

IWASAWA THEORY OF AUTOMORPHIC REPRESENTATIONS
OF $GL_{2n}$ AT NON-ORDINARY PRIMES

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Abstract. Let $\Pi$ be a cuspidal automorphic representation of $GL_{2n}(\mathbb{A}\mathbb{Q})$ and let $p$
be an odd prime at which $\Pi$ is unramified. In a recent work, Barrera, Dimitrov and
Williams constructed possibly unbounded $p$-adic $L$-functions interpolating complex
$L$-values of $\Pi$ in the non-ordinary case. Under certain assumptions, we construct
two bounded $p$-adic $L$-functions for $\Pi$, thus extending an earlier work of Rockwood
by relaxing the Pollack condition. Using Langlands local-global compatibility, we
define signed Selmer groups over the $p$-adic cyclotomic extension of $\mathbb{Q}$ attached
to the $p$-adic Galois representation of $\Pi$ and formulate Iwasawa main conjec-
tures in the spirit of Kobayashi’s plus and minus main conjectures for $p$-supersingular
elliptic curves.

1. Introduction

A brief history on non-ordinary Iwasawa theory of elliptic curves and
elliptic modular forms. Let $p$ be a fixed odd prime and $f = \sum_{m\geq 1} a_m q^m$
a cus-
pidal elliptic modular form. When $f$ has good ordinary reduction at $p$, Mazur and
Swinnerton-Dyer, as well as Manin, constructed a $p$-adic $L$-function attached to $f$
which is a bounded measure on $\Gamma := \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$, interpolating complex $L$-values
of $f$ twisted by Dirichlet characters on $\Gamma$ (see [MSD74, Man73]). When $f$
has good non-ordinary reduction at $p$, the situation is quite different. While the aforemen-
tioned works on the ordinary case can be extended to the non-ordinary case, the
resulting $p$-adic $L$-functions are not necessarily bounded (see [AV75, Vis76]).

In [Pol03], Pollack showed that when $a_p = 0$, there is a very elegant way
to decompose the $p$-adic $L$-functions attached to $f$, using the so-called plus and minus
logarithms, into bounded measures by exploiting the symmetry between the two
roots of the Hecke polynomial of $f$ at $p$. These bounded measures were utilized by
Kobayashi to formulate the so-called plus and minus Iwasawa main conjectures in
[Kob03] when $f$ corresponds to an elliptic curve defined over $\mathbb{Q}$. More precisely,
Kobayashi defined the so-called plus and minus Selmer groups over $\mathbb{Q}(\mu_{p^\infty})$. Using Kato’s Euler system constructed in [Kat04], he showed that their Pontryagin duals are torsion over the Iwasawa algebra of $\Gamma$ and related the characteristic ideals to Pollack’s bounded $p$-adic $L$-functions. Kobayashi’s work has been generalized to higher weight elliptic modular forms when $a_p = 0$ by the first named author of this article (see [Lei11]). The works of Pollack and Kobayashi have been simultaneously generalized to the $a_p \neq 0$ case by Sprung [Spr12] (for elliptic curves) and by Lei–Loeffler–Zerbes [LLZ10, LLZ11] (for higher weight elliptic modular forms).

**Automorphic results in this paper.** Let $n \geq 1$ be an integer and $\Pi$ a regular algebraic, essentially self-dual, cuspidal automorphic representation of $GL_{2n}(\mathbb{A}_Q)$ that is unramified at $p$. When $\Pi$ admits a Shalika model and is ordinary at $p$, Dimitrov, Januszewski and Raghuram generalized earlier results of Ash and Ginzburg in [AG94] to construct a bounded $p$-adic $L$-function interpolating the complex $L$-values of $\Pi$ twisted by Dirichlet characters on $\Gamma$ (see [DJR20]). This construction has been further generalized to the non-ordinary case in a recent work of Barrera, Dimitrov and Williams [BDW21]. As in the case of elliptic modular forms, when $\Pi$ is non-ordinary at $p$, the resulting $p$-adic $L$-functions are distributions that are possibly unbounded.

Thanks to results from local Langlands program, there exists a compatible family $\rho_{\Pi,\lambda,i} : G_\mathbb{Q} \rightarrow GL_{2n}(E(\Pi)_{\lambda})$ of continuous $\lambda$-adic representations where $\lambda$ runs through the finite places of a number field $E(\Pi)$ (see Section 2.1 below). From now on, we fix a prime $\lambda$ of $E(\Pi)$ lying above $p$ with ring of integers $\mathcal{O}$ and write $F = E(\Pi)_{\lambda}$. By enlarging $F$ if necessary, we shall assume that $F$ contains all the Satake parameters of $\Pi$ at $p$. Let $V_{\Pi}$ be the dual representation $\rho_{\Pi,\lambda,i}^*$. We fix a $G_\mathbb{Q}$-stable lattice $T_{\Pi}$ inside $V_{\Pi}$ and write $F_{\Pi} = \text{Hom}_{cts}(T_{\Pi}, F/\mathcal{O}(1))$. The main results of this paper can be summarized as follows.

**Proposition A** (Proposition 3.3). Suppose that the hypotheses in Hyp 3.2 hold. The local representation $V_{\Pi|G_\mathbb{Q}}$ is of the form

$$
\begin{pmatrix}
\chi_{\text{cyc}}^{h_1} \theta_1 & * & \cdots & * & * & * & \cdots & * & * \\
0 & \chi_{\text{cyc}}^{h_2} \theta_2 & \cdots & * & * & * & \cdots & * & * \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \chi_{\text{cyc}}^{h_{n-1}} \theta_{n-1} & * & * & \cdots & * & * \\
0 & \cdots & \cdots & 0 & \chi_{\text{cyc}}^{h_n} & \cdots & \cdots & * & * \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & 0 & \chi_{\text{cyc}}^{h_{n+2}} \theta_{n+2} & \cdots & * & * \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \chi_{\text{cyc}}^{h_{2n-1}} \theta_{2n-1} & * \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & \chi_{\text{cyc}}^{h_{2n}} \theta_{2n}
\end{pmatrix}
$$
where \( \chi_{\text{cyc}} \) is the \( p \)-adic cyclotomic character, \( h_i \) are the Hodge–Tate weights of \( V_{\Pi} \) and \( \theta_i \) are unramified characters on \( G_{\mathbb{Q}_p} \).

Proposition [A] generalizes a result of Ghate–Kumar in [GK11] where they showed that the local representation attached to \( \Pi \) is upper-triangular in the ordinary case. The conditions in Hyp 3.2 assert certain relations between the Hodge–Tate weights and Satake parameters of \( \Pi \) at \( p \) and that the Newton and Hodge filtrations of the Dieudonné module are in general position. More specifically, if \( \alpha_1, \ldots, \alpha_{2n} \) are the Satake parameters ordered by \( \text{ord}_p(\alpha_1) \geq \text{ord}_p(\alpha_2) \geq \cdots \geq \text{ord}_p(\alpha_{2n}) \), we are assuming that

\[
\text{ord}_p(\alpha_i) = h_i, \ i = n + 2, \ldots, 2n;
\]
\[
\text{ord}_p(\alpha_{n+1}) > h_{n+1} \text{ and } \alpha_n \neq \alpha_{n+1}.
\]

These conditions ensure that the local representation is close to being ordinary, admitting at most a 2-dimensional non-ordinary sub-quotient. It allows us to modify previous works on non-ordinary Iwasawa theory for elliptic modular forms to the current setting. In particular, we prove:

**Theorem B** (Theorem 4.15). Suppose that the hypotheses Hyp 3.2 and Hyp 4.1 hold. Let \( \mathcal{L}_p^{(\alpha)} \) and \( \mathcal{L}_p^{(\beta)} \) be certain twists of the \( p \)-adic \( L \)-functions of Barrera, Dimitrov and Williams (see Definition 4.14). There exist bounded \( p \)-adic \( L \)-functions \( \mathcal{L}_p^{\#}, \mathcal{L}_p^\flat \in \mathcal{O}[[\Gamma]] \otimes F \) such that

\[
\left( \begin{array}{c}
\mathcal{L}_p^{(\alpha)} \\
\mathcal{L}_p^{(\beta)}
\end{array} \right) = Q^{-1} M'_{\log} \left( \begin{array}{c}
\mathcal{L}_p^{\#} \\
\mathcal{L}_p^\flat
\end{array} \right),
\]

where \( Q \) and \( M'_{\log} \) are certain \( 2 \times 2 \) matrices with coefficients in the distribution algebra on \( \Gamma \), defined in Definitions 4.10 and 4.8 respectively. Furthermore, at least one of the two \( p \)-adic \( L \)-functions \( \mathcal{L}_p^{\#} \) and \( \mathcal{L}_p^\flat \) is non-zero.

The conditions in Hyp 4.1 assert that the unramified characters appearing in the 1-dimensional sub-quotients in Proposition [A] are non-trivial and that the 2-dimensional sub-quotient in the middle is non-ordinary and satisfies the Fontaine–Laffaille condition. The former condition simplifies some of our calculations with local cohomology groups and can potentially be removed with some extra work, whereas the latter is crucial in order for us to apply previous results on Wach modules theory in our construction of the matrix \( M'_{\log} \).

Theorem 13 generalizes a prior result of Rockwood [Roc22]. One of the main hypotheses in [Roc22] is the Pollack condition, which says that \( \alpha_n + \alpha_{n+1} = 0 \), where \( \alpha_j \) is the \( j \)-th Satake parameter of \( \Pi \) at \( p \), ordered according to their \( p \)-adic valuations. This Pollack condition is analogous to the condition \( a_p = 0 \) for elliptic
modular forms, which means that the two roots of its Hecke polynomial at $p$ add up to zero. Our main effort in this article is to construct bounded $p$-adic $L$-functions without assuming the Pollack condition.

Note that in [BDW21], the authors in fact constructed $p$-adic $L$-functions for automorphic representations over a totally real field. However, the Wach module theory that we rely on is only available for unramified extensions of $\mathbb{Q}_p$. It will be interesting to study a generalization of Theorem B for automorphic representations over a totally real field where $p$ is unramified. Assuming the local representation of $\Pi$ satisfies analogous conditions of Hyp. 4.1, one will probably have to consider a $2^d \times 2^d$ matrix in the decomposition, where $d$ is the number of primes lying above $p$. Another natural question one may ponder is to what extent can one relax the conditions in Hyp. 3.2 studying representations that are further from being ordinary at $p$. The results on $\text{GL}_2 \times \text{GL}_2$ in [BLLV19] suggest that further generalizations may be possible when the local representation can be described using smaller 1-dimensional and 2-dimensional representations upon constructing a larger matrix. We plan to study these questions in the future.

Signed Iwasawa main conjectures for automorphic representations. The explicit description of the local representation $V_\Pi$ given by Proposition A allows us to define signed Coleman maps over $\mathbb{Q}_p(\mu_{p^\infty})$, generalizing previous works on elliptic modular forms and elliptic curves (e.g. [Kob03, Spr12, Lei11, LLZ10]). We utilize the kernels of these Coleman maps as local conditions at $p$ to define signed Selmer groups for $\Pi$ over $\mathbb{Q}(\mu_{p^\infty})$ (see Definition 5.3) and formulate Iwasawa main conjectures connecting the Pontryagin duals of the signed Selmer groups with the bounded $p$-adic $L$-functions $L_\#_p, L_\flat_p$ given by Theorem B (see Conjecture 5.5).

The structure of this article is as follows. In Section 2, we introduce notation and collect preliminary results that will be used in the rest of the article. We study in Section 3 the structure of the local Galois representation $V_\Pi$ and prove Proposition A. We then make use of Proposition A to define bounded signed Coleman maps and study their interpolation properties in Section 3. Once this is done, we construct the bounded $p$-adic $L$-functions $L_\#_p, L_\flat_p$ given in Theorem B and explain how to recast the work of Rockwood in [Roc22] in terms of the $p$-adic $L$-functions we construct. Section 5 deals with the construction of signed Selmer groups which are conjecturally cotorsion modules over the cyclotomic Iwasawa algebra. We finish the article with a discussion on the signed Iwasawa main conjectures and how they are related to the weak Leopoldt conjecture and Perrin-Riou’s conjecture on the existence of Euler systems for $\Pi$. 
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2. Preliminaries

In this section, we review results on automorphic representations and local global compatibility. We also recall Perrin-Riou maps for crystalline representations. This will lay the ground work for the construction of signed Coleman maps and other related objects in subsequent sections.

