RIEMANN-HILBERT APPROACH TO THE SIX-VERTEX MODEL

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Abstract. The six-vertex model, or the square ice model, with domain wall boundary conditions (DWBC) has been introduced and solved for finite $n$ by Korepin and Izergin. The solution is based on the Yang-Baxter equations and it represents the free energy in terms of an $n \times n$ Hankel determinant. Paul Zinn-Justin observed that the Izergin-Korepin formula can be re-expressed in terms of the partition function of a random matrix model with a nonpolynomial interaction. We use this observation to obtain the large $n$ asymptotics of the six-vertex model with DWBC. The solution is based on the Riemann-Hilbert approach. In this paper we review asymptotic results obtained in different regions of the phase diagram.

1. Six-vertex model

The six-vertex model, or the model of two-dimensional ice, is stated on a square lattice with arrows on edges. The arrows obey the rule that at every vertex there are two arrows pointing in and two arrows pointing out. This rule is sometimes called the ice-rule. There are only six possible configurations of arrows at each vertex, hence the name of the model, see Fig. 1.

![Arrow configurations](image)

**Figure 1.** The six arrow configurations allowed at a vertex.

We will consider the domain wall boundary conditions (DWBC), in which the arrows on the upper and lower boundaries point into the square, and the ones on the left and right
boundaries point out. One possible configuration with DWBC on the $4 \times 4$ lattice is shown on Fig. 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{An example of $4 \times 4$ configuration.}
\end{figure}

The name of the square ice comes from the two-dimensional arrangement of water molecules, $H_2O$, with oxygen atoms at the vertices of the lattice and one hydrogen atom between each pair of adjacent oxygen atoms. We place an arrow in the direction from a hydrogen atom toward an oxygen atom if there is a bond between them. Thus, as we already noticed before, there are two in-bound and two out-bound arrows at each vertex.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{The corresponding ice model.}
\end{figure}

For each possible vertex state we assign a weight $w_i, i = 1, \ldots, 6$, and define, as usual, the partition function, as a sum over all possible arrow configurations of the product of the vertex weights,

$$Z_n = \sum_{\text{arrow configurations } \sigma} w(\sigma), \quad w(\sigma) = \prod_{x \in V_n} w_{\sigma(x)} = \prod_{i=1}^{6} w_i^{N_i(\sigma)},$$

(1.1)
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where $V_n$ is the $n \times n$ set of vertices, $\sigma(x) \in \{1, \ldots, 6\}$ is the vertex configuration of $\sigma$ at vertex $x$, according to Fig. 1, and $N_i(\sigma)$ is the number of vertices of type $i$ in the configuration $\sigma$. The sum is taken over all possible configurations obeying the given boundary condition. The Gibbs measure is defined then as

$$\mu_n(\sigma) = \frac{w(\sigma)}{Z_n}.$$  \hspace{1cm} (1.2)

Our main goal is to obtain the large $n$ asymptotics of the partition function $Z_n$.

In general, the six-vertex model has six parameters: the weights $w_i$. However, by using some conservation laws we can reduce these to only two parameters. Any fixed boundary conditions impose some conservation laws on the six-vertex model. In the case of DWBC, they are

$$N_1(\sigma) = N_2(\sigma), \quad N_3(\sigma) = N_4(\sigma), \quad N_5(\sigma) = N_6(\sigma) + n.$$  \hspace{1cm} (1.3)

This allows us to reduce to the case

$$w_1 = w_2 \equiv a, \quad w_3 = w_4 \equiv b, \quad w_5 = w_6 \equiv c.$$  \hspace{1cm} (1.4)

Then by using the identity,

$$Z_n(a, a, b, b, c, c) = c^{n^2} Z_n \left( \frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 1, 1 \right),$$

we can reduce to the two parameters, $\frac{a}{c}$ and $\frac{b}{c}$. For details on how we make this reduction, see, e.g., the works [1] of Allison and Reshetikhin, [15] of Ferrari and Spohn, and [7] of Bleher and Liechty.

2. Phase diagram of the six-vertex model

Introduce the parameter

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}.$$  \hspace{1cm} (2.1)

The phase diagram of the six-vertex model consists of the following three regions: the ferroelectric phase region, $\Delta > 1$; the anti-ferroelectric phase region, $\Delta < -1$; and, the disordered phase region, $-1 < \Delta < 1$, see, e.g., [27]. In these three regions we parameterize the weights in the standard way: in the ferroelectric phase region,

$$a = \sinh(t - \gamma), \quad b = \sinh(t + \gamma), \quad c = \sinh(2|\gamma|), \quad 0 < |\gamma| < t,$$

in the anti-ferroelectric phase region,

$$a = \sinh(\gamma - t), \quad b = \sinh(\gamma + t), \quad c = \sinh(2\gamma), \quad |t| < \gamma,$$

and in the disordered phase region,

$$a = \sin(\gamma - t), \quad b = \sin(\gamma + t), \quad c = \sin(2\gamma), \quad |t| < \gamma.$$  \hspace{1cm} (2.4)

The phase diagram of the model is shown on Fig. 4.

