Time-Fractional Allen–Cahn Equations: Analysis and Numerical Methods

Qiang Du¹ · Jiang Yang²,³ · Zhi Zhou⁴

Received: 8 July 2020 / Revised: 7 October 2020 / Accepted: 15 October 2020 / Published online: 2 November 2020
© Springer Science+Business Media, LLC, part of Springer Nature 2020

Abstract
In this work, we consider a time-fractional Allen–Cahn equation, where the conventional first order time derivative is replaced by a Caputo fractional derivative with order \(\alpha \in (0, 1)\). First, the well-posedness and (limited) smoothing property are studied, by using the maximal \(L^p\) regularity of fractional evolution equations and the fractional Grönwall’s inequality. We also show the maximum principle like their conventional local-in-time counterpart, that is, the time-fractional equation preserves the property that the solution only takes value between the wells of the double-well potential when the initial data does the same. Second, after discretizing the fractional derivative by backward Euler convolution quadrature, we develop several unconditionally solvable and stable time stepping schemes, such as a convex splitting scheme, a weighted convex splitting scheme and a linear weighted stabilized scheme. Meanwhile, we study the discrete energy dissipation property (in a weighted average sense), which is important for gradient flow type models, for the two weighted schemes. In addition, we prove the fractional energy dissipation law for the gradient flow associated with a convex free energy. Finally, using a discrete version of fractional Grönwall’s inequality and maximal \(\ell^p\) regularity, we prove that the convergence rates of those time-stepping schemes are \(O(\tau^\alpha)\) without any extra regularity assumption on the solution. We also present extensive numerical results to support our theoretical findings and to offer new insight on the time-fractional Allen–Cahn dynamics.

Keywords Time-fractional Allen–Cahn · Regularity · Time stepping scheme · Energy dissipation · Error estimate

Mathematics Subject Classification 65M70 · 65R20
1 Introduction

Classical phase-field models are diffuse interface models that have found numerous applications in diverse research areas, e.g., hydrodynamics [4,33,40], material sciences [2,9], biology [18,42,51] and image processing [38,52], to name just a few. Recently, there are also many studies on nonlocal phase field models involving spatially nonlocal interactions [3,5–7,15,16,22,23,45,49], see [13,14] for more extensive reviews of the literature. Historically, nonlocal interactions in phase field models expressed mathematically in terms of integral operators have been noted in the work of van der Waals [50], see discussions made in [39]. In fact, one can deduce the usual differential equation form of the local phase field energy from the nonlocal version via the so-called Landau expansion [24], under the usual assumption on the smooth and slowly varying nature of the phase field variables. Meanwhile, it has been reported that the presence of nonlocal operators in either time [12,31,48] or space [1,8,20,45] in phase field equations may change diffusive dynamics significantly.

In this paper, we focus on phase field models involving time nonlocality. More specifically, we consider the following time-fractional Allen–Cahn equation [31,48]:

\[
\begin{aligned}
\partial_t^\alpha u - \kappa^2 \Delta u &= -F'(u) =: f(u) \quad \text{in } \Omega \times (0, T), \\
u(x, t) &= 0 \quad \text{in } \partial\Omega \times (0, T), \\
u(x, 0) &= u_0 \quad \text{in } \Omega,
\end{aligned}
\]

where the function \( F(u) \) is bistable, e.g.,

\[ F(s) = \frac{1}{4}(1 - s^2)^2 \]

represents a double-well potential [14]. Here \( \Omega \) is a smooth domain of \( \mathbb{R}^d \) with \( d = 1, 2, 3 \) with boundary \( \partial\Omega \). The operator \( \partial_t^\alpha \) denotes the Caputo-type fractional derivative of order \( \alpha \in (0, 1) \) in time, which is a typical example of nonlocal operators and defined by [29, p. 70]

\[ \partial_t^\alpha u(x, t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} \partial_s u(x, s) ds. \]

Time-fractional PDEs have attracted some attention in the modeling of anomalous diffusion, e.g., protein diffusion within cells [21], contaminant transport in groundwater [30], and thermal diffusion in media with fractal geometry [36]. For the time-fractional Allen–Cahn model (1.1), some numerical studies have been presented [31]. It was reported there that the order \( \alpha \) affects significantly the relaxation time to the equilibrium and the sharpness of the interface. Their simulations were based on a fully discrete scheme for (1.1) that uses a piecewise linear interpolation to discretize the fractional derivative, and the Fourier spectral method for spatial discretization, along with a fast algorithm to reduce the computational complexity and the storage requirement. However, no stability analysis nor convergence theory was presented. Very recently, Chen et al. studied a time-fractional molecular beam epitaxy (MBE) model numerically using a similar fully discrete scheme [12]. They observed that both the energy decay rate and the coarsening rate satisfy a power-law determined by the fractional order. The first theoretical work regarding the energy stability for the model (1.1) was given by Tang et al. in [48], where they showed that the energy at a latter time must be bounded by the initial energy, i.e.,

\[ E(u(t)) \leq E(u_0), \quad \text{where } E(u) = \int_{\Omega} \frac{\kappa^2}{2} |\nabla u(x)|^2 + F(u(x)) \, dx. \]
On the discrete level, they proposed a stabilized time stepping scheme which maintains this kind of stability. Note that the original local-in-time integer-order Allen–Cahn is a gradient flow, so the free energy $E(u(t))$ is decreasing. Nevertheless, except for the boundedness of the energy (1.3), the energy dissipation has not been discussed at neither continuous nor discrete levels. Moreover, none of the aforementioned works study the well-posedness and the regularity of solutions. Sharp error estimates of those time stepping schemes are also unavailable in the literature, due to the lack of studies on the smoothing property of the equation. Let us also add that a related time-fractional Cahn-Hilliard model has also been studied numerically in [31,48], where the energy dissipation rate is reported to satisfy a power-law $t^{-\alpha/3}$. Indeed, in-depth studies of these time-fractional phase field type models remain fairly scarce.

The goal of the present work is to provide more extensive studies of nonlocal-in-time phase field models, and to develop and analyze stable numerical schemes. We present a rigorous analysis of the model (1.1) with smooth initial data, provide sharp regularity estimate, investigate time stepping schemes, establish the optimal error estimates, and study the discrete energy dissipation law. In comparison with the classical phase-field equation, the analysis of the nonlocal (time-fractional) model becomes more challenging, since the nonlocal fractional (differential) operator does not preserve some of the rules enjoyed by their local counterpart, e.g., the product rule and the chain rule. Below, we describe the main contributions of the paper and how the paper is organized.

The first contribution is the study of the well-posedness and regularity of the model (1.1), presented in Sect. 2. Nonlinear time-fractional diffusion equations with globally Lipschitz continuous potential term has already been investigated in [27]. However, the loss of global Lipschitz continuity in the time-fractional phase-field Eq. (1.1) will result in additional complications. To overcome the added difficulty, we apply the energy argument, the maximal $L^p$ regularity and fractional Grönwall’s inequality, to prove the existence and uniqueness of the weak solution, which satisfies (Theorem 2.1)

$$\|\partial_t^\alpha u\|_{L^p(0,T;L^2(\Omega))} + \|\Delta u\|_{L^p(0,T;L^2(\Omega))} \leq c, \quad \text{for any } p \in [2, 2/\alpha).$$

provided that $u_0 \in H^1_0(\Omega)$. Furthermore, the regularity will be improved so that $u \in L^\infty((0, T) \times \Omega)$ under the assumption that $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$. These results, in combination with the argument in [27, Theorem 3.1], lead to sharp regularity estimates. One application of those regularity estimates is to prove the maximum bound principle in the sense that if $|u_0| \leq 1$ then the solution $u$ shares the same pointwise maximum norm bound at all time. This is a property enjoyed by the conventional local-in-time Allen Cahn equation. Note that such a principle has been shown in [48], but it was under certain a priori regularity assumptions that have not been rigorously confirmed previously.

Our second contribution is to develop several unconditionally solvable and stable schemes in Sect. 3. The first scheme is motivated by the classical convex splitting method (CS). This scheme satisfies the discrete maximum norm bound principle. Moreover, it satisfies the energy stability, i.e.,

$$E(u_n) \leq E(u_0) \quad \text{for all } n \geq 1.$$

However, the energy dissipation law of the CS scheme is hard to establish due to the nonlocal effect of the fractional derivative. This motivates us to develop two other schemes, named as weighted convex splitting (WCS) scheme and linear weighted stabilized (LWS) scheme. Both schemes satisfy the discrete maximum bound principle as well as a weighted energy
dissipated law, i.e.,
\[ E(u_n) \leq E(u_{n,\alpha}), \quad \text{for all } n \geq 1, \]
where \( u_{n,\alpha} \) is a convex combination of \( u_0, u_1, \ldots, u_{n-1} \). In cases where the free energy adopted under consideration is convex (e.g., the linear problem or time-fractional Allen–Cahn with \( |u_n| \geq \frac{\sqrt{3}}{3} \)), the weighted energy stability indicates the fractional energy dissipation law as
\[
\bar{\partial}_t^\alpha E(u_n) \leq 0,
\]
where the operator \( \bar{\partial}_t^\alpha \) is defined by (3.1). This is consistent with the energy decay property on the continuous level (cf. Proposition 2.1), and that of classical gradient flow (\( \alpha = 1 \)), \( \bar{\partial}_t^1 E(u_n) \leq 0 \) (cf. Remark 3.2).

As an additional contribution to the analytical study, we show the convergence of the proposed time-stepping schemes in Sect. 4. The pointwise-in-time errors are derived by applying the regularity estimates in Sect. 2, the discrete fractional Grönwall’s inequality established in [27] and discrete maximal \( \ell^p \) regularity derived in [26], as well as some novel stability estimates proved in Sect. 4. In particular, under the assumption that \( u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \) and \( |u_0(x)| \leq 1 \), we prove that all those three time stepping solutions satisfy, under no additional regularity assumptions, that

\[
\max_{1 \leq n \leq N} \|u(t_n) - u_n\|_{L^2(\Omega)} \leq c \tau^\alpha.
\]
where \( c \) denotes a generic constant depending on the \( \alpha, u_0, T \) and \( \kappa \), but always independent of \( \tau \) and any smoothness of \( u \).

Finally, in Sect. 5, we present some numerical experiments to confirm the theoretical findings and to offer new insight on the time-fractional Allen–Cahn dynamics. Numerically, we observe that the relaxation time to the steady state gets longer for smaller \( \alpha \), and the time-fractional Allen–Cahn equation converge to some slow mean curvature flow. Those phenomena await further theoretical understanding in the future.

Throughout this paper, unless otherwise noted, the notation \( c \) denotes a generic constant, which may depend on \( \alpha, \kappa, \) and \( T \) but independent of the step size \( \tau \). Moreover, it does not depend on \( u \) except for \( u_0 \).

### 2 Solution Theory of the Time-Fractional Allen–Cahn Equations

In this section, we study the well-posedness of the time-fractional Allen–Cahn Eq. (1.1) and prove the sharp regularity. The argument for nonlinear fractional diffusion equation with globally Lipschitz continuous potential has been investigated in [27]. To overcome the lack of global Lipschitz property of the nonlinear potential \( f(s) \), it is important to show \( u \in L^\infty((0, T) \times \Omega) \) in the first place. To this end, we need the following variant of Grönwall’s inequality. The proof is included for completeness.

**Lemma 2.1** Suppose that \( y \) is nonnegative and \( y \) satisfies the inequality
\[
\partial_t^\alpha y(t) \leq \beta y(t) + \sigma(t),
\]
where the function \( \sigma \in L^\infty(0, T) \) and the constant \( \beta > 0 \). Then
\[
y(t) \leq c_T \left(y(0) + \|\sigma\|_{L^\infty(0,T)}\right).
\]
where the constant $c_T$ is independent of $\sigma$ and $y$, but may depend on $\alpha$, $\beta$ and $T$.

