Metric Dichotomies

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Abstract. These are notes from talks given at ICMS, Edinburgh, 4/2007 (“Geometry and Algorithms workshop”) and at Bernoulli Center, Lausanne 5/2007 (“Limits of graphs in group theory and computer science”). We survey the following type of dichotomies exhibited by certain classes $X$ of finite metric spaces: For every host space $H$, either all metrics in $X$ embed almost isometrically in $H$, or the distortion of embedding some metrics of $X$ in $H$ is unbounded.

1. Problem statement and motivation

In these notes we examine dichotomy phenomena exhibited by certain classes $X$ of finite metric spaces. When attempting to embed the metrics in $X$ in any given host spaces $H$, either all of them embed almost isometrically, or there are some metrics in $X$ which are very poorly embedded in $H$. To make this statement precise we define the distortion of metric embeddings.

Given a mapping between metric spaces $f: X \rightarrow H$, define the Lipschitz norm of $f$ to be $\|f\|_{\text{Lip}} = \sup_{x \neq y} d_H(f(x), f(y))/d_X(x, y)$. The distortion of injective mapping $f$ is defined as $\text{dist}(f) = \|f\|_{\text{Lip}} \cdot \|f^{-1}\|_{\text{Lip}}$, where $f^{-1}$ is defined on $f(X)$. The “least distortion” in which $X$ can be embedded in $H$ is defined as $c_H(X) = \inf \{ \text{dist}(f) | f: X \rightarrow H \}$. This is a measure of the faithfulness possible when representing $X$ using a subset of $H$.

We formalize the discussion above as follows:

Definition 1 (Qualitative Dichotomy). A class of finite metric spaces $X$ has the qualitative dichotomy property if for any host space $H$, either

- $\sup_{X \in X} c_H(X) = 1$; or
- $\sup_{X \in X} c_H(X) = \infty$.

Remark 1. As defined in Def. 1, the dichotomy is with respect to all metric spaces as hosts. It is possible to extend the definition to be with respect to all sets of metric spaces as hosts. That is, for a set of metric spaces $H$, define $c_H(X) = \inf_{H \in H} c_H(X)$, and replace the use of “$c_H(X)$” in Def. 1 with “$c_H(X)$”. This extension, however, is inconsequential and the two definitions are equivalent. This follows from the proof of Theorem 1.6 in [33], which implies that for any set of metric spaces $H$, there exists a metric $\hat{H}$, such that for any finite metric space $X$, $c_{\hat{H}}(X) = c_H(X)$.

A dichotomy theorem for $X$ can be interpreted as a form of rigidity of $X$: Small deformations of all the spaces in $X$ is impossible.

We will also be interested in stronger dichotomies — of a quantitative nature — in which the unboundedness condition of the distortion is replaced with quantitative estimates on the rate in which it tends to infinity as a function of the size of the
metric space. I.e., by asymptotic lower bounds on the sequence

\[ D_N(H, \mathcal{X}) = \sup \{ c_H(X) : X \in \mathcal{X}, |X| \leq N \}. \]

The question of identifying such dichotomies was first explicitly raised by Arora et. al. [3]. They were motivated by a question from the theory of combinatorial approximation algorithms, where bounded distortion embeddings have become a basic tool. When dealing with algorithmically hard problem on a metric data \( X \in \mathcal{X} \), some algorithms first embed \( X \) into a better understood metric space \( H \), \( e : X \to H \), and then solve the algorithmic problem on \( e(X) \). This approach is used, for example, in [21, 5, 16, 15, 14]. For this approach to work:

1. \( H \) should be simple enough to make the algorithmic problem tractable.
2. \( e(X) \) should be close to \( X \).

Metric dichotomies draw limits on this approach when “closeness” is measured in terms of the distortion. Dichotomy means that either \( H \) already (essentially) contains \( \mathcal{X} \), and therefore cannot be understood better than \( \mathcal{X} \), or \( H \) does not approximate some metrics in \( \mathcal{X} \) very well. The algorithmic point of view also motivates the interest in quantitative dichotomies: When dealing with finite objects, slowly growing approximation ratios are also useful, and can be ruled out by quantitative dichotomies.

Matoušek [23] studied a closely related notion, which he called bounded distortion (bd-) Ramsey. Simplifying his definitions a bit, a class of finite metric space \( \mathcal{X} \) is called bd-Ramsey. If for every \( K > 1, \varepsilon > 0 \), and \( X \in \mathcal{X} \), there exists \( Y \in \mathcal{X} \) such that for any host space \( H \), and any embedding \( f : Y \to H \), if \( \text{dist}(f) \leq K \), then there exists \( g : X \to Y \) such that \( \text{dist}(g) \leq 1 + \varepsilon \), and \( \text{dist}(f|_{g(X)}) \leq 1 + \varepsilon \).

As observed in [3], the bd-Ramsey property implies qualitative dichotomy.

**Proposition 1.** If a class of finite metric spaces \( \mathcal{X} \) is bd-Ramsey then it has the qualitative dichotomy property.

**Proof.** Fix a host space \( H \), and suppose that

\[ \sup_{Y \in \mathcal{X}} c_H(Y) < \infty. \]

Fix \( X \in \mathcal{X} \), and \( \varepsilon \in (0, 1/2) \), and let \( K = 1 + \sup_{Y \in \mathcal{X}} c_H(Y) \). Pick \( Y \in \mathcal{X} \) that satisfies the bd-Ramsey condition. By (1), there exists \( f : Y \to H \) such that \( \text{dist}(f) \leq K \). By the bd-Ramsey property, there exists \( g : X \to Y \) such that \( \text{dist}(g) \leq 1 + \varepsilon \), and \( \text{dist}(f|_{g(X)}) \leq 1 + \varepsilon \), and so \( c_H(X) \leq \text{dist}(g) \cdot \text{dist}(f|_{g(X)}) \leq 1 + 3\varepsilon \). Since this is true for any \( \varepsilon \in (0, 1/2) \), we conclude that \( c_H(X) = 1 \). As this is true for any \( X \in \mathcal{X} \), we conclude that \( \sup_{X \in \mathcal{X}} c_H(X) = 1 \). \( \Box \)

**Remark 2.** All the dichotomies results in this note are actually bd-Ramsey results.

Matoušek’s study of bd-Ramsey phenomena [23] is partially motivated by a general theme in the geometric theory of Banach spaces to translate notions and results from the linear theory of finite dimensional Banach spaces to finite metric spaces. One such example is a theorem of Maurey, Pisier, and Krivine [29, 18] (see also [27] and [4, Ch. 12]) which implies that if a normed space \( H \) contains linear images of \( c_p^n \) for any \( n \) with uniformly bounded distortion, then \( H \) contains linear images of \( c_p^n \) for any \( n \) almost isometrically. More precisely, For every \( t \in \mathbb{N} \), \( \varepsilon > 0 \), \( K \geq 1 \), and \( p \in [1, \infty] \), there exists \( n = n(t, \varepsilon K, p) \) such that if there exists a linear mapping \( f : c_p^n \to H \), with \( \text{dist}(f) \leq K \), then there exists a linear mapping \( g : c_p^n \to c_p^n \) such that both \( \text{dist}(g) \leq 1 + \varepsilon \), and \( \text{dist}(f|_{g(c_p^n)}) \leq 1 + \varepsilon \). The bd-Ramsey property is a similar property, without the linear structure.
When studying metric dichotomy for a given class $\mathcal{X}$ of metric spaces, it is beneficial to work with a structured dense subclass $\mathcal{Y} \subset \mathcal{X}$.

