On pure derived and pure singularity categories

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Abstract
Firstly, we compare the bounded derived categories with respect to the pure-exact and the usual exact structures, and describe bounded derived category by pure-projective modules, under a fairly strong assumption on the ring. Then, we study Verdier quotient of bounded pure derived category modulo the bounded homotopy category of pure-projective modules, which is called a pure singularity category since we show that it reflects the finiteness of pure-global dimension of rings. Moreover, invariance of pure singularity in a recollement of bounded pure derived categories is studied.

Key Words: derived category, pure derived category, pure singularity category.
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1. Introduction
Recall that a short exact sequence $E = 0 \to M_1 \to M_2 \to M_3 \to 0$ of left $R$-modules is pure-exact provided that for any right $R$-module $N$ the induced sequence $N \otimes_R E$ is exact. Purity is the basis of a relative homological theory, which leads to concepts such as pure-projective modules, pure-projective resolutions, pure derived functors $\text{Pext}(\cdot, \cdot)$, pure-projective dimension etc., see for example [10, 11, 12, 14, 18, 22, 24]. It is also worth noting that a proof of the flat cover conjecture may be obtained by studying filtrations of a module by pure submodules [4].

It is well known that in the sense of Neeman [16], $D(R)$ is the derived category of the exact category $(R\text{-Mod}, \mathcal{E})$, where $\mathcal{E}$ is the collection of all short exact sequences of $R$-modules. Let $\mathcal{E}_{\text{pur}}$ be the collection of all short pure-exact sequences, then $(R\text{-Mod}, \mathcal{E}_{\text{pur}})$ is an exact category. Recently, Zheng and Huang introduced in [25] the pure derived category $D_{\text{pur}}(R)$ as the derived category of the exact category $(R\text{-Mod}, \mathcal{E}_{\text{pur}})$.

In this paper, we firstly intend to compare the bounded derived categories of these two exact structures, and to describe bounded derived category by pure-projective modules. Our first main result (Theorem 3.1) shows that: If the subcategory $\mathcal{P}\mathcal{P}$ of pure-projective $R$-modules is closed under kernels of epimorphisms, then there are triangle-equivalences:

$$D^b(R) \simeq D^b_{\text{pur}}(R)/K^b_{\text{ac}}(\mathcal{P}\mathcal{P}) \simeq K^{-,\text{ppb}}(\mathcal{P}\mathcal{P})/K^b_{\text{ac}}(\mathcal{P}\mathcal{P}),$$
where $K^{-,ppb}(\mathcal{P}\mathcal{P})$ is the homotopy category of upper bounded complexes of pure-projective modules which is bounded in the sense of pure cohomology, and $K^0_{ac}(\mathcal{P}\mathcal{P})$ is the homotopy category of bounded and exact (=acyclic) complexes of pure-projective modules. Some conditions for the assumption that $\mathcal{P}\mathcal{P}$ is closed under kernels of epimorphisms are given in Remark 3.2.

Recall that for a noetherian ring $R$, the category $D_{sg}(R)$ is defined to be a Verdier quotient of bounded derived category of finitely generated modules $D^b(R\text{-mod})$ modulo the bounded homotopy category of finitely generated projective modules $K^b(R\text{-proj})$; it was also studied by Buchweitz [7] under the name of “stable derived category”. Since $D_{sg}(R) = 0$ if and only if $R$ has finite global dimension (i.e. $D_{sg}(R)$ reflects homological singularity of the ring $R$), this quotient triangulated category is called the singularity category of $R$ after Orlov [17]. For any ring $R$, Beligiannis [3] studied the singularity category $D^b(R)/K^b(\mathcal{P})$, where $D^b(R)$ is the bounded derived category of any $R$-modules, and $K^b(\mathcal{P})$ is the bounded homotopy category of any projective $R$-modules. See also the literature [26].

We are inspired to study the pure version of singularity category. The pure singularity category $D_{psg}(R)$ is defined (Definition 4.1) as Verdier quotient $D^b_{par}(R)/K^b(\mathcal{P}\mathcal{P})$. We remark that the notion of pure singularity category seems to be reasonable since $D_{psg}(R) = 0$ if and only if the pure-global dimension of $R$ is finite, see Proposition 4.4. Moreover, we show that for any rings $A$, $B$ and $C$, if $D^b_{par}(A)$ admits a recollement $D^b_{par}(B) \rightleftharpoons D^b_{par}(A) \rightleftharpoons D^b_{par}(C)$ relative to $D^b_{par}(B)$ and $D^b_{par}(C)$, then $D_{psg}(A) = 0$ if and only if $D_{psg}(B) = 0 = D_{psg}(C)$; see Theorem 4.5.

