Impact Mechanics of Elastic Structures with Point Contact

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Abstract

This paper introduces a modeling framework that is suitable to resolve singularities of impact phenomena encountered in applications. The method involves an exact transformation that turns the continuum, often partial differential equation description of the contact problem into a delay differential equation. The new form of the physical model highlights the source of singularities and suggests a simple criterion for regularity. To contrast singular and regular behavior the impacting Euler-Bernoulli and Timoshenko beam models are compared.
I. INTRODUCTION

Impact mechanics is a great concern for engineers, thus many models were developed to understand this phenomena \[16\]. Even simple models of impact \[7\] lead to complicated predictions, like infinite chatter \[11\], period adding bifurcations, chaos \[1\] and non-deterministic motion \[10\]. The more elaborate models, however can suffer from convergence problems as either the time step of the solution decreases \[23\] or the resolved degrees of freedom is increased \[9\]. This signals the need for a better modeling framework for impact phenomena.

Most state-of-the-art impact models are finite dimensional and predict infinite contact forces. This happens because at impact the contact point abruptly changes its velocity and has an infinite acceleration. If the mass of the contact point is not zero, this infinite acceleration requires an infinite contact force. One example is the standard modal description \[5\] of linear structures. Each vibration mode is associated with a non-zero modal mass. Truncation of the modal expansion describes the motion as if a finite set of rigid bodies were coupled with springs and dampers. The arising infinite forces are difficult to handle and are the source of singularities. In contrast, contact points of elastic bodies have infinitesimally small mass. This means that the contact force can stay finite despite infinite accelerations. In this paper we show that finite contact forces are possible under general conditions.

In some models, impact is treated as a discrete-time event. Since impacting bodies must not overlap, the velocity state of the bodies must be altered at contact, so that their subsequent motion avoids overlap momentarily. One such change of velocities can described by the coefficient of restitution (CoR) model that stipulates that the incident velocity and a rebound velocity of the contact points are opposite and linearly related. This still leaves a great deal freedom in choosing the rest of the rebound velocities of the structure, therefore the CoR model must be coupled with additional rules, which can be based on momentum balances \[13, 21\] or collocation \[22\]. The main weakness of the CoR method is that it leads to high-frequency chatter that is proportional to the highest natural frequency of the system. This phenomenon restricts the highest natural frequencies that can be included in the model, because numerical simulation would becomes prohibitively slow. There are a few extensions to the CoR method that avoid high-frequency chatter \[11, 14\], however they do not fix the cause of the problem which is the presence of infinite or undefined contact forces. To illustrate this point we compare our method to a CoR model \[21\] and show how chatter
is eliminated when the contact force is well-defined.

Impact can also be modeled by calculating the contact force using impulse response functions of the elastic structures at the contact point. This approach is expected to be more accurate, however convergence can be as troublesome as for CoR models. For example, Wang and Kim [23] found that in the limit of zero time-step of their algorithm, the contact force diverges when an Euler-Bernoulli beam impacts an elastic St Venant rod. This divergence is related to the approximation of the impulse-response function with finitely many vibration modes. Similar approaches were used by other authors [4, 6, 24] using time stepping algorithms and finitely many vibration modes; in some cases the shape of the contact force was assumed [8], which avoids divergence.

In this paper we use a similar technique to impulse-response functions, and transform the governing equation of the impacting mechanical system to a delay differential equation. The memory term of our equation can be thought of as the convolution integral with the impulse response function. Depending on the properties of the memory term the model is either singular or regular. In case of a regular model the finite contact force is uniquely calculated from a delay-differential equation.

II. MECHANICAL MODEL

We analyze impact mechanics through a converging series expansion of the continuum problem. We use infinitely many vibration modes \( x_k(t) \) to recover the entire motion of the structure. Through this expansion the displacement of any point \( \chi \) of the elastic body can be written as an infinite sum

\[
u(t, \chi) = \sum_{k=1}^{\infty} \psi_k(\chi) x_k(t),
\]

where \( \psi_k(\chi) \) are the mode shapes of the structure [31]. If the motion of the structure is decoupled into non-resonant modes of vibration, the equation of motion can be written as

\[
\ddot{x}(t) + 2D\Omega\dot{x}(t) + \Omega^2 x(t) = f_e(t) + nf_c(t),
\]

