A computational approach for solving second-order nonlinear ordinary differential equations by means of Laguerre series

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Abstract
In this work, a novel efficient numeric procedure for obtaining the approximate solution of a class of second-order nonlinear ordinary differential equations is presented which play a significant part in science and engineering branches. The technique is based on matrix equations and collocation points with truncated Laguerre series. The acquired approximate solutions subject to initial conditions are obtained in terms of Laguerre polynomials. Also, some examples together with error analysis techniques are acquired to demonstrate the efficacy of the present method, and the comparisons are made with current studies.

Keywords: Laguerre series, nonlinear ordinary differential equations, collocation method.

1. Introduction
Nonlinear differential equations play important role in many fields of engineering, science and even in mathematical models in social sciences. Some real phenomena examples are given for mathematical models for urban growth, modeling learning theories in education and psychology, reaction rates in chemistry, optional pricing in economics so on. Moreover, nonlinear models in biology are important to explain by mathematical models. In order to solve these type of equations

Nonlinear differential equations with initial and boundary value conditions are significant problems and they are important in many fields such as engineering, astrophysics, physical sciences. In recent years, in order to get the solutions of these problems both analytically and numerically some techniques have been introduced. These type of equations are of great importance on applied sciences [3]. Due to this reason, the second-order nonlinear ordinary differential equations are considered:

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Geliş Tarihi: 11.06.2019, Kabul Tarihi: 19.09.2019
\[
\sum_{k=0}^{2} P_k(x)y^{(k)}(x) + \sum_{p=0}^{2} \sum_{q=0}^{N} Q_{pq}(x)y^{(p)}(x)y^{(q)}(x) = g(x), \quad 0 \leq x \leq b < \infty
\] (1)

with the mixed conditions

\[
\sum_{k=0}^{1} \left( a_{kj}(x)y^{(k)}(0) + b_{kj}(x)y^{(k)}(b) \right) = \lambda_j, \quad j = 0, 1
\] (2)

where the functions \( P_k(x), Q_{pq}(x), \) and \( g(x) \) are defined \( 0 \leq x \leq b < \infty; \) \( a_{kj}, b_{kj}, \) and \( \lambda_j \) are appropriate and real constants; \( y(x) \) is the unknown function to be computed. For this purpose, approximate solution of the problem (1)-(2) is assumed as

\[
y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n(x)L_N(x), \quad 0 \leq x \leq b < \infty
\] (3)

where \( L_N(x) \) delineates the Laguerre polynomials;

\[
L_N(x) = \sum_{k=0}^{N} \frac{(-1)^k}{k!} \left( \frac{n}{k} \right) x^k, \quad n \in \mathbb{N}, \quad 0 \leq x < \infty
\] (4)

and \( a_n, (n = 0, 1, \ldots, N) \) are unknown coefficients with regard to the Laguerre polynomials, and \( N \in \mathbb{Z}^+ \) and \( N \geq 2. \)

### 2. Operational Matrix Relations

In this section, operational matrix relations are given for finding the approximate solution with regards to Laguerre polynomials in the form (3). For this purpose, Eq. (1) is considered in the form of two parts: the linear ordinary part and the nonlinear quadratic part, respectively as

\[
L[y(x)] + N_2[y(x)] = g(x)
\] (5)

where

\[
L[y(x)] = \sum_{k=0}^{2} P_k(x)y^{(k)}(x)
\]

and

\[
N_2[y(x)] = \sum_{p=0}^{2} \sum_{q=0}^{N} Q_{pq}(x)y^{(p)}(x)y^{(q)}(x)
\].

Now, each terms of the Eq.(1) are presented by the matrix forms [2]. So, the linear ordinary part in Eq.(5) in the matrix form is shown as

\[
y(x) = L(x)A.
\]

\[
y^{(1)}(x) = L(x)CA,
\] (6)

\[
y^{(2)}(x) = L(x)C^2A
\]
where
\[
\mathbf{L}(x) = [L_0(x) \quad L_1(x) \quad \ldots \quad L_N(x)],
\]
\[
\mathbf{C} = [c_{mn}],
\]
\[
c_{mn} = \begin{cases} 
-1, & m < n \\
0, & m \geq n
\end{cases},
\]
\[
\mathbf{A} = [a_0 \quad a_1 \quad \ldots \quad a_N]^T.
\]

