An $\mathcal{N}=2$ Supersymmetric Membrane Flow

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We find $M$-theory solutions that are holographic duals of flows of the maximally supersymmetric ($\mathcal{N}=8$) scalar-fermion theory in $(2+1)$ dimensions. In particular, we construct the $M$-theory solution dual to a flow in which a single chiral multiplet is given a mass, and the theory goes to a new infra-red fixed point. We also examine this new solution using $M2$-brane probes. The $(2+1)$-dimensional field theory fixed-point is closely related to that of Leigh and Strassler, while the $M$-theory solution is closely related to the corresponding IIB flow solution. We recast the IIB flow solution in a more geometric manner and use this to obtain an Ansatz for the $M$-theory flow. We are able to generalize our solution further to obtain flows with del Pezzo sub-manifolds, and we give an explicit solution with a conifold singularity.
1. Introduction

Holographic RG flows have been fairly widely studied in using $D3$ branes in IIB supergravity, but considerably less has been done for large $N$ theories on branes of other dimensions. There are several fairly obvious reasons for this, but probably the primary reason is that IIB supergravity is dual, on the $D3$-brane, to a very interesting theory: $\mathcal{N}=4$ supersymmetric Yang-Mills. Moreover, one can study flows of this theory in which some, or all of the supersymmetry is broken, and the resulting field theory on the brane exhibits some very non-trivial quantum behavior. The other maximally supersymmetric holographic field theories proposed in \cite{1} arise in M-theory, and correspond to superconformal theories on either a stack of $M5$-branes or on a stack of $M2$-branes. Here we focus on the latter, and this is primarily because the theory on the brane is a renormalizable, $(2 + 1)$-dimensional field theory that is closely related to $\mathcal{N}=4$ Yang-Mills theory in $(3 + 1)$ dimensions.

The field theory on the worldvolume of a single $M2$-brane is a conformally invariant $\mathcal{N}=8$ supersymmetric theory (16 supersymmetries) with eight scalar fields and eight fermions. As required by superconformal invariance in three-dimensions, there is an $SO(8)$ $\mathcal{R}$-symmetry under which the scalars, fermions and supersymmetries transform as the $8_v$, $8_c$ and $8_s$, respectively. On a collection of $(N + 1)$ $M2$-branes, there is an $SO(8)$-invariant theory with $8(N + 1)$ scalar degrees of freedom, corresponding to the transverse positions of $(N + 1)$ $M2$-branes, as well as their superpartners. One of these $\mathcal{N}=8$ multiplets corresponds to the free theory describing the center-of-mass motion of the system and is decoupled. The remaining degrees of freedom parameterize a moduli space $(\mathbb{R}^8)^N/S_{N+1}$. At the fixed points in the moduli space, the theory is an interacting superconformal field theory \cite{2}. This field theory also arises as a UV limit of the Kaluza-Klein reduction of $\mathcal{N}=4$ supersymmetric Yang-Mills theory on a circle. The extra scalars in three-dimensions come from the components of the gauge fields along the circle and a Wilson line parameter around the circle.

In supergravity, or M-theory, the maximally supersymmetric solution dual to the large $N$, brane vacuum configuration is the compactification of M-theory on $AdS_4 \times S^7$. There is a consistent truncation \cite{3,4} of this supergravity theory to the massless sector, that is, to gauged $\mathcal{N}=8$ supergravity in four dimensions \cite{5}. This gauged supergravity theory in four dimensions contains 70 scalar fields, and these are holographically dual to the (traceless) bilinears in the scalars and fermions:

\begin{equation}
O^{IJ} = \text{Tr} \left( X^I X^J \right) - \frac{1}{8} \delta^{IJ} \text{Tr} \left( X^K X^K \right), \quad I, J, \ldots = 1, \ldots, 8
\end{equation}

\begin{equation}
P^{AB} = \text{Tr} \left( \lambda^A \lambda^B \right) - \frac{1}{8} \delta^{AB} \text{Tr} \left( \lambda^C \lambda^C \right), \quad A, B, \ldots = 1, \ldots, 8,
\end{equation}

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where $O^{IJ}$ transforms in the $35_v$ of $SO(8)$, and $P^{AB}$ transforms in the $35_c$. Thus, gauged $\mathcal{N}=8$ supergravity in four dimensions can be used to study mass perturbations, and a uniform subsector of the Coulomb branch of the $\mathcal{N}=8$ field theory on the $M2$-brane. The gauged $\mathcal{N}=8$ supergravity in four dimensions thus plays a very analogous role to the gauged $\mathcal{N}=8$ supergravity in five dimensions. There is, however, a significant difference: the Yang-Mills theory has a freely choosable (dimensionless) coupling constant and $\theta$-angle, and these are dual to a pair of scalars in the five-dimensional gauged supergravity theory. The scalar-fermion theory on the $M2$ branes has no free coupling: In particular, three-dimensional super Yang-Mills theory flows to the UV interacting superconformal fixed point, the gauge coupling is driven to infinity $[2]$. There are thus no supergravity fields dual to a coupling: there are only masses and vevs in the dual of the four-dimensional gauged supergravity.

The fact that the conformal scalar-fermion theory is necessarily strongly coupled makes it, a priori, hard to analyze. However, we understand its holographic dual as well as we understand that of $\mathcal{N}=4$ Yang-Mills theory. More to the point, the general (non-conformal) $\mathcal{N}=8$ scalar-fermion theory has played a very interesting role in helping us understand softly broken $\mathcal{N}=4$ Yang-Mills, that is the so-called $\mathcal{N}=1^*$ theories. In a beautiful paper $[6]$ Dorey argued that one could compute a quantum exact superpotential for the softly broken scalar-fermion theory considered as a reduction of softly broken $\mathcal{N}=4$ Yang-Mills on a circle. Even more remarkably, it was argued that this superpotential was independent of the radius of the circle, and thus gave quantum exact information about ground states and domain walls of the original $\mathcal{N}=4$ Yang-Mills theory. This contention was strongly supported by the fact that the superpotential exactly reproduced the known quantum ground states structure of the $\mathcal{N}=1^*$ theories $[7]$. The role of the modular $S$-duality group was manifest in the results of $[6]$, and was subsequently used to great effect in $[8]$ to obtain exact modular expressions for various ground-state vevs. This also proved to be a powerful tool in probing the brane description of the $\mathcal{N}=1^*$ flows of Yang-Mills theory $[6][7][8]$.

In terms of branes, the link between the scalar-fermion theory and Yang-Mills theory can be implemented rather directly: One simply compactifies $D3$-branes on a circle and T-dualizes. The result is a uniform distribution of $D2$-branes on the T-dual circle in IIA supergravity. This can then be lifted to a distribution of $M2$ branes on a torus in M-theory. A distribution of such branes can then be analyzed in terms of a Coulomb branch deformation of a set of localized $M2$-branes. If one considers a set of $D3$-branes in the large $N$ limit then, because of the cosmological factor, the radius of the compactifying
circle becomes very large in the UV, and very small in the IR. The fact that the radius of the circle vanishes in the IR means that one must work with the T-dual description to understand properly the infra-red behavior. In this paper we will examine the link between the large $N$ Yang-Mills and the large $N$ scalar-fermion theory by looking more closely at links between the holographically dual theories. We will do this in a context that is not part of the $\mathcal{N}=1^*$ flows of \cite{9,10,6,8}, but instead will look at the M-theory analog of the Leigh-Strassler fixed point \cite{12}.

The dual supergravity theories are the gauged $\mathcal{N}=8$ supergravity theories in five (for $D3$ branes) and four (for $M2$ branes) dimensions. The scalar fields of these theories live on an $E_6(6)/USp(8)$ and an $E_7(7)/SU(8)$ coset respectively. These groups will play a significant role in this paper, and it is important to understand their interrelationship. First, $E_6(6)$ commutes with an $SO(1,1)$ in $E_7(7)$. This extra non-compact scalar may be identified with the relative scale of $S^5$ and the circle upon which the $D3$ branes are wrapped. The group, $E_6(6)$, contains a particularly important subgroup, $SL(6,\mathbb{R}) \times SL(2,\mathbb{R})$. The non-compact scalars in $SL(6,\mathbb{R})$ lie in the $20'$ of $SO(6)$ and are holographically dual to scalar bilinears in the $\mathcal{N}=4$ Yang-Mills theory. If one turns on scalars in the $20'$ only, the round $S^5$ upon which the IIB theory is compactified is deformed ellipsoidally to the surface:

$$\{x \in \mathbb{R}^P: x^T S^T S x = 1\},$$

(1.2)

where $P = 6$ and $S \in SL(6,\mathbb{R})$. The non-compact scalars in $SL(2,\mathbb{R})$ correspond to the gauge coupling and $\theta$-angle, and are thus moduli of the supergravity theory as well.

