Some generalizations on the univalence of an integral operator and quasiconformal extensions

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SOME GENERALIZATIONS ON THE UNIVALENCE OF AN INTEGRAL OPERATOR AND QUASICONFORMAL EXTENSIONS

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Abstract. By using the method of Loewner chains, we establish some sufficient conditions for the analyticity and univalency of functions defined by an integral operator. Also, we refine the result to a quasiconformal extension criterion with the help of Beckers’s method.

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1. INTRODUCTION

Let \( \mathcal{A} \) the class of functions \( f \) which are analytic in the open unit disk \( \mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \} \) with \( f(0) = f'(0) - 1 = 0 \). We denote by \( \mathcal{U}_r \) the open disk \( \{ z \in \mathbb{C} : |z| < r \} \), where \( 0 < r \leq 1 \), by \( \mathbb{U} = \mathbb{U}_1 \) the open unit disk of the complex plane and by \( I \) the interval \([0, \infty)\).

Let \( k \) be constant in \([0, 1)\). Then a homeomorphism \( f \) of \( G \subset \mathbb{C} \) is said to be \( k \)-quasiconformal, if \( \partial_\sigma f \) and \( \partial_\tau f \) in the distributional sense are locally integrable on \( G \) and fulfill the inequality \( |\partial_\sigma f| \leq k |\partial_\tau f| \) almost everywhere in \( G \). If we do not need to specify \( k \), we will simply call \( f \) quasiconformal.

Three of the most important and known univalence criteria for analytic functions defined in the open unit disk were obtained by Nehari [14], Ozaki-Nunokawa [17] and Becker [3]. Some extensions of these three criteria were given by [15, 16, 21–25]. Furthermore a lot of univalence criteria have been obtained by different authors (see also [7–9]).

In the present investigation, we will obtain a number of new criteria for the functions defined by the integral operator \( \mathcal{F}_\beta(z) \). Also, we obtain a refinement to a quasiconformal extension criterion of the main result.

2. PRELIMINARIES

Before proving our main theorem we present a brief summary of the method of Loewner chains and quasiconformal extension criterion.

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A function $L(z,t) : U \times [0, \infty) \to \mathbb{C}$ is said to be subordination chain (or Loewner chain) if:

(i) $L(z,t)$ is analytic and univalent in $U$ for all $t \geq 0$.
(ii) $L(z,t) \prec L(z,s)$ for all $0 \leq t \leq s < \infty$, where the symbol ” $\prec$ ” stands for subordination.

To prove our results, we will need the following theorem due to Ch. Pommerenke [20].

**Theorem 1.** Let $L(z,t) = a_1(t)z + a_2(t)z^2 + ..., a_1(t) \neq 0$ be analytic in $U_r$ for all $t \in I$, locally absolutely continuous in $I$, and locally uniform with respect to $U_r$. For almost all $t \in I$, suppose that

$$
\frac{\partial L(z,t)}{\partial z} = p(z,t) \frac{\partial L(z,t)}{\partial t}, \quad \forall z \in U_r
$$

where $p(z,t)$ is analytic in $U$ and satisfies the condition $\Re p(z,t) > 0$ for all $z \in U$, $t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in $U_r$, then for each $t \in I$, the function $L(z,t)$ has an analytic and univalent extension to the whole disk $U$.

The method of constructing quasiconformal extension criteria is based on the following result of Becker (see [3], [4] and also [5]).

**Theorem 2.** Suppose that $L(z,t)$ is a Loewner chain for which the function $p(z,t)$ given in (2.1) satisfies the condition

$$
p(z,t) \in \mathcal{U}(k) := \left\{ w \in \mathbb{C} : \left| \frac{w-1}{w+1} \right| \leq k \right\}
$$

$$
= \left\{ w \in \mathbb{C} : \left| \frac{1+k^2}{1-k^2} \right| \leq \frac{2k}{1-k^2}, \quad (0 \leq k < 1) \right\}
$$

for all $z \in U$ and $t \geq 0$. Then $L(z,t)$ admits a continuous extension to $\overline{U}$ for each $t \geq 0$ and the function $F(z,\zeta)$ defined by

$$
F(z,\zeta) = \begin{cases} 
L(z,0), & \text{if } |z| < 1 \\
\frac{L(z,|\zeta| \log |z|)}{|z|}, & \text{if } |z| \geq 1 
\end{cases}
$$

is a $k$–quasiconformal extension of $L(z,0)$ to $\mathbb{C}$.

