On the Berezinian of a moduli space of curves in $\mathbb{P}^{n|n+1}$

Michael Movshev*
IAS
Princeton, NJ 0854, USA
January 26, 2022

Abstract
A supermanifold $\mathbb{P}^{3|4}$ is a target space for twistor string theory. In this note we identify a line bundle of holomorphic volume elements $\text{Ber} \mathcal{M}_g(\mathbb{P}^{3|4})$ defined on the moduli space of curves of genus $g$ in $\mathbb{P}^{3|4}$ with a pullback of a line bundle defined on $\mathcal{M}_g(pt)$. We also give some generalizations of this fact.

1 Introduction
A twistor string theory (TST) was introduced in [6] as a tool for studying perturbative expansion of amplitudes in $D = 4, N = 4$ Yang-Mills theory (SYM). It was shown later in [1] that the full TST contains conformal supergravity and its connection with SYM is not quite clear. However there is a hope that a modification of TST is possible that enables us to relate it to perturbative SYM theory.

*This material is partially based upon work supported by the National Science Foundation under agreement No. DMS-0111298.
It is desirable to achieve a better understanding of the main ingredients of TST. One of such ingredients is an integration measure for loop diagrams. In \[7\] we read:

...the proper definition of the integration measure for loop diagrams in twistor space (that is, for $D$-instanton configurations of positive genus) remains unclear.

In this note we clarify a situation with the measure.

In TST the target manifold is a complex projective superspace $P^{3|4}$. We will work in a slightly more general context, when the target is $P^{n|n+1}$.

Suppose $V$ is equal to $\mathbb{C}^{n+1}$. We denote by $\Pi$ an operation of parity reversion that acts upon $\mathbb{Z}_2$-graded linear spaces. A mathematical formulation of the problem can be outlined as follows. We consider a manifold

$$P^{n|n+1} = V \times \Pi V \setminus \{0\}/\mathbb{C}^*$$

(1)

The moduli space of algebraic smooth curves in $P^{n|n+1}$ of genus $g$ is denoted by $\mathcal{M}_g(P^{n|n+1})$. It is a (nonconnected) superorbifold. The problem is to compute Berezinian of $\mathcal{M}_g(P^{n|n+1})$ in simple terms.

The moduli space $\mathcal{M}_g(P^{n|n+1})$ admits a projection

$$n : \mathcal{M}_g(P^{n|n+1}) \to \mathcal{M}_g$$

(2)

where $\mathcal{M}_g = \mathcal{M}_g(pt)$. The space $\mathcal{M}_g$ carries several line bundles: $\lambda$ is a determinant line bundle. It will be described more explicitly in section (2). There is also the canonical line bundle $K = \Omega^{3g-3}_{\mathcal{M}_g}$ - a bundle of holomorphic volume forms. There is a classical identity $\lambda \otimes \lambda = K$ (see \[4\], \[5\]).

**Proposition 1** Denote by $n^*$ a pullback operation by the map $n$.

If $g = 0$, then $\text{Ber}_n(\mathcal{M}_0(P^{n|n+1}))$ is trivial.

For $g = 1$ $\text{Ber}_n(\mathcal{M}_1(P^{n|n+1}))$ is equal to $n^* \lambda \otimes \lambda$.

$\text{Ber}_n(\mathcal{M}_g(P^{n|n+1})) = n^* \lambda \otimes \lambda$, $g \geq 2$.

The note is organized as follows: in section (2) we compute $\text{Ber}_n(\mathcal{M}_g(P^{n|n+1}))$ and prove proposition (1) using simple homological algebra. In section (3) we consider possible generalizations of the construction from section (2).
2 Computation of $\text{Ber}\mathcal{M}_g(P^{n|n+1})$

The manifold $P^{n|n+1}$ splits, the splitting is induced by projection $p : V + IV \rightarrow V$. We use the same notation for projection

$$P^{n|n+1} \rightarrow P^n \quad (3)$$

The following classical algebro-geometric construction is essential for our computation.

