PROBING THE BOUNDARIES OF THE HADRONIC PHASE THROUGH A STRANGENESS INCLUDING STATISTICAL BOOTSTRAP MODEL (S-SBM)

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Abstract

A recently constructed strangeness-including Statistical Bootstrap Model (S-SBM), which defines the limits of the hadronic phase and provides for a phase beyond, is further extended so as to include a factor $\gamma_s$ that describes strangeness suppression. The model is then used to analyse the multiplicity data from collision experiments in which the colliding entities form isospin symmetric systems, the primary focus being on $S+S$ interactions (NA35 collaboration). An optimal set of thermodynamical variables is extracted through a fit to both the inclusive $4\pi$ and midrapidity data. The assumption that the measured particles originate from a thermally and partial-chemically equilibrated source described by the S-SBM is satisfactorily established. The proximity of the thermodynamical variables extracted from the $S+S$ data to the limits of the hadron phase is systematically investigated. Finally, experimental data from $p+\bar{p}$ collisions (UA5 collaboration) are similarly analysed.

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1. Introduction

The description of multiparticle hadronic states which emerge from a collision process at very high energies can, most economically, be conducted by adopting thermalization hypotheses pervading their evolution course. It is, in fact, the quintessence of the statistical mechanical approach to a given system with a large number of degrees of freedom that it transcribes a hopelessly complicated microscopic description into a corresponding concise one which employs a handful of thermodynamical variables (e.g. temperature, volume, chemical potentials and the like).

Naturally, the ultimate goal of any attempt to address issues involving multiparticle production in high energy collisions is to account for the dynamics which give rise to the experimentally recorded profile of the final state of the system. For the particular case of heavy ion collisions, on which the bulk of our attention will be focused in this work, the question of fundamental interest is whether the overall collision process has gone through a deconfined phase and, if yes, in what way can one pick signatures of this occurrence in the composition of the final state.

Given the above general remarks, let us now become more concrete and consider the basic thermodynamical scenario which has been adopted by the pioneers of this approach [1-3]. To describe a system that has been produced by one (or more) high energy density source(s), one assumes a dynamical evolution that reaches freeze-out points (thermal, chemical) at which corresponding thermodynamical quantities acquire their final equilibrium values, reflected in the observed particle species and multiplicities.

The statistical mechanical analysis evaluating the experimental data can be conducted in the spirit: (a) Taking for granted the validity of the thermal picture and looking for possible discrepancies which provide signals of something interesting having taken place, (b) testing the reliability of a given thermal model, (c) a combination thereof. In any case, it is particle multiplicities (or ratios) which provide concrete numerical input for the theoretical analysis. Such information addresses itself to the chemical composition of the system, which means that chemical potentials become a key ingredient in this type of description. In this context, one must also consider the prospect of amending the equilibrium scheme by inserting parameters which account for partial equilibrium conditions with respect to a given
quantity. This possibility will, in fact, prominently enter the present analysis in connection with saturation of strangeness phase space.

Theoretical confrontations of thermal hadron production in various types of experiments, ranging from $e^+e^-$ to nucleus-nucleus collisions, have been conducted within the framework of a statistical model corresponding to an ideal hadron gas with results that, \textit{a posteriori} at least, justify the premises of the whole approach [1-11]. However, such a model has a built-in assumption that the hadronic world has no bounds and it does not anticipate the existence of a new phase, beyond the hadronic one, when a critical point in the space of thermodynamical parameters is reached. By contrast, the statistical bootstrap model (SBM), originally introduced by Hagedorn [12], see also [13-16], \textit{does} anticipate the end of the hadronic domain and, under certain conditions on model parameters [17], is consistent with the existence of a new phase beyond the hadronic one. Given this state of affairs, as well as the fact that the quantum number of strangeness plays a central role in revealing the quark-gluon plasma (QGP) phase, we have recently constructed an extension of the SBM so as to incorporate Strangeness [18,19], henceforth to be referred to as S-SBM. The model exhibits attractive properties consistent with what one anticipates from the transition between the hadronic and the new, presumably QGP, phase of matter.

At present, the S-SBM treats $u$ and $d$ quarks on the same footing so that a single fugacity corresponds to both of them. This amounts to considering only isospin symmetric systems, i.e. systems for which the number of $u$-quarks minus $\bar{u}$-quarks is held equal to the number of $d$-quarks minus $\bar{d}$-quarks. Then it is easy to prove that

$$n_u - n_{\bar{u}} = n_d - n_{\bar{d}} \Leftrightarrow \lambda_u = \lambda_d \equiv \lambda_q ,$$

(1)

where the $n$’s stand for quark numbers and the $\lambda$’s for corresponding fugacities\footnote{Our notational conventions regarding fugacity labels will be specified in the next section.}. The above relation, in the presence of the condition that the total strangeness vanishes, is equivalent to

$$B = 2Q \Leftrightarrow \lambda_Q = 1 ,$$

(2)

where $Q$ is the total electric charge and $\lambda_Q$ is the corresponding fugacity. So in the aforementioned situation we need not consider the existence of the additional fugacity $\lambda_Q$ and the present form of S-SBM ascribes precisely to these systems.
In this paper we intend to confront, primarily, experimental data coming from isospin symmetric $S + S$ collisions (NA35 experiment at CERN) within the framework of the S-SBM. For purposes of completeness, we shall also use our methodology for the evaluation of $p + \overline{p}$ data (UA5 experiment at CERN). We shall be addressing ourselves to both hadron multiplicities and corresponding particle ratios which involve strange particles, assuming partial strangeness saturation. To this end, we shall introduce into our scheme the additional parameter $\gamma_s$, which will be discussed in the following section and can be identified as the “fugacity” pertaining to the number of strange plus antistrange quarks.

Our analysis will cover both the inclusive $4\pi$ set of data and the, more restricted, midrapidity region. A fine tuning due to corrections from Bose/Fermi statistics will also be taken into account, given that the SBM scheme adopts the Boltzmann distribution (maximum entropy content). Our purpose is twofold: first to find out whether a thermal description based on the S-SBM accounts for the experimental data in a satisfactory manner, and second to find the proximity of the data to the critical surface which sets the limits of the hadronic phase.

The organisation of the paper is the following. In the next section we shall consider the ramifications brought by the inclusion of the $\gamma_s$ parameter into the S-SBM scheme. In Section 3 we shall proceed to present the methodology by which we propose to conduct our analysis of observed particle multiplicities. The actual examination of the experimental data, both for the $S + S$ and $p + \overline{p}$ cases, will be presented in Section 4. A $\chi^2$ fit will be employed to assess our theoretical predictions. We shall devote the last section to a discussion-evaluation of our results, paying special attention to the issue of whether and/or how close the $S + S$ data come to revealing the attainment of the QGP phase, always according to our model. Two appendices (A and B) are devoted to technical matters concerning routines used in our numerical calculations, while in appendix C we discuss the particle ratios vs particle multiplicities issue.
2. Partial strangeness equilibrium in the S-SBM

We shall begin this section by presenting, in an outline form, the S-SBM construction of Ref. [18,19]. The underlying assumption in that work was that the dynamical development of the hadronic system formed after a high energy collision is characterised by chemical equilibrium throughout. Subsequently, we shall consider the generalisation of this scheme brought about by allowing for only partial strangeness equilibrium.

The two basic ingredients going into any SBM scheme are the following:

(a) The so-called bootstrap equation, which incorporates in a self-consistent manner the internal dynamics operating in a relativistic multiparticle system.

(b) A statistical mechanical account of the said system in terms of a suitably defined partition function (equilibrium mode of description)

The bootstrap equation refers to a spectrum function and receives input from the set of all known hadrons through the so-called input function. Relativistic invariance in the casting of the system leads to a term of the form \(B(p^2)\tau(p^2, b, s)\), where \(B(p^2)\) is a kinetic factor and \(\tau(p^2, b, s)\) the naturally defined spectrum function which exhibits dependence on baryon (b) and strangeness (s) numbers. Rearrangements of the kind

\[
B(p^2)\tau(p^2, b, s) = \tilde{B}(p^2)\tilde{\tau}(p^2, b, s)
\]

(3)
define different versions of the bootstrap scheme depending on whether the \(\tilde{\tau}\)-function carries all or part of the dynamics acting internally. In the latter case, the \(\tilde{B}\)-factor can also assume a dynamical role.

The particular choice specifying the S-SBM has been extensively discussed in [18,19], on the basis of the decisive physical advantages it exhibits [20], is

\[
B(p^2) = \frac{2V^\mu p_\mu}{(2\pi)^3} = \frac{2Vm}{(2\pi)^3},
\]

(4)

where the last expression refers to the rest frame of the particle/fireball. The above selection simply implies that \(B(p^2)\), which takes the form \(B(m^2)\) in the rest frame, remains a purely kinematical quantity.

Setting \(B(m^2) = H_0 m^2\) one arrives at the notable relation

\[
H_0 = \frac{2}{(2\pi)^3 4B}
\]

(5)
which links the bootstrap scheme parameter $H_0$ with the MIT bag constant $B$, the latter entering through the relation $\frac{1}{4B} = \frac{V}{m}$. It turns out that $H_0$ is directly proportional to the maximum value of the critical temperature $T_0$.

A Laplace transformation which takes the set of variables $(p, b, s)$ to the set $(\beta, \lambda_B, \lambda_S)$ introduces the inverse four-temperature, along with baryon and strangeness fugacities, into the bootstrap scheme. One is thereby led to the construction of a partition function $Z(\beta, V, \lambda_B, \lambda_S)$ in the fireball rest frame. In this way a statistical mechanical description of the system is introduced, giving the final form to the S-SBM.

The analysis conducted in Ref. [18,19], with respect to the S-SBM, took place in the 3-dimensional space of the thermodynamical variables $(T, \mu_q, \mu_s)$, where $\mu_q$ and $\mu_s$ stand for up-down and strange quark chemical potentials. In this space, the following two surfaces were considered:

(a) The critical surface specified by one of the conditions

$$\varphi(T_{cr}, \mu_{q_{cr}}, \mu_{s_{cr}}; H_0) = \ln 4 - 1, \quad G(T_{cr}, \mu_{q_{cr}}, \mu_{s_{cr}}; H_0) = \ln 2, \quad (6)$$

where $\varphi$ corresponds to the input and $G$ to the mass spectrum-containing function entering the bootstrap equation. This surface constitutes the earmark of the bootstrap scheme, which separates it from the ideal hadron gas model. It signifies the termination of the hadron phase, since the bootstrap equation does not possess a physically meaningful solution beyond the critical surface. In other words, the critical surface sets the boundaries of the hadronic world in the space $(T, \mu_q, \mu_s)$.

