GUT Relations from String Theory Compactifications

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Abstract

Wilson line on a non-simply connected manifold is a nice way to break SU(5) unified symmetry, and to solve the doublet–triplet splitting problem. This mechanism also requires, however, that the two Higgs doublets are strictly vector-like under all underlying gauge symmetries, and consequently there is a limit in a class of modes and their phenomenology for which the Wilson line can be used. An alternative is to turn on a non-flat line bundle in the U(1) direction on an internal manifold, which does not have to be non-simply connected. The U(1) gauge field has to remain in the massless spectrum, and its coupling has to satisfy the GUT relation. In string theory compactifications, however, it is not that easy to satisfy these conditions in a natural way; we call it U(1) problem. In this article, we explain how the problem is solved in some parts of moduli space of string theory compactifications. Two major ingredients are an extra strongly coupled U(1) gauge field and parametrically large volume for compactification, which is also essential in accounting for the hierarchy between the Planck scale and the GUT scale. Heterotic-M theory vacua and F-theory vacua are discussed. This article also shows that the toroidal orbifold GUT approach using discrete Wilson lines corresponds to the non-flat line-bundle breaking above when orbifold singularities are blown up. Thus, the orbifold GUT approach also suffers from the U(1) problem, and this article shows how to fix it.
1 Introduction

The gauge coupling unification of the minimal supersymmetric standard model (MSSM) is the biggest (phenomenological) motivation to study supersymmetric unified theories. The SU(5)$_{\text{GUT}}$ unified symmetry is broken down to the standard-model gauge group SU(3)$_C \times$ SU(2)$_L \times$ U(1)$_Y$ without reducing the rank of the gauge group, when an expectation value is turned on for a scalar field in the SU(5)$_{\text{GUT}}$ adjoint representation.

For higher-dimensional supersymmetric theories such as geometric compactification of the superstring theory, there always exists SU(5)$_{\text{GUT}}$ gauge field with polarization pointing to the directions of internal manifold, and a Wilson line in the U(1)$_Y$ direction can play the role of the D = 4 scalar field in the adjoint representation. The Wilson lines can be introduced only in a manifold $Z$ with a non-trivial homotopy group $\pi_1(Z) \neq \{1\}$. The Wilson lines in the U(1)$_Y$ direction, or equivalently the flat bundles, break the SU(5)$_{\text{GUT}}$ symmetry, get rid of gauge bosons in the off-diagonal blocks from the massless spectrum and allow the spectrum of coloured Higgs multiplets to be different from that of Higgs doublets.

Since those goals can be achieved also by line bundles that are not flat, one could think of compactification on a simply connected manifold with a line bundle turned on in the U(1)$_Y$ direction, instead. Many models fall into this category, including toroidal orbifold compactification [4, 5, 6, 7, 8] and SU(5) × U(1)$_Y$ bundle compactification of Heterotic $E_8 \times E_8'$ string theory [11, 12] and Calabi–Yau orientifold compactification models of Type IIB string theory [13, 14, 15, 16].

The problem of this approach with non-flat line bundles is that U(1)$_Y$ gauge field in the SU(5)$_{\text{GUT}}$ symmetry (and hence U(1)$_{\text{QED}}$) generically does not remain massless. This problem can be avoided by starting from a gauge group larger than SU(5)$_{\text{GUT}}$, such as U(6) in a model of Type IIB compactification [13], or $E_8 \times E_8$ in Heterotic compactification [12]. The massless U(1)$_Y$ gauge field below the Kaluza–Klein scale is a linear combination of the ordinary U(1)$_Y$ gauge field in the SU(5)$_{\text{GUT}}$ gauge group and an additional U(1) symmetry contained in the larger gauge group. The gauge coupling constant of the low-energy U(1)$_Y$ gauge field is, however, weakened due to the mixture of the additional U(1) gauge field, and the successful prediction of the gauge coupling unification is lost. The primary goal of this note is to show that the gauge coupling unification is restored in certain region (limit) of moduli space.

We are not only trying to explore just another class of string vacua with successful gauge

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1 Models leaving SU(3)$_C \times$ SU(2)$_L$ (and some U(1)'s) symmetry have been discussed in free-fermion formalism as well [9]. Relation between the free-fermion formalism and orbifold compactification is studied in [10].
coupling unification. Note that Wilson lines can be a solution to the doublet–triplet splitting problem only when a pair of Higgs doublets \( H_u \) and \( H_d \) is completely vector like under the underlying gauge symmetry such as \( E_8 \) (and in fact, \( E_8 \) is the only candidate of the underlying gauge symmetry if we assume \( SU(5)_{\text{GUT}} \) unification and the vector-like nature of \( H_u \) and \( H_d \); see [17]). In the Heterotic \( E_8 \times E_8 \) string theory, for instance, the Higgs multiplets \( H(5) \) and \( \bar{H}(\bar{5}) \) may originate from \( H^1(Z; \wedge^2 V_5) \cong H^2(Z; \wedge^2 V_5^\times) \) and \( H^1(Z; \wedge^2 V_5) \), respectively, where \( Z \) is a Calabi–Yau 3-fold, \( V_5 \) is a rank-5 vector bundle in one of \( E_8 \) and \( \bar{V}_5 = V_5^\times \) its dual bundle. In this case, \( H(5) \supset H_u \) and \( \bar{H}(\bar{5}) \supset H_d \) are vector-like not only under \( SU(5)_{\text{GUT}} \) but also under the structure group \( SU(5) \). A flat bundle \( \mathcal{L}_Y \) can be turned on in the \( U(1)_Y \) direction, when \((Z, V_5)\) has an isometry group \( \Gamma \) that acts freely on \( Z \). The index theorem says that

\[
\#H_u - \#H_d = \chi(Z/\Gamma; \wedge^2 V_5 \otimes \mathcal{L}_Y^{1/2}) = \frac{1}{\#\Gamma} \chi(Z; \wedge^2 V_5),
\]

\[
\#H_c(3) - \#\bar{H}_c(\bar{3}) = \chi(Z/\Gamma; \wedge^2 V_5 \otimes \mathcal{L}_Y^{1/3}) = \frac{1}{\#\Gamma} \chi(Z; \wedge^2 V_5),
\]

and hence coloured Higgs multiplets can be absent in the low-energy spectrum (that is, \( \#H_c = 0 \) and \( \#\bar{H}_c = 0 \)), while we have a pair of massless Higgs doublets, \( \#H_u = \#H_d = 1 \).

If \( H_u \) and \( H_d \) originate from bundles that are not dual, on the other hand, the index theorem has to be applied separately for the bundle of \( H_u \) and that of \( H_d \). Suppose, say, that they are identified with cohomology groups \( H^1(Z/\Gamma; U_{H_u} \otimes \mathcal{L}_Y^{1/2}) \) and \( H^1(Z/\Gamma; U_{H_d} \otimes \mathcal{L}_Y^{1/2}) \) for some bundles \( U_{H_u} \) and \( U_{H_d} \) on \( Z \), respectively. Now, \( H_u \) and \( H_d \) are not vector-like under the structure groups of the vector bundles. In this case, \( \#H_u \) and \( \#H_d \) are directly related to the Euler characteristics \(-\chi(Z/\Gamma; U_{H_u} \otimes \mathcal{L}_Y^{1/2})\) and \(-\chi(Z/\Gamma; U_{H_d} \otimes \mathcal{L}_Y^{1/2})\). If the symmetry breaking of \( SU(5)_{\text{GUT}} \) were due to a flat bundle \( \mathcal{L}_Y \) in the \( U(1)_Y \) direction, then the Euler characteristic of the bundle in the doublet parts and the triplets part cannot be different,

\[
\chi(Z/\Gamma; U_{H_u} \otimes \mathcal{L}_Y^{1/2}) = \chi(Z/\Gamma; U_{H_u} \otimes \mathcal{L}_Y^{1/3}) = \chi(Z; U_{H_u})/\#\Gamma,
\]

\[
\chi(Z/\Gamma; U_{H_d} \otimes \mathcal{L}_Y^{-1/2}) = \chi(Z/\Gamma; U_{H_d} \otimes \mathcal{L}_Y^{1/3}) = \chi(Z; U_{H_d})/\#\gamma,
\]

because flat bundles do not contribute to the Euler characteristics. Thus, if there is a pair of Higgs doublets \( H_u \) and \( H_d \) in low energy spectrum, and there is only a pair, then there is also a pair of Higgs triplets at low energies. Gauge coupling unification is no longer expected in the presence of this additional triplets in the spectrum. If the \( SU(5)_{\text{GUT}} \) symmetry is broken by a non-flat line bundle in the \( U(1)_Y \) direction, however, the chirality in the doublet part and the triplet part can be different, and there can be no triplets at low energies; such compactifications.
are consistent with the gauge coupling unification. (hereafter, whenever we say a line bundle in this article, it is meant to be non-flat unless specifically mentioned as a flat bundle.)

Models with non-vector-like two Higgs doublets has a natural mechanism to bring dimension-5 proton decay operators under control \cite{17, 18}. A pair of Higgs multiplets being completely vector-like is the essence of the dimension-5 proton decay problem, and hence this problem is always an issue for the SU(5)\textsubscript{GUT} symmetry breaking using the Wilson line. Although the dimension-5 operators can be eliminated by imposing an extra discrete symmetry for this special purpose, probability of finding such a symmetry in a landscape of vacua is very small.\footnote{There would hardly be an anthropic argument for such a discrete symmetry, because it seems there is nothing wrong with a proton life time of order $10^{28}$ years. cf. \cite{19}.} Thus, there exists a phenomenological motivation to study the SU(5)\textsubscript{GUT} symmetry breaking due to a line bundle in the U(1)\textsubscript{Y} direction.

This article is organized as follows. Section \ref{sec:2.1} explains why it is difficult in wide class of string compactification to get a massless U(1)\textsubscript{Y} gauge field while maintaining the gauge-coupling unification. We see in section \ref{sec:2.2} however, that this generic problem can be solved by assuming an extra strongly coupled U(1) gauge theory; the disparity between the strongly coupled U(1) sector and the visible perturbative SU(5)\textsubscript{GUT} sector can be attributed to a parametrically large volume of compactification, which also accounts for the hierarchy between the unification scale and the Planck scale \cite{20, 21}.\footnote{An $E_7$-type underlying symmetry is essential in obtaining the Yukawa couplings as explained in \cite{17}, but not in the SU(5)\textsubscript{GUT} symmetry breaking. Thus, although presentation of \cite{20, 21} uses Type IIB string theory, it does not mean that the idea cannot be extended to F-theory.} This observation is elaborated in sections \ref{sec:3} and \ref{sec:4} by using the compactifications of Heterotic string and F-theory, respectively. Along the way, we will also see that the idea of containing U(1)\textsubscript{Y} flux in a local region in the internal space \cite{22} is useful in bringing threshold corrections under control. Presentation of \cite{22} (and orbifold-GUT papers that followed) is based exclusively on toroidal orbifold compactification (of the Heterotic $E_8 \times E_8$ string theory), but we find a way to implement the idea in general string theory compactifications.

The appendix, which constitutes a big part of this paper, is somewhat independent from the main text of this article. It explains how the toroidal orbifold compactification is understood as certain limits of Calabi–Yau compactification. Heterotic orbifold-GUT approach in the last several years often make use of “discrete Wilson lines” in breaking the SU(5)\textsubscript{GUT} symmetry, and the primary purpose of the appendix is to clarify the meaning of discrete Wilson lines of toroidal orbifold compactification in terms of Calabi–Yau compactification. The discrete Wilson lines in toroidal orbifolds are totally different from the Wilson lines associated with
finite discrete homotopy group $\pi_1(Z)$ of non-simply connected Calabi–Yau $Z$. They should be understood as special cases (and special corners of moduli space) of non-flat line bundles in the $U(1)_Y$ direction on Calabi-Yau compactifications.

Thus, orbifold GUT models also suffer from the $U(1)_Y$ problem in section 2.1 and this problem is solved as we explain in this article. Because the idea of orbifold GUT has received attention for the last several years from much wider community, the appendix is pedagogically presented. The appendix A.2 shows that the “continuous Wilson lines” in toroidal orbifold compactifications corresponds to vector-bundle moli of smooth Calabi–Yau compactifications, and has nothing to do with Wilson lines associated with $\pi_1(Z) \sim \mathbb{Z}$.\footnote{Although the contents of the appendix A.2 is irrelevant to the main text, we include the contents of the appendix A.2 in this note, because little effort beyond the appendix A.1 is necessary, yet we expect that some people are interested in geometric interpretation of various aspects of toroidal orbifolds.}

As we were finishing this work, an article [23] was posted on the web, which also discusses $SU(5)_{\text{GUT}}$ breaking due to a line bundle in the $U(1)_Y$ direction. There, an idea of [15] in perturbative Type IIB string theory is generalized to F-theory compactifications, and explicit examples of geometry are given. Thus, a solution to the $U(1)_Y$ problem in this article (and in [20, 21]) is different from those in [15, 23]. We have also learnt that Donagi and Wijnholt have been working on a related subject ([65]).

2 The $U(1)_Y$ Problem and an Idea to Solve It

2.1 The $U(1)_Y$ Problem

2.1.1 Massless $U(1)$ Gauge Field

Let us first consider the Heterotic $E_8 \times E_8$ theory compactified on a Calabi–Yau 3-fold $Z$ with vector bundles $V_5$ and $L_Y$ turned on in one of $E_8$. The structure group of $V_5$ is $SU(5)_{\text{bdl}}$, whose commutant in the $E_8$ symmetry is the $SU(5)_{\text{GUT}}$ symmetry. The line bundle $L_Y$ is in the $U(1)_Y \subset SU(5)_{\text{GUT}}$ direction. The $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry of the standard model is the commutant of the bundle structure group $SU(5) \times U(1)_Y$. The gauge fields of the non-Abelian part of the unbroken symmetry, $SU(3)_C \times SU(2)_L$, remain massless below the Kaluza–Klein scale.

The $U(1)_Y$ gauge field, however, does not remain massless [24, 11, 25]. The $D = 10$ action of the Heterotic string theory contains the kinetic term of the $B$-field

$$S = -\frac{1}{4\kappa^2} \int d^{10}x \sqrt{g_{10}} e^{-2\phi}|H|^2; \quad H = dB^{(2)} - \frac{\alpha'}{4} \left( \text{tr}_{E_8 \times E_8} \left( AF - \frac{2}{3} AAA \right) - \omega_{\text{grav}} \right), \quad (5)$$

The appendix A.2 is pedagogically presented. The appendix A.2 shows that the “continuous Wilson lines” in toroidal orbifold compactifications corresponds to vector-bundle moli of smooth Calabi–Yau compactifications, and has nothing to do with Wilson lines associated with $\pi_1(Z) \sim \mathbb{Z}$.
where $\omega_{\text{grav}}$ is the Chern–Simons 3-form of gravity. Fluctuations of the $B$-field of the form $b^k \omega_k$ are massless in the Kaluza–Klein reduction, where $b^k$ ($k = 1, \cdots, h^{1,1}$) are $D = 4$ scalar fields and $\omega_k$ form a basis of $H^{1,1}(Z)$ of a compact Calabi–Yau 3-fold $Z$. Their kinetic terms in the $D = 4$ effective theory are of the form

$$G_{kl} d^4 x \mathcal{L} = G_{kl}(\partial b^k - Q_k^A)(\partial b^l - Q_l^A); \quad c_1(L_Y) \propto \omega_k Q^k,$$

where $G_{kl}$ is a metric on the Kähler moduli space, and $A$ is the $U(1)_Y$ gauge field. Thus, a linear combination of these $B$-field fluctuations is absorbed to be the longitudinal mode of the $U(1)_Y$ gauge field. The kinetic term above also contains the mass term of the $U(1)_Y$ gauge field. Thus, whether the bundle $L_Y$ is flat ($c_1(L_Y) \propto \langle dA \rangle = 0$) or not leads to a big difference in phenomenology.

The same problem exists in Type IIB Calabi–Yau orientifold compactification. Let us consider the Type IIB string theory compactified on a Calabi–Yau 3-fold $X$ with a holomorphic involution $\mathcal{I}$; the Calabi–Yau 3-fold is modded by an orientifold projection associated with $\mathcal{I}$; $D7$-branes are wrapped on holomorphic 4-cycles, so that $\mathcal{N} = 1$ supersymmetry is preserved in $D = 4$ effective theory. If 5 $D7$-branes are wrapped on a holomorphic 4-cycle $\Sigma$ of $X$, the $SU(5)_{\text{GUT}}$ gauge field propagates on $\Sigma$. Suppose that a line bundle $L_Y$ is turned on on $\Sigma$ in the $U(1)_Y$ direction in $SU(5)_{\text{GUT}}$ symmetry. Then the $SU(5)_{\text{GUT}}$ symmetry is broken to $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry of the standard model. Although the $SU(3)_C \times SU(2)_L$ part of the gauge field remains massless in this Type IIB compactification as well, the $U(1)_Y$ gauge field does not. The Wess–Zumino action on $\Sigma$ contains

$$S_{\text{CS};\Sigma} = \int_{\mathbb{R}^{3,1} \times \Sigma} dC \text{ tr } e^{\frac{F}{2\pi}} \propto \int_{\mathbb{R}^{3,1}} A \wedge d c^m \int_{\Sigma} \omega_m \wedge c_1(L_Y) + \cdots,$$

where $D = 4$ 2-form fields $c^m$ describe massless fluctuations of the Ramond–Ramond 4-form field $C^{(4)} \sim c^m \omega_m$. $A$ is the $U(1)_Y$ gauge field. Thus, a linear combination of the $D = 4$ Hodge dual of the 2-forms $c^m$ is absorbed to be the longitudinal mode of the $U(1)_Y$ gauge field. The $U(1)_Y$ gauge field becomes massive, and so does the QED gauge field. This is a problem in the context of large volume compactification, e.g. in toroidal orbifolds and e.g. in orientifolded Calabi–Yau 3-folds in general.

These phenomena in the Heterotic theory and Type IIB theory are related by the string duality. It is the $B$-field fluctuation of the form $b^k \omega_k \propto Q^k \omega_k$ in the Heterotic theory that is

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5In addition to this generalized Green–Schwarz couplings of Kähler moduli chiral multiplets at tree level, there is also a 1-loop coupling for the dilation chiral multiplet [29, 30, 31, 26]. Coefficients of the tree-level generalized Green–Schwarz couplings are worked out in [27, 12] in the Heterotic $E_8 \times E_8$ string theory. (For $SO(32)$ Heterotic string theory, see [28].)
mixed with the $U(1)_Y$ gauge field. Roughly speaking, it corresponds to a fluctuations of the Ramond–Ramond 2-form field $C^{(2)} \sim c^k \omega_k \propto Q^k \omega_k$ in the Type I string theory, where $c^k$ are $D = 4$ scalar fields, and then to $C^{(4)} \sim c^k (\ast \omega_k) \propto Q^k (\ast \omega_k)$ in Type IIB string theory, where the Hodge dual $\ast$ is taken in a complex 2-fold $B$ that the Heterotic and Type IIB string theory share in the duality.

The above argument, however, does not mean that it is impossible to obtain a massless $U(1)$ gauge field in the low-energy spectrum. Each line bundle in a compactification leaves a $U(1)$ gauge field, and each massless fluctuation of the $B$-field or Ramond–Ramond field couples to a linear combination of those $U(1)$ gauge field through the generalized Green–Schwarz mechanism $[29]$. If there is an abundant supply of $U(1)$ gauge fields compared with the number of the bulk moduli fields, the $U(1)$ gauge fields with no moduli-field counterpart remain massless.

