Stability of Non-asymptotically flat thin-shell wormholes in generalized dilaton-axion gravity

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Abstract

We construct a new type of thin-shell wormhole for non-asymptotically flat charged black holes in generalized dilaton-axion gravity inspired by low-energy string theory using cut-and-paste technique. We have shown that this thin shell wormhole is stable. The most striking feature of our model is that the total amount of exotic matter needed to support the wormhole can be reduced as desired with the suitable choice of the value of a parameter. Various other aspects of thin-shell wormhole are also analyzed.

1 Introduction

The study of traversable wormholes and thin-shell wormholes are very interesting subject in recent years as it is known that wormholes represent shortcut in space-time, though the observational evidence of wormholes has not been possible still now. At first, Morris and Thorne [1] elaborated the structure of traversable wormhole with two mouths and a throat. The traversable wormhole was obtained as a solution of Einstein’s equations

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having two asymptotically flat regions. These flat regions are connected by a minimal surface area, known as the throat satisfying the flare-out condition [2]. This type of wormhole would allow the travel between two parts of the same universe or between two different universes. Morris and Thorne [1] had the idea to make the conversion of a wormhole traversing space into a wormhole traversing time. In the Morris and Thorne model of wormhole the violation of positive energy condition is unavoidable. The throat of the wormhole is held open with exotic matter. Since it is very difficult to deal with the exotic matter (violation of positive energy condition) Visser [3], proposed a way to minimize the usage of exotic matter by applying cut-and-paste technique on a black hole to construct a new class of spherically symmetric wormholes, known as thin-shell wormhole in which the exotic matter is concentrated within the joining shell. This shell can act as the wormhole throat. Using the Darmois-Israel formalism[4], the surface stress-energy tensor components within the shell i.e. at the throat could be determined. Visser’s approach is very simple for theoretical construction of wormhole and perhaps also practical because it minimizes the amount of exotic matter required and therefore this approach was adopted by various authors to construct thin shell wormholes [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31].

Recently, Mazharimousavi et al [32] have studied d-dimensional non-asymptotically flat thin-shell wormholes in Einstein-Yang-Mills-Dilaton gravity where as Rahaman et al [33] have obtained a new class of stable (2+1) dimensional non-asymptotically flat thin-shell wormhole. In this investigation, we are searching stable non-asymptotically flat thin-shell wormholes in generalized dilaton-axion gravity.

In 2005, Sur, Das and SenGupta[34] discovered new black hole solutions considering gravity coupled with dilaton (ϕ), Kalb-Ramond axion (ζ) scalar fields and a Maxwell field with arbitrarily coupled to the scalars. They considered the generalized action in four dimensions as:

\[
I = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} \partial_\mu \epsilon^{\mu} \epsilon_{\nu} - \frac{\omega(\phi)}{2} \partial_\mu \zeta \partial^\mu \zeta - \alpha(\phi, \zeta) F_{\mu}^{\nu} F_{\mu}^{\nu} \right].
\]

(1)

where \( \kappa = 8\pi G \), R is the curvature scalar and \( F_{\mu}^{\nu} \) is the Maxwell field tensor where two massless scalar or pseudo scalar fields are represented by \( \phi \) and \( \zeta \) which are coupled to the Maxwell field with the functional dependence by the functions \( \alpha(\phi, \zeta) \) and \( \beta(\phi, \zeta) \).

