Lowest-order electron-electron and electron-muon scattering in a strong magnetic field

Abhishek Tiwari†1 and Binoy Krishna Patra† 2

† Department of Physics, Indian Institute of Technology Roorkee, Roorkee 247 667, India

Abstract

In this work we have investigated how the much studied scattering processes in vacuum at the lowest-order, viz. electron-muon (e-µ) scattering in both s- and t-channel, Bhabha scattering, and Møller scattering, have been modified in the presence of a strong magnetic field (|eB| >> m², m is the mass of electron or muon). For that purpose, we have first calculated the square of the matrix element by summing over the spin states, using the spinors in the presence of a strong magnetic field and then obtain the crosssection by integrating over the available phase space for the final states and averaging over the initial states. The first noticeable observation in the spin-summed of the matrix element squared is that the interference term between s- and t-channel and t- and u-channel in Bhabha and Møller scattering, respectively are missing in the presence of strong magnetic field. We have found that in the presence of strong magnetic field, the crosssection of e⁻e⁺ → µ⁻µ⁺ annihilation process in the lowest order decreases inversely proportional to the fourth power of the center-of-mass energy (√s), compared to the inversely proportional to the square of center-of-mass energy in vacuum alone. Like in vacuum, the crosssection for e-µ scattering in t channel, i.e. for e⁻µ⁻ → e⁻µ⁻ process, even in a strong magnetic field too diverges but the finite part decreases with √s much faster than in vacuum alone. Similarly the crosssections for both Bhabha and Møller scattering at the lowest-order diverges in the infrared limit. However, the finite term decreases with √s much faster than the vacuum alone. In addition there is finite negative contribution, which is independent of √s and decreases with the magnetic field.

PACS: 12.39.-x,11.10.St,12.38.Mh,12.39.Pn 12.75.N, 12.38.G

Keywords: Dirac spinor, electron-electron scattering, electron-muon scattering, strong magnetic

1abhi7phy@gmail.com
2binoyfph@iitr.ac.in
field, crosssection, Mandelstam and magnetic Mandelstam variables

1 Introduction

The relativistic heavy-ion collider (RHIC) at the Brookhaven National Laboratory, USA, with the center-of-mass energy, $\sqrt{s} = 200$ GeV per nucleon in Au - Au collisions and the large hadron collider (LHC) at the European Organization for Nuclear Research, Geneva, with $\sqrt{s} = 2.76$ TeV per nucleon in Pb-Pb collisions may have produced intensely strong magnetic field at very early stages of collisions, when the event is off-central [1, 2, 3, 4, 5]. Depending on the centralities, the strength of the magnetic field may reach between $m^2$ ($\sim 10^{18}$ Gauss) at RHIC [6] to $15 m^2$ at LHC [7]. At extreme cases it may reach values of $50 m^2$ at LHC. A very strong magnetic field ($\sim 10^{23}$ Gauss) may have existed in the early universe during the electroweak phase transition due to the gradients in Higgs field [8] or at the core of magnetars [9].

Thus we are motivated in this work to study the different processes in electron-muon and electron-electron scattering in the lowest order in the presence of strong magnetic field. However, the above processes in the lowest as well as higher order are well studied theoretically in vacuum [10, 11, 12] with the solutions of Dirac equation in vacuum, i.e. with the free Dirac spinors for positive and negative energy and their corresponding completeness relation. However, the scenario in the presence of strong magnetic field is different because the form of the Dirac spinors are going to change, where, apart from the momentum dependence, the spinors also depend on the spatial coordinates due to the gauge used to solve the Dirac equation in magnetic field. As a result, the completeness relations are also going to change. For the sake of simplicity we assume the magnetic field to be uniform and stationary. Moreover the strength of the magnetic field is strong enough so that only the lowest Landau level is sufficient for the calculation of the matrix element and the crosssection.

This paper is divided into following sections. We first nomenclature the four-momentum, Mandelstam variables, magnetic Mandelstam variables, suitable for the description in a strong magnetic field and then revisit the Dirac equation in a strong and homogeneous magnetic field in section 2.1 and 2.2, respectively. Using those notations, we calculate the spin-summed matrix element squared for the electron-muon scattering $s$-channel (annihilation process) and $t$- channel,
Bhabha scattering and Møller scattering in sections 2.3-2.5, respectively. In section 3, we first revisit for the formula for calculating the crosssection by constructing the Lorentz invariant phase space, flux factor, energy-momentum conserving Dirac-Delta function etc. in the presence of strong magnetic field and then using the the matrix elements for the above processes from the above sections 2.3-2.5, the corresponding crosssections have been evaluated in sections 3.1-3.4, respectively. Finally we conclude our results and discussion in section 4.

2  Electron-Electron and Electron-Muon Scattering in Strong Magnetic field

Our aim in this section is to calculate the square of matrix element for the electron-muon scattering in both s- and t-channel, Bhabha scattering, and Møller scattering in a strong homogeneous magnetic field. For the sake of simplicity we work in the extreme relativistic limit, where we neglect the masses of electrons as well as muons. In the presence of magnetic field the form of spinor and hence the form of electron propagator is changed but the form of photon propagator remains the same. So first we are going to revisit the Dirac equation in the strong magnetic field to obtain the form of positive energy and negative energy spinors and their completeness relations.

2.1  Notations

The dynamics of an electron in a magnetic field is factorized into transverse and longitudinal
plane with respect to the direction of magnetic field, as a result its momentum \( \mathbf{p} \) is separated into perpendicular and longitudinal components with respect to the direction of magnetic field (say, \( z \)-direction). Hence the dispersion relation is modified quantum mechanically into

\[
E_n(p_z) = \sqrt{p_z^2 + m_f^2 + 2n|eB|},
\]

(1)

In a strong magnetic field \(|eB| \gg m_f^2\), electrons prefer to lie in the lowest Landau level, hence the electron momentum becomes purely longitudinal \([13]\), i.e \( \mathbf{p}_\perp \approx 0 \). The aforesaid observation motivates to construct the following kinematic variables, viz. momentum, Mandelstam variables etc. which will be advantageous to express matrix element squared, crosssection etc. in a strong magnetic field. Thus using the following convention of the metric tensor

\[
g^{\mu\nu} = (1, -1, -1, -1),
\]

\[
g^{\mu\nu\perp} = (0, -1, -1, 0)
\]

and \( g^{\mu\nu\parallel} = (1, 0, 0, -1) \),

we will first denote the four-momentum for a generic Feynman diagram in Figure 1,

\[
p_\perp^\mu = (0, p^1, p^2, 0) = (0, p_x, p_y, 0),
\]

(3)

\[
p_\parallel^\mu = (p^0, 0, 0, p^3) = (E_0, 0, 0, p_z),
\]

(4)

\[
tilde{p}_\parallel^\mu = (\tilde{p}^0, 0, 0, \tilde{p}^3) = (p^3, 0, 0, p^0),
\]

(5)

therefore the usual Mandelstam variables take the following form:

\[
s = (p_\parallel + k_\parallel)^2 = (P_\parallel + K_\parallel)^2,
\]

(6)

\[
t = (p_\parallel - P_\parallel)^2 = (K_\parallel - k_\parallel)^2,
\]

(7)

\[
u = (p_\parallel - K_\parallel)^2 = (P_\parallel - k_\parallel)^2.
\]

(8)

We define some new variables \( s_p, s_k, s_P, s_K, t_p, t_P, t_K, t_k, u_p, u_k, u_P, u_K \), dubbed as the magnetic Mandelstam variables, which are defined as

\[
s_p = (\tilde{p}_\parallel + k_\parallel)^2, \quad s_k = (\tilde{k}_\parallel + p_\parallel)^2,
\]

(9)

\[
s_P = (\tilde{P}_\parallel + K_\parallel)^2, \quad s_K = (\tilde{K}_\parallel + P_\parallel)^2,
\]

(10)

\[
t_p = (\tilde{p}_\parallel - P_\parallel)^2, \quad t_P = (\tilde{P}_\parallel - p_\parallel)^2,
\]

(11)

\[
t_K = (\tilde{K}_\parallel - k_\parallel)^2, \quad t_k = (\tilde{k}_\parallel - K_\parallel)^2,
\]

(12)

\[
u_p = (\tilde{p}_\parallel - K_\parallel)^2, \quad u_K = (\tilde{K}_\parallel - p_\parallel)^2,
\]

(13)

\[
u_P = (\tilde{P}_\parallel - k_\parallel)^2, \quad u_k = (\tilde{k}_\parallel - P_\parallel)^2.
\]

(14)
Although the form of the magnetic Mandelstam variables (or the half-tilde Mandelstam variables, where one momentum is tilde) are same as the Mandelstam variables but they are completely different from them. In the extreme relativistic limit, they satisfy the following relations among themselves

\[
\begin{align*}
    s_p &= -s_k, & s_P &= -s_K, \\
    t_p &= -t_P, & t_K &= -t_k, \\
    u_p &= -u_K, & u_P &= -u_k.
\end{align*}
\]

Furthermore we use other notations for the full-tilde Mandelstam variables, which are defined as

\[
\begin{align*}
    \tilde{s} &= (\tilde{p}_\parallel + \tilde{k}_\parallel)^2 = (\tilde{P}_\parallel + \tilde{K}_\parallel)^2, \\
    \tilde{t} &= (\tilde{p}_\parallel - \tilde{P}_\parallel)^2 = (\tilde{K}_\parallel - \tilde{k}_\parallel)^2, \\
    \tilde{u} &= (\tilde{p}_\parallel - \tilde{K}_\parallel)^2 = (\tilde{P}_\parallel - \tilde{k}_\parallel)^2.
\end{align*}
\]

We can directly relate the full-tilde Mandelstam variables to the Mandelstam variables as

\[
s = -\tilde{s}, \quad t = -\tilde{t}, \quad u = -\tilde{u}.
\]

### 2.2 Dirac Spinors in a strong magnetic field

The methods of Ritus eigenfunction [14] along with the Schwinger Proper-time formalism [15] are commonly used to solve the Dirac equation of charged fermions in the presence of a constant magnetic field. There are different ways which have been adopted in the literature [16, 17, 18] to obtain the spinor in a magnetic field, however, we have mainly adopted to solve the Dirac equation in a constant external field from Ref. [19].

For the sake of simplicity, we assume a static and homogeneous magnetic field, which is along the \(z\)-direction, \(B = B\hat{z}\). Such a magnetic field can be obtained from a vector potential \(A^\mu = (0, 0, Bx, 0)\). The choice of vector potential is not unique as the same magnetic field can also be obtained from a symmetric potential given by \(A^\mu = (0, -\frac{By}{2}, \frac{Bx}{2}, 0)\). Thus the positive energy Dirac spinors with the gauge \(A^\mu = (0, -By, 0, 0)\) are given by the shifted coordinate, \(\xi = \sqrt{eB} \left( y - \frac{p_y}{eB} \right)\)
Similarly the negative energy Dirac spinors with $\tilde{\xi} = \sqrt{eB} \left( y + \frac{p_y}{eB} \right)$ are given by [20, 19]

\[
U_-(y, n, p_y) = N \begin{pmatrix} \frac{p_z}{E_n + m} I_{n-1}(\tilde{\xi}) \\ \sqrt{2neB} I_n(\tilde{\xi}) \\ I_{n-1}(\tilde{\xi}) \\ 0 \end{pmatrix}; \quad V_+(y, n, p_y) = N \begin{pmatrix} 0 \\ \frac{\sqrt{2neB}}{E_n + m} I_{n-1}(\tilde{\xi}) \\ -\frac{p_z}{E_n + m} I_n(\tilde{\xi}) \\ I_n(\tilde{\xi}) \end{pmatrix}
\]

where the normalization constant ($N$) is $N = \sqrt{E_n + m}$ and the symbol, $p_y$ denotes the absence of the $y$-component of momentum in the spinors. The energy eigenvalues are given by the above Landau quantization (1), where $n$ denotes the Landau levels and the energy eigenfunctions, $I_n(\xi)$ are expressed in terms of Hermite polynomials, $H_n(\xi)$

\[
I_n(\xi) = \frac{\sqrt{eB}}{n!2^n\sqrt{\pi}} e^{-\xi^2/2} H_n(\xi),
\]

with the properties: $I_{-1}(\xi) = 0$ and $I_0^2(\xi) = 1$. As mentioned earlier, in a strong magnetic field, only the lowest Landau level ($n=0$) is populated.

We can now calculate the spin sums for the particles ($P_U$) and anti-particles ($P_V$) in the presence of external magnetic field as [20, 19]

\[
P_U(y, y', n, p_y) = \sum_{s} U_s(y, n, p_y) \overline{U}_s(y', n, p_y)
\]  

\[= \frac{1}{2} \left[ \{ m(1 + \Sigma_z) + \gamma_\parallel - \gamma_\parallel \gamma_5 \} I_{n-1}(\xi) I_{n-1}(\xi') + \{ m(1 - \Sigma_z) + \gamma_\parallel - \gamma_5 \gamma_\parallel \} I_n(\xi) I_n(\xi') - \sqrt{2neB} (\gamma_1 - i\gamma_2) I_n(\xi) I_{n-1}(\xi') - \sqrt{2neB} (\gamma_1 + i\gamma_2) I_{n-1}(\xi) I_n(\xi') \right],
\]

(25)
\[ P_V(y, \tilde{y}, n, p_s) = \sum_s V_s(\tilde{y}, n, p_s) \bar{V}_s(y, n, p_s) \]

\[ = \frac{1}{2} \left\{ -m(1 + \Sigma_z) + \slashed{p} - \gamma_5 \gamma_5 \right\} I_{n-1}(\tilde{\xi}) I_{n-1}(\tilde{\xi}') + \sqrt{2} n e B (\gamma_1 + i \gamma_2) I_n(\tilde{\xi}) I_{n-1}(\tilde{\xi}') \]

where \( \Sigma_z = i \gamma^1 \gamma^2 \), \( \slashed{p} = p^0 \gamma^0 - p^3 \gamma^3 \), \( \tilde{\slashed{p}} = p^3 \gamma^0 - p^0 \gamma^3 \) and \( m \) is the mass of electron.

2.3 Matrix element for electron-muon scattering at the lowest-order

There are two different processes for electron-muon scattering, viz. \( (e^- e^+ \rightarrow \mu^- \mu^+) \) and \( (e^- \mu^- \rightarrow e^- \mu^-) \) represented by \( s \) and \( t \)-channel diagrams, respectively. We will first evaluate the matrix element (ME) for the \( s \)-channel diagram and calculate the \( |\text{ME}|^2 \) by summing over the spin states. Finally we calculate the same for the \( t \)-channel process.

