Hamiltonian reduction of the $U_{EM}(1)$ gauged three flavour WZW model

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Abstract

The three-flavour Wess-Zumino model coupled to electromagnetism is treated as a constraint system using the Faddeev-Jackiw method. Expanding into series of powers of the Goldstone boson fields and keeping terms up to second and third order we obtain Coulomb-gauge Hamiltonian densities.

\footnote{On leave from the Tbilisi State University, Tbilisi, Georgia.}
1 Introduction

The Faddeev-Jackiw method [1] provides a simple and straightforward way to deal with constraint systems without having to distinguish between primary and secondary, first class and second class constraints. A lot of work has been done in this field by several authors. The equivalence of the Faddeev-Jackiw approach to the Dirac’s method is discussed in [2, 3]. Its extension in supersymmetry is given in [4, 5]. Application of the approach to the light cone quantum field theory is given in [6, 7], and to hidden symmetries in [8]. Further elaboration and applications of the method in different cases is given in [9]. A formulation, intimately related to the FJ approach, for constructing unconstraint hamiltonians starting from general first order lagrangians is given in [10].

In a previous paper [11] we used the Faddeev-Jackiw method [1] to treat the two-flavour WZW model coupled to electromagnetism [12, 13]. In this work we extend this analysis to the SU(3) case. The gauge invariant action under the electromagnetic group gauge transformations is given by

\[ \Gamma_{\text{eff}}(U, A_\mu) = \Gamma_{\text{EM}}(A_\mu) + \Gamma_\sigma(U, A_\mu) + \Gamma_{\text{WZW}}(U) + \Gamma_{\text{WZW}}(U, A_\mu) , \]  

where the first term is the action of the free electromagnetic field and the sum of the rest three terms constitute the action of the gauged Wess-Zumino-Witten model

\[ \Gamma_{\text{EM}}(A_\mu) = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} , \]

\[ \Gamma_\sigma(U, A_\mu) = -\frac{f_\pi^2}{16} \int d^4x \text{tr} (R_\mu R^\mu) \]

\[ = -\frac{f_\pi^2}{16} \int d^4x \text{tr} (r_\mu r^\mu) + \frac{if_\pi^2 e}{8} \int d^4x A_\mu \text{tr} [Q(r_\mu - l_\mu)] \]

\[ + \frac{f_\pi^2 e^2}{8} \int d^4x A_\mu A^\mu \text{tr} (Q^2 - U^\dagger QUQ) , \]

\[ \Gamma_{\text{WZW}}(U) = -\frac{iN_c}{240\pi^2} \int d^dxe^{ijkm} \text{tr} (l_i l_j l_k l_m) , \]

\[ \Gamma_{\text{WZW}}(U, A_\mu) = -\frac{N_c e}{48\pi^2} \int d^4x e^{\mu\nu\alpha\beta} A_\mu \text{tr} [Q(r_\nu r_\alpha r_\beta + l_\nu l_\alpha l_\beta)] \]

\[ + \frac{iN_c e^2}{24\pi^2} \int d^4x e^{\mu\nu\alpha\beta} A_\mu (\partial_\nu A_\alpha) \text{tr} [Q^2 (r_\beta + l_\beta) + \frac{1}{2} QU^\dagger QU r_\beta + \frac{1}{2} QUQU^\dagger l_\beta] , \]

where \( U = \exp (2i\theta_\alpha \lambda_\alpha / f_\pi) \), \( a = 1, ..., 8 \) is an element of SU(3)

\[ r_\mu = U^\dagger \partial_\mu U \, , \, \, R_\mu = U^\dagger D_\mu U \, , \, \, l_\mu = (\partial_\mu U)U^\dagger \, , \, \, L_\mu = (D_\mu U)U^\dagger \].

See Appendix for notation.
The model describes successfully the low energy dynamics of photon and the eight Goldstone bosons including their interactions related to the axial anomaly.