2. Preliminary

Throughout this paper, $R$ denotes a ring with unity, and $R\text{-Mod}$ the category of left $R$-modules. A short exact sequence $E = 0 \to M_1 \overset{f}{\to} M_2 \overset{g}{\to} M_3 \to 0$ in $R\text{-Mod}$ is called pure-exact if for any right $R$-module $N$, the induced sequence $0 \to M_1 \otimes_R N \to M_2 \otimes_R N \to M_3 \otimes_R N \to 0$ remains exact. In this case, $f$ is called pure monic and $g$ is called pure epic. By Cohn’s theorem (see [21] Theorem 3.65)), the above sequence is pure-exact if and only if $0 \to \text{Hom}_R(F, M_1) \to \text{Hom}_R(F, M_2) \to \text{Hom}_R(F, M_3) \to 0$ is exact for any finitely presented $R$-module $F$. Moreover, it turns out that the above exact sequence is pure-exact if and only if the associated short exact sequence of Pontryagin duals $0 \to \text{Hom}_\mathbb{Z}(M_3, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}_\mathbb{Z}(M_2, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}_\mathbb{Z}(M_1, \mathbb{Q}/\mathbb{Z}) \to 0$ is split (see [12] Theorem 6.4). It is worth noting that the sequence $E$ is pure-exact if and only if every finite system of linear equations $y_i = \sum_{j \in I} r_{ij} x_j$ with $y_i \in M_1$, $r_{ij} \in R$, $i \in I$, which has a solution $(x_j) \in M^I_2$ also has a solution in $M^I_1$, where $I$ and $J$ are finite index sets.

Recall that an exact structure on an abelian category $\mathcal{A}$ is a class $\mathcal{E}$ of kernel-cokernel pairs $A \overset{f}{\leftarrow} B \overset{g}{\to} C$ which satisfies some axioms, see for example [8]. In the kernel-cokernel pair $(f, g)$, $f$ is called an admissible monic and $g$ is called an admissible epic. Let $\mathcal{E}_{par}$ denote the class of all short pure-exact sequences, then $(R\text{-Mod}, \mathcal{E}_{par})$ is an exact category.

Let $X = \cdots \to X^{i-1} \overset{d_{i-1}}{\to} X^i \overset{d_i}{\to} X^{i+1} \to \cdots$ be an exact complex of modules. Then $X$ is said to be pure-exact at degree $n$ if the short exact sequence $0 \to K^n \to X^n \to K^{n+1} \to 0$ is
pure-exact, where $K^n = \text{Ker}d^n$. $X$ is called pure-exact (=pure-acyclic) if it is pure-exact at all degree $n$.

An $R$-module $P$ is called pure-projective if $\text{Hom}_R(P, -)$ is exact on all pure-exact sequences. Thus projective modules and finitely presented modules are pure-projective. Warfield [24] showed that a module $P$ is pure-projective if and only if it is a direct summand of a direct sum of finitely presented modules. We use $\mathcal{P}$ and $\mathcal{PP}$ to denote the class of all projective modules and pure-projective modules, respectively.

Let $\mathcal{C}$ be a full subcategory of an additive category $\mathcal{A}$, and $X \in \mathcal{A}$. Recall that a morphism $f : C \to X$ with $C \in \mathcal{C}$ is a $\mathcal{C}$-precover (or, right $\mathcal{C}$-approximation) of $X$, if $\text{Hom}_\mathcal{A}(C', C) \to \text{Hom}_\mathcal{A}(C', X)$ is surjective for each $C' \in \mathcal{C}$. If each object $X \in \mathcal{A}$ admits a $\mathcal{C}$-precover, then $\mathcal{C}$ is said to be precovering (or, contravariantly finite) in $\mathcal{A}$. Any module $M$ has a pure-projective precover, for example, if $M$ is expressed as the colimit $\lim_{\to} M_i$ of a directed system of finitely presented modules, then the canonical map $\bigoplus_i M_i \to M$ is a pure epimorphism (a pure-projective precover) (see [19, §II.1.1.3]); see also [24] for the existence of pure-projective precovers. Thus, $\mathcal{PP}$ is precovering, and any module $M$ has a pure-projective resolution, i.e. a pure-exact complex $\cdots \to P_1 \to P_0 \to M \to 0$ with each $P_i$ pure-projective.

We always identify an $R$-module $M$ with the complex concentrated in degree 0. For a complex $X$ and an integer $n$, $X[n]$ denotes the complex $X$ shifting $n$ degree to the left, that is, $X[n]^m = X^{n+m}$ and $d^m_X[n] = (-1)^n d^m_X$. Given two $R$-complexes $X$ and $Y$, the complex $\text{Hom}_R(X, Y)$ is defined with $\text{Hom}_R(X, Y)^n = \prod_{k \in \mathbb{Z}} \text{Hom}_R(X^k, Y^{k+n})$, and with differential $d^n(f^k) = (d^{k+n}_Y f^k - (-1)^n f^{k+1} d^k_X)_{k \in \mathbb{Z}}$ for $f = (f^k) \in \text{Hom}_R(X, Y)^n$.

A cochain map $f : X \to Y$ of complexes is a family of morphisms $f = (f^n : X^n \to Y^n)_{n \in \mathbb{Z}}$ of $R$-modules satisfying $d^k_Y f^n = f^{n+1} d^k_X$ for all $n \in \mathbb{Z}$. Cochain maps $f, g : X \to Y$ are called homotopic, denoted $f \sim g$, if there exists a family of morphisms $(s^n : X^n \to Y^{n-1})_{n \in \mathbb{Z}}$ of $R$-modules, satisfying $f^n - g^n = d^{n+1}_X s^n + s^{n+1}_Y d^n_X$ for all $n \in \mathbb{Z}$. A map $f : X \to Y$ of complexes is called a quasi-isomorphism if it induces isomorphic homology groups, and $f$ is called a homotopy equivalence if there exists a cochain map $g : Y \to X$ such that $gf \sim \text{Id}_X$ and $fg \sim \text{Id}_Y$. For $\text{Con}(f)$ we mean the mapping cone of $f$, which is defined with $\text{Con}(f)^n = X^{n+1} \oplus Y^n$ and with differential $d^n_{\text{Con}(f)} = \begin{pmatrix} -d^{n+1}_X & 0 \\ f^{n+1} & d^n_Y \end{pmatrix}$ for all $n \in \mathbb{Z}$. Note that $f : X \to Y$ is a quasi-isomorphism if and only if $\text{Con}(f)$ is an exact complex.

