COMPUTATION OF KONTSEVICH WEIGHTS OF CONNECTION AND CURVATURE GRAPHS FOR SYMPLECTIC POISSON STRUCTURES

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ABSTRACT. We give a detailed explicit computation of weights of Kontsevich graphs which arise from connection and curvature terms within the globalization picture as in [12] for the special case of symplectic manifolds. We will show how the weights for the curvature graphs can be explicitly expressed in terms of the hypergeometric function as well as by a much simpler formula combining it with the explicit expression for the weights of its underlined connection graphs. Moreover, we consider the case of a cotangent bundle, which will simplify the curvature expression significantly.

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1. Introduction

1.1. Motivation. In [17] Kontsevich proved that the differential graded Lie algebra (DGLA)\(^1\) of multivector fields on an open subset \(M \subset \mathbb{R}^d\) is \(L_\infty\)-quasi-isomorphic to the DGLA\(^2\) of multidifferential operators on functions on \(M\), i.e. there exists an \(L_\infty\)-quasi-isomorphism

\[
\mathcal{U} : T_{poly}(M) \to D_{poly}(M),
\]

such that its zeroth Taylor component \(\mathcal{U}^{(0)}\) is given by the Hochschild–Kostant–Rosenberg map. This result is known as the formality theorem. If one restricts to the case of bivector fields and bidifferential operators, one can recover deformation quantization for Poisson manifolds. The resulting star product was also constructed in [17] by an explicit formula. In [7, 5, 12] a globalization picture was presented for this star product on any Poisson manifold \(M\), including the construction of the local star product by using techniques of field theory, in particular the Poisson Sigma Model [16, 20, 4]. In [12] this construction was extended to manifolds with boundary as in the BV-BFV formalism [8, 9, 10] which is a perturbative quantum gauge formalism compatible with cutting and gluing. A similar approach, as the one presented by Fedosov in [13] for symplectic manifolds, was used, by considering notions of formal geometry. In particular, one starts with a formal exponential map \(\phi\) on the manifold \(M\) and constructs a flat connection \(D_G\), called the classical Grothendieck connection, on the completed symmetric algebra of the cotangent bundle \(\hat{\text{Sym}}(T^*M)\). This construction can be deformed to the Weyl bundle \(\hat{\text{Sym}}(T^*M)[[\hbar]]\) and, as it was shown in [5, 6, 12], it induces a similar equation as the one in Fedosov’s construction. In [12] it was shown that the different terms of this equation are given by a certain class of graphs. We want to give an explicit computation of the weights for these graphs. Let \(G_{n,m}\) denote the set of all admissible graphs as in [17] with \(n\) vertices in the bulk of the upper half-plane \(\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}\) and \(m\) vertices on \(\mathbb{R}\). Define a map

\[
\mathcal{U}_\Gamma : \bigwedge^n T_{poly}(M) \to D_{poly}(M)[1-n]
\]

using the \(L_\infty\)-morphism \(\mathcal{U}\). Let \(\pi\) be a Poisson structure on \(\mathbb{R}^d\) and let \(\xi, \zeta\) be any two vector fields on \(\mathbb{R}^d\). Let us define

\[
P(\pi) := \sum_{n \geq 0} \sum_{\Gamma \in G_{n,2}} \frac{\hbar^n}{n!} w_\Gamma \mathcal{U}_\Gamma(\pi \wedge \cdots \wedge \pi),
\]

\[
A(\xi, \pi) := \sum_{n \geq 0} \sum_{\Gamma \in G_{n+1,1}} \frac{\hbar^n}{n!} w_\Gamma \mathcal{U}_\Gamma(\xi \wedge \pi \wedge \cdots \wedge \pi),
\]

\[
F(\xi, \zeta, \pi) := \sum_{n \geq 0} \sum_{\Gamma \in G_{n+2,0}} \frac{\hbar^n}{n!} w_\Gamma \mathcal{U}_\Gamma(\xi \wedge \zeta \wedge \pi \wedge \cdots \wedge \pi),
\]

where \(w_\Gamma \in \mathbb{R}\) denotes the Kontsevich weight of the graph \(\Gamma\). The term (3) represents Kontsevich’s star product, (4) represents the deformed Grothendieck connection \(D_G := D_G + O(\hbar)\) (see construction below) and (5) its curvature. Let us emphasize a bit more on the formal geometry construction.

---

\(^1\)Endowed with the zero differential and the Schouten–Nijenhuis bracket.

\(^2\)Endowed with the Hochschild differential and the Gerstenhaber bracket.
1.2. Notions of Formal Geometry. Let $M$ be a smooth manifold and let $\phi: U \to M$ be a map where $U \subset TM$ is an open neighbourhood of the zero section. For $x \in M$, $y \in T_x M \cap U$ we write $\phi_x(y) := \phi(x, y)$. We say that $\phi$ is a generalized exponential map if for all $x \in M$ we have that $\phi_x(0) = x$, and $d\phi_x(0) = id_{T_xM}$. In local coordinates we can write

$$\phi^j_i(x, y) = x^i + y^i + \frac{1}{2} \phi^i_{x,jk} y^j y^k + \frac{1}{3!} \phi^i_{x,jkt} y^j y^k y^t + \cdots$$

where the $x^i$ are coordinates on the base and the $y^i$ are coordinates on the fibers. We identify two generalized exponential maps if their jets agree to all orders. A formal exponential map is an equivalence class of generalized exponential maps. It is completely specified by the sequence of functions $(\phi^j_i(x, \cdots y))_{k=0}^\infty$. By abuse of notation, we will denote equivalence classes and their representatives by $\phi$. From a formal exponential map $\phi$ and a function $f \in C^\infty(M)$, we can produce a section $\sigma \in \Gamma(\mathfrak{Sym}(T^*M))$ by defining $\sigma_x = T \phi^*_x f$, where $T$ denotes the Taylor expansion in the fiber coordinates around $y = 0$ and we use any representative of $\phi$ to define the pullback. We denote this section by $T \phi^*_x f$, it is independent of the choice of representative, since it only depends on the jets of the representative.

As it was shown [5, 3, 11], one can define a flat connection $D_G$ on $\mathfrak{Sym}(T^*M)$ with the property that $D_G \sigma = 0$ if and only if $\sigma = T \phi^*_x f$ for some $f \in C^\infty(M)$. As already mentioned before, this connection is called the classical Grothendieck connection. In fact, $D_G = d_x + L_R$ where $R \in \Omega^1(M, \text{Der}(\mathfrak{Sym}(T^*M)))$ is a 1-form with values in derivations of $\mathfrak{Sym}(T^*M)$, which we identify with $\Gamma(TM \otimes \mathfrak{Sym}(T^*M))$. We have denoted by $d_x$ the de Rham differential on $M$ and by $L$ the Lie derivative. In coordinates we have

$$R(\sigma)_k = - \frac{\partial \sigma}{\partial y^j} \left( \left( \frac{\partial \phi}{\partial y} \right)^{-1} \right)^j_k \frac{\partial \phi^k}{\partial x^j}. \tag{7}$$

Define $R(x, y) := R(\sigma)_k(x, y) dx^k$, $R_k(x, y) := R^j_k(x, y) \frac{\partial}{\partial y^j}$, $R^j(x, y) := R^j_k(x, y) dx^k$, and

$$R^j_k = - \left( \frac{\partial \phi}{\partial y} \right)^{-1}_j_k \frac{\partial \phi^k}{\partial x^j} = -\delta^j_k + O(y). \tag{8}$$

For $\sigma \in \Gamma(\mathfrak{Sym}(T^*M))$, $L_R \sigma$ is given by the Taylor expansion (in the $y$ coordinates) of

$$-d_y \sigma \circ (d_y \phi)^{-1} \circ d_x \phi: \Gamma(TM) \to \Gamma(\mathfrak{Sym}(T^*M)),$$

where we denote by $d_y$ the de Rham differential on the fiber. This shows that $R$ does not depend on the choice of coordinates. One can generalize this also for any fixed vector $\xi = \xi^i(x) \frac{\partial}{\partial x^i} \in T_x M$, instead of just considering the de Rham differential $d_x$, by

$$D_G^\xi = \xi + \hat{\xi}, \tag{9}$$

where

$$\hat{\xi}(x, y) = \iota_\xi R(x, y) = \xi^i(x) R^j_k(x, y) \frac{\partial}{\partial y^j}. \tag{10}$$

Here $\xi^i(x)$ would replace the 1-form part $dx^i$.

This paper is based on the master thesis [19].
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2. Computation of Kontsevich Weights

Let \((M, \omega)\) be a symplectic manifold regarded as a Poisson manifold with Poisson structure \(\pi\) induced by the symplectic form \(\omega\). Moreover let \(\phi: TM \supset U \rightarrow M\) be a formal exponential map and denote by \(T\) the Taylor expansion in fiber coordinates around zero. Anticipating the computation of the star product \(P(T\phi^*\pi)\), the connection 1-form \(A(R, T\phi^*\pi)\) and its curvature 2-form \(F(R, R, T\phi^*\pi)\) as in [12], we will explicitly compute the Kontsevich weights of three families of graphs in this section. Throughout the paper we use the harmonic angle function

\[
\varphi(u, v) = \arg \left( \frac{v - u}{\bar{v} - \bar{u}} \right) = \frac{1}{2i} \log \left( \frac{(v - u)(\bar{v} - \bar{u})}{(v - \bar{u})(\bar{v} - u)} \right) \tag{11}
\]

which measures the angle on \(\mathbb{H} \cup \mathbb{R}\) as depicted in Figure 1.

