Extra force from an extra dimension. Comparison between brane theory, STM and other approaches

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Abstract

We investigate the question of how an observer in 4D perceives the five-dimensional geodesic motion. We consider the interpretation of null and non-null bulk geodesics in the context of brane theory, space-time-matter theory (STM) and other non-compact approaches. We develop a “frame-invariant” formalism that allows the computation of the rest mass and its variation as observed in 4D. We find the appropriate expression for the four-acceleration and thus obtain the extra force observed in 4D. Our formulae extend and generalize all previous results in the literature. An important result here is that the extra force in brane-world models with $\mathbb{Z}_2$-symmetry is continuous and well defined across the brane. This is because the momentum component along the extra dimension is discontinuous across the brane, which effectively compensates the discontinuity of the extrinsic curvature. We show that brane theory and STM produce identical interpretation of the bulk geodesic motion. This holds for null and non-null bulk geodesics. Thus, experiments with test particles are unable to distinguish whether our universe is described by the brane world scenario or by STM. However, they do discriminate between the brane/STM scenario and other non-compact approaches. Among them the canonical and embedding approaches, which we examine in detail here.

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1 Introduction

The concept that our world may be embedded in a universe that possesses more than four dimensions has a long and distinguished history. In theoretical physics, it can be traced back to the pioneers works of Kaluza and Klein [1-2] who interpreted the electromagnetic field as a geometrical effect of a hidden fifth dimension. Currently, theories of the Kaluza-Klein type in many dimensions are used in different branches of physics. Superstrings (10D) and supergravity (11D) are well known examples [3].

In higher-dimensional gravity theories, the scenario is that matter fields are confined to our four-dimensional universe, a 3-brane, in a 1 + 3 + d dimensional spacetime, while gravity propagates in the extra d dimensions as well [4-8]. In these theories there are several motivations for the introduction of extra dimensions. Among them to resolve the differences between gravity and quantum field theory and ultimately unify all forces of nature. Also, as providing possible solutions to the hierarchy and the cosmological constant problems [7-8].

The idea of extra dimensions is also inspired by the vision that matter in 4D is purely geometric in nature. In space-time-matter theory (STM) one large extra dimension is needed in order to get a consistent description, at the macroscopic level, of the properties of the matter as observed in 4D [9-13]. The mathematical support of this theory is given by a theorem of differential geometry due to Campbell and Magaard [14-17].

Although these theories have different motivations for the introduction of extra dimensions, they confront similar challenges. From a theoretical viewpoint, they have to predict observable effects from the extra dimensions. From an experimental viewpoint, the vital issue is the discovery of new physical phenomena, which could unambiguously be associated with the existence of extra dimensions.

A possible way of testing for new physics coming from extra dimensions is to examine the dynamics of test particles. In practice this means to search for deviations from the universal “free fall” in 4D. For that reason the geodesic motion on 5D manifolds and 4D submanifolds has been a subject of intensive investigations [16-29]. Two main results have emerged from the dimensional reduction of geodesics in 5D. Firstly, that the free motion in 5D is observed in 4D as being under the influence of a non-gravitational force, if the velocity of the test particle has non-zero component along the extra dimension. Secondly, since the extra force has a component which is parallel to the particle’s four-velocity, the rest mass is observed to vary with time.

These results are important in view of their potential experimental/observational relevance. However, their interpretation and the new physics related to them, is not clear yet. In fact, despite some successful applications, the implementation of these results in the context of brane theory and STM has lead to a number of statements and conclusions that we believe should be reconsidered.

One of them is that the extra force cannot be implemented directly in brane-world models, in the RS2 scenario [21-23]. A related statement is that the extra force is zero in brane-world cosmological models with $\mathbb{Z}_2$ symmetry.

In this work we bring a positive perspective to this topic. We demonstrate that, in brane-world models with $\mathbb{Z}_2$-symmetry, the extra force is continuous and well defined across the brane. We show, by means of explicit examples, that the extra force in cosmological models with $\mathbb{Z}_2$ symmetry is not necessarily zero. We also illustrate how the force and mass, as observed on the three-brane, depend on whether the bulk motion is along null or non-null geodesics.

Another new discovery in this work is that brane theory and STM produce the same results for test particles as observed in 4D. Consequently, for the computation of the extra force and mass we can ignore the details of whether the bulk geodesic motion is interpreted on the non-singular hypersurface of STM or on the singular hypersurface of brane theory. This equivalence has nothing to do with the dynamics in 4D, but it is a result of the assumption that test particles move along five-dimensional geodesics in both theories. From an observational viewpoint, this means that experiments measuring the extra force acting on test particles are not able to discriminate whether our universe is described by the brane world scenario or by STM. In order to settle this point, a self-consistent analysis of the combination of physical, astrophysical and cosmological effects like in Ref. [20] should be made.

We also elucidate some important issues related to the interpretation of STM in 4D. We refer to the concept that the extra force can be made to disappear by changing the parameterization of the metric. This seems to be related to the geodesic approach where the mass of the particle depends on the affine parameters used to describe the motion in 5D and 4D [21].

In this work we show how the rest mass as well as the extra force as observed in 4D crucially depend on the
method we use to identify the 4D metric from the 5D one. In particular we point out that, unlike the case of brane theory, each bulk metric in STM can be used to generate at least six different expressions for the mass and force as observed in 4D (for null and non-null bulk geodesics). This wealth of interpretations is not a consequence of changing any parameter in the bulk metric, but it is an attribute of STM, where the number of physical restrictions in the theory is not in general sufficient to determine the properties in 4D.

We also clarify the question of whether the extra force is a pure consequence of the fact that the bulk metrics in brane theory and STM are allowed to depend on the extra coordinate. This is certainly true when the metric along the fifth dimension is flat. However, in general we find that a large extra dimension does not necessarily imply the existence of an extra non-gravitational force. Conversely, in general a compact extra dimension does not preclude the existence of an extra force.

The structure of the paper is as follows: In section 2 we recall the definition of the relativistic force four-vector in covariant and contravariant components. We also recall some requirements on the covariant derivative in 4D. In section 3 we present the bulk metric and develop a “frame-invariant” formalism that allows the computation of rest mass and its variation as observed in 4D. In section 4 we expound some technical problems which arise when the covariant derivative in 5D is used in 4D. Next, we define the appropriate covariant derivative in 4D and find the four-acceleration, which satisfies physical conditions. This allows us to find the four-force in agreement with the definition given in section 2. Throughout the discussion we consider the interpretation of null and non-null geodesic motion in the bulk. In sections 5 and 6 we apply our formalism to the brane-world scenario and STM respectively. In section 7 we discuss the canonical metric and the foliating approach as alternative interpretations to STM. Finally, in section 8 we give a summary.

2 Definition of force in 4D

Here we present the definition of relativistic four-force that we are going to use throughout this paper. In four dimensions the motion of a test particle is described by its four velocity

\[ u^\mu = \frac{dx^\mu}{ds}, \quad u_\mu u^\mu = 1. \]  

(1)

The four-momentum of a particle of rest mass \( m_0 \) is defined as

\[ p^\mu = m_0 u^\mu, \quad p_\mu = m_0 u_\mu. \]  

(2)

In special relativity, in Cartesian coordinates the four-force acting on a test particle is given by

\[ F^\mu = \frac{dp^\mu}{ds} = \frac{d}{ds} (m_0 u^\mu). \]  

(3)

Thus,

\[ \frac{F^\mu}{m_0} = \frac{du^\mu}{ds} + \frac{u^\mu}{m_0} \frac{dm_0}{ds}. \]  

(4)

If the rest mass of the particle is constant along its motion, then the 4D-force is orthogonal to the four velocity, i.e., \( F^\mu u_\mu = 0 \). Otherwise, the four-force has a component parallel to the four-velocity such that \( F^\mu u_\mu = \frac{dm_0}{ds} \).

