Thomas Ernst

$q$-ANALOGUES OF GENERAL REDUCTION FORMULAS
BY BUSCHMAN AND SRIVASTAVA
AND AN IMPORTANT $q$-OPERATOR REMINDING
OF MACROBERT

Abstract. We find four $q$-analogues of general reduction formulas from Buschman and Srivastava together with some special cases, e.g. $q$-analogues of reduction formulas for Appell- and Kampé de Fériet functions. A proper $q$-analogue of the notation $\triangle(l; \lambda)$ by MacRobert, Meijer and Srivastava is given, and the definition of $q$-hypergeometric series is generalized accordingly.

1. Introduction

The umbral method for $q$-calculus [2] - [8], consisting of logarithmic $q$-shifted factorials, the tilde operator, a comfortable notation for $q$-powers, the symbol for real infinity, equivalent to the zero in Gasper-Rahman [10], the $q$–Kampé de Fériet function, compare with [3], are the main ingredients in this new method, which will increase our knowledge of $q$-calculus, advocated in the beginning of the last century by the late Cambridge student, reverend F. H. Jackson. All the topics above are not new; they have been presented in the book [9].

In this article, the important notation $\triangle(l; \lambda)$ of MacRobert [11], Meijer and Srivastava [15] for a certain array of $l$ parameters is given its proper $q$-analogue with the aid of a generalized tilde operator; in this paper we only consider the cases $l = 2, 3$, but a general definition is given. A deep knowledge of the $\triangle(l; \lambda)$ operator is necessary to grasp the subtleties of multiple hypergeometric functions. This $\triangle$-operator has a very long history in the field of special functions, in particular in India, which we will come back to in later papers.

2000 Mathematics Subject Classification: Primary 33D70; Secondary 33D15.

Key words and phrases: Buschman and Srivastava reduction formulas, MacRobert $\triangle(l; \lambda)$ operator; reduction formulas for Appell functions.
Buschman and Srivastava [1] have proved a great number of double series identities with general terms. We will find $q$-analogues of most of these formulas like in [3]; the method of proof will be similar except that we now use the $q$-Dixon- and $q$-Watson summation formulas. Some of the obtained formulas are symmetric in two variables, just as in the undeformed case. We pick out a form of these formulas, which converges nicely for small values of $x$. A list of different formulations of the Buschman–Srivastava formulas and their $q$-analogues in various journals and books is given, for better orientation.

This paper is organized as follows: In this section we give a general introduction. In section 2, four $q$-analogues of Buschman–Srivastava formulas are given. In section 3, we apply the Buschman–Srivastava formulas to find $q$-analogues of reduction formulas for Appell and Kampé de Fériet functions; the $\Delta$ operator appears only in the Heine function. In other papers, the $\Delta$ operator can appear also in the $q$-Kampé de Fériet function.

**Definition 1.** The power function is defined by $q^a \equiv e^{a \log(q)}$. Let $\delta > 0$ be an arbitrary small number. We will use the following branch of the logarithm: $-\pi + \delta < \text{Im}(\log q) \leq \pi + \delta$. This defines a simply connected space in the complex plane.

The variables $a, b, c, \ldots \in \mathbb{C}$ denote certain parameters. The variables $i, j, k, l, m, n, p, r$ will denote natural numbers except for certain cases where it will be clear from the context that $i$ will denote the imaginary unit.

Let the $q$-shifted factorial be defined by

$$\langle a; q \rangle_n \equiv \begin{cases} 1, & n = 0; \\ \prod_{m=0}^{n-1} (1 - q^{a+m}) & n = 1, 2, \ldots. \end{cases}$$

Since products of $q$-shifted factorials occur so often, to simplify them we shall frequently use the more compact notation

$$\langle a_1, \ldots, a_m; q \rangle_n \equiv \prod_{j=1}^{m} \langle a_j; q \rangle_n.$$

Let the $\Gamma_q$-function be defined in the unit disk $0 < |q| < 1$ by

$$\Gamma_q(x) \equiv \frac{\langle 1; q \rangle_\infty}{\langle x; q \rangle_\infty} (1 - q)^{1-x}.$$

