LUNA-VUST INVARIANTS OF COX RINGS
OF SPHERICAL VARIETIES

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Abstract. Given the Luna-Vust invariants of a spherical variety, we determine the
Luna-Vust invariants of the spectrum of its Cox ring. In particular, we deduce an
explicit description of the divisor class group of the Cox ring. Moreover, we obtain a
combinatorial proof of the fact that the spherical skeleton determines the Cox ring.
This is a known fact following also from the description of the Cox ring due to Brion.
Finally, we show that a conjecture on spherical skeletons can be reduced to the case of
a factorial affine spherical variety with a fixed point.

1. Introduction

Let $X$ be a normal irreducible algebraic variety over an algebraically closed field $K$ of
characteristic zero and assume that the divisor class group $\text{Cl}(X)$ is finitely generated.
The Cox ring of $X$ is the graded $K$-algebra defined as
$$\mathcal{R}(X) := \bigoplus_{[D] \in \text{Cl}(X)} \Gamma(X, \mathcal{O}(D)),$$
where some care has to be taken in order to define the multiplication law. In order to
obtain a well-defined Cox ring, it is in general necessary to assume that $X$ has only
constant invertible global functions, i.e. $\Gamma(X, \mathcal{O}^*) = K^*$. If the Cox ring $\mathcal{R}(X)$
is finitely generated, then its spectrum $\overline{X} := \text{Spec} \mathcal{R}(X)$ is a normal irreducible affine algebraic
variety with the action of a quasitorus $\mathbb{T}$ having character group $\chi(\mathbb{T}) \cong \text{Cl}(X)$. Moreover,
there exists an open subset $\tilde{X} \subseteq \overline{X}$ with complement of codimension at least 2 with a
good quotient $\pi: \tilde{X} \to X$ for the $\mathbb{T}$-action. For details, we refer to the comprehensive
reference [ADHL15], in particular to Sections 1.4 and 1.6.

In this paper, we consider the case where $X$ is a spherical variety, i.e. where $X$ is
-equipped with the action of a connected reductive algebraic group $G$ such that $X$
contains a dense orbit for a Borel subgroup $B \subseteq G$. The Cox ring of a spherical variety has been
described and in particular shown to be finitely generated by Brion in [Bri07].

In order to obtain a $G$-action on the Cox ring, there are two approaches. If we want to
obtain a canonical action, the definition of the Cox ring has to be altered. This method
was introduced by Brion in [Bri07 Section 4], where an equivariant version of the Cox
ring of a spherical variety is defined. It is graded by the group $\text{Cl}^G(X)$ of isomorphism
classes of $G$-linearized divisorial sheaves on $X$ instead of the usual class group $\text{Cl}(X)$.

On the other hand, using [ADHL15 Section 4.2], it is possible to find a surjective
homomorphism of connected reductive algebraic groups $\overline{G} \to G$ and a $\overline{G}$-action on $\overline{X}$
(the spectrum of the usual Cox ring) such that $\overline{X}$ becomes a spherical $\overline{G}$-variety and the

2010 Mathematics Subject Classification. 14M27, 14L30.
The Cox ring $\mathcal{R}(X)$ is factorial. In this case, we can define a function $f_\lambda \in K[\lambda]$ that there exists a regular function $f_\lambda \in K[\lambda]$ with $\text{div} s = \text{div} f_\lambda$. As we always assume $K[\lambda]^* = K^*$, the regular function $f_\lambda$ is determined up to a constant factor, hence its weight $\lambda \in \mathcal{M}$ is uniquely determined. In this case, we can define $\varphi(X)$ as follows.
Definition 1.2. Let $X$ be a factorial affine spherical $G$-variety with $K[X]^* = K^*$. We define
\[
\varphi(X) := \sup \left\{ \sum_{D \in \Delta} (\rho(D), \vartheta) : \vartheta \in (\lambda + T) \cap \text{cone}(M^+) \right\} - \text{rank } X.
\]

We will see in Remark [5.4] that this definition is in agreement with the more general definition [GH15a, Definition 5.3]. Then the restriction of [GH15a, Conjecture 5.5] to the setting of factorial affine varieties is the following:

Conjecture 1.3. Let $X$ be a factorial affine spherical $G$-variety with a $G$-fixed point. Then we have
\[
\varphi(X) \leq \dim X - \text{rank } X,
\]
where equality holds if and only if $X$ is isomorphic to an affine space.

In fact, this involves no loss of generality:

Theorem 1.4. Conjecture 1.3 is equivalent to [GH15a, Conjecture 5.5].

2. The Luna-Vust invariants of the spectrum of the Cox ring

We continue to use the notation of the previous section. Let $T \subseteq B$ be a maximal torus. We denote by $R \subseteq \mathfrak{X}(T) \cong \mathfrak{X}(B)$ the root system of the reductive group $G$ with respect to $T$ and by $S \subseteq R$ the set of simple roots corresponding to $B$. For every $\alpha \in S$ we denote by $P_\alpha \subseteq G$ the corresponding minimal parabolic subgroup containing $B$.

We denote by $T(S)$ the power set of $S$. The set $\Delta$ of $B$-invariant prime divisors in $X$ is equipped with the map $\varsigma : \Delta \to T(S)$ defined by $\varsigma(D) := \{ \alpha \in S : P_\alpha \cdot D \neq D \}$. From a combinatorial point of view, we will consider $\Delta$ as an abstract finite set equipped with the two maps $\rho$ and $\varsigma$. Note that a $B$-invariant prime divisor $D \in \Delta$ is $G$-invariant, i.e. $D \in \Delta \setminus D$, if and only if $\varsigma(D) = \emptyset$.

We now recall some notation from [GH15a, Section 6]. As always, we assume that the spherical variety $X$ has only constant invertible global functions, i.e. $\Gamma(X, \mathcal{O}^*) = K^*$. We denote by $\mathcal{R}(X)$ the Cox ring of $X$. We write $\overline{X} := \text{Spec } \mathcal{R}(X)$ and denote by $\hat{X} \subseteq \overline{X}$ the open subset (with complement of codimension at least 2) such that there exists a good quotient $\pi : \hat{X} \to X$ for the action of the quasitorus $T := \text{Spec } K[\mathcal{O}(X)]$.