The manifold $P^n$ is a space of complex lines $\{[l]\}$ through the origin in $V$. The manifold $P^n$ carries the Hopf line bundle $O(-1)$. Its total space is formed by pairs $\{([l], a)\mid a \in [l] \subset V\}$. The total space of $O(-1)$ is embedded as a subspace into a direct product $P^n \times V$. The embedding produces a short exact sequence of bundles

$$0 \rightarrow O \rightarrow V \rightarrow T(-1) \rightarrow 0 \quad (4)$$

where $T$ is the tangent bundle $1$ of $P^n$. It can be easily verified fiberwise. After appropriate twist the short exact sequence (4) becomes

$$0 \rightarrow O \rightarrow V(1) \rightarrow T \rightarrow 0 \quad (5)$$

An alternative description of $P^{n|n+1}$ is available. There is an isomorphism $P^{n|n+1} = \Pi IV(1)$. The manifold $\Pi IV(1)$ is obtained from the total space of $V(1)$ by the reversal of parity of the fibers.

This isomorphism can be verified. The quotient $\Pi$ can be identified with $V\setminus 0 \times IV/\mathbb{C}^*$ because embedding $V\setminus 0 \times IV \subset V \times IV \setminus 0$ is an isomorphism (if we replace $IV$ by some even space, this becomes false). We can conclude that $V \times IV \setminus \{0\}$ is a total space of a sum of several copies of a line bundle with $\Pi$ applied fiberwise. This line bundle is $V\setminus 0 \times \Pi \mathbb{C}/\mathbb{C}^*$, which by definition is equal to $\Pi O(1)$.

We adopt the following notations: a vector bundle $\mathcal{F}$ over an algebraic manifold $M$ has a space of global sections denoted by $H^0(M, \mathcal{F})$. A linear space

$^{\dagger}$It is a standard practice in algebraic geometry to denote a tensor product of sheaves $\mathcal{F} \otimes O(1)^{\otimes i}, i \in \mathbb{Z}$ by $\mathcal{F}(i)$. An operation of tensor multiplication of $\mathcal{F}$ on $O(i)$ is called twist.
of $i$-th cohomology of $\mathcal{F}$ computed through, say, Čech resolution is denoted by $H^i(M, \mathcal{F})$. A fibration $p : M \to N$ enables us to transfer vector bundles from $M$ to $N$ by an operation of direct image $p_* \mathcal{F}$. By definition the fiber of $p_* \mathcal{F}$ at $x \in N$ is equal to $H^0(p^{-1}(x), \mathcal{F})$. Similarly a fiber of $R^i \mathcal{F}$ at $x$ is $H^i(p^{-1}(x), \mathcal{F})$.

A vector bundle $R^i \mathcal{F}$ is called a higher direct image. There is a caveat to the above definition. The dimensions of $H^i(p^{-1}(x), \mathcal{F})$ can change from point to point. Thus $R^i p_* \mathcal{F}$ is a vector bundle only in favorable circumstances but typically it is only a sheaf.

Suppose $M$ is a supermanifold which splits, i.e. there is a fibration $p : M \to M_{rd}$ identical on $M_{rd}$ ($M_{rd} \subset M$ is the underlying ordinary manifold). Let $T_M$ be the tangent bundle of $M$. The bundle $T_M$ restricted on $M_{rd}$ splits into a sum of even and odd parts $T^0 + T^1$. There is an isomorphism of $T^0$ over $M_{rd}$ with the tangent bundle $T_{M_{rd}}$. To simplify notations we drop the subscript $T = T_{M_{rd}}$. If we reverse the parity of the fibers of $T^1$ we get an ordinary vector bundle $E$ over $M_{rd}$. According to \cite{3} the total space of $\Pi E$ is (noncanonically) isomorphic to $M$ and

$$\text{Ber}M \cong p^*(\text{det}^{-1}T \otimes \text{det}E)$$

\textbf{Definition 2} We denote by $\text{det}$ the top exterior power of a vector space (bundle).

