The Anderson–Weber strategy is not optimal for symmetric rendezvous search on $K_4$

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8 July 2009

Abstract

We consider the symmetric rendezvous search game on a complete graph of $n$ locations. In 1990, Anderson and Weber proposed a strategy in which, over successive blocks of $n - 1$ steps, the players independently choose either to stay at their initial location or to tour the other $n - 1$ locations, with probabilities $p$ and $1 - p$, respectively. Their strategy has been proved optimal for $n = 2$ with $p = 1/2$, and for $n = 3$ with $p = 1/3$. The proof for $n = 3$ is very complicated and it has been difficult to guess what might be true for $n > 3$. Anderson and Weber suspected that their strategy might not be optimal for $n > 3$, but they had no particular reason to believe this and no one has been able to find anything better. This paper describes a strategy that is better than Anderson–Weber for $n = 4$. However, it is better by only a tiny fraction of a percent.

1 The Anderson–Weber strategy

In the symmetric rendezvous search game on $K_n$ (the completely connected graph on $n$ vertices) two players are initially placed at two distinct vertices (called locations). The game is played in discrete steps and at each step each player can either stay where he is or move to a different location. The players share no common labelling of the locations. Our aim is to find a (randomizing) strategy such that if both players independently follow this strategy then they minimize the expected number of steps until they first meet. Rendezvous search games of this type were first proposed by Steve Alpern in 1976. They are simple to describe, and have received considerable attention in the popular press as they model problems that are familiar in real life. They are notoriously difficult to analyse.

The Anderson–Weber strategy is a mixed strategy that proceeds in blocks of $n - 1$ steps. Players begin at distinct locations, called their home locations. In each successive block a player either stays at his home location, with probability $p$, or makes a randomly chosen tour of his $n - 1$ non-home locations, doing this with probability $1 - p$. The motivation for the strategy comes from the wait-for-mommy strategy that is optimal in an asymmetric version of the problem. With probability $2p(1 - p)$ the players play the wait-for-mommy strategy over the first $n - 1$ steps and so rendezvous in expected time $(n + 1)/2$.

Anderson and Weber (1990) proved that the above strategy is optimal for the game on $K_2$, with $p = 1/2$, and conjectured that it should be optimal for $K_3$, with $p = 1/3$. This

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was finally proved by Weber (2006), who established a strong AW property (SAW) that AW minimizes $E[\min\{T, k\}]$ for all $k$. Anderson and Weber suspected that their strategy might not be optimal for $n > 3$, but they had no particular reason to believe this and no one has been able to find any strategy that is better. Indeed, AW has been shown optimal amongst 2–Markov strategies. Fan (2009) showed that AW minimizes $P(T > 2)$ and $E[\min\{T, 2\}]$. He also found that AW is not optimal on $K_4$ if players have the extra information that the location can be viewed as being arranged on a circle and the players are given a common notion of clockwise. However, the question as to whether or not AW is optimal has remained open for the case in which there is no such special extra information. Fan writes, ‘The author believes that SAW still holds on $K_4$, and so AW strategy is still optimal’. We were inclined to agree, but now find that AW can be bettered. For more background to the problem see Weber (2006).

Let us begin by reprising the AW strategy for the symmetric rendezvous game on $K_4$. We assume that there is no special knowledge (such as a common notion of clockwise on a circle). The AW strategy is a 3–Markov strategy that repeats in blocks of 3 steps. In each successive block of 3 steps, each player, independently, remains at his home location with probability $p$, or does a random chosen tour of his 3 non-home locations, with probability $1 - p$. This leads to rendezvous in an expected number steps $ET$, where

$$ET = p^2 \times (3 + ET) + 2p(1 - p) \times 2 + (1 - p)^2 \times \left( \frac{1}{2}(16/9) + \frac{1}{2}(3 + ET) \right)$$

$$= \frac{43 - 14p + 25p^2}{9(1 + 2p - 3p^2)}.$$

This is explained as follows.

1. If both stay home they do not meet.
2. If one stays home, while the other tours, then they meet in expected time 2.
3. If both tour, then they meet with probability $1/2$, and conditional on meeting they meet in expected time $16/9$.

