ON FREMDERVECTORS: VECTORS ORTHOGONAL TO THEIR IMAGES UNDER LINEAR TRANSFORMATIONS

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Abstract. Geometrically, the eigenvectors of a square matrix \( A \) are not rotated by \( A \). Here we consider vectors that are rotated \( \pi/2 \) by \( A \); that is, vectors orthogonal to their images. We call these vectors fremdervectors of \( A \) and discuss conditions for their existence. We also define fremdervalues, scalars \( z \) such that \( zI - A \) has a fremdervector, and discuss several known applications for fremdervectors.

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1. Introduction. Determining the eigenvalues and eigenvectors of a square matrix (or linear transformation, more generally) has become a standard topic of introductory linear algebra courses [8]. Algebraically, the eigenvalues and eigenvectors satisfy the canonical equation

\[
Ax = \lambda x,
\]

where \( A \in \mathbb{C}^{n \times n} \) is the matrix, \( \lambda \in \mathbb{C} \) is one of its eigenvalues, and \( x \in \mathbb{C}^n \) is an associated eigenvector. Geometrically, an eigenvector can be interpreted as a vector that is not rotated by \( A \); we make no distinction between “parallel” and “anti-parallel” directions.

Since the 1960s, Gustafson and co-workers [4] have more deeply examined the rotation of vectors by \( A \) and asked the related question, what vector is rotated the most by \( A \)? Such a vector is called an antieigenvector of \( A \), and its antieigenvalue is the angle of rotation. As discussed in [4], antieigenvalues lead to a matrix trigonometry and have found numerous applications in, e.g., numerical analysis and finance. Additionally, \( A \)'s antieigenvalues have a close relationship with its eigenvectors when \( A \) is Hermitian and positive definite.

In this work we go one step beyond antieigenvector analysis and consider the properties that \( A \) must satisfy for it to have an antieigenvalue of \( \pi/2 \). That is, which \( A \) rotate a vector to be orthogonal to its image? We thus seek to find \( A \) and \( x \) satisfying

\[
(x, Ax) = 0,
\]

where we assume the usual inner product for complex vectors. As a play on words, we call such \( x \) fremdervectors. The word \textit{eigen} means “own” in German (the image vector belongs to its eigenvector’s one-dimensional subspace), whereas \textit{fremder} means “stranger” (the fremdervector and its image share no common component).

Properties of the pencil \( zI - A \) help develop the notion of a fremdervalue. Recall that eigenvalues can alternatively be defined as values \( z \) that make \( zI - A \) singular. We define a fremdervalue of \( A \) as \( z \) such that \( zI - A \) has an antieigenvalue of \( \pi/2 \).

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The remainder of this note discusses properties that $A$ must satisfy to have fremdervectors and for $z$ to be a fremdervalue. Section 2 begins by motivating fremdervectors with a brief, and likely incomplete, overview of existing applications. We then summarize some pertinent aspects of matrix theory in section 3. Results on the existence of fremdervectors and fremdervalues for several cases are presented in section 4. We finally conclude in section 5 with a short discussion on calculating fremdervalues and fremdervectors.

2. Known Applications. Fremdervectors and fremdervalues are fundamentally interesting quantities that also have applications. Here we list a few, knowing that there are probably many others of which we are unaware. These applications do not typically consider the geometric interpretation of fremdervectors, but use them nevertheless.

One application is in computer graphics. Many solid bodies of interest can be represented by quadric surfaces, and it may be desirable to find the intersection of two such surfaces [5]. As we will discuss in section 5, this problem can be recast to use fremdervectors.

Our idea for fremdervectors originated in our research on nanotechnology, where we aim to understand how electric current flows through molecules. These systems are sufficiently small that quantum mechanics is required [2] and the system’s properties are described by a linear operator (the Hamiltonian). Eigenvectors of the Hamiltonian relate to maxima in the electric conductance. We recently showed that certain minima in the conductance correspond to fremdervectors of the Hamiltonian [7].

We know of two other applications for fremdervectors in quantum mechanics. The first is in spectroscopy [1], where transitions between two quantum states are measured. In some cases the quantum state must change in response to the spectroscopic setup; such a quantum state is a fremdervector of the transition matrix. The second is in condensed matter physics, where the existence of surface states in a material can be framed in terms of fremdervectors [3].

3. Preliminaries. Throughout this note we will denote the spectrum of $A$ by $\lambda(A)$. Several basic results from matrix theory will also be useful in our discussion. We state them here without proof, referencing any standard text on linear algebra (e.g., [8]) for more information.

