MAPPING CLASS GROUP IS GENERATED BY FOUR INVOLUTIONS

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Abstract. We prove that the mapping class group of a closed connected orientable surface of genus at least three is generated by four involutions.

1. Introduction

The mapping class group $\text{Mod}(\Sigma_g)$ of a closed connected orientable surface $\Sigma_g$ is defined as the group of isotopy classes of orientation-preserving self-diffeomorphisms of $\Sigma_g$. We are interested in generating $\text{Mod}(\Sigma_g)$ by a small number of involutions.

The group $\text{Mod}(\Sigma_g)$ cannot be generated by two involutions, because, for example, it contains nonabelian free groups. Thus any generating set consisting of involutions must contain at least three elements. The purpose of this paper is to prove that the group $\text{Mod}(\Sigma_g)$ can be generated by four involutions, leaving the question in [1] whether it can be generated by three involutions open.

Theorem 1. For $g \geq 3$, the mapping class group $\text{Mod}(\Sigma_g)$ is generated by four involutions.

After the works of Luo and Brendle-Farb as explained below, Kassabov [10] obtained the above result for $g \geq 7$. He also proved results for lower genus mapping class groups.

For homological reasons, the groups $\text{Mod}(\Sigma_1)$ and $\text{Mod}(\Sigma_2)$ cannot be generated by involutions. Since there is a surjective homomorphism from $\text{Mod}(\Sigma_g)$ onto the symplectic group $\text{Sp}(2g, \mathbb{Z})$, the latter group is also generated by four involutions for $g \geq 3$.

Corollary 2. For $g \geq 3$, the symplectic group $\text{Sp}(2g, \mathbb{Z})$ is generated by four involutions.

Here is a brief history of the problem of finding the generating sets for $\text{Mod}(\Sigma_g)$ with various properties. Dehn [4] obtained a generating
set for Mod(Σg) consisting of 2g(g − 1) Dehn twists. About a quarter century later, Lickorish [9] showed that it can be generated by 3g − 1 Dehn twists, and Humphries [6] reduced this number to 2g + 1. He showed, moreover, that 2g + 1 is minimal: the cardinality of every generating set of Mod(Σg) consisting of Dehn twists is at least 2g + 1.

If one does not require generators to be Dehn twists, by using the generating set obtained by Lickorish, it is easy to find a generating set consisting of an element of order g and three Dehn twists. It is also possible to get a generating set with three elements (see Corollary 6 below). Lu [12] obtained a generating set consisting of three elements, two of which are of finite order. A minimal generating set was first obtained by Wajnryb [19] who proved that Mod(Σg) can be generated by two elements; one is of order 4g + 2 and the other is a product of a right and a left Dehn twist about disjoint curves. In [11], we showed that Mod(Σg) is generated by an element of order 4g + 2 and a Dehn twist, improving Wajnryb’s result.

The interest in the problem of finding generating sets for the mapping class group consisting of finite order elements goes back to 1971: Maclachlan [14] proved that Mod(Σg) is generated by conjugates of two elements of orders 2g + 2 and 4g + 2. He deduced from this that the moduli space is simply-connected. McCarthy and Papadopoulos [16] showed that for g ≥ 3 the group Mod(Σg) is generated normally by a single involution; i.e., it is generated by an involution and its conjugates. Luo [13] observed that every Dehn twist is a product of six involutions for g ≥ 3. It then follows from the fact that the mapping class group is generated by 2g + 1 Dehn twists, the group Mod(Σg) can be generated by 12g + 6 involutions. Luo also asked whether it is possible to generate the mapping class group by torsion elements, where the number of generators is independent of the genus (and boundary components in the case the surface has boundary).

