The Wigner function negative value domains and energy function poles of the harmonic oscillator

E. E. Perepelkin1,2,4 · B. I. Sadovnikov1 · N. G. Inozemtseva2,3 · E. V. Burlakov1,2

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Abstract
For a quantum harmonic oscillator, an explicit expression that describes the energy distribution as a coordinate function is obtained. The presence of the energy function poles is shown for the quantum system in domains where the Wigner function has negative values.

Keywords Wigner function · Moyal equation · Vlasov equation · Quantum harmonic oscillator

1 Introduction

The Wigner function [1] is one of the most effective tools for describing quantum systems in phase space. The Wigner function is a quasi-probability density set in phase space. It is widely used in quantum tomography [2], quantum communication and cryptography [3], quantum informatics [4], and signal processing problems [5]. The traditional definition of the Wigner function is given in terms of the density matrix of a quantum system:

$$W(\vec{r}, \vec{p}, t) = \frac{1}{(2\pi\hbar)^3} \int_{(\infty)}^{(\infty)} \exp \left( -\frac{i\vec{p} \cdot \vec{r}}{\hbar} - \frac{\vec{p} t}{2} \frac{\hat{\rho}(t) \vec{r} - \frac{3}{2} \vec{p}}{2} \right) d^3 s.$$ (1)

For a quantum system described by the wave function $\Psi(\vec{r}, t)$ or $\tilde{\Psi}(\vec{p}, t)$, the integration of the function $W$ over the space of momenta and coordinates gives the correct relations for the functions $|\Psi|^2$ and $|\tilde{\Psi}|^2$, respectively:

$$\int_{(\infty)}^{(\infty)} W(\vec{r}, \vec{p}, t) d^3 p = |\Psi(\vec{r}, t)|^2, \quad \int_{(\infty)}^{(\infty)} W(\vec{r}, \vec{p}, t) d^3 r = |\tilde{\Psi}(\vec{p}, t)|^2.$$ (2)

This shows that this definition (1) is consistent with a natural property inherent in classical distribution functions. However, both estimations for the integral (1) and direct calculation of the Wigner functions according to this definition show that it can take negative values. This fact does not allow us to consider it as a rigorous distribution function. The question of the negativity of the quasi-probability density has been discussed in many works [6–9].

As an alternative to the Wigner function for some classes of problems, various variants of $P, Q$—distribution functions are phenomenologically constructed [10–13]. The $Q$—function is positive in the phase space. The $P$—function is positive for coherent and temperature states and can be negative for the Fock and entangled states. A property similar to (2) no longer holds for $P, Q$—functions.

Thus, in the quantum case, the use of any distribution function on the phase space leads to the violation of one or the other requirements of the classical probability theory. The problems regarding the search for positive
quasi-distribution functions as well as possible interpretations of negative quasi-probability values remain to be challenging today.

It should be noted that a quantum system in the phase space can be described in terms of the formalism of the Vlasov infinite self-linked chain of equations [14, 15]. This approach clearly demonstrates the relationship between classical and quantum physics. Below are basic equations from the Vlasov formalism for quantum mechanics in the phase space which lie in the background of the present paper.

A.A. Vlasov obtained an infinite self-linked chain of equations for the distribution density functions of higher kinematic quantities \( f_1(\vec{r}, t), f_2(\vec{r}, \vec{v}, t), f_3(\vec{r}, \vec{v}, \vec{v}, t), \ldots \) [16–18]. Let us consider the first two equations from the infinite self-linked Vlasov equations chain for the probability density distribution functions \( f_1(\vec{r}, t) \) and \( f_2(\vec{r}, \vec{v}, t) \):

\[
\frac{\partial f_1(\vec{r}, t)}{\partial t} + \text{div} \left[ \left( \vec{v} \right) f_1(\vec{r}, t) \right] = 0, \tag{3}
\]

\[
\frac{\partial f_2(\vec{r}, \vec{v}, t)}{\partial t} + \text{div} \left[ \left( \vec{v} \vec{v} \right) f_2(\vec{r}, \vec{v}, t) \right] + \text{div} \left[ \left( \vec{v} \vec{v} \right) f_2(\vec{r}, \vec{v}, t) \right] = 0, \tag{4}
\]

where

\[
f_1(\vec{r}, t) = \int_{(\infty)} f_2(\vec{r}, \vec{v}, t) d^3v, \quad N(t) = \int_{(\infty)} f_1(\vec{r}, t) d^3r, \tag{5}
\]

\[
f_1(\vec{r}, t) \langle \vec{v} \rangle (\vec{r}, t) = \int_{(\infty)} \vec{v} f_2(\vec{r}, \vec{v}, t) d^3v, \quad f_2(\vec{r}, \vec{v}, t) \langle \vec{v} \rangle (\vec{r}, \vec{v}, t) = \int_{(\infty)} \vec{v} f_2(\vec{r}, \vec{v}, \vec{v}, t) d^3v.
\]

