WEAK-TYPE ESTIMATES FOR THE BERGMAN PROJECTION ON THE POLYDISC AND THE HARTOGS TRIANGLE

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ABSTRACT. In this paper, we investigate the weak-type regularity of the Bergman projection. The two domains we focus on are the polydisc and the Hartogs triangle.

For the polydisc we provide a proof that the weak-type behavior is of “$L \log L$” type. This result is likely known to the experts, but does not appear to be in the literature.

For the Hartogs triangle we show that the operator is of weak-type $(4,4)$; settling the question of the behavior of the projection at this endpoint. At the other endpoint of interest, we show that the Bergman projection is not of weak-type $(\frac{4}{3}, \frac{4}{3})$ and provide evidence as to what the correct behavior at this endpoint might be.

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1. Introduction

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. Let $L^2(\Omega)$ denote the space of square-integrable functions with respect to the Lebesgue measure $dV$ on $\Omega$. Let $A^2(\Omega)$ denote the subspace of square-integrable holomorphic functions. The Bergman projection $P$ is the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$. Associated with $P$, there is a unique function $K_\Omega$ on $\Omega \times \Omega$ such that for any $f \in L^2(\Omega)$:

$$P(f)(z) = \int_\Omega K_\Omega(z; \overline{w}) f(w) dV(w). \quad (1.1)$$

Let $P^+$ denote the absolute Bergman projection defined by:

$$P^+(f)(z) = \int_\Omega |K_\Omega(z; \overline{w})| f(w) dV(w). \quad (1.2)$$

Mapping properties of $P$ have been an object of considerable interest for many years. By its definition, the Bergman projection is a $L^2$ bounded operator. It is natural to consider the regularity of $P$ in other settings. Using known estimates for the Bergman kernel, $L^p$ regularity results have been obtained in various settings. See [Fel74, PS77, McN89, McN94a, NRSW88, McN94a, McN94b, MS94, CD06, EL08, BS12, Huo18]. In all these results, the domain needs to satisfy certain nice boundary conditions. On some other domains, the projection has only a finite range of mapping regularity. See for example [KP08, BS12, Zey13, CZ16, EM16, EM17, Che17] for recent progress along this line.

Among the results mentioned above, there are mainly two techniques adopted. One is to use the Schur’s test (see for example [Zhu05]), where boundedness can be deduced from analyzing the behavior of the absolute Bergman projection on a certain test function $h$. In many cases, one can choose $h$ to be the distance function to the boundary of the domain $\Omega$.
or the Bergman kernel on the diagonal. The second approach is to show that the (absolute) Bergman projection satisfies certain weak-type estimate. For example, if the projection is of weak-type \((1, 1)\), then its \(L^2\) regularity together with interpolation theorem implies the \(L^p\) regularity for \(1 < p \leq 2\). Since the Bergman projection is self-adjoint, \(L^p\) regularity for \(1 < p \leq 2\) yields \(L^p\) regularity for \(1 < p < \infty\).

While both techniques are powerful tools on obtaining \(L^p\) regularity results, the Schur’s test is unable to tell the weak-type regularity of the operator near the endpoint of its \(L^p\) range. In this paper, we choose the polydisc and the Hartogs triangle as two classical examples and investigate the weak-type regularity of the Bergman projection on them.

The polydisc serves as a simple example where the Bergman kernel function is of a product form. It is well known that the weak-type behavior of the classical operators in the multi-parameter setting could be very different from the one-parameter case. For instance, the double Hilbert transform \(H_1H_2\) on \(\mathbb{R}^2\) and the Hilbert transform \(H\) on \(\mathbb{R}\) behave differently near \(L^1\): \(H\) is of weak-type \((1,1)\) while \(H_1H_2\) is of weak-type \(L \log^+ L\). See for example [Fe72]. By the same reason, one should expect the weak-type regularity of the Bergman projection on \(\mathbb{C}^n\) to be different from the Bergman projection on \(\mathbb{D}\).

The Hartogs triangle \(\mathbb{H}\), on the other hand, is a classical model where the projection has only limited \(L^p\) regularity. It was shown by Chakrabarti and Zeytuncu in [CZ16] that the Bergman projection on the Hartogs triangle is \(L^p\)-regular if and only if \(\frac{4}{3} < p < 4\). Since the Hartogs triangle is biholomorphically equivalent to \(\mathbb{D} \times \mathbb{D}\)\(\setminus\{0\}\), the \(L^p\) boundedness of the Bergman projection on \(\mathbb{H}\) can also be related to the regularity of the projection on the weighted space \(L^p(\mathbb{D}^2, |z_2|^{2-p})\). From this perspective, both the product structure of \(\mathbb{D}^2\) and the weight \(|z_2|^{2-p}\) may affect the weak-type regularity of the projection near \(L^{\frac{4}{3}}\) and \(L^4\).

We summarize our results about the Bergman projection \(P\) as follows:

1. On the bidisc, \(P\) is not of weak-type \((1,1)\). (Theorem 3.1)
2. On the polydisc \(\mathbb{D}^n\), \(P\) is of weak-type \(L(\log^+ L)^{n-1}\). (Theorem 3.7)
3. On the Hartogs triangle \(\mathbb{H}\), \(P\) is not of weak-type \((4/3, 4/3)\). (Theorem 4.1)
4. On the Hartogs triangle \(\mathbb{H}\), \(P\) is of weak-type \((4, 4)\). (Theorem 4.2)
5. For any \(\epsilon > 0\), \(P\) on \(\mathbb{H}\) is bounded from \(L^{\frac{4}{3}}(\mathbb{H}, |z_2|^{-\epsilon})\) to \(L^{\frac{4}{3}, \infty}(\mathbb{H})\). (Theorem 4.6)

Results (1) and (2) above are not surprising from a multi-parameter analysis perspective, and hence could be known to people. Since we couldn’t find them in the literature, we decide to put them here. As a consequence of Result (4), the projection \(P\) is bounded from the Lorentz space \(L^{4/3, 1}(\mathbb{H})\) to \(L^{4/3}(\mathbb{H})\). See Remark 4.4. Also, we provide refinements of Result (5). See Theorems 4.7 and 4.9.

The paper is organized as follows: In Section 2, we recall the definition of the Hartogs triangle and provide lemmas that will be used in the paper. In Section 3, we consider weak-type estimates for the Bergman projection on the bidisc and the polydisc. We give an example in the proof of Theorem 3.1 to show that the projection \(P\) is not of weak-type \((1,1)\). We show in Theorem 3.7 that \(P\) on the polydisc \(\mathbb{D}^n\) is of weak-type \(L(\log^+ L)^{n-1}\). In Section 4, we state and prove weak-type results for the Bergman projection on the Hartogs triangle.