Examples of quasiconformal extension criteria can be found in [1], [2], [6], [13], [19] and more recently in [10–12].

### 3. Main Results

In this section, using Theorem 1, we obtain certain sufficient conditions for the univalence of an integral operator.
Theorem 3. Let \( m \) be a positive real number and let \( \alpha, \beta \) be complex numbers such that \( \Re \alpha < 1/2, \Re \beta > 0 \) and \( f \in \mathbb{A} \). Let \( g \) and \( h \) be two analytic functions in \( U, g(z) = 1 + b_1 z + \ldots, h(z) = c_0 + c_1 z + \ldots \) If the following inequalities

\[
\left| \frac{f'(z)}{g(z) - \alpha} - \frac{m-1}{2} \right| < \frac{m+1}{2}, \tag{3.1}
\]

and

\[
\left| \left( \frac{f'(z)}{g(z) - \alpha} - 1 \right) |z|^\beta (m+1) + \left( 1 - |z|^{\beta (m+1)} \right) \left[ 2z^\beta \frac{f'(z)h(z)}{g(z) - \alpha} + \frac{1}{\beta} \frac{zg'(z)}{g(z) - \alpha} \right] \right| \leq \frac{m+1}{2} \tag{3.2}
\]

are true for all \( z \in U \), then the function \( F_\beta(z) \) defined by

\[
F_\beta(z) = \left[ \beta \int_0^z u^{\beta-1} f'(u) du \right]^{1/\beta} \tag{3.3}
\]

is analytic and univalent in \( U \), where the principal branch is intended.

Proof. We shall prove that there exists a real number \( r; r \in (0,1] \) such that the function \( \mathcal{L} : \mathbb{U}_r \times I \to \mathbb{C} \), defined formally by

\[
\mathcal{L}(z,t) = \left[ \beta \int_0^{e^{-t} z} u^{\beta-1} f'(u) du + \frac{(e^{\beta m t} - e^{-\beta t})}{1 + (e^{\beta m t} - e^{-\beta t})} e^\beta \left( g(e^{-t} z) - \alpha \right) \right]^{1/\beta} \tag{3.4}
\]

is analytic in \( \mathbb{U}_r \) for all \( t \in I \).

Because \( f \in \mathbb{A} \) we have

\[
f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots, \quad \forall z \in U.
\]

Let us denote by

\[
\varphi_1(z,t) = \beta \int_0^{e^{-t} z} u^{\beta-1} f'(u) du. \tag{3.5}
\]

We obtain \( \varphi_1(z,t) = (e^{-t} z)^\beta + \frac{2 \beta a_2}{\beta + 1} (e^{-t} z)^{\beta+1} + \ldots \) and we observe that

\[
\varphi_1(z,t) = z^\beta \varphi_2(z,t) \tag{3.6}
\]
where
\[
\varphi_2(z, t) = e^{-\beta t} + \sum_{n=2}^{\infty} \frac{n^\beta}{n + \beta - 1} a_n e^{-(n + \beta - 1) t} z^{n-1}.
\] (3.7)

The function \(\varphi_2\) is analytic in \(U\) for all \(t \in I\), since
\[
\lim_{n \to \infty} \left| \frac{n^\beta}{n + \beta - 1} a_n e^{-(n + \beta - 1) t} \right| = e^{-t} \lim_{n \to \infty} \sqrt{\left| a_n \right|}.
\]

It is clear that if \(z \in U\), then \(e^{-t} z \in U\) for all \(t \in I\) and because \(f'(0) = 1\), there exists a disk \(U_{r_1}, 0 < r_1 \leq 1\) in which \(f'(e^{-t} z) \neq 0\) for all \(t \geq 0\).

From the analyticity of \(f\) it follows that the function \(\varphi_3\) is also analytic in \(U_{r_1}\), where
\[
\varphi_3(z, t) = 1 + \left( e^{\beta m t} - e^{-\beta t} \right) z^\beta h(e^{-t} z).
\] (3.8)

We have \(\varphi_3(0, t) = 1\) and then there exists a disk \(U_{r_2}, 0 < r_2 \leq r_1\) in which \(\varphi_3(z, t) \neq 0\) for all \(t \geq 0\).

