Highly-damped quasi-normal frequencies for piecewise Eckart potentials

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Highly-damped quasi-normal frequencies are very often of the form \( \omega_n = (\text{offset}) + in \) (gap). We investigate the genericity of this phenomenon by considering a model potential that is piecewise Eckart (piecewise Pöschl–Teller), and developing an analytic “quantization condition” for the highly-damped quasi-normal frequencies. We find that this \( \omega_n = (\text{offset}) + in \) (gap) behaviour is generic but not universal, with the controlling feature being whether or not the ratio of the rates of exponential falloff in the two asymptotic directions is a rational number. These observations are of direct relevance to any physical situation where highly-damped quasi-normal modes (damped modes) are important — in particular (but not limited to) to black hole physics, both theoretical and observational.

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In many diverse branches of physics one is interested in studying potentials that enjoy suitable falloff conditions at spatial infinity, and it is generally observed that such potentials lead to quasi-normal modes (QNMs, damped modes) with associated quasi-normal frequencies (QNFs). To help gain semi-analytic understanding of this phenomenon we investigate the QNFs of a piecewise Eckart (Pöschl–Teller) potential [1–4]. We are particularly interested in understanding the

\[
\omega_n = (\text{offset}) + in \ (\text{gap})
\]

behaivour that has been encountered in very many different analyses, often in the context of black hole physics, but by no means limited to black hole physics [5–38].

The specific model we are interested in is

\[
-\psi''(x) + V(x) \psi(x) = 0,
\]

with a piecewise Eckart potential [39]

\[
V(x) = \begin{cases} 
V_{0-} \sech^2(x/b_-) & \text{for } x < 0; \\
V_{0+} \sech^2(x/b_+) & \text{for } x > 0.
\end{cases}
\]

The standard case is \( V_{0-} = V_{0+} = V_0 \), with \( b_- = b_+ = b \), so that \( V(x) = V_0 \sech^2(x/b) \). A related model where \( V_{0-} \neq V_{0+} \) but with \( b_+ \neq b_- \) has been explored by Suneeta [40], but our current model is more general, and we will take the analysis much further. We start by imposing quasi-normal boundary conditions (outgoing radiation boundary conditions) [39, 41]

\[
\psi_+(x \to +\infty) \to e^{-i\omega x}; \quad \psi_-(x \to -\infty) \to e^{i\omega x}.
\]

On each half line \((x < 0, \text{ and } x > 0)\) the exact wavefunction (see especially page 405 of [41]) is:

\[
\psi_{\pm}(x) = e^{\mp i\omega x} F_2 \left( \frac{1}{2} + \alpha_{\pm}, \frac{1}{2} - \alpha_{\pm}, 1 + ib_{\pm}\omega, z \right),
\]

where

\[
\alpha = \begin{cases} 
\pm \frac{1}{2} - \frac{V_0 b^2}{4} & \text{for } V_0 b^2 < 1/4; \\
\pm i\sqrt{\frac{1}{4} - \frac{V_0 b^2}{4}} & \text{for } V_0 b^2 > 1/4;
\end{cases}
\]

and \( z = 1/(1 + e^{\pm 2x/b}) \). The key step in matching these two exact wavefunctions at \( x = 0 \) is to calculate the logarithmic derivative. Using the Leibnitz rule and the chain rule one evaluates \( \psi_+'(0)/\psi_+(0) \) as

\[
\mp i\omega \frac{1}{2b_{\pm}} \ln \left\{ \frac{2 F_1 \left( \frac{1}{2} + \alpha_{\pm}, \frac{1}{2} - \alpha_{\pm}, 1 + ib_{\pm}\omega, z \right)}{2 F_1 \left( \frac{1}{2} + \alpha_{\pm}, \frac{1}{2} - \alpha_{\pm}, 1 + ib_{\pm}\omega, z \right)} \right\}.
\]

