Analytical Helmholtz Decomposition and Potential Functions for many n-dimensional unbounded vector fields

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Abstract: We present a Helmholtz Decomposition for many n-dimensional, continuously differentiable vector fields on unbounded domains that do not decay at infinity. Existing methods are restricted to fields not growing faster than polynomially and require solving n-dimensional volume integrals over unbounded domains. With our method only one-dimensional integrals have to be solved to derive gradient and rotation potentials.

Analytical solutions are obtained for smooth vector fields $f(x)$ whose components are separable into a product of two functions: $f_k(x) = u_k(x_k) \cdot v_k(x_{\neq k})$, where $u_k(x_k)$ depends only on $x_k$ and $v_k(x_{\neq k})$ depends not on $x_k$. Additionally, an integer $\lambda_k$ must exist such that the $2\lambda_k$-th integral of one of the functions times the $\lambda_k$-th power of the Laplacian applied to the other function yields a product that is a multiple of the original product. A similar condition is well-known from repeated partial integration: If the shifting of derivatives yields a multiple of the original integrand, the calculation can be terminated.

Also linear combinations of such vector fields can be decomposed. These conditions include periodic and exponential functions, combinations of polynomials with arbitrary integrable functions, their products and linear combinations, and examples such as Lorenz or Rössler attractor.

Keywords: Helmholtz Decomposition, Fundamental Theorem of Vector Calculus, Gradient and Rotation Potentials, Unbounded Domains, Poisson Equation.

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1. Introduction

The Helmholtz Decomposition splits a sufficiently smooth vector field $f(x)$ into an irrotational (curl-free) and a solenoidal (divergence-free) vector field. This ‘Fundamental Theorem of Vector Calculus’ is indispensable for many problems in mathematical physics (Dassios and Lindell, 2002; Kustepeli, 2016; Sprössig, 2009; Tran-Cong, 1993), but is also used in animation, computer vision or robotics (Bhatia et al., 2013), or for describing ‘quasi-potential’ landscapes and Lyapunov functions for high-dimensional non-gradient systems (Suda, 2019; Zhou et al., 2012).
The challenge is to derive the potential $G(x)$ and the rotation field $r(x)$ such that:

$$f(x) = \text{grad } G(x) + r(x),$$  \hspace{1cm} (1)

$$\text{div } r(x) = 0.$$  \hspace{1cm} (2)

Only if $f(x)$ is curl-free, path integration yields the potential $G(x)$. In other cases, the Poisson equation has to be solved:

$$\Delta G(x) = \text{div } f(x).$$  \hspace{1cm} (3)

On bounded domains, a unique solution is guaranteed by appropriate boundary conditions (Chorin and Marsden, 1990; Schwarz, 1995). Numerical methods based on finite elements or difference methods, Fourier or wavelet domains have been developed to derive $L^2$-orthogonal decompositions. This concept has been extended as Hodge or Helmholtz–Hodge Decomposition to compact Riemannian manifolds (Bhatia et al., 2013).

To solve the Poisson equation on unbounded domains, the standard approach as summarized in Sec. 2 is limited to fields decaying sufficiently fast and requires to calculate numerically infinite integrals over $\mathbb{R}^n$ for each point $x$. This paper provides an alternative, applicable to many unboundedly growing vector fields. Only one-dimensional integrals have to be solved to derive gradient and rotation potentials. Sec. 3 introduces our notation. Sec. 4 illustrates our method using an exponentially growing field. Sec. 5 states two theorems, a special case for fields with only one non-zero component and a more general case for linear combinations of the former which includes fields with more than one non-zero component. Three corollaries provide simplified versions for special cases, including linear functions. Sec. 6 concludes.

2. Literature Review

On unbounded domains as in the classical Helmholtz decomposition in $\mathbb{R}^3$, solving Eq. (3) leads to volume integrals over the entire space. The Helmholtz Decomposition in $\mathbb{R}^n$ proceeds analogously to $\mathbb{R}^3$ (Glötzl and Richters, 2021), see Figure 1. Starting from a twice continuously differentiable vector field $f$, the scalar source density $\gamma(x)$ and the rotation densities $\rho_{ij}(x)$ are calculated. To determine the ‘gradient potential’ $G(x)$ and the antisymmetric ‘rotation potentials’ $R_{ij}(x) = -R_{ji}(x)$, the Newton Integral operator $\int$ convolves these densities with the fundamental solutions of the Laplace equation, performing a multiple integral over $\mathbb{R}^n$:

$$G(x) = \int_{\mathbb{R}^n} \gamma(x) = \int_{\mathbb{R}^n} K(x - \xi) \text{div } f(\xi) d\xi^n,$$  \hspace{1cm} (4)

$$R_{ij}(x) = \int_{\mathbb{R}^n} \rho_{ij}(x) = \int_{\mathbb{R}^n} K(x - \xi) \overline{\text{ROT}_{ij}} f(\xi) d\xi^n,$$  \hspace{1cm} (5)

with the Newtonian kernel

$$K(x, \xi) = \begin{cases} \frac{1}{2\pi} \log |x - \xi| & n = 2 \\ \frac{(\xi^2)^{1/2} n(n-2-n)\pi}{n(2-n)n} |x - \xi|^{2-n} & n \neq 2 \end{cases}$$  \hspace{1cm} (6)

and the rotation operator

$$\overline{\text{ROT}_{ij}} f(x) = \frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i}.$$  \hspace{1cm} (7)
Figure 1: For unbounded domains, the usual approach derives scalar and rotation densities from the vector field \( f \). These densities are convolved with the fundamental solutions of the Laplace equation to derive scalar and rotation potentials (‘Newton Integrals’ \( \int \)). This integration over the entire space requires the vector field \( f \) to decay sufficiently fast at infinity. Our approach allows to drop this condition by directly calculating the potentials. Figure adapted from (Glötzl and Richters, 2021).

