DUAL NATURE OF THE RICCI SCALAR, CREATION OF SPINLESS AND SPIN-1/2 PARTICLES AS WELL AS A NEW COSMOLOGICAL SCENARIO I

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In earlier papers [11-17], it is found that the Ricci scalar behaves in a dual manner (i) like a matter field and (ii) like a geometrical field. In ref.[17], inhomogeneous cosmological models are derived using dual roles of the Ricci scalar. The essential features of these models is capability of these to exhibit gravitational effect of compact objects also in an expanding universe. Here, production of spinless and spin-1/2 particles are demonstrated in these models.

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1. Introduction

Consideration of higher-derivative gravity is not new and it is a strong candidate in the context of string theories and Einstein-Gauss-Bonnet gravity. In the context of the early universe, use of higher-derivative gravity started from more than three decades ago. In 1977, Stelle considered it and pointed out that, in contrast to square of Weyl tensor, introduction of $R^2$ or $f(R)$ terms did not lead to the ghost problem [1]. Here $R$ is the Ricci scalar and $f(R)$ is the linear combination of $R^2$ terms subject to the condition $\lim_{R \to 0} f(R)/R = 0$ [2]. From these theories, it is found that the Ricci scalar behaves as a physical field also [3-10].

While quantizing gravity (quantizing components of the metric tensor), this theory has problem at the perturbation level where ghost terms appear in the Feynman propagator of gravitons. Around 25 years back, Starobinsky noted that if coupling constants in the action of higher-derivative gravity are taken properly, ghost terms do not appear and only one scalar particle, represented by the Ricci scalar, is obtained with positive energy as well as positive squared mass [6,7]. He called this particle as “scalaron” [6,7]. In papers [1-8], gravitational constant $G$ is either taken equal to unity or lagrangian density is taken as $\frac{1}{16\pi G}(R + \text{higher - derivative terms})$. As a result, $(\text{mass})^2$ of the Ricci scalar, does not depend on the gravitational constant $G$. As $G$ has a very important role in gravity, in papers [11 - 17] as well as here, lagrangian density is taken as $(\frac{1}{16\pi G}R + \text{higher - derivative terms})$ leading to a drastic change where $(\text{mass})^2$ of $R$ depends on $G$ also. In these papers [11 - 17], choosing coupling constants of the higher-derivative gravitational action
in a suitable manner, it is found that the Ricci scalar behaves like a physical particle, called as “riccion”, with positive \( (\text{mass})^2 \) without any ghost problem. This feature leads to manifestation of Ricci scalar in dual manner (like a geometrical field as well as a scalar particle). It is important to mention that riccion is a scalar particle (having only one mode unlike a graviton having five modes), but it is different from scalar mode of graviton. Detailed discussion on this difference is given in Appendix A. Riccion is given by a scalar field \( \tilde{R} = \eta R \), where \( \eta \) is a constant having length dimension. A “riccion” is different from a “scalaron” in two ways (i) mass dimension of riccion is one like other scalar fields used in a physical theory, such as quintessence, inflaton and Higg’s particle, whereas mass dimension of scalaron is two and (ii) \( (\text{mass})^2 \) of scalaron does not depend on \( G \), whereas riccion \( (\text{mass})^2 \) depends on \( G \).

In refs.[11-17], some interesting results are obtained using dual role of the Ricci scalar. One of the important consequences of the same is derivation of models of the early universe using a physical method of phase transition and spontaneous symmetry breaking. In ref.[11], spatially homogeneous cosmological models were derived using these methods and it was shown that universe should bounce at the critical temperature \( \sim 10^{18}\text{GeV} \). In ref.[15], employing these methods, models of the early universe are derived exhibiting inhomogeneity and anisotropy at small scales. It is found that these models are asymptotically homogeneous and isotropic.

The theory of gravity is manifested through geometry of the space-time. Two types of space-times are used for manifestation of gravitational interactions. One type is static space-times representing geometry around compact
objects e.g., Schwarzschild space-time, Riessner-Nordström space-time and others. The other type of space-times are homogeneous and inhomogeneous models representing the expanding universe. Former type of models do not give any information about the expanding universe and the latter type of models are unable to account for gravitational effect of individual objects in the expanding universe. So, models exhibiting local gravitational effect of an individual object as well as expansion of the universe simultaneously are required to discuss evolution of the universe in a more natural way. Using dual roles of the Ricci scalar, such cosmological models are derived in the reference [17], giving the first two stages of the beginning of the early universe.

In the present paper, production of spinless and spin-1/2 particles is demonstrated in the inhomogeneous models derived in the reference [17]. Contributions of these particles to the development of the universe will be discussed in paper II.

The paper is organized as follows. For convenience of the reader, a review of reference [17] is given in section 2.

In section 3, taking interaction of riccions with another scalar field \( \Phi \), production of spinless particles is demonstrated. In section 4, creation of spin-1/2 particles has been shown. Cosmological consequences of these created particles are planned to be discussed in the second paper of the series.

Natural units \((\hbar = c = 1)\) are used throughout the paper. Here \( \hbar \) and \( c \) have their usual meaning. For the sake of convenience, it is given that 

\[
1 \text{GeV} = 1.16 \times 10^{13} \text{K} = 1.78 \times 10^{-24} \text{ g}, \ 1 \text{ GeV}^{-1} = 1.97 \times 10^{-14} \text{ cm} = 6.58 \times 10^{-25} \text{ Sec.}
\]
2.(a) Dual role of the Ricci scalar and riccion

In this section, a brief review of the reference [17] is given. Here, the gravitational action with thermal correction is taken as

$$S_g = \int d^4x \sqrt{-g} \left[ -\frac{R}{16\pi G} + 4\lambda\alpha T^2 R + \alpha R^2 - 16\eta^2 \lambda\alpha (R^3 + 9\Box R^2) + \frac{\eta^2}{48}R^3 \right],$$

(2.1)

where \(\alpha\) and \(\lambda\) are dimensionless coupling constants. Here \(T\) is the temperature, \(G\) is the gravitational constant and \(\eta\) is another constant with length dimension, which is used for dimensional corrections only [11-17]. The invariance of \(S_g\), given by eq.(2.1), under transformations \(g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}\) yields the gravitational field equations

$$\left[ -\frac{1}{16\pi G} + 4\lambda\alpha T^2 \right] (R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) + \alpha (2R_{\cdot\mu\nu} - 2g_{\mu\nu}\Box R$$

$$- \frac{1}{2}g_{\mu\nu}R^2 + 2RR_{\cdot\mu\nu}) - 16\eta^2 \lambda\alpha \{3R_{;\mu\nu}^2 - 3g_{\mu\nu}\Box R^2$$

$$+ 9(-\frac{1}{2}g_{\mu\nu}\Box R^2 + 2R_{\cdot\mu\nu}\Box R + R_{;\mu\nu}^2 - \frac{1}{2}g_{\mu\nu}R^3 + 3R^2 R_{\cdot\mu\nu}) \right] = 0,$$

(2.2)

where

$$\Box = \frac{1}{\sqrt{-g}} \partial_{\mu} \left[ \sqrt{-g} g^{\mu\nu} \partial_{\nu} \right].$$

(2.3)

General anastaz for space-time is given as

$$dS^2 = g_{\mu\nu}dx^\mu dx^\nu.$$  

(2.4)
Trace of field equations (2.3) yields

\[
\left[ -\frac{1}{32\pi G} + 2\lambda T^2 \right] R + \alpha(8\Box R + \frac{1}{2} R^2) + 8\eta^2 \lambda \alpha R^3 = 0.
\] (2.5)

\(R\), being a linear combination of second-order derivatives and square of first-order derivatives of \(g_{\mu\nu}\), has mass dimension 2, whereas the same for scalar fields in known theories is 1. So, eq.(2.5) is multiplied by \(\eta\) (having length dimension) and \(\eta R\) is recognized \(\tilde{R}\). Thus \(\tilde{R}\), having mass dimension 1, represents a scalar particle. This scalar particle is called “riccion”. In the case of scalaron, \(\eta\) is dimensionless and its magnitude is 1 [6]. So, it is obtained that

\[
\Box \tilde{R} + \frac{1}{16} R\tilde{R} + \frac{1}{8} \left[ -\frac{1}{32\pi G\alpha} + 2\lambda T^2 \right] \tilde{R} + \lambda\eta^2 \tilde{R}^3 = 0
\] (2.6)

If \(\tilde{R}\) is a basic physical field, there should be an action \(S_{\tilde{R}}\) yielding eq.(2.6) for invariance of \(S_{\tilde{R}}\), under transformations \(\tilde{R} \rightarrow \tilde{R} + \delta \tilde{R}\).

In what follows, \(S_{\tilde{R}}\) is obtained. If such an action exists, one can write

\[
\delta S_{\tilde{R}} = -\int d^4x \sqrt{-g} \delta \tilde{R} \left[ \Box + \frac{1}{16} R \tilde{R} + \frac{1}{8} \left[ -\frac{1}{32\pi G\alpha} + 2\lambda T^2 \right] \tilde{R} + \lambda\eta^2 \tilde{R}^3 \right] (2.7a)
\]

which yields eq.(2.6) if \(\delta S_{\tilde{R}} = 0\) under transformations \(\tilde{R} \rightarrow \tilde{R} + \delta \tilde{R}\).

