On the weighted average number of subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ with $mn \leq x$

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Abstract

Let $\mathbb{Z}_m$ be the additive group of residue classes modulo $m$. For any positive integers $m$ and $n$, let $s(m, n)$ and $c(m, n)$ denote the total number of subgroups and cyclic subgroups of the group $\mathbb{Z}_m \times \mathbb{Z}_n$, respectively. Define

$$\tilde{D}_s(x) = \sum_{mn \leq x} s(m, n) \log \frac{x}{mn}$$

and

$$\tilde{D}_c(x) = \sum_{mn \leq x} c(m, n) \log \frac{x}{mn}.$$

In this paper, we study the asymptotic behaviour of functions $\tilde{D}_s(x)$ and $\tilde{D}_c(x)$.

1 Introduction and main result

Let $\mathbb{Z}_m$ be the additive group of residue classes modulo $m$. Let $\mu$, $\tau$ and $\phi$ be the Möbius function, the divisor function and the Euler totient function, respectively. For any positive integers $m$ and $n$, $s(m, n)$ and $c(m, n)$ denote the total number of subgroups and cyclic subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$, respectively. The properties of the subgroups of the group $\mathbb{Z}_m \times \mathbb{Z}_n$ were studied by Hampejs, Holighaus, Tóth and Wiesmeyr in [1]. We recall that $\gcd(m, n)$ is the greatest common divisor of $m$ and $n$. The authors deduced formulas for $s(m, n)$ and $c(m, n)$, using a simple elementary method. They showed that

$$s(m, n) = \sum_{a|m, b|n} \gcd(a, b) = \sum_{d|\gcd(m, n)} \phi(d) \tau \left( \frac{m}{d} \right) \tau \left( \frac{n}{d} \right) = \sum_{d|\gcd(m, n)} d \tau \left( \frac{mn}{d^2} \right).$$

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and
\[
c(m, n) = \sum_{\substack{a|\gcd(m,n/b) \leq 1 \atop b|a}} \gcd(a, b) = \sum_{\substack{a|\gcd(m,n/b) \leq 1 \atop b|a}} \phi(\gcd(a, b))
\]
\[
= \sum_{d|\gcd(m,n)} (\mu * \phi)(d) \tau \left( \frac{m}{d} \right) \tau \left( \frac{n}{d} \right) = \sum_{d|\gcd(m,n)} \phi(d) \tau \left( \frac{mn}{d^2} \right).
\]

Here, as usual, the symbol $*$ denotes the Dirichlet convolution of two arithmetical functions $f$ and $g$ defined by $(f * g)(n) = \sum_{d|n} f(d)g(n/d)$, for every positive integer $n$. Suppose $x > 0$ is a real number. Define
\[
S^{(1)}(x) := \sum_{m,n \leq x} s(m, n), \quad S^{(2)}(x) := \sum_{m,n \leq x, \gcd(m,n) > 1} s(m, n)
\]
\[
S^{(3)}(x) := \sum_{m,n \leq x} c(m, n), \quad S^{(4)}(x) := \sum_{m,n \leq x, \gcd(m,n) > 1} c(m, n)
\]

The functions $S^{(2)}(x)$ and $S^{(4)}(x)$ represent the number of total subgroups, and cyclic subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$, respectively, having rank two, with $m, n \leq x$. W.G. Nowak and L. Tóth [5] studied the above functions and proved that
\[
S^{(j)}(x) = x^2 \left( \sum_{r=0}^{3} A_{j,r}(\log x)^r \right) + O \left( x^{\frac{11}{2} + \epsilon} \right),
\]
where $A_{j,r}$ $(1 \leq j \leq 4, 0 \leq r \leq 3)$ are computable constants. Moreover, they showed that the double Dirichlet series of the functions $s(m, n)$ and $c(m, n)$ can be represented by the Riemann zeta function. Later, the above error term has been improved by Tóth and Zhai [8] to $O \left( x^{\frac{2}{3}} (\log x)^{\frac{11}{2}} \right)$.

