Non Archimedean Pseudodifferential Equations of Klein-Gordon Type and Quantum Scalar Fields

W. A. Zúñiga-Galindo

Abstract. In this article we introduce a new class of non Archimedean pseudodifferential equations of Klein-Gordon type whose solutions can be easily quantized using the machinery of the second quantization. We study the Cauchy problem for these equations. We present a ‘semi-formal’ construction, on the \( p \)-adic Minkowski space, of the neutral and charged quantum scalar fields having a nonzero \( p \)-adic number as mass parameter.

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1. Introduction

In the 80’s I. Volovich proposed that the world geometry in regimes smaller that the Planck scale might be non Archimedean \[21, 22\]. This hypothesis conducts naturally to consider models involving geometry and analysis over \( \mathbb{Q}_p \), the field of \( p \)-adic numbers, \( p \) being a rational prime \( \geq 2 \). Since then, a big number of articles have appeared exploring these and related themes, see e.g. \[4, 19\] Chapter 6 and the references therein. The problem of constructing non Archimedean analogues of infinite dimensional quantum systems such as quantum fields or scattering theory has been proposed by several authors see e.g. \[20, 19\] Chapter 6. It is well-known that the construction of (Euclidean) quantum fields is reduced to the construction of certain probability measures on Euclidean space-time. For free fields the measure is Gaussian and for interacting fields non-Gaussian measures are required. In a seminal article \[13\] A. N. Kochubei and M. R. Sait-Ametov constructed such non-Gaussian measures on the space \( S'(\mathbb{Q}_p^n) \) of distributions on \( \mathbb{Q}_p^n \).

In this article we introduce a new class of non Archimedean pseudodifferential equations of Klein-Gordon type whose solutions can be easily quantized using the machinery of the second quantization. We work on the \( p \)-adic Minkowski space which is the quadratic space \( (\mathbb{Q}_p^4, Q) \) where \( Q(x) = k_0^2 - k_1^2 - k_2^2 - k_3^2 \). Our starting point is a result of Rallis-Schiffman that asserts the existence of a unique measure on \( V_t = \{ k \in \mathbb{Q}_p^4 : Q(k) = t \} \) which is invariant under the orthogonal group \( O(Q) \) of \( Q \), see Proposition 1 or \[14\]. By using Gel’fand-Leray differential forms, we reformulate this results in terms of Dirac distributions \( \delta(Q(k) - t) \) invariant under \( O(Q) \), see Remark 2 and Lemma 1. We introduce a notion of positivity, see Definition 1.

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that allows us to introduce positive and negative mass shells $V_{m^2}^+$ and $V_{m^2}^-$, here $m$ is the mass parameter which is taken to be a nonzero $p$-adic number. The mass shells can be describe as $V_{m^2}^\pm = \{ (k_0,k) \in \mathbb{Q}_p^2 : k_0 = \pm \sqrt{k \cdot k + m^2}, \text{ for } k \in U_{m^2} \}$ where $\sqrt{k \cdot k + m^2}$ denotes a $p$-adic analytic function on $U_{m^2}$ which is an open closed subset of $\mathbb{Q}_p^3$, see Lemma 2. The restriction of $\delta (Q(k) - t)$ to $V_{m^2}^\pm$ gives two distributions $\delta_\pm (Q(k) - t)$ which are invariant under $\mathcal{L}_\pm^\wedge$, the ‘Lorentz proper group’, see Definition 2 and that satisfy $\delta (Q(k) - t) = \delta_+ (Q(k) - t) + \delta_- (Q(k) - t)$, see Lemma 4. The $p$-adic Klein-Gordon type pseudodifferential operators introduced here have the form

$$\left( \Box_{\alpha,m} \varphi \right)(x) = \mathcal{F}^{-1}_{k \rightarrow x} \left[ (Q(k) - m^2)^{\alpha} \mathcal{F}_{k \rightarrow \varphi} \right], \alpha > 0, m \in \mathbb{Q}_p \setminus \{ 0 \},$$

where $\mathcal{F}$ denotes the Fourier-Minkowski transform. We solve the Cauchy problem for these operators, see Theorem 11. The equations $\left( \Box_{\alpha,m} \varphi \right)(t, x) = 0$ have many similar properties to the classical Klein-Gordon equations, see e.g. 2, 3, 7, 16. These equation admit plane waves as weak solutions, see Lemma 5. The distributions $a\mathcal{F}^{-1}[\delta_\pm (Q(k) - m^2)] + b\mathcal{F}^{-1}[-\delta_\pm (Q(k) - m^2)]$, $a, b \in \mathbb{C}$, are weak solutions of these equations, see Proposition 2 the locally constant functions

$$\varphi(t, x) = \int_{U_{m^2}} \chi_p(x \cdot k) \left\{ \chi_p(-t \omega(k)) \phi_+(k) + \chi_p(t \omega(k)) \phi_-(k) \right\} d^3k,$$

where $\chi_p(\cdot)$ denotes the standard additive character of $\mathbb{Q}_p$ and $\phi_\pm$ are locally constant functions $\phi_\pm$ with support in $U_{m^2}$, are weak solutions of these equations. These last solutions can be quantized using the machinery of the second quantization. More precisely, the construction of a neutral (real) quantum scalar field with mass parameter $m \in \mathbb{Q}_p^\wedge$ can be carry out using the machinery of the second quantization starting with $\mathcal{H} = L^2(V_{m^2}^\wedge, d\lambda_{m^2})$, the state space for a single spin-zero particle of mass $m$, see Section 6 and 5, 6, 15. The construction of the charged complex field presented for instance in 6 Section 5.2 can be carry out in the non-Archimedean setting by using the positive and negative mass shells $V_{m^2}^\pm$ and $V_{m^2}^-$ each equipped with the measure $d\lambda_{m^2}(k)$, see Section 6 and 6.

We worked along the article with a fixed quadratic form defined on $\mathbb{Q}_p$, however, the techniques used and the results are valid for a large class of quadratic forms defined on a locally compact field of characteristic different from 2. Finally, we want to mention that other type of wave equations were introduced by A. N. Kochubei in 11, 12.

2. Preliminaries

Along this article $p$ will denote a prime number different from 2. The field of $p$–adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p$–adic norm $| \cdot |_p$, which is defined as

$$| x |_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-\gamma} & \text{if } x = p^\gamma \frac{a}{b}, \end{cases}$$

where $a$ and $b$ are integers coprime with $p$. The integer $\gamma := \text{ord}(x)$, with $\text{ord}(0) := +\infty$, is called the $p$–adic order of $x$. Any $p$–adic number $x \neq 0$ has a unique expansion $x = p^{\text{ord}(x)} \sum_{j=0}^{\infty} x_j p^j$, where $x_j \in \{ 0, 1, 2, \ldots, p - 1 \}$ and $x_0 \neq 0$. Thus any nonzero $p$–adic number can written uniquely as $x = p^{\text{ord}(x)} ac(x)$, where $ac(x)$,
the angular component of \( x \), is a unit i.e. \( |ac(x)|_p = 1 \). For a unit \( a = \sum_{i=0}^\infty a_i p^i \), \( a_0 \neq 0 \), we define \( \tau := a_0 \in \mathbb{F}_p \), where \( \mathbb{F}_p \) denotes the field of \( p \) elements. We also define the fractional part of \( x \in \mathbb{Q}_p \), denoted \( \{x\}_p \), as the rational number

\[
\{x\}_p = \begin{cases} 0 & \text{if } x = 0 \text{ or } \text{ord}(x) \geq 0 \\ p^{\text{ord}(x)} \sum_{j=0}^{-\text{ord}(x)-1} x_j p^j & \text{if } \text{ord}(x) < 0.
\end{cases}
\]

Set \( \chi(y) = \exp(2\pi i \{y\}_p) \) for \( y \in \mathbb{Q}_p \). The map \( \chi(\cdot) \) is an additive character on \( \mathbb{Q}_p \), i.e. a continuous map from \( \mathbb{Q}_p \) into \( S \) (the unit circle) satisfying \( \chi(y_0 + y_1) = \chi(y_0)\chi(y_1) \).

We extend the \( p \)-adic norm to \( \mathbb{Q}_p^n \) by taking

\[
||x||_p := \max_{1 \leq i \leq n} |x_i|_p, \quad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.
\]

For \( \gamma \in \mathbb{Z} \), denote by \( B_0^p(a) = \{x \in \mathbb{Q}_p^n : ||x - a||_p \leq p^\gamma \} \) the ball of radius \( p^\gamma \) with center at \( a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n \), and take \( B_0^p(0) := B^p_0 \). Note that \( B_0^p(a) = B_0^p(a_1) \times \cdots \times B_0^p(a_n) \), where \( B_0^p(a_i) := \{x \in \mathbb{Q}_p : |x_i - a_i|_p \leq p^\gamma \} \) is the one-dimensional ball of radius \( p^\gamma \) with center at \( a_i \in \mathbb{Q}_p \). The ball \( B_0^p(0) \) is equal to the product of \( n \) copies of \( B_0(0) := \mathbb{Z}_p \), the ring of \( p \)-adic integers.

A complex-valued function \( \varphi \) defined on \( \mathbb{Q}_p^n \) is called locally constant if for any \( x \in \mathbb{Q}_p^n \) there exist an integer \( l(x) \in \mathbb{Z} \) such that

\[
\varphi(x + x') = \varphi(x) \quad \text{for } x' \in B_{l(x)}^n.
\]

A function \( \varphi : \mathbb{Q}_p^n \to \mathbb{C} \) is called a Bruhat-Schwartz function (or a test function) if it is locally constant with compact support. The \( \mathbb{C} \)-vector space of Bruhat-Schwartz functions is denoted by \( S(\mathbb{Q}_p^n) \). Let \( S'(\mathbb{Q}_p^n) \) denote the set of all functionals (distributions) on \( S(\mathbb{Q}_p^n) \). All functionals on \( S(\mathbb{Q}_p^n) \) are continuous.

