Almost complex structures on connected sums of complex projective spaces

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Abstract

We show that the $m$-fold connected sum $m \# \mathbb{CP}^{2n}$ admits an almost complex structure if and only if $m$ is odd.

1 Introduction

A complex structure on a real vector bundle $F$ over a connected CW complex $X$ is a complex vector bundle $E$ over $X$ such that its underlying real vector bundle $E_{\mathbb{R}}$ is isomorphic to $F$. A stable complex structure on $F$ is an complex structure on $F \oplus \varepsilon^d$, where $\varepsilon^d$ is the $d$-dimensional real vector bundle over $X$. For $X$ a manifold we say that $X$ has an almost complex structure (respectively stable almost complex structure) if its tangent bundle admits an almost complex structure (respectively stable complex structure). Motivated by the question in [6] we consider in this paper the $m$-fold connected sum of complex projective spaces $m \# \mathbb{CP}^{2n}$.

As shown by Hirzebruch [4, Kommentare, p. 777], a necessary condition for the existence of an almost complex structure on a $4n$-dimensional compact manifold $M$ is the congruence $\chi(M) \equiv (-1)^n \sigma(M) \mod 4$, where $\chi(M)$ is the Euler characteristic and $\sigma(M)$ the signature of $M$. Thus, for even $m$, the connected sums above cannot carry an almost complex structure. We will show that for odd $m$ they do admit almost complex structures, thus showing

Theorem 1.1. The $m$-fold connected sum $m \# \mathbb{CP}^{2n}$ admits an almost complex structure if and only if $m$ is odd.

In odd complex dimensions, the connected sums $m \# \mathbb{CP}^{2n+1}$ are Kähler, since $\mathbb{CP}^{2n+1}$ admits an orientation reversing diffeomorphism and therefore $m \# \mathbb{CP}^{2n+1}$ is diffeomorphic to $\mathbb{CP}^{2n+1} \# (m-1)\mathbb{CP}^{2n+1}$ which is a blow–up of $\mathbb{CP}^{2n+1}$ in $m-1$ points, hence Kähler. Furthermore Theorem 1.1 is known for $n = 1$ and $n = 2$, see [1] and [10] respectively. In both cases the authors use general results on the existence of almost complex structures on manifolds of dimension 4 and 8 respectively.

We will prove Theorem 1.1 as follows. In [12, Theorem 1.1] or in [14, Theorem 1.7] the authors showed

Theorem 1.2. Let $M$ be a closed smooth $2d$-dimensional manifold. Then $TM$ admits an almost complex structure if and only if it admits a stable almost complex structure $E$ such that $c_d(E) = c(M)$, where $c_d$ is the $d$–th Chern class of $E$ and $c(M)$ is the Euler class of $M$.

In Section 2 we will describe the full set of stable almost complex structures in the reduced $K$–theory of $m \# \mathbb{CP}^{2n}$. In Section 3 we give, for odd $m$, an explicit example of a stable almost...
complex structure to which Theorem 1.2 applies. In Section 4 we give another argument for
the nonexistence of an almost complex structure for even \( m \) using Theorem 1.2.

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2 Stable almost complex structures on \( m\#\mathbb{CP}^{2n} \)

For a CW complex \( X \) let \( K(X) \) and \( KO(X) \) denote the complex and real \( K \)-groups respectively.
Moreover we denote by \( \tilde{K}(X) \) and \( \tilde{KO}(X) \) the reduced groups. Let \( r: K(X) \to KO(X) \) denote
the real reduction map, which can be restricted to a map \( \tilde{K}(X) \to \tilde{KO}(X) \). We denote the
restricted map again with \( r \). A real vector bundle \( F \) over \( X \) has a stable almost complex
structure if there is a an element \( y \in \tilde{K}(X) \) such that \( r(y) = F - \dim F \). Since \( r \) is a group
homomorphism, the set of all stable almost complex structures of \( F \) is given by

\[
y + \ker r \subset \tilde{K}(X),
\]

where \( y \) is such that \( r(y) = F - \dim F \). Let \( c: KO(X) \to K(X) \) denote the complexification
map and \( t: K(X) \to K(X) \) the map which is induced by complex conjugation of complex vector
bundles. The maps \( t \) and \( c \) are ring homomorphisms, but \( r \) preserves only the group structure.
The following identities involving the maps \( r, c \) and \( t \) are well known

\[
c \circ r = 1 + t: K(X) \to K(X),
\]

\[
r \circ c = 2: KO(X) \to KO(X).
\]

We will write \( \tilde{y} = t(y) \) for an element \( y \in K(X) \).

For two oriented manifolds \( M \) and \( N \) of same dimension \( d \), we denote by \( M\#N \) the connected
sum of \( M \) with \( N \) which inherits an orientation from \( M \) and \( N \). The stable tangent bundle
of \( M\#N \) is induced by \( p_M^*(TM) \oplus p_N^*(TN) \) in \( KO(M\#N) \), where \( p_M: M\#N \to M \) and
\( p_N: M\#N \to N \) are the smooth collapsing maps to each factor. Hence \( T(M\#N) - d = TM + TN - 2d \) in \( KO(M\#N) \), where \( TM \) and \( TN \) denote the elements in \( \tilde{KO}(M\#N) \) induced
by \( p_M^*(TM) \) and \( p_N^*(TN) \) respectively. This shows that if \( M \) and \( N \) admit stable almost complex
structures so does \( M\#N \) (cf. [3]). For \( M = N = \mathbb{CP}^{2n} \) we consider the natural orientation
induced by the complex structure of \( \mathbb{CP}^{2n} \).