![Figure 1. Illustration of the angle function \(\varphi\).](image)

The propagator used in the computation of the Kontsevich weights is then simply given by \(d\varphi(u, v)\) and is usually called the Kontsevich propagator. Now let \(\Gamma \in G_{n,m}\) be an admissible graph with \(n\) vertices of first type, \(m\) vertices of second type and \(2n + m - 2\) edges. We use this propagator to compute the Kontsevich weight [17] \(w_{\Gamma}\) of \(\Gamma\) as

\[
w_{\Gamma} = \int_{\bar{C}_{n,m}} \omega_{\Gamma}. \tag{12}
\]

Here \(\bar{C}_{n,m}\) denotes the Fulton–MacPherson/Axelrod–Singer (FMAS) compactification [15, 1] of the configuration space \(C_{n,m}\) of \(n\) points in \(\mathbb{H}\) and \(m\) points on \(\mathbb{R}\) modulo scaling and translation. Let us briefly recall the construction of the needed configuration spaces. We define the open configuration space

\[
\text{Conf}_{n,m} = \{ (x_1, \ldots, x_n, q_1, \ldots, q_m) \in \mathbb{H}^n \times \mathbb{R}^m \mid x_i \neq x_j, \forall i \neq j, q_1 < \ldots < q_m \}. \tag{13}
\]
The 2-dimensional real Lie group of orientation preserving affine transformations of the real line

\[ G^{(1)} = \{ z \mapsto az + b \mid a, b \in \mathbb{R}, a > 0 \} \]

acts freely on Conf\(_{n,m}\). One can check that the quotient space \( C_{n,m} := \text{Conf}_{n,m}/G^{(1)} \) is in fact a smooth manifold of dimension \( 2n + m - 2 \). The differential form \( \omega_\Gamma \) on \( \bar{C}_{n,m} \) is given by

\[ \omega_\Gamma = \frac{1}{(2\pi)^{2n+m-2}} \bigwedge_{\text{edges } e} d\phi_e, \]

where the wedge product is over all \( 2n + m - 2 \) edges \( e \) of the graph \( \Gamma \). Let \( n \geq 2 \) and define

\[ \text{Conf}_n = \{ (x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_i \neq x_j, \forall i \neq j \}. \]

We have an action on \( \text{Conf}_n \) by the 3-dimensional Lie group

\[ G^{(2)} = \{ z \mapsto az + b \mid a \in \mathbb{R}, b \in \mathbb{C}, a > 0 \}. \]

Again, one can check that the quotient space \( C_n := \text{Conf}_n/G^{(2)} \) is a smooth manifold of dimension \( 2n - 3 \). Also here, we will denote its FMAS compactification by \( \bar{C}_n \). We refer to [17] for a more detailed construction. To simplify the notation we will use graphical language, where the figure below corresponds to a factor of \( d(\varphi(u,v)^n)/(2\pi)^n \) in \( w_\Gamma \). If there is no \( n \) above the arrow it simply means that \( n = 1 \).

![Graphical representation](image)

We know that the dimension of the configuration space \( C_{n,m} \) is \( 2n + m - 2 \), and since we work on a symplectic manifold \( M \) (with Darboux coordinates around each point \( x \in M \)), a vertex of first type is either a vertex representing the tensor \( T\phi^*_x\pi \), which we will call a \( T\phi^*_x\pi \)-vertex, with precisely two outgoing and no incoming edges, or a vertex representing the 1-form \( R \), which we will call an \( R \)-vertex, with precisely one outgoing edge and arbitrarily many incoming edges [12, 18]. So we may write \( n = p + r \), where \( p \) is the number of \( T\phi^*_x\pi \)-vertices and \( r \) is the number of \( R \)-vertices. We then have that \( \text{deg}(\omega_\Gamma) = 2p + r \), and in order for the integral (12) not to vanish, we must have that \( \omega_\Gamma \) is a top form, i.e. that \( 2n + m - 2 = 2p + r \). This then implies that

\[ r + m = 2 \]

So we have to distinguish three different cases, namely \( (r, m) = (2, 0) \), \( (r, m) = (1, 1) \) and \( (r, m) = (0, 2) \), which we will treat separately in what follows.

**Remark 2.1.** Actually, we will see below that all the integrals over the non-compactified configuration spaces \( C_{n,m} \) of the graphs we are considering converge and are thus finite. So it is not necessary to work with the compactifications.
2.1. Case 1: No Boundary Vertices. We will first treat the case \((r, m) = (2, 0)\), i.e. the case where we have no boundary vertices and exactly two \(R\)-vertices. In that case we get a family of graphs \((\Gamma_n)_{n \geq 0}\), where \(\Gamma_n\) is the graph with \(n\) wedges as in Figure 2(a) (stemming from \(n\) \(T\phi_x^*\pi\)-vertices) attached to the wheel as in Figure 2(b) (stemming from the two \(R\)-vertices).

Examples of the graphs \(\Gamma_n\) are given in Figure 3 below for \(n = 0, 1, 2\).

The Kontsevich weight of the graph \(\Gamma_n\) for \(n \geq 0\) is given by

\[
\omega_{\Gamma_n} = \frac{1}{(2\pi)^{2n+2}} \int_{C_{n+2,0}} d\phi(x,y)d\phi(y,x)d\phi(z_1,x)d\phi(z_1,y)\cdots d\phi(z_n,x)d\phi(z_n,y).
\]

**Remark 2.2.** We will omit the wedge product if it is clear. Moreover, for \(n = 0\) we simply set \(d\phi(z_1, x)d\phi(z_1, y)\cdots d\phi(z_n, x)d\phi(z_n, y) = 1\) in the integral above.

**Remark 2.3.** The sign of the weight \(\omega_{\Gamma}\) depends on the ordering of the edges of the graph \(\Gamma\) (i.e. the ordering of the propagator 1-forms in the integrand), and thus the ordering must always be specified. Throughout this whole section, we will stick to the ordering given in (19).

The goal now is to compute (19) explicitly. We will do this in several steps, mainly using Stokes’ theorem as in [21].
2.1.1. Step 1. In a first step, we want to integrate out the wedges. More precisely, for a wedge as in Figure 2(a) we want to compute the corresponding integral

\[
\frac{1}{(2\pi)^2} \int_{z \in \mathbb{H}\setminus\{x,y\}} d\varphi(z,x)d\varphi(z,y),
\]

i.e. we want to integrate out \( z \) (with \( x, y \in \mathbb{H} \) fixed). To do this we make a branch cut such that \( \varphi(z,x) \in (0,2\pi) \) and use Stokes' theorem

\[
\int_{z \in \mathbb{H}\setminus\{x,y\}} d\varphi(z,x)d\varphi(z,y) = \int_{\partial} \varphi(z,x)d\varphi(z,y),
\]

where \( \partial \) is the boundary of the integration domain depicted in Figure 4 below.

\[
\begin{aligned}
C & \quad C_2 \\
C_1 & \quad y \\
B_+ & \quad B_- \\
H_+ & \quad H_-
\end{aligned}
\]

**Figure 4.** Boundary \( \partial \) of the integration domain: \( C \) is the half-circle at infinity, \( B_+ \) and \( B_- \) are infinitesimally close together, the circles \( C_1 \) and \( C_2 \) have infinitesimal radius and \( H_+ \cup H_- \) is the real line.

Now using (11) we can discuss the different boundary components:

- On \( H_+ \cup H_- \): \( z \in \mathbb{R} \) and hence \( d\varphi(z,y) = \text{darg}(1) = 0 \)
- On \( B_+ \): \( \varphi(z,x) = 2\pi \)
- On \( B_- \): \( \varphi(z,x) = 0 \)
- On \( C_1 \): \( z = x + \varepsilon e^{-i\theta} \) for \( \varepsilon \to 0 \) \( \implies d\varphi(z,y) = \text{darg} \left( \frac{y-x}{y-x} \right) = 0 \)
- On \( C_2 \): \( z = y + \varepsilon e^{-i\theta} \) for \( \varepsilon \to 0 \) and \( \theta \in [0,2\pi] \) \( \implies \varphi(z,x) \to \varphi(y,x), d\varphi(z,y) = -d\theta \)
- On \( C \): \( z = Re^{i\theta} \) for \( R \to \infty \) and \( \theta \in [0,\pi] \) \( \implies \varphi(z,x) = \varphi(z,y) = 2\theta, d\varphi(z,y) = 2d\theta \)
We then finally get
\begin{equation}
\int_{z \in H \setminus \{x,y\}} d\varphi(z, x)d\varphi(z, y) = \int d\varphi(z, x)d\varphi(z, y) = 2\pi \int_{B_+} d\varphi(z, y) + \int_{0}^{2\pi} 4\theta d\theta - \varphi(y, x) \int_{0}^{2\pi} d\theta
\end{equation}
\begin{equation}
= 2\pi(\varphi(y, x) - \varphi(y, x) + [x; y]\pi),
\end{equation}
where
\begin{equation}
[x; y] = \begin{cases} +1, & \text{if Re}(x) > \text{Re}(y) \\ -1, & \text{if Re}(x) < \text{Re}(y) \end{cases}
\end{equation}
Dividing the result (22) by \((2\pi)^2\) (as in (20)), we get
\begin{equation}
\frac{1}{2\pi}(\varphi(x, y) - \varphi(y, x)) \pm \frac{1}{2},
\end{equation}
which agrees with the result given in [2, Lemma 3.3].