As in the two-flavour case we expand the U fields into series of powers of the Goldstone boson fields $\theta_a$. Then using the expression for $\Gamma_{\text{eff}}(U, A_\mu)$ given in (1) we obtain lagrangian densities containing terms initially up to two and afterwards up to three Goldstone boson fields. These lagrangian densities are constrained and we treat them using the Faddeev-Jackiw approach. The general case which involves any number of Goldstone bosons is still out of hand.

2 Expanding up to second order in the Goldstone boson fields

We expand the U field into series of powers of the Goldstone boson fields

$$U = 1 + \frac{2i}{f_\pi} \theta_a \lambda_a + \ldots, \quad a = 1, \ldots, 8$$

and we substitute back into the expression (1) for the effective action. The resulting lagrangian density with terms up to second order in $\theta_a$ is given by

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{EM}} + \mathcal{L}^{(2)}_{\sigma} + \mathcal{L}^{(2)}_{WZW} + O(\theta^3), \quad (3)$$

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

$$\mathcal{L}^{(2)}_{\sigma} = \frac{1}{2} \left( \partial_\mu \theta_a \right) \left( \partial^\mu \theta_a \right) + e A_\mu \left( \theta_2 \partial_\mu \theta_1 - \theta_1 \partial_\mu \theta_2 + \theta_5 \partial_\mu \theta_4 - \theta_4 \partial_\mu \theta_5 \right)$$

$$+ \frac{e^2}{2} A_\mu A^\mu \left( \theta_2^2 + \theta_4^2 + \theta_5^2 \right),$$

$$\mathcal{L}^{(2)}_{WZW} = -\frac{N_c e^2}{12\pi^2 f_\pi} \epsilon^{\mu\nu\alpha\beta} A_\mu \left( \partial_\nu A_\alpha \right) \left( \partial_\beta \theta_3 + \frac{1}{\sqrt{3}} \partial_\beta \theta_8 \right).$$

We can easily check that the $U_{\text{EM}}(1)$ gauge invariance is not lost. In the two flavour case the expression for the corresponding lagrangian density can be deduced from (3) by omitting those terms which include powers or derivatives of $\theta_4$ up to $\theta_8$. We should also notice that the term $\Gamma_{WZW}(U)$ of the effective action given in (2) which is zero in the two flavour case, contributes, in the three flavour case, terms involving five or more Goldstone boson fields. In the non-covariant notation, (3) can be written as follows

$$\mathcal{L}_{\text{eff}} = -E \cdot \dot{A} + \frac{1}{2} \dot{\theta}_a^2 + e A_0 (\theta_2 \dot{\theta}_1 - \theta_1 \dot{\theta}_2 + \theta_5 \dot{\theta}_4 - \theta_4 \dot{\theta}_5) + A_0 \nabla \cdot E$$

$$+ \frac{e^2}{2} \left( A_0^2 - A^2 \right) \left( \theta_1^2 + \theta_2^2 + \theta_4^2 + \theta_5^2 \right) - \frac{1}{2} \left( E^2 + B^2 + (\nabla \theta_a)^2 \right)$$

$$+ e A \cdot \left( \theta_2 \nabla \theta_1 - \theta_1 \nabla \theta_2 + \theta_5 \nabla \theta_4 - \theta_4 \nabla \theta_5 \right) + \frac{N_c e^2}{6\pi^2 f_\pi} (E \cdot B) \left( \theta_3 + \frac{1}{\sqrt{3}} \theta_8 \right), \quad (4)$$
where $E = -\dot{A} - \nabla A_0$, $B = \nabla \times A$ are the electric and magnetic fields. We see that the dynamical variables in (4) which span the configuration space of the lagrangian density are the three components of the vector potential and the eight Goldstone boson fields. The scalar potential $A_0$ will be treated not as a dynamical variable but as a Lagrange multiplier since there is no time derivative of it in (4). Now according to Faddeev and Jackiw [1] we must construct out of (4), which is second order in time derivatives of the $\theta_a$ fields, a new lagrangian density which is first order in time derivatives of all the dynamical variables. This can be accomplished by enlarging the configuration space so that it includes the whole phase space of the model. The enlarged configuration space coordinates will include the three vector potential components, the eight $\theta_a$ fields and the corresponding canonical momenta.