**Proof** We define an axillary function $w(t)$ such that

$$w(t) = E_{\alpha,1}(-\beta t^\alpha) y(0) + \int_0^t s^{\alpha-1} E_{\alpha,a}(-\beta s^\alpha) \sigma (t - s) \, ds$$

where the $E_{\alpha,b}(\cdot)$ denotes the Mittag-Leffler function [29]. Then the function $w(t)$ satisfies the fractional initial value problem

$$\frac{\partial^\alpha}{\partial t^\alpha} w(t) = \beta w(t) + \sigma (t), \quad \text{with } w(0) = y(0),$$

Using the positivity and boundedness of the Mittag-Leffler function, we obtain that

$$w(t) \leq c_1 y(0) + c_2 \int_0^t s^{\alpha-1} \sigma (t - s) \, ds \leq c_T y(0) + \|\sigma\|_{L^\infty(0,T)},$$

and the desired result follows from the comparison principle, i.e., $y(t) \leq w(t).$ □

In order to study the well-posedness and regularity, we use the Bochner space. For a Banach space $B$, we define

$$L^r(0, T; B) = \{ u(t) \in B \text{ for a.e. } t \in (0, T) \text{ and } \|u\|_{L^r(0,T;B)} < \infty \},$$

for any $r \geq 1$, and the norm $\| \cdot \|_{L^r(0,T;B)}$ is defined by

$$\|u\|_{L^r(0,T;B)} = \left\{ \left( \int_0^T \|u(t)\|^r_B dt \right)^{1/r} \right\}, \quad r \in [1, \infty),$$

$$\|u\|_{L^\infty(0,T;B)} = \text{esssup}_{t \in (0,T)} \|u(t)\|_B, \quad r = \infty.$$ 

We also use extensively the Bochner-Sobolev spaces $W^{s,p}(0, T; B)$. For any $s \geq 0$ and $1 \leq p < \infty$, we denote by $W^{s,p}(0, T; B)$ the space of functions $v : (0, T) \to B$, with the norm defined by interpolation. Equivalently, the space is equipped with the quotient norm

$$\|v\|_{W^{s,p}(0,T;B)} := \inf \| \tilde{v} \|_{W^{s,p}(\mathbb{R};B)}, \quad (2.1)$$

where the infimum is taken over all possible extensions $\tilde{v}$ that extend $v$ from $(0, T)$ to $\mathbb{R}$. For any $0 < s < 1$ and $1 \leq p < \infty$, one can define the Sobolev–Slobodeckii seminorm $| \cdot |_{W^{s,p}(0,T;L^2(\Omega))}$ by

$$|v|^p_{W^{s,p}(0,T;L^2(\Omega))} := \left( \int_0^T \int_0^T \frac{\|v(t) - v(\xi)\|^p_B}{|t - \xi|^{1+ps}} \, dt \, d\xi \right), \quad (2.2)$$

and the full norm $\| \cdot \|_{W^{s,p}(0,T;B)}$ by

$$\|v\|_{W^{s,p}(0,T;B)} = \|v\|_{L^p(0,T;B)} + |v|^p_{W^{s,p}(0,T;B)}.$$

Now we are ready to prove the well-posedness.

**Theorem 2.1** For any $u_0 \in H^1_0(\Omega)$, there exists a unique weak solution

$$u \in W^{\alpha,p}(0, T; L^2(\Omega)) \cap L^p(0, T; H^2(\Omega) \cap H^1_0(\Omega)), \quad \text{for any } p \in [2, 2/\alpha).$$

Moreover, if $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, the solution

$$u \in L^\infty((0, T) \times \Omega).$$
Proof Step 1. Following the routine of the Galerkin method, let \( \{\lambda_j\}_{j=1}^{\infty} \) and \( \{\phi_j\}_{j=1}^{\infty} \) be respectively the eigenvalues and the \( L^2(\Omega) \)-orthonormal eigenfunctions of the negative Laplace operator \( -\Delta \) on the domain \( \Omega \) with the homogeneous Dirichlet boundary condition. For every \( N \in \mathbb{N} \), by setting \( X_N = \text{span}\{\phi_j\}_{j=1}^{N} \), we consider the finite dimensional problem: find \( u^N \in X_N \) such that

\[
(\partial_t^\alpha u^N, v) - \kappa^2(\nabla u^N, \nabla v) = (f(u^N), v) \quad \forall \ v \in X_N \text{ and } u^N(0) = P_N u_0,
\]
where \( P_N \) is a \( L^2 \)-projection from \( L^2(\Omega) \) onto \( X_N \) by

\[
(u, v) = (P_N u, v) \quad \forall \ v \in X_N.
\]

The existence and uniqueness of a local solution to the finite dimensional problem (2.3) can be proved by the Banach fixed point theorem, by noting that \( f(s) \) is smooth and hence locally Lipschitz continuous [29, Section 42]. Then by the energy argument, we let \( v = u^N \) in (2.3) and using the fact that

\[
\frac{1}{2} \partial_t^\alpha \|u^N(t)\|_{L^2(\Omega)}^2 \leq (\partial_t^\alpha u^N(t), u^N(t)).
\]

Then we obtain the following estimate

\[
\partial_t^\alpha \|u^N(t)\|_{L^2(\Omega)}^2 + 2\kappa^2 \|\nabla u^N(t)\|_{L^2(\Omega)}^2 + 2\|u^N(t)\|_{L^4(\Omega)}^4 \leq 2\|u^N(t)\|_{L^2(\Omega)}^2.
\]

which together with the fractional Gronwall’s inequality in Lemma 2.1 yield that

\[
\|u^N(t)\|_{L^2(\Omega)} \leq c_T \|u^N(0)\|_{L^2(\Omega)} \leq c_T \|u_0\|_{L^2(\Omega)},
\]

with a constant \( c_T \) independent of \( N \). That means \( u^N \in C([0, T); L^2(\Omega)) \), and hence the solution is a global solution.

Step 2. Now we prove that \( u^N \) converges to a weak solution of the time-fractional Allen–Cahn Eq. (1.1) as \( N \to \infty \). Now we repeat the energy argument by taking \( v = -\Delta u^N \) in (2.3), and use the fact that

\[
-\left(u^N - (u^N)^3, \Delta u_N\right) = \|\nabla u^N\|_{L^2(\Omega)}^2 - 3\|u^N\|_{L^4(\Omega)}^4 \leq \|\nabla u^N(t)\|_{L^2(\Omega)}^2.
\]

Then we arrive at the estimate

\[
\partial_t^\alpha \|\nabla u^N(t)\|_{L^2(\Omega)}^2 + 2\kappa^2 \|\Delta u^N(t)\|_{L^2(\Omega)}^2 + 6\|u^N\|_{L^4(\Omega)}^4 \leq 2\|\nabla u^N(t)\|_{L^2(\Omega)}^2.
\]

By Lemma 2.1, it holds that

\[
\|\nabla u^N\|_{L^2(0,T;L^2(\Omega))} \leq c_T \|\nabla u^N\|_{L^2(\Omega)} \leq c_T \|\nabla u_0\|_{L^2(\Omega)},
\]

with a constant \( c_T \) independent of \( N \). As a result, the Sobolev embedding inequality leads to that \( u^N \in L^\infty(0, T; L^6(\Omega)) \) for \( \Omega \subset \mathbb{R}^d \) with \( d \leq 3 \), and hence

\[
\|f(u^N)\|_{L^\infty(0,T;L^2(\Omega))} \leq c.
\]

Now under the assumption that \( u_0 \in H^1_0(\Omega) \), we apply the maximal \( L^p \) regularity [26] to obtain

\[
\|\partial_t^\alpha u_N\|_{L^p(0,T;L^2(\Omega))} + \|\Delta u_N\|_{L^p(0,T;L^2(\Omega))} \leq c_{p,\kappa}, \quad \text{for all } p \in [2, 2/\alpha),
\]

where the constant \( c_{p,\kappa} \) is independent of \( N \). This further implies

\[
u^N \in W^{\alpha,p}(0, T; L^2(\Omega)) \cap L^p(0, T; H^2(\Omega) \cap H^1_0(\Omega)),
\]
which is compactly embedded in $C([0, T]; L^2(\Omega))$ for $p \in (1/\alpha, 2/\alpha)$. Therefore, there exist function $u$ and a subsequence, which we denote as $\{u_N\}$ again, such that

$$ u_N \rightharpoonup u \quad \text{weak-* in} \quad L^\infty(0, T; H^1_0(\Omega)), $$

$$ \partial_t^\alpha u_N \rightharpoonup \partial_t^\alpha u \quad \text{weakly in} \quad L^p(0, T; L^2(\Omega)), $$

$$ u_N \rightharpoonup u \quad \text{weakly in} \quad L^p(0, T; H^2(\Omega) \cap H^1_0(\Omega)), $$

$$ u_N \rightharpoonup u \quad \text{strongly in} \quad C([0, T]; L^2(\Omega)), $$

as $N \to 0$. Consequently, we can pass to the limit in (2.3) and the function $u$ satisfies

$$ (\partial_t^\alpha u, v) + \kappa^2 (\nabla u, \nabla v) = (f(u), v) \quad \text{for all} \quad v \in H^1_0(\Omega). \tag{2.5} $$

Moreover, by the strong convergence of $u_N$ in $C([0, T]; L^2(\Omega))$, we know that $u_N(0)$ converges to $u_0$ in $L^2(\Omega)$ and so $u(0) = u_0$. Therefore, the fractional Allen–Cahn Eq. (1.1) admits a weak solution in $W^{\alpha,p}(0, T; L^2(\Omega)) \cap L^p(0, T; H^2(\Omega) \cap H^1_0(\Omega))$ with $p \in [2, 2/\alpha)$.

**Step 3.** Next, we prove the $L^\infty((0, T) \times \Omega)$ bound under the assumption on initial data. The preceding argument implies that $f(u) \in L^\infty(0, T; L^2(\Omega))$. Here we assume that $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, and apply the maximal $L^p$ regularity again to obtain that

$$ \|\partial_t^\alpha u\|_{L^p(0, T; L^2(\Omega))} + \|\Delta u\|_{L^p(0, T; L^2(\Omega))} \leq c, \quad \text{for all} \quad p \in (1, \infty), $$

which implies that

$$ u \in W^{\alpha,p}(0, T; L^2(\Omega)) \cap L^p(0, T; H^2(\Omega) \cap H^1_0(\Omega)). $$

Therefore, by means of the real interpolation with sufficiently large $p$, we obtain that

$$ u \in L^\infty((0, T) \times \Omega). $$

**Step 4.** Finally, we prove the uniqueness of the weak solution. Assume that $u_1$ and $u_2$ are two weak solution of (1.1). Then $w = u_1 - u_2$ satisfies

$$ (\partial_t^\alpha w(t), v(t)) + \kappa^2 (\nabla w(t), v(t)) = (f(u_1) - f(u_2), v(t)) \quad \text{for all} \quad v \in H^1_0(\Omega) $$

with $w(0) = 0$. By taking $v = w(t)$ and using the facts that

$$ (f(u_1) - f(u_2))(u_1 - u_2) = (u_1 - u_2)^2(1 - u_1^2 - u_1u_2 - u_2^2) \leq (u_1 - u_2)^2 $$

and

$$ (\partial_t^\alpha w, w) \geq \frac{1}{2} \partial_t^\alpha \|w(t)\|^2 $$

we obtain that

$$ \partial_t^\alpha \|w(t)\|_{L^2(\Omega)}^2 \leq 2 \|w(t)\|_{L^2(\Omega)}^2 $$

with $w(0) = 0$. Applying the fractional Grönwall’s inequality in Lemma 2.1, we conclude that $w \equiv 0$, i.e., $u_1 = u_2$. Note that, the uniqueness of the weak solution in turn implies the limit of the convergent subsequence taken in the Step 2 is independent of the choice of the subsequence, thus we get the convergence of the whole sequence $u_N$ to the unique weak solution. \hfill \square

With the help of Theorem 2.1, we know that $u \in L^\infty((0, T) \times \Omega)$ and hence $f(u)$ is Globally Lipschitz continuous. Then the regularity theory follows from the argument in [27, Theorem 3.1]. To this end, we reformulate the fractional Allen–Cahn equation by

$$ \partial_t^\alpha u + Au = u - u^3 \quad \text{for} \quad (x, t) \in \Omega \times (0, T) \quad \text{and} \quad u(0) = u_0, \tag{2.7} $$

