Balanced Boolean functions that can be evaluated so that every input bit is unlikely to be read

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Abstract

A Boolean function of \( n \) bits is balanced if it takes the value 1 with probability \( \frac{1}{2} \). We exhibit a balanced Boolean function with a randomized evaluation procedure (with probability 0 of making a mistake) so that on uniformly random inputs, no input bit is read with probability more than \( \Theta\left(\frac{n^{-1/2}}{\sqrt{\log n}}\right) \). We give a balanced monotone Boolean function for which the corresponding probability is \( \Theta\left(n^{-1/3} \log n\right) \). We then show that for any randomized algorithm for evaluating a balanced Boolean function, when the input bits are uniformly random, there is some input bit that is read with probability at least \( \Theta\left(n^{-1/2}\right) \). For balanced monotone Boolean functions, there is some input bit that is read with probability at least \( \Theta\left(n^{-1/3}\right) \).

1 Results

Suppose that a randomized algorithm evaluates a Boolean function \( f \) on \( n \) input bits. When the input bits are uniformly random, let \( \delta_i = \Pr[\text{bit } i \text{ gets read}] \), where the probability is over the randomness of the input as well as the internal randomness of the algorithm. Let \( \delta = \max_i \delta_i \) be the maximum probability that a particular bit is read. How small can \( \delta \) be for a balanced Boolean function (one that takes on the values 0 and 1 equally often)? What if the Boolean function is monotone (i.e., \( f(x) \geq f(y) \) whenever \( x_i \geq y_i \) for all \( i = 1, 2, \ldots, n \))?

An obvious lower bound for \( \delta_i \) is \( I_i(f) \), the influence of the bit \( i \). Recall that the influence \( I_i = I_i(f) \) of the \( i \)th input bit on \( f \) is defined to be the probability, for a uniformly random input, that changing the bit changes the value of the function. Many readers are familiar with the majority function and the fact that the influence of each bit is small, only \( O\left(n^{-1/2}\right) \). But on most inputs the numbers of 0’s and 1’s are nearly balanced, and any algorithm that reliably evaluates majority will typically read \( \Theta(n) \) of the input bits, so that \( \delta = \Theta(1) \).

There are a couple of other balanced Boolean functions that some readers may think of. The tribes function on \( n = m 2^m \) input bits is defined by partitioning the set of bits into “tribes” of size \( m \), and the function takes the value 1 if and only if there is at least one tribe where all the bits are 1. The influence of each bit on the tribes function is only \( \Theta(\log n/n) \) \textbf{[BOL89]} (which is as small as \( \max_i I_i(f) \) can be \textbf{[KKL88]}), but any algorithm that reliably evaluates the tribes function will typically read \( \Theta(n/\log n) \) of the input bits,
so that $\delta = \Theta(1/\log n)$. For the dictatorship function, where one bit determines the output, very few bits need to be read, but the dictator bit needs to be read, so $\delta = 1$. The tribes function and the majority are two examples of symmetric functions; they are invariant under a group acting transitively on the input bits. Many other boolean functions of interest are symmetric. Examples include recursive majority and percolation crossings in a torus. For symmetric functions there is an algorithm $A$ computing $f$ that reads on average $m$ bits if and only if there is an algorithm $A'$ computing $f$ with $\delta \leq m/n$. (To go from $A$ to $A'$, one just permutes the bits by a uniformly-random element from the automorphism group before applying $A$. The other direction is obvious, since the expected number of bits that $A'$ reads is at most $\delta n$.) Thus, for symmetric functions, estimates on $\delta$ are equivalent to estimates on the expected number of bits that need to be read.

In the next section we give some nearly optimal constructions of Boolean functions that may be evaluated so that every input bit is unlikely to be read:

**Theorem 1.**

1. There is a balanced Boolean function with an algorithm that always correctly evaluates it on any input and for which $\delta = \Theta(n^{-1/2} \sqrt{\log n})$.

2. There is a balanced Boolean function with an algorithm that correctly evaluates it on most inputs most of the time and for which $\delta = \Theta(n^{-1/2})$.

3. There is a balanced monotone Boolean function with an algorithm that always correctly evaluates it on any input and for which $\delta = \Theta(n^{-1/3} \log n)$.