For a detailed discussion on \( p \)-adic analysis the reader may consult \[1, 10, 18, 20\].

2.1. Fourier transform on locally compact groups. Let \( G \) be a locally compact Abelian group and \( G^* \), the dual group of \( G \) consisting of the group \( Hom(G, \mathbb{C}^\times) \) of all continuous homomorphisms of \( G \) into the group \( \mathbb{C}^\times \) of complex numbers of absolute value 1. We recall that \( G^* \cong G \), Pontryagin duality. For \( g \in G \) and \( g^* \in G^* \), we write \( g^*(g) = \langle g, g^* \rangle \). Let \( dg \) be the Haar measure on \( G \) and \( L^1(G) \) the space of complex valued functions which are integrable on \( G \). For \( \varphi \in L^1(G) \), the Fourier transform \( \varphi^* \) is defined by

\[
\varphi^*(g^*) = \int_G \varphi(g) \langle g, g^* \rangle \, dg \quad \text{for } g^* \in G^*.
\]

Then \( \varphi^* \in L^\infty(G^*) \). Let \( L(G) \) be the space of continuous functions \( \varphi \) in \( L^1(G) \) whose Fourier transform \( \varphi^* \) is in \( L^1(G^*) \). Then \( \varphi \to \varphi^* \) is a bijection of \( L(G) \) on \( L(G^*) \). The Haar measure \( dg^* \) on \( G^* \) can be normalized so that \( \langle \varphi^* \rangle^*(g) = \varphi(-g) \) for every \( \varphi \in L(G) \) and \( g \in G \). This is just the Fourier inversion theorem and the measure \( dg^* \) is said to be dual of \( dg \).

Let \( E \) be a finite dimensional vector space over \( \mathbb{Q}_p \) and \( \chi_p \) a non trivial additive character of \( \mathbb{Q}_p \) as before. Let \( [x, y] \) be a symmetric non-degenerate \( \mathbb{Q}_p \)-bilinear form on \( E \times E \). Thus \( Q(e) := [e, e] \in E \) is a non-degenerate quadratic form on \( E \). We identify \( E \) with is algebraic dual \( E^* \) by means of \([,] \). We now identify the dual
group of \((E, +)\) with \(E^*\) by taking \(\langle e, e^* \rangle = \chi_p ([e, e^*])\) where \([e, e^*]\) is the algebraic duality. The Fourier transform takes the form

\[
\hat{\varphi} (y) = \int_E \varphi (x) \chi_p ([x, y]) \, dx \text{ for } u \in L^1 (E),
\]

where \(dx\) is a Haar measure on \(E\). The measure \(dx\) can be normalized uniquely in such manner that \((\hat{\varphi}) (x) = \varphi (-x)\) for every \(\varphi\) belonging to \(\mathcal{L} (E)\). We say that \(dx\) is a self-dual measure relative to \(\chi_p ([\cdot, \cdot])\).

For further details about the material presented in this section the reader may consult [23].

2.2. The \(p\)-adic Minkowski space. We take \(E\) to be the \(\mathbb{Q}_p\)-vector space of dimension 4. By fixing a basis we identify \(E\) with \(\mathbb{Q}_p^4\) considered as a \(\mathbb{Q}_p\)-vector space. For \(x = (x_0, x_1, x_2, x_3) := (x, \mathbf{0})\) and \(y = (y_0, y_1, y_2, y_3) := (y, \mathbf{0})\) in \(\mathbb{Q}_p^4\) we set

\[
[x, y] := x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3 := x_0 y_0 - \mathbf{x} \cdot \mathbf{y},
\]

which is a symmetric non-degenerate bilinear form. From now on, we use \([x, y]\) to mean the bilinear form \((2.2)\). Then \((\mathbb{Q}_p^4, \mathbb{Q})\), with \(Q(x) = [x, x]\) is a quadratic vector space and \(Q\) is a non-degenerate quadratic form on \(\mathbb{Q}_p^4\). We will call \((\mathbb{Q}_p^4, \mathbb{Q})\) the \(p\)-adic Minkowski space.

On \((\mathbb{Q}_p^4, \mathbb{Q})\), the Fourier transform takes the form

\[
\mathcal{F} \varphi (k) = \int_{\mathbb{Q}_p^4} \chi_p ([x, k]) \varphi (x) \, d^4 x \text{ for } \varphi \in L^1 (\mathbb{Q}_p^4),
\]

where \(d^4 x\) is a self-dual measure for \(\chi_p ([\cdot, \cdot])\), i.e. \(\mathcal{F} [\mathcal{F} \varphi] (x) = \varphi (-x)\) for every \(\varphi\) belonging to \(\mathcal{L} (\mathbb{Q}_p^4)\). Notice that \(d^4 x\) is equal to a positive multiple of the normalized Haar measure on \(\mathbb{Q}_p^4\), i.e. \(d^4 x = C d^4 \mu (x)\).

**Remark 1.** (i) We set the usual Fourier transform \(\mathfrak{F}\) to be

\[
\mathfrak{F} [\varphi] [k] := \int_{\mathbb{Q}_p^4} \chi_p (x_0 k_0 + x_1 k_1 + x_2 k_2 + x_3 k_3) \varphi (x) \, d^4 \mu (x) \text{ for } \varphi \in L^1 (\mathbb{Q}_p^4),
\]

where \(d^4 \mu (x)\) is the normalized Haar measure of \(\mathbb{Q}_p^4\). The connection between \(\mathcal{F}\) and \(\mathfrak{F}\) is given by the formula

\[
\mathfrak{F} ([\mathcal{F} \varphi] (x_0, x_1, x_2, x_3)] = C \varphi (x_0, -x_1, -x_2, -x_3),
\]

which is equally valid for integrable functions as well as distributions.

(ii) Note that \(C = 1\), i.e. \(d^4 x = d^4 \mu (x)\). Indeed, take \(\varphi (x)\) to be the characteristic function of \(\mathbb{Z}_p^4\), now

\[
\varphi (x) = \mathcal{F} [\mathcal{F} \varphi] (x) = C \mathcal{F} \varphi \mathcal{F} \varphi \mathcal{F} \varphi \mathcal{F} \varphi = C \varphi (k_0, k),
\]

therefore \(C = \pm 1\).
2.3. Invariant measures under the orthogonal group $O(Q)$. We set $Q(x) = [x,x]$ as before. We also set

$$G := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$ 

Then $Q(x) = x^TGx$, where $T$ denotes the transpose of a matrix. The orthogonal group of $Q(x)$ is defined as

$$O(Q) = \{ \Lambda \in GL_4 (\mathbb{Q}_p) : [\Lambda x, \Lambda y] = [x,y] \} = \{ \Lambda \in GL_4 (\mathbb{Q}_p) : \Lambda^TGA = G \}.$$ 

Notice that any $\Lambda \in O(Q)$ satisfies $\det \Lambda = \pm 1$. We consider $O(Q)$ as a $p$-adic Lie subgroup of $GL_4 (\mathbb{Q}_p)$ which is a $p$-adic Lie group.

For $t \in \mathbb{Q}_p^\times$, we set

$$V_t := \{ k \in \mathbb{Q}_p^4 : Q(k) = t \}.$$ 

**Proposition 1** (Rallis-Schiffman, [14] Proposition 2-2). The orthogonal group $O(Q)$ acts transitively on $V_t$. On each orbit $V_t$ there is a measure which is invariant under $O(Q)$ and unique up to multiplication by a positive constant.

For each $t \in \mathbb{Q}_p^\times$, let $d\mu_t$ be a measure on $V_t$ invariant under $O(Q)$. Since $V_t$ is closed in $\mathbb{Q}_p^4$, it is possible to consider $d\mu_t$ as a measure on $\mathbb{Q}_p^4$ supported on $V_t$.

2.4. $p$-adic analytic manifolds. We give a brief review on $p$-adic manifolds in the sense of Serre. For further details the reader may consult [9], [17].

We denote by

$$\mathbb{Q}_p \langle \langle y-a \rangle \rangle := \mathbb{Q}_p \langle \langle y_1-a_1, \ldots, y_n-a_n \rangle \rangle$$

the ring of convergent power series around $a \in \mathbb{Q}_p^n$. Let $U \subset \mathbb{Q}_p^n$ be an open set and let $f : U \to \mathbb{Q}_p$ be a function. We say that $f$ is a $p$-adic analytic function on $U$ if for every point $a \in U$ there exists an element $f_a \in \mathbb{Q}_p \langle \langle y-a \rangle \rangle$ such that $f(y) = f_a(y)$ for $y$ belonging to a small open set that contains $a$. Then $f$ is differentiable and its partial derivatives are $p$-adic analytic functions on $U$, see e.g. [9] Section 2.1. Now if $f : U \to \mathbb{Q}_p^n$ is a mapping with $f = (f_1, \ldots, f_m)$, we say $f$ is a $p$-adic analytic mapping on $U$ if each $f_i$ is a $p$-adic analytic function on $U$.

Let $X$ be a topological Hausdorff space. A pair $(U, \phi_U)$, where $U$ is a nonempty open subset of $X$ and $\phi_U$ is a homeomorphism from $U$ to $\phi_U(U) \subset \mathbb{Q}_p^n$, with $n$ fixed, is called a chart. Furthermore, we take $\phi_U(x) = (x_1, \ldots, x_n)$ for a variable point $x$ of $U$. An atlas for $X$ is a family of compatible charts $\{(U, \phi_U)\}$ covering $X$. There is a natural equivalence relation on atlases, like in the real case. By picking an atlas, in an equivalence class, for $X$, we equip $X$ with an structure of $n$-dimensional $p$-adic analytic manifold.

