The classical Liouville property says that all bounded harmonic functions in $\mathbb{R}^n$, that is, all bounded functions satisfying $\Delta f = 0$, are constant. In this paper, we obtain necessary and sufficient conditions on the symbol of a Fourier multiplier operator $m(D)$, such that the solutions $f$ to $m(D)f = 0$ are Lebesgue a.e. constant (if $f$ is bounded) or coincide Lebesgue a.e. with a polynomial (if $f$ is polynomially bounded). The class of Fourier multipliers includes the (in general non-local) generators of Lévy processes. For generators of Lévy processes, we obtain necessary and sufficient conditions for a strong Liouville theorem where $f$ is positive and grows at most exponentially fast. As an application of our results above, we prove a coupling result for space-time Lévy processes.
that convolution and $\Delta$ commute or, as we do here, understands $\Delta f$ as a Schwartz distribution, that is, $\Delta f$ is defined as the continuous linear form $\phi \mapsto \langle f, \Delta \phi \rangle$ for any $\phi \in C_c^\infty(\mathbb{R}^n)$ (the smooth functions with compact support) or $\phi \in S(\mathbb{R}^n)$ (the rapidly decreasing smooth functions).

Recently, Alibaud et al. [1] and, independently, two of us [3] gave proofs providing necessary and sufficient conditions ensuring an analogue of the Liouville property (1) for infinitesimal generators of Lévy processes. The proof in [1] combines harmonic analysis and further methods from group theory, while the approach in [3] uses mainly probabilistic arguments; the latter proof also yields the strong Liouville property where, in the appropriate analogue to (1), $f \in L^\infty(\mathbb{R}^n)$ is relaxed to $f \geq 0$ and at most exponential growth at infinity. Sufficient conditions for a ‘polynomial’ Liouville property (if $f$ is polynomially bounded, then $f$ coincides a.e. with a polynomial) are due to Kühn [14].

In the present paper, we give a very short and purely analytic proof for both the Liouville property and the polynomial Liouville property for Lévy generators and — as it turns out — a much larger class of Fourier multiplier operators. In fact, the necessary and sufficient condition for the Liouville property is that $\xi = 0$ is the only zero of the multiplier $m(\xi)$. For generators of Lévy processes, we refine the strong Liouville result proved in [3] and we establish a further probabilistic interpretation of the Liouville property for Lévy generators in terms of coupling and space-time harmonic functions.

Notation. Most of our notation is standard or self-explanatory. We write $\mathcal{F}\phi(\xi) = \hat{\phi}(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \phi(x) \, dx$ and $\mathcal{F}^{-1} u(x) = \int_{\mathbb{R}^n} e^{i\xi \cdot x} u(\xi) \, d\xi$ for the Fourier and the inverse Fourier transform. We denote by $(T_h \phi)(x) := \phi(x - h)$ the shift by $h \in \mathbb{R}^n$, $\tilde{\phi}(x) := \phi(-x)$ is the reflection at the origin, and $\Lambda(x) := (1 + |x|^2)^{1/2}$ is the standard weight function.

1 | THE PROOF OF THE LIOUVILLE THEOREM FROM [3] REVISITED

We will need a few notions from probability theory and stochastic processes, which can be found in Sato [17] or Jacob [10,11] and [12], but the essential ingredient is the structure of the infinitesimal generator, see below. A Lévy process is a stochastic process $(X_t)_{t \geq 0}$ with values in $\mathbb{R}^n$ and sample paths $t \mapsto X_t(\omega)$ that are for almost all $\omega$ right-continuous with finite left-hand limits; moreover, the random variables $X_{t_k} - X_{t_{k-1}}$, $0 = t_0 < t_1 < \cdots < t_m$, $m \in \mathbb{N}$, are stochastically independent (independent increments) and each increment $X_{t_k} - X_{t_{k-1}}$ has the same distribution as $X_{t_k - t_{k-1}}$ (stationary increments). The fact that we are looking at a process with independent and stationary increments means that the distribution of $X_t$ (for any fixed $t > 0$) characterises the whole process; moreover, $X_t$ is necessarily infinitely divisible, so that its characteristic function (inverse Fourier transform) is of the form

$$\mathbb{E} e^{i\xi X_t} = e^{-t \psi(\xi)}, \quad \xi \in \mathbb{R}^n, \ t > 0, \quad (2)$$

where the characteristic exponent $\psi(\xi)$ is uniquely given by the Lévy–Khintchine formula

$$\psi(\xi) = -ib \cdot \xi + \frac{1}{2} \xi \cdot Q \xi + \int_{0<|y|<1} \left( 1 - e^{iy \cdot \xi} + i y \cdot \xi \right) \nu(dy) + \int_{|y|\geq1} \left( 1 - e^{iy \cdot \xi} \right) \nu(dy). \quad (3)$$
The so-called Lévy triplet \((b, Q, \nu)\) comprising a vector \(b \in \mathbb{R}^n\), a symmetric, positive semi-definite matrix \(Q \in \mathbb{R}^{n \times n}\), and a measure \(\nu\) such that \(\int_{y \neq 0} \min \{1, |y|^2\} \nu(dy) < \infty\), characterises \(\psi\) uniquely. For example, taking \(b = 0, Q = \text{id}, \nu \equiv 0\), we get Brownian motion with its characteristic exponent \(\psi(\xi) = \frac{1}{2} |\xi|^2\), and the choice \(b = 0, Q = 0, \nu(dy) = c_\alpha |y|^{-n-\alpha} dy\) with a suitable constant \(c_\alpha\) yields a rotationally symmetric \(\alpha\)-stable process with characteristic exponent \(|\xi|^\alpha\), \(0 < \alpha < 2\).

Since Lévy processes are Markov processes, their transition behaviour can be described by a transition semi-group \(P_t f(x) = \mathbb{E} f(X_t + x)\) which, in turn, is uniquely characterised by the infinitesimal generator \(L f := \frac{d}{dt}P_t f|_{t=0}\) (in the Banach space \(C_\infty(\mathbb{R}^n)\) of all continuous functions vanishing at infinity, say). Now the key point is the following observation.

**Fact 1.** *The infinitesimal generator \(L = L_\psi\) of a Lévy process with characteristic exponent \(\psi(\xi)\) is on \(C_\infty(\mathbb{R}^n)\) a Fourier multiplier operator with symbol \(-\psi(\xi)\), that is,*

\[
L_\psi \phi(x) = -\psi(D)\phi(x) = P_{\xi \to x}^{-1} \left( -\psi(\xi)\hat{\phi}(\xi) \right), \quad \phi \in C_\infty(\mathbb{R}^n),
\]

*and, combining this with (3),*

\[
L_\psi \phi(x) = b \cdot \nabla \phi(x) + \frac{1}{2} \nabla \cdot Q \nabla \phi(x)
\]

\[
+ \int_{0<|y|<1} (\phi(x+y) - \phi(x) - y \cdot \nabla \phi(x))\nu(dy) + \int_{|y|\geq 1} (\phi(x+y) - \phi(x))\nu(dy).
\]

Note that we get \(L_\psi = \frac{1}{2} \Delta\), if \(\psi(\xi) = \frac{1}{2} |\xi|^2\) (Brownian motion) and \(L_\psi = -(-\Delta)^{\alpha/2}, 0 < \alpha < 2\), if \(\psi(\xi) = |\xi|^\alpha\) (stable process).

Our first aim is to give a purely analytic proof of the following result.

**Theorem 1** Liouville [1, 3]. *Let \(\psi\) be the characteristic exponent of a Lévy process and denote by \(\psi(D)\) the corresponding Fourier multiplier operator. Suppose \(f \in L^\infty(\mathbb{R}^n)\) is such that \(\psi(D)f = 0\) as a distribution, that is,*

\[
\langle f, \hat{\psi}(D)\phi \rangle = 0 \quad \text{for all } \phi \in C_\infty(\mathbb{R}^n).
\]

*If \(\{\eta \in \mathbb{R}^n \mid \psi(\eta) = 0\} = \{0\}\), then \(f \equiv \text{const}\) Lebesgue almost everywhere.

*Conversely, if \(f \equiv \text{const}\) Lebesgue almost everywhere for every \(f \in L^\infty(\mathbb{R}^n)\) satisfying (6), then \(\{\eta \in \mathbb{R}^n \mid \psi(\eta) = 0\} = \{0\}\).*

A formal proof of the implication

\[
\{\eta \in \mathbb{R}^n \mid \psi(\eta) = 0\} = \{0\}
\]

\[\Rightarrow \text{ every bounded solution of } \psi(D)f = 0 \text{ is a.e. constant}\]

is very easy. Indeed, if \(\psi(D)f = 0\), then \(\psi(\eta)\hat{f}(\eta) \equiv 0\). Hence,

\[
\text{supp} \hat{f} \subset \{\eta \in \mathbb{R}^d \mid \psi(\eta) = 0\} = \{0\}.
\]
Then, $\hat{f} = \sum_{|\alpha| \leq N} c_{\alpha} \delta^{\alpha}$ for some $N \in \mathbb{Z}_+$ and $c_{\alpha} \in \mathbb{C}$. Hence, $f$ is a polynomial. If it is bounded, it has to be constant. This proof can be made rigorous if $\psi$ is $C^\infty$-smooth (see [3, Theorem 3.2]). However, $\psi$ may be continuous but nowhere differentiable (see [3, Remark 3.3]), in which case defining the product of $\psi$ and the distribution $\hat{f}$ is by no means trivial.

Our analytic proof of Theorem 1 is based on the following standard result, which is known from the proof of Wiener’s Tauberian theorem, cf. Rudin [16, Theorem 9.3].

