Formation of a Satellite Navigation System Using X-Ray Pulsars

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Abstract

The ephemeris of a pulsar is stable for a long time, which allows navigation based on pulsar orientation to be vector feasible. The formation of a satellite navigation model using the orientation vector of an X-ray pulsar signal is presented in this paper. To obtain the time difference of arrival (TDOA), a new estimation method is constructed, which can measure the photon sequence of an X-ray pulsar signal and is based on the fast Fourier transform (FFT). Next, three new observation variables are constructed. The variables are satellite phase incrementation; the angle between the satellite baseline and the pulsar direction vector; and the angle between the plane spanned by three satellite baselines and the pulsar direction vector. All three variables, along with the TDOA of the X-ray pulsar signal, are utilized to determine the orbit. The position and velocity of the satellite formations are estimated by using the adaptive divided difference filter (ADDF) to eliminate nonlinearity. Several simulation cases are designed to verify the proposed method.

Key words: X-ray pulsar – autonomous navigation – adaptive divided difference filter – Satellite baseline

Online material: color figures

1. Introduction

Throughout the solar system, pulsars, which are a type of high-speed spinning neutron star, are useful in acquiring a spacecraft’s altitude, location, and time (Guo et al. 2018; Sheikh & Pines 2006; Xiaolin et al. 2018). Navigation using pulsars, particularly X-ray pulsar navigation (XNAV; Li et al. 2018; Sheikh 2005), has become a popular autonomous navigation method. However, most spacecraft still rely, directly or indirectly, on a ground station to obtain their position parameters, which increases the ground station’s burden. Thus, autonomous navigation using pulsars is becoming attractive (Anderson & Pines 2013; Emadzadeh & Speyer 2011; Liang et al. 2016).

Pulsar signals have a stable period, and their time of arrival (TOA) in the solar system barycentric (SSB) reference frame can be predicted accurately by a phase prediction model (Sheikh 2005; Wang et al. 2017). Thus, when a detector captures a signal, the recorded phase is translated to the SSB reference frame to be compared with the results of the phase-prediction model (Ashby & Howe 2006). In general, the phase deviation is proportional to the position error of the detector. If several pulsar phase deviations are observed, the position error can be calculated using time transformation functions. This procedure reflects the fundamental principle of the traditional XNAV. However, this method is subject to a weak signal, a phase model error and the relativity effect, which results in low precision of the traditional XNAV.

Several methods have been proposed to ameliorate these deficiencies. Some focus on the improvement in the accuracy of the weak signal phase comparison of pulsars (Zhang & Xu 2011). Nonlinear filter technologies have been proposed to eliminate estimation noise (Rong et al. 2016). System errors arising from relativity are addressed in several papers (Ashby & Howe 2006). However, these methods are simply variations of the traditional XNAV, and no additional information has been added to the navigation system. In some situations, such as in relative navigation, the time difference of arrival (TDOA) is adopted as a new variable between two spacecraft, and an X-ray pulsar can be used to solve for this variable. Emadzadeh provides an X-ray pulsar TDOA measurement method for relative navigation, and Liliu builds a system measurement model by combining relative pulsar-based and inter-satellite range measurements (Emadzadeh & Speyer 2011; Feng et al. 2013). XiongKai utilizes the projection of the inter-satellite pseudo-range between two spacecraft onto the direction of the pulsar as additional information to improve navigation accuracy (Xiong et al. 2009). These methods focus on the TDOA measurement between two satellites. However, these methods can be applied to more common situations. Therefore, in this paper, a method for satellite formation navigation is
proposed that fully utilizes the space vector of formation for flying satellites to obtain autonomous navigation.

Because the TDOA is one of the main measurements in the proposed method, a fast-phase comparison of the pulsar signal, which will be used to acquire the TDOA of the pulsar signals and is based on FFT, is proposed. Next, three new variables are introduced: the projection of the phase increment onto the direction of the pulsar radiation that results from the satellite motion; the angle between the inter-satellite baseline and the pulsar direction; and the angle between the plane spanned by three satellite baselines. Then, these measurements will be integrated into the ADDF to enhance the filter performance. The ADDF handles the nonlinear filter problem well, and it is found that numerical simulation has its advantages (Dey et al. 2015). The advantages of the proposed method can be summarized as follows: (1) the phase increment is used to reduce the time transform error of the spacecraft at the SSB; (2) the TDOA between satellites is used to derive the angle between the satellite baseline and the pulsar position vector, which can be used as an additional measurement; and (3) the angle between the plane and the pulsar position vector is used to improve navigation performance, which is expected to be more stable than pulsar phase evolution model and can provide high performance in navigation. In addition, compared with traditional celestial navigation, the main difference is the X-ray pulsar can be used to determine the inter-satellite baseline, so, it is perfect for formation of a satellite navigation system.

This paper is organized as follows: the basic principles of XNAV are presented in Section 2; the fast-phase comparison of the signal and the navigation model using one star, two stars and three stars is introduced in Section 3; the filtering algorithm of formation flying satellites is discussed in Section 4; the analysis and filter results are verified in Section 5; and concluding remarks are presented in Section 6.

2. The Basic Principle Of XNAV

As shown in Figure 1., \( O_{SSB} \) is chosen as the origin of the coordinate frame. The \( X_{SSB} \) axis points to the vernal point standard, defined by J2000.0. The direction of \( Y_{SSB} \) is determined according to the right-hand rule, and the solar system barycenter coordinates \( O_{SSB} - X_{SSB} Y_{SSB} Z_{SSB} \) are set up in the celestial equator plane. \( O_E \) and \( O_S \) denote the mass center of Earth and the Sun, respectively. PSR and SC denote the X-ray pulsar and the spacecraft, respectively. \( R_{SC}, R_E \) and \( R_{SC_E} \) represent the position vectors of the spacecraft, Earth, and the spacecraft relative to Earth in the solar system barycenter coordinate system, respectively. \( n \) denotes the direction of the pulsar radiation signal. \( \alpha \) is the right ascension, and \( \lambda \) is the declination. Thus, \( n \) is defined by

\[
 n = [\cos \alpha \cos \lambda \cos \alpha \sin \lambda \sin \alpha]^T.
\]

Because the distance between the solar system and the pulsar is tens of thousands of light years away, \( n \) is considered a constant. Figure 1. illustrates that if the relativity effect is ignored, the basic equation of XNAV is

\[
t_{SSB} - t_{SC} = n \cdot R_{SC} + \delta_{SC},
\]

where \( t_{SC} \) is the TOA of the pulsar signal recorded by an atomic clock on the spacecraft. \( t_{SSB} \) is the TOA of the same pulsar signal in the SSB coordinate system, which can be accurately forecast by the pulsar timing model (Chen et al. 2017)

\[
\phi(t_{SSB}) = \phi(t_0) + f(t_{SSB}) \phi(t_{SSB}).
\]