The following notation will prove convenient, since many of our formulas contain exponents with upper and lower indices, which become less legible in the Gasper–Rahman notation.

$$\text{QE}(x) \equiv q^x.$$
The operator
\[ \tilde{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C} \]
is defined by
\[ a \mapsto a + \pi i \frac{\log q}{\log q}. \]
By (5) it follows that
\[ \langle a; q \rangle_n = \prod_{m=0}^{n-1} (1 + q^{a+m}). \]
Assume that \((m, l) = 1\), i.e. \(m\) and \(l\) relatively prime. The operator
\[ \tilde{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C} \]
is defined by
\[ a \mapsto a + \frac{2\pi i m}{l \log q}. \]
We will also need another generalization of the tilde operator.

\[ k\langle \tilde{\alpha}; q \rangle_n \equiv \prod_{m=0}^{n-1} \left( \sum_{i=0}^{k-1} q^{i(a+m)} \right). \]

This leads to the following \(q\)-analogue of [12, p.22, (2)].

**Theorem 1.1** ([6]).

\[ \langle a; q \rangle_{kn} = \prod_{m=0}^{k-1} \left( \frac{a + m}{k} \right)_n \times_k \left( \frac{\tilde{a} + m}{l} ; q \right)_n. \]

**Definition 2.** A \(q\)-analogue of a notation due to Thomas MacRobert (1884–1962) [11, p. 135] and Srivastava [15]. This notation was also often used for the Meijer G-function and the Fox H-function \((q = 1)\).

\[ \langle \triangle(q; l; \lambda); q \rangle_n \equiv \prod_{m=0}^{l-1} \left( \frac{\lambda + m}{l} ; q \right)_n \times_l \left( \frac{\lambda + m}{l} ; q \right)_n. \]

When \(\lambda\) is a vector, we mean the corresponding product of vector elements. When \(\lambda\) is replaced by a sequence of numbers separated by commas, we mean the corresponding product as in the case of \(q\)-shifted factorials. The last factor in (10) corresponds to \(l^m\).
1.1. Definition of the q-Kampé de Fériet function. We will give a definition reminding of [10], which allows easy confluence to diminish the dimension in (12), and has the advantage of being symmetric in the variables. Furthermore, $q$ is allowed to be a vector and the full machinery of tilde operators and $q$-additions will be used.

In the following two definitions we put

$$
\hat{a} \equiv a \lor \tilde{a} \lor \sum_k \tilde{a} \lor \triangle (q; l; \lambda).
$$

The following definition is a $q$-analogue of [16, (24), p. 38], in the spirit of Srivastava.

**Definition 3.** Let

$$(a), (b), (g_i), (h_i), (a'), (b'), (g'_i), (h'_i)$$

have dimensions

$$A, B, G_i, H_i, A', B', G'_i, H'_i.$$ 

Let

$$1 + B + B' + H_i + H'_i - A - A' - G_i - G'_i \geq 0, \ i = 1, \ldots, n.$$ 

Then the generalized $q$-Kampé de Fériet function is defined by

$$
\Phi_{B+B':H_1+H'_1;\ldots;H_n+H'_n}^{A+A':G_1+G'_1;\ldots;G_n+G'_n} \left[ (a) : (\hat{g}_1); \ldots; (\hat{g}_n) | q; \bar{x} || (a') : (\hat{g}'_1); \ldots; (\hat{g}'_n) \right] \\
\left[ (b) : (\hat{h}_1); \ldots; (\hat{h}_n) | (b') : (\hat{h}'_1); \ldots; (\hat{h}'_n) \right] \\
\equiv \sum_{m} \frac{\langle (\hat{a}); q_0 \rangle_m (a') (q_0, m) \prod_{j=1}^{n} ((\hat{g}_j); q_j)_{m_j} ((\hat{g}'_j); (q_j, m_j) x_j^{m_j})}{\langle (\hat{b}); q_0 \rangle_m (b') (q_0, m) \prod_{j=1}^{n} ((\hat{h}_j); q_j)_{m_j} ((\hat{h}'_j); (q_j, m_j) (1; q_j)_{m_j})} \\
\times (-1)^{\sum_{j=1}^{n} m_j} \left( 1 + H_j + H'_j - G_j - G'_j + B + B' - A - A' \right) \\
\times \text{QE} \left( B + B' - A - A' \left( \begin{array}{c} m \\ 2 \end{array} \right) ; q_0 \right) \prod_{j=1}^{n} \text{QE} \left( 1 + H_j + H'_j - G_j - G'_j \left( \begin{array}{c} m_j \\ 2 \end{array} \right) ; q_j \right).}
$$

