Biological Experiments Based on Fractional Integral Equations

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Abstract. This paper deals with modeling of mathematical biological experiments using the iterative fractional integral equations following type

\[ w(u) = h(u) + \int_{u_0}^{u} \frac{(u - r)^\beta}{\Gamma(\beta + 1)} K(r, w(r)))dr \] (1)

where $u_0, u \in [a, b], w, h \in C[a, b], K \in C([a, b] \times [a, b]).$ We propose that the mathematical model (1) containing the iterative integral of fractional order that is the best method in the studying this field. We establish the existence and uniqueness solutions for fractional iterative integral equation by using the technique function $h$ non-expansive mappings. Also, we show the results of the system of fractional iterative integral equation by using the technique of non-expansive operators.

1. Introduction

The mathematical biological model containing the fractional iterative integral equations that deals with equations affecting our lives directly or indirectly as growth bacteria or infective disease processes based on fractional integral equation. This study can be applied to obtain the beneficial bacteria that are used in factories to make some of the things that we use in our lives such as yogurt. Iterative equations are the equivalent of the growth of bacteria so that through the equations we control the number of bacteria in the product or control the spread of infective disease [1, 2, 3].

Mathematics plays an important role in all scientific fields such as physics, engineering, biology and medicine through equations that study and deal with these fields [4, 6, 7].

Majority the existence or existence and uniqueness theorems for fractional integral equations are generally obtained through the technique of fixed point, for example, the fixed point theorem of Schauder principle or contractions of the mapping principle. One of the major significant tool to establish the existence or uniqueness theorems for fractional integral equations in the mapping principle of classical Banach contraction or some of its generalizations paraphernalia [4, 6, 8, 9, 10, 11, 12, 13, 14].
This paper proposes that the mathematical model (1) containing the iterative integral of fractional order that is the best method in the studying this field. We establish the existence and uniqueness solutions for fractional iterative integral equation by using the technique function $h$ non-expansive mappings. Also, we show the results of the system of fractional iterative integral equation by using the technique of non-expansive operators.

1.1. Preliminaries
In this section, we introduce some important concepts, which are useful in the sequel [15, 4, 8, 19].

**Definition 1:** The integral operator is defined as

$$I^\alpha_ah(u) = \frac{1}{\Gamma(\alpha)} \int_0^u \frac{\psi(\beta)}{(u - \beta)^{1-\alpha}} d\beta, \quad \alpha > 0.$$  \hspace{1cm} (2)

**Definition 2:** Let $(Z,d)$ a metric space and $H: Z \rightarrow Z$ is a mapping said to be an $\delta$-contraction if there exit $\delta \in [0, 1)$ so that

$$d(Hz, Hy) \leq \delta d(z, y), \forall z, y \in Z.$$  

The application is called non expansive when $\gamma = 1$.

**Definition 3:** Let $A$ a normed linear space and $Q \subset A$ is a convex and $H: Q \rightarrow Q$ is a self-mapping. In view of $w_0 \in Q$ and the real numbers $\xi \in [0, 1]$, $w_m$ is a sequence introduced by the formula

$$w_{m+1} = (1 - \xi)w_m + \xi Hw_m, \quad m = 0, 1, \ldots.$$  

In general, the above referred as Krasnoselskij iteration or Krasnoselskij-Mann iteration.

2. Main findings
In this section, we focus on two problems which are the study of sufficient conditions for the existence of a growth bacteria based on fractional iterative integral equations and system of fractional iterative integral equations. The advantage of this study is to control the number of bacteria in the product such as in yogurt making or to control the spread of infective disease. On the other hand, if there is any mistake in the initial condition we can not control the spread of infective disease. We use the mathematical model of the iterative integral of fractional order to study the existence and uniqueness of solutions.

2.1. The fractional iterative integral equations
In this section, we study the of the spread of infective disease in Yemen Country such as tuberculosis. The fractional iterative integral equations is the best way to study the spread of infective disease and control it. We examine the theorems of the existence and uniqueness solution of fractional iterative integral equations by using the technique function $h$ non-expansive mappings to control spread of tuberculosis disease between people.