(b) The $<S> = 0$ surface which imposes a strangeness neutrality assumption on the S-SBM construction and is specified by

$$\int_\beta^\infty x^3 \frac{\partial G(x, \lambda_q, \lambda_s; H_0)}{\partial \lambda_s} dx = 0. \quad (7)$$

As already mentioned in the introduction, our aim in this paper is to extend the framework of the S-SBM so as to allow for partial strangeness equilibrium before confronting experimental data (particle multiplicities or ratios). This is most conveniently done by introducing the variable $\gamma_s$ which is, in fact, a fugacity related to the number of $s$-quarks plus $\bar{s}$-quarks [21] (we shall henceforth call this number $|s| \equiv n_s + n_{\bar{s}}$):

$$\gamma_s \equiv \lambda_{|s|} = \exp(\mu_{|s|}/T), \quad (8)$$

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The bootstrap equation in this generalised model reads ($g_{bs|s|}$ denotes degeneracy factors applicable to the given set of labels)

$$\tilde{B}(p^2)\tilde{\tau}(p^2, b, s, |s|) = g_{bs|s|}\tilde{B}(p^2)\delta_0(p^2 - m_{bs|s|}^2) + \sum_{n=2}^{\infty} \frac{1}{n!} \int \delta^4(p - \sum_{i=1}^{n} p_i) \cdot \sum_{\{b_i\}} \delta_K(b - \sum_{i=1}^{n} b_i) \sum_{\{s_i\}} \delta_K(s - \sum_{i=1}^{n} s_i) \sum_{\{|s|_i\}} \delta_K(|s| - \sum_{i=1}^{n} |s|_i) \prod_{i=1}^{n} \tilde{B}(p_i^2)\tilde{\tau}(p_i^2, b_i, s_i, |s|_i) d^4 p_i \ .$$ (9)

In the above equation, baryon number is denoted by “$b$” and strangeness by “$s$”. Then we can perform in (9) four Laplace transformations which lead to the replacement (after going to the rest frame of the system)

$$(p^\mu, b, s, |s|) \longrightarrow (\beta, \lambda_B, \lambda_S, \gamma_s) \quad (10)$$

where $\lambda_B$ and $\lambda_S$ are the fugacities corresponding to baryon number and strangeness, respectively. Since we are accustomed to working with fugacities corresponding to quarks, we can equivalently use the set defined by

$$\lambda_q = \lambda_B^{1/3}, \quad \lambda_s^{neq} = \gamma_s \lambda_B^{1/3} \lambda_S^{-1}, \quad \lambda_{\pi}^{neq} = \gamma_s \lambda_B^{-1/3} \lambda_S \ .$$ (11)

The up and down quark fugacity is denoted by $\lambda_q$, whereas $\lambda_s^{neq}$ and $\lambda_{\pi}^{neq}$ denote the $s$-quark and $\pi$-quark fugacities respectively. The index $neq$ means that the chemical equilibrium connected with strangeness is, in principle, not achieved. The factor $\gamma_s$ entering (11) can, then, be given by

$$\gamma_s^2 = \lambda_s^{neq} \lambda_{\pi}^{neq} \ .$$ (12)

In order to stay in agreement with the conventions of the quark fugacities used in [18,19], as well with the ones used by other authors, we shall employ the set ($\lambda_q, \lambda_s, \gamma_s$). Now $\lambda_s \equiv \lambda_B^{1/3} \lambda_S^{-1}$ is the fugacity of the $s$-quark which corresponds to the chemical equilibrium of strangeness.

The bootstrap equation acquires the form

$$\varphi(T, \lambda_q, \lambda_s, \gamma_s) = 2G(T, \lambda_q, \lambda_s, \gamma_s) - \exp[G(T, \lambda_q, \lambda_s, \gamma_s)] + 1 \ ,$$ (13)

with the functions $\varphi$ and $G$ given by

$$\varphi(T, \lambda_q, \lambda_s, \gamma_s; H_0) = 2\pi H_0 T \sum_{a} \alpha_a \sum_{i} g_{ai} m_{ai}^3 K_1(m_{ai}/T) \ ,$$ (14)
\[ G(T, \lambda_q, \lambda_s, \gamma_s; H_0) = 2\pi H_0 T \int_0^\infty m^3 \tau_0(m^2, \lambda_q, \lambda_s, \gamma_s) K_1(m/T) dm^2, \]  

(15)

where the subscript “0” on \( \tau \) signifies the S-SBM choice given by eq. (4). The fugacity \( \lambda_a \) corresponds to the existing hadronic families that are used as input in (14) (light unflavoured mesons, kaons, \( N & \Delta, \Lambda & \Sigma, \Xi \) and \( \Omega \) Baryons). It runs over all particles and antiparticles and obeys the equality

\[ \lambda_a = \gamma_s^{n_s+n_\bar{s}} \lambda_q^{n_q-n_\bar{q}} \lambda_s^{n_s-n_\bar{s}}, \]  

(16)

where \( n_i \) corresponds to the number of the \( i \)-th flavour quark which is included in the “a” hadron. For the case of the light unflavoured mesons which have the quark content of the form \( c_1 q\bar{q} + c_2 s\bar{s} \) \( (c_1 + c_2 = 1) \) the corresponding fugacity equals

\[ \lambda_a = c_1 + c_2 \gamma_s^2. \]  

(17)

For the evaluation of the coefficients \( c_1 \) and \( c_2 \) we have used [22]. At the same time the partition function is amended by the addition of the fugacity variable \( \lambda_{\mid s\mid} \) or, equivalently, the chemical potential \( \mu_{\mid s\mid}. \)

The critical and \( <S> = 0 \) surfaces are now determined by equations similar to those given by (6). The only difference is that \( \gamma_s \) is also included as a variable:

\[ \varphi(T_{cr}, \mu_{q cr}, \mu_{s cr}, \gamma_{s cr}; H_0) = \ln 4 - 1, \quad G(T_{cr}, \mu_{q cr}, \mu_{s cr}, \gamma_{s cr}; H_0) = \ln 2, \]  

(18)

and

\[ \int_\beta x^3 \frac{\partial G(x, \lambda_q, \lambda_s, \gamma_s; H_0)}{\partial \lambda_s} dx = 0. \]  

(19)

For the S-SBM, as constructed in Refs [18,19], \( T_0 \), the critical temperature for vanishing chemical potentials, can be taken as the only free thermodynamical parameter of the model, in the place of \( H_0. \) \( T_0 \) is determined by the relation

\[ \varphi(T_0, \lambda_q = 1, \lambda_s = 1; H_0) = \ln 4 - 1, \]  

(20)

and, as can be seen, is function of \( H_0 \) only.

In the present case the last equation becomes

\[ \varphi(T_0, \lambda_q = 1, \lambda_s = 1, \gamma_s; H_0) = \ln 4 - 1, \]  

(21)
and $T_0$ is a function not only of $H_0$, but of $\gamma_s$ as well. It should be noted, on the other hand, that the bag constant $B$ continues to be in one to one correspondence with $H_0$, as can be seen from (5).

The dependence of $T_0$ on $\gamma_s$ complicates the numerical analyses needed to determine the profile of the critical surface in the considered extension of the S-SBM. Nevertheless, the relevant study has been carried out and the results are presented in a series of figures.

Figure 1a depicts two characteristic intersections of the critical surface. One is with the $\mu_s = 0$ MeV and the other is with the $\mu_s = 500$ MeV plane, for different values of $\gamma_s$. The unphysical value $\gamma_s = 1.5^{\pm}$ has been chosen to show the effect of the increase of $\gamma_s$ beyond unity. Figure 1b depicts the same situation as the previous one, but the intersections of the critical surface are with the planes $\mu_q = 0$ MeV and $\mu_q = 300$ MeV. As it can be seen, a decrease in the value of $\gamma_s$ below 1 leads to an expansion of the critical surface. The opposite happens for the increase of $\gamma_s$ above 1. Figure 2 shows the variation of the critical temperature $T_0$ with $\gamma_s$. The decrease of $\gamma_s$ leads to the increase of $T_0$ and vice versa. Figure 3 displays the $< S > = 0$ surface for the S-SBM and for the “ideal hadron gas”. The corresponding merging with the critical surface, in the S-SBM case, for two characteristic values of $\gamma_s$ (1 and 0.5) is also depicted. The above curves are plotted for constant fugacity $\lambda_q (\mu_q = 0.4)$.

In figures 1-3 a specific value of $H_0$ has been used which leads to $T_0 = 183$ MeV for $\gamma_s = 1$, corresponding to the maximum value of the bag constant, $B^{1/4} = 235$ MeV, according to [19]. By comparison, figures 4a and 4b present the intersections of the $< S > = 0$ with the critical surface for two values of $H_0$ (one leads to $T_0 = 150$ MeV and the other to $T_0 = 183$ MeV, for $\gamma_s = 1$). Fig. 4a shows the projections of the aforementioned intersections on the $\mu_s - \mu_q$ plane, while Fig. 4b the projections on the $\mu_s - T$ plane.

In conclusion, the extension of the S-SBM to allow for partial strangeness equilibration is feasible and its impact on the model is simply to shift the critical and $< S > = 0$ surfaces as $\gamma_s$ changes values. Its inclusion in the S-SBM scheme provides an extra (thermodynamical) degree of freedom, widening its scope of applicability.

\footnote{The strangeness production is usually found to be suppressed, so physical values for $\gamma_s$ should be less than 1.}
3. Methodology for Examining Experimental Data

Our confrontation of experimental data has as its focal point particle multiplicities. Particle ratios will also be studied at a subsequent stage. Our task will be to determine the thermodynamic parameters \((V, T, \lambda_q, \lambda_s, \gamma_s)\) which best fit the \((N)\) particle multiplicities measured in a given experiment. In principle, this is equivalent to determining the parameters \((T, \lambda_q, \lambda_s, \gamma_s)\) which best fit a group of \((N - 1)\) independent particle ratios we can form from the \(N\) measured multiplicities. Although more complicated, we shall prefer the first case, where it is possible, for the reasons discussed in Appendix C.