\hfill \copyright \ Springer
where $A = -k^2 \Delta$ with homogeneous Dirichlet boundary condition. Let $\| \cdot \|_{L^2(\Omega) \to L^2(\Omega)}$ be the operator norm on the space $L^2(\Omega)$. Then the operator $A$ satisfies the following resolvent estimate

$$\|(z + A)^{-1}\|_{L^2(\Omega) \to L^2(\Omega)} \leq c_{\phi,\kappa} |z|^{-1}, \quad \forall z \in \Sigma_{\phi}, \forall \phi \in (0, \pi),$$

where for $\phi \in (0, \pi)$, $\Sigma_{\phi} := \{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \phi \}$. This further implies

$$\|(z^{\alpha} + A)^{-1}\|_{L^2(\Omega) \to L^2(\Omega)} \leq c_{\phi,\alpha,\kappa} |z|^{-\alpha}, \quad \forall z \in \Sigma_{\phi}, \forall \phi \in (0, \pi),$$

$$\|A(z^{\alpha} + A)^{-1}\|_{L^2(\Omega) \to L^2(\Omega)} \leq c_{\phi,\alpha,\kappa}, \quad \forall z \in \Sigma_{\phi}, \forall \phi \in (0, \pi).$$

Now we consider the solution representation of the following linear equation

$$\partial_t^{\alpha} u(t) + A u(t) = g(t),$$

with the homogeneous Dirichlet boundary condition and $u(0) = u_0$. By means of Laplace transform, denoted by $\hat{\cdot}$, we obtain

$$z^{\alpha} \hat{u}(z) + A \hat{u}(z) = z^{\alpha-1} u_0 + \hat{g}(z),$$

which together with (2.8) implies $\hat{u}(z) = (z^{\alpha} - A)^{-1}(z^{\alpha-1} u_0 + \hat{g}(z)).$ By inverse Laplace transform and convolution rule, the solution $u(t)$ to (2.9) is given by

$$u(t) = F(t)u_0 + \int_0^t E(t-s)g(s)ds,$$

where the operators $F(t) : X \to X$ and $E(t) : X \to X$ are defined by

$$F(t) := \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}} e^{zt} z^{\alpha-1}(z^{\alpha} + A)^{-1} dz$$

and $E(t) := \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}} e^{zt} (z^{\alpha} + A)^{-1} dz,$

respectively. Clearly, we have $E(t) = F'(t)$. The contour $\Gamma_{\theta,\delta}$ is defined by

$$\Gamma_{\theta,\delta} = \{ z \in \mathbb{C} : |z| = \delta, |\arg z| \leq \theta \} \cup \{ z \in \mathbb{C} : z = \rho e^{\pm \pi i}, \rho \geq \delta \},$$

oriented with an increasing imaginary part, where $\theta \in (\pi/2, \pi)$ is fixed.

The following lemma gives the smoothing properties of $F(t)$ and $E(t)$, which are important in the regularity estimate. (i), (ii) and (iii) has been proved in [28, Lemma 2.2] and (iv) was given in [27, Lemma 3.2].

**Lemma 2.2** For the operators $F$ and $E$ defined in (2.10), the following properties hold.

(i) $\|F(t)v\|_{L^2(\Omega)} + t^{\alpha}\|AF(t)v\|_{L^2(\Omega)} \leq c\|v\|_{L^2(\Omega)}, \quad \forall t \in (0, T]$,

(ii) $t^{1-\alpha}\|F'(t)v\|_{L^2(\Omega)} + t\|AF'(t)v\|_{L^2(\Omega)} + t^{1-\beta\alpha}\|A^{-\beta}F'(t)v\|_{L^2(\Omega)} \leq c\|v\|_{L^2(\Omega)}, \quad \forall t \in (0, T]$,

(iii) $t^{1-\alpha}\|E(t)v\|_{L^2(\Omega)} + t^{2-\alpha}\|E'(t)v\|_{L^2(\Omega)} + t\|AE(t)v\|_{L^2(\Omega)} \leq c\|v\|_{L^2(\Omega)}, \quad \forall t \in (0, T]$,

(iv) $F(t)$ and $E(t) : L^2(\Omega) \to H^\gamma(\Omega) \cap H^\delta(\Omega)$ is continuous with respect to $t \in (0, T]$.

**Theorem 2.2** Let $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$. Then the time-fractional Allen–Cahn Eq. (1.1) has a unique solution $u$ such that for $s \in [0, 1)$

$$u \in C^\alpha([0, T]; L^2(\Omega)) \cap C([0, T]; H^2(\Omega) \cap H^1_0(\Omega)), \quad \partial_t^{\alpha} u \in C([0, T]; L^2(\Omega)), \quad (2.13)$$

$$\Delta u \in C((0, T]; H^{2\alpha}(\Omega)), \quad \|\Delta^{1+s} u(t)\|_{L^2(\Omega)} \leq c t^{-s\alpha} \quad \text{for} \ t \in (0, T], \quad (2.14)$$

$$\partial_t u(t) \in C((0, T]; H^{2\alpha}(\Omega)) \quad \text{and} \quad \|\Delta^\alpha u(t)\|_{L^2(\Omega)} \leq c t^{\alpha(1-s)-1} \quad \text{for} \ t \in (0, T]. \quad (2.15)$$

The constant $c$ above depends on $\|Au_0\|_{L^2(\Omega)}$ and $T$. Springer
**Proof** The regularity estimate (2.13) with \( p = 2 \) has already been proved in [27, Theorem 3.1]. The result in case that \( p \in (2, \infty) \) follows from the same argument and the resolvent estimate (2.8). To show other estimates, we apply the representation (2.10) and smoothing properties in Lemma 2.2. Specifically,

\[
A^{1+s} u(t) = A^s F(t)(\Delta u_0) + \int_0^t A^s E(t-y)[A(u - u^3)(y)]dy.
\]

Then we apply Lemma 2.2 and obtain that

\[
\|A^{1+s} u(t)\|_{L^2(\Omega)} \leq c t^{-s\alpha} \|A u_0\|_{L^2(\Omega)} + \int_0^t y^{(1-s)\alpha-1} \|A(u - u^3)(y)\|_{L^2(\Omega)} dy.
\]

Now applying the fact that \( \Delta u^3 = 3u|\nabla u|^2 + 3u^2 \Delta u, \ u \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega)) \) from (2.13) and \( u \in L^\infty((0, T) \times \Omega) \) by Theorem 2.1 we have

\[
\|A((2u - u^3))\|_{L^2(\Omega)} \leq c \|\Delta u\|_{L^2(\Omega)} + c \|u|\nabla u|^2\|_{L^2(\Omega)} + c \|u^2 \Delta u\|_{L^2(\Omega)} \leq c_T,
\]

with a constant \( c_T \) independent of \( t \). Therefore, we obtain that

\[
\|A^{1+s} u(t)\|_{L^2(\Omega)} \leq c t^{-s\alpha}.
\]

Now we turn to (2.15). The case that \( s = 0 \) has be confirmed in [27, Theorem 3.1]. In general,

\[
A^s u_t(t) = A^{s-1} F'(t)(A u_0) + \frac{d}{dt} \int_0^t A^s E(y)[(u - u^3)(t-y)]dy
\]

\[
= A^{s-1} F'(t)(A u_0) + A^s E(t)(2u - u^3)(0) + \int_0^t A^s E(y)[(u_t - 3u^2 u_t)(t-y)]dy
\]

The first term could be bounded using Lemma 2.2 (ii)

\[
\|A^{s-1} F'(t)(A u_0)\|_{L^2(\Omega)} \leq c t^{1-(s-1)\alpha-1} \|\Delta u_0\|_{L^2(\Omega)}. \tag{2.16}
\]

For the second term, we use Lemma 2.2 (iii) to have

\[
\|A^s E(t)(2u - u^3)(0)\|_{L^2(\Omega)} \leq c t^{1-(s-1)\alpha-1} \|2u_0 - u^3_0\|_{L^2(\Omega)} \leq c t^{1-(s-1)\alpha-1} \|\Delta u_0\|_{L^2(\Omega)}. \tag{2.17}
\]

Similarly, the third term follows analogously

\[
\int_0^t \|A^s E(y)(2u_t - 3u^2 u_t)(t-y)\|_{L^2(\Omega)} dy \leq c \int_0^t y^{\alpha-1} \|A^s u_t(t-y)\|_{L^2(\Omega)} dy.
\]

This together with (2.16) and (2.17), and the Grönwall’s inequality lead to the desired result.

Finally, the continuity follows directly from the continuity of the solution operators \( E(t) \) and \( F(t) \), i.e., Lemma 2.2 (iv).

Next, we prove the maximum principle in sense that if \( |u_0| \leq 1 \), then the solution \( u(t) \) is also bounded by 1 in \( L^\infty(\Omega) \). Note that the maximum principle was showed in [48] under certain a priori strong regularity assumptions on the solution \( u \), which have not been rigorously proven yet.

**Lemma 2.3** Assume that \( f \in C[0, T] \cap C^1(0, T) \) and \( f \) attains its minimum (maximum) at \( t_0 \in (0, T) \). Then there holds that

\[
\delta^s_t f(t_0) \leq (\geq) 0.
\]


Theorem 2.3 Let \( u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \). Further, we assume that \( |u_0(x)| \leq 1 \), then the solution of the time-fractional Allen Cahn Eq. (1.1) with homogeneous boundary conditions satisfies the maximum principle that
\[
|u(x, t)| \leq 1 \quad \text{for all} \quad (x, t) \in \Omega \times (0, T].
\]

Proof Assume that the minimum of \( u \) is smaller than \(-1\) and achieved at \((x_0, t_0) \in \Omega \times (0, T)\). Then by the regularity (2.13) and (2.15) in Theorem 2.2, and Lemma 2.3, we have that
\[
\partial_t^{\alpha} u(x_0, t_0) \leq 0.
\]
Meanwhile, via applying Theorem 2.2, we have that \( \Delta u \) is continuous in \( \overline{\Omega} \) and hence
\[
\Delta u(x_0, t_0) \geq 0.
\]
Therefore, we arrive at the result
\[
0 \geq \partial_t^{\alpha} u(x_0, t_0) - \Delta u(x_0, t_0) = u(x_0, t_0) - u(x_0, t_0)^3 > 0,
\]
which leads to a contradiction. The upper bound of the solution can be proved using the same way.

Remark 2.1 We are also interested in the energy bound of the solution. Suppose that the solution satisfies \( u \in W^{1, \frac{2}{\sigma}}(\Omega) \cap L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \), then the stability of energy can be directly observed by taking \( v = u_t \) in (2.5) and integrate over \((0, T)\),
\[
\int_0^T (\partial_t^{\alpha} u(t), u_t(t)) + (\nabla u(t), \nabla u_t(t)) \, dt = \int_0^T (f(u(t)), u_t(t)) \, dt
\]
Using the regularity assumption and applying the fact that (see [48, Corollary 2.1.] or [35, Lemma 3.1])
\[
\int_0^T (\partial_t^{\alpha} u(t), u_t(t)) \, dt \geq 0,
\]
we obtain that
\[
\frac{1}{2} \int_0^T \frac{d}{dt} \int_\Omega |\nabla u(t)|^2 \, dx \, dt + \int_0^T \frac{d}{dt} \int_\Omega F(u(t)) \, dx \, dt \leq 0.
\]
and hence we derive the energy bound that
\[
E(u(T)) \leq E(u_0).
\]
This property has been proved in [48] without confirmation of the regularity assumption. By Theorem 2.2, one can easily verify those assumption, provided that \( u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \).

In case that energy functional is convex, we are able to prove the following stronger result, i.e., a fractional energy dissipation law.

