Data-driven Robust LQR with Multiplicative Noise via System Level Synthesis

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Abstract—This paper aims to develop a data-driven method for solving the closed-loop state-feedback control of a discrete-time LQR problem for systems affected by multiplicative norm bounded model uncertainty. To synthesize a tractable robust state feedback policy, first, we adopt the recently developed system-level synthesis (SLS) framework to reformulate the LQR control design closed-loop system responses rather than the control gain. In many situations, however, the solution to this worst-case optimization problem may be too conservative since it sets out to enforce the design constraints for every possible value of the uncertainty. To deal with this issue, we reformulate this optimization problem as a chance-constrained program (CCP), where the guarantees are not expressed as deterministic satisfaction against all possible uncertainty outcomes but rather expressed as guarantees against uncertainty outcomes. To approximately solve the CCP without prior knowledge of how the uncertainties in the system matrices are described, we employ the so-called scenario approach, which provides probabilistic guarantees based on a finite number of samples and results in a convex optimization program with moderate computational complexity. Finally, numerical simulations are presented to illustrate the theoretical findings.

Index Terms—Multiplicative Model Uncertainty, System Level Synthesis, Scenario Approach.

I. INTRODUCTION

LINEAR quadratic regulator (LQR) problem is one of the most well-known problems in the classical optimal control literature [1], [2], which has been widely used in various applications such as aerospace, robotics, finance, and so forth. The aim of the LQR problem is to design a state-feedback controller that minimizes a convex quadratic cost related to the control of a linear dynamical system. Ever since the LQR problem emerged, its robustness to uncertainties was questioned as a fundamental issue because of the inherent presence of uncertainties in practice. The robust LQR problem is investigated in an anthology of papers, see [3]–[6], and references therein. The min-max model predictive control (MPC) framework [7]–[9] also provides approximate robust solutions for constrained LQR problems. Although most of the work on robust LQR problems has considered additive noise, its counterpart, which is called multiplicative noise [10]–[13], has been investigated far less. However, robustness of LQR problems for systems with multiplicative noise has great importance from a practical point of view. This is because, in essence, it provides us with an intuition of how to formulate a controller which could stabilize a linear dynamical system in the presence of stochastic dynamics perturbations. Moreover, by doing so one can explicitly incorporate model uncertainty and inherent stochasticity, which can lead to the improvement of robustness properties of the controller.

The recently introduced System Level Synthesis (SLS) framework [5], [6], [14] provides powerful tools to explore the trade-off between conservatism and computational complexity to deal with the robust LQR problem more efficiently. The core concept behind the SLS framework scheme is that it transforms the control design over the linear feedback control gains to closed-loop system responses and provides an explicit link between them. The robust form of the SLS framework [6], [15]–[18] allows for an explicit mapping from the model uncertainty to the system behavior, providing an explicit characterization of the joint effects of additive disturbances and model errors (i.e., multiplicative noise) when solving robust LQR problem [10], [19]. The rationale behind the SLS framework is both practical and applicable to many settings, and since its beginning, there have been many extensions developed such as works on MPC [17], [20]–[23], dynamic programming [24], data-driven adaptive control [6], [16], and so forth. However, the SLS framework for solving the LQR problem for systems with multiplicative noise is not considered in the literature, despite its importance.

Robust LQR control approaches for a system with multiplicative noise require optimizing the performance function for the worst-case system realization. This, however, can be overly conservative, especially if the support of the uncertainty is infinite, unknown or high (e.g., random Gaussian multiplicative noise). One alternative approach to remedy this problem would be to interpret robustness in a probabilistic sense, in which the guarantees of constraints fulfillment are intended in the probabilistic sense (satisfying most uncertain instances) rather than the deterministic sense (satisfaction against all possible uncertain outcomes). That is, constraint violation is allowed with a low probability. This leads to stochastic optimization problems, which are typically called chance-constrained programming problems [25]–[29]. Chance-constrained programming (CCP) has a wide range of applications, e.g., in finance [30], control [31], and so forth. With the exception of a few special cases [32], however, CCP problems are computationally intractable (i.e., NP-hard) as they entail the calculations of multi-dimensional probability integrals [33]. The scenario approach [34]–[37] is a simple yet promising

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method for approximately solving chance-constrained optimization. To this aim, the scenario approach employs a dataset with some samples (so-called scenarios) from the set of uncertain parameters and requires the constraints to be satisfied for each scenario. A prominent feature of the scenario approach is its generality and tractability, as well as the fact that it requires no assumptions apart from constraint convexity.