**s-Channel Process:**

The electron-muon scattering in \( s \)-channel represents the \( e^- e^+ \rightarrow \mu^- \mu^+ \) process by the following Feynman diagram at the lowest-order in Figure 2, where \( U \) and \( \bar{U} \) denote the Dirac spinors for the incoming and the outgoing fermions, respectively whereas \( V \) and \( \bar{V} \) in Figure 2 represent the
spinors for the outgoing and the incoming anti-fermions, respectively. Therefore the invariant amplitude for the $e^- e^+ \rightarrow \mu^- \mu^+$ process at the lowest-order is given by the matrix element,

$$-iM_s(e^- e^+ \rightarrow \mu^- \mu^+) = [\overline{V}(y_B, k_y) i e \gamma^\mu U(y_A, p_y)] \left(-i g^{\mu \nu} \overline{q} \gamma^\nu V(Y_D, K_y)\right).$$  \hspace{1cm} (27)

We will now calculate the square of the matrix element, $|M_s|^2$ and then sum over the spin-states. It is convenient and easy to solve if we separately write the spin sums for electron and muon

$$|M_s|^2(e^- e^+ \rightarrow \mu^- \mu^+) = \frac{e^4}{q^2} L_{e}^{\mu \nu} L_{\mu \nu}^{\text{muon}},$$  \hspace{1cm} (28)

where the $L_{e}^{\mu \nu}$ and $L_{\mu \nu}^{\text{muon}}$ are given by

$$L_{e}^{\mu \nu} = \sum_{e \text{ spins}} [\overline{V}(y_B, k_y) \gamma^\mu U(y_A, p_y)] [\overline{V}(y_B, k_y) \gamma^\nu U(y_A, p_y)]^*, \hspace{1cm} (29)$$

$$L_{\mu \nu}^{\text{muon}} = \sum_{\mu \text{ spins}} [\overline{U}(Y_C, P_y) \gamma^\mu V(Y_D, K_y)] [\overline{U}(Y_C, P_y) \gamma^\nu V(Y_D, K_y)]^*, \hspace{1cm} (30)$$

respectively.

Using the properties of gamma matrices, like

$$\gamma^{\mu \dagger} = \gamma^0 \gamma^\mu \gamma^0 \text{ and } \gamma^0 \gamma^0 = I,$$

and the cyclic property of trace, we further simplify $L_{e}^{\mu \nu}$ as

$$L_{e}^{\mu \nu} = Tr[P_V(y_B, k_y) \gamma^\mu P_U(y_A, p_y) \gamma^\nu].$$  \hspace{1cm} (31)

The above spin-sums, $P_V$ and $P_U$ for the positron and the electron \[20, 19\] in strong magnetic field \(i.e.\) for the lowest Landau levels, $n = 0$ can be calculated as

$$P_V(y_B, n = 0, k_y) = \frac{1}{2} I_0^2(\xi_B) \left[-m(1 - \Sigma_z) + k_{||} - \gamma_5 \tilde{k}_{||}\right], \hspace{1cm} (32)$$

$$P_U(y_A, n = 0, p_y) = \frac{1}{2} I_0^2(\xi_A) \left[m(1 - \Sigma_z) + p_{||} - \gamma_5 \tilde{p}_{||}\right], \hspace{1cm} (33)$$

respectively.

Thus, after calculating the traces \[^3\] the tensor at the electron vertex is simplified into

$$L_{e}^{\mu \nu} = \left[p_{||}^\mu k_{||}^\nu + p_{||}^\nu k_{||}^\mu - (p_{||} \cdot k_{||}) g^{\mu \nu} + \tilde{p}_{||}^\mu \tilde{k}_{||}^\nu + \tilde{p}_{||}^\nu \tilde{k}_{||}^\mu - (\tilde{p}_{||} \cdot \tilde{k}_{||}) g^{\mu \nu}\right] - 2m^2(g^{\mu \nu} - g^{\mu \perp} g^{\nu \perp}).$$  \hspace{1cm} (35)

\[^3\] which is calculated in Appendix A
In a similar way the tensor at the muon vertex is also calculated as

\[ L_{\mu\nu}^{\text{muon}} = [K_{||\mu}P_{||\nu} + K_{||\nu}P_{||\mu} - (K_{||\cdot}P_{||\cdot})g_{\mu\nu} + \bar{K}_{||\mu}\bar{P}_{||\nu} + \bar{K}_{||\nu}\bar{P}_{||\mu} - (\bar{K}_{||\cdot}\bar{P}_{||\cdot})g_{\mu\nu}] - 2M^2(g_{\mu\nu} - g_{\mu\cdot\nu\cdot}). \]  

(36)

However in the extreme relativistic limit, where the mass terms could be neglected, the above squared matrix element (28) becomes simplified and is given by the short-hand notation

\[ |\mathcal{M}_\mu|^2 (e^-e^+ \rightarrow \mu^-\mu^+) = \frac{e^4}{q^4} (T^\mu\nu R_{\mu\nu} + \tilde{T}^\mu\nu \tilde{R}_{\mu\nu} + \bar{T}^\mu\nu \bar{R}_{\mu\nu}), \]  

(37)

where \( T^\mu\nu, R_{\mu\nu}, \tilde{T}^\mu\nu, \tilde{R}_{\mu\nu} \) are defined by

\[ T^\mu\nu = p_{||}^\mu k_{||}^\nu + p_{||}^\nu k_{||}^\mu - (p_{||}\cdot k_{||})g_{\mu\nu}, \]  

(38)

\[ \tilde{T}^\mu\nu = \bar{p}_{||}^\mu \bar{k}_{||}^\nu + \bar{p}_{||}^\nu \bar{k}_{||}^\mu - (\bar{p}_{||}\cdot \bar{k}_{||})g_{\mu\nu}, \]  

(39)

\[ R_{\mu\nu} = K_{||\mu}P_{||\nu} + K_{||\nu}P_{||\mu} - (K_{||\cdot}P_{||\cdot})g_{\mu\nu}, \]  

(40)

\[ \tilde{R}_{\mu\nu} = \bar{K}_{||\mu}\bar{P}_{||\nu} + \bar{K}_{||\nu}\bar{P}_{||\mu} - (\bar{K}_{||\cdot}\bar{P}_{||\cdot})g_{\mu\nu}. \]  

(41)

Thus the products are calculated as

\[ T^\mu\nu R_{\mu\nu} = 2\left( (p_{||}\cdot K_{||})(k_{||}\cdot P_{||}) + (p_{||}\cdot P_{||})(k_{||}\cdot K_{||}) \right), \]  

(42)

\[ \tilde{T}^\mu\nu \tilde{R}_{\mu\nu} = 2\left( (\bar{p}_{||}\cdot \bar{K}_{||})(\bar{k}_{||}\cdot \bar{P}_{||}) + (\bar{p}_{||}\cdot \bar{P}_{||})(\bar{k}_{||}\cdot \bar{K}_{||}) \right), \]  

(43)

\[ T^\mu\nu \bar{R}_{\mu\nu} = 2\left( (p_{||}\cdot \bar{K}_{||})(\bar{k}_{||}\cdot \bar{P}_{||}) + (p_{||}\cdot \bar{P}_{||})(\bar{k}_{||}\cdot \bar{K}_{||}) \right), \]  

(44)

\[ \tilde{T}^\mu\nu \bar{R}_{\mu\nu} = 2\left( (\bar{p}_{||}\cdot \bar{K}_{||})(\bar{k}_{||}\cdot \bar{P}_{||}) + (\bar{p}_{||}\cdot \bar{P}_{||})(\bar{k}_{||}\cdot \bar{K}_{||}) \right). \]  

(45)

Thus after substituting the products, the matrix element squared (37) for the electron-muon scattering for s-channel becomes

\[ |\mathcal{M}_\mu|^2 (e^-e^+ \rightarrow \mu^-\mu^+) = \frac{2e^4}{q^4} \left( (p_{||}\cdot K_{||})(k_{||}\cdot P_{||}) + (p_{||}\cdot P_{||})(k_{||}\cdot K_{||}) + (p_{||}\cdot \bar{K}_{||})(k_{||}\cdot \bar{P}_{||}) + (p_{||}\cdot \bar{P}_{||})(k_{||}\cdot \bar{K}_{||}) + (p_{||}\cdot K_{||})(\bar{k}_{||}\cdot \bar{P}_{||}) + (p_{||}\cdot \bar{P}_{||})(\bar{k}_{||}\cdot K_{||}) + (\bar{p}_{||}\cdot K_{||})(\bar{k}_{||}\cdot P_{||}) + (\bar{p}_{||}\cdot P_{||})(\bar{k}_{||}\cdot K_{||}) \right). \]  

(46)

Using the notations for the Mandelstam and magnetic Mandelstam variables mentioned in equations (6)-(8) and (9)-(14), respectively, the above matrix element squared becomes

\[ |\mathcal{M}_\mu|^2_{B\neq0}(e^-e^+ \rightarrow \mu^-\mu^+) = \frac{e^4}{2q^4} \left[ u^2 + t^2 + u_K u_P + t_P t_K + u_P u_K + t_P t_K + \bar{u}^2 + \bar{t}^2 \right] \]

\[ = \frac{e^4}{8s^2} \left[ u^2 + t^2 + u_P u_K + t_P t_K \right]. \]  

(47)
For the sake of completeness, the $e$-$\mu$ scattering in $s$-channel, i.e. $(e^- e^+ \rightarrow \mu^- \mu^+)$ process in vacuum can be calculated as [12]
\[
|\mathcal{M}_s|^2_{B=0} (e^- e^+ \rightarrow \mu^- \mu^+) = \frac{2e^4}{s^2} [u^2 + t^2].
\] (48)

**$t$-Channel Process:**

The electron-muon scattering in the $t$-channel, i.e. $e^- \mu^- \rightarrow e^- \mu^-$ process in the lowest-order is represented by the following Feynman diagram in Figure 3. We will now evaluate the matrix element for it as

\[
\mathcal{M}_t(e^- \mu^- \rightarrow e^- \mu^-) = \frac{-e^2}{q^2} [\overline{U}(Y_C, P_y) \gamma^\mu U(y_A, p_y)] [\overline{V}(y_B, k_y) \gamma_\mu V(Y_D, K_y)].
\] (49)

As we know that the $e$-$\mu$ scattering in $s$-channel, i.e. $e^- e^+ \rightarrow \mu^- \mu^+$ process is related to the $e$-$\mu$ scattering in $t$-channel, i.e. $e^- \mu^- \rightarrow e^- \mu^-$ process by the crossing symmetry. This facilitates us to obtain the spin-summed squared matrix element for the $t$-channel directly from the $e$-$\mu$ scattering in $s$-channel in following way.

As mentioned earlier, in the strong magnetic field (along the $z$-direction), the spinors do not have the spatial $y$ dependence and the perpendicular component of momentum ($p_\perp$) becomes vanishingly small. Therefore the matrix element in $s$-channel (27) can be rewritten as
\[
\mathcal{M}_s(e^- e^+ \rightarrow \mu^- \mu^+) = \frac{-e^2}{(p_\parallel + k_\parallel)^2} [\overline{V}(k_\parallel) \gamma^\mu U(p_\parallel)] [\overline{U}(P_\parallel) \gamma_\mu V(K_\parallel)].
\] (50)

Figure 3: $t$-channel
Now if we interchange the momentum \(k\) with \(-P\) due to the crossing symmetry, the above matrix element becomes

\[
M_\alpha(e^- e^+ \rightarrow \mu^- \mu^+) = \frac{-e^2}{(p - P)^2} [\overline{V}(-P)\gamma \mu U(p)] [\overline{V}(-k)\gamma \mu V(K)].
\] (51)

Using the Feynman-Stückelberg interpretation, where by reversing the direction of momentum (say, \(p\)) in the spinors, \(\overline{V}(-p)\) becomes \(\overline{U}(p)\) and \(\overline{U}(-p)\) becomes \(\overline{V}(p)\), the above \(s\)-channel matrix element gives the desired \(t\)-channel matrix element

\[
M_t(e^- \mu^- \rightarrow e^- \mu^-) = \frac{-e^2}{(p - P)^2} [\overline{U}(P)\gamma \mu U(p)] [\overline{V}(k)\gamma \mu V(K)].
\] (52)

Therefore the spin-summed matrix element squared for the \(t\)-channel diagram (Figure 3) is easily derived as

\[
|\overline{M}_t|_{B \neq 0}^2(e^- \mu^- \rightarrow e^- \mu^-) = \frac{2e^4}{q^4} \left[ (p \cdot k)(P \cdot K) + (p \cdot K)(P \cdot k) + (p \cdot \tilde{k})(P \cdot \tilde{K}) + (p \cdot \tilde{K})(P \cdot k) + (\tilde{p} \cdot k)(\tilde{P} \cdot K) + (\tilde{p} \cdot K)(\tilde{P} \cdot k) \right],
\] (53)

which, in turn, will be expressed in terms of the Mandelstam variables from the \(s\)-channel expression (47) by interchanging \(s\) with \(t\), \(u_k\) with \(u_P\), \(s_k\) with \(t_P\), and \(s_P\) with \(t_k\) and is given by

\[
|\overline{M}_t|_{B \neq 0}^2(e^- \mu^- \rightarrow e^- \mu^-) = \frac{e^4}{4q^4} \left[ s^2 + u^2 + sKsK + uKuk + sPsP + uPup + s^2 + u^2 \right] = \frac{e^4}{t^2} \left[ s^2 + u^2 + sPsP + uPup \right].
\] (54)

However, the above squared-matrix element in vacuum is [12]

\[
|\overline{M}_t|_{B = 0}^2(e^- \mu^- \rightarrow e^- \mu^-) = \frac{2e^4}{t^2} \left[ s^2 + u^2 \right].
\] (55)

### 2.4 Matrix Element for Bhabha Scattering: \(e^- e^+ \rightarrow e^- e^+\)

There are possible \(s\)- and \(t\)-channel diagrams, which contribute to the Bhabha scattering, in Figure(s) 4 and 5, respectively in the lowest order. Therefore the matrix element for the Bhabha scattering in the lowest order is

\[
M(e^- e^+ \rightarrow e^- e^+) = M_\alpha(e^- e^+ \rightarrow e^- e^+) + M_t(e^- e^+ \rightarrow e^- e^+),
\] (56)
where the $s$- and $t$-channel contributions to the matrix element in a strong magnetic field are given by

\[ \mathcal{M}_s(e^- e^+ \rightarrow e^- e^+) = -\frac{e^2}{q_1^2} \left[ \mathcal{V}(y_B, k_y) \gamma^\mu U(y_A, p_y) \right] \left[ \mathcal{U}(Y_C, P_y) \gamma^\nu V(Y_D, K_y) \right], \]

\[ \mathcal{M}_t(e^- e^+ \rightarrow e^- e^+) = -\frac{e^2}{q_2^2} \left[ \mathcal{U}(Y_C, P_y) \gamma^\mu U(y_A, p_y) \right] \left[ \mathcal{V}(y_B, k_y) \gamma^\nu V(Y_D, K_y) \right], \]

respectively. Hence the total matrix element squared becomes

\[ |\mathcal{M}|^2 = |\mathcal{M}_s|^2 + |\mathcal{M}_t|^2 + \mathcal{M}_s^* \mathcal{M}_t. \]  

(59)

Similar to the electron-muon scattering in the $s$-channel \((46)\), we can now directly write the spin-summed squared matrix element of the Bhabha scattering for the $s$-channel in extreme relativistic limit

\[ |\mathcal{M}_s|^2(e^- e^+ \rightarrow e^- e^+) = \frac{2e^4}{q_1^4} \left[ (p_\parallel \cdot K_\parallel)(k_\parallel \cdot P_\parallel) + (p_\parallel \cdot P_\parallel)(k_\parallel \cdot K_\parallel) + (p_\parallel \cdot \tilde{K}_\parallel)(k_\parallel \cdot \tilde{P}_\parallel) \right. \]

\[ + (p_\parallel \cdot \tilde{P}_\parallel)(k_\parallel \cdot \tilde{K}_\parallel) + (\tilde{p}_\parallel \cdot K_\parallel)(k_\parallel \cdot P_\parallel) + (\tilde{p}_\parallel \cdot P_\parallel)(k_\parallel \cdot K_\parallel) \]

\[ + (\tilde{p}_\parallel \cdot \tilde{K}_\parallel)(k_\parallel \cdot \tilde{P}_\parallel) + (\tilde{p}_\parallel \cdot \tilde{P}_\parallel)(k_\parallel \cdot \tilde{K}_\parallel) \Bigg], \]