The canonical momenta conjugate to $A$ are the three component of $-E$, and those conjugate to $\theta_a$ are given by

\[
\begin{align*}
p_1 &= \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{\theta}_1} = \dot{\theta}_1 + e A_0 \theta_2 , \\
p_2 &= \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{\theta}_2} = \dot{\theta}_2 - e A_0 \theta_1 , \\
p_3 &= \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{\theta}_3} = \dot{\theta}_3 , \\
p_4 &= \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{\theta}_4} = \dot{\theta}_4 + e A_0 \theta_5 , \\
p_5 &= \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{\theta}_5} = \dot{\theta}_5 - e A_0 \theta_4 , \\
p_6 &= \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{\theta}_6} = \dot{\theta}_6 , \\
p_7 &= \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{\theta}_7} = \dot{\theta}_7 , \\
p_8 &= \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \dot{\theta}_8} = \dot{\theta}_8 .
\end{align*}
\]

So the resulting, first order in time derivatives, lagrangian density is given by

\[
\mathcal{L}_{\text{eff}} = -E \cdot \dot{A} + p_a \dot{\theta}_a - H^{(2)}(E, A, p_a, \theta_a) - A_0(\rho^{(2)} - \nabla \cdot E) + O(\theta^3) ,
\]

\[
H^{(2)} = \frac{1}{2}[E^2 + B^2 + (\nabla \theta_a)^2 + p_a^2] + \frac{e^2}{2}A^2(\theta_1^2 + \theta_2^2 + \theta_4^2 + \theta_5^2) + eA \cdot (\theta_1 \nabla \theta_2 - \theta_2 \nabla \theta_1 + \theta_4 \nabla \theta_5 - \theta_5 \nabla \theta_4) - \frac{N_{\text{c}} e^2}{6\pi^2 f \tau} (E \cdot B)(\theta_3 + \frac{1}{\sqrt{3}} \theta_8) ,
\]

\[
\rho^{(2)} = e(p_2 \theta_1 - p_1 \theta_2 + p_5 \theta_4 - p_4 \theta_5) .
\]

The expression $(p_a \dot{\theta}_a - E \cdot \dot{A})dt$, called the canonical one-form, can be written up to a total time derivative as

\[
\frac{1}{2} \xi^i \omega_{ij} d\xi^j ,
\]
where
\[ \xi^i = A_i , \quad i = 1, ..., 3 \]
\[ \xi^i = \theta_i , \quad i = 4, ..., 11 \]
\[ \xi^i = -E_i , \quad i = 12, ..., 14 \]
\[ \xi^i = p_i , \quad i = 15, ..., 22 \]

and \( \omega_{ij} \) is the symplectic \( 22 \times 22 \) matrix
\[ \omega_{ij} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \]

As we mentioned before \( A_0 \) is the Lagrange multiplier. The expression \( \rho^{(2)} - \nabla \cdot E \) which is multiplied by \( A_0 \) is the only constraint in (5). The next step is to solve the equation of the constraint
\[ \nabla \cdot E - \rho^{(2)} = 0 , \quad (7) \]
and incorporate the solution into the expression (5) for the lagrangian density. In order to do that we decompose the electric field \( E \) and the vector potential \( A \) into transverse and longitudinal components
\[ E = E_T + E_L , \quad A = A_T + A_L , \]
\[ \nabla \cdot E_T = 0 , \quad \nabla \times E_L = 0 , \quad \nabla \cdot A_T = 0 , \quad \nabla \times A_L = 0 . \]