2.1. RAESDC automorphic representations. We say that $\Pi$ is an RAESDC (regular algebraic, essentially self-dual, cuspidal) representation of $\text{GL}_{2n}(\mathbb{A}_Q)$ if $\Pi$ is a cuspidal automorphic representation such that:

1. The contragredient $\Pi^\vee$ of $\Pi$ satisfies $\Pi^\vee \cong \Pi \otimes \chi$ for some Hecke character $\chi : \mathbb{A}_Q^\times / \mathbb{Q}^\times \to \mathbb{C}^\times$;

2. Writing $\Pi = \Pi_\infty \otimes \Pi_f$, the infinite part $\Pi_\infty$ has the same infinitesimal character as some irreducible algebraic representation of $\text{GL}_{2n}(\mathbb{R})$.

Let $\mu = (\mu_1, ..., \mu_{2n}) \in \mathbb{Z}^{2n}$ satisfy the dominant weight condition

$$\mu_1 \geq \cdots \geq \mu_{2n}.$$ 

Let $E_\mu$ be the irreducible algebraic representation of $\text{GL}_{2n}(\mathbb{R})$ with highest weight $\mu$. We say that a RAESDC automorphic representation has weight $\mu$ if $\Pi_\infty$ has the same infinitesimal character as $E_\mu^\vee$. Let $\mathfrak{g}_\infty = \text{Lie}(\text{GL}_{2n}(\mathbb{R}))$ and $K_\infty$ be the product of a maximal compact subgroup of the real Lie group $\text{GL}_{2n}(\mathbb{R})$ with the center of
GL_{2n}(\mathbb{R})$. The algebraic regularity condition in (2) above holds if and only if $\Pi_\infty$ is cohomological, i.e.

$$H^q(\mathfrak{g}_\infty, K_\infty; \Pi_\infty \otimes E_\mu) \neq 0$$

for some degree $q$. Here $H^q(\mathfrak{g}_\infty, K_\infty; \Pi_\infty \otimes E_\mu)$ is the space of $(\mathfrak{g}_\infty, K_\infty)$-cohomology in degree $q$ (see [BW80, §I.5]). In this case, $\mu$ also satisfies the purity condition

$$(2.1) \quad \mu_i + \mu_{2n+1-i} = w, \quad \text{for } i = 1, \ldots, n \text{ and some } w \in \mathbb{Z}.$$  

Henceforth we assume that $\Pi$ is the transfer of a globally generic cuspidal automorphic representation of $\text{GSpin}_{2n+1}(\mathbb{A}_\mathbb{Q})$. This functorial transfer has been established for unitary globally generic cuspidal automorphic representations by Asgari–Shahidi in [AS06, Theorem 1.1] in its weak form and in [AS14, Corollary 4.25] at every place. For non-unitary representations $\Pi$, this transfer is discussed in [GR14, p. 686]. Moreover, the transfer of a globally generic cuspidal automorphic representation of $\text{GSpin}_{2n+1}(\mathbb{A}_\mathbb{Q})$ admits a global Shalika model in the sense of Grobner–Raghuram and are essentially self dual (see [GR14, Proposition 3.14]).

2.2. Strict local-global compatibility. Let $\rho : G_\mathbb{Q} \to \text{GL}_{2n}(\mathbb{Q}_\ell)$ be a Galois representation of the absolute Galois group $G_\mathbb{Q}$ of $\mathbb{Q}$. Assume that $\rho$ is geometric, that is, it is unramified outside a finite set of primes of $\mathbb{Q}$ and its restrictions to the decomposition groups at primes above $\ell$ are potentially semistable in the sense of Fontaine (see for example [FO, §6.3]). Let $WD_p$ be the Weil–Deligne group at $p$. For a geometric representation, one can define a Weil–Deligne representation $WD_p : G_\mathbb{Q} \to \text{GL}_{2n}(\mathbb{Q}_\ell)$ up to conjugacy. This definition is classical for $p \neq \ell$ and comes from Deligne–Grothendieck whereas for $p = \ell$, it is due to Fontaine. A concise survey of both constructions with references are given in [Tay04, p. 77-79].

**Definition 2.1.** [Tay04, p. 81–82], [GK11, Section 2] A $\mathbb{Q}$-rational, strictly compatible (or strongly compatible) system of geometric representations $(\rho_\ell)$ of $G_\mathbb{Q}$ is a collection of data consisting of:

1. For each prime $\ell$ and each embedding $i : \mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}_\ell$, a continuous, semisimple representation $\rho_\ell : G_\mathbb{Q} \to \text{GL}_{2n}(\overline{\mathbb{Q}}_\ell)$ that is geometric.
2. For each prime $p$ of $\mathbb{Q}$, a Frobenius semisimple representation $r_p : WD_p \to \text{GL}_{2n}(L)$ such that
   - $r_p$ is unramified for all $p$ outside a finite set.
   - For each $\ell$, the Frobenius semisimple Weil-Deligne representation $WD_p \to \text{GL}_{2n}(\overline{\mathbb{Q}}_\ell)$ associated to $\rho_\ell|_{G_{\mathbb{Q}p}}$ is conjugate to $r_p$ via the embedding $i : \mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}_\ell$.
   - There exists a multiset of integers $H$ such that for each prime $\ell$ and each embedding $i : \mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}_\ell$, the the multiset of Hodge–Tate weights of $\rho_\ell|_{G_{\mathbb{Q}\ell}}$ is $H$. 


The following conjecture is given in [Tay04 Conjecture 3.4] (see also [GK11 Conjecture 3.3]).

**Conjecture 2.2.** Suppose \( \Pi \) is a cuspidal automorphic form on \( GL_{2n}(A_Q) \) with infinitesimal character \( \chi_H \) where \( H \) is a multiset of distinct integers. Then there is a strictly compatible system of Galois representations \((\rho_{\Pi,\ell}, \ell)\) associated to \( \Pi \) with Hodge–Tate weights \( H \) such that the local-global compatibility holds for all primes.

Here, local-global compatibility means that the underlying Frobenius semi-simplified Weil–Deligne representation at \( p \) in the compatible system (which is independent of the residue characteristic \( \ell \) of the coefficients by hypothesis) corresponds to \( \Pi_p \) via the local Langlands correspondence.

There is a significant evidence towards this conjecture for RASEDC representations, thanks to the works of Clozel, Harris, Taylor for \( p \neq \ell \) [CHT08], and Barnet-Lamb, Gee, Geraghty and Taylor for \( p = \ell \) [BGGT14].

**Convention 2.3.** We normalize the Hodge–Tate weights on \( G_{\mathbb{Q}_p} \)-representations so that the cyclotomic character has Hodge–Tate weight +1.

**Theorem 2.4.** Let \( \iota : \mathbb{Q}_\ell \cong \mathbb{C} \). Suppose \( \Pi \) is an RASEDC automorphic representation of \( GL_{2n}(A_Q) \) of weight \( \mu \) as in Section 2.1. Then there is a number field \( E(\Pi) \) and a compatible system \( \rho_{\Pi,\lambda,\iota} : G_{\mathbb{Q}} \to GL_{2n}(E(\Pi)_\lambda) \) of continuous \( \lambda \)-adic representations where \( \lambda \) runs through finite places of \( E(\Pi) \), such that

1. If \( p \) is coprime to the prime \( \ell \) dividing the norm \( N_{E(\Pi)/\mathbb{Q}}(\lambda) \), we have
   \[
   WD(\rho_{\Pi,\lambda,\iota}|_{G_{\mathbb{Q}_p}})^{ss} = \left( r_\ell(\Pi_p \circ \iota)^\vee (1 - 2n) \right)^{ss}
   \]
   where \( r_\ell \) is the reciprocity map defined in [HT01].

2. If \( p \) divides \( N_{E(\Pi)/\mathbb{Q}}(\lambda) \), and if \( \Pi_p \) is unramified, the representation \( \rho_{\Pi,\lambda,\iota}|_{G_{\mathbb{Q}_p}} \) is crystalline with Hodge–Tate weights \( -h_i = -(\mu_i + 2n - i) \) for \( i = 1, \ldots, 2n \) and each of these Hodge–Tate weights have multiplicity one. (the minus signs arise since the weights are the negatives of the jumps in the Hodge filtration on the associated filtered \( \varphi \)-module \( \mathbb{D}_{\text{cris}}(\rho_{\Pi,\lambda,\iota}|_{G_{\mathbb{Q}_p}}) \) constructed by Fontaine, see [FO, Definition 6.29]). Furthermore, the characteristic polynomial of \( \varphi \) equals the characteristic polynomial of the geometric Frobenius at \( p \) of the Weil–Deligne representation \( r_p(\Pi_p \circ \iota)^\vee (1 - 2n) \). (the minus signs arise since the weights are the negatives of the jumps in the Hodge filtration on the associated filtered \( \varphi \)-module \( \mathbb{D}_{\text{cris}}(\rho_{\Pi,\lambda,\iota}|_{G_{\mathbb{Q}_p}}) \) constructed by Fontaine, see [FO, Definition 6.29]).

3. The local-global compatibility holds at primes \( p \) dividing \( N_{E(\Pi)/\mathbb{Q}}(\lambda) \). That is,
   \[
   i WD(\rho_{\Pi,\lambda,\iota}|_{G_{\mathbb{Q}_p}})^{ss} = \text{rec}(\Pi_p \otimes | \det \frac{1-2s}{2})^{ss},
   \]
   where \( WD(\rho_{\Pi,\lambda,\iota}|_{G_{\mathbb{Q}_p}}) \) is the Weil–Deligne representation associated to \( \rho_{\Pi,\lambda,\iota}|_{G_{\mathbb{Q}_p}} \) and \( \text{rec} \) is the local Langlands correspondence [HT01].
Proof. For (1) and (2), see Chenevier–Harris [CH13, Theorem 4.2] and Geraghty [Ger19, Proposition 2.27]. See also [GK11, Theorem 3.5]. The case of \( p \) being coprime to \( N_{E(\Pi)/\mathbb{Q}}(\lambda) \) is due to Clozel–Harris–Taylor [CHT08, Proposition 4.3.1]. For (3), see [BGGT14, Theorem A]. \( \square \)

Remark 2.5. As \( 2n \) is even, we note that \( \text{trace}(\rho_{\Pi,\lambda,i}(c)) = 0 \), where \( c \) is the complex conjugation (see [CLH16, Theorem 1.1]).

2.3. The Perrin-Riou map for local representations. Let \( \Gamma = \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \cong \Delta \times \Gamma_1 \), where \( \Delta \cong \mathbb{Z}/(p-1)\mathbb{Z} \) and \( \Gamma_1 \cong \mathbb{Z}_p \). We choose a topological generator \( \gamma \) of \( \Gamma_1 \). We fix a finite extension \( F \) of \( \mathbb{Q}_p \), whose ring of integers is denoted by \( O \).

We write \( \Lambda(\Gamma) = O[[\Gamma]] \) and \( \Lambda(\Gamma_1) = O[[\Gamma_1]] \) for the Iwasawa algebra of \( \Gamma \) and \( \Gamma_1 \) over \( O \) respectively. We consider \( \Lambda(\Gamma) \) and \( \Lambda(\Gamma_1) \) as subrings of \( H(\Gamma) \) and \( H(\Gamma_1) \), which are the rings of power series \( f \in F[[X]] \) (respectively \( f \in F[[\Delta]] \)) which converge on the open unit disc \( |X| < 1 \) in \( \mathbb{C}_p \), where \( | \cdot | \) denotes the \( p \)-adic norm on \( \mathbb{C}_p \), normalized by \( |p| = p^{-1} \). For any real number \( r \geq 0 \), we write \( H_r(\Gamma) \) and \( H_r(\Gamma_1) \) for the set of power series \( f \) in \( H(\Gamma) \) and \( H(\Gamma_1) \) respectively satisfying \( \sup_t p^{-i(r)} \| f \|_{\rho_t} < \infty \), where \( \rho_t = p^{-i(r)}(p^{r-1}) \) and \( \| f \|_{\rho_t} = \sup_{|z| \leq \rho_t} |f(z)| \). It is common to write \( f = O(\log^r_p) \) when \( f \) satisfies this condition.

Given an integer \( i \), we write \( T^i \) for the \( \mathbb{Q}_p \)-algebra automorphism of \( H(\Gamma) \) defined by \( \sigma \mapsto \chi^{i}_{\text{cyc}}(\sigma)\sigma \) for \( \sigma \in \Gamma \). We set \( u := \chi^{i}_{\text{cyc}}(\gamma) \). If \( m \geq 1 \) is an integer, we define \( \Phi_m \) to be the \( p^m \)-th cyclotomic polynomial in \( 1 + X \), namely \( \frac{(1 + X)^{p^m} - 1}{(1 + X)^{p^{m-1}} - 1} \). We let \( \log_p = \log_p(1 + X) \in H(\Gamma_1) \) denote the \( p \)-adic logarithm. We also define for an integer \( i \), the element

\[
\ell_i = \frac{\log_p}{\log_p^m(u)} - i = \frac{T^{-i}(\log_p)}{\log_p(u)} \in H(\Gamma_1)
\]

For \( i \geq 0 \), we define the product

\[
\tilde{\ell}_i = \prod_{j=0}^{i-1} \ell_j.
\]

Note that for \( i = 0 \), we have \( \tilde{\ell}_0 = 1 \).