In curvilinear coordinates the metric of the spacetime is described by a symmetric tensor \( g_{\mu\nu} \). In such coordinates, the appropriate generalization of (4) is

\[ \frac{F^\mu}{m_0} = \frac{D^{(4)} u^\mu}{ds} + \frac{u^\mu}{m_0} \frac{dm_0}{ds}, \]  

(5)

where \( D^{(4)} \) denotes the covariant differential calculated in 4D, i.e.,

\[ D^{(4)} g_{\mu\nu} = 0. \]  

(6)

The indexes of four-vectors and four-tensors are lowered and raised with the aid of \( g_{\mu\nu} \). For instance,

\[ u_\mu = g_{\mu\nu} u^\nu, \quad F_\mu = g_{\mu\nu} F^\nu. \]  

(7)
Consequently,
\[ u_\mu D^{(4)} u^\mu = u^\mu D^{(4)} u_\mu = 0, \] (8)
and the covariant components of the four-force are given by
\[ \frac{F_\mu}{m_0} = \frac{D^{(4)} u_\mu}{ds} + \frac{u_\mu}{m_0} \frac{d m_0}{ds}, \] (9)
In this way
\[ F_\mu u_\mu = \frac{d m_0}{ds}, \] (10)
is valid not only in Cartesian coordinates, but in all coordinate systems. We will use these properties in section 4.

3 Motion in higher dimensions

In the Randall-Sundrum brane-world scenario and other non-compact Kaluza-Klein theories, the motion of test particles is higher-dimensional in nature. In other words, all test particles travel on five-dimensional geodesics but observers, who are bounded to spacetime, have access only to the 4D part of the trajectory. From a mathematical viewpoint, this means that the equations governing the motion in 4D are projections of the 5D equations on the 4D-hypersurfaces orthogonal to some vector field \( \psi^A \). The corresponding projector can be written as
\[ h_{AB} = \gamma_{AB} - \epsilon \psi^A \psi_B, \] (11)
where \( \gamma_{AB} \) is the five-dimensional metric and the factor \( \epsilon \) can be \(-1 \) or \(+1 \) depending on whether the extra dimension is spacelike or timelike, respectively. In what follows we will consider the background 5D metric
\[ ds^2 = \gamma_{\mu\nu}(x^\rho, y) dx^\mu dx^\nu + \epsilon \Phi^2(x^\rho, y) dy^2, \] (12)
where \( \gamma_{\mu\nu} \) is the metric induced in 4D. The vector \( \psi^A \), orthogonal to spacetime is given by
\[ \psi^A = (0, 0, 0, 0, \Phi^{-1}), \quad \psi_A \psi^A = \epsilon. \] (13)
In order to obtain the four-dimensional interpretation of the geodesic motion in 5D, we have to decide how to identify the physical or observable spacetime metric from the induced one. In brane-world theory and STM the spacetime metric \( g_{\mu\nu} \) is commonly identified with \( \gamma_{\mu\nu} \). However, in some approaches the physical metric in 4D is assumed to be conformally related to the induced one, viz.,
\[ ds^2 = \Omega(y) g_{\mu\nu}(x^\rho, y) dx^\mu dx^\nu + \epsilon \Phi^2(x^\rho, y) dy^2, \]
\[ = \Omega(y) ds^2 + \epsilon \Phi^2(x^\rho, y) dy^2, \] (14)
where \( \Omega(y) \) is called “warp” factor and satisfies the obvious condition that \( \Omega > 0 \). This line element is more general than the Randall-Sundrum metric, the so-called canonical metric, and encompasses all the metrics generally used in brane-world and STM theories. The object of this section is to examine the motion of test particles in the background metric \( \Omega \).
In order to facilitate the discussion and make the presentation self-consistent, we will give a brief review of our formalism [28] for the effective rest mass \( m_0 \), and its variation along the observed trajectory in 4D, as an effect caused by the motion (momentum) along the extra dimension. Some technical details of the discussion depend on whether the test particle in 5D is massive or massless. We therefore approach these two cases separately.
3.1 Massive particles in 5D

Let us consider a massive test particle moving in a five-dimensional manifold with metric $\gamma_{AB}$. The momentum $P^A$ of such a particle (extending the dynamics of test particles from 4D to 5D) is defined in the usual way, namely,

$$P^A = M^{(5)} \left( \frac{dx^\mu}{dS}, \frac{dy}{dS} \right),$$

(15)

where $M^{(5)} > 0$ is the constant five-dimensional mass of the particle and $U^A = (dx^\mu/dS, dy/dS)$ is the velocity in 5D. Thus $U^A U_A = 1$ and

$$P^A P_A = M^{2(5)}.$$  

(16)

We note that five-dimensional indexes are lowered and raised with the aid of the 5D metric $\gamma_{AB}$.

The five-dimensional motion is perceived by an observer in 4D as the motion of a particle with four-momentum $p_\mu$. Consequently, the effective rest mass in 4D is given by

$$p_\alpha p^\alpha = m_0^2,$$  

(17)

where the four-dimensional indexes are lowered and raised by the spacetime metric $g_{\mu\nu}$. Because of the absence of cross terms in (14), the 4D components of $P_A$ and $P^A$ (i.e., $A = 0, 1, 2, 3$) are already “projected” onto spacetime. Namely,

$$p_\mu = h_{\mu A} P^A = h_{\mu\nu} P^\nu = \Omega g_{\mu\nu} P^\nu = P_\mu.$$  

(18)

Thus from (16) we get

$$m_0^2 + \Omega(y) P_4 P^4 = \Omega(y) M^{2(5)}.$$  

(19)

Therefore, the relation between the rest mass in 4D and 5D is given by

$$m_0 = \sqrt{\Omega M^{(5)}} \left[ 1 + \frac{\epsilon \Phi^2}{\Omega} \left( \frac{dy}{ds} \right)^2 \right]^{-1/2}.$$  

(20)

This equation is the five-dimensional counterpart to $m = m_0 [1 - v^2]^{-1/2}$, for the variation of particle’s mass due to its motion in spacetime. It shows how the nature of the extra dimension and the motion in 5D affect the rest mass measured in 4D. It allows us to conclude that $m_0$ depends on (i) the mass of the particle in 5D, (ii) the character of motion in 5D, i.e. on $dy/ds$, and (iii) the nature of the extra coordinate, i.e., whether it is spacelike or timelike.

3.1.1 Variation of rest mass for $M^{(5)} \neq 0$

From (19) it follows that if the trajectory in 5D lies entirely on a hypersurface $y = const$, i.e. if $P^4 = P_4 = 0$, then the observed mass in 4D is constant. The opposite happens if the five-dimensional motion has non vanishing velocity along $y$. In this case the rest masses, measured by an observer in 4D, in general vary along the trajectory.

