consider $H^r$-mappings
\[ \mathbb{T}^d \ni \varphi \mapsto A(\varphi) \] (2.30)
with $A(\varphi)$ a symmetric pseudo-differential operators of order $m$ in $\mathcal{A}_m$. We can then decompose $A$ in Fourier writing
\[ A(\varphi) = \sum_{l \in \mathbb{Z}^d} A(l) e^{i l \cdot \varphi} \] (2.31)
with $A(l)$ a pseudo-differential operators of order $m$ in $\mathcal{A}_m$. We give the following definition.

**Definition 2.3.** Let $m \in \mathbb{R}$, $r > d/2$. We denote by $\mathcal{A}_{m,r}$ the Fréchet space of mapping $\mathbb{T}^d \ni \varphi \mapsto A(\varphi) \in \mathcal{A}_m$ that have $H^r$ on $\mathbb{T}^d$. We endow $\mathcal{A}_{m,r}$ with the family of semi-norms
\[ \left( N_{m,r,p}^\gamma(A) \right)^2 := \sum_{\ell \in \mathbb{Z}^d} \langle \ell \rangle^{2r} N_{m,p}^2(A(l)), \quad p \geq 1. \] (2.32)
Consider a Lipschitz family $\mathcal{O} \ni \omega \mapsto A(\omega) \in \mathcal{A}_{m,r}$ where $\mathcal{O}$ is a compact subset of $\mathbb{R}^d$, $d \geq 1$. For $\gamma > 0$ we define the Lipschitz semi-norms (recall (2.16)) as
\[ N_{m,r,p}^{\gamma,\mathcal{O}}(A) := N_{m,r,p}^{\sup,\mathcal{O}}(A) + \gamma N_{m,r,p}^{\text{lip},\mathcal{O}}(A) \] (2.33)
We denote by $\mathcal{A}^{\gamma,\mathcal{O}}_{m,r}$ the Fréchet space of families of pseudo differential operators $A(\omega) \in \mathcal{A}_{m,r}$ endowed with the family of semi-norms $\{ N_{m,r,p}^{\gamma,\mathcal{O}} \}_{p \geq 1}$.

Similarly we define the corresponding class of $\rho$-smoothing operators $R(\omega, \varphi)$, $H^r$ in $\varphi$ and Lipschitz in $\omega$.

**Definition 2.4.** Let $\rho \in \mathbb{R}$ and $r > d/2$. We denote by $\mathcal{R}_{\rho,r}$ the Fréchet space of $\rho$-smoothing $H^r$-mapping $\mathbb{T}^d \ni \varphi \mapsto R(\varphi) \in \mathcal{L}(H^s, H^{s+\rho})$ for all $s \in \mathbb{R}$ endowed with the family of semi-norms
\[ |R|_{\rho,r,s}^2 := \sum_{\ell \in \mathbb{Z}^d} \langle \ell \rangle^{2r} \|R(l)\|^2_{\mathcal{L}(H^s, H^{s+\rho})} \quad s \in \mathbb{R}. \] (2.34)
Consider a family $\mathcal{O} \ni \omega \mapsto R(\omega) \in \mathcal{R}_{\rho,r}$ where $\mathcal{O}$ is a compact subset of $\mathbb{R}^d$, $d \geq 1$. For $\gamma > 0$ we denote by $\mathcal{R}^{\gamma,\mathcal{O}}_{\rho,r}$ the Fréchet space of families of pseudo differential operators $R(\omega) \in \mathcal{R}_{\rho,r}$ endowed with the family of semi-norms $\{ N_{\rho,r,p}^{\gamma,\mathcal{O}} \}_{p \in \mathbb{N}}$ defined by (recall (2.16))
\[ |R|_{\rho,r,s}^{\gamma,\mathcal{O}} := |R|_{\rho,r,s}^{\sup,\mathcal{O}} + \gamma |R|_{\rho,r,s}^{\text{lip},\mathcal{O}}. \] (2.35)
We notice that by (2.21) we have $\mathcal{A}_{m,r} \subset \mathcal{R}_{-m,r}$.

**Lemma 2.5.** Let $r > d/2$, $m, \rho \in \mathbb{R}$ and consider $R \in \mathcal{R}^{\gamma,\mathcal{O}}_{\rho,r}$ and $A \in \mathcal{A}^{\gamma,\mathcal{O}}_{m,r}$. Then, for any $s \in \mathbb{R}$, there are $C = C(s, r) > 0$, $p(s, m) > 0$ such that
\[ \|Ah\|_{\gamma,s-m,r} \leq C N_{m,r,p(s,m)}^{\gamma,\mathcal{O}}(A) \|h\|_{h_{s,r}^{\gamma,\mathcal{O}}}, \] (2.36)
\[ \|Rh\|_{h_{s+\rho,r}^{\gamma,\mathcal{O}}} \leq C |R|_{\rho,r,s}^{\gamma,\mathcal{O}} \|h\|_{h_{s,r}^{\gamma,\mathcal{O}}}, \] (2.37)
for any $h \in h_{s,r}^{\gamma,\mathcal{O}}$. 

Proof. We start by proving the (2.37) for the norm $\| \cdot \|_{h_{\alpha+p},r}$. Recalling (2.13) we have

$$
\| Rh \|_{h_{\alpha+p},r} \leq \sum_{l' \in \mathbb{Z}^d} (l')^{2r} \left( \sum_{l' \in \mathbb{Z}^d} \| R(l-l') h(l') \|_{H^{\alpha+r}} \right)^2
$$

$$
\leq \sum_{l' \in \mathbb{Z}^d} (l')^{2r} \left( \sum_{l' \in \mathbb{Z}^d} \| R(l-l') \|_{L(H^s;H^{\alpha+r})} \| h(l') \|_{H^s} \right)^2
$$

$$
\leq \sum_{l' \in \mathbb{Z}^d} (l')^{2r} \left( \sum_{l' \in \mathbb{Z}^d} \langle l-l' \rangle^r \| R(l-l') \|_{L(H^s;H^{\alpha+r})} \langle h(l') \rangle \| h(l') \|_{H^s} \right)^2
$$

$$
+ \sum_{l' \in \mathbb{Z}^d} (l')^{2r} \left( \sum_{l' \in \mathbb{Z}^d} \langle l-l' \rangle^r \| R(l-l') \|_{L(H^s;H^{\alpha+r})} \langle h(l') \rangle \| h(l') \|_{H^s} \right)^2.
$$

Hence, by using the Cauchy-Schwartz inequality, we get

$$
\| Rh \|_{h_{\alpha+p},r} \leq C \sum_{l',l' \in \mathbb{Z}^d} (l')^{2r} \| R(l-l') \|_{L(H^s;H^{\alpha+r})}^2 \langle h(l') \rangle \| h(l') \|_{H^s}^2
$$

$$
\leq C \| h \|_{h_{\alpha+p},r}^2 \| R \|_{r,s}^2,
$$

which implies the (2.37) for the norm $\| \cdot \|_{h_{\alpha+p},r}$. The Lipschitz bound on the norm $\| \cdot \|_{h_{\alpha+p},r}$ and the (2.36) follows similarly. 

\square

In the following Lemma we state some properties and estimates\(^3\) that will be proved in Appendix A.1.

Lemma 2.6. Let $A, B$ are pseudo-differential operators in $A^{\gamma,0}_{m,r}$ and $A^{\gamma,0}_{n,r}$. For any $p \geq 1$ there exist constants $C = C(r, m, n, p)$ and $q = q(r, m, n, p)$ which is increasing in $p$ such that

(i) $AB, BA \in A^{\gamma,0}_{m+n,r}$ and

$$
\mathcal{N}^{\gamma,0}_{m+n,r,p}(AB), \mathcal{N}^{\gamma,0}_{m+n,r,p}(BA) \leq C \mathcal{N}^{\gamma,0}_{m,r,q}(A) \mathcal{N}^{\gamma,0}_{n,r,q}(B).
$$

(ii) $[A, B] \in A^{\gamma,0}_{m+n-1,r}$ and

$$
\mathcal{N}^{\gamma,0}_{m+n-1,r,p}([A, B]) \leq C \mathcal{N}^{\gamma,0}_{m,r,q}(A) \mathcal{N}^{\gamma,0}_{n,r,q}(B).
$$

(iii) Let $\omega \in \mathbb{R}^d$, then $\omega \cdot \partial \varphi A \in A_{m,r-1}$ and

$$
\mathcal{N}^{\gamma,0}_{m,r-1,p}(\omega \cdot \partial \varphi A) \leq C \mathcal{N}^{\gamma,0}_{m,r,p}(A).
$$

If furthermore $\omega$ satisfies, for some $\alpha > d - 1$,

$$
|\omega \cdot l| > \frac{\gamma}{|l|^\alpha}, \quad \forall \ l \in \mathbb{Z}^d \setminus \{0\},
$$

and $r - 2\alpha - 1 > d/2$ then $(\omega \cdot \partial \varphi)^{-1} A \in A_{m,r-(2\alpha+1)}$ and

$$
\mathcal{N}^{\gamma,0}_{m,r-(2\alpha+1),p}( (\omega \cdot \partial \varphi)^{-1} A) \leq C \gamma^{-1} \mathcal{N}^{\gamma,0}_{m,r,p}(A).
$$

\(^3\)Estimates (2.38), (2.39) and (2.43) are written taking into account Remark 2.2. For instance (2.39) should be interpreted as: for any $\rho \geq 0$ there exists $S$ a $\rho$-smoothing operator such that for any $p \geq 1$ for any $s \in \mathbb{R}$ there are constants $C = C(m, n, p, r, \rho) > 0, q = q(m, n, p, r, \rho) \geq 1$ such that

$$
\mathcal{N}^{\gamma,0}_{m+n-1,r,p}([A, B] - S) \leq C \mathcal{N}^{\gamma,0}_{m,r,q}(A) \mathcal{N}^{\gamma,0}_{n,r,q}(B), \ |S|_{\rho,s} \leq C \mathcal{N}^{\gamma,0}_{m,r,q}(A) \mathcal{N}^{\gamma,0}_{n,r,q}(B).
$$
Consider an operator of the form
\[ L = L(\varphi, \omega) := \omega \cdot \partial_\varphi + iM(\varphi), \]
where \( M(\varphi) \) is some map \( \mathbb{T}^d \ni \varphi \mapsto M(\varphi) \in \mathcal{L}(H^s; H^{s+m}) \), for some \( m \in \mathbb{R} \). We shall study how the operator \( L \) in (2.45) conjugates under the map \( \Phi_S \) defined as
\[ \Phi_S := (\Phi^T_S)|_{\tau = 1}, \quad \Phi^T_S := e^{i\tau S} = \sum_{p=0}^{\infty} \frac{1}{p!}(iS)^p \]
where \( S(\varphi) \) is some map \( \mathbb{T}^d \ni \varphi \mapsto S = S(\varphi) \in \mathcal{L}(H^s; H^{s+m'}) \), for some \( m' \in \mathbb{R} \). For the well-posedness of a map of the form (2.46) we refer to Lemma A.6 in Appendix A.3.

By using the Lie series expansions we have
\[ L^+ = L^+(\varphi) := \Phi_S \circ L \circ \Phi_S^{-1} = \omega \cdot \partial_\varphi + iM^+(\varphi) \]
where \( M^+(\varphi) = M_1^+(\varphi) + M_2^+(\varphi) \) with, for any \( q \in \mathbb{N} \),
\[ iM_1^+(\varphi) := \Phi_S \circ iM \circ \Phi_S^{-1} = iM + \sum_{p=1}^{q} \frac{1}{p!} \text{ad}^p_{iS}(iM) + \frac{1}{q!} \int_0^1 (1 - \tau)^q \Phi_S \text{ad}^{q+1}_{iS}(iM) \Phi_S^{-1} d\tau, \]
and
\[ iM_2^+(\varphi) := \Phi_S \circ \omega \cdot \partial_\varphi \circ \Phi_S^{-1} - \omega \cdot \partial_\varphi = -i\omega \cdot \partial_\varphi S - \sum_{p=2}^{q} \frac{1}{p!} \text{ad}^p_{iS}^{-1}(i\omega \cdot \partial_\varphi S) + \frac{1}{q!} \int_0^1 (1 - \tau)^q \Phi_S \text{ad}^q_{iS}(i\omega \cdot \partial_\varphi S) \Phi_S^{-1} d\tau, \]
where we defined \( \text{ad}^0_{iS}(M) = M \) and
\[ \text{ad}^q_{iS}(M) = \text{ad}^{q-1}_{iS}([S, M]), \quad [S, M] = SM - MS. \]

**Remark 2.7. (Hamiltonian structure)** We remark that, if the operator \( S \) in and \( M \) are Hermitian, then by Lemma 2.9 in [FG19], we have that also the operator \( M^+ \) in (2.47) is Hermitian.

### 2.5. Linear operators and matrices.

According to the orthogonal splitting
\[ L^2(S^n; \mathbb{C}) = \bigoplus_{k \in \mathbb{N}} E_k, \]
we identify a linear operator acting on \( L^2(M; \mathbb{C}) \) with its matrix representation \( A := \left( A_{[k]}^{[k']} \right)_{k, k' \in \mathbb{N}} \in \mathcal{L}(h^0) \) (recall 2.10) with blocks \( A_{[k]}^{[k']} \in \mathcal{L}(E_k; E_k) \). Notice that each block \( A_{[k]}^{[k']} \) is a \( d_k \times d_k' \):
\[ A_{[k]}^{[k']} := \left( A_{k,j'}^{k,j} \right)_{j = 1, \ldots, d_k, j' = 1, \ldots, d_{k'}}. \]
The action of the operator $A$ on functions $u(x)$ as in (2.8) of the space variable in $L^2(\mathbb{R}^n; \mathbb{C})$ is given by

$$(Au)(x) = \sum_{k \in \mathbb{N}} (Az)[k] \cdot \Phi[k](x), \quad z[k] \in \mathbb{C}^{d_k}, \quad (Az)[k] = \sum_{j \in \mathbb{N}} A[j]_{[k]} z[j]. \quad (2.53)$$

Given $s, s' \in \mathbb{R}$ we denote by $\mathcal{L}(H^s, H^{s'})$ the space of linear bounded operators from $H^s$ to $H^{s'}$ endowed with the standard operator norm $\| \cdot \|_{\mathcal{L}(H^s, H^{s'})}$.

In this paper we also consider regular $\varphi$-dependent families of linear operators

$$\mathbb{T}^d \ni \varphi \mapsto A = A(\varphi) = \sum_{l \in \mathbb{Z}^d} A(l) e^{il\varphi}$$

where $A(l)$ are linear operators in $\mathcal{L}(H^s, H^{s'})$, for any $l \in \mathbb{Z}^d$. We also regard $A$ as an operator acting on functions $u(\varphi, x)$ of space-time as

$$(Au)(\varphi, x) = (A(\varphi)u(\varphi, \cdot))(x).$$

More precisely, expanding $u$ as in (2.12), we have

$$(Au)(\varphi, x) = \sum_{l \in \mathbb{Z}^d, k \in \mathbb{N}} (Az)[k](l)e^{il\varphi}\Phi[k](x), \quad (Az)[k](l) = \sum_{p \in \mathbb{Z}^d, k' \in \mathbb{N}} A[k']_{[k]}(l-p)z[k'](p). \quad (2.55)$$

Relation (2.53) shows that, in order to define operators that conserve the $H^s$ regularity in space we need to assume some decay of $\|A[k']\|_{L^2(\mathbb{L}^2)}$ with respect to $|k - k'|$. That is the reason for the following definition first introduced in [BPT11] for (i) and in [BCP15] for (ii).