More recently, Sui and Liu [6] considered the sum of $s(m, n)$ and of $c(m, n)$ in the Dirichlet region $\{(m, n) : m, n \leq x\}$. Define
\[
D_s(x) := \sum_{mn \leq x} s(m, n) \quad \text{and} \quad D_c(x) := \sum_{mn \leq x} c(m, n),
\]
the authors obtained two asymptotic formulas of $D_s(x)$ and $D_c(x)$ by using the method of exponential sums. They proved that
\[
D_s(x) = xP_4(\log x) + O \left( x^{2/3} (\log x)^6 \right)
\]
and
\[
D_c(x) = xR_4(\log x) + O \left( x^{2/3} (\log x)^6 \right),
\]
where $P_4(u)$ and $R_4(u)$ are polynomials in $u$ of degree 4 with the leading coefficients $1/(8\pi^2)$ and $3/(4\pi^4)$, respectively. Put
\[
\Delta_s(x) := D_s(x) - xP_4(\log x), \quad \Delta_c(x) := D_c(x) - xR_4(\log x),
\]
Sui and Liu also studied the upper bound of the mean-square estimate of \( \Delta_s(x) \) and \( \Delta_c(x) \) and guessed that \( \Delta_s(x), \Delta_c(x) \ll x^{41/72+\varepsilon} \) hold on average. Moreover, they conjectured that \( \Delta_s(x), \Delta_c(x) \ll x^{1/2+\varepsilon} \).

In this paper, we study the weighted average of \( s(m, n) \) and \( c(m, n) \) with weight concerning logarithms. Let

\[
\tilde{D}_s(x) = \sum_{mn \leq x} s(m, n) \log \frac{x}{mn} \quad \text{and} \quad \tilde{D}_c(x) = \sum_{mn \leq x} c(m, n) \log \frac{x}{mn},
\]

then, we have the following results.

**Theorem 1.** Let the notation be as above. For any positive real number \( x > 2 \), we have

\[
\tilde{D}_s(x) = x \tilde{P}_4(\log x) + O \left( x^{1/2} \log x \right),
\]

and

\[
\tilde{D}_c(x) = x \tilde{R}_4(\log x) + O \left( x^{1/2} \log x \right),
\]

where \( \tilde{P}_4(u) \) and \( \tilde{R}_4(u) \) are polynomials in \( u \) of degree 4 with computable coefficients.

## 2 Auxiliary results

In order to prove our main result, we first show some necessary lemmas.

**Lemma 1.** For every \( z, w \in \mathbb{C} \) with \( \Re(z), \Re(w) > 1 \), we have

\[
\sum_{m,n=1}^{\infty} s(m, n) \frac{z^m w^n}{m^zn^w} = \frac{\zeta^2(z) \zeta^2(w) \zeta(z+w-1)}{\zeta(z+w)},
\]

and

\[
\sum_{m,n=1}^{\infty} c(m, n) \frac{z^m w^n}{m^zn^w} = \frac{\zeta^2(z) \zeta^2(w) \zeta(z+w-1)}{\zeta^2(z+w)}.
\]

**Proof.** The proof can be found in \[5, Theorem 1\]. \( \square \)

**Lemma 2.** For \( t \geq t_0 > 0 \) uniformly in \( \sigma \), we have

\[
\zeta(\sigma + it) \ll \begin{cases} 
\frac{t^{(3-4\varepsilon)}}{\log(\sigma)} & (0 \leq \sigma \leq \frac{1}{2}), \\
\frac{t^{(1-\sigma)}}{\log(\sigma)} & \left( \frac{1}{2} \leq \sigma \leq 1 \right). 
\end{cases}
\]

Moreover, for \( \sigma > 1 \) we have

\[
\zeta(\sigma + it) \ll \min \left( \frac{1}{\sigma - 1}, \log(|t| + 2) \right)
\]

and

\[
\zeta^{-1}(\sigma + it) \ll \min \left( \frac{1}{\sigma - 1}, \log(|t| + 2) \right).
\]
Proof. The first estimate follows immediately from [7, Theorem II.3.8]. The second and third estimates can be found in [6].