**Theorem 2.** If $f \in L^\infty(\mathbb{R}^n)$, $Y$ is a linear subspace of $L^1(\mathbb{R}^n)$, and

$$ f \ast g = 0 \quad \text{for every } g \in Y, $$

then the set

$$ Z(Y) := \bigcap_{y \in Y} \{ \xi \in \mathbb{R}^n \mid \hat{\psi}(\xi) = 0 \} $$

contains the support of the tempered distribution $\hat{f}$.

Theorem 2 says, in essence, that for $f \in L^\infty(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, the condition $f \ast g = 0$ implies that $\hat{g}(\xi) = 0$ for all $\xi$ in the support of $\hat{f}$.

**Proof of Liouville’s Theorem 1.** Since $\psi$ is the characteristic exponent of a Lévy process, $\hat{\psi}(D)$ (as well as $\psi(D)$) is a continuous operator from $C^\infty_0(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, see, for example, [18, Lemma 3.4] or the discussion in Example 4 below. Thus, the dual pairing in (6) is well defined.

Suppose $\{ \eta \in \mathbb{R}^n \mid \psi(\eta) = 0 \} = \{0\}$. Notice that $T_y \phi \in C^\infty_0(\mathbb{R}^n)$ for every $\phi \in C^\infty_0(\mathbb{R}^n)$ and all $y \in \mathbb{R}^n$. Therefore, we see from (6) that

$$ \left( f \ast \hat{\psi}(D) \phi \right)(y) = \langle f, T_y \hat{\psi}(D) \phi \rangle = \langle f, \hat{\psi}(D)(T_y \phi) \rangle = 0 \quad \text{for all } y \in \mathbb{R}^n. $$

Hence,

$$ f \ast \hat{\psi}(D) \phi = 0 \quad \text{for all } \phi \in C^\infty_0(\mathbb{R}^n). $$

It is easy to see that

$$ \bigcap_{\phi \in C^\infty_0(\mathbb{R}^n)} \left\{ \eta \in \mathbb{R}^n \mid \hat{\psi}(D) \phi(\eta) = 0 \right\} = \bigcap_{\phi \in C^\infty_0(\mathbb{R}^n)} \left\{ \eta \in \mathbb{R}^n \mid \hat{\psi}(D) \phi(-\eta) = 0 \right\} $$

$$ = \bigcap_{\phi \in C^\infty_0(\mathbb{R}^n)} \left\{ \eta \in \mathbb{R}^n \mid \psi(\eta) \hat{\phi}(-\eta) = 0 \right\} $$

$$ = \{ \eta \in \mathbb{R}^n \mid \psi(\eta) = 0 \} = \{0\}. $$

Applying Wiener’s theorem (Theorem 2) with

$$ Y := \left\{ \hat{\psi}(D) \phi \mid \phi \in C^\infty_0(\mathbb{R}^n) \right\} \subset L^1(\mathbb{R}^n), $$

then $f \ast \hat{\psi}(D) \phi = 0$ for every $\phi \in C^\infty_0(\mathbb{R}^n)$.
we conclude that the support of the tempered distribution \( \hat{f} \) is contained in \( \{0\} \). Therefore, there exist \( N \in \mathbb{N}_0 \) and \( c_{\alpha} \in C (\mathbb{N}_0^n, |\alpha| \leq N) \) such that

\[
\hat{f} = \sum_{|\alpha| \leq N} c_{\alpha} \partial^\alpha \delta_0,
\]

see, for example, [16, Theorems 6.24 and 6.25]. Inverting the Fourier transform, and using the assumption that \( f \) is bounded, shows that \( f \equiv c_0 \) Lebesgue a.e.

For the converse direction, we assume the contrary, that is, that the zero-set \( \{ \eta \in \mathbb{R}^n \mid \psi(\eta) = 0 \} \) contains some \( \gamma \neq 0 \). Then, \( f(x) := e^{i\gamma \cdot x} \) satisfies \( \psi(D_x) e^{i\gamma \cdot x} = e^{i\gamma \cdot x} \psi(\gamma) = 0 \) for all \( x \in \mathbb{R}^n \).

Thus, \( f \) is a non-constant solution, and we are done. We note in passing that since \( \psi(-\gamma) = \psi(\gamma) \), \( -\gamma \in \{ \psi = 0 \} \), and we can even get a real-valued solution:

\[
2 \psi(D_x) \cos(\gamma \cdot x) = \psi(D_x) (e^{i\gamma \cdot x} + e^{-i\gamma \cdot x}) = e^{i\gamma \cdot x} \psi(\gamma) + e^{-i\gamma \cdot x} \psi(-\gamma) = 0.
\]

Remark 1. With a bit more effort, see [3] or [1], one can show that all bounded solutions in the converse direction of Theorem 1 are necessarily periodic. Since \( \{ \psi = 0 \} \) is a closed subgroup of the additive group \( (\mathbb{R}^n, +) \), the periodicity group of all bounded solutions is given by the dual lattice \( \{ \psi = 0 \}^\perp := \{ x \in \mathbb{R}^n \mid e^{x \cdot y} = 1, \forall y \in \{ \psi = 0 \} \} \). The proof in [1] actually shows that a Lévy generator \( \psi(D) \) has the Liouville property if, and only if, \( \{ \psi = 0 \}^\perp = \mathbb{R}^n \).

The above proof of Theorem 1 extends without change to Fourier multiplier operators that map \( C^\infty_c(\mathbb{R}^n) \) into \( L^1(\mathbb{R}^n) \).

**Theorem 3.** Let \( m \in C(\mathbb{R}^n) \) be such the Fourier multiplier operator

\[
C^\infty_c(\mathbb{R}^n) \ni \phi \mapsto m(D)\phi := F^{-1}(\hat{m}\hat{\phi})
\]

maps \( C^\infty_c(\mathbb{R}^n) \) into \( L^1(\mathbb{R}^n) \). Suppose \( f \in L^\infty(\mathbb{R}^n) \) is such that \( m(D)f = 0 \) as a distribution, that is,

\[
\langle f, m(D)\phi \rangle = 0 \quad \text{for all } \phi \in C^\infty_c(\mathbb{R}^n). \tag{9}
\]

If \( \{ \eta \in \mathbb{R}^n \mid m(\eta) = 0 \} \subset \{0\} \), then \( f \equiv \text{const} \) Lebesgue almost everywhere.

Conversely, if \( f \equiv \text{const} \) Lebesgue almost everywhere for every complex-valued \( f \in L^\infty(\mathbb{R}^n) \) satisfying (9), then \( \{ \eta \in \mathbb{R}^n \mid m(\eta) = 0 \} \subset \{0\} \). If \( m(\eta) = 0 \) implies that \( m(-\eta) = 0 \), then it is enough to consider real-valued \( f \in L^\infty(\mathbb{R}^n) \) satisfying (9).

Remark 2. If \( \{ \eta \in \mathbb{R}^n \mid m(\eta) = 0 \} \subset \{0\} \), that is, \( \{ \eta \in \mathbb{R}^n \mid m(\eta) = 0 \} = \emptyset \), then \( f \equiv 0 \) is the only constant solution of the equation \( m(D)f = 0 \).

**Example 4.** Let us give a few examples of multipliers satisfying the key assumption of Theorem 3. The following multipliers \( \kappa \) are such that \( \kappa(D) \) maps \( C^\infty_c(\mathbb{R}^n) \) into \( L^1(\mathbb{R}^n) \).

(a) \( \kappa \) is a linear combination of terms of the form \( ab \), where \( a \) is the Fourier transform of a finite Borel measure on \( \mathbb{R}^n \), and all partial derivatives \( \partial^\alpha b \) with \( |\alpha| \leq n+1 \) are polynomially bounded.
Indeed: Let \( b_N := b \Lambda^{-2N}, N \in \mathbb{R}^n \). For a sufficiently large \( N \), all partial derivatives \( \partial^\alpha \varphi \) with \( |\alpha| \leq n + 1 \) belong to \( L^1(\mathbb{R}^n) \). Then \( x^\alpha \mathcal{F}^{-1}(b_N) \in L^\infty(\mathbb{R}^n), |\alpha| \leq n + 1 \). Hence,
\[
\mathcal{F}^{-1}(b_N), |x|^{n+1} \mathcal{F}^{-1}(b_N) \in L^\infty(\mathbb{R}^n), j = 1, \ldots, n
\]
\[
\Rightarrow (1 + |x|)^{n+1} \mathcal{F}^{-1}(b_N) \in L^\infty(\mathbb{R}^n) \Rightarrow \mathcal{F}^{-1}(b_N) \in L^1(\mathbb{R}^n).
\]

Let \( \phi \in C_\infty(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \). We have \( b_N(D)\phi = (2\pi)^{-n} \mathcal{F}^{-1}(b_N) * \phi \), and it follows from Young’s inequality
\[
\|(F^{-1}b_N) * \phi\|_{L^1} \leq \|F^{-1}b_N\|_{L^1} \|\phi\|_{L^1}
\]
that \( b_N(D) \) maps \( C_\infty(\mathbb{R}^n) \) into \( L^1(\mathbb{R}^n) \). Since the differential operator \( \Lambda^{2N}(D) \) maps \( C_\infty(\mathbb{R}^n) \) into itself, and \( (ab)(D) = a(D)b_N(D)\Lambda^{2N}(D) \), it is left to show that \( a(D) \) maps \( L^1(\mathbb{R}^n) \) into itself. Since \( a = F\mu \) for a finite Borel measure \( \mu \),
\[
(a(D)g)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} g(x - y) \mu(dy), \quad x \in \mathbb{R}^n, \quad g \in \mathbb{R}^n,
\]
and
\[
\|a(D)g\|_{L^1} \leq (2\pi)^{-n} \mu(\mathbb{R}^n) \|g\|_{L^1} \quad \text{for all} \quad g \in L^1(\mathbb{R}^n).
\]