Proposition 2.1 Assume that (1.1) has a unique solution, and the energy functional \( E(\cdot) \) is convex (i.e., \( F(s) \) is a convex function). Then there holds that
\[
\partial_t^{\alpha} E(u(t)) \leq 0.
\]
Proof Extend \( u(t) \) by \( u(t) = u_0 \) for all \( t < 0 \). Then the Caputo derivative can be reformulated as
\[
\partial_t^\alpha u(t) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{u(t) - u(s)}{(t-s)^{\alpha+1}} \, ds.
\]
Then we multiply \(-\partial_t^\alpha u(t)\) on both sides of the Eq. (1.1), integrate over \( \Omega \), use (2.6) and obtain
\[
-\|\partial_t^\alpha u(t)\|_{L^2(\Omega)}^2 \geq \frac{1}{2} \partial_t^\alpha \|u(t)\|_{L^2(\Omega)}^2 + (F'(u(t)), \partial_t^\alpha u(t)).
\]
In case that \( F \) is convex, then for any \( t, s \in \mathbb{R} \), we have
\[
(F'(u(t)), u(t) - u(s)) \geq \int_{\Omega} F(u(t)) - F(u(s)) \, dx.
\]
and hence we arrive at
\[
-\|\partial_t^\alpha u(t)\|_{L^2(\Omega)}^2 \geq \frac{1}{2} \partial_t^\alpha \|u(t)\|_{L^2(\Omega)}^2 + \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{(F(u(t), u(t) - u(s))}{(t-s)^{\alpha+1}} \, ds
\]
\[
\geq \frac{1}{2} \partial_t^\alpha \|u(t)\|_{L^2(\Omega)}^2 + \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{F(u(t)) - F(u(s))}{(t-s)^{\alpha+1}} \, dx \, ds
\]
\[
= \partial_t^\alpha \left( \frac{1}{2} \|u(t)\|_{L^2(\Omega)}^2 + \int_{\Omega} F(u(t)) \, dx \right) = \partial_t^\alpha E(u(t)).
\]

3 Numerical Schemes for the Time-Fractional Allen–Cahn Equation

In this section, we propose time stepping schemes and study some quantitative properties, such as discrete maximum principle and energy dissipation, which are important for phase field models. To this end, we discretize the Caputo fractional derivative by using the convolution quadrature [34] generated by backward Euler method (BE-CQ), which is commonly known as the Grünwald-Letnikov scheme in the literature [37]. We divide the interval \([0, T]\) into a uniform grid with a time step size \( \tau = T/N \), \( N \in \mathbb{N} \), so that \( 0 = t_0 < t_1 < \ldots < t_n = T \), and \( t_n = n \tau, n = 0, \ldots, N \). Then the Caputo fractional derivative is approximated by
\[
\tilde{\partial}_t^\alpha u_n := \tau^{-\alpha} \sum_{j=1}^{n} \omega_{n-j} (u_j - u_0), \quad \text{where} \quad \omega_j = (-1)^j \frac{\alpha(\alpha - 1) \ldots (\alpha - j + 1)}{j!}.
\]

3.1 Convex Splitting Scheme and the Energy Stability

The first method is the convex splitting scheme (CS). For given \( u_0, u_1, \ldots, u_{n-1} \), we find \( u_n \) by solving a nonlinear elliptic problem
\[
\tilde{\partial}_t^\alpha u_n - \kappa^2 \Delta u_n = u_n - (u_n)^3, \quad \text{for} \ 1 \leq n \leq N,
\]
with the homogeneous Dirichlet boundary condition. This simple time stepping method is a nonlinear scheme inspired by the standard convex splitting method for general gradient
flows, which handles the convex part $u^3$ implicitly and replaces the concave part $u_n$ with $u_{n-1}$. Meanwhile, the CS scheme (3.2) can be written as the nonlinear elliptic problem

$$
(I - \tau^\alpha k^2 \Delta) u_n + \tau^\alpha (u_n)^3 = u_{n,\alpha} + \tau^\alpha u_{n-1}.
$$

(3.3)

where $u_{n,\alpha}$ denotes the fractional extrapolation (3.10). Since left part is monotone with respect to $u_n$. By the implicit function theorem, there exists a unique solution and hence the CS scheme (3.2) is uniquely solvable.

The following theorem states that the CS scheme satisfies the discrete maximum principle. This is an important and well-known property of the standard CS scheme of the classical Allen–Cahn equation. More investigations on maximum principle preserving schemes for integer order Allen–Cahn equations can be found in [16,17,44,47].

**Theorem 3.1** The CS scheme (3.2) satisfies the discrete maximum principle unconditionally, i.e.,

$$
\|u_0\|_{L^\infty(\Omega)} \leq 1 \implies \|u_n\|_{L^\infty(\Omega)} \leq 1 \quad \text{for all } n \geq 1.
$$

**Proof** We prove the discrete maximum principle by induction. To this end, we assume that $\|u_k\|_{L^\infty(\Omega)} \leq 1$ for all $k < n$. Note that the weights $\{\omega_j\}$ satisfies the property that

(i) $\omega_0 > 0$ and $\omega_j < 0$ for $j \geq 1$, and (ii) $\sum_{j=0}^{n} \omega_j > 0$, for $n \geq 1$.

Thus $u_{n,\alpha}$ is a convex combination of $u_0, u_1, \ldots, u_{n-1}$, and hence $\|u_{n,\alpha}\|_{L^\infty(\Omega)} \leq 1$. Therefore, we have $\|u_{n,\alpha} + \tau^\alpha u_{n-1}\|_{L^\infty(\Omega)} \leq 1 + \tau^\alpha$. Besides, the operator $I - \tau^\alpha k^2 \Delta$ is positive and $(u^n)^3$ has the same sign of $u^n$. Then it follows immediately that $\|u^n\|_{\infty} \leq 1$ by the monotonicity and (3.3).

**Remark** Besides the discrete maximum principle, we prove the energy stability of the time-stepping scheme (3.2). To this end, we use the following lemma on the backward Euler convolution quadrature (BE-CQ). A similar result has been proved in [48, Lemma 3.1] for the time stepping method using piecewise linear polynomial interpolation, known as L1 approximation in the literature [32,46]. However, the convolution quadrature does not satisfy the algebraic property like the L1 scheme. So here we apply a different approach, say discrete time Fourier transform.

**Lemma 3.1** The BE-CQ satisfies the following positivity property

$$
\sum_{j=1}^{N} (\bar{\partial}_\tau^\alpha u_n, \bar{\partial}_\tau^1 u_n) \geq 0.
$$

**Proof** First, we assume that $u_0 = 0$. Then by setting $y_n = \bar{\partial}_\tau^1 u_n$ for $n \geq 1$ and $y_0 = 0$, the definition of convolution quadrature yields that

$$
\sum_{j=1}^{N} (\bar{\partial}_\tau^\alpha u_n, \bar{\partial}_\tau^1 u_n) = \sum_{j=1}^{N} (\tilde{\partial}_\tau^\alpha - 1 y_n, y_n).
$$

where

$$
\tilde{\partial}_\tau^\alpha - 1 y_n = \tau^{1-\alpha} \sum_{j=0}^{n} \omega_{n-j} y_j \quad \text{with } \omega_j = (-1)^j \frac{(\alpha - 1) \ldots (\alpha - j)}{j!}.
$$

(3.4)
To show this nonnegativity property, we extend \( \{ y_n \}_{n=0}^N \) to \( \{ y_n \}_{n=-\infty}^\infty \) and \( \{ \omega_n \}_{n=0}^N \) to \( \{ \omega_n \}_{n=-\infty}^\infty \) by zero extension. Then the discrete fractional derivative can be written as

\[
\tilde{\partial}_\tau^{\alpha-1} u_n = \tau^{1-\alpha} \sum_{j=-\infty}^{\infty} \omega_n^{(\alpha-1)} y_j
\] (3.5)

From now on, we denote the discrete time Fourier transform \( \tilde{[u_n]}(\zeta) \) by

\[
\tilde{[u_n]}(\zeta) = \sum_{n=-\infty}^{\infty} u_n e^{-in\zeta}.
\] (3.6)

Then by the Parseval’s theorem we have

\[
\sum_{j=0}^{N} (\tilde{\partial}_\tau^{\alpha-1} y_n, y_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (|\tilde{\partial}_\tau^{\alpha-1} y_n(\zeta)|^2) d\zeta
\]

By the property of discrete time Fourier transform (3.6) and the discrete convolution (3.4), we have

\[
\sum_{j=0}^{N} (\tilde{\partial}_\tau^{\alpha-1} y_n, y_n) = \frac{\tau^{1-\alpha}}{2\pi} \int_{-\pi}^{\pi} (1 - e^{-i\zeta})^{\alpha-1} |\tilde{y_n}(\zeta)|^2 d\zeta
\]

In case that \( u_0 \neq 0 \), we let \( v_n = u_n - u_0 \) and note that

\[
\sum_{j=1}^{N} (\tilde{\partial}_\tau^\alpha u_n, \tilde{\partial}_\tau^1 u_n) = \sum_{j=1}^{N} (\tilde{\partial}_\tau^\alpha v_n, \tilde{\partial}_\tau^1 v_n).
\]

The repeating the argument for \( v_n \) leads to the desired result. That completes the proof of the lemma.

Then the next theorem states that the CS scheme (3.2) is energy stable.

**Theorem 3.2** Suppose that \( u_0 \in H_0^1(\Omega) \). Then the CS scheme (3.2) satisfies the energy stability

\[
E(u_n) \leq E(u_0), \quad \text{for all } n \geq 1.
\]

**Proof** Taking \( L^2 \)-inner product of (3.2) with \( -(u_n - u_{n-1}) \) yields

\[
-(\tilde{\partial}_\tau^\alpha u_n, u_n - u_{n-1}) = \kappa^2 (\nabla u_n, \nabla u_n - \nabla u_{n-1}) + (u_n)^2 - u_{n-1}, u_n - u_{n-1})
\]

Here we note that

\[
\kappa^2 (\nabla u_n, \nabla u_n - \nabla u_{n-1}) = \frac{\kappa^2}{2} \| \nabla u_n \|^2_{L^2(\Omega)} - \frac{\kappa^2}{2} \| \nabla u_{n-1} \|^2_{L^2(\Omega)} + \frac{\kappa^2}{2} \| \nabla (u_n - u_{n-1}) \|^2_{L^2(\Omega)}
\]

Meanwhile, the fundamental inequality

\[
(a^3 - b, a - b) \geq \frac{1}{4} (a^2 - 1)^2 - \frac{1}{4} (b^2 - 1)^2
\] (3.7)
yields that
\[
\left( (u_n)^3 - u_{n-1}, u_n - u_{n-1} \right) \geq \frac{1}{4} \| (u_n)^2 - 1 \|_{L^2(\Omega)}^2 - \frac{1}{4} \| (u_{n-1})^2 - 1 \|_{L^2(\Omega)}^2.
\] (3.8)

Therefore, we arrive at the estimate
\[
- (\bar{\partial}_\alpha \tau u_n, u_n - u_{n-1}) \geq E(u_n) - E(u_{n-1})
\]

Summing up both sides for \( n = 1, \ldots, N \), we obtain that
\[
- \tau \sum_{n=1}^{N} (\bar{\partial}_\alpha \tau u_n, \tilde{\partial}_\alpha^1 u_n) \geq E(u_N) - E(u_0).
\]

Using Lemma 3.1, we have that
\[
E(u_N) - E(u_0) \leq 0
\]
which completes the proof. \( \Box \)

Remark 3.1 While the above result shows the energy bound, little is known about the energy dissipation law, which is important to the gradient flow models. For the convex splitting scheme of the conventional Allen–Cahn equation, it is well-known that the energy at each time level is decreasing, i.e., \( E(u_n) \leq E(u_{n-1}) \) for all \( n \geq 1 \), whose validity has not been established for the fractional case. Therefore, to develop a time stepping scheme which satisfies some energy dissipation principles is a very interesting and important task, which is explored in the next part.

3.2 Fractional Weighted Schemes and Energy Dissipation Law

To derive some novel time stepping schemes satisfying discrete energy dissipation laws, we introduce the following backward fractional interpolation
\[
I_\alpha u(t_n) = - \sum_{j=1}^{n-1} \omega_{n-j} (u(t_{n-j}) - u(0)) + \omega_0 u(0),
\] (3.9)
which is independent of \( u(t_n) \). Then it is easy to see that
\[
u(t_n) - I_\alpha u(t_n) = \sum_{j=1}^{n} \omega_{n-j} (u(t_j) - u(0)) = \tau^\alpha \bar{\partial}_\alpha \tau u(t_n),
\] (3.10)
which means the approximation error is of order \( O(\tau^\alpha) \).