where \( x = (x_1, x_2, \ldots)^T \), the mass matrix is assumed to be the identity, \( \Omega = \text{diag}(\omega_1, \omega_2, \ldots) \) and \( D = \text{diag}(D_1, D_1, \ldots) \), \( f_e(t) \) represents the external force, and \( f_c(t) \) is the contact force. We assume that the natural frequencies scale according to \( \omega_k = \omega_1 k^\alpha \), for \( k \gg 1 \). Vector \( n \)
in equation (1) represents the contribution of the modes to the motion of the contact point
\[ y(t) = n \cdot x(t) \]
with
\[ n = (\psi_1(\chi^*), \psi_2(\chi^*), \ldots)^T, \]
where \( \chi^* \) represents the contact point. The method described in this paper is not restricted
to modal equations (1), a more general description can be found in [17].

A. Approximating the contact force

To better understand the impact process we first approximate the contact force assuming
that the impact is infinitesimally short. We assume a single structure that interacts with a
rigid stop. Contact occurs at \( t_0 \) if \( y(t_0) = 0 \). As a first step we calculate a constant contact
force that keeps the stop penetrating the structure after time \( \delta t \), that is, \( y(t_0 + \delta t) = 0 \), which
forms a boundary value problem. After solving equation (1) the contact force becomes
\[ f_c = -C \delta t^{\frac{1}{\alpha}} (n \cdot \dot{x}(t_0^+)), \]
where \( \dot{x}(t_0^+) \) is the vector of modal velocities just before the impact and \( 0 < C < \infty \) is a
constant. When \( \delta t \to 0 \) the velocity of the contact point reverses and that corresponds to a
unit CoR. The details of the calculation can be found in the Appendix.

When evaluating the contact force there are three cases as \( \delta t \to 0 \) at the onset of contact.
If \( \alpha < 1 \) the contact force becomes zero, if \( \alpha = 1 \), the contact force is a finite constant and
for \( \alpha > 1 \) the contact force tends to infinity. This simple result implies that for a finite
contact force at least a subsequence of the natural frequencies \( \omega_k \) must scale at most linearly
as \( k \to \infty \).

For a system composed of an elastic body which strikes a rigid stop, the impact should
change the momentum of the elastic body. Our calculation shows that the change of mo-
mentum of the elastic body is zero, i.e., \( \lim_{\delta t \to 0} (f_c \delta t) = 0 \) as the contact time tends to zero.
This implies that the impact must occur during a non-zero and finitely long time-interval.

III. MODEL TRANSFORMATION

To accurately calculate the contact force as a function of time in the continuum problem
we transform the infinite dimensional system (1) into a delay equation. Delay terms can nat-
urally arise from traveling wave solutions of partial differential equations [15,18]. However
dispersion might prevent one to write down such solutions. Instead we use the Mori-Zwanzig formalism as is described for mechanical systems in [17] and obtain a time-delay model.

Our aim is to find a self-contained equation that exactly describes the evolution of \( y(t) = (n \cdot \dot{x}(t), n \cdot x(t))^T \). We call \( y \) the vector of resolved variables. The first step in the process is to transform (1) into a first-order form

\[
\dot{z}(t) = Rz(t) + \begin{pmatrix} 0 \\
 f_c(t) \end{pmatrix} + f(t), \quad R = \begin{pmatrix} 0 & I \\
 -\Omega^2 & -2D\Omega \end{pmatrix},
\]

(3)

where \( f(t) = (0, f_c(t))^T \). Note that \( R \) can also represent any convergent expansion of the continuum problem, including numerical schemes such as finite difference methods and \( y \) can include any finite number of variables [17]. To arrive at a model that describes the evolution of the resolved variables \( y \), we construct a projection with a finite dimensional range with the help of the matrices

\[
V = \begin{pmatrix} n^T & 0 \\
0 & n^T \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} m & 0 \\
0 & m \end{pmatrix},
\]

where the vector \( m \) is chosen such that \( m \cdot n = 1 \) and its components obey \( m_j = 0 \) for \( j > M < \infty \). To simplify our analysis we also assume that the columns of \( W \) span an invariant subspace of \( R \). The resolved variables now can be expressed as \( y = Vz \), and the projection and the reciprocal projection matrices become \( S = WV \) and \( Q = I - S \), respectively. Further, we assume that the initial condition of (3) is specified at \( t = 0 \) and that there is no contact force initially. According to [17], with this notation the governing equation for the resolved variables becomes

\[
\frac{d}{dt}y(t) = Ay(t) + L^\infty f_c(t) + \int_0^t \ddr L(\tau) \frac{d}{d\tau} [f_c(t - \tau)] + g(t),
\]

