Higher-order corrections to the short-pulse equation

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Abstract
Using renormalization group techniques, we derive an extended short-pulse equation as an approximation to a nonlinear wave equation. We investigate the new equation numerically and show that the new equation captures efficiently higher-order effects on pulse propagation in cubic nonlinear media. We illustrate our findings using one- and two-soliton solutions of the first-order short-pulse equation as initial conditions in the nonlinear wave equation.

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1. Introduction
The theory of electromagnetic wave propagation in dispersive nonlinear media such as optical fibers is one of the key elements of the information age in the 20th century. Maxwell’s equations present a complete description of the classic (non-quantized) electromagnetic field. In the presence of nonlinear effects, however, a general solution to Maxwell’s equations seems impossible and approximate solutions are commonly used to explain phenomena. These approximate solutions are often constructed using simpler partial differential equations that model the solutions of the original wave equation. This is in particular useful if the original wave equation contains nonlinear terms that make any analytical approach difficult. The reduced models are often simpler or even solvable.

For the particular case of pulse propagation in cubic nonlinear media, the cubic nonlinear Schrödinger equation (NLSE) presents a slowly varying amplitude approximation that has been extremely successful in nonlinear optics [1]. The NLSE belongs to the small class of integrable partial differential equations that can be solved by inverse scattering transform [28]. It admits solitary wave solutions that show stable propagation in Maxwell’s equations as long as one is working in regimes where solutions of the NLSE are close to solutions of Maxwell’s equations. The extraordinary stability properties of optical solitons over long distances have made it possible to use them as bit carriers in fiber-optic communications [10].
The derivation of the NLSE assumes a small parameter which is the ratio of pulse width and period of the carrier wave. This small parameter is used in creating an asymptotic expansion, typically using multi-scale expansions or the renormalization group (RG) method. In the regime of ultra-short pulses, where the pulse lengths shorten, the NLSE approximation becomes less accurate [18, 25]. It is possible to increase the accuracy of the NLSE approximation by incorporating higher-order corrections. Physically, these next-order correction terms model important effects as Raman delay, higher-order dispersion, or self-steepening of the pulse [1].

Advances in the generation of ultra-short pulses allow pulse widths in the range of 10 fs or below. The nonlinear optical pulse compression technique even makes it possible to generate pulses of about 6 fs [11]. Femtosecond laser systems generating short pulses have led to diverse interests, studies, and applications of ultra-short pulses such as extremely high intensity laser–matter interactions, characterization of high-speed electronic and optoelectronic devices and systems, optical communications, medical imaging as well as ultrafast phenomena in solid-state, chemical and biological materials [27].

Recently, the so-called short-pulse equation (SPE) was proposed [23] to describe nonlinear pulse dynamics exploiting a scaling designed to describe the behavior of pulses in the ultra-short limit. Sakovich and Sakovich showed that the SPE is integrable [19], and possesses exact one-solitary wave [20], later multi-solitary wave solutions [15] were found as well. This development led to an intensive research on the SPE over the years. The bi-Hamiltonian structure [2, 3], conserved quantities [9] and the periodic solutions [16] of the SPE were analyzed. Generalizations as the vector SPE [17, 22] and the regularized short-pulse equation (RSPE) [8] were studied. For the RSPE, the existence of multi-pulses [14], traveling waves, and solitary wave solutions [8, 7] were confirmed.

Note that there is a variety of models aside from the SPE that have been developed in order to describe ultra-short pulses in different type of media. A detailed comparison of all models is beyond the scope of the current paper, most models assume certain properties of the susceptibility of the excited media in which the pulse is propagating. For instance, if all resonant frequencies of the material are much larger than the characteristic frequency of the pulse, a modified Korteweg–de Vries equation model can be derived [26]. In the current paper, we extend the analysis of ultra-short pulses in silica in the infrared regime, for which the SPE was originally derived [23].

It is natural to ask whether the solitary wave solutions of SPE show stable propagation in Maxwell’s equations when chosen as initial conditions in the ultra-short-pulse regime, similar to the soliton solutions of NLSE in the broad pulse regime. Numerical studies show that this is indeed the case [13]. Observed over longer distances, however, the solitary wave solutions of SPE show slow distortions when propagating in Maxwell’s equations. This behavior is expected, as the SPE is only an approximation to Maxwell’s equations. Similar to the NLSE, it is therefore worthwhile to consider higher-order contributions in order to obtain a better approximation to Maxwell’s equation.

