The effect of interdependence on the percolation of interdependent networks

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Abstract

Two stochastic models are proposed to generate a system composed of two interdependent scale-free (SF) or Erdős-Rényi (ER) networks where interdependent nodes are connected with exponential or power-law relation, as well as different dependence strength, respectively. Each subnetwork grows through the addition of new nodes with constant accelerating random attachment in the first model but with preferential attachment in the second model. Two subnetworks interact with multi-support and undirectional dependence links. The effect of dependence relations and strength between subnetworks are analyzed in the percolation behavior of fully interdependent networks against random failure, both theoretically and numerically, and as a result, for both relations: interdependent SF networks show a second-order percolation phase transition and increased dependence strength decreases the robustness of the system, whereas, interdependent ER networks show the opposite results. In addition, power-law relation between networks yields greater robustness than exponential one at given dependence strength.

Keywords: Interdependent networks; Cascading failures; Interdependence; Percolation
1 Introduction

Nowadays, with enhanced development of modern technology, the interaction between networks becomes increasingly intensive and complicated [1, 2, 3]. Examples of interdependent networks are ubiquitous and include, subway network and airport network in transportation system, bank network and company network in economy system, communication network and power grid network in infrastructure system, and so forth. In these interdependent networks, the failures of nodes in one subnetwork generally will lead to the failure of dependent nodes in the other subnetworks [4, 5, 6, 7, 8, 9]. This may happen recursively and might lead to a cascade of failures. Understanding how robustness is affected by the interdependence between subnetworks becomes one challenge when designing resilient systems. Very recently, several studies presented a theoretical framework for studying the process of cascading failures in interdependent networks and showed that interdependencies significantly increase the vulnerability of the entire networks to random attack [10, 11, 12, 13]. In addition, the first-order phase transition presented in interdependent networks is totally different from the second-order phase transition occurred in isolated network.

Most existing studies have focused almost exclusively on random interdependent networks in which the interdependent nodes are randomly connected, which is at odds with real complex systems. Taking the Italian power grid and communication networks as an example [4, 10, 14], it is very common that a central communication station depends on a central power station and vice versa. Similarly, well-connected seaports are found more likely to depend on well-connected airports in Ref. [15] where positive correlation exists between the interaction of subnetworks. Based on this feature, interdependence with correlation, not random, has attracted much attention in the robustness of interdependent networks currently. Parshani [15] and Cho [16] have shown similar result that the positive correlated interdependence enhances the robustness of networks, respectively. Buldyrev et.al [17] have analytically investigated the situation with one simple correlation that all pairs of interdependent nodes have the same degree. In addition, Ref.[6] and Ref.[18] have discussed the interdependence relation represented by Poisson distribution and power-law distribution in stochastic models, respectively. Furthermore, the effect of the dependence strength between subnetworks also plays the key role in the percolation of interdependent networks. Ref.[10] has found that when the dependence strength is reduced, the percolation tran-
sition becomes second-order transition at a critical coupling strength, which enhanced the robustness of the system. How and to what extent the relation of interdependence between subnetworks might influence the entire system’s structure and function are still not well known.

In present work, discussing the effect of different dependence relation and dependence strength on the robustness of interacting system under random attack is our focus and motivation. Two types of relations are generated by two stochastic growing network models whereby the origin of relations is explained. One is that interdependent nodes randomly depend with each other with exponential degree distribution, the other is that they preferentially depend with each other with power-law degree distribution. In addition, two interdependent scale-free (SF) and Erdős-Rényi (ER) networks are also created in these two models, respectively. Besides, the influences of dependence relations and coupling strength of multi-support, undirectional dependence links on the robustness of networks are theoretically analyzed and simulated. As a result, it is found that, (1) two different interdependence links could be generated by the addition of dependence links; (2) for interdependent SF networks and ER networks, different types of phase transition and opposite effects of dependence strength are presented; (3) for the effect of interdependence, power law distribution of dependence degree yields higher robustness than exponential one with given dependence strength.