Let $X$ be an $R$-complex. It follows immediately that $X$ is pure-exact if $\text{Hom}_R(P, X)$ is exact for each pure-projective module $P$. A cochain map $f : X \to Y$ of $R$-complexes is a pure-quasi-isomorphism provided that $\text{Hom}_R(P, f)$ is a quasi-isomorphism for all $P \in \mathcal{PP}$, or equivalently, $\text{Con}(f)$ is a pure-exact complex. It is easy to see that every pure-exact complex is exact, and every pure-quasi-isomorphism is a quasi-isomorphism.

It is well known that the derived category is a Verdier quotient of the homotopy category with respect to the thick triangulated subcategory of exact complexes. In general, given a triangulated
subcategory $\mathcal{B}$ of a triangulated category $\mathcal{K}$, in the Verdier quotient $\mathcal{K}/\mathcal{B} = S^{-1}\mathcal{K}$, where $S$ is the compatible multiplicative system determined by $\mathcal{B}$, each morphism $f : X \to Y$ is given by an equivalent class of right fractions $a/s$ presented by $X \xleftarrow{s} Z \xrightarrow{a} Y$.

We use “$\cong$” to denote the isomorphisms of objects, and “$\simeq$” to denote the equivalences of categories. As usual, $K(R)$ is the homotopy category of $R$-complexes, and $D(R)$ is the derived category of $R$; and furthermore, we use the superscript $* \in \{-, +, b\}$ to denote the corresponding subcategories with conditions of bounded above, bounded below and bounded, respectively. We denote by $K^*_ac(R)$ and $K^*_pac(R)$ the homotopy category consisting of exact (acyclic) complexes and pure-exact (pure-acyclic) complexes, respectively. Clearly, $K^*_ac(R)$ is a thick triangulated subcategory of $K^*(R)$. Since pure-acyclic complexes are closed under summands, it follows from Rickard’s criterion (see [20] Proposition 1.3 or [16] Criterion 1.3)] that $K^*_pac(R)$ is a thick triangulated subcategory of $K^*_ac(R)$, and hence of $K^*(R)$. We remark that the terminology ”thick” in [20] is “épaisse” in French. For $* \in \{\text{blank}, -, b\}$, let $K^*(\mathcal{P})$ and $K^*(\mathcal{PP})$ be respectively the homotopy category of complexes of projective modules and pure-projective modules.

3. Pure derived categories and derived categories

Zheng and Huang [25] defined the pure derived category $D^b_{\text{par}}(R) := K^*(R)/K^*_pac(R)$ as a Verdier quotient of $K^*(R)$ modulo the thick subcategory $K^*_pac(R)$, where $* \in \{\text{blank}, -, b\}$. In fact, $D^b_{\text{par}}(R)$ is the derived category of the exact categories $(R\text{-Mod}, \mathcal{E}_{\text{par}})$ in the sense of Neeman [16], where $\mathcal{E}_{\text{par}}$ is the collection of all short pure-exact sequences of $R$-modules.

In this section, we intend to compare $D^b(R)$ and $D^b_{\text{par}}(R)$, and to describe $D^b(R)$ by pure-projective modules. The main result of this section is stated below, where

$$K^{-,\text{ppb}}(\mathcal{PP}) := \left\{ X \in K^-(\mathcal{PP}) \mid \text{there exists } n = n(X) \in \mathbb{Z}, \text{ such that } H^i\text{Hom}_R(P, X) = 0, \forall i \leq n, \forall P \in \mathcal{PP} \right\},$$

and $K^b_{\text{ac}}(\mathcal{PP})$ is the homotopy category of bounded exact complexes of pure-projective modules.

**Theorem 3.1.** Let $R$ be a ring. If the subcategory $\mathcal{PP}$ of pure-projective $R$-modules is closed under kernels of epimorphisms, then there are triangle-equivalences

$$D^b(R) \simeq D^b_{\text{par}}(R)/K^b_{\text{ac}}(\mathcal{PP}) \simeq K^{-,\text{ppb}}(\mathcal{PP})/K^b_{\text{ac}}(\mathcal{PP}).$$

Next, some examples of rings such that the subcategory $\mathcal{PP}$ of pure-projective $R$-modules is closed under kernels of epimorphisms will be given. However, due to the following counterexamples suggested by M. Hrbek, the assumption is fairly strong. For example, if the ring $R$ is not coherent, then for any finitely presented module $F$, its finitely generated, by not finitely presented, submodule $G$ will give a counterexample, as the epimorphism $F \to F/G$ has the kernel $G$ which is not pure-projective. It follows from [5] that for an artin algebra $A$, maximal submodules of pure-projective modules are pure-projective if and only if $A$ is of finite representation.
type, and then there are counterexamples of the above assumption for artin algebras of non-finite representation type. We look forward to weakening this assumption.

