Research Article

Multiplicity Results for Variable-Order Nonlinear Fractional Magnetic Schrödinger Equation with Variable Growth

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In this paper, we prove the multiplicity of nontrivial solutions for a class of fractional-order elliptic equation with magnetic field. Under appropriate assumptions, firstly, we prove that the system has at least two different solutions by applying the mountain pass theorem and Ekeland’s variational principle. Secondly, we prove that these two solutions converge to the two solutions of the limit problem. Finally, we prove the existence of infinitely many solutions for the system and its limit problems, respectively.

1. Introduction

In this paper, we consider the multiplicity of nontrivial solutions of the following concave-convex elliptic equation involving variable-order nonlinear fractional magnetic Schrödinger equation:

\[
\begin{cases}
\left(-\Delta^{\alpha}_{A}\right) u + V_{A}(x)u = f(x)|u|^{p(x)-2}u + g(x)|u|^{q(x)-2}u \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

(1)

where \(N \geq 1\); \(s(\cdot) : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (0, 1)\) is a continuous function; \(\Omega\) is a bounded subset in \(\mathbb{R}^N\) with \(N > 2s(x, y)\) for all \((x, y) \in \Omega \times \Omega\); \(\left(-\Delta^{\alpha}_{A}\right)\) is the variable-order fractional magnetic Laplace operator; the potential \(V_{A}(x) = \lambda V^{+}(x) - V^{-}(x)\) with \(V^{\pm} = \max \{\pm V, 0\}\); \(\lambda > 0\) is a parameter; magnetic field \(A \in \mathcal{C}^{0,\alpha}(\mathbb{R}^N, \mathbb{R}^N)\) with \(\alpha \in (0, 1]\); \(f, g > 0\) are two bounded nonnegative measurable function; \(p, q \in C(\Omega)\); and \(u : \mathbb{R}^N \rightarrow C\). In [1], the fractional magnetic Laplacian has been defined as

\[
\left(-\Delta^{\alpha}_{A}\right) u(x) = \lim_{r \to 0} \int_{B_r(x)} \frac{u(x) - e^{i(x-y) \cdot A(x+y)/2} u(y)}{|x-y|^{N+2\alpha}} \, dy.
\]

(2)

for \(x \in \mathbb{R}^N\). In [2], the variable-order fractional magnetic Laplace \(\left(-\Delta^{\alpha}_{A}\right)\) is defined as follows: for each \(x \in \mathbb{R}^N\),

\[
\left(-\Delta^{\alpha}_{A}\right) \phi(x) = 2P.V \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y)}{|x-y|^{N+2s(x, y)}} \, dy,
\]

(3)

along any \(\phi \in C_0^\infty(\Omega)\). Inspired by them, we define the variable-order fractional magnetic Laplacian \(\left(-\Delta^{\alpha}_{A}\right)\) as follows: for each \(x \in \mathbb{R}^N\),

\[
\left(-\Delta^{\alpha}_{A}\right) u(x) = \lim_{r \to 0} \int_{B_r(x)} \frac{u(x) - e^{i(x-y) \cdot A(x+y)/2} u(y)}{|x-y|^{N+2s(x, y)}} \, dy.
\]

(4)

Since \(s(\cdot)\) is a function, magnetic field \(A \in \mathcal{C}^{0,\alpha}(\mathbb{R}^N, \mathbb{R}^N)\) with \(\alpha \in (0, 1]\), we see that operator \(\left(-\Delta^{\alpha}_{A}\right)\) is variable order fractional magnetic Laplace operator. Especially, when \(s(\cdot) \equiv \text{constant}\), \(\left(-\Delta^{\alpha}_{A}\right)\) reduce to the usual fractional magnetic Laplace operator. When \(s(\cdot) \equiv \text{constant}\), \(\lambda \equiv 0, \left(-\Delta^{\alpha}_{A}\right)\) reduce to the usual fractional Laplace operator. Very recently, for \(\lambda = 1, p(x), q(x) \equiv \text{constant}\), and \(\lambda = 0\); in [3], under appropriate assumptions, the authors obtained the multiplicity and concentration of the positive solution of the following indefinite semilinear elliptic equations involving concave-convex nonlinearities by the variational method:
\(-\Delta u + V_A(x)u = f(x)|u|^{\gamma-2}u + g(x)|u|^{\sigma-2}u \text{ in } \mathbb{R}^N, \\
u \geq 0 \text{ in } \mathbb{R}^N. \tag{5}\)

For \(s(\cdot) = \alpha, p(x), q(x) \equiv \text{constant}, \) and \(A = 0;\) in \([4],\) the authors obtained the existence, multiplicity, and concentration of nontrivial solutions for the following indefinite fractional elliptic equation by using Nehari manifold decomposition:

\[-(\Delta)^a u + V_A(x)u = \alpha(u|^{p(x)-2}u + b(x)|u|^{\sigma-2}u \text{ in } \Omega, \\
u = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \tag{6}\]

When \(A = 0, V^- (x) = 0, \) and \(f(x), g(x) \equiv \text{constant},\) the authors in \([2],\) give some sufficient conditions to ensure the existence of two different weak solutions and use the variational method and the mountain pass theorem to obtain the two weak solutions of problem (12) which converge to two solutions of its limit problems and the existence of infinitely many solutions to its limit problem:

\[-(\Delta)^a u + V_A(x)u = \alpha|u|^{p(x)-2}u + \beta|u|^{q(x)-2}u \text{ in } \Omega, \\
u = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \tag{7}\]

For \(s(\cdot) = s, p(x), q(x) \equiv \text{constant},\) in \([1],\) the authors study the existence of solutions for the following equation on the whole space by using the method of Nehari manifold decomposition, and obtain some sufficient conditions for the existence of nontrivial solutions of the following equation:

\[-(\Delta)^a u + V_A(x)u = f(x)|u|^{\gamma-2}u + g(x)|u|^{\sigma-2}u \text{ in } \mathbb{R}^N. \tag{8}\]

In recent years, with the continuous deepening of research, the fractional magnetic problem has attracted extensive attention of researchers. More and more researchers have studied the solvability of the fractional magnetic problem

\[
\inf_{u \in H^{\alpha}_0(\Omega, C) \setminus \{0\}} \int_{\mathbb{R}^N} \left( |u(x) - \phi(x,x)|^{2} + 2 \int_{\mathbb{R}^N} |u(y)|^{2} \right) \frac{dx}{V(x)} \geq \vartheta_0, \tag{9}\]

or all \(\lambda > 0,\) where \(H^{\alpha}_0(\Omega, C)\) is the Hilbert space related to magnetic field \(A\) (see Section 2). For the variable exponents \(p, q\), we assume that \(p, q \in C(\Omega)\) and satisfy the following assumption:

\((H_1): 2 < p(x) < 2N/(N - 2s(x, x)) \) for all \(x \in \Omega, \)

\((H_2): 1 < q(x) \leq 2\) for all \(x \in \Omega. \)

In addition, we assume that \(f, g\) satisfy the following assumption:

\((H_3): f, g : \mathbb{R}^N \to [0, \infty)\) are bounded nonnegative measurable function such that \(f > 0, g > 0\) on open interval \(\Omega_f, \Omega_g \subset \Omega\) and

\[
||f||_{\infty} = ||f||_{L^{\infty}(\mathbb{R}^N)} \leq \frac{r^2(2 - q^*)^2(\theta_0 - 1)}{\max \left\{ C_p^r, C_p^s \right\} 2\theta_0(p^* - q^*)}, \\
||g||_{\infty} = ||g||_{L^{\infty}(\mathbb{R}^N)} \leq \frac{q^r(p^*-2)(\theta_0 - 1)}{\max \left\{ C_q^r, C_q^s \right\} 2\theta_0(p^* - q^*)} + \frac{(2 - q^*)(\theta_0 - 1)}{2A_1^r(\theta_0 - 1)}(2-q^*)(r^2(2 - q^*)^2)^{1/2}. \tag{10}\]

Based on the hypothesis \((S_2),\) we can give the following definition of weak solutions for problem (1).
Lebesgue space is exponent: \( L_\lambda^{r(x)}(\Omega, \mathbb{C}) \) for any \( \lambda \geq 1 \) for each \( r(x) \).

In this section, we will describe the first main result as follows.

**Theorem 2.** Assume that \((S_1), (S_2), (V_1) - (V_2), \) and \((H_1) - (H_2)\) hold. Let \( N > 2s \). Then, the problem (1) allows at least two different solutions for all \( \lambda > 0 \).

**Theorem 3.** Let \( u_1^1 \) and \( u_2^1 \) be two solutions obtained in Theorem 2. Then, \( u_1^1 \longrightarrow u_1 \) and \( u_2^1 \longrightarrow u_2 \) in \( H_\lambda^{r(x)}(\Omega, \mathbb{C}) \) as \( \lambda \longrightarrow \infty \), where \( u_1 \neq u_2 \) are two nontrivial solutions of the following problem:

\[
\begin{align*}
\lambda^{-} u - V^-(x)u &= f(x)|u|^{p(x)-2}u + g(x)|u|^{q(x)-2}u \quad \text{in } \Omega_\lambda, \\
&= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega_\lambda.
\end{align*}
\]

**Remark 4.** In general, if \( s(\cdot) : \mathbb{R}^N \mapsto (0, 1) \) is a continuous function, magnetic field \( A \in C_0(\mathbb{R}^N, \mathbb{R}^N) \) with \( \alpha \in (0, 1] \), then the variable-order fractional magnetic Sobolev space can be defined as follows: for each \( u \in C_0^\infty(\Omega, \mathbb{C}), \)

\[
\|u\|_{L_\lambda^{r(x)}(\Omega, \mathbb{C})}^\gamma = \mathfrak{R} \int_{\mathbb{R}^N} \frac{|u(x)|^{r(x)}}{|x - y|^{N+2s(x,y)}} \, dx dy,
\]

along any \( v \in C_0^\infty(\Omega, \mathbb{C})\).

### 2. Preliminaries and Notations

In this section, we first give the definition of the variable exponential Lebesgue space. Secondly, we define variable-order fractional magnetic Sobolev spaces and prove the compact conditions between them. Finally, we give the variational setting for problem (1) and theorems that will be used later.