Then (7) implies that
\[ E_L = \frac{\nabla \nabla^2 \rho^{(2)}}{\nabla^2} \equiv -\frac{1}{4\pi} \int d^3y \frac{1}{|y-x|} \nabla \rho^{(2)}(y) . \quad (8) \]

Substituting into (5) the expression of \( E_L \) given in (8) we get (apart from a total divergence)
\[ \mathcal{L}_{eff} = -E_T \cdot \dot{A}_T + \rho^{(2)} \frac{\nabla}{\nabla^2} \cdot \dot{A}_L + p_a \dot{\theta}_a - \frac{1}{2} [E_T^2 + B^2 - \rho^{(2)} \frac{1}{\nabla^2} \rho^{(2)} + (\nabla \theta_a)^2 + p_a^2] \\
- e A \cdot (\theta_1 \nabla \theta_2 - \theta_2 \nabla \theta_1 + \theta_4 \nabla \theta_5 - \theta_5 \nabla \theta_4) - \frac{e^2}{2} A^2 (\theta_1^2 + \theta_2^2 + \theta_4^2 + \theta_5^2) \\
+ \frac{N_c e^2}{6\pi^2 f_\pi} [E_T \cdot B + (\nabla \rho^{(2)}) \cdot B] (\theta_3 + \frac{1}{\sqrt{3}} \theta_8) . \quad (9) \]

We see that the canonical one-form in (9) has lost the standard form as given in (6) because of the term \( \rho^{(2)} \frac{\nabla}{\nabla^2} \cdot \dot{A}_L dt \). According to Darboux’s theorem \[ \square \] we can perform coordinate transformations so that the canonical one-form gets back the standard form. This coordinate transformations called Darboux’s transformations can be written as follows
where $\alpha = e\nabla \cdot A_L$. We substitute (10) into the expression (9) for the lagrangian density and we find out that all the terms which include powers or derivatives of $A_L$ cancel out exactly. Only terms with the physical transverse components of the vector potential survive, and the canonical one-form in (9) acquires the standard form. The resulting lagrangian density is given by

$$L_{\text{eff}} = -E_T \cdot \dot{A}_T + p_a \dot{\theta}_a - H_C^{(2)} ,$$

where

$$H_C^{(2)} = \frac{1}{2} [ E_T^2 + B^2 - \rho^{(2)} - \frac{1}{4} \nabla (\theta_a) + p_3^2 ] + eA_T \cdot \left( \theta_1 \nabla \theta_2 - \theta_2 \nabla \theta_1 + \theta_4 \nabla \theta_5 - \theta_5 \nabla \theta_4 \right)$$

$$+ \frac{e^2}{2} A_T^2 (\theta_1^2 + \theta_2^2 + \theta_4^2 + \theta_5^2) - \frac{N_\pi e^2}{6\pi^2 f_\pi^2} [ E_T \cdot B + \frac{\nabla}{\nabla^2} \rho(2) \cdot B ] \left( \theta_3 + \frac{1}{\sqrt{3}} \theta_8 \right) ,$$

is the expression for the hamiltonian density of the model. We see that without mentioning gauge fixing the solution of the constraint lead to a Coulomb gauge hamiltonian density. (Note that $\mathbf{B} = \nabla \times \mathbf{A}_T$).

### 3 Keeping third order terms

We now proceed with the expansion and keep terms up to the third order in the Goldstone boson fields. We obtain the following expression for the resulting $U_{\text{EM}}(1)$ gauge invariant effective lagrangian density

$$L_{\text{eff}} = L_{\text{EM}} + L_{\sigma}^{(2)} + L_{WZW}^{(2)} + L_{WZW}^{(3)} + O(\theta^4) ,$$

where the expression for $L_{\sigma}^{(2)}$ and $L_{WZW}^{(2)}$ are given in (3) and