Denote by $\mathcal{M}_{g,k}(\mathbb{P}^{n|n+1})$ the moduli space of smooth connected algebraic curves in $\mathbb{P}^{n|n+1}$ with $k$ distinct marked points. The orbifold $\mathcal{M}_{g,k}(\mathbb{P}^{n|n+1})$ splits over $\mathcal{M}_{g,k}(\mathbb{P}^n)$:

$$p : \mathcal{M}_{g,k}(\mathbb{P}^{n|n+1}) \to \mathcal{M}_{g,k}(\mathbb{P}^n)$$

The splitting comes from the projection \cite{3}.

Let us find vector bundles $T$ and $E$ in a context of the pair $\mathcal{M}_g(\mathbb{P}^n) \subset \mathcal{M}_g(\mathbb{P}^{n|n+1})$.

A linear space $T_\psi$ is the tangent vector space at a point $\psi \in \mathcal{M}_g(\mathbb{P}^n)$. The tangent space to the moduli of curves in a manifold at a point represented by
a curve is equal to the space of section of the normal bundle to the curve. For \( \mathbf{P}^n \) it means the following: let \( \psi : \Sigma \to \mathbf{P}^n, \psi \in \mathcal{M}_g(\mathbf{P}^n) \). There is a map \( D : T_\Sigma \to \psi^*T_{\mathbf{P}^n} \) defined by Jacobian of \( \psi \). Denote by \( N \) the normal bundle. It completes the map \( D \) to a short exact sequence:

\[
0 \to T_\Sigma \to \psi^*T_{\mathbf{P}^n} \to N \to 0 \tag{8}
\]

The tangent space \( T_\psi \) is equal to \( H^0(N) = H^0(\Sigma, N) \). Similarly we define \( E_\psi \) as \( H^0(\psi^*V(1)) \). According to the formula (6) the fiber of \( \text{Ber}_g(\mathbf{P}^n|_n + 1) \) at a point \( \psi \) is equal to \( p^*(\det^{-1}H^0(N) \otimes \det H^0(\psi^*V(1))) \).

The group \( H^0(N) \) is a part of a long exact sequence

\[
0 \to H^0(T_\Sigma) \to H^0(\psi^*T_{\mathbf{P}^n}) \to H^0(N) \to H^1(T_\Sigma) \to H^1(\psi^*T_{\mathbf{P}^n}) \to H^1(N) \to 0 \tag{9}
\]

It gives an isomorphism of fibers of determinant line bundles:

\[
det^{-1}H^0(N) \otimes \det H^0(\psi^*V(1)) =
\]

\[
= \det^{-1}H^0(\psi^*T_{\mathbf{P}^n}) \otimes \det H^0(\psi^*V(1)) \otimes \det^{-1}H^1(T_\Sigma) \otimes \det H^0(T_\Sigma) \]

\[
\otimes \det H^1(\psi^*T_{\mathbf{P}^n}) \otimes \det^{-1}H^1(\psi^*V(1)) \otimes \det H^0(T_\Sigma) \tag{10}
\]

Machinery of higher direct images is a convenient tool that enables us to see how various vector spaces involved in the formula (10) vary in families. A universal curve \( \mathcal{M}_{g,1}(\mathbf{P}^n) \) has evaluation and forgetting maps

\[
ev : \mathcal{M}_{g,1}(\mathbf{P}^n) \to \mathbf{P}^n
\]

\[
q : \mathcal{M}_{g,1}(\mathbf{P}^n) \to \mathcal{M}_g(\mathbf{P}^n)
\]

A fiber of \( q \) is equal to a curve \( \Sigma \). A vector space \( H^0(\psi^*T_{\mathbf{P}^n}) \) is the fiber at \( \psi \in \mathcal{M}_g(\mathbf{P}^n) \) of the bundle \( q_*ev^*\mathbf{P}^n \). A space \( H^0(\psi^*V(1)) \) is the fiber of \( q_*ev^*V(1) \) at the same point.

A pullback transforms an exact sequence of vector bundles to an exact sequence. Denote a pullback of the exact sequence (5) by the map \( ev \) by

\[
0 \to \mathcal{O} \to E_2 \to E_1 \to 0 \tag{12}
\]

\(^2\)In the future to simplify notations we suppress \( \Sigma \) in \( H^i(\Sigma, \cdot) \)
We use the isomorphism $ev^* O = O$.

