Gribov problem, contact terms and Čech-De Rham cohomology in 2D topological gravity

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ABSTRACT

We point out that averages of equivariant observables of 2D topological gravity are not globally defined forms on moduli space, when one uses the functional measure corresponding to the formulation of the theory as a 2D superconformal model. This is shown to be a consequence of the existence of the Gribov horizon and of the dependence of the observables on derivatives of the super-ghost field. By requiring the absence of global BRS anomalies, it is nevertheless possible to associate global forms to correlators of observables by resorting to the Čech-De Rham notion of form cohomology. To this end, we derive and solve the “descent” of local Ward identities which characterize the functional measure. We obtain in this way an explicit expression for the Čech-De Rham cocycles corresponding to arbitrary correlators of observables. This provides the way to compute and understand contact terms in string theory from first principles.

CERN-TH/95-242
GEF-TH/95-8
September 1995

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1. Introduction

In the modern formulation of closed string theory, \(n\)-point amplitudes of order \(g\) are integrals of closed top-forms over \(\mathcal{M}_{g,n}\) – the moduli space of Riemann surfaces of genus \(g\) and \(n\) marked points. These top-forms are correlators of operators that belong in the BRS cohomology of the underlying 2D quantum field theory. \(\mathcal{M}_{g,n}\) is non-compact: its points at “infinity” correspond to degenerate surfaces with nodes. “Contact” terms are usually referred to as the contributions to the string amplitudes coming from those singular geometries.

The familiar example illustrating the role of contact terms in string theory is the dilaton equation, a low-energy recursion relation expected to be valid in any closed string theory:\[\int_{\mathcal{M}_{g,n+1}} \left\langle \sigma_1^{(0)}(x) \prod_{i=1}^{n} O_i^{(0)}(x_i) \right\rangle = (2g - 2 + n) \int_{\mathcal{M}_{g,n}} \left\langle \prod_{i=1}^{n} O_i^{(0)}(x_i) \right\rangle.\] (1.1)

In Eq. (1.1) the \(O_i^{(0)}(x_i)\) are generic observables with values in the 0-forms on the world-sheet. The (zero-momentum) dilaton operator \(\sigma_1^{(0)}(x)\) is the element of the cohomology of the BRS operator \(s\) which is obtained from the world-sheet Euler 2-form \(\sigma_1^{(2)} = R^{(2)}\) via the so-called “descent” equations:

\[s \sigma_1^{(2)} = d \sigma_1^{(1)}, \quad s \sigma_1^{(1)} = d \sigma_1^{(0)}, \quad s \sigma_1^{(0)} = 0.\] (1.2)

Equation (1.1) is usually understood \[1\], [2], [3] by interpreting the \(2g - 2\) in the r.h.s. as the “bulk” part of the amplitude. The \(n\) term is instead seen to arise from contacts between \(\sigma_1^{(0)}(x)\) and the operators \(O_i^{(0)}(x_i)\) when the modulus associated with the point \(x\) approaches the punctures \(x_i\).

Topological string theories describing non-critical strings with \(c \leq 1\) have an infinite number of dilaton-like observables – the gravitational descendants – whose correlators obey recursion relations [4], [5], [6] which generalize the dilaton equation (1.1). These recursion relations can be interpreted, in much the same way as the dilaton equation, by decomposing physical amplitudes into “bulk” (usually simple to evaluate) and “contact” parts (typically much more difficult to derive from first principles) [6]–[8]. Contact terms have been shown to encode much of the dynamics of topological string models describing large-\(N\) 2D Yang-Mills theory as well [9]. Finally, the study of contact terms in models of topological gravity
coupled to $N = 2$ topological models has also led to the discovery of the “holomorphic” anomaly [11].

It seems fair to say that the world-sheet, field theoretical understanding of contact terms in (topological) string theories has so far remained heuristic. In concrete situations, contact terms have been computed mainly through consistency requirements and/or comparison with known a priori results. It would seem reasonable to expect that contact terms be determined unambiguously by gauge-invariance considerations. Instead, even the derivation of the dilaton equation (the best understood among the recursion relations of string theory) as presented in [3], involves certain arbitrary choices – emphasized, for example, in [11].

In the present article, we consider the issue of contact terms in the context of 2D topological gravity. Here the situation is somewhat paradoxical. The solution of this model presented in Ref. [6] starts from a Lagrangian describing a free superconformal theory and from a set of physical observables. Averages of these observables vanish identically in the corresponding functional measure. The whole non-trivial content of the theory lies in contacts terms whose structure is derived from factorization arguments rather than from a direct evaluation of the functional integral.

It seems natural to wonder if (and how) the information about such contacts is encoded in the original – vanishing – functional measure, or if extra, hidden choices, beyond that of the Lagrangian, are necessary to determine the values of the contacts.

In this paper we show that the physical principle that fixes contact terms ambiguities is the vanishing of global BRS anomalies. This requirement turns out to be non-trivial since we will prove that averages of observables – though generally vanishing in the functional measure considered – are not globally defined on $M_{g,n}$. The technical novelty of our analysis with respect to previous treatments of topological gravity is formulating the quantum theory in a general covariant background gauge. This allows us to probe the dependence of correlators on the gauge-slice.

A familiar feature of non-Abelian gauge theories is the existence the Gribov horizon, that is the lack of a global gauge-slice. The Gribov horizon is the locus in orbit space where the Fadeev-Popov measure degenerates. This would coincide, in the case of topological gravity, with moduli space itself, since the antighost zero modes cause the Fadeev-Popov measure relative to the local fields to vanish identically. In order to gauge-fix these zero modes, the action of BRS operator needs to be extended to global quantum mechanical
variables associated to the moduli and supermoduli. Since string theorists might be unacquainted with such an explicitly BRS-invariant procedure to gauge-fix the antighost zero modes, we review it in section 2.

The new Fadeev-Popov measure, including both the local fields and the global quantum mechanical variables, is generically non-degenerate. The Gribov horizon associated to it has codimension one in moduli space. This implies that the functional integral defines correlators of observables which are \textit{local} closed top-forms on $M_{g,n}$. In order that the correlators have physical meaning, however, such locally defined forms must be local restrictions of forms which are globally defined. For this to be true the observables must satisfy suitable conditions. In closed string theory this condition is known, in the context of the conformal gauge, as the $b_0^-$ equivariance condition \cite{2}. In a covariant framework the $b_0^-$ condition is equivalent to the request that observables be reparametrization-invariant \cite{12}. Nevertheless, in the case of topological gravity, functional averages of equivariant (i.e. reparametrization-covariant) observables are not in general globally defined. We will demonstrate this by deriving the Ward identities associated to \textit{finite} reparametrizations of the background gauge. This phenomenon is originated by the dependence of the observables on derivatives of the super-ghost field.

Even if functional averages of equivariant observables are not globally defined, it is still possible to associate to them globally defined forms by resorting to the Čech-De Rham notion of form cohomology. The idea is to derive from our “anomalous” Ward identities a chain of descendant identities defining a local cocycle of the Čech-De Rham complex of $M_{g,n}$. A well-known construction of cohomology theory leads from this local cocycle to a globally defined form. The integral of the globally defined form receives contributions not only from the original local top-form (which vanishes in the superconformal gauge), but also from the tower of local forms of lower degree that solve the chain of Ward identities. The contact terms are, in this view, precisely the contributions of the descendant local forms to the globally defined integral. They compensate the lack of global definition of the original top-form correlator.

The paper is organized as follows: in section 2, we review the definition of the theory and of the observables, and present the construction of the Lagrangian in a reparametrization-covariant framework. Essentially, this Lagrangian is the background-gauge version of the Lagrangian of Ref. \cite{6} which was written in the superconformal gauge. In section 3, we derive and solve the Ward identities relative to finite reparametrizations
of the background metric. In section 4, we recall the notion of Čech-De Rham cohomology and exhibit its relevance to the case at hand. By solving the Čech-De Rham chain of Ward identities we arrive at an explicit expression for the Čech-De Rham cocycles associated to arbitrary correlators of observables. In section 5, we show that the Čech-De Rham expression for correlators we obtained from the local Ward identities agrees with the algebro-geometric definition of correlators in terms of intersection numbers of cohomology classes on moduli space. In sections 6 and 7 we apply our general formulas to some simple correlators to verify the agreement with the dilaton and puncture equations of Refs. [4], [6]. The goal of this exercise is to show that in this framework the computation of contact terms does not involve ambiguities or arbitrary choices, but is completely combinatoric.

2. Lagrangian and BRS identities

Two-dimensional topological gravity [13], [14] is a topological quantum field theory characterized by the following nilpotent BRS transformation laws [15], [16], [6]:

\[ s g_{\mu\nu} = L_c g_{\mu\nu} + \psi_{\mu\nu} \]
\[ s c^\mu = \frac{1}{2} L_c c^\mu + \gamma^\mu \]
\[ s \gamma^\mu = L_c \gamma^\mu, \]

(2.1)

where \( g_{\mu\nu} \) is the two-dimensional metric, \( \psi_{\mu\nu} \) is the gravitino field, \( c^\mu \) is the ghost vector field and \( \gamma^\mu \) is the superghost vector field; \( L_c \) and \( L_\gamma \) denote the action of infinitesimal diffeomorphisms with parameters \( c^\mu \) and \( \gamma^\mu \) respectively.

A class of observables, local in the fields \( g_{\mu\nu}, \psi_{\mu\nu}, c^\mu \) and \( \gamma^\mu \), can be constructed [15], [16], [18], [19], [6], [12] starting from the Euler two-form

\[ \sigma^{(2)} = \frac{1}{8\pi} \sqrt{g} R \epsilon_{\mu\nu} dx^\mu \wedge dx^\nu, \]

(2.2)

where \( R \) is the two-dimensional scalar curvature and \( \epsilon_{\mu\nu} \) is the antisymmetric numeric tensor defined by \( \epsilon_{12} = 1 \). Since \( s \) and the exterior differential \( d \) on the two-dimensional world-sheet anti-commute, the two-form in Eq. (2.2) gives rise to the descent equations:

\[ s \sigma^{(2)} = d \sigma^{(1)} \]
\[ s \sigma^{(1)} = d \sigma^{(0)} \]
\[ s \sigma^{(0)} = 0. \]

(2.3)
The 0-form $\sigma^{(0)}$ and the 1-form $\sigma^{(1)}$ are computed to be

$$\sigma^{(0)} = \frac{1}{4\pi} \sqrt{g} \epsilon_{\mu\nu} \left[ \frac{1}{2} c^\mu c^\nu R + c^\mu D_\rho (\psi_{\mu\rho} - g_{\mu\rho} \psi_\sigma) + D^\mu \gamma^\nu - \frac{1}{4} \psi_{\rho\lambda} \psi_{\mu\nu} \right]$$

$$\sigma^{(1)} = \frac{1}{4\pi} \sqrt{g} \epsilon_{\mu\nu} [c^\nu R + D_\rho (\psi_{\mu\rho} - g_{\mu\rho} \psi_\sigma)] \, dx^\mu. \tag{2.4}$$

$\sigma^{(0)}$ is a non-trivial class in the cohomology of $s$ acting on the space of the reparametrization-covariant tensor fields. One can verify explicitly that such cohomology is in one-to-one correspondence with the semirelative state BRS cohomology defined on the state space of the theory, quantized on the infinite cylinder in the conformal gauge.\[12\]

Since the superghosts $\gamma^\mu$ are commutative, one can build an infinite tower of cohomologically non-trivial operators by taking arbitrary powers of $\sigma^{(0)}$:

$$\sigma^{(0)}_n \equiv (\sigma^{(0)})^n \tag{2.5}$$

with $n = 0, 1, \ldots$ The corresponding 2-forms

$$\sigma^{(2)}_n = n(\sigma^{(0)}_n)^{n-1} \sigma^{(2)} + \frac{n(n-1)}{2} (\sigma^{(0)}_n)^{n-2} \sigma^{(1)} \wedge \sigma^{(1)} \tag{2.6}$$

all belong in the $s$-cohomology modulo $d$ on the space of the reparametrization-covariant tensor fields.

In order to evaluate correlators of observables $\sigma_n$, the choice of a Lagrangian is required. The theory being topological, the choice of a Lagrangian amounts to fixing the gauge.