Eq.(2.7a) is re-written as

\[
\delta S_{\tilde{R}} = \int d^4x \sqrt{-g} \delta \left[ \frac{1}{2} \theta^\mu \tilde{R} \partial_\mu (\delta \tilde{R}) - \frac{1}{48\eta} R\tilde{R}^2 - V^T(\tilde{R}) \right], (2.7b)
\]

where

\[
V^T(\tilde{R}) = -\frac{1}{2} m^2 (\tilde{R}^2 + \frac{1}{12} T^2) + \frac{1}{4} \lambda \tilde{R}^4 + \frac{1}{8} \lambda T^2 \tilde{R}^2 - \frac{\pi^2}{90} T^4 (2.7c)
\]

with
\[ m^2 = \frac{1}{256\pi G\alpha}. \quad (2.7d) \]

\( R, \tilde{R} \) and \( d^4x\sqrt{-g} \) are invariant under co-ordinate transformations. So, \( R(x) = R(X), \tilde{R}(x) = \tilde{R}(X) \) and \( d^4x\sqrt{-g} = d^4X \), where \( X^i(i = 0, 1, 2, 3) \) are local and \( x^i(i = 0, 1, 2, 3) \) are global coordinates. Moreover,

\[ \Box = g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \frac{1}{2} g^{mn} \frac{\partial g}{\partial x^i} g^{ij} \frac{\partial}{\partial x^j} + \frac{\partial g^{ij}}{\partial x^i} \frac{\partial}{\partial x^j} = \frac{\partial^2}{\partial X^i \partial X^j} \]

in a locally inertial co-ordinate system, where \( g_{ij} = \eta_{ij} \) (components of Minkowskian metric) and \( g_{ij} = 0 \) (comma (,) stands for partial derivative).

Thus, in a locally inertial co-ordinate system,

\[
\delta S_{\tilde{R}} = \int d^4X \delta \left[ \frac{1}{2} \partial^{\mu} \tilde{R} \partial_{\mu} (\delta \tilde{R}) - \frac{1}{48\eta} R\tilde{R}^2 - V^T(\tilde{R}) \right] = \delta \int d^4X \left[ \frac{1}{2} \partial^{\mu} \tilde{R} \partial_{\mu} (\delta \tilde{R}) - \frac{1}{48\eta} R\tilde{R}^2 - V^T(\tilde{R}) \right].
\]

(2.8)

Employing principles of covariance and equivalence in eq.(2.8)

\[ \delta S_{\tilde{R}} = \delta \int d^4x \left\{ \sqrt{-g} \left[ \frac{1}{2} \partial^{\mu} \tilde{R} \partial_{\mu} (\delta \tilde{R}) - \frac{1}{48\eta} R\tilde{R}^2 - V^T(\tilde{R}) \right] \right\} \]

yielding the action for riccion as

\[ S_{\tilde{R}} = \int d^4x \left\{ \sqrt{-g} \left[ \frac{1}{2} \partial^{\mu} \tilde{R} \partial_{\mu} (\delta \tilde{R}) - \frac{1}{48} R\tilde{R}^2 - V^T(\tilde{R}) \right] \right\}. \quad (2.9) \]

It is important to mention here that \( \tilde{R} \) is different from other scalar fields due to dependence of \((\text{mass})^2\) for \( \tilde{R} \) on the gravitational constant and the coupling constant \( \alpha \) which is given by the eq.(2.1). Moreover, it emerges from geometry of the space-time.
Further, action for a scalar $\phi$ and a Dirac spinor $\psi$ are taken as

$$S_{(m)} = \int d^4x \left\{ \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi^* - \frac{1}{2} \Lambda \bar{\Phi} \Phi^* \right] + \frac{1}{2} \int d^4x \sqrt{-g} \left[ \bar{\psi} i \gamma^\mu D_\mu \psi - \sigma \bar{\Phi} \bar{\Phi} + c.c. \right] \right\}, \quad (2.10)$$

where $\Lambda$ and $\sigma$ are dimensionless coupling constants. Here $\bar{\psi} = \psi^\dagger \gamma^0$, $D_\mu$ are covariant derivatives. $\gamma^\mu$ are curved space Dirac matrices and $\bar{\gamma}^\mu$ being flat space Dirac matrices.

So, the total action is

$$S = S_R + S_{m}$$

$$= \int d^4x \left\{ \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu (\delta \bar{R})^2 - \frac{1}{48} \bar{R} \bar{R}^2 - V_T(\bar{R}) \right] + \int d^4x \left\{ \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi^* - \frac{1}{2} \Lambda \bar{\Phi} \Phi^* \right] + \frac{1}{2} \int d^4x \sqrt{-g} \left[ \bar{\psi} i \gamma^\mu D_\mu \psi - \sigma \bar{\Phi} \bar{\Phi} + c.c. \right] \right\}, \quad (2.11)$$

(b) Models of the early universe inspired by dual role of the Ricci scalar

Here models of the early universe are obtained in vacuum states of riccion, given by Higgs like potential (2.7c) [11, 17, 18] employing the condition

$$\frac{\partial V_T}{\partial \bar{R}} = 0,$$
or
\[-m^2 \ddot{R} + \lambda \dot{R}^2 + \frac{1}{4} \lambda T^2 \ddot{R} = 0, \quad (2.12)\]

This equation yields turning points of $V^T(\tilde{R})$ as
\[\tilde{R} = 0 \quad (2.13a)\]

and
\[\tilde{R} = \pm \frac{1}{2} \sqrt{(T_c^2 - T^2)}, \quad (2.13b)\]

where
\[T_c = \frac{2m}{\sqrt{\lambda}} \quad (2.13c)\]

These states can be written in terms of vacuum expectation value of $\tilde{R}$ as $< \tilde{R} > = 0$ and $< \tilde{R} > = (1/2) \sqrt{(T_c^2 - T^2)}$. It shows that as long as $T \geq T_c$, the field $\tilde{R}$ remains confined to the state $< \tilde{R} > = 0$. But when temperature goes down such that $T \ll T_c$, $\tilde{R}$ tunnels through the temperature barrier $T = T_c$ and settles in the state $< \tilde{R} > = (1/2)T_c$ when $T \ll T_c$. The state $< \tilde{R} > = 0$ is called the false vacuum and $< \tilde{R} > = (1/2)T_c$ corresponds to the true vacuum state. Moreover, symmetry is intact in the state $< \tilde{R} > = 0$, but it is broken spontaneously at $< \tilde{R} > = (1/2)T_c$ and energy is released as latent heat in the form of radiation with density given as
\[V^T(< \tilde{R} >= 0) - V^T(< \tilde{R} >= \frac{1}{2} T_c) = \frac{m^4}{4\lambda}. \quad (2.14)\]

The line-element for the cosmological model of the early universe is taken as
\[dS^2 = dt^2 - a^2(t) f^2(r) \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\psi^2) \right], \quad (2.15a)\]
where \( t \) is the cosmic time, \( 0 \leq r \leq \infty \), \( 0 \leq \theta \leq \pi \) and \( 0 \leq \psi \leq 2\pi \).

Looking at the line-element, given by (2.15a), one may think that the function \( f \) can be absorbed by redefining \( r \) as new coordinates \( \tilde{r} \) such that

\[
f^2(r) \left[ dr^2 + r^2(d\theta^2 + \sin^2 \theta d\psi^2) \right] = d\tilde{r}^2 + \tilde{r}^2(d\Theta^2 + \sin^2 \Theta d\Psi^2) \tag{2.15b}
\]

It is important to mention here that redefinition of coordinates are not valid at points where \( f = 0 \) or \( \infty \), as such points lead to spatial singularity (singularity at points on \( t = \text{constant} \) hypersurface). So, the absorption of \( f \) in \( r \) coordinates is possible only when \( f \neq 0 \) and \( f < \infty \) at all points of \( t = \text{constant} \) hypersurface. It means that only after getting explicit definition of \( f \), it is possible to decide about \( f \). This aspect will be looked into below.