Finally, consider the limit \((x, y) \to (p, q)\) for \(p, q \in \mathbb{R}\) with \(p < q\). Using (24) and the fact that \(\varphi(p, q) = 2\pi\) and \(\varphi(q, p) = 0\), we can compute the Kontsevich weight of the graph below.

We get that the integral of the form representing this graph over \(C_{1,2}\) is equal to \(\frac{1}{2}\), which agrees with the result in [17, Section 6.4.3].

2.1.2. Step 2. In a second step we want to compute the weight of the graph

where \(n, m \geq 1\) and we use the notation introduced above, i.e. we want to explicitly compute the integral
\begin{equation}
\frac{1}{(2\pi)^{n+m}} \int_{C_{1,2}} d\varphi(x, y)^m d\varphi(y, x)^n.
\end{equation}
As before we make the branch cut such that \(\varphi(x, y) \in (0, 2\pi)\) and use Stokes’ theorem
\begin{equation}
\int_{y \in \mathbb{H} \setminus \{x\}} d\varphi(y, x)^m d\varphi(y, x)^n = \int_{\partial} \varphi(x, y)^m d\varphi(y, x)^n
\end{equation}
with boundary \(\partial\) of the integration domain depicted in Figure 5 below.

Again, we discuss the different boundary components:
Figure 5. Boundary $\partial$ of the integration domain: $C_- \cup C_+$ is the half-circle at infinity, $B_+$ and $B_-$ are infinitesimally close together, the circle $C_1$ has infinitesimal radius and $H$ is the real line.

- On $H$: $y \in \mathbb{R} \implies d\varphi(y, x) = \text{darg}(1) = 0$
- On $B_- \cup B_+$: $d\varphi(y, x) = 0$
- On $C_-$: $y = \Re e^{i\theta}$ for $R \to \infty \implies \varphi(x, y) = 2\pi$
- On $C_+$: $y = \Re e^{i\theta}$ for $R \to \infty \implies \varphi(x, y) = 0$
- On $C_1$: $y = x + \varepsilon e^{-i\theta}$ for $\varepsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{3\pi}{2}) \implies \varphi(x, y) = \frac{3\pi}{2} - \theta$ and

$$\varphi(y, x) = \begin{cases} \frac{\pi}{2} - \theta, & \text{for } \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ \frac{5\pi}{2} - \theta, & \text{for } \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \end{cases}$$

With this, we compute the integral

\begin{align*}
& \int_{y \in H \setminus \{x\}} d\varphi(x, y)^m d\varphi(y, x)^n = \int_{\partial} \varphi(x, y)^m d\varphi(y, x)^n = (2\pi)^m \int_{C_-} \varphi(x, y)^n d\varphi(y, x)^n + \int_{C_1} \varphi(x, y)^m d\varphi(y, x)^n \\
& = (2\pi)^m \pi^n - n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{3\pi}{2} - \theta \right)^m \left( \frac{\pi}{2} - \theta \right)^{n-1} d\theta - n \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left( \frac{3\pi}{2} - \theta \right)^m \left( \frac{5\pi}{2} - \theta \right)^{n-1} d\theta.
\end{align*}
Now we use the substitution \( a = \frac{\pi}{2} - \theta \) to compute

\[
\frac{\pi}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{3\pi}{2} - \theta \right)^m \left( \frac{\pi}{2} - \theta \right)^{n-1} d\theta = \int_0^\pi (\pi + a)^m a^{n-1} da = \sum_{k=0}^{m} \left( \frac{m}{k} \right) \pi^k \int_0^\pi a^{m+n-k-1} da
\]

\[
= \sum_{k=0}^{m} \left( \frac{m}{k} \right) \frac{\pi^{m+n}}{m+n-k}.
\]  

Similarly, we use the substitution \( a = \frac{3\pi}{2} - \theta \) to compute

\[
\frac{3\pi}{2} \int_{-\frac{3\pi}{2}}^{\frac{3\pi}{2}} \left( \frac{5\pi}{2} - \theta \right)^m \left( \frac{\pi}{2} - \theta \right)^{n-1} d\theta = \int_0^\pi a^m (\pi + a)^{n-1} da = \sum_{k=0}^{n-1} \left( \frac{n-1}{k} \right) \pi^k \int_0^\pi a^{m+n-k-1} da
\]

\[
= \sum_{k=0}^{n-1} \left( \frac{n-1}{k} \right) \frac{\pi^{m+n}}{m+n-k}.
\]

Putting everything together we get

\[
\int_{y \in \mathbb{H}\setminus\{x\}} d\varphi(x, y)^m d\varphi(y, x)^n = \left( 2^m - \sum_{k=0}^{m} \left( \frac{m}{k} \right) \frac{n}{m+n-k} - \sum_{l=0}^{n-1} \left( \frac{n-1}{l} \right) \frac{n}{m+n-l} \right) \pi^{m+n}.
\]  

It is not hard to see that for \( n = 1 \) the above formula simplifies to

\[
\int_{y \in \mathbb{H}\setminus\{x\}} d\varphi(x, y)^m d\varphi(y, x) = 2^m \left( 1 - \frac{2}{m+1} \right) \pi^{m+1},
\]

which agrees with the result in \([21, \text{Section 4}]\).

2.1.3. Step 3. In a third step we want to compute an integral similar to (25), but with an additional factor \([x; y]\) as defined in (23). So we want to compute the integral

\[
\frac{1}{(2\pi)^{n+m}} \int_{y \in \mathbb{H}\setminus\{x\}} [x; y] d\varphi(x, y)^m d\varphi(y, x)^n
\]

As usual, we use Stokes’ theorem

\[
\int_{y \in \mathbb{H}\setminus\{x\}} [x; y] d\varphi(x, y)^m d\varphi(y, x)^n = \int_{y \in \mathbb{H}\setminus\{x\}} d\varphi(x, y)^m d\varphi(y, x)^n - \int_{y \in \mathbb{H}\setminus\{x\}} d\varphi(x, y)^m d\varphi(y, x)^n
\]

\[
= \int_{\partial_+} \varphi(x, y)^m d\varphi(y, x)^n - \int_{\partial_-} \varphi(x, y)^m d\varphi(y, x)^n
\]

with boundaries \( \partial_+ \) and \( \partial_- \) of the integration domain depicted in Figure 6 below.
As before, we discuss the different boundary components:

- On $H_+ \cup H_-$: $d\varphi(y, x) = 0$
- On $B_\pm \cup L_\pm$: $d\varphi(y, x) = 0$
- On $C_-: \varphi(x, y) = 2\pi$
- On $C_+: \varphi(x, y) = 0$
- On $D_-: y = x + \varepsilon e^{-i\theta}$ for $\varepsilon \to 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \implies \varphi(x, y) = \frac{3\pi}{2} - \theta, \varphi(y, x) = \frac{\pi}{2} - \theta$
- On $D_+: y = x + \varepsilon e^{-i\theta}$ for $\varepsilon \to 0$ and $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \implies \varphi(x, y) = \frac{3\pi}{2} - \theta, \varphi(y, x) = \frac{5\pi}{2} - \theta$

With this we compute the integral

$$\int \left[ x; y \right] d\varphi(y, x)^m d\varphi(y, x)^n = \int_{\partial_+} \varphi(x, y)^m d\varphi(y, x)^n - \int_{\partial_-} \varphi(x, y)^m d\varphi(y, x)^n$$

$$= -n \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\frac{3\pi}{2} - \theta)^m (\frac{5\pi}{2} - \theta)^{n-1} d\theta - (2\pi)^m \pi^n + n \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\frac{3\pi}{2} - \theta)^m (\frac{\pi}{2} - \theta)^{n-1} d\theta$$

$$= \left( -2^m + \sum_{k=0}^{m} \binom{m}{k} \frac{n}{m+n-k} - \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{n}{m+n-l} \right) \pi^{m+n},$$

where we used (28) and (29) in the last step.