$$L_{WZW}^{(3)} = -\frac{N_\pi e}{3\pi^2 f_\pi^3} \epsilon^{\mu\nu\alpha\beta}(\partial_\mu A_\nu)(\theta_1 \partial_\alpha \theta_2 - \theta_2 \partial_\alpha \theta_1 + \theta_4 \partial_\alpha \theta_5 - \theta_5 \partial_\alpha \theta_4)(\partial_\beta \theta_3 + \frac{1}{\sqrt{3}} \partial_\beta \theta_8)$$

$$- \frac{N_\pi e}{\sqrt{3}\pi^2 f_\pi^3} \epsilon^{\mu\nu\alpha\beta}(\partial_\mu A_\nu)(\theta_7 \partial_\alpha \theta_6 - \theta_6 \partial_\alpha \theta_7) \partial_\beta \theta_8$$

$$+ \frac{N_\pi e^2}{18\pi^2 f_\pi^3} \epsilon^{\mu\nu\alpha\beta}(\partial_\mu A_\nu)(\partial_\alpha A_\beta) \left\{ [4(\theta_1^2 + \theta_2^2) + 5(\theta_4^2 + \theta_5^2)] \theta_3 \right. + 2\sqrt{2} \theta_1 \theta_2 + \theta_3 \theta_8 \left. + 2(\theta_1 \theta_5 - \theta_2 \theta_4) \theta_7 + (\theta_1 \theta_4 + \theta_2 \theta_5) \theta_6 \right\}$$

$$- \frac{N_\pi e^2}{3\pi^2 f_\pi^3} \epsilon^{\mu\nu\alpha\beta}(\partial_\nu A_\alpha)(\theta_3 + \frac{1}{\sqrt{3}} \theta_8) \partial_\beta (\theta_1^2 + \theta_2^2 + \theta_4^2 + \theta_5^2) .$$
We see that only $\Gamma_{WZW}(U, A_\mu)$ (the part of the Wess-Zumino-Witten action which describes the anomalous interaction of the Goldstone bosons with the photons) gives extra contribution at this order. Next we derive the expressions for the canonical momenta $p_a$ conjugate to the Goldstone boson fields $\theta_a$. We substitute into (13) and we get the following expression for the effective lagrangian density in the enlarged configuration space

$$\mathcal{L}_{\text{eff}} = -\mathbf{E} \cdot \dot{\mathbf{A}} + p_a \dot{\theta}_a - H^{(2)} - H^{(3)} - A_0(\rho^{(2)} + \rho^{(3)} - \nabla \cdot \mathbf{E}) + O(\theta^4), \quad (14)$$

where the expression for $H^{(3)}$ is given in (12) and

$$H^{(3)} = -\frac{N_c e}{3\pi^2 f_\pi^3} \left[ \mathbf{E} \times \nabla (\theta_3 + \frac{1}{\sqrt{3}} \theta_8) - (p_3 + \frac{1}{\sqrt{3}} p_8) \mathbf{B} \right] \cdot (\theta_1 \nabla \theta_2 - \theta_2 \nabla \theta_1 + \theta_4 \nabla \theta_5 - \theta_5 \nabla \theta_4) - \frac{N_c e}{\sqrt{3} \pi^2 f_\pi^3} (\mathbf{E} \times \nabla \theta_8 - p_8 \mathbf{B}) \cdot (\theta_7 \nabla \theta_6 - \theta_6 \nabla \theta_7)$$

$$+ \frac{N_c e^2}{9\pi^2 f_\pi^3} (\mathbf{E} \cdot \mathbf{B}) \{ [4(\theta_1^2 + \theta_2^2) + 5(\theta_4^2 + \theta_5^2)]\theta_3$$

$$+ \sqrt{3}[2(\theta_1^2 + \theta_2^2) + \theta_4^2 + \theta_5^2]\theta_8 + 2[(\theta_1 \theta_2 - \theta_2 \theta_1)\theta_7 + (\theta_4 \theta_5 - \theta_5 \theta_4)\theta_6] \}$$