2.4.1. Gel’fand-Leray differential forms. Since $\nabla Q(k) \neq 0$ for any $k \in V_t$, by using the non-Archimedean implicit function theorem one verifies that $V_t$ is a $p$-adic closed submanifold of codimension 1. We construct an atlas for $V_t$ as follows. Pick $b \in V_t$. We may assume, after renaming the coordinates if necessary, that $\frac{\partial Q}{\partial k_0}(b) \neq 0$. Then by applying the implicit function theorem, see e.g. [9] Section
2.1, there exist an open compact set \( U = U' \times U'' \) containing \( b \) and a \( p \)-adic analytic function \( h (k_1, k_2, k_3) : U'' \to U' \) such that

\[
V_i \cap U = \{ (k_0, k_1, k_2, k_3) \in U : k_0 = h (k_1, k_2, k_3) \text{ with } (k_1, k_2, k_3) \in U'' \}.
\]

We set \( (V_i \cap U, \Psi_{V_i \cap U}) \) with

\[
\Psi_{V_i \cap U} : \quad V_i \cap U \to \mathbb{Q}_p^3
\]

\[
(k_0, k_1, k_2, k_3) \quad \Psi_{V_i \cap U} \quad (k_1, k_2, k_3).
\]

Then \( \{(V_i \cap U, \Psi_{V_i \cap U})\} \) is an atlas for \( V_i \). Notice that

\[
\Psi_{V_i \cap U}^{-1} (k_1, k_2, k_3) = (h (k_1, k_2, k_3), k_1, k_2, k_3).
\]

The condition \( \nabla Q (k) \neq 0 \) for any \( k \in V_i \), implies the existence of a \( p \)-adic analytic differential form \( \lambda_i \) on \( V_i \) satisfying

\[
dk_0 \wedge dk_1 \wedge dk_2 \wedge dk_3 = dQ (k) \wedge \lambda_i.
\]

A such form is typically called a Gel'fand-Leray form. The differential form \( \lambda_i \) is not unique but its restriction to \( V_i \) is independent of the choice of \( \lambda_i \), see e.g. [8, Chap. III, Sect. 1-9, [9, Section 7.4 and 7.6, [25]. We denote the corresponding measure as \( \lambda_i (A) = \int_A d\lambda_i \) for an open compact subset \( A \) of \( V_i \).

The notation \( (k_0, \ldots, \hat{k}_l(j), \ldots, k_3) \) means omit the \( l \)th coordinate. We now describe this measure in a suitable chart. We may assume that \( V_i \) is a countable disjoint union of submanifolds of the form

\[
V_i^{(j)} := \left\{ (k_0, \ldots, k_3) \in \mathbb{Q}_p^3 : k_l(j) = h_j \left( k_0, \ldots, \hat{k}_l(j), \ldots, k_3 \right) \right\}
\]

where \( h_j \left( k_0, \ldots, \hat{k}_l(j), \ldots, k_3 \right) \) is a \( p \)-adic analytic function on some open compact subset \( V_j \) of \( \mathbb{Q}_p^3 \), and \( \frac{\partial Q}{\partial k_l(j)} (z) \neq 0 \) for any \( z \in V_i^{(j)} \). If \( A \) is a compact open subset contained in \( V_i^{(j)} \), then

\[
\lambda_i (A) = \int_{h_j^{-1}(A)} \frac{d k_0 \ldots d \hat{k}_l(j) \ldots d k_3}{\left| \frac{\partial Q}{\partial k_l(j)} (k) \right|_p},
\]

where we are identifying the set \( A \subset V_i^{(j)} \) with the set of all the coordinates of the points of \( A \), which is a subset of \( \mathbb{Q}_p^3 \), and \( h_j^{-1} (A) \) denotes the subset of \( \mathbb{Q}_p^3 \) consisting of the points \( (k_0, k_1, k_2, k_3) \) such that \( k_l(j) = h_j \left( k_0, \ldots, \hat{k}_l(j), \ldots, k_3 \right) \) for \( (k_0, \ldots, \hat{k}_l(j), \ldots, k_3) \in A \).

**Remark 2.** (i) Let \( \mathcal{T} (V_i) \) denote the family of all compact open subsets of \( V_i \). Then \( \lambda_i \) is a additive function on \( \mathcal{T} (V_i) \) such that \( \lambda_i (A) \geq 0 \) for every \( A \in \mathcal{T} (V_i) \). By Carathéodory's extension theorem \( \lambda_i \) has a unique extension to the \( \sigma \)-algebra generated by \( \mathcal{T} (V_i) \). We also note that the measure \( \lambda_i \) is supported on \( V_i \).

(ii) Let \( \mathbf{S} (V_i) \) denote the \( \mathbb{C} \)-vector space generated by the characteristic functions of the elements of \( \mathcal{T} (V_i) \). The fact that \( \lambda_i \) is a positive additive function on
\[ T(V_t) \text{ is equivalent to say that} \]
\[ S(V_t) \rightarrow \mathbb{C} \]
\[ \varphi \rightarrow \int_{\mathbb{Q}_p^4} \varphi(k) d\lambda_t(k) \]

is a positive distribution. We can identify the measure \( d\lambda_t \) with a distribution on \( \mathbb{Q}_p^4 \) supported on \( V_t \).

(iv) Some authors use \( \delta(Q(k) - t) \) or \( \delta(Q(k) - t) d^4k \) to denote the measure \( d\lambda_t \). We will use \( \delta(Q(k) - t) \).

Remark 3. Let \( G_0 \) be a subgroup of \( GL_4(\mathbb{Q}_p) \). Let \( \varphi \in S(\mathbb{Q}_p^4) \) and let \( \Lambda \in G_0 \). We define the action of \( \Lambda \) on \( \varphi \) by putting
\[ (\Lambda \varphi)(x) = \varphi(\Lambda^{-1}x), \]
and the action of \( \Lambda \) on a distribution \( T \in S'(\mathbb{Q}_p^4) \) by putting
\[ (\Lambda T, \varphi) = (T, \Lambda^{-1} \varphi). \]

We say that \( T \) is invariant under \( G_0 \) if \( \Lambda T = T \) for any \( \Lambda \in G_0 \).

Lemma 1. With the above notation, we have \( d\mu_t = Ad\lambda_t \) for some positive constant \( A \).

Proof. By Remark 2 and Proposition 1, it is sufficient to show that the distribution \( \delta(Q(k) - t) \) is invariant under \( O(Q) \), i.e.
\[ \int_{V_t} \varphi(\Lambda k) d\lambda_t(k) = \int_{V_t} \varphi(k) d\lambda_t(k) \]
for any \( \Lambda \in O(Q) \) and \( \varphi \in S(\mathbb{Q}_p^4) \). Since \( V_t \) is invariant under \( \Lambda \), it is sufficient to show that \( d\lambda_t(k) = d\lambda_t(y) \) under \( k = \Lambda^{-1}y \), for any \( \Lambda \in O(Q) \). To verify this fact we note that
\[ dk_0 \wedge dk_1 \wedge dk_2 \wedge dk_3 = (\det \Lambda^{-1}) dy_0 \wedge dy_1 \wedge dy_2 \wedge dy_3 \text{ and } dQ(k) = dQ(y) \]
under \( k = \Lambda^{-1}y \). Now by (2.4) and the fact that the restriction of \( \lambda_t \) to \( V_t \) is unique we have \( \lambda_t(k) = (\det \Lambda^{-1}) \lambda_t(y) \) on \( V_t \), i.e. \( d\lambda_t(k) = d\lambda_t(y) \) under \( k = \Lambda^{-1}y \) on \( V_t \).

2.4.2. Some additional results on \( \delta(Q(k) - t) \). We now take \( t = m^2 \) with \( m \in \mathbb{Q}_p^\times \). Notice that \( V_{m^2} \) has infinitely many points and that \( (k_0, k) \in V_{m^2} \) if and only if \( \langle -k_0, k \rangle \in V_{m^2} \). In order to exploit this symmetry we need a ‘notion of positivity’ on \( \mathbb{Q}_p \). To motivate our definitions consider \( a \in \mathbb{Q}_p^\times \) with \( p \)-adic expansion
\[ a = a_{-n} p^{-n} + a_{-n+1} p^{-n+1} + \cdots + a_0 + a_1 p + \cdots = p^{-n} ac(a) \]
with \( a_{-n} \neq 0 \) and \( a_j \in \{0, 1, \ldots, p-1\} \). Then the \( p \)-adic expansion of \(-a\) equals
\[ (p-a_{-n}) p^{-n} + (p-1-a_{-n+1}) p^{-n+1} + \cdots + (p-1-a_0) + (p-1-a_1) p + \cdots = p^{-n} \{ (p-a_{-n}) + (p-1-a_{-n+1}) + \cdots + (p-1-a_0) p^n + \cdots \} = p^{-n} ac(-a). \]

Thus changing the sign of \( a \) is equivalent to changing the sign of its angular component. On the other hand, the equation \( x^2 = a \) has two solutions if and only if
$n$ is even and $\left( \frac{a-n}{p} \right) = 1$, here $\left( \frac{\cdot}{p} \right)$ denotes the Legendre symbol. The condition $\left( \frac{a-n}{p} \right) = 1$ means that the equation $z^2 \equiv a-n \mod p$ has two solutions, say $\pm z_0$, because $p \neq 2$, with $z_0 \in \{1, \ldots, \frac{p-1}{2}\}$ and $-z_0 \in \{\frac{p+1}{2}, \ldots, p-1\}$.