**Proposition 2.** Suppose that $\mathcal{Y} \subset \mathcal{X}$ and $\mathcal{Y}$ is dense in $\mathcal{X}$, i.e., for every $X \in \mathcal{X}$, $c_\mathcal{Y}(X) = 1$. Then if $\mathcal{Y}$ has a metric dichotomy (either qualitative or quantitative) then $\mathcal{X}$ has the same dichotomy.

**Proof.** Since $\mathcal{Y} \subset \mathcal{X}$, for any host space $H$, $D_N(H,\mathcal{Y}) \geq D_N(H,\mathcal{X})$. On the other hand, if $\sup_{Y \in \mathcal{Y}} c_H(Y) = 1$, then $1 \leq \sup_{X \in \mathcal{X}} c_H(X) \leq \sup_{Y \in \mathcal{Y}} c_H(Y) \cdot \sup_{X \in \mathcal{X}} c_\mathcal{Y}(X) = 1$. \qed

Table 1 lists the classes of finite metric spaces which we will deal with and their dense regular subclasses used in the proofs. The proofs of the density are standard.

| Metric class (Finite subsets of $\mathcal{X}$) | Dense structured subclass ($n \in \mathbb{N}$) | Shorthand ($n \in \mathbb{N}$) |
|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| $\mathbb{R}$                               | $\{0, \ldots, n\}$                        | $P_n$                                      |
| $L_1$                                      | $\{(0,1)^n, \| \cdot \|_1\}$              | $\{0,1\}^n$                               |
| $L_{\infty}$ (i.e., $\mathcal{MET}$)       | $\{(1, \ldots, n)^n, \| \cdot \|_{\infty}\}$ | $[n]_{\infty}$                            |
| tree metrics                               | $\{(0,1)^{\leq n}, \text{tree distance}\}$ | $B_n$                                      |

**Theorem 1.** The following classes of finite metric spaces have the qualitative dichotomy property:

1. Finite subsets of $\mathbb{R}$,
2. For any $p \in [1, \infty]$, the class of finite subsets of $L_p$,
3. Finite equilateral spaces.

Here we just outline the proof for finite subsets of $L_p$, which is a nice demonstration of the linearization technique for Lipschitz mappings of normed spaces: Given a Lipschitz map, find a point of differentiability. The differential is a linear map with the same Lipschitz norm. Now apply a result from the linear theory. In our case, the linear result is the Maurey, Pisier, and Krivine theorem, and the differentiability argument is due to Kirchheim.

**Sketch of a proof of Theorem 1.** Item 2. Fix a host space $H$, and $p \geq 1$, and assume that there exists $K \in [1, \infty)$ such that any finite subset of $S \subset L_p$ embeds in $H$, $f_S : S \to H$, and $\text{dist}(f_S) \leq K$. We fix a finite $S \subset L_p$, $\varepsilon > 0$ and want to prove that $c_H(S) \leq 1 + \varepsilon$. It is known (see Sec. 11.2) that $S$ can be isometrically embedded in $l_p^n$, for $t = \left(\frac{|S|}{2}\right)$. Let $n = n(p,t,K,\varepsilon)$ be chosen as in the Maurey-Pisier-Krivine theorem discussed in Section 1.

The argument of the proof goes roughly as follows: Use a compactness argument to conclude that there exists an embedding $\tilde{f} : l_p^n \to H$ whose distortion at most $K$. Since this embedding is in particular Lipschitz, use a differentiation argument to find a point of “differentiability”. The differential is a “linear mapping” of $l_p^n$ whose distortion is at most $K$, and thus by the Maurey-Pisier-Krivine theorem alluded
to above, there exists a $t$-dimensional subspace of $\ell^p_n$ which is almost isometric to $\ell^t_p$ for which that mapping is almost isometry into $H$. This would finish the proof, since $S$ is isometrically embeddable in $\ell^t_p$.

The above argument has two major difficulties: 1) Since $H$ is not compact, the required “compactness argument” is false. 2) The notions of linear mapping and derivative when the target is general metric space, are not clear.

The first difficulty is addressed using a compactness argument, similar to Rado’s Lemma (see [23] Lemmas 3.4, & 4.4), which implies there exists a metric space $\tilde{H}$, and an embedding $\tilde{f}: \ell^t_p \to \tilde{H}$ such that $\text{dist}(\tilde{f}) \leq K$, and moreover, for any finite $T \subset \ell^t_p$, and $\delta > 0$, there exists $R$, $T \subset R \subset \ell^t_p$ such that $\tilde{f}(R)$ distorts the distance in $f_B(R)$ by at most a factor of $1 + \varepsilon$.

The second difficulty is addressed using a metric differentiation theorem of Kirchheim [17]. It implies that there exists $x_0 \in \mathbb{R}^n$, and a pseudo-norm on $||\cdot||$ on $\mathbb{R}^n$ such that for every $h, k \in \ell^t_p$, $d_{H}(\tilde{f}(x_0+h), \tilde{f}(x_0+k)) = ||h-k||+o(||h||_p+||k||_p)$. We conclude that $||\cdot||$ is a norm on $\mathbb{R}^n$ whose Banach-Mazur distance from the $\ell^t_p$ norm is at most $K$. Furthermore on a ball $B$ small enough around $x_0$ in $\ell^t_p$, $||\cdot||$ is $1 + \varepsilon$ approximation to the metric on $\tilde{f}(B)$.

Hence, by translating and rescaling $S$ we can assume it is inside $B$, and thus we can view $\tilde{f}$ as an approximate mapping between $\ell^t_p$ and $(\mathbb{R}^n, ||\cdot||)$. At this point Maurey-Pisier+Krivine theorem can be applied rigorously.