In X-ray pulsar navigation, \( t_{SC} \) is the proper time in the local frame with the spacecraft as the origin. \( t_{SSB} \) is predicted using the pulsar timing model established in the SSB reference frame with the timescale of barycentric coordinate time (TCB). Thus, it is necessary to transform the proper time into the coordinate time of TCB. Only in this situation can one compare the pulse arrival time measured by detectors with the pulse arrival time predicted by the timing model. Ignoring the gravity of the planets in the solar system, except for the Sun and the higher order terms, the TOA transfer equation from the spacecraft to the SSB can be given as (Anderson & Pines 2013; Ma et al. 2017)

\[
t_{SSB} - t_{SC} = \frac{n \cdot r_{SC}}{c} + \frac{1}{2cD_0}[(n \cdot r_{SC})^2 - ||r_{SC}||^2] + 2(n \cdot b)(n \cdot r_{SC}) - 2(b \cdot r_{SC}) + \frac{2\mu_S}{c^3} \ln \left| \frac{n \cdot r_{SC} + n \cdot b + ||r_{SC}|| + ||b||}{n \cdot b + ||b||} \right|,
\]

Figure 1. Geometric relationships among pulsars, Earth, the Sun, and a spacecraft in SSB.

(A color version of this figure is available in the online journal.)
where \( D_0 \) is the distance between the first pulsar and the SSB, \( r_{5C} \) is the spacecraft position relative to the SSB, and \( b \) is the position of the mass center of the Sun relative to the SSB. In the traditional XNAV method, the difference TOA between the observed pulse and the predicted pulse along the three different pulsar directions is computed. However, the traditional methods lack the space vector, i.e., the angle between the inter-satellite baseline and the pulsar direction; therefore, traditional XNAV does not possess good accuracy.

### 3. Proposed Navigation Model

Three navigation models are presented in this section. The first model uses the phase increment at different times to increase the navigation performance. The second model utilizes the angle between the baseline and the pulsar radiation vector. The third method enhances the satellite formation navigation performance. These three methods measure the phase difference between two X-ray pulsar signal sequences from different X-ray detectors. Thus, the X-ray pulsar signal model and a proposed high efficient phase comparison method are introduced first in this section.

#### 3.1. Phase Estimation Method

The use of the X-ray Pulsar signal has appeared in several papers (Anderson et al. 2015; Zhang et al. 2014b), and in them, the arrival photon sequence of the X-ray pulsar at the SSB is measured by a cyclostationary nonhomogeneous Poisson process (CNHPP). The exact phase evolution at any instant in the reference frame of the SSB can be represented by (1) the previous estimation of the phase evolution at a reference time \( t_0 \); (2) an estimation of the pulse frequency \( f \); and (3) the multiple derivatives \( f^{(m)} \) of the pulse frequency, which appears in Equation (3). Generally, \( f^{(m)} \) of \( m \geq 2 \) are very small (approximately \( 10^{-12} - 10^{-10} \)). Thus, the pulsar signal is assumed to be periodic over the course of several hours or days. The period is defined to be \( 1/f \).

If the X-ray pulsar photon sequence is observed over the time interval \( \Delta t \), and the number of photons detected in the interval \( \Delta t \) is defined by \( N_{\Delta t} = X(t + \Delta t) - X(t) \), then the number of photons in the time interval can be modeled by the Poisson distribution.

Two other models of the X-ray pulsar signal that use the Poisson distribution are presented in (Emadzadeh & Speyer 2010) and (Zhang et al. 2014b). Based on these models, the existing phase comparison methods using the X-ray pulsar can be divided into two categories. They are the epoch folding methods and the direct methods, which measure the initial phase of the photon sequence using the maximum-likelihood estimation (Emadzadeh & Speyer 2010; Taylor 1992; Zhang et al. 2014a). In this paper, a weighted direct phase estimation algorithm for the photon sequence in the frequency domain is presented.

Let \( (t_0, T_{\text{obs}}) \) denote the observation time interval, where \( t_0 \) is set to 0 for simplicity. Let \( t_i \) represent the TOA of the \( i \)-th photon. Let \( n \) denote the number of detected photons, and let \( \{t_i\}_{i=1}^n \) denote a random sequence in increasing order \( \{t_1, t_2, t_3, \ldots, t_n\} \), such that

\[
 t_0 < t_1 < t_2 < t_3, \ldots, < t_n \leq T_{\text{obs}}
\]  

To use the Fourier transform, the photon sequence is measured in the time interval \( \Delta t \). The sequence is

\[
s(t) = \sum_{j=0}^{N} c(t, \Delta t) \delta(t + j\Delta t),
\]

where \( c(t, \Delta t) \) represents the number of photons in the interval whose start time is \( t \), and \( \delta t \) is the Dirac function. The Fourier transform of \( s(t) \) is

\[
 S_1(\omega) = \int_{-\infty}^{T_{\text{obs}}} s(t) e^{-j\omega t} dt.
\]

Let \( s(t - \tau) + n_2(t) \) denote any other sequence, where \( \tau \) is the time delay with respect to the original \( s(t) \). The Fourier transform can be written as

\[
 S_2(\omega) = \int_{-\infty}^{T_{\text{obs}}+\tau} (s(t - \tau) + n_2(t)) e^{-j\omega t} dt.
\]

There is

\[
 S_2(\omega) = S_1(\omega) e^{-j\omega \tau} - \int_{-\infty}^{T_{\text{obs}}} n(t) e^{-j\omega t} dt,
\]

where \( \int_{-\infty}^{T_{\text{obs}}} n(t) e^{-j\omega t} dt = \int_{-\infty}^{T_{\text{obs}}} n_2(t - \tau) e^{-j\omega (t - \tau)} dt - \int_{-\infty}^{T_{\text{obs}}} n_1(t) e^{-j\omega t} dt \) because we assume \( \int_{-\infty}^{T_{\text{obs}}} n(t) e^{-j\omega t} dt \) that the noise \( n(t) \) is a stationary distribution. In these equations, \( \tau \) denotes the time delay between the two sequences. \( S_1(\omega) \) and \( S_2(\omega) \) are calculated via the FFT. The flux density of the background noise is constant, and is calculated via the Poisson distribution. We assume that \( n(t) \) is white noise; therefore \( \int_{-\infty}^{T_{\text{obs}}} n(t) e^{-j\omega t} dt \) is constant and the time delay \( \tau \) is proportional to the frequency \( \omega \). A weighted average method to solve for \( \tau \) is proposed, and the details of this method can be found in (Jiao et al. 2016).

