RIEMANNIAN METRIC REPRESENTATIVES OF THE
STIEFEL-WHITNEY CLASSES

SANTIAGO R. SIMANCA

Abstract. If $M$ is a closed manifold, and $K$ is a smooth triangulation of $M$, Whitney proved that all of the Stiefel-Whitney classes are specified as cochains on the dual cell complex $(K')^*$ assigning the value $1$ mod $2$ to each dual cell. We provide the pair $(M, K)$ with an arbitrary Riemannian metric $g$, and use Whitney’s criteria to show that there are associated representatives of all the Stiefel-Whitney classes $w_1(M), \ldots, w_n(M)$. The representative of $w_1(M)$ is determined by $\det g_{ij}$, the $g_{ij}$s computed in a frame that is locally defined at each dual 1-cell; the representatives of the even classes $w_{2k}(M)$ are determined by the Chern-Gauss-Bonnet density $2k$-form of locally defined totally geodesic oriented $2k$-manifolds with boundary associated to each dual $2k$-cell; and the representatives of the odd classes $w_{2k+1}(M)$ are determined by the hypersurface area form of the boundary sphere of a locally defined totally geodesic oriented $(2k + 1)$ manifold with boundary associated to each dual $(2k + 1)$-cell. If $(M, J, g)$ is Hermitian, we prove that the metric representative of $w_{2k}(M)$ so obtained is the $\mathbb{Z}/2$ reduction of the $k$-th Chern class $c_k(M, J)$ induced by the coefficient homomorphism, and that the metric representative of any odd degree class $w_{2k+1}(M)$ so obtained is trivial in cohomology.

1. The Stiefel-Whitney classes

A smooth triangulation $K$ of a closed manifold $M$ suffices to specify all of the Stiefel-Whitney classes of $M$. For if $K'$ is the barycentric subdivision of $K$, and $(K')^*$ is the cell complex dual to $K'$, then $w_i(M)$ is represented by the mod 2 cochain that assigns $1$ to each dual $i$-cell. This remarkably beautiful, and simple characterization of these cohomology classes was announced by Whitney in [12], but his proof never appeared in print. In 1970, Cheeger provided the only known written proof of such to date [2]. In 1971, Halperin and Toledo gave a proof by characterizing the Stiefel-Whitney homology classes [5], an equivalent version that had been conjectured much earlier by Stiefel [10], and which was the starting point of Whitney, who dualized the statement to the cohomology classes. Sullivan used the homology version to define Stiefel-Whitney classes for more general spaces [11].

When we endow a manifold with a Riemannian metric, the manifold acquires a geometry that we attempt to understand for various reasons, one of which is to be able to try to read off from it the topology. If a canonical metric exists, and determining if this is the case is a very worthwhile, and usually hard problem in its own right, we choose this metric to make the geometry of the manifold the best, so that we can then read off its topology with ease. But the topology of the manifold

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remains fixed, no matter its geometric shape, and any choice of the metric must lead to the same readings of the topology, if at all.

We provide the triangulated manifold \((M, K)\) with an arbitrary Riemannian metric \(g\). A dual \(i\)-cell in \((K')^*\) corresponds to a unique \((n-i)\)-simplex \(\sigma_{n-i}\) in \(K\) that is a face of an \(n\)-simplex \(\sigma_n\) in \(K\), and a sequence of simplices \(\sigma_j, j = n-i, \ldots, n\), where \(\sigma_{j+1}\) succeeds \(\sigma_j, j = n-i, \ldots, n-1\), in the natural order of the simplices of \(K'\). We firstly use the metric to fix compatibly a local positive orientation for all of the simplices in this chain, and relying on Whitney’s characterization of the Stiefel-Whitney classes, we then prove the following:

- Given a 1-cell determined by a simplex \(\sigma_{n-1}\), the 1-cochain defined by computing \(\det g_{ij}\) in a basis of \(TM\) that fixes compatibly the local positive orientation of the simplex represents the first Stiefel-Whitney class \(w_1(M)\).
- Given a 2-cell determined by a simplex \(\sigma_{n-2}\), the fiber of the normal bundle \(\nu(\hat{\sigma}_{n-2})\) of \(\hat{\sigma}_{n-2} \subset M\) through the barycenter determines a local totally geodesic positively oriented smooth 2-disk with boundary. If \(r\) is the intrinsic Ricci tensor of this disk, the evaluation of \(\frac{1}{2\pi^2}r\) as a 2-form over the closure of the 2-disk defines a 2-cochain that represents the second Stiefel-Whitney class \(w_2(M)\).
- Given a 3-cell determined by a simplex \(\sigma_{n-3}\), the fiber of the normal bundle \(\nu(\hat{\sigma}_{n-3})\) of \(\hat{\sigma}_{n-3} \subset M\) through the barycenter determines a local totally geodesic positively oriented smooth 3-disk with boundary a 2-sphere. If \(d\sigma\) is the area form on this sphere, and \(\varepsilon\) is its radius, the limit as \(\varepsilon \searrow 0\) of the evaluation of \(d\sigma/4\pi\varepsilon^2\) over the 2-sphere cycle defines a 3-cochain that represents the third Stiefel-Whitney class \(w_3(M)\).

It becomes clear how to extend the last argument to produce representatives of the odd degree classes \(w_{2k+1}(M), k > 1\). The extension of the second case to finding representatives of all the even degree classes \(w_{2k}(M), k > 1\), is a bit more elaborate, but becomes clear also once we notice that the role that the intrinsic Ricci tensor \(r\) plays in the argument is exactly that played by the curvature 2-form of the Levi-Civita connection of the intrinsic metric on the totally geodesic oriented disk. The extension is then accomplished by using the \(k\)-fold product of this local curvature form in the role that \(r\) plays in the case of \(w_2\). Up to a suitable constant, this is nothing but the locally defined density of the Chern-Gauss-Bonnet theorem for \(2k\)-manifolds with boundary.

