THE NONLINEAR SCHRÖDINGER EQUATION AND CONSERVATION LAWS

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Abstract. The purpose of this short note is to show some results using the solution to the nonlinear equation of Schrödinger, (NLS), is used the Conservation Laws, that through them is obtained inequality that are of utmost importance, also show a variation of the theorem discussed in [8], classical local smoothing estimate in one dimension, and the Principle of Least Action, in Lagrangian mechanics. The Nonlinear Schrödinger Equation is a nonlinear partial differential equation, and where the solution is a complex-valued function of $d$-dimensional. We focus our attention on working in one dimension, that is to say, $d = 1$. We consider the existence of a solution to (NLS) and it meets the conditions to be a smooth function of a complex variable.

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1. Introduction

Let $u$ be a complex-valued function where $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ and let $d \geq 1$, we consider the Cauchy problem for the cubic nonlinear Schrödinger (NLS) equation,

\[
\begin{align*}
    iu_t + \Delta_{\mathbb{R}^{d+1}} u &= |u|^2 u \\
    u(0, x) &= u_0(x).
\end{align*}
\]

Let $u_0$ be a complex-valued function of $\mathbb{R}^d$ the initial condition, and $u_0(x) \in H^s_x(\mathbb{R}^2)$ where $H^s_x(\mathbb{R}^2)$ is a Sobolev space. We have to $\Delta$ is the Laplacian operator $\sum_{i=1}^d \partial^2_{x_i}$.

The purpose of this paper is to show some results about the relationship between conservation laws, to be more precise the conservation of mass and energy, and some applications in specific cases in physical systems. Also show some estimates of this equation and relationships with physical equations. Therefore, in this paper we use the solution to the nonlinear Schrödinger equation to get to such outcomes.

The paper is organized as follows, in section 2 we express some relations with both the laws of mass conservation, momentum and energy. We also show a variation of the theorem proved by the author in [8], Local classical smoothing estimate

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in one dimension, using the Poincaré inequality. Finally in Section 3, we will express some results and computation in one dimension in Lagrangian mechanics, using the concept of energy conservation, and the Principle of Least Action.

2. Conservation Laws

A simple physical interpretation of the function \( u(t, x) \) solution of the nonlinear Schrödinger equation is the conservation of density, momentum and energy over time, taking into account if we that \(|u(t, x)|^2\) as density of particles\(^1\) in a space \( x \) and time \( t \). More precisely if we use a tensor that defines this,

\[
F_{00} = |u(t, x)|^2
\]

\[
F_{j0} = Im(u_x(t, x)\overline{u(t, x)})
\]

\[
F_{jk} = Re(u_x(t, x)u_{xj}(t, x)) - \frac{1}{4}\delta_{jk}\Delta(|u(t, x)|^2) + \lambda \frac{p-1}{p+1}\delta_{jk}|u(t, x)|^{p+1}
\]

for all \( j, k = 1, \ldots, d \), where \( \delta_{jk} \) is the Kronecker delta, and \( \lambda \geq 1 \). If we use (NLS) equation and the above equations it is possible to get the following equation,

\[
\partial_t F_{00} + \partial_x F_{0j} = 0, \quad \partial_t F_{j0} + \partial_{xk} F_{jk} = 0.
\]

The equation (2.1) implies the mass density, so that when integrated over a region\(^2\) space is possible to get the mass, therefore the equation is,

\[
m(t) = \int_{\mathbb{R}^d} |u(t, x)|^2 dx
\]

analogous form of the equation (2.2) (momentum density) is possible to get the moment to integrate this equation also on a region of space,

\[
p(t) = -\int_{\mathbb{R}^d} Im(u_x(t, x)\overline{u(t, x)})dx
\]

if we compute the total energy of a system at time \( t \), we have,

\[
E(t) = T(t) + V(t)
\]

where respectively \( T \) and \( V \) are the kinetic energy and potential energy. In our case we have,

\[
T(t) = \frac{1}{2} \int |\nabla u|^2(t, x) dx \quad \text{and} \quad V(t) = \frac{2\lambda}{p+1} \int |u(t, x)|^{p+1}
\]

On which \( \lambda \in \mathbb{R} \) y \( p \) the order of the nonlinear term in this case is \( p = 3 \).

We can pose our argument in which the energy conservation equation that involves using before, and if we derive with respect to time it should take the value of zero, which is obtained the principle of stationarity, for example that any system

\(^1\)It is also worth thinking like total probability function, which could be an interpretation as a probability distribution function, which is widely used in quantum mechanics, or statistical mechanics.