(4)

where \( A = VRW \), the memory kernel is a function of bounded variation,

\[
L(\tau) = \int_0^\tau \left( Ve^{RQ\theta}(0, n)^T - L^\infty \right) d\theta,
\]

\[
L^\infty = AVR^{-1}(0, n)^T,
\]

and the forcing term is

\[
g(t) = VRQe^{Rt}(x(0), \dot{x}(0))^T + \int_0^t (VR - AV)e^{R\tau} f(t - \tau) d\tau.
\]

The integral in (4) is meant in the Riemann-Stieltjes sense. This means that discontinuities of \( L(\tau) \) at \( \tau_i \) represent discrete values of \( \frac{d}{dt}f_c(t - \tau_i) \).

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IV. CONTACT FORCE CALCULATION

In what follows, we interpret the meaning of (4) for impact problems. In the simple case when impact occurs with a stationary stop the contact point must have both a constant position $y_1 = \bar{y}_1$ and zero velocity $y_2 = 0$. This means that at the time of initial contact the acceleration of the contact point becomes infinite as the velocity resets to zero. Despite the infinite acceleration, the zero mass of the contact point guarantees a finite contact force. We show that in the transformed model the zero mass of contact point is equivalent to the condition

$$\lim_{\tau \to 0^+} [L(\tau)]_2 \neq 0.$$  \hspace{1cm} (5)

In other words, if condition (5) is satisfied the contact force is finite. This is a similar, but more general condition that has been derived by Wang and Kim [23]. For multiple resolved coordinates equation (5) must hold for the coordinates that experience a discontinuity at impact, e.g., the velocities. To show that our conclusion holds we integrate equation (4) through the initial contact to get

$$y(t^+_0) - y(t^-_0) = L^+ (f_c(t^+_0) - f_c(t^-_0)),$$

where $L^+ = \lim_{\tau \to 0^+} L(\tau)$ and $t^-_0$ signals a limit from the left and $t^+_0$ a limit from the right of the impact. Because the contact force before the impact $f_c(t^-_0) = 0$ and the velocity of the contact point after the impact $y_2(t^+_0) = 0$, it follows that the initial contact force is

$$f_c(t^+_0) = -\frac{y_2(t^-_0)}{[L^+]_2}. \hspace{1cm} (6)$$

During contact $y(t) = \bar{y} = (\bar{y}_1, 0)^T$ is constant, which can be substituted into (4) to find out the contact force. Rearranging the resulting equation yields

$$[L^+]_2 \frac{d}{dt} f_c(t) = -[L^\infty f_c(t) + A\bar{y} + g(t)]_2 - \int_{0^+}^{t} d\tau [L(\tau)]_2 \frac{d}{dt} [f_c(t - \tau)]. \hspace{1cm} (7)$$

Equation (7) is a delay-differential equation that contains the history of $f_c(t)$, which means that previous impacts have a great influence on the evolution of the contact force. Clearly, the contact force is not a continuous function, therefore the derivative of its history can become infinite. A short calculation shows that a jump of magnitude $f_c^{\text{jump}}$ of $f_c$ at $t - \tau^*$ contributes a finite value $\frac{d}{d\tau} L(\tau^*) f_c^{\text{jump}}$ to the integral in (7).
In conservative systems where shock waves are present, e.g., for an undamped string, \( L(\tau) \) has further isolated discontinuities. Let \( \tau_d \neq 0 \) be the position of such a discontinuity of \( L(\tau) \). If at time \( t_1 = t_0 + \tau_d \) the two bodies are in contact the contact force develops a further jump. This can be seen by integrating equation (7) for the infinitesimal time interval \([t_1^-, t_1^+])\). The integral of the integral on the right side of (7) becomes