A particular example is the characteristic exponent of a Lévy process
\[
\psi(\xi) = -ib \cdot \xi + \frac{1}{2} \xi \cdot Q_\xi + \int_{|y|<1} \left( 1 - e^{iy \cdot \xi} + iy \cdot \xi \right) \nu(dy) + \int_{|y|\geq1} \left( 1 - e^{iy \cdot \xi} \right) \nu(dy)
\]
\[
= : \psi_0(\xi) - \int_{|y|\geq1} e^{iy \cdot \xi} \nu(dy).
\]
The last term in the above formula is the Fourier transform of a finite Borel measure. The smoothness of \( \psi_0 \) follows immediately from the differentiation lemma for parameter-dependent integrals and the observation that the integrand of the formally differentiated function \( \partial^\alpha \psi_0 \) can be bounded by \( \text{const} \cdot |y|^2 \), which is \( \nu \)-integrable on \( \{y \in \mathbb{R}^n : 0 < |y| < 1\} \). This bound also shows that \( |\partial^\alpha \psi_0(\xi)| \leq c_0(1 + |\xi|^2) \) if \( |\alpha| = 0 \), \( |\partial^\alpha \psi_0(\xi)| \leq c_1(1 + |\xi|) \) if \( |\alpha| = 1 \), and \( |\partial^\alpha \psi_0(\xi)| \leq c_\alpha \) if \( |\alpha| = \ell \geq 2 \); thus, we get polynomial boundedness of \( \psi_0 \) and its derivatives. A fully worked-out proof can be found in [3, Lemma 4] as well as [10, Lemma 3.6.22, Theorem 3.7.13].

(b) Any \( \kappa \) of the form
\[
\kappa(\xi) = \sum_{|\alpha| \leq s} c_\alpha \frac{\alpha! |\alpha| \xi^\alpha}{|\alpha|!} + \int_{0 < |y| < 1} \left[ 1 - e^{iy \cdot \xi} + \sum_{|\alpha| = 0}^{2s-1} \frac{i\alpha! |\alpha| y^{\alpha |\xi|^2} \nu(dy) + \int_{|y|\geq1} \left( 1 - e^{iy \cdot \xi} \right) \nu(dy)}
\]
with \( s \in \mathbb{N}, c_\alpha \in \mathbb{R}, \) and a measure \( \nu \) on \( \mathbb{R}^n \setminus \{0\} \) such that \( \int_{y \neq 0} \min \{1, |y|^{2s}\} \nu(ds) < \infty \). (As usual, for any \( \alpha \in \mathbb{N}_0^n \) and \( \xi \in \mathbb{R}^n \), we define \( \alpha! := \prod_{1}^{n} \alpha_k! \) and \( \xi^\alpha := \prod_{1}^{n} \xi_k^{\alpha_k} \). The proof of this assertion goes along the lines of Part a).
Functions of this type appear naturally in positivity questions related to generalised functions, see, for example, Gelfand and Vilenkin [7, Chapter II.4] or Wendland [21]. Some authors call the function $-\kappa$ (under suitable additional conditions on $c_\alpha$'s) a *conditionally positive definite function*. Note that $s = 1$ is just the Lévy–Khintchine formula (3).

## 2 THE LIOUVILLE THEOREM FOR POLYNOMIALLY BOUNDED FUNCTIONS

We are now going to show that our argument used in the proof of Theorems 1 and 3 extends to polynomially bounded functions $f$ in (6).

To simplify the presentation, we use the function $\Lambda(x) = \left(1 + |x|^2\right)^{1/2}$ as well as the following function spaces. Let $\beta \geq 0$, and

$$
L^1_\beta(\mathbb{R}^n) := \left\{ g \in L^1(\mathbb{R}^n) \mid \| g \|_{L^1_\beta} := \int_{\mathbb{R}^n} |g(x)| \Lambda(x)^\beta \, dx < \infty \right\},
$$

$$
L^\infty_{-\beta}(\mathbb{R}^n) := \left\{ f \in L^\infty_{\text{loc}}(\mathbb{R}^n) \mid \| f \|_{L^\infty_{-\beta}} := \left\| \Lambda^{-\beta} f \right\|_{L^\infty} = \text{ess sup}_{x \in \mathbb{R}^n} \Lambda(x)^{-\beta} |f(x)| < \infty \right\}.
$$

Obviously, $L^\infty_{-\beta}(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$, $S(\mathbb{R}^n) \subset L^1_\beta(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, and $L^1_\beta(\mathbb{R}^n)$ is a convolution algebra;

$$
A_\beta := \left\{ c \delta_0 + g \mid c \in \mathbb{C}, g \in L^1_\beta(\mathbb{R}^n) \right\}
$$

is $L^1_\beta(\mathbb{R}^n)$ with a unit attached, cf. Rudin [16, 10.3(d), 11.13(e)].

We need the following analogue of Theorem 2 for the pair $(L^1_\beta(\mathbb{R}^n), L^\infty_{-\beta}(\mathbb{R}^n))$.

**Theorem 5.** If $f \in L^\infty_{-\beta}(\mathbb{R}^n)$, $Y$ is a linear subspace of $L^1_\beta(\mathbb{R}^n)$, and

$$
f \ast g = 0 \quad \text{for every } g \in Y,
$$

then the set

$$
Z(Y) := \bigcap_{g \in Y} \{ \xi \in \mathbb{R}^n \mid \hat{g}(\xi) = 0 \}
$$

contains the support of the tempered distribution $\hat{f}$.

**Proof.** Pick any $\xi_0 \in \mathbb{R}^n \setminus Z(Y)$. There exists some $g \in Y$ such that $\hat{g}(\xi_0) = 1$. Since $\hat{g}$ is continuous, there is a neighbourhood $V = V(\xi_0)$ of $\xi_0$ such that $|\hat{g}(\xi) - 1| < 1/2$ for all $\xi \in V$.

To prove the theorem, it is sufficient to show that $\hat{f} = 0$ in $V$, or, equivalently, that $\langle \hat{f}, \hat{\nu} \rangle = 0$ for every $\nu \in S(\mathbb{R}^n)$ whose Fourier transform $\hat{\nu}$ has its support in $V$. Since

$$
\langle \hat{f}, \hat{\nu} \rangle = (2\pi)^{-n} \langle f, \nu \rangle = (2\pi)^{-n} (f \ast \nu)(0),
$$

it is sufficient to prove that $f \ast \nu(0) = 0$. 


Take $\phi \in C_c^\infty(V)$ such that $0 \leq \phi \leq 1$, and $\phi(\xi) = 1$ for every $\xi$ in the support of $\hat{\phi}$. Since we have $\mathcal{F}^{-1} \phi \in S(\mathbb{R}^n)$, there exists an element $u \in A_\beta$ such that

$$\tilde{u} = \phi \widehat{g} + 1 - \phi.$$  

It is easy to see that $\text{Re} \tilde{u}(\xi) > 1/2$ for all $\xi \in \mathbb{R}^n$. Then, there exists some function $w \in A_\beta$ such that $\tilde{w} = 1/\tilde{u}$, see, for example, [5, Theorems 1.41 and 2.11] or [6, Theorem 1.3]. Hence,

$$\tilde{v} = \tilde{w} \tilde{u} \tilde{v} = \tilde{w} \phi \widehat{g} \tilde{v} = \widehat{g} \tilde{w} \phi \tilde{v}.$$  

From this we get $v = g \ast G$ for some $G \in L^1_\beta(\mathbb{R}^n)$. Iterating the standard Peetre inequality $\Lambda(x - y) \leq \sqrt{2} \Lambda(x) \Lambda(y)$, we see that

$$\Lambda(x - y)^\beta \leq 2^\beta \Lambda(y)^\beta \Lambda(z)^\beta$$  

for all $x, y, z \in \mathbb{R}^n$. This yields

$$\int_{\mathbb{R}^n} |f(x - y)| \left( \int_{\mathbb{R}^n} |g(y - z)||G(z)| \, dz \right) \, dy \leq 2^\beta \Lambda(x)^\beta \int_{\mathbb{R}^n} |f(x - y)| \Lambda(x - y)^{-\beta} \left( \int_{\mathbb{R}^n} |g(y - z)| \Lambda(y - z)^\beta \Lambda(z)^\beta |G(z)| \, dz \right) \, dy,$$

$$\leq 2^\beta \Lambda(x)^\beta \lVert f \rVert_{L^{\infty}_{\beta}} \lVert g \rVert_{L^1_{\beta}} \lVert G \rVert_{L^1_{\beta}} < \infty,$$

and the Fubini–Tonelli theorem implies $f \ast (g \ast G) = (f \ast g) \ast G$. Finally,

$$f \ast v = f \ast (g \ast G) = (f \ast g) \ast G = 0 \ast G = 0. \quad \Box$$

We can now state and prove the Liouville theorem for polynomially bounded functions.

**Theorem 6** (Liouville property for polynomially bounded functions). Let $m \in C(\mathbb{R}^n)$ be such that the Fourier multiplier operator

$$C_c^\infty(\mathbb{R}^n) \ni \phi \mapsto \tilde{m}(D)\phi := \mathcal{F}^{-1}(\tilde{m}\hat{\phi})$$

maps $C_c^\infty(\mathbb{R}^n)$ into $L^1_{\beta}(\mathbb{R}^n)$. Suppose $f \in L^\infty_{\beta}(\mathbb{R}^n)$ is such that $m(D)f = 0$ as a distribution, that is,

$$\langle f, \tilde{m}(D)\phi \rangle = 0 \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n). \quad (12)$$

If $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subset \{0\}$, then $f$ coincides Lebesgue a.e. with a polynomial of degree at most $\lfloor \beta \rfloor$.