In order to find the observed variation of $m_0$ we have to evaluate $dm_0/ds$. This requires the computation of $dP^4/ds$ and $dP_4/ds$, which can be easily done from the geodesic equation in 5D,

$$\frac{dU^A}{dS} + K^A_{BC} U^B U^C = 0,$$  

(21)

where $U^A = (dx^\mu/dS, dy/dS)$ is the five-velocity and $K^A_{BC}$ is the Christoffel symbol formed with the 5D metric $\gamma_{AB}$. We also have to use the relationship

$$\frac{dS}{M^{(5)}} = \Omega \frac{ds}{m_0},$$  

(22)

which follows from (14) and (20).
Thus, setting $A = 4$ in (21) we obtain
\[
\frac{1}{m_0} \frac{dP^4}{ds} = \frac{\epsilon}{2\Omega \Phi^2} \partial (\Omega g_{\mu \nu}) u^\mu u^\nu - \frac{2u^\mu}{\Omega \Phi} \frac{\partial \Phi}{\partial x^\mu} \left( \frac{dy}{ds} \right) - \frac{1}{\Omega \Phi} \frac{\partial \Phi}{\partial y} \left( \frac{dy}{ds} \right)^2,
\]
where $u^\mu = (dx^\mu/ds)$ is the usual four-velocity of the particle. Also, for the covariant component we get
\[
\frac{1}{m_0} \frac{dP_4}{ds} = \frac{1}{2\Omega} \frac{\partial (\Omega g_{\mu \nu})}{\partial y} u^\mu u^\nu + \frac{\epsilon \Phi}{\Omega} \frac{\partial \Phi}{\partial y} \left( \frac{dy}{ds} \right)^2,
\]
(24)

Now, taking derivative of (19) and using the above expressions (23) we obtain the variation of the effective rest mass as follows
\[
\frac{1}{m_0} \frac{dm_0}{ds} = -\frac{1}{2} u^\mu u^\nu \frac{\partial g_{\mu \nu}}{\partial y} \left( \frac{dy}{ds} \right) + \frac{\epsilon \Phi u^\mu}{\Omega} \frac{\partial \Phi}{\partial x^\mu} \left( \frac{dy}{ds} \right)^2.
\]
(25)

### 3.2 Massless particles in 5D

Let us now consider massless test particles, $M(5) = 0$, moving in the five-dimensional metric (14). The motion of such particles is along isotropic geodesics, which in five-dimensions requires $dS = 0$. Therefore,
\[
\Omega ds^2 = -\epsilon \Phi^2 dy^2.
\]
(26)

It is clear that the signature of the extra dimension plays an important role here. In particular, null geodesics in 5D appear as timelike paths in 4D only if the following two conditions are met simultaneously: (i) the extra dimension is spacelike, and (ii) the particle in its five-dimensional motion has $P^4 \neq 0$. Otherwise, a null geodesic in 5D is observed as a lightlike particle in 4D.

In the case where $M(5) = 0$, the derivatives $M(5)/dS$ in (15) have to be replaced by $d/d\lambda$, where $\lambda$ is the parameter along the null 5D geodesic [32]. Thus, from (19), with $M(5) = 0$, $\epsilon = -1$ and $P^4 = dy/d\lambda$, we obtain
\[
m_0 = \pm \sqrt{\Omega} \frac{dy}{d\lambda} = \frac{\sqrt{\Omega} P_4}{\Phi} > 0.
\]
(27)

It is important to mention that $P_4$ is independent of $\lambda$, which means that the mass calculated from (27) is unaffected by the parameterization along the five-dimensional null geodesic. This can be clearly illustrated in terms of the five-dimensional action $S$, in which case $m_0 = (\sqrt{\Omega}/\Phi)|\partial S/\partial y|$. We will come back to this point in sections 5 and 7.

#### 3.2.1 Variation of rest mass for $M(5) = 0$

From (26), for a spacelike extra coordinate ($\epsilon = -1$), it follows that\(^1\) $dy = \pm (\sqrt{\Omega}/\Phi) dx$. Therefore,
\[
d\lambda = \left( \frac{\Omega}{m_0} \right) ds.
\]
(28)

From this and the 4-component of the geodesic equation we obtain
\[
\frac{1}{m_0} \frac{dP_4}{ds} = \frac{1}{2\Omega} \frac{\partial (\Omega g_{\mu \nu})}{\partial y} u^\mu u^\nu - \frac{1}{\Phi} \frac{\partial \Phi}{\partial y}.
\]
(29)

Consequently, the variation of rest mass for a spacelike extra coordinate ($\epsilon = -1$) and $M(5) = 0$ is obtained from (27), as
\[
\frac{1}{m_0} \frac{dm_0}{ds} = \pm \sqrt{\Omega} \frac{\partial g_{\mu \nu}}{\partial y} u^\mu u^\nu - \frac{u^\mu}{2\Phi} \frac{\partial \Phi}{\partial x^\mu}.
\]
(30)

\(^1\)When taking the roots we choose the signs in such a way that $m_0 > 0$ and $d\lambda/ds > 0$. 

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We note that although the mathematical description of massless particles in 5D differs from that of massive particles in 5D, the last two equations (for \( M(5) = 0 \)) can be readily obtained from (24) and (25) (for \( M(5) \neq 0 \)) just by setting \((dy/ds) = \pm (\sqrt{\Omega/\Phi}) \) and \( \epsilon = -1 \).

We emphasize that for a timelike extra dimension \((\epsilon = +1)\), there is only one physical possibility. Namely, massless particles in 5D are perceived as massless particles in 4D. In addition, their motion is confined to hypersurfaces \( y = \text{const} \).

To summarize, a bulk test particle moving freely in a five-dimensional manifold is observed in 4D as a test particle with variable rest, as given by (25) or (30). We would like to emphasize that this is not an artifact of a poor choice of coordinates or parameter used in the geodesic description, but it is a genuine four-dimensional manifestation of the extra dimension.

4 Dynamics of test particles from 5D to 4D

So far we have only used the fourth component of the five-dimensional geodesic equation (21). We now turn our attention to the spacetime components of that equation.

Setting \( A = \mu \) in (21) and using

\[
U^\mu = \frac{m_0}{\Omega M(5)} u^\mu, \quad U_\mu = \frac{m_0}{M(5)} u_\mu,
\]

we find

\[
\frac{Du^\mu}{ds} \equiv \frac{du^\mu}{ds} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = \left( \frac{1}{2} u^\nu u^\rho - g^{\nu\rho} \right) u^\lambda \frac{\partial g_{\nu\lambda}}{\partial y} \left( \frac{dy}{ds} \right) + \frac{\epsilon \Phi}{\Omega} \left[ \Phi_{\nu\rho} - u^\nu u^\rho \phi \right] \left( \frac{dy}{ds} \right)^2,
\]

where \( \Gamma^\mu_{\alpha\beta} \) is the Christoffel symbol calculated with the spacetime metric \( g_{\alpha\beta} \).

For the covariant components of the four-velocity we get,

\[
\frac{Du_\mu}{ds} = \frac{du_\mu}{ds} - \Gamma^{\beta}_{\mu\alpha} u^\alpha u_\beta = \frac{1}{2} u_\mu u^\lambda \frac{\partial g_{\lambda\rho}}{\partial y} \left( \frac{dy}{ds} \right) + \frac{\epsilon \Phi}{\Omega} \left[ \Phi_{\nu\mu} - u_\nu \phi \phi \right] \left( \frac{dy}{ds} \right)^2.
\]

Thus, for any given five-dimensional metric (14), we can always (at least in principle) solve the above equations to find

\[
u^\mu = u^\mu(s),
\]

and the observed trajectory in 4D

\[
x^\mu = x^\mu(s).
\]

From (32) and (33) it is clear that the geodesic motion in the five-dimensional manifold is observed in 4D to be under the influence of extra non-gravitational forces.

At first glance one would identify these extra forces with the terms on the right hand side of (32) and (33), and this is indeed the usual approach. However, this identification faces two problems.

First of all, while the observed force should be a four-vector, the quantities \( Du^\mu/ds \) and \( Du_\mu/ds \) given above do not represent the contravariant and covariant component of any four-dimensional vector. In order to see this, let us notice that

\[
u_\mu \left( \frac{Du^\mu}{ds} \right) \neq u^\mu \left( \frac{Du_\mu}{ds} \right) \neq 0.
\]

Clearly this is not what we expect in 4D, which is given by (3) as a result of \( u_\nu u^\nu = 0 \).