\[
\int_{t_1^-}^{t_1^+} \int_{t}^{t_1^+} \mathrm{d} \tau [L(\tau)]_2 \frac{\mathrm{d}}{\mathrm{d} \tau} [f_c(t-\tau)] \mathrm{d} t = \left[ \int_{t_0^+}^{t} \mathrm{d} \tau [L(\tau)]_2 f_c(t-\tau) \right]_{t_1^-}^{t_1^+}, \tag{8}
\]

while the other terms are continuous and their integral vanishes. The right side of equation (8) is regular because all of its terms are finite. The discontinuity of \( L(\tau) \) at \( \tau_d \) contributes \((L(\tau_d^+) - L(\tau_d^-)) f_c(t-\tau_d)\) to the integral (8). Note that \( t_1 - \tau_d = t_0 \), therefore (8) evaluates to \((L(\tau_d^+) - L(\tau_d^-)) f_c(t_0^+)(\text{as } f_c(t_0^-) = 0)\), which means that the contact force at \( t_1 \) also develops a further discontinuity

\[
L^+ (f_c(t_1^+) - f_c(t_1^-)) = (L(\tau_d^+) - L(\tau_d^-)) f_c(t_0^+).
\]

### A. Numerical solution of the reduced model

The non-smooth delay-differential equations (4, 6, 7) are somewhat unusual and therefore standard numerical techniques are not directly applicable to them. In what follows, we use a simple explicit Euler method and the rectangle rule of numerical integration to approximate the solution of (4) for cases when \([L^+] \neq 0\). We assume that time is quantized in \( \varepsilon \) chunks, so that \( y_q = y(q\varepsilon) \), \( f_{c,q} = f_c(q\varepsilon) \), where \( q = 0, 1, 2, \ldots \). If there is no contact and consequently no contact force \((f_{c,q} = 0)\), the only unknown is the state variable \( y_q \). Therefore the evolution of the resolved variables is given by

\[
y_{q+1} = y_q + \varepsilon \left( Ay_q + g(q\varepsilon) \right) + \sum_{j=0}^{q-1} (L_{j+1} - L_j) (f_{c,j+1} - f_{c,j}), \tag{9}
\]

where \( L_j = L(j\varepsilon) \) and \( L_0 = L^+ \). Contact of the impacting bodies is detected, when \([y_{q+1}] \leq \overline{y}_1\). In this case the resolved variables are kept constant with \( y_{q+1} = \overline{y} \). Also, equation (6) is applied at the onset of contact, so that the initial contact force becomes

\[
f_{c,q+1} = - \frac{[y_q]_2}{[L^+]_2}. \tag{10}
\]
The subsequent values of the contact force are calculated by

\[
f_{c,q+1} = f_{c,q} - \varepsilon \left[ L_+ \right]_2 \left[ L_\infty f_{c,q} + A \dot{y} + g(q\varepsilon) \right]_2 - \frac{1}{\left[ L_+ \right]_2^2} \sum_{j=0}^{q-1} \left[ L_{j+1} - L_j \right]_2 (f_{c,q-j} - f_{c,q-j-1}).
\]

which is the discretized counterpart of (11). If \( f_{c,q+1} \) as predicted by equation (11) becomes negative, we set \( f_{c,q+1} = 0 \) and continue the calculation with (9).

V. IMPACTING CANTILEVER BEAM MODELS

Our theory sets a criterion in the form of equation (5) for the regularity of the mechanical model, which can be used to test different models of elastic structures. We consider the example of a cantilever beam in Fig. 1 described by two different models. Through our calculation it becomes clear why the Euler-Bernoulli model often used in impact models \[6, 9, 24\] exhibits signs of singularity.

As the underlying elastic structure, consider the Euler-Bernoulli cantilever beam model

\[
\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^4 u}{\partial \xi^4} + \psi_2 f_c(t), \quad u(t,0) = \frac{\partial u(t,\xi)}{\partial \xi} \Big|_{\xi=0} = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} \Big|_{\xi=1} = 0,
\]

with \( \frac{\partial^2 u(t,\xi)}{\partial \xi^2} \big|_{\xi=1} = f_c(t) \), where \( u(t,\xi) \) represents the deflection of the beam. The natural frequencies of (12) are determined by the equation \( 1 + \cos \sqrt{\omega_k} \cosh \sqrt{\omega_k} = 0 \), while the mode shape values at the end of the beam are described by

\[
\mathbf{n} = (2, -2, 2, -2, \ldots)^T.
\]