Conversely, if every complex-valued $f \in L^\infty_{\beta}(\mathbb{R}^n)$ satisfying (12) coincides Lebesgue a.e. with a polynomial, then $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subset \{0\}$. If $m(\eta) = 0$ implies that $m(-\eta) = 0$, then it is enough to consider real-valued $f \in L^\infty_{\beta}(\mathbb{R}^n)$ satisfying (12).

---

1 The equality $v = g \ast G$ with $G \in L^1(\mathbb{R}^n)$ is derived in the proof of [16, Theorem 9.3]. Unfortunately, this version does not seem sufficient for the proof of the equality $f \ast (g \ast G) = (f \ast g) \ast G$ when $f \in L^\infty_{\beta}(\mathbb{R}^n)$. 

Proof. If we replace in the proof of Theorem 2 \( \psi \mapsto m, L^1(\mathbb{R}^n) \mapsto L^1_\beta(\mathbb{R}^n) \) and \( L^\infty(\mathbb{R}^n) \mapsto L^\infty_{-\beta}(\mathbb{R}^n) \), we can follow the argument line-by-line up to the point where we get

\[
\hat{f} = \sum_{|\alpha| \leq N} c_\alpha \tilde{\delta}_0.
\]

Again, we invert the Fourier transform and use the polynomial boundedness of \( f \) to see that \( f \) coincides Lebesgue a.e. with a polynomial of degree less or equal than \( |\beta| \). The converse statement follows from that in Theorem 3. Indeed, if every function \( f \in L^\infty_{-\beta}(\mathbb{R}^n) \) satisfying (12) coincides Lebesgue a.e. with a polynomial, then the same is true for any such \( f \) in the space \( L^\infty(\mathbb{R}^n) \subseteq L^\infty_{-\beta}(\mathbb{R}^n) \). Since the only polynomials contained in \( L^\infty(\mathbb{R}^n) \) are constants, one can apply the converse statement in Theorem 3. \( \square \)

Example 7. In this example, we discuss the multipliers from Example 4 in the setting of Theorem 6. The following conditions ensure that \( \mathcal{A}(D) \) maps \( C^\infty_c(\mathbb{R}^n) \) into \( L^1_\beta(\mathbb{R}^n) \).

(a) \( \kappa \) is a linear combination of terms of the form \( ab \), where \( a = \mathcal{T} \mu \), \( \mu \) is a finite Borel measure on \( \mathbb{R}^n \) such that \( \int \Lambda_\beta(y) \mu(dy) < \infty \), and all partial derivatives \( \partial_\beta^\alpha b \) with \( |\alpha| \leq |n + \beta| + 1 \) are polynomially bounded.

Indeed: Let \( b_N := b \Lambda^{-2N} \), \( N \in \mathbb{R}^n \). For a sufficiently large \( N \), all partial derivatives \( \partial_\beta^\alpha b_N \) with \( |\alpha| \leq |n + \beta| + 1 \) belong to \( L^1(\mathbb{R}^n) \). Then, \( x^\alpha \mathcal{T}^{-1}(b_N) \in L^\infty(\mathbb{R}^n) \), \( |\alpha| \leq |n + \beta| + 1 \). Hence,

\[
\mathcal{T}^{-1}(b_N) \in L^\infty(\mathbb{R}^n), \quad j = 1, \ldots, n
\]

\[
\Rightarrow (1 + |x|)^{|n+\beta|+1} \mathcal{T}^{-1}(b_N) \in L^\infty(\mathbb{R}^n) \Rightarrow \mathcal{T}^{-1}(b_N) \in L^1_\beta(\mathbb{R}^n).
\]

Let \( \phi \in C^\infty_c(\mathbb{R}^n) \subset L^1_\beta(\mathbb{R}^n) \). We have \( b_N(D)\phi = (2\pi)^{-n} \mathcal{T}^{-1}(b_N) * \phi \), and it follows from Young’s and Peetre’s inequalities

\[
\left\| (\mathcal{T}^{-1}b_N)*\phi \right\|_{L^1_\beta} = \left\| \Lambda_\beta((\mathcal{T}^{-1}b_N)*\phi) \right\|_{L^1} \leq 2\beta/2 \left\| \Lambda_\beta \mathcal{T}^{-1}b_N * (\Lambda_\beta \phi) \right\|_{L^1} \leq 2\beta/2 \left\| \Lambda_\beta \mathcal{T}^{-1}b_N \right\|_{L^1} \left\| \Lambda_\beta \phi \right\|_{L^1} = 2\beta/2 \left\| \mathcal{T}^{-1}b_N \right\|_{L^1_\beta} \left\| \phi \right\|_{L^1_\beta}
\]

that \( b_N(D) \) maps \( C^\infty_c(\mathbb{R}^n) \) into \( L^1_\beta(\mathbb{R}^n) \). Since the differential operator \( \Lambda^{2N}(D) \) maps \( C^\infty_c(\mathbb{R}^n) \) into itself, and \( (ab)(D) = a(D)b \Lambda^{2N}(D) \), it is left to show that \( a(D) \) maps \( L^1_\beta(\mathbb{R}^n) \) into itself. Using Peetre’s inequality again, we deduce from

\[
(a(D)g)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} g(x-y) \mu(dy), \quad x \in \mathbb{R}^n, \ g \in \mathbb{R}^n,
\]

that

\[
(2\pi)^n \left\| a(D)g \right\|_{L^1_\beta(\mathbb{R}^n)}
\]
\[
\leq \int_{\mathbb{R}} \Lambda^\beta(x) \int_{\mathbb{R}} \Lambda^\beta(y) \mu(\text{d}y) \mu(\text{d}x) \leq 2^{\beta/2} \int_{\mathbb{R}} |g(z)| \Lambda^\beta(z) \text{d}z \int_{\mathbb{R}} \Lambda^\beta(y) \mu(\text{d}y)
= 2^{\beta/2} \int_{\mathbb{R}} \Lambda^\beta(y) \mu(\text{d}y) \|g\|_{L^1_\beta}.
\]

With a bit more effort, one can actually show that
\[
\int \Lambda^\beta(y) \mu(\text{d}y)<\infty
\]
is also a necessary condition for \(a(D)\) to map \(C_\infty(\mathbb{R}^n)\) into \(L^1_\beta(\mathbb{R}^n)\) (see [4, Theorem 3] for a proof).

A particular example is the characteristic exponent \(\kappa = \psi\) of a Lévy process such that the Lévy measure has finite moments of order \(\beta\) (cf. Example 4a)).

(b) Any \(\kappa\) of the form
\[
\kappa(\xi) = \sum_{|\alpha|=0}^{2s} c_\alpha \frac{|\alpha|!}{\alpha!} \xi^\alpha + \int_{|y|<1} \left(1 - e^{iy \cdot \xi} + \sum_{|\alpha|=0}^{2s-1} \frac{|\alpha|!}{\alpha!} y^\alpha \xi^\alpha \right) \nu(\text{d}y) + \int_{|y|\geq 1} \left(1 - e^{iy \cdot \xi} \right) \nu(\text{d}y)
\]
with \(s \in \mathbb{N}\), \(c_\alpha \in \mathbb{R}\), and a measure \(\nu\) on \(\mathbb{R}^n \setminus \{0\}\) such that \(\int_{|y|<1} |y|^{2s} \nu(\text{d}y) + \int_{|y|\geq 1} |y|^\beta \nu(\text{d}y) < \infty\).

It is not difficult to extend Theorem 6 from solutions of the equation \(m(D)f = 0\) to solutions of \(m(D)f = p\), where \(p\) is a polynomial.

**Corollary 1.** Let \(m \in C(\mathbb{R}^n)\) be as in Theorem 6, and let \(p\) be a polynomial. Suppose \(f \in L^\infty_\beta(\mathbb{R}^n)\) is such that
\[
\langle f, \tilde{m}(D)\phi \rangle = \langle p, \phi \rangle \quad \text{for all } \phi \in C_\infty(\mathbb{R}^n).
\]  
If \(\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subset \{0\}\), then \(f\) coincides Lebesgue a.e. with a polynomial of degree at most \(\lfloor \beta \rfloor\). Conversely, if there exists \(f \in L^\infty_\beta(\mathbb{R}^n)\) satisfying (13), and every such \(f\) coincides Lebesgue a.e. with a polynomial, then it holds that \(\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subset \{0\}\).

**Proof.** Pick \(k \in \mathbb{N}\) such that \(2k\) is greater than the degree of the polynomial \(p\). Then, \(\Delta^k p = 0\). Set \(m_k(\xi) := |\xi|^{2k} m(\xi)\); clearly, \(m_k \in C(\mathbb{R}^n)\). Since \(\Delta^k\) maps \(C_\infty(\mathbb{R}^n)\) continuously into itself, \(\tilde{m}_k(D) = \tilde{m}(D)\Delta^k\) maps \(C_\infty(\mathbb{R}^n)\) into \(L^1_\beta(\mathbb{R}^n)\). Moreover,
\[
\langle f, \tilde{m}_k(D)\phi \rangle = \langle f, \tilde{m}(D)(\Delta^k\phi) \rangle = \langle p, \Delta^k\phi \rangle = \langle \Delta^k p, \phi \rangle = \langle 0, \phi \rangle = 0
\]
for all \(\phi \in C_\infty(\mathbb{R}^n)\). Now we can apply Theorem 6 with \(m_k\) in place of \(m\). Note that \(\{m_k = 0\} = \{m = 0\} \cup \{0\}\), that is, \(\{m_k = 0\} \subset \{0\}\) if, and only if, \(\{m = 0\} \subset \{0\}\).