**Definition 2.8. ($s$-decay norm)**

(i) We define the $s$-decay norm of a matrix $A \in \mathcal{L}(H^s; H^s)$ as

$$[A]_s^2 := \sum_{h \in \mathbb{N}} \langle h \rangle^{2s} \sup_{|k - k'| = h} \|A[k]\|_{L^2(\mathbb{L}^2)}^2 \quad (2.56)$$

where $\| \cdot \|_{L^2(\mathbb{L}^2)}$ is the $L^2$-operator norm in $\mathcal{L}(E_{k'}, E_k)$.

(ii) Consider a map $\mathbb{T}^d \ni \varphi \mapsto A = A(\varphi) \in \mathcal{L}(H^s; H^s)$. We define its decay norm as

$$[A]_s^2 := \sum_{l \in \mathbb{Z}^d, h \in \mathbb{N}} \langle l, h \rangle^{2s} \sup_{|k - k'| = h} \|A[k'](l)\|_{L^2(\mathbb{L}^2)}^2 \quad (2.57)$$

We denote by $\mathcal{M}_s$ the space matrices with finite $s$-decay norm $[\cdot]_s$.

(iii) Consider a Lipschitz family $O \ni \omega \mapsto A(\omega) \in \mathcal{M}_s$ where $O$ is a compact subset of $\mathbb{R}^d, d \geq 1$. For $\gamma > 0$ we define the Lipschitz decay norm as

$$[A]_{s,O}^\gamma := \sup_{\omega \in O} \frac{[A]_{s,O} + \gamma [A]_{lip,O}}{\sup_{\omega \in O} A(\omega)} \quad (2.58)$$

We denote by $\mathcal{M}_{s,O}^\gamma$ the space of families of Lipschitz mapping $\omega \mapsto A(\omega) \in \mathcal{M}_s$ with finite $[\cdot]_{s,O}^\gamma$-norm.

**Remark 2.9.** The $s$-decay norm (2.57) link the regularity in space and the regularity in $\phi$ (i.e. in time). In fact for $s$ integer we have

$$\mathcal{M}_s = \cap_{p + q \leq s} H^p(\mathbb{T}^d, \mathcal{L}(H^q(\mathbb{M}^n), H^l(\mathbb{M}^n))).$$
We define the Lipschitz norm as $\|A\|_{\beta,s} := \sup_{\|\cdot\|_{\beta,s}}\|A\|_{\beta,s}$. For any $s > (d + n)/2$ the following holds:
(i) there is $C = C(s) > 0$ such that (recall (2.14), (2.15))

$$\|A\|_{\beta,s} \leq C(s)\|A\|_{\beta,s} + C(s)\|A\|_{\beta,s}, \quad (2.59)$$

for any $h \in \ell_2$;
(ii) there is $C = C(s) > 0$ such that

$$\|AB\|_{\beta,s} \leq C(s)\|A\|_{\beta,s} + C(s)\|A\|_{\beta,s}, \quad (2.60)$$

(iii) for $N > 0$ we define (recall (2.54)) the matrix $\Pi_N A$

$$(\Pi_N A)(l) := \begin{cases} A(l), & l \in \mathbb{Z}^d, k, k' \in \mathbb{N}, \quad |l| \leq N, \\ 0, & \text{otherwise}, \end{cases}$$

(2.61)

One has

$$[(\Id - \Pi_N)A]_{\beta,s} \leq CN^{-2\beta}A_{\beta,s}, \quad (2.62)$$

for some $C = C(s) > 0$.

Similar bounds holds also replacing $\|\cdot\|_{\beta,s}$ with the norms $\|\cdot\|_{\gamma,s}^{\ell_2}$, $\|\cdot\|_{\gamma,s}^{\ell_2}$ respectively (see (2.20), (2.58)).

Proof. Items (i) and (ii) follow by lemmata 2.6, 2.7 in [BCP15]. Item (iii) follows by the definition of the norm in (2.57). □

We will also need a class of matrices that take into account a notion of regularization.

Definition 2.12. Define the diagonal $\varphi$-independent operator $D$, acting on $z \in \ell_2$ (see (2.14)), as

$$Dz := \text{diag}_{l \in \mathbb{Z}^d, k \in \mathbb{N}}(\lambda_k)z = (\lambda_kz(l))_{l \in \mathbb{Z}^d, k \in \mathbb{N}}. \quad (2.63)$$

For $\beta \in \mathbb{R}$ we define the norm $\|\cdot\|_{\beta,s}^D$ of a matrix $A$ in (2.54) as

$$\|A\|_{\beta,s}^D := \|D^\beta A\|_{\beta,s} + \|AD^\beta\|_{\beta,s}. \quad (2.64)$$

We denote by $\mathcal{M}_{\beta,s}$ the space of maps $\mathbb{T} \ni \varphi \mapsto A = A(\varphi) \in \mathcal{L}(L^2)$ with finite $\|\cdot\|_{\beta,s}$-norm.

Consider a family $\mathcal{O} \ni \omega \mapsto A(\omega) \in \mathcal{M}_{\beta,s}$ where $\mathcal{O}$ is a compact subset of $\mathbb{R}^d, d \geq 1$. For $\gamma > 0$ we define the Lipschitz norm as

$$\|A\|_{\beta,s}^{\gamma} := \|A\|_{\beta,s}^{\gamma} + \gamma(\sup_{\omega \in \mathcal{O}} [A(\omega)]_{\beta,s} + \gamma \sup_{\omega_1, \omega_2 \in \mathcal{O}} |A(\omega_1) - A(\omega_2)|_{\beta,s}). \quad (2.65)$$

We denote by $\mathcal{M}_{\beta,s}^{\gamma}$ the space of families of matrices $A(\omega)$ with finite $\|\cdot\|_{\beta,s}^{\gamma}$-norm.

For properties of matrices in $\mathcal{M}_{\beta,s}^{\gamma}$ we refer to Appendix A.2 and in particular Lemma A.4 stating a tame property for the norm given by (2.65).
**Definition 2.13. (Block-diagonal matrices)** We say that $A(\varphi)$ is block-diagonal if and only if $A(k')_{[k]}(\varphi) = 0$ for any $k \neq k'$ and any $\varphi \in \mathbb{T}^d$.

We notice that operators commuting with $K_0$ have matrices that are block-diagonal: let $Z$ be such that

$$[K_0, Z] = 0.$$  \hspace{1cm} (2.66)

Since

$$[H_0, Z]_{[k']} = (\lambda k' - \lambda k)Z_{[k']}^{[k']} \forall k, k',$$

condition (2.66) implies that the matrix $(Z_{[k]}^{[k']})_{k,k' \in \mathbb{N}}$ representing the operator $Z$ is block-diagonal according to Definition 2.13.

2.6. **Link between pseudo-differential operators and matrices.** To a linear operator $R$ we associate its matrix representation still denoted $R$ through the formula

$$R_{[k]}^{[k']} = \int_{\mathbb{R}^d} R\Phi[k]\Phi'[k'] dx.$$ \hspace{1cm} (2.67)

In the following we show that the decay norm $[\cdot]_{\beta,s}$ (see Definitions 2.8 and 2.12) is well designed to capture the smoothing property.

**Lemma 2.14.** Fix $s > (d + n)/2$ and $\beta \geq 0$. Assume that $R \in \mathcal{R}_{\rho,s}$ with $\rho \geq s + \beta + 1/2$ and that $R$ is symmetric then $R \in \mathcal{M}_{\beta,s}$. Moreover, there exists a constant $C = C(s, \rho, \beta)$ such that

$$[R]_{\beta,s} \leq C|R|_{\rho,s,s}.$$ \hspace{1cm} (2.68)

If $R \in \mathcal{R}_{\rho,s}$ then the bound (2.68) holds with the norms $[\cdot]_{\beta,s}, |\cdot|_{\rho,s,s}$ replaced by the norms $[\cdot]_{\beta,s}, |\cdot|_{\rho,s,s}$.

**Proof.** We have for $k \in \mathbb{Z}^d$

$$\|R_{[k]}^{[k']}(\ell)\|_{L^2} = |\langle D^{\rho+s} R(\ell) \Phi[k], D^{-\rho-s} \Phi[k'] \rangle|$$

$$\leq \|D^{\rho+s} R(\ell) \Phi[k]\|_{L^2} \| \Phi[k']\|_{L^2} \langle k' \rangle^{-\rho-s}$$

$$\leq \|R(\ell)\|_{L(H^s, H^{s+\rho})} \|\Phi[k]\|_{H^s} \|k\|^{-\rho-s}$$

$$\leq \|R(\ell)\|_{L(H^s, H^{s+\rho})} \langle k \rangle^s \|k\|^{-\rho-s}$$

where we used that, for $s \in \mathbb{R}$ (recall (2.11)),

$$\|\Phi[k,j]\|_{H^s} \sim \|K_0^{s} \Phi[k,j]\|_{L^2} = \lambda_k^s \sim \langle k \rangle^s.$$

Similarly, since $R$ is symmetric,

$$\|R_{[k]}^{[k']}(\ell)\|_{L^2} \leq \|R(\ell)\|_{L(H^s, H^{s+\rho})} \langle k \rangle^s \|k\|^{-\rho-s},$$

therefore we get

$$\|R_{[k]}^{[k']}(\ell)\|_{L^2} \leq \min \left( \langle k \rangle^s \|k\|^{-\rho-s}, \langle k \rangle^s \|k\|^{-\rho-s} \right) \|R(\ell)\|_{L(H^s, H^{s+\rho})}.$$
So, by definition, we get using that \( \langle h, \ell \rangle \leq \langle \ell, h \rangle \),
\[
\| [D^\beta R]_s \|^2 = \sum_{h \in \mathbb{N}, \ell \in \mathbb{Z}^d} \langle h, \ell \rangle^{2s} \sup_{|k-k'|=h} \| (D^\beta R)^{[k']}_{[k]} (\ell) \|_{L^2(\mathbb{L})}^2 \\
\leq \sum_{\ell \in \mathbb{Z}^d} \langle \ell \rangle^{2s} \| R(\ell) \|_{L^2(H^s, H^{s+\rho})} \sum_{h \in \mathbb{N}} \langle h \rangle^{2s} \sup_{|k-k'|=h} \langle k \rangle^{2\beta} \min \left( \langle k' \rangle^{\beta} \langle k' \rangle^{-\rho-s}, \langle k \rangle^{\beta} \langle k' \rangle^{-\rho-s} \right) \\
\leq 2^{2\rho-2\beta} |R|_{\rho,s,s}^2 \sum_{h \in \mathbb{N}} \langle h \rangle^{2s+2\beta-2\rho}
\]
where we used that if \( |k-k'| = h \) then \( \max(|k|, |k'|) \geq h/2 \). A similar estimates holds true for \( \| RD^\beta \|_s \) and thus for
\[
\| R \|_{\beta,s} = \| D^\beta R \|_s + \| RD^\beta \|_s.
\]
Following a similar reasoning one gets the Lipschitz bounds. \( \square \)

3. REGULARIZATION PROCEDURE

Let us consider \( 0 < \delta < 1, r > d/2 \) and the operator
\[
\mathcal{F} = \mathcal{F}(\omega) := \omega \cdot \partial_{\varphi} + i(\Delta_{\varphi} + V(\varphi)), \quad V \in A_{\delta,r}.
\]
We also assume that the operator \( V \) is self-adjoint. Let us define the diophantine set \( \mathcal{O}_0 \subseteq [1/2, 3/2]^d \) by
\[
\mathcal{O}_0 := \{ \omega \in [1/2, 3/2]^d : |\omega \cdot l| \geq \frac{4\gamma}{|l|^2}, \forall l \in \mathbb{Z}^d \}, \quad \tau := d + 1.
\]

The aim of this section is to prove the following result.

**Theorem 3.1. (Regularization)** Let \( \rho_0 \geq 0, 0 < \delta < 1 \) and \( r_0 > d/2 \). There is \( r_* = r_*(\delta, \rho_0, r_0) \) such that, for \( r > r_* \) and \( S \geq s_0 > n/2 \), there exist \( p = p(S, \rho_0) \geq 1 \) and \( 0 < \varepsilon_* = \varepsilon_*(S, \rho_0) \) such that the following holds. If
\[
\gamma^{-1} N_{\delta,r,p}(V) \leq \varepsilon_*,
\]
then there is, for any \( \varphi \in T^d \), for any \( \omega \in \mathcal{O}_0 \), a bounded and invertible map \( \Phi \in L(H^s, H^s) \) for any \( s \in [s_0, S] \) such that
\[
\mathcal{F}_+ := \Phi \mathcal{F} \Phi^{-1} := \omega \cdot \partial_{\varphi} + i(\Delta_{\varphi} + Z + R),
\]
where \( Z \in A_{\delta}^{\gamma,\mathcal{O}_0} \) is independent of \( \varphi \), \( Z \) is Hermitian and
\[
[Z, K_0] = 0,
\]
\( R(\varphi) \) is a Hermitian \( \rho_0 \)-smoothing operator in \( R_{\rho_0, r_0}^{\gamma,\mathcal{O}_0} \).

Furthermore \( Z = Z_1 + Z_2 \) with \( Z_1 \in A_{\delta} \) is independent of \( \omega \in \mathcal{O}_0 \), and \( Z_2 \in A_{2\delta-1}^{\gamma,\mathcal{O}_0} \).

Moreover the following estimates holds: for any \( s \in [s_0, S] \) there exits \( q = q(s, \rho_0) \geq p \) and \( C = C(s, \rho_0) > 0 \) such that
\[
N_{\delta,s}(Z_1) + N_{\delta,2\delta-1,s}(Z_2) \leq CN_{\delta, r, q}(V),
\]
\[
|R|_{\rho_0, r_0, s} \leq CN_{\delta, r, q}(V),
\]
\[
\sup_{\varphi \in T^d} \| \Phi^{\pm 1} \|_{H^s, H^{s+\delta}} \leq CN_{\delta, r, q}(V),
\]
\[
\sup_{\varphi \in T^d} \| \Phi^{\pm 1} \|_{H^s, H^s} \leq 1 + CN_{\delta, r, q}(V).
\]
As explained in the introduction this Theorem will be demonstrated by an iterative procedure alternating an averaging step according to the periodic flow of $K_0$ (section 3.1) and a step of eliminating the time dependence of the averaged term (section 3.2). The iteration is detailed in section 3.3.

3.1. Averaging procedure. For $A \in \mathcal{A}_m$, $m \in \mathbb{R}$, we denote for $\tau \in [0, 2\pi]$

$$A(\tau) := e^{-i\tau K_0} A e^{i\tau K_0}$$

(3.10)

and

$$\langle A \rangle := \int_0^{2\pi} A(\tau) d\tau,$$

(3.11)

the average of $A$ along the flow of $K_0$.

We notice that $\langle A \rangle$ belongs to $\mathcal{A}_m$ and if $A$ is Hermitian then $\langle A \rangle$ is Hermitian. Let $O \subset O_0$ (see (3.2)) and consider the operator

$$G = \omega \cdot \partial_\varphi + i M (\varphi); \quad M (\varphi) := \Delta_g + W + A (\varphi) + R (\varphi)$$

(3.12)

where $W \in A_0^\gamma, 0 < \delta < 1$, is independent of time and commutes with $K_0, A \in A_{\delta', r}^\gamma$ for some $\delta' \leq \delta$ and $R (\varphi) \in R_{\rho, r}^\gamma$ (see Def. 2.4). We also assume that $M (\varphi)$ is Hermitian $\forall \varphi \in \mathbb{T}^d$.