We consider the following integral equation of fractional order:

$$w(u) = h(u) + \int_{u_0}^u \frac{(u - s)^\beta}{\Gamma(\beta + 1)} K(s, w(w(s)))ds$$  \hspace{1cm} (3)

where $u_0, u \in [a, b], w, h \in C[a, b], K \in C([a, b] \times [a, b])$. We denote $E_u = \max \{u - a, b - u\}, \forall u \in [a, b]$, and

$$C_{\ell, \beta} = \left\{ w \in C([a, b] \times [a, b]) : |w(s_1) - w(s_2)| \leq \ell, \frac{|s_1 - s_2|^\beta}{\Gamma(\beta + 1)}, \forall s_1, s_2 \in [a, b] \right\}, \quad \ell > 0$$  \hspace{1cm} (4)

**Theorem 4:** Presume that the following conditions are satisfied:-
(b1) \( K \in C([a, b] \times [a, b]); \)

(b2) \( \exists \ell_1 > 0, \exists \ell_2 > 0 \) so that
\[
|K(r, z_1) - K(r, z_2)| \leq \ell_1|z_1 - z_2|, \quad \forall r, z_n, \in [a, b]
\] (5)

and
\[
|h(s_1) - h(s_2)| \leq \ell_2|s_1 - s_2|, \quad \forall s_1, s_2 \in [a, b]
\] (6)

(b3) If \( \ell \) is the Lipschitz constant contained in 5, thus
\[
M = \max \{|k(r, z)| : (r, z) \in [a, b], \} \text{ and } 2M + \ell_2 \leq \ell,
\]

(b4) If \( \exists u_0 \in [a, b] \) so that \( |h(u)| \leq h(u_0) \) then, either:
\[
(b4i) \quad M.E_{u_0} \leq E_{w_0}, \quad w_0 = h(u_0), \text{ or}
\]
\[
(b4ii) \quad u_0 = 0, \quad M \frac{(T)^{\beta}}{\Gamma(\beta+1)} \leq b - w_0, \quad K(t, \Upsilon) \geq 0, \quad \forall (r, \Upsilon) \in [a, b], \text{ where } T = \max[a, b] \text{ or}
\]
\[
(b4iii) \quad u_0 = b, \quad M \frac{(T)^{\beta}}{\Gamma(\beta+1)} \leq w_0 - a, \quad K(t, \Upsilon) \geq 0, \quad \forall (r, \Upsilon) \in [a, b],
\]

(b5) \( \ell_1, \frac{\ell_1}{\Gamma(\beta+1)} (\ell + 1) E_{u_0} \leq 1. \)

Thus there exist at least one of the solutions of 3 in \( C_{\ell_1} \), that can be approximated by iteration Krasnoselskij
\[
w_{m+1}(u) = (1 - \eta)w_m(u) + \eta \int_{s_0}^{s} \frac{(u - r)^{\beta}}{\Gamma(\beta+1)} K(r, w_m(w_m(r)))dr, \quad u \in [a, b], \quad m \geq 1
\]
in which \( \eta \in (0, 1) \) and \( w_1 \in C_{\ell_1} \) is arbitrary.

**Proof.** In the proving, we use the same technique which introduced in [16]. We consider the operator of integral \( H : C_{\ell_1, \beta} \rightarrow C[a, b] \) introduced by
\[
(Hw)(s) = h(s) + \int_{s_0}^{s} \frac{(s - r)^{\beta}}{\Gamma(\beta+1)} K(r, w(w(r)))dr, \quad s, u_0 \in [a, b],
\]
Obviously, \( w \in C_{\ell_1, \beta} \) is a solution of 3 if only if \( w \) is a fixed point of \( H \), that is \( w = Hw \). We show that \( C_{\ell_1, \beta} \) is an invariant set with regard to \( H \), i.e., we have \( H(C_{\ell_1, \beta}) \subset C_{\ell_1, \beta} \). If the condition \( (b4i) \) achieves, therefore any \( u_0 \in C_{\ell_1, \beta} \) and \( s \in [a, b] \), we obtain
\[
|(Hw)(s)| = h(s) + \int_{s_0}^{s} \frac{(s - r)^{\beta}}{\Gamma(\beta+1)} K(r, w(w(r)))dr
\]
\[
\leq |h(s)| + \int_{s_0}^{s} \frac{(s - r)^{\beta}}{\Gamma(\beta+1)} K(r, w(w(r)))dr
\]
\[
\leq h(u_0) + M \frac{(s - u_0)^{\beta}}{\Gamma(\beta+1)}
\]
\[
\leq h(u_0) + M.E_{u_0}
\]
\[
= h(u_0) + M.E_{u_0} \leq b
\]

