Disorder-driven quantum transition in relativistic semimetals: functional renormalization via the porous medium equation

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In the presence of randomness, a relativistic semimetal undergoes a quantum transition towards a diffusive phase. A standard approach relates this transition to the $U(N)$ Gross-Neveu model in the limit of $N \to 0$, while rare events were argued to be relevant close to the transition. In this work we reconcile previous studies by developing a functional renormalization group amenable to include non-perturbative effects. We show that the previously considered fixed point is indeed infinitely unstable, confirming the necessity to describe fluctuations beyond the Gaussian approximation. Furthermore, the disorder distribution renormalizes following the so-called porous medium equation. We find that the transition is controlled by a non-analytic fixed point drastically different from the fixed point of the $U(N)$ Gross-Neveu model. We relate a self-similar solution of the porous medium equation to a mechanism of generation of a finite density of states at the nodal point responsible for the transition.

Introduction. - Three dimensional (3D) semimetals in which bands cross linearly at isolated nodal points (two bands for Weyl [1, 2] or four bands for Dirac [3–5]) attracted considerable interest since their recent discovery. The behavior of electrons around such crossing points is described by relativistic equations of motion. As a consequence these materials exhibit a wealth of phenomena among which we can mention the chiral anomaly manifested by a large negative magnetoresistance and anomalous quantum oscillations due to surface Fermi arcs [6]. The effect of disorder on the spectral and transport properties of these materials is also remarkable (for recent review see [7]). Dimensional arguments suggest that weak disorder can be neglected close to the band crossing point [8–11]: the density of states (DOS) remains vanishing quadratically at the nodal point (up to exponentially small corrections due to rare events [12–14]), the optical conductivity vanishes linearly with frequency, and the transport is pseudoballistic. However, a strong disorder drives the semimetal into a diffusive phase (if we neglect scattering between different nodal points) with a finite DOS [15], finite zero-frequency optical conductivity [16] and diffusive transport [17] at zero energy.

The disorder-driven quantum transition from a single-cone Weyl semimetal to a diffusive phase is a transition of a new type, which is different from the Anderson transition [7]. Its understanding has triggered many different efforts, in particular applying standard perturbative renormalization group (RG) methods using both replica and SUSY approaches [18–21]. These studies imply that the transition is controlled by a perturbative in $\varepsilon = d - 2$ fixed point (FP) of the $d$-dimensional $U(N)$ Gross-Neveu (GN) model taken in the unusual limit of a vanishing number of fermion flavors $N \to 0$. In this limit the standard chiral symmetry breaking transition of the GN model for interacting massless fermions translates into the generation of a finite DOS at the nodal point of a disordered relativistic semimetal. However, the description offered by this approach is not satisfactory. On the other hand numerous numerical simulations of the semimetal-diffusive metal transition [22–26] demonstrate quantitative discrepancies with the predictions of the RG studies of the GN model [18–20]. On the other hand rare event effects, not included in the GN approach, were argued to play some role around the phase transition [12–14]. The physics of phenomena involving rare fluctuations usually requires the use of more sophisticated renormalization methods, in which one has to follow renormalization of the whole distribution rather than just a few cumulants and thus go beyond the Gaussian approximation [27].

It is the purpose of the present Letter to develop such a functional renormalization group (FRG) description of the disorder driven semimetal-diffusive metal transition. Through the use of such technique, we show that the conventional FP of the GN model previously used to describe the transition is dramatically unstable with respect to weak non-Gaussian disorder, typically present in real materials or numerical simulations. To that end we derive the flow equation for the characteristic function of the renormalized disorder distribution, perturbatively in $2 + \varepsilon$ dimensions. Quite remarkably, this flow equation can be cast in the form of the well-known porous medium equation (PME) [28]. Building on the extensive mathematical studies of this non-linear diffusion equation, we find that the FP governing the phase transition is crucially different from the GN FP and non-analytic. Moreover, the non-linear diffusion description which stems from the FRG flow not only describes the transition but also elucidates the non-analytic mechanism of the DOS generation at the nodal point which is not accessible within the usual RG treatment.

Model. - We start from the imaginary time action of relativistic fermions moving in a $d$-dimensional space in...
the presence of an external potential $V(r)$

$$S = i \int d^d x d\tau \bar{\psi}(x, \tau) (\partial_\tau - i\gamma_j \partial_j + V(x)) \psi(x, \tau), \quad (1)$$

where $\bar{\psi}$ and $\psi$ are independent Grassmann fields and $\tau$ is the imaginary time. $\gamma_j$ are elements of a Clifford algebra satisfying the anticommutation relations: $\gamma_j \gamma_k + \gamma_k \gamma_j = 2\delta_{jk} \mathbb{I}$ ($j, k = 1, ..., d$), which reduce to the Pauli matrices $\gamma_j = \sigma_j$ in $d = 3$. The disorder potential is assumed to be uncorrelated in space, and thus, $\langle W(\omega) \rangle$ goes from $m$ to 0. The renormalized Green’s function corresponding to action (2) is

$$G_{\alpha\beta}(k) = \delta_{\alpha\beta}/(\gamma_j k_j - i\omega - iW'(0)),$$

where $\bar{\Theta}(x) = \sum_{\alpha=1}^{N} \bar{\psi}_\alpha(x)\psi_\alpha(x)$ is the local density of fermions.

Renormalization. To derive the FRG equations we use the effective average action formalism developed by Wetterich [30] together with $\varepsilon = d - 2$ expansion. Introducing the IR cutoff in the form of mass $m$ we obtain the flow equation for the characteristic function [31]

$$-m\partial_m W(\Theta) = 2m^z \left( \Theta W'(\Theta)W''(\Theta) - \tilde{N} W'(\Theta)^2 \right) \quad (3)$$

where $\tilde{N} = \frac{N}{m} \text{tr} \mathbb{I}$. $\Theta$ is the expectation value of $\Theta(x)$, and $m$ goes from $m_0$ to 0. The renormalized Green’s function corresponding to action (2) is $G_{\alpha\beta}(k) = \delta_{\alpha\beta}/(\gamma_j k_j - i\omega - iW'(0))$. For physically relevant disorder distributions the bare characteristic function $W(\Theta)$ is analytic and satisfies $W'(0) = 0$. Hence, the bare DOS given by $\rho(\omega) = -1/\pi \text{Im} \int_k G_{\alpha\alpha}(k, \omega)$ vanishes at zero energy [14]. However, as we will see later the renormalized characteristic function can develop a cusp at the origin, and thus, generate a non-vanishing DOS at zero energy.

