A Theory of Ex Post Rationalization

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Abstract

People rationalize their past choices, even those that were mistakes in hindsight. We propose a formal theory of this behavior. The theory predicts that sunk costs affect later choices. Its model primitives are identified by choice behavior and it yields tractable comparative statics. (JEL: D11, D81, D91)

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1 Introduction

People rationalize their past choices. We look back on our lives and try to make sense of what we have done. Upon realizing that, in hindsight, we have made a mistake, we can adapt our goals, attitudes, or beliefs to justify the decision.

Classical economic theory rules out rationalization. It assumes that people make forward-looking choices according to fixed preferences, rather than adapting their preferences to rationalize past decisions. Introspection, common sense, and psychological research all suggest that the classical approach omits a key aspect of human decision-making.\footnote{We review psychological research on rationalization in Section 2.}

How can economic models capture \textit{ex post} rationalization? We develop a theory that accommodates this behavior. To fix ideas, consider the following example from Thaler (1980).\footnote{The ticket cost $40 in Thaler’s example; we have raised the price due to inflation.}

Example 1: Bob pays $100 for a ticket to a basketball game to be played 60 miles from his home. On the day of the game there is a snowstorm. He decides to go anyway. If the ticket had been free-of-charge, he would have stayed home.

The $100 that Bob paid is a sunk cost. It is not worth going to the basketball game during a snowstorm. In hindsight, it was a mistake to have bought the ticket.\footnote{Of course, it may have been \textit{ex ante} optimal to buy the ticket. We use “mistake” as shorthand for “\textit{ex post} sub-optimal action”.

But if Bob goes to the game, then he can avoid acknowledging the mistake, by exaggerating his enthusiasm for basketball or by downplaying the hazards of driving through a snowstorm. If he stays home, then he is inarguably worse off than if he had not bought a ticket in the first place.

There are two key ingredients for this behavior. First, Bob must have made a choice that was an \textit{ex post} mistake. Hence, this modified example is far-fetched:

Example 2: Bob receives a free ticket to a basketball game and loses $100 due to an unusually high utility bill. On the day of the game there is a snowstorm. He decides to go anyway. If he had not lost the $100, he would have stayed home.

Second, there must be plausible preferences that, if adopted, would justify Bob’s earlier decision. To illustrate this, we replace the physical consequences in Example 1 with monetary gains and losses.

Example 3: Bob pays $100 for a financial option that can only be exercised on the day of the basketball game. It yields $180 if exercised in
good weather and loses $20 if exercised in a snowstorm. On the day of the
game there is a snowstorm. He decides to exercise the option anyway, for a
net loss of $100 + $20.

Example 3 is unnatural because there is no way for Bob to rationalize his initial pur-
chase. Letting the option expire results in a net loss of $100, whereas exercising the
option results in a net loss of $120. More money is better, so Bob has to acknowledge
the mistake and cut his losses.

Even at high stakes, decision-makers sometimes rationalize sunk costs rather than
acknowledge error. For instance, a senior Irish Republican Army leader was asked in
1978 whether the years of violent resistance had been worth it. He replied, “Virtually
nothing has been achieved. We can’t give up now and admit that men and women who
were sent to their graves died for nothing.” (Smith, 1997, p. 225)

Motivated by these examples, we propose a theory about agents who seek to ra-
tionalize their past choices by adapting their preferences. We model an agent facing a
decision problem with this structure:

1. The agent chooses action $a_1$ from menu $A_1$.

2. The agent learns the state of the world $s \in S$.

3. The agent chooses action $a_2$ from menu $A_2(a_1, s)$, which can depend on his first
action and the state.

A utility function takes as arguments $a_1, a_2,$ and $s$. The agent’s material utility
function is denoted $u$. The agent may adopt any utility function in the set $\mathcal{V}$, which
we call rationales. $u$ and $\mathcal{V}$ are primitives of the model. We assume that $u \in \mathcal{V}$.

We start by describing the agent’s choice from menu $A_2(a_1, s)$, after having chosen
$a_1$ from menu $A_1$ and learned that the state is $s$. The agent chooses action $a_2$ from
menu $A_2(a_1, s)$ and rationale $v$ from $\mathcal{V}$ to maximize total utility, that is

$$(1 - \gamma) u(a_1, a_2, s) + \gamma \left[ v(a_1, a_2, s) - \max_{\hat{a}_1 \in A_1} \max_{\hat{a}_2 \in A_2(\hat{a}_1, s)} v(\hat{a}_1, \hat{a}_2, s) \right],$$

where $\gamma \in [0, 1]$ is the weight on rationalization utility. Rationalization utility is the
negative of ex post regret, assessed under the chosen rationale. By construction, ratio-
nalization utility is no more than 0.

\footnote{For further reading on how rationalizations by combatants prolonged the Troubles, see Chapter 3 of Alonso (2007).}
Observe that if \( a_1 \) is \textit{ex post} materially optimal, that is

\[
\max_{\hat{a}_2 \in A_2(a_1, s)} u(a_1, \hat{a}_2, s) = \max_{\hat{a}_1 \in A_1} \max_{\hat{a}_2 \in A_2(a_1, s)} u(\hat{a}_1, \hat{a}_2, s),
\]

then the theory predicts that the agent chooses \( a_2 \) to maximize material utility, which yields rationalization utility 0 for rationale \( v = u \). Moreover, if the menu \( A_1 \) is a singleton then \( a_1 \) is \textit{ex post} materially optimal. Hence, the theory departs from the classical prediction only when the agent has made an \textit{ex post} mistake.

When the agent has made an \textit{ex post} mistake, he may be able to increase rationalization utility by choosing rationale \( v \neq u \). By construction, \( a_2 \) maximizes a weighted sum of his material utility \( u(a_1, a_2, s) \) and his chosen rationale \( v(a_1, a_2, s) \), which distorts his choices compared to the classical benchmark.

We now apply the model to the earlier examples. The rationales \( V \) are parameterized by \( \theta \in [0, 400] \). Utility function \( v_\theta \) specifies that the agent gets \( \theta \) utils for attending the game, \(-200\) utils for driving through a snowstorm, and \(-p\) utils for paying \( p \) dollars. Material utility is \( u = v_{180} \), so a classical agent (\( \gamma = 0 \)) is willing to pay \$180 to attend the game in good weather, but will stay home in a snowstorm.

In Example 1, the menu \( A_1 \) has two alternatives; the agent can get a ticket and lose \$100 or he can decline. After buying the ticket for \$100, the agent learns that there is a snowstorm. If he stays home, then his material utility is \(-100\), and his rationalization utility is \(-100 - \max\{\theta - 200 - 100, 0\} \). It is optimal to choose \( \theta \leq 300 \), yielding total utility \((1 - \gamma)(-100) + \gamma(100) = -100\). By contrast, if the agent attends the game, then his material utility is \( 180 - 200 - 100 = -120 \) and his rationalization utility is \( \theta - 200 - 100 - \max\{\theta - 200 - 100, 0\} \). Now it is optimal to choose \( \theta \geq 300 \), for rationalization utility of 0 and total utility of \((1 - \gamma)(-120)\). By attending the game and exaggerating his enthusiasm, the agent is able to increase rationalization utility at the cost of material utility. For \( \gamma > \frac{1}{6} \), his total utility is strictly higher when he attends the game.

Suppose instead that the ticket was free-of-charge. Then staying home in a snowstorm leads to no regret under the agent’s material utility function \( u \). Hence, the agent maximizes total utility by adopting rationale \( v = u \) and staying home. The agent’s behavior exhibits sunk-cost effects; his choice on the day of the basketball game depends on upfront costs that he cannot recover.

In Example 2, the agent has no choice initially, so the menu \( A_1 \) contains only one alternative: the agent gets a ticket and loses \$100. This is trivially \textit{ex post} optimal, so the agent maximizes total utility by staying home in the snowstorm. Hence, removing unchosen alternatives from the menu \( A_1 \) can alter the agent’s later choice from \( A_2 \).

In Example 3, we have taken the agent’s material utility for the outcomes in Ex-
ample 1 and converted utils to dollars. The agent bought the financial option for $100. Exercising the option in good weather yields $180, and exercising it in a snowstorm loses $20. But every rationale agrees about money, so there is no room to reduce regret and the agent does not exercise the option in a snowstorm.

Perhaps Bob’s behavior is not due to rationalization, but due to an error in reasoning. Bob falls for the sunk cost fallacy: If he buys the ticket and stays home, then he has wasted $100, whereas if he goes to the game then it was not a waste. But the same logic applies to Example 3—if Bob buys the financial option and does not use it, then he has wasted $100. Why cannot he avoid waste by exercising the option (and losing a further $20)? In the present theory, sunk-cost effects arise when one can rationalize incurring past costs. This provides an explanation for why people commit the sunk cost fallacy in some situations and not in others.\footnote{Of course, this is only a partial account of the psychology at work. Imas (2016) finds that lab subjects take on more risk after paper losses and less risk after realized losses. The present theory neglects this dependence on framing.}

To complete the model, we specify the agent’s behavior when choosing from the first menu $A_1$. At this point, the agent has no earlier choices to rationalize, so we assume that the agent evaluates choices from $A_1$ according to his expected material utility under some prior on the states $S$. This depends on the agent’s beliefs about his future choice from $A_2$. A naïf believes he will maximize material utility when choosing from $A_2$; a sophisticate correctly foresees his own choices from $A_2$.

How is the modeler to specify material utility $u$ and the rationales $V$? In some situations, we can use standard restrictions on the preferences agents may plausibly hold. For instance, for an agent choosing between money lotteries, we could assume that $V$ is a class of preferences with constant relative risk aversion. Similarly, for an agent bidding in an auction, we could assume that the rationales $V$ have different valuations for the object, but are all quasi-linear in money.

If we do not make a priori restrictions on $V$, can we nonetheless deduce the rationales? We prove that the model primitives $u$, $V$, and $\gamma$ are fully identified by choice behavior. That is, suppose we start with finitely many outcomes, and each utility function depends on the outcome and the state. We then construct objective lotteries over outcomes, and extend utility functions by taking expectations. The agent faces decision problems of this form:

1. The agent selects a menu $M$ from a collection of menus of lotteries.
2. The agent learns the state of the world $s \in S$.
3. The agent chooses a lottery from $M$.

Under a regularity condition, we prove that whenever the agent’s choice correspondence is consistent with material utility $u$ and rationales $V$, those primitives are unique up to
a state-specific affine transformation. Moreover, the weight $\gamma$ on the rationalization utility is identified. Hence, statements about the agent’s rationales can be reduced to statements about the agent’s choice behavior.

Next, we impose more structure to yield comparative statics, by assuming that first actions, second actions, and rationales are complements. Let the first actions, the second actions, and parameter set $\Theta$ be totally ordered sets. We assume that the rationales $\mathcal{V}$ have the form $\{w(a_1, a_2, \theta, s) : \theta \in \Theta\}$, for some function $w$ that is supermodular in $(a_1, a_2, \theta)$. For instance, this includes the rationales we posited for Example 1, if we impose that buy a ticket is a higher action than don’t buy a ticket, and that go to the game is a higher action than stay home. It also includes time-separable utility functions, of the form $w(a_1, a_2, \theta, s) = w_1(a_1, \theta, s) + w_2(a_2, \theta, s)$, with $w_1$ supermodular in $(a_1, \theta)$. We assume that the menu $A_2(a_1)$ is monotone non-decreasing in $a_1$.

We prove that if the rationalizer’s first action was ex post too high, then his second action is distorted upwards compared to the classical benchmark. Symmetrically, if the rationalizer’s first action was ex post too low, then his second action is distorted downwards compared to the classical benchmark, as happens in Example 4.

This result yields comparative statics for a variety of settings. It predicts sunk-cost effects in multi-part projects — when first-period effort and second-period effort are complements, the agent responds to first-period cost shocks by exaggerating the value of the project and raising second-period effort. It predicts that agents repeatedly facing identical decisions will have ‘sticky’ choice behavior, responding too little to new information. In particular, lab subjects who make incentivized reports of priors and posteriors will report posteriors biased towards their priors, and will underweight informative signals compared to subjects who report only posteriors.

Finally, we study the effect of unchosen time-1 alternatives on time-2 choice. In the classical model, such alternatives are irrelevant for time-2 behavior. By contrast, for a rationalizer facing a supermodular decision problem, raising the unchosen time-1 alternatives lowers the agent’s time-2 choices. This prediction compares cleanly to the zero effect predicted by the classical model.

2 Literature review

Various economic theories allow that agents choose their beliefs or preferences. In particular, Yariv (2005), Acharya et al. (2018), and a preprint of Bernheim et al. (2021) model agents who change their beliefs or preferences to align with past actions.

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6 Even classical models of state-dependent utility are only unique up to a state-specific affine transformation, because beliefs and utilities are not separately identified.

7 For examples, see Rabin (1994), Rotemberg (1994), Becker and Mulligan (1997), Akerlof and Kranton (2000), Brunnermeier and Parker (2005), and Bernheim et al. (2021).
Unlike these theories, *ex post* rationalization is about choosing preferences to reduce regret. Regret depends not only on what the agent chose, but also on what else she could have chosen. Similarly, Suzuki (2019) studies an agent who chooses a project, observes a signal, and may ignore the signal to justify her choice of project.

An *ex post* rationalizer distorts today’s choices so as to justify yesterday’s choices. By contrast, regret theory posits that the agent makes choices today so as to reduce regret tomorrow, and evaluates regret according to a fixed utility function (Savage, 1951; Loomes and Sugden, 1982, 1987; Bell, 1982; Sarver, 2008).

Many experimental paradigms in psychology have found evidence of rationalization. Cushman (2020) writes, “Among psychologists, [rationalization] is one of the most exhaustively documented and relentlessly maligned acts in the human repertoire.” For brevity, we focus on two topics that pertain closely to the present theory: cognitive dissonance and confabulation.

Cognitive dissonance theory posits that people adapt their cognitions so as to achieve internal consistency (Festinger, 1957). It predicts that the more a person suffers to obtain achieve some result, the more they will value that result. For instance, some experiments find that unexpectedly severe initiation rituals cause new members to evaluate the group more positively, a result that is also consistent with *ex post* rationalization (Aronson and Mills, 1959; Gerard and Mathewson, 1966).

Instead of accurately recalling the reasons for their choices, people sometimes generate *post hoc* explanations and sincerely believe them. Such confabulation has been documented in split-brain patients (Gazzaniga, 1967, 2005) and in ordinary people manipulated to misremember what they chose (Johansson et al., 2005, 2006). Nisbett and Wilson (1977) find that subjects are often unaware of the effects of experimental stimuli on their behavior, and offer spurious explanations when queried by experimenters.

The present theory posits that the agent seeks to justify her choice *ex post*. She does not adopt the perspective that her choice, while mistaken in hindsight, was sensible given what she knew at the time. People may evaluate decisions from the *ex post* perspective because of hindsight bias. That is, after an event has occurred, people are overconfident that they could have predicted it in advance (Fischhoff, 1975; Blank et al., 2007), and even misremember their own *ex ante* predictions, falsely believing that they predicted what came to pass (Fischhoff and Beyth, 1975; Fischhoff, 1977).

Sunk-cost effects have been documented in many settings, but there is no consensus

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8 Harmon-Jones and Harmon-Jones (2007) review experimental paradigms testing the theory of cognitive dissonance. However, one of these—the free choice paradigm of Brellm (1956)—is confounded by self-selection bias and requires caution in interpreting the results (Chen and Risen, 2010; Risen and Chen, 2010).

9 These include business decisions (Staw, 1976; McCarthy et al., 1993; Schoorman, 1988; Staw et al., 1997), usage of durable goods (Ho et al., 2018), professional sports (Staw and Hoang, 1995; Camerer and Weber, 1999; Keefer, 2017), and auctions (Herrmann et al., 2015; Augenblick, 2016). Some studies do not find evidence of sunk-cost effects (Ashraf et al., 2010; Friedman et al., 2007; Ketel et al., 2016; Negrini et al., 2017).
as to their causes. Some studies have offered explanations based on prospect theory (Thaler, 1980; Arkes and Blumer, 1985). These explanations depend on losses relative to a reference point, whereas the present theory predicts that sunk-cost effects depend on whether the agent made a choice to incur those costs. Other studies have suggested that sunk-cost effects may be rational due to reputation concerns (Prendergast and Stole, 1996; McAfee et al., 2010), limited memory (Baliga and Ely, 2011), or self-signaling motives (Hong et al., 2019). Closest to our explanation, Staw (1980) argues that sunk-cost effects arise because people “re-evaluate alternatives and outcomes to make it appear that that they have acted in a competent or intelligent manner”.

Recent work finds that sunk-cost effects depend on whether the decision-maker was responsible for incurring those costs, as predicted by ex post rationalization. Martens and Orzen (2021) study the effect of sunk costs on follow-up investment decisions, in a laboratory experiment. They find that sunk-cost effects increase when subjects are responsible for the initial investment decision. Similarly, Guenzel (2021) finds that for high-stakes corporate acquisitions: exogenous cost shocks (occurring after the acquisition decision) decrease the acquiring company’s willingness to divest, but this effect is substantially reduced if the CEO who led the acquisition steps down.

Most directly, we build on ideas from Eyster (2002) and Ridout (2020).