We proceed with recalling some basic facts on complex projective spaces. Let \( H \) be the
tautological line bundle over \( \mathbb{CP}^d \) and let \( x \in H^2(\mathbb{CP}^d; \mathbb{Z}) \) be the generator, such that the total
Chern class \( c(H) \) is given by \( 1 + x \). The cohomology ring of \( \mathbb{CP}^d \) is isomorphic to \( \mathbb{Z}[x]/(x^{d+1}) \).
The \( K \) and \( KO \) theory of \( \mathbb{CP}^d \) are completely understood. Let \( \eta := H - 1 \in \tilde{K}(\mathbb{CP}^d) \) and
\( \eta_R := r(\eta) \in \tilde{KO}(\mathbb{CP}^d) \). Then we have

**Theorem 2.1** (cf. [11] Theorem 3.9, [2] Lemma 3.5, [3] p. 170 and [13] Proposition 4.3).

(a) \( K(\mathbb{CP}^d) = \mathbb{Z}[\eta]/(\eta^{d+1}) \). The following sets of elements are an integral basis of \( K(\mathbb{CP}^d) \)

\[
(i) \ 1, \eta, \eta(\eta + \tilde{\eta}), \ldots, \eta(\eta + \tilde{\eta})^{r-1}, (\eta + \tilde{\eta}), \ldots, (\eta + \tilde{\eta})^r, \text{ and also, in case } d \text{ is odd, } \\
\eta^{2r+1} = \eta(\eta + \tilde{\eta})^r.
\]

\[
(ii) \ 1, \eta, \eta(\eta + \tilde{\eta}), \ldots, \eta(\eta + \tilde{\eta})^{r-1}, (\eta - \tilde{\eta})(\eta + \tilde{\eta}), \ldots, (\eta - \tilde{\eta})(\eta + \tilde{\eta})^{r-1}, \text{ and also, in case } d \text{ is odd, } \\
\eta^{2r+1}
\]

where \( r \) is the largest integer \( \leq d/2 \).
(b) (i) if \( d = 2r \) then \( KO(\mathbb{C}P^d) = \mathbb{Z}[\eta_R]/(\eta_R^{r+1}) \)
(ii) if \( d = 4r + 1 \) then \( KO(\mathbb{C}P^d) = \mathbb{Z}[\eta_R]/(\eta_R^{2r+1}, 2\eta_R^{2r+2}) \)
(iii) if \( d = 4r + 3 \) then \( KO(\mathbb{C}P^d) = \mathbb{Z}[\eta_R]/(\eta_R^{2r+2}) \).

(c) The complex stable tangent bundle is given by \((2n+1)\bar{\eta} \in \tilde{K}(\mathbb{C}P^{2n})\) and the real stable tangent bundle is given by \( r((2n+1)\bar{\eta}) \in \tilde{KO}(\mathbb{C}P^{2n}) \).

(d) The kernel of the real reduction map \( r: \tilde{K}(\mathbb{C}P^d) \to \tilde{KO}(\mathbb{C}P^d) \) is freely generated by the elements

\[
\begin{align*}
(i) & \quad \eta - \bar{\eta}, (\eta - \bar{\eta})(\eta + \bar{\eta}), \ldots, (\eta - \bar{\eta})(\eta + \bar{\eta})^{d-1}, \text{ if } d \text{ is even}, \\
(ii) & \quad \eta - \bar{\eta}, (\eta - \bar{\eta})(\eta + \bar{\eta}), \ldots, (\eta - \bar{\eta})(\eta + \bar{\eta})^{r-1}, 2\eta^d, \text{ if } d = 4r + 1, \\
(iii) & \quad \eta - \bar{\eta}, (\eta - \bar{\eta})(\eta + \bar{\eta}), \ldots, (\eta - \bar{\eta})(\eta + \bar{\eta})^{r-1}, \eta^d, \text{ if } d = 4r + 3.
\end{align*}
\]

Next we would like to describe the integer cohomology ring of \( m\# \mathbb{C}P^{2n} \). For that we introduce the following notation: Let \( \Lambda \) denote either \( \mathbb{Z} \) or \( \mathbb{Q} \). We define an ideal \( R_d(X_1, \ldots, X_m) \) in \( \Lambda[X_1, \ldots, X_m] \), where \( X_1, \ldots, X_m \) are indeterminants, as the ideal generated by the following elements

\[
X_i \cdot X_j, \quad i \neq j, \quad i \neq d \neq j, \\
X_i^d - X_j^d, \quad i \neq j, \\
X_{j+1}^d, \quad j = 1, \ldots, m.
\]

Hence we have

\[
H^*(m\# \mathbb{C}P^d; \Lambda) \cong \Lambda[x_1, \ldots, x_m]/R_d(x_1, \ldots, x_m)
\]

(1)
where \( x_j = p_j^*(x) \in H^2(m\# \mathbb{C}P^d; \Lambda) \), for \( x \in H^2(\mathbb{C}P^d; \Lambda) \) defined as above and \( p_j: m\# \mathbb{C}P^d \to \mathbb{C}P^d \) the projection onto the \( j \)-th factor. Note that \( p_j \) induces a monomorphism on cohomology.