To demonstrate how one can recover the conventional FP of the $U(N)$ GN model we rewrite the FRG equation (3) in dimensionless form by substituting $W(\Theta) = m^{z+\varepsilon} w(\theta)$ and $\Theta = m^{1+\varepsilon} \theta$. This gives

$$-m\partial_m w(\theta) = (2 + \varepsilon) w(\theta) - (1 + \varepsilon) \theta w'(\theta) + 2 \left( \theta w'(\theta) w''(\theta) - \tilde{N} w'(\theta)^2 \right). \quad (4)$$

The $U(N)$ GN model corresponds to the model (2) with $W(\Theta)$ being a quadratic function, so that the conventional FP of the GN model can be easily identified with $w^*(\theta) = \varepsilon \theta^2/[8(1 - N)]$. If we restrict $w(\theta)$ to a quadratic function the FP has only one unstable direction (the amplitude) and thus it describes a transition for $d > 2$. To check the stability in functional space we linearize the flow (4) in the vicinity of this conventional FP. Expanding the characteristic function in a Taylor series we find that its derivatives $w^{(n)}(0)$ are the coupling constants coupled to the scaling operators $\theta^n$ corresponding to the higher cumulants of disorder distribution. Using (4) we can calculate their scaling dimensions $[w^{(n)}(0)] = 2 + \varepsilon - n(1 + \varepsilon) + \varepsilon n(2N - n)/(2N - 2)$ which are in agreement with diagrammatic [20] and conformal field theory [32] results. The coupling constant is relevant if its scaling dimension is positive. Hence in the limit of $N \rightarrow 0$, which describes the disordered relativistic semimetal, infinitely many relevant operators corresponding to higher order cumulants of the disorder distribution are identified signaling the relevance of rare configurations of disorder at the transition.

**Zero N limit and porous medium equation.** - Since in the limit of $N \rightarrow 0$ the conventional GN FP becomes unstable in infinitely many directions, it cannot control a continuous transition. Nevertheless, it is premature to conclude that the transition is smeared out or first order. A direct numerical integration of the rescaled flow equation (4), however, failed to find any physical FP different from the conventional one. As we will see below, this can be explained by the fact that the FP we are looking for has a non-analytical behavior at the origin additionally to the absence of boundary conditions at infinity. Notice, however, that if the large $\theta$ asymptotics of the FP $w^*(\theta)$ was known then the whole FP could be computed by numerical integration of Eq. (4). Fortunately, introducing the "time" $t = (m_0^d - m^e)/\varepsilon$, the "coordinate" $r = \sqrt{2\tilde{N}}$ and the "density profile" $u = W'(\theta)$ we can rewrite the unrescaled flow equation (3) in the form of a 2D non-linear diffusion equation

$$2\partial_t u(r,t) = \frac{1}{r} \partial_r r \partial_r u^2(r,t) = \Delta u^2(r,t) \quad (5)$$

with the superimposed radial symmetry. Since $m$ changes from $m_0$ to 0 one has to stop the evolution of the density profile $u(r,t)$ at the maximal observation time $T_0 = m_0^e/\varepsilon$. Equation (5) is the 2D PME which has been intensively studied by mathematicians for several decades [28]. Due to the presence of degeneracy points (regions where $u = 0$ and thus vanishing diffusion constant) the PME exhibits remarkable non-linear phenomena. They include finite velocity propagation of fronts separating the regions with zero and non-zero $u$ [33], waiting times before the front starts to move [34], and self-focusing solutions describing shrinking of holes in the support of $u$ [35] with post-focusing accumulation of diffusing particles [36]. Following the route paved by these mathematical studies we look for a backward self-similar
solution (BSS) to Eq. (5) which has the form

$$u(r, t) = (T - t)^{2\delta - 1} F(\zeta), \quad \zeta = \frac{r}{(T - t)\tau}.$$  \hfill (6)

The self-similar solutions to the PME play a special role since they lead to a universal large time behavior. It is straightforward to identify the BSS (6) with a FP solution \(w(\theta)\) to the rescaled FRG equation (4) setting \(\delta = (1 + \varepsilon)/(2\varepsilon), T = T_0\) and \(F(\zeta) = \varepsilon w' (\zeta^2/(2\varepsilon^2))\). Then the rescaled FRG equation (4) becomes

$$\partial_\tau F(\zeta) = F(\zeta) \left[ F''(\zeta) + \frac{1}{\zeta} F'(\zeta) \right] + F'(\zeta)^2 - \delta_\varepsilon F'(\zeta) + (2\delta - 1) F(\zeta),$$  \hfill (7)

where we have defined \(\tau = -\ln(T - t)\) such that \(\varepsilon \partial_\tau = -m \partial_m\). For a BSS \(F(\zeta)\) the r.h.s of Eq. (7) identically vanishes. The conventional GN FP corresponds to the BSS with \(F(\zeta) = \zeta^2/8\). One may get impression that re-writing the FRG equation (4) in the form (7) is just a beautiful mathematical trick which connects two a priori unrelated problems. However, there is much more to it than that. Indeed, while the BSS (6) translates into a FP at \(T = T_0\), as we will see below, its dependence on \(T\) also provides an explicit expression for the flow of the whole disorder distribution along a single unstable direction. Moreover, the non-trivial BSS (6) can be captured by the phase-plane formalism [28] which is a powerful tool for analysis of the PME (5). To that end we define the phase variables \(Z\) and \(Y\) as \(F(\zeta) = -\zeta^2 Z(\zeta)\) and \(Y(\zeta) = -2(2Z(\zeta) + \zeta Z'(\zeta))\). They satisfy autonomous first order differential equations [31] whose solution for \(\delta = 1\), i.e. \(d = 3\), is shown in Fig 1. In the phase-plane \((Z, Y)\) the BSS is represented by an integral curve which connects the singular point \((0, 0)\) controlling the large \(\zeta\) behavior and a limiting cycle around the singular point \((-\frac{1}{2}, \frac{1}{2})\) corresponding to the conventional GN FP (see Inset of Fig 1). Although the function \(Z(\zeta)\) is infinitely oscillating at the origin, the corresponding profile function \(F(\zeta)\) is surprisingly monotonic as one can see in Fig. 2. It grows as \(F(\zeta) \sim \zeta^{2-1/\delta}\) for large \(\zeta\) and is strongly non-analytic at \(\zeta = 0\). This explains why the non-trivial FP can be easily overlooked when solving numerically the FRG equation (4). The new non-analytic FP exists only for \(\delta > \delta_c \approx 0.8563265\), i.e. only below the critical dimension \(d_c \approx 3.4\), and thus controls the transition in \(d = 3\).

**Stability analysis.** - We now study the stability of the new non-analytic FP. To that end we add to the BSS (6) a time dependent perturbation \(F(\zeta; \tau) = F(\zeta) + \phi(\zeta)e^{\lambda \tau}\), where \(\lambda\) is the stability eigenvalue and \(\phi\) is the corresponding eigenfunction. Substituting it into Eq. (7) and linearizing around the BSS we arrive at

$$-Z\phi + [2Y - \delta] \phi + [2\delta - 1 - \lambda + 2Y + \dot{Y}] \phi = 0,$$  \hfill (8)

where the dots stand for the logarithmic derivatives, \(\dot{X} \equiv dX/d\ln \zeta\). In order to obtain the stability spectrum of the FRG FP one has to impose the boundary condition at \(\zeta = 0\) using additional physical arguments [37]. Here we look for perturbations originating from higher order cumulants. Choosing \(\phi_n(\zeta) = \zeta^n f_n(\zeta), n = 2, 4, 6, \ldots\) such that the functions \(f_n(\zeta)\) are bounded for \(\zeta \to 0\) but not necessarily analytic we render the spectrum discrete. Numerical solution of Eq. (8) shows that only the eigenvalue corresponding to \(n = 2\) is positive, and thus, the non-analytic FP we have found indeed is a critical FP describing the disorder driven transition [31].