#### 3.2. Single Satellite Model

More than three pulsars are needed to calculate the three-dimensional position of a spaceship. XNAV does not have the precision to perform this calculation due to its phase evolution error, time transformation error, etc. Therefore, the phase increment between spacecraft that is projected onto the direction of the pulsar at different times will be considered in the observation. Thus, the phase increment alleviates common errors as phase evolution errors and high precision is achieved. As shown in Figure 2, the phase increment at times \( t_1 \) and \( t_2 \) of
the satellite’s orbit is

\[ \Delta \phi = \phi_{t_2} - \phi_{t_1}, \]

where \( \phi_{t_1} \) and \( \phi_{t_2} \) denote the pulsar signal phase at times \( t_1 \) and \( t_2 \), respectively. Let \( \mathbf{r}_{ssb0} \) and \( \mathbf{r}_{ssb1} \) represent the position vectors of the satellite at \( t_1 \) and \( t_2 \) in the SSB reference frame, respectively, and let \( \| \mathbf{r}_{ssb0} \| \) and \( \| \mathbf{r}_{ssb1} \| \) denote the modules of \( \mathbf{r}_{ssb0} \) and \( \mathbf{r}_{ssb1} \). According to Equation (4), there is

\[ \Delta \phi = \frac{n \cdot \delta \mathbf{r}}{c} - \frac{1}{2cD_0}[(n \cdot (\mathbf{r}_{ssb0} + \mathbf{r}_{ssb1}))(n \cdot \delta \mathbf{r}) - 2(n \cdot \delta \mathbf{r})(n \cdot \delta \mathbf{r}) - 2(\mathbf{b} \cdot \delta \mathbf{r}) - \frac{2\mu_0}{c^3} \ln \left( 1 + \frac{n \cdot \delta \mathbf{r} + \| \delta \mathbf{r} \|}{\| n \cdot \mathbf{r}_{ssb0} + n \cdot \mathbf{b} + \| \mathbf{r}_{ssb0} \| + \| \mathbf{b} \|} \right), \]

where \( \delta \mathbf{r} = \mathbf{r}_{ssb1} - \mathbf{r}_{ssb0} \). In Equation (11), the distance between the satellite and SSB \( r = \| \mathbf{r} \| \) is on the order of \( 10^7 \) km, and the distance between the center of the Sun and the SSB, \( b = \| \mathbf{b} \| \), is on the order of \( 10^8 \) km. However, the distance between the pulsar and the SSB, \( D_0 \), is on the order of \( 10^{14} \) km. Therefore, \( \left( \| \mathbf{r}_{ssb0} \| + \| \mathbf{r}_{ssb1} \| \right) \| \delta \mathbf{r} \| /D_0 \) and \( (n \cdot \delta \mathbf{r})(n \cdot \delta \mathbf{r})/D_0 \) are negligible. Because \( \delta \mathbf{r} < \mathbf{r}_{ssb0} < D_0 \), the third term in Equation (11) is eliminated. Thus, Equation (11) can finally be written as

\[ \Delta \phi = \frac{1}{cP} n \cdot (r_{ssb1} - r_{ssb0}). \]

### 3.3. Two Satellites Model

Traditional XNAV methods use three pulsars to achieve autonomous navigation, but fail to make full use of the space vector information from the satellite formation. Double-satellite formation navigation, which is discussed in this paper, adopts the traditional concept of pulsar navigation. At the same time, the basic geometry of the two satellite formations, namely, the inter-satellite baseline information, is integrated into the navigation model to obtain more precise navigation information. Because the direction of the inter-satellite baseline is changing with the movement of the satellite, the angle between the direction vector of the pulsar and the inter-satellite baseline at different times is always changing according to the positions of the two satellites. The use of this information is expected to improve the position estimation precision of the traditional XNAV model. The geometric representation of this method is shown in Figure 3.

In Figure 3, \( \mathbf{b} \) is the Sun’s coordinate in the SSB reference frame, \( \mathbf{r}_{ssb} \) is Earth’s coordinate in the SSB reference frame. \( \mathbf{A} \) is the coordinate of the satellite formation SA in the SSB

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**Figure 2.** Geometric model of single star navigation. (A color version of this figure is available in the online journal.)
reference frame. B is the coordinate of the satellite formation SB in the SSB reference frame. \( \mathbf{r}_{EA} \) is the vector of the SA in Earth’s inertia coordinate system. \( \mathbf{r}_{EB} \) is the vector of the SB in Earth’s inertia coordinate system, \( \mathbf{r}_{AB} \) is the vector of the inter-satellite baseline of the angle between the reference satellite SA and the companion satellite SB. \( \mathbf{n}_i \) is the pulsar radiation direction and \( \alpha_i \) is the angle between the inter-satellite baseline and the pulsar radiation direction. As can be seen in the figure, if the distance between the satellites, which can be acquired by the inter-satellite link, is known, and the TDOA between the satellites is measured by the pulsar signal, \( \alpha_i \) is defined by

\[
\alpha_i = h(\mathbf{r}_{AB}, t) = \arccos \frac{\mathbf{n}_i \cdot \mathbf{r}_{AB}}{||\mathbf{r}_{AB}||} + n_\alpha, \tag{13}
\]

where \( n_\alpha \) is the observation noise. In addition, the projection of the satellite phase increment in the direction of a different pulsar orientation can be added into the observations to obtain a more accurate location. The details will be given in Section 4.

3.4. Three Satellites Model

In this section, a model of a three satellite formation navigation method using pulsars that will complement the traditional XNAV is proposed. A new observation variable, namely, the angle between the plane formed by three satellites and the pulsar direction vector, will be derived. The geometric model corresponding to this method is shown in Figure 4.