Reference $[12]$ considered an $SU(5) \times U(1)_Y \times U(1)_2$-bundle compactification of the Heterotic $E_8 \times E'_8$ string theory. The $SU(5) \times U(1)_Y$ bundle is in one of $E_8$, and another line bundle has a structure group $U(1)_2$ in $E'_8$. The first Chern classes of the two line bundles are chosen to be parallel in $H^{1,1}(Z)$, so that the gauge fields of both $U(1)_Y$ and $U(1)_2$ couple to the one and the same linear combination of the $B$-field fluctuations: $B^{(2)} \propto c_1(L_Y) \propto c_1(L_2)$. This $B$-field fluctuation absorbs only a linear combination of the two massless $U(1)$ gauge fields, and the other combination remains massless. This gauge field, which is a linear combination of gauge fields in the visible $E_8$ and the hidden $E'_8$, can be identified with the massless hypercharge gauge field. The ratio of the hypercharges of the fields in the visible sector is determined by the charges of the original $U(1)_Y \subset SU(5)_{GUT}$ gauge field; hence the standard explanation of the hypercharge quantization in $SU(5)$ unified theories—the original motivation of unified theories—is maintained.

The $\mathbb{C}^3/\mathbb{Z}_3$ model in Type IIB string theory in $[13]$ breaks an $SU(6)$ symmetry by turning

\[ h^{1,1}(Z) \] Kähler moduli chiral multiplets and one dilaton chiral multiplet. Under the Heterotic–F-theory duality, an elliptic-fibred $Z$ on a base 2-fold $B$ is mapped to a K3-fibred Calabi-Yau 4-fold $X'$ on $B$. Heterotic compactification has an F-theory dual only when line bundles are trivial in the elliptic fibre direction (if they had non-trivial first Chern classes in the fibre direction, vector bundles would not be stable in the small fibre limit). Thus, the Kähler moduli multiplet associated with the size of the elliptic fibre does not participate in the generalized Green–Schwarz mechanism. So, $(h^{1,1}(Z) - 1) = h^{1,1}(B)$ Kähler moduli chiral multiplets and the dilaton chiral multiplet can absorb massless $U(1)$ gauge fields in the Heterotic compactification. On the other hand, the Type IIB compactification has $h^{1,1}(X) = h^{1,1}(B) + 1$ chiral multiplets containing fluctuations of the Ramond–Ramond 4-form or 2-form. Thus, the same number of massless gauge fields are absorbed in both descriptions; otherwise those two descriptions were not dual!
The SU(6) symmetry is broken down to SU(3) × SU(2) × U(1) × U(1), the non-Abelian part of which is identified with those of the standard model gauge group. The chiral multiplet that describes the blow-up of the \( \mathbb{C}^3/\mathbb{Z}_3 \) singularity (and hence the size of the \( \mathbb{C}P^2 \) cycle) absorbs a linear combination of the two U(1) gauge fields, and the other linear combination remains massless. This massless gauge field can be identified with that of the hypercharge. Models in [14, 15, 16] adopt essentially the same strategy in maintaining a massless U(1) gauge field in the low-energy spectrum. One should keep in mind that how many massless U(1) gauge field remains massless is a global issue.

### 2.1.2 Normalization of the Hypercharges

The overall normalization of hypercharges—not just the quantized ratio among them—is also an important prediction of supersymmetric unified theories. The SU(5)_{\text{GUT}} GUT’s predict that

\[
\frac{1}{(5/3)\alpha_Y} = \frac{1}{\alpha_{\text{GUT}}} = \frac{1}{\alpha_C} = \frac{1}{\alpha_L},
\]

which is called the GUT relation. The factor \( (5/3) \) in the denominator comes from

\[
q_Y = \text{diag} \left( -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2} \right), \quad \text{tr}(q_Y^2) = \frac{5}{3},
\]

In this article we imply \( \text{tr} = T_R^{-1}\text{tr}_R \) for any representations \( R \) and, in particular, \( \text{tr} = 2\text{tr}_F \) for fundamental representations of SU(\( N \)) symmetries, \( \text{tr} = \text{tr}_{\text{vect.}} \) for vector representations of SO(2\( N \)) symmetries and \( \text{tr} = (1/30)\text{tr}_{\text{adj.}} \) for adjoint representations of \( E_8 \) and SO(32).

Now, when considering the idea of section 2.1.1 to maintain a massless U(1) gauge field at low energies, the low-energy U(1) gauge symmetry is not exactly the same as the U(1) hypercharge of SU(5) unified theories. Let us first pick up an example in the Heterotic string compactification that we mentioned above. The linear combination of U(1) gauge fields that becomes massive is

\[
\left( db^k - \frac{1}{4\pi} \text{tr}(q_Y^2)Q_Y^k A_Y - \frac{1}{4\pi} \text{tr}(q_Z^2)Q_Z^k A_Z \right)^2,
\]

\footnote{The fractional D3-branes at the \( \mathbb{C}^3/\mathbb{Z}_3 \) singularity are not just D7-branes wrapped on the vanishing 4-cycle isomorphic to \( \mathbb{C}P^2 \). One of the three fractional D3-branes at this singularity should be interpreted as a two anti-D7-branes wrapped on the vanishing cycle with a rank-2 vector bundle turned on \([36]\). Thus, this model does not immediately fit to the discussion so far that is based on large-volume compactification. However, we only discuss symmetry breaking pattern and counting of massless U(1) gauge fields, and in that context, the difference between anti-D7 branes and D7-branes does not make an essential difference. The same is true for other models such as those in [14, 15, 16].}
where
\[
\left( \frac{F}{2\pi} \right)_{L_Y} = q_Y c_1(L_Y) = q_Y Q^k_Y \omega_k, \quad \left( \frac{F}{2\pi} \right)_{L_2} = q_2 Q^k_Y \omega_k. \tag{11}
\]

The assumption that \( c_1(L_Y) \propto c_1(L_2) \) in \( H^{1,1}(Z) \) allows us to express the first Chern classes by using the same set of linear combination coefficients \( Q^k_Y \). Gauge fields \( A_Y \) and \( A_2 \) have kinetic terms
\[
\mathcal{L} = -\frac{1}{16\pi\alpha} \text{tr}(q_Y^2) F^2_Y - \frac{1}{16\pi\alpha'} \text{tr}(q_2^2) F^2_2 \tag{12}
\]
in the effective Lagrangian in \( D = 4 \), where \( \alpha \) and \( \alpha' \) are effective fine structure constants in the visible and hidden sectors, and hence the canonically normalized gauge fields \( \tilde{A}_Y \) and \( \tilde{A}_2 \) are obtained from \( A_Y \) and \( A_2 \) by rescaling them by \( \sqrt{\frac{4\pi\alpha}{\text{tr}(q_Y^2)}} \) and \( \sqrt{\frac{4\pi\alpha'}{\text{tr}(q_2^2)}} \), respectively. Thus, the canonically normalized massive vector field \( A_{\text{massive}} \) and its orthogonal complement \( \tilde{A}_Y \) are given in terms of \( \tilde{A}_Y \) and \( \tilde{A}_2 \) by
\[
\begin{pmatrix}
A_{\text{massive}} \\
\tilde{A}_2
\end{pmatrix} = \frac{1}{\sqrt{\alpha \text{tr}(q_Y^2) + \alpha' \text{tr}(q_2^2)}} \begin{pmatrix}
\sqrt{\alpha \text{tr}(q_Y^2)} & \sqrt{\alpha' \text{tr}(q_2^2)} \\
\sqrt{\alpha' \text{tr}(q_2^2)} & -\sqrt{\alpha \text{tr}(q_Y^2)}
\end{pmatrix} \begin{pmatrix}
\tilde{A}_Y \\
\tilde{A}_2
\end{pmatrix}. \tag{13}
\]

It is \( A_Y \) that remains massless in low-energy effective theory. Fields in the visible sector are coupled to the massless gauge field \( A_Y \) through the original hypercharge gauge field \( \tilde{A}_Y \):
\[
\partial - iq_Y \sqrt{\frac{4\pi\alpha}{\text{tr}(q_Y^2)}} \tilde{A}_Y \rightarrow \partial - iq_Y \sqrt{\frac{4\pi\alpha}{\text{tr}(q_Y^2)}} \sqrt{\frac{\alpha'}{\alpha \text{tr}(q_Y^2) + \alpha' \text{tr}(q_2^2)}} A_Y. \tag{14}
\]

Thus, the gauge coupling constant of this massless hypercharge gauge field is given by
\[
\frac{1}{\text{tr}(q_Y^2)\alpha_Y} = \frac{1}{\alpha} + \frac{1}{\alpha' \text{tr}(q_2^2)}; \tag{15}
\]

The above discussion is essentially the same as calculating the QED coupling constant in the Weinberg–Salam model. In the weakly coupled Heterotic \( E_8 \times E_8' \) string theory, the gauge coupling constants of the visible and hidden sector \( E_8 \), namely, \( \alpha = \alpha_{\text{GUT}} = \alpha_{E_8} \) and \( \alpha' = \alpha_{E_8}' \) are the same at the tree level, and hence the second term in (15) makes the hypercharge coupling constant weaker by of order 100% [12]. The GUT relation (8) is not satisfied at all.

Let us now take an example of [13] in Type IIB string local singularity. There, \( U(1)_Y \) massless gauge field comes essentially from a subgroup of \( U(6) \) generated by
\[
q_{U(6)} = \text{diag} \left( \begin{array}{cccccc}
-\frac{1}{3}, & -\frac{1}{3}, & -\frac{1}{3}, & -\frac{1}{2}, & -\frac{1}{2}, & -1
\end{array} \right). \tag{16}
\]
When all the six fractional D3-branes are assumed to have the same gauge coupling constant, the massless gauge field has a coupling constant given by

$$\frac{1}{\text{tr}(q_Y^2)\alpha_{U(6)}} = \frac{\text{tr}(q_{U(6)}^2)}{\text{tr}(q_Y^2)\alpha_{C,L}} = \frac{11}{5} \alpha_{C,L}.$$ \hspace{1cm} (17)

This is much smaller than those of $SU(3)_C \times SU(2)_L$, and this is because of the extra last entry of (16).

In summary, when the $SU(5)_{GUT}$ symmetry is broken by a line bundle in the $U(1)_Y$ direction, the $U(1)_Y$ gauge field tends to be massive by absorbing the Kähler moduli along the direction of the first Chern class of the line bundle. By considering compactification with multiple line bundles, however, it is possible to keep a massless $U(1)$ gauge field, under which the ratio of the charges of the standard-model particles is that of the hypercharges. The overall normalization of the new hypercharges, or equivalently the gauge coupling constant of the new massless hypercharge gauge field, is different from the standard prediction of $SU(5)_{GUT}$ unified theories. We call it the $U(1)_Y$ problem.

2.2 Solving the $U(1)_Y$ Problem with a Strongly Coupled $U(1)$ Gauge Field

Gauge coupling constants are functions of moduli fields in string theory, and hence the GUT relation may be satisfied somewhere in the moduli space. Since we know that the first term in (15) satisfies the GUT relation, it is clear that the GUT relation is satisfied approximately, if the contribution from the second term in (15) is negligible compared with the first term. In other words, as long as the extra $U(1)$ gauge symmetry that mixes into the hypercharge is strongly coupled at the compactification scale, the effective gauge coupling constant of hypercharge at

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*It is still possible to maintain the GUT relation in Type IIB compactification, if we give up Georgi–Glashow SU(5) unification. For example, one can take $q_Y = q_{B-L}/2 - q_R$, where $q_{B-L}$ is a generator of $U(1)_{B-L} \subset SU(3)_C \times U(1)_{B-L}SU(4)_C$, and $q_R$ that of $U(1)_R \subset SU(2)_R$. One could imagine that the gauge coupling constants of all of $SU(4)_C \times SU(2)_L \times SU(2)_R$ are the same, if $4 + 2 + 2$ D7-branes are wrapped on one and the same holomorphic 4-cycle with parametrically large volume. If this is the case, then the GUT relation is satisfied because $\text{tr}((q_{B-L}/2 - q_R)^2) = 5/3$. (This fact was exploited in a Type IIA model.) In order to obtain appropriate spectrum, however, some of the 7-branes forming $SU(4)_C \times SU(2)_L \times SU(2)_R$ have to be anti-7-branes, because there are sum rules in the net chirality of various representation if they are all D7-branes. Thus, the volume of the 4-cycle has to be comparable to the string length, and there, values of $B$-fields integrated over various 2-cycles also have significant contributions to gauge couplings of (subgroups of) $SU(4)_C \times SU(2)_L \times SU(2)_R$. Thus, it is not obvious whether such quiver standard model in Type IIB string theory naturally predicts the GUT relation. (cf [16])"
Figure 1: This figure, borrowed from [39], shows renormalization-group evolution of the three gauge coupling constants of the MSSM. Supersymmetry partners of the Standard-Model particles are assumed to be around 100 GeV–1 TeV, and 2-loop renormalization group equation was used for calculation. $\pm 2\sigma$ error bar associated with the measurements of the QCD coupling is shown as the three parallel trajectories for $1/\alpha_3$. (See [39] for more details.)

low-energy is not very much different from the ordinary prediction of SU(5)$_{\text{GUT}}$ unified theories. There are such field-theory models in the literature (eg. [38]).

As one can see in Figure 1, the three gauge coupling constants of the minimal supersymmetric standard model do not unify exactly at any energy scale around the GUT scale; at the energy scale $M_{2-3}$ in the figure, where $\alpha_C$ and $\alpha_L$ are equal, $(5/3) \times \alpha_Y$ is different from the others by 2–4%. Thus, the contribution from the second term in $(3/5)/\alpha_Y$ is phenomenologically acceptable. Furthermore, the extra contribution is supposed to be positive in $1/\alpha$, which is really the case if the deviation from the GUT relation is due to the mixing with an extra strongly coupled U(1) gauge field. We will see in the following sections that the extra U(1) is strongly coupled and hence the extra contribution to $1/\alpha_Y$ is small enough for some classes of string vacua in certain region of its moduli space.

Now one might wonder what is the point of maintaining the SU(5) unification. This is
certainly a legitimate question. Unified theories can predict one of the three gauge coupling constants of $SU(3)_C \times SU(2)_L \times U(1)_Y$ in terms of the other two, because there are only 2 parameters—the GUT scale and the unified gauge coupling constant. What is the point of considering a unified framework if one allows oneself to introduce an extra (moduli) parameter that change the $U(1)_Y$ gauge coupling? Predictability on the gauge coupling constants seems to be lost. As we will see in the following sections, this is actually not the case. In the Heterotic–M-theory compactification, the hidden sector gauge coupling is strong, due to the warping in the 11-th direction. In F-theory compactifications, which is motivated (as opposed to the perturbative Type IIB Calabi–Yau orientifold compactification) by the up-type Yukawa couplings [17], the dilaton vev cannot be small everywhere in the internal manifold. Thus, having an extra strongly coupled U(1) gauge theory is extremely natural. Parametrically large volume for compactification is required in order to account for the little hierarchy between the GUT scale and the Planck scale, and a parametrically large volume to string length ratio can render the visible sector $SU(5)_{GUT}$ weakly coupled, in contrast to other strongly coupled sectors $SU(5)_{GUT}$ [20, 21].

From a perspective of phenomenology, the framework with a unified $SU(5)$ and a strongly coupled extra U(1) symmetries says more than just having $SU(3)_C \times SU(2)_L \times U(1)_Y$ massless gauge field at low energy with the GUT relation. The GUT gauge bosons exist around the energy scale of the gauge coupling unification, leading to dimension-6 proton decay. Since the rate of dimension-6 decay is proportional to the fourth power of the unification scale, the rate, and the proton lifetime is very sensitive to where the unification scale really is. If we take a closer look at where the “unification scale” is, it is important to note that the extra contribution to $(3/5)/\alpha_Y$ is always positive. Thus, “the unification scale” is more likely to be around $M_{2-3}$ in Figure 1 than $M_{1-2} \simeq 2 \times 10^{16}$ GeV conventionally referred to as the GUT scale. Although one has to take account of threshold corrections and non-perturbative corrections in order to determine the GUT gauge boson mass (or the Kaluza–Klein scale) precisely, it is unlikely that the scale is as high as $M_{1-2}$ without an accidental cancellation between the threshold/non-perturbative corrections and the tree-level deviation from the GUT relation. This implies that the proton decay may be faster considerably than estimation based on $M_{1-2}$ as the GUT scale. All the statements above on proton decay is valid whether the framework is implemented in the Heterotic–M-theory or in F-theory compactifications. See also related comments in the following sections.
3 Heterotic-M Theory Vacua

The Heterotic $E_8 \times E_8'$ string theory is compactified on a Calabi–Yau 3-fold $Z$ to yield a D = 4 effective theory with $\mathcal{N} = 1$ supersymmetry. Vector bundles $V_1$ and $V_2$ have to be turned on in both visible and hidden $E_8$ symmetries, so that

$$c_2(V_1) + c_2(V_2) = c_2(TZ).$$

(18)

Apart from special cases,

$$\int_Z J \wedge \left( c_2(V_1) - \frac{1}{2} c_2(TZ) \right) = - \int_Z J \wedge \left( c_2(V_2) - \frac{1}{2} c_2(TZ) \right)$$

(19)

does not vanish for a Kähler form $J$ of the Calabi–Yau 3-fold $Z$. When (19) is not zero, it is known (as we review later) that the gauge coupling of one of the two $E_8$ gauge groups is stronger than that of the other $E_8$. For a large string coupling, $g_s$, the difference becomes significant, and in the limit of the largest possible $g_s$, one of the gauge couplings of D = 4 effective theory is really strongly coupled \[40, 41\]. Thus, if the $E_8$ gauge group with the weaker gauge coupling is identified the visible sector, $\alpha_{E_8} = \alpha_{\text{GUT}}$, and the other $E_8'$ symmetry is strongly coupled, and $1/\alpha_{E_8'}$ in (15) is small; the GUT relation is maintained approximately. The purpose of this section is to check if this idea really works.

3.1 In Language of the Weak Coupling Heterotic String Theory

A vector bundle $V_5$ whose structure group is $\text{SU}(5)_{\text{bd}} \subset E_8$ breaks the $E_8$ symmetry down to the commutant of the $\text{SU}(5)_{\text{bd}}$, $\text{SU}(5)_{\text{GUT}}$. The $\text{SU}(5)_{\text{GUT}}$ symmetry is further broken down to $\text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y$ by turning on a line bundle $L_Y$ in the hypercharge direction. The $E_8$ super Yang–Mills fields of D = 10 Heterotic string theory yield all the gauge and matter multiplets except just one, $\text{U}(1)_Y$ vector multiplet. The $\text{U}(1)_Y$ symmetry may remain unbroken as a global symmetry, but the gauge field absorbs a fluctuation of the $B$-field, and becomes massive. Whether the $\text{SU}(5)_{\text{GUT}}$ symmetry is broken by a flat bundle or by a line bundle makes a big difference [11].