(60)

which in terms of Mandelstam and Magnetic Mandelstam variables can be written as

\[ |\mathcal{M}_s|^2(e^- e^+ \rightarrow e^- e^+) = \frac{e^4}{s^2} \left[ u^2 + t^2 + u_p u_k + t_p t_k \right]. \]  

(61)
The contribution for the Bhabha scattering in $t$-channel can be obtained in extreme relativistic limit, in analogy with $e$-$\mu$ scattering for $t$-channel in (53),

$$\overline{|\mathcal{M}_t|^2(e^-e^+ \rightarrow e^-e^+)} = \frac{2e^4}{q_1^2q_2^2} \left[ (p_{\parallel} \cdot k_{\parallel})(P_{\parallel} \cdot K_{\parallel}) + (p_{\parallel} \cdot K_{\parallel})(P_{\parallel} \cdot k_{\parallel}) + (p_{\parallel} \cdot \tilde{K}_{\parallel})(P_{\parallel} \cdot \tilde{k}_{\parallel}) + (\tilde{p}_{\parallel} \cdot \tilde{k}_{\parallel})(\tilde{P}_{\parallel} \cdot \tilde{K}_{\parallel}) + (\tilde{p}_{\parallel} \cdot K_{\parallel})(\tilde{P}_{\parallel} \cdot \tilde{k}_{\parallel}) \right],$$

which can be expressed in terms of Mandelstam variables as

$$\overline{|\mathcal{M}_t|^2(e^-e^+ \rightarrow e^-e^+)} = \frac{e^4}{t^2} \left[ s^2 + u^2 + s_p s_P + u_p u_P \right].$$

The interference term is given by

$$\mathcal{M}_s \mathcal{M}_t^* = \frac{e^4}{q_1^2q_2^2} \left[ \overline{V(y_B,k_y)\gamma^\mu U(y_A,p_y)} \right] \left[ \overline{U(Y_C,P_y)\gamma_\mu V(Y_D,K_y)} \right] \times \left[ \overline{V(Y_D,K_y)\gamma_\nu V(y_B,k_y)} \right] \left[ \overline{U(y_A,p_y)\gamma^\nu U(Y_C,P_y)} \right].$$

(64)

However, in the strong magnetic field limit, all the spatial ($y$) dependence of the Dirac spinors are gone and also the $p_\perp$ is zero, so the above interference term is rewritten as

$$\mathcal{M}_s \mathcal{M}_t^* = \frac{e^4}{q_1^2q_2^2} \left[ \overline{V(k_{\parallel})\gamma^\mu U(p_{\parallel})} \right] \left[ \overline{U(P_{\parallel})\gamma_\mu V(K_{\parallel})} \right] \times \left[ \overline{V(K_{\parallel})\gamma_\nu V(k_{\parallel})} \right] \left[ \overline{U(p_{\parallel})\gamma^\nu U(P_{\parallel})} \right],$$

(65)

which becomes, after summing over the spin states

$$\overline{\mathcal{M}_s \mathcal{M}_t^*} = \frac{e^4}{q_1^2q_2^2} Tr \left[ P_{\gamma^\nu}(k_{\parallel})\gamma^\mu P_{\nu}(p_{\parallel})\gamma^\nu P_{\mu}(P_{\parallel})\gamma_\mu P_{\nu}(K_{\parallel})\gamma_\nu \right].$$

(66)

Using the property of $\gamma$-matrices, $\gamma^\mu \bar{\psi} \gamma^\nu \gamma^\rho \psi = -2\bar{\psi} \gamma^\rho \psi$, the above interference term can be further simplified in terms of Mandelstam and the magnetic Mandelstam variables

$$\overline{\mathcal{M}_s \mathcal{M}_t^*} = 0.$$

(67)

Finally using the Mandelstam and magnetic Mandelstam variables, the matrix element squared for the Bhabha scattering is obtained by the $s$- (61) and $t$-channel (63) contributions only in a strong magnetic field

$$\overline{|\mathcal{M}|^2}_{B \neq 0}(e^-e^+ \rightarrow e^-e^+) = |\mathcal{M}_s|^2 + |\mathcal{M}_t|^2$$

$$= \frac{e^4}{s^2} \left[ u^2 + t^2 + u_p u_k + t_p t_k \right] + \frac{e^4}{t^2} \left[ s^2 + u^2 + s_p s_P + u_p u_P \right].$$

(68)

\(^4\text{Calculated in Appendix B}\)
The above crucial observation in a strong magnetic field can be understood as follows: In vacuum, Bhabha scattering in $s$ channel gives the forward peak whereas in $t$-channel it gives the backward peak. In the presence of strong magnetic field, the dynamics of the electron is restricted to one dimension so the interference of two peaks in vacuum may not be feasible in the presence of strong magnetic field.

However, in vacuum, the interference term does not vanish, thus the above matrix element squared for Bhabha scattering in the absence of strong magnetic field is given by \[|\mathcal{M}|^2_{B=0}(e^-e^+ \rightarrow e^-e^+) = \frac{2e^4}{s^2} \left[ u^2 + t^2 \right] + \frac{4e^4u^2}{ts} + \frac{2e^4}{t^2} \left[ s^2 + u^2 \right]. \] (69)

### 2.5 Matrix Element for Möller Scattering

The Feynman diagrams for the Möller scattering ($e^-e^- \rightarrow e^-e^-$) are shown below in Figure (s) 6 and 7, which are related to the Bhabha scattering ($e^-e^+ \rightarrow e^-e^+$) by the crossing symmetry, namely by simply crossing the incoming positron to outgoing positron

\[
e^- (p) + e^+ (k) \rightarrow e^- (P) + e^+ (K), \tag{70}
\]
\[
e^- (p) + e^- (-K) \rightarrow e^- (P) + e^- (-k). \tag{71}
\]

As we know already, in strong magnetic field, the interchange of momenta due to the crossing symmetry is effectively translated in terms of their longitudinal component only. Thus the momentum exchange between $k_\parallel$ and $K_\parallel$ in (70)-(71) helps to obtain the matrix element for the
Møller scattering in u and t-channel by identifying the same for the Bhabha scattering in s- and t-channels \((57)-(58)\), respectively.

The above interchange of momenta can be equivalently expressed in terms of the Mandelstam and magnetic Mandelstam variables as

\[
s(= (p_\parallel + k_\parallel)^2) \rightarrow u(= (p_\parallel - K_\parallel)^2)
\]

\[
u(= (p_\parallel - K_\parallel)^2) \rightarrow s(= (p_\parallel + k_\parallel)^2)
\]

\[
t_k(= (\tilde{k}_\parallel - K_\parallel)^2) \rightarrow t_K(= (k_\parallel - \tilde{K}_\parallel)^2)
\]

\[
s_k(= (p_\parallel + \tilde{k}_\parallel)^2) \rightarrow u_K(= (p_\parallel - \tilde{K}_\parallel)^2).
\]

Therefore the above crossing symmetry \((72)-(75)\) helps us to obtain the squared matrix element for the Møller scattering from the Bhabha scattering \((68)\), namely

\[
|\mathcal{M}|^2_{B \neq 0}(e^- e^- \to e^- e^-) = |\mathcal{M}_u|^2_{B \neq 0}(e^- e^- \to e^- e^-) + |\mathcal{M}_t|^2_{B \neq 0}(e^- e^- \to e^- e^-),
\]

where the u and t-channel matrix element squared are given by

\[
|\mathcal{M}_u|^2_{B \neq 0}(e^- e^- \to e^- e^-) = \frac{e^4}{u^2} \left[ s^2 + t^2 + s_p s_K + t_p t_K \right],
\]

\[
|\mathcal{M}_t|^2_{B \neq 0}(e^- e^- \to e^- e^-) = \frac{e^4}{t^2} \left[ u^2 + s^2 + u_p u_P + s_p s_P \right],
\]

respectively. However, the above matrix element in vacuum can also be calculated as \([12]\)

\[
|\mathcal{M}|^2_{B = 0}(e^- e^- \to e^- e^-) = \frac{2e^4}{u^2} \left[ s^2 + t^2 \right] + \frac{4e^4 s^2}{tu} + \frac{2e^4}{t^2} \left[ s^2 + u^2 \right].
\]

\section{3 Crosssection}

Let us illustrate the usual procedure to compute the crosssection from the transition amplitude for the above mentioned processes in the presence of strong magnetic field. For that we choose a generic \(e^- \mu^- \to e^- \mu^-\) process in Figure 8, for which we will calculate the transition amplitude.

The solution of the Dirac equation for the \(e^-\) in an external magnetic field in the \(z\)-direction is given by \([20, 19]\)

\[
\Psi(X, p) = U(y, n, p_y) e^{-i p_y \cdot X},
\]

where \(U\) is the \(e^-\) spinor in the presence of an external magnetic field, \(n\) labels the Landau levels, \(X^\mu = (t, x, y, z)\) and \(p^\mu = (E, p_x, 0, p_z)\), where the \(y\)-component of momentum is missing. The
transition matrix element for the above process is thus given by

\[ T_{fi} = \int (\overline{U_D} i e\gamma^\mu U_B) \left(\frac{-i}{q^2}\right) (\overline{U_C} i e\gamma^\mu U_A) e^{i(p^D - p^C - p^A - p^B) \cdot X} d^4 X \quad (80) \]

\[ \equiv \int -i \mathcal{M} e^{i(p^D - p^C - p^A - p^B) \cdot X} d^4 X. \quad (81) \]

Therefore the matrix element is given by

\[ -i \mathcal{M} = (\overline{U_D} i e\gamma^\mu U_B) \left(\frac{-i}{q^2}\right) (\overline{U_C} i e\gamma^\mu U_A), \quad (82) \]

where the main problem arises that \( \mathcal{M} \) now becomes a function of \( y \) in the presence of an external magnetic field, hence we cannot take it outside the integration. We circumvent this issue by taking the strong magnetic field limit, where all the \( y \) dependence in \( \mathcal{M} \) is gone. The transition matrix element, in the strong magnetic field becomes

\[ T_{fi} = -i \mathcal{M} (2\pi)^4 \delta^4(p^D - p^C - p^A - p^B), \quad (83) \]

and the transition rate per unit volume is given by

\[ W_{fi} = \frac{|T_{fi}|^2}{TV} = (2\pi)^4 |\mathcal{M}|^2 \delta^4(p^D - p^C - p^A - p^B). \quad (84) \]

Therefore using the definition of crosssection

\[ \text{crosssection} = \frac{W_{fi}}{\text{(initial flux)} \times \text{(number of final states)}}. \]
the crosssection is given by
\[ d\sigma = \frac{(2\pi)^4 |\mathcal{M}|^2 \delta^4(p^D_y + p^C_y - p^A_y - p^B_y)}{F} d^4p^C d^4p^D. \]  
(85)

All the particles A, B, C and D are real particles so they must satisfy the on-shell mass condition, which in the strong magnetic field becomes
\[ p^2_\parallel = m^2 \]  
[13] with \( p_\parallel^\mu = (E, 0, 0, p_z). \) This gives the crosssection
\[ d\sigma = \frac{(2\pi)^4 |\mathcal{M}|^2 \delta^4(p^D_y + p^C_y - p^A_y - p^B_y)}{F} \frac{d^3p^C}{(2\pi)^3 2E_C} \frac{d^3p^D}{(2\pi)^3 2E_D}. \]  
(86)

To calculate the unpolarized crosssection, we need to average over the quantum states of incoming particles and sum over the final states, therefore we replace the above matrix element squared
\[ |\mathcal{M}|^2 \rightarrow \frac{1}{(2s_A + 1)(2s_B + 1)} \sum_{\text{all states}} |\mathcal{M}|^2, \]  
(87)
where \( s_A \) and \( s_B \) are the spin of incoming particles. We have already summed over the particle states in the matrix element squared, denoted as \( |\mathcal{M}|^2 \), so we just need to divide \( |\mathcal{M}|^2 \) by the degeneracy factor, 4, to get the desired crosssection.

Now we have all the ingredients to compute the crosssection for the aforesaid processes with the corresponding matrix element squared.

### 3.1 Electron-Muon scattering in s channel: Annihilation Process \((e^- e^+ \rightarrow \mu^- \mu^+)\)

With the help of (86) the crosssection for the electron-muon scattering in s-channel \((e^- e^+ \rightarrow \mu^- \mu^+)\) in Figure 2 with the matrix element (47) is

\[ d\sigma_{B \neq 0}^{s}(e^- e^+ \rightarrow \mu^- \mu^+) = \frac{(2\pi)^4}{F} |\mathcal{M}_s|^2 \delta^4(e^- e^+ \rightarrow \mu^- \mu^+) \delta^4(K_y + P_y - k_y) \frac{d^3P}{(2\pi)^3 2E_P} \frac{d^3K}{(2\pi)^3 2E_K}. \]

\[ = \frac{|\mathcal{M}_s|^2}{(2\pi)^4} \delta^4(e^- e^+ \rightarrow \mu^- \mu^+) \delta^3(E_P + E_K - E_p - E_k) \delta^4(K_y + P_y - k_y) \frac{d^3P}{E_P} \frac{d^3K}{E_K}. \]

In the center-of-mass (cm) frame, \( \mathbf{p} + \mathbf{k} = 0 \), this also implies, \( \mathbf{p}_y + \mathbf{k}_y = 0 \). Thus then
crosssection becomes

\[
d\sigma^\mu_{B\neq 0}(e^- e^+ \rightarrow \mu^- \mu^+) = \frac{\vert M_s \vert^2}{(2\pi)^2 4F} \delta(E_P + E_K - E_P - E_k) \delta^3(K_y + P_y) \frac{d^3 P}{E_P} \frac{d^3 K}{E_K}.
\]

In the presence of magnetic field, the momentum integration gets factorized into parallel and perpendicular components with respect to the direction of magnetic field (z-direction), where the integral over \(d^2 P_{\perp}\) in strong magnetic field limit becomes \((\int_0^{\mid eB \mid} d^2 P_{\perp} = \pi \mid eB \mid)\). Thus the total crosssection becomes by integrating over the parallel component of momentum\( (P_z)\)

\[
\sigma^\mu_{B\neq 0}(e^- e^+ \rightarrow \mu^- \mu^+) = \int_{-\infty}^{\infty} \frac{\pi \mid eB \mid \vert M_s \vert^2}{(2\pi)^2 4F} \delta(E_P + E_K - E_P - E_k) \frac{dP_z}{E_K E_P}.
\]

In a strong magnetic field limit \((eB >> m^2, n = 0)\), the perpendicular component \((\perp)\) of the momentum is zero \([13]\) so the particles can only move in \(z\) direction. They can either move in \(+ve\) \(z\) direction or in \(-ve\) \(z\) direction. Accordingly the four momentum dot product can be written as

\[
p \cdot k = \begin{cases} E_P E_k - \mid p \mid \mid k \mid, & \text{if } p \text{ and } k \text{ are in the same direction} \\ E_P E_k + \mid p \mid \mid k \mid, & \text{if } p \text{ and } k \text{ are in the opposite direction}. \end{cases}
\]

In extreme relativistic limit we can neglect the dot product where the momenta are in the same direction compared to them in opposite direction. The diagram for the process in the center-of-mass frame (for \(\theta = 0\) degree, where \(\theta\) is the angle between \(p\) and \(P\)) is drawn in the Figure 9. For our case \(\theta\) (scattering angle) can have two values 0 and 180 degree, for \(\theta = 0\) the \(t\) variables in eq (47) are negligible as compared to all \(u\) variables whereas for \(\theta = 180\) the \(u\) variables are negligible as compared to all \(t\) variables. Thus the squared matrix element eq (47) at high energy for \(\theta = 0\) degree gets simplified

\[
\vert M_s \vert^2_{B\neq 0}(e^- e^+ \rightarrow \mu^- \mu^+) = \frac{e^4}{4s^2} \left[ u^2 + u_p u_k \right],
\]

and for \(\theta = 180\) degree, it becomes

\[
\vert M_s \vert^2_{B\neq 0}(e^- e^+ \rightarrow \mu^- \mu^+) = \frac{e^4}{4s^2} \left[ t^2 + t_p t_k \right].
\]

Let us denote \(\mid p \mid = \mid k \mid = p_t\) and \(\mid P \mid = \mid K \mid = p_f\) so \(E_p\) and \(E_k\) become the same (say, \(E_i\)) whereas \(E_P\) and \(E_K\) become equal (say, \(E_f\)). As a consequence the square of the matrix element for \(\theta = 0\) degree comes out to be same as for the \(\theta = 180\) degree. Hence

\[
\vert M_s \vert^2(e^- e^+ \rightarrow \mu^- \mu^+) = \frac{2 e^4}{s^2} E_f^2 \left[ E_i + p_t \right]^2,
\]
and the flux factor for collinear collision is

\[
F = |v_p - v_k| 2E_p E_k
= 4p_i \sqrt{s},
\]  

where \( \sqrt{s} = E_p + E_k \).