When \(M\) carries a complex structure \(J\), and \((M, J, g)\) is Hermitian, by the definition of the metric representative of \(w_{2k}(M)\), we see that this agrees with the mod 2 reduction of the \(k\)-th Chern class \(c_k(M, J)\) induced by the homomorphism \(\mathbb{Z} \to \mathbb{Z}/2\). By studying the relationship between the local positive orientation of the dual cells, and the two orientations on \(M\) determined by \(J\), and its conjugate \(\overline{J}\), we see that the metric representative of \(w_{2k+1}(M)\) is trivial in cohomology.

2. Smooth triangulations, barycentric subdivisions, and dual cell complexes: The class \(w_0(M)\)

We let \(M = M^n\) be a smooth closed \(n\)-dimensional manifold, and \(K\) be a locally finite smooth simplicial complex triangulation of \(M\), so the underlying polytope of \(K\) is \(M\). We recall quickly the notions of barycentric subdivision \(K'\) of \(K\), and dual cell complex \((K')^*\). We refer the reader to [8] for details.
The barycentric subdivision $K'$ of any triangulation $K$ is a naturally oriented simplicial complex. The simplices of $K'$ are all of the form
\[ \hat{\sigma}_1 \hat{\sigma}_2 \ldots \hat{\sigma}_k \]
where $\sigma_1, \ldots, \sigma_k$ are simplices of $K$ such that $\sigma_1 \supset \sigma_2 \supset \cdots \supset \sigma_k$. Here, for any simplex $\sigma$ of $K$, we denote by $\hat{\sigma}$ its barycenter in $K'$, and $\sigma \supset \sigma_j$ means that the simplex $\sigma_j$ is a proper face of the simplex $\sigma$. The vertices of $K'$ are ordered by decreasing dimension of the simplices of the triangulation $K$ of which they are the barycenters. This ordering induces a linear ordering of the vertices of each simplex of $K'$.

Given a simplex $\sigma$ in $K$, the union of all open simplices of $K'$ of which $\hat{\sigma}$ is the initial vertex is the interior $\hat{\sigma}$ of $\sigma$. The block $D(\sigma)$ dual to $\sigma$ is the union of all the open simplices of $K'$ of which $\hat{\sigma}$ is the final vertex. The closed block $\overline{D}(\sigma)$ is the closure of $D(\sigma)$, and coincides with the union of all simplices of $K'$ of which $\hat{\sigma}$ is the final vertex. It is the polytope of a subcomplex of $K'$. We let $\hat{D}(\sigma) = \overline{D}(\sigma) \setminus D(\sigma)$.

The collection $\{D(\sigma)\}_{\sigma \in K}$ of all dual blocks is pairwise disjoint, and their union equals $M$. This collection is therefore a cell complex that we denote by $(K')^*$. An $i$-cell in $(K')^*$ is determined by an $(n-i)$-simplex $\sigma_{n-i}$ in $K$, and is the interior of a simplex of the form $e^i_{\sigma_{n-i}} = [\hat{\sigma}_{n-i}, \ldots, \hat{\sigma}_{n}] \subset \overline{D}(\sigma_{n-i})$ in $K'$, where $\sigma_n \supset \cdots \supset \sigma_{n-i}$.

Notice that if $\sigma$ is a $k$-simplex of $K$, then [64 Theorem 64.1]:

1. $\overline{D}(\sigma)$ is the polytope of a subcomplex of $K'$ of dimension $n-k$.
2. $D(\sigma)$ is the union of the blocks $D(\tau)$ for which $\tau$ has $\sigma$ as a proper face.
3. These blocks have dimension less than $n-k$.
4. If $H_i(K, K \setminus \hat{\sigma}) \cong \mathbb{Z}$ for $i = n$ and vanishes otherwise, then $(\overline{D}(\sigma), \hat{D}(\sigma))$ has the homology of an $(n-k)$-cell modulo its boundary.

By [7 Theorems 1.4 and 1.5], any triangulation $K$ is isotopic to a smooth triangulation whose dual cells form a smooth cell decomposition of $M$. Thus, our assumption above that $K$ has this property does not restrict the generality of our work, and so in [64 above, we actually have that $(\overline{D}(\sigma), \hat{D}(\sigma))$ is topologically an $(n-k)$-cell modulo its boundary.

For the convenience of the exposition, we specify the following cases:

1. If $d_0$ is a dual 0-block, there exists an $n$-simplex $\sigma_n$ in $K$ such that $\overline{d_0} = \overline{D}(\sigma_n) = \hat{\sigma}_n$.
   Thus, any 0-cell $e^0_{\sigma_n}$ in $(K')^*$ is determined by an $n$-simplex $\sigma_n$ in $K$, and it is its barycenter $\hat{\sigma}_n$ in $K'$. Since the vertices in $K'$ have the discrete topology, $\partial \sigma_n = e^0_{\sigma_n}$.
2. If $d_1$ is a dual 1-block, then there exists an $(n-1)$-simplex $\sigma_{n-1}$ in $K$ such that $d_1 = D(\sigma_{n-1})$. Now, $\sigma_{n-1}$ is a face of exactly two $n$ simplices $\sigma_n$ and $\sigma_n^1$, and we have $\overline{d_1} = \overline{D}(\sigma_{n-1}) = \{ [\hat{\sigma}_n^0, \hat{\sigma}_{n-1}], [\hat{\sigma}_n^1, \hat{\sigma}_{n-1}] \}$.
   (Notice that $\sigma_{n-1}$ could appear with any sign in the boundaries $\partial e^0_{\sigma_n}$ and $\partial e^1_{\sigma_n}$.) Thus, any 1-cell $e^1_{\sigma_{n-1}}$ in $(K')^*$ is determined by an $(n-1)$-simplex $\sigma_{n-1}$ in $K$, and it is given by the interior of a 1-simplex of the form $[\hat{\sigma}_n, \hat{\sigma}_{n-1}]$ in $K'$, where $\sigma_n$ is an $n$-simplex in $K$ such that $\sigma_n \supset \sigma_{n-1}$.
3. If $d_2$ is a dual 2-block, then there exists an $(n-2)$-simplex in $K$ such that $d_2 = D(\sigma_{n-2})$. Given any $n$-simplex $\sigma_n$ that has $\sigma_{n-2}$ as a face, there
exists an \((n - 1)\)-simplex \(\sigma_{n-1}\) such that
\[
\sigma_n \succ \sigma_{n-1} \succ \sigma_{n-2},
\]
and we have that
\[
\overline{d}_2 = \overline{D}(\sigma_{n-2}) = \cup_{\sigma_{n-1} \succ \sigma_{n-2}} \{ [\hat{\sigma}_n, \hat{\sigma}_{n-1}, \hat{\sigma}_{n-2}] \}
\]
is the union of all such \(2\)-simplices. Thus, any \(2\)-cell \(e^2_{\sigma_{n-2}}\) in \((K')^*\) is determined by an \((n - 2)\)-simplex \(\sigma_{n-2}\) in \(K\), and it is given by the interior of a \(2\)-simplex of the form \([\hat{\sigma}_n, \hat{\sigma}_{n-1}, \hat{\sigma}_{n-2}]\) in \(K'\), where \(\sigma_n\) and \(\sigma_{n-1}\) are simplices in \(K\) such that \(\sigma_n \succ \sigma_{n-1} \succ \sigma_{n-2}\).