The gauge fixing choice is of course dependent on the gauge freedom, which in turn depends on the particular correlation function being computed. Indeed, if all the involved observables correspond to the integral over the Riemann surface of 2-forms such as $\sigma^{(2)}_n$, the gauge freedom corresponds to the whole set of supercoordinate transformations on the Riemann surface. But if the observables involve the local operator $\sigma^{(0)}_n(x)$ at some point $x$, we have to restrict the gauge freedom to supercoordinate transformations that leave this point fixed; correspondingly, the fields $c$ and $\gamma$ must vanish at this point. This, of course, transforms the Riemann surface into a punctured surface. We must also remember that in the case of a sphere the gauge group should not include the isometries of the sphere; therefore the gauge supercoordinate transformations should leave three distinguished points
fixed; some or all of these points could possibly coincide with the position of some local operator. An analogous remark holds true for the torus.

In the following we shall limit our analysis to the case of local operators, therefore the fields $c$ and $\gamma$ vanish in a set of points including those where the observables are sitting. In this situation the coordinates of the fixed points, with the exclusion of three of them in the case of the sphere, and of one of them for the torus, are among the moduli of the punctured surface. One should also notice that, due to the local vanishing of $c$ and $\gamma$, the 0-form $\sigma^{(0)}$ reduces to:

$$\sigma^{(0)} \rightarrow \frac{1}{4\pi} \sqrt{g} \epsilon_{\mu\nu} \left( D^\mu \gamma^\nu - \frac{1}{4} \psi^\mu \psi^{\mu\nu} \right).$$  \hspace{1cm} (2.7)

Let $\mathcal{M}_{g,n}$ be the moduli space of two-dimensional Riemann surfaces of a given genus $g$, and let $m = (m^i)$, with $i = 1, \ldots, 6g - 6 + 2n$, be local coordinates on $\mathcal{M}_{g,n}$. Fixing the gauge means choosing a background metric $\eta_{\mu\nu}(x;m)$ for each gauge equivalence class of metrics corresponding to the point $m$ of $\mathcal{M}_{g,n}$.

It is convenient to decompose $\eta_{\mu\nu}$ as follows:

$$\eta_{\mu\nu}(x;m) \equiv \sqrt{\hat{\eta}} \hat{\eta}_{\mu\nu}(x;m) \equiv e^{\bar{\phi}} \hat{\eta}_{\mu\nu}(x;m), \hspace{1cm} \text{with} \hspace{1cm} \det(\hat{\eta})_{\mu\nu} = 1; \hspace{0.5cm} \hat{g}_{\mu\nu} \text{ is given by the analogous definition for } g_{\mu\nu}. \hspace{0.5cm} \text{We also introduce the tensor density}$$

$$\hat{\psi}^{\mu\nu} \equiv \sqrt{\hat{g}} (\psi^{\mu\nu} - \frac{1}{2} g^{\mu
u} \psi^\sigma), \hspace{1cm} \text{in correspondence with the traceless part of the gravitino field.}$$

$\eta_{\mu\nu}$ defines a gauge-slice on the field space whose associated Lagrangian reads as follows [20], [21], [6]:

$$\mathcal{L} = s \left[ \frac{1}{2} b_{\mu\nu}(\hat{\psi}^{\mu\nu} - \hat{\eta}^{\mu\nu}) + \frac{1}{2} \beta_{\mu\nu}(\hat{\psi}^{\mu\nu} - d_p \hat{\eta}^{\mu\nu}) + \chi \partial_{\mu}(\hat{g}^{\mu\nu} \partial_{\nu}(\varphi - \bar{\varphi})) \right]. \hspace{1cm} (2.9)$$

In Eq. (2.9) we have introduced the “exterior-derivative” operator

$$d_p \equiv p^i \frac{\partial}{\partial m^i};$$

where $p^i$ are the anticommuting supermoduli, with $i = 1, \ldots, 6g - 6 + 2n$, the superpartners of the commuting moduli $m^i$; $b_{\mu\nu}, \beta_{\mu\nu}$ and $\chi$ are the antighost fields, with ghost numbers $-1, -2,$ and 0 respectively. Their BRS transformation laws are given by

$$sb_{\mu\nu} = \Lambda_{\mu\nu}, \hspace{1cm} s\Lambda_{\mu\nu} = 0$$

$$s\beta_{\mu\nu} = L_{\mu\nu}, \hspace{1cm} sL_{\mu\nu} = 0$$

$$s\chi = \mathcal{L}_c \chi + \pi, \hspace{1cm} s\pi = \mathcal{L}_c \pi - \mathcal{L}_\gamma \chi, \hspace{1cm} (2.10)$$
where $\Lambda_{\mu\nu}$, $L_{\mu\nu}$ and $\pi$ are Lagrangian multipliers.

At first sight it would seem that the Levi-Civita connection in the covariant derivative acting on $\gamma$ in (2.7), causes $\sigma^{(0)}(x)$ to depend on the Liouville field $\varphi(x)$ defined in Eq. (2.8). However, the vanishing of $\gamma$ at point $x$ kills the term in $\sigma^{(0)}$ dependent on the connection; therefore the local observables $\sigma^{(0)}_{\alpha}$ are actually invariant under Weyl transformations, that is under $\varphi(x)$ translations. Considering now the path integral that defines the correlation functions and integrating out the Liouville field and its superpartner (whose respective functional determinants cancel by supersymmetry) one obtains a functional measure which depends on the reduced, Weyl-invariant, background metric $\hat{\eta}_{\mu\nu}(x; m)$ but not on $\bar{\varphi}(x)$.

The moduli and the supermoduli are quantum mechanical variables, which should be integrated over to obtain the gauge-invariant correlators. It is therefore natural that they transform under the action of the BRS operator $s$, according to the following transformation laws \[22, 20\]:

\[
\begin{align*}
    s\, m^i &= C^i & s\, C^i &= 0 \\
    s\, p^i &= -\Gamma^i & s\, \Gamma^i &= 0
\end{align*}
\tag{2.11}
\]

where $C^i$ and $\Gamma^i$ are respectively anticommuting and commuting Lagrange multipliers.

It is easy to check that the action of $s$ on the background metric in the Lagrangian (2.9) produces exactly the antighost insertions necessary to gauge-fix the degeneracy associated to the zero-modes of the antighost fields.

The Lagrangian in Eq. (2.9), written out in extended form, reads:

\[
L = \frac{1}{2} \Lambda_{\mu\nu} (\hat{g}^{\mu\nu} - \hat{\eta}^{\mu\nu}) + \frac{1}{2} L_{\mu\nu} (\hat{\psi}^{\mu\nu} - d_p \hat{\eta}^{\mu\nu}) - \frac{1}{2} b_{\mu\nu} L c \hat{g}^{\mu\nu} - \frac{1}{2} \beta_{\mu\nu} L \gamma \hat{g}^{\mu\nu}
+ \frac{1}{2} \hat{\psi}^{\mu\nu} \left[ (L c \beta)_{\mu\nu} + b_{\mu\nu} + 2 \partial_\mu \chi \partial_\nu (\varphi - \bar{\varphi}) \right]
+ \pi \partial_\mu (\bar{g}^{\mu\nu} \partial_\nu (\varphi - \bar{\varphi})) - \chi \partial_\mu (\hat{g}^{\mu\nu} \partial_\nu \psi')
+ \frac{1}{2} \beta_{\mu\nu} d_{\Gamma} \hat{\eta}^{\mu\nu} + \frac{1}{2} b_{\mu\nu} d_{C} \hat{\eta}^{\mu\nu} + \frac{1}{2} \beta_{\mu\nu} d_{p} d_{C} \hat{\eta}^{\mu\nu} + \chi \partial_\mu (\hat{g}^{\mu\nu} \partial_\nu d_{C} \bar{\varphi}),
\]

where

\[
\psi' \equiv \bar{D}_\sigma c^\sigma + \frac{1}{2} \psi^\sigma, \quad \tag{2.13}
\]

and where the notation $d_{C} \equiv C^i \frac{\partial}{\partial m^i}$ and $d_{\Gamma} \equiv \Gamma^i \frac{\partial}{\partial m^i}$, has been introduced.

Integrating out the Lagrange multipliers $\Lambda_{\mu\nu}$, $L_{\mu\nu}$, $\pi$ and $\chi$ forces the metric and the gravitino field to take their background values,

\[
\hat{g}^{\mu\nu} \rightarrow \hat{\eta}^{\mu\nu}, \quad \varphi \rightarrow \bar{\varphi}, \quad \hat{\psi}^{\mu\nu} \rightarrow d_p \hat{\eta}_{\mu\nu}, \quad \frac{1}{2} \psi^\sigma + \bar{D}_\sigma c^\sigma \rightarrow d_{C} \bar{\varphi}, \quad \tag{2.14}
\]
and the Lagrangian becomes
\[
\mathcal{L}' = \frac{1}{2} \left[ -b_{\mu\nu} \mathcal{L}_c \tilde{\eta}^\mu{}^\nu - \beta_{\mu\nu} \mathcal{L}_\gamma \tilde{\eta}^\mu{}^\nu + d_p \tilde{\eta}^\mu{}^\nu (\mathcal{L}_c \beta)_{\mu\nu} \\
+ b_{\mu\nu} (d_C \tilde{\eta}^\mu{}^\nu - d_p \tilde{\eta}^\mu{}^\nu) + \beta_{\mu\nu} d_{\Gamma} \tilde{\eta}^\mu{}^\nu + \beta_{\mu\nu} d_p d_C \tilde{\eta}^\mu{}^\nu \right].
\] (2.15)

In the following we will repeatedly make use of the fact that, when the observables do not contain the antighost zero modes \(b^{(i)}\), integrating them out introduces into the correlators the factor
\[
\prod_i \delta(C^i - p^i). \quad (2.16)
\]

If moreover there are no antighost zero modes \(\beta^{(i)}\) and no antighost fields \(b_{\mu\nu}\) in the observables, one can integrate out \(\beta^{(i)}\) as well. This produces a further factor
\[
\prod_i \delta(\Gamma^i). \quad (2.17)
\]

We will consider correlators of the operator-valued 0-forms \(\sigma^{(0)}_{i_k}\) obtained by functional averaging with respect to the local quantum fields and the \(C^i\) and \(\Gamma^i\) multipliers, collectively denoted by \(\Phi\), but not with respect to the moduli and supermoduli \(m^i\) and \(p^i\):
\[
Z(m^i; p^i) \equiv \left\langle \prod_k \sigma^{(0)}_{i_k}(P_{i_k}) \right\rangle
\equiv \int [d\Phi] e^{-S(\Phi; m^i, p^i)} \prod_k \sigma^{(0)}_{i_k}(P_{i_k}). \quad (2.18)
\]

The 0-form \(\sigma^{(0)}_{i_k}\) sits on the point \(P_{i_k}\) of the world-sheet manifold; therefore \(Z(m^i; p^i)\) also depends on the choice of the \(P_{i_k}\)'s. We are not considering this dependence explicitly since it will disappear after moduli integration.

Because of ghost-number conservation, \(Z(m^i; p^i)\) is a monomial of the anticommuting supermoduli:
\[
Z(m^i; p^i) = Z_{i_1 \ldots i_N}(m^i)p^{i_1} \ldots p^{i_N}, \quad (2.19)
\]
where \(N\) is the total ghost number of observables \(\sigma^{(0)}_{i_k}\):
\[
N = \sum_k (\text{ghost} \ # \ \sigma^{(0)}_{i_k}) = 2 \sum_k i_k. \quad (2.20)
\]

Under a reparametrization \(\tilde{m}^i = \tilde{m}^i(m)\) of the local coordinates \(m^i\) on the moduli space \(\mathcal{M}_{g,n}\), the supermoduli transform as:
\[
p^i = \frac{\partial \tilde{m}^i}{\partial m^j} p^j.
\]
We thus identify the anticommuting supermoduli with the differentials on the moduli space, i.e. \( p^i \to dm^i \). Correspondingly, the function \( Z(m^i; p^i) \) of the moduli and supermoduli can be thought of as an \( N \)-form on the moduli space \( M_{g,n} \), at least locally on \( M_{g,n} \). The question of whether or not such a local form extends to a globally defined form on \( M_{g,n} \) will be discussed in the next section.

Assume for the moment that the form \( Z(m^i; p^i) \) is globally defined on \( M_{g,n} \). Whenever the following ghost number selection rule is satisfied,

\[
N = 2 \sum_k i_k = 6g - 6 + 2n,
\]  

(2.21)

\( Z(m^i; p^i) \) defines a measure on \( M_{g,n} \), which can be integrated to produce some number. The collection of these numbers encodes some, at least, of the gauge-invariant contents of 2D topological gravity.