Then the function
\[
\varphi_4(z, t) = \varphi_2(z, t) + \left( e^{\beta m t} - e^{-\beta t} \right) \left( g(e^{-t} z) - \alpha \right) \varphi_3(z, t)
\] (3.9)
is also analytic in \(U_{r_2}\) and \(\varphi_4(0, t) = (1 - \alpha) e^{\beta m t} + \alpha e^{-\beta t}\). From \(\Re \alpha < 1/2, \Re \beta > 0\) we deduce that \(\varphi_4(0, t) \neq 0\) for all \(t \in I\). Therefore, there exists a disk \(U_{r'}, 0 < r' \leq r_2\) in which \(\varphi_4(0, t) \neq 0\) for all \(t \in I\) and we can choose an analytic branch of \([\varphi_4(z, t)]^{1/\beta}\), denoted by \(\varphi_5(z, t)\). We choose the uniform branch which is equal to \(a_1(t) = \left[ (1 - \alpha) e^{\beta m t} + \alpha e^{-\beta t} \right]^{1/\beta} \) at the origin, and for \(a_1(t)\) we get \(\lim_{t \to \infty} |a_1(t)| = \infty\). Moreover, we have \(a_1(t) \neq 0\) for all \(t \geq 0\).

From (3.4)-(3.9) it follows that the relation (3.4) can be written as
\[
\mathcal{L}(z, t) = z \varphi_5(z, t)
\] (3.10)
and hence we obtain that the function \(\mathcal{L}(z, t)\) is analytic in \(U_r\),
\[
\mathcal{L}(z, t) = a_1(t) z + \ldots, \forall z \in U_r, \forall t \in I.
\]
\(\mathcal{L}(z, t)\) is an analytic function in \(U_r\) for all \(t \in I\) and then it follows that there is a number \(r_3, 0 < r_3 < r\) and a positive constant \(K = K(r_3)\) such that
\[
\left| \frac{\mathcal{L}(z, t)}{a_1(t)} \right| < K, \forall z \in U_{r_3}, t \geq 0.
\]
Then, by Montel’s theorem, it follows that \(\left\{ \frac{\mathcal{L}(z, t)}{a_1(t)} \right\}_{t \geq 0}\) is a normal family in \(U_{r_3}\).

From (3.10) we have
\[
\frac{\partial \mathcal{L}(z, t)}{\partial t} = z \frac{\partial \varphi_5(z, t)}{\partial t}.
\] (3.11)
It is clear that \( \frac{\partial \phi(z,t)}{\partial t} \) is an analytic function in \( U_{r_3} \) and then \( \frac{\partial \mathcal{L}(z,t)}{\partial t} \) is also an analytic function in \( U_{r_3} \). Then, for all fixed numbers \( T > 0 \) and \( r_4, 0 < r_4 < r_3 \), there exists a constant \( K_1 > 0 \) (which depends on \( T \) and \( r_4 \)) such that

\[
\left| \frac{\partial \mathcal{L}(z,t)}{\partial t} \right| < K_1, \quad \forall z \in U_{r_4} \text{ and } t \in [0, T].
\]

Therefore, the function \( \mathcal{L}(z,t) \) is locally absolutely continuous in \( [0, \infty) \) and is locally uniform with respect to \( U_{r_4} \).

Since \( \frac{\partial \mathcal{L}(z,t)}{\partial t} \) is analytic in \( U_{r_4} \), from (3.11) it follows that there is a number \( r_0 \), \( 0 < r_0 < r_4 \), such that \( \frac{1}{\varepsilon} \frac{\partial \mathcal{L}(z,t)}{\partial t} \neq 0, \forall z \in U_{r_0} \), so the function

\[
p(z,t) = z \frac{\partial \mathcal{L}(z,t)}{\partial z} \frac{\partial \mathcal{L}(z,t)}{\partial t}
\]

is analytic in \( U_{r_0} \) for all \( t \geq 0 \).