The phase diagram and the Bethe-Ansatz solution of the six-vertex model for periodic and anti-periodic boundary conditions are thoroughly discussed in the works of Lieb [23]-[26], Lieb, Wu [27], Sutherland [29], Baxter [1], Batchelor, Baxter, O’Rourke, Yung [3]. See also the work of Wu, Lin [32], in which the Pfaffian solution for the six-vertex model with periodic boundary conditions is obtained on the free fermion line, $\Delta = 0$. 

Figure 4. The phase diagram of the model, where $F$, $AF$ and $D$ mark ferroelectric, antiferroelectric, and disordered phases, respectively. The circular arc corresponds to the so-called “free fermion” line, where $\Delta = 0$, and the three dots correspond to 1-, 2-, and 3-enumeration of alternating sign matrices.

3. Izergin-Korepin determinantal formula

The six-vertex model with DWBC was introduced by Korepin in [20], who derived an important recursion relation for the partition function of the model. This lead to a beautiful determinantal formula of Izergin and Korepin [17], for the partition function of the six-vertex model with DWBC. A detailed proof of this formula and its generalizations are given in the paper of Izergin, Coker, and Korepin [18]. When the weights are parameterized according to (2.4), the formula of Izergin-Korepin is

$$Z_n = \frac{(ab)^n}{(\prod_{k=0}^{n-1} k!)^2} \tau_n,$$

where $\tau_n$ is the Hankel determinant,

$$\tau_n = \det \left( \frac{d^{i+k-2} \phi}{dt^{i+k-2}} \right)_{1 \leq i,k \leq n},$$

and

$$\phi(t) = \frac{c}{ab}.$$

Observe that $a,b,c$ have different parameterizations (2.2)–(2.4) in different phase regions. An elegant derivation of the Izergin-Korepin determinantal formula from the Yang-Baxter equations is given in the papers of Korepin and Zinn-Justin [21] and Kuperberg [22].
One of the applications of the determinantal formula is that it implies that the partition function $\tau_n$ solves the Toda equation,

$$\tau_n'' - \tau_n'^2 = \tau_{n+1}\tau_{n-1}, \quad n \geq 1, \quad ('') = \frac{\partial}{\partial t}, \quad (3.4)$$

cf. [28]. This was used by Korepin and Zinn-Justin [21] to derive the free energy of the six-vertex model with DWBC, assuming some Ansatz on the behavior of subdominant terms in the large $n$ asymptotics of the free energy.

4. The six-vertex model with DWBC and a random matrix model

Another application of the Izergin-Korepin determinantal formula is that $\tau_n$ can be expressed in terms of a partition function of a random matrix model. The relation to the random matrix model was obtained and used by Zinn-Justin [33]. It can be derived as follows. Consider first the disordered phase region.

4.1. Disordered phase region. For the evaluation of the Hankel determinant, it is convenient to use an integral representation of the function

$$\phi(t) = \frac{\sin 2\gamma}{\sin(\gamma - t)\sin(\gamma + t)}, \quad (4.1)$$

namely, to write it in the form of the Laplace transform,

$$\phi(t) = \int_{-\infty}^{\infty} e^{\lambda t} m(\lambda) d\lambda, \quad (4.2)$$

where

$$m(\lambda) = \frac{\sinh \frac{\lambda}{2}(\pi - 2\gamma)}{\sinh \frac{\lambda}{2}\pi}. \quad (4.3)$$

Then

$$\frac{d^i\phi}{dt^i} = \int_{-\infty}^{\infty} \lambda^i e^{\lambda t} m(\lambda) d\lambda, \quad (4.4)$$

and by substituting this into the Hankel determinant, (3.2), we obtain that

$$\tau_n = \int \prod_{i=1}^{n} [e^{t\lambda_i}m(\lambda_i)d\lambda_i] \det(\lambda_i^{i+k-2})_{1 \leq i, k \leq n} \quad (4.5)$$

Consider any permutation $\sigma \in S_n$ of variables $\lambda_i$. From the last equation we have that

$$\tau_n = \int \prod_{i=1}^{n} [e^{t\lambda_i}m(\lambda_i)d\lambda_i] (-1)^\sigma \det(\lambda_i^{k-1})_{1 \leq i, k \leq n} \prod_{i=1}^{n} \lambda_i^{i-1}. \quad (4.6)$$

By summing over $\sigma \in S_n$, we obtain that

$$\tau_n = \frac{1}{n!} \int \prod_{i=1}^{n} [e^{t\lambda_i}m(\lambda_i)d\lambda_i] \Delta(\lambda)^2, \quad (4.7)$$
where \( \Delta(\lambda) \) is the Vandermonde determinant,

\[
\Delta(\lambda) = \det(\lambda_i^{k-1})_{1 \leq i,k \leq n} = \prod_{i<k}(\lambda_k - \lambda_i).
\]

(4.8)

Equation (4.7) expresses \( \tau_n \) in terms of a matrix model integral. Namely, if \( m(x) = e^{-V(x)} \), then

\[
\tau_n = \frac{\prod_{n=1}^{n-1} n!}{\pi^{n(n-1)/2}} \int dM e^{Tr[M-V(M)]},
\]

(4.9)

where the integration is over the space of \( n \times n \) Hermitian matrices. The matrix model integral can be solved, furthermore, in terms of orthogonal polynomials.