**Remark 3.2.** (1) It follows immediately from [23, Corollary 4.3] that for \( * \in \{ \text{blank}, -, +, b \} \), there is an equivalence of triangulated categories \( D^*(R) \cong D^*_\text{pur}(R)/(K^*_{ac}(R)/K^*_{pac}(R)) \). Moreover, it is easy to see that \( D^*(R) \cong D^*_\text{pur}(R) \) if and only if \( K^*_{ac}(R) \cong K^*_{pac}(R) \), if and only if \( \mathcal{PP} = \mathcal{P} \). It is clear that when \( R \) is von Neumann regular, these conditions hold.

(2) Let \( R \) be a commutative ring. It follows from [11, Theorem 4.3] that every \( R \)-module is pure-projective if and only if \( R \) is an artinian principal ideal ring, if and only if \( R \) is a generalized uniserial quasi-Frobenius ring. In this case \( \mathcal{PP} \) is obviously closed under kernels of epimorphisms. The rings with the above properties are also called pure-semisimple, and are characterized in [12, Theorem 8.4].

(3) Recall that Kulikov proved in 1945 that subgroups of finitely generated abelian groups are again direct sums of finitely generated groups. It follows from a version of Kulikov’s theorem [6, Theorem 2.1] that there are some rings with Kulikov property (the heredity of pure-projectivity), i.e. any submodule of pure-projective module is pure-projective. For example, consider the oriented cycle \( \Gamma_n : 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n \) of length \( n \geq 1 \), and then the path algebra \( k\Gamma_n \) of this quiver over some field \( k \) admits Kulikov property as the category of \( k\Gamma_n \)-modules is isomorphic to the category of \( k \)-linear representations of \( \Gamma_n \); the \( k \)-linearization \( k[N] \) of the ordered set \( N \) has Kulikov property; let \( R \) be a Dedekind domain which is not a field, then \( R \) has Kulikov property; see [6, pp. 34] and [12, Example 11.19].

In order to prove Theorem 3.1 we need to make some preparations. Let \( K^{-b}(\mathcal{P}) \) be the homotopy category of upper bounded complexes of projective modules with only finitely many non-zero cohomologies.

**Lemma 3.3.** Let \( X \in K^{-b}(\mathcal{P}) \). Then there exists a quasi-isomorphism \( X \to P \) with \( P \in K^{-\text{ppb}}(\mathcal{PP}) \).

**Proof.** For \( X \in K^{-b}(\mathcal{P}) \), there is an integer \( n \in \mathbb{Z} \) such that \( H^i(X) = 0 \) for \( i \leq n \). For \( \text{Ker}d^n_X \), there is a pure-projective resolution \( \cdots \to P^{n-2} \to P^{n-1} \to \text{Ker}d^n_X \to 0 \). By a version of comparison theorem, we can get a cochain map

\[
\begin{array}{cccccccc}
X = & \cdots & X^{n-2} & X^{n-1} & X^n & \cdots \\
\downarrow f & & & & \downarrow d^n_X & & \\
P = & \cdots & P^{n-2} & P^{n-1} & X^n & \cdots \\
\end{array}
\]

From the construction, it is direct that \( f \) is a quasi-isomorphism and \( P \in K^{-\text{ppb}}(\mathcal{PP}) \). \( \square \)

**Lemma 3.4.** Let \( X \in K^{-b}(\mathcal{P}) \) and \( G \in K^{-\text{ppb}}(\mathcal{PP}) \). Then for any cochain map \( g : X \to G \), there is a quasi-isomorphism \( f : X \to P \) with \( P \in K^{-\text{ppb}}(\mathcal{PP}) \), and a cochain map \( h : P \to G \), such that \( g \sim hf \).
Proof. For $X \in K^{-b}(P)$ and $G \in K^{-opp}(PP)$, there exist integers $n(X)$ and $n(G)$, such that $H^i \text{Hom}_R(M, X) = 0$ for any $i \leq n(X)$ and any $M \in P$, and $H^j \text{Hom}_R(N, G) = 0$ for any $j \leq n(G)$ and any $N \in PP$. Let $n = \min\{n(X), n(G)\}$. By the construction in the above lemma, we get a quasi-isomorphism $f : X \to P$, where $P \in K^{-opp}(PP)$ with $P^i = X^i$ for $i \geq n$.

For any $i \geq n$, $f^i = Id_{X^i}$, and let $h^i = g^i$. Since $G \in K^{-opp}(PP)$, the sequence

$$0 \to \text{Ker}d^n_G \to G^{n-1} \to \text{Ker}d^n_G \to 0$$

is pure-exact, and then it remains exact by applying $\text{Hom}_R(P^{n-1}, -)$. It follows from

$$d^n_G h^n = d^n_G g^n d^n_G = g^{n+1} d^{n+1}_G d^{n-1}_G = 0$$

that $h^n d^{n-1}_P \in \text{Hom}_R(P^{n-1}, \text{Ker}d^n_G)$. This yields a morphism $h^{n-1} \in \text{Hom}_R(P^{n-1}, G^{n-1})$ such that $d^n_G h^{n-1} = h^n d^{n-1}_P$. Inductive, we get morphisms $h^j : P^j \to G^j$ ($j < n$) such that $d^{j-1}_G h^{j-1} = h^j d^{j-1}_P$. Hence we have a cochain map $h : P \to G$.