In order to prove that the function \( p(z,t) \) has an analytic extension with positive real part in \( U \) for all \( t \geq 0 \), it is sufficient to prove that the function \( w(z,t) \) defined in \( U_{r_0} \) by

\[
w(z,t) = \frac{p(z,t) - 1}{p(z,t) + 1}
\]

can be extended analytically in \( U, |w(z,t)| < 1 \) for all \( z \in U \) and \( t \geq 0 \).

After some calculations we obtain:

\[
w(z,t) = \frac{2}{m+1} G(z,t) - \frac{m-1}{m+1}, \quad (3.12)
\]

where

\[
G(z,t) = e^{-\beta(m+1)t} \left( \frac{f'(e^{-t}z)}{g(e^{-t}z) - \alpha} - 1 \right)
\]

\[
+ \left( 1 - e^{-\beta(m+1)t} \right) \left[ 2e^{-\beta t} \beta \frac{f'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z) - \alpha} + \frac{e^{-t}z}{\beta} g'(e^{-t}z) - \frac{h'(e^{-t}z)}{g(e^{-t}z) - \alpha} \right]
\]

\[
+ e^{-\beta t} \beta \left( 1 - e^{-\beta(m+1)t} \right)^2 \times \left[ e^{-\beta t} \beta \frac{f'(e^{-t}z)h^2(e^{-t}z)}{g(e^{-t}z) - \alpha} + \frac{e^{-t}z}{\beta} \frac{h(e^{-t}z)g'(e^{-t}z)}{g(e^{-t}z) - \alpha} - h'(e^{-t}z) \right]. \quad (3.13)
\]

for \( z \in U \) and \( t \geq 0 \).
The inequality $|w(z, t)| < 1$ for all $z \in \mathcal{U}$ and $t \geq 0$, where $w(z, t)$ defined by (3.12), is equivalent to

$$\left| G(z, t) - \frac{m-1}{2} \right| < \frac{m+1}{2}, \quad \forall z \in \mathcal{U} \text{ and } t \geq 0. \quad (3.14)$$

Define

$$\mathcal{H}(z, t) = G(z, t) - \frac{m-1}{2}, \quad \forall z \in \mathcal{U} \text{ and } t \geq 0. \quad (3.15)$$

In view of (3.1) and (3.2), from (3.13) and (3.15) we have

$$|\mathcal{H}(z, 0)| = \left| \left( \frac{f'(z)}{g(z) - \alpha} - 1 \right) - \frac{m-1}{2} \right| < \frac{m+1}{2}. \quad (3.16)$$

Let $t > 0, z \in \mathcal{U} - \{0\}$. In this case the function $\mathcal{H}(z, t)$ is analytic in $\overline{\mathcal{U}}$ because $|e^{-t}z| < e^{-t} < 1$, for all $z \in \overline{\mathcal{U}}$. Using the maximum principle for $z \in \mathcal{U}$ and $t > 0$ we have

$$|\mathcal{H}(z, t)| < \max_{|\xi| = 1} |\mathcal{H}(\xi, t)| = |\mathcal{H}(e^{i\theta}, t)|,$$

where $\theta = \theta(t)$ is a real number.

Let $u = e^{-t}e^{i\theta}$. We have $|u| = e^{-t}$ and $e^{-\beta(m+1)t} = (e^{-t})^{\beta(m+1)} = |u|^{\beta(m+1)}$. From (3.13), we have

$$|\mathcal{G}(e^{i\theta}, t)| = |u|^{\beta(m+1)} \left( \frac{f'(u)}{g(u) - \alpha} - 1 \right)$$

$$+ \left( 1 - |u|^{\beta(m+1)} \right) \left[ 2u^{\beta} \frac{f'(u)h(u)}{g(u) - \alpha} + \frac{u}{\beta} \frac{g'(u)}{g(u) - \alpha} \right]$$

$$+ \frac{u^{\beta} \left( 1 - |u|^{\beta(m+1)} \right)^2}{|u|^{\beta(m+1)}}$$

$$\times \left[ u^{\beta} \frac{f'(u)h^2(u)}{g(u) - \alpha} + \frac{u}{\beta} \left( \frac{h(u)g'(u)}{g(u) - \alpha} - h'(u) \right) \right] - \frac{m-1}{2}. \quad (3.17)$$

Since $u \in \mathcal{U}$, the inequality (3.2) implies that

$$|\mathcal{H}(e^{i\theta}, t)| \leq \frac{m+1}{2}, \quad (3.17)$$

and from (3.16) and (3.17) it follows that the inequality (3.14)

$$|\mathcal{H}(z, t)| = \left| G(z, t) - \frac{m-1}{2} \right| < \frac{m+1}{2}$$

is satisfied for all $z \in \mathcal{U}$ and $t \in I$. Therefore $|w(z, t)| < 1$, for all $z \in \mathcal{U}$ and $t \geq 0$.