Finite subsets of finite (but larger than one) dimensional normed space is a natural class of metric spaces that does not have the dichotomy property:

**Proposition 3.** Fix $d > 1$, and some norm, $||\cdot||$, on $\mathbb{R}^d$. Then the class of finite subsets of $(\mathbb{R}^d, ||\cdot||)$ does not have qualitative dichotomy.

**Sketch of a proof.** It is possible to construct another norm $||\cdot||$ on $\mathbb{R}^d$, whose Banach-Mazur distance from $||\cdot||$ is some $B > 1$. I.e., for any linear mapping $T: (\mathbb{R}^d, ||\cdot||) \to (\mathbb{R}^d, ||\cdot||)$, $||T|| \cdot ||T^{-1}|| \geq B > 1$, and the inequality is tight for some $T$. We take $H = (\mathbb{R}^d, ||\cdot||)$, and so $c_H((\mathbb{R}^d, ||\cdot||)) \leq B$. On the other hand, assume for the sake of contradiction that there exists $A < B$ such that any finite subset of $(\mathbb{R}^d, ||\cdot||)$ can be embedded in $H$ with distortion at most $A$. By a compactness argument there exists an embedding of the unit ball of $(\mathbb{R}^d, ||\cdot||)$ in $(\mathbb{R}^d, ||\cdot||)$ with distortion at most $A$. Next, by Rademacher differentiation theorem there exists a point of differentiability in this embedding. The differential is a linear mapping whose distortion is at most $A$, which is a contradiction.

3. Dichotomy for subsets of the line

In the next two sections we discuss quantitative dichotomies and sketch direct proofs. We begin with finite subsets of $\mathbb{R}$.

**Theorem 2.** For every metric space $H$, either

- $c_H(A) = 1$, for every finite $A \subset \mathbb{R}$; or
- There exists $\beta > 0$, such that $c_H(P_n) \geq \Omega(n^\beta)$, where $P_n$ is the $n$-point path metric.

As discussed above, Matoušek showed a qualitative dichotomy for finite subsets of the line, based on differentiation argument. His proof actually gives the

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1 To see it, notice that the Banach-Mazur distance between $\ell^2_d$ and $\ell^2_4$ is $\sqrt{\frac{d}{4}} > 1$, and therefore by the triangle inequality any other $d$-dimensional norm must be at distance at least $\sqrt{\frac{d}{4}} > 1$ from one of them. By John’s Theorem (see e.g. [25], Sec. 13.4), the distance is at most $d$. 
same quantitative bounds as in Theorem 2. Here, following [32], we sketch a somewhat different proof which conveys the approach to prove the more complicated quantitative dichotomies for finite subsets of \( L_1 \), and for all finite metric spaces.

The general approach in those proofs is to define an “isomorphic” inequality, and prove sub-multiplicativity. This approach — originated in the work of Pisier [35] — is used in Banach space theory quite often.

We proceed to prove Theorem 2. We first choose an appropriate inequality that captures the distortion of embedding \( P_n \) in \( H \). Let \( \Psi_n(H) \) be the infimum over \( \Psi > 0 \) such that for every \( f : P_n \to H \),

\[
d_H(f(0), f(n)) \leq \Psi n \max_{i=0, \ldots, n-1} d_H(f(i), f(i+1)).
\]

**Lemma 4.** For every metric space \( H \), and \( m, n \in \mathbb{N} \),

1. \( \Psi_n(H) \leq 1 \).
2. \( c_H(P_n) \geq 1/\Psi_n(H) \).
3. If \( \Psi_n(H) = 1 \), then \( c_H(P_n) = 1 \).
4. \( \Psi_{mn}(H) \leq \Psi_m(H) \cdot \Psi_n(H) \).

Before proceeding with the proof of lemma 4, let us see how Theorem 2 is derived.

**Proof of Theorem 2.** We will prove the dichotomy to \( (P_n)_n \) (the path metrics), which by Prop. 2 is sufficient. Fix a host space \( H \).

- If for every \( n \in \mathbb{N} \), \( \Psi_n(H) = 1 \), then \( c_H(P_n) = 1 \).
- If there exists \( n_0 \) for which \( \Psi_{n_0}(H) = \eta < 1 \), then let \( \beta > 0 \) be such that \( n_0 - \beta = \eta \), and from the submultiplicativity, \( \Psi_{n_0}(H) \leq \eta^k = (n_0)^{-\beta} \), and so \( c_H(P_{n_0}) \geq (n_0)^{-\beta} \).

**Proof of Lemma 4.**

1. Follows from the triangle inequality.
2. Fix \( f : P_n \to H \), and \( \Psi > \Psi_n(H) \). Plugging the Lipschitz norms into (2),

\[
\frac{n}{\|f^{-1}\|_{\text{Lip}}} \leq d_H(f(0), f(n)) \leq \Psi n \max_{i=0, \ldots, n-1} d_H(f(i), f(i+1)) \leq \Psi n \|f\|_{\text{Lip}},
\]

so \( \text{dist}(f) = \|f\|_{\text{Lip}} \cdot \|f^{-1}\|_{\text{Lip}} \geq 1/\Psi \). Since this is true for any \( \Psi > \Psi_n(H) \),

\[
\text{dist}(f) \geq 1/\Psi_n(H).
\]

3. If \( \Psi_n(H) = 1 \), then for any \( \varepsilon \in (0, 1/2n) \), there exists \( f : P_n \to H \) for which

\[
n \max_{i=0, \ldots, n-1} d_H(f(i), f(i+1)) \geq d_H(f(0), f(n-1)) \\
\geq (1-\varepsilon)n \max_{i=0, \ldots, n-1} d_H(f(i), f(i+1)). \quad (3)
\]

Let \( A = \max_{i=0, \ldots, n-1} d_H(f_i, f_{i+1}) \), and for \( i > j \),

\[
(i-j)A \geq d_H(f(i), f(j)) \geq d_H(f(0), f(n)) - d_H(f(j), f(0)) - d_H(f(n), f(i)) \geq (1-\varepsilon)nA - jA - (n-i)A = (i-j-\varepsilon n)A.
\]

This means that \( \text{dist}(f) \leq 1 + 2\varepsilon n \), which implies that \( c_H(P_n) = 1 \).

4. Fix \( f : P_{mn} \to H \). Define \( g : P_n \to H \), by \( g(i) = f(im) \). Applying (2) to \( g \), we obtain

\[
d_H(f(0), f(mn)) \leq (\Psi_n(H) + \varepsilon)n \max_{i=0, \ldots, n-1} d_H(f(im), f((i+1)m)). \quad (4)
\]

Next, define \( h_i : P_m \to H \), \( h_i(j) = f(im + j) \), and apply (2) for each \( h_i \), and so

\[
d_H(f(im), f((i+1)m)) \leq (\Psi_m(H) + \varepsilon)m \max_{j=0, \ldots, m-1} d_H(f(im + j), f(im + j + 1)). \quad (5)
\]
Combining \(c_H\) with \(\Omega\), and we conclude the claim.