9 An unbroken subgroup of this $E_8$ symmetry may lead to dynamical supersymmetry breaking. The energy scale of the supersymmetry breaking $\Lambda_{\text{DSB}}$ is, however, determined by a combination $(2\pi/b_0 \alpha_{E_8'})$ where $b_0$ is the 1-loop beta function of the gauge coupling of the unbroken symmetry; the coupling $\alpha_{E_8'}$ alone does not determine the scale. Thus, the supersymmetry breaking scale can be much lower than the Kaluza–Klein scale when this hidden sector is nearly conformal, $b_0 \approx 0$. In model-building in F-theory, there is no such tight relation between the supersymmetry breaking scale and the deviation from the GUT relation. This may be regarded as a motivation for model building in F-theory.
| multiplets | $Q$ | $\mathcal{U}$ | $\mathcal{E}$ | $\mathcal{D}$ | $L$ | $H_u$ | $H_d$ |
|-----------|-----|-------------|-------------|-------------|-----|-------|-------|
| bundles   | $V_5 \otimes L^{\frac{1}{2}}$ | $V_5 \otimes L^{-\frac{1}{2}}$ | $V_5 \otimes L$ | $\wedge^2 V_5 \otimes L^{\frac{1}{2}}$ | $\wedge^2 V_5 \otimes L^{-\frac{1}{2}}$ | $\wedge^2 V_5 \otimes L^{\frac{1}{2}}$ | $\wedge^2 V_5 \otimes L^{-\frac{1}{2}}$ |

Table 1: Vector bundles of chiral multiplets in supersymmetric standard models. For a realistic model, the vector bundle $V_5$ cannot be generic; otherwise, there is a problem of dimension-4 proton decay. For example, a $\mathbb{Z}_2$ symmetry (matter parity or R-parity) or an extension structure removes virtually all the dimension-4 proton decay operators [17, 42, 18]. We do not go into details because such extra structures of the bundle $V_5$ are not essential to the gauge coupling unification, the main theme of this article. We used a notation $L$ for $L_Y$ in the text to save space in this table.

References [11, 12] proposed a solution to this problem. Here, we briefly review the construction of [12] in order to set the notation in this article.

The (weakly coupled) $E_8 \times E_8'$ Heterotic string theory is compactified on a Calabi–Yau 3-fold $Z$, whose $\pi_1(Z)$ does not have to be non-trivial. A vector bundle $V_1$ is turned on in the visible sector $E_8$, which consists of a rank-5 vector bundle $V_5$ and a line bundle $L$. The $D = 10$ $E_8$ super Yang–Mills multiplet yields all the chiral multiplets necessary in supersymmetric standard model; see Table 1. $SU(3)_C \times SU(2)_L$ gauge fields remain massless. A vector bundle $V_2$ in the “hidden sector” $E_8'$ should contain a line bundle $L_2$ (and possibly another bundle $V'$ whose structure group commutes with the $U(1)_2$ structure group of $L_2$) which satisfies

$$c_1(L_2) \propto c_1(L_Y) \in H^{1,1}(Z). \quad (20)$$

We set the normalization of the generator $q_2$ for $L_2$ as in [11], using $c_1(L_Y)$. The second Chern classes are given by

$$c_2(V_1) = c_2(V_5) - \frac{\text{tr}(q_2^2)}{4} c_1(L_Y)^2, \quad (21)$$

$$c_2(V_2) = c_2(V') - \frac{\text{tr}(q_2^2)}{4} c_1(L_Y)^2, \quad (22)$$

and they have to satisfy the consistency condition [18]. An explicit example of a Calabi–Yau 3-fold $Z$ and vector bundles on it is found in [12]. In order to obtain the spectrum of supersymmetric standard model, bundles introduced so far have to satisfy

$$\int_Z c_1(L_Y) \wedge c_2(TZ) = 0, \quad \int_Z c_1(L_Y) \wedge c_2(V_5) = 0, \quad \int_Z c_1(L_Y)^3 = 0. \quad (23)$$

Dimensional reduction of a Calabi–Yau compactification leaves a dilaton chiral multiplet $S$.
and $h^{1,1}(Z)$ Kähler moduli chiral multiplets $T^k$ ($k = 1, \cdots, h^{1,1}(Z)$):

\[
S = \frac{M_G^2 \alpha'}{4} \left( \frac{1}{e^{2\phi}} \langle \text{vol}(Z) \rangle - i a \right),
\]

\[
T^k = \frac{1}{2\pi} \left( \alpha^k + i b^k \right) \quad (k = 1, \cdots, h^{1,1}(Z)),
\]

where $\phi$ and $a$ are dilaton fluctuation and model-independent axion of the Heterotic string theory; $M_G \simeq 2.4 \times 10^{18}$ GeV is given by

\[
\frac{M_G^2}{2} = \frac{\langle \text{vol}(Z) \rangle}{2\kappa^2_0 g_s^2} = \frac{\langle \text{vol}(Z) \rangle}{(2\pi)^7 \alpha' g_s^4}.
\]

\[
\alpha^k \text{ and } b^k \text{ parametrize the metric and } B\text{-field on } Z \text{ by}
\]

\[
J = l_s^2 \alpha^k \omega_k, \quad B = l_s^2 b^l \omega_l,
\]

where $\omega_k$ ($k = 1, \cdots, h^{1,1}(Z)$) are basis of $H^{1,1}(Z)$, and $J$ is a Kähler form.\(^{10}\)

The kinetic term of the $B$-field contains

\[
\left| \left( dB^{(2)} - \frac{\alpha'}{4} (\omega_{YM1} + \omega_{YM2}) \right) \right|^2 \rightarrow \left| \left( db^k - \frac{Q_Y^k}{4\pi} \left( \text{tr}(q_Y^2)A_Y + \text{tr}(q_2^2)A_2 \right) \right) \omega_k l_s^2 \right|^2,
\]

where $A_Y$ and $A_2$ are gauge fields associated with the generators $q_Y$ and $q_2$, respectively. A linear combination of vector multiplets, $V_{\text{massive}} \equiv \text{tr}(q_Y^2)V_Y + \text{tr}(q_2^2)V_2$, enters the Kähler potential as in

\[
K = -M_G^2 \ln \left( \frac{1}{3!} \int_Z J J \bar{J} \right), \quad \bar{J} = -\pi l_s^2 \omega_k \left( T^k + T^{k\dagger} + \frac{Q_Y^k}{8\pi^2} V_{\text{massive}} \right),
\]

and becomes massive. On the other hand, these vector multiplets do not have a similar coupling with the dilaton in the Kähler potential; although they could enter the Kähler potential as in

\[
K = -\ln \left( S + S^\dagger + \frac{Q^S}{32\pi^2} V_{\text{massive}} \right), \quad Q^S = \int_Z c_1(L_Y) \left( c_2(V_1) - \frac{1}{2} c_2(TZ) \right),
\]

\[\text{vol}(Z) = (1/3!) \int_Z J^3 \text{ in this definition. Note that the Kähler form in } \text{ is } \omega = -ig_{\alpha\beta}dz^\alpha \wedge d\bar{z}^\beta, \text{ different by a factor } -1.\]
$Q^S$ is proportional to $U(1)_1$−[non-Abelian]$^2$ mixed anomalies with $SU(3)_C$ (and $SU(2)_L$) as the non-Abelian gauge group (see also footnote 14), and hence vanishes in vacua with spectra of supersymmetric standard models.

Since only one linear combination, $V_{\text{massive}}$ becomes massive, another linear combination of the gauge fields $A_Y$ and $A_2$ remains massless. All the particles in Table I are charged under this massless $U(1)$ gauge symmetry through its $A_Y$ component, and hence the ratio of the $U(1)$ charges remains the same. This massless $U(1)_{\tilde{Y}}$ vector field is regarded as the hypercharge gauge field of the Standard Model [11, 12]. The only problem of this solution is that the gauge coupling constants of $SU(3)_C \times SU(2)_L \times U(1)_Y$—given as functions of moduli $S$ and $T^k$—do not satisfy (generically) the GUT relation. To see this, note that the gauge kinetic term of the two $U(1)$ gauge fields $A_Y$ and $A_2$ is

$$- \frac{1}{4} \begin{pmatrix} F_Y \\ F_2 \end{pmatrix} \text{Re} \left( \begin{array}{c} \text{tr}(q_Y^2) \left( S + T - \frac{\text{tr}(q_Y^2)}{3} A \right) \\ \frac{\text{tr}(q_Y^2) \text{tr}(q_2^2)}{6} A \\ \text{tr}(q_2^2) \left( S - T - \frac{\text{tr}(q_2^2)}{3} A \right) \end{array} \right) \begin{pmatrix} F_Y \\ F_2 \end{pmatrix} \right) \quad (33)$$

in the large volume limit, where

$$T \equiv \frac{1}{4} T^k \int_Z \omega_k \wedge \left( c_2(V_1) - \frac{1}{2} c_2(TZ) \right), \quad (34)$$

$$A \equiv \frac{1}{4} T^k \int_Z \omega_k \wedge c_1(L_Y)^2. \quad (35)$$

We only consider $\text{Re}A \propto \int_Z J \wedge c_1(L_Y)^2 = 0$ for simplicity for the moment. Following the process described in section 2, one can see that the massless linear combination is

$$A_Y \propto \sqrt{\frac{\text{Re}(S + T)}{\text{Re}(S - T)}} A_Y - \sqrt{\frac{\text{Re}(S - T)}{\text{Re}(S + T)}} A_2, \quad (36)$$

and the gauge coupling constant is given by

$$\frac{3}{g_Y^2} = \text{Re}(S + T) + \frac{\text{tr}(q_Y^2)}{\text{tr}(q_2^2)} \text{Re}(S - T). \quad (37)$$

Note that $1/g_C^2$ and $1/g_L^2$ in the visible sector are given by

$$\frac{1}{g_C^2} = \frac{1}{g_L^2} = \text{Re} f = \text{Re}(S + T) \quad (38)$$

11Later, we will see that it is an important assumption necessary for the gauge coupling unification.
in the large volume limit. When the hidden sector has an unbroken non-Abelian symmetry group, its gauge coupling constant is given by

\[
\frac{1}{g'^2} = \text{Re} f' = \text{Re} \left( S + \frac{1}{4} T^k \int_Z \omega_k \wedge \left( c_2(V_2) - \frac{1}{2} c_2(TZ) \right) \right) = \text{Re}(S - T). \tag{39}
\]

The U(1)$_Y$ gauge coupling in (37) is given just as the discussion in section 2.1. In the weakly coupled Heterotic string theory, the tree-level coupling Re$S$ dominates, with 1-loop corrections $\propto \text{Re} T$ being subleading. Thus, ignoring $\text{Re} T$ in

\[
\frac{3}{g'^2} = \left( 1 + \frac{\text{tr}(q_2^2)}{\text{tr}(q_2^2)} \right) \text{Re} S + \left( 1 - \frac{\text{tr}(q_3^2)}{\text{tr}(q_3^2)} \right) \text{Re} T, \tag{40}
\]

\[
\approx \left( 1 + \frac{\text{tr}(q_2^2)}{\text{tr}(q_2^2)} \right) \frac{1}{g'^2 c_L}, \tag{41}
\]

the GUT relation is badly violated; the factor in the parenthesis on the right-hand side is different from 1 by of order unity for the model in [12]. If the 1-loop threshold correction, the second term in (11), were to partially cancel the tree level gauge coupling so that the gauge coupling constants of the MSSM apparently satisfy the GUT relation, it sounds very artificial. This is the Heterotic-string version of the U(1)$_Y$ problem.

Reference [11] points out that the GUT relation is maintained approximately if $\text{tr}(q_2^2)$ is chosen much larger than $\text{tr}(q_3^2)$. While this is true, we will see in the following, that the approximate GUT relation is actually maintained even if $\text{tr}(q_2^2)$ and $\text{tr}(q_3^2)$ are comparable.

### 3.2 In the Strongly Coupled Heterotic-M Theory

#### 3.2.1 Strongly Coupled Hidden Sector

When the gauge coupling constant in the hidden sector is way stronger than that of the visible sector for some reason, the second term of (15) and (37) is negligible, and the GUT relation is approximately satisfied; that was the idea of section 2, phrased in the context of the Heterotic string theory.

Such a disparity between the gauge coupling constants naturally happen in strongly coupled Heterotic $E_8 \times E_8$ string theory. The Bianchi identity of the NS–NS 2-form field requires that the total sum of the second Chern classes vanish, but they are not necessarily distributed equally to the visible and hidden sector. In general, (19) does not vanish, and the asymmetric distribution of the second Chern classes provide sources for the configuration of the Ramond–
Ramond 3-form field in the bulk of the Heterotic-M theory.

$$G_{\alpha\bar{\beta}\gamma\delta} = \begin{cases} \frac{(2\pi)^2}{\sqrt{2\pi}} \frac{(4\pi)}{(4\pi)}^{2/3} (c_2(V_1) - \frac{1}{2}c_2(TZ))_{\alpha\bar{\beta}\gamma\delta} & \text{(for } 0 < x_{11} < \pi \rho), \\ -\frac{(2\pi)^2}{\sqrt{2\pi}} \frac{(4\pi)}{(4\pi)}^{2/3} (c_2(V_1) - \frac{1}{2}c_2(TZ))_{\alpha\bar{\beta}\gamma\delta} & \text{(for } -\pi \rho < x_{11} < 0). \end{cases}$$

(42)

Coefficients are taken from [43]. The non-zero 4-form field strength of the Ramond–Ramond field in the bulk, in turn, becomes the source of metric. The metric of $D = 11$ gravity is expanded as

$$ds^2 = e^{b(x_{11})} dx^2 + 2(g_{\alpha\bar{\beta}} + h_{\alpha\bar{\beta}}) dz^\alpha d\bar{z}^{\bar{\beta}} + e^{k(x_{11})} dx_{11}^2,$$

(43)

and at the linear order in $\kappa^{2/3}$, first order deformation $b(x_{11})$ and $h(x_{11}, z, \bar{z})$ follow the equations

$$\partial_{11} b = \frac{\sqrt{2}}{24} \alpha,$$

(44)

$$\partial_{11} h_{\alpha\bar{\beta}} = -\frac{1}{\sqrt{2}} \left( i \Theta_{\alpha\bar{\beta}} - \frac{1}{12} \alpha g_{\alpha\bar{\beta}} \right).$$

(45)

Here, $\Theta_{\alpha\bar{\beta}} := 2ig^{\delta\gamma}G_{\alpha\bar{\gamma}\beta\delta}$ and $\alpha = 2ig^{\delta\gamma}\Theta_{\alpha\bar{\gamma}}$ as in [40]. If

$$\Theta_{\alpha\bar{\beta}} \propto g_{\alpha\bar{\beta}},$$

(46)

the last one above becomes $\partial_{11} h_{\alpha\bar{\beta}} = -\sqrt{2}/24\alpha g_{\alpha\bar{\beta}}$, and hence the $(g_{\alpha\bar{\beta}} + h_{\alpha\bar{\beta}})$ part is of the form $e^f g_{\alpha\bar{\beta}}$ with $f$ satisfying $\partial_{11} f = -\sqrt{2}/24\alpha$ [41]. $k(z, \bar{z})$ should have the same $(z, \bar{z})$ dependence as $f$, and its $x_{11}$ can be chosen so that $k = f$ [40] [41]. Thus, the metric has the warped structure [41]:

$$ds^2 = e^{-f(x_{11})} e^{-\frac{2\phi}{\alpha}} dx^2 + e^{f(x_{11})}(e^{-\frac{2\phi}{\alpha} g_{\alpha\bar{\beta}}} dz^\alpha d\bar{z}^{\bar{\beta}} + e^{\frac{4\phi}{\alpha}} dx_{11}^2),$$

(47)

where

$$e^{f(x_{11})} = \left( 1 - \frac{\alpha}{8\sqrt{2}} x_{11} \right)^\frac{2}{3}.$$  

(48)

The volume of Calabi–Yau 3-fold varies over $x_{11}$, and in particular, decreases monotonically. It follows that

$$\frac{\text{vol}(Z)|_{x_{11}=\pi \rho}}{\text{vol}(Z)|_{x_{11}=0}} = \left( 1 - \frac{\alpha}{8\sqrt{2}} \pi \rho \right)^2,$$

(49)

and the gauge coupling constants of the visible and hidden sectors in $D = 4$ effective theory are given by

$$\frac{1}{\alpha_{\text{GUT}}} = \frac{\text{vol}(Z)|_{x_{11}=0}}{(4\pi \kappa^2)^{\frac{2}{3}}}, \quad \frac{1}{\alpha_{\text{hidden}}} = \frac{\text{vol}(Z)|_{x_{11}=\pi \rho}}{(4\pi \kappa^2)^{\frac{2}{3}}} = \left( \frac{1 - \frac{\alpha}{8\sqrt{2}} \pi \rho}{\alpha_{\text{GUT}}} \right)^2.$$  

(50)
Larger volume at \( x_{11} = 0 \) makes the visible sector coupling weaker, while the hidden sector coupling remains strong \([31][41][41]\).

The expression for the two gauge coupling constants in the weakly coupled Heterotic theory, \( (38) \) and \( (39) \) captures the warped factor effect. Indeed,

\[
\frac{1}{g^2} - \frac{1}{g'^2} = \frac{1}{4\pi} \left( \frac{\text{vol}(Z)|_{x_{11}=0}}{(4\pi\kappa^2)^{\frac{4}{3}}} - \frac{\text{vol}(Z)|_{x_{11}=\pi\rho}}{(4\pi\kappa^2)^{\frac{4}{3}}} \right) \approx \frac{1}{4\pi} \frac{\text{vol}(Z)|_{x_{11}=0}}{(4\pi\kappa^2)^{\frac{4}{3}}} \frac{\pi\rho}{4\sqrt{2} \alpha},
\]

\[
= \frac{1}{4\pi} \frac{\text{vol}(Z)|_{x_{11}=0}}{(4\pi\kappa^2)^{\frac{4}{3}}} \frac{\pi\rho}{4\sqrt{2}} \left( \text{volume form} \right),
\]

\[
= \frac{1}{4\pi} \frac{1}{(4\pi\kappa^2)^{\frac{2}{3}} \sqrt{2}} \int_Z J \wedge G \propto \frac{1}{4\pi l_s^2} \int_Z J \wedge \left( c_2(V_1) - \frac{1}{2} c_2(TZ) \right)
\]

(51)
in Heterotic-M theory language agrees with the result of weakly coupled Heterotic string theory, \( \text{Re}(S + T) - \text{Re}(S - T) = 2\text{Re}T \) (up to a proportionality factor)\(^{12}\) Here, higher order \( O(\kappa^4) \) corrections are ignored.

Although the perturbative expansion of the Heterotic string theory is not reliable for \( g_s > 1 \), the gauge kinetic function is protected by holomorphicity. Only the tree and 1-loop level contributions exist, apart from non-perturbative corrections. They are given by \( S \pm T \) at this level, and the holomorphicity of \( f \) and \( f' \) guarantees that their expressions are right as the perturbative part even in the strong coupling regime. It is true that the physical gauge coupling constants receive higher loop corrections despite the holomorphicity of \( \mathcal{N} = 1 \) supersymmetry. However, such corrections arise only through the rescaling of the vector supermultiplets (\( U(1)_Y \) and \( SU(3)_C \times SU(2)_L \) in this case) and super-Weyl transformation in rewriting Lagrangian in the Einstein frame. The former only involve \( \ln(g_Y) \) and \( \ln(g_C) = \ln(g_L) \) and are always small, while the latter is universal to all the gauge coupling constants. Thus, these corrections, which correspond to higher loops, are not the concern for us.