2. Preliminaries

Let \(\mathbb{D}\) denote the unit disc in \(\mathbb{C}\). The Bergman kernel on \(\mathbb{D}\) is given by,

\[
K_\mathbb{D}(z; w) = \frac{1}{\pi(1 - zw)^2}, \quad \text{for } z, w \in \mathbb{D}.
\]
Since the Bergman kernel on the product domain $\Omega_1 \times \Omega_2$ equals the product of the kernel functions on $\Omega_1$ and $\Omega_2$, the Bergman kernel on the polydisc $\mathbb{D}^n$ is given by,

$$K_{\mathbb{D}^n}(z; \bar{w}) = \prod_{j=1}^{n} \frac{1}{\pi(1 - z_j \bar{w}_j)^2}, \quad \text{for } z, w \in \mathbb{D}^n. \quad (2.2)$$

The Hartogs triangle $\mathbb{H}$ is defined by $\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}$. Let $\mathbb{D}^*$ denote the punctured disc $\mathbb{D}\setminus\{0\}$. The mapping $(z_1, z_2) \mapsto (\frac{z_1}{z_2}, z_2)$ is a biholomorphism from $\mathbb{H}$ onto $\mathbb{D} \times \mathbb{D}^*$. The biholomorphic transformation formula (see [Kra01]) then implies that

$$K_{\mathbb{H}}(z_1, z_2; \bar{w}_1, \bar{w}_2) = \frac{1}{z_2 \bar{w}_2} K_{\mathbb{D} \times \mathbb{D}^*} \left( \frac{z_1}{z_2}, \frac{2 \bar{w}_1}{z_2}, \frac{2 \bar{w}_2}{z_2} \right) = \frac{1}{z_2 \bar{w}_2} K_{\mathbb{D} \times \mathbb{D}^*} \left( \frac{z_1}{z_2}, \frac{\bar{w}_1}{\bar{w}_2}, \frac{\bar{w}_2}{\bar{w}_2} \right) = \frac{1}{\pi^2 z_2 \bar{w}_2 (1 - \frac{z_1 \bar{w}_1}{z_2 \bar{w}_2})^2 (1 - z_2 \bar{w}_2)^2}. \quad (2.3)$$

The second equality sign above holds since $A^2(\mathbb{D} \times \mathbb{D}^*)$ and $A^2(\mathbb{D}^2)$ are identical.

Given functions of several variables $f$ and $g$, we use $f \lesssim g$ to denote that $f \leq Cg$ for a constant $C$. If $f \lesssim g$ and $g \lesssim f$, then we say $f$ is comparable to $g$ and write $f \approx g$. We reference below the Forelli-Rudin estimate. See for example [Zhu05] for its proof.

**Lemma 2.1** (Forelli-Rudin). Let $\sigma$ denote Lebesgue measure on the unit circle $S^1 \subset \mathbb{C}$. For $\epsilon < 1$ and $w \in \mathbb{D}$, let

$$a_{\epsilon, \delta}(w) = \int_{\mathbb{D}} \frac{(1 - |\eta|^2)^{-\epsilon}}{|1 - w \bar{\eta}|^{2-\epsilon - \delta}} dV(\eta), \quad (2.4)$$

and let

$$b_{\delta}(w) = \int_{S^1} \frac{1}{|1 - w \eta|^{1-\delta}} d\sigma(\eta). \quad (2.5)$$

Then

1. for $\delta > 0$, both $a_{\epsilon, \delta}$ and $b_{\delta}$ are bounded on $\mathbb{D}$;
2. for $\delta = 0$, both $a_{\epsilon, \delta}(w)$ and $b_{\delta}(w)$ are comparable to the function $-\log(1 - |w|^2)$;
3. for $\delta < 0$, both $a_{\epsilon, \delta}(w)$ and $b_{\delta}(w)$ are comparable to the function $(1 - |w|^2)^{\delta}$.

We also recall the weighted inequalities by Bekollé and Bonami in [BB78] for $P$ and $P^+$ on the unit disk:

**Lemma 2.2** (Bekollé-Bonami). Let $T_z$ denote the Carleson tent over $z$ in $\mathbb{D}$ defined as below:

- $T_z := \left\{ w \in \mathbb{D} : \frac{|1 - \bar{w}|}{|z|} < 1 - |z| \right\}$ for $z \neq 0$, and
- $T_z := \mathbb{D}$ for $z = 0$.

Let the weight $\mu$ be a positive, locally integrable function on the $\mathbb{D}$. Let $1 < p < \infty$. Then the following conditions are equivalent:

1. $P : L^p(\mathbb{D}, \mu) \mapsto L^p(\mathbb{D}, \mu)$ is bounded;
2. $P^+ : L^p(\mathbb{D}, \mu) \mapsto L^p(\mathbb{D}, \mu)$ is bounded;
3. The Bekollé-Bonami constant $B_p(\mu)$ is finite where:

$$B_p(\mu) := \sup_{z \in \mathbb{D}} \frac{\int_{T_z} \mu(w) dV(w)}{\int_{T_z} dV(w)} \left( \frac{\int_{T_z} \mu^{-\frac{1}{p-1}}(w) dV(w)}{\int_{T_z} dV(w)} \right)^{p-1}.$$
We end this section by recalling the definitions of the weak $L^p$ space, weak-type $(p,p)$, and the $L(\log^+ L)^k$ space. Given a subset $U$ in the domain $\Omega$ and let $\mu$ be a measure on $\Omega$. We use the notation $\mu(U)$ to denote the $\mu$-measure of $U$. When $\mu$ is the Lebesgue measure, we will simply write $|U|.$

**Definition 2.3.** Let $(X, \mu)$ be a measure space. For $0 < p < \infty$, the weak $L^p$ space $L^{p,\infty}(X, \mu)$ is defined as the set of all $\mu$-measurable functions $f$ such that

$$
\|f\|_{L^{p,\infty}} = \inf \left\{ C > 0 : \mu(\{x \in X : |f(x)| > \lambda\}) \leq \frac{C^p}{\lambda^p} \text{ for all } \lambda > 0 \right\} < \infty. \tag{2.6}
$$

**Definition 2.4.** Let $(X, \mu)$ and $(Y, \nu)$ be two measure spaces. Let $0 < p < \infty$ and $0 < q < \infty$. An operator $T$ that is said to be of weak-type $(p,q)$ if $T$ is bounded from $L^p(X, \mu)$ to $L^{q,\infty}(Y, \nu)$, i.e. for any $f \in L^p(X, \mu)$ and any $\lambda > 0$,

$$
\nu(\{y \in Y : |T(f)(y)| > \lambda\}) \lesssim \frac{\|f\|_{L^p(X, \mu)}^q}{\lambda^q}. \tag{2.7}
$$

**Definition 2.5.** Set $\log^+ x := \begin{cases} 0 & x = 0 \\ \max\{0, \log x\} & x > 0. \end{cases}$ Let $L^p(\log^+ L)^k(\Omega)$ be the set of all functions $f$ on $\Omega$ satisfying $\int_{\Omega} |f(z)|^p (\log^+ |f(z)|)^k dV < \infty$. We define the Orlicz space $L^p(\log^+ L)^k(\Omega)$ to be the linear hull of $L^p(\log^+ L)^k(\Omega)$ with the norm

$$
\|f\|_{L^p(\log^+ L)^k(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} |f(z)|/\lambda^p \left( \log^+ |f(z)|/\lambda \right)^k dV(z) \leq 1 \right\}.
$$

We say an operator $T$ is of weak-type $L^p(\log^+ L)^k$ on $\Omega$ if for any $f \in L^p(\log^+ L)^k(\Omega)$ and any $\lambda > 0$,

$$
|\{z \in \Omega : |T(f)(z)| > \lambda\}| \lesssim \frac{\|f\|_{L^p(\log^+ L)^k(\Omega)}^p}{\lambda^p}. \tag{2.8}
$$

For more details about the Orlicz space, see for example [RR91].