In this paper, we aim to develop a data-driven method for solving the discrete-time LQR problem for systems affected by multiplicative noise can be expressed probabilistically or

\[ \Pr(X) \]

respectively. Bold characters denote discrete-time signals. It is first shown that the closed-loop system responses for a system can be compactly written as [5]

\[ x_{t+1} = Ax_t + Bu_t + W_t \]

where \( x_t \in \mathbb{R}^n \), \( u_t \in \mathbb{R}^m \), and \( W_t \in \mathbb{R}^n \) is an exogenous disturbance process. Let \( x_0 \equiv W_{[-1]} \) and \( W_{[-1,T]} \equiv 0 \) for all \( t \geq 0 \). We assume that the pair \((A,B)\) is controllable. Considering the system given in (1), now, the finite-horizon LQR problem is formulated as

\[ \text{min}_{u_t} \quad \mathbb{E} \left[ \sum_{t=0}^{T-1} x_t^\top Q x_t + u_t^\top R u_t + x_T^\top Q_F x_T \right] \]

subject to

\[ x_{t+1} = Ax_t + Bu_t + W_t \]

\[ u_t = K x_t \]

where \( Q, Q_F \geq 0 \) and \( R > 0 \). Utilizing a time-varying state-feedback control law \( u_t = K_t x_t \), the closed-loop dynamics can be compactly written as [5]

\[ x_{[0,T-1]} = Z(A + BK) x_{[0,L-1]} + W_{[0,T-2]} \]

where

\[ A := \text{blkdiag}(A_1, A_2, \ldots, A_k) \]

and

\[ B := \text{blkdiag}(B_1, B_2, \ldots, B_N) \]

is the block-downshift operator (i.e., a matrix with identity matrices along the first block subdiagonal and zeros elsewhere), and

\[ K \in \mathcal{L}_{TV}^{m \times n} \]

denotes the block matrix operator for the linear causal time-varying state-feedback controller. Recasting (3), the closed-loop map from disturbance to state and control input is given by [5]

\[ x_{[0,T-1]} := \Phi_{u,T} W_{[-1,T-2]} = K(I - Z(A + BK))^{-1} W_{[-1,T-2]} \]

\[ u_{[0,T-1]} := \Phi_{u,T} W_{[-1,T-2]} = K(I - Z(A + BK))^{-1} W_{[-1,T-2]} \]

where \( \Phi_{u,T} \) are two block-lower triangular matrices called as the closed-loop system response, and one realization of the controller is given by \( K = \Phi_{u,T}^{-1} \in \mathcal{L}_{TV}^{T \times m \times n} \).

The following proposition will be helpful in the rest of the development.

**Proposition 1:** [5] Consider the system (1) with the state-feedback control law \( K \in \mathcal{L}_{TV}^{T \times m \times n} \), i.e., \( u_{[0,T-1]} = K x_{[0,T-1]} \).

Then, the following statements hold:

1. The following affine subspace

\[ \Phi_u = I \]
with $\Phi_x \in L_{TV}^{T,n \times n}$ and $\Phi_u \in L_{TV}^{T,m \times n}$, parameterizes all possible closed-loop system responses from $u[-1, T-2] \to [x_0(T-1), u(0, T-1)]$.

2: The controller $K := \Phi_x \Phi_u^{-1}$ is internally stabilizing and leads to the desired closed-loop responses $\Phi_x$, provided that the operators $\Phi_x$ and $\Phi_u$ satisfy $\Phi_u$.