\(^2\)Here the region will be determined according to the value contained \( d \), according to this you will have the dimensions of the space.
physically isolated (no interaction with any other system) remains unchanged with
time. Also it is clear that we can cite another example in case the energy equation
\( E(t) = 0 \), by taking this value is obtained so trivial that \( T(t) = -V(t) \). As we
know the law of conservation of energy is,
\[
E(u(t)) = \int \frac{1}{2} |\nabla u(t, x)|^2 + \frac{1}{4} |u(t, x)|^4 \, dx
\]
Another interesting use of conservation laws is used in [8], the author uses the
conservation laws to prove a theorem which implies a local soft test estimates in
one dimension (Theorem 2.1). The author uses the fundamental theorem of calculus
and mass conservation for a proof. We will follow the same path and in the last step
we use the inequality Poincaré, in order to reach the following result.

2.1. Main Result. In this section we show some results we obtained. The
inequalities in the study of the NLS equation are of great interest, the author in [8]
(see theorem [2.1]) gives a new demonstration to Theorem textit Local classical
smoothing estimate in one dimension, we show a variant to this theorem and use
the same tools in this case.

**Theorem 2.2.** Let \( u : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) be a solution to the Schrödinger equation for
d = 1. Then, exists a constant \( C = C(d) \) such that
\[
(2.8) \quad \int \text{Im}(u_x \overline{u})(t, 0) \, dt \leq C \|\nabla u(x)\|_{L^2_x(\mathbb{R})}^2
\]

**Proof.** The proof involves using the tools of conservation of mass and the funda-
mental theorem of calculus, we apply the Poincaré inequality to the term \( \|u(x)\|_{L^2_x(\mathbb{R})} \leq
C \|\nabla u(x)\|_{L^2_x(\mathbb{R})}^2 \), can be followed in further detail the author [3]. □

3. ONE-DIMENSION LAGRANGIAN MECHANICS

For describe the motion of a system with a finite number of degrees of freedom
results are more efficient methods of Lagrange and Hamilton. The advantage of
the methods is the universality that allows for any dynamic equations describing
system. It is not our aim to give a definition extensively on this topic but rather to give
some examples of its application. In the previous chapter was defined one energy
functional \( E(u(t)) \), here we use the concept of Lagrangian. Then the Lagrangian
\( L \) is defined by the following functional,
\[
L(u) = \int \frac{1}{2} |\nabla u(t, x)|^2 - \frac{1}{4} |u(t, x)|^4 \, dx
\]

**Proposition 3.1.** Let \( u : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) be a solutions a NLS, let \( x \) a function of
space type \( x = x(t) \), then applying the Euler-Lagrange equation is obtained,
\[
\partial_t u \|u(x)\|_{L^2_x(\mathbb{R})}^2 = -\frac{1}{4} \int |u(x)|^4 \, dx
\]

**Proof.** The proof of this corollary is easy since the Euler-Lagrange equation we
obtain the second law of Newton, therefore, is \( L \) the Lagrangian, the Euler-Lagrang
equations is,
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial (\partial_t x)} \right) - \frac{\partial L}{\partial x} = 0
\]
therefore is obtained by developing the equation (3.3),
(3.4) \[ m \partial_t x = -V(x) \]

so trivial result is obtained by replacing (2.5) and the potential energy in the previous equation. □

3.2. The Action. The action is a quantity that involves the product of the energy involved in a physical process and the time that this takes. It is greatly used in classical and relativistic mechanics, this concept was extended to quantum mechanics, obtaining a quantum theory of fields. The time evolution equations or equations of motion of all classical fields can be derived from the principle of least action for fields. According to this principle for the whole region of space-time can define a functional, called the action functional, such that the real fields and their derivatives are a minimum of the functional. That can be defined in functional form,

(3.5) \[ S(u(t,x)) := \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u(t,x)|^2 - \frac{1}{4} |u(t,x)|^4 \, dx \, dt \]

According to the principle of least action, is a basic assumption of classical and relativistic mechanics to describe the evolution over time of the state of motion of a particle as a physical field. We know that every particle from the point of view of quantum field theory, is the fluctuation of some hypothetical field. Here we express the variational principle only where \( \delta S(u(t,x)) = 0 \). This would imply finding the minimum path would make a particle along its trajectory with respect to two points.

One of the properties attributed to it to the Lagrangian in quantum field theory is that it is invariant under transformations, this implies that under a transformation of rotation, translational, and so on, this remain invariant. This invariance is accomplished after adding fields, which are called gauge fields. To way of conclusion a future paper an analysis of this question that would involve use these tools and relate the concept of invariance of the Lagrangian.

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3The Variational Principle, ie \( \delta S(u(t,x)) = 0 \), is possible to obtain the Euler-Lagrange equations.
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