To ensure that the ‘Newton Integrals’ (Eqs. 4–6) used to calculate the ‘Newton Potentials’ \( G(x) \) and \( R_{ij}(x) \) converge, \( f(x) \) must decay faster than \( 1/|x| \) for \( |x| \to \infty \). Replacing \( K(x, \xi) \) by \( K(x, \xi) - K(0, \xi) \) makes these integrals converge if \( f(x) \) decays faster than \( x^{-\delta} \) for some \( \delta > 0 \) and \( |x| \to \infty \) (Blumenthal, 1905; Gregory, 1996; Gurtin, 1962). By choosing more complicated kernel functions, this method can also be applied for functions that grow slower than a polynomial \( x^q \) with \( q > 0 \) (Petrascheck, 2015; Tran-Cong, 1993).

From the potentials, a ‘gradient field’ \( g(x) \) as gradient of \( G \) and a ‘rotation field’ \( r(x) \) as divergence of the antisymmetric second-rank tensor \( R = [R_{ij}] \) can be calculated (Glötzl and Richters, 2021):

\[
g(x) := \text{grad} \ G(x) = \left[ \partial_{x_k} G(x); \ 1 \leq k \leq n \right],
\]

\[
r(x) := \text{ROT} \ R(x) := \left[ \sum_{j=1}^{n} \partial_{x_j} R_{kj}(x); \ 1 \leq k \leq n \right].
\]

In sum, they yield the original vector field and provide a Helmholtz Decomposition of \( f(x) \), as \( r \) satisfies \( \text{div} \ r(x) = 0 \), and \( f = g + r \) (see proof in Glötzl and Richters, 2021).
3. Notation and Definitions

Square brackets \([f; \, \mathbf{1}_{k \leq n}]\) indicate a n-dimensional vector. We denote the partial derivative \(\partial_{x_j}\), the k-th partial derivative \(\partial^k_{x_j}\), the Laplace operator \(\Delta\) and the Laplace to the power of \(p\) with \(\Delta^p\), with the convention that \(\partial^0_{x_j}f = \Delta^0f = f\).

We denote the antiderivative of a scalar field \(f_i\) with respect to \(x_j\) as:

\[
\mathcal{A}_{x_j} f_i(x) := \int_0^{x_j} f_i(\xi)d\xi. \tag{10}
\]

The \(p\)-th antiderivative of a scalar \(f_i\) with respect to \(x_j\), using the convention that \(\mathcal{A}_{x_j}^p f = f\), is given by the Cauchy formula for repeated integration or the Riemann–Liouville integral (Cauchy, 1823; Riesz, 1949):

\[
\mathcal{A}_{x_j}^p f_i(x) := \mathcal{A}_{x_j}^{p-1} \mathcal{A}_{x_j} f_i(x) = \frac{1}{(p-1)!} \int_0^{x_j} (x_j - t_j)^{p-1} f_i(t) dt. \tag{11}
\]

The \(p\)-th partial derivative of the \(p\)-th antiderivative yields the original function:

\[
\partial_{x_j} \mathcal{A}_{x_j}^p f_i(x) = \partial_{x_j}^p \mathcal{A}_{x_j}^p f_i(x) = f_i(x). \tag{12}
\]

4. Examples with exponentially diverging vector field

In \(\mathbb{R}^3\), for the exponentially growing vector field

\[
f(x_1, x_2, x_3) = \begin{bmatrix} f_1(x), 0, 0 \end{bmatrix} = \begin{bmatrix} e^{ax_1}e^{bx_2}, 0, 0 \end{bmatrix}, \tag{13}
\]

with \(a \neq b\), the Newton Integrals according to Eqs. (4–6) diverge and a Helmholtz Decomposition based on Newton Potentials does not exist. As an alternative, we present two approaches (a) and (b) to derive the gradient potential \(G(x)\) and the rotation potentials with an analytical calculation. Denote \(u_1(x_1) = e^{ax_1}\) and \(v_1(x_2) = e^{bx_2}\), such that \(f_1(x) = u_1(x_1) \cdot v_1(x_2)\).

For approach (a), to get \(\partial_{x_1} G(x) = f_1(x)\), choose

\[
G(x) = \left( \mathcal{A}_{x_1} u_1(x_1) \right) \left( v_2(x_2) \right) = \frac{1}{a} e^{ax_1}e^{bx_2} \tag{14}
\]

by integrating the first component. This results in

\[
g(x) = \left[ \partial_{x_1} G(x), \, \mathbf{1}_{1 \leq k \leq 3} \right] = \begin{bmatrix} u_1(x_1) v_2(x_2), \mathcal{A}_{x_1} u_1(x_1) \left( \partial_{x_2} v_2(x_2) \right), 0 \end{bmatrix} = \begin{bmatrix} e^{ax_1}e^{bx_2}, \frac{b}{a} e^{ax_1}e^{bx_2}, 0 \end{bmatrix}. \tag{15}
\]

with an unwanted term in the component \(g_2(x)\). To get rid of this part, choose

\[
R_{12}(x) = -R_{21}(x) = \mathcal{A}_{x_1} g_2(x) = \left( \mathcal{A}_{x_1}^2 u_1(x_1) \right) \left( \partial_{x_2} v_2(x_2) \right) = \frac{b}{a^2} e^{ax_1}e^{bx_2}, \tag{16}
\]

\[
R_{13}(x) = R_{31}(x) = R_{23}(x) = R_{32}(x) = 0, \tag{17}
\]

such that

\[
r(x) = \begin{bmatrix} \frac{b^2}{a^2} e^{ax_1}e^{bx_2}, -\frac{b}{a} e^{ax_1}e^{bx_2}, 0 \end{bmatrix}. \tag{18}
\]
This results in
\[ g(x) + r(x) = \left[ (1 + \frac{b^2}{a^2}) e^{ax_1} e^{bx_2}, 0, 0 \right] = \left( 1 + \frac{b^2}{a^2} \right) f(x). \] (19)