The stable tangent bundle of \( m\# \mathbb{C}P^{2n} \) in \( \tilde{KO}(m\# \mathbb{C}P^{2n}) \) is represented by

\[
(2n+1) \sum_{j=1}^{m} r(\bar{\eta}_j)
\]

where \( \eta_j := p_j^*(\eta) \in \tilde{K}(\mathbb{C}P^{2n}) \) and \( r: \tilde{K}(m\# \mathbb{C}P^{2n}) \to \tilde{KO}(m\# \mathbb{C}P^{2n}) \) is the real reduction map.

Hence the set of stable almost complex structures on \( m\# \mathbb{C}P^{2n} \) is given by

\[
(2n+1) \sum_{j=1}^{m} \bar{\eta}_j + \ker r,
\]

(2)
For \( k \in \mathbb{N} \) and \( j = 1, \ldots, m \), set \( w_j^k = p_j^*(H)^k - p_j^*(H)^{-k} \), \( e_j^{n-1} = n_j(\eta_j + \bar{\eta}_j)^{n-1} \) and \( \omega = \eta_1^{2n} \).

**Proposition 2.2.** The kernel of \( r: \tilde{K}(m\# \mathbb{C}P^{2n}) \to \tilde{KO}(m\# \mathbb{C}P^{2n}) \) is freely generated by

\[
(a) \quad \{w_j^k : k = 1, \ldots, n-1, j = 1, \ldots, m\} \cup \{e_1^{n-1} - e_j^{n-1} : j = 2, \ldots, m\} \cup \{2e_1^{n-1} - \omega\}, \text{ for } n \text{ even},
\]

\[
(b) \quad \{w_j^k : k = 1, \ldots, n, j = 1, \ldots, m\}, \text{ for } n \text{ odd}.
\]
Proof. Consider the cofiber sequence
\[ \bigvee_{j=1}^{m} \mathbb{C}P^{2n-1} \xrightarrow{i} m \# \mathbb{C}P^{2n} \xrightarrow{\pi} S^{4n}. \] (3)

Note that the line bundle \( i^*p_j^*(H) \) is the tautological line bundle over the \( j \)-th summand of \( \bigvee_{j=1}^{m} \mathbb{C}P^{2n-1} \) and the trivial bundle on the other summands, since the first Chern classes are the same. For the reduced groups we have
\[ \tilde{K} \left( \bigvee_{j=1}^{m} \mathbb{C}P^{2n-1} \right) \cong \bigoplus_{j=1}^{m} \tilde{K} \left( \mathbb{C}P^{2n-1} \right) \]
and \( i^*p_j^*(\eta) \) generates the \( j \)-th summand of the above sum according to Theorem 2.1. The long exact sequence in \( K \)-theory of the cofibration (3) is given by
\[ \cdots \to \tilde{K}^{-1} \left( \bigvee_{j=1}^{m} \mathbb{C}P^{2n-1} \right) \to \tilde{K} \left( S^{4n} \right) \to \tilde{K} \left( m \# \mathbb{C}P^{2n} \right) \to \tilde{K} \left( \bigvee_{j=1}^{m} \mathbb{C}P^{2n-1} \right) \to \tilde{K}^{1} \left( S^{4n} \right) \to \cdots \] (4)

From Theorem 2 in [3] we have \( \tilde{K}^{-1} \left( \mathbb{C}P^{2n-1} \right) = 0 \), hence \( \tilde{K}^{-1} \left( \bigvee_{j=1}^{m} \mathbb{C}P^{2n-1} \right) = 0 \) and from Bott periodicity we deduce \( \tilde{K}^{1} \left( S^{4n} \right) = 0 \). So we obtain a short exact sequence
\[ 0 \to \tilde{K} \left( S^{4n} \right) \xrightarrow{\pi^*} \tilde{K} \left( m \# \mathbb{C}P^{2n} \right) \xrightarrow{j^*} \tilde{K} \left( \bigvee_{j=1}^{m} \mathbb{C}P^{2n-1} \right) \to 0. \]

which splits, since the involving groups are finitely generated, torsion free abelian groups. Let \( \omega_{\mathbb{C}} \) be the generator of \( \tilde{K} \left( S^{4n} \right) \), then the set
\[ \{ \pi^*(\omega_{\mathbb{C}}) \} \cup \left\{ \eta_j^k : j = 1, \ldots, m, k = 1, \ldots, 2n-1 \right\} \]
is an integral basis of \( \tilde{K} \left( m \# \mathbb{C}P^{2n} \right) \). We claim that \( \eta_j^{2n} = \pi^*(\omega_{\mathbb{C}}) \) for all \( j \). Indeed, the elements \( \eta_j^{2n} \) lie in the kernel of \( \pi^* \), hence there are \( k_j \in \mathbb{Z} \) such that \( \eta_j^{2n} = k_j \cdot \pi^*(\omega_{\mathbb{C}}) \).

