THE ORLICZ INEQUALITY FOR MULTILINEAR FORMS

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Abstract. The Orlicz (ℓ_2, ℓ_1)-mixed inequality states that
\[ \left( \sum_{j_1=1}^{n} \left( \sum_{j_2=1}^{n} |A(e_{j_1}, e_{j_2})| \right) \right)^{\frac{1}{2}} \leq \sqrt{2} \|A\| \]
for all bilinear forms \( A : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K} \) and all positive integers \( n \), where \( \mathbb{K}^n \) denotes \( \mathbb{R}^n \) or \( \mathbb{C}^n \) endowed with the supremum norm. In this paper we extend this inequality to multilinear forms, with \( \mathbb{K}^n \) endowed with \( \ell_p \) norms for all \( p \in [1, \infty) \).

1. Introduction

The origins of the theory of summability of multilinear forms and absolutely summing multilinear operators are probably associated to Orlicz (ℓ_2, ℓ_1)-mixed inequality published in the 1930’s (see [8, page 24]). It states that
\[ \left( \sum_{j_1=1}^{n} \left( \sum_{j_2=1}^{n} |A(e_{j_1}, e_{j_2})| \right)^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \|A\| \]
for all bilinear forms \( A : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K} \), and all positive integers \( n \). Here and henceforth \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \) and \( \mathbb{K}^n \) is endowed with the supremum norm. We also represent by \( e_k \) the canonical vectors in a sequence space and
\[ \|A\| := \sup \{|A(x, y)| : \|x\| \leq 1 \text{ and } \|y\| \leq 1\} . \]
An equivalent formulation is the following:
\[ \left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} |A(e_{j_1}, e_{j_2})| \right)^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \|A\| \]
for all continuous bilinear forms \( A : c_0 \times c_0 \to \mathbb{K} \). The exponents in (1) are optimal in the sense that, fixing the exponent 1, the exponent 2 cannot be replaced by smaller exponents (nor the exponent 1 can be replaced by smaller exponents) keeping the constant independent of \( n \). The Orlicz inequality is closely related to Littlewood’s (ℓ_1, ℓ_2)-mixed inequality (see [8, page 23]), which asserts that
\[ \left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} |A(e_{j_1}, e_{j_2})|^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \|A\| \]
for all continuous bilinear forms \( A : c_0 \times c_0 \to \mathbb{K} \). Again, the exponents are optimal in the same sense above described. Combining these two inequalities, and using the Hölder inequality for mixed sums we recover Littlewood’s 4/3 inequality:
\[ \left( \sum_{j_1,j_2=1}^{\infty} |A(e_{j_1}, e_{j_2})|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \sqrt{2} \|A\| \]

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for all continuous bilinear forms $A : c_0 \times c_0 \to \mathbb{K}$. For recent results on absolutely summing linear and multilinear operators we refer the interested reader to [6, 11, 17] and the references therein.

The exponent $4/3$ from the previous inequality cannot be replaced by smaller exponents keeping the constant independent of $n$. The constant $\sqrt{2}$ is optimal (in all the three inequalities) when $\mathbb{K} = \mathbb{R}$, but the optimal constants when $\mathbb{K} = \mathbb{C}$ are unknown.

In 1934 Hardy and Littlewood [10] (see also [13]) pushed the subject further, extending the above results to bilinear forms defined on $\ell_p$ spaces (when $p = \infty$ we consider $c_0$ instead of $\ell_\infty$). The investigation of extensions of the Hardy–Littlewood inequalities to multilinear forms were initiated by Praciano-Pereira [16] in 1981 and intensively investigated since then (see, for instance, [1, 2, 5, 7, 12, 13, 14, 15]), but there are still several open problems regarding the optimal exponents and optimal constants involved.