Integrating \( u_1(x_1) \) twice and taking the derivative \( \partial_{x_2} v_1(x_2) \) yielded the function \( C_1 u_1(x_1) v_1(x_2) \) with \( b^2/a^2 = C_1 \in \mathbb{R} \setminus \{1\} \), a multiple \( C_1 \) of \( f_1(x) \). This allows to multiply all potentials by \( 1/(1 - C_1) \) to get fields that add to \( f \) and provide a Helmholtz Decomposition of \( f \).

Approach (b) works inversely by taking derivatives of \( u_1(x_1) \) and integrating \( v_1(x_2) \): Start by choosing
\[ R_{12}(x) = -R_{21}(x) = u_1(x_1)(\mathcal{A}_{x_2} v_2(x_2)) = \frac{1}{b} e^{ax_1} e^{bx_2}, \] (20)
\[ R_{13}(x) = R_{31}(x) = R_{23}(x) = R_{32}(x) = 0, \] (21)
\[ r(x) = \left[ e^{ax_1} e^{bx_2}, -\frac{a}{b} e^{ax_1} e^{bx_2}, 0 \right]. \] (22)

To correct for the unwanted term in the second component \( r_2(x) \), choose
\[ G(x) = \mathcal{A}_{x_2} r_2(x) = (\partial_{x_2} u_1(x_1))(\mathcal{A}_{x_2}^2 v_2(x_2)) = -\frac{a}{b^2} e^{ax_1} e^{bx_2}, \] (23)
\[ g(x) = \left[ -\frac{a^2}{b^2} e^{ax_1} e^{bx_2}, \frac{a}{b} e^{ax_1} e^{bx_2}, 0 \right]. \] (24)

This results in
\[ g(x) + r(x) = \left[ (1 - \frac{a^2}{b^2}) e^{ax_1} e^{bx_2}, 0, 0 \right] = \left( 1 - \frac{a^2}{b^2} \right) f(x). \] (25)

Integrating \( v_1(x_2) \) twice and taking the derivative \( \partial_{x_1} u_1(x_1) \) yielded a function \( C'_1 u_1(x_1) v_1(x_2) \) with \( a^2/b^2 = C'_1 \in \mathbb{R} \setminus \{1\} \). This allows to multiply all potentials by \( 1/(1 - C'_1) \) to get fields that add to \( f \) and provide a Helmholtz Decomposition of \( f \).

Another example is the vector field
\[ f(x) = [e^{x_1} + e^{x_2}, e^{x_2} - e^{x_1}]. \] (26)

The first part \( [e^{x_1}, e^{x_2}] \) is rotation-free and the second part \( [e^{x_2}, -e^{x_1}] \) is divergence free, the corresponding gradient and rotation potentials can be guessed by integration for each part. Because of linearity of all the operators, the total potentials, gradient and rotation fields are:
\[ G(x) = e^{x_1} + e^{x_2}, \] (27)
\[ R_{12}(x) = R_{21}(x) = e^{x_1} + e^{x_2}, \] (28)
\[ R_{kk}(x) = 0, \] (29)
\[ g(x) = [+e^{x_1}, +e^{x_2}], \] (30)
\[ r(x) = [+e^{x_2}, -e^{x_1}]. \] (31)

5. Helmholtz Decomposition Theorems for many unbounded vector fields

Theorem 1 formalizes the process described for the above examples for fields with only one non-zero component. Theorem 2 extends it to linear combinations, which levis the restriction to one-component fields. The intuition for the theorem is that we try to compensate terms created by the gradient of the scalar potential by the choice of appropriate rotation potentials, and inversely.
**Theorem 1** (Special case for fields with one non-zero component).

Let \( f \in C^1(\mathbb{R}^n, \mathbb{R}^n) \) be a vector field with only one non-zero component \( f_k(x) \) that satisfies conditions 1 (separability) and either 2a or 2b (partial integrability). Then, a potential matrix \( F \), a gradient potential \( G \) and rotation potentials \( R = [R_{ij}] \) can be specified such that the rotation-free gradient field \( g = \text{grad} G \) and the divergence-free rotation field \( r = \text{ROT} R \) yield a Helmholtz Decomposition of \( f = g + r \).

**Condition 1: Separability:** The component \( f_k(x) = u_k(x_k) \cdot v_k(x_{\neq k}) \) is separable into a product of \( u_k(x_k) \) that depends only on the ‘own’ coordinate \( x_k \) and \( v_k(x_{\neq k}) \) that depends only on the ‘foreign’ coordinates \( x_j \) with \( j \neq k \).

**Condition 2: Partial integrability:** It has to be possible to integrate one of the two functions \( u_k \) and \( v_k \) and differentiate the other until a multiple \( C_k \) of the original function is obtained. This technique is known from repeated integration by parts (Polyanin and Chernoutsan, 2010, p. 173). Two cases can be distinguished:

**Condition 2a: Integrate ‘own’ and differentiate ‘foreign’ coordinates:** It exists \( \lambda_k \in \mathbb{N} \) and \( C_k \in \mathbb{R} \setminus \{1\} \) such that \( 2\lambda_k \) antiderivatives of \( u_k(x_k) \) can be determined and

\[
(-1)^{\lambda_k} \left( \partial_{x_k}^{\lambda_k + 1} u_k(x_k) \right) \left( \Delta^{\lambda_k} v_k(x_{\neq k}) \right) = C_k u_k(x_k)v_k(x_{\neq k}).
\]

Define the \( n \times n \) potential matrix \( F \in C^2(\mathbb{R}^n, \mathbb{R}^n) \) as:

\[
F_{ij}(x) := \sum_{p_i=0}^{\lambda_k-1} \frac{(-1)^{p_i}}{1 - C_k} \partial_{x_i} \left( \partial_{x_i}^{p_i + 1} u_k(x_k) \right) \left( \Delta^{p_i} v_k(x_{\neq k}) \right),
\]

for \( i, j \neq k \).