2 The first model

In both two models, there are two types of links among the nodes: connectivity links (intra-links in each subnetwork) that enable the nodes to function cooperatively as a network, and dependence links (cross-links between subnetworks) that bind the failure of one subnetwork node to the failure of other subnetwork nodes. These two kinds of links correspond to two kinds of degree of each node in networks, connectivity degree (\(k_{con}\)) and dependence degree (\(k_{dep}\)), respectively. The first model of two interdependent scale free (SF) networks is built by the following considerations.

Initially, both subnetworks A and B contain \(m_0\) nodes and \(n_0\) connectivity links, without dependence links between subnetworks. At each time step \(t\), two new nodes are introduced simultaneously, one belonging to subnetwork A and the other belonging to subnetwork B. The new node joining to subnetwork A with \(m_A\) links added, preferentially attaches \(1 - q_A\) fraction of its
links as connectivity links to pre-existing nodes in subnetwork A. The rate of acquiring a link replies on the degrees of pre-existing nodes in subnetwork A. And then this new node randomly or preferentially attaches $q_A$ fraction of links as dependence links to pre-existing nodes in subnetwork B. In other words, the connectivity degree and the dependence degree of the new node joining to subnetwork A are equal to $m_A(1-q_A)$ and $m_A q_A$ at time step $t$ through different addition methods, respectively. The similar process is executed when a new node joins to subnetwork B, where the new node has $m_B$ links added from which $1-q_B$ fraction of them randomly connect pre-existing nodes in subnetwork B and $q_B$ fraction of them randomly or preferentially connect to pre-existing nodes in subnetwork A, and its connectivity degree and dependence degree are equal to $m_B(1-q_B)$, $m_B q_B$, respectively. $q_A$ and $q_B$ are defined as the strength of dependence between two subnetworks. Larger $q_A (q_B)$ means the more dependence links between subnetworks or the more intensively two subnetworks depend on each other. The process ends when the size of both subnetworks increases up to $N$. In fact, through this model, the subnetworks A and B generated are equivalent to the classical random graph studied by Barabási-Albert with power-law degree distribution ($p(k_{con})$), and thereby named two interdependent SF networks. Two dependence relations between interdependent nodes are represented by the degree distribution of dependence links $p(k_{dep})$. One is exponential distribution with general form [19]

$$p(k_{dep}) = \frac{1}{mq + 1} \left( \frac{mq}{mq + 1} \right)^{k_{dep} - mq};$$ (1)

$mq \geq 1$, with random dependence between subnetworks with supposition $m_A = m_B = m$ and $q_A = q_B = q$, and the other is power-law distribution $p(k_{dep}) \sim k_{dep}^{-3}$ [20] with preferential dependence between subnetworks.

The iterative process of cascading failures is initiated by randomly removing a fraction $1-p$ of nodes from subnetwork A and all edges linked to them. When nodes in subnetwork A fail, the interdependent nodes in subnetwork B also fail. Specially, we suppose that only the nodes in the giant component with at least one dependence link remain functional, which leads to the further failure in the first subnetwork. This dynamic process ends with no further node failure in the system. The cascade of failures in small interdependent networks with $N = 7$ is demonstrated in Fig.1.

The dynamics of cascading failures is performed as following and $g_A$ and $g_B$ are defined as the fraction of nodes belonging to the giant component
Figure 1: (Color online) Description of the process of cascading failures in two fully interdependent networks. Black lines represent connectivity links, and blue lines represent dependence links. The white nodes represent the survival nodes, the red nodes represent the attacked nodes, the green nodes represent the ones separate from the giant component of networks, the blue nodes represent the ones without dependence links. Initially, nodes 4 and 5 (red in A) are attacked and removed from subnetwork A. **Stage 1 in A**: node 6 (green in A) is removed because it does not belong to the giant component of subnetwork A. **Stage 1 in B**: nodes 3 and 6 (blue in Stage 1 in A) are removed because they lose all their dependence links, nodes 5 and 7 (green in Stage 1 in A) are removed because of separation from the giant component of subnetwork B. The similar process is carried out in stage 2. Note that: the failure of node 4 (blue in Stage 2 in A) results in two giant components with same size in subnetwork B. In this case, we randomly choose one giant component to fail as node 1 marked in green in Stage 2 in A. After two stages, the interdependent networks reaches a stable state, since no further failure occurs in networks.