In this paper, we use \( |\Omega| \) to represent \( n \)-dimensional Lebesgue measure of a measurable set \( \Omega \subset \mathbb{R}^N \). In addition, for each \( a \in \mathbb{C} \), we will use \( \mathfrak{R}a \) to represent the real part of \( a \) and \( \overline{a} \) to represent the complex conjugate of \( a \). Let \( N \geq 1 \) and \( \Omega \subset \mathbb{R}^N \) be a nonempty set. A measurable function \( r : \overline{\Omega} \longrightarrow [1, \infty) \) is called a variable exponent, and we define \( r^+ = \text{esssup}_{\mathbb{R}^N} r(x), \) \( r^- = \text{essinf}_{\mathbb{R}^N} r(x). \) If \( r^+ \) is finite, then the exponent \( r \) is said to be bounded. The variable exponent Lebesgue space is

\[
L^{r(x)}(\Omega, \mathbb{C}) = \left\{ u : \Omega \longrightarrow \mathbb{C} \text{ is a measurable function} ; \begin{array}{l}
|u(x)|^{r(x)} \, dx < \infty \\
|\nabla u(x)|^{r(x)} \, dx < \infty
\end{array} \right\}
\]

with the Luxemburg norm

\[
\|u\|_{L^{r(x)}(\Omega, \mathbb{C})} = \inf \left\{ \mu > 0 : \int_{\mathbb{R}^N} \left( \frac{|u(x)|}{\mu} \right)^{r(x)} \, dx \leq 1 \right\},
\]

where \( \mathfrak{R}a \) is the real part of \( a \) and \( \overline{a} \) is the complex conjugate of \( a \). Let \( N \geq 1 \) and \( \Omega \subset \mathbb{R}^N \) be a nonempty set. A measurable function \( r : \overline{\Omega} \longrightarrow [1, \infty) \) is called a variable exponent, and we define \( r^+ = \text{esssup}_{\mathbb{R}^N} r(x), \) \( r^- = \text{essinf}_{\mathbb{R}^N} r(x). \) If \( r^+ \) is finite, then the exponent \( r \) is said to be bounded. The variable exponent Lebesgue space is

\[
L^{r(x)}(\Omega, \mathbb{C}) = \left\{ u : \Omega \longrightarrow \mathbb{C} \text{ is a measurable function} ; \begin{array}{l}
|u(x)|^{r(x)} \, dx < \infty \\
|\nabla u(x)|^{r(x)} \, dx < \infty
\end{array} \right\}
\]

with the Luxemburg norm

\[
\|u\|_{L^{r(x)}(\Omega, \mathbb{C})} = \inf \left\{ \mu > 0 : \int_{\mathbb{R}^N} \left( \frac{|u(x)|}{\mu} \right)^{r(x)} \, dx \leq 1 \right\},
\]

where

\[
\mathfrak{R}a = \text{real part of } a, \quad \overline{a} = \text{complex conjugate of } a.
\]

For bounded exponent, the dual space \( L^{r(x)}(\Omega, \mathbb{C})' \) can be identified with \( L^{\tilde{r}(x)}(\Omega, \mathbb{C}) \), where the conjugate exponent \( \tilde{r}(x) \) is defined by \( r^+ = r(1 - r) \). If \( 1 < r^- < r^+ < \infty \), then the variable exponent Lebesgue space \( L^{r(x)}(\Omega, \mathbb{C}) \) is a separable and reflexive. In particular,

\[
L^2(\Omega, \mathbb{C}) = \left\{ u : \Omega \longrightarrow \mathbb{C} \text{ is a measurable function} ; \begin{array}{l}
\int_\Omega |u(x)|^2 \, dx < \infty
\end{array} \right\}
\]

with the scalar product

\[
(u, v)_{L^2(\Omega, \mathbb{C})} = \mathfrak{R} \int_\Omega uv \, dx
\]

By Lemma 3.2.20 of [10] and \( \|u\|_{L^{r(x)}(\Omega, \mathbb{C})} = \|u\|_{L^{r(x)}(\Omega, \mathbb{R})} \), we know that in the variable exponent Lebesgue space, Hölder inequality is still valid. For all \( u \in L^{r(x)}(\Omega, \mathbb{C}), v \in L^{s(x)}(\Omega, \mathbb{C}) \) with \( r(x) \in (1, \infty) \), the following inequality holds:
\[ \int \Omega |u| \, |v| \, dx \leq \left( \frac{1}{r} + \frac{1}{r'} \right) \|u\|_{L^{r}(\Omega; C)} \|v\|_{L^{r'}(\Omega; C)} \leq 2 \|u\|_{L^{r}(\Omega; C)} \|v\|_{L^{r'}(\Omega; C)}. \]  

(19)

Let \( \Omega \) be a nonempty open subset of \( \mathbb{R}^{N} \), and let \( s(\cdot) : \mathbb{R}^{N} \times \mathbb{R}^{N} \to (0, 1) \) be a measurable function, and there exist two constants \( 0 < s_{0} < s_{1} < 1 \) such that \( s_{0} < s(x, y) < s_{1} \) for all \( (x, y) \in \mathbb{R}^{N} \times \mathbb{R}^{N} \). Set

\[ H^{s(\cdot)}(\Omega, C) = \left\{ u \in L^{2}(\Omega, C) : \left( \int_{\Omega, \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s(x, y)}} \, dx \, dy \right)^{\frac{1}{2}} < \infty \right\}. \]  

(20)

Equip \( H^{s(\cdot)}(\Omega, C) \) with the norm

\[ \|u\|_{H^{s(\cdot)}(\Omega, C)} = \|u\|_{L^{2}(\Omega, C)} + \left( \int_{\Omega, \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s(x, y)}} \, dx \, dy \right)^{\frac{1}{2}}. \]  

(21)

Especially, if \( s(\cdot) \equiv \text{constant} \), then the space \( H^{s}(\Omega, C) \) is the usual variable fractional Sobolev space \( H^{s}(\Omega, C) \).

\textbf{Lemma 8.} Let \( \Omega \) be a smooth bounded subset of \( \mathbb{R}^{N} \) and let \( s(\cdot) : \mathbb{R}^{N} \times \mathbb{R}^{N} \to (0, 1) \) satisfying \((S_{1})\) and \( r : \Omega \to (1, \infty) \) satisfying \( 1 \leq r \leq 2N/(N - 2s(x, x)) \). Then, there exists \( \tilde{C}_{r} = \widetilde{C}(N, r^{s}, s^{+}, s^{-}) > 0 \) such that

\[ \|u\|_{L^{r}(\Omega; C)} \leq \tilde{C}_{r} \|u\|_{H^{s(\cdot)}(\Omega, C)}, \]  

for any \( u \in H^{s(\cdot)}(\Omega, C) \). That is, the embedding \( H^{s(\cdot)}(\Omega, C) \hookrightarrow L^{r(\cdot)}(\Omega, C) \) is continuous. Moreover, \( H^{s(\cdot)}(\Omega, C) \hookrightarrow L^{r(\cdot)}(\Omega, C) \) is compact.

\textbf{Proof.} By Theorem 2.1 of [2], we know that \( H^{s(\cdot)}(\Omega, \mathbb{R}) \hookrightarrow L^{r(\cdot)}(\Omega, \mathbb{R}) \) is continuous and compact, there exists \( \tilde{C}_{r} = \widetilde{C}(N, r^{s}, s^{+}, s^{-}) > 0 \) such that \( \|u\|_{L^{r}(\Omega; \mathbb{R})} \leq \tilde{C}_{r} \|u\|_{H^{s}(\Omega; \mathbb{R})} \). Then, for any \( u \in H^{s}(\Omega, C) \), we have

\[ \|u\|_{L^{r}(\Omega; C)} = \|u\|_{L^{r}(\Omega; \mathbb{R})} \leq \tilde{C}_{r} \|u\|_{H^{s}(\Omega; \mathbb{R})} \leq \tilde{C}_{r} \|u\|_{H^{s}(\Omega; C)}, \]  

which induces the following norm \( \|u\|_{H^{s}(\Omega; C)} = \langle u, u \rangle_{H^{s}(\Omega; C)}^{1/2} \). Hence, \( H^{s(\cdot)}_{0, a}(\Omega, C) \) generalizes to the variable-order fractional Sobolev space (see [2]) and the magnetic framework the space introduced in [9]. Next, we state and prove some properties of space \( H^{s(\cdot)}_{0, a}(\Omega, C) \), which will be useful in the sequel.

\[ \|u\|_{L^{r}(\Omega; C)} \leq \|u\|_{L^{r}(\Omega; \mathbb{R})} \leq \tilde{C}_{r} \|u\|_{H^{s}(\Omega; \mathbb{R})} \]  

\[ \leq \tilde{C}_{r} \|u\|_{H^{s}(\Omega; C)} \leq \tilde{C}_{r} \|u\|_{H^{s}(\Omega; C)} \]  

(22)

Hence, the embedding \( H^{s(\cdot)}(\Omega, C) \hookrightarrow L^{r(\cdot)}(\Omega, C) \) is continuous and compact.

Let \( A : \mathbb{R}^{N} \to \mathbb{R}^{N} \) be a continuous function and \( A \in L_{\text{loc}}^{s(\cdot)}(\mathbb{R}^{N}, \mathbb{R}^{N}) \). For a function \( u : \mathbb{R}^{N} \to C \), define

\[ \|u\|_{H^{s}(\Omega; C)}^{2} = \left( \int_{\mathbb{R}^{N}} \int_{\mathbb{R}^{N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s(x, y)}} \, dx \, dy \right)^{1/2} \]  

(23)

and the corresponding norm denoted by \( \|u\|_{H^{s}(\Omega; C)}^{2} = \|u\|_{H^{s}(\Omega; C)}^{2} \). We consider the space \( H^{s}(\Omega; C) \) of measurable functions \( u : \mathbb{R}^{N} \to C \) such that \( \|u\|_{H^{s}(\Omega; C)} < \infty \); then, \( (H^{s(\cdot)}, \cdot)_{H^{s(\cdot)}(\Omega; C)} \) is a Hilbert space. Define \( H^{s(\cdot)}_{A}(\mathbb{R}^{N}, C) \) as the closure of \( C^{0}_{c}(\mathbb{R}^{N}, C) \) in \( H^{s(\cdot)} \); then, \( H^{s(\cdot)}_{A}(\mathbb{R}^{N}, C) \) is a Hilbert space. Especially, if \( A = 0 \), then the space \( H^{s(\cdot)}_{A}(\mathbb{R}^{N}, C) \) is the variable-order fractional Sobolev space \( H^{s(\cdot)}(\mathbb{R}^{N}, C) \); if \( A = 0 \) and \( s(\cdot) \equiv \text{constant} \), then the space \( H^{s(\cdot)}_{A}(\mathbb{R}^{N}, C) \) is the usual fractional Sobolev space \( H^{s}(\mathbb{R}^{N}, C) \).