Then we propose the so-called weighted convex splitting (WCS) scheme, reading that for given \( u_0, u_1, \ldots, u_{n-1} \), we look for the function \( u_n \) satisfying
\[
\bar{\partial}_\alpha \tau u_n - \kappa^2 \Delta u_n = u_{n,\alpha} - (u_n)^3, \quad \text{for } 1 \leq n \leq N,
\] (3.11)
with the homogeneous Dirichlet boundary condition, where \( u_{n,\alpha} \) denotes the fractional extrapolation (3.9). In this nonlinear scheme, we handle the convex part \( u^3 \) implicitly and replaces the concave part \( u \) with the fractional extrapolation \( u_{n,\alpha} \). The WCS scheme (3.11) can be written as the following nonlinear elliptic problem
\[
(I - \tau^\alpha \kappa^2 \Delta) u_n + \tau^\alpha (u_n)^3 = (1 + \tau^\alpha) u_{n,\alpha}
\] (3.12)
Since the left hand side is monotone with respect to $u_n$, the unique solvability follows directly from the implicit function theorem.

The next time stepping scheme, called linear weighted stabilized (LWS) scheme, is developed as

$$
\bar{\partial}_\tau^\alpha u_n - \kappa^2 \Delta u_n = u_{n,\alpha} - (u_{n,\alpha})^3 - S (u_n - u_{n,\alpha}), \quad \text{for } 1 \leq n \leq N,
$$

(3.13)

where $S > 0$ is a stabilization constant. Such the scheme is linear because $u_{n,\alpha}$ is independent of $u_n$. Then at each time level, the scheme (3.13) requires solving the elliptic problem

$$
((1 + \tau^\alpha S)I - \tau^\alpha \kappa^2 \Delta) u_n = (1 + (S + 1)\tau^\alpha)u_{n,\alpha} - \tau^\alpha (u_{n,\alpha})^3
$$

with the homogeneous Dirichlet boundary condition, which is uniquely solvable.

Next, we intend to show that both of schemes (3.11) and (3.13) satisfy the discrete maximum principle.

**Theorem 3.3** The WCS scheme (3.11) satisfies the discrete maximum principle unconditionally, i.e.,

$$
\|u_0\|_{L^\infty(\Omega)} \leq 1 \implies \|u_n\|_{L^\infty(\Omega)} \leq 1 \text{ for all } n \geq 1.
$$

Meanwhile, the LWS scheme (3.13) satisfies the discrete maximum principle with the constraint

$$
S + \tau^{-\alpha} \geq 2,
$$

where $S$ is the stabilization constant in (3.13).

**Proof** We prove the discrete maximum principle by induction. Recall that $u_{n,\alpha}$ is a convex combination of $u_0, u_1, \ldots, u_{n-1}$, and hence $\|u_{n,\alpha}\|_{L^\infty(\Omega)} \leq 1$. Again, we rewrite the WCS scheme (3.11) into the form (3.12) and note that the operator $I - \tau^\alpha \kappa^2 \Delta$ is positive. Meanwhile, $(u_n)^3$ has the same sign of $u^n$. Hence, it follows immediately that $\|u^n\|_\infty \leq 1$ by the monotonicity.

Now we turn to the LWS scheme (3.13), we rewrite the time stepping scheme as

$$
((S + \tau^{-\alpha})I - \kappa^2 \Delta) u_n = (1 + S + \tau^{-\alpha}) u_{n,\alpha} - (u_{n,\alpha})^3.
$$

The right side in the form of $h(x) = (1 + c)x - x^3$ with the constant $c = S + \tau^{-\alpha}$ and $x \in [-1, 1]$. Under the assumption that $c = S + \tau^{-\alpha} \geq 2$, the function $h(x)$ is monotonically increasing as well as

$$
\min_{x \in [-1, 1]} h(x) = h(-1) = -c \quad \text{and} \quad \max_{x \in [-1, 1]} h(x) = h(1) = c,
$$

due to $h'(x) = (1 + c) - 3x^2 \geq 0$. As a result, we have

$$
\left\| (1 + S + \tau^{-\alpha}) u_{n,\alpha} - (u_{n,\alpha})^3 \right\|_{L^\infty(\Omega)} \leq S + \tau^{-\alpha},
$$

which together with the property that

$$
\left\| (S + \tau^{-\alpha})I - \kappa^2 \Delta \right\|_{L^\infty(\Omega) \to L^\infty(\Omega)} \leq (S + \tau^{-\alpha})^{-1}
$$

complete the proof of the theorem. \qed

For the classical gradient flow ($\alpha = 1$), we usually aim to find energy decay preserving schemes, seeing [43]. Next we intend to study such energy dissipation property for time-fractional Allen–Cahn equations.
Theorem 3.4  The WCS scheme (3.11) satisfies the weighted energy stability unconditionally, i.e.,

\[ E(u_n) \leq E(u_{n,\alpha}), \quad \text{for all } n \geq 1. \]

The LWS scheme (3.13) also satisfies the weighted energy stability with the constraint \( S + \tau^{-\alpha} \geq 2 \).

**Proof**  Taking \( L^2 \)-inner product of (3.11) with \(- (u_n - u_{n,\alpha})\) yields

\[
- \frac{1}{\Gamma(2 - \alpha) \tau^\alpha} \| u_n - u_{n,\alpha} \|_{L^2}^2 = \kappa^2 (\nabla u_n, \nabla u_n - \nabla u_{n,\alpha}) + ((u_n)^3 - u_{n,\alpha}, u_n - u_{n,\alpha}).
\]

By a fundamental equality

\[
a(a - b) = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a - b)^2,
\]

and a fundamental inequality

\[
(a^3 - b, a - b) \geq \frac{1}{4} (a^2 - 1)^2 - \frac{1}{4} (b^2 - 1)^2,
\]

we have

\[
\frac{\kappa^2}{2} \| \nabla u_n \|_{L^2}^2 - \frac{\kappa^2}{2} \| \nabla u_{n,\alpha} \|_{L^2}^2 + \frac{1}{4} \| (u_n) - 1 \|^2_{L^2} - \frac{1}{4} \| (u_{n,\alpha}) - 1 \|^2_{L^2}
\leq - \frac{1}{\Gamma(2 - \alpha) \tau^\alpha} \| u_n - u_{n,\alpha} \|_{L^2}^2 - \frac{\kappa^2}{2} \| \nabla u_n - \nabla u_{n,\alpha} \|_{L^2}^2.
\]

Thus, for the WCS scheme (3.11) the weighted energy stability \( E(u_n) \leq E(u_{n,\alpha}) \) holds unconditionally.

For the LWS scheme (3.13), we follow the similar technique, we can derive

\[
\kappa^2 (\nabla u_n, \nabla u_n - \nabla u_{n,\alpha}) + ((u_n,\alpha)^3 - u_{n,\alpha}, u_n - u_{n,\alpha}) + (S + \tau^{-\alpha}) \| u_n - u_{n,\alpha} \|^2_{L^2} = 0.
\]

We again use the fundamental equality

\[
a(a - b) = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a - b)^2,
\]

and obtain the following inequality

\[
(b^3 - b)(a - b) + (a - b)^2 \geq \frac{1}{4} (a^2 - 1)^2 - \frac{1}{4} (b^2 - 1)^2, \quad \forall a, b \in [-1, 1].
\]

As in last theorem we have shown the discrete maximum principle, it follows above inequality that

\[
((u_n,\alpha)^3 - u_{n,\alpha}, u_n - u_{n,\alpha}) + (S + \tau^{-\alpha}) \| u_n - u_{n,\alpha} \|^2_{L^2}
\geq \frac{1}{4} \| (u_n) - 1 \|^2_{L^2} - \frac{1}{4} \| (u_{n,\alpha}) - 1 \|^2_{L^2},
\]

whenever \( S + \tau^{-\alpha} \geq 2 \). This ends up with

\[
\frac{\kappa^2}{2} \| \nabla u_n \|^2_{L^2} - \frac{\kappa^2}{2} \| \nabla u_{n,\alpha} \|^2_{L^2} + \frac{1}{4} \| (u_n) - 1 \|^2_{L^2} - \frac{1}{4} \| (u_{n,\alpha}) - 1 \|^2_{L^2} \leq 0,
\]

which is exactly the desired result. \(\Box\)
Remark 3.2 It is seen that the weighted energy stability can be extended to free energy in more general forms. We emphasize that if the free energy is given convex, the weighted energy stability indicates the fractional energy dissipation law as

$$\tilde{\partial}^\alpha_t E(u_n) \leq 0,$$

(3.14)

which is in agreement with the energy law on the continuous level (cf. Proposition (2.1)).

In addition, (3.14) is consistent with the energy decay property of classical gradient flow (\(\alpha = 1\)), i.e., \(\tilde{\partial}^1_t E(u_n) \leq 0\). At the continuous level, it is possible to derive the fractional energy dissipation law for the linear subdiffusion model

$$\partial_t^\alpha u - \kappa^2 \Delta u = 0.$$  

(3.15)

By taking \(L^2\)-inner product of (3.15) with \(-\partial_t^\alpha u\), we derive that

$$0 \geq -\|\partial_t^\alpha u(t)\|^2_{L^2(\Omega)} = \kappa^2 (\nabla u(t), \partial_t^\alpha \nabla u(t)) \geq \partial_t^\alpha \left( \frac{1}{2} \|\nabla u(t)\|^2_{L^2(\Omega)} \right) = \partial_t^\alpha E(u(t)),
$$

which is consistent with the discrete energy dissipation law (3.14). Unfortunately, the free energy associated with Allen–Cahn equation is nonlinear and not always convex with respect to \(u\). So far, we are unable to prove the fractional energy dissipation law but many numerical examples have already verified it computationally in the literature. Seeking the theoretical proof of the fractional energy dissipation law for the time fractional Allen–Cahn equations with a more general nonlinear free energy remains a very interesting and meaningful future work.

4 Error Analysis of Time Stepping Schemes

In this section, we derive the error analysis of the time stepping schemes proposed in Sect. 3. This requires some preliminary estimate for linear problem (2.9) and the implicit BE-CQ scheme

$$\tilde{\partial}^\alpha_t u_n + Au_n = g(t_n) \quad \text{for} \ 1 \leq n \leq N,$$

(4.1)

with given initial condition \(u_0\), where \(A = -\kappa^2 \Delta\) with the homogeneous Dirichlet boundary condition. The following lemma gives the error estimate, which has been developed in [25].

Lemma 4.1 Let \(u_0 \in H^2(\Omega) \cap H^1_0(\Omega)\) and \(u(t)\) be the solution of the linear time-fractional evolution Eq. (2.9). Then \([u_n]_t\), the solutions of implicit BE-CQ scheme (4.1) satisfies

$$\|u_n - u(t_n)\|_{L^2(\Omega)} \leq c \tau^{\alpha-1} \|Au_0 + g(0)\|_{L^2(\Omega)} + c \tau \int_0^{t_n} (t_n - s)^{\alpha-1} \|g'(s)\|_{L^2(\Omega)} ds.$$

For \(1 \leq p \leq \infty\) and \(X\) being a Banach space, we denote by \(\ell^p(X)\) the space of sequences \(v^n \in X, n = 0, 1, \ldots\), such that \(\|(v^n)_{n=0}^\infty\|_{\ell^p(X)} < \infty\), where

$$\|(v^n)_{n=0}^\infty\|_{\ell^p(X)} := \begin{cases} \left( \sum_{n=0}^\infty \tau \|v^n\|_X^p \right)^{1/p} & \text{if} \ 1 \leq p < \infty, \\ \sup_{n \geq 0} \|v^n\|_X & \text{if} \ p = \infty. \end{cases}$$
For a finite sequence \( v^n \in X, n = 0, 1, \ldots, m \), we denote \( \|(v^n)^m_{n=0}\|_{L^p(X)} := \|(v^n)_{n=0}^\infty\|_{L^p(X)} \), by setting \( v^n = 0 \) for \( n > m \). The next two lemma shows that the BE-CQ scheme (4.1) satisfies the discrete fractional Grönwall’s inequality and discrete maximal \( \ell^p \) regularity, whose complete proof is given in [27, Theorem 2.2] and [26, Theorem 5].