According to [ADHL15, Theorem 4.2.3.2], there exist a connected reductive algebraic group $G'$, a finite epimorphism $\varepsilon : G' \to G$, and a $G'$-action on $\hat{X}$ (hence also on $\overline{X}$) commuting with the $T'$-action such that $\pi : \hat{X} \to X$ becomes $G'$-equivariant for the action of $\overline{G} := G' \times T^0$ on $X$ via $\varepsilon : \overline{G} \to G$. There is a Borel subgroup $B \subseteq \overline{G}$ and a maximal torus $T' \subseteq B$ such that $\varepsilon(B) = B$ and $\varepsilon(T) = T$. We identify the corresponding root system and set of simple roots of $G$ and $\overline{G}$ via $\varepsilon$.

The $\overline{G}$-variety $\overline{X}$ is spherical, and we use the same notation as we have introduced for $X$, but we add a line over the respective symbol. For instance, we have the weight lattice $\overline{M}$, the set of $B$-invariant prime divisors $\overline{\Delta}$, and the valuation cone $\overline{V}$.

We have a natural injective pullback map $\pi^* : M \to \overline{M}$ and the corresponding dual map $\pi_* : \overline{N} \to N$. For $D \in \Delta$ the canonical section $1_D \in \Gamma(X, \mathcal{O}(D)) \to \mathcal{R}(X)$ is $B$-semi-invariant and is defined in $\mathcal{R}(X)$ up to a constant factor. We denote its weight by $e_D \in \overline{M}$.
Proposition 2.1. The weights $e_D$ for $D \in \Delta$ generate the weight monoid $\mathcal{M}^+$ of the affine spherical variety $\overline{X} = \text{Spec} \mathcal{R}(X)$. Moreover, they form a $\mathbb{Z}$-basis of $\mathcal{M}$.

Proof. According to [ADHL15, Theorem 4.5.4.6], the $e_D$ are linearly independent and generate the weight monoid $\mathcal{M}^+$ in $\mathcal{M}_\mathbb{Q}$, which implies $\mathcal{R}(X)^* = \Gamma(\overline{X}, \mathcal{O}^*) = K^*$. □

Proposition 2.2. We have $\mathcal{R}(X)^* = K^*$.

Proof. According to Proposition 2.1, the cone spanned by the weight monoid $\mathcal{M}^+$ in $\mathcal{M}_\mathbb{Q}$ is strictly convex, which implies $\mathcal{R}(X)^* = \Gamma(\overline{X}, \mathcal{O}^*) = K^*$. □

We denote by $(e^*_D)_{D \in \Delta}$ the basis of the lattice $\mathcal{N}$ which is dual to the basis $(e_D)_{D \in \Delta}$ of the lattice $\mathcal{M}$.

Proposition 2.3. For every $D \in \Delta$ we have $\pi_*(e^*_D) = \rho(D)$.

Proof. Let $f_\chi \in K(X)$ be $B$-semi-invariant of weight $\chi \in \mathcal{M}$. Then we have
\[
\text{div } f_\chi = \sum_{D \in \Delta} \langle \rho(D), \chi \rangle D.
\]
According to [ADHL15, Proposition 1.6.2.1], we have $\pi_*(f_\chi) = \text{div } 1_D$, so that
\[
\text{div } \pi_*(f_\chi) = \sum_{D \in \Delta} \langle \rho(D), \chi \rangle \text{div } 1_D.
\]
As we have $\mathcal{R}(X)^* = K^*$ by Proposition 2.2, it follows that $\pi_*(f_\chi)$ coincides with $\prod_{D \in \Delta} 1_D^{\langle \rho(D), \chi \rangle}$ up to a constant factor, i.e. we have
\[
\pi_*(\chi) = \sum_{D \in \Delta} \langle \rho(D), \chi \rangle e_D.
\]
Consequently, for every $D \in \Delta$ we have $\langle \pi_*(e^*_D), \chi \rangle = \langle e_D, \pi_*(\chi) \rangle = \langle \rho(D), \chi \rangle$. □

Lemma 2.4. Every $\overline{B}$-semi-invariant $f \in K(\overline{X})$ is also $T$-semi-invariant.

Proof. Let $f \in K(\overline{X})$ be a $\overline{B}$-semi-invariant rational function. For every $t \in T$ the rational function $t \cdot f$ is $\overline{B}$-semi-invariant of the same weight as $f$ (because the actions of $\overline{G}$ and $T$ commute). Hence $t \cdot f = cf$ for some constant factor $c \in K^*$. □

Remark 2.5. We denote by $\Sigma$ the uniquely determined linearly independent set of primitive elements in $\mathcal{M}$ such that $T = \text{cone}(\Sigma)$. The elements of $\Sigma$ are called the spherical roots of $X$. It is sometimes more useful to consider the set $\Sigma^{sc}$ of spherically closed spherical roots, where the spherical roots are possibly given a different length (for details, see, for instance, [GH15a, Section 2]). In this paper, however, we are mostly concerned with the following subsets, which are the same for $\Sigma$ and $\Sigma^{sc}$:
\[
\Sigma^{a} := \Sigma^{sc} \cap S = \Sigma \cap S,
\]
\[
\Sigma^{2a} := \Sigma^{sc} \cap 2S = \Sigma \cap 2S.
\]

Proposition 2.6. We have $\pi^{-1}_*(\mathcal{V}) = \mathcal{V}$. 4
Proof. According to [Kno91 Corollary 1.5], we have \( \pi_\ast(V) = V \). It remains to verify \( \pi_\ast^{-1}(0) \subseteq V \). Let \( u \in \pi_\ast^{-1}(0) \) and \( \gamma \in \Sigma \). Let \( f_\gamma \in K(X) \) be a \( \mathcal{B} \)-semi-invariant rational function of weight \( \gamma \in \mathcal{M} \). As \( \gamma \) is a linear combination of simple roots with nonnegative coefficients (see, for instance, [Tim11 Table 30.2]), the rational function \( f_\gamma \) is \( \mathbb{T} \)-invariant. It follows from Lemma 2.3 that some power \( f_\gamma^k \) is \( \mathbb{T} \)-invariant, hence \( f_\gamma^k \in K(X) \), i.e. we have \( k \gamma = \pi^\ast(v) \) for some \( v \in \mathcal{M} \). It follows that we have \( \langle u, k \gamma \rangle = \langle u, \pi^\ast(v) \rangle = \langle \pi_\ast(u), v \rangle = \langle 0, v \rangle = 0 \), which implies \( \langle u, \gamma \rangle = 0 \). From the equality \( V = -\mathbb{T}^\vee = -\cone(\Sigma)^\vee \), we now obtain \( \pi_\ast^{-1}(0) \subseteq V \). □

Corollary 2.7. We have \( \pi^\ast(T) = \mathcal{T} \).