Since all the conditions of Theorem 1 are satisfied, we obtain that the function $\mathcal{L}(z, t)$ has an analytic and univalent extension to the whole unit disk $\mathcal{U}$, for all $t \in I$. 


For \( t = 0 \) we have \( \mathcal{L}(z, 0) = \mathcal{F}_\beta(z) \), for \( z \in U \) and therefore, the function \( \mathcal{F}_\beta(z) \) is analytic and univalent in \( U \).

For \( g = f' \) in Theorem 3, we obtain another univalence criterion as follows.

**Corollary 1.** Let \( m \) be a positive real number and let \( \alpha, \beta \) be complex numbers such that \( \Re \alpha < 1/2, \Re \beta > 0 \) and \( f \in \mathcal{A} \). Let \( h \) be an analytic functions in \( U \), \( h(z) = c_0 + c_1 z + \ldots \). If the following inequalities

\[
\left| \frac{f'(z)}{f''(z) - \alpha} - \frac{m + 1}{2} \right| < \frac{m + 1}{2}, \tag{3.18}
\]

and

\[
\left| \left( \frac{f'(z)}{f''(z) - \alpha} - 1 \right) |z|^\beta(m+1) \right.
\]

\[
+ \left( 1 - |z|^{\beta(m+1)} \right) \left[ 2z^\beta \frac{f'(z)h(z)}{f''(z) - \alpha} + \frac{1}{\beta} \frac{zf''(z)}{f''(z) - \alpha} \right]
\]

\[
+ \frac{z^{\beta+1} \left( 1 - |z|^{\beta(m+1)} \right)^2}{|z|^\beta(m+1)} \left[ 2z^\beta \frac{f'(z)h(z)}{f''(z) - \alpha} + \frac{1}{\beta} \left( \frac{f''(z)h(z)}{f''(z) - \alpha} - h'(z) \right) \right] \tag{3.19}
\]

\[
\leq \frac{m + 1}{2} \tag{3.20}
\]

are true for all \( z \in U \), then the function \( \mathcal{F}_\beta(z) \) defined by (3.3) is analytic and univalent in \( U \), where the principal branch is intended.

If we choose \( h = f'' \) in Corollary 1, we have another univalence criterion as follows.

**Corollary 2.** Let \( m \) be a positive real number and let \( \alpha, \beta \) be complex numbers such that \( \Re \alpha < 1/2, \Re \beta > 0 \) and \( f \in \mathcal{A} \). Let \( h \) be an analytic functions in \( U \), \( h(z) = c_0 + c_1 z + \ldots \). If the following inequalities

\[
\left| \frac{f'(z)}{f''(z) - \alpha} - \frac{m + 1}{2} \right| < \frac{m + 1}{2}. \tag{3.21}
\]

and

\[
\left| \left( \frac{f'(z)}{f''(z) - \alpha} - 1 \right) |z|^\beta(m+1) \right.
\]

\[
+ \left( 1 - |z|^{\beta(m+1)} \right) \left[ 2z^\beta \frac{f'(z)h(z)}{f''(z) - \alpha} + \frac{1}{\beta} \frac{zf''(z)}{f''(z) - \alpha} \right]
\]

\[
+ \frac{z^{\beta+1} \left( 1 - |z|^{\beta(m+1)} \right)^2}{|z|^\beta(m+1)} \left[ 2z^\beta \frac{f'(z)h(z)}{f''(z) - \alpha} + \frac{1}{\beta} \left( \frac{f''(z)h(z)}{f''(z) - \alpha} - h'(z) \right) \right] \tag{3.22}
\]

\[
\leq \frac{m + 1}{2} \tag{3.23}
\]


\[ z^{\beta+1} \left( 1 - |z|^{\beta(m+1)} \right)^2 \left[ \frac{z^{\beta-1} f'(z)h^2(z)}{f'(z) - \alpha} + \frac{1}{\beta} \left( \frac{f''(z)h(z)}{f'(z) - \alpha} - h'(z) \right) \right] \] (3.22)

\[ \leq \frac{m - 1}{2} \] (3.23)

are true for all \( z \in U \), then the function \( F_\beta(z) \) defined by (3.3) is analytic and univalent in \( U \), where the principal branch is intended.