2.1.4. Putting everything together. Finally, we are able to compute the Kontsevich weight (19) of the graphs $\Gamma_n$ described at the beginning of Section 2.1. Integrating over $z_i$ for
\[ * = 1, \ldots, n, \text{ and applying the result for (22) obtained in the first step we get} \]
\[ (35) \]
\[
\begin{align*}
    w_{\Gamma_n} &= \frac{1}{(2\pi)^{2n+2}} \int_{C_{n+1,0}} d\varphi(x,y)d\varphi(y,x)d\varphi(z_1,x)d\varphi(z_1,y) \cdots d\varphi(z_n,x)d\varphi(z_n,y) \\
    &= \frac{1}{(2\pi)^{n+2}} \int_{y \in \mathbb{H} \setminus \{x\}} (\varphi(x,y) - \varphi(y,x) + [x;y]_n)^n d\varphi(x,y)d\varphi(y,x) \\
    &= \frac{1}{(2\pi)^{n+2}} \sum_{k=0}^{n-1} \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} (-1)^l \int_{y \in \mathbb{H} \setminus \{x\}} \varphi(x,y)^{n-k-l} \varphi(y,x)^l ([x;y]_n)^k d\varphi(x,y)d\varphi(y,x) \\
    &= \frac{1}{2n+2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} (-1)^l \pi^{n-k+2(n-k-l+1)(l+1)} \int_{y \in \mathbb{H} \setminus \{x\}} [x;y]_n^k d\varphi(x,y)^{n-k-l+1} d\varphi(y,x)^{l+1}.
\end{align*}
\]

Note that for even \( k \) we have
\[ (36) \]
\[
\int_{y \in \mathbb{H} \setminus \{x\}} [x;y]^k d\varphi(x,y)^{n-k-l+1} d\varphi(y,x)^{l+1} = \int_{y \in \mathbb{H} \setminus \{x\}} d\varphi(x,y)^{n-k-l+1} d\varphi(y,x)^{l+1} + 2^{n-k-l+1} - \sum_{r=0}^{n-k-l+1} \binom{n-k-l+1}{r} l+1 n-k-r+2 - \sum_{s=0}^{l} \binom{l}{s} \frac{l+1}{n-k-s+2} \pi^{n-k+2},
\]
where we have used (30). Similarly, for odd \( k \) we get
\[ (37) \]
\[
\int_{y \in \mathbb{H} \setminus \{x\}} [x;y]^k d\varphi(x,y)^{n-k-l+1} d\varphi(y,x)^{l+1} = \int_{y \in \mathbb{H} \setminus \{x\}} [x;y] d\varphi(x,y)^{n-k-l+1} d\varphi(y,x)^{l+1} + 2^{n-k-l+1} - \sum_{r=0}^{n-k-l+1} \binom{n-k-l+1}{r} l+1 n-k-r+2 - \sum_{s=0}^{l} \binom{l}{s} \frac{l+1}{n-k-s+2} \pi^{n-k+2},
\]
where we have used (34).

We will now simplify the expressions we got. We start by observing a few things:
First of all, we clearly have that
\[ (38) \]
\[
d\varphi(x,y)^{n-k-l+1} d\varphi(y,x)^{l+1} = -d\varphi(y,x)^{l+1} d\varphi(x,y)^{n-k-l+1}.
\]
Similarly, we also have that
\[ (39) \]
\[
[x;y] = -[y;x].
\]
Furthermore, we can obviously swap \( x \) and \( y \) in the integral and get the same result, i.e.
\[ (40) \]
\[
\int_{y \in \mathbb{H} \setminus \{x\}} [x;y]^m d\varphi(x,y)^{n-k-l+1} d\varphi(y,x)^{l+1} = \int_{x \in \mathbb{H} \setminus \{y\}} [y;x]^m d\varphi(y,x)^{n-k-l+1} d\varphi(x,y)^{l+1}.
\]
Now assume that $n$ is even. Applying (38), (39) and (40) to the last line of (35), it follows that

$$w_{\Gamma_n} = \frac{1}{2^{n+2}} \sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{\frac{n-k}{2}} \left( \frac{1}{\pi^{n-k+2} \left( \frac{n-k}{2} \right)^2} \right) \int_{y \in \mathcal{H}\setminus \{x\}} \varphi(x,y)^{\frac{n-k}{2}+1} \varphi(y,x)^{\frac{n-k}{2}+1} \, d\varphi(x,y)^m \varphi(y,x)^m = 2^m - \sum_{k=0}^{m} \binom{m}{k} \frac{m}{2m-k} - \sum_{l=0}^{m-1} \binom{m-1}{l} \frac{m}{2m-l} = 2^m - \sum_{k=0}^{m-1} \left( \binom{m}{k} + \binom{m-1}{k} \right) \frac{m}{2m-k} - 1 = 2^m - \sum_{k=0}^{m-1} \binom{m}{k} - 1 = 2^m - \sum_{k=0}^{m} \binom{m}{k} = 0.$$

Plugging this result into (41) with $m = \frac{n-k}{2}$ we finally get that

$$w_{\Gamma_n} = 0 \quad \text{for even } n \geq 0.$$

For $n$ odd the different terms in the last line of (35) do not cancel anymore. Instead, we will try to write (30) and (34) more compactly. To do this, let us introduce the so-called hypergeometric function $\mathbf{2}_1F_1(a,b;c;z)$. It is defined by the series

$$\mathbf{2}_1F_1(a,b;c;z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k$$

for $z \in \mathbb{C}$ with $|z| < 1$, where $(a)_k$ is the Pochhammer symbol given by

$$(a)_k = \begin{cases} 1, & \text{if } k = 0 \\ a(a+1) \cdots (a+k-1), & \text{if } k > 0 \end{cases}$$

It is not hard to see that the series terminates if either $a$ or $b$ is a non-positive integer. In that case the hypergeometric function reduces to a polynomial and can therefore also be defined for $|z| \geq 1$.

We can now use the hypergeometric function to write

$$\sum_{k=0}^{m} \binom{m}{k} \frac{1}{m+n-k} = \mathbf{2}_1F_1(-m,-m-n;1-m-n;-1),$$

and

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{m+n-k} = \mathbf{2}_1F_1(1-n,-m-n;1-m-n;-1).$$
This allows us to write (30) as
\[
\int_{y \in H \setminus \{x\}} d\varphi(x,y)^m d\varphi(y,x)^n \\
= \left( 2^m - \frac{n}{m+n} \left( _2F_1(-m, -m-n; 1-m-n; -1) + _2F_1(1-n, -m-n; 1-m-n; -1) \right) \right) \pi^{m+n},
\]
and (34) as
\[
\int_{y \in H \setminus \{x\}} [x;y] d\varphi(x,y)^m d\varphi(y,x)^n \\
= \left( -2^m + \frac{n}{m+n} \left( _2F_1(-m, -m-n; 1-m-n; -1) - _2F_1(1-n, -m-n; 1-m-n; -1) \right) \right) \pi^{m+n}.
\]
Plugging those results into (35) we finally get for all \( n \geq 0 \)
\[
w_{\Gamma_n} = \frac{1}{2^{n+2}} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} \frac{(-1)^l}{(n-k-l+1)(l+1)} (-1)^k 2^{n-k-l+1} \\
- \frac{l+1}{n-k+2} \left( _2F_1(-l, -n+k-2; -n+k-1; -1) \\
+ (-1)^k _2F_1(-n+k+l-1, -n+k-2; -n+k-1; -1) \right).
\]
The Kontsevich weights of the first few graphs are given in Table 1 below.

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------|---|---|---|---|---|---|---|---|---|---|
| \( w_{\Gamma_n} \) | 0 | 1/24 | 0 | 1/320 | 0 | 1/2688 | 0 | 1/18432 | 0 | 1/12649 |

Table 1. Kontsevich weights of the graphs \( \Gamma_n \) for \( n = 0, 1, \ldots, 9 \)

As a sanity check we have the following: For \( n = 0 \) the graph \( \Gamma_0 \) is just a wheel with two vertices (see Figure 3(a)) and its weight is zero according to [17, Lemma 7.3]. For \( n = 1 \) the graph \( \Gamma_1 \) is just a wheel with two spokes pointing outward (see Figure 3(b)) and its weight is \( 1/24 \) according to [21, Proposition 1.1]. So at least for \( n = 0, 1 \) our formula (50) for the Kontsevich weights \( w_{\Gamma_n} \) produces the correct values.

2.2. **Case 2: One Boundary Vertex.** We will now treat the case \((r, m) = (1, 1)\), i.e. the case where we have one boundary vertex and one \( R \)-vertex. In that case we get a family of graphs \( (\Upsilon_n)_{n \geq 0}\), where \( \Upsilon_n \) is the graph with \( n \) wedges as in Figure 7(a) (stemming from \( n \ U \phi^*_x \pi \)-vertices) attached to the graph containing a single edge from the \( R \)-vertex to the boundary vertex as in Figure 7(b) below.
Examples of the graphs \( \Upsilon_n \) are given in Figure 8 below for \( n = 0, 1, 2 \).
The Kontsevich weight of the graph $\Upsilon_n$ for $n \geq 0$ is given by
\begin{equation}
    w_{\Upsilon_n} = \frac{1}{(2\pi)^{2n+1}} \int_{C_{n+1,1}} d\varphi(z, x) d\varphi(z_1, x) d\varphi(z_1, q) \cdots d\varphi(z_n, x) d\varphi(z_n, q).
\end{equation}

Remark 2.4. As before, the ordering of the edges of the graph $\Upsilon_n$ specified in (51) above determines the sign of $w_{\Upsilon_n}$. Throughout this whole section we will stick to this ordering.

Again, the goal is to compute (51) explicitly. As before, we will do this in several steps.

