Generalized Bézier Curves Based on Bernstein-Stancu-Chlodowsky Type Operators

Kejal Khatri∗ and Vishnu Narayan Mishra

ABSTRACT: In this paper, we use the blending functions of Bernstein-Stancu-Chlodowsky type operators with shifted knots for construction of modified Chlodowsky Bézier curves. We study the nature of degree elevation and degree reduction for Bézier Bernstein-Stancu-Chlodowsky functions with shifted knots for \( t \in [\gamma_n, n\gamma_n + \delta] \).

We also present a de Casteljau algorithm to compute Bernstein Bézier curves with shifted knots. The new curves have some properties similar to Bézier curves. Furthermore, some fundamental properties for Bernstein Bézier curves are discussed. Our generalizations show more flexibility in taking the value of \( \gamma \) and \( \delta \) and advantage in shape control of curves. The shape parameters give more convenience for the curve modelling.

Key Words: Bernstein-Stancu-Chlodowsky type operators, Bézier curves, Degree elevation, de Casteljau algorithm, Shape parameters.

Contents

1 Introduction

2 Properties of the Bernstein-Stancu-Chlodowsky functions

2.1 Theorem

2.2 Theorem

2.3 Theorem

3 Bernstein-Stancu-Chlodowsky Bézier curves

3.1 Theorem

4 Shape control of Bernstein-Stancu-Chlodowsky Bézier curves

5 Future work

1. Introduction

Bézier curves were developed by Casteljau [4] and Bézier [3], and have been applied to many computer-aided design (CAD) applications. While their origin can be traced back to the design of car body shapes. A Bézier curve is defined in terms of a set of control points, though it only considers global information i.e. it does not consider local information and calculates the curve points in a linear recursive approach starting with the edges of the control polygon. Frequently, there is a large gap between the Bézier curve and its control polygon, which restricts the maximum length of a curve segment. While strategies such as degree elevation, composite Bézier curves, refinement and subdivision reduce this gap, they also increase the number of control points. A higher-degree Bézier curve obviously provides a better shape representation.

In this problem, we generalize some of the very well-known Bézier curve techniques by using a generalization of the Bernstein basis, called the Bernstein-Stancu-Chlodowsky basis. S.N. Bernstein [2] in 1912, who first introduced his famous operators \( B_n : C[0,1] \rightarrow C[0,1] \) defined for any \( n \in \mathbb{N} \) and for any function \( f \in C[0,1] \)

\[
B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f \left( \frac{k}{n} \right), \quad 0 \leq x \leq 1. \tag{1.1}
\]

∗ Corresponding author

2010 Mathematics Subject Classification: Primary: 65D17; Secondary: 41A10, 41A25, 41A36.
Submitted January 29, 2020. Published April 13, 2020

Typeset by \$T_{\LaTeX}\$ style.
© Soc. Paran. de Mat.
and named it Bernstein polynomials to prove the Weierstrass theorem [16] and Bernstein polynomials possess many remarkable properties and has various applications in many areas such as approximation theory, numerical analysis, computer-aided geometric design, and solutions of differential equations due to its fine properties of approximation.

In computer aided geometric design (CAGD), Bernstein polynomials and its variants are used in order to preserve the shape of the curves or surfaces. One of the most important curve in CAGD [29] is the classical Bézier curve [3] constructed with the help of Bernstein basis functions. Other works related to different generalization of Bernstein polynomials and Bézier curves and surfaces can be found in [6-7, 8, 13-14, 17-18, 22-28].