4. There is a balanced monotone Boolean function with an algorithm that correctly evaluates it on most inputs most of the time and for which $\delta = \Theta(n^{-1/3})$.

The constructions of Theorem 1 are optimal except possibly for the factors of $\sqrt{\log n}$ and $\log n$, as the lower bounds that we prove in Sections 3 and 4 show:

**Theorem 2.**

1. If an algorithm correctly computes a balanced Boolean function $f : \{-1, 1\}^n \to \{-1, 1\}$ on most inputs most of the time, then $\delta \geq \Theta(n^{-1/2})$. More generally and more precisely, $\Pr[\text{algorithm is wrong}] \geq \frac{1}{8} \Var(f) - \frac{1}{4} n \delta^2$, holds regardless if $f$ is balanced.

2. If an algorithm correctly computes a balanced monotone Boolean function on all inputs all of the time, then $\delta \geq \Theta(n^{-1/3/2})$. More generally and more precisely, if $f$ is monotone but not necessarily balanced, then $\Var(f) \leq \delta^{3/2} n^{1/2}$.

## 2 Constructions

The constructions are based on directed percolation on certain graphs, and were inspired in part by coupling-from-the-past on generic Markov chains [PW98] and in part by Radford Neal’s circular coupling [Nea02], though background on these topics is not required to understand the constructions.
2.1 Directed percolation on the wrapped extended butterfly

The examples we construct are based on a directed graph. There are various different choices that might work here, and the framework we have chosen, the wrapped extended butterfly, offers an explicit description and reasonably clear proofs.

Define the wrapped extended butterfly $\Omega_{H,W}$ to be a directed graph with $H \times W$ vertices, where $H$ is even and often a power of 2. The coordinates of a vertex are $(h, t)$ where $h$ and $t$ are integers with $0 \leq h < H$ and $0 \leq t < W$. Each vertex has in-degree two and out-degree two. The vertex $(h, t)$ has directed edges leading to $(2h \mod H, t + 1 \mod W)$ and $(2h + 1 \mod H, t + 1 \mod W)$. When $H = 2^d$ and $W = d$, the undirected unwrapped version of this graph is sometimes known as the “omega network” or “shuffle network” [Lei92, Chapter 3.8.1], which is isomorphic to the usual butterfly [Lei92, Chapter 3.2.1] that is sometimes used in parallel computing architectures. The extended butterfly (when $H = 2^d$ and $W = 5d$) has also been used in the construction of holographic proof systems [PS94]. We will for convenience take $H = 2^d$, but for our constructions $W$ will be much larger, $W = \Theta(d2^d)$ for one construction and $W = \Theta(2^{d/2})$ for the other. There are many graphs that we could have used in our constructions, but with the wrapped extended butterfly, the calculations come out nicely.

We will refer to the set of vertices with last coordinate $t$ as the “$t$th time slice” (where “time” is periodic), and the vertices with the same first coordinate as “points”.

We will consider two ensembles of random subgraphs of the wrapped extended butterfly. In the non-monotone ensemble, which we use for the non-monotone construction, each vertex has exactly one of its two out-going edges included in the subgraph, and an input bit determines the least significant bit (i.e., the parity) of the first coordinate of the destination vertex. We think of the $n = HW$ input bits as being associated with the vertices. In the monotone ensemble, which we use for the monotone construction, both possible out-going edges from a vertex have an input bit to determine whether or not the edge is present in the subgraph. Here the expected out-degree of each vertex is 1. A total of $n = 2HW$ input bits (associated with the edges) are used.

In the various constructions we will be interested in the directed cycles that may exist in this random subgraph defined by the input bits. A Las Vegas algorithm may determine the cycles that exist in the random subgraph by picking a uniformly random time slice and following all the paths forward in time until the starting time slice is reached. No additional bits of the input need to be read. Many of the paths merge early on, so following them all does not require as many reads as it might at first appear. As we shall see later, a Monte Carlo algorithm that works most of the time for most inputs may instead follow the paths starting from a smaller random set of vertices.