A particular motivation of the present work is the fact that, for realistic parameters, the parameter $\epsilon$ linking the SPE and Maxwell’s equations is estimated to be small but not very small. A detailed analysis of parameters for silica [23] shows that, for a pulse of three to four cycles, $\epsilon \approx 0.2$. Therefore, contributions coming from higher-order correction terms are important.

The derivation of such higher-order corrections to the SPE is the objective of the present paper. In the next section, following the previous work [6], we use the RG method in order to derive the higher-order short-pulse equation (HSPE). Section 3 presents a detailed numerical study of the new equation, comparing results to the original SPE and to Maxwell’s nonlinear wave equations.
2. Derivation of the higher-order SPE

In this work, we analyze a one-dimensional nonlinear wave equation describing the propagation of pulses in nonlinear cubic media given by

$$u_{xx} = u_t + \chi_0 u + \chi_3 (u^3)_t,$$

(1)

where the $\chi_0$ and $\chi_3$ model the linear and nonlinear response of the medium to the applied field. Equation (1) is derived directly from Maxwell’s equations under certain assumptions, for details we refer to previous work [13, 23]. One main assumption in the derivation is that the Fourier transform of the linear susceptibility $\hat{\chi}^{(1)}(\omega)$ of the dielectric material can be approximated by the form

$$\hat{\chi}^{(1)}(\omega) \approx a - \frac{b}{\omega^2}$$

and that the nonlinear response of the material is assumed to be instantaneous. For silica, for example, (2) can be found by a fit of this functional form to the experimentally obtained linear susceptibility in the infrared range with wavelengths of 1600–3000 nm [23]. There are many extensions of the model possible—for example the inclusion of higher-order dispersive terms in the above expansion. In particular, the case with an additional dispersive term has been studied recently [24].

We assume that we are sufficiently far away from material resonances and also that the pulse spectrum is bounded away from $\omega = 0$. In numerical simulations, we therefore implement the susceptibilities as Fourier multipliers: Let $\hat{u}$ denote the Fourier transform of $u$, then we compute numerically the terms on the rhs of (1) using

$$\hat{u}_{xx} + \omega^2 \left(1 - \frac{\omega_p^2}{\omega^2}\right) \hat{u}$$

we assume that $\omega > \omega_p$ where $\omega_p = \sqrt{\chi_0}$ is the plasma frequency.

We are interested in asymptotic solutions of the form

$$u(x,t) = \epsilon A_0(\phi, x_1, x_2, \ldots) + \epsilon^2 A_1 + \cdots, \quad \phi = \frac{t-x}{\epsilon}, \quad x_n = \epsilon^nx$$

(4)

and it has been shown [23] that, at the leading order, the evolution equation for $A_0$ is given by

$$-2\partial_\phi \partial_t A_0 = \chi_0 A_0 + \chi_3 \partial_\phi^3 A_0^3.$$  

(5)

Note that the scaling used in (4) is different from the scaling leading to the cubic NLSE that can be derived from (1) using a slowly varying amplitude approximation. For details we refer to [6] in which both approximations are compared.

We also note that equation (5) can be written after a simple transform [21] as,

$$U_{XY} = U + \frac{1}{6}U_{XX}^3.$$  

(6)

In the following, we compute higher-order corrections of this expansion. Although it is possible to do this using a multi-scale expansion [13], we prefer to present the derivation using the RG method. The RG method, which was developed as a tool for asymptotic analysis [4, 5], has already been successfully applied to the nonlinear wave equation (1) to derive the SPE (5) [6] at the leading order. This method has two main advantages over the method of multiple scales: (a) algebraic calculations in the case of the RG treatment are simpler than the ones appearing
in the multiple-scale expansions especially if higher-order corrections are considered, and (b) the introduction of a particular multi-scale-ansatz is not required a priori for the reason that the RG equation naturally gives rise to the scales that need to be introduced for a consistent asymptotic expansion.

We first reproduce quickly the SPE and then proceed to the next order. We apply a coordinate transformation in the form of

\[ u(x,t) = B(\phi, x) \]  

(7)

where \( \phi \) and \( x \) are given in (4). Substituting (7) in (1), we obtain

\[ \frac{2}{\epsilon} B_{\phi x} = \chi_0 B - B_{xx} + \frac{1}{\epsilon^2} \chi_3 (B^3)_{\phi\phi}. \]

(8)

Note that the introduction of the coordinate \( \phi \) corresponds to uni-directional propagation of the pulse. At this point, however, this transformation is exact and, hence, takes into account also counter-propagating waves.