One easily finds that the minimum of $ET$ is achieved by taking

$$p = \frac{1}{4} \left( 3\sqrt{681} - 77 \right) \approx 0.321983$$

and then

$$ET = \frac{1}{12} \left( 15 + \sqrt{681} \right) \approx 3.42466.$$

**2 A strategy better than Anderson–Weber on $K_4$**

We now explain how the AW strategy can be bettered. Suppose player I has location 1 as his home, and player II has location 2 as home. We might imagine that each player labels his non-home locations as $a, b, c$, and so a tour of his non-home locations is one of six possible
tours: abc, acb, bac, bca, cab, cba. In the case that player I has \((a, b, c) = (2, 3, 4)\) and player II has \((a, b, c) = (1, 3, 4)\) we can compute the matrix

\[
B = \begin{pmatrix}
2 & X & 3 & X & 2 \\
X & 2 & X & 2 & 3 & X \\
3 & X & 1 & 1 & X & X \\
X & 2 & 1 & 1 & X & X \\
X & 3 & X & X & 1 & 1 \\
2 & X & X & X & 1 & 1
\end{pmatrix}
\]

where we have ordered the rows and columns to correspond to abc, acb, bac, bca, cab, cba. A number entry indicates the step at which players meet when they meet, and X indicates that they do not meet. There are 36 such matrices, over which we must average, for each possible pair of assignments by players I and II, of \((2, 3, 4)\) and \((1, 3, 4)\), respectively, to \((a, b, c)\).

Let us begin by noting that if a player stays home for three steps and meeting does not occur, then the other player must also have been staying home. Similarly, if a player tours for three steps and meeting does not occur, then the other player must also have been touring (and their tours not meeting). Thus after any \(3k\) steps (a multiple of 3) each player knows exactly how many times both have been touring.

Whenever a player makes a tour in the AW strategy he chooses his tour at random (independently of previous tours). We show how to improve AW introducing some dependence between tours. Let us adopt a notation in which the first tour a player makes is labelled \(A\). The second distinct tour a player makes is labelled \(B\), and so on. So, for example, \(AAB\) means that on his first three tours, a player (i) first makes a random tour, (ii) second makes the same tour as his first tour, (iii) and third makes a tour chosen randomly from amongst the 5 tours he has not yet tried.

Let us consider first a modified problem in which at each so-called ‘t–step’ each player makes a tour of his non-home locations. In this modified problem no player stays home for a t–step. We wish to minimize the expected number of t–steps until the players meet. At the first t–step both players do \(A\) and the probability of meeting is \(1/2\). If a 1–Markov strategy is employed, so successive t–steps are chosen at random, then the expected number of t–steps until meeting occurs is 2.

Over the first two t–steps, the players can do either \(AA\) or \(AB\). The matrix for not meeting is

\[
P_2 = \begin{pmatrix}
\frac{1}{2} & \frac{1}{3} \\
\frac{1}{5} & \frac{13}{50}
\end{pmatrix}
\]

One can check that \(P_2 \succ 0\) (i.e., \(P_2\) is positive definite). thus for a 2–Markov strategy we would be solving

\[
ET = x^\top (J + P_2 ET)x
\]

where \(J\) is a \(2 \times 2\) matrix filled with 1s. This has a minimum value of \(ET = 2\), when we take \(x^\top = (1/6, 5/6)\). This means that, restricting to 2–Markov strategies, tours should be chosen at random.

Similarly, over the first three t–steps, the players can do \(AAA, AAB, ABA, ABB, ABC\).
The matrix for not meeting is

\[ P_3 = \begin{pmatrix}
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{10} \\
\frac{1}{5} & \frac{3}{20} & \frac{1}{25} & \frac{1}{25} & \frac{1}{100} \\
\frac{1}{5} & \frac{1}{10} & \frac{2}{25} & \frac{1}{25} & \frac{1}{50} \\
\frac{1}{5} & \frac{2}{25} & \frac{1}{25} & \frac{1}{50} & \frac{1}{100} \\
\frac{1}{20} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{50}
\end{pmatrix}. \]

Again, \( P_3 \succeq 0 \), and \( (x_{AAA}, x_{AAB}, x_{ABA}, x_{ABB}, x_{ABC}) = (1/36, 5/36, 5/36, 5/36, 20/36) \) is optimal in the sense of minimizing the solution of

\[ ET = x^\top (J + P_2 + P_3 ET)x, \]

where \( J \) is now \( 5 \times 5 \) and \( P_2 \) is expanded to the appropriate \( 5 \times 5 \) matrix. Thus amongst 3–Markov strategies, tours should also be chosen at random.

However, over four t–steps things turn out differently. There are now 15 possible strategies: \( AAAA, AAAB, AABA, AABB, AABC, ABAA, ABAB, ABAC, ABBA, ABBB, ABBC, ABCA, ABCB, ABCC, ABCD \). The matrix for not meeting can be computed to be

\[ P_4 = \begin{pmatrix}
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} & \frac{1}{20} & \frac{1}{100} \\
\frac{1}{5} & \frac{1}{10} & \frac{1}{25} & \frac{1}{25} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} \\
\frac{1}{5} & \frac{1}{25} & \frac{1}{50} & \frac{1}{25} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} \\
\frac{1}{5} & \frac{1}{25} & \frac{1}{50} & \frac{1}{25} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} \\
\frac{1}{20} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} \\
\frac{1}{5} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} \\
\frac{1}{20} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} \\
\frac{1}{5} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} \\
\frac{1}{5} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} & \frac{1}{25} \\
\frac{1}{20} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} \\
\frac{1}{20} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} \\
0 & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100} & \frac{1}{100}
\end{pmatrix} \]