Using the expression of the flux factor \( F \) and the squared matrix element \( |\mathcal{M}_s|^2 \), the crosssection \( (90) \) can be rewritten as

\[
\sigma_{s \neq 0}^s(e^-e^+ \rightarrow \mu^-\mu^+) = \frac{e^4|eB|}{16\pi} \int_{-\infty}^{\infty} \frac{\pi e^4|eB| [E_i + p_i]^2}{s^2 p_i \sqrt{s}} \delta(W - \sqrt{s}) \frac{E_f^2}{p_f W} dW,
\]  

where \( W = E_p + E_K \).

With the further approximation: \( p_f \approx E_f \) and \( p_i \approx E_i \), the crosssection for the e-\( \mu \) scattering in \( s \)-channel\(^5\), \( i.e. \) for the annihilation process (\( e^-e^+ \rightarrow \mu^-\mu^+ \)) in the lowest order takes the final form as a function of the center-of-mass energy

\[
\sigma_{B \neq 0}^s(e^-e^+ \rightarrow \mu^-\mu^+) = \frac{\pi \alpha^2|eB|}{s^2}.
\]  

The approximations, \( p_f \approx E_f \) and \( p_i \approx E_i \) only hold good at extremely high energies, where the masses (order of MeV) can be neglected.

\(^5\)Detailed calculation of crosssection for the process \( e^-e^+ \rightarrow \mu^-\mu^+ \) is given in Appendix C.
For the sake of comparison, the crosssection for the same in vacuum in the lowest order is \[ \sigma_{B=0}(e^- e^+ \rightarrow \mu^- \mu^+) = \frac{4\pi\alpha^2}{3s}. \] (98)

One thus immediately infer that in the presence of strong magnetic field, the crosssection is inversely proportional to the fourth power of the center-of-mass energy while in the absence of magnetic field, \(\sigma_{B=0}\) is inversely proportional to the square of the center-of-mass energy.

To see the effect of strong magnetic field on the annihilation process, we have plotted a variation of the crosssection with the center-of-mass energy in the presence and absence of magnetic field in Fig-10, where \(M\) is the mass of muon. From the above graph, it is evident that the strong magnetic field suppresses the scattering, which can be understood by the fact that the availability of phase space in the presence of magnetic field is reduced drastically. Moreover we have also found that for a fixed center-of-mass energy, as we increase the magnetic field, \(\sigma\) increases linearly.

### 3.2 Electron-Muon scattering in \(t\) channel: \(e^- \mu^- \rightarrow e^- \mu^-\) process

The crosssection for the electron-muon scattering in \(t\) channel, \(i.e.\) for the \(e^- \mu^- \rightarrow e^- \mu^-\) process diverges at \(t = 0\), since the matrix element in \(t\)-channel diagram has a pole at \(t = 0\). Let us now examine below how the pole \(t = 0\) translates into the momentum variable in the final state. The
Mandelstam variables, \( t \) and \( u \) for the Figure 3 are defined as
\[
\begin{align*}
t^2 &= 4|p|^2|P|^2(\cos \theta - 1)^2 = 4|p|^2|P|^2 \left( \frac{P_z}{|P|} - 1 \right)^2 = 4p_i^2 p_f^2 \left( \frac{P_z}{p_f} - 1 \right)^2, \\
u^2 &= 4|p|^2|K|^2(\cos \theta + 1)^2 = 4|p|^2|K|^2 \left( \frac{P_z}{|K|} + 1 \right)^2 = 4p_i^2 p_f^2 \left( \frac{P_z}{p_f} + 1 \right)^2,
\end{align*}
\]
where we denote the momenta \(|p|\) and \(|k|\) by \(p_i\) and the momenta \(|P|, |K|\) by \(p_f\).

As mentioned earlier, in the strong magnetic field limit, electrons occupy only the lowest Landau levels \((n = 0)\), so the lower limit of the transverse momentum becomes vanishing small, i.e. \(P_\perp \sim 0\). Therefore, the momentum, \(p_f = |P| = \sqrt{P_\perp^2 + P_z^2}\) simply becomes \(P_z\), hence the pole \(t = 0\) appears. Therefore, using the matrix element for \(e-\mu\) scattering for \(t\)-channel (54) as well a lower cut-off, \(\epsilon_B\) to the transverse momentum, \(P_\perp\) the crosssection (86) looks as
\[
\sigma_{B\neq 0}^i(e^-\mu^- \rightarrow e^-\mu^-) = \int_{\epsilon_B}^{\sqrt{\epsilon_B}} dP_\perp \int_{-\infty}^{\infty} dP_z e^4 4t^2 \left[ s^2 + u^2 + s_k s_K + u_k u_f \right] \frac{\delta(W - \sqrt{s})}{(2\pi)^4 F} \frac{P_z}{E_f},
\]
with the notations: \(W = E_P + E_K, \sqrt{s} = E_p + E_k, E_p = E_k = E_\ell \) and \(E_K = E_P = E_f\). Thus after integrating the momentum integrations over \(dP_\perp\) and \(dP_z\), the crosssection for the \(e^-\mu^- \rightarrow e^-\mu^-\) process in the lowest order is \(6\)
\[
\sigma_{B\neq 0}^i(e^-\mu^- \rightarrow e^-\mu^-) = \frac{\pi \alpha^2}{2s^2} |eB| - \frac{2\pi \alpha^2}{|eB|} + 2\pi \alpha^2 \lim_{\epsilon_B \rightarrow 0} \left[ \frac{1}{\epsilon_B} \right],
\]
which implies a divergence in \(\epsilon_B \rightarrow 0\) limit.

For the sake of comparison, we calculate the same crosssection in the extreme relativistic limit in the absence of strong magnetic field using matrix element from (55) for the \(e-\mu\) scattering in the \(t\)-channel
\[
\sigma_{B=0}^i(e^-\mu^- \rightarrow e^-\mu^-) = -\frac{4\pi \alpha^2}{s} + \frac{4\pi \alpha^2}{s} \lim_{\epsilon_V \rightarrow 0} \left[ \frac{2}{\epsilon_V} + \ln \left( \frac{\epsilon_V}{2} \right) \right].
\]
This also shows a divergence in the lower limit of \(\epsilon_V\), i.e. \(\epsilon_V \rightarrow 0\), which, in turn, arises due to the lower limit of scattering angle (between \(p\) and \(P\)) as
\[
\epsilon_V = 1 - \cos \theta_0.
\]
However, the above divergences in the presence and the absence of strong magnetic field are related to each other and we can geometrically derive\(7\) an equation using the fact that \(P_\perp = P_z \tan \theta\), which is
\[
\frac{1}{\epsilon_V} = \frac{3}{2} + \frac{s}{2\epsilon_B}.
\]
\(6\)Calculated in Appendix D
\(7\)Calculated at the end of Appendix D
Hence, the above relation helps us to compare the crosssection in vacuum with the crosssection in the presence of magnetic field

\[
\sigma^t_{B=0}(e^-\mu^- \rightarrow e^-\mu^-) = \frac{8\pi\alpha^2}{s} + \frac{4\pi\alpha^2}{s} \lim_{\epsilon_V \rightarrow 0} \left[ \ln \left( \frac{\epsilon_V}{2} \right) \right] + 4\pi\alpha^2 \lim_{\epsilon_B \rightarrow 0} \left[ \frac{1}{\epsilon_B^2} \right]. \quad (106)
\]

One thus finds that the logarithmic divergence in vacuum disappears due to the presence of external magnetic field and apart from that in magnetic field there is an another finite term which is independent of \(s\) but decreases with the increasing magnetic field.

### 3.3 Bhabha Scattering: \(e^-e^+ \rightarrow e^-e^+\)

The crosssection for the Bhabha Scattering, \(i.e.\) for the \(e^-e^+ \rightarrow e^-e^+\) processes in the lowest-order can be obtained from the definition (86), with the matrix element (68). In the presence of strong magnetic field, \(\sigma(e^-e^+ \rightarrow e^-e^+)\) can be decomposed into \(s\)- and \(t\)-channel contribution due to the vanishing interference term

\[
\sigma_{B\neq 0}(e^-e^+ \rightarrow e^-e^+) = \sigma^s_{B\neq 0}(e^-e^+ \rightarrow e^-e^+) + \sigma^t_{B\neq 0}(e^-e^+ \rightarrow e^-e^+), \quad (107)
\]

where the \(s\)- and \(t\)-channel contribution are given by

\[
\sigma^s_{B\neq 0}(e^-e^+ \rightarrow e^-e^+) = \frac{e^4}{16\pi s^2} |eB|, \quad (108)
\]

\[
\sigma^t_{B\neq 0}(e^-e^+ \rightarrow e^-e^+) = \frac{e^4}{32\pi s^2} |eB| + \frac{e^4}{8\pi} \lim_{\epsilon_B \rightarrow 0} \left[ \frac{1}{\epsilon_B^2} - \frac{1}{|eB|} \right], \quad (109)
\]

respectively. Therefore the total crosssection (107) for the Bhabha scattering yields

\[
\sigma_{B\neq 0}(e^-e^+ \rightarrow e^-e^+) = \frac{3\pi\alpha^2}{2s^2} |eB| - \frac{2\pi\alpha^2}{|eB|} + 2\pi\alpha^2 \lim_{\epsilon_B \rightarrow 0} \left[ \frac{1}{\epsilon_B^2} \right]. \quad (110)
\]

To isolate the effect of strong magnetic field, we have also calculated the same at the lowest-order in vacuum only using the matrix element from (69), where the interference term is nonzero unlike the former case and is given by

\[
\sigma_{B=0}(e^-e^+ \rightarrow e^-e^+) = \sigma^s_{B=0} + \sigma^\text{interference}_{B=0} + \sigma^t_{B=0} = \left[ \frac{4\pi\alpha^2}{3s} \right] - \left[ \frac{4\pi\alpha^2}{s} \lim_{\epsilon_V \rightarrow 0} \ln \frac{\epsilon_V}{2} \right] + \left[ \frac{4\pi\alpha^2}{s} \lim_{\epsilon_V \rightarrow 0} \left( \frac{2}{\epsilon_V} - 1 + \ln \frac{\epsilon_V}{2} \right) \right] = -\frac{8\pi\alpha^2}{3s} + \frac{8\pi\alpha^2}{s} \lim_{\epsilon_V \rightarrow 0} \left[ \frac{1}{\epsilon_V} \right]. \quad (111)
\]
Again using the relation \((105)\), we can write the above crosssection in terms of the parameter \((\epsilon_B)\) in the presence of strong magnetic field, which causes the divergence

\[
\sigma_{B=0}(e^- e^+ \rightarrow e^- e^+) = \frac{4\pi\alpha^2}{3s} + 4\pi\alpha^2 \lim_{\epsilon_B \to 0} \left[ \frac{1}{\epsilon_B^2} \right].
\]

If we leave the divergent part and compare only with the finite part then we find that apart from a constant magnetic field dependent term, the crosssection in vacuum at the lowest order decreases with the center-of-mass energy \((\sqrt{s})\) slower than the same in the presence of strong magnetic field. However, the presence of strong magnetic field does not alter the degree of divergence.

### 3.4 Møller Scattering: \(e^- e^- \rightarrow e^- e^-\)

The crosssection for the Møller scattering at the lowest order can be obtained from its matrix element \((78)\), which is factorizable into \(u\) and \(t\)-channel contributions due to the vanishing interference term. The crosssection for the \(t\)-channel matrix element in extreme relativistic limit is same as the crosssection for electron-muon scattering \((102)\) in \(t\)-channel \((e^- \mu^- \rightarrow e^- \mu^-)\) process. So we are left with the crosssection due to the \(u\)-channel and can be calculated from its matrix element \((77)\) in a strong magnetic field

\[
\sigma_{B \neq 0}^u = \int_{-\infty}^{\infty} dP_z \int_0^{\sqrt{eB}} dP_{\perp} |M_u|^2 (e^- e^- \rightarrow e^- e^-) \frac{\delta(W - \sqrt{s}) P_{\perp}}{(2\pi)^4 F E_f^2},
\]

with the notations: \(W = E_P + E_K\), \(\sqrt{s} = E_p + E_k\), \(E_p = E_k = E_i\) and \(E_K = E_P = E_f\). In the above integral, the matrix element has a pole at \(u = 0\), which could be translated in terms of momentum variable as follows: In the presence of strong magnetic field, the lower limit of the transverse momentum \((P_{\perp})\) becomes vanishingly small. As a result, the momentum, \(P = (\sqrt{P_{\perp}^2 + P_z^2})\) comes out to be \(\pm P_z\), which give rise \(t = 0\) and \(u = 0\) poles, defined in \((99)\) and \((100)\), respectively and can be circumvented by taking a lower cut-off to the lower limit of transverse momentum \((P_{\perp})\) integration.