We assume the expansion

\[ B(\phi, x) = \epsilon \Lambda_0(\phi, x) + \epsilon^2 \Lambda_1(\phi, x) + \cdots. \]

(9)

Substituting the expansion (9) into (8), the terms of \( O(1) \) yield

\[ -2\Lambda_0_{\phi x} = 0. \]

(10)

This implies \( \Lambda_0 \) is independent of \( x \), i.e., \( \Lambda_0 = \Lambda_0(\phi) \). On the other hand, the terms of \( O(\epsilon) \) yield

\[ -2\Lambda_1_{\phi x} = \chi_0 \Lambda_0 + \chi_3 (\Lambda_0(\phi)^3)_{\phi\phi}. \]

(11)

Therefore, we find the second-order approximate solution

\[ B^{(2)}(\phi, x) = \epsilon \Lambda_0(\phi, x) + \epsilon^2 \Lambda_1(\phi, x) + \cdots. \]

(12)

Note that a secular term appears on the equation (12) which corresponds to the term proportional to \( x \). In order to remove this secular term, we consider the term \( \Lambda_0(\phi) - \frac{\epsilon^2}{2} x (\int_{-\infty}^{\phi} \chi_0 \Lambda_0(\phi) d\phi + \chi_3 (\Lambda_0(\phi)^3)_{\phi\phi}) \) as the Taylor expansion of a function \( V_0(\phi, x) \) about \( x = 0 \). Thus, we need to find \( V_0(\phi, x) \) which satisfies

\[ V_0(\phi, x = 0) = \Lambda_0, \]

(13)

\[ \frac{\partial V_0}{\partial x} = -\frac{\epsilon}{2} \left( \int_{-\infty}^{\phi} \chi_0 V_0(\phi) d\phi + \chi_3 (V_0(\phi)^3)_{\phi\phi} \right). \]

(14)

This equation introduces a new scale \( \epsilon x \). Let us define \( x_1 = \epsilon x \) as in equation (4) then we obtain

\[ V_0(\phi, x_1 = 0) = \Lambda_0, \]

(15)

\[ \frac{\partial V_0}{\partial x_1} = -\frac{1}{2} \left( \int_{-\infty}^{\phi} \chi_0 V_0(\phi) d\phi + \chi_3 (V_0(\phi)^3)_{\phi\phi} \right). \]

(16)

Solving equation (16) provided that the initial condition (15) is satisfied, we can express the rhs of equation (12) as the Taylor expansion of \( V_0 \). Hence, we finally obtain the second-order approximate solution, \( B^{(2)}(\phi, x) = \epsilon V_0 \). Furthermore, equation (16) yields

\[ (V_0)_{\phi x_1} = -\frac{1}{2} \chi_0 V_0 - \frac{1}{2} \chi_3 (V_0^3)_{\phi\phi}. \]

(17)

This is the SPE.
For higher-order corrections, we follow similar steps. First, we need to collect higher-order terms in $\epsilon$. The usual way of obtaining these is by assuming the ansatz $B = \epsilon \Lambda_0 + \epsilon^2 \Lambda_1 + \epsilon^3 \Lambda_2 + \cdots$ and collecting appropriated terms. However, this will lead to highly complicated algebraic calculations. Here, we approach this problem by assuming a different ansatz. Since we have already obtained the second-order approximation of the solution, $B^{(2)}$, we assume

$$B(\phi, x) = B^{(2)} + \epsilon^{k+1} \Lambda_k(\phi, x).$$

Plugging the equation (18) into equation (8), we find that the next higher-order nontrivial contribution comes from the term $-B^{(k)}$ in (8), corresponding to $-\epsilon^3 (V_0)^{3}_{x_1 x_1}$. Hence we need to choose $k = 3$. Then, $O(\epsilon^5)$ terms yield

$$-2(\Lambda_3)_{xx} = -(V_0)_{x_1 x_1}. \quad (19)$$

Before we proceed with (19), let us define $N(V_0) = -(V_0)^{3}_{x_1 x_1}$ and rewrite this term. From the SPE (17), it follows that

$$-(V_0)^{3}_{x_1 x_1} = \frac{1}{2} \chi_0 \int_{-\infty}^\phi V_0(\phi) \, d\phi + \frac{1}{2} \chi_3 V_0(\phi)^3. \quad (20)$$