We assume that no factors in the denominator are zero. We assume that $(a')(q_0, m), (g'_j)(q_j, m_j), (b')(q_0, m), (h'_j)(q_j, m_j)$ contain factors of the form $\langle a(\bar{k}); q \rangle_k, (s); q, (s(\bar{k}); q) \text{ or QE}(f(\bar{m})).$ 

**Definition 4.** Generalizing Heine’s series we shall define a $q$ hypergeometric series by

$$
p + p' \phi_{r + r'} \left[ \hat{a}_1, \ldots, \hat{a}_p | q; z \right] \prod_i f_i(k) \\
\hat{b}_1, \ldots, \hat{b}_r \mid \prod_j g_j(k) \\
\equiv \sum_{k=0}^{\infty} \langle \hat{a}_1; q \rangle_k \ldots \langle \hat{a}_p; q \rangle_k \langle 1, \hat{b}_1; q \rangle_k \ldots \langle \hat{b}_r; q \rangle_k \left[ (-1)q^{k(2)} \right]_{1+r+r'-p-p'} z^k \prod_i f_i(k) \prod_j g_j(k).$$
We assume that the $f_i(k)$ and $g_j(k)$ contain $p'$ and $r'$ factors of the form $\langle a(k); q \rangle_k$ or $\langle s(k); q \rangle_k$ respectively. In case of $\triangle(q;l; \lambda)$, the index is adapted accordingly. When we have a sequence of elements $a_i$, we can denote them by $(A)$.  

1.2. Two lemmata In the following three proofs we will use the finite $q$-Dixon theorem.

**Theorem 1.2.** [3, p. 210 (39)]

\[
\begin{aligned}
\phi_3 \left[ \begin{array}{c}
-2k, b, c, 1-k
\end{array} \right]_{\langle 1-k-b-c \rangle}
\end{aligned}
\]

\[
\equiv \sum_{j=0}^{2k} \binom{2k}{j}_q \frac{\langle b, c, 1-k-q \rangle_j (1-1/j^2 q^{3k-b-c})}{\langle 1-2k-b, 1-2k-c, -k \rangle_j}
\]

\[
= \frac{(1-2k-b-c, 1-k-b; q)_k}{(1-2k-c, 1-k-b; q)_k} \left( \frac{1}{2}; q^2 \right)_k
\]

\[
\phi_3 \left[ \begin{array}{c}
-k, b, c, 1-k
\end{array} \right]_{\langle 1-k-b-c \rangle}
\]

\[
= 0, \text{ } k \text{ odd.}
\]

In another proof we will use a $q$-analogue of the Watson formula [1].

**Theorem 1.3.** [6, p. 170 (43)]

\[
\phi_3 \left[ \begin{array}{c}
\frac{c}{2}, \frac{c}{2}, a, -N
\end{array} \right]_{\langle N \rangle} = \frac{\langle 1+a-c, 1+a; q^{2} \rangle_N}{\langle 1-a, 1+c; q^{2} \rangle_N}, \quad \text{if } N \text{ even;}
\]

\[
= 0, \quad \text{if } N \text{ odd.}
\]

2. $q$-analogues of Buschman–Srivastava double sums

The Buschman–Srivastava paper [1] was a landmark for the studies of multiple $q$-hypergeometric series. Some of these formulas had previously been published in other form by Shanker and Saran [13]. Srivastava and Jain [17] have found $q$-analogues of some of these formulas, some of which are included in the book [9]. The following table summarizes the connection between the various formulas and the methods of proof; the four references are in chronological order.