As an artifact of the above observation, the \(u\)-channel matrix element squared in Møller scattering \((77)\) in the momentum interval, \(P_z \in (-\infty, 0]\) is mapped into \(t\)-channel matrix element \((78)\) in the momentum interval \(P_z \in [0, \infty)\) and vice versa

\[
\frac{|M_u|^2}{P_z \in (-\infty, 0]} = \frac{|M_t|^2}{P_z \in [0, \infty)}, \quad (114)
\]

\[
\frac{|M_u|^2}{P_z \in [0, \infty)} = \frac{|M_t|^2}{P_z \in (-\infty, 0]}, \quad (115)
\]
As a consequence the crosssections from both channels come out to be same except for the fact that in $t$-channel, the crosssection peaks at the forward angle ($\theta = 0$) while the $u$-channel peaks at $\theta = 180$. Thus the crosssection for the ($e^-e^- \rightarrow e^-e^-$) scattering in the lowest-order is obtained by doubling the crosssection in $t$ channel (102)

$$\sigma_{B\neq 0}(e^-e^- \rightarrow e^-e^-) = \frac{\pi \alpha^2}{s^2} |eB| - \frac{4\pi \alpha^2}{|eB|} + 4\pi \alpha^2 \lim_{\epsilon_B \rightarrow 0} \left[ \frac{1}{\epsilon_B^2} \right]. \quad (116)$$

The crosssection for the Møller scattering in vacuum in the lowest order using the matrix element from (79) can be easily calculated as

$$\sigma_{B=0}(e^-e^- \rightarrow e^-e^-) = \sigma_{B=0}^u + \sigma_{B=0}^{\text{interference}} + \sigma_{B=0}^t$$

$$= \left[ \frac{4\pi \alpha^2}{s} \lim_{\epsilon_V \rightarrow 0} \left( \frac{2}{\epsilon_V} - 1 + \ln \frac{\epsilon_V}{2} \right) \right] - \left[ \frac{8\pi \alpha^2}{s} \lim_{\epsilon_V \rightarrow 0} \ln \frac{\epsilon_V}{2} \right] + \left[ \frac{4\pi \alpha^2}{s} \lim_{\epsilon_V \rightarrow 0} \left( \frac{2}{\epsilon_V} - 1 + \ln \frac{\epsilon_V}{2} \right) \right] \quad (117)$$

$$= -\frac{8\pi \alpha^2}{s} + \frac{16\pi \alpha^2}{s} \lim_{\epsilon_V \rightarrow 0} \left[ \frac{1}{\epsilon_V} \right], \quad (118)$$

which can be compared with the result in a strong magnetic field by replacing $\epsilon_V$ in terms of $\epsilon_B$ through the relation (105)

$$\sigma_{B=0}(e^-e^- \rightarrow e^-e^-) = \frac{16\pi \alpha^2}{s} + 8\pi \alpha^2 \lim_{\epsilon_B \rightarrow 0} \left[ \frac{1}{\epsilon_B^2} \right]. \quad (119)$$

Like other processes the $s$-dependence in vacuum is being modified due to the presence of strong magnetic field whereas the diverging component in vacuum ($\frac{1}{\epsilon_B^2}$) becomes halved due to the magnetic field.

4 Results and Discussions

We have revisited the lepton-lepton scattering at the lowest order in an additional presence of strong magnetic field. The recent observations at the ultra relativistic heavy ion collisions at Relativistic Heavy-ion Collider and Large Hadron Collider, where a very strong magnetic field up to $10^{18} - 10^{20}$ Gauss is expected to be produced for the noncentral events, motivates us to revisit the above processes in a strong magnetic field. In particular, we have calculated the crosssection for electron-muon ($e-\mu$) scattering in both $s$- and $t$-channel, Bhabha scattering, and Møller scattering in the presence of a strong magnetic field ($|eB| >> m^2$, $m$ is the mass of electron or muon).
that purpose, using the Dirac spinor in strong magnetic field, we have first calculated the square of the matrix element and then summed over the final spin states. We have found that unlike in vacuum, the interference term in Bhabha scattering between $s$- and $t$-channel and in Møller scattering between $t$- and $u$-channel contribution in the presence of strong magnetic field vanishes. Secondly we have illustrated the usual procedure to compute the crosssection from the transition amplitude for the above mentioned processes in the presence of strong magnetic field. We have noticed that the $e$-$\mu$ scattering in $s$-channel, i.e. $e^-e^+ \rightarrow \mu^-\mu^+$ process gets suppressed due to the presence of strong magnetic field. More precisely the crosssection at a fixed magnetic field is inversely proportional to the fourth power of the center-of-mass energy compared to the vacuum alone, where $\sigma$ is inversely proportional to the square of the center-of-mass energy. However, for a fixed center-of-mass energy, the crosssection increases with the magnetic field.

On the other hand the $e$-$\mu$ scattering in lowest order for $t$-channel, i.e. $e^-\mu^- \rightarrow e^-\mu^-$ process also diverges like in vacuum but in the presence of strong magnetic field, the logarithmic divergence in vacuum disappears and the infrared divergence remains. As far as finite terms are concerned, the crosssection decreases faster like the $s$-channel. However, there is a negative term, which is independent of the center-of-mass energy and decreases with the magnetic field. The above observation is also found in Bhabha and Møller scattering in the strong magnetic field. The divergence in Bhabha scattering in vacuum arises at the lower limit of the incident angle whereas the same in the presence of strong magnetic field arises due to the lower limit of transverse momentum. However the above divergences in the presence and in the absence of strong magnetic field inter-related to each other and we have derived the relation among these two divergences geometrically so that we can compare the crosssections in both cases at the same footing.

However, the above mentioned processes have also been studied extensively up to higher-order in vacuum (i.e. in the absence of magnetic field), as a result the divergences appeared have been controlled by the higher-order corrections. For an example, the collinear divergence in the Bhabha scattering had been cured by adding the corrections to the tree-level vacuum result by the radiative corrections in Ref. [21, and references therein], the $O(\alpha^3)$ corrections [22], the $O(\alpha^4)$ corrections with the full mass dependence [10] etc., where the contributions from the higher-order diagrams have been calculated and the IR and UV divergences were regularized by the dimensional regularization scheme.
5 Acknowledgement

We thank to Mr. Mujeeb Hasan, Mr. Bhaswar Chatterjee, Ms. Shubhalaxmi Rath and Mr. Jitendra Pal for their help from time to time.

References

[1] I. A. Shovkovy, Lect. Notes Phys. 871, 13 (2013).
[2] M. D’Elia, Lect. Notes Phys. 871, 181 (2013).
[3] K. Fukushima, Lect. Notes Phys. 871, 241 (2013).
[4] N. Muller, J. A. Bonnet, and C. S. Fisher, Phys. Rev. D 89, 094023 (2014).
[5] V. A. Miransky and I. A. Shovkovy, Phys. Rep. 576, 1-209 (2015).
[6] D. Kharzeev, L. McLerran, and H. Warringa, Nucl. Phys. A 803, 227 (2008).
[7] V. Skokov, A. Illarionov, and V. Toneev, Int. J. Mod. Phys. A 24, 5925 (2009).
[8] T. Vachaspati, Phys. Lett. B 265, 258 (1991).
[9] R. C. Duncan and C. Thompson, Astrophys. J. Lett. 392, L9 (1992).
[10] R. Bonciani, A. Ferroglia Phys. Rev. D 72, 056004 (2005).
[11] A. Ilyichev, V. Zykunov Phys. Rev. D 72, 033018 (2005).
[12] F. Halzen, A. D. Martin, Quarks and Leptons: An Introductory Course in Modern Particle Physics.
[13] V. P. Gusynin and Andrei V. Smilga, Phys. Lett. B 450, 267 (1999).
[14] V. I. Ritus, Annals Phys. 69, 555 (1972).
[15] J. S. Schwinger, Phys. Rev. 82, 664 (1951).
[16] P. Mészáros, Radiation from Magnetized Neutron Stars, Theoretical Astrophysics (University of Chicago, Chicago, 1992).
A Matrix Element for electron-muon scattering: $s$-channel process

This appendix contains detailed calculation of the squared matrix element for the process $e^-e^+ \rightarrow \mu^-\mu^+$. We start with the matrix element for the Feynman diagram in Figure 2:

$$\mathcal{M}_0(e^-e^+ \rightarrow \mu^-\mu^+) = \frac{-e^2}{q^2} \left[ \mathcal{V}(y_B,k_y)\gamma^\mu U(y_A,p_y) \right] \left[ \mathcal{U}(Y_D,K_y)\gamma_\mu \mathcal{V}(Y_C,P_y) \right].$$

In order to find the cross section we have to take the square of the modulus of $\mathcal{M}$ and then sum over the spin states by factorizing into tensors at the electron (e) and muon (muon) vertex

$$|\mathcal{M}_0|^2(e^-e^+ \rightarrow \mu^-\mu^+) = \frac{e^4}{q^4} L^{\mu\nu}_e L^{\mu\nu}_{\mu\nu}, \quad (120)$$

where $L^{\mu\nu}_e$ and $L^{\mu\nu}_{\mu\nu}$ are given by

$$L^{\mu\nu}_e = \sum_{e \text{ spins}} \left[ \mathcal{V}(y_B,k_y)\gamma^\mu U(y_A,p_y) \right] \left[ \mathcal{V}(y_B,k_y)\gamma^\nu U(y_A,p_y) \right]^*,$$

$$L^{\mu\nu}_{\mu\nu} = \sum_{\mu \text{ spins}} \left[ \mathcal{U}(Y_C,P_y)\gamma_\mu \mathcal{V}(Y_D,K_y) \right] \left[ \mathcal{U}(Y_C,P_y)\gamma_\nu \mathcal{V}(Y_D,K_y) \right]^*.$$. 
To calculate the complex conjugate of $\nabla(y_B, k)\gamma^\nu U(y_A, p)$, we start with the fact that it is a $1 \times 1$ matrix therefore its complex conjugate is equal to its hermitian conjugate,

$$[[\nabla(y_B, k)\gamma^\nu U(y_A, p)]^\dagger = [\nabla(y_B, k)\gamma^\nu U(y_A, p)]^\dagger$$

where we have used some properties of gamma matrices like $\gamma^\mu = \gamma^0 \gamma^\mu \gamma^0$ and $\gamma^0 \gamma^0 = I$. Applying these simplification the $L_{e}^{\mu\nu}$ becomes

$$L_{e}^{\mu\nu} = \sum_{s, s'} [\nabla(y_B, k)\gamma^\mu U(y_A, p)\overline{U}(y_A, p)\gamma^\nu V(y_B, k)]$$

where $s$ and $s'$ denote the spin states of electron and positron respectively.

To simplify $L_{e}^{\mu\nu}$, we begin with explicitly writing the above equation in terms of individual matrix elements, which tells us that the above equation is a $1 \times 1$ matrix. Therefore we can calculate $L_{e}^{\mu\nu}$ by taking the trace of above equation. We then take the trace above equation and use the cyclic property of trace to form the completeness condition. Thus the above equation becomes

$$L_{e}^{\mu\nu} = Tr\left[\sum_{s, s'} V^{s}(y_B, k)\overline{V}^{s'}(y_B, k)\gamma^\mu \sum_{s} U^{s}(y_A, p)\overline{U}^{s}(y_A, p)\gamma^\nu \right]$$

where $P_V$ and $P_U$ are the spin sums of the positron and the electron, respectively [20, 19]. In strong magnetic field these spin sums are given by

$$P_V(y_B, n = 0, k) = \sum_{s'} V^{s'}(y_B, k)\overline{V}^{s'}(y_B, k) = \frac{1}{2} I_{0}^{2}(\xi_B) [ - m(1 - \Sigma_z) + \gamma^i K^i - \gamma^5 K^5 ]$$

$$P_U(y_A, n = 0, p) = \sum_{s} U^{s}(y_A, p)\overline{U}^{s}(y_A, p) = \frac{1}{2} I_{0}^{2}(\xi_A) [ m(1 - \Sigma_z) + \gamma^i \tilde{p}^i - \gamma^5 \tilde{p}^5 ]$$

where $p^i = p^0 \gamma^0 - p^3 \gamma^3, \tilde{p}^i = p^3 \gamma^0 - p^0 \gamma^3$ and $m$ is the mass of electron with $I_{0}^{2}(\xi) = 1$.

To further simplify $L_{e}^{\mu\nu}$ in a convenient way, let us denote $A = (1 - \Sigma_z), K = K^i - \gamma^5 K^5$ and
\[ P = \slashed{p} - \gamma_5 \slashed{\bar{p}}, \text{ and in this way } L^\mu_{\nu} \text{ can be rewritten as} \]

\[
L^\mu_{\nu} = \frac{1}{4} Tr \left[ (-mA + K)\gamma^\mu (mA + P)\gamma^\nu \right] \
= \frac{1}{4} Tr \left[ -m^2 (A\gamma^\mu A\gamma^\nu) + m(K\gamma^\mu A\gamma^\nu - A\gamma^\mu P\gamma^\nu) + (K\gamma^\mu P\gamma^\nu) \right].
\]

The coefficient of \( m \) in the above equation contains odd number of gamma matrices, hence their trace vanishes and the above equation becomes

\[
L^\mu_{\nu} = \frac{1}{4} Tr \left[ -m^2 (A\gamma^\mu A\gamma^\nu) + (K\gamma^\mu P\gamma^\nu) \right]. \quad (122)
\]

Above equation contains two terms. We first simplify the first term \( A\gamma^\mu A\gamma^\nu \) and calculate its trace. Using the value of \( A = 1 - \Sigma_z \), we simplify the first term in the above equation as

\[
A\gamma^\mu A\gamma^\nu = (1 - \Sigma_z)\gamma^\mu (1 - \Sigma_z)\gamma^\nu
= (1 - i\gamma^1\gamma^2)\gamma^\mu (1 - i\gamma^1\gamma^2)\gamma^\nu \quad (\Sigma_z = i\gamma^1\gamma^2)
= \gamma^\mu\gamma^\nu - i\gamma^\mu\gamma^1\gamma^2\gamma^\nu - i\gamma^1\gamma^2\gamma^\mu\gamma^\nu - \gamma^1\gamma^2\gamma^\mu\gamma^1\gamma^2\gamma^\nu,
\]

where the fourth term \( (\gamma^1\gamma^2\gamma^\mu\gamma^1\gamma^2\gamma^\nu) \) in the above equation can be further simplified, using the property \( \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^\mu\nu \), which is

\[
\gamma^1\gamma^2\gamma^\mu\gamma^1\gamma^2\gamma^\nu = 2\gamma^\mu\gamma^1\gamma^2\gamma^\nu - \gamma^\mu\gamma^\nu;
\]

where \( \gamma^\mu\nu = (0, \gamma^1, \gamma^1, 0) \).