Definition 2.6. Let \( T \) be a \( G_{\mathbb{Q}_p} \)-stable \( O \)-lattice of a finite dimensional \( F \)-linear crystalline representation \( V \) of \( G_{\mathbb{Q}_p} \) with non-negative Hodge–Tate weights such that \( V \) has no sub-quotient isomorphic to the trivial representation \( F \).

i) We define \( H^1_{Iw}(\mathbb{Q}_p, T) \) to be the inverse limit \( \lim\downarrow H^1(\mathbb{Q}_p(\mu_{p^m}), T) \), where the connecting maps are corestrictions.
ii) We write \( N(T) \) for the Wach module of \( T \) (see for example [Ber04, §II.1] for the precise definitions; it is a filtered module over the ring \( \mathcal{O}[[\pi]] \), where \( \pi \) can be regarded as a formal variable equipped with an action of \( \varphi \) and \( \Gamma \) given by \( \varphi(\pi) = (1 + \pi)^p - 1 \) and \( \sigma(\pi) = (1 + \pi)^{\chi_{\text{cycl}}(\sigma)} - 1 \) for \( \sigma \in \Gamma \)). We have the integral Dieudonné module \( D_{\text{cris}}(T) \) given by \( N(T) \) modulo \( \pi \).

iii) We write \( L_T : H^1_{\text{Iw}}(\mathbb{Q}_p, T) \to H(\Gamma) \otimes D_{\text{cris}}(T) \) for the Perrin-Riou map as defined in [LLZ11, §3.1] and [LZ14, Appendix B] (see also our discussion below for a review of this map).

Let \( \psi \) denote the left inverse of \( \varphi \) as given in [Ber03, §I.2]. We write \( B_{\text{rig}}^+ \) for the ring of power series in \( F[[\pi]] \) that converge on the open unit disk. Let \( t = \log_p(1 + \pi) \in B_{\text{rig}}^+ \) and \( q = \varphi(\pi)/\pi \). We recall that the Mellin transform \( m \) sending 1 to \( 1 + \pi \) induces the \( \Lambda(\Gamma) \)-isomorphisms

\[ \Lambda(\Gamma) \sim \to \mathcal{O}[[\pi]]_{\psi=0}, \quad H(\Gamma) \sim \to (B_{\text{rig}}^+)_{\psi=0}. \]

Note that the condition \( \psi = 0 \) simply cuts out the distributions supported on \( \Gamma = \mathbb{Z}_p^\times \subset \mathbb{Z}_p \) which is where these natural Mellin transform isomorphisms come from.

We fix a \( \mathbb{Z}_p \) basis \( e_1 \) for the \( G_{\mathbb{Q}_p} \)-representation \( \mathbb{Z}_p(1) \). Given an integer \( k \), we write \( e_k = e_1^k \). We have the identifications

\[ H^1_{\text{Iw}}(\mathbb{Q}_p, T(k)) = H^1_{\text{Iw}}(\mathbb{Q}_p, T) \cdot e_k, \]
\[ D_{\text{cris}}(T(k)) = D_{\text{cris}}(T) \cdot t^{-k}e_k, \]
\[ N(T(k)) = N(T) \cdot \pi^{-k}e_k. \]

We recall from [Ber03, Theorem A.3] and [BB08, §1.3] that the assumption that the Hodge–Tate weights are non-negative implies that there is an isomorphism of \( \Lambda(\Gamma) \)-modules

\[ H^1_{\text{Iw}}(\mathbb{Q}_p, T) \cong N(T)^{\psi=1} \]

via the Herr complex. Here, the superscript \( \psi = 1 \) signifies the kernel of the morphism \( \psi - 1 \). The Perrin-Riou map \( L_T : H^1_{\text{Iw}}(\mathbb{Q}_p, T) \to H(\Gamma) \otimes D_{\text{cris}}(T) \) in Definition 2.6 can be defined via this isomorphism composed with

\[ N(T)^{\psi=1} \xrightarrow{1-\varphi} (B_{\text{rig}}^+)_{\psi=0} \otimes D_{\text{cris}}(T) \xrightarrow{\text{m}^{-1} \otimes 1} H(\Gamma) \otimes D_{\text{cris}}(T). \]

1We recall that \( \pi \) is in the ring of of Witt vectors of \( \lim_{\xrightarrow{\longleftarrow}} \mathbb{Z} \subset \mathbb{C}_p \) given by \( [1, \zeta_p, \zeta_p^2, \ldots] - 1 \), where \( \zeta_p \) is a primitive \( p^n \)-th root of unity in \( \mathbb{C}_p \) such that \( \zeta_p^{p^n-1} = \zeta_p^1 \). It is used to define Fontaine’s period \( t = \log(1 + [\pi]) \in \mathbb{B}_{\text{dR}}^+ \).
We will also need the following notations while constructing Selmer groups in Section 5.

i) Given a \( O \)-module \( M \), we write \( M^\vee = \text{Hom}_{cts}(M, F/\mathcal{O}) \) for its Pontryagin dual.

ii) For a finitely generated torsion \( \Lambda(\Gamma_1) \)-module \( M \), let \( \text{char}_{\Lambda(\Gamma_1)}(M) \) denote the characteristic ideal of \( M \).

iii) For a Dirichlet character \( \eta : \Delta \rightarrow \mathbb{Z}^\times \), we define \( e_\eta \) to be the idempotent attached to \( \eta \) given by

\[
\frac{1}{p-1} \sum_{\sigma \in \Delta} \eta(\sigma)^{-1} \sigma \in \mathbb{Z}_p[\Delta].
\]

iv) Given a \( \Lambda(\Gamma) \)-module \( M \) and a character \( \eta \) as above, we define its \( \eta \)-isotypic component to be \( M^\eta = e_\eta \cdot M \), which we consider as a \( \Lambda(\Gamma_1) \)-module.

v) Given an element \( f = \sum_{k \geq 0, \sigma \in \Delta} a_{k,\sigma} \cdot \sigma \cdot (\gamma - 1)^k \in \mathcal{H}(\Gamma) \), where \( a_{k,\sigma} \in F \) and \( \gamma \) is a topological generator of \( \Gamma_1 \), we may decompose \( f \) into \( \sum_{\eta \in \Delta} f_\eta \), where \( f_\eta = e_\eta \cdot f \in \mathcal{H}(\Gamma)^\eta \). Furthermore, we may identify \( f_\eta \) with an element of \( \mathcal{H}(\Gamma_1) \) given by

\[
\sum_{k \geq 0} \left( \sum_{\sigma \in \Delta} a_{k,\sigma} \eta(\sigma) \right) (\gamma - 1)^k.
\]

3. Structure of non-ordinary local Galois representation at \( p \)

Fix an integer \( n \geq 1 \) and set \( G = \text{GL}_{2n} \). Let \( T \) be the standard diagonal split torus of \( G \) and \( B \) the upper triangular Borel subgroup. Now suppose \( \Pi \) is a RAESDC automorphic representation of \( \text{GL}_{2n}(\mathbb{A}_\mathbb{Q}) \) which is the transfer of a globally generic cuspidal automorphic representation of \( \text{GSpin}_{2n+1}(\mathbb{A}_\mathbb{Q}) \) and suppose \( \Pi_p \) is unramified. Let us suppose, there is an unramified character \( \chi_p : T(\mathbb{Q}_p) \rightarrow \mathbb{C}^\times \)

such that

\[
\Pi_p = \text{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(1 \cdot |z|^{2n-1/2} \chi_p).
\]

We define the Satake parameters at \( p \) to be the values \( \alpha_i = \chi_{p,i}(p) \), where the \( \chi_{p,i} \)'s denote the projections to the diagonal entries. After choosing an isomorphism \( \mathbb{Q}_p \cong \mathbb{C} \), the indices \( i \) are ordered so that \( \text{ord}_p(\alpha_1) \geq \text{ord}_p(\alpha_2) \geq \cdots \geq \text{ord}_p(\alpha_{2n}) \). For all \( i \), we also have

\[
\alpha_i \alpha_{2n+1-i} = \lambda
\]

for fixed \( \lambda \) with \( p \)-adic valuation \( 2n - 1 + w \). As in the introduction, we take \( F = E(\Pi) \lambda \), where \( \lambda \) is a prime of \( E(\Pi) \) above \( p \). By extending scalars if necessary, we assume that all \( \alpha_i \) are contained in \( F \).
Lemma 3.1. When $p = \ell$, the eigenvalues of $\varphi$ on $D_{\text{cris}}(\rho_{\Pi,\lambda,\iota}|G_{\mathbb{Q}_p})$ are given by $\alpha_i$ for $i = 1, \ldots, 2n$.

Proof. This is a direct consequence of Theorem 2.4. □

Henceforth, we shall write $V_{\Pi}$ for the representation $\rho^*_{\Pi,\lambda,\iota}$. By an abuse of notation, we may write $V_{\Pi}$ to denote $V_{\Pi}|G_{\mathbb{Q}_p}$ when no confusion arises. We recall from Theorem 2.4 that the jumps in the Hodge filtration of $V_{\Pi}$ are $-(\mu_i + 2n - i)$. Therefore, the $\varphi$-eigenvalues on $D_{\text{cris}}(V_{\Pi})$ are given by $\alpha_i^{-1}$.

We work under the following hypotheses.

**Hyp 3.2.** We assume that the Hodge–Tate weights and the Satake parameters of $\Pi$ at $p$ satisfy the following hypotheses.

- (M.Slo) $\text{ord}_p(\alpha_i) = h_i$, $i = n + 2, \ldots, 2n$;
- (N.ord) $\text{ord}_p(\alpha_{n+1}) > h_{n+1}$ and $\alpha_n \neq \alpha_{n+1}$;
- (G.Po) The Newton filtration on $D_{\text{cris}}(V_{\Pi})$ is in general position with respect to the Hodge filtration.

The hypothesis (M.Slo) is the “minimal slope” hypothesis in [Roc22, §3]. As the Newton polygon of $\Pi$ lies on or above the Hodge polygon of $\Pi$ and their end points coincide (see Hida’s work [Hid98, Section 8.2]), it is easy to see from (2.1) and (3.1) that (M.Slo) implies

$$\text{(3.2)} \quad \text{ord}_p(\alpha_i) = h_i \text{ for } i = 1, \ldots, n - 1.$$

Hence the Newton and the Hodge polygons coincide at all points except the $n$-th point.

Note that the inequality $\text{ord}_p(\alpha_{n+1}) > h_{n+1}$ in (N.ord) implies that

$$h_n > \text{ord}_p(\alpha_n), \text{ord}_p(\alpha_{n+1}) > h_{n+1}$$

(once again thanks to (2.1) and (3.1)). We remark that the hypothesis (N.ord) is a weaker condition than the Pollack condition $\alpha_n + \alpha_{n+1} = 0$ considered in [Roc22]. Indeed, the Pollack condition has the consequence that $\text{ord}_p(\alpha_n) = \text{ord}_p(\alpha_{n+1}) = \frac{1}{2}(h_n + h_{n+1})$, which is a special case of $\text{ord}_p(\alpha_{n+1}) > h_{n+1}$.

Hypothesis (G.Po) is [GK11, Assumption 3.6], which means that given a $d$-dimensional $\varphi$-stable subspace of $D_{\text{cris}}(V_{\Pi})$, the jumps in its Hodge filtration occur at the first $d$ jumps in the Hodge filtration of $D_{\text{cris}}(V_{\Pi})$. As discussed in op. cit., (G.Po) is expected to hold generically.

We are now ready to prove Proposition [A].
Proposition 3.3. Assume that Hyp 3.2 holds. The local representation \( V_\Pi|_{\mathbb{G}_{\mathbb{Q}_p}} \) is isomorphic to a representation of the form

\[
\begin{pmatrix}
\chi_{\text{cyc}}^{h_1} \theta_1 & * & ... & * & * & * & ... & * & * \\
0 & \chi_{\text{cyc}}^{h_2} \theta_2 & ... & * & * & * & ... & * & * \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & ... & 0 & \chi_{\text{cyc}}^{h_{n-1}} \theta_{n-1} & * & * & ... & * & * \\
0 & ... & 0 & 0 & \chi_{\text{cyc}}^{h_{n+1}} \theta_{n+2} & ... & * & * & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & ... & 0 & 0 & 0 & ... & 0 & \chi_{\text{cyc}}^{h_{2n-1}} \theta_{2n-1} & * \\
0 & ... & 0 & 0 & 0 & ... & 0 & \chi_{\text{cyc}}^{h_{2n}} \theta_{2n} & *
\end{pmatrix},
\]

where \( \chi_{\text{cyc}} \) denotes the \( p \)-adic cyclotomic character, \( \theta_i \) are unramified characters on \( \mathbb{G}_{\mathbb{Q}_p} \) and the middle \( 2 \times 2 \) square, given by the symbol \( \times \), is a two-dimensional subquotient \( V'_\Pi \) of \( V_\Pi \) whose Hodge–Tate weights are \( h_n \) and \( h_{n+1} \). Furthermore, the \( \varphi \)-eigenvalues on \( \mathbb{D}_{\text{cris}}(V'_\Pi) \) are \( \alpha^{-1}_n \) and \( \alpha^{-1}_{n+1} \).