Using the above action, Sur, Das and SenGupta[34] have found two types of black-hole solutions, classified as asymptotically flat and asymptotically non-flat. For our thin-shell wormhole solution, we have considered non asymptotically flat metric given by

\[
ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + h(r)d\Omega^2.
\]

(2)

where

\[
f(r) = \frac{(r - r_+)(r - r_-)}{r^2 \left( \frac{2m}{r} \right)^{2n}}.
\]

(3)

and

\[
h(r) = r^2 \left( \frac{2r_0}{r} \right)^{2n}.
\]

(4)
The parameter $n$ is a dimensionless constant lies in the range $0 < n < 1$. The various parameters for non asymptotically flat metric are given by

$$r_\pm = \frac{1}{(1-n)} \left[ m \pm \sqrt{m^2 - (1-n) \frac{K_2}{4}} \right],$$

(5)

$$r_0 = \frac{1}{16m_0} \left( \frac{K_1}{n} - \frac{K_2}{1-n} \right),$$

(6)

$$m_0 = m - (2n-1)r_0,$$

(7)

$$K_1 = 16nr_0^2,$$

(8)

$$K_2 = 4(1-n)r_+r_-,$$

(9)

$$m = \frac{1}{16r_0} \left( \frac{K_1}{n} - \frac{K_2}{1-n} \right) + (2n-1)r_0.$$  

(10)

where $m$ is the mass of the black hole and the inner, outer event horizons and curvature singularity represented by the parameters $r_+, r_-$ and $r = r_0$ respectively. The parameters obey the restriction $r_0 < r_- < r_+$.  

Employing the above black hole solution, we have constructed a new thin-shell wormhole. We have shown that this thin shell wormhole is stable. The most striking feature of our model is that the total amount of exotic matter needed to support the wormhole can be reduced as desired by choosing the parameter $n$ very close to unity. Various other aspects of thin-shell wormhole have also been analyzed.

We have organized the work as follows: In section 2, we have provided mathematical formulation for constructing a thin-shell wormhole from charged black hole in generalized dilaton-axion gravity. In section 3, we have discussed the effect of gravitational field on wormhole i.e. attractive or repulsive nature of the wormhole. In section 4, we have discussed about the equation of state relating pressure and density at the wormhole throat and in section 5, we have calculated total amount of exotic matter. Linearized stability analysis has been discussed in section 6. Finally, we have made a conclusion about the work.

## 2 Construction of thin-shell wormhole

For the construction of thin-shell wormhole, at first, we remove two identical copies from four-dimension spacetime of the black hole given in equation (2) region with radius $r \leq a$. Here we have assumed $a > r_+$ to avoid all types of singularity. Thus we have two copies of regions:

$$\mathcal{M}^\pm \equiv \{ r = a : a > r_+ \}.$$  

(11)
Now, we stick them together at the hypersurface, $\Sigma = \Sigma^\pm$, to get a geodesically complete manifold $M = M^+ \cup M^-$. Here, two regions are connected by minimal surface area, called throat with radius $a$.

The induced 3-metric on the hypersurface is given by

$$ds^2 = -d\tau^2 + h[r(\tau)]d\Omega^2.$$  (12)

where $\tau$ represents the proper time on the junction surface. To obtain the surface stress-energy tensor $S^i_j = \text{diag} (-\sigma, P, P)$, we use the Lanczos equations [4], which reduce to

$$\sigma = -\frac{1}{4\pi} \frac{h'(a)}{h(a)} \sqrt{f(a) + \dot{a}^2},$$  (13)

and

$$P_\theta = P_\phi = P = \frac{1}{8\pi} \frac{h'(a)}{h(a)} \sqrt{f(a) + \dot{a}^2} + \frac{1}{8\pi} \frac{f'(a) + 2\dot{a}}{\sqrt{f(a) + \dot{a}^2}}.$$  (14)

where $\sigma$ and $P$ represents the surface energy density and surface pressures respectively. To understand the dynamics of the wormhole, we consider that the throat as function of proper time i.e. $a = a(\tau)$ and over dot and prime denote the derivatives with respect to $\tau$ and $a$. For a static configuration of radius $a$ (assuming $\dot{a} = 0$ and $\ddot{a} = 0$), we obtain the values of the surface energy density and the surface pressures as

$$\sigma = -\frac{(1 - n)(a - r_+)(a - r_-)}{a^{2-n}K},$$  (15)

$$P = \frac{2a - r_+ - r_-}{4Ka^{1-n}},$$  (16)

where

$$K = 2\pi (2r_0)^n \sqrt{(a - r_+)(a - r_-)}.$$  (17)

From Eqs.(15) and (16), we see that the energy density $\sigma$ is negative, however the pressure $p$ is positive depending on the position of the throat and on the parameters $r_0$, $r_-$ and $r_+$ defining on wormhole. Here, matter distribution within the shell of the wormhole violates the weak energy condition.