On the other hand the Timoshenko beam model is represented by

\[
\frac{\partial^2 u}{\partial t^2} = \beta \gamma \left( \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial \phi}{\partial \xi} \right) + \psi_2 f_c(t),
\]

\[
\frac{\partial^2 \phi}{\partial t^2} = \beta \frac{\partial^2 \phi}{\partial \xi^2} + \beta^2 \gamma \left( \frac{\partial u}{\partial \xi} - \phi \right),
\]

\[
u(t,0) = \phi(t,0) = \phi(t,1) = 0,
\]

Figure 1: Impacting cantilever beam.
Figure 2: (color online) Graph of $[L(\tau)]_2$ for the Euler-Bernoulli and the Timoshenko beam models. The inset shows that the Timoshenko beam model converges to a function where $[L^+]_2 \neq 0$, while the Euler-Bernoulli model is singular since its $L(\tau)$ is continuous. The parameters are $\beta = 4800$, $\gamma = 1/4$.

and $\partial / \partial \xi u(t, 1)|_{\xi=1} - \phi(t, 1) = f_e(t)$, where $f_e(t)$ is an external forcing through the second mode shape $\psi_2$, and $\phi$ is the rotation angle of the cross-section of the beam. The resolved coordinates in both cases are $y_1 = u(t, 1)$ and $y_2 = \partial / \partial t u(t, 1)$. In case of the Timoshenko beam, modal decomposition in the form of (1) is not practical, instead we use Chebyshev collocation [19] to discretize the system with $N$ number of collocation points. This is possible, since our formulation is not restricted to modal decomposition, the matrix $R$ can represent any form of discretization. The governing equations [12,14] are conservative, thus to obtain a decaying solution we add modal damping ratios $D_k = 1/10$ to both systems [12,14] after being discretized. The result of our calculation is shown in Fig. 2. It can be seen that for the Euler-Bernoulli model $[L(\tau)]_2$ is continuous while for the Timoshenko model $[L(\tau)]_2$ is discontinuous. This result is in accordance with the fact that the scaling exponent of the highest natural frequencies for the Euler-Bernoulli model is $\alpha = 2$ and for the Timoshenko model $\alpha = 1$ [20]. This means that the Euler-Bernoulli model is not suitable for impact calculations.
Figure 3: (color online) Vibrations of the impacting Timoshenko beam using two impact models. The rigid stop as illustrated in Fig. 1 is placed at $y_1 = -0.05$, and the forcing is $f_e(t) = 30 \cos(13t)$ through the second mode. Trajectories of the reduced model (4,6,7) are shown in dark (blue) and the solution of the CoR model (1,15) is represented by light (green) lines for comparison. The time step used to solve the reduced model is $\varepsilon = 3.5 \times 10^{-5}$ and the number of collocation points used to solve the CoR model is $N = 20$. Panels (a,b) show that the solution converges to a periodic orbit. A single period of the solution is illustrated in panels (c,d).

A. The Timoshenko model

In the previous section we have shown that the Timoshenko beam model (11) satisfies our criterion (5), which guarantees that equations (4,6,7) are well defined and that the contact force is finite during impact. To confirm that this is indeed the case we simulate the impacting cantilever beam with the Timoshenko beam model (14) as illustrated in figure 1. The rigid stop is placed at $y_1 = -0.05$, so that the beam contacts the stop when the position of its tip reaches $u(t,1) = -0.05$. In all our simulations we use the initial conditions $u(0,\xi) = \partial/\partial \alpha |u(t,\xi)|_{t=0} = 1.056\psi_1(\xi)$, where $\psi_1$ is the first mode shape of the structure and it is normalized by $\psi_1(1) = 1$. We also force the beam through its second mode with $f_e(t) = 30 \cos(13t)$. The numerical solution is obtained using equations (9,10,11). In addition to our method we also simulate the dynamics using the CoR model described in [21]. This comparison highlights that high-frequency chatter is eliminated when our method is used.
Figure 4: (color online) A sequence of two impacts within a period of the periodic solution in Fig. 3. The solution of the CoR model (1,15) as shown by the light (green) lines is highly oscillatory. The dark (blue) lines show that the reduced model (4,6,7) is similarly accurate and eliminates the high-frequency chatter of the CoR model. (a,b). Insets (d,e) show the small-scale dynamics of the CoR model. The contact force is finite and continuos after the initial contact (c).