**Remark 2.6.** It is worth noting that when $\nu$ is a finite measure and $\lambda$ is chosen to be small, inequalities (2.7) and (2.8) trivially holds. Since all the domains involved in this paper are bounded and hence have finite Lebesgue measure, we only need to check (2.7) for large $\lambda$ to prove the weak-type results.

3. **Weak-type estimates for the Bergman projection on the polydisc**

The results in this section are not surprising from a multi-parameter analysis perspective, and could be known to experts. Since we couldn’t find them in the literature, we decide to put them here with their proofs.

**Theorem 3.1.** The Bergman projection $P$ on the bidisc $\mathbb{D}^2$ is not of weak-type $(1,1)$.

**Proof.** By (2.7), it suffices to show that there exists a parameter family of integrable functions $\{f_s\}$ on $\mathbb{D}^2$ satisfying the inequality below:

$$
|\{(z_1, z_2) \in \mathbb{D}^2 : |P(f_s)(z_1, z_2)| > \lambda\}| \geq \frac{C_s \|f_s\|_{L^1}}{\lambda}, \tag{3.1}
$$

where $C_s$ can be arbitrarily large. For $1 > s > 0$, we set

$$
f_s(w) = (1 - s^2)^4|1 - sw_1|^{-4}|1 - sw_2|^{-4} = \pi^4(1 - s^2)^4 K_{\mathbb{D}^2}(s, s; \bar{w}_1, \bar{w}_2).$$
Hence \( \| f_s \|_{L^1} = \pi^4(1-s^2)^4 K_{D^2}(s, s; s, s) = \pi^2 \). On the other hand, it is easy to see that

\[
P(f_s)(z_1, z_2) = \int_{\mathbb{D}^2} \frac{(1-s)^2(1-s\bar{w}_1)^2(1-s\bar{w}_2)^2}{\pi^2(1-z_1\bar{w}_1)(1-z_2\bar{w}_2)^2} \frac{1}{(1-s\bar{w}_1)^2(1-s\bar{w}_2)^2} dV(w_1, w_2)
\]

\[
= P\left( \frac{(1-s)^2(1-s\bar{w}_1)^2(1-s\bar{w}_2)^2}{(1-z_1\bar{w}_1)(1-z_2\bar{w}_2)^2} \right) (s, s)
\]

\[
= \frac{(1-s)^2(1-s^2)(1-s^2)^2}{(1-sz_1)^2(1-sz_2)^2} = (1-sz_1)^{-2}(1-sz_2)^{-2}.
\]  

(3.2)

Therefore \( \{(z_1, z_2) \in \mathbb{D}^2 : |P(f_s)(z_1, z_2)| > \lambda \} = \{(z_1, z_2) \in \mathbb{D}^2 : |1-sz_1|^2 |1-sz_2|^2 > \lambda \} \).

Set \( U_{\lambda, s} = \{(z_1, z_2) \in \mathbb{D}^2 : |1-sz_2|^2 < |1-sz_1|^2 \lambda^{-1} \text{ and } 2|1-sz_1| < (1-s)^{-1} \lambda^{-1/2} \} \). Then

\[
|\{(z_1, z_2) \in \mathbb{D}^2 : |1-sz_1|^2 |1-sz_2|^2 > \lambda \}| \\
\geq |U_{\lambda, s}| = \int_{\mathbb{D}^2} \mathbb{1}_{|1-sz_2|^2 < |1-sz_1|^2 \lambda^{-1/2}} \int_{|1-sz_2|^2 < |1-sz_1|^2 \lambda^{-1/2}} dV(z_2) dV(z_1).
\]  

(3.3)

By a change of the variable \( z_2 = \frac{i-w_2}{1+w_2} \), we have

\[
\int_{\mathbb{D}^2} \mathbb{1}_{|1-sz_2|^2 < |1-sz_1|^2 \lambda^{-1/2}} dV(z_2)
\]

\[
= \int_{\{w_2 \in \mathbb{C} : \text{Im}(w_2) > 0, \frac{|1-s+i(w_2)|}{|1+w_2|^2} < \frac{1}{|1-sz_1|^2 \lambda^{-1/2}} \}} \frac{4}{|i+w_2|^4} dV(w_2).
\]  

(3.4)

When \( |w_2| < 1 \), we have \( |i+w| \approx 1 \) and \( |(1-s)i+(1+s)w_2| \leq (1+s)|w_2|+(1-s) \).

Combining these inequalities with the fact that \( 2|1-sz_1| < (1-s)^{-1} \lambda^{-1/2} \) for \((z_1, z_2) \in U_{\lambda, s} \), there holds

\[
\int_{\{w_2 \in \mathbb{C} : \text{Im}(w_2) > 0, \frac{|1-s+i(w_2)|}{|1+w_2|^2} < \frac{1}{|1-sz_1|^2 \lambda^{-1/2}} \}} \frac{4}{|i+w_2|^4} dV(w_2)
\]

\[
\geq \int_{\{w_2 \in \mathbb{C} : \text{Im}(w_2) > 0, \frac{|1-s+i(w_2)|}{|1+w_2|^2} < \frac{1}{|1-sz_1|^2 \lambda^{-1/2}} \}} \frac{4}{|i+w_2|^4} dV(w_2)
\]

\[
\geq \frac{1}{|1-sz_1|^2 \lambda^{-1/2} -(1-s)} 2 \geq |1-sz_1|^{-2} \lambda^{-1}.
\]  

(3.5)

Applying inequalities (3.5) and Lemma 2.1 to (3.3) and choose \( \lambda = 16^{-1} (1-s)^{-2} \) yield

\[
|U_{\lambda, s}| \geq \int_{\mathbb{D}^2} |1-sz_1|^{-2} \lambda^{-1} dV(z_1)
\]

\[
= \int_{\mathbb{D}} |1-sz_1|^{-2} \lambda^{-1} dV(z_1) \approx 1 \lambda \log \left( \frac{1}{1-s^2} \right).
\]  

(3.6)

Thus

\[
|\{(z_1, z_2) \in \mathbb{D}^2 : |P(f_s)(z_1, z_2)| > \lambda \}| \geq \frac{1}{\lambda} \log \left( \frac{1}{1-s^2} \right) = \frac{(- \log (1-s))\|f_s\|_{L^1}}{\pi^2 \lambda}.
\]

Since \( (- \log(1-s)) \) approaches \( \infty \) as \( s \) tends to 1, (3.1) holds and the proof is complete. \( \square \)
Remark 3.2. Using the same example in the proof of Theorem 3.1, one can also show that Theorem 3.1 holds true for the polydisc case.

The positive result for the weak-type estimate of the Bergman projection is a consequence of the following two theorems from [DHZZ01].

Theorem 3.3 ([DHZZ01, Theorem 1.1]). The Bergman projection is of weak-type $(1,1)$ on the unit disc.

Theorem 3.4 ([DHZZ01, Theorem 1.3]). The Bergman projection is a bounded operator from $L \log^+ L(\mathbb{D})$ to $L^1(\mathbb{D})$.

Theorem 3.5. The Bergman projection on the bidisc $\mathbb{D}^2$ is of weak-type $L \log^+ L$.

