On the Hyper-Kloosterman Codes Over Galois Rings

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Abstract

The Hyper-Kloosterman code was first defined over finite fields by Chinen-Hiramatsu, see (Chinen, & Hiramatsu, 2001). In the present paper we define the Hyper-Kloosterman codes over Galois rings \( R(p', m) \). We show that this code is the trace of linear code over \( R(p', m) \). By the Hyper-Kloosterman sums over Galois rings, we determine the Hamming weight of any codeword of this code over Galois rings.

Keywords: characters sums, Hyper-Kloosterman sums, Hyper-Kloosterman codes, trace codes, Galois rings

1. Introduction

It all starts with a paper by Kloosterman in 1926 (Kloosterman, 1926), in his study of certain positive definite integral quadratic forms. In his paper, Kloosterman, introduced certain exponential sum (since then known as the Kloosterman sum). Since then Kloosterman sums have enjoyed much attention of the finite fields. Some of this interest is due to their applications in cryptography and coding theory; see for example: the search for the number of solution of certain equations over finite fields, the distribution of values of Kloosterman sums, the divisibility properties of classical binary quadratic forms. In his paper, Kloosterman, introduced certain exponential sum (since then known as the Kloosterman sum).

2. Preliminaries

Some preliminaries on Galois rings are given below. For more details, the reader is referred to (Shuqin, & Han, 2004). Let \( e \geq 1 \) be a fixed integer, \( p \) a prime number. A monic polynomial \( h(x) \in \mathbb{Z}_p[x] \) is said to be a basic irreducible polynomial of degree \( m \) if \( h(x) \mod p \in \mathbb{Z}_p[x] \) is a monic irreducible polynomial of degree \( m \). Galois ring \( R_{e,m} = GR(p', m) \) is the unique unramified extension of degree \( m \) over \( \mathbb{Z}_p \) and can be written as

\[
R_{e,m} = GR(p', m) = \mathbb{Z}_p[X]/(h(X))
\]

where \( h(x) \) is a basic irreducible polynomial of degree \( m \) over \( \mathbb{Z}_p \). The ring \( R_{e,m} \) is a local ring with unique maximal ideal \( pR_{e,m} \). The unit set \( R_{e,m}^* = R_{e,m}/pR_{e,m} \) in \( R_{e,m} \) is a multiplicative group of order

\[
|pR_{e,m}^*| = (p^m - 1)p^{m(e-1)} = p^{me} - p^{m(e-1)}.
\]

The set \( R_{e,m}^* \) always contains a cyclic group of order \( p^m - 1 \). In analogy with finite fields, we will call an element a primitive element of the Galois ring \( R_{e,m} \) if it is a generator for this cyclic group. Let \( \gamma_{e,m} \) denote a primitive element in \( R_{e,m} \). Let \( T = \{1, \gamma_{e,m}, \gamma_{e,m}^2, ..., \gamma_{e,m}^{p-2}\} \). The set \( T^* \) is called Teichmuller system. Let \( \mathcal{T} = T^* \cup \{0\} = \{0, 1, \gamma_{e,m}, \gamma_{e,m}^2, ..., \gamma_{e,m}^{p-2}\} \).

For any element \( z \in R_{e,m} \) the \( p \)-adic expansion has giving by:

\[
z = z_0 + p z_1 + ... + p^{e-1} z_{e-1}
\]

where \( z_i \in T^* \).

Let \( \tau \) the Frobenius map of \( R_{e,m} \) over \( \mathbb{Z}_p \) given by

\[
\tau(z) = z_0^p + z_1^p + ... + z_{e-1}^p
\]
where \( z = \sum_{i=0}^{l-1} p^i z_i \in R_{e,m} \) and \( z_i \in T \). As we know \( \tau \) is the generator of Galois group of \( R_{e,m}/\mathbb{Z}_{p^m} \) which is a cyclic group of order \( m \). The trace mapping \( tr_{e,m} : R_{e,m} \rightarrow \mathbb{Z}_{p^m} \) is defined via
\[
tr_{e,m}(x) = x + \tau(x) + \ldots + \tau^{m-1}(x)
\]
for \( x \in R_{e,m} \).