The CoR model that we are using for comparison describes the impact as an infinitesimally short process. During the impact an impulse is applied at the contact point that alters the velocity state of the body. The magnitude of this impulse is determined by the desired rebound velocity of the contact point which is $-C_R$ times the incident velocity. Assuming that the equation of motion is in the form of (1) and impact occurs at $t_0$, the after-impact velocity of the structure is

$$\dot{x}(t_0^+) = \left(I - (1 + C_R) \frac{n \otimes n}{n^2}\right) \dot{x}(t_0^-),$$  \hspace{1cm} (15)$$

where $\otimes$ means the outer product between vectors. The position of the structure remains the same throughout the impact: $x(t_0^+) = x(t_0^-)$. The CoR model is physically questionable because it treats the impact as an infinitesimally short event. It can however, reproduce most experimental observations [12]. The source of high-frequency chatter can be explained by equation (15), which stipulates that the change in modal velocities is proportional to a
constant times vector \( \mathbf{n} \). Elements of \( \mathbf{n} \) corresponding to high frequency modes are of the same magnitude as for low frequency modes (e.g., see equation (13)), which means that after an impact the tip of the beam acquires a high frequency vibration. Due to this vibration another impact is likely to follow shortly and repeatedly, which results in chatter that has roughly the same frequency as the highest vibration mode of the structure. This is illustrated by the light (green) lines in figure 4.

In contrast to the CoR method (1,15) our method as solved by equations (9,10,11) eliminates chatter and produces a much smoother result, which are shown by dark (blue) lines Figs. 3 and 4. On the larger scale in figure 3 the two solutions roughly coincide, while on the smaller scale in figure 4 the high frequency chatter is apparent and would increase in frequency if more vibration modes or collocation points on the beam were used. This high frequency chatter can stall numerical simulations, while the time-step in our method is not affected by the inclusion of higher natural frequencies.

The contact force in the CoR model is infinite at times of contact and hence it cannot be calculated. On the other hand our method allows the calculation of the contact force, which is finite as shown in figure 4(c). The contact force even becomes a smooth function of time after the onset of contact.

VI. CONCLUSIONS

In this paper we introduced a new way of modeling the impact mechanics of elastic structures. With our method regularity of the model can be predicted and a finite and piecewise continuous contact force can be calculated. The key to this result is that the delay equation description preserves the infinite dimensional nature of the mechanics and the zero mass of the contact point. The results presented in this paper open a significant number of new avenues of research. Models that show non-deterministic behavior such as the Painlevé paradox [10] might be regularized through our method. The strong dependence of dynamical phenomena on the number of underlying dimensions [3] could also be eliminated, since our framework considers all the infinite dimensions. Further, the bifurcation theory of non-smooth delay equations requires attention in order to understand the implications of our regularized impact mechanics, especially in how far it is an improvement over finite dimensional models.
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Appendix A: Contact force asymptotics

In this appendix we approximate the contact force at the onset of an impact. We assume that an impact takes place at \( t = t_0 \). To help the notation we define \( \Box^- = \Box(t_0) \) and \( \Box^+ = \Box(t_0 + \delta t) \), where \( \Box \) stands for any dependent variable. We aim to calculate a constant contact force \( f_c \) that allows the elastic body to overlap with the rigid stop for an exactly \( \delta t \) long time interval. As \( \delta t \) tends to zero the overlap is removed, hence the calculated \( f_c \) force tends to the actual contact force.