**Proof.** Let $P_1$ and $P_2$ denote the Bergman projection in variable $z_1$ and $z_2$ respectively. Then the Bergman projection $P$ on the bidisc $\mathbb{D}^2$ equals $P_1 \circ P_2$. By Theorems 3.3, 3.4 and Fubini’s theorem, we have for all $f \in L^1(\mathbb{D}^2)$

$$|\{(z_1, z_2) \in \mathbb{D}^2 : |P(f)(z_1, z_2)| > \lambda\}|$$

$$= |\{(z_1, z_2) \in \mathbb{D}^2 : |P_1 \circ P_2(f)(z_1, z_2)| > \lambda\}|$$

$$= \int_\mathbb{D} |\{z_1 \in \mathbb{D} : |P_1 \circ P_2(f)(z_1, z_2)| > \lambda\}|dV(z_2) \lesssim \frac{\|P_2(f)\|_{L^1}}{\lambda} \lesssim \frac{\|f\|_{L \log^+ L}}{\lambda}.$$

By slightly modifying the proof of Theorem 3.4, one also obtains the following theorem:

**Theorem 3.6.** For $k \in \mathbb{N}$, the Bergman projection is bounded from $L(\log^+ L)^{k+1}(\mathbb{D})$ to $L(\log^+ L)^k(\mathbb{D})$.

**Proof.** It suffices to show that the Bergman projection is bounded from the unit sphere of $L(\log^+ L)^{k+1}(\mathbb{D})$ to $L(\log^+ L)^k(\mathbb{D})$. Given $f \in L(\log^+ L)^{k+1}(\mathbb{D})$ with $\|f\|_{L(\log^+ L)^{k+1}(\mathbb{D})}$ equal to 1, the definition of the Orlicz norm $\|\cdot\|_{L(\log^+ L)^{k+1}(\mathbb{D})}$ implies:

$$1 = \|f\|_{L(\log^+ L)^{k+1}(\mathbb{D})} = \int_\mathbb{D} |f(z)| \left(\log^+ |f(z)|\right)^{k+1} dV(z).$$

If $\|P(f)\|_{L(\log^+ L)^k(\mathbb{D})} \leq 1$, then $\|P(f)\|_{L(\log^+ L)^{k+1}(\mathbb{D})} \leq \|f\|_{L(\log^+ L)^{k+1}(\mathbb{D})}$ and the theorem is proved. We turn to consider the case when $\|P(f)\|_{L(\log^+ L)^k(\mathbb{D})} = \lambda > 1$. We show that in this case, the estimate $\|P(f)\|_{L(\log^+ L)^k(\mathbb{D})} \lesssim \|f\|_{L(\log^+ L)^{k+1}(\mathbb{D})}$ still holds.

For a fixed $t > 0$, we set

$$f_1(z) = \begin{cases} f(z) & |f(z)| > t \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_2(z) = \begin{cases} 0 & |f(z)| > t \\ f(z) & \text{otherwise}. \end{cases}$$

Then $f(z) = f_1(z) + f_2(z)$. For a function $g$ on $\mathbb{D}$ and a fixed $t > 0$, let $g_*(t)$ denote the distribution function: $g_*(t) := |\{z \in \mathbb{D} : g(z) > t\}|$. Since the Bergman projection is $L^2$ bounded by its definition and of weak-type $(1,1)$ by Theorem 3.3, we have

$$(P(f))_*(t) = |\{z \in \mathbb{D} : |P(f)(z)| > t\}|$$

$$\leq |\{z \in \mathbb{D} : |P(f_1)(z)| > \frac{t}{2}\}| + |\{z \in \mathbb{D} : |P(f_2)(z)| > \frac{t}{2}\}|$$

$$\lesssim \frac{\|f_1\|_{L^1}}{t} + \frac{\|f_2\|_{L^2}}{t^2} = \int_0^\infty f_*(s)ds + \int_0^\infty \int_0^s f_*(s)ds ds \frac{2}{t^2}.$$
Multiplying both sides of the inequality by \((\log t)^k + k(\log t)^{k-1}\) and integrating them from 1 to \(\infty\) yields:

\[
\int_1^\infty \left( P(f) \right)_+ (t)((\log t)^k + k(\log t)^{k-1}) dt \\
\leq \int_1^\infty \left( \frac{f''(s)}{t} + \frac{2 f'_0 s f'_s(s)}{t^2} \right) dt \\
= \int_1^\infty \left( \frac{(\log s)^{k+1}}{k+1} + (\log s)^k f'_s(s) ds + 2 \int_0^\infty \int_{\max\{0, s\}}^\infty \left( \frac{(\log t)^k + k(\log t)^{k-1}}{t^2} \right) s f'_s(s) dtds \right) \\
\leq \int_1^\infty \left( \frac{(\log s)^{k+1}}{k+1} + (\log s)^k f'_s(s) ds + \int_0^\infty \int_{\max\{0, s\}}^\infty \left( \frac{(\log t)^k + 1}{t^2} \right) s f'_s(s) dtds. \tag{3.7} \right.
\]

Since \(\int_{\max\{1, s\}}^\infty \frac{(\log t)^k + 1}{t^2} dtds = -(\frac{1}{t} + \sum_{j=0}^k \frac{k!}{j!}(\log t)^j)|_{\max\{1, s\}}\), there holds

\[
\int_0^\infty \int_{\max\{0, s\}}^\infty \left( \frac{(\log t)^k + 1}{t^2} \right) s f'_s(s) dtds \lesssim \int_D |f(z)|((\log^+ |f(z)|)^k) dV(z).
\]

We claim that \(\int_D |f(z)|((\log^+ |f(z)|)^k) dV(z) \lesssim 1\). Assume the claim is true. Then applying this claim and the fact that \(\int_1^\infty (\frac{(\log s)^{k+1}}{k+1} + (\log s)^k) f'_s(s) ds = \|f\|_{L(\log^+ L)^{k+1}}\) into (3.7) yields the estimate:

\[
\int_1^\infty \left( P(f) \right)_+ (t)((\log t)^k + k(\log t)^{k-1}) dt \lesssim \int_D |f(z)|((\log^+ |f(z)|)^k) dV(z). \tag{3.8} \right.
\]

Since \(\|P(f)\|_{L(\log^+ L)^k} = \lambda\), there holds that

\[
\int_D \left| P \left( \frac{f}{\lambda} \right) (z) \right| \left( \log^+ \left| P \left( \frac{f}{\lambda} \right) (z) \right| \right)^k dV(z) = 1.
\]

Thus

\[
\int_D |P(f)(z)|((\log^+ |P(\frac{f}{\lambda})(z)|)^k) dV(z) = \lambda = \|P(f)\|_{L(\log^+ L)^k(D)}. \right.
\]

Note also that \(\lambda > 1\). Therefore

\[
\|P(f)\|_{L(\log^+ L)^k(D)} = \int_D |P(f)(z)|((\log^+ |P(\frac{f}{\lambda})(z)|)^k) dV(z) \\
\leq \int_D |P(f)(z)|((\log^+ |P(f)(z)|)^k) dV(z) \\
= \int_1^\infty \left( P(f) \right)_+ (t)((\log t)^k + k(\log t)^{k-1}) dt. \tag{3.9} \right.
\]

Combining inequalities (3.8) and (3.9) yields the desired estimate:

\[
\|P(f)\|_{L(\log^+ L)^k(D)} \lesssim \|f\|_{L(\log^+ L)^{k+1}}. \right.
\]