**Condition 2b: Differentiate ‘own’ and integrate one ‘foreign’ coordinate:** \( v_k(x_m) \) depends only on one coordinate \( x_m \) with \( m \neq k \), and it exists \( \lambda_k \in \mathbb{N} \) and \( C_k \in \mathbb{R} \setminus \{1\} \) such that \( 2\lambda_k \) antiderivatives of \( v_k(x_m) \) can be determined, and

\[
(-1)^{\lambda_k} \left( \partial_{x_k}^{\lambda_k + 1} u_k(x_k) \right) \left( \Delta^{\lambda_k} v_k(x_m) \right) = C_k u_k(x_k)v_k(x_m).
\]

Define the \( n \times n \) potential matrix \( F \in C^2(\mathbb{R}^n, \mathbb{R}^n) \) as:

\[
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\]

for \( j = k, m \), and \( F_{ij}(x) := 0 \) otherwise.

**Definition of potentials and vector fields:** Define the ‘gradient potential’ \( G \in C^2(\mathbb{R}^n, \mathbb{R}) \) and the ‘rotation potential’ matrix \( R \in C^2(\mathbb{R}^n, \mathbb{R}^n) \) as:

\[
G(x) := \sum_i F_{ii}(x) = \text{Tr} F_{ij}(x),
\]

\[
R_{ij}(x) := F_{ij}(x) - F_{ji}(x) = F - F^T.
\]

Note that \( R_{ij}(x) = -R_{ji}(x) \) is antisymmetric, and \( R_{kk}(x) = 0 \).

Calculating the rotation-free gradient field \( g(x) = \text{grad} G(x) \) and the divergence-free rotation field \( r(x) = \text{ROT} R(x) \) according to Eqs. (8–9) yields a Helmholtz Decomposition of \( f \) with \( f = g + r \):

\[
g(x) = \text{grad} \text{Tr} F_{ij}(x) = \left[ \partial_{x_k} \sum_i F_{ii}(x); \ 1 \leq k \leq n \right],
\]

\[
r(x) = \left[ \sum_{j=1}^n \partial_{x_j} (F_{kj}(x) - F_{jk}(x)); \ 1 \leq k \leq n \right].
\]
Theorem 2 (General case for linear combinations of the special case).

This theorem removes the restriction of Theorem 1 to fields that are non-zero in only one component and extends it further to field components that contains sums. A vector field \( f' \in C^1(\mathbb{R}^n, \mathbb{R}^m) \) that is a linear combination of vector fields satisfying Theorem 1 can be decomposed into two vector fields, one rotation-free gradient field \( g' \) and one divergence-free rotation field \( r' \). The decomposition can be obtained by deriving the potentials resp. the vectors \( g \) and \( r \) for each field separately and adding them.

The proof of Theorem 2 follows directly from Theorem 1 by linearity of all the operators.

Proof of Theorem 1: Since \( g(x) \) is defined as gradient of some potential \( \text{Tr}_i F_{ij}(x) \), it is known to be rotation-free, thus \( \text{ROT} g(x) = \text{curl} g(x) = 0 \), and \( \text{div} r(x) = 0 \) if defined as \( \text{ROT} R(x) \) for some antisymmetric potential \( R(x) \) (see Prop. 2–3 in Glötzl and Richters, 2021). Note that in the conventional Helmholtz Decomposition, the fact that \( R_{ij} \) is antisymmetric is guaranteed before the Newton Integration on the level of the rotation densities.

It remains to be shown that \( f = g + r \) for both Conditions 2a and 2b.

For Condition 2a that integrates the ‘own’ and differentiates the ‘foreign’ coordinates, gradient and rotation potentials yield:

\[
G(x) = \sum_{p_k=0}^{\lambda_k-1} \frac{(-1)^{p_k}}{1 - C_k} \left( \mathcal{A}_{x_k}^{2p_k+1} u_k(x_k) \right) \left( \Delta^{p_k} v_k(x_{\neq k}) \right),
\]

\[
R_{kj}(x) = \sum_{p_k=0}^{\lambda_k-1} \frac{(-1)^{p_k}}{1 - C_k} \left( \mathcal{A}_{x_k}^{2p_k+2} u_k(x_k) \right) \left( \partial_j \Delta^{p_k} v_k(x_{\neq k}) \right),
\]

\[
R_{jk}(x) = -R_{kj}(x) \quad \text{for} \; j \neq k, \quad \text{and} \; R_{ij}(x) = 0 \quad \text{otherwise}.
\]

