Rainbow matchings in properly-colored hypergraphs

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Abstract

A hypergraph $H$ is properly colored if for every vertex $v \in V(H)$, all the edges incident to $v$ have distinct colors. In this paper, we show that if $H_1, \ldots, H_s$ are properly-colored $k$-uniform hypergraphs on $n$ vertices, where $n \geq 3k^2s$, and $e(H_i) > \left(\binom{n}{k} - \binom{n-s+1}{k}\right)$, then there exists a rainbow matching of size $s$, containing one edge from each $H_i$. This generalizes some previous results on the Erdős Matching Conjecture.

Keywords: rainbow matching, properly-colored hypergraphs

1 Introduction

A $k$-uniform hypergraph is a pair $H = (V, E)$, where $V = V(H)$ is a finite set of vertices, and $E = E(H) \subseteq \binom{V}{k}$ is a family of $k$-element subsets of $V$ called edges. A matching in a hypergraph $H$ is a collection of vertex-disjoint edges. The size of a matching is the number of edges in the matching. The matching number $\nu(H)$ is the maximum size of a matching in $H$. In 1965, Erdős [4] asked to determine the maximum number of edges that could appear in a $k$-uniform $n$-vertex hypergraph $H$ with matching number $\nu(H) < s$, for given integer $s \leq \frac{n}{k}$. He conjectured that the problem has two extremal constructions. The first one is a hyper-clique consisting of all the $k$-subsets on $ks - 1$ vertices. The other one is a $k$-uniform hypergraph on $n$ vertices containing all the edges intersecting a fixed set of $s - 1$ vertices. Erdős posed the following conjecture:

Conjecture 1.1 ([4]) Every $k$-uniform hypergraph $H$ on $n$ vertices with matching number $\nu(H) < s \leq \frac{n}{k}$ satisfies $e(H) \leq \max\{\binom{ks-1}{k}, \binom{n}{k} - \binom{n-s+1}{k}\}$.

The case $s = 1$ is the classic Erdős–Ko–Rado Theorem [9]. The graph case ($k = 2$) was verified in [5] by Erdős and Gallai. The problem seems to be significantly harder for hypergraphs. When $k = 3$, Frankl, Rödl and Ruciński [11] proved the conjecture for $s \leq \frac{n}{4}$. Luczak and Mieczkowska [14]

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proved it for sufficiently large \( s \). The \( k = 3 \) case was finally settled by Frankl [8]. For general \( k \), a short calculation shows that when \( s \leq \frac{n}{k+1} \), we always have \((\binom{n}{k}) - (\frac{n-s+1}{k}) > (\frac{ks-1}{k})\). For this range, the second construction is believed to be optimal. Erdős [4] proved the conjecture for \( n \geq n_0(k, s) \). Bollobás, Daykin and Erdős [2] proved the conjecture for \( n > \frac{3k^3}{s} \), which was further improved to \( n \geq \frac{3k^2 s}{\log k} \) by Frankl, Luczak and Mieczkowska [10]. On the other hand, in an unpublished note, Füredi and Frankl proved the conjecture for \( n \geq \frac{5}{3}sk - \frac{2}{3}s \). Currently the best range is \( n \geq \frac{3}{2}sk - 2s \) by Frankl and Kupavskii [9].

In this paper, we consider a generalization of Erdős Matching Conjecture to properly-colored hypergraphs. A hypergraph \( H \) is properly colored if for every vertex \( v \in V(H) \), all edges incident to \( v \) are colored differently. A rainbow matching in a properly-colored hypergraph \( H \) is a collection of vertex disjoint edges with pairwise different colors. The size of a rainbow matching is the number of edges in the matching. The rainbow matching number, denoted by \( \nu_r(H) \), is the maximum size of a rainbow matching in \( H \). Motivated by the Erdős Matching Conjecture, we consider the following problem: how many edges can appear in a properly-colored \( k \)-uniform hypergraph \( H \) such that its rainbow matching number satisfies \( \nu_r(H) < s \leq \frac{n}{k} \)? In fact, it is called Rainbow Turán problem and is well studied in [13]. Note that here if we let \( H \) be rainbow, that is, every edge of \( H \) receives distinct colors, then we obtain the original Erdős Matching Conjecture.