Gadjiev and Gorhanalizadeh [9] introduced the following construction of Bernstein- Stancu type polynomials with shifted knots:

\[ S_{n,\gamma,\delta}(f; x) = \left( \frac{n + \delta_2}{n} \right) \sum_{k=0}^{n} \binom{n}{k} \left( x - \frac{\gamma_2}{n + \delta_2} \right)^k \left( \frac{n + \gamma_2}{n + \delta_2} - x \right)^{n-k} f \left( \frac{k + \gamma_1}{n + \delta_1} \right) \]  

(1.2)

where \( \frac{\gamma_2}{n + \delta_2} \leq x \leq \frac{n + \gamma_2}{n + \delta_2} \) and \( \gamma_k, \delta_k (k = 1, 2) \) are positive real numbers provided \( 0 \leq \gamma_1 \leq \gamma_2 \leq \delta_1 \leq \delta_2 \). It is clear that for \( \gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 0 \), then these operators reduces to the classical Bernstein operators. The classical Bernstein-Chlodowsky polynomials have the following form

\[ C_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{x}{b_n} \right)^k \left( 1 - \frac{x}{b_n} \right)^{n-k} f \left( \frac{k}{b_n} \right), \]  

(1.3)

where \( 0 \leq x \leq b_n \) and \( \{ b_n \}_{n \geq 1} \) is a positive increasing sequence with the properties

\[ \lim_{n \to \infty} b_n = \infty, \quad \lim_{n \to \infty} \frac{b_n}{n} = 0. \]

These polynomials were introduced by Chlodowsky [5] as a generalization of Bernstein polynomials on an unbounded set. Aral et al. [1] defined Bernstein-Stancu- Chlodowsky polynomials which are generalization of \( S_{n,\gamma,\delta}(f; x) \) as:

\[ S_{n,\gamma,\delta}(f; x) = \left( \frac{n + \delta_2}{n} \right) \sum_{k=0}^{n} \binom{n}{k} \left( \frac{x}{b_n} - \frac{\gamma_2}{n + \delta_2} \right)^k \left( \frac{n + \gamma_2}{n + \delta_2} - \frac{x}{b_n} \right)^{n-k} f \left( \frac{k + \gamma_1}{n + \delta_1} \right), \]  

(1.4)

where \( \frac{\gamma_2}{n + \delta_2} b_n \leq x \leq \frac{n + \gamma_2}{n + \delta_2} b_n, \gamma_k, \delta_k (k = 1, 2) \) are positive real numbers provided \( 0 \leq \gamma_1 \leq \gamma_2 \leq \delta_1 \leq \delta_2 \) and \( \{ b_n \}_{n \geq 1} \) is a positive increasing sequence such that

\[ \lim_{n \to \infty} b_n = \infty, \quad \lim_{n \to \infty} \frac{b_n}{n} = 0. \]

1. **Case 1.** Take \( b_n = 1 \), then (1.4) reduces to (1.2).

2. **Case 2.** Take \( \gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 0 \), then (1.4) gives (1.3).

3. **Case 3.** Combined Case 1 and Case 2, we get classical Bernstein operators (1.1).

In recent years, generalization of the Bézier curve with shape parameters has received continuous attention. Several authors were concerned with the problem of changing the shape of curves and surfaces, while keeping the control polygon unchanged and thus they generalized the Bézier curves in [12, 13-14, 25].

Recently, Mishra, et al. [20] studied on inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators and various generalization of Szász - Mirakjan operators have been studied by Mishra et al. [21], Gandhi et al. [11] and Mishra and Gandhi [19] and Gairola et al. [10] have discussed approximation properties of linear positive operators.

In 2017, Khatri and Mishra [15] introduced Generalized Szász-Mirakyan operators involving Brenke type...
2. Properties of the Bernstein-Stancu-Chlodowsky functions

The Bernstein-Stancu-Chlodowsky functions are introduced as

\[
H_{n,\gamma,\delta}^k(s) = \binom{n}{k} \left(\frac{n+\delta}{n}\right)^n \left(\frac{s - \gamma}{b_n - n + \delta}\right)^k \left(n + \gamma - s\right)^{n-k}, \quad (2.1)
\]

where \(\frac{\gamma}{n+\delta}b_n \leq s \leq \frac{n+\gamma}{n+\delta}b_n\) and \(\gamma, \delta\) are positive real numbers provided \(0 \leq \gamma \leq \delta\).