Using geometrical aspect of the Ricci scalar, one can compute \( \tilde{R} \) in the background geometry of the line-element, given by eq.(2.15), as

\[
\tilde{R} = \eta \left[ 6 \left\{ \ddot{a} + \left( \frac{\dot{a}}{a} \right)^2 \right\} - \frac{1}{a^2 r^2} \left\{ \frac{2}{f^4} \left\{ \frac{\partial^2 f^2}{\partial r^2} + \frac{2}{r} \frac{\partial f^2}{\partial r} \right\} + \frac{3}{2 f^6} \left( \frac{\partial f^2}{\partial r} \right)^2 \right\} \right], \tag{2.16}
\]

which is linear sum of the homogeneous part, given as \( 6 \eta \left\{ \ddot{a} + \left( \frac{\dot{a}}{a} \right)^2 \right\} \) as well as the inhomogeneous part containing derivatives of \( f^2 \). \( \langle \tilde{R} \rangle \), being vacuum expectation value of \( \tilde{R} \), is homogeneous. So, in the vacuum states \( \langle \tilde{R} \rangle = 0 \) and \( \langle \tilde{R} \rangle = \pm (1/2)T_c \),

\[
\frac{1}{a^2 r^2} \left[ \frac{2}{f^4} \left\{ \frac{\partial^2 f^2}{\partial r^2} + \frac{2}{r} \frac{\partial f^2}{\partial r} \right\} + \frac{3}{2 f^6} \left( \frac{\partial f^2}{\partial r} \right)^2 \right] = 0, \tag{2.17}
\]

**First stage of the early universe**

Coming to the physical aspect of \( \tilde{R} \), one finds that at \( T \geq T_c \), \( \tilde{R} \) stays in the false vacuum state \( \tilde{R} = \langle \tilde{R} \rangle = 0 \). So, from eq.(2.16), one obtains the
differential equation
\[ \frac{\dot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 = 0 \] (2.18)
which yields the solution \[ a^2 = a_c^2 + \frac{|t|}{t_{pl}}, \] (2.19)
where \( t_{pl} \) is the Planck time. At this point, a question arises whether \( a_c = 0 \) or it is non-zero. In what follows, this question is answered.

Components of energy-momentum tensor for matter field \( \tilde{R} \) can be defined as \[ T_{\mu\nu} = 2 \frac{1}{\sqrt{-g}} \frac{\delta S_{\tilde{R}}}{\delta g^{\mu\nu}} \] which is obtained as
\[
T_{\mu\nu} = \partial_\mu \tilde{R} \partial_\nu \tilde{R} - g_{\mu\nu} \left[ \frac{1}{2} \partial^\gamma \tilde{R} \partial_\gamma \tilde{R} - V^T(\tilde{R}) \right] - 2\eta \tilde{R}_{\mu\nu} \Box \tilde{R} + 2\eta m^2 \tilde{R} R_{\mu\nu} \\
- 2\eta \lambda \tilde{R}^3 R_{\mu\nu} - \frac{1}{2} \eta \lambda T^2 \tilde{R} R_{\mu\nu} + 4\eta m^2 \tilde{R} R_{\mu\nu} \Box \tilde{R} - 4\eta \lambda \tilde{R}^3 + 4\eta \lambda g_{\mu\nu} \Box \tilde{R} - \lambda T^2 \tilde{R} R_{\mu\nu} + \lambda T^2 g_{\mu\nu} \Box \tilde{R}. \] (2.20)

In the false vacuum state \( \tilde{R} = < \tilde{R} > = 0 \), one obtains \[ T_{\mu\nu} U^\mu U^\nu < 0, \] (2.21)
where \( T_{\mu\nu} \) are given by eq.(2.20) and \( U^\mu \) are velocities normalized as
\[ U^\mu U_\mu = +1. \]
The result, given by the inequality (2.21), shows that the energy condition is violated in the false vacuum state when \( T \geq T_c \) \[11,15,17\]. It means that
the universe will bounce without any encounter with temporal singularity (singularity in time). As a result,

\[ a_c \neq 0. \quad (2.22) \]

The partial differential equation (2.17) yields an exact solution

\[ f^2 = \left[ 1 - \frac{T_0}{r} \right]^{4/7}. \quad (2.23) \]

Thus, in the false vacuum state, model of the early universe is obtained as given by eq.(2.15a) with \( a^2(t) \) and \( f^2(r) \) defined by eqs.(2.19) and (2.23) respectively.

The expansion of the cosmological model, derived above, is adiabatic. So, entropy will remain conserved, which implies that the average uniform temperature will fall as

\[ T \propto \left[ a_c^2 + \langle t | /t_p, l \rangle \right]^{-1/2}. \quad (2.24) \]

**Second stage of the early universe**

When \( T \ll T_c \)

\[ \tilde{R} =< \tilde{R} >= \pm (1/2)T_c. \]

In this state also, one obtains two differential equations

\[ 6\eta \left\{ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right\} = \pm \frac{1}{2}T_c \quad (2.25) \]

and the other differential equation is the equation given by eq.(2.17). Eq.(2.25) yields two cases. The first case is

\[ 6\eta \left\{ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right\} = \frac{1}{2}T_c, \quad (2.26) \]
yielding the solution

\[ a^2(t) = \sqrt{\frac{24}{\eta T_c}} \sinh \left[ (t - t_{1e}) \sqrt{\frac{T_c}{6\eta}} + D \right] \]
\[ = a_{1e}^2 \cosh^{-1} \left[ \left( t - t_{1e} \right) \sqrt{\frac{T_c}{6\eta}} + D \right] \]
\[ \simeq a_{1e}^2 \exp \left[ \left( t - t_{1e} \right) \sqrt{\frac{T_c}{6\eta}} \right] \]

(2.27a)

with \( a_{1e} = a(t = t_{1e}) \) and

\[ D = \cosh^{-1} \left[ a_{1e}^2 \sqrt{\frac{\eta T_c}{24}} \right]. \]

Here \( t_{1e} \) is the time at the end of first stage of the universe and \( a_{1e} \) is the corresponding scale factor, given as.

\[ a(t) \simeq a_{1e} \exp \left[ \left( t - t_{1e} \right) \sqrt{\frac{T_c}{24\eta}} \right] \]

(2.27b)

The second case is

\[ 6\eta \left( \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right) = -\frac{1}{2} T_c \]

(2.28)

which integrates to

\[ a^2(t) = a_{10}^2 \sinh \left[ \left( t - t_0 \right) \sqrt{\frac{T_c}{6\eta}} + \frac{\pi}{2} \right]. \]

(2.29)

In true vacuum states also, \( \phi \) is given by eq.(2.23).

But \( a(t) \), given by eqs. (2.27), shows that the cosmological model will expand more rapidly in the state \( \bar{R} = \bar{\bar{R}} = (1/2)T_c \) compared to the state \( \bar{R} = \bar{\bar{R}} = 0 \). The expansion, in the state \( \bar{R} = \bar{\bar{R}} = -(1/2)T_c \), is
given by eq.(2.29), which shows that \(-a_{10}^2 \leq a^2 \leq a_{10}^2\) leading to imaginary \(a(t)\) also, which is unphysical. So the state \(<\tilde{R}> = -(1/2)T_c\) will not be considered any more.

Thus one finds that vacuum states \(\tilde{R} = <\tilde{R}> = 0\) and \(\tilde{R} = <\tilde{R}> = (1/2)T_c\) lead to cosmological models

\[
dS^2 = dt^2 - \left[ a_c^2 + \left| \frac{t}{t_{Pl}} \right| \right] \left[ 1 - \frac{r_0}{r} \right]^{4/7} \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\psi^2) \right] \tag{2.30}
\]

at temperature \(T \geq T_c\) and

\[
dS^2 = dt^2 - a_{1e}^2 \exp \left( t - t_{1e} \right) \sqrt{\frac{T_c}{6\eta}} \left[ 1 - \frac{r_0}{r} \right]^{4/7} \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\psi^2) \right] \tag{2.31}
\]

when \(T \ll T_c\).

Divergence of scalar polynomials of the curvature tensors at points \(r = 0\) and \(r = r_0\) shows that models ( given by eqs.(2.30) and (2.31)) are singular at these points [21-22]. Moreover, these models are inhomogeneous at small scales i.e. when \(r\) is comparable to \(r_0\) and homogeneous when \(r \gg r_0\).

The model (2.31) exhibits accelerated expansion of the universe which is consistent with consequences of recent experiments Ia Supernova [24] and WMAP [20].

It is interesting to note that models, given by eqs.(2.30) and (2.31), reduce to homogeneous Robertson-Walker type models as \(r \to \infty\). As far as spatial singularity is concerned, at a particular instant of time, these models can be compared with Schwarzschild space-time [19, 21]

\[
dS^2 = \left[ 1 - \frac{2GM}{r} \right] dt^2 - \left[ 1 - \frac{2GM}{r} \right]^{-1} dr^2 - r^2 \left[ d\theta^2 + \sin^2 \theta d\psi^2 \right] \tag{2.34}
\]

It is remarked that the Schwarzschild space-time given by eq.(2.34) is singular at \(r = 0\) and \(r = 2GM\), but only \(r = 0\) is the real physical singularity and
the other point is a co-ordinate singularity. In models, given by eqs.(2.30) and (2.31), both points \( r = 0 \) and \( r = r_0 \) exhibit real singularities as mentioned above. Physically \( r_0 \) signifies radius of particles or compact objects dominating the expanding universe. Further investigations are carried out for \( r > r_0 \).