Introduce monic polynomials \( P_k(x) = x^k + \ldots \) orthogonal on the line with respect to the weight

\[
w(x) = e^{tx}m(x),
\]

(4.10)

so that

\[
\int_{-\infty}^{\infty} P_j(x)P_k(x)e^{tx}m(x)dx = h_k \delta_{nm}.
\]

(4.11)

Then it follows from (4.7) that

\[
\tau_n = \prod_{k=0}^{n-1} h_k.
\]

(4.12)

The orthogonal polynomials satisfy the three term recurrence relation,

\[
xP_k(x) = P_{k+1}(x) + Q_k P_n(x) + R_k P_{k-1}(x),
\]

(4.13)

where \( R_n \) can be found as

\[
R_n = \frac{h_k}{h_{k-1}}.
\]

(4.14)

see, e.g., [30]. This gives that

\[
h_k = h_0 \prod_{j=1}^{k} R_j,
\]

(4.15)

where

\[
h_0 = \int_{-\infty}^{\infty} e^{tx}m(x)dx = \frac{\sin(2\gamma)}{\sin(\gamma + t)\sin(\gamma - t)}. \]

(4.16)

By substituting (4.15) into (4.12), we obtain that

\[
\tau_n = h_0^n \prod_{k=1}^{n-1} R_{n-k}.
\]

(4.17)
4.2. **Ferroelectric phase.** In the ferroelectric phase, the parameters \(a, b,\) and \(c\) are parameterized by (2.2). We consider the case \(\gamma > 0\), which corresponds to the region \(b > a + c\) in the phase diagram. The case \(\gamma < 0\) is similar, and \(a\) and \(b\) should be exchanged in that case. The function \(\phi\) is the Laplace transform of a discrete measure supported on the positive integers:

\[
\phi(t) = \frac{\sinh(2\gamma)}{\sinh(t+\gamma) \sinh(t-\gamma)} = 4 \sum_{l=1}^{\infty} e^{-2tl} \sinh(2\gamma l).
\]  

(4.18)

Then, similar to (4.7), we find that

\[
\tau_n = \frac{2^{n^2}}{n!} \sum_{l_1, \ldots, l_n=1}^{\infty} \prod_{i=1}^{n} \left[ 2e^{-2tl_i} \sinh(2\gamma l_i) \right].
\]  

(4.19)

This is the partition function for a discrete version of a Hermitian random matrix model, often called a *discrete orthogonal polynomial ensemble* (DOPE), and can also be solved in terms of orthogonal polynomials. The appropriate polynomials in this case are the monic polynomials \(P_n(l) = l^n + \cdots\) with the orthogonality

\[
\sum_{l=1}^{\infty} P_j(l) P_k(l) w(l) = h_k \delta_{jk}, \quad w(l) = 2e^{-2tl} \sinh(2\gamma l) = e^{-2tl+2\gamma l} - e^{-2tl-2\gamma l}.
\]  

(4.20)

Then it follows from (4.19) that

\[
\tau_n = 2^{n^2} \prod_{k=0}^{n-1} h_k.
\]  

(4.21)

4.3. **Critical line between disordered and ferroelectric phase.** When the parameters \(a, b,\) and \(c\) are such that \(b - a = c\), (so \(\Delta = 1\) in (2.1)), the Izergin-Korepin formula is not directly applicable. However, we may consider a limiting case of the orthogonal polynomial formula (4.21). On the critical line

\[
\frac{b}{c} - \frac{a}{c} = 1,
\]  

(4.22)

we fix a point,

\[
\frac{a}{c} = \frac{\alpha - 1}{2}, \quad \frac{b}{c} = \frac{\alpha + 1}{2}; \quad \alpha > 1,
\]  

(4.23)

and consider the partition function

\[
Z_n = Z_n \left( \frac{\alpha - 1}{2}, \frac{\alpha - 1}{2}, \frac{\alpha + 1}{2}, \frac{\alpha + 1}{2}, 1, 1 \right).
\]  

(4.24)

Consider the limit of (4.21) as

\[
t, \gamma \to +0, \quad \frac{t}{\gamma} \to \alpha.
\]  

(4.25)

Observe that in this limit,

\[
\frac{a}{c} = \frac{\sinh(t-\gamma)}{\sinh(2\gamma)} \to \frac{\alpha - 1}{2}, \quad \frac{b}{c} = \frac{\sinh(t+\gamma)}{\sinh(2\gamma)} \to \frac{\alpha + 1}{2}.
\]  