| Ref. | Proof          | Equation no. |
|------|----------------|--------------|
| [13] | $q$-Vandermonde | [3](62), (66) |
| [14] | finite Bailey-Daum | [3](68) |
| [1]  | finite Bailey-Daum | [3](81) |
| b p. 10 | finite $q$-Dixon | (16) |
We are now going to find a number of general double sums. Since the convergence problem is rather delicate, we try to choose the most proper form with respect to an arbitrary q-power. Sometimes we add this q-power afterwards, to save space in the proof. In the following, a statement like \( a \neq k \) will mean \( a \neq k, k \in \mathbb{N} \). Everywhere the symbol \( \{C_n\}_{n=0}^{\infty} \) denotes a bounded sequence of complex numbers. It is assumed that both sides converge. Note that the formulas (18), (20) and (23) are symmetric in two variables.

**Theorem 2.1.** A q-analogue of Buschman, Srivastava [1, p. 437 (2.7)].

\[
\sum_{m,n} C_{m+n}x^{m+n}(-1)^n \langle g; q \rangle_m \langle g, 1 - m/n; q \rangle_n \text{QE} \left( -\frac{n}{2} - \frac{mn}{2} \right) \\
\langle 1, h; q \rangle_m \langle 1, h, -m/n; q \rangle_n \\
= \sum_{N=0}^{\infty} C_{2N}x^{2N} \langle g, h - g, 1 - N; q \rangle_N q^{(n/2) + N}g, -h \neq k.
\]

**Proof.**

\[
\text{LHS} = \sum_{N,n} C_Nx^N (-1)^n \langle g; q \rangle_{N-n} \langle g, 1 - N/2; q \rangle_n \text{QE} \left( \left( \frac{n}{2} \right) - \frac{nN}{2} \right) \\
\langle 1, h; q \rangle_{N-n} \langle 1, h, -N/2; q \rangle_n \\
= \sum_{N=0} \frac{C_Nx^Nq^{N(g-h)}(-g + 1 - N)}{\langle 1, -h + 1 - N; q \rangle_N} \\
\sum_n (-1)^n \binom{N}{n} q \langle g, -h + 1 - N, 1 - N/2; q \rangle_n \text{QE} \left( \left( \frac{n}{2} \right) + n(h - g - N/2) \right) \\
\text{by (14)} \\
= \sum_{N=0} C_{2N}x^{2N} q^{2N(g-h)}(-g+1-2N; q)q_{2N} \Gamma_q \left[ 1-2N-g, h, 1-N, h-g+N \right] \\
\langle 1, -h+1-2N; q \rangle_{2N} \\
= \sum_{N=0} C_{2N}x^{2N} \langle h-g, 1-N, \tilde{g}; q \rangle_N \langle -g+1-2N; q \rangle_{2N} \langle 1/2; q \rangle_{2N} \text{QE} \left( \left( \frac{2N}{2} \right) + 2Ng \right) \\
\langle h, \tilde{h}, \tilde{h} + 1/2, \tilde{h} + 1/2, 1-N-g; q \rangle_N \langle 1; q \rangle_{2N} \\
= \sum_{N=0} C_{2N}x^{2N} \langle g, h-g, 1-N, \tilde{g}; q \rangle_N \langle 1, \tilde{h}, \tilde{h} + 1/2, \tilde{h} + 1/2, 1-N-g; q \rangle_N = \text{RHS}. \]

\[\square\]
Theorem 2.2. A $q$-analogue of [1, p. 438 (2.8)].

\begin{equation}
\sum_{m,n} C_{m+n}x^{m+n}(-1)^n\langle g; h; q \rangle_m\langle g, h, 1 - \frac{m+n}{2}; q \rangle_n\langle 1; q \rangle_m\langle 1, -\frac{m+n}{2}; q \rangle_n = \sum_{N=0}^{\infty} C_{2N}x^{2N}\frac{\langle g, h, 1 - N, g + h, g + h + 1, g + h + 1; q \rangle_N q^{N(\frac{1}{2})}}{\langle 1, 1 + g + h; q \rangle_N}, -h - g \neq k.
\end{equation}

Proof. We prove an equivalent formula.