Now consider the following diagram:

It is easy to see that $g^i - h^i f^i = 0$ for $i \geq n$. Since $G$ is exact at the degree less than $n$, the sequence $0 \to \text{Ker}d^n_G \to G^{n-2} \to \text{Ker}d^n_G \to 0$ is exact. Note that $d^n_G (g^n - h^n f^n) = 0$, then $g^{n-1} - h^{n-1} f^{n-1} \in \text{Hom}_R(X^{n-1}, \text{Ker}d^{n-1}_G)$. Since $X^{n-1}$ is projective, there exists a map $s^{n-1} : X^{n-1} \to G^{n-2}$ such that $g^{n-1} - h^{n-1} f^{n-1} = d^{n-2}_G s^{n-1}$. By induction we get homotopy maps $s : X \to G[-1]$ with $s^i = 0$ for any $i \geq n$. Hence $g - hf : X \to G$ is null homotopy. This completes the proof.

Lemma 3.5. Let $P \in K^{-}(PP)$. If $P$ is pure-exact, then $P = 0$ in $K(R)$.

Proof. Without loss of generality, we assume that

$$P = \cdots \to P^{-2} \overset{d^{-2}}{\to} P^{-1} \overset{d^{-1}}{\to} P^0 \to 0.$$ 

Set $P^0 = \text{Ker}d^0$, and consider the sequences $0 \to \text{Ker}d^{i-1} \to P^{i-1} \to \text{Ker}d^i \to 0$, $i \leq 0$. These sequences are pure-exact with each module pure-projective, and hence they are split. This yields that $P^i \cong \text{Ker}d^i \oplus \text{Ker}d^{i+1}$, and $P$ is a direct sum of contractible complexes of the form $\cdots \to 0 \to \text{Ker}d^{i} \to \text{Ker}d^i \to 0 \to \cdots$. Thus $P = 0$ in $K(R)$.

Lemma 3.6. Let $P \in K^{-opp}(PP)$. If $PP$ is closed under kernels of epimorphisms and $P$ is acyclic, then $P \in K_{ac}^b(PP)$. 
Proof. For $P \in \mathbf{K}^{−,\text{ppb}}(\mathcal{P}\mathcal{P})$, there exists an integer $n \in \mathbb{Z}$ such that $\text{H}^i\text{Hom}_R(Q, P) = 0$ for any $i \leq n$ and any pure-projective module $Q$. Let

$$P' = \cdots \rightarrow 0 \rightarrow 0 \rightarrow \text{Im}d^n \rightarrow P^{n+1} \rightarrow P^{n+2} \rightarrow \cdots,$$

$$P'' = \cdots \rightarrow P^{n-2} \rightarrow P^{n-1} \rightarrow \text{Ker}d^n \rightarrow 0 \rightarrow 0 \rightarrow \cdots.$$  

We have an exact sequence of complexes $0 \rightarrow P'' \xrightarrow{f} P \xrightarrow{g} P' \rightarrow 0$. By the assumption, $\mathcal{P}\mathcal{P}$ is closed under kernels of epimorphisms, and $P$ is acyclic and upper bounded, then $\text{Im}d^n$ and $\text{Ker}d^n$ are pure-projective. Moreover, $\text{H}^i\text{Hom}_R(Q, P) = 0$ for any pure-projective module $Q$. This implies that the sequence $0 \rightarrow \text{Ker}d^n \rightarrow P^n \rightarrow \text{Im}d^n \rightarrow 0$ is pure-exact, and moreover, it is split. Thus, the sequence of complexes $0 \rightarrow P'' \rightarrow P \rightarrow P' \rightarrow 0$ is split degree-wise, and then there is a distinguished triangle in the homotopy category $\mathbf{K}(R)$:

$$P'' \xrightarrow{f} P \xrightarrow{g} P' \xrightarrow{h} P''[1].$$

Note that $h = 0$, so the triangle is split and $P = P' \oplus P''$. Since $P \in \mathbf{K}^{−,\text{ppb}}(\mathcal{P}\mathcal{P})$, we have $P'' \in \mathbf{K}^-(\mathcal{P}\mathcal{P})$; moreover, $P''$ is pure-exact, and then $P'' = 0$ by Lemma 3.5. Hence $P \cong P' \in \mathbf{K}_{\text{ac}}^b(\mathcal{P}\mathcal{P})$.  

Now we are in a position to prove the main result of this section.

Proof of Theorem 3.1. Let $\text{Inc} : \mathbf{K}^{−,\text{ppb}}(\mathcal{P}\mathcal{P}) \rightarrow \mathbf{K}^-(R)$ be the embedding functor, and $Q : \mathbf{K}^-(R) \rightarrow \mathbf{D}^-(R)$ be the canonical localization functor. We denote the composition functor by $\eta : \mathbf{K}^{−,\text{ppb}}(\mathcal{P}\mathcal{P}) \rightarrow \mathbf{D}^-(R)$. Since $\eta(\mathbf{K}_{\text{ac}}^b(\mathcal{P}\mathcal{P})) = 0$, by the universal property of quotient functor we have an unique triangle functor

$$\overline{\eta} : \mathbf{K}^{−,\text{ppb}}(\mathcal{P}\mathcal{P}) / \mathbf{K}_{\text{ac}}^b(\mathcal{P}\mathcal{P}) \rightarrow \mathbf{D}^-(R).$$

Clearly, $\text{Im}\overline{\eta} \subseteq \mathbf{D}^b(R)$. By Lemma 3.3, for any $X \in \mathbf{K}^{−,b}(\mathcal{P}) \simeq \mathbf{D}^b(R)$, there exists $P \in \mathbf{K}^{−,\text{ppb}}(\mathcal{P}\mathcal{P})$ such that $X = \overline{\eta}(P)$. Hence the triangle functor $\overline{\eta} : \mathbf{K}^{−,\text{ppb}}(\mathcal{P}\mathcal{P}) / \mathbf{K}_{\text{ac}}^b(\mathcal{P}\mathcal{P}) \rightarrow \mathbf{D}^b(R)$ is dense.