**Lemma 3.2.** Let $r > d/2, 0 < \delta < 1, \delta' \leq \delta$ there exists $S \in A_{\delta', r}^\gamma$ such that for any $n > n/2$ and $\rho \geq 0$ there exists $p = p(s, \rho) \geq 1$, an increasing function of $s$, and $0 < \varepsilon_0 = \varepsilon_0 (s, \rho) \rho$ such that if

$$\gamma^{-1} N_{\delta', r, p}^\gamma (A) \leq \varepsilon_0, \quad \gamma^{-1} N_{\delta', r, p}^\gamma (W) \leq 1$$

(3.13)

the symplectic change of variable $\Phi_S = e^{i S (\varphi)}$ belongs to $\mathcal{L} (H^s, H^s)$ and we have

$$G^+ := \Phi_S \circ G \circ \Phi_S^{-1} = \omega \cdot \partial_\varphi + i M^+ (\varphi)$$

(3.14)

$$M^+ (\varphi) := \Delta_g + W + \langle A (\varphi) \rangle + A^+ (\varphi) + R^+ (\varphi)$$

(3.15)

where $\langle A (\varphi) \rangle$ is defined as in (3.11), $A^+ \in A_{\delta', r}^\gamma$ and $R^+ \in R_{\rho, r}^\gamma$. The operator $M^+ (\varphi)$ is Hermitian $\forall \varphi \in \mathbb{T}^d$.

Moreover there exists $C = C (s, \rho)$ such that

$$N_{\delta', r, p, s}^\gamma (S) \leq C N_{\delta', r, p}^\gamma (A)$$

(3.16)

$$\sup_{\varphi \in \mathbb{T}^d} \| \Phi_S (\varphi) \|_{\mathcal{L} (H^s, H^s)} \leq 1 + C N_{\delta', r, p}^\gamma (A)$$

(3.17)

$$\sup_{\varphi \in \mathbb{T}^d} \| \Phi_S (\varphi) - \text{Id} \|_{\mathcal{L} (H^s, H^s)} \leq C N_{\delta', r, p}^\gamma (A) \quad \forall \tau \in [0, 1].$$

(3.18)

$$N_{\delta', r, p, s}^\gamma (A^+) \leq C N_{\delta', r, p}^\gamma (A)$$

(3.19)

$$\| R^+ |_{\rho, r, s} \| \leq C \| R |_{\rho, r, s} \| + C N_{\sigma, \rho}^\gamma (A).$$

(3.20)

**Proof.** The idea comes from [Wein77, CdV79] and were extensively used in [BGRM19]. It consists to average with respect to the flow of $K_0$ (see (3.11)) which is periodic since its spectrum is included in $\mathbb{N} + \lambda$ (see (2.1)).

Let us define $Y = \frac{1}{2\pi} \int_0^{2\pi} \tau (A - \langle A \rangle) (\tau) d\tau$. Then $Y \in A_{\delta', r}^\gamma$ and by integration by parts we verify that $Y$ solves the homological equation

$$i[K_0, Y] = A - \langle A \rangle.$$ 

(3.21)

Then we define

$$S = \frac{1}{4} (Y K_0^{-1} + K_0^{-1} Y).$$

(3.22)
and we note that $S \in A_{J-1,r}^{\rho}$ is a pseudo-differential operator of order $\delta - 1 \leq 0$. Moreover, by using Lemma 2.6 we deduce the estimate (3.16). By applying Lemma A.6 we obtain estimates (3.17) and (3.18) (see (A.16), (A.17)). By an explicit computation we also get

$$i[K^2, S] = A - \langle A \rangle - \frac{1}{4}[[A, K_0], K_0^{-1}] .$$

(3.23)

To study the conjugate of $L$ in (3.12) under the map $\Phi_S$ defined as in (2.46) with $S$ in (3.22) we use the Lie expansions (2.48) and (2.49) for some $q \in \mathbb{N}$ large to be chosen later. Recalling the splitting (2.1)-(2.2) we have by (2.48)

$$\Phi_S \circ M \circ \Phi_S^{-1} \equiv iK^2 + iQ_0 + iW + i(A) + \frac{i}{4}[[A, K_0], K_0^{-1}]$$

(3.24)

$$- i[Q_0 + W + A, iS] + \sum_{j=2}^q \frac{1}{j!} \text{ad}^j_{iS}(i\Delta_g + iW + iA)$$

(3.25)

$$+ \frac{1}{q!} \int_0^1 (1 - \tau)^{q}e^{i\tau S}\text{ad}^{q+1}_{iS}(i\Delta_g + iW + iA)e^{-i\tau S}d\tau$$

(3.26)

$$+ i\Phi_S \circ R \circ \Phi_S^{-1} .$$

(3.27)

Taking into account the time contribution given by (2.49) we obtain that the conjugate $\Phi_S \circ G \circ \Phi_S^{-1}$ has the form (3.14)-(3.15) where

$$iA^+ = \frac{i}{4}[[A, K_0], K_0^{-1}] - i[Q_0 + W + A, iS]$$

(3.28)

$$+ \sum_{j=2}^q \frac{1}{j!} \text{ad}^j_{iS}(i\Delta_g + iW + iA) - \sum_{p=1}^q \frac{1}{p!} \text{ad}^{p-1}_{iS}(i\omega \cdot \partial_\varphi S)$$

and

$$iR^+ = \frac{1}{q!} \int_0^1 (1 - \tau)^{q}e^{i\tau S}\text{ad}^{q+1}_{iS}(i\Delta_g + iW + iA)e^{-i\tau S}d\tau$$

(3.29)

$$+ \frac{1}{q!} \int_0^1 (1 - \tau)^{q}\Phi_S^q\text{ad}^{q}_{iS}(i\omega \cdot \partial_\varphi S)\Phi_S^{-q}d\tau ,$$

$$+ i\Phi_S \circ R \circ \Phi_S^{-1} .$$

We need to prove the bounds (3.19)-(3.20). We start by studying the remainder $R^+$ in (3.29). To simplify the notation we shall write $a \lesssim b$ to denote $a \leq Cb$ for some constant $C = C(s, \rho)$.

Using the smallness condition (3.13), we have that the third summand in (3.29) is a $\rho$-smoothing operator satisfying (3.20) by Lemma A.8.

By items (ii), (iii) of Lemma 2.6 we have (up to smoothing remainder and for some $p$ depending on $s$ and $\rho$)

$$N_{\delta', r, s}^{\gamma, O}(\text{ad}_{iS}(i\Delta_g + iW + iA)) + N_{\delta' - 1, r - 1, s}^{\gamma, O}(\omega \cdot \partial_\varphi S) \lesssim N_{\delta', r, p}^{\gamma, O}(A) .$$

(3.30)

By iterating the estimate above and using the smallness condition (3.13) we deduce, for $1 \leq j \leq q$ and for some $p$ depending on $s, \rho, q$,

$$N_{\delta' - 2(j-1), r, s}^{\gamma, O}(\text{ad}_{iS}(i\Delta_g + iW + iA)) + N_{(j+1)\delta' - 1 - 2j, r - 1, s}^{\gamma, O}(\text{ad}_{iS}(\omega \cdot \partial_\varphi S)) \lesssim N_{\delta', r, p}^{\gamma, O}(A) .$$

(3.31)
The sequences $j\delta' - 2(j - 1)$ and $(j + 1)\delta' - 1 - 2j$ are decreasing since $\delta' \leq 1$. Hence, by choosing $q$ large enough, the integrands in (3.29) are $q$-smoothing operator (with arbitrary $q$) conjugated by the flow $e^{i\tau S}$. Therefore by Lemma A.8 all the expressions in (3.29) are smoothing remainders satisfying (3.20) for some $p$ depending on $s$ and $\rho$.

Let us now consider the terms in (3.28). First of all we have

\[
\mathcal{N}_{\delta' - 2, r, s}([A, K_0], K_0^{-1}) \lesssim \mathcal{N}_{\delta', r, N_1}([A, K_0])\mathcal{N}_{-1, N_1}(K_0^{-1})
\]

\[
\lesssim \mathcal{N}_{\delta', r, 0}(A)\mathcal{N}_{1, N_1}(K_0)\mathcal{N}_{-1, N_1}(K_0^{-1})
\]

\[
\lesssim \mathcal{N}_{\delta', r, 0}(A),
\]

for some constant $N_1 \leq N \leq p$ depending only on $s, \rho$. In the same way (recalling also (3.13)) we have

\[
\mathcal{N}_{\delta' + \delta' - 2, r, s}((Q_0 + W + A, iS)) \lesssim \mathcal{N}_{\delta' - 1, r, p}(S)(\mathcal{N}_{0, p}(Q_0) + \mathcal{N}_{\delta', p}(W))
\]

\[
+ \mathcal{N}_{\delta' - 1, r, p}(S)\mathcal{N}_{\delta', r, p}(A)
\]

\[
\lesssim \mathcal{N}_{\delta', r, p}(A).
\]

The other summands in (3.28) can be estimated by using (3.30) and (3.31). This proves the (3.19). \qed

3.2. Time elimination. Let us consider the operator $L^+$ in (3.14)-(3.15) obtained after an average step (see Lemma 3.2). The aim of this section is to eliminate the time dependence (i.e. the dependence with respect to $\varphi$) in the term $\langle A(\varphi) \rangle$ in (3.15). First we introduce the pseudo-differential operator $T = T(\varphi)$ defined as

\[
T(\varphi) = \sum_{0 \neq l \in \mathbb{Z}^d} \frac{e^{i\varphi}}{\omega \cdot l} \langle A(l) \rangle.
\]

We have the following Lemma.

**Lemma 3.3.** Let $r \geq 5d/2 + 9/2$ and $\omega \in \mathcal{O}_0$ (see (3.2)). Then the operator $T$ in (3.32) belongs to $\mathcal{A}_{\delta', r - (2r + 1)}$ is Hermitian, commutes with the operator $K_0$. Moreover it solves the equation

\[
\langle A(\varphi) \rangle - \omega \cdot \partial_{\varphi} T = \langle A(0) \rangle,
\]

and satisfies

\[
\mathcal{N}_{\delta', r - (2r + 1), s}(T) \leq C\mathcal{N}_{\delta', r, p}(A).
\]

Furthermore, setting $\Phi_T^\tau := e^{i\tau T(\varphi)}$, we have that for any $s > d/2$ there are constants $C, p$ (depending only on $s$ and $\rho$) such that if (3.13) holds then

\[
sup_{\varphi \in \mathbb{T}_d} \|\Phi_T^\tau\|_{\mathcal{L}(H^s, H^s)} \leq 1 + C\mathcal{N}_{\delta', r, p}(A)
\]

\[
(3.35)
\]

\[
sup_{\varphi \in \mathbb{T}_d} \|\Phi_T^\tau - \text{Id}\|_{\mathcal{L}(H^s, H^{s - \delta'})} \leq C\mathcal{N}_{\delta', r, p}(A) \quad \forall \tau \in [0, 1].
\]

(3.36)

**Proof.** The operator $T$ is Hermitian and commutes with $K_0$ thanks to the properties of $\langle A \rangle$. The fact that $T$ solves (3.33) is obtained by an explicit computation. The bound (3.34) follows by item (iii) of Lemma 2.6. Finally applying Lemma A.6 we obtain the estimates (3.35)-(3.36) (see (A.18) and (A.19)). \qed

In the following lemma we study how the operator $G_+^+$ in (3.14)-(3.15) changes under the map $\Phi_T^\tau$ defined by Lemma 3.3. We have to distinguish the cases $\delta'$ strictly positive or $\delta'$ less or equal zero.
Lemma 3.4. Let $\delta' \leq 0$ and $r > 2\tau + 2 + d/2$. Let us define $\delta_1 := \delta + \delta' - 1$ and $\Phi_T := \Phi_1^T$. Then the conjugated operator $G_1 := \Phi_T \circ G^+ \circ \Phi_T^{-1}$ has the form

$$G_1 = \omega \cdot \partial_\varphi + iM_1(\varphi)$$

(3.37)

where

$$M_1(\varphi) := \Delta_g + W_1 + A_1(\varphi) + R_1(\varphi)$$

(3.38)

where

$$W_1 = W + \int_{\mathbb{T}^d} \langle A(\varphi) \rangle d\varphi,$$

(3.39)

is independent of $\varphi \in \mathbb{T}^d$, $A_1 \in \mathcal{A}^\gamma_{\delta_1,r-2\tau-2}$ and $R_1 \in \mathcal{R}^\gamma_{\rho,r-2\tau-2}$. The operator $M_1(\varphi)$ is Hermitian $\forall \varphi \in \mathbb{T}^d$.

Moreover for any $s > d/2$ there exist $p = p(s, \rho)$ and $C = C(s, \rho)$ such that if (3.13) holds then

$$|R_1|^{\gamma^O_{\delta_1,r-2\tau-2,s}}(A_1) \leq C|N^{\gamma^O_{\delta_1,r-2\tau-2,s}}(A)| + N^{\gamma^O_{\delta_1,r,s}}(A).$$

(3.40)

$$|R_1|^{\gamma^O_{\delta_1,r-2\tau-2,s}}(A_1) \leq C|N^{\gamma^O_{\delta_1,r-2\tau-2,s}}(A)| + N^{\gamma^O_{\delta_1,r,s}}(A).$$

(3.41)

Proof. Notice that, since $T$ in (3.32) commutes with $K_0$, then $\Phi_T \circ K_0^2 \circ \Phi_T^{-1} = K_0^2$. By using the expansions (2.48)-(2.49) and since $T$ solves (3.33) we have that the conjugate $L_1$ has the form (3.37)-(3.38) with $W_1$ as in (3.39) and

$$iA_1 := iA^+ + \sum_{j=1}^q \frac{1}{j!} \text{ad}_{IT}^j(iQ_0 + iW + i\langle A(\varphi) \rangle + iA^+)$$

(3.42)

$$- \sum_{j=2}^q \frac{1}{j!} \text{ad}_{IT}^{j-1}(i\omega \cdot \partial_\varphi T),$$

$$iR_1 := i\Phi_T \circ R^+ \circ \Phi_T^{-1}$$

$$+ \frac{1}{q!} \int_0^1 (1 - \tau)^q \Phi_T^{-1} \text{ad}_{IT}^{q+1}(iQ_0 + iW + i\langle A(\varphi) \rangle + iA^+) \Phi_T \tau d\tau$$

(3.43)

and where $q \in \mathbb{N}$ is a large constant to be chosen later. We now estimate the different terms in (3.42), (3.43).

By (2.39) we have for some $p' = p'(s, \rho)$

$$\mathcal{N}^{\gamma^O_{\delta_1,\rho-2\tau-2,s}}(\text{ad}_{IT}(i\omega \cdot \partial_\varphi T)) \lesssim \mathcal{N}^{\gamma^O_{\delta_1,\rho-2\tau-2,s'}}(T) \mathcal{N}^{\gamma^O_{\delta_1,\rho-2\tau-2,s'}}(\omega \cdot \partial_\varphi T).$$

On the other hand we have by (2.40)

$$\mathcal{N}^{\gamma^O_{\delta_1,\rho-2\tau-2,s'}}(\omega \cdot \partial_\varphi T) \lesssim \mathcal{N}^{\gamma^O_{\delta_1,\rho,1-\rho'}}(T),$$

thus using (3.44) we deduce

$$\mathcal{N}^{\gamma^O_{\delta_1,\rho-2\tau-2,s}}(\text{ad}_{IT}(i\omega \cdot \partial_\varphi T)) \lesssim (\mathcal{N}^{\gamma^O_{\delta_1,\rho}}(A))^2 \lesssim \mathcal{N}^{\gamma^O_{\delta_1,\rho}}(A).$$

(3.44)

Similarly we prove

$$\mathcal{N}^{\gamma^O_{\delta_1,\rho-2\tau-2,s}}(\text{ad}_{IT}(Q_0)) + \mathcal{N}^{\gamma^O_{\delta_1,\rho-2\tau-2,s}}(\text{ad}_{IT}(W + \langle A(\varphi) \rangle))$$

(3.45)

$$+ \mathcal{N}^{\gamma^O_{\delta_1,\rho-2\tau-2,s}}(\text{ad}_{IT}(A^+)) \lesssim \mathcal{N}^{\gamma^O_{\delta_1,\rho}}(A).$$

Notice that, since $0 < \delta < 1$, the highest order pseudo-differential operator among the ones estimated in (3.44), (3.45) is the one of order $\delta + \delta' - 1 < \delta'$. By the estimates above, by choosing the constant $q \in \mathbb{N}$
large enough with respect to \( \rho \) and by reasoning as in the proof of Lemma 3.2 one gets the estimates (3.40), (3.41). In particular, since \( \delta' \leq 0 \) we shall use Lemma A.8 in order to estimate the conjugates of smoothing operator under the flow \( \Phi_T \).