**Corollary 3.** Let \( m \) be a positive real number and let \( \alpha, \beta \) be complex numbers such that \( \Re \alpha < 1/2, \Re \beta > 0 \) and \( f \in A \). If the following inequalities

\[ \left| \frac{f''(z)}{f'(z) - \alpha} - \frac{m + 1}{2} \right| < \frac{m + 1}{2}, \] (3.24)

and

\[ \left| \left( \frac{f'(z)}{f'(z) - \alpha} - 1 \right) |z|^{\beta(m+1)} + \left( 1 - |z|^{\beta(m+1)} \right) \left[ \frac{1}{\beta} \frac{z f''(z)}{f'(z) - \alpha} - \frac{m - 1}{2} \right] \right| \leq \frac{m + 1}{2} \] (3.25)

are true for all \( z \in U \), then the function \( F_\beta(z) \) defined by (3.3) is analytic and univalent in \( U \), where the principal branch is intended.

**Proof.** It results from Corollary 1 with \( g = f' \) and \( h = 0 \). \( \square \)

If we consider \( g(z) = f', h(z) = -\frac{1}{2} f'' \), \( \alpha = 0, \beta = 1 \) in Theorem 3, we obtain another univalence criterion as follows.

**Corollary 4.** Let \( m \) be a positive real number and \( f \in A \). If the following inequality

\[ \left| \frac{z^2 (1 - |z|^{m+1})^2}{|z|^{m+1}} \left( \frac{1}{2} \{ f ; z \} \right) - \frac{m - 1}{2} \right| \leq \frac{m + 1}{2} \] (3.26)

where

\[ \{ f ; z \} = \left( \frac{f''(z)}{f'(z)} \right)^2 - \frac{1}{2} \left( \frac{f'(z)}{f'(z)} \right)^2 \]

is true for all \( z \in U \), then the function \( f(z) \) is analytic and univalent in \( U \), where the principal branch is intended.

Setting \( \alpha = 0 \) in Corollary 3 we have another univalence criterion as follows.
Corollary 5. Let $m$ be a positive real number and let $\beta$ be complex number such that $\Re \beta > 0$ and $f \in A$. If the following inequality
\[
\left| \frac{1 - |z|^\beta}{\beta} \right| \left( \frac{z f''(z)}{f'(z)} \right) - \frac{m-1}{2} \leq \frac{m+1}{2}
\] (3.27)
is true for all $z \in \mathcal{U}$, then the function $F_\beta(z)$ defined by (3.3) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Corollary 6. Let $m$ be a positive real number and let $\beta$ be complex number with $\Re \beta > 0$ and $f \in A$. If the following inequality
\[
\left| \frac{1 - |z|^{m+1}}{\frac{m+1}{\Re \beta}} \right| \left( \frac{z f''(z)}{f'(z)} \right) \leq 1
\] (3.28)
is true for all $z \in \mathcal{U}$, then the function $F_\beta(z)$ defined by (3.3) is analytic and univalent in $\mathcal{U}$, where the principal branch is intended.

Proof. It can be proved (see [18]) that for $z \in \mathcal{U} \setminus \{0\}$, $\Re \beta > 0$ and $m \in \mathbb{R}_+$
\[
\left| \frac{1 - |z|^\beta}{\beta} \right| \leq \frac{1 - |z|^{m+1}}{\frac{m+1}{\Re \beta}}.
\]
For $m \geq 1$, we have
\[
\left| \frac{1 - |z|^\beta}{\beta} \right| \left( \frac{z f''(z)}{f'(z)} \right) - \frac{m-1}{2} \leq \frac{m+1}{2}
\]
\[
\leq \left| \frac{1 - |z|^{m+1}}{\frac{m+1}{\Re \beta}} \right| \left( \frac{z f''(z)}{f'(z)} \right) + \frac{m-1}{2}
\]
\[
\leq 1 + \frac{m-1}{2} = \frac{m+1}{2}.
\]
Since inequalities (3.1) and (3.2) are satisfied, making use of Theorem 3, we can conclude that the function $F_\beta$ is analytic and univalent in $\mathcal{U}$. \qed

Putting $g(z) = \left( \frac{f(z)}{z} \right)^2$, $h(z) = 0$, $\alpha = 0$, in Theorem 3, we get the univalence criterion as follows.