Using Proposition I, the LQR problem (I) can be reformulated in terms of the closed-loop system responses $\{\Phi_x, \Phi_u\}$ as follows $[5]$

$$\min_{\Phi_x, \Phi_u} \left\| \left[ \begin{array}{c} \Phi_x \\ \Phi_u \end{array} \right] \right\|_F$$

subject to $\left[ \begin{array}{c} I - Z(A\Phi_x) \\ -ZB(\Phi_u) \end{array} \right] = I$ \hspace{1cm} (6)

where $Q := I_L \otimes Q$ and $R := I_L \otimes R$.

**Proposition 2:** [39] For a random vector $X \in \mathbb{R}^n$ with cumulative distribution $F(\cdot)$,

$$P\{\|X - \mathbb{E}[X]\| \geq \varepsilon\} \leq \frac{\text{Var}(X)}{\varepsilon^2} \quad \forall \varepsilon > 0 \quad (7)$$

and

$$\text{Var}(X) \overset{\text{def}}{=} \int_{V \in \mathbb{R}^n} \|V - \mathbb{E}[X]\|^2 dF(V) \quad (8)$$

Now, let the matrices $A$ and $B$ be uncertain with input- and state-multiplicative noise $\delta_t$ as follow

$$x_{t+1} = A(\delta_t)x_t + B(\delta_t)u_t + W_t, \quad \forall \delta_t \in \Delta \quad (9)$$

with $A(\delta_t) := A_0 + \sum_{i=1}^{n_\delta} \delta(i)A_i \in \mathbb{R}^{n \times n}$ and $B(\delta_t) := B_0 + \sum_{i=1}^{n_\delta} \delta(i)B_i \in \mathbb{R}^{n \times m}$. Let, at each time $t$, $\delta_t := [\delta_t^{(1)} I_{n \times n} \ldots \delta_t^{(n_\delta)} I_{n \times n}]_{n \times n_\delta} \in \mathbb{R}^{n \times n_\delta}$, where $\delta_t^{(i)} \in \mathbb{R}$, be i.i.d. copy of a square integrable random variable $\delta$ distributed according to probability space $(\Delta, \mathcal{F}, P_\delta)$, and $x_0 = \mathcal{W}_{-1}$ and $W_t = 0, \forall t \geq 0$. Moreover, let $A_0 := \left[ \begin{array}{c} A_0^T \\ A_1^T \ldots A_{n_\delta}^T \end{array} \right] \in \mathbb{R}^{m \times n_\delta}$, $B_0 := \left[ \begin{array}{c} B_0^T \\ B_1^T \ldots B_{n_\delta}^T \end{array} \right] \in \mathbb{R}^{m \times m}$.

We can now state the problem of interest as follows

$$\min_{\Phi_x, \Phi_u} \left\| \left[ \begin{array}{c} \Phi_x \\ \Phi_u \end{array} \right] \right\|_F$$

subject to $\left[ \begin{array}{c} I - Z(A(\delta_t)) \\ -ZB(\delta_t) \end{array} \right] = I$ \hspace{1cm} (10)

where $A(\delta_t) := \text{blkdiag}(A(\delta_t), \ldots, A(\delta_t), 0)$ and $B(\delta_t) := \text{blkdiag}(B(\delta_t), \ldots, B(\delta_t), 0)$.

### III. MAIN RESULT

Routine calculations show that

$$x = (I_{nT \times nT} - Z(A + BK))^{-1}w \quad (11)$$

where the stacked states $x_{0:T} = x = [x_0, x_1, \ldots, x_T]^T$, inputs $u_{0:T} = u = [u_0, u_1, \ldots, u_T]^T$, disturbances $W_{0:T} = w = [x_0, w_0, w_1, \ldots, w_{T-1}]^T$, and causal linear time-varying state-feedback controller $K \in L_{TV}^{T,n \times n}$, and

$$A = \left[ \begin{array}{c} I_{T-1} \\ \vdots \\ I_1 \end{array} \right] \otimes A_0 + \left[ \begin{array}{c} \Delta \\ \vdots \\ \Delta \end{array} \right] (I_T \otimes A) \quad (12)$$

$$B = \left[ \begin{array}{c} I_{T-1} \\ \vdots \\ I_1 \end{array} \right] \otimes B_0 + \left[ \begin{array}{c} \Delta \\ \vdots \\ \Delta \end{array} \right] (I_T \otimes B) \quad (13)$$