Invoking the differential identity

\[
\frac{d}{dz} \left\{ 2 F_1 \left( a, b, c, z \right) \right\} = c - 1 - 2 F_1 \left( a, b, c - 1, z \right) - 2 F_1 \left( a, b, c, z \right),
\]

we see

\[
\psi_+'(0)/\psi_+(0) = \mp i\omega \frac{2 F_1 \left( \frac{1}{2} + \alpha_{\pm}, \frac{1}{2} - \alpha_{\pm}, 1 + ib_{\pm}\omega, z \right)}{2 F_1 \left( \frac{1}{2} + \alpha_{\pm}, \frac{1}{2} - \alpha_{\pm}, 1 + ib_{\pm}\omega, z \right)}.
\]

Now using Bailey’s theorem

\[
2 F_1 \left( a, 1 - a, c, \frac{1}{2} \right) = \frac{\Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{c + a}{2} \right)}{\Gamma \left( \frac{c + a + 1}{2} \right) \Gamma \left( \frac{c - a + 1}{2} \right)},
\]

we have the exact result

\[
\psi_+'(0)/\psi_+(0) = \mp \frac{2}{b_{\pm}} \frac{\Gamma \left( \frac{a + i\omega b_{\pm}}{2} + \frac{3}{4} \right) \Gamma \left( -\frac{a + i\omega b_{\pm}}{2} + \frac{1}{4} \right)}{\Gamma \left( \frac{a + i\omega b_{\pm}}{2} + \frac{1}{4} \right) \Gamma \left( -\frac{a + i\omega b_{\pm}}{2} + \frac{3}{4} \right)}.
\]

To obtain a more tractable result it is extremely useful to use the reflection formula

\[
\Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin(\pi z)},
\]

to derive

\[
\frac{\Gamma \left( \frac{a + i\omega b_{\pm}}{2} + \frac{3}{4} \right)}{\Gamma \left( \frac{a + i\omega b_{\pm}}{2} + \frac{1}{4} \right)} \times \tan \left( \pi \frac{\alpha_{\pm} + i\omega b_{\pm}}{2} + \frac{1}{4} \right).
\]
This leads to the exact result
\[
\frac{\psi'_\pm(0)}{\psi_\pm(0)} = \pm \frac{2}{b_\pm} \frac{\Gamma\left(\frac{\alpha_\pm - i\omega b_\pm}{2} + \frac{3}{4}\right) \Gamma\left(\frac{\alpha_\pm + i\omega b_\pm}{2} + \frac{3}{4}\right)}{\Gamma\left(\frac{\alpha_\pm - i\omega b_\pm}{2} + \frac{1}{4}\right) \Gamma\left(\frac{\alpha_\pm + i\omega b_\pm}{2} + \frac{1}{4}\right)} \times \tan\left(\pi \left[\frac{\alpha_\pm + i\omega b_\pm}{2} + \frac{1}{4}\right]\right) 
\times \tan\left(\pi \left[-\frac{\alpha_\pm + i\omega b_\pm}{2} + \frac{1}{4}\right]\right).
\] (14)

If \(\omega\) has a large positive imaginary part, then the Gamma function arguments above tend towards the positive real axis, a region where the Gamma function is smooth — all potential poles in the logarithmic derivative have been isolated in the trigonometric functions. We now use the trigonometric identity
\[
\tan A \tan B = \frac{\cos(A - B) - \cos(A + B)}{\cos(A - B) + \cos(A + B)}.
\] (15)

to rewrite this as
\[
\frac{\psi'_\pm(0)}{\psi_\pm(0)} = \pm \frac{2}{b_\pm} \frac{\Gamma\left(\frac{\alpha_\pm - i\omega b_\pm}{2} + \frac{3}{4}\right) \Gamma\left(\frac{\alpha_\pm + i\omega b_\pm}{2} + \frac{3}{4}\right)}{\Gamma\left(\frac{\alpha_\pm - i\omega b_\pm}{2} + \frac{1}{4}\right) \Gamma\left(\frac{\alpha_\pm + i\omega b_\pm}{2} + \frac{1}{4}\right)} \times \frac{\cos(\pi\alpha_\pm) + \sin(i\pi\omega b_\pm)}{\cos(\pi\alpha_\pm) - \sin(i\pi\omega b_\pm)}.
\] (16)

The exact junction condition we wish to apply is
\[
\frac{\psi'_+(0)}{\psi_+(0)} = \frac{\psi'_-(0)}{\psi_-(0)}.
\] (17)