The components of the gradient field \( g(x) \) and the rotation field \( r(x) \) are, using Eqs. (38–39):

\[
g_k = \sum_{p_k=0}^{\lambda_k-1} \frac{(-1)^{p_k}}{1 - C_k} \left( \mathcal{A}_{x_k}^{2p_k+1} u_k(x_k) \right) \left( \Delta^{p_k} v_k(x_{\neq k}) \right),
\]

\[
g_{j\neq k} = \sum_{p_k=0}^{\lambda_k-1} \frac{(-1)^{p_k}}{1 - C_k} \left( \mathcal{A}_{x_k}^{2p_k+1} u_k(x_k) \right) \left( \partial_j \Delta^{p_k} v_k(x_{\neq k}) \right),
\]

\[
r_k = \sum_{p_k=0}^{\lambda_k-1} \frac{(-1)^{p_k}}{1 - C_k} \left( \mathcal{A}_{x_k}^{2p_k+2} u_k(x_k) \right) \left( \Delta^{p_k} v_k(x_{\neq k}) \right)
\]

\[
= \sum_{p_k=0}^{\lambda_k-1} \frac{(-1)^{p_k}}{1 - C_k} \left( \mathcal{A}_{x_k}^{2p_k} u_k(x_k) \right) \left( \Delta^{p_k} v_k(x_{\neq k}) \right),
\]

\[
r_{j\neq k} = \sum_{p_k=0}^{\lambda_k-1} \frac{(-1)^{p_k}}{1 - C_k} \left( \mathcal{A}_{x_k}^{2p_k+1} u_k(x_k) \right) \left( \partial_j \Delta^{p_k} v_k(x_{\neq k}) \right).
\]

In the \( k \)-component of \( r \), the additional \( \Delta \) arises out of the sum over \( \partial_j, \partial_k \). This sum was simplified by a shift of the sum index.

\[
g_{j\neq k} + r_{j\neq k} = 0 = f_{j\neq k}
\]
as the sums cancel out. For \( g_k + r_k \), only the terms with equal index \( p_k \) cancel out, and it remains the term with \( p_k = 0 \) of \( g_k \) and the term with \( p_k = \lambda_k \) of \( r_k \). With the definitions for \( \mathcal{A}^0 \) and \( \Delta^0 \) in Sec. 3 and Eq. (32), it holds:

\[
g_k + r_k = \frac{1}{1 - C_k} u_k(x_k)v_k(x_{\neq k}) - \frac{(-1)^{k_i}}{1 - C_k} \left( \mathcal{A}_{x_k}^{2 \lambda_k} u_k(x_k) \right) \left( \Delta^{\lambda_k} v_k(x_{\neq k}) \right)
\]

\[
= \frac{1}{1 - C_k} \left( u_k(x_k)v_k(x_{\neq k}) - C_k u_k(x_k)v_k(x_{\neq k}) \right)
\]

\[
= u_k(x_k)v_k(x_{\neq k}) = f_k(x),
\]

which proves that \( g(x) + r(x) = f(x) \). □

For **Condition 2b** that differentiates the ‘own’ and integrates one ‘foreign’ coordinate, gradient and rotation potentials yield:

\[
G(x) = \sum_{p_k=0}^{\lambda_k-1} \frac{(-1)^{p_k}}{1 - C_k} \left( \partial_{x_k}^{2p_k+1} u_k(x_k) \right) \left( \mathcal{A}_{x_m}^{2p_k+2} v_k(x_m) \right),
\]

\[
R_{km}(x) = \sum_{p_k=0}^{\lambda_k-1} \frac{(-1)^{p_k}}{1 - C_k} \left( \partial_{x_k}^{2p_k+1} u_k(x_k) \right) \left( \mathcal{A}_{x_m}^{2p_k+1} v_k(x_m) \right),
\]

\[
R_{mk}(x) = -R_{km}, \text{ and } R_{ij}(x) = 0 \text{ otherwise.}
\]

The components of the vector fields yield:

\[
g_k = \sum_{p_k=0}^{\lambda_k-1} \frac{(-1)^{p_k}}{1 - C_k} \left( \partial_{x_k}^{2p_k+2} u_k(x_k) \right) \left( \mathcal{A}_{x_m}^{2p_k+2} v_k(x_m) \right),
\]

\[
r_k = \sum_{p_k=0}^{\lambda_k-1} \frac{(-1)^{p_k}}{1 - C_k} \left( \partial_{x_k}^{2p_k+1} u_k(x_k) \right) \left( \mathcal{A}_{x_m}^{2p_k+1} v_k(x_m) \right),
\]

\[
g_m = \sum_{p_k=0}^{\lambda_k-1} \frac{(-1)^{p_k}}{1 - C_k} \left( \partial_{x_k}^{2p_k+1} u_k(x_k) \right) \left( \mathcal{A}_{x_m}^{2p_k+1} v_k(x_m) \right),
\]

\[
r_m = \sum_{p_k=0}^{\lambda_k-1} \frac{(-1)^{p_k}}{1 - C_k} \left( \partial_{x_k}^{2p_k+1} u_k(x_k) \right) \left( \mathcal{A}_{x_m}^{2p_k+1} v_k(x_m) \right),
\]

\[
g_i = r_i = 0 \text{ for } i \neq k, m.
\]

Summing \( f = r + g \), most of the terms cancel. Note that the term \( \frac{1}{1 - C_k} f_k \) is provided by the rotation field, not the gradient field as with **Condition 2a**.

\[
f_i = g_i + r_i = 0 \text{ for } i \neq k,
\]

\[
g_k + r_k = -\frac{1}{1 - C_k} \left( -(-1)^{k_i} \left( \partial_{x_k}^{2\lambda_k} u_k(x_k) \right) \left( \mathcal{A}_{x_m}^{2\lambda_k} v_k(x_m) \right) \right) + \frac{1}{1 - C_k} \left( u_k(x_k) \right) \left( v_k(x_m) \right)
\]

\[
= \frac{1}{1 - C_k} \left( u_k(x_k)v_k(x_{\neq k}) - C_k u_k(x_k)v_k(x_{\neq k}) \right)
\]

\[
= u_k(x_k)v_k(x_{\neq k}) = f_k(x),
\]

which proves that \( g(x) + r(x) = f(x) \). □
Corollary 3 (Fields linear in ‘foreign’ coordinates).