\begin{equation}
\sum_{m,n} C_{m+n}x^{m+n}(-1)^n\langle g, h; q \rangle_m\langle g, h, 1 - \frac{m+n}{2}; q \rangle_n\langle 1; q \rangle_m\langle 1, -\frac{m+n}{2}; q \rangle_n = \sum_{N,n} C_N x^N(-1)^n\langle g, h; q \rangle_{N-n}\langle g, h, 1 - \frac{N}{2}; q \rangle_n\langle 1; q \rangle_{N-n}\langle 1, -\frac{N}{2}; q \rangle_n \times \frac{\langle -N^2 - n^2 - (m+n)(g+h) \rangle}{\langle 1; q \rangle_{N-n}\langle 1, -\frac{N}{2}; q \rangle_n} = \sum_{N,n} C_N x^N(-1)^n
\end{equation}

\begin{equation}
\frac{\langle N \rangle_q\langle -g+1-N,-h+1-N; q \rangle_N\langle g, h, 1 - \frac{N}{2}; q \rangle_n\langle 1; q \rangle_N\langle -h+1-N,-g+1-N,1-N; q \rangle_n}{(\frac{n}{2})_n + n(1-h-g-\frac{3N}{2})} = \sum_{N,n} C_N x^N(-1)^n
\end{equation}

by (14)

\begin{equation}
\sum_{N=0}^{\infty} C_{2N}x^{2N}\langle 1-N, g + h, 1 - 2N - g - h; q \rangle_N\langle 1 - 2N - g, 1 - 2N - h; q \rangle_{2N}\frac{1}{2}\langle q^2 \rangle_N = \sum_{N=0}^{\infty} C_{2N}x^{2N}\langle 1-N, g + h + N; q \rangle_N\langle g, h; q \rangle_{2N}\langle N + h, 1 - N - g, g + N, 1, q \rangle_N\langle N + h, 1 - N - g, g + N, 1, q \rangle_N
\end{equation}

\begin{equation}
= \sum_{N=0}^{\infty} C_{2N}x^{2N}\langle g, h, 1 - N, \triangle(q; 2; g + h); q \rangle_N\langle g, h; q \rangle_{2N}\langle 1, 1, g + h; q \rangle_N = \sum_{N=0}^{\infty} C_{2N}x^{2N}\langle g, h, 1 - N, q; 2; g + h; q \rangle_N\langle g, h; q \rangle_{2N}\langle 1, 1, g + h; q \rangle_N = \sum_{N=0}^{\infty} C_{2N}x^{2N}\langle g, h, 1 - N, \triangle(q; 2; g + h); q \rangle_N\langle g, h; q \rangle_{2N}\langle 2 \rangle_q + n(g+h)).
\end{equation}

Finally, multiply $C_n$ by $\langle (\frac{n}{2}) + n(g+h) \rangle$. \qed
THEOREM 2.3. A $q$-analogue of [1, p. 438 (2.9)].

\[ (20) \sum_{m,n} C_{m+n}x^{m+n}(-1)^m \left( 1 - \frac{m+n}{2} ; q \right)_n \text{QE} \left( -\frac{n}{2} - \frac{3mn}{2} + \frac{(m+n)^2}{4} \right) \]

\[ = \sum_{N=0}^{\infty} C_{2N}x^{2N} \left( 1 - \nu + \sigma ; q \right)_{3N} \left( 1 - N ; q \right)_N (-1)^N, \nu, \sigma + 1 \neq -k. \]

Proof.

\[ (21) \quad \text{LHS} \]

\[ = \sum_{N=0}^{\infty} \sum_{n=0}^{N} C_{N}x^{N}(-1)^{N-n} \left( 1 - \frac{N}{2} ; q \right)_n \text{QE} \left( 3 \left( \frac{n}{2} \right) - \frac{3nN}{2} + n + \frac{N^2}{4} \right) \]

\[ = \sum_{N=0}^{\infty} C_{N}x^{N} q^{ \frac{N^2}{2} } (-1)^{N} \sum_{n=0}^{N} \left( -N, -\nu + 1 - N, -\sigma + 1 - N, 1 - \frac{N}{2} ; q \right)_n \]

\[ \times \text{QE} \left( 3 \left( \frac{n}{2} \right) - \frac{3nN}{2} + n + \frac{N^2}{4} \right) \]

\[ = \sum_{N=0}^{\infty} C_{2N}x^{2N} \left( 1 - N, 1 - 2N - \nu, 2N + \nu + \sigma - 1 ; q \right)_{N} \left( \frac{1}{2} ; q^2 \right)_N \]

\[ \left. \langle 1, \nu, -\sigma + 1 - N ; q \right)_N \]

where we have used (14) for the $q$-Dixon theorem. \[ \blacksquare \]