Proof. Given that \( h_i = \mu_i + 2n - i \) and \( \mu_1 \geq \cdots \geq \mu_{2n} \), we have

\[
h_1 > \cdots > h_{2n}.
\]

This, together with (3.2) and (M.Slo) imply that

\[
\text{ord}_p(\alpha_1) > \cdots > \text{ord}_p(\alpha_{n-1}) > \text{ord}_p(\alpha_n) = \text{ord}_p(\alpha_{n+1}) = \text{ord}_p(\alpha_{n+2}) > \cdots > \text{ord}_p(\alpha_{2n}).
\]

Therefore, the characteristic polynomial of \( \varphi|_{\mathbb{D}_{\text{cris}}(V_\Pi)} \) factors into the product

\[
Q(X) = \prod_{1 \leq i \leq n-1 \atop n+2 \leq i \leq 2n} (X - \alpha^{-1}_i),
\]

where \( Q(X) \) is a monic quadratic polynomial defined over \( F \) whose roots are \( \alpha^{-1}_n \) and \( \alpha^{-1}_{n+1} \). This tells us that for \( 1 \leq i \leq n-1 \) and \( n+2 \leq i \leq 2n \), the \( \varphi \)-eigensubspace of \( \mathbb{D}_{\text{cris}}(V_\Pi) \) for the eigenvalue \( \alpha^{-1}_i \), which we denote by \( E_i \), is a one-dimensional \( F \)-subspace of \( \mathbb{D}_{\text{cris}}(V_\Pi) \). Let \( E' \) denote the subspace of \( \mathbb{D}_{\text{cris}}(V_\Pi) \) given by the kernel of \( Q(\varphi) \). Since the \( \alpha_i \)'s are all distinct, we have

\[
\mathbb{D}_{\text{cris}}(V_\Pi) = E' \oplus \bigoplus_{1 \leq i \leq n-1 \atop n+2 \leq i \leq 2n} E_i.
\]
We define the following subspaces of $\mathbb{D}_{\text{cris}}(V_{\Pi})$:

$$D_i = \bigoplus_{j=1}^{i} E_j, \quad 1 \leq i \leq n - 1,$$

$$D_n = D_{n+1} = D_{n-1} \oplus E',$$

$$D_i = D_n \oplus \bigoplus_{j=n+2}^{i} E_j, \quad n + 2 \leq i \leq 2n.$$

This gives the filtration

$$0 =: D_0 \subsetneq D_1 \subsetneq \cdots \subsetneq D_{n-1} \subsetneq D_n = D_{n+1} \subsetneq D_{n+2} \subsetneq \cdots \subsetneq D_{2n-1} \subsetneq D_{2n} = \mathbb{D}_{\text{cris}}(V_{\Pi}).$$

Then, by the hypothesis (G.Po) and (3.2), for all $1 \leq i \leq 2n$, $D_i$ is an admissible filtered $(\varphi, N)$-module with $t_N(D_i) = t_H(D_i)$ (we refer the reader to [GK11, §2] for the notation and terminology being used here; note that we are taking the fields $E$ and $F$ in op. cit. to be $F$ and $\mathbb{Q}_p$ here). Consequently, $t_N(D_i/D_{i-1}) = t_H(D_i/D_{i-1})$.

The equivalence of categories between admissible filtered $(\varphi, N)$-modules and $G_{\mathbb{Q}_p}$-representations proved by Colmez–Fontaine [CF00] (see also [GK11, Theorem 2.4]) tells us that for each $i$, $D_i = \mathbb{D}_{\text{cris}}(V_i)$ for some $G_{\mathbb{Q}_p}$-sub-representation $V_i$ of $V_{\Pi}$ such that

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = V_{n+1} \subsetneq V_{n+2} \subsetneq \cdots \subsetneq V_{2n-1} \subsetneq V_{2n} = V_{\Pi},$$

where $V_i/V_{i-1}$ one-dimensional with Hodge–Tate weight $h_i$ for $i = 1, 2, \ldots, n-1, n+2, \ldots, 2n$ and $V_n/V_{n-1}$ is 2-dimensional with Hodge–Tate weights $h_n$ and $h_{n+1}$. This finishes the proof of the proposition.

**Remark 3.4.** Note that when $n = 2$, the representations studied in Proposition 3.3 have exactly the same form as those considered in [Urb05, Corollary 1(i)].

### 4. Construction of bounded Coleman maps and $p$-adic $L$-functions

We study $\Lambda(\Gamma)$-valued Coleman maps and a certain logarithmic matrix attached to $T_{\Pi}$. This allows us to prove Theorem 3.3.

#### 4.1. Perrin-Riou maps and Coleman maps

Let $V_{\Pi}$ be as in previous sections. Recall that $T_{\Pi}$ is a $G_{\mathbb{Q}}$-stable lattice of $V_{\Pi}$. We continue to assume that Hyp 3.2 holds. The main goal of this section is to define bounded Coleman maps, decomposing the Perrin-Riou map attached to $T_{\Pi}$. If $V_i$ is one of the sub-representations in the proof of Proposition 3.3, we write $T_i$ to be the $\mathcal{O}$-lattice $V_i \cap T_{\Pi}$ inside $V_i$. Our construction of Coleman maps relies on patching together the Coleman maps for the sub-quotients $T_i/T_{i-1}$, $i = n, \ldots, 2n$. 


We work under the following hypotheses.

**Hyp 4.1.** From now on, we assume the following additional hypotheses:

- **(Pos)** \( h_{2n} \geq 0 \) and the characters \( \theta_i, i = n + 2, \ldots, 2n \) in Proposition 3.3 are non-trivial;
- **(FL)** \( p > h_n - h_{n+1} > 1 \).

Note that the hypothesis (FL) implies that the two-dimensional representation \( V'_\Pi \) given in Theorem 3.3 satisfies the Fontaine–Laffaille condition. It also ensures that the \( p \)-adic \( L \)-functions we consider are non-zero. This condition excludes \( p \) from being even. With extra work, it is possible that the case \( p = 2 \) may be treated separately if one does not insist on proving the non-triviality of the \( p \)-adic \( L \)-functions.

Let \( T = T_i/T_j \), where \( i, j \in \{0, 1, \ldots, 2n\} \) with \( j < i \). The hypothesis (Pos) implies that there is an isomorphism of \( \Lambda(\Gamma) \)-modules

\[
H^1_{\text{Iw}}(\mathbb{Q}_p, T) \cong \mathbb{N}(T)^{\psi=1},
\]

given by Herr’s complex (see [Ber03, Theorem A.3] and [BB08, §1.3]).

**Proposition 4.2.** Let \( i \in \{n + 2, \ldots, 2n\} \). The image of \( L_{T_i/T_{i-1}} \) as defined in Definition 2.6 lands inside \( \tilde{\ell}_{h_i} \Lambda(\Gamma) \otimes D_{\text{cris}}(T_{i}/T_{i-1}) \), where \( \tilde{\ell}_{h_i} \) is given by (2.2).

**Proof.** Let us write in this proof \( T = T_i/T_{i-1} \). Recall from Proposition 3.3 that \( T = \mathcal{O}(\chi_{\text{cyc}}^{h_i} \theta_i) \). Since the Hodge–Tate weight of \( \mathcal{O}(\theta_i) \) is 0, it follows that \( \mathcal{L}_{\mathcal{O}(\theta_i)} \) has image in \( \Lambda(\Gamma) \otimes D_{\text{cris}}(\mathcal{O}(\theta_i)) \). We recall from [LZ14, §4.4] that

\[
\left( \prod_{j=1}^{h_i} \ell_{-j} \right) \mathcal{L}_{\mathcal{O}(\theta_i)}(z) = (\text{Tw}^{h_i} \otimes 1) (\mathcal{L}_{T}(z \otimes e_{h_i})) \cdot t^{h_i} e_{-h_i}
\]

for \( z \in H^1_{\text{Iw}}(\mathbb{Q}_p, \mathcal{O}(\theta_i)) \). A direct calculation shows that

\[
\text{Tw}^{-h_i} \left( \prod_{j=1}^{h_i} \ell_{-j} \right) = \tilde{\ell}_{h_i}.
\]

Therefore, the image of \( \mathcal{L}_{T} \) lies inside

\[
\tilde{\ell}_{h_i} \Lambda(\Gamma) \otimes D_{\text{cris}}(\mathcal{O}(\theta_i)) \cdot t^{-h_i} e_{h_i} = \tilde{\ell}_{h_i} \Lambda(\Gamma) \otimes D_{\text{cris}}(T)
\]

as required. \( \square \)

**Definition 4.3.** For \( i \in \{n + 2, \ldots, 2n\} \), fix an \( \mathcal{O} \)-basis \( \omega_i \) of \( D_{\text{cris}}(T_i/T_{i-1}) \) (which is necessarily a \( \varphi \)-eigenvector, with \( \varphi(\omega_i) = \alpha_i^{-1} \omega_i \)). Proposition 4.2 allows us to define

\[
\text{Col}_{\omega_i} : H^1_{\text{Iw}}(\mathbb{Q}_p, T_i/T_{i-1}) \rightarrow \Lambda(\Gamma)
\]
to be the unique $\Lambda(\Gamma)$-morphism satisfying $L_{T_i/T_{i-1}}(z) = \text{Col}_{\omega_i}(z)\tilde{\ell}_{h_i}\omega_i$. We also define
\[
L_{\omega_i} = \tilde{\ell}_{h_i}\text{Col}_{\omega_i}.
\]

**Lemma 4.4.** For $i \in \{n+2,\ldots,2n\}$, the maps $\text{Col}_{\omega_i}$ are injective.

**Proof.** Note that $\mathbb{N}(T_i/T_{i-1})^{\neq 1} \subset H^0(\mathbb{Q}_p(\mu_p^{\infty}), T_i/T_{i-1})$, which is zero, thanks to our hypothesis that $\theta_i \neq 1$ (as given in (Pos)). Therefore, the argument in [LZ14, proof of Proposition 4.10] shows that $L_{T_i/T_{i-1}}$ is injective. Consequently, $\text{Col}_{\omega_i}$ is also injective. □

We now turn our attention to the two-dimensional sub-quotient $T_n/T_{n-1}$. For notational simplicity, we shall write $T_{\Pi}$ for $T_n/T_{n-1}$. We shall consider the Tate twist $T_n' := T_{\Pi}(-h_{n+1})$.

**Lemma 4.5.** The image of $L_{T_n'}$ lies inside $\tilde{\ell}_{h_{n+1}}H(\Gamma) \otimes \mathbb{D}_{\text{cris}}(T_{\Pi}')$.

**Proof.** Note that the Hodge–Tate weights of $T_n'$ are 0 and $h_n - h_{n+1} \geq 0$. The image of $L_{T_n'}$ lies inside $H(\Gamma) \otimes \mathbb{D}_{\text{cris}}(T_{\Pi}')$. Similar to the proof of Proposition 4.2, we have
\[
\prod_{i=1}^{h_{n+1}} \ell_{-i} L_{T_{\Pi}'}(z) = (\text{Tw}^{h_{n+1}} \otimes 1) (L_{T_{\Pi}'}(z \otimes e_{h_{n+1}})) \cdot t^{h_{n+1}}e_{-h_{n+1}}
\]
for $z \in H^1_{Iw}(\mathbb{Q}_p, T_{\Pi}')$, which tells us that the image of $L_{T_{\Pi}'}$ lies inside
\[
\text{Tw}^{-h_{n+1}} \prod_{i=1}^{h_{n+1}} \ell_{-i} H(\Gamma) \otimes \mathbb{D}_{\text{cris}}(T_{\Pi}') = \tilde{\ell}_{h_{n+1}} H(\Gamma) \otimes \mathbb{D}_{\text{cris}}(T_{\Pi}'),
\]
as required. □

Our goal is to define bounded Coleman maps on $H^1_{Iw}(\mathbb{Q}_p, T_{\Pi}')$. We do so by first defining such maps on $H^1_{Iw}(\mathbb{Q}_p, T_{\Pi}')$ and then twist these maps appropriately.