We turn to prove the claim. By Hölder’s inequality, there holds:

\[
\int_D |f(z)|((\log^+ |f(z)|)^k) dV(z) \leq \left( \int_D |f(z)|((\log^+ |f(z)|)^{k+1}) dV(z) \right)^{\frac{1}{k+1}} \left( \int_D |f(z)| dV(z) \right)^{\frac{k}{k+1}} \\
= \left( \int_D |f(z)| dV(z) \right)^{\frac{1}{k+1}}.
\]
Since

\[
\int_D |f(z)|dV(z) = \int_{\{z \in D: |f(z)| < e\}} |f(z)|dV(z) + \int_{\{z \in D: |f(z)| \geq e\}} |f(z)|dV(z)
\]

\[
\leq e|D| + \int_{\{z \in D: |f(z)| \geq e\}} |f(z)|(\log^+ |f|)^{k+1}dV(z) \leq e\pi + 1 \approx 1,
\]

there holds \(\int_D |f(z)|(\log^+ |f(z)|)^{k}dV(z) \lesssim 1\) and the proof is complete. \(\square\)

Theorem 3.6 together with the argument in the proof of Theorem 3.5 then gives the weak-type estimate for the Bergman projection on the polydisc:

**Theorem 3.7.** The Bergman projection on the polydisc \(\mathbb{D}^n\) is of weak-type \(L(\log^+ L)^{n-1}\).

4. Weak-type estimates for the Bergman projection on \(\mathbb{H}\)

**Theorem 4.1.** The Bergman projection on the Hartogs triangle is not of weak-type \((\frac{4}{3}, \frac{4}{3})\).

**Proof.** For a constant \(p > \frac{4}{3}\), let \(p' = \frac{p}{p-1}\) denote its conjugate index. Set \(f_p(z) = \overline{z}_2z_2|^{-p'}\).

Then

\[
\|f_p\|^{\frac{4}{L_4^+}}_{L_4^+} = \int_{\mathbb{H}} |z_2|^{4(1-p')/p}dV(z_1, z_2) = \int_{\mathbb{H}^2} |z_2|^{2p-4}dV(z_1, z_2) = \pi^2\frac{(p-1)}{4(p-\frac{4}{3})}.
\] (4.1)

Given \((z_1, z_2) \in \mathbb{H},\)

\[
|P(f_p)(z_1, z_2)| = \int_{\mathbb{H}^2} \sum_{a+b \geq -1, a \geq 0} \frac{(z_1\overline{w}_1)^a(z_2\overline{w}_2)^b}{\|w_1^aw_2^b\|^2_{L_2}} \overline{w}_2|w_2|^{-p'}dV(w_1, w_2).
\] (4.2)

Since the Hartogs triangle is a Reinhardt domain, it’s easy to check using polar coordinates that \(\int_{\mathbb{H}} \overline{w}_1^aw_2^b\overline{w}_2|w_2|^{-p'}dV(w_1, w_2) \neq 0\) if and only if \(a = 0\) and \(b = -1\). Thus

\[
|P(f_p)(z_1, z_2)| = \left|\int_{\mathbb{H}} \frac{1}{z_2\overline{w}_2\|w_2^{-1}\|^2_{L_2}} \overline{w}_2|w_2|^{-p'}dV(w_1, w_2)\right|
\]

\[
= \pi^2 \left|\int_{\mathbb{H}} \frac{1}{z_2|w_2|^{-p'}dV(w_1, w_2)}\right| = \frac{p-1}{3(p-4)|z_2|}.
\] (4.3)

Note that for \(\frac{p-1}{3(p-4)|z_2|} < 1,\)

\[
|\{(z_1, z_2) \in \mathbb{H} : |P(f_p)(z_1, z_2)| > \lambda\}| = \int_{\{(z_1, z_2) \in \mathbb{H} : |z_2| < \frac{p-1}{3(p-4)|z_2|}\}} dV(z_1, z_2)
\]

\[
= \int_{\{(z_1, z_2) \in \mathbb{D}^2 : |z_2| < \frac{p-1}{3(p-4)|z_2|}\}} |z_2|^2dV(z_1, z_2)
\]

\[
= \frac{\pi^2}{4} \left(\frac{p-1}{3(p-4)|z_2|}\right)^4 \approx \frac{\|f_p\|^{4/3}_{L_4^+}}{\lambda^{4/3}} \left(\frac{1}{(p-3/4)\lambda}\right) \approx \frac{\|f_p\|^{4/3}_{L_4^+}}{\lambda^{4/3}}.
\] (4.4)

Setting \(p = 4/3 + \lambda^{-9/10}\), then \((p-1)(3p-4)^{-1}\lambda^{-1}\) still goes to 0 as \(\lambda\) tends to \(\infty\). Hence \((p-1)(3p-4)^{-1}\lambda^{-1} < 1\) holds. On the other hand, \((p-4/3)^{-3}\lambda^{-8/3} = \lambda^{1/30}\), which is blowing up as \(\lambda\) tends to \(\infty\). Therefore, the weak-type estimate

\[
|\{(z_1, z_2) \in \mathbb{H} : |P(f_p)(z_1, z_2)| > \lambda\}| \lesssim \frac{\|f_p\|^{4/3}_{L_4^+}}{\lambda^{4/3}}
\]

fails and the Bergman projection on \(\mathbb{H}\) is not of weak-type \((4/3, 4/3)\). \(\square\)
Theorem 4.2. The Bergman projection on the Hartogs triangle is of weak-type $(4,4)$.

Proof. Let $f$ be an arbitrary function in $L^4(\mathbb{H})$. Then
\[
\|f\|_{L^4}^4 = \int_{\mathbb{H}} |f(z_1, z_2)|^4 dV(z_1) dV(z_2)
\]
\[
= \int_{\mathbb{D}^2} |f(z_1, z_2)|^4 |z_2|^2 dV(z_1) dV(z_2)
\]
\[
= \int_{\mathbb{D}^2} |z_2 f(z_1, z_2)|^4 |z_2|^{-2} dV(z_1) dV(z_2). 
\] (4.5)

Set $g(z_1, z_2) = z_2 f(z_1, z_2)$. Then $g \in L^4(\mathbb{D}^2, |z_2|^{-2} dV)$ and $\|g\|_{L^4(\mathbb{D}^2, |z_2|^{-2} dV)} = \|f\|_{L^4(\mathbb{H})}$.