(4.26)
By (1.5), (3.1), and (4.12), we have

$$Z_n\left(\frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 1, 1\right) = \left[\frac{2\sinh(t - \gamma)\sinh(t + \gamma)}{\sinh(2\gamma)}\right]^{n^2} \prod_{k=0}^{n-1} \frac{h_k}{(k!)^2}.$$  (4.27)

To deal with limit (4.25) we need to rescale the orthogonal polynomials $P_k(l)$. Introduce the rescaled variable,

$$x = 2tl - 2\gamma l,$$  (4.28)

and the rescaled limiting weight,

$$w_{\alpha}(x) = \lim_{t, \gamma \to +0, \frac{t}{l} \to \alpha} (e^{-2tl+2\gamma l} - e^{-2tl-2\gamma l}) = e^{-x} - e^{-rx}, \quad r = \frac{\alpha + 1}{\alpha - 1} > 1.$$  (4.29)

Consider monic orthogonal polynomials $P_j(x; \alpha)$ satisfying the orthogonality condition,

$$\int_0^\infty P_j(x; \alpha)P_k(x; \alpha)w_{\alpha}(x)dx = h_{k,\alpha}\delta_{jk}.$$  (4.30)

To find a relation between $P_k(l)$ and $P_k(x; \alpha)$, introduce the monic polynomials

$$\tilde{P}_k(x) = \delta^k P_k(x/\delta),$$  (4.31)

where

$$\delta = 2t - 2\gamma,$$  (4.32)

and rewrite orthogonality condition (4.11) in the form

$$\sum_{l=1}^{\infty} \tilde{P}_j(l\delta)\tilde{P}_k(l\delta)w_{\alpha}(l\delta)\delta = \delta^{2k+1}h_k\delta_{jk},$$  (4.33)

which is a Riemann sum for the integral in orthogonality condition (4.30). Therefore,

$$\lim_{t, \gamma \to +0, \frac{t}{l} \to \alpha} \tilde{P}_k(x) = P_k(x; \alpha), \quad \lim_{t, \gamma \to +0, \frac{t}{l} \to \alpha} \delta^{2k+1}h_k = h_{k,\alpha}.$$  (4.34)

Thus, if we rewrite formula (4.27) as

$$Z_n\left(\frac{a}{c}, \frac{a}{c}, \frac{b}{c}, \frac{b}{c}, 1, 1\right) = \left[\frac{2\sinh(t - \gamma)\sinh(t + \gamma)}{\sinh(2\gamma)}\right]^{n^2} \prod_{k=0}^{n-1} \frac{\delta^{2k+1}h_k}{(k!)^2},$$  (4.35)

we can take limit (4.23). In the limit we obtain that

$$Z_n = Z_n\left(\frac{\alpha - 1}{2}, \frac{\alpha - 1}{2}, \frac{\alpha + 1}{2}, \frac{\alpha + 1}{2}, 1, 1\right) = \left(\frac{\alpha + 1}{2}\right)^{n^2} \prod_{k=0}^{n-1} \frac{h_{k,\alpha}}{(k!)^2}.$$  (4.36)
4.4. **Antiferroelectric phase.** In the antiferroelectric phase, the parameters $a, b,$ and $c$ are parameterized by (2.3), and the function

$$\phi(t) = \frac{\sinh(2\gamma)}{\sinh(\gamma - t) \sinh(\gamma + t)}, \quad |t| < \gamma,$$

is the Laplace transform of a discrete measure supported on the integers:

$$\phi(t) = \frac{\sinh(2\gamma)}{\sinh(\gamma - t) \sinh(\gamma + t)} = 2 \sum_{l=-\infty}^{\infty} e^{2tl-2\gamma|l|}.$$ 

Then

$$\tau_n = \frac{2^{n^2}}{n!} \sum_{l_1,\ldots,l_n=-\infty}^{\infty} \Delta(l)^2 \prod_{i=1}^{n} e^{2tl_i-2\gamma|l_i|}. \quad (4.39)$$

This is again the partition function of a DOPE, and we introduce the discrete monic polynomials $P_n(l) = l^n + \ldots$ via the orthogonality condition

$$\sum_{l=-\infty}^{\infty} P_j(l) P_k(l) w(l) = h_k \delta_{jk}, \quad w(l) = e^{2tl-2\gamma|l|}. \quad (4.40)$$

Then it follows from (4.39) that

$$\tau_n = 2^{n^2} \prod_{k=0}^{n-1} h_k. \quad (4.41)$$

4.5. **Critical line between the antiferroelectric and disordered phases.** When the parameters $a, b,$ and $c$ are such that $a + b = c,$ (so $\Delta = -1$ in (2.1)), the Izergin-Korepin formula is not directly applicable, and we must consider a limiting case of the orthogonal polynomial formula (4.41). On the critical line

$$\frac{a}{c} + \frac{b}{c} = 1,$$ 

we fix a point,

$$\frac{a}{c} = \frac{1 - \alpha}{2}, \quad \frac{b}{c} = \frac{1 + \alpha}{2}, \quad -1 < \alpha < 1,$$ 

and consider the partition function

$$Z_n = Z_n \left( \frac{1 - \alpha}{2}, \frac{1 - \alpha}{2}, \frac{1 + \alpha}{2}, \frac{1 + \alpha}{2}, 1, 1 \right). \quad (4.44)$$