(d) If \(d_3\) is a dual \(3\)-cell, then there exists an \((n - 3)\)-simplex in \(K\) such that \(d_3 = D(\sigma_{n-3})\). Given any \(n\)-simplex \(\sigma_n\) that has \(\sigma_{n-3}\) as a face, there are simplices \(\sigma_{n-1}\) and \(\sigma_{n-2}\) of dimensions \(n - 1\) and \(n - 2\), respectively, such that
\[
\sigma_n \succ \sigma_{n-1} \succ \sigma_{n-2} \succ \sigma_{n-3},
\]
and we have that
\[
\overline{d}_3 = \overline{D}(\sigma_{n-3}) = \cup_{\sigma_{n-1} \succ \sigma_{n-2} \succ \sigma_{n-3}} \{ [\hat{\sigma}_n, \hat{\sigma}_{n-1}, \hat{\sigma}_{n-2}, \hat{\sigma}_{n-3}] \}.
\]

2.1. The Stiefel-Whitney class \(w_0(M)\). The Stiefel-Whitney class \(w_0(M)\) is defined axiomatically as the unit element \(1 \in H^0(M; \mathbb{Z}/2)\) [\text{Axiom 1, p. 37}] by \([\check{a}]\) above, a dual \(0\)-cell is a simplex of the form \(e^0_{\sigma_n} = \hat{\sigma}_n\) in \(K'\), \(\sigma_n\) an \(n\)-simplex in \(K\). Thus, the \(0\)-cochain
\[
(1) \quad e^0_{\sigma_n} = \hat{\sigma}_n \mapsto w_0(\hat{\sigma}_n) := 1 \mod 2
\]
represents \(w_0(M)\).

3. Smoothly triangulated Riemannian manifolds: The classes \(w_1(M)\), \(w_2(M)\), and \(w_3(M)\)

We endow the triangulated manifold manifold \(M\) with an arbitrary Riemannian metric \(g\). The Riemann curvature tensor of \(g\) is \(R^g(X, Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla^g_{[X, Y]})Z\), where \(\nabla^g\) the Levi-Civita connection of \(g\). It is usually expressed as a the \((0, 4)\) tensor \(g(R^g(X, Y)Z, W)\). The Ricci tensor \(r_g(X, Y)\) of \(g\) is the trace of the map \(L \rightarrow R^g(L, X)Y\). The scalar curvature \(s_g\) is the metric trace of \(r_g\). If \((M, J, g)\) is Hermitian, the curvature 2-form \(\Omega^g(X, Y)\) of the connection \(\nabla^g\) is the trace of the map \(L \rightarrow g(R^g(L, JX)Y)\).

Given any dual \(i\)-cell \(e^i_{\sigma_n \succ \cdots \succ \sigma_{n-i}}\) in \((K')^*\) with closure \(\overline{e}^i_{\sigma_n \succ \cdots \succ \sigma_{n-i}} = [\sigma_n, \ldots, \sigma_{n-i}]\), \(\sigma_n \succ \cdots \succ \sigma_{n-i}\), we use the metric \(g\) to fix its local orientation, and to regularize the corners and boundary faces of the block \(\overline{D}(\sigma_{n-i})\), as follows.

We let \(\nu(\hat{\sigma}_{n-i})\) denote the normal bundle of \(\hat{\sigma}_{n-i} \subset \sigma_n\). We have the local Whitney sum decomposition \(T \sigma_n|_{\hat{\sigma}_{n-i}} = T \hat{\sigma}_{n-i} \oplus \nu(\hat{\sigma}_{n-i})\). Since simplices are contractible, their tangent bundles are trivial. Using the metric, we can choose a positively oriented orthonormal frame \(\{e_1, \ldots, e_{n-i}\}\) for \(T \hat{\sigma}_{n-i}\), and then extend it inductively to a positively oriented orthonormal frame \(\{e_1, \ldots, e_n\}\) of \(T \sigma_n|_{\hat{\sigma}_{n-i}}\).
such that, for each $j = 1, \ldots, i$, \{e_1, \ldots, e_{n-i+j}\} is a positively oriented orthonormal frame for the tangent bundle $T\sigma_{n-i+j} |_{\hat{\sigma}_{n-i}}$ over $\hat{\sigma}_{n-i}$ of the $(n-i+j)$-th simplex in between. By this construction, \{e_{n-i+1}, \ldots, e_n\} is a positively oriented orthonormal basis for the fibers of the normal bundle $\nu(\hat{\sigma}_{n-i})$. We say that the basis \{e_1, \ldots, e_n\} fixes compatibly the local positive orientation of the cell $e_{\sigma_{n-i}}$. When the manifold $M$ is oriented, we always choose the positive orientation of the starting \{e_1, \ldots, e_{n-i}\} in the construction to agree with the orientation on it induced by that of $M$. In this manner, the local notion of positive orientation defined by the basis \{e_1, \ldots, e_n\}, and that of $M$ as a manifold, agree with each other.