The direct image functor by a map $q$ transforms the short exact sequence into a long exact sequence of vector bundles:

$$
0 \to q_* O \to q_* E_2 \to q_* E_1 \to R^1 q_* O \to R^1 q_* E_2 \to R^1 q_* E_1 \to 0
$$

(13)

It generalizes a long exact sequence in cohomology.

Some comments are in order on the sheaves that appear in the last formula.

The fiber of $q_* O$ at $(\Sigma, \psi)$ is a space of global holomorphic functions over $\Sigma$. It is exhausted by constants for compact $\Sigma$. Thus $q_* O = O$. The fiber of $R^1 q_* O$ at the same point is a space dual to the space of holomorphic differentials on $\Sigma$ (Serre duality). It does not depend on the map $\psi$; the vector bundle $R^1 q_* O$ originates on $M_\Sigma$. This observation will be used later in this note. A fiber of $R^1 q_* E_2$ is equal to $H^1(\psi^* V(1))$. The latter is zero upon using Kodaira vanishing theorem and the fact that $O(1)$ is ample. Due to exactness of the sequence (13) the sheaf $R^1 q_* E_1$ also vanishes. This argument eliminates terms $det H^1(\psi^* T_{P^n}) \otimes det^{-1} H^1(\psi^* V(1))$ in the formula (10).

Considerations of the previous paragraph imply that the long exact sequence (13) gives rise to a canonical isomorphism.

$$
det(R^1 q_* O)^{-1} = det(q_* E_2) \otimes det^{-1}(q_* E_1)
$$

(14)

The LHS of (14) is a pullback from $M_\Sigma$.

Let $T_{rel}$ be a line bundle of vector fields tangential to the fibers of projection $q$. The reader can easily identify the fiber of $q_* T_{rel}$ over $\Sigma$ with $H^0(T_{\Sigma})$ and the fiber of $R^1 q_* T_{rel}$ with $H^1(T_{\Sigma})$.

Suppose $g \geq 2$. Then for any $\Sigma H^0(T_{\Sigma}) = 0$ and we can rewrite $det^{-1} T \otimes det(E)$ using equations (10) and (14) in terms independent of $P^n$:

$$
det^{-1} T \otimes det(E) = det^{-1}(R^1 q_* O) \otimes det^{-1}(R^1 q_* T_{rel})
$$

(15)

According to [4] and [5] there is an isomorphism between certain line bundles on $M_\Sigma$. Let $\lambda = det^{-1}(R^1 q_* O)$ and $K = det^{-1}(R^1 q_* T_{rel})$, then $K = \lambda^{13}$ and $Ber M_\Sigma(P^{n|n+1}) = \lambda^{14}$. 

6
If \( g = 0 \), additional terms in the formula for Berezinian (according to isomorphism (10)) are present: \( \text{Ber} = \det(R^1q_*\mathcal{O})^{-1} \otimes \det^{-1}(R^1q_*T_{rel}) \otimes \det(R^0q_*T_{rel}) \).

If \( g = 0 \), then \( R^1q_*\mathcal{O} \) and \( R^1q_*T_{rel} \) vanish (direct computation). The group \( H^0(T_\Sigma) \) (a fiber of \( R^0q_*T_{rel} \)) is isomorphic to the Lie algebra \( \mathfrak{sl}_2 \). The group \( \text{Aut}(\Sigma) \) acts trivially on linear space \( \det(\mathfrak{sl}_2) \). We conclude that \( \det(R^1q_*\mathcal{O})^{-1} \otimes \det^{-1}(R^1q_*T_{rel}) \otimes \det(R^0q_*T_{rel}) \) is trivial.