Lemma 3. We have
\[ \int_{1}^{T} \frac{|\zeta(1/2 + it)|^4}{|\zeta(1 + 2it)|} \, dt \ll T(\log T)^4. \] (5)

Proof. The proof of this result can be deduced from [3, Proposition 2] when \( k = 1 \).

3 Proof of Theorem 1

Our proof is similar in spirit to the proof of Theorem 1.1 in [6]. Both proofs are based on the residue theorem and the classical method to estimate the integrals. Here, we only prove our theorem for the function \( \tilde{D}_s(x) \) is similar. Suppose that the double Dirichlet series of \( s(m,n) \)

\[ \alpha(s) = \sum_{m,n \geq 1} \frac{s(m,n)}{(mn)^s}, \]

has abscissa of convergence \( \sigma_c \). Applying Riesz typical means, (see [4, Chapter 5: (5.21) and (5.22)]), show that

\[ \alpha(s) = s^2 \int_{1}^{\infty} \tilde{D}_s(x)x^{-s-1} \, dx \]

and that

\[ \tilde{D}_s(x) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{\alpha(s)x^s}{s^2} \, ds, \]

when \( x > 0 \) and \( \sigma_0 > \max(0, \sigma_c) \). Using Lemma 1 with \( \sigma_0 = 1 + 1/\log x \), we have

\[ \tilde{D}_s(x) := \sum_{m,n \leq x} s(m,n) \log \frac{x}{mn} = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{\zeta^4(s)\zeta(2s-1)}{\zeta(2s)} \frac{x^s}{s^2} \, ds. \]

Let \( T \geq 1 \) be a larger parameter, then the above equality can be rewritten as follows

\[ \tilde{D}_s(x) = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta^4(s)\zeta(2s-1)}{\zeta(2s)} \frac{x^s}{s^2} \, ds + O\left(\frac{x^{1+\varepsilon}}{T}\right), \]

where we used Lemma 2 to estimate

\[ \frac{1}{2\pi i} \int_{\sigma_0 \pm iT}^{\sigma_0 \pm i\infty} \frac{\zeta^4(s)\zeta(2s-1)}{\zeta(2s)} \frac{x^s}{s^2} \, ds \ll \int_{T}^{+\infty} \frac{\zeta^4(\sigma_0 \pm it)}{|\zeta(2\sigma_0 - 1 \pm 2it)|} \frac{x^{\sigma_0}}{t^2} \, dt \]

\[ \ll x^{\sigma_0 + \varepsilon} \int_{T}^{+\infty} \frac{dt}{t^2} \ll \frac{x^{\sigma_0 + \varepsilon}}{T}. \]
We consider a rectangle \( D \) in the \( s \) plane with vertices at the points \( \frac{1}{2} - iT, \sigma_0 - iT, \sigma_0 + iT \) and \( \frac{1}{2} + iT \), where \( T \geq 1 \). Notice that our function

\[
\frac{\zeta^4(s) \zeta(2s - 1)}{\zeta(2s)} x^s \frac{s^2}{s^2}
\]

has a pole at \( s = 1 \) of order 5. Then, by the residue theorem we find that

\[
R(x, T) = I_1(x, T) - I_2(x, T) - I_3(x, T) + I_4(x, T),
\]

where

\[
R(x, T) = \text{Res}_{s=1} \frac{\zeta^4(s) \zeta(2s - 1)}{\zeta(2s)} x^s \frac{s^2}{s^2},
\]

\[
I_1(x, T) = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta^4(s) \zeta(2s - 1)}{\zeta(2s)} \frac{x^s}{s^2} ds,
\]

\[
I_2(x, T) = \frac{1}{2\pi i} \int_{\frac{1}{2} + iT}^{\frac{1}{2} + iT} \frac{\zeta^4(s) \zeta(2s - 1)}{\zeta(2s)} \frac{x^s}{s^2} ds,
\]

\[
I_3(x, T) = \frac{1}{2\pi i} \int_{\frac{1}{2} - iT}^{\frac{1}{2} - iT} \frac{\zeta^4(s) \zeta(2s - 1)}{\zeta(2s)} \frac{x^s}{s^2} ds,
\]

\[
I_4(x, T) = \frac{1}{2\pi i} \int_{\frac{1}{2} - iT}^{\frac{1}{2} - iT} \frac{\zeta^4(s) \zeta(2s - 1)}{\zeta(2s)} \frac{x^s}{s^2} ds.
\]