2.1. Theorem

The Bernstein-Stancu-Chlodowsky functions possess the following properties:

1. Non-negativity:
\[
H_{n,\gamma,\delta}^k(s) \geq 0, \quad k = 0, 1, \ldots, n, \quad \frac{\gamma}{n+\delta}b_n \leq s \leq \frac{n+\gamma}{n+\delta}b_n.
\]

2. Partition of unity:
\[
\sum_{k=0}^{n} H_{n,\gamma,\delta}^k(s) = 1, \quad \frac{\gamma}{n+\delta}b_n \leq s \leq \frac{n+\gamma}{n+\delta}b_n.
\]

3. End-point property:
\[
H_{n,\gamma,\delta}^k\left(\frac{\gamma}{n+\delta}b_n\right) = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0, \end{cases}
\]
\[
H_{n,\gamma,\delta}^k\left(\frac{n+\gamma}{n+\delta}b_n\right) = \begin{cases} 1, & \text{if } k = n, \\ 0, & \text{if } k \neq n, \end{cases}
\]

Clearly both side end point property holds.

4. Reducibility: when \(\gamma = \delta = 0\), \(b_n = 1\) formula (2.1) reduces to the classical Bernstein bases on \([0, 1]\).

Proof: All these property can be proved easily from equation (2.1). Fig. 1 represents the Bernstein-Stancu-Chlodowsky functions of degree 3 for \(\gamma = 6, \delta = 8\) and \(b_n = (n)^{1/3}\). Here, we can see that sum of blending functions is always unity and also satisfies end-point interpolation property. If \(\gamma = \delta = 0, b_n = 1\) it gives classical Bernstein basis on \([0, 1]\) which is presented in Fig. 2.

Apart from the basic properties above, Bernstein-Stancu-Chlodowsky functions also satisfy the following recurrence relations, as for the classical Bernstein basis.
K. Khatri and V. N. Mishra

2.2. Theorem

Each Bernstein-Stancu-Chlodowsky functions of degree \( n \) is a linear combination of two Bernstein-Stancu-Chlodowsky functions of degree \( n + 1 \).

\[
H_{n,\gamma,\delta}^k(s) = \left( \frac{n + 1 - k}{n + 1} \right) H_{n+1,\gamma,\delta}^k(s) + \left( \frac{k + 1}{n + 1} \right) H_{n+1,\gamma,\delta}^{k+1}(s),
\]

where \( \frac{\gamma}{n+1} b_n \leq s \leq \frac{\gamma+\delta}{n+1} b_n \) and \( \gamma, \delta \) are positive real numbers \( 0 \leq \gamma \leq \delta \).

Proof:

\[
\left( \frac{n}{n + \delta} \right) H_{n,\gamma,\delta}^k = H_{n+1,\gamma,\delta}^k \left( \frac{n + \gamma}{n + \delta} - \frac{s}{b_n} \right) + \left( \frac{s}{b_n} - \frac{\gamma}{n + \delta} \right) H_{n+1,\gamma,\delta}^k(s)
\]

\[
= I_1 + I_2,
\]

where

\[
I_1 = \left( \frac{n + \gamma - s}{n + \delta} \right) \left( \frac{n + \delta}{n} \right)^n \left( \frac{s}{b_n} - \frac{\gamma}{n + \delta} \right)^k \left( \frac{n + \gamma - s}{n + \delta} \right)^{n-k}
\]

and

\[
I_2 = \left( \frac{n}{n + \delta} \right) \left( \frac{n + 1 - k}{n + 1} \right) H_{n+1,\gamma,\delta}^k(s).
\]
Similarly, 
\begin{align*}
I_2 &= \left( \frac{s}{b_n} - \frac{\gamma}{n + \delta} \right) \left( \frac{n}{k} \right)^n \left( \frac{s}{b_n} - \frac{\gamma}{n + \delta} \right)^k \left( \frac{n + \gamma - s}{b_n} \right)^{n-k} \\
&= \left( \frac{n}{n + \delta} \right)^{(k+1)} H^{k+1}_{n+1, \gamma, \delta}(s).
\end{align*}

By putting the values of $I_1$ and $I_2$ in (2.3), we get required result (2.2).