Therefore, using the second derivative of $V_0$,

$$-(V_0)_{x_1 x_1} = \frac{1}{2} \chi_0 \int_{-\infty}^\phi (V_0)_{x_1 x_1}(\phi) \, d\phi + \frac{1}{2} \chi_3 V_0(\phi)^3_{x_1}, \quad (21)$$

and the SPE (17) along with some algebraic manipulation, we obtain $N(V_0)$,

$$N(V_0) = -\frac{\chi_0^2}{4} \int_{-\infty}^\phi \int_{-\infty}^\phi V_0(\phi, \phi') \, d\phi \, d\phi' - \chi_0 \chi_3 V_0^3 - \frac{3}{2} \chi_0 \chi_3 \int_{-\infty}^\phi V_0(\phi) \, d\phi$$

$$- \frac{3}{4} \chi_3^2 V_0^2 V_0^3_{x_1} - \frac{3}{4} \chi_3^2 V_0^3_{x_1}. \quad (22)$$

Clearly, $N(V_0)$ involves higher-order linear and nonlinear terms, a more detailed discussion will follow in the next section. Coming back to (19), we obtain the fourth-order approximate solution

$$B^{(4)}(\phi, x) = \epsilon \left( V_0(\phi, x_1) - \frac{\epsilon^3}{2} \int_{-\infty}^\phi N(V_0) \, d\phi' \right). \quad (23)$$

In order to combine all the possible secular terms, we rewrite equation (23) using equations (13), (14) and the Taylor expansion of $V_0$. This yields

$$B^{(4)}(\phi, x) = \epsilon \left( \Lambda_0(\phi) - \frac{\epsilon}{2} \left( \int_{-\infty}^\phi \chi_0 A(\phi) \, d\phi + \chi_3 \Lambda_0(\phi)^3_{\phi} \right) \right)$$

$$- \frac{\epsilon^3}{2} \int_{-\infty}^\phi N(\Lambda_0) \, d\phi' + O(\epsilon^5). \quad (24)$$

Again, to remove the secular terms which are proportional to $x$, we now need to find $A(\phi, x)$ satisfying

$$A(\phi, x = 0) = \Lambda_0(\phi), \quad (25)$$

$$\frac{\partial A}{\partial x} = -\frac{\epsilon}{2} \left( \int_{-\infty}^\phi \chi_0 A(\phi) \, d\phi + \chi_3 \Lambda_0(\phi)^3_{\phi} \right) - \frac{\epsilon^3}{2} \int_{-\infty}^\phi N(A) \, d\phi'. \quad (26)$$

The term that is proportional to $\epsilon^3$ leads us to introduce a new scale $x_3 = \epsilon^3 x$ in addition to $x_1 = \epsilon x$. From equation (26), we also find

$$\frac{\partial A}{\partial x_3} = -\frac{1}{2} \left( \int_{-\infty}^\phi \chi_0 A(\phi) \, d\phi + \chi_3 \Lambda_0(\phi)^3_{\phi} \right) - \frac{\epsilon^2}{2} \int_{-\infty}^\phi N(A) \, d\phi'. \quad (27)$$
Denoting \( \kappa_1 = x_1 \), we finally obtain
\[
-2A_{x\phi} = \chi_0 A + \chi_3 (A^3)_{\phi\phi} + \epsilon^2 N(A).
\]
(28)
This is the HSPE, and \( A(\kappa, \phi) \) is the magnitude of the electric field following the introduction of the new variable \( \kappa \). It is obvious that this equation is an extension of the SPE (5) with additional terms on the right-hand side. The additional operator \( N(A) \), represents the higher-order corrections of the SPE approximation.

3. Numerical analysis

In order to show that the HSPE captures the effects beyond the SPE approximation, we compare the solutions of the original nonlinear wave equation (1) to the solutions of the SPE and the HSPE. As initial conditions, we use one- and two-soliton solutions of the SPE. We first summarize very briefly the structure of these solutions and then discuss their evolutions in the HSPE given by (28) in comparison to the nonlinear wave equation (1). Both equations are numerically solved using standard techniques. For the nonlinear wave equation we use a pseudo-spectral leap-frog scheme in Fourier space, the SPE and the HSPE are solved using a fourth-order exponential time-differencing method [12] in Fourier space. For the solution of the nonlinear wave equation it is essential to implement the cut-off in the susceptibility operators as in (3). For the HSPE, this can be done in a similar fashion. In the examples considered here, we have found only a very little difference when neglecting the cut-off and treating the operators simply as scalars. However, for consistency, the cut-off was implemented in all simulations.