$\Delta = \text{diag}(\delta_0, \delta_1, \ldots, \delta_{T-1}) \in \mathbb{R}^{(T-1) \times (T-1) n_\delta}$

It follows from $[11] - [14]$ that

$$\Phi_x = (I_{nT \times nT} - Z(A_0 + B_0 K))^{-1} \left( I - \left[ \begin{array}{c} \Delta \\ \vdots \\ \Delta \end{array} \right] (I_T \otimes A) (I_T \otimes B K) \right)^{-1} \quad (15)$$

$$\Phi_u = K(I_{nT \times nT} - Z(A_0 + B_0 K))^{-1} \left( I - \left[ \begin{array}{c} \Delta \\ \vdots \\ \Delta \end{array} \right] (I_T \otimes A) (I_T \otimes B K) \right)^{-1} \quad (16)$$

which leads to

$$\left[ \begin{array}{c} x \\ u \end{array} \right] = \left[ \begin{array}{c} \hat{\Phi}_x \\ \hat{\Phi}_u \end{array} \right] \tilde{w} \quad (17)$$

where $\tilde{w} = (I + \Delta)^{-1}w$ and

$$\hat{\Phi}_x = (I_{nT \times nT} - Z(A_0 + B_0 K))^{-1} \quad (18)$$

$$\hat{\Phi}_u = K(I_{nT \times nT} - Z(A_0 + B_0 K))^{-1} \quad (19)$$

$$\Delta = \frac{-Z \left[ \begin{array}{c} \Delta \\ \vdots \\ \Delta \end{array} \right] (I_T \otimes A) (I_T \otimes B K)}{I_{nT \times nT} - Z(A_0 + B_0 K)} \quad \in \mathbb{R}^{nT \times nT} \quad (20)$$

**Theorem 1:** Let $\hat{\Delta}$ be defined as $\hat{\Delta}$, and suppose that the controller $K = \hat{\Phi}_u \hat{\Phi}_x$ achieves the system response

$$\left[ \begin{array}{c} x \\ u \end{array} \right] = \left[ \begin{array}{c} \hat{\Phi}_x \\ \hat{\Phi}_u \end{array} \right] (I_{nT \times nT} + \hat{\Delta})^{-1} w \quad (21)$$

If $(I + \Delta_i, 0)$ exists for $i = 1, \ldots, T$, then $\{\Phi_x, \Phi_u\}$ satisfy

$$\left[ I - ZA_0 \quad -ZB_0 \right] \left[ \begin{array}{c} \hat{\Phi}_x \\ \hat{\Phi}_u \end{array} \right] = I_{nT \times nT} + \Delta \quad (22)$$

**Proof.** Noting that

$$K = \hat{\Phi}_u \hat{\Phi}_x^{-1} = \hat{\Phi}_u (I_{nT \times nT} + \hat{\Delta})^{-1} (\hat{\Phi}_x (I_{nT \times nT} + \hat{\Delta})^{-1})^{-1} \quad (23)$$

Now, using Proposition II one has

$$\left[ I - ZA_0 \quad -ZB_0 \right] \left[ \begin{array}{c} \hat{\Phi}_x \\ \hat{\Phi}_u \end{array} \right] (I_{nT \times nT} + \hat{\Delta})^{-1} = I \quad (24)$$
since \((I + \Delta^i,0)\) exists for \(i = 1, \ldots, T\), (24) is equivalent to
\[
\begin{bmatrix}
I - ZA_0 & -ZB_0
\end{bmatrix}
\begin{bmatrix}
\Phi_x \\
\Phi_u
\end{bmatrix} = I_{nT \times nT} + \Delta 
\tag{25}
\]
This completes the proof.