Eyster (2002) studies a two-period model with an agent who wishes to reduce ex post regret, assessed according to material utility, but limits attention to alternative first actions that are ‘consistent with’ the chosen second action. To illustrate, the theory of Eyster (2002) would explain Example 1 by positing that if Bob attends the game, then only buy a ticket is consistent, so he feels no regret. On the other hand, if Bob stays home, then both buy a ticket and don’t buy a ticket are consistent, so he feels regret for having bought the ticket. One limitation of this approach is that the modeler’s intuitions about consistency may vary with how the actions are framed – stay home seems consistent with don’t buy a ticket, but stay home with a ticket in hand does not seem consistent with don’t buy a ticket, so Bob can also avoid regret by staying home with a ticket in hand. Our present approach overcomes this framing objection. Instead of a frame-dependent consistency relation, our main primitive is a set of rationales, i.e. post hoc reasons for the agent’s choice, and these are identified from choice data.

Ridout (2020) studies a model of one-shot choice, with an agent who has a set of ‘justifiable’ preferences and a material preference, and is constrained to choose only alternatives that maximize some justifiable preference. As in the present study, the agent sometimes fails to maximize his material preference, but the mechanism is different. Ridout’s agent may forego an alternative he prefers because he does not consider his material preference justifiable. By contrast, our agent’s material preference belongs

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For a meta-analysis, see Roth et al. (2015)
to the set of rationales, so there is no distortion in the absence of past mistakes.

3 Statement of theory

In our model, an agent chooses an action from a menu, then learns the state of the world, and then finally chooses an action from a second menu, which can depend on the first action and the state.

We now define the model primitives. \( A_1 \) denotes the first actions; \( A_2 \) denotes the second actions; and \( S \) denotes the states of the world, with representative elements \( a_1, a_2, \) and \( s \), respectively.

A decision problem \( D \equiv (A_1, A_2, F) \) consists of

1. a first-period menu \( A_1 \subseteq A_1 \),
2. and a second-period menu correspondence \( A_2 : A_1 \times S \rightarrow A_2 \).
3. a prior over states \( F \in \Delta S \).

We require that \( A_1 \) and \( A_2 \) be non-empty.

A utility function is a function \( v : A_1 \times A_2 \times S \rightarrow \mathbb{R} \). The rationales are denoted \( V \); these are a set of utility functions that the agent may adopt to justify her actions. The agent’s material utility function is denoted \( u \), and we assume that \( u \in V \).

The set of rationales \( V \) captures the preferences that the agent regards as reasonable. For instance, the rationales could specify the agent’s utility from consuming a good or service. Alternatively, rationales could specify the agent’s subjective beliefs about some payoff-relevant event, with the observed state \( s \) being a noisy signal about that event.

We start by describing choice in the second period. The agent facing decision problem \( D \) has chosen \( a_1 \) from menu \( A_1 \) and learned that the state is \( s \). She chooses \( a_2 \in A_2(a_1, s) \) and \( v \in V \) to maximize

\[
U_D(a_2, v | a_1, s) \equiv (1 - \gamma) u(a_1, a_2, s) + \gamma \left[ v(a_1, a_2, s) - \max_{\hat{a}_1 \in A_1} \max_{\hat{a}_2 \in A_2(\hat{a}_1, s)} v(\hat{a}_1, \hat{a}_2, s) \right] \tag{1}
\]

for parameter \( \gamma \in [0, 1] \). Equation (1) states that the agent places weight \((1 - \gamma)\) on material utility, and \( \gamma \) on rationalization utility. Rationalization utility measures how close her course of action is to the ex post optimum under her chosen rationale \( v \). When \( a_1 \) was ex post sub-optimal according to \( u \), rationalization utility might be increased by adopting rationale \( v \neq u \). This distorts the agent’s choice of \( a_2 \), which maximizes \((1 - \gamma)u(a_1, a_2, s) + \gamma v(a_1, a_2, s)\).
We restrict attention to decision problems for which the relevant maxima are well-defined. This is implied, for instance, if every \( v \in V \) is continuous in the actions, and the sets \( A_2(a_1, s) \), \( \{(a_1', a_2') : a_1' \in A_1 \text{ and } a_2' \in A_2(a_1')\} \), and \( V \) are compact.

We discuss some natural benchmarks for first-period behavior. A na"ïf chooses \( a_1 \) to maximize \( \mathbb{E}_F[u(a_1, a_2^*(a_1, s), s)] \) where \( a_2^*(a_1, s) \) is a selection from

\[
\arg\max_{a_2 \in A_2(a_1, s)} u(a_1, a_2, s).
\]

A sophisticate chooses \( a_1 \) to maximize \( \mathbb{E}_F[u(a_1, \tilde{a}_2(a_1, s), s)] \) where \( \tilde{a}_2(a_1, s) \) is a selection from

\[
\arg\max_{a_2 \in A_2(a_1, s)} \max_{v \in V} U_D(a_2, v \mid a_1, s). \tag{2}
\]

Naïfs and sophisticates both maximize expected material utility \textit{ex ante}, albeit with different beliefs about \textit{ex post} behavior. If choice correspondence \textit{(2)} is non-singleton, one may select between personal equilibria as in Köszegi and Rabin (2006).

Another natural benchmark is the empathetic sophisticate, whose first action maximizes expected total utility, \textit{i.e.}

\[
\mathbb{E}_F \left[ \max_{a_2 \in A_2(a_1, s)} \max_{v \in V} U_D(a_2, v \mid a_1, s) \right].
\]

This agent both correctly foresees his \textit{ex post} behavior and desires to reduce regret when choosing \textit{ex ante}.

We have interpreted the theory as capturing individual psychological motives, but it has another interpretation in the context of organizations. On this interpretation, the theory describes a rational agent who is rewarded for past performance, but can influence the principal’s criteria for performance evaluation. The agent’s need to defend past decisions generates sunk-cost effects. This formalizes an observation by Staw (1980) about the perverse incentives of retrospective performance evaluation.

### 3.1 Discussion of modeling choices

Plausibly, the agent’s rationalization motive depends on the kind of \textit{ex ante} uncertainty she faced. Choosing a risky investment is not like choosing a bet in roulette. It is easier to remember the \textit{ex ante} perspective when evaluating choices with objective risks. By contrast, people are more likely to say, “I should have known it!” for decisions that involved subjective uncertainty or required deliberation to weigh competing considerations. Our model abstracts from this nuance, representing uncertainty using only a distribution over states. However, we interpret the scope of the theory to be confined...
to those kinds of uncertainty which seem predictable in hindsight.\footnote{We suggest that experimental tests of the theory use forms of uncertainty that require the subject to exercise judgment, rather than objective risks such as coin flips or dice rolls. Experimenters might also consider designs that give subjects the illusion of control (Langer, 1975; Presson and Benassi, 1996).}

For the theory to depart from the classical prediction, the available rationales $\mathcal{V}$ must be limited. For instance, if $\mathcal{V}$ includes a ‘stoic’ rationale that is indifferent between all action sequences, then the second term in (1) can always be set to zero, and the theory predicts material utility maximization. Thus, the theory’s novel predictions depend on plausible restrictions on the rationales that the agent can adopt.

For specific applications, we suggest that $\mathcal{V}$ should be some standard class of preferences for the setting under consideration, if such standards exist. This serves to prevent ad hoc explanations and to make the theory a portable extension of existing models, in the sense of Rabin (2013). Other economic exercises also take restrictions on preferences as given. For instance, in mechanism design, positive results often depend on restricting the agent’s preferences to lie within certain \textit{a priori} limits (Hurwicz, 1972). As another example, structural estimation methods often require functional-form restrictions on preferences.

Nonetheless, because the theory’s predictions depend on $\mathcal{V}$, its general applicability depends on whether $\mathcal{V}$ can be identified from choice behavior. We take up this challenge in the next section.

4 Identification of preference parameters

In this section, we find that the theory’s preference parameters are identified from choice behavior, by extending the environment to include objective lotteries.

4.1 Extending the environment

Earlier, in Section 3, we described decision problems in terms of choosing a first action $a_1$ and then a second action $a_2$. However, we can equivalently conceive of the agent first choosing between menus of ‘outcomes’, and then as selecting an outcome from the chosen menu.\footnote{This is structurally similar to the decision problems of Gul and Pesendorfer (2001), though the theories are different. In Gul and Pesendorfer (2001), the agent is tempted at time 2, which affects the time-1 choice. Under the present theory, the agent desires to rationalize the time-1 choice, which affects the time-2 choice.} For instance, the action buy a ticket is equivalent to a menu with two outcomes, attend the game & pay $100 and stay home & pay $100. The action don’t buy a ticket is equivalent to the singleton menu consisting of the outcome stay home & pay nothing.\footnote{If the correspondence $A_2$ depends on the state $s$, we can represent this with state-contingent menus of outcomes.}
For the purposes of identification, it saves notation to work with outcomes instead of actions. Of course, if we map action sequences to outcomes and identify the model defined over outcomes, then we also identify the model defined over action sequences.

We start with a finite set of outcomes $Z$. The lotteries over outcomes are denoted $\Delta(Z)$. For any set $B$, let $\mathcal{K}(B)$ denote the collection of nonempty subsets of $B$. Let $\mathcal{K}_f(B)$ denote the collection of finite nonempty subsets of $B$.

The agent faces decision problems of this form:

1. At $t = 1$, the agent selects a menu $M$ from a finite collection of menus $\mathcal{M} \subset \mathcal{K}_f(\Delta(Z))$.
2. The agent learns the state $s$.
3. At $t = 2$, the agent chooses a lottery $q$ from the selected menu $M$.

We take as data the agent’s time-2 choice correspondence in each state. Having selected menu $M$ from collection $\mathcal{M}$ and learned that the state is $s$, the agent’s choices from $M$ are denoted $c_s^2(M | M)$, which satisfies $c_s^2(M | M) \subseteq M$. Observe that the agent can, by choosing differently at time-1, achieve any lottery in $\bigcup M$. We use $\mathcal{U}$ to denote the set of all functions from $Z$ to $\mathbb{R}$, and extend these to $\Delta(Z)$ by taking expectations.

**Definition 4.1 (Representation).** $(\gamma, u^s, \mathcal{V}^s) \in [0, 1] \times \mathcal{U} \times \mathcal{K}(\mathcal{U})$ represents choice correspondence $c_s^2$ if

$$c_s^2(M | \mathcal{M}) = \arg\max_{q \in M} \left\{ (1 - \gamma)u^s(q) + \gamma \max_{v^s \in \mathcal{V}^s} \left\{ v^s(q) - \max_{\hat{q} \in \bigcup \mathcal{M}} v^s(\hat{q}) \right\} \right\}$$

for all $\mathcal{M} \subset \mathcal{K}_f(\Delta(Z))$ and all $M \in \mathcal{M}$.

Our identification exercise relies only on time-2 choice behavior, so the resulting theorem applies equally to naïfs, sophisticates, and empathetic sophisticates.

By taking the choice correspondence $c_s^2$ as data, we are asking whether the preference parameters are identified from a complete description of time-2 choice behavior. That is, even setting aside practical questions of elicitation, are the parameters pinned down by knowing how the agent would choose, for each menu $M$ and each collection $\mathcal{M}$?

An important caveat is that we observe $c_s^2(M | \mathcal{M})$ even if choosing $M$ from $\mathcal{M}$ is sub-optimal from the ex ante perspective. Our interpretation of this is that the agent trembles at time 1 and chooses each menu with some small probability, and in such cases $c_s^2(M | \mathcal{M})$ captures how the agent chooses after a slip of the hand, as in Selten (1975). We discuss how to relax this requirement in Section 4.4.

As with classical models of state-dependent utility, the agent’s behavior in both periods remains the same if we scale up the utility functions in state $s$ and scale down
the prior probability on \( s \) to compensate. Hence, in general, utility can at best be identified up to a state-specific affine transformation. We will shortly state conditions under which this is possible.

4.2 Identification result

For the identification result, we restrict attention to choice correspondences with regular representations. In the definition that follows, we treat each element of \( U \) as a point in \( \mathbb{R}^Z \).

**Definition 4.2 (Regularity).** \((\gamma, u^s, V^s) \in [0,1] \times U \times K(U)\) is regular if it satisfies the following conditions:

1. \( 0 < \gamma < 1 \).
2. \( V^s \) is compact, convex, and non-singleton.
3. \( u^s \) is in the relative interior of \( V^s \).
4. (Non-redundancy) There do not exist distinct \( v^s, \tilde{v}^s \in V^s \) such that \( v^s \) is a positive affine transformation of \( \tilde{v}^s \).\(^{13}\)

Clause 3 of Definition 4.2 means that if the agent can distort his rationale in one direction, then he can distort his rationale (at least slightly) in the opposite direction.

Non-redundancy requires that \( V^s \) does not contain extra rationales that are irrelevant for behavior. In particular, consider an agent with rationales \( v^s \) and \( \alpha v^s + \beta \), for \( \alpha \geq 1 \) and \( \beta \in \mathbb{R} \). By inspection of (1), switching from \( \alpha v^s + \beta \) to \( v^s \) weakly increases total utility, so deleting \( \alpha v^s + \beta \) has no effect on choice behavior. Non-redundancy requires that \( V^s \) includes at most one rationale in each positive affine equivalence class.

Regularity rules out that the agent always maximizes material utility \( u^s \) in state \( s \).\(^{14}\) In that case, it is not possible to fully identify the rationales.

Suppose that the agent’s second-period choice correspondence in state \( s \) has a regular representation. The next theorem states that this representation is unique up to a positive affine transformation.

**Theorem 4.3.** For any state \( s \), if choice correspondence \( c^s_2 \) has regular representations \((\gamma, u^s, V^s)\) and \((\hat{\gamma}, \hat{u}^s, \hat{V}^s)\), then \( \hat{\gamma} = \gamma \) and there exist scale factor \( \alpha > 0 \) and additive

---

\(^{13}\) \( V^s \) satisfies non-redundancy if and only if no two distinct utilities in \( V^s \) represent the same preference relation on \( \Delta(Z) \).

\(^{14}\) For instance, regularity rules out the ’stoic’ rationale \( \bar{v}^s \) that is indifferent between all outcomes. Suppose \( \bar{v}^s \in V^s \). By \( V^s \) non-singleton, \( V^s \) contains some rationale \( v^s \neq \bar{v}^s \). By convexity, we have \( .5v^s + .5\bar{v}^s \in V^s \). But \( .5v^s + .5\bar{v}^s \) is a positive affine transformation of \( v^s \), contradicting non-redundancy.
constants $\beta : \mathcal{V}^s \to \mathbb{R}$ such that

$$\hat{u}^s = \alpha u^s + \beta(u^s),$$

$$\hat{V}^s = \{\alpha v^s + \beta(v^s) : v^s \in \mathcal{V}^s\}.$$ 

The proof is in Appendix A.

### 4.2.1 Proof sketch for Theorem 4.3

The key step in the proof is to recover $\mathcal{V}^s$ from choice data; we sketch the method here.

An example illustrates the main idea. The agent is contemplating a two-part project. There are three outcomes: decline yields 0 utils, start & quit yields $-s$ utils, and start & finish yields $\theta - 1 - s$ utils. The available rationales have $\theta \in [0, 3]$.

Let us suppose that $s = 1$, and fix some fully mixed lottery $q$ (over outcomes), which is in the unit 3-simplex. Figure 1 depicts the indifference curves that pass through this point, for rationales with $\theta = 0$, $\theta = 2$, and $\theta = 3$. The shaded area consists of lotteries that every rationale regards as weakly worse than $q$. This set is a convex cone with vertex at $q$, and its supporting hyperplanes at $q$ are the rationales’ indifference curves through $q$. Thus, we can recover the preferences represented by the rationales by recovering the shaded area.

Returning to generality, let us fix some fully mixed lottery $q$ and some state $s$. Let us consider the lotteries that every rationale regards as weakly worse than $q$, that is

$$\bar{W}_{\text{inner}}(q,s) \equiv \bigcap_{v^s \in \mathcal{V}^s} \{r \in \Delta(Z) : v^s(q) \geq v^s(r)\}$$

which corresponds to the shaded area in Figure 1. Since each rationale evaluates lotteries according to expected utility, (3) is a convex cone with vertex at $q$. It will turn out that the cone’s supporting hyperplanes at $q$ pin down $\mathcal{V}^s$.

Recall from Example 2 in Section 1 that a rationalizer’s second-period choice can depend on the availability of unchosen first-period alternatives. We use this phenomenon to fully identify $\bar{W}_{\text{inner}}(q,s)$. Let $x$ be a lottery. We say that $x$ matters for $q$ (in state $s$) if there exist lotteries $r$ and $y$ such that

1. after choosing menu $\{q,r\}$ and foregoing $\{y\}$, and learning the state is $s$, the agent chooses $q$,

2. but after choosing menu $\{q,r\}$ and foregoing $\{y\}$ and $\{x\}$, and learning the state is $s$, the agent does not choose $q$.

Intuitively, the availability of $x$ at time 1 increases the regret cost of choosing $q$ from menu $\{q,r\}$ at time 2, which alters the agent’s choice. Figure 2 illustrates.
Figure 1: Indifference curves for lotteries over outcomes when \( s = 1 \). Arrows indicate direction of increasing utility for each rationale.

Notice that if \( x \in \bar{W}_{\text{inner}}(q, s) \), then every rationale weakly prefers \( q \) to \( x \) in state \( s \), so the availability of \( x \) does not ever alter the regret cost of choosing \( q \). It follows that \( x \) does not matter for \( q \) in state \( s \). Thus, \( x \notin \bar{W}_{\text{inner}}(q, s) \) if \( x \) matters for \( q \) in state \( s \).