**Lemma 4.2** Let \( \tilde{\partial}_x^\alpha \) denote the BE-CQ given by (3.1). If \( \alpha \in (0, 1) \) and \( p \in (1/\alpha, \infty) \), and a sequence \( v^n \in X, n = 0, 1, 2, \ldots, \), with \( v^0 = 0 \), satisfies

\[
\|(v^n)_{n=1}^m\|_{L^p(X)} \leq c_0 \|(v^n)_{n=1}^m\|_{L^p(X)} + \sigma, \quad \forall 0 \leq m \leq N,
\]

for some positive constants \( \kappa \) and \( \sigma \), then there exists a \( \tau_0 > 0 \) such that for any \( \tau < \tau_0 \) there holds

\[
\|(v^n)_{n=1}^N\|_{L^\infty(X)} + \|(\tilde{\partial}_x^\alpha v^n)_{n=1}^N\|_{L^p(X)} \leq c \sigma,
\]

where the constants \( c \) and \( \tau_0 \) are independent of \( \sigma, \tau, N, X \) and \( v^n \), but may depend on \( \alpha, p, c_0 \), and \( T \).

**Lemma 4.3** The BE-CQ scheme (4.1) with \( u_0 = 0 \) has the following maximal \( \ell^p \)-regularity

\[
\|(\tilde{\partial}_x^\alpha u^n)_{n=1}^m\|_{L^p(L^2(\Omega))} + \|(Au^n)_{n=1}^m\|_{L^p(L^2(\Omega))} \leq c \|(f^n)_{n=1}^m\|_{L^p(L^2(\Omega))},
\]

where the constant \( c \) is independent of \( N, \tau \).

### 4.1 Error Analysis of the CS Scheme

With the help of error estimate in Lemma 4.1, the discrete fractional Grönwall’s inequality in Lemma 4.2, the discrete maximal \( \ell^p \) regularity in Lemma 4.3 and the regularity estimate in Sect. 2, we can derive the error analysis of the CS scheme (3.2). To this end, we use the splitting that

\[
u(t_n) - u_n = (u(t_n) - v_n) + (v_n - u_n) =: \theta_n + \rho_n. \quad (4.2)
\]

where \( v_n \) satisfies the time stepping scheme

\[
\tilde{\partial}_x^\alpha v_n + Av_n = f(u(t_n)), \quad \text{with} \quad v_0 = u_0. \quad (4.3)
\]

**Lemma 4.4** Let \( u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \) and \( u(t) \) be the solution of the time-fractional Allen–Cahn Eq. (1.1). Then \( \theta_n = u(t_n) - v_n \), where \( v_n \) is the solutions of (4.3), satisfies

\[
\max_{1 \leq n \leq N} \|\theta_n\|_{L^2(\Omega)} \leq cT^{\alpha}. \quad (4.4)
\]

where the constant \( c \) may depends on \( \alpha, \kappa, T \) but is independent of \( \tau \) and \( u \) except for \( u_0 \).

**Proof** Using Lemma 4.1, we have the following estimate that

\[
\|u_n(t_n) - v_n\|_{L^2(\Omega)} \leq cT^{\alpha-1} \tau \left( \|Au_0 + g(u_0)\|_{L^2(\Omega)} \right) + c \tau \int_0^{t_n} (t_n - s)^{\alpha-1} \|f'(u(s))\|_{L^2(\Omega)} ds.
\]

Using the fact that \( u \in L^\infty((0, T) \times \Omega) \) by Theorem 2.1, we have \( \|f'(u(s))\|_{L^2(\Omega)} \leq c \), and hence

\[
\|f'(u(s))\|_{L^2(\Omega)} \leq c \|\partial_s u(s)\|_{L^2(\Omega)} \leq c s^{\alpha-1}.
\]
where the last inequality is given by (2.15). As a result, we derive that
\[
\|u_h(t_n) - u_h^n\|_{L^2(\Omega)} \\
\leq c \tau_n^{\alpha - 1} + \tau \int_0^{t_n} (t_n - s)^{\alpha - 1} s^{\alpha - 1} ds \\
\leq c \tau (t_n^{\alpha - 1} + t_n^{2\alpha - 1}) \\
\leq c \tau t_n^{\alpha - 1} \\
\leq c \tau^\alpha.
\]

The next lemma gives the bound of \(\rho_n\).

**Lemma 4.5** Suppose that \(u_0 \in H^2(\Omega) \cap H^1_0(\Omega)\) and \(|u_0| \leq 1\). Let \(v_n\) be the solution of (4.3) and \(u_n\) be the solution of the CS scheme (3.2). Then \(\rho_n = v_n - u_n\) satisfies
\[
\max_{1 \leq n \leq N} \|\rho_n\|_{L^2(\Omega)} \leq c \tau^\alpha.
\]
where the constant \(c\) may depend on \(\alpha, \kappa, T\) but is independent of \(\tau\) and \(u\) except for \(u_0\).

**Proof** We note that \(\rho_n\) satisfies the following discrete problem
\[
\tilde{\partial}_t^\alpha \rho_n + A \rho_n = (u(t_n) - u_{n-1}) + (u(t_n)^3 - (u_n)^3), \quad \text{with } \rho_0 = 0.
\]
By applying the discrete maximal \(\ell^p\)-regularity in Lemma 4.3, we obtain that for all \(1 < p < \infty\):
\[
\|(\tilde{\partial}_t^\alpha \rho_n)^m\|_{\ell^p(L^2(\Omega))} \\
\leq c \|(u(t_n) - u_{n-1})^m\|_{\ell^p(L^2(\Omega))} + c \|(u(t_n)^3 - (u_n)^3)^m\|_{\ell^p(L^2(\Omega))} = \sum_{i=1}^2 I_i.
\]
Using the regularity estimate (2.15), we have an estimate for \(I_1\)
\[
I_1 \leq c \|(u(t_n) - u_{n-1})^m\|_{\ell^p(L^2(\Omega))} + c \|(u(t_n) - u(t_{n-1}))^m\|_{\ell^p(L^2(\Omega))} \\
\leq c \tau^\alpha + c \|(\rho_n)^m\|_{\ell^p(L^2(\Omega))} + c \left|\int_{t_{n-1}}^{t_n} u'(s) ds\right|^m_{\ell^p(L^2(\Omega))} \\
\leq c \tau^\alpha + c \|(\rho_n)^m\|_{\ell^p(L^2(\Omega))}.
\]
Finally, using the fact that \(|u(x, t)| \leq 1\) and \(|u^n| \leq 1\) respectively by Theorems 2.3 and 3.1, we derive the bound for the second term
\[
I_2 \leq c \|(u(t_n) - u_n)^m\|_{\ell^p(L^2(\Omega))} \leq c \|(\theta_n)^m\|_{\ell^p(L^2(\Omega))} + c \|(\rho_n)^m\|_{\ell^p(L^2(\Omega))} \\
\leq c \tau^\alpha + c \|(\rho_n)^m\|_{\ell^p(L^2(\Omega))}.
\]
Combining the preceding three estimates, we arrive at
\[
\|(\tilde{\partial}_t^\alpha \rho_n)^m\|_{\ell^p(L^2(\Omega))} \leq c \tau^\alpha + c \|(\rho_n)^m\|_{\ell^p(L^2(\Omega))},
\]
By choosing \(p > 1/\alpha\) and applying the discrete Grönwall’s inequality in Lemma 4.2, we obtain
\[
\max_{1 \leq n \leq N} \|\rho_n\|_{L^2(\Omega)} \leq c \tau^\alpha.
\]
This completes the proof of the lemma.

Combining Lemmas 4.4 and 4.5, we have the error estimate for the CS scheme (3.2).
Theorem 4.1 Suppose that \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \) and \( |u_0| \leq 1 \). Let \( u \) be the solution of the time-fractional Allen–Cahn Eq. (1.1). Then \( u_n \), the solution of the CS scheme (3.2), satisfies the error estimate

\[
\max_{1 \leq n \leq N} \| u(t_n) - u_n \|_{L^2(\Omega)} \leq c \tau^\alpha.
\] (4.6)

where the constant \( c \) may depends on \( \alpha, \kappa, T \) but is independent of \( \tau \) and \( u \) except for \( u_0 \).

4.2 Error Analysis of the Weighted Time Stepping Scheme

Now we turn to the error estimate for the WCS scheme (3.11) and the LWS scheme (3.13). Besides the useful tools applied in the previous section, we also need the following lemma for the bound of \( \bar{\partial}_\tau^\alpha u(t_n) \).

Lemma 4.6 Let \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \) and \( u \) be the solution of the time-fractional Allen–Cahn Eq. (1.1). Meanwhile, let \( \bar{\partial}_\tau^\alpha \) denote the BE-CQ given by (3.1). Then we have for \( n \geq 1 \)

\[
\max_{1 \leq n \leq N} \| \bar{\partial}_\tau^\alpha u(t_n) \|_{L^2(\Omega)} \leq c.
\]

Proof By setting \( y(t) = u(t) - u_0 \), then we have

\[
\bar{\partial}_\tau^\alpha u(t_n) = \bar{\partial}_\tau^\alpha y(t_n) = \bar{\partial}_\tau^{\alpha-1} \psi_n = \sum_{j=0}^{n} \omega_{n-j} \psi_j,
\]

where \( \psi_n = \bar{\partial}_\tau^1 y(t_n) \), for \( n = 1, \ldots, N \) and \( \psi_0 = 0 \). By the regularity estimate (2.15), we have

\[
\| \psi_1 \|_{L^2(\Omega)} \leq \tau^{-1} \int_0^\tau \| u'(s) \|_{L^2(\Omega)} \, ds \leq \tau^{-1} \int_0^\tau \| u'(s) \|_{L^2(\Omega)} \, ds \leq c \tau^{-1} \int_0^\tau s^{\alpha-1} \, ds \leq c \tau^{\alpha-1}.
\]

Meanwhile, for \( n \geq 2 \), we derive that

\[
\| \psi_n \|_{L^2(\Omega)} \leq \tau^{-1} \int_{t_{n-1}}^{t_n} \| u'(s) \|_{L^2(\Omega)} \, ds \leq \tau^{-1} \int_{t_{n-1}}^{t_n} \| u'(s) \|_{L^2(\Omega)} \, ds \leq c \tau^{-1} \int_{t_{n-1}}^{t_n} s^{\alpha-1} \, ds \leq c \tau^{\alpha-1}.
\]

Finally, find a bound for the weights \( \omega_n = \prod_{j=1}^{n} \left( 1 - \frac{\alpha}{j} \right) \) in (3.4). By the trivial inequality \( \ln(1 + x) \leq x \) for \( x > -1 \), we derive

\[
\ln \omega_n^{(\alpha-1)} = \sum_{j=1}^{n} \ln \left( 1 - \frac{\alpha}{j} \right) \leq -\alpha \sum_{j=1}^{n} j^{-1} \leq -\alpha \ln(n + 1).
\]

which indicates that for \( n \geq 0 \) satisfy the estimate that

\[
0 < \omega_n^{(\alpha-1)} < (n + 1)^{-\alpha}.
\]
Therefore,
\[
\|\tilde{\partial}_\tau^\alpha u(t_n)\|_{L^2(\Omega)} \leq \tau^{1-\alpha} \sum_{j=2}^n \omega_n^{(\alpha-1)} \|\psi_j\|_{L^2(\Omega)} + \tau^{1-\alpha} \omega_n^{(\alpha-1)} \|\psi_1\|_{L^2(\Omega)}
\]
\[
\leq c \sum_{j=2}^n (n-j+1)^{-\alpha} (j-1)^{\alpha-1} + cn^{-\alpha} \leq c.
\]
where the constant \(c\) is independent of \(n\). This completes the proof.

\[\square\]

**Theorem 4.2** Suppose that \(u_0 \in H^2(\Omega) \cap H^1_0(\Omega)\) and \(|u_0| \leq 1\). Let \(u\) be the solution of the time-fractional Allen–Cahn Eq. (1.1). Then \(u_n\), the solution of the WCS scheme (3.11), satisfies
\[
\max_{1 \leq n \leq N} \|u(t_n) - u_n\|_{L^2(\Omega)} \leq c\tau^\alpha.
\]
where the constant \(c\) may depends on \(\alpha, \kappa, T\) but is independent of \(\tau\) and \(u\) except for \(u_0\).