We plots $\sigma$ and $P$ versus $a$ in Figs.(1-2) for such wormholes whose radii fall within the range of 6 to 16 km, keeping the restriction on the parameters $r_0$, $r_-$ and $r_+$ defining on wormhole. Here, matter distribution within the shell of the wormhole violates the weak energy condition.

Note that the weak energy condition requires that $\sigma > 0$ and $\sigma + P > 0$. These state that the energy density is positive and the pressure is not too large compared to the energy density. The null energy condition $\sigma + P > 0$ is a special case of the latter and implies that energy density can be negative if there is a compensating positive pressure. The figure 3 indicates in the present case that the null energy condition is satisfied.
Figure 1: For every branch of curves, colors black, blue and pink represent for $n = 0.7$, 0.5 & 0.1 and $r_+ = 8, 7 & 6$ respectively. For every combination of $n$ and $r_+$, we draw three different sets when $(r_- = 5, r_0 = 3)$, $(r_- = 5, r_0 = 2)$ and $(r_- = 4, r_0 = 3)$ which are shown by solid, dot-dash and dotted curves respectively.

Figure 2: For the branch of curves, the description of the figure is same as in Fig1.
Figure 3: For the branch of curves, the description of the figure is same as in Fig1.

3 The gravitational field

In this section we have analyzed the nature of the gravitational field of the wormhole constructed from charged black holes in generalized dilaton-axion gravity and calculate the observer’s four-acceleration $A^\mu = u^\mu u^\nu$, where $u^\nu = dx^\nu/d\tau = (1/\sqrt{f(r)}, 0, 0, 0)$. Only non-zero component for the line element in Eq.(1), is given by

$$A^r = \Gamma^r_{tt} \left( \frac{dt}{d\tau} \right)^2 = \frac{r^{2n-3}}{(2r_0)^{2n}} \left[ nr^2 - \left( n - \frac{1}{2} \right) (r_+ + r_-)r + (n - 1)r_+ r_- \right].$$

(18)

The equation of motion of a test particle is given by

$$\frac{d^2 r}{d\tau^2} = -\Gamma^r_{tt} \left( \frac{dt}{d\tau} \right)^2 = -A^r.$$

(19)

when it is radially moving and initially at rest.

For $A^r = 0$, we get the geodesic equation. From the figure 4, we note that the wormhole is attractive as $A^r > 0$ outside the event horizon.
4 An equation of state

To know the exact nature of the matter distribution in the shell, we calculate equation of state (EoS) which is given by

\[
P = w = -\frac{a(2a - r_+ - r_-)}{4(1 - n)(a - r_+)(a - r_-)}. \tag{20}
\]

The plot for \(w\) (fig 5) indicates that matter distribution in the shell is phantom energy type. For \(a \to \infty\) i.e. if the wormhole throat is very very large, then \(w \to -\frac{1}{2(1-n)}\). In that case, EoS solely depends on the parameter \(n\) only. One can note that for the following range of \(n\), \(\frac{1}{2} < n < \frac{3}{2}\), \(w\) lies on \(-1 < w < -\frac{1}{3}\). This indicates the matter distribution is quintessence like. However, for \(n > \frac{3}{2}\), \(w < -1\) i.e. the matter distribution is phantom energy type as \(w < -1\).