The situation is similar, but different for $M2$ branes. The group $E_{7(7)}$ has a maximal non-compact subgroup $SL(8,\mathbb{R})$. This group contains the abovementioned $SL(6,\mathbb{R}) \times SL(2,\mathbb{R})$ in the obvious manner. Indeed the latter commutes with an $SO(1,1)$ generator defined in $SL(8,\mathbb{R})$ by:

$$h \equiv \text{diag}(1,1,1,1,1,1,-3,-3).$$

(1.3)

Moreover, the subgroup of $E_{7(7)}$ that commutes with $h$ is precisely $E_6(6)$. While the embedding of the groups is extremely straightforward, the gauged supergravity theories do not embed directly into one another. This is because the minimal couplings are rather different ($SO(6)$ vs. $SO(8)$), and as a result the supersymmetrization proceeds differently, and in particular the supergravity potentials are not easily related. The $SL(2,\mathbb{R})$ subgroup of $SL(8,\mathbb{R})$ is in no way special in the four-dimensional gauged supergravity; all 35

\footnote{There are however, some rather interesting results that can be obtained in this area \cite{13}.}
scalars are on the same footing. If one turns on scalars in the $35_v$ only, the round $S^7$ upon which the M theory is compactified is deformed to the ellipsoidal surface defined by $(1.2)$ with $P = 8$.

The foregoing supergravity picture leads to a generalized class of $F$-theory compactifications. Recall that M-theory on a $T^2$ is dual to IIB on $S^1$, and that the complex structure modulus of the $T^2$ is dual to the dilaton and axion, while the Kähler modulus of the $T^2$ is dual to the radius of the $S^1$ [14,15]. Thus we can identify the M-theory $T^2$ with the torus of $F$-theory, and this generalizes to elliptically fibered Calabi-Yau manifolds. From the discussion of the $S^7$ compactification of M-theory we see a very similar structure: The dilaton and axion parameters of the IIB theory have been folded into the 7-metric. To be more precise, combining the $S^7$ with the radial coordinate, one has a compactification of M-theory on an 8-manifold. The shape of the metric in two directions of this 8-manifold is parametrized by the $SL(2,\mathbb{R})$ of the IIB theory, and the scale of this 2-metric is dual to the radius of the circle upon which the IIB $D3$-branes are wrapped. The 8-manifold may well be a singular elliptic fibration, and the full 11-metric is a warped product, but the compactification of M-theory on the 8-manifold, and of the IIB theory on the 7-manifold of the radial coordinate and $S^5 \times S^1$, is a generalization of the usual F-theory and M-theory story.

In this paper we will examine these ideas for the $\mathcal{N}=1$ supersymmetric flow in which one chiral multiplet is given a mass, and in the infra-red this field may be “integrated out” to leave a non-trivial conformal fixed point theory. The holographic version of this has been extensively studied for the flow of $\mathcal{N}=4$ Yang-Mills to the $\mathcal{N}=1$ supersymmetric Leigh-Strassler fixed point [12–19,11]. In section 2 we will review the ten-dimensional solution the IIB supergravity that corresponds to this flow [11], and we will rewrite the solution in a more geometrically transparent manner. We will show that the solution of [11] is a generalization of the compactification solutions of [20], and by recasting it in this way we see how to create more solutions in the same class. In particular, it will lead to a natural Ansatz for M-theory compactifications. This Ansatz is further supported by calculating the T-dual of the IIB solution to obtain a solution of IIA supergravity and its lift it to M-theory.

Our primary purpose here is to examine the analog of the Leigh-Strassler flow in the scalar-fermion theory in $(2+1)$ dimensions. The supergravity critical point was found long ago [21–23], and it has an $SU(3) \times U(1)$ symmetry, and $\mathcal{N}=2$ supersymmetry (8 supersymmetries) in the bulk. This critical point has several amusing features and
was studied in [23]. More recently this supergravity solution was studied from the four-dimensional perspective in the context of RG flows of the scalar-fermion theory [24,25]. Indeed, these authors found the $\mathcal{N} = 2$ supersymmetric RG flow solution from the $\mathcal{N} = 8$ superconformal point. In section 3 we summarize the relevant results of [22,21,23,24], and then in section 4 we will use the ideas of section 2 to construct the lift of the four-dimensional solution to M-theory. We find that the deformed geometry of $S^7$ in the lift contains a $\mathbb{CP}^2$ (upon which the $SU(3)$ acts transitively). In the same spirit as [20], we find that this $\mathbb{CP}^2$ can be replaced by any Einstein-Kähler space, and in particular by $S^2 \times S^2$. This leads to a solution with a conifold singularity. More generally, we argue that the solution of [11] and the solutions presented here are examples of a general class of solutions, and that in $2n+1$ dimensions, there will be a solution to Einstein’s equations with an $n$-form potential, and in which there is a $2(n-1)$-dimensional (real) Einstein-Kähler submanifold.

Finally, at the end of section 4 we compute the potential, and metric on the moduli space of an M2-brane probe in our solutions. The results are very similar to those of the D3-brane probe of the corresponding solution of IIB supergravity [26]. Section 5 contains some final remarks on the structure of our new solutions.

2. The geometry of the $\mathcal{N}=1$ Supersymmetric IIB flow

2.1. The flow in ten dimensions

We first recall some of the results of [18] and [11]. The holographic form of the Leigh-Strassler flow is described in five-dimensional supergravity by two scalars, denoted $\alpha$ and $\chi$. These are respectively dual to the operators:

$$ \mathcal{O}_1 \equiv \text{Tr}(-X_1^2 - X_2^2 - X_3^2 - X_4^2 + 2 X_5^2 + 2 X_6^2), $$
$$ \mathcal{O}_3 \equiv \text{Tr}(\lambda_3 \lambda_3) + h.c. . \quad (2.1) $$

The five-dimensional metric is taken to be:

$$ ds_{1,4}^2 = dr^2 + e^{2A(r)} (\eta_{\mu\nu} dx^\mu dx^\nu) . \quad (2.2) $$

\footnote{Remember that the brane is (2+1)-dimensional, and so the supersymmetry parameters have two components, not four.}
The equations describing the flow are then:

\[
\frac{d\alpha}{dr} = \frac{1}{2L} \frac{\partial W}{\partial \alpha}, \quad \frac{d\chi}{dr} = \frac{1}{L} \frac{\partial W}{\partial \chi}, \quad \frac{dA}{dr} = -\frac{2}{3L} W, \quad (2.3)
\]

where \( L \) is the radius of the \( AdS_5 \), and \( W \) is the superpotential:

\[
W = \frac{1}{4} \rho^4 \left( \cosh(2\chi) - 3 \right) - \frac{1}{2\rho^2} \left( \cosh(2\chi) + 1 \right), \quad (2.4)
\]

where \( \rho \equiv e^{\alpha} \).

This can then be lifted to a solution of the IIB theory in which the ten-dimensional metric is given by:

\[
ds_{10}^2 = \Omega^2 ds_{4,5}^2 + ds_5^2, \quad (2.5)
\]

where \( ds_5^2 \) is the metric on the deformed \( S^5 \) and \( \Omega \) is the warp-factor. The IIB dilaton and axion are constant, but there is a non-trivial \( B \)-field. The general metric Ansatz \([16,11]\) is given by computing a non-trivial metric on \( \mathbb{R}^6 \) and projecting it onto the unit \( S^5 \). Let \( x^I, I = 1, \ldots, 6 \) be Cartesian coordinates on \( \mathbb{R}^6 \) with \( S^5 \) defined by \( \sum(x^I)^2 = 1 \), then for the flow defined above we have:

\[
ds_5^2 = L^2 \frac{\text{sech} \chi}{\xi} (dx^I Q^{-1}_{IJ} dx^J) + L^2 \frac{\sinh \chi \tanh \chi}{\xi^3} (x^I J_{IJ} dx^J)^2. \quad (2.6)
\]

In this equation \( Q \) is a diagonal matrix with \( Q_{11} = \ldots = Q_{44} = \rho^{-2} \) and \( Q_{55} = Q_{66} = \rho^4 \), \( J \) is an antisymmetric matrix with \( J_{14} = J_{23} = J_{65} = 1 \), and \( \xi^2 = x^I Q_{IJ} x^J \). The warp factor is simply

\[
\Omega^2 = \xi \cosh \chi. \quad (2.7)
\]

We thus see that the metric is a combination of ellipsoidal squashing of the \( S^5 \), and a stretching of the Hopf fiber.