As long as we are primarily focussed on the highly damped QNFs \((\text{Im}(\omega) \to \infty)\) we can employ Stirling’s approximation to deduce
\[
\frac{\Gamma\left(z + \frac{1}{2}\right)}{\Gamma(z)} = \sqrt{\pi} \left[1 + O\left(\frac{1}{z}\right)\right] ; \quad \text{Re}(z) \to \infty.
\] (18)

Therefore
\[
\frac{\Gamma\left(\frac{\alpha_\pm - i\omega b_\pm}{2} + \frac{3}{4}\right)}{\Gamma\left(\frac{\alpha_\pm - i\omega b_\pm}{2} + \frac{1}{4}\right)} = \sqrt{\text{Im}(\omega b_\pm)} \frac{1}{\sqrt{2}} \left[1 + O\left(\frac{1}{\text{Im}(\omega b_\pm)}\right)\right].
\] (19)

This allows us to deduce an approximate junction condition for the asymptotic QNFs
\[
\frac{\cos(\pi\alpha_+) + \sin(i\pi\omega b_+)}{\cos(\pi\alpha_-) - \sin(i\pi\omega b_-)} = \frac{-\cos(\pi\alpha_-) + \sin(i\pi\omega b_-)}{\cos(\pi\alpha_-) - \sin(i\pi\omega b_-)},
\] (20)

which is accurate up to fractional corrections of order \(O(1/\text{Im}(\omega b_\pm))\). This asymptotic QNF condition can be rewritten in any one of the equivalent forms:
\[
\sin(-i\pi\omega b_+) \sin(-i\pi\omega b_-) = \cos(\pi\alpha_+) \cos(\pi\alpha_-); \quad (21)
\sinh(\pi\omega b_+) \sinh(\pi\omega b_-) = -\cos(\pi\alpha_+) \cos(\pi\alpha_-); \quad (22)
\cos(-i\pi\omega[b_+ - b_-]) - \cos(-i\pi\omega[b_+ + b_-]) = 2 \cos(\pi\alpha_+) \cos(\pi\alpha_-); \quad (23)
cosh(\pi\omega[b_+ - b_-]) - \cosh(\pi\omega[b_+ + b_-]) = 2 \cos(\pi\alpha_+) \cos(\pi\alpha_-). \quad (24)
\]

Now suppose \(b_+/b_-\) is rational, that is
\[
b_+/b_- = p_/p_-, \quad (25)
\]

and suppose we define \(b_*\) by
\[
b_+ = p_+ b_*; \quad b_- = p_- b_*; \quad b_* = \text{hcf}(b_+, b_-), \quad (26)
\]

then the asymptotic QNF condition is given by
\[
\sin(-i\pi\omega p_+ b_*) \sin(-i\pi\omega p_- b_*) = \cos(\pi\alpha_+) \cos(\pi\alpha_-). \quad (27)
\]

If \(\omega_*\) is any specific solution of this equation, then
\[
\omega_n = \omega_* - \frac{\ln b_*}{b_*} = \omega_* + \ln(\frac{1}{b_+} \frac{1}{b_-}), \quad (28)
\]

will also be a solution. To characterize all the solutions, consider the set of QNFs for which
\[
\text{Im}(\omega) < 1/b_*, \quad (29)
\]

and label them as
\[
\omega_{n,a} \quad a \in \{1, 2, 3 \ldots N\}. \quad (30)
\]

Then the set of QNFs decomposes into a set of “families”
\[
\omega_{n,a} = \omega_{0,a} + \frac{\ln b_*}{b_*} \quad (31)
\]

with \(a \in \{1, 2, 3 \ldots N\}\) and \(n \in \{0, 1, 2, 3 \ldots \}\), and where \(N\) is yet to be determined. But for rational \(b_+/b_-\) we can rewrite the QNF condition as
\[
\cos(-i\pi\omega p_+ b_+) \cos(-i\pi\omega p_- b_-) = 2 \cos(\pi\alpha_+) \cos(\pi\alpha_-). \quad (32)
\]