Remarkably, the relevant eigenvalue and eigenfunction can be identified from general symmetry considerations.

Let \(u(r, t)\) be a BSS with profile \(F(\zeta)\) and waiting time \(T\). Owing to the time-translation invariance of the PME (5) we can shift \(T \to T + \Delta T\) to obtain another
BSS: \( u(r, t) \rightarrow u(r, t) + \Delta T \partial_T u(r, t) \). From Eq. (6) we find that \( \partial_T u(r, t) = (T - t)^{2\zeta - 1} \phi_2(\zeta) e^T \) where
\[
\phi_2(\zeta) = (2\delta - 1)F(\zeta) - \delta \xi F'(\zeta).
\]

It is straightforward to see that (9) is the eigenfunction of Eq. (8) which corresponds to the only positive eigenvalue \( \lambda_2 = 1 \). Thus, \( T_0 - T \) is the only relevant parameter which controls the transition. Taking into account the relation between \( \tau \) and \( m \) we can find the correlation length exponent \( \nu^{-1} = \varepsilon \lambda_2 \). Although the non-analytic FP can be always expressed as a BSS (6) with \( T = T_0 \), higher loop order corrections to the critical exponents are expected.

**Post-focusing regime and DOS generation.** - The inverse waiting time \( T^{-1} \) determined by the full bare disorder distribution \([38]\) turns out to be a natural measure of the disorder strength. If the bare disorder is weak \( (T > T_0) \) the system is in the semimetal phase, while for strong disorder \( (T < T_0) \) it is in the diffusive phase. The disorder is critical for \( T = T_0 \). In order see how the DOS at the zero energy is generated by the FRG flow the BSS (6) of PME (5) corresponding to \( T < T_0 \) has to be continued analytically from \( t < T \) to \( t > T \). Recalling the asymptotic behavior \( F(\zeta) \sim \zeta^{2-1/\delta} \) for \( \zeta \rightarrow \infty \) we find by the continuity that \( u(r, t = T) = c^* r^{(2\zeta - 1)/\delta} (1 + o(1)) \) for \( r \rightarrow 0 \). In the post-focusing regime, i.e. for \( t > T \), the fictional particles, whose nonlinear diffusion is described by the PME (5), start to accumulate at the origin. This is described by a forward self-similar solution (FSS) \([36]\):
\[
\phi_2(\zeta) = (2\delta - 1)F(\zeta) - \delta \xi F'(\zeta).
\]

The FSS (10) can be found using the same phase-plane formalism \([31]\). It implies that \( \tilde{F}(0) = \text{const} \) and \( \tilde{F}(\tilde{\zeta}) \sim \tilde{\zeta}^{2-1/\delta} \) for \( \tilde{\zeta} \rightarrow \infty \) (see Fig. 2). Since \( W' (0^+) = u(0, t) \neq 0 \) in the post-focusing regime, the FSS (10) describes the diffusive phase of relativistic fermions and allows one to compute explicitly the DOS at zero energy. We find that close to the transition, i.e. for \( T_0 - T \ll T_0 \) the DOS at zero energy is given by \( \rho(0) \sim (T_0 - T)^{\beta} \) with the order parameter critical exponent \( \beta = 2\delta - 1 \). Assuming that the hyperscaling relation \( \beta = (d - \nu) \nu \) is not broken we obtain the dynamic critical exponent as \( z = 1 + \varepsilon + O(\varepsilon^2) \).

**Conclusions and outlook.** - We have developed a functional renormalization group approach to the semimetal-diffusive metal transition in disordered Weyl fermions. We have shown that the previously studied fixed point corresponding to the Gaussian distribution of disorder is unstable, demonstrating the relevance of rare fluctuations at this transition. This instability manifests itself as the proliferation of infinite number of higher order cumulants in the running disorder distribution, even if starting from a pure Gaussian distribution \([31]\). We have found that the transition is controlled by a new non-analytic fixed point distribution. In order to study this transition we have established a connection between our one-loop FRG equation and the celebrated non-linear diffusion equation (PME). Using the phase-plane formalism we have obtained the self-similar solution, controlling the transition, and identified the mechanism of spontaneous generation of a finite DOS at zero energy in the diffusive phase. This is a remarkable result in view of the lack of a mean-field description of the semimetal-diffusive metal transition similar to that for the GN model in the limit of large \( N \) \([39]\). More generally, our approach may resolve some ambiguities encountered in other problems such as non-linear sigma models \([40–42]\) where existence of infinitely many relevant (high-gradient) operators has been shown at the conventional FP in \( 2 + \varepsilon \) dimensions \([43]\).

Let us finally comment on the effect of spatial correlations of disorder on the possible avoidance of this transition, in relation with recent works \([12–14]\). In this Letter we have restricted our consideration to uncorrelated disorder. The existence of the continuous transition is connected to the presence of degenerate points in the FRG flow equation rewritten as an effective nonlinear diffusion equation. In these points the diffusion is suppressed until reaching a finite waiting time which introduces a diverging length scale in the problem rendering it critical. The disorder correlations introduced in \([12]\) as a cutoff to regularize the instanton solutions at small scales, can destabilize the degenerate points of the FRG equation. In this case the waiting time phenomenon of the non-linear diffusion is superimposed with a very slow creep-like motion which generates a finite but exponentially small DOS in the semimetal phase as predicted by the instanton calculations \([12]\). However, despite the formal avoidance of the transition, the critical properties of the underlying FP are expected to dominate the experimentally relevant typical value of physical properties. In particular, one expects that the critical wave functions exhibit multifractality at the transition in a similar fashion to the Anderson transition \([44]\). While the multifractal spectrum expected at the conventional FP was found in \([20, 45]\), the spectrum associated with the non-analytic FP identified in this Letter remains to be computed.

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SUPPLEMENTAL MATERIAL

The Supplemental Material is organized as follows. In Sec. A we derive the Nonperturbative functional RG (NPRG) equation directly in \( d \) dimensions. Using these results in Sec. B we obtain the flow equation which is perturbative in \( \varepsilon = d - 2 \). In Sec. C we show how the FRG flow equation can be mapped onto the inviscid Burgers equation in the limit of large \( N \). In Sec. D we explain the difference between the classical and weak (or integral) solutions. In Sec. E we give the details of the phase-plane analysis of the PME which allows us to find a BSS corresponding to a non-analytic FP of the FRG equation. Section F contains details of the stability analysis. In Sec. G the phase-plane formalism is used for studying the post-focusing regime which describes generation of the DOS at zero energy in the diffusive phase.