Remark 2.8. Let \( 2\alpha \in \Sigma_\alpha^2 \). According to Corollary 2.7, the extremal ray spanned by \( 2\alpha \) in \( T \) is also an extremal ray of \( \mathcal{T} \). We then have either \( 2\alpha \in \Sigma_\alpha^2 \) (which happens if \( \alpha \notin \mathcal{M} \)) or \( \alpha \in \Sigma_\alpha^a \) (which happens if \( \alpha \in \mathcal{M} \)).

Definition 2.9. We define \( S := \{ \alpha \in S : 2\alpha \in \Sigma_\alpha^2 \text{ and } \alpha \in \mathcal{M} \} = \Sigma_\alpha^a \setminus \Sigma_\alpha^a \), which is the set of simple roots appearing in the second case of Remark 2.8.

Remark 2.10. The following general combinatorial properties of the colors explain the importance of the set \( S \). For details, we refer, for instance, to [BL11 Section 1.1]. If \( 2\alpha \in \Sigma_\alpha^2 \), then there is exactly one color \( D \) with \( \alpha \in \varsigma(D) \), and such a colors are said to be of type \( 2a \). If \( \alpha \in \Sigma_\alpha^a \), then there are exactly two colors \( D \) with \( \alpha \in \varsigma(D) \), and such colors are said to be of type \( a \).

If \( 2\alpha \in \Sigma_\alpha^2 \), then \( \alpha \in \varsigma(D) \) implies \( \varsigma(D) = \{ \alpha \} \) and \( \varsigma^{-1}(\{ \alpha \}) = \{ D \} \). In particular, we obtain a bijection between the set \( S \) and the set \( D^S := \{ D \in D : \varsigma(D) \cap S \neq \emptyset \} \). We expect that every color in \( X \) belonging to \( D^S \) should be replaced by two colors in \( \overline{X} \), which will be confirmed in Proposition 2.13.

Lemma 2.11. For every \( D' \in \Delta \) the element \( \overline{p}(D') \in \overline{N} \) is primitive.

Proof. There exists an effective \( \mathbb{T} \)-invariant Weil divisor \( \delta \) on \( \overline{X} \) such that \( D' \) has multiplicity 1 in \( \delta \). Indeed, since \( D' \) is \( \overline{B} \)-invariant and \( \mathbb{T}^0 \subseteq \overline{B} \), we could choose \( \delta := \sum_{D' \in \Delta} D' \) or, alternatively, \( \delta := \sum_{D' \in \Delta} D' \). As every \( \mathbb{T} \)-invariant Weil divisor on \( \overline{X} \) is principal (see [ADHL13 Proposition 1.5.3.3]), there exists a \( \overline{B} \)-semi-invariant regular function \( f_\delta \in C(\overline{X}) = \mathcal{R}(X) \) with \( \div f_\delta = \delta \) of some weight \( \chi \in \mathcal{M} \). As \( D' \) has multiplicity 1 in \( \div f_\delta \), we have \( \langle \overline{p}(D'), \chi \rangle = 1 \). □

Lemma 2.12. For every \( D' \in \Delta \) the element \( \overline{p}(D') \in \overline{N} \) lies in some extremal ray of \( \cone(\mathcal{M}^+) \).\( ^\vee = \cone(\mathcal{E}^+_D : D \in \Delta) \subseteq \overline{N} \).

Proof. For every \( D \in \Delta \) we have \( \langle \overline{p}(D'), e_D \rangle \geq 0 \), and, by [ADHL13 Proposition 1.6.2.1], we have \( \pi^\ast(D) = \div 1_D \). Let \( D_1, D_2 \in \Delta \) with \( \langle \overline{p}(D'), e_{D_1} \rangle > 0 \) and \( \langle \overline{p}(D'), e_{D_2} \rangle > 0 \). As \( 1_{D_1} \) and \( 1_{D_2} \) are \( \text{Cl}(X) \)-prime (see [ADHL13 Proposition 1.5.3.5]), we have

\[
\pi^{-1}(D_1) = \sup \div 1_{D_1} = \mathbb{T} \cdot D' = \sup \div 1_{D_2} = \pi^{-1}(D_2),
\]

i.e. we have \( D_1 = D_2 \), hence \( \overline{p}(D') \) lies in some extremal ray of \( \cone(\mathcal{M}^+) \). □

Proposition 2.13. The \( \overline{B} \)-invariant prime divisors in \( \overline{X} \) are described as follows:
(1) For every \( D \in \mathcal{D}^S \) there exist two distinct colors \( D', D'' \in \mathcal{D} \subseteq \overline{\Delta} \) with 
\[
\overline{\rho}(D') = e_D^* \quad \text{and} \quad \overline{\tau}(D') = \overline{\tau}(D'') = \overline{\varsigma}(D).
\]
Moreover, we have \( \overline{\pi}^*(D) = D' + D'' \).

(2) For every \( D \in \Delta \setminus \mathcal{D}^S \) there exists a divisor \( D' \in \overline{\Delta} \) with
\[
\overline{\rho}(D') = e_D^* \quad \text{and} \quad \overline{\tau}(D') = \overline{\varsigma}(D).
\]
Moreover, we have \( \overline{\pi}^*(D) = D' \).

(3) We have
\[
\sum_{D' \in \overline{\Delta}} D' = \sum_{D \in \Delta} \overline{\pi}^*(D).
\]
This means that all divisors \( D' \) and \( D'' \) occurring in (1) and (2) above are pairwise distinct and that every divisor in \( \overline{\Delta} \) appears as some \( D' \) or \( D'' \).