Let \( f(x) \) have only one non-zero component
\[
f_k(x) = u_k(x_k)v_k(x_{\neq k})
\]
and let \( v_k(x_{\neq k}) \) be linear in all ‘foreign’ coordinates \( x_i \) with \( i \neq k \). This implies \( \Delta v_k(x_{\neq k}) = 0 \), and Condition 2a is satisfied with \( \lambda_k = 1 \) and \( C_k = 0 \). The equations simplify to:
\[
F_{k,j}(x) = \partial_{x_j}(\mathcal{A}_{x_k}^2 u_k(x_k))v_k(x_{\neq k}),
\]
\[
F_{i,j}(x) = 0 \quad \text{for} \quad i \neq k,
\]
\[
G(x) = (\mathcal{A}_{x_k} u_k(x_k))v_k(x_{\neq k}).
\]
\[
R_{k,j}(x) = (\mathcal{A}_{x_k}^2 u_k(x_k))\left(\partial_{x_j} v_k(x_{\neq k})\right) = -R_{j,k} \quad \text{for} \quad j \neq k,
\]
\[
g_k(x) = f_k(x),
\]
\[
g_{j\neq k}(x) = -(\mathcal{A}_{x_k} u_k(x_k))\left(\partial_{x_j} v_k(x_{\neq k})\right),
\]
\[
r_{j\neq k}(x) = -\frac{1}{2}(\mathcal{A}_{x_k} u_k(x_k))\left(\partial_{x_j} v_k(x_{\neq k})\right).
\]

Corollary 4 (Fields linear in ‘own’ coordinate).

Let \( f(x) \) have only one non-zero component
\[
f_k(x) = u_k(x_k)v_k(x_m),
\]
with \( u_k(x_k) \) a linear function of its ‘own’ coordinate \( x_k \) and \( v_k(x_m) \) dependent only on one ‘foreign’ coordinate \( x_m \). This implies \( \Delta u_k(x) = 0 \), and Condition 2b is satisfied with \( \lambda_k = 1 \) and \( C_k = 0 \). The equations simplify to:
\[
F_{k,j}(x) = \partial_{x_j} u_k(x_k)\left(\mathcal{A}_{x_m}^2 v_k(x_m)\right) \quad \text{for} \quad j = k, m,
\]
\[
F_{i,j}(x) = 0 \quad \text{otherwise},
\]
\[
G(x) = (\partial_{x_k} u_k(x_k))\left(\mathcal{A}_{x_m}^2 v_k(x_m)\right),
\]
\[
R_{km}(x) = u_k(x_k)\left(\mathcal{A}_{x_m} v_k(x_m)\right),
\]
\[
R_{mk}(x) = -R_{km}, \quad \text{and} \quad R_{i,j}(x) = 0 \quad \text{otherwise}.
\]

Corollary 5 (Linear vector fields).

If \( f(x) \) is a linear vector field
\[
f(x) = Mx,
\]
\[
f_k(x) = \sum_{j=1}^n M_{kj}x_j,
\]
with a \( n \times n \) matrix \( M = [M_{ij}] \), the easiest way to derive a Helmholtz Decomposition is to split the vector \( f \) into two parts using Theorem 2, one containing the ‘own’ coordinate and one the ‘foreign’ coordinates:

\[
f^{(1)}_k(x) = M_{kk}x_k, \quad (74)
\]

\[
f^{(2)}_k(x) = \sum_{j \neq k} M_{kj}x_j. \quad (75)
\]

The components of the first part \( f^{(1)} \) satisfy Condition 2a with \( u_k(x_k) = M_{kk}x_k, \ v_k(x_{\neq k}) = 1, \lambda_k = 1 \) and \( C_k = 0 \), which yields using Corollary 3:

\[
F^{(1)}_{kk}(x) = \frac{1}{2}M_{kk}x_k^2, \quad F^{(1)}_{ij}(x) = 0 \text{ for } i \neq j, \quad (76)
\]

\[
G^{(1)}(x) = \sum_{k=1}^{n} \frac{1}{2}M_{kk}x_k^2, \quad (77)
\]

\[
R^{(1)}_{ij}(x) = 0 \quad \forall \ i, j. \quad (78)
\]

All the summands \( M_{kj}x_j \) of the components of the second part \( f^{(2)} \) satisfy Condition 2b with \( u_k(x_k) = 1, \ v_k(x_j) = M_{kj}x_j, \lambda_k = 1, \ C_k = 0 \), which yields using Theorem 2 and Corollary 4:

\[
F^{(2)}_{ij}(x) = \frac{1}{2}M_{ij}x_i^2 \text{ for } i \neq j, \quad F^{(2)}_{kk}(x) = 0, \quad (79)
\]

\[
G^{(2)}(x) = 0, \quad (80)
\]

\[
R^{(2)}_{ij}(x) = \frac{1}{2}M_{ij}x_i^2 - \frac{1}{2}M_{ji}x_j^2. \quad (81)
\]

These results show that \( g = f^{(1)} \) and \( r = f^{(2)} \) was already a Helmholtz Decomposition of \( f \). The total potentials are:

\[
F_{ij}(x) = \frac{1}{2}M_{ij}x_i^2, \quad (82)
\]

\[
G(x) = \sum_{k=1}^{n} \frac{1}{2}M_{kk}x_k^2, \quad (83)
\]

\[
R_{ij}(x) = \frac{1}{2}M_{ij}x_i^2 - \frac{1}{2}M_{ji}x_j^2. \quad (84)
\]

6. Discussion

The theorems presented in this paper allows to analytically derive a Helmholtz Decomposition and scalar and rotation potentials for many unbounded vector fields on unbounded domains. This decomposition is not unique, because adding a harmonic function \( H(x) \) with \( \Delta H(x) = 0 \) to the scalar potential, and correcting \( r \) to maintain \( r + g = f \) yields

\[
g'(x) = \text{grad}(G(x) + H(x)) \quad \text{and} \quad r'(x) = r(x) - \text{grad} H(x) \quad (85)
\]

that are also a Helmholtz Decomposition of \( f(x) \). By the choice of this harmonic function \( H \), additional boundary conditions can be satisfied. In some cases, harmonic functions can be chosen such that the decomposition is orthogonal with \( \langle g(x), r(x) \rangle = 0 \) \( \forall \ x \) (Suda, 2020).