We consider a tubular neighborhood of $\hat{\sigma}_{n-i}$ in $M$. For some $\varepsilon > 0$, this neighborhood is obtained by applying the exponential map to vectors of norm less than $\varepsilon$ lying in $\nu_p(\hat{\sigma}_{n-i})$, $p \in \hat{\sigma}_{n-i}$, the fiber base. We choose $\varepsilon$ sufficiently small so that this action of the exponential map can be extended by continuity, with continuous inverse, to vectors of norm less or equal than $\varepsilon$. We denote by $E^\varepsilon(\hat{\sigma}_{n-i})$ the resulting tubular neighborhood. It is the total space of a fiber bundle over $\hat{\sigma}_{n-i}$ whose fibers are geodesic open $i$-disks of radius $\varepsilon$ centered at the base points. We denote the fiberwise closure of $E^\varepsilon(\hat{\sigma}_{n-i})$ by $\overline{E}(\hat{\sigma}_{n-i})$, and the fiberwise boundary of this by $\partial E(\hat{\sigma}_{n-i})$. They are the total spaces of fiber bundles over $\hat{\sigma}_{n-i}$ by closed $i$-disks centered at the base points, and $(i-1)$-spheres, respectively.

The fiber of $E^\varepsilon(\hat{\sigma}_{n-i})$ through the barycenter $\hat{\sigma}_{n-i}$, $D^\varepsilon_{\hat{\sigma}_{n-i}}$, is an open $i$-disk with center at $\hat{\sigma}_{n-i}$ whose closure $\overline{D}^\varepsilon_{\hat{\sigma}_{n-i}}$ is a manifold with boundary, the fiber of $\overline{E}(\hat{\sigma}_{n-i})$ over $\hat{\sigma}_{n-i}$. We have that $\partial \overline{D}^\varepsilon_{\hat{\sigma}_{n-i}}$ is the fiber of $\partial E^\varepsilon(\hat{\sigma}_{n-i})$ over the said barycenter base point. By construction, the pair $(\overline{D}^\varepsilon_{\hat{\sigma}_{n-i}}, \partial \overline{D}^\varepsilon_{\hat{\sigma}_{n-i}})$ is homotopically equivalent to the block pair $(\overline{T}(\sigma_{n-i}), \partial(\sigma_{n-i}))$. We have just smoothed out the corners and edges of the latter. We provide this fiber $D^\varepsilon_{\hat{\sigma}_{n-i}}$, which is totally geodesic in $M$, with the metric induced by $g$ on it, which we call $g_{\hat{\sigma}_{n-i}}$. We call the pair $(D^\varepsilon_{\hat{\sigma}_{n-i}}, g_{\hat{\sigma}_{n-i}})$ the smooth $\varepsilon$-Riemannian $\sigma_{n-i}$ block.

Under the inclusion map $i : \hat{\sigma}_{n-i} \rightarrow M$, the pull-back bundle $i^*TM$ decomposes as the Whitney sum $i^*TM = TD^\varepsilon_{\hat{\sigma}_{n-i}} \oplus \nu(D^\varepsilon_{\hat{\sigma}_{n-i}})$. Since parallel transport preserves inner products, we can extend the basis \{e_1, \ldots, e_n\} of $T\sigma_n |_{\hat{\sigma}_{n-i}}$ above, which fixes compatibly the local positive orientation of the cell $e^\varepsilon_{\sigma_{n-i}}$, to an orthonormal frame \{v_1, \ldots, v_n\} for $TM$ defined in a neighborhood of $D^\varepsilon_{\hat{\sigma}_{n-i}}$, and such that, over $D^\varepsilon_{\hat{\sigma}_{n-i}}$, \{v_{n-i+1}, \ldots, v_n\} is an orthonormal basis of $TD^\varepsilon_{\hat{\sigma}_{n-i}}$. We say that this positively oriented orthonormal frame \{v_1, \ldots, v_n\} is compatible with the local positive orientation of the smooth $\varepsilon$-Riemannian $\sigma_{n-i}$ block $(D^\varepsilon_{\hat{\sigma}_{n-i}}, g_{\hat{\sigma}_{n-i}})$. When $M$ is oriented, by construction, the positive orientation of \{v_1, \ldots, v_n\}, and that of $M$ agree with each another.

3.1. **The Stiefel-Whitney class** $w_1(M)$. By (2.1(b)) above, a dual 1-cell $e^1_{\sigma_{n-1}}$, determined by an $(n-1)$-simplex $\sigma_{n-1}$ in $K$, has closure given by a simplex of the form

$$\overline{e^1_{\sigma_{n-1}}} = [\hat{\sigma}_{n}, \hat{\sigma}_{n-1}]$$

in $K'$, where $\sigma_n$ is an $n$-simplex in $K$ such that $\sigma_n \succ \sigma_{n-1}$. 
that by Gauss’ theorem for geodesic triangles \[4\], and a limiting procedure, we conclude

\[ w_1^2(e_{\sigma_{n-1}}) \mapsto \omega_1^2(e_{\sigma_{n-1}}) := [\det(g(e_i, e_j)(\hat{\sigma}_{n-1}))] \mod 2, \]

is a well defined \( \mathbb{Z}/2 \) 1-cochain.

**Theorem 1.** The cochain \( w_1^2 \) in (2) represents the first Stiefel-Whitney class \( w_1(M) \).

**Proof.** By construction, we have that \( \det(g(e_i, e_j)(p)) = 1 \) for any \( p \in \sigma_{n-1} \subset \sigma_n \).