### 3. Hyper-Kloosterman Codes

**Definition 3.1** In a similar way to the Hyper-Kloosterman codes over Galois fields defined for the first time by Chinen-Hiramatsu, we define the hyper-Kloosterman code \( C_l(p^r, m) \) of degree \( l \rightarrow 1 \), \( l \geq 2 \), \( l \) is an integer, over Galois ring, by the image of the map:
\[
\varphi_1 : R_{e,m}^l \rightarrow \mathbb{Z}_{p^m}^{(p^{mr} - p^{r(e-1)} - 1)^{-1}}
\]
where
\[
\varphi_1(a) = \{ Tr(a, x) \}_{x \in (R_{e,m}^e)^{-1}}
\]
where
\[
Tr(a, x) = tr_{e,m}(a_1 x_1 + a_2 x_2 + \ldots + a_{l-1} x_{l-1} + a_l (x_1 \ldots x_{l-1})^{-1})
\]
\( a = (a_1, \ldots, a_l) \in R_{e,m}^l \). Therefore \( C_l(p^r, m) \) is a linear code.

**Remark 3.1** The length of a code \( C_l(p^r, m) \) is a power of the order of the group of invertible elements, \( R_{e,m}^e \cdot (p^{mr} - p^{r(e-1)} - 1)^{-1} = (R_{e,m})^{-1} \).

**Proposition 3.1** The code \( C_l(p^r, m) \) is a linear code over \( \mathbb{Z}_{p^m} \).

**Proof**
\( \varphi_1 \) is a module homomorphism and so his image is a sub-module of \( \mathbb{Z}_{p^m}^{(p^{mr} - p^{r(e-1)} - 1)^{-1}} \). Therefore \( C_l(p^r, m) \) is a linear code.

**Definition 3.2** For a code \( C \) over \( R_{e,m} \), we denote the trace code of \( C \) (over \( \mathbb{Z}_{p^m} \)) by \( tr_{e,m} C \)
\[
tr_{e,m} C := (tr_{e,m} c_1, \ldots, tr_{e,m} c_n)(c_1, \ldots, c_n) \in C
\]

**Proposition 3.2** The code \( C_l(p^r, m) \) is a trace code over Galois ring.

**Proof** Let \( \alpha_1 \) the map defined by:
\[
\alpha_1 : R_{e,m}^l \rightarrow R_{e,m}^e \cdot (p^{mr} - p^{r(e-1)} - 1)^{-1}
\]
\[
\alpha_1(a) = (\alpha_1 x)_{x \in (R_{e,m}^e)^{-1}}
\]
where
\( l \geq 2 \), \( l \) is an integer, \( (a, x) = a_1 x_1 + a_2 x_2 + \ldots + a_{l-1} x_{l-1} + a_l (x_1 \ldots x_{l-1})^{-1} \), \( a = (a_1, \ldots, a_l) \in R_{e,m}^l \) and \( x = (x_1, \ldots, x_{l-1}) \in (R_{e,m}^e)^{-1} \). We see easily \( \alpha_1 \) is a module homomorphism, and so his image is a sub-module. Let \( C_l(p^r, m) = Im(\alpha_1) \), the image of \( \alpha_1 \). \( C_l(p^r, m) \) is a linear code over Galois ring \( R_{e,m} \). By definition of \( \alpha_1 \), we obtained \( C_l(p^r, m) = tr_{e,m} (C_l(p^r, m)) \).

In the following, \( C_l(p^r, m) \) denote the code defined in the previous proof. That is to say:
\[
C_l(p^r, m) = tr_{e,m} (C_l(p^r, m))
\]

Construction of the generator matrix \( C_l(p^r, m) \) of \( C_l(p^r, m) \), where \( C_l(p^r, m) = tr_{e,m} (C_l(p^r, m)) \).

Let \( f_1, f_2, \ldots, f_{(p^{mr} - p^{r(e-1)} - 1)} \) be a fixed ordering of the elements of
\[
(R_{e,m}^e)^{-1} = \{ f_1, \ldots, f_{(p^{mr} - p^{r(e-1)} - 1)} \}
\]
Let \( f_j \in (R_{e,m}^e)^{-1} \), so
\[
f_j = (f_{1,j}, f_{2,j}, \ldots, f_{l-1,j})
\]
Moreover we define $I(x)$ for $x = (x_1, x_2, ..., x_{l-1}) \in (R_{e,m}^*)^{l-1}$ by:

$$I(x) = (x_1, x_2, ..., x_{l-1})^{-1}$$

Let $f^T_j$ the transpose of the vector $f_j$.