More generally, let \( H_1, \ldots, H_s \) be properly-colored \( k \)-uniform hypergraphs on \( n \) vertices, a rainbow matching of size \( s \) in \( H_1, \ldots, H_s \) is a collection of vertex disjoint edges \( e_1, \ldots, e_s \) with pairwise different colors, where \( e_1 \in E(H_1), \ldots, e_s \in E(H_s) \). For simplicity, we call it an \( s \)-rainbow matching. Then what is the minimum \( M \), such that by assuming \( e(H_i) > M \) for every \( i \), it guarantees the existance of an \( s \)-rainbow matching?

In this paper, we prove the following result, which generalizes Theorem 1.2 and Theorem 3.3 of [12].

**Theorem 1.2** Let \( H_1, \ldots, H_s \) be properly-colored \( k \)-uniform hypergraphs on \( n \) vertices. If \( n \geq 3k^2 s \) and every \( e(H_i) > \frac{n}{k} - (\frac{n-s+1}{k}) \), then there exists an \( s \)-rainbow matching in \( H_1, \ldots, H_s \).

### 2 Preliminary results

In this section, we list some preliminary results about “rainbow” hypergraphs, which is a special case of properly-colored hypergraphs. In the next section, we will prove our main theorem with the help of these results. A hypergraph \( H \) is rainbow if the colors of any two edges in \( E(H) \) are different. From now on, when we say an edge \( e \) is disjoint from a collection of edges, it means that not only \( e \) is vertex-disjoint from those edges, but it also has a color different from the colors of
all these edges. We start by the following lemma for graphs. Note that here although each $G_i$ is rainbow, a color may appear in more than one $G_i$'s.

**Lemma 2.1** Let $G_1, \ldots, G_s$ be rainbow graphs on $n$ vertices. If $n \geq 5s$ and $e(G_i) > \binom{n}{2} - \binom{n-s+1}{2}$, then there exists an $s$-rainbow matching in $G_1, \ldots, G_s$.

**Proof.** We do induction on $s$. The base case $s = 1$ is trivial. For every vertex $v \in V(G_i)$ and $j \neq i$, let $G^j_i$ be the subgraph of $G_j$ induced by the vertex set $V(G_j) \setminus \{v\}$. Since there are at most $n - 1$ edges containing $v$ in $E(G_j)$, we have $e(G^j_i) \geq e(G_j) - (n - 1) > \binom{n}{2} - \binom{n-s+1}{2} - (n - 1) = \binom{n-1}{2} - \binom{n-1-(s-1)+1}{2}$. By induction, there exists an $(s-1)$-rainbow matching $\{e_j\}_{j \neq i}$ in $\{G^j_i\}_{j \neq i}$, which spans $2(s-1)$ vertices. So if some $G_i$ has a vertex $v$ with degree greater than $3(s-1)$, then there exists an edge $e$ in $G_i$ which contains $v$ and disjoint from the edges of the $(s-1)$-rainbow matching, which produces an $s$-rainbow matching. Hence we may assume that the maximum degree of each $G_i$ is at most $3(s-1)$.

Now pick an arbitrary edge $uv$ in $G_1$. Assume the color of $uv$ is $c(uv)$. Then we delete the vertices $u, v$ and the edge colored by $c(uv)$ in $G_2, \ldots, G_s$. Denote the resulting graphs by $G'_2, \ldots, G'_s$.

We can see that when $n \geq 5s$, for each $i \in \{2, \ldots, s\}$, we have $e(G'_i) > \binom{n}{2} - \binom{n-s+1}{2} - 2.3(s-1)-1 > \binom{n-2}{2} - \binom{n-2-(s-1)+1}{2}$. By induction on $s$, there exists an $(s-1)$-rainbow matching in the graphs $G'_2, \ldots, G'_s$. Taking these $s-1$ edges with the edge $uv$, we obtain an $s$-rainbow matching in $G_1, \ldots, G_s$.

**Lemma 2.2** Let $H_1, \ldots, H_s$ be rainbow $k$-uniform hypergraphs on $n$ vertices. If $n \geq 3k^2s$ and $e(H_i) > \binom{n}{k} - \binom{n-s+1}{k}$, then there exists an $s$-rainbow matching in $H_1, \ldots, H_s$.