For the first Vlasov equation (3), the following Hamilton-Jacobi equation is valid [14, 19]:

\[
-\hbar \frac{\partial \chi_1}{\partial t} = \frac{m}{2} \left| \langle \vec{v} \rangle \right|^2 + e \chi_1 = H_1, \tag{6}
\]

\[
e^\chi_1 = U_1 + Q_1 + \frac{e^2}{2m} |\vec{A}|^2, \quad Q_1 = \frac{\alpha_1}{\beta_1} \frac{\Delta |\Psi_1|}{|\Psi_1|} = -\hbar^2 \frac{\Delta |\Psi_1|}{2m |\Psi_1|}, \tag{7}
\]

where \( \vec{A} \) is the vector potential; \( U_1 \) is the potential from the Schrödinger equation; \( Q_1 \) is the phase of wave function \( \Psi_1 \); \( f_1 = |\Psi_1|^2; \alpha_1 = -\frac{\hbar}{2m}, \beta_1 = \frac{1}{\hbar} \); and quantity \( Q_1 \) is the quantum potential from the de Broglie–Bohm theory of the “pilot wave” [20–23]. The quantum potential \( Q_1 \) allows us to determine the quantum pressure tensor \( P^{ij}_1 \):

\[
-\frac{1}{f_1} \frac{\partial P^{ij}_1}{\partial x^j} = 2\alpha^2_1 \frac{\partial}{\partial x^i} \left( \frac{1}{\sqrt{f_1}} \frac{\partial^2 \sqrt{f_1}}{\partial x^i \partial x^j} \right) = 2\alpha_1 \beta_1 \frac{\partial Q_1}{\partial x^i}. \tag{8}
\]

The motion equations are as follows [14, 19]:

\[
\frac{d}{dt} \langle \vec{v} \rangle = -\gamma_1 \left( \vec{E}_1 + \langle \vec{v} \rangle \times \vec{B}_1 \right), \tag{9}
\]

\[
\vec{E}_1 = -\frac{\partial}{\partial t} \vec{A}_1 - \nabla \chi_1, \quad \vec{B}_1 = \frac{\partial \vec{A}_1}{\partial t} - \frac{3}{2} \nabla \times \vec{E}_1.
\]

The vector fields \( \langle \vec{v} \rangle (\vec{r}, t) \) and \( \langle \vec{v} \vec{v} \rangle (\vec{r}, \vec{v}, t) \) correspond to the speed and acceleration of the probability flows. The function \( N(t) \) determines the number of particles in the system, which can be non-integer [16]. For a constant number of particles \( (N = \text{const}) \), the value \( N \) is used as a normalizing factor when calculating the total probability. The distribution function \( f_2(\vec{r}, \vec{v}, \vec{v}, t) \) satisfies the third Vlasov equation. Note that the variables \( \vec{r}, \vec{v}, \vec{v}, \vec{v}, \ldots \) are independent kinematic quantities.

It is shown [19] that the first Vlasov equation (3) and the first principles (on the base of the Helmholtz theory for a vector field \( \langle \vec{v} \rangle (\vec{r}, t) \)) can be used to derive the Schrödinger equation for a scalar particle in electromagnetic field.

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\]
with the Vlasov–Moyal approximation [16] of the vector acceleration field \( \vec{\ddot{v}}(\vec{r}, \vec{v}, t) \):

\[
\langle \vec{\ddot{v}} \rangle = \sum_{n=0}^{+\infty} \left( -\frac{1}{m} \frac{\partial}{\partial n} \left( \frac{E}{n+1} \right) \right) \frac{1}{m^2 + 1} \frac{\partial^2 f_2}{\partial v^2} \frac{n+1}{2} ! \frac{\partial^2 f_2}{\partial v^2} \frac{n+1}{2} !
\]

(13)

This implies a direct connection between the Wigner function and the second Vlasov quasi-distribution function \( f_2(\vec{r}, \vec{v}, t) \). However, the Vlasov formalism has its advantages. Firstly, it is based on the first principle—the probability conservation law, and does not contain phenomenological constructions such as the phenomenological Wigner function (1) definition. Secondly, it applies to the distribution functions depending on kinematic quantities of all orders \( \vec{r}, \vec{v}, \vec{v}, \vec{v}, ... \) And this circumstance can be used to generalize quantum mechanics to the case of higher-order kinematic quantities [14].

The well-known Wigner function of a harmonic oscillator with potential 2 has the following form:

\[
f_{2,n}(x, v) = \frac{(-1)^n m}{\pi \hbar} e^{-\frac{p^2}{2m} + \frac{m \omega^2 x^2}{2}} L_n \left( \frac{2m}{\hbar \omega} (v^2 + \omega^2 x^2) \right),
\]

(14)

where \( L_n \) are the Laguerre polynomials. The function \( f_{2,n}(x, v) \) is related to the Wigner function from its definition (1) as \( W_n(x, p) = \frac{1}{m} f_{2,n}(x, \frac{p}{m}) \).

Note that function (14) can be written as \( f_{2,n}(x, v) = F_n(\varepsilon(x, mv)) \):

\[
F_n(\varepsilon) = \frac{(-1)^n m}{\pi \hbar} e^{-\frac{p^2}{2m} + \frac{m \omega^2 x^2}{2}} L_n(4\varepsilon),
\]

(15)

where

\[
\varepsilon(x, p) = \frac{1}{\hbar \omega} \left( \frac{p^2}{2m} + \frac{m \omega^2 x^2}{2} \right).
\]

(16)

The second Vlasov equation (4) for a harmonic oscillator takes the form:

\[
v \frac{\partial f_{2,n}(x, v)}{\partial x} - \omega^2 x \frac{\partial f_{2,n}(x, v)}{\partial v} = \int_{-\infty}^{+\infty} f_{2,n}(x, v) dv = f_{1,n}(x).
\]

(17)

Averaging over the velocity space of the Vlasov–Moyal approximation (12) will give the classical Vlasov approximation [16] for equation (10):

\[
\langle \vec{\dot{v}} \rangle = \frac{1}{m} \frac{\partial U}{\partial x}.
\]

(18)

It was shown [14] that function (14) can be obtained not only from the definition (1), but also, directly, from the evolution equation of the quasi-distribution function which can be represented by either the Moyal equation (12) or the second Vlasov equation (17) with Vlasov–Moyal approximation (13). Regardless of the obtaining method, the Wigner function (14) allows one to perform quantum averaging in the same way as it is done in classical statistical physics. Thus, in the case of quantum mechanics, one can formally introduce analogs of such quantities as potential and kinetic energy. Despite the seeming formality of such constructions, the results obtained can be interpreted in an interesting way.