For the sake of simplicity, we shall use the same notation from [1]:

$$X_p := \begin{cases} \ell_p, & \text{if } p \in [1, \infty) \\ c_0, & \text{if } p = \infty \end{cases}$$

and, when $q = \infty$, the sum $\left( \sum_{j} \|x_j\|^q \right)^{1/q}$ shall represent the supremum of $\|x_j\|$. We also assume that $1/0 = \infty$ and $1/\infty = 0$ and denote the conjugate index of $s$ by $s^*$, i.e., $1/s + 1/s^* = 1$. One of the main goals of this line of research is to find the optimal values of the exponents $s_1, \ldots, s_m$ and of the constants $C(\mathbb{K})_{p_1, \ldots, p_m}$ satisfying

$$\left( \sum_{j=1}^{\infty} \left( \sum_{j_m=1}^{\infty} \left( \sum_{j_{m-1}=1}^{\infty} \cdots \left( \sum_{j_1=1}^{\infty} |A(e_{j_1}, \ldots, e_{j_m})|^{s_{m-1}} \right)^{s_{m-2}} \cdots \right)^{s_2} \right)^{s_1} \right)^{1/s} \leq C(\mathbb{K})_{p_1, \ldots, p_m} \|A\|$$

for all continuous $m$-linear forms $A : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$. The answer is known in several cases (see [1, 5, 18] and the references therein), but a complete solution is still unknown. In this note we shall be interested in investigating the optimal exponents $s_1, \ldots, s_m$. It is simple to prove that the optimal exponent $s_m$ associated to the sum $\sum_{j_m=1}^{\infty}$ is $p_m^*$. Our main result provides the optimal exponents $s_1, \ldots, s_m$ in the case that $s_m = p_m^*$.

From now on, let $r \geq 2$, and let $s_1, \ldots, s_m \in [1, \infty]$. Let us define $\delta_{s_k, \ldots, s_m}$ and $\lambda_{s_k, \ldots, s_m}$ by

$$\delta_{s_k, \ldots, s_m} := \frac{1}{\max \left\{ 1 - \frac{1}{s_k} + \cdots + \frac{1}{s_m}, 0 \right\}}$$

and

$$\lambda_{s_k, \ldots, s_m} := \frac{1}{\max \left\{ \frac{1}{r} - \frac{1}{s_k} + \cdots + \frac{1}{s_m}, 0 \right\}}$$

for all positive integers $m$ and $k = 1, \ldots, m$. Note that when $1/s_k + \cdots + 1/s_m \geq 1$ we have

$$\delta_{s_k, \ldots, s_m} = \infty$$

and, also, when $1/s_k + \cdots + 1/s_m \geq \frac{1}{r}$ we have

$$\lambda_{s_k, \ldots, s_m} = \infty.$$

Our main result is, in some sense, a generalization of the the Orlicz inequality. In fact, if we consider the very particular case $(m, p_1, p_2) = (2, \infty, \infty)$ and $\sigma$ as the identity map in its statement, we recover the Orlicz inequality:

**Theorem 1.1.** Let $m \geq 2$ be an integer and $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ be a bijection. If

$$(q_1, \ldots, q_{m-1}) \in (0, \infty)^{m-1},$$

$$(p_1, \ldots, p_m) \in [1, \infty]^m,$$

the following assertions are equivalent:
(1) There is a constant $C_{p_1, \ldots, p_m} \geq 1$ such that
\[
\left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_m=1}^{\infty} |A(e_{j_1}, \ldots, e_{j_m})|^{p_{\sigma(m)}} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C_{p_1, \ldots, p_m} \|A\|
\]
for all continuous $m$-linear forms $A : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$.

(2) The exponents $q_1, \ldots, q_{m-1}$ satisfy
\[
q_1 \geq \delta_{p_{\sigma(1)}}, \ldots, q_{m-1} \geq \delta_{p_{\sigma(m-1)}},
\]
where $\mu = \min\{p_{\sigma(m)}, 2\}$.

2. Preliminary results

Let $2 \leq q < \infty$ and $0 < s < \infty$. Recall that a Banach space $X$ has cotype $q$ if there is a constant $C > 0$ such that, no matter how we select finitely many vectors $x_1, \ldots, x_n \in X$,
\[
\left( \sum_{j=1}^{n} \|x_j\|^q \right)^{\frac{1}{q}} \leq C \left( \int_{[0,1]} \left( \sum_{j=1}^{n} r_j(t)x_j \right)^2 dt \right)^{1/2},
\]
where $r_j$ denotes the $j$-th Rademacher function. The infimum of the cotypes of $X$ is denoted by $\cot X$.