Thus \( A\gamma^\mu A\gamma^\nu \) can be rewritten as

\[
A\gamma^\mu A\gamma^\nu = 2\gamma^\mu\gamma^\nu - i\gamma^\mu\gamma^1\gamma^2\gamma^\nu - i\gamma^1\gamma^2\gamma^\mu\gamma^\nu - 2\gamma^\mu\gamma^\nu.
\]

Using the trace properties like \( Tr(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\delta) = 4[g^\mu\nu g^\lambda\delta - g^\mu\lambda g^\nu\delta + g^\mu\delta g^\nu\lambda] \) and \( Tr(\gamma^\mu\gamma^\nu) = 4g^\mu\nu \), the trace of \( A\gamma^\mu A\gamma^\nu \) becomes

\[
Tr[A\gamma^\mu A\gamma^\nu] = 8(g^\mu\nu - g^\mu\nu), \quad (123)
\]

where \( g^\mu\nu \) is \( g^\mu\nu\nu = (0, -1, -1, 0) \). Therefore the above trace can be written as

\[
Tr[A\gamma^\mu A\gamma^\nu] = 8(g^\mu\nu - g^\mu\nu). \quad (124)
\]

The second term in \( (122) \) can be simplified as

\[
K\gamma^\mu P\gamma^\nu = \left( \slashed{k} - \gamma_5 \slashed{\bar{k}} \right)\gamma^\mu \left( \slashed{\bar{p}} - \gamma_5 \slashed{\bar{p}} \right)\gamma^\nu
= \left( \slashed{k}\gamma^\mu - \gamma_5 \slashed{\bar{k}}\gamma^\mu \right) \left( \slashed{\bar{p}}\gamma^\nu - \gamma_5 \slashed{\bar{p}}\gamma^\nu \right)
= \slashed{k}\gamma^\mu \slashed{\bar{p}}\gamma^\nu - \gamma_5 \slashed{\bar{k}}\gamma^\mu \slashed{\bar{p}}\gamma^\nu - \gamma_5 \slashed{\bar{k}}\gamma^\mu \slashed{\bar{p}}\gamma^\nu - \gamma_5 \slashed{\bar{k}}\gamma^\mu \slashed{\bar{p}}\gamma^\nu
= \slashed{k}\gamma^\mu \slashed{\bar{p}}\gamma^\nu - \gamma_5 \slashed{\bar{k}}\gamma^\mu \slashed{\bar{p}}\gamma^\nu - \gamma_5 \slashed{\bar{k}}\gamma^\mu \slashed{\bar{p}}\gamma^\nu + \slashed{k}\gamma^\mu \slashed{\bar{p}}\gamma^\nu \quad (125).
\]
The above equation have four terms. The trace of the first and the last terms is given by the equations

\[
Tr(\kappa^\mu \bar{\rho}^\nu) = 4[p_\mu^\nu k_\mu^\nu + p_\mu^\nu \bar{k}_\mu^\nu - (p_\mu \cdot \bar{k}_\mu) g^\mu^\nu], \tag{126}
\]

\[
Tr(\bar{k}^\mu \gamma^\nu \bar{\rho}^\nu) = 4[\bar{p}_\mu^\nu \kappa_\mu^\nu + \bar{p}_\mu^\nu \bar{k}_\mu^\nu - (\bar{p}_\mu \cdot \kappa_\mu) g^\mu^\nu]. \tag{127}
\]

For the trace of second and third term in (125), which are the standard traces, but to compare these with each other, we first expand the terms and then calculate the traces. The second term in (125) can be simplified as

\[
\gamma_5 k^\mu \bar{\rho}^\nu = \gamma_5 [k^0 \gamma^0 - k^3 \gamma^3] \gamma^\mu (p^0 \gamma^3 - p^3 \gamma^0) \gamma^\nu
\]

\[
= \gamma_5 [(k^0 \gamma^0 \gamma^\mu - k^3 \gamma^3 \gamma^\mu)(p^0 \gamma^3 \gamma^\nu - p^3 \gamma^0 \gamma^\nu)]
\]

\[
= \gamma_5 [k^0 p^0 (\gamma^0 \gamma^\mu \gamma^3 \gamma^\nu) - k^0 p^3 (\gamma^0 \gamma^\mu \gamma^0 \gamma^\nu) - k^3 p^0 (\gamma^3 \gamma^\mu \gamma^3 \gamma^\nu)
\]

\[
+ k^3 p^3 (\gamma^3 \gamma^\mu \gamma^0 \gamma^\nu)]
\]

\[
= [k^0 p^0 (\gamma_5 \gamma^0 \mu \gamma^3 \gamma^\nu) - k^0 p^3 (\gamma_5 \gamma^0 \mu \gamma^0 \gamma^\nu) - k^3 p^0 (\gamma_5 \gamma^3 \mu \gamma^3 \gamma^\nu)
\]

\[
+ k^3 p^3 (\gamma_5 \gamma^3 \mu \gamma^0 \gamma^\nu)]. \tag{128}
\]

Now we calculate the trace of above equation, which is

\[
Tr(\gamma_5 k^\mu \bar{\rho}^\nu) = 4i[k^0 p^0 \epsilon^{03\nu} - k^0 p^3 \epsilon^{03\nu} - k^3 p^0 \epsilon^{30\nu} + k^3 p^3 \epsilon^{30\nu}]
\]

\[
= 4i[k^0 p^0 \epsilon^{03\nu} + k^3 p^3 \epsilon^{30\nu}]
\]

\[
= 4i(k^0 p^0 - k^3 p^3) \epsilon^{03\nu},
\]

where we use the trace property

\[
Tr(\gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\lambda) = 4i \epsilon^{\mu \nu \sigma \lambda},
\]

where \( \epsilon^{\mu \nu \sigma \lambda} \) is defined as

\[
\begin{cases}
  1, & \text{for } \mu, \nu, \sigma, \lambda \text{ an even permutation of 0,1,2,3}, \\
  1, & \text{for } \mu, \nu, \sigma, \lambda \text{ an odd permutation of 0,1,2,3}, \\
  0, & \text{if two indices are same}.
\end{cases}
\]

Similar to the second term, the third term in (125) can be simplified as

\[
\gamma_5 \bar{k}^\mu \gamma^\nu \bar{\rho}^\nu = \left[k^0 p^0 (\gamma_5 \gamma^3 \gamma^\mu \gamma^3 \gamma^\nu) - k^0 p^3 (\gamma_5 \gamma^3 \gamma^\mu \gamma^0 \gamma^\nu) - k^3 p^0 (\gamma_5 \gamma^0 \gamma^\mu \gamma^3 \gamma^\nu)
\]

\[
+ k^3 p^3 (\gamma_5 \gamma^0 \gamma^\mu \gamma^0 \gamma^\nu) \right].
\]

Thus the trace of above equation becomes

\[
Tr(\gamma_5 \bar{k}^\mu \gamma^\nu) = 4i[k^0 p^0 \epsilon^{30\nu} - k^0 p^3 \epsilon^{30\nu} - k^3 p^0 \epsilon^{03\nu} + k^3 p^3 \epsilon^{03\nu}]
\]

\[
= 4i[k^0 p^0 \epsilon^{30\nu} + k^3 p^3 \epsilon^{03\nu}]
\]

\[
= -4i(k^0 p^0 - k^3 p^3) \epsilon^{03\nu}.
\]

30
We can see that the trace of second and fourth term in Eq.(125) cancel each other. Now we substitute the trace value of first and fourth term in Eq.(125), which gives the trace of (125) as

$$
T_r(K \gamma^\mu P \gamma^\nu) = 4 \left[ p_\parallel^\mu k_\parallel^\nu + p_\parallel^\nu k_\parallel^\mu - (p_\parallel \cdot k_\parallel) g^\mu\nu + \tilde{p}_\parallel^\mu \tilde{k}_\parallel^\nu + \tilde{p}_\parallel^\nu \tilde{k}_\parallel^\mu - (\tilde{p}_\parallel \cdot \tilde{k}_\parallel) g^\mu\nu \right].
$$

(129)

To get the final form of $L_\epsilon^{\mu\nu}$, we substitute the value from (124) and (129) in (122). Thus $L_\epsilon^{\mu\nu}$ becomes,

$$
L_\epsilon^{\mu\nu} = \left[ p_\parallel^\mu k_\parallel^\nu + p_\parallel^\nu k_\parallel^\mu - (p_\parallel \cdot k_\parallel) g^\mu\nu + \tilde{p}_\parallel^\mu \tilde{k}_\parallel^\nu + \tilde{p}_\parallel^\nu \tilde{k}_\parallel^\mu - (\tilde{p}_\parallel \cdot \tilde{k}_\parallel) g^\mu\nu \right] - 2m^2(g^{\mu\nu} - g^{\mu_1\nu_1}).
$$

(130)

In a similar way the muonic vertex part can also be calculated as

$$
L^{\alpha\beta}_{\text{muon}} = \left[ K_{\parallel \mu} P_{\parallel \nu} + K_{\parallel \nu} P_{\parallel \mu} - (K_{\parallel} \cdot P_{\parallel}) g_{\mu\nu} + \tilde{K}_{\parallel \mu} \tilde{P}_{\parallel \nu} + \tilde{K}_{\parallel \nu} \tilde{P}_{\parallel \mu} - (\tilde{K}_{\parallel} \cdot \tilde{P}_{\parallel}) g_{\mu\nu} \right] - 2M^2(g_{\mu\nu} - g_{\mu_1\nu_1}).
$$

(131)

Let us denote $L^{\mu\nu}_\epsilon = (T^{\mu\nu} + \tilde{T}^{\mu\nu})$ and $L^{\alpha\beta}_{\text{muon}} = (R_{\mu\nu} + \tilde{R}_{\mu\nu})$, where

$$
T^{\mu\nu} = p_\parallel^{\mu} k_\parallel^{\nu} + p_\parallel^{\nu} k_\parallel^{\mu} - (p_\parallel \cdot k_\parallel) g^{\mu\nu},
$$

(132)

$$
\tilde{T}^{\mu\nu} = \tilde{p}_\parallel^{\mu} \tilde{k}_\parallel^{\nu} + \tilde{p}_\parallel^{\nu} \tilde{k}_\parallel^{\mu} - (\tilde{p}_\parallel \cdot \tilde{k}_\parallel) g^{\mu\nu},
$$

(133)

$$
R_{\mu\nu} = K_{\parallel \mu} P_{\parallel \nu} + K_{\parallel \nu} P_{\parallel \mu} - (K_{\parallel} \cdot P_{\parallel}) g_{\mu\nu},
$$

(134)

$$
\tilde{R}_{\mu\nu} = \tilde{K}_{\parallel \mu} \tilde{P}_{\parallel \nu} + \tilde{K}_{\parallel \nu} \tilde{P}_{\parallel \mu} - (\tilde{K}_{\parallel} \cdot \tilde{P}_{\parallel}) g_{\mu\nu}.
$$

(135)

Therefore the spin summed squared matrix element (120) becomes,

$$
|\mathcal{M}_s|^2 (e^- e^+ \rightarrow \mu^- \mu^+) = \frac{e^4}{4q^4}(T^{\mu\nu}R_{\mu\nu} + T^{\mu\nu}\tilde{R}_{\mu\nu} + \tilde{T}^{\mu\nu}R_{\mu\nu} + \tilde{T}^{\mu\nu}\tilde{R}_{\mu\nu}),
$$

(136)

where we neglect the mass of electron as well as the mass of muon because of the reason that we are working in the extreme relativistic limit. Each term in the above equation can be calculated as

$$
T^{\mu\nu}R_{\mu\nu} = \left[ p_\parallel^{\mu} k_\parallel^{\nu} + p_\parallel^{\nu} k_\parallel^{\mu} - (p_\parallel \cdot k_\parallel) g^{\mu\nu} \right] \left[ K_{\parallel \mu} P_{\parallel \nu} + K_{\parallel \nu} P_{\parallel \mu} - (K_{\parallel} \cdot P_{\parallel}) g_{\mu\nu} \right]
$$

$$
= (p_\parallel \cdot K_{\parallel})(k_\parallel \cdot P_{\parallel}) + (p_\parallel \cdot P_{\parallel})(k_\parallel \cdot K_{\parallel}) - (p_\parallel \cdot k_\parallel)(K_{\parallel} \cdot P_{\parallel})
$$

$$
+(p_\parallel \cdot K_{\parallel})(k_\parallel \cdot K_{\parallel}) + (p_\parallel \cdot P_{\parallel})(k_\parallel \cdot K_{\parallel}) - (p_\parallel \cdot k_\parallel)(K_{\parallel} \cdot P_{\parallel})
$$

$$
-(p_\parallel \cdot K_{\parallel})(K_{\parallel} \cdot P_{\parallel}) - (p_\parallel \cdot k_\parallel)(K_{\parallel} \cdot P_{\parallel}) + 4(p_\parallel \cdot k_\parallel)(K_{\parallel} \cdot P_{\parallel})
$$

$$
= 2\left[ (p_\parallel \cdot P_{\parallel})(k_\parallel \cdot K_{\parallel}) + (p_\parallel \cdot K_{\parallel})(k_\parallel \cdot P_{\parallel}) \right].
$$

Similarly the other terms can also be calculated as

$$
\tilde{T}^{\mu\nu}R_{\mu\nu} = 2\left[ (\tilde{p}_\parallel \cdot K_{\parallel})(\tilde{k}_\parallel \cdot P_{\parallel}) + (\tilde{p}_\parallel \cdot P_{\parallel})(\tilde{k}_\parallel \cdot K_{\parallel}) \right],
$$

(137)

$$
T^{\mu\nu}\tilde{R}_{\mu\nu} = 2\left[ (p_\parallel \cdot K_{\parallel})(\tilde{k}_\parallel \cdot \tilde{P}_{\parallel}) + (p_\parallel \cdot \tilde{P}_{\parallel})(k_\parallel \cdot K_{\parallel}) \right],
$$

(138)

$$
\tilde{T}^{\mu\nu}\tilde{R}_{\mu\nu} = 2\left[ (\tilde{p}_\parallel \cdot \tilde{K}_{\parallel})(\tilde{k}_\parallel \cdot \tilde{P}_{\parallel}) + (\tilde{p}_\parallel \cdot \tilde{P}_{\parallel})(\tilde{k}_\parallel \cdot \tilde{K}_{\parallel}) \right].
$$

(139)
Thus the spin summed matrix element becomes,
\[
|M_{s}^2|_{B \neq 0}(e^- e^+ \rightarrow \mu^- \mu^+) = \frac{2e^4}{q^4} \left[ (p_\parallel \cdot K_\parallel)(k_\parallel \cdot P_\parallel) + (p_\parallel \cdot P_\parallel)(k_\parallel \cdot K_\parallel) + (p_\parallel \cdot \tilde{K}_\parallel)(k_\parallel \cdot \tilde{P}_\parallel) + (\tilde{p}_\parallel \cdot K_\parallel)(\tilde{k}_\parallel \cdot P_\parallel) + (\tilde{p}_\parallel \cdot P_\parallel)(\tilde{k}_\parallel \cdot K_\parallel) + (\tilde{p}_\parallel \cdot \tilde{K}_\parallel)(\tilde{k}_\parallel \cdot \tilde{P}_\parallel) + (\tilde{p}_\parallel \cdot \tilde{P}_\parallel)(\tilde{k}_\parallel \cdot \tilde{K}_\parallel) \right].
\] (140)

This can be rewritten using Mandelstam and Magnetic Mandelstam variables
\[
|M_{s}^2|_{B \neq 0}(e^- e^+ \rightarrow \mu^- \mu^+) = \frac{e^4}{2q^2} \left[ u^2 + t^2 + u_K u_p + t_p t_K + u_p u_k + t_p t_k + \bar{u}^2 + \bar{t}^2 \right]
\] (141)

**B Cross term in Bhabha Scattering**

We provide here the calculation of interference term of Bhabha scattering. We start from the matrix element for Bhabha scattering for s- and t-channel (figure 4 and 5)
\[
\mathcal{M}_s(e^- e^+ \rightarrow e^- e^+) = \frac{-e^2}{q_1} \left[ \bar{V}(y_B, k, k') \gamma^\mu U(y_A, p) \right] \left[ \bar{U}(Y_C, P_\parallel) \gamma_\mu V(Y_D, K_\parallel) \right],
\] (142)
\[
\mathcal{M}_t(e^- e^+ \rightarrow e^- e^+) = \frac{-e^2}{q_2} \left[ \bar{U}(Y_C, P_\parallel) \gamma^\mu U(y_A, p) \right] \left[ \bar{V}(y_B, k, k') \gamma_\mu V(Y_D, K_\parallel) \right].
\] (143)