The converse also holds. Suppose \( x \notin \bar{W}_{\text{inner}}(q, s) \). We want to show that \( x \) matters for \( q \) in \( s \), that is, to find lotteries \( r \) and \( y \) such that offering \( \{x\} \) in addition to \( \{q, r\} \) and \( \{y\} \) at time 1 causes the agent to switch away from \( q \) at time 2. Figure 1 illustrates the construction. Assume that material utility has \( \theta = 2 \). Lottery \( q \) is weakly materially better than both \( r \) and \( y \), so the agent will choose \( q \) from \( \{q, r\} \) after foregoing \( \{y\} \).\(^{15}\)

\(^{15}\text{q and r yield identical total utility in this case. We could break the tie by perturbing r, at the cost of a} \)
The rationale with $\theta = 3$ prefers $r$ to all three alternatives, so the agent can always choose $r$ without regret. Every rationale strictly prefers at least one of $y$ or $x$ to $q$, so choosing $q$ results in positive regret when $y$ and $x$ were available. Since $q$ and $r$ have equal material utility, the agent will not choose $q$ from $\{q, r\}$ after foregoing $\{y\}$ and $\{x\}$.

We find that similar constructions work for all regular representations and for all $x \notin \bar{W}_{\text{inner}}(q, s)$. This yields the lemma that $x \notin \bar{W}_{\text{inner}}(q, s)$ if and only if $x$ matters for $q$ in state $s$ (Lemma A.3). This lemma establishes that $\bar{W}_{\text{inner}}(q, s)$ is identified from choice behavior. With this in hand, we then prove that the supporting hyperplanes of $\bar{W}_{\text{inner}}(q, s)$ at $q$ are the indifference curves of the utility functions in $V^s$. Thus, the preferences represented by the rationales can be deduced from choice data.

The preference represented by material utility function $u^s$ is easy to deduce, as the agent will always choose the materially best alternative from a binary menu $\{q, r\}$ when he was not offered any other menu in the first period. We leave further details (including the scaling of rationales relative to material utility) to Appendix A.

4.3 Empirical content

The identification procedure discussed above can also be used to study the general empirical content of the theory—that is, the predictions that follow directly from the existence of a regular representation, without any further assumptions about material utility or rationales.

One simple prediction regards the agent’s behavior when he has not made an ex post mistake. As noted in the last section, we can use binary menus to fully recover the agent’s material preferences. This allows us to check whether a given first-period choice is an ex post mistake. We can then check whether the agent maximizes his material preference after any ex post optimal first-period choice, as the theory predicts.

A more interesting prediction uses Lemma A.3, discussed in the previous section. Suppose that the agent chooses $r$ over $q$ in state $s$ when no other alternatives were available in the first period. We have seen that this implies $u^s(r) > u^s(q)$. Since material utility $u$ is one of the rationales, it cannot be that every rationale weakly prefers $q$ to $r$. By Lemma A.3, $r$ matters for $q$. In summary: if the agent would choose $r$ over $q$ in the second period when no other alternatives were available in the first period, then the sunk cost of foregoing $r$ in the first period can, under some circumstances, induce the decision maker to forego $q$ in the second period.

The converse does not hold, as $r$ may matter for $q$ in state $s$ even if $u^s(q) > u^s(r)$. Intuitively, $r$ alone cannot make it costly to choose $q$, but $r$ in combination with other,
better alternatives can. This is not to say that any alternative can matter for \( q \). In the proof of Theorem 4.3, we show that the set \( \bar{W}_{\text{inner}}(q, s) \) must be nonempty. Applying Lemma A.3 delivers another prediction: there exist alternatives that do not matter for \( q \).

### 4.4 Identification with only on-path choices

One plausible objection to our identification procedure is that some pairs \((M, \mathcal{M})\) could be off-path, meaning that the agent would not choose \( M \) from \( \mathcal{M} \) at time 1. (The off-path pairs depend on whether the agent is a naïf, a sophisticate, or an empathetic sophisticate, as well as on the agent’s prior.) If we rule out trembles, can we still elicit the counterfactual choice \( c^2_s(M \mid \mathcal{M}) \)?

We can elicit \( c^2_s(M \mid \mathcal{M}) \) using only on-path choices, provided that we expand the state space to include a payoff-irrelevant horse race, and offer choices between race-contingent menus.

The construction is as follows: We expand the state space to be \( S \times \{h, h'\} \), where \( h \) and \( h' \) represent different results for the horse race. By assumption, for all rationales \( v \), we have \( v^{s,h} = v^{s,h'} = v^s \). A race-contingent menu is a function \( N : \{h, h'\} \to K_f(\Delta(Z)) \). The agent now faces decision problems of this form:

1. At \( t = 1 \), the agent selects a race-contingent menu \( N \) from a finite collection of such menus \( \mathcal{N} \).
2. The agent learns the state \( s \) and the race result \( \rho \in \{h, h'\} \).
3. At \( t = 2 \), the agent chooses a lottery \( q \) from the menu \( N(\rho) \).

Suppose we wish to elicit \( c^2_s(M \mid \mathcal{M}) \) for some arbitrary pair \((M, \mathcal{M})\). We construct two race-contingent menus, \( N \) and \( N' \), such that \( N(h) = N'(h') = M \) and \( N(h') = N'(h) = \bigcup \mathcal{M} \). We then offer the agent, at time 1, the collection \( \mathcal{N} = \{N, N'\} \).

By construction, \( c^2_s(M \mid \mathcal{M}) \) is equal to the agent’s choice from \( N(h) \) in state \((s, h)\) and also equal to the agent’s choice from \( N'(h') \) in state \((s, h')\), that is

\[
c^2_s(M \mid \mathcal{M}) \equiv \arg\max_{q \in M} \left\{ (1 - \gamma)u^s(q) + \gamma \max_{v^s \in \mathcal{V}_s} \left\{ v^s(q) - \max_{\hat{q} \in \bigcup \mathcal{M}} v^s(\hat{q}) \right\} \right\} \\
= c^s_{2,h}(N(h) \mid \mathcal{N}) \equiv \arg\max_{q \in N(h)} \left\{ (1 - \gamma)u^s(q) + \gamma \max_{v^s \in \mathcal{V}_s} \left\{ v^s(q) - \max_{\hat{q} \in N(h) \cup N'(h)} v^s(\hat{q}) \right\} \right\} \\
= c^s_{2,h'}(N'(h') \mid \mathcal{N}) \equiv \arg\max_{q \in N'(h')} \left\{ (1 - \gamma)u^s(q) + \gamma \max_{v^s \in \mathcal{V}_s} \left\{ v^s(q) - \max_{\hat{q} \in N'(h') \cup N'(h)} v^s(\hat{q}) \right\} \right\}.
\]

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16 This prediction is easy to test if we assume structure on that rationales, such as that they are strictly increasing in money: \( r \) should not matter for \( q \) if every rationale regards \( r \) as worse than \( q \).

17 Savage (1972, p. 90) argued that some kinds of events are payoff-irrelevant: “Consider, for example, a lottery in which numbered tickets are drawn from a drum. It seems clear that for an ordinary person the outcome of the lottery is utterly irrelevant to his life, except through the rules of the lottery itself.”
If the agent chooses $N$, then we observe $c_s^2(M \mid M)$ in state $(s, h)$. Otherwise, he chooses $N'$, and we observe $c_{s'}^2(M \mid M)$ in state $(s, h')$.

In summary, in the expanded model with race-contingent menus, the model primitives are identified as in Theorem 4.3 even if we restrict the data to exclude off-path pairs $(N, N')$.  

5 Comparative statics for complements

At this point, the reader may be expecting an axiomatic treatment of ex post rationalization. However, our aim is not to remedy the known falsifications of expected utility theory, but rather to expand the theory to include rationalization as a motive for decision-making.

“All models are wrong, but some are useful” (Box, 1979). The axioms of expected utility theory are falsified by the Allais paradox (Allais, 1953), and yet the theory is a clean benchmark that delivers correct comparative statics in many natural settings. Ex post rationalization builds on expected utility theory and inherits many of the same falsifications.  

We think the theory is useful if it captures a commonsense aspect of human behavior. To assess this, we will supply the theory with psychologically plausible rationales and examine its predictions about choice. Hence, rather than pursuing a general axiomatization, we instead develop comparative statics given some standard restrictions on preferences.

We now assume that the rationales $\mathcal{V}$ have the form \{w(a_1, a_2, \theta, s) : \theta \in \Theta\}, for some set $\Theta$ and some function $w$.  

We use $\theta^*$ to denote the parameter value that corresponds to material utility, so $u(a_1, a_2, s) = w(a_1, a_2, \theta^*, s)$. To ease notation, we assume that $A_2$ does not depend on $s$. Hence, the agent facing some decision problem $D$, having chosen $a_1$ from menu $A_1$ and observed state $s$, chooses $a_2 \in A_2(a_1)$ and $\theta \in \Theta$ to maximize

$$U_D(a_2, \theta \mid a_1, s) \equiv (1 - \gamma)w(a_1, a_2, \theta^*, s) + \gamma \left[ w(a_1, a_2, \theta, s) - \max_{\hat{a}_1 \in A_1} \max_{\hat{a}_2 \in A_2(\hat{a}_1)} w(\hat{a}_1, \hat{a}_2, \theta, s) \right].$$

We assume that $w$ and $U_D$ have non-empty maxima with respect to $(a_1, a_2, \theta)$, and

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18 If we rule out contingent menus, is the model still identified by on-path behavior? Some further assumption is required. To illustrate, suppose that material utility $u$ does not depend on the state. Then the agent never makes an ex post mistake, so on-path behavior does not identify $\gamma$ or $\mathcal{V}$. However, if we assume that the agent’s ex ante preferences over $\Delta(Z)$ are ‘sufficiently misaligned’ with her ex post preferences, then the model is still identified by on-path behavior. We omit this result for brevity.

19 In particular, it reduces to expected utility theory for naïfs and sophisticates facing one-shot choices, i.e. singleton $A_2$.

20 On its own, this assumption is without loss of generality.
similarly for subsets of these arguments.

We will assume that the choice variables are complements—the marginal return of raising one variable is non-decreasing in the other variables. All our results cover the case of \( A_1, A_2, \) and \( \Theta \) totally ordered, and \( w \) supermodular in \( (a_1, a_2, \theta) \). If additionally \( w \) is differentiable, then this reduces to the requirement that the cross partial derivatives are all non-negative. However, we state our results under weaker order-theoretic assumptions to expand their scope.

We now state some standard definitions; for more detail see Milgrom and Shannon (1994). Suppose \( X \) and \( Y \) are partially ordered sets. Function \( f : X \times Y \times S \to \mathbb{R} \) has increasing differences between \( x \) and \( y \) if for all \( \tilde{x} \leq \tilde{x}', \) all \( \tilde{y} \leq \tilde{y}' \), and all \( s \), we have
\[
f(\tilde{x}, \tilde{y}', s) - f(\tilde{x}, \tilde{y}, s) \leq f(\tilde{x}', \tilde{y}', s) - f(\tilde{x}', \tilde{y}, s).
\]
Suppose \( X \) is a lattice and \( Y \) is an arbitrary set. Function \( f : X \times Y \to \mathbb{R} \) is supermodular in \( x \) if for all \( \tilde{x}, \) all \( \tilde{x}' \), and all \( y \), we have
\[
f(\tilde{x}, y) + f(\tilde{x}', y) \leq f(\tilde{x} \land \tilde{x}', y) + f(\tilde{x} \lor \tilde{x}', y).
\]

Given two lattices \( X \) and \( Y \), we order \( X \times Y \) according to the component-wise order. Given any lattice, we order subsets \( X \) and \( Y \) with the strong set order, writing \( X \ll Y \) if for any \( x \in X \) and \( y \in Y \), we have \( x \land y \in X \) and \( x \lor y \in Y \). Given a partially ordered set \( X \) and a lattice \( Y \), we say that a correspondence \( J : X \Rightarrow Y \) is monotone non-decreasing if \( x \leq x' \) implies that \( J(x) \ll J(x') \).

In decision problems with complements, Topkis’s theorem implies that raising the first action, ceteris paribus, raises the classical agent’s second action. For rationalizers, we find that raising the first action raises both the second action and the rationale.

**Proposition 5.1.** Let \( A_1 \) be a partially ordered set, let \( A_2 \) be a lattice and let \( \Theta \) be totally ordered. Suppose that \( w \) has increasing differences between \( a_1 \) and \( (a_2, \theta) \) and is supermodular in \( (a_2, \theta) \). Suppose that \( A_2(a_1) \) is monotone non-decreasing. For any decision problem \( D \) and any state \( s \), the correspondence
\[
\arg\max_{a_2 \in A_2(a_1)} U_D(a_2, \theta \mid a_1, s)
\]
is monotone non-decreasing in \( a_1 \).

The proof is in Appendix B.1. The conclusion of Proposition 5.1 implies that the

\[21\]The relation \( \leq \) is reflexive, so this implies that for all \( x \), \( J(x) \) is a sublattice of \( Y \).
agent’s chosen actions,

$$\arg\max_{a_2 \in A_2(a_1)} \left\{ \max_{\theta \in \Theta} U_D(a_2, \theta \mid a_1, s) \right\},$$

are monotone non-decreasing in $a_1$.

The next theorem shows that under appropriate complementarities, the theory yields systematic deviations from the classical benchmark. In the theory, sunk-cost effects are one instance of a more general phenomenon, stated as follows.

**Theorem 5.2.** Let $A_1$ be a partially ordered set, let $A_2$ be a lattice and let $\Theta$ be totally ordered. Suppose that $w$ has increasing differences between $a_1$ and $(a_2, \theta)$ and is supermodular in $(a_2, \theta)$. Suppose that $A_2(a_1)$ is monotone non-decreasing. If the agent’s time-1 choice was ex post weakly higher than optimal, i.e. $\bar{a}_1 \geq a_1^*$ for

$$a_1^* \in \arg\max_{a_1 \in A_1} \left\{ \max_{a_2 \in A_2(a_1)} w(a_1, a_2, \theta^*, s) \right\},$$

then the agent’s time-2 choice is weakly higher than materially optimal, i.e.

$$\arg\max_{a_2 \in A_2(\bar{a}_1)} \left\{ \max_{\theta \in \Theta} U_D(a_2, \theta \mid \bar{a}_1, s) \right\} \succ \arg\max_{a_2 \in A_2(\bar{a}_1)} w(\bar{a}_1, a_2, \theta^*, s),$$

(4)

and for any selection $\bar{a}_2$ from the left-hand side of (4), there exists $\bar{\theta} \geq \theta^*$ such that

$$\bar{a}_2 \in \arg\max_{a_2 \in A_2(\bar{a}_1)} U_D(a_2, \bar{\theta} \mid \bar{a}_1, s).$$

Symmetrically, if the agent’s time-1 choice was ex post weakly lower than optimal, then the agent’s time-2 choice is weakly lower than materially optimal, and there exists $\bar{\theta} \leq \theta^*$ such that $\bar{a}_2 \in \arg\max_{a_2 \in A_2(\bar{a}_1)} U_D(a_2, \bar{\theta} \mid \bar{a}_1, s).

The proof is in Appendix B.2.

Whether first actions, second actions, and rationales are complements depends not only on the agent’s preferences, but also on the structure of the decision problem. As is usual for comparative statics results, applying Theorem 5.2 requires good judgement in defining orders.

Now we consider changing the unchosen alternatives at $t = 1$. Observe that for a classical agent, unchosen alternatives from the first menu have no effect on second-period choice. Under complements, the next theorem predicts that increasing the first
menu (in the strong set order), while leaving the first choice unchanged, decreases the rationalizer’s second-period choice as well as her rationale.

**Theorem 5.3.** Let $A_1$ and $A_2$ be lattices and let $\Theta$ be totally ordered. Suppose that $w$ is supermodular in $(a_1, a_2, \theta)$. Suppose that the correspondence $A_2$ satisfies

\[
a_2 \in A_2(a_1) \text{ and } a_2' \in A_2(a_1') \implies a_2 \land a_2' \in A_2(a_1 \land a_1') \text{ and } a_2 \lor a_2' \in A_2(a_1 \lor a_1'). \tag{5}
\]

Take any $A_1, A_1' \subseteq A_1$ such that $A_1 \ll A_1'$. Let $D$ and $D'$ denote the decision problems with $A_1$ and $A_1'$ respectively. For any $\bar{a}_1 \in A_1 \cap A_1'$ and any $s$ we have

\[
\arg\max_{a_2 \in A_2(\bar{a}_1)} U_D(a_2, \theta | \bar{a}_1, s) \gg \arg\max_{a_2 \in A_2(\bar{a}_1)} U_{D'}(a_2, \theta | \bar{a}_1, s). \tag{6}
\]

The proof is in Appendix B.3.

An intuition for Theorem 5.3 is that when we add high actions to the first menu, the agent has to rationalize foregoing those actions, so he lowers $\theta$ and hence lowers $a_2$. Similarly, when we remove low actions from the first menu, the agent no longer has to rationalize foregoing those actions, so he lowers $\theta$ and hence lowers $a_2$.

Condition (5) is stronger that $A_2$ monotone non-decreasing. It is implied by $A_1$ totally ordered and $A_2$ monotone non-decreasing. Alternatively, it is implied by constant $A_2$.

Note that (6) implies that the rationalizer’s chosen actions decrease, i.e.

\[
\arg\max_{a_2 \in A_2(\bar{a}_1)} \left\{ \max_{\theta \in \Theta} U_D(a_2, \theta | \bar{a}_1, s) \right\} \gg \arg\max_{a_2 \in A_2(\bar{a}_1)} \left\{ \max_{\theta \in \Theta} U_{D'}(a_2, \theta | \bar{a}_1, s) \right\}.
\]

This provides a testable prediction of the theory that does not require us to separately identify the adopted rationales or the materially optimal benchmark.