Following \([11]\), we introduce complex coordinates on this \( \mathbb{R}^6 \)

\[
u^1 = x^1 + i x^4, \quad \nu^2 = x^2 + i x^3, \quad \nu^3 = x^5 - i x^6, \quad (2.8)
\]

and then reparametrize them using an \( SU(2) \) group action:

\[
\begin{pmatrix} \nu^1 \\ \nu^2 \end{pmatrix} = \cos \theta M(\alpha_1, \alpha_2, \alpha_3) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \nu^3 = e^{-i\phi} \sin \theta, \quad (2.9)
\]
where $\alpha_1, \alpha_2, \alpha_3$ are Euler angles. Associated to $M$ are the left-invariant 1-forms: $\sigma_k = \text{Tr}(dM \cdot M^{-1} J_k)$. Explicitly these are:

$$
\begin{align*}
\sigma_1 &= \cos(\alpha_3) \, d\alpha_1 + \sin(\alpha_1) \sin(\alpha_3) \, d\alpha_2, \\
\sigma_2 &= \sin(\alpha_3) \, d\alpha_1 - \sin(\alpha_1) \cos(\alpha_3) \, d\alpha_2, \\
\sigma_1 &= d\alpha_3 + \cos(\alpha_1) \, d\alpha_2.
\end{align*}
$$

(2.10)

and they satisfy $d\sigma_i = \frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k$

Finally, define:

$$
X_1(r, \theta) = \cos^2 \theta + \rho(r)^6 \sin^2 \theta.
$$

(2.11)

and then the warp-factor, $\Omega$ is given by:

$$
\Omega = \rho^{-\frac{1}{2}} (\cosh \chi)^{\frac{1}{2}} X_1^T.
$$

(2.12)

and the ten-dimensional metric can be diagonalized in terms of the following frames:

$$
\begin{align*}
e^{\mu+1} &= \rho^{-\frac{1}{2}} (\cosh \chi)^{\frac{1}{2}} X_1^T e^A \, dx^\mu, \quad \mu = 1, \ldots, 4, \\
e^5 &= \rho^{-\frac{1}{2}} (\cosh \chi)^{\frac{1}{2}} X_1^T \, dr, \\
e^6 &= \rho^{\frac{3}{2}} (\cosh \chi)^{\frac{1}{2}} X_1^{-\frac{3}{4}} \left( \sin^2 \theta \, d\phi + \frac{1}{2} \cos^2 \theta \, \sigma_3 \right), \\
e^7 &= \rho^{\frac{3}{2}} (\cosh \chi)^{-\frac{1}{2}} X_1^\frac{1}{4} \, d\theta, \\
e^8 &= L \rho^{\frac{3}{2}} (\cosh \chi)^{-\frac{1}{2}} X_1^\frac{1}{4} \sin \theta \cos \theta \left( d\phi - \frac{1}{2} \sigma_3 \\
&\quad + (1 - \rho^6) X_1^{-1} \left( \sin^2 \theta \, d\phi + \frac{1}{2} \cos^2 \theta \, \sigma_3 \right) \right), \\
e^9 &= \frac{1}{2} L \rho^{\frac{3}{2}} (\cosh \chi)^{-\frac{1}{2}} X_1^{-\frac{1}{4}} \cos \theta \, \sigma_1, \\
e^{10} &= \frac{1}{2} L \rho^{\frac{3}{2}} (\cosh \chi)^{-\frac{1}{2}} X_1^{-\frac{1}{4}} \cos \theta \, \sigma_2.
\end{align*}
$$

(2.13)

It is worth noting that for the round $S^5$, $e^7, \ldots, e^{10}$ define the $\mathbb{CP}^2$ base of the Hopf fibration, and the 1-form:

$$
\omega \equiv \left( \sin^2 \theta \, d\phi + \frac{1}{2} \cos^2 \theta \, \sigma_3 \right),
$$

(2.14)

is that of the fiber.

This frame basis is slightly simpler than that of [11], but the $B$-fields become much simpler in this system:

$$
\mathcal{B} \equiv \frac{1}{2} B_M dy^M \wedge dy^P = \frac{i}{2} \, e^{-i\phi} \sinh \chi (e^7 - i e^8) \wedge (e^9 - i e^{10}).
$$

(2.15)
What is apparent here, and was not noted in [11], is that the $B$-field is “doubly-null” in frames. That is, it is the wedge of two complex frames whose norm is zero. The metric above appears to have an almost complex structure defined by

$$
J = e^5 \wedge e^6 + e^7 \wedge e^8 + e^9 \wedge e^{10},
$$

and if this, or some multiple of it, were integrable, then the field $B$ would carry two holomorphic indices.

We thus see that the foregoing solution is a generalization of the compactification technique of [20,27]. This technique was used to create non-trivial compactifications based upon any Einstein-Kähler manifold of complex dimension $n$. The idea was to introduce a tensor gauge field, $A$, of the form

$$
A = e^{-ik\phi} d\zeta_1 \wedge \ldots \wedge d\zeta_n,
$$

where the $\zeta_j$ are complex coordinates on the $n$-fold, and $\phi$ is the coordinate on the $U(1)$ fibration defined via the Kähler structure. There is generically no globally defined holomorphic $(n,0)$-form on the $n$-fold (unless $c_1 = 0$), but for suitable choice of $k$, (2.17) yields a globally defined form on the total space of the fibration. It was shown in [20,27] that one could find interesting anti-de Sitter compactifications of higher-dimensional supergravities using a background that involves (2.17), and in which the metric is that of the total space of the fibration but with a “stretched” fiber.

One can thus view the solutions of [19,11] as a generalization of this class in which the original Einstein-Kähler space is also ellipsoidally deformed, and the space-time metric can be that of a flow (2.2) and not just anti-de Sitter space. The solutions of [20,27] were also generically non-supersymmetric, whereas the whole point of these ellipsoidally squashed solutions is that they are supersymmetric.

As we will see, recasting the solution of [11] in the foregoing manner will lead to fairly generalizations in M-theory.
2.2. The T-dual of the ten-dimensional flow

We now compactify the $y \equiv x^3$ coordinate in the brane to a circle of radius $R_1$ and follow the standard procedure for construction the T-dual of a string background [28,29]. The transformation of the metric is elementary, and simply replaces $g_{22} \to (g_{22})^{-1}$. The IIB dilaton is constant, and so the IIA dilaton background is:

$$e^{2\tilde{\phi}} = (g_{22})^{-1} = \frac{\rho e^{-2A(r)}}{X_1 \cosh(\chi)} = e^{-2A(r)} \Omega^2.$$  

(2.18)

Since there are no components of $B_{\mu\nu}$ in the $y$-direction, and there are no off-diagonal elements of the metric involving $y$, the $B^{NS}_{MN} \equiv \Re(B_{MN})$ field is unchanged. The Ramond-Ramond field, $B^{RR}_{MN}$ similarly has no component in the $y$ direction, and so its T-dual gives $A^{(3)}_{MNy} = B^{RR}_{MN}$. The $A^{(4)}$-field does have a component in the $y$-direction, and its $T$-dual merely involves dropping this $y$ index. Thus we get the following expressions for the IIA backgrounds in terms of the IIB fields:

$$A^{(3)}_{MNy} = \Im m(B_{MN}), \quad A^{(3)}_{\mu\nu\rho} = A^{(4)}_{\mu\nu\rho y}.$$  

(2.19)

Technically, for a non-zero, but constant axion field, $C^{(0)}$ there is a corresponding constant IIA background vector field, $A_y = C^{(0)}$.