This corresponds to taking a limit of the Izergin-Korepin formula in the antiferroelectric phase as $t, \gamma \to 0,$ and $t/\gamma = \alpha.$ Introduce the rescaled variable,

$$x = -2tl + 2\gamma l,$$ 

and the rescaled limiting weight,

$$w_\alpha(x) = \lim_{t,\gamma \to 0, \frac{t}{\gamma} \to \alpha} e^{2tl-2\gamma|l|} = \begin{cases} e^{-x}, & x \geq 0 \\ e^{rx}, & x < 0 \end{cases} \quad (4.46)$$

where

$$r = \frac{1 + \alpha}{1 - \alpha} > 0. \quad (4.47)$$
Consider monic orthogonal polynomials $P_j(x; \alpha)$ satisfying the orthogonality condition,
\[
\int_{\mathbb{R}} P_j(x; \alpha) P_k(x; \alpha) w_\alpha(x) \, dx = h_{k,\alpha} \delta_{jk},
\] (4.48)
which can be obtained from the polynomials (4.40) by taking the appropriate scaling limit as $t, \gamma \to 0$, and $t/\gamma = \alpha$. Similar to (4.36), we obtain
\[
Z_n = Z_n \left( \frac{\alpha - 1}{2}, \frac{\alpha - 1}{2}, \frac{\alpha + 1}{2}, 1, 1 \right) = \left( \frac{1 + \alpha}{2} \right)^{n^2} \prod_{k=0}^{n-1} \frac{h_{k,\alpha}}{(k!)^2}.
\] (4.49)

5. LARGE $n$ ASYMPTOTICS OF $Z_n$

The asymptotic evaluation of $Z_n$ in the different regions of the phase diagram thus reduces to asymptotic evaluation of different systems of orthogonal polynomials. In general, this may be done by formulating the orthogonal polynomials as the solution to a $2 \times 2$ matrix valued Riemann-Hilbert problem as in [16]. One may then perform the steepest descent analysis of Deift and Zhou [13]. In the case that the weight of orthogonality is a continuous one on $\mathbb{R}$, this analysis was performed for weights of the form $\exp(-nV(x))$ for a very general class of analytic potential functions $V(x)$ in [12]. The analysis was adapted to the case that the orthogonality is with respect to a discrete measure in [2] and [10]. The steepest descent analysis yields the following results in the different regions of the phase diagram.

5.1. Disordered phase.

**Theorem 5.1.** (See [5].) Let the weights $a, b$, and $c$, in the six-vertex model with DWBC be parameterized as in (2.4). Then, as $n \to \infty$, the partition function $Z_n$ has the asymptotic expansion
\[
Z_n = C n^\kappa F^{n^2} \left( 1 + O(n^{-\varepsilon}) \right), \quad \varepsilon > 0,
\] (5.1)
where
\[
F = \frac{\pi ab}{2 \gamma \cos \left( \frac{\pi t}{2\gamma} \right)}, \quad \kappa = \frac{1}{12} - \frac{2\gamma^2}{3\pi(\pi - 2\gamma)},
\] (5.2)
and $C > 0$ is a constant.

This proves the conjecture of Zinn-Justin, and it gives the exact value of the exponent $\kappa$. Let us remark, that the presence of the power-like factor $n^\kappa$ in the asymptotic expansion of $Z_n$ in (5.1) is rather unusual from the point of view of random matrix models. Also, in the one-cut case the usual large $n$ asymptotics of $Z_n$ in a non-critical random matrix model is the so called “topological expansion”, which gives $Z_n$ as an asymptotic series in powers of $1/n^2$ (see e.g. [14] and [6]). In this case the asymptotic expansion goes over, in general, non-integer inverse powers of $n$ (see [5]).

It is noteworthy that, as shown in [11], asymptotic formula (5.1) remains valid on the borderline between the disordered and antiferroelectric phases. In this case $\kappa = \frac{1}{12}$, which corresponds to $\gamma = 0$. 


5.2. Ferroelectric phase. Bleher and Liechty [7], [8] obtained the large $n$ asymptotics of $Z_n$ in the ferroelectric phase, $\Delta > 1$, and also on the critical line between the ferroelectric and disordered phases, $\Delta = 1$. In the ferroelectric phase we use parameterization (2.2) for $a, b$ and $c$. The large $n$ asymptotics of $Z_n$ in the ferroelectric phase is given by the following theorem.

**Theorem 5.2.** (See [7].) Let the weights $a$, $b$, and $c$ in the six-vertex model with DWBC be parameterized as in (2.2) with $t > \gamma > 0$. For any $\varepsilon > 0$, as $n \to \infty$,

$$Z_n = C G_n F_n^2 \left[1 + O(e^{-n^{1-\varepsilon}})\right],$$

(5.3)

where $C = 1 - e^{-4\gamma}$, $G = e^{\gamma-t}$, and $F = b$.