5 The total amount of exotic matter

We observe that matter distribution at throat is phantom energy type i.e. matter at throat is exotic. Here, we evaluate the total amount of exotic matter which can be quantified by the integral \[10, 11, 12, 13, 14, 15, 17\]

\[
\Omega_\sigma = \int [\rho + p] \sqrt{-g} d^3x. \tag{21}
\]
Figure 5: We draw the figure for $w$ with different values of $n$ assuming $r_+ = 4$, $r_- = 3$. Figure indicates the matter distribution is phantom energy type.

where $g$ represents the determinant of the metric tensor.

Now, by using the radial coordinate $R = r - a$, we have

$$\Omega_{\sigma} = \int_0^{2\pi} \int_0^\pi \int_{-\infty}^{\infty} [\rho + p] \sqrt{-g} dR d\theta d\phi. \quad (22)$$

For the infinitely thin shell it does not exert any radial pressure and using $\rho = \delta(R)\sigma(a)$ we have,

$$\Omega_{\sigma} = \int_0^{2\pi} \int_0^\pi [\rho \sqrt{-g}]_{r=a} d\theta d\phi = 4\pi h(a)\sigma(a) = -\frac{(1-n)K}{a^n}. \quad (23)$$

where $K$ is given in Eq.(17).

It is to be noted that the total amount of exotic matter needed can be reduced as small as desired by choosing the parameter $n$ very close to unity (see figure 6). Also, the exotic matter can be minimized by taking the value of $a$ very close to the location of the outer event horizon (see figure 7).

6 Linearizing method

Now, we are trying to find the local stability of the configuration under small perturbation around the static solution at $a = a_0$. So, rearranging the Eq.(13) we can write

$$a^2 + V(a) = 0, \quad (24)$$
Figure 6: The plot for $\Omega_\sigma$ versus $n$, for fixed value of $a = 10$. For the branch of curves, the description of the figure is same as in Fig1.

Figure 7: The plot for $\Omega_\sigma$ versus $a$. For the branch of curves, the description of the figure is same as in Fig1.
which represents the equation of motion of the shell, where the potential \( V(a) \) is defined as

\[
V(a) = f(a) - 16\pi^2 \left[ \sigma(a) \frac{h(a)}{h'(a)} \right]^2.
\]  
(25)

Using the Taylor series expansion for \( V(a) \) around the static solution \( a_0 \), we get

\[
V(a) = V(a_0) + V'(a_0)(a - a_0) + \frac{1}{2} V''(a_0)(a - a_0)^2 + O[(a - a_0)^3],
\]  
(26)

where the prime denotes the derivative with respect to \( a \).

For our stability analysis, we start with the energy conservation equation. Now using Eqs. (15) and (16), preserving the symmetry of linearizing radial perturbations, the energy conservation equation defined as

\[
\frac{d}{d\tau} (\sigma A) + P \frac{dA}{d\tau} = \left\{ [h'(a)]^2 - 2h(a)h''(a) \right\} \frac{\dot{a}\sqrt{f(a) + \dot{a}^2}}{2h(a)},
\]  
(27)

where \( A = 4\pi h(a) \) denotes the area of the wormhole throat.

From Eq. (25), one can get

\[
\frac{d}{da} [\sigma h(a)] + P \frac{d}{da} [h(a)] = -\left\{ [h'(a)]^2 - 2h(a)h''(a) \right\} \frac{\sigma}{2h'(a)},
\]  
(28)

which finally gives

\[
h(a)\sigma' + h'(a)(\sigma + P) + \left\{ [h'(a)]^2 - 2h(a)h''(a) \right\} \frac{\sigma}{2h'(a)} = 0,
\]  
(29)

The second order derivative for the potential can be defined as

\[
V''(a) = f''(a) + 16\pi^2 \times \left\{ \left[ \frac{h(a)}{h'(a)} \right] \sigma'(a) + \left( 1 - \frac{h(a)h''(a)}{[h'(a)]^2} \right) \sigma(a) \right\}
\times \left[ \sigma(a) + 2P(a) + \frac{h(a)}{h'(a)} \sigma(a) \right] \left[ \sigma'(a) + 2p'(a) \right],
\]  
(30)