### 5.1 Applications of results

We examine some natural decision problems that satisfy the assumptions of Proposition 5.1, Theorem 5.2, and Theorem 5.3.

#### 5.1.1 The sunk-cost effect for two-part projects

The agent chooses effort levels $a_1 \in [0, 1]$ and $a_2 \in [0, 1]$. The project succeeds with probability $a_1 a_2$, he receives a reward valued at $\theta \geq 0$ if it succeeds, and he pays effort cost $c_1(a_1, s) + c_2(a_2)$, for continuous non-decreasing cost functions $c_1$ and $c_2$, where $s$
is a real-valued cost shock for first-period effort. Hence

\[ w(a_1, a_2, \theta) = \theta a_1 a_2 - c_1(a_1, s) - c_2(a_2) \]

The agent chooses \( a_1 \) before learning \( s \), so the materially optimal choice of \( a_2 \) does not depend on the realized \( s \). Theorem 5.2 implies that when \( s \) has a high enough realization, so that \( a_1 \) was \textit{ex post} too high, the rationalizer’s choice of \( a_2 \) is distorted upwards compared to the classical benchmark. Hence, the agent persists more than is materially optimal for projects that turned out to be unexpectedly costly.

### 5.1.2 Encountering the same problem twice

The agent faces a decision problem, chooses an action, then learns the state, and then faces the same problem again. That is, \( A_1 = A_2 = A \) and

\[ w(a_1, a_2, \theta, s) = \phi(a_1, \theta, s) + \phi(a_2, \theta, s), \]

for some function \( \phi : A \times \Theta \times S \rightarrow \mathbb{R} \) that is supermodular in \((a, \theta)\). Let \( a^*(s) \) be an \textit{ex post} optimal choice in state \( s \), i.e. \( a^*(s) \in \arg\max_a \phi(a, \theta^*, s) \).

Material utility is time-separable, so upon learning the state, a classical agent’s time-2 choice is \( \arg\max_a \phi(a, \theta^*, s) \); his second-period choice does not depend on his first-period choice. By contrast, Theorem 5.2 implies that a rationalizer chooses \( a_2 \) to maximize \((1 - \gamma)\phi(a_2, \theta^*, s) + \gamma \phi(a_2, \bar{\theta}, s)\), with \( \bar{\theta} \geq \theta^* \) when \( a_1 \geq a^*(s) \) and \( \bar{\theta} \leq \theta^* \) when \( a_1 \leq a^*(s) \). Attempting to rationalize the earlier decision creates a link between otherwise-separate decisions, pulling the rationalizer’s second-period choice away from \( a^*(s) \) in the direction of his initial choice \( a_1 \). Thus, the rationalizer’s choice is ‘stickier’ than a classical agent’s choice, responding less to the new information about the state.

### 5.1.3 Belief elicitation

In many laboratory experiments, subjects provide point estimates of some quantity, then learn some information, and finally report updated estimates. They are paid for one decision drawn at random, so they encounter the same problem twice, in the sense of Section 5.1.2.

The outcome \( Y \) is a real-valued random variable with countable support. The subject makes an incentivized report of \( \mathbb{E}[Y] \), then observes a signal \( X \) with known conditional distribution \( g(x \mid y) \), then makes an incentivized report of \( \mathbb{E}[Y \mid X = x] \). The available rationales are priors on \( Y \); these are a set of probability mass functions

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23 Azrieli et al. (2018) study the merits of paying one decision drawn at random.

24 A parallel construction works if \( Y \) has support in some interval and each available rationale is an atomless distribution with strictly positive density.
indexed by $\theta$, denoted $(h_\theta)_{\theta \in \Theta}$. We assume that this set is totally ordered by the monotone likelihood ratio property (MLRP) (Milgrom, 1981), that is, for any $\theta > \theta'$ and any $y > y'$

$$h_\theta(y) h_{\theta'}(y') > h_{\theta'}(y) h_\theta(y').$$

This restriction is without loss of generality if $Y$ is a Bernoulli random variable, i.e. when the agent is being asked to report the probability of some event. We assume that each $h_\theta$ has full support, so that no rationale is ruled out by some signal realization.

Given prior $h_\theta$ and signal realization $x$, we denote the posterior probability mass function $h_\theta(y \mid X = x)$. The agent reports $a_1$, then observes the signal realization, then reports $a_2$. For each report $a_t$, the agent faces quadratic loss (conditional on the signal realization), resulting in the payoff

$$\phi(a_t, \theta, x) = - \sum_y (a_t - y)^2 h_\theta(y \mid X = x).$$

This captures the interim expected utility of a risk-neutral agent facing a quadratic scoring rule. It also captures the interim expected utility of an agent with general risk preferences facing an appropriate binarized scoring rule (Hossain and Okui, 2013).

Given the same signal realization, MLRP-ordered priors induce posteriors that are ordered by first-order stochastic dominance (Milgrom, 1981; Klemens, 2007). Thus, if $\theta > \theta'$ then $h_\theta(y \mid X = x)$ first-order stochastically dominates $h_{\theta'}(y \mid X = x)$. It follows that $\phi$ is supermodular in $(a_t, \theta)$.

Our analysis in Section 5.1.2 implies that when $a_1 \geq E[Y \mid X = x]$, then the agent’s reported posterior beliefs are distorted upwards, $a_2 \geq E[Y \mid X = x]$. Such preference for consistency in belief elicitation is folk wisdom amongst experimenters.

### 5.1.4 Consumption under two-part tariffs

Consider a consumer facing two-part tariffs, each consisting of a lump-sum payment $L$ and a per-unit price $p$ (both non-negative), as in Thaler (1980). The consumer faces a finite list of such tariffs, denoted $(L_k, p_k)_{k \in K}$. Without loss of generality, we assume that the list contains no dominated tariffs and no duplicates. We order the tariffs so that $L_1 < L_2 < \cdots$ and $p_1 > p_2 > \cdots$.

The timing is as follows:

\[\begin{align*}
25\text{Also in that case, when } a_1 \in \arg\max_{a \in A_1} \{\phi(a, \theta', x)\} \text{ for some } \theta', \text{ then } a_2 \leq a_1. \text{ (If } a_2 > a_1, \text{ then } \phi(a_1, \theta'^*, x) > \phi(a_2, \theta'^*, x), \text{ in which case by reducing } a_2 \text{ to } a_1 \text{ and adopting the rationale } \theta'^*, \text{ the agent can achieve higher material utility and zero rationalization utility, which contradicts the optimality of } a_2.\text{) This assumption is satisfied whenever the rationales include all full-support priors on } Y.}
\]

\[\begin{align*}
26\text{See Falk and Zimmermann (2018) for experiments finding that laboratory subjects report beliefs that are distorted towards their prior reports.}
\]
1. The consumer chooses a tariff \((L_k, p_k)\) from the list or declines.

2. The consumer learns the taste shock \(s \in [0, 1]\).

3. If the consumer chose a tariff, the consumer chooses quantity \(q \in [0, 1]\).

The set of rationales is indexed by \(\Theta = [0, 1]\). The consumer’s utility from tariff \((L, p)\) under rationale \(\theta\) is

\[
u(q, s, \theta, L, p) = s\psi(q, \theta) - pq - L\]

where \(\psi\) is continuous and supermodular in both arguments, and \(\psi(0, \theta) = 0\) for all \(\theta\). The utility from declining is 0.

Consider a list comprised of tariffs \((L_1, p_1)\) and \((L_2, p_2)\), with \(L_1 < L_2\) and \(p_1 > p_2\). By Proposition 5.1, changing the chosen tariff from \((L_1, p_1)\) to \((L_2, p_2)\) weakly raises the quantity consumed, for every realization of the taste shock.

Observe that for a classical consumer, once a tariff has been chosen, the lump-sum \(L\) is sunk and has no effect on the quantity demanded. By contrast, (Thaler, 1980) proposes that when a consumer responds to sunk costs, raising the lump-sum payment can increase the quantity demanded. We formalize this observation in the context of our model.

Take any per-unit price \(p\) and taste shock \(s\). Take any \(L < L'\) such that

\[
\max_{q \in [0,1]} \{s\psi(q, \theta^*) - pq - L\} \geq 0 > \max_{q \in [0,1]} \{s\psi(q, \theta^*) - pq - L'\}.
\]

If the consumer was offered only tariff \((L, p)\) and accepted, then in state \(s\) this was not an \textit{ex post} mistake, so he demands the materially optimal quantity. By contrast, if he was offered only tariff \((L', p)\) and accepted, then his first choice was \textit{ex post} too high. Theorem 5.2 implies that his demand is weakly higher after \((L', p)\) than after \((L, p)\). Moreover, it is strictly higher for various simple functional forms, such as \(\psi(q, \theta) = \theta \sqrt{q}\) with interior \(\theta^*\). Thus, high enough lump-sum payments can raise demand compared to the material optimum, provided that they do not cause the consumer to decline the tariff.

Finally, let us take compare two lists, one of which is produced by truncating the other from above. That is, we have \((L_k, p_k)^{K}_{k=1}\) and \((L_k, p_k)^{K'}_{k=1}\), for \(K < K'\). Suppose we fix the agent’s chosen tariff at \((L_j, p_j)\) for \(j \leq K\) and switch from the full list \((L_k, p_k)^{K'}_{k=1}\) to the truncated list \((L_k, p_k)^{K}_{k=1}\). By Theorem 5.3, this change weakly increases demand in every state. This suggests that a firm selling to rationalizing consumers may find it beneficial to withdraw options with high lump-sums and low marginal prices, especially if those options are seldom chosen.
6 Extensions

In our model, all uncertainty is resolved before the second action. If instead the agent observes a random variable \( X \) correlated with the state, then it is natural to stipulate that she assesses rationalization utility conditional on the signal realization. This can be accommodated by transforming the state space and utility functions, so our previous results also apply to decisions with noisy signals. Namely, given each signal realization \( x \) and each rationale \( v \), we define

\[
\bar{v}(a_1, a_2, x) \equiv \mathbb{E}_s[v(a_1, a_2, s) \mid X = x],
\]

and transform the objective function (1) by substituting \( \bar{v} \) for \( v \) and \( x \) for \( s \). Moreover, each rationale can represent a different prior about \( s \), so that the agent can rationalize her actions by adjusting her beliefs, as in Section 5.1.3.

More subtly, the agent’s first action could determine not only her payoffs, but also what signal she sees. In this case, sophisticated rationalizers have a novel motive for information avoidance: If the sophisticate avoids new information, then \textit{ex ante} optimal actions are also \textit{ex post} optimal, so there is no loss of material utility from rationalizing behavior.

The theory can be extended beyond two periods. To do so, we stipulate that at each time \( t \), the agent chooses some rationale \( v \), and compares the expected utility of her strategy under \( v \) to the expected utility of the interim-optimal strategy for \( v \), with both expectations conditional on the information available at \( t \). At each \( t \), she chooses a continuation strategy that maximizes a weighted sum of expected material utility and this generalized rationalization utility. For sophisticates, we restrict the continuation strategies to be consistent with future rationalizing behavior.

Should we additionally require that the chosen rationales are stable over time? That is, if the agent acts on Monday, Tuesday, and Wednesday, can she adopt one rationale on Tuesday and then a different rationale on Wednesday? The theory does not forbid such inconsistent rationalizations, but it does weigh against them, because Tuesday’s rationale affects Tuesday’s action, which then affects Wednesday’s rationale. If we directly required Tuesday’s rationale to be equal to Wednesday’s rationale, then the theory’s predictions would vary with the addition of ‘dummy periods’ with singleton action menus. In the interest of invariance, we do not impose this requirement.

7 Conclusion

Standard economic theory holds that people make decisions to satisfy their preferences. By contrast, the present theory allows that people adopt preferences to rationalize their
past decisions.

Decision-makers appear to rationalize their choices *ex post*, in situations ranging from investment to warfare. In the Iliad, Odysseus urges the Achaeans to persist, arguing that after nine years spent besieging Troy, “what a humiliation it would be / to hold out so long, then sail home empty-handed” (Fagles, 1990, 2.348-49).

Having come this far, it is for the reader to decide whether the exercise was worthwhile.

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A Proof of Theorem 4.3

We establish some notation.

- Since we will only compare choices within a given state, we drop the superscript $s$ throughout.
- For any $G \subseteq \mathcal{U}$, let $G_{\text{pref}}$ denote the set of preferences with representations in $G$.
- For any set $X$, let $\text{co}(X)$ denote the convex hull, and let $\bar{X}$ denote the closure.\footnote{When $X$ is a set of preferences, a preference $\preceq \in \text{co}(X)$ if some representation of $\preceq$ is a convex combination of representations of preferences in $S$. A non-constant preference $\preceq \in \bar{X}$ if some representation of $\preceq$ is the limit of representations of preferences in $X$.}
- For any $q \in \Delta(Z)$ and any $M \in \mathcal{K}_f(\Delta(Z))$ such that $q \in M$, let

$$U(q | M) \equiv (1 - \gamma)u(q) + \gamma \max_{v \in \mathcal{V}} \left( v(q) - \max_{\tilde{q} \in \bar{M}} v(\tilde{q}) \right).$$

- For any $q \in \Delta(Z)$, let

$$W_{\text{inner}}(q) \equiv \bigcap_{v \in \mathcal{V}} \{ r \in \Delta(Z) : v(q) > v(r) \}$$

$$W_{\text{outer}}(q) \equiv \bigcap_{v \in \mathcal{V}} \{ r \in \Delta(Z) : ((1 - \gamma)u + \gamma v)(q) > ((1 - \gamma)u + \gamma v)(r) \}.$$

The next lemma establishes some properties of $W_{\text{inner}}(q)$ and $W_{\text{outer}}(q)$.

**Lemma A.1.** For any regular representation $(\gamma, u, \mathcal{V})$ and any $q \in \text{int}(\Delta(Z))$, $W_{\text{inner}}(q)$ and $W_{\text{outer}}(q)$ are open and nonempty, and satisfy

$$\bar{W}_{\text{inner}}(q) = \bigcap_{v \in \mathcal{V}} \{ r \in \Delta(Z) : v(q) \geq v(r) \} \quad (7)$$

$$\bar{W}_{\text{outer}}(q) = \bigcap_{v \in \mathcal{V}} \{ r \in \Delta(Z) : ((1 - \gamma)u + \gamma v)(q) \geq ((1 - \gamma)u + \gamma v)(r) \}. \quad (8)$$

**Proof.** For openness of $W_{\text{inner}}(q)$: Consider a sequence $r_i \to r$ such that $r_i \notin W_{\text{inner}}(q)$ for all $i$. By definition of $W_{\text{inner}}(q)$, for each $r_i$, there exists some $v_i \in \mathcal{V}$ such that $v_i(r_i) \geq v_i(q)$. Since $\mathcal{V}$ is compact, we can pass to a convergent subsequence $\tilde{v} \in \mathcal{V}$. We must have $\tilde{v}(r) \geq \tilde{v}(q)$, so $r \notin W_{\text{inner}}(q)$. The argument for openness of $W_{\text{outer}}(q)$ is the same since $\{(1 - \gamma)u + \gamma v : v \in \mathcal{V}\}$ inherits compactness from $\mathcal{V}$.

For nonemptiness of $W_{\text{inner}}(q)$: Fix $q \in \Delta(Z)$ and $R \in \mathcal{K}_f(\Delta(Z))$. Suppose there does not exist $v \in \mathcal{V}$ such that $v(q) \geq \max_{r \in R} v(r)$. We show that there exists $\tilde{r} \in \text{co}(R)$ such that $v(q) < v(\tilde{r})$ for all $v \in \mathcal{V}$. For each $v \in \mathcal{V}$, we have some $r \in R$ such that $v(r) < v(q)$. It without loss to assume that $v(q) = 0$ for all $v \in \mathcal{V}$. We want to find a...
set of weights $\alpha$ such that
\[ \sum_{r \in R} \alpha(r)v(r) > 0 \] (9)
for all $v \in V$. Enumerate the elements of $R$: $(r_1, \ldots, r_{|R|})$. For each $v \in V$, let $v_R$ denote the vector with $v(r_i)$ as its $i$-th entry. Let $V_R$ denote $\{v_R : v \in V\}$. As with $V$, $V_R$ is nonempty, compact and convex. Let $N \equiv \mathbb{R}_{\leq 0}^{|R|}$, which is nonempty, closed and convex. Notice that no element of $V_R$ can be less than or equal to 0. (Otherwise, some $v \in V$ would rank $q$ weakly higher than each member of $R$.) Thus, $V_R$ and $N$ are disjoint, and we can apply the separating hyperplane theorem. This delivers a nonzero $\alpha \in \mathbb{R}^{|R|}$ and $\psi \in \mathbb{R}$ such that $\alpha' n < \psi < \alpha' v_R$ for all $n \in N$, $v_R \in V_R$. Since the zero vector belongs to $N$, we must have $\psi > 0$. Suppose the $i$th element of $\alpha$ is strictly negative. By choosing $n$ with a sufficiently negative number in $i$th position and zeros elsewhere, we get $\alpha' n > \psi$, a contradiction. Thus, each element of $\alpha$ is weakly positive.

If we rescale $\alpha$ to a unit sum, we still have $\alpha' v_R > 0$ for all $v_R \in V_R$. This inequality is equivalent to (9).