Before we prove the next formula, we remind the reader that the following $q$-analogue of [1, p. 440 (3.10)] has been found by Srivastava and Jain [17, p.217, 2.2]:

\[ (22) \sum_{m,n=0}^{\infty} C_{m+n}x^{m+n}(-1)^n \langle a, \tilde{a}; q \rangle_m \langle b, \tilde{b}; q \rangle_n \]

\[ \langle 1, 2a; q \rangle_m \langle 1, 2b; q \rangle_n \]

\[ = \sum_{N=0}^{\infty} C_{2N}x^{2N} \langle a + b, a + b; q \rangle_{2N} \]

\[ \langle 1, a + \frac{1}{2}, b + \frac{1}{2}, a + b; q^2 \rangle_N \]

THEOREM 2.4. Another $q$-analogue of [1, p. 440 (3.10)].

\[ (23) \sum_{m,n=0}^{\infty} C_{m+n}x^{m+n}(-1)^n \text{QE} \left( \left( \frac{m}{2} \right) - ng \right) \langle g; q \rangle_m \langle h, \tilde{h}; q \rangle_n \]

\[ \langle 1, 2g; q \rangle_m \langle 1, 2h, -m - g + 1 - n ; q \rangle_n \]

\[ = \sum_{N=0}^{\infty} C_{2N}x^{2N} \left( h + g + N, \frac{h + g}{2}, \frac{h + g}{2}, \frac{h + g + 1}{2}, \frac{h + g + 1}{2}; q \right)_N \text{QE} \left( \left( \frac{2N}{2} \right) \right) \]

\[ \langle g + h, \tilde{g}, g + \frac{1}{2}, \tilde{g} + \frac{1}{2}, h + \frac{1}{2}, \tilde{h} + \frac{1}{2}, g + N, h + \frac{1}{2}, \tilde{1}, 1; q \rangle_N \]
Proof. We prove the equivalent formula

\[ \sum_{m,n=0}^{\infty} C_{m+n}x^{m+n}(-1)^n \text{QE} \left( -\left( \frac{n}{2} \right) - mn + mg \right) \langle g; q \rangle_m \langle h, \tilde{h}; q \rangle_n \]

(24) \[ = \sum_{N=0}^{\infty} C_{2N}x^{2N} \langle h + g + N, g + h; \tilde{g}, \tilde{h}; \tilde{q} \rangle_{2N} \left( \frac{g + h}{2}, \frac{g + h + 1}{2}, \frac{g + h + 1}{2}; q \right)_{N} q^{2gN}. \]

Finally, multiply \( C_n \) by \( \text{QE} \left( \left( \frac{n}{2} \right) - ng \right) \).

3. Reduction formulas

We now specialize the very general formulas to reduction formulas for Appell- and Kampé de Fériet functions. We will need the \( \Delta \) notation, since otherwise there will not be enough space to write out the formulas.

**Theorem 3.1.** A \( q \)-analogue of a reduction formula for the second Appell function.
\[ (26) \quad \sum_{m,n=0}^{\infty} \langle \lambda; q \rangle_{m+n}x^{m+n}(-1)^n \langle g; q \rangle_{m} \langle g, 1 - \frac{m+n}{2}; q \rangle_n \text{QE} \left( \frac{-mn-n}{2} \right) \]

\[ \langle 1, h; q \rangle_{m} \langle 1, h, -\frac{m+n}{2}; q \rangle_n \]

\[ = 7\phi_{7} \left[ \frac{\lambda}{\langle \lambda; q \rangle_{k}} \right] \]

Proof. Put \( C_k = \langle \lambda; q \rangle_{k} \) in (16). \( \blacksquare \)

**Remark 1.** The righthand sides of formulas (26) and (28) converge quicker than the LHS because of the \( q \)-power with negative exponent, the double sum and the minus sign on the left. The other formulas in this section have similar properties.