$$- \frac{N_c e^2}{3\pi^2 f_\pi^3} (\mathbf{E} \times \mathbf{A}) \cdot [\nabla (\theta_1^2 + \theta_2^2 + \theta_4^2 + \theta_5^2)](\theta_3 + \frac{1}{\sqrt{3}} \theta_8)$$

$$- \frac{2N_c e^2}{3\pi^2 f_\pi^3} (\mathbf{A} \cdot \mathbf{B})(p_1 \theta_1 + p_2 \theta_2 + p_4 \theta_4 + p_5 \theta_5)(\theta_3 + \frac{1}{\sqrt{3}} \theta_8)$$

$$- \frac{N_c e}{3\pi^2 f_\pi^3} [\mathbf{B} \cdot \nabla (\theta_3 + \frac{1}{\sqrt{3}} \theta_8)](p_2 \theta_1 - p_1 \theta_2 + p_4 \theta_4 - p_4 \theta_5)$$

$$- \frac{N_c e}{\sqrt{3} \pi^2 f_\pi^3} (\mathbf{B} \cdot \nabla \theta_8)(p_6 \theta_7 - p_7 \theta_6), \quad (15)$$

$$\rho^{(2)} = e(p_2 \theta_1 - p_1 \theta_2 + p_5 \theta_4 - p_4 \theta_5),$$

$$\rho^{(3)} = -\frac{N_c e^2}{3\pi^2 f_\pi^3} \nabla \cdot [\mathbf{B}(\theta_1^2 + \theta_2^2 + \theta_4^2 + \theta_5^2)](\theta_3 + \frac{1}{\sqrt{3}} \theta_8).$$

The scalar potential $A_0$ is again the Lagrange multiplier and the equation of the constraint is given by

$$\nabla \cdot \mathbf{E} - (\rho^{(2)} + \rho^{(3)}) = 0. \quad (16)$$

The equation (16) has similar structure as in the previous case (7) so we proceed similarly. We decompose the electric field $\mathbf{E}$ and the vector potential $\mathbf{A}$ into transverse and longitudinal components. Then (16) implies

$$\mathbf{E_L} = \frac{\nabla}{\nabla^2} (\rho^{(2)} + \rho^{(3)}) \equiv -\frac{1}{4\pi} \int d^3 y \frac{1}{|y - x|} \nabla (\rho^{(2)}(y) + \rho^{(3)}(y)). \quad (17)$$
Next we substitute the expression for $\mathbf{E}_L$ given in (17) into (14) and we obtain a lagrangian density whose canonical one-form is given (apart from a total divergence) by

$$-\mathbf{E}_T \cdot \dot{\mathbf{A}}_T + \rho^{(2)} \frac{\nabla}{\nabla^2} \cdot \dot{\mathbf{A}}_L + \rho^{(3)} \frac{\nabla}{\nabla^2} \cdot \dot{\mathbf{A}}_L + p_a \dot{\theta}_a ,$$

(18)

This expression must be diagonalized so that it acquires the standard form (6). In order to do this we proceed in two steps as in the SU(2) case [11]. First we perform the Darboux’s transformations given in (10) which lead to partial diagonalization of (18). The new lagrangian density has a canonical one-form given by

$$-\mathbf{E}_T \cdot \dot{\mathbf{A}}_T + \rho^{(3)} \frac{\nabla}{\nabla^2} \cdot \dot{\mathbf{A}}_L + p_a \dot{\theta}_a ,$$

The term $\rho^{(3)} \frac{\nabla}{\nabla^2} \cdot \dot{\mathbf{A}}_L$ can be written as follows apart from a total time derivative

$$\rho^{(3)} \frac{\nabla}{\nabla^2} \cdot \dot{\mathbf{A}}_L = -\frac{N_e e^2}{3\pi^2 f_\pi} [\nabla \phi \times \mathbf{A}_L] \cdot \dot{\mathbf{A}}_T + (\mathbf{B} \cdot \mathbf{A}_L) \dot{\phi} ,$$

where $\phi = (\theta_1^2 + \theta_2^2 + \theta_4^2 + \theta_5^2)(\theta_3 + \frac{1}{\sqrt{3}} \theta_8)$.