Let \( \widetilde{ch} : \tilde{K} (X) \to \tilde{H} (X; \mathbb{Q}) \) denote the Chern character for a finite CW complex \( X \), then \( \widetilde{ch} \) is a monomorphism for \( X = m \# \mathbb{C}P^d \) (since \( \tilde{H}^*(m \# \mathbb{C}P^d; \mathbb{Z}) \) has no torsion, cf. [7]) and an isomorphism for \( X = S^d \) onto \( \tilde{H}^*(S^d; \mathbb{Z}) \) embedded in \( \tilde{H}^*(S^d; \mathbb{Q}) \). Using the notation of (4) we have
\[ \widetilde{ch}(\eta_j^{2n}) = (x_j^2)^2 = x_j^{2n}, \]
and using the naturality of \( \widetilde{ch} \)
\[ \widetilde{ch}(\pi^*(\omega_{\mathbb{C}})) = \pi^*(\widetilde{ch}(\omega_{\mathbb{C}})) = \pm x_j^{2n} \]
since \( \pi^* \) is an isomorphism on cohomology in dimension \( 2n \). We can choose \( \omega_{\mathbb{C}} \) such that \( \widetilde{ch}(\pi^*(\omega_{\mathbb{C}})) = x_j^{2n} \). This shows \( k_j = 1 \) for all \( j \) and \( \tilde{K} \left( m \# \mathbb{C}P^{2n} \right) \) is freely generated by
\[ \left\{ \eta_j^k : j = 1, \ldots, m, k = 1, \ldots, 2n-1 \right\} \cup \left\{ \eta_1^{2n} \cdot \cdots \cdot \eta_m^{2n} \right\}. \]
Hence \( K(m \# \mathbb{C}P^{2n}) = \mathbb{Z}[\eta_1, \ldots, \eta_m]/R_{2n}(\eta_1, \ldots, \eta_m) \). Since \( p_j^*(H) \otimes p_j^*(\overline{H}) \) is the trivial bundle we compute the identity
\[ \eta_j = \frac{-\eta_j}{1+\eta_j} = -\eta_j + \eta_j^2 - \cdots + \eta_j^{2n}. \]
The ring $\mathbb{Z}[\eta_1, \ldots, \eta_m]/R_{2n}(\eta_1, \ldots, \eta_m)$ is isomorphic to
\[
\left( \bigoplus_{j=1}^{m} \mathbb{Z}[\eta_j]/(\eta_j^{2n+1}) \right) / \langle \eta_j^{2n} - \eta_i^{2n} : j \neq i \rangle
\]
and from Theorem 2.1 the set $\Gamma_j$ which contains the elements
\[
\eta_j, \eta_j(\eta_j + \bar{\eta}_j), \ldots, \eta_j(\eta_j + \bar{\eta}_j)^{n-1} \\
\eta_j - \bar{\eta}_j, (\eta_j - \bar{\eta}_j)(\eta_j + \bar{\eta}_j), \ldots, (\eta_j - \bar{\eta}_j)(\eta_j + \bar{\eta}_j)^{n-1}
\]
together with $\{1\}$ is an integral basis of $\mathbb{Z}[\eta_j]/(\eta_j^{2n+1})$. Thus the set $\Gamma_1 \cup \ldots \cup \Gamma_m \subset \widetilde{K}(m\#\mathbb{CP}^2)$ generates the group $\widetilde{K}(m\#\mathbb{CP}^2)$. Observe that
\[
(\eta_j + \bar{\eta}_j)^k = 2\eta_j(\eta_j + \bar{\eta}_j)^{k-1} - (\eta_j - \bar{\eta}_j)(\eta_j + \bar{\eta}_j)^{k-1},
\]
thus
\[
\eta_j^{2n} = (\eta_j + \bar{\eta}_j)^n = 2\eta_j(\eta_j + \bar{\eta}_j)^{n-1} - (\eta_j - \bar{\eta}_j)(\eta_j + \bar{\eta}_j)^{n-1}.
\]
We set $\omega := \eta_j^{2n}$ for any $j = 1, \ldots, m$ and
\[
e_j^k := \eta_j(\eta_j + \bar{\eta}_j)^k, \quad j = 1, \ldots, m, \quad k = 0, \ldots, n-1 \\
f_j^k := (\eta_j - \bar{\eta}_j)(\eta_j + \bar{\eta}_j)^k, \quad j = 1, \ldots, m, \quad k = 0, \ldots, n-2
\]
and in virtue of relation (6) the set
\[
B := \{\omega\} \cup \{e_j^k : j = 1, \ldots, m, k = 0, \ldots, n-1\} \cup \{f_j^k : j = 1, \ldots, m, k = 0, \ldots, n-2\}
\]
is an integral basis of $\widetilde{K}(m\#\mathbb{CP}^2)$.

We proceed with the computation of $KO(m\#\mathbb{CP}^2)$. We have a long exact sequence for $\widetilde{KO}$-theory like in [4]. From Theorem 2 in [3] we deduce $\widetilde{KO}^{-1}(\mathbb{CP}^2) = 0$ and therefore $\widetilde{KO}^{-1}\left(\bigoplus_{j=1}^{m} \mathbb{CP}^2\right) = 0$. Moreover $\widetilde{KO}^{-1}(S^{4n}) = \widetilde{KO}^{-1}(S^{8l}) = \widetilde{KO}(S^{8l+7}) = 0$ by Bott periodicity. Hence we obtain a short exact sequence
\[
0 \longrightarrow \widetilde{KO}(S^{4n}) \longrightarrow \widetilde{KO}(m\#\mathbb{CP}^2) \longrightarrow \widetilde{KO}(\bigoplus_{j=1}^{m} \mathbb{CP}^{2n-1}) \longrightarrow 0.
\]
Now we have to distinguish between the cases where $n$ is even or odd. We first assume that $n = 2l$. In that case the ring $KO(\mathbb{CP}^{2n-1})$ is isomorphic to $\mathbb{Z}[\eta_R]/(\eta_R^{2l})$, see Theorem 2.1 (b). Hence all groups in (7) are torsion free. Therefore the kernel of $r: \widetilde{K}(m\#\mathbb{CP}^2) \to \widetilde{K}(m\#\mathbb{CP}^2)$ is the same as the kernel of
\[
\varphi := c \circ r = 1 + t: \widetilde{K}(m\#\mathbb{CP}^2) \to \widetilde{K}(m\#\mathbb{CP}^2)
\]
since $r \circ c = 2$ and thus $c$ is a monomorphism of the torsion free part of $\widetilde{KO}(m\#\mathbb{CP}^2)$. Next we compute a basis of $\ker \varphi$. Using relation (5) we have $\varphi(\omega) = 2\omega$, $\varphi(e_j^k) = 2e_j^k - f_j^k$ and $\varphi(f_j^k) = 0$, thus if
\[
y = \lambda \omega + \sum_{j=1}^{m} \sum_{k=0}^{n-1} \lambda_j^k e_j^k
\]
then \( \varphi(y) = 0 \) if and only if \( \lambda_k^j = 0 \) for \( j = 1, \ldots, m, \ k = 1, \ldots, n - 2 \) and 
\[
\sum_{j=1}^{m} \lambda_j^{n-1} + 2\lambda = 0.
\]