It is equally straightforward to lift this solution to M-theory. Let $\varphi$ be the extra circle of radius $R_2$, then the corresponding M-theory metric is:

$$ds_{11}^2 = e^{-\frac{2}{3}\tilde{\phi}} ds_{11A}^2 + e^{\frac{2}{3}\tilde{\phi}} (d\varphi + A)^2.$$  

(2.20)

The radii $R_1$, $R_2$ and the dilaton field thus parametrize the metric moduli of the torus, $T^2$ defined by $(y, \varphi)$. The tensor gauge fields lift to eleven dimensions in the obvious manner:

$$A^{(3)}_{M\varphi} = \Re(B_{MN}) , \quad A^{(3)}_{MNy} = \Im m(B_{MN}) , \quad A^{(3)}_{\mu\nu\rho} = A^{(4)}_{\mu\nu\rho y}.$$  

(2.21)

Putting this all together, we arrive at the following metric in eleven dimensions:

$$ds_{11}^2 = e^{\frac{2}{3}A(r)} \Omega^\frac{2}{3} (dt^2 + e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu) + L^2 e^{\frac{2}{3}A(r)} \Omega^{-\frac{2}{3}} \cosh^4(\chi) \omega^2$$

$$+ e^{-\frac{2}{3}A(r)} \Omega^{-\frac{2}{3}} (ds_2^2 + \frac{1}{4} L^2 \rho^2 e^{2A(r)} \cos^2 \theta \left( \sigma_1^2 + \sigma_2^2 \right))$$

$$+ \frac{L^2 \Omega^\frac{2}{3} e^{\frac{2}{3}A(r)}}{\rho^2 \cosh^2(\chi)} \left(d\theta^2 + \sin^2 \theta \cos^2 \theta \left( d\phi - \frac{1}{2} \sigma_3 + (1 - \rho^6) X_1^{-1} \omega \right)^2 \right),$$  

(2.22)

where $ds_2^2$ is the metric on the flat torus, and we now have $\mu, \nu, \ldots = 0, 1, 2.$
The tensor gauge field background becomes:
\[
A_{\mu \nu \rho}^{(3)} = A_{\mu \nu y}^{(4)}, \quad A_{MNz}^{(3)} = B_{MN},
\]
where we have introduced the natural complex coordinate \(z = \varphi + (C_0 + ie^{-\Phi})y\) on the torus \(T^2\), and we have been more careful in incorporating the effect of the non-trivial flat metric on \(T^2\).

The basic structure of the \(F\)-theory compactification is now more evident. By virtue of the warp-factors, the \(T^2\) fiber is most naturally paired with the \(S^2\) upon which the \(SU(2)\) isometry acts. The internal tensor gauge field, \(A^{(3)}\), is obtained by adding the holomorphic index of the \(T^2\) to the “\((2, 0)\)” structure of the field, \(B_{MN}\). It is precisely this structure that we generalize in subsequent sections.

One should, of course, remember that the foregoing solution represents a uniform distribution of \(M2\) branes spread over the \(T^2\). This solution is a function of the radial coordinate, \(r\), of \(\mathbb{R}^6\), and not \(\mathbb{R}^8\). In subsequent sections we will be looking for \(M2\)-brane solutions that are localized in \(\mathbb{R}^8\), and the solution above should be some kind of generalized “Coulomb branch” of the more localized brane distributions. Based on our experience of other Coulomb branch flows, the results above suggest that the more localized \(M2\)-brane solutions should have a similar natural complex structure to the metric and to the \(B\)-field background. We will indeed find that this is the case.

3. The holographic RG flow in four dimensions

The analogue of the LS flow in four dimensions is the \(N = 2\) supersymmetric RG flow in \(\mathcal{N} = 8\), four-dimensional gauged supergravity constructed in [24, 25]. The two scalar fields, \(\lambda\) and \(\lambda'\), in this flow parametrize an \(SU(3) \times U(1)\) invariant subspace of the full scalar manifold \(E_{7(7)}/SU(8)\). The explicit dependence of the scalar 56-bein,
\[
\mathcal{V}(\lambda, \lambda') = \begin{pmatrix}
u_{ij}^{ab} & v_{ijcd} \\ u_{klab} & u_{kld}^{cd} \end{pmatrix},
\]
on \(\lambda\) and \(\lambda'\) fields has been obtained in [21, 25]. We recall that all indices \(i, j\) and \(a, b\) in (3.1) run from 1 to 8 and correspond to the realization of \(E_{7(7)}\) in the \(SU(8)\) basis.

\[\text{The } SU(8) \text{ basis corresponds to the use of } 8_v \text{-indices of } SO(8). \text{ In this basis, the scalars (35_v) and pseudoscalars (35_c) are represented by self-dual and ati-self-dual 4-forms. To compute the metric and to compare operators in the dual SCFT, it is more convenient to use the } SL(8, \mathbb{R}) \text{ basis. This is a triality rotation of the } SU(8) \text{ basis in which the } 8_v \text{-indices are converted to } 8_c \text{-indices using gamma matrices.}\]
We refer the reader to either [23], or the Appendix of [24] for the explicit results that we will use here.

The structure of the scalar sector of the $\mathcal{N}=8$ supergravity is encoded in the $SU(8)$ T-tensor $T$:

$$T^{kij} = (u^{ij} + v^{iab}) (u^{lm} u^{km} - v^{lm} v^{km}).$$

In particular, the superpotential, $W$, for the flow is found as one of the eigenvalues of the symmetric tensor

$$A^i_j \equiv -\frac{4}{21} T^m_{ijm}, \quad W = A^i_i = A^8_8.$$

To preserve the analogy with the five-dimensional flow, we introduce new fields

$$\alpha = \frac{\lambda}{4\sqrt{2}}, \quad \chi = \frac{\lambda'}{\sqrt{2}},$$

and define $\rho = e^{\alpha}$. Indeed, upon rotation to the $SL(8,\mathbb{R})$ basis $(x^{IJ}, y^{IJ})$ the generator, $Q$, corresponding to the $\alpha$ field is diagonal in $SL(8,\mathbb{R})$

$$Q = \left( \begin{array}{cc} t^{[ij]}_{[ij]} & 0 \\ 0 & t^{[kl]}_{[kl]} \end{array} \right),$$

where

$$t^{[ij]}_{[ij]} = q^i + q^j, \quad t^{[kl]}_{[kl]} = q_i + q_j, \quad q_i = -q_i,$$

and

$$q^i = \frac{1}{2} (-1, +3, +3, -1, -1, -1, -1, -1).$$

An explicit realization of the $SO(8)$ matrices $\Gamma_{IJ}$ we use here can be obtained as follows: Starting with the $SO(7)$ gamma matrices in Appendix C.1 of [18] we define

$$\Gamma_I = -i \Gamma^\text{FGPW}_I, \quad \Gamma_7 = -i \Gamma^\text{FGPW}_0, \quad I = 1, \ldots, 6.$$
which is the counterpart of the similar result in five dimensions. In subsequent sections we will make an \(SO(8)\) rotation of this so as to place the two +3 eigenvalues in the last two entries of \(q^i\).

In terms of new fields the superpotential is \([24]\):

\[
W(\alpha, \chi) = \frac{1}{8} \rho^6 (\cosh(2\chi) - 3) - \frac{3}{8} \frac{1}{\rho^2} (\cosh(2\chi) + 1),
\]

and the supergravity potential is then given by:

\[
P(\alpha, \chi) = \frac{4}{L^2} \left[ \frac{1}{6} \left( \frac{\partial W}{\partial \alpha} \right)^2 + \left( \frac{\partial W}{\partial \chi} \right)^2 - 3 W^2 \right].
\]

The study of supersymmetric flows closely parallels the discussion in \([18]\). One considers a metric of the form (2.2), but now with \(\mu, \nu = 0, 1, 2\). One then finds that the supersymmetric flow equations are given by \([24]\):

\[
A'(r) = -\frac{2}{L} W,
\]

and

\[
\frac{d\rho}{dr} = \frac{1}{8L} \frac{(\cosh(2\chi) + 1) + (\cosh(2\chi) - 3)\rho^8}{\rho},
\]

\[
\frac{d\chi}{dr} = \frac{1}{2L} \frac{(\rho^8 - 3)\sinh(2\chi)}{\rho^2}.
\]

From this it is evident that there is a supersymmetric critical point at \(\rho = 3^{1/8}, \cosh(2\chi) = 2\). At this point we have \(W = -\frac{1}{2}3^{3/4}\). This is the \(\mathcal{N} = 2\) supersymmetric critical point found in \([21]\), and studied in \([23]\). The flow that we are primarily interested in is the one that starts at the \(\mathcal{N} = 8\) point, and finishes at the non-trivial \(\mathcal{N} = 2\) supersymmetric point.

It is a useful exercise to determine the operators which are dual to the supergravity fields \(\rho\) and \(\chi\). In \([23]\), the scalar expectation value, \(\rho\), was given by a self-dual form written in the \(SU(8)\) basis. An \(SO(8)\)-triality rotation of this self-dual form to the \(SL(8, \mathbb{R})\) basis results in an \(8 \times 8\), symmetric, traceless matrix from which one can read off the dual operator: \(O^{(\rho)} = O^{77} + O^{88}\), where \(O^{IJ}\) was defined in (1.1). The pseudoscalar \(\chi\) is given by an anti-self-dual form which, after a triality rotation is found to be dual to \(\mathcal{P}(\chi) = \mathcal{P}^{33} - \mathcal{P}^{44}\). Despite the apparently disparate indices, these two operators do indeed lie in the same supermultiplet on the brane.
4. Generalizing the supersymmetric flow

Our goal is ultimately to construct the lift to $M$-theory of the solution described in section 3. We will, however, proceed rather more generally and start by abstracting some ideas from the flow described in section 2. We will thus be first led to an Ansatz for $S^{2n+1}$, and we will then implement it on an $S^7$ in $M$-theory.