**Definition 4.6.**

i) We fix an $\mathcal{O}$-basis $\{\omega_n^\circ, \omega_{n+1}^\circ\}$ of $\mathbb{D}_{\text{cris}}(T_{\Pi}')$ that is adapted to the filtration of $\mathbb{D}_{\text{cris}}(T_{\Pi}')$ in the sense that $\omega_n^\circ$ is an $\mathcal{O}$-basis of $\text{Fil}^{-h_{n+1}}\mathbb{D}_{\text{cris}}(T_{\Pi}')$ (see [Ber04, §V.2], where this condition is discussed). We also fix $\varphi$-eigenvectors $\omega_n, \omega_{n+1} \in \mathbb{D}_{\text{cris}}(T_{\Pi}') \otimes \mathcal{O} F$ with $\varphi(\omega_i) = \alpha_i^{-1}\omega_i$ for $i = n$ and $n+1$.

ii) For $i \in \{n, n+1\}$, we write $\varpi_i^\circ = \omega_i^\circ \cdot t^{h_{n+1}}e_{-h_{n+1}} \in \mathbb{D}_{\text{cris}}(T_{\Pi}')$ and $\varpi_i = \omega_i \cdot t^{h_{n+1}}e_{-h_{n+1}} \in \mathbb{D}_{\text{cris}}(T_{\Pi}') \otimes \mathcal{O} F$. 

iii) Let \( \{x_n, x_{n+1}\} \) be the Wach module basis of \( \mathbb{N}(T''_\Pi) \) given by \[\text{Proposition V.2.3 of [Ber04]} \] (which applies to \( T''_\Pi \) thanks to our hypotheses \((\text{N.ord})\) and \((\text{FL})\)) lifting \( \{\varpi_n, \varpi_{n+1}\} \) (meaning that \( x_i \) is sent to \( \varpi_i \) under the natural projection \( \mathbb{N}(T''_\Pi) \to \mathbb{D}_{\text{cris}}(T''_\Pi) \)).

iv) We define the change of basis matrices \( M \in \mathbb{M}_{2\times 2}(\mathbb{B}_{\text{rig}}^+) \) and \( Q \in \mathbb{M}_{2\times 2}(\mathbb{F}) \) given by the relations
\[
\begin{pmatrix}
x_n & x_{n+1}
x_n & x_{n+1}
\end{pmatrix} = \begin{pmatrix}
\varpi_n & \varpi_{n+1}
\varpi_n & \varpi_{n+1}
\end{pmatrix} M,
\begin{pmatrix}
\varpi_n & \varpi_{n+1}
\varpi_n & \varpi_{n+1}
\end{pmatrix} = \begin{pmatrix}
\varpi_n & \varpi_{n+1}
\varpi_n & \varpi_{n+1}
\end{pmatrix} Q.
\]

v) We define the logarithmic matrix
\[
M'_{\text{log}} = m^{-1} ((1 + \pi) A \varphi(M)) \in \mathcal{H}(\Gamma),
\]
where \( A \in \mathbb{M}_{2\times 2}(\mathbb{F}) \) denotes the matrix of \( \varphi \) with respect to \( \{\varpi_n, \varpi_{n+1}\} \).

vi) We define the logarithmic matrix
\[
M''_{\text{log}} = m^{-1} \begin{pmatrix}
0 & -p^{2h_{n+1}}(\alpha_n \alpha_{n+1})^{-1} \\
1 & p^{h_{n+1}}(\alpha_n^{-1} + \alpha_{n+1}^{-1})
\end{pmatrix},
\]
where \( \beta_n, \beta_{n+1} \in \mathbb{F}^\times \).

vii) We also define, for \( i = n, n + 1, \) the \( \Lambda(\Gamma) \)-morphisms
\[
\mathcal{L}_{\varpi_i} : H^1_{\text{Iw}}(\mathbb{Q}_p, T''_\Pi) \to \mathcal{H}_{\text{ord}_p(\alpha_i) - h_{n+1}}(\Gamma)
\]
given by the relation
\[
\mathcal{L}_{T''_\Pi}(z) = \mathcal{L}_{\varpi_n}(z) \varpi_n + \mathcal{L}_{\varpi_{n+1}}(z) \varpi_{n+1}.
\]

Remark 4.7. The proof of \[\text{[LLZ17, Lemma 3.1]} \] tells us that we may choose \( \varpi_{n+1} \) to be \( \varphi(\varpi_n) \), which we shall do in the rest of the article. In this case, \( A \) is given by
\[
\begin{pmatrix}
0 & -p^{2h_{n+1}}(\alpha_n \alpha_{n+1})^{-1} \\
1 & p^{h_{n+1}}(\alpha_n^{-1} + \alpha_{n+1}^{-1})
\end{pmatrix}.
\]
A direct calculation shows that the matrix \( Q \) is of the form
\[
\begin{pmatrix}
-p^{h_{n+1}}\alpha_n^{-1} \beta_n & -p^{h_{n+1}} \alpha_n^{-1} \beta_{n+1} \\
\beta_n & \beta_{n+1}
\end{pmatrix},
\]
\]
where \( \beta_n, \beta_{n+1} \in \mathbb{F}^\times \).

We now define Coleman maps for \( T'_\Pi \) by taking appropriate twists of those defined for \( T''_\Pi \).
Definition 4.8.

i) For \( i \in \{n, n+1\} \), we define the Coleman map

\[
\Col_{\omega_i} : H^1_{Iw}(\mathbb{Q}_p, T'_\Pi) \rightarrow \Lambda(\Gamma)
\]

as the composition

\[
H^1_{Iw}(\mathbb{Q}_p, T'_\Pi) \xrightarrow{e^{-h_{n+1}}} H^1_{Iw}(\mathbb{Q}_p, T''_\Pi) \xrightarrow{\Col_{\omega_i}} \Lambda(\Gamma) \xrightarrow{\Tw^{h_{n+1}}} \Lambda(\Gamma).
\]

ii) We define \( M'_{\log} \) to be \( \Tw^{h_{n+1}}(M''_{\log}) \).

iii) As in Definition 4.6, we also define, for \( i = n, n+1 \), the \( \Lambda(\Gamma) \)-morphisms

\[
\mathcal{L}_{\omega_i} : H^1_{Iw}(\mathbb{Q}_p, T'_\Pi) \rightarrow \mathcal{H}_{\text{ord}_p(\alpha_i)}(\Gamma)
\]

given by the relation

\[
\mathcal{L}_{T'_\Pi}(z) = \mathcal{L}_{\omega_n}(z)\omega_n + \mathcal{L}_{\omega_{n+1}}(z)\omega_{n+1}.
\]

Remark 4.9.

i) The maps \( \Col_{\omega_i} \) are non-zero and their images can be described explicitly by products of certain linear factors (see [LLZ11, Theorem 5.10]).

ii) The calculations in [BL21] (see particularly (9) in op. cit.) show that

\[
\left( \begin{array}{c} \mathcal{L}_{\omega_n} \\ \mathcal{L}_{\omega_{n+1}} \end{array} \right) = Q^{-1}M''_{\log} \left( \begin{array}{c} \Col_{\omega_n} \\ \Col_{\omega_{n+1}} \end{array} \right).
\]

Then (4.1) gives that

\[
\left( \begin{array}{c} \mathcal{L}_{\omega_n} \\ \mathcal{L}_{\omega_{n+1}} \end{array} \right) = \tilde{\ell}_{h_{n+1}} Q^{-1}M''_{\log} \left( \begin{array}{c} \Col_{\omega_n} \\ \Col_{\omega_{n+1}} \end{array} \right).
\]

We now define certain projections of the Perrin-Riou map and Coleman maps using wedge products. We expect that the former will give rise to the \( p \)-adic \( L \)-functions of Barrera–Dimitrov–Williams when applied to appropriate cohomology classes, whereas the latter will give rise to certain bounded \( p \)-adic \( L \)-functions (see Remark 5.11 for more details). The latter will also allow us to define signed Selmer groups, which is the content of §5 below.

Definition 4.10.

i) We set

\[
\alpha = \alpha_{n+1}\alpha_{n+2} \cdots \alpha_{2n}, \quad \beta = \alpha_{n}\alpha_{n+2} \cdots \alpha_{2n},
\]
For \( \lambda \in \{\alpha, \beta\}\)\(^2\) we write

\[
\rho_\lambda = \operatorname{ord}_p(\lambda) - \sum_{i=n+1}^{2n} h_i.
\]

ii) For \( i \in \{n+2, \ldots, 2n\} \), let \( \Pr_i : H^1_{Iw}(\mathbb{Q}_p, T_{\Pi}) \to H^1_{Iw}(\mathbb{Q}_p, T_i/T_{i-1}) \) denote the map induced by the projection \( T_{\Pi} \to T_i/T_{i-1} \) as given in Proposition 3.3. For \( i \in \{n, n+1\} \), we define \( \Pr_i : H^1_{Iw}(\mathbb{Q}_p, T_{\Pi}) \to H^1_{Iw}(\mathbb{Q}_p, T'_i) \) similarly.

iii) For \( \lambda \in \{\alpha, \beta\} \), we define the \( \Lambda(\Gamma) \)-morphism

\[
\mathcal{L}_\omega^{(\lambda)} : \bigwedge^n H^1_{Iw}(\mathbb{Q}_p, T_{\Pi}) \to \mathcal{H}_{r_\lambda}(\Gamma),
\]

\[
z_1 \wedge \cdots \wedge z_n \mapsto \left( \prod_{i=n+1}^{2n} \tilde{\ell}_{h_i} \right)^{-1} \det(\mathcal{L}_{\omega_i} \circ \Pr_i(z_j)),
\]

where the subscript \( \omega \) represents our choice of \( \varphi \)-eigenvectors \( \{\omega_i : i = n, n+1, \ldots, 2n\} \) and the subscripts \( i \) in the determinant run through the same subscripts in the product that defines \( \lambda \). (The fact that the determinant is divisible by the product of \( \tilde{\ell}_{h_i} \) follows from Proposition 4.2 and Lemma 4.5.)

iv) We define similarly for \( \bullet \in \{\# , \flat\} \) the Coleman maps

\[
\text{Col}_\omega^{\bullet} : \bigwedge^n H^1_{Iw}(\mathbb{Q}_p, T_{\Pi}) \to \Lambda(\Gamma),
\]

\[
z_1 \wedge \cdots \wedge z_n \mapsto \det(\text{Col}_{\omega_i}^{\bullet} \circ \Pr_i(z_j) | \text{Col}_{\omega_{i+2}}^{\bullet} \circ \Pr_{n+2}(z_j) | \cdots | \text{Col}_{\omega_{2n}}^{\bullet} \circ \Pr_{2n}(z_j)),
\]

where \( i = n \) for \( \bullet = \# \) and \( i = n + 1 \) for \( \bullet = \flat \).

Remark 4.11. It follows from (4.2) that

\[
\begin{pmatrix}
\mathcal{L}_\omega^{(\alpha)} \\
\mathcal{L}_\omega^{(\beta)}
\end{pmatrix} = Q^{-1}M'_\log \begin{pmatrix}
\text{Col}_\omega^{\#} \\
\text{Col}_\omega^{\flat}
\end{pmatrix}.
\]

We finish this subsection by presenting the interpolation formulae of \( \mathcal{L}_\omega^{(\lambda)} \).

For \( i \in \{n+2, \ldots, 2n\} \), we write \( \omega'_i \) for the dual basis of \( \omega_i \) under the natural pairing

\[
\langle -,- \rangle : \mathcal{D}_{\text{cris}}(T_i/T_{i-1}) \times \mathcal{D}_{\text{cris}}((T_i/T_{i-1})^*(1)) \to \mathcal{D}_{\text{cris}}(\mathcal{O}(1)) = \mathcal{O} \cdot t^{-1} e_1 \to \mathcal{O}.
\]

\(^2\)Let \( U_p \) be the Hecke operator corresponding to the diagonal matrix \( \begin{pmatrix} pI_n & 0 \\ 0 & I_n \end{pmatrix} \), where \( I_n \) denotes the \( n \times n \) identity matrix, then \( \lambda \) is an integrally normalized \( U_p \)-eigenvalue on cohomology of \( \Pi \). We thank Chris Williams for explaining this to us.
We write \( \{\omega'_n, \omega'_{n+1}\} \) for the dual basis of \( \{\omega_n, \omega_{n+1}\} \) under the natural pairing
\[
\langle - , - \rangle : \mathbb{D}_{\text{cris}}(T_n/T_{n-1}) \otimes F \times \mathbb{D}_{\text{cris}}((T_n/T_{n-1})^*(1)) \otimes F \to \mathbb{D}_{\text{cris}}(F(1)) = F \cdot t^{-1} e_1 \to F.
\]
Note that \( \varphi(\omega'_i) = p^{-1} \alpha_i \omega'_i \).