Before we proceed, let us also note that, for the numerical simulation, it is useful to cast the HSPE in a slightly different form,
\[
A_\kappa = -\frac{\chi_0}{2} \int_{-\infty}^{\phi} A(\kappa, \hat{\phi}) d\hat{\phi} - \frac{\chi_2}{2} (A^3)_{\phi\phi} - \frac{\epsilon^2}{2} \int_{-\infty}^{\phi} N(A(\kappa, \hat{\phi})) d\hat{\phi},
\]
(29)
where we write the integral of the operator \( N(A) \) involving the higher-order correction terms as
\[
-\int_{-\infty}^{\phi} N(A(\kappa, \hat{\phi})) d\hat{\phi} = \frac{\chi_0^2}{4} \int_{-\infty}^{\phi} \int_{-\infty}^{\phi_1} \int_{-\infty}^{\phi_2} \int_{-\infty}^{\phi_3} A(\kappa, \phi_3) + \frac{\chi_0 \chi_3}{4} \int_{-\infty}^{\phi} \int_{-\infty}^{\phi_1} A(\kappa, \phi_1) A(\kappa, \phi_2) A(\kappa, \phi_3) + \frac{3 \chi_2^2}{4} \int_{-\infty}^{\phi} \int_{-\infty}^{\phi_1} A(\kappa, \phi_1)^2 (A(\kappa, \phi_2)^3)_{\phi}. (30)
\]
Here, we define
\[
L_1(A) = \frac{\chi_0^2}{4} \int_{-\infty}^{\phi} \int_{-\infty}^{\phi_1} \int_{-\infty}^{\phi_2} A(\kappa, \phi_1),
\]
(31)
\[
N_1(A) = \frac{\chi_0 \chi_3}{4} \int_{-\infty}^{\phi} \int_{-\infty}^{\phi_1} A(\kappa, \phi_1)^2,
\]
(32)
\[
N_2(A) = \frac{3 \chi_0 \chi_3}{4} A(\kappa, \phi_1)^2 \int_{-\infty}^{\phi} \int_{-\infty}^{\phi_1} A(\kappa, \phi_2),
\]
(33)
\[
N_3(A) = \frac{3 \chi_2^2}{4} A(\kappa, \phi_1)^2 (A(\kappa, \phi_2)^3)_{\phi}.
\]
(34)
In the next subsection, we will compare the linear term \( L_1 \) to the nonlinear terms \( N_1, N_2, N_3 \). Also note that numerical evaluation of these terms can be done by using the fast Fourier transform and that the singularity at \( \omega = 0 \) is removed by introducing the cut-off.
3.1 Propagation of one-pulse solutions

The analytical one-soliton solution of the SPE is found \[20\]

\[
U = \frac{4mn \sin \psi \sinh \theta + n \cos \psi \cosh \theta}{m^2 \sin^2 \psi + n^2 \cosh^2 \theta}
\]

\[35\]

\[
X = Y + 2mn \frac{m \sin 2\psi - n \sinh 2\theta}{m^2 \sin^2 \psi + n^2 \cosh^2 \theta}
\]

\[36\]

with

\[
\theta = m(Y + T), \quad \psi = n(Y - T), \quad n = \sqrt{1 - m^2}.
\]

\[37\]

Here, we only consider non-singular solutions, which implies that the soliton parameter \(m\) satisfies the condition \(0 < m < \sin \frac{\pi}{8} \approx 0.383 \ [20]\).

In our numerical simulations, we set \(m = 0.35\) in order to have strong nonlinear effects, but no singularities. We also choose \(\epsilon = 0.4\) since we are mainly interested in higher-order effects. Therefore, we expect that the solutions of the nonlinear wave equation and the SPE will show differences already after fairly short propagation distances.

We set a propagation distance \(x = 51.2\), and susceptibilities \(\chi_0 = 2\), \(\chi_3 = 1/3\). Using an SPE soliton (35) as the initial condition, we numerically integrate the SPE (6), the HSPE (28), and the full nonlinear wave equation (1). Figure 1 demonstrates the numerical solutions at the distance \(x = 51.2\) of the SPE, the HSPE, and the nonlinear wave equation. Clearly, if an SPE soliton (35) is taken as the initial condition, the HSPE provides a much better approximation to the nonlinear wave equation than the SPE.