It is easy to show that the action of the BRS operator \( s \) on the quantum fields \( \Phi \) translates into the action of the exterior differential \( d \equiv p^i \partial_i \) on the forms \( Z(m^i; p^i) \). More precisely, one can prove the following Slavnov-Taylor identities:

\[
(i) \quad s O(\Phi') = 0 \Rightarrow d_p \langle O(\Phi') \rangle = 0
\]

\[
(ii) \quad O(\Phi') = s X(\Phi') \Rightarrow \langle O(\Phi') \rangle = d_p \langle X(\Phi') \rangle,
\]

(2.22)

where \( O(\Phi') \) and \( X(\Phi') \) are operators that depend on the fields \( \Phi' \) other than \( C^i \) and \( \Gamma^i \) and do not contain the antighost zero modes and the antighost field \( b \).

Let us prove, for example (\( i \)). Denote by \( \langle O' \rangle' \) the average with respect to the local fields \( \Phi' \) only, without integrating over \( C^i \) and \( \Gamma^i \). Then

\[
\left[ d_C - \Gamma^i \frac{\partial}{\partial p^i} \right] \langle O(\Phi') \rangle' = \left\langle \sum_{\Phi'} s \Phi' \frac{\delta S}{\delta \Phi'} O(\Phi') \right\rangle' = 0.
\]

(2.23)

Since, under the state conditions, \( \langle O' \rangle' \) is proportional to \( \prod_i \delta(C^i - p_i)\delta(\Gamma^i) \), by integrating both sides of Eq. (2.23) with respect to the multipliers \( C^i \) and \( \Gamma^i \), one obtains (\( i \)). The proof of (\( ii \)) is analogous.
3. The main Ward identity

The background metric $\eta_{\mu\nu}(x; m)$ cannot be chosen to be a everywhere continuous function of $m$. In fact $\eta_{\mu\nu}(x; m)$ is a section of the gauge bundle over $M_{g,n}$ defined by the space of two-dimensional metrics on a surface of given genus and $n$ punctures. This bundle is non-trivial and therefore does not admit a global section. It follows that $\eta_{\mu\nu}(x; m)$ must be a local section of the bundle of two-dimensional metrics. Let $\{U_a\}$ be a covering of $M_{g,n}$ of open neighbourhoods of $M_{g,n}$, with $\bigcup_a U_a = M_{g,n}$. The background gauge is defined by a collection $\{\eta_a^{\mu\nu}(x; m)\}$ of two-dimensional metrics, with each $\eta_a^{\mu\nu}(x; m)$ locally defined, as a function of $m$, on $U_a$. The functional average in Eq. (2.18) defines local closed forms $Z_a(m^i; p^i)$, on each open set $U_a$.

The collection of local forms $\{Z_a(m^i; p^i)\}$ corresponds to a globally defined form $Z(m^i; p^i)$ on $M_{g,n}$ if and only if

$$Z_a(m^i; p^i) = Z_b(m^i; p^i) \quad \text{on} \quad U_{ab} \equiv U_a \cap U_b. \quad (3.1)$$

We will see that for general operators and background metrics, $\{Z_a(m^i; p^i)\}$ does not satisfy Eq. (3.1), but rather an equation of the more general form:

$$Z_a(m^i; p^i) - Z_b(m^i; p^i) = d_p S_{ab}(m^i; p^i) \quad \text{on} \quad U_{ab} \equiv U_a \cap U_b. \quad (3.2)$$

When $S_{ab}(m^i; p^i)$ in Eq. (3.2) is non-vanishing, $\{Z_a(m^i; p^i)\}$ does not define a global form $Z(m^i; p^i)$, which could be integrated over $M_{g,n}$ to give a gauge-invariant expectation value. Under such circumstances, one has to resort to the Čech notion of form cohomology in order to obtain a gauge-invariant definition of the “integral” of $\{Z_a(m^i; p^i)\}$ over $M_{g,n}$. This will be illustrated in the next section. Here, we will derive a general formula for the $S_{ab}(m^i; p^i)$ appearing in the Ward identity (3.2).

Two background metrics $\eta_a^{\mu\nu}$ and $\eta_b^{\mu\nu}$ are related on the intersection $U_{ab}$ by a combination of a diffeomorphism and a Weyl transformation. One should remember however that we are considering in Eq. (2.18) expectation values of observables that involve only the 0-forms (2.7) which, as explained in the previous section, depend on the reduced, Weyl-invariant, background metric $\hat{\eta}^{\mu\nu}(x; m)$ but not on $\bar{\varphi}(x)$. Therefore we only need to consider the Ward identities relative to diffeomorphisms of the background reduced metric $\hat{\eta}^{\mu\nu}(x; m)$.
Let therefore $O$ be the observable

$$O \equiv \prod_{i=1}^{n} \sigma_i^{(0)}(x_i) \quad (3.3)$$

of total ghost number $\sum_i 2n_i = 2N$. The Ward identity that we will prove in this section reads as:

$$\left( \delta \langle O \rangle \right)_{ab} \equiv \langle O \rangle_b - \langle O \rangle_a = d_p \int_0^1 dt \int d[\Phi] e^{-S_{ab}(t)} (I_b - I_a) O(c, \gamma, \hat{g}, \hat{\psi}). \quad (3.4)$$

Let us first explain the notation used in Eq. (3.4). $\langle O \rangle_a$ and $\langle O \rangle_b$ denote the functional averages of $O$ with backgrounds $\hat{\eta}_0^{\mu\nu}$ and $\hat{\eta}_b^{\mu\nu}$ respectively.

$S_{ab}(t)$ is the following action depending on the real parameter $t$

$$S_{ab}(t) \equiv S_0 - s \int d^2 x \beta_{\mu\nu} \left[ t(L_{\hat{v}_b} \hat{\eta}_0)^{\mu\nu} + (1 - t)(L_{\hat{v}_a} \hat{\eta}_0)^{\mu\nu} \right], \quad (3.5)$$

where we introduced a third background $\hat{\eta}_0^{\mu\nu}$ and the corresponding action $S_0$. $\hat{\eta}_0^{\mu\nu}$ is related to the backgrounds $\hat{\eta}_a^{\mu\nu}$ and $\hat{\eta}_b^{\mu\nu}$ by two diffeomorphisms:

$$x_a \to x_0(x_a; m), \quad x_b \to x_0(x_b; m), \quad (3.6)$$

which may in general depend on $m$ and which define the following forms $\hat{v}_a$ and $\hat{v}_b$ on $\mathcal{M}_{g,n}$ with values in the vector fields on the world-sheet:

$$\hat{v}_a \equiv p^i v_{ia}^{\mu} \equiv \partial_i x_0^{\mu}(x_a; m) \big|_{x_a = x_a(x_0; m)} \quad \hat{v}_b \equiv p^i v_{ib}^{\mu} \equiv \partial_i x_0^{\mu}(x_b; m) \big|_{x_b = x_b(x_0; m)}. \quad (3.7)$$

Finally $I_a$ and $I_b$ are operators which shift the ghost field $\gamma^\mu$ by $\hat{v}_a$ and $\hat{v}_b$, i.e.

$$I_a \equiv - \int d^2 x \hat{v}_a^{\mu}(x) \frac{\delta}{\delta \gamma^\mu}(x), \quad (3.8)$$

and analogously for $I_b$.

Before plunging into the derivation of our Ward identity (3.4) we will specialize it to the case when $O = \sigma_i^{(0)}$. Since the general observable has the product structure (3.3), the action of $I_a$ on it will be expressible in terms of the action on the operator $\sigma_i^{(0)}$. If we introduce the transition matrix

$$(M_a)^{\nu}_{\mu} \equiv \frac{\partial x_0^{\nu}}{\partial x_a^\mu} \quad (3.9)$$

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and define analogously the matrix $M_b$, the Ward identity (3.4) becomes, for $O = \sigma_1^{(0)}$,

\[
(\delta \langle \sigma_1^{(0)} \rangle)_{ab} = d_p \langle I_b \sigma_1^{(0)} \rangle_0 - d_p \langle I_a \sigma_1^{(0)} \rangle_0 \\
= \frac{1}{4\pi} d_p \epsilon_{\mu\nu} \hat{\eta}^\nu_\lambda (M_b^{-1} d_p M_b - M_a^{-1} d_p M_a)_\lambda^\mu.
\] (3.10)

It is convenient to take the coordinates $x_0(x_a; m)$ to be isothermal complex coordinates relative to the complex structure $m$. That is, let $x_0(x_a; m) = (Z_m(z_a, \bar{z}_a; m), \bar{Z}_m(z_a, \bar{z}_a; m))$, with $x_a = (z_a, \bar{z}_a)$ and

\[
dZ_m \otimes d\bar{Z}_m = |\lambda_a|^2 (dz_a + \mu_a d\bar{z}_a) \otimes (d\bar{z}_a + \bar{\mu}_a dz_a),
\] (3.11)

where $\mu_a$ and $\lambda_a$ are corresponding Beltrami differentials and integrating factors. Then, the transition matrix $M_a$ becomes

\[
(M_a)_\nu^\mu = \left( \begin{array}{cc} \lambda_a & \bar{\lambda}_a \bar{\mu}_a \\ \lambda_a \mu_a & \bar{\lambda}_a \\ \end{array} \right),
\] (3.12)

and the action of the operator $I_a$ on $\sigma_1^{(0)}$ reads

\[
f_a(x; m) \equiv \langle I_a \sigma_1^{(0)}(x) \rangle_0 = \frac{1}{4\pi i} d_p \log \frac{\lambda_a}{\bar{\lambda}_a} + \frac{1}{4\pi i} \frac{\mu_a d_p \bar{\mu}_a - \bar{\mu}_a d_p \mu_a}{1 - |\mu_a|^2}.
\] (3.13)

Eq. (3.4) implies that $\{ Z_a(m^i; p^i) \}$ is the restriction of a globally defined form only for those backgrounds $\{ \hat{\eta}_a \}$ for which $\hat{v}_a = \hat{v}_b$. This condition is equivalent to the requirement that $\{ \hat{\eta}_a \}$ be a modular invariant global section on the space of reduced metrics over Teichmüller space [20, 21]. By this we mean that values of the section at points related by modular transformations must differ by diffeomorphisms which are independent of the Teichmüller point. In special cases – for example, on $M_{1,1}$ – sections $\{ \hat{\eta}_a \}$ with this property can be found. Correspondingly, the local forms $\{ Z_a(m^i; p^i) \}$ associated to the correlator $\langle \sigma_1 \rangle_{g=0}$ match on the various patches of moduli space to give a globally defined form [21]. However, for $M_{g,n}$ with $g$ and $n$ generic, local sections with this property do not exist. In fact the existence of a section with the stated property would imply the existence of a homomorphism from the mapping class group into the group of diffeomorphisms.

Actually, the absence of modular invariant sections is fortunate since the local expectation values $\langle O \rangle_a$ vanish for almost all observables, as we will explain in detail in the next section. Or, turning things around, the fact that correlators of topological gravity are not trivial proves indirectly that there is, generally, no section for which $\langle O \rangle_a$ glue together
to produce a globally defined form. One must therefore live with the general situation in which the r.h.s. of Eq. (3.4) does not vanish. In the next section we will explain how to recover a global form starting from (3.4) and from a tower of descendant Ward identities.