3. Creation of spinless particles

Invariance of \( S \), given by eq.(2.11) under transformation \( \Phi \rightarrow \Phi + \delta \Phi \) leads to the Klein - Gordon equation

\[
[\Box + M^2_b] \Phi = 0,
\]

where

\[
M^2_b = \Lambda < \tilde{R} >^2.
\]

In the background geometry of the line - element (2.15 a) with eq.(2.23), the equation (3.1) is re-written as

\[
\frac{\partial^2 \Phi}{\partial t^2} + \frac{3 \dot{a}}{a} \frac{\partial \Phi}{\partial t} - \frac{1}{a^2} \left[ 1 - \frac{r_0}{r} \right]^{-4/7} \frac{\partial^2 \Phi}{\partial r^2} - \frac{1}{r^2 a^2} \left[ 1 - \frac{r_0}{r} \right]^{-6/7} \frac{\partial \Phi}{\partial r} \left[ r^2 \left( 1 - \frac{r_0}{r} \right)^{2/7} \right] \frac{\partial \Phi}{\partial r} - \frac{1}{a^2 r^2} \left[ 1 - \frac{r_0}{r} \right]^{-4/7} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) \Phi + M^2_b \Phi = 0, \tag{3.3}
\]

where dot (.) over the variable denotes the derivative with respect to time.

The solution for this equation is taken as

\[
\Phi = \left[ (2\pi)^{1/2} r \left( 1 - \frac{r_0}{r} \right)^{3/7} \right]^{-1} \sum_{k,l,m} A_{klm} \Psi_{klm}(t) Y_{lm}(\theta, \psi) e^{ikr} + c.c., \tag{3.4}
\]

where \( r > r_0, m = -l \cdots + l, l = 1, 2, 3, \cdots \) and \( -\infty < k < \infty \). \( Y_{lm}(\theta, \psi) \) satisfies the equation

\[
\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \psi^2} \right] Y_{lm} = -l(l + 1) Y_{lm} \tag{3.5a}
\]
with normalization as
\[
\int \sin \theta d\theta d\psi Y_{lm}Y_{l'm'} = \delta_{ll'}\delta_{mm'}.
\] (3.5b)

Connecting eq.(3.3) with eq.(3.4) and using eq.(3.5), it is obtained that
\[
e^{ik.r}\left[\ddot{\Psi}_{klm} + \frac{3\dot{a}}{a}\Psi_{klm} + \left(\frac{k^2}{a^2} + \frac{l(l+1)}{a^2r^2} + \frac{6r_0(2r - r_0)}{7a^2r^2(r - r_0)^2}\right)\left[1 - \frac{r_0}{r}\right]^{-4/7}
\]
\[
+ M_b^2\right)\Psi_{klm} = 0.
\] (3.6)

Using the convolution theorem [25] and integrating over \(r\), eq.(3.6) looks like
\[
\int_{r_0+\epsilon}^{\infty} e^{ik.r} dr \left[\ddot{\Psi}_{klm} + \frac{3\dot{a}}{a}\Psi_{klm} + \left(\frac{k^2}{a^2\eta} + \frac{l(l+1)}{a^2\eta} \int_{r_0+\epsilon}^{r_0+\epsilon} e^{-ik.y}\left[1 - \frac{r_0}{y}\right]^{-4/7} dy + \frac{6r_0}{a^2\eta} \int_{r_0+\epsilon}^{\infty} \frac{2y - r_0}{y^2(y - r_0)^2}\left[1 - \frac{r_0}{y}\right]^{-4/7} dy + M_b^2\right)\Psi_{klm} = 0.
\] (3.7)

Here \(\epsilon\) is an extremely small positive real number.

Details of the evaluation of integrals with respect to \(y\) are given in the Appendix B. Using these results, in eq.(3.7), one obtains
\[
\ddot{\Psi}_{klm} + \frac{3\dot{a}}{a}\Psi_{klm} + \left[\frac{X_{klm}}{a^2} + M_b^2\right]\Psi_{klm} = 0,
\] (3.8a)

where
\[
X_{klm} = (1/a^2\eta)e^{3/7}(r_0 + \epsilon)^{4/7}\cos\{k(r_0 + \epsilon)\}\left[k^2 + \frac{l(l+1)}{(r_0 + \epsilon)^2}
\]
\[
- \frac{12}{7}r_0 e^{-2}(r_0 + \epsilon)^{-1} + \frac{6}{7}r_0^2 e^{-2}(r_0 + \epsilon)^{-2}\right].
\] (3.8b)

Case 1: The case of state \(< \hat{R} >= 0\)
Using $a(t)$ from the line-element (2.30) for this state, the equation (3.8) is written as

$$\ddot{\Psi}^{klm} + \frac{3}{2(t + a_{10}^2 t_P)} \dot{\Psi}^{klm} + \frac{X_{klm} t_P}{(t + t_P a_{10}^2)} \Psi^{klm} = 0,$$  \hspace{1cm} (3.9)

which yields the solution for $t > 0$ as

$$\Psi^{klm} = \tau^{-1/2} \left[ A J_{-1/2}(\tau) + B Y_{-1/2}(\tau) \right],$$  \hspace{1cm} (3.10a)

where $A$ and $B$ are integration constants. Here $\tau$ is defined as

$$\tau = \sqrt{t_P X_{klm}(t + t_P a_{10}^2)}$$ \hspace{1cm} (3.10b)

showing that $\tau \to \infty$ when $t \to \infty$.

For large $\tau$,

$$J_{-1/2}(\tau) \simeq \frac{cos(\tau)}{\sqrt{\pi \tau / 2}} \quad \text{and} \quad Y_{-1/2}(\tau) \simeq \frac{sin(\tau)}{\sqrt{\pi \tau / 2}}$$

So, when $\tau$ is large,

$$\Psi^{klm} = \left[ \pi \tau^2 / 2 \right]^{-1/2} [A \cos \tau + B \sin \tau]$$

$$= \left[ \frac{\tau^3}{2t_P X_{klm}} \right]^{1/2} \tau^{-1} \left[ (1 + i)e^{-i\tau} + (1 - i)e^{i\tau} \right]$$

\hspace{1cm} (3.11)

using the normalization condition

$$(\Phi^{kln}, \Phi^{kln}) = 1 = - (\Phi^{*}_{kln}, \Phi^{*}_{kln}),$$  \hspace{1cm} (3.12a)

where

$$\Phi^{kln} = \left[ (2\pi \eta^{-1} W)^{1/2} r \left( 1 - \frac{r_0}{r} \right)^{1/7} \right]^{-1} \Psi^{klm}(t)e^{ikr}Y_{lm}$$  \hspace{1cm} (3.12b)
and the scalar product is defined as

\[ (\Phi_{klm}, \Phi_{klm}) = -i \int_{r_0+\epsilon}^{\infty} \int_0^\pi \int_0^{2\pi} \sqrt{-g_{\Sigma}} d\Sigma \left[ \Phi_{klm} \partial_t \Phi_{k'l'm'}^* \right] \]

\[ - (\partial_t \Phi_{klm}) \Phi_{k'l'm'}^* |Y_{lm}(\theta, \psi)|^2, \tag{3.12c} \]

where \( \Sigma \) is the \( t = t_1 \) hypersurface, \( \sqrt{-g_{\Sigma}} = r^2 \left[ 1 - \frac{r_0}{r} \right]^{6/7} \) and \( d\Sigma = \sin \theta dr d\theta d\psi \).

Thus

\[ \Psi_{klm}^{\text{out}} = \left[ \frac{\tau^3}{2tpX_{klm}} \right]^{1/2} \tau^{-1} (1 + i) e^{-i\tau}. \tag{3.13} \]

For \( t < 0 \), eq.(3.9) yields the solution

\[ \Psi_{klm} = \tilde{\tau}^{-1/2} \left[ A' J_{-1/2}(\tilde{\tau}) + B' Y_{-1/2}(\tilde{\tau}) \right], \tag{3.14a} \]

where \( A' \) and \( B' \) are integration constants. Here \( \tilde{\tau} \) is defined as

\[ \tilde{\tau} = -\sqrt{tpX_{klm}(-t + t_0 a_0^2)} \tag{3.14b} \]

showing that \( \tilde{\tau} \to -\infty \) when \( t \to -\infty \).