Introducing hadron fugacities \(\lambda_i\) into our partition function via an extension of the form

\[
Z(V, \beta, \lambda_q, \lambda_s, \gamma_s) \rightarrow Z(V, \beta, \lambda_q, \lambda_s, \gamma_s, \ldots, \lambda_i, \ldots) ,
\]

we have

\[
N_i^{\text{thermal}} = \left. \left( \lambda_i \frac{\partial \ln Z(V, \beta, \lambda_q, \lambda_s, \gamma_s, \ldots, \lambda_i, \ldots)}{\partial \lambda_i} \right) \right|_{\ldots=\lambda_i=\ldots=1} ,
\]

or equivalently

\[
N_i^{\text{thermal}}(V, T, \lambda_q, \lambda_s, \gamma_s) = \frac{VT^3}{4\pi^3 \hbar} \int_0^T \frac{1}{y^5} \frac{1}{2 - \exp[G(y, \lambda_q, \lambda_s, \gamma_s)]} \left. \frac{\partial \varphi(y, \lambda_q, \lambda_s, \gamma_s, \ldots, \lambda_i, \ldots)}{\partial \lambda_i} \right|_{\ldots=\lambda_i=\ldots=1} \, dy (24)
\]

where \(N_i^{\text{thermal}}\) stands for the number of particles of type \(i\) coming directly from the collision. To this we must add the number of particles of the same species resulting through secondary production processes. Accordingly, the number \(N_i^{\text{theory}}\), representing the theoretical prediction, is given by

\[
N_i^{\text{theory}} = N_i^{\text{thermal}} + \sum_j b_{ij} N_j^{\text{thermal}} ,
\]

where the \(b_{ij}\) are are branching ratios corresponding to the decay of resonance \(j\) into a particle of type \(i\).

As can be seen from (24), the system’s volume enters in each particle multiplicity as a common multiplicative factor. This does not change if finite volume corrections are introduced to account for particle size, as in [14]. In such a case, the volume \(V\) entering in (24) should be replaced by the free volume \(\Delta\), given by

\[
\Delta = \frac{V}{1 + \varepsilon_{pt}(T, \{\lambda\})/4B} .
\]
In the above relation, \( \varepsilon_{pt} \) is the energy density of a system of point particles which is a function of the same variables \((T, \{\lambda\})\) as those characterising the system with the extended particles \((\{\lambda\} \text{ denotes all fugacities collectively})\).

The actual determination of \( V \) itself is of no importance to our analysis. It will only be viewed as a parameter which serves to fit the data. So we need not consider volume corrections. Moreover, the analysis can be simplified (e.g. evaluation of the first and second derivatives with respect to the temperature) if we choose \( VT^3/4\pi^3 \) as our free parameter in the place of \( V \). We shall, therefore, adopt this choice in the following.

Our first task is to determine an optimal set of values for the thermal parameters, i.e. a collection \((VT^3/4\pi^3, T, \lambda_q, \lambda_s, \gamma_s)\) which best fits the experimentally measured multiplicities. A primary concern is the imposition of the constraint \( < S > = 0 \), which, according to the S-SBM scheme, amounts to enforcing eq. (7). This constraint does not allow, of course, the consideration of all the above parameters as being free (one of them has to be determined by the rest). Our way of ensuring the constraint is by introducing an extra parameter \( l \) which plays the role of a Lagrange multiplier.

Putting the above considerations together we form the function

\[
\chi^2(VT^3/4\pi^3, T, \lambda_q, \lambda_s, \gamma_s, l) = \sum_{i=1}^{N} \left[ \frac{N_i^{\text{exp}} - N_i^{\text{theory}}(VT^3/4\pi^3, T, \lambda_q, \lambda_s, \gamma_s)}{\sigma_i} \right]^2 + l \int_{\beta}^{\infty} x^2 \frac{\partial G(x, \lambda_q, \lambda_s, \gamma_s)}{\partial \lambda_s} dx ,
\]

where \( i \) runs over all hadrons measured in the experiment and \( \sigma_i \) is the corresponding experimental error. Our aim will be to determine the values of the parameters \((VT^3/4\pi^3, T, \lambda_q, \lambda_s, \gamma_s, l)\) which minimise the above function. The problem amounts to solving a system of six equations of the form

\[
\frac{\partial \chi^2(x_1, \ldots, x_6)}{\partial x_i} = 0 \quad (i = 1, \ldots, 6) ,
\]

where the \( x_i \) run over the set of parameters \((VT^3/4\pi^3, T, \lambda_q, \lambda_s, \gamma_s, l)\).

The above system can be solved with a generalisation of the Newton-Raphson method to a multidimensional space. By this procedure we have to evaluate the second partial

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\footnote{We have denoted the function by \( \chi^2 \) for obvious reasons: It provides the measure of a chi-square fit.}
derivatives of $\chi^2$ with respect to its parameters. The method is quite sufficient when the point, which represents the optimised parameter values, is well inside the domain of the hadronic phase. On the other hand, it becomes very ineffective when this point is near the critical surface (or, worse, outside)\textsuperscript{4}. This occurs because, when we begin to search for the solution with the Newton-Raphson method by giving an initial starting point, the subsequent points, through which the function passes during the evaluation procedure, in general oscillate. In the multidimensional space this oscillation may become very strong. When the optimal point happens to be near the critical surface it is quite likely that the aforementioned oscillations take us to points which lie on the outside. This will lead, of course, to a failure of the method. In view of this problem we have devised a different strategy of computation which does enable us to locate the desired minimum no matter how close it is to the critical surface. The relevant procedure is analyzed in detail in Appendix A.

Another issue which is of importance to our analysis is whether the experimental data lead to an optimal point which lies inside or outside the critical surface. It would be of great use if we could know beforehand, so that we do not try to reach an elusive “inside” point. To this end, we have developed (see Appendix B) a method through which we can determine whether the minimum lies inside our outside the critical surface without actually evaluating its exact location. We can also estimate what percentage of the region, as defined by the experimental errors of the data (in particle ratios), is inside and what outside the critical surface.

A final matter of methodological concern pertains to the relevance of Bose/Fermi statistics effects and the impact they might have on our analysis. As it turns out (see next section) the source of greatest worry, as far as discrepancies between predicted and experimental values are concerned, are the (negative) pion multiplicities, in particular for the $S + S$ experiment. The problem is that the direct inclusion of quantum particle statistics is quite difficult to be accommodated by the SBM scheme [16], whose formulation is based on the Boltzmann approximation. So, in order to gain an approximate evaluation of the error that is due to the omission of the Bose/Fermi statistics we turn to the Ideal Hadron Gas (IHG), always with the use of the grand canonical ensemble.

\textsuperscript{4}Recall that the SBM equations have analytic solutions only in the region inside the critical surface.
The method that we shall adopt consists of finding the optimised set of parameters $(VT^3/4\pi^3, T, \lambda_q, \lambda_s, \gamma_s)$ from the S-SBM theory for the $S + S$ data, subject, of course, to the constraint $< S >_{S-BM} = 0$. Then we determine the set $(VT^3/4\pi^3, T, \lambda_q, \lambda'_s, \gamma_s)$ which represents an optimal point in the Ideal Hadron Gas (IHG), formulated in the Boltzmann approximation and in the grand canonical ensemble (a constraint of the form $< S >_{IHG} = 0$ is naturally imposed). For the latter collection of parameters we calculate the particle multiplicities $(N_{IHG})_i$ that we are interested in. At this point we switch to Bose/Fermi statistics as per in the IHG. For this case (BF) and for the set of parameters $(VT^3/4\pi^3, T, \lambda_q, \lambda''_s, \gamma_s)$ which verify the constraint $< S >_{BF} = 0$, we calculate, once again, the desired particle multiplicities $(N_{BF})_i$. From the last two evaluations it follows that the relevant error in the estimation of the $i$th multiplicity, when we neglect the quantum statistics in the IHG, is $f_i = [(N_{IHG})_i - (N_{BF})_i]/(N_{IHG})_i$. Our working assumption is that about the same amount of error persists in the case of the SBM. So any correction factor for this situation will be taken to be $\simeq 1 + f_i$. This correction factor will only be used for the $h^-$ multiplicity in the $S + S$ experiment and, as it will be seen, is small (of the order of 3%).

4. Analysis of the Experimental Data

In this Section we shall present the results of the application of the methodology we just described to experimental data. Our primary emphasis will be placed on particle multiplicities extracted from the NA35 $S + S$ experiment at CERN. We shall, for purposes of checking the consistency of our scheme, also confront data from the $p + \bar{p}$ experiment UA5 at CERN.

We shall display our results on multiplicities in a series of Tables where a $\chi^2$ fit with respect to the optimal set of thermodynamical parameters will be given as well. A series of the graphical presentations of our results will be depicted on the $(T, \mu_q)$ plane. We shall first set $\gamma_s$ to its value determined from the $\chi^2$ fit. We also drop the dependence on the volume $V$, otherwise we would have to make a specific choice before being able to display our results on a two dimensional plane. Making our plots as independent as possible on fitted parameters, we shall turn to particle ratios. The ratios that will be used will be chosen
via a procedure that will be described in Appendix C. For a given particle ratio $x$, with experimental value $x^{exp}$, we shall plot, for fixed $\gamma_s$, the projections on the ($T, \mu_q$) plane of the lines $x(T, \mu_q, \mu_s, \gamma_s) = x^{exp} + \delta x^{exp}$ and $x(T, \mu_q, \mu_s, \gamma_s) = x^{exp} - \delta x^{exp}$. This will be done for each the ratio. On the same figure we shall trace the projection on the ($T, \mu_q$) plane of the intersection of the surface $< S > = 0$ with the critical surface (calculated for the same value of $\gamma_s$), which sets the limit of the hadronic domain. Finally, the fitted values of $T$ and $\mu_q$, which result from the multiplicities, will also be indicated, along with the errors that result from the $\chi^2$ fit, represented by a solid circle on each graph. In this way we can conclude whether all the bands that are formed from the ratios and their errors have a common overlapping region. Such an occurrence would verify that the measured multiplicities are consistent with a system in thermal and (partial) chemical equilibrium described by the S-SBM and would help us evaluate the corresponding $T$ and $\mu_q$. More interestingly, we can determine how close to the limit of the hadronic domain the common region of the particle ratios' bands is and to what extent that region lies either in part, or as a whole, outside this boundary. Finally, we shall be in position to compare the results from the $\chi^2$ fit and the graphical analysis from the particle ratios, so that we can evaluate the degree of their complementarity.

In the case of $p + \bar{p}$ experiment similar graphs will also be given, with the difference that they will be depicted on the ($T, \gamma_s$) plane, while no choice for one of the relevant parameters will be needed in order to draw the bands which correspond to the particle ratios.

4.1. S + S experiment (NA35)

The study of $S + S$ experimental data, which pertain to an isospin symmetric nucleus-nucleus colliding system, will be conducted first in the full phase space and subsequently at midrapidity.

4.1.1. Full phase space data

We deal, first, with the multiplicities that cover the full phase space. The $4\pi$ data from the $S + S$ experiment at SPS (200 GeV/nucleon) [23] are listed in Table 1, along with experimental errors. With $B - \overline{B}$ we denote the baryon – antibaryon multiplicity. In the
In this analysis, the free phenomenological parameter of the model $H_0$ is fixed so that $T_0$ attains, for $\gamma_s = 1$, its “maximum” value $T_0 = 183$ MeV (see [19]). The theoretically calculated multiplicities are listed in the third column of Table 1. For the theoretical estimate of the $h^-$ multiplicity, we have used a correction factor $f = 1.0255$ to account for the effect of quantum statistics. The evaluation of this factor has already been elucidated in Section 3.