On the critical line between the ferroelectric and disordered phases we use the parameterization $b = a + 1$, $c = 1$. The main result here is the following asymptotic formula for $Z_n$.

**Theorem 5.3.** (See [8].) As $n \to \infty$,

$$Z_n = C n^{\kappa} G^{\sqrt{n}} F^{n^2} [1 + O(n^{-1/2})],$$

(5.4)

where $C > 0$,

$$\kappa = \frac{1}{4}, \quad G = \exp \left[-\zeta \left(\frac{3}{2}\right) \sqrt{\frac{a}{\pi}}\right],$$

(5.5)

and

$$F = b.$$  

(5.6)

Notice that in both Theorem 5.2 and in Theorem 5.3, the limiting free energy $F$ is the weight $b$. The ground state in this phase is unique and is achieved when there is exactly one $c$-type vertex in each row and column, and the rest of the vertices are of type $b$. That is, the diagonal consists of type 5 vertices while above the diagonal all vertices are type 3 and below all vertices are type 4. The weight of the ground state is $b n^2 (c/b)^n$, and thus the free energy in the ferroelectric phase is completely determined by the ground state. This is a reflection of the fact that local fluctuations from the ground state can take place only in a thin neighborhood of the diagonal. The conservation laws (1.3) forbid local fluctuations away from the diagonal.

5.3. Antiferroelectric phase. The large $n$ asymptotics in the antiferroelectric phase were obtained by Bleher and Liechty in [9]. They are given in the following theorem. In this theorem $\vartheta_1$ and $\vartheta_4$ are the Jacobi theta functions with elliptic nome $q = e^{-\pi^2/2\gamma}$ (see e.g., [31]), and the phase $\omega$ is given as

$$\omega = \frac{\pi}{2} \left(1 + \frac{t}{\gamma}\right).$$

(5.7)

**Theorem 5.4.** (See [9].) Let the weights $a$, $b$, and $c$ in the six-vertex model with DWBC be parameterized as in (2.2). As $n \to \infty$,

$$Z_n = C \vartheta_4 (n \omega) F^{n^2} (1 + O(n^{-1})),$$

(5.8)
where $C > 0$ is a constant, and

$$F = \frac{\pi ab\theta'(0)}{2\gamma \theta_1(\omega)}. \quad (5.9)$$

In contrast to the disordered phase, note the lack of a power like term. In contrast to the ferroelectric phase, notice that the free energy depends transcendentally on the weight of the ground state configuration. Only in the limit as $\gamma \to \infty$, which can be regarded as the low temperature limit, does the weight of the ground state become dominant. For a discussion of this limit, see [33].

6. THE RIEMANN-HILBERT APPROACH

All the above asymptotic results are obtained in the Riemann-Hilbert approach, but the concrete asymptotic analysis of the Riemann-Hilbert problem is quite different in the different phase regions. Let us discuss it.

6.1. DISORDERED PHASE REGION. To apply the Riemann-Hilbert approach, we introduce a rescaled weight as

$$w_n(x) = w\left(\frac{nx}{\gamma}\right). \quad (6.1)$$

It can be written as

$$w_n(x) = e^{-nV_n(x)}, \quad (6.2)$$

where

$$V_n(x) = -\zeta x - \frac{1}{n} \ln \frac{\sinh \left(\frac{n}{2\gamma} - 1\right)x}{\sinh \left(\frac{n\pi x}{2\gamma}\right)}, \quad \zeta = \frac{t}{\gamma}. \quad (6.3)$$

The external potential $V_n(x)$ is real analytic for any finite $n$, but it has logarithmic singularities on the imaginary axes, which accumulate to the origin as $n \to \infty$. In fact, the limiting external potential,

$$\lim_{n \to \infty} V_n(x) = V(x) = -\zeta x + |x|, \quad (6.4)$$

is not analytic at $x = 0$. The Riemann-Hilbert approach developed in [5] is based on opening of lenses whose boundary approaches the origin as $n \to \infty$. This turns out to be possible due to the fact that the density of the equilibrium measure $\rho_n(x)$ for the external potential $V_n(x)$ diverges logarithmically at the origin as $n \to \infty$, and as a result, the jump matrix on the boundary of the lenses converges to the unit matrix (for details see [5]). The calculation of subdominant asymptotic terms in the partition function as $n \to \infty$ is the central difficult part of the work [5], and it is done by an asymptotic analysis of the solution to the Riemann-Hilbert problem near the turning points and near the origin.