It appeared in language of weakly coupled Heterotic string theory that a fine-tuning between the tree-level contribution to the gauge coupling \( \text{Re}S \) and 1-loop \( \text{Re}T \) is necessary for the approximate GUT relation. We have seen, however, that the 1-loop \( \text{Re}T \) to the visible sector and \( -\text{Re}T \) to the hidden sector corresponds to the warped factor in the 11-th direction in language of Heterotic M-theory. The warped metric is a consequence of asymmetric distribution of the second Chern class (instanton numbers). Once we have such a geometric meaning,

\[
\frac{\alpha_{\text{hidden}}}{\alpha_{\text{GUT}}} = \frac{\text{vol}(Z)|_{x_{11}=\pi\rho}}{\text{vol}(Z)|_{x_{11}=0}},
\]

(52)

\(^{12}\)They should agree without a proportionality factor, but we have not succeeded in clarifying relation among various conventions in the literature.
and a hierarchy is easily generated between the two gauge coupling constants, unless the “instanton numbers” are distributed precisely the same in the visible and hidden sectors. Thus, actually the approximate GUT relation does not require a fine-tuning; we can understand it as a natural consequence of dynamics of Ramond–Ramond field and metric in the 11th direction.

This is still a predictive framework of GUT. Conventional unified theories use two continuous parameters, $M_{\text{GUT}}$ and $\alpha_{\text{GUT}}$, to fit two gauge coupling constants, e.g., $\alpha_C$ and $\alpha_Y$, and predict the last one, e.g., $\alpha_L$. Now, in this framework, three continuous parameters are involved, namely, $\kappa^2$, $\rho$ and the compactification scale $\text{vol}(Z)|_{x_{11}=0}$, but there are four observable data that are given by those parameters, namely the three gauge coupling constants $\alpha_{C,L}$, and $\alpha_Y$, and the Planck scale. When the three parameters are use to fit $\alpha_{C,L}$ and the Planck scale, this framework predicts that $\alpha_Y$ is quite close to $\alpha_{C,L}$ at the unification scale, and is a little smaller. We know that this prediction is consistent with the precise measurement of the Standard Model gauge couplings at LEP. See Figure 1.

Note that it is not necessary to assume (46) for the disparity between the gauge coupling constants of unbroken non-Abelian symmetries in the visible and hidden sectors; the running of $\text{vol}(Z)$ along the $x_{11}$ direction is always given by $\propto (1-\alpha/8\sqrt{x_{11}})^2$, whether (46) is satisfied or not. However, we keep this assumption because we need another phenomenological requirement, namely $A \propto \int_Z c_1(L_Y)^2 \wedge J = 0$. As one can see from (34–35), $A$ can potentially be of order of $T$. Even if warped metric in the $x_{11}$ direction accounts for why $\text{Re}(S-T) \ll \text{Re}(S+T)$, non-vanishing $A \approx \mathcal{O}(S,T)$ in the kinetic mixing matrix (33) invalidates the scenario in this section. The Kähler form is expanded as in (27), and the coefficients $\alpha_k(x_{11})$ would run differently in the $x_{11}$ direction, if (46) were not satisfied. If $\alpha_k$’s change their ratio among them over the interval $x_{11} \in [0, \pi \rho]$, then $A$ will not vanish even if it does somewhere in the interval. Thus, in order to impose that $A = 0$, we assume (46).

This may not be a problem because $A$ is of order $\kappa^2$ to begin with, and the running effect of $A$ in $x_{11}$ comes only in another $\kappa^2$ order, hence in the next-to-next-to-leading order, $\mathcal{O}(\kappa^4)$. But, for making an error in safe side as well as for simplicity, we maintain the assumption (46) in what follows.

---

$^{13}$A = 0 when the volume of certain cycle vanishes, as we discuss later. In this sufficient condition for $A = 0$, some Kähler moduli are chosen to be zero. If the running of $\alpha_k$ is totally arbitrary, as oppose to the case (46) when $\partial_1 \alpha_k \propto \alpha_k$, some of $\alpha_k$, already chosen to be zero may run into negative value. The Heterotic M theory compactification in this case is geometric in part of the interval of $x_{11} \in [0, \pi \rho]$, while possibly non-geometric for the rest of the interval. Such a situation is avoided when (46) is satisfied.
3.2.2 Generalized Green–Schwarz Coupling in the Heterotic-M Theory

Just like $\alpha^k$, the coefficients of the Kähler form, run in $x_{11}$ when (19) does not vanish, the zero modes from the Ramond–Ramond 3-form field $C^{(3)}$, i.e., $b^k$ in (27), also have non-trivial wavefunction along the $x_{11}$ direction [45]. Thus, one has to check whether the generalized Green–Schwarz coupling (10) of $D=4$ effective theory is modified or not; the discussion so far on the gauge coupling unification is based on an assumption that only the gauge coupling constants $1/g^2$ and $1/g^{'2}$ are affected by the warping geometry, but the linear combination coefficients of the generalized Green–Schwarz coupling (10) are not.

It is sufficient to see the coefficients of the the cross terms of (10), now in the warped compactification of the Heterotic M theory. The cross term originates from the interaction

$$\frac{1}{2\sqrt{2}\pi\kappa^2}\left(\frac{\kappa}{4\pi}\right)^{\frac{3}{2}} \int_{\text{11D}} \tilde{C}^{(6)} \wedge (J_1 \delta(x_{11}) + J_2 \delta(x_{11} - \pi \rho)),$$

(53)

where

$$J_1 = \text{tr}_1 \left( \frac{F}{2\pi} \right)^2 - \frac{1}{2} \text{tr} \left( \frac{R}{2\pi} \right)^2, \quad J_2 = \text{tr}_2 \left( \frac{F}{2\pi} \right)^2 - \frac{1}{2} \text{tr} \left( \frac{R}{2\pi} \right)^2.$$

(54)

$\tilde{C}^{(6)}$ is related to $C^{(3)}$ via $d\tilde{C}^{(6)} = *_{11D} dC^{(3)}$. The interaction above yields the source term to the Bianchi identities

$$dG^{(4)} = -\frac{1}{2\sqrt{2}\pi}\left(\frac{\kappa}{4\pi}\right)^{\frac{3}{2}} \left( \delta(x_{11}) J_1 + \delta(x_{11} - \pi \rho) J_2 \right).$$

(55)

The wavefunction of the zero modes from $C^{(3)}$ have the form [45]

$$C^{(3)} = \omega_k \wedge dx_{11} e^{f(x_{11})/2} b^k(x^\mu) + \cdots.$$

(56)

Here, we maintained only the modes in the chiral multiplets $T^k$, dropping the one in $S$, because $Q^S = 0$ and we are interested in the generalized Green–Schwarz interaction involving the Kähler moduli chiral multiplets. Now, we take the Hodge dual of this zero-mode wavefunctions. They are

$$d\tilde{C}^{(6)} = (\epsilon_{\mu\nu\lambda\kappa} \partial^\mu b^k(x) dx^\nu dx^\lambda dx^\kappa) \wedge (*_6 \omega_k) + \cdots,$$

(57)

where $*_6$ is the Hodge dual on a Calabi–Yau 3-fold $Z$ with the unwarped Kähler metric $g_{\alpha\beta}$. The warped factor $e^{f(x_{11})/2}$ in (56) is cancelled and disappears in $\tilde{C}^{(6)}$ after taking the Hodge dual. Thus, the coefficients of the cross term in (10), which arises from (53), are not suppressed or enhanced by the warped factor $e^{f(x_{11})}$. Therefore, the discussion until section 3.2.1 does not have to be altered.
3.3 Phenomenological Aspects

3.3.1 Fayet–Iliopoulos Parameters and a Global U(1) Symmetry

Let us take a brief look at Fayet–Iliopoulos parameters of those U(1) symmetries. They are given by

\[ \xi_Y = \text{tr} \left( q_Y^2 \right) \frac{M_G^2}{32\pi^2} \left( \frac{2\pi l_s^2}{\text{vol}(Z)} \int_Z c_1(L_Y) \wedge J \wedge J - \frac{g_Y^2}{2} e^{2\tilde{\phi}_Y} Q^S_Y \right), \] \hspace{1cm} (58)

\[ \xi_2 = \text{tr} \left( q_2^2 \right) \frac{M_G^2}{32\pi^2} \left( \frac{2\pi l_s^2}{\text{vol}(Z)} \int_Z c_1(L_Y) \wedge J \wedge J - \frac{g_Y^2}{2} e^{2\tilde{\phi}_Y} Q^S_2 \right), \] \hspace{1cm} (59)

where \( e^{-2\tilde{\phi}_Y} = e^{-2\tilde{\phi}_{\text{vol}(Z)}} / \langle \text{vol}(Z) \rangle \), and they enter in the \( D=4 \) effective theory as

\[ \mathcal{L} = -\frac{1}{2g^2} D_Y^2 - \frac{1}{2g'^2} D_2^2 + D_Y \left( \xi_Y + q_Y \phi^i \phi \right) + D_2 \xi_2. \] \hspace{1cm} (60)

The auxiliary fields \( D_Y \) and \( D_2 \) are rotated just as the vector fields \( A_Y \) and \( A_2 \) are, and the Fayet–Iliopoulos parameters are also re-organized accordingly. Thus, Fayet–Iliopoulos parameters of the \( U(1)_{\text{massive}} \) and \( U(1)_{\tilde{Y}} \) vector multiplets are given by linear combination of \( \xi_Y \) and \( \xi_2 \).

Zero modes from the visible sector—denoted by \( \phi \) above—carry charges under the massless \( U(1)_{\tilde{Y}} \) and massive \( U(1) \), and if there are zero modes from the hidden sector charged under the \( U(1)_2 \) symmetry, then they are also charged under the both. If the Fayet–Iliopoulos parameter of the massive \( U(1) \) does not vanish, and if it is absorbed by vev’s of chiral multiplets, then their vev’s break the \( U(1)_{\tilde{Y}} \) symmetry as well. Thus, the Fayet–Iliopoulos parameters of both \( U(1)_{\text{massive}} \) and \( U(1)_{\tilde{Y}} \) have to vanish, and so do \( \xi_Y \) and \( \xi_2 \) (at the supersymmetric limit).

Geometry of Calabi–Yau 3-fold and vector bundles on it has to be arranged so that just the matter spectrum of the supersymmetric standard model arise from the visible sector. Thus, the \( U(1)_{\tilde{Y}} \left[ \text{SU}(3)_C \right]^2 \) and \( U(1)_Y \left[ \text{SU}(2)_L \right]^2 \) mixed anomalies vanish. It is known that the coefficient of the one-loop Fayet–Iliopoulos parameters \( Q^S \) of (possibly anomalous) \( U(1) \) symmetries are proportional to the \( U(1)-[\text{non-Abelian}]^2 \) mixed anomaly in low-energy effective theories of the Heterotic \( E_8 \times E_8' \) string theory\(^{14} \) and hence \( Q^S \) vanishes for \( \xi_Y \). Without the 1-loop term, the

\(^{14} \) Reference \cite{16} argues based on field theory that 1-loop Fayet–Iliopoulos parameters are proportional to \( U(1)-[\text{gravity}]^2 \) anomalies of low-energy spectrum, but this argument implicitly assumes that quadratically divergent contributions from any one of massless chiral multiplets are regularized exactly in the same way. It is very subtle, however, to discuss cancellation among divergent quantities, and it is more appropriate to study this issue (Fayet–Iliopoulos parameter) in a UV finite framework such as string theory. In a compactification of Heterotic \( SO(32) \) string theory with an \( SU(3) \) vector bundle, Fayet–Iliopoulos parameter of a \( U(1) \) vector multiplet was
tree-level term should also vanish in order for $\xi_Y$ to vanish. Thus,

$$\int_Z c_1(L_Y) \wedge J \wedge J = 0.$$  \hfill (61)

It also follows from this condition that $Q^S_2 = 0$ by requiring $\xi_2 = 0$. All of this argument ignores all the non-perturbative (and stringy) corrections to the Fayet–Iliopoulos parameters.

### 3.3.2 Orbifold GUT and Beyond

**Localized $U(1)_Y$ Breaking**

Two assumptions that are essential in maintaining the gauge coupling unification are

$$\int_Z J \wedge J \wedge c_1(L) = 0, \quad \int_Z J \wedge c_1(L)^2 = 0.$$ \hfill (62)

The first one comes from the stability condition of the vector bundle $V_1$ (also from requiring the vanishing Fayet–Iliopoulos parameters $\xi_{2,Y}$), and the second one was introduced right after (35) in order to bring the 1-loop threshold corrections under control. These conditions are derived in the supersymmetric and large-volume limit.

Suppose that $c_1(L_Y)$ is given by

$$c_1(L_Y) = \sum_I n_I D_I,$$ \hfill (63)

where $D_I$ are divisors of a Calabi–Yau 3-fold $Z$, and $n_I$ coefficients. The first equation of (62) becomes

$$\int_Z J^2 \wedge (\sum_I n_I D_I) = \sum_I n_I \int_{D_I} J^2 = 0.$$ \hfill (64)

calculated explicitly, and it turned out to be proportional to $U(1)$-[gravity]$^2$ indeed \cite{30, 31}. Reference \cite{28} further showed that this is true for Calabi–Yau 3-fold compactifications of Heterotic $SO(32)$ string theory with generic (supersymmetry preserving) vector bundles. In compactifications of Heterotic $E_8 \times E_8'$ string theory, however, \cite{27} showed that the 1-loop Fayet–Iliopoulos parameters $Q^S$ are proportional to $U(1)$-[non-Abelian]$^2$ mixed anomalies. $Q^S$ does not have to be proportional to $U(1)$-[gravity]$^2$ anomalies, because various massless multiplets originate from cohomology groups of vector bundles in various representations, and UV divergent contributions to Fayet–Iliopoulos parameters from those multiplets are not regularized (cut-off and made UV-finite) exactly in the same way.

Section 3 of \cite{30} argues, however, that the 1-loop Fayet–Iliopoulos parameters (i.e. $Q^S$) are proportional to $U(1)$-[gravity]$^2$ anomalies in compactifications of Heterotic $E_8 \times E_8'$ string theory as well. We have not yet clarified how the two apparently contradicting statements from \cite{30} and \cite{27} are related. In this article, we adopted the statement in \cite{27}. 

22
This condition is satisfied, if all the $D_I$'s that appear in (63) have vanishing sizes, for example.

The second equation of (62) becomes

$$\int_Z J \wedge c_1(L)^2 = \sum_I \sum_J n_{I\alpha} n_{J\beta} \int_{D_I \cdot D_J} J = 0. \quad (65)$$

If all the curves $D_I \cdot D_J \neq \phi$ have vanishing volumes, then the second condition is also satisfied.

For an example, $T^6/\mathbb{Z}_3$ orbifold has 27 isolated vanishing exceptional divisors, each of which is isomorphic to $\mathbb{C}P^2$. Another example is $WP_{1,1,3,3} \supset (9)$, which also contains 3 isolated $\mathbb{C}^3/\mathbb{Z}_3$ singularities, and hence 3 such divisors each of which is isomorphic to $\mathbb{C}P^2$. Reference [22] argued that containing a source of $SU(5)_{\text{GUT}}$ symmetry breaking into an orbifold singularity brings the threshold correction under control. Indeed, we found that the 1-loop threshold corrections to the $U(1)_Y$ gauge coupling is proportional to $A$, and hence this correction is made small when $A = 0$ [stringy correction would remain, but it will not have a large-volume enhancement]. Thus, we largely confirm their claim that the 1-loop threshold correction can be made small when the symmetry breaking is confined to orbifold singularities. By now, we see that (62) is the generalized version of the idea of [22], and it is obvious that the global geometry does not have to be a toroidal orbifold, as long as (62) are satisfied. This generalization should allow much more variety in the choice of geometry.

**Naive Dimensional Analysis**

There are a couple of different sources that give rise to a small deviation from the GUT relation. As we have seen, one of such sources was the mixing with an extra massless strongly coupled gauge field. The extra contribution to the gauge coupling $(3/5)/g^2_Y$ is suppressed relatively to the leading contribution $\simeq 1/g^2_C \simeq 1/g^2_L$ by a factor of order

$$\frac{\text{vol}(Z)|_{x_{11}=\pi \rho}}{\text{vol}(Z)|_{x_{11}=0}} \gtrsim \frac{\alpha'^3}{\text{vol}(Z)|_{x_{11}=0}}. \quad (66)$$

Since the observed values of the Planck scale, GUT scale and the unified gauge coupling constant suggest that the $\text{vol}(Z)|_{x_{11}=\pi \rho}$ is almost close to $\alpha'^3$ [40, 41, 47], the inequality above is almost saturated in the reality, and it can be quite small.

Only supergravity approximation (large-volume limit) was used in (63) in the expression for the threshold corrections to the gauge coupling constants. There will be extra stringy contributions, which cannot be captured by supergravity approximation. Since there are literatures on the threshold corrections to the gauge kinetic functions, results in such references can be used to obtain a precise estimate of how large they are (to the level of whether some power of $\pi$ is involved or not). This article does not cover such calculation, however. Instead,
we just assume in this article that they are of order unity, because there is no characteristic scales other than the string scale for such contributions. Since we consider a situation where \( \text{Re} S \sim \text{Re} T \sim R^2/\alpha' \), the order-one stringy and possibly SU(5)\(_{\text{GUT}}\)-breaking corrections to the gauge coupling are relatively

\[
\frac{\mathcal{O}(1)}{\text{Re} T} \sim \frac{\alpha'}{R^2},
\]

compared with the leading term \( \text{Re}(S + T) \). Therefore, this correction is more important than \( \mathcal{O}(1/\alpha' R^2)^3 \), which may be of order \( \mathcal{O}((\alpha'/R^2)^3) \). Orbifold calculations may be useful, as we mentioned above, in obtaining more precise estimate of the stringy corrections to the GUT relation.

### 3.3.3 Dimension-6 Proton Decay

Here is a remark on dimension-6 proton decay. As for the process of determining the Kaluza–Klein scale from observables, we do not have much to add to what we already wrote at the end of section 2.2. The dimension-6 proton decay operators are generated after massive gauge bosons. Two vertices, each of which involves two fermion zero modes and one massive gauge boson, are combined together. The three-point vertex comes from the covariant derivative interaction of the gaugino kinetic term. Quarks and leptons come from a part of gaugino in the adjoint representation of \( E_8 \), and Kaluza–Klein tower of off-diagonal gauge bosons in SU(5)\(_{\text{GUT}}\) is also a part of \( E_8 \) gauge filed on 10 dimensions. The coefficient of the three-point vertex is calculated by overlap integration over the Calabi–Yau 3-fold for the compactification.

In toroidal orbifold compactification, (fermion) zero-mode from untwisted sector (bulk) has absolutely flat wavefunctions, while that of the Kaluza–Klein gauge bosons are Fourier modes on the torus. The overlap integration involving two untwisted-sector fermions and one Kaluza–Klein gauge boson vanishes, and the Kaluza–Klein gauge bosons do not induce a transition between zero-mode fermions from the untwisted sectors. Although such predictions appear in the literature from phenomenology community, they should hold only for toroidal orbifold compactifications. In general Calabi–Yau 3-fold compactification of the Heterotic string theory, wavefunctions of zero-modes of chiral multiplets are identified with elements of bundle-valued cohomology groups on a Calabi–Yau 3-fold, and they are not absolutely flat on a curved manifold. Products of two cohomology group elements multiplied by a higher harmonic function do not vanish generically, after being integrated over a Calabi–Yau manifold. Branching fractions of various decay modes of a proton can be generation dependent, but more detailed geometric data is necessary in order to calculate branching fractions for individual models of Heterotic string compactification.
3.4 Digression: Landscape of Unified Theories

Our presentation has consisted in considering the Georgi–Glashow SU(5)$_{\text{GUT}}$ unified theories and study how to break the SU(5)$_{\text{GUT}}$ symmetry down to the Standard-Model SU(3)$_C \times SU(2)_L \times U(1)_Y$. There are other types of unified theories, among which flipped SU(5) model and Patti–Salam model will be the most famous. We could have studied how to construct such unified theories, and then consider how to break those unified symmetries.