A. NONPERTURBATIVE FUNCTIONAL RG APPROACH

Here we give an overview of the derivation of the RG flow equation for the characteristic function starting from the Wetterich effective average action approach [1]. All the notions are fairly standard and technical and more details can be found in [2, 3] where the account is given of the GN model, which is formally very similar to the problem we study. One has to define the generating functional using the replicated bare level action (2), with an addition of the infrared regulator [4]

\[
\Delta S_k[\bar{\psi}, \psi] = \int d^d x d^d y \bar{\psi}^i_{\alpha}(x) R_{\alpha,\beta}(x - y) \psi^j_\beta(y),
\]  

(S.1)

which includes \( k \) as the cutoff wave vector and \( \alpha \) and \( \beta \) are replica indices. The regulator modifies the bare action by decoupling the fast modes \( (q > k) \) from the slow \( (q < k) \) ones by giving a large mass to the latter and leaving the first unaffected. With the bare action modified in this way and with the addition of external sources, one can construct a \( k \) dependent generating functional as a path integral

\[
Z_k[\bar{B}], B = \int D\bar{\psi} D\psi e^{-S - \Delta S_k + \int d^d x \bar{\psi}(x) B(x) + \int d^d x \bar{B}(x) \psi(x)},
\]  

(S.2)

where we used a shortcut notation \( \bar{\psi} \psi := \sum_{i,\alpha} \bar{\psi}^i_\alpha \psi^i_\alpha \), and similarly for the terms with external sources.

To define the effective average action \( \Gamma_k \), explicitly dependent on the cutoff \( k \), one performs the Legendre transform on the logarithm of the generating functional

\[
\Gamma_k[\bar{\Psi}, \Psi] + \ln(Z_k[\bar{B}], B) = \int d^d x \bar{\Psi}(x) B(x) + \int d^d x \bar{B}(x) \Psi(x) + \Delta S_k[\bar{\Psi}, \Psi]
\]  

(S.3)

where \( \Psi = \langle \psi \rangle \) and \( \bar{\Psi} = \langle \bar{\psi} \rangle \) are formal ensemble averages of \( \psi \) and \( \bar{\psi} \), respectively.

To derive the exact flow equation for the effective action (S.3) we first combine the two fermion fields in a 4-component spinor \( (\bar{\Psi}_a, \Psi_a) \) and then construct the regulated second derivatives matrix \( \Gamma^{(2)} \) as [1]

\[
\Gamma^{(2)}_{\alpha,\beta}(x, y) + R_{k;\alpha,\beta}(x - y) = \left( \frac{3}{3} \frac{\delta}{\delta \bar{\Psi}_a(x)} (\bar{\Gamma} + \Delta S_k) \frac{3}{3} \frac{\partial}{\partial \Psi_a(x)} (\Gamma + \Delta S_k) \frac{3}{3} \frac{\delta}{\delta \bar{\Psi}_a(y)} \frac{3}{3} \frac{\partial}{\partial \Psi_a(y)} (\Gamma + \Delta S_k) \frac{3}{3} \right).
\]  

(S.4)

Here we have omitted the spinor indices and used the notation of the left and right derivatives, which is useful when considering the action depending on Grassmann (anticommuting) variables. We calculate the “full” \( \bar{\Psi}, \Psi \)-dependant exact propagator by inverting the expression (S.4) as

\[
\mathcal{G}_{k;\alpha,\beta}(x - y) = (\Gamma^{(2)} + R_k)^{-1}_{\alpha,\beta}(x - y).
\]  

(S.5)

The exact flow equation for the effective action can be written as [1]

\[
\partial_s \Gamma_s = -\frac{1}{2} \text{Tr} \left\{ (\partial_s R_k) \mathcal{G}_k \right\},
\]  

(S.6)

where the symbol \( \text{Tr} \) stands for the trace over all indices and integration over space. In Eq. (S.6) we have also introduced the RG time \( s = \ln(k/\Lambda) \) going from 0 for the microscopic theory to \( -\infty \) for a theory where all the degrees
of freedom have been integrated out. Here $\Lambda$ is the UV cutoff, e.g. the width of the Brillouin zone. The minus in Eq. (S.6) is the usual extra minus associated with a fermionic loop.

To make the flow equation for $\Gamma$ tractable, one needs an approximation scheme for the effective average action, since the flow equation (S.6) can not be solved exactly. The most simple approximation is the so-called local potential approximation (LPA) in which one takes into account the nonlocal space dependence only in the gradient term of the single replica term in the bare action (2). It is known that already this approximation is able to properly capture the long-wavelength scaling behavior of many systems [5]. We start from the following ansatz for the effective average action

$$\Gamma_k = \int dx i\bar{\Psi}(x)(-i\gamma^\beta \partial_\beta - i\omega)\Psi(\omega + W_k(\theta(x))). \quad \text{(S.7)}$$

where $\theta(x)$ is the local density as in Eq. (2) of the main paper and $W_k$ is the renormalized characteristic function.

After a tedious, but straightforward derivation, which will be detailed elsewhere, one arrives at the following RG flow equation for the characteristic function

$$\partial_t W_s = -\frac{1}{2} \int_q \partial_q R_k(q) \left\{ \frac{2(q\cdot\tilde{q})N\text{tr}I}{\tilde{q}^2 + B^2} + \frac{4B(q\cdot\tilde{q})\Theta W''(\theta)}{(\tilde{q}^2 + B^2)[\tilde{q}^2 + B^2 + 2B\Theta W''(\theta)]} \right\}. \quad \text{(S.8)}$$

In Eq. (S.8), $B = \omega + W'(\theta)$, $I$ is the unity matrix in the spinor space (e.g. for a single 3D Weyl cone $\text{tr}I = 2$) and $\tilde{q}$ is the momentum modified by the regulator, defined in a similar way as in e.g. Ref. [5]. It is easy to that even if the bare disorder has a pure Gaussian distribution with $W(\theta) \sim \theta^2$ the higher order cumulants will be ultimately generated by the FRG flow due to the non-trivial denominator in the second term of Eq. (S.8).

B. PERTURBATIVE VERSION OF THE FLOW EQUATION

Analysis of Eq. (S.8) is rather complicated. In order to gain the physics insight we will restrict ourselves to functional but perturbative in $\varepsilon = d - 2$ RG equation. In principal this equation can derived by computing the diagrams similar to those introduced in Supplemental material of paper [6]. However, instead of direct computing diagrams for a field theory with infinitely many coupling constants represented by the high order cumulants one can use the NPRG flow equation (S.8) as a starting point to derive the one-loop perturbative FRG equation in $d = 2 + \varepsilon$. To that end we rewrite Eq. (S.8) in an infinitesimal form replacing the cutoff function $\partial_q R_k(q)$ by imposing dimensional regularization [7]. This gives the one-loop correction to the characteristic function

$$\delta W(\theta) = 2 \int_q \left\{ \tilde{N} - \frac{W'(\theta)^2 \tilde{N}}{q^2 + B^2} + \frac{q^2 \Theta W'(\theta) W''(\theta)}{(q^2 + B^2)[q^2 + B^2 + 2B\Theta W''(\theta)]} \right\}. \quad \text{(S.9)}$$