This implies that the set
\[
\{f^k_j : j = 1, \ldots, m, \ k = 0, \ldots, n - 2\} \cup \{e_1^{n-1} - e_j^{n-1} : j = 2, \ldots, m\} \cup \{2e_1^{n-1} - \omega\},
\]
is an integral basis of \( \ker \varphi \). Note that from (6) we have \( 2e_1^{n-1} - \omega = (\eta_1 - \bar{\eta}_1)(\eta_1 + \bar{\eta}_1)^{n-1} \). By an inductive argument we see that
\[
(\eta_j - \bar{\eta}_j)(\eta_j + \bar{\eta}_j)^k w_j^{k+1} + \text{linear combinations of } w_j^1, \ldots, w_j^k
\]
and
\[
e_j^{n-1} - e_j^{n-1} = \eta_1^{2n-1} - \eta_j^{2n-1}.
\]
Thus an integral basis of the kernel, in case \( n \) is even, is given by
\[
\{w^k_j : j = 1, \ldots, m, \ k = 1, \ldots, n - 1\} \cup \{w_1^n\} \cup \{\eta_1^{2n-1} - \eta_j^{2n-1} : j = 2, \ldots, m\}.
\]
Now let us assume that \( n = 2l + 1 \). Consider the commutative diagram
\[
\begin{array}{cccccc}
0 & \longrightarrow & \widehat{K}(S^{4n}) & \stackrel{\pi^*}{\longrightarrow} & \widehat{K}(m#CP^{2n}) & \stackrel{i^*}{\longrightarrow} & \widehat{K}(\vee_{j=1}^{m} CP^{2n-1}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widehat{KO}(S^{4n}) & \stackrel{\pi^*}{\longrightarrow} & \widehat{KO}(m#CP^{2n}) & \stackrel{i^*}{\longrightarrow} & \widehat{KO}(\vee_{j=1}^{m} CP^{2n-1}) & \longrightarrow & 0
\end{array}
\]
The map \( r_S : \widehat{K}(S^{8l+4}) \rightarrow \widehat{KO}(S^{8l+4}) \) is an isomorphism and therefore \( i^*|_{\ker r_\#} : \ker r_\# \rightarrow \ker r_\vee \) is an isomorphism, hence the rank of \( \ker r_\# \) is \( mn \). We see that the set
\[
\{f^k_j : j = 1, \ldots, m, \ k = 0, \ldots, n - 2\} \cup \{2e_j^{n-1} : j = 1, \ldots, m\} \cup \{\omega\}
\]
is an integral basis of \( (i^*)^{-1} (\ker r_\vee) \), which follows because \( e_j^{n-1} = \eta_j^{2n-1} - (n-1)\omega \) and the structure of the kernel of \( r_\vee \), see Theorem 2.1 (d) (ii). The elements \( f^k_j \) for \( j = 1, \ldots, m \) and \( k = 0, \ldots, n - 2 \) lie in the kernel of \( r_\# \). Let
\[
y = \lambda \omega + \sum_{j=1}^{m} \lambda_j^{n-1}2e_j^{n-1}
\]
for \( \lambda, \lambda_j^{n-1} \in \mathbb{Z} \) and suppose \( r_\#(y) = 0 \). From \( \varphi(\omega) = 2\omega \) and \( \varphi(e_j^{n-1}) = (\eta_j + \bar{\eta}_j)^n = \eta_j^{2n} = \omega \) it follows that
\[
\lambda + \sum_{j=1}^{m} \lambda_j^{n-1} = 0.
\]
Hence \( \ker r_\# \) is freely generated by the elements \( f^k_j \) and \( 2e_j^{n-1} - \omega \). Observe from (6) that \( 2e_j^{n-1} - \omega = (\eta - \bar{\eta})(\eta + \bar{\eta})^{n-1} \). Thus in case of \( n \) odd we deduce like in (8) that the kernel of \( r_\# \) is freely generated by \( w_j^k \) for \( j = 1, \ldots, m \) and \( k = 1, \ldots, n \). ■
Hence by Equation (2), stable almost complex structures of $m\#\mathbb{CP}^{2n}$ for $n$ even are given by elements of the form

$$y = (2n + 1) \sum_{i=1}^{m} \bar{a}_i + \sum_{j=1}^{m} \sum_{k=1}^{n-1} a_j^k w_k^j + a^n w^n + \sum_{j=2}^{m} b_j (\eta_1^{2n-1} - \eta_2^{2n-1}).$$  \hspace{1cm} (9)$$