2.2.1. Step 1. For a wedge as in Figure 7(a) we want to compute the corresponding integral
\begin{equation}
    \frac{1}{(2\pi)^2} \int_{z \in \mathbb{H} \setminus \{x\}} d\varphi(z, x) d\varphi(z, q),
\end{equation}
i.e. we want to integrate out $z$ (with $x, q \in \mathbb{H}$ fixed). The computation is almost the same as the one we have already done in Section 2.1.1 above: Again we make a branch cut such that $\varphi(z, x) \in (0, 2\pi)$ and use Stokes’ theorem
\begin{equation}
    \int_{z \in \mathbb{H} \setminus \{x\}} d\varphi(z, x) d\varphi(z, q) = \int_{\partial} \varphi(z, x) d\varphi(z, q),
\end{equation}
where \( \partial \) is the boundary of the integration domain depicted in Figure 9 below.

\[
\begin{align*}
\int_{\partial} d\varphi(z, x) d\varphi(z, q) &= \int_{\partial} \varphi(z, x) d\varphi(z, q) = 2\pi \int_{B_+} d\varphi(z, q) + \frac{\pi}{2} \int_{-\pi}^{0} 4\theta d\theta - 2\varphi(q, x) \int_{\pi}^{0} d\theta \\
&= 2\pi (\varphi(x, q) - \varphi(q, x) + [x; q])
\end{align*}
\]

We can then compute the integral

\[
[x; q] = \begin{cases} 
+1, & \text{if } \text{Re}(x) > q \\
-1, & \text{if } \text{Re}(x) < q
\end{cases}
\]
Dividing the result (54) by \((2\pi)^2\) we get
\[
\frac{1}{2\pi} (\varphi(x,q) - \varphi(q,x)) \pm \frac{1}{2},
\]
which agrees with the result in [21, Lemma 5.3].

Finally, observe that one obtains (54) by simply taking the limit \(y \to q \in \mathbb{R}\) in (22).

2.2.2. Step 2. In a second step we want to compute the integral
\[
\frac{1}{(2\pi)^{m+n}} \int_{C_{1,1}} \varphi(q,x)^m d\varphi(x,q)^n
\]
for \(n \geq 1\) and \(m \geq 0\).

First note that \(C_{1,1}\), shown in Figure 10 below, is a smooth manifold of dimension 1 which is homeomorphic to an open interval and \(\overline{C_{1,1}}\) is homeomorphic to a closed interval.

**Remark 2.5.** We work with the standard orientation on \(C_{1,1}\), which is induced by the standard orientation on the plane \(\mathbb{R}^2\).

![Figure 10. The manifold \(C_{1,1}\) is the product of a (fixed) single point \(q\) on the real line and an open half circle](image)

It is not hard to see that the boundary \(\partial C_{1,1}\) is just a two-element set. More precisely \(\partial C_{1,1} = \{(s,q), (t,q)\}\) with \(s < q\) and \(t > q\) (for a more detailed treatment, see [17, 12, 10]).

But now we have to make a branch cut such that \(\varphi(q,x) \in (0, 2\pi)\). Then the boundary \(\partial\) of the integration domain, depicted in Figure 11 below, contains four points, namely
\[
\partial = \{(s,q), (t,q), (q,y_+), (q,y_-)\},
\]
where \(y\) is the point on the half circle directly above \(q\), i.e. with \(\text{Re}(y) = q\), and \(y_+\) and \(y_-\) are the limits \(x \to y\) on the half circle from the left (i.e. from the region \(\text{Re}(x) < q\) of the half-circle) and from the right (i.e. from the region \(\text{Re}(x) > q\)) respectively.

Finally, using Stokes’ theorem and the fact that \(d\varphi(q,x) = 0\) for \(q \in \mathbb{R}\), we get
\[
\int_{C_{1,1}} \varphi(q,x)^m d\varphi(x,q)^n = \int_{\partial} \varphi(q,x)^m \varphi(x,q)^n
\]
\[
= \varphi(q,s)^m \varphi(s,q)^n - \varphi(q,y_+)^m \varphi(y_+,q)^n + \varphi(q,y_-)^m \varphi(y_-,q)^n - \varphi(q,t)^m \varphi(t,q)^n
\]
\[
= \begin{cases} (2\pi)^n, & \text{if } m = 0 \\
2^m \pi^{m+n}, & \text{if } m > 0 \end{cases}
\]
2.2.3. Step 3. Let us start with writing (54) differently as

$$2\pi(\varphi(x,q) - \varphi(q,x) + \pi[x;q]) = 2\pi(\varphi(x,q) - \varphi(q,x) - \pi + 2\pi(x;q))$$

where

$$\left(x; q\right) = \begin{cases} +1, & \text{if } \Re(x) > q \\ 0, & \text{if } \Re(x) < q \end{cases}$$

In this step we then want to compute an integral similar to (57), but with an additional factor \( (x; y) \) as defined above. So we want to compute

$$\frac{1}{(2\pi)^{m+n}} \int_{C_{1,1}} (x; q)\varphi(x, q)^m d\varphi(x, q)^n$$

for \( n \geq 1 \) and \( m \geq 0 \).

As before we use Stokes’ theorem and find that

$$\int_{C_{1,1}} (x; q)\varphi(x, q)^m d\varphi(x, q)^n = \int_{C_{1,1}}^{\Re(x) > q} \varphi(q, x)^m d\varphi(x, q)^n$$

$$= \varphi(q, y)^m \varphi(y, q)^n - \varphi(q, t)^m \varphi(t, q)^n = 2^m \pi^{m+n}$$

for all \( m \geq 0 \) and all \( n \geq 1 \).

2.2.4. Putting everything together. Now we can use the results from steps 1-3 to compute the Kontsevich weight \((51)\) of the graphs \(\Upsilon_n\), \(n \geq 0\), described at the beginning of Section 2.2.
Integrating over $z_i$ for $i = 1, \ldots, n$, and applying (60), we get

\[
w_{\mathcal{Y}_n} = \frac{1}{(2\pi)^{2n+1}} \int_{C_{n+1,1}} d\varphi(x, q) d\varphi(z_1, x) d\varphi(z_1, q) \cdots d\varphi(z_n, x) d\varphi(z_n, q)
\]

\[
= \frac{1}{(2\pi)^{n+1}} \int_{C_{n+1,1}} (\varphi(x, q) - \varphi(q, x) - \pi + 2\pi(x; q))^n d\varphi(x, q)
\]

\[
= \frac{1}{(2\pi)^{n+1}} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \sum_{s=0}^{n-k-l} \binom{n}{k} \binom{n-k}{l} \binom{n-k-l}{s} (-1)^{l+s} \int_{C_{n+1,1}} \varphi(x, q)^{n-k-l-s} \varphi(q, x)^s (2\pi(x; q))^k d\varphi(x, q)
\]

\[
= \sum_{k=0}^{n} \sum_{l=0}^{n-k} \sum_{s=0}^{n-k-l} \binom{n}{k} \binom{n-k}{l} \binom{n-k-l}{s} \frac{(-1)^{l+s}}{2^{n-k+1}\pi^{n-k-l+1}(n-k-l-s+1)} \int_{C_{n+1,1}} (x; q)^k \varphi(q, x)^s d\varphi(x, q)^{n-k-l-s+1}.
\]

(64)

We note that for $k = 0$ we have

\[
\int_{C_{n+1,1}} \varphi(q, x)^s d\varphi(x, q)^{n-l-s+1} = \begin{cases} 
(2\pi)^{n-l+1}, & \text{if } s = 0 \\
2^s\pi^{n-l+1}, & \text{if } s > 0
\end{cases}
\]

(65)

where we have used (59). Similarly, for $k \geq 1$ we have

\[
\int_{C_{n+1,1}} (x; q)^k \varphi(q, x)^s d\varphi(x, q)^{n-k-l-s+1} = \int_{C_{n+1,1}} (x; q) \varphi(q, x)^s d\varphi(x, q)^{n-k-l-s+1} = 2^s\pi^{n-k-l+1},
\]

(66)

where we have used (63).

Plugging the above two results into the last line of (64) we get

\[
w_{\mathcal{Y}_n} = \sum_{k=0}^{n} \sum_{l=0}^{n-k} \sum_{s=0}^{n-k-l} \binom{n}{k} \binom{n-k}{l} \binom{n-k-l}{s} \frac{(-1)^{l+s}}{2^{n-k-s+1}(n-k-l-s+1)} =: A(n)
\]

(67)

\[- \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^{l}}{2^{n+1}(n-l+1)} + \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^{l}}{2^{l(n-l+1)}} =: B(n) \quad \text{and} \quad C(n)\]

where we have used (63).
As shown in Appendix A, we have that
\[ A(n) = \frac{(-1)^n}{2^{n+1}(n + 1)}, \]
\[ B(n) = \frac{(-1)^n}{2^{n+1}(n + 1)}, \]
\[ C(n) = 1 + \frac{(-1)^n}{2^{n+1}(n + 1)}. \]

Hence, we finally have
\[ w_{\Upsilon_n} = \frac{1 + (-1)^n}{2^{n+1}(n + 1)}, \quad n \geq 0. \]

(69)

In particular, we see that
\[ w_{\Upsilon_n} = 0 \quad \text{for odd } n \geq 1. \]

(70)

The Kontsevich weights of the first few graphs are given in Table 2 below.

| n  | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
|----|----|----|----|----|----|----|----|----|----|----|
| \( w_{\Upsilon_n} \) | 1  | 0  | \( \frac{1}{12} \) | 0  | \( \frac{1}{80} \) | 0  | \( \frac{1}{348} \) | 0  | \( \frac{1}{2304} \) | 0  |

Table 2. Kontsevich weights of the graphs \( \Upsilon_n \) for \( n = 0, 1, \ldots, 9 \)

As a sanity check we have the following: For \( n = 0 \) the graph \( \Upsilon_0 \) is just a single edge as in Figure 7(b) and its weight is zero according to [17, Section 6.4.3]. For \( n = 1 \) the graph \( \Upsilon_1 \) is just a single edge with one wedge attached as in Figure 8(b) and its weight is 0 according to [14, Appendix B]. For \( n = 2 \) the graph \( \Upsilon_2 \) is a single edge with two wedges attached as in Figure 8(c) and its weight is \( \frac{1}{12} \) according to [21, Appendix A]. So at least for \( n = 0, 1, 2 \) our formula (69) for the Kontsevich weights \( w_{\Upsilon_n} \) produces the correct values.

