BAKER-SPRINDŽUK CONJECTURES
FOR COMPLEX ANALYTIC MANIFOLDS

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Abstract. We show a large class of analytic submanifolds of $\mathbb{C}^n$ to be strongly extremal. This generalizes V. Sprindžuk’s solution of the complex case of Mahler’s Problem, and settles complex analogues of conjectures made in the 1970s by Baker and Sprindžuk. The proof is based on a variation of quantitative nondivergence estimates for quasi-polynomial flows on the space of lattices.

1. Introduction

The circle of problems that the present paper belongs to dates back to the 1930s, namely, to K. Mahler’s work on a classification of transcendental real and complex numbers. For a polynomial $P(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$, let us denote by $h_P$ the height of $P$, that is, $h_P \overset{\text{def}}{=} \max_{i=0,\ldots,n} |a_i|$. It can be easily shown using Dirichlet’s Principle that for any $z \in \mathbb{C}$ and any $n \in \mathbb{N}$ there exists a positive constant $c(n, z)$ such that

\begin{equation}
|P(z)| < c(n, z) h_P^{-v} \quad \text{for infinitely many } P \in \mathbb{Z}[\cdot] \quad \text{with } \deg P \leq n,
\end{equation}

where $v = \frac{n-1}{2}$, and one can take $v = n$ if $z \in \mathbb{R}$. Mahler’s Conjecture [M], proved in 1964 by V. Sprindžuk [S1, S2, S3], states that for almost every $z \in \mathbb{C}$ (resp. $z \in \mathbb{R}$), the values $v = \frac{n-1}{2}$ (resp. $v = n$) in (1.1) cannot be increased. Loosely put, this result show that almost all real/complex numbers are ‘as far from being algebraic as they could possibly be’.

Let us restate the aforementioned results by defining the **Diophantine exponent** $\omega(z)$ of $z \in \mathbb{C}^n$ by

\begin{equation}
\omega(z) \overset{\text{def}}{=} \sup \left\{ v > 0 \left| \left| z \cdot q + p \right| \leq \|q\|^{-v} \right. \text{for infinitely many } q \in \mathbb{Z}^n, \ p \in \mathbb{Z} \right. \right\},
\end{equation}

where $\|q\|$ stands for $\max_i |q_i|$. Dirichlet’s Principle and the Borel-Cantelli Lemma imply that $\omega(z)$ is:

- not less than $\frac{n-1}{2}$ for all $z \in \mathbb{C}^n$, and equal to $\frac{n-1}{2}$ for almost all $z \in \mathbb{C}^n$;
- not less than $n$ for all $z \in \mathbb{R}^n$, and equal to $n$ for almost all $z \in \mathbb{R}^n$.

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Let us use the notation $\mathbb{K}$ for $\mathbb{R}$ or $\mathbb{C}$. Following a terminology introduced by Sprindžuk, say that a map $f$ from an open subset $U$ of $\mathbb{K}^d$ to $\mathbb{K}^n$ is extremal if the Diophantine exponent of $f(x)$ is for almost every $x \in U$ equal to that of a generic point of $\mathbb{K}^n$ (that is, to $n$ if $\mathbb{K} = \mathbb{R}$ or to $\frac{n-1}{2}$ if $\mathbb{K} = \mathbb{C}$), and that a smooth submanifold of $\mathbb{K}^n$ is extremal if so are all its parametrizing maps. Sprindžuk’s result therefore states that the curve

$$\mathcal{M} = \{(z, z^2, \ldots, z^n) \mid z \in \mathbb{K}\} \subset \mathbb{K}^n$$

is extremal. Thus a natural generalization of Mahler’s Problem is to look for general conditions sufficient for the extremality of a manifold/map.

Another extension arises if one replaces $\|q\|$ in (1.2) by the ‘geometric mean’ of the components of $q$. More precisely, define the multiplicative Diophantine exponent $\omega^\times(z)$ of $z \in \mathbb{C}^n$ by

$$\omega^\times(z) \overset{\text{def}}{=} \sup \left\{ v > 0 \mid |z \cdot q + p| \leq (\Pi_+(q))^{-v/n} \text{ for infinitely many } q \in \mathbb{Z}^n, p \in \mathbb{Z} \right\},$$

where

$$\Pi_+(q) = \prod_{i=1}^{n} |q_i|_+ \text{ and } |q|_+ = \max(|q|, 1).$$

Since $\Pi_+(q)$ is not greater than $\|q\|^n$ for any nonzero $q$, one has $\omega^\times(z) \geq \omega(z)$ for any $z \in \mathbb{C}^n$. However, one can show by a Borel-Cantelli-type argument that the two exponents agree for almost all $z$; more precisely, that $\omega^\times(z)$ is equal to $\frac{n-1}{2}$ for almost all $z \in \mathbb{C}^n$ and to $n$ for almost all $z \in \mathbb{R}^n$.