4.1. Stretching and squashing spheres

The metric we ultimately want is an ellipsoidally squashed sphere with a stretched Hopf fiber. To set our notation, and explain this terminology, we begin with a brief review of the Hopf fibration and its stretching.

Introduce Cartesian coordinates, $x^I$, $I = 1, \ldots, 2n + 2$, on $\mathbb{R}^{2n+2}$ and think of $S^{2n+1}$ as defined by the surface $\sum_I (x^I)^2 = 1$. Now define complex coordinates:

$$z^1 = x^1 + ix^2, \ldots, z^{n+1} = x^{2n+1} + ix^{2n+2},$$

and an associated Kähler form, $J_{IJ}$, with:

$$J_{12} = J_{34} = \ldots = J_{2n+1, 2n+2} = 1.$$  

Introduce projective coordinates, $\zeta_j$, and the Hopf fiber angle, $\psi$, via:

$$z^i = \zeta^i z^{n+1}, \quad i = 1, \ldots, n; \quad z^{n+1} = (1 + \zeta^i \zeta^i)^{-1/2} e^{i\psi}.$$  

In these coordinates the metric on $S^{2n+1}$ becomes:

$$ds^2 = (d\psi + A_n)^2 + ds^2_{FS(n)},$$

where $ds^2_{FS(n)}$ is the Fubini-Study metric on $\mathbb{CP}^n$, and $A_n$ is the potential for the Kähler form on $\mathbb{CP}^n$. More explicitly, we have:

$$ds^2_{FS(n)} = \frac{d\zeta^i d\bar{\zeta}_i}{1 + \zeta^i \zeta^i}, \quad (\zeta^i d\bar{\zeta}_i)(\bar{\zeta}_j d\zeta^j) = \frac{(\zeta^i d\bar{\zeta}_i)(\bar{\zeta}_j d\zeta^j)}{(1 + \zeta^i \zeta^i)^2},$$

and

$$A_n = -\frac{i}{2} \frac{\zeta^i d\zeta^i - \zeta^i d\bar{\zeta}_i}{1 + \zeta^i \zeta^i},$$

One may also verify the following rather useful identities:

$$ds^2_{FS(n)} = (dx)^2 - (x J dx)^2, \quad d\psi + A_n = x J dx.$$
The metric on the stretched sphere \( S^{2n+1} \) is thus given by:

\[
\begin{align*}
\frac{ds^2(\chi)}{ds^2_{FS(n)}} &= \left( (dx)^2 - (xJdx)^2 \right) + \cosh^2(\chi)(xJdx)^2 \\
&= \frac{ds^2_{FS(n)}}{\cosh^2(\chi)} (d\psi + A(n))^2 , \tag{4.8}
\end{align*}
\]

The parameter, \( \chi \), represents the stretching factor, with \( \chi = 0 \) corresponding to the round sphere. The isometry of \( S^{2n+1} \) is, of course, \( SO(2n + 2) \), and stretching the Hopf fiber breaks this to \( U(n + 1) \).

The metrics we wish to construct not only have this stretched fiber, but are also ellipsoidally squashed, and this further reduces the isometry to \( SU(n) \times U(1)^2 \). The ellipsoidal squashing is done by following the construction of (2.6), and by once again making a non-trivial metric in \( \mathbb{R}^{2n+2} \) and then projecting it onto the unit \( S^{2n+1} \). Let \( Q \) be a diagonal matrix

\[
Q = \text{diag}(\rho^{-2}, \ldots, \rho^{-2}, \rho^{2n}, \rho^{2n}), \tag{4.9}
\]

The metric on the deformed \( \mathbb{R}^{2n+2} \) is then given by:

\[
\begin{align*}
\frac{ds^2(\rho, \chi)}{ds^2_{FS(n-1)}} &= dx^I Q^{-1}_{IJ} dx^J + \frac{\sinh^2(\chi)}{\xi^2} (x^I J_I dx^J)^2 , \tag{4.10}
\end{align*}
\]

where \( \xi^2 = x^I Q_{IJ} x^J \). The metric (2.6) is simply \( L^2(\xi \cosh \chi)^{-1} ds^2(\rho, \chi) \) with \( n = 2 \). This class of metrics, for \( \chi = 0 \), were also obtained in the study of consistent truncations in [31].

To write the metric (4.10) in terms of intrinsic coordinates on \( S^{2n+1} \), we split the coordinates according to the eigenvalues of \( Q \) by setting:

\[
\begin{align*}
x^i &= \cos \mu \ u^i, \quad i = 1, \ldots, 2n; \quad x^{2n+1,2} = \sin \mu \ v^{1,2} , \tag{4.11}
\end{align*}
\]

where \( u^i \) parametrize a unit \( S^{2n-1} \) and \( v^{1,2} \) a unit circle. Using (4.11) and (1.3), we verify that

\[
\begin{align*}
vJdv &= d\psi , \quad (uJdu) = d\psi + \frac{A(n)}{\cos^2 \mu} . \tag{4.12}
\end{align*}
\]

It is now straightforward to verify that the metric (4.10) can be written in the following diagonal form

\[
\begin{align*}
\frac{ds^2(\rho, \chi)}{ds^2_{FS(n-1)}} &= \left( \rho^{-4} \xi^2 d\mu^2 + \rho^2 \cos^2 \mu \ ds^2_{FS(n-1)} + \xi^{-2} \omega^2 \right) + \cosh^2 \chi \xi^{-2} (d\psi + A(n))^2 . \tag{4.13}
\end{align*}
\]
where

\[ \omega = \frac{1}{2}(\rho^4 - \rho^{-4}) \sin(2\mu) d\psi + \rho^4 \tan \mu A_{(n)}. \]

Comparing (4.13) with (4.8), we see that the non-trivial squashing deforms the metric on the \( \mathbb{C}P^n \) base and rescales the Hopf fiber, but preserves the \( \mathbb{C}P^{n-1} \), whose symmetry group is \( SU(n) \). It should be remembered that \( A_{(n)} \) is the vector potential for the Kähler structure on \( \mathbb{C}P^n \), and if one decomposes it into components on \( \mathbb{C}P^{n-1} \), then another angular coordinate, \( \tilde{\psi} \), emerges in \( A_{(n)} \). There are thus two \( U(1) \) symmetries, namely rotations in \( \psi \) and \( \tilde{\psi} \).

This leads to a rather natural Ansatz for an \( n \)-form potential on the deformed \( S^7 \):

\[ C^{(n)} = F(\chi, \rho) e^{i(\kappa_1 \psi + \kappa_2 \tilde{\psi})} (e_1 - ie_2) \wedge \ldots \wedge (e_{2n-1} - ie_{2n}), \quad (4.14) \]

In this equation the \( e_j \) are an orthonormal frame such that \( (e_{2j-1} - ie_{2j}) \), \( j = 1, \ldots, n-1 \) are holomorphic frames on \( \mathbb{C}P^{n-1} \), \( F \) is generically an arbitrary function of \( \rho \) and \( \chi \) (but we will specify it more completely below), and \( \kappa_a \) are constants to be determined by requiring that \( C^{(n)} \) be globally defined, and further fixed by the determining the unbroken \( U(1) \) symmetry.

4.2. The \( \mathcal{N}=2 \) supersymmetric flow in M-theory

Given the known results for consistent truncation of gauged \( \mathcal{N}=8 \) supergravity in four dimensions [3,4], we can obtain the metric of deformed 7-sphere compactification of M-theory rather directly. To be explicit, we have:

\[ ds_{11}^2 = \Delta^{-1} (dr^2 + e^{2A(r)}(\eta_{\mu\nu}dx^\mu dx^\nu)) + ds_7^2, \quad (4.15) \]

where \( \mu, \nu = 0, 1, 2 \). The inverse metric on \( S^7 \) is given by [4]:

\[ \Delta^{-1} g^{pq} = (K_{MN})^p (\Gamma_{MN})^{ab}(u_{ij}^{ab} + v_{ijab})(u^{ij}_{\cd} + v^{ij_{cd}})(\Gamma_{PQ})^{cd} (K_{PQ})^q, \quad (4.16) \]

where \( K_{MN} = x^M \partial_N - x^N \partial_M \), and \( x^I \) are coordinates in \( R^8 \). As usual, the warp factor, \( \Delta \), is defined by:

\[ \Delta \equiv \sqrt{\det(\tilde{g}_{mp} \tilde{g}^{pq})}, \quad (4.17) \]

where the inverse metric, \( \tilde{g}^{pq} \), is that of the “round” \( S^7 \). One can compute \( \Delta \) by taking the determinant of both sides of (4.16). In (4.16) we have also inserted \( \Gamma \)-matrices so as to triality rotate the \( SO(8) \) Killing vectors. This has the effect of changing the \( SU(8) \)
indices on \( u, v \) to those the \( SL(8, \mathbb{R}) \) basis of \( E_{7(7)} \), and it in this basis that the ellipsoidal squashing is more directly visible.