Now, recall that
\[
\Delta = \text{diag}(\delta_0, \delta_1, \ldots, \delta_{T-1}) \in \mathbb{R}^{(T-1) \times (T-1) \times n_{
}}
\tag{26}
\]
where \(\delta_i := \begin{bmatrix} \delta_i^{(1)} & \cdots & \delta_i^{(n_u)} \end{bmatrix} \in \mathbb{R}^{n_u \times n_u}\), one can rewrite \(\Delta\) as \(\Delta = Z(\text{blkdiag}(\Delta, 0))\Theta\) where \(\Theta = \Theta_1 \Theta_2 \in \mathbb{R}^{n_{\text{out}} \times nT \times T}\) and
\[
\Theta_1 := (- (I_T \otimes A) - (I_T \otimes B)K) \in \mathbb{R}^{n_{\text{out}} \times T \times nT} 
\tag{27}
\]
\[
\Theta_2 := (I_{nT \times nT} - Z (A_0 + B_0K))^{-1} \in \mathbb{R}^{n_{\text{out}} \times nT \times T} 
\tag{28}
\]

**Theorem 2:** Let \(\Delta\) be defined as (20), and suppose that \(\hat{K} = \Phi_x, \Phi_u\) achieves the system response (21). Then system responses \(\hat{\Phi}_X, \hat{\Phi}_u\) satisfies the change constraint
\[
\begin{aligned}
\mathbb{P}\left\{ \| \text{vec} \left[ \begin{bmatrix} I - ZA_0 & -ZB_0 \end{bmatrix} \begin{bmatrix} \Phi_x \\
\Phi_u
\end{bmatrix} - I \right] \|_F \geq \varepsilon \right\} \leq 1/{\varepsilon}^2 \left( (\Theta)^T \otimes I_{T \times nT} \right)^T \Sigma(\text{vec}(\Theta))^T (\Theta)^T \otimes I_{T \times nT}, \forall \varepsilon > 0
\end{aligned}
\tag{29}
\]
where \(\Sigma(\text{vec}(\Theta)) = \text{Var} \{ \text{vec}(\Theta) \}, \text{ and } \Sigma(\text{vec}(\Theta)) = \text{vec}(\Theta) \in \mathbb{R}^{T \times n_{\text{out}}}\), which is a random vector defined on \((\Omega, \mathcal{F}, \mathbb{P})\).

**Proof.** Note that \(\text{vec}(ABC) = (C^T \otimes A) \text{vec}(B)\). Therefore,
\[
\begin{aligned}
\text{vec} \left[ \begin{bmatrix} I - ZA_0 & -ZB_0 \end{bmatrix} \begin{bmatrix} \Phi_x \\
\Phi_u
\end{bmatrix} - I \right] = \left( (\Theta)^T \otimes I \right) \Sigma(\text{vec}(\Theta)) = \mu
\end{aligned}
\tag{30}
\]
Now, let
\[
\begin{aligned}
\mathbb{E} \{ \left( (\Theta)^T \otimes I_{T \times nT} \right) \Sigma(\text{vec}(\Theta)) \} = \mu
\end{aligned}
\tag{31}
\]
\[
\forall \varepsilon > 0, \text{ which implies }
\begin{aligned}
\mathbb{P}\left\{ \| \text{vec} \left[ \begin{bmatrix} I - ZA_0 & -ZB_0 \end{bmatrix} \begin{bmatrix} \Phi_x \\
\Phi_u
\end{bmatrix} - I \right] \|_F \geq \varepsilon \right\} \leq \frac{\text{Var} \{ \left( (\Theta)^T \otimes I \right) \Sigma(\text{vec}(\Theta)) \}}{\varepsilon^2}
\end{aligned}
\tag{32}
\]
Invoking Proposition 2 one has
\[
\begin{aligned}
\mathbb{P}\left\{ \| \text{vec} \left[ \begin{bmatrix} I - ZA_0 & -ZB_0 \end{bmatrix} \begin{bmatrix} \Phi_x \\
\Phi_u
\end{bmatrix} - I \right] \|_F \geq \varepsilon \right\} \leq 1/{\varepsilon}^2 \left( (\Theta)^T \otimes I_{T \times nT} \right)^T \Sigma(\text{vec}(\Theta))^T (\Theta)^T \otimes I_{T \times nT}, \forall \varepsilon > 0
\end{aligned}
\tag{33}
\]
Substituting \(\mu = 0\) into (33) gives (29), which completes the proof.