In the next Lemma we study the case in which the generator \( T \) of Lemma 3.3 has order \( \delta' > 0 \).

**Lemma 3.5.** Let \( 0 < \delta' \leq \delta \). Let us define \( \delta_1 := \delta + \delta' - 1 \) and \( \Phi_T := \Phi_T^{\delta_1} \). Fix moreover \( r_1 > d/2 \) and \( \rho_1 \geq 0 \) and assume \( r > \max(r_1 + d/2, 2r + 2 + d/2) \) and \( \rho \geq \rho_1 + \delta' r_1 + 1 \). Then the conjugated operator \( G_1 := \Phi_T \circ G^+ \circ \Phi_T^{-1} \) (see (3.14)) has the form (3.37), (3.38), (3.39), is independent of \( \varphi \in T^d \), \( A_1 \in A_{\delta_1,r-2r-2}^{\gamma_1,O} \) and \( R_1 \in R_{\rho_1,r_1}^{\gamma_1,O} \). The operator \( M_1(\varphi) \) is Hermitian \( \forall \varphi \in T^d \).

Moreover for any \( s \in \mathbb{R} \) there exist \( p = p(s, \rho) \) and if (3.13) holds then \( C = C(s, \rho) \) such that

\[
\begin{align*}
N_{\delta_1,r-2r-2,s}^\gamma(O)(A) & \leq C N_{\delta,s}^\gamma(O)(A) \quad (3.46) \\
| R_1 |^p_{\rho_1,r_1,s} & \leq C | R |^p_{\rho,r,s} + N_{\delta,\rho,p}^\gamma(O) \quad (3.47)
\end{align*}
\]

**Proof:** One reasons as in the proof of Lemma A.8. The difference is in estimating the remainder \( R_1 \) in (3.43). Since the generator \( T \) is of order \( \delta' > 0 \) one has to apply Lemma A.7 (instead of Lemma A.8) which provides estimates (3.47) instead of the (3.41).

---

3.3. **Proof of Theorem 3.1.** In this section we give the proof of Theorem 3.1 which is based on an iterative application of Lemmata of the previous section. Recalling (3.1) we set

\[ G_0 := \mathcal{F} = \omega \cdot \partial \varphi + i \Delta_g + i V. \]

The operator \( G_0 \) above has the form (3.12) with

\[
\begin{align*}
\mathcal{O} & = \mathcal{O}_0, \quad W = 0, \quad R = 0, \quad \mathcal{A}(\varphi) = V(\varphi), \quad \delta' = \delta.
\end{align*}
\]

Since \( V \) is \( C^\infty \), \( r > d/2 \) can be chosen arbitrary large. We will chose it later in function of the order \( \delta \), of the final regularity \( r_0 \) and the smoothness \( \rho_0 \) prescribed by (3.7).

Lemma 3.2 provides \( p_1(S) \) such that if \( p \geq p_1(S) \) in (3.3) then (3.13) holds for any \( s \in [s_0, S] \). By applying Lemma 3.2 to \( G_0 \) we obtain a symplectic map \( \Phi_{S_0} \) such that (see (3.14))

\[
\begin{align*}
\tilde{G}_0 & := \Phi_{S_0} \circ G_0 \circ \Phi_{S_0}^{-1} = \omega \cdot \partial \varphi + i \Delta_g + i \langle V(\varphi) \rangle + i \tilde{A}_0 + i \tilde{R}_0, \\
\tilde{A}_0 & \in A_{\delta_1-1,r-1}^{\gamma_1,O}, \quad \tilde{R}_0 \in R_{\rho,r}^{\gamma_1,O},
\end{align*}
\]

with \( \rho > 0 \) arbitrary to be chosen later and where \( \langle V(\varphi) \rangle \) is defined as in (3.11). We apply Lemma 3.5 to the operator given by (3.49) with \( \rho_1 \sim \rho_0 \) of Theorem 3.1 and \( r_1 > d/2 \) (to be chosen later) provided that \( \rho \) and \( r \) are sufficiently large \((\rho > \rho_0 + \delta r_1 + 1 \text { and } r > \max(r_1 + d/2, 2r + 2 + d/2))\). Hence we obtain a symplectic map \( \Phi_{T_0} \)

\[
\begin{align*}
G_1 & := \Phi_{T_0} \circ \tilde{G}_0 \circ \Phi_{T_0}^{-1} = \Phi_{T_0} \circ \Phi_{S_0} \circ G_0 \circ \Phi_{S_0}^{-1} \circ \Phi_{T_0}^{-1} \\
& = \omega \cdot \partial \varphi + i \Delta_g + i W_1 + i A_1 + i R_1
\end{align*}
\]

with \( W_1 := \int_{T^d} \langle V(\varphi) \rangle d\varphi \), \( A_1 \in A_{\delta_1-1,r_1}^{\gamma_1,O} \), \( R_1 \in R_{\rho,r_1}^{\gamma_1,O} \) and estimates (3.46), (3.47) are satisfied for all \( s \in [s_0, S] \) provided \( p \geq p_2(p_1, S) \) depending on \( p_1 \) and \( S \) (and still increasing in \( S \)).

We notice that \( W_1 \) is independent of \( \varphi \) and of the parameters \( \omega \in \mathcal{O}_0 \).

Now we want to iterate this procedure.

Let us first consider the case \( 0 < \delta \leq 1/2 \). Then \( 2\delta - 1 \leq 0 \) and hence, form now on, we will apply iteratively Lemmata 3.2 and 3.4 (instead of Lemma 3.5).

\footnote{Actually Theorem 3.1 will be applied only in this case.}
We introduce the following parameters: for \( n \geq 1 \) we set
\[
\delta_n = (n+1)\delta - n, \quad r_n = r_1 - n(2\tau + 2), \quad q_n = q_0 \circ q_{n-1}
\] (3.50)
where \( q_0(\cdot) = q_1(\cdot, S) \) is the composition of the two functions \( s \mapsto p(s) \) given by Lemmata 3.2 and 3.4 and \( q_1 = p_2 \circ p_1 \). We notice that \( q_n \) is an increasing function of \( S \).

Then applying Lemmata 3.2 and 3.4 iteratively, there exist symplectic changes of variables \( \{\Phi_{S_n}\}_n \) and \( \{\Phi_{T_n}\}_n \) such that, setting \( \Phi_n := \Phi_{T_n} \circ \Phi_{S_n} \), we have
\[
G_{n+1} := \Phi_n \circ G_n \circ \Phi_n^{-1} = \omega \cdot \partial_\varphi + i\Delta_g + iW_{n+1} + A_{n+1} + R_{n+1}
\] (3.51)
where \( W_n \) is pseudo-differential operator independent of \( \varphi \) of order \( \delta \) commuting with \( K_0; A_n \) is pseudo-differential operator of order \( \delta_n \); \( R_n \) is \( \rho_0 \)-smoothing operator. Moreover, by estimates (3.19), (3.20) and (3.40), (3.41) we get
\[
\mathcal{N}_{r,\delta,\varphi}^0(W_n) + \mathcal{N}_{r,\delta,\varphi}^0(A_n) + |R_n|_{\rho_0,\varphi} \leq C N_{\delta,r,q_n}(V) \quad \text{for all } s \in [s_0, S].
\] (3.52)

We perform \( N = N(\rho_0, \delta) \) steps of this procedure in order to get \( \delta_N = \delta - N(1 - \delta) \leq -\rho_0 \). This require to choose \( r_1 \) (and hence \( r \)) sufficiently large. More precisely, we want \( r_N \geq r_0 \), the prescribed regularity, and thus in view of (3.50) \( r_1 \geq N(2\tau + 2) + r_0 \). Then recalling that we need \( r > \max(r_1 + d/2, 2\tau + 2 + d/2) \) we have to chose
\[
r > \max(N(2\tau + 2) + r_0 + d/2, 2\tau + 2 + d/2) := r_\ast(\delta, \rho_0, r_0).
\]
Moreover the constant \( \rho \) appearing in (3.49) should be chosen in such a way
\[
\rho \geq \rho_0 + \delta r_1 + 1 \geq \rho_0 + 2N(\tau + 1) + r_0.
\]
Therefore the operator \( G_N \), defined as in (3.51), has the form (3.4) with \( Z := W_N \). We notice that \( W_1 := \int_V d\varphi \in A_\delta \) is independent of \( \varphi \) and that \( W_N - W_1 \in \mathcal{A}_{\delta,1}^{\gamma_0} \), which leads to the desired splitting \( Z = Z_1 + Z_2 \). The bounds (3.6), (3.7) follows by (3.52) with \( q = q_N \). The estimates (3.8), (3.9) follows by composition and estimates (3.17), (3.18), (3.35) and (3.36).

The case \( 1/2 \leq \delta < 1 \) requires to apply Lemmata 3.2 and 3.3 iteratively to construct \( \tilde{A}_n \in \mathcal{A}_{\delta_n,\tilde{r}_n} \) with \( \delta_n = 2\delta_{n-1} - 1 \) and \( \delta_0 = \delta \), until \( \delta_n \) became negative. Then we can apply the second procedure using Lemmata 3.2 and 3.4 as in the previous case.

4. KAM Reducibility

In this section we will prove an abstract KAM Theorem for a matrix operator of the form
\[
L_0 = L_0(\varphi; \varphi) := \omega \cdot \partial_\varphi + i(\Delta_g + Z_0 + R_0(\varphi)).
\] (4.1)

To precise our hypothesis on \( L_0 \) we define the following constants
\[
b := 6d + 15n + 23, \quad \tau = d + 1, \quad \rho = 5n + 3,
\]
\[
\gamma \in (0, 1), \quad 0 \leq \kappa \leq 1 \quad s_0 > \frac{d + n}{2}, \quad S \geq s_0 + b.
\] (4.2)

In this section we assume:

(A1) the matrix \( Z_0 \) is Hermitian, block diagonal, independent of \( \varphi \) and Lipschitz in \( \omega \in \mathcal{O} \subseteq \mathcal{O}_0 \equiv \mathcal{O}_0(\gamma, \tau) \) (see (3.2)). Furthermore, denoting \( (\mu_{k,j}^{(0)})_{j=1,\ldots,d_k} \) the eigenvalues of the block \( (Z_0)^{(k)}_{[k]} \), we assume that there exists \( \kappa \geq 0 \) such that (recall that \( c_0 \) is defined in (2.7))
\[
|\mu_{k,j}^{(0)}(\omega)| \leq \frac{c_0}{2}|k|, \quad \omega \in \mathcal{O}, \ k \in \mathbb{N}, \ j = 1, \ldots, d_k,
\] (4.3)
\[
\|Z_0^{(k)}\|_{\mathcal{L}(\ell^2)} \leq \frac{1}{4}(k)^{-\kappa}, \quad k \in \mathbb{N}.
\] (4.4)
Lemma 4.2. The operator $R_0$ is in $\mathcal{M}^{\gamma,\mathcal{O}}_{\rho,s}$ (see Def. 2.12) and is Hermitian.

Let us define
\[ \epsilon := \gamma^{-1}\|R_0\|^{\gamma,\mathcal{O}}_{\rho,s_0+b}. \]  
(4.5)

We shall prove the following.

Theorem 4.1. (Reducibility) Let $s \in [s_0, S-b]$. There exist positive constants $\epsilon_0 = \epsilon_0(s), C = C(s)$ such that, if
\[ \epsilon \leq \epsilon_0, \]  
(4.6)
then there is a set $\mathcal{O}_\epsilon \subseteq \mathcal{O}$ with
\[ \text{meas}(\mathcal{O} \setminus \mathcal{O}_\epsilon) \leq C\gamma \]  
(4.7)
such that the following holds. For any $\omega \in \mathcal{O}_\epsilon$ there are

(i) (Normal form) a matrix $Z_\infty = Z_0 + \hat{Z}_\infty$ with $\hat{Z}_\infty \in \mathcal{M}^{\gamma,\mathcal{O}}_{\rho,s}$ which is $\varphi$-independent, Hermitian and block-diagonal;

(ii) (Conjugacy) a bounded and invertible map $\Phi_\infty = \Phi_\infty(\omega, \varphi) : H^s \to H^s$ such that for all $\varphi \in \mathbb{T}^d$, for all $\omega \in \mathcal{O}_\epsilon$,
\[ L_\infty := \Phi_\infty L_0 \Phi_\infty^{-1} := \omega \cdot \partial_\varphi + i(\Delta_g + Z_\infty). \]  
(4.8)

Moreover we have
\[ \sup_{\varphi \in \mathbb{T}^d} \|\Phi_{\infty}^{\pm 1}(\varphi) - \text{Id}\|_{\mathcal{L}(H^s;H^s)} \leq C\gamma^{-1}\|R_0\|^{\gamma,\mathcal{O}}_{\rho,s+b}, \quad \forall \omega \in \mathcal{O}_\epsilon, \]  
(4.9)
\[ \|\hat{Z}_\infty\|^{\gamma,\mathcal{O}}_{\rho,s} \leq C\|R_0\|^{\gamma,\mathcal{O}}_{\rho,s+b}. \]  
(4.10)

4.1. The KAM step. The proof of Theorem 4.1 is based on an iterative scheme. In this section we show how to perform one step of the iteration. We consider an operator
\[ L := \omega \cdot \partial_\varphi + i(\Delta_g + Z + R), \]  
(4.11)
where $Z = Z_0 + Z_2$ is Hermitian with $Z_0$ satisfying (A1) and $Z_2 \in \mathcal{M}^{\gamma,\mathcal{O}}_{\rho,s}$ for all $s \in [s_0, S]$ and for some $\mathcal{O} \subseteq \mathcal{O}_0$ (see (3.2)). The remainder $R$ satisfies (A2), i.e. belongs to $\mathcal{M}^{\gamma,\mathcal{O}}_{\rho,s}$ for all $s \in [s_0, S]$ and is Hermitian.

4.1.1. Control of the small divisors. Let us denote by $\mu_{k,j}, k \in \mathbb{N}$ and $j = 1, \ldots, d_k$ (see 2.3), the eigenvalues of the block $(\Delta_g + Z)^{\lfloor k \rfloor}_{\lfloor k \rfloor}$. First of all we prove the following.