6.2. FERROELECTRIC PHASE REGION. In the ferroelectric region, the measure of orthogonality is a discrete one on $\mathbb{N}$. To apply the Riemann-Hilbert approach to discrete orthogonal polynomials, we need to rescale both the weight and the lattice that supports the measure so that the mesh of the lattice goes to zero as $n \to \infty$. Introduce the rescaled lattice and weight

$$L_n = \left(\frac{2t}{n}\right) \mathbb{N}, \quad w_n(x) = e^{-nx(1-\zeta)} \left(1 - e^{-4nx}\right) = e^{-nV_n(x)} , \quad (6.5)$$
where
\[ V_n(x) = x(1 - \zeta) - \frac{1}{n} \log \left(1 - e^{-2nx\zeta}\right), \quad 0 < \zeta = \frac{\gamma}{t} < 1. \] (6.6)

Then the orthogonality condition (4.20) can be written as
\[ \sum_{x \in L_n} P_j \left(\frac{nx}{2t}\right) P_k \left(\frac{nx}{2t}\right) w_n(x) = h_k \delta_{jk}. \] (6.7)

Notice that, as \( n \to \infty \), \( V_n(x) \) has the limit
\[ \lim_{n \to \infty} V_n(x) = x(1 - \zeta), \] (6.8)

which would indicate that, in the large \( n \) limit, the polynomials (4.20) behave as polynomials orthogonal on \( \mathbb{N} \) with a simple exponential weight. These polynomials are a special case of the classical Meixner polynomials, and there are exact formulae for their recurrence coefficients (see e.g., [19]). The monic Meixner polynomials which concern us are defined from the orthogonality condition
\[ \sum_{l=1}^{\infty} Q_j(l)Q_k(l)q^l = h_k^Q \delta_{jk}, \quad q = e^{2\gamma - 2t}, \] (6.9)

and the normalizing constants are given exactly as
\[ h_k^Q = \frac{(k!)^2 q^{k+1}}{(1 - q)2^{k+1}}. \] (6.10)

Up to the constant factor, Theorem 5.2 can therefore be proven by showing that \( h_k \) and \( h_k^Q \) are asymptotically close as \( n \to \infty \). More precisely, it is shown in [7] that as \( k \to \infty \), for any \( \varepsilon > 0 \),
\[ h_k = h_k^Q \left(1 + O \left( e^{-k^{1-\varepsilon}} \right) \right). \] (6.11)

6.3. Antiferroelectric region. In the antiferroelectric region, the orthogonal polynomials are with respect to a discrete weight, and we rescale the weight in (4.40) and the integer lattice as
\[ L_n = \left(\frac{2\gamma}{n}\right) \mathbb{Z}, \quad w_n(x) = e^{-nV(x)}, \quad V(x) = |x| - \zeta x, \quad \zeta = \frac{t}{\gamma} < 1, \] (6.12)

so that the orthogonality condition (4.40) can be written as
\[ \sum_{x \in L_n} P_j \left(\frac{nx}{2\gamma}\right) P_k \left(\frac{nx}{2\gamma}\right) w_n(x) = h_k \delta_{jk}. \] (6.13)

The mesh of the lattice \( L_n \) is \( 2\gamma/n \), which places an upper constraint on the equilibrium measure, which is the limiting distribution of zeroes of the orthogonal polynomials. This upper constraint is realized. The equilibrium measure, which has density \( \rho(x) \), is supported on a single interval \([\alpha, \beta]\), but within that interval is an interval \([\alpha', \beta']\) on which \( \rho(x) \equiv 1/2\gamma \). This interval is called the saturated region, and it separates the single band of support \([\alpha, \beta]\) into the two analytic bands \([\alpha, \alpha']\) and \([\beta', \beta]\). Thus in effect we have a “two-cut” situation, which is the source of the quasi-periodic factor \( \vartheta_4(n\omega) \) in Theorem 5.4.
In principle, a problem could come from the fact that the potential $V(x)$ is not analytic at the origin. However, it turns out that this point of nonanalyticity is always in the saturated region and therefore does not present a problem in the steepest descent analysis.

As previously noted, there is no power-like term in the asymptotic formula for $Z_n$ in the antiferroelectric phase. The Riemann-Hilbert approach to orthogonal polynomials generally gives an expansion of the normalizing constants $h_n$ in inverse powers of $n$. In the two-cut case, the coefficients in this expansion may be quasi-periodic functions of $n$. For the orthogonal polynomials (4.40) it is a tedious calculation involving the Jacobi theta functions to show that the term of order $n^{-1}$ vanishes in the expansion of $h_n$, which then implies the absence of the power-like term in $Z_n$.

References

[1] D. Allison and N. Reshetikhin, Numerical study of the 6-vertex model with domain wall boundary conditions, Ann. Inst. Fourier (Grenoble) 55 (2005) 1847-1869.

[2] J. Baik, T. Kriecherbauer, K. T.-R. McLaughlin, and P.D. Miller, Discrete orthogonal polynomials. Asymptotics and applications. Ann. Math. Studies 164. Princeton University Press. Princeton and Oxford, 2007.

[3] M.T. Batchelor, R.J. Baxter, M.J. O'Rourke, and C.M. Yung, Exact solution and interfacial tension of the six-vertex model with anti-periodic boundary conditions. J. Phys. A 28 (1995) 2759–2770.

[4] R. Baxter, Exactly solved models in statistical mechanics, Academic Press, San Diego, CA.