The present paper has the following structure. In Sect. 1, we consider the obtaining of explicit expressions for the average (in coordinate/velocity) kinematic quantities corresponding to the “kinetic” and “potential” energy of a quantum harmonic oscillator. In Sect. 2, it is shown that the average values of the energies \( \langle \varepsilon \rangle, \langle x \rangle, \langle \dot{\varepsilon} \rangle, \langle v \rangle \) have poles located in the regions where the Wigner function takes negative values. The energy poles \( \langle \varepsilon \rangle, \langle x \rangle \) form infinite potential barriers which number increases along with the growth of the state number \( n \) of the quantum system. The conclusion section contains an interpretation of the main results. The details of mathematical transformations are presented in “Appendix.”

### 2 §1 Average kinematic values calculation

Knowing the distribution function \( f_{2,n}(x, v) \) (13), it is possible to calculate the average value of the energy \( E \) is given by expressions (1.2) and (1.3) are equal for each state, the standard deviation energy (1.3) is the same. In this case, the distances between the average values \( E_n \) of the energy according to expressions (1.2) and (1.3) are equal. Thus, from the standpoint of quantum mechanics various (continuous spectrum) energy values \( \varepsilon \) “exist” in the phase space, but the set of their average values (according to the Wigner distribution function) is countable and coincides with the eigenvalues of the Hamiltonian. The use of phase space allows for a visual interpretation of the relationship between classical and quantum mechanics.
x, velocity v, or over the entire phase space (over both variables x, v). Let us find the values of $\langle x^2 \rangle$, $\langle v^2 \rangle$, $\langle x^2 v^2 \rangle$, $\langle x^2 \rangle$ for different states n of the system which are described by the distribution function $f_{2,n}(x,v)$ (13). Note that due to the symmetry of the distribution function $f_{2,n}(x,v)$, the average values of $\langle x \rangle = \langle x \rangle = 0$ and $\langle v \rangle = \langle v \rangle = 0$

For convenience purposes, we introduce designations of $\sigma^2_x = \frac{\hbar}{2m\nu_0}$ and $\sigma^2_v = \frac{\hbar\nu_0}{2m}$, which correspond to the standard deviation for the ground state ($n = 0$) of a harmonic oscillator. The quantities $\sigma_x, \sigma_v$ are related to the Heisenberg uncertainty principle

$$\sigma_x \sigma_v = |\alpha| = \frac{\hbar}{2m}, \quad \omega = \frac{\sigma_v}{\sigma_x}. \tag{1.4}$$

Using the quantities $\sigma_x$ and $\sigma_v$, the expression for the energy (15) and the representations of the distribution functions (5) and (13) can be rewritten in the form:

$$2\varepsilon(x,p) = \varepsilon(x,v) = \frac{v^2}{2\sigma^2_v} + \frac{x^2}{2\sigma^2_x}. \tag{1.5}$$

Performing calculations, we find $\langle v^2 \rangle$ [Appendix A]:

$$\langle v^2 \rangle_{v,n}(x,v) = \sigma^2_v \sum_{k=0}^{n} C_k H_k^2\left(\frac{v^2}{\sigma^2_v}\right), \tag{1.6}$$

where $C_k = (-1)^k \sum_{j=0}^{k} \frac{1 + \eta(j)}{2} \frac{H^2_{2k-2j}(0) + 2(k-j)H_{2k-2j-1}(0)}{2^{k-j}(k-j)!},$

and

$$\tilde{C}_k = (-1)^k \sum_{j=0}^{k} \frac{1 + \eta(j)}{2} \frac{H^2_{2k-2j}(0) - 2(k-j)H_{2k-2j-1}(0)}{2^{k-j}(k-j)!}.$$

Note that $\eta(s)$ is the Heaviside function. To calculate the coefficients $\tilde{C}_{2k}$, it is convenient to use the properties of the zeros of the Hermite polynomials:

$$H_{2k}^2(0) = \frac{(2k)!}{k!^2}, \quad H_{2k+1}^2(0) = 0. \tag{1.7}$$

The index v in expression (1.6) indicates the averaging over the space of velocities, and the index n corresponds to the state number of the quantum harmonic oscillator.

Due to the symmetry of the function $f_{2,n}$ with respect to the variables x and v, similarly as with expression (1.6) we can obtain an expression for $\langle x^2 \rangle_{x,n}$ when averaging over the coordinate space x [Appendix A]:

$$\langle x^2 \rangle_{x,n}(v) = \sigma^2_x \sum_{k=0}^{n} C_k L_{n-k}^2\left(\frac{v^2}{\sigma^2_v}\right). \tag{1.8}$$

3 §2 The Energy function poles distribution in the phase space

Let us analyze the obtained distributions (1.6) and (1.8). A peculiarity of expressions (1.6) and (1.8) is the presence of poles for the coordinate and velocity, respectively. Figure 1 shows the oscillator energy dependence

$$\langle \varepsilon \rangle_{v,n}(x,v) = \frac{v^2}{2\sigma^2_v} + \frac{x^2}{2\sigma^2_x}. \tag{2.1}$$

![Fig. 1 Energy density distribution $\langle \varepsilon \rangle_{v,n}(x)$](image-url)
Figure 3 demonstrates a formal overlay of the graphs of the density of probability distribution \( f_{1,n}(x) \) and the density of energy distribution \( \langle \varepsilon \rangle_{v,n}(x) \) for the states \( n = 0, 1, 2, 3 \), respectively.