Proof. It follows from Lemmas 2.11 and 2.12 that for every \( D' \in \overline{\Delta} \) there exists \( D \in \Delta \) with \( \overline{\rho}(D') = e_D^* \).

Let \( D \in \Delta \). According to [ADHL15, Proposition 1.6.2.1], we have \( \overline{\pi}^*(D) = \text{div} \ 1_D \).

For every \( D' \in \overline{\Delta} \) we have \( \nu_{D'}(1_D) = (\overline{\rho}(D'), e_D) \). In view of Lemmas 2.11 and 2.12 this means \( \nu_{D'}(1_D) = 1 \) if \( \overline{\rho}(D') = e_D^* \) and \( \nu_{D'}(1_D) = 0 \) otherwise. In other words, we have
\[
\overline{\pi}^*(D) = \sum_{D' \in \overline{\Delta}, \overline{\rho}(D') = e_D^*} D'.
\]

Let \( D' \in \overline{\Delta} \) with \( \overline{\rho}(D') = e_D^* \). We want to show \( \overline{\tau}(D') = \overline{\varsigma}(D) \). By removing the \( \mathcal{G} \)-orbits of codimension at least 2 from \( X \), we obtain an open \( \mathcal{G} \)-subvariety \( X_1 \subseteq X \) and a geometric quotient \( \pi_1: X_1 \rightarrow X_1 \) since \( X_1 \) is smooth, in particular \( \mathcal{Q} \)-factorial (see [ADHL15, Corollary 1.6.2.7]). We identify the \( \mathcal{B} \)-invariant prime divisors in \( X_1 \) (resp. in \( \pi_1^{-1}(X_1) \)) with those in \( X \) (resp. in \( \overline{X} \)). Let \( \alpha \notin \overline{\varsigma}(D) \). Then \( \overline{\mathcal{P}}_\alpha \) stabilizes \( \pi_1^{-1}(D) \) and hence, as \( \overline{\mathcal{P}}_\alpha \) is connected, the connected component \( D' \subseteq \pi_1^{-1}(D) \), i.e. \( \alpha \notin \overline{\tau}(D') \).

Now let \( \alpha \notin \overline{\tau}(D') \). As \( 1_D \) is \( \text{Cl}(X) \)-prime (see [ADHL15, Proposition 1.5.3.5]), we have \( T \cdot D' = \sup \text{div} \ 1_D = \pi_1^{-1}(D) \), hence \( \pi_1(D') = D \), and therefore \( \alpha \notin \overline{\varsigma}(D) \).

It remains to show that for a fixed \( D \in \Delta \) the number of \( D' \in \overline{\Delta} \) with \( \overline{\rho}(D') = e_D^* \) is exactly 2 in the case \( D \in \mathcal{S} \) and exactly 1 otherwise. First note that we have
\[
\text{cone}(\overline{\rho}(D') : D' \in \overline{\Delta}) = \text{cone}(\mathcal{M}^+) = \text{cone}(e_D^* : D \in \Delta),
\]
so that the number of \( D' \in \overline{\Delta} \) with \( \overline{\rho}(D') = e_D^* \) is always at least 1. For \( \mathcal{G} \)-invariant prime divisors \( D' \), i.e. in the case \( \overline{\tau}(D') = \overline{\varsigma}(D) = \emptyset \), it follows from the Luna-Vust theory of spherical embeddings that the number of \( D' \) with \( \overline{\rho}(D') = e_D^* \) is at most 1. On the other hand, if the prime divisor \( D \) is a color, then the claimed number of \( D' \) with \( \overline{\rho}(D') = e_D^* \) follows from the general combinatorial properties of the colors (see, for instance, [BL11, Section 1.1]). \( \square \)

**Theorem 2.14.** We have \( \text{Cl}(\overline{X}) \cong \mathbb{Z}^S \). In particular, the Cox ring \( \mathcal{R}(X) \) is factorial if and only if \( S = \emptyset \).

**Proof.** This follows directly from the description of the divisor class group of a spherical variety given in [Bri07, Proposition 4.1.1]: If \( \Delta_1 \subseteq \overline{\Delta} \) is a subset with \( \overline{\rho}|_{\Delta_1} \) injective and \( \overline{\rho}(\Delta_1) \) a \( \mathbb{Z} \)-basis of \( \mathcal{N} \), then the set \( \overline{\Delta} \setminus \Delta_1 \) freely generates \( \text{Cl}(\overline{X}) \). \( \square \)
**Remark 2.15.** If the divisor class group $\text{Cl}(X)$ is free, then the finitely generated Cox ring of an arbitrary normal variety is factorial (see [ADHL15 Proposition 1.5.2.5]). In particular, for spherical varieties, we obtain that $\mathcal{S} \neq \emptyset$ implies that $\text{Cl}(X)$ is not free.

3. **The spherical skeleton determines the Cox ring**

According to [GH15a, Section 5], we may assign to the spherical variety $X$ a combinatorial object called its *spherical skeleton*. We continue to use the notation of the previous section. We define $\Lambda_Q := \text{span}_Q \mathcal{T} \subseteq \mathcal{M}_Q$ and write $\rho_Q : \Delta \rightarrow \mathcal{N}_Q$ for the composition of $\rho : \Delta \rightarrow \mathcal{N}$ with the natural map $\mathcal{N} \rightarrow \mathcal{N}_Q$. Note that we have $\mathcal{T} \subseteq \Lambda_Q \subseteq \text{span}_Q R$. We will abbreviate the notation $\rho_Q(D)|_{\Lambda_Q}$ by $\varsigma(D)$.

**Definition 3.1.** The *spherical skeleton* of $X$ is the tuple $\mathcal{R}_X := (R, S, \mathcal{T}, \Delta)$ where $\Delta$ is treated as an abstract finite set equipped with the maps $\varsigma : \Delta \rightarrow \Lambda_Q$ and $\varsigma : \Delta \rightarrow \mathcal{P}(S)$.

This definition is slightly different, but contains exactly the same information as the definition given in [GH15a Section 5]. It can be seen in the proof of [GH15a Theorem 6.11] that every spherical skeleton $\mathcal{R}$ as defined in [GH15a Section 5] can be obtained as $\mathcal{R}_X$ for some spherical variety $X$ with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$.  