The second delicate point here is that

\[
g_{\mu\nu} \left( \frac{Du^\nu}{ds} \right) = \frac{du^\nu}{ds} - u^\lambda \frac{\partial g_{\nu\lambda}}{\partial y} \left( \frac{dy}{ds} \right).
\]
Thus, condition (10) is not satisfied if we identify the r.h.s. of (32) and (33) with the contravariant and covariant components of the extra forces. On the other hand, if $Du^\mu/ds$ and $Du_\mu/ds$ were the contravariant and covariant components of a four-vector, they would comply with $g_{\mu\nu}(Du^\mu/ds) = (Du_\nu/ds)$.

Third, the appropriate definition of force should involve the change of particle’s momentum, as in (3) and (4).

4.1 The four-acceleration

Thus, the direct identification of $(Du^\mu/ds)$ and $(Du_\mu/ds)$ with the contravariant and covariant components of the extra non-gravitational force is questionable. Meanwhile, the correct definition of force (per unit mass) in 4D, free of the problems mentioned above, was discussed in section 2. It contains two terms; one of them is $(u^\mu/m_0)(dm_0/ds)$, which has already been obtained in (30) and/or (34). The other term has yet to be found; it is the four-acceleration $D(4)u^\mu/ds$, which is the same for all test particles regardless of their mass.

The vectorial nature of the force in (3) and (4), is assured by the fact $D(4)g_{\mu\nu} = 0$. On the other hand, $Dg_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial y}$ dy. This means that the operator $D$ defined in (32) and (33) is not the appropriate covariant differential calculated in 4D. In order to construct the appropriate differential in 4D, let us notice that

$$\frac{\partial u^\mu}{\partial y} = -\frac{1}{2}u^\mu u^\rho \frac{\partial g_{\rho\lambda}}{\partial y}, \quad u^\rho \frac{\partial u_\mu}{\partial y} = \frac{1}{2}u^\mu u^\rho \frac{\partial g_{\rho\lambda}}{\partial y},$$

which can be easily shown in the comoving frame of reference. Using these expressions, from (32), (33) and (38) we get

$$u_\mu \left[D \frac{ds}{dy} \frac{\partial}{\partial y} \right] u^\mu = 0, \quad u^\rho \left[D \frac{ds}{dy} \frac{\partial}{\partial y} \right] u_\mu = 0, \quad \left[D \frac{ds}{dy} \frac{\partial}{\partial y} \right] g_{\mu\nu} = 0.$$

If we compare these expressions with (6) and (8) it is clear that a suitable definition for $D(4)$ is given by

$$\frac{D(4)}{ds} = \left[D \frac{ds}{dy} \frac{\partial}{\partial y} \right].$$

For the case of more general metrics, $D(4)$ can also be defined, but this requires the introduction of the appropriate projectors.

With this definition we have

$$u_\mu \frac{D(4)u^\mu}{ds} = 0, \quad u^\rho \frac{D(4)u_\mu}{ds} = 0, \quad \frac{D(4)g_{\mu\nu}}{ds} = 0.$$

As a consequence the acceleration is a four-vector. Namely, from (32) we get

$$\frac{D(4)u^\mu}{ds} = (u^\mu u^\rho - g^{\mu\rho}) u_\lambda \frac{\partial g_{\rho\lambda}}{\partial y} \left(\frac{dy}{ds}\right) + \frac{c\Phi}{\Omega} \left[\Phi_{;\mu} - u^\mu u^\rho \Phi_{;\rho}\right] \left(\frac{dy}{ds}\right)^2. $$

On the other hand, from (33) we obtain

$$\frac{D(4)u_\mu}{ds} = (u_\mu u^\rho - \delta^\rho_\mu) u_\lambda \frac{\partial g_{\rho\lambda}}{\partial y} \left(\frac{dy}{ds}\right) + \frac{c\Phi}{\Omega} \left[\Phi_{;\mu} - u_\mu \Phi_{;\rho} u^\rho\right] \left(\frac{dy}{ds}\right)^2. $$

Clearly $g_{\mu\nu}(D(4)u^\mu/ds) = (D(4)u_\nu/ds)$ as required for the correct vectorial behavior of the four-acceleration.

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2We note that in the case of Kaluza-Klein theories with the so-called “cylinder” condition, (i.e. $\partial g_{\mu\nu}/\partial y = 0$) the quantities $(Du^\mu/ds)$ and $Du_\mu/ds$ do represent the contravariant and covariant components of a four-vector, namely the four-acceleration.

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4.2 The extra force observed in 4D

Collecting results we obtain the explicit form of the extra force as follows.

Massive particles in 5D: Using the definitions \( \Phi \) and \( \frac{\Omega}{\sqrt{\Phi}} \), we find that a massive bulk test particle \((M_5 \neq 0)\) moving freely in a five-dimensional manifold is observed in 4D as a massive particle \((m_0 \neq 0)\) moving under the influence of the force

\[
\frac{1}{m_0} F^\mu = \frac{D^{(4)}u^\mu}{ds} + \frac{u^\mu}{m_0} \frac{dm_0}{ds} = \frac{e \Phi}{\Omega} \Phi^\mu \left( \frac{dy}{ds} \right)^2 + \left( \frac{1}{2} u^\mu u^\rho - g^{\mu\rho} \right) u^\lambda \frac{\partial g_{\rho\lambda}}{\partial y} \frac{dy}{ds}.
\]

(45)

The covariant components are

\[
\frac{1}{m_0} F^\mu_\parallel = \frac{D^{(4)}u^\mu}{ds} + \frac{u^\mu}{m_0} \frac{dm_0}{ds} = \frac{e \Phi}{\Omega} \Phi_\mu \left( \frac{dy}{ds} \right)^2 + \left( \frac{1}{2} u^\mu u^\rho - \delta^\mu_\rho \right) u^\lambda \frac{\partial g_{\rho\lambda}}{\partial y} \frac{dy}{ds}.
\]

(46)

We see that the extra force is made up of three distinct contributions, viz.,

\[ F^\mu = F^\mu_\perp + F^\mu_\parallel + F^\mu_\parallel, \]

(47)

where

\[
\frac{1}{m_0} F^\mu_\perp = \frac{e \Phi}{\Omega} \Phi^\mu \left( \frac{dy}{ds} \right)^2,
\]

(48)

\[
\frac{1}{m_0} F^\mu_\parallel = [u^\mu u^\rho - g^{\mu\rho}] u^\lambda \frac{\partial g_{\rho\lambda}}{\partial y} \frac{dy}{ds},
\]

(49)

and

\[
\frac{1}{m_0} F^\mu_\parallel = u^\mu \left[ - \frac{1}{2} u^\lambda u^\rho \frac{\partial g_{\lambda\rho}}{\partial y} \frac{dy}{ds} + \frac{e \Phi u^\lambda}{\Omega} \frac{\partial \Phi}{\partial x^\lambda} \left( \frac{dy}{ds} \right) \right].
\]

(50)

All these terms have to be evaluated along the trajectory \( \Phi = \text{const} \). We note that \( F^\mu_\perp \) and \( F^\mu_\parallel \) are orthogonal to the four-velocity, while \( F^\mu_\parallel \) is parallel to it. They crucially depend on the motion along the extra dimension. In particular, if the 5D motion is confined to hypersurfaces with \( y = \text{const} \), then \( F^\mu_\perp = F^\mu_\parallel = F^\mu_\parallel = 0 \), identically.