The following result was proved in [5]:

Theorem 2.1. (see [5]) Let $(q_1, \ldots, q_m) \in (0, \infty)^m$, and $Y$ be an infinite-dimensional Banach space with cotype $\cot Y$. If
\[
\frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{\cot Y},
\]
then the following assertions are equivalent:

(a) There is a constant $C_{p_1, \ldots, p_m} \geq 1$ such that
\[
\left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_m=1}^{\infty} \|A(e_{j_1}, \ldots, e_{j_m})\|^{q_{m-1}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C_{p_1, \ldots, p_m} \|A\|
\]
for all continuous $m$-linear operators $A : X_{p_1} \times \cdots \times X_{p_m} \to Y$.

(b) The exponents $q_1, \ldots, q_m$ satisfy
\[
q_1 \geq \lambda_{\cot Y}^{p_1}, \ldots, q_{m-1} \geq \lambda_{\cot Y}^{p_{m-1}}, q_m \geq \lambda_{\cot Y}^{p_m}.
\]

We need the following extension of the previous theorem, relaxing the hypothesis (3). Besides, below we have $(q_1, \ldots, q_m) \in (0, \infty)^m$ while in Theorem 2.1 we have $(q_1, \ldots, q_m) \in (0, \infty)^m$.

Theorem 2.2. Let $(q_1, \ldots, q_m) \in (0, \infty]^m$, $(p_1, \ldots, p_m) \in [1, \infty]^m$ and $Y$ be an infinite-dimensional Banach space with cotype $\cot Y$. The following assertions are equivalent:

(a) There is a constant $C_{p_1, \ldots, p_m} \geq 1$ such that
\[
\left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_m=1}^{\infty} \|A(e_{j_1}, \ldots, e_{j_m})\|^{q_{m-1}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C_{p_1, \ldots, p_m} \|A\|
\]
for all continuous $m$-linear operators $A : X_{p_1} \times \cdots \times X_{p_m} \to Y$.

(b) The exponents $q_1, \ldots, q_m$ satisfy
\[
q_1 \geq \lambda_{\cot Y}^{p_1}, \ldots, q_{m-1} \geq \lambda_{\cot Y}^{p_{m-1}}, q_m \geq \lambda_{\cot Y}^{p_m}.
\]
Proof. We begin by proving the direct implication. We just need to consider the case

\[ \frac{1}{p_1} + \cdots + \frac{1}{p_m} \geq \frac{1}{\cot Y}, \]

since the other case is covered by Theorem 2.1. By the Maurey–Pisier factorization result (see [9, pages 286, 287]), the Banach space \( Y \) finitely factors the formal inclusion \( \ell_{\cot Y} \to \ell_\infty \), i.e., there are universal constants \( C_1, C_2 > 0 \) such that, for all \( n \), there are vectors \( z_1, \ldots, z_n \in Y \) satisfying

\[ C_1 \left\| (a_j)_{j=1}^n \right\|_\infty \leq \left\| \sum_{j=1}^n a_j z_j \right\| \leq C_2 \left( \sum_{j=1}^n |a_j| \cot Y \right)^{1/\cot Y}, \]

for all sequences of scalars \( (a_j)_{j=1}^n \). Consider the continuous \( m \)-linear operator \( A_n : X_{p_1} \times \cdots X_{p_m} \to Y \) given by

\[ A_n(x^{(1)}, \ldots, x^{(m)}) = \sum_{j=1}^n x_j^{(1)} x_j^{(2)} \cdots x_j^{(m)} z_j. \]