[5] P.M. Bleher and V.V. Fokin, Exact solution of the six-vertex model with domain wall boundary conditions. Disordered phase. Commun. Math. Phys. 268 (2006), 223–284.

[6] P. Bleher and A. Its, Asymptotics of the partition function of a random matrix model. Ann. Inst. Fourier 55 (2005), 1943–2000.

[7] P.M. Bleher and K. Liechty, Exact solution of the six-vertex model with domain wall boundary conditions. Ferroelectric phase, Commun. Math. Phys. 286 (2009), 777–801.

[8] P.M. Bleher and K. Liechty, Exact solution of the six-vertex model with domain wall boundary condition. Critical line between ferroelectric and disordered phases, J. Statist. Phys. 134 (2009), 463–485.

[9] P.M. Bleher and K. Liechty, Exact solution of the six-vertex model with domain wall boundary conditions. Antiferroelectric phase. Commun. Pure Appl. Math. 63 (2010), 779–829.

[10] P.M. Bleher and K. Liechty, Uniform Asymptotics for Discrete Orthogonal Polynomials with respect to Varying Exponential Weights on a Regular Infinite Lattice. Int. Math. Res. Not. (2010) 0: rq081v2-rq081.

[11] N.M. Bogoliubov, A.M. Kitaev, and M.B. Zvonarev, Boundary polarization in the six-vertex model, Phys. Rev. E 65 (2002), 026126.

[12] P.A. Deift, T. Kriecherbauer, K.T-R. McLaughlin, S. Venakides, and Z. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. Commun. Pure Appl. Math. 52 (1999), 1335-1425.

[13] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, Ann. Math. 137 (1993), 295–368.

[14] N.M. Ercolani and K.T.-R. McLaughlin, Asymptotics of the partition function for random matrices via Riemann-Hilbert techniques and applications to graphical enumeration, Int. Math. Res. Not., 14 (2003), 755–820.

[15] P.L. Ferrari and H. Spohn, Domino tilings and the six-vertex model at its free fermion point, J. Phys. A: Math. Gen. 39 (2006) 1029710306.

[16] A.S. Fokas, A.R. Its, and A.V. Kitaev. The isomonodromy approach to matrix models in 2D quantum gravity. Comm. Math. Phys. 147 (1992), 395–430.

[17] A. G. Izergin, Partition function of the six-vertex model in a finite volume. Sov. Phys. Dokl. 32 (1987), 878.
[18] A. G. Izergin, D. A. Coker, and V. E. Korepin, Determinant formula for the six-vertex model. *J. Phys. A*. **25** (1992), 4315.

[19] R. Koekoek, P.A Lesky, R. Swarttouw, *Hybergeometric orthogonal polynomials and their q-analogues*, Springer, 2010.

[20] V. E. Korepin, Calculation of norms of Bethe wave functions. *Commun. Math. Phys.* **86** (1982), 391-418.

[21] V. Korepin and P. Zinn-Justin, Thermodynamic limit of the six-vertex model with domain wall boundary conditions, *J. Phys. A* **33** No. 40 (2000), 7053.

[22] G. Kuperberg, Another proof of the alternating sign matrix conjecture. *Int. Math. Res. Not.* (1996), 139-150.

[23] E. H. Lieb, Exact solution of the problem of the entropy of two-dimensional ice. *Phys. Rev. Lett.* **18** (1967) 692.

[24] E. H. Lieb, Exact solution of the two-dimensional Slater KDP model of an antiferroelectric. *Phys. Rev. Lett.* **18** (1967) 1046-1048.

[25] E. H. Lieb, Exact solution of the two-dimensional Slater KDP model of a ferroelectric. *Phys. Rev. Lett.* **19** (1967) 108-110.

[26] E. H. Lieb, Residual entropy of square ice. *Phys. Rev.* **162** (1967) 162.

[27] E. H. Lieb and F. Y. Wu, Two dimensional ferroelectric models, in *Phase Transitions and Critical Phenomena*, C. Domb and M. Green eds., vol. 1, Academic Press (1972) 331-490.

[28] K. Sogo, Toda molecule equation and quotient-difference method. *Journ. Phys. Soc. Japan* **62** (1993), 1887.

[29] B. Sutherland, Exact solution of a two-dimensional model for hydrogen-bonded crystals. *Phys. Rev. Lett.* **19** (1967) 103-104.

[30] G. Szegö, *Orthogonal Polynomials*. Fourth edition. Colloquium Publications, vol. 23, AMS, Providence, RI, 1975.

[31] E.T. Whittaker and G.N. Watson, *A course of modern analysis*. Fourth edition, reprinted. Cambridge University Press, 2000.

[32] F.Y. Wu and K.Y. Lin, Staggered ice-rule vertex model. The Pfaffian solution. *Phys. Rev. B* **12** (1975), 419-428.

[33] P. Zinn-Justin, Six-vertex model with domain wall boundary conditions and one-matrix model. *Phys. Rev. E* **62** (2000), 3411–3418.

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