In Figure 4., SA is the reference satellite. The two remaining satellites, denoted by SB and SC, are companion satellites. \( \mathbf{n}_i \) is the pulsar radiation direction. If the projection of onto the plane ABC is denoted by \( \alpha’ \), and the projection of \( \angle PAC \) onto the plane ABC is denoted by \( \beta’ \), then

\[
\alpha' + \beta' = \omega. \tag{14}
\]

Thus, the angle \( \eta \) between the ABC plane and \( \mathbf{n}_i \) is deduced. The angle \( \eta \) is the new observation variable proposed in this paper to enhance the XNAV’s location of the satellite formation. The lengths of AB, AC and BC are measured with the help of an intersatellite link. Using the law of cosines, \( \cos \omega \) is given as

\[
\cos \omega = \frac{\mathbf{AB}^2 + \mathbf{AC}^2 - \mathbf{BC}^2}{2 \cdot \mathbf{AB} \cdot \mathbf{AC}}. \tag{15}
\]

It is not difficult to obtain \( \alpha = \arccos(\mathbf{AP}/\mathbf{AB}) \), where AP can be measured by the X-ray pulsar signal propagation delay between SA and SB. Note that the X-ray pulsar signal delay between A and B yields the equation \( l = \text{TDOA} \cdot c \), where \( c \) is the speed of light. The angle \( \beta \) is obtained similarly. With the known variables \( \alpha, \beta, d, \) and \( c \), the angle \( \beta \) can be calculated by

\[
\cos \alpha = \frac{l}{\mathbf{AB}} = \frac{d}{l} \quad \text{yields} \quad d = \frac{l^2}{\mathbf{AB}}, \tag{16}
\]

\[
\cos \beta = \frac{\mathbf{AK}}{\mathbf{AC}} = \frac{c}{\mathbf{AK}} \quad \text{yields} \quad c = \frac{\mathbf{AK}^2}{\mathbf{AC}}. \tag{17}
\]
Using the equations $\beta' = \arccos(d/e)$, $\beta'' = \arccos(c/e)$, $e = l \cos \eta$ and Equation (14), the following equation can be obtained:

$$\arccos \frac{l^2/AB}{l \cos \eta} + \arccos \frac{AK^2/AC}{l \cos \eta} = \omega. \tag{18}$$

Solving Equation (18) yields the observation $\eta$:

$$\eta = \arccos \frac{\sqrt{\left(\frac{l}{AB}\right)^2 + \left(\frac{AK}{AC}\right)^2} - 2 \cos \omega \cdot \frac{l}{AB} \cdot \frac{AK}{AC}}{/(1 - \cos^2 \omega)}. \tag{19}$$

In this paper, the X-ray pulsar light time and $\eta$ are utilized to improve the navigation precision of the three formation satellites.

Projecting the satellite phase angle onto the direction of the pulsar radiation improves the navigation precision of satellites formation. The projection of the satellite phase increment is the phase increment vector multiplied by the pulsar direction vector.

### 4. Iteration Filter

#### 4.1. Orbit Dynamic Model

The state vector $X$ of the satellite consists of the position vector $r = [x, y, z]^T$ and the velocity vector $v = v_S = [v_x, v_y, v_z]^T$. The state vector of the satellite can be expressed as

$$X = [r \ v]^T. \tag{20}$$

The dynamic model of the nonlinear system can be expressed as

$$\dot{X} = f(X(t), t) + w(t), \tag{21}$$

where $f$ represents the nonlinear dynamic function of the state vector, and $w(t)$ is the state noise. To obtain the forecast value, the state noise is ignored. The derivative of the state vector can be expressed as

$$\dot{X} = [\dot{r} \ \dot{v}]^T = [v \ a]^T. \tag{22}$$

Integrating Equation (22) on both sides, the following equation is obtained:

$$X(t) = X(t_0) + \int_{t_0}^t f(X(\tau), \tau)d\tau. \tag{23}$$
The initial state of the satellite is known:
\[ X(t_0) = X_0 = [r_0, v_0]^T. \] (24)

The acceleration can be calculated. By solving Equation (23), the state of the satellite at any time in the future can be automatically calculated. However, due to the presence of many higher order terms, it is difficult to obtain an exact analytical solution for the satellite state. Thus, the numerical integration of the initial state of the satellite and the dynamic equation of the satellite is used to obtain the state information of the satellite at a certain time in the future. This model will be considered within the context of the ADDF in following section.

4.2. ADDF Filter Model

The satellite state estimation problem consists of estimating the state of the satellite according to the selected criteria and the observed signals. The purpose of the state estimation is to smooth the past state, filter the current state and forecast the future state of the target. The state of the satellite is represented by its position, its velocity and its acceleration. Due to the nonlinearity of the system, an adaptive divided difference Kalman filter (ADDF), which has the advantage of the Unscented Kalman Filter (UKF) with a second order Taylor series approximation, is applied. Furthermore, it can estimate a noise covariance matrix and adjust the matrix depending on the level of the process noise. The nonlinear system can be expressed as
\[
\begin{align*}
   x(k+1) &= f(x(k), k) + w(k) \\
   y(k) &= h(x(k), k) + v(k)
\end{align*}
\] (25)
where \( f \in \mathbb{R}^{n \times 1} \) is transfer equation of the nonlinear state of \( x(k) \), \( h \in \mathbb{R}^{m \times 1} \) is nonlinear observation function of \( x(k) \), \( w(k) \) in \( \mathbb{R}^{n \times 1} \) is Gaussian noise of system, \( v(k) \) in \( \mathbb{R}^{m \times 1} \) is observation noise. \( w(k) \) and \( v(k) \) are white noise with mean zero and satisfy
\[
\begin{align*}
   &E\{w(k)w^T(j)\} = Q(k)\delta_{ij} \\
   &E\{v(k)v^T(j)\} = R(k)\delta_{ij} \\
   &E\{w(k)v^T(j)\} = 0
\end{align*}
\] (26)
where \( \delta_{ij} \) is the Kronecker Delta function. The details of ADDF can be found in the Appendix-A, and the numerical simulation is presented in Section 5.

4.3. Single Satellite Navigation Model

A positioning method is proposed for a single satellite using three pulsars according to the dynamic model of the satellite orbit. The method uses the ADDF filter to estimate the state of the satellite; the state of the system is
\[ X_S = [r_0, v_0]^T, \] (27)
where \( r_0, v_0 \) is the position and velocity of the reference satellite in the inertial coordinate system of Earth.

In X-ray pulsar navigation, photon arrival time at the spacecraft is converted into the time in the SSB reference frame. According to Equation (4), the time transformation equation is
\[
\begin{align*}
   t_{SSB}^i - t_{SC}^i &= \frac{n_i \cdot r_{ssb0}}{c} + \frac{1}{2cD_0}[(n_i \cdot r_{ssb0})^2 - \|r_{ssb0}\|^2] \\
   &+ 2(n_i \cdot b)(n_i \cdot r_{ssb0}) - 2(b \cdot r_{ssb0}) \\
   &+ \frac{2\mu_S}{c^3} \ln \left| \frac{n_i \cdot r_{ssb0} + n_i \cdot b + \|r_{ssb0}\| + \|b\|}{n_i \cdot b + \|b\|} \right|,
\end{align*}
\] (28)
where \( i \) denotes the \( i \)-th pulsar, and \( r_{ssb0} \) is the position vector of the reference satellite in the SSB reference frame. Equation (28) defines the components of the measurement equation.