Using this formula, we find the following form for the metric on \( S^7 \):

\[
ds^2_7 = \Delta^{\frac{1}{2}} ds^2(\rho, \chi),
\]

where \( ds^2(\rho, \chi) \) is given by (4.10), \( \xi^2 = x^I Q_{IJ} x^J \), and

\[
\Delta = (\xi \cosh \chi)^{-\frac{4}{3}}.
\]

In particular, we recover a metric that is conformally related to one of the metrics described in the previous subsection.

We now introduce a spherical parametrization of this metric in a manner closely analogous to [11]. The coordinates \( u^i \) and \( v^a \) of (4.11) are replaced according to:

\[
\begin{align*}
    u^1 + i u^2 &= \sin \theta \cos(\frac{1}{2} \alpha_1) e^{\frac{i}{2} (\alpha_2 + \alpha_3)} e^{i(\phi + \psi)}, \\
    u^3 + i u^4 &= \sin \theta \sin(\frac{1}{2} \alpha_1) e^{-\frac{i}{2} (\alpha_2 - \alpha_3)} e^{i(\phi + \psi)}, \\
    u^5 + i u^6 &= \cos \theta e^{i(\phi + \psi)}, \\
    v^1 + i v^2 &= e^{i\psi}.
\end{align*}
\]

The coordinate, \( \psi \), is that of the Hopf fiber on \( S^7 \), while \( \psi + \phi \) is the Hopf fiber coordinate of the \( S^5 \) defined by the \( u^i \).

The left-invariant 1-forms are given by (2.10), and one can easily rewrite the metric on \( \mathbb{C} \mathbb{P}^2 \) in terms of them:

\[
ds^2_{FS(2)} = d\theta^2 + \frac{1}{4} \sin^2 \theta \left( \sigma_1^2 + \sigma_2^2 + \cos^2 \theta \sigma_3^2 \right). \tag{4.21}
\]

Similarly, the metric in \( \mathbb{C} \mathbb{P}^3 \) may be written:

\[
ds^2_{FS(3)} = d\mu^2 + \cos^2 \mu \left( d_{FS(2)}^2 + \sin^2 \mu (d\phi + \frac{1}{2} \sin^2 \theta \sigma_3)^2 \right). \tag{4.22}
\]
Using these coordinates, we find the following set of frames for the eleven-dimensional metric (4.13):

\[ e^\mu + 1 = e^A (\cosh \chi)^{2/3} \frac{X^{1/3}}{\rho^{2/3}} \, dx^\mu, \quad \mu = 0, 1, 2, \]
\[ e^4 = (\cosh \chi)^{2/3} \frac{X^{1/3}}{\rho^{2/3}} \, dr, \]
\[ e^5 = a (\text{sech} \chi)^{1/3} \frac{X^{1/3}}{\rho^{8/3}} \, d\mu, \]
\[ e^6 = a (\text{sech} \chi)^{1/3} \frac{\rho^{4/3}}{X^{1/6}} \cos \mu \, d\theta, \]
\[ e^7 = \frac{a}{2} (\text{sech} \chi)^{1/3} \frac{\rho^{4/3}}{X^{1/6}} \cos \mu \sin \theta \, \sigma_1, \]
\[ e^8 = \frac{a}{2} (\text{sech} \chi)^{1/3} \frac{\rho^{4/3}}{X^{1/6}} \cos \mu \sin \theta \, \sigma_2, \]
\[ e^9 = \frac{a}{4} (\text{sech} \chi)^{1/3} \frac{\rho^{4/3}}{X^{1/6}} \cos \mu \sin(2\theta) \, \sigma_3, \]
\[ e^{10} = \frac{a}{2} (\text{sech} \chi)^{1/3} \frac{\rho^{16/3}}{X^{2/3}} \sin(2\mu) \left[ (1 - \frac{1}{\rho^8}) \, d\psi + (d\phi + \frac{1}{2} \sin^2 \theta \, \sigma_3) \right], \]
\[ e^{11} = a (\cosh \chi)^{2/3} \frac{\rho^{4/3}}{X^{2/3}} [d\psi + \cos^2 \mu (d\phi + \frac{1}{2} \sin^2 \theta \, \sigma_3)], \]

where the constant, \( a \), will be fixed momentarily, and

\[ X(r, \mu) \equiv \cos^2 \mu + \rho(r)^8 \sin^2 \mu. \]

In computing the Ricci tensor we use the equations of motion (3.11) and (3.12). For \( \rho = 1 \) and \( \chi = 0 \) we must recover the \( AdS_4 \times S^7 \) solution in which one has:

\[ R_{AB} = \frac{6}{L^2} \text{diag} (2, -2, -2, -2, 1, \ldots, 1), \quad \text{(4.24)} \]

where \( A, B \) are frame indices. This fixes the constant, \( a \), according to:

\[ a = L, \quad \text{(4.25)} \]

that is the radii of the \( AdS_4 \) and \( S^7 \) are \( L/2 \) and \( L \), respectively. We also find that the general Ricci tensor has only two non-vanishing off-diagonal components: \( R_{45} \) and \( R_{1011} \). This tensor also satisfies obvious symmetries due to the Poincaré and \( SU(3) \) invariance.
but it also satisfies a non-trivial identity parallel to the one found in [11]. Thus we find in general that:

\[
R_{11} = -R_{22} = -R_{33} = 2R_{66} = 2R_{77} = 2R_{88} = 2R_{99},
\]

where all the indices are frame indices.

For the antisymmetric field \( F^{(4)} \) we take an ansatz similar to that of [11], and motivated by (2.17). First note that for \( \chi = 0 \) and \( \rho = 1 \) the internal metric of (4.23) contains a \( \mathbb{CP}^3 \) factor, and that the Kähler form of this, when written in terms of frames, is

\[
\mathcal{J} = \frac{1}{2} L^2 dA_{(3)} = e^5 \wedge e^{10} + e^6 \wedge e^9 + e^7 \wedge e^8,
\]

which implies that the natural basis of the holomorphic 1-forms consists of

\[
e^5 - ie^{10}, \quad e^6 - ie^9, \quad e^7 - ie^8.
\]

We thus take the internal part of \( A^{(3)} \) to be the real part of:

\[
C^{(3)} \equiv c \sinh \chi e^{i(\kappa_1 \psi + \kappa_2 \phi)} (e^5 - ie^{10}) \wedge (e^6 - ie^9) \wedge (e^7 - ie^8),
\]

where \( c, \kappa_1 \) and \( \kappa_2 \) are some real constants. This is of the form (4.14), and the arbitrary function \( F(\rho, \chi) \) is now fixed by the proper choice of frames, and through comparison with (2.15).

As is implied by (2.23), the tensor \( A^{(3)} \) also has a space-time part that is very similar to the Ansatz for the \( A^{(4)} \)-tensor in the IIB theory. We therefore take:

\[
A^{(3)} = \tilde{W}(r, \mu) e^{3A(r)} dx^0 \wedge dx^1 \wedge dx^2 + (C^{(3)} + (C^{(3)})^\ast),
\]

where \( \tilde{W}(r, \mu) \) is a “geometric superpotential” to be determined.

The equations of motion (in the conventions of [20]) are:

\[
R_{MN} + R g_{MN} = \frac{1}{3} F^{(4)}_{M^P Q^R} F^{(4) P^Q R^S}, \quad d \ast F^{(4)} = F^{(4)} \wedge F^{(4)},
\]

where \( \ast \) is defined using \( \epsilon^{1 \cdots 11} = 1 \).

Starting with the Einstein equations, one finds that the right-hand side has generically non-vanishing off-diagonal terms whereas the corresponding components of the Ricci tensor vanish. These off-diagonal components can be made to vanish by setting:

\[
\kappa_1 = -4, \quad \kappa_2 = -3.
\]
The \((10,11)\)-component determines \( c \) (up to a sign), and so we have

\[
c = \frac{1}{4}.
\] (4.32)

The \((4,4)\), \((4,5)\) and \((5,5)\) components may be used to determine \( \tilde{W} \) (again, up to a sign), and we find

\[
\tilde{W}(r, \mu) = \frac{1}{4 \rho^2} \left[ (\cosh(2\chi) + 1) \cos^2 \mu - \rho^8 (\cosh(2\chi) - 3) \sin^2 \mu \right].
\] (4.33)

With these values for the constants and using (4.33), we find that all of the equations of motion of \( M \)-theory are indeed satisfied.