The method is applicable to all vector fields described by multivariate polynomial functions. Each of the terms of a polynomial satisfies both Conditions 2a and 2b with \( C_k = 0 \) and \( \lambda_k \) not higher than the total degree (sum of exponents) of the polynomial – but the choice of the appropriate method
Applying Condition 2a is practical for vector fields such as \( f(x) = [x_1^{100}, x_2, 0, 0] \), while applying Condition 2b is better for vector fields such as \( f(x) = [x_1, x_2^{100}, 0, 0] \), because in the better case \( \lambda_k = 1 \) is sufficient, while the worse approach requires \( \lambda_k = 51 \). This is similar to the choice for the integration by parts which function to integrate and which to differentiate.

Further applications include exponential functions, sine or cosine functions, and their product with polynomials. For example: \( f(x) = [\cos(wx_1) \exp(ax_3), 0, 0] \) with \( u_1(x_1) = \cos(wx_1) \) and \( v_1(x_{x_1}) = \exp(ax_3) \). The appendix explains the process step-by-step for some polynomials, exponential functions, trigonometric functions, and two examples from complex system theory that require the application of the general theorem for linear combinations, the Rössler and Lorenz attractors.

Thanks to its versatility and the possibility of obtaining analytical solutions for Helmholtz decomposition, this method may prove helpful for problems of vector analysis, theoretical physics, and complex systems theory.
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A. Examples

A.1. Example 1 with Multipolynomial using Condition 2a

\[ f(x) = \left[ x_1^2 x_2^2, 0, 0 \right], \text{ with } a \in \mathbb{R}^+ . \tag{86} \]

This vector field satisfies Conditions 1 and 2a with

\[ u_1(x_1) = x_1^a, \quad \mathcal{A}^1_{\lambda_1} u_1(x_1) = \frac{x_1^{a+1}}{a+1}, \quad \mathcal{A}_{\lambda_1}^0 u_1(x_1) = \frac{x_1^{a+p} a!}{(a+p)!}, \tag{87} \]

and \( v_1(x_{\neq 1}) = x_2^2 x_3, \quad \Delta v(x_{\neq 1}) = 2 x_2^2 + 2 x_3^2, \quad \Delta^2 v(x_{\neq 1}) = 0, \quad \Delta^3 v(x_{\neq 1}) = 0 . \tag{88} \]

As \( \Delta^3 v(x_{\neq 1}) = 0 \), chose \( \lambda_1 = 3, C_1 = 0 \) and calculate:

\[ F_{11}(x) = G(x) = \frac{x_1^{a+1}}{a+1} \cdot x_2^2 x_3 - \frac{x_1^{a+3} a!}{(a+3)!} (2 x_2^2 + 2 x_3^2) + \frac{x_1^{a+5} a!}{(a+5)!} \cdot 8, \tag{89} \]

\[ F_{1i}(x) = R_{1i}(x) = -R_{1i}(x) = -\frac{x_1^{a+2}}{(a+1)(a+2)} \cdot \partial_{x_i} (x_2^2 x_3^2) + \frac{x_1^{a+4} a!}{(a+4)!} \cdot 4 x_i, \tag{90} \]

\[ F_{ij}(x) = 0 \text{ and } R_{ij}(x) = 0 \text{ otherwise} . \tag{91} \]

The Helmholtz Decomposition \( f(x) = g(x) + r(x) \) is given by:

\[ g(x) = \left[ \frac{x_1^{a+2}}{a+1} \cdot x_2^2 x_3^2 \frac{x_1^{a+2}}{a+1} \cdot x_2^2 x_3^2 \right] \cdot (2 x_2^2 + 2 x_3^2) + \frac{x_1^{a+4} a!}{(a+4)!} \cdot 8, \tag{92} \]

\[ r(x) = \left[ \frac{x_1^{a+2}}{a+1} \cdot x_2^2 x_3^2 \frac{x_1^{a+2}}{a+1} \cdot x_2^2 x_3^2 \right] + \left[ \frac{x_1^{a+2}}{a+1} \cdot x_2^2 x_3^2 \frac{x_1^{a+2}}{a+1} \cdot x_2^2 x_3^2 \right] \cdot (2 x_2^2 + 2 x_3^2) - \frac{x_1^{a+4} a!}{(a+4)!} \cdot 4 \tag{93} \]

A.2. Example 2 with Multipolynomial using Condition 2b

\[ f(x) = \left[ x_1^2 x_2^2, 0, 0 \right] \text{ with } u_1(x_1) = x_1^2 \text{ and } v(x_2) = x_2^2. \tag{94} \]

As \( \partial_1^2 u_1(x_1) = 2 \) and \( \partial_2^1 u_1(x_1) = 0 \), Conditions 1 and 2b are satisfied with \( \lambda_1 = 2 \) and \( C_1 = 0 \). This implies:

\[ F_{11}(x) = G(x) = \frac{2 x_1 x_2^{a+2}}{(a+1)(a+2)}, \tag{95} \]

\[ F_{12}(x) = R_{12}(x) = -R_{21}(x) = \frac{2 x_2^{a+3} a!}{(a+3)!} \cdot \frac{x_1^{a+1}}{a+1}, \tag{96} \]

\[ F_{ij}(x) = 0 \text{ and } R_{ij}(x) = 0 \text{ otherwise} . \tag{97} \]