The measurement equation is
\[
Y_{SS} = H_{SS} + V_{SS},
\] (29)
where \( H_{SS} = [h_{SSi}, i = 1, \ldots, N]^T \) is an N-dimensional vector, and \( N \) is the number of pulsars. \( h_{SSi} = h_{SS}(X_S(t), t) = c(t_{SSB}^i - t_{SC}^i) \) indicates the \( i \)-th pulsar observation. \( V_{SSi} = [V_{SSi}] \) is the observation noise with mean zero and standard variance \( \sigma_i \). The standard deviation \( \sigma_i \) is calculated as
\[
\sigma_{SSi} = \frac{Wc\sqrt{[B_X + F_X(1 - p_f)]A_{obs}d + F_XAp_1t_{obs}}}{2F_XAp_1t_{obs}},
\] (30)
where \( B_X \) is X-ray radiation background traffic, \( F_X \) is X-ray pulsar radiation photon flux, \( p_f \) is the pulsar period, \( A \) is detector area, \( p_f \) is ratio of average radiation flux to the total radiation flux in a period, \( t_{obs} \) is observation time, \( W \) is pulsar width, \( d \) is ratio of pulsar width to pulsar period, and \( c \) is speed of light.

If the satellite is in BCRS, according to Equation (12), the measurement equation of the incremental phase of the satellite at the two different times can be expressed as
\[
Y_{SB} = H_{SB} + V_{SB},
\] (31)
where \( H_{SB} = [h_{SBi}, i = 1, \ldots, N]^T \) is an N-dimensional vector, and \( N \) is the number of pulsars. \( h_{SBi} = h_{SB}(X_S(t), t) = n_i \cdot (r_{ssb1} - r_{ssb0})/(cF) \), and \( n_i \) is the direction of the pulsar. \( V_{SBi} = [V_{SBi}] \) represents the observation noise and is the Gaussian white noise with mean zero and variance \( \sigma_{SBi}^2 \). The observation of the single satellite navigation by X-ray pulsar is
\[
Y_S = H_S + V_S,
\] (32)
where \( H_S = [H_{SS}^T, H_{SB}^T]^T \) is the measurement matrix, \( V_S = [V_{SS1}, V_{SS2}, V_{SS3}, V_{SB1}, V_{SB2}, V_{SB3}]^T \) is the observation noise matrix, and the variance matrix of \( V_S(t) \) is represented as \( R_S = \text{diag}(\sigma_{SS1}^2, \sigma_{SS2}^2, \sigma_{SS3}^2, \sigma_{SB1}^2, \sigma_{SB2}^2, \sigma_{SB3}^2) \). \( \sigma_{SS}^2 \) is the...
variance of the time observation of the $i$-th pulsar, and $\sigma_{S_{i;i}}^2$ is the variance of the incremental phase.

### 4.4. Double Satellites Filter Model

In this paper, the state of satellite formation is obtained by means of the ADDF, and the state of satellite formation is

$$X_D = [r_0, v_0, \mathbf{n}_1, \mathbf{v}_1]^{T}, \quad (33)$$

where $r_0$, $v_0$ are the position and velocity vector of the reference satellite in the inertial coordinate system of Earth, $\mathbf{n}_1$, $\mathbf{v}_1$ are the position and velocity vector of the companion satellite in the inertial coordinate system.

The timing observation model, which is similar to the single satellite navigation model, is expressed as

$$Y_{Da} = H_{Da} + V_{Da}, \quad (34)$$

where $H_{Da} = [h_{Dai}, i = 1, \ldots, N]^{T}$ is an N-dimensional vector, and $N$ is the number of pulsars. $H_{Dai} = h_{Dai}(X_D(t), t) = \Delta T_j = c(t_{SSB}^{(i)} - t_{SC}^{(i)}), i = 1, 2, 3; j = 1, 2$ and $V_{Da} = [v_{Da}]$ is the observation noise with mean zero and variance $\sigma_{D_i;i}$.

As in the single satellite navigation model, consider the phase increments of the satellite at the two different times. Then, the measurement equation can be expressed as

$$Y_{Db} = H_{Db} + V_{Db}, \quad (35)$$

where $H_{Db} = [h_{Db}, i = 1, \ldots, N]^{T}$ is an N-dimensional vector, and $N$ is the number of pulsars. $H_{Db} = h_{Db}(X_D(t), t) = \Delta \varphi_j = \mathbf{n}_1 \cdot (\mathbf{r}_{\text{abl}} - \mathbf{r}_{\text{slido}})/(cP) \quad (i = 1, 2, 3; j = 1, 2)$. $\mathbf{n}_1$ is the direction of the pulsar, $V_{Db} = [v_{Db}]$ represents the observation noise, which is the Gaussian white noise with mean zero and variance $\sigma_{D_i;i}$.

Suppose that the angle between the inner-satellite baseline and the X-ray pulsar direction is $Z_i = \alpha_i$, $i = 1, 2, 3$, where $\alpha_i$ denotes the coordinate axes and the observation noise is $V_{Dc} = [v_{Dc}]$. Then, the measurement equation is expressed as

$$Y_{Dc} = \arccos \left( \frac{\mathbf{n}_1 \cdot \mathbf{r}_{AB}}{|\mathbf{r}_{AB}|} \right) + v_{Dc}, \quad (36)$$

where $\mathbf{n}_1$ is the direction of the pulsar, $|\mathbf{r}_{AB}|$ is computed from the inter-satellite baseline, $v_{Dc}$ is the composite error from the inter-satellite baseline measurement and the pulse arrival time that is recorded by the detector.