We define $F_+^p = \left\{ 1, \ldots, \frac{p-1}{2} \right\} \subset \mathbb{F}_p^\times$ and $F_-^p = \left\{ \frac{p+1}{2}, \ldots, p-1 \right\} \subset \mathbb{F}_p^\times$.

Motivated by the above discussion we introduce the following notion of positivity.

**Definition 1.** We say that $a \in \mathbb{Q}_p^\times$ is positive if $ac(a) \in F_+^p$, otherwise we declare $a$ to be negative. We will use the notation $a > 0$, in the first case, and $a < 0$ in the second case.

The reader must be aware that this notion of positivity is not compatible with the arithmetic operations on $\mathbb{F}_p$ neither on $\mathbb{Q}_p^\times$ because these fields cannot be ordered.

We now define the mass shells as follows:

$V_{m^2}^+ = \left\{ (k_0, k) \in V_{m^2} : k_0 > 0 \right\}$ and $V_{m^2}^- = \left\{ (k_0, k) \in V_{m^2} : k_0 < 0 \right\}$.

Hence

$$V_{m^2} = V_{m^2}^+ \sqcup V_{m^2}^- \sqcup \left\{ (k_0, k) \in V_{m^2} : k_0 = 0 \right\}. \quad (2.8)$$

Notice that

$$V_{m^2}^+ \to V_{m^2}^-$$

$$(k_0, k) \to (-k_0, k)$$

is a bijection. We define

$$\Pi : \quad \mathbb{Q}_p^4 \to \mathbb{Q}_p^3$$

$$(k_0, k) \to k,$$

and $\Pi (V_{m^2}^+) = \Pi (V_{m^2}^-) := U_{Q,m}$. Given $k \in U_{Q,m}$, there are two p-adic numbers, $k_0 > 0$ and $-k_0 < 0$, such that $(k_0, k), (-k_0, k) \in V_{m^2}$, thus we can define the following two functions:

$$U_{Q,m} \to \mathbb{Q}_p^\times, \quad U_{Q,m} \to \mathbb{Q}_p^\times$$

$$k \to \sqrt{k \cdot k + m^2} =: k_0, \quad k \to -\sqrt{k \cdot k + m^2} =: -k_0.$$

Furthermore, we obtain the following description of the sets $V_{m^2}^\pm$:

$$V_{m^2}^\pm = \left\{ (k_0, k) \in \mathbb{Q}_p^4 : k_0 = \pm \sqrt{k \cdot k + m^2}, \text{ for } k \in U_{Q,m} \right\}. \quad (2.9)$$

**Lemma 2.** With the above notation the following assertions hold:

(i) $U_{Q,m}$ is an open subset of $\mathbb{Q}_p^3$;

(ii) the functions $\pm \sqrt{k \cdot k + m^2}$ are p-adic analytic on $U_{Q,m}$;

(iii) $U_{Q,m}$ is p-adic bianaalytic equivalent to each $V_{m^2}^\pm$, and $V_{m^2}^\pm$ are open subsets of $\mathbb{Q}_p^3$.
where $d^3k$ is the normalized Haar measure of $\mathbb{Q}_p^3$;
\[(iv)\quad \int_{\{(k_0,k)\in V^+_m: k_0=0\}} \varphi (k_0,k) d\lambda_{m^2} (k_0,k) = 0 \text{ for any } \varphi (k_0,k) \in \mathcal{S} \left( \mathbb{Q}_p^3 \right),\]

PROOF. Take a point $(k_0,k) \in V^+_m$, then $(k_0,k) \in V^+_i$ for some $j$, see (2.5), thus there exist an open compact subset $U_+ = U'_+ \times U''_+$ containing $(k_0,k)$ and a $p$-adic analytic function $h_+ : U''_+ \to U'_+$ such that
\[(2.10)\quad V^+_m \cap U = \{(k_0,k) : k_0 = h_+ (k) \text{ with } k \in U''_+ \}.
\]
Now by (2.3), we have
\[h_+ (k) \mid_{U''_+} = \sqrt{k_k + m^2} \mid_{U''_+},\]
which implies that $\sqrt{k_k + m^2}$ is a $p$-adic analytic function on $U_{Q,m}$, which is an open subset of $\mathbb{Q}_p^3$ since it is the union of all the $U''_+$ which are open. In this way we established (i)-(ii).

We now prove (iii). By (ii)
\[U_{Q,m} \to \quad V^+_m,\]
\[k \to \quad (\pm \sqrt{k_k + m^2},k) =: i_\pm (k)\]
are $p$-adic bianalytic mappings, and by (i) $V^+_m$ are open subsets of $\mathbb{Q}_p^3$.

The formulas (iv)-(v) follow from (2.6) by a direct calculation. \hfill \Box

LEMMA 3. (i) Each of the spaces $\mathcal{S}(V^\pm_m)$ is isomorphic to $\mathcal{S}(U_{Q,m})$ as $\mathbb{C}$-vector space.

(ii) If $\phi : U_{Q,m} \to \mathbb{C}$ is a function with compact support, then
\[\int_{U_{Q,m}} \phi (k) \frac{d^3k}{\sqrt{k_k + m^2}} = \int_{V^+_m} (\phi \circ i^-_\pm) (k) d\lambda_{m^2} (k_0,k).\]

(iii) If $m \in \mathbb{Q}_p^*$ and $\varphi : \mathbb{Q}_p^3 \to \mathbb{C}$ is a function with compact support, then
\[\int_{V^+_m} \varphi (k) d\lambda_{m^2} (k) = \int_{U_{Q,m}} \varphi \left( \sqrt{k_k + m^2},k \right) d^3k_{U_{Q,m}} \mid_{U_{Q,m}} + \int_{U_{Q,m}} \varphi \left( -\sqrt{k_k + m^2},k \right) d^3k_{U_{Q,m}} \mid_{U_{Q,m}}.
\]

PROOF. (i) Let $\phi_\pm$ be a function in $\mathcal{S}(V^\pm_m)$, by applying Lemma 2 (iii), we have $\phi_\pm \circ i_\pm \in \mathcal{S}(U_{Q,m})$. Conversely, if $\varphi \in \mathcal{S}(U_{Q,m})$, then, by Lemma 2 (iii), $\varphi \circ i^-_\pm \in \mathcal{S}(V^+_m)$. (ii) The formula follows from (ii) by applying Lemma 2 (iv). (iii) The formula follows from (2.8) by applying Lemma 2 (iii)-(iv)-(v). \hfill \Box
Remark 4. We set

\[ \delta_\pm (Q (k) - m^2) := \delta (Q (k) - m^2) \bigg|_{V^\pm_{m^2}}. \]

If we take \( \Lambda_0 := \begin{bmatrix} -1 & 0 \\ 0 & I_{3 \times 3} \end{bmatrix} \in O(Q) \), then

\[ \delta_- (Q (k) - m^2) = \Lambda_0 \delta_+ (Q (k) - m^2). \]

Notice that instead of \( \Lambda_0 \) we can use any \( \Lambda \) satisfying \( \Lambda (V^+_{m^2}) = V^-_{m^2} \).

Now formula (2.11) can written as

\[ \delta (Q (k) - m^2) = \delta_+ (Q (k) - m^2) + \delta_- (Q (k) - m^2). \]

2.4.3. The \( p \)-adic restricted Poincaré group.

Definition 2. We define the \( p \)-adic restricted Lorentz group \( \mathcal{L}^+_p \) to be the largest subgroup of \( SO(Q) \) such that \( \mathcal{L}^+_p (V^\pm_{m^2}) = V^\pm_{m^2} \). The \( p \)-adic restricted Poincaré group \( \mathcal{P}^+_p \) is defined to be the semi-direct product of \( (\mathbb{Q}_p^4, +) \) and \( \mathcal{L}^+_p \).

Notice that \( \mathcal{L}^+_p \) is a non-trivial subgroup of \( SO(Q) \). Indeed, take \( \Lambda \) in

\[ SO(3) = \{ R \in GL_3 (\mathbb{Q}_p) : R^T = R^{-1}, \det R = 1 \}, \]

and define

\[ \bar{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & \Lambda \end{bmatrix}. \]

Then

\[ (\bar{\Lambda})^T = \begin{bmatrix} 1 & 0 \\ 0 & \Lambda^{-1} \end{bmatrix} \quad \text{and} \quad (\bar{\Lambda})^T G \bar{\Lambda} = G, \text{ i.e. } \bar{\Lambda} \in SO(Q), \]

and since \( \bar{\Lambda} k \cdot \bar{\Lambda} k = k \cdot k \) we have \( \bar{\Lambda} (V^\pm_{m^2}) = V^\pm_{m^2} \).

At the moment, we do not know if \( \mathcal{L}^+_p = \{ 1 \} \times SO(3) \). It seems that this depends on \( \mathbb{Q}_p \), which can be replaced for any locally compact field of characteristic different from 2.

Lemma 4. The distributions \( \delta_\pm (Q (k) - m^2) \) are invariant under \( \mathcal{L}^+_p \).