and for $n$ odd

$$y = (2n + 1) \sum_{i=1}^{m} \bar{a}_i + \sum_{j=1}^{m} \sum_{k=1}^{n} a_j^k w_k^j$$  \hspace{1cm} (10)$$

for $a_j^k, b_j \in \mathbb{Z}$. For Theorem 1.2 we have to compute the $(2n)$-th Chern class $c_{2n}(E)$ of a vector bundle $E$ representing an element of the form (9) and (10). Let $\eta_1^{2n-1} - \eta_2^{2n-1}$ denote also a vector bundle over $m\#\mathbb{CP}^{2n}$ which represents the element $\eta_1^{2n-1} - \eta_2^{2n-1}$ in $\tilde{K}(m\#\mathbb{CP}^{2n})$. The total Chern class of $\eta_1^{2n-1} - \eta_2^{2n-1}$ can be computed through the Chern character: We have

$$\tilde{c}(\eta_1^{2n-1} - \eta_2^{2n-1}) = \tilde{c}(\eta_1)^{2n-1} - \tilde{c}(\eta_2)^{2n-1} = x_1^{2n-1} - x_2^{2n-1}.$$ 

The elements of degree $k$ in the Chern character are given by $\nu_k(c_1, \ldots, c_k)/k!$ where $\nu_k$ are the Newton polynomials. The coefficient in front of $c_k$ in $\nu_k(c_1, \ldots, c_k)$ is $k$ (see [9], p 195) and the other terms are products of Chern classes of lower degree, hence the only non-vanishing Chern class is given by

$$c_{2n-1}(\eta_1^{2n-1} - \eta_2^{2n-1}) = (2n - 2)! (x_1^{2n-1} - x_2^{2n-1}).$$

Thus the total Chern class of a vector bundle $E$ representing an element of the form (9) is given by

$$c(E) = (1 - (x_1 + \ldots + x_m))^{2n+1}$$

$$\cdot \left(1 + \frac{n x_1}{1 - n x_1}\right)^a \prod_{j=2}^{m} \left(1 + (2n - 2)! (x_1^{2n-1} - x_2^{2n-1})\right)^{b_j} \prod_{j=1}^{m} \prod_{k=1}^{n-1} \left(1 + k x_j\right)^{a_j^k}$$

and for (10)

$$c(E) = (1 - (x_1 + \ldots + x_m))^{2n+1} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(1 + k x_j\right)^{a_j^k}$$

where the coefficient in front of $x_1^{2n} = \ldots = x_m^{2n}$ is equal to $c_{2n}(E)$.

**Remark 2.3.** Note that for $m = 1$ (and complex projective spaces of arbitrary dimension) this total Chern class was already computed by Thomas, see [13, p. 130].

### 3 The case $m$ odd

We now describe an explicit stable almost complex structure on $m\#\mathbb{CP}^{2n}$, where $m = 2u + 1$, for which the assumptions of Theorem 1.2 are satisfied, thereby producing an almost complex structure on $m\#\mathbb{CP}^{2n}$. We choose, in the notation of (9) and (10), $a_j^k = 2$ for $j = 1, \ldots, u$ and $k = 1$, and all other coefficients 0. Then the top Chern class is as desired:

**Proposition 3.1.** Let $m = 2u + 1$ be an odd number. In the cohomology ring of $m\#\mathbb{CP}^{2n}$, the coefficient $c_{2n}$ of $x_1^{2n} = \cdots = x_k^{2n}$ of the class

$$c = (1 - (x_1 + \cdots + x_{2u+1}))^{2n+1} \prod_{r=1}^{u} \left(1 + \frac{x_r}{1 - x_r}\right)^2$$

for $u \geq 1$.
is
\[ c_{2n} = m(2n - 1) + 2 = \chi(m\#\mathbb{C}P^{2n}). \]

**Proof.** As \( x_i \cdot x_j = 0 \) for \( i \neq j \), we have
\[
(1 - (x_1 + \cdots + x_{2u+1}))^{2n+1} = \sum_{j_0=0}^{2n+1} (-1)^{j_0} \binom{2n+1}{j_0} x_1^{j_0} + \cdots + x_{2u+1}^{j_0}.
\]

Thus,
\[
c = \prod_{r=1}^{u} (1 - x_r)^{2n-1} (1 + x_r)^2 \prod_{s=u+1}^{2n+1} (1 - x_s)^{2n+1}.
\]
The factors \((1 - x_s)^{2n+1}\) contribute \(2n + 1\) to \(c_{2n}\), whereas the factors \((1 - x_r)^{2n-1}(1 + x_r)^2\) contribute \(2n - 3\). Thus,
\[
c_{2n} = u(2n - 3) + (u + 1)(2n + 1) = (2u + 1)(2n - 1) + 2 = \chi((2u + 1)\#\mathbb{C}P^{2n}).
\]

\[\blacksquare\]

## 4 The case \( m \) even

As already explained in the introduction, the congruence \( \chi(M) \equiv (-1)^n \sigma(M) \mod 4 \) immediately shows the other implication of Theorem 1.1. In this section we give an alternative argument, using only Theorem 1.2 and the description of the total Chern class of a general stable almost complex structures we obtained in Section 2. For \( k \in \mathbb{N} \) and \( a \in \mathbb{Z} \) consider the rational function
\[
\left( \frac{1 + kx}{1 - kx} \right)^a
\]
and its power series
\[
\sum_{m \geq 0} p^k_m(a)x^m.
\]

**Proposition 4.1.** The coefficients \( p^k_m(a) \) are even for all \( k \in \mathbb{N} \), \( a \in \mathbb{Z} \) and \( m > 0 \). Moreover we have
\[
\frac{1}{2} \cdot p^k_m(a) \equiv ka \quad \text{mod} \ 2
\]
for \( m \geq 1 \).