2.3. Case 3: Two Boundary Vertices. Finally, we will treat the case \((r, m) = (0, 2)\), i.e. the case where we have two boundary vertices and no \( R \)-vertex. In that case we get a family of graphs \( (\Lambda_n)_{n \geq 0} \), where \( \Lambda_n \) is the graph with \( n \) wedges as in Figure 12 (stemming from \( n \) \( T \phi^+ \pi \)-vertices) attached to the two boundary vertices.

![Figure 12](image-url)

Figure 12. Graphs in the case \((r, m) = (0, 2)\) consist of wedges attached to the two boundary vertices

Examples of the graphs \( \Lambda_n \) are given in Figure 13 below for \( n = 0, 1, 2 \).
The Kontsevich weight of the graph $\Lambda_n$ for $n \geq 0$ is given by

$$w_{\Lambda_n} = \frac{1}{(2\pi)^{2n}} \int_{C_{n,2}} d\varphi(z_1,p)d\varphi(z_1,q) \cdots d\varphi(z_n,p)d\varphi(z_n,q).$$

(71)

**Remark 2.6.** For $n = 0$ we simply set $d\varphi(z_1,p)d\varphi(z_1,q) \cdots d\varphi(z_n,p)d\varphi(z_n,q) = 1$ in the integral above.

**Remark 2.7.** As before, the ordering of the edges of the graph $\Lambda_n$ specified in (71) above determines the sign of $w_{\Lambda_n}$. Throughout this whole section we will stick to this ordering.

**Remark 2.8.** Since we work with the configuration space $\text{Conf}_{n,m}$ as in (13), and in particular with the quotient $C_{n,m} = \text{Conf}_{n,m}/G^{(1)}$, it follows that $C_{0,2}$ is a single point (and not a two-element set).

Finally, our goal is to compute (51) explicitly for the given family of graphs. However, this time the computation is much easier and shorter than before. For the boundary vertices $p, q \in \mathbb{R}$ with $p < q$ we have already computed the Kontsevich weight of a wedge as in Figure 12 at the end of Section 2.1.1. Our result was

$$\frac{1}{(2\pi)^2} \int_{C_{1,2}} d\varphi(z,p)d\varphi(z,q) = \frac{1}{2}.$$  

(72)

For the sake of completeness and to make sure that we get the same result, let us nonetheless do a direct computation. For a wedge as in Figure 12, we compute the corresponding integral

$$\frac{1}{(2\pi)^2} \int_{\mathbb{H}} d\varphi(z,p)d\varphi(z,q),$$

(73)

with $p, q \in \mathbb{R}$, $p < q$ fixed. As before, we use Stokes’ theorem

$$\int_{\mathbb{H}} d\varphi(z,p)d\varphi(z,q) = \int_{\partial} \varphi(z,p)d\varphi(z,q),$$

(74)

where $\partial$ is the boundary of the integration domain depicted in Figure 14 below. As usual, let us have a look at the different boundary components:
Figure 14. Boundary $\partial$ of the integration domain: $C$ is the half-circle at infinity, the half circles $C_1$ and $C_2$ have infinitesimal radius and $H_1 \cup H_2 \cup H_3$ is the real line.

- On $H_1 \cup H_2 \cup H_3$: $z \in \mathbb{R}$ and hence $d\varphi(z, q) = 0$
- On $C_1$: $z = p + \epsilon e^{-i\theta}$ for $\epsilon \to 0 \implies d\varphi(z, q) = \text{darg}(1) = 0$
- On $C_2$: $z = q + \epsilon e^{-i\theta}$ for $\epsilon \to 0 \implies \varphi(z, p) \to \varphi(q, p) = 0$
- On $C$: $z = Re^{i\theta}$ for $R \to \infty$ and $\theta \in [0, \pi] \implies \varphi(z, p) = \varphi(z, q) = 2\theta$

We can then compute the integral

$$\int_{z \in \mathbb{H}} d\varphi(z, p) d\varphi(z, q) = \int_{\partial} \varphi(z, p) d\varphi(z, q) = \int_{0}^{\pi} 4\theta d\theta = 2\pi^2,$$

which indeed agrees with (72) after dividing by $(2\pi)^2$.

With this result at hand, it is now easy to compute the Kontsevich weight of the graph $\Lambda_n$ for $n \geq 1$. We get

$$w_{\Lambda_n} = \frac{1}{(2\pi)^2n} \int_{C_{n,2}} d\varphi(z_1, p) d\varphi(z_1, q) \cdots d\varphi(z_n, p) d\varphi(z_n, q) = \frac{1}{(2\pi)^2n} (2\pi^2)^n = \frac{1}{2^n}.$$  

For $n = 0$ it is not hard to see that

$$w_{\Lambda_0} = \int_{C_{0,2}} 1 = 1,$$
since \(C_{0,2}\) is a single point (cf Remark 2.8). So all in all we finally have
\[
(78) \quad w_{\Lambda_n} = \frac{1}{2n}, \quad n \geq 0.
\]

2.4. **Another Approach for the Explicit Computation of \(w_{\Upsilon_n}\) and \(w_{\Upsilon_{n'}}\).** We want to give a more fast and explicit approach for the computation of the weights \(w_{\Upsilon_n}\) and \(w_{\Upsilon_{n'}}\). The following approach has the advantage that it doesn’t require the use of the hypergeometric function for \(w_{\Upsilon_n}\) but rather gives an explicit expression in terms of \(w_{\Upsilon_{n'}}\). First note that
\[
(79) \quad d\varphi(z, x) = \frac{1}{4\pi i} d \log \left( \frac{(z - x)(z - \bar{x})}{(\bar{z} - x)(\bar{z} - \bar{x})} \right), \quad \forall z, x \in \mathbb{H} \cup \mathbb{R}.
\]
Moreover, recall from [2, Lemma 5.3] the formula
\[
(80) \quad \int_{z \in \mathbb{H}} d\varphi(z, x) \wedge d\varphi(z, y) = \frac{1}{2\pi i} \log \left( \frac{x - \bar{y}}{y - \bar{x}} \right), \quad \forall x, y \in \mathbb{H} \cup \mathbb{R}.
\]
Integrating out the \(z_i\) variables in \(w_{\Upsilon_n}\) using (80), we get
\[
(81) \quad w_{\Upsilon_n} = \int_{C_{1,1}} \left( \frac{1}{2\pi i} \log \left( \frac{x - p}{p - \bar{x}} \right) \right)^n \frac{1}{2\pi i} d \log \left( \frac{x - p}{p - \bar{x}} \right) = \frac{1}{n + 1} \int_{C_{1,1}} \left( \frac{1}{2\pi i} d \log \left( \frac{x - p}{p - \bar{x}} \right) \right)^{n+1}
\]
because \(d \log (\bar{x} - p) = d \log (p - \bar{x})\). Now recall that \(C_{1,1} = (\mathbb{H} \times \mathbb{R})/(\mathbb{R}^+ \ltimes \mathbb{R})\) is isomorphic to \(\mathbb{R}\); for instance every point in the quotient can be represented uniquely by a pair \((i, p)\) with \(i\) the imaginary unit and \(p \in \mathbb{R}\). Hence we get
\[
(82) \quad w_{\Upsilon_n} = \frac{1}{n + 1} \int_{-\infty}^{\infty} dp \left( \frac{1}{2\pi i} \log \left( \frac{i - p}{p + i} \right) \right)^{n+1} \bigg|_{p = -\infty}^{p = \infty}
\]
\[
= \left( \frac{1}{2\pi i} \log \left( \frac{i - p}{p + i} \right) \right)^{n+1} \bigg|_{p = -\infty}^{p = \infty}
\]
\[
= \frac{1 + (-1)^n}{2^{n+1}(n + 1)},
\]
where we have used the boundary values
\[
(83) \quad \lim_{p \to \pm \infty} \log \left( \frac{i - p}{p + i} \right) = \lim_{p \to \pm \infty} \log \left( -1 + \frac{2i}{p} + O(1/p^2) \right) = \pm i\pi.
\]
Now, using (80) and integrating out the \(z_i\) variables of \(w_{\Upsilon_n}\), we get
\[
(84) \quad w_{\Upsilon_n} = \int_{C_{2,0}} \left( \frac{1}{2\pi i} \log \left( \frac{x - \bar{y}}{z - \bar{x}} \right) \right)^n d\varphi(x, y) \wedge d\varphi(y, x).
\]
Note that \(d\varphi(x, y) = d\varphi(y, x) + \frac{1}{2\pi i} d \log \left( \frac{x - y}{y - \bar{x}} \right)\), and thus
\[
(85) \quad w_{\Upsilon_n} = \int_{C_{2,0}} \left( \frac{1}{2\pi i} \log \left( \frac{x - \bar{y}}{y - \bar{x}} \right) \right)^n \frac{1}{2\pi i} d \log \left( \frac{x - \bar{y}}{y - \bar{x}} \right) \wedge d\varphi(y, x)
\]
\[
= \frac{1}{n + 1} \int_{C_{2,0}} d \left[ \left( \frac{1}{2\pi i} \log \left( \frac{x - \bar{y}}{y - \bar{x}} \right) \right)^{n+1} d\varphi(y, x) \right].
\]
By Stokes’ theorem, the integral is reduced to the boundary of Kontsevich’s eye $\mathcal{C}_{2,0}$ [17, Section 5.2]. This boundary has three components:

- The iris of the eye which is isomorphic to $S^1$, corresponding to the collision $x \to y$. This gives zero contribution, since $\log \left( \frac{x-y}{y-x} \right) = 0$ for $x = y$.