Corollary 7. Let $m$ be a positive real number and let $\beta$ be complex number such that $\Re \beta > 0$ and $f \in A$. If the following inequalities
\[
\left| \frac{z^2 f'(z)}{f^2(z)} \right| - \frac{m+1}{2} < \frac{m+1}{2}
\] (3.29)
and
\[
\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| + \frac{2(1 - |z|^\beta)}{\beta} \left( \frac{f(z)}{f(z)} - 1 \right) - \frac{m - 1}{2} \leq \frac{m + 1}{2}
\]
(3.30)
are true for all \( z \in \mathcal{U} \), then the function \( \mathcal{F}_\beta(z) \) defined by (3.3) is analytic and univalent in \( \mathcal{U} \), where the principal branch is intended.

**Corollary 8.** Let \( m \) be a positive real number and \( f \in \mathcal{A} \). If the following inequality
\[
\left| z \left( 1 - |z|^{m+1} \right) \left( \frac{2f''(z)}{f'(z)} + \frac{f''(z)}{f'(z)} \right) + \frac{z^2(1 - |z|^{m+1})^2}{|z|^{m+1}} \left( \frac{(f''(z))^2}{f'(z)} + (f''(z))^2 - f'''(z) \right) - \frac{m - 1}{2} \right| \leq \frac{m + 1}{2}
\]
(3.31)
is true for all \( z \in \mathcal{U} \), then the function \( f(z) \) is analytic and univalent in \( \mathcal{U} \), where the principal branch is intended.

**Proof.** It results from Corollary 2 with \( \alpha = 0, \beta = 1 \).

**Remark 1.**
1. Putting \( g(z) = f'(z), h(z) = 0, \alpha = 0, \beta = m = 1 \) in Theorem 3, we have Becker’s criterion [3].
2. If we consider \( g(z) = f'(z), h(z) = -\frac{1}{2} \frac{f''(z)}{f'(z)}, \alpha = 0, \beta = m = 1 \) in Theorem 3, we obtain the univalence criterion due to Nehari [14].
3. Setting \( g(z) = \left( \frac{f(z)}{z} \right)^2, h(z) = \frac{1}{z} - \frac{f(z)}{z^2}, \alpha = 0, \beta = m = 1 \) in Theorem 3, we get the univalence criterion due to Ozaki-Nunokawa [17].
4. For \( g(z) = f'(z), h(z) = \frac{1}{z} - \frac{f(z)}{z^2}, \alpha = 0, \beta = m = 1 \) in Theorem 3, we arrive at Goluzin’s criterion for univalence [9].
5. For \( m = 1 \) in Corollary 6, we obtain the univalence criterion due to Pascu [18].
6. If we consider \( g(z) = f'(z), h(z) = 0, \beta = 1 \) in Theorem 3, we have results of Raducanu et al. [23].
7. Putting \( \alpha = 0, \beta = m = 1 \) in Theorem 3, we get the univalence criterion due to Ovesea-Tudor and Owa [16].

**Example 1.** Let the function
\[
f(z) = \frac{z}{1 - \frac{z^2}{2}}.
\]
(3.32)
Then $f$ is univalent in $\mathcal{U}$ and the function

$$\mathcal{F}_2(z) = \left( 2 \int_0^z uf'(u)du \right)^{\frac{1}{2}} \tag{3.33}$$

is analytic and univalent in $\mathcal{U}$.

Infact, from equality (3.29) for $m = 1$, we have

$$\frac{z^2 f'(z)}{f^2(z)} - 1 = \frac{z^2}{2}. \tag{3.34}$$

It is clear that the condition (3.29) of the Corollary 7 is satisfied for $m = 1$, and then the function $f$ is univalent in $\mathcal{U}$.