Now we proceed with the second step. We perform the following Darboux’s transformations

$$\mathbf{E}_T \rightarrow \mathbf{E}_T - \frac{N_e e^2}{3\pi^2 f_\pi} \nabla [\theta_2 (\theta_3 + \frac{1}{\sqrt{3}} \theta_8) + \frac{1}{\sqrt{3}} \sqrt{3} \mathbf{A}_L] \times \mathbf{A}_L ,$$

$$p_i \rightarrow p_i + \frac{N_e e^2}{3\pi^2 f_\pi} (\mathbf{B} \cdot \mathbf{A}_L) (\theta_3 + \frac{1}{\sqrt{3}} \theta_8) \theta_i , \quad i = 1, 2, 4, 5$$

$$p_3 \rightarrow p_3 + \frac{N_e e^2}{3\pi^2 f_\pi} (\mathbf{B} \cdot \mathbf{A}_L) (\theta_1^2 + \theta_2^2 + \theta_4^2 + \theta_5^2) ,$$

$$p_8 \rightarrow p_8 + \frac{N_e e^2}{3\pi^2 f_\pi} (\mathbf{B} \cdot \mathbf{A}_L) (\theta_1^2 + \theta_2^2 + \theta_4^2 + \theta_5^2) ,$$

$$\mathbf{A}_T \rightarrow \mathbf{A}_T , \quad p_6 \rightarrow p_6 , \quad p_7 \rightarrow p_7 , \quad \theta_a \rightarrow \theta_a , \quad a = 1, ..., 8 .$$

These transformations complete the diagonalization of the canonical one-form and we end up with a lagrangian density where the longitudinal part of the vector potential $\mathbf{A}_L$ cancels out exactly

$$\mathcal{L} = -\mathbf{E}_T \cdot \dot{\mathbf{A}}_T + p_a \dot{\theta}_a - H^{(2)}_C - H^{(3)}_C + O(\theta^4) .$$

(19)

The expression for $H^{(2)}_C$ is given in (12) and

$$H^{(3)}_C = -\frac{N_e e}{3\pi^2 f_\pi} [\mathbf{E}_T \times \nabla (\theta_3 + \frac{1}{\sqrt{3}} \theta_8) - (p_3 + \frac{1}{\sqrt{3}} p_8) \mathbf{B}] \cdot (\theta_5 \nabla \theta_2 - \theta_2 \nabla \theta_1 + \theta_4 \nabla \theta_5 - \theta_5 \nabla \theta_4)

- \frac{N_e e}{\sqrt{3}\pi^2 f_\pi^3} (\mathbf{E}_T \times \nabla \theta_8 - p_8 \mathbf{B}) \cdot (\theta_7 \nabla \theta_6 - \theta_6 \nabla \theta_7)

+ \frac{N_e e^2}{9\pi^2 f_\pi^3} (\mathbf{E}_T \cdot \mathbf{B}) \{[4(\theta_1^2 + \theta_2^2) + 5(\theta_4^2 + \theta_5^2)] \theta_3 .