- The upper eyelid, corresponding to the limit $|x| \to \infty$. This gives no contribution, since $d\varphi(y, x) = 0$ in the limit.

- The lower eyelid, corresponding to the collision $x \to \bar{x}$ onto the real line. The lower eyelid is isomorphic to $C_{1,1}$ and by comparison with the integrand of $w_{\Upsilon_{n+1}}$ in (81) with $p := x = \bar{x}$, we get

$$w_{\Gamma_{n}} = \frac{1}{n+1} w_{\Upsilon_{n+1}} = \frac{1 - (-1)^n}{2^{n+2}(n+1)(n+2)}.$$

One can easily check that this formula produces the same values as in Table 1.

3. Including the Weights

3.1. The Product $P(T\phi^*\pi)$. Let $\sigma, \tau \in \Gamma(\widehat{\text{Sym}}(T^*M)[[\hbar]])$ be sections and let $x \in M$. Using the Kontsevich weights computed above, we get

$$P(T\phi^*_x\pi)(\sigma_x \otimes \tau_x) = \sum_{n=0}^{\infty} \frac{\hbar^n}{2^{2n}n!} (T\phi^*_x\pi)^{i_1j_1} \cdots (T\phi^*_x\pi)^{i_nj_n} (\sigma_x)_{i_1 \cdots i_n} (\tau_x)_{j_1 \cdots j_n}$$

where we sum over all the indices $i_1, \ldots, i_n, j_1, \ldots, j_n$. Moreover, we use the notation, where the indices on the right of the comma denote derivatives with respect to the corresponding variable, e.g.

$$R_{i_1i_2 \cdots i_k} := \partial_{i_1} \cdots \partial_{i_k} R_i.$$

3.2. The Connection 1-form $A(R, T\phi^*\pi)$. In Section 2.2 we have obtained the Kontsevich weights

$$w_{\Upsilon_{n}} = \frac{1 + (-1)^n}{2^{n+1}(n+1)}, \quad n \geq 0$$

of the family of graphs $(\Upsilon_{n})_{n \geq 0}$. Let $\sigma \in \Gamma(\widehat{\text{Sym}}(T^*M)[[\hbar]])$ be a section and fix $x \in M$. For $R$ as in Section 1.2 we set $R_x(y) := R(x, y)$ and $(R_x)^k(y) := R^k(x, y)$. Using the Kontsevich weights above, we get

$$A(R_x, T\phi^*_x\pi)(\sigma_x) = dx^i A \left( (R_x)_i^k \frac{\partial}{\partial y^k}, T\phi^*_x\pi \right)(\sigma_x)$$

$$= dx^i \sum_{n=0}^{\infty} \frac{\hbar^n}{2^{2n}n!} \frac{1 + (-1)^n}{2^{n+1}(n+1)} (T\phi^*_x\pi)^{i_1j_1} \cdots (T\phi^*_x\pi)^{i_nj_n} (R_x)^{k}_{i_1 \cdots i_n} (\sigma_x)_{j_1 \cdots j_n}$$

where we again sum over all indices $i, k, i_1, \ldots, i_n, j_1, \ldots, j_n$.

This allows us to write down an explicit expression for the deformed Grothendieck connection,
namely

\begin{equation}
(D_G)_x = dx + A(R_x, T\phi_x^\pi) = \left( \frac{\partial}{\partial x^i} + A((R_x)_i, T\phi_x^\pi) \right) dx^i
\end{equation}

\begin{equation}
= \left( \frac{\partial}{\partial x^i} + \sum_{n=0}^{\infty} \frac{\hbar^n}{2^n n! 2^{n+1}(n+1)} (T\phi_x^\pi)^{i_1j_1} \cdots (T\phi_x^\pi)^{i_nj_n} (R_x)^k_{i_1i_2\cdots i_n} \frac{\partial^{n+1}}{\partial y^{j_n} \cdots \partial y^j} \right) dx^i
\end{equation}

3.3. The Curvature 2-form $F(R, R, T\phi^\pi)$. In Section 2.1 we have obtained the Kontsevich weights in terms of the hypergeometric function and gave a more explicit formula in Section 2.4

\begin{equation}
w_{\Gamma_n} = \frac{1 - (-1)^n}{2^{n+2}(n+1)(n+2)}, \quad n \geq 0
\end{equation}

for the family of graphs $(\Gamma_n)_{n \geq 0}$. Using these weights above we then get for $x \in M$

\begin{equation}
F(R_x, R_x, T\phi_x^\pi) = dx^i \wedge dx^j F((R_x)_i, (R_x)_j, T\phi_x^\pi)
\end{equation}

\begin{equation}
= dx^i \wedge dx^j \sum_{n=0}^{\infty} \frac{\hbar^n}{2^n n! 2^{n+2}(n+1)(n+2)} (T\phi_x^\pi)^{i_1j_1} \cdots (T\phi_x^\pi)^{i_nj_n} (R_x)^{k_{i_1i_2\cdots i_n}} (R_x)_{j_{j_1j_2\cdots j_n}}
\end{equation}

where, as usual, we sum over the indices $i, j, k, l, i_1, \ldots, i_n, j_1, \ldots, j_n$.  

3.4. A Fedosov-type Equation for Poisson Manifolds. We can now write down the modified deformed Grothendieck connection as defined in [5, 12], namely

\begin{equation}
\overline{D}_G = D_G + [\gamma, \_],
\end{equation}

where the deformed Grothendieck connection $D_G$ is explicitly given by (91), the star product is explicitly given by (87), $[\_ , \_ ]_*$ denotes the star commutator and $\gamma \in \Omega^1(M, \widehat{\text{Sym}}(T^*M)[[\hbar]])$ is such that the following Fedosov-type equation holds:

\begin{equation}
F^M + D_G \gamma + \gamma \star \gamma = 0,
\end{equation}

with Weyl curvature $F^M = F(R, R, T\phi^\pi)$ explicitly given by (93). This equation appears in the globalization construction for deformation quantization of Poisson manifolds. The existence of such a $\gamma$ was given in [5, 6]. Let us emphasize a bit more on this existence result. Since $\gamma$ takes values in $\text{Sym}(T^*M)[[\hbar]]$ we may write

\begin{equation}
\gamma = \gamma_0 + \hbar \gamma_1 + \hbar^2 \gamma_2 + \ldots
\end{equation}

Similarly, for the deformed Grothendieck connection we may write

\begin{equation}
D_G = D_G + \hbar^2 D_2 + \hbar^4 D_4 + \ldots
\end{equation}

where $D_G = d + L_R$ is the classical Grothendieck connection and where we have used that the Kontsevich weights (89) satisfy $w_{\Gamma_0} = 1$ and $w_{\Gamma_n} = 0$ for all odd $n \geq 1$. Finally, for the curvature we can write

\begin{equation}
F^M = \hbar F_1 + \hbar^3 F_3 + \hbar^5 F_5 + \ldots
\end{equation}

where we have used that the Kontsevich weights (92) satisfy $w_{\Gamma_n} = 0$ for all even $n \geq 0$. This now allows us to decompose Equation (95) into a system of equations depending on the
order of $h$.
In order $h^0$ we get the equation
\begin{equation}
D_G \gamma_0 = 0,
\end{equation}
which, according to Section 1.2, can be solved by $\gamma_0 = T\phi^* f$ for some smooth function $f \in C^\infty(M)$, since the cohomology of the classical Grothendieck connection $H^*_D G(\Gamma(\text{Sym}(T^*M)))$ is concentrated in degree zero (by the Poincaré Lemma) and
\begin{equation}
H^0_D G(\Gamma(\text{Sym}(T^*M))) \cong T\phi^* C^\infty(M) \cong C^\infty(M).
\end{equation}
In order $h^1$ we get the equation
\begin{equation}
F_1 + D_G \gamma_1 + (\gamma_0 \star \gamma_0)_1 = 0.
\end{equation}
Using the Bianchi identity we see that $D_G F_1 = 0$ and by Equation (99) it also immediately follows that $D_G (\gamma_0 \star \gamma_0)_1 = 0$. So we get that $D_G \gamma_1$ is equal to a $D_G$-closed form, but the corresponding cohomology group is trivial. Hence it follows that $D_G \gamma_1$ is equal to a $D_G$-exact form, and thus it is possible to find a $\gamma_1$ that solves Equation (101).
By induction, one can show that in each order $h^k$ for $k \geq 1$, $D_G \gamma_k$ is equal to a $D_G$-closed and hence $D_G$-exact form depending on the lower order coefficients of $F^M$ and $\gamma$. In particular it follows that there exists a $\gamma_k$ solving the equation for the corresponding order.