The multiplicative analogue of the notion of extremality is usually referred to as strong extremality. Let us say that a map $f : U \to \mathbb{K}^n$ is strongly extremal if $\omega^\times(f(x))$ is for almost every $x \in U$ equal to $\frac{n-1}{2}$ or $n$ for $\mathbb{K} = \mathbb{C}$ of $\mathbb{R}$ respectively, with a similar definition for strongly extremal submanifolds of $\mathbb{K}^n$.

Identifying extremal and strongly extremal manifolds has been one of the central issues of metric Diophantine approximation for the last 40 years. However most of the activity revolved around the case $\mathbb{K} = \mathbb{R}$. In his 1980 survey of the field [S4], Sprindžuk conjectured that a real analytic submanifold $\mathcal{M}$ of $\mathbb{R}^n$ is strongly extremal whenever it is not contained in any proper affine subspace of $\mathbb{R}^n$; or, equivalently, that a real analytic map $f = (f_1, \ldots, f_n)$ is strongly extremal if

$$1, f_1, \ldots, f_n \text{ are linearly independent over } \mathbb{R}.$$
via a method involving dynamics on the space of lattices\footnote{See \cite{Bere} for an alternative proof and \cite{BKM, BBKM, K} for further developments.}. In contrast, very few developments took place in the complex case since Sprindžuk’s original work. The only paper on non-polynomial extremal submanifolds of $\mathbb{C}^n$ known to the author is \cite{V}, where it is shown that a complex analytic map $f: U \to \mathbb{C}^n$, $U \subset \mathbb{C}$, such that

\begin{equation}
(1.5) \quad 1, f_1, \ldots, f_n \text{ are linearly independent over } \mathbb{C},
\end{equation}

is extremal when $n = 3$. Earlier, it was proved in \cite{Ko} that for arbitrary $n$ and $f$ satisfying (1.5), one has $\omega(f(z)) \leq 2n^2 + n - 3$ for almost all $z$.

In the present paper we fill this gap by proving

**Theorem 1.1.** Let $U \subset \mathbb{C}^d$ be an open subset, and let $f: U \to \mathbb{C}^n$ be a complex analytic map such that (1.4) holds. Then $f$ is strongly extremal.

**Remark 1.2.** A ‘slicing trick’ originally due to A. Pyartli ([P], see also [S4]) shows that it is enough to prove the above theorem for $d = 1$.

**Remark 1.3.** Note that assumption (1.4) is weaker than (1.5). For example, it follows from Theorem 1.1 that a straight line $\{ (z, iz) \mid z \in \mathbb{C} \} \subset \mathbb{C}^2$ is strongly extremal.

**Remark 1.4.** Note also that in the paper \cite{KM1} Sprindžuk’s conjecture has been proved in a stronger infinitesimal form, where one replaces the analyticity of $f$ by existence of a certain number of derivatives, and (1.4) by the so-called ‘non-degeneracy’ condition, equivalent to (1.4) when $f$ is real analytic. On the other hand, it is clearly impossible to relax the analyticity assumption in the complex case. For example,

$$f: z = x + iy \mapsto (z, z^2 \bar{z}) = (x + iy)(1, x^2 + y^2)$$

is a map which is polynomial in $x, y$, satisfies (1.5), but is not extremal.

**Remark 1.5.** In this paper we do not touch the subject of so-called Khintchine-Groshev-type theorems on complex manifolds, where $\|q\|^{-u}$ in the right hand side of the inequality in (1.2) is replaced by an arbitrary function of $\|q\|$. Results of this type exist for complex polynomials [BD, BernV] and analytic curves in $\mathbb{C}^3$ [BereV]. It seems plausible that an approach of this paper can lead to Khintchine-Groshev-type results for maps $f$ as in Theorem 1.1.

**Remark 1.6.** A separate, and also quite natural, problem is to consider small values of $|z \cdot q + p|$ where both $p$ and the components of $q$ are Gaussian integers, that is, integer points of the field $\mathbb{C}$. Or, more generally, one can take two global fields $K$ with $[K : L] < \infty$, let $k$ be a completion of $K$ with respect to some valuation (Archimedean or not), and define Diophantine exponents (multiplicative or not) of $z \in k^n$ with respect to integer points $L_2$ of $L$ by looking at small values of $|z \cdot q + p|$ where $p \in L_2$ and $q \in (L_2)^n$. Note that Sprindžuk’s book \cite{S3} solves the analogues of Mahler’s Problem for $L = K$ being either $\mathbb{Q}$ or the field of formal power series over a finite field. A dynamical approach to problems of this type, including
the $S$-arithmetic versions where one is allowed to consider several completions at the same time, is now being developed in [KT].

