BIRATIONAL TRANSFORMATIONS BELONGING TO GALOIS POINTS FOR A CERTAIN PLANE QUARTIC

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ABSTRACT. In this note, we study birational transformations belonging to Galois points for a certain plane quartic curve. In fact, we see that they can be extended to Cremona transformations. In particular, we determine their conjugacy class and show that they are all conjugate to linear transformations.

1. Introduction

In this note, we study birational transformations belonging to Galois points for a certain plane quartic curve. First, we briefly recall the notion of Galois points. Let $k$ be the field of complex numbers $\mathbb{C}$. We fix it as the ground field of our discussion. Let $C$ be an irreducible curve in $\mathbb{P}^2$ of degree $d$ ($d \geq 3$), and $k(C)$ be the function field of $C$. Taking a point $P$ of $\mathbb{P}^2$, we consider the projection $\pi_P : C \dashrightarrow \mathbb{P}^1$, which is the restriction of the projection $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ with center $P$. Then, we obtain the field extension induced by $\pi_P$, i.e., $\pi_P^* : k(\mathbb{P}^1) \hookrightarrow k(C)$. By putting $K_P = \pi_P^*(k(\mathbb{P}^1))$, we have the following definition.

Definition 1. The point $P$ is called a Galois point for $C$ if the field extension $k(C)/K_P$ is Galois. Then, we put $G_P = \text{Gal}(k(C)/K_P)$, which we call a Galois group at $P$. In particular, a Galois point $P$ is called a smooth Galois point if $P \in \text{Reg}(C)$, where $\text{Reg}(C)$ is the open subset of all non-singular points of $C$.

The notion of Galois point for $C$ was introduced by Yoshihara (cf. [8]) to study the structure of the field extension $k(C)/k$ from geometrical viewpoint.

Definition 2. We denote by $\delta(C)$ the number of smooth Galois points on $C$.

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We have studied Galois points for some cases (cf. [3], [6], [7], [8], [11], [12], [13], etc.). For example, we have determined \( \delta(C) \) and we have found the characterization of \( C \) having Galois points. On the other hand, when \( P \) is not a Galois point, we have determined the monodromy group and constructed the variety corresponding to the Galois closure.

The objective of this note is to investigate the action of \( G_P \) on \( C \). Specifically, our problem is stated as follows.

**Problem 1.** Study the properties of \( \sigma \).

For this problem, we first state the following theorem.

**Theorem 1** (Yoshihara, [11], [12]). Let \( V \) be a smooth hypersurface in \( \mathbb{P}^n \) of degree \( d \) (\( d \geq 4 \)). Then, every element \( \sigma \in G_P \) is a restriction of a projective transformation of \( \mathbb{P}^n \). In other words, there exists \( \tilde{\sigma} \in \text{PGL}(n+1, k) \) such that \( \tilde{\sigma}|_V = \sigma \). Furthermore, \( G_P \) is a cyclic group.

The conditions of smoothness and \( d \geq 4 \) are both required for the theorem to hold. Hence, in this note, we study the problem for a singular plane quartic curve \( C \). In particular, we study the case in which \( C \) has a simple cusp of multiplicity three.

Suppose that \( C \) is a quartic with a simple cusp of multiplicity three. Then, it is well known that \( C \) is projectively equivalent to one of the following (cf. [9]):

(a) \( X^4 - X^3Y + Y^3Z = 0 \),
(b) \( X^4 - Y^3Z = 0 \).

In the case of (a), \( C \) has two flexes of order one. Indeed \( Q_1 = (0 : 1 : 0) \) and \( Q_2 = (8 : 16 : 3) \) are flexes. Then, the tangent lines at these flexes meet \( C \) at \( P_1 = (1 : 1 : 0) \) and \( P_2 = (8 : -16 : 3) \), respectively. We can easily check that \( \pi_{P_i} \) is a totally ramified triple covering. Hence, we see that \( P_i \) is a smooth Galois point for \( C \) (\( i = 1, 2 \)). By the condition of flexes of \( C \), there is no other smooth Galois point. Hence, \( \delta(C) = 2 \). In the case of (b), we have a smooth Galois point \( P_3 = (0 : 1 : 0) \), which is a flex of order two. Since there is no more flex, \( \delta(C) = 1 \).

**Remark 1.** The curve of type (b) has a Galois point \( P_4 = (1 : 0 : 0) \in \mathbb{P}^2 \setminus C \). We call such a Galois point an *outer Galois point*. We can easily check that \( P_4 \) is a unique outer Galois point for \( C \) of type (b).
The objective of this note is to investigate the properties of birational transformations belonging to $P_1$, $P_2$, and $P_3$. We note that the Galois groups at $P_i$ are all isomorphic to the cyclic group of order three, $\mathbb{Z}_3$ ($i = 1, 2, 3$). By setting $G_{P_i} = \langle \sigma_i \rangle$ ($i = 1, 2, 3$), we state our main results as follows.

**Theorem 2.** All the transformations $\sigma_1$, $\sigma_2$, and $\sigma_3$ are (restrictions of) Cremona transformations. Furthermore, $\sigma_1$ and $\sigma_2$ are conjugate to linear transformations. In addition, $\sigma_3$ is a linear transformation.

**Remark 2.** By referring to [4], [1], we have the classification of the conjugacy classes of elements of order three in Bir$(\mathbb{P}^2)$ as follows.