Brendle and Farb [11] answered Luo’s question affirmatively by proving that 6 involutions, and also 3 torsion elements, generate the mapping class group for g ≥ 3. Kassabov [10] improved this result further, proving that Mod(Σg) is generated by 4 (resp. 5 and 6) involutions if g ≥ 7 (resp. g ≥ 5 and g ≥ 3). We proved in [11] that the minimal number of torsion generators of Mod(Σg) is 2, by showing that the mapping class group Mod(Σg) can be generated by two elements of order 4g + 2. See also [15], [5] and [8] for generating the mapping class group by torsion elements of various orders.
In order to prove the main results of their papers, Brendle-Farb and also Kassabov write a Dehn twist as a product of four involutions, and use a generating set consisting of a torsion element and three Dehn twists. It is easier to write a product of two opposite Dehn twists about disjoint curves as a product of two involutions. Therefore, Wajnryb’s generating set, consisting of an element of order $4g + 2$ and a product of two opposite Dehn twists, looks like a good candidate to use in order to find a small number of involution generators. However, the element of order $4g + 2$ cannot be written as a product of two (orientation–preserving) involutions, because it follows from [2] that the group $\text{Mod}(\Sigma_g)$ does not contain a dihedral subgroup of order $8g + 4$. In order to implement this idea, we find a generating set consisting of a rotation of order $g$ and two more elements each of which can be written as a product of two involutions.

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2. Background and results on mapping class groups

In this article we will only consider closed surfaces. Let $\Sigma_g$ be a closed connected oriented surface of genus $g$ embedded in $\mathbb{R}^3$, as illustrated in Figure 1 in such a way that it is invariant under the two rotations, $\rho_1$ and $\rho_2$, by $\pi$ about the indicated lines. The mapping class group $\text{Mod}(\Sigma_g)$ of $\Sigma_g$ is the group of isotopy classes of orientation–preserving self–diffeomorphisms of $\Sigma_g$.

Throughout the paper diffeomorphisms are considered up to isotopy. Likewise, curves are considered up to isotopy. Simple closed curves are denoted by the lowercase letters $a, b, c$, with indices, while the right Dehn twists about them by the corresponding capital letters $A, B, C$. Occasionally, we will write $t_a$ for the right Dehn twist about $a$. For the composition of diffeomorphisms, we use the functional notation: $fh$ means that $h$ is applied first. The notations $a_i, b_i, c_i$ and $d_1, d_2$ will always denote the curves shown in Figure 1.

Let us review the basic relations among Dehn twists that we need below. For the proofs the reader is referred to [3]. First of all, if $f : \Sigma_g \to \Sigma_g$ is a diffeomorphism and if $a$ is a simple closed curve on $\Sigma_g$, then $ft_a f^{-1} = t_{f(a)}$. 

Commutativity: If $a$ and $b$ are two disjoint simple closed curves on $\Sigma_g$, then $AB = BA$.

Lantern relation: This relation was discovered by Dehn \cite{4} in 1930s, and rediscovered and popularized by Johnson \cite{7} in 1979. Suppose that $x_1, x_2, x_3, x_4$ are pairwise disjoint simple closed curves on $\Sigma_g$ bounding a sphere $S$ with four boundary components. Let us choose a point $P_i$ on $x_i$ for each $i = 1, 2, 3, 4$. For $j = 1, 2, 3$, choose a properly embedded arc $\gamma_j$ on $S$ connecting $P_4$ and $P_j$, so that they are disjoint in the interior of $S$. Suppose that in a small neighborhood of the point $P_4$, the arcs are read as $\gamma_1, \gamma_2, \gamma_3$ in the clockwise order. Let $y_j$ be the boundary component of a regular neighborhood of $x_4 \cup \gamma_j \cup x_j$ lying on $S$. Then the Dehn twists about the seven curves $x_1, x_2, x_3, x_4, y_1, y_2, y_3$ satisfy the lantern relation

$$X_1X_2X_3X_4 = Y_1Y_2Y_3.$$  

It was proved by Dehn \cite{4} that the mapping class group $\text{Mod}(\Sigma_g)$ is generated by Dehn twists about finitely many nonseparating simple closed curves. Lickorish also obtained the same result and showed in \cite{9} that it is generated by the Dehn twists

$$A_1, A_2, \ldots, A_g, B_1, B_2, \ldots, B_g, C_1, C_2, \ldots, C_{g-1}$$

about the curves shown in Figure 1. We state this fact as a theorem by adding one more Dehn twist for the sake of symmetry.