The Helmholtz Decomposition \( f(x) = g(x) + r(x) \) is given by:

\[ g(x) = \left[ \frac{2 x_1^{a+2}}{(a+1)(a+2)} + \frac{2 x_1 x_2^{a+2}}{a+1}, 0 \right], \tag{98} \]

\[ r(x) = \left[ x_1^2 x_2^2 + \frac{2 x_2^{a+2}}{(a+1)(a+2)} - \frac{2 x_1 x_2^{a+2}}{a+1}, 0 \right]. \tag{99} \]
A.3. Example 3 with cosine and exponential function using Condition 2a

\[ f(x) = \left[ \cos(wx_1) \exp(ax_3), 0, 0 \right]. \]  

(100)

This implies \( u_1(x_1) = \cos(wx_1) \) and \( v_1(x_{x_1}) = \exp(ax_3) \). For \( \lambda_1 = 1 \) and \( C_1 = \frac{w^2}{w^2} \), Condition 2a is satisfied:

\[
\frac{1}{w^2} \cos(wx_1) a^2 \exp(ax) = (-1)^{d_i} \left( J_{d_i}^{2d_i} u_1(x_1) \right) \left( \Delta_i^{d_i} v_1(x_{x_1}) \right) = C_i u_1(x_1) v_1(x_{x_1}).
\]

(101)

Therefore, for \( a \neq w \), set:

\[
F_{1i}(x) = G(x) = \frac{1}{w} \sin(wx_1) \exp(ax_3),
\]

(102)

\[
F_{13}(x) = R(x) = -R_{31}(x) = -\frac{a}{w^2} \cos(wx_1) \exp(ax_3),
\]

(103)

\[
F_{ij}(x) = 0 \text{ and } R_{ij}(x) = 0 \text{ otherwise.}
\]

(104)

The Helmholtz Decomposition \( f(x) = g(x) + r(x) \) is given by:

\[
g(x) = \left[ \frac{1}{1 - a^2/w^2} \cos(wx_1) \exp(ax_3), 0, \frac{+a/w}{1 - a^2/w^2} \sin(wx_1) \exp(ax_3) \right],
\]

(105)

\[
r(x) = \left[ \frac{-a^2/w^2}{1 - a^2/w^2} \cos(wx_1) \exp(ax_3), 0, \frac{-a/w}{1 - a^2/w^2} \sin(wx_1) \exp(ax_3) \right].
\]

(106)

A.4. Example 4: Rössler attractor

The Rössler attractor, a classic example from complex system theory (Peitgen et al., 1992; Rössler, 1976), is given by:

\[ f(x) = \left[ -(x_2 + x_3), x_1 + ax_2, b + (x_1 - c)x_3 \right]. \]

(107)

Splitting the sums, deriving the potentials, and adding them yields:

\[
G(x) = \frac{a}{2} x_2^2 + bx_3 - \frac{c}{2} x_3^2 + \frac{1}{2} x_1 x_3^2,
\]

(108)

\[
R_{12}(x) = -\frac{1}{2} x_2^2 - \frac{1}{2} x_3^2,
\]

(109)

\[
R_{13}(x) = -\frac{1}{2} x_3^2 - \frac{1}{2} x_3^2,
\]

(110)

\[
R_{23}(x) = 0.
\]

(111)

The Helmholtz Decomposition \( f(x) = g(x) + r(x) \) is given by:

\[
g(x) = \text{grad } G(x) = \left[ \frac{1}{2} x_3^2, +ax_2, b - cx_3 + x_1 x_3 \right],
\]

(112)

\[
r(x) = \text{ROT } R(x) = \left[ -x_2 - x_3 - \frac{1}{2} x_3^2, +x_1, 0 \right].
\]

(113)
A.5. Example 5: Lorenz attractor

The Lorenz attractor, another classic example from complex system theory (Lorenz, 1963; Peitgen et al., 1992), is given by:

\[
f(x) = [a(x_2 - x_1), x_1(b - x_3) - x_2, x_1x_2 - cx_3].
\]  

(114)

Splitting the sums, deriving the potentials, and adding them yields:

\[
G(x) = -\frac{a}{2}x_1^2 - \frac{1}{2}x_2^2 - \frac{c}{2}x_3^2,
\]  

(115)

\[
R_{12}(x) = -\frac{a}{2}x_2^2 + \frac{b}{2}x_1^2 - \frac{1}{4}x_1^2x_3 - \frac{1}{12}x_3^3,
\]  

(116)

\[
R_{13}(x) = \frac{1}{4}x_1^2x_2 + \frac{1}{12}x_3^2,
\]  

(117)

\[
R_{23}(x) = \frac{1}{4}x_1x_2^2 + \frac{1}{4}x_1x_2 + \frac{1}{6}x_1^3.
\]  

(118)

The Helmholtz Decomposition \(f(x) = g(x) + r(x)\) is given by:

\[
g(x) = \text{grad} \ G(x) = [-ax_1, -x_2, -cx_3],
\]  

(119)

\[
r(x) = \text{ROT} \ R(x) = [ax_2, bx_1 - x_1x_3, x_1x_2].
\]  

(120)

The Lorenz system contains a square gradient potential, pushing the dynamics into the direction of the origin. This is responsible for the stable fixed point at the origin for some parameters. If the rotation field \(r\) becomes ‘strong’ enough, it can push the dynamics away from this fixed point, creating a strange attractor.