The quantitative dichotomy in Theorem 2 is tight for finite subsets of the line: For any \(\beta \in (0, 1]\), there exists \(H_\beta\) such that \(c_{H_\beta}(F_n) = \Theta(n^\beta)\). For \(\beta \in (0, 1)\), \(H_\beta\) can be taken as the real line with the usual metric to the power of \(1 - \beta\). For \(\beta = 1\), \(H_1\) can be taken as the ultrametric defined on \(\{0, 1\}^N\), with the distance function \(\rho(x, y) = 2^{-|\text{lcp}(x, y)|}\), where \text{lcp} is the longest common prefix of the two sequences.

4. Dichotomies for finite subsets of \(L_1\), and \(L_\infty\)

The proofs of the quantitative dichotomies for subsets of \(L_1\) and subsets of \(L_\infty\) use the same general approach taken in Section 3: we write inequalities for which we can prove a lemma similar to Lemma 4 but replacing paths with Hamming cubes (for subsets of \(L_1\)) and grids with the \(L_\infty\) distance (for subsets of \(L_\infty\)).

In both cases the hard part in the proof seems to be coming up with the inequality. However, in contrast to path metrics, the proofs of the lemmas analogous to Lemma 4 (especially item (3)) are technical and lengthy. We will therefore omit all these details and concentrate on the inequalities.

4.1. Finite subsets of \(L_1\). The argument given here is essentially from a paper of Bourgain, Milman, and Wolfson [9] on metric type.

**Theorem 3.** [9] For every metric space \(H\), either

- \(c_H(X) = 1\), for every finite \(X \subset L_1\); or
- There exists \(\beta > 0\), such that \(c_H(\{0, 1\}^n, \|\cdot\|_1) \geq \Omega(n^\beta)\).

Similarly to the dichotomy of subsets of \(\mathbb{R}\), we use an inequality to guide the proof: Let \((e_i)_{i=1}^n\) denote the standard basis of \(\{0, 1\}^n\), and \(1 = \sum_i e_i\). Let \(T_n(H)\) be the infimum over \(T > 0\) such that for every \(f : \{0, 1\}^n \rightarrow H\),

\[
\mathbb{E}_{x \in \{0, 1\}^n} d_H(f(x), f(x + 1))^2 \leq T^2n \sum_{i=1}^n \mathbb{E}_{x \in \{0, 1\}^n} d_H(f(x), f(x + e_i))^2,
\]

where the operator \(\mathbb{E}\) means averaging.

Inequality (6) was chosen to “capture” the distortion of embeddings the Hamming cubes in \(H\), and have the sub-multiplicativity property (in \(n\)). It is a variant of the metric-type inequality from [9]. The connection (and motivation) to the type property is expanded upon in Section 4.3. Formally, we can prove a lemma analogous to Lemma 4.

**Lemma 5.** For every metric space \(H\), and \(m, n \in \mathbb{N}\),

1. \(T_m(H) \leq 1\).
2. \(c_H(\{0, 1\}^n) \geq 1/T_n(H)\).
3. If \(T_n(H) = 1\), then \(c_H(\{0, 1\}^n) = 1\).
4. \(T_m(H) \leq T_m(H) \cdot T_n(H)\).

Using Lemma 5, the proof of Theorem 3 is the same as the proof of Theorem 2 replacing references to \(\Psi_n(H)\) with \(T_n(H)\), and the path metric with the Hamming cube.

Regarding the quantitative tightness of Theorem 3. It is known [12] that \(c_{\ell_2}(\{0, 1\}^n, \|\cdot\|_1) = \sqrt{n}\), and that any \(N\)-point subset of \(L_1\) is \(O(\sqrt{\log N \log \log N})\) embeddable in \(\ell_2\) [4, 10, 2]. I do not know much more.

**Question 1.** Does there exist \(\beta \in (0, 1/2)\) and a metric space \(H\) such that \(1 < D_N(H, 2\ell_2) = O((\log N)^\beta)\)? If so, is it true for every \(\beta > 0\)?

\(^2\)The paper [9] does not discuss dichotomy, but rather a non-linear analogue for Pisier theorem for type-1. As we shall see in Section 4.3, from that result it is easy to obtain the dichotomy.
4.2. Finite metric spaces. Next, we consider $\mathcal{MET}$, the set of all finite metric spaces which is equal to the set of finite subsets of $\ell_\infty$.

**Theorem 4.** [33] For every metric space $H$, either
- $\sup_{X \in \mathcal{MET}} c_H(X) = 1$; or
- There exists $\beta > 0$, such that $c_H((\{1, \ldots, n\}^n, \|\cdot\|_\infty) \geq \Omega(n^\beta)$, where $[n]_\infty$ is the $\{1, \ldots, n\}$ grid with the $\ell_\infty$ distance.

Again, we use an inequality (derived from the metric cotype inequality [33]) to guide the proof. Denote by $\Gamma_n(H)$ the infimum over $\Gamma > 0$ such that for every $m \in \mathbb{N}$, and every $f : \mathbb{Z}_m^n \to H$,

$$\sum_{i=1}^n \mathbb{E}_{x \in \mathbb{Z}_m^n} d_H(f(x), f(x + ne_j))^2 \leq \Gamma^2 \cdot n^2 \cdot n \mathbb{E}_{\varepsilon \in \{-1, 0, 1\}^n} \mathbb{E}_{x \in \mathbb{Z}_m^n} d_H(f(x), f(x + \varepsilon))^2$$  \(7\)

(the additions “$x + ne_j$”, and “$x + \varepsilon$” are in $\mathbb{Z}_m^n$). Inequality (7) is designed to capture the distortion of embedding $\{1, \ldots, n\}$ with the $\ell_\infty$ distance in $H$. In this context, it seems more natural to average $\varepsilon$ over all $\{-1, 0, 1\}^n$ which are the distance 1 in the $\ell_\infty$ metric. However, this choice would complicate the proof of the submultiplicativity property. Note that the metric induced by the graph $\mathbb{Z}_m^n$ with the $\{-1\}^n$ edges contains an isometric copy of $\{1, \ldots, n/4\}^n$ with the $\ell_\infty$ metric. Also, the design choice of the universal quantifier on $m$ (instead of say fixing $m = n$) was done to make the proof of the submultiplicativity easy. The connection to the cotype property of Banach spaces is expanded upon in Section 4.3. As before, we use a lemma analogous to Lemma 3.