We now use the dimension regularization properties in order to simplify Eq. (S.9). Since $\int_q 1 = 0$ in dimensional regularization, the first term in r.h.s. of Eq. (S.9) vanishes. To one-loop order only the poles in $\varepsilon$ should be kept in the rest two terms. We find

$$\int_q \frac{1}{q^2 + B^2} = -S_d \frac{B^\varepsilon}{\varepsilon} + O(\varepsilon). \quad \text{(S.10)}$$

and

$$\int_q \frac{q^2}{(q^2 + B^2)(q^2 + B^2 + \text{const}B)} = -S_d \frac{B^\varepsilon}{\varepsilon} + O(1), \quad \text{(S.11)}$$

where $S_d$ is the area of $d$-dimensional sphere divided by $(2\pi)^d$. It is easy to see that $B$ plays the role of IR cutoff in Eqs. (S.10) and (S.11). Replacing it by mass $m$ for convenience and collecting all the terms we arrive at

$$\delta W(\theta) = -2S_d \frac{m^\varepsilon}{\varepsilon} \left[ \Theta W'(\theta) W''(\theta) - W'(\theta)^2 \tilde{N} \right]. \quad \text{(S.12)}$$

One can now identify the terms in Eq. (S.12) with the contributions coming from the one-loop diagrams in the perturbative loop expansion. To compute the one-loop perturbative FRG flow equation we follow Refs. [8, 9] and define the renormalized characteristic function as $W_R(\theta) = W(\theta) + \delta W(\theta)$. Taking a derivative $-m\partial_m$ on both sides and reexpressing the obtained equation in terms of the renormalized characteristic function $W_R$ we obtain the flow equation (3) from the main paper (after including $S_d$ into redefinition of $W_R$).
C. LARGE N LIMIT AND BURGERS EQUATION

Here we discuss how the conventional GN FP describes the chiral transition of the N-flavor interacting relativistic massless fermions in the limit of large N. Since this transition is driven by repulsive interactions, the GN FP $w^*$ is negative in our convention. We find that for $d > 2$ the only relevant coupling constant is the strong strength of fermion repulsion $-w''(0)$ which controls the transition. The corresponding positive eigenvalue gives the correlation length exponent $\nu$. The non-analytic nature of the transition can be revealed by introducing $y(r) = W'(\Theta)$, $\Theta = 4N r$ and the "time" $t = (m_0^2 - m^2)/\varepsilon$, directly in the unrescaled flow equation (3) in the main text. Then, in the limit $N \to \infty$, it transforms into the inviscid Burgers equation [10]

$$\partial_t y(r) + y(r)y'(r) = 0,$$

which can be solved by the method of characteristics. The reasonable bare interactions between fermions correspond to the initial condition with an odd function $y_0(r)$ satisfying $y_0(0) = 0$. After waiting (breaking) time $T^* = 1/|y_0'(0)|$ the "velocity" profile $y(r)$ develops a shock exactly at the origin $r = 0$. The appearance of the discontinuity $y(0^+) = -y(0^-) > 0$ preserves the single-value property of the classical solution and transforms it into the so-called weak (or integral) solution (see Section D). The shock leads to $W'(0^+) \neq 0$, and thus, to the dynamical fermion mass generation and spontaneous chiral symmetry breaking. However, since $0 \leq m \leq m_0$ one has to stop the evolution of the profile $y(r)$ at the maximal observation time $T_0 = m_0^2/\varepsilon$. Thus, we conclude that the system is in the symmetric phase if the bare interactions are so small that the waiting time $T^*$ is larger than the maximal observation time $T_0$. If $T^* < T_0$ then the system is in the symmetry broken phase and the shock size at final time $T_0$ gives the value of the order parameter.

A similar mechanism was recently proposed for the chiral symmetry breaking transition in the 4D Nambu-Jona-Lasinio model [11].

D. WEAK VS. STRONG SOLUTION

The important physical implications obtained from the flow equation for the characteristic function $W$ detailed in the main text, rely on the spontaneous generation of a cusp at the origin. This is possible only if we extend the definition of solution to the FRG flow equation from the classical strong solution to the so-called “weak solution” [12]. Contrary to the strong solution a weak solution may contain discontinuities and may not be differentiable. To define a weak solution to a PDE one usually needs to reformulate the problem in an integral form.

As an example, let us consider the Burgers equation (S.13) which has a form of a conservation law. The solution to the first order PDE (S.13) constructed by the method of characteristics becomes a multivalued function beyond the so-called breaking time. The multivalued parts can be eliminated by inserting shocks using the equal area rule which is a result of conservation. In other words the integral of the discontinuous weak solution with shock must be the same as the integral of the auxiliary multivalued solution. In order to avoid the necessity of using the equal area rule let us integrate the Burgers equation (S.13) convoluted with a smooth bounded test function $\phi(r,t)$. Then, after integrating by parts we arrive at

$$\int_0^\infty dt \int_{-\infty}^\infty dr \left( y(r,t) \frac{\partial \phi(r,t)}{\partial t} + \frac{1}{2} y(r,t)^2 \frac{\partial \phi(r,t)}{\partial r} \right) + \int_{-\infty}^\infty dr \ y(r,0)\phi(r,0) = 0,$$

This equation has no explicit derivatives of $y(r, t)$. By definition, a weak solution to PDE (S.13) is a function $y(r, t)$ which satisfies the integral equation (S.14) for any smooth and bounded test function $\phi(r, t)$. The weak solution $y(r, t)$ does not have to be differentiable everywhere. Any strong solution is also a weak solution but not vice versa. A weak solution to the FRG equation can exist and describe infrared physical quantities even when the flow equation does not have a strong solution [11]. The same arguments are also applied to the PME [13].

E. THE NON-ANALYTIC FIXED POINT: PHASE PLANE FORMALISM

We now look for the BSSs of the form (6) to the PME (5). The corresponding profile functions $F(\zeta)$ satisfy the nonlinear ODE

$$F'(\zeta) \left[ F''(\zeta) + \frac{1}{\zeta} F'(\zeta) \right] + F'(\zeta)^2 - \delta \zeta F'(\zeta) + (2\delta - 1) F(\zeta) = 0.$$  

(S.15)
In order to solve the ODE (S.15) we will use the phase-plane formalism [13]. We introduce the phase variables \( Z \) and \( Y \),

\[
F(\zeta) = -\zeta^2 Z(\zeta), \quad Y(\zeta) = -(2Z(\zeta) + \zeta Z'(\zeta)).
\]  

(S.16)

The phase variables satisfy the autonomous first order differential equations

\[
\dot{Z} + 2Z + Y = 0,
\]

(S.17)

\[
Z\dot{Y} - \dot{Z}(\delta - Y) + (4Y - 1)Z = 0,
\]

(S.18)

where \( \dot{X} \equiv dX/d\ln \zeta \). Dividing Eq. (S.18) by Eq. (S.17) we obtain an equation for \( Y(Z) \)

\[
\frac{dY}{dZ} = \frac{Z(2\delta - 1) + 2YZ + (\delta - Y)Y}{Z[2Z + Y]}.
\]  

(S.19)

Note, however, that \( Y(Z) \) is not a single value function [14] for arbitrary \( \delta \). The relation between \( \zeta \) and \( Z \) is then given by

\[
\frac{d\ln \zeta}{dZ} = -\frac{1}{2Z + Y}.
\]  

(S.20)