**Proof** To derive the error, we use the splitting (4.2) and note that the estimate for \(\theta_n\) has been given in the Lemma (4.1). Then \(\rho_n\) satisfies the time stepping problem
\[
\tilde{\partial}_\tau^\alpha \rho_n + A\rho_n = (u(t_n) - u_{n,\alpha}) + (u(t_n)^3 - (u_n)^3), \quad \text{with} \quad \rho_0 = 0.
\]
Using the discrete maximal \(\ell^p\)-regularity in Lemma 4.3, we obtain that for all \(1 < p < \infty\):
\[
\|(\tilde{\partial}_\tau^\alpha \rho_n)^m\|_{\ell^p(L^2(\Omega))} \leq c\|(u(t_n) - u_{n,\alpha})^m\|_{\ell^p(L^2(\Omega))} + c\|(u(t_n)^3 - (u_n)^3)^m\|_{\ell^p(L^2(\Omega))} =: I_1 + I_2.
\]
The second term could be bounded using the same argument in Lemma 4.5, with the help of the (discrete) maximum principle and regularity estimate (2.15). In particular, we have
\[
I_2 \leq c\tau^\alpha + c\|(\rho_n)^m\|_{\ell^p(L^2(\Omega))}.
\]
Now we turn to the first term. Using the definition of fractional weighted extrapolation (3.9), we have
\[
I_1 \leq \tau^\alpha \|(\tilde{\partial}_\tau^\alpha u(t_n))^m\|_{\ell^p(L^2(\Omega))} + \left\|\left(\sum_{i=1}^{n-1} \omega_{n-i}(u_i - u(t_i))\right)^m\right\|_{\ell^p(L^2(\Omega))}.
\]
Using the Young’s inequality of discrete convolution, we arrive at
\[
I_1 \leq \tau^\alpha \|(\tilde{\partial}_\tau^\alpha u(t_n))^m\|_{\ell^p(L^2(\Omega))} + c\|(u_n - u(t_n))^m\|_{\ell^p(L^2(\Omega))}
\]
Then Lemma 4.6 leads to
\[
I_1 \leq c\tau^\alpha + c\|(u_n - u(t_n))^m\|_{\ell^p(L^2(\Omega))} \leq c\tau^\alpha + c\|(\rho_n)^m\|_{\ell^p(L^2(\Omega))} \quad (4.7)
\]
Combining the preceding two estimates, we arrive at
\[
\|(\tilde{\partial}_\tau^\alpha \rho_n)^m\|_{\ell^p(L^2(\Omega))} \leq c\tau^\alpha + c\|(\rho_n)^m\|_{\ell^p(L^2(\Omega))}.
\]
By choosing \(p > 1/\alpha\) and applying the discrete Grönwall’s inequality in Lemma 4.2, we obtain
\[
\max_{1 \leq n \leq N} \|\rho_n\|_{L^2(\Omega)} \leq c\tau^\alpha.
\]
This completes the proof of the lemma. \[\square\]
Similar argument may derive the error estimate for the LWS scheme (3.13) with smooth initial data.

**Corollary 4.1** Suppose that \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \) and \( |u_0| \leq 1 \). Let \( u \) be the solution of the time-fractional Allen–Cahn Eq. (1.1). Then \( \{u_n\} \), the solution of the LWS scheme (3.13), satisfies the error estimate

\[
\max_{1 \leq n \leq N} \|u(t_n) - u_n\|_{L^2(\Omega)} \leq c \tau^\alpha.
\]

(4.8)

where the constant \( c \) may depends on \( \alpha, \kappa, T \) but is independent of \( \tau \) and \( u \) except for \( u_0 \).

**Proof** To derive the error, we use the usual splitting (4.2) and note that the estimate for \( \partial_n \) has been given in the Lemma (4.4). Then \( \rho_n \) satisfies the time stepping problem

\[
\tilde{\partial}_\tau \rho_n + A \rho_n = (u(t_n) - u_{n,\alpha}) + (u(t_n)^3 - (u_{n,\alpha})^3) + S(u_n - u_{n-\alpha})
\]

with \( \rho_0 = 0 \). Using the discrete maximal \( \ell^p \)-regularity in Lemma 4.3, as well as the fact that \( |u(x,t)| \leq 1 \) and \( |u_{n,\alpha}(x)| \leq 1 \) for all \( n \geq 0 \), we obtain that for all \( 1 < p < \infty \):

\[
\| (\tilde{\partial}_\tau \rho_n)^m \|_{\ell^p(L^2(\Omega))} \leq c \| (u(t_n) - u_{n,\alpha})_n \|_{\ell^p(L^2(\Omega))} + \| (u_n - u_{n-\alpha})_n \|_{\ell^p(L^2(\Omega))}
\]

The estimate for the first term has been given by (4.7). Now we apply Lemma 4.6 to obtain that

\[
I_2 = c \| (\tilde{\partial}_\tau u_n)^m \|_{\ell^p(L^2(\Omega))}
\]

\[
\leq c \tau^\alpha \| (\tilde{\partial}_\tau (u(t_n) - u_n))^m \|_{\ell^p(L^2(\Omega))} + c \tau^\alpha \| (\tilde{\partial}_\tau (u(t_n) - u_n))^m \|_{\ell^p(L^2(\Omega))}
\]

Then the discrete (in time) inverse inequality yields that

\[
I_2 \leq c \tau^\alpha + c \| (u(t_n) - u_n)^m \|_{\ell^p(L^2(\Omega))} \leq c \tau^\alpha + c \| (\rho_n)^m \|_{\ell^p(L^2(\Omega))}.
\]

Finally, the preceding estimates together with the discrete Grönwall’s inequality in Lemma 4.2 result in the desired result. \( \square \)

**Remark 4.1** In this paper, we only present the results associated with the homogeneous Dirichlet boundary condition. In fact, all the argument could be applied to get similar conclusions for other types of boundary conditions, e.g., periodic boundary condition and Neumann boundary condition, since the analysis only depends on the abstract operator \( A \) and its resolvent estimate.

5 Numerical Results

In this section, we present some numerical experiments to confirm the theoretical findings and to offer new insight on the time-fractional Allen–Cahn dynamics.

**Example 5.1** In the first example, we use a two-dimensional problem to support the derived convergence rate of the proposed numerical schemes. To this end, we let \( \Omega = (0, 1)^2 \) and \( \kappa = 0.5 \), and take the fractional power \( \alpha \) to be 0.4, 0.6, 0.8, respectively. We use the smooth initial condition \( u_0(x, y) = x(1 - x) y(1 - y) \).
In the computation, the central finite difference discretization is used for the spatial directions with a sufficiently small $h = 1/200$ in each direction so that the time discretization errors are dominant. Here we present the error of numerical solution of the CS (3.2), WCS (3.11) and LWS schemes (3.13), in Table 1. Since the exact solution is unknown, to numerically evaluate pointwise-in-time temporal error $e_t$, we compute

$$e_t \approx \max_{1 \leq n \leq N} \| u^n - u^{2n}_\tau \|_{L^2(\Omega)}.$$ 

Numerical experiments show that all the three schemes are convergent with rate $O(\tau^\alpha)$. This observation fully supports our theoretical result in Theorems 4.1 and 4.2, and Corollary 4.1.

**Example 5.2** We now consider the one-dimensional fractional Allen–Cahn equation in $[0, 2\pi]$ with zero Dirichlet boundary conditions. We fix $\kappa = 0.1$ and take the fractional power $\alpha$ to be 0.5, 0.7, 0.9, respectively. The dynamics corresponding to the classical Allen–Cahn equation ($\alpha = 1$) is also computed for comparison. The central finite difference method is again used for the spatial discretization with $N = 2^7$, and the time step $\tau = 10^{-2}$.

With the initial condition specified as $u(0, x) = 0.05 \sin(x)$, we present snapshots of solutions at different times and the time evolution of energy in Fig. 1. We observe that for the time fractional case, the evolution is slower than that of the classical Allen–Cahn. In particular, we observe that the dynamic is slower for the smaller $\alpha$, especially for long time, e.g. after $T = 10$, where the solution of classical Allen–Cahn almost arrives at the steady state but the fractional one with $\alpha = 0.5$ is far from the steady state. But in all the cases, the energy decays monotonically, and moreover, the following energy dispassion law

$$\frac{\partial}{\partial t} E(t_n) \leq 0, \quad \text{for all } n \geq 1,$$

holds as expected.

**Example 5.3** Next, we consider the two-dimensional fractional Allen–Cahn equation in $(0, 2\pi)^2$ with zero Dirichlet boundary conditions. We fix $\kappa = 0.1$ and take the fractional power $\alpha$ to be 0.5, 0.7, 0.9 respectively. The classical case with $\alpha = 1$ is again also solved for comparison. The central finite difference method is used for the discretization in spatial space with $N = 2^7$ in each direction, and the time step $\tau = 5 \times 10^{-3}$. In the experiment, we test the random initial value with zero mean $u(0, x, y) = 0.05(2 \times \text{rand}(x, y) - 1)$, where $\text{rand}(\cdot)$ denotes a random number in $[0, 1]$. 

---

**Table 1** Example 5.1: $e_t$ with $T = 1$, $\tau = T/N$, $N = k \times 10^4$, and $h = 1/200$

| Scheme | $\alpha \backslash k$ | 1     | 2     | 4     | 8     | 16    | Rate     |
|--------|----------------------|-------|-------|-------|-------|-------|----------|
| CS     | 0.4                  | 6.01e−4 | 4.50e−4 | 3.36e−4 | 2.51e−4 | 1.87e−4 | 0.42  (0.40) |
|        | 0.6                  | 1.38e−4 | 8.78e−5 | 5.64e−5 | 3.65e−5 | 2.37e−5 | 0.63  (0.60) |
|        | 0.8                  | 2.51e−5 | 1.26e−5 | 6.51e−6 | 3.62e−6 | 2.04e−6 | 0.84  (0.80) |
| WCS    | 0.4                  | 4.84e−4 | 3.76e−4 | 2.90e−4 | 2.22e−4 | 1.70e−4 | 0.38  (0.40) |
|        | 0.6                  | 1.32e−4 | 8.54e−5 | 5.54e−5 | 3.60e−5 | 2.35e−5 | 0.62  (0.60) |
|        | 0.8                  | 3.52e−5 | 1.79e−5 | 9.72e−6 | 5.29e−6 | 2.89e−6 | 0.85  (0.80) |
| LWS    | 0.4                  | 5.70e−4 | 4.35e−4 | 3.29e−4 | 2.48e−4 | 1.86e−4 | 0.41  (0.40) |
|        | 0.6                  | 1.70e−4 | 1.10e−4 | 7.16e−5 | 4.67e−5 | 3.05e−5 | 0.62  (0.60) |
|        | 0.8                  | 6.71e−5 | 3.76e−6 | 2.10e−6 | 1.18e−6 | 6.64e−6 | 0.83  (0.80) |
In Fig. 2, the numerical solutions look almost the same before \( T = 5 \) for all four cases, which corresponds to the phase separation period. After that the solutions begin to evolve differently, with the dynamics entering the long time scale phase coarsening. This is further verified by the energy evolution. Numerically, we can conclude that the equations with different fractional orders have similar dynamics for the phase separation but the coarsening dynamics becomes slower for the smaller fractional orders. Again, from the numerical
experiments, we observe the fractional energy dispassion law
\[
\tilde{\partial}_t^\alpha E(u_n) \leq 0, \quad \text{for all } n \geq 1.
\]

**Example 5.4** It is well-known that the classical Allen–Cahn equation converges to the mean curvature flow. In this example, we intend to numerically compare the difference between the dynamics of the integer order AC equation and the ones of the time-fractional Allen–Cahn equation. To this end, we continue the two-dimensional problem with \( \kappa = 0.1 \). We test two experiments with initial data representing different interfaces, i.e., a circle (I) and a dumbbell (II), and observe how the phase parameter and the energy evolve for different values of \( \alpha \).
The results for the first experiment are presented in Fig. 4, and those for the second one are shown in Fig. 5, respectively. In both of these tests, it is seen that the relaxation time changes with different $\alpha$. For the circular case, after a short time for phase reordering, all three circles shrink as time goes on. Particularly, the area of the circle in the classical Allen–Cahn equation shrinks linearly, which essentially illustrate the dynamics of mean curvature flow in the two-dimension. For the time-fractional Allen–Cahn model, the area shrinking rate seems to exhibit a power-law scaling, which is reminiscent to the classical case. The detailed behavior is not presented here. For the dumbbell experiment, in all three cases it is observed that two balls shrink first while the neck in between keeps its flatness, where the curvature is zero. It is reasonable to conjecture that the dynamics of time-fractional AC equation may converge to some kind of non-local in time mean curvature flow, at least for the simple structure cases. This will be an interesting topic in our future studies.