**Lemma 6.** For every metric space $H$, and $m, n \in \mathbb{N}$,
1. $\Gamma_n(H) \leq 1$, for all even $n$.
2. $c_H((\{1, \ldots, n/4\}^n, \|\cdot\|_\infty) \geq 1/\Gamma_n(H)$.
3. If $\Gamma_n(H) = 1$, then $c_H((\{1, \ldots, n/4\}^n, \|\cdot\|_\infty) = 1$.
4. $\Gamma_m(H) \leq \Gamma_m(H) \cdot \Gamma_n(H)$.

Theorem 4 is deduced similarly to Theorems 2 and 3 but now using Lemma 6. A sketch of the proof of Lemma 6 can be found in [30, Sec. 2]. The complete proof appears in [33, Sec. 6].

I do not know much about the quantitative tightness of Theorem 4.

**Question 2.** Does there exist $\beta \in (0, 1)$ and a metric space $H$ for which $1 < D_N(H, \mathcal{MET}) = O((\log N)^\beta)$? If so, is it true for every $\beta \in (0, 1)$?

Personally, the dichotomy of $\mathcal{MET}$ seems to me the most natural problem in these notes, and Question 2 the most fundamental open problem.

Bourgain’s embedding theorem [7] and the matching lower bound [21] implies that $D_N(\ell_2, \mathcal{MET}) = \Theta((\log N)$). Essentially, all examples of families of metrics having logarithmic distortion when embedded into Hilbert space, are families of expander graphs with a logarithmic diameter. Matoušek proved that expanders have logarithmic distortion when embedded in $L_p$, for any $p \in [1, \infty)$. Recently, Lafforgue [19] has exhibited classes of expanders with logarithmic distortion when embedded in $B$-convex Banach spaces — spaces with type greater than 1.

In view of the seemingly surprising fact of the metric dichotomy, it is natural to ask which monotone $f : \mathbb{N} \to \mathbb{N}$ has a metric space $H$ such that $D_N(H, \mathcal{MET}) = \Theta^*(f(N))$ (here $\Theta^*$ can “hide” polylogarithmic multiplicative factors). From Bourgain embedding theorem we know that $f(N) = \log N$ is achievable using Hilbert space. Matoušek [26] showed that for every even $d$, $D_N(\ell_2^d, \mathcal{MET}) = \Theta^*(N^{2/d})$. Furthermore, in the full version of [32] it is shown that for every $\varepsilon \in (0, 1]$, there exists a metric space $H_\varepsilon$, for which $D_N(H_\varepsilon, \mathcal{MET}) = \Theta(N^\varepsilon)$. This leaves us with a concrete question:
Question 3. Does there exist $H$ for which $D_N(H,\mathcal{MET}) \in \omega(\log N) \cap N^{o(1)}$?

Comparing the results in this section with those in Section 2 we also ask:

Question 4. Is there a (substantial) quantitative dichotomy for finite subsets of $L_p$, and in particular for $L_2$?

4.3. Metric type and cotype and non-linear Maurey-Pisier theorems.

A normed space is said to have type $p$, $1 \leq p \leq 2$, with constant $T$, if for every finite family $x_1, \ldots, x_n \in X$,

$$\left( \mathbb{E}_{\varepsilon \in \{-1,1\}^n} \left| \sum_{j=1}^n \varepsilon_j x_j \right|^p_X \right)^{\frac{1}{p}} \leq T \left( \sum_{j=1}^n \|x_j\|^p \right)^{\frac{1}{p}} \tag{8}$$

and cotype $q$, $2 \leq q \leq \infty$, with constant $C$, if for every finite family $x_1, \ldots, x_n \in X$,

$$C \left( \mathbb{E}_{\varepsilon \in \{-1,1\}^n} \left| \sum_{j=1}^n \varepsilon_j x_j \right|^q_X \right)^{\frac{1}{q}} \geq \left( \sum_{j=1}^n \|x_j\|^q \right)^{\frac{1}{q}} \tag{9}$$

The theory around these notions was developed since the 70’s, with fascinating results. The interested reader may consult \[34\, 27\] and references therein. Here we are interested in one aspect of this theory, called Maurey-Pisier Theorem.

A closely related conditions are equal norms type and cotype: A normed space is said to have equal norm (en) type $p$, $1 \leq p \leq 2$, with constant $\hat{T}$ if for every finite family $x_1, \ldots, x_n \in X$,

$$\mathbb{E}_{\varepsilon \in \{-1,1\}^n} \left| \sum_{j=1}^n \varepsilon_j x_j \right|^2_X \leq \hat{T}^2 n^{\frac{2}{p}} - 1 \sum_{j=1}^n \|x_j\|^2 \tag{10}$$

Similarly, a normed space is said to have equal norms (en) cotype $q$, $2 \leq q \leq \infty$, with constant $\hat{C}$, if for every finite family $x_1, \ldots, x_n \in X$,

$$\hat{C}^2 n^{1-\frac{2}{q}} \mathbb{E}_{\varepsilon \in \{-1,1\}^n} \left| \sum_{j=1}^n \varepsilon_j x_j \right|^2_X \geq \sum_{j=1}^n \|x_j\|^2 \tag{11}$$

Type/cotype and en-type/cotype are closely related\[3\]. Type $p$ implies en-type $p$ which implies type $p - \varepsilon$ for every $\varepsilon > 0$. Cotype $q$ implies en-cotype $q$ which implies cotype $q + \varepsilon$ for every $\varepsilon > 0$ (see \[37\]).

We observe that any normed space has en-type 1 and en-cotype $\infty$, as these inequalities follow from the triangle inequality (and Cauchy-Schwarz). A normed space has type $> 1$ if and only if it has en-type $> 1$, and cotype $< \infty$ if and only if it has en-cotype $< \infty$. Maurey and Pisier \[28\] proved:

Theorem 5. A normed space $X$ does not have (en-)cotype $\infty$ if and only if for every $n \in \mathbb{N}$, and $\eta > 0$, $\ell_\infty^n$ can be linearly embedded in $X$ with distortion at most $1 + \eta$.

Pisier \[35\] proved an analogous result for type:

Theorem 6. A normed space $X$ does not have (en-)type $> 1$ if and only if for every $n \in \mathbb{N}$, and $\eta > 0$, $\ell_1^n$ can be linearly embedded in $X$ with distortion at most $1 + \eta$.

Those results imply dichotomy results for the class of finite dimensional subspaces.