Let $P_1, P_2 \in \mathbf{K}^{−,\text{ppb}}(\mathcal{P}\mathcal{P})$ and $\alpha/s : P_1 \leftarrow Y \rightarrow P_2$ be a morphism in $\mathbf{D}^b(R)$, where $s : Y \rightarrow P_1$ is a quasi-isomorphism with $Y \in \mathbf{K}^-(R)$ and $\alpha : Y \rightarrow P_2$ is a morphism in $\mathbf{K}^-(R)$. Let $t : X \rightarrow Y$ be a dg-projective resolution of $Y$, then $t$ is a quasi-isomorphism and we can let $X \in \mathbf{K}^{−,b}(\mathcal{P})$.

For $X \in \mathbf{K}^{−,b}(\mathcal{P})$ and $P_1, P_2 \in \mathbf{K}^{−,\text{ppb}}(\mathcal{P}\mathcal{P})$, there exist integers $n(X), n(P_1)$ and $n(P_2)$, such that $\text{H}^i\text{Hom}_R(M, X) = 0$ for any $i \leq n(X)$ and any $M \in \mathcal{P}$, and $\text{H}^j\text{Hom}_R(N, P_k) = 0$ for any $j \leq n(P_k)$ and any $N \in \mathcal{P}\mathcal{P}$ ($k = 1, 2$). Let $n = \min\{n(X), n(P_1), n(P_2)\}$. We consider pure-projective resolution of $\text{Ker}d^n_X$, and it follows from the above construction that there is a complex $P \in \mathbf{K}^{−,\text{ppb}}(\mathcal{P}\mathcal{P})$ and a quasi-isomorphism $f : X \rightarrow P$, such that for morphisms $st : X \rightarrow P_1$ and
αt : X → P2, we have morphisms g1 : P → P1 and g2 : P → P2 satisfying st ∼ g1f and αt ∼ g2f. Then we have the following commutative diagram in K−(R):

\[
\begin{array}{c}
\text{Y} \\
\downarrow s \downarrow t \\
P_1 \xrightarrow{st} X \xrightarrow{αt} P_2 \\
\downarrow g_1 \downarrow f \\
P_1 \xrightarrow{g_2} P_2
\end{array}
\]

where the double arrowed morphisms mean quasi-isomorphisms. Note that g1 is also a quasi-isomorphism, hence the mapping cone Con(g1) is acyclic. Moreover, Con(g1) ∈ K−b(PPP), and it follows from Lemma 3.6 that Con(g1) ∈ K−b(PPP). By the definition of right fraction, we have g2/g1 ∈ HomK−(PPP)/K−b(PPP)(P1, P2) and α/s = g2/g1 = \overline{η}(g2/g1). This implies that the functor \overline{η} is full.

It remains to prove \overline{η} is faithful. Because the triangle functor \overline{η} is full, by [20, p.446] it suffices to show that it sends non-zero objects to non-zero objects. Suppose P ∈ K−b(PPP) and \overline{η}(P) = 0, then P is acyclic and it follows from Lemma 3.6 that P ∈ K−b(PPP). Hence \overline{η} is faithful. This completes the proof. □

4. Pure singularity categories

By [25, Theorem 3.6 (1)], for any ring R there is a triangle-equivalence D−b(R) ∼= K−b(PPP), and it is trivial that K−b(PPP) is a thick subcategory of K−b(PPP). In this section, we introduce and study the pure singularity category, as a version of the singularity category with respect to pure-exact structure. Note that relative singularity categories under another conditions have also been studied, see for example [1, 9, 15].

**Definition 4.1.** For a ring R, the pure singularity category is defined to be the Verdier quotient

\[
D_{psg}(R) := D_{pur}(R)/K^b(PPP) \simeq K^{-ppb}(PPP)/K^b(PPP).
\]

Warfield [24] showed that any module M has a pure-projective resolution P → M → 0. In [11] Griffith defined Pext^i_R(M, N) := H^iHom_R(P, N). Analogous to the setting of derived category, the following shows that the relative derived functors of Hom with respect to pure-projective modules can be interpreted as the morphisms in the corresponding relative derived category with respect to pure-projective modules.