**Theorem 3.2.** A \( q \)-analogue of a reduction formula for the third Appell function. By using vectors, this formula can easily be extended to a \( q \)-analogue of \([16, p. 31 (48)]\).

\[ (27) \quad \sum_{m,n=0}^{\infty} x^{m+n}(-1)^n \langle g, h; q \rangle_{m} \langle g, h, 1 - \frac{m+n}{2}; q \rangle_n \text{QE} \left( \frac{-n+mn}{2} \right) \]

\[ \langle \mu; q \rangle_{m+n} \langle 1; q \rangle_{m} \langle 1, -\frac{m+n}{2}; q \rangle_n \]

\[ = 7\phi_{7} \left[ \frac{g, h, \Delta(q; 2; g+h)}{\Delta(q; 2; \mu), g+h, \hat{1}, \infty} \right] \]

Proof. Put \( C_k = \frac{1}{\langle \mu; q \rangle_{k}} \) in (18). \( \blacksquare \)

**Corollary 3.3.** A \( q \)-analogue of \([1, p. 439 (3.4)]\) and \([16, p. 31 (46)]\)

\[ (28) \quad \Phi_{p;2;3}^{p;1;2} \left[ \begin{array}{c} \vec{\lambda} : g, \infty; g, \infty \langle x, -xq^{-\frac{1}{2}} \rangle_{\langle 1 - \frac{m+n}{2}; q \rangle_n q^{-\frac{mn}{2}}} \\ \vec{\mu} : h, h \end{array} \right] \]

\[ = 6+4p\phi_{6+4p} \left[ \Delta(q; 2; \vec{\lambda}), g, h - g, 3\infty \langle q; -x^2q^2 \rangle_{\langle 1 - \frac{\mu}{q} \rangle_n q^{-\frac{mn}{2}}} \right] \]

Proof. Put \( C_n = \frac{\langle \vec{\lambda}; q \rangle_{n}}{\langle \mu; q \rangle_{n}} \) in (16). \( \blacksquare \)
COROLLARY 3.4. A $q$-analogue of [1, p. 439 (3.5)]

\begin{equation}
(29) \quad \Phi_{p;1;2}^{p;1;2} \left[ \left( \begin{array}{c}
\vec{\lambda} : g; h, \vec{h} \\
\vec{\mu} : 2g; 2h \end{array} \right) |q; -x, x| \right] \frac{q^{mn}}{(m + g; q)_n} = 5 + 4p \phi_{8 + 4p} \left[ \begin{array}{c}
\Delta(q; 2; \vec{\lambda}, g + h) \\
\Delta(q; 2; \vec{\mu}), g + h, g + \frac{1}{2}, h + \frac{1}{2}, g + \frac{1}{2}, h + \frac{1}{2}, \tilde{g}
\end{array} \right] \frac{|q; x^2q||h + g + k; q)_k}{(k + g; q)_k}.
\end{equation}

Proof. Put $C_n = \frac{\langle \vec{x}; q \rangle_n}{\langle \vec{\mu}; q \rangle_n}$ in (23). ■

THEOREM 3.5. A $q$-analogue of [1, p. 439 (3.8)] and [16, p. 32 (50)]

\begin{equation}
(30) \quad \sum_{m,n} \langle \vec{\lambda}; q \rangle_{m+n} x^{m+n} (-1)^m \langle 1 - \frac{m+n}{2}; q \rangle_n \frac{q^{mn}}{(m + g; q)_n} \frac{\text{QE} \left( - \frac{n}{2} - \frac{3mn}{2} + \frac{(m + n)^2}{4} \right)}{\langle \vec{\mu}; q \rangle_{m+n} \langle 1, \nu, \sigma, -\frac{m+n}{2}; q \rangle_n \langle 1, \nu, \sigma; q \rangle_m} = 16 + 4p \phi_{15 + 4p} \left[ \begin{array}{c}
\Delta(q; 2; \vec{\lambda}), \Delta(q; 3; \nu + \sigma - 1), 9 \infty \frac{|q; x^2q||1 - k; q)_k}{(k + g; q)_k} \\
\Delta(q; 2; \vec{\mu}, \nu, \sigma, \nu + \sigma - 1), \nu, \sigma, \tilde{1}
\end{array} \right].
\end{equation}