Our choice of Georgi–Glashow SU(5)$_{\text{GUT}}$ is not without a reason. The electroweak mixing angles in the quark sector are all small, but those in the lepton sector are large (apart from the last one yet to be measured). In Pati–Salam type unified theories, the quark doublets and lepton doublets are contained in a common irreducible representation of the unified gauge group. In order to obtain the qualitative pattern of the electroweak mixing stated above, one generically needs to have Yukawa couplings that heavily involve the source of symmetry breaking of the Pati–Salam gauge group. In the flipped SU(5) model with Froggatt–Nielsen (or Abelian flavour symmetry) type Yukawa matrices, not all the Yukawa eigenvalues and mixing angles come out right either, meaning presumably that the Yukawa couplings heavily involve symmetry breaking of the flipped SU(5) symmetry. The Georgi–Glashow SU(5)$_{\text{GUT}}$ symmetry does not have this problem, and it can be a fairly well approximate symmetry (to some extent) in Yukawa couplings of quarks and leptons.

In field-theory model building, different types of unified theories are just different models. It is a matter of which model provides better approximation to the reality. From the perspective of (landscape of) string theory, however, things begin to look a little different. If the moduli space of various Calabi–Yau manifolds and vector bundles are interconnected, there may

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15It is known that Yukawa couplings follow such a pattern in certain region of the moduli space; examples include small torus fibered compactification and near-orbifold-limit region.

16In most generic compactification of Heterotic string theory, it may not be an important issue whether or not SU(5)$_{\text{GUT}}$ (or some other unified-theory gauge group) is a good approximate symmetry with respect to Yukawa couplings. Gauge field background is introduced in the U(1)$_Y$ direction, and zero modes (cohomology group elements) in the same irreducible representation of SU(5)$_{\text{GUT}}$ but with different U(1)$_Y$ charges have different zero-mode wavefunctions, and hence the Yukawa couplings are not expected to satisfy relations that would have followed from SU(5)$_{\text{GUT}}$ symmetry. In Heterotic vacua with F-theory dual and also in F-theory vacua, however, zero modes from a common SU(5)$_{\text{GUT}}$ representation are localized in a common matter curve, and flavour properties associated with fields in representations such as 10 or 5 may be attributed to some properties of matter curves of corresponding representations. Thus, it is not a meaningless question which unified symmetry is a good approximation in Yukawa matrices. A relevant discussion is found in [15].

17Note, however, that we are not interested in dynamical (or cosmological) transitions between vacua in this article. Thus, we are not concerned about whether there is a topological barriers within the moduli space. Note also that the connectedness of landscape of vacua depends on the “sea level”—how much symmetry breaking one allows when one goes from one vacuum to another.
not actually be a definite distinction between various types of unified theories. From one vacuum in one type of unified theory to another in a different type of unified theory, it may be possible to deform continuously over the moduli space (before introducing fluxes). Low-energy observables such as Yukawa eigenvalues and mixing angles are functions of moduli, and they change continuously until they look phenomenologically qualitatively different. Thus, any types of unified theories in landscape of string vacua cannot be absolutely “wrong”; it is just a matter of how far those vacua are from ours. String landscape accommodates hundreds of models of unified theories, and may set a stage to discuss dynamical selection of models of unified theories. String landscape works as a unified theory of unified theories.

In what follows, we study the relation between Georgi–Glashow SU(5) and flipped SU(5) unified theories in string landscape. We will be very crude in that we do not restrict ourselves to a partial moduli space where matter parity is preserved, or to a moduli space where vector bundles have appropriate extension structure.

Both the Georgi–Glashow SU(5) gauge group and the flipped SU(5)′ gauge group can be embedded in a common SO(10) model. Thus, it is easiest to see how those theories are obtained by breaking SO(10) symmetry. Georgi–Glashow SU(5)_{GUT} symmetry is the commutant of a U(1)_{χ} in a maximal torus, specified by a charge vector \( q_{χ} \) in the Cartan subalgebra. The gauge group of the flipped SU(5), SU(5)′ × U(1)_{χ′} is the commutant of U(1)_{χ′} generated by \( q_{χ′} \). Those two theories share a rank-5 Cartan subalgebra of SO(10), and the charge vectors are related by

\[
q_{χ} = \text{diag}(2, 2, 2, 2, 2), \quad q_{Y} = \text{diag}\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right), \quad (68)
\]

\[
q_{χ′} = \text{diag}(2, 2, 2, -2, -2), \quad q_{Y′} = \text{diag}\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{2}, -\frac{1}{2}\right). \quad (69)
\]

Those charge vectors satisfy

\[
\begin{pmatrix}
q_{χ′} \\
q_{Y′}
\end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & -24 \\ -1 & -1 \end{pmatrix} \begin{pmatrix}
q_{χ} \\
q_{Y}
\end{pmatrix}. \quad (70)
\]

In order to obtain Georgi–Glashow SU(5) unified theories in compactification of the Heterotic \( E_8 \times E_8 \) string theory, we can begin with an SU(4) vector bundle \( V_4 \) and a line bundle \( L_χ \). By further turning on vev’s in zero modes \( H^1(Z; V_4 \otimes L_χ^{-5}) \) and \( H^1(Z; V_4^c \otimes L_χ^5) \), one obtains an SU(5) bundle, leaving unbroken Georgi–Glashow SU(5)_{GUT} symmetry. Vev’s in the zero modes are regarded as deformation of the vector bundle, since those cohomology groups describe the deformation of the bundles. The Georgi–Glashow SU(5)_{GUT} symmetry can be broken down to SU(3)_{C} × SU(2)_{L} (and U(1)_{Y}) when a line bundle \( L_Y \) is turned on in the direction specified by \( q_Y \).
The flipped SU(5) theories are obtained in Heterotic string compactification\footnote{In the flipped SU(5) unified theories, one needs to assume that the gauge coupling constant of \( U(1)_{\chi'} \) is the same as that of \( SU(5)' \) in order to obtain the GUT relation after the symmetry breaking due to the vev. This assumption seems to be satisfied when they are obtained through compactification of string theory containing \( SO(10) \) gauge group, because the \( U(1)_{\chi'} \) symmetry originates from the same \( SO(10) \) gauge group. However, a line bundle in the \( U(1)_{\chi'} \) direction removes the massless \( U(1)_{\chi'} \) gauge field from the spectrum, just like in the case of \( U(1)_{Y} \) gauge field. Thus, an extra gauge field has to be obtained through a line bundle sharing the same first Chern class with \( L_{\chi'} \). In order to maintain the approximate GUT relation, the gauge coupling of the combined massless \( U(1) \) gauge field should be almost the same as that of \( U(1)_{\chi'} \). This is achieved when the extra \( U(1) \) gauge field has a large coupling constant, just like in sections 2 and 3. The same idea works for the flipped SU(5) unified theories as well.} by turning on the same SU(4) bundle \( V \) and a line bundle \( L_{\chi'} \) in the direction specified by \( q_{\chi}' \). Furthermore, vev’s are turned on within zero modes \( 10' \)'s = \( H^{1}(Z; V_{4} \otimes L_{\chi'}^{-1} \otimes L_{Y}^{-1}) \) and its conjugate, so that the \( SU(5)' \times U(1)_{\chi'} \) symmetry is broken down to \( SU(3)_{C} \times SU(2)_{L} \times U(1)_{Y} \). \( L_{Y} \) is a trivial bundle in the flipped SU(5) models; we included \( L_{Y} \) in the expression above to clarify where the chiral multiplets in the \( 10' \) representation vev’s should develop.) Vev’s in these chiral multiplets correspond to deformation of the vector bundle. The structure group is enlarged.

Because of the relation among the charge vectors above, we find a translation

\[
L_{\chi} \leftrightarrow L_{\chi}^{\frac{1}{5}} \otimes L_{Y}^{-\frac{1}{5}}, \quad L_{Y} \leftrightarrow L_{Y}^{\frac{24}{5}} \otimes L_{Y}^{-\frac{1}{5}}.
\]

(71)

Thus, the deformation of the bundle in the flipped SU(5) unified theories \( H^{1}(Z; V_{4} \otimes L_{\chi}^{-1} \otimes L_{Y}) \) is actually the same deformation as the one in Georgi–Glashow SU(5) unified theories, \( H^{1}(Z; V_{4} \otimes L_{-5}) \). When one talks of flipped SU(5) unified theories, one usually assumes that the Kaluza–Klein scale is higher than the unification scale, where the vev in \( 10' \) breaks the symmetry. As the vev increases, and it becomes comparable to the Kaluza–Klein scale, however, it is more appropriate to treat the vev as a part of vector bundle moduli. In the large vev limit of the flipped SU(5) unified theories, a rank-5 vector bundle breaks \( SU(5) \subset E_{8} \) containing \( SU(4) \) and \( U(1)_{\chi} \), and a line bundle still remains, with the structure group of the \( U(1) \) bundle set in the direction

\[
q_{\chi}' \equiv \frac{1}{5} q_{Y} \quad \text{(mod } q_{\chi})\text{).}
\]

(72)

Thus, this is nothing but the Georgi–Glashow SU(5) unified theories with a line bundle in the \( U(1)_{Y} \) direction.

\section{F-theory Vacua}

The \( SU(5)_{\text{GUT}} \) symmetry can be broken by turning on a line bundle in the \( U(1)_{Y} \) direction. The line bundle is given by a 2-form field strength tensor of a gauge field on the D7-brane world

\begin{thebibliography}{10}
\end{thebibliography}
volume in the perturbative Type IIB string theory, and in F-theory vacua in general, essentially
the same thing is expressed by a four-form field strength borrowing language of M-theory.

The $U(1)_Y$ problem exists for such models, just like we already explained in section 2.1 in
Type IIB models, and the Green–Schwarz coupling that makes the $U(1)_Y$ gauge field a mass
term is rephrased from the Chern–Simons interaction on the D7-brane worldvolume in the Type
IIB description to the Chern–Simons term in the eleven-dimensional supergravity.

The $U(1)_Y$ problem can be, in principle, solved by allowing an extra $U(1)$ gauge symmetry
to mix with the $U(1)_Y$ gauge field contained in the $SU(5)_{GUT}$ symmetry; the extra $U(1)$ gauge
field has to be strongly coupled so that the deviation from the GUT relation is not too large.
Note that the unification between the $SU(2)_L$ and $SU(3)_C$ gauge coupling constants is already
achieved, by wrapping two and three D7-branes (or by just having a locus of $A_4$ singularity)
on a common holomorphic 4-cycle.

The extra $U(1)$ gauge field may arise from an extra D7-brane (or from an extra 7-brane
locus in F-theory in general). In order to obtain a little hierarchy between the GUT scale
(Kaluza–Klein scale) and the Planck scale of the $D = 4$ effective theory, the volume of $A_4$
singularity is chosen to be parametrically large in string scale units. Since the gauge kinetic
function $1/g^2$ is roughly proportional to the volume of the 4-cycle a D7-brane is wrapped in the
Type IIB string theory, the unified gauge coupling constant $1/g^2_{GUT} \sim 1/g^2_L$ is small. An effective
theory below the Kaluza–Klein scale becomes perturbative, just like we expect the MSSM to
be. On the other hand, if the extra 7-brane is wrapped on a 4-cycle whose volume is of order
one in string scale units, its gauge kinetic function remains small, and the gauge theory on the
7-brane is strongly coupled. Thus, the deviation of the $U(1)_Y$ coupling from the GUT relation
is (positive, in $\Delta(1/g^2)$, and) small as long as the extra $U(1)$ gauge theory is strongly coupled.
This picture dates back to [21] (and further to [20]), where fractional D3-branes were used as
the extra 7-brane; fractional D3-braes are known to be wrapped (possibly anti-) D7-branes or
D5-branes depending on a nature of singularity. Why some 4-cycle has a parametrically large
volume, and some do not is a question associated with stabilization of Kähler moduli. Thus,
the $U(1)_Y$ problem is translated into a problem of moduli stabilization. Since the doublet and
triplet part of the Higgs multiplets are regarded as global holomorphic sections of different line
bundles (they differ by $L_{Y}^{\otimes 5}$), the massless spectrum of doublets and triplets can be different,
giving a solution to the doublet–triplet splitting problem.

There may be threshold corrections to the gauge kinetic functions of order

$$\int_{\Sigma} J \wedge c_1(L_Y), \quad \int_{\Sigma} c_1(L_Y)^2.$$

(73)
The first term should vanish because it is the stability condition (or the Fayet–Iliopoulos D-term parameter of the $U(1)_Y$ symmetry). There may be a threshold corrections of the order of the second term above to the gauge coupling $1/g_Y^2$, but it is small by a factor of $\alpha''^2/\text{vol}(\Sigma) \sim \alpha''^2/R^4$ compared with the leading order term. Thus, the threshold correction does not affect the GUT relation seriously.

As we discussed at the end of section 2.2, dimension-6 proton decay is expected to be fast. Furthermore, in F-theory vacua, there may be an extra enhancement in the decay rate, because the amplitude receives an UV-divergent enhancement factor when matter multiplets are localized in the extra dimensions [49]. The enhancement factor depends on the number of codimensions in which matter multiplets are localized relatively to gauge fields, and if there is an UV-divergent factor indeed, then string theory calculation has to be involved in making an estimate of the form factor, just like in [50]. It is an interesting open problem what the enhancement factor will be in F-theory models.

**Note added in version 3:** This enhancement factor was studied in [65] after the first version of this article. The prediction of the enhancement factor is indeed one of the most important results of F-theory phenomenology; it is not mere a hindsight explanation of known parameters of the Standard Model, but it does change the prediction of observables in experiments in the future (i.e., that is physics!). Furthermore, the enhancement factor directly originates from localization of quarks and leptons in internal dimensions relatively to gauge fields, and hence is clearly an effect that is absent in field theory models purely in 3+1 dimensions (i.e., that is string theory!). Note also that the enhancement factor is very robust, in that it does not depend on details of how $SU(5)_{\text{GUT}}$ symmetry is broken. It is (and the following result is) applied to the $SU(5)_{\text{GUT}}$ breaking scenario discussed in this article, as well as to the scenario in case of non-surjective pull-back of 2-forms, $i^* : H^2(B_3) \rightarrow H^2(S)$, in [66, 23, 65]. Unfortunately the paper [65] considered that dominant contribution to (86) and (91) comes from a subset where the Green function $G_{\text{int}}$ diverges, but this subset is measure zero. In this version of our article we present our study on the enhancement factor by evaluating (86) and (91) carefully, and we obtain predictions on dimension-6 proton decay that are qualitatively different from those of [65].

Before we begin to study the enhancement factor in F-theory compactifications, let us briefly review the essence of [49] in M-theory compactifications on manifolds with $G_2$ holonomy. In $G_2$ holonomy compactifications with $SU(5)$ unification, $SU(5)_{\text{GUT}}$ gauge fields propagate on a 3-cycle $Q$, and charged matter fields such as those in $\mathbf{10}$ and $\overline{\mathbf{5}}$ representations are localized at isolated points $\vec{y}_i$ in $Q$; here, we use $\vec{y}$ as coordinates of $Q$, and $x^\mu$ for the Minkowski space
$\mathbb{R}^{3,1}$. Charged matter fields are coupled to the gauge field as

$$
\int_{\mathbb{R}^{3,1}} d^4x \ J_\mu^i(x) A_\mu(x, \vec{y}_i),
$$

and the gauge field has a kinetic term

$$
- \frac{1}{4g^2_7} \int_{\mathbb{R}^{3,1}} d^4x \int_Q d^3y \ \text{tr}(F_{MN}F^{MN}).
$$

The gauge field $A_M$ and current $J^\mu$ are given 1 and 3 mass dimensions, respectively, and the gauge coupling $g^2_7$ have $-3$ mass dimensions. The proton decay amplitude through the gauge-boson exchange is given by

$$
d^4x \ d^4x' \ g^2_7 J^\mu_i(x) J^\nu_j(x') \ \eta_{\mu\nu} G(x - x'; \vec{y}_i, \vec{y}_j).
$$

Here, $G(x - x'; \vec{y}_i, \vec{y}_j)$ is a Green function on $\mathbb{R}^{3,1} \times Q$, and is approximately

$$
G(x - x'; \vec{y}_i, \vec{y}_j) \sim \frac{1}{\text{vol}(Q)} \sum \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2 - |\vec{q}|^2} e^{-i\vec{q} \cdot (x-x')} e^{i\vec{q} \cdot (\vec{y}_i - \vec{y}_j)}.
$$

$\vec{q}$ labels eigenmodes of Laplace operator on $Q$, and their eigenvalues (Kaluza–Klein mass-square) and eigenfunctions are denoted by $|\vec{q}|^2$ and $e^{i\vec{q} \cdot \vec{y}}/\sqrt{\text{vol}(Q)}$, respectively. The momentum transfer $p^\mu$ in $\mathbb{R}^{3,1}$ direction is only of order 1 GeV in proton decay process. Since $p^2$ in the propagator is much smaller than the Kaluza–Klein masses $|\vec{q}|^2$, $p^2$ can be ignored and dropped from the propagator. Carrying out $d^4p$ integration, one obtains dimension-6 operators in an effective theory:

$$
d^4x \ J^\mu_i(x) J_{\mu\nu}(x) \left[ g^2_7 G_{\text{int}}(\vec{y}_i, \vec{y}_j) \right] \sim d^4x \ J^\mu_i(x) J_{\mu\nu}(x) \left[ g^2_7 \frac{1}{|\vec{y}_i - \vec{y}_j|} \right].
$$

$G_{\text{int}}$ is the Green function on the internal space $Q$. The coefficient in the square bracket has mass dimension $-2$.

For two different currents $i \neq j$, the effective (mass)$^{-2}$ parameter is of order

$$
\frac{g^2_7}{R} \sim \frac{g^2_7/R^2}{1/R^2} \sim \frac{g^2_{\text{GUT}}}{M^2_{\text{GUT}}},
$$

where $|\vec{y}_i - \vec{y}_j|$ is set to a typical Kaluza–Klein radius of $Q$, $R$, and we assumed that the SU(5)$_{\text{GUT}}$ symmetry is broken by topological gauge field configuration on $Q$ (such as Wilson line), and hence $M_{\text{GUT}} \sim M_{\text{KK}} \sim 1/R$. Thus, there is no enhancement compared with typical proton mass.
decay amplitude through gauge-boson exchange in 4D field theory models. For the same current, $i = j$, however, $1/|\vec{y}_i - \vec{y}_j|$ diverges. The amplitude diverges (linearly), because all the Kaluza–Klein modes with arbitrary large momentum $q$ equally contribute without cancellation in (77) in case $\vec{y}_i = \vec{y}_j$. In reality, however, localized charged matter fields have certain form factor, or put differently, intersecting D6-branes effectively have certain “thickness”. The Kaluza–Klein momentum sum in (77) is effectively cut-off at around $|q| \sim M_*$, where $M_*$ is the string scale, and the effective (mass)$^{-2}$ parameter becomes

$$g_7^2 M_* \sim \frac{g_7^2/R^3}{1/R^2} (RM_*) \sim \frac{g_{\text{GUT}}^2}{M_{\text{GUT}}^2} \times \left( \frac{M_*}{M_{\text{GUT}}} \right). \quad (80)$$

The effective dimension-6 operator for the same current, $10^i 10 j 10^j$ in the effective Kähler potential, has a coefficient enhanced by $(M_*/M_{\text{KK}})$ [49].