Therefore,

\[ w_1^2(e_{\sigma_{n-1}}) = 1 \mod 2. \]

The theorem follows by Whitney’s characterization of \( w_1(M) \). \( \square \)

### 3.2. The Stiefel-Whitney class \( w_2(M) \)

By \([3, c]\) above, a dual 2-cell \( e_{\sigma_{n-2}}^2 \), determined by an \((n-2)\)-simplex \( \sigma_{n-2} \) in \( K \), has closure that is given by a simplex of the form

\[ \tau_{\sigma_{n-2}} = [\hat{\sigma}_n, \hat{\sigma}_{n-1}, \hat{\sigma}_{n-2}] \]

in \( K' \), where \( \sigma_n \) and \( \sigma_{n-1} \) are simplices in \( K \) such that \( \sigma_n \succ \sigma_{n-1} \succ \sigma_{n-2} \). The pair \((\overline{D}(\sigma_{n-2}), D(\sigma_{n-2}))\) has the homology of a 2-cell modulo its boundary.

We choose any smooth \( \varepsilon \)-Riemannian \( \sigma_{n-2} \) block \((D_{\sigma_{n-2}}^\varepsilon, g_{\sigma_{n-2}})\), and an orthonormal frame \( \{v_1, \ldots, v_n\} \) for \( TM \) that is compatible with its local positive orientation. Thus, \( D_{\sigma_{n-2}}^\varepsilon \), is an open 2-disk, with center at \( \hat{\sigma}_{n-2} \), whose closure is a manifold with boundary, the smooth pair \( (\overline{D}_{\sigma_{n-2}}^\varepsilon, \partial D_{\sigma_{n-2}}^\varepsilon) \) is homotopically equivalent to the block pair \((\overline{D}(\sigma_{n-2}), D(\sigma_{n-2}))\), and the orthonormal frame \( \{v_1, \ldots, v_n\} \) for \( TM \) is defined in a neighborhood of \( D_{\sigma_{n-2}}^\varepsilon \), and is such that, over \( D_{\sigma_{n-2}}^\varepsilon \), \( \{v_{n-1}, v_n\} \) is an orthonormal basis of \( T D_{\sigma_{n-2}}^\varepsilon \). We define a complex structure \( J_{\sigma_{n-2}} \) on this 2-disk by setting \( J_{\sigma_{n-2}} v_{n-1} := v_n \). This complex structure induces the same local positive orientation on the disk that it had already.

By construction, \((D_{\sigma_{n-2}}^\varepsilon, g_{\sigma_{n-2}})\) is a totally geodesic submanifold of \( M \). Hence, by Gauss’ equation, the intrinsic and extrinsic Riemann curvature tensors are the same, and this implies the relation

\[ r_{g_{\sigma_{n-2}}}(X, Y) = r_g(X, Y) - \sum_{i=1}^{n-2} g(R^g(v_i, X)Y, v_i) \]

between the intrinsic and extrinsic Ricci tensors (see \([3, \S 2]\)). Since for dimensional reasons, we have that

\[ r_{g_{\sigma_{n-2}}}(X, Y) = \frac{s_{g_{\sigma_{n-2}}}}{2} g(X, Y), \]

by Gauss’ theorem for geodesic triangles \([4]\), and a limiting procedure, we conclude that

\[ \int_{\sigma_{n-2}} \frac{s_{g_{\sigma_{n-2}}}}{2} g_{\sigma_{n-2}}(J_{\sigma_{n-2}} v_{n-1}, v_{n-1}) d\mu_{g_{\sigma_{n-2}}} = \int_{D_{\sigma_{n-2}}^\varepsilon} \frac{s_{g_{\sigma_{n-2}}}}{2} d\mu_{g_{\sigma_{n-2}}} = 2\pi. \]
Therefore,
\begin{equation}
 e^2_{σ_n-2} \mapsto w^2_2(e^2_{σ_n-2}) := \frac{1}{2\pi} \int_{D_{σ_n-2}^2} \left( r_g(Jσ_{n-2}v_{n-1}, v_{n-1}) - \sum_{i=1}^{n-2} g(R^g(v_i, Jσ_{n-2}v_{n-1}, v_{n-1}, v_i)) dμ_{g_{σ_n-2}} \right) \mod 2
\end{equation}
is a well defined \( \mathbb{Z}/2 \) 2-cochain.

**Theorem 2.** The cochain \( w^2_2 \) in (4) represents the second Stiefel-Whitney class \( w_2(M) \).

**Proof.** We have that
\begin{equation}
 \frac{1}{2\pi} \int_{D_{σ_n-2}^2} \left( r_g(Jσ_{n-2}v_{n-1}, v_{n-1}) - \sum_{i=1}^{n-2} g(R^g(v_i, Jσ_{n-2}v_{n-1}, v_{n-1}, v_i)) dμ_{g_{σ_n-2}} \right) = 1.
\end{equation}
Therefore,
\begin{equation}
 e^2_{σ_n-2} \mapsto w^2_2(e^2_{σ_n-2}) = 1 \mod 2.
\end{equation}
The theorem follows by Whitney’s characterization of \( w_2(M) \).  