Then we define matrix representations of nonlinear quadratic part as
\[
[(y(x))^2] = \mathbf{L}(x)\tilde{\mathbf{L}}(x)\tilde{\mathbf{A}},
\]
\[
[y^{(1)}(x)y(x)] = \mathbf{L}(x)\mathbf{CL}(x)\tilde{\mathbf{A}},
\]
\[
\left[y^{(1)}(x)^2\right] = \mathbf{L}(x)\mathbf{CL}(x)\tilde{\mathbf{C}}\tilde{\mathbf{A}},
\]
\[
[y^{(2)}(x)y^{(1)}(x)] = \mathbf{L}(x)\mathbf{C}^2\tilde{\mathbf{L}}(x)\tilde{\mathbf{C}}\tilde{\mathbf{A}},
\]
\[
[y^{(2)}(x)y(x)] = \mathbf{L}(x)\mathbf{C}^2\tilde{\mathbf{L}}(x)\tilde{\mathbf{A}},
\]
\[
\left[y^{(2)}(x)^2\right] = \mathbf{L}(x)\mathbf{C}^2\tilde{\mathbf{L}}(x)\mathbf{C}^2\tilde{\mathbf{A}};
\]
and
\[
\tilde{\mathbf{L}}(x) = \text{diag} [\mathbf{L}(x) \quad \mathbf{L}(x) \quad \ldots \quad \mathbf{L}(x)],
\]
\[
\tilde{\mathbf{C}} = \text{diag} [\mathbf{C} \quad \mathbf{C} \quad \ldots \quad \mathbf{C}],
\]
\[
\tilde{\mathbf{A}} = [a_0\mathbf{A} \quad a_1\mathbf{A} \quad \ldots \quad a_N\mathbf{A}]^T.
\]

3. Method of Solution

By putting the collocation points
\[
x_i(x) = \frac{b}{N}i, \quad i = 0, 1, \ldots, N, \quad 0 \leq x_0 < x_1 < \cdots < x_N \leq b < \infty
\]
into Eq.(1), for \(i = 0, 1, \ldots, N\) thus the fundamental matrix equation is obtained as
\[
\sum_{k=0}^{2} P_k(x_i)y^{(k)}(x_i) + \sum_{p=0}^{2} \sum_{q=0}^{p} Q_{pq}(x_i)y^{(p)}(x_i)y^{(q)}(x_i) = g(x_i)
\]
or
\[
\sum_{k=0}^{2} P_k\mathbf{Y}^{(k)} + \sum_{p=0}^{2} \sum_{q=0}^{p} Q_{pq}\mathbf{Y}^{(p,q)} = \mathbf{G}
\]
where
\[ P_k = diag \{ P_k(x_0) \ P_k(x_1) \ ... \ P_k(x_N) \}, \]
\[ Q_{pq} = diag \{ Q_{pq}(x_0) \ Q_{pq}(x_1) \ ... \ Q_{pq}(x_N) \}, \]
and
\[ \mathbf{Y}_k = \begin{bmatrix} y^{(k)}(x_0) \\ y^{(k)}(x_1) \\ \vdots \\ y^{(k)}(x_N) \end{bmatrix}, \quad \mathbf{Y}^{(p,q)} = \begin{bmatrix} y^{(p)}(x_0)y^{(q)}(x_0) \\ y^{(p)}(x_1)y^{(q)}(x_1) \\ \vdots \\ y^{(p)}(x_N)y^{(q)}(x_N) \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}. \]