Then we form the matrix as follows:

$$\overline{G}_l(p^e, m) = \begin{pmatrix} f^T_1 & f^T_2 & \ldots & f^T_{l-1} \\
I(f_1) & I(f_2) & \ldots & I(f_{l-1}) \end{pmatrix}$$

**Proposition 3.3** The matrix $\overline{G}_l(p^e, m)$ is a generator matrix of the code $\overline{C}_l(p^e, m)$

**Proof**

Let $a = (a_1, ..., a_l) \in R_{e,m}^l$

$$a\overline{G}_l(p^e, m) = (a_1, ..., a_l) \begin{pmatrix} f^T_1 & f^T_2 & \ldots & f^T_{l-1} \\
I(f_1) & I(f_2) & \ldots & I(f_{l-1}) \end{pmatrix}$$

$$= (a_1, ..., a_l) \begin{pmatrix} f_{1,1} & \ldots & f_{1,l} \\
\vdots & \ddots & \vdots \\
(f_{l,1} \ldots f_{l,1})^{-1} & \ldots & (f_{l,l} \ldots f_{l,l})^{-1} \end{pmatrix}$$

$$= [a_1 f_{1,1} + \ldots + a_l f_{l,1}, a_1 f_{1,2} + \ldots + a_l f_{l,2}, \ldots, a_1 f_{1,l} + \ldots + a_l f_{l,l}]$$

Thus, the matrix $\overline{G}_l(p^e, m)$ is a generator matrix of the code $\overline{C}_l(p^e, m)$.

**Remark 3.2** The matrix $\overline{C}_l(p^e, m)$ is the type $(l, (p^me - p^me^{-1})^{l-1}); l$-lengths and $(p^me - p^me^{-1})^{l-1}$-columns.

4. Hamming Weights of the Codewords of $\overline{C}_l(p^e, m)$

**Definition 4.1** Let $C$ a code of length $n$. $x = x_1 x_2 \ldots x_n$ and $y = y_1 y_2 \ldots y_n$ two codewords on code $C$. The Hamming distance of $x$ and $y$, is given by

$$d(x, y) = \# \{i : x_i \neq y_i \}.$$ 

The Hamming weight $W_H$ of a codeword $x = x_1 x_2 \ldots x_n$ of $C$ is the number of non-zero $x_i$ for $1 \leq i \leq n$.

$$W_H(x) = \# \{i : x_i \neq 0 \}$$

**Definition 4.2** Additive characters over Galois rings.

An additive character of $R_{e,m}$ is a homomorphism from the additive group of $R_{e,m}$ to $\mathbb{C}^*$; the multiplicative group of complex field. We define $\psi(a) = \exp(\pi i r(a, e)/p^e)$ for any given element $a$ in $R_{e,m}$ and ”$\exp$” denote the exponential function ; It is easily seen that $\psi$ is an additive character of $R_{e,m}$ called the canonical additive character. For $b \in R_{e,m}$, define $\psi_b(a) = \psi(ba)$: $a \in R_{e,m}$, $\psi$ is also an additive character. In fact, we have:

**Proposition 4.1** ($\psi_b | b \in R_{e,m}$) consists of all the additive characters of $R_{e,m}$.

We present a well-known result, sometimes called the orthogonality of characters, as a proposition for later reference.

**Proposition 4.2** Let $b \in R_{e,m}$; $\psi$ be the canonical additive character of $R_{e,m}$. Then:

$$\sum_{a \in R_{e,m}} \psi_b(a) = \begin{cases} p^me & \text{if } a = 0 \\ 0 & \text{if } a \neq 0 \end{cases}$$

**Definition 4.3** For $l \geq 2$ and any $a \in R^l_{e,m}$ the hyper-Kloosterman sums of degree $l - 1$ are defined by

$$K_{e,m}(a, p) = \sum_{x \in R^l_{e,m}} \exp \left( \frac{2\pi i}{p^e} Tr(a, x) \right)$$
where $Tr(a, x) = tr_{r,m}(a_1x_1 + a_2x_2 + \ldots + a_{l-1}x_{l-1} + a_l(x_1x_2\ldots x_{l-1})^{-1})$ for $a = (a_1, a_2, \ldots, a_l)$ and $x = (x_1, x_2, \ldots, x_{l-1})$, and $tr_{r,m} = trace_{R_{r,m}/Z_{p^e}}$

The Kloosterman sums have been used by several authors to evaluate the Hamming weights of a certain linear code $C(q)$, called the Kloosterman code (see for example (Gilles, 1989) and (Jacques, 1989) ).