**Proof.** Define \( g(x) = \left( \frac{1 + kx}{1 - kx} \right)^a \), then
\[
g(x) = \left( \frac{1 + kx}{1 - kx} \right)^a = (f_\varepsilon(x))^{[a]}
\]
where \( f_\varepsilon(x) = \left( \frac{1 + kx}{1 - kx} \right)^\varepsilon \) and \( \varepsilon = 1 \) if \( a \geq 0 \) and \( \varepsilon = -1 \) if \( a < 0 \). We have
\[
p^k_m(a) = \frac{1}{m!} g^{(m)}(0) = \frac{1}{m!} (f_\varepsilon^{[a]})^{(m)}(0)
\]
hence $f^{(m)}(0) = 2\varepsilon^m k^m m! \ (m > 0)$. Recall that for functions $f_1, \ldots, f_l$ we have the generalized Leibniz rule

$$(f_1 \cdots f_l)^{(m)} = \sum_{|j|=m, j \in \mathbb{N}^l} \binom{m}{j} \prod_{1 \leq t \leq m} f_t^{(j_t)}$$

where $j = (j_1, \ldots, j_l), |j| = \sum_{t=1}^l j_t$ and $\binom{m}{j} = \frac{m!}{j_1! \cdots j_l!}$. Applying this to $g$ we obtain

$$p_k^m(a) = \frac{1}{m!} \sum_{|j|=m, j \in \mathbb{N}^{|a|}} \binom{m}{j} \prod_{1 \leq t \leq |a|} f_t^{(j_t)}(0). \tag{11}$$

Define a map $\nu: \mathbb{N}^{|a|} \to \mathbb{N}$ such that $\nu(j)$ is the cardinality of non–zero entries of $j$. Now, Equation (11) can be rewritten as

$$p_k^m(a) = \sum_{|j|=m, j \in \mathbb{N}^{|a|}} 2^{\nu(j)} \varepsilon^{|j|} = \varepsilon^m k^m \sum_{|j|=m, j \in \mathbb{N}^{|a|}} 2^{\nu(j)}.$$ 

From now on we assume without loss of generality that $a$ is positive. We will establish a recursive formula for $p_k^m(a)$. Let $E_a^t = \{ j \in \mathbb{N}^a : |j|=m \}$, then we can express $E_a^{m+1}$ as the disjoint union of the following three sets

$$A_{m+1}^{m-1} = \left\{ j \in \mathbb{N}^a : (j_1, \ldots, j_{m-1}, 0), \sum_{t=1}^{m-1} j_t = m+1 \right\}$$

$$\bigcup_{l=1}^{m} B_{m+1-l}^{m-1} = \bigcup_{l=1}^{m} \left\{ j \in \mathbb{N}^a : (j_1, \ldots, j_{m-1}, m+1-l), \sum_{t=1}^{m-1} j_t = l \right\}$$

$$C_{m+1}^1 = \{(0, \ldots, 0, m+1)\}$$

hence

$$E_a^{m+1} = A_{m+1}^{m-1} \cup C_{m+1}^1 \cup \bigcup_{l=1}^{m} B_{m+1-l}^{m-1}.$$ 

With this decomposition we obtain

$$p_k^m(a) = k^{m+1} \sum_{|j|=m+1, j \in \mathbb{N}^a} 2^{\nu(j)} = k^{m+1} \sum_{j \in E_a^{m+1}} 2^{\nu(j)}$$

$$= k^{m+1} \left( \sum_{j \in A_{m+1}^{m-1}} 2^{\nu(j)} + \sum_{j \in C_{m+1}^1} 2^{\nu(j)} + \sum_{l=1}^{m} \sum_{j \in B_{m+1-l}^{m-1}} 2^{\nu(j)+1} \right)$$

$$= p_k^m(a-1) + 2k^{m+1} + \sum_{l=1}^{m} 2k^{m+1-l} p_l^k(a-1).$$

Repeating this identity $(a-1)$–times we end up with

$$p_k^m(a) = p_{m+1}(1) + 2k^{m+1}(a-1) + 2 \sum_{l=1}^{a-1} \sum_{s=1}^{m-l} k^{m+1-l} p_l^s(s)$$

$$= 2 \left( ak^{m+1} + \sum_{l=1}^{a-1} \sum_{s=1}^{m-l} k^{m+1-l} p_l^s(s) \right)$$

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since $p_{m+1}(1) = 2k^{m+1}$. This shows that $p_{m+1}^k(a)$ is even for all $k, m, a \in \mathbb{N}$. Moreover it follows that
\[
\frac{1}{2} \cdot p_{m+1}^k(a) \equiv ka \mod 2
\]
since $p(k)$ are even for all $1 \leq l \leq m$, $1 \leq s \leq a - 1$ and $k \in \mathbb{N}$. □