2.3. Theorem

Each Bernstein-Stancu-Chlodowsky functions of degree $n$ is a linear combination of two Bernstein-Stancu-Chlodowsky functions of degree $n - 1$.

\[ H^k_{n, \gamma, \delta}(s) = \left( \frac{s}{b_n} - \frac{\gamma}{n + \delta} \right) \left( \frac{n + \delta}{n} \right) H^{k-1}_{n-1, \gamma, \delta}(s) + \left( \frac{n + \gamma - s}{b_n} \right) H^k_{n-1, \gamma, \delta}(s), \tag{2.4} \]

where \( \frac{\gamma}{n + \delta} b_n \leq s \leq \frac{n + \gamma}{n + \delta} b_n \) and \( \gamma, \delta \) are positive real numbers \( 0 \leq \gamma \leq \delta \).

**Proof:** We use the Pascal-type relations, we have

\[ H^k_{n, \gamma, \delta}(s) = \left( \frac{n}{k} \right)^n \left( \frac{s}{b_n} - \frac{\gamma}{n + \delta} \right)^k \left( \frac{n + \gamma - s}{b_n} \right)^{n-k} \]

\[ = \left\{ \left( \frac{n-1}{k-1} \right) + \left( \frac{n-1}{k} \right) \right\} \left( \frac{n + \delta}{n} \right)^n \left( \frac{s}{b_n} - \frac{\gamma}{n + \delta} \right)^k \left( \frac{n + \gamma - s}{b_n} \right)^{n-k} \]

\[ + \left( \frac{n-1}{k} \right) \left( \frac{n + \delta}{n} \right)^n \left( \frac{s}{b_n} - \frac{\gamma}{n + \delta} \right)^k \left( \frac{n + \gamma - s}{b_n} \right)^{n-k} \]

\[ = \left( \frac{s}{b_n} - \frac{\gamma}{n + \delta} \right) \left( \frac{n + \delta}{n} \right) H^{k-1}_{n-1, \gamma, \delta}(s) + \left( \frac{n + \gamma - s}{b_n} \right) H^k_{n-1, \gamma, \delta}(s). \]

3. Bernstein-Stancu-Chlodowsky Bézier curves

We define the Bernstein-Stancu-Chlodowsky Bézier curves of degree \( n \) using the Bernstein-Stancu-Chlodowsky functions as the following:

\[ R(s) = \sum_{k=0}^n R_k H^k_{n, \gamma, \delta}(s). \tag{3.1} \]

where \( R_k, R_j(j = 0, 1, \ldots, n) \). \( R_k \) are control points. Joining up adjacent points \( R_k, k = 0, 1, 2, \ldots, n \) to obtain a polygon which is called the control polygon of Bernstein-Stancu-Chlodowsky Bézier curves.

3.1. Theorem

The end-point property of derivative:

\[ R'(\frac{\gamma}{n + \delta} b_n) = \left( \frac{n + \delta}{b_n} \right) (R_1 - R_0) \left( \frac{n-1 + \delta}{n-1} \right)^{n-1} \left( \frac{n-1 + \gamma - \gamma}{n-1 + \delta - \gamma} \right)^{n-1-k} \tag{3.2} \]

\[ R'(\frac{n + \gamma}{n + \delta} b_n) = \left( \frac{n + \delta}{b_n} \right) (R_n - R_{n-1}) \left( \frac{n-1 + \delta}{n-1} \right)^{n-1} \left( \frac{n+\gamma - \gamma}{n+\delta - \gamma} \right)^{n-1} \tag{3.3} \]
i.e. Bernstein-Stancu-Chlodowsky Bézier curves are tangent to fore-and-aft edges of its control polygon at end points.