of subnetwork A and B, respectively [5]. After the initial removal of $1 - p$ fraction of nodes in subnetwork A, the remaining fraction of subnetwork A nodes is $\psi_1' = p$. The remaining functional part of subnetwork A contains a fraction $\psi_1 = \psi_1' g_A(\psi_1')$ of the network nodes. Since the number of dependence links $k_{dep}^B$ of each node in subnetwork B is multiple and a random number, the probability that a node in subnetwork B has no dependence links in subnetwork A is $\mu_1^B = \sum_{k_{dep}^B} p^B(k_{dep}^B)(1 - \psi_1)^{k_{dep}^B} = \hat{G}^B(1 - \psi_1)(\hat{G}^B)$ the generating function of degree distribution $p^B(k_{dep}^B))$. Accordingly, the
remaining size of subnetwork B is \( \phi'_1 = 1 - \mu_1^B \), and the fraction of nodes in the giant component of subnetwork B is \( \phi_1 = \phi'_1 g_B(\phi'_1) \). Following this method, the sequence of giant components, \( \psi_n \) and \( \phi_n \), and that of the remaining fractions of nodes, \( \psi'_n \) and \( \phi'_n \), at each stage of the cascading failures are constructed as following:

\[
\begin{align*}
\psi'_1 &= p, \quad \psi = \psi'_1 g_A(\psi'_1), \quad \phi'_0 = 1, \\
\phi'_1 &= 1 - \tilde{G}^B(1 - pg_A(\psi'_1)), \quad \phi_1 = \phi'_1 g_B(\phi'_1), \\
&\ldots, \\
\psi'_n &= p[1 - \tilde{G}^A(1 - g_B(\phi'_{n-1}))], \quad \psi_n = \psi'_n g_A(\psi'_n), \\
\phi'_n &= 1 - \tilde{G}^B(1 - pg_A(\psi'_n)), \quad \phi_n = \phi'_n g_B(\phi'_n).
\end{align*}
\] (2)

The final size of each subnetwork at the end of the cascade process can be represented by \( \psi'_n, \phi'_n \) at the limit of \( n \to \infty \). This limit satisfies the equations \( \psi'_n = \psi'_{n+1} \) and \( \phi'_n = \phi'_{n+1} \) since the cluster is not further fragmented. An exact analytical solution can be obtained using the formalism of generating functions. According to Refs.[21, 22], the generating functions of the degree distributions of subnetworks A and B, \( G_{A0}(x) = \sum_{k_{con}^A} p^A(k_{con}^A)x^{k_{con}^A} \) and \( G_{B0}(x) = \sum_{k_{con}^B} p^B(k_{con}^B)x^{k_{con}^B} \) are introduced. Analogously, the generating functions of the underlying branching processes, \( G_{A1}(x) = G'_{A0}(x)/G_{A0}(1) \) and \( G_{B1}(x) = G'_{B0}(x)/G_{B0}(1) \) are also introduced. As the random removal of fraction \( 1 - p \) of nodes will change the degree distribution of the remaining nodes, so the generating functions of the new distribution are equal to generating functions of the original distribution with the argument \( x \) replaced by \( 1 - p(1 - x) \) [23]. The fraction of nodes that belong to the giant component after the removal of \( 1 - p \) nodes is \( g_A(p) = 1 - G_{A0}[1 - p(1 - f^A)] \), where \( f^A = f^A(p) \) satisfies a transcendental equation \( f^A = G_{A1}[1 - p(1 - f^A)] \).

As the theoretical analysis of generating function with power-law distribution is not available in the first model, we just present the numerical result here with \( N = 10^4, m = 5 \) in simulations.