Massless particles in 5D: According to our discussion in section 3.2, massive 4D-particles, which travel on timelike paths \((ds^2 > 0)\), can also move on null paths in 5D provided the extra dimension is spacelike \((\epsilon = -1)\). The bulk geodesic motion of a massless particle \((M_5 = 0)\) with \( dy/ds \neq 0 \) is observed in 4D as the motion of a massive particle \((m_0 \neq 0)\) under the influence of the force given by \( \Phi = 1 \) and/or \( \Phi = \Omega \) with

\[
\frac{dy}{ds} = \pm \frac{\sqrt{\Omega}}{\Phi}.
\]

(51)

4.3 \( \Phi = \text{const} \)

In brane-world theory and STM, many authors choose to work in a Gaussian normal coordinate system based on our brane/spacetime. This choice might be convenient because it makes \( \Phi = 1 \), but is not necessary\(^3\). We note that in this case the quantities \( F^\mu/m_0 \) and \( Du^\mu/ds \) yield the same result. This coincidence, however, does not mean that \( Du^\mu/ds \) represents the correct definition of the extra force when \( \Phi = \text{const} \). This is because even now \( Du^\mu \) does not behave like a “regular” four-vector. Indeed, any four-vector \( A^\mu \) must satisfy the relation \( DA^\mu = g_{\mu\nu}DA^\nu \).

\[
DA^\mu = g_{\mu\nu}DA^\nu,
\]

(52)

On the other hand, \( Du^\mu \neq g_{\mu\nu}Du^\nu \), regardless of the choice of \( \Phi \).

---

\(^3\)The choice \( \Phi = 1 \) is not a requirement of the field equations, it is an external condition, namely, \( A^B = \psi^B_\psi \psi^C = 0 \). In brane theory a variable scalar field \( \Phi \) entails the possibility of variable fundamental physical “constants” \( \Phi \).
The extra force in the Brane-world scenario

In order to evaluate the observed quantities in 4D we have to identify the metric of the spacetime. However, there are distinct approaches to determine the 4D-geometry from a given five-dimensional manifold. In this section we examine the mass and extra force in spacetime as prescribed by the brane-world scenario. Our purpose is to show that, in brane-world models with $Z_2$-symmetry, the extra force is continuous and well defined across the brane. We illustrate this result with an example.

In the brane-world scenario our spacetime is identified with a singular hypersurface (or 3-brane), say $\Sigma$, orthogonal to the 5D vector field $\psi^A = (0, 0, 0, 0, \Phi^{-1})$. The effective equations for gravity on a 3-brane were obtained by Shiromizu et al [35]. In their approach the physical metric $g_{\mu \nu}$ is identified with the induced metric $\gamma_{\mu \nu}$ (this is equivalent to setting $\Omega = 1$) on the brane, which is fixed at some $y = y_0$.

The extra force has a term which is proportional to the first derivatives of the metric with respect to the extra coordinate. These derivatives can be written in terms of $K_{\alpha \beta}$, the extrinsic curvature of hypersurfaces $y = \text{const}$. Namely,

$$K_{\alpha \beta} = \frac{1}{2} \mathcal{L}_\psi g_{\alpha \beta} = \frac{1}{2\Phi} \frac{\partial g_{\alpha \beta}}{\partial y}, \quad K_{A4} = 0.$$  (53)

In the brane-world scenario the metric is continuous across $\Sigma$, but the extrinsic curvature $K_{\mu \nu}$ is discontinuous. In view of this, the general belief is that in this scenario cannot be implemented directly. Some authors argue that the effective equations in 4D should be obtained by taking the mean values of the extrinsic curvature across $\Sigma$.

However, for the calculation of the force the important term is the product of the extrinsic curvature times $dy/ds$, not $K_{\mu \nu}$ alone, i.e.\footnote{As mentioned above, in this approach $g_{\mu \nu} = \gamma_{\mu \nu}$. Therefore, in this section we set $\Omega = 1$.}

$$\frac{1}{m_0} F^\mu = e\Phi \Phi^\mu \left( \frac{dy}{ds} \right)^2 + \Phi u^\lambda (u^\mu u^\rho - 2g^{\mu \rho}) K_{\rho \lambda} \left( \frac{dy}{ds} \right).$$  (54)

Most brane-world models assume that the universe is invariant under the $Z_2$ transformation $y \rightarrow -y$, about our brane.\footnote{Since the action depends only on the coordinates of the particle in 5D, it follows that $P_A = -\partial S/\partial x^A$ is independent of the parameterization along the bulk geodesic.}

Namely,

$$dS^2 = g_{\mu \nu}(x^\rho, +y)dx^\mu dx^\nu + e\Phi^2 (x^\rho, +y)dy^2, \quad \text{for } y \geq 0$$

$$dS^2 = g_{\mu \nu}(x^\rho, -y)dx^\mu dx^\nu + e\Phi^2 (x^\rho, -y)dy^2, \quad \text{for } y \leq 0.$$  (55)

Thus

$$K_{\mu \nu} |_{\Sigma^+} = -K_{\mu \nu} |_{\Sigma^-}.$$  (56)

Let us now consider the bulk geodesic motion of test particles. It can be studied by means of the Hamilton-Jacobi equation, which in 5D is given by

$$\gamma^{AB} \left( \frac{\partial S}{\partial x^A} \right) \left( \frac{\partial S}{\partial x^B} \right) = M^2_{(5)},$$  (57)

where $S$ is the five-dimensional action and the metric is given by 55. It is clear that the solution of this equation in the bulk satisfies

$$S^{(+)} = S(x^\rho, +y), \quad \text{for } y \geq 0$$

$$S^{(-)} = S(x^\rho, -y), \quad \text{for } y \leq 0.$$  (58)

The covariant components of the four-momentum, according to 15, are given by

$$p_\mu = P_\mu = -\left( \frac{\partial S}{\partial x^\mu} \right) |_{\Sigma}.$$  (59)
They do not depend on whether we use $S^{(+)\,\prime}$ or $S^{(-)\,\prime}$ for their calculation. However, $P_4$ does depend on that; it changes its sign across the brane. Namely, since

$$P_4^{(\pm)} = -\frac{\partial S^{(\pm)}}{\partial y}, \quad (60)$$

from (60) it follows that $P_4^{(+)\,\prime} = -P_4^{(-)\,\prime}$. Now, using $P^A = \gamma^{AB} P_B$ and $P^A = M_{(5)} dx^A / dS$, or $P^A = dx^A / d\lambda$ for $M_{(5)} = 0$, we get

$$\frac{dy}{ds} = \left(\frac{\gamma^{44} u^0}{\gamma^{00} P_0}\right) P_4. \quad (61)$$

Since the metric is continuous across the brane, and the four-momentum as well as the four-velocity are independent on which side of the brane we are using, it follows that

$$\left(\frac{dy}{ds}\right)_{\Sigma^+} = -\left(\frac{dy}{ds}\right)_{\Sigma^-}. \quad (62)$$

Therefore, in a $\mathbb{Z}_2$-symmetric universe the product $K_{\mu\nu}dy/ds$ is continuous across the brane, viz,

$$K_{\mu\nu}(\Sigma^+\,dy/ds)_{\Sigma^+} = K_{\mu\nu}(\Sigma^-\,dy/ds)_{\Sigma^-}. \quad (63)$$

This means that the force (45) and/or (54\,\prime) is perfectly well defined in a $\mathbb{Z}_2$-symmetric universe, i.e., we get the same result regardless of whether we calculate it from “above” or “bellow” the brane.