Since the quantities in the square brackets are 1 × 1 matrices, we can rearrange them to form the completeness condition. Taking the sum over the spin states, the interference term, in the strong magnetic field limit, becomes
\[
|\mathcal{M}_s \mathcal{M}_s^\dagger| = \frac{e^4}{q_1^2 q_2^2} \sum_{all \text{ states}} \left[ \bar{V}(k_\parallel, k) \gamma^\mu U(p_\parallel) \right] \left[ \bar{U}(p_\parallel) \gamma^\nu U(P_\parallel) \right] \left[ \bar{U}(P_\parallel) \gamma_\mu V(K) \right] \left[ \bar{V}(K) \gamma_\nu V(k) \right]
\] = \frac{e^4}{q_1^2 q_2^2} Tr \left[ P_V(k_\parallel) \gamma^\mu P_U(p_\parallel) \gamma^\nu P_U(P_\parallel) \gamma_\mu P_V(K) \gamma_\nu \right]
\] = \frac{e^4}{q_1^2 q_2^2} Tr \left[ (k_\parallel - \gamma_5 k_\parallel') \gamma^\mu (p_\parallel - \gamma_5 p_\parallel) \gamma^\nu (P_\parallel - \gamma_5 \tilde{P}_\parallel) \gamma_\mu (K_\parallel - \gamma_5 \tilde{K}_\parallel') \gamma_\nu \right]
\] = \frac{e^4}{q_1^2 q_2^2} Tr \left[ (k_\parallel - \gamma_5 k_\parallel') \gamma^\mu (p_\parallel \gamma^\nu P_\parallel - \gamma_5 \tilde{P}_\parallel) \gamma_\mu (K_\parallel - \gamma_5 \tilde{K}_\parallel) \gamma_\nu \right]
We further simplify the above equation by using the property $\gamma^\mu \phi \bar{b} \gamma_\mu = -2 \phi \bar{b} \phi$

\[
\mathcal{M}_t \mathcal{M}_t = -\frac{2e^4}{q_1 q_2^2} Tr \left[ (k_\| - \gamma_5 \tilde{k}_\|) (p_\| \gamma^\nu p_\| + \gamma_5 \tilde{P}_\| \gamma^\nu \tilde{p}_\| + \gamma_5 \tilde{P}_\| \gamma^\nu \tilde{p}_\|) (k_\| - \gamma_5 \tilde{k}_\|) \right]
\]
\[
= -\frac{2e^4}{q_1 q_2^2} Tr \left[ (k_\| - \gamma_5 \tilde{k}_\|) \left( \tilde{P}_\| \gamma^\nu \tilde{p}_\| k_\| \gamma_\nu + \gamma_5 \tilde{P}_\| \gamma^\nu \tilde{p}_\| k_\| \gamma_\nu - \gamma_5 \tilde{P}_\| \gamma^\nu \tilde{p}_\| k_\| \gamma_\nu + \tilde{P}_\| \gamma^\nu \tilde{p}_\| k_\| \gamma_\nu - \gamma_5 \tilde{P}_\| \gamma^\nu \tilde{p}_\| k_\| \gamma_\nu \right) \right]
\]
\[
= -\frac{2e^4}{q_1 q_2^2} Tr \left[ (k_\| - \gamma_5 \tilde{k}_\|) \left( \tilde{P}_\| \gamma^\nu \tilde{p}_\| k_\| \gamma_\nu + \gamma_5 \tilde{P}_\| \gamma^\nu \tilde{p}_\| k_\| \gamma_\nu + \gamma_5 \tilde{P}_\| \gamma^\nu \tilde{p}_\| k_\| \gamma_\nu + \tilde{P}_\| \gamma^\nu \tilde{p}_\| k_\| \gamma_\nu + \gamma_5 \tilde{P}_\| \gamma^\nu \tilde{p}_\| k_\| \gamma_\nu \right) \right].
\]

Using the property $\gamma^\mu \phi \bar{b} \gamma_\mu = 4a \cdot b$, the above equation can be further simplified

\[
\mathcal{M}_t \mathcal{M}_t = -\frac{8e^4}{q_1 q_2^2} Tr \left[ (k_\| - \gamma_5 \tilde{k}_\|) \left\{ \tilde{P}_\| (p_\| \cdot K_\|) + \gamma_5 \tilde{P}_\| (p_\| \cdot \tilde{K}_\|) + \gamma_5 \tilde{P}_\| (p_\| \cdot K_\|) + \tilde{P}_\| (p_\| \cdot \tilde{K}_\|) \right\} \right]
\]
\[
= -\frac{8e^4}{q_1 q_2^2} Tr \left[ k_\| \tilde{P}_\| (p_\| \cdot K_\|) - \gamma_5 k_\| \tilde{P}_\| (p_\| \cdot \tilde{K}_\|) - \gamma_5 k_\| \tilde{P}_\| (p_\| \cdot K_\|) + k_\| \tilde{P}_\| (p_\| \cdot \tilde{K}_\|) \right.
\]
\[
- \gamma_5 k_\| \tilde{P}_\| (p_\| \cdot K_\|) + k_\| \tilde{P}_\| (p_\| \cdot K_\|) + k_\| \tilde{P}_\| (p_\| \cdot K_\|) - \gamma_5 k_\| \tilde{P}_\| (p_\| \cdot \tilde{K}_\|) \]
\[
- \gamma_5 k_\| \tilde{P}_\| (p_\| \cdot \tilde{K}_\|) + k_\| \tilde{P}_\| (p_\| \cdot \tilde{K}_\|) + k_\| \tilde{P}_\| (p_\| \cdot K_\|) - \gamma_5 k_\| \tilde{P}_\| (p_\| \cdot \tilde{K}_\|) \]
\[
+ k_\| \tilde{P}_\| (p_\| \cdot K_\|) - \gamma_5 k_\| \tilde{P}_\| (p_\| \cdot \tilde{K}_\|) - \gamma_5 k_\| \tilde{P}_\| (p_\| \cdot K_\|) + k_\| \tilde{P}_\| (p_\| \cdot \tilde{K}_\|). \]
\]

To simplify the above equation, we use some trace property of gamma matrices like $Tr(\gamma_5 \phi \bar{b}) = 0$ and $Tr(\phi \bar{b}) = 4a \cdot b$. Thus the above equation becomes

\[
\mathcal{M}_t \mathcal{M}_t = -\frac{32e^4}{q_1^2 q_2^2} \left[ (k_\| \cdot P_\|) (p_\| \cdot K_\|) + (k_\| \cdot \tilde{P}_\|) (p_\| \cdot \tilde{K}_\|) + (k_\| \cdot P_\|) (p_\| \cdot \tilde{K}_\|) + (k_\| \cdot \tilde{P}_\|) (p_\| \cdot K_\|) \right]
\]
\[
+ (k_\| \cdot P_\|) (p_\| \cdot \tilde{K}_\|) + (k_\| \cdot \tilde{P}_\|) (p_\| \cdot K_\|) + (k_\| \cdot P_\|) (p_\| \cdot K_\|) + (k_\| \cdot \tilde{P}_\|) (p_\| \cdot \tilde{K}_\|). \]
\]

With the help of Mandelstam and magnetic Mandelstam variables, this can be further simplified

\[
\mathcal{M}_t \mathcal{M}_t = -\frac{8e^4}{q_1 q_2^2} \left[ u^2 + u_K u_P + \tilde{u} u + u_P u_P + u_K u_K + \tilde{u} \tilde{u} + u_K u_P + \tilde{u} \tilde{u} \right]
\]
\[
= -\frac{8e^4}{q_1 q_2^2} \left[ u_K u_P + u_P u_P + u_K u_K \right]
\]
\[
= -\frac{8e^4}{q_1 q_2^2} \left[ u_K u_P - u_K u_P - u_P u_K + u_P u_K \right]
\]
\[
= 0.
\]
C Electron-Muon scattering in s channel: $e^- e^+ \rightarrow \mu^- \mu^+$

This appendix contains the detailed calculation of crosssection for the process $(e^- e^+ \rightarrow \mu^- \mu^+)$. We start from the differential crosssection for the process $e^- e^+ \rightarrow \mu^- \mu^+$ (Figure 2) using the Eq-(86) with the matrix element (47)

$$
d\sigma^s_{B \neq 0}(e^- e^+ \rightarrow \mu^- \mu^+) = \frac{(2\pi)^4}{F} |\mathcal{M}_s|^2 (e^- e^+ \rightarrow \mu^- \mu^+) \delta^4(K_y + P_y - k_y - p_y) \int \frac{d^3 P}{(2\pi)^3 E_P} \int \frac{d^3 K}{(2\pi)^3 E_K} 
$$

In center-of-mass frame, $p + k = 0$, which also implies $p_y + k_y = 0$. Thus $d\sigma^s_{B \neq 0}(e^- e^+ \rightarrow \mu^- \mu^+)$ becomes

$$
d\sigma^s_{B \neq 0}(e^- e^+ \rightarrow \mu^- \mu^+) = \frac{|\mathcal{M}_s|^2}{(2\pi)^2 4F} \delta(E_P + E_K - E_p - E_k) \delta^3(K_y + P_y - k_y - p_y) \int \frac{d^3 P}{E_P} \int \frac{d^3 K}{E_K E_P} 
$$

In strong magnetic field $d^3 P$ split into $d^2 P_\perp$ and $dP_z$, where the integral over $d^2 P_\perp$ in strong magnetic field limit becomes $\int_0^{\pi eB} d^2 P_\perp = \pi |eB|$. This simplify the crosssection as

$$
\sigma^s_{B \neq 0}(e^- e^+ \rightarrow \mu^- \mu^+) = \int_{-\infty}^{\infty} \frac{\pi |eB| |\mathcal{M}_s|^2}{(2\pi)^2 4F} \delta(E_P + E_K - E_p - E_k) dP_z/E_K E_P.
$$

(144)

In strong magnetic field, $P_\perp \sim 0$, so the particles are restricted to move in the direction of magnetic field. They can either move in $+ve$ z direction or in $-ve$ z direction. Accordingly the four momentum dot product can be written as :-

$$
p.k = \begin{cases} 
E_p E_k - |p||k|, & \text{if } p \text{ and } k \text{ are in the same direction}, \\
E_p E_k + |p||k|, & \text{if } p \text{ and } k \text{ are in the opposite direction}.
\end{cases}
$$

(145)

In extreme relativistic limit, first type of dot product is negligible compared to the second type of dot product. For $\theta = 0$, $\theta$ is the scattering angle, the $t$ variables in eq (47) are negligible as compared to all $u$ variables and for $\theta = 180$ the $u$ variables are negligible as compared to all $t$ variables.
Thus for the one-dimensional scattering at high energy for \( \theta = 0 \) degree

\[
|M_1|^2 (e^- e^+ \rightarrow \mu^- \mu^+) = \frac{e^4}{4 \pi^2} \left[ u^2 + u_p u_k \right],
\]

and for \( \theta = 180 \) degree

\[
|M_1|^2 (e^- e^+ \rightarrow \mu^- \mu^+) = \frac{e^4}{4 \pi^2} \left[ t^2 + t_p t_k \right].
\]

Let us denote \(|p| = |k| = p_i\) and \(|P| = |K| = p_f\) then \(E_p = E_k = E_i\) and \(E_K = E_P = E_f\). Using these energies and momenta, the value of \(u, u_p, u_k\) can be calculated as

\[
u = -2k \cdot P
\]

\[
= -2[E_i E_f + p_i p_f].
\]

Therefore the square of \(u\) becomes

\[
u^2 = 4[E_i^2 E_f^2 + p_i^2 p_f^2 + 2E_i E_f p_i p_f].
\]

Similarly \(u_p, u_k\) and their product become

\[
u_p = -2\bar{p} \cdot K = -2[\bar{p}^0 K^0 - \bar{p}^3 K^3]
\]

\[
= -2[p_x E_K + E_p K_x]
\]

\[
= -2[p_i E_f + E_i p_f],
\]

\[
u_k = -2k \cdot P = -2[p_i E_f + E_i p_f],
\]

\[
u_p u_k = 4[p_i^2 E_f^2 + E_i^2 p_f^2 + 2p_i E_f E_i p_f].
\]

The addition of \(u^2\) and \(u_p u_k\) can be simplified, using the approximation \(p_f \sim E_f\), which holds good in the extreme relativistic limit. Thus \(u^2 + u_p u_k\) becomes

\[
u^2 + u_p u_k = 4 \left[ p_i^2 E_f^2 + E_i^2 p_f^2 + E_i^2 E_f^2 + p_i^2 p_f^2 + 4p_i E_f E_i p_f \right]
\]

\[
= 4 \left[ p_i^2 E_f^2 + E_i^2 (E_f^2 - M^2) + E_i^2 E_f^2 + p_i^2 (E_f^2 - M^2) + 4p_i \sqrt{(E_f^2 - M^2)E_i E_f} \right]
\]

\[
= 4 \left[ p_i^2 E_f^2 + E_i^2 E_f^2 + E_i^2 E_f^2 + p_i^2 E_f^2 + 4p_i E_f E_i E_f \right]
\]

\[
= 8E_f^2 \left[ p_i^2 + E_i^2 + 4p_i E_i \right]
\]

\[
= 8E_f^2 [E_i + p_i]^2.
\]
Thus for $\theta = 0$ degree the square of the matrix element (which comes out to be same as for the $\theta = 180$ degree) becomes

$$\left|\mathcal{M}_{n}\right|^2 (e^- e^+ \rightarrow \mu^- \mu^+) = \frac{e^4}{4s^2} \left[u^2 + u_p u_k\right] = \frac{2e^4}{s^2} E_f^2 [E_i + p_i]^2,$$

(148)

and the flux factor for the collinear collision becomes

$$F = |v_p - v_k| 2E_p 2E_k$$

$$= \left[|v_p| + |v_k|\right] 2E_p 2E_k \quad \text{(for collinear collision)}$$

$$= 4E_p E_k \left[\frac{|p|}{E_p} + \frac{|k|}{E_k}\right]$$

$$= 4 \left[|p|E_k + |k|E_p\right]$$

$$= 4p_i \sqrt{s}. \quad \text{($\sqrt{s} = E_p + E_k$)}$$

Therefore, using the expressions of $F$ and $\left|\mathcal{M}\right|^2$, eq-(144) can be rewritten as

$$\sigma^n_{B \neq 0}(e^- e^+ \rightarrow \mu^- \mu^+) = \int_{-\infty}^{\infty} \frac{2\pi e^4 |eB| E_f^2 [E_i + p_i]^2}{s^2 (2\pi)^2 16p_i \sqrt{s}} \delta (E_P + E_K - \sqrt{s}) \frac{dp_z}{E_f^2}$$

$$= 2 \int_{0}^{\infty} \frac{\pi e^4 |eB| [E_i + p_i]^2}{s^2 (2\pi)^2 8p_i \sqrt{s}} \delta (E_P + E_K - \sqrt{s}) dp_f, \quad (149)$$

where the factor of 2 comes due to the fact that the value of spin averaged squared matrix element is same for $P_z \in [0, \infty)$ and $P_z \in (-\infty, 0]$ which corresponds to $\theta = 0$ and $\theta = 180$ degree respectively.