Proof. Consider first \( \delta_\pm (Q (k) - m^2) \). Take \( \Lambda \in \mathcal{L}^+_p \), then

\[ (\Lambda \delta_\pm (Q (k) - m^2), \varphi) = \int_{V^\pm_{m^2}} \varphi (\Lambda k) d\lambda_{m^2} (k) = \int_{V^\pm_{m^2}} \varphi (k) d\lambda_{m^2} (k) \]

because \( \Lambda (V^\pm_{m^2}) = V^\pm_{m^2} \) and \( d\lambda_{m^2} \) is invariant under any element of \( O(Q) \), see proof of Lemma [\textbf{4}]. \( \square \)

3. A non-Archimedean analog of the Klein-Gordon Equation

Given a positive real number \( \alpha \) and a nonzero \( p \)-adic number \( m \), we define the pseudodifferential operator

\[ S (\mathbb{Q}_p^4) \rightarrow C (\mathbb{Q}_p^4, \mathbb{C}) \cap L^2 (\mathbb{Q}_p^4) \]

\[ \varphi \rightarrow \Box_{\alpha, m} \varphi, \]

where \( (\Box_{\alpha, m} \varphi) (x) := F_{k \rightarrow x}^{-1} \left[ \left| [k, k] - m^2 \right|_p^{\alpha} F_{x \rightarrow k} \varphi \right]. \)
We set \( \mathcal{E}_{Q,m}(\mathbb{Q}_p^4) := \mathcal{E}_{Q,m} \) to be the subspace of \( \mathbb{S}'(\mathbb{Q}_p^4) \) consisting of the distributions \( T \) such that the product \( \left| [k,k] - m^2 \right| x^\alpha \mathcal{F}T \) exists in \( \mathbb{S}'(\mathbb{Q}_p^4) \), here \( \left| [k,k] - m^2 \right| x^\alpha \) denotes the distribution \( \varphi \mapsto \int_{\mathbb{Q}_p^4} \left| [k,k] - m^2 \right| x^\alpha \varphi(k) d^4k \). Notice that \( \mathcal{E}(\mathbb{Q}_p^4) \), the space of locally constant functions, is contained in \( \mathcal{E}_{Q,m} \). We consider \( \mathcal{E}_{Q,m} \) as topological space with the topology inherited from \( \mathbb{S}'(\mathbb{Q}_p^4) \).

 Definition 3. A weak solution of

\[
\square_{\alpha,m} T = S, \text{ with } S \in \mathbb{S}'(\mathbb{Q}_p^4),
\]

is a distribution \( T \in \mathcal{E}_{Q,m}(\mathbb{Q}_p^4) \) satisfying (3.1).

For a subset \( U \) of \( \mathbb{Q}_p^4 \) we denote by \( 1_U \) its characteristic function.

Lemma 5. Let \( T, S \in \mathbb{S}'(\mathbb{Q}_p^4) \). The following assertions are equivalent:

(i) there exists \( W \in \mathbb{S}'(\mathbb{Q}_p^4) \) such that \( TS = W \);

(ii) for each \( x \in \mathbb{Q}_p^4 \), there exists an open compact subset \( U \) containing \( x \) so that for each \( k \in \mathbb{Q}_p^4 \):

\[
\mathcal{F}[1_U W](k) := \int_{\mathbb{Q}_p^4} \mathcal{F}[1_U T](l) \mathcal{F}[1_U S](k-l) d^4l
\]

exists.

Proof. Any distribution is uniquely determined by its restrictions to any countable open covering of \( \mathbb{Q}_p^4 \), see e.g. [20] p. 89. On the other hand, the product \( TS \) exists if and only if \( \mathcal{F}[T] \ast \mathcal{F}[S] \) exists, and in this case \( \mathcal{F}[TS] = \mathcal{F}[T] \ast \mathcal{F}[S] \), see e.g. [20] p. 115. Assume that \( TS = W \) exists and take a countable covering \( \{U_i\}_{i \in \mathbb{N}} \) of \( \mathbb{Q}_p^4 \) by open and compact subsets, then \( TS|_{U_i} = W|_{U_i} \) i.e. \( 1_{U_i} TS = 1_{U_i} W \). We recall that the product of a finite number of distributions involving at least one distribution with compact support is associative and commutative, see e.g. [15] Theorem 3.19, then

\[
1_{U_i} TS = 1_{U_i} (1_{U_i} TS) = (1_{U_i} T)(1_{U_i} S) = T|_{U_i} S|_{U_i} = W|_{U_i}.
\]

Now for each \( x \in \mathbb{Q}_p^4 \), there exists an open compact subset \( U_i \) containing \( x \) such that \( \mathcal{F}[T|_{U_i}] \ast \mathcal{F}[S|_{U_i}] = \mathcal{F}[1_{U_i} T] \ast \mathcal{F}[1_{U_i} S] = \mathcal{F}[1_{U_i} W] \). Conversely, if for each \( x \) there exist an open compact subset \( U_i \) containing \( x \) (from this we get countable subcovering of \( \mathbb{Q}_p^4 \) also denoted as \( \{U_i\}_{i \in \mathbb{N}} \) such that \( \mathcal{F}[T|_{U_i}] \ast \mathcal{F}[S|_{U_i}] = \mathcal{F}[W|_{U_i}] \) i.e. \( T|_{U_i} S|_{U_i} = W|_{U_i} \) exists, then \( TS = W \).

Corollary 1. If \( TS \) exists, then \( \text{supp}(TS) \subseteq \text{supp}(T) \cap \text{supp}(S) \).

Proof. Since \( x \notin \text{supp}(S) \), there exists a compact open set \( U \) containing \( x \) such \( (S, \varphi) = 0 \) for any \( \varphi \in \mathbb{S}(U) \), hence \( 1_U S = 0 \), and \( \mathcal{F}[1_U T] \ast \mathcal{F}[1_U S] = 0 = \mathcal{F}[1_U W] \), i.e. \( W|_{U_i} = 0 \), which means \( x \notin \text{supp}(W) \).

Remark 5. Lemma 5 and Corollary 1 are valid in arbitrary dimension. These results are well-known in the Archimedean setting, see e.g. [14] Theorem IX.43, however, such results do not appear in the standard books of p-adic analysis [11], [10], [15], [20].
Remark 6. (i) Let $\Omega$ denote the characteristic function of the interval $[0, 1]$. Then $\Omega \left(p^{-\frac{3}{p}} \|x\|_p\right)$ is the characteristic function of the ball $B_j^{(n)}(0)$. We recall definition of the product of two distributions. Set $\delta_j(x) := p^{-\frac{3}{p}} \Omega \left(p^j \|x\|_p\right)$ for $j \in \mathbb{N}$. Given $T, S \in \mathcal{S}'(\mathbb{Q}_p^4)$, their product $TS$ is defined by

$$(TS, \varphi) = \lim_{j \to +\infty} (S, (T \ast \delta_j) \varphi)$$

if the limit exists for all $\varphi \in \mathcal{S}(\mathbb{Q}_p^4)$.

(ii) We assert that

$$\left(\|k, k\|_p - m^2 \right)^{\alpha} FT, \varphi = \left(FT, \|k, k\|_p - m^2 \right)^{\alpha}$$

for any $T \in \mathcal{E}_{\Omega, m}(\mathbb{Q}_p^4)$ and any $\varphi \in \mathcal{S}(\mathbb{Q}_p^4)$. Indeed, by using the fact that $V_{m^2}$ has $d^4$-measure zero, we have

$$\|k, k\|_p - m^2 \gamma \ast \delta_j(k) = \left(\|y, y\|_p - m^2 \gamma, \delta_j(k - y)\right)$$

$$= p^{\delta j} \int_{k + (p^j \mathbb{Z}_p)^4} \|y, y\|_p - m^2 \gamma d^4 y$$

$$= p^{\delta j} \int_{k + (p^j \mathbb{Z}_p)^4 \setminus V_{m^2}} \|y, y\|_p - m^2 \gamma d^4 y = \|k, k\|_p - m^2 \gamma$$

for $j$ big enough depending on $k$. Then

$$(\|k, k\|_p - m^2 \gamma FT, \varphi) = \lim_{j \to +\infty} \left(FT(k), \|k, k\|_p - m^2 \gamma \ast \delta_j(k) \right) \varphi(k)$$

$$= \left(FT(k), \|k, k\|_p - m^2 \gamma \varphi(k)\right).$$

Lemma 6. A distribution $T \in \mathcal{E}_{\Omega, m}(\mathbb{Q}_p^4)$ is a $\varphi$ weak solution of $\square_{\alpha, m} T = 0$ if and only $\text{supp} FT \subseteq V_{m^2}$.

Proof. Suppose that $\text{supp} FT \subseteq V_{m^2}$, then by Corollary 1 we have

$$\text{supp} \left(\|k, k\|_p - m^2 \gamma FT\right) \subseteq \text{supp} (FT) \cap \text{supp} \left(\|k, k\|_p - m^2 \gamma \right) = \emptyset$$

because $\|k, k\|_p - m^2 \gamma = \|k, k\|_p - m^2 \gamma 1_{\mathbb{Q}_p^4 \setminus V_{m^2}}$ in $\mathcal{S}'(\mathbb{Q}_p^4)$ ($V_{m^2}$ has $d^4$-measure zero) and $\text{supp} \left(\|k, k\|_p - m^2 \gamma 1_{\mathbb{Q}_p^4 \setminus V_{m^2}}\right) \subseteq \mathbb{Q}_p^4 \setminus V_{m^2}$, therefore $\|k, k\|_p - m^2 \gamma FT = 0$.