**Proposition 4.12.** Let \( k \) be an integer such that \( h_{n+1} \leq k \leq h_n - 1 \) and \( \theta \) a finite character on \( \Gamma \) of conductor \( p^m \cdot 1 \). For \( \lambda \in \{\alpha, \beta\} \) and \( z = z_1 \wedge \cdots \wedge z_n \in \bigwedge^n H^1_{\text{tr}}(\mathbb{Q}_p, T_H) \), we have
\[
\mathcal{L}^{(\lambda)}(z)(\lambda^k_{\text{cyc}} \theta) = \left( \prod_{i=n+1}^{2n} \ell_{\theta}(\lambda^k_{\text{cyc}} \theta) \right)^{-1} \det \left( \frac{k! p^{(m+1)k}}{\alpha_i^m \tau(\theta)} \langle \exp^*(e_\theta z_i \cdot e_{-k}) \cdot t^{-k} e_k, \omega'_i \rangle \right),
\]
where \( \tau(\theta) \) is the Gauss sum of \( \theta \) and \( e_\theta \) represents the element \( \sum_{\sigma \in \Gamma / \Gamma^{p^m-1}} \theta(\sigma) \sigma \) in the group ring \( \mathbb{Z}_p[\mu_{p^m-1}] / \Gamma^{\ell_{\text{cyc}}} \), the subscript \( i \) in the determinant is indexed as in Definition \( 4.10 \), the element \( z_{i,j} \) denotes \( \text{Pr}_i(z_j) \) and \( \exp^* \) signifies the Bloch–Kato dual exponential map on \( H^1(\mathbb{Q}_p(\mu_{p^m}), T_i/T_{i-1}) \) for \( i \geq n+2 \) (or \( H^1(\mathbb{Q}_p(\mu_{p^m}), T_H) \) otherwise).

**Proof.** It follows from [LZ11, Theorem B.5] that for \( i \in \{n, n+2, \ldots, 2n\} \) and \( z \in H^1_{\text{tr}}(\mathbb{Q}_p, T_i/T_{i-1}) \), we have
\[
\mathcal{L}_{T_i/T_{i-1}}(z)(\lambda^k_{\text{cyc}} \theta) = \frac{j! p^{(m+1)k}}{\tau(\theta)} \varphi^m(\exp^*(e_\theta z \cdot e_{-k}) \cdot t^{-k} e_k).
\]
Therefore,
\[
\mathcal{L}_{\omega_i}(z)(\lambda^k_{\text{cyc}} \theta) = \frac{j! p^{(m+1)k}}{\tau(\theta)} \varphi^m(\exp^*(e_\theta z \cdot e_{-k}) \cdot t^{-k} e_k, \omega'_i)
\]
\[
= \frac{j! p^{(m+1)k}}{\tau(\theta)} \langle \exp^*(e_\theta z \cdot e_{-k}) \cdot t^{-k} e_k, (p \varphi)^{-m}(\omega'_i) \rangle
\]
\[
= \frac{j! p^{(m+1)k}}{\tau(\theta)} \langle \exp^*(e_\theta z \cdot e_{-k}) \cdot t^{-k} e_k, \alpha_i^{-m}(\omega'_i) \rangle
\]
and the result follows. \( \square \)

### 4.2. Construction of bounded \( p \)-adic \( L \)-functions.

We introduce certain notions related to \( p \)-adic \( L \)-functions in order to prove Theorem 4.3. We set
\[
\text{Crit}(\Pi) = \{ j \in \mathbb{Z} : \mu_{n+1} \leq j \leq \mu_n \} = \{ j \in \mathbb{Z} : h_{n+1} - n + 1 \leq j \leq h_n - n \}.
\]
By [DJR20, Section 1.1], the half integers \( j + 1/2 \) for \( j \in \text{Crit}(\Pi) \) are precisely the critical points of the \( L \)-function \( L(\Pi, s) \). In a recent work of Barrera–Dimitrov–Williams [BDW21], \( p \)-adic \( L \)-functions \( L_\rho^{(\lambda)} \in H_{\tau_\lambda}(\Gamma) \) are constructed for \( \lambda \in \{\alpha, \beta\} \),
where $\alpha$ and $\beta$ are defined as in Definition 4.10. For an integer $j \in \text{Crit}(\Pi)$ and a finite character $\theta$ on $\Gamma$ of conductor $p^m > 1$, we have the interpolation formula:

$$L_p^{(\lambda)}(\chi^j_{\text{cyc}} \theta) = \frac{c_{j,\theta}}{\lambda^m} L(\Pi \otimes \theta, j + 1/2),$$

where $c_{j,\theta}$ is a constant independent of the choice of $\lambda$ (but depends on $j$ and $\theta$).

**Remark 4.13.** For $\lambda \in \{\alpha, \beta\}$, recall $r_\lambda$ from Definition 4.10. Our assumptions (M.Slo) and (N.ord) combine to give

$$\text{ord}_p(\lambda) < \mu_n - \mu + 1,$$

which is precisely the small slope bound condition, under which the construction of $p$-adic $L$-functions of Barrera–Dimitrov–Williams is valid unconditionally.\(^3\)

In order to relate these $p$-adic $L$-functions to the Perrin-Riou maps and logarithmic matrix studied in the previous section, we introduce the following twisted $p$-adic $L$-functions.

**Definition 4.14.** For $\lambda \in \{\alpha, \beta\}$ as in Definition 4.10, the twisted $p$-adic $L$-functions are defined by

$$L_p^{(\lambda)} = T_{w^{-1}} L_p^{(\lambda)}.$$

In particular, for $h_n + 1 \leq k \leq h_n - 1$ and $\theta$ a finite character on $\Gamma$,

$$L_p^{(\lambda)}(\chi^k_{\text{cyc}} \theta) = L_p^{(\lambda)}(\chi^j_{\text{cyc}} \theta) = \frac{c_{j,\theta}}{\lambda^m} L(\Pi \otimes \theta, j + 1/2),$$

where $j = k - n + 1 \in \text{Crit}(\Pi)$.

We now prove Theorem 4.15.

**Theorem 4.15.** There exist signed $p$-adic $L$-functions $L_p^\#$, $L_p^\flat \in H_0(\Gamma) = \Lambda(\Gamma) \otimes F$ such that

$$(L_p^{(\alpha)} \rho_p^{(\beta)}) = Q^{-1} M'_{\log} \left( L_p^\# \right).$$

Furthermore, at least one of the two signed $p$-adic $L$-functions, $L_p^\#$ and $L_p^\flat$, is non-zero.

**Proof.** By [BL21] Proposition 2.11], the matrix $Q^{-1} M'_{\log}$ satisfies the following property. For each $\mu \in \{\alpha_n, \alpha_n+1\}$, suppose that we are given $F_\mu \in H_{\text{ord}_p(\lambda)-h_n+1}(\Gamma)$ such that for all $j \in \{0, \ldots, h_n - h_n+1 - 1\}$ and all Dirichlet characters $\theta$ of conductor $p^m > 1$,

$$F_\mu(\chi^j_{\text{cyc}} \theta) = \mu^{-m} \times c_{j,\theta}$$

\(^3\)We thank Chris Williams for pointing this out to us.
for some constant $c_{j,\theta} \in \overline{\mathbb{Q}}_p$ that is independent of the choice of $\lambda$. Then there exist $F_\#, F_\flat \in \mathcal{H}_0(\Gamma)$ such that
\[
\begin{pmatrix} F_{\alpha_n} \\ F_{\alpha_{n+1}} \end{pmatrix} = Q^{-1} M'' \log \cdot \begin{pmatrix} F_\# \\ F_\flat \end{pmatrix}.
\]

For $\lambda \in \{\alpha, \beta\}$, let us write
\[
\mathcal{L}^{(\lambda)}_p = Tw^{-h_{n+1}} \mathcal{L}^{(\lambda)}_p.
\]
Then, we deduce from (4.3) that there exist $\mathcal{L}^{\#}_p, \mathcal{L}^{\flat}_p \in \mathcal{H}_0(\Gamma)$ such that
\[
\begin{pmatrix} \mathcal{L}^{(\alpha)}_p \\ \mathcal{L}^{(\beta)}_p \end{pmatrix} = Q^{-1} M'' \log \left( \begin{pmatrix} \mathcal{L}^{\#}_p \\ \mathcal{L}^{\flat}_p \end{pmatrix} \right).
\]
Hence, we may take
\[
\mathcal{L}^{\bullet} = Tw^{-h_{n+1}} \mathcal{L}^{\bullet}_p
\]
for $\bullet \in \{\#, \flat\}$, proving (4.4).

Our hypothesis (FL) implies that $|\text{Crit}(\Pi)| > 1$. As in [Roc22] proof of Proposition 3.13, there exists $j \in \text{Crit}(\Pi)$ such that $L(\Pi \otimes \theta, j + 1/2) \neq 0$ by [JS77, (1.3)]. In particular, both $\mathcal{L}^{(\lambda)}_p$ are non-zero. Consequently, $\mathcal{L}^{\#}_p$ and $\mathcal{L}^{\flat}_p$ cannot be simultaneously zero by (4.4).

After rescaling the periods in the construction of $\mathcal{L}^{(\alpha)}_p$ and $\mathcal{L}^{(\beta)}_p$ in [BDW21], one may assert that $\mathcal{L}^{\#}_p, \mathcal{L}^{\flat}_p \in \Lambda(\Gamma)$. However, it is unclear to us whether there is an optimal choice of such periods.

4.3. **Reformulation of Rockwood’s result.** In this section, we give an explicit description of the matrix $Q^{-1} M'' \log$ under the Pollack condition, that is
\[
\alpha_n + \alpha_{n+1} = 0.
\]
This allows us to recast Rockwood’s plus and minus $p$-adic $L$-functions obtained in [Roc22] in the framework of Theorem 4.15.

As we have already mentioned, (Pol) is a special case of the hypothesis (N.ord). For an integer $m \geq 1$, we define Pollack’s half logarithms
\[
\begin{align*}
\log^+_{p,m} &= \prod_{j=0}^{m-1} \frac{1}{p} \text{Tw}^{-j} \left( \prod_{i=1}^{\infty} \frac{\Phi_{2i}(X)}{p} \right), \\
\log^-_{p,m} &= \prod_{j=0}^{m-1} \frac{1}{p} \text{Tw}^{-j} \left( \prod_{i=1}^{\infty} \frac{\Phi_{2i-1}(X)}{p} \right).
\end{align*}
\]
We recall from [Roc22 §3] that there exist two bounded measures $L^\pm_p \in \mathcal{H}_0(\Gamma)$ such that

$$L^\pm_p = \frac{L_p^{(\alpha)} \pm L_p^{(\beta)}}{\text{Tw}^{h_{n+1} - m_1} log_{p,h_n-h_{n+1}}^\pm}.$$  

The construction of these $p$-adic $L$-functions follows closely the work of Pollack in [Pol03], where these functions were defined for normalized cuspidal eigen-newforms $f$ on $\text{GL}_2$ with $a_p(f) = 0$. Let us define $\mathcal{L}^\pm_p$ to be $\text{Tw}^{n-1} L^\pm_p$. Then, by definition,

$$\mathcal{L}^\pm_p = \frac{L_p^{(\alpha)} \pm L_p^{(\beta)}}{\text{Tw}^{n+1} log_{p,h_n-h_{n+1}}^\pm},$$

which we may rewrite as an matrix equation:

$$\left(\begin{array}{c}
L_p^{(\alpha)} \\
L_p^{(\beta)}
\end{array}\right) = \frac{1}{2} \left(\begin{array}{cc}
\text{Tw}^{h_{n+1}} \log_{p,h_n-h_{n+1}}^+ & \text{Tw}^{h_{n+1}} \log_{p,h_n-h_{n+1}}^-
\text{Tw}^{h_{n+1}} \log_{p,h_n-h_{n+1}}^+ & -\text{Tw}^{h_{n+1}} \log_{p,h_n-h_{n+1}}^-
\end{array}\right) \left(\begin{array}{c}
L_p^+ \\
L_p^-
\end{array}\right).$$  

(4.5)

We shall recast (4.5) in terms of the logarithmic matrix $Q^{-1}M^\prime_{\log}$. Under (Pol), the trace of the action of $\varphi$ on $\mathbb{D}^{\text{cris}}(T^\prime_{\Pi})$ is zero. As in [LLZ10, Proposition 5.10], the work of Berger–Li–Zhu [BLZ03] allows us to choose appropriate bases of $\mathbb{D}^{\text{cris}}(T^\prime_{\Pi})$ and $\mathbb{N}(T^\prime_{\Pi})$ so that the matrix $M^\prime_{\log}$ is of the form

$$M^\prime_{\log} = \left(\begin{array}{cc}
c^+ \log_{p,h_n-h_{n+1}}^+ & c^- \log_{p,h_n-h_{n+1}}^-
0 & 0
\end{array}\right),$$

where $c^\pm \in \mathcal{H}_0(\Gamma_1)^\times$. Furthermore, we may choose $\beta_n = \beta_{n+1} = 1$ in the matrix $Q$ (see Remark 4.7), then

$$Q = \left(\begin{array}{cc}
p^{h_{n+1}} \alpha_n^{-1} & -p^{h_{n+1}} \alpha_n^{-1}
1 & 1
\end{array}\right).$$

We deduce that

$$Q^{-1}M^\prime_{\log} = \frac{1}{2} \left(\begin{array}{cc}
d^+ \text{Tw}^{h_{n+1}} \log_{p,h_n-h_{n+1}}^+ & d^- \text{Tw}^{h_{n+1}} \log_{p,h_n-h_{n+1}}^-
\text{Tw}^{h_{n+1}} \log_{p,h_n-h_{n+1}}^+ & -d^- \text{Tw}^{h_{n+1}} \log_{p,h_n-h_{n+1}}^-
\end{array}\right)$$

for some $d^\pm \in \mathcal{H}(\Gamma_1)^\times$. Therefore, we may rewrite (4.5) as

$$\left(\begin{array}{c}
\mathcal{L}_p^{(\alpha)} \\
\mathcal{L}_p^{(\beta)}
\end{array}\right) = \frac{1}{2} Q^{-1}M^\prime_{\log} \left(\begin{array}{c}
\frac{1}{d^+} \mathcal{L}_p^+ \\
\frac{1}{d^-} \mathcal{L}_p^-
\end{array}\right).$$  

We conclude this section by the following remark.