Notice that \( (D_{σ_n-2}^2, Jσ_{n-2}, g_{σ_n-2}) \) is Kähler, and that the curvature form of the Levi-Civita connection \( \nabla^{g_{σ_n-2}} \) is
\begin{equation}
 Ω^{g_{σ_n-2}} = r_{g_{σ_n-2}}(Jσ_{n-2}v_{n-1}, v_n) dμ_{g_{σ_n-2}} = \frac{s_{g_{σ_n-2}}}{2} dμ_{g_{σ_n-2}}.
\end{equation}
When \( (M^{2m}, J, g) \) is Hermitian, if \( i : D_{σ_n-2}^2 \to M \) is the inclusion map, by construction we have that the tensors \( i^*J \) and \( Jσ_{n-2} \), and \( i^*dμ_g \) and \( dμ_{g_{σ_n-2}} \), coincide, respectively, and we have that
\begin{equation}
 i^*Ω^g = (r_g(Jv_{n-1}, v_n) - \sum_{i=1}^{n-2} g(R^g(v_i, Jv_{n-1})v_n, v_i))(i^*dμ_g).
\end{equation}
In this case, (3) simply reads [9, Eq. (3)]
\begin{equation}
 Ω^{g_{σ_n-2}} = i^*Ω^g.
\end{equation}

**Corollary 3.** Suppose that \( (M, J, g) \) is a Hermitian manifold. Then the cochain \( w^2_2 \) in (4) is the \( \mathbb{Z}/2 \) reduction of the first Chern class \( c_1(M, J) \).

**Proof.** As integral classes, we know that
\begin{equation}
 c_1 = \frac{1}{2\pi}[Ω^g].
\end{equation}
Since (3) is the fact that for totally geodesic 2-submanifolds, identity (5) holds between the curvature forms of the intrinsic and extrinsic Levi-Civita connections, the result follows by the ensuing definition of \( w^2_2 \) derived from this identity.

The Chern classes were introduced by Chern in 1946 [3], just a year after making seminal contributions to the then understanding of the Gauss-Bonnet theorem, the reason why it is known nowadays as the Chern-Gauss-Bonnet theorem. Notice that when \( (M, J, g) \) is a Kähler manifold, if \( ρ_g(X, Y) = r_g(JX, Y) \) the Ricci form of the metric \( g \), \( Ω^g = ρ_g \), and we have that \( c_1 = \frac{1}{2\pi}[ρ_g] \). The fact that in this case \( (1/2π)ρ_g \) represents \( c_1 \) is basically proven in that paper of Chern, an excellent testimony to the power of Chern’s ideas since the study of both, complex and Kähler structures, were then in their infancies.
3.3. The Stiefel-Whitney class \( w_3(M) \).

By \( \{ \text{(d)} \} \) above, a dual 3-cell \( e_3^{\sigma_{n-3}} \),
determined by an \((n - 3)\)-simplex \( \sigma_{n-3} \) in \( K \), has as closure a simplex of the form
\[
e_3^{\sigma_{n-3}} = [\hat{\sigma}_n, \hat{\sigma}_{n-1}, \hat{\sigma}_{n-2}, \sigma_{n-3}]
\]
in \( K' \), where \( \sigma_n, \sigma_{n-1}, \) and \( \sigma_{n-2} \) are simplices in \( K \) such that \( \sigma_n \succ \sigma_{n-1} \succ \sigma_{n-2} \succ \sigma_{n-3} \).
The pair \((\bar{D}(\sigma_{n-3}), \hat{D}(\sigma_{n-3}))\) has the homology of a 3-cell modulo its boundary.

We choose any smooth \( \varepsilon \)-Riemannian \( \sigma_{n-3} \) block \((D_{\sigma_{n-3}}, g_{\sigma_{n-3}})\). Thus, \( D_{\sigma_{n-3}} \),
is an open 3-disk, with center at \( \hat{\sigma}_{n-3} \), whose closure is a manifold with boundary,
and the smooth pair \((\bar{D}_{\sigma_{n-3}}, \partial \bar{D}_{\sigma_{n-3}})\) is homotopically equivalent to the block pair
\((\bar{D}(\sigma_{n-3}), \hat{D}(\sigma_{n-3}))\).

By continuity, we extend the metric \( g_{\sigma_{n-3}} \) on the oriented 3-disk \( D_{\sigma_{n-3}} \) to a metric on \( \bar{D}_{\sigma_{n-3}} \).
It induces a volume form \( d\mu_{g_{\sigma_{n-3}}} \) on the oriented closed disk,
and a compatibly oriented area form \( d\sigma_{g_{\sigma_{n-3}}} \) on the 2-sphere \( \partial \bar{D}_{\sigma_{n-3}} \).

Since
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{\partial \bar{D}_{\sigma_{n-3}}} d\sigma_{g_{\sigma_{n-3}}} = 4\pi,
\]
the total solid angle subtended by \( \partial \bar{D}_{\sigma_{n-3}} \), we have that
\[
e_3^{\sigma_{n-3}} \mapsto w_3^g(e_3^{\sigma_{n-3}}) := \lim_{\varepsilon \to 0} \frac{1}{4\pi\varepsilon^2} \int_{\partial \bar{D}_{\sigma_{n-3}}} d\sigma_{g_{\sigma_{n-3}}} \mod 2
\]
is a well defined \( \mathbb{Z}/2 \) 3-cochain.

**Theorem 4.** The cochain \( w_3^g \) in \( \{ \text{d} \} \) represents the third Stiefel-Whitney class \( w_3(M) \).

**Proof.** By construction \( w_3^g(e_3^{\sigma_{n-3}}) = 1 \mod 2 \). The result follows by Whitney’s characterization of \( w_3(M) \).

\( \square \)

4. Metric representatives of higher degree Stiefel-Whitney classes

The argument in \( \{ \text{b+2} \} \) for \( w_2(M) \) generalizes to produce representatives of the even degree classes \( w_{2k}(M), k > 1 \).
The argument in \( \{ \text{b} \} \) for \( w_3(M) \) generalizes to produce representatives of the odd degrees classes \( w_{2k+1}(M), k > 1 \).
We prove these assertions here.

For convenience, we denote by \( \omega_d \) the volume of the unit \( d \)-sphere \( S^d \) in Euclidean space \( \mathbb{R}^{d+1} \).