Proof. Put $C_n = \frac{\langle \vec{x}; q \rangle_n}{\langle \vec{\mu}; q \rangle_n}$ in (20). The $\Delta(q; 3; \nu + \sigma - 1)$ corresponds to six $q$-shifted factorials, this explains the $9 \infty$. ■

This last formula is the crown of our efforts in this section, and beautifully unites the notation used so far. The formula [16, p. 32 (50)] is also the last one in the corresponding chapter. We will come back to more $q$-analogues from [16] in later papers.

4. Discussion

We would like to remind that the umbral notation is equivalent to Gasper and Rahman [10]; however, the $\Delta(q; l; \lambda)$ operator cannot be readily expressed in their notation. The same goes for the factor $\langle 1 - k; q \rangle_k$, which elucidates the integration property in $q$-calculus. There are more comments at the end of the article [3].

References

[1] R. G. Buschman, H. M. Srivastava, Series identities and reducibility of Kampé de Fériet functions, Proc. Cambridge Philos. Soc. 91 (1982), 435–440.
[2] T. Ernst, A method for $q$-calculus, J. Nonlinear Math. Physics 10 (2003), 487–525.
[3] T. Ernst, *Some results for q-functions of many variables*, Rend. di Padova Sem. Mat. Univ. 112 (2004), 199–235.

[4] T. Ernst, *q-Generating functions for one and two variables*, Simon Stevin 12 (2005), 589–605.

[5] T. Ernst, *A renaissance for a q-umbral calculus*, Proceedings of the International Conference Munich, Germany 25–30 July 2005. (World Scientific, 2007).

[6] T. Ernst, *Some new formulas involving $\Gamma_q$ functions*, Rend. di Padova Sem. Mat. Univ. 118 (2007), 159–188.

[7] T. Ernst, *The different tongues of q-calculus*, Proceedings of Estonian Academy of Sciences 57 no. 2 (2008), 81–99.

[8] T. Ernst, *q-calculus as operational algebra*, Proceedings of Estonian Academy of Sciences 58 no. 2 (2009), 73–97.

[9] T. Ernst, *Handbook for q-analysis*, submitted.

[10] G. Gasper, M. Rahman. *Basic Hypergeometric Series*, Cambridge, 1990.

[11] T. M. MacRobert, *The multiplication formula for the gamma function and E-function series*, Math. Ann. 139 (1959), 133–139.

[12] E. D. Rainville, *Special Functions*, Reprint of 1960 first edition, Chelsea Publishing Co., Bronx, N.Y., 1971.

[13] O. Shanker, S. Saran, *Reducibility of Kampé de Fériet function*, Ganita 21 (1970), 9–16.

[14] H. M. Srivastava, *On the reducibility of Appell’s function $F_4$*, Canad. Math. Bull. 16 (1973), 295–298.

[15] H. M. Srivastava, *A note on certain identities involving generalized hypergeometric series*, Nederl. Akad. Wetensch. Indag. Math. 41, (1979) , 191–201.

[16] H. M. Srivastava, P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood, New York, 1985.

[17] H. M. Srivastava, V. K. Jain, *q-series identities and reducibility of basic double hypergeometric functions*, Canad. J. Math. 38 (1986), 215–231.

[18] H. M. Srivastava, P. W. Karlsson, *Transformations of multiple q-series with quasi-arbitrary terms*, J. Math. Anal. Appl. 231, (1999), 241–254.

DEPARTMENT OF MATHEMATICS
UPPSALA UNIVERSITY
P.O. BOX 480
SE-751 06 UPPSALA, SWEDEN
E-mail: Thomas.Ernst@math.uu.se

Received July 17, 2010.