2.2 The nonmonotone random subgraph ensemble

Since the subgraphs in the nonmonotone ensemble have out-degree 1 at each vertex, there must be at least one directed cycle. The Boolean function can be any sort of balanced function of these cycles. For example, we could take the lexicographically smallest vertex that is in a cycle and in time slice 0, and take its bit.

**Lemma 3.** Provided $H = 2^d$ and $W > d$, this Boolean function is exactly balanced.
Proof. Because the wrapped extended butterfly is symmetric under the operation that flips the interpretation of the bits at a given time slice, a random subgraph is isomorphic to the subgraph obtained by flipping the bits at a time slice and suitably permuting the bits at the next $d$ time slices.

Remark: If we wish the Boolean function to be symmetric in its input bits while remaining exactly balanced, we can do this via a trick that places four input bits $b_{h,t,0}, \ldots, b_{h,t,3}$ at each vertex $(h,t)$. The parity $(b_{h,t,0} \oplus \cdots \oplus b_{h,t,3})$ of these four bits determines which outgoing edge from vertex $(h,t)$ is used. These four bits also determine another bit $b'_{h,t}$ that is symmetric in the four bits and independent of parity: this bit $b'_{h,t}$ is 1 iff there are exactly one or two ones among $(b_{h,t,0}, b_{h,t,1}, b_{h,t,2}, b_{h,t,3})$ and these are consecutive in the cyclic order. We can take our Boolean function $f$ to be the XOR of these $b'_{h,t}$ bits at all vertices that lie in a cycle.

For any $H = 2^d$ and $W > d$, the Boolean function is symmetric in the input bits and exactly balanced.

Lemma 4. In the nonmonotone ensemble, the probability that the Las Vegas algorithm reads a bit is at most $O(W^{-1} \log H + H^{-1})$.

Proof. Let $p_t$ be the number of vertices whose bits are read in a time slice $t$ time units after the initial time slice. If we run the $p$ particles independently forward $\log_2 H$ time steps from these locations, then their resulting locations would be independent and uniformly random. Under this independent random walks dynamics, when two particles reach a vertex, their outgoing edges may be different. If instead we consider coalescing dynamics, each vertex has one outgoing edge (and these outgoing edges are chosen uniformly independently), and all the particles arriving at the vertex always use that same edge. We may couple the coalescing dynamics with the independent random walks dynamics in such a way that the occupied locations at the end of the walks in the coalescing dynamics is a subset of the locations in the independent random walks dynamics. Suppose that two independent uniformly random particles in the same time slice do non-coalescing independent random walks on the wrapped extended butterfly for $s$ steps each, where $s < W$, and let $N$ be the number of collisions, that is, vertices visited by both. Then $\mathbb{E}[N] = s/H$. Moveover, given that the particles visited the same vertex, the probability that they will be at the same vertex $t$ time slices later is $\max\{2^{-t}, 1/H\}$. Consequently, $\mathbb{E}[N|N > 0] < 2 + s/H$, and therefore $\Pr[N > 0] = \mathbb{E}[N]/\mathbb{E}[N|N > 0] > s/(2H + s)$. Regardless of what the two particles do, the probability that a third particle (started from an independent and uniformly random location in their initial time slice) collides with at least one of them is at most $2s/H$. Thus the probability that a given pair of particles collide but that neither one collides with another particle is at least

$$\frac{s}{2H + s} \left[1 - \frac{2s}{H}\right]^{p_t}.$$

Under the coalescing dynamics, the probability of this event can only be larger. The number of such events is at most the reduction in the number of particles, so we find

$$\mathbb{E}[p_{t+\log_2 H+s}|p_t] \leq p_t - \left(\frac{p_t}{2}\right) \frac{s}{2H + s} \left[1 - \frac{2s}{H}\right]^{p_t}.$$
Taking $s \approx H/p_t$, we see that the expected reduction (that is, $\mathbb{E}[p_t - p_t + \log_2 n + s | p_t]$) is $\Theta(p_t)$. Let $T_k$ be the number of time slices for which $2^k < p_t \leq 2^{k+1}$. From the above, we see that $\mathbb{E}[T_k] = O(H/2^k + \log H)$. Thus $\sum_k 2^{k+1} \mathbb{E}[T_k] \leq O(H \log H)$. After the paths have all coalesced, there is one path that continues for $O(W)$ steps. Since the expected number of input bits read is $O(H \log H + W)$, and each of the $HW$ input bits is read with the same probability, we obtain the desired bound on $\delta$. 