In order to define weak solutions of problem (1), we introduce the functional space

\[ H^{s(\cdot)}_{0, a}(\Omega, C) = \left\{ u \in H^{s(\cdot)}(\mathbb{R}^{N}, C) : u = 0 \text{ a.e. in } \mathbb{R}^{N} \setminus \Omega \right\}, \]  

(24)

equipping \( H^{s(\cdot)}_{0, a}(\Omega, C) \) with the scalar product

\[ \langle u, v \rangle_{H^{s}(\Omega; C)} = \left( \int_{\mathbb{R}^{N}} \left( u(x) - e^{i(x-y)A(x,y)/2} u(y) \right) \overline{v(x)} e^{i(y-y)A(x,y)/2} v(y) \right) \, dx \, dy, \]  

(25)

\textbf{Lemma 6.} There exists a constant \( C_{2} > 0 \), depending only on \( N, s_{1} \) and \( \Omega \), such that

\[ \|u\|_{H^{s}(\Omega; C)}^{2} \leq \|u\|_{H^{s}(\Omega; C)}^{2} \leq C_{2} \|u\|_{H^{s}(\Omega; C)} \]  

(26)

defining \( u \in H^{s(\cdot)}_{0, a}(\Omega, C) \). Thus, \( \|u\|_{H^{s(\cdot)}_{0, a}(\Omega, C)} \) is an equivalent norm of \( H^{s(\cdot)}_{0, a}(\Omega, C) \).
Proof. For any \( u \in H^{s}_{0,M}(\Omega, \mathbb{C}) \), by Lemma 3.1 in [9], we have the pointwise diamagnetic inequality

\[
|u(x) - e^{i(x-y) \cdot A((x+y)/2)}u(y)| \geq \|u(x) - |u(y)|\|	ext{, for a.e. } x, y \in \mathbb{R}^N,
\]

(27)

from which we immediately have

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - e^{i(x-y) \cdot A((x+y)/2)}u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \geq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^2 - |u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy
\]

\[
\int_{\Omega} \left( \int_{\Omega} \frac{|u(y)|^2}{|x-y|^{N+2s}} \, dy \right) \, dx 
\]

\[
\geq \int_{\Omega} \left( \int_{\Omega} \frac{|u(y)|^2}{|2r|^{N+2s}} \, dy \right) \, dx
\]

\[
\geq \frac{1}{(2r)^{N+2s}} \int_{B_r \setminus \Omega} \left( \int_{\Omega} |u(y)|^2 \, dy \right) \, dx
\]

\[
= \frac{|B_r \setminus \Omega|}{(2r)^{N+2s}} \|u\|^2_{L^2(\Omega, \mathbb{C})}.
\]

(28)

Thus, we obtain

\[
\|u\|^2_{L^2(\Omega, \mathbb{C})} \leq \frac{(2r)^{N+2s}}{|B_r \setminus \Omega|} \int_{\mathbb{R}^N} \frac{|u(x) - e^{i(x-y) \cdot A((x+y)/2)}u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy
\]

\[
= C_1 \|u\|^2_{H^s(\Omega, \mathbb{C})},
\]

\[
\|u\|^2_{L^2(\Omega, \mathbb{C})} \leq \|u\|^2_{L^2(\Omega, \mathbb{C})} + \|u\|^2_{H^s(\mathbb{R}^N, \mathbb{C})}
\]

\[
\leq C_1 \|u\|^2_{H^s(\Omega, \mathbb{C})} + \|u\|^2_{H^s(\mathbb{R}^N, \mathbb{C})}
\]

\[
= C_2 \|u\|^2_{H^s(\Omega, \mathbb{C})},
\]

(30)

where \( C_1 = (2r)^{N+2s}/|B_r \setminus \Omega| \) and \( C_2 = C_1 + 1 \).

In addition,

\[
\|u\|^2_{H^s_{0,M}(\Omega, \mathbb{C})} \leq \int_{\Omega} \frac{|u(x)|^2}{|x-y|^{N+2s}} \, dx \int_{\mathbb{R}^N} \frac{|u(x) - e^{i(x-y) \cdot A((x+y)/2)}u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy
\]

\[
= \|u\|^2_{H^s_{0,M}(\mathbb{R}^N, \mathbb{C})}.
\]

(31)

Combining the above two aspects, we have

\[
\|u\|^2_{H^s_{0,M}(\Omega, \mathbb{C})} \leq \|u\|^2_{H^s_{0,M}(\mathbb{R}^N, \mathbb{C})} \leq C_2 \|u\|^2_{H^s_{0,M}(\Omega, \mathbb{C})},
\]

(32)

which implies that \( \|u\|_{H^s_{0,M}(\Omega, \mathbb{C})} \) is the equivalent norm of a norm \( \|u\|_{H^s_{0,M}(\mathbb{R}^N, \mathbb{C})} \).

**Lemma 2.3.** Let \( \Omega \) be a bounded subset of \( \mathbb{R}^N \). Assume that \( s(\cdot) \colon \mathbb{R}^N \times \mathbb{R}^N \rightarrow (0, 1) \) is a continuous function satisfying \( s_1 \) and \( r : \Omega \rightarrow (1, \infty) \) is a continuous function satisfying \( 1 \leq r \leq 2N/(N - 2s(x, x)) \). If \( u \in H^{s}_{0,0}(\Omega, \mathbb{C}) \), then

\[
H^{s}_{0,0}(\Omega, \mathbb{C}) \hookrightarrow H^{s}(\Omega, \mathbb{C})
\]

is continuous and

\[
H^{s}_{0,0}(\Omega, \mathbb{C}) \hookrightarrow L^r(\Omega, \mathbb{C})
\]

is compact, that is, there exists \( C_r = C(N, r^s, s^s, s^s) > 0 \) such that

\[
\|u\|_{L^r(\Omega, \mathbb{C})} \leq C_r \|u\|^s_{H^{s}_{0,0}(\Omega, \mathbb{C})}.
\]

(34)

Proof. For any \( u \in H^{s}_{0,0}(\Omega, \mathbb{C}) \), we have

\[
\|u\|^2_{H^{s}_{0,0}(\Omega, \mathbb{C})} = \int_{\Omega} |u(x)|^2 \, dx + \int_{\Omega} \frac{|u(x) - e^{i(x-y) \cdot A((x+y)/2)}u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy
\]

\[
\leq \int_{\Omega} |u(x)|^2 \, dx
\]

\[
+ 2 \int_{\Omega} \frac{|u(x) - e^{i(x-y) \cdot A((x+y)/2)}u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy
\]

\[
= 2\|u\|^2_{H^s_{0,0}(\Omega, \mathbb{C})} + 2J,
\]

(36)
where

\[
J = \int_{\Omega} \int_{\Omega} \left| u(y) \right|^2 \frac{e^{i(x-y) \cdot A((x+y)/2)} - 1}{|x-y|^{N+2s}} dxdy \leq \int_{\Omega} |u(y)|^2 \left( \int_{\Omega \cap |x-y|>1} \frac{1}{|x-y|^{N+2s}} dx \right) dy + \int_{\Omega} |u(y)|^2 \left( \int_{\Omega \cap |x-y|\leq1} \frac{1}{|x-y|^{N+2s}} dx \right) dy = J_1 + J_2.
\]

(37)

Since \( |e^u - 1| \leq 2 \), we have

\[
J_1 \leq 4 \int_{\Omega} |u(y)|^2 \left( \int_{\Omega \cap |x-y|>1} \frac{1}{|x-y|^{N+2s}} dx \right) dy \leq 4 \int_{\Omega} |u(y)|^2 \left( \int_{\Omega \cap |x-y|>1} \frac{1}{|x-y|^{N+2s}} dx \right) dy \leq 4 \int_{\Omega} |u(y)|^2 dx = C_4 \| u \|^2_{\mathcal{E}(\Omega, \mathbb{C})}.
\]

(38)

In view of \( \Omega \) which is bounded, there exists a compact set \( K \subset \mathbb{R}^N \) such that \( \Omega \subset K \). By Lemma 2.2 of [11], we know that \( A \) is locally bounded and \( K \subset \mathbb{R}^N \) is compact, \( |e^{i(x-y) \cdot A((x+y)/2)} - 1|^2 \leq C_4 |x-y|^2 \), for \( |x-y| \leq 1 \), \( x, y \in K \).

Thus, we obtain

\[
J_2 \leq \int_{K} |u(y)|^2 \left( \int_{\Omega \cap |x-y|\leq1} \frac{|e^{i(x-y) \cdot A((x+y)/2)} - 1|^2}{|x-y|^{N+2s}} dx \right) dy \leq \int_{K} |u(y)|^2 \left( \int_{\Omega \cap |x-y|\leq1} \frac{C_4}{|x-y|^{N+2s}} dx \right) dy \leq \int_{K} |u(y)|^2 \left( \int_{\Omega \cap |x-y|\leq1} \frac{1}{|x-y|^{N+2s-1}} dx \right) dy \leq C_5 \int_{K} |u(y)|^2 dy = C_5 \| u \|^2_{\mathcal{E}(\Omega, \mathbb{C})}.
\]

Equations (36)–(39) together with Lemma 6, we have

\[
\| u \|^2_{H^s_0(\Omega, \mathbb{C})} \leq 2 \| u \|^2_{H^s(\Omega, \mathbb{C})} + 2C_5 \| u \|^2_{\mathcal{E}(\Omega, \mathbb{C})}
\]

(40)

which implies that the embedding \( H^s_0(\Omega, \mathbb{C}) \hookrightarrow H^s(\Omega, \mathbb{C}) \) is continuous. In addition, by Lemma 5, we know that \( H^s(\Omega, \mathbb{C}) \hookrightarrow L^r(\Omega, \mathbb{C}) \) is compact. Therefore, the embedding \( H^s_0(\Omega, \mathbb{C}) \hookrightarrow L^r(\Omega, \mathbb{C}) \) is compact.