Any solution of Eq. (S.19) determines an integral curve in the phase plane \((Z, Y)\), which represents a BSS of a certain kind. Thus, any FP of the FRG equation (4) corresponds to a particular integral curve in phase plane \((Z, Y)\). A single integral curve passes through any regular point of the phase plane while a curve corresponding to a FP of the FRG flow has to satisfy certain boundary conditions at its ends. To find out which curve corresponds to a physical FP one needs to know the behavior in the vicinity of singular points of system (S.17)-(S.18). Since \( F(\zeta) > 0 \) we are interested in the region \( Z \leq 0 \) which is shown in the left panel of Fig. 3. In this region the system (S.17)-(S.18) has three singular points \( O: (0, 0), A: (0, \delta) \) and \( B: (-\frac{1}{2}, \frac{1}{2}) \). It is easy to see that the singular point \( B \) is itself an integral curve which corresponds to the canonical GN FP independently of \( \delta(\zeta) \) and which is given by

\[
F(\zeta) = \frac{\zeta^2}{8}, \quad w(\theta) = \frac{1}{8} \zeta^2 \theta^2 \quad (\tilde{N} = 0).
\]  

(S.21)

The singular points \( A \) and \( B \) control the behavior of \( F(\zeta) \) at small \( \zeta \), while the point \( O \) describes its asymptotic behavior for \( \zeta \to \infty \). It turns out that the new non-analytic FP corresponds to the integral curve which connects singular point \( B \) (or limiting cycle around it) and point \( O \) (see Fig. 3). However, numerical integration of Eqs. (S.17) and (S.18) shows that this curve exists only for \( \delta > \delta_c \approx 0.8563265 \). At \( \delta = \delta_c \) the curve passes from \( O \) directly to the third singular point \( A \) instead of spiraling endlessly around point \( B \). Linearizing the flow (S.17)-(S.18) around point \( B \) we find that there are three critical values of \( \delta \): \( \delta_c = 1 \pm 1/\sqrt{2} \) and \( \delta_0 = 1 \).

(i) The point \( B \) is an unstable node for \( \delta > \delta_c \), so that the curve goes straight to the point \( B \). In this case \( Y(Z) \) is a single-valued function which could be found from numerical integration of Eq. (S.19). In this case it has a Taylor expansion around \( B \) which is given by

\[
B: \quad Y(Z) = \frac{1}{4} - \alpha_1 \left( Z + \frac{1}{8} \right) + O \left( \left( Z + \frac{1}{8} \right)^2 \right) \quad \text{with} \quad \alpha_1 = -2 \left( \sqrt{2/\sqrt{2}} - 4\delta + 1 - 2\delta + 1 \right).
\]  

(S.22)
Although $Y(Z)$ is a single-value analytic function for $\delta > \delta_c$ the corresponding FP is non-analytic. Substituting (S.22) into Eq. (S.20) and integrating the resulting ODE we find that the resulting FP functions $Z(\zeta)$ and $F(\zeta)$ are non-analytic at the origin
\[
Z(\zeta) \approx -\frac{1}{8} + C_1 \zeta^{-2+\alpha_1}, \quad F(\zeta) \approx \frac{1}{8} \zeta^2 - C_1 \zeta^{\alpha_1} \quad \text{for} \quad \zeta \to 0. \quad (S.23)
\]
Here $C_1$ is an arbitrary constant which comes from the integration of Eq. (S.20). While the function $Y(Z)$ is unique for any fixed $\delta \geq \delta_c$, the function $F(\zeta)$ has a free parameter which fixes the value of $C_1$ in Eq. (S.23). This is related to existence of zero eigenvalue in the stability spectrum that is discussed in the next section.

(ii) The point $B$ is an unstable focus for $\delta_0 < \delta < \delta_c$, so that the integral curve spirals finite time towards $B$ until reaching it. In this case $Y(Z)$ is a multi-valued function so that the functions $Z(\zeta)$ and $F(\zeta)$ are strongly non-analytic at the origin.

(iii) The point $B$ is a stable focus for $\delta_c < \delta < \delta_0$. The integral curve spirals infinitely many times around $B$ reaching a limiting cycle. This insures a very strong non-analyticity of the functions $Z(\zeta)$ and $F(\zeta)$ at the origin.

For $\delta < \delta_c$ the point $B$ represents the solutions which asymptotically satisfy boundary condition $F(\zeta) \sim \zeta^2$ at $\zeta \to 0$, but the function $F(\zeta)$ is not analytic at $\zeta = 0$. In order to impose non-vanishing boundary conditions for $F(\zeta)$ at large $\zeta$ the integral curve has to pass through the point $O$ which represents the behavior for $\zeta \to \infty$. For $Z < 0$ it is a saddle point so that it can be reached only along a single direction that determines the asymptotic behavior of $F(\zeta)$ at infinity. Expanding around $O$ we obtain
\[
O: \quad Y(Z) = \frac{(1-2\delta)Z}{\delta} + \frac{2(2\delta-1)^2Z^2}{\delta^3} + O(Z^3), \quad (S.24)
\]
\[
Z(\zeta) \approx C_2 \zeta^{-1/\delta}, \quad F(\zeta) \approx C_2 \zeta^{2-1/\delta} \quad \text{for} \quad \zeta \to \infty, \quad (S.25)
\]
where $C_2$ could be related to $C_1$ by matching of both asymptotic solutions (S.23) and (S.25). The integral curves connecting the point $O$ with point $B$ (or a limit cycle around $B$) are shown in the left panel of Fig. 3 for several values of $\delta \geq \delta_c$ corresponding to different regimes. The corresponding function $F(\zeta)$ representing the new non-analytic FP of the FRG is shown in the right panel of Fig. 3. The function $F(\zeta)$ is strongly non-analytic at the origin but surprisingly grows monotonically despite the fact that $Z(\zeta)$ could be oscillating function (see Fig. 1 in the main text). Once $\delta$ goes to $\delta_c$ the integral curve approaches the curve connecting the singular points $O$ and $A$. It corresponds to the self-focusing solution for the initial function $u$ with a finite hole in the support around the origin which shrinks to zero in a finite time $T$. There is no such non-analytic FP for $\delta < \delta_c$, i.e. above the critical dimension $d_c \approx 3.4$ (notice that it is only a one-loop estimation).