The optimal choice is $H = \Theta(\sqrt{n/\log n})$ and $W = \Theta(\sqrt{n \log n})$, giving $\delta = O(n^{-1/2} \sqrt{\log n})$, which gives part 1 of Theorem 1.

To obtain part 2 of Theorem 1 we instead take $W = H$. The Las Vegas algorithm would have $\delta = O(n^{-1/2} \log n)$, but we can save the factor of $\log n$ by settling for an algorithm that reports the correct answer most of the time on most inputs. Rather than locating the set of cycles by trying all possible starting points at a random time slice and following them forward $W$ steps, we instead pick only $m$ points from the uniformly selected time slice $t_0$, and follow them forward until they cycle. Here, $m$ is large, but depends only on the required reliability of the algorithm. Given $t_0$, the $m$ points are selected at random, uniformly and independently.

**Lemma 5.** Assuming $W = H$, the probability that this algorithm reads any specific bit is $O(m/H)$

**Proof.** We prove this in the case $m = 1$, which is clearly sufficient. Moreover, by symmetry, it is enough to show that the expected number of vertices visited is $O(W)$. We start with a single particle, and follow its path. Let $X$ be the number of vertices visited. If the path did not enter its previously visited vertices after $s_0$ steps, where $s_0 \geq W$, then the conditioned probability that it will hit itself in the next $s + \log_2 H$ steps is at least $s/(2H + s)$, by the argument giving the lower bound for $\Pr[N > 0]$ in the proof of Lemma 4. Taking $s = H - \log_2 H$, say, we find that $\Pr[X > (j + 1)W | X > jW] < 3/4$ for every $j = 1, 2, \ldots$. Consequently, $\mathbb{E}[X] = O(W)$. 

To complete the argument, we need to show that with probability tending to 1 as $m \to \infty$ (uniformly in $H$), the algorithm finds all open cycles. This follows from the following lemma.

**Lemma 6.** Assuming $W = H$, the expected number of open cycles that are undiscovered by the algorithm is $O(1/(m + 1))$.

**Proof.** Let $\{v_0, v_1, \ldots, v_m\}$ be a set of $m + 1$ uniformly chosen vertices in the $t_0$ time slice, which are chosen independently given $t_0$. Let $\Gamma$ be the union of the paths of length $W/2$ starting at $\{v_1, v_2, \ldots, v_m\}$. We are going bound the probability for the event $\mathcal{A}$ that $v_0$ is on an open cycle that does not intersect $\Gamma$. Let $\gamma$ be the path of length $W/2$ starting at $v_0$.

If we were to follow all $H$ particles at time slice $t_0$ forward in time, let $\tau_j$ be the number of time steps before there are at most $j$ particles. Because $\Pr[\tau_j \geq \lfloor 2\mathbb{E}[\tau_j] \rfloor] \leq 1/2$ and successive blocks of $\lfloor 2\mathbb{E}[\tau_j] \rfloor$ time slices are independent, $\Pr[\tau_j \geq \ell \lfloor 2\mathbb{E}[\tau_j] \rfloor] \leq 1/2^\ell$. Provided $j$ is not too large ($j \leq H/(\log H \log \log H)$), our bound for $\mathbb{E}[T_k]$ in Lemma 4 implies that $\mathbb{E}[\tau_j] = O(H/j)$, so we can take $\ell = \Theta(j)$ to find that $\Pr[\tau_j > H/2]$ decays geometrically in $j$. 