Our proof of Theorem 1.1 is based on a variation of methods of [KM1] and [BKM]. In the next section we describe the reduction of the theorem to a chain of statements involving discrete subgroups of Euclidean spaces, and in §3 take care of the final link of that chain.

2. LATTICES AND MEASURE ESTIMATES

In order to prove Theorem 1.1, one needs to take any $v > \frac{n-1}{2}$ and show that the set of $z \in f(U)$ for which there exist infinitely many $(p, q) \in \mathbb{Z}^{n+1}$ with

\[ |z \cdot q + p| \leq (\Pi_+(q))^{-v/n} \]

is null with respect to the pushforward of the Lebesgue measure on $U$. Writing $z = x + iy$ and perhaps slightly changing $v$, one can replace (2.1) with

\[ \max(|x \cdot q + p|, |y \cdot q|) \leq (\Pi_+(q))^{-v/n} . \]

Our first step is to rephrase (2.2). Choose $\beta > 0$, define

\[ r = \Pi_+(q)^{-\beta} , \]

and then define $t = (t_1, \ldots, t_n) \in \mathbb{R}_+^n$ by

\[ |q_i|_+ = re^{t_i} , \quad i = 1, \ldots, n . \]

Let us denote the sum of the components of $t$ by $t$ (the latter notation will be used throughout the paper, so that whenever $t$ and $t$ appear in the same formula, $t$ will stand for $\sum_{i=1}^n t_i$). Then (2.2a) and (2.2b) imply that

\[ \Pi_+(q) = r^n e^t = (\Pi_+(q))^{-n\beta} e^t , \]

hence $\Pi_+(q) = e^{\frac{1}{1+n\beta} t}$ and

\[ r = e^{-\frac{\beta}{1+n\beta} t} . \]

This allows us to write the right hand side of (2.2) as

\[ (\Pi_+(q))^{-v/n} = e^{-\frac{v}{1+n\beta} t} = e^{-\frac{\beta}{1+n\beta} t} e^{-\frac{v-\beta}{n(1+n\beta)} t} = re^{-at} , \]

where

\[ a = \frac{v - n\beta}{n(1 + n\beta)} \iff \beta = \frac{v - an}{n(1 + an)} . \]

Now recall that we still have a freedom to choose either $\beta$ or $a$. At this point we choose $a$ in order to let $\beta$ tend to 0 as $v$ tends to its critical value $\frac{n-1}{2}$. That is, we let $a = \frac{n-1}{2n}$, which yields

\[ \beta = \frac{2v - n + 1}{n(n + 1)} , \]

and, in view of (2.4),

\[ r = e^{-\gamma t} , \quad \text{where } \gamma = \frac{2v - n + 1}{2n(v + 1)} . \]

We summarize the above computation\(^2\) as

\[^2\text{A similar argument can be found in [KM1, §2], [KM2, §9], [K, §5].}\]
Lemma 2.1. Let $v > \frac{n-1}{2}$, $x, y \in \mathbb{R}^n$ and $(p, q) \in \mathbb{Z}^{n+1}$ be such that (2.2) holds. Define $\beta$ by (2.5a), $r$ by (2.3a) and $t$ by (2.3b). Then

\begin{equation}
\tag{2.5a}
e^{-\frac{n-1}{2}t} \max (|x \cdot q + p|, |y \cdot q|) \leq r
\end{equation}

and

\begin{equation}
\tag{2.3a}
e^{-t_i}|q_i| \leq r, \quad i = 1, \ldots, n;
\end{equation}

moreover, $r$ and $t$ are related via (2.5b).

Now discrete subgroups of $\mathbb{R}^{n+2}$ enter naturally to provide a concise form for inequalities (2.6ab). For $z = x + iy \in \mathbb{C}^n$ define

\begin{equation}
\tag{2.7}
u_z = \begin{pmatrix} 1 & 0 & x^T \\ 0 & 1 & y^T \\ 0 & 0 & I_n \end{pmatrix} \in \text{SL}_{n+2}(\mathbb{R}),
\end{equation}

and for $t \in \mathbb{R}_+$ let

\begin{equation}
\tag{2.8}
g_t = \text{diag}(e^{-\frac{n-1}{2}t}, e^{-\frac{n-1}{2}t}, e^{-t_1}, \ldots, e^{-t_n}) \in \text{GL}_{n+2}(\mathbb{R})
\end{equation}

(note that $\det(g_t) = e^{-t/n}$). Then (2.6ab) can be rewritten as $\|g_t u_z \begin{pmatrix} p \\ 0 \\ q \end{pmatrix}\| \leq r$, where $\| \cdot \|$ stands for the $l^\infty$ norm on $\mathbb{R}^{n+2}$. Recall that for any discrete subgroup $\Lambda$ of $\mathbb{R}^m$, $m \in \mathbb{N}$, one defines $\delta(\Lambda)$ to be the norm of a nonzero element of $\Lambda$ with the smallest norm, that is,

$$\delta(\Lambda) \overset{\text{def}}{=} \inf_{v \in \Lambda \setminus \{0\}} \|v\|.$$