Now define \(z = \exp(\pi\omega b_+ p_+ p_-)\), then the QNF condition is
\[
z^{p_+ p_-} [z^{p_+ p_-} - z^{p_+ p_-}] = 4 \cos(\pi\alpha_+) \cos(\pi\alpha_-). \quad (33)
\]

Equivalently
\[
z^{2(p_+ p_-)} - z^{2p_{\text{max}}} + 4 \cos(\pi\alpha_+) \cos(\pi\alpha_-) z^{p_+ p_-} = -z^{2p_{\text{min}}} + 1 = 0. \quad (34)
\]

This is a polynomial of degree \(N = 2(p_+ + p_-)\), so it has exactly \(N\) roots \(z_a\) (occurring in complex conjugate pairs). Thus the QNFs are
\[
\omega_{n,a} = \frac{\ln(z_a)}{\pi b_*} + \frac{\ln b_*}{b_*}, \quad (35)
\]

with the imaginary part of the logarithm lying in \([0, 2\pi]\), and where \(a \in \{1, 2, 3 \ldots N\}\) and \(n \in \{0, 1, 2, 3 \ldots \}\).

So for rational \(b_+/b_-\) with \(b_+/b_- = p_+/p_-\) we have exactly \(N = 2(p_+ + p_-)\) families of equi-spaced QNFs.
all with with gap $i/b_\ast$ and with (typically distinct) offsets $\ln(z_n)/(\pi b_\ast)$. That is: \textit{Arbitrary rational ratios of $b_+ / b_- \text{ automatically imply the $\omega_n = (offset) + in \text{ (gap)}$ behaviour.}}

Now in contrast suppose $b_+ / b_- \text{ is irrational, that is} \quad b_* = \text{hcf}(b_+, b_-) = 0. \quad (36)$

Then the “families” each have only one element

\[ \omega_{0,a} \quad a \in \{1, 2, 3 \ldots \infty\}. \quad (37) \]

That is, there will be no “pattern” in the QNFs, and they will not regularly spaced. (Conversely, if there is a “pattern” then $b_+ / b_- \text{ is rational.}) \text{ Stated more formally: Suppose we have at least one family of equi-spaced QNFs such that} \]

\[ \omega_n = \omega_0 + inK, \quad (38) \]

then $b_+ / b_- \text{ is rational.} \text{ To see this: If we have a family of QNFs of the form given in equation } \text{(38) then we know that} \forall n \geq 0 \]

\[ \cos(A + nJ) - \cos(B + nL) = \cos(A) - \cos(B), \quad (40) \]

and realize that this also implies

\[ \cos(A + \lfloor n \rfloor J) - \cos(B + \lfloor n \rfloor L) = \cos(A) - \cos(B), \quad (41) \]

and

\[ \cos(A + \lfloor n \rfloor J) - \cos(B + \lfloor n \rfloor L) = \cos(A) - \cos(B). \quad (42) \]

Now appeal to the trigonometric identity

\[ \cos(A + \lfloor n \rfloor J) + \cos(A + nJ) = 2 \cos(J) \cos(A + \lfloor n \rfloor J), \quad (43) \]

to deduce

\[ \cos(J) \cos(A + \lfloor n \rfloor J) - \cos(L) \cos(B + \lfloor n \rfloor L) = \cos(A) - \cos(B). \quad (44) \]

That is, $\forall n \geq 0 \text{ we have both } \text{(41) and (42). The first of these equations asserts that all the points} \]

\[ \left( \cos(A + \lfloor n \rfloor J), \cos(B + \lfloor n \rfloor L) \right) \quad (45) \]

lie on the straight line of slope 1 that passes through the point $(0, \cos B - \cos A)$. The second of these equations asserts that these same points also lie on the straight line of slope $\cos(J)/\cos(L)$ that passes through the point $(0, \cos B - \cos A)/\cos(L)$. We then argue as follows:

\begin{enumerate}
\item[i)] If $\cos J \neq \cos L \text{ then these two lines are not parallel and so meet only at a single point, let us call it} (\cos A, \cos B), \text{ whence we deduce} \]

\[ \cos(A + [n + 1]J) = \cos A, \quad \cos(B + [n + 1]L) = \cos B. \quad (46) \]