Using above approximations for Bessel’s functions and normalization prescription, it is obtained that for \( t \to -\infty \)

\[ \Psi_{klm}^{\text{in}} = \left[ \frac{\tilde{\tau}^3}{2tpX_{klm}} \right]^{1/2} \tilde{\tau}^{-1} (1 + i) e^{i\tilde{\tau}}. \tag{3.15} \]

Since \( \Phi_{klm}^{\text{out}} \) and \( \Phi_{klm}^{\text{in}} \) both belong to the same Hilbert space, so one write

\[
\Phi = \sum_{k,l,m} \left[ A_{klm}^{\text{in}} \Phi_{klm}^{\text{in}} Y_{lm}(\theta, \psi) + A_{klm}^{\text{in}^*} \Phi_{klm}^{\text{in}} Y_{lm}(\theta, \psi) \right] \\
= \sum_{k,l,m} \left[ A_{klm}^{\text{out}} \Phi_{klm}^{\text{out}} Y_{lm}(\theta, \psi) + A_{klm}^{\text{out}^*} \Phi_{klm}^{\text{out}} Y_{lm}(\theta, \psi) \right].
\]
As a result, one obtains
\[ \Phi^{\text{out}}_{klm} = \alpha_{klm} \Phi^{\text{in}}_{klm} + \beta_{klm} \Phi^{*\text{in}}_{klm}, \] (3.16a)

where \( \alpha_{klm} \) and \( \beta_{klm} \) are Bogoliubov coefficients satisfying the condition [18, 26-27]
\[ |\alpha_{klm}|^2 - |\beta_{klm}|^2 = 1. \] (3.16c)

The in- and out- vacuum states are defined as
\[ A^{\text{in}}_{klm} |\text{in}\> = 0 = A^{\text{out}}_{klm} |\text{out}\>. \] (3.17a, b)

Moreover,
\[ A^{\text{out}}_{klm} = \alpha_{klm} A^{\text{in}}_{klm} + \beta_{klm} A^{*\text{in}}_{klm}. \] (3.17c)

Connecting eqs. (3.11) - (3.17), it is obtained that
\[ \alpha_{klm} = \frac{1}{2} e^{-i(\tau + \tilde{\tau})} \left[ \left( \frac{t_1 + a_{10}^2 t_P}{-t_1 + a_{10}^2 t_P} \right)^{1/4} + \left( \frac{-t_1 - a_{10}^2 t_P}{t_1 + a_{10}^2 t_P} \right)^{1/4} \right], \]
\[ \beta_{klm} = \frac{1}{2} e^{-i(\tau + \tilde{\tau})} \left[ \left( \frac{t_1 + a_{10}^2 t_P}{-t_1 + a_{10}^2 t_P} \right)^{1/4} - \left( \frac{-t_1 - a_{10}^2 t_P}{t_1 + a_{10}^2 t_P} \right)^{1/4} \right]. \] (3.18a, b)

These results yield
\[ \beta_{klm} \simeq i \frac{a_{10}^2 t_P}{t_1} e^{-i(\tau_1 + \tilde{\tau}_1)} \] (3.19a)

implying
\[ |\beta_{klm}|^2 \simeq \left[ \frac{a_{10}^2 t_P}{t_1} \right]^2 = \left[ \frac{5 t_P}{t_1} \right]^2. \] (3.19b)

Eq. (3.19b) shows that when \( t_1 \) is sufficiently larger than \( 5 t_P \), \( |\beta_{klm}|^2 = 0 \). It means that spinless particles will be created in the state \( \langle \tilde{R} >= 0 \) only.
when \( t_1 \leq 5t_p \). For \( t_1 \) sufficiently larger than \( 5t_p \), there will be no production of scalar particles, in this state.

Anomalous terms emerge when vacuum expectation value of the energy-momentum tensor components \( T_{\mu\nu} \) is evaluated for \( \Phi \), if dimension of the space-time is an even integer. These terms get cancelled for the difference of vacuum expectation values \( T_{\mu\nu} \) in in- and out-states. Thus \( T_{\mu\nu} \) for created particles is defined as [28]

\[
< T_{\mu\nu} > = < \text{out}|T_{\mu\nu}|\text{out} > - < \text{in}|T_{\mu\nu}|\text{in} > . \tag{3.20a}
\]

\( T_{\mu\nu} \) for \( \phi \) are obtained from the action (2.11) as

\[
T_{\mu\nu} = \partial_\mu \Phi^* \partial_\nu \Phi - 2\eta \Lambda [R_{\mu\nu} + \{(\tilde{R}\Phi^*\Phi)_{;\mu\nu} - g_{\mu\nu} \Box (\tilde{R}\Phi^*\Phi)\} - \frac{1}{2} g_{\mu\nu} [\partial^\rho \Phi^* \partial_\rho \Phi - \Lambda \tilde{R}^2 \Phi^* \Phi]] \tag{3.20b}
\]

with its trace

\[
T = -(1 - 12\eta \Lambda \tilde{R}) \partial^\mu \Phi^* \partial_\mu \Phi - 12\eta \Lambda^2 < \tilde{R} >^3 \Phi^* \Phi. \tag{3.20c}
\]

Now trace of \( T_{\mu\nu} \) for created particle-antiparticle pairs is obtained as

\[
< T > = < \text{out}|T|\text{out} > - < \text{in}|T|\text{in} > \tag{3.20d}
\]

Eqs.(3.12b), (3.13), (3.15) and (3.20 d) lead to trace of created particles in the state \(< \tilde{R} > = 0 \) as

\[
< T_1(b) > = 0. \tag{3.21}
\]

Case 2: The case of state \(< \tilde{R} > = \frac{1}{2} T_c \)
Using $a(t)$ from the line-element (2.31) for this state, the equation (3.8) is written as

$$
\ddot{\Psi}_{klm} + \frac{3}{2} \sqrt{\frac{T_c}{6\eta}} \dot{\Psi}_{klm} + \left[ \frac{X_{klm}}{a_0^2} e^{-(t-t_{1e})\sqrt{T_c/6\eta}} + M_b^2 \right] \Psi_{klm} = 0,
$$

(3.22)

where $M_b^2 = \frac{1}{4} \Lambda T_c^2$ using the definition of $M_b$ from eq.(3.2). The equation (3.22) yields the solution for $t > 0$ as

$$
\Psi_{klm} = \left[ -\frac{\pi}{\sinh 2\pi \alpha} \right]^{1/2} \left[ \frac{6\eta}{T_c} \right]^{1/4} e^{-\frac{3}{4}(t-t_{1e})\sqrt{T_c/6\eta}} \times
\nonumber
\right]
J_{\pm 2i\alpha} \left( \gamma_{klm} e^{\frac{3}{4}(t-t_{1e})\sqrt{T_c/6\eta}} \right),

(3.23a)

where

$$
\alpha^2 = \frac{6\eta}{T_c} \left[ \frac{1}{4} \Lambda T_c^2 - \frac{3T_c}{32\eta} \right]
$$

(3.23b)

and

$$
\gamma_{klm}^2 = \frac{24\eta X_{klm}}{a_0^2 T_c}.
$$

(3.23c)

For $t < 0$, the equation (3.22) yields the solution as

$$
\Psi_{klm} = \left[ -\frac{\pi \alpha}{\eta \sinh(2\pi \alpha)} \right]^{1/2} \left[ \frac{6\eta}{T_c} \right]^{1/4} e^{\frac{3}{4}(t-t_{1e})\sqrt{T_c/6\eta}} \times
\nonumber
\right]
J_{\pm 2i\alpha} \left( \gamma_{klm} e^{\frac{3}{4}(t-t_{1e})\sqrt{T_c/6\eta}} \right),

(3.24)

Using the approximation of

$$
J_n(x) \approx \frac{x^n}{2^n \Gamma(1 + n)}
$$

for small $x$ in eqns.(3.23) and (3.24), for $t \to \infty$

$$
\Psi_{klm}^{\text{out}} = \left[ -\frac{\pi \alpha}{\eta \sinh(2\pi \alpha)} \right]^{1/2} \left[ \frac{6\eta}{T_c} \right]^{1/4} \left( \frac{1}{\Gamma(1 \pm 2i\alpha)} \right) \left( \gamma_{klm} \right)^{\pm 2i\alpha} \times
\nonumber
$$

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\[ e^{-(t-t_e)(\frac{3}{4}+i\alpha)}\sqrt{T_c/6\eta}, \] 

(3.25a)

and for \( t \to -\infty \)

\[ \Psi_{in}^{klm} = \left[ -\frac{\pi\alpha}{\eta\sinh 2\pi \alpha} \right]^{1/2} \left[ \frac{6\eta}{T_c} \right]^{1/4} \left( \frac{1}{\Gamma(1 \mp 2i\alpha)} \right) \left( \frac{\gamma_{klm}}{2} \right)^{\mp 2i\alpha} \times \]

\[ e^{(t-t_{in})(\frac{3}{4}+i\alpha)}\sqrt{T_c/6\eta}, \] 

(3.25b)

Using \( \Psi_{out}^{klm} \) and \( \Psi_{in}^{klm} \), one obtains

\[ \alpha_{klm} = -i \left( \frac{3}{2} \mp 2i\alpha \right) \] 

(3.26a)

and

\[ \beta_{klm} = -\frac{3i}{2} \] 

(3.26b)

implying that

\[ |\beta_{klm}|^2 = \frac{9}{4}, \] 

(3.26c)

which ensures creation of spinless particles. Connecting eqs.(3.16c) and (3.26b)

\[ |\alpha_{klm}|^2 = \frac{13}{4} \] 

(3.26d)

Moreover, the absolute probability for no particle creation is given as

\[ |<\text{out}|\text{in}|>|^2 = \prod_{klm} |\alpha_{klm}|^{-2} = \frac{4}{9 + 16\alpha^2}. \] 

(3.27)

Eqs.(3.26 d) and (3.27) yield

\[ \alpha^2 = 1/4. \] 