### 5.1 Homogeneous cosmology in brane-world

We now study the geodesic motion in a five-dimensional bulk space with three-dimensional isotropy and homogeneity. Our goal here is to provide an explicit example of the above discussion. The metric may be written as

$$dS^2 = N^2(t,y) dt^2 - A^2(t,y) \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] + \epsilon \Phi^2(t,y) dy^2, \quad (64)$$

where $y$ is the coordinate along the extra-dimension and $t, r, \theta$ and $\phi$ are the usual coordinates for a spacetime with spherically symmetric spatial sections. In spherically symmetric fields test particles move on a single “plane” passing through the center. We take this plane as the $\theta = \pi/2$ plane. Thus, the Hamilton-Jacobi equation (57) for the metric (64) is

$$\frac{1}{N^2} \left( \frac{\partial S}{\partial t} \right)^2 - \frac{1}{A^2} \left[ \left( \frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial S}{\partial \phi} \right)^2 \right] + \frac{\epsilon}{\Phi^2} \left( \frac{\partial S}{\partial y} \right)^2 = M_{(5)}^2. \quad (65)$$

Since $\phi$ is a cyclic coordinate, it is clear that the action separates as

$$S = S_1(t,y) + S_r(r) + L\phi, \quad (66)$$

where $L$ is the angular momentum. Thus, we obtain

$$\frac{1}{N^2} \left( \frac{\partial S_1}{\partial t} \right)^2 - \frac{k^2}{A^2} + \frac{\epsilon}{\Phi^2} \left( \frac{\partial S_1}{\partial y} \right)^2 = M_{(5)}^2, \quad (67)$$

and

$$\left( \frac{dS_r}{dr} \right)^2 + \frac{L^2}{r^2} = k^2 \geq 0, \quad (68)$$

where $k$ is the separation constant.
If $k = 0$, then the particle in its five-dimensional motion remains at rest in space. In this case $u^\mu = \delta^\mu_0 / N$, and

$$F^\mu_{\Phi \perp} = F^\mu_{g \perp} = 0.$$  \hspace{1cm} (69)

Consequently, in this situation only the extra force $F^\mu_{||} = u^\mu d\rho_0 / ds$ would be observable in 4D. In general, in any other case with $k \neq 0$ the forces $F^\mu_{\Phi \perp}, F^\mu_{g \perp}$ will be non-zero.

In order to illustrate the equations for mass and force, we consider the Ricci-flat five-dimensional metric

$$dS^2 = \frac{\Lambda y^2}{3} \left\{ dt^2 - e^{2\sqrt{\Lambda}/3t} \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] \right\} - dy^2.$$ \hspace{1cm} (70)

This metric belongs to the family of separable solutions found by the present author [9]. It exhibits the distinctive features in cosmology and is simple enough as to allow the integration of (67).

Here we cannot set the brane at $y = 0$. We set it at $y = y_0 = \sqrt{3/\Lambda}$ and impose the $Z_3$ symmetry under the transformation $y \to y_0^2 / y$ (see [21] and references therein). The appropriate bulk background is

$$dS^2 = \frac{y_0^2}{y^2} \left\{ dt^2 - e^{2\sqrt{\Lambda}/3t} \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] \right\} - dy^2,$$ \hspace{1cm} (71)

and

$$dS^2 = \frac{y_0^2}{y^2} \left\{ dt^2 - e^{2\sqrt{\Lambda}/3t} \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] \right\} - \frac{y_0^4}{y^4} dy^2,$$ \hspace{1cm} (72)

for $y \geq 0$ and $y \leq 0$, respectively. The metric at the brane, located at $y = y_0$, is the usual de Sitter metric in 4D,

$$ds^2 = dt^2 - e^{2\sqrt{\Lambda}/3t} \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right].$$ \hspace{1cm} (73)

Thus,

$$K^\mu_+ = \frac{1}{y_0}, \hspace{1cm} K^\mu_- = -\frac{1}{y_0}.$$ \hspace{1cm} (74)

### 5.1.1 Non-null bulk geodesics

Let us consider the five-dimensional motion with $k = 0$. In this way we isolate the effects of the extra dimension from the effects due to the motion in spacetime. From (67) (with $\epsilon = -1$), using (71) and (72) we get

$$S_1^{(+)} = -M(5)y \sinh \sqrt{\frac{\Lambda}{3}}t, \hspace{1cm} S_1^{(-)} = -M(5)\frac{y_0^2}{y} \sinh \sqrt{\frac{\Lambda}{3}}t.$$ \hspace{1cm} (75)

The four-momentum is a well defined quantity, viz.,

$$p_\mu = -\left( \frac{\partial S}{\partial x_\mu} \right)_\Sigma = \delta^\mu_0 M(5) \cosh \sqrt{\frac{\Lambda}{3}}t.$$ \hspace{1cm} (76)

Now using

$$P^A = M(5) \frac{dx^A}{dS} = \gamma^{AB} P_B = -\gamma^{AB} \frac{\partial S}{\partial x^B},$$ \hspace{1cm} (77)

we obtain

$$P^A = \left( M(5) \frac{y_0}{y} \cosh \sqrt{\frac{\Lambda}{3}}t, \hspace{0.5cm} 0, \hspace{0.5cm} 0, \hspace{0.5cm} -M(5) \sinh \sqrt{\frac{\Lambda}{3}}t \right),$$ \hspace{1cm} (78)

for $y \geq 0$, and

$$P^A = \left( M(5) \frac{y}{y_0} \cosh \sqrt{\frac{\Lambda}{3}}t, \hspace{0.5cm} 0, \hspace{0.5cm} 0, \hspace{0.5cm} +M(5) \frac{y_0^2}{y} \sinh \sqrt{\frac{\Lambda}{3}}t \right),$$ \hspace{1cm} (79)

---

6 The sign in $S$ is chosen in such a way that the energy be positive, viz., $P_0 = -\partial S / \partial t > 0$. 

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for \( y \leq 0 \). From these expressions we get

\[
\left( \frac{dy}{dt} \right)_{\Sigma^+} = -\tanh \sqrt{\frac{\Lambda}{3}} t, \quad \left( \frac{dy}{dt} \right)_{\Sigma^-} = +\tanh \sqrt{\frac{\Lambda}{3}} t.
\] (80)

In the present case \( u^\mu = \delta_0^\mu \), so that \( dt/ds = 1 \). Thus, from (74) we obtain

\[
K_{tt}(\Sigma^+) \left( \frac{dy}{ds} \right)_{\Sigma^+} = K_{tt}(\Sigma^-) \left( \frac{dy}{ds} \right)_{\Sigma^-} = -\sqrt{\frac{\Lambda}{3}} \tanh \sqrt{\frac{\Lambda}{3}} t.
\] (81)

Consequently, the extra force as observed on the brane (54) is given by

\[
\frac{F^\mu}{m_0} = \delta_0^\mu \sqrt{\frac{\Lambda}{3}} \tanh \sqrt{\frac{\Lambda}{3}} t.
\] (82)

The rest mass measured on the brane is given by (20) as

\[
m_0 = M(5) \cosh \sqrt{\frac{\Lambda}{3}} t.
\] (83)

It is clear that these expressions are consistent with (25).

5.1.2 Null bulk geodesics

If \( M(5) = 0 \), then the bulk motion is along null geodesics. In this case the 5D action is given by

\[
S^{(+)}_{1(Null)} = Cy_0 e^{-\sqrt{\Lambda/3} t}, \quad S^{(-)}_{1(Null)} = C \frac{y_0^2}{y} e^{-\sqrt{\Lambda/3} t}.
\] (84)

where \( C \) is a constant of integration. The corresponding four-momentum as observed on the brane (located at \( y = y_0 = \sqrt{3/\Lambda} \)) is

\[
p_\mu = \delta_0^\mu C e^{-\sqrt{\Lambda/3} t}.
\] (85)

Consequently, for the mass we obtain

\[
m_0 = p_\mu u^\mu = Ce^{-\sqrt{\Lambda/3} t}.
\] (86)

The extra force, per unit mass, due to variation of rest mass is

\[
\frac{F^\mu}{m_0} = -\delta_0^\mu \sqrt{\frac{\Lambda}{3}}
\] (87)

Identical results can be obtained from an analysis similar to the one in section 5.1.1. Here \( (dy/ds)_{\Sigma^+} = -(dy/ds)_{\Sigma^-} = +1 \). Thus, using (74), from (54) we recover (87). We also note that the mass of the particle can be obtained by evaluating (19), with \( M(5) = 0 \), from either side of the brane.

6 The extra force in STM

In this section we compare the rest mass and extra force as predicted by STM and brane theory. We will show that, although these theories give distinct prescriptions for the geometry of the spacetime, they lead to identical results for the mass and force as observed in 4D.