PROOF OF EQ. (3.4). Let $\Phi_0(x), \Phi_a(x)$ and $\Phi_b(x)$ denote collectively the quantum fields in the coordinate systems $x_0, x_a$ and $x_b$ respectively; $\Phi_0(x_0)$ and $\Phi_a(x_a)$ are related by a diffeomorphism $x_0 = x_0(x_a;m)$, which generally depends on $m$. The same diffeomorphism relates the background $\hat{\eta}_{\mu\nu}^0$ and $\hat{\eta}_{\mu\nu}^a$:

$$
\hat{\eta}_{\mu\nu}^a(x_a;m) = \frac{1}{\det(\frac{\partial x_0^\sigma}{\partial x_a^\rho})} \frac{\partial x_0^{\mu}}{\partial x_a^\sigma} \frac{\partial x_0^{\nu}}{\partial x_a^\rho} \hat{\eta}_{\sigma\rho}^0(x_0;m).
$$

(3.14)

From Eq. (3.14) one derives the transformation law for the derivatives of the background with respect to the moduli:

$$
\partial_i \hat{\eta}_{\mu\nu}^a(x_a;m) \equiv \frac{\partial \hat{\eta}_{\mu\nu}^a}{\partial m^i} = \frac{1}{\det(\frac{\partial x_a^\sigma}{\partial x_0^\rho})} \frac{\partial x_a^{\mu}}{\partial x_0^\sigma} \frac{\partial x_a^{\nu}}{\partial x_0^\rho} (\partial_i \hat{\eta}_{\sigma\rho}^0(x_0;m) + (L_{\psi} \hat{\eta}_0)^{\sigma\rho}(x_0;m)).
$$

(3.15)

Let $S_0$ denote the action defined via the background metric $\hat{\eta}_{\mu\nu}^0$, that is:

$$
S_0 = s \int d^2 x_0 \left[ b_{\mu\nu} (\hat{g}_{\mu\nu} - \hat{\eta}_{\mu\nu}^0)(x_0) + \beta_{\mu\nu} (\hat{\psi}_{\mu\nu} - d_p \hat{\eta}_{\mu\nu}^0)(x_0) \right]
$$

$$
= \int d^2 x_0 \left[ \Lambda_{\mu\nu} (\hat{g}_{\mu\nu} - \hat{\eta}_{\mu\nu}^0)(x_0) + L_{\mu\nu} (\hat{\psi}_{\mu\nu} - d_p \hat{\eta}_{\mu\nu}^0)(x_0) \right]
$$

$$
- b_{\mu\nu} (\hat{\psi}_{\mu\nu} + (L_c \hat{g})^{\mu\nu} - d_c \hat{\eta}_{\mu\nu}^0)(x_0)
$$

$$
+ \beta_{\mu\nu} ((L_c \hat{\psi})^{\mu\nu} - d_c d_p \hat{\eta}_{\mu\nu}^0 + d \Gamma \hat{\eta}_{\mu\nu}(x_0)),
$$

(3.16)

where it should be recalled that

$$
d_p = p_i \frac{\partial}{\partial m^i}, \quad d_c = c^i \frac{\partial}{\partial m^i}, \quad d \Gamma = \Gamma^i \frac{\partial}{\partial m^i}.
$$

The crucial identity relating $S_0$ to $S_a$ is

$$
S_a(\Phi_a) = \left[ S_0 - s \int d^2 x_0 \beta_{\mu\nu}(L_c \hat{\psi})^{\mu\nu}(x_0) \right] (c_0 - \tilde{v}_a; \Lambda_0 - L_{\tilde{v}_a} b_0; L_0 - \mathcal{L}_{\tilde{v}_a} \beta_0; \Phi'_0),
$$

(3.17)

where $\Phi'_0$ denote the fields in the $x_0$ coordinate system other than $c^i$, $\Lambda_{\mu\nu}$ and $L_{\mu\nu}$, and we defined $\tilde{v}_a \equiv \mathcal{C}^i v_a^i$.

The identity (3.17) can be verified directly. A more conceptual way to understand it is to recall that the BRS operator $s$ includes an exterior derivative with respect to the
moduli and that the relation between the fields $\Phi_a$ and the fields $\Phi_0$ involves $m$-dependent transition functions. Therefore the $s$-variation of a field in the $x_a$ reference frame does not transform covariantly when expressed in terms of the fields in the $x_0$ reference system. If one writes $s\Phi_a = \hat{\Phi}_a$ for the $s$-variation of the fields in the $x_a$ reference system, one finds, by a calculation identical to the one leading to Eq. (3.15), that

$$s \Phi_0 = \hat{\Phi}_0 - \mathcal{L}_{\tilde{v}_a} \Phi_0,$$  \hspace{1cm} (3.18)

where $\hat{\Phi}_0$ is obtained from $\hat{\Phi}_a$ by means of a general coordinate transformation. To give an example,

$$s \hat{g}_\mu^\nu = (\mathcal{L}_{c_0} \hat{g}_0)^\mu^\nu + \hat{\psi}_0^\mu^\nu - \mathcal{L}_{\tilde{v}_a} \hat{g}_0.$$  \hspace{1cm} (3.18)

Comparing Eq. (3.18) with the BRS transformation laws, Eqs. (2.1), (2.10), one sees that the Lie derivative in the r.h.s. of Eq. (3.18) can be reabsorbed in the operator $s$ by means of the following redefinition of the $c$ ghosts and the Lagrangian multipliers:

$$\tilde{c}^\mu = c_0^\mu - \tilde{v}_a^\mu, \quad \tilde{\Lambda}_{\mu^\nu} = \Lambda_{\mu^\nu} - (\mathcal{L}_{\tilde{v}_a} b_0)^\mu^\nu, \quad \tilde{L}_{\mu^\nu} = L_{\mu^\nu} - (\mathcal{L}_{\tilde{v}_a} \beta_0)^\mu^\nu.$$  \hspace{1cm} (3.19)

For example, $s \hat{g}_0^\mu^\nu = (\mathcal{L}_{c_0 - \tilde{v}_a} \hat{g}_0)^\mu^\nu + \hat{\psi}_0^\mu^\nu$.

Thus, taking into account the transformation law of the background Eq. (3.15), one obtains for the action in the background $\hat{\eta}_a$:

$$S_a = s \int d^2 x_0 \left[ (b_0)^\mu^\nu (\hat{g}_0^\mu^\nu - \hat{\eta}_0^\mu^\nu)(x_0) + (\beta_0)^\mu^\nu (\hat{\psi}_0^\mu^\nu - d_p \hat{\eta}_0^\mu^\nu - (\mathcal{L}_{\tilde{v}_a} \hat{\eta}_0)^\mu^\nu)(x_0) \right],$$  \hspace{1cm} (3.20)

where the operator $s$ acts on the fields $\Phi_0$ according to Eq. (3.18). But since, after the redefinitions in Eqs. (3.19), the action of $s$ on the fields $\Phi_0$ is the standard one, Eq. (3.20) is equivalent to the identity (3.17).

From now on, let us restrict ourselves to observables $O$ which are reparametrization-invariant and independent from the Lagrangian multipliers $\Lambda_{\mu^\nu}$ and $L_{\mu^\nu}$. The functional average $\langle O \rangle_a$ in the background $\hat{\eta}_a$ can be rewritten in terms of the averages in the background $\hat{\eta}_0$ by means of the identity (3.17) and after a suitable change of integration variables:

$$\langle O(c, \gamma, \hat{g}, \hat{\psi}^\mu^\nu) \rangle_a = \int d[\Phi] e^{-S_0 + s \int d^2 x \beta_{\mu^\nu} (\mathcal{L}_{\tilde{v}_a} \hat{\eta}_0)^\mu^\nu} O(c + \tilde{v}_a, \gamma, \hat{g}, \hat{\psi}^\mu^\nu).$$  \hspace{1cm} (3.21)

In a similar way, we derive

$$\langle O(c, \gamma, \hat{g}, \hat{\psi}^\mu^\nu) \rangle_b = \int d[\Phi] e^{-S_0 + s \int d^2 x \beta_{\mu^\nu} (\mathcal{L}_{\tilde{v}_b} \hat{\eta}_0)^\mu^\nu} O(c + \tilde{v}_b, \gamma, \hat{g}, \hat{\psi}^\mu^\nu).$$  \hspace{1cm} (3.22)
At this point we need to spell out a further crucial requirement that must be satisfied by the observable \( O \), that is

\[
O(c + \tilde{v}_a, \gamma, \hat{g}, \hat{\psi}^{\mu\nu}) = O(c, \gamma, \hat{g}, \hat{\psi}^{\mu\nu}).
\]  

(3.23)

The coordinate of the points of the world-sheet where the local observables are inserted do not change under the reparametrizations (3.6), i.e. \( \tilde{v}_a(x_i) = 0 \), at the points \( x_i \) where the \( \sigma_n^{(0)}(x) \) are considered. Therefore the condition (3.23) is satisfied whenever the observables do not depend on derivatives of the ghost \( c \). This is of course the case for the \( \sigma_n^{(0)} \), which are the observables studied in this paper. Eq. (3.23) is the \textit{equivariant} condition in the present context and is equivalent to the \( b_0^{-} \)-condition of the operator formalism [12].

Thus, assuming Eq. (3.23), we arrive at

\[
\langle O \rangle_b - \langle O \rangle_a = \int_0^1 dt \int d[\Phi] \frac{d}{dt} e^{-S_{ab}(t)} O \\
= \int_0^1 dt \int d[\Phi] [s \int d^2 x \beta_{\mu\nu}((\mathcal{L}_{\tilde{v}_a} \hat{\eta}_0)^{\mu\nu} - (\mathcal{L}_{\tilde{v}_a} \hat{\eta}_0)^{\mu\nu})] e^{-S_{ab}(t)} O \\
= \int_0^1 dt \int d[\Phi] s \left[ \int d^2 x \beta_{\mu\nu}((\mathcal{L}_{\tilde{v}_a} \hat{\eta}_0)^{\mu\nu} - (\mathcal{L}_{\tilde{v}_a} \hat{\eta}_0)^{\mu\nu}) e^{-S_{ab}(t)} O \right],
\]

(3.24)

where \( S_{ab}(t) \) is the interpolating action defined in Eq. (3.3).

Notice that the functional integral of an \( s \)-variation would vanish if only the action of \( s \) on the local fields (Eqs. (2.1), (2.10)) were considered. However, since \( s \) acts also on moduli and supermoduli (Eq. (2.11)), a non-vanishing result is obtained:

\[
(\delta \langle O \rangle)_{ab} = \int_0^1 dt \int \prod_i dC_i \prod_j d\Gamma^j \left( \frac{dC}{-\Gamma^k} \frac{\partial}{\partial p^k} \right) \\
\int d[\Phi'] \int d^2 x \beta_{\mu\nu}((\mathcal{L}_{\tilde{v}_a} \hat{\eta}_0)^{\mu\nu} - (\mathcal{L}_{\tilde{v}_a} \hat{\eta}_0)^{\mu\nu}) e^{-S_{ab}(t)} O,
\]

(3.25)

where \( d[\Phi'] \) denotes the functional integral with respect to the fields other than \( C_i \) and \( \Gamma^j \). As stated before, \( O \) does not depend on \( \Lambda_{\mu\nu} \) and thus one can safely substitute \( \hat{g}^{\mu\nu} \) with \( \hat{\gamma}^{\mu\nu} \) in the functional integral in the r.h.s. of Eq. (3.25). Moreover, \( I_a S_{ab}(t) = \int d^2 x \beta_{\mu\nu}(\mathcal{L}_{\tilde{v}_a} \hat{g})^{\mu\nu} \) and \( I_b S_{ab}(t) = \int d^2 x \beta_{\mu\nu}(\mathcal{L}_{\tilde{v}_a} \hat{g})^{\mu\nu} \), where \( I_a \) and \( I_b \) are the operators defined in Eq. (3.8). Therefore

\[
(\delta \langle O \rangle)_{ab} = -\int_0^1 dt \int \prod_i dC_i \prod_j d\Gamma^j \left( \frac{dC}{-\Gamma^k} \frac{\partial}{\partial p^k} \right) \int d[\Phi'] \left[ (I_a - I_a) e^{-S_{ab}(t)} \right] O \\
= \int_0^1 dt \int \prod_i dC_i \prod_j d\Gamma^j \left( \frac{dC}{-\Gamma^k} \frac{\partial}{\partial p^k} \right) \int d[\Phi'] e^{-S_{ab}(t)} (I_b - I_a) O.
\]

(3.26)
Finally, since \( O \) is independent of \( \Lambda_{\mu\nu}, L_{\mu\nu}, b_{\mu\nu} \) and \( \beta_{\mu\nu} \), the functional integral in the last formula is proportional to \( \prod_i \delta(p^i - C^i)\delta(\Gamma^i) \) and hence one arrives at the Ward identity (3.4).

4. Čech-De Rham cohomology and contact terms

We have seen in the previous section that, for a generic background gauge \( \{\tilde{\eta}_a\} \) with \( m \)-dependent transition functions \( M_a(x;m) \), the collection of local forms \( \{Z_a(m^i;p^i)\} \) is not the restriction to the cover \( \{U_a\} \) of a form globally defined on \( \mathcal{M}_{g,n} \). We want to show in this section that the Ward identities (3.4), together with a whole chain of “descendant” Ward identities, contain the necessary geometrical data to define a global closed form on the moduli space.