F. STABILITY OF THE NON-ANALYTIC FIXED POINT

We now study the linear stability of the BSS solution $F(\zeta)$ using the flow equation (7) which we recall here
\[
\partial_\tau F(\zeta; \tau) = F(\zeta; \tau) \left[ F''(\zeta; \tau) \right] + F' \left[ \frac{1}{\zeta} F'(\zeta; \tau) \right] + F' \left[ (2\delta - 1) F'(\zeta; \tau) + (2\delta - 1) F(\zeta; \tau) \right], \quad (S.26)
\]
where we define a new time $\tau = -\ln(T - t)$ which has nothing to do with the imaginary time in the action (1). To that end we linearize Eq. (S.26) around the BSS $F(\zeta)$ by adding perturbation $\phi$ as $F(\zeta; \tau) = F(\zeta) + \phi(\zeta)e^{\lambda \tau}$ and expanding to the linear order in $\phi$. This gives
\[
F(\zeta)\phi''(\zeta) + \left[ \frac{F(\zeta)}{\zeta} + 2F'(\zeta) - \delta \zeta \right] \phi'(\zeta) + \left[ 2\delta - 1 - \lambda + F''(\zeta) + \frac{F'(\zeta)}{\zeta} \right] \phi(\zeta) = 0. \quad (S.27)
\]
By inspecting the different terms in Eq. (S.27) we find that the eigenfunctions should satisfies boundary conditions $\phi(0) = 0$ and $\phi(\zeta) \sim \zeta^{(2\delta - 1 -\lambda)/\delta}$ for $\zeta \to \infty$. However, it turns out that without additional conditions the spectrum of Eq. (S.27) is continuous except for the case $\delta = \delta_c$. The focusing solution with $\delta = \delta_c$ (see the grey dashed line in the right panel of Fig. 3) belongs to the so-called BSSs of the second kind in the classification of Barenblatt and Zeldovich [13]. In this case the boundary conditions for $\phi(\zeta)$ deduced directly from the flow equation (S.27) are enough to render the spectrum discrete. The stability of this BSS of the second kind has been already studied in the mathematical literature [15–17]. Despite the fact that this solution is a marginal case in our problem let us outline the known results in order to get intuition about stability in the general case. It was shown that this BSS describes the generic disappearance of holes in the support (for 2D case) even if one starts from a non-radial initial
configuration $u(\vec{r}, t = 0)$. Notice that here we need a weaker stability since the radial symmetry is imposed by the problem. It was found that the spectrum has only three non-negative eigenvalues. The eigenvalue $\lambda = 1$ is related to the property that shifting the focusing time $T$ one arrives at another BSS. The eigenvalue $\lambda = \delta_c$ corresponds to a non-radial perturbation (forbidden in our problem) which shifts the focusing point from the origin $\vec{r} = 0$ to a finite $\vec{r}$. The eigenvalue $\lambda = 0$ corresponds to the fact that one can redefine $\zeta$ as $\zeta = r/[b(T - t)^{1/2}]$ with arbitrary $b$ and there is a freedom in the choice of $b$.

For a BSS of first kind ($\delta > \delta_c$) the boundary condition at $\zeta = 0$ has to be imposed using additional physical arguments [18]. The natural choice is considering the perturbations which resemble the presence of higher order cumulants in the bare disorder distribution. Since $F(\zeta) = \varepsilon u'(\zeta^2/(2\zeta^2))$ we impose

$$\phi_n(\zeta) = \zeta^n f_n(\zeta), \quad n = 2, 4, 6, ... \tag{S.28}$$

where functions $f_n(\zeta)$ are bounded for $\zeta \to 0$ but not necessarily analytic. Conditions (S.28) render the spectrum discrete.

Before we consider the whole spectrum let us first show that some eigenfunctions $\phi(\zeta)$ and the corresponding eigenvalues $\lambda$ can be found from general symmetry considerations similar to the case of BSS of second kind. Indeed, $u(r, t)$ given by Eq. (6) is a BSS solution to the PME (5) for arbitrary waiting time $T$. Thus, we have that

$$u(r, t) \to u(r, t) + \frac{\partial u(r, t)}{\partial T} \Delta T \tag{S.29}$$

is also a BSS for small $\Delta T$. Computing the derivative in Eq. (S.29) we find

$$\frac{\partial u(\vec{r}, t)}{\partial T} = (T - t)^{2\delta - 2} [2(\delta - 1)F(\zeta; \tau) - \delta \zeta \partial_\zeta F(\zeta; \tau) - \partial_\tau F(\zeta; \tau)]$$

$$= (T - t)^{2\delta - 1} [(2\delta - 1)F(\zeta; \tau) - \delta \zeta \partial_\zeta F(\zeta; \tau) - \partial_\tau F(\zeta; \tau)] e^\tau. \tag{S.30}$$

Substituting the BSS $F(\zeta; \tau) \to F(\zeta)$ we identify the eigenvalue $\lambda_2 = 1$ which corresponds to the eigenfunction

$$\phi_2(\zeta) = (2\delta - 1)F(\zeta) - \delta \zeta F'(\zeta). \tag{S.31}$$

It is easy to check that the function (S.31) indeed has the asymptotic behavior $\phi_2(\zeta) = \zeta^2 f_2(\zeta)$ where $f_2(\zeta)$ is bounded but non-analytical at $\zeta = 0$.

The FRG equation (4) has a property that if $w(\theta)$ is a FP then $b^2 w(\theta/b)$ is also a FP. Thus, the FRG equation has a line of FPs parameterized by $b$. In general this could result in scaling behavior with non-universal values of critical exponents. However, as we will see below the values of the critical exponents do not change along the line and they are universal at least to one-loop order. In terms of the PME this symmetry implies that if $u(\zeta; \tau)$ is a BSS to the PME then $b^2 u(r/b; t)$ is also a BSS. Using the infinitesimal form of this transformation

$$u(rb^{-1}, t) \to u(rb^{-1}, t) + \frac{\partial u(rb^{-1}, t)}{\partial b} \Delta b \tag{S.32}$$

we find

$$\frac{\partial u(rb^{-1}, t)}{\partial b} = (T - t)^{2\delta - 1} \frac{\partial}{\partial b} b^2 F(\zeta b^{-1}; \tau) = b(T - t)^{2\delta - 1} [2F(\zeta b^{-1}; \tau) - F'(\zeta b^{-1}; \tau) \zeta b^{-1}]. \tag{S.33}$$

Thus, the flow has zero eigenvalue $\lambda = 0$ which corresponds to the eigenfunction

$$\phi(\zeta) = 2F(\zeta) - \zeta F'(\zeta). \tag{S.34}$$

Note, however, that the eigenfunction (S.34) does not fulfill the additional condition (S.28) and thus does not affect the stability properties.

In order to study the properties of the whole spectrum we now rewrite Eq. (S.27) in terms of phase-variable $Z, Y$ as

$$-\zeta^2 Z(\zeta) \phi''(\zeta) + [-\zeta Z(\zeta) + 2\zeta Y(\zeta) - \delta \zeta] \phi'(\zeta) + [2\delta - 1 - \lambda + 2Y(\zeta) + \zeta Y'(\zeta)] \phi(\zeta) = 0, \tag{S.35}$$

where $\phi' = \partial_\zeta \phi(\zeta)$. It is convenient to introduce to $l = \ln \zeta$ and $\phi = \partial_l \phi(l)$ that gives Eq. (9), i.e.

$$-Z\phi + [2Y - \delta] \phi + [2\delta - 1 - \lambda + 2Y + \dot{Y}] \phi = 0. \tag{S.36}$$
we obtain
\[ \phi \]
Substituting the main text using scaling dimensions of the composite operators \( \theta \)
we can rewrite Eqs. (S.36) as
\[ \frac{1}{8}(2\tilde{n} - 2)^2 + \left[ \frac{1}{2} - \delta \right](2\tilde{n} - 2) + \left[ 2\delta - 1 - \lambda \right] = 0. \]  
(S.37)

Solving Eq. (S.37) we find that \( \epsilon \lambda = 2 + \epsilon - \tilde{n}(1 + \epsilon) + \frac{\epsilon \tilde{n}^2}{2} \) which coincides with the scaling dimensions \( [w^{(n)}(0)] \) found in the main text in the limit of \( N \to 0 \). Thus the canonical FP is indeed infinitely unstable in \( d > 2 \).