Figure 3 shows that the poles actually “break” the potential well into several potential wells, each of which formally has its own oscillatory process. Indeed, in Fig. 2 there are no poles for the ground state \( (n = 0) \). Therefore, there is only one initial potential well in which there is a Gaussian probability density distribution \( f_{1,0} \). Having \( n = 1 \), the kinetic energy has a pole at zero (Figs. 1 and 3), which leads to the presence of an energy “barrier” and the division of the distribution function \( f_{1,0} \) into two symmetric distributions relative to zero in the form of a distribution function \( f_{1,1} \) (Fig. 3). A similar situation is observed for states with \( n = 2, 3 \ldots \) (Fig. 3).

For a harmonic oscillator, the transition from one quantum state with a number \( n \) to a state with a number \( n + 1 \) is associated with the presence of negative values in the Wigner function.

It is in the domain of negative values that the kinetic (1.6) and potential energies (1.8) have poles which lead to energy barriers (Figs. 1, 2, and 3) and formally “create” one more oscillator.

Let us calculate the standard deviations \( \langle \langle x^2 \rangle \rangle_n \) and \( \langle \langle v^2 \rangle \rangle_n \) [Appendix B]:

\[
\langle \langle v^2 \rangle \rangle_n = \sigma_v^2(2n + 1), \quad \langle \langle x^2 \rangle \rangle_n = \sigma_x^2(2n + 1). \quad (2.2)
\]

In expression (2.2), it can be seen that for the ground state \( (n = 0) \), the standard deviations \( \sqrt{\langle \langle v^2 \rangle \rangle_0} = \sigma_v \) and \( \sqrt{\langle \langle x^2 \rangle \rangle_0} = \sigma_x \) or in other words coincide with the values \( \sigma_v \) and \( \sigma_x \) introduced formally above (1.4). Consequently, notation (1.4) has a clear interpretation. As the state number \( n \) increases, the quantities (2.2) grow. Knowing \( \langle \langle x^2 \rangle \rangle_n \) and \( \langle \langle v^2 \rangle \rangle_n \), we can calculate the total average energy of the harmonic oscillator (1.2) in the state \( n \):

\[
E_n = \frac{m v^2}{2} + \frac{m o^2 x^2}{2},
\]

\[
E_n = \frac{m}{2} \sigma_v^2(2n + 1) + \frac{m o^2}{2} \sigma_x^2(2n + 1) = m \left( n + \frac{1}{2} \right) (\sigma_v^2 + o^2 \sigma_x^2),
\]

\[
E_n = h o \left( n + \frac{1}{2} \right), \quad (2.3)
\]

where relations (1.4) are taken into account. The resulting expression (2.3) completely coincides with expression...
Thus, despite the incorrectness of reasoning about the kinetic and potential energy separately from the standpoint of the Heisenberg uncertainty principle such contradictions are leveled out for quantum mechanics in the phase space.

4 Conclusions

From a physical point of view and intuition, one can be under an illusion that the presence of the poles of the energy density $\langle \tilde{\varepsilon} \rangle_{v,n}(x)$ is stipulated by “possible” poles of quantum potential $Q(x)$ (6). The denominator of quantum potential (6) is obviously represented by the function $f_{1,n}(x)$ that goes to zero at the poles of function $\langle \tilde{\varepsilon} \rangle_{v,n}(x)$ That is why it would be logical to assume that the quantum potential $Q(x)$ itself will have poles at these points. But this really is not true. It can be shown as follows. Since the Wigner function $f_{2,n}(x,v)$ (13) is even, the velocity of average probability flow is expressed as $\langle \tilde{v} \rangle(x) = 0$ because of

\[
\int_{-\infty}^{+\infty} v f_{2,n}(x,v) dv = 0.
\]

Taking into account (3.1), (7), and (17), the motion equation (10) takes the following form:

\[
U_1(x) + Q(x) = \text{const.}
\]

where $U_1(x) = \frac{m \omega^2 x^2}{2}$. Thus, quantum potential $Q(x)$ is an analytical function having no poles for any quantum state $n$. The constant value in expression (3.2) according to the Hamilton–Jacobi equation (5) represents the energy $E_n$ (2.3).

\[
Q_n(x) = -\frac{m \omega^2 x^2}{2} + \hbar \omega \left( n + \frac{1}{2} \right).
\]

This fact (3.3) can also be derived by substituting the function $f_{1,n}(x)$ in the expression for quantum potential $Q(x)$ (6) [Appendix C]. From a mathematical point of view, both the denominator and the numerator of the quantum potential $Q(x)$ around the zero points of $f_{1,n}(x)$ are infinitesimals of the same order. Thus, the quantum potential does not “feel” energy density poles $\langle \tilde{\varepsilon} \rangle_{v,n}(x)$ and even more of the function $\langle \tilde{\varepsilon} \rangle_{v,n}(v)$.