The following result shows that the polynomial \(f\) appearing in Theorem 6 has degree at most 1 in the case where \(n = 1\) and \(m\) is the characteristic exponent of a Lévy process.

**Corollary 2.** Let \(\psi : \mathbb{R} \to \mathbb{C}\) be the characteristic exponent of a Lévy process with Lévy triplet \((Q, b, \nu)\). Suppose that there exists a \(\beta > 0\) such that \(\int_{|y|>1} |y|^\beta \nu(\text{d}y) < \infty\). If \(\{\xi \in \mathbb{R} \mid \psi(\xi) = 0\} = \{0\}\) and \(f \in L^\infty_\beta(\mathbb{R})\) is a weak solution of the equation \(\psi(D)f = 0\), then \(f(x) = c_1 x + c_0\) with some constants \(c_1\) and \(c_0\).
Proof. It follows from Theorem 6 that \( f \) is a polynomial of degree at most \([\beta]\). According to Lemma 2.4 in [15], the degree of \( f \) is less than or equal to 2. The lemma deals with solutions of the equation \( \psi(D)f = \text{const} \). In the case of \( \psi(D)f = 0 \), its proof (see, in particular, the last paragraph of the proof) shows that the degree of \( f \) is actually less than or equal to 1. \( \square \)

The above result does not hold in the multi-dimensional case \( n \geq 2 \). Note that in the case \( n = 1 \), \( f'' = 0 \Rightarrow f(x) = ax + b \), while in the case \( n = 2 \), \( \Delta f = 0 \) has polynomial solutions of any degree, for example, \( \Re(x_1 + ix_2)^k, k \in \mathbb{N} \).

3 THE LIOUVILLE THEOREM FOR SLOWLY GROWING FUNCTIONS

Theorem 5 covers bounded functions \( f \), while Theorem 6 is about functions whose growth is compared with the growth of a polynomial. This leaves a gap where \( f \) grows slower than a polynomial, for example, at a logarithmic scale. To deal with this case, we need the notion of a measurable, locally bounded, submultiplicative function, that is, a locally bounded measurable function \( h : \mathbb{R}^n \to (0, \infty) \) satisfying

\[
    h(x + y) \leq c h(x) h(y) \quad \text{for some } c \geq 1 \text{ and all } x, y \in \mathbb{R}^n.
\]

Without loss of generality, we will always assume that \( h \geq 1 \); otherwise, we would replace \( h(x) \) by \( h(x) + 1 \). Typical examples of submultiplicative functions are \((1 + |x|)^\beta, \Lambda(x)^\beta, e^{\alpha|x|^{\beta}}\) for \( \beta \in [0, 1] \), and \( \log^\beta(|x|+e) \) for \( \beta > 0 \). Observe that every submultiplicative function is exponentially bounded. For further details, see [17, Section 25] or [13, Section II.§1].

Fix some locally bounded submultiplicative \( h \) that satisfies, in addition,

\[
    \lim_{|x| \to \infty} \Lambda^{-k}(x) h(x) = 0 \quad \text{for some } k \in \mathbb{N}. \quad (14)
\]

The condition (14) implies the so-called GRS (Gelfand–Raikov–Shilov)-condition, see [2] and [6] for details. We replace the pair \((L^1_\beta(\mathbb{R}^n), L^\infty_{-\beta}(\mathbb{R}^n))\) by

\[
L^1_h(\mathbb{R}^n) := \left\{ g \in L^1(\mathbb{R}^n) \mid \|g\|_{L^1_h} := \int_{\mathbb{R}^n} |g(x)|h(x) \, dx < \infty \right\},
\]

\[
L^\infty_{h^{-1}}(\mathbb{R}^n) := \left\{ f \in L^\infty_{\text{loc}}(\mathbb{R}^n) \mid \|f\|_{L^\infty_{h^{-1}}} := \|h^{-1} f\|_{L^\infty} = \text{ess sup}_{x \in \mathbb{R}^n} h(x)^{-1} |f(x)| < \infty \right\}
\]

and use, instead of \( A_\beta \),

\[
A_h := \{ c \delta_0 + g \mid c \in \mathbb{C}, g \in L^1_h(\mathbb{R}^n) \}.
\]

The family \( A_h \) is a convolution algebra and, due to the GRS-condition, an element of \( u \in A_h \) is invertible if, and only if, it is invertible in \( A_1 \), where \( A_1 \) is the algebra with \( h = 1 \). One can replace the Peetre inequality by \( h(x - y) \leq c h(x) h(-y) \), which is a direct consequence of the submultiplicativity of \( h \), and then get by iteration \( h(x - y) \leq c^2 h(x) h(-(y - z)) h(-z) \). This allows
one to repeat the arguments in the proof of Theorem 5 and arrive at a version of this theorem with $L^1_{\beta} \rightarrow L^1_{\tilde{h}}$ and $L^1_{-\beta} \rightarrow L^{\infty}_{\tilde{h}}$. If $\tilde{m}(D)$ maps $C^\infty_\mathcal{C}(\mathbb{R}^n)$ into $L^1_{\tilde{h}}(\mathbb{R}^n)$, one can apply this version of Theorem 5 with

$$Y := \left\{ \tilde{m}(D)\phi \mid \phi \in C^\infty_\mathcal{C}(\mathbb{R}^n) \right\} \subset L^1_{\tilde{h}}(\mathbb{R}^n).$$

This results in the following analogue of Theorem 6 for slowly growing functions.

**Theorem 8** (Liouville property for slowly growing functions). Let $m \in C(\mathbb{R}^n)$ be such that the Fourier multiplier operator

$$C^\infty_\mathcal{C}(\mathbb{R}^n) \ni \phi \mapsto \tilde{m}(D)\phi := F^{-1}(\tilde{m}\hat{\phi})$$

maps $C^\infty_\mathcal{C}(\mathbb{R}^n)$ into $L^1_{\tilde{h}}(\mathbb{R}^n)$. Suppose $f \in L^\infty_{h^{-1}}(\mathbb{R}^n)$ is such that $m(D)f = 0$ as a distribution, that is,

$$\langle f, \tilde{m}(D)\phi \rangle = 0 \quad \text{for all } \phi \in C^\infty_\mathcal{C}(\mathbb{R}^n).$$

(15)

If $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subset \{0\}$, then $f$ coincides Lebesgue a.e. with a polynomial $p \in L^\infty_{h^{-1}}(\mathbb{R}^n)$.

Conversely, if $f$ coincides Lebesgue a.e. with a polynomial for every $f \in L^\infty(\mathbb{R}^n)$ satisfying (15), then it holds that $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subset \{0\}$.

Theorem 8 can be used for functions of the form $h(x) = \Lambda^\beta(x)\log(e + |x|)^\alpha$ for $\alpha, \beta \geq 0$. If we use $h(x) = \log(e + |x|)$, we see that in the setting of Theorem 8, every solution $m(D)f = 0$ is a.e. constant.

We want to point out that without the boundedness condition (14), the function $f$ is not necessarily a tempered distribution. For example, if we choose $h(x) = e^{a|x|^\gamma}$, then $f$ need not be a tempered distribution, and our method above would not work. In the next section, we discuss functions, which might not define a tempered distributions, but are positive. The condition that $\tilde{m}(D)$ maps $C^\infty_\mathcal{C}(\mathbb{R}^n)$ into $L^1_{\tilde{h}}(\mathbb{R}^n)$ is essential for Theorem 8. Recently, we learned from M. Kwasnicki (private communication) and the paper by T. Grzywny and M. Kwasnicki [8, Theorem 1.9.(c) & Theorem 4.1] that there is a multiplier given by a Lévy process, that is, $m = \psi$, admitting a very slowly growing non-constant function $u$ such that $\psi(D)u = 0$. Note that this multiplier does not satisfy the mapping property required for Theorem 8.

### 4 | The Liouville Theorem for Rapidly Growing Functions

We will now turn to the case where the solution of $\psi(D)f = 0$ is locally bounded and positive. It is well known that

$$f \geq 0, \quad \Delta f = 0 \quad \Rightarrow \quad f \equiv \text{const.}$$

This is usually called the strong Liouville property. One cannot expect this property to hold for general Fourier multiplier operators discussed in Sections 2 and 3 or even for higher order partial
differential operators. Indeed,

\[ f(x) := |x|^2 = x_1^2 + \cdots + x_n^2 \]

is a non-constant nonnegative polynomially bounded solution of the equation \( \Delta^2 f = 0 \). Here, \( \Delta^2 = \psi(D) \), \( \psi(\xi) = |\xi|^4 \) and \( \{ \xi \in \mathbb{R}^n \mid \psi(\xi) = 0 \} = \{0\} = \{ \eta \in \mathbb{R}^n \mid \psi(-i\eta) = 0 \} \) (cf. Theorem 10 below). So, we consider in this section Fourier multiplier operators \( \psi(D) \) that generate positivity preserving operator semi-groups \( \{ e^{-t\psi(D)} \}_{t \geq 0} \), that is, such that \( \psi \) are the characteristic exponents of Lévy processes, see Section 1 and Example 4. Even within this class of operators, the Laplacian is a special case, and the strong Liouville property does not hold for more general second-order partial differential operators \( \mathcal{L} \) without restrictions on the growth rate of a solution \( f \) of the equation \( \mathcal{L} f = 0 \). Let

\[ \mathcal{L} u(x) = \frac{1}{2} \nabla \cdot Q \nabla u(x) + b \cdot \nabla u(x), \]

where \( Q \in \mathbb{R}^{n \times n} \) is a positive semi-definite matrix, and \( b \in \mathbb{R}^n \setminus \{0\} \). Then, \( \mathcal{L} f = 0 \) has non-constant nonnegative solutions. Indeed, if \( Q \neq 0 \), there exists \( c_0 \in \mathbb{R}^n \) such that \( c_0 \cdot Qc_0 > 0 \). Since \( b \neq 0 \), there exists \( c \) in a neighbourhood of \( c_0 \) such that \( c \cdot Qc > 0 \) and \( b \cdot c \neq 0 \). Let

\[ f(x) := e^{\tau c \cdot x}, \quad \text{where} \quad \tau = -\frac{2b \cdot c}{c \cdot Qc} \neq 0. \]