The values of the optimised thermal parameters ($VT^3/4\pi^3, T, \lambda_q, \gamma_s$) which are calculated from our $\chi^2$ fit are listed in the second column of Table 3, along with their errors. Here we have to note that in the collision of two $S$ nuclei it is not evident how many nucleons really interact, even for central collisions, so the baryon number is not known a priori. This can be seen from the $B-\overline{B}$ multiplicity in Table 1, which is the net baryon number per interaction. It is measured to be $54 \pm 3$, whereas the total baryon number of the two incident $S$ nuclei is 64. So $\mu_q$ has to be left as a free parameter to be determined by the fit. The aforementioned set comprises the true collection of free parameters which remain on account of the $<S>$ constraint. Accordingly, the degrees of freedom (dof) of the $\chi^2$ fit are $9 - 4 = 5$, given that we have 9 experimental points. The least value of $\chi^2$ which is achieved by this fit is also listed in the same column. For completeness we have included the value of $\lambda_s$, which does not belong to the independent set of parameters, as well as those of the chemical potentials $\mu_q$ and $\mu_s$, which result from the calculated fugacities.

The analysis with the particle ratios is depicted in Fig. 5, where we have set $\gamma_s = 0.664$. The chosen particle ratios are listed in Table 2, along with their experimental values and errors; they will be employed throughout our treatment of the full phase space data for the $S + S$ experiment. In Fig. 7 we depict the value of $(N_{\text{exp}} - N_{\text{theory}})/\sigma_{\text{exp}}$, that is the number of standard deviations between experimental and theoretical values. The particular way of displaying our results was so chosen because there is great difference in the order
of magnitude of the measured multiplicities. The points of our fit are denoted by empty squares.

**S + S (NA35) 4π phase space**

| Particles | Experimental Data | Calculated with $h^-$ | Calculated without $h^-$ (Case B) |
|-----------|------------------|-----------------------|----------------------------------|
| $K^+$     | 12.5 ± 0.4       | 12.581                | 12.706                           |
| $K^-$     | 6.9 ± 0.4        | 7.4590                | 6.6332                           |
| $K^0_s$   | 10.5 ± 1.7       | 9.8106                | 9.3791                           |
| $\Lambda$ | 9.4 ± 1.0        | 7.8385                | 9.6767                           |
| $\bar{\Lambda}$ | 2.2 ± 0.4 | 1.3720                | 2.0156                           |
| $\bar{p}$ | 1.15 ± 0.4       | 1.9994                | 1.5117                           |
| $p - \bar{p}$ | 21.2 ± 1.3 | 22.849                | 21.529                           |
| $B - \bar{B}$ | 54 ± 3       | 53.544                | 52.348                           |
| $h^-$     | 98 ± 3           | 94.086$^a$            | 71.227$^b$                       |

$^a$ A correction factor 1.0255 has been included for the effect of Bose statistics.

$^b$ A correction factor 1.0171 has been included for the effect of Bose statistics. This multiplicity is not included in the fit.

Table 1. Experimentally measured full phase space multiplicities in the NA35 $S + S$ experiment and their theoretically fitted values by S-SBM, with the inclusion of the $h^-$ multiplicity and without it (case B).

It is evident that the fit, where all the multiplicities are included, is not so good, as can be witnessed from the relatively large value of $\chi^2$ (16.73, with 5 degrees of freedom). This is similar to the result of Becattini ($\chi^2/dof = 17.2/5$) [9] and of Sollfrank ($\chi^2/dof = 11.6/4$) [10], who have conducted a fit to the multiplicity data from NA35 $S + S$ (with $h^-$ included) for the ideal hadron gas case, formulated in the canonical ensemble.

Looking at Fig. 5 one can see that there is no common overlapping region between the particle ratios. The fitted value for the pair $(T, \mu_q)$ seems to lie inside the hadronic phase,
albeit near its limits. An analysis conducted along the lines described in Appendix B leads, for the values of particle ratios listed in Table 2, to a probability of 92.97% (238 points out of 256) for lying inside and 7.03% (18 points out of 256) for lying outside the hadronic phase. Another point of note is that the strange sector is greatly suppressed, leading to $\gamma_s = 0.66$.

| Particle Ratios used | Experimental Values | Calculated without $h^-$ (Case A) |
|----------------------|---------------------|-----------------------------------|
| $K^-/K^+$            | $0.552 \pm 0.037$   | 0.52811                           |
| $K_s^0/K^+$          | $0.84 \pm 0.14$     | 0.74268                           |
| $\Lambda/K^+$        | $0.752 \pm 0.084$   | 0.73901                           |
| $\overline{\Lambda}/K^+$ | $0.176 \pm 0.032$ | 0.14413                           |
| $\bar{p}/K^+$        | $0.092 \pm 0.032$   | 0.12044                           |
| $p - \bar{p}/K^+$    | $1.70 \pm 0.12$     | 1.7191                           |
| $B - \overline{B}/K^+$ | $4.32 \pm 0.28$  | 4.1593                           |
| $h^-/K^+$            | $7.84 \pm 0.35^c$   | $5.8036^d$                        |

- $^c$ This ratio is not included in the fit.
- $^d$ A correction factor 1.0199 has been included for the effect of Bose statistics.

Table 2. Particle ratios from the experimentally measured full phase space multiplicities in the $S + S$ experiment and their theoretically fitted values by S-SBM, without the inclusion of the $h^-/K^+$ particle ratio (case A).

4.1.1.b. Analysis with the $h^-$ excluded

We now perform the same analysis, but without explicit input from the $h^-$ multiplicity. The latter can, of course, be calculated theoretically but with the optimal parameter set which results from the fit to the rest of the multiplicities. It is now determined that, for the same choice of $H_0$, the values of the thermal parameters lead outside the hadronic domain. Not being able to locate the point where $\chi^2$ attains its absolute minimum in this case, we

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This probability will be referred to as $P_{INSIDE}$ and its value will be listed to the appropriate Tables.
search for that point on the critical surface which optimizes the value of $\chi^2$. Clearly, this is the closest we can get to the absolute minimum. In order to carry out such an analysis we have to turn to the method described in Appendix B. We designate this procedure as case A. The theoretically calculated particle ratios are listed in the last column of Table 2 and are plotted with empty triangles in Fig. 8. The parameters that represent the location of the aforementioned least value of $\chi^2$ are listed in the second column of Table 3. Values for the volume have not been entered in this column, since particle ratios were used in the analysis. Note also that the fitted parameters are not accompanied by errors, which cannot be evaluated due to the fact that we are not at the absolute minimum value of $\chi^2$. In Fig. 6 we have plotted the bands for the particle ratios that have been employed in the analysis, while the single point designates the location of the least value of $\chi^2$ in the hadronic phase (on the critical surface).

| S + S (NA35) 4π phase space |
|-----------------------------|
| Fitted Parameters | Fitted with $h^-$ | Fitted without $h^-$ |
| Case A | Case B |
|---|---|---|
| $T$ (MeV) | 169.7 ± 9.1 | 176.7 | 184.3 ± 5.1 |
| $\lambda_q$ | 1.538 ± 0.064 | 1.609 | 1.613 ± 0.079 |
| $\gamma_s$ | 0.664 ± 0.056 | 0.909 | 0.96 ± 0.16 |
| $VT^3/4\pi^3$ | 1.61 ± 0.51 | - | 0.61 ± 0.31 |
| $\chi^2$/dof | 16.73 / 5 | 3.06 / 4 $^e$ | 2.62 / 4 |
| $\lambda_s$ | 1.089 ± 0.056 | 1.010 | 0.989 ± 0.052 |
| $\mu_q$ (MeV) | 73.1 ± 8.0 | 84.0 | 88.2 ± 9.3 |
| $\mu_s$ (MeV) | 14.4 ± 8.8 | 1.7 | −1.9 ± 9.7 |
| $P_{INSIDE}$ | 92.97% | 25.78% | - |

$^e$ It is the minimum of $\chi^2$ within the Hadron Gas with $T_0 = 183$ MeV (for $\gamma_s = 1$), not the absolute minimum.

Table 3. Results of the analysis by S-SBM of the experimental data from $S + S$ experiment (4π phase space), with the inclusion of the $h^-$ multiplicity and without it (cases A and B).
What can be seen from the last analysis is that the quality of the fit is very good. The value of $\chi^2$ (though not its absolute minimum) is very small (3.06 with 4 degrees of freedom), compared to the previous one. All the calculated particle ratios that enter the fit are in very good agreement with the experimental values, as can be seen from the second column of Table 2 and Fig. 8. The last ratio, however, which includes the $h^-$ multiplicity is very far from its experimental measurement. One can draw the conclusion that, for the thermal parameters coming out of this fit, the theoretically calculated number for the $h^-$’s (mostly $\pi^-$’s) is much less than the experimentally recorded one.

From Fig. 6 one can see that bands for all the particle ratios, except for the one with the $h^-$ multiplicity, converge to the optimal point. If the hadronic phase were allowed to occupy more space, one could infer that a good overlapping region would be formed. We evaluated the probability of being outside and inside the critical surface. It turns out that there is a 25.78% chance for being inside (33 points out of 128) and 74.22% for being outside (95 points out of 128). Finally, one notices that the strange sector is almost fully saturated, as $\gamma_s = 0.909$.