To calculate this constant \( f_c \) one needs to solve

\[
0 = \mathbf{n} \cdot \mathbf{x}^+
\]

for \( f_c \). Due to the linearity of equation (1), the motion depends linearly on the contact force. Therefore expanding the constraint \( 0 = \mathbf{n} \cdot \mathbf{x}^+ \) at \( f_c = 0 \) we get the exact equation

\[
0 = \mathbf{n} \cdot \mathbf{x}^+_{f_c=0} + f_c \sum_{k=1}^{N} \psi_k(x^*) \frac{\partial x^+_k}{\partial f_c}.
\]  (A1)

The derivatives in (A1) are calculated in closed form as

\[
\frac{\partial x^+_k}{\partial f_c} = -\frac{\psi_k(x^*)}{\omega_k^2} \left( e^{-D_k \omega_k \delta t} \frac{\sin \left( \omega_k \sqrt{1 - D_k^2 \delta t} \right)}{\sqrt{1 - D_k^2}} + e^{-D_k \omega_k \delta t} \cos \left( \omega_k \sqrt{1 - D_k^2 \delta t} \right) - 1 \right).
\]  (A2)

When calculating the derivatives (A2) for \( \delta t = 10^{-5} \), we get a vanishing sequence as is illustrated in Fig. 5(a).

We assume that before the impact, the structure has a smooth motion so that the displacement without the contact force can be approximated as

\[
\mathbf{n} \cdot \mathbf{x}^+_{f_c=0} \approx \delta t \mathbf{n} \cdot \dot{x}^-.
\]
Figure 5: (a) Coefficients \( A_k \) for \( \delta t = 10^{-5}, \ D_k = 0 \) and \( \omega_k = (k\pi - \frac{\pi}{2})^2 \). (b) The sum of the infinite series of coefficients \( A_k \) is shown by the continuos (blue) line. The dashed (red) line is the estimate of \( N \), in Eqn. (A4) that overestimates the sum.

To quantify how the derivatives in Eqn. (A2) vanish we asymptotically expanded them as

\[
\frac{\partial x_k^+}{\partial f_c} = \frac{1}{2} \psi_k(\chi^*) \delta t^2 \left(1 - \frac{2}{3} D_k \omega_k \delta t - \frac{1 - 4 D_k^2}{12} \omega_k^2 \delta t^2 + \cdots\right),
\]

Substituting these two estimates into (A1) we get the contact force

\[
f_c \approx \frac{-2n \cdot \dot{x}^-}{\delta t \sum_{k=1}^{N} \psi_k^2(\chi^*)} = \frac{-2n \cdot \dot{x}^-}{N \delta t \psi_k^2(\chi^*)},
\]

up to the leading order, where the limit \( N \) of the summation equals the smallest \( k \) for which \( A_k \) vanishes, and \( \bar{\psi}_k^2(\chi^*) \) is the average value of \( \psi_k^2(\chi^*) \), for \( k \leq N \). \( N \) can be calculated as the smallest \( k \) such that the expanded coefficients become small

\[
\frac{1}{\delta t^2} \frac{\partial x_k^+}{\partial f_c} \approx 1 - \frac{2}{3} D_k \omega_k \delta t - \frac{1 - 4 D_k^2}{12} \omega_k^2 \delta t^2 < \eta \ll 1.
\]

Using the asymptotic scaling of the natural frequencies \( \omega_k = \omega_0 k^\alpha \) and assuming zero damping \( (D_k = 0) \) we get

\[
N > \left( \frac{2\sqrt{3} - 3\eta}{\omega_0} \right)^{\frac{1}{\alpha}} \delta t^{-\frac{1}{\alpha}},
\]

therefore the restoring force is

\[
f_c = -C \delta t^{\frac{1}{\alpha} - 1} n \cdot \dot{x}^-,
\]

where \( C \approx 2 \left( \frac{2\sqrt{3} - 3\eta}{\omega_0} \right)^{-\frac{1}{\alpha}} \bar{\psi}_k^2(\chi^*)^{-1} \). In order to check validity of our estimate of \( N \) we plotted the sum of derivatives in (A2) and compared to the estimate (A4) in Fig. 5(b).
As the last step we calculate the change in modal velocities

\[ \dot{x}_k^+ = \dot{x}_k^- - C\delta t^{\frac{1}{\alpha} - 1} n \cdot \dot{x}^- \frac{\partial \dot{x}_k^+}{\partial f_c} \]

\[ \approx \dot{x}_k^- - \psi_k(\chi^*) C\delta t^{\frac{1}{\alpha}} n \cdot \dot{x}^- , \]

which means that if \( \delta t \to 0 \), there is no change in individual modal velocities, except when \( \alpha \to \infty \), that is the case of a rigid body. However, when calculating the velocity of the impacting point after the impact we have

\[ n \cdot \dot{x}^+ = -n \cdot \dot{x}^- . \]

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