Note that
\[
|P(f)(z_1, z_2)| = \left| \int_{\mathbb{D}^2} \frac{f(w_1, w_2)}{\pi^2 z_2 w_2 (1 - \frac{z_1 w_1}{z_2 w_2})^2 (1 - \frac{z_2 w_2}{z_2 w_2})^2} dV(w_1, w_2) \right|
\]
\[
= \left| \int_{\mathbb{D}^2} \frac{w_2 f(w_1 w_2, w_2)}{\pi^2 z_2 (1 - \frac{z_1 w_1}{z_2 w_2})^2 (1 - \frac{z_2 w_2}{z_2 w_2})^2} dV(w_1, w_2) \right| = \frac{1}{|z_2|} \left| P_{\mathbb{D}^2}(g) \left( \frac{z_1}{z_2}, z_2 \right) \right|. 
\] (4.6)

Then there holds
\[
\left| \left\{ (z_1, z_2) \in \mathbb{H} : |P(f)(z_1, z_2)| > \lambda \right\} \right|
\]
\[
= \int_{\left\{ (z_1, z_2) \in \mathbb{D}^2 : |P_{\mathbb{D}^2}(g)(z_1, z_2)| > \lambda \right\}} dV(z_1, z_2)
\]
\[
= \int_{\left\{ (z_1, z_2) \in \mathbb{D}^2 : |P_{\mathbb{D}^2}(g)(z_1, z_2)| > \lambda \right\}} |z_2|^2 dV(z_1, z_2)
\]
\[
= \int_{\left\{ (z_1, z_2) \in \mathbb{D}^2 : |z_2| < \frac{1}{2} \text{ and } |P_{\mathbb{D}^2}(g)(z_1, z_2)| > \lambda \right\}} |z_2|^2 dV(z_1, z_2)
\]
\[
+ \int_{\left\{ (z_1, z_2) \in \mathbb{D}^2 : |z_2| > \frac{1}{2} \text{ and } |P_{\mathbb{D}^2}(g)(z_1, z_2)| > \lambda \right\}} |z_2|^2 dV(z_1, z_2). 
\] (4.7)

We first consider the integral with $|z_2| \leq \frac{1}{2}$. When $|z_2| \leq \frac{1}{2}$, the Bergman kernel $K_{\mathbb{D}^2}$ satisfies:
\[
|K_{\mathbb{D}^2}(z_1, z_2; w_1, w_2)| = \frac{1}{\pi^2 |1 - z_1 w_1|^2 |1 - z_2 w_2|^2} \approx \frac{1}{\pi |1 - z_1 w_1|^2} = |K_{\mathbb{D}}(z_1; \bar{w}_1)|. 
\] (4.8)

Therefore,
\[
|P_{\mathbb{D}^2}(g)(z_1, z_2)| \leq \left| \int_{\mathbb{D}^2} \frac{g(w_1, w_2)}{\pi^2 (1 - z_1 w_1)^2 (1 - z_2 w_2)^2} dV(w_1, w_2) \right|
\]
\[
\leq \int_{\mathbb{D}^2} \frac{|g(w_1, w_2)|}{\pi^2 |1 - z_1 w_1|^2 |1 - z_2 w_2|^2} dV(w_1, w_2)
\]
\[
\leq \int_{\mathbb{D}} |g(w_1, w_2)| dV(w_2) \frac{dV(w_1)}{\pi |1 - z_1 \bar{w}_1|^2}
\]
\[
= P_{\mathbb{D}}^+ \left( \int_{\mathbb{D}} |g(w_1, w_2)| dV(w_2) \right)(z_1). 
\] (4.9)
Set $G(w_1) = \int_D |g(w_1, w_2)| dV(w_2)$. Then there exists a constant $C$ such that,

$$\int \left\{ (z_1, z_2) \in \mathbb{D}^2 : |z_2| \leq \frac{1}{2} \text{ and } \frac{1}{|z_2|^2} P_{\mathbb{D}^2}(g)(z_1, z_2) > \lambda \right\} |z_2|^2 dV(z_1, z_2)$$

$$\leq \int \left\{ (z_1, z_2) \in \mathbb{D}^2 : |z_2| \leq \frac{1}{2} \text{ and } \frac{1}{|z_2|^2} P_{\mathbb{D}^2}^+(G)(z_1) > C\lambda \right\} |z_2|^2 dV(z_1, z_2)$$

$$\leq \int_D \int \left\{ z_2 \in \mathbb{D} : |z_2| \leq \frac{1}{2} \text{ and } \frac{1}{z_2} P_{\mathbb{D}^2}^+(G)(z_1) > |z_2|^2 dV(z_2) \right\} |z_2|^2 dV(z_1)$$

$$\leq \int_0^\infty \int_D \left\{ \frac{1}{z_2} P_{\mathbb{D}^2}^+(G)(z_1) \right\} r^3 dr dz_1 \leq \int_D \left( \frac{P_{\mathbb{D}^2}^+(G)(z_1)}{\lambda^4} \right)^4 dV(z_1) = \frac{\|P_{\mathbb{D}^2}^+(G)\|_{L^4(\mathbb{D})}^4}{\lambda^4}. \quad (4.10)$$

Applying Hölder’s inequality,

$$\int \mathbb{D} G(w_1)^4 dV(w_1) \leq \int \mathbb{D} \int \mathbb{D} |g(w_1, w_2)|^4 dV(w_2) dV(w_1) = \|g\|^4_{L^4(\mathbb{D}^2)} \leq \|g\|^4_{L^4(\mathbb{D}^2, |w_2|^{-2})}. \quad (4.11)$$

Thus $G(w_1)$ is in $L^4(\mathbb{D})$. By (4.11) and the $L^p$ boundedness of $P_{\mathbb{D}^2}^+$, inequality (4.10) yields

$$\int \left\{ (z_1, z_2) \in \mathbb{D}^2 : |z_2| \leq \frac{1}{2} \text{ and } \frac{1}{|z_2|^2} P_{\mathbb{D}^2}(g)(z_1, z_2) > \lambda \right\} |z_2|^2 dV(z_1, z_2) \lesssim \frac{\|g\|_{L^4(\mathbb{D}^2, |w_2|^{-2})}^4}{\lambda^4} = \frac{\|f\|_{L^4(\mathbb{D})}^4}{\lambda^4}. \quad (4.12)$$

Now we turn to the integral in (4.7) with $|z_2| > \frac{1}{2}$. By $|z_2| > \frac{1}{2}$, there holds $\frac{1}{|z_2|} < 2$ and

$$\int \left\{ (z_1, z_2) \in \mathbb{D}^2 : |z_2| > \frac{1}{2} \text{ and } \frac{1}{|z_2|^2} P_{\mathbb{D}^2}(g)(z_1, z_2) > \lambda \right\} |z_2|^2 dV(z_1, z_2)$$

$$\leq \int \left\{ (z_1, z_2) \in \mathbb{D}^2 : P_{\mathbb{D}^2}(g)(z_1, z_2) > \lambda \right\} |z_2|^2 dV(z_1, z_2)$$

$$\leq \frac{2^4}{\lambda^4} \int_0^\infty \int \left( \frac{P_{\mathbb{D}^2}(g)(z_1, z_2)}{\lambda} \right)^4 |z_2|^2 dV(z_1, z_2)$$

$$\leq \frac{2^4}{\lambda^4} \int_0^\infty \int \left( \frac{P_{\mathbb{D}^2}(g)(z_1, z_2)}{\lambda} \right)^4 dV(z_1, z_2). \quad (4.13)$$

Since $P_{\mathbb{D}^2}$ is also $L^p$ bounded for $1 < p < \infty$, there holds

$$\frac{2^4}{\lambda^4} \int_0^\infty \int \left( \frac{P_{\mathbb{D}^2}(g)(z_1, z_2)}{\lambda} \right)^4 dV(z_1, z_2) \lesssim \frac{\|g\|_{L^4(\mathbb{D}^2)}^4}{\lambda^4} \leq \frac{\|g\|_{L^4(\mathbb{D}^2, |w_2|^{-2})}^4}{\lambda^4} = \frac{\|f\|_{L^4(\mathbb{D})}^4}{\lambda^4}. \quad (4.14)$$