In STM our four-dimensional world is embedded in a five-dimensional spacetime, which is a solution of the five-dimensional Einstein’s equations in vacuum. The extra dimension is not assumed to be compactified, which allows us to obtain the properties of matter as a consequence of the large extra dimension.
An important similarity between brane theory and STM is that in both schemes our four-dimensional spacetime is identified with a fixed hypersurface \( \Sigma \) (defined by the equation \( y = y_0 = \text{constant} \)), and the metric in 4D is taken to be the induced one, viz., \( g_{\alpha\beta}(x^\mu) = \gamma_{\alpha\beta}(x^\mu, y_0) \). The main difference is that this hypersurface is singular in brane theory and non-singular in STM. However, the effective matter content of spacetime is the same whether calculated from STM equations or from the \( \mathbb{Z}_2 \)-symmetric brane perspective.

Although STM and brane theory have different physical motivations for the introduction of a large extra dimension, they share the same working scenario and are equivalent in many respects. In particular, STM includes the so-called local high-energy corrections, and non-local Weyl corrections typical of brane-world scenarios [31].

We now proceed to show that both prescriptions, brane-theory and STM, lead to the same expressions for the rest mass and extra force as observed in 4D. This is clear from the fact that in both theories all relevant quantities for the calculation of the extra force are continuous across \( \Sigma \). Indeed, in STM there are no discontinuities, and in brane theory \( K_{\mu\nu}dy/ds \) is continuous across the brane, despite of the fact that each quantity; \( K_{\mu\nu} \) and \( dy/ds \) is discontinuous separately.

As an illustration, let us again consider the 5D metric [70]. In STM the bulk metric is the same in both sides of \( \Sigma \), which we locate at \( y = y_0 = \sqrt{3/\Lambda} \). Thus, in our calculation we can use either [71] or [72]. If we choose [71], then \( (\partial g_{tt}/\partial y)_{\Sigma} = 2\sqrt{\Lambda/3} \) (in this prescription \( \Omega = 1 \)). The results observed in 4D depend on whether the motion in the bulk is along non-null or null geodesics.

In the case of non-null bulk geodesic motion we have \( (dy/ds)_{\Sigma} = -\tanh \frac{\sqrt{\Lambda/3}}{t} \). Thus, when evaluating the mass from [70] and the extra force from [65], we obtain the same results as in brane theory, namely [82] and [83]. In the case of null bulk geodesics we have \( (dy/ds)_{\Sigma} = 1 \). Therefore, we recover the results [86] and [87] obtained on the brane.

Thus, the mass and extra force perceived by an observer in 4D are independent of whether the bulk geodesic motion is interpreted on the non-singular hypersurface \( \Sigma \) of STM or on the singular hypersurface (located at \( y = y_0 \)) of brane theory. These two 5D theories produce indistinguishable results for test particles as observed in 4D.

7 The extra force in other non-compact theories

The aim of this section is to show, by means of an explicit example, how the results for mass and force as observed in 4D severely depend on the way we separate the spacetime from the extra dimension.

With this aim, here we consider two different approaches, which are alternative interpretations of STM. They hold a different view regarding the identification of spacetime. Namely, in these approaches the geometry of the 4D spacetime is identified with the *entire* foliation orthogonal to the 5D vector field \( \psi^A = (0, 0, 0, 0, \Phi^{-1}) \), instead of a fixed hypersurface \( \Sigma \).

In order to illustrate these interpretations, we go back to the 5D Ricci-flat manifold [70]. First, we revisit the bulk geodesic motion with \( k = 0 \). Second, we interpret the bulk geodesic motion as observed in 4D.

Since there are no discontinuities, the action throughout the bulk is given by

\[
S_1 = -M_{(5)}y \sinh \sqrt{\frac{\Lambda}{3}}t. \tag{88}
\]

From [84] it follows that

\[
U^A = \left( \frac{dt}{dS}, 0, 0, \frac{dy}{dS} \right) = \left( \sqrt{\frac{3}{\Lambda y}} \cosh \sqrt{\frac{\Lambda}{3}}t, 0, 0, 0, -\sinh \sqrt{\frac{\Lambda}{3}}t \right). \tag{89}
\]

Consequently,

\[
y = \frac{\bar{y}_0}{\cosh \sqrt{\frac{4}{3}}t}, \tag{90}
\]

where \( \bar{y}_0 \) is a constant of integration. Therefore, according to [88] the four-momentum observed in this approach is given by

\[
p_\mu = (\bar{y}_0 M_{(5)} \sqrt{\frac{\Lambda}{3}}, 0, 0, 0). \tag{91}
\]
In the case of null geodesics in 5D the action is given by
\[ S^{(\text{Null})}_1 = C y e^{-\alpha \sqrt{\Lambda / 3} t}, \] (92)
where \( \alpha = \pm 1 \) and \( C \) is a constant of integration. In (77) we replace \( M_{(5)} dS / d\lambda \) by \( d/d\lambda \), where \( \lambda \) is the parameter along the null geodesic, and obtain
\[ P^A = \left( \frac{dt}{d\lambda}, 0, 0, 0, \frac{dy}{d\lambda} \right) = \left( \sqrt{\frac{3}{\Lambda}} \frac{\alpha C}{y} e^{-\alpha \sqrt{\Lambda / 3} t}, 0, 0, 0, C e^{-\alpha \sqrt{\Lambda / 3} t} \right). \] (93)
In this case
\[ y^{(\text{Null})} = \bar{y}_0 e^{\alpha \sqrt{\Lambda / 3} t}, \] (94)
and the four-momentum as observed in 4D is
\[ p_\mu = (\alpha C \bar{y}_0 \sqrt{\frac{\Lambda}{3}}, 0, 0, 0). \] (95)
It is important to note that (94) and (95) are independent of the choice of geodesic parameter \( \lambda \).

For the interpretation of the bulk geodesic motion as observed in 4D we have to identify the metric of the physical spacetime. We will consider two approaches.

### 7.1 Canonical approach

In this approach the metric in the bulk is simplified by using all five available coordinate degrees of freedom to set \( \gamma_{\mu 4} = 0 \) and \( \Phi = 1 \). Besides, the physical metric in 4D is assumed to be conformally related to the induced one. The warp factor is taken as \( \Omega = (y/L)^2 \). Namely,
\[ ds^2 = \frac{y^2}{L^2} g_{\mu \nu}(x^\alpha, y) dx^\mu dx^\nu - dy^2. \] (96)
This metric is usually called canonical metric [12]. Here \( L \) is a constant of length, which in cosmological solutions is identified with the cosmological constant via \( L = \sqrt{3/\Lambda} \).

We note that the Ricci-flat metric (70) has the canonical form (96). Thus, in the canonical metric approach the geometry of the spacetime is determined by (68) and the warp factor is \( \Omega = \Lambda y^2 / 3 \). Since \( \partial g_{\mu \nu} / \partial y = 0 \), from (45) it follows that
\[ F^\mu = 0, \quad (\mu = 0, 1, 2, 3), \] (97)
in this interpretation. This is consistent with the fact that here the rest mass is constant, which is a consequence of (26). In order to get \( m_0 \) we can use (41). Namely,
\[ m_0 = p_\mu u^\mu = \bar{y}_0 M_{(5)} \sqrt{\frac{\Lambda}{3}}. \] (98)
This also can be obtained by direct substitution of (91) into (20).