Fig. 2 shows the effect of different dependence relations, exponential and power-law relation in the function of \( \psi_\infty \), the fraction of nodes in giant component of subnetwork A, after a random attack with different dependence strength \( q \). We find two common points for both relations: (1) \( \psi_\infty \) has similar tendency against \( p \) with different dependence strength \( q \). It smoothly decreases to zero at critical point \( p_c > 0 \) characterizing a second-order phase transition. This result differs from the general known result the first-order
phase transition discovered in coupled networks [4, 10]; (2) with the increasing of dependence strength $q$, the value of critical point $p_c$ increases, which implies the decreasing of resilience of networks. The potential reasons for this may be that since the sum of connectivity link and dependence link per new node at each time step is constant, the larger dependence strength $q$ means the less the connectivity links (the smaller mean connectivity degree) and the more the dependence links of each node in each subnetwork, or stronger interdependence between subnetworks. Smaller mean connection degree quickens the fragmentation of individual network and hubs in one network can depend on weak (low-degree) nodes in the other network and vice-versa, and then the strong interdependence leads to accelerated cascades of failures.

Figure 2: (Color online) The dependence of giant components $\psi_\infty$ of subnetwork A with different dependence strength and relations on $p$ at $N = 10^4$ and $m = 5$. For two relations, $\psi_\infty$ changes continuously from a finite value to zero at critical threshold $p_c$, characterizing the second-order phase transition occurred in the system, and the value of $p_c$ increases with the increasing of $q$ in both cases.

The discrepancy of effects caused by two relations is shown in Fig. 3. The critical point $p_c$ is an increasing function of dependence strength $q$. When $q$ is close to zero corresponding to the extreme case that there is no interdependence between subnetworks, $p_c$ attends to zero and goes back to the classical case that the single scale-free network has critical percolation value $p_c = 0$ under random failure. For weak dependence strength $q$ around $q = 0.2$, the same $p_c$ is found for two dependence relations. In the range of $q > 0.2$, the value of $p_c$ for power-law relation is always smaller than that for exponential relation, which demonstrates that power-law distribution of dependence degree yields greater robustness than exponential one when certain dependence strength $q$ is given. This result could be attributed to the possibility that power-law relation between the dependent nodes could suppress the phe-
nomenon of hubs in one network becoming vulnerable by being dependent on weak nodes in the other network, when the dependence strength arrives at certain critical threshold. In addition, this finding strengthens the conclusions of recent studies [15, 16, 17] that coupled networks with positively correlated degrees of dependent nodes are always more robust than randomly coupled networks.

Figure 3: (Color online) The tendency of critical threshold $p_c$ on dependence strength $q$ with two dependence relations. When $q$ is close to zero, interdependent SF networks becomes single scale-free network where the critical threshold $p_c = 0$ is obtained under random attack in percolation. For small $q$ around 0.2, networks with both relations have the same $p_c$ under cascading failures. For $q > 0.2$, networks with power-law relation between dependent nodes have smaller $p_c$ than those with exponential one, which indicates that power-law distribution of dependent degrees yields greater robustness of system than exponential one.

3 The second model

Two interdependent ER networks is generated in second model, which is the difference from the first model. The common place between two models is that subnetworks depend on each other with two identical relations. In second model, more attention is paid on the theoretical analysis of the effect brought by interdependence. This model is constructed as following.

Initially, both subnetworks A and B contain $m_0$ nodes and $n_0$ connectivity links, without dependence links between subnetworks. At each time step $t$, two new nodes, one belonging to subnetwork A and the other belonging
to B, are introduced simultaneously. Connectivity links will be created by
the scenario of constant acceleration (See Ref.[19] for details). It is processed
as follows: for subnetwork A, connectivity links between the new node and
pre-existing nodes are established randomly with probability $s$ satisfying the
requirement that the expected number of links for the new node is equal
to $st$. For the addition of dependence links, there are two approaches like
those in the first model: randomly or preferentially connect the new node
belonging to subnetwork A to $DL$ pre-existing nodes in subnetwork B, which
will generate the exponential and power-law distribution of dependence de-
grees, respectively. $DL$ is defined as the strength of dependence between two
subnetworks like $q$ in the first model. For subnetwork B, similar process is
carried out in the creation of connecting and dependence links. Through this
model, the subnetworks A and B generated are equivalent to the classical
random graph studied by Erdős-Rényi with Poisson degree distribution, and
thereby named two interdependent ER networks. When the size of subnet-
works A and B increases to $N$, the process of building two interdependent
ER networks is concluded.