For calculating \( R(x, T) \), we recall that the Laurent series expansion of the Riemann zeta function at \( s = 1 \) is given by

\[
\zeta(s) = \frac{1}{s - 1} + \gamma_0 + \gamma_1(s - 1) + \gamma_2(s - 1)^2 + \gamma_3(s - 1)^3 + \cdots,
\]

where the constants \( \gamma_n \) are often called the Stieltjes constants or generalized Euler constants (see [2]). In particular, \( \gamma_0 = 0.5772156649 \cdots \) is the well-known Euler constant. Using the above expansion, the function \( f(s) = \zeta^4(s) \zeta(2s - 1) \) can be written as

\[
f(s) := \zeta^4(s) \zeta(2s - 1) = \frac{1}{(s - 1)^3} + \frac{3\gamma_0}{(s - 1)^4} + \frac{7\gamma_0^2 - 4\gamma_1}{(s - 1)^5}
\]

\[
+ \frac{8\gamma_0^3 - 18\gamma_0\gamma_1 + 3\gamma_2}{(s - 1)^6}
\]

\[
+ \frac{\frac{2}{5} \gamma_0^4 - 30\gamma_0^2 \gamma_1 + 11\gamma_1^2 + 13\gamma_0 \gamma_2 - \frac{2}{7} \gamma_3}{s - 1}
\]

\[
+ \left( \gamma_0^5 - 22\gamma_0^3 \gamma_1 + 36\gamma_0^2 \gamma_1^2 + 21\gamma_0^2 \gamma_2 - 15\gamma_1 \gamma_2 - 7\gamma_0 \gamma_3 + \frac{3}{4} \gamma_4 \right) + O(s - 1).
\]
Let \( g(s, x) = x^s s^{-2} \zeta^{-1}(2s) \), then we have
\[
R(x, T) := \text{Res}_{s=1} f(s)g(s, x) = \frac{1}{2 \times 4!} g^{(4)}(1, x) + \frac{4}{4!} \times 3 \gamma_0 \times g^{(3)}(1, x)
+ \frac{12}{4!} \times (7 \gamma_0^2 - 4 \gamma_1) \times g^{(2)}(1, x)
+ \frac{24}{4!} \times (8 \gamma_0^3 - 18 \gamma_0 \gamma_1 + 3 \gamma_2) \times g'(1, x)
+ \frac{24}{4!} \times \left( \frac{9}{2} \gamma_0^4 - 30 \gamma_0^2 \gamma_1 + 11 \gamma_1^2 + 13 \gamma_0 \gamma_2 - \frac{5}{3} \gamma_3 \right) \times g(1, x),
\]
where \( g^{(i)}(1, x) \) denotes the \( i \)-th derivative of the function \( g(s, x) \) with respect to \( s \) at \( s = 1 \). By careful calculations, we find that
\[
g^{(4)}(1, x) = \frac{6}{\pi^2} x \log x \cdot \frac{48}{\pi^4} (\pi^2 + 6 \zeta'(2)) x \log x
+ \frac{216}{\pi^6} \left( -4 \pi^2 \zeta''(2) + 48 \zeta'(2)^2 + 8 \pi^2 \zeta'(2) + \pi^4 \right) x \log x
+ \frac{576}{\pi^8} \left( -432 \zeta''(2)^3 + 72 \pi^2 \zeta'(2) \zeta''(2) - 72 \pi^2 \zeta'(2)^2 - \pi^6 \right) x \log x
+ \frac{576}{\pi^4} \left( -2 \zeta^{(3)}(2) - 9 \zeta'(2) + 6 \zeta''(2) \right) x \log x
+ \frac{144}{\pi^6} \left( 20736 \zeta'(2)^4 + 1728 \pi^2 \zeta'(2)^2 (2 \zeta'(2) - 3 \zeta''(2)) + 5 \pi^8 \right) x
+ \frac{6912}{\pi^6} \left( 3 \zeta''(2)^2 + 9 \zeta'(2)^2 - 4 \zeta'(2) (3 \zeta''(2) - \zeta^{(3)}(2)) \right) x
+ \frac{576}{\pi^4} \left( - \zeta^{(4)}(2) + 4 \zeta'(2)^3 - 9 \zeta''(2) + 12 \zeta'(2) \right) x,
\]
\[
g^{(3)}(1, x) = \frac{6}{\pi^2} x \log x \cdot \frac{36}{\pi^4} (6 \zeta'(2) + \pi^2) \log x
+ \frac{108}{\pi^6} \left( 48 \zeta'(2)^2 + 4 \pi^2 (2 \zeta'(2) - \zeta''(2)) + \pi^4 \right) x \log x
- \frac{144}{\pi^8} \left( 432 \zeta'(2)^3 + 72 \pi^2 \zeta'(2) \zeta''(2) - \zeta'(2) \zeta''(2) + \pi^6 \right) x
- \frac{144}{\pi^4} \left( 9 \zeta'(2) - 6 \zeta''(2) + 2 \zeta^{(3)}(2) \right) x,
\]
\[
g^{(2)}(1, x) = \frac{6}{\pi^2} x \log x \cdot \frac{24}{\pi^4} (6 \zeta'(2) + \pi^2) \log x
+ \frac{36}{\pi^6} \left( 48 \zeta'(2)^2 + 4 \pi^2 (2 \zeta'(2) - \zeta''(2)) + \pi^4 \right) x
\]
and that
\[
g'(1, x) = \frac{6}{\pi^2} x \log x - \frac{12}{\pi^4} (6 \zeta'(2) + \pi^2) x,
\]
\[
g(1, x) = \frac{x}{\zeta(2)}.
\]
Thus, the function $R(x, T)$ becomes