Now, using the parameter \( \beta^2 = \frac{4P}{\sigma} \), which is normally interpreted as the speed of sound, the above expression can be written as

\[
V''(a) = f''(a) - 8\pi^2 \times \left\{ \left[ \sigma(a) + 2P(a) \right] \right^2
+ 2\sigma(a) \left[ \left( \frac{3}{2} - \frac{h(a)h''(a)}{[h'(a)]^2} \right) \sigma(a) + P(a) \right] \left[ 1 + 2\beta^2 \right],
\]  
(31)
The solution gives a stable configuration if $V(a)$ process a local minima at $a_0$, in other words, $V''(a_0) > 0$. Now using the condition $V(a_0) = 0$ and $V'(a_0) = 0$, we solve for $\beta^2$ as

$$\beta^2 = -\frac{1}{2} + \frac{\frac{f''}{8\pi^2} - (\sigma + 2P)^2}{4\sigma \left( \frac{3}{2} - \frac{hh''}{|h|^2} \right) \sigma + P}.$$  \hfill (32)

Now, we are trying find the stable region with the help of graphical representation due to complexity of the expression of $V''(a_0) > 0$. In Figs. 8-11, we find the possible range of $a_0$, where $V(a_0) = 0$ possess a local minima. We plot the figures 8 and 9 for the parameters $r_0 = 2$, $r_- = 3$ and $r_+ = 4$ and different values of $n=0.5$ and $0.7$, which show that graphs don’t possess any stable configuration as $\beta^2$ lies outside $(0,1)$. However, if we choose the the parameters $r_0 = 2$, $r_- = 1$, $r_+ = 6$ and $n=0.5$, respectively, then the stable region is more closer to the normal range of $\beta^2$. This indicates that the increase of the difference between inner and outer event horizons, stable region may fall to the normal range of $\beta^2$. Note that this criteria holds when we are dealing with real matter. However, according Poisson and Visser \cite{5} the interpretation of $\beta^2$ should be relaxed when dealing with exotic matter. As a result, the values of $\beta^2$ may fall out side $(0,1)$. Since the stability criteria is $V''(a_0) > 0$ and our figures 8-11 indicate the stable regions where $V''(a_0) > 0$, therefore, our thin shell wormholes are stable.
Figure 9: The stability region is above the curve on the left and below the curve on the right for n=0.7.

Figure 10: The stability region is above the curve on the left and below the curve on the right when n=0.5 and \( r_0 = 0.5 \), \( r_- = 1 \) and \( r_+ = 6 \), i.e. plots for large difference between inner and outer event horizon.
Figure 11: The stability region is above the curve on the left and below the curve on the right when $n=0.9$ and $r_0 = 0.5$, $r_- = 1$ and $r_+ = 6$, i.e. plots for large difference between inner and outer event horizon.

7 Conclusions

In this work, we have provided a new type of thin-shell wormhole applying the cut-and-paste technique on charged black hole in generalized dilaton-axion gravity. The metric which we have used is not asymptotically flat, therefore, the the wormholes are not asymptotically flat. The matter confined within the shell of the wormhole violates the weak energy condition. The matter distribution in the shell is of phantom energy type. However, when the wormhole throat is very very large, then for the following range of $n$, $\frac{1}{4} < n < \frac{3}{4}$, mater distribution is quentessence like. It is interesting to note that the total amount of exotic matter needed can be reduced as small as desired by choosing the parameter $n$ very close to unity. Finally, to show the stability, we have performed linearized stability analysis around the static solution. Since the matter distribution within the shell is exotic type, therefore, according to Poisson and Visser [5], we can relax the range of $\beta^2$. Here the stability regions are shown graphically.

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