As a result, problem (10) can be rewritten as
\[
\begin{aligned}
\begin{array}{l}
\min_{\Phi_x, \Phi_u} \left\{ \left\| \begin{bmatrix} Q^{1/2} \\
R^{1/2}
\end{bmatrix} \begin{bmatrix} \Phi_x \\
\Phi_u
\end{bmatrix} (I + \Delta) - I \right\|_F \right\}
\end{array}
\end{aligned}
\tag{34}
\]
or equivalently,
\[
\begin{aligned}
\min_{\Phi_x, \Phi_u} \left\{ \left\| \begin{bmatrix} Q^{1/2} \\
R^{1/2}
\end{bmatrix} \begin{bmatrix} \Phi_x \\
\Phi_u
\end{bmatrix} - I \right\|_F \right\}
\end{aligned}
\tag{35}
\]
After some manipulation, one has
\[
\begin{aligned}
\begin{array}{l}
\min_{\Phi_x} \left\{ \left\| \begin{bmatrix} Q^{1/2} \\
R^{1/2}
\end{bmatrix} \begin{bmatrix} \Phi_x \\
\Phi_u
\end{bmatrix} - I \right\|_F \right\}
\end{array}
\end{aligned}
\tag{36}
\]
where
\[
\begin{aligned}
\Delta = \begin{bmatrix} \Delta_{1,0} & \cdots & \Delta_{T,0} \\
\vdots & \ddots & \vdots \\
\Delta_{T,0} & \cdots & \Delta_{T,0}
\end{bmatrix} \begin{bmatrix} \Phi_x \\
\Phi_u
\end{bmatrix} - I
\end{aligned}
\tag{37}
\]
where \(\Pi_{k,m} = \Phi_x^{k,m} - A_0 \Phi_x^{k-1,m-1} - B_0 \Phi_x^{k-1,m-1}\).

One has
\[
\begin{aligned}
\begin{array}{l}
\min_{\Phi_x} \left\{ \left\| \begin{bmatrix} Q^{1/2} \\
R^{1/2}
\end{bmatrix} \begin{bmatrix} \Phi_x \\
\Phi_u
\end{bmatrix} - I \right\|_F \right\}
\end{array}
\end{aligned}
\tag{38}
\]
which implies
\[
\begin{aligned}
\begin{array}{l}
\min_{\Phi_x, \Phi_u} \left\{ \left\| \begin{bmatrix} Q^{1/2} \\
R^{1/2}
\end{bmatrix} \begin{bmatrix} \Phi_x \\
\Phi_u
\end{bmatrix} - I \right\|_F \right\}
\end{array}
\end{aligned}
\tag{39}
\]

where \(\Pi_{k,m} = \Phi_x^{k,m} - A_0 \Phi_x^{k-1,m-1} - B_0 \Phi_x^{k-1,m-1}\).