Let \( 2N \) be the degree of the local forms \( Z_a(m^i;p^i) \), i.e. the ghost number of the observable \( O(c,\gamma,\tilde{g},\tilde{\psi}) \). It is convenient to introduce formal anticommuting variables \( \xi^a \) and to collect the local forms \( Z_a(m^i;p^i) \) into one single object:

\[
Z_0^{(2N)} = \sum_a \xi^a Z_a(m^i;p^i). \tag{4.1}
\]

\( Z_0^{(2N)} \) is an element of \( C^0(\mathcal{M}_{g,n};\Omega^{(2N)}) \), the space of 0-cochains with values in the local 2\( N \)-forms of \( \mathcal{M}_{g,n} \). One defines analogously \( C^q(\mathcal{M}_{g,n};\Omega^{(p)}) \) as the space of \( q \)-cochains \( Z_q^{(p)} \):

\[
Z_q^{(p)} = \sum_{a_0 < a_1 < \ldots < a_q} Z_{a_0 \ldots a_q}^{(p)} \xi^{a_0} \xi^{a_1} \ldots \xi^{a_q}, \tag{4.2}
\]

with values in the local \( p \)-forms \( Z_{a_0 \ldots a_q}^{(p)} \) defined on the \( q \)-fold intersections \( U_{a_0 a_1 \ldots a_q} \equiv U_{a_0} \cap U_{a_1} \ldots \cap U_{a_q} \). (In Eq. (4.2) one is assuming that the set where the indexes \( a_0, a_1, \ldots, a_q \) take values is ordered.)

By introducing the nilpotent coboundary operator

\[
\delta \equiv \sum_a \xi^a, \tag{4.3}
\]

one can recast the Ward identity (3.4) in a compact form

\[
\delta Z_0^{(2N)} = d_p Z_1^{(2N-1)}. \tag{4.4}
\]
Since $d_p$ and the coboundary operator $\delta$ anticommute, Eq. (4.4) implies the existence of the familiar chain of descent equations:

\[
\begin{align*}
\delta Z_0^{(2N)} &= d_p Z_1^{(2N-1)} \\
\delta Z_1^{(2N-1)} &= d_p Z_2^{(2N-2)} \\
&\cdots \\
\delta Z_{2N-1}^{(1)} &= d_p Z_2^{(0)} \\
\delta Z_2^{(0)} &= 0,
\end{align*}
\]  

(4.5)

where $Z_q^{(2N-q)}$ is a $q$-cochain with values in the space of local $(2N - q)$-forms defined on the $q$-fold intersections $U_{a_0a_1...a_q}$.

Let $\{Z_0^{(2N)}, Z_1^{(2N-1)}, \ldots, Z_{2N-1}^{(1)}, Z_{2N}^{(0)}\}$ be a solution of the chain of Ward identities in Eq. (4.3), and consider the following object:

\[
Z_D = Z_0^{(2N)} + Z_1^{(2N-1)} + \cdots + Z_{2N}^{(0)}. 
\]

(4.6)

$Z_D$ is a cocycle of the Čech-De Rham complex $C^*(\mathcal{M}_{g,n}; \Omega^*) \equiv \oplus_{p,q \geq 0} C^q(\mathcal{M}_{g,n}; \Omega^p)$. This is the complex equipped with the nilpotent operator $D \equiv d_p - \delta$. (Recall that the variables $\xi^a$ anticommute by definition with the exterior differential $d_p$.) In fact it is easily proved that the Ward identities (4.3) are equivalent to the single cocycle equation

\[
D Z_D = 0. 
\]

(4.7)

A well-known result of cohomology theory is that the cohomology of the Čech-De Rham complex is isomorphic to the De Rham cohomology of globally defined forms [23]. This means that given a solution of the Ward identities (4.3), (4.7) one can construct a globally defined closed form on $\mathcal{M}_{g,n}$. The equivalence of Čech-De Rham cohomology and De Rham cohomology rests on the existence of the homotopy operator

\[
K \equiv \sum_a \rho^a(m) \frac{\partial}{\partial \xi^a},
\]

(4.8)

where $\rho^a(m)$ is a partition of unity on $\mathcal{M}_{g,n}$; $K$ has the obvious but fundamental property

\[
\{K, \delta\} = 1,
\]

(4.9)
so that it acts on cocycles as the inverse of $\delta$. One can use $K$ to solve the last of the equations in (4.3), and write

$$Z^{(0)}_{2N} = \delta K Z^{(0)}_{2N}. \tag{4.10}$$

Plugging this expression into the equation before last in (4.3), one obtains:

$$\delta(Z^{(1)}_{2N-1} + d_p K Z^{(0)}_{2N}) = 0, \tag{4.11}$$

which again can be solved in terms of the homotopy operator:

$$Z^{(1)}_{2N-1} = -d_p K Z^{(0)}_{2N} + \delta K Z^{(1)}_{2N-1} + \delta K d_p K Z^{(0)}_{2N}. \tag{4.12}$$

Continuing in this way, it is possible to climb the chain (4.5) up to the top, and to derive:

$$\delta \left( \sum_{q=0}^{2N} (d_p K)^q Z^{(2N-q)}_q \right) = 0. \tag{4.13}$$

Equation (4.13) implies that the components $(Z^{\text{global}}_0)_a$ of the following 0-cocycle

$$Z^{\text{global}}_0 = \sum_a \xi^a (Z^{\text{global}}_0)_a = Z^{(2N)}_0 + d_p K Z^{(2N-1)}_1 + \cdots + (d_p K)^{2N} Z^{(0)}_{2N} \tag{4.14}$$

are the restrictions to $U_a$ of a globally defined form $Z^{\text{global}}$ on $\mathcal{M}_{g,n}$.

Suppose now that the degree $2N$ of the globally defined form $Z^{\text{global}}$ equals the dimension $6g - 6 + 2n$ of $\mathcal{M}_{g,n}$. Then, $Z^{\text{global}}$ can be integrated over $\mathcal{M}_{g,n}$ and this integral can be directly expressed in terms of the solutions of the Ward identities (4.3). Let $\{C_a\}$ be a cell decomposition of $\mathcal{M}_{g,n}$, with $C_a \subset U_a$, and let $C_{a_0 a_1 \ldots a_q} \equiv \cap_{i=1}^q C_{a_i}$ of codimension $q$ in $\mathcal{M}_{g,n}$ oriented in such a way that the boundary of a cell $\partial C_{a_0 a_1 \ldots a_q}$ satisfies:

$$\partial C_{a_0 a_1 \ldots a_q} = \bigcup_b C_{a_0 a_1 \ldots a_q b}, \tag{4.15}$$

where we have introduced the convention that $C_{a_0 a_1 \ldots a_q}$ is antisymmetric in its indices in the sense that it changes orientation when exchanging a pair of indices. We have defined in this way $q$-chains of cells of codimension $q$ that are adjoint to the $q$-cochains defined above. Indeed, given a $q$-chain and a $q$-cochain, we can define the integral:

$$\int_{C_q} Z^{(2N-q)}_q \equiv \sum_{a_0 < a_1 \ldots < a_q} \int_{C_{a_0 a_1 \ldots a_q}} \left( Z^{(2N-q)}_q \right)_{a_0 a_1 \ldots a_q}. \tag{4.16}$$

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Taking into account the anticommuting variables $\xi^a$ introduced above, it is clear that this integral can be interpreted as a combined ordinary and Berezin integral, that is:

$$
\int_{C_q} = \sum_{a_0 < a_1 \ldots < a_q} \int_{C_{a_0 a_1 \ldots a_q}} d\xi^{a_q} \ldots d\xi^{a_0}.
$$

(4.17)

Exploiting this formalism and the identity

$$
\delta \int_{C_q} + (-1)^q \int_{C_q} \delta = \frac{1}{q!} \sum_{a_0 a_1 \ldots a_q} \int_{C_{a_0 a_1 \ldots a_q}} d\xi^{a_q-1} \ldots d\xi^{a_0},
$$

it is not difficult to prove that

$$
\int_{C_q} d_p Z_{q}^{(2N-q)} = (-1)^{q+1} \int_{C_{q+1}} \delta Z_{q}^{(2N-q)}.
$$

(4.19)

Using Eqs. (4.14) and (4.19) the integral of the globally defined form $Z^{global}$ is expressed in terms of the integrals of the cochains satisfying the Ward identities (4.5):

$$
\int_{M_{g,n}} Z^{global} = \sum_{q=0}^{2N} (-1)^q \sum_{a_0 < a_1 \ldots < a_q} \int_{C_{a_0 a_1 \ldots a_q}} \left( Z_{q}^{(2N-q)} \right)^{a_0 a_1 \ldots a_q}.
$$

(4.20)

In the previous section we derived the following expression for $Z_{1}^{(2N-1)}$:

$$
(Z_{1}^{(2N-1)})_{ab} = \int_0^1 dt \int d\Phi e^{-S_{ab}(t)} (I_b - I_a) O.
$$

(4.21)

By repeated use of the main Ward identity (3.4) one can derive an analogous formula for the $Z_{q}^{(2N-q)}$.

Let us introduce the interpolating action

$$
S_{a_0 \ldots a_q (t_0, \ldots , t_q)} = S_0 - s \sum_{k=0}^q t_k \int d^2 x \beta_{\mu\nu} (L_{\hat{v}_{ak} \hat{\eta}_0})^\mu\nu,
$$

(4.22)

depending on the $q+1$ real parameters $t_k$. Let us also use the notation

$$
\langle \langle O \rangle \rangle_{a_0, \ldots , a_q} = \int_0^1 \prod_{k=0}^q dt_k \delta \left( \sum_{k=0}^q t_k - 1 \right) \int d\Phi e^{-S_{a_0 \ldots a_q (t_0, \ldots , t_q)}} O
$$

$$
= \int_0^1 dt_0 \int_0^{1-t_0} dt_1 \ldots \int_0^{1-t_0 - \ldots - t_{q-2}} dt_{q-1} \int d\Phi e^{-S_{a_0 \ldots a_q (t_0, \ldots , t_q-1, \sum_{k=0}^{q-1} t_k)}} O,
$$

(4.23)
for the functional average of the observable $O$ with the interpolating action (4.22) together with the integration over the parameters $t_k$ with the measure specified above.

In Appendix A, we prove the following formula for the components of the cochain $Z^{(2N-q)}_q$:

$$
(Z^{(2N-q)}_q)_{a_0 \ldots a_q} = \sum_{k=0}^q (-1)^k \langle \langle I_{a_0} \ldots \hat{I}_{a_k} \ldots I_{a_q} O \rangle \rangle_{a_0 \ldots a_q}, \tag{4.24}
$$

where the check mark above $I_{a_k}$ means that this term should be omitted.

It is clear from this expression that the last non-vanishing cochain in the descent (4.3) is $Z^{(N)}_N$, since each of the operators $I_{a_k}$ eats up one $\gamma$ field. From Eq. (4.24) we have

$$
(Z^{(N)}_N)_{a_0, \ldots, a_N} = \sum_{l=0}^N (-1)^l \langle \langle I_{a_0} \ldots \hat{I}_{a_l} \ldots I_{a_N} O \rangle \rangle_{a_0, \ldots, a_N}. \tag{4.25}
$$

Since $I_{a_0} \ldots \hat{I}_{a_l} \ldots I_{a_N} O$ does not contain the gravitino field but only the metric field, its functional average with the interpolating action (4.22) is independent of the $t_k$ parameters. Moreover

$$
\int_0^1 \prod_{k=1}^N dt_k \delta \left( \sum_{k=0}^N t_k - 1 \right) = \frac{1}{N!}. \tag{4.26}
$$

Therefore

$$
Z^{(N)}_N = \delta S^{(N)}_{N-1}, \tag{4.27}
$$

where $S^{(N)}_{N-1}$ is the $(N-1)$-cochain with values in the $N$-forms

$$
S^{(N)}_{N-1} = \frac{1}{N!} \sum_{a_1 < \ldots < a_N} \xi^{a_1} \ldots \xi^{a_N} \langle I_{a_1} \ldots I_{a_N} O \rangle_0 = \left( \frac{1}{N!} \right)^2 \sum_{a_1, \ldots, a_N} \xi^{a_1} \ldots \xi^{a_N} \langle I_{a_1} \ldots I_{a_N} O \rangle_0 = \left( \frac{1}{N!} \right)^2 (-1)^{\binom{N-1}{2}} \langle I^N O \rangle_0, \tag{4.28}
$$

and we introduced the operator $I \equiv \sum_a \xi^a I_a$.

Recalling the product structure (3.3) of the observables, one finally obtains

$$
S^{(N)}_{N-1} = \frac{1}{N!} (-1)^{\binom{N-1}{2}} \prod_i f(x_i)^{n_i}, \tag{4.29}
$$

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where \( f(x_i) \) is the 0-cochain with values in the local 1-forms of \( \mathcal{M}_{g,n} \) defined by Eq. (3.13), that is

\[
f(x_i) \equiv \sum_a \xi^a \langle I_a \sigma_1^{(0)}(x_i) \rangle_0 = \frac{1}{4\pi} \sum_a \xi^a \text{tr} \left( \epsilon M^{-1}_a d_p M_a \right)
= \frac{1}{4\pi i} \sum_a \xi^a \left[ d_p \log \frac{\lambda_a}{\lambda_a} + \frac{\mu_a d_p \bar{\mu}_a - \bar{\mu}_a d_p \mu_a}{1 - |\mu_a|^2} \right].
\]

(4.30)

It should be noted at this point that the expectation value of the superghost \( \gamma \) vanishes in the functional measure that we are considering (Eq. (2.12)). The reason is that the field \( \beta \) conjugate to \( \gamma \) appears only in the supercurrent \( L_c \beta \). However, this latter term is zero inside any correlator since it is linear in \( c \) and no antighost \( b \) is present either in the measure or in the observables.