4.1. The Stiefel-Whitney class \( w_{2k}(M) \).

A dual \( 2k \)-cell \( e_{\sigma_{n-2k}}^{2k} \), determined by an \((n - 2k)\)-simplex \( \sigma_{n-2k} \) of \( K \), has closure a \( 2k \)-simplex of the form
\[
e_{\sigma_{n-2k}}^{2k} = [\hat{\sigma}_n, \ldots, \hat{\sigma}_{n-2k}]
\]
in \( K' \), where \( \sigma_n \succ \sigma_{l+1} \succ \ldots \succ \sigma_{n-2k} \). The block pair \((\bar{D}(\sigma_{n-2k}), \hat{D}(\sigma_{n-2k}))\)
has the homology of a \( 2k \)-pair modulo its boundary.

We choose any smooth \( \varepsilon \)-Riemannian \( \sigma_{n-2k} \) block \((D_{\sigma_{n-2k}}, g_{\sigma_{n-2k}})\), and an orthonormal frame \( \{v_1, \ldots, v_n\} \) for \( TM \) that is compatible with its local positive orientation.
Then, \( D_{\sigma_{n-2k}} \) is an open oriented \( 2k \)-disk with center at \( \hat{\sigma}_{n-2k} \) whose closure is a manifold with boundary.
The smooth pair \((\bar{D}_{\sigma_{n-2k}}, \partial \bar{D}_{\sigma_{n-2k}})\) is homotopically equivalent to the block pair \((\bar{D}(\sigma_{n-2k}), \hat{D}(\sigma_{n-2k}))\). The orthonormal frame
\{v_1, \ldots, v_n\} \text{ for } TM \text{ is defined in a neighborhood of } D^\varepsilon_{\sigma_{n-2k}} \text{, and is such that, over } D^\varepsilon_{\sigma_{n-2k}}, \{v_{n-2k+1}^{j}\}_{j=1}^{2k} \text{ is a positive orthonormal basis of } T D^\varepsilon_{\sigma_{n-2k}}. \text{ We define a complex structure } J_{\sigma_{n-2k}} \text{ on this } 2k\text{-disk by setting } J_{\sigma_{n-2k}}v_{n-2k+(2j+1)} := v_{n-2k+(2j+2)}, j = 0, \ldots, k-1. \text{ This complex structure induces the same local positive orientation on the disk that it had already.}

By construction, \( (D^\varepsilon_{\sigma_{n-2k}}, g_{\sigma_{n-2k}}) \) is a totally geodesic submanifold of \( M \), and with the metric extended continuously to the closed disk, \( (\overline{D}^\varepsilon_{\sigma_{n-2k}}, J_{\sigma_{n-2k}}, g_{\sigma_{n-2k}}) \) is Hermitian. We let \( \Omega^{\sigma_{n-2k}} \) be the curvature 2-form of the Levi-Civita connection \( \nabla^{\sigma_{n-2k}} \). Then, by the Chern-Gauss-Bonnet theorem for polyhedral manifolds of Allendoerfer and Weil [4], we may conclude that

\[
\frac{2k}{(2k)!} \int_{D^\varepsilon_{\sigma_{n-2k}}} \Omega^{\sigma_{n-2k}} \wedge \cdots \wedge \Omega^{\sigma_{n-2k}} = \frac{(2\pi)^k}{(2k-1)(2k-3) \cdots 3 \cdot 1} = \frac{\omega_{2k}}{2},
\]

and therefore, the expression

\[
(7) \quad \varepsilon_{\sigma_{n-2k}}^{2k} \mapsto w_{2k}^\sigma(\varepsilon_{\sigma_{n-2k}}^{2k}) := \frac{2^{k+1}}{\omega_{2k}(2k)!} \int_{D^\varepsilon_{\sigma_{n-2k}}} \Omega^{\sigma_{n-2k}} \wedge \cdots \wedge \Omega^{\sigma_{n-2k}} \text{ mod 2}
\]

is a well defined \( \mathbb{Z}/2 \) 2k-cochain that assigns 1 mod 2 to the dual cell \( \varepsilon_{\sigma_{n-2k}}^{2k} \).

**Theorem 5.** The cochain \( w_{2k}^\sigma \) in \( \mathbb{Z}/2 \) represents the 2k-th Stiefel-Whitney class \( w_{2k}(M) \).

If \( (M^{n=2m}, J, g) \) is Hermitian, and \( i : \overline{D}^\varepsilon_{\sigma_{n-2k}} \to M \) is the inclusion map, by construction we have that the tensors \( i^*J \) and \( J_{\sigma_{n-2k}} \), and \( i^*d\mu_g \) and \( d\mu_{g_{\sigma_{n-2k}}} \), coincide, respectively, and we have that \( i^*\Omega^g = \Omega^{\sigma_{n-2k}} \). Thus, the identity

\[
(8) \quad \Omega^{\sigma_{n-2k}} \wedge \cdots \wedge \Omega^{\sigma_{n-2k}} = (i^*\Omega^g) \wedge \cdots \wedge (i^*\Omega^g)
\]

of the k-fold products of the intrinsic and extrinsic curvature forms holds.

**Corollary 6.** Suppose that \( (M, J, g) \) is Hermitian. Then the cochain \( w_{2k}^\sigma \) in \( \mathbb{Z}/2 \) is the \( k \)-th Chern class \( c_k(M, J) \).

4.2. **The Stiefel-Whitney class \( w_{2k+1}(M) \).** A dual \( (2k+1) \)-cell \( \varepsilon_{\sigma_{n-2k}}^{2k+1} \), determined by an \( (n-2k-1) \)-simplex \( \sigma_{n-2k-1} \) of \( K \), has closure a \( (2k+1) \)-simplex of the form

\[
\overline{\varepsilon}_{\sigma_{n-2k-1}}^{2k+1} = [\hat{\sigma}_n, \hat{\sigma}_{n-1}, \ldots, \hat{\sigma}_{n-2k-1}]
\]

in \( K' \), where any pair \( \sigma_{j+1}, \sigma_j \) of consecutive intermediate simplices are such that \( \sigma_{j+1} \supset \sigma_j \). The block pair \( (\overline{D}(\sigma_{n-2k-1}), \overline{D}(\sigma_{n-2k-1})) \) has the homology of a \( (2k+1) \)-cell modulo its boundary.