where \(\Pi_{k,m} = \Phi_x^{k,m} - A_0 \Phi_x^{k-1,m-1} - B_0 \Phi_x^{k-1,m-1}\). Therefore, one can show that
\[
\begin{aligned}
\begin{array}{l}
\min_{\Phi_x} \left\{ \left\| \begin{bmatrix} Q^{1/2} \\
R^{1/2}
\end{bmatrix} \begin{bmatrix} \Phi_x \\
\Phi_u
\end{bmatrix} - I \right\|_F \right\}
\end{array}
\end{aligned}
\tag{40}
\]
which implies
\[
\begin{aligned}
\begin{array}{l}
\min_{\Phi_x, \Phi_u} \left\{ \left\| \begin{bmatrix} Q^{1/2} \\
R^{1/2}
\end{bmatrix} \begin{bmatrix} \Phi_x \\
\Phi_u
\end{bmatrix} - I \right\|_F \right\}
\end{array}
\end{aligned}
\tag{41}
\]
where \(\mathbf{1}^n_{T \times nT}\) and \(\mathbf{1}^{nT \times nT}\) denote the \(nT \times nT\)-dimensional low-triangular matrix and \(nT \times nT\)-dimensional column vector whose components are all one, respectively. Now, using (41), P4 can be reformulated as
\[
\begin{aligned}
\begin{array}{l}
\min_{\Phi_x, \Phi_u} \left\{ \left\| \begin{bmatrix} Q^{1/2} \\
R^{1/2}
\end{bmatrix} \begin{bmatrix} \Phi_x \\
\Phi_u
\end{bmatrix} - I \right\|_F \right\}
\end{array}
\end{aligned}
\tag{42}
\]
which can be rewritten as the following joint chance-constrained programming (JCCP) problem:
\[
\begin{aligned}
\begin{array}{l}
\min_{\Phi_x, \Phi_u} \left\{ \left\| \begin{bmatrix} Q^{1/2} \\
R^{1/2}
\end{bmatrix} \begin{bmatrix} \Phi_x \\
\Phi_u
\end{bmatrix} - I \right\|_F \right\}
\end{array}
\end{aligned}
\tag{43}
\]

We assume, for simplicity, that $\Delta_{i,j} \in \Delta_s, \forall i,j = 0, ..., T$. Now, the robust counterpart of $P_6$ is:

$$\begin{align*}
\min_{\Phi_x, \Phi_u} & \quad \max_{\Delta_{i,j} \in \Delta_s} \sqrt{R_{i,j}} \left\| \begin{bmatrix} Q_{i,j}^{1/2} & R_{i,j}^{1/2} \end{bmatrix} \begin{bmatrix} \Phi_x & \Phi_u \end{bmatrix} \right\|_F \\
\text{s.t.} & \quad \{A_{i,j} \leq \frac{2\varepsilon}{\sqrt{N}} T_i, i \geq j, i,j = 1, ..., T \} \geq 1 - \epsilon
\end{align*}$$

(45)

Note that the uncertain constraints in $P_7$ (45) are linear inequalities, but involve an infinite number of constraints, since $\Delta$ is uncountable. As a data-driven relaxation of $P_7$ with a finite ($N$-dimensional) set of all possible realizations. Note that this approach results in a less conservative solution compared to a robust approach. To this end, $P_7$ should be first rephrased in epigraph form [40] as

$$\begin{align*}
\min_{\Phi_x, \Phi_u, \alpha} & \quad \alpha \\
\text{s.t.} & \quad \sqrt{R_{i,j}} \left\| \begin{bmatrix} Q_{i,j}^{1/2} & R_{i,j}^{1/2} \end{bmatrix} \begin{bmatrix} \Phi_x & \Phi_u \end{bmatrix} \right\|_F \leq \alpha \\
& \quad \{A_{i,j} \leq \frac{2\varepsilon}{\sqrt{N}} T_i, i \geq j, i,j = 1, ..., T \} \geq 1 - \epsilon
\end{align*}$$

(46)

Now, the main idea is to use the following sampled counterpart (i.e., the scenario-based problem) instead of the hard optimization $P_8$.

$$\begin{align*}
\Phi_x^{SC}, \Phi_u^{SC} = \arg\min_{\Phi_x, \Phi_u, \alpha} & \quad \alpha \\
\text{s.t.} & \quad \sqrt{R_{i,j}} \left\| \begin{bmatrix} Q_{i,j}^{1/2} & R_{i,j}^{1/2} \end{bmatrix} \begin{bmatrix} \Phi_x & \Phi_u \end{bmatrix} \right\|_F \leq \alpha \\
& \quad \{A_{i,j} \leq \frac{2\varepsilon}{\sqrt{N}} T_i, i \geq j, i,j = 1, ..., T \} \geq 1 - \epsilon
\end{align*}$$

(47)

where $\Delta_k, k = 1, ..., N$ are i.i.d. samples extracted. Moreover, the near-optimal solutions $\Phi_x^{SC}$ and $\Phi_u^{SC}$ of this optimization problem are random variables that depend on the random extractions $\Delta_1, ..., \Delta_N$.