Then, \( f > 0 \) is non-constant and

\[
\mathcal{L} f(x) = \frac{1}{2} \nabla \cdot Q e^{\tau c \cdot x} + b \cdot \nabla e^{\tau c \cdot x} = \left( \frac{1}{2} \tau^2 c \cdot Qc + \tau b \cdot c \right) e^{\tau c \cdot x} = \tau \left( \frac{1}{2} \tau c \cdot Qc + b \cdot c \right) e^{\tau c \cdot x} = 0.
\]

If \( Q = 0 \), it is sufficient to take any \( c \in \mathbb{R}^n \setminus \{0\} \) such that \( b \cdot c = 0 \). Then,

\[ f(x) := e^{c \cdot x} \]

is positive, non-constant and

\[ \mathcal{L} f(x) = b \cdot \nabla e^{c \cdot x} = (b \cdot c)e^{c \cdot x} = 0. \]

It follows from the above that one needs to put appropriate boundedness restrictions on \( f \). Note also that while local boundedness of \( f \) is sufficient to ensure that \( \langle f, \mathcal{F}(D)\phi \rangle \) is well defined when \( \psi(D) \) is a local operator, that is, a partial differential operator, one needs boundedness restrictions on \( f \) to ensure that \( \psi(D)f \) has a meaning when \( \psi(D) \) is non-local.

Our current proof is a refinement of our result in [3, Theorem 17], and we focus here on the role of the upper bound. The key ingredient in the proof of [3, Theorem 17] is the equivalence of \( -\psi(D)f = 0 \) and \( e^{-t\psi(D)} f = f \), which is used to get a Choquet representation of all positive solutions \( f \geq 0 \). In order to define \( \psi(D)f \) or \( e^{-t\psi(D)} \) as a distribution, we need a bound \( f \leq g \) and an integrability condition on the measure \( \nu \) appearing in the Lévy–Khintchine representation (3) of \( \psi \), see the discussion in [3]. Here, we will concentrate on the role of the upper bound \( g \) in the
proof of the strong Liouville property which was glossed-over in our presentation in [3, Theorem 17]; this explains, in particular, in which directions \( \psi \) can be extended from \( \mathbb{R}^n \) into \( C^n = \mathbb{R}^n + i\mathbb{R}^n \).

Let \( g : \mathbb{R}^n \to [1, \infty) \) be a locally bounded, measurable submultiplicative function. We need to describe the directional growth behaviour of \( g \). For \( \omega \in S^{n-1} = \{ x \in \mathbb{R}^n \mid |x| = 1 \} \), let

\[
g_\omega(r) := g(r\omega), \quad r \geq 0.
\]

Set

\[
\beta(\omega) := \inf_{r > 0} \frac{\ln g_\omega(r)}{r} = \lim_{r \to \infty} \frac{\ln g_\omega(r)}{r}.
\]

Since \( \ln g_\omega \) is subadditive, the infimum is, in fact, a limit; moreover, the function \( \beta : S^{n-1} \to \mathbb{R} \) is continuous, see [9, Theorem 7.13.2].

Applying [13, Chapter II, Theorem 1.3] to the function \( v(t) := g_\omega(\ln t), \ t > 0 \), we conclude that

\[
g_\omega(r) \geq e^{\beta(\omega)r} \quad \text{for all } r > 0,
\]

and that for every \( \varepsilon > 0 \), there is some \( r_\varepsilon > 0 \) such that

\[
g_\omega(r) \leq e^{(\beta(\omega)+\varepsilon)r} \quad \text{for all } r > r_\varepsilon.
\]

Let

\[
\Pi_g := \{ \xi \in \mathbb{R}^n \mid \xi \cdot \omega \leq \beta(\omega) \quad \text{for all } \omega \in S^{n-1} \}.
\]

Example 9.

(a) If \( g(x) = (1 + |x|)^{\lambda} \) with \( \lambda \geq 0 \), or \( g(x) = e^{\alpha |x|^\gamma} \) with \( \alpha \geq 0, \gamma \in [0, 1) \), then we have \( \beta(\omega) \equiv 0 \) and \( \Pi_g = \{0\} \).

(b) If \( g(x) = e^{\alpha |x|} \) with \( \alpha > 0 \), then \( \beta(\omega) \equiv \alpha \) and

\[
\Pi_g = \{ \eta \in \mathbb{R}^n \mid |\eta| \leq \alpha \}.
\]

(c) If \( g(x) := \max \{e^{x_1}, 1\} \), then \( \beta(\omega) = \max \{\omega_1, 0\} \), and it is easy to see that

\[
\Pi_g = \left\{ \xi \in \mathbb{R}^n \mid \xi_1 \omega_1 + \sum_{j=2}^d \xi_j \omega_j \leq \max \{\omega_1, 0\} \quad \text{for all } \omega \in S^{n-1} \right\}
\]

\[
= \{ \xi = (\xi_1, 0, \ldots, 0) \in \mathbb{R}^n \mid \xi_1 \omega_1 \leq \max \{\omega_1, 0\} \quad \text{for all } \omega_1 \in [-1, 1]\}
\]

\[
= \{ \xi = (\xi_1, 0, \ldots, 0) \in \mathbb{R}^n \mid \xi_1 \in [0, 1] \},
\]

that is, \( \Pi_g \) is the one-dimensional interval \([0, 1] \times \{0\} \times \cdots \times \{0\} \).
We will need to extend $\psi$ into a strip in $\mathbb{C}^n$. This can be achieved by (2) or (3), provided that the measure $\nu$ is sufficiently well behaved. It follows from (16) and (18) that if $\eta \in \Pi_g$, then
\[
\left| e^{i(\xi - i \eta) y} \right| = e^{\eta y} \leq g(y) \quad \text{for all } y \in \mathbb{R}^n.
\]
So, if one assumes, as we do in the next theorem, that $\int_{|y| \geq 1} g(y) \nu(dy) < \infty$, then it follows from (3) that $\psi(\xi - i \eta)$ is well defined.

**Theorem 10** (Strong Liouville property). Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be the characteristic exponent of a Lévy process with Lévy triplet $(b, Q, \nu)$. Let $g : \mathbb{R}^n \to [1, \infty)$ be a locally bounded, measurable submultiplicative function such that $\int_{|y| \geq 1} g(y) \nu(dy) < \infty$. Every measurable, positive and $g$-bounded ($0 \leq f \leq g$) weak solution $f$ of the equation $\psi(D)f = 0$ is constant if, and only if,
\[
\{\xi \in \mathbb{R}^n \mid \psi(\xi) = 0\} = \{0\} = \{\eta \in \Pi_g \mid \psi(-i \eta) = 0\}.
\]

**Proof.** Sufficiency of (19). Assume that the first equality in (19) holds, $0 \leq f \leq g$, and $\psi(D)f = 0$. Theorem 10 is a refinement of the corresponding result [3, Theorem 17] that takes care of the nature of the extension of $\psi$ into the complex plane. Rather than reproducing the full proof here, we quote the structural part of the solution to $\psi(D)f = 0$ from [3, Theorem 17]: there exists a measure $\rho$ with support in $E := \{\eta \in \mathbb{R}^n \mid \psi(-i \eta) = 0\}$ such that
\[
f(x) = \int_E e^{i \xi \cdot \rho}(d\xi).
\]
We will show that supp $\rho \subset \{\eta \in \Pi_g \mid \psi(-i \eta) = 0\}$; thus, the second equality in (19) proves that $f \equiv \rho(0)$ a.s.

Suppose that there exists some $\xi^0 \in \text{supp } \rho \setminus \Pi_g$. This means that there is some $\omega_0 \in S^{n-1}$ such that $\xi^0 \cdot \omega^0 > \beta(\omega^0)$. Take any $0 < \epsilon < \frac{1}{2}(\xi^0 \cdot \omega^0 - \beta(\omega^0))$ and consider the open ball $B_\epsilon := B_\epsilon(\xi^0)$ of radius $\epsilon$ centred at $\xi^0$. Let $x = r\omega_0, r > 0$. Then,
\[
x \cdot \xi = x \cdot \xi^0 + x \cdot (\xi - \xi^0) > (\xi^0 \cdot \omega^0) r - r|\xi - \xi^0| > (\xi^0 \cdot \omega^0 - \epsilon)r \quad \text{for all } \xi \in B_\epsilon,
\]
and there exists some $r_0 > 0$, depending only on $g, \omega_0$ and $\epsilon$ (see (17)), such that
\[
\frac{e^{x \cdot \xi}}{g_{\omega_0}(r)} > \frac{e^{(\xi^0 \cdot \omega^0 - \epsilon)r}}{g_{\omega_0}(r)} \geq \frac{e^{(\xi^0 \cdot \omega^0 - \epsilon)r}}{e^{\beta(\omega_0) + \epsilon)r}} = e^{(\xi^0 \cdot \omega^0 - \beta(\omega^0) - 2\epsilon)r} \quad \text{for all } r \geq r_0.
\]
Since $\xi^0 \in \text{supp } \rho$, we know that $\rho(B_\epsilon) > 0$. Thus,
\[
\frac{f(x)}{g(x)} = \frac{\int_E e^{i \xi \cdot \rho}(d\xi)}{g(x)} \geq \frac{\int_{B_\epsilon} e^{i \xi \cdot \rho}(d\xi)}{g_{\omega_0}(r)} \geq \frac{e^{(\xi^0 \cdot \omega^0 - \beta(\omega^0) - 2\epsilon)r}}{g_{\omega_0}(r)} \rho(B_\epsilon) = e^{(\xi^0 \cdot \omega^0 - \beta(\omega^0) - 2\epsilon)r} \rho(B_\epsilon) \to \infty \quad \text{as } r \to \infty.
\]
This contradicts the bound $0 \leq f \leq g$, and we conclude that $\text{supp } \rho \setminus \Pi_g = \emptyset$, and, therefore, $\text{supp } \rho \subset \{\eta \in \Pi_g \mid \psi(-i \eta) = 0\}$. 