**Proposition 4.2.** Let M, N be any R-modules. Then Pext^i_R(M, N) ∼= Hom_{D_{pur}(R)}(M, N[i]).

**Proof.** Let P → M → 0 be a pure-projective resolution of M. Viewing M as a complex concentrated in degree zero, then P → M is a pure-quasi-isomorphism, and P ∼= M in D_{pur}(R). Hence
we have
\[ \text{Pext}_R^i(M, N) = H^i \text{Hom}_R(P, N) \]
\[ = \text{Hom}^i_{K(R)}(P, N[i]) \]
\[ \cong \text{Hom}^i_{D_{\text{pur}}(R)}(P, N[i]) \]
\[ \cong \text{Hom}^i_{D_{\text{pur}}^b(R)}(M, N[i]) \]
where the last two isomorphisms hold by [25, Proposition 3.1(1)] and [25, Proposition 3.2]. □

**Definition 4.3.** ([11]) Let \( M \) be an \( R \)-module. The pure-projective dimension \( p.pd(M) \) of \( M \) is defined to be the smallest positive integer \( n \) such that \( \text{Pext}_R^{n+1}(M, N) = 0 \), and set \( p.pd(M) = \infty \) if no such \( n \) exists. For any ring \( R \), the pure-global dimension \( p.pgdim(R) \) is defined as the supremum of the pure-projective dimensions of all \( R \)-modules.

Note that pure-projective dimension of a module \( M \) can be also defined by the shortest length of pure-projective resolutions of \( M \). It is a routine job to show that these two definitions of pure-projective dimension coincide. For any ring \( R \), the supremum of the pure-projective dimensions of all \( R \)-modules is equal to the supremum of the pure-injective dimensions of all \( R \)-modules.

Recall that \( D_{\text{sg}}(R) = 0 \) if and only if the global dimension of \( R \) is finite. We have the following pure version, which implies that the notion of “pure singularity” seems to be reasonable.

**Proposition 4.4.** Let \( R \) be a ring. Then \( D_{\text{psg}}(R) = 0 \) if and only if \( p.pgdim(R) \) is finite.

**Proof.** If the pure-global dimension \( p.pgdim(R) \) of \( R \) is finite, then it follows from [25, Theorem 4.7] that for any \( X \in D_{\text{pur}}^b(R) \), there exists a pure-quasi-isomorphism \( P \to X \) with \( P \in K^b(\mathcal{P}\mathcal{P}) \). Then \( D_{\text{pur}}^b(R) \cong K^b(\mathcal{P}\mathcal{P}) \), and hence \( D_{\text{psg}}(R) = 0 \).

It remains to prove the necessity. Suppose \( D_{psg}(R) = 0 \). Let \( M \) be an \( R \)-module. Then \( M = 0 \) in \( D_{psg}(R) \), and there exists some \( X \in K^b(\mathcal{P}\mathcal{P}) \) such that \( M \cong X \) in \( D_{\text{pur}}^b(R) \). We denote this isomorphism by a right fraction \( \alpha / f : M \leftarrow \cdots Y \to X \), where \( f \) and \( \alpha \) are pure-quasi-isomorphisms. Then there is a triangle \( Y \xrightarrow{\alpha} X \to \text{Con}(\alpha) \to Y[1] \) in \( K(R) \) with \( \text{Con}(\alpha) \) pure-exact. By applying \( \text{Hom}_{K(R)}(X, -) \) to it, we get an exact sequence
\[ \text{Hom}_{K(R)}(X, Y) \to \text{Hom}_{K(R)}(X, X) \to \text{Hom}_{K(R)}(X, \text{Con}(\alpha)). \]

It follows from [25, Lemma 2.9(1)] that \( \text{Hom}_{K(R)}(X, \text{Con}(\alpha)) = 0 \), so there is a cochain map \( \beta : X \to Y \) such that \( \alpha \beta \) is homotopic to \( \text{Id}_X \). Thus we get a pure quasi-isomorphism \( f \beta : X \to M \).

Consider the soft truncation \( X_0 = \cdots \to X^{-2} \to X^{-1} \to \text{Ker}d_X^0 \to 0 \) of \( X \). Then there is a pure quasi-isomorphism \( f \beta \iota : X_0 \to M \), where \( \iota : X_0 \to X \) is a natural embedding. Since \( X \in K^b(\mathcal{P}\mathcal{P}) \), we assume that there is an integer \( n \) such that \( X^i = 0 \) for any \( i > n \). Note that the sequence
\[ 0 \to \text{Ker}d_X^n \to X^0 \to \cdots \to X^{n-1} \to X^n \to 0 \]
is pure-exact with each \( X^i \in \mathcal{P} \mathcal{P} \), and it follows that \( \text{Ker} d^i_0 \) is also pure-projective. Thus \( M \) has a bounded pure-projective resolution \( X_0 \rightarrow M \), and then \( \text{p.pd}(M) < \infty \). This yields the desired assertion. \( \square \)

The notion of recollement of triangulated categories was introduced by Beilinson, Bernstein and Deligne [2] with an idea that one category can be viewed as being “glued together” from two others. Let \( \mathcal{T}', \mathcal{T} \) and \( \mathcal{T}'' \) be triangulated categories. We say that \( \mathcal{T} \) admits a recollement relative to \( \mathcal{T}' \) and \( \mathcal{T}'' \), if there exist six triangulated functors as in the following diagram

\[
\begin{array}{ccc}
\mathcal{T}' & \xrightarrow{i^*} & \mathcal{T} \\
\downarrow{j^*} & & \downarrow{j_*} \\
\mathcal{T}'' & \xleftarrow{i_*} & \mathcal{T}
\end{array}
\]

such that

1. \((i^*, i_*) \), \((i, i^!), \) \((j^!, j^*) \) and \((j^*, j_*) \) are adjoint pairs;
2. \(i_*, j_* \) and \( j! \) are full embedding;
3. \( j^* i_* = 0; \)
4. for each \( X \in \mathcal{T} \), there are distinguished triangles

\[
i_* i^!(X) \rightarrow X \rightarrow j_* j^!(X) \rightarrow i_* i^!(X)[1],
\]

\[
\rightarrow X \rightarrow i_* i^!(X) \rightarrow j_* j^!(X)[1].
\]