If \( M_{(5)} = 0 \), then the motion is along null geodesics in 5D. From (41) and (94) we obtain \( dt / ds = \alpha \). Therefore, from (95)
\[ m_0 = \bar{y}_0 C \sqrt{\frac{\Lambda}{3}}. \] (99)
7.2 Induced-metric approach

In this approach the metric of the spacetime is identified with the one induced on the set of hypersurfaces orthogonal to the 5D vector field \( \psi^A \), which is given by

\[
g_{\mu\nu}(x^\rho, y) = h^A_{\mu} h^B_{\nu} \gamma_{AB}(x^\rho, y),
\]

(100)

where \( h_{AB} \) is the projector introduced in (11). We note that in brane theory as well as STM the spacetime is fixed at some \( y = y_0 = \text{const} \). In the present approach, which is also called “foliating” approach \([29]\), the geometry of the spacetime is determined by the whole family of orthogonal hypersurfaces.

In the case under consideration the metric of the hypersurfaces orthogonal to the 5D vector field \( \psi^A = (0, 0, 0, 0, 1) \) is given by

\[
ds^2 = \frac{\Lambda y^2}{3} \left\{ dt^2 - e^{2\sqrt{\Lambda/3}t} \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \right\}.
\]

(101)

For this metric \( u_\mu = \delta_\mu^0 \sqrt{\Lambda/3y} \). Since \( p_\mu = m_0 u_\mu \), from (91) it follows that

\[
m_0 = M(5) \cosh \sqrt{\frac{\Lambda}{3}} t,
\]

(102)

where we have evaluated \( y \) along the trajectory by using (90). In this approach \( \Omega = 1 \), \( dt/ds = \left( \sqrt{\frac{3}{\Lambda}} / \bar{y}_0 \right) \cosh \sqrt{\Lambda/3} t \), and \( dy/ds = - \tanh \sqrt{\Lambda/3} t \). Consequently, from our general equation (20) we get the same result as in (102), as expected.

Taking derivatives in (102) we get

\[
F_\| \mu = \frac{\delta_\mu^0}{2\bar{y}_0} \sqrt{\frac{3}{\Lambda}} \sinh 2 \sqrt{\frac{3}{\Lambda}} t.
\]

(103)

It is easy to verify that this result is consistent with (50).

In order to avoid misunderstanding, we should mention that the 4D metric contains no \( y \). It is obtained from (101) after we substitute (90) into it. Namely,

\[
ds^2 = \frac{\Lambda y^2}{3 \cosh^2 \sqrt{\Lambda/3}t} \left\{ dt^2 - e^{2\sqrt{\Lambda/3}t} \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \right\}.
\]

(104)

This line element is distinct from the one in the brane-world and STM interpretation, which is given by (73). As a consequence, (83) and (102) are distinct functions of the proper time.

For the case of null geodesic motion in the bulk metric (70), in the induced-metric approach we have \( dy^2 = ds^2 \). Taking \( dy/ds = 1 \), from (94) we get \( dt/ds = (\sqrt{3/\Lambda} / \bar{y}_0) e^{-\alpha \sqrt{\Lambda/3} t} \). Thus, using (95) we obtain

\[
m_0 = C e^{-\alpha \sqrt{\Lambda/3} t} = \frac{(\alpha C \bar{y}_0)}{s},
\]

(105)

where \( s \) is the proper time. Taking derivatives we find the extra force as

\[
\frac{F_\| \mu}{m_0} = -\delta_\mu^0 \sqrt{\frac{3}{\Lambda}} \frac{1}{s^2}.
\]

(106)

This can be easily corroborated from (100). The geometry in 4D, is given by (101) evaluated at \( y = y^{(null)} \) from (94).

The discussion of this section clearly shows that the rest mass as well as the extra force as observed in 4D depend on (i) the method we use to identify the 4D metric from the 5D one, (ii) the nature of the geodesic motion in 5D and (iii) the motion in 3-space.
8 Summary and final comments

The aim of this work has been to present a clear and general discussion of how an observer in 4D interprets the geodesic motion in a five-dimensional bulk space. Here we have provided a unified methodology for the discussion of the mass and extra force as observed in 4D. Our method presents a number of advantages over other studies in the literature. First, it can successfully be applied to compactified Kaluza-Klein theory, brane world, STM, and other non-compact theories in 5D. Second, the whole discussion is free of the subtle details associated with the choice of affine parameters used to describe the motion in 4D and 5D. Third, it works equally well for non-null and null geodesics in the bulk (the latter involves the change of \(dy/\text{d}s\) by \(\pm \sqrt{\Omega/\Phi}\).

In the scenario of compactified Kaluza-Klein theory (with the cylinder condition) the extra force reduces to

\[
\frac{1}{m_0} F_\mu = \frac{\epsilon \Phi \mu}{\Omega} \left( \frac{dy}{ds} \right)^2, \quad \frac{1}{m_0} F_\mu = -\frac{\Phi \mu}{\Phi}.
\]

for non-null and null bulk geodesics, respectively. We remind the reader that in the case of null bulk geodesics the extra coordinate has to be spacelike (\(\epsilon = -1\)), otherwise the particles observed in 4D are massless. The above equations show that the existence of an extra force is not a prerogative of theories with large extra dimensions like brane theory and STM. Conversely, there are 5D metrics with explicit dependence on the extra coordinate, which show constant rest mass and no extra force, when they are interpreted in the context of the brane/STM scenario. In these metrics the constancy of the rest mass is a consequence of the mutual cancelation of the mass-change induced by the term \((\partial g_{\mu\nu}/\partial y)u^\mu u^\nu\) and the one induced by the scalar field.

In the brane world scenario with \(Z_2\)-symmetry, we have shown that the extra force is continuous and well defined across the brane. This is an effect of the required symmetry. In fact, in such a scenario the momentum component along the extra dimension changes its sign across the brane, which effectively compensates the discontinuity of the extrinsic curvature. This is an important result because if our universe is described by the brane world scenario, then it has to have \(Z_2\) symmetry. Indeed, if the \(Z_2\)-symmetry is dropped, then there is an extra term in the Friedmann equation [14]. This term is constrained by the condition that standard cosmology is in place by the time of nucleosynthesis. In other words, the effects associated with the lack of \(Z_2\) symmetry must decrease with time. Which means that the extra term should be small enough at the time of nucleosynthesis and negligible today. This is why brane-world models without \(Z_2\) symmetry (at late times) seem to be of no observational significance today.

In the original interpretation of STM, our four-dimensional spacetime was identified with a fixed hypersurface \(\Sigma\) (defined by the equation \(y = y_0 = \text{constant}\)), and the metric in 4D was taken to be the induced one. With this identification, we find that brane world theory as well as STM lead to equal results for the mass and extra force as observed on the three-brane/spacetime. This means that observations made with particles cannot help us to distinguish whether we live on a singular or regular brane/STM hypersurface. This result is compatible with previous investigations where we showed that these two theories are equivalent to each other, although they look very different at first sight [31].

Subsequent interpretations of STM use the so-called canonical metric, in which the geometry of the 4D-spacetime is taken to be conformally related to the induced metric. This approach is not equivalent to the brane/STM scenario discussed above. This is illustrated by our example in section 7.1, which shows that, unlike the observations made on the brane, the rest mass is constant and consequently there is no extra force.

Another alternative approach which deserves consideration is the one where the geometry of the 4D spacetime is determined not by a fixed hypersurface \(y = \text{const}\), but by the whole family of hypersurfaces orthogonal to the extra dimension. This brings to mind the situation where the motion of a test particle is described from two distinct frames of reference. For instance, the comoving and some other non-comoving frame. Certainly the observed quantities in these frames are different. The corresponding similarity in 5D is clear. The observations made in the fixed brane/STM \((y = \text{const})\) should be different from those made on the “moving” \((y \neq \text{const})\) brane. This explains the results in our example in section 7.2.

To conclude, the two leading five-dimensional theories, namely brane world and STM, predict identical results for test particles as observed in 4D. However, other approaches seem to be possible. The discovery of new physical phenomena, unmistakable related to extra dimensions, is a challenge for these theories.
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