(3.28)

Connecting eqs.(3.2), (3.23b) and (3.28), it is obtained that
Using the convolution theorem, eq.(3.12b) can be written as

\[
\Phi_{klm}(t,r,\theta,\phi) = \left[(2\pi)^{-1/2}e^{-ikr} \int_{r_0 + \epsilon}^{\infty} \left\{ y \left( 1 - \frac{r_0}{y} \right) \right\}^{-1} e^{iky} dy \right] \Psi_{klm}(t) Y_{lm}(\theta,\phi)
\]

\[
= (2\pi)^{-1/2}e^{-ikr} \frac{\epsilon^{4/7}}{(r_0 + \epsilon)^{11/7}} \cos[k(r_0 + \epsilon)] \Psi_{klm}(t) Y_{lm}(\theta,\phi).
\]

(3.30)

Connecting eqs.(2.27 a), (3.20 d), (3.25) and (3.30) as well as taking average over \( \theta \) and \( \phi \), trace of the energy-momentum tensor for created particles, in the state \( \bar{R} = (1/2)T_c \), is obtained at \( t = t_{2e} \) as

\[
T_{2(b)} = \sum_{klm} (1/2\pi) \left( \frac{6\eta}{T_c} \right)^{1/2} \sinh[(3/4)\sqrt{\frac{T_c}{6\eta}}(t_{2e} - t_{1e})]
\]

\[
\times \frac{\epsilon^{8/7}}{(r_0 + \epsilon)^{22/7}} \cos^2 k(r_0 + \epsilon) \left[ (13/16) \frac{T_c}{6\eta} (6\eta \Lambda T_c - 1) - (3/2)\eta \Lambda^2 T_c^3 \right]
\]

\[
= \left( \frac{2}{3\pi} \right) \text{Big} \left( \frac{6\eta}{T_c} \right)^{1/2} \sinh[(3/4)\sqrt{\frac{T_c}{6\eta}}(t_{2e} - t_{1e})]
\]

\[
\times \frac{\epsilon^{8/7}}{(r_0 + \epsilon)^{22/7}} \cos^2 k(r_0 + \epsilon) \left[ (13/16) \frac{T_c}{6\eta} (6\eta \Lambda T_c - 1) - (3/2)\eta \Lambda^2 T_c^3 \right].
\]

(3.31)

Here, summation is done with the help of the Riemann zeta function defined as \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \). Moreover,

\[
\sum_{klm} \cos^2[k(r_0 + \epsilon)] = \sum_{klm} [1 - k^2(r_0 + \epsilon)^2 + \cdots]
\]
\[ \sum_{klm} 1 = \sum_{kl} (2l + 1) \]
\[ = \sum_k [4\zeta(-1) + 2\zeta(0)] = -2(4/3)\zeta(0) = 4/3 \]

as \( \zeta(-2m) = 0 \). Also \( \zeta(0) = -\frac{1}{2} \) and \( \zeta(-1) = -\frac{1}{12} \) obtained through the analytic continuation.

4. Creation of spin-1/2 particles

Using invariance of \( S \), given by eq.(2.11) under transformation \( \psi \rightarrow \psi + \delta \psi \), the Dirac equation is obtained as

\[ \left( i\gamma^\mu D_\mu - M_f \right) \psi = 0, \quad (4.1a) \]

where

\[ M_f = \sigma \tilde{R}. \quad (4.1b) \]

Here, \( D_\mu = \partial_\mu - \Gamma_\mu \) with +

\[ \gamma^\mu = e^\mu_a \tilde{\gamma}^a, \quad (4.2a) \]

where \((a, \mu = 0, 1, 2, 3)\) and \( e^\mu_a \) are defined through

\[ e^\mu_a e^\nu_b g_{\mu\nu} = \eta_{ab}. \quad (4.2b) \]

Here \( \eta_{ab} \) are Minkowskian metric tensor components and \( g_{\mu\nu} \) are metric tensor components in curved space-time. \( \text{c.c.} \) stands for complex conjugation.

Dirac matrices \( \gamma^\mu \) in curved space-time satisfy the anti-commutation rule [18]

\[ \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad (4.2c) \]
and Dirac matrices $\tilde{\gamma}^{a}$ in Minkowskian space-time satisfy the anti-commutation rule
\[ \{ \tilde{\gamma}^{a}, \tilde{\gamma}^{b} \} = 2\eta^{ab}. \] (4.2d)

$\Gamma_{\mu}$ are defined as
\[ \Gamma_{\mu} = -\frac{1}{4}(\partial_{\mu}e^{a}_{\rho} + \Gamma^{a}_{\mu}e^{\sigma}_{a})g_{\nu\rho}e^{b}_{\mu}\tilde{\gamma}^{b}\tilde{\gamma}^{a}. \] (4.2e)

In general, solution of the Dirac equation $\psi$ can be written as
\[ \psi = \sum_{s=\pm 1} \sum_{k} \left( b_{k,s}\psi_{I_{k},s} + d^{\dagger}_{-k,-s}\psi_{II_{k},s} \right), \] (4.3a)
\[ \psi^{\dagger} = \sum_{s=\pm 1} \sum_{k} \left( \overline{\psi}_{I_{k},s}\gamma^{0}b^{\dagger}_{k,s} + \overline{\psi}_{II_{k},s}\gamma^{0}d_{k,s} \right), \] (4.3b)

where
\[ \psi_{I_{k},s} = \left[ (2\pi\eta)^{1/2}r \left( 1 - \frac{r_{0}}{r} \right) \right]^{1/7} \sum_{l,m} f_{klm,s}(t)Y_{lm}(\theta, \phi)e^{ik.r}u_{s} \] (4.4a)

and
\[ \psi_{II_{k},s} = \left[ (2\pi\eta)^{1/2}r \left( 1 - \frac{r_{0}}{r} \right) \right]^{1/7} \sum_{l,m} g_{klm,s}(t)Y_{lm}(\theta, \phi)e^{-ik.r}\hat{u}_{s}. \] (4.4b)

with $g_{klm,s}(t) = f_{-klm,s}(t)$.

In eqs.(4.4)
\[ u^{T}_{1} = (1000), \quad u^{T}_{-1} = (0100) \]
\[ \hat{u}^{T}_{1} = (0010), \quad \hat{u}^{T}_{-1} = (0001), \] (4.5a, b, c, d)

where the index $(T)$ stands for transpose of the column matrices $u_{\pm s}$ and $\hat{u}_{\pm s}$. In eqs.(4.4), $m = -l, \cdots, +l; l = 1, 2, 3, \cdots$ and $-\infty < k < \infty$. 

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Using eqs.(3.5) $\psi$ is normalized as

$$
(\psi_{k,s}, \psi_{k',s'}) = \int \Sigma d^3x \bar{\psi}_{k,s} \gamma^0 \psi_{k',s'}
= \delta_{kk'} \delta_{ss'}.
$$

(4.7)

In eqs.(4.3), $b^\dagger_{k,s}(b_{k,s})$ are creation (annihilation) operators for the positive - energy particles and $d^-_{k,s}(d^\dagger_{-k,s})$ are creation (annihilation) operators for the negative - energy particles (anti-particles).

Connecting eqs.(4.2) and (4.4), it is obtained that

$$
(i\gamma^\mu D_\mu - M_f)\psi_{I_k,s} = 0,
$$

(4.8a)

$$
(i\gamma^\mu D_\mu - M_f)\psi_{II,s} = 0,
$$

(4.8b)

Now using the operator $(-i\gamma^\mu D_\mu - M_f)$ from the left in eq.(4.8a) as

$$
(-i\gamma^\mu D_\mu - M_f)(i\gamma^\mu D_\mu - M_f)\psi_{I_k,s} = 0,
$$

it is obtained that [17]

$$
(\Box + (4\eta)^{-1} < \tilde{R} > + \sigma^2 < \tilde{R} >^2 )\psi_{I_k,s} = 0,
$$

(4.9a)

where $\tilde{R} = \eta R$ and

$$
\Box = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left[ \sqrt{-gg^{\mu\nu}} \frac{\partial}{\partial x^\nu} \right].
$$

(4.9b)

Similarly, from eq.(4.7b), it is obtained that

$$
(\Box + (4\eta)^{-1} < \tilde{R} > + \sigma^2 < \tilde{R} >^2 )\psi_{II,s} = 0.
$$

(4.10)

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When temperature $T$ is not very much below $T_c$ one obtains the vacuum state $<\tilde{R}> = 0$, where eqs.(4.7) and (4.8) reduce to

$$\Box \psi_{I,k,s} = 0 = \Box \psi_{II,k,s} \quad (4.11a, b)$$

Particles are created due to conformal symmetry breaking, which is caused by the mass term. In the state $<\tilde{R}> = 0$, there is no term to break this symmetry. So, production of spin-1/2 particles are not possible in this state.

As given in section 2, one obtains the state $<\tilde{R}> = \frac{1}{2}T_c$ when the temperature falls sufficiently below $T_c$. In this state, the universe obeys the geometry given by the line-element (2.31).