Proposition 7. For any normed space $H$, either

\[3\] The LHS are equivalent, up to a constant factor, by Kahane inequality. In the RHS, type/cotype condition implies the equal norms variant by Hölder inequality, and they are equal when all the $x_i$ has the same norm, hence the name.
• For every $\varepsilon > 0$, any finite dimensional normed space $X$ can be linearly embedded in $H$ with distortion $1 + \varepsilon$.

• There exists $\beta > 0$, such that for every $n \in \mathbb{N}$, and linear embedding $f : \ell^n_\infty \rightarrow H$, $\text{dist}(f) = \Omega(n^\beta)$.

**Proof.** If $H$ does not have finite en-cotype, then by Theorem 5 any $\ell^n_\infty$ can be linearly embedded in $H$ with distortion $1 + \varepsilon$, for every $\varepsilon > 0$. Combining this with the elementary observation that for any finite dimensional normed space can be $(1 + \varepsilon)$-embedded in $\ell^n_\infty$, for some $n \in \mathbb{N}$ we obtain the first bullet.

If, on the other hand, $H$ has some finite en-cotype $q < \infty$, then for any linear embedding $f : \ell^n_\infty \rightarrow H$,

$$\hat{C} \| f \| = \hat{C} \| f \| \left( E \left\| \sum_{j=1}^{n} \varepsilon_j e_j \right\|_\infty^2 \right)^{\frac{1}{2}} \geq \hat{C} \left( E \left\| \sum_{j=1}^{n} \varepsilon_j f(e_j) \right\|_H^2 \right)^{\frac{1}{2}} \geq n^{\frac{1}{2} \frac{1}{2}} \left( \sum_{j=1}^{n} \| f(e_j) \|_H^2 \right)^{\frac{1}{2}} \geq \| f^{-1} \| n^{\frac{1}{2} \frac{1}{2}} \left( \sum_{j=1}^{n} \| e_j \|_2^2 \right)^{\frac{1}{2}} = n^{\frac{1}{2}} \cdot |f^{-1}|,$$

which implies that $\text{dist}(f) = \Omega(n^{1/q})$. □

A similar dichotomy can be proved for $\ell^n_1$, using Theorem 6.

Theorems 3 and 6 are proved using linear analogues of Lemmas 5 and 6.

Indeed, (6) is a natural non-linear analogue of (10). One can view (10) as (6) restricted to linear mappings of the cube (and $\hat{T}$ as a substitute to $T^{1 - \frac{1}{2}}$). Variant of (6) was suggested by Enflo [13] as a non-linear version of type (Ineq. 8), and the (almost) equivalence to (en-)type was proved in [9, 36, 51].

The connection between en-cotype 11, and 7 is less apparent: It is shown in [33] that the following metric property is equivalent to en-cotype $q$ in Banach spaces: There exists $\tilde{C} > 0$ such that for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for every $f : \mathbb{Z}_m^n \rightarrow H$,

$$\sum_{i=1}^{n} E_{x \in \mathbb{Z}_m^n} d_H(f(x), f(x + m \varepsilon e_j))^2 \leq \tilde{C} m^2 n^{1 - \frac{1}{2}} \sum_{x \in \{0, \pm 1\}^n} E_{x \in \mathbb{Z}_m^n} d_H(f(x), f(x + \varepsilon))^2.$$

(12)

Inequality (12) is “close in spirit” to (7). Rigorously, if we change (7) by replacing the “$n$” in the LHS with “$n^m$”, and the “$n^2$” on the RHS with “$n^{2\varepsilon}$”, then it is proved in [33] that (12) is satisfied for some $q < \infty$ if and only if $\limsup_{n \rightarrow \infty} \Gamma_n(H) < 1$ (where we $\Gamma_n(H)$ is defined according to the modifications of (7) we have just suggested).

5. Tree metrics

The class of finite tree metrics does not have the dichotomy property.

**Theorem 7.** [32] For any $B > 4$, there exists a metric space $H$ such that $\sup_T c_H(T) = B$, where $T$ ranges over the finite tree metrics.

It is natural to ask whether there is a dichotomy between constant distortions and say $\log^* N$ distortions. We believe no such dichotomy exists. For complete binary trees we can prove:

\footnote{The sub-multiplicativity argument originates in the work of Pisier [35] on type 1. The original proof of the cotype $\infty$ in [28] uses a more complicated argument.}

\footnote{Actually, the paper [32] proves a variant of the above statement: the equivalence between cotype and metric cotype. But the arguments are the same.}
Theorem 8. For any $\delta \in (0, 0.001)$, and for any sequence $s(n)$ satisfying (i) $s(n)$ is non-decreasing; (ii) $s(n)/n$ is non-increasing (iii) $4 < s(n) \leq O(\delta \log n / \log \log n)$, there exists a metric space $H$ and $n_0$ such that for every $n \geq n_0$, $(1 - \delta)s(n) \leq c_H(B_n) \leq s(n)$. Here $B_n$ is the (metric on) unweighted complete binary tree of depth $n$.

Question 5. Is there an extension of Theorem 8 to all tree metrics, with dependence on the size of the metric? For example “For every $\delta > 0$, and $s(n)$ as in Theorem 8, there exists a metric space $H$ such that for every tree metric $T$, $(1 - \delta)s(\log N) \leq D_N(H, trees) \leq s(\log N)$”.

Theorem 7 is a corollary of Theorem 8, when substituting $s(n) = B_n$, and using the fact that the complete binary trees are “dense” in the finite tree metrics, in the sense of Prop. 2. The rest of the section is devoted to a non-quantitative sketch of the argument in the proof of Theorem 7. The more complicated (and complete) proof of Theorem 8 can be found in [32].

Denote $\eta = 1/B$. To define $H_\eta$, consider the infinite binary tree $B_\infty$ with the tree metric on it, and contract the “horizontal” distances by a factor of $B$. More precisely Let $h(x)$ be the depth of $x \in B_\infty$, i.e. the distance from the root of $B_\infty$. Assuming $h(y) \geq h(x)$, the distance $d_\eta(x, y)$, for $x, y \in B_\infty$ is defined as

$$d_\eta(x, y) = h(y) - h(x) + 2(h(x) - h(lca(x, y))) \cdot \eta. \quad (13)$$

It is not hard to check that:

Proposition 8.

1. $d_\eta$ is a metric on $B_\infty$.
2. $c_{d_\eta}(B_\infty) \leq \eta^{-1}$. Indeed the identity mapping does not expand distances, and contracts them by factor of at most $1/\eta$.

Thus $H_\eta = (B_\infty, d_\eta)$ is our candidate host space for proving Theorem 7. We are left to show that $\lim_{n \to \infty} c_{H_\eta}(B_n) = \eta^{-1}$. Our approach follows Matoušek’s proof [24] of:

Theorem 9. $C_{L_2}(B_n) \geq \Omega(\sqrt{\log n})$.