The zero value of the function $f_{1,n}(x)$ at $x_k$ means that (5)
\[ f_{1,n}(x_k) = \int_{-\infty}^{+\infty} f_{2,n}(x_k, v) dv = 0. \] (3.4)

Let us assume that the Wigner function \( f_{2,n}(x_k, v) \) is non-negative \((f_{2,n}(x_k, v) \geq 0)\) (Fig. 4). Then, from expression (3.4) it means that \( f_{2,n}(x_k, v) = 0 \) almost everywhere for \( v \in (-\infty, +\infty) \). Solution (15) \( f_{2,n}(x, v) = F_n(\epsilon(x, v)) \) of the Moyal equation (or the second Vlasov equation) (17) can be represented via characteristics \( \epsilon = \text{const} \) (16). Hence, the function \( f_{2,n}(x, v) \) is non-zero over the whole phase space outside the ellipse.

\[
\frac{1}{\hbar \omega} \left( \frac{p^2}{2m} + \frac{m \omega^2 x^2}{2} \right) = \epsilon(x_k, 0) = \frac{\max^2}{2\hbar} = \epsilon(0, v_k) = \frac{v^2}{2\hbar}\omega,
\] (3.5)

and is only not equal to zero inside the phase ellipse (3.5) (Fig. 4). Thus, this is a contradiction, showing that the Wigner function \( f_{2,n}(x, v) \) takes negative values along the line \( x = x_k (v = v_k) \).

The position of poles of the expressions \( \langle \vec{F}\rangle_{\epsilon,n}(v) / \langle \epsilon \rangle_{\epsilon,n}(v) \) in the domains where the corresponding Wigner function \( f_{2,n}(x, 0)/f_{2,n}(0, v) \) has negative values is associated with the presence of zeros for the functions \( f_{1,n}(x) = |\Psi_n(x)|^2 / f_{1,n}(v) = |\Psi_n(v)|^2 \). This statement will remain true for a quantum system with a higher-order polynomial potential [16, 18, 25]. In the case of the second degree potential, the position of the energy poles \( \langle \vec{F}\rangle_{\epsilon,n}(x) / \langle \epsilon \rangle_{\epsilon,n}(x) \) is determined by the zeros of the Hermite polynomials (1.5) and (1.7).

Applying the asymptotic methods to the Hermite and Langer polynomials, it is possible to show that zero values of \( f_{1,n}(x) \) function are located within the range of negative values of \( f_{2,n}(x, 0) \) function. Using Stirling’s approximation [26]:

\[
L_n(x) = \frac{e^{x^2}}{\sqrt{\pi(nx)^{1/4}}} \sin \left( 2\sqrt{n\pi + \frac{\pi}{4}} \right) + O(n^{-3/4}). \] (3.6)

\[
e^{-\frac{x^2}{2}} H_n(x) \sim \left( \frac{2m}{e} \right)^{n/2} \sqrt{2} \cos \left( x\sqrt{2n} - \frac{\pi n}{2} \right) \left( 1 - \frac{x^2}{2n+1} \right). \]

After simple manipulations, we can estimate the location of zero points of the polynomials \( H_n \left( \frac{x}{\sigma_n^2} \right) \) and \( L_n \left( \frac{x^2}{\sigma_n^2} \right) \), respectively:

\[
x_k = \frac{\pi \sigma_n}{2\sqrt{n}} \left\{ \begin{array}{ll}
2k+1, & k = 0, \ldots, n - 1, \ n - even \\
2k, & k = 0, \ldots, n - 1, \ n - odd
\end{array} \right. \] (3.7)

\[
x_l = \frac{\pi \sigma_l}{2\sqrt{n}} \left( l + \frac{3}{4} \right), \ l = 0, \ldots, n - 1.
\]

From (3.7), it is obvious that between the two zero values of \( f_{1,n}(x_k) \) function, there are always two zero values of \( f_{2,n}(x, 0) \) function, which comply with Fig. 2

**Appendix A**

From equations (6), (8), (9), (10), (17) and (16) for a harmonic oscillator, it follows that

\[
\frac{1}{\hbar} \frac{\partial P}{\partial x} = \frac{1}{\hbar} \int_{-\infty}^{+\infty} \frac{\partial f_{2,n}}{\partial x} dv = -\frac{1}{m} \frac{\partial U_1}{\partial x} = -\omega^2 x,
\]

where in the one-dimensional case (9) the notation of \( P_{\mu,\lambda} \) is changed to \( P \)

\[
\frac{x}{\sigma^2} \int_{-\infty}^{+\infty} \frac{\partial F_n}{\partial x} dv = -\omega^2 x f_{1,n} = -\omega^2 x \int_{-\infty}^{+\infty} F_n(\epsilon) dv.
\]

\[
0 = \int_{-\infty}^{+\infty} (v^2 F_n' + \sigma_n^2 \omega^2 F_n) dv = \int_{-\infty}^{+\infty} \left( v^2 \frac{\partial S_{2,n}}{\partial \epsilon} + \sigma_n^2 \omega^2 \right) F_n(\epsilon) dv = f_{1,n} \left( v^2 \frac{\partial S_{2,n}}{\partial \epsilon} + \sigma_n^2 \omega^2 \right).
\]
\[
\left< v^2 \frac{\partial S_{2,n}}{\partial \varepsilon} + \sigma^2 \frac{\partial^2 S_{2,n}}{\partial x^2} \right> = 0, \quad \left< v^2 \frac{\partial^2 S_{2,n}}{\partial \varepsilon^2} \right> = -\sigma^2 \omega^2, \quad \tag{A.1}
\]

where \( S_{2,n} = \text{Ln}f_{2,n} \). Rewriting condition (7) in the form
\[
P_{\mu,k} = -\alpha^2 f_{1,n} \frac{d}{d\varepsilon} f_{1,n}, \quad S_{1,n} = \text{Ln}f_{1,n},
\]
we obtain
\[
\frac{1}{f_{1,n}} \int_{-\infty}^{\infty} v^2 f_{2,n}(x) dv = -\alpha^2 \frac{\partial^2 S_{1,n}}{\partial \varepsilon^2}. \tag{A.2}
\]