To complete the analysis without the $h^-$’s we perform another kind of fit, referred to as case B. This time we allow for the hadronic phase to occupy more space, so that the point of absolute minimum $\chi^2$ falls inside this phase. Such an arrangement can be achieved via the unphysical situation wherein $H_0$ is set so that $T_0 = 193$ MeV\footnote{With this value the absolute minimum is just enclosed in the hadronic phase.} for $\gamma_s = 1$. From the relevant study we can gain a better feeling of the quality of the fit without $h^-$ and to what extend full strangeness is attained. Our results indicate that when we vary $H_0$, which amounts to shifting the critical surface, the minimum value of $\chi^2$ remains basically constant. All the fitted thermal parameters, except for $T$ and $VT^3/4\pi^3$, undergo very slight changes. Our numerical results for the fit of case B give $\chi^2/dof = 2.62/4$ and $\gamma_s = 0.96 \pm 0.16$, i.e. the strange sector is fully saturated. The calculated values of the multiplicities are listed in Table 1 and plotted as empty triangles in Fig. 7. One notices again the difference between the experimental and the theoretical values of $h^-$. The optimised parameters for this fit are listed in Table 3 (last column).
4.1.2. Midrapidity data

We now turn to the analysis of multiplicity data restricted to the midrapidity region. The data are taken from Ref. [23,3] and are listed in Table 4. The rapidity interval is $2 < y < 3$ for all multiplicities, except for the $\bar{p}$ which are measured in the interval $3 < y < 4$. This poses no problem since the system is symmetric around $y = 3$.

| Particles | Experimental Data | Calculated with $h^-$ | Calculated without $h^-$ |
|-----------|-------------------|------------------------|--------------------------|
| $K^+$     | $3.2 \pm 0.5$     | 3.7659                 | 3.2874                   |
| $K^-$     | $2.2 \pm 0.5$     | 2.4957                 | 2.0594                   |
| $\Lambda$ | $2.05 \pm 0.2$    | 1.9184                 | 2.0593                   |
| $\bar{\Lambda}$ | $0.57 \pm 0.2$ | 0.33815               | 0.53988                  |
| $p - \bar{p}$ | $3.2 \pm 1.0$ | 4.3518               | 3.0375                   |
| $h^-$     | $26 \pm 1$        | $25.544^f$             | $16.617^g$               |

$^f$ A correction factor 1.031 has been included for the effect of Bose statistics.

$^g$ A correction factor 1.021 has been included for the effect of Bose statistics. This multiplicity is not used in the fit.

Table 4. Experimentally measured midrapidity multiplicities in the $S + S$ experiment and their theoretically fitted values by S-SBM, with and without the inclusion of the $h^-$ multiplicity.

4.1.2.a. Analysis with the $h^-$ included

Once again we perform a $\chi^2$ fit with the $h^-$ included. The theoretical multiplicities are listed in Table 4 and plotted in Fig. 11. For the $h^-$ a correction factor $f = 1.031$ has been used to account for the effect of the Bose/Fermi statistics. The optimal set of values for the thermal parameters of this fit are listed in Table 6. The quality of the fit is, once again, not so good: $\chi^2/dof = 5.59/3$. This also becomes evident from Fig. 9, where we have plotted the ratios. From this graph we can see that there is no common overlapping area of the particle
rations. The location of the minimum of $\chi^2$ is observed to be well inside the hadronic domain. We actually find 100\% probability for the point representing the experiment to be inside the hadronic sector (64 points are inside out of 64). Strangeness is also found suppressed, as $\gamma_s = 0.78 \pm 0.12$.

### S + S (NA35) midrapidity

| Particle Ratios used for the fit with $h^-$ | Experimental Values | Particle Ratios used for the fit without $h^-$ | Experimental Values |
|--------------------------------------------|---------------------|-----------------------------------------------|---------------------|
| $K^+/h^-$                                  | $0.123 \pm 0.020$   | $K^+/$                                        | $1.56 \pm 0.29$    |
| $K^-/h^-$                                  | $0.0846 \pm 0.020$  | $K^-/$                                        | $1.07 \pm 0.27$    |
| $\Lambda/h^-$                              | $0.0788 \pm 0.0083$ | $\bar{\Lambda}/\Lambda$                     | $0.28 \pm 0.10$    |
| $\bar{\Lambda}/h^-$                       | $0.0219 \pm 0.0077$ | $\bar{p}/\Lambda$                            | $0.195 \pm 0.052$  |
| $p - \bar{p}/h^-$                          | $0.154 \pm 0.0039$  |                                              |                     |
| $p/h^-$                                    | $0.123 \pm 0.039$   | $h^-/$                                        | $12.7 \pm 1.3^g$   |

$^g$ This ratio is not included in the fit.

Table 5. Particle ratios from the experimentally measured midrapidity multiplicities in the $S + S$ experiment, used in the analysis with and without the $h^-$ multiplicity.

#### 4.1.2.b. Analysis with the $h^-$ excluded

A different situation from the one described above arises when the $h^-$ are excluded. The results from the fit are listed in Tables 4 and 6, whilst the particle ratios that have been used are exhibited in Table 5. All the multiplicities, except $h^-$, are consistent with a system in thermal and chemical equilibrium as described by the S-SBM. This occurrence can be directly inferred from the very small value of $\chi^2/dof = 0.172/2$.

In Fig. 10 we depict the situation for the particle ratios in the $(T, \mu_q)$ plane. All the particle ratios, except the one that includes the $h^-$ multiplicity, form a common overlapping area which is partly within the hadronic domain. We find that the probability for a point belonging to this area to lie inside the hadron phase is 81.25\% (26 points are inside out of 32) and outside 18.75\% (6 points are outside out of 32). Note, also, that strangeness is
almost oversaturated: $\gamma_s = 1.15 \pm 0.26$. Of course, the error is such that the calculated $\gamma_s$ is consistent with the maximum possible value of $\gamma_s = 1$. Note, on the other hand, that Sollfrank et al. in [3] have found $\gamma_s = 1.19$ for the $S+S$ data at midrapidity. The study used the ideal hadron gas formulated in the grand canonical ensemble, with the cut to the input mass spectrum set at 2000 MeV. In our study, we have used as input all hadrons with mass up to 2400 MeV.

| Fitted Parameters | Fitted with $h^-$ | Fitted without $h^-$ |
|-------------------|-------------------|----------------------|
| $T$ (MeV)         | 156.7 ± 8.1       | 171.7 ± 9.4          |
| $\lambda_q$      | 1.485 ± 0.081     | 1.450 ± 0.075        |
| $\gamma_s$       | 0.78 ± 0.12       | 1.15 ± 0.26          |
| $VT^3/4\pi^3$    | 0.62 ± 0.15       | 0.22 ± 0.13          |
| $\chi^2$/dof     | 5.59 / 3          | 0.172 / 2            |
| $\lambda_s$      | 1.149 ± 0.050     | 1.045 ± 0.055        |
| $\mu_q$ (MeV)    | 61.9 ± 9.2        | 63.8 ± 9.6           |
| $\mu_s$ (MeV)    | 21.7 ± 6.9        | 7.6 ± 9.0            |
| $P_{\text{INSIDE}}$ | 100%              | 81.25%               |

Table 6. Results of the S-SBM analysis of the NA35 $S+S$ experiment (midrapidity), with and without the inclusion of the $h^-$ multiplicity.

**4.1.3. General observations on the $S+S$ data analysis**

Some general remarks are in order. Let us start by commenting on the correction factors due to Bose/Fermi statistics. This correction is worth taking into account only for the $h^-$ multiplicity. To demonstrate our claim, we consider the magnitude of the error of omitting the correct statistics by evaluating it for each multiplicity taken from the $S-S$ experiment. Moreover, we choose that set of parameters which leads to the greatest correction factor. The results appear in Table 7.
Table 7. Evaluation of the largest error in the analysis of the $S + S$ data due to the omission of the Bose/Fermi statistics. In the evaluation reference to the Ideal Hadron Gas model is made.

The analysis for both the full phase space and midrapidity give a fit which is not so good when the $h^-$ are included, whilst all other multiplicities are very well fitted when the $h^-$ are excluded. We conclude that the thermal parameters (temperature, chemical potentials and $\gamma_s$) are more accurate when evaluated without the $h^-$ multiplicity. The measured $\pi^-$ from the experiment are found to be, in both cases, a lot more compared to the theoretical predictions. The strangeness suppression factor $\gamma_s$ is also near its maximum value 1 in the absence of the $h^-$ multiplicity.

Next, we comment on the fact that the fitted temperature consistently drops when the $h^-$ are included in the analysis. One may think that this should not happen, since the increase in temperature should lead to the increase of particle multiplicity. This is true, of course, provided the volume of the system is held fixed. In the fit we have performed the volume is not known; it simply enters as a free parameter. Thus what really is important is not the actual but the relative magnitude of the different multiplicities. When the $h^-$ are increased much more than the other multiplicities, then the ratios of the latter over $h^-$ are decreased. Such a situation corresponds to lower temperature and probably greater volume (which, we repeat, was irrelevant to our analysis).
A final remark concerns the fact that at midrapidity the evaluated quarkchemical potential $\mu_q$ is found to be somewhat less compared to full phase space case. This is expected, because the midrapidity region has smaller baryon number than the fragmentation region \cite{[23]} (The full baryon number in the latter case is included in the $4\pi$ region). What is not so expected is that the fitted temperatures are lower at midrapidity than in $4\pi$ rapidity regime. This can be explained by the difference that exists in the fitted values of $\gamma_s$. The temperature and $\gamma_s$ are strongly correlated: increase in $\gamma_s$ leads to decrease of temperature and vice versa (see Fig. 2). The fitted $\gamma_s$ is found to be greater at midrapidity and that leads to a drop of the fitted temperature. Had we, on the other hand, calculated the temperature in full phase space and at midrapidity with $\gamma_s$ fixed to the same value for both cases, then we would have found greater temperature for the midrapidity case.

\subsection{\(p + \bar{p}\) experiment (UA5)}

We shall, now, concern ourselves with another isospin symmetric system, namely $p + \bar{p}$ collisions. An analysis of the multiplicity data of the relevant UA5 experiment has been performed by Becattini in \cite{[8]}, based on a canonical formulation of the ideal hadron gas. Our analysis will be performed through the grand canonical formulation of the S-SBM. The data we shall use are taken from \cite{[24,8]} and are listed in Table 8, for the different center of mass energies.

The examination of this system calls for a qualitatively different approach as compared to the $S + S$ case. In the present situation, a $p$ always interacts with a $\bar{p}$. Thus the total baryon number is identically zero. We can thereby set the additional constraint on our system specified by $<B>=0$. The $<S>=0$ constraint still exists, as well. These two equations can easily be solved analytically for the fugacities and their solution gives

$$\lambda_q = \lambda_s = 1 \Leftrightarrow \mu_q = \mu_s = 0 ,$$

no matter what the value of $\gamma_s$ is. So we are only left with 3 free parameters to evaluate from the fit and no constraint to fulfill. These parameters are $(VT^3/4\pi^3, T, \gamma_s)$. In a similar manner as in the previous subsections, we perform a $\chi^2$ fit. In Table 8 we list the theoretically calculated multiplicities for the different energies and have plotted them in Fig. 15. In Table
8 we also list the ratios that we depict on the \((T, \gamma_s)\) plane in Figs. 12, 13 and 14. Our fitted parameters are tabulated in Table 9.