If \( g = 1 \), then the line bundles \( R^i q_* T^\otimes_k \) are one-dimensional for all \( i = 0, 1 \) and \( k \in \mathbb{Z} \). Denote \( \Omega_{rel} = T^*_{rel} \). Using Serre duality we conclude that
\[
\det(R^1q_*\mathcal{O})^{-1} \otimes \det^{-1}(R^1q_*T_{rel}) \otimes \det(q_*T_{rel}) = q_*\Omega_{rel} \otimes q_*\Omega^2_{rel} \otimes q_*T_{rel} = q_*\Omega^2_{rel}
\]

Thus we have an isomorphism
\[
\lambda^2 = \det^{-2}R^1q_*\mathcal{O} = q_*\Omega^2_{rel} = \text{Ber}_M1(P^{n|n+1}) \quad (16)
\]

3 Computation with targets more general then \( P^{n|n+1} \)

In this section we extend proposition (14) to wider class of manifolds.

Suppose \( M \) is a compact Kähler manifold. Let \( \bigoplus_{i+j=k} H^{i,j} \) be the Hodge decomposition of \( k \)-th de Rham cohomology; \( h^{i,j} = \dim H^{i,j} \). Denote \( h^{1,1} \) by \( h \). We have an identification \( H^{1,1} = H^1(M, \Omega^1) \). The last vector space is the Dolbeault cohomology of the vector bundle of holomorphic differentials. The isomorphisms \( H^1(M, \Omega^1) = Ext^1(\mathcal{O}, \Omega^1) = Ext^1(T, \mathcal{O}) \) are standard in algebraic geometry (see [2]). The linear space \( Ext^1(\ldots) \) is a group of extensions of vector bundles. The group \( Ext^1(T, \mathcal{O}) \) classifies extensions \( 0 \to \mathcal{O} \to X \to T \to 0 \). Denote by \( \mathbb{C}^h \) the group \( H^1(M, \Omega^1) \). There is a universal extension
\[
0 \to \mathbb{C}^h \to E \to T \to 0 \quad (17)
\]
Denote by \( W(M) \) a split supermanifold. It is equal to the total space of \( E \) with the reversed parity of the fibers. It is convenient to impose some restrictions on \( M \):
Condition  For any holomorphic curve \( \psi : \Sigma \to M \) a linear space \( H^1(\Sigma, \psi^* E) \) vanishes.

If \( E \) has a filtration with ample adjoint quotients the above condition is fulfilled. One can find many examples of such \( M \) in the class of toric varieties. A space \( \mathbb{P}^n \) considered in section [2] is one of them. We have \( h(\mathbb{P}^n) = 1, E = \mathbb{C}^{n+1}(1) \).

Proposition 3  Suppose \( M \) satisfies above assumption. The Berezinian of \( \mathcal{M}_g(W(M)) \) is equal to \( n^* \lambda^{13+h}, g \geq 2 \).

If \( g = 1 \), then \( \text{Ber} \mathcal{M}_g(W(M)) \) is equal to \( n^* \lambda^{h+1} \).

If \( g = 0 \), then \( \text{Ber} \mathcal{M}_g(W(M)) \) is trivial.

Proof. The same as for \( \mathbb{P}^n \). ■

3.1  Acknowledgements

This work was written while author was visiting Institut Mittag-Leffler (2004) and Institute for Advanced Study (2005-2007). The author wishes to thank these institutions for hospitality. He also would like to thank L. Mason, A. Schwarz and E. Witten for useful discussions.

References

[1] N. Berkovits, E. Witten Conformal Supergravity in Twistor-String Theory JHEP 0408 (2004) 009

[2] P Griffiths, J Harris Principles of algebraic geometry 1978 - Wiley New York.

[3] Yu. Manin Gauge Field theory and Complex Geometry Springer-Verlag 1988.

[4] D. Mumford Stability of Projective Varieties L’Enseignement mathematique 1977
[5] AA Belavin, VG Knizhnik *Complex geometry and the theory of quantum strings* Zhurnal Eksperimental’noi i Teoreticheskoi Fiziki, 1986

[6] E. Witten *Perturbative Gauge Theory as a String Theory in Twistor Space* Commun.Math.Phys. 252 (2004) 189-258

[7] E. Witten *Parity Invariance For Strings In Twistor Space* hep-th/0403199