**Proof:** Let

\[
R(s) = \sum_{k=0}^{n} R_k H_{n,\gamma,\delta}^k(s)
\]

\[
= \sum_{k=0}^{n} R_k \left( \binom{n}{k} \left( \frac{n+\delta}{n} \right)^n \left( \frac{s}{b_n} - \frac{\gamma}{n+\delta} \right)^k \left( \frac{n+\gamma}{n+\delta} - \frac{s}{b_n} \right)^{n-k} \right)
\]

or

\[
R(s) = U(s).
\]

Now, on differentiating both side with respect to \(s\), we get

\[
R'(s) = U'(s).
\]

\[
B_k^n(s) = \left( \binom{n}{k} \left( \frac{n+\delta}{n} \right)^n \left( \frac{s}{b_n} - \frac{\gamma}{n+\delta} \right)^k \left( \frac{n+\gamma}{n+\delta} - \frac{s}{b_n} \right)^{n-k} \right),
\]

then

\[
U(s) = \sum_{k=0}^{n} R_k B_k^n(s).
\]

\[
(B_k^n(s))' = \left( \binom{n}{k} \left( \frac{n+\delta}{n} \right)^n \left( \frac{s}{b_n} - \frac{\gamma}{n+\delta} \right)^k \left( \frac{n+\gamma}{n+\delta} - \frac{s}{b_n} \right)^{n-k} \right)
\]

\[
- \left( \binom{n}{k} \left( \frac{n+\delta}{n} \right)^n \left( \frac{s}{b_n} - \frac{\gamma}{n+\delta} \right)^{k-1} \left( \frac{n+\gamma}{n+\delta} - \frac{s}{b_n} \right)^{n-k-1} \right)
\]

\[
= \left( \frac{n+\delta}{b_n} \right) [B_{k-1}^{n-1}(s) + B_k^{n-1}(s)],
\]

then

\[
U'(s) = \sum_{k=0}^{n} R_k (B_k^n(s))'.
\]

Now

\[
U'\left( \frac{\gamma}{n+\delta} b_n \right) = R'\left( \frac{\gamma}{n+\delta} b_n \right) = \left( \frac{n+\delta}{b_n} \right) (R_1 - R_0) B_0^{n-1} \left( \frac{\gamma}{n+\delta} b_n \right)
\]

and

\[
R'\left( \frac{\gamma}{n+\delta} b_n \right) = \left( \frac{n+\delta}{b_n} \right) (R_1 - R_0) \left( \frac{n-1+\delta}{n-1} \right)^{n-1} \left( \frac{n-1+\gamma}{n-1+\delta} - \frac{\gamma}{n+\delta} \right)^{n-k-1}.
\]

Similarly, we get

\[
U'\left( \frac{n+\gamma}{n+\delta} b_n \right) = R'\left( \frac{n+\gamma}{n+\delta} b_n \right) = \left( \frac{n+\delta}{b_n} \right) (R_n - R_{n-1}) B_0^{n-1} \left( \frac{n+\gamma}{n+\delta} b_n \right)
\]

and

\[
R'\left( \frac{n+\gamma}{n+\delta} b_n \right) = \left( \frac{n+\delta}{b_n} \right) (R_n - R_{n-1}) \left( \frac{n-1+\delta}{n-1} \right)^{n-1} \left( \frac{n+\gamma}{n+\delta} - \frac{\gamma}{n-1+\delta} \right)^n.
\]
Degree elevation and de Casteljau algorithm

Degree elevation

Bernstein-Stancu-Chlodowsky Bézier curves have a degree elevation algorithm that is similar to that possessed by the classical Bézier curves. To increase the flexibility of a given curve, we use the technique of degree elevation. A degree elevation algorithm calculates a new set of control points by taking a convex combination of the old set of control points which retains the old end points.

\[ R(s) = \sum_{k=0}^{n} R_k H_{n,\gamma,\delta}^k(s) \]

and

\[ R(s) = \sum_{k=0}^{n+1} R^*_k H_{n+1,\gamma,\delta}^k(s). \]

where

\[ R^*_k = \left( \frac{k}{n+1} \right) R_{k-1} + \left( 1 - \frac{k}{n+1} \right) R_k, \quad k = 0, 1, \ldots, n+1, \quad R_{-1} = R_{n+1} = 0. \] (3.4)