By (6) and the Hölder inequality we have

\[ \|A_n\| = \sup_{\|x^{(1)}\|_{p_1} \leq 1, \ldots, \|x^{(m)}\|_{p_m} \leq 1} \left\| \sum_{j=1}^n x_j^{(1)} \cdots x_j^{(m)} z_j \right\| \leq C_2 \left( \sum_{j=1}^n |x_j^{(1)}| \cdots |x_j^{(m)}| \cot Y \right)^{1/\cot Y} \]

\[ \leq \sup_{\|x^{(1)}\|_{p_1} \leq 1, \ldots, \|x^{(m)}\|_{p_m} \leq 1} C_2 \left( \sum_{j=1}^n |x_j^{(1)}| \cdots |x_j^{(m)}| \right)^{1/\cot Y} \]

\[ \leq C_2. \]

Note that, by (7), we have

\[ \left( \sum_{j=1}^n \left( \sum_{j_2=1}^n \cdots \left( \sum_{j_m=1}^n \|A_n(e_{j_1}, \ldots, e_{j_m})\|_{q_m}^{\frac{q_{m-1}}{q_m}} \right)^{\frac{q_{m-1}}{q_{m}} \cdot \frac{q_1}{q_m}} \right) = \left( \sum_{j=1}^n \|z_j\|_{q_1} \right)^{\frac{1}{q_1}}. \]

Thus, by (6) we conclude that

\[ \left( \sum_{j=1}^n \left( \sum_{j_2=1}^n \cdots \left( \sum_{j_m=1}^n \|A_n(e_{j_1}, \ldots, e_{j_m})\|_{q_m}^{\frac{q_{m-1}}{q_m}} \right)^{\frac{q_{m-1}}{q_m} \cdot \frac{q_1}{q_m}} \right)^{\frac{1}{q_1}} \right)^{\frac{1}{p_1}} \geq C_1 n^{\frac{1}{q_1}}. \]

Combining the previous inequality with (4) and (8) we conclude that

\[ C_1 n^{1/q_1} \leq C_{p_1, \ldots, p_m}^Y C_2. \]

Thus, since \( n \) is arbitrary, we have

\[ q_1 = \infty = \lambda_{p_1, \ldots, p_m} \cot Y. \]

If

\[ \frac{1}{p_1} + \cdots + \frac{1}{p_m} \geq \frac{1}{\cot Y}, \]
for all $i$, the proof is immediate. Otherwise, let $i_0 \in \{2, 3, \ldots, m\}$ be the smallest index such that

\[
\frac{1}{p_{i_0}} + \cdots + \frac{1}{p_m} < \frac{1}{\cot Y},
\]

\[
\frac{1}{p_{i_0 - 1}} + \cdots + \frac{1}{p_m} \geq \frac{1}{\cot Y}.
\]

If $i_0 = 2$, note that by (9) we have

\[
(10) \quad \sup_{j_1} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_m=1}^{\infty} \|A(e_{j_1}, \ldots, e_{j_m})\|^m \right)^{\frac{q_{m-1}}{m}} \cdots \right) \leq \frac{C^Y_{p_1, \ldots, p_m}}{\sqrt[2]{\sqrt[2]{\cdots \sqrt[2]{\cdots}}}}
\]

for all continuous $m$-linear operators $A : X_{p_1} \times \cdots \times X_{p_m} \to Y$. From (10) it is simple to show that

\[
\left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_m=1}^{\infty} \|A(e_{j_2}, \ldots, e_{j_m})\|^m \right)^{\frac{q_{m-1}}{m}} \cdots \right) \leq C^Y_{p_1, \ldots, p_m} \|A\|,
\]

for all continuous $(m - 1)$-linear operators $A : X_{p_2} \times \cdots \times X_{p_m} \to Y$. Since

\[
\frac{1}{p_2} + \cdots + \frac{1}{p_m} < \frac{1}{\cot Y},
\]

by Theorem 2.1 we conclude that

\[
q_2 \geq \lambda_{\cot Y}^{p_2, \ldots, p_m}, q_3 \geq \lambda_{\cot Y}^{p_3, \ldots, p_m}, \ldots, q_{m-1} \geq \lambda_{\cot Y}^{p_{m-1}, p_m}, q_m \geq \lambda_{\cot Y}^{p_m}.
\]