Now fix some $q \in \text{int}(\Delta(Z))$, and suppose that $W_{\text{inner}}(q) = \emptyset$. Since there is no $r \in \Delta(Z)$ such that $v(q) > v(r)$ for all $v \in V$, there cannot be any $\bar{r} \in \Delta(Z)$ such that $v(\bar{r}) > v(q)$ for all $v \in V$. Fix any $R \in \mathcal{K}_f(\Delta(Z))$ such that $q \in \text{int}(\text{co}(R))$. By the previous argument, there must be some $v \in V$ such that $v(q) \geq \max_{r \in R} v(r)$. Since $q \in \text{int}(\text{co}(R))$, $v$ must be a constant. By non-redundancy and $|V| > 1$, $V$ must also contain some non-constant utility $v'$. Since $V$ is convex, it contains all convex combinations of $v'$ and $v$, which violates non-redundancy. Conclude that $V$ does not contain a constant preference, so $W_{\text{inner}}(q) \neq \emptyset$. Moreover, $W_{\text{inner}}(q) \subseteq W_{\text{outer}}(q)$, so $W_{\text{outer}}(q) \neq \emptyset$.

For (7): Fix some $q \in \text{int}(\Delta(Z))$ and some $r \in \Delta(Z)$ such that $v(q) \geq v(r)$ for all $v \in V$. We want to show that $r$ is a limit point of $W_{\text{inner}}(q)$. Let $\epsilon_k \equiv \frac{1}{2k}$ for $k \in \mathbb{N}$. By $q$ interior, $W_{\text{inner}}(q)$ is non-empty. Let $x$ be some element of $W_{\text{inner}}(q)$, and define $q_k \equiv \epsilon_k x + (1 - \epsilon_k)q$. Observe that $q_k \to q$ and for all $k$, $q_k \in W_{\text{inner}}(q)$. For all $k$ and all $v \in V$, we have
\[ v(q) > \epsilon_k v(q_k) + (1 - \epsilon_k)v(r) = v(\epsilon_k q_k + (1 - \epsilon_k)r) \]
and thus $\epsilon_k q_k + (1 - \epsilon_k)r \in W_{\text{inner}}(q)$. Moreover, $\epsilon_k q_k + (1 - \epsilon_k)r \to r$, so $r$ is a limit point of $W_{\text{inner}}(q)$.

Now fix some $r \in W_{\text{inner}}(q)$. We have a sequence $r_k \to r$ such that, for each $k$, $v(q) > v(r_k)$ for all $v \in V$. By continuity of $v$, $v(q) \geq v(r)$ for all $v \in V$. The same arguments work for (8).

Next, we show that $W_{\text{inner}}(q)$ can be fully identified from choice data. Since the
agent’s $t = 2$ choices depend on the collection $\mathcal{M}$ only via $\bigcup \mathcal{M}$, we abuse notation and write $c_2(M | \mathcal{M})$ as $c_2(M | \bigcup \mathcal{M})$.

**Definition A.2.** Fix $q, x \in \Delta(Z)$. Say that $x$ **matters for** $q$ if there exist $y, r \in \Delta(Z)$ such that

\[
q \in c_2(q, r \mid q, r, y) \\
q \notin c_2(q, r \mid q, r, y, x).
\]

**Lemma A.3.** For any regular representation $(\gamma, u, \mathcal{V})$, any $q \in \text{int}(\Delta(Z))$, and any $x \in \Delta(Z)$, we have that $x \notin \bar{W}_{\text{inner}}(q)$ if and only if $x$ matters for $q$.

**Proof.** Suppose that $x \notin \bar{W}_{\text{inner}}(q)$. We show that $x$ matters for $q$.

First, suppose that $q \in \bar{W}_{\text{inner}}(x)$. Let $v^*$ be any member of $\mathcal{V}$ that maximizes $v^*(q) - v^*(x)$, which exists by compactness of $\mathcal{V}$. We have

\[
U(q \mid q, x) = (1 - \gamma)u(q) + \gamma(v^*(q) - v^*(x)).
\]

Since $u$ is in the relative interior of $\mathcal{V}$, it is without loss to assume that $v^* \neq u$. Since $q$ is interior and $v^* \neq u$, by non-redundancy there exists $r \in \Delta(Z)$ such that $u(q) = u(r)$ and $v^*(r) > v^*(q)$. We can always choose $r$ close enough to $q$ that $v^*(x) > v^*(r)$. Then,

\[
U(r \mid q, r, x) \geq (1 - \gamma)u(q) + \gamma(v^*(r) - v^*(x)) \\
> (1 - \gamma)u(q) + \gamma(v^*(q) - v^*(x)) \\
= U(q \mid q, x) \\
\geq U(q \mid q, r, x) \\
\implies q \notin c_2(q, r \mid q, r, x).
\]

Since $u(q) = u(r)$,

\[
U(q \mid q, r) = (1 - \gamma)u(q) = U(r \mid q, r) \\
\implies q \in c_2(q, r \mid q, r).
\]

Conclude that $x$ matters for $q$. (To fit the definition, set $y$ equal to $q$ or $r$.)

Now we consider the case $q \notin \bar{W}_{\text{inner}}(x)$. Since $x \notin \bar{W}_{\text{inner}}(q)$, there exists $v^* \in \mathcal{V}$ such that $v^*(x) > v^*(q)$. It is without loss to assume that $v^*$ is extreme. Fix $\hat{y} \in W_{\text{inner}}(q)$, which is nonempty by Lemma A.1. Since $q$ is interior, $q + \epsilon(q - \hat{y}) \in \Delta(Z)$ for $\epsilon > 0$ sufficiently small. For all such $\epsilon$, let $\tilde{y}_\epsilon := q + \epsilon(q - \hat{y})$. Since $\hat{y} \in W_{\text{inner}}(q)$,
we have $q \in W_{\text{inner}}(\bar{y}_\epsilon)$ for all $\epsilon$ such that $\bar{y}_\epsilon$ is defined. For all such $\epsilon$, let

$$\lambda_\epsilon := \frac{v^*(\bar{y}_\epsilon) - v^*(q)}{v^*(x) - v^*(\bar{y}_\epsilon)}.$$

Since $\bar{y}_\epsilon \to q$ and $v^*(x) > v^*(q)$, we have $\lim_{\epsilon \to 0} \lambda_\epsilon = 0$. Since $q$ is interior, $\bar{y}_\epsilon + \lambda_\epsilon (\bar{y}_\epsilon - x) \in \Delta(Z)$ for $\epsilon > 0$ sufficiently small. Let $y_\epsilon := \bar{y}_\epsilon + \lambda_\epsilon (\bar{y}_\epsilon - x)$ for all such $\epsilon$. By construction, $v^*(y_\epsilon) = v^*(q)$ for all $\epsilon$ such that $y_\epsilon$ is defined.

For every $\epsilon$ for which $y_\epsilon$ is defined, $\bar{y}_\epsilon$ is a convex combination of $y_\epsilon$ and $x$. Since every $v \in \mathcal{V}$ has $v(\bar{y}_\epsilon) > v(a)$ for all $\epsilon$ such that $y_\epsilon$ is defined, we have $\max_{y_\epsilon,x} v > v(q)$ for all $v \in \mathcal{V}$ and all such $\epsilon$. This implies

$$U(q \mid q, y_\epsilon, x) < (1 - \gamma)u(q).$$

for all $\epsilon > 0$ for which $y_\epsilon$ is defined.

Since $q \notin W_{\text{inner}}(x)$, there exists $\bar{v} \in \mathcal{V}$ such that $\bar{v}(q) > \bar{v}(x)$. It is without loss to assume that $\bar{v}$ is extreme. We claim that there exists $r \in \Delta(Z)$ such that

$$u(r) = u(q)$$
$$v^*(r) \leq v^*(q)$$
$$\bar{v}(r) > \bar{v}(q).$$

Suppose not. Then, $v^*$ and $\bar{v}$ must agree when restricted to $\{r \in \Delta(Z) : u(r) = u(q)\}$. But this implies that $v^* \in \text{co}(\{\bar{v}, u\})$ or that $\bar{v} \in \text{co}(\{v^*, u\})$. The first case contradicts extremeness of $v^*$, and the second contradicts extremeness of $\bar{v}$.

Since $\bar{v}(r) > \bar{v}(q) > \bar{v}(x)$, and since $\lim_{\epsilon \to 0} y_\epsilon = q$,

$$\bar{v}(r) = \max_{q,r,y_\epsilon,x} \bar{v}$$

for all $\epsilon$ sufficiently small. This implies

$$U(r \mid q, r, y_\epsilon, x) = (1 - \gamma)u(r)$$
$$U(r \mid q, r, y_\epsilon) = (1 - \gamma)u(r)$$
for all $\epsilon$ sufficiently small. Since $U(q | Q) \leq U(q | R)$ whenever $q \in R \subset Q$,

$$U(q | q, y, x) \leq U(q | q, y, x)$$

$$< (1 - \gamma)u(q)$$

$$= (1 - \gamma)u(r)$$

$$= U(r | q, r, y, x)$$

$$\implies q \notin c_2(q, r | q, r, y, x)$$

for all $\epsilon$ sufficiently small. Since $v^*(q) = v^*(y) \geq v^*(r)$ for all $\epsilon$ such that $y$ is defined,

$$U(q | q, y) = (1 - \gamma)u(q)$$

$$= (1 - \gamma)u(r)$$

$$= U(r | q, r, y)$$

$$\implies q \in c_2(q, r | q, r, y)$$

for all $\epsilon$ sufficiently small. Conclude that $x$ matters for $q$.

The next lemma shows that we can use $W_{inner}(q)$ to fully recover $V_{pref}$.

**Lemma A.4.** Fix any regular representation $(\gamma, u, V)$ and any $q \in int(\Delta(Z))$.

1. For any $G_{pref} \subset U_{pref}$ such that $W_{inner}(q) = \bigcap_{\succeq \in G_{pref}} \{ r \in \Delta(Z) : q \succ r \}$,

$$\bar{co}(G_{pref}) = \{ \succeq \in U_{pref} : q \succ r \ for \ all \ r \in W_{inner}(q) \}.$$

2. $V_{pref} = \{ \succeq \in U_{pref} : q \succ r \ for \ all \ r \in W_{inner}(q) \}$.

**Proof.** First part: We show that $\succeq^* \in co(G_{pref})$ if $q \succ r$ for all $r \in W_{inner}(q) \setminus \{q\}$. By definition of $G_{pref}$, we have

$$\{ r \in \Delta(Z) : r \succeq^* q \} \setminus \{ q \} \subset \bigcup_{\succeq \in V_{pref}} \{ r \in \Delta(Z) : r \succ q \}.$$

Fix some $\epsilon > 0$ such that the $\epsilon$-ball $B_\epsilon(q) \subset int(\Delta(Z))$. We have

$$\{ r \in \Delta(Z) : r \succeq^* q \} \setminus B_\epsilon(q) \subset \bigcup_{\succeq \in V_{pref}} \{ r \in \Delta(Z) : r \succ q \}.$$

By the Heine-Borel theorem, there exists some finite $\hat{G}_{pref} \subset G_{pref}$ such that

$$\{ r \in \Delta(Z) : r \succeq^* q \} \setminus B_\epsilon(q) \subset \bigcup_{\succeq \in \hat{G}_{pref}} \{ r \in \Delta(Z) : r \succ q \}. \quad (10)$$
This implies

$$\{ r \in \Delta(Z) : r \succeq^* q \} \setminus \{ q \} \subset \bigcup_{\succeq \in \hat{G}_{\text{pref}}} \{ r \in \Delta(Z) : r \succ q \}. \quad (11)$$

Suppose not. Then, there must be some \( r \in B_e(q) \setminus \{ q \} \) such that \( r \succ^* q \), but \( r \succ q \) for all \( \succeq \in \hat{G}_{\text{pref}} \). For some \( \lambda > 0 \), we will have \( r + \lambda(r - q) \in \Delta(Z) \setminus B_e(q) \). By linearity, \( r + \lambda(r - q) \succ^* q \), but \( r + \lambda(r - q) \not\succ q \) for all \( \succeq \in \hat{G}_{\text{pref}} \). Since \( r + \lambda(r - q) \not\in B_e(q) \), this contradicts (10).

Assign \( \succeq^* \) a representation \( v^* \) such that \( v^*(q) = 0 \). For each \( \succeq \in \hat{G}_{\text{pref}} \), assign a representation \( v \) such that \( v(q) = 0 \), and let \( \hat{G} \) denote the resulting set of utilities. By Farkas’ lemma, we can find \( \alpha > 0 \) such that \( \alpha v^* \in \text{co}(\hat{G}) \) provided there does not exist \( \rho \in \mathbb{R}^{|Z|} \) such that \( \rho^t v^* < 0 \) and \( \rho^t v \geq 0 \) for all \( v \in S \). Suppose there exists some such \( \rho \). Consider the case \( \sum_{k;\rho_k > 0} \rho_k \geq \sum_{k;\rho_k < 0} (-\rho_k) \). Normalize \( \rho \) by dividing each \( \rho_k \) by \( \sum_{k;\rho_k > 0} \rho_k \). (Since \( \rho \neq 0 \), this term must be strictly positive.) We have

$$\sum_{k;\rho_k > 0} \rho_k v^*(q_k) < \sum_{k;\rho_k < 0} (-\rho_k)v^*(q_k)$$

$$\sum_{k;\rho_k > 0} \rho_k v(q_k) \geq \sum_{k;\rho_k < 0} (-\rho_k)v(q_k) \text{ for all } v \in S.$$  

Since the normalization ensures that \( \sum_{k;\rho_k > 0} \rho_k = 1 \), the left-hand side is the valuation of a lottery, which we label \( \rho^+ \). Since \( v^*(q) = v(q) = 0 \),

$$v^*(\rho^+) = \sum_{k;\rho_k < 0} (-\rho_k)v^*(q_k) + \left( 1 - \sum_{k;\rho_k < 0} (-\rho_k) \right) v^*(q)$$

$$v(\rho^+) = \sum_{k;\rho_k < 0} (-\rho_k)v(q_k) + \left( 1 - \sum_{k;\rho_k < 0} (-\rho_k) \right) v(q) \text{ for all } v \in \hat{G}.$$  

Now the right-hand side is also the valuation of a lottery, which we label \( \rho^- \). We have

$$v^*(\rho^+) < v^*(\rho^-)$$

$$v(\rho^+) \geq v(\rho^-) \text{ for all } v \in \hat{G}.$$  

For \( \epsilon > 0 \) sufficiently small, we have

$$v^*(q) < v^*(q + \epsilon(\rho^- - \rho^+))$$

$$v(q) \geq v(q + \epsilon(\rho^- - \rho^+)) \text{ for all } v \in \hat{G}.$$  

This contradicts (11). An exactly similar argument covers the case \( \sum_{k;\rho_k > 0} \rho_k \geq \sum_{k;\rho_k < 0} -\rho_k \).
\[ \sum_{k: \rho_k < 0} (\rho_k). \] Conclude that \( av^* \in \text{co}(\hat{G}) \) for some \( \alpha > 0 \), so \( \succsim^* \in \text{co}(\hat{G}_{\text{pref}}) \). Since \( \hat{G}_{\text{pref}} \subset G_{\text{pref}}, \succsim^* \in \text{co}(G_{\text{pref}}) \).

Now take any \( \succsim \in U_{\text{pref}} \) such that \( q \succ r \) for all \( r \in W_{\text{inner}}(q) \). Suppose that there is some \( \succsim^* \in U_{\text{pref}} \) such that \( q \succ^* r \) for all \( r \in W_{\text{inner}}(q) \setminus \{q\} \). We already showed that \( \succsim^* \in \text{co}(G_{\text{pref}}) \). Assign a representation \( \tilde{v} \), and assign \( \succsim^* \) a representation \( v^* \). Fix any sequence \( \epsilon_i \to 0 \) such that \( \epsilon_i \in (0, 1) \) for each \( i \). Each \( \epsilon_i v^* + (1 - \epsilon_i) \tilde{v} \) represents some \( \succsim_i \in U_{\text{pref}} \). For each \( i, q \succ_i r \) for all \( r \in W_{\text{inner}}(q) \setminus \{q\} \). Thus, \( \succsim_i \in V_{\text{pref}} \) for all \( i \). Since the utilities that represent \( \succsim_i \) converge to \( \tilde{v} \), which represents \( \succsim \), we have \( \succsim \in \text{co}(G_{\text{pref}}) \).

Finally, suppose there is no \( \succsim^* \in U_{\text{pref}} \) such that \( q \succ^* r \) for all \( r \in W_{\text{inner}}(q) \setminus \{q\} \). For any \( J \subset Z \), we will use the superscript \( J \) to denote the restriction of a set or preference to \( \Delta(J) \). We will find \( J^* \subset Z \) such that \( |J^*| \geq 3 \) for which there exists an EU preference \( \succsim^* \) on \( \Delta(J^*) \) with the following property: for any \( q^* \in \Delta(J^*) \),

\[ r^* \in W_{\text{inner}}(q^*) \setminus \{q^*\} \implies q^* \succ^* r^*. \tag{12} \]

Since there is no preference with this property when \( J^* = Z \), there must be \( q^0, r^0 \in \Delta(Z) \) such that

\[ r^0 \in W_{\text{inner}}(q^0) \setminus \{q^0\} \]
\[ q^0 + (q^0 - r^0) \in W_{\text{inner}}(q^0) \setminus \{q^0\}. \]

This implies \( r^0 \sim q^0 \) for all \( \succsim \in G_{\text{pref}} \). Since \( r^0 \neq q^0 \), we can always relabel the elements of \( Z \) so that \( r^0(z_1) \neq q^0(z_1) \). Let \( J^1 \equiv Z \setminus \{z_1\} \), and check whether there is any preference satisfying (12) with \( J^1 \) in place of \( J^* \). If not, then we can repeat the argument, deriving a new indifference condition \( q^1 \sim r^1 \) and setting \( J^2 \equiv Z \setminus \{z_1, z_2\} \).