(ii) Stability of the non-analytic FP. – We now check the stability of the non-analytic FP. To that end we switch from \( \phi(l) \), \( Z(l) \) and \( Y(l) \) to \( \phi(Z) \) and \( Y(Z) \). Using that
\[ \frac{\dot{\phi}}{\dot{t}} = \frac{\partial \phi}{\partial Z} \dot{Z}, \quad \frac{\ddot{\phi}}{\dot{t}^2} = \frac{\partial^2 \phi}{\partial Z^2} + \left( \frac{\dot{Z}}{Z} \right)^2 \frac{\partial \phi}{\partial Z}, \]  
(S.38)

we can rewrite Eqs. (S.36) as
\[ -Z(2Z + Y)^2 \frac{\partial^2 \phi(Z)}{\partial Z^2} - \left[ 4Z^2 + 8Y Z - Z + Y^2 \right] \frac{\partial \phi(Z)}{\partial Z} + \left[ \frac{Y}{Z} \right] (Y - \delta - \lambda) \phi(Z) = 0. \]  
(S.39)

which should be solved simultaneously with Eq. (S.19) We now introduce the (inverse) logarithmic derivative of \( \phi(Z) \) as \( U(Z) = \frac{\phi'(Z)}{\phi(Z)} \). Then we arrive at the systems of three first order ODEs
\[ \frac{dY(Z)}{dZ} = \frac{Z(2\delta - 1) + 2Y(Z)Z + (\delta - Y(Z))Y(Z)}{Z[2Z + Y(Z)]}, \]  
(S.40)
\[ \frac{dU(Z)}{dZ} = 1 + \frac{4Z^2 + 8Y(Z)Z - Z + Y^2}{Z(2Z + Y(Z))^2} U(Z) - \frac{Y(Z)(Y(Z) - \delta - \lambda Z)}{Z^2(2Z + Y(Z))^2} U(Z)^2, \]  
(S.41)
\[ \frac{d\ln \zeta}{dZ} = -\frac{1}{2Z + Y(Z)}. \]  
(S.42)

Note that functions \( Y(Z) \) and \( U(Z) \) are single valued only for \( \delta > \delta_+ \) so that the stability analysis is more simple in this case. For the sake of simplicity we restrict our consideration here to this case, but the obtain conclusions are also applied for \( \delta_+ < \delta < \delta_+ \). We first study the asymptotic behavior of the eigenfunctions at small and large \( \zeta \). Expanding around \( 0 \), i.e. for \( \zeta \to \infty \) and thus \( Z, Y \to 0 \), we find
\[ O: U(Z) = \frac{Z}{\lambda + 1 - 2\delta} - \frac{Z^2 (16\delta^3 - 8\delta^2(\lambda + 1) - 4\delta + \lambda^2 + 2\lambda + 2)}{\delta^2(\lambda + 1 - 2\delta)^2} + O(Z^3), \]  
(S.43)
\[ \phi(Z) \approx C_3 Z^{-(2\delta - 1 - \lambda)}, \quad \phi(\zeta) \approx C_3' \zeta^{(2\delta - 1 - \lambda)/\delta}, \]  
(S.44)
Expansion around point $B$ ($Z_B = -\frac{1}{8}, V_B = \frac{1}{4}$), i.e. for $\zeta \to 0$, which exists only for $\delta > \delta_+$, reads

$$B: \quad U(Z) = \beta_1 \left( Z + \frac{1}{8} \right) + O \left( \left( Z + \frac{1}{8} \right)^2 \right), \quad (S.45)$$

$$\beta_1 = \frac{1}{1 + \delta (2\delta - 3) + (\delta - \frac{1}{2})\sqrt{4(\delta - 2)\delta + 2} - \frac{\sqrt{4(\delta - 2)\delta + 2\lambda + 2}}{\sqrt{4(\delta - 2)\delta + 2 - 2\delta}}}, \quad (S.46)$$

$$\phi(Z) \approx C_4 \left( Z + \frac{1}{8} \right)^{1/\beta_1}, \quad \phi(\zeta) \approx C_4' \zeta^{(\alpha_1 - 2)/\beta_1}, \quad (S.47)$$

where $C_4$ and $C_4'$ are related by a cumbersome formula. The condition (S.28) can be now written as

$$\frac{\alpha_1 - 2}{\beta_1} = n, \quad n = 2, 4, 6... \quad (S.48)$$

One can check that $(\alpha_1 - 2)/\beta_1 = 2$ for $\lambda = 1$ and any $\delta > \delta_+$, that is in consistence with Eq. (S.31). The first several eigenfunctions $U(Z)$ and eigenvalues computed numerically for $\delta = 3$ ($\varepsilon = 0.2$) using Eqs. (S.40) are shown in Fig. 4.

### G. THE POST-FOCUSING REGIME

In the post-focusing regime, i.e. for $T < t < T_0$, the time evolution of the system can be described by a FSS of type (11) such that $\tilde{F}(\zeta = 0) > 0$. Thus, this regime describes the diffusive phase of the relativistic fermions. The corresponding profile function $\tilde{F}(\zeta)$ is the solution to

$$\tilde{F}(\zeta) \left[ \tilde{F}''(\zeta) + \frac{1}{\zeta} \tilde{F}'(\zeta) \right] + \tilde{F}'(\zeta)^2 + \delta \tilde{F}(\zeta) - (2\delta - 1)\tilde{F}(\zeta) = 0. \quad (S.49)$$

Equation (S.49) can be analyzed using the same phase variables $Z$ and $Y$ but now in the semiplane $Z > 0$. The phase variables satisfy the same systems of autonomous first order differential equations (S.17) and (S.18), however, the relation with $\tilde{F}(\zeta)$ is now given by

$$\tilde{F}(\zeta) = \zeta^2 Z(\zeta), \quad Y(\zeta) = -(2Z(\zeta) + \zeta Z'(\zeta)). \quad (S.50)$$

The FSS in the post-focusing regime corresponds to the integral curve connecting the singular point $O$, which describes the asymptotics for $\zeta \to \infty$, and the singular point $D$, $(Z_D = \infty, Y_D = (1 - 2\delta)/2)$, which describes the behavior at $\zeta = 0$. Expansion around point $O$ is still given by Eq. (S.24) with $Z > 0$ while expansion around point $D$ reads

$$D: \quad Y(Z) = \frac{1}{2} (1 - 2\delta) + \frac{8\delta^2 - 6\delta + 1}{16Z} - \frac{(2\delta - 1)(1 - 4\delta)^2}{192Z^2} + \frac{\delta(2\delta - 1)(1 - 4\delta)^2}{1536Z^3} + O(Z^{-4}). \quad (S.51)$$
Using these expansions we numerically integrate Eqs. (S.17) and (S.18). The resulting integrals curves in the plane $(Z, Y)$ and the corresponding profile functions $\tilde{F}(\tilde{z})$ in the post-focusing regime are shown in Fig. 5 for several values of $\delta$.

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