**Remark 4.16.** Relying on the explicit description of Pollack’s half logarithms, Rockwood proved in [Roc22 Proposition 3.13] that both $\mathcal{L}_p^+$ and $\mathcal{L}_p^-$ are non-zero, which is stronger than the last assertion of Theorem 4.15. It is not clear to us how to prove...
5. Iwasawa main conjectures and cohomological classes

In this section, we define the so-called signed Selmer groups using the Coleman maps we defined in Section 4.1 and formulate Iwasawa main conjectures relating them to the $p$-adic $L$-functions studied in Section 4.2.

5.1. Definitions of Selmer groups. Let us first recall the definitions of the Bloch–Kato Selmer group and the fine Selmer groups of Coates–Sujatha over a number field studied in [BK90] and [CS05] respectively.

**Definition 5.1.**

i) Let $\Sigma$ be the set of prime numbers $\ell$ where $T_{\Pi}$ has bad reduction, as well as the prime $p$ and the archimedean prime in $\mathbb{Q}$. Let $K$ be a number field and write $K_\Sigma$ for the maximal extension of $K$ that is unramified outside $\Sigma$.

ii) Write $T_{\Pi}^\dagger = \text{Hom}_{cts}(T_{\Pi}, F/\mathcal{O}(1))$. The Bloch–Kato Selmer group of $T_{\Pi}^\dagger$ over $K$ is defined as

$$\text{Sel}_p(T_{\Pi}^\dagger/K) := \text{Ker} \left( H^1(K_\Sigma/K, T_{\Pi}^\dagger) \to \prod_v \frac{H^1(K_v, T_{\Pi}^\dagger)}{H^1(K_v, T_{\Pi}^\dagger)} \right),$$

where $v$ runs through all the places of $K$ dividing $\Sigma$ (see [BK90], Section 3) for the definition of the Bloch–Kato subgroups $H^1_f(K_v, T_{\Pi}^\dagger)$.

iii) The fine Selmer group of $T_{\Pi}^\dagger$ over $K$ is defined as

$$\text{Sel}_p^0(T_{\Pi}^\dagger/K) := \text{Ker} \left( H^1(K_\Sigma/K, T_{\Pi}^\dagger) \to \prod_v H^1(K_v, T_{\Pi}^\dagger) \right),$$

where $v$ runs through all the places of $K$ dividing $\Sigma$.

iv) If $L$ is an infinite algebraic extension of $\mathbb{Q}$, we may define $\text{Sel}_p(T_{\Pi}^\dagger/L)$ and $\text{Sel}_p^0(T_{\Pi}^\dagger/L)$ by the direct limits $\varinjlim \text{Sel}_p(T_{\Pi}^\dagger/K)$ and $\varinjlim \text{Sel}_p^0(T_{\Pi}^\dagger/K)$ respectively, where $K$ runs over finite sub-extensions of $L$.

Notice that if $L$ is an infinite algebraic extension inside $\mathbb{Q}_\Sigma$, we have

$$\text{Sel}_p(T_{\Pi}^\dagger/L) = \text{Ker} \left( H^1(\mathbb{Q}_\Sigma/L, T_{\Pi}^\dagger) \to \prod_v \frac{H^1(L_w, T_{\Pi}^\dagger)}{H^1_f(L_w, T_{\Pi}^\dagger)} \right),$$
where \( w \) runs through all places of \( L \) dividing \( \Sigma \) and \( H_j^1(L_w, T^{\dagger}_\Pi) \) is given by \( \lim_{\rightarrow} H_j^1(K_v, T^{\dagger}_\Pi) \), where \( K \) runs through all finite sub-extensions of \( L \) and \( v \) denotes the place of \( K \) lying below \( w \). Similarly,

\[
\text{Sel}_p^0(T^{\dagger}_\Pi/L) = \text{Ker} \left( H^1(\mathbb{Q}_\Sigma/L, T^{\dagger}_\Pi) \to \prod_v H^1(L_w, T^{\dagger}_\Pi) \right).
\]

We now give an alternative description of the dual fine Selmer group \( \text{Sel}_p^0(T^{\dagger}_\Pi/K)^\vee \).

By [PR95, Section A.3], we have the Poitou–Tate exact sequence

\[
\bigoplus_{v \in \Sigma} H^0(K_v, T^{\dagger}_\Pi) \to H^2(K_\Sigma/K, T^{\dagger}_\Pi) \to H^1(K_\Sigma/K, T^{\dagger}_\Pi) \to \bigoplus_{v \in \Sigma} H^1(K_v, T^{\dagger}_\Pi).
\]

Therefore, the fine Selmer group sits inside the following exact sequence:

\[
\bigoplus_{v \in \Sigma} H^0(K_v, T^{\dagger}_\Pi) \to H^2(K_\Sigma/K, T^{\dagger}_\Pi) \to \text{Sel}_p^0(T^{\dagger}_\Pi/K) \to 0.
\]

Taking Pontryagin duals and using the fact that \( H^0(K_v, T^{\dagger}_\Pi)^\vee \cong H^2(K_v, T^{\dagger}_\Pi) \), we obtain

\[
(5.1) \quad \text{Sel}_p^0(T^{\dagger}_\Pi/K)^\vee = \text{Ker} \left( H^2(K_\Sigma/K, T^{\dagger}_\Pi) \to \bigoplus_{v \in \Sigma} H^2(K_v, T^{\dagger}_\Pi) \right).
\]

Set \( \mathbb{Q}_\infty = \mathbb{Q}(\mu_{p\infty}) \) and \( \mathbb{Q}_{p,k} = \mathbb{Q}_p(\mu_{p^k}) \). We now define the signed Selmer groups of \( T^{\dagger}_\Pi \) over \( \mathbb{Q}_\infty \).

**Definition 5.2.**

i) For \( \bullet \in \{\# , \♭\} \), define the direct sum counterpart of \( \text{Col}_\omega^\bullet \) to be \( \overline{\text{Col}}_\omega^\bullet : H^1_{lw}(\mathbb{Q}_p, T^{\dagger}_\Pi) \to \Lambda(\Gamma)^{\oplus n} \),

\[
z \mapsto \text{Col}_{\omega_i}^\bullet \circ \text{Pr}_i \oplus \bigoplus_{j=n+2}^{2n} \text{Col}_{\omega_j} \circ \text{Pr}_j,
\]

where \( i = n \) for \( \bullet = \# \) and \( i = n + 1 \) for \( \bullet = \♭ \). We write

\[
H^1_{lw}(\mathbb{Q}_p, T^{\dagger}_\Pi)^\bullet = \text{ker} \left( \overline{\text{Col}}_\omega^\bullet \right).
\]

ii) Local Tate duality gives a perfect pairing

\[
(5.2) \quad H^1_{lw}(\mathbb{Q}_p, T^{\dagger}_\Pi) \times H^1(\mathbb{Q}_p(\mu_{p\infty}), T^{\dagger}_\Pi) \to F/O.
\]
We define $H^1_{\bullet}(\mathbb{Q}_p(\mu_p), T_{\Pi}^\dagger)$ to be the orthogonal complement of $H^1_{\text{Iw}}(\mathbb{Q}_p, T_{\Pi})^\bullet$ under the Tate pairing (5.2).

We can now finally define the “signed” Selmer groups for $T_{\Pi}^\dagger$ as follows.

**Definition 5.3.** Let $\bullet \in \{\#, \flat\}$. We define $\text{Sel}_{\bullet}(T_{\Pi}^\dagger/\mathbb{Q}_\infty)$ to be the kernel of

$$H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, T_{\Pi}^\dagger) \to \frac{H^1(\mathbb{Q}_p(\mu_p), T_{\Pi}^\dagger)}{H^1_{\bullet}(\mathbb{Q}_p(\mu_p), T_{\Pi})} \times \prod_{v \mid p} H^1(\mathbb{Q}_{\infty,v}, T_{\Pi}^\dagger)$$

where the last product runs through places of $\mathbb{Q}_\infty$ dividing $\Sigma$ but not $p$.

**Lemma 5.4.** For $\bullet \in \{\#, \flat\}$ the $\Lambda(\Gamma)$-module $\text{Sel}^\bullet_p(T_{\Pi}^\dagger/\mathbb{Q}_\infty)^\flat$ is finitely generated.

**Proof.** This follows from the fact that $H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, T_{\Pi}^\dagger)^\flat$ is finitely generated over $\Lambda(\Gamma)$ (which is a result of Greenberg [Gre89, Proposition 3]).

We finish this subsection with an alternative description of $H^1_{\text{Iw}}(\mathbb{Q}_p, T_{\Pi})^\bullet$. Note that

$$\ker(\text{Col}^\bullet) = \ker(\text{Col}_{\omega_i} \circ \text{Pr}_j) \cap \bigcap_{j=n+2}^{2n} \ker(\text{Col}_{\omega_j} \circ \text{Pr}_j)$$

where $i$ is given as in Definition 5.2(i). For $n + 2 \leq j \leq 2n$, recall from that $\ker(\text{Col}_{\omega_j}) = 0$. This implies that $z \in H^1_{\text{Iw}}(\mathbb{Q}_p, T_{\Pi})$ lies inside $\ker(\text{Col}_{\omega_j} \circ \text{Pr}_j)$ if and only if its projection in $H^1_{\text{Iw}}(\mathbb{Q}_p, T_j/T_{j-1})$ is zero. Considering the filtration

$$T_{n+1} \subset T_{n+2} \subset \cdots \subset T_{2n},$$

we see that

$$\bigcap_{j=n+2}^{2n} \ker(\text{Col}_{\omega_j} \circ \text{Pr}_j) = \text{Image}(H^1_{\text{Iw}}(\mathbb{Q}_p, T_{n+1}) \to H^1_{\text{Iw}}(\mathbb{Q}_p, T_{n+1})).$$

It remains to study $\ker(\text{Col}_{\omega_i} \circ \text{Pr}_j)$. Let us write $\text{Pr}'$ for the natural projection $H^1_{\text{Iw}}(\mathbb{Q}_p, T_{\Pi}) \to H^1_{\text{Iw}}(\mathbb{Q}_p, T_{\Pi})'$. Then,

$$\ker(\text{Col}_{\omega_i} \circ \text{Pr}_j) = \{x \in H^1_{\text{Iw}}(\mathbb{Q}_p, T_{\Pi}) : \text{Pr}'(z) \in \ker(\text{Col}_{\omega_i})\}.$$

Therefore, we deduce that

$$H^1_{\text{Iw}}(\mathbb{Q}_p, T_{\Pi})'^{\#} = \{x \in \text{Image}(H^1_{\text{Iw}}(\mathbb{Q}_p, T_{\Pi}) \to H^1_{\text{Iw}}(\mathbb{Q}_p, T_{\Pi})): \text{Pr}'(z) \in \ker(\text{Col}_{\omega_i})\},$$

$$H^1_{\text{Iw}}(\mathbb{Q}_p, T_{\Pi})'^{\flat} = \{x \in \text{Image}(H^1_{\text{Iw}}(\mathbb{Q}_p, T_{\Pi}) \to H^1_{\text{Iw}}(\mathbb{Q}_p, T_{\Pi})): \text{Pr}'(z) \in \ker(\text{Col}_{\omega_{i+1}})\}.$$

When $\alpha_n = -\alpha_{n+1}$, (that is, when the Pollack condition holds), we may describe $\ker\text{Col}_{\omega_n}$ and $\ker\text{Col}_{\omega_{n+1}}$ explicitly in terms of certain jumping conditions.
on the classes in $H^1(Q_{p,k}, T^\dagger_\Pi)$, similar to Kobayashi’s plus and minus Selmer conditions for elliptic curves studied in [Kob03] as well as their generalizations to elliptic modular forms studied in [Lei11].