We repeat the process until we arrive at the desired \( J^* \). To see why some such \( J^* \) must exist, suppose that we have iterated the process to \( J^{n-3} = \{z_{n-2}, z_{n-1}, z_n\} \).

Suppose there is no preference satisfying (12) with \( J^{n-3} \) in place of \( J^* \). This can only happen if \( W_{\text{inner}}(J^{n-3}) \) is a half-plane, which in turn only happens if \( |V_{\text{pref}}^{J^{n-3}}| = 1 \). Since each preference in \( V_{\text{pref}} \) is pinned down by its restriction to \( V_{\text{pref}}^{J^{n-3}} \) (and the indifference conditions), we have \( |V_{\text{pref}}| = 1 \). This contradicts non-redundancy and \( |V| > 1 \). Conclude that \( J^* \) exists.

Fix any \( q^* \in \text{int}(\Delta(J^*)) \). The first part of the proof implies, for any EU preference \( \succsim J^* \) on \( \Delta(J^*) \),

\[ \succsim J^* \in G_{\text{pref}}^{J^*} \iff q^* \succ J^* \succ^* r^* \text{ for all } r^* \in W_{\text{inner}}(q^*). \tag{13} \]

Take any \( \succsim \in U_{\text{pref}} \) such that, for all \( q \in Z, q \succ r \) for all \( r \in W_{\text{inner}}(q) \). By (13), the
restriction of \( \succeq \) to \( J^* \) belongs to \( G^*_\text{pref} \). Thus, there is some preference in \( G_{\text{pref}} \) that agrees with \( \succeq \) in its restriction to \( J^* \). This preference is pinned down by the indifference conditions derived at the last step: it must be indifferent between \( q^0 \) and \( r^0 \), \( q^1 \) and \( r^1 \), and so on. Notice that \( \succeq \) must satisfy all these indifference conditions. (For instance, \( q^0 \sim r^0 \) because \( r^0 \in \bar{W}_{\text{inner}}(q^0) \) implies \( q^0 \succeq r^0 \), while \( q^0 + (q^0 - r^0) \in \bar{W}_{\text{inner}}(q^0) \) implies \( r^0 \succeq q^0 \). An exactly similar argument covers any remaining indifference conditions.)

Conclude that \( \succeq \in G_{\text{pref}} \).

Second part: By definition,

\[
W_{\text{inner}}(q) = \bigcap_{\succ \in \mathcal{V}_{\text{pref}}} \{ r \in \Delta(Z) : q \succ r \}.
\]

By the first part, \( \succeq \in \bar{c}(\mathcal{V}_{\text{pref}}) \) if and only if \( q \succ r \) for all \( r \in W_{\text{inner}}(q) \). Since \( \mathcal{V}_{\text{pref}} \) is closed and convex, \( \succeq \in \mathcal{V}_{\text{pref}} \) if and only if \( q \succ r \) for all \( r \in W_{\text{inner}}(q) \).

We now recover \( \succeq_u \), the preference relation represented by the material utility function. For any \( M \in K_f(\Delta(Z)) \), we have \( c_2(M \mid M) = \text{argmax}(M, \succeq_u) \). For any \( q, r \in \Delta(Z) \), we have

\[
q \succeq_u r \iff q \in c_2(q, r \mid q, r).
\]

This allows us to recover \( \succeq_u \). As usual, the representation of \( \succeq_u \) is pinned down up to a positive affine transformation.

We now show that, for any two representations with the same \( u \) and \( \gamma \), for any \( \succeq \in \mathcal{V}_{\text{pref}} \), the utilities that represent \( \succeq \) differ (at most) by an additive constant. For this step, we need to know \( W_{\text{outer}}(q) \). The following lemma shows that \( W_{\text{outer}}(q) \) can be fully recovered from choice data.

**Definition A.5.** For \( q, x \in \Delta(Z) \), lottery \( x \) is never chosen over lottery \( q \) if there is no \( y \in \Delta(Z) \) such that

\[
\{ x \} = c_2(q, x \mid q, x, y).
\]

**Lemma A.6.** For any regular representation \((\gamma, u, \mathcal{V})\) and any \( q, x \in \text{int}(\Delta(Z)) \): \( x \in W_{\text{outer}}(q) \) if and only if, for all \( \epsilon \in (0, 1] \), \( cx + (1 - \epsilon)q \) is never chosen over \( q \).

**Proof.** First, we show that if \( x \in W_{\text{outer}}(q) \), then \( x_\epsilon \) is never chosen over \( q \) for any \( \epsilon \in (0, 1] \). If \( x \in W_{\text{outer}}(q) \), then \( x_\epsilon \in W_{\text{outer}}(q) \) as well. Thus, it suffices to show that if \( x \in W_{\text{outer}}(q) \), then \( x \) is never chosen over \( q \). Let \( v' \) denote an element of \( \text{argmax}_{v \in \mathcal{V}} \{ v(x) - \text{max}_{q, x, y} v \} \), which is nonempty by continuity and compactness. We...
U(x \mid q, x, y) = (1 - \gamma)u(x) + \gamma \left( v'(x) - \max_{q,x,y} v' \right) \\
\leq (1 - \gamma)u(q) + \gamma \left( v'(q) - \max_{q,x,y} v' \right) \\
\leq U(q \mid q, x, y),

which is the desired result.

Now we show: if \( x \notin \bar{W}_{outer}(q) \), then there is some \( \epsilon \in (0, 1] \) for which \( x_\epsilon \) is sometimes chosen over \( q \). This is clearly the case if \( u(x) > u(q) \). Suppose \( u(x) = u(q) \).

By assumption, there is some \( v \in V \) such that \( ((1 - \gamma)u + \gamma v)(x) > ((1 - \gamma)u + \gamma v)(q) \), so \( v(x) > v(q) \). Choose any \( \hat{y} \) such that \( q \in W_{inner}(\hat{y}) \). Since \( v(x) > v(q) \), we have

\[ v(x) > v(\epsilon \hat{y} + (1 - \epsilon)q) \]

for \( \epsilon \) close enough to 0. Let \( y \equiv \epsilon \hat{y} + (1 - \epsilon)q \) for some such \( \epsilon \). Since \( v(x) > \max_{y,q} v \),

\[ U(x \mid q, x, y) = (1 - \gamma)u(x) = (1 - \gamma)u(q). \]

Since \( q \in W_{inner}(\hat{y}) \), we have \( q \in W_{inner}(y) \), so

\[ U(q \mid q, x, y) < (1 - \gamma)u(q) = U(x \mid q, x, y). \]

Conclude that \( x \) is sometimes chosen over \( q \).

Now we cover the final case: \( u(x) < u(q) \). Let

\[ G \equiv \{ v \in V : ((1 - \gamma)u + \gamma v)(q) \geq ((1 - \gamma)u + \gamma v)(x) \}. \]

We want to find \( \hat{y} \) on the boundary of \( W_{inner}(q) \) such that \( v(q) > v(\hat{y}) \) for all \( v \in G \). Suppose there is no such \( \hat{y} \). Then,

\[ W_{inner}(q) = \bigcup_{\hat{z} \in \bar{G}_{\text{pref}}} \{ r \in \Delta(Z) : q \succ r \}. \]

By Lemma A.4, \( G_{\text{pref}} = \bar{\text{co}}(G_{\text{pref}}) = V_{\text{pref}} \). That is, every \( v \in V \) represents the same preference as some \( g \in G \subseteq V \). By non-redundancy, we must have \( v = g \), so \( V = G \).

But since \( ((1 - \gamma)u + \gamma v)(x) > ((1 - \gamma)u + \gamma v)(q) \) for some \( v \in V \), this cannot be the case. Conclude that there exists \( \hat{y} \) on the boundary of \( W_{inner}(q) \) such that \( v(q) > v(\hat{y}) \) for all \( v \in G \). Since \( \hat{y} \in \bar{W}_{inner}(q) \), we must have \( v(q) = v(\hat{y}) \) for some \( v \in V \setminus G \). For some \( \hat{\delta} > 0 \), we have \( q + \hat{\delta}(q - \hat{y}) \in \Delta(Z) \). Set \( y \equiv q + \hat{\delta}(q - \hat{y}) \) for some such \( \hat{\delta} \). Notice
that
\[ v(y) > v(q) \text{ for all } v \in G \]
\[ v(q) = v(y) \text{ for some } v \in V \setminus G. \]

Since no \( v \in V \) prefers \( q \) to both \( x \) and \( y \), we have
\[
U(q \mid q, x, y) = (1 - \gamma)u(q) + \gamma \left( v_{\epsilon}(q) - \max_{x, y} v_{\epsilon} \right)
\]
for all \( \epsilon > 0 \), where \( v_{\epsilon} \in \arg \max_{v \in V} \{ v(q) - \max_{x, y} v \} \). Take a sequence \( \{ \epsilon_k \} \to 0 \).

Since \( V \) is compact, we can pass to a convergent subsequence of \( v_{\epsilon_k} \). Denote the limit \( v_0 \). Fix some \( v^* \in V \) such that \( v^*(q) = v^*(y) \). We have
\[
v_{\epsilon_k}(q) - v_{\epsilon_k}(y) \geq v_{\epsilon_k}(q) - \max_{x, y} v_{\epsilon_k}
\]
\[
= \max_{v \in V} \left\{ v(q) - \max_{x, y} v \right\}
\]
\[
\geq v^*(q) - \max_{x, y} v^*
\]
\[
= \min \{ v^*(q) - v^*(x_{\epsilon_k}), v^*(q) - v^*(y) \}
\]
\[
= \min \{ \epsilon_k(v^*(q) - v^*(x)), 0 \}. \]

Moreover, the limit of the left-hand side is
\[
\lim_{k \to \infty} (v_{\epsilon_k}(q) - v_{\epsilon_k}(y)) = v_0(q) - v_0(y)
\]
and the limit of the right-hand side is
\[
\lim_{k \to \infty} \min \{ \epsilon_k(v^*(q) - v^*(x)), 0 \} = 0
\]
so \( v_0(q) \geq v_0(y) \), which implies \( v_0 \notin G \), and hence \((1 - \gamma)u + \gamma v_0)(x) > ((1 - \gamma)u + \gamma v_0)(q) \). For \( k \) sufficiently large, the same will be true with \( v_{\epsilon_k} \) in place of \( v_0 \). Using this fact,
\[
U(q \mid q, x_{\epsilon_k}, y) = (1 - \gamma)u(q) + \gamma \left( v_{\epsilon_k}(q) - \max_{x_{\epsilon_k}, y} v_{\epsilon_k} \right)
\]
\[
\leq (1 - \gamma)u(q) + \gamma (v_{\epsilon_k}(q) - v_{\epsilon_k}(x_{\epsilon_k}))
\]
\[
= (1 - \gamma)(1 - \epsilon_k)u(q) + \epsilon_k [(1 - \gamma)u(q) + \gamma (v_{\epsilon_k}(q) - v_{\epsilon_k}(x))]
\]
\[
< (1 - \gamma) [(1 - \epsilon_k)u(q) + \epsilon_k u(x)]
\]
\[
= (1 - \gamma)u(x_{\epsilon_k})
\]
for $k$ sufficiently large. Since $v_0(x) > v_0(q) = v_0(y)$, we have

$$U(x_{\epsilon k} | q, x_{\epsilon k}, y) = (1 - \gamma)u(x_{\epsilon k}).$$

Conclude that, for some $\epsilon \in (0, 1]$, $x_{\epsilon}$ is sometimes chosen over $q$. $\square$

Fix some $q \in \text{int}(\Delta(Z))$, and let

$$H_{\text{pref}} \equiv \{ \succsim \in \mathcal{V}_{\text{pref}} : q \sim r \text{ for some } r \in \bar{W}_{\text{inner}}(q) \text{ s.t. } u(q) > u(r) \}.$$

We will show that, for any two representations with the same $u$, for any $\succsim \in H_{\text{pref}}$, the utilities that represent $\succsim$ differ (at most) by an additive constant.

**Lemma A.7.** Suppose $(\gamma, u, \mathcal{V})$ is regular. Fix some $\succsim \in H_{\text{pref}}$ and some $q \in \text{int}(\Delta(Z))$. A representation $v$ of $\succsim$ belongs to $\mathcal{V}$ only if

$$((1 - \gamma)u + \gamma v)(q) > ((1 - \gamma)u + \gamma v)(r) \text{ for all } r \in W_{\text{outer}}(q)\quad (14)$$

$$((1 - \gamma)u + \gamma v)(q) = ((1 - \gamma)u + \gamma v)(r) \text{ for some } r \in W_{\text{outer}}(q) \text{ s.t. } u(q) > u(r).\quad (15)$$

*Proof.* Each $v \in \mathcal{V}$ must satisfy (14) by definition of $W_{\text{outer}}(q)$. For (15), suppose that $\hat{v} \in \mathcal{V}$ represents $\succsim \in H_{\text{pref}}$ but $((1 - \gamma)u + \gamma \hat{v})(q) > ((1 - \gamma)u + \gamma \hat{v})(r)$ for all $r \in \bar{W}_{\text{outer}}(q)$ such that $u(q) > u(r)$. Then, there exists some $\alpha > 1$ such that

$$((1 - \gamma)u + \alpha \gamma \hat{v})(q) > ((1 - \gamma)u + \alpha \gamma \hat{v})(r) \text{ for all } r \in W_{\text{outer}}(q).\quad (16)$$

We can use exactly the same arguments in the proof of Lemma A.4 to show the following: any preference on $\Delta(Z)$ that strictly prefers $q$ to everything in $W_{\text{outer}}(q)$ must have a representation in $\{(1 - \gamma)u + \gamma v : v \in \mathcal{V}\}$. By (16), the preference represented by $(1 - \gamma)u + \alpha \gamma \hat{v}$ satisfies this condition. Thus, there exist $\beta_1 > 0$ and $\beta_2 \in \mathbb{R}$ such that

$$\beta_1((1 - \gamma)u + \alpha \gamma \hat{v}) + \beta_2 \in \{(1 - \gamma)u + \gamma v : v \in \mathcal{V}\}.$$

This implies

$$(\beta_1 - 1) \left( \frac{1 - \gamma}{\gamma} \right) u + \alpha \beta_1 \hat{v} + \frac{\beta_2}{\gamma} \in \mathcal{V},\quad (17)$$

\footnote{This may not be obvious. For a proof, see Appendix A.1.}
which is conveniently rewritten

\[
\left( \beta_1 \left( \frac{1}{\gamma} - 1 + \alpha \right) - \left( \frac{1}{\gamma} - 1 \right) \right) \times \left( \frac{(\beta_1 - 1) \left( \frac{1}{\gamma} - 1 \right)}{(\beta_1 \left( \frac{1}{\gamma} - 1 + \alpha \right) - \left( \frac{1}{\gamma} - 1 \right))} u + \frac{\alpha \beta_1}{\beta_1 \left( \frac{1}{\gamma} - 1 + \alpha \right) - \left( \frac{1}{\gamma} - 1 \right)} \hat{v} \right) + \frac{\beta_2}{\gamma} \in \mathcal{V}. \tag{18}
\]

Suppose \( \beta_1 \geq 1 \). Since \( \mathcal{V} \) is convex and both \( u \) and \( \hat{v} \) belong to it,

\[
\frac{(\beta_1 - 1) \left( \frac{1}{\gamma} - 1 \right)}{\beta_1 \left( \frac{1}{\gamma} - 1 + \alpha \right) - \left( \frac{1}{\gamma} - 1 \right)} u + \frac{\alpha \beta_1}{\beta_1 \left( \frac{1}{\gamma} - 1 + \alpha \right) - \left( \frac{1}{\gamma} - 1 \right)} \hat{v} \in \mathcal{V}. \tag{19}
\]

Together with non-redundancy, (18) and (19) imply that

\[
\beta_1 \left( \frac{1}{\gamma} - 1 + \alpha \right) - \left( \frac{1}{\gamma} - 1 \right) = 1.
\]

Rearranging gives

\[
\frac{1}{\gamma} (\beta_1 - 1) + \beta_1 (\alpha - 1) = 0.
\]

But since \( \alpha > 1 \) and \( \beta_1 \geq 1 \), this cannot hold.