Next, we give the variational setting for problem (1). For \( \lambda > 0 \), we need the following scalar product and norm:

\[
\langle u, v \rangle_\lambda := \mathfrak{Re} \int_{\mathbb{R}^N} \frac{(u(x) - e^{i(x-y) \cdot A((x+y)/2)} u(y)) \cdot (\overline{v(x)} e^{i(x-y) \cdot A((x+y)/2)} v(y))}{|x-y|^{N+2s(x,y)}} dxdy + \mathfrak{Re} \int_{\Omega} V^* u \overline{v} dx, \| u \|_\lambda := \langle u, u \rangle_\lambda^{1/2}.
\]

(41)

Let \( E = \{ u \in H^s_0(\Omega, \mathbb{C}): \int_{\Omega} V^* u^2 dx < \infty \} \) be equipped with the norm \( \| u \|_E = \| u \|_1 \) (that is, \( \lambda = 1 \) in \( \| u \|_\lambda \)). Obviously, \( \| u \|_E \leq \| u \|_\lambda \) for \( \lambda \geq 1 \). Set \( E_1 = (E, \| u \|_1) \). Moreover, for \( r(x) \in (1, 2N/(N-2s(x,x))) \), we can get

\[
\int_{\Omega} |u(x)|^{r(x)} dx \leq \max \left\{ \| u \|^2_{H^s_0(\Omega, \mathbb{C})}, \| u \|^2_{\mathcal{E}(\Omega, \mathbb{C})} \right\} \leq \max \left\{ C_r \| u \|^2_{H^s_0(\Omega, \mathbb{C})}, C_r \| u \|^2_{\mathcal{E}(\Omega, \mathbb{C})} \right\} \leq \max \left\{ C_r \| u \|^2_{\lambda, 1}, C_r \| u \|^2_{\lambda, \infty} \right\}.
\]

(42)

For simplicity, we let \( \| u \|^2_{\lambda, V} = \| u \|^2_{H^s_0(\Omega, \mathbb{C})} + \int_{\Omega} V u^2 dx \). Therefore, by condition (V_4),

\[
\| u \|^2_{\lambda, V} \geq \| u \|^2_{H^s_0(\Omega, \mathbb{C})} + \int_{\Omega} V u^2 dx.
\]

In fact, one can verify that \( \Psi_\lambda \) is well-defined of class \( C^1 \) in \( E_\lambda \) and

\[
\Psi_\lambda(u) = \frac{1}{2} \| u \|^2_{\lambda, 1} - \frac{1}{2} \int_{\Omega} V^* u^2 dx
\]

(44)

\[
- \int_{\Omega} \left( \frac{f(x)}{p(x)} |u|^{p(x)} + \frac{g(x)}{q(x)} |u|^{q(x)} \right) u \overline{u} (x) dx
\]

\[
= \frac{1}{2} \| u \|^2_{\lambda, 1} - \int_{\Omega} \left( \frac{f(x)}{p(x)} |u|^{p(x)} + \frac{g(x)}{q(x)} |u|^{q(x)} \right) u \overline{u} (x) dx.
\]
for all \( u, v \in E_A \). Therefore, if \( u \in E_A \) is a critical point of \( \Psi_A \), then \( u \) is a solution of problem (1).

Now we give the theorems that we need later.

**Theorem 8** (see [2, 12]). Let \( X \) be a real infinite dimensional Banach space and \( I \in C^1(X) \) a functional satisfying the \( (PS)_c \) condition as well as the following three properties:

1. \( I(0) = 0 \), and there exists two constants \( \rho, \delta > 0 \) such that \( I(u) \geq \delta \) for all \( u \in X \) with \( \|u\| = \rho \).
2. \( I \) is even.
3. For all finite dimensional subspaces \( Y \subset X \), there exists \( R = R(Y) > 0 \) such that \( I(u) \leq 0 \) for all \( u \in X \setminus B_R(Y) \), where \( B_R(Y) = \{ u \in Y : \|u\| \leq R \} \). Then, \( I \) poses an unbounded sequence of critical values characterized by a minimax argument

**Theorem 9** (see [13]). Let \( X \) be a real Banach space and \( I \in C^1(X, \mathbb{R}) \). Suppose \( I \) satisfies the \( (PS)_c \) condition, which is even and bounded from below, and \( I(0) = 0 \). If for any \( k \in \mathbb{N} \), there exists a \( k \)-dimensional subspace \( X_k \) of \( X \) and \( \rho_k > 0 \) such that \( \sup I < 0 \), then at least \( X_{k+1} \) is separable Hilbert space. Let \( E_1 = \text{span}\{e_1\} \), then \( E_1 \) is a separable Hilbert space. Let \( X_1 = \text{span}\{e_1\} \), then \( E_1 = \oplus_{i \in I} X_i \). We define

\[
Y_k = \oplus_{i = 1}^k X_i, \quad Z_k = \overline{\oplus_{i = k+1}^\infty X_i}.
\]  

Then, \( Y_k \) is a sequence of critical values which have the form

\[
\xi_k = \inf_{\Gamma_k} \max_{\Gamma_k} I(u),
\]

where \( \Gamma_k = \{ \eta \in C(B_k, E) : \eta \text{ is equivariant and } \eta|_{\partial B_k} = \text{id} \} \).

**Theorem 11** (see [15], dual fountain theorem). Suppose that \( I \in C^1(E, \mathbb{R}) \) satisfying \( I(-u) = I(u) \). Assume that for every \( k \geq k_0 \), there exist \( \gamma_k > 0 \) such that

\[
(D_1): I(0) = 0, \quad (D_2): a_k = \inf \{ I(u) : u \in Z_k, \|u\| = r_k \} \geq 0, \quad (D_3): b_k = \max \{ I(u) : u \in Z_k, \|u\| = \gamma_k \} < 0,
\]

for all \( u \in E_k \). Suppose that \( I \) satisfies \( (PS)_c \) condition for every \( c > 0 \).

3. **Proof of Theorem 1**

In this part, we first recall that definitions of functional \( \Psi_A \) satisfies the \( (PS)_c \) condition and \( (PS)_c^* \) condition in \( E_1 \) at the level \( c \in E \) and use the usual mountain pass theorem (see [2]) to find a \( (PS)_c \) sequence in \( E_1 \). Second, we show that functional \( \Psi_A \) satisfies the \( (PS)_c^* \) condition in \( E_1 \) at the level \( c < c_0 \). Finally, we give the proof of problem (1).

**Definition 12** (see [2]). Let \( I \in C^1(E, \mathbb{R}) \) and \( c \in \mathbb{R} \). The functional \( I \) satisfies the \( (PS)_c \) condition if for any sequence \( \{u_n\} \subset E \) such that \( I(u_n) \rightharpoonup c \) and \( I'(u_n) \rightharpoonup 0 \) as \( n \to \infty \) admits a strongly convergent subsequence in \( E \).

**Definition 13** (see [16]). Let \( I \in C^1(E, \mathbb{R}) \) and \( c \in \mathbb{R} \). The functional \( I \) satisfies the \( (PS)_c^* \) condition (with respect to \( Y_n \)) if for any sequence \( \{u_n\} \subset E \) such that \( u_n \in E \), \( I'(u_n) \rightharpoonup c \) and \( I'(Y_n u_n) \rightharpoonup 0 \) as \( n \to \infty \) admits a strongly convergent subsequence in \( E \).

**Remark 14**. From Remark 2.1 in [16], we get that the \( (PS)_c \) condition means the \( (PS)_c \) condition.

**Theorem 15** (Theorem 3.1, [2]). Let \( E \) be a real Banach space and \( I \in C^1(E, \mathbb{R}) \) with \( I(0) = 0 \). Suppose that

1. there exist \( \delta > 0 \) and \( \rho > 0 \) such that \( I(u) \geq \delta \) for each \( u \in E \) subject to \( \|u\| = \rho \).

Then, \( I \) has an unbounded sequence of critical values which have the form

\[
\xi_k = \inf_{\Gamma_k} \max_{\Gamma_k} I(u),
\]

where \( \Gamma_k = \{ \eta \in C(B_k, E) : \eta \text{ is equivariant and } \eta|_{\partial B_k} = \text{id} \} \).
(ii) there exists \( e \in E \) with \( \|e\|_E > \rho \) such that \( I(e) < 0 \)

Define \( \Gamma = \{ \gamma \in C([0,1], E); \gamma(0) = 1, \gamma(1) = e \} \). Then,

\[
e = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \geq \delta,
\]

and there exists a \((PS)_\epsilon\) sequence \( \{u_n\}_n \subset E \).

In order to obtain our main results by using the mountain pass theorem, we first prove that \( \Psi_\lambda \) satisfies the mountain pass geometry (i) and (ii).

**Lemma 16. Assume that \((S)_\ast\), \((V)_\ast\), and \((H)_\ast\) hold. Then, for each \( \lambda > 0 \), there exists \( \rho > 0 \) and \( \tau > 0 \) such that**

\[
\Psi_\lambda(u) > \tau \quad \text{for all} \quad u \in E_\lambda \quad \text{with} \quad \|u\|_\lambda = \rho.
\]

**Proof.** In view of (42) and the fractional Sobolev inequality, for each \( u \in E_\lambda \), one has

\[
\int_\Omega \left( \frac{f(x)}{p(x)} |u|^{p(x)} + \frac{g(x)}{q(x)} |u|^{q(x)} \right) dx \\
\leq \left\| \frac{f}{p(x)} \right\|_{\infty} \int_\Omega |u|^{p(x)} dx + \left\| \frac{g}{q(x)} \right\|_{\infty} \int_\Omega |u|^{q(x)} dx \\
\leq \left\| \frac{f}{p(x)} \right\|_{\infty} \max \left\{ C_p^\ast \|u\|_\lambda^{p(x)}, C_p^\ast \|u\|_\lambda^{q(x)} \right\} \\
+ \left\| \frac{g}{q(x)} \right\|_{\infty} \max \left\{ C_q^\ast \|u\|_\lambda^{p(x)}, C_q^\ast \|u\|_\lambda^{q(x)} \right\},
\]

where \( C_p^\ast, C_q^\ast \) are two constants of embedding from variable-order fractional Sobolev space \( H^{\theta(x)}_{\ast\ast}(\Omega, \mathbb{C}) \) to \( L^{p(x)}(\Omega, \mathbb{C}) \) and \( L^{q(x)}(\Omega, \mathbb{C}) \), respectively. Making use of (43) and (50), we obtain that

\[
\Psi_\lambda(u) = \frac{1}{2} \|u\|_\lambda^2 - \int_\Omega V - u^2 dx \\
- \int_\Omega \left( \frac{f(x)}{p(x)} |u|^{p(x)} + \frac{g(x)}{q(x)} |u|^{q(x)} \right) dx \\
\geq \frac{1}{2} \|u\|_\lambda^2 - \int_\Omega \left( \frac{f(x)}{p(x)} |u|^{p(x)} + \frac{g(x)}{q(x)} |u|^{q(x)} \right) dx \\
\geq \frac{1}{2} \left( 1 - \frac{\theta_0 - 1}{2\theta_0} \right) \|u\|_\lambda^2 - \left\| \frac{f}{p(x)} \right\|_{\infty} \max \left\{ C_p^\ast, C_p^\ast \right\} \|u\|_\lambda^{p(x)} \\
- \left\| \frac{g}{q(x)} \right\|_{\infty} \max \left\{ C_q^\ast, C_q^\ast \right\} \|u\|_\lambda^{q(x)} \\
\geq \frac{\theta_0 - 1}{2\theta_0} \|u\|_\lambda^2 - A_1 \|u\|_\lambda^{p(x)} - A_2 \|u\|_\lambda^{q(x)} \\
= \|u\|_\lambda^{q(x)} \left( \frac{\theta_0 - 1}{2\theta_0} \|u\|_\lambda^{2-q(x)} - A_1 \|u\|_\lambda^{p(x)-q(x)} - A_2 \right),
\]

for each \( u \in E_\lambda \) with \( \|u\|_\lambda \geq 1 \).