Let us denote $W = E_P + E_K$ which with the help of energy eigenvalue equation in the strong magnetic field limit becomes

$$W = \sqrt{p_f^2 + M^2} + \sqrt{p_f^2 + M^2},$$

therefore the derivative $dW$ or $dp_f$ can be calculated as

$$dW = \frac{p_f W}{E_P E_K} dp_f,$$

or,$$dp_f = \frac{E_f^2}{p_f W} dW.$$

To solve the Dirac Delta function we substitute the value of $dp_f$ in (149). Thus the crosssection becomes,

$$\sigma^n_{B \neq 0}(e^- e^+ \rightarrow \mu^- \mu^+) = \int_{0}^{\infty} \frac{\pi e^4 |eB| [E_i + p_i]^2}{s^2 (2\pi)^2 4p_i \sqrt{s}} \delta (W - \sqrt{s}) \frac{E_f^2}{p_f W} dW.$$
With the help of the approximations: $p_f \approx E_f$ and $p_i \approx E_i$, the crosssection can be further simplified as

$$
\sigma_{B \neq 0}^\delta(e^+ e^- \to \mu^- \mu^+) = \int_0^\infty \frac{\pi e^4 |eB|^2 [2E_i]^2}{s^2 (2\pi)^4 E_i \sqrt{s}} \delta(W - \sqrt{s}) \frac{E_f}{2W} dW
$$

$$
= \int_0^\infty \frac{\pi e^4 |eB|^2}{s^2 (2\pi)^2} \frac{\sqrt{s}}{\sqrt{s}} \delta(W - \sqrt{s}) \frac{W}{2W} dW
$$

$$
= \int_0^\infty \frac{\pi e^4 |eB|^2}{4s^2 (2\pi)^2} \delta(W - \sqrt{s}) dW
$$

$$
= \frac{\pi e^4 |eB|^2}{4s^2 (2\pi)^2}.
$$

Using $e^2 = 4\pi \alpha$ the above result can be rewritten in terms of $\alpha$, which is

$$
\sigma_{B \neq 0}^\delta(e^+ e^- \to \mu^- \mu^+) = \frac{\alpha^2 \pi |eB|}{s^2}.
$$

(150)

D  **Electron-Muon scattering in $t$ channel: $e^- \mu^- \to e^- \mu^-$**

This appendix provides the calculation of the crosssection for the $(e^- \mu^- \to e^- \mu^+)$ process. We start from the fact that the matrix element for $t$ channel diagram has a pole at $t = 0$, which can be easily justified by observing the equations below and the pole is arrived due to the lower limit of $P_\perp$, i.e. $P_\perp = 0$ ( $|P| = \sqrt{P_x^2 + P_y^2}$ and also $|p| = |k| = p_i$ and $|P| = |K| = p_f$).

$$
t^2 = 4|p|^2 P_F^2 (\cos \theta - 1)^2 = 4|p|^2 |P|^2 \left( \frac{P_z}{|P|} - 1 \right)^2 = 4p_i^2 p_f^2 \left( \frac{P_z}{p_f} - 1 \right)^2,
$$

$$
u^2 = 4|p|^2 |K|^2 (\cos \theta + 1)^2 = 4|p|^2 |K|^2 \left( \frac{P_z}{|K|} + 1 \right)^2 = 4p_i^2 p_f^2 \left( \frac{P_z}{p_f} + 1 \right)^2.
$$

To deal with this problem, we apply a lower cut off, $\epsilon_B$ ($\epsilon_B \to 0$) to $P_\perp$. Let us denote $W = E_P + E_K$, $\sqrt{s} = E_p + E_k$, $E_p = E_k = E_i$ and $E_K = E_P = E_f$. Thus $d\sigma_{B \neq 0}^t$ can be written as

$$
d\sigma_{B \neq 0}^t(e^- \mu^- \to e^- \mu^-) = \frac{|\mathcal{M}|^2 (e^- \mu^- \to e^- \mu^-)}{(2\pi)^2 4F} \delta(W - \sqrt{s}) \frac{P_{\perp} dP_{\perp} d\phi dP_z}{E_f^2},
$$

$$
= \frac{e^4}{4t^2} \left[ s^2 + u^2 + s_p s_P + u_p u_P \right] \frac{\delta(W - \sqrt{s})}{(2\pi)4F} \frac{P_{\perp} dP_{\perp} dP_z}{E_f^2}.
$$

The above equation, with the help of $t$ and $u$, can be rewritten as

$$
d\sigma_{B \neq 0}^t = \int_{-\infty}^{\infty} dP_z \frac{e^4}{16p_i^2 p_f^2 \left( \frac{p_z}{p_f} - 1 \right)^2} \left[ s^2 + 4p_i^2 p_f^2 \left( \frac{P_z}{p_f} + 1 \right)^2 + s_p s_P + u_p u_P \right] \frac{\delta(W - \sqrt{s})}{(2\pi)4F} \frac{P_{\perp} dP_{\perp}}{E_f^2}.
$$

37
The squared matrix element has different values for the different directions of \( P_z \), so we split the integral into two parts, which gives

\[
\begin{align*}
  d\sigma_{B \neq 0}^t &= \int_{-\infty}^{\infty} dP_z \frac{e^4}{16p_f^2 P_f^2 \left( \frac{P_z}{p_f} - 1 \right)} \left[ s^2 + 4p_f^2 P_f^2 \left( \frac{P_z}{p_f} + 1 \right)^2 + s_p p_s + u_p u_P \right] \frac{\delta(W - \sqrt{s}) P_\perp dP_\perp}{(2\pi)4F \, E_f^2} \\
  &+ \int_{0}^{\infty} dP_z \frac{e^4}{16p_f^2 P_f^2 \left( \frac{P_z}{p_f} - 1 \right)} \left[ s^2 + 4p_f^2 P_f^2 \left( \frac{P_z}{p_f} + 1 \right)^2 + s_p p_s + u_p u_P \right] \frac{\delta(W - \sqrt{s}) P_\perp dP_\perp}{(2\pi)4F \, E_f^2}.
\end{align*}
\]

The first integral in the above equation, the \( u \) variables are negligible compared to \( s \) variables, thus \( d\sigma_{B \neq 0}^t \) becomes,

\[
\begin{align*}
  d\sigma_{B \neq 0}^t &= \int_{-\infty}^{\infty} dP_z \frac{e^4}{16p_f^2 P_f^2 \left( \frac{P_z}{p_f} - 1 \right)} \left[ s^2 + s_p s_P \right] \frac{\delta(W - \sqrt{s}) P_\perp dP_\perp}{(2\pi)4F \, E_f^2} \\
  &+ \int_{0}^{\infty} dP_z \frac{e^4}{16p_f^2 P_f^2 \left( \frac{P_z}{p_f} - 1 \right)} \left[ s^2 + 4p_f^2 P_f^2 \left( \frac{P_z}{p_f} + 1 \right)^2 + s_p s_P + u_p u_P \right] \frac{\delta(W - \sqrt{s}) P_\perp dP_\perp}{(2\pi)4F \, E_f^2}.
\end{align*}
\]

We can see that the second integral is the source of divergence and in a way we have separated the divergent term from the finite piece. After a little bit simplification, the above equation becomes

\[
\begin{align*}
  d\sigma_{B \neq 0}^t &= \int_{-\infty}^{\infty} dP_z \frac{e^4}{16p_f^2 (-P_z - p_f)^2} \left[ s^2 + s_p s_P \right] \frac{\delta(W - \sqrt{s}) P_\perp dP_\perp}{(2\pi)4F \, E_f^2} \\
  &+ \int_{0}^{\infty} dP_z \frac{e^4}{16p_f^2 (P_z - p_f)^2} \left[ s^2 + 4p_f^2 (P_z + p_f)^2 + s_p s_P + u_p u_P \right] \frac{\delta(W - \sqrt{s}) P_\perp dP_\perp}{(2\pi)4F \, E_f^2}.
\end{align*}
\]

As we discussed earlier that \( P_\perp \sim 0 \) in the strong magnetic field. Therefore the above differences in the momenta can be simplified by the approximations: \( P_z - p_f \simeq -\frac{p_f^2}{2P_z} \) and \( P_z + p_f \simeq 2P_z \), hence the above integral can be rewritten as

\[
\begin{align*}
  d\sigma_{B \neq 0}^t &= \int_{0}^{\infty} dP_z \frac{e^4}{64p_f^2 P_z^2} \left[ s^2 + s_p s_P \right] \frac{\delta(W - \sqrt{s}) P_\perp dP_\perp}{(2\pi)4F \, E_f^2} \\
  &+ \int_{0}^{\infty} dP_z \frac{e^4 P_z^2}{4P_f^2 P_\perp^2} \left[ s^2 + 16p_f^2 P_z^2 + s_p s_P + u_p u_P \right] \frac{\delta(W - \sqrt{s}) P_\perp dP_\perp}{(2\pi)4F \, E_f^2}.
\end{align*}
\]
We have separated the integral in functions of $P_z$ and $P_\perp$. Next we integrate over $P_\perp$, by applying a lower cut off to the $P_\perp$ where it causes the divergence. Thus the above equation simplifies as

$$\sigma_{B \neq 0}^t = \int_0^\infty dP_z \frac{e^4}{64p_f^2P_z^2} \left[ s^2 + s_p s_p \right] \frac{\delta(W - \sqrt{s})}{(2\pi)4FE_f^2} \int_0^{\sqrt{|eB|}} P_\perp dP_\perp$$

$$+ \int_0^\infty dP_z \frac{e^4P_z^2}{4p_f^2} \left[ s^2 + 16p_z^2P_z^2 + s_p s_p + u_p u_p \right] \frac{\delta(W - \sqrt{s})}{(2\pi)4FE_f^2} \lim_{\epsilon \rightarrow 0} \int_{\epsilon B}^{\sqrt{|eB|}} \frac{dP_\perp}{P_\perp^3},$$

$$= \int_0^\infty dP_z \frac{e^4}{64p_f^2P_z^2} \left[ 16E_i^4 + 16k_z E_i K_z E_f \right] \frac{\delta(W - \sqrt{s}) |eB|}{2(2\pi)4FE_f^2}$$

$$+ \int_0^\infty dP_z \frac{e^4P_z^2}{4p_f^2} \left[ 16E_i^4 + 16p_z^2P_z^2 + 16k_z E_i K_z E_f + u_p u_p \right] \frac{\delta(W - \sqrt{s})}{(2\pi)4FE_f^2} \lim_{\epsilon \rightarrow 0} \left[ \frac{1}{\epsilon^2} - \frac{1}{|eB|} \right].$$

Since the perpendicular component of momentum is very small compared to the $z$ component of momentum, we approximate $P_z = K_z \simeq p_f$. Therefore $\sigma_{B \neq 0}^t$ becomes

$$\sigma_{B \neq 0}^t = \int_0^\infty dp_f \frac{e^4}{64p_f^2p_f^2} \left[ 16E_i^4 + 16p_i E_i p_f E_f \right] \frac{\delta(W - \sqrt{s}) |eB|}{32\pi p_i \sqrt{s}E_f^2} \frac{1}{2}$$

$$+ \int_0^\infty \left[ dp_f \frac{e^4p_f^2}{4p_f^2} \left\{ 16E_i^4 + 16p_f^2p_f^2 + 16p_i E_i p_f E_f + 4(p_i E_f + E_i p_f)^2 \right\} \right.$$  

$$\times \frac{\delta(W - \sqrt{s})}{32\pi p_i \sqrt{s}E_f^2} \lim_{\epsilon \rightarrow 0} \left\{ \frac{1}{\epsilon^2} - \frac{1}{|eB|} \right\}.$$  

To deal with the Dirac Delta function, we write $dp_f$ in terms of $dW$ i.e. $dp_f = \frac{E_f^2}{p_f W} dW$ and approximate $p_i \simeq E_i$ and $p_f \simeq E_f$. Thus the above integral becomes

$$\sigma_{B \neq 0}^t = \int_0^\infty dW \frac{e^4}{8E_f W} \left[ E_i^2 + E_f^2 \right] \frac{\delta(W - \sqrt{s})}{32\pi E_i \sqrt{s}E_f^2} |eB|$$

$$+ \int_0^\infty \frac{E_f}{W} dW e^4 \left[ E_i^2 + 3E_f^2 \right] \frac{\delta(W - \sqrt{s})}{8\pi E_i \sqrt{s}} \lim_{\epsilon \rightarrow 0} \left[ \frac{1}{\epsilon^2} - \frac{1}{|eB|} \right].$$

39
Using \( E_f = W/2 \) and \( E_i = \sqrt{s}/2 \), the above integral can be further simplified as

\[
\sigma_{B\neq0}^t = \int_0^\infty dW \frac{e^4}{16W^2} \left[ s + W^2 \right] \frac{\delta(W - \sqrt{s})}{4\pi s W^2} |eB|
\]

\[
+ \int_0^\infty dW e^4 \left[ s + 3W^2 \right] \frac{\delta(W - \sqrt{s})}{32\pi s} \lim_{\epsilon_B \to 0} \left[ \frac{1}{\epsilon_B^2} - \frac{1}{|eB|} \right].
\]

To obtain the crosssection, we integrate over \( dW \) by using the property of Dirac Delta function. Thus the above equation becomes

\[
\sigma_{B\neq0}^t = \frac{e^4}{32\pi s^2} |eB| + \frac{e^4}{8\pi} \lim_{\epsilon_B \to 0} \left[ \frac{1}{\epsilon_B^2} - \frac{1}{|eB|} \right]
\]

\[
= \frac{\pi\alpha^2}{2s^2} |eB| - \frac{2\pi\alpha^2}{|eB|} + 2\pi\alpha^2 \lim_{\epsilon_B \to 0} \left[ \frac{1}{\epsilon_B^2} \right].
\]

**Relation between the divergence of vacuum and magnetic field**

Starting from the relation \( P_\perp = P_z \tan \theta \), where \( P_\perp \) is the momentum of the particle in the transverse direction, we can calculate

\[
\frac{1}{P_\perp^2} = \frac{1}{P_z^2} \tan^2 \theta = \frac{1}{P_z^2} \left[ \cos^2 \theta \right].
\]

Let us denote \( \cos \theta = x \), which simplify the above equation as

\[
\frac{1}{P_\perp^2} = \frac{1}{P_z^2} \left[ \frac{x^2}{1 - x^2} \right]
\]

\[
= \frac{x^2}{2P_z^2} \left[ \frac{1}{1 + x} + \frac{1}{1 - x} \right]
\]

\[
= \frac{1}{2P_z^2} \left[ \frac{x^2}{1 + x} + \frac{x^2}{1 - x} \right]
\]

\[
= \frac{1}{2P_z^2} \left[ \frac{x^2}{1 + x} + \frac{x^2}{1 - x} \right] - \frac{1}{1 + x} + \frac{1}{1 - x}
\]

\[
= \frac{1}{2P_z^2} \left[ \frac{x^2}{1 + x} - (1 + x) + \frac{1}{1 - x} \right].
\]

We set a lower cut off to \( \theta, \theta_0 \to 0 \), which gives \( x_0 \to 1 \), for \( x_0 = \cos \theta_0 \). Thus above equation becomes

\[
\frac{1}{P_\perp^2} = \frac{1}{2P_z^2} \left[ \frac{1}{1 + 1} - (1 + 1) + \lim_{x_0 \to 1} \frac{1}{1 - x_0} \right]
\]

\[
= \frac{1}{2P_z^2} \left[ -\frac{3}{2} + \lim_{x_0 \to 1} \frac{1}{1 - x_0} \right].
\]
where we set $x_0 = 1$ in those terms which don’t cause the divergence.

In the strong magnetic field $|\mathbf{P}| \approx P_z$ and in the extreme relativistic limit, $P_z \approx E_P$. With the help of these approximations, we can approximate $P_z^2 \approx s/4i$, which thus simplifies the above equation as

$$\frac{1}{P_{\perp}^2} = \frac{2}{s} \left[ -\frac{3}{2} + \lim_{x_0 \to 1} \frac{1}{1-x_0} \right]$$

$$= \frac{2}{s} \left[ -\frac{3}{2} + \lim_{\cos \theta_0 \to 1} \frac{1}{1-\cos \theta_0} \right],$$

$$\frac{1}{\epsilon_B^2} = \frac{2}{s} \left[ -\frac{3}{2} + \frac{1}{\epsilon_V} \right],$$

or

$$\frac{1}{\epsilon_V} = \frac{s}{2\epsilon_B^2} + \frac{3}{2},$$

where $\epsilon_B$ is the lower cut off on $P_{\perp}$ and $\epsilon_V = 1 - \cos \theta_0$. 

41