**Remark 1:** It is worth noting that $P_9$ is efficiently solvable since it is now a convex optimization problem with a finite number of constraints.

The following standard assumption is routinely made in the literature on the scenario approach [41]–[43].

**Assumption 1:** $\forall N \geq N_{SC}^*$, $P_9$ has a unique optimal solution $\Phi_{SC}^* := \{\Phi_x^{SC}, \Phi_u^{SC}\}$.

**Theorem 3:** Under Assumption 1 and given $\beta \in (0, 1)$, if the number of scenarios $N$ satisfies the relation

$$N \geq N_{SC}^* = \left\lceil \frac{4}{\beta} \log \frac{1}{1-\beta} + n(n+n)T^2 \right\rceil$$

(48)

then $\Phi_{SC}^*$ satisfies the chance-constrained program [47] with confidence $(1 - \beta)$.

**Proof.** The proof follows from the key results in [37] (i.e., Theorem 1 and Corollary 1) and [44] (i.e., Proposition 2.1).

**Remark 2:** Note that Theorem 3 states that the solution $\Phi_{SC}^*$ is feasible for all the constraints in $P_9$ with high probability $(1 - \beta)$, except possibly for those in a set having probability measure smaller than $\epsilon \in (0, 1)$ [45]. Despite any probability distribution on the uncertainties, Theorem 3 provides a promising tool to compute a sufficient number of scenarios $N_{SC}^*$ that guarantee a certain level of robustness. In practice, the $\beta$ value can be fixed at a pretty small number (e.g., $10^{-7}$), without increasing the number of scenarios significantly [41].

## IV. Simulation

The efficiency of the proposed algorithm is verified using the following linear dynamical system given by

$$x_{t+1} = (0.8 + \varepsilon_t)x_t + 0.5u_t$$

and the performance function is chosen as $J = \sum_{t=0}^{T} (u_t^2 + \varepsilon_t^2)$. The i.i.d. random variables $\{\varepsilon_t\}_{t=1}^{T}$ are generated by a truncated normal distribution with mean $\mu = 0$ and covariance $\sigma = 0.5$.

The trajectory of state and the designed control input for the case of 1000 scenario samples are displayed in Figs. [1a] and [1b], respectively. Fig. [1a] shows the performance as a function of number of scenarios. Choosing more scenario samples, as seen in Fig. [1a], reduces the mean and variance of performance significantly and, as a result, will lead to better results.

## V. Conclusion

In this paper, a data-driven method is developed for solving the closed-loop state-feedback control of a discrete-time LQR problem for systems affected by multiplicative norm bounded model uncertainty. To synthesize a tractable robust state feedback policy, first, we adopted the recently developed system-level synthesis framework to recast the problem at hand over system responses. Then, we reformulated this optimization problem as a chance-constrained program (CCP) in which the guarantees are intended in a probabilistic sense. To approximately solve the CCP without the requirement of knowing the probabilistic distribution of the uncertainty in the system dynamics matrices, we utilized the scenario approach that provides probabilistic guarantees based on a finite number of samples and results in a convex optimization program with moderate computational complexity. Finally, numerical simulations were presented to illustrate the theoretical findings.

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Fig. 1: The performance, state, and control trajectories. (a) The performance function versus the number of scenarios $N$. (b) System state trajectory, and (c) control input $u$. The shaded area and dashed line represent the results between [10%, 90%] quantiles and the means across 25 independent experiments.

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