### 4.3. Brane probes

It is relatively straightforward to perform a brane probe calculation of the supergravity solution presented above, and the results are directly parallel those of [26,32]. This is perhaps not surprising since dimensional reduction and T-dualization of the IIB solution effectively adds one more complex scalar, and thereby extends the Coulomb moduli space.

The \( M2 \)-brane calculation is very similar to the \( D \)-brane probe calculation. One starts with an action:

\[
S = \int d^3 \sigma \left[ \sqrt{-\det(\tilde{g})} + \frac{1}{4} \tilde{A}^{(3)} \right].
\] (4.34)

where \( \tilde{g} \) and \( \tilde{A}^{(3)} \) denote the pull-back of the metric and the 3-form onto the membrane. The normalization of the \( A^{(3)} \)-term in (4.34) is twice the usual normalization since this is the normalization that we have used in the eleven-dimensional equations of motion. As usual, we consider a probe that is parallel to the source membranes, and assume that it is traveling at a small velocity transverse to its world-volume. This calculation produces a potential, \( V \), and a kinetic term for the brane probe. If the potential vanishes, then the kinetic term provides us with a metric on the corresponding moduli space, and this metric has the form \( h_{ab} = \delta^{1/2} (g_{00})^{-1} g_{ab} \), where \( g_{MN} \) is the eleven-dimensional metric, \( a, b \) index coordinates transverse to the \( M2 \)-branes, and \( \delta \) is the determinant of the projection of \( g_{MN} \) parallel to the brane.

We find the following expression for the potential:

\[
V = e^{3A(r)} (\Delta^{-\frac{4}{3}} - 2 \tilde{W}) = 2 e^{3A(r)} \rho^6 \sinh^2 \chi \sin^2 \mu,
\] (4.35)

which is very similar to that found in [29].
This potential vanishes for $\mu = 0$, and on this subspace we have the following metric on the 6-dimensional moduli space transverse to the branes:

$$ds^2 = \Delta^{-\frac{3}{2}} e^A \left[ dr^2 + L^2 \rho^4 sech^2 \chi ds^2_{FS(2)} + L^2 \rho^4 (d\psi + d\phi + \frac{1}{2} \sin^2 \theta \sigma_3)^2 \right].$$

As one approaches the critical point one can introduce a new radial coordinate, $u \sim e^{\frac{1}{2} A(r)}$ to obtain the following asymptotic form of the metric:

$$ds^2 \sim du^2 + \frac{3}{2} u^2 ds^2_{FS(2)} + \frac{9}{4} u^2 (d\psi + d\phi + \frac{1}{2} \sin^2 \theta \sigma_3)^2.$$  

This form is very similar of that found in [26]: In the latter D3-brane probe calculation, the asymptotic metric had a similar conical singularity at $u = 0$, with the $\mathbb{CP}^2$ replaced by $S^2$, and the “stretching factors” $\frac{3}{2}$ and $\frac{9}{4} = (\frac{3}{2})^2$ replaced by $\frac{4}{3}$ and $\frac{16}{9} = (\frac{4}{3})^2$. The meaning of these conical singularities in moduli space has yet to be adequately explained, but they may indicate that the supergravity coordinates are not appropriate to the correct description of the moduli space. We understand that a forthcoming paper [33] will greatly elucidate this issue.

4.4. Generalizations and conjectures

The construction of solutions of [20] was done in two steps: the first was to obtain a solution on a deformed $S^7$, and the second step was to generalize the result to the canonical $U(1)$ bundle over an arbitrary Einstein-Kähler manifold. One can obviously try to do the same thing here. It is not clear whether one could replace the $\mathbb{CP}^3$ in our construction by an arbitrary Einstein-Kähler 3-fold. On the other hand it does seem very plausible that one could replace the $\mathbb{CP}^2$ parametrized by $\theta, \alpha_j$ by an arbitrary Einstein-Kähler 2-fold. This is because this space is homogeneous and presumably the eleven-dimensional Ricci tensor only depends upon whether the 2-fold is Einstein, and if there is a canonical $U(1)$ bundle whose total space can also be made into an Einstein space.

Einstein metrics on Kähler manifolds have been fairly extensively studied. In particular, for 2-folds there are the obvious ones: $\mathbb{CP}^2$ and $S^2 \times S^2$, and also the less obvious: there are Einstein-Kähler metrics on the del Pezzo surfaces, $P_k$, for $k \geq 3$ (for further discussion of this see, for example, [34–36]). The total spaces of the canonical $U(1)$ bundles over these spaces can be made into an Einstein-Sasaki manifold, and have been used for compactifications of supergravity without fluxes. The trivial case is the sphere, $S^{2n+1}$, as a $U(1)$ bundle over $\mathbb{CP}^n$. A less trivial example in the $T^{1,1}$ space used in [27] to compactify IIB supergravity. This space is, of course, a $U(1)$ bundle over $S^2 \times S^2$, and even more
significantly, it preserves some of the supersymmetry of the IIB theory. One can include fluxes in this compactification in much the same manner as \cite{20}. Indeed, (1.14) generalizes in an obvious manner. Moreover, if the Einstein metric on the $U(1)$ fibration leads to a supersymmetric compactification without fluxes, then one might hope that inclusion of a flux might preserve some supersymmetry.

While we have not tested all of these ideas, we have at least considered the metric defined from (4.23), but with $\mathbb{C}P^2$ replaced by $S^2 \times S^2$. To be more precise, we consider frames of the form (4.23), but with:

\begin{align}
    e^6 &= ab (\text{sech}\chi)^{1/3} \frac{\rho^{4/3}}{X^{1/6}} \cos \mu \, d\theta_1, \\
    e^7 &= ab (\text{sech}\chi)^{1/3} \frac{\rho^{4/3}}{X^{1/6}} \cos \mu \sin \theta_1 \, d\varphi_1, \\
    e^8 &= ab (\text{sech}\chi)^{1/3} \frac{\rho^{4/3}}{X^{1/6}} \cos \mu \, d\theta_2, \\
    e^9 &= ab (\text{sech}\chi)^{1/3} \frac{\rho^{4/3}}{X^{1/6}} \cos \mu \sin \theta_2 \, d\varphi_2, \\
    e^{10} &= \frac{a}{2} (\text{sech}\chi)^{1/3} \frac{\rho^{16/3}}{X^{2/3}} \sin(2\mu) \left[ -\rho^{-8} \, d\psi + e \, A \right], \\
    e^{11} &= a (\cosh \chi)^{2/3} \frac{\rho^{4/3}}{X^{2/3}} \left[ \sin^2 \mu \, d\psi + e \cos^2 \mu \, A \right].
\end{align}

where

\[ A = (d\psi + d\phi + \cos \theta_1 \, d\varphi_1 + \cos \theta_2 \, d\varphi_2) \]

The frames, $e^6, \ldots, e^9$ are proportional to those of $S^2 \times S^2$. Observe that the Hopf fiber connections $\cos \theta_j \, d\varphi_j$ appear in $e^{10}$ and $e^{11}$ with equal weight, and thus $S^2 \times S^2$ along with the coordinate $(\phi + \psi)$ map out the $T^{1,1}$ space. The value of $a, b$ and $e$ can be fixed so as to make the Ricci tensor have the form (4.24) for $\chi = 0$ and $\rho = 1$. Indeed, this fixes these constants to:

\[ a = L, \quad b = \frac{1}{\sqrt{6}}, \quad e = \frac{1}{3}. \]

These values of $b$ and $e$ are precisely those that give the $T^{1,1}$ space.

First, we check that, when written with frame indices, the Ricci tensor of this new metric is identical to that defined by (4.23) provided that one uses exactly the same equations of motion for $\chi$ and $\rho$. This supports the conjecture that the Ricci tensor is indeed
independent of the particular Einstein-Kähler 2-fold. Secondly, one can obviously generalize the Ansatz for $A^{(3)}$. In fact, we find that a minor modification of (4.29), where the geometric superpotential, $\tilde{W}(r, \mu)$, is kept the same while
\[
C^{(3)} = -\frac{1}{4} \sinh \chi e^{-\phi - 2\psi} (e^5 - ie^{10}) \wedge (e^6 + ie^7) \wedge (e^8 + ie^9),
\] (4.41)
yields a solution to the equations of motion. We find it quite intriguing that the same flow in four dimensions yields two different solutions to the equations of the 11-dimensional supergravity.