So if one denotes

\begin{equation}
\tag{2.9}\Lambda \overset{\text{def}}{=} \left\{ \begin{pmatrix} p \\ 0 \\ q \end{pmatrix} \bigg| p \in \mathbb{Z}, q \in \mathbb{Z}^n \right\},
\end{equation}

the following is straightforward:

Corollary 2.2. Let $v > \frac{n-1}{2}$ and $z \in \mathbb{C}^n$ be such that (2.2) holds for infinitely many $(p, q) \in \mathbb{Z}^{n+1}$. Then there exists an unbounded set of $t \in \mathbb{R}_+$ such that

\begin{equation}
\tag{2.10}\delta(g_t u_z \Lambda) \leq e^{-\gamma t},
\end{equation}

where $\gamma$ is as in (2.5b).
Corollary 2.3. Let \( z \in \mathbb{C}^n \) be such that for some \( v > \frac{n-1}{2} \) one has (2.2) for infinitely many \((p, q) \in \mathbb{Z}^{n+1}\). Then there exists \( \gamma > 0 \) such that (2.10) holds for infinitely many \( t \in \mathbb{Z}^n_+ \).

Proof. Straightforward from (2.5b) and the fact that the ratio of \( \delta(g_{t^*}) \) and \( \delta(g_{s^*}) \) is uniformly bounded from both sides when \( \|t - s\| < 1 \). □

In fact, the converse to Corollary 2.3 is also true, and can be proved by an argument from [K, §5].

In the next corollary and thereafter, \(|\cdot|\) stands for Lebesgue measure.

Corollary 2.4. Let \( f \) be a map from an open subset \( U \) of \( \mathbb{C} \) to \( \mathbb{C}^n \). Suppose that for almost every \( z_0 \in U \) there exists a neighborhood \( B \subset U \) of \( z_0 \) such that for any \( \gamma > 0 \) one has

\[
(2.11) \quad \sum_{t \in \mathbb{Z}^n_+} \left| \{ z \in B \mid \delta(g_{t}u_{f(z)}\Lambda) \leq e^{-\gamma t} \} \right| < \infty.
\]

Then \( f \) is strongly extremal.

Proof. In view of the Borel-Cantelli Lemma, it follows from (2.11) that for almost all \( z \) of the form \( f(z) \), \( z \in B \), (2.10) is satisfied for at most finitely many \( t \in \mathbb{Z}^n_+ \). Corollary 2.3 then implies that for any \( v > \frac{n-1}{2} \), almost all \( z \) as above satisfy (2.1) for at most finitely many \((p, q) \in \mathbb{Z}^{n+1}\), that is, \( \{ z \in B \mid \omega^\times(f(z)) \geq v \} = 0 \). □

In view of the above corollary and Remark 1.2, to prove Theorem 1.1 it suffices to show that for any complex analytic \( f : U \to \mathbb{C}^n \), \( U \subset \mathbb{C} \), satisfying (1.4), one can find a neighborhood \( B \subset U \) of almost every \( z_0 \in U \) such that for any \( \gamma > 0 \) and any \( t \in \mathbb{Z}^n_+ \) it is possible to estimate the measure of sets

\[
(2.12) \quad \{ z \in B \mid \delta(g_{t}u_{f(z)}\Lambda) \leq e^{-\gamma t} \}
\]

so that (2.11) holds. Observe that at this point it makes no difference if one replaces the \( l^\infty \) norm used to define \( \delta(\cdot) \) by any other norm, and we are going to switch to the Euclidean one from now on.

In order to state the main estimate that will be used to bound the measure of sets (2.12), we need to introduce some additional terminology. If \( \Gamma \) is a discrete subgroup of \( \mathbb{R}^m \), we define the rank of \( \Gamma \) to be the dimension of \( \mathbb{R} \Gamma \), and denote by \( ||\Gamma|| \) the covolume of \( \Gamma \), that is, the volume of the quotient space \( \mathbb{R} \Gamma / \Gamma \). For \( \Lambda \) as above, we denote by \( \mathcal{S}(\Lambda) \) the set of all nonzero subgroups of \( \Lambda \). The key ingredient in what follows is, for fixed \( t \in \mathbb{Z}^n_+ \) and \( \Gamma \in \mathcal{S}(\Lambda) \), keeping track of the covolumes of subgroups \( g_{t}u_{f(z)}\Gamma \) as functions of \( z \). In particular, it will be useful to see that all those functions share a certain property, referred to in [KM1] and subsequent papers as being \((C, \alpha)\)-good.