**Remark 3.1.** Note that in order to globalize Kontsevich’s star product one may be tempted to define a bullet product
\begin{equation}
(f \bullet g)(x) := (P(T\phi^* \pi)(T\phi^* f \otimes T\phi^* g))(x; 0).
\end{equation}
This is indeed a well-defined global product on $C^\infty(M)[[h]]$, but it is in general not associative. To make this product associative one has to introduce a quantization map (see e.g. [5])
\begin{equation}
\rho: H^0_D G(\Gamma(\text{Sym}(T^*M))) \to H^0_D G(\Gamma(\text{Sym}(T^*M)[[h]]))
\end{equation}
which then again leads to the global star product
\begin{equation}
f \star_M g := (\rho^{-1}(\rho(T\phi^* f) \star \rho(T\phi^* g)))_{g=0}.
\end{equation}
Here $\star$ denotes Kontsevich’s star product and $\star_M$ its global version on $M$. Using the weights, we can also get an explicit expression for the bullet product (102) by
\begin{equation}
(P(T\phi^* \pi)(T\phi^* f \otimes T\phi^* g))(x; 0)
= \left(\sum_{n=0}^{\infty} \frac{h^n}{2^n n!} (T\phi_x^* \pi)^{i_1 j_1} \cdots (T\phi_x^* \pi)^{i_n j_n} (T\phi_x^* f)_{j_1 \cdots i_n} (T\phi_x^* g)_{j_1 \cdots j_n} \right)(0).
\end{equation}

**3.5. The Lifted Curvature 2-form $F(\mathcal{R}, \mathcal{R}, T\phi^* \pi)$.** Let $M$ be a smooth manifold and let $\phi: TM \to M$ be a formal exponential map and consider the lift $\overline{\phi}: TN \to N$ to the cotangent bundle $N = T^*M$. We set $x = (q, p) \in N$ and $y = (\bar{q}, \bar{p}) \in T_x N$. Note that this is a particular case of a canonical symplectic manifold.
We will consider the lifted vector fields $\mathcal{R}$ to the cotangent case, which induce lifted interaction vertices within the Feynman graphs which appear in the computation of the connection 1-form
and its curvature 2-form and see how these terms simplify. First we note that $A(\overline{R}, T\phi^* \pi)$ is still given by
\begin{equation}
A(\overline{R}_x, T\phi^*_x \pi)(\sigma_x) = dx^i \sum_{n=0}^\infty \frac{h^n}{2^n n!} \frac{1 + (-1)^n}{2^{n+1}(n + 1)} (T\phi^*_x \pi)^{i_1j_1} \cdots (T\phi^*_x \pi)^{i_nj_n} (\overline{R}_x)^{k_{i_1 \cdots i_n}}_{j_1 \cdots j_n},
\end{equation}
The simplification in this case is a small one: All summands containing a term $(\overline{R}_x)^{k_{i_1 \cdots i_n}}_{j_1 \cdots j_n}$ with more than one derivative with respect to $\overline{p}$ will vanish [18].

For the case of the curvature 2-form $F^N$ the simplification is more interesting. Since for each non-vanishing coefficient $(T\phi^*_x \pi)^{ij}$ one of the two outgoing edges is always representing a $\overline{q}$-derivative and the other corresponding edge representing a $\overline{p}$-derivative (since we work with Darboux coordinates around $x \in N$), we see that the sum in (93) terminates at $n = 2$. Or put differently, we only have to consider the graphs $\Gamma_n$ up to $n = 2$, i.e. with at most two wedges attached to the wheel consisting of two $\overline{R}$-vertices (cf. Figure 2 in Section 2.1). Moreover, since the Kontsevich weights (92) are, up to $n = 2$, given by $w_{\Gamma_0} = 0$, $w_{\Gamma_1} = \frac{1}{24}$ and $w_{\Gamma_2} = 0$, we get
\begin{equation}
F^N_x = F(\overline{R}_x, \overline{R}_x, T\phi^*_x \pi) = \frac{h}{48} (T\phi^*_x \pi)^{rs} (\overline{R}_x)^k_{i_1\cdots i_n} (\overline{R}_x)^l_{j_1\cdots j_n} dx^i \wedge dx^j,
\end{equation}
where we sum over the indices $i, j, r, s, k, l$ and where again summands containing a term $(\overline{R}_x)^{k_{i_1 \cdots i_n}}_{j_1 \cdots j_n}$ with more than one derivative with respect to $\overline{p}$ vanish. So in the case of a cotangent bundle we get a much simpler expression for the Weyl curvature $F^N$.

Appendix A. Binomial Sums

Here we will treat the binomial sums appearing in the expression (67) and show that we indeed get the results stated in (68).

A.1. $B(n)$. Let us start with
\begin{equation}
B(n) = \sum_{l=0}^n \binom{n}{l} \frac{(-1)^l}{2^n l! (n - l + 1)^2}.
\end{equation}
Using the well known identity
\begin{equation}
\sum_{l=0}^n (-1)^l \binom{n + 1}{l} = \sum_{l=0}^n (-1)^l \binom{n}{l} \frac{n + 1}{n - l + 1} = (-1)^n
\end{equation}
it immediatley follows that
\begin{equation}
B(n) = \frac{(-1)^n}{2^{n+1}(n + 1)}.\)

A.2. $C(n)$. Let us continue with
\begin{equation}
C(n) = \sum_{l=0}^n \binom{n}{l} \frac{(-1)^l}{2^l (n - l + 1)}.\)

Using the identity (109), we can write

\[(n + 1)C(n) = \sum_{l=0}^{n} \binom{n+1}{l} \left(-\frac{1}{2}\right)^l.\]

Using the Binomial theorem we then find

\[\sum_{l=0}^{n} \binom{n+1}{l} \left(-\frac{1}{2}\right)^l = \left(\frac{1}{2}\right)^{n+1} - \left(-\frac{1}{2}\right)^{n+1},\]

and hence

\[C(n) = \frac{1 + (-1)^n}{2^{n+1}(n+1)}.\]

A.3. \(A(n)\). Finally, let us treat the case

\[A(n) = \sum_{k=0}^{n} \sum_{l=0}^{n-k} \sum_{s=0}^{n-k-l} \binom{n}{k} \binom{n-k}{l} \binom{n-k-l}{s} (-1)^{l+s} 2^{n-k-s+1} (n-k-l-s+1).\]

Write

\[A(n) = \sum_{k=0}^{n} \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} (-1)^{l} \sum_{s=0}^{n-k-l} \binom{n-k-l}{s} (-2)^s (n-k-l-s+1).\]

We first treat the innermost sum: Set \(m = n - k - l\). Then

\[\sum_{s=0}^{n-k-l} \binom{n-k-l}{s} (-2)^s = \sum_{s=0}^{m} \binom{m}{s} (-2)^s.\]

Using \(\binom{m+1}{s} = \binom{m}{s} \binom{m+1}{s+1}\), we find that

\[\sum_{s=0}^{m} \binom{m}{s} (-2)^s = \frac{1}{m+1} \sum_{s=0}^{m} \binom{m+1}{s} (-2)^s.\]

Applying the Binomial theorem we get that

\[\sum_{s=0}^{m} \binom{m+1}{s} (-2)^s = (-1)^{m+1}(1 - 2^{m+1}).\]

Plugging all of this into (116), we find

\[A(n) = \sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{l} (-1)^{l} \binom{n-k-l+1}{n-k-l+1} 2^{n-k+1} (1 - 2^{n-k-l+1})\]

\[= \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n-k+1}}{2^{n-k+1}} \sum_{l=0}^{n-k} \binom{n-k}{l} \frac{1}{n-k-l+1} (1 - 2^{n-k-l+1}).\]

Moreover, we have

\[\sum_{l=0}^{n-k} \binom{n-k}{l} \frac{1}{n-k-l+1} = \frac{1}{n-k+1} \sum_{l=0}^{n-k} \binom{n-k+1}{l} = \frac{1}{n-k+1} (2^{n-k+1} - 1),\]
and
\[ \sum_{l=0}^{n-k} \binom{n-k}{l} \frac{2^{n-k-l+1}}{n-k-l+1} = \frac{1}{n-k+1} \sum_{l=0}^{n-k} \binom{n-k+1}{l} 2^{n-k-l+1} = \frac{1}{n-k+1} (3^{n-k+1} - 1). \]

Hence
\begin{align*}
A(n) &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k+1} \frac{2^{n-k+1}}{n-k+1} (3^{n-k+1} - 3^{n-k+1}) \\
&= (-1)^{n+1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{k}}{n-k+1} \left( 1 - \left( \frac{3}{2} \right)^{n-k+1} \right). 
\end{align*}

(123)

Now
\[ \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{k}}{n-k+1} = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} (-1)^{k} = \frac{(-1)^{n}}{n+1}, \]

and
\begin{align*}
\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{k}}{n-k+1} \left( \frac{3}{2} \right)^{n-k+1} &= \frac{(-1)^{n+1}}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} \left( -\frac{3}{2} \right)^{n-k+1} \\
&= \frac{1}{2^{n+1}(n+1)} + \frac{(-1)^{n}}{n+1}.
\end{align*}

(125)

Finally, we get
\[ A(n) = (-1)^{n+1} \left( \frac{(-1)^{n}}{n+1} - \frac{1}{2^{n+1}(n+1)} - \frac{(-1)^{n}}{n+1} \right) = \frac{(-1)^{n}}{2^{n+1}(n+1)}. \]

(126)

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