Table 8. Experimentally measured in full phase space multiplicities and particle ratios from the \(p + \bar{p}\) experiment and corresponding theoretical predictions by S-SBM.

| Particles | Experimental Data | Calculated Values | Particle Ratios used | Experimental Values |
|-----------|-------------------|-------------------|----------------------|---------------------|
| \(\sqrt{s} = 200\text{ GeV}\) \(p + \bar{p}\) (UA5) \(4\pi\) phase space | | | | |
| Charged | 21.4 ± 0.4 | 21.379 | \(K^0_s/\text{Charged}\) | 0.0350 ± 0.0043 |
| \(K_s^0\) | 0.75 ± 0.09 | 0.77497 | \(n/\text{Charged}\) | 0.0350 ± 0.0047 |
| \(n\) | 0.75 ± 0.10 | 0.77828 | \(\Lambda/\text{Charged}\) | 0.0107 ± 0.0028 |
| \(\Lambda\) | 0.23 ± 0.06 | 0.19934 | \(\Xi^-/\text{Charged}\) | 0.00070 ± 0.00070 |
| \(\Xi^-\) | 0.015 ± 0.015 | 0.01251 | | |
| \(\sqrt{s} = 546\text{ GeV}\) \(p + \bar{p}\) (UA5) \(4\pi\) phase space | | | | |
| Charged | 29.4 ± 0.3 | 29.392 | \(K^0_s/\text{Charged}\) | 0.0381 ± 0.0027 |
| \(K_s^0\) | 1.12 ± 0.08 | 1.1325 | \(\Lambda/\text{Charged}\) | 0.0090 ± 0.0019 |
| \(n\) | 0.265 ± 0.055 | 0.29660 | \(\Xi^-/\text{Charged}\) | 0.00170 ± 0.00051 |
| \(\Xi^-\) | 0.050 ± 0.015 | 0.01984 | | |
| \(\sqrt{s} = 900\text{ GeV}\) \(p + \bar{p}\) (UA5) \(4\pi\) phase space | | | | |
| Charged | 35.6 ± 0.9 | 35.472 | \(K^0_s/\text{Charged}\) | 0.0385 ± 0.0038 |
| \(K_s^0\) | 1.37 ± 0.13 | 1.4288 | \(n/\text{Charged}\) | 0.0281 ± 0.0057 |
| \(n\) | 1.0 ± 0.2 | 1.1426 | \(\Lambda/\text{Charged}\) | 0.0107 ± 0.0023 |
| \(\Lambda\) | 0.38 ± 0.08 | 0.32629 | \(\Xi^-/\text{Charged}\) | 0.00098 ± 0.00056 |
| \(\Xi^-\) | 0.035 ± 0.020 | 0.02236 | | |

From Figs. 12-15 it can be inferred that a generally good fit exists. The minimum values of \(\chi^2\) are similar to those found by Becattini in [8]. The overlapping region of the particle ratios is well inside the hadronic domain, as expected. The only discrepancy comes from the
Ξ⁻ multiplicity for $\sqrt{s} = 546$ GeV which has an experimental value much greater than its theoretical prediction. This does not necessarily show something interesting. Finally, from the fitted parameters we can see that the temperature is almost constant with respect to the center of mass energy and that $\gamma_s$ appears to be almost constant as well.

| Fitted Parameters | $\sqrt{s} = 200$ GeV | $\sqrt{s} = 546$ GeV | $\sqrt{s} = 900$ GeV |
|-------------------|----------------------|----------------------|----------------------|
| $T$ (MeV)         | $159.0 \pm 6.5$      | $159.3 \pm 6.3$      | $153.2 \pm 6.9$      |
| $\gamma_s$        | $0.422 \pm 0.058$    | $0.454 \pm 0.027$    | $0.481 \pm 0.054$    |
| $VT^3/4\pi^3$     | $0.296 \pm 0.051$    | $0.398 \pm 0.064$    | $0.56 \pm 0.10$      |
| $\chi^2/dof$      | $0.448 / 2$          | $4.398 / 1$          | $1.583 / 2$          |

Table 9. Results of the S-SBM analysis of the data from $p + \bar{p}$ experiment.

5. Assessments and Conclusions

In this paper we have performed an S-SBM study of particle multiplicities, recorded in very high energy collisions in which isospin symmetry holds. On the theoretical side, we adopted the thermal description of the multiparticle system under partial chemical equilibrium conditions.

The relevant construction involved five thermodynamical parameters subject to the constraint $< S > = 0$. Our first concern was to determine an optimal set of values for these parameters through a $\chi^2$ fit to the experimental particle multiplicities. This formed the basis of our numerical computations which had dual aim. On one hand to assess the viability of our thermal description and, on the other, to locate the source of the multi-hadron system, on the phase diagramme, i.e. inside or outside the hadronic domain.

The most striking observation, pervading our overall analysis, concerns the role of (negative) pion multiplicities. A drastic change transpires, with respect to a $\chi^2$ fit of our results, between the cases with and without the inclusion of pions, irrespective of adjustments made for quantum particle statistics. Full chemical equilibration is consistently achieved when
pions are not included. Typical is the case of the $4\pi S + S$ data analysis which shows compatibility with a value for $\gamma_s$ close to unity, i.e. with complete strangeness saturation, when the pions are not considered, whereas $\gamma_s$ drops to a considerably lower value when they are. The implications from analysis involving particle ratios are similar. Consistency with an equilibrium thermal behaviour is far better when experimental input from the $h^-$ is excluded. A similar situation arises in thermal analyses based on ideal hadron gas models, see e.g. [10].

One would like to understand the situation better, e.g. whether it is the freeze-out process responsible, or it is a signal that the location of the source of the hadrons is outside the hadronic domain. The latter case brings out the one special feature of the S-SBM, namely that it presents a critical surface beyond which the hadronic phase gives its place to a new one. Our studies involving particle ratios have established criteria for identifying the possible location of sources for the multiparticle hadronic state beyond this critical surface. The viability of such a possibility, assuming reasonable values for the bag constant ($B^{1/4} = 235$ MeV) and/or the maximum critical temperature $T_0 = 183$ MeV, has come up in relation to our analysis of data from the $S + S$ experiment NA35 at CERN [23]. In particular, the case of the $4\pi$ particle ratios (excluding the $h^-$) indicates that about $3/4$ of the particle-emitting state resides beyond the critical surface. This information, despite its semi-quantitative relevance, cannot be overlooked or ignored. We evaluate it as a strong sign of reaching and entering into the sought-for deconfined region beyond the hadron gas.

Perhaps a more quantitative way by which to address the same issue is through entropy considerations. It was clear from our analysis that the $\pi^-$ measured by the experiment are much more than we calculated theoretically. Sollfrank in [10] has pointed out that these particles may originate from a high entropy phase [25-27].

In order to shed some light into the notable discrepancy between the observed high $h^-$ multiplicity in the $S + S$ data and the one predicted by S-SBM, we examine the entropy content, since it depends mostly on the $h^-$ multiplicity. We calculate this quantity for the thermodynamical parameters obtained from the fit to the $4\pi S + S$ data, Table 3. We remind that this fit produces a minimum point just beyond the assumed limit of the hadronic phase ($T_0 = 183$ MeV, $B^{1/4} = 235$ MeV).
Table 10. Calculated entropy for the fitted parameters of the $4\pi S + S$ data, case B.

Table 10 presents the results of the respective calculations for the ideal Hadron Gas, S-SBM and QGP formalisms. We note that the S-SBM value is within $(71 - 78)\%$ of the QGP one for strange quark mass in the range $150 - 300$ MeV and tends towards it. In contrast, the IHG value is within $(55 - 57)\%$ of the QGP value and about $77\%$ of the S-SBM one. Noting that the predicted to the observed $h^{-}$ multiplicity is about $73\%$ of the experimentally recorded value, we may infer that the $S + S$ interaction at $200$ GeV/nucleon may have just (initiated) a deconfinement phase transition and the $h^{-}$ excess comes from an early stage of a deconfined quark matter state[6].

In future work we intend to extend the S-SBM even more, by including the electric charge fugacity, $\lambda_Q$. This will enable us to take fully into account the difference in the number of the $u$ and $d$ quark of the initial colliding nuclei. In this way the analysis of a greater variety of heavy-ion experiments, including $Pb + Pb$ collisions, will become possible.

Acknowledgement

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[6] We have suggested in [5] that, between the hadron phase and the ideal QGP one there is a deconfined quark matter phase with massive interacting quarks.
Appendix A

As mentioned in Section 3, in performing our analysis we have to find a set of parameters \((VT^3/4\pi^3, T, \lambda_q, \lambda_s, \gamma_s)\), in the context of the S-SBM, that corresponds to the minimum value of \(\chi^2\). In doing so, through a multidimensional generalisation of the Newton-Raphson method, we come across oscillations of points produced by the method during the evaluation procedure, irrespective of whether these lie inside the critical surface. We are, therefore, in need of another method for locating the minimum of \(\chi^2\) that fully takes into account the restriction of the region where our functions are analytically determined. Of course we have to locate the minimum under the condition that the constraint \(< S >= 0\) is fulfilled (recall that our way of fulfilling this constraint was by the introduction of an additional parameter which played the role of a Lagrange Multiplier).

Due to the fact that the critical surface is defined by the equation \(\varphi(T, \lambda_q, \lambda_s, \gamma_s) = \ln 4 - 1\), there is no particular value of one of the parameters through which we can tell when the critical surface is approached. A combination of values of all the parameters is needed. The solution to our problem calls for dropping one of the parameters in terms of which we have mapped the space enclosed by the critical surface and replacing it by \(\varphi\). We choose to replace the temperature \(T\). As far as the fulfilment of the \(< S >= 0\) constraint is concerned, we propose to try to find the minimum without ever leaving the \(< S >= 0\) surface. That means that one more parameter will no longer be considered as free, preferably \(\lambda_s\). Its value at every point will be determined by the rest of the parameters.

The method can be summarised as follows. From the original set of parameters we choose to describe \(\chi^2\) as a function of \((VT^3/4\pi^3, \varphi, \lambda_q, \gamma_s)\) when multiplicities are the experimental input and of \((\varphi, \lambda_q, \gamma_s)\) when the input are the ratios. In the following we consider the case of the ratios. We start by an initial point which is inside the critical surface, denoted by \((\varphi^0, \lambda_q^0, \gamma_s^0)\). Such a point is easy to specify: it is enough for it to satisfy \(\varphi^0 \leq \ln 4 - 1\). In order to determine the value of \(\chi^2\) at this initial point we have to find the corresponding fugacity \(\lambda_s\). For this reason we solve the system of the two equations:

\[
\varphi(T, \lambda_q^0, \lambda_s^0, \gamma_s^0) = \varphi^0
\]

\[
< S > (T, \lambda_q^0, \lambda_s, \gamma_s^0) = 0.
\]
The above system is solved numerically through a proper Newton-Raphson method. Let us stress that our designation of a starting point \((T_{\text{start}}, \lambda_{s,\text{start}})\) to the method is of paramount importance. If, again, we are led to points outside the critical surface then the present method will not solve our problem. To this end, it is very useful to observe that the critical surface around the value of \(\mu_s = 0\) is almost perpendicular to the \((T, \mu_q)\) plane (see Fig. 9 in [19]). That means that whether we are outside or inside the critical surface will be determined by the rest of parameters and not by \(\lambda_s\). So it is not important what starting point we shall give to \(\lambda_s\), e.g. a value \(\lambda_{s,\text{start}} = 1\) will not cause any trouble. The starting value for the temperature, on the other hand, needs more caution. This can be given by solving numerically the equation:

\[
\varphi(T_{\text{start}}, \lambda_0^q, \lambda_{s,\text{start}}, \gamma_0^s) = \varphi^0.
\]  

(32)

Once the desired \(T\) and \(\lambda_s\) are specified \(\chi^2\) can be subsequently evaluated. It is noted that the described routine has to be repeated every time the function \(\chi^2\) is evaluated.