The statement above can be derived from Theorem (2.2). When \( \gamma = \delta = 0 \) and \( b_n = 1 \) formula (3.4) reduces to the degree evaluation formula of the classical Bézier curves. If we let \( R = (R_0, R_1, \ldots, R_n)^T \) denote the vector of control points of the initial Bernstein-Stancu-Chlodowsky Bézier curves of degree \( n \), and \( R^{(1)} = (R_0^*, R_1^*, \ldots, R_{n+1}^*) \) the vector of control points of the degree elevated Bernstein-Stancu-Chlodowsky Bézier curves of degree \( n+1 \), then we can represent the degree elevation procedure as:

\[ R^{(1)} = T_{n+1} R, \] (3.5)

where

\[ T_{n+1} = \frac{1}{n+1} \begin{bmatrix} n+1 & 0 & \cdots & 0 & 0 \\ n+1-n & n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & n+1-2 & 2 & 0 \\ 0 & 0 & \cdots & n+1-1 & 1 \\ 0 & 0 & \cdots & 0 & n+1 \end{bmatrix} \]

For any \( l \in \mathbb{N} \), the vector of control points of the degree elevated Bernstein-Stancu-Chlodowsky Bézier curves of degree \( n + l \) is:

\[ R^{(1)} = T_{n+1} \cdots T_{n+l} T_{n+1} R. \] (3.6)

As \( l \to 0 \), the control polygon \( R^{(1)} \) converges to a Bernstein-Stancu-Chlodowsky Bézier curves.

De Casteljau algorithm

Bernstein-Stancu-Chlodowsky Bézier curves of degree \( n \) can be written as two kinds of linear combination of two Bernstein-Stancu-Chlodowsky Bézier curves of degree \( n-1 \), and we can get the two selectable algorithms to evaluate Bernstein-Stancu-Chlodowsky Bézier curves. The algorithms can be expressed as:

Algorithm 1.

\[
\begin{align*}
R^0_k(s) & = R^0_k, \quad k = 0, 1, 2, \ldots, n, \\
R^r_k(s) & = \frac{n+\gamma}{n} \left( \frac{s}{\gamma} \right) R^r_{k+1}(s) + \frac{n+\delta}{n} \left( \frac{n+\gamma}{n+\delta} - \frac{s}{\gamma} \right) R^r_{k-1}(s), \\
r & = 1, \ldots, n, \quad k = 0, 1, 2, \ldots, n-r, \quad \frac{\gamma}{n+\delta} b_n \leq s \leq \frac{n+\gamma}{n+\delta} b_n, \quad 0 \leq \gamma \leq \delta.
\end{align*}
\] (3.7)

\[ R(s) = \sum_{k=0}^{n-1} R^1_k(s) = \sum_{r=0}^{n-1} R^r_0(s) H_{n-r,\gamma,\delta}^k(s) = \sum_{r=0}^{n-1} R^r_0(s). \] (3.8)
It is clear that the results can be obtained from Theorem (2.3). When \( \gamma = \delta = 0 \) and \( b_n = 1 \), formula (3.7) and (3.8) recover the de Casteljau algorithms of classical Bézier curves. Let \( R^0 = (R_0, R_1, ..., R_n)^T \), \( R^r = (R^r_0, R^r_1, ..., R^r_{n-r})^T \), then de Casteljau algorithm can be expressed as:

Algorithm 2.

\[
R^r(s) = M_r(s)...M_2(s)M_1(s)R^0, \quad (3.9)
\]

where \( M_r(s) \) is a \((n - r + 1) \times (n - r + 2)\) matrix and

\[
M_r(s) = \frac{n + \delta}{n} \begin{bmatrix}
\frac{n + \gamma}{n + \delta} & \frac{s}{b_n} & \frac{\gamma}{n + \delta} & 0 & 0 \\
0 & \frac{n + \gamma}{n + \delta} & \frac{s}{b_n} & \frac{\gamma}{n + \delta} & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \frac{n + \gamma}{n + \delta} & \frac{s}{b_n} & \frac{\gamma}{n + \delta} \\
0 & 0 & \cdots & \frac{n + \gamma}{n + \delta} & \frac{s}{b_n} & \frac{\gamma}{n + \delta}
\end{bmatrix}
\]