We choose any smooth \( \varepsilon \)-Riemannian \( \sigma_{n-2k-1} \) block \( D^\varepsilon_{\sigma_{n-2k-1}}, g_{\sigma_{n-2k-1}} \). Thus, \( D^\varepsilon_{\sigma_{n-2k-1}} \), is an open \( (2k+1) \)-disk, with center at \( \hat{\sigma}_{n-2k-1} \), and whose closure is a manifold with boundary. The smooth pair \( (\overline{D}_{\sigma_{n-2k-1}}, \partial \overline{D}_{\sigma_{n-2k-1}}) \) is homotopically equivalent to the block pair \( (\overline{D}(\sigma_{n-2k-1}), \overline{D}(\sigma_{n-2k-1})) \).

The continuously extended metric \( g_{\sigma_{n-2k-1}} \) to \( \overline{D}_{\sigma_{n-2k-1}} \) induces a volume form \( d\mu_{g_{\sigma_{n-2k-1}}} \) on it, and a compatibly oriented hypersurface area form \( d\sigma_{g_{\sigma_{n-2k-1}}} \) on the \( 2k \)-sphere \( \partial \overline{D}_{\sigma_{n-2k-1}} \). Since we have

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{2k}} \int_{\overline{D}_{\sigma_{n-3}}} d\sigma_{g_{\sigma_{n-3}}} = \frac{\pi^k}{(2k-1)(2k-3) \cdots 3 \cdot 1} = \omega_{2k},
\]
the total solid angle subtended by $\partial D_{\sigma_n-2k-1}$), the expression

$$
(9) \quad \epsilon_{\sigma_n-2k-1}^{2k+1} \mapsto w_2^{2k+1}(: \epsilon_{\sigma_n-2k-1}^{2k+1}) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{2k+2}} \int_{\partial D_{\sigma_n-2k-1}} d\sigma_{\sigma_n-2k-1} \mod 2
$$

is a well defined $\mathbb{Z}/2 (2k+1)$-cochain assigning the value 1 in $\mathbb{Z}/2$ to the dual cell $e_{\sigma_n-2k-1}^{2k+1}$.

**Theorem 7.** The cochain $w_2^{2k+1}$ in (9) represents the $(2k+1)$th Stiefel-Whitney class $w_{2k+1}(M)$.

All the odd dimensional dual cells come in pairs: Given the cell $e_{\sigma_n-2k-1}^{2k+1}$ associated to the chain of simplices $\sigma_n \succ \sigma_{n-1} \succ \cdots \succ \sigma_{2k-1}$, there is exactly one other $(2k+1)$-cell $\epsilon_{\sigma_n-2k-1}^{2k+1}$ determined by $\sigma_{2k+1}$ that is associated to a chain of simplices of the form $\hat{\sigma}_n \succ \sigma_{n-1} \succ \cdots \succ \sigma_{2k-1}$, where all but the first of the simplices in the latter chain are the same as those in the former one. For the $(n-1)$-simplex $\sigma_n$ in the first chain is a face of exactly two simplices of top dimension, $\sigma_n$, and a second one, which we call $\hat{\sigma}_n$, and use it to form the second chain. Notice that if $(M, K)$ is an oriented homology $n$-manifold, and we orient all the $n$-simplices in $K$ so that $\gamma = \sum \sigma_n$ is the cycle 1 in $H_n(M; \mathbb{Z})$, and orient the other simplices of $K$ arbitrarily, then the orientations of $\sigma_n$ and $\hat{\sigma}_n$ are such that the boundary $(n-1)$-chain $\partial \sigma_n + \partial \hat{\sigma}_n$ has coefficient zero on $\sigma_{n-1}$. Thus, if $\partial \sigma_n$ has coefficient 1 on $\sigma_{n-1}$, then $\partial \hat{\sigma}_n$ has coefficient $-1$ on it, or vice versa. (Naturally, when $M$ is an oriented manifold, the orientation of $M$ as a homological $n$-manifold that we use, and the orientation of $M$ as a manifold, agree with each other.)

Let us assume that $(M^{n=2m}, J, g)$ is Hermitian. If $k = 0$, since $w_1^2$ maps any dual 1-cell to 1, and the local positive orientations of simplices are all compatible with the orientation on $M$, we conclude that the local compatible positive orientations of $e_{\sigma_n-1}^1$ and $e_{\hat{\sigma}_n-1}^1$ must be, in turn, compatible with each other, and since this conclusion is independent of $\sigma_{n-1}$, compatible with the local positive orientation of any $(n-1)$-dual cell. Therefore, $w_1^2 = 1$ in cohomology. By an induction on $k$, working on the $(n-2k)$-skeleton at the time (skeleton on which $J$ induces a natural orientation compatible with that of $M$ as a whole), we handle all the possible choices of intermediate simplices larger than $\sigma_{n-2k+1}$ in the chain $\sigma_n \succ \sigma_{n-1} \succ \cdots \succ \sigma_{2k-1}$ for a given $\sigma_{2k-1}$, and conclude that any $w_{2k+1}^2 = 1$ in cohomology also.

**Theorem 8.** Suppose that $(M, J, g)$ is Hermitian. Then the cochain $w_2^{2k+1}$ in (9) is trivial in cohomology, and so any odd degree Stiefel-Whitney classes $w_{2k+1}(M)$ is trivial.

Notice that starting step in the induction argument above is really dependent on the orientation of $M$ only, so if we apply it to the cochain $w_1^2$ in (2) for oriented manifolds of any dimension, we would conclude then that $w_1^2$ is trivial in cohomology, and therefore, so would be $w_1(M)$.

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DEPARTMENT OF MATHEMATICS, COURANT INSTITUTE OF MATHEMATICAL SCIENCES, 251 MERCER ST., NEW YORK, NY 10012
E-mail address: srs2@cims.nyu.edu