Lemma 4.2. One has
\[ \sup_{k \in \mathbb{N}} \langle k \rangle^n \mu_{\lfloor k \rfloor}^{\text{lip},\mathcal{O}} \leq \frac{1}{4} + [Z_2]^{\text{lip},\mathcal{O}}_{\kappa,s_0}. \]  
(4.12)

Proof: By Corollary A.7 in [FG19] the Lipschitz variation of the eigenvalues of an Hermitian matrix is controlled by the Lipschitz variation of the matrix. Then, in view of hypothesis (A1), we get
\[ |\mu_{\lfloor k \rfloor}^{\text{lip},\mathcal{O}}_0| \leq \|Z_0\|_{\lfloor k \rfloor}^{\text{lip},\mathcal{O}}_{\mathcal{L}(L^2)} + \|Z_2\|_{\lfloor k \rfloor}^{\text{lip},\mathcal{O}}_{\mathcal{L}(L^2)} \leq \langle k \rangle^{-\kappa}\left(\frac{1}{4} + [Z_2]^{\text{lip},\mathcal{O}}_{\kappa,s_0}\right) \]  
and the (4.12) follows. \(\square\)
We define the set $\mathcal{O}_+ \subseteq \mathcal{O}$ of parameters $\omega$ for which we have a good control of the small divisors. We set, for $N \geq 1$,

$$\mathcal{O}_+ \equiv \mathcal{O}_+ (\gamma, N) := \left\{ \omega \in \mathcal{O} : |\omega \cdot l + \mu_{k,j} - \mu_{k',j'}| \geq \frac{2\gamma}{N\tau(k,k')^{2n+2}}, \right.$$  

$$|l| \leq N, \quad k, k' \in \mathbb{N}, \quad j = 1, \ldots, d_k, \quad j' = 1, \ldots, d_{k'}, \quad (l, k, k') \neq (0, k, k) \right\}. \quad (4.13)$$

We have the following.

**Lemma 4.3.** Assume that

$$[Z_2]_{\kappa,0}^{\mathcal{O},\infty}b \leq \gamma/8 \quad (4.14)$$

for some $0 < \gamma \leq \frac{c_0}{8}$ (see (2.7)) then we have

$$\text{meas}(\mathcal{O} \setminus \mathcal{O}_+ (\gamma, N)) \leq C\gamma N^{-1} \quad (4.15)$$

for some constant $C > 0$ depending only on $d$.

**Proof.** We write

$$\mathcal{O} \setminus \mathcal{O}_+ = \bigcup_{l \in \mathbb{Z}^d, |l| \leq N} \bigcup_{k, k' \in \mathbb{N}} R_{l;k,k'}^{j,j'}$$

where

$$R_{l;k,k'}^{j,j'} := \left\{ \omega \in \mathcal{O} : |\omega \cdot l + \mu_{k,j} - \mu_{k',j'}| \leq \frac{2\gamma}{N\tau(k,k')^{2n+2}} \right\}.$$  

Notice that when $l = 0$ and $k \neq k'$ then $R_{l;k,k'}^{j,j'} = \emptyset$ for all $j, j'$. Indeed in such case we get using (4.13), (2.7) and (4.14)

$$|\lambda_k + \mu_{k,j} - \lambda_{k'} + \mu_{k',j'}| \geq \frac{c_0}{2} (k + k') - 2[Z_2]_{\kappa,0}^{\mathcal{O},\infty}b \geq \frac{c_0}{2} - \gamma \geq 2\gamma.$$

Let us now consider the case $l \neq 0$. We give the estimate of the measure of a single bad set $R_{l;k,k'}^{j,j'}$. Let us consider the Lipschitz function

$$f(\omega) = \omega \cdot l + \mu_{k,j}(\omega) - \mu_{k',j'}(\omega) = \omega \cdot l + g(\omega).$$

Using condition (4.14) we have that Lemma 4.2 implies that (recall that $l \neq 0$)

$$|g|^2_{lip,\mathcal{O}} \leq \frac{1}{2}.$$

Then Lemma 5.2 in [FG19] implies that $\text{meas}(R_{l;k,k'}^{j,j'}) \leq \frac{C\gamma}{N\tau(k,k')^{2n+2}}$ for some constant $C > 0$ depending only on $d$. Finally by (2.3) we have that

$$d_k d_{k'} \leq (k, k')^{2(n-1)}.$$

Hence

$$\text{meas}(\mathcal{O} \setminus \mathcal{O}_+) \leq C \sum_{l \in \mathbb{Z}^d, 0 < |l| \leq N} \sum_{k, k' \in \mathbb{N}} R_{l;k,k'}^{j,j'} \leq C \sum_{l \in \mathbb{Z}^d, 0 < |l| \leq N} \sum_{k, k' \in \mathbb{N}} 2 \frac{\gamma}{N\tau(k,k')^{2n+2}} \leq C \gamma N^{-1},$$

since $\tau = d + 1$. \qed
4.1.2. Resolution of the Homological equation. In this section we solve the following homological equation equation

\[-i\omega \cdot \partial_{\varphi} S + [iS, \Delta_g + Z] + R = \text{Diag} R + Q\]  \hspace{1cm} (4.16)

where \(Q\) is some remainder to be determined and

\[
\text{Diag} R = \left( (\text{Diag} R)^{[k']^{[k]} (l)} \right)_{l \in \mathbb{Z}^d, k, k' \in \mathbb{N},},
\]

\[
(\text{Diag} R)^{[k']^{[k]} (l)} := 0 \quad \text{for } l \neq 0, k, k' \in \mathbb{N} \text{ or } l = 0, k \neq k',
\]  \hspace{1cm} (4.17)

\[
(\text{Diag} R)^{[k']^{[k]} (0)} := A_{[k']^{[k]} (0)}, \quad \text{otherwise}.
\]

Lemma 4.4. (Homological equation) Let \(R \in \mathcal{M}_{\rho, s}^{\gamma, O} \) for \(s \in [s_0, S], \rho \) in (4.2). For any \(\omega \in \mathcal{O}_+ \equiv \mathcal{O}_+ (\gamma, N)\) (defined in (4.13)) there exist Hermitian matrices \(S, Q\) solving equation (4.16) and satisfying

\[
[S]_{\rho, s} = \frac{N^{2\gamma + 1}}{\gamma} [R]_{\rho, s}^{\gamma, O},
\]

\[
[D^{\pm \rho} SD^{\mp \rho}]_{\rho, s} = \frac{N^{2\gamma + \rho + 1}}{\gamma} [R]_{\rho, s}^{\gamma, O},
\]

\[
[Q]_{\rho, s} = [R]_{\rho, s}^{\gamma, O} N^{-b},
\]

\[
[Q]_{\rho, s}^{\gamma, O} = [R]_{\rho, s}^{\gamma, O}.
\]  \hspace{1cm} (4.18)

Proof. For \(N > 0\) we define (recall (2.54)) the matrix \(\Pi_N R\) as

\[
(\Pi_N R)^{[k']^{[k]} (l)} := \begin{cases} R^{[k']^{[k]} (l)}, & l \in \mathbb{Z}^d, k, k' \in \mathbb{N}, \quad |l| \leq N, \\ 0, & \text{otherwise} \end{cases}
\]  \hspace{1cm} (4.20)

Then we set

\[
Q = (1 - \Pi_N) R \hspace{1cm} (4.21)
\]

By Lemma [A.4] and since the regularity in \(\varphi\) has been fixed at \(r = b\), one deduces the estimates (4.19). Moreover, recalling (4.17), we have that equation (4.16) is equivalent to

\[
\mathcal{G}(l, k, k', \omega) S^{[k']^{[k]} (l)} + (\Pi_N R)^{[k']^{[k]} (l)} = 0
\]  \hspace{1cm} (4.22)

for any \(l \in \mathbb{Z}^d, k, k' \in \mathbb{N}\) with \((l, k, k') \neq (0, k, k)\) where the operator \(\mathcal{G}(l, k, k', \omega)\) is the linear operator acting on complex \(d_k \times d_k\)-matrices as

\[
\mathcal{G}(l, k, k', \omega) A := -i \left[ \omega \cdot l + (\Delta_g + Z)^{[k']^{[k]} (l)} \right] A + i A (\Delta_g + Z)^{[k']^{[k]} (l)}.
\]  \hspace{1cm} (4.23)

Now, since \((\Delta + Z)^{[k]}_{[k]}\) is Hermitian, there is a orthogonal \(d_k \times d_k\)-matrix \(U_{[k]}\) such that

\[
U_{[k]}^T (\Delta_g + Z)^{[k]}_{[k]} U_{[k]} = D_{[k]} := \text{diag}_{j=1, \ldots, d_k} \mu_{k, j},
\]

where \(\mu_{k, j}\) are the eigenvalues of the \(k\)-th block. By setting

\[
\widehat{S}^{[k']^{[k]} (l)} := U_{[k]}^T S^{[k']^{[k]} (l)} U_{[k]}, \quad \widehat{R}^{[k']^{[k]} (l)} := U_{[k]} R^{[k']^{[k]} (l)} U_{[k]}
\]

equation (4.22) reads

\[
-i \left( \omega \cdot l + D_{[k]} \right) \widehat{S}^{[k']^{[k]} (l)} + i \widehat{S}^{[k']^{[k]} (l)} D_{[k]} + (\Pi_N R)^{[k']^{[k]} (l)} = 0.
\]  \hspace{1cm} (4.24)
For $\omega \in \mathcal{O}_+$ (see (4.13)) the solution of (4.24) is given by (recalling the notation (2.52))

$$
\tilde{S}_{k,j}^{k',j'}(l) :=
\begin{cases}
-\frac{i \tilde{R}_{k,j}^{k',j'}(l)}{\omega \cdot l + \mu_{k,j} - \mu_{k',j}'}, & |l| \leq N, \\
0, & |k - k'| \leq N, (l,k,k') \neq (0,k,k'),
\end{cases}
$$

(4.25)

Since $R$ is Hermitian it is easy to check that also $S$ is Hermitian. Using the (4.12), (4.4) we deduce

$$
\|S\|_{\infty} \leq \gamma^{-1} ||R||_{\infty}^{2} N^{2n+2}.
$$

(4.26)

Then, by denoting by $\|\cdot\|_\infty$ the sup-norm of a $d_k \times d_k'$-matrix, we deduce

$$
\|S^{[k']}_{[k]}(l)\|_{\mathcal{L}(L^2)} = \|\tilde{S}^{[k']}_{[k]}(l)\|_{\mathcal{L}(L^2)} \leq \sqrt{d_k d_k'} \|\tilde{S}^{[k']}_{[k]}(l)\|_\infty
\leq \gamma^{-1} \|R^{[k']}_{[k]}(l)\|_{\mathcal{L}(L^2)} N^{2n+2}.
$$

(4.27)

We now estimate the decay norm of the matrix $S$. We have

$$
\|S\|_s \leq \gamma^{-2} N^{2\tau} \sum_{l,h} \langle l, h \rangle^{2s} \sup_{|k-k'|=h} \|R^{[k']}_{[k]}(l)\|_{\mathcal{L}(L^2)}^2 (k,k') 6n+2
\leq \gamma^{-2} N^{2\tau} \sum_{l,h} \langle l, h \rangle^{2s} \sup_{|k-k'|=h} \|\rho R^{[k']}_{[k]}(l)\|_{\mathcal{L}(L^2)}^2 (k,k') 6n+2 - 2\rho
$$

(4.28)

$$
+ \gamma^{-2} N^{2\tau} \sum_{l,h} \langle l, h \rangle^{2s} \sup_{|k-k'|=h} \|\rho R^{[k']}_{[k]}(l)\|_{\mathcal{L}(L^2)}^2 (k,k') 6n+2 - 2\rho
$$

$$
\leq s \gamma^{-2} N^{2\tau} \|R\|_{\rho,s}^2,
$$

provided that $\rho \geq 3n+1$ which is true thanks to the choices in (4.2). Hence the bound (A.3) in Lemma A.1 implies

$$
\|D^{\pm \rho} S D^{\mp \rho}\|_s \leq s \gamma^{-1} N^{\tau + \rho} \|R\|_{\rho,s}.
$$

(4.29)

To obtain (4.18), it remains to estimate the Lipschitz variation of the matrix $S$. We reason as in the proof of item (iii) of Lemma 2.6. To simplify the notation, for any $l \in \mathbb{Z}^d$, $k,k' \in \mathbb{Z}$, $j = 1, \ldots, d_k$ and $j' = 1, \ldots, d_{k'}$, we set

$$
d(\omega) := i(\omega \cdot l + \mu_{k,j}(\omega) - \mu_{k',j'}(\omega)), \quad \forall \omega \in \mathcal{O}_+.
$$

(4.30)

By (4.25) we have that, for any $\omega_1, \omega_2 \in \mathcal{O}_+$

$$
\tilde{S}_{k,j}^{k',j'}(\omega_1; l) - \tilde{S}_{k,j}^{k',j'}(\omega_2; l) = \frac{\tilde{R}_{k,j}^{k',j'}(\omega_1; l) - \tilde{R}_{k,j}^{k',j'}(\omega_2; l)}{d(\omega_1)}
+ \frac{d(\omega_1) - d(\omega_2)}{d(\omega_1) d(\omega_2)} \tilde{R}_{k,j}^{k',j'}(\omega_2; l).
$$

Using the (4.12), (4.4) we deduce

$$
\frac{|d(\omega_1) - d(\omega_2)|}{|\omega_1 - \omega_2|} \leq |l|, \quad \forall \omega_1, \omega_2 \in \mathcal{O}_+, \omega_1 \neq \omega_2.
$$
Therefore, recalling (4.13), (2.16) and reasoning as in (4.26), (4.27), we get
\[
\|S_{[k]}^{[k']}(I)\|_{L^2(L^2)}^{lip,O} \lesssim N^{-1} N^2 \left(\begin{array}{c} k' \end{array}\right)^{3n+1} + \gamma^{-1} N^{2r+1} \left(\begin{array}{c} k' \end{array}\right)^{5n+3} + \|R_{[k]}^{[k']}(I)\|_{sup, O}^{lip}.
\]
Finally, reasoning as in (4.28) and using (2.37), we deduce
\[
\|S_{[s]}^{lip,O} \|_{sup} \leq s^{-1} N^{2r+1} [R]_{\rho,s}^{\gamma,O} + \|R_{[s]}^{\gamma,O} \|_{\rho,s}^{sup,O},
\]
provided that \( \rho \geq 5n + 3 \), which is true by (4.2). Combining (4.28) and (4.31) (recall (2.58)) we get the first bound in (4.18). The second one follows by (A.5) in Lemma A.1.

**Lemma 4.5.** There is \( C(s) > 0 \) (depending only on \( s \geq s_0 \)) such that, if
\[
\gamma^{-1} C(s) N^{2r+1} [R]_{\rho,s_0}^{\gamma,O} \leq \frac{1}{2},
\]
then the map \( \Phi = e^{iS} = \text{Id} + \Psi \), with \( S \) given by Lemma 4.4 satisfies
\[
[S]_{\rho,s}^{\gamma,O} \leq s^{-1} N^{2r+1} [R]_{\rho,s}^{\gamma,O}.
\]

**Proof.** By (4.18) and (4.32) we have that
\[
C(s) [S]_{s_0}^{\gamma,O} \leq 1/2,
\]
which implies the (A.6). Hence the (4.33) follows by Lemma A.3.

**4.1.3. The new remainder.** In this subsection we study the conjugate of the operator \( L \) under the map \( \Phi \) given by Lemma 4.5. We first define the new normal form \( Z_+ \) as
\[
Z_+ := Z + \text{Diag} R.
\]
We have the following.