If $i_0 = 3$, we consider

\[A(x^{(1)}, \ldots, x^{(m)}) = x_1^{(1)} \sum_{j=1}^{n} x_j^{(2)} \cdots x_j^{(m)} z_j\]

and we can imitate the previous arguments to conclude that

\[
q_2 = \infty = \lambda_{\cot Y}^{p_2, \ldots, p_m},
\]

and hence

\[
(11) \quad \sup_{j_1, j_2} \left( \sum_{j_3=1}^{\infty} \cdots \left( \sum_{j_m=1}^{\infty} \|A(e_{j_1}, \ldots, e_{j_m})\|^m \right)^{\frac{q_{m-1}}{m}} \cdots \right) \leq \frac{C^Y_{p_1, \ldots, p_m}}{\sqrt[2]{\sqrt[2]{\cdots \sqrt[2]{\cdots}}}} \|A\|,
\]

for all continuous $m$-linear operators $A : X_{p_1} \times \cdots \times X_{p_m} \to Y$. Again, it is plain that

\[
\left( \sum_{j_3=1}^{\infty} \cdots \left( \sum_{j_m=1}^{\infty} \|A(e_{j_3}, \ldots, e_{j_m})\|^m \right)^{\frac{q_{m-1}}{m}} \cdots \right) \leq C^Y_{p_1, \ldots, p_m} \|A\|,
\]

for all continuous $(m - 2)$-linear operators $A : X_{p_3} \times \cdots \times X_{p_m} \to Y$. Since

\[
\frac{1}{p_3} + \cdots + \frac{1}{p_m} < \frac{1}{\cot Y},
\]

by Theorem 2.1 we have

\[
q_3 \geq \lambda_{\cot Y}^{p_3, \ldots, p_m}, q_4 \geq \lambda_{\cot Y}^{p_4, \ldots, p_m}, \ldots, q_{m-1} \geq \lambda_{\cot Y}^{p_{m-1}, p_m}, q_m \geq \lambda_{\cot Y}^{p_m}.
\]

We conclude the proof in a similar fashion for $i_0 = 4, \ldots, m$.

Now we prove the reverse implication. The case

\[
\frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{\cot Y}
\]
is encompassed by Theorem 2.1. So, we shall consider
\[
\frac{1}{p_1} + \cdots + \frac{1}{p_m} \geq \frac{1}{\cot Y}.
\]

If
\[
\frac{1}{p_i} + \cdots + \frac{1}{p_m} \geq \frac{1}{\cot Y}
\]
for all \(i\), the proof is immediate. Otherwise, let \(i_0 \in \{2, \ldots, m\}\) be the smallest index such that
\[
\frac{1}{p_{i_0}} + \cdots + \frac{1}{p_m} < \frac{1}{\cot Y}.
\]

We need to prove that there is a constant \(C_{p_1, \ldots, p_m}^Y \geq 1\), such that
\[
\sup_{j_1, \ldots, j_{i_0-1}} \left( \sum_{j_{i_0} = 1}^{\infty} \left( \cdots \left( \sum_{j_m = 1}^{\infty} \| A(e_{j_1}, \ldots, e_{j_m}) \|^{\lambda_{p_m-1} p_m / \lambda_{\cot Y}^2 Y} \right)^{q_{i_0-1}} \right)^{q_{i_0}} \right)^{1 / q_{i_0}} \leq C_{p_1, \ldots, p_m}^Y \| A \|
\]
for
\[
q_{i_0} \geq \lambda_{p_{i_0-1} \ldots p_m}, \ldots, q_m \geq \lambda_{\cot Y}^2 Y.
\]