Hence we also obtain the inequality

$$\int \left\{ (z_1, z_2) \in \mathbb{D}^2 : |z_2| > \frac{1}{2} \text{ and } \frac{1}{|z_2|^2} P_{\mathbb{D}^2}(g)(z_1, z_2) > \lambda \right\} |z_2|^2 dV(z_1, z_2) \lesssim \frac{\|f\|_{L^4(\mathbb{D})}^4}{\lambda^4}. \quad (4.14)$$

Applying (4.12) and (4.14) to (4.7) yields the desired weak-type (4,4) estimate. \hfill \Box

**Remark 4.3.** It can be shown that if the Bergman projection $P$ on a weighted space $L^p(\mathbb{D}^2, \mu)$ is of weak-type $(p, p)$, then $P$ is bounded on $L^p(\mathbb{D}^2, \mu)$. The idea of the proof can be found in Theorem 1 in [RTW17] and Theorem 1.2 in [HW19]. Theorem 4.2, on the other hand, shows a different phenomenon in the Hartogs triangle case: the Bergman projection on $\mathbb{H}$ is of weak-type $(4, 4)$ but not $L^4$-bounded. This difference is caused by the fact that while $L^4(\mathbb{D}^2, |z_2|^{-2})$ and $L^4(\mathbb{H})$ are isometrically equivalent via the mapping $(z_1, z_2) \rightarrow (z_1 z_2, z_2)$ between $\mathbb{D} \times \mathbb{D}\{0\}$ and $\mathbb{H}$, the weak spaces $L^{4,\infty}(\mathbb{D}^2, |z_2|^{-2})$ and $L^{4,\infty}(\mathbb{H})$ are not.
Remark 4.4. Since the Bergman projection $P$ is self-adjoint, a duality argument together with Theorem 4.2 implies that $P$ is bounded from the Lorentz space $L^{4/3,1}(\mathbb{H})$ to $L^{4/3}(\mathbb{H})$. See for example Theorem 1.4.16 in [Gra14].

Remark 4.5. Theorems 4.1 and 4.2 also provide an alternative proof of the $L^p$-regularity result for the Bergman projection on the Hartogs triangle: by interpolation, the weak type $(4, 4)$ and $L^2$ regularity of the Bergman projection implies that the projection is $L^p$ bounded for $p \in [2, 4)$. Then a duality argument yields the $L^p$ regularity for $p \in (4/3, 4)$. Since the projection is not of weak-type $(4/3, 4)$, it’s not $L^{4/3}$ bounded, and hence not $L^p$ bounded for $p \notin (4/3, 4)$. Therefore the Bergman projection on $\mathbb{H}$ is $L^p$ bounded if and only if $p \in (4/3, 4)$.

Using the same idea of the proof of Theorem 4.2, one can obtain the following weak-type estimate for $P$ near $L^{\frac{4}{3}}$. Here we provide another proof using Lemma 2.2.

**Theorem 4.6.** For any $\epsilon > 0$, the Bergman projection $P$ on the Hartogs triangle satisfies the following weak-type estimate:

$$
|\{(z_1, z_2) \in \mathbb{H} : |P(f)(z_1, z_2)| > \lambda\}| \lesssim \frac{\|f\|_{L^{4/3}(\mathbb{H}, |z_2|^{-\epsilon})}^{4/3}}{\lambda^{4/3}}.
$$

**Proof.** We claim that the Bergman projection is bounded on $L^{\frac{4}{3}}(\mathbb{H}, |z_2|^{-\epsilon})$. Then the desired estimate holds:

$$
|\{(z_1, z_2) \in \mathbb{H} : |P(f)(z_1, z_2)| > \lambda\}| \leq \int_{\{(z_1, z_2) \in \mathbb{H} : |P(f)(z_1, z_2)| > \lambda\}} |z_2|^{-\epsilon} dV(z_1, z_2)
$$

$$
\leq \int_{\mathbb{H}} \frac{|P(f)(z_1, z_2)|^{4/3}}{\lambda^{4/3}} |z_2|^{-\epsilon} dV(z_1, z_2)
$$

$$
\lesssim \frac{\|f\|_{L^{4/3}(\mathbb{H}, |z_2|^{-\epsilon})}^{4/3}}{\lambda^{4/3}}.
$$

To prove the claim, we recall that for any given $f \in L^{\frac{4}{3}}(\mathbb{H}, |z_2|^{-\epsilon})$, the induced function $g(w_1, w_2) := w_2 f(w_1, w_2, w_2)$ is in $L^{\frac{4}{3}}(\mathbb{D}^2, |z_2|^{\frac{4}{3} - \epsilon})$. Moreover,

$$
|P(f)(z_1, z_2)| = \int_{\mathbb{H}} \frac{f(w_1, w_2)}{|z_2|^2} dV(w_1, w_2)
$$

$$
= \int_{\mathbb{D}^2} \frac{w_2 f(w_1, w_2, w_2)}{|z_2|^2} dV(w_1, w_2) = \frac{1}{|z_2|} \left| P_{\mathbb{D}^2}(g) \left( \frac{z_1}{z_2}, z_2 \right) \right|.
$$

Then it is easy to see that the two operator norms $\|P\|_{L^{\frac{4}{3}}(\mathbb{H}, |z_2|^{-\epsilon})}$ and $\|P_{\mathbb{D}^2}\|_{L^{\frac{4}{3}}(\mathbb{D}^2, |z_2|^{\frac{4}{3} - \epsilon})}$ are identical. We first show that $P_{\mathbb{D}}$ is bounded on $L^{\frac{4}{3}}(\mathbb{D}, |w|^{\frac{4}{3} - \epsilon})$. Recall the Carleson tent $T_z$ in Lemma 2.2. When $|z| > \frac{1}{2}$, the function $|w| \approx 1$ for all $w \in T_z$. Hence for $|z| > \frac{1}{2}$, we have

$$
\int_{T_z} |w|^{\frac{4}{3} - \epsilon} dV(w) \left( \frac{1}{V(T_z)} \right)^p \lesssim 1.
$$

For $|z| \leq \frac{1}{2}$, the Lebesgue measure of $|T_z| \approx 1$. Thus for $p = \frac{4}{3}$, there holds

$$
\int_{T_z} |w|^{\frac{4}{3} - \epsilon} dV(w) \left( \frac{1}{V(T_z)} \right)^p \lesssim 1.
$$
Theorem 4.7. For any $\epsilon > \frac{1}{3}$, the Bergman projection $P$ on the Hartogs triangle satisfies the following weak-type estimate:

$$\left| \{(z_1, z_2) \in \mathbb{H} : |P(f)(z_1, z_2)| > \lambda \} \right| \lesssim \frac{\|f\|_{L^{4/3}(\mathbb{H}, (\log |z_2| + 1)^\epsilon)}^{4/3}}{\lambda^{4/3}}. \quad (4.21)$$