4. Shape control of Bernstein-Stancu-Chlodowsky Bézier curves

The Bernstein-Stancu-Chlodowsky Bézier curves is generated by setting \( \gamma = 6, \delta = 8 \) (red lines), the classical Bézier curve generated by setting \( \gamma = 0, \delta = 0, b_n = 1 \) (dashed blue lines). From fig. 3, Bernstein-Stancu-Chlodowsky Bézier curves move close to the control polygon approximately same as classical Bézier curves. Similarly, in order to construct closed, we can set \( R_n = R_0 \). The Bernstein-Stancu-Chlodowsky Bézier curves is generated by setting \( \gamma = 6, \delta = 8 \) (red line), the classical Bézier curve is generated by setting \( \gamma = 0, \delta = 0, b_n = 1 \) (dashed blue line). From fig. 4, Bernstein-Stancu-Chlodowsky Bézier curves is closer to the control polygon than classical Bézier curves.
Generalized Bézier Curves Based on Bernstein-Stancu-Chlodowsky Type Operators

5. Future work

Bernstein-Stancu-Chlodowsky Bézier curves share most properties of classical Bézier curves. Moreover, the shape of Bernstein-Stancu-Chlodowsky Bézier curves can be adjusted by altering the value of shape parameters. In the future, we will construct Bernstein-Stancu-Chlodowsky Bézier surfaces and will discuss some fundamental properties for Bernstein-Stancu-Chlodowsky Bézier surfaces, study de Casteljau algorithm and degree evaluation properties for surfaces. Similarly, we will determine q-analogue of Bernstein-Stancu-Chlodowsky Bézier curves and surfaces. We will also explain de Casteljau algorithm and degree evaluation properties for curves and surfaces. We also hope to construct generalizations of classical rational Bézier curves and surfaces based on these operators.

Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

Acknowledgments

The authors would like to express their deep gratitude to the anonymous learned referees of the journal for their valuable suggestions and constructive comments for better improvement of the manuscript. The first author Kejal Khatri acknowledges the Department of Atomic Energy, National Board Higher Mathematics, Mumbai, India for supporting this research article, DAE Ref. Number: 2/40(58)/2015/R&D-II/13262 on dated 29/09/2015.

References

1. Ali Aral, Onur ökten and Tuncer Acar, A note on bernstein-stancu-chlodowsky operators, Kirikkale University, Faculty of Science and Arts, Department of Mathematics, YahSihan, Kirikkale, Turkey, 2012.
2. S. Bernstein, Demonstration du théoréme de Weierstrass fondé sur le calcul de probabilités. Commun. Soc. Math. Kharkow, 13 (1) (1912) 1-2.
3. P.E. Bézier, Numerical Control-Mathematics and applications, John Wiley and Sons, London, 1972.
4. P. De Casteljau, Outillage Méthodes Calcul, Citroën, 1959.
5. I. Chlodowsky, Sur le développement des fonctions définies dans un interval infinie série de polynomes de S.N. Bernstein, Compositio Math. 4 (1937) 380-392.
6. Cetin Disibuyuk, and Halil Oruc, Tensor Product q-Bernstein Polynomials, BIT Numerical Mathematics, Springer 48 (2008) 689-700.
7. Cetin Disibuyuk, Tensor Product q -Bernstein Bézier Patches, Lecture Notes in Computer Science, 2009.
8. Rida T. Farouki and V.T. Rajan, Algorithms for polynomials in Bernstein form, Computer Aided Geometric Design, 5 (1) 1988.
9. A.D. Gadjiev and A.M. Ghorbanalizadeh, *Approximation properties of a new type Bernstein-Stancu polynomials of one and two variables*, Appl. Math. Comput. 216 (3) (2010) 890-901.

10. A.R. Gairola, Deepmala and L.N. Mishra, *On the q-derivatives of a certain linear positive operators*, Iranian Journal of Science & Technology, Transactions A: Science (2017). DOI 10.1007/s40995-017-0227-8.

11. R.B. Gandhi, Deepmala and V.N. Mishra, *Local and global results for modified Szász - Mirakjan operators*, Math. Method. Appl. Sci. (2016). DOI: 10.1002/mma4171.