First, $A$ is normal if it has a complete set of orthogonal eigenvectors. Second, if $A$ is Hermitian (i.e., self-adjoint, $A = A^\dagger$), then it is normal and its eigenvalues are real. In such a case, $A$ is classified as positive (semi-)definite if all eigenvalues are positive (non-negative), negative (semi-)definite if all eigenvalues are negative (non-positive), or indefinite if $A$ has both positive and negative eigenvalues. Third, we extend a similar classification scheme to skew-Hermitian matrices, $A = -A^\dagger$, which are normal and have purely imaginary eigenvalues. We use the signs of the eigenvalues’ imaginary parts to determine positive definiteness, etc. Finally, an arbitrary matrix $A$ can be decomposed into a Hermitian part, $B = (A + A^\dagger)/2$, and a skew-Hermitian part, $C = (A - A^\dagger)/2$, such that $A = B + C$.

4. The Existence of Fremdervectors and Fremdervalues. There are several trivial fremdervectors to mention before proceeding. One is $x = 0$, which clearly satisfies Eq. (1). Any $x \in \ker(A)$ also satisfies Eq. (1), as does any $x \in \ker(A^\dagger)$. In all three of these cases, the angle of rotation between $x$ and either $Ax$ or $A^\dagger x$ is not well-defined, and might best be regarded as 0 because $x$ is an eigenvector of $A$.

**Definition 4.1.** Any $x \in \ker(A) \cup \ker(A^\dagger)$ is a trivial fremdervector of $A$. 
The remainder of our discussion will focus on nontrivial fremdervectors. Lemma 4.2 provides one general result.

**Lemma 4.2.** Let $x$ be a fremdervector of $A$ and let $y \in \ker(A) \cap \ker(A^\dagger)$. Then $x + y$ is also a fremdervector of $A$.

**Proof.** The proof is straightforward using Eq. (1). \[ \square \]

### 4.1. Fremdervectors

The simplest place to begin is with Hermitian matrices.

**Theorem 4.3.** Let $A$ be Hermitian. Then $A$ has a nontrivial fremdervector if and only if it is indefinite.

**Proof.** Lemma 4.2 implies that the trivial fremdervectors of $A$ form a subspace. Without loss of generality, let $x$ be a nontrivial fremdervector with no component in this subspace, and let $\lambda_j$ and $\varphi_j$ be the corresponding eigenvalues and normalized eigenvectors of $A$. Write $x = \sum_j c_j \varphi_j$ such that, from Eq. (1),

\begin{equation}
0 = \langle x, A x \rangle = \sum_{j=1}^n |c_j|^2 \lambda_j.
\end{equation}

Equation (2) has nontrivial solutions if and only if $A$ has at least one negative eigenvalue and at least one positive eigenvalue such that the expansion coefficients can be chosen to cancel the positive and negative terms. \[ \square \]

A similar result holds for skew-Hermitian matrices.

**Corollary 4.4.** Let $A$ be skew-Hermitian. Then $A$ has a nontrivial fremdervector if and only if it is indefinite (that is, $A$ has at least one eigenvalue with positive imaginary part and at least one eigenvalue with negative imaginary part).

**Proof.** From Eq. (1), the fremdervector $x$ must satisfy

\[ 0 = \langle x, A x \rangle = \langle x, iA x \rangle. \]

Because $iA$ is Hermitian, the result follows from Theorem 4.3. \[ \square \]

The investigation of nontrivial fremdervectors for arbitrary matrices is very similar to that of Hermitian matrices. The key difference is that we first decompose the matrix into its Hermitian and skew-Hermitian parts.

**Theorem 4.5.** Let $A$ have Hermitian part $B$ and skew-Hermitian part $C$. If $x$ is a nontrivial fremdervector of $A$, then the following three conditions must hold.

(i) $x$ is a fremdervector of $B$.

(ii) $x$ is a fremdervector of $C$.

(iii) At least one of $B$ and $C$ is indefinite.

**Proof.** From Eq. (1) and $A = B + C$,

\[ 0 = \langle x, A x \rangle = \langle x, B x \rangle + \langle x, C x \rangle. \]

Notice that $\langle x, B x \rangle$ is real and $\langle x, C x \rangle$ is purely imaginary. Thus, both terms must be 0 independently, meaning $x$ is a fremdervector of both $B$ and $C$.