**Lemma 4.6.** (New normal form) We have that \( Z_+ \) in (4.35) is \( \varphi \)-independent, Hermitian and block-diagonal, and satisfies
\[
[Z_+ - Z]_{\rho,s}^{\gamma,O} \leq s [R]_{\rho,s}^{\gamma,O}.
\]

**Proof.** It follows by construction.

**Lemma 4.7.** (The new remainder) Assume that the smallness condition (4.32) holds true. Then one has
\[
L_+ := \Phi L \Phi^{-1} := \omega \cdot \partial \varphi + i(\Delta_g + Z_+ + R_+)
\]
where \( Z_+ \) is the normal form given by (4.35) and the new remainder \( R_+ \) is Hermitian and satisfies for all \( s \in [s_0, S - b] \)
\[
[R_+]_{\rho,s}^{\gamma,O} \leq s N^{-b} [R]_{\rho,s+b}^{\gamma,O} + \gamma^{-1} N^{2r+\rho+1} [R]_{\rho,s+b}^{\gamma,O} [R]_{\rho,s}^{\gamma,O} + [R]_{\rho,s+b}^{\gamma,O} [R]_{\rho,s}^{\gamma,O}.
\]

**Proof.** Using the Lie expansions (2.48) and (2.49) we get
\[
L_+ := \Phi L \Phi^{-1} = \omega \cdot \partial \varphi + i(\Delta_g + Z) + i R + i [iS, \Delta_g + Z] - i \omega \cdot \partial \varphi S + i \sum_{p \geq 1} \frac{i^p}{p!} \text{ad}_S^p (R)
\]
Hence, equations (4.16), (4.35) lead to the following formula:
\[
R_+ = Q + \tilde{R}_+.
\]
with $Q := (1 - \Pi_N)R$ satisfying (4.19) and
\[
\tilde{R}_+ := \sum_{p \geq 2} \frac{1}{p!} \text{ad}_S^{p-1} \left( \text{Diag} R + Q - R \right) + \sum_{p \geq 1} \frac{1}{p!} \text{ad}_S^p (R) . \tag{4.40}
\]
Thus, in order to prove (4.38) we need to estimate $\tilde{R}_+$. Consider (for instance) the composition operator $SR$. In order to control the $[\ ]^\gamma_{\rho,s,+}$-norm we shall bound the decay norm of $D^\rho SR$. The estimates for for $SRD^\rho$ is the same. We have that
\[
\left[ D^\rho SR \right]^\gamma_{\rho,s,+} \leq \left[ D^\rho SD^\rho \right]^\gamma_{\rho,s,+} \left[ D^\rho R \right]^\gamma_{\rho,s,0} + \left[ D^\rho SD^\rho \right]^\gamma_{\rho,s,0} \left[ D^\rho R \right]^\gamma_{\rho,s,0} \tag{4.41}
\]
\[
\leq s \gamma^{-1} N^{2r+\rho+1} \left[ R \right]^\gamma_{\rho,s} \left[ R \right]^\gamma_{\rho,s,0} \tag{4.42}
\]
The commutator $[S, R]$ satisfies the same bound as (4.41). Therefore, by (4.41), (4.32), formula (4.40), the smallness assumption (4.32), and reasoning as in the proof of Lemma A.3 we get the (4.38) and (4.39). □

4.2. Iteration and Convergence. In this section we introduce a new constant
\[
a := b - 2 = 6d + 15n + 18 . \tag{4.42}
\]
For $N_0 \geq 1$ we define the sequence $(N_\nu)_{\nu \geq 0}$ by
\[
N_\nu := N_0^{\lambda \nu} , \ \nu \geq 0
\]
with $\lambda := 3/2$ and we set $N_{-1} = 1$. The proof of Theorem 4.1 is based on the following iterative lemma.

**Proposition 4.8. (Iteration)** Let $s \in [s_0, S - b]$. There exist $C(s) > 0$ and $N_0 \equiv N_0(s) \geq 1$ such that, if (recall (4.5))
\[
C(s) N_0^{2r+\rho+1} \leq \frac{1}{2} , \tag{4.43}
\]
then we may construct recursively sets $\mathcal{O}_\nu \subset \mathcal{O}_0$ and operators, defined for $\omega \in \mathcal{O}_\nu$,
\[
L_\nu := L_\nu(\omega) := \omega \cdot \partial_\omega + i(\Delta_g + Z_\nu + R_\nu) \tag{4.44}
\]
such that the following properties are satisfied for all $\nu \in \mathbb{N}$:

**S1.** There is a Lipschitz family of symplectic maps $\Phi_\nu(\varphi) = \Phi_\nu(\varphi, \omega) := \text{Id} + \Psi_\nu(\varphi) \in \mathcal{L}(H^s, H^s)$ defined on $\mathcal{O}_\nu$ such that, for $\nu \geq 1$,
\[
L_\nu := L_\nu L_{\nu-1} \Phi_\nu^{-1} , \tag{4.45}
\]
and, for $s \in [s_0, S - b]$,
\[
[\Psi_\nu]_{\rho,s} \leq \gamma^{-1} \left[ R_0 \right]_{\rho,s,b} N_{\nu-1}^{2r+2} N_{\nu-2}^{-a} . \tag{4.46}
\]

**S2.** The operator $Z_\nu = Z_0 + Z_{\nu,2}$ where $Z_{\nu,2}$ is $\varphi$-independent, block-diagonal and Hermitian. Moreover it satisfies
\[
[Z_\nu - Z_{\nu-1}]_{\rho,s} \leq \left[ R_0 \right]_{\rho,s,b} N_{\nu-2}^{-a} . \tag{4.47}
\]
Moreover there is a sequence of Lipschitz function
\[
\mu_{(\nu)}^{(k)} : \mathcal{O}_0 \to \mathbb{R}^{d_k} , \ \ k \in \mathbb{N}
\]
such that, for $\omega \in \mathcal{O}_\nu$, the functions $\mu_{(\nu)}^{(k)}$, for $j = 1, \ldots, d_k$, are the eigenvalues of the block
\[
(\Delta + Z_\nu)_{(k)}^{(j)} .
\]
Lemma 4.6, applied with \((\ref{4.5})\) follows by Lemma 4.2 and \((\ref{4.51})\). Moreover, by Kirchbraun Theorem, there is an extension 
\[O \ni \gamma \rightarrow N_\nu \ni \mu_{k,j} \rightarrow \mu_{k,j}^{(\nu)}\] 
We proceed by induction. We first verify the inductive step. So we assume that conditions \(\text{(Si)}_j\), 
\(i = 1, 2, 3, 4\), hold for \(1 \leq j \leq \nu\). We shall prove that they holds for \(\nu \gg \nu + 1\).

We define the set \(O_{\nu+1}\) as in \((\ref{4.13})\) with \(O \ni O_{\nu}, N \ni N_\nu, \mu_{k,j} \ni \mu_{k,j}^{(\nu)}\). Using the \((\ref{4.47})\) for \(\nu \gg \nu + 1\), we have that 
\[
\begin{align*}
[O_{\nu+1}]_{\gamma,0}^{[O_{\nu}]_{\gamma,0}} & \leq \frac{\gamma,0}{\gamma,0} \frac{\gamma,0 + 1}{\gamma,0 + 1} \frac{\gamma,0 + 1}{\gamma,0 + 1} \\
& \leq \sum_{j=0}^{\nu-1} \frac{\gamma,0}{\gamma,0} \frac{\gamma,0 + 1}{\gamma,0 + 1} \frac{\gamma,0 + 1}{\gamma,0 + 1} \\
& \leq \sum_{j=0}^{\nu-1} 2^{-j}
\end{align*}
\]

for \(N_0 \geq 1\) large enough. Hence condition \((\ref{4.14})\) is satisfied for \(\nu \gg 0\) small enough, i.e.\(N_0\) large enough (recall \((\ref{4.35})\)). Therefore Lemma \(4.3\) implies that \((\ref{4.50})\) holds for the set \(O_{\nu+1}\) which is the \((\text{S4}_\nu)_{\nu+1}\).

We define the new normal form \(Z_{\nu+1}\) as (recall \((\ref{4.35})\))
\[Z_{\nu+1} := Z_{\nu} + \text{Diag}(R_{\nu})\,.
\]

Lemma \(4.6\) applied with \(R \gg R_{\nu}\), together with the estimates \((\ref{4.49})\), implies the estimate \((\ref{4.47})\). Let \(\mu_{[k]}^{(\nu+1)}\) be the eigenvalues of the block \((\Delta + Z_{\nu+1})_{[k]}\) which are defined on the set \(O_{\nu+1}\). The bounds \((\ref{4.48})\) follows by Lemma \(4.2\) and \((\ref{4.51})\). Moreover, by Kirchbraun Theorem, there is an extension \(\mu_{[k]}^{(\nu+1)}\) of \(\mu_{[k]}^{(\nu+1)}\) to the whole set \(O_0\) with the same Lipschitz norm. This prove the \((\text{S2})_{\nu+1}\).

Then we want to construct a map \(\Phi_{\nu+1} = \text{Id} + \Psi_{\nu+1}\). First by the inductive hypothesis \((\ref{4.49})\) we deduce that (with \(C(s)\) given in Lemma \(4.3\))
\[
C(s)^{-1} N_\nu^{2\tau + 1} \bigl[ R_{\nu} \bigr]_{\gamma,0}^{[O_{\nu}]_{\gamma,0}} \leq C(s)^{-1} C(s)^{-1} N_\nu^{2\tau + 1} N_{\nu-1}^{\gamma,0} \bigl[ R_0 \bigr]_{\gamma,0}^{[O_{\nu}]_{\gamma,0}} \leq C(s) \epsilon N_\nu^{2\tau+1} \cdot \frac{\gamma,0}{\gamma,0} \frac{\gamma,0 + 1}{\gamma,0 + 1} \frac{\gamma,0 + 1}{\gamma,0 + 1} \leq \frac{1}{2}
\]

for \(\epsilon\) small enough and since \(2\tau + 1 - \frac{\gamma,0}{\gamma,0} \frac{\gamma,0 + 1}{\gamma,0 + 1} \frac{\gamma,0 + 1}{\gamma,0 + 1} \leq 0\). Hence the smallness condition \((\ref{4.32})\) holds true. We then apply Lemmata \(4.4\) and \(4.5\) with \(R \gg R_\nu\) and \(O_+ \gg O_{\nu+1}\) and construct a map \(\Phi_{\nu+1} = \text{Id} + \Psi_{\nu+1}\).

Furthermore using \((\ref{4.33})\), \((\ref{4.49})\) at rank \(\nu\), \(N_{\nu-1} = N_\nu^{-2/3}\) and \(2\tau + 1 - \frac{\gamma,0}{\gamma,0} \frac{\gamma,0 + 1}{\gamma,0 + 1} \frac{\gamma,0 + 1}{\gamma,0 + 1} \leq -1\), we obtain the estimate \((\ref{4.46})\) at rank \(\nu + 1\). This proves the \((\text{S1})_{\nu+1}\).

We finally set 
\[L_{\nu+1} := \Phi_{\nu+1} L_\nu \Phi_{\nu+1}^{-1} = \omega \cdot \partial \nu + i(\Delta + Z_{\nu+1} + R_{\nu+1})\]
where the remainder $R_{\nu+1}$ is given by Lemma \[\ref{lem:remainder}.\] We have

\[
\begin{align*}
[R_{\nu+1}]_{\rho,s}^{\gamma,C_{\nu}} & \leq s N_{\nu}^{-b}[R_{\nu}]_{\rho,s}^{\gamma,C_{\nu}} + \gamma^{-1} N_{\nu}^{2\tau+\rho+1}[R_{\nu}]_{\rho,s}^{\gamma,C_{\nu}} [R_{0}]_{\rho,s}^{\gamma,C_{0}} \\
& \leq s [R_{0}]_{\rho,s}^{\gamma,C_{0}} N_{\nu}^{-b-1} + \gamma^{-1} [R_{0}]_{\rho,s}^{\gamma,C_{0}} N_{\nu}^{2\tau+\rho+1-\frac{2}{3}a} \\
& \leq N_{\nu}^{-a}[R_{0}]_{\rho,s}^{\gamma,C_{0}}
\end{align*}
\tag{4.54}
\]

for $N_0$ large enough where we used that $\gamma^{-1} [R_{0}]_{\rho,s}^{\gamma,C_{0}} \leq 1$ (thanks to \[\ref{lem:smallness}.\]) and

\[b \geq a + 2, \quad 2\tau + \rho + 1 - \frac{1}{3}a \leq -1.\]

The latter condition is implied by the choice of $a$ in \[\ref{lem:remainder}.\] recalling the \[\ref{eq:smallness}.\] The \[\ref{eq:remainder}.\] is the first estimate in \[\ref{eq:inductive}.\] at step $\nu+1$. We now give the estimate in “high” norm. We have

\[
\begin{align*}
[R_{\nu+1}]_{\rho,s}^{\gamma,C_{\nu}} & \leq s [R_{\nu}]_{\rho,s}^{\gamma,C_{\nu}} + \gamma^{-1} N_{\nu}^{2\tau+\rho+1}[R_{\nu}]_{\rho,s}^{\gamma,C_{\nu}} [R_{0}]_{\rho,s}^{\gamma,C_{0}} \\
& \leq s [R_{0}]_{\rho,s}^{\gamma,C_{0}} N_{\nu-1} \left(1 + \gamma^{-1} [R_{0}]_{\rho,s}^{\gamma,C_{0}} N_{\nu}^{2\tau+\rho+1} N_{\nu-1} \right) \\
& \leq s N_{\nu} [R_{0}]_{\rho,s}^{\gamma,C_{0}}
\end{align*}
\tag{4.55}
\]

for $N_0$ large enough depending on $s$ and thanks to fact that $3\tau + \frac{3}{2}\rho + \frac{1}{2} - a \leq 0$. This is the \(S3\)\(_{\nu+1}\).

Now we have to verify the initial step: $\nu = 1$. \(S2\) and \(S4\) are proved exactly in the way as in the inductive step. Now to proceed we have to construct $\Phi_1$ but now \[\ref{eq:inductive}.\] becomes

\[
C(s)\gamma^{-1} N_{0}^{2\tau+1+1}[R_{0}]_{\rho,s}^{\gamma,C_{0}} \leq C(s)\epsilon N_{0}^{2\tau+1} \leq \frac{1}{2}
\tag{4.56}
\]

which is less than $\frac{1}{2}$ for $\epsilon$ and $N_0$ satisfying \[\ref{eq:smallness}.\].

Furthermore using \[\ref{eq:remainder}.\] we obtain

\[
[\Psi]_{\rho,s}^{\gamma,O} \leq C(s)\gamma^{-1} N_{0}^{2\tau+1+1}[R_{0}]_{\rho,s}^{\gamma,O} \leq \gamma^{-1} N_{0}^{2\tau+2} [R_{0}]_{\rho,s}^{\gamma,O}
\]

for $N_0$ large enough. This proves the \(S1\)\(_{\nu+1}\).