Bourgain [8] proved Theorem 9 first, and there are subsequent proofs [22, 20]. For our purpose, Matoušek’s argument seems the most appropriate, we therefore outline his proof of Theorem 9.

A $\delta$-fork is a quadruple $(x, y, z, w)$ such that both $(x, y, z)$, and $(x, y, w)$ are $1 + \delta$ equivalent to the metric $(0, 1, 2)$ (where $x$ is mapped to 0 and $y$ is mapped...
It is not hard to see that in Hilbert space (and more generally, 2-uniform convex spaces), if \((x, y, z, w)\) is a \(\delta\)-fork then \(\|z - w\| \leq O(\sqrt{\delta})\|x - y\|\).

Matoušek’s approach is to assume toward a contradiction that there exists a Lipschitz embedding \(f : B_n \to L_2\) such that \(\text{dist}(f) \leq c\sqrt{\log n}\), and use this assumption to find a 3-leaf star \((x, y, z, w)\) in \(B_n\) whose center is \(y\) such that \((f(x), f(y), f(z), f(w))\) is a \(\delta\) fork for \(\delta \approx 1/\log n\). This implies a large contraction of the distance between \(z\) and \(w\), which is a contradiction to the assumed upper bound on the distortion.

Consider the first part of Matoušek’s proof: Finding a star in \(B_n\) whose image is \(\delta\)-fork. It is proved along the following lines: Call a metric embedding \(f : B_n \to H\), \(A\)-vertically faithful if \(\|f\|_{\text{Lip}} \leq 1\), and for every \(x, y \in B_n\) in which \(x\) is an ancestor of \(y\), \(d_X(f(x), f(y)) \geq d_{B_n}(x, y)/A\). It turns out (as proved by Matoušek) that when considering only the vertical distances in \(B_n\), this class has the BD-Ramsey property, or the dichotomy property. In other words:

**Lemma 9.** For every \(t \in \mathbb{N}\), \(\delta > 0\), and \(A > 1\), there exists \(n = n(t, \delta, A)\), such that for any host space \(H\), and \(A\)-vertically faithful embedding \(f : B_n \to H\), there exists a subset \(C \subset B_n\) which is isometric to \(B_t\), and \(f(C)\) is \(1 + \delta\)-vertically faithful to \(B_t\).

Note that for \(t = 2\), \(f(C)\) contains a copy of \(\delta\)-fork (actually, two copies). We should also mention that, not surprisingly, the (simple) proof of Lemma 9 uses the BD-Ramsey property of the path metrics as proved in Section 3.

Since the part of finding a \(\delta\)-fork is independent of the range of the embedding, it makes sense to use it on embedding into \((B_\infty, d_\eta)\). Examining possible \(\delta\)-forks \((x, y, z, w)\) inside \(H_\eta\), the two configurations in Fig. 3 contracts the distance between \(z\) and \(w\) by at least \(1/(O(\delta) + \eta)\) factor, which is what we are looking for.

However this is not the whole story! There are other types of \(\delta\) forks embedded in \(H_\eta\). For example type II in Fig. 4 can be even made 0-fork, but with very small contraction of the tips. This means that the approach that attempts to show large
contraction of δ-forks in \( H_\eta \) will not work. There are also other configurations of “bad” δ-forks, such as type I, III, and IV in Fig. 4.

![Figure 4. Forks in which the distance between the prongs (z and w) do not contract. In type II, x is a descendant of the forking point y, which is deeper (in \( B_\infty \)) than the prongs z and w.](image)

It turns out that the situation is not that bad. The four types of “bad forks” are the only ones that exist.

**Lemma 10.** Every δ-fork in \( H_\eta \) is close to one of the 6 types of forks in Figures 3 and 4, up to distortion of \( 1 + O(\delta) \).

The proof of Lemma 10 is a tedious and contains long case analyses (not to mention the need to properly define the configurations in Figures 3 and 4). But having it, it is reasonable to assume that a slight generalization of the tip contraction argument for δ-fork would be true in \( H_\eta \). Indeed, we show that

**Lemma 11.** Any \( 1 + \delta \) vertically faithful embedding of \( B_4 \) in \( H_\eta \), must have distortion at least \( 1/(O(\delta) + \eta) \).

Notice that this lemma is sufficient to prove a lower bound on the distortion of \( B_n \) in \( H_\eta \), by using Lemma 9 with \( t = 4 \).

In order to prove Lemma 11 we view \( 1 + \delta \) vertically faithful embedding of \( B_4 \) as a collection of δ-forks “glued” together in prong-to-handle fashion. For this purpose, it is helpful to analyze what are the possible configurations of \( 1 + \delta \) embedding of 4-point paths, \( \{0, 1, 2, 3\} \), in \( H_\eta \). There are essentially only three different configurations, as depicted in Fig. 5.

At this point we can do a “syntactic” case analysis of how the δ-forks of Fig. 4 can be glued together into \( 1 + \delta \) vertically faithful embedding of \( B_4 \), using the “rules” enforced by the configuration of 4-point paths described in Fig. 5. Doing this lead to the inevitable conclusion that a fork of a type described in Fig. 3 must appear in the embedding of \( B_4 \), leading to the conclusion that the embedding of \( B_4 \) must have a large contraction, and hence a large distortion. □

**Acknowledgments.** This work was supported by an Israel Science Foundation (ISF) grant no. 221/07, and a US-Israel Bi-national Science Foundation (BSF) grant no. 2006009.

The author thanks Assaf Naor for his help in assimilating the subject while collaborating on the papers [33, 32]. He also thanks Assaf Naor and the anonymous

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6A more careful examination reveals that the configurations labeled type I, III, IV in Fig. 4 can be \( O(\eta) \)-fork at best. Hence, by taking \( \delta \ll \eta \), we can rule out their existence as δ-forks. This approach, however, will fail to prove the more general result of Theorem 8 in which \( \eta \) is no longer a constant.
Figure 5. The possible configurations of 4-point path $(x_0, x_1, x_2, x_3)$. In type $B$, for example, $x_1$ is an ancestor of $x_0$, $x_3$ is an ancestor of $x_2$, and $x_1$ and $x_2$ have the same depth in $B_\infty$.

referee for commenting on an earlier version of these notes, which helped improving the presentation. The figures in these notes are adapted from [32].

Finally, the author wish to thank the organizers of the ICMS “Geometry and Algorithms workshop”, (Edinburgh, 4/2007) and the organizers of “Limits of graphs in group theory and computer science semester” in the Bernoulli Center (Lausanne 5/2007) for inviting him to give a talk on which these notes are based.