We show condition (iii) by contradiction. Suppose that neither $B$ nor $C$ is indefinite. From the proof of Theorem 4.3, $B$ must then be positive or negative semi-definite such that $x$ is a trivial fremdervector of $B$. $x$ is similarly a trivial fremdervector of $C$. Finally, $x \in \ker(B) \cap \ker(C)$ implies $x \in \ker(A)$ such that $x$ is a trivial fremdervector of $A$. Thus, at least one of $B$ and $C$ must be indefinite for $A$ to possess a nontrivial fremdervector. \[ \square \]
4.2. Fremdervales. We have so far considered the properties $A$ must satisfy to have nontrivial fremdervectors. Let us now shift focus to the fremdervales of $A$ and discuss bounds for their existence.

**Theorem 4.6.** Let $B$ and $C$ be the Hermitian and skew-Hermitian parts of $A$, respectively. If $z$ is a fremdervale of $A$, then the following three conditions hold.

(i) $\min \lambda(B) \leq \text{Re}(z) \leq \max \lambda(B)$.

(ii) $\min \text{Im}(\lambda(C)) \leq \text{Im}(z) \leq \max \text{Im}(\lambda(C))$.

(iii) Conditions (i) and (ii) cannot both be strict equalities. For instance, if $\text{Re}(z) = \min \lambda(B)$, then $\min \text{Im}(\lambda(C)) < \text{Im}(z) < \max \text{Im}(\lambda(C))$.

**Proof.** By definition, $z$ is a fremdervale of $A$ if $zI - A$ has a nontrivial fremdervector. Invoking Theorem 4.5, we thus need to consider the definiteness of the Hermitian and skew-Hermitian parts of $zI - A$.

$\text{Re}(z)I - B$ is the Hermitian part of $zI - A$, and cannot be positive or negative definite by Theorem 4.5. Thus, we have condition (i). In a similar fashion, $i\text{Im}(z)I - C$ is the skew-Hermitian part of $zI - A$ and leads to condition (ii). Finally, condition (iii) of Theorem 4.5 requires $\text{Re}(z)I - B$ and/or $\text{Im}(z)I - C$ to be indefinite, which implies condition (iii).

**Corollary 4.7.** Let $A$ be normal. If $z$ is a fremdervale of $A$, then the following three conditions hold.

(i) $\min \text{Re}(\lambda(A)) \leq \text{Re}(z) \leq \max \text{Re}(\lambda(A))$.

(ii) $\min \text{Im}(\lambda(A)) \leq \text{Im}(z) \leq \max \text{Im}(\lambda(A))$.

(iii) Conditions (i) and (ii) cannot both be strict equalities (similar to Theorem 4.6).

**Proof.** Let $B$ and $C$ be the Hermitian and skew-Hermitian parts of $A$, respectively. This corollary follows directly from the facts that $\lambda(B) = \text{Re}(\lambda(A))$ and $\lambda(C) = i\text{Im}(\lambda(A))$ when $A$ is normal.

**Theorem 4.6** and Corollary 4.7 provide bounds for fremdervales. In the event $A$ is (skew-)Hermitian, we can also show the existence of fremdervales.

**Corollary 4.8.** Let $A$ be Hermitian. $z$ is a fremdervale of $A$ if and only if $z \in \mathbb{R}$ and $\min \lambda(A) < z < \max \lambda(A)$. A similar statement using imaginary parts holds if $A$ is skew-Hermitian instead.

**Proof.** ($\Rightarrow$) Appealing to Corollary 4.7, $\min \lambda(A) \leq \text{Re}(z) \leq \max \lambda(A)$ and $0 \leq \text{Im}(z) \leq 0$, with at most one equality holding. Clearly $\text{Im}(z) = 0$, implying that $z \in \mathbb{R}$ and $\min \lambda(A) < z < \max \lambda(A)$. ($\Leftarrow$) $zI - A$ is indefinite when $\min \lambda(A) < z < \max \lambda(A)$, which implies the existence of a nontrivial fremdervector (Theorem 4.3). Similar logic shows the corresponding statement when $A$ is skew-Hermitian.

5. Finding Fremdervectors and Fremdervales. The previous section detailed the existence of fremdervectors and fremdervales of a matrix $A$. We conclude this note with a brief discussion on calculating fremdervectors and fremdervales. We hope this note inspires the development of additional tools.