**Proposition 4.2.** Let $m$ be an even number. Consider, in the cohomology ring of $m\#\mathbb{C}P^{2n}$ the coefficient $c_{2n}$ of $x_{1}^{2n} = \cdots = x_{m}^{2n}$ of the total class $c$, which for odd $n$ is of the form
\[
c = (1 - (x_1 + \cdots + x_m))^{2n+1} \prod_{r=1}^{m} \prod_{k=1}^{n} \left( \frac{1 + kx_r}{1 - kx_r} \right)^{a_r^k}
\]
and for even $n$ of the form
\[
c = (1 - (x_1 + \cdots + x_m))^{2n+1} \cdot \left( \frac{1 + nx_1}{1 - nx_1} \right)^{a_1^n} \prod_{j=2}^{m} (1 + (2n - 2)! (x_{2j-1}^{2n-1} - x_j^{2n-1})) \prod_{j=1}^{n} \prod_{k=1}^{m} \left( \frac{1 + kx_j}{1 - kx_j} \right)^{a_j^k}
\]
Then $c_{2n}$ is even and
\[
\frac{1}{2} \cdot c_{2n} \equiv \frac{m}{2} \mod 2.
\]

On the other hand, the Euler characteristic of $m\#\mathbb{C}P^{2n}$ is $\chi(m\#\mathbb{C}P^{2n}) = m(2n - 1) + 2$, hence
\[
\frac{1}{2} \chi(m\#\mathbb{C}P^{2n}) = \frac{1}{2} m(2n - 1) + 1 \equiv \frac{m}{2} + 1 \mod 2.
\]

**Proof.** We consider the case $n$ odd first. As in the proof of Proposition 3.1 we compute
\[
(1 - (x_1 + \cdots + x_m))^{2n+1} = \sum_{r=1}^{m} \sum_{j=0}^{2n+1} (-1)^j \binom{2n+1}{j} x_r^j = \sum_{r=1}^{m} (1 - x_r)^{2n+1}.
\]
This implies
\[
c_{2n} = \sum_{r=1}^{m} \sum_{|j|=2n,j \in \mathbb{N}^{n+1}} (-1)^{j_0} \binom{2n+1}{j_0} \prod_{i=1}^{n} p_{j_i}^i(a_i^j)
\]
where $j = (j_0, j_1, \ldots, j_n) \in \mathbb{N}^{n+1}$. Because the summand for $j = (2n, 0, \ldots, 0)$ is $2n + 1$,
\[
c_{2n} = m(2n + 1) + \sum_{r=1}^{m} \sum_{|j|=2n,j \in \mathbb{N}^{n+1},j_0<2n} (-1)^{j_0} \binom{2n+1}{j_0} \prod_{i=1}^{n} p_{j_i}^i(a_i^j).
\]
By Proposition 1.1 we have that $p_{j_i}^i(a_i^j)$ is even as long as $j_i > 0$, so the products $\prod_{i=1}^{n} p_{j_i}^i(a_i^j)$ are even, and because $m$ is even as well, so is $c_{2n}$. We divide the expression by 2, and use that
\( \frac{1}{2} p_j^i (a_i^j) \equiv i \cdot a_i^j \mod 2 \) by Proposition 4.1

\[
\frac{1}{2} c_{2n} \equiv \frac{m}{2} + \sum_{r=1}^{m} \sum_{l=1}^{n} \sum_{j_0+j_r=2n, j_r \neq 0}^{n} \left( \frac{2n+1}{j_0} \right) \frac{1}{2} p_j^i (a_i^j) \mod 2 \\
\equiv \frac{m}{2} + \sum_{r=1}^{m} \sum_{l=1}^{n} \sum_{j_0+j_r=2n, j_r \neq 0}^{n} \left( \frac{2n+1}{j_0} \right) la_r^l \mod 2 \\
\equiv \frac{m}{2} + \sum_{r=1}^{m} \sum_{l=1}^{n} la_r^l \sum_{j_0=0}^{2n-1} \left( \frac{2n+1}{j_0} \right) \mod 2 \\
\equiv \frac{m}{2} + \sum_{r=1}^{m} \sum_{l=1}^{n} la_r^l \left( 2^{2n+1} - 2(n+1) \right) \mod 2 \\
\equiv \frac{m}{2} \mod 2.
\]

This shows the claim for odd \( n \). For even \( n \), the structure of \( c_{2n} \) is similar. The main difference is the appearance of the factor

\[
\prod_{j=2}^{m} (1 + (2n-2)! (x_1^{2n-1} - x_j^{2n-1}))^{b_j} = \prod_{j=2}^{m} (1 + b_j (2n-2)! (x_1^{2n-1} - x_j^{2n-1})) \\
= 1 + (2n-2)! \sum_{j=2}^{m} b_j (x_1^{2n-1} - x_j^{2n-1}).
\]

Note that the prefactor \( (2n-2)! \) is, except for \( n = 2 \), divisible by 4, which directly implies that this factor has no effect on \( \frac{1}{2} \cdot c_{2n} \mod 2 \) for \( n \geq 4 \). But this is true for arbitrary \( n \): by Proposition 4.1, the degree one coefficients \( p_j^i (a) \) of the factors of the form \( \left( \frac{1+b_k}{1+k} \right)^a \) are even, so multiplied with \( (2n-2)! \) they always result in a number divisible by 4. Further, we have

\[
(1 - (x_1 + \cdots + x_m))^{2n+1} = 1 - (2n+1)(x_1 + \cdots + x_m) + \cdots,
\]

and

\[
(x_1 + \cdots + x_m) (x_1^{2n-1} - x_j^{2n-1}) = 0,
\]

so no further summands result from multiplication with the factor \( (1 - (x_1 + \cdots + x_m))^{2n+1} \).

In total, the result \( \frac{1}{2} \cdot c_{2n} \equiv \frac{m}{2} \mod 2 \) remains true for even \( n \).

The above proposition together with Theorem 1.2 shows that \( m\# \text{CP}^{2n} \) admits no almost complex structure for \( m \) even.

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