Set

\[
A_1 = \left\| \frac{f}{p(x)} \right\|_{\infty} \max \left\{ C_p^\ast, C_p^\ast \right\} \left( \frac{p}{p-2} \right),
\]

\[
A_2 = \left\| \frac{g}{q(x)} \right\|_{\infty} \max \left\{ C_q^\ast, C_q^\ast \right\} \left( \frac{q}{q-2} \right).
\]

Define \( \Phi_1(t) : [0, \infty) \rightarrow \mathbb{R} \) as follows

\[
\Phi_1(t) = \frac{\theta_0 - 1}{2\theta_0} t^{2-q(x)} - A_1 t^{p(x)-q(x)} - A_2.
\]

As long as

\[
A_2 t^{2-q(x)} \left( \frac{p(x)-q(x)}{2(\theta_0-1)(p(x)-2)} \right)^{\frac{q(x)}{q(x)-2}},
\]

that is,

\[
\left( \frac{2-q(x)}{2(\theta_0-1)} \right)^{\frac{q(x)}{q(x)-2}} \left( \frac{p(x)-q(x)}{2(\theta_0-1)(p(x)-2)} \right)^{\frac{q(x)}{q(x)-2}},
\]

we can easily show that for \( t = T = \left( 1 - \frac{\theta_0 - 1}{2\theta_0} \right) \left( \frac{p}{p-2} \right) \), we have

\[
\Phi_1(T) = \max_{t \geq 0} \Phi_1(t) > 0.
\]

By

\[
\left\| \frac{f}{p(x)} \right\|_{\infty} \max \left\{ C_p^\ast, C_p^\ast \right\} \left( \frac{p}{p-2} \right),
\]

it is easy to derive that

\[
T = \left( 1 - \frac{\theta_0 - 1}{2\theta_0} \right) \left( \frac{p}{p-2} \right)^{\frac{1}{(p(x)-2)}} \geq 1.
\]

By letting \( \rho = T > 0 \) and \( \tau = \Phi(T) > 0 \), we can easily get

\[
\Psi_\lambda(u) > \tau \quad \text{for all} \quad u \in E_\lambda \quad \text{with} \quad \|u\|_\lambda = \rho.
\]
Lemma 17. Suppose that $(S_1), (V_1) - (V_d)$, and $(H_1) - (H_3)$ hold. Then, there exists $e \in E_\lambda$ with $\|e\|_{\ell} > \rho$ such that $\Psi_{\lambda}(e) < 0$ for all $\lambda > 0$, where $\rho > 0$ is given by Lemma 16.

Proof. Notice that $f : R^N \to [0, \infty)$ is a bounded nonnegative measurable function such that $f > 0$ on open interval $\Omega_\lambda \subset \Omega$; then, we can select $w_0 \in E_\lambda$ such that

$$\|w_0\|_{\lambda} = 1 \text{ and } \int_{\Omega} f(x)|w_0(x)|^{p(x)} dx > 0.$$  \hfill (61)

For all $t \geq 1$, combining (43) with (44), we have

$$\Psi_{\lambda}(tw_0) = \frac{1}{2}\|tw_0\|_{\lambda}^2 - \frac{1}{2}\int_{\Omega} V^{-1}(tw_0)^2 dx$$

$$- \int_{\Omega} \left( f(x)\frac{|tw_0|^{p(x)}}{p(x)} + g(x)\frac{|tw_0|^{q(x)}}{q(x)} \right) dx$$

$$\geq \frac{1}{2}\|tw_0\|_{\lambda}^2 - \frac{1}{2}\int_{\Omega} f(x)|w_0|^{p(x)} dx$$

$$\geq \frac{1}{2}\|w_0\|_{\lambda}^2 - \frac{p}{p^*} \int_{\Omega} f(x)|w_0|^{p(x)} dx.$$  \hfill (62)

Since $p^* > 2$, there exists $t^* \geq 1$ large enough such that $\|t^*w_0\|_{\lambda} > \rho$ and $\Psi_{\lambda}(t^*w_0) < 0$. By letting $e = t^*w_0$, we can easily reach the conclusion.

Define

$$c_\lambda = \inf_{\gamma \in \Gamma_{t^*}} \max_{0 \leq t \leq 1} \Psi_{\lambda}(\gamma(t)), \text{ and } c(\Omega_\lambda) = \inf_{\gamma \in \Gamma_{t^*}} \max_{0 \leq t \leq 1} \Psi_{\lambda}[\Gamma_{t^*}(\Omega_\lambda, c)](\gamma(t)), \hfill (63)$$

where $\Psi_{\lambda}[\Gamma_{t^*}(\Omega_\lambda, c)]$ is a restriction of $\Psi_{\lambda}$ on $H_{t^*}^{0,1}(\Omega_\lambda, C)$ and

$$\Gamma_1 = \{ \gamma \in C([0, 1], E_\lambda) : \gamma(0) = 0, \gamma(1) = e \},$$

$$\Gamma_2 = \{ \gamma \in C([0, 1], H_{t^*}^{0,1}(\Omega_\lambda, C)) : \gamma(0) = 0, \gamma(1) = e \}. \hfill (64)$$

Observe that

$$\Psi_{\lambda}[\Gamma_{t^*}(\Omega_\lambda, c)](u) = \frac{1}{2}\|u\|_{\lambda}^2 - \frac{1}{2}\int_{\Omega} V^{-1}u^2 dx$$

$$- \int_{\Omega} \left( f(x)\frac{|u|^{p(x)}}{p(x)} + g(x)\frac{|u|^{q(x)}}{q(x)} \right) dx.$$  \hfill (65)

for all $u \in H_{t^*}^{0,1}(\Omega_\lambda, C)$. Obviously, $c(\Omega_\lambda)$ is independent of $\lambda$. From the proofs of Lemma 16 and Lemma 17, we can easily derive that $\Psi_{\lambda}[\Gamma_{t^*}(\Omega_\lambda, c)]$ satisfies the mountain pass geometry.

Since $H_{t^*}^{0,1}(\Omega_\lambda, C) \subset E_\lambda$ for all $\lambda > 0$, we have $0 < \tau \leq c_\lambda \leq c(\Omega_\lambda)$ for all $\lambda > 0$. Evidently, for any $t \in [0, 1], \tau \in \Gamma_2$.

Consequently, there exists $c_0 > 0$ such that

$$c(\Omega_\lambda) \leq \max_{0 \leq t \leq 1} \Psi_{\lambda}(te) \leq c_0 < \infty,$$  \hfill (66)

being $p^* > 2$. Then,

$$0 < \tau \leq c_\lambda \leq c(\Omega_\lambda) < c_0.$$  \hfill (67)

Lemma 18. Assume that $(S_1), (V_1) - (V_d)$, and $(H_1) - (H_3)$ hold. Then, $\Psi_{\lambda}$ satisfies the (PS)$_c$ condition in $E_\lambda$ for all $c < c_0$ and $\lambda > 0$.

Proof. Assume that $\{u_n\}$ be a (PS)$_c$ sequence in $E_\lambda$ with $c < c_0$, then, $\{u_n\} \in Y_\lambda$, $\Psi_{\lambda}(u_n) \to c_\lambda$ and $\Psi'_{\lambda}(u_n) \to 0$ as $n \to \infty$. It follows from (42) and (43) and the Hölder inequality that

$$c_\lambda + O(1)\|u_n\|_{\lambda} \geq \Psi_{\lambda}(u_n) - \frac{1}{p} \int \frac{f(x)|w_0|^{p(x)}}{p(x)}|u_n|^{p(x)} dx - \frac{1}{p^*} \|u_n\|_{p^*}^2$$

$$- \int \int f(x)|w_0|^{p(x)}|u_n|^{q(x)} dx \leq \frac{1}{2} \|u_n\|_{\lambda}^2 - \frac{1}{p} \|u_n\|_{p^*}^2$$

$$\leq \frac{1}{2} \|u_n\|_{\lambda}^2 - \frac{1}{p} \|u_n\|_{p^*}^2 \leq \frac{1}{2} \|u_n\|_{\lambda}^2 - \frac{1}{p} \|u_n\|_{p^*}^2$$

$$\geq \frac{1}{2} \|u_n\|_{\lambda}^2 - \frac{1}{p} \|u_n\|_{p^*}^2 \leq \frac{1}{2} \|u_n\|_{\lambda}^2 - \frac{1}{p} \|u_n\|_{p^*}^2$$

$$\geq \frac{1}{2} \|u_n\|_{\lambda}^2 - \frac{1}{p} \|u_n\|_{p^*}^2 \leq \frac{1}{2} \|u_n\|_{\lambda}^2 - \frac{1}{p} \|u_n\|_{p^*}^2$$

for all $\lambda > 0$. In view of Lemma 16, Lemma 17, and Theorem 15, it is easy to get that for all $\lambda > 0$, there exists $\{u_n\} \subset E_\lambda$ such

$$\Psi_{\lambda}(u_n) \to c_\lambda > 0,$$

$$\Psi'_{\lambda}(u_n) \to 0, \text{ as } n \to \infty.$$  \hfill (68)

On the contrary, we suppose that $\{u_n\}$ is not bounded in $E_\lambda$. Then, there exists a subsequence still denoted by
\{u_n\} \text{ such that } \|u_n\|_{\lambda} \to \infty \text{ as } n \to \infty. \text{ Then, it follows from (69) that}

\[
\frac{\varepsilon_\lambda}{\|u_n\|_{\lambda}} + o(1) \frac{1}{\|u_n\|_{\lambda}} \geq \frac{p' - 2 \delta_0 - 1}{2p'} - \left(\frac{1}{q'} - \frac{1}{p'}\right) \cdot \|g\|_{\infty} \max\left\{C_{\delta}^\Psi \|u_n\|^{q'-2}_{\lambda}, C_{\delta}^\Psi \|u_n\|^{q'-2}_{\lambda}\right\},
\]

which leads to a contradiction. Hence, \{u_n\} is bounded in \(E_\lambda\) for all \(\lambda > 0\). Therefore, there exist a subsequence of \{u_n\} still denoted by \{u_n\} and \(u_0\) in \(E_\lambda\) such that

\[
\begin{align*}
u_n & \to u_0 \text{ in } E_\lambda, \quad u_n \to u_0 \text{ a.e. in } \Omega, \\
\|u_n\|_{r'(x)-2} & \to |u_0|_{r'(x)-2} \text{ in } L^{r'(x)}(\Omega, \mathbb{C}),
\end{align*}
\]

where \(r'(x) = r(x)/r(x) - 1\). The next step is to show \(u_n \to u_0\) in \(E_\lambda\). By Lemma 7, we can get \(u_n \to u_0\) in \(L^{r(x)}(\Omega, \mathbb{C})\). Thus,

\[
\lim_{n \to \infty} \int_\Omega |u_n - u_0|^{p(x)}dx = 0,
\]

\[
\lim_{n \to \infty} \int_\Omega |u_n - u_0|^{p(x)}dx = 0.
\]