**Remark 3.2.** It is possible to recover the set of spherically closed spherical roots $\Sigma^{sc}$ (as well as the subsets $\Sigma^s$ and $\Sigma^{sa}$, see Remark 2.5, but not the ordinary set of spherical roots $\Sigma$) from the spherical skeleton $\mathcal{R}_X$. Recall that $\Sigma^{sc}$ is a linearly independent set of generators for the cone $\mathcal{T}$ and that we have $\langle \rho(\Delta), \Sigma^{sc} \rangle \subseteq \mathbb{Z}$.

**Definition 3.3.** Two spherical skeletons $\mathcal{R}_1 := (R_1, S_1, \mathcal{T}_1, \Delta_1)$ and $\mathcal{R}_2 := (R_2, S_2, \mathcal{T}_2, \Delta_2)$ are said to be isomorphic, written $\mathcal{R}_1 \cong \mathcal{R}_2$, if there exists an isomorphism of root systems $\phi_R : R_1 \rightarrow R_2$ with $\phi_R(S_1) = S_2$ and $\phi_R(\mathcal{T}_1) = \mathcal{T}_2$ as well as a bijection $\phi_\Delta : \Delta_1 \rightarrow \Delta_2$ such that for every $D \in \Delta_1$ we have $\varsigma_1(D) = \varsigma_2(\phi_\Delta(D)) \circ \phi_R|_{\Lambda_Q}$ and $\phi_R(\varsigma_1(D)) = \varsigma_2(\phi_\Delta(D))$.

An important property of the spherical skeleton $\mathcal{R}_X$ is that it determines the Cox ring $\mathcal{R}(X)$ considered as a non-graded $\mathbb{K}$-algebra. This follows from the explicit description of the Cox ring of a spherical variety due to Brion (see [Br07 4.3.2]). We will now give another proof of this fact.

By replacing $G$ with a finite cover, we may assume $G = G^{ss} \times C$ where $G^{ss}$ is semisimple simply-connected and $C$ is a torus. For such actions we consider the following notion.

**Definition 3.4 ([AB04 Definition 4.4]).** The action of $G = G^{ss} \times C$ on the spherical variety $X$ is called *smart* if the kernel of the action is finite, $C$ acts faithfully, and the natural map $C \rightarrow \text{Aut}_G(X)^0$ is an isomorphism.

The following observation is joint work with Johannes Hofscheier.

**Lemma 3.5.** The following statements are equivalent:

1. The $G$-action on $X$ is smart.
2. We have $\text{dim } \Lambda_Q = \text{rank}(\mathcal{M} \cap \mathcal{X}(B^{ss}))$.

**Proof.** See [Hof15 Lemma 4.10.0.8].

For $i \in \{1, 2\}$, let $G_i := G^{ss}_i \times C_i$ be a connected reductive group as above. Let $R_i$ be the root system of $G_i$ with respect to the maximal torus $T_i := T^{ss}_i \times C_i$. Moreover, let
Then there exists an isomorphism of algebraic groups.

Theorem 3.8. 

We add the appropriate index to the respective symbol. For instance, we have the weight

As we have

it follows from Lemma 2.4 that the lattice

and replacing

inclusion

divide

C

can use [Los09, Theorem 1.2].

use [Kno91, Section 4]. If

equivariant isomorphism

Remark 3.7. 

If there exists a isomorphism

isomorphism

X\_i\^o \to X\_2\^o

such that

The assumption that the G\_i-action on X\_i is smart is required for the following result, where we denote by X\_i\^o the open G\_i-orbit in X\_i.

Proposition 3.6. Assume that there exist

(1) an isomorphism of root systems \( \phi_R: R_2 \to R_1 \) with \( \phi_R(\mathfrak{s}_2) = \mathfrak{s}_1 \),

(2) an isomorphism of lattices \( \psi: \mathcal{M}_2 \to \mathcal{M}_1 \) with \( \psi|\mathfrak{t}_2 = \phi_R|\mathfrak{t}_2 \) and \( \psi(\mathfrak{t}_2) = \mathfrak{t}_1 \), and

(3) a bijection \( \tau: \mathcal{D}_2 \to \mathcal{D}_1 \) such that for every \( \mathcal{D} \in \mathcal{D}_2 \) we have \( \rho_2(\mathcal{D}) = \rho_1(\tau(\mathcal{D})) \circ \psi \)

and \( \phi_R(\mathfrak{s}_2(\mathcal{D})) = \mathfrak{s}_1(\tau(\mathcal{D})) \).

Then there exists an isomorphism of algebraic groups \( G_1 \to G_2 \) and a \( G_1\)-\( G_2 \)-equivariant isomorphism \( \phi: X\_1\^o \to X\_2\^o \) such that \( \phi^\circ = \psi \).

Proof. The proof is similar to the proof of [Hof15 Proposition 4.11.0.4].

Remark 3.7. If there exists a \( G_1\)-\( G_2 \)-equivariant isomorphism \( X\_1\^o \to X\_2\^o \), one of the following methods can be used in order to determine whether it extends to a \( G_1\)-\( G_2 \)-equivariant isomorphism \( X\_1 \to X\_2 \): If the colored fans of \( X\_1 \) and \( X\_2 \) are known, one can use [Kno91 Section 4]. If \( X\_1 \) and \( X\_2 \) are affine and the weight monoids are known, one can use [Los99 Theorem 1.2].

As we have assumed \( G_i = G_i\^s \times C_i \), we have \( \overline{C}_i = G_i\^s \times C_i' \times T\_i^0 \) where \( C_i' \to C_i \) is a finite cover. Moreover, since we have assumed that \( C_i \) acts faithfully on \( X_i \), we may divide \( C_i' \times T\_i^0 \) through the kernel of its action to obtain a torus \( \overline{C}_i \) together with an inclusion \( T\_i^0 \to \overline{C}_i \). Finally, as we have assumed that the action of \( G_i \) on \( X_i \) is smart, replacing \( G_i \) by \( G_i\^s \times \overline{C}_i \), we obtain a smart action of \( \overline{C}_i \) on \( X_i \) by Lemma 3.5 because it follows from Lemma 2.4 that the lattice \( \mathcal{M} \cap \mathfrak{X}(B_i\^s) \) is of finite index in \( \mathcal{M} \cap \mathfrak{X}(B_i) \).