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\[ H_C^{(2)} = \frac{1}{2} |E_T|^2 + B^2 - \rho^{(2)} \frac{1}{\sqrt{2}} \rho^{(2)} + p_{\pi^\pm}^2 + p_{\eta_8}^2 + 2p_{\pi^+}p_{\pi^-} + 2p_{K^0}p_{K^-} + 2p_{\pi^0}p_{\bar{K}^0} + \frac{1}{\sqrt{3}} (\nabla \pi^\pm)^2 + (\nabla \eta_8)^2 + 2(\nabla \pi^+) \cdot (\nabla \pi^-) + 2(\nabla K^+) \cdot (\nabla K^-) + 2(\nabla K^0) \cdot (\nabla \bar{K}^0)] \]
\[ - i e A_T \cdot (\pi^+ \nabla \pi^- - \pi^- \nabla \pi^+ + K^+ \nabla K^- - K^- \nabla K^+) + e^2 A_T^2 (\pi^+ \pi^- + K^+ K^-) \]
\[ - \frac{N_{e^2}}{6 \pi^2 f_{\pi}^2} [E_T \cdot B + (\nabla \rho^{(2)}) \cdot B] (\pi^0 + \frac{1}{\sqrt{3}} \eta_8) , \]

\[ H_C^{(3)} = \frac{i N_e e}{3 \pi^2 f_{\pi}^2} [E_T \times \nabla (\pi^0 + \frac{1}{\sqrt{3}} \eta_8) - (p_{\pi^0} + \frac{1}{\sqrt{3}} p_{\eta_8}) B] \cdot (\pi^+ \nabla \pi^- + K^+ \nabla K^-) \]
\[ + \frac{i N_e e}{\sqrt{3} \pi^2 f_{\pi}^2} (E_T \times \nabla \eta_8 - p_{\eta_8} B) \cdot (\bar{K}^0 \nabla \bar{K}^0) \]
\[ + \frac{N_{e^2}}{9 \pi^2 f_{\pi}^2} (E_T \cdot B) [(4 \pi^+ \pi^- + 5 K^+ K^-) \pi^0 + \sqrt{3} (2 \pi^+ \pi^- + K^+ K^-) \eta_8 + 2 \sqrt{2} p_{K^0} K^-] \]
\[ - \frac{N_{e^2}}{3 \pi^2 f_{\pi}^2} (E_T \times A_T) \cdot [\nabla (\pi^+ \pi^- + K^+ K^-)] (\pi^0 + \frac{1}{\sqrt{3}} \eta_8) \]
\[ - \frac{2 N_{e^2}}{3 \pi^2 f_{\pi}^2} (A_T \cdot B) (p_{\pi^+} \pi^+ + p_{K^0} K^+) (\pi^0 + \frac{1}{\sqrt{3}} \eta_8) \]
\[ + \frac{i N_{e^2}}{3 \pi^2 f_{\pi}^2} [B \cdot \nabla (\pi^0 + \frac{1}{\sqrt{3}} \eta_8)] (p_{\pi^+} \pi^+ + p_{K^0} K^+) \]
\[ + \frac{i N_{e^2}}{\sqrt{3} \pi^2 f_{\pi}^2} (B \cdot \nabla \eta_8) p_{\pi^0} K^0 + \text{h.c.} , \]

\[ \rho^{(2)} = -ie(\pi^+ \pi^- - p_{\pi^-} \pi^+ + p_{K^0} K^+ - p_{K^-} K^-) . \]
4 Conclusion

In this work we applied the Faddeev and Jackiw method for constrained systems to the $U_{EM}(1)$ gauged three flavour WZW model. We expanded the U-field into series of powers of the Goldstone boson fields and we limited ourselves to lagrangian densities with terms up to the third power in $\theta_a$. After treating these constrained langrangian densities using the FJ approach we ended up with unconstrained Coulomb-gauge hamiltonians, as in the case of QED \[\text{[1]}. The general case which involves any number of Goldstone boson fields is currently under investigation.

5 Appendix

Our metric is $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, $Q = \text{diag}(2/3, -1/3, -1/3)$ is the charge matrix, $D_\mu = \partial_\mu + ieA_\mu[Q]$, denote the covariant derivative. $\lambda_a$, $a = 1, \ldots, 8$ are the SU(3) generators $\text{tr}(\lambda_a\lambda_b) = 2\delta_{ab}$. We choose $e > 0$ so that the electric charge of the electron is $-e$. We define $e^{0123} = 1$.

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