$$R(x, T) = x \sum_{r=0}^{4} B_r (\log x)^r,$$

where

$$B_4 = \frac{1}{8\pi^2},$$

$$B_3 = \frac{3\gamma - 1}{\pi^2} - \frac{6}{\pi^4} \zeta'(2),$$

$$B_2 = -\frac{18}{\pi} \zeta''(2) + \frac{216}{\pi^4} \zeta'(2)^2 + \frac{36}{\pi^4} (-3\gamma + 1) \zeta'(2) + \frac{3}{2\pi^2} (3 + 14\gamma_0 - 12\gamma_0 - 8\gamma_0),$$

$$B_1 = -\frac{5184}{\pi^2} \zeta'(2) + \frac{864}{\pi^2} \zeta'(2) (-\zeta'(2) + 3\gamma_0 \zeta'(2) + \zeta''(2))$$

$$- \frac{12}{\pi^4} (42\gamma_0^3 \zeta'(2) + \zeta'(2) (9 - 24\gamma_1) - 6\zeta''(2) + 18\gamma_0 (-2\zeta'(2) + \zeta''(2)) + 2 \zeta'(2))^2$$

$$+ \frac{6}{\pi^2} (8\gamma_0^3 - 14\gamma_0 + 9\gamma_0 - 8\gamma_0\gamma_1 + 8\gamma_1 + 3\gamma_2 - 2),$$

$$B_0 = \frac{1}{\pi^2} (15 - 96\gamma_0^3 + 27\gamma_4 - 72\gamma_1 + 66\gamma_2^2 - 18\gamma_1^3 - 7 + 10\gamma_1) - 36\gamma_2)$$

$$+ \frac{1}{\pi} (6\gamma_0 (-12 + 36\gamma_1 + 13\gamma_2 - 10\gamma_3) - \frac{5184}{\pi^2}\zeta'(2)^2 (-2\zeta'(2) + 6\gamma_0 \zeta'(2) + 3 \zeta''(2))$$