It follows that expectation values \( \langle O(c, \gamma, \hat{g}, \hat{\psi}) \rangle_a = \langle \prod_k \sigma_{n_k}^{(0)} \rangle_a \) all vanish, with the exception of those containing exclusively dilatons \( \sigma_1^{(0)} \) and puncture operators. (Such exceptional correlators can only occur in geni 0 and 1.) In fact, these exceptional correlators also vanish if one takes a gauge for which \( d_p \hat{\eta}_a(x_i; m) = 0 \) at the points \( x_i \) where the observables are inserted.

In other words the “bulk” term of the generic correlator – i.e. the top-component \( Z^{(2N)}_0 \) of the Čech-De Rham cocycle \( Z_D \) – vanishes with the given action for 2D topological gravity. However, our analysis makes it clear that this is not a gauge-invariant statement: the “bulk” of the correlators is not a globally defined form on moduli space. The higher order cochains \( Z^{(2N-q)}_q \) in Eq. (4.24) solving the chain of Ward identities (4.5) should be included in the evaluation of integral (4.20) over moduli space of the the gauge-invariant global form \( Z^{\text{global}} \). It is worth emphasizing that in our framework the “contact” terms – i.e. the contribution of the higher-order cochains to the integrated gauge-invariant correlators – have been uniquely determined by solving the local Ward identities (3.4), (4.5) characterizing the functional measure of the theory.

The solution (4.24) of the Ward identities (4.5) simplifies considerably in a gauge in which \( d_p \hat{\eta}_a(x_i; m) = 0 \). In this case both the top-component \( Z^{(2N)}_0 \) and all the descendant cochains \( Z^{(2N-q)}_q \) vanish, with the exception of \( Z^{(N)}_N \). Therefore the integrated gauge-invariant correlators

\[
\int_{\mathcal{M}_{g,n}} Z^{\text{global}} = (-1)^N \frac{1}{N!} \sum_{a_0 < a_1 \ldots < a_N} \int_{C_{a_0 \ldots a_N}} (Z^{(N)}_N)_{a_0 a_1 \ldots a_N}
= (-1)^{N(N+1)/2} \frac{1}{N!} \sum_{a_0 < a_1 \ldots < a_N} \int_{C_{a_0 \ldots a_N}} (\delta S^{(N)}_{N-1})_{a_0 \ldots a_N},
\]

(4.31)
are rewritten, after taking into account Eq. (4.19):

\[
\int_{\mathcal{M}_{g,n}} Z^{\text{global}} = (-1)^{N(N-1)} \frac{1}{N!} \sum_{a_1 < \ldots < a_N} \int_{\partial C_{a_1 \ldots a_N}} (S^{(N)}_{N-1})_{a_1 \ldots a_N}.
\]  

Moreover \(S^{(N)}_{N-1}\) in Eq. (4.32) is \(d_p\)-closed since the 0-cochains with values in the 1-forms \(f(x_i)\) are \(d_p\)-closed when \(d_p\hat{\eta}_a(x_i; m) = 0:\)

\[
f(x_i) = \frac{1}{4\pi i} d_p \log \frac{\lambda_a}{\lambda_a} (x_i; m).
\]  

Thus, only the cochains \(C_{a_1 \ldots a_N}\) whose interiors contain points where some \(\lambda_{a_k}\) vanish contribute to the sum (4.32). The integrating factors \(\lambda_{a_k}\) have zeros at those points in moduli space which correspond to degenerate Riemann surfaces. We conclude that, in this gauge, the computation of globally defined correlators can be localized at the boundary of moduli space of Riemann surfaces.

5. Relation to the algebro-geometric formulation

The observables of topological gravity are expected to correspond to certain cohomology classes of \(\hat{\mathcal{M}}_{g,n}\) (the compactification of \(\mathcal{M}_{g,n}\) obtained adding curves with double points) introduced by Morita [24], Mumford [25] and Miller [26]. The intersection numbers of such classes have been computed by Kontsevitch [27]. In this section we will show that the Čech-De Rham cocycles (4.24) we obtained by solving the local Ward identities of 2D topological gravity are indeed equivalent to the Mumford-Morita-Miller classes.

Consider the holomorphic line bundles \(L_i\) over \(\hat{\mathcal{M}}_{g,n}\) whose fibers are the cotangent bundles at the \(n\) marked points \(x_i\) of the Riemann surface. Let \(c_1(L_i)\) be the first Chern class of \(L_i\). The expectation [4] is that the global form corresponding to \(\langle \sigma^{(0)}(x_i) \rangle\) be cohomologous to \(c_1(L_i)^n\). Indeed, the Čech-De Rham cocycle \(\{Z^{(2)}_0, Z^{(1)}_1, Z^{(0)}_2\}\) corresponding to \(\sigma^{(0)}(x_i)\) is, by virtue of our general formula (4.24):

\[
(Z^{(2)}_0)_a = \frac{1}{2\pi i} \frac{d_p \mu_a \wedge d_p \bar{\mu}_a}{1 - |\mu_a|^2},
\]

\[
(Z^{(1)}_1)_{ab} = \langle I_b \sigma^{(0)}_1(x_i) \rangle_0 - \langle I_a \sigma^{(0)}_1(x_i) \rangle_0 = \frac{1}{4\pi i} \left( d_p \log \frac{\lambda_b}{\lambda_a} - d_p \log \frac{\lambda_a}{\lambda_b} \right) + \frac{\mu_b d_p \bar{\mu}_b - \bar{\mu}_b d_p \mu_b}{1 - |\mu_b|^2} - \frac{\mu_a d_p \bar{\mu}_a - \bar{\mu}_a d_p \mu_a}{1 - |\mu_a|^2},
\]

\[
(Z^{(0)}_2)_{abc} = 0.
\]
The corresponding global 2-form $Z^{(2)}$ can be integrated over any 2-cycle $C^{(2)}$ of $M_{g,n}$. If $C_a$ is a cell decomposition of $C^{(2)}$ one has, according to formula (4.20):

$$\int_{C^{(2)}} Z^{(2)} = \sum_a \int_{C_a} (Z_0^{(2)})_a - \sum_{a<b} \int_{C_{ab}} (Z_1^{(1)})_{ab}$$

$$= \sum_a \int_{C_a} \frac{1}{2\pi i} \frac{d_p \mu_a \wedge d_p \bar{\mu}_a}{(1 - |\mu_a|^2)^2}$$

$$+ \sum_a \int_{\partial C_a} \frac{1}{4\pi i} \left[ d_p \log \frac{\lambda_a}{\bar{\lambda}_a} + \frac{\mu_a d_p \bar{\mu}_a - \bar{\mu}_a d_p \mu_a}{1 - |\mu_a|^2} \right]$$

$$= \sum_a \int_{\partial C_a} \frac{1}{4\pi i} d_p \log \frac{\lambda_a}{\bar{\lambda}_a}.$$  \hspace{1cm} (5.2)

The crucial assumption in Eq. (5.2) was the smoothness of the Beltrami differentials $\mu_a(x_i; m)$ at all points in $C^{(2)}$, including the points corresponding to surfaces with nodes. This justifies applying Stokes theorem to cancel $\int_{\partial C_a} Z_1^{(1)}$. This regularity condition is natural from the field theoretical point of view, since $\mu_a$ is the expectation value of the reduced metric field $\hat{g}^{\mu\nu}$. From the mathematical point of view, smoothness of the complex structure at the marked points is a feature of the Mumford-Deligne compactification $\bar{M}_{g,n}$ of moduli space.

If $M_{g,n}$ were smooth the integral in Eq. (5.2) would equal the number of zeros and poles of $\lambda_a$ on $C^{(2)}$ taken with opposite signs. Since $M_{g,n}$ is an orbifold, this means that the zeros and the poles might have fractional weights corresponding to the order of the orbifold singularity. At any rate the integral in Eq. (5.2) coincides with the integral $\int_{C^{(2)}} c_1(\mathcal{L}_i)$. For, a holomorphic section of $\mathcal{L}_i$ is given by $dZ_m = \lambda_a (dz + \mu_a d\bar{z})(x_i; m)$. Since the zeros and poles of this holomorphic section are precisely the zeros and poles of $\lambda_a$, $\int_{C^{(2)}} c_1(\mathcal{L}_i) = \int_{C^{(2)}} Z^{(2)}$ for any $C^{(2)}$.

6. $\langle \sigma_1 \sigma_0^3 \rangle_{g=0}$

In this expression, and in the analogous ones describing the other examples we label by $\sigma_0$ the fixed points of the Riemann surface that are not associated with any operator. Hence by $\langle \sigma_1 \sigma_0^3 \rangle_{g=0}$ we mean the vacuum expectation value of $\sigma_1$ on a sphere with four fixed points. This is a 2-form on $M_{0,4}$. Let $x_1$ be the point on the Riemann sphere where $\sigma_1(x_1)$ is inserted, $m$ the associated complex modulus and $P_0 = 0$, $P_1 = 1$, and $P_2 = \infty$ the points where the $\sigma_0$ operators are located.
Consider a cell decomposition $C_a$, with $a = 0, 1, 2, 3$ of $M_{0,4}$ where $C_0, C_1$ and $C_2$ are disks surrounding $P_0, P_1,$ and $P_2$ respectively, and $C_3$ is the closure of the complement in $M_{0,4}$ of $\bigcup_{a=0,1,2}C_a$. In Appendix B, we review the “plumbing fixture” construction and show explicitly that one can choose $\mu_a(x_1; m) = 0$ and $f_a(x_1) = \frac{1}{4\pi i} \log \frac{\lambda_a}{\bar{\lambda}_a}(x_1; m)$ such that

$$f_a(x_1) = d\theta_a, \quad \text{for} \quad a = 0, 1, 2, \quad f_3(x_1) = 0,$$

where $\theta_a \equiv \arg(m - P_a)$, for $a = 0, 1$ and $\theta_2 \equiv \arg(\frac{1}{m})$.

Taking into account the orientation of $C_2$ with respect to $C_0$ and $C_1$, one obtains from Eq. (5.2)

$$\int_{M_{0,4}} \langle \sigma_1 \sigma_3 \rangle = 1 + 1 - 1 = 1,$$

in agreement with the dilaton equation.

7. $\langle \sigma_1^2 \sigma_0^3 \rangle_{g=0}$ and $\langle \sigma_2 \sigma_4^4 \rangle_{g=0}$

The non-vanishing correlators on $M_{0,5}$ are

$$Z^{(4)} \equiv \langle \sigma_1^2 \sigma_0^3 \rangle \quad \text{and} \quad \tilde{Z}^{(4)} \equiv \langle \sigma_2 \sigma_4^4 \rangle.$$

$M_{0,5}$ is parametrized by two complex coordinates $m_1$ and $m_2$ representing the positions on the complex sphere of the two punctures $x_1$ and $x_2$. Let us choose a gauge for which $\mu(x_i; m) = 0$ for $i = 1, 2$. Thus, as explained in section 4, the non-vanishing terms of the Čech-De Rham towers are

$$Z_2^{(2)} = \frac{1}{2} \delta(f(x_1)f(x_2)) \quad \text{and} \quad \tilde{Z}_2^{(2)} = \frac{1}{2} \delta(f(x_1)^2).$$

Let us denote by $N_{1\alpha}$ the hypersurfaces of $M_{0,5}$, of complex codimension 1 characterized by the equation $m_1 = P_\alpha$, with $\alpha = 0, 1, 2$ and $P_0 = 0, P_1 = 1$ and $P_2 = \infty$. Analogously, let $N_{2\alpha}$ be the hypersurface characterized by the equation $m_2 = P_\alpha$. Finally, let $\tilde{N}$ be the hypersurface with $m_1 = m_2$.

A generic point in $N_{1\alpha} \cup N_{2\alpha} \cup \tilde{N}$ represents a 5-punctured complex sphere with one node. The nine points $m_{\alpha\beta} \in M_{0,5}$ with $m_1 = P_\alpha$ and $m_2 = P_\beta$ where two or three of these hypersurfaces intersect correspond to 5-punctured Riemann spheres with two nodes. As explained in section 4, the computation of the correlators can be localized around such points.