Now suppose \( \beta_1 < 1 \). By (17), we know there exists \( \bar{v} \in \mathcal{V} \) such that

\[
\bar{v} = (\beta_1 - 1) \left( \frac{1}{\gamma} - 1 \right) u + \alpha \beta_1 \hat{v} + \beta_2
\]

\[
\implies \frac{1}{\gamma (1 - \beta_1) + \beta_1} \bar{v} + \frac{(1 - \beta_1) \left( \frac{1}{\gamma} - 1 \right)}{\frac{1}{\gamma} (1 - \beta_1) + \beta_1} u = \frac{\alpha \beta_1}{\frac{1}{\gamma} (1 - \beta_1) + \beta_1} \hat{v} + \frac{\beta_2}{\frac{1}{\gamma} (1 - \beta_1) + \beta_1}.
\]

Since \( u \) and \( \bar{v} \) both belong to \( \mathcal{V} \), and since \( \mathcal{V} \) is convex,

\[
\frac{1}{\gamma (1 - \beta_1) + \beta_1} \bar{v} + \frac{(1 - \beta_1) \left( \frac{1}{\gamma} - 1 \right)}{\gamma (1 - \beta_1) + \beta_1} u \in \mathcal{V}.
\]

Since this utility represents the same preference on \( \Delta(Z) \) that \( \hat{v} \) does, and since \( \hat{v} \in \mathcal{V} \), non-redundancy implies

\[
\frac{1}{\gamma (1 - \beta_1) + \beta_1} \bar{v} + \frac{(1 - \beta_1) \left( \frac{1}{\gamma} - 1 \right)}{\gamma (1 - \beta_1) + \beta_1} u = \hat{v}. \tag{20}
\]

Since \( \hat{v} \) represents a preference in \( H_{\text{pref}} \), by definition of \( H_{\text{pref}} \) there exists \( r \in \tilde{W}_{\text{inner}}(q) \).
Lemma A.8. For any regular representation \( (\gamma, u, V) \),

1. \( \overline{\text{co}}(H_{\text{pref}}) = V_{\text{pref}} \).
2. \( \overline{\text{co}}(H) = V \).

Proof. We first show that \( \overline{\text{co}}(H_{\text{pref}}) = V_{\text{pref}} \). By Lemma A.4, it suffices to show

\[
W_{\text{inner}}(q) = \bigcap_{\gamma \in H_{\text{pref}} \cup \{\gamma_u\}} \{ r \in \Delta(Z) : q \succ r \}. \tag{21}
\]

Since \( H_{\text{pref}} \cup \{\gamma_u\} \subseteq V_{\text{pref}} \), the left-hand side of (21) is a subset of the right-hand side. Suppose the set inclusion is strict. Since the right-hand side is convex, it includes a boundary point \( y \) of \( W_{\text{inner}}(q) \). Notice that \( u(q) > u(y) \). Since \( W_{\text{inner}}(q) \) is an open convex cone, there exists some utility \( \hat{v} \) such that \( \hat{v}(q) = \hat{v}(y) > \hat{v}(r) \) for all \( r \in W_{\text{inner}}(q) \). By Lemma A.4, \( \hat{v} \) represents a preference that belongs to \( V_{\text{pref}} \). Call this preference \( \hat{\gamma} \). Since \( q \sim y \) for some \( y \in \overline{W}_{\text{inner}}(q) \) such that \( u(q) > u(y) \), \( \hat{\gamma} \in H_{\text{pref}} \). But this contradicts the assumption that \( y \) belongs to the right-hand side of (21).

such that \( u(q) > u(r) \) but \( \hat{v}(q) = \hat{v}(r) \). Plugging this into (20) and rearranging gives

\[
\hat{v}(q) - \hat{v}(r) = (\beta_1 - 1) \left( \frac{1}{\gamma} - 1 \right) (u(q) - u(r)) < 0.
\]

We have \( \hat{v}(q) < \hat{v}(r) \) for some \( r \in \overline{W}_{\text{inner}}(q) \), which contradicts \( \hat{v} \in V \). Conclude that (15) cannot be violated. \( \square \)

Fix some \( \gtrsim \in H_{\text{pref}} \) and an arbitrary representation \( \hat{v} \) of \( \gtrsim \). Suppose that \( \alpha \hat{v} + \beta \) and \( \alpha' \hat{v} + \beta' \) represent \( \gtrsim \) in \( (\gamma, u, V) \) and \( (\gamma, u, V') \) respectively. By Lemma A.7,

\[
((1 - \gamma)u + \alpha \eta)(q) = ((1 - \gamma)u + \alpha \gamma)(r) \text{ for some } r \in \overline{W}_{\text{outer}}(q) \text{ s.t. } u(q) > u(r)

((1 - \gamma)u + \alpha' \gamma)(q) = ((1 - \gamma)u + \alpha' \gamma)(r') \text{ for some } r' \in \overline{W}_{\text{outer}}(q) \text{ s.t. } u(q) > u(r').
\]

If \( \alpha' > \alpha \), we have

\[
(1 - \gamma)(u(q) - u(r)) = \alpha \gamma(\hat{v}(r) - \hat{v}(q)) < \alpha' \gamma(\hat{v}(r) - \hat{v}(q))
\]

\[
\implies ((1 - \gamma)u + \alpha' \gamma)(q) < ((1 - \gamma)u + \alpha' \gamma)(r).
\]

Since \( r \in \overline{W}_{\text{outer}}(q) \), this contradicts (14). The case \( \alpha > \alpha' \) is exactly the same, so we have \( \alpha = \alpha' \). Conclude that, conditional on \( u \) and \( \gamma \), two representations of \( \gtrsim \) differ at most by an additive constant.

Now we extend the argument from \( H_{\text{pref}} \) to the remainder of \( V_{\text{pref}} \). Let

\[
H \equiv \{ v \in V : v \text{ represents } \gtrsim \in H_{\text{pref}} \}.
\]

Lemma A.8. For any regular representation \( (\gamma, u, V) \),

1. \( \overline{\text{co}}(H_{\text{pref}}) = V_{\text{pref}} \).
2. \( \overline{\text{co}}(H) = V \).

Proof. We first show that \( \overline{\text{co}}(H_{\text{pref}}) = V_{\text{pref}} \). By Lemma A.4, it suffices to show

\[
W_{\text{inner}}(q) = \bigcap_{\gamma \in H_{\text{pref}} \cup \{\gamma_u\}} \{ r \in \Delta(Z) : q \succ r \}. \tag{21}
\]

Since \( H_{\text{pref}} \cup \{\gamma_u\} \subseteq V_{\text{pref}} \), the left-hand side of (21) is a subset of the right-hand side. Suppose the set inclusion is strict. Since the right-hand side is convex, it includes a boundary point \( y \) of \( W_{\text{inner}}(q) \). Notice that \( u(q) > u(y) \). Since \( W_{\text{inner}}(q) \) is an open convex cone, there exists some utility \( \hat{v} \) such that \( \hat{v}(q) = \hat{v}(y) > \hat{v}(r) \) for all \( r \in W_{\text{inner}}(q) \). By Lemma A.4, \( \hat{v} \) represents a preference that belongs to \( V_{\text{pref}} \). Call this preference \( \hat{\gamma} \). Since \( q \sim y \) for some \( y \in \overline{W}_{\text{inner}}(q) \) such that \( u(q) > u(y) \), \( \hat{\gamma} \in H_{\text{pref}} \). But this contradicts the assumption that \( y \) belongs to the right-hand side of (21).
Next, we show that \( \text{co}(H \cup \{u\}) = V \). Since \( V \) is closed and convex and \( H \cup \{u\} \subseteq V \), we have \( \text{co}(H \cup \{u\}) \subseteq V \). Now take any \( v \in V \). By the first part, \( v \) represents a preference in \( \text{co}(H_{\text{pref}} \cup \{u\}) \). Every such preference has a representation in \( \text{co}(H \cup \{u\}) \). By non-redundancy, that representation can only be \( v \). Thus, \( V \subseteq \text{co}(H \cup \{u\}) \).

Fix any \( v^* \in \text{ext}(V) \). Since \( V = \text{co}(H \cup \{u\}) \), \( v^* \) can be written

\[
v^* = \lim_{i \to \infty} v^i,
\]

where each \( v^i \in \text{co}(H \cup \{u\}) \). In turn, each \( v^i \) can be written

\[
v^i = \left(1 - \sum_{j=1}^{j^i} \lambda_j^i\right) u + \sum_{j=1}^{j^i} \lambda_j^i h_j^i
\]

where the \( \lambda_j^i \) are positive and sum to no more than unity, and each \( h_j^i \in H \). Pass to a subsequence such that \( \lim_{i \to \infty} \sum_{j=1}^{j^i} \lambda_j^i \) converges, and let \( \lambda^* \) denote the limit. If \( \lambda^* = 0 \), then \( v^* = u \). This contradicts \( u \in \text{relint}(V) \), so \( \lambda^* > 0 \). Substituting for each \( v^i \) gives

\[
v^* = (1 - \lambda^*)u + \lim_{i \to \infty} \sum_{j=1}^{j^i} \lambda_j^i h_j^i
\]

\[
= (1 - \lambda^*)u + \lim_{i \to \infty} \left( \sum_{j=1}^{j^i} \lambda_j^i \times \sum_{j=1}^{j^i} \left( \frac{\lambda_j^i}{\sum_{k=1}^{j^i} \lambda_k^i} \right) h_j^i \right)
\]

\[
= (1 - \lambda^*)u + \lambda^* \lim_{i \to \infty} \sum_{j=1}^{j^i} \left( \frac{\lambda_j^i}{\sum_{k=1}^{j^i} \lambda_k^i} \right) h_j^i
\]

\[
= (1 - \lambda^*)u + \lambda^* \bar{h} \text{ for some } \bar{h} \in \text{co}(H).
\]

Since \( u, \bar{g} \in V \), both can be written as convex combinations of extremes of \( V \). Since \( u \) is non-extreme, it must place weight on at least two distinct extremes. If \( \lambda^* < 1 \), then the same is true of \( v^* \). Since \( v^* \) is extreme, it must be that \( \lambda^* = 1 \), so \( v^* \in \text{co}(H) \). Since \( v^* \) was an arbitrary member of \( \text{ext}(V) \), we have \( V = \text{co}(H) \). This implies \( V_{\text{pref}} = \text{co}(H_{\text{pref}}) \).

Fix any \( \succsim \in V_{\text{pref}} \). Suppose that \( v \) and \( v' \) represent \( \succsim \) in \( (\gamma, u, V) \) and \( (\gamma, u, V') \) respectively. Since \( V = \text{co}(H) \), each \( v \in V \) can be written

\[
v = \lim_{i \to \infty} v^i
\]
where each \( v^i \in \text{co}(H) \). In turn, each \( v^i \) can be written

\[
v^i = \sum_{j=1}^{J^i} \lambda^i_j h^i_j
\]

where \( \lambda^i \) is a vector of weights and \( h^i \) is a vector of utilities in \( H \). Substituting for each \( v_i \), we have

\[
v = \lim_{i \to \infty} \sum_{j=1}^{J^i} \lambda^i_j h^i_j
\]

where each \( h^i_j \in H \). We already showed that, for each \( h^i_j \in H \), there exists a constant \( \beta^i_j \) such that \( h^i_j + \beta^i_j \in H' \). Since \( V' \) is convex, it must contain each \( \sum_{j=1}^{J^i} \lambda^i_j (h^i_j + \beta^i_j) \in V' \). Since \( V' \) is compact, we can pass to a convergent subsequence of \( \sum_{j=1}^{J^i} \lambda^i_j (h^i_j + \beta^i_j) \). Let \( v^\infty \) denote the limit. Since \( V' \) is closed, we have \( v^\infty \in V' \). Moreover,

\[
v^\infty - v = \lim_{i \to \infty} \sum_{j=1}^{J^i} \lambda^i_j \beta^i_j \in \mathbb{R}
\]

so \( v^\infty \) represents \( \succ \). Since \( v^\infty \) and \( v' \) both represent \( \succ \) and belong to \( V' \), non-redundancy implies that they are identical. Substituting \( v' \) for \( v^\infty \) gives

\[
v' - v = \lim_{i \to \infty} \sum_{j=1}^{J^i} \lambda^i_j \beta^i_j \in \mathbb{R},
\]

so, conditional on \( u \) and \( \gamma \), \( v' \) and \( v \) differ by an additive constant at most.

Finally, we show that \( \gamma \) is identified. Suppose that \( (\gamma, u, V) \) and \( (\gamma', u', V') \) are both regular representations. Since we can always construct a new regular representation by dividing \( u' \) and \( V' \) by a positive constant and adding another constant to \( u' \), it is without loss to assume that \( u = u' \). Since \( V'_{\text{pref}} \) and \( W'_{\text{inner}} \) do not differ across representations, we must also have \( H'_{\text{pref}} = H_{\text{pref}}' \). Fix some \( \succ \in H_{\text{pref}} \). Suppose that \( h \) and \( h' \equiv \alpha h + \beta \) represent \( \succ \) in \( H \) and \( H' \) respectively. By Lemma A.7, we must have

\[
((1 - \gamma') u + \gamma' h')(q) = ((1 - \gamma') u + \gamma' h')(r') \text{ for some } r' \in \text{\( W'_{\text{outer}}(q) \) s.t. } u(q) > u(r').
\]

This implies

\[
((1 - \gamma) u + \gamma kh)(q) = ((1 - \gamma) u + \gamma kh)(r') \tag{22}
\]

where

\[
k \equiv \frac{\gamma'}{\gamma} \frac{1 - \gamma}{1 - \gamma'}.
\]
Again by Lemma A.7, we must have

\[(1 - \gamma)u + \gamma h)(q) = ((1 - \gamma)u + \gamma h)(r) \text{ for some } r \in \tilde{W}_{\text{outer}}(q) \text{ s.t. } u(q) > u(r). \tag{23}\]

We showed below Lemma A.7 that (22) and (23) together imply \(k = 1\). Solving for \(\alpha\) gives

\[h' = \frac{1 - \gamma'}{1 - \gamma} h + \beta.\]

Since \(V = \text{co}(H)\) and \(u \in V\), we have

\[u = \sum_i \lambda_i h_i\]

for some weight vector \(\lambda\) and some vector \(h\) of utilities in \(H\). By the previous argument, \(h_i \in H\) implies

\[\frac{1 - \gamma'}{1 - \gamma} h_i + \beta_i \in H'.\]

for some constant \(\beta_i\). Since \(V'\) is convex,

\[\sum_i \lambda_i \left( \frac{1 - \gamma'}{1 - \gamma} h_i + \beta_i \right) = \frac{1 - \gamma'}{1 - \gamma} u + \sum_i \lambda_i \beta_i \in H'.\]

Since \(u \in V'\), non-redundancy implies

\[\frac{1 - \gamma'}{1 - \gamma} = 1,\]

which in turn implies \(\gamma = \gamma'\).

A.1 Proof of claim in Lemma A.7

Lemma A.9. Fix a regular representation \((\gamma, u, V)\) and \(q \in \text{int}(\Delta(Z))\). Suppose that \(\hat{v} \in V\) represents \(\succcurlyeq \in H_{\text{pref}}\), but \(((1 - \gamma)u + \gamma \hat{v})(q) > ((1 - \gamma)u + \gamma \hat{v})(r)\) for all \(r \in \tilde{W}_{\text{outer}}(q)\) such that \(u(q) > u(r)\). Then, there exists some \(\alpha > 1\) such that (16) holds.

Proof. The result is clearly true if \(u = \hat{v}\), so suppose \(u \neq \hat{v}\). Let

\[X \equiv \{ r \in \Delta(Z) : u(q) = u(r) \text{ and } \hat{v}(q) = \hat{v}(r) \}.\]

Let

\[Y \equiv \text{co}(X \cup \tilde{W}_{\text{outer}}(q)).\]

Using the fact that \(X\) and \(\tilde{W}_{\text{outer}}(q)\) are compact and convex, it is straightforward to show that \(Y\) is closed.
Since \( u \neq \hat{v} \) and \( q \in \text{int}(\Delta(Z)) \), by non-redundancy there must exist \( r^* \in \text{int}(\Delta(Z)) \) such that

\[
\begin{align*}
u(q) &> u(r^*) \quad \\
((1 - \gamma)u + \gamma \hat{v})(q) &> ((1 - \gamma)u + \gamma \hat{v})(r^*). \quad 
\end{align*}
\]

Suppose \( r^* \in Y \). Then, by definition of \( X \) and \( \bar{W}_{\text{outer}}(q) \),

\[
r^* \in \text{co}\{r \in Y : ((1 - \gamma)u + \gamma \hat{v})(q) = ((1 - \gamma)u + \gamma \hat{v})(r)\}.
\]

Moreover, \( r^* \) must place positive weight on some \( \bar{r} \in \bar{W}_{\text{outer}}(q) \) with \( u(q) > u(\bar{r}) \) as well as \( ((1 - \gamma)u + \gamma \hat{v})(q) = ((1 - \gamma)u + \gamma \hat{v})(\bar{r}) \). This contradicts the assumption about \( \hat{v} \). Conclude that \( r^* \notin Y \).

We extend \( Y \) beyond the simplex as follows:

\[
Y^{\text{ext}} \equiv \left\{ y \in \mathbb{R}^Z : \sum_Z y(z) = 1 \text{ and } \epsilon y + (1 - \epsilon)a \in Y \text{ for some } \epsilon > 0 \right\}.
\]

It is straightforward to show that \( Y^{\text{ext}} \) inherits closedness and convexity from \( Y \). Moreover, since \( r^* \in \text{int}(\Delta(Z)) \) and \( Y \) and \( Y^{\text{ext}} \) agree on \( \Delta(Z) \), \( r^* \notin Y^{\text{ext}} \). By the separating hyperplane theorem, there exists some utility \( b \) and some constant \( \bar{b} \) such that

\[
b(r^*) > \bar{b} \geq b(y) \text{ for all } y \in Y^{\text{ext}}.
\]

Suppose that \( b(x) > b(q) \) for some \( x \in X \). We have \( x + \lambda(x - q) \in Y^{\text{ext}} \) for all \( \lambda > 0 \). But for \( \lambda \) sufficiently large, \( b(x + \lambda(x - q)) > \bar{b} \), which contradicts (24). Now suppose \( b(x) < b(q) \) for some \( x \in X \). By definition of \( x \), we must have \( q + \epsilon(q - x) \in X \) for \( \epsilon > 0 \) sufficiently small. Thus, \( q + \lambda(q - x) \in Y^{\text{ext}} \) for all \( \lambda > 0 \). But for \( \lambda \) sufficiently large, \( b(q + \lambda(q - x)) > \bar{b} \), which contradicts (24). Conclude that \( b(q) = b(x) \) for all \( x \in X \). This implies that \( b \) can be written as a linear combination of \((1 - \gamma)u + \gamma \hat{v}\) and \( u \): for some \( \alpha_1, \alpha_2 \in \mathbb{R} \),

\[
b = \alpha_1((1 - \gamma)u + \gamma \hat{v}) + \alpha_2(u).
\]

Since \( b(r^*) > b(q) \) but \((1 - \gamma)u + \gamma \hat{v})(r^*) = ((1 - \gamma)u + \gamma \hat{v})(q) \) and \( u(q) > u(r^*) \), we must have \( \alpha_2 < 0 \).