Let us now study the enhancement factor in F-theory compactifications. We begin with the $10^i 10 j 10^j$ operator. In supersymmetric F-theory compactifications, chiral multiplets in the SU(5)$_{\text{GUT}-10}$ representation correspond to holomorphic sections $f_i$ of a line bundle on the matter curve $\bar{c}_{(10)}$ in a complex surface $S$ of $A_4$ singularity [67, 68]. Here, $i = 1, 2, 3$ is now the generation index of chiral multiplets in the SU(5)$_{\text{GUT}-10}$ representation. We take the coordinates on $S$ as $y_{1,2}$ and $w_{1,2}$, where $y_{1,2}$ correspond to normal directions of the matter curve $\bar{c}_{(10)}$, and $w_{1,2}$ to coordinates on the curve. The gauge fields on $S$ and the chiral zero modes in the $10$ representation couple as

$$\int_{\mathbb{R}^{3,1}} d^4x \int_S d^2y d^2w \ J_{ji}^\mu(x) \chi_i(y, w) \chi_j^\ast(y, w) A_\mu(x, y, w), \quad (81)$$

where $J_{ji}^\mu(x)$ is a dimension-3 current $\bar{\lambda}_j \sigma^\mu \lambda_i$ on $\mathbb{R}^{3,1}$, where $\lambda_i(x)$ and $\lambda_j(x)$ are fermions in the effective theory corresponding to the zero modes $f_i(w)$ and $f_j(w)$. $A_\mu(x, y, w)$ is the gauge field on $S$, and is assigned a mass-dimension 1. Its kinetic term is

$$- \frac{1}{4g_8^2} \int_{\mathbb{R}^{3,1}} d^4x \int_S d^2y d^2w \ \text{tr}(F_{MN} F^{MN}). \quad (82)$$

$\chi_{i,j}(y, w)$ are the zero-mode wavefunctions on $S$, corresponding to $f_{i,j}$ (see [69]). Their approximate form, as well as their normalization, are

$$\chi_{i,j} \sim e^{-M_*^2 |y|^2} f_{i,j}(w) \left( \frac{M_*}{\sqrt{\text{vol}(\bar{c}_{(10)})}} \right). \quad (83)$$
Thus, the proton decay amplitude becomes
\[ d^4x d^4x' J_{ji}^\mu(x) J_{\mu lk}(x') \]
\[ \int_S d^2y d^2w \int_S d^2y' d^2w' g_8^2 \chi_i(y, w) \chi_j^*(y, w) \chi_k(y', w') \chi_l(y', w') G(x - x'; y, y', w, w'), \] (84)
and the approximate form of the Green function is
\[ G(x - x'; y, y', w, w') \sim \frac{1}{\text{vol}(S)} \sum_{\vec{k}, \vec{q}} \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2 - |\vec{k}|^2 - |\vec{q}|^2} e^{-ip(x - x')} e^{i\vec{q} \cdot (\vec{y} - \vec{y}')} e^{i\vec{k} \cdot (\vec{w} - \vec{w}')}. \] (85)

\( p^2 \) in the propagator is negligible (just like in the case of M-theory compactifications), and the \( d^4p \) integration can be carried out. Thus, we obtain a dimension-6 operator
\[ d^4x J_{ji}^\mu(x) J_{\mu lk}(x) \left[ \int_S d^2y d^2w \int_S d^2y' d^2w' g_8^2 (\chi_j^* \chi_i)(y, w) (\chi_l^* \chi_k)(y', w') G_{\text{int}}(y, y', w, w') \right]. \] (86)

To evaluate the effective (mass)\(^{-2}\) parameter in the square bracket, we proceed as follows. Because of the fact that the 10-representation fields are localized along the curve, or equivalently because of the exponentially falling off wavefunctions \( \chi_{i,j,k,l} \) in (83), dominant contribution to the amplitude comes from a region where \( \vec{y} \) and \( \vec{y}' \) are close to each other (and also to the matter curve where \( \vec{y} \sim \vec{y}' \sim \vec{0} \)), very large \( \vec{q} \) can contribute in (85). On the other hand, all the zero modes in 10-representation are characterized by holomorphic (and hence smooth) sections \( f_{i,j,k,l}(w) \) of a line bundle (without a torsion component), and only low-lying Kaluza–Klein momenta \( \vec{k} \) can contribute after \( d^2w d^2w' \) integration\(^{19}\). Thus, we ignore \( |\vec{k}|^2 \) and keep only \( |\vec{q}|^2 \) in the denominator of (85). Now, summation in \( \vec{k} \) can be carried out, and \( G_{\text{int}} \) becomes proportional to \( \delta^2(\vec{w} - \vec{w}') \). The Kaluza–Klein momentum sum in \( \vec{q} \) yields a logarithmic divergence, which is cut-off at \( |\vec{q}| \sim M_* \) because of the thickness of the Gaussian wavefunction in (83). In the end, the amplitude looks
\[ d^4x J_{ji}^\mu(x) J_{\mu lk}(x) \left[ \int_S d^2w \left( f_{x}^* f_{x} f_{x}^* f_{x} \right)(w) \times \ln \left( \frac{M_*^2}{M_{KK}^2} \right) \right]. \] (87)

Since the factor
\[ \frac{g_8^2}{\text{vol}(c(10))} \sim \frac{g_8^2/R^4}{1/R^2} \sim \frac{g_{\text{GUT}}^2}{M_{\text{GUT}}^2}, \] (88)
\[ ^{19}\text{This is where our analysis is different from that of [65].} \]
is the usual effective (mass)\(^{-2}\) scale of the dimension-6 proton decay operator, the enhancement factor in F-theory compactifications is \(\ln(M_*/M_{\text{GUT}})^2\). This logarithmic enhancement factor\(^{20}\) originates from the fact that chiral multiplets in the SU(5)\(_{\text{GUT}}\)-10 representation are localized relatively to the SU(5)\(_{\text{GUT}}\) gauge fields in real two dimensions.

Finally, we study the enhancement factor associated with the effective dimension-6 proton decay operator \(10^\dagger j10 \bar{5}_a\). Here, \(a, b\) are generation indices. Chiral multiplets in the SU(5)\(_{\text{GUT}}\)-\(\bar{5}\) representation are also described by holomorphic sections \(h_{a,b}\) of a line bundle on a curve \(\tilde{c}(\bar{5})\), which is obtained by resolving all the double point singularities of the matter curve \(\bar{c}(\bar{5})\) at the codimension-3 loci of enhanced \(D_6\) singularity\(^{70}\) (called type (d) points there). In light of Heterotic–F theory duality, \(\bar{5}\)'s may well be described only as sections of a sheaf on \(\tilde{c}(\bar{5})\), and the sheaf may not be torsion free or locally free, in principle. Whether such a localized component exists in the sheaf on the curve \(\tilde{c}(\bar{5})\), and hence in the zero modes, is crucial for the analysis of the enhancement factor in proton decay amplitude. Reference\(^{70}\) concluded, however, that there is not a localized component at all; all the zero modes are described by smooth sections \(h_{a,b}\) on the covering matter curve \(\tilde{c}(\bar{5})\).

The current of zero modes of \(\bar{5}\)'s couple to the SU(5)\(_{\text{GUT}}\) gauge field through

\[
\int_{\mathbb{R}^{3,1}} d^4x \int_S d^2y d^2w \ J_{ba}^\mu(x) \chi_b^\dagger(y,w) \chi_a(y,w) A_{\mu}(x,y,w).
\]  

(89)

Here, \(J_{ba}^\mu\) is a current on \(\mathbb{R}^{3,1}\) that consists of fermions \(\lambda_{a,b}\) corresponding to the zero modes \(h_{a,b}\), and \(\chi_{a,b}\) are their zero-mode wavefunctions (see\(^{69}\)). The dominant contribution to the \(10^\dagger 105^\dagger 5\) decay amplitude most likely comes from a region around intersection points of the two matter curves, \(\tilde{c}(10)\) and \(\tilde{c}(\bar{5})\). Although there are two different types of intersection points (type (a) and type (d) points in the classification of\(^{70}\)), the difference will not matter for the proton decay that takes place within SU(5)\(_{\text{GUT}}\). Thus, here, we assume that the matter curve \(\tilde{c}(10)\) is along \(y = 0\) (locally), and \(\tilde{c}(\bar{5})\) along \(w = 0\). Then, the wavefunction \(\chi_{a,b}\) becomes approximately

\[
\chi_{a,b}(y, w) \sim e^{-M_2^2 |w|^2} h_{a,b}(y) \times \left( \frac{M_*}{R} \right).
\]  

(90)

Repeating the same process as before, one finds that the effective dimension-6 operator is

---

\(^{20}\) If one ignores the difference between \(M_*\), \(1/l_s\) or \(1/\sqrt{\alpha'}\), and sets \(g_s \sim 1\), then \((M_*/M_{\text{GUT}})^4 \sim 1/\alpha_{\text{GUT}} \sim 24\). Thus, the enhancement factor in the amplitude is of order \(\ln(1/\sqrt{\alpha_{\text{GUT}}}) \sim \ln 5\). This result differs from the \((1/\alpha_{\text{GUT}})^{1/2}\) enhancement (which corresponds to quadratic divergence) in\(^{65}\). Quantitatively, \(\ln \sqrt{1/\alpha_{\text{GUT}}} \sim \ln 5\) is about a factor 4 smaller than \(1/\sqrt{\alpha_{\text{GUT}}} \sim 5\) in the decay amplitude, and the decay rate based on logarithmic enhancement is about an order of magnitude smaller than that based on quadratic enhancement.
given by
\[
d^4x \ J_{jl}^\mu(x)J_{l\mu a}(x) \int_S d^2y d^2w \int_S d^2y' d^2w' \ g_8^2
\]
\[
\frac{1}{R^4} \sum_{\vec{k}, \vec{q}} (M_*/R)^4 (f^*_j f_i)(w)e^{-M_2^2|w|^2}(h^*_a h_a)(y')e^{-M_2^2|w'|^2} \frac{i}{|\vec{k}|^2 + |\vec{q}|^2} e^{i\vec{k}(\vec{a}-\vec{w})} e^{i\vec{q}(\vec{a}-\vec{y})}.
\]  
(91)

Since \(h_{a,b}(y')\) and \(f_{i,j}(w)\) are zero modes, and are smooth everywhere along the matter curves, only Kaluza–Klein gauge bosons with low-lying Kaluza–Klein momenta \(\vec{q}\) AND \(\vec{k}\) couple to both \(\bar{5}\)'s and \(10\)'s. \[\text{[Remember that we have } d^2y'd^2w \text{ integration in the expression above.}]\]

Thus, the infinite sum in \(\vec{q}\) and \(\vec{k}\) is effectively dropped, \(\vec{q}\) and \(\vec{k}\) replaced by \(1/R\), and we find that the effective coefficient of the dimension-6 operator is of order
\[
\frac{g_8^2}{R^2} \sim \frac{g_8^2}{R^4} \sim \frac{g_{\text{GUT}}^2}{M_{\text{GUT}}^2}.
\]
(92)

Therefore, the prediction of the \(10_j^\dagger 10, \bar{5}_a \bar{5}\) dimension-6 proton decay in F-theory compactifications is just as the same as that of the ordinary GUT dimension-6 proton decay. There is no particular enhancement factor for this mode [22] this is essentially because only low-lying Kaluza–Klein modes can couple to both zero modes in the \(10\) representation and those in \(\bar{5}\).

To conclude, \(\Delta K = 10_j^\dagger 10, 10_l^\dagger 10_k\) dimension-6 proton decay amplitude has a logarithmic enhancement. The enhancement factor is \(\ln(M_*/M_{\text{KK}})^2\), which is roughly \(\ln 5 \sim 1.6\), where we used \(1/\alpha_{\text{GUT}} \sim 25\), and ignored a difference among \(M_*\), \(1/l_5\) and \(1/\sqrt{\alpha}\). On the other hand, \(\Delta K = 10_j^\dagger 10, \bar{5}_a \bar{5}\) amplitude is dominated by low-lying Kaluza–Klein gauge bosons, and is not enhanced. Thus, the decay rates to left-handed positively charged leptons \((\ell^+_L)\) in \(10\) are enhanced typically by a factor of \(1.6^2 \sim (2 \sim 3)\), relatively to rates of decay to right-handed positively charged leptons, \((\ell^+_R)\), or to right-handed anti-neutrinos, \(\bar{\nu}_R\) in \(\bar{5}\) (c.f. [49]). It should be noted, however, that the decay amplitudes have generation-dependent factors [23].

\[
\frac{1}{\text{vol}(\tilde{c}(10))} \int d^2w(f^*_j f_i f^*_k f_l)(w) \quad \text{and} \quad \frac{1}{\text{vol}(\tilde{c}(10))} \int d^2w(f^*_j f_i)(w) \quad \frac{1}{\text{vol}(\tilde{c}(5))} \int d^2y'(h^*_a h_a)(y')
\]
(93)

[21] This is where our study differs from that in [65].
[22] This conclusion differs from the result, \(1/\sqrt{\alpha_{\text{GUT}}}\) enhancement, in [65].
[23] These expressions should not be taken literally. It should be reminded that \(f_k\) and \(f_l\) for the current \(J^\mu_k\) correspond to zero modes in different irreducible representations of the Standard Model gauge group, although they are in the same irreducible representation \(10\) under SU(5)$_{\text{GUT}}$. Thus, \(f_k\) is not necessarily the same as \(f_l\) even when \(k = l\). The same is true for the zero modes \(h_a\) and \(h^*_a\) in the current \(J^\nu_a\) for fermions in the SU(5)$_{\text{GUT}}$–\(\bar{5}\) representation. It should also be clear from the discussion in the main text that \(e^{i\vec{k} \cdot \vec{a}}\) and \(e^{i\vec{q} \cdot \vec{y}}\) with low-lying Kaluza–Klein momenta \(\vec{k}\) and \(\vec{q}\) are omitted from the second factor.
for the $10^j_1 10^j_1 10^k_1$ and $10^j_1 5^j_1 5^a_1$ processes, respectively, and these factors may well be more important for individual decay modes than the logarithmic enhancement factor. Thus, the logarithmic enhancement of decay rates to charged leptons should be regarded only as a tendency predicted among all the decay modes.

As for the total decay rate of proton through the gauge-boson exchange, the enhancement remains only logarithmic, and is of order a factor of 2–3. It is not even clear whether this enhancement is more important than the yet to be (and hard to be) calculated factors in (93), which may result in suppression. More important is a fact that the total decay rate is proportional to $M_{GUT}^4$, and that the value of $M_{GUT}$ still has a large uncertainty, ranging from, say, $10^{15.7}$ GeV to $10^{16.5}$ GeV (see Figure 1). The decay rate for $M_{GUT} = 10^{15.75}$ GeV is three orders of magnitude larger than that for $M_{GUT} = 10^{16.5}$ GeV $\simeq 3 \times 10^{16}$ GeV. In the scenario of SU(5)$_{GUT}$ symmetry breaking discussed in this article, $M_{GUT}$ tends to be small, because of the tree-level correction to the gauge couplings, and hence the decay rate tends to be large. Such model-dependence is more important in the total decay rate than the logarithmic enhancement that is applied to all the F-theory models of SU(5)$_{GUT}$. Thus, the total decay rate can be used in discriminating various models, and the ratio of rates of decays to charged leptons to rates of decays to anti-neutrinos can be used to see whether charged matter fields are localized in internal space dimensions or not.

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**A Interpretation of “Wilson Lines” in Toroidal Orbifolds**

Model building using toroidal orbifold has a long history that dates back to 1980’s. Geometry of orbifolds is understood as certain limits of Calabi–Yau manifold. It has been known since early days (e.g. [53, 52]) that toroidal orbifold compactifications of the Heterotic theory corresponds to some limits in the moduli space of compactifications given by a Calabi–Yau manifold and a vector bundle on it. Toroidal orbifolds in the context of Type IIB orientifold with D7-branes
and O7-planes are little more involved in its interpretation as limits of smooth Calabi–Yau orientifold, yet some works have already been done.

In the appendix of this article, we clarify how one should interpret “Wilson lines” in toroidal orbifold compactifications of the Heterotic string theory in terminology of smooth Calabi–Yau compactifications. Toroidal orbifold models using discrete Wilson lines gained a renewed attention triggered by an activity that followed papers on $S^1/Z_2 \times Z_2'$ orbifold GUT [55, 22, 56]. We will see in Heterotic theory compactification that the toroidal orbifolds with “discrete Wilson lines” are also understood as some limits of compactifications described by smooth Calabi–Yau manifold and a vector bundle on it. The discrete Wilson lines in toroidal orbifolds are not Wilson lines (or flat bundles) on smooth Calabi–Yau $Z$ associated with a discrete homotopy group $\pi_1(Z)$, but rather they correspond to turning on line bundles on a Calabi–Yau with the U(1) structure group of the line bundles chosen differently at different vanishing cycles buried at orbifold singularities.

Once one adopts the interpretation above, then the SU(5)$_{GUT}$ symmetry breaking in Heterotic toroidal orbifold compactifications (with or without discrete Wilson lines) are regarded as special cases of the material discussed in the main text. Thus, as we discuss in the appendix A.1.3 (and as one can understand as special cases of the discussion in section 2.1), so-called the toroidal orbifold GUT’s in Heterotic string theory also suffer from the U(1)$_Y$ problem.

In the literature of toroidal orbifolds, another terminology “continuous Wilson line” is also found. Although the continuous Wilson lines have nothing to do with the main theme of this article, we take this opportunity (in the appendix A.2) to clarify that the “continuous Wilson lines” in Heterotic toroidal orbifold correspond to a part of vector bundle moduli in smooth Calabi–Yau compactification.

### A.1 Discrete Wilson Lines

Since our motivation is to understand what the “discrete Wilson lines” really are, we do not have to work on a very realistic model. Simple examples that illustrate the point will be better suited for our purpose. Thus, we use $T^4/Z_k$ orbifolds instead of $T^6/Z_N$ orbifolds, and provide interpretations of discrete Wilson lines in terms of compactification on K3 surfaces with vector bundles on them. K3 compactification [$T^4/Z_k$ in orbifold limits] has an advantage

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24 Some of fractional D3-branes are interpreted as anti D7-breens with a vector bundle on them, and such fractional D3-branes do not remain stable when vanishing cycle is blown up to be large.

25 See e.g. [57] for recent studies on this subject.
over CY₃ [resp. T⁶/Zₙ] compactification in that index theorem can calculate the massless spectrum of vector bundle moduli in addition to that of charged multiplets, so that we can compare the number of vector bundle moduli of smooth manifolds with that of orbifolds. We also use the Heterotic SO(32) string theory, instead of E₈ × E₈', because we are not trying to analyse geometry of specific toroidal orbifolds to be used for semi-realistic models, but we try to understand what the discrete Wilson lines are. For that purpose, difference in the choice of gauge group is not a big deal. We calculate the massless spectrum both in K₃+bundle compactification and in toroidal orbifolds and confirm that the results do agree. The agreement shows that the K₃+bundle interpretation is correct for the toroidal orbifolds of the Heterotic theory, and at the same time tells us the geometric meaning of twisted sector fields.