**Proof.** We do induction on both $k$ and $s$. According to Lemma 2.1 the case $k = 2$ holds for every $s$ and $n \geq 5s$. And for every $k$, the case $s = 1$ is trivial. We first consider the situation when some $H_i$ has a vertex $v$ with degree greater than $k(s-1)\binom{n-2}{k-2} + s - 1$. For every vertex $v \in V(H_i)$ and $j \neq i$, let $H^j_i$ be the subgraph of $H_j$ induced by the vertex set $V(H_j) \setminus \{v\}$. Since there are at most $\binom{n-1}{k-1}$ edges containing $v$ in $E(H_j)$, we have $e(H^j_i) \geq e(H_j) - \binom{n-1}{k-1} > \binom{n}{k} - \binom{n-s+1}{k} - \binom{n-1}{k} = \binom{n-1}{k} - \binom{n-1-(s-1)+1}{k}$. By inductive hypothesis for the case $(n-1, k, s-1)$, there exists an $(s-1)$-rainbow matching $\{e_j\}_{j \neq i}$ in $\{H^j_i\}_{j \neq i}$, which spans $k(s-1)$ vertices. So if some $H_i$ has a vertex $v$ with degree greater than $k(s-1)\binom{n-2}{k-2} + s - 1$, then there exists an edge $e$ in $E(H_i)$ which contains $v$ and disjoint from the edges of the $(s-1)$-rainbow matching, which produces an $s$-rainbow matching. Hence we may assume that the maximum degree in each hypergraph $H_i$ is at most $k(s-1)\binom{n-2}{k-2} + s - 1$.  

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By induction on $s$, we know that for every $i$ there exists an $(s-1)$-rainbow matching in the hypergraphs $\{H_j\}_{j \neq i}$, spanning $k(s-1)$ vertices. If for some $i$, the $s$-th largest degree of $H_i$ is at most $2(s-1)\left(\frac{n-2}{k-2}\right) + s - 1$, then the sum of degrees of these $k(s-1)$ vertices in $H_i$ is at most

$$(s-1)[k(s-1)\left(\frac{n-2}{k-2}\right) + s - 1] + (s-1)(k-1)[2(s-1)\left(\frac{n-2}{k-2}\right) + s - 1] = (3k-2)(s-1)^2\left(\frac{n-2}{k-2}\right) + (s-1)^2k.$$ 

Since $n \geq 3k^2s$, we have $e(H_i) > \binom{n}{k} > \binom{n-s+1}{k} > (s-1)^2(3k-\frac{1}{2})\left(\frac{n-2}{k-2}\right) > (3k-2)(s-1)^2\left(\frac{n-2}{k-2}\right) + (s-1)^2k + s - 1$, which guarantees the existence of an edge in $H_i$ which is disjoint from the previous $(s-1)$-rainbow matching in $\{H_j\}_{j \neq i}$, which produces an $s$-rainbow matching. So we may assume that each $H_i$ contains at least $s$ vertices with degree above $2(s-1)\left(\frac{n-2}{k-2}\right) + s - 1$.

Now we may greedily select distinct vertices $v_i \in V(H_i)$, such that for each $1 \leq i \leq s$, the degree of $v_i$ in $H_i$ exceeds $2(s-1)\left(\frac{n-2}{k-2}\right) + s - 1$. Consider all the subsets of $V(H_i) \setminus \{v_1, \ldots, v_s\}$ which together with $v_i$ form an edge of $H_i$. Denote the $(k-1)$-uniform hypergraph by $H'_i$. Then $e(H'_i) > 2(s-1)\left(\frac{n-2}{k-2}\right) + s - 1 - (s-1)\left(\frac{n-2}{k-2}\right) > \binom{n-s}{k-1} - \binom{n-2s+1}{k-1}$. By the inductive hypothesis for the case $(n-s,k-1,s)$, there exists an $s$-rainbow matching $\{e_i\}_{1 \leq i \leq s}$ in $\{H'_i\}_{1 \leq i \leq s}$. Taking the edges $e_i \cup \{v_i\}$, we obtain an $s$-rainbow matching in $\{H_i\}_{1 \leq i \leq s}$.

3 Main Theorem

In this section we prove our main result, Theorem 1.2 using induction and Lemma 2.2.

Proof. We split our proof into two cases.