Theorem 3.8. Assume \( \mathcal{R}_{X\_1} \cong \mathcal{R}_{X\_2} \). Then there exists a \( G_1\)-\( G_2 \)-equivariant isomorphism \( \phi: X\_1 \to X\_2 \).

Proof. As we have \( \mathcal{R}_{X\_1} \cong \mathcal{R}_{X\_2} \), there exist maps \( \phi_R: R_2 \to R_1 \) and \( \phi_\Delta: \Delta_2 \to \Delta_1 \) as in Definition 3.3. The map \( \psi: \mathcal{M}_2 \to \mathcal{M}_1 \) induced by the \( e_D \to e_{\phi_\Delta(D)} \) for \( D \in \Delta_2 \) (notation as in Section 2) together with the map \( \tau := \phi_\Delta|\mathfrak{t}_2 \) satisfies the assumptions of Proposition 3.6. Consequently, we obtain a \( G_1\)-\( G_2 \)-equivariant isomorphism \( \phi: X\_1\^o \to X\_2\^o \) such that \( \phi^\circ = \psi \). As we have \( \psi(\mathcal{M}_2) = \mathcal{M}_1 \), we can extend \( \phi \) to a \( G_1\)-\( G_2 \)-equivariant isomorphism \( \phi: X\_1 \to X\_2 \) by [Los99 Theorem 1.2].

Corollary 3.9. The spherical skeleton \( \mathcal{R}_X \) up to isomorphism determines the Cox ring \( \mathcal{R}(X) \) up to isomorphism of (non-graded) K-algebras.

4. Some properties of spherical skeletons

We continue to use the notation of the previous section.
Remark 4.1. The set $S$ can be obtained directly from the spherical skeleton $\mathcal{R}_X$. First, note that the set $\frac{1}{2}\Sigma^{2a}$ consists of exactly those $\alpha \in S \cap T$ (or, equivalently, exactly those $\alpha \in S$ such that $\text{cone}(\alpha)$ is an extremal ray of $T$) such that there exists exactly one $D \in \Delta$ with $\alpha \in \zeta(D)$. Now we abstractly define $\mathcal{M}$ to be the lattice with basis $(e_D)_{D \in \Delta}$. The surjective map $\overline{\mathcal{N}}_Q \rightarrow \Lambda^*_Q$ with $e_D^* \mapsto \zeta(D)$ induces a dual inclusion $\Lambda_Q \rightarrow \mathcal{M}_Q$. Then $\alpha \in \frac{1}{2}\Sigma^{2a}$ lies in $S$ if and only if we have $\alpha \in \mathcal{M}$ via the above inclusion (in which case $\alpha$ will be a primitive element in $\mathcal{M}$).

Remark 4.2. The spherical skeleton $\mathcal{R}_X$ can be obtained from $\mathcal{R}_X$ using Proposition 2.6 and Proposition 2.13. In fact, the only change is that every color $D$ (which is always of type $2a$) is replaced by two distinct colors $D', D''$ (of type $a$) with $\zeta(D') := \zeta(D'') := \zeta(D)$.

Recall from [GH15a, Definition 5.1] that the spherical skeleton $\mathcal{R}_X$ is said to be complete if $\text{cone}(\zeta(D) : D \in \Delta) = \Lambda^*_Q$. We always have $\mathfrak{m}$.

**Proposition 4.3.** The spherical skeleton $\mathcal{R}_X$ is complete if and only if the spherical skeleton $\mathcal{R}_X$ is complete.

**Proof.** This follows immediately from Remark 4.2.

**Proposition 4.4.** The spherical skeleton $\mathcal{R}_X$ is complete if and only if $\mathcal{X}$ contains a fixed point for $G$.

**Proof.** “$\Rightarrow$”: If $\mathcal{X}$ contains a fixed point, we can argue as in the proof of [GH15a, Lemma 7.3] to show that $\mathcal{R}_X$ is complete. Hence $\mathcal{R}_X$ is complete by Proposition 4.3.

“$\Leftarrow$”: If $\mathcal{X}$ does not contain a fixed point, it follows from $\Gamma(\mathcal{X}, O^*) = K^*$ that we have $\text{relint}(\text{cone}(\zeta_D : D \in \Delta)) \cap \mathcal{V} = \emptyset$ in $\mathcal{N}_Q$. It follows that there exists a separating hyperplane given by $\langle \cdot, v \rangle = 0$ for some $v \in \mathcal{M}_Q$ such that

$$\langle \text{cone}(\zeta_D^* : D \in \Delta), v \rangle \subseteq Q_{\geq 0} \quad \text{and} \quad \langle \mathcal{V}, v \rangle \subseteq Q_{\leq 0},$$

in particular $\langle \mathcal{V} \cap (-\mathcal{V}), v \rangle = \{0\}$, i.e. $v \in \mathcal{N}_Q$. Therefore we have

$$\langle \text{cone}(\zeta_D : D \in \Delta), v \rangle \subseteq Q_{\geq 0},$$

hence $\mathcal{R}_X$ is not complete, and $\mathcal{R}_X$ is not complete by Proposition 4.3.

**Definition 4.5.** We say that the spherical skeleton $\mathcal{R}_X$ is factorial if $S = \emptyset$, i.e. if the Cox ring $\mathcal{R}(X)$, which is uniquely determined by $\mathcal{R}_X$ according to Corollary 3.9, is a factorial ring.

**Proposition 4.6.** The following statements are equivalent:

1. The Cox ring $\mathcal{R}(X)$ is a factorial ring.
2. We have $\mathcal{R}_X \cong \mathcal{R}_X$.

**Proof.** If $\mathcal{R}(X)$ is a factorial ring, then, according to Theorem 2.14, we have $S = \emptyset$, in which case Remark 4.2 yields $\mathcal{R}_X \cong \mathcal{R}_X$.