Also, we consider the ensuing matrix forms of the nonlinear quadratic part from (7)
\[ \mathbf{Y}^{(0,0)} = \mathbf{L}^{\phi_0} \mathbf{A}, \quad \mathbf{Y}^{(1,0)} = \mathbf{L}^{\psi_0} \mathbf{A}, \quad \mathbf{Y}^{(1,1)} = \mathbf{L}^{\phi_1} \mathbf{A}, \]
\[ \mathbf{Y}^{(2,0)} = \mathbf{L}^{\phi_2} \mathbf{A}, \quad \mathbf{Y}^{(2,1)} = \mathbf{L}^{\psi_2} \mathbf{A}, \quad \mathbf{Y}^{(2,2)} = \mathbf{L}^{\phi_2} \mathbf{A}; \]
\[ \mathbf{L}^{\phi_0} = \begin{bmatrix} \mathbf{L}(x_0) & \mathbf{L}(x_1) \\ \mathbf{L}(x_N) & \mathbf{L}(x_N) \end{bmatrix}, \quad \mathbf{L}^{\psi_0} = \begin{bmatrix} \mathbf{L}(x_0) & \mathbf{C} \mathbf{L}(x_0) \\ \mathbf{L}(x_1) & \mathbf{C} \mathbf{L}(x_1) \end{bmatrix}, \quad \mathbf{L}^{\phi_1} = \begin{bmatrix} \mathbf{L}(x_0) & \mathbf{C}^2 \mathbf{L}(x_0) \\ \mathbf{L}(x_1) & \mathbf{C}^2 \mathbf{L}(x_1) \end{bmatrix}, \]
\[ \mathbf{L}^{\psi_1} = \begin{bmatrix} \mathbf{L}(x_0) & \mathbf{C}^2 \mathbf{L}(x_0) \\ \mathbf{L}(x_1) & \mathbf{C}^2 \mathbf{L}(x_1) \end{bmatrix}, \quad \mathbf{L}^{\phi_2} = \begin{bmatrix} \mathbf{L}(x_0) & \mathbf{C}^2 \mathbf{L}(x_0) \\ \mathbf{L}(x_1) & \mathbf{C}^2 \mathbf{L}(x_1) \end{bmatrix}, \quad \mathbf{L}^{\psi_2} = \begin{bmatrix} \mathbf{L}(x_0) & \mathbf{C}^2 \mathbf{L}(x_0) \\ \mathbf{L}(x_1) & \mathbf{C}^2 \mathbf{L}(x_1) \end{bmatrix}. \]

Then the fundamental matrix equation can be shown as
\[ \sum_{k=0}^{2} P_k \mathbf{L} \mathbf{A} + \sum_{p=0}^{2} \sum_{q=0}^{p} Q_{pq} \mathbf{L}_{pq}^* \mathbf{A} = \mathbf{G} \]
or
\[ \mathbf{W} \mathbf{A} + \mathbf{V} \mathbf{A} = \mathbf{G} \quad (9) \]
where
\[ \mathbf{W} = \sum_{k=0}^{2} P_k \mathbf{L} \mathbf{A} = [w_{ij}]_{(N+1) \times (N+1)}, \]
\[ \mathbf{V} = \sum_{p=0}^{2} \sum_{q=0}^{p} Q_{pq} \mathbf{L}_{pq}^* \mathbf{A} = [v_{ij}]_{(N+1) \times (N+1)}, \quad i, j = 0, 1, ..., N. \]

Furthermore, fundamental matrix equation (9) is written in the augmented matrix form
\[ [\mathbf{W}; \mathbf{V}; \mathbf{G}] \quad (10) \]
If the identical procedure is used for the mixed conditions (2), then

for \( j = 0 \), \( \mathbf{u}_0 = [y(0)] = \mathbf{L}(0) \),

for \( j = 1 \), \( \mathbf{u}_1 = [y^{(1)}(0)] = \mathbf{L}(0)\mathbf{c} \).

Also, briefly

\[
\mathbf{u} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \end{bmatrix}_{2 \times (N+1)}, \quad \mathbf{o}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{2 \times (N+1)^2}, \quad \lambda = \begin{bmatrix} \lambda_2 \\ \lambda_3 \end{bmatrix}: [\mathbf{u}; \; \mathbf{o}_2; \; \lambda].
\] (11)

Therefore, so as to get the solution of the problem (1)-(2) the row matrices (11) are replaced by the appropriate two rows of the augmented matrix (10). Thus, the new augmented matrix \([\mathbf{w}; \mathbf{\tilde{v}}; \mathbf{\tilde{g}}]\) is constructed and by having the solution of the system, required Laguerre coefficients are calculated. Accordingly, requisite approximate solution is is occurred in the Eq. (3).

4. Error Analysis

Here, a short introduction for the error analysis of the Laguerre polynomial solution (3) is held to demonstrate the exactitude of the method. Error function is defined by \( x = x_\alpha, \alpha = 0,1, ... \)

\[
E_N(x_\alpha) = \left| y(x_\alpha) - \sum_{k=0}^{2} P_k(x_\alpha) y_N^{(k)}(x_\alpha) - \sum_{p=0}^{2} \sum_{q=0}^{N} Q_{pq}(x_\alpha) y_N^{(p)}(x_\alpha) y_N^{(q)}(x_\alpha) + g(x_\alpha) \right| \cong 0
\]

where \( E_N(x_\alpha) \leq 10^{-k\alpha} = 10^{-k}, (k \in \mathbb{Z}^+) \) is recommended, then the truncation limit \( N \) is escalated till having enough small prescribed \( 10^{-k} \) for the value of disparity \( E_N(x_\alpha) \) at each of the point [3].