According to the dynamics of cascading failures described in the first
model, for interdependent ER networks, the problem can be solved explicitly,
since $G_0(x)$ and $G_1(x)$ have the same simple form $G_0(x) = G_1(x) = e^{(k)(x-1)}$
[21]. Supposing that the average degree of subnetwork A is $\langle k \rangle = a$, and
for subnetwork B, one gets $\langle k \rangle = b$. Thus, from $g_A(\psi'_\infty) = 1 - f^A$ and
$g_B(\phi'_\infty) = 1 - f^B$, both $g^A(\psi'_\infty) = 1 - e^{-a\psi_\infty}$ and $g^B(\phi'_\infty) = 1 - e^{-b\phi_\infty}$ are
reduced. According to the definitions in Eqs.(2) at the limit of $n \rightarrow \infty$, the giant components of subnetwork A and B with generating functions of
dependence relations $\tilde{G}^A$ and $\tilde{G}^B$ at the stable state are obtained:

$$\psi_\infty = p[1 - \tilde{G}^A(e^{-b\phi_\infty})](1 - e^{-a\psi_\infty}),$$
(3)

$$\phi_\infty = [1 - \tilde{G}^B(1 - p(1 - e^{-a\psi_\infty}))](1 - e^{-b\phi_\infty}).$$
(4)

(1) In the case of exponential dependence relation between subnetworks, according to the definition of $\tilde{G}^A$, $\tilde{G}^B$ and Eq.(1), Eqs.(3) and (4) become:

$$\psi_\infty = p[1 - \frac{1}{-DL + e^{b\phi_\infty} + DLe^{a\phi_\infty}}](1 - e^{-a\psi_\infty}),$$
(5)

$$\phi_\infty = [1 + \frac{-e^{a\psi_\infty} - p + pe^{a\psi_\infty}}{e^{a\psi_\infty} - DLp + DLe^{a\psi_\infty}}](1 - e^{-b\phi_\infty}).$$
(6)

(2) In the case of power-law dependence relation between subnetworks, based
on the simulation result, the degree distributions of dependence degrees with
different link addition $DL$ are found to have the same factor of $\sim k_{dep}^{-3}$ over the central range of degree and have various minimum degree $k_{min}^{dep}$. Hence the formation of degree distribution $p(k_{dep}) = (k_{dep}^{min}/k_{dep})^2 - (k_{min}^{dep}/(k_{dep} + 1))^2$, behaving asymptotically as $2(k_{min}^{dep})^2/k_{dep}^3$, is used in the calculation of $\tilde{G}^A$ and $\tilde{G}^B$. In addition, $DL = 1, 2, 8$ correspond to $k_{min}^{dep} = 1, 2, 8$ in the degree distributions, respectively. Similarly, along the definition of $\tilde{G}^A$ and $\tilde{G}^B$, Eqs. (3) and (4) become:

$$\psi_\infty = p[\text{PolyGamma}[2, k_{dep}^{min}] + 2(e^{-b\phi_\infty})k_{dep}^{min}$$
$$\text{HurwitzLerchPhi}[e^{-b\phi_\infty}, 3, k_{dep}^{min}][1 - e^{-a\psi_\infty}]$$
$$/\text{PolyGamma}[2, k_{dep}^{min}],$$