**Theorem 4.1** For any codeword $\varphi(a) \in C_l(p^e, m)$, the weight of $\varphi_i(a)$ is given by:

$$W_H(\varphi_i(a)) = (p^{me} - p^{m(e-1)})^{l-1} - \frac{1}{p^e} \sum_{A=0}^{p^e-1} K_{r,m}(Aa, p)$$

**Proof.**

Let ”$\exp$” denote the exponential function. For all $a \in R_{r,m}^*$ and all $x \in (R_{r,m}^*)^{l-1}$, $\exp(Tr(a, x))$ is a $p^e$-th root of unity, and is equal to 1 or not according to $Tr(a, x) = 0$ or not. So we have:

$$\sum_{x \in (R_{r,m}^*)^{l-1}} \exp(\lambda Tr(a, x)) = \begin{cases} p^e & \text{if } Tr(a, x) = 0 \\ 0 & \text{if } Tr(a, x) \neq 0 \end{cases}$$

Therefore

$$\#\{x \in (R_{r,m}^*)^{l-1}, Tr(a, x) = 0\} = \frac{1}{p^e} \sum_{\lambda \in (Z_{p^e})^*} \sum_{x \in (R_{r,m}^*)^{l-1}} \exp(\lambda Tr(a, x))$$

$$= \frac{1}{p^e} \sum_{\lambda \in (Z_{p^e})^*} \sum_{x \in (R_{r,m}^*)^{l-1}} \exp(\lambda Tr(a, x))$$

$$= \frac{1}{p^e} \sum_{\lambda \in (Z_{p^e})^*} K_{r,m}(Aa, p).$$

We know that, the unit set $R_{r,m}^* = R_{r,m} \setminus PR_{r,m}$ in $R_{r,m}$ is a multiplicative group of order

$$\#R_{r,m}^* = (p^m - 1)p^{me-1} = p^{me} - p^{m(e-1)}$$

So the weight $W_H(\varphi_i(a))$ for any codeword $\varphi_i(a)$ is obtained by:

$$W_H(\varphi_i(a)) = \#(R_{r,m}^*)^{l-1} - \#\{x \in (R_{r,m}^*)^{l-1}, Tr(a, x) = 0\}$$

$$= (p^{me} - p^{m(e-1)})^{l-1} - \frac{1}{p^e} \sum_{A=0}^{p^e-1} K_{r,m}(Aa, p).$$

**Example 4.1** For $l = 2$, $p = 2$, $e = 2$ and $m = 1$ we obtained $R_{r,m} = Z_4$ and the trace map $tr_{r,m}$ become identity map from $Z_4^*$. $Z_4^* = Z_4 \setminus 2Z_4 = \{1, 3\}$. Without harming the generality, let’s $x_1 = 1$ and $x_2 = 3$ So $x_1^{-1} = x_1 = 1$ and $x_2^{-1} = x_2 = 3$. $C_2(4, 1)$ it is the image of the map:

$$\varphi_2 : \begin{array}{c} Z_4^2 \\ a = (a_1, a_2) \end{array} \rightarrow \begin{array}{c} Z_4^2 \\ \varphi_2(a) = (a_1x_1 + a_2x_1^{-1}, a_1x_2 + a_2x_2^{-1}) = (a_1 + a_2, 3a_1 + 3a_2) \end{array}$$

We have:

$$\varphi_2(0, 0) = (0, 0)$$
$$\varphi_2(0, 1) = (1, 3)$$
$$\varphi_2(0, 2) = (2, 2)$$
$$\varphi_2(0, 3) = (3, 1)$$
\[
\varphi_2(1, 3) = (0, 0) \\
\varphi_2(1, 2) = (3, 1) \\
\varphi_2(1, 1) = (2, 2) \\
\varphi_2(1, 0) = (1, 3) \\
\]

... etc

We obtained
\[
C_2(4, 1) = \{(0, 0); (2, 2); (1, 3); (3, 1)\}
\]

5. Conclusion and Perspectives

We have generalized the hyper-Kloosterman codes over Galois rings. This code is seen as a trace of a linear code over galois rings. We get the Hamming weight of a codeword by the hyper-kloosterman sums. The first trivial example gives the 1-quasi-cyclic or cyclic codes. We can therefore continue to study the properties of this code, quasy-cyclic properties and others.

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