By Theorem 2.1, we know that for any fixed vectors \(e_{j_1}, \ldots, e_{j_{i_0-1}}\), there is a constant \(C_{p_{i_0}, \ldots, p_m}^Y \geq 1\), such that
\[
\sup_{j_1, \ldots, j_{i_0-1}} \left( \sum_{j_{i_0} = 1}^{\infty} \left( \cdots \left( \sum_{j_m = 1}^{\infty} \| A(e_{j_1}, \ldots, e_{j_m}) \|^{\lambda_{p_m-1} p_m / \lambda_{\cot Y}^2 Y} \right)^{q_{i_0-1}} \right)^{q_{i_0}} \right)^{1 / q_{i_0}} \leq C_{p_{i_0}, \ldots, p_m}^Y \| A \|
\]
for all continuous \(m\)-linear operators \(A: X_{p_1} \times \cdots \times X_{p_m} \to Y\). Then,
\[
\sup_{j_1, \ldots, j_{i_0-1}} \left( \sum_{j_{i_0} = 1}^{\infty} \left( \cdots \left( \sum_{j_m = 1}^{\infty} \| A(e_{j_1}, \ldots, e_{j_m}) \|^{\lambda_{p_m-1} p_m / \lambda_{\cot Y}^2 Y} \right)^{q_{i_0-1}} \right)^{q_{i_0}} \right)^{1 / q_{i_0}} \leq C_{p_{i_0}, \ldots, p_m}^Y \| A \|
\]
for all continuous \(m\)-linear operators \(A: X_{p_1} \times \cdots \times X_{p_m} \to Y\).

To conclude the proof we just need to remark that
\[
\sup_{j_1, \ldots, j_{i_0-1}} \left( \sum_{j_{i_0} = 1}^{\infty} \left( \cdots \left( \sum_{j_m = 1}^{\infty} \| A(e_{j_1}, \ldots, e_{j_m}) \|^{\lambda_{p_m-1} p_m / \lambda_{\cot Y}^2 Y} \right)^{q_{i_0-1}} \right)^{q_{i_0}} \right)^{1 / q_{i_0}} \leq \sup_{j_1, \ldots, j_{i_0-1}} \left( \sum_{j_{i_0} = 1}^{\infty} \left( \cdots \left( \sum_{j_m = 1}^{\infty} \| A(e_{j_1}, \ldots, e_{j_m}) \|^{\lambda_{p_m-1} p_m / \lambda_{\cot Y}^2 Y} \right)^{q_{i_0-1}} \right)^{q_{i_0}} \right)^{1 / q_{i_0}}
\]
provided

\[ q_{i_0} \geq \lambda_{\cot Y}^{p_{i_0}, \ldots, p_m}, \ldots, q_m \geq \lambda_{\cot Y}^{p_m}. \]

\[ \square \]

3. Proof of Theorem 1.1

Throughout this proof the adjoint of a Banach space \( X \) will be denoted by \( X^* \). To simplify the notation we will consider \( \sigma(j) = j \) for all \( j \); the other cases are similar. Let \( \mathcal{L}^m (X_{p_1}, \ldots, X_{p_m}; Y) \) denote the space of all continuous \( m \)-linear operators from \( X_{p_1} \times \cdots \times X_{p_m} \) to \( Y \). By the canonical isometric isomorphism

\[ \Psi: \mathcal{L}^m (X_{p_1}, \ldots, X_{p_m}; \mathbb{K}) \to \mathcal{L}^{m-1} (X_{p_1}, \ldots, X_{p_{m-1}}; (X_{p_m})^*) \]

and duality in \( X_{p_m} \), note that, if \( R \in \mathcal{L}^m (X_{p_1}, \ldots, X_{p_m}; \mathbb{K}) \), we have

\[ (x_1, \ldots, x_{m-1}, e_n) = \Psi(R) (x_1, \ldots, x_{m-1}) (e_n) = (\Psi(R) (x_1, \ldots, x_{m-1}))_n. \]

We start off by proving (1)\( \Rightarrow \) (2). Let us suppose that there is a constant \( C_{p_1, \ldots, p_m} \geq 1 \) such that

\[ (13) \quad \left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \ldots \left( \sum_{j_{m-1}=1}^{\infty} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^{p_m} \right)^{\frac{q_{m-1}}{q_m}} \right)^{\frac{q_m}{q_{m-1}}} \right)^{\frac{1}{q_1}} \leq C_{p_1, \ldots, p_m} \| T \|
\]

for all continuous \( m \)-linear forms \( T: X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K} \).