and
\[
|(Hw)(s)| \geq |h(s)| - \left| \int_{s_0}^{s} \frac{(s - r)^{\beta}}{\Gamma(\beta+1)} K\big(r, w(w(r))\big)dr \right|
\]
\[
\geq -h(u_0) - M \frac{(s - u_0)^{\beta}}{\Gamma(\beta+1)}
\]
\[ \geq -h(u_0) - M.E_{u_0} \geq -h(u_0) - E_{w_0} \geq a \]

Hence, \((Hw)s \in C[a, b]\) for every \(w \in C_{\ell, \beta}\). We prove that \(Hw \in C_{\ell, \beta}\) for all \(w \in C_{\ell, \beta}\).

Therefore, for \(\forall s_1, s_2 \in [a, b]\), we have

\[
|Hw(s_1) - Hw(s_2)| \leq |h(s_1) - h(s_2)| + \left| \int_{u_0}^{s_1} \frac{(s - r)^\beta}{\Gamma(\beta + 1)} K(r, w(w(r)))dr - \int_{u_0}^{s_2} \frac{(s - r)^\beta}{\Gamma(\beta + 1)} K(r, w(w(r)))dr \right|
\]

\[
\leq |h(s_1) - h(s_2)| + \int_{s_1}^{s_2} \frac{(u - r)^\beta}{\Gamma(\beta + 1)} K(r, w(w(r)))dr
\]

\[
\leq \ell_2 \frac{|s_1 - s_2|}{\Gamma(\beta + 1)} + 2M \frac{|s_1 - s_2|}{\Gamma(\beta + 1)} \leq \ell_2 \frac{|s_1 - s_2|^\beta}{\Gamma(\beta + 1)} + 2M \frac{|s_1 - s_2|^\beta}{\Gamma(\beta + 1)} \leq \ell. \frac{|s_1 - s_2|^\beta}{\Gamma(\beta + 1)}
\]

Thus, \(Hw \in C_{\ell, \beta}\) for every \(w \in C_{\ell, \beta}\). In the same manner we treat the cases \((b_{iii})\) and \((b_{iv})\). Let \(w, z \in C_{\ell, \beta}\), and \(s \in [a, b]\). Then

\[
|Hw(s) - Hz(s)| = \left| \int_{u_0}^{s} \frac{(s - r)^\beta}{\Gamma(\beta + 1)} K(r, w(w(r)))dr - \int_{u_0}^{s} \frac{(s - r)^\beta}{\Gamma(\beta + 1)} K(r, z(z(r)))dr \right|
\]

\[
\leq \int_{u_0}^{s} \left| \frac{(s - r)^\beta}{\Gamma(\beta + 1)} [K(r, w(w(r))) - K(r, z(z(r)))] \right|dr
\]

\[
\leq \ell_1 \int_{u_0}^{s} \frac{(s - r)^\beta}{\Gamma(\beta + 1)} |w(w(r)) - z(z(r))|dr
\]

\[
\leq \ell_1 \int_{u_0}^{s} \frac{(s - r)^\beta}{\Gamma(\beta + 1)} \left[ |w(w(r)) - z(z(r))| + |w(z(r)) - z(z(r))| \right]dt
\]

\[
\leq \ell_1 \int_{u_0}^{s} \frac{(s - r)^{\beta - 1}}{\Gamma(\beta)} \left[ \ell. |w(r) - z(r)| + |w(z(r)) - z(z(r))| \right]dr
\]

Now, taking the norm, we have

\[
\|(Hw) - (Hz)\|_{C[a, b]} \leq \ell_1 \frac{(T)^\beta}{\Gamma(\beta + 1)} (\ell + 1) E_{u_0} \| w - z \|_{C[a, b]},
\]

which the presumption given \((b_4)\) proves that \(H\) is non-expansive therefore continues, so the Schauder fixed point theorem means that Eq.3, has at least one fixed point (has solution), and from the Corollary 2.6 and 2.7 in [17] can be approximated this solutions by Krasnoselskii iteration.