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At the points \( m_{\alpha\beta} \) with \( \alpha \neq \beta \), two hypersurfaces \( N_{1\alpha} \) and \( N_{2\beta} \) intersect. Near each of these six points, there are only two relevant cells of real codimension two, with torus topology,

\[
C_{0AB} = \{ |m_1 - P_\alpha| = \epsilon, \ |m_2 - P_\beta| = \epsilon' \} \tag{7.3}
\]

\[
C'_{0CD} = \{ |m_1 - P_\alpha| = \epsilon', \ |m_2 - P_\beta| = \epsilon \}
\]

where \( \epsilon < \epsilon' \).

\( C_{0AB} \) is the triple intersection of cells \( C_0, C_A \) and \( C_B \) for which the 0-cochain, with values in the 1-forms, \( f_a(x_i) = \frac{1}{4\pi i} d_a \log \frac{\lambda_a}{\lambda_\alpha} (x_i; m) \), can be chosen to be

\[
f_0(x_1) = f_0(x_2) = 0 \\
f_A(x_1) = d\theta_1, \quad f_A(x_2) = d\theta_2 \\
f_B(x_1) = d\theta_1, \quad f_B(x_2) = 0, \tag{7.4}
\]

with \( \theta_1 \equiv \arg(m_1 - P_\alpha) \) and \( \theta_2 \equiv \arg(m_2 - P_\beta) \). \( C_{0CD} \) is the triple intersection with \( f_0(x_i), f_C(x_i) \) and \( f_D(x_i) \) defined as in Eq. (7.4) after exchanging \( x_1 \) and \( x_2 \).

From Eq. (7.2) one derives the components of \( Z_2^{(2)} \) and \( \tilde{Z}_2^{(2)} \) on \( C_{0AB} \) and \( C_{0CD} \):

\[
\begin{align*}
(Z_2^{(2)})_{0AB} &= (Z_2^{(2)})_{0CD} = \frac{1}{2} d\theta_1 \wedge d\theta_2 \\
(\tilde{Z}_2^{(2)})_{0AB} &= (\tilde{Z}_2^{(2)})_{0CD} = 0. \tag{7.5}
\end{align*}
\]

Near each of the points \( m_{\alpha\alpha} \) there is, beyond two cells analogous to \( C_{0AB} \) and \( C_{0CD} \), a third relevant cell,

\[
C_{0FD} = \{ |m_1 - m_2| = \epsilon, \ |m_2 - P_\alpha| = \epsilon' \}, \tag{7.6}
\]

which is the triple intersection of cells \( C_0, C_F \) and \( C_E \), where

\[
\begin{align*}
f_0(x_1) &= f_0(x_2) = 0 \\
f_E(x_1) &= d\theta_1, \quad f_E(x_2) = d\theta_2 \\
f_F(x_1) &= f_F(x_2) = d\theta_3, \tag{7.7}
\end{align*}
\]

where \( \theta_3 \equiv \arg(m_1 - m_2) \). From Eq. (7.7) one derives the components of \( Z_2^{(2)} \) and \( \tilde{Z}_2^{(2)} \) on \( C_{0EF} \):

\[
\begin{align*}
(Z_2^{(2)})_{0EF} &= \frac{1}{2} (f_E(x_1)f_F(x_2) + f_F(x_1)f_E(x_2)) = \frac{1}{2} (d\theta_1 + d\theta_2) \wedge d\theta_3 \\
&= d\theta_2 \wedge d\theta_3 \tag{7.8}
\end{align*}
\]

\[
(\tilde{Z}_2^{(2)})_{0EF} = \frac{1}{2} (2f_E(x_1)f_F(x_1)) = d\theta_2 \wedge d\theta_3,
\]

25
where we took into account that \((d\theta_1 - d\theta_2) \wedge d\theta_3 = 0\) on \(C_{0EF}\).

To sum the contributions of the various cells to the integrated correlator, we need only to take into account the relative orientations of the triple intersections at each of the nine points \(m_{\alpha\beta}\).

Let us first consider \(Z^{(4)} = \langle \sigma_1^2 \sigma_3^3 \rangle\). The union of cells of type \(C_{0AB} \cup C_{0CD}\) give \(\pm 1\) for each of the nine points \(m_{\alpha\beta}\). It is straightforward to verify that the contribution is 1 for the five points \(m_{\alpha\beta}\) with \(\alpha, \beta = 0, 1\) and \(\alpha = \beta = 2\), and \(-1\) for the other four points. The total is 1. The cells of type \(C_{0EF}\) give instead 1 for \(m_{\alpha\alpha}\) with \(\alpha = 0, 1\) and \(-1\) for \(\alpha = 2\), for a total of 1. Thus,

\[
\int_{\mathcal{M}_{0,5}} \langle \sigma_1^2 \sigma_3^3 \rangle = 2, \tag{7.9}
\]

in agreement with the dilaton equation.

Only cells of type \(C_{0EF}\) contribute instead to \(\tilde{Z}^{(4)} = \langle \sigma_2 \sigma_4^4 \rangle\), the ones near \(m_{00}\) and \(m_{11}\) giving 1 each and the cell near \(m_{22}\) giving \(-1\). Hence,

\[
\int_{\mathcal{M}_{0,5}} \langle \sigma_2 \sigma_4^4 \rangle = 1, \tag{7.10}
\]

as predicted by the puncture equation.

8. Conclusions

We have elucidated an intriguing question arising in the context of 2D topological gravity. The Lagrangian that leads to a free superconformal model also leads to a functional measure, for which averages of all equivariant observables vanish, locally on the moduli space. The non-trivial content of the theory should be encoded in contact terms sitting at the boundary of moduli space, but it was not easy to see, in the usual framework, how to compute these contact terms directly from the functional integral. We showed that the local Ward identities characterizing the dependence of the functional measure on the background gauge do capture the contact terms at the infinity in moduli space. Indeed the distinction between “bulk” and “contacts” is not a gauge-invariant one. Contacts are required precisely for restoring gauge invariance of the (possibly vanishing) “bulk” part of the correlators. We have shown that the “bulk” of correlators of equivariant observables is not globally defined because of the non-trivial dependence of the observables on the superghost.
We think that our analysis, beyond providing an intellectually satisfying understanding of contact terms in the particular context of 2D topological gravity, has a general validity and therefore might prove useful in situations where contacts have not yet been computed. We are thinking for example of the exotic type topological string models relevant for 2D QCD [9] or of superstring models. We also expect that our approach should clarify the nature of the holomorphic anomaly discovered in the context of topological gravity coupled to \( N = 2 \) superconformal models [10].

Finally, and more speculatively, the mechanism to implement global BRS invariance that we have analysed in 2D topological gravity might be of some relevance to analogous global non-perturbative issues associated with the Gribov horizon in 4D non-Abelian gauge theories.

Acknowledgements
It is a pleasure to thank R. Collina and R. Stora for many useful discussions. We also would like to thank E. Verlinde for interesting comments and for bringing to our attention Ref. [28], in which the relevance of Čech-De Rham cohomology in the context of superstring had been pointed out. This work is partially supported by the ECPR, contract SC1-CT92-0789.

Appendix A. Proof of Eq. (4.24)
In this appendix we prove formula (4.24) for the components of the cochain \( Z^{(2N-q)}_q \):

\[
\left( Z^{(2N-q)}_q \right)_{a_0...a_q} = \sum_{k=0}^{q} (-1)^k \langle \langle I_{a_0} ... \tilde{I}_{a_k} ... I_{a_q} O \rangle \rangle_{a_0...a_q},
\]

where the check mark above \( I_{a_k} \) means that this term should be omitted.

We need to show that \( \delta Z^{(2N-q)}_q = d_p Z^{(2N-q-1)}_q \). One has

\[
\left( \delta Z^{(2N-q)}_q \right)_{a_0...a_{q+1}} = \sum_{l=0}^{q+1} (-1)^l \left( Z^{(2N-q)}_q \right)_{a_0...\tilde{a}_l...a_{q+1}}
\]

\[
\begin{align*}
&= \sum_{l=0}^{q+1} (-1)^l \langle \langle \left( \sum_{k=0}^{l-1} (-1)^k I_{a_0} ... \tilde{I}_{a_k} ... \tilde{I}_{a_l} ... I_{a_{q+1}} \right) \rangle \rangle_{a_0...\tilde{a}_l...a_{q+1}} \\
&\quad - \sum_{k=l+1}^{q+1} (-1)^k I_{a_0} ... \tilde{I}_{a_l} ... \tilde{I}_{a_k} ... I_{a_{q+1}} \rangle \rangle_{a_0...\tilde{a}_l...a_{q+1}} \\
&\quad - \langle \langle I_{a_0} ... \tilde{I}_{a_k} ... \tilde{I}_{a_l} ... I_{a_{q+1}} O \rangle \rangle_{a_0...\tilde{a}_l...a_{q+1}} \\
&\quad - \langle \langle I_{a_0} ... \tilde{I}_{a_k} ... \tilde{I}_{a_l} ... I_{a_{q+1}} O \rangle \rangle_{a_0...\tilde{a}_k...a_{q+1}}.
\end{align*}
\]

(A.2)
According to the Eq. (4.23), the functional measure associated with the right-hand side of (A.2), that is the measure corresponding to $\langle \langle \ldots \rangle \rangle a_0 \ldots a_{q+1} - \langle \langle \ldots \rangle \rangle a_0 \ldots \hat{a}_k \ldots a_{q+1}$, for a suitable choice of the $t$ variables, can be written in the form

$$
\int_0^1 \prod_{i=0, i \neq k, l}^{q+1} dt_i \, dt'(t' + \sum_{i=0, i \neq k, l}^{q+1} t_i - 1) \left( e^{-S_{a_0 \ldots \hat{a}_k \ldots a_{q+1}}(t_0, \ldots, t', \ldots, t_{q+1})} - e^{-S_{a_0 \ldots \hat{a}_k \ldots a_{q+1}}(t_0, \ldots, \hat{t}_k, \ldots, t_{q+1})} \right),
$$
(A.3)

Taking into account that

$$
\frac{\partial}{\partial t_k} S_{a_0 \ldots \hat{a}_k \ldots a_{q+1}}(t_0, \ldots, t_k, \ldots, t_{q+1}) = -s \int d^2 x \beta_{\mu \nu} \left( \mathcal{L}_{\tilde{v}_a k} \tilde{\eta}_0 \right)_{\mu \nu},
$$
(A.4)

we transform (A.3) into the interpolating measure:

$$
\int_0^1 \prod_{i=0, i \neq k, l}^{q+1} dt_i \, dt'(t' + \sum_{i=0, i \neq k, l}^{q+1} t_i - 1) \int_0^1 dt' \left[ s \int d^2 x \beta_{\mu \nu} \left( \mathcal{L}_{\tilde{v}_a k} \tilde{\eta}_0 - \mathcal{L}_{\tilde{v}_a l} \tilde{\eta}_0 \right)_{\mu \nu} \right] e^{-S_{a_0 \ldots \hat{a}_k \ldots a_{q+1}}(t_0, \ldots, (1-u)t', \ldots, ut' \ldots, t_{q+1})},
$$
(A.5)

which, after the relabelling $(1-u)t' \rightarrow t_k$ and $u t' \rightarrow t_l$, becomes

$$
\int_0^{1/q+1} \prod_{i=0}^{q+1} dt_i \left( \sum_{i=0}^{q+1} t_i - 1 \right) \left[ s \int d^2 x \beta_{\mu \nu} \left( \mathcal{L}_{\tilde{v}_a k} \tilde{\eta}_0 - \mathcal{L}_{\tilde{v}_a l} \tilde{\eta}_0 \right)_{\mu \nu} \right] e^{-S_{a_0 \ldots \hat{a}_k \ldots a_{q+1}}(t_0, \ldots, t_{q+1})}.
$$
(A.6)