Let

\[
b^* \equiv (1 - \epsilon)((1 - \gamma)u + \gamma \hat{v}) + \epsilon b
\]
for $\epsilon > 0$ small enough that $\epsilon(1 - \alpha_1 - \alpha_2) < 1$. We can rewrite $b^*$ as follows:

$$b^* = (1 - \gamma)(1 - \epsilon(1 - \alpha_1 - \alpha_2))u + \gamma(1 - \epsilon(1 - \alpha_1))\hat{v}$$

$$= (1 - \gamma)u + \gamma\frac{1 - \epsilon(1 - \alpha_1)}{1 - \epsilon(1 - \alpha_1 - \alpha_2)}\hat{v}$$

where the final equality uses $\alpha_2 < 0$. Since $b^*$ is a convex combination of $(1 - \gamma)u + \gamma v$ and $b$, and since both of these utilities weakly prefer $q$ to everything in $\hat{W}_{outer}(q)$, (16) holds.

\[ \square \]

## B Proofs of comparative statics results

### B.1 Proof of Proposition 5.1

In this proof, we fix the state $s$ and the decision problem $D$ with $A_2(a_1)$ monotone non-decreasing, and we suppress the dependence of $w$ on $s$ to reduce notation. We define $\tilde{U}(a_1, a_2, \theta) \equiv U_D(a_2, \theta | a_1, s)$.

**Lemma B.1.** If $w$ has increasing differences between $a_1$ and $(a_2, \theta)$, then $\tilde{U}$ has increasing differences between $a_1$ and $(a_2, \theta)$.

**Proof.** Take any $a_1' \geq a_1$ and any $(a_2', \theta') \geq (a_2, \theta)$. Canceling terms, we have

\[
\tilde{U}(a_1', a_2', \theta') - \tilde{U}(a_1', a_2, \theta) - \left[ \tilde{U}(a_1, a_2', \theta') - \tilde{U}(a_1, a_2, \theta) \right] \\
= (1 - \gamma) \left[ w(a_1', a_2', \theta') - w(a_1, a_2, \theta) \right] \tag{25}
\]

By increasing differences for $w$, the term multiplied by $(1 - \gamma)$ and the term multiplied by $\gamma$ are both non-negative, so the right-hand side of (25) is non-negative. \[ \square \]

**Lemma B.2.** If $w$ is supermodular in $(a_2, \theta)$, then $\tilde{U}$ is supermodular in $(a_2, \theta)$.

**Proof.** Take any $(a_2, \theta)$ and $(a_2', \theta')$. By substitution, we have

\[
\tilde{U}(a_1, (a_2, \theta) \wedge (a_2', \theta')) + \tilde{U}(a_1, (a_2, \theta) \vee (a_2', \theta')) \\
- \tilde{U}(a_1, a_2, \theta) - \tilde{U}(a_1, a_2', \theta') \\
= (1 - \gamma) \left[ w(a_1, a_2 \wedge a_2', \theta^*) + w(a_1, a_2 \vee a_2', \theta^*) \right] \\
- \left[ w(a_1, a_2, \theta^*) - w(a_1, a_2', \theta^*) \right] \\
+ \gamma \left[ w(a_1, (a_2, \theta) \wedge (a_2', \theta')) + w(a_1, (a_2, \theta) \vee (a_2', \theta')) \right] \\
- \left[ w(a_1, a_2, \theta) - w(a_1, a_2', \theta') \right] - \gamma \Upsilon, \tag{26}
\]

\[ \square \]
for
\[
\gamma \equiv \max_{\hat{a}_1 \in A_1 \atop \hat{a}_2 \in A_2(\hat{a}_1)} w(\hat{a}_1, \hat{a}_2, \theta \land \theta') + \max_{\hat{a}_1 \in A_1 \atop \hat{a}_2 \in A_2(\hat{a}_1)} w(\hat{a}_1, \hat{a}_2, \theta \lor \theta')
\]
\[
- \max_{\hat{a}_1 \in A_1 \atop \hat{a}_2 \in A_2(\hat{a}_1)} w(\hat{a}_1, \hat{a}_2, \theta) - \max_{\hat{a}_1 \in A_1 \atop \hat{a}_2 \in A_2(\hat{a}_1)} w(\hat{a}_1, \hat{a}_2, \theta').
\]

By $\Theta$ totally ordered, $\gamma = 0$. Thus, supermodularity of $w$ in $(a_2, \theta)$ implies that the right-hand side of (26) is non-negative. \hfill \Box

Lemma B.1, Lemma B.2, and Topkis’s theorem yield Proposition 5.1. \hfill \Box

### B.2 Proof of Theorem 5.2

Suppose $\bar{a}_1$ was *ex post* too high. We prove that if some action $\bar{a}_2$ and some rationale $\bar{\theta}$ maximizes total utility, then there exists $\theta \geq \theta^*$ such that $(\bar{a}_2, \theta)$ maximizes total utility. Thus, when we are only concerned with actions $a_2$ that maximize total utility, it is without loss of generality to restrict the rationales to be at least $\theta^*$. An argument using Topkis’s theorem then yields Theorem 5.2.

In this proof, we fix the state $s$ and the decision problem $D$ with $A_2(a_1)$ monotone non-decreasing, and we suppress the dependence of $w$ on $s$ to reduce notation. As before, we define $\tilde{U}(a_1, a_2, \theta) \equiv U_D(a_2, \theta \mid a_1, s)$.

Let $a_1^*$ be as defined in Theorem 5.2 and let $\bar{a}_1 \geq a_1^*$. We now define
\[
\bar{\Theta} \equiv \arg\max_{\bar{\theta} \in \Theta} \max_{a_2 \in A_2(\bar{a}_1)} \tilde{U}(\bar{a}_1, a_2, \theta),
\]
\[
\bar{\Theta}_\geq \equiv \{ \theta \in \bar{\Theta} : \theta \geq \theta^* \}.
\]

Observe that
\[
\arg\max_{a_2 \in A_2(\bar{a}_1)} \max_{\theta \in \Theta} \tilde{U}(\bar{a}_1, a_2, \theta) = \bigcup_{\bar{\theta} \in \Theta_\geq} \arg\max_{a_2 \in A_2(\bar{a}_1)} \tilde{U}(\bar{a}_1, a_2, \bar{\theta}).
\]

**Lemma B.3.** Under the assumptions of Theorem 5.2, we have
\[
\bigcup_{\bar{\theta} \in \Theta_\geq} \arg\max_{a_2 \in A_2(\bar{a}_1)} \tilde{U}(\bar{a}_1, a_2, \bar{\theta}) = \bigcup_{\theta \in \Theta_\geq} \arg\max_{a_2 \in A_2(\bar{a}_1)} \tilde{U}(\bar{a}_1, a_2, \theta)
\]

**Proof.** Take any
\[
(\bar{a}_2, \bar{\theta}) \in \arg\max_{a_2 \in A_2(\bar{a}_1)} \tilde{U}(\bar{a}_1, a_2, \theta).
\]

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We will show that
\[(\bar{a}_2, \bar{\theta} \lor \theta^*) \in \arg\max_{\theta \in \Theta} \tilde{U}(\bar{a}_1, a_2, \theta).\]

Let us take any \(a_2^* \in \arg\max_{a_2 \in A_2(a_1^*)} w(a_1^*, a_2, \theta^*).\) By \(a_1^*\) ex post optimal, we have
\[
\tilde{U}(a_1^*, a_2^*, \theta^*) = (1 - \gamma) \max_{a_1 \in A_1} w(a_1, a_2, \theta^*) = \max_{\theta \in \Theta} \tilde{U}(a_1^*, a_2, \theta) = \max_{a_2 \in A_2(a_1^*)} \tilde{U}(a_1^*, a_2, \theta). \tag{30}
\]

By \(a_1^* \leq \bar{a}_1, (29), (30),\) and Proposition 5.1, we have
\[
(a_2^* \land \bar{a}_2, \theta^* \land \bar{\theta}) \in \arg\max_{\theta \in \Theta} \tilde{U}(a_1^*, a_2, \theta). \tag{31}
\]

By (31) and then (30), we have
\[
(1 - \gamma)w(a_1^*, a_2^* \land \bar{a}_2, \theta^*) \geq \tilde{U}(a_1^*, a_2^* \land \bar{a}_2, \theta^* \land \bar{\theta}) \\
\geq \tilde{U}(a_1^*, a_2^*, \theta^*) = (1 - \gamma) \max_{a_1 \in A_1} w(a_1, a_2, \theta^*).
\]

This implies that the action sequence \((a_1^*, a_2^* \land \bar{a}_2)\) yields no regret under rationale \(\theta^*\), and thus
\[
(a_2^* \land \bar{a}_2, \theta^*) \in \arg\max_{\theta \in \Theta} \tilde{U}(a_1^*, a_2, \theta). \tag{32}
\]

By \(a_1^* \leq \bar{a}_1, (29), (32),\) and Proposition 5.1, we have
\[
(\bar{a}_2, \bar{\theta} \lor \theta^*) = (\bar{a}_2, \bar{\theta}) \lor (a_2^* \land \bar{a}_2, \theta^*) \in \arg\max_{\theta \in \Theta} \tilde{U}(\bar{a}_1, a_2, \theta).
\]

Our argument holds for any \((\bar{a}_2, \bar{\theta})\) satisfying (29), which yields (28). \qed

Lemma B.4. Under the assumptions of Theorem 5.2, we have
\[
\bigcup_{\bar{\theta} \in \bar{\Theta}_\geq} \arg\max_{a_2 \in A_2(\bar{a}_1)} \tilde{U}(\bar{a}_1, a_2, \bar{\theta}) \gg \arg\max_{a_2 \in A_2(\bar{a}_1)} w(\bar{a}_1, a_2, \theta^*). \tag{33}
\]

Proof. Take any \(\bar{\theta} \in \bar{\Theta}_\geq\). We define
\[
g_{\bar{\theta}}(a_2, \gamma) \equiv (1 - \gamma)w(\bar{a}_1, a_2, \theta^*) + \gamma w(\bar{a}_1, a_2, \bar{\theta}).
\]

By \(\bar{\theta} \geq \theta^*\) and \(w\) supermodular, \(g_{\bar{\theta}}(a_2, \gamma)\) is supermodular. Thus, by Topkis’s theorem,
argmax_{a_2 \in A_2(\bar{a}_1)} g_{\bar{\theta}}(a_2, \gamma) is monotone non-decreasing in \gamma. Thus we have,

\[
\argmax_{a_2 \in A_2(\bar{a}_1)} \bar{U}(\bar{a}_1, a_2, \bar{\theta}) = \argmax_{a_2 \in A_2(\bar{a}_1)} g_{\bar{\theta}}(a_2, \gamma) \geq \argmax_{a_2 \in A_2(\bar{a}_1)} g_{\bar{\theta}}(a_2, 0) = \argmax_{a_2 \in A_2(\bar{a}_1)} w(\bar{a}_1, a_2, \theta^*).  \tag{34}
\]

(34) holds for all \bar{\theta} \in \bar{\Theta}_{\geq}, which implies (33). \hfill \square

By (27), Lemma B.3, and Lemma B.4, we have

\[
\argmax_{a_2 \in A_2(\bar{a}_1)} \max_{\theta \in \Theta} \bar{U}(\bar{a}_1, a_2, \theta) \geq \argmax_{a_2 \in A_2(\bar{a}_1)} w(\bar{a}_1, a_2, \theta^*). \tag{35}
\]

Moreover, by Lemma B.3, for any \( \bar{a}_2 \) in the left-hand side of (35), there exists \( \bar{\theta} \geq \theta^* \) such that

\[
\bar{a}_2 \in \argmax_{a_2 \in A_2(\bar{a}_1)} \bar{U}(\bar{a}_1, a_2, \bar{\theta}).
\]

which completes the proof of Theorem 5.2 for the case \( \bar{a}_1 \geq a_1^* \).

Finally, note that if \( w \) has increasing differences and is supermodular with respect to some order relations on \( \mathcal{A}_1, \mathcal{A}_2, \) and \( \Theta \), then it has increasing differences and is supermodular with respect to the inverse orders on \( \mathcal{A}_1, \mathcal{A}_2, \) and \( \Theta \). Similarly, if \( A_2(a_1) \) is monotone non-decreasing with respect to some order relations on \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \), then it is also monotone non-decreasing with respect to the inverse orders. Thus, our proof covers the case \( \bar{a}_1 \leq a_1^* \). \hfill \square

B.3 Proof of Theorem 5.3

In this proof, we fix the state \( s \) and suppress the dependence of \( U \) and \( w \) on \( s \) to reduce notation.

We first establish that switching from \( D' \) to \( D \) increases the difference from raising \( a_2 \) and \( \theta \).

**Lemma B.5.** Under the assumptions of Theorem 5.3, for any \((\bar{a}_2, \bar{\theta}) \leq (\bar{a}_2', \bar{\theta}')\), we have

\[
U_{D'}(\bar{a}_2', \bar{\theta}' | \bar{a}_1) - U_{D'}(\bar{a}_2, \bar{\theta} | \bar{a}_1) \leq U_D(\bar{a}_2', \bar{\theta}' | \bar{a}_1) - U_D(\bar{a}_2, \bar{\theta} | \bar{a}_1). \tag{36}
\]

**Proof.** For \( \gamma = 0 \), we have (36) trivially. Otherwise, by substitution and then some
algebra, (36) reduces to

\[
\max_{a_1 \in A'_1} \max_{a_2 \in A_2(a_1)} w(a_1, a_2, \tilde{\theta}) + w(a_1, a_2, \tilde{\theta}') \\
\leq \max_{a_1 \in A_1} \max_{a_2 \in A_2(a_1)} w(a_1, a_2, \tilde{\theta}) + w(a_1, a_2, \tilde{\theta}').
\]

Let us define

\[
(\hat{a}_1, \hat{a}_2) \in \arg\max_{a_1 \in A'_1} \max_{a_2 \in A_2(a_1)} w(a_1, a_2, \tilde{\theta}),
\]

\[
(\tilde{a}_1, \tilde{a}_2) \in \arg\max_{a_1 \in A_1} \max_{a_2 \in A_2(a_1)} w(a_1, a_2, \tilde{\theta}'),
\]

By \( A_1 \ll A'_1 \), we have \( \hat{a}_1 \wedge \tilde{a}_1 \in A_1 \) and \( \hat{a}_1 \vee \tilde{a}_1 \in A'_1 \). By (5), we have \( \hat{a}_2 \wedge \tilde{a}_2 \in A_2(\hat{a}_1 \wedge \tilde{a}_1) \) and \( \hat{a}_2 \vee \tilde{a}_2 \in A_2(\hat{a}_1 \vee \tilde{a}_1) \). Thus, we have

\[
w(\hat{a}_1 \wedge \tilde{a}_1, \hat{a}_2 \wedge \tilde{a}_2, \tilde{\theta}) \leq \max_{a_1 \in A_1} \max_{a_2 \in A_2(a_1)} w(a_1, a_2, \tilde{\theta})
\]

\[
w(\hat{a}_1 \vee \tilde{a}_1, \hat{a}_2 \vee \tilde{a}_2, \tilde{\theta}') \leq \max_{a_2 \in A_2(a_1)} w(a_1, a_2, \tilde{\theta}'),
\]

Combining inequalities yields

\[
\max_{a_1 \in A'_1} \max_{a_2 \in A_2(a_1)} w(a_1, a_2, \tilde{\theta}) + w(a_1, a_2, \tilde{\theta}')
\]

\[
= w(\hat{a}_1, \hat{a}_2, \tilde{\theta}) + w(\tilde{a}_1, \tilde{a}_2, \tilde{\theta}')
\]

\[
\leq w(\hat{a}_1 \wedge \tilde{a}_1, \hat{a}_2 \wedge \tilde{a}_2, \tilde{\theta}) + w(\hat{a}_1 \vee \tilde{a}_1, \hat{a}_2 \vee \tilde{a}_2, \tilde{\theta}')
\]

by supermodularity

\[
\leq \max_{a_1 \in A_1} \max_{a_2 \in A_2(a_1)} w(a_1, a_2, \tilde{\theta}) + \max_{a_1 \in A'_1} \max_{a_2 \in A_2(a_1)} w(a_1, a_2, \tilde{\theta}'),
\]

which implies (36).

By Lemma B.2, \( U \) is supermodular in \((a_2, \theta)\). Thus, by Lemma B.5 and Topkis’s theorem, we have Theorem 5.3. 

\[\square\]