2.2. Theorems of existence and uniqueness of system of fractional iterative integral equations

In this section, we study the spread of infective diseases in Yemen Country such as tuberculosis and Syphilis. The system of fractional iterative integral equations is the best way to study the spread of infective diseases and control them.

We examine the theorems of the existence and uniqueness of system of fractional iterative integral equations by using the technique of non-expansive operators to control spread of those diseases between people.
We consider a system of fractional iterative integral equations of the form:

\[
\begin{aligned}
w_1(u) &= h_1(u) + \int_0^u \frac{(u-r)^\beta}{\Gamma(\beta+1)} k_1(r, w_1(r), w_2(r), w_1(w_1(r)))dr, \\
\frac{w_2(u)}{u} &= h_2(u) + \int_0^u \frac{(u-r)^\beta}{\Gamma(\beta+1)} k_2(r, w_1(r), w_2(r), w_2(w_2(r)))dr,
\end{aligned}
\]

(7)

we consider \(C_{\ell,\beta}([a, b], [a, b]^2)\), as the subspace of the Banach space \((C([a, b], R^2), ||.||_C)\), endowed with the norm \(||u||_C = (||u_1||, ||u_2||)\), where \(||u_j|| = \max_{a \leq u \leq b} |u_j(s)|\), \(j = 1, 2\), introduce the set

\[
C_{\ell,\beta}([a, b], [a, b]^2) = \left\{ (w_1, w_2) \in C([a, b], [a, b]^2) : |w_j(u_1) - w_j(u_2)| \leq \ell \left| \frac{u_1 - u_2}{\Gamma(\beta + 1)} \right|, \forall u_1, u_2 \in [a, b], j = 1, 2 \right\}
\]

where \(\ell > 0\) is fixed, and denote \(E_u = \max\{u - a; b - u\}\), for all \(u \in [a, b]\).

**Theorem 5**: Suppose that

(a1) \(h \in C([a, b], [a, b]^2)\).

(a2) \(K \in C([a, b]^2, R^2)\).

(a3) There exists \(\ell_m\), so that

\[|K_j(t, x_1, y_1; z_1) - K_j(t, x_2, y_2; z_2)| \leq \ell_m (|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|),\]

for every \(t, x_j, y_j; z_j \in [a; b]; j = 1; 2\); and there exists \(\ell > 0\), so that

\[|h_j(u_1) - h_j(u_2)| \leq \ell \left| \frac{u_1 - u_2}{\Gamma(\beta + 1)} \right|, \forall u_1, u_2 \in [a, b], j = 1, 2,\]

(a4) \(M + \ell \leq \ell\), where indicate

\[M = \max_{0 \leq j \leq 2} \{ |K_j(t, x_j, y_j; z_j)| : t, x_j, y_j; z_j \in [a, b] \}\]

(a5) Have \(|h_j(u)| \leq |h_j(u_0)|\), for everys \(u \in [a, b], j = 1; 2\), and one of the following conditions achieves:

(a5i) \(M, E_{w_0} \leq E_{w_0}, w_0 = \max(h_1(u_0), h_2(u_0))\);

(a5ii) \(u_0 = a, M \left| \frac{(b-a)^\beta}{\Gamma(\beta+1)} \right| \leq b - E_{w_0}, K(t, x_j, y_j; z_j) \geq 0; \forall t, x_j, y_j; z_j \in [a, b], j = 1; 2,\)

(a5iii) \(u_0 = b, M \left| \frac{(b-a)^\beta}{\Gamma(\beta+1)} \right| \leq E_{w_0} - a, K(t, x_j, y_j; z_j) \geq 0; \forall t, x_j, y_j; z_j \in [a, b], j = 1; 2,\)

(a6) \(\ell_m, (\ell + 1), E_{w_0} \leq 1\).