Making use of Hölder inequality, we can obtain

\[
\begin{align*}
\int_\Omega \left|f(x)\left(|u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right)(u_n-u_0)\right|dx \\
\leq \|f\|_{\infty} \int_\Omega \left(|u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right)\left|u_n-u_0\right|dx \\
= \|f\|_{\infty} \int_\Omega \left(|u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right)\left|u_n-u_0\right|dx \\
= \|f\|_{\infty} \int_\Omega \left(|u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right)\left|u_n-u_0\right|dx \\
\leq \|f\|_{\infty} \left(\int_\Omega \left(|u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right)^{\frac{p(x)}{(p(x)-1)/p(x)}}dx\right)^{\frac{p(x)-1)}{p(x)}} \cdot \left(\int_\Omega |u_n - u_0|^{p(x)}dx\right)^{\frac{1}{p(x)}}.
\end{align*}
\]

Combining (H3), (71), and (72), we have

\[
\lim_{n \to \infty} \int_\Omega f(x)\left(|u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right)(u_n-u_0)dx = 0.
\]

Similarly, we have

\[
\lim_{n \to \infty} \int_\Omega g(x)\left(|u_n|^{q(x)-2}u_n - |u_0|^{q(x)-2}u_0\right)(u_n-u_0)dx = 0.
\]

We notice that \(u_n \to u_0\) in \(E_\lambda\) and \(\Psi'_\lambda(u_n) \to 0\); then, we obtain

\[
\lim_{n \to \infty} \left(\Psi'_\lambda(u_n) - \Psi'_\lambda(u_0), u_n - u_0\right) = 0.
\]

Therefore,

\[
\begin{align*}
o(1) &= \left\langle \Psi'_\lambda(u_n) - \Psi'_\lambda(u_0), u_n - u_0\right\rangle \\
&= \left\langle u_n - u_0, u_n - u_0\right\rangle - \Re \int_\Omega \left(f(x)|u_n|^{p(x)-2}u_n + g(x)|u_0|^{q(x)-2}u_0\right)(u_n-u_0)dx \\
&= (u_n - u_0, u_n - u_0) - \Re \int_\Omega f(x)|u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0)(u_n-u_0)dx \\
&\quad - \Re \int_\Omega g(x)|u_0|^{q(x)-2}u_0)(u_n-u_0)dx
\end{align*}
\]

which means that

\[
\lim_{n \to \infty} \|u_n - u_0\|_{\lambda,V} = 0.
\]

It follows from (43) that \(\lim_{n \to \infty} \|u_n - u_0\|_\lambda = 0\).

**Proof of Theorem 2.** In view of Lemma 16, Lemma 17, and Theorem 15, we can easily infer that for all \(\lambda > 0\), there exists a (PS)\(_c\) sequence \{\(u_n\)\} for \(\Psi'_\lambda\) on \(E_\lambda\). It derives from Lemma 18 and \(0 < c_1 < c_2(\Omega_\lambda) < c_0\) for all \(\lambda > 0\) that there exists a subsequence of \{\(u_n\)\} still denoted by \{\(u_n\)\} and \(u_1^\lambda \in E_\lambda\) such that \(u_n \to u_1^\lambda\) in \(E_\lambda\). Furthermore, \(\Psi'_\lambda(u_n) \to c_1 \geq r\) and \(u_1^\lambda\) is a solution of problem (1).

The next step is to prove that system (1) has another solution. Set

\[
\tilde{c}_\lambda = \inf \left\{\Psi'_\lambda(u) : u \in E_{\lambda}^\rho\right\},
\]

where \(E_{\lambda}^\rho = \{u \in E_\lambda : \|u\|_{\lambda} < \rho\}\) and \(\rho > 0\) is given by Lemma 16. Then, \(\tilde{c}_\lambda < 0\) for all \(\lambda > 0\). For this purpose, we first prove there exists \(v_0 \in E_{\lambda}\) such that \(\Psi'_\lambda(\sigma v_0) < 0\) for all \(\sigma > 0\).
sufficiently small. Let \( v_0 \in H^2_0(\Omega, \mathbb{C}) \) such that \( \int_{\Omega} g(x) |v_0|^2(x) \, dx > 0 \). Making use of the hypothesis \((H_2)\) and \((43)\), we obtain that for \( \sigma \in (0, 1) \) small enough,

\[
\Psi_\lambda(\sigma v_0) = \frac{\sigma^2}{2} \|v_0\|^2 - \frac{1}{2} \int_{\mathbb{R}^n} V^-(\sigma v_0)^2 \, dx \\
- \int_{\Omega} \left( \frac{f(x)}{p(x)} |\sigma v_0|^p(x) + \frac{g(x)}{q(x)} |\sigma v_0|^q(x) \right) \, dx \\
\leq \frac{\sigma^2}{2} \|v_0\|^2 - \sigma^p \int_{\Omega} \frac{f(x)}{p(x)} |v_0|^p \, dx \\
- \sigma^q \int_{\Omega} \frac{g(x)}{q(x)} |v_0|^q \, dx \\
\leq \frac{\sigma^2}{2} \|v_0\|^2 - \frac{\sigma^q}{q^*} \int_{\Omega} g(x) |v_0|^q \, dx < 0.
\]

Consequently, there exists \( v_0 \in E_\lambda \) such that \( \Psi_\lambda(\sigma v_0) < 0 \) for all \( \sigma > 0 \) sufficiently small.

Applying Lemma 16 and the Ekeland variational principle to \( B_\rho \), there exists a sequence \( \{u_n\} \) such that

\[
\bar{c}_\lambda \leq \Psi_\lambda(u_n) \leq \bar{c}_\lambda + \frac{1}{n}, \\
\Psi_\lambda(v) \geq \Psi_\lambda(u_n) - \left\| u_n - v \right\|_\lambda \frac{1}{n},
\]

for all \( v \in B_\rho \). Now we prove that \( \left\| u_n \right\|_\lambda < \rho \) for \( n \) large. On the contrary, we suppose that \( \left\| u_n \right\|_\lambda = \rho \) for infinitely many \( n \). Without loss of generality, we can suppose that \( \left\| u_n \right\|_\lambda = \rho \) for \( n \in N \). It follows from Lemma 16 that

\[
\Psi_\lambda(u_n) \geq \tau > 0.
\]

Combining with \((82)\), we obtain \( \bar{c}_\lambda \geq \tau > 0 \), which is contradictory with \( \bar{c}_\lambda < 0 \). Next, we prove \( \Psi'_\lambda(u_n) \rightarrow 0 \) in \( E^*_\lambda \) as \( n \rightarrow \infty \). Let

\[
y_n = u_n + \sigma v, \quad \text{for all } v \in B_1 := \{ u \in E_\lambda : \|u\|_\lambda = 1 \},
\]

where \( \sigma > 0 \) small enough such that \( 2\sigma \rho + \sigma^2 \leq \rho^2 - \|u_n\|_\lambda^2 \) for fixed \( n \) large. Then,

\[
\|y_n\|_\lambda^2 = \|u_n\|_\lambda^2 + 2\sigma \rho(u_n, v)_\lambda + \sigma^2 \\
\leq \|u_n\|_\lambda^2 + 2\sigma \rho + \sigma^2 = \rho^2,
\]

which implies that \( y_n \in B_\rho \). Hence, by using \((50)\), we get

\[
\Psi'_\lambda(y_n) \geq \Psi'_\lambda(u_n) - \left\| u_n - y_n \right\|_\lambda \frac{1}{n},
\]

that is,

\[
\Psi'_\lambda(u_n + \sigma v) - \Psi'_\lambda(u_n) \geq -\frac{1}{n}.
\]

Set \( \sigma \rightarrow 0^+ \); we obtain \( \left\langle \Psi'_\lambda(u_n), v \right\rangle \geq -1/n \) for each fixed \( n \) large. Similarly, choosing \( \sigma < 0 \) and \( |\sigma| \) small enough and repeating the process above, we can easily get that

\[
\left\langle \Psi'_\lambda(u_n), v \right\rangle \leq \frac{1}{n},
\]

for each fixed \( n \) large.

In short, we have

\[
\lim_{n \to \infty} \sup_{v \in B_1} |\Psi'_\lambda(u_n), v| = 0,
\]

which immediately concludes that \( \Psi'_\lambda(u_n) \rightarrow 0 \) in \( E^*_\lambda \) as \( n \rightarrow \infty \). Therefore, \( \{u_n\} \) is a \((PS)_\epsilon\) sequence for the functional \( \Psi_\lambda \). Making use of a similar proof as Lemma 18, there exists \( u^*_1 \) such that \( u_n \rightarrow u^*_1 \) in \( E_\lambda \). Therefore, we obtain a nontrivial solution \( u^*_1 \) of problem \((1)\) satisfying

\[
\Psi_\lambda(u^*_1) \leq \xi < 0, \\
\|u^*_1\|_\lambda < \rho.
\]

Hence, it is easy to conclude that

\[
\Psi_\lambda(u^*_1) = \bar{c}_\lambda \leq \xi < 0 < \tau < c_\lambda = \Psi_\lambda(u^*_1), \quad \text{for all } \lambda > 0,
\]

which completes the proof.

4. Proof of Theorem 3

In this section, we mainly give the proof of Theorem 3. In addition, inspired by [2, 17], we obtain the method to prove Theorem 3.