Then we set

\[
L_1 := \Phi_1 L_0 \Phi_1^{-1} = \omega \cdot \partial_{\rho} + i(\Delta_{g} + Z_1 + R_1)
\tag{4.57}
\]

where the remainder $R_1$ is given by Lemma \[\ref{lem:remainder}.\] We have

\[
\begin{align*}
[R_{1}]_{\rho,s}^{\gamma,O} & \leq s N_{0}^{-b}[R_{0}]_{\rho,s}^{\gamma,O} + \gamma^{-1} N_{0}^{2\tau+\rho+1}[R_{0}]_{\rho,s}^{\gamma,O} [R_{0}]_{\rho,s}^{\gamma,O} \\
& \leq s [R_{0}]_{\rho,s}^{\gamma,O} N_{0}^{-b-1} + \gamma^{-1} [R_{0}]_{\rho,s}^{\gamma,O} N_{0}^{2\tau+\rho+1} \\
& \leq N_{0}^{-a}[R_{0}]_{\rho,s}^{\gamma,O}
\end{align*}
\tag{4.58}
\]

for $N_0$ large enough where we used \[\ref{eq:smallness}.\] and $b \geq a + 2$. The \[\ref{eq:remainder}.\] is the first estimate in \[\ref{eq:inductive}.\] at step 1, the other is proved similarly.

\[\square\]

**Proof of Theorem \[\ref{thm:main}.\]** Consider the operator $L_0$ in \[\ref{eq:operator}.\] The smallness condition \[\ref{eq:smallness}.\] implies the \[\ref{eq:smallness}.\], hence Proposition \[\ref{prop:smallness}.\] applies. We define the set

\[
O_{\epsilon} \equiv O_{\infty} := \cap_{\nu \geq 0} O_{\nu}
\tag{4.59}
\]
By the measure estimate (4.50) we deduce (4.7). For any \( \omega \in O_\infty, \nu \geq 0 \), we define (see (4.45), (4.46)) the map
\[ \Phi_{\nu+1} := \Phi_1 \circ \Phi_2 \circ \cdots \Phi_{\nu+1} = \Phi_{\nu} \Phi_{\nu+1} = \Phi_{\nu}(\text{Id} + \Psi_{\nu+1}). \] (4.60)
We want to prove that \((\Phi_{\nu})_{\nu \geq 1}\) converges in \(M_s^{\gamma, O_\infty}\). Let us define
\[ \delta_s^{(\nu)} := [\Phi_{\nu}]_{s}^{\gamma, O_\infty}. \] (4.61)
We have
\[ \delta_s^{(\nu+1)} \leq \delta_s^{(\nu)} (1 + C[\Psi_{\nu+1}]_{s, 0}^{\gamma, O_\infty}) \leq \delta_s^{(\nu)} (1 + C\epsilon N_{\nu}^{-1}). \] (4.62)
By iterating the (4.62) we get, for any \( \nu \),
\[ [\Phi_{\nu}]_{s, 0}^{\gamma, O_\infty} \leq (1 + [\Psi_{1}]_{s, 0}^{\gamma, O_\infty}) \Pi_{j \geq 1} (1 + C\epsilon N_{\nu}^{-1}) \leq 2 \] (4.63)
where we used the (4.66) to estimate \([\Psi_{1}]_{s, 0}^{\gamma, O_\infty}\) and we take \(N_0\) large enough.

The high norm of \(\Phi_{\nu+1}\) is estimated by
\[ [\Phi_{\nu}](\nu+1) \leq [\Phi_{\nu}](\nu) (1 + (s[\Psi_{\nu+1}]_{s, 0}^{\gamma, O_\infty}) + C(s)[\Psi_{\nu+1}]_{s, 0}^{\gamma, O_\infty} [\Phi_{\nu}]_{s, 0}^{\gamma, O_\infty} \leq [\Phi_{\nu}](\nu) (1 + C(s)\epsilon N_{\nu}^{-1}) + \epsilon_{\nu}. \] (4.64)
where
\[ \epsilon_{\nu} := C(s)\gamma^{-1} [R_0]_{\gamma, 0, s+b} N_{\nu}^{-1}. \]
By iterating (4.64), using \(\Pi_{j \geq 0} (1 + C(s)\epsilon N_{\nu}^{-1}) \leq 2\) for \(N_0\) large enough, we obtain
\[ [\Phi_{\nu}]_{s, 0}^{\gamma, O_\infty} \leq [\Phi_{1}]_{s, 0}^{\gamma, O_\infty} + 2 \sum_{j \geq 1} \epsilon_j \leq 1 + C(s)\gamma^{-1} [R_0]_{\gamma, 0, s+b}. \] (4.65)
Then we have
\[ [\Phi_{\nu+1} - \Phi_{\nu}]_{s, 0}^{\gamma, O_\infty} \leq [\Phi_{\nu}]_{s, 0}^{\gamma, O_\infty} [\Phi_{\nu+1}]_{s, 0}^{\gamma, O_\infty} \leq [\Phi_{\nu}]_{s, 0}^{\gamma, O_\infty} \Pi_{j \geq 1} (1 + \gamma^{-1} [R_0]_{\gamma, 0, s+b} N_{\nu}^{-1} + \gamma^{-1} [R_0]_{\gamma, 0, s+b} N_{\nu}^{-1} \leq [\Phi_{\nu}]_{s, 0}^{\gamma, O_\infty} \Pi_{j \geq 1} (1 + \gamma^{-1} [R_0]_{\gamma, 0, s+b} N_{\nu}^{-1}. \] (4.66)
Now fix \(s \in [s_0, s - b]\), since by hypothesis (A2), \(R_{0} \in M_{s+b}^{\gamma, O_\infty}\), we deduce from the last estimate that \((\Phi_{\nu})_{\nu \geq 0}\) is a Cauchy sequence in \(M_{s}^{\gamma, O_\infty}\). Hence \(\Phi_{\nu} \to \Phi_{\infty} \in M_{s}^{\gamma, O_\infty}\). Furthermore writing
\[ \|\Phi_{\nu} - \text{Id}\|_{L(H^s, H^s)} \leq \sum_{j \geq 2} \|\Phi_j - \Phi_{j-1}\|_{L(H^{s}, H^{s})} + \|\Psi_1\|_{L(H^{s}, H^{s})} \]
we deduce by (4.66) and Lemma A.2 that \(\Phi_{\infty}\) satisfies (4.9). The estimate on \(\Phi_{\infty}^{-1} - \text{Id}\) follows by using Neumann series and reasoning as in the proof of Lemma A.3. By (4.47) we deduce that \(Z_{\nu, 2}\) is a Cauchy sequence in \(M_{s+b}^{\gamma, O_\infty}\). Hence we set
\[ Z_{\infty} = Z_{0} + Z_{\infty, 2} := Z_{0} + \lim_{\nu \to \infty} Z_{\nu}, \] (4.67)
The (4.10) follows again by (4.47). We also notice that (4.49) implies that \(R_{\nu} \to 0\) in \(M_{s+b}^{\gamma, O_\infty}\). Now by applying iteratively the (4.45) we have that
\[ L_{\nu} = \Phi_{\nu} L_{0} \Phi_{\nu}^{-1}. \]
Hence, passing to the limit, we get \(L_{\nu} \to L_{\infty}\) of the form (4.8) with \(Z_{\infty}\) given by (4.67).
5. Proof of Theorem 1.1

In this short section we merge the two previous sections to prove the reducibility of the Schrödinger equation (LS): Theorem 1.1.
We recall that equation (LS) has the form
\[ \partial_t u = -i(\Delta_g + \varepsilon W(\omega t))u \]
where \( \varphi \mapsto W(\varphi) \) is a \( C^\infty \) map from \( \mathbb{T}^d \) to \( A_\delta, \delta \leq 1/2 \), and thus \( W \in A_{\delta,r} \) for any \( r > d/2 \). Its reducibility rely on the reducibility of the operator \( F \) in (3.1) with \( V(\varphi) = \varepsilon W(\varphi) \). Roughly speaking we want to apply Theorem 3.1 to regularize \( F \) in such a way operator \( F \) is transformed into the operator \( F_+ \) in (3.4). Then we apply Lemma 2.14 to control the remainder \( R \) in (3.4) in \( s \)-decay norm. This allows, for \( \varepsilon \) small enough, to apply the reducibility Theorem 4.1 and to conclude.

To justify all these steps we have to carefully follow the parameters and the smallness conditions. First we fix \( \alpha \in (0, 1) \) and \( \gamma = \varepsilon^\alpha, \delta \leq 1/2, s > n/2 \) and \( W \) belonging to all the \( A_{\delta,r} \) with \( r > d/2 \). Then we fix \( \rho, \beta, \tau \) as in (4.2), we set \( \kappa = 2\delta - 1 \) and we fix \( s_0 > n/2 \) and \( S \) such that \( s \) and \( s + \beta \) belong to \([s_0, S]\) and \( S \geq p(\delta, 0) \) (see (2.21)). Finally we set
\[ \rho_0 = S + \rho + \frac{1}{2}, \quad r_0 = S. \]
With these values of \( \rho_0, r_0 \), Theorem 3.1 provide us with \( \varepsilon_s = \varepsilon_s(S, n, d, \delta), r_s = r_s(S, n, d, \delta) \) and \( p = p(S, n, d, \delta) \) such that if \( r > r_s \) and
\[ \gamma^{-1}N_{\delta,\rho,p}(\varepsilon W) < \varepsilon_s(n, d, \delta) \] (5.1)
then we can apply Theorem 3.1 to \( F \) with \( V = \varepsilon W \). Since \( W \) belongs to \( A_{\delta,r} \) for any \( r > d/2 \), (5.1) is satisfied for \( \varepsilon \) small enough. So there exists \( \Phi(\varphi) \in \mathcal{L}(H^s, H^s) \) such that (see (3.4))
\[ \Phi F \Phi^{-1} = F_+ = \omega \cdot \partial_\varphi + i(\Delta_g + Z + R). \]
Further we knows that \( R \in \mathcal{R}^{\gamma,\mathcal{C}_0}_{\rho_0,r_0} \) and
\[ |R|^{\gamma,\mathcal{C}_0}_{\rho_0,\alpha} \leq CN_{\delta,\rho,p}(\varepsilon W). \] (5.2)
Now we apply Lemma 2.14 to conclude that \( R \in \mathcal{M}^{\gamma,\mathcal{C}_0}_{\rho,S} \) and
\[ |R|^{\gamma,\mathcal{C}_0}_{\rho,S} \leq CN_{\delta,\rho,p}(\varepsilon W). \] (5.3)

We notice that the operator \( F_+ \) has the same form of the operator \( L_0 \) in (4.1) with \( Z_0 = Z, R_0 = R \) and \( \mathcal{O} = \mathcal{O}_0 \) (see (4.2)). The remainder \( R_0 \) satisfies the assumption (A2) by the discussion above. Notice also that, in view of by (5.3), the constant \( \epsilon \) given by (4.2) satisfies
\[ \epsilon \leq N_{\delta,\rho,p}(W)\epsilon^{1-\alpha} \] (5.4)
and thus the smallness condition (4.6) is satisfied provided that \( \varepsilon \) is small enough. We now prove that \( Z_0 \) satisfies assumption (A1) with \( \kappa := 2\delta - 1 \). First we note that, since \( \delta \leq 1/2 \) then \( \kappa \leq 0 \). Moreover, by Theorem 3.1 we have that \( Z_0 = Z = Z_1 + Z_2 \) with \( Z_1 \in A_\delta \) independent of \( \omega \in \mathcal{O}_0 \), and \( Z_2 \in A_{\delta,2\delta - 1} \). Estimate (3.6) implies that for all \( s \in [s_0, S] \)
\[ N_{\delta,s}(Z) \leq CN_{\delta,\rho,p}(\varepsilon W) \leq CN_{\delta,\rho,p}(W)\varepsilon^{1-\alpha}. \]
Since \( S \geq p(\delta, 0) \) we deduce by (2.21) that
\[ \|Z\|_{L^2(H^{-\delta})} \leq CN_{\delta,\rho,p}(W)\varepsilon^{1-\alpha} \leq \frac{c_0}{2}. \]
for $\varepsilon$ small enough which in turn implies that

$$|\mu_{k,j}^{(0)}(\omega)| \leq \frac{c_0}{2} |k|^\delta$$

and thus (4.3) holds true. Furthermore since $Z_1$ does not depend on $\omega$, we have

$$\|(Z_0)_{[k]}\|_{L^2} = \|(Z_2)_{[k]}\|_{L^2}$$

and thus (3.6) implies also (4.4) for $\varepsilon$ small enough.

Hence all the hypothesis of Theorem 4.1 are satisfied for $L_0 = F$ and this theorem provides a set of frequencies $O_\varepsilon$ such that, for $\omega \in O_\varepsilon$, there is a map $\Phi_\infty$ satisfying (4.9) such that $L_0 \equiv F$ transforms into $L_\infty$ in (4.8).

By (4.7) we have

$$\text{meas}(O_0 \cap O_\varepsilon) \leq C\gamma,$$

for some constant $C > 0$ depending on $s$. It is also known that (recall (3.2))

$$\text{meas}([1/2, 3/2] \cap O_\varepsilon) \leq C\gamma.$$

Therefore, recalling that we set $\gamma = \varepsilon^\alpha$ we have that the (1.1) holds. For $\omega \in O_\varepsilon$ we set

$$\Psi(\omega t) := \Phi_\infty(\omega t) \circ \Phi(\omega t).$$

By construction the function $v := \Psi(\omega t)u$ satisfies the equation (1.4) with $\varepsilon Z \to Z_\infty$ in (4.8). Moreover, by (3.8), (3.9), (4.9) and (5.2), we have

$$\sup_{\varphi \in \mathbb{T}^d} \|\Psi^{\pm 1}(\varphi) - \text{Id}\|_{L(H^s,H^{s-\delta})} \leq \gamma^{-1} C_s \varepsilon,$$

$$\sup_{\varphi \in \mathbb{T}^d} \|\Psi^{\pm 1}(\varphi)\|_{L(H^s,H^{s})} \leq 1 + \gamma^{-1} C_s \varepsilon,$$

for some $C_s > 0$. This concludes the proof.

APPENDIX A. TECHNICAL LEMMATA

A.1. Proof of Lemma 2.6

Proof of Lemma 2.6 The bounds (2.38), (2.39) and (2.43) can be deduced by using the properties of the semi-norm in (2.21)-(2.24) and the definition in (2.32). We give the proof the bound (2.42) of item (iii). The bound (2.40) is similar. We have that

$$B(\omega) := (\omega \cdot \partial_\varphi)^{-1} A(\omega) = \sum_{0 \neq l \in \mathbb{Z}^d} \frac{1}{|l^{\omega}|} e^{il \cdot \varphi} A(\omega; l).$$

(A.1)

Thus

$$(N_{m,r-\alpha,p}(\omega \cdot \partial_\varphi)^{-1} A)^2 \leq \sum_{0 \neq l \in \mathbb{Z}^d} \frac{1}{|l^{\omega}|^2} \langle l \rangle^{2(r-\alpha)} N_{m,p}^2(A(l)) \leq \sum_{l \in \mathbb{Z}^d \setminus \{0\}} \langle l \rangle^{2(r-\alpha)} |l|^{2\alpha} N_{m,p}^2(A(l)) \leq \frac{C}{s^2} \sum_{0 \neq l \in \mathbb{Z}^d \setminus \{0\}} \langle l \rangle^{2r} N_{m,p}^2(A(l)) = \frac{C}{s^2} (N_{m,r,p}(A))^2.$$
To estimate $N_{m,r-(2\alpha+1),p}^{\gamma,O}(B)$ (see \eqref{eq:2.16}) we reason as follows. We first note that

$$B(\omega_1) - B(\omega_2) = \sum_{0 \neq l \in \mathbb{Z}^d} \frac{1}{i\omega_1 \cdot l} e^{il \cdot \varphi} \left( A(\omega_1;l) - A(\omega_2;l) \right)$$

$$+ \sum_{0 \neq l \in \mathbb{Z}^d} \frac{(\omega_1 - \omega_2) \cdot l}{i(\omega_1 \cdot l)(\omega_2 \cdot l)} e^{il \cdot \varphi} A(\omega_2;l).$$

Moreover, by using \eqref{eq:2.61} and that $O$ is compact, we have

$$\left| \frac{(\omega_1 - \omega_2) \cdot l}{i(\omega_1 \cdot l)(\omega_2 \cdot l)} \right| \leq C \frac{1}{\gamma^2} |l|^{2\alpha+1} |\omega_1 - \omega_2|.$$ 

Therefore reasoning as in \eqref{eq:A.2} we get

$$\frac{N_{m,r-(2\alpha+1),p}^{\gamma,O}(B(\omega_1) - B(\omega_2))}{|\omega_1 - \omega_2|} \lesssim \frac{1}{\gamma} N_{m,r,p}^{\text{lip,}O}(A) + \frac{1}{\gamma^2} N_{m,r,p}^{\text{sup,}O}(A). \quad (A.3)$$