$$- \frac{12}{\pi^4} (48\gamma_0^3 \zeta'(2) + 6 (-2 + 8\gamma_1 + 3\gamma_2) \zeta'(2) + 9\zeta''(2) - 24\gamma_1 \zeta''(2) + 42\gamma_0 (-2\zeta'(2) + \zeta''(2))$$

$$+ \frac{12}{\pi^4} (6\gamma_0 (9 (-1 + 2\gamma_1) \zeta'(2) + 6\zeta''(2) - 2 \zeta'(2)) + 4 \zeta'(2))^2 - \zeta''(2))$$

$$+ \frac{144}{\pi^6} (42\gamma_0^3 \zeta'(2)^2 + (9 - 24\gamma_1) \zeta'(2)^2 - 36\gamma_0 \zeta'(2) (\zeta'(2) - \zeta''(2)) + 3 \zeta''(2)^2)$$

$$+ \frac{144}{\pi^6} (4 \zeta'(2) (-3 \zeta''(2) + 3 \zeta''(2))) + \frac{62208}{\pi^10} \zeta'(2)^4.$$

Again, we use Lemma 2 to estimate the function $I_2(x, T)$

$$I_2(x, T) = \frac{1}{2\pi^4} \left( \int_{\frac{1}{2}}^{1} + \int_{\frac{1}{2}}^{\frac{\sigma_0}{2}} \right) \frac{\zeta^4(\sigma + iT)\zeta(2\sigma - 1 + 2iT)^{x^{\alpha + iT}}d\sigma}{(2\sigma + 2iT)(\sigma + iT)^2} x^{\sigma + iT} d\sigma$$

$$\ll \frac{\log T}{T^2} \left( \int_{\frac{1}{2}}^{\frac{1}{2}} + \int_{\frac{1}{2}}^{\frac{\sigma_0}{2}} \right) |\zeta(\sigma + iT)|^4 |\zeta(2\sigma - 1 + 2iT)| x^\sigma d\sigma$$

$$\ll T^{\frac{1}{2}} (\log T)^6 \int_{\frac{1}{2}}^{\frac{1}{2}} \left( \frac{x}{T^{\frac{s}{4}}} \right)^\sigma d\sigma + (\log T)^6 \int_{\frac{1}{2}}^{\sigma_0} \left( \frac{x}{T^{\frac{s}{2}}} \right)^\sigma d\sigma$$

$$\ll \left( \frac{x}{T^2} + (\frac{x}{T^2})^{\frac{1}{2}} + (\frac{x}{T^2})^{\frac{1}{2}} \right) (\log T)^6.$$

choosing $T = x$, we deduce that $I_2(x, T) \ll x^{-1/3}(\log x)^6$. A similar argument shows that the function $I_4(x, T)$ is estimated by $x^{-1/3}(\log x)^6$.
Next we estimate $I_3(x, T)$. Using Lemmas 2 and 3 we find that

$$I_3(x, T) = \frac{1}{2\pi} \int_{-T}^{T} \frac{\zeta^4(\frac{1}{2} + it)\zeta(2it)}{\zeta(1 + 2it)(\frac{1}{2} + it)^2} x^{\frac{1}{2} + it} dt$$

$$\ll x^{\frac{1}{2}} x^{\frac{1}{2}} \int_{1}^{T} \frac{|\zeta(\frac{1}{2} + it)|^4 |\zeta(2it)|}{|\zeta(1 + 2it)| t^2} dt$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \sum_{k \leq \log T} \frac{1}{(2^k)^2} \int_{2^{k-1}}^{2^k} \frac{|\zeta(\frac{1}{2} + it)|^4}{|\zeta(1 + 2it)|} dt$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \sum_{k \leq \log T} \frac{1}{(2^k)^2} k^4$$

$$\ll x^{\frac{1}{2}} + x^{\frac{1}{2}} \sum_{k \leq k_0} \frac{k^4}{(2^k)^k} + x^{\frac{1}{2}} \sum_{k_0 < k \leq \log T} 1$$

$$\ll x^{\frac{1}{2}} \log T.$$

Combining the above results with (6), we get the desired conclusion.

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