Let $Y$ be the number of particles starting at $\{v_0, \ldots, v_m\}$ that have not merged with any other particle after $W/2$ steps; we have $\mathbb{E}[Y] \leq O(1)$. By symmetry $\Pr[\gamma \cap \Gamma = \emptyset | Y] = Y/(m+1)$, so in fact $\Pr[\gamma \cap \Gamma = \emptyset] \leq O(1/(m+1))$. Given $\gamma$ and $\Gamma$, the probability that the continuation of $\gamma$ hits $v_0$ on first return to the $t_0$ time slice is exactly $1/H$. Moreover, the same argument as in Lemma 5 shows that given $\gamma$ and $\Gamma$ the probability that the continuation of $\gamma$ hits $v_0$ at the $j$th return to time slice $t_0$ (but not sooner) is bounded by $O((3/4)^j/H)$. Thus, we have $\Pr[\mathcal{A}] \leq O(1/(m+1)H))$. Since the number of open cycles that are disjoint from $\Gamma$ is at most the number of vertices in the $t_0$ time slice that are on such cycles, this proves the lemma.

### 2.3 The monotone random subgraph ensemble

Since the subgraphs in the monotone random subgraph ensemble have random out-degree, we are not assured that there is a directed cycle, and when $W \gg \sqrt{H}$ it is even unlikely for there to be a directed cycle. This suggests that we take our Boolean function to be the existence of a directed cycle, or what turns out to be a little simpler to analyze, the existence of a directed cycle of length $W$.

**Lemma 7.** In the monotone ensemble, the probability that the Las Vegas reads a bit is at most $(4 + o(1))W - 1 \log W$.

**Proof.** Let $v$ be the starting vertex of the edge assigned to the given input bit, and suppose that the initial time slice was $t$ steps before $v$. Let $G_v$ be the subgraph of the wrapped extended butterfly which consists of all directed paths of length at most $W$ terminating at $v$. Each edge of $G_v$ occurs in the random subgraph with probability $1/2$. There is an obvious graph covering of $G_v$ by the binary tree of depth $W$. An upper bound for the probability that $v$ is explored conditioned on $t$ is therefore the probability that the root of the binary tree is connected to some vertex at level $t$ in a subgraph where each edge is included with probability $1/2$ independently. (See, e.g., [CR85] or [BS96, Theorem 1].) Let $S_t$ be this latter probability. We have $S_{t+1} = S_t - \frac{1}{4} S_t^2$, so for large $t$, $S_t \sim 4/t$. The probability that a vertex is explored is upper bounded by

$$\frac{1}{W} \sum_{t=0}^{W-1} S_t = (4 + o(1)) \frac{\log W}{W}.$$ 

In the monotone ensemble, let us consider the cycles that wind around exactly once (cycles with length $W$). Assuming $W \geq d$, these cycles are in bijective correspondence with bit strings of length $W$: given a cycle, the bit string is formed by the parities of the vertical coordinates, and given a bit string, the vertical coordinates are determined by groups of $d$ consecutive bits (in circular order). Let $N_{H,W}$ be the number of such cycles that occur in the graph. Since there are $2^W$ possible such cycles, each occurring with probability $2^{-W}$, we have $\mathbb{E}[N_{H,W}] = 1$.

**Remark:** As we shall see, the interesting regime is when $W = \Theta(\sqrt{H})$. It is not hard to show $\lim_{1 \leq W \leq \sqrt{H}} \Pr[N_{H,W} > 0] = 1 - 1/e$ and $\lim_{1 \leq \sqrt{H} \leq W} \Pr[N_{H,W} > 0] = 0$. Presumably $\lim_{H \to \infty} \Pr[N_{H,c\sqrt{H}} > 0]$ is a continuous function of $c$, in which case there is some particular positive value of $c$ that we can pick so that $\Pr[N_{H,c\sqrt{H}} > 0] = 1/2$, so that our Boolean
function is nearly balanced. $\lim_{H \to \infty} \Pr[N_{H,c\sqrt{H}} > 0]$ is an interesting function of $c$, but not one that we shall explore here. For our purposes it suffices to use a second moment estimate.

