Abstract

Quantum entanglement entropy has a geometric character. This is illustrated by the interpretation of Rindler space or black hole entropy as entanglement entropy. In general, one can define a “geometric entropy”, associated with an event horizon as a boundary that concentrates a large number of quantum states. This allows one to connect with the “density matrix renormalization group” and to unveil its connection with the theory of quantum information. This renormalization group has been introduced in condensed matter physics in a heuristic manner, but it can be conceived as a method of compression of quantum information in the presence of a horizon. We propose generalizations to problems of interest in cosmology.

1 Entanglement entropy in some relevant geometries

Entanglement or nonseparability refers to the existence of quantum correlations between two sets of degrees of freedom of a physical system that can be considered as subsystems. It is natural that two (sub)systems in interaction are entangled and that they still are entangled after their interaction has ceased. Particularly interesting situations arise when two entangled systems become causally disconnected because of an event horizon.

1.1 Introduction: entanglement entropy

The entanglement of two parts of a quantum system can be measured by the von Neumann entropy. This is defined in terms of the density matrices of either part. Let us consider, for later convenience, one part as “left” or “interior” and another as “right” or “exterior”, in a yet imprecise sense. Then, let us represent states belonging to the left (or interior) with small letters and states belonging to the right (or exterior) with capital letters. A basis for the global states (left plus right) is \( \{ |a\rangle \} \otimes \{ |A\rangle \} \). By representing the ground state in this basis as

\[
|0\rangle = \sum_{aA} \psi_{aA} |a\rangle \otimes |A\rangle,
\]
we define a coefficient matrix $\psi_{aA}$. Then we have two different density matrices:

$$
\rho_R = \frac{\psi^\dagger \psi}{\text{Tr} \, \psi^\dagger \psi}, \quad \rho_L = \frac{\psi^* \psi^T}{\text{Tr} \, \psi^* \psi^T}.
$$

(2)

Correspondingly, we have two von Neumann entropies:

1. $S_R = -\text{Tr} \, (\rho_R \ln \rho_R)$
2. $S_L = -\text{Tr} \, (\rho_L \ln \rho_L)$

Now it is important to recall the “symmetry theorem”, which states that both entropies are equal, $S_R = S_L$ (this can be proved in several ways, the most popular one appealing to the Schmidt decomposition of the entangled state. The equality of entropies implies that they are associated with properties shared by both parts, that is, with (quantum) correlations.

More generally, two non-interacting parts of a quantum system can be originally in respective mixed states. After their interaction, which we describe as an arbitrary unitary evolution of the composite system, the initial density matrix $\rho_L \otimes \rho_R$ has evolved to $\rho'_{LR}$. It is easy to see that the partial traces $\rho'_L$ and $\rho'_R$, in general, have von Neumann entropies $S'_L$ and $S'_R$ such that $S'_L + S'_R \geq S_L + S_R$ [1]. Of course, if the initial state is pure $S_L = S_R = 0$.

### 1.2 Field theory half-space density matrix

Let us now consider the quantum system to be a chain of coupled oscillators. Moreover, we shall chiefly work in the continuum limit, where the concepts and mathematical expressions are more transparent, in spite of dealing with non-denumerable sets of degrees of freedom (we will return to a discrete chain to describe the density matrix renormalization group algorithm). The action for this model, namely, a one-dimensional scalar field, is

$$
A[\varphi(x,t)] = \int dt \, dx \left( \frac{1}{2} \left[ (\partial_t \varphi)^2 - (\partial_x \varphi)^2 \right] - V(\varphi) \right),
$$

(3)

where $\varphi$ is the field.

Let us obtain a path integral representation for the density matrix on the half-line of a system that is in its ground state [2, 3, 4]. In the continuum limit, the half-line density matrix is a functional integral,

$$
\rho[\varphi_R(x), \varphi'_R(x)] = \int D\varphi_L(x) \, \psi_0[\varphi_L(x), \varphi_R(x)] \psi_0^*[\varphi_L(x), \varphi'_R(x)],
$$

(4)

where the subscripts refer to the left or right position of the coordinates with respect to the boundary (the origin). Now, we must express the ground-state wave-function as a path integral,

$$
\psi_0[\varphi_L(x), \varphi_R(x)] = \int D\varphi(x, t) \, \exp (-A[\varphi(x, t)]),
$$

(5)

where $t \in (-\infty, 0]$ and with boundary conditions $\varphi(x, 0) = \varphi_L(x)$ if $x < 0$, and $\varphi(x, 0) = \varphi_R(x)$ if $x > 0$. The conjugate wave function is given by the same path integral and
boundary conditions but with \( t \in [0, \infty) \). Substituting into Eq. (4) and performing the integral over \( \varphi_L(x) \), one can express \( \rho(\varphi_R, \varphi'_R) \) as a path integral over \( \varphi(x, t) \), with \( t \in (-\infty, \infty) \), and boundary conditions \( \varphi_R(x, 0+) = \varphi'_R(x) \), \( \varphi_R(x, 0-) = \varphi_R(x) \). In other words, \( \rho(\varphi_R, \varphi'_R) \) is represented by a single path integral covering the entire plane with a cut along the positive semiaxis, where the boundary conditions are imposed.

### 1.3 Angular quantization and Rindler space

In Euclidean two-dimensional field theory, the generator of rotations in the \((x, t)\) plane is given by

\[
\mathcal{L} = \int dx \left( tT_{11} - xT_{00} \right),
\]

in terms of the components of the stress tensor computed from the action (3). To simplify, one can evaluate it at \( t = 0 \). Let us consider a free action \((V = 0)\). In the Schrödinger representation, we should replace the momentum \( \Pi = \partial_t \varphi \) with \( \Pi(x) = i \delta/\delta \varphi(x) \). However, as in canonical quantization, one rather uses the second-quantization method, which diagonalizes the Hamiltonian by solving the classical equations of motion and quantizing the corresponding normal modes. Let us recall that, in canonical quantization, if we disregard anharmonic terms, the classical equations of motion in the continuum limit become the Klein-Gordon field equation, giving rise to the usual Fock space. Not surprisingly, the eigenvalue equation for \( \mathcal{L} \) leads to the Klein-Gordon equation in polar coordinates in the \((x, t)\) plane.

The free field wave equation in polar coordinates,

\[
(\Delta + m^2)\varphi = \left( \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + m^2 \right) \varphi = 0,
\]

can be solved by separating the angular variable: it becomes a Bessel differential equation in the \( r \) coordinate with complex solutions \( I_{\pm i \ell}(mr) \), \( \ell \) being the angular frequency. We have a continuous spectrum, which becomes discrete on introducing boundary conditions.
One of them must be set at a short distance from the origin, to act as an ultraviolet regulator \[2, 3, 4\], necessary in the continuum limit.

Therefore, the second-quantized field is (on the positive semiaxis \(t = 0 \Leftrightarrow \phi = 0, x = r\))

\[
\varphi(x) = \int \frac{d\ell}{2\pi} \frac{b_\ell I_{i\ell}(m x) + b^{\dagger}_\ell I_{-i\ell}(m x)}{\sqrt{2 \sinh(\pi \ell)}},
\]

where we have introduced annihilation and creation operators and where the term that appears in the denominator is just for normalization, to ensure that those operators satisfy canonical commutations relations. There is an associated Fock space built by acting with \(b^{\dagger}_\ell\) on the “vacuum state”. These states constitute the spectrum of eigenstates of \(\mathcal{L}\), which adopts the form \(\mathcal{L} = \int d\ell \ell b^{\dagger}_\ell b_\ell\) (where the integral is replaced with a sum for discrete \(\ell\)).

The type of quantization just exposed was first introduced in the context of quantization in curved space, in particular, in Rindler space. Rindler space is just Minkowski space (therefore, not curved) in coordinates such that the time is the proper time of a set of accelerated observers. Its interesting feature is the appearance of an event horizon, which implies that the ground state (the Minkowski vacuum) is a mixed (thermal) state \[5\]. The connection with black hole entropy and Hawking radiation is explained in the next section.

It is pertinent to note that the functions \(I_{\pm i\ell}(m x)\) have wave-lengths that increase with \(x\). It is illustrative to represent a real “angular wave”,

\[
K_{i\ell}(m x) = \frac{i \pi}{2 \sinh(\pi \ell)} [I_{i\ell}(m x) - I_{-i\ell}(m x)].
\]

This solution is oscillatory for \(x < \ell/m\), with a wavelength proportional to \(x\), and decays exponentially for \(x > \ell/m\) (Fig. 1). The fact that the wavelength vanishes at \(x = 0\) is to be expected from the Rindler space viewpoint, because it corresponds to the horizon.

### 1.4 Black hole entropy

Let us consider the Schwarzschild geometry in the Kruskal-Szekeres coordinates \(u, v\), defined by

\[
u = 16M^2 \left(\frac{r}{2M} - 1\right) \exp \left(\frac{r}{2M} - 1\right), \quad \frac{u}{v} = \exp \frac{t}{2M}.
\]

The horizon is given by \(u = 0\) or \(v = 0\), like in the Rindler geometry of the previous section. Then, if we further define

\[
Z = u + v, \quad T = u - v,
\]

these coordinates behave like the ordinary Minkowskian coordinates. Moreover, for small \(u\) or \(v\) (or the large \(M\) limit), the curvature can be neglected and the geometry becomes locally the one of Rindler space. Therefore, we could define radial coordinates \(Z = \rho \cosh \tau\), \(T = \rho \sinh \tau\), and perform a radial quantization like we did in the preceding section.

Once established that the geometry near the black-hole horizon is locally the Rindler geometry of the preceding section, we can readily transfer the form of the density matrix
of a scalar field therein, where we now ignore (trace over) the degrees of freedom inside the horizon. Hence, we can define a von Neumann entropy associated with this density matrix. Furthermore, in so doing, we can appreciate that the concept of black-hole entropy takes a new meaning: in addition to being of quantum origin, this entropy is related with shared properties between interior and exterior, namely, with the horizon. In addition, the radial vacuum is a thermal state with respect to the original Schwarzshild coordinates, giving rise to Hawking radiation [5].

2 Quantum information and RG transformations

2.1 Information theory and maximum entropy principle

The entropy concept appeared in Thermodynamics but only took a truly fundamental meaning with the advent of information theory. In this theory, entropy is just missing information, while information itself is often called negentropy. To recall basic definitions, the information attached to an event that occurs with probability \( p_n \) is \( I_n = -\ln p_n \). Hence, the average information (per event) of a source of events is \( S(\{p_n\}) = \sum_n p_n I_n = -\sum_n p_n \ln p_n \).

The previous definitions, given by Shannon in his theory of communication, seem unrelated to thermodynamic entropy as a property of a physical system. However, according to the foundations of Statistical Mechanics on Probability Theory (the Gibbs concept of ensembles), a clear relation can be established, as done by Jaynes [6]. Jaynes made connection with the Bayesian philosophy of probability theory, in which the concept of “a priori” knowledge is crucial. Indeed, although the exact microscopic state of a system with many degrees of freedom may be unknown, one has some “a priori” knowledge given by the known macroscopic variables. This is a particular case of Jaynes’ adaptation of the
Bayesian probability theory, which postulates that the best probability distribution to be attributed to a stochastic event is such that it incorporates only the “a priori” knowledge about the event and nothing else. This postulate amounts to Jaynes’ maximum entropy principle: given some constraints, one must find the maximum entropy probability distribution (density matrix, in the quantum case) compatible with those constraints, usually, by implementing them via Lagrange multipliers. In particular, more constraints mean less missing information and lead to less entropy.

2.2 Quantum information

The concepts of Shannon’s classical theory of communication have quantum analogues [1, 7]. Nevertheless, the quantum theory of communication is richer (and less intuitive!). Indeed, the key new notion in the quantum theory is entanglement (already described in the foregoing): if a state (an event) is entangled with the environment, we have the type of purely quantum phenomena to which the EPR paradox is associated.

Schumacher posed the problem of communication of an entangled state [7]. (The technical name is transposition, since the copy of a quantum state is not possible: no-cloning theorem [1].) His conclusion is that the von Neumann entropy of the state is the quantity that determines the fidelity of the transposition: it is possible to transpose the state with near-perfect fidelity if the signal can carry at least that information.

The fidelity is simply defined as the overlap between normalized states, namely, $|\langle \psi | \psi' \rangle|^2$. It is directly related with the natural distance in the space of rays, that is, the angle between rays. The geometrical meaning of this distance is best perceived by considering the complex projective space of rays, where it is called the Fubini-Study metric [8]. Schumacher maximization of the fidelity uses the Schmidt decomposition of the entangled state [7].

2.3 Renormalization group and information theory

The problem of transforming one quantum state into another while preserving its fundamental features is reminiscent of an operation performed with the quantum renormalization group (in this connection, see Ref. [11]). This operation does not intend to reach near-perfect fidelity but it is desirable that it reach as much fidelity as possible. We shall see that a particular formulation of the renormalization group, namely, the density matrix renormalization group, comes close to applying the concepts of quantum information theory, and in a similar way to Rindler space quantization [13]. In fact, the construction of the density matrix renormalization group algorithm is based on the Schmidt decomposition, in parallel to Schumacher fidelity maximization.

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1 It is very interesting to realize that the distance between the distribution probabilities defined by the outcomes of possible measurements of observables that distinguish the states $\psi$ and $\psi'$ coincides with the statistical distance in the classical sense of Fisher [9, 10].
Let us consider an entangled quantum state $|\psi\rangle$ for which we seek an optimal reduced representation $|\tilde{\psi}\rangle$ (a sort of quantum coding). That is, we must find a projection

$$|\psi\rangle = \sum_{aA} \psi_{aA} |a\rangle \otimes |A\rangle \rightarrow |\tilde{\psi}\rangle = \sum_{iA} \psi_{iA} |i\rangle \otimes |A\rangle$$

(9)

to a subspace spanned by reduced basis $\{i\}$ such that the distance $S = |||\tilde{\psi}\rangle - |\psi\rangle||^2$ is minimized. This is the problem that S. White met in looking for an improved “real space” quantum renormalization group algorithm: In a renormalization group step, one must reduce the number of block states, but in a way that accounts for the influence of the rest of the system (that is, with no introduction of arbitrary boundary conditions) [12]. This amounts to consider the entanglement of the block and to formulate the above described problem. White found the solution in terms of the singular value decomposition, a well-known numerical algorithm, equivalent to the Schmidt decomposition of the entangled state and, hence, leading to discarding the smallest eigenvalues of the block density matrix; in other words, $|i\rangle$ are the eigenstates of $\rho$ with largest eigenvalues. The proper formulation of the density matrix renormalization group in terms of quantum information concepts has been provided recently [14].

### 2.3.1 Density matrix renormalization group algorithm

Method for a 1D quantum system on a chain (e.g., a chain of oscillators):

1. Select a sufficiently small “soluble” block $[0, L]$:

   ![Diagram of a 1D chain with a block selection]

2. Reflect the block on the origin:

   ![Diagram of a reflected block]

3. Compute the ground state.

4. Compute the density matrix of the block $[0, L]$.

5. Discard eigenstates with smallest eigenvalues.

6. Add one site next to the origin.

7. Go to 2.

One has to adjust this procedure in such a way that the iteration keeps the Hilbert space size approximately constant. The procedure can be performed algebraically for a chain of coupled harmonic oscillators [13]. Otherwise, it has to be performed numerically. In the continuum limit, the connection with the scalar field theory half-space density matrix previously calculated is clear. In particular, the property of “angular waves” of having vanishing wave-length at $x = 0$ pointed out above provides an explanation of the form in which the density matrix renormalization group solves the boundary condition problem, that is, because of the concentration of quantum states near the boundary. To be precise, using the eigenfunctions of $\mathcal{L}$ instead of free waves, we have a basis in which the region close to $x = 0$ (the boundary point) is more accurately represented than the region far from it when we cut off the higher $\ell$ eigenfunctions.
3 Generalizations and applications

3.1 Geometric entropy

We have seen that the half line density matrix of a field theory has a geometric interpretation in Rindler space. Furthermore, the entropy of black holes can be understood as a generalization to a more complicated geometry. We may wonder if further generalizations are possible.

In this connection, we recall the notion of “geometric entropy”, introduced by C. Callan and F. Wilczek [4], as the entropy “associated with a pure [global] state and a geometrical region by forming the pure state density matrix, tracing over the field variables inside the region to create an ‘impure’ density matrix”. Of course, their motivation was the earlier suggestion that back-hole entropy is of quantum-mechanical entanglement origin [2]. They proposed a generalization to different topologies but, actually, they only computed the Rindler space case, discussing the divergence of the entropy at the horizon [4] (the UV divergence of this type of entropy had been discussed in general in Ref. [2]).

A different notion of geometric entropy can be deduced by purely geometrical means from the presence of horizons, namely, as associated with a spacetime topology that does not admit a trivial Hamiltonian foliation [15]. This type of topology prevents unitary evolution and leads to mixed states.

In fact, it is only the second type of entropy that embodies the famous “one-quarter area law” for black holes, due to its origin in purely relativistic concepts (this was demonstrated for an earlier relativistic notion of entropy in Ref. [16]). On the contrary, the quantum notion of geometric entropy for a field theory involves UV divergences and needs renormalization before a comparison with the relativistic notion can be made (see the discussion in Ref. [17]).

3.2 Application to cosmology

The generalization of the concept of black-hole entropy to the de Sitter space and, hence, to cosmology is relatively old [18]. Of course, the concept of entropy in its traditional thermodynamical sense has been crucial in explaining the dynamics of inflation (now a standard paradigm), namely, in accounting for the reheating process (entropy generation). However, the relations between the traditional view and the one associated to quantum entanglement may lead to further insight, when they are properly formulated in the context of quantum information theory.

In particular, we can regard the generation of entropy and fluctuations in de Sitter spacetime as a fundamentally quantum process leading to the celebrated Harrison-Zeldovich scale invariant spectrum of Gaussian fluctuations. If we consider the initial state for inflation as a pure quantum state (that some theory of quantum gravity will hopefully characterize some day), the de Sitter space horizon induces decoherence of the modes which cross it, irrespective of the actual inflaton dynamics and, therefore, of the details of the reheating process. This decoherence consists of a randomization of the phases
of the present quantum fields and naturally produces thermal Gaussian fluctuations. Moreover, the symmetry of de Sitter space implies that each mode has the same physical size as it crosses the horizon, leading to the Harrison-Zeldovich power spectrum:

\[ P(k) \equiv \left| \frac{\delta \rho}{\rho_0} \right|_k^2 \propto k. \]

Actually, this spectrum is given by the equipartition theorem at the corresponding Hawking temperature \( T = H/(2\pi) \) \( (H = \Lambda/3) \) [18].

In this reasoning, one could also consider the initial state to be mixed (e.g., a thermal state) and apply the argument for entropy growth exposed in subsection 1.1.

One may wonder if a density-matrix type renormalization group could help with the dynamics. As long as the fluctuations are Gaussian, we are in a similar situation to the case of harmonic oscillators commented above, which makes any renormalization group superfluous. However, as is well known, the gravitational instability produces non-linear evolution and leads to phase correlations. Therefore, the ideas presented here may be useful in setting up a renormalization group for the study of non-Gaussian fluctuations and its non-linear evolution. In particular, the Wilson or exact renormalization group irreversibility properties and can be connected with other methods of analysis of the non-linear evolution (for preliminary steps in that direction, see Ref. [19]).

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