Taking into account (3.34), the condition (3.30) of Corollary 7 becomes for $\beta = 2, m = 1$,

$$\left| \frac{z^2}{2} |z|^4 + \left( 1 - |z|^4 \right) \frac{2z^2}{2 - z^2} \right| \leq \frac{|z|^6}{2} + 2 \left( 1 - |z|^4 \right) |z|^{2}$$

$$= \frac{1}{2} \left( 4|z|^2 - 3|z|^6 \right) < 1$$

because the greatest value of the function $g(x) = 4x^2 - 3x^6$, for $x \in [0, 1]$ is taken for $x = \sqrt{\frac{2}{3}}$ and $g(\sqrt{\frac{2}{3}}) = \frac{24}{27}$. Therefore the function $\mathcal{F}_2(z)$ defined by (3.33) is analytic and univalent in $\mathcal{U}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{$f(z) = \frac{-z}{1-z^2}$}
\end{figure}
4. Quasiconformal Extension Criterion

In this section we will generalize the univalence condition given in Theorem 3 to a quasiconformal extension criterion.

**Theorem 4.** Let $m$ be a positive real number and let $\alpha, \beta$ be complex numbers such that $\Re \alpha < 1/2$, $\Re \beta > 0$, $f \in \mathcal{A}$ and $k \in [0, 1)$. Let $g$ and $h$ be two analytic functions in $\mathcal{U}$, $g(z) = 1 + b_1 z + ...$, $h(z) = c_0 + c_1 z + ...$. If the following inequalities

\[
\begin{align*}
&\left| \frac{f'(z)}{g(z) - \alpha} - \frac{m+1}{2} \right| < k \frac{m+1}{2}, \quad \tag{4.1} \\
&\left| f'(z) - 1 \right| \left| z \right|^{\beta(m+1)} \\
&+ \left( 1 - \left| z \right|^{\beta(m+1)} \right) \left[ 2z^\beta \frac{f'(z)h(z)}{g(z) - \alpha} + \frac{1}{\beta} z \frac{g'(z)}{g(z) - \alpha} \right] \\
&\leq k \frac{m+1}{2} \tag{4.2}
\end{align*}
\]

is true for all $z \in \mathcal{U}$, then the function $F_\beta(z)$ given by (3.3) has a $k$–quasiconformal extension to $\mathbb{C}$. 

\[
\begin{align*}
F_2(z) = \left( 4 \int_0^z \frac{2+u^2}{(2-u^2)^2} \, du \right)^{\frac{1}{2}}
\end{align*}
\]
Proof. Set
\[ \mathcal{L}(z, t) = \left[ \beta \int_0^z u^{\beta-1} f'(u) du + \frac{e^{\beta mt} - e^{-\beta t}}{1 + (e^{\beta mt} - e^{-\beta t}) z^\beta h(e^{-t}z)} \right]^{1/\beta} \] (4.3)
In the proof of Theorem 3 has been shown that the function \( \mathcal{L}(z, t) \) given by (4.3) is a subordination chain in \( \mathcal{U} \). Then we have
\[
\left| \frac{p(z, t) - 1}{p(z, t) + 1} \right| = \left| \frac{2}{m + 1} \left( e^{-\beta(m+1)t} \left( \frac{f'(e^{-t}z)}{g(e^{-t}z) - \alpha} + \frac{e^{-t}z}{\beta} \frac{g'(e^{-t}z)}{g(e^{-t}z) - \alpha} \right) \right) + \left( 1 - e^{-\beta(m+1)t} \right) \left[ e^{-\beta t} \frac{f'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z) - \alpha} + \frac{e^{-t}z}{\beta} \frac{g'(e^{-t}z)}{g(e^{-t}z) - \alpha} \right] \right| \]
\[
\times \left[ e^{-\beta t} \frac{f'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z) - \alpha} + \frac{e^{-t}z}{\beta} \left( \frac{h(e^{-t}z)g'(e^{-t}z)}{g(e^{-t}z) - \alpha} - h'(e^{-t}z) \right) \right] \left| - \frac{m - 1}{m + 1} \right| \leq k. \] (4.4)
The right hand of (4.4) always less than or equal to \( k \) from (4.2) and therefore \( \mathcal{F}_\beta \) can be extended to \( k \) quasiconformal mapping to \( \mathbb{C} \) by Theorem 1 and Theorem 2.

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