The subsequent steps involve the performance of a number of one parameter minimisations. A lot of routines exist for this task (see for example [28]). We consider the function \(\chi^2(\varphi, \lambda_q^1, \gamma_s^1)\), which means that we hold fixed the parameters \(\lambda_q = \lambda_q^1\) and \(\gamma_s = \gamma_s^1\), and proceed with its minimisation. Suppose we find that its minimum is located at \(\varphi = \varphi^2\). Then we minimise \(\chi^2(\varphi^2, \lambda_q, \gamma_s^1)\) with respect to \(\lambda_q\). Let \(\lambda_q^2\) be the location of this minimum. We repeat the routine for \(\chi^2(\varphi^2, \lambda_q^2, \gamma_s)\), this time with respect to \(\gamma_s\), which gives us \(\gamma_s^2\). Of course, the point \((\varphi^2, \lambda_q^2, \gamma_s^2)\) will not give the location of the absolute minimum, so that the whole procedure will have to be repeated until \(\chi^2\) is efficiently reduced. Note that, during our trials to find minima for one of the parameters \(\lambda_q\) or \(\gamma_s\), we are not in danger of getting out of the critical surface because \(\varphi\) is held fixed to a value less than \(\ln 4 - 1\). When we try to find a better \(\varphi\), with \(\lambda_q\) and \(\gamma_s\) fixed, we can get out of the critical surface only if the method of one parameter minimisation gives a value to \(\varphi\) greater than \(\ln 4 - 1\). Usually, when this happens it means that the absolute minimum of \(\chi^2\) is outside the critical surface. The method described in the following appendix will certify this fact.

The method exhibited above is very slow compared to the generalised Newton-Raphson one, especially if the parameter \(VT^3/4\pi^3\) is added (in the multiplicity case). But it will
locate the desired minimum, even if it is near the critical surface, provided, of course, that it lies inside this surface. It can also be used to guide us near to the optimised parameters of the minimum, so the oscillation of the “test” points of Newton-Raphson is suppressed and we can reach the minimum with no danger of getting outside the region of analyticity.

Appendix B

In this appendix we shall present the method which allows us to conclude whether a given set of experimental data leads to thermal parameters which define a point inside or outside the critical surface. Because this method is rather slow we shall use as experimental input particle ratios and not multiplicities, reducing, in this way, by one our free parameters.

The method consists of, firstly, locating the minimum value of $\chi^2$ on the intersection of the critical surface with the $< S > = 0$ surface. This intersection is no longer a single curve, as in [18,19]. Given that the new parameter $\gamma_s$ has been introduced into the S-SBM, the intersection is described by two variables which we choose to be $(\lambda_q, \gamma_s)$. For a given value of this pair the corresponding $\lambda_{s\, cr}$ can be found by numerically solving the generalisation of eq. (49) in [19]:

$$\int_1^0 \frac{dz}{z - 2} \left[ \frac{\partial \varphi(y, \lambda_{q\, cr}, \lambda_{s\, cr}, \gamma_{s\, cr})}{\partial \lambda_s} \right] y^5 \cdot \frac{\partial \varphi(y, \lambda_{q\, cr}, \lambda_{s\, cr}, \gamma_{s\, cr})}{\partial y} = 0. \quad (33)$$

Then the value of multiplicities prior to decays, which enter the theoretical value of the ratios, is given by (see eq. (21)):

$$N_i^{thermal}(V, T_{cr}, \lambda_{q\, cr}, \lambda_{s\, cr}, \gamma_{s\, cr}) = \frac{VT_{cr}^3}{4\pi^3 H_0} \cdot \int_1^0 \frac{dz}{z - 2} \left[ \frac{\partial \varphi(y, \lambda_{q\, cr}, \lambda_{s\, cr}, \gamma_{s\, cr}, \ldots, \lambda_i, \ldots)}{\partial \lambda_{i\, cr}} \right] y^5 \cdot \frac{\partial \varphi(y, \lambda_{q\, cr}, \lambda_{s\, cr}, \gamma_{s\, cr})}{\partial y} \right|_{z = 2 - \exp[G(y, \lambda_{q\, cr}, \lambda_{s\, cr}, \gamma_{s\, cr})]} = 0. \quad (34)$$

Note that $V$ and $T_{cr}$ need not to be known since they cancel in the ratios.

Knowing how to evaluate $\chi^2$ on the intersection we perform, as in Appendix A, as many steps as required, each of which consists of two successive one parameter minimisations. In
the end, we obtain a point $A^m = (\lambda^m_q, \gamma^m_s)$ which corresponds to the minimum value of $\chi^2$ on the aforementioned intersection.

To proceed further we think that, if the absolute minimum of $\chi^2$ lies inside the critical surface, then by moving away from $A^m$ towards the interior there should be some direction where $\chi^2$ decreases. On the contrary, if the absolute minimum lies on the outside, the minimum value of $\chi^2$ within the hadronic domain should lie on its limits. Any move from this location, in any direction inside the critical surface, should lead to the increase of $\chi^2$.

With these considerations in mind, we perform the following check. We form a grid of 9 points: $A^{ij} = (\lambda^m_q + \delta_i, \gamma^m_s + \delta_j)$, where $\delta_i = -\delta_1, 0, \delta_1$ and $\delta_j = -\delta_2, 0, \delta_2$, with $\delta_1$ and $\delta_2$ small numbers. The points $A^{ij}$ are taken on a surface very close to the critical one, which is almost parallel to it. Such a surface can be determined by $\varphi = \ln{4} - 1 - \delta_3$, with $\delta_3$ a small positive number. The points $A^{ij}$ are located within the hadronic phase, very close to $A^m$ but in different directions from it. We evaluate $\chi^2$ for all $A^{ij}$. If one of the points $A^{ij}$ is found to lead to value of $\chi^2$ less than the one it has at $A^m$, then the absolute minimum of $\chi^2$ is considered to be inside the critical surface. If, on the contrary, all the points $A^{ij}$ possess greater values for $\chi^2$ than the one at $A^m$, then we surmise that the absolute minimum lies outside.

With the method we just described we can evaluate the probability that an experimental point, with its experimental errors, is inside or outside the critical surface. Explicitly, suppose that we have $n$ ratios given by the experiment. One of these ratios, e.g. $x_1$, takes the experimental value $x_1^\text{exp} \pm \delta x_1^\text{exp}$ and so is somewhere in the interval $(x_1^\text{exp} - \delta x_1^\text{exp}, x_1^\text{exp} + \delta x_1^\text{exp})$. By the same token, a second ratio $x_2$ can lie in the interval $(x_2^\text{exp} - \delta x_2^\text{exp}, x_2^\text{exp} + \delta x_2^\text{exp})$ and so on. By taking one value for every ratio from its corresponding allowed interval, we form an $n$-valued point which is within the error margins of our data. This point can be fed as experimental input to $\chi^2$ and with the previous method we can find out whether it leads to an absolute minimum inside the hadron gas phase or not. We can, similarly, consider other points, forming different combinations among the allowed values of each ratio. The ratio of the number of points that are outside (inside) to the whole number of points considered give the probability that the data lead us outside (inside) the critical surface.

In practice it takes a lot of time to process one point. So we only enter two possible values
of every ratio at the limits of its interval. Thus the points that we consider as experimental input in $\chi^2$ are:

$$(x_1^{\text{exp}} + i_1 \delta x_1^{\text{exp}}, x_2^{\text{exp}} + i_2 \delta x_1^{\text{exp}}, \ldots x_n^{\text{exp}} + i_n \delta x_n^{\text{exp}}),$$

with $i_1 = \pm 1, i_2 = \pm 1, \ldots, i_n = \pm 1$. This accounts to processing $2^n$ points for every experiment that provides $n$ particle ratios.

**Appendix C**

In principle, it is equivalent to use as experimental input in a $\chi^2$ fit a number $N$ of measured particle multiplicities or a number of $N-1$ independent particle ratios. There are numerous combinations of $N-1$ ratios that can be formed from the same set of multiplicities. The way we proceed in practice is the following. After performing the $\chi^2$ fit with the multiplicities from the $S+S$ data, we conduct a number of $\chi^2$ fits using different sets of ratios. We found a great deal of difference among these results as both the optimised parameters and the minimum value of $\chi^2$ exhibited puzzling variations.

Before we analyse the situation let us recall the basic reasons for which it is needed to deal with particle ratios. Firstly, to locate the possible overlapping region of the bands of the particle ratios drawn on the $(T, \mu_q)$ plane. Secondly, to evaluate the least value of $\chi^2$ on the two-dimensional intersection of the critical surface with the $<S>=0$ surface when the absolute minimum lies outside the critical surface. Thirdly, to estimate the probability that the experimental measurements lead to an “inside” or an “outside” point. It follows that we cannot avoid facing the question which set of particle ratios to use in a way that it leads us to results as close as possible to those represented by the multiplicities.

Translating the multiplicities to a set of ratios does not change anything as far as the centroid values are concerned. What is changed are the accompanying errors. The errors associated with the ratios depend on the way the multiplicities are coupled together to form them. To clarify this point, let us suppose that we choose the multiplicity $x_m$ with the greatest relevant error ($e_m = \delta x_m/x_m = \text{max}$) and couple it with all the rest of the multiplicities. Let us also suppose that among them exists a multiplicity with very small
relevant error \( (e_n = \delta x_n/x_n) \). This multiplicity will now enter in the ratio \( x_n/x_m \) and the relevant error will be \( e_{nm} = \sqrt{e_n^2 + e_m^2} \gg e_n \). When the multiplicity fit is performed, \( x_n \) plays an important role in the evaluation of the fitted parameters, since it has a very narrow interval for allowed values. The \( \chi^2 \) fit has to arrange the free parameters so that \( x_n^{\text{theory}} \) comes close to \( x_n \) and the contribution to \( \chi^2 \) from this multiplicity is not too great\(^8\). On the contrary, the big relevant error to the ratio that includes \( x_n \) has the effect that the fit does not really take the smallness of \( e_n \) too much into account. Therefore, in order for the results of the fit with the ratios not to come out too distorted, we have to pay attention not to upset the connection of every multiplicity to its error. One simple way to do this is to pick that multiplicity with the smallest relevant error and form ratios by coupling the rest of multiplicities with it.