$$\phi_\infty = [\text{PolyGamma}[2, k_{dep}^{min} + 2[1 + (-1 + e^{-a\psi_\infty})p]k_{dep}^{min}$$
$$\text{HurwitzLerchPhi}[1 + (-1 + e^{-a\psi_\infty})p, 3, k_{dep}^{min}][1 - e^{-b\phi_\infty}]$$
$$/\text{PolyGamma}[2, k_{dep}^{min}],$$

where $\text{PolyGamma}[n, z]$ gives the $n^{th}$ derivative of the digamma function $x^n(z), x(z) = \Gamma'(z)/\Gamma(z)$, and $\text{HurwitzLerchPhi}[z, s, a]$ gives the Hurwitz-Lerch transcendent $\Phi(z, s, a) = \sum_{k=0}^{\infty} z^k/(k + a)^s$.

In the first case (1), in the limit of $DL \to \infty$, the giant component of two interdependent ER networks will not depend on each other and the percolation theory of single network $\psi_\infty = p(1 - e^{-a\psi_\infty})$ is recovered, which is comparable with the result of reference [6] where the Poisson degree distribution was given between interdependent nodes. The solutions of system with Eqs. (5) and (6) can be graphically presented by the intersection of the curves $\phi_\infty(\psi_\infty)$ and $\psi_\infty(\phi_\infty)$. The trivial solutions correspond to $\psi_\infty = \phi_\infty = 0$ and the nontrivial solutions in the critical case can be found from the tangential condition $\frac{d\psi_\infty(\phi_\infty)}{d\phi_\infty}(\psi_\infty) = 1$, corresponding to the single point of two curves. Together with Eqs. (5) and (6), the critical value of the parameters $a, b, DL, p$ can be reduced when three of them are fixed. Here, with the assumption of $a = b$, we get the expression of critical threshold $p_c$ above which two interdependent ER networks have non-zero giant components:

$$p_c = \frac{5a + 7aDL - 4DL + 2aDL^2 - 2DL^2}{2a(1 + DL)(a + (a - 1)DL)}.$$  \hspace{1cm} (8)

When $a$ is fixed and $DL \to \infty$, the above equation will become: $p_c = 1/a$, which is the critical threshold of random percolation for single ER network
In the second case (2), the exact theoretical results for networks with power-law dependence relation are not available, so the numerical simulation results will be given below.

3.1 Numerical simulations

In this section, the theoretical results discussed in above section are compared with results of numerical simulations. In all simulations, we have $N = 10^5$ and $a = b = 4$. In Fig. 4, the giant components of two interdependent ER networks with two dependence relations as a function of $p$, the fraction of nodes in subnetwork A needed to be preserved at the beginning of the cascading failures is shown. In panels (a) and (b), for exponential dependence relation, as $DL$ increases, the critical value of $p_c^A$ ($p_c^B$) is close to 0.25 eventually, the critical threshold value of random percolation of a single ER network with average degree 4, and then a second-order phase transition will be shown with infinite $DL$. For finite $DL$, however, $\psi_\infty$ and $\phi_\infty$ behave as the first-order phase transition characterized by discontinuously changing from nonzero fraction to zero, which differs from the second-order phase transition occurred in the first model with interdependent SF networks. It suggests that enhanced dependence strength between subnetworks leads to more robust performance and the change from first-order phase transition to second-order phase transition. In addition, this simulation result agrees well with the prediction of Eqs. (5) and (6). In panels (c) and (d), for power-law dependence relation, similar tendency of giant components on $p$ is found. Nevertheless, there is a little deviation between the prediction and the simulation in the case of $DL = 1$. The actual degree distribution of dependence degrees in simulation has fat-tail deviating from the distribution predicted by the theory, which causes that nodes with large degree in fat-tail, or with more dependence links make them still functional under larger fraction of nodes randomly attacked in subnetwork A. So this possibly results in the critical value $p_c$ in simulation is smaller than that in prediction. In addition, as $DL$ approaches to infinity, $\phi_\infty$ in panels (b) and (d) converges to a Heaviside step function, $H(p - p_c)$, which discontinuously changes from one for $p > p_c$ to zero for $p < p_c$ and $p_c = 0.25$. The potential explanation for this phenomenon is that two subnetworks will connect fully with each other as the dependence links between them increase to infinity (actually increase up to the size of subnetwork in the simulation). When $p < p_c$, the giant component of subnetwork A disappears, so $\phi_\infty$ is close to zero, and when
$p > p_c$, subnetwork B is almost a complete network as most of its nodes have dependence links from subnetwork A, so $\phi_\infty$ is close to one.