Consider a continuous \((m-1)\)-linear operator \( A: X_{p_1} \times \cdots \times X_{p_{m-1}} \to (X_{p_m})^* \). Then, using our hypothesis, we have

\[ (14) \quad \left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \ldots \left( \sum_{j_{m-1}=1}^{\infty} \| A(e_{j_1}, \ldots, e_{j_{m-1}}) \|^{q_{m-1}}_{(X_{p_m})^*} \right)^{\frac{q_m-1}{q_m}} \right)^{\frac{q_m}{q_{m-1}}} \right)^{\frac{1}{q_2}} \]

\[ \leq C_{p_1, \ldots, p_m} \| A \|
\]

for all continuous \((m-1)\)-linear operators \( A: X_{p_1} \times \cdots \times X_{p_{m-1}} \to (X_{p_m})^* \). Since \((X_{p_m})^* \) has cotype max\( \{p_m, 2\} \), by Theorem 2.2, the exponents \( q_1, \ldots, q_{m-1} \) in (2.2) satisfy

\[ (15) \quad q_1 \geq \lambda_{\max \{p_m, 2\}}, \ldots, q_{m-1} \geq \lambda_{\max \{p_m, 2\}}. \]

Since

\[ 1 - \frac{1}{\max \{p_m, 2\}} = \frac{1}{\mu} \]
we have

\[
\lambda_{\max\{p_n, 2\}}^{p_1, \ldots, p_m, -1} = \max \left\{ \frac{1}{\max\{p_n, 2\}} - \left( \frac{1}{p_1} + \cdots + \frac{1}{p_m} \right), 0 \right\} \\
= \max \left\{ 1 - \left( \frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{p} \right), 0 \right\} \\
= \delta_{p_1, \ldots, p_{m-1}, p}
\]

for all \( i \in \{1, \ldots, m-1\} \). Then, (15) can be re-stated as

\[
q_1 \geq \delta_{p_1, \ldots, p_{m-1}, p}, q_2 \geq \delta_{p_2, \ldots, p_{m-1}, p}, \ldots, q_{m-1} \geq \delta_{p_{m-1}, p}
\]

and the proof is done.

(2)\(\Rightarrow\)(1). If the exponents \( q_1, \ldots, q_{m-1} \) satisfy

\[
q_1 \geq \delta_{p_1, \ldots, p_{m-1}, p}, q_2 \geq \delta_{p_2, \ldots, p_{m-1}, p}, \ldots, q_{m-1} \geq \delta_{p_{m-1}, p},
\]

we have, again, that the exponents \( q_1, \ldots, q_{m-1} \) satisfy

\[
q_1 \geq \lambda_{r_1, \ldots, p_{m-1}}, q_2 \geq \lambda_{r_2, \ldots, p_{m-1}}, \ldots, q_{m-1} \geq \lambda_{r_{m-1}, p},
\]

with \( r = \cot (X_p)^* \). Thus, by Theorem 2.2, there is a constant \( C_{p_1, \ldots, p_{m-1}}^{(X_p)'} \geq 1 \) such that

\[
\left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_{m-1}=1}^{\infty} \| T(e_{j_1}, \ldots, e_{j_{m-1}}) \|_{(X_p)^*}^{q_{m-1}} \right) \right) \right)^{\frac{1}{q_1}} \leq C_{p_1, \ldots, p_{m-1}}^{(X_p)'} \| T \|
\]

for all continuous \( m \)-linear operators \( T : X_{p_1} \times \cdots \times X_{p_{m-1}} \rightarrow (X_p)^* \).

We thus have

\[
\left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_{m-1}=1}^{\infty} \| A(e_{j_1}, \ldots, e_{j_{m-1}}) \|_{p_m}^{\frac{q_{m-2}}{q_m-1}} \right) \right) \right)^{\frac{1}{p_1}} \leq C_{p_1, \ldots, p_{m-1}}^{(X_p)'} \| A \|
\]

and the proof is done.

Remark 3.1. The determination of the exact values of the constants involved in our main theorem is probably a difficult task, as it happens with the Hardy–Littlewood inequalities (see [3, 4] and the references therein). However when we are restricted to the bilinear case, with \( p_1 = p_2 = \infty \) and \( \sigma \) as the identity map, it is not difficult to check that we recover the constant \( \sqrt{2} \) from the Orlicz inequality.
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