Proof of Theorem 3. For each sequence \( \{\lambda_n\} \) such that \( 1 \leq \lambda_n \rightarrow \infty \) as \( n \rightarrow \infty \), set \( u_n^{(i)} \) to be the critical points of \( \Psi_{\lambda_n} \) obtained in Theorem 2, where \( i = 1, 2 \). Therefore, one has

\[
\Psi_{\lambda_n}(u_n^{(2)}) \leq \xi < 0 < \tau < c_{\lambda_n} = \Psi_{\lambda_n}(u_n^{(1)}), \quad c_0,
\]

\[
\Psi_{\lambda_n'}(u_n^{(1)}) = \Psi_{\lambda_n'}(u_n^{(2)}) = 0
\]

(92)
Indeed, combining \( \Omega \overset{\lambda}{\rightarrow} n, \Omega \),

\[
\Psi_{\lambda_n}(u_n^{(i)}) = \frac{1}{2}\|u_n^{(i)}\|_{L^2}^2 - \frac{1}{2} \int_{\Omega} V^{-}(u_n^{(i)})^2 \, dx
- \frac{f(x)u_n^{(i)}(p(x) - x)}{\Omega p(x)u_n^{(i)}(p(x))} - \left( g(x)u_n^{(i)}(q(x)) \right) \, dx
- \frac{1}{2}\|u_n^{(i)}\|_{L^2}^2 - \frac{1}{2} \int_{\Omega} V^{-}(u_n^{(i)})^2 \, dx
- \frac{g(x)u_n^{(i)}(q(x))}{\Omega q(x)u_n^{(i)}(q(x))} \geq \frac{1}{2} \frac{\Omega - 1}{\Omega} \|u_n^{(i)}\|_{L^2}^2
- \frac{1}{q^2} \|u_n^{(i)}\|_{L^2}^2
\]

We notice that

\[
\lim_{n \to \infty} \left\langle \Psi_{\lambda_n}'(u_n^{(i)}), u_n^{(i)} \right\rangle = \lim_{n \to \infty} \left\langle \Psi_{\lambda_n}'(u_n^{(i)}), u_n^{(i)} \right\rangle = 0. 
\]

Therefore, we have

\[
\|u_n^{(i)}\|_{L^2}^2 = \int_{\Omega} V^{-}(u_n^{(i)})^2 \, dx
+ \left( \int f(x)u_n^{(i)}(p(x)) + g(x)u_n^{(i)}(q(x)) \, dx + o(1) \right),
\]

\[
\left\langle u_n^{(i)}, u_n^{(i)} \right\rangle_{\lambda_n} = \Re \int_{\Omega} V^{-}(u_n^{(i)})^2 \, dx
+ \Re \left( \int f(x)u_n^{(i)}(p(x)) + g(x)u_n^{(i)}(q(x)) \, dx + o(1) \right),
\]

By (97)–(101), we have

\[
\lim_{n \to \infty} \|u_n^{(i)}\|_{\lambda_n}^2 = \lim_{n \to \infty} \left\langle u_n^{(i)}, u_n^{(i)} \right\rangle_{\lambda_n} = \|u_n^{(i)}\|_{H_{0,n}^{1}(\Omega,\mathbb{C})}^2. 
\]

On the other hand, the weak lower semicontinuity of norm yields that

\[
\|u_n^{(i)}\|_{H_{0,n}^{1}(\Omega,\mathbb{C})} \leq \operatorname{liminf}_{n \to \infty} \|u_n^{(i)}\|_{H_{0,n}^{1}(\Omega,\mathbb{C})} \leq \operatorname{limsup}_{n \to \infty} \|u_n^{(i)}\|_{H_{0,n}^{1}(\Omega,\mathbb{C})} \leq \|u_n^{(i)}\|_{H_{0,n}^{1}(\Omega,\mathbb{C})}.
\]

To sum up, we can see that

\[
\lim_{n \to \infty} \|u_n^{(i)}\|_{H_{0,n}^{1}(\Omega,\mathbb{C})} \leq \|u_n^{(i)}\|_{H_{0,n}^{1}(\Omega,\mathbb{C})}.
\]

By Proposition 3.32 of [18], we can obtain that \( u_n^{(i)} \overset{w}{\to} u^{(i)} \) in \( H_{0,0}(\Omega, \mathbb{C}) \). We notice that \( \lim_{n \to \infty} \left\langle \Psi_{\lambda_n}'(u_n^{(i)}), v \right\rangle = 0 \), for any \( v \in C_0^0(\Omega_0, \mathbb{C}) \). Hence,

\[
\Re \left( \int_{\mathbb{R}^N} \frac{(u_n^{(i)}(x) - e^{(x,y),n}(xy)^{1/2})u_i(0)}{|x-y|^{N+2d(y)}} \, dx \right) = 0
- \Re \left( \int_{\Omega} V^{-}(u_n^{(i)})^2 \, dx + \int_{\Omega} f(x)u_n^{(i)}(p(x) - x)^2 \, dx + g(x)u_n^{(i)}(q(x) - x)^2 \, dx \right) = 0.
\]
Since the density of $C_0^\infty(\Omega_0, \mathbb{C})$ in $H_0^1(\Omega_0, \mathbb{C})$, we can obtain that $u^{(i)}$ is a weak solution of problem (12).

Together with (92), $u^{(i)} = 0$ a.e. in $\mathbb{R}^N \setminus (V^*)^{-1}(0)$ and the constants $\xi, \tau$ are independent of $\lambda$, we have

\[
\frac{1}{2} \left\| \frac{d}{d\lambda} u^{(1)} \right\|^2_{H_0^1(\Omega_0, \mathbb{C})} - \frac{1}{2} \int_{\Omega_0} V' \left( \frac{d}{d\lambda} u^{(1)} \right)^2 \, dx \\
= \frac{1}{2} \int_{\Omega_0} \left( \frac{f(x)}{p(x)} \right) \left| \frac{d}{d\lambda} u^{(1)} \right|^p \, dx + \frac{\xi}{\tau} \left( \frac{d}{d\lambda} u^{(1)} \right)^\tau \right) \, dx \geq 0, \\
\frac{1}{2} \left\| \frac{d}{d\lambda} u^{(2)} \right\|^2_{H_0^1(\Omega_0, \mathbb{C})} - \frac{1}{2} \int_{\Omega_0} V' \left( \frac{d}{d\lambda} u^{(2)} \right)^2 \, dx \\
- \int_{\Omega_0} \left( \frac{f(x)}{p(x)} \right) \left| \frac{d}{d\lambda} u^{(2)} \right|^p \, dx + \frac{\xi}{\tau} \left( \frac{d}{d\lambda} u^{(2)} \right)^\tau \right) \, dx \leq 0,
\]

which implies that $u^{(i)} \neq 0$ and $u^{(1)} \neq u^{(2)}$.

Now we consider the case where $f(x), g(x) \equiv$ constants; that is

\[
\begin{cases}
(-\Delta)^{\frac{1}{2}} u + V_\lambda(x) u = a|u|^{p(x)-2}u + b|u|^{q(x)-2}u \text{ in } \Omega, \\
u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where $a, b$ are two nonnegative constants. Correspondingly, the energy functional $\Psi_\lambda: E_\lambda \rightarrow \mathbb{R}$ is

\[
\Psi_\lambda(u) = \frac{1}{2} \left\| u \right\|^2_{L^2(\Omega)} - \frac{1}{2} \int_{\Omega} V - u^2 \, dx \\
- \int_{\Omega} \left( \frac{a}{p(x)} \right) |u|^{p(x)} \, dx + \frac{b}{q(x)} \left| u^{q(x)} \right| \, dx \\
- \frac{1}{2} \left\| u \right\|^2_{L^2(\Omega)} - \int_{\Omega} \left( \frac{a}{p(x)} \right) |u|^{p(x)} \, dx + \frac{b}{q(x)} \left| u^{q(x)} \right| \, dx.
\]

Next, we mainly prove the existence of infinitely many solutions to problem (107) by using four different methods.

**Theorem 19.** Assume that $(S)_\lambda$, $(S)'_\lambda$, $(V)'_\lambda$–$(V)_\lambda$, and $(H)'_\lambda$–$(H)_\lambda$ hold. Let $N \geq 2$. Then, problem (107) has infinitely many solutions.

**Proof.** Method 1: It is easy to verify that functional $\Psi_\lambda$ is even and satisfies $\Psi_\lambda(0) = 0$. Furthermore, Lemma 18 shows that functional $\Psi_\lambda$ is bounded from below in $E_\lambda$ and satisfies the (PS) condition. For any $k \in \mathbb{N}$ and $\rho_k > 0$, let $S_{\rho_k} = \{ u \in E_\lambda : \| u \|_\lambda = \rho_k \}$; then, for any $u \in S_{\rho_k}$, one has

\[
\Psi_\lambda(u) = \frac{1}{2} \left\| u \right\|^2_{L^2(\Omega)} - \int_{\Omega} \left( \frac{a}{p(x)} \right) |u|^{p(x)} \, dx - \int_{\Omega} \left( \frac{b}{q(x)} \right) |u|^{q(x)} \, dx \\
\leq \frac{1}{2} \left\| u \right\|^2_{L^2(\Omega)} - \frac{b}{q^*} \int_{\Omega} |u|^{q(x)} \, dx \\
\leq \frac{1}{2} \left\| u \right\|^2_{L^2(\Omega)} - \frac{b}{q^*} \min \left\{ \left\| u \right\|^q_{L^2(\Omega)}, \left\| u \right\|^q_{L^2(\Omega)} \right\}.
\]

We find that there exists $C_{\lambda, \rho_k} > 0$ such that $\| u \|_{\Omega, \xi} \geq C_{\lambda, \rho_k} \| u \|_\lambda$, since all norms are equivalent on finite dimensional Banach space. Then, by $1 < q(x) < 2$, it gains

\[
\sup_{u \in X_{\lambda}^* \cap S_{\rho_k}} \Psi_\lambda(u) \leq \frac{1}{2} \left\| u \right\|^2_{L^2(\Omega)} - \frac{b}{q^*} \min \left\{ \left\| u \right\|^q_{L^2(\Omega)}, \left\| u \right\|^q_{L^2(\Omega)} \right\}.
\]

Letting $\| u \|_\lambda = \rho_k$ small enough, we can obtain $\sup_{u \in X_{\lambda}^* \cap S_{\rho_k}} \Psi_\lambda(u) < 0 = \Psi_\lambda(0)$. Furthermore, we assert that (ii) of Theorem 9 does not work. In fact, (109) means $\Psi_\lambda(u) < 0$ since $\Psi_\lambda(u) < 0$ as $t$ small enough with the given $\varepsilon \in E_\lambda$. Thus, by Theorem 9, we get that problem (1) has a sequence of solutions $\{ u_k \}$ with $\| u_k \|_\lambda \rightarrow 0$ as $k \rightarrow \infty$. In short, problem (1) has infinitely many solutions for all $\lambda > 0$.