- linear transformation
- automorphisms of special del Pezzo surfaces of degree 3
- automorphisms of special del Pezzo surfaces of degree 1

The Cremona group Bir$(\mathbb{P}^2)$ is the group of birational transformations of $\mathbb{P}^2$. This is a classical object in algebraic geometry. Hence, there are many results on Bir$(\mathbb{P}^2)$, e.g., [2], [3]. However, we have obtained very few results on birational transformations belonging to Galois points, e.g., [6], [7], [13]. In particular, it is difficult to determine when a birational transformation extends to a Cremona transformation.

2. **Proofs**

First, we prove the case for type (a).

**Claim 1.** Suppose that $C$ is a curve of type (a). Then, there exists a linear automorphism $A$ of $C$ such that $A(P_1) = P_2$.

**Proof.** By putting $A = \begin{pmatrix} 16 & -8 & 0 \\ 0 & -16 & 0 \\ 4 & -1 & 16 \end{pmatrix}$, we can easily check that $A(C) = C$ and $A(P_1) = P_2$.

Therefore, it is sufficient to study on $\sigma_1$. We can obtain a concrete representation of $\sigma_1$ as follows.

**Claim 2.** $\sigma_1$ is represented as $(X : Y : Z) \mapsto (XY : Y((\omega - 1)X + \omega Y) : Z((\omega - 1)X + \omega Y))$, up to projective coordinate change, where $\omega$ is a primitive cubic root of unity.

**Proof.** By taking a suitable projective coordinate, we may assume that $P_1$ is translated to $P'_1 = (1 : 0 : 0)$. Then, $\pi_{P'_1}$ is represented as $(X : Y : Z) \mapsto (Y : Z)$ and $C$ is defined by $(X + Y)^3Z - X^3Y = 0$. 

By putting $x = X/Z$, $y = Y/Z$, we can obtain the field extension induced by $\pi_{P_1'}$ as

$$k(C) \subseteq k(x, y) \cup k(\mathbb{P}^1) \subseteq k(y)$$

where $k(x, y)/k(y)$ is given by the equation $(x + y)^3 - x^3y = 0$. Then, we obtain $\left(\frac{x + y}{x}\right)^3 = y$, i.e., $\left(1 + \frac{y}{x}\right)^3 = y$. We put $G_{P_1'} = \langle \sigma_1' \rangle$.

Hence, we infer that $\sigma_1' \left(1 + \frac{y}{x}\right) = \omega \left(1 + \frac{y}{x}\right)$. Thus, we conclude that $\sigma_1'(x) = \frac{yx}{\omega x - x + \omega y}$. Since $\sigma_1'(x : y : 1) = (\sigma_1'(x) : y : 1)$ and $x = X/Z, y = Y/Z$, we get the claim. We note that $\sigma_1'$ is conjugate to $\sigma_1$ up to projective coordinate change. □

By using the above representation of $\sigma_1'$, we can prove the following.

**Claim 3.** $\sigma_1$ is conjugate to a linear transformation.

**Proof.** By the above claim, $\sigma_1'$ is represented as $\sigma_1'(x) = \frac{yx}{\omega x - x + \omega y}$. We may consider $\sigma_1'$ as an element

$$M_{\sigma_1'} = \begin{pmatrix} y & 0 \\ \omega & 0 \\ 1 & 0 \end{pmatrix} \in \text{PGL}(2, k(y))$$

Then, by putting $P = \begin{pmatrix} -y & 0 \\ 1 & 1 \end{pmatrix}$, we have

$$P^{-1}M_{\sigma_1'}P = \begin{pmatrix} y & 0 \\ 0 & \omega y \end{pmatrix}.$$

Assuming $\bar{\sigma}_1$ to be a transformation defined by $P^{-1}M_{\sigma_1'}P$, we have $\bar{\sigma}_1(x) = \frac{yx}{\omega y} = \omega^2x$. Then, we conclude that $\sigma_1$ is conjugate to $\bar{\sigma}_1$, which is a linear transformation defined by $\begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. □

**Remark 3.** The transformation defined by $P$ above is a Cremona transformation $(X : Y : Z) \mapsto (-XY : Y(X + Z) : Z(X + Z))$.

Thus, we prove the theorem for type (a).
Next, we prove the case for type (b). By putting \( x = X/Y, y = Z/Y \), we obtain the field extension \( \pi^*_P \) as \( k(x, y)/k(y) \), where \( x^3 - y = 0 \).

Then, it is easy to see that \( \sigma_3 \) is represented as
\[
\begin{pmatrix}
\omega & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \omega
\end{pmatrix}.
\]

Thus, we complete the theorem.

3. Problems

Finally, we raise some problems.

**Problem 2.** Let \( P \) be a Galois point for \( C \).

1. Find the condition when the birational transformation belonging to \( P \) can be extended to a Cremona transformation (cf. \[7], \[13]\).

2. Suppose that the birational transformation belonging to \( P \) is extended to a Cremona transformation. Then, is this a de Jonquières transformation? If this is not true, find the condition when that transformation becomes de Jonquières.

3. Investigate the relation among Galois points, the decomposition group and the inertia group. The decomposition group is the group of Cremona transformations that preserves \( C \), and the inertia group is the group of Cremona transformations that fix \( C \). That is,
\[
\text{Dec}(C) = \{ \varphi \in \text{Bir}(\mathbb{P}^2) | \varphi|_C : C \to C \},
\]
\[
\text{Ine}(C) = \{ \varphi \in \text{Bir}(\mathbb{P}^2) | \varphi|_C = \text{id}_C \}.
\]

Then, does the element of \( \text{Dec}(C) \) (resp. \( \text{Ine}(C) \)) preserve the Galois point \( P \) for \( C \)?

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