For $\epsilon \leq \frac{1}{3}$, Theorem 4.7 does not hold anymore. Below we provide an example to illustrate the failure of the estimate (4.21) when $\epsilon = \frac{1}{3}$.
Set $f_p(z) = \overline{z}_2|z_2|^{-p'}(-\log |z_2| + 1)^{-1}$. Then
\[
\|f_p\|_{L^{4/3}(\mathbb{H},(\log |z_2|+1)^{1/3})}^{4/3} = \int_\mathbb{H} |z_2|^{\frac{4}{3}(1-p')}(-\log |z_2| + 1)^{-1}dV(z_1, z_2)
= \int_{\mathbb{D}^2} |z_2|^{\frac{4}{3}(1-p')}(-\log |z_2| + 1)^{-1}dV(z_1, z_2)
= 2\pi^2 \int_0^1 x^{\frac{4}{3}+\frac{4}{3}(1-p')}(-\log x + 1)^{-1}dx
= 2\pi^2 e^{4+\frac{4}{3}(1-p')}E_1(4 + \frac{4}{3}(1-p')),
\]
where $E_1$ is the exponential integral defined by
\[
E_1(x) = \int_x^\infty t^{-1}e^{-t}dt.
\]
Note that (AS64, p. 229, 5.1.20)
\[
\frac{1}{2}e^{-x}log \left(1 + \frac{2}{x}\right) < E_1(x) < e^{-x}log \left(1 + \frac{1}{x}\right).
\]
Therefore as $p \to \frac{4}{3}$, there holds $4 + \frac{4}{3}(1-p') \to 0$ and
\[
E_1 \left(4 + \frac{4}{3}(1-p')\right) \approx \log \left(\frac{1}{4 + \frac{4}{3}(1-p')}\right) \approx \log \left(\frac{1}{3p - 4}\right). \tag{4.23}
\]
Substituting this back into (4.22) yields $\|f_p\|_{L^{4/3}(\mathbb{H},(\log |z_2|+1)^{1/3})}^{4/3} \approx \log \left(\frac{1}{3p - 4}\right)$. On the other hand,
\[
|\{f_p\}(z_1, z_2)| = \left|\int_\mathbb{H} \frac{1}{z_2 \overline{w}_2 \|w_2^{-1}\|_L^{2}} \overline{w}_2|w_2|^{-p'}(-\log |z_2| + 1)^{-1}dV(w_1, w_2)\right|
= \pi^{-2} \left|\int_\mathbb{H} \frac{1}{z_2 |w_2|^{-p'}(-\log |z_2| + 1)^{-1}dV(w_1, w_2)\right| \approx \frac{1}{|z_2|} \log \left(\frac{1}{3p - 4}\right). \tag{4.24}
\]
Therefore, when $\log \left(\frac{1}{3p - 4}\right) \frac{1}{\lambda}$ is small, we have
\[
|\{(z_1, z_2) \in \mathbb{H} : |\{f_p\}(z_1, z_2)| > \lambda\}|
\geq \int \{\{(z_1, z_2) \in \mathbb{H} : |z_2| < \log \left(\frac{1}{3p - 4}\right)\}\} dV(z_1, z_2)
= \int \{\{(z_1, z_2) \in \mathbb{D}^2 : |z_2| < \log \left(\frac{1}{3p - 4}\right)\}\} |z_2|^2dV(z_1, z_2)
= \pi^2 \left(\log \left(\frac{1}{3p - 4}\right)\right) \frac{1}{\lambda} \approx \frac{\|f_p\|_{L^{4/3}}^{4/3}}{\lambda^{4/3}} \left(\log \left(\frac{1}{p - 4/3}\right)\right) \frac{1}{\lambda^{8/3}}. \tag{4.25}
\]
Setting $p = 4/3 + \exp \left\{-\lambda^{9/10}\right\}$, then $\log \left(\frac{1}{3p - 4}\right) \frac{1}{\lambda}$ still goes to 0 as $\lambda$ tends to $\infty$. Hence $\frac{p-1}{(3p-4)\lambda} < 1$ holds. Note that $\left(\log \left(\frac{1}{p - 4/3}\right)\right)^3 \frac{1}{\lambda^{30}} = \lambda^{1/30}$, which is blowing up as $\lambda$ tends to $\infty$. Therefore, the weak-type estimate
\[
|\{(z_1, z_2) \in \mathbb{H} : |\{f_p\}(z_1, z_2)| > \lambda\}| \lesssim \frac{\|f\|_{L^{4/3}(\mathbb{H},(\log |z_2|+1)^{1/3})}^{4/3}}{\lambda^{4/3}} \text{ fails for } \epsilon = 1/3.
\]
Remark 4.8. Estimate \((4.21)\) in Theorem 4.7 is a consequence of the finite integral in \((4.20)\) and the Bekollé-Bonami theory on the unit disc. The integrand \(|z|^{-2}(-\log |z| + 1)\alpha\) blows up at a slower speed near the origin than \(|z|^{-2}\) and hence is in \(L^1(D)\). Similarly, one can construct an integrable function \(|z|^{-2}(-\log |z| + 1)^{-1}(\log(-\log |z| + 1) + 1)\) with \(\alpha < -1\) from \(|z|^{-2}(-\log |z| + 1)^{-1}\). Iterating this process, we obtain a sequence of functions \(\{f_{\alpha,j}(z)\}\) in \(L^1(D)\) where \(f_{\alpha,j}(z) = |z|^{-2}h_j^\alpha(z)\prod_{k=1}^{j-1}h_k^j(z)\) with \(\alpha < -1\) and \(h_j(z)\) defined as follows:

1. \(h_1(z) = -\log |z| + 1;\)
2. \(h_{j+1}(z) = \log(h_j(z) + 1) + 1\) for \(j > 0.\)

Then repeating the argument for \((4.17)\) and \((4.18)\) yields that the Bekollé-Bonami constant \(B^j_4(f_{\alpha,j}) < \infty\). Using this fact, \((4.21)\) can be generalized as below:

**Theorem 4.9.** Let \(f_{\alpha,j}\) be defined as above. For any \(\alpha < -1\), the Bergman projection \(P\) on the Hartogs triangle satisfies the following weak-type estimate:

\[
|\{(z_1, z_2) \in \mathbb{H} : |P(f)(z_1, z_2)| > \lambda\}| \lesssim \frac{\|f\|_{L^{4/3}(\mathbb{H}(|z_1|^2f_{\alpha,j}(z_2))^{-1/3})}^4}{\lambda^{4/3}}. \tag{4.26}
\]

Remark 4.10. Despite Theorems 4.6, 4.7 and 4.9, a sharp weak-type estimate for \(P\) near \(L^{4/3}\) is still unknown to us. One of our guesses is the weak-type \(L^{4/3}(\log^+ L)^\alpha\) estimate. For \(p > 4/3\) and \(\alpha > 0\), there holds \(\|\tilde{z}_2|z_2|^{-p}\|_{L^{4/3}(\log^+ L)^\alpha} \lesssim (p - 4/3)^{-\alpha-1}\). Then an argument as in the proof of Theorem 4.1 would imply that the projection \(P\) is not of weak-type \(L^{4/3}(\log^+ L)^\alpha\) for \(\alpha < 1/3\). For \(\alpha = 1/3\), the estimate holds for \(f_p(z) = \tilde{z}_2|z_2|^{-p}\) which is served as a counterexample in the proof of Theorem 4.1. Hence we suspect that the Bergman projection is bounded from \(L^{4/3}(\log^+ L)^{1/3}(\mathbb{H})\) to \(L^{4/3,\infty}(\mathbb{H})\). We hope to further investigate it in the future.

**Remark 4.11.** \(L^p\) regularity of the Bergman projection has also been studied on various generalizations of the Hartogs triangle. See for instance [EM16][EM17][Che17]. It would be interesting to study the weak-type regularity of the Bergman projection in those settings.

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