Method 2. To start with, we assert that for any finite dimensional subspace $X$ of $E_\lambda$, there exists $r_j = r_j(X)$ such that $\Psi_\lambda(u) < 0$ for all $u \in E_\lambda \setminus B_{r_j}(X)$, where $B_{r_j}(X) = \{ u \in E_\lambda : \| u \|_\lambda < r_j \}$. Indeed, for each $i \geq 1$, we can easily get that

\[
\Psi_\lambda(tu) = \frac{t^2}{2} \left\| u \right\|^2_{L^2(\Omega)} - \frac{t^2}{2} \int_{\Omega} \left( \frac{a}{p(x)} \right) |tu|^{p(x)} \, dx - \int_{\Omega} \left( \frac{b}{q(x)} \right) |tu|^{q(x)} \, dx \\
\leq \frac{t^2}{2} \left\| u \right\|^2_{L^2(\Omega)} - \frac{at^p}{p^*} \int_{\Omega} |tu|^{p(x)} \, dx \\
\leq \frac{t^2}{2} \left\| u \right\|^2_{L^2(\Omega)} - \frac{at^p}{p^*} \min \left\{ \left\| u \right\|^p_{L^2(\Omega)}, \left\| u \right\|^p_{L^2(\Omega)} \right\}.
\]

We observe that there exists $C_X > 0$ such that $\| u \|_{L^2(\Omega)} \geq C_X \| u \|_\lambda$, since all norms are equivalent on finite dimensional Banach space $X$. Then, by $p^* > 2$, it gains

\[
\Psi_\lambda(tu) \leq \frac{t^2}{2} \left\| u \right\|^2_{L^2(\Omega)} - \frac{at^p}{p^*} \min \left\{ C_X^p, C_X^p \right\} \| u \|_\lambda^p \quad \rightarrow -\infty \quad \text{as} \quad t \rightarrow \infty.
\]

Thus, there exists $r_j > 0$ large enough such that $\Psi_\lambda(u) < 0$ for all $u \in E_\lambda$, with $\| u \|_\lambda = r_j$ and $r_j \geq r_1$. Consequently, the assertion is valid.

From Lemma 18, we know that $\Psi_\lambda$ satisfies the (PS) condition for any $c \in \mathbb{R}$. Obviously, $\Psi_\lambda(0) = 0$ and $\Psi_\lambda$ is an even functional. In short, it follows from Theorem 8 that there
exists an unbounded sequence of solutions of problem (1) for all $\lambda > 0$.

**Lemma 19** (see Lemma 4.1, [10]). Let $1 < r(x) < 2N/(N - 2s(x, x))$ for all $x \in \Omega$. For any $k \in \mathbb{N}$, define

$$
\beta_k := \left\{ \|u\|_{L^{0,0}(\Omega, C)} : u \in Z_k, \|u\|_\lambda = 1 \right\}.
$$

Then, $\beta_k \rightarrow 0$, as $k \rightarrow \infty$.

**Method 3.** By Remark 14 and Lemma 18, we know that $\Psi_\lambda$ satisfies the (PS)$_s$ condition for any $c \in \mathbb{R}$. To start with, we will prove $(D_3)$ is satisfied. It follows from (36) and (43) that

$$
\Psi_\lambda(u) = \frac{1}{2} \|u\|^2_{0, \lambda} - \frac{\|
abla u\|^2}{\int_\Omega \nabla u \cdot \nabla u} - \frac{1}{2} \int_\Omega \left( \frac{a}{p(x)} |u|^{p(x)} + \frac{b}{q(x)} |u|^{q(x)} \right) dx
$$

Choosing $\gamma_k = (\frac{(\theta_0 - 1)}{8a\beta_k^p})^{1/(p - 2)}$. Combing with Lemma 20, we know that $\beta_k \rightarrow 0$ as $k \rightarrow \infty$. Thus, one has $\gamma_k \rightarrow +\infty$ as $k \rightarrow \infty$ and

$$
\Psi_\lambda(u) \geq \gamma_k \left( \frac{3}{8} \delta_0 - \frac{1}{8} \|y_k\|^{2-q} - \frac{b}{q} \beta_k^p \right) \rightarrow +\infty,
$$

as $k \rightarrow \infty$. In conclusion, $(D_3)$ is fulfilled. It is easy to check that satisfying $\Psi_\lambda(-u) = \Psi_\lambda(u)$. Thus, by Theorem 10, we can obtain that problem (1) has infinitely many solutions for all $\lambda > 0$.

**Method 4.** First, we will show that $(D_4)$ is fulfilled. By (42) and (43), one has

$$
\Psi_\lambda(u) \geq \frac{1}{2} \|u\|^2_{0, \lambda} - \frac{a}{p} \max \left\{ C_p^{(0)} \|u\|^{p(x)}_{\Omega, \lambda}, \frac{C_p^{(0)} \|u\|^{p(x)}_{\Omega, \lambda}}{\|u\|^{p(x)}_{\Omega, \lambda}} \right\}
$$

Choosing $\gamma_k = \frac{(\theta_0 - 1)}{8a\beta_k^p}^{1/(p - 2)}$. Combing with Lemma 20, we know that $\beta_k \rightarrow 0$ as $k \rightarrow \infty$. Hence, there exists $k$ such that $r_k \leq \gamma_k < \gamma_k$. Consequently, for $k \geq k$ and $u \in Z_k$ with $\|u\|_{\lambda} = 1$, we can obtain that $\Psi_\lambda(u) \geq 0$. Next, we will show $(D_4)$ is fulfilled. For any $u \in Y_k$, $\|u\|_{\lambda} = \gamma_k$ with $r_k > \gamma_k > 0$, we get

$$
\Psi_\lambda(u) \geq \gamma_k \left( \frac{3}{8} \delta_0 - \frac{1}{8} \|y_k\|^{2-q} - \frac{b}{q} \beta_k^p \right) \rightarrow +\infty.
$$

Choosing $\gamma_k = \frac{8b\delta_0 \beta_k^p}{3q(\theta_0 - 1)}$. We have

$$
\Psi_\lambda(u) \geq \gamma_k \left( \frac{3}{8} \delta_0 - \frac{1}{8} \|y_k\|^{2-q} - \frac{b}{q} \beta_k^p \right) \rightarrow +\infty.
$$

Since $1 < q^* < q^* < 2 < p^* < 2N/(N - 2s(x, x))$, we can choose $r_k \in (0, 1)$ small enough such that for all $u \in E_{\lambda}^{(1)}$ with $\|u\|_{\lambda} \leq r_k$,

$$
\frac{1}{2} \|u\|^2_{0, \lambda} \geq \frac{a}{p} \max \left\{ C_p^{(0)} \|u\|^{p(x)}_{\Omega, \lambda}, \frac{C_p^{(0)} \|u\|^{p(x)}_{\Omega, \lambda}}{\|u\|^{p(x)}_{\Omega, \lambda}} \right\}
$$

hold. Then,

$$
\Psi_\lambda(u) \geq \frac{3}{8} \delta_0 - \frac{1}{8} \|y_k\|^{2-q} - \frac{b}{q} \beta_k^p \rightarrow +\infty.
$$

Choosing $\gamma_k = (\frac{(\theta_0 - 1)}{8a\beta_k^p})^{1/(p - 2)}$. Combing with Lemma 20, we know that $\beta_k \rightarrow 0$ as $k \rightarrow \infty$. Hence, there exists $k$ such that $r_k \leq \gamma_k < \gamma_k$. Consequently, for $k \geq k$ and $u \in Z_k$ with $\|u\|_{\lambda} = 1$, we can obtain that $\Psi_\lambda(u) \geq 0$. Next, we will show $(D_4)$ is fulfilled. For any $u \in Y_k$, $\|u\|_{\lambda} = \gamma_k$ with $r_k > \gamma_k > 0$, we get

$$
\Psi_\lambda(u) \leq \frac{1}{2} \|u\|^2_{0, \lambda} - \frac{1}{2} \int_{\mathbb{R}^N} \int_\Omega \nabla u \cdot \nabla u dx
$$

Choosing $\gamma_k = \frac{(\theta_0 - 1)}{8a\beta_k^p}^{1/(p - 2)}$. Combing with Lemma 20, we know that $\beta_k \rightarrow 0$ as $k \rightarrow \infty$. Thus, one has $\gamma_k \rightarrow +\infty$ as $k \rightarrow \infty$ and

$$
\Psi_\lambda(u) \geq \gamma_k \left( \frac{3}{8} \delta_0 - \frac{1}{8} \|y_k\|^{2-q} - \frac{b}{q} \beta_k^p \right) \rightarrow +\infty,
$$

as $k \rightarrow \infty$. In conclusion, $(D_4)$ is fulfilled. It is easy to check that satisfying $\Psi_\lambda(-u) = \Psi_\lambda(u)$. Thus, by Theorem 10, we can obtain that problem (1) has infinitely many solutions for all $\lambda > 0$.

$$
\Psi_\lambda(u) \geq \gamma_k \left( \frac{3}{8} \delta_0 - \frac{1}{8} \|y_k\|^{2-q} - \frac{b}{q} \beta_k^p \right) \rightarrow +\infty,
$$

as $k \rightarrow \infty$. In conclusion, $(D_4)$ is fulfilled.
We find that there exists $C_{Y_k} > 0$ such that

\[ \|u\|_{H^s(\Omega)} \geq C_{Y_k} \|u\|_{A}, \quad \text{and} \quad \|u\|_{L^{p_c}(\Omega)} \geq C_{Y_k} \|u\|_{A}, \]

since all norms are equivalent on finite dimensional Banach space $Y_k$. Then,

\[
\Psi_A(u) \leq \frac{1}{2} \|u\|_A^2 - \frac{a}{p^*} \min \left\{ C_{Y_k}^p \|u\|_A, C_{Y_k}^p \|u\|_A^p \right\}
- \frac{b}{q} \min \left\{ C_{Y_k}^q, C_{Y_k}^q \right\} \|u\|_A^q
\leq \frac{1}{2} \|u\|_A^2 - \frac{a}{p^*} \min \left\{ C_{Y_k}^p, C_{Y_k}^p \right\} \|u\|_A^p
- \frac{b}{q} \min \left\{ C_{Y_k}^q, C_{Y_k}^q \right\} \|u\|_A^q < 0,
\]

as $y_k > 0$ small enough. Now we check $(D_3)$ is fulfilled. It follows from $(D_2)$ that for $k \geq k$ and $u \in Z_k$ with $\|u\|_A \leq r_k$,

\[
\Psi_A(u) \geq \frac{3}{8} \delta_0 - \frac{1}{\delta_0} \|u\|_A^2 - \frac{b}{q} \beta_k^q \|u\|_A^q
\geq - \frac{b}{q} \beta_k^q \|u\|_A^q \geq - \frac{b}{q} \beta_k^q r_k^q,
\]

thanks to $\beta_k \to 0$ as $k \to \infty$. Thus, one has $r_k \to 0$ as $k \to \infty$. Thus, $(D_3)$ is also satisfied. In conclusion, by Theorem 11, we can obtain that problem (1) has infinitely many solutions for all $\lambda > 0$.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Authors’ Contributions**

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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