Necessity of (19). Suppose that every measurable, positive and $g$-bounded weak solution $f$ of $\psi(D)f = 0$ is constant. If the first equality in (19) does not hold, then there exists $\xi^0 \in \{ \xi \in \mathbb{R}^d : \psi(\xi) = 0 \} \setminus \{0\}$. Since $\psi(-\xi^0) = \psi(\xi^0) = 0$ and $\psi(0) = 0$, one has $-\xi^0, 0 \in \{ \xi \in \mathbb{R}^d : \psi(\xi) = 0 \}$. Let $e_{\pm \xi^0}(x) := e^{\pm i \xi^0 \cdot x}$. Then, $\psi(D)e_{\pm \xi^0} = \psi(\pm \xi^0)e_{\pm \xi^0} = 0$ and $\psi(D)1 = \psi(0) = 0$. Hence,

$$f(x) := \frac{1}{2}(1 + \cos(\xi^0 \cdot x))$$

is a non-constant measurable, positive and $g$-bounded weak solution $f$ of $\psi(D)f = 0$. This contradiction proves the first equality in (19).

Suppose now the second equality in (19) does not hold, that is, there exists a non-zero element $\theta \in \{ \eta \in \Pi_g \mid \psi(-i\eta) = 0 \}$. From the definition of $\Pi_g$, we see that, for every $x = r\omega$, $\omega \in \mathbb{S}^{n-1}$ and $r \geq 0$,

$$f(x) := e^{\theta \cdot x} = e^{(\theta \cdot \omega)r} \leq g_{\theta}(r) = g(r\omega) = g(x),$$

see (16). So, $f$ is non-constant, positive and $g$-bounded. Since $\psi(D)f = \psi(-i\theta)f \equiv 0$, we get again a contradiction.

5 COUPLING

In this section, we want to establish a connection between our Livouville theorem (Theorem 1), the coupling property of Lévy processes and the notion of space-time harmonic functions. Throughout, $(X_t)_{t \geq 0}$ is a Lévy process starting at 0 with characteristic exponent $\psi$, see Section 1.

A coupling of the Lévy process $(X_t)_{t \geq 0}$ with values in $\mathbb{R}^n$ is any Markov process $((Z^x_t,Z^y_t))_{t \geq 0}$ taking values in $\mathbb{R}^{2n}$ such that each of the marginal processes $(Z^x_t)_{t \geq 0}$ has the same finite-dimensional distributions as $(X^x_t)_{t \geq 0}$, $X^x_t := X_t + z$, $z = x,y$. The coupling is usually realised on a new probability space $(\Omega, \mathcal{F}, \mathbb{P} = \mathbb{P}(x,y), F)$. The Lévy process $(X_t)_{t \geq 0}$ has the (exact) coupling property, if for some coupling with $x \neq y$, the trajectories of the processes $(Z^x_t)_{t \geq 0}$ and $(Z^y_t)_{t \geq 0}$ meet with probability 1 in finite time; this is the coupling time $\tau = \tau_{x,y}$. Intuitively, both processes $(Z^x_t)_{t \geq 0}$ and $(Z^y_t)_{t \geq 0}$ run on the same probability space, move together and cannot (statistically) be distinguished from each other or from $(X_t + z)_{t \geq 0}$.

Coupling techniques provide powerful tools to study the regularity of the operator semi-group $x \mapsto P_t f(x) = \mathbb{E} f(X^x_t)$, the existence of invariant (stationary) measures for $(X^x_t)_{t \geq 0,x \in \mathbb{R}^n}$ and many further properties, see the discussion in [20]. It was shown in [19, Theorem 4.1] that a Lévy process has the coupling property if, and only if, the transition probability $p_t(dy)$ of the Lévy process $(X_t)_{t \geq 0}$ has an absolutely continuous component for some $t \geq 0$.

If $(X^x_t)_{t \geq 0,x \in \mathbb{R}^n}$ is a Lévy process (or a general Markov process with generator $A_x$), the so-called space-time process given by $((s + t,X^x_t))_{t \geq 0,(s,x) \in [0,\infty) \times \mathbb{R}^n}$ is again a Lévy process (resp., Markov process), and its semi-group is given by $Q_t u(s,x) = \mathbb{E}(s + t,X^x_t)$. Thus, the infinitesimal generator is of the form $\frac{d}{ds} - \psi(D_x)$ (resp. $\frac{d}{ds} + A_x$), and we are naturally led to the notion of space-time harmonic functions and the space-time Liouville property.

Definition 1. Let $(X^x_t)_{t \geq 0,x \in \mathbb{R}^n}$ be a Lévy process.
(a) A function $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is **space-time harmonic**, if one has $f(s, x) = \mathbb{E}_x f(s + t, X_t) = \mathbb{E} f(s + t, X_t + x)$ for every $t, s \geq 0$ and every $x \in \mathbb{R}^n$.

(b) The process has the **space-time Liouville property**, if every measurable and bounded space-time harmonic function is constant.

**Remark 3.**

(a) The space-time Liouville property is known to be equivalent to the exact coupling property for Markov processes, see [20, Theorem 4.1, p. 205].

(b) We call a function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ satisfying $f(s, x) = \mathbb{E}_x f(s + t, X_t)$ for every $t \geq 0$ and every $(s, x) \in \mathbb{R}^{n+1}$ also space-time harmonic.

(c) Notice that the notions of ‘space-time harmonicity’ and ‘exact coupling’ are pointwise defined notions, which do not allow for exceptional sets. This is the main difficulty when we want to compare our notion of the Liouville property and the space-time Liouville property.

For a Lévy triplet $(b, Q, \nu)$ with Lévy process $X = (X_t)_{t \geq 0}$ in $\mathbb{R}^n$ and a vector $\eta \in \mathbb{R}^n$, we introduce the notation

$$b^n := \eta \cdot b + \int \eta \cdot y \left( 1_{(0,1)}(|\eta \cdot y|) - 1_{(0,1)}(|y|) \right) \nu(dy).$$

This is the vector in the Lévy triplet of $\eta \cdot X_t$, see [17, Proposition 11.10]. We need a few auxiliary lemmas preparing the proof of the main result of this section, Theorem 11.

**Lemma 1.** Let $\psi : \mathbb{R}^d \to \mathbb{C}$ be the characteristic exponent of a Lévy process with Lévy triplet $(b, Q, \nu)$. If $\xi \in \mathbb{R}^n$ is such that $\psi(\xi) \in i\mathbb{R}$, then $\psi(\xi) = -ib\xi$.

**Proof.** Let

$$E_\xi := \left\{ y \in \mathbb{R}^n \mid 1 - e^{iy \cdot \xi} = 0 \right\} = \left\{ y \in \mathbb{R}^n \mid y \cdot \xi \in 2\pi \mathbb{Z} \right\},$$

$$E_0^\xi := \left\{ y \in \mathbb{R}^n \mid y \cdot \xi = 0 \right\} = \left\{ \xi \right\}^\perp.$$

Since $\text{Re} \left( 1 - e^{iy \cdot \xi} \right) \geq 0$, and $\text{Re} \left( 1 - e^{iy \cdot \xi} \right) = 0$ if, and only if, $1 - e^{iy \cdot \xi} = 0$, it follows from the fact that $\psi(\xi) \in i\mathbb{R}$ and (3) that $\nu(\mathbb{R}^n \setminus E_\xi) = 0$ and $Q\xi = 0$.