Suppose now that $\|k, k\|_p - m^2 \gamma FT = 0$. By contradiction, assume that $\text{supp} FT \nsubseteq V_{m^2}$. Then, there exists $k_0 \in \mathbb{Q}_p^4 \setminus V_{m^2}$ and an open compact subset $U \subseteq \mathbb{Q}_p^4 \setminus V_{m^2}$ containing $k_0$ such that $\left(FT, 1_U\right) \neq 0$. By using Remark 6(ii) and by shrinking $U$ if necessary,

$$\left(\|k, k\|_p - m^2 \gamma FT, 1_U\right) = \left(FT, \|k, k\|_p - m^2 \gamma 1_U\right)$$

$$= \|k_0, k_0\|_p - m^2 \gamma \left(FT, 1_U\right) \neq 0,$n

contradicting $\|k, k\|_p - m^2 \gamma FT = 0$. \qed
Remark 7. Let $\varphi \in S \left( \mathbb{Q}_p^1 \right)$ and let $\Lambda \in \mathcal{L}_+^1$, a Lorentz transformation. We have
\[
\mathcal{F} [ \varphi (\Lambda x) ] (k) = \mathcal{F} [ \varphi ] (\Lambda k)
\]
and
\[
\Lambda \mathcal{F} [T] = \mathcal{F} [\Lambda T].
\]
Hence the Fourier transform preserves invariance under $O(Q)$.

Proposition 2. The distributions
\[
\mathcal{F} [T] (k) = a \delta_+ (Q (k) - m^2) + b \delta_- (Q (k) - m^2), \quad a, b \in \mathbb{C},
\]
are weak solutions of $\Box_{\alpha,\beta} T = 0$ invariant under $\mathcal{L}_+^1$.

Proof. By Remark 7, it is sufficient to show that $\delta_+ (Q (k) - m^2)$ are invariant solutions of $\mathcal{F} [T] = 0$, which follows from Lemmas 4-6. □

At this point we should mention that a similar results to Lemmas 4-6 and Proposition 2 are valid for the Archimedean Klein-Gordon equation, see e.g. [2] or Chapter IV.

4. The Cauchy Problem for the non-Archimedean Klein-Gordon Equation

In this section we study the Cauchy problem for the $p$-adic Klein-Gordon equations.

4.1. Twisted Vladimirov pseudodifferential operators. Let $\mathbb{C}_1^\times$ denote the multiplicative group of complex numbers having modulus one as before. Let $\pi_1 : \mathbb{Z}_p^\times \to \mathbb{C}_1^\times$ be a non-trivial multiplicative character of $\mathbb{Z}_p^\times$ with positive conductor $k$, i.e. $k$ is the smallest positive integer such that $\pi_1 |_{1+p^k \mathbb{Z}_p} = 1$. Some authors call a such character a unitary character of $\mathbb{Z}_p^\times$. We extend $\pi_1$ to $\mathbb{Q}_p^\times$ by putting $\pi_1 (x) := \pi_1 (ac (x))$. A quasicharacter of $\mathbb{Q}_p^\times$ (some authors use multiplicative character) is a continuous homomorphism from $\mathbb{Q}_p^\times$ into $\mathbb{C}^\times$. Every quasicharacter has the form $\pi_s (x) = \pi_1 (x) |x|_p^{s-1}$ for some complex number $s$.

The distribution associated with $\pi_s (x)$ has a meromorphic continuation to the whole complex plane given by
\[
(\pi_s (x), \varphi (x)) = \int_{\mathbb{Z}_p} \pi_1 (x) |x|_p^{s-1} \{ \varphi (x) - \varphi (0) \} dx + \int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} \pi_1 (x) |x|_p^{s-1} \varphi (x) dx,
\]
see e.g. [20] p. 117.

On the other hand,
\[
\mathfrak{F} [\pi_s] (\xi) = \Gamma_p (s, \pi_1) \pi_1^{-1} (\xi) |\xi|_p^{-s}, \text{ for any } s \in \mathbb{C},
\]
where
\[
\Gamma_p (s, \pi_1) = p^k a_{p,k} (\pi_1),
\]
\[
a_{p,k} (\pi_1) = \int_{\mathbb{Z}_p^\times} \pi_1 (t) \chi_p (p^{-k} t) dt \quad \text{and} \quad |a_{p,k} (\pi_1)| = p^{-\frac{k}{2}},
\]
see e.g. [20] p. 124.
Another useful formula is the following:

(4.3)

\[
\pi_s(x, \phi(x)) = \int_{\mathbb{Q}_p} \pi_1(x) \left\{ \phi(x) - \phi(0) \right\} \frac{dx}{|x|^{s+1}_p}, \text{ for } \text{Re}(s) > 0, \text{ and } \phi \in S(\mathbb{Q}_p).
\]

The formula follows from (4.1) by using that

\[
(4.4)
\int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} \pi_1(x) \frac{dx}{|x|^{s+1}_p} = \left( \sum_{j=1}^{\infty} p^{-j s} \right) \int_{\mathbb{Z}_p^\times} \pi_1(y) dy = 0.
\]

Given \( \alpha > 0 \), we define the twisted Vladimirov operator by

\[
(\tilde{D}_x^\alpha \varphi)(x) = \mathcal{F}_{k \to x} \left( \pi_1^{-1}(k)|k|_p^\alpha \mathcal{F}_{x \to k}(\varphi) \right) \text{ for } \varphi \in S(\mathbb{Q}_p).
\]

Notice that

\[
S(\mathbb{Q}_p) \to C(\mathbb{Q}_p, \mathbb{C}) \cap L^2
\]

is a well-defined linear operator.

**Lemma 7.** For \( \alpha > 0 \) and \( \varphi \in S(\mathbb{Q}_p) \), the following formula holds:

(4.5)

\[
(\tilde{D}_x^\alpha \varphi)(x) = \frac{1}{\Gamma_p(-\alpha, \pi_1)} \int_{\mathbb{Q}_p} \pi_1(y) \left\{ \varphi(x - y) - \varphi(x) \right\} \frac{dy}{|y|^{\alpha+1}_p}.
\]

**Proof.** By using (4.2), we have

\[
(\tilde{D}_x^\alpha \varphi)(x) = \mathcal{F}_{k \to x} \left( \pi_1^{-1}(k)|k|_p^\alpha \mathcal{F}_{x \to k}(\varphi) \right) \varphi(x) = \frac{1}{\Gamma_p(-\alpha, \pi_1)} \pi_{-\alpha}(x) \varphi(x)
\]

\[
= \frac{1}{\Gamma_p(-\alpha, \pi_1)} \int_{\mathbb{Z}_p^\times} \pi_1(y) \left\{ \varphi(x - y) - \varphi(x) \right\} \frac{dy}{|y|^{\alpha+1}_p}
\]

\[
= \frac{1}{\Gamma_p(-\alpha, \pi_1)} \int_{\mathbb{Z}_p^\times} \pi_1(y) \left\{ \varphi(x - y) - \varphi(x) \right\} \frac{dy}{|y|^{\alpha+1}_p} + \frac{1}{\Gamma_p(-\alpha, \pi_1)} \int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} \pi_1(y) \varphi(x - y) \frac{dy}{|y|^{\alpha+1}_p},
\]

where we used (4.4). \( \square \)

Note that the right-hand side of (4.5) makes sense for a wider class of functions. For instance, for \( \mathcal{E}_\alpha(\mathbb{Q}_p) \), the \( \mathbb{C} \)-vector space of locally constant functions \( u(x) \) satisfying

\[
\int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} \frac{|u(x)|}{|x|^{\alpha+1}_p} dx < \infty.
\]

Another useful formula is the following:
Lemma 8. For $\alpha > 0$, we have
\[
\pi_1^{-1}(x) |x|^\alpha_p = \frac{1}{\Gamma_p(-\alpha, \pi_1)} \int_{\mathbb{Q}_p} \pi_1(y) \left\{ \chi_p(yx) - 1 \right\} \frac{dy}{|y|_p^{\alpha+1}} \text{ in } S' (\mathbb{Q}_p).
\]

Proof. The formula follows from (4.2) and (4.3). The proof is a simple variation of the proof given for the case in which $\pi_1$ is the trivial character, see e.g. [10].

From now on we put
\[
\pi_1(x) := i \left( \frac{ac(x)}{p} \right) \text{ for } x \in \mathbb{Z}_p^*,
\]
where \( \left( \frac{a}{p} \right) \) denotes the Legendre symbol. Note that $k = 1$ and $\pi_1(x) \in \{ \pm i \}$, furthermore,
\[
\left( \frac{-1}{p} \right) = \begin{cases} 
1, & \text{if } p-1 \text{ is divisible by } 4 \\
-1, & \text{if } p-3 \text{ is divisible by } 4.
\end{cases}
\]

4.2. The Cauchy Problem for the $p$-adic Klein-Gordon Equation. In this section we take $x_0 = t$ and $(x_0, x) = (t, x) \in \mathbb{Q}_p \times \mathbb{Q}_p^3$. Our goal is to study the following Cauchy problem:

\[
\begin{cases}
(\Box_{\alpha, m} u)(t, x) = J(t, x), & J(t, x) \in S(\mathbb{Q}_p^4) \quad \text{(A)} \\
u(t, x)|_{t=0} = \Psi_0(x), & \psi_0 \in S(\mathbb{Q}_p^3) \text{ and } \mathfrak{F}^{-1}[\psi_0] \in S(U_{Q,m}) \quad \text{(B)} \\
\tilde{D}_x^a u(t, x)|_{t=0} = \Psi_1(x), & \psi_1 \in S(\mathbb{Q}_p^3) \text{ and } \mathfrak{F}^{-1}[\psi_1] \in S(U_{Q,m}) \quad \text{(C)}.
\end{cases}
\]