In what follows, creation of spin-1/2 particles is investigated in the model given by this line-element. In the background geometry of this model, for every $k, l, m$ and $s$, eq.(4.9) looks like

$$e^{ik \cdot r} \left[ \ddot{f}_{klm,s} + \frac{3 \dot{a}}{a} \dot{f}_{klm,s} + \left\{ \frac{k^2}{a^2} + \frac{l(l+1)}{a^2 r^2} + \frac{6 r_0 (2r-r_0)}{7 a^2 r^2 (r-r_0)^2} \right\} \left[ 1 - \frac{r_0}{r} \right]^{-4/7} + \tilde{M}_f^2 \right] f_{klm,s} = 0 \quad (4.12a)$$

connecting eqs.(4.4a) and (4.9). Here

$$\tilde{M}_f^2 = (4 \eta)^{-1} <\tilde{R}> + \sigma <\tilde{R}>^2 = \frac{T_c}{8 \eta} + \frac{\sigma^2 T_c^2}{4}. \quad (4.12b)$$

Using the convolution theorem [25] as above and integrating over $r$, one obtains that

$$\int_{r_0+\epsilon}^{\infty} e^{ik \cdot r} dr \left[ \ddot{f}_{klm,s} + \frac{3 \dot{a}}{a} \dot{f}_{klm,s} + \left\{ \frac{k^2}{a^2 \eta} \int_{r_0+\epsilon}^{\infty} e^{-ik \cdot y} \left[ 1 - \frac{r_0}{y} \right]^{-4/7} dy \right\} \right] f_{klm,s} = 0$$
\[ l(l + 1) \int_{r_0 + \epsilon}^{\infty} y^{-2} \left( 1 - \frac{r_0}{y} \right)^{-3/7} e^{-ikny} dy + \frac{6r_0}{7a^2\eta} \times \]
\[ \int_{r_0 + \epsilon}^{\infty} \frac{(2y - r_0)}{y^2(y - r_0)^2} \left( 1 - \frac{r_0}{y} \right)^{-3/7} e^{-ikny} dy + \tilde{M}_f^2 f_{klm,s} \right] = 0. \quad (4.13) \]

Details of the evaluation of integrals with respect to \( y \) are given in the Appendix B. Using these results, in eq.(4.13), one obtains

\[ \ddot{f}_{klm,s} + \frac{3}{2} \sqrt{\frac{T_c}{6\eta}} \dot{f}_{klm,s} + \left[ \frac{X_{klm}}{a_{10}^2} e^{-\frac{1}{2}(t-t_{1z})}\sqrt{T_c/6\eta} + \tilde{M}_f^2 \right] f_{klm,s} = 0, \quad (4.14a) \]

where

\[ X_{klm,s} = \left( \frac{1}{a^2\eta} \right) e^{3/7}(r_0 + \epsilon)^{4/7} \cos \left\{ k(r_0 + \epsilon) \right\} \left[ k^2 + \frac{l(l + 1)}{(r_0 + \epsilon)^2} \right. \]
\[ - \left. \frac{12}{7} r_0 \epsilon^{-2}(r_0 + \epsilon)^{-1} + \frac{6}{7} r_0^2 \epsilon^{-2}(r_0 + \epsilon)^{-2} \right]. \quad (4.14b) \]

Eqs.(4.14) yield the solution for \( t > 0, \)

\[ f_{klm,s} = C_1 e^{\pm \frac{i}{2}(t-t_{1z})}\sqrt{T_c/6\eta} J_{\pm 2\alpha} \left( \gamma_{klm,s} e^{-\frac{1}{2}(t-t_{1z})}\sqrt{T_c/6\eta} \right), \quad (4.15a) \]

where \( C_1 \) is a normalization constant,

\[ \gamma_{klm,s}^2 = \frac{24\eta X_{klm,s}}{a_{10}^2 T_c} \quad (4.15b) \]

and

\[ \alpha^2 = \frac{6\eta}{T_c} \left[ \tilde{M}_f^2 - \frac{3T_c}{32\eta} \right] = \frac{6\eta}{T_c} \left[ \frac{1}{4} \sigma^2 T_c^2 + \frac{5T_c}{32\eta} \right] \quad (4.15c) \]

using the definition of \( \tilde{M}_f^2 \) from eq.(4.12b).

Connecting eqs.(4.5a) and (4.15)

\[ \psi_{Ik,s} = \left[ (2\pi\eta)^{1/2} \left( 1 - \frac{r_0}{r} \right)^{1/7} \right]^{-1} \sum_{l,m} C_1 e^{-\frac{1}{2}(t-t_{1z})}\sqrt{T_c/6\eta} \times \]
\[ J_{\pm 2\alpha} \left( \gamma_{klm,s} e^{-\frac{1}{2} (t-t_{1e}) \sqrt{T_c/6\eta}} \right) Y_{lm}(\theta, \phi) e^{i k \cdot r \hat{u}_s}. \] (4.16)

Normalization of \( \psi_{Ik,s} \) is done using the rule (4.7c) at the \( t = t_{1e} \) hypersurface denoted as \( \Sigma \) onwards. As a result, \( C_1 \) is obtained as

\[ C_1 = \left| J_{\pm 2\alpha}(\gamma_{klm,s}) \right|^{-1}. \] (4.17)

Thus

\[ \psi_{Ik,s} = \left[ (2\pi\eta)^{1/2} r \left( \frac{T_0}{r} \right)^{1/7} \right]^{-1} \sum_{l,m} \left| J_{\pm 2\alpha}(\gamma_{klm,s}) \right|^{-1} e^{-\frac{3}{4} (t-t_{1e}) \sqrt{T_c/6\eta}} \times \]

\[ J_{\pm 2\alpha} \left( \gamma_{klm,s} e^{-\frac{1}{2} (t-t_{1e}) \sqrt{T_c/6\eta}} \right) Y_{lm}(\theta, \phi) e^{i k \cdot r \hat{u}_s}. \] (4.18)

Similarly

\[ \psi_{IIk,s} = \left[ (2\pi\eta)^{1/2} r \left( \frac{T_0}{r} \right)^{1/7} \right]^{-1} \sum_{l,m} \left| J_{\pm 2\alpha}(\gamma_{-klm,s}) \right|^{-1} e^{-\frac{3}{4} (t-t_{1e}) \sqrt{T_c/6\eta}} \times \]

\[ J_{\pm 2\alpha} \left( \gamma_{-klm,s} e^{-\frac{1}{2} (t-t_{1e}) \sqrt{T_c/6\eta}} \right) Y_{lm}(\theta, \phi) e^{-i k \cdot r \hat{u}_s}. \] (4.19)

Asymptotics of these solutions, when \( t \to \infty \), yield

\[ \psi_{\text{out},Ik,s} = \left[ (2\pi\eta)^{1/2} r \left( \frac{T_0}{r} \right)^{1/7} \right]^{-1} \sum_{l,m} \left| J_{\pm 2\alpha}(\gamma_{klm,s}) \right|^{-1} e^{-\frac{3}{4} (t-t_{1e}) \sqrt{T_c/6\eta}} \times \]

\[ \left[ \Gamma(1 - 2i\alpha) \right]^{-1} \left( \frac{\gamma_{klm,s}}{2} \right)^{-2i\alpha} e^{+i \alpha(t-t_{1e}) \sqrt{T_c/6\eta}} Y_{lm}(\theta, \phi) e^{i k \cdot r \hat{u}_s}. \] (4.20a)

and

\[ \psi_{\text{out},IIk,s} = \left[ (2\pi\eta)^{1/2} r \left( \frac{T_0}{r} \right)^{1/7} \right]^{-1} \sum_{l,m} \left| J_{\pm 2\alpha}(\gamma_{-klm,s}) \right|^{-1} e^{-\frac{3}{4} (t-t_{1e}) \sqrt{T_c/6\eta}} \times \]

\[ \left[ \Gamma(1 - 2i\alpha) \right]^{-1} \left( \frac{\gamma_{-klm,s}}{2} \right)^{-2i\alpha} e^{+i \alpha(t-t_{1e}) \sqrt{T_c/6\eta}} Y_{lm}(\theta, \phi) e^{-i k \cdot r \hat{u}_s}. \] (4.20b)

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For $t \to -\infty$, one obtains

$$
\psi_{in}^{Ik,s} = \left[ (2\pi \eta)^{1/2} r \left( 1 - \frac{r_0}{r} \right)^{-1/7} \right]^{-1} \sum_{l,m} \left| J_{\pm 2i\alpha} (\gamma_{klm,s}) \right|^{-1/2} \left( e^{\frac{2}{T_c/\eta}} \right) \times
$$

$$
\left[ \Gamma(1 + 2i\alpha) \right]^{-1} \left( \frac{\gamma_{klm,s}}{2} \right)^{-2i\alpha} \left[ \gamma_{klm,s} \right]^{2i\alpha} e^{-i\alpha(t-t_{1e})} \sqrt{T_c/\eta} \frac{Y_{lm}(\theta, \phi)}{\gamma_{klm,s}} e^{ik_r u_s}. \tag{4.21a}
$$

and

$$
\psi_{out}^{Ik,s} = \left[ (2\pi \eta)^{1/2} r \left( 1 - \frac{r_0}{r} \right)^{-1/7} \right]^{-1} \sum_{l,m} \left| J_{\pm 2i\alpha} (\gamma_{klm,s}) \right|^{-1/2} \left( e^{\frac{2}{T_c/\eta}} \right) \times
$$

$$
\left[ \Gamma(1 + 2i\alpha) \right]^{-1} \left( \frac{\gamma_{klm,s}}{2} \right)^{-2i\alpha} \left[ \gamma_{klm,s} \right]^{2i\alpha} e^{-i\alpha(t-t_{1e})} \sqrt{T_c/\eta} \frac{Y_{lm}(\theta, \phi)}{\gamma_{klm,s}} e^{-ik_r u_s}. \tag{4.21b}
$$

In eqs. (4.20) - (4.21), $\Gamma(x)$ stands for the gamma function of $x$.