In the most general case, a fremdervector sits at the intersection of two quadric hypersurfaces. Equation (2) shows that the fremdervectors of the Hermitian part of $A$ describe a quadric hypersurface when they are interpreted as displacement vectors from the origin. A similar result holds for the skew-Hermitian part. Because a fremdervector of $A$ must be a fremdervector of both the Hermitian and skew-Hermitian parts of $A$ by Theorem 4.5, the fremdervectors describe the intersection of
two quadric hypersurfaces. This problem is generally discussed in [6]. Furthermore, computer graphics has a longstanding interest in solving this problem [5] — usually in 3 or 4 dimensions — and has developed algorithms that might be adaptable to the fremdervector problem.

If \( A \) is normal, the situation simplifies because the quadric surfaces from \( A \)'s Hermitian and skew-Hermitian parts share principal axes. In this case, the problem of finding fremdervectors reduces to finding nonnegative solutions of a linear system.

**Theorem 5.1.** Let \( A \) be normal with eigenvalues \( \lambda_1, \ldots, \lambda_n \). Every solution of the problem

\[
0 = \sum_{j=1}^{n} d_j \text{Re}(\lambda_j),
\]

\[
0 = \sum_{j=1}^{n} d_j \text{Im}(\lambda_j),
\]

subject to \( d_j \geq 0 \) corresponds to a fremdervector of \( A \).

**Proof.** Let \( \varphi_j \) be a normalized eigenvector of \( A \) associated with eigenvalue \( \lambda_j \) and let \( B \) and \( C \) be the Hermitian and skew-Hermitian parts of \( A \), respectively. Because \( A \) is normal, \( A, B, \) and \( C \) are simultaneously diagonalizable. Moreover, the spectrum of \( B \) is \{Re(\lambda_j)\}; likewise, \{iIm(\lambda_j)\} is the spectrum of \( C \).

Suppose \( x = \sum_j c_j \varphi_j \) is a fremdervector of \( A \). From Theorem 4.5, we have that \( x \) is a fremdervector of both \( B \) and \( C \). Mirroring logic from the proof of Theorem 4.3,

\[
0 = \langle x, Bx \rangle = \sum_{j=1}^{n} |c_j|^2 \text{Re}(\lambda_j).
\]

Similarly,

\[
0 = \langle x, Cx \rangle = \sum_{j=1}^{n} |c_j|^2 i\text{Im}(\lambda_j) = \sum_{j=1}^{n} |c_j|^2 \text{Im}(\lambda_j).
\]

Rewriting \( d_j = |c_j|^2 \) in the above equations gives a linear system where we want nonnegative solutions.

Theorem 5.1 simplifies if \( A \) is Hermitian or skew-Hermitian because Eq. (3a) or (3b), respectively, is trivially satisfied. Furthermore, adding the equation \( \sum_j d_j = 1 \) to the system excludes the trivial solution \( x = 0 \) without loss of generality.

One last result is somewhat of a corner case that comes from our research in nanotechnology [7]. The conductance minima mentioned in section 2 are captured by a generalized eigenvalue problem that may have fremdervector solutions. The key condition is that the skew-Hermitian part of \( A \) is negative semi-definite, and the present discussion readily generalizes to cases where either the Hermitian or skew-Hermitian part is positive or negative semi-definite.

**Theorem 5.2.** Let \( B \) and \( C \) be the Hermitian and skew-Hermitian parts of \( A \), respectively, with \( C \) either positive or negative semi-definite. Furthermore, let \( P \) be an orthogonal projector onto ker(\( C \)). If \( x \in \ker(\( C \)) \) and \( z \) are an eigenvector/eigenvalue pair of the generalized eigenvalue problem

\[
PBPx = zPx,
\]

...
then \( z \) is a fremdervalue of \( A \) and \( x \) is a fremdervector. A similar result holds if \( B \) is positive or negative semi-definite instead of \( C \).

**Proof.** Notice that Eq. (4) is Hermitian such that \( z \in \mathbb{R} \). Then, from Theorem 4.5, any fremdervector \( x \) of \( zI - A \) must be a fremdervector of \( C \), requiring \( x \in \ker(C) \). Thus, \( x = Px \). Finally, \( x \) and \( z \) satisfying Eq. (4) implies

\[
\langle x, (zI - A)x \rangle = \langle x, (zI - B - C)x \rangle \\
= \langle x, (zI - B)x \rangle \\
= \langle Px, (zI - B)Px \rangle \\
= \langle x, P(zI - B)Px \rangle \\
= z \langle x, Px \rangle - \langle x, PBPx \rangle \\
= 0.
\]

Hence, \( x \) is a fremdervector of \( A \) and \( z \) is a fremdervalue.

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