**Theorem 3. (Dehn-Lickorish)** The mapping class group $\text{Mod}(\Sigma_g)$ is generated by the set $\{A_1, A_2, \ldots, A_g, B_1, B_2, \ldots, B_g, C_1, C_2, \ldots, C_g\}$.

As was shown by Humphries \cite{6}, the Dehn twists $A_3, A_4, \ldots, A_g, C_g$ can be removed from the above generating set, leaving $2g + 1$ Dehn twists. He also proved that it is minimal, in the sense that $2g$ or less Dehn twists cannot generate $\text{Mod}(\Sigma_g)$ if $g \geq 2$.

As a corollary to Dehn-Lickorish Theorem, it is easy to show that $\text{Mod}(\Sigma_g)$ can be generated by four elements, and also by three elements. We state it as the next corollary. To this end, let $R$ be the rotation by $2\pi/g$ of $\Sigma_g$ represented in Figure 1 so that $R = \rho_1\rho_2$. Thus, $R(a_k) = a_{k+1}$, $R(b_k) = b_{k+1}$ and $R(c_k) = c_{k+1}$.

**Corollary 4.** The mapping class group $\text{Mod}(\Sigma_g)$ is generated

(i) by the four elements $R, A_1, B_1, C_1$, and also
(ii) by the three elements $R, A_1, A_1B_1C_1$.  


Proof. The claim (i) follows from Theorem 3 and the fact that $R^k$ maps the curves $a_1, b_1$ and $c_1$ to the curves $a_{k+1}, b_{k+1}$ and $c_{k+1}$, respectively. Because, for example, it follows from $R^k(a_1) = a_{k+1}$ that $A_{k+1} = R^kA_1R^{-k}$. The claim (ii) then follows from (i) together with the fact that $A_1B_1C_1(a_1) = b_1$ and $A_1B_1C_1(b_1) = c_1$.

Although we will not directly use the above corollary, we stated it anyway, because a version of it will be proved in the next section and will be used in our proof of Theorem 1.

3. Two new generating sets for $\text{Mod}(\Sigma_g)$.

In this section, we obtain two new generating sets for the mapping class group. The first one will be the first generating set in Corollary 4 where each Dehn twist is replaced by a product of two opposite Dehn twists. The new generating sets will allow us to generate the mapping class group by four involutions.

Recall that $R$ denotes the $2\pi/g$–rotation of $\Sigma_g$ represented in Figure 1. It is a torsion element of order $g$ in the group $\text{Mod}(\Sigma_g)$. Our first generating set is given in the next theorem. For the proof of it, following the idea of Wajnryb in [19], we employ the lantern relation. The second generating set is given as a corollary to the theorem.

**Theorem 5.** If $g \geq 3$, then the mapping class group $\text{Mod}(\Sigma_g)$ is generated by the four elements $R, A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1}$. 

![Figure 1. The curves $a_i, b_i, c_i, d_i$, the rotation $R$ and the involutions $\rho_1$ and $\rho_2$ on the surface $\Sigma_g$.](image)
Proof. Let $G$ denote the subgroup of $\text{Mod}(\Sigma_g)$ generated by the set
\[ \{ R, A_1A_2^{-1}, B_1B_2^{-1}, C_1C_2^{-1} \} . \]

Let $\mathcal{I}$ denote the set of isotopy classes of nonseparating simple closed curves on the surface $\Sigma_g$. We define a subset $\mathcal{G}$ of $\mathcal{I} \times \mathcal{I}$ as
\[ \mathcal{G} = \{ (a, b) : AB^{-1} \in G \} . \]

It is clear that
\begin{itemize}
  \item (symmetry) if $(a, b) \in \mathcal{G}$ then $(b, a) \in \mathcal{G}$,
  \item (transitivity) if $(a, b)$ and $(b, c)$ are in $\mathcal{G}$ then so is $(a, c)$, and
  \item ($G$–invariance) if $(a, b) \in \mathcal{G}$ and $F \in G$, then $(F(a), F(b)) \in \mathcal{G}$.
\end{itemize}

Thus $\mathcal{G}$ is an equivalence relation on $\mathcal{I}$ invariant under the action of $G$. The last property follows from $Ft_a t_b^{-1} F^{-1} = t_F(a) t_F(b)$.