Substitute the distribution functions (14) into expressions (A.1) and (A.2). Let us start with expression (A.1).
\[
S_{2,n} = \text{Ln}f_{n}(\varepsilon) = \text{Ln}B_n - \varepsilon + \text{Ln}L_n(2\varepsilon), \quad \frac{\partial S_{2,n}}{\partial \varepsilon} = -1 + 2 \frac{L'_n(2\varepsilon)}{L_n(2\varepsilon)}, \tag{A.3}
\]
where \( B_n = \frac{(-1)^n}{2\pi \sigma_n^2} \). Averaging expression (A.3) using (A.1), we obtain
\[
\sigma^2 \omega^2 = \frac{B_n}{f_{1,n}} \int_{-\infty}^{\infty} v^2 e^{-\varepsilon^2} \left( L_n(2\varepsilon) + 2L^{(1)}_{n-1}(2\varepsilon) \right) dv. \tag{A.4}
\]
where we take into account that \( L'_n = L'_{n-1} - L_{n-1} \),
\[
\sum_{\mu=0}^{(\mu+1)} \frac{L^{(\mu)}_n}{L'_{n-1}} \text{Considering the expression}
\]
\[
L^{(\mu)}(x) = L^{(\mu+1)}(x) - L^{(\mu+1)}(x) \text{ which at } \mu = 0 \text{ will be}
\]
\[
L_n(x) = L^{(1)}_n(x) - L^{(1)}_{n-1}(x), \text{ expression (A.4) will take the form:}
\]
\[
\sigma^2 \omega^2 = \frac{B_n}{f_{1,n}} \int_{-\infty}^{\infty} v^2 e^{-\varepsilon^2} \left( L^{(1)}_n(2\varepsilon) + L^{(1)}_{n-1}(2\varepsilon) \right) dv \tag{A.5}
\]

The generalized Laguerre polynomials satisfy the relations

\[
J_k = (-1)^k \frac{\sqrt{\pi}}{2^k k!} H^2_k(0) + (-1)^k \frac{\sqrt{\pi}}{2^k (k-1)!} H^2_{k-1}(0) + 2J_{k-2} - 2 \int_{-\infty}^{\infty} \tau^2 e^{-\tau^2} L'_{k-2}(2\tau^2) d\tau \tag{A.11}
\]

Then, let us carry out a similar substitution procedure for the integrals \( J_{k-2} \) and \( J_{k-3} \):

\[
J_k = (-1)^k \frac{\sqrt{\pi}}{2^k k!} H^2_k(0) + (-1)^k \frac{\sqrt{\pi}}{2^k (k-1)!} H^2_{k-1}(0) + \frac{\sqrt{\pi}}{2^{k-2} (k-2)!} H^2_{k-2}(0) - \int_{-\infty}^{\infty} \tau^2 e^{-\tau^2} L'_{k-3}(2\tau^2) d\tau. \tag{A.12}
\]
\[ J_k = (-1)^k \frac{\sqrt{\pi}}{2^{k+1}k!} H_k^2(0) + (-1)^k \frac{\sqrt{\pi}}{2^{k-1}(k-1)!} H_{k-1}^2(0) + \sum_{s=1}^{\infty} \frac{(-1)^s}{2^{s-1}(k-s)!} H_{k-s}^2(0) + \sum_{s=1}^{\infty} 2J_{k-s} - 2 \int_{-\infty}^{+\infty} t^2 e^{-t^2} L_{k-s}(2t^2) dt. \] (A.13)

The expression for \( J_0 \) has the form

\[ J_0 = \int_{-\infty}^{+\infty} t^2 e^{-t^2} J_0(2t^2) dt = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sqrt{2\pi} \sqrt{2} \frac{\sqrt{2}}{2}. \] (A.14)

Let us consider the even \( (k = 2m) \) and odd \( (k = 2m + 1) \) values for the expression \( J_k \) Proceeding with the iterative procedure expression (A.13) for \( J_{2m} \) takes the form:

\[ J_{2m} = \frac{\sqrt{\pi}}{2^{2m+1}(2m)!} H_{2m}^2(0) + \sqrt{\pi} \sum_{s=1}^{2m} \frac{H_{2m-s}^2(0)}{2^{2m-s}(2m-s)!}. \] (A.15)

Similarly, for \( J_{2m+1} \) we obtain

\[ J_{2m+1} = \frac{\sqrt{\pi}}{2^{2m+2}(2m+1)!} H_{2m+1}^2(0) - \sqrt{\pi} \sum_{s=1}^{2m+1} \frac{H_{2m+1-s}^2(0)}{2^{2m+1-s}(2m+1-s)!}. \] (A.16)