5. Numerical Experiments

In this section, an example will be given to show applicability of our method. All the calculations and plots are done by using Maple18 and MatlabR2014b.
Example 5.1. [4]

Second-order nonlinear ordinary differential equation with quadratic terms with initial conditions is considered as an illustrative example:

\[ y''(x) + 2y'(x) + y(x) + y^2(x) - y''(x)y'(x) = 12 \exp(x) + 2; \quad y(0) = 3, y'(0) = 2. \tag{12} \]

Problem’s exact solution is \( y(x) = 2 \exp(x) + 1. \)

![Figure 2. Comparison of the \( x > 0 \) of Example 5.1. for different \( N \) values](image)

| \( x \) | \(|E_2|\) | \(|E_4|\) | \(|E_5|\) |
|---|---|---|---|
| 0.0 | 0.000000 | 0.000000 | 0.000000 |
| 0.1 | 0.341836E-4 | 0.530766E-4 | 0.450128E-6 |
| 0.2 | 0.280551E-3 | 0.281048E-4 | 0.194988E-5 |
| 0.3 | 0.971761E-3 | 0.563571E-4 | 0.318105E-5 |
| 0.4 | 0.236493E-3 | 0.671672E-4 | 0.339696E-5 |
| 0.5 | 0.474425E-2 | 0.525094E-4 | 0.36512E-5 |
| 0.6 | 0.842376E-2 | 0.476701E-4 | 0.530256E-5 |
| 0.7 | 0.137505E-2 | 0.162679E-3 | 0.552450E-5 |
| 0.8 | 0.211081E-1 | 0.617047E-3 | 0.104534E-4 |
| 0.9 | 0.309206E-1 | 0.177815E-2 | 0.808003E-4 |
| 1.0 | 0.436563E-1 | 0.420369E-2 | 0.282554E-3 |

6. Conclusion

In this study, a computational procedure depending on Laguerre polynomials has been proposed in spite of solving a class of second-order nonlinear ordinary differential equations having quadratic terms numerically. Furthermore, the error analysis is explained and applied to determine the reliability of the method. The technique has been tested on illustrative example which has been shown by figure and table. The method has significant importance such as; the present method has short and concise computing procedure by writing the algorithm in Maple18, has sufficient results when \( N \) is chosen large enough and the method also can be extended on other studies [5].

Author’s Contributions

The author developed the theoretical formalism, performed the numerical calculations and performed the simulations. Besides, the author made substantial contributions to conception and design, participated in drafting the article or revising it critically for important intellectual content. Finally, the author gave final approval of the version to be submitted and any revised version.
Statement of Conflicts of Interest

No potential conflict of interest was reported by the author.

Statement of Research and Publication Ethics

The author declares that this study complies with Research and Publication Ethics.

References

[1] Fried I. 1979. Numerical Solution of Differential Equations. Academic Press, New York.
[2] Jones D.S., Plank M., Sleeman B.D. 2009. Differential Equations and Mathematical Biology. Chapman and Hall/CRC.
[3] Bracken A.J., Tuckwell H.C. 1992. Simple mathematical models for urban growth, Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences, 438 (1902): 171-181.
[4] Murray J.D. 1989. Mathematical Biology. Springer, Berlin.
[5] Gürbüz B., Sezer M. 2016. Laguerre polynomial solutions of a class of initial and boundary value problems arising in science and engineering fields. Acta Physica Polonica A, 129 (1): 194-197. DOI:10.12693/APhysPolA.130.194.
[6] Gürbüz B., Sezer M. 2017. A numerical solution of parabolic-type Volterra partial integro-differential equations by Laguerre collocation method. International Journal of Applied Physics and Mathematics, 7 (1): 49-58.
[8] Jordan D.W., Smith P. 2007. Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers. 4th Edition, Oxford University Press, New York.
[9] King A.C., Billingham J., Otto S.R. 2003. Differential Equations: Linear, Nonlinear, Ordinary, Partial. Cambridge University Press, New York.