Notice that by the very definition of $\mathcal{G}$, the set $\mathcal{G}$ contains the three pairs $(a_1, a_2), (b_1, b_2)$ and $(c_1, c_2)$. Since
\[ R^{k-1}(\alpha_1, \alpha_2) = (\alpha_k, \alpha_{k+1}) \]
for every $\alpha \in \{ a, b, c \}$, the pairs $(a_k, a_{k+1}), (b_k, b_{k+1})$ and $(c_k, c_{k+1})$ are contained in $\mathcal{G}$. It follows from the transitivity that the pairs $(a_i, a_j), (b_i, b_j)$ and $(c_i, c_j)$ are also contained in $\mathcal{G}$ for all $i, j$. (Here, by abusing the notation we write $f(a, b)$ for $(f(a), f(b))$, and all indices are integers modulo $g$.)

Since
\[ A_1A_2^{-1}B_1B_2^{-1}(a_1, a_3) = (b_1, a_3) \]
and
\[ B_1B_2^{-1}C_1C_2^{-1}(b_1, a_3) = (c_1, a_3) , \]
we conclude that the pairs $(a_i, b_j), (a_i, c_j), (b_i, c_j)$ are all contained in the set $\mathcal{G}$ as well.

As a result of this, we get that if $X, Y \in \{ A_i, B_i, C_i \}$, then the mapping class $XY^{-1} \in G$ is contained in the subgroup $G$.

We now use the lantern relation to conclude that $G$ contains one, and hence all, of the Dehn twist generators given in Dehn-Lickorish theorem, Theorem 3.

It is easy to check that the diffeomorphism
\[ (B_2A_1^{-1})(C_1A_1^{-1})(A_1A_2^{-1})(C_2A_1^{-1}) \]
maps $(b_2, a_1)$ to $(d_1, a_1)$, and the diffeomorphism
\[ (B_3A_1^{-1})(C_2A_1^{-1})(A_3A_1^{-1})(B_3A_1^{-1}) \]
maps \((d_1, a_1)\) to \((d_2, a_1)\). Since both of these two diffeomorphisms are contained in the subgroup \(G\), it follows now from the transitivity and \(G\)-invariance of \(\mathcal{G}\) that \(D_1A_{i}^{-1}\) and \(D_2C_{i}^{-1}\) are contained in \(G\).

Note that \(a_1, c_1, c_2, a_3\) bound a sphere with four boundary components. The Dehn twists about these four curves and about the curves \(a_2, d_1, d_2\) given in Figure 1 satisfy the lantern relation

\[A_1C_1C_2A_3 = A_2D_1D_2,\]

which may be rewritten as

\[A_3 = (A_2C_2^{-1})(D_1A_{i}^{-1})(D_2C_{i}^{-1}).\]

Since each term on the right-hand side is contained in \(G\), the Dehn twist \(A_3\) is also contained in \(G\). Now from the fact that \(A_iA_{i}^{-1}, B_iA_{i}^{-1}, C_iA_{i}^{-1}\) are in \(G\), we conclude that \(G\) contains all generators \(A_i, B_i\) and \(C_i\) of \(\text{Mod}(\Sigma_g)\) given in Theorem 3. Consequently, \(G = \text{Mod}(\Sigma_g)\).

This concludes the proof of the theorem.

**Corollary 6.** If \(g \geq 3\), then the mapping class group \(\text{Mod}(\Sigma_g)\) is generated by the three elements \(R, A_1A_{2}^{-1}, A_1B_1C_1C_2^{-1}B_3^{-1}A_3^{-1}\).