### A.1.1 Spectrum of Smooth-Manifold Compactification

Let us consider a Heterotic SO(32) string theory compactified on a K₃ manifold Z, with a vector bundle V turned on. The D = 10 supergravity multiplet reduces to

- D = 6 supergravity multiplet and a D = 6 tensor multiplet, containing D = 6 metric, one 2-form field and one scalar.
- h¹¹(Z) = 20 hypermultiplets, containing 3 × 19 real scalars describing the deformation of the metric of Z, 22 scalars obtained by integrating B-field over the 22 2-cycles of Z, and one more scalar [59].

When the structure group of the vector bundle V is SO(2r) ⊂ SO(32), SO(32 − 2r) is the unbroken symmetry, and the SO(32)-adjoint representation decomposes into

\[
\mathfrak{so}(32)\text{-adj.} \rightarrow (1, \mathfrak{so}(32 − 2r)\text{-adj.}) + (\mathfrak{so}(2r)\text{-adj.}, 1) + (\text{vect.}, \text{vect.}).
\]  

The multiplicity of hypermultiplets is calculated by indices

\[
- \frac{1}{2} \int_Z \text{ch}_R(V) \hat{A}(TZ) = T_R I_V - (\text{dim.} R) \int_Z \frac{c_2(TZ)}{24} = 24 T_R - (\text{dim.} R),
\]

where \(T_R\) is a Dynkin index[^26], \(I_V \equiv -(2T_R)^{-1} \int_Z \text{ch}_{2,R}(V)\) is the instanton number of the bundle V, and \(I_V = \int_Z c_2(TZ) = 24\) is used at the last equality. The D = 10 SO(32) vector multiplet reduces to

- one D = 6 SO(32 − 2r) vector multiplet

[^26]: \(T_R\) is 1 for vector representations and \(2r - 2\) for adjoint representations of SO(2r), and 1/2 for fundamental representations and \(N\) for adjoint representations of SU(N).
• (24 − 2r) hypermultiplets of SO(32 − 2r)-vector representation,
• 24(2r − 2) − r(2r − 1) hypermultiplets of vector bundle moduli.

Let us check the Higgs cascade, as in the analysis of [60, 61]. As one of hypermultiplets in the vector representation develops an expectation value, the unbroken symmetry becomes SO(32 − 2(2r+1)). Each hypermultiplet in the [32 − 2r]-dimensional vector representation reduces into a hypermultiplet in the [32 − 2(2r+1)]-dimensional vector representation of SO(32 − 2(2r+1)) unbroken symmetry and 2 singlets. The Higgs mechanism associated with the symmetry breaking SO(32 − 2r) → SO(32 − 2(2r+1)) absorbs 2 hypermultiplets in the SO(32 − 2(2r+1))-vector representation and one singlet. Thus, [(24 − 2r) − 2] = [24 − 2(2r+1)] hypermultiplets in the vector representation are left after the symmetry breaking, which agrees with the result of SO(2(r+1))-bundle compactification. Similarly, the number of singlet hypermultiplet—vector bundle moduli—increases by 2 × (24 − 2r − 1), because two singlets arise from one hypermultiplet in the vector representation, but one hypermultiplet is absorbed by a Higgsed vector multiplet. Thus, the number of vector bundle moduli hypermultiplets becomes [24(2(r+1) − 2) − (r + 1)(2(r+1) − 1)] after the symmetry breaking SO(32 − 2r) → SO(32 − 2(2r+1)), which also agrees with the result of SO(2(r+1)) bundle compactification. Moduli spaces of different unbroken symmetry and different structure group are continuously connected through this Higgs cascade process.

Case with r = 2, however, needs a separate treatment, because the structure group of a rank-4 vector bundle SO(4) ≃ SU(2) × SU(2) is not a simple group. The rank-4 bundle is a tensor product V ≃ V₁ ⊗ V₂, and the instanton number is given by

\[ I_V = I_{V_1} + I_{V_2}. \] (96)

One can see that the numbers of SO(28)-vector and singlet hypermultiplets given above are correct also for the r = 2 cases, if I_{V_1} and I_{V_2} are both non-zero. If the instanton number is only in either one of SU(2), say, I_{V_2} = 0, however, the unbroken symmetry group is SU(2) × SO(28), and there are \([T_{V_1}, V_1] = (24 − 4)/2 = 10\) hypermultiplets in the (2, 28) representation and 2 × 24 − 3 = 45 vector bundle moduli.

A.1.2 Spectrum of Orbifold Compactification

Let us now calculate massless spectra of some of T^4/Z_k orbifolds, and compare them with what we have got from the field-theory calculation. The Heterotic SO(32) string theory is described by bosons on the worldsheet, X^\mu (\mu = 0, 1, 2, 3), Z^A, \xi^A (A = 1, 2), right-moving

38
fermions, ψμ, ψA, ψ̄, and left-moving fermions, λI, λI (I = 1, · · · , 16). Toroidal orbifolds $T^4/Z_k$ (k = 2, 3, 4, 6) are quotients $\mathbb{C}^2/(Z_k \langle \sigma \rangle \rtimes \Lambda)$, where Λ is a rank-4 lattice in $\mathbb{C}^2$ whose basis consists of 4 vectors $e^A_a$ (a = 1, 2, 3, 4) and σ is an SU(2) ⊂ SO(4) rotation on $\mathbb{C}^2$, satisfying $\sigma^k = \text{id}$. The worldsheet fields $Z^A$ and $\overline{Z}^A$ transform under the generators of the space group $Z_k \rtimes \Lambda$ as

$$\tau_a : Z^A \rightarrow Z^A + e^A_a; \quad \sigma : Z^A \rightarrow e^{2\pi i v^A} Z^A,$$

$$\tau_a : \overline{Z}^A \rightarrow \overline{Z}^A + e^A_a; \quad \sigma : \overline{Z}^A \rightarrow e^{-2\pi i v^A} \overline{Z}^A,$$

where $\tau_a$ (a = 1, 2, 3, 4) are translation along the vectors $e_a$, and $e^A_a$ are complex conjugates of $e^A_a$. $\sigma$ is a generator of rotation on the complex coordinates, and $v^A = (1/k, -1/k)$. Other fields on the worldsheet transform under the translation and rotation as

$$\tau_a : \psi^A \rightarrow \psi^A, \quad \sigma : \psi^A \rightarrow e^{2\pi i v^A} \psi^A,$$

$$\tau_a : \overline{\psi}^A \rightarrow \overline{\psi}^A, \quad \sigma : \overline{\psi}^A \rightarrow e^{-2\pi i v^A} \overline{\psi}^A,$$

$$\tau_a : \lambda^I \rightarrow e^{2\pi i W^I_a \lambda^I}, \quad \sigma : \lambda^I \rightarrow e^{2\pi i V^I} \lambda^I,$$

$$\tau_a : \overline{\lambda}^I \rightarrow e^{-2\pi i W^I_a \overline{\lambda}^I}, \quad \sigma : \overline{\lambda}^I \rightarrow e^{-2\pi i V^I} \overline{\lambda}^I,$$

all of $\beta_a \equiv \text{diag}(e^{2\pi i W^I_a}, e^{-2\pi i W^I_a})$ (a = 1, 2, 3, 4) and $\gamma_{\sigma} \equiv \text{diag}(e^{2\pi i V^I}, e^{-2\pi i V^I})$ in SO(32) acting on $(\lambda^I, \overline{\lambda}^I)$ commute each other; although they do not have to commute as long as those matrices satisfy the algebra of $\tau_a$ and $\sigma$ in the space group, we only consider the simplest cases here.

When $\text{diag}(W^I_a, -W^I_a) \neq 0$, $W^I_a$ are called discrete Wilson lines. In toroidal compactification, $2\pi W^I_a = A^I_A e^A_a + A^I_A e^A_a$ are the Wilson lines along the four independent topological 1-cycles of $T^4$. But, in (the blow up of) toroidal orbifolds $T^4/Z_k$ (k = 2, 3, 4, 6), there are no topological 1-cycles. Thus, there is no way the “Wilson lines” have anything to do with a flat bundle associated with a non-trivial homotopy group. The “Wilson lines” $W^I_a$ in toroidal orbifolds are allowed to take only discrete values, because of the algebraic relation between $\sigma$ and $\tau_a$ (and of the relation between $\gamma_{\sigma}$ and $\beta_a$), and hence they are called “discrete Wilson lines” in the literature of toroidal orbifold compactifications, but they are not Wilson lines associated

$^{27}$Slightly more complicated examples—non-diagonal $\gamma_{\sigma}$—will be discussed in the appendix.

$^{28}$The Euler number of a simply connected K3 manifold is 24, and the Euler number of the resolved $T^4/Z_k$ should be $24/#\pi_1(T^4/Z_k)$, where the resolution of $T^4/Z_k$ were to have a non-trivial homotopy group. The Euler number of the blow up of $T^4/Z_k$ can be calculated (See e.g., [62] for how to calculate the Euler number of toroidal orbifolds,) and is known to be 24 for all of $k = 2, 3, 4, 6$. Thus, all the $T^4/Z_k$'s have trivial homotopy groups. It is also possible to confirm that they are simply connected, by explicitly looking at the geometry of $A_{k-1}$-type ALE space expressed as $S^1$-fibration over a real three-dimensional space.
with a non-trivial homotopy group $\pi_1(Z)$ that is allowed to take discrete values because of the discreteness of $\omega_1(Z)$.

**Cases Without Discrete Wilson Lines**

Now that the notation is set, let us compute the massless spectrum of toroidal orbifolds. We discuss only $T^4/\mathbb{Z}_2$ and $T^4/\mathbb{Z}_3$ orbifolds for simplicity. As a warming up, we start with cases without discrete Wilson lines. Toroidal orbifolds with the discrete Wilson lines are discussed later.

One has to choose $\gamma_\sigma = \text{diag}(e^{2\pi V^I}, e^{-2\pi i V^I})$ so that

$$\frac{1}{2} \left[ \sum_A |v^A|(1 - |v^A|) - \sum_I V^I(1 - V^I) \right] \equiv 0 \pmod{\frac{1}{k}\mathbb{Z}} \quad (103)$$

for a consistency on the spectrum of a $\sigma$-twisted sector $[52]$. For the $T^4/\mathbb{Z}_2$ orbifold ($k = 2$), solutions are

$$V^I_{r=2} = \frac{1}{2} (1, 1, 0, \ldots, 0), \quad (104)$$

$$V^I_{r=6} = \frac{1}{2} (1, \ldots, 1, 0, \ldots, 0). \quad (105)$$

The spectrum is summarized as follows:

- $r = 2$
  - Untwisted sector
    - $D = 6$ sugra and tensor multiplets,
    - 4 hypermultiplets from $D = 10$ metric and $B$-field,
    - $\text{SU}(2) \times \text{SU}(2) \times \text{SO}(28)$ vector multiplet,
    - $(2, 2, 28)$ hypermultiplet.
  - Twisted sector $\times 16$
    - $(1, 2, 1)$ 4 half hypermultiplets,
    - $(2, 1, 28)$ half hypermultiplet.

- $r = 6$
  - Untwisted sector
    - $D = 6$ sugra and tensor multiplets,

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29We choose $0 \leq V^I \leq 1$ for $I = 1, \ldots, 16$. 

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\* 4 hypermultiplets from \( D = 10 \) metric and \( B \)-field
\* \( \text{SO}(12) \times \text{SO}(20) \) vector multiplet,
\* \( (12,20) \) hypermultiplet.

Twisted sector \( \times 16 \)
\* \( (\text{spin},1) \) half hypermultiplet.

In the \( r = 2 \) case, the symmetry is broken down to \( \text{SU}(2) \times \text{SO}(28) \), if the \( (1,2,1) \) half hypermultiplets develop expectation values. There are \( (1/2) \times 16 \) hypermultiplets in the \( (2,28) \) representation, and there are 2 from the untwisted sectors; there are 10 hypermultiplets in the \( (2,28) \) representation as a whole in the toroidal orbifold calculation. This agrees with the field theory prediction in section A.1.1 for the case with instantons contained only in one of \( \text{SU}(2) \)'s in \( \text{SO}(2r = 4) \simeq \text{SU}(2) \times \text{SU}(2) \).

The twisted sectors and untwisted sector contribute to \( \text{SU}(2) \times \text{SO}(28) \)-singlet moduli hypermultiplets by \( 4 \times 16 - 3 \) and 4, respectively, and there are 65 as a whole. They correspond to \( 3 \times 16 - 3 = 45 \) vector bundle moduli and \( 16 + 4 = 20 = h^{1,1}(K3) \) bulk moduli, as we obtained in section A.1.1 for the case with an \( \text{SU}(2) \) bundle. Roughly speaking, each twisted sector has 4 moduli hypermultiplets, and one of them describes the blow up of the \( C^2/Z_2 \) singularity. Thus, the remaining three twisted sector hypermultiplets (at each \( C^2/Z_2 \) fixed point) describe deformation of the vector bundle.

In the \( r = 6 \) case, expectation values can be given to the hypermultiplets in the \( \text{SO}(12)-\text{spin} \) representation, so that the the \( \text{SO}(12) \) symmetry is completely Higgsed. Let us compare the massless spectra of toroidal orbifold and field-theory prediction in such a situation, only the \( \text{SO}(20) \) gauge symmetry is left unbroken. The \( \text{SO}(20)-\text{vect} \) hypermultiplets arise only from the untwisted sector, and there are 12 as a whole, once again in agreement with the field-theory result, \( 24 - 2r = 12 \), in section A.1.1. The twisted and untwisted sectors yield \( 16 \times 16 - 66 \) and 4 moduli multiplets, and there are \( 190 + 4 = 194 \) moduli in toroidal orbifold calculation. This number of moduli is equal to the number of vector bundle moduli, \( 24 \times 10 - 66 = 174 \), and the number of \( K3 \)-moduli \( 16 + 4 = 20 \) combined.

Thus, the number of moduli and the multiplicity of \( \text{SO}(32 - 2r)-\text{vect} \) hypermultiplets are calculated both by field theory and by orbifold, and they agree. In the two examples of \( T^4/Z_2 \)
orbifolds we studied, the geometry $T^4/\mathbb{Z}_2$ is regarded as a particular limit of a K3 manifold, where 16 2-cycles are collapsed. An example with $V_{\tau=6}^I$ is obtained by taking a limit further in the moduli space of SO(2$\tau=12$) vector bundle on the K3-manifold, a limit where the structure group is reduced from SO(12) until the SO(12) symmetry is enhanced. Likewise, the toroidal orbifold compactification with $V_{\tau=2}^I$ can be approached from a field theory compactification, by taking a limit in the moduli space of SU(2) $\subset$ SO(4) vector bundle. It is a limit where the structure group is reduced from SU(2) until the SU(2) symmetry is enhanced and restored.

Let us also see examples of $T^4/\mathbb{Z}_3$ orbifolds. For the $T^4/\mathbb{Z}_3$ orbifold ($k=3$), solutions to the consistency condition (103) are

$$V_{\tau=2}^I = \frac{1}{3} (1, 1, 0, \cdots, 0), \quad (106)$$
$$V_{\tau=5}^I = \frac{1}{3} (1, \cdots, 1, 0, \cdots, 0), \quad (107)$$
$$V_{\tau=8}^I = \frac{1}{3} (1, \cdots, 1, 0, \cdots, 0). \quad (108)$$

The massless spectra of those models are:

- Untwisted sector of $V_{\tau=2,5,8}^I$ models
  - D = 6 sugra and tensor multiplets,
  - 2 hypermultiplets from D = 10 metric and B-field,
  - SU($\tau$) $\times$ SO($32-2\tau$) $\times$ U(1) vector multiplet,
  - $(r,\text{vect.})^1 + (\wedge^2 r,1)^2$ hypermultiplets,

- Twisted sectors $\times 9$
  - $r = 2$: $(2,28)^1 + 2 \times (1,1)^2 + 5 \times (1,1)^0$ hypermultiplets,
  - $r = 5$: $(1,\text{vect.}) + 2 \times (5,1)^1 + (\wedge^2 5,1)^{-3}$ hypermultiplets,
  - $r = 8$: $(\wedge^2 8,1)^{-2} + 2 \times (1,1)^0$ hypermultiplets.

By turning on expectation values in hypermultiplets in the $(\wedge^2 r,1)$ representation, the symmetry can be broken down to SO($32-2\tau$) for the case $r = 5,8$ [to SU(2) $\times$ SO(28) for $r = 2$]. One can explicitly check that there are $24-2\tau$ hypermultiplets in the SO($32-2\tau$)-$\text{vect.}$ representation [($24-2\tau)/2$ in the SU(2) $\times$ SO(28)-(2,$\text{vect.}$) representation], in agreement with the field-theory calculation. The number of singlet moduli are also equal to the sum of the vector bundle moduli and $h^{1,1} = 20$ K3 moduli. Since the two singlet hypermultiplets in the
untwisted sector are genuine $K3$ moduli, remaining 18 are from the 9 twisted sectors. Thus, roughly speaking, each twisted sector at $\mathbb{C}^2/\mathbb{Z}_3$ has two hypermultiplets for the $K3$ moduli and all the other singlet hypermultiplets in each twisted sector correspond to the vector bundle moduli. This is in good agreement because two 2-cycles emerge from the blow up of a $\mathbb{C}^2/\mathbb{Z}_3$ singularity (see footnote 31). The geometry $T^4/\mathbb{Z}_3$ orbifold is a limit of a $K3$-manifold, where $2 \times 9$ 2-cycles are collapsed. The toroidal orbifold compactifications with $V^I_{r=2,5,8}$ in (106, 108) are obtained on top of such a singular “manifold”, by taking a limit in the moduli space of $SU(2)$, $SO(10)$ and $SO(16)$-bundle compactification. For the cases with $r = 5$ and 8, this is a limit where the structure groups are reduced from $SO(10)$ ($r = 5$) and $SO(16)$ ($r = 8$) to $U(1)$, so that $SU(5)$ and $SU(8)$ symmetries are restored. Thus, the vector bundles have become line bundles at the orbifold limit. The $U(1)$ structure group of the line bundles is also restored as a global symmetry there, but we will argue in the appendix A.1.3 that a massless gauge field of the $U(1)$ symmetry does not remain in the spectrum.

**Cases With Discrete Wilson Lines**

Let us now look at toroidal orbifold compactifications with discrete Wilson lines $W^I_a \neq 0$. Only a couple of examples are examined in the following, and we think that it is enough to see that such compactifications are also nothing more than special limits of geometric smooth-manifold compactification.

Suppose that an orbifold $T^4/\mathbb{Z}_k$ is a quotient of $\mathbb{C}^2$ by a space group generated by a rotation $\sigma$ ($\sigma^k = \text{id}$) and translations $\tau_a$ ($a = 1, 2, 3, 4$). Associated with each element of the space group, say, $\tau_a^m \circ \sigma^n$, is a ($\tau_a^m \circ \sigma^n$)-twisted sector, quantized states of worldsheet fields satisfying a boundary condition $\Psi(\sigma + 2\pi) = (\tau_a^m \circ \sigma^n)(\Psi)(\sigma)$, where $\Psi$ denotes worldsheet fields, $Z, \overline{Z}, \psi, \overline{\psi}, \lambda$ and $\overline{\lambda}$. In the presence of non-trivial discrete Wilson lines, 32 left-moving fermions are twisted by a matrix

$$\gamma^m_n \cdot \beta^m_a = \text{diag} \left( e^{2\pi i (nV^I + m^aW^I_a)}, e^{-2\pi i (nV^I + m^aW^I_a)} \right).$$

(109)

A consistency condition corresponding to (103) should be satisfied for each twisted sector, where $V^I$ in (103) is replaced by $nV^I + m^aW^I_a \mod \mathbb{Z}$, chosen in an interval $[0 : 1]$ for the ($\tau_a^m \circ \sigma^n$)-twisted sector.