Combining \eqref{eq:A.2}, \eqref{eq:A.3} and recalling \eqref{eq:2.17}, \eqref{eq:A.1} we obtained

$$N_{m,r-(2\alpha+1),p}^{\gamma,O}(B) \lesssim \frac{1}{\gamma} N_{m,r,p}^{\text{sup,}O}(A) + \gamma \left( N_{m,r,p}^{\text{lip,}O}(A) + \frac{1}{\gamma^2} N_{m,r,p}^{\text{sup,}O}(A) \right)$$

$$\lesssim \frac{1}{\gamma} \left( N_{m,r,p}^{\text{sup,}O}(A) + \gamma N_{m,r,p}^{\text{lip,}O}(A) \right)$$

which is bound \eqref{eq:2.42}. \hfill \Box

\section{Properties of the $s$-decay norm.}

In this appendix $s_0$ is some fixed number satisfying $s_0 > (d+n)/2$.

\begin{lemma}
Let $\alpha > 0$. Then (recall \eqref{eq:2.63}, \eqref{eq:2.57})

$$\| D^{\pm \alpha} A D^{\mp \alpha} \|_{s}^{\gamma,O} \lesssim \| A \|_{s+\alpha}^{\gamma,O}, \quad (A.4)$$

$$\| D^{\pm \alpha} (\Pi_N A) D^{\mp \alpha} \|_{s}^{\gamma,O} \lesssim N^{\alpha} \| A \|_{s}^{\gamma,O}. \quad (A.5)$$

\end{lemma}

\begin{proof}
The bounds \eqref{eq:A.4}, \eqref{eq:A.5} follow by reasoning as in the proof of Lemma A.1 in \cite{FG19} and using the \eqref{eq:2.61}. \hfill \Box

\begin{lemma}
Let $A$ be a matrix as in \eqref{eq:2.54} with finite $\| \cdot \|_s$-norm (see \eqref{eq:2.57}). Then (recall \eqref{eq:2.56}) one has

$$\| A(\varphi) \|_{\mathcal{L}(H^s,H^s)} \lesssim_s |A(\varphi)|_s \lesssim_s \| A \|_{s+s_0}, \quad \forall \varphi \in \mathbb{T}^d.$$ \hfill \Box

\end{lemma}

\begin{lemma}
Assume that

$$C(s) [A]_{s_0}^{\gamma,O} \lesssim 1/2 \quad (A.6)$$

for some large $C(s) > 0$ depending on $s \geq s_0$. Then the map \( \Phi := \text{Id} + \Psi \) defined as

$$\Phi := e^{iA} := \sum_{p \geq 0} \frac{1}{p!} (iA)^p, \quad (A.7)$$

satisfies

$$\| \Psi \|_{s}^{\gamma,O} \lesssim_s \| A \|_{s}^{\gamma,O}, \quad (A.8)$$

\end{lemma}
Proof. For any \( n \geq 1 \), using (2.60), we have, for some \( C(s) > 0 \),
\[
[A^n]_{s_0} \leq [C(s_0)]^{n-1} [A]_{s_0}^n, \\
[A^n]_s \leq n[C(s)[A]_{s_0}]^{n-1} C(s)[A]_s, \quad \forall s \geq s_0.
\]
The same holds also for the norm \([\cdot]_{s}^{\gamma, O}\). Hence
\[
[\Psi]_{s}^{\gamma, O} \leq [A]_{s}^{\gamma, O} \sum_{p \geq 1} \frac{C(s)p}{p!} ([A]_{s_0}^{\gamma, O})^{p-1},
\]
for some (large) \( C(s) > 0 \). By the smallness condition (A.6) one deduces the bounds (A.8). \(\square\)

Lemma A.4. Let \( \alpha, \beta \in \mathbb{R} \). Then
\[
[AM]_{\alpha+\beta, s}^{\gamma, O} \leq [A]_{\alpha, s+|\beta|}^{\gamma, O} [M]_{\beta, s+|\alpha|}^{\gamma, O} + [A]_{\alpha, s+|\beta|}^{\gamma, O} [M]_{\beta, s+|\alpha|}^{\gamma, O},
\]
(A.9)
\[
(\text{Id} - \Pi_N) [M]_{\beta, s}^{\gamma, O} \leq N^{-s} [M]_{\beta, s}^{\gamma, O}, \quad s \geq 0.
\]
(A.10)
Moreover, if \( \alpha \leq \beta < 0 \) then
\[
[AM]_{\beta, s}^{\gamma, O} \leq [A]_{\alpha, s}^{\gamma, O} [M]_{\beta, s}^{\gamma, O} + [A]_{\alpha, s}^{\gamma, O} [M]_{\beta, s}^{\gamma, O}.
\]
(A.11)
Proof. To prove (A.9) one reasons as in Lemma A.3 in [FG19]. The (A.11) and (A.10) follow by Lemma A.1. \(\square\)

Lemma A.5. One has
\[
\|D^\beta Ah\|_{\ell_s} \leq s [A]_{\beta, s} \|h\|_{\ell_{s_0}} + [A]_{\beta, s_0} \|h\|_{\ell_s},
\]
(A.12)
for any \( h \in \ell_s \) (see (2.15)) and \( \beta \in \mathbb{R} \).
Proof. One reasons as in Lemma A.4 in [FG19]. \(\square\)

A.3. Flows of pseudo differential operators.

Lemma A.6. Fix \( m \leq 0, 0 \leq \delta \leq 1, r > d/2 \) and \( \rho \geq 0 \) and consider \( S_1 \in A_{m, r}^{\gamma, O} \) and \( S_2 \in A_{\delta, r}^{\gamma, O} \) (see Definition 2.3). Assume also that
\[
[S_2, K_0] = 0, \quad \langle S_2 h, v \rangle = \langle h, S_2 v \rangle
\]
(A.13)
where \( \langle \cdot, \cdot \rangle \) is the standard \( L^2 \) scalar product. Let us define
\[
\Phi_1^\tau := \Phi_1^\tau (\varphi) := e^{\tau iS_1}, \quad \Phi_2^\tau := \Phi_2^\tau (\varphi) := e^{\tau iS_2}.
\]
(A.14)
For any \( s \geq 0 \) there are \( \varepsilon_0, C, p > 0 \) such that, for any \( 0 < \varepsilon \leq \varepsilon_0 \), if
\[
N_{m, r, p}^{\gamma, O}(S_1) + N_{\delta, r, p}^{\gamma, O}(S_2) \leq \varepsilon,
\]
(A.15)
then the following holds true:
(i) the map \( \Phi_1^\tau \) satisfies
\[
\sup_{\varphi \in \mathcal{T}_d} \| \Phi_1^\tau (\varphi) - \text{Id} \|_{L(H^s; H^{s-m})} \leq C N_{m, r, p}^{\gamma, O}(S_1),
\]
(A.16)
\[
\sup_{\varphi \in \mathcal{T}_d} \| (\partial^k \Phi_1^\tau (\varphi)) \|_{L(H^s; H^s)} \leq C N_{\delta, r, p}^{\gamma, O}(S_1), \quad 0 \leq k \leq r,
\]
(A.17)
for any \( \tau \in [0, 1] \);
(ii) the map $\Phi_2^\tau$ satisfies

$$
\sup_{\varphi \in \mathbb{T}^d} \|\Phi_2^\tau(\varphi)\|_{L^2(H^s, H^{s-\delta})} \leq (1 + CN_{\delta, r, p}(S_2)),
$$
(A.18)

$$
\sup_{\varphi \in \mathbb{T}^d} \|(\Phi_2^\tau(\varphi) - \text{Id})\|_{L^2(H^s, H^{s-\delta})} \leq CN_{\delta, r, p}(S_2),
$$
(A.19)

$$
\sup_{\varphi \in \mathbb{T}^d} \|(\partial_\varphi^k \Phi_2^\tau(\varphi))\|_{L^2(H^s, H^{s-k\delta-1})} \leq CN_{\delta, r, p}(S_2), \quad 1 \leq k \leq r,
$$
(A.20)

for any $\tau \in [0, 1]$ and any $\omega \in \mathcal{O}$. Moreover the following bounds on the Lipschitz norm hold true:

$$
\sup_{\varphi \in \mathbb{T}^d} \|\Phi_2^\tau(\varphi)\|_{H^s} \leq (1 + CN_{\delta, r, p}(S_2)),
$$
(A.21)

$$
\sup_{\varphi \in \mathbb{T}^d} \|(\Phi_2^\tau(\varphi) - \text{Id})\|_{H^s} \leq CN_{\delta, r, p}(S_2),
$$
(A.22)

$$
\sup_{\varphi \in \mathbb{T}^d} \|(\partial_\varphi^k \Phi_2^\tau(\varphi))\|_{H^s} \leq CN_{\delta, r, p}(S_2), \quad 1 \leq k \leq r,
$$
(A.23)

for any $\tau \in [0, 1]$.

**Proof.** We shall prove the result for the map $\Phi_2^\tau$. The estimates on $\Phi_1^\tau$ can be obtained in the same way. Notice that the operator $\Phi_2^\tau$ solves the problem

$$
\begin{cases}
\partial_\tau \Phi_2^\tau(\varphi) = iS_2(\varphi)\Phi_2^\tau(\varphi) \\
\Phi_2^0(\varphi) = \text{Id}.
\end{cases}
$$
(A.24)

Using (A.24) and the assumption (A.13) one can check that

$$
\partial_\tau \|\Phi_2^\tau h\|_{H^s}^2 = 0 \quad \Rightarrow \quad \|\Phi_2^\tau h\|_{H^s} \leq \|h\|_{H^s},
$$

for any $\tau \in [0, 1]$, $h \in H^s$ and $\varphi \in \mathbb{T}^d$. This is the (A.18). Let us now define

$$
\Gamma^\tau(\varphi) := \Phi_2^\tau(\varphi) - \text{Id}.
$$

It solve the problem

$$
\partial_\tau \Gamma^\tau(\varphi) = iS_2(\varphi)\Gamma^\tau(\varphi) + iS_2(\varphi), \quad \Gamma^0(\varphi) = 0.
$$

By Duhamel formula have

$$
\Gamma^\tau(\varphi) h = \int_0^\tau \Phi_2^\tau(\varphi) \Phi_{2}^{-\sigma}(\varphi) iS_2(\varphi) h d\sigma.
$$

Therefore the bound (A.19) follows by (A.18) and the estimates on $S_2$. Similarly the operator $(\partial_\varphi \Phi_2^\tau)(\varphi)$ satisfies

$$
\begin{cases}
\partial_\tau (\partial_\varphi \Phi_2^\tau)(\varphi) = iS_2(\varphi)(\partial_\varphi \Phi_2^\tau)(\varphi) + i(\partial_\varphi S_2)(\varphi) \Phi_2^\tau(\varphi), \\
(\partial_\varphi \Phi_2^0)(\varphi) = 0.
\end{cases}
$$
(A.25)

We have that

$$
\sup_{\varphi \in \mathbb{T}^d} \|(\partial_\varphi S_2)(\varphi)\Phi_2^\tau(\varphi) h\|_{H^{s-\delta}} \lesssim \|h\|_{H^s} N_{\delta, r, p}^\gamma(\mathcal{O}, S_2)
$$

by (A.18) and the fact that $r > d/2$. Hence, using Duhamel formula and the (A.18), we deduce the (A.20) for $k = 1$. The (A.20) for $k > 1$ can be obtained in the same way by differentiating (A.25). The Lipschitz bounds (A.21)-(A.23) follows reasoning as in the estimates of $\partial_\varphi \Phi_2^\tau(\varphi)$. The bounds (A.16), (A.17) can be deduced reasoning as done above and using the fact that the generator $iS_1(\varphi)$ is a *bounded* pseudodifferential operator.

□
Lemma A.7. Let $r_1 \geq 0$ and $r > r_1 + d/2$, $\delta > 0$, $\rho_1 > 0$, $\rho := \rho_1 + \delta r_1 + 1$ and consider $R \in \mathcal{R}^{\gamma,0}_{\rho,r}$ (see Definition 2.4). Consider also the map $\Phi_2(\varphi) := \Phi_2^\gamma(\varphi)|_{\tau=1}$, where $\Phi_2^\gamma(\varphi)$ is given in Lemma A.6. Then $G_2(\varphi) := \Phi_2(\varphi)R(\varphi)\Phi_2^{-1}(\varphi)$ belongs to $\mathcal{R}^{\gamma,0}_{\rho_1,r_1}$. Moreover for any $s \geq 0$ there exist $p$ and $C$ such that

$$|G_2|_{\rho_1,r_1,s} \leq |R|_{\rho,r,s}^\gamma(1 + C\mathcal{N}_{m,r,p}(S_2)),$$

(A.26)

Proof. We need to prove that the map $\varphi \mapsto \Gamma(\varphi)$ is in $H^{r_1}(\mathbb{T}^d; \mathcal{L}(H^s; H^{s+\rho_1}))$. We note that

$$|G_2|_{\rho_1,r_1,s} \lesssim \sum_{k=0}^{r_1} \sup_{\varphi \in \mathbb{T}^d} \| (\partial_k^\varphi G_2)(\varphi) \|_{\mathcal{L}(H^{s_1}; H^{s+\rho_1})}$$

(A.27)

We estimate separately each summand in (A.27). First of all notice that, by the definition of the norm in (2.34) and the fact that $r > r_1 + d/2$, one has

$$\sup_{\varphi \in \mathbb{T}^d} \| \partial_k^\varphi (R(\varphi)) \|_{\mathcal{L}(H^{s_1}; H^{s+\rho})} \lesssim \| R \|_{\rho,r,s}.$$

(A.28)

Hence the summand in (A.27) with $k_1 = k_3 = 0$ is trivially bounded by the right hand side in (A.26). If at least one between $k_1, k_2$ is different from zero we have, for any $h \in \mathbb{H}^s$,

$$\| (\partial_k^\varphi \Phi_2)(\varphi) (\partial_k^\varphi R)(\varphi) (\partial_k^\varphi \Phi_2^{-1})(\varphi) h \|_{H^{s_1+\rho_1}}$$

(A.29)

$$\lesssim \mathcal{N}_{\delta,r,p}(S_2) \| (\partial_k^\varphi R)(\varphi) (\partial_k^\varphi \Phi_2^{-1})(\varphi) h \|_{H^{s_1+\rho_1+k_1\delta}}$$

(A.30)

$$\lesssim \mathcal{N}_{\delta,r,p}(S_2) \| R \|_{\rho,r,s} \| (\partial_k^\varphi \Phi_2^{-1})(\varphi) h \|_{H^{s_1+\rho_1+k_1\delta-\rho}}$$

(A.31)

Notice that $+\rho_1 + (k_1 + k_3)\delta - \rho \leq 0$ since $k_1 + k_3 \leq r_1$ and that the estimate above is uniform in $\varphi \in \mathbb{T}^d$. Hence, together with the (A.27), it implies the (A.26) for the norm $| \cdot |_{\rho_1,r_1,s}$. The Lipschitz bounds are obtained similarly taking into account the extra loss of derivatives appearing in the estimates (A.21)–(A.23).

Similarly we prove in the bounded case:

Lemma A.8. Let $r_1 \geq 0$ and $r > r_1 + d/2$, $\rho > 0$ and consider $R \in \mathcal{R}^{\gamma,0}_{\rho,r}$. Consider also the map $\Phi_1(\varphi) := \Phi_1^\gamma(\varphi)|_{\tau=1}$, where $\Phi_1^\gamma(\varphi)$ is given in Lemma A.6. Then $G_1(\varphi) := \Phi_1(\varphi)R(\varphi)\Phi_1^{-1}(\varphi)$ belongs to $\mathcal{R}^{\gamma,0}_{\rho,r}$. Moreover for any $s \geq 0$ there exist $p$ and $C$ such that

$$|G_1|_{\rho,r_1,s} \leq C |R|_{\rho,r,s}^\gamma (1 + \mathcal{N}_{m,r,p}(S)),$$

(A.29)

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