6 Conclusion

In this paper, we study a time-fractional Allen–Cahn equation, where the conventional first-order time-derivative is replaced by a fractional derivative with order $\alpha \in (0, 1)$, resulting in history dependent nonlocal-in-time dynamics. The well-posedness, solution regularity, and maximum principle are proved, by using some useful tools such as the maximal $L^p$ regularity of fractional evolution equations and the fractional Gröwall’s inequality. Then we developed three unconditionally solvable and stable time stepping schemes, i.e., convex splitting scheme, weighted convex splitting scheme and linear weighted stabilized scheme. The energy dissipation property (in a weighted average sense) were discussed for the last two time stepping schemes. Finally, we show the convergence of those time stepping schemes, where the error is of order $O(\tau^\alpha)$ without any extra regularity assumption on the solution.
It is promising to extend our argument to the general nonlocal-in-time phase field model

\[ D_\rho u(t) - \kappa^2 \Delta u = -F'(u), \]

with historical initial data, where the nonlocal operator is given by

\[ D_\rho u(t) = \int_0^\infty (u(t) - u(t-s))\rho(s) \, ds, \]

which contains many popular examples, such as Caputo fractional derivative we discussed in this paper, tempered fractional derivative [41], distributed-order derivative [11], or nonlocal operator with a finite nonlocal horizon [13,19], even though the theoretical analysis of those models remain fairly scarce, and those technical tools, such as (discrete) maximal regularity and weighted Grönwall’s inequality, await rigorous study. It would be of high interests to see whether these different nonlocal dynamics can correspond to some natural processes thus leading to interesting applications Finally, the simulated dynamics of the nonlocal-in-time Allen–Cahn equations observed in numerical experiments reveal interesting insight on
Fig. 5 (Example 5.4 (II)) Comparisons between classical AC and factional AC nonlocal-in-time curvature dependent dynamics, which awaits further theoretical studies in the future.

References

1. Ainthworth, M., Mao, Z.: Analysis and approximation of a fractional Cahn–Hilliard equation. SIAM J. Numer. Anal. 55(4), 1689–1718 (2017)
2. Allen, S.M., Cahn, J.W.: A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. Acta Metall. 27(6), 1085–1095 (1979)
3. Akagi, Goro, Schimperna, Giulio, Segatti, Antonio: Fractional Cahn–Hilliard, Allen–Cahn and porous medium equations. J. Diff. Equ. 261(6), 2935–2985 (2016)
4. Anderson, D.M., McFadden, G.B., Wheeler, A.A.: Diffuse-interface methods in fluid mechanics. Annu. Rev. Fluid Mech. 30(1), 139–165 (1998)
5. Antil, Harbir, Bartels, Sören: Spectral approximation of fractional PDEs in image processing and phase field modeling. Comput. Methods Appl. Math. 17(4), 661–678 (2017)
6. Bates, P.W.: On some nonlocal evolution equations arising in materials science. Nonlinear Dyn. Evol. Equ. 48, 13–52 (2006)
7. Bates, W.P., Sarah, B., Jianlong, H.: Numerical analysis for a nonlocal Allen–Cahn equation. Int. J. Numer. Anal. Model. 6(1), 33–49 (2009)
8. Caffarelli, L., Roquejoffre, J.-M., Savin, O.: Nonlocal minimal surfaces. Commun. Pure Appl. Math. 63(9), 1111–1144 (2010)
9. Cahn, J.W., Hilliard, J.E.: Free energy of a nonuniform system. I. Interfacial free energy. J. Chem. Phys. 28(2), 258–267 (1958)
10. Clarke, S., Vvedensky, D.D.: Origin of reflection high-energy electron–diffraction intensity oscillations during molecular-beam epitaxy: A computational modeling approach. Phys. Rev. Lett. 58(21), 2235 (1987)
11. Chechkin, A.V., Gorenflo, R., Sokolov, I.M.: Retarding subdiffusion and accelerating superdiffusion governed by distributed-order fractional diffusion equations. Phys. Rev. E 66, 046129 (2002)
12. Chen, L., Zhao, J., Cao, W., Wang, H., Zhang, J.: An accurate and efficient algorithm for the time-fractional molecular beam epitaxy model with slope selection. Comput. Phys. Commun. 245, 106842 (2019)
13. Du, Q.: Nonlocal Modeling, Analysis and Computation. In: CBMS-NSF regional conference series, vol. 94. SIAM, (2019)
14. Du, Q., Feng, X.-B.: The phase field method for geometric moving interfaces and their numerical approximations. arXiv preprint arXiv:1902.04924 (2019)
15. Du, Q., Ju, L., Li, X., Qiao, Z.: Stabilized linear semi-implicit schemes for the nonlocal Cahn–Hilliard equation. J. Comput. Phys. 363, 39–54 (2018)
16. Du, Q., Ju, L., Li, X., Qiao, Z.: Maximum principle preserving exponential time differencing schemes for the nonlocal Allen–Cahn equation. J. Numer. Anal. 57, 875–898 (2019)
17. Du, Q., Liu, L., Li, X., Qiao, Z.: Maximum bound principles for a class of semilinear parabolic equations and exponential time differencing schemes. SIAM Review (2020). arXiv preprint arXiv:2005.11465
18. Du, Q., Liu, C., Wang, X.: A phase field approach in the numerical study of the elastic bending energy for vesicle membranes. J. Comput. Phys. 198(2), 450–468 (2004)
19. Du, Q., Yang, J., Zhou, Z.: Analysis of a nonlocal-in-time parabolic equation. Discrete Contin. Dyn. Syst. Ser. B 22(2), 339–368 (2017)
20. Du, Q., Yang, J.: Asymptotically compatible Fourier spectral approximations of nonlocal Allen–Cahn equations. SIAM J. Numer. Anal. 54(3), 1899–1919 (2016)
21. Golding, I., Cox, E.C.: Physical nature of bacterial cytoplasm. Phys. Rev. Lett. 96(9), 098102 (2006)
22. Guan, Z., Lowengrub, J., Wang, C., Wise, S.: Second order convex splitting schemes for periodic nonlocal Cahn–Hilliard and Allen–Cahn equations. J. Comput. Phys. 277, 48–71 (2014)
23. Gui, C., Zhao, M.: Traveling wave solutions of Allen–Cahn equation with a fractional laplacian. Annales de l’Institut Henri Poincare C Non Linear Anal. 32(4), 785–812 (2015)
24. Lev Davidovich Landau and Evgenii Mikhailevich Lifshitz: Course of Theoretical Physics. Elsevier, Amsterdam (2013)
25. Jin, B., Lazarov, R., Zhou, Z.: Two fully discrete schemes for fractional diffusion and diffusion-wave equations with nonsmooth data. SIAM J. Sci. Comput. 38(1), A146–A170 (2016)
26. Jin, B., Li, B., Zhou, Z.: Discrete maximal regularity of time-stepping schemes for fractional evolution equations. Numer. Math. 138(1), 101–131 (2018)
27. Jin, B., Li, B., Zhou, Z.: Numerical analysis of nonlinear subdiffusion equations. SIAM J. Numer. Anal. 56(1), 1–23 (2018)
28. Jin, B., Li, B., Zhou, Z.: Subdiffusion with a time-dependent coefficient: analysis and numerical solution. Math. Comput. 88(319), 2157–2186 (2019)
29. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
30. Kirchner, J.W., Feng, X., Neal, C.: Fractal stream chemistry and its implications for contaminant transport in catchments. Nature 403(6769), 524 (2000)
31. Liu, H., Cheng, A., Wang, H., Zhao, J.: Time-fractional Allen–Cahn and Cahn–Hilliard phase-field models and their numerical investigation. Comput. Math. Appl., 76(8):1876–1892
32. Lin, Y., Xu, C.: Finite difference/spectral approximations for the time-fractional diffusion equation. J. Comput. Phys. 225(2), 1533–1552 (2007)
33. Liu, C., Shen, J.: A phase field model for the mixture of two incompressible fluids and its approximation by a fourier-spectral method. Phys. D 179(3–4), 211–228 (2003)
34. Lubich, C.: Convolution quadrature and discretized operational calculus I. Numer. Math. 52(2), 129–145 (1988)
35. Mustapha, K., Abdallah, B., Furati, K.M.: A discontinuous Petrov–Galerkin method for time-fractional diffusion equations. SIAM J. Numer. Anal. 52(5), 2512–2529 (2014)
36. Nigmatullin, R.: The realization of the generalized transfer equation in a medium with fractal geometry. Phys. Status Solidi (b) 133(1), 425–430 (1986)
37. Oldham, K.B., Spanier, J.: The Fractional Calculus. Academic Press, New York (1974)
38. Paragios, N. Mellina-Gottardo, O., Ramesh, V.: Gradient vector flow fast geodesic active contours. In: Proceedings Eighth IEEE International Conference on Computer Vision. ICCV 2001, volume 1, pages 67–73. IEEE (2001)
39. Pismen, L.M.: Nonlocal diffuse interface theory of thin films and the moving contact line. Phys. Rev. E 64(2), 021603 (2001)
40. Qian, T., Wang, X.-P., Sheng, P.: Molecular scale contact line hydrodynamics of immiscible flows. Phys. Rev. E 68(1), 016306 (2003)
41. Sabzikar, F., Meerschaert, M.M., Chen, J.: Tempered fractional calculus. J. Comput. Phys. 293, 14–28 (2015)
42. Shao, D., Rappel, W.J., Levine, H.: Computational model for cell morphodynamics. Phys. Rev. Lett. 105(10), 108104 (2010)
43. Shen, J., Xu, J., Yang, J.: A new class of efficient and robust energy stable schemes for gradient flows. SIAM Rev. 61(3), 474–506 (2019)
44. Shen, J., Tang, T., Yang, J.: On the maximum principle preserving schemes for the generalized Allen–Cahn equation. Commun. Math. Sci. 14(6), 1517–1534 (2016)
45. Song, F., Xu, C., Karniadakis, G.: A fractional phase-field model for two-phase flows with tunable sharpness: algorithms and simulations. Comput. Methods Appl. Mech. Eng. 305, 376–404 (2016)
46. Sun, Z.-Z., Wu, X.: A fully discrete scheme for a diffusion wave system. Appl. Numer. Math. 56(2), 193–209 (2006)
47. Tang, T., Yang, J.: Implicit-explicit scheme for the Allen–Cahn equation preserves the maximum principle. J. Comput. Math. 34(5), 471–481 (2016)
48. Tang, T., Yu, H., Zhou, T.: On energy dissipation theory and numerical stability for time-fractional phase field equations. SIAM J. Sci. Comput. 41(6), A3757–A3778 (2019)
49. Valdinoci, Enrico: A fractional framework for perimeters and phase transitions. Milan J. Math. 81(1), 1–23 (2013)
50. Van der Waals, J.D.: Thermodynamic theory of capillarity assuming steady density change. J. Phys. Chem. 13(1), 657–725 (1894)
51. Wise, S.M., Lowengrub, J.S., Frieboes, H.B., Cristini, V.: Three-dimensional multispecies nonlinear tumor growth I: model and numerical method. J. Theor. Biol. 253(3), 524–543 (2008)
52. Xu, C., Prince, J.L.: Snakes, shapes, and gradient vector flow. IEEE Trans. Image Process. 7(3), 359–369 (1998)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Affiliations

Qiang Du1 · Jiang Yang2,3 · Zhi Zhou4

✉ Jiang Yang
yangj7@sustech.edu.cn

Qiang Du
qd2125@columbia.edu

Zhi Zhou
zhizhou@polyu.edu.hk, zhizhou0125@gmail.com

1 Department of Applied Physics and Applied Mathematics, Columbia University, New York, NY 10027, USA
2 Department of Mathematics, SUSTech International Center for Mathematics, Southern University of Science and Technology, Shenzhen, China
3 Guangdong Provincial Key Laboratory of Computational Science and Material Design, Southern University of Science and Technology, Shenzhen, China
4 Department of Applied Mathematics, The Hong Kong Polytechnic University, Kowloon, Hong Kong