To compute $\mathbb{E}[N_{H,W}^2]$, we are interested in pairs of circular bit strings. Some pairs of cycles share edges, and for these pairs the probability that both cycles occur will be larger than $2^{-2W}$. Define a merge time to be a time slice where the two cycles coincide on the outgoing edge but not the incoming edge, and a split time to be a time slice where the two cycles coincide on the incoming edge but not the outgoing edge. The splits and merges alternate, and after a split, the next merge cannot occur within the next $d$ time slices, but there are no other constraints on when the splits and merges may occur. Suppose there are $\ell$ splits and $\ell$ merges — these occur at distinct times, so there are at most $2\binom{W}{2\ell}$ ways to select the times during which the two cycles share an edge. (When $\ell = 0$, there is one way to select the merge and split times, but still two ways to decide which edges agree between the cycles.) Given the times during which the cycles share edges, how many pairs of cycles satisfy these constraints? There are $2^W$ ways to select the first cycle, each occurring with probability $2^{-W}$. Every merge specifies the preceding $d + 1$ bits in the second cycle, and the bit for every shared edge as well as every edge at a split time is also specified. The number of ways to pick the second string is at most $2^{W-S}/(2H)^\ell$, where $S$ is the number of shared edges between the two cycles. The probability that the second cycle occurs given that the first one does is $2^{-W+S}$. The expected number of pairs of cycles with $\ell$ splits and $\ell$ merges is at most

$$\frac{2^W 2^\ell}{(2\ell)! (2H)^\ell}.$$ 

We may sum over all $\ell$ to conclude $\mathbb{E}[N_{H,W}^2] \leq \exp(W/\sqrt{2H}) + \exp(-W/\sqrt{2H})$. Since $\Pr[N_{H,W} > 0] \geq \mathbb{E}[N_{H,W}]^2/\mathbb{E}[N_{H,W}^2]$, we have

$$\Pr[N_{H,W} > 0] \geq \frac{1}{\exp(W/\sqrt{2H}) + \exp(-W/\sqrt{2H})}.$$ 

We can pick that value of $c$ for which $1/(e^c + e^{-c}) = 1 - 1/\sqrt{2}$ ($c \approx 1.12838$) and set $W = \lfloor c\sqrt{2H} \rfloor$. Instead of asking about the existence of a cycle of length $W$, we instead ask if there is a cycle that passes through a suitable set of vertices at the first time slice. The probability that a particular vertex is part of a cycle is at most $1/H$, the probability that a vertex is part of a cycle while the lexicographically smaller ones aren’t will be even smaller, so by adjusting the size of the set of vertices we can ensure $1 - 1/\sqrt{2} \leq \Pr[\text{suitable cycle exists}] \leq 1 - 1/\sqrt{2} + 1/H$. If a cycle passes through the last node in the suitable set, then call it a marginally suitable cycle, otherwise call it completely suitable: $\Pr[\text{completely suitable cycle exists}] \leq 1 - 1/\sqrt{2}$. We may repeat this experiment twice (with new bits for the second experiment) while taking $H = \Theta(n^{2/3})$ and $W = \lfloor c\sqrt{2H} \rfloor$ in both experiments. If there is a completely suitable cycle in either experiment, the Boolean function takes the value 1, if there is no suitable cycle in either experiment, the Boolean function takes the value 0. In the third scenario, where there is a marginally suitable cycle but not a completely suitable cycle, we may tweak the definition of the Boolean function so that it is monotone and exactly balanced. This third scenario occurs with probability $O(n^{-2/3})$, so it can increase $\delta$ by at most $O(n^{-2/3})$, so we still have $\delta \leq O(n^{-1/3} \log n)$, giving us part 3 of Theorem 1.
For part 4 of Theorem 1, we can use the same Boolean function that we used for part 3, and use the same choice of $W$ and $H$. As before, we can save the factor of $\log n$ by choosing a typically smaller collection of starting points. Let $m$ be a large constant that depends only on the required reliability of the algorithm. The algorithm chooses a random set $S$ of vertices in the extended butterfly, where each vertex is in $S$ with probability $m/H$, independently. The algorithm then explores all the open paths of length at most $2W$ starting at the vertices in $S$.

As above, $\delta$ can be bounded by a corresponding process on the binary tree of depth $2W$. In this case, $\delta$ is bounded by the expected number of “selected” vertices in the percolation component of the root, where each vertex is selected with probability $m/H$ independently from other vertices and from the percolation process. Since the expected number of vertices in the percolation cluster of the root and at distance $s$ from other vertices and from the percolation process. Since the expected number of vertices is bounded by the expected number of “selected” vertices in the percolation cluster of the root and at distance $s$ from the root is 1 (when $s \leq 2W$), we get $\delta \leq 2W m/H = O(m n^{-1/3})$.