The observation of the X-ray pulsar navigation method for double star formation flying is

$$Y_D = H_D + V_D, \quad (37)$$

where $H_D = [H_{Da}, H_{Db}, H_{Dc}]^{T}$ is the measurement matrix and $V_D$ is the observation noise matrix, which can also be expressed as

$$V_D = [V_{Da}, V_{Db}, V_{Dc}]^{T}, \quad (38)$$

where

$$V_{Da} = [v_{D1i}, v_{D2i}, v_{D3i}, v_{D12i}, v_{D13i}, v_{D23i}],$$

$$V_{Db} = [v_{D21i}, v_{D32i}, v_{D12i}, v_{D13i}, v_{D23i}],$$

$$V_{Dc} = [v_{D13i}, v_{D23i}, v_{D33i}].$$

The variance matrix of $V_D$ is represented as

$$R_D = \begin{bmatrix} R_{Da} & R_{D,\varphi} \end{bmatrix} \begin{bmatrix} R_{Da} & R_{D,\varphi} \end{bmatrix}^{T}, \quad (39)$$

where

$$R_{Da} = \text{diag}(\sigma_{Da1}^2, \sigma_{Da2}^2, \sigma_{Da3}^2, \sigma_{Da4}^2, \sigma_{Da5}^2),$$

$$R_{D,\varphi} = \text{diag}(\sigma_{D,\varphi1}^2, \sigma_{D,\varphi2}^2, \sigma_{D,\varphi3}^2, \sigma_{D,\varphi4}^2, \sigma_{D,\varphi5}^2).$$

### 4.5. Triple Satellites Filter Model

The angle between the plane consisting of the three star satellite formation and the direction of the pulsar is introduced as a new observation in the triple-star filter model, which is similar to the double-star filter model. Considering the three star formation UKF based on the X-ray pulsar, the state of satellite formation is defined as

$$X_T = [r_0, v_0, \mathbf{n}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2]^{T}, \quad (40)$$

where $r_0$, $v_0$ are the position and velocity of the reference satellite $A$ in the inertial coordinate system of Earth, and $\mathbf{n}_1$, $\mathbf{v}_1$, $\mathbf{r}_2$, $\mathbf{v}_2$ are the position and velocity of companion satellite $B$ and companion satellite $C$, respectively, in the inertial coordinate system.

Similar to the single satellite navigation model, the timing observation model can be expressed as

$$Y_{Ta} = H_{Ta} + V_{Ta}, \quad (41)$$

where $H_{Ta} = h_{Ta}(X_T(t), t) = \Delta T_j = c(t_{SSB}^{(i)} - t_{SC}^{(i)}), \quad (i = 1, 2, 3; j = 1, 2, 3)$ and $V_{Ta} = [v_{Ta}]$ is the observation noise with mean zero and variance $\sigma_{Ta}$.

As in the single satellite navigation model, consider the phase increment of the satellite at two different times. Then the measurement equation can be expressed as

$$Y_{Th} = H_{Th} + V_{Th}, \quad (42)$$

where $H_{Th} = h_{Th}(X_T(t), t) = \Delta \varphi_j = \mathbf{n}_1 \cdot (\mathbf{r}_{\text{abl}} - \mathbf{r}_{\text{slido}})/(cP) \quad (i = 1, 2, 3; j = 1, 2, 3)$. $\mathbf{n}_1$ is the direction of the pulsar and $V_{Th} = [v_{Th}]$ represents the observation noise, which is the Gaussian white noise with mean zero and variance $\sigma_{T,i;i}$.

Similar to Section 3.5, the equation can be rewritten as

$$Y_{Tc} = H_{Tc} + V_{Tc}, \quad (43)$$

where

$$H_{Tc} = h_{Tc}(X_T(t), t) = \arccos \left( \left( \frac{1}{|\mathbf{r}_{AB}|} \right)^2 + \left( \frac{\mathbf{AK}}{|\mathbf{AC}|} \right)^2 - 2 \cos \omega \cdot \frac{l}{|\mathbf{AB}|} \cdot \frac{\mathbf{AK}}{|\mathbf{AC}|} \right)/(1 - \cos^2 \omega).$$
and $V_T = [v_{T1}, v_{T2}, v_{T3}, v_{T4}, v_{T5}]$ is the observation noise with mean zero and variance $\sigma_{v_{T}}$. The observation of the triple-satellite formation based on X-ray pulsar navigation is

$$Y_T = H_T + V_T$$

(44)

where $H_T = [H_{Ta}, H_{Tb}, H_{Tc}]^T$ is the measurement matrix and $V_T$ is the observation noise matrix, which can be described as

$$V_T = [V_{Ta}, V_{Tb}, V_{Tc}]^T$$

(45)

where

$$V_{Ta} = [v_{T1}, v_{T2}, v_{T3}, v_{T4}, v_{T5}]$$

$$V_{Tb} = [v_{T6}, v_{T7}, v_{T8}, v_{T9}, v_{T10}]$$

and the variance matrix of $V_T$ is represented as

$$R_T = [R_{T1}, R_{T2}, R_{T3}]$$

(46)

where

$$R_{T1} = \text{diag}(\sigma_{T1}^2, \sigma_{T2}^2, \sigma_{T3}^2, \sigma_{T4}^2, \sigma_{T5}^2)$$

$$R_{T2} = \text{diag}(\sigma_{T6}^2, \sigma_{T7}^2, \sigma_{T8}^2, \sigma_{T9}^2, \sigma_{T10}^2)$$

$$R_{T3} = \text{diag}(\sigma_{T11}^2, \sigma_{T12}^2, \sigma_{T13}^2)$$

$\sigma_{T1}, \sigma_{T2}, \sigma_{T3}$ are the standard deviations of the satellite positions in different directions.

5. Simulation and Results

There are three issues that need to be resolved in this section. The first issue involves the performance of the multi-satellite formation navigation using pulsar timing observations rather than single satellites. The second issue involves the effectiveness of the XNAV method enhanced by the phase increment and inter-satellite baseline vector angle in twin-star formation. The final issue concerns the effectiveness of the XNAV method and whether it is enhanced by the phase increment and the angle between the radiating direction of the pulsar and the plane formulated by three satellites.

In the simulation, three kinds of satellites, GPS BIIA-10, MEGSAT-1, and GPS BIIA-4, are used. The GPS BIIA-10 is chosen as the reference orbit, while MEGSAT-1 and GPS BIIA-4 are the companion orbits. The parameters of the pulsar and the orbits utilized for numerical simulation in this paper are presented in Tables 1 and 2.

5.1. Simulation Results using Timing Data

The filtering parameters of the single satellite based on X-ray pulsar navigation are shown below. The satellite orbit is GPS BIIA-10. The initial state error of the satellite is

$$\delta X_S(0) = [2 \text{ km } 2 \text{ km } 2 \text{ km } 10 \text{ m s}^{-1} 10 \text{ m s}^{-1} 10 \text{ m s}^{-1}]^T.$$  

(47)

The noise covariance matrix of the initial condition is set to

$$Q_S = \text{diag}(p_1^2, p_1^2, p_1^2, p_2^2, p_2^2, p_2^2),$$

(48)

where $p_1 = 20 \text{ m}$, $p_2 = 1 \text{ m s}^{-1}$. The pulsar timing observation noise is set to

$$R_S = [(0.109 \text{ km})^2, (0.325 \text{ km})^2, (0.344 \text{ km})^2].$$

(49)

The noise variance matrix of the satellite phase increment is

$$R_{S_p} = [(3e-4)^2, (3e-4)^2, (3e-4)^2].$$

(50)

The observation model in Section 3.2 and the navigation model in Section 4.3 are used here, and the filtering method is ADDF, whose details are contained in the Appendix. The simulation results of the signal star navigation with ADDF with the above parameters are shown in Figure 5.