12. Li-Wen Hana, Ying Chua and Zhi-Yu Qiu, *Generalized Bézier curves and surfaces based on Lupas q-analogue of Bernstein operator*, Journal of Computational and Applied Mathematics 261 (2014) 352-363.

13. Khalid Khan, D.K. Lobiyal and Adeem Kilicman, *A de Casteljau Algorithm for Bernstein type Polynomials based on (p,q)-integers*, arXiv 1507.04110 (2015).

14. Khalid Khan and D.K. Lobiyal, *Bézier curves based on Lupas (p,q)-analogue of Bernstein functions in CAGD*, Journal of Computational and Applied Mathematics 317 (2017) 458-477.

15. Kejal Khatri and V.N. Mishra, *Generalized Szász-Mirakyan operators involving Brenke type polynomials*, Applied Mathematics and Computation 324 (2018) 228-238. https://doi.org/10.1016/j.amc.2017.11.049.

16. P.P. Korovkin, *Linear operators and approximation theory*, Hindustan Publishing Corporation, Delhi, 1960.

17. A. Lupas, *A q-analogue of the Bernstein operator*, Seminar on Numerical and Statistical Calculus, University of Cluj-Napoca 9 (1987) 85-92.

18. N.I. Mahmudov and P. Sabancgil, *Some approximation properties of Lupas q-analogue of Bernstein operators*, arXiv:1012.4245v1 [math.FA] 20 Dec 2010.

19. V.N. Mishra and R.B. Gandhi, *Simultaneous approximation by Szász-Mirakjan-Stancu-Durrmeyer type operators*, Periodica Mathematica Hungarica 74 (1) (2017) 118–127. doi:10.1007/s10998-016-0145-0.

20. V.N. Mishra, K. Khatri, L.N. Mishra and Deepmala, *Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators*, Journal of Inequalities and Applications (2013) 2013:586. DOI:10.1007/s10998-016-0145-0.

21. V.N. Mishra, R.B. Gandhi and F. Nasaireh, *Simultaneous approximation by Szász-Mirakjan-Durrmeyer-type operators*, Bollettino dell’Unione Matematica Italiana 8 (4) (2016) 297-305.

22. M. Mursaleen, K.J. Ansari and A. Khan, *On (p,q)-analogue of Bernstein Operators*, Applied Mathematics and Computation, 266 (2015) 874-882.

23. M. Mursaleen, K.J. Ansari and Asif Khan, *Some Approximation Results by (p,q)-analogue of Bernstein-Stancu Operators*, Applied Mathematics and Computation, 264 (2015) 392-402.

24. M. Mursaleen and Asif Khan, *Generalized q-Bernstein-Schurer Operators and Some Approximation Theorems*, Journal of Function Spaces and Applications Volume 2013, Article ID 719834, 7 pages http://dx.doi.org/10.1155/2013/719834

25. Halil Oruk and George M. Phillips, *q-Bernstein polynomials and Bézier curves*, Journal of Computational and Applied Mathematics 151 (2003) 1-12.

26. Sofiya Ostrovska, *On the Lupas q-analogue of the Bernstein operator*, Rocky mountain journal of mathematics 36 (5) 2006.

27. G.M. Phillips, *Bernstein polynomials based on the q-integers*, The heritage of P.L. Chebyshev, J. Numer. Math., 4 (1997) 511-518.

28. G.M. Phillips, *A survey of results on the q-Bernstein polynomials*, IMA Journal of Numerical Analysis, 2009.

29. Thomas W. Sederberg, *Computer Aided Geometric Design Course Notes*, Department of Computer Science Brigham Young University, October 9, 2014.

Kejal Khatri,  
Govt. College Simalwara 314403,  
Dungarpur, Rajasthan, India.  
E-mail address: kejal0909@gmail.com

and

Vishnu Narayan Mishra,  
Department of Mathematics,  
Indira Gandhi National Tribal University,  
Lalpur, Amarkantak, Anuppur,  
Madhya Pradesh 484 887, India.  
E-mail address: vishnunarayann Mishra@gmail.com and vishnu_narayann Mishra@yahoo.co.in