Therefore, the system (7) has at least one solution in \(C_{\ell,\beta}([a, b], [a, b]^2)\).

**Proof**: It follows from Lemma 1 in ([18]) that \(\emptyset \neq C_{\ell,\beta}([a, b], [a, b]^2) \subset (C([a, b], R^2), ||.||_C)\) is a compact and convex. For any \(w \in C_{\ell,\beta}([a, b], [a, b]^2)\) and \(s \in [a, b]\), we consider the integral operator

\[
H : C_{\ell,\beta}([a, b], [a, b]^2) \rightarrow C([a, b], [a, b]^2),
\]

introduced by

\[
(Hw)(s) = (H_1w)(s), (H_2w)(s) = \left( h_1(s) + \int_a^s \frac{(s-r)^\beta}{\Gamma(\beta+1)} K(r, w_1(r), w_2(r), w_1(w_1(r)))dr, h_2(s) + \int_a^s \frac{(s-r)^\beta}{\Gamma(\beta+1)} K(r, w_1(r), w_2(r), w_2(w_2(r)))dr \right).
\]
The set of fixed points of $H$ is the set of solutions of the 7. Now, show that $C_t([a, b], [a, b]^2)$ is an invariant set with respect to $G$, which means that $H(C_t([a, b], [a, b]^2)) \subset C([a, b], [a, b]^2)$. By $u \leq w, u, w \in R^2$, we realize that $u_j \leq w_j$, for every $j = 1, 2$. Have

$$|(Hw)(s)| = |(H_1w)(s), (H_2w)(u)| = \left| h_1(s) + \int_a^s \frac{(s-r)^\beta}{\Gamma(\beta+1)} K_1(r, w_1(r), w_2(r), w_1(w_1(r))) \right| dr, |h_2(s)|$$

$$+ \int_a^s \frac{(s-r)^\beta}{\Gamma(\beta+1)} K_2(r, w_1(r), w_2(r), w_2(w_2(r))) \right| dr$$

$$\leq \left( |h_1(u_0)| + M \left| \frac{s-u_0}{\Gamma(\beta+1)} \right| |h_2(u_0)| + M \left| \frac{s-u_0}{\Gamma(\beta+1)} \right| \right)$$

$$\leq \left( |h_1(u_0)| + M.E_{u_0}, |h_2(u_0)| + M.E_{u_0} \right)$$

and

$$|Hw(s)| = |(H_1w)(s), (H_2w)(s)| \geq \left( |h_1(s) - \int_a^s \frac{(s-r)^\beta}{\Gamma(\beta+1)} K_1(r, w_1(r), w_2(r), w_1(w_1(r))) \right| dr, |h_2(s)|$$

$$- \int_a^s \frac{(s-r)^\beta}{\Gamma(\beta+1)} K_2(r, w_1(r), w_2(r), w_2(w_2(r))) \right| dr$$

$$\geq \left( |h_1(u_0)| - M \left| \frac{u-s_0}{\Gamma(\beta+1)} \right| |g_2(s_0)| - M \left| \frac{u-s_0}{\Gamma(\beta+1)} \right| \right)$$

$$\geq \left( |g_1(s_0)| - M.E_{s_0}, |g_2(u_0)| - M.E_{u_0} \right)$$

$$\geq \left( |h_1(u_0)| - E_{u_0}, |h_2(u_0)| - E_{u_0} \right)$$.