On the other hand, the ring $\mathcal{R}(X)$ is always factorial since, according to Theorem 2.14, the divisor class group $\text{Cl}(\mathcal{X})$ is free. Hence, if $\mathcal{R}(X)$ is not factorial, then we have $\mathcal{R}(\mathcal{X}) \ncong \mathcal{R}(\mathcal{X})$, in particular $\mathcal{R}_X \ncong \mathcal{R}_X$. 

**Corollary 4.7.** We always have $\mathcal{R}_X = \mathcal{R}_X$. 

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Remark 4.8. The implication "(1)⇒(2)" of Proposition 4.6 generalizes [GH15a, Proposition 6.6]. Note that, in contrast to the proof of [GH15a, Proposition 6.6], the proof of Proposition 4.6 does not depend on the list in [Gag15, Section 2].

5. THE CONJECTURE ON SPHERICAL SKELETONS

We continue to use the notation of the previous section. Recall that a multiplicity-free space is a vector space equipped with a linear action of a connected reductive algebraic group such that it is also a spherical variety.

Definition 5.1 ([GH15a, Definition 10.1]). We say that the spherical skeleton $\mathcal{R}_X$ is linear if $\mathcal{R}_X \cong \mathcal{R}_V$ for some multiplicity-free space $V$.

Remark 5.2. A linear spherical skeleton is complete and factorial.

Brion and Luna introduced certain coefficients $m_D \in \mathbb{Z}_{>0}$ for $D \in \Delta$ to describe the anticanonical divisor of any spherical variety (see [Bri97, Theorem 4.2] and [Lun97, 3.6]). With $s \in \Gamma(X,\omega_X^\vee)$ as in Section 1, they can be defined by

$$\text{div } s = \sum_{D \in \Delta} m_D D.$$ 

For the next definition, note that the coefficients $m_D$ only depend on the spherical skeleton $\mathcal{R}_X$ (see [Bri97, 4.2], see also [GH15b, Section 5]).

Definition 5.3 ([GH15a, Definition 5.3]). We set

$$Q^* := \bigcap_{D \in \Delta} \{ v \in \Lambda_Q : \langle \epsilon(D), v \rangle \geq -m_D \}$$ 

and define

$$\varphi(\mathcal{R}_X) := \sup \left\{ \sum_{D \in \Delta} (m_D - 1 + \langle \epsilon(D), \vartheta \rangle) : \vartheta \in Q^* \cap T \right\} \in \mathbb{Q}_{\geq 0} \cup \{\infty\}.$$ 

Remark 5.4. When $X$ is affine, factorial, and has a fixed point, it follows by substituting $\vartheta$ with $\vartheta - \lambda$ and using the facts $\langle \rho(D), \lambda \rangle = m_D$ for $D \in \Delta$ as well as $\sum_{D \in \Delta} 1 = \text{rank } X$ that Definition 1.2 is in agreement with Definition 5.3.

We denote by $P \subseteq G$ the parabolic subgroup which is the stabilizer of the open $B$-orbit in $X$. Then $P$ is determined by the set $\varsigma(\Delta)$. In particular, $\dim G/P$ depends only on the spherical skeleton $\mathcal{R}_X$.

Conjecture 5.5 ([GH15a, Conjecture 5.5]). Let $\mathcal{R}$ be a complete spherical skeleton. Then we have

$$\varphi(\mathcal{R}) \leq \dim G/P,$$

where equality holds if and only if $\mathcal{R}$ is linear.

Proposition 4.4 and Remark 5.4 show that Conjecture 1.3 is equivalent to Conjecture 5.5 if we only consider spherical skeletons $\mathcal{R}$ which are factorial. Hence, if we can show that factoriality may be assumed without loss of generality, we obtain Theorem 1.4.
Lemma 5.6. For every complete spherical skeleton \( \mathcal{R} \) there exists a factorial complete spherical skeleton \( \mathcal{R}' \) such that \( \varphi(\mathcal{R}) \leq \varphi(\mathcal{R}') \) and \( \mathcal{R}' \) differs from \( \mathcal{R} \) in the following way: For every \( D_{2\alpha} \in \mathcal{D}^S \) with \( \varsigma(D_{2\alpha}) = \{ \alpha \} \)

1. either a color \( D' \) with \( c(D') := c(D_{2\alpha}) \) and \( \varsigma(D') := \varsigma(D_{2\alpha}) \)
2. or a \( \mathcal{G} \)-invariant divisor \( D' \) with \( \langle c(D'), 2\alpha \rangle := -1 \) and \( \langle c(D'), \Sigma^\text{sc} \setminus \{ 2\alpha \} \rangle := \{ 0 \} \)

is added. In particular, \( \dim G/P \) remains unchanged.

Proof. Let \( D_{2\alpha} \in \mathcal{D}^S \) with \( \varsigma(D_{2\alpha}) = \{ \alpha \} \). Let \( \alpha^* \in \Lambda_0^1 \) be the element which is uniquely determined by \( \langle \alpha^*, 2\alpha \rangle = -1 \) and \( \langle \alpha^*, \Sigma^\text{sc} \setminus \{ 2\alpha \} \rangle = \{ 0 \} \). We define

\[
\Theta := \left\{ \vartheta \in \mathcal{Q}^* \cap \mathcal{T} : \varphi(X) = \sum_{D \in \Delta} (m_D - 1 + \langle c(D), \vartheta \rangle) \right\}.
\]

Our first aim is to show that there exists \( \vartheta' \in \Theta \) such that \( \langle c(D_{2\alpha}), \vartheta' \rangle \geq 0 \) or \( \langle \alpha^*, \vartheta' \rangle = 0 \).

Let \( \vartheta \in \Theta \). If \( \langle c(D_{2\alpha}), \vartheta \rangle \geq 0 \), we define \( \vartheta' := \vartheta \).