Therefore we can rewrite (A.2) as follows:

$$
(\delta Z^{(2N-q)})_{a_0 \ldots a_{q+1}} = \sum_{k<l} (-1)^{l+k} \left\langle \left\langle \left[ s \int d^2 x \beta_{\mu \nu} \left( \mathcal{L}_{\tilde{v}_a k} \tilde{\eta}_0 - \mathcal{L}_{\tilde{v}_a l} \tilde{\eta}_0 \right)_{\mu \nu} \right] I_{a_0} \ldots \tilde{I}_a \ldots \tilde{I}_l \ldots I_{a_{q+1}} O \right\rangle \right\rangle_{a_0 \ldots a_{q+1}}
$$

$$
+ \sum_{k<l} (-1)^{l+k} \left\langle \left\langle \left[ \left( I_{a_k} - I_{a_l} \right) S_{a_0 \ldots a_{q+1}} \right] s I_{a_0} \ldots \tilde{I}_a \ldots I_{a_{q+1}} O \right\rangle \right\rangle_{a_0 \ldots a_{q+1}}
$$
(A.7)

where we first made use of the Slavnov-Taylor identity (2.22) (ii), then substituted $\tilde{\eta}^{\mu \nu}$ with $\hat{\eta}^{\mu \nu}$ and finally took into account the equations

$$
I_{a_{k,l}} S_{a_0 \ldots a_{q+1}} = \int d^2 x \beta_{\mu \nu} \left( \mathcal{L}_{\tilde{v}_a k,l} \hat{\eta} \right)_{\mu \nu}.
$$
(A.8)
Integrating by parts the operators $I_{a_k}$ and $I_{a_l}$ in (A.7) we obtain

\[
(\delta Z_{q}^{(2N-q)})_{a_0...a_{q+1}} = \sum_{k<l} (-1)^{l+k} d_p \langle\langle (I_{a_k} - I_{a_l}) I_{a_0}...\hat{I}_{a_k} \\
...\hat{I}_{a_l}...I_{a_{q+1} O})\rangle_{a_0...a_{q+1}} \\
+ \sum_{k<l} (-1)^{l+k} \langle\langle (I_{a_k} - I_{a_l}) s I_{a_0}...\hat{I}_{a_k} \\
...\hat{I}_{a_l}...I_{a_{q+1} O})\rangle_{a_0...a_{q+1}} \\
= \sum_{k<l} (-1)^{l+k} \langle\langle \{s, I_{a_k}\} - \{s, I_{a_l}\}\rangle_{a_0...\hat{I}_{a_k} \\
...\hat{I}_{a_l}...I_{a_{q+1} O})\rangle_{a_0...a_{q+1}}.
\] (A.9)

Now, $\{s, I_{a}\}$ is easily computed to be:

\[
\int d^2 x \left[ (\Gamma^i v^\mu_i(x) - d_C \hat{v}^\mu_a(x) + L_c \hat{v}^\mu_a(x)) \frac{\delta}{\delta \gamma^\mu}(x) \\
- \hat{v}^\mu_a(x) \frac{\delta}{\delta \gamma^\mu}(x) + L_{\bar{v}_a} g^\mu\nu(x) \frac{\delta}{\delta \psi^\mu\nu}(x) \right].
\] (A.10)

which is a functional partial differential operator that commutes with any $I_b$. Hence, we obtain from (A.9) the desired result:

\[
(\delta Z_{q}^{(2N-q)})_{a_0...a_{q+1}} = \sum_{l} (-1)^l \sum_{k=0}^{l-1} (-1)^k \langle\langle \{s, I_{a_k}\}\rangle_{a_0...\hat{I}_{a_k} \\
...\hat{I}_{a_l}...I_{a_{q+1} O})\rangle_{a_0...a_{q+1}} \\
+ \sum_{l} (-1)^l \sum_{k=l+1}^{q+1} (-1)^{k+1} \langle\langle \{s, I_{a_k}\}\rangle_{a_0...\hat{I}_{a_k} \\
...\hat{I}_{a_l}...I_{a_{q+1} O})\rangle_{a_0...a_{q+1}} \\
= \sum_{l} (-1)^l \langle\langle s (I_{a_0}...\hat{I}_{a_l}...I_{a_{q+1} O})\rangle\rangle_{a_0...a_{q+1}} \\
= d_p \sum_{l} (-1)^l \langle\langle I_{a_0}...\hat{I}_{a_l}...I_{a_{q+1} O}\rangle\rangle_{a_0...a_{q+1}},
\] (A.11)

where, in the second term in the right-hand-side of the first identity, we have exchanged $k$ with $l$ and the order of the two sums.

Appendix B. The plumbing fixture

We consider a cover of $\mathcal{M}_{0,4}$ consisting of three open disks $U_a$, with $a = 0, 1, 2$, centred around the three punctures $P_a$. The radius of these disks will be determined in
the following. A cell decomposition is obtained from this cover by taking $C_a$ with $a = 0, 1, 2$ to be proper closed subsets of $\mathcal{U}_a$, and $C_3$ to be the closure of the complement of $C_0 \cup C_1 \cup C_2$.

On the open patches $\mathcal{U}_a$ we consider maps $\Phi_m^{(a)} : (z_a, \bar{z}_a) \mapsto (Z_m, \bar{Z}_m)$

$$\Phi_m^{(a)} = h_a^{-1} \circ F_{q_a(m)} \circ h_a,$$

which carry from the coordinates $x_a$ to isothermal coordinates (3.11); $h_a(z_a)$ are conformal maps of the complex sphere, and $F_q(z, \bar{z})$ is defined by

$$F_q(z, \bar{z}) \equiv z q^{\theta_R(1-|z|^2)}. \quad \text{(B.2)}$$

$\theta_R(x)$ is a regularized step function, interpolating smoothly between 0 and 1:

$$\theta_R(x) = \begin{cases} 0, & \text{if } x \leq 0; \\ 1, & \text{if } x \geq 1 - C, \end{cases} \quad \text{(B.3)}$$

with $0 < C < 1$. $F_q$ is a smoothened version of the quasi-conformal map which defines the plumbing-fixture construction.

The plumbing-fixture parameters $q_a(m)$ are determined in terms of the holomorphic $h_a$:

$$q_a(m) \equiv \frac{h_a(m)}{h_a(x_1)}, \quad \text{(B.4)}$$

where $x_1$ is the position where the $\sigma_1$ operator is inserted. $h_a$ are chosen to be:

$$h_a(z) = \begin{cases} z, & \text{if } a = 0; \\ z - 1, & \text{if } a = 1; \\ \frac{1}{2z-1}, & \text{if } a = 2. \end{cases} \quad \text{(B.5)}$$

It is necessary that

$$\Phi_m^{(a)}(P_a) = P_a \quad \text{and} \quad \Phi_m^{(a)}(x_1) = m \quad \text{(B.6)}$$

for the maps $\Phi_m^{(a)}$ to define isothermal coordinates corresponding to the complex structure $m$. It is straightforward to verify that Eqs. (B.4) are satisfied as long as

$$\max \left\{ |x_1|^2, |x_1 - 1|^2, \frac{1}{|2x_1 - 1|^2} \right\} \leq C. \quad \text{(B.7)}$$

We can meet this condition by choosing, for concreteness, $x_1 = 1/2 + i\sqrt{2}/2$ and $C = 3/4$.

The open sets $\mathcal{U}_a$ of $\mathcal{M}_{0,4}$ on which the maps $\Phi_m^{(a)}$ are invertible are defined by the equations

$$0 < |q_a(m)| < R_0 \equiv \exp \left[ \frac{1}{\sup_{x \geq 0} (2x\theta_R(1-x))} \right]. \quad \text{(B.8)}$$
It is important that $R_0 > 1$ for $\cup_a \mathcal{U}_a$ to cover the whole $\mathcal{M}_{0,4}$. It is not difficult to check that in our concrete example we can take $R_0 = 2/\sqrt{3}$. Then, the three open sets which cover $\mathcal{M}_{0,4}$ are explicitly

$$\mathcal{U}_0 = \{m : |m| < 1\}, \quad \mathcal{U}_1 = \{m : |m - 1| < 1\}, \quad \mathcal{U}_2 = \{m : |m - 1/2| > \frac{\sqrt{3}}{2\sqrt{2}}\}. \quad (B.9)$$

The Beltrami differentials $\mu_a(z_a, \bar{z}_a; m) = \frac{\partial \Phi^{(a)}_m}{\partial \bar{z}_a} / \frac{\partial \Phi^{(a)}_m}{\partial z_a}$ vanish identically at the point $x_1$:

$$\mu_a(x_1; m) = 0, \quad \forall m \in \mathcal{U}_a. \quad (B.10)$$

The integrating factors $\lambda_a(z_a, \bar{z}_a; m) = \frac{\partial \Phi^{(a)}_m}{\partial z_a}$ evaluated at the point $x_1$ are

$$\lambda_a(x_1; m) = \frac{(\log h_a)'_{z=x_1}}{(\log h_a)'_{z=m}} = \begin{cases} m_{x_1} & \text{for } a = 0; \\ m_{x_1-1} & \text{for } a = 1; \\ m_{x_1-\frac{1}{2}} & \text{for } a = 2. \end{cases} \quad (B.11)$$

Thus, by taking $C_a \subset \mathcal{U}_a$ for $a = 0, 1$ and $2$, one obtains $f_a(x_1) = d\theta_a$, as stated in Eq. (6.1). On the complement of $C_0 \cup C_1 \cup C_2$, $\lambda_a(x_1; m)$ has neither zeros nor poles, and hence we can equivalently set $f_3(x_1) = 0$. 
References

[1] J. Polchinsky, Nucl. Phys. B 307 (1988) 61.
[2] P. Nelson, Phys. Rev. Lett. 62 (1989) 993.
[3] J. Distler and P. Nelson, Commun. Math. Phys. 138 (1991) 273.
[4] E. Witten, Nucl. Phys. B 340 (1990) 281.
[5] R. Dijkgraaf and E. Witten, Nucl. Phys. B 342 (1990) 486.
[6] E. Verlinde and H. Verlinde, Nucl. Phys. B 348 (1991) 457.
[7] K. Li, Nucl. Phys. B 354 (1990) 711 and 725.
[8] A. Losev, hep-th/9211089, Theor. Math. Phys 95 (1993) 595.;
D. Ghoshal and S. Mukhi, hep-th/9312189, Nucl. Phys. B 425 (1994) 173;
A. Hanany, Y. Oz and R. Plesser, hep-th/9401030, Nucl. Phys. B 425 (1994) 150;
Y. Lavi, Y. Oz and J. Sonnenschein, hep-th/9406056, Nucl. Phys. B 431 (1994) 223;
D. Ghoshal, C. Imbimbo and S. Mukhi, hep-th/9410034, Nucl. Phys. B 440 (1995) 355.
[9] S. Cordes, G. Moore and S. Ramgoolam, hep-th/9402107, Yale preprint YCTP-P23-93, 1994.
[10] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, hep-th/9302103, Nucl. Phys. B 405 (1993) 279.
[11] O. Bergman and B. Zwiebach, hep-th/9411047, Nucl. Phys. B 441 (1995) 76.
[12] C.M. Becchi, R. Collina and C. Imbimbo, hep-th/9311097, Phys. Lett. B 322 (1994) 79.
[13] J. Labastida, M. Pernici and E. Witten, Nucl. Phys. B 310 (1988) 611.
[14] D. Montano and J. Sonnenschein, Nucl. Phys. B 313 (1989) 258.
[15] R. Myers and V. Periwal, Nucl. Phys. B 333 (1990) 536.
[16] R. Brooks, D. Montano and J. Sonnenschein, Phys. Lett. B 214 (1988) 91.
[17] R. Myers, Nucl. Phys. B 343 (1990) 705.
[18] D. Montano and J. Sonnenschein, Nucl. Phys. B 324 (1989) 348.
[19] L. Baulieu and I.M. Singer, Commun. Math. Phys. 135 (1991) 253.
[20] C.M. Becchi, R. Collina and C. Imbimbo, hep-th/9406093, CERN and Genoa preprints CERN-TH 7302/94, GEF-Th 6/1994, Symmetry and Simplicity in Theoretical Physics, Proceedings of the Symposium for the 65-th Birthday of Sergio Fubini, Turin, 1994 (World Scientific, Singapore, 1994).
[21] C. Imbimbo, Nucl. Phys. (Proc. Suppl.) B 41 (1995) 302.
[22] L. Baulieu and M. Bellon, Phys. Lett. B 202 (1988) 67.
[23] R. Bott and L.W. Tu, Differential Forms in Algebraic Topology (Springer-Verlag, New York, 1982).
[24] S. Morita, Invent. Math. 90 (1987) 551.
[25] D. Mumford, “Towards an enumerative geometry of the moduli space of curves”, in *Arithmetic and Geometry*, Michael Artin and John Tate, eds. (Birkhäuser, Basle, 1983).

[26] E. Miller, J. Diff. Geom. **24** (1986) 1.

[27] M. Kontsevich, Commun. Math. Phys. **147** (1992) 1.

[28] H. Verlinde, Utrecht preprint THU-87/26, 1987, unpublished.