In order to numerically analyze the scaling of the error, we performed numerical simulations over a range of values of \(\epsilon\) and considered the renormalized \(L^2\)-norm

\[
\|\Delta u\| = \frac{1}{\epsilon} \|\Delta u\|_{L^2},
\]

where \(\Delta u\) is the difference between the SPE (or HSPE) solution and the solution of Maxwell’s equations. From the above considerations, we would estimate that the error between the SPE
and Maxwell’s equations grows roughly with a third power of $\epsilon$, whereas the error between the HSPE and Maxwell’s equations should show quintic scaling. Figure 2 shows the scaling of the error for both approximations on a log–log plot. The slopes of the regression lines indicate that the error scales even slightly better than expected, at least for the range of parameters under consideration.

We now consider the nonlinear operator (22) that presents the higher-order contributions. Although, we cannot make a general statement, our numerical experiments show that for large $m$-values close to singularity, surprisingly, the linear term seems to be dominant.

Figure 3 shows all four terms, $L_1, N_1, N_2$, and $N_3$ for the Sakovich soliton with $m = 0.35$ and, clearly, the nonlocal dispersive term is much stronger than the nonlinear terms. This numerical observation leads to a truncated HSPE, that only involves the linear term and, therefore, is much easier to compute and to analyze:

$$ -2 \alpha \phi = \chi_0 A + \chi_2 (A') \phi - \epsilon^2 \chi_2^2 \frac{\chi_0^2}{4} \int_{-\infty}^{\phi} d\phi' \int_{-\infty}^{\phi'} A(\phi) \, d\phi'. $$

(38)

Figure 4 shows the numerical solutions of the HSPE and the truncated HSPE in comparison to the numerical solution of the SPE, again with a Sakovich soliton being the initial condition. This figure shows that the difference in the performance of truncated HSPE and HSPE is very small and the truncated HSPE is still a better approximation to the full nonlinear wave equation than the SPE. This implies that, for practical purposes in some cases, the truncated HSPE might provide sufficient additional accuracy.

3.2 Multi-soliton propagation

Multi-soliton solutions of the SPE have been derived [15] through a systematic procedure from breather solutions of the sine-Gordon equation. The parametric multi-soliton solution can be expressed in the compact form as

$$ U(X, T) = 2i \left( \ln \frac{f'}{f} \right)_T, \quad X(Y, T) = Y - 2(\ln f')_T + d $$

(39)
Figure 3. Comparison of the four correction terms from equation (30), clockwise starting at the upper left figure: $L_1$, $N_1$, $N_2$, and $N_3$, for a Sakovich soliton as initial condition at $x = 0$.

Figure 4. Snapshot of the evolution of the Sakovich soliton in the HSPE (solid line), the truncated HSPE (dashed line) and the SPE (dotted line).
with

\[
f = \sum_{\mu=0,1} \exp \left[ \sum_{j=1}^{N} \mu_j \left( \chi_j + \frac{\pi}{2} \right) + \sum_{1 \leq j < k \leq N} \mu_j \mu_k Y_{jk} \right]
\]

\[
f' = \sum_{\mu=0,1} \exp \left[ \sum_{j=1}^{N} \mu_j \left( \chi_j - \frac{\pi}{2} \right) + \sum_{1 \leq j < k \leq N} \mu_j \mu_k Y_{jk} \right]
\]

\[
\xi_j = p_j T + \frac{1}{p_j} (j = 1, 2, \ldots, M)
\]

\[
e^{\gamma_p} = \left( \frac{p_j - p_k}{p_j + p_k} \right)^2, \quad (j, k = 1, 2, \ldots, M; j \neq k)
\]

\[
p_{2j-1} = p^*_{2j-1} \equiv a_j + \sqrt{2} b_j, \quad a_j > 0, \quad b_j > 0, \quad (j = 1, 2, \ldots, M)
\]

\[
\xi_{2j-1,0} = \xi^*_{2j-1,0} \equiv \lambda_j + \sqrt{2} i \mu_j, \quad (j = 1, 2, \ldots, M)
\]