5.2. Iwasawa Main Conjectures. We can now formulate the following “signed” Iwasawa Main Conjecture.

**Conjecture 5.5.** For $\bullet \in \{\# , \flat\}$ and $\eta$ a Dirichlet character modulo $p$, the $\Lambda(\Gamma_1)$-module $\text{Sel}_p^\bullet(T^\dagger_\Pi/Q_{\infty})^{\vee, \eta}$ is torsion and there exists an integer $k(\bullet, \eta) \geq 0$, depending on $\eta$ and $\bullet$, such that

$$\text{char}_{\Lambda(\Gamma_1)}(\text{Sel}_p^\bullet(T^\dagger_\Pi/Q_{\infty})^{\vee, \eta}) = (\pi^{k(\bullet, \eta)}(\mathcal{L}^\bullet_p/\mathcal{I}_{\bullet, \eta}),$$

where $\pi$ is a uniformizer of $F$ and $\mathcal{I}_{\bullet, \eta}$ denotes a generator of $\text{char}_{\Lambda(\Gamma_1)}(\text{coker}(\text{Col}_{\omega})^\eta)$. 

**Remark 5.6.** We expect that $\mathcal{L}^\bullet_p$ to be inside the image of $\text{Col}_{\omega}$, which explains the presence of the term $\mathcal{I}_{\bullet, \eta}$. See Remark 5.11 in the next section for more details. The appearance of $k(\bullet, \eta)$ is due to the possible non-integrality of the $p$-adic $L$-functions. For any given $\bullet$ and $\eta$, one may set this constant to be zero after normalizing the periods in the construction of Barrera–Dimitrov-Williams’ $p$-adic $L$-functions. However, it is not clear to us whether this can be done simultaneously for all choices of $\bullet$ and $\eta$. 

We conclude this subsection by saying a few words on why we expect $\text{Sel}_p^\bullet(T^\dagger_\Pi/Q_{\infty})^{\vee, \eta}$ to be torsion over $\Lambda(\Gamma_1)$. We recall the following weak Leopoldt conjecture for $T^\dagger_\Pi$ (see [Gre89, Conjecture 3]).

**Conjecture 5.7.** The second cohomology group $H^2(Q_{\Sigma}/Q_{\infty}, T^\dagger_\Pi)$ is zero.

By Remark 2.5 and [PR95, Proposition 1.3.2], Conjecture 5.7 is equivalent to

$$\text{rank}_{\Lambda(\Gamma_1)}(H^1(Q_{\Sigma}/Q_{\infty}, T^\dagger_\Pi)^{\vee, \eta} = n.$$ 

As in [Gre89] proof of Proposition 6],

$$\text{rank}_{\Lambda(\Gamma_1)} \left( \prod_{v \mid p} \frac{H^1(Q_{\infty,v}, T^\dagger_\Pi)}{H^1(Q_{\infty,v}, T^\dagger_\Pi)} \right)^{\vee, \eta} = 0.$$ 

By duality, we have

$$\text{rank}_{\Lambda(\Gamma_1)} \left( \frac{H^1(Q_{p, \mu_{p, \infty}}, T^\dagger_\Pi)}{H^1(Q_{p, \mu_{p, \infty}}, T^\dagger_\Pi)} \right)^{\vee, \eta} = \text{rank}_{\Lambda(\Gamma_1)} \left( \text{Ker}(\text{Col}_{\omega})^\eta \right) = n,$$

since $H^1_{Iw}(Q_{p}, T^\dagger_\Pi)^{\eta}$ is of rank $2n$ over $\Lambda(\Gamma_1)$ (see [PR94, Proposition in §3.2.1]) and the image of each Coleman map in the direct sum defining $\text{Col}_{\omega}$ is non-zero (see...
Lemma 4.4 and Remark 4.9(i)). Therefore, Sel_p(T^\dagger/\Q_\infty) is the kernel of a morphism from a \Lambda(\Gamma_1)-module of corank conjecturally n to a \Lambda(\Gamma_1)-module of corank n, which is a necessary (but not sufficient) condition to be cotorsion. This is in line with the Greenberg Selmer group for p-ordinary representations studied in [Gre89].

5.3. Conjectural Euler systems. We discuss in this section how the existence of an Euler system for the representation T_\Pi would allow us to obtain evidence towards Conjectures 5.5. We emphasize that the discussion in this section is mostly speculative. Throughout, we fix \bullet \in \{\#, \flat\} and a Dirichlet character \eta modulo p.

Definition 5.8. For i = 1, 2, define the \Lambda(\Gamma)-module

H^i(T_\Pi) = \varprojlim H^i(\Q_\Sigma/\Q(\mu_p^k), T_\Pi).

Perrin-Riou formulated the following conjecture on the existence of Euler systems in [PR98]:

Conjecture 5.9. Let N be the set of integers of the form mp^k where m is a square-free product of integers that are coprime to the conductor of T_\Pi. There exists a system of cohomological classes

\left\{ c_r \in \bigwedge^n H^1(\Q(\mu_r), T_\Pi) : r \in N \right\}

where \eta = Gal(\Q(\mu_r)/\Q) satisfying a precise norm relation as m varies. Furthermore, the p-localization of c_r is related to the complex L-values of T_\Pi^\ast(1) twisted by characters on \eta under the Bloch–Kato dual exponential map. Furthermore, as r varies, these classes are compatible under corestriction maps, up to multiplication by explicit Euler factors.

Remark 5.10. When the representation comes from elliptic modular forms, the existence of Euler systems is known; thanks to the work of Kato [Kat04]. There are also results on the existence of rank-one Euler systems (classes lying inside H^1(\Q(\mu_r), T_\Pi), rather than in a wedge product) when G = GSp(4) in the ordinary case (see [LSZ22, LZ20]) and G = GL(2) \times GL(2) in both ordinary and non-ordinary cases (see [LLZ14, KLZ20, KLZ17, LZ16]).

Remark 5.11. Note that Conjecture 5.9 predicts the existence of a special element z = z_1 \wedge \cdots \wedge z_n \in \bigwedge^n H^1(T_\Pi). It seems reasonable to expect the following equality to hold:

L_\infty^{(\lambda)}(\text{loc}(z)) \overset{?}{=} L_p^{(\lambda)},
\( \lambda \in \{ \alpha, \beta \} \). Here loc denotes the localization map

\[ H^1(T_\Pi) \rightarrow H^1_{lw}(\mathbb{Q}_p, T_\Pi), \]

which we extend to the wedge products. In fact, Proposition 4.12 gives us a hint on how the classes should be related to complex \( L\)-values under localizations and the Bloch–Kato dual exponential map. For the rest of the article, we assume such an element \( z \) does exist. Then (5.3) would imply that

\[ \text{Col}^\bullet_{\Psi}(\text{loc}(z)) = \mathcal{L}_p^\bullet \]

and it would give an alternative and more direct proof of Theorem B. Under various technical hypotheses of the Euler system machinery, it would also give the inclusion \( \supset \) of Conjecture 5.5.

The Poitou–Tate exact sequence in [PR95, Proposition A.3.2] gives the following exact sequence

\[
\begin{align*}
H^1(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^k}), T_\Pi) & \rightarrow H^1(\mathbb{Q}_{p,k}, T_\Pi)^\bullet \oplus \bigoplus_{v|p, v \in \Sigma} H^1(\mathbb{Q}(\mu_{p^k}), T_\Pi) \\
& \rightarrow \text{Sel}^\bullet_{p}(T_\Pi/\mathbb{Q}(\mu_{p^k}))^\vee \\
& \rightarrow H^2(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^k}), T_\Pi) \rightarrow \bigoplus_{v \in \Sigma} H^2(\mathbb{Q}(\mu_{p^k}), T_\Pi).
\end{align*}
\]

Upon taking inverse limits, [Kat04, Section 17.10] tells us that the modules in the second term of (5.3) vanish for \( v \nmid p \). Hence, after taking the \( \eta \) component and inverse limits\(^4\), we obtain the following exact sequence

\[
\begin{align*}
H^1(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_{p^k}), T_\Pi) & \rightarrow H^1(\mathbb{Q}_{p,k}, T_\Pi)^\bullet \oplus \bigoplus_{v|p, v \in \Sigma} H^1(\mathbb{Q}(\mu_{p^k}), T_\Pi) \\
& \rightarrow \text{Sel}^\bullet_{p}(T_\Pi/\mathbb{Q}(\mu_{p^k}))^\vee \\
& \rightarrow \text{Sel}^0_{p}(T_\Pi/\mathbb{Q}(\mu_{p^k}))^\vee \rightarrow 0.
\end{align*}
\]

Let us write

\[ Z(T_\Pi) := \text{Span}_{\Lambda(\Gamma)} \{ z_j \}_{j=1}^n \subset H^1(T_\Pi). \]

\(^4\)Since the \( \mathcal{O} \)-modules appearing in the Poitou–Tate exact sequence are finitely generated, the Mittag–Leffler condition is satisfied.
Then (5.5) gives the following exact sequence:

\[(5.6) \quad 0 \to \frac{H^1(T_{\Pi})^\eta}{Z(T_{\Pi})^\eta} \to \text{Image} \left( \frac{\text{Col}^\bullet}{\text{Col} \circ \text{loc}(Z(T_{\Pi}))^\eta} \right) \to \text{Sel}_p^0(T_{\Pi}/Q_\infty)^{\vee,\eta} \to \text{Sel}_0^0(T_{\Pi}/Q_\infty)^{\vee,\eta} \to 0.\]

In particular, we see that the equality of characteristic ideals in Conjecture 5.5 is, up to a power of \(\pi\), equivalent to

\[(5.7) \quad \text{char}_{\Lambda(\Gamma_1)} \left( \frac{H^1(T_{\Pi})^\eta}{Z(T_{\Pi})^\eta} \right) \cong \text{char}_{\Lambda(\Gamma_1)} \left( \text{Sel}_p^0(T_{\Pi}/Q_\infty)^{\vee,\eta} \right).\]

We note that \(\text{Sel}_p^0(T_{\Pi}/Q_\infty)^{\vee,\eta} \hookrightarrow H^2(T_{\Pi})^\eta\) and is hence torsion whenever \(H^2(T_{\Pi})^\eta\) is torsion. As \(T_{\Pi}\) is a finitely generated \(\mathbb{Z}_p\)-module, we have

\[\lim_{\leftarrow} \bigoplus_{v \mid p, v \in \Sigma} H^0(\kappa(v), H^1((\mathbb{Q}(\mu_{p^k})_v)_{nr}, T_{\Pi})) = 0,\]

where \((\mathbb{Q}(\mu_{p^k})_v)_{nr}\) is the maximal unramified extension of \(\mathbb{Q}(\mu_{p^k})_v\) and \(\kappa(v)\) is the residue field at \(v\). By an argument similar to [Kur02, p. 217], we have

\[\text{char}_{\Lambda(\Gamma_1)} \left( \text{Sel}_p^0(T_{\Pi}/Q_\infty)^{\vee,\eta} \right) = \text{char}_{\Lambda(\Gamma_1)} \left( H^2(T_{\Pi})^\eta \right).\]

Hence (5.7) is equivalent to

\[\text{char}_{\Lambda(\Gamma_1)} \left( \frac{H^1(T_{\Pi})^\eta}{Z(T_{\Pi})^\eta} \right) \cong \text{char}_{\Lambda(\Gamma_1)} \left( H^2(T_{\Pi})^\eta \right).\]

This is analogous to Kato’s Iwasawa main conjecture without \(p\)-adic zeta functions formulated for elliptic modular forms (see [Kat04, Conjecture 12.10]).

**Remark 5.12.** In this remark, we suppose that Conjecture 5.7 holds. By Remark 2.5 and [PR95, Prop. 1.3.2], \(H^2(T_{\Pi})^\eta\) would be \(\Lambda(\Gamma_1)\)-torsion, whereas the \(\Lambda(\Gamma_1)\)-rank of \(H^1(T_{\Pi})^\eta\) would be \(n\). Then (5.5) tells us that the torsionness of \(\text{Sel}_p^0(T_{\Pi}/Q_\infty)^{\vee,\eta}\) is in fact equivalent to the existence of an element \(c_{p,\infty} \in \bigwedge^n H^1(T_{\Pi})^\eta\) such that its image under \(\text{Col}^\bullet \circ \text{loc}\) is non-zero.

**Data availability statement**

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.
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