Case 1: $k = 2$. Now $H_1, \ldots, H_s$ are properly-colored graphs. We do induction on $s$. The base case $s = 1$ is trivial. For every vertex $v \in V(H_i)$ and $j \neq i$, let $H'_i$ be the subgraph of $H_j$ induced by the vertex set $V(H_j) \setminus \{v\}$. Since there are at most $n-1$ edges containing $v$ in $E(H_j)$, we have $e(H'_i) \geq e(H_j) - (n-1) > \binom{n}{2} - \binom{n-s+1}{2} - (n-1) = \binom{n-1}{2} - \binom{n-1}{2} - (s-1)\binom{n-1}{2}$. By induction, there exists an $(s-1)$-rainbow matching $\{e_j\}_{j \neq i}$ in $\{H'_i\}_{j \neq i}$, which spans $2(s-1)$ vertices. So if some $H_i$ has a vertex $v$ of degree greater than $3(s-1)$, then there exists an edge $e$ in $H_i$ which contains $v$ and disjoint from the edges of the $(s-1)$-rainbow matching, which produces an $s$-rainbow matching. Hence we may assume the maximum degree in each $H_i$ is at most $3(s-1)$.

For every color $c$ in $H_i$ and $j \neq i$, let $H'_i$ be the subgraph of $H_j$ obtained by deleting all the edges colored by $c$ in $E(H_j)$. Since each $H_j$ is properly colored, there are at most $\binom{n}{2}$ edges colored by $c$ in $E(H_j)$. So $e(H'_i) \geq e(H_j) - \binom{n}{2} - \binom{n-s+1}{2} - \binom{n-1}{2} - (s-1)\binom{n-1}{2}$. By induction, there exists an $(s-1)$-rainbow matching $\{e_j\}_{j \neq i}$ in $\{H'_i\}_{j \neq i}$, which spans $2(s-1)$ vertices $u_1, \ldots, u_{2(s-1)}$. Also since $H_i$ is properly colored, it has at most one edge containing each $u_j$ and colored by $c$. So if the number of edges in $H_i$ colored by $c$ is greater than $2(s-1)$, then there exists an edge $e$ in
Taking the edges colored, we can see that each vertex $v$ in $H_i$ has a vertex of degree greater than $k(s-1)\binom{n-2}{k-2}+s-1$. For every vertex $v \in H_i$ and $j \neq i$, let $H_i^j$ be the subgraph of $H_j$ induced by the vertex set $V(H_j) \setminus \{v\}$. Since there are at most $\binom{n-1}{k-1}$ edges containing $v$ in $E(H_j)$, we have $e(H_i^j) \geq e(H_j) - \binom{n-1}{k-1} > \binom{n}{k} - \binom{n-1}{k-1} - \binom{n-1}{k-1} = \binom{n}{k} - \binom{(n-1)-(s-1)+1}{k}$. By induction, there exists an $(s-1)$-rainbow matching $\{e_j\}_{j \neq i}$ in $\{H_i^j\}_{j \neq i}$, which spans $k(s-1)$ vertices. So if some $H_i$ has a vertex $v$ with degree greater than $k(s-1)\binom{n-2}{k-2}+s-1$, then there exists an edge $e$ in $E(H_i)$ which contains $v$ and disjoint from the edges of the $(s-1)$-rainbow matching, which produces an $s$-rainbow matching. Hence we may assume the maximum degree in each hypergraph $H_i$ is at most $k(s-1)\binom{n-2}{k-2}+s-1$.

Case 2: $k \geq 3$. We do induction on $s$. The case $s = 1$ is trivial. We first consider the situation when some $H_i$ has a vertex of degree greater than $k(s-1)\binom{n-2}{k-2}+s-1$. For every vertex $v \in H_i$ and $j \neq i$, let $H_i^j$ be the subgraph of $H_j$ induced by the vertex set $V(H_j) \setminus \{v\}$. Since there are at most $\binom{n-1}{k-1}$ edges containing $v$ in $E(H_j)$, we have $e(H_i^j) \geq e(H_j) - \binom{n-1}{k-1} > \binom{n}{k} - \binom{n-1}{k-1} - \binom{n-1}{k-1} = \binom{n}{k} - \binom{(n-1)-(s-1)+1}{k}$. By induction, there exists an $(s-1)$-rainbow matching $\{e_j\}_{j \neq i}$ in $\{H_i^j\}_{j \neq i}$, which spans $k(s-1)$ vertices. So if some $H_i$ has a vertex $v$ with degree greater than $k(s-1)\binom{n-2}{k-2}+s-1$, then there exists an edge $e$ in $E(H_i)$ which contains $v$ and disjoint from the edges of the $(s-1)$-rainbow matching, which produces an $s$-rainbow matching. Hence we may assume the maximum degree in each hypergraph $H_i$ is at most $k(s-1)\binom{n-2}{k-2}+s-1$.