The proof of the theorem is completed by showing that this algorithm finds all open cycles of length $W$ with high probability. This follows from the following lemma.

**Lemma 8.** Assuming $\sqrt{m} < W = \Theta(\sqrt{H})$, the expected number of open cycles of length $W$ unexplored by this algorithm is at most $c^{-\sqrt{m}}$ for some constant $c \in (0,1)$.

**Proof.** Fix some cycle $\gamma$ of length $W$. Let $A$ be the event that $\gamma$ is open and undetected by the algorithm. Set $W_0 = \lfloor W/\sqrt{m} \rfloor$, and let $\gamma_0$ be a subpath of $\gamma$ with $W_0$ vertices. Let $N$ be the number of open paths of length at most $W_0$ that start at a vertex in $S$, end at a vertex in $\gamma_0$, and are otherwise disjoint from $\gamma$. We will now prove that $\Pr[N > 0] = \Theta(1)$ by a second moment argument.

Fix some $k \in \{0, 1, 2, \ldots, W_0\}$. The number of paths of length $k$ that end at a vertex in $\gamma_0$ but are otherwise disjoint from $\gamma$ is $\Theta(W_0 2^k)$. Thus

$$\mathbb{E}[N] = \sum_{k=0}^{W_0} \Theta(W_0 2^k) 2^{-k} \frac{m}{H} = \Theta(1).$$

The estimation of the second moment is similar to the one done above for $N_{H,W}$. We can bound $\mathbb{E}[N^2]$ by considering separately the four cases where the pair of paths have the same or different starting points and the same or different ending points. We enumerate the pairs of paths by tracing them backwards from $\gamma_0$. Let $\ell$ be the number of splits (merges in reverse), and let $k_1$ and $k_2$ be the lengths of the two paths. We find $\mathbb{E}[N^2]$ is at most

$$\sum_{k_1=0}^{W_0} W_0 2^{k_1} 2^{-k_1} \frac{m}{H} \times \left[ \sum_{k_2=0}^{W_0} W_0 2^{k_2} 2^{-k_2} \frac{m}{H} \sum_{\ell \geq 0} \left( \frac{W_0}{2\ell} \right) \frac{1}{H^\ell} + \sum_{k_2=0}^{W_0} 2^{k_2} 2^{-k_2} \frac{m}{H} \sum_{\ell \geq 0} \left( \frac{W_0}{2\ell + 1} \right) \frac{1}{H^\ell} \right] + \sum_{k_2=0}^{W_0} 2^{k_2} 2^{-k_2} \sum_{\ell \geq 1} \left( \frac{W_0}{2\ell - 1} \right) \frac{1}{H^\ell} + \sum_{k_2=k_1}^{W_0} 2^{k_2} 2^{-k_2} \sum_{\ell \geq 0} \left( \frac{W_0}{2\ell} \right) \frac{1}{H^\ell} = O(1),$$

and thus $\Pr[N > 0] = \Theta(1)$. 

We may conclude that for every $k = 0, 1, \ldots, \lfloor \sqrt{m}/2 \rfloor - 1$ the probability that $\gamma$ is hit in a time slice $t \in [2kW_0, (2k+1)W_0)$ by an open path of length at most $W_0$ starting in $S$ is $\Theta(1)$. Since these events are independent, the probability that $\gamma$ is not visited by an open path of length at most $W_0$ starting in $S$ is $c^{-\sqrt{m}}$ for some constant $c \in (0, 1)$. The proof is complete by noting that there are $2^W$ possible cycles $\gamma$, and each is open with probability $2^{-W}$.