\[
\theta_j = a_j (Y + c_j T) + \lambda_j, \quad (j = 1, 2, \ldots, M)
\]

\[
\chi_j = b_j (Y - c_j T) + \mu_j, \quad (j = 1, 2, \ldots, M)
\]

\[
c_j = \frac{1}{a_j^2 + b_j^2}, \quad (j = 1, 2, \ldots, M), \quad (40)
\]

where \(p_j\) and \(\xi_{j0}\) are arbitrary parameters such that \(p_j \neq \pm p_k\) for \(j \neq k\), \(i = \sqrt{-1}\), \(N\) is an arbitrary positive integer, and \(M = N/2\) is the number representing the multi-soliton solutions (one soliton, two solitons, etc). If \(N = 4\) and \(M = 2\) are chosen, one can generate the two-soliton solution. The condition for a single-valued nonsingular multi-breather solution is

\[
0 < \sum_{j=1}^{M} \frac{a_j}{b_j} < \sqrt{2} - 1. \quad (41)
\]

In order to investigate collisions of SPE-solitons in the HSPE and the nonlinear wave equation, we use a two-soliton solution as the initial condition. The parameters are chosen \(a_1 = 0.1, b_1 = 0.5, a_2 = 0.16, b_2 = 0.8, d = 0, \lambda_1 = 10, \lambda_2 = 0, \mu_1 = 0, \mu_2 = 0\). Here, the propagation distance needs to be much longer in order to capture the entire collision. Therefore, we choose \(\epsilon = 0.2\) and a propagation distance \(x = 375\). At such large distances, higher-order terms start to influence the propagation as it can be seen in figure 5. Clearly, the HSPE approximates the solution of the nonlinear wave equation much better than the SPE. However, it seems to be necessary to incorporate even higher-order terms in the approximation derived in section 2. Extending the results of section 2, it is easy to see that the next linear term in the expansion is given by an integral operator involving four integrations of the form

\[
\epsilon^4 M(A) = \epsilon^4 \frac{X_0^3}{8} \int_{-\infty}^{\phi_1} \int_{-\infty}^{\phi_2} \int_{-\infty}^{\phi_3} \int_{-\infty}^{\phi_4} \frac{d\phi_1}{\phi_1} \frac{d\phi_2}{\phi_2} \frac{d\phi_3}{\phi_3} \frac{d\phi_4}{\phi_4} A(\alpha, \phi_4) \quad (42)
\]

yielding the equation

\[
-2A_{x\phi} = \chi_0 A + \chi_3 (A^3)_{\phi_4} + \epsilon^2 N(A) + \epsilon^4 M(A). \quad (43)
\]

In a heuristic way, this additional term can be understood in the following way: consider again equation (8), we see that higher-order contributions will arise from \(-B_{xx}\) in (8),
in particular a term \(-2A_{x_3}\phi\) which will be renormalized by introducing a scale \(x_5\). From equation (28), we find that \(2A_{x_3}\phi = A_{x_3}\phi\). Therefore, keeping only linear terms, we obtain

\[
A_{x_3} \approx \frac{\lambda_0^2}{8} \int_{-\infty}^{\phi} d\phi_1 \int_{-\infty}^{\phi_1} d\phi_2 \int_{-\infty}^{\phi_2} d\phi_3 A
\]

and hence,

\[
A_{x_3,\phi_1} \approx -\frac{\lambda_0^3}{16} \int_{-\infty}^{\phi} d\phi_1 \int_{-\infty}^{\phi_1} d\phi_2 \int_{-\infty}^{\phi_2} d\phi_3 \int_{-\infty}^{\phi_4} d\phi_4 A
\]

yielding the additional linear term in the HSPE (43).
Indeed, figure 6 shows that there is a considerable improvement for the evolution of two-soliton solution when incorporating this additional linear term on the rhs of equation (43). Without showing the corresponding figure, we also note that, as expected, there is improvement for the approximation of the one-soliton solution.

4. Conclusion

Using the renormalization group method, we derived higher-order correction terms to the short-pulse equation (SPE). We showed numerically that the incorporation of these terms can considerably improve the accuracy of the SPE approximation when describing solutions to Maxwell’s equations. This was shown using one- and two-soliton SPE solutions as initial conditions. Even the incorporation of one additional linear term in the SPE approximation can lead to considerable improvement, and the extension to higher-order terms is straightforward. A precise study of the effect of the higher-order terms on the solitary waves is subject to future studies.

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