Comparing (A.15) and (A.16), we obtain a general expression for \( J_k \)

\[ J_k = (-1)^k \sqrt{\pi} \left\{ \frac{1}{2^{k+1}k!} H_k^2(0) + \sum_{s=1}^{k} \frac{H_{k-s}^2(0)}{2^{k-s}(k-s)!} \right\}. \] (A.17)

Expression (A.19) can be rewritten in a compact form using the Heaviside function

\[ \eta(s) = \begin{cases} 0, & s = 0, \\ 1, & s > 0. \end{cases} \] (A.20)

Using (A.20), expression (A.19) takes the following form:

\[ \frac{(-1)^n}{2^{n+1}n!} H_n^2 \left( \frac{x}{\sqrt{2\sigma}} \right) = \sum_{k=0}^{n} \tilde{C}_k L_{n-k} \left( \frac{x^2}{\sigma^2} \right), \] (A.21)

where

\[ \tilde{C}_k = (-1)^k \frac{1 + \eta(s) H_{k-s}^2(0) - 2(k-s)H_{k-s-1}^2(0)}{2^{k-s}(k-s)!}. \]

Now let us consider expression (A.2). First of all, we calculate the expression \( \frac{\partial^2 S_1}{\partial x^2} \) in (A.2)

\[ \frac{\partial^2 S_1}{\partial x^2} = -\frac{1}{\sigma^2} \left( 1 - \frac{H_n''}{H_n} + \left( \frac{H_n'}{H_n} \right)^2 \right). \] (A.22)

And find \( \langle \psi^2 \rangle \)

\[ \langle \psi^2 \rangle \]

To transform expression (A.7), let us calculate the sum \( J_k + J_{k-1} \) using (A.17)

\[ J_k + J_{k-1} = (-1)^k \sqrt{\pi} \left\{ \frac{H_k^2(0) - 2kH_{k-1}^2(0)}{2^{k+1}k!} + \sum_{s=1}^{k-1} \frac{H_{k-s}^2(0) - 2(k-s)H_{k-s-1}^2(0)}{2^{k-s}(k-s)!} \right\}. \] (A.18)

Substituting (A.18) into (A.7), we get

\[ \frac{(-1)^n}{2^{n+1}n!} H_n^2 \left( \frac{x}{\sqrt{2\sigma}} \right) = \frac{1}{2} L_n \left( \frac{x^2}{\sigma^2} \right) + \sum_{k=1}^{n} \frac{(-1)^k}{2^{k+1}k!} \frac{H_k^2(0) - 2kH_{k-1}^2(0)}{2^{k+1}k!} + \sum_{s=1}^{k} \frac{H_{k-s}^2(0) - 2(k-s)H_{k-s-1}^2(0)}{2^{k-s}(k-s)!}. \] (A.19)
where relations (1.4) are taken into account. Substituting (C.1) and (C.3) into (6), we get

\[
Q_n(x) = -\frac{\hbar^2}{2m} \frac{\sqrt{f_{1,n}(x)\cdot\text{sgn}H_n}}{\sqrt{f_{1,n}}} [y^2 - (2n + 1)]
\]

where relations (1.4) are taken into account.

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**References**

1. Wigner, E.P.: On the quantum correction for thermodynamic equilibrium. Phys. Rev. 40, 749–759 (1932)
2. Smitey, D.T., Beck, M., Raymer, M.G., Faridani, A.: Measurement of the Wigner distribution and the density matrix of a light mode using optical homodyne tomography: application to squeezed states and the vacuum. Phys. Rev. Lett. 70, 1244–1247 (1993)
3. Casado, A., Guerra, S., Plácido, J.: Wigner representation for experiments on quantum cryptography using two-photon polarization entanglement produced in parametric down-conversion. J. Phys. B. Mol. Opt. Phys. 41, 045501 (2008)
4. Andersen, U., Neergaard-Nielsen, J., van Loock, P., et al.: Hybrid discrete- and continuous-variable quantum information. Nature Phys. 11, 713–719 (2015)
5. Claassen, T.A.C.M., Mecklenbräuker, W.F.G.: The Wigner distribution—a tool for time-frequency signal analysis II: discrete-time Hermite polynomials $H_n(y) = 2nH_{n-1}(y)$ and the orthogonality condition, we obtain the following expression:

\[
\left\langle x^2 \right\rangle_n = \frac{a^2}{\sigma_x^2} + \frac{a^2}{\sigma_y^2} \frac{1}{2n!} \frac{\sqrt{2\pi\sigma_y}}{\sqrt{2\pi\sigma_x}} e^{-\frac{1}{2}(2n+1)}
\]

where relation (1.4) is taken into account. By virtue of the symmetry of expressions (1.6) and (1.8), we can rewrite the expression \(\left\langle x^2 \right\rangle_n\) in a similar way

\[
\left\langle x^2 \right\rangle_n = \sigma_x^2(2n + 1).
\]