Namely, if \( C \) and \( \alpha \) are positive numbers and \( V \) is a subset of \( \mathbb{R}^d \), one says that a function \( f : V \to \mathbb{R} \) is \((C, \alpha)\)-good on \( V \) if for any ball \( B \subset V \) and any \( \varepsilon > 0 \) one has

\[
\left| \{ x \in B \mid |f(x)| < \varepsilon \cdot \sup_{x \in B} |f(x)| \} \right| \leq C \varepsilon^\alpha |B|.
\]

Later we will need the following facts:
Lemma 2.5. (a) Suppose that $f_1, \ldots, f_k$ are $(C, \alpha)$-good on $V$; then the function $(f_1^2 + \cdots + f_k^2)^{1/2}$ is $(k^{\alpha/2}C, \alpha)$-good on $V$.

(b) Let $f$ be a real analytic map from a connected open subset $U$ of $\mathbb{R}^d$ to $\mathbb{R}^n$. Then for any $x_0 \in U$ there exists a neighborhood $V \subset U$ of $x_0$ and positive $C, \alpha$ such that any linear combination of $1, f_1, \ldots, f_n$ is $(C, \alpha)$-good on $V$.

Proof. The first statement is elementary and an easy consequence of parts (b), (c) of [BKM, Lemma 3.1]. For the second assertion, which is based on the work done in [KM1], see [K, Corollary 3.3]. □

Now we can state the crucial estimate that the whole proof hinges upon. It is a special case of [BKM, Theorem 6.2]. We remark that it is proved by a variation of a combinatorial construction used by Margulis in the 1970s [Ma] and then by S. G. Dani in the 1980s [D] to establish and quantitatively describe non-divergence of unipotent flows on homogeneous spaces.

Theorem 2.6. Fix $d, k, m \in \mathbb{N}$. Let $\Lambda$ be a discrete subgroup of $\mathbb{R}^m$ of rank $k$, and let a ball $B = B(x_0, r_0) \subset \mathbb{R}^d$ and a continuous map $H : \tilde{B} \to \text{GL}_m(\mathbb{R})$ be given, where $\tilde{B}$ stands for $B(x_0, 3^kr_0)$. Take $C, \alpha > 0$, $0 < \rho \leq 1$, and assume that for any $\Gamma \in S(\Lambda)$,

(i) the function $x \mapsto \|H(x)\Gamma\|$ is $(C, \alpha)$-good on $\tilde{B}$, and

(ii) $\sup_{x \in B} \|H(x)\Gamma\| \geq \rho$.

Then for any positive $\varepsilon \leq \rho$ one has

$$|\{x \in B \mid \delta(H(x)\Lambda) < \varepsilon\}| \leq c_{d,k}C \left(\frac{\varepsilon}{\rho}\right)^\alpha |B|,$$

where $c_{d,k}$ is a constant depending only on $d$ and $k$ (and explicitly computed in [KM1] and [BKM]).

Corollary 2.7. Let $U \subset \mathbb{C}$ be an open subset, and let $f : U \to \mathbb{C}^n$ be a continuous map. Keep the notation (2.7), (2.8), (2.9). Assume that for almost every $z_0 \in U$ one can find balls $B = B(z_0, r_0) \subset \tilde{B} = B(z_0, 3^{n+1}r_0) \subset U$ and constants $C, \alpha, \rho > 0$ such that for any $t \in \mathbb{R}^n_+$ and any $\Gamma \in S(\Lambda)$ the following holds:

(2.13) $\forall t \in \mathbb{R}^n_+ \forall \Gamma \in S(\Lambda)$ the function $z \mapsto \|g_tF(z)\Gamma\|$ is $(C, \alpha)$-good on $\tilde{B}$, and

(2.14) $\forall t \in \mathbb{R}^n_+ \forall \Gamma \in S(\Lambda) \sup_{z \in B} \|g_tF(z)\Gamma\| \geq \rho$.

Then $f$ is strongly extremal.

Proof. Under the assumptions (2.13) and (2.14) above, the previous theorem, with $d = 2$, $k = n + 1$, $m = n + 2$ and $H(z) = g_tF(z)$ for fixed $t \in \mathbb{Z}^n_+$, forces the measure of every set (2.12) to be not greater than $c_{d,k}C\rho^{-\alpha}|B|e^{-\alpha t}$ whenever $t$ is far enough from zero. This implies (2.11), and hence, in view of Corollary 2.4, the strong extremality of $f$. □

In the next section we show how one can write down explicit expressions for the covolume of subgroups of the form $g_tF(z)\Gamma$ for any $\Gamma \in S(\Lambda)$, and, assuming (1.4) and the analyticity of $f$, verify conditions (2.13) and (2.14).
3. Exterior products and covolume estimates

In this section we keep the notation introduced in (2.7)–(2.9), and prove the following:

**Lemma 3.1.** Let $U \subset \mathbb{C}$ be an open subset, and let $f : U \to \mathbb{C}^n$ be a complex analytic map. Then:

(a) for any $z \in U$ there exists a ball $\tilde{B} \subset U$ centered at $z$ and $C, \alpha > 0$ such that (2.13) holds;

(b) if, in addition, (1.4) is satisfied, then for every ball $B \subset U$ there exists $\rho > 0$ such that (2.14) holds.