But then both $J$ and $L$ must be multiples of $2\pi$, and so $\cos J = 1 = \cos L \text{ contrary to hypothesis.} \]

\item[ii)] If $\cos J = \cos L \neq 1 \text{ then we have both}$

\[ \cos(A + [n + 1]J) - \cos(B + [n + 1]L) = \cos(A) - \cos(B), \quad (47) \]

and

\[ \cos(J) \left[ \cos(A + [n + 1]J) - \cos(B + [n + 1]L) \right] = \cos(A) - \cos(B). \quad (48) \]

but these are two parallel lines, both of slope 1, that never intersect unless $\cos(J) = 1$. Thus $\cos J = 1 = \cos L \text{ contrary to hypothesis.} \]

\item[iii)] We therefore conclude that both $J$ and $L$ must be multiples of $2\pi$, so that in particular $\cos J = 1 = \cos L$ (in which case the QNF condition is certainly satisfied).

But now

\[ \left| b_+ - b_- \right|/(b_+ + b_-) = J/L \in \mathbb{Q}, \quad (49) \]

and therefore

\[ b_+ / b_- \in \mathbb{Q}. \quad (50) \]

That is: \textit{Rational ratios of $b_+ / b_- \text{ are implied by the $\omega_n = (offset) + in \text{ (gap)}$ behaviour.} \text{ Thus we have demonstrated that the $\omega_n = (offset) + in \text{ (gap)}$ behaviour is generic but not universal, and is intimately related to the rationality (or otherwise) of the ratio of the $c$-folding parameters $b_{\pm}$.} \text{ We have also checked that the analysis sketched above satisfies several appropriate consistency checks and has suitable well-behaved limits [39]. A particularly important case (not dealt with in [39]) is to consider the situation where one side of the potential exhibits power law (rather than exponential) falloff. For example, for black hole physics a particularly common situation is} \]

\[ V(x) \rightarrow V_0 + \frac{a^2}{(x + a)^2} \text{ for } x \rightarrow +\infty. \quad (51) \]

The exact wavefunctions for this potential can be written down in terms of Bessel functions, and in the limit of highly damped QNFs one can easily see

\[ \frac{\psi_+^{(0)}}{\psi_0^{(0)}} \rightarrow Im(\omega), \quad (52) \]

leading to the very simple asymptotic QNF condition

\[ 1 = \frac{\cos(\pi a_-) + \sin(\pi \omega b_-)}{\cos(\pi a_-) - \sin(\pi \omega b_-)}. \quad (53) \]

whence, in this particular situation with a one-sided exponential falloff one asymptotically has

\[ \omega_n = \frac{ln}{b_-}. \quad (54) \]
This observation is useful in that it indicates that one-sided exponential falloff can be treated via a minor variant of the analysis in [33], and a power law falloff in the potential exhibits behaviour qualitatively similar to the limit \( b_+ \to \infty \). (As it should on physical grounds.)

Turning to specific applications in black hole physics: Will the general case \( a_0 = (\text{offset}) + \text{in (gap)} \) behaviour discussed above extend to more “realistic” astrophysical or de Sitter black holes? Consider a “wavepacket” centered near the peak of the Regge–Wheeler (Zerelli) potential that is built up out of highly damped modes. While the initial short-time behaviour of the wavepacket is likely to be sensitive to the details of the Regge–Wheeler (Zerelli) potential, such a wavepacket will quickly damp out and spread out towards both \( r_s \to -\infty \) and \( r_+ \to +\infty \), so that the wavepacket will penetrate regions where our piecewise Eckart model potential should be a good approximation to the true potential. We should therefore expect the results of our semi-analytic model to be qualitatively (but not necessarily quantitatively) accurate for estimating the asymptotic QNFs of “realistic” black holes. Because of the way the asymptotic potential near asymptotic infinity — indeed if we completely forget the black hole motivation, it is already of considerable mathematical and physical interest that we have a nontrivial extension of the Eckart potential for which the QNFs are asymptotically exactly solvable — one could in principle loop back to Eckart’s original article and start asking questions about tunnelling probabilities for electrons encountering such piecewise Eckart barriers.

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