**Appendix B**

Let us calculate

\[
\left\langle x^2 \right\rangle_n = \int_{-\infty}^{\infty} f_{1,n}(x)[x^2]_n dx = \frac{a^2}{\sigma_x^2} - \frac{a^2}{\sigma_y^2} + \frac{a^2}{\sigma_y^2} \frac{1}{2n!} \frac{\sqrt{2\pi\sigma_y}}{\sqrt{2\pi\sigma_x}} \left[ e^{-\frac{1}{2}(2n+1)} \left( \frac{\sqrt{2\pi\sigma_x}}{\sqrt{2\pi\sigma_y}} \right)^{2n+1} \right] dx + \frac{a^2}{\sigma_y^2} \frac{1}{2n!} \frac{\sqrt{2\pi\sigma_x}}{\sqrt{2\pi\sigma_y}} \left[ e^{-\frac{1}{2}y^2} H_n(y) H_n(y) dy \right]
\]

and taking into account the differentiation formula for the Hermite polynomials $H_n(y) = 2nH_{n-1}(y)$ and the orthogonality condition, we obtain the following expression:

\[
\left\langle x^2 \right\rangle_n = \frac{a^2}{\sigma_x^2} + \frac{a^2}{\sigma_y^2} \frac{1}{2n!} \frac{\sqrt{2\pi\sigma_x}}{\sqrt{2\pi\sigma_y}} \left[ e^{-\frac{1}{2}(2n+1)} \left( \frac{\sqrt{2\pi\sigma_y}}{\sqrt{2\pi\sigma_x}} \right)^{2n+1} \right] dx + \frac{a^2}{\sigma_y^2} \frac{1}{2n!} \frac{\sqrt{2\pi\sigma_x}}{\sqrt{2\pi\sigma_y}} \left[ e^{-\frac{1}{2}y^2} H_n(y) H_n(y) dy \right]
\]

\[
\left\langle x^2 \right\rangle_n = \sigma_x^2(2n + 1).
\]

**Appendix C**

Let us derive the quantum potential (6) for the function $f_{1,n}$ (1.5):

\[
\sqrt{f_{1,n}} = \sqrt{c_n} e^{-\frac{1}{2}y^2} H_n(y) \text{sgn}H_n,
\]

\[
\left(\sqrt{f_{1,n}}\right)_{xx} = \frac{\sqrt{c_n}}{\sigma^2} e^{-\frac{1}{2}y^2} \text{sgn}H_n \left[ 2n(n-1)H_{n-2} - 2nyH_{n-1} + \frac{1}{2}H_n \right].
\]

where $c_n^{-1} = 2n! \frac{\sqrt{2\pi\sigma_y}}{\sqrt{2\pi\sigma_x}}$. Taking recurrent relationship $H_{n+2} - 2yH_{n+1} + (2n+1)H_n = 0$ into account, expression (C.2) will take the following form:

\[
\left(\sqrt{f_{1,n}}\right)_{xx} = \frac{\sqrt{c_n}}{\sigma^2} e^{-\frac{1}{2}y^2} H_n \text{sgn}H_n \left[ y^2 - (2n + 1) \right]
\]

\[
Q_n(x) = -\frac{\hbar^2}{2m} \frac{\sqrt{f_{1,n}\cdot\text{sgn}H_n}}{\sqrt{f_{1,n}}} \left[ y^2 - (2n + 1) \right] = -\frac{m\omega^2x^2}{2} + \hbar\alpha \left( n + \frac{1}{2} \right),
\]

where relations (1.4) are taken into account.
12. Glauber, R.J.: Photon correlations. Phys. Rev. Lett. 10, 84–86 (1963)
13. Sudarshan, E.C.G.: Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams. Phys. Rev. Lett. 10, 277–279 (1963)
14. Perepelkin, E.E., Sadovnikov, B.I., Inozemtseva, N.G.: The quantum mechanics of high-order kinematic values. Ann Phys. 401, 59–90 (2019)
15. Perepelkin, E.E., Sadovnikov, B.I., Inozemtseva, N.G.: Paradigm of infinite dimensional phase space Understanding the Schrödinger Equation: Some Non-Linear Perspectives United States, p. 330. Nova Science Publishers (2020)
16. Perepelkin, E.E., Sadovnikov, B.I., Inozemtseva, N.G., Burlakov, E.V.: Wigner function of a quantum system with polynomial potential. J. Statist. Mech. Theory Exp. 2020 053105 (2020)
17. Vlasov, A.A.: Many-Particle Theory and Its Application to Plasma. Gordon and Breach, New York (1961)
18. Sadovnikov, B.I., Inozemtseva, N.G., Perepelkin, E.E.: Generalized phase space and conservative systems. Dokl. Math. 88(1), 457–459 (2013)
19. Perepelkin, E.E., Sadovnikov, B.I., Inozemtseva, N.G.: The properties of the first equation of the Vlasov chain of equations. J. Statist. Mech. Theory Exp. 2015 P05019 (2015)
20. Bohm, D.: A suggested interpretation of the quantum theory in terms of “hidden” variables I and II. Phys. Rev. 85, 166–193 (1952)
21. Bohm, D., Hiley, B.J., Kaloyerou, P.N.: An ontological basis for the quantum theory. Phys. Rep. 144, 321–375 (1987)
22. Bohm, D., Hiley, B.J.: The Undivided Universe: An Ontological Interpretation of Quantum Theory. Routledge, London (1993)
23. de Broglie, L.: Une interpretation causale et non lineaire de la mecanique ondulatoire: la theorie de ladouble solution. Gauthiers-Villars, Paris (1956)
24. Moyal E. Quantum mechanics as a statistical theory Proceedings of the Cambridge Philosophical Society,1949. 45. 99–124
25. Perepelkin, E.E., Sadovnikov, B.I., Inozemtseva, N.G., Burlakov, E.V.: Explicit form for the kernel operator matrix elements in eigenfunction basis of harmonic oscillator. J. Statist. Mech. Theory Exp. 2020 023109 (2020)
26. Borwein, D., Borwein, J.M., Crandall, R.E.: Effective Laguerre asymptotics. SIAM J. Numer. Anal. 46(6), 3285–3312 (2008). https://doi.org/10.1137/07068031X

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