\section{Lower bound for balanced Boolean functions}

Suppose that a randomized algorithm approximately computes a Boolean function. Let $A(r, z)$ denote the output of the algorithm when it uses coins $r = r_1r_2\cdots$ on input $z = z_1z_2\cdots z_n$. As usual we let $\delta_i = \Pr[\text{bit } i \text{ gets read}]$. Consider two independent runs of the algorithm (i.e., using independent coins $r = r_1r_2\cdots$ and $s = s_1s_2\cdots$) on independent inputs $x = x_1x_2\cdots x_n$ and $y = y_1y_2\cdots y_n$. Let $N$ be the number of bit positions that are read by both of these independent runs:

$$\Pr[N > 0] \leq \mathbb{E}[N] = \sum_i \delta_i^2 \leq n\delta^2.$$

We define $z = z_1z_2\cdots z_n$ by

$$z_i = \begin{cases} x_i & \text{if algorithm with coins } r \text{ and input } x \text{ reads bit } i \\ y_i & \text{otherwise.} \end{cases}$$

The vector $z$ is uniformly random and independent of $r$ and $s$. Of course $A(r, z) = A(r, x)$. In the event that $N = 0$, we have $A(s, z) = A(s, y)$. Since $A(r, x)$ and $A(s, y)$ are i.i.d.,

$$\Pr[A(r, z) = A(s, z)] \leq \Pr[A(r, x) = A(s, y)] + \Pr[N > 0] \leq \Pr[A(r, x) = -1]^2 + \Pr[A(r, x) = 1]^2 + n\delta^2.$$

Set $w = \Pr[A(r, x) \neq f(x)]$, $p = \Pr[f(x) = 1]$ and $p' = \Pr[A(r, x) = 1]$. If $A(r, z) \neq A(s, z)$, then either $A(r, z) \neq f(z)$ or $A(s, z) \neq f(z)$. Thus,

$$w \geq \frac{1}{2}(1 - p'^2 - (1 - p')^2 - n\delta^2) = p' - p^2 - \frac{1}{2}n\delta^2.$$

Now note that the absolute value of the $p$-derivative of $\Var(f)/4 = p - p^2$ is bounded by 1. Thus, $p' - p^2 \geq \frac{1}{4} \Var(f) - |p - p'| \geq \frac{1}{4} \Var(f) - w$. Consequently,

$$2w \geq \frac{1}{4} \Var(f) - \frac{1}{2}n\delta^2,$$

as required.
4 Lower bound for balanced monotone Boolean functions

Suppose that \( f : \{-1, 1\}^n \to \mathbb{R} \) is some function, and \( A \) is a randomized algorithm calculating \( f \) (exactly and always). Recall the definition of the Fourier coefficients

\[
\hat{f}(S) := \mathbb{E}\left[ f(x) \prod_{i \in S} x_i \right], \quad S \subset \{1, 2, \ldots, n\}.
\]

When \( f \) is monotone and takes values in \( \{-1, 1\} \), we clearly have

\[
I_i(f) = \hat{f}(\{i\}).
\]

We will need the inequality

\[
\sum_{i=1}^n \hat{f}(\{i\}) \leq \sqrt{n} \delta. \tag{1}
\]

This inequality holds for \( f \) taking values in \( \{-1, 1\} \), even if \( f \) is not monotone. It is obtained by combining the first two displayed inequalities in the proof of Theorem 1 in [OS04] (although the algorithms discussed there are deterministic, the proof applies to random algorithms as well). Alternatively, the case \( k = 1 \) of the inequality

\[
\sum_{|S| = k} \hat{f}(S)^2 \leq \delta k \|f\|_2^2 \tag{2}
\]

from [SS04] gives

\[
\left( \sum_{i=1}^n \hat{f}(\{i\}) \right)^2 \leq n \sum_{i=1}^n \hat{f}(\{i\})^2 \leq n \delta,
\]

implying (1). (Although [2] does not assume that \( f \) is monotone or boolean, in the last step we assumed that \( f \) takes values in \( \{-1, 1\} \) to drop the factor \( \|f\|_2^2 \).)

Another inequality that we need to quote (valid for boolean but not necessarily monotone \( f \)) is

\[
\text{Var}(f) \leq \sum_{i=1}^n \delta_i I_i(f). \tag{3}
\]

See [OSSS04].

Now assume that \( f : \{-1, 1\}^n \to \{-1, 1\} \) is monotone. Since \( I_i(f) = \hat{f}(\{i\}) \), inequalities (3) and (1) give

\[
\text{Var}(f) \leq \delta^3/2 n^{1/2},
\]

which proves part 2 of Theorem [2]

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