Figure 4: (Color online) The dependence of giant components $\psi_\infty$ and $\phi_\infty$ of two interdependent ER networks with two different dependence relations on $p$. In all cases, $N = 10^5$, $a = b = 4$, $DL = 1, 2, 8$. In panels (a) and (b), for the system with exponential dependence relation, as $DL$ increases, the giant components discontinuously change from finite value to zero at critical threshold $p_c \to 0.25$ that is the critical point of random percolation of a single ER network for infinite $DL$. The simulation results (symbols) agree well with analytical results (red lines). In panels (c) and (d), for the system with power-law dependence relation, similar tendency is found in the dependence of giant components of both networks on parameter $p$. There is a little deviation between the analytical results and simulations in the case of $DL = 1$. In addition, as $DL \to \infty$, $\phi_\infty$ in panels (b) and (d) converges to a Heaviside step function.

In order to compare the influence caused by various dependence relations on the percolation behavior of networks, the critical threshold $p_c$ as a function of $DL$ is plotted in Fig. 5 where the theory is found agreeing well with the simulation result. As $DL$ increases, the critical threshold $p_c$ decreases in both relations. In addition, the value of $p_c$ for power-law relation is always smaller than that for exponential relation with different $DL$ in simulations. In theory,
however, there is an exception in weak dependence strength $DL = 1$ where the critical threshold $p_c$ for power-law relation (marked by red line) is larger than that for exponential relation (marked by black line). Similar exception is found in the case of $q \leq 0.2$ in Fig. 2 in the first model, so it comes to a strong conclusion that when the dependence strength between networks is larger than one certain value, the system with power-law relation behaves more robust against random failure than networks with exponential relation.

![Figure 5: (Color online) Comparison of robustness of networks with two different dependence relations at $N = 10^5$ and $a = b = 4$. The critical threshold $p_c$ decreases with the increasing of $DL$ and approaches the critical value 0.25 of random percolation of a single network with infinite $DL$. The value of $p_c$ for power-law relation is always smaller than that for exponential relation in simulations (symbols), which suggests that the networks with former relation is more robust to random failure than networks with latter relation. However, there is an exception in theoretical result (red line) at $DL = 1$ with power-law relation. Weak dependence strength between subnetworks may be responsible for this.](image)

4 Conclusions and discussion

In present work, based on two network models, we developed a framework for studying the effect of dependence relation and strength in the percolation of two fully interdependent SF and ER networks, subject to random attack. The addition of dependence links with random and preferential attachment between subnetworks in two stochastic models results in the exponential and power-law distribution of dependence degree, respectively. For both dependence relations, we find that, 1) in two interdependent SF networks,
dependence strength makes the system more vulnerable and the system only goes through the second-order phase transition; 2) in two interdependent ER networks, the opposite results are found that increased dependence strength can enhance the robustness of system and the system shows a first-order phase transition. In addition, when the dependence strength is given in excess of certain value, power-law relation yields greater robustness than exponential one, which strengthens the known conclusion that correlated coupled system always has more robustness than randomly coupled system. The accurate theoretical analysis needed to be provided to improve this result in future.

The models studied here can help to further understand the design of real-world interdependent networks where comprise more complex dependence relations. Through adjusting the parameters of models, they could also have the flexibility to represent a variety of interdependent complex systems. Moreover, inspired by Ref.[11], this work could be extended by taking the inter-connectivity links between subnetworks into consideration and more plentiful behavior might be found in the percolation phase transition of coupled networks.

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