Theorem 1. Assume that $p-3$ is divisible by 4. The Cauchy problem (4.6) has a weak solution given by

\[
u(t, x) = E_\alpha(t, x) * J(t, x) + \\
\int_{\mathbb{U}_{Q,m}} \chi_p \left( -t \sqrt{k \cdot k + m^2} + x \cdot k \right) u_+(k) \frac{d^3k}{|k \cdot k + m^2|^p} + \\
\int_{\mathbb{U}_{Q,m}} \chi_p \left( t \sqrt{k \cdot k + m^2} + x \cdot k \right) u_-(k) \frac{d^3k}{|k \cdot k + m^2|^p},
\]

where $E_\alpha(t, x)$ is a distribution on $S(\mathbb{Q}_p^4)$ satisfying

\[
[k, k] - m^2|_p \mathcal{F} t \to k_0 [E_\alpha(t, x)] = 1 \text{ in } S' (\mathbb{Q}_p^4),
\]

\[
u(t, x) = \frac{1}{2} \left\{ \sqrt{k \cdot k + m^2} \mathfrak{F}^{-1}[\psi_0](k) - i\pi_1 \left( \sqrt{k \cdot k + m^2} \mathfrak{F}^{-1}[\psi_1](k) \right) \right\},
\]
and

\[ u_-(k) = \frac{1}{2} \left\{ \sqrt{k \cdot k + m^2}_p \tilde{\delta}^{-1}[\psi_1](k) + \frac{i\pi_1 \left( \sqrt{k \cdot k + m^2}_p \right)}{\sqrt{k \cdot k + m^2}_p} \tilde{\delta}^{-1}[\psi_1](k) \right\}. \]

**Proof.** Like in the classical case a solution of (4.10)-(A) is computed as \( u_0(t,x) + u_1(t,x) \) with \( u_0(t,x) \) a particular solution of (4.6)-(A) and \( u_1(t,x) \) a general solution of (4.9)\(^{16}\)

\[ (\Box_{\alpha,m} u_1)(t,x) = 0. \]

The existence of a fundamental solution for (4.9)-(A), i.e. a distribution \( E_{\alpha}(t,x) \) is a distribution on \( S(Q^4_m) \) such that \( E_{\alpha}(t,x) * J(t,x) \) is a weak solution of (4.6)-(A), was established in (24), see also (26). This fundamental solution satisfies (4.8).

We now show that

\[ u_1(t,x) := \int_{U_{Q,m}} \chi_p \left( -t\sqrt{k \cdot k + m^2} + x \cdot k \right) u_+(k) \frac{d^3k}{\sqrt{k \cdot k + m^2}_p} + \int_{U_{Q,m}} \chi_p \left( t\sqrt{k \cdot k + m^2} + x \cdot k \right) u_-(k) \frac{d^3k}{\sqrt{k \cdot k + m^2}_p}, \]

is a weak solution of (4.10). By applying Lemmas 23\(^8\) we have

\[ \int_{U_{Q,m}} \chi_p \left( \mp t\sqrt{k \cdot k + m^2} + x \cdot k \right) u_\pm(k) \frac{d^3k}{\sqrt{k \cdot k + m^2}_p} \]

\[ = \int_{V^\pm_m} \chi_p \left( [k_0,k],(-t,-x) \right) u_\pm(i^{-1}_\pm(k)) d\lambda m^2(k_0,k) \]

\[ = \mathcal{F}^{-1}_{(k_0,k) \rightarrow (t,x)} \left[ (u_\pm \circ i^{-1}_\pm)(k) \delta_\pm(Q(k) - m^2) \right], \]

whence

\[ \mathcal{F}_{(t,x) \rightarrow (k_0,k)} [u_1(t,x)] = \left[ (u_+ \circ i^{-1}_+)(k) \delta_+(Q(k) - m^2) \right] \]

\[ + \left[ (u_- \circ i^{-1}_-)(k) \delta_-(Q(k) - m^2) \right]. \]

By Lemma 8\(^9\) \( u_1(t,x) \) is a weak solution of (4.9), if

\[ \text{supp} \left( u_\pm \circ i^{-1}_\pm \right) (k) \delta_\pm(Q(k) - m^2) \subseteq V^\pm_m. \]

This last condition is verified by applying Corollary 1 and the fact that

\[ \text{supp} \left( u_\pm \circ i^{-1}_\pm \right) \subseteq V^\pm_m \subseteq V_m^\pm \text{ and supp} \left( \delta_\pm(Q(k) - m^2) \right) \subseteq V^\pm_m \subseteq V_m^\pm. \]

The verification of (4.9)-(B) is straightforward. To verify (4.9)-(C) we proceed as follows. By using Lemma 7\(^6\) Fubini’s theorem and Lemma 8\(^8\) we get the following
energy solutions which have $\phi$ replace $\chi$.

The solution (5.1) with initial condition $\square$ is a weak solution of (4.9). Furthermore, the parameter $\alpha$ does not have any influence on the solutions of (4.9).

Now Condition (4.9)-(C) follows from the previous formula. $\square$

Note that the condition ‘$p$–3 is divisible by 4’ is required only to establish (4.9)-(C). Furthermore, the parameter $\alpha$ does not have any influence on the solutions of (4.9).

Like in the Archimedean case the non-Archimedean Klein-Gordon equations admit plane waves.

**Lemma 9. Existence of plane waves.** Let $(E, p) \in V^\pm_{m^2}, i.e. E = \pm \sqrt{p \cdot p + m^2}$. Then $u(t, x) = \chi_p \left( ((t, x), (E, p)) \right)$ is a weak solution of $(\Box_{\alpha, m}) u(t, x) = 0$.

**Proof.** (i) Since $\mathcal{F}_{(t,x)\rightarrow(k_0,k)} [u(t,x)] = \delta (k_0 - E, k - p)$, the results follows from Lemma 6. $\square$

5. Further Results on the $p$-adic Klein-Gordon Equation

In this section we change the notation slightly to get close of the usual notation in quantum field theory. This will facilitate the comparison with the classical results and constructions.

Set $\omega (k) := \sqrt{k \cdot k + m^2}$ for $k \in U_{Q,m}$.

The function $\omega (k)$ is a $p$-adic analytic function on $U_{Q,m}$, cf. Lemma 2 and $\omega (k) \neq 0$ for any $k \in U_{Q,m}$. Then, by Taylor formula, $|\omega (k)|_p$ is a locally constant function on $U_{Q,m}$, and if $\phi_\pm \in S (U_{Q,m})$, then $|\omega (k)|_{p}^{\frac{1}{2}} \phi_\pm (k) \in S (U_{Q,m})$.

Then

$$\phi (t, x) = \int_{U_{Q,m}} \chi_p (-x \cdot k) \left\{ \chi_p (-t \omega (k)) \phi_+ (k) + \chi_p (t \omega (k)) \phi_- (k) \right\} d^3 k$$

is a weak solution of $(\Box_{\alpha, m}) \phi (t, x) = 0$ for any $\phi_\pm \in S (U_{Q,m})$. Note that if we replace $\chi_p (-x \cdot k)$ by $\chi_p (x \cdot k)$ in (5.1) we get another weak solution.

As in the classical case, for the quantum interpretation we specialize to positive energy solutions which have $\phi_- (k) = 0$. Then the solution is determined by a single complex valued initial condition. The solution (5.1) with initial condition
\[ \Psi \in L^2_{Q,m} := L^2 \left\{ \{ \Phi \in L^2 (Q^3_m) : \text{supp} (\Phi \chi) \subseteq U_{Q,m} \} , d^3 k \right\} \]

where the condition \( \text{supp}(\Phi \chi) \subseteq U_{Q,m} \) means that there exists a function \( \Phi' \) in the equivalence class containing \( \Phi \chi \) such that \( \text{supp}(\Phi') \subseteq U_{Q,m} \). is given by

\[ (5.2) \quad \Psi (t, x) = \int_{U_{Q,m}} \chi_p (-t \omega (k) - x \cdot k) (\tilde{\Phi} \chi) (k) d^3 k. \]

Set

\[ U (t) : L^2_{Q,m} \rightarrow L^2 (Q^3_p) \]

\[ \Psi \rightarrow \Psi (t, x) = \tilde{\Phi}_{k \rightarrow x}^{-1} (\chi_p (-t \omega (k)) \tilde{\Phi} \chi_p (-t \omega (k)) \chi_p \tilde{\Phi} \chi_p \tilde{\Phi} \chi_p) \]

for \( t \in Q_p \).

**Lemma 10.** (i) \( U (t) \), \( t \in Q_p \) is a group of unitary operators on \( L^2_{Q,m} \).

(ii) \( \lim_{t \rightarrow 0} U (t) = I \).

**Proof.** (i) It is straightforward to verify that \( U (t) U (t') = U (t + t') \) and \( U (0) = I \). Finally, if \( \langle \cdot , \cdot \rangle \) denotes the standard \( L^2 (Q^3_p) \)-inner product, we have

\[ \langle U (t) \Phi , U (t) \Psi \rangle = \langle \tilde{\Phi} (U (t) \Phi) , \tilde{\Phi} (U (t) \Psi) \rangle \]

\[ = \langle \chi_p (-t \omega (k)) \tilde{\Phi} \chi_p (-t \omega (k)) \tilde{\Phi} \chi_p \tilde{\Phi} \chi_p \tilde{\Phi} \chi_p \rangle = \langle \Phi , \Psi \rangle \]

for any \( \Phi , \Psi \in L^2_{Q,m} \).