In the in- as well as out- region,

the decomposed form of $\psi$ can be written as [4]

$$
\psi = \sum_{s=\pm 1} \sum_{k} \left( b_{in}^{k,s} \psi_{in}^{I(k,s)} + d_{in}^{k,s} \psi_{in}^{II(-k,-s)} \right)
$$

$$
= \sum_{s=\pm 1} \sum_{k} \left( b_{out}^{k,s} \psi_{out}^{I(k,s)} + d_{out}^{k,s} \psi_{out}^{II(-k,-s)} \right),
$$

(4.22a, b)

as both in- and out-spinors belong to the same Hilbert space. The in- and out-vacuum states are defined as

$$
b_{k,s}^{in} |\text{in} >= d_{k,-s}^{in} |\text{in} >= 0 \tag{4.23a, b}
$$

and

$$
b_{k,s}^{out} |\text{out} >= d_{k,-s}^{out} |\text{out} >= 0 \tag{4.23c, d}
$$
Bogoliubov transformations are given as \[3,4\]

\[
\begin{align*}
b_{k,s}^{\text{out}} &= b_{k,s}^{\text{in}} \alpha_{k,s} + d_{-k,-s}^{\text{fin}} \beta_{k,s} \\
b_{k,s}^{\text{out}}^* &= \alpha_{k,s}^* b_{k,s}^{\text{in}} + \beta_{k,s}^* d_{-k,-s}^{\text{fin}} \\
d_{-k,-s}^{\text{out}} &= b_{k,s}^{\text{in}} \alpha_{k,s} + d_{-k,-s}^{\text{fin}} \beta_{k,s} \\
d_{-k,-s}^{\text{out}}^* &= \alpha_{k,s}^* b_{k,s}^{\text{in}} + \beta_{k,s}^* d_{-k,-s}^{\text{fin}}.
\end{align*}
\]

(4.24a, b, c, d)

Connecting eqs.(2.22)-(2.24), one obtains [22]

\[
|\alpha_k|^2 + |\beta_k|^2 = \sum_s |\alpha_{k,s}|^2 + |\beta_{k,s}|^2 = 1, \quad (4.25)
\]

where

\[
\alpha_{k,s} = \int_\Sigma \sqrt{-g} \bar{\psi}^{\text{out}} \tilde{\gamma}_I \psi^{\text{in}} I (k,s)
\]

(4.26a)

and

\[
\beta_{k,s} = \int_\Sigma \sqrt{-g} \bar{\psi}^{\text{out}} \tilde{\gamma}_II \psi^{\text{in}} I (k,s)
\]

(4.26b)

Connecting eqs.(4.20a), (4.21a) and (4.26a)

\[
\begin{align*}
\alpha_{k,s} &= \frac{1}{2} \left[ \Gamma(1 - 2i\alpha) \right]^{-2} \left( \frac{\gamma_{klm,s}}{2} \right)^{-4i\alpha} \left| J_{\pm 2i\alpha} (\gamma_{klm,s}) \right|^{-2} \\
&= \frac{\sinh(2\pi\alpha)}{(4\pi\alpha)} \left[ \Gamma(1 - 2i\alpha) \right]^{-2}.
\end{align*}
\]

(4.27a)

Similarly eqs.(4.20b), (4.21b) and (4.26b)

\[
\begin{align*}
\beta_{k,s} &= \frac{1}{2} \left[ \Gamma(1 + 2i\alpha) \right]^{-2} \left( \frac{\gamma_{klm,s}}{2} \right)^{-4i\alpha} \left| J_{\pm 2i\alpha} (\gamma_{klm,s}) \right|^{-2} \\
&= \frac{\sinh(2\pi\alpha)}{(4\pi\alpha)} \left[ \Gamma(1 + 2i\alpha) \right]^{-2}.
\end{align*}
\]

(4.27b)
Conditions (4.25) and eqs. (4.27) yield an useful result

\[ |\alpha_{k,s}|^2 = |\beta_{k,s}|^2 = \frac{1}{4} \]

(4.28)
suggesting that \( \alpha \) should be very small. Using the symmetry \( k \rightarrow -k \) caused by symmetry of the model under \( r \rightarrow -r \).

The relative probability of creation of a particle - antiparticle pair is given as

\[ \omega_{klm,s} = \left| \frac{\beta_{k,s}}{\alpha_{k,s}} \right|^2 = 1. \]

(4.29)

Absolute probability of the creation of particle - antiparticle pairs requires the total probability of creating \( 0, 1, 2, \cdots \) pairs to be unity [5], which means that

\[ N_{klm,s}(1 + \omega_{klm,s} + \omega_{klm,s}^2 + \cdots) = 1, \]

where \( N_{klm,s} \) is the probability of creation of no pair of particle-antiparticle. Using eq.(4.29), it is obtained that

\[ N_{klm,s} = \frac{1}{1 + 1 + 1 + \cdots} = 0. \]

It shows that probability of vacuum to remain vacuum is

\[ |\langle \text{out} | \text{in} \rangle|^2 = \prod_{klm,s} N_{klm,s} = 0. \]

(4.30)

implying certainty of creation of particle-antiparticle pairs.

The action \( S \), given by eq.(2.11), yields components of energy - momentum tensor for \( \psi \) field as [22,27]

\[ T_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \]
\[
= i\bar{\psi}\gamma_\mu D_\nu \psi - \sigma \eta R^0_{\mu\nu}\bar{\psi}\psi - \frac{1}{2} g_{\mu\nu}\bar{\psi}(i\gamma^\rho D_\rho - \sigma \tilde{R})\psi \\
- \sigma \eta (D_\mu D_\nu - g_{\mu\nu}\Box)\bar{\psi}\psi + c.c.
\]

(4.31)

Connecting eqs.(3.20a) and (4.31), energy density of created particles in the state \(\tilde{R} = 1/2T_c\) is obtained as

\[
\rho = < T^0_0 > = < \text{in} | - i\bar{\psi}\gamma^0 \partial_0 - \sigma \eta R^0_0\bar{\psi}\psi + 2\sigma \tilde{R}\bar{\psi}\psi + c.c.|\text{in} > \\
- < \text{out} | - i\bar{\psi}\gamma^0 \partial_0 - \sigma \eta R^0_0\bar{\psi}\psi + 2\sigma \tilde{R}\bar{\psi}\psi + c.c.|\text{out} > \\
= < \text{in} | - i\bar{\psi}\gamma^0 \partial_0 + 1/2\sigma T_c\bar{\psi}\psi + c.c.|\text{in} > \\
- < \text{out} | - i\bar{\psi}\gamma^0 \partial_0 + 1/2\sigma T_c\bar{\psi}\psi + c.c.|\text{out} >
\]

(4.32)

using \(R^0_0 = 3\ddot{a}/a = T_c/2\eta\).

In- and out- \(\psi\), given by eqs.(4.20) and (4.21), recast eq.(4.32) as

\[
\rho = \sum_{klm} \frac{8\alpha}{\eta} \frac{\epsilon^{8/7}}{(r_0 + \epsilon)^{22/7}} \cos^2[k(r_0 + \epsilon)] \sinh[(3/2)(t - t_{1e})\sqrt{T_c/6\eta}]
\]

(4.33)

using convolution theorem for \(\psi\) as given in the preceding section.

Similarly trace of stress - tensor for created spin-1/2 particles is given as

\[
< T > = < \text{in}|T_{\mu\nu}|\text{in} > - < \text{out}|T_{\mu\nu}|\text{out} > = 0,
\]

(4.34)

using In- and out- solutions for \(\psi\), given by eqs.(4.20) and (4.21).

Eq.(4.28) shows number density of created spin-1/2 particles as
\[ |\beta_k|^2 = \frac{1}{2} \]  \hspace{1cm} (4.35)

yielding energy density of created particles also as

\[ \rho = \frac{1}{2} M_f. \]  \hspace{1cm} (4.36)

\[ |\beta_k|^2 \] gives number of created particles per unit volume during