**Proof.** Let us denote by \(H\) the subgroup of \(\text{Mod}(\Sigma_g)\) generated by the set

\[\{R, A_1A_{2}^{-1}, A_1B_1C_1C_2^{-1}B_3^{-1}A_3^{-1}\}.\]

It suffices to prove that \(H\) contains \(B_1B_2^{-1}\) and \(C_1C_2^{-1}\).

It follows from

- \(R(a_1, a_2) = (a_2, a_3)\),
- \(A_1B_1C_1C_2^{-1}B_3^{-1}A_3^{-1}(a_1, a_2) = (b_1, a_2)\),
- \(A_1B_1C_1C_2^{-1}B_3^{-1}A_3^{-1}(b_1, a_2) = (c_1, a_2)\),
- \(R(b_1, a_2) = (b_2, a_3)\) and
- \(R(c_1, a_2) = (c_2, a_3)\)

that the elements

- \(A_2A_{3}^{-1}\),
- \(B_1A_{2}^{-1}\),
- \(C_1A_{2}^{-1}\),
- \(B_2A_{3}^{-1}\) and
- \(C_2A_{3}^{-1}\)

are contained in \(H\). Now we have

\[B_1B_2^{-1} = B_1A_{2}^{-1} \cdot A_2A_{3}^{-1} \cdot A_3B_2^{-1} \in H\] and
\[C_1C_2^{-1} = C_1A_{2}^{-1} \cdot A_2A_{3}^{-1} \cdot A_3C_2^{-1} \in H.\]
It now follows from Theorem 5 that $H = \text{Mod}(\Sigma_g)$, completing the proof of the corollary.

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4. INVOLUTION GENERATORS

Recall that an involution in a group $G$ is an element of order 2. If $\rho$ is an involution in $G$ conjugating $x$ to $y$, then

$$(\rho xy^{-1})^2 = \rho xy^{-1} \rho xy^{-1} = yx^{-1}xy^{-1} = 1.$$ 

Thus we have the following lemma.

**Lemma 7.** If $\rho$ is an involution in a group $G$ and if $x$ and $y$ are elements in $G$ satisfying $\rho x \rho = y$, then $\rho xy^{-1}$ is an involution.

Consider now the surface $\Sigma_g$ of genus $g \geq 3$. Since $\rho_2(a_1) = a_2$, $\rho_1(a_1) = a_3$, $\rho_1(b_1) = b_3$ and $\rho_1(c_1) = c_2$, we have

$$\rho_2 A_1 \rho_2 = A_2 \quad \text{and} \quad \rho_1 A_1 B_1 C_1 \rho_1 = A_3 B_3 C_2.$$ 

It follows now from Lemma 7 that

$$\rho_2 A_1 A_2^{-1} \quad \text{and} \quad \rho_1 A_1 B_1 C_1 C_2^{-1} B_3^{-1} A_3^{-1}$$

are involutions.

Finally, we state our main result.

**Theorem 8.** If $g \geq 3$, then the mapping class group $\text{Mod}(\Sigma_g)$ is generated by the four involutions $\rho_1, \rho_2, \rho_2 A_1 A_2^{-1}, \rho_1 A_1 B_1 C_1 C_2^{-1} B_3^{-1} A_3^{-1}$.

**Proof.** The proof follows at once from Corollary 6 and the fact that $R = \rho_1 \rho_2$.

Since the first homology groups of $\text{Mod}(\Sigma_1)$ and $\text{Mod}(\Sigma_2)$ are isomorphic to the cyclic group $\mathbb{Z}_{12}$ and $\mathbb{Z}_{10}$ respectively, these two mapping class groups cannot be generated by involutions. In fact, the group $\text{Mod}(\Sigma_1)$ is isomorphic to $\text{SL}(2, \mathbb{Z})$ and $-I$ is the only element of order two in $\text{SL}(2, \mathbb{Z})$, where $I$ denotes the identity matrix. It was shown by Stukow [17] that the subgroup of $\text{Mod}(\Sigma_2)$ generated involutions is of index five.
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