We have checked in practice that the fit to the ratios chosen with the abovementioned logic really lead to the same results (fitted parameters and value of \( \chi^2 \)) as for the multiplicity fit. In the case of the 4\( \pi \) data from \( S + S \) we find that the \( K^+ \) and the \( h^- \) multiplicities have the smalest relevant error (\( \simeq 3\% \)). In order to have the same set of multiplicities to both of our fits (with and without \( h^- \)) we choose the \( K^+ \) and form their ratios with the rest. In the case of the midrapidity data from \( S + S \) the \( h^- \) has the smallest relevant error (\( \simeq 3.8\% \)), so it was chosen to form the ratios with the rest, in the case of the fit with the inclusion of \( h^- \). When \( h^- \) is excluded from the fit, the smallest error lies with \( \Lambda \) (\( \simeq 9.8\% \)) and thus this multiplicity was chosen. In the same way the \textit{charged} multiplicity was chosen for all the fits to \( p + \bar{p} \) data.

\(^8\)The dependence of \( \chi^2 \) on the relevant error can be seen if we arrange its terms as

\[
\chi^2 = \sum_i \left(\frac{1 - x_i^{\text{theory}}/x_i^{\exp}}{\sigma_i^{\exp}/x_i^{\exp}}\right)^2 = \sum_i \left(\frac{1 - x_i^{\text{theory}}/x_i^{\exp}}{e_i}\right)^2.
\]
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Figure Captions

Figure 1 (a) Intersections of planes of constant $s$-quark chemical potential $\mu_s$ with the critical surface $\varphi(T, \mu_q, \mu_s, \gamma_s) = \ln 4 - 1$ for $T_0 = 183$ MeV (for $\gamma_s = 1$).
(b) Intersections of planes of constant $q$-quark chemical potential $\mu_q$ with the critical surface $\varphi(T, \mu_q, \mu_s, \gamma_s) = \ln 4 - 1$ for $T_0 = 180$ MeV (for $\gamma_s = 1$).

Figure 2 Variation of the critical temperature, $T_0$, for zero chemical potentials $\mu_q$ and $\mu_s$ with $\gamma_s$. $T_0$ is set to be 183 MeV when $\gamma_s = 1$.

Figure 3 Projection on the plane $(T, \mu_s)$ of intersections of planes of constant $q$-quark fugacity $\lambda_q (\mu_q/T = 0.4)$ with the surface $<S>=0$ for two values of $\gamma_s$. $T_0$ is set to be 183 MeV when $\gamma_s = 1$.

Figure 4 (a) Variation of the projection on the plane $(\mu_q, \mu_s)$ of the intersection of the critical surface and the surface $<S>=0$ with $\gamma_s$, for two different values of $T_0$ (for $\gamma_s = 1$).
(b) Variation of the projection on the plane $(T, \mu_s)$ of the intersection of the critical surface and the surface $<S>=0$ with $\gamma_s$, for two different values of $T_0$ (for $\gamma_s = 1$).

Figure 5 Experimental Particle Ratios in the $(T, \mu_q)$ plane for $S+S$ experiment measured in $4\pi$ phase space. $\gamma_s$ is set to 0.664. The point and the cross are from the $\chi^2$ fit with the $h^-$. The thick solid line represents the limits of the hadronic phase (HG) as they are set by S-SBM. The other line corresponding to the ratio $K_s^0/K^+$ lies at the domain of negative $\mu_q$, for $T > 100$ MeV.

Figure 6 Experimental Particle Ratios in the $(T, \mu_q)$ plane for $S+S$ experiment measured in $4\pi$ phase space. $\gamma_s$ is set to 0.909. The point represented by the solid circle corresponds to the location of the least value within the hadron gas of $\chi^2$, without the $h^-$. The thick solid line represents the limits of the hadronic phase (HG) as they are set by S-SBM. The other line, pertaining to the ratio $K_s^0/K^+$, lies in the domain of negative $\mu_q$, for $T > 100$ MeV.
**Figure 7** Comparison between the experimentally measured multiplicities in $4\pi$ phase space and the theoretically calculated values in the fit with $h^-$ and without $h^-$ (case B) for the $S + S$ experiment. The difference is measured in units of the relevant experimental error.

**Figure 8** Comparison between the experimentally measured particle ratios in $4\pi$ phase space and the theoretically calculated values without $h^-$ (case A) for the $S + S$ experiment. The difference is measured in units of the relevant experimental error.

**Figure 9** Experimental Particle Ratios in the $(T, \mu_q)$ plane for $S + S$ experiment measured in midrapidity. $\gamma_s$ is set to 0.78. The point and the cross are from the $\chi^2$ fit with the $h^-$. The thick solid line represents the limits of the hadronic phase (HG) as they are set by S-SBM.

**Figure 10** Experimental Particle Ratios in the $(T, \mu_q)$ plane for $S + S$ experiment measured in midrapidity. $\gamma_s$ is set to 1.15. The point and the cross are from the $\chi^2$ fit without the $h^-$. The thick solid line represents the limits of the hadronic phase (HG) as they are set by S-SBM.

**Figure 11** Comparison between the experimentally measured multiplicities in midrapidity and the theoretically calculated values in the fit with $h^-$ and without $h^-$ for the $S + S$ experiment. The difference is measured in units of the relevant experimental error.

**Figure 12** Experimental Particle Ratios in the $(T, \gamma_s)$ plane for $p + \overline{p}$ experiment for $\sqrt{s} = 200$ GeV measured in $4\pi$ phase space. The point represented by the solid circle corresponds to the $\chi^2$ fit. The thick solid line represents the limits of the hadronic phase (HG), as they are set by S-SBM.

**Figure 13** Experimental Particle Ratios in the $(T, \gamma_s)$ plane for $p + \overline{p}$ experiment for $\sqrt{s} = 546$ GeV measured in $4\pi$ phase space. The point represented by the solid circle corresponds to the $\chi^2$ fit. The thick solid line represents the limits of the hadronic phase (HG), as they are set by S-SBM.
Figure 14 Experimental Particle Ratios in the \((T, \gamma_s)\) plane for \(p + \bar{p}\) experiment for \(\sqrt{s} = 900\) GeV measured in \(4\pi\) phase space. The point represented by the solid circle corresponds to the \(\chi^2\) fit. The thick solid line represents the limits of the hadronic phase (HG), as they are set by S-SBM.

Figure 15 Comparison between the experimentally measured (solid circles) particle ratios in \(4\pi\) phase space and the theoretically calculated values (empty squares) for the \(p + \bar{p}\) experiment and for \(\sqrt{s} = 200\) GeV, 546 GeV and 900 GeV. The difference is measured in units of the relevant experimental error.
Fig. 1a

- - - - $\gamma_s = 0.5$

$\gamma_s = 1.0$

$\gamma_s = 1.5$

$\mu_s = 0$ MeV

$\mu_s = 500$ MeV
Fig. 1b
Fig. 2
Intersections of the plane $\mu q/T=0.4$ with the critical surface $\varphi(T,\mu q,\mu s) = \ln 4 - 1$.

\[ \gamma_s = 1.0 \]
\[ \gamma_s = 0.5 \]
\[ \mu_{\text{cr}} \text{ (MeV)} \]

\[ \gamma_s = 1.0 \]

\[ \gamma_s = 0.5 \]

To $= 150$ MeV

To $= 183$ MeV

q-quark Critical Chemical Potential, $\mu_{q\text{cr}}$ (MeV)

Fig. 4a
S + S (NA35), full phase space, $\gamma_S = 0.66$

**Fig. 5**

- $K^-/K^+$
- $K_S^0/K^+$
- $\Lambda/K^+$
- $\bar{\Lambda}/K^+$
- $\bar{p}/K^+$
- $p - \bar{p}/K^+$
- $B - \bar{B}/K^+$
- $h^-/K^+$

**Diagram Description**

- The graph illustrates the phase space of the S + S (NA35) system with $\gamma_S = 0.66$.
- The temperature (T) is plotted on the y-axis in MeV, ranging from 100 to 200.
- The q-quark chemical potential ($\mu_q$) is plotted on the x-axis in MeV, ranging from 0 to 200.
- Various lines and contours represent different particle ratios, indicating phase transitions and stability regions.

**Legend**

- $K^-/K^+$
- $K_S^0/K^+$
- $\Lambda/K^+$
- $\bar{\Lambda}/K^+$
- $\bar{p}/K^+$
- $p - \bar{p}/K^+$
- $B - \bar{B}/K^+$
- $h^-/K^+$
Fig. 6

S + S (NA35), full phase space, $\gamma_S = 0.91$

- $K^-/K^+$
- $K_s^0/K^+$
- $\Lambda/K^+$
- $\bar{\Lambda}/K^+$
- $\bar{p}/K^+$
- $p - \bar{p}/K^+$
- $B - \bar{B}/K^+$
- $h^-/K^+$

Temperature, $T$ (MeV)

$q$-quark Chemical Potential, $\mu_q$ (MeV)
Fig. 7

S + S (NA35), full phase space

- Experimental data
- Fitted with h^-
- Fitted without h^- (B)

\[ \frac{N_{ex} - N_{th}}{\sigma_{ex}} \]
Fig. 8
S + S (NA35), midrapidity, $\gamma_s = 0.78$

Fig. 9
S + S (NA35), midrapidity, $\gamma_S = 1.15$

Fig. 10
$S + S$ (NA35), midrapidity

\[ \frac{N_{\text{ex}} - N_{\text{th}}}{\sigma_{\text{ex}}} \]

- ● Experimental data
- □ Fitted with $h^-$
- △ Fitted without $h^-$

Fig. 11
$p + \bar{p}$ (UA5), $\sqrt{s} = 200$ GeV

Fig. 12
$p + \bar{p}$ (UA5), $\sqrt{s} = 546$ GeV

Temperature, $T$ (MeV)

Fig. 13
$p + \bar{p} \text{ (UA5), } \sqrt{s} = 900 \text{ GeV}$

Fig. 14
Fig. 15

$p + \bar{p} \text{ (UA5), } \sqrt{s} = 200 \text{ GeV}$

$p + \bar{p} \text{ (UA5), } \sqrt{s} = 546 \text{ GeV}$

$p + \bar{p} \text{ (UA5), } \sqrt{s} = 900 \text{ GeV}$