By induction on $s$, we know that for every $i$ there exists an $(s-1)$-rainbow matching in the hypergraphs $\{H_j\}_{j \neq i}$, spanning $k(s-1)$ vertices. If for some $i$, the $s$-th largest degree of $H_i$ is at most $2(s-1)\binom{n-2}{k-2}+s-1$, then the sum of degrees of these vertices in $H_i$ is at most

$$
(s-1)[k(s-1)\binom{n-2}{k-2}+s-1]+(s-1)(k-1)[2(s-1)\binom{n-2}{k-2}+s-1] = (3k-2)(s-1)^2\binom{n-2}{k-2}+(s-1)^2 k.
$$

On the other hand, the maximum degree of the subgraph of $H_i$ by deleting these $k(s-1)$ vertices is at most $s-1$, otherwise we can find an $s$-rainbow matching. Since $n \geq 3ks$, we have $e(H_i) > \binom{n}{k} - \binom{n-1}{k-1} > (s-1)^2(3k-1)\binom{n-2}{k-2} > (3k-2)(s-1)^2\binom{n-2}{k-2} + (s-1)^2 k + \frac{(s-1)[n-k(s-1)]}{k}$, which guarantees the existence of an edge in $H_i$ disjoint from the previous $(s-1)$-rainbow matching in $\{H_j\}_{j \neq i}$, which produces an $s$-rainbow matching. So we may assume that each $H_i$ contains at least $s$ vertices with degree above $2(s-1)\binom{n-2}{k-2}+s-1$.

Now we may greedily select distinct vertices $v_i \in V(H_i)$, such that for each $1 \leq i \leq s$, the degree of $v_i$ in $H_i$ exceeds $2(s-1)\binom{n-2}{k-2}+s-1$. Consider all the subsets of $V(H_i) \setminus \{v_1, \ldots, v_s\}$ which together with $v_i$ form an edge of $H_i$. Denote the $(k-1)$-uniform hypergraph by $H_i^s$. Since each $H_i$ is properly colored, we can see that each $H_i^s$ is rainbow and $e(H_i^s) > 2(s-1)\binom{n-2}{k-2} + s-1 - (s-1)\binom{n-2}{k-2} > \binom{n}{k} - \binom{n-2s+1}{k-1}$. By Lemma 222 there exists an $s$-rainbow matching $\{e_i\}_{1 \leq i \leq s}$ in $\{H_i^s\}_{1 \leq i \leq s}$. Taking the edges $e_i \cup \{v_i\}$, we obtain an $s$-rainbow matching in $\{H_i\}_{1 \leq i \leq s}$. ■
4 Concluding Remarks

In this short note, we propose a generalization of the Erdős hypergraph matching conjecture to finding rainbow matchings in properly-colored hypergraphs, and prove Theorem 1.2 for $s < n/(3k^2)$. The following conjecture seems plausible.

**Conjecture 4.1** There exists constant $C > 0$ such that if $H_1, \ldots, H_s$ are properly-colored $k$-uniform hypergraphs on $n$ vertices, with $n \geq Cks$ and every $e(H_i) > {n \choose k} - {n-s+1 \choose k}$, then there exists an $s$-rainbow matching in $H_1, \ldots, H_s$.

Recall that for the special case when each $H_i$ is identical and rainbow, Frankl and Kupavskii [9] were able to verify it for $C = 5/3$. However the proof relies on the technique of shifting, while the property of a hypergraph being properly colored may not be preserved under shifting.

It is tempting to believe that Erdős Matching Conjecture can be extended to properly-colored hypergraphs for the entire range of $s$, that is, once the number of edges in each hypergraph exceeds the maximum of $({n \choose k} - {n-s+1 \choose k}$ and $({ks-1 \choose k}$, then one can find an $s$-rainbow matching. However this is false in general, a simple construction is by taking $s = 2$ and $n = 2k$. The maximum of these two expressions is $({2k-1 \choose k}$, while one can let $H_1$ be a rainbow $K_{2k}$ with an edge coloring $c_1$, and $H_2$ be on the same vertex set with edge coloring $c_2$, such that $c_2(e) = c_1([2k] \setminus e)$. Then clearly each $H_i$ contains $({2k \choose k} > ({2k-1 \choose k}$ edges and there is no 2-rainbow matching. It would be interesting to find constructions for $s$ close to $n/k$, and formulate a complete conjecture for properly-colored hypergraphs.

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