By presumption $(a4)$ conclude that each vector component $|(Hw)(s)|$ is in $[a, b]$. So $Hw \in C([a, b], [a, b]^2)$. We show that $Hw \in C_t([a, b], [a, b]^2)$, for every $w = (u_1, u_2) \in C_t([a, b], [a, b]^2)$. For any $s_1, s_2 \in [a, b]$, we get

$$|Gv(u_1) - Gv(u_2)| = |(G_1v(u_1) - G_1v(u_2), G_2v(u_1) - G_2v(u_2))|$$

$$\leq \left( |g_1(u_1) - g_1(u_2)| + \int_{u_1}^{u_2} \left| \frac{(u-r)^\beta}{\Gamma(\beta+1)} K_1(r, v_1(r), v_2(r), v_1(v_1(r))) \right| dr, |g_1(u_1)|$$

$$- g_1(u_2)| + \int_{u_1}^{u_2} \left| \frac{(u-r)^\beta}{\Gamma(\beta)} K_2(r, v_1(r), v_2(r), v_2(v_2(r))) \right| dr$$

$$\leq \left( |\ell_1 + M| \frac{|u_1 - u_2|^\beta}{\Gamma(\beta+1)}, |\ell_2 + M| \frac{|u_1 - u_2|^\beta}{\Gamma(\beta+1)} \right)$$

that shows by presumption $(a3)$, that $Hw \in C_t([a, b], [a, b]^2)$, for every $w \in C_t([a, b], [a, b]^2)$. For any $w, z \in C_t([a, b], [a, b]^2)$, $w = (w_1, w_2), z = (z_1, z_2)$, we get

$$|Hw(z) - Gz(z)| = |(H_1w(s) - H_1z(s)), |H_2z(s) - H_2z(s))|$$

$$\leq \left( \int_{u_0}^s \left| \frac{(s-r)^\beta}{\Gamma(\beta+1)} \left( K_1(r, w_1(r), w_2(r), w_1(w_1(r))) - K_1(r, z_1(r), z_2(r), z_1(z_1(r))) \right) \right| dr$$

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The function $u$ for any $s \in [0,1]$, and $v \in C([0,1] \times [0,1])$ belonging to the set

$$C_{1,0.5} = \left\{ v \in C([0,1] \times [0,1]) : |v(u_1) - v(u_2)| \leq \frac{|u_1 - u_2|}{\Gamma(\frac{1}{2} + 1)} \right\},$$

for any $u_1, u_2 \in [0,1]$ that, in given our notes, that means $\ell = 1$, It contains $a = 0, b = 1, s_0 = 0.5$, hence

$$E_{s_0} = \max \{s_0 - a, b - s_0\} = \max \{0.5 - 0, 1 - 0.5\} = 0.5,$$

$$T = \max \{0, 1\} = 1, \quad \Gamma\left(\frac{1}{2} + 1\right) = \frac{3}{2} = 0.8863.$$

The function $g(s, y) = -1 + 0.8863y$ is Lipschitzian with the Lipschitz constant $\ell_1 = 0.8863$.

Then, it contains

$$\ell_1 \frac{1}{0.8863}|1 + 1|E_{s_0} = 1,$$

therefore, the condition $(b_5)$ in Theorem 2.1 is satisfied. From Theorem 2.1 initial-value problem 8 having at least one solution in $C_{1,0.5}$, can be approximated as iteration Krasnoselskii

$$v_{m+1}(s) = (1 - \sigma)v_m(s) + \sigma g(s) + \sigma \int_{s_0}^{s} \frac{(s - r)^{0.5}}{\Gamma(1.5)}(v_m(v_m(r)))dr, \quad s \in [0,1], \quad m \geq 1, \quad s > r,$$

where $\sigma = \frac{1}{2} \in (0, 1), g(s) = -1$, and $v_1 = \frac{1}{2} \in C_{1,\frac{1}{2}}$ is arbitrary. Then

$$v_{m+1}(s) = 0.5v_m(s) - 0.5 + 0.5 \int_{0}^{s} \frac{(s - r)^{0.5}}{\Gamma(1.5)}(v_m(v_m(r)))dr, \quad s \in [0,1], \quad m \geq 1, \quad s > r.$$
3. Conclusion
From the above development, we conclude that the iterative integral equations can be extended into fractional integral equation. Moreover, as future work, one can investigate the existence of the following equation
\[
\begin{align*}
\left\{ \begin{array}{l}
w(s) &= h(s, w(s), w(w(s))) + f(s, w(s)) \int_0^s (s-u)^\beta \Gamma(\beta + 1) K(u, w(u), w(w(u))) du \\
w(0) &= w_0,
\end{array} \right.
\end{align*}
\]

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