In view of Corollary 2.7 and Remark 1.2, this lemma immediately implies Theorem 1.1.

**Proof.** Fix $t \in \mathbb{R}_n$ and $\Gamma \in \mathcal{S}(\Lambda)$ of rank $k$, where $1 \leq k \leq n + 1$. Without loss of generality we will order the components of $t$ so that $t_1 \leq \cdots \leq t_n$.

It will be convenient to denote the standard basis of $\mathbb{R}^{n+2}$ by $\{e_0, e_*, e_1, \ldots, e_n\}$, so that $\Lambda$ as in (2.9) is equal to $Ze_0 + Ze_1 + \cdots + Ze_n$. We will describe $\Gamma$ by means of its representing element from the exterior algebra of $\mathbb{R}^{n+2}$. Recall that $w \in \Lambda^k(\mathbb{R}^{n+2})$ is said to represent $\Gamma$ if

$$w = v_1 \wedge \cdots \wedge v_k,$$

where $v_1, \ldots, v_k$ form a basis of $\Gamma$.

Clearly an element representing $\Gamma$ is defined up to a sign, and the covolume $||\Gamma||$ of $\Gamma$ is equal to the Euclidean norm (with respect to the standard Euclidean structure extended to $\Lambda(\mathbb{R}^{n+2})$) of $w$.

For brevity we will suppress the variable $z$ whenever it does not cause confusion. Write $f = g + ih$, and with some abuse of notation identify $g$ and $h$ with vector-functions $\begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} : U \to \mathbb{R}^{n+2}$. Then it is immediate from (2.7) that for any $v \in \mathbb{R}^{n+2}$ one has

$$u_f v = v + (v \cdot g)e_0 + (v \cdot h)e_* .$$

(3.1)

Our goal now is to choose an orthonormal set with respect to which it is convenient to compute coordinates of $u_f w$, where $w$ represents $\Gamma$. We closely follow an approach developed in [BKM], where statements similar to (2.13) and (2.14) were established to prove the convergence case of the Khintchine-Groshev Theorem for non-degenerate manifolds.

First choose an orthonormal subset $v_1, \ldots, v_{k-1}$ of $\mathbb{R}\Gamma$ such that each $v_i$, $i = 1, \ldots, k - 1$, is orthogonal to $e_0$. Then, if $\mathbb{R}\Gamma$ does not contain $e_0$, choose $v_0 \in \mathbb{R}e_0 \oplus \mathbb{R}\Gamma$ such that $\{e_0, v_0, v_1, \ldots, v_{k-1}\}$ is an orthonormal basis of $\mathbb{R}e_0 \oplus \mathbb{R}\Gamma$, and represent $\Gamma$ by

$$w = (ae_0 + bv_0) \wedge v_1 \cdots \wedge v_{k-1} = ae_0 \wedge v_1 \cdots \wedge v_{k-1} + bv_0 \wedge v_1 \cdots \wedge v_{k-1},$$

(3.2)

where $a^2 + b^2 \geq 1$. If $\mathbb{R}\Gamma$ does contain $e_0$, then $\{e_0, v_1, \ldots, v_{k-1}\}$ is already a basis of $\mathbb{R}e_0 + \mathbb{R}\Gamma$, so (3.2) is valid with $b = 0$ and $v_0$ taken to be any unit vector orthogonal to $\mathbb{R}e_* + \mathbb{R}\Gamma$. 

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Combining (3.1) and (3.2), one gets

\[(3.3) \quad \mu_f w = \left( ae_0 + b (v_0 + (v_0 \cdot g)e_0 + (v_0 \cdot h)e_0) \right) \wedge_{i=1}^{k-1} (v_i + (v_i \cdot g)e_0 + (v_i \cdot h)e_0) \]

From there one can see that every coordinate of \( u_f w \) with respect to the basis

\[(3.4) \quad \left\{ \bigwedge_{i=0}^{k-1} v_i, \ e_0 \wedge \bigwedge_{s \neq i} v_s, \ e_s \wedge \bigwedge_{s \neq i} v_s, \ e_0 \wedge e_s \wedge \bigwedge_{s \neq i,j} v_s \right\}
\]

of \( \bigwedge^k (\mathbb{R} e_0 \oplus \mathbb{R} e_* \oplus \mathbb{R} \Gamma \oplus \mathbb{R} v_0) \) is a linear combination of functions

\[
1, \ v_i \cdot g, \ v_i \cdot h, \begin{vmatrix} v_i \cdot g & v_j \cdot g \\ v_i \cdot h & v_j \cdot h \end{vmatrix},
\]

hence a linear combination of 1 and the components of the map \( \tilde{f} : U \to \mathbb{R}^{n(n+3)/2} \) given by

\[
\tilde{f} \overset{\text{def}}{=} \left( g, \ h; \left| \begin{array}{cc} g_i & g_j \\ h_i & h_j \end{array} \right|, 1 \leq i < j \leq n \right).
\]