Otherwise, we have \( \langle c(D_{2\alpha}), \vartheta \rangle < 0 \). Since we have \( \alpha \in \mathcal{S} \), for every \( D \in \Delta \) the integer \( \langle c(D), 2\alpha \rangle \) is divisible by 2. In this case, it is possible to see by inspecting [GH15a, Definition 2.3 and Table 1] that the following statements hold:

1. There exists exactly one \( \gamma \in \Sigma^\text{sc} \) with \( \langle c(D_{2\alpha}), \gamma \rangle < 0 \).
2. We have \( \langle c(D_{2\alpha}), \gamma \rangle = -1 \).
3. Let \( D^* := \{ D \in \Delta : \langle c(D), \gamma \rangle > 0 \} \cup \{ D_{2\alpha} \} \). We have
   a. \( \sum_{D \in D^*} \langle c(D), 2\alpha \rangle = 0 \),
   b. \( \sum_{D \in D^*} \langle c(D), \gamma \rangle = 1 \),
   c. \( \langle c(D), 2\alpha \rangle \leq 0 \) and \( \langle c(D), \gamma \rangle \leq 0 \) for every \( D \in \Delta \setminus D^* \).
4. It follows from the completeness of \( \mathcal{R} \) that there exists some \( D^* \in \Delta \setminus D^* \) such that at least one of the inequalities \( \langle c(D^*), 2\alpha \rangle \leq 0 \) and \( \langle c(D^*), \gamma \rangle \leq 0 \) is strict. Recall that \( \langle c(D^*), 2\alpha \rangle < 0 \) implies \( \langle c(D^*), 2\alpha \rangle \leq -2 \) since \( \alpha \in \mathcal{S} \).

We define \( v_1 := -2\alpha \) and \( v_2 := -2\alpha - 2\gamma \). It follows from statements (3) and (4) that we have \( \sum_{D \in \Delta} \langle c(D), v_i \rangle \geq 0 \) for \( i \in \{ 1, 2 \} \). Consequently, for every

\[
(\lambda_1, \lambda_2) \in \Psi := \left\{ (\lambda_1, \lambda_2) \in \mathbb{R}_{\geq 0}^2 : \vartheta + \lambda_1 v_1 + \lambda_2 v_2 \in \mathcal{Q}^* \cap \mathcal{T} \right\}
\]

we have \( \vartheta + \lambda_1 v_1 + \lambda_2 v_2 \in \Theta \). Now it is not difficult to see that there exists \( (\lambda_1, \lambda_2) \in \Psi \) such that \( \vartheta' := \vartheta + \lambda_1 v_1 + \lambda_2 v_2 \) satisfies \( \langle \alpha^*, \vartheta' \rangle = 0 \) by first maximizing \( \lambda_1 \) and then maximizing \( \lambda_2 \) (note that \( \langle c(D_{2\alpha}), v_2 \rangle = 0 \) by statement (2)).

In the first case, i.e. in the case \( \langle c(D_{2\alpha}), \vartheta \rangle \geq 0 \), let \( \mathcal{R}' \) be obtained from \( \mathcal{R} \) by adding a color \( D' \) with \( c(D') := c(D_{2\alpha}) \) and \( \varsigma(D') := \varsigma(D_{2\alpha}) \). Otherwise, i.e. in the case \( \langle \alpha^*, \vartheta \rangle = 0 \), let \( \mathcal{R}' \) be obtained from \( \mathcal{R} \) by adding a \( \mathcal{G} \)-invariant divisor \( D' \) with \( c(D') := \alpha^* \). In both cases, we have \( \vartheta' \in (\mathcal{Q}^* \setminus \{ \alpha^* \}) \cap \mathcal{T} \) and \( \langle c(D'), \vartheta' \rangle \geq 0 \), so that \( \varphi(\mathcal{R}') \geq \varphi(\mathcal{R}) \).

Moreover, we have \( S' = S \setminus \{ \alpha \} \). Replace \( \mathcal{R} \) by \( \mathcal{R}' \) and repeat until \( S' = \emptyset \).

Theorem 5.7. In Conjecture 5.5 we may assume without loss of generality that the spherical skeleton \( \mathcal{R}_X \) is factorial.

Proof. We assume that Conjecture 5.5 holds for spherical skeletons which are factorial.

Let \( \mathcal{R} \) be an arbitrary complete spherical skeleton. Let \( \mathcal{R}' \) be a complete factorial spherical skeleton as in Lemma 5.6. Then we have \( \varphi(\mathcal{R}) \leq \varphi(\mathcal{R}') \leq \dim G/P \). If \( \mathcal{R} \) is linear, then \( \mathcal{R} \) is factorial, hence we have \( \varphi(\mathcal{R}) = \dim G/P \).
Now assume $\wp(\mathcal{R}) = \dim G/P$. Since we have $\wp(\mathcal{R}) \leq \wp(\mathcal{R}') \leq \dim G/P$, we obtain $\wp(\mathcal{R}) = \wp(\mathcal{R}') = \dim G/P$, so that $\wp(\mathcal{R}')$ is linear. We may assume without loss of generality that $\mathcal{R}'$ is a factor from the list in [Gag15, Section 2].

Consider the possible situations for $\mathcal{R} \not\cong \mathcal{R}'$ according to Lemma 5.6. Situation (2) can only occur if $\mathcal{R}'$ is (5) from the list in [Gag15, Section 2] and $\mathcal{R}$ is obtained from $\mathcal{R}'$ by removing the $G$-invariant prime divisor. In this case, $\mathcal{R}$ is not complete.

On the other hand, situation (1) can only occur if $\mathcal{R}'$ is the entry (21), (22), or (38) from the list in [Gag15, Section 2]. We denote the corresponding spherical skeletons by $\mathcal{R}'_{21}$, $\mathcal{R}'_{22}$, and $\mathcal{R}'_{38}$. A straightforward calculation now yields

\[
\wp(\mathcal{R}'_{21}) = 1, \quad \wp(\mathcal{R}'_{22}) = 2n - 3, \quad \wp(\mathcal{R}'_{38}) = 4n - 4,
\]
\[
\wp(\mathcal{R}_{21}) = 0, \quad \wp(\mathcal{R}_{22}) = n - 2, \quad \wp(\mathcal{R}_{38}) = 2n - 3,
\]

where the notation is as in [Gag15, Section 2].

It follows that $\wp(\mathcal{R}) = \wp(\mathcal{R}') = \dim G/P$ implies $\mathcal{R} \cong \mathcal{R}'$, hence $\mathcal{R}$ is linear. □

Acknowledgements

The author would like to thank Johannes Hofscheier for several helpful comments and discussions.

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