(ii) Notice that

\[ \| U (t) \Phi - \Phi \|_{L^2} = \| \{ \chi_p (-t \omega (k)) - 1 \} \tilde{\Phi} \|_{L^2} \leq 2 \| \Phi \|_{L^2} \]

for any \( \Phi \in L^2_{Q,m} \). By the Dominated Convergence Theorem, we have

\[ \lim_{t \rightarrow 0} \| U (t) \Phi - \Phi \|_{L^2} = \lim_{t \rightarrow 0} \| \{ \chi_p (-t \omega (k)) - 1 \} \tilde{\Phi} \|_{L^2} = 0 \text{ for any } \Phi \in L^2_{Q,m}. \]

We now consider the effects of space-time translations by \( a = (a_0, a) \in Q_p^4 \) and rotations \( R \) in \( SO (3) \). The transformations \( \{ a, R \} \) act on \( Q_p^4 \) naturally and they form a group, the semi-direct product \( Q_p^4 \ltimes SO (3) \), with group law given by

\[ \{ a, R \} \{ a', R' \} = \{ a + Ra', RR' \}. \]

We attach to each \( \{ a, R \} \) the operator

\[ (U_0 (a, R) \psi) (x) := \tilde{\Phi}_{k \rightarrow x}^{-1} [\chi_p (((a_0, a) , (\omega (k) , k))) (\tilde{\Phi}_{k \rightarrow x} \psi) (R^{-1} k)] \]

for \( \psi \in L^2_{Q,m} \).

**Lemma 11.** (i) The correspondence \( \{ a, R \} \rightarrow U_0 (a, R) \) gives rise a unitary representation of \( Q_p^4 \ltimes SO (3) \) in \( L^2_{Q,m} \).

(ii) For the wave functions \( \{ \psi, \chi \} \), we have

\[ (U_0 (a, R) \Psi (t, \cdot))(x) = \Psi (t - a_0, R^{-1} (x - a)). \]

**Proof.** (i) The calculations involved are similar to the ones required in the proof of Lemma 10(i). (ii) It is a straightforward calculation. \( \square \)
6. Non-Archimedean Quantum Scalar Fields

In this section, we present a ‘semi-formal’ construction of the neutral (real) quantum scalar field with mass parameter \( m \in \mathbb{Q}_p \times \). We follow closely the presentation given in [6, Section 5.2] for the classical case. The state space for a single spin-zero particle of mass \( m \) is

\[
\mathcal{H} = L^2 \left( V_{m^2}^+, d\lambda_{m^2} \right)
\]

where

\[
d\lambda_{m^2}(k) = \frac{d^3k}{|\omega(k)|_p}, \quad \omega(k) = \sqrt{k \cdot k + m^2} \quad \text{for} \ k \in U_{Q,m}.
\]

Using well-known results on theory of Fock spaces, we construct the Boson Fock space \( F_s(H) \) on which we have annihilation and creation operators \( A(v), A^\dagger(v) \) for \( v \in \mathcal{H} \), see e.g. [6, Section 4.5].

We define \( R : \mathcal{S}(\mathbb{Q}_p^4) \to \mathcal{H} \) by

\[
R\varphi = \hat{\varphi} \mid_{V_{m^2}^+}
\]

where \( \hat{\varphi} \) is the ‘Fourier-Minkowski’ transform of \( \varphi \), see (2.3). The quantum field \( \Phi \) is defined as a real distribution on \( \mathbb{Q}_p^4 \) with values in the space of operators on the finite-particle Fock space \( F_0^s(H) \), i.e. an \( \mathbb{R} \)-linear map that takes a real-valued Bruhat-Schwartz function \( \varphi \) on \( \mathbb{Q}_p^4 \) to an operator \( \Phi(\varphi) \) on \( F_0^s(H) \) defined by

\[
(6.1) \quad \Phi(f) = \frac{1}{\sqrt{2}} \left\{ A(R\varphi) + A(R\varphi)^\dagger \right\}.
\]

One can extend \( \Phi \) to a \( \mathbb{C} \)-linear map on the complex Bruhat-Schwartz functions by taking

\[
\Phi(\varphi + i\rho) = \Phi(\varphi) + i\Phi(\rho).
\]

**Remark 8.** \( \Phi \) is a distribution solution of the \( p \)-adic Klein-Gordon equation, that is

\[
\Phi(\Box_{\alpha,m}\varphi) = 0 \quad \text{for any} \ \varphi \in \mathcal{S}(\mathbb{Q}_p^4)
\]

because \(|Q(k) - m^2|^\alpha_p \hat{\varphi} = 0\) on \( V_{m^2}^+ \).

We now define

\[
J : L^2 \left( V_{m^2}^+, d\lambda_{m^2} \right) \to L^2 \left( U_{Q,m}, d^3k \right)
\]

by

\[
(Ju)(k) = \frac{1}{\sqrt{\left|\omega(k)\right|_p}} u\left( \omega(k), k \right).
\]

By Lemma 2-(iv), we have

\[
\int_{U_{Q,m}} |(Ju)(k)|^2 d^3k = \int_{U_{Q,m}} |u\left( \omega(k), k \right)|^2 \frac{d^3k}{|\omega(k)|_p} = \int_{V_{m^2}^+} |u(k_0,k)|^2 d\lambda_{m^2}(k_0,k),
\]

hence \( J \) is a unitary map, and we can identify \( L^2 \left( V_{m^2}^+, d\lambda_{m^2} \right) \) with \( L^2 \left( U_{Q,m}, d^3k \right) \). By the theory of Fock spaces, we have an induced unitary map

\[
F(J) : F_s \left( L^2 \left( V_{m^2}^+, d\lambda_{m^2} \right) \right) \to F_s \left( L^2 \left( U_{Q,m}, d^3k \right) \right),
\]

see e.g. [6, Section 4.5]. We use \( F(J) \) to transfer the annihilation, creation and field operators to \( L^2 \left( U_{Q,m}, d^3k \right) \) in the following form:
\[ a(v) := F(J)A(J^{-1}v) F(J)^{-1} \]
\[ a^\dagger(v) := F(J)A(J^{-1}v)^\dagger F(J)^{-1}, \text{ for } v \in L^2(U_{Q,m},d^3k), \]
and
\[ \phi(\varphi) := F(J)\Phi(\varphi) F(J)^{-1} \text{ for } \varphi \in S(Q_p^4). \]

We will use \( A^\dagger(J^{-1}v) \) instead of \( A(J^{-1}v)^\dagger \). We now interpret \( a^\dagger \) as an operator-valued distribution on \( U_{Q,m} \) by restricting its argument to \( S(U_{Q,m}). \) Following the presentation given in section 5.2 in Folland’s book, we now adopt ‘the notational fiction’ that \( a^\dagger \) is a function, thus
\[ a^\dagger(v) = \int_{U_{Q,m}} a^\dagger(k) v(k) d^3k. \]
Similarly
\[ a(v) = \int_{U_{Q,m}} a(k) \overline{v(k)} d^3k. \]

By using the Canonical Commutation Relations:
\[
\left[ A(v), A(w)^\dagger \right] = \langle v, w \rangle I \quad \text{and} \quad [A(v), A(w)] = [A(v)^\dagger, A(w)^\dagger] = 0,
\]
for any \( v, w \in \mathcal{H} \), see e.g. [6 (4.48)], we have
\[
\int_{U_{Q,m}U_{Q,m}} [a(k), a^\dagger(k')] u(k)v(k') d^3kd^3k' = [a(u), a^\dagger(v)] = \langle u, v \rangle I = 
\left( \int_{U_{Q,m}} \int_{U_{Q,m}} \delta(k-k') u(k) v(k') d^3k d^3k' \right) I,
\]
which in colloquial distribution language can be expressed as
\[ [a(k), a^\dagger(k')] = \delta(k-k'). \]

In a similar form we get
\[ [a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0. \]

On the other hand, for \( \varphi \in S(Q_p^4) \) we have
\[ \phi(\varphi) = \frac{1}{\sqrt{2}} F(J) \left[ A(R\varphi) + A(R\varphi)^\dagger \right] F(J)^{-1} = \frac{1}{\sqrt{2}} [a(JR\varphi) + a^\dagger(JR\varphi)] \]
and
\[ JR\varphi(k) = \frac{1}{\sqrt{\omega(k)}} \hat{\varphi}(\omega(k), k), \]
so
\[
\phi(\varphi) = \int_{U_{Q,m}} \frac{1}{\sqrt{2|\omega(k)|_p}} \left( \bar{\varphi}(\omega(k), k) a(k) + \varphi(\omega(k), k) a^\dagger(k) \right) d^3k
\]

\[
= \int_{U_{Q,m}} \int_{\mathbb{Q}_p^3} \sqrt{2|\omega(k)|_p} \left\{ \chi_p([k, -x]) a(k) + \chi_p([k, x]) a^\dagger(k) \right\} \varphi(x) d^4x d^3k,
\]

which can be rewritten in ‘the notational fiction’ as
\[
\phi(x) = \int_{U_{Q,m}} \frac{1}{\sqrt{2|\omega(k)|_p}} \left\{ \chi_p([k, -x]) a(k) + \chi_p([k, x]) a^\dagger(k) \right\} d^3k.
\]

Now by ‘downgrading \(a(k), a^\dagger(k)\) to functions in \(S(U_{Q,m})\)’ and by multiplying them by \(\sqrt{2|\omega(k)|_p}\) (which produces two functions in \(S(U_{Q,m})\)), we get that \(\phi(x)\) is a weak solution of the \(p\)-adic Klein-Gordon equation.

**Remark 9.** (i) The construction of the charged complex field presented in [6] Section 5.2 can be carry out in the same form in the non-Archimedean setting by using the positive and negative mass shells \(V^{+}_{m^2}\) and \(V^{-}_{m^2}\) each equipped with the measure \(d\lambda_{m^2}(k)\).

(ii) A formal treatment of the Canonical Commutation Relations, as presented for instance in [15], demands further mathematical developments, for instance, the development of the wave front sets theory for the \(p\)-adic Klein-Gordon equation.

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Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, Departamento de Matemáticas- Unidad Querétaro, Libramiento Norponiente #2000, Fracc. Real de Juriquilla. Santiago de Querétaro, Qro. 76230, México

E-mail address: wazuniga@math.cinvestav.edu.mx