It now turns out that \( P_4 \) has a negative eigenvalue. The AW strategy would be to choose tours at random, which gives

\[
x^\top = \left( P_{AAAA}, P_{AAAB}, P_{AABA}, P_{AABB}, P_{AABC}, P_{ABAA}, P_{ABAB}, P_{ABAC}, P_{ABBA}, P_{ABBB}, P_{ABBC}, P_{PABCA}, P_{PABCB}, P_{PABCC}, P_{PABCD} \right) = \frac{1}{63} (1, 5, 5, 20, 5, 5, 20, 5, 20, 20, 20, 20, 60). \]
Solving \( ET = x^\top (J + P_2 + P_3 + P_4 \cdot ET)x \), we find \( ET = 2 \), as we expect. However consider \( y^\top = \left( 0, \frac{1}{12}, \frac{1}{12}, 0, 0, \frac{1}{12}, 0, 0, 0, \frac{1}{12}, 0, 0, 0, 0, 0, \frac{2}{3} \right) \).

Solving \( ET = y^\top (J + P_2 + P_3 + P_4 \cdot ET)y \) gives \( ET = 2 - \frac{23}{16200} = 1.99858 \). Thus, rendezvous occurs in a smaller expected number of t–steps than it does under AW. This happens when players use a mixed 4–Markov strategy of doing AAAB, AABA, ABAA, ABBB, each with probability 1/12, and ABCD with probability 2/3. This corresponds to choosing tours for the first two t–steps at random, but then making the choice of tours at the 3rd and 4th t–step depend on the tours taken at the 1st and 2nd t–step. The choice of \( y \) is not unique. It has been choose to be simple, containing many 0s, and it was found by using the fact that the eigenvector of \( P_4 \) having a negative eigenvalue is of a pattern \((\alpha, \beta, \gamma, \delta, \beta, \gamma, \delta, \gamma, \beta, \delta, \delta, \delta, \delta, \epsilon)\) for some irrational \( \alpha, \beta, \gamma, \delta, \epsilon \).

The above makes it very plausible that we can find a strategy that is better than AW on \( K_4 \). We now need to do some careful calculations. We consider a 12–Markov strategy consisting of 4 t–steps. In each t-step a player remains home with probability \( p \), and tours with probability \( 1 - p \). When he makes tours, he does so in an manner that achieves the distribution previously described. That is, any 1st and 2nd tours are made at random, but 3rd and 4th tours are made so that these are consistent with the distribution over 4 tours being AAAB, AABA, ABAA, ABBB, each with probability 1/12, and ABCD with probability 2/3. If at the end of 12 steps the players have not met then the strategy restarts, forgetting about the number of previous t-steps on which players made non-meeting tours.

We found it easiest to calculate the expected meeting time by attaching a probability to each possible 12–step paths that the strategy might take. There are 1585 possible paths which have nonzero probability. We computed the step at which players meet, or event that they do not meet, for each of the \( 1585 \times 1585 \) possibilities, and averaged these using the appropriate probabilities. The calculations are intricate, but can be checked in various ways to provide confidence that no mistake has been made. It turns out that the expected meeting time is

\[
ET = -\frac{227773p^8 + 582884p^7 - 1329319p^6 + 1737938p^5 - 1941235p^4 + 1420688p^3 - 998669p^2 + 389834p - 217648}{3(82001p^8 - 218608p^7 + 327728p^6 - 315256p^5 + 215870p^4 - 104656p^3 + 36128p^2 - 8008p - 15199)}.
\]

Taking \( p = \frac{1}{4} (3\sqrt{681} - 77) \), which is the optimal value for the AW strategy, we find that the new strategy produces an expected meeting time that is less than that of AW by

\[
\frac{243(75041961207 + 4700853101 \sqrt{681})}{327540887401488016} \approx 0.000146683.
\]

The tiny improvement is due to the fact that when both players do four t-tours (which happens with probability \((1 - p)^4\)), the new strategy gives a greater probability that the players meet than does AW. It would be possible to make the new strategy even better, by choosing \( p \) slightly differently, or indeed making it depend on the number of tours that have been taken so far over which players have not met. We could also do better by not restarting after 12 steps. However, our aim is not to try to find the best strategy for \( K_4 \), which still seems very difficult, but simply to show that AW is not optimal. This we have now done.
References

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[3] R. R. Weber The optimal strategy for symmetric rendezvous on $K_3$. *arXiv:0906.5447v1*, 2006.