The same can be said about every coordinate of \( g_t u_f w \) with respect to any orthonormal basis of \( \bigwedge^k (\mathbb{R}^{n+2}) \) containing the one given by (3.4). Since all the components of \( \tilde{f} \) are real analytic functions, it follows from Lemma 2.5(b) that for any \( z \in U \) there exists a ball \( \tilde{B} \subset U \) centered at \( z \) and \( C', \alpha > 0 \) such that every linear combination of 1 and the components of \( \tilde{f} \) is \((C', \alpha)-good\) on \( \tilde{B} \). Part (a) of this lemma then immediately follows from Lemma 2.5(a).

Part (b) requires some more work. Let us state the following auxiliary result:

**Lemma 3.2.** Let \( B \subset \mathbb{C} \) be a nonempty ball.

(a) Suppose that \( f = (f_1, \ldots, f_n) \) is an \( n \)-tuple of real-valued functions on \( B \) satisfying (1.4). Then there exists \( \rho_1 = \rho_1(B, f) > 0 \) such that for any \( \mathbf{c} = (c_0, c_1, \ldots, c_n) \) with \( \| \mathbf{c} \| = 1 \) one has

\[
\sup_{z \in B} |c_0 + c_1 f_1(z) + \cdots + c_n f_n(z)| \geq \rho_1.
\]

(b) Suppose that \( \mathcal{F} \) is a compact (in \( C^0 \) topology) family of pairs of complex-valued functions \((\varphi_1, \varphi_2)\) analytic in \( B \) such that one is not a real multiple of another. Then there exists \( \rho_2 = \rho_2(B, \mathcal{F}) > 0 \) such that

\[
|\text{Im}(\varphi_1 \varphi_2)| \geq \rho_2 \quad \forall (\varphi_1, \varphi_2) \in \mathcal{F}.
\]

**Proof.** The first statement follows from the standard compactness argument applied to the set of functions \( \left\{ c_0 + c_1 f_1 + \cdots + c_n f_n \mid \| \mathbf{c} \| = 1 \right\} \). The same kind of argument shows that if the second assertion does not hold, one must have \( \text{Im}(\varphi_1 \varphi_2) = 0 \) for some \((\varphi_1, \varphi_2) \in \mathcal{F}\). Hence the ratio of \( \varphi_1(z) \) and \( \varphi_2(z) \) is real for all \( z \in B \),
which, due to the complex analyticity of \( \phi_1 \) and \( \phi_2 \), can happen only if \( \phi_1/\phi_2 \) is a constant. \( \square \)

Now let us consider two cases. If \( k = \dim(\mathbb{R} \Gamma) = 1 \), equality (3.3) gives

\[
u_t w = (a + b(v_0 \cdot g))e_0 + b v_0 + b(v_0 \cdot h)e_* ,
\]

hence

\[
(3.5) \quad \| g_t u_r w \| \geq \| g_t u_r w \cdot e_0 \| = e^{\frac{n-1}{n}} |a + b(v_0 \cdot g)| .
\]

The right hand side of (3.5) is a linear combination of \( 1, g_1, \ldots, g_n \) (which, due to (1.4), are linear independent over \( \mathbb{R} \)) with big enough coefficients, therefore its supremum on any ball \( B \subset \mathbb{C} \) is uniformly (in \( a, b, v_0 \) and \( t \)) bounded from below by \( \rho_1(B, g) \) due to Lemma 3.2(a).

The argument in the case \( k \geq 1 \) is different. Namely, we define the family

\[
(3.6) \quad \mathcal{F} \overset{\text{def}}{=} \{ (u_1 \cdot f, a + b u_2 \cdot f) \mid a^2 + b^2 = 1, \ u_1 \perp u_2 \in \mathbb{R}^n, \ \| u_1 \| = \| u_2 \| = 1 \} ,
\]

which clearly satisfies all the assumptions of Lemma 3.2(b), and put \( \rho = \rho_2(B, \mathcal{F}) \).