If $y \in E_\xi \setminus E_0^\xi$, then $|y \cdot \xi| \geq 2\pi > 1$. So, $1_{(0,1)}(|\xi \cdot y|) = 0$ if $y \in E_\xi$. Hence,

$$\psi(\xi) = -ib \cdot \xi + \int_{\mathbb{R}^n} \left[ 1 - e^{iy \cdot \xi} + iy \cdot \xi 1_{(0,1)}(|y|) \right] \nu(dy)$$

$$= -ib \cdot \xi + \int_{E_\xi} \left[ 1 - e^{iy \cdot \xi} + iy \cdot \xi 1_{(0,1)}(|y|) \right] \nu(dy)$$

$$= -ib \cdot \xi + \int_{E_\xi} \left[ 1 - e^{iy \cdot \xi} + iy \cdot \xi (1_{(0,1)}(|y|) - 1_{(0,1)}(|\xi \cdot y|)) \right] \nu(dy).$$
\[ = -ib \cdot \xi + i \int_{\mathbb{R}^d} y \cdot \xi \left[ 1_{(0,1)}(|y|) - 1_{(0,1)}(|\xi \cdot y|) \right] \nu(dy) \]

\[ = -ib \cdot \xi + i \int_{\mathbb{R}^n} y \cdot \xi \left[ 1_{(0,1)}(|y|) - 1_{(0,1)}(|\xi \cdot y|) \right] \nu(dy) = -ib \xi. \]

\[ \square \]

**Lemma 2.** Let \( n, d \in \mathbb{N}, \) \( \psi_1 : \mathbb{R}^d \to \mathbb{C} \) and \( \psi_2 : \mathbb{R}^n \to \mathbb{C} \) be characteristic exponents of two Lévy processes with Lévy triplets \((b_1, Q_1, \nu_1)\) and \((b_2, Q_2, \nu_2)\). Then,

\[ \{(\eta, \xi) \in \mathbb{R}^d \times \mathbb{R}^n \mid \psi_1(\eta) + \psi_2(\xi) = 0\} = \{\eta \in \psi_1^{-1}(i\mathbb{R}), \ \xi \in \psi_2^{-1}(i\mathbb{R}) \mid b_1^\eta + b_2^\xi = 0\}. \]

**Proof.** Since \( \text{Re} \psi_i \geq 0 \) for \( i = 1, 2 \) (see (3)), a necessary condition for \( \psi_1(\eta) + \psi_2(\xi) = 0 \) is \( \text{Re} \psi_1(\eta) = 0 \) and \( \text{Re} \psi_2(\xi) = 0 \). In this case, \( \psi_1(\eta) = -ib_1^\eta \) and \( \psi_2(\xi) = -ib_2^\xi \) (see Lemma 1), and the equality \( \psi_1(\eta) + \psi_2(\xi) = 0 \) is equivalent to \( b_1^\eta + b_2^\xi = 0 \).

\[ \square \]

**Remark 4.** Assume that \( d = 1 \) and \( \psi_1(\eta) = -ib\eta \) for some \( b \in \mathbb{R} \setminus \{0\} \). We see easily that the set \( \{(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n \mid \psi_1(\eta) + \psi_2(\xi) = 0\} \) is equal to \( \{(0, 0)\} \) if, and only if, \( \psi_2^{-1}(i\mathbb{R}) = \{0\} \).

**Lemma 3.** Let \( f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R} \) be a space-time harmonic function for the Lévy process \((X_t)_{t \geq 0}\). Then, there exists a unique extension \( \tilde{f} : (\mathbb{R} \setminus \mathbb{N}) \times \mathbb{R}^n \to \mathbb{R} \), which is still space-time harmonic.

**Proof.** Assume that \( f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R} \) is such that

\[ f(s, x) = \mathbb{E} f(s + t, x + X_t) \text{ for all } s, t \geq 0, \ x \in \mathbb{R}^n. \]

We define for \( s > 0 \) and \( x \in \mathbb{R}^n \)

\[ f(-s, x) := \mathbb{E} f(0, x + X_s). \]

Let \( s > t > 0 \). By the Markov property and the stationary increments property of \( X_t \), we obtain that

\[ \mathbb{E} f(-s + t, x + X_t) = \int \mathbb{E} f(0, (x + X_t) + y)|_{y=X_{s-t}} \, d\mathbb{P} = \mathbb{E} f(0, (x + X_t) + (X_s - X_t)) = \mathbb{E} f(0, x + X_s) =: f(-s, x). \]

Now let \( t > s > 0 \). Again, by the Markov property and the stationarity of the increments, we see that

\[ \mathbb{E} f(-s + t, x + X_t) = \mathbb{E} f(t - s, x + (X_t - X_s) + X_s) = \mathbb{E} f(0, x + X_s) =: f(-s, x). \]

\[ \square \]

Recall that a Lévy process has the strong Feller property — that is, \( x \mapsto \mathbb{E} u(x + X_t) \) is a continuous function for every bounded measurable \( u \) — if, and only if, the transition probability \( \mathbb{P}(X_t \in dy) \) is absolutely continuous w.r.t. Lebesgue measure, cf. Jacob [10, Lemma 4.8.20].
Lemma 4. Let \((X_t)_{t \geq 0}\) be a Lévy process with characteristic exponent \(\psi_X\) and let \((S_t)_{t \geq 0}\) be an independent subordinator\(^*\) with Laplace transform \(\mathbb{E}e^{-xS_t} = e^{-tf_S(x)}, x \geq 0\). If both \(X\) and \(S\) are strong Feller processes, then the subordinated space-time process \((S_t, X_{S_t})\) is a strong Feller Lévy process; its characteristic exponent is given by
\[
\mathfrak{f}_S(-i\tau + \psi_X(\xi)), \quad (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n.
\] (20)

Proof. The process \(Y = (Y_t)_{t \geq 0} := ((S_t, X_{S_t}))_{t \geq 0}\) can be seen as subordination of the space-time Lévy process \(((t, X_t))_{t \geq 0}\), from which we conclude that \(Y\) is indeed a Lévy process. Furthermore, \(Y_1\) has for every \(t > 0\) a transition density \(p_{Y_t}(s, x)\), which is given by
\[
p_{Y_t}(s, x) = p_{X_s}(x)p_{S_t}(s)|_{(0, \infty)}(s).
\]
This implies that \(Y\) has the strong Feller property. That the symbol of \(Y\) is given by (20) is a direct consequence of the subordination of the process \(((t, X_t))_{t \geq 0}\):
\[
\mathbb{E}\left[e^{i\mathfrak{r}_S + i\xi X_{S_t}}\right] = \mathbb{E}\left[\left(e^{i\mathfrak{r}_r + i\xi X_r}\right)\bigg|_{r = S_t}\right] = \mathbb{E}\left[e^{-(i\tau + \psi_X(\xi))S_t}\right] = e^{-tf_S(-i\tau + \psi_X(\xi))}.
\]

Theorem 11. Let \((X_t)_{t \geq 0}\) be a strong Feller Lévy process with characteristic exponent \(\psi_X\). Then, the following assertions are equivalent:

(a) \((X_t)_{t \geq 0}\) has the (exact) coupling property,
(b) \((X_t)_{t \geq 0}\) has the space-time Liouville property,
(c) \((t, X_t)_{t \geq 0}\) has the Liouville property (as in Theorem 1),
(d) \(\{\xi \in \mathbb{R}^n \mid \psi_X(\xi) \in i\mathbb{R}\} = \{0\}\).

Proof. (a)\(\Leftrightarrow\)(b) is due to Thorisson [20, Theorem 4.5, p. 205].
(c)\(\Leftrightarrow\)(d) is due to Theorem 1, (the proof of) Lemma 4 for \(f_S(\lambda) = \lambda\) and Remark 4.
(b)\(\Rightarrow\)(c): let \(u\) be a bounded measurable function such that \(\left(\frac{d}{ds} - \psi_X(D_x)\right)u = 0\) in the sense of distributions. If \(P_t\) is the semi-group generated by \(\frac{d}{ds} - \psi(D_x)\), we know from the relation between semi-group and generator that
\[
P_tu(s, x) = u(s, x) + \int_0^t P_r\left(\frac{d}{ds} - \psi_X(D_x)\right)u(s, x)dr = u(s, x)
\]
t > 0 and all \((s, x) \in \mathbb{R} \times \mathbb{R}^n\) in the sense of distributions. Since \(u(s, x) = \mathbb{E}u(s + t, x + X_t)\) does not depend on \(t > 0\), we have
\[
u(s, x) = \int_0^\infty \mathbb{E}u(s + r, x + X_r)P(S_t \in dr) = \mathbb{E}u(s + S_t, x + X_{S_t}),
\]
where \(S = (S_t)_{t \geq 0}\) is a 1/2-stable subordinator.\(^\dagger\) By Lemma 4, \((S_t, X_{S_t})\) has again the strong Feller property, and hence, we choose \(u\) to be continuous, and the equality \(P_tu(s, x) = u(s, x)\) holds.

\(^*\)A subordinator is a one-dimensional Lévy process with increasing sample paths.
\(^\dagger\)A 1/2-stable subordinator is a subordinator with Laplace exponent \(f_S(x) = \sqrt{x}\). The transition probability of the random variable \(S_t, t > 0\), is given by the Lévy distribution \(r \mapsto t(2\pi)^{-1/2}r^{-3/2}e^{-t^2/2r}, r > 0\), see [17, Example 40.14].
pointwise for every \((s, x) \in \mathbb{R} \times \mathbb{R}^n\); in particular, \(u\) is a bounded and continuous function. Setting \(f(t, x) := u(t, x)\), it is clear that \(f\) is space-time harmonic, hence constant.

\((c) \Rightarrow (b)\): We have to show that any space-time harmonic function \(f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}\) is constant. Fix such an \(f\) and construct, as in Lemma 3, its unique extension to the negative real line; this extension is still space-time harmonic. By a similar argument as before, we see that \(f(s, x) = \mathbb{E} f(s + S_t, x + X_{S_t})\) pointwise for every \((s, x) \in \mathbb{R} \times \mathbb{R}^n\), where \(S = (S_t)_{t \geq 0}\) is again a \(1/2\)-stable subordinator. We conclude that \(f\) is continuous. The characteristic exponent of the process \((S_t, X_{S_t})\) is given by \(\sqrt{-i\tau + \psi_X(\xi)}\) by Lemma 4. As \((t, X_t)_{t \geq 0}\) has the Liouville property, we know that \(\tau - \psi_X(\xi) = 0\) if, and only if, \((\tau, \xi) = (0, 0)\), from which we conclude that \(\sqrt{-i\tau + \psi_X(\xi)} = 0\) if, and only if, \((\tau, \xi) = (0, 0)\). In view of Theorem 1, \((S_t, X_{S_t})_{t \geq 0}\) has the Liouville property; therefore, \(f\) is a.e. constant, and, as \(f\) is continuous, \(f\) is constant.

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