As regards the geometry, we first note that the brane-probe calculation for this new solution will be virtually identical to that of the previous subsection, merely with $\mathbb{CP}^2$ replaced by $S^2 \times S^2$. Perhaps more interesting is that at the other extreme, i.e. at $\mu = \frac{\pi}{2}$, as opposed to $\mu = 0$, the metric defined using (4.38) has a conifold singularity, and the $S^7$ degenerates to the conifold times $S^1$. It would seem reasonable to conjecture that the solution defined by the metric based upon $T^{1,1}$ could be related to the T-dual of the compactification solution of [27]. This would imply that it would also be related to the non-trivial fixed point of the holographic flow of [27]. Adding a flux to this solution, and ellipsoidally squashing would then represent a flow away from this non-trivial fixed point. One might further attempt to generalize this to ADE singularities as in [38], and presumably some of these might be related to using Einstein-Kähler metrics on del Pezzo surfaces.

4.5. Consistent truncation and simplifying the Ansatz

In [32] a number of formulae for the consistent truncation of the IIB theory were conjectured, and in particular the Ansatz for $A^{(4)}$ was given in terms of a geometric superpotential. We find that we can generalize this directly to M-theory.

We start by converting to the $SL(8, \mathbb{R})$ basis, and introducing rotated vielbeins
\[
U^{ij}_{IJ} = u_{ij}^{ab}(\Gamma_{IJ})^{ab}, \quad V^{ij}_{IJ} = v_{ij}^{ab}(\Gamma_{IJ})^{ab},
\]
\[
U^{ij}_{I} = u_{ij}^{ab}(\Gamma_{I})^{ab}, \quad V^{ij}_{I} = v_{ij}^{ab}(\Gamma_{I})^{ab},
\]
Define:
\[
A_{ijI} = U^{ij}_{IJ} + V^{ij}_{IJ}, \quad B_{ijI} = i(U^{ij}_{IJ} - V^{ij}_{IJ}),
\]
\[
C^{ij}_{IJ} = U^{ij}_{IJ} + V^{ij}_{IJ}, \quad D^{ijI} = -i(U^{ij}_{IJ} - V^{ij}_{IJ}),
\]
22
and observe that up to a constant, $k$, we have:

$$T_{ikj} = k C_{LM}^{ij} A_{lmJK} D^{kmKI} \delta^L_I \delta^M_J - B_{lm}^{JK} C^{kmKI} \delta^L_I \delta^M_J,$$

In this expression we have deliberately inserted explicit Kronecker $\delta$’s as we are going to want to think of $\delta_I^J$ as $SL(8, \mathbb{R})$ covariant, but $\delta^{IJ}$ as a metric in a particular $SL(8, \mathbb{R})$ frame. The general idea of (32) is to introduce geometric analogues of the $T$-tensor by replacing $\delta^{IJ}$ by $x^I x^J$, but leaving $\delta_I^J$ alone.

Now recall that the $A_1$-tensor is defined by

$$A_{ij}^1 = T_{im}^{mj}.$$ 

It is diagonal along the flow considered in section 3, and indeed the superpotential is read off from $W \propto A_{177}^1 = A_{88}^1$. Modifying the $A_1$ tensor as outlined above leads to the geometric $A_1$ tensor, which we will denote by $\tilde{A}_1$. We then find that

$$\tilde{A}_{177} + \tilde{A}_{88}^1 \propto \frac{1}{12 \rho^2} \left[ 2 \cosh^2 \chi \cos^2 \mu + \rho^8 (3 - \cosh(2\chi)) \sin^2 \mu \right] \equiv \tilde{W},$$

which is exactly the geometric superpotential introduced in (4.29) and (4.33).

There are also simplifications that can be made in writing the Ansatz for $C^{(3)}$. Define complex coordinates in $\mathbb{R}^8$ by:

$$z_1 = x^1 - ix^2, \quad z_2 = x^3 - ix^4, \quad z_3 = x^5 - ix^6, \quad z_4 = x^7 - ix^8.$$ 

These coordinates, $z^i$, are non-holomorphic, linear functions of the coordinates defined in (4.20): $z^1 = \cos \mu(u^1 - iu^2)$, $z^2 = \cos \mu(w^3 - iu^4)$, $z^3 = \cos \mu(w^5 - iu^6)$, $z^4 = \sin \mu(v^1 - iv^2)$. Using these new coordinates on $S^7$, we may write the internal background gauge field more simply as:

$$C_{(3)} = \frac{i L^3}{4 \sqrt{3}} \tan \frac{\chi}{X} \left( 3 z^{[1} d z^{2} \wedge d z^{3]} \wedge d z^4 - \rho^4 z^4 d z^1 \wedge d z^2 \wedge d z^3 \right).$$

5. Final comments

We have shown that the solution of (11) that represents the RG flow of $\mathcal{N} = 4$ Yang-Mills theory to the Leigh-Strassler fixed point can be recast in a more natural form. In particular, one sees that this solution represents a significant generalization of the class of solutions described in (20). This observation enables us to generalize easily the
supersymmetric flow solution to $M$-theory. The Ansatz we obtain is further supported by considering the T-dual, and lift to $M$-theory, of the IIB-flow of [11]. The key ingredients in the Ansatz are to first write the “round” compactification in terms of a $U(1)$ fibration over an Einstein-Kähler manifold. One then ellipsoidally squashes the Einstein-Kähler base, stretches the $U(1)$ fiber, and introduces a tensor gauge potential that proportional to the holomorphic “volume form” on the base. The latter tensor is really a non-trivial section of a bundle on the base, but is globally defined on the total-space of the bundle provided it is given the proper $U(1)$ charge. This formulation enables us to immediately generalize our solution from $S^7$ to manifolds that have del Pezzo spaces as sub-manifolds. In particular we obtain an explicit solution with a conifold singularity.

The holographic dual of the $\mathcal{N} = 2$ supersymmetric flow in $M$-theory is an $\mathcal{N} = 2$ supersymmetric flow of the $\mathcal{N} = 8$ scalar-fermion theory in $(2 + 1)$-dimensions. This strongly coupled theory must therefore have a $\mathcal{N} = 2$ supersymmetric fixed point that is completely analogous to the Leigh-Strassler fixed point of $\mathcal{N} = 4$ Yang-Mills theory. One might have naively expected this from field theory in that one can obtain the $(2 + 1)$-dimensional theory from trivial dimensional reduction of $\mathcal{N} = 4$ Yang-Mills theory on a circle. Alternatively, one could emulate the arguments of Leigh and Strassler directly in $(2 + 1)$ dimensions. What makes this less obvious is that the $(2 + 1)$-dimensional theory is strongly coupled even at the UV fixed point. Moreover, trivial dimensional reduction of the Yang-Mills theory will yield a theory at a different point in the Coulomb branch moduli space compared to the holographic UV fixed point in $M$-theory: As we saw, reduction and T-dualization leads to a uniform distribution of branes on a torus, and not to a set of localized branes. It was not a priori clear that this difference in moduli would be consistent with the flows to non-trivial fixed points. In terms of the gauged supergravity, the potentials of gauged four- and five-dimensional supergravity are rather different, and it was far from clear that a flow in one theory would be directly convertible into a flow in the other. Our results show that, at least for these supersymmetric flows, the naive expectations are in fact correct. It would be very interesting to probe the extent of this correspondence: The indications from field theory [6,8] are very favorable, at least for the ground state structure and domain walls. The supergravity version of this story is somewhat murkier, but should be clarified in the near future [13].

There are now many solutions of IIB supergravity that correspond to RG flows in four-dimensional field theories. For those solutions constructed directly in ten dimensions, the initial Ansatz and the final solution usually exploit details of some non-trivial topology, such as a non-trivial cycle and a non-trivial flux upon it. On the other hand, flows generated
by masses or vevs from a maximally supersymmetric field theory, such as the one considered here or those of \cite{39,40,41}, generally have no non-trivial topology to exploit. As a result, the ten- or eleven-dimensional geometry is rather hard to characterize. The results here at least shed a little more light on the geometric structure of these “non-topological” flows. One feels that the half-maximal supersymmetric flows (such as that of \cite{40}) should have a particularly simple geometric characterization. So far this has eluded us, but such a characterization could be very useful: As was shown in \cite{41,42} the $\mathcal{N} = 2$ flow of \cite{40} represents only one point on the continuum moduli space of the large $N$, Seiberg-Witten effective action. It would be very interesting to find the general solution with all the moduli, and understanding the supergravity geometry is probably crucial to doing this.

Acknowledgements

We would like to thank Jaume Gomis for many discussions, and Clifford Johnson for his comments and for giving us a preliminary copy of \cite{33}. This work was supported in part by funds provided by the DOE under grant number DE-FG03-84ER-40168.
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