A Study on Using Hierarchical Basis Error Estimates in Anisotropic Mesh Adaptation for the Finite Element Method

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Summary. A common approach for generating an anisotropic mesh is the $M$-uniform mesh approach where an adaptive mesh is generated as a uniform one in the metric specified by a given tensor $M$. A key component is the determination of an appropriate metric which is often based on some type of Hessian recovery. This study discusses the use of a hierarchical basis error estimator for the development of an anisotropic metric tensor needed for the adaptive finite element solution. A global hierarchical basis error estimator is employed to obtain reliable directional information. Numerical results for a selection of different applications show that the method performs comparable with existing metric tensors based on Hessian recovery and can provide even better adaptation to the solution if applied to problems with gradient jumps and steep boundary layers.

Keywords: mesh adaptation, anisotropic mesh, finite element, a posteriori estimate, hierarchical basis, variational problem, anisotropic diffusion

1 Introduction

A common approach for generating an anisotropic mesh is the $M$-uniform mesh approach based on generation of a quasi-uniform mesh in the metric space defined by a symmetric and strictly positive definite metric tensor $M$. A scalar metric tensor will lead to an isotropic mesh while a full metric tensor will generally result in an anisotropic mesh. In this sense, the mesh generation procedure is the same for both isotropic and anisotropic mesh generation. A key component of the approach is the determination of an appropriate metric often based on some type of error estimates.

Typically, the appropriate metric tensor depends on the Hessian of the exact solution of the underlying problem, which is often unavailable in practical computation, thus requiring the recovery of an approximate Hessian from the computed solution. A number of recovery techniques are used for this purpose, for example the gradient recovery technique by Zienkiewicz and Zhu
the technique based on the variational formulation by Dolejší [17], or the quadratic least squares fitting (QLS) proposed by Zhang and Naga [33]. Generally speaking, Hessian recovery methods work well when exact nodal function values are provided (e.g. interpolation problems), but unfortunately they do not provide an accurate recovery when applied to linear finite element approximations on non-uniform meshes, as pointed out by the author in [28]. Recently, conditions for asymptotically exact gradient and convergent Hessian recovery from a hierarchical basis error estimator have been given by Ovall [31]. His result is based on superconvergence results by Bank and Xu [7, 8], which require the mesh to be uniform or almost uniform: assumptions which are usually violated by adaptive meshes.

Hence, a convergence of adaptive algorithms based explicitly on the Hessian recovery cannot be proved in a direct way, even if their application is quite successful in practical computations [17] [26] [29]. This explains the recent interest in anisotropic adaptation strategies based on some type of a posteriori error estimates. For example, Cao et al. [12] studied two a posteriori error estimation strategies for computing scalar monitor functions for use in adaptive mesh movement; Apel et al. [5] investigated a number of a posteriori strategies for computing error gradients used for directional refinement; and Agouzal et al. [1] [2] [3] and Agouzal and Vassilevski [4] proposed a new method for computing metric tensors to minimize the interpolation error provided that an edge-based error estimate is given.

Recently, Huang et al. [24] presented a mesh adaptation method based on hierarchical basis error estimates (HBEE). The new framework is developed for the linear finite element solution of a boundary value problem of a second-order elliptic partial differential equation (PDE), but it is quite general and can easily be adopted to other problems. A key idea in the new approach is the use of the globally defined HBEE for the reliable directional information: globally defined error estimators have the advantage that they contain more directional information of the solution; error estimation based on solving local error problems, despite its success in isotropic mesh adaptation, do not contain enough directional information, which is global in nature; moreover, Dobrowolski et al. [16] have pointed out that local error estimates can be inaccurate on anisotropic meshes.

The objective of this article is to study the application of this new anisotropic adaptation method to different problems. A brief description of the method is provided in Sect. 2. An example of its application to a boundary value problem of a second-order elliptic PDE is given in Sect. 3 for heat conduction in a thermal battery with large and orthotropic jumps in the material coefficients. Section 4 presents an anisotropic metric tensor for general variational problems developed by Huang et al. [25] using the HBEE and the underlying variational formulation and gives a numerical example for a non-quadratic variational problem. The metric tensor is completely a posteriori:

\footnote{A Sandia National Laboratories benchmark problem.}