References

[1] Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2006, Miami, Florida, USA, January 22–26, 2006. ACM Press, 2006.
[2] Sanjeev Arora, James R. Lee, and Assaf Naor. Euclidean distortion and the sparsest cut [extended abstract]. In STOC’05. Proceedings of the 37th Annual ACM Symposium on Theory of Computing, pages 553–562, New York, 2005. ACM.
[3] Sanjeev Arora, László Lovász, Ilan Newman, Yuval Rabani, Yuri Rabinovich, and Santosh Vempala. Local versus global properties of metric spaces. In SODA [1], pages 41–50.
[4] Sanjeev Arora, Satish Rao, and Umesh Vazirani. Expander flows, geometric embeddings and graph partitioning. In Proceedings of the 36th Annual ACM Symposium on Theory of Computing, pages 222–231 (electronic), New York, 2004. ACM.
[5] Yair Bartal. Probabilistic approximation of metric spaces and its algorithmic applications. In 37th Annual Symposium on Foundations of Computer Science (Burlington, VT, 1996), pages 184–193. IEEE Comput. Soc. Press, Los Alamitos, CA, 1996.
[6] Yoav Benyamini and Joram Lindenstrauss. Geometric nonlinear functional analysis. Vol. 1, volume 48 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2000.
[7] Jean Bourgain. On Lipschitz embedding of finite metric spaces in Hilbert space. Israel J. Math., 52(1-2):46–52, 1985.
[8] Jean Bourgain. The metrical interpretation of superreflexivity in Banach spaces. Israel J. Math., 56(2):222–230, 1986.
[9] Jean Bourgain, Vitali Milman, and Haim Wolfson. On type of metric spaces. Trans. Amer. Math. Soc., 294(1):295–317, 1986.
[10] Shuchi Chawla, Anupam Gupta, and Harald Räcke. Embeddings of negative-type metrics and an improved approximation to generalized sparsest cut. In Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 102–111 (electronic), New York, 2005. ACM.
[11] Michel Marie Deza and Monique Laurent. Geometry of cuts and metrics, volume 15 of Algorithms and Combinatorics. Springer-Verlag, Berlin, 1997.
[12] Per Enflo. On the nonexistence of uniform homeomorphisms between $L_p$-spaces. Ark. Mat., 8:103–105 (1969), 1969.
[13] Per Enflo. On infinite-dimensional topological groups. In Séminaire sur la Géométrie des Espaces de Banach (1977–1978), pages Exp. No. 10–11, 11. École Polytech., Palaiseau, 1978.
[14] Jittat Fakcharoenphol, Satish Rao, and Kunal Talwar. A tight bound on approximating arbitrary metrics by tree metrics. J. Comput. System Sci., 60(3):485–497, 2004.
[15] Uriel Feige. Approximating the bandwidth via volume respecting embeddings. *J. Comput. System Sci.* 60(3):510–539, 2000.

[16] Naveen Garg, Goran Konjevod, and R. Ravi. A polylogarithmic approximation algorithm for the group Steiner tree problem. *J. Algorithms*, 37(1):66–84, 2000. Ninth Annual ACM-SIAM Symposium on Discrete Algorithms (San Francisco, CA, 1998).

[17] Bernd Kirchheim. Rectifiable metric spaces: local structure and regularity of the Hausdorff measure. *Proc. Amer. Math. Soc.* 121(1):113–123, 1994.

[18] Jean-Louis Krivine. Sous-espaces de dimension finie des espaces de Banach réticulés. *Ann. of Math. (2)* 104(1):1–29, 1976.

[19] Vincent Lafforgue. Un renforcement de la propriété (T), 2007. Available at http://www.institut.math.jussieu.fr/~vlafforg/Trenforce.pdf.

[20] James R. Lee, Assaf Naor, and Yuval Peres. Trees and Markov convexity. In *SODA* 1, pages 1028–1037.

[21] Nathan Linial, Eran London, and Yuri Rabinovich. The geometry of graphs and some of its algorithmic applications. *Combinatorica*, 15(2):215–245, 1995.

[22] Nathan Linial and Michael Saks. The Euclidean distortion of complete binary trees. *Discrete Comput. Geom.*, 29(1):19–21, 2003.

[23] Jiří Matoušek. Ramsey-like properties for bi-Lipschitz mappings of finite metric spaces. *Comment. Math. Univ. Carolin.*, 33(3):451–463, 1992.

[24] Jiří Matoušek. On embedding trees into uniformly convex Banach spaces. *Israel J. Math.*, 114:221–237, 1999.

[25] Jiří Matoušek. *Lectures on discrete geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.

[26] Jiří Matoušek. Bi-Lipschitz embeddings into low-dimensional euclidean spaces. *Comment. Math. Univ. Caroliniae*, 31:589–600, 1990.

[27] Bernard Maurey. Type, cotype and $K$-convexity. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1299–1332. North-Holland, Amsterdam, 2003.

[28] Bernard Maurey and Gilles Pisier. Un théorème d’extrapolation et ses conséquences. *C. R. Acad. Sci. Paris Sér. A-B*, 277:A39–A42, 1973.

[29] Bernard Maurey and Gilles Pisier. Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach. *Studia Math.*, 58(1):45–90, 1976.

[30] Manor Mendel and Assaf Naor. Metric cotype. In *SODA* 1, pages 79–88.

[31] Manor Mendel and Assaf Naor. Scaled Enflo type is equivalent to Rademacher type. *Bull. London Math. Soc.*, 39(3):493–498, 2007.

[32] Manor Mendel and Assaf Naor. Markov convexity and local rigidity of distorted metrics. In *Proceedings of the 24th Annual ACM Symposium on Computational Geometry*, 2008.

[33] Manor Mendel and Assaf Naor. Metric cotype. *Ann. of Math.*, to appear. arXiv:math/0506201v3.

[34] Vitali D. Milman and Gideon Schechtman. *Asymptotic theory of finite-dimensional normed spaces*, volume 1200 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With an appendix by M. Gromov.

[35] Gilles Pisier. Sur les espaces de Banach qui ne contiennent pas uniformément de $l^n$. *C. R. Acad. Sci. Paris Sér. A-B*, 277:A991–A994, 1973.

[36] Gilles Pisier. Probabilistic methods in the geometry of Banach spaces. In *Probability and analysis (Varenna, 1985)*, volume 1206 of *Lecture Notes in Math.*, pages 167–241. Springer, Berlin, 1986.

[37] Nicole Tomczak-Taeggerman. *Banach-Mazur distances and finite-dimensional operator ideals*, volume 38 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Longman Scientific & Technical, Harlow, 1989.

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