To prove that (2.14) holds with this value of \( \rho \), we need to fine-tune the choice of the orthonormal set \( \{ v_0, \ldots, v_{k-1} \} \). Namely, we will pay special attention to the vector \( e_n \), which is the eigenvector of \( g_t \) with one of the smallest eigenvalues (recall that \( t_1 \leq \cdots \leq t_n \) by our assumption). We do it by first choosing an orthonormal set \( v_1, \ldots, v_{k-2} \in \mathbb{R} \Gamma \) such that each \( v_i, \ i = 1, \ldots, k-2 \), is orthogonal to both \( e_0 \) and \( e_n \). Then choose \( v_{k-1} \) orthogonal to \( v_i, \ i = 1, \ldots, k-2 \), and to \( e_0 \) (but in general not to \( e_n \)). After that choose \( v_0 \) either to complete \( \{ e_0, v_1, \ldots, v_{k-1} \} \) to an orthonormal basis of \( \mathbb{R} e_0 \oplus \mathbb{R} \Gamma \) or (in case \( \Gamma \) contains \( e_0 \)) to be any unit vector orthogonal to \( \mathbb{R} e_* \oplus \mathbb{R} \Gamma \).

With this in mind, denote by \( W \) the subspace \( e_0 \wedge e_* \wedge \bigwedge_{2}^{k-2} (\mathbb{R}^{n+2}) \) of \( \bigwedge^k (\mathbb{R}^{n+2}) \) (that is, the set of elements corresponding to \( k \)-dimensional subspaces of \( \mathbb{R}^{n+2} \) containing both \( e_0 \) and \( e_* \)), and write (3.3) in the form

\[
u_t(z)w = w'(z) + e_0 \wedge e_* \wedge w''(z) ,
\]

where \( w'(z) \) is orthogonal to \( W \) and \( w''(z) \in \bigwedge_{2}^{k-2} ((\mathbb{R} e_0 \oplus \mathbb{R} e_*)^\perp) \). Since both \( e_0 \) and \( e_* \) are eigenvectors of \( g_t \), both \( W \) and its orthogonal complement are \( g_t \)-invariant. Therefore it will suffice to show that

\[
\sup_{z \in B} \| g_t(e_0 \wedge e_* \wedge w''(z)) \| \geq \rho ,
\]

or, equivalently,

\[
\sup_{z \in B} \| g_t w''(z) \| \geq e^{-\frac{n-1}{n}} \rho .
\]

Next consider the product \( e_n \wedge w''(z) \). We claim that it is enough to show that

\[
(3.7) \quad \| e_n \wedge w''(z) \| \geq \rho \quad \text{for some} \ z \in B .
\]
Indeed, since $e_n$ is an eigenvector of $g_t$ with eigenvalue $e^{-t_n}$, for any $z \in B$ the norm of $g_t(e_n \wedge w''(z))$ is not greater than $e^{-t_n} \|g_t w''(z)\|$. Therefore, since the smallest eigenvalue of $g_t$ on $\bigwedge^{k-1}(\mathbb{R}^{n+2})$ is equal to $e^{-(t_{n-k+1}+\cdots+t_n)}$ (here we set $t_0 = \frac{n-1}{2n}$, so that the above statement holds for $k = n + 1$ as well), the norm of $g_t w''(z)$ is not less than

$$e^{t_n} \|g_t(e_n \wedge w''(z))\| \geq e^{t_n} e^{-(t_{n-k+1}+\cdots+t_n)} \|e_n \wedge w''(z)\|$$

for some $z \in B$, by (3.7)

as required. Thus it remains to prove (3.7). Using (3.3), it is possible to write down coefficients in the decomposition of $w''(z)$ as a linear combination of elements of the form $\bigwedge_{s \neq i,j} v_s$. We are going to do it for the term containing $v_1 \wedge \cdots \wedge v_{k-2}$. Namely, one can write

$$e_n \wedge w'' = \pm \begin{vmatrix} v_{k-1} \cdot g & a + b v_0 \cdot g \\ v_{k-1} \cdot h & b v_0 \cdot h \end{vmatrix} e_n \wedge v_1 \wedge \cdots \wedge v_{k-2}$$

+ other terms where one or two of $v_i$, $i = 1, \ldots, k - 2$, are missing.

From the orthogonality of the two summands in (3.8) it follows that $\|e_n \wedge w''\|$ is not less than the norm of the first summand, which, in view of $e_n$ being orthogonal to $v_i$, $i = 1, \ldots, k - 2$, is equal to the absolute value of the coefficient in front of $e_n \wedge v_1 \wedge \cdots \wedge v_{k-2}$. The latter can be written as

$$\sqrt{a^2 + b^2} \text{Im}(\varphi_1 \varphi_2)$$

where

$$\varphi_1 = v_{k-1} \cdot f \quad \text{and} \quad \varphi_2 = (a^2 + b^2)^{-1/2}(a + b v_0 \cdot f).$$

It is immediate that $(\varphi_1, \varphi_2)$ belongs to $\mathcal{F}$ as defined in (3.6). Since $a^2 + b^2 \geq 1$, this completes the proof of (3.7) with $\rho = \rho_2(B, \mathcal{F})$.

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