The generators of the space group satisfy algebraic relations such as

$$\tau_a^m \circ \sigma = \sigma \circ \tau_a^m.$$  

(110)

The twist matrices $\gamma_\sigma$, $\beta^m_a$ and $\beta^{m'}_a$ corresponding to the generators $\sigma$, $\tau_a^m$ and $\tau_a^{m'}$ should
also satisfy corresponding relations

\[ m_a W_a^I \equiv m'_a W'_a \mod 2\pi \mathbb{Z}. \]  (111)

\((\tau^a \circ \sigma)\)-twisted sector is localized at a fixed point \(x\) satisfying \(\sigma^n x + m a e_a = x\). Because of \(\pm \Lambda\) ambiguity in the fixed points \(x\), twisted sectors are grouped into \(\Lambda/(\sigma - \text{id})\Lambda\). Consistency conditions like (109) have to be satisfied for \(nV^I + m a W_a^I\) for each one of \(\Lambda/(\sigma^n \text{id})\Lambda\).

**Example A:** The following choice of the discrete Wilson line is consistent with \(T^4/\mathbb{Z}_2\) orbifold with \(V_{r=2}^I\) in (104):

\[ W_1^I = \frac{1}{2} (1, \cdots, 1, 0, \cdots, 0), \quad W_{2,3,4}^I = 0. \]  (112)

Eight twisted sectors have a twist vector \(V^I\), while eight others have \((V + W_1)\); they are given \((\mod Z)\) by

\[ V_{r=2}^I \equiv \frac{1}{2} (2, 1, 0, 0, \cdots, 0), \quad (V_{r=2} + W_1)^I \equiv \frac{1}{2} (2, 0, 0, 1, 1, 0, \cdots, 0). \]  (113)

The unbroken symmetry is \(\text{SO}(4) \times \text{SO}(4) \times \text{SO}(24)\) at the orbifold limit, but it can be broken down to \(\text{SO}(24)\) by turning on vev’s in some of hypermultiplets. Each fixed point has one massless hypermultiplet in the \(\text{SO}(24)\)-vect. representation, while such multiplet is absent in the untwisted sector. Thus, there are overall 16 hypermultiplets in the vector representation, which agrees with the multiplicity \((24 - 2r)\) in the case of \(r = 4\) of the smooth-manifold calculation, with an \(\text{SO}(8)\) bundle and \(\text{SO}(24)\) unbroken symmetry. The massless spectrum calculated through the orbifold technique have 136 \(\text{SO}(24)\) singlets, after the \(\text{SO}(4) \times \text{SO}(4)\) symmetry breaking absorbs 12 hypermultiplets. This agrees with the sum of the number of vector bundle moduli, \(24(2r - 2) - r(2r - 1) = 116\), and of the K3 moduli, 20. Thus, this orbifold compactification can be regarded as a limit of smooth K3 manifold compactification with a rank-4 bundle. Even a toroidal orbifold with non-trivial discrete Wilson line is regarded as a limit of a smooth-manifold compactification with a vector bundle. Not only the moduli spaces of rank-2, 5, 6, 8 bundles but also that of rank-4 bundle contains an orbifold point.

**Example B:** The \(T^4/\mathbb{Z}_2\) orbifold with the twist \(V_{r=2}^I\) in (103) is also consistent with the following discrete Wilson lines:

\[ W_1^I = \frac{1}{2} (2, 0, 1, 1, 0, 0, 0, \cdots, 0), \quad W_2^I = \frac{1}{2} (2, 0, 1, 1, 0, 1, 0, \cdots, 0), \quad W_{3,4}^I = 0. \]  (114)

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\(^{32}\)The unbroken symmetries at fixed points (in twisted sectors) are determined by the twist vectors associated with the fixed points. The symmetry group at fixed points with the twist \(V^I\) and those with \((V + W_1)^I\) are different subgroups of \(\text{SO}(32)\), though they are both \(\text{SO}(4) \times \text{SO}(28)\).
The sixteen fixed points of $T^4/Z_2$ are classified into 4 groups of four fixed points, and each group has its own twist vector given by

$$(V + W_2)^I \equiv \frac{1}{2} (2,0,0,0,1,1,0,\cdots,0), \quad (V + W_1 + W_2)^I \equiv \frac{1}{2} (2,2,2,2,10,\cdots) \quad (115)$$

$$V^I \equiv \frac{1}{2} (2,2,2,2,10,\cdots), \quad (V + W_1)^I \equiv \frac{1}{2} (0,0,1,1,0,0,\cdots) \quad (116)$$

The unbroken symmetry is $SO(4) \times SO(4) \times SO(4) \times SO(20)$ at the orbifold limit, which can be broken down to $SO(20)$ by giving vev’s in some of hypermultiplets. Massless hypermultiplets in the $SO(20)$ - {f vect.} representation are not found in the untwisted sector or in the four twisted sectors with the twist vector $(V + W_1 + W_2)^I$. The twelve other twisted sectors, whose twist vectors are $V$, $V + W_1$ and $V + W_2$, have one massless $SO(20)$ - {f vect.} hypermultiplet each, and there are twelve as a whole. This multiplicity agrees with the smooth-manifold calculation of the rank-6 bundle, $[24 - 2r] = 12$. The orbifold calculation yields 194 $SO(20)$-singlet hypermultiplets (after Higgsing $SO(4) \times SO(4) \times SO(4)$), which agrees with the sum of 174 vector bundle moduli and 20 K3 moduli of the smooth-manifold calculation.

Thus, the $T^4/Z_2$ orbifold with $V^I_{r=6}$ and $W^I = 0$ and with $V^I_{r=2}$ and $W^I$ in (114) are both regarded as special limits in the moduli space of $SO(12)$ vector bundle, limits where the structure group is reduced and the unbroken symmetry is enhanced. In the case with $V^I_{r=6}$ and $W^I = 0$, instantons are squeezed in the $U(1)$ generated by a charge vector $q = \text{diag}(V^I_{r=6}, -V^I_{r=6})$ at all the 16 collapsed 2-cycles. In the case with $V^I_{r=2}$ and the discrete Wilson lines in (114), however, they are squeezed in a $U(1)$ subgroup generated by $q = \text{diag}((V^I_{r=2} + m^a W_a)^I, -(V^I_{r=2} + m^a W_a)^I)$ at the collapsed 2-cycles at $(m^a e_a)/2$; the charge vector $q$ can be different at different collapsed 2-cycles.

**Example C:** One can introduce discrete Wilson lines in a $T^4/Z_3$ orbifold with the twist $V^I_{r=2}$ as follows:

$$W^I_1 = W^I_2 = \frac{1}{3} (2,0,0,1,1,1,0,\cdots) \quad (117)$$

$$W^I_{3,4} = 0$$

The nine fixed points (twisted sectors) are grouped into 3 sets of three fixed points (twisted
sectors) whose twist vectors are

\[
\sigma \text{-twisted : } V^I_{r=2} = \frac{1}{3}(1, 1, 0, 0, 0, \ldots, 0),
\]

\[
(\tau_1 \cdot \sigma) \text{-twisted : } (V^I_{r=2} + W_1)^I = \frac{1}{3}(1, 1, 1, 1, 0, \ldots, 0),
\]

\[
(\tau_1 \cdot \tau_2 \cdot \sigma) \text{-twisted : } (V^I_{r=2} + W_1 + W_2)^I = \frac{1}{3}(1, 1, 2, 2, 0, \ldots, 0).
\]

The unbroken symmetry at the orbifold limit is SU(2) \times SU(3) \times SO(22) \times U(1) \times U(1), but all the factors other than SO(22) can be Higgsed away. The total number of massless hypermultiplets in the SO(22)-vect. representation is 14, in agreement with the smooth-manifold result for a rank 5 bundle (and the unbroken SO(22) symmetry). The number of SO(22)-singlet hypermultiplets of this toroidal orbifold also agrees with the smooth-manifold calculation.

The T^4/Z_3 orbifold with V_{r=5} and W_a = 0 and with V_{r=2} and W_a given in Equation (117) are both special points of the moduli space of K3 compactification with a rank-5 vector bundle. The SO(10) instantons are squeezed in U(1) subgroups generated by \( q = \text{diag}(V_{r=5}, -V_{r=5}) \) on all of nine collapsed C_2/Z_3 singularities of a K3 manifold in the case without a discrete Wilson line, whereas they are squeezed in 3 different U(1) subgroups at 3 different groups of C^2/Z_3 singularities in the case with the discrete Wilson lines (117):

\[
U(1) \text{ along } q = \text{diag}(V_{r=2}, -V_{r=2}) \text{ at } \frac{m}{3}(e_3 + e_4),
\]

\[
q = \text{diag}((V_{r=2} + W_1), -(V_{r=2} + W_1)) \text{ at } \frac{1}{3}(2e_1 + e_2 + m(e_3 + e_4)),
\]

\[
q = \text{diag}((V_{r=2} + W_1 + W_2), -(V_{r=2} + W_1 + W_2)) \text{ at } \frac{1}{3}(e_1 + 2e_2 + m(e_3 + e_4)).
\]

The moduli space of rank-5 bundle compactification contains more orbifold points than the two explicitly described above; W_4^I = W_5^I can be multiplied by a factor of 2, and W_3^I = W_4^I can also be non-zero. At the toroidal orbifold limits with non-trivial discrete Wilson lines, the U(1) subgroups in which the instantons are squeezed (that is, the structure group of line bundles) can be different from one singularity to another. Variety of the choice of W_d correspond to the variety of finding such U(1) subgroups in which the instantons are squeezed. Apart from that, there is no essential difference between toroidal orbifolds with or without discrete Wilson lines. They are all special limits of a simply-connected K3-manifold compactification with a vector bundle on it.

\footnote{They come from two from each fixed point at \( m(e_3 + e_4)/3 \), one from each fixed point at either \( (2e_1 + e_2 + m(e_3 + e_4))/3 \) or \( (e_1 + 2e_2 + m(e_3 + e_4))/3 \) and 2 from the untwisted sector.}
A.1.3 Discussion

We have seen that the toroidal orbifold compactifications of the Heterotic theory corresponds to special points in the moduli space of compactifications with Calabi–Yau and a vector bundle on it. At the orbifold points, the structure group of the vector bundle is reduced and an unbroken symmetry is enhanced. This interpretation holds true regardless of “discrete Wilson lines” are used or not. If the twist vectors $V^I$’s and $W^a_I$’s are arranged so that U(1) symmetries are left, then the structure group of the bundle contains the U(1) symmetries. That is, the bundle contains line bundles at such orbifold limits.

It is important to note that U(1) symmetries in effective theory below the Kaluza–Klein scale does not imply that the low-energy spectrum has corresponding massless vector field. This is why we put all the U(1) factors in brackets in the examples of toroidal orbifolds in the appendix A.1.2. If we label multiple U(1) factors of the structure group at various fixed points by $a, b$, then the effective lagrangian contains

$$\begin{align*}
-\frac{1}{2g^2}C_{ab}(\partial A^a)(\partial A^b) + \frac{1}{2}G_{kl}(T, T^\dagger)Q^k_a A^a Q^l A^b,
\end{align*}$$

(124)

where the second term comes from (6). At the orbifold limits, U(1) symmetries in the directions spanned by $q$’s may be preserved as global symmetries, but the gauge fields acquire mass terms from the second term in the effective action above. The $B$-field fluctuations $b^k \omega_k$ in (6) will be played by twisted sector fields in orbifold language. Multiple U(1) gauge fields acquire large masses from the Stuckelberg form interactions in Heterotic compactifications, and toroidal orbifolds with or without “discrete Wilson lines” are not exceptions.

If orbifold projection conditions are to be used in breaking the $SU(5)_{GUT}$ symmetry down to $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry in Heterotic theory, then such toroidal orbifold compactifications are regarded as limits where vector bundle has a structure group $U(1) \times U(1) \times \cdots \subset E_8$. Whether the discrete Wilson lines are used or not does not make an essential difference in this argument. The charge vector for the hypercharge $q_Y$ is not orthogonal to all the charge vectors of the structure groups of the line bundles above. Thus, the $U(1)_Y$ vector field also has a large mass term through (6, 124) in such toroidal orbifolds.

The Stuckelberg coupling with the dilaton chiral multiplet vanishes in compactifications of the Heterotic $E_8 \times E_8'$, as long as the $U(1)_Y[SU(3)_C]^2$ mixed gauge anomaly vanishes. The $U(1)_Y$ symmetry can be preserved (approximately) as a global symmetry in low-energy effective theory,

\footnote{It is desirable to confirm by explicit orbifold calculations that such couplings do exist, but we do not do this in this article.}
if the vev's of vector bundle moduli (twisted/untwisted sector fields) are chose appropriately. Yet, there may not be massless U(1) \_Y gauge field in the low-energy spectrum, because of the Stuckelberg coupling with the Kähler chiral multiplets (twisted sector fields).

The main text of this article also proposes an idea of how to get out of this U(1) \_Y problem. If one can find a U(1) symmetry in the hidden sector \( E'_8 \) that is a structure group of a line bundle of compactification, and if the first Chern class of the U(1) line bundle is the same as that of U(1) \_Y, then their linear combination U(1) \( \tilde{Y} \) remains massless. If the hidden sector is strongly coupled, then the gauge coupling constant of U(1) \( \tilde{Y} \) still satisfies the GUT relation approximately.

A.2 Continuous Wilson Lines as Vector Bundle Moduli

Example D: Let us study a following example, to see the claim in the title of this subsection. We consider \( T^4/\mathbb{Z}_3 \) orbifold, and take

\[
\gamma_\sigma = \tilde{\gamma}_\sigma \oplus \tilde{\gamma}_\sigma^{T-1}, \quad \tilde{\gamma}_\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \oplus 1_{13 \times 13},
\]

(125)

\[
\beta_{a=1} = \tilde{\beta}_{a=1} \oplus \tilde{\beta}_{a=1}^{T-1}, \quad \tilde{\beta}_{a=1} = \text{diag}(e^{i\alpha}, e^{i\beta}, e^{-i(\alpha+\beta)}) \oplus 1_{13 \times 13},
\]

(126)

\[
\beta_{a=1} = \tilde{\beta}_{a=1} \oplus \tilde{\beta}_{a=1}^{T-1}, \quad \tilde{\beta}_{a=2} = \text{diag}(e^{-i(\alpha+\beta)}, e^{i\alpha}, e^{i\beta}) \oplus 1_{13 \times 13}.
\]

(127)

Those matrices for the orbifold twists are chosen so that they satisfy algebraic relations

\[
\sigma \circ \tau_{a=1} \circ \sigma^{-1} = \tau_{a=2} \quad \rightarrow \quad \gamma_\sigma^{-1} \cdot \beta_{a=1} \cdot \gamma_\sigma = \beta_{a=2},
\]

(128)

\[
\sigma \circ \tau_{a=2} \circ \sigma^{-1} = (\tau_{a=1} + \tau_{a=2})^{-1} \quad \rightarrow \quad \gamma_\sigma^{-1} \cdot \beta_{a=2} \cdot \gamma_\sigma = (\beta_{a=2} \cdot \beta_{a=1})^{-1}.
\]

(129)

These relations are satisfied for any values of \( \alpha, \beta \in \mathbb{R} \), and hence this is called the continuous Wilson lines. Certainly the matrix \( \beta_{a=1,2} \) are the ordinary Wilson lines on torus \( T^4 \), in the absence of orbifold projection by \( \mathbb{Z}_3 \). This is a typical situation where we have a continuous Wilson line. Although the continuous Wilson lines \( (\alpha, \beta) \) are introduced only in one of the two complex planes of \( T^4 \) for simplicity, continuous Wilson lines can be introduced for the other complex plane, too. Thus, there are four real-scalar degrees of freedom in the continuous Wilson lines in this example.

This example should correspond to a SU(3) \( \subset \) SO(6) \( \subset \) SO(32) bundle compactification on a K3 manifold, which leaves U(1) \times SO(26) unbroken symmetry. Therefore, one should have

- D=6 supergravity multiplet and a D=6 tensor multiplet,
• $h^{1,1} = 20$ hypermultiplets coming from moduli of K3,
• D=6 SO(26) vector multiplet,
• 18 hypermultiplets in the vector representation of SO(26),
• 18 SO(26)-singlet hypermultiplets that are charged under the U(1) symmetry, and
• 64 completely neutral hypermultiplets coming from vector bundle moduli.

The spectrum can also be calculated using the standard techniques in toroidal orbifolds. Each one of the twisted sectors localized at $9 \mathbb{C}^2/\mathbb{Z}_3$ singularity contribute to the spectrum of hypermultiplets by 2 in the vector representation, 2 in the U(1) charged ones, and 9 in the U(1) neutral ones. Thus, all of the hypermultiplets in the 4th and 5th items in the above list are accounted for in the orbifold calculation. The $9 \times 9$ neutral hypermultiplets account for $9 \times 2$ of the K3 moduli hypermultiplets and $9 \times 7$ of the vector bundle moduli; each $\mathbb{C}^2/\mathbb{Z}_3$ singularity has two hypermultiplet worth of resolution/deformation degrees of freedom. Among the twisted-sector spectrum of neutral hypermultiplets, two are still missing in the moduli of K3, and one in the bundle moduli.

Gravitational part of the untwisted sector gives rise to two neutral hypermultiplets, and hence all the 20 hypermultiplets for the K3 moduli are recovered from toroidal orbifold calculation. The SU(3)-adjoint part of the untwisted sector leaves one massless hypermultiplets, and this is identified with the remaining one vector bundle moduli. This hypermultiplet takes values in the diagonal entries of $3 \times 3$ matrix in the basis that diagonalises $\tilde{\beta}$ as in (125–127).

One can further see from the orbifold calculation that two more hypermultiplets become massless if $\alpha = \beta = 0$, and the unbroken symmetry is enhanced to $\text{SO}(26) \times \text{U}(1) \times \text{U}(1) \times \text{U}(1)$. This phenomenon is better understood in a frame that diagonalizes the twisting matrix $\tilde{\gamma}_\sigma$ rather than $\tilde{\beta}_{a=1,2}$. Generators of $\tilde{\beta}_{a=1,2}$ and the hypermultiplets from the untwisted sector take their values now in off-diagonal entries of the $3 \times 3$ matrix of adjoint SU(3), and the symmetry breaking $\text{U}(1) \times \text{U}(1) \times \text{U}(1) \rightarrow \text{U}(1)$ is understood as the Higgs mechanism due to the vev in the untwisted-sector hypermultiplet. Put another way, vev’s in the untwisted sector hypermultiplet correspond to deformation of vector bundle that enlarges the structure group from $\text{U}(1) \times \text{U}(1)$ to SU(3). That is, the continuous Wilson line (and the vev’s of the untwisted sector hypermultiplets) studied in this example corresponds a part of vector bundle moduli explained above.

Continuous Wilson lines exist in cases where the twisting matrix $\gamma_\sigma$ acts as permutation. When a basis is chosen so that $\gamma_\sigma$ is diagonal, generators of the continuous Wilson lines $\beta$ becomes off-diagonal, and the off-diagonal vev’s enlarge the structure group of vector bundle.
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51
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54
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