It should be noted that in Figure 5., the acronym “STIP” indicates a single-satellite formation with a timing and incremental phase. The term “ST” indicates a single-satellite timing observation. The abscissa corresponds to the number of iterations of the filtering process, and the ordinate corresponds to the position or velocity estimation error or the cumulative error of the GPS_BIIA-10 orbit. From the simulation results, it can be seen that the multi-information
The fusion method has better performance than the timing observation method alone. The improvement in the position estimation error is greater than that of the velocity estimation error. Table 3 also shows an improvement in the precision of the position and a velocity of 58% and 29%, respectively, in the reference satellite. This result is because the phase increment involves the calculation of the phase difference at two different times, which allows common errors to be deduced via relativity and phase evolution errors. Thus, the filtering performance is improved through the use of phase increments. The results also illustrate that the filter obtains better results as more information is added to the model. Therefore, in the next section, further simulations are presented, which improve satellite navigation performance through the addition of more effective observations and information.

5.2. Simulation Results of Double Star Navigation

The filtering parameters of the double star navigation formation calculation are set as follows. In the simulation of double star satellite formation based on X-ray pulsar navigation, the reference satellite orbit uses the GPS BIIA-10, and the orbit of the associated satellite is MEGSAT-1. The initial state error of satellite formation is set to

\[
\delta X_D(0) = [2 \text{ km} \ 2 \text{ km} \ 2 \text{ km} \ 10 \text{ m s}^{-1} \ 10 \text{ m s}^{-1} \ 10 \text{ m s}^{-1} \\
2 \text{ km} \ 2 \text{ km} \ 10 \text{ m s}^{-1} \ 10 \text{ m s}^{-1} \ 10 \text{ m s}^{-1} \ 10 \text{ m s}^{-1}]^T.
\]

The noise covariance matrix of the initial error is set to

\[
Q_D = \text{diag}(p_1^2, p_1^2, p_2^2, p_2^2, p_1^2, p_1^2, p_2^2, p_2^2, p_1^2, p_1^2, p_2^2, p_2^2).
\]
where \( p_1 = 20 \text{ m} \) and \( p_2 = 1 \text{ m s}^{-1} \). The pulsar timing observation noise is set to

\[
R_{D_1} = [(0.109 \text{ km})^2, (0.325 \text{ km})^2, (0.344 \text{ km})^2, (0.109 \text{ km})^2, (0.325 \text{ km})^2, (0.344 \text{ km})^2].
\]  

(53)

The noise variance matrix of angle deviation between the pulsar direction and inter-satellite baseline is set to

\[
R_{D_0} = [(9e-7)^2, (9e-7)^2, (9e-7)^2].
\]  

(54)

The noise variance matrix of the satellite phase increment is

\[
R_{D_p} = [(3e-4)^2, (3e-4)^2, (3e-4)^2, (3e-4)^2, (3e-4)^2, (3e-4)^2].
\]  

(55)

In the case of double satellite formation, the baseline of the two satellites will be used to improve the performance. The proposed method will be compared to the traditional method, which uses the TOA of the pulsar as the input. The results are shown in Figures 6–7. Figure 5 consists of two plots
corresponding to the reference satellite and two plots corresponding to the adjoint satellite.

In Figures 6–7, the abscissa corresponds to the number of iterations of the filtering process, and the ordinate corresponds to the position or velocity estimation error or the cumulative error of the different orbits. The term “DTIPVA” indicates a double-satellite formation with timing, an incremental phase and the angle between the

### Table 4

| Methods        | Average Position Error/km | Position Error Standard Variance/km | Average Velocity Error/(m s⁻¹) | Velocity Error Variance/(m s)² |
|----------------|---------------------------|-------------------------------------|-------------------------------|-------------------------------|
| Reference satellite-A | DTIPVA 0.3713            | 0.0770                              | 0.6984                        | 0.1579                        |
|                 | DT 0.8777                |                                     |                               |                               |
| Adjoint satellite-B | DTIPVA 0.3657            | 0.0787                              | 0.7128                        | 0.1822                        |
|                 | DT 0.8741                |                                     | 1.0272                        | 0.3022                        |

**Figure 7.** Navigation precision of adjoint satellite-B with different methods.
(A color version of this figure is available in the online journal.)
satellite baseline vector and the radiation direction of the X-ray pulsar: The term “DT” indicates a double-satellite formation with timing observation. From the figures, it can be seen that the multi-information fusion method has a better performance than the timing observation method alone. The improvement of the position estimation error is greater than that of the velocity estimation error because the satellite velocity itself changes only slightly. Table 4 shows an improvement in the precision of the position and a velocity of 65% and 29%, respectively, in the reference satellite. The position precision is improved in the adjoint satellite calculation by 58%, and the velocity is improved by 30%, as shown in Figures 6–7. In Table 4, it can be seen that the position estimation accuracy improvement is nearly 60%, and the accuracy improvement in the velocity calculation is approximately 20% compared to the traditional XPNA.

5.3. Simulation Results of Triple Star Navigation

In the simulation of the three star satellite formation based on X-ray pulsar navigation, the reference satellite (A) orbit uses the GPS BIIA-5, accompanied satellite (B) orbit the MEGSAT, satellite (C) orbit the GPS_BII-04. The filtering parameters of the triple star formation navigation are set as follows. The initial state error of satellite formation is set to

$$\delta X_T(0) = [2 \text{ km} \ 2 \text{ km} \ 10 \text{ m s}^{-1}10 \text{ m s}^{-1}10 \text{ m s}^{-1}$$

$$2 \text{ km}2 \text{ km}2 \text{ km}10 \text{ m s}^{-1}10 \text{ m s}^{-1}10 \text{ m s}^{-1}]^T.$$  (56)

The noise covariance matrix of the initial error is set to

$$Q_T = \text{diag}(p_1^2, p_1^2, p_1^2, p_2^2, p_2^2, p_2^2, p_1^2, p_1^2, p_2^2, p_2^2, p_2^2, p_2^2, p_2^2, p_2^2, p_2^2).$$  (57)
where \( p_1 = 20 \text{ m}, p_2 = 1 \text{ m s}^{-1} \). The pulsar timing observation noise is set to
\[
R_t = \begin{bmatrix}
(0.109 \text{ km})^2, (0.325 \text{ km})^2, (0.344 \text{ km})^2,
(0.109 \text{ km})^2, (0.325 \text{ km})^2, (0.344 \text{ km})^2,
(0.109 \text{ km})^2, (0.325 \text{ km})^2, (0.344 \text{ km})^2.
\end{bmatrix} (58)
\]

The noise variance matrix of the angle between the radiating direction of pulsars and the plane determined by the satellites in formation is set to
\[
R_{t\theta} = \begin{bmatrix}
(9e-7)^2, (9e-7)^2, (9e-7)^2
\end{bmatrix}. (59)
\]

The noise variance matrix of satellite phase increment is
\[
R_{t\varphi} = \begin{bmatrix}
(3e-4)^2, (3e-4)^2, (3e-4)^2, (3e-4)^2, (3e-4)^2, (3e-4)^2,
(3e-4)^2, (3e-4)^2, (3e-4)^2.
\end{bmatrix} (60)
\]

In the simulation, three different satellites are adopted to verify the method proposed in this study. Two methods are applied to estimate the state of navigation in real time. In Figures 8–10 and Table 5, the term “TRTIPVA” denotes that the timing, the incremental phase and the angle between the radiating direction vector of pulsars and the plane comprise the three satellite information fusion algorithms in three satellites formations. The term “TRT” denotes the traditional timing observation. From Figures 8–10, it can be seen that the position and velocity estimation error of the method with TRTIPVA are smaller than those of the traditional timing method. The position accumulative error decreases as the number of iterations increases. The improvement in the position estimation error is greater than that of the velocity estimation error. From the statistical simulation data, Table 5, shows the precision of the position and the velocity of the reference satellite with TRTIPVA are improved by 51% and 19%, respectively, compared with the TRT.
observation, while the precision of the position and the velocity of the two adjoint satellites is improved by 50% and 20%, respectively. As shown in Figures 8–10 and Table 5, the accuracy with the improved method is nearly 50% for the position and 20% for the velocity compared with the traditional XPNA by employing ADDF.

Note that the performance in the TRT case is better than that in the DT case or the ST case. Further, the performance in

---

**Table 5**

Position and Velocity Estimation Error for Three Satellite Formations

|                  | Methods | Average Position Error/km | Position Error Standard Variance/km | Average Velocity Error/(m s\(^{-1}\)) | Velocity Error Variance/(m s\(^{-2}\)) |
|------------------|---------|---------------------------|-------------------------------------|--------------------------------------|--------------------------------------|
| Reference satellite_A | TRTIPVA | 0.3599                    | 0.0453                              | 0.6288                               | 0.0954                               |
|                  | TRT     | 0.7373                    | 0.2177                              | 0.7795                               | 0.1545                               |
| Adjoint satellite_B | TRTIPVA | 0.3631                    | 0.0532                              | 0.6499                               | 0.1060                               |
|                  | TRT     | 0.7390                    | 0.2179                              | 0.8119                               | 0.2120                               |
| Adjoint satellite_C | TRTIPVA | 0.3629                    | 0.0516                              | 0.6406                               | 0.0962                               |
|                  | TRT     | 0.7375                    | 0.2181                              | 0.7814                               | 0.1565                               |
the DT case is better than in the ST case. This finding is because, if more TDOAs are integrated in the ADDF, as has been done in the TRT and DT cases, the total information of the system has increased, thus improving the overall performance.

6. Conclusion

Several methods are presented in this paper to improve the performance of XNAV with respect to satellite formation navigation. The concept of a FFT-based phase comparison method is introduced. A novel method that employs the phase increment, the baseline of double satellites, and the plane formed by three satellites is proposed. The main advantages of using the phase increment are that common error can be deduced and that topology can adopt the orientation vector of the pulsar as an additional piece of information. In addition, the nonlinear ADDF is used as an iteration filter method. The proposed method achieves better results than the traditional methods do. Numerical experiments verify the performance of the new method. When the ADDF is adopted as the multi-information fusion method, the precision is improved by more than 50% for the position estimates and by more than 20% for the velocity estimates, which aids the spacecraft in achieving autonomous X-ray-based navigation with better accuracy. Furthermore, if the angle between the orientation method and the baseline of the two satellites is added to the ADDF, the accuracy is increased by another 12%. The angle between the pulsar orientation vector and the plane formed by three satellites increases the precision by approximately 9%. In summary, the proposed method contributes to the enhancement of the performance of XNAV, and details of the model should be further addressed in a future study.

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Appendix

The algorithm procedures for Adaptive Divided Difference Filter are as follows:

1. The initial state estimate is given as $\bar{x}_0$ and $P_0$ is the covariance matrix.

2. Time updating steps

A set of Sigma points of state with a length of $2n+1$ is selected and the sampling strategy is

$$
\begin{align*}
\chi_0 &= x_{k-1} \\
\chi_i &= x_{k-1} + h\bar{S}_{i,k-1}, & i = 1 \cdots n \\
\chi_{i+n} &= x_{k-1} - h\bar{S}_{i,k-1}, & i = 1 \cdots n
\end{align*}
$$

where $i = 1 \cdots n$, interval length is $h = \sqrt{3}$ for Gaussian noise distribution, $\bar{S}_{i,k-1}$ is the Cholesky factor of $P_{i,k-1}$, namely $P_{i,k-1} = S_{i,k-1}S_{i,k-1}^T$. The $x_{k-1}$ is obtained as

$$
x_{k-1} = [X_0, \chi_1, \chi_{i+n}]
$$

3. Measurement updating steps

$$
P_{i,k-1} = S_{i,k-1} \cdot \bar{S}_{i,k-1}^T
$$

$$
\begin{align*}
\chi_0 &= \bar{x}_{i,k-1} \\
\chi_i &= \bar{x}_{i,k-1} + h\bar{S}_{i,k-1}, & i = 1 \cdots n \\
\chi_{i+n} &= \bar{x}_{i,k-1} - h\bar{S}_{i,k-1}, & i = 1 \cdots n
\end{align*}
$$

$$
x_{i,k-1} = [X_0, \chi_i, \chi_{i+n}]
$$

Then a recursive calculation from the beginning $K = 1$ can be conducted.
4. Steps for DDF adaption,

\[ C_k^\theta(k) = \frac{1}{N} \sum_{j=k-N+1}^{k} \tilde{\vartheta}_k(j) \tilde{\vartheta}_k^T(j), \]

\[ Q_k = K_k C_k^\theta(k) K_k^T, \]

where \( \tilde{\vartheta}_k \) denotes the difference between the actual measurement and the estimate of measurement, \( N \) is the length of a sliding estimation window.

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