**Theorem 4.5.** Let \( A, B \) and \( C \) be rings. Assume that \( D^b_{\text{pur}}(A) \) admits the following recollement

\[
\begin{array}{ccc}
D^b_{\text{pur}}(B) & \xrightarrow{i^*} & D^b_{\text{pur}}(A) \\
\downarrow{i^!} & & \downarrow{j_*} \\
D^b_{\text{pur}}(C)
\end{array}
\]

Then \( D_{\text{psg}}(A) = 0 \) if and only if \( D_{\text{psg}}(B) = 0 = D_{\text{psg}}(C) \).

**Proof.** Since \( D^b_{\text{pur}}(B) \) and \( D^b_{\text{pur}}(C) \) can be fully embedded into \( D^b_{\text{pur}}(A) \), it is clear that the finiteness of \( \text{p.gldim}(A) \) implies the finiteness of both \( \text{p.gldim}(B) \) and \( \text{p.gldim}(C) \). By Proposition 4.4, it follows that \( D_{\text{psg}}(B) = 0 = D_{\text{psg}}(C) \) if \( D_{\text{psg}}(A) = 0 \).

For the converse, it suffices to prove that if \( B \) and \( C \) are of finite pure-global dimension, then \( \text{p.gldim}(A) < \infty \). Let \( M \) and \( N \) be any \( A \)-modules. The above recollement induces the following distinguished triangles in \( D^b_{\text{pur}}(A) \):

\[
j_* j^!(M) \rightarrow M \rightarrow i_* i^!(M) \rightarrow j_* j^!(M)[1],
\]

\[
\rightarrow N \rightarrow j_* j^!(N) \rightarrow i_* i^!(N)[1].
\]

We abbreviate \( \text{Hom}_{D^b_{\text{pur}}(A)}(\_ \_ \_ , \_ \_ \_) \) with \( D^b_{\text{pur}}(A)(\_ \_ \_ , \_ \_ \_ ) \). By applying \( \text{Hom}_{D^b_{\text{pur}}(A)}(\_ \_ \_ , i_* i^!(N)) \) and \( \text{Hom}_{D^b_{\text{pur}}(A)}(\_ \_ \_ , j_* j^!(N)) \) to the first triangle, we get the following long exact sequences:

\[
\cdots \rightarrow D^b_{\text{pur}}(A)(i_* i^!(M), i_* i^!(N)[n]) \rightarrow D^b_{\text{pur}}(A)(M, i_* i^!(N)[n]) \rightarrow D^b_{\text{pur}}(A)(j_* j^!(M), i_* i^!(N)[n]) \rightarrow \cdots
\]
\[ \cdots \rightarrow D_{\text{par}}^b(A)(i_*i^*(M), j_*j^*(N)[n]) \rightarrow D_{\text{par}}^b(A)(M, i_*j^*(N)[n]) \rightarrow D_{\text{par}}^b(A)(j_*j^*(M), j_*j^*(N)[n]) \rightarrow \cdots \]

We have \( D_{\text{par}}^b(A)(j_*j^*(M), i_*i^*(N)[n]) \cong D_{\text{par}}^b(C)(j^*(M), j^*i_*i^*(N)[n]) = 0 \) for every \( n \in \mathbb{Z} \) since \( j^*i_* = 0 \); and similarly, \( D_{\text{par}}^b(A)(i_*i^*(M), j_*j^*(N)[n]) = 0 \) follows by the adjunction \((j^*, j_*)\). Since \( p_{\text{gldim}}(B) < \infty \), \( D_{\text{par}}^b(B) \cong K^b(\mathcal{P}) \). Noting that \( i^*(M) \) and \( i^*(N) \) lie in \( K^b(\mathcal{P}) \), one has \( \text{Hom}_{D_{\text{par}}^b(B)}(i^*(M), i^*(N)[n]) = 0 \) for \( n > 0 \); moreover, \( i_* \) is a full embedding and then \( D_{\text{par}}^b(A)(i_*i^*(M), i_*i^*(N)[n]) = 0 \). Similarly, since \( p_{\text{gldim}}(C) < \infty \), it follows that \( D_{\text{par}}^b(A)(j_*j^*(M), j_*j^*(N)[n]) = 0 \) for \( n > 0 \).

Now we apply \( \text{Hom}_{D_{\text{par}}^b(A)}(M, -) \) to the second triangle, and obtain the following long exact sequence

\[ \cdots \rightarrow D_{\text{par}}^b(A)(M, i_*i^*(N)[n]) \rightarrow D_{\text{par}}^b(A)(M, N[n]) \rightarrow D_{\text{par}}^b(A)(M, j_*j^*(N)[n]) \rightarrow \cdots \]

By the above argument, we have \( D_{\text{par}}^b(A)(M, i_*i^*(N)[n]) = 0 = D_{\text{par}}^b(A)(M, j_*j^*(N)[n]) \) for \( n > 0 \). Then \( \text{Pext}^n_{A}(M, N) \cong \text{Hom}_{D_{\text{par}}^b(A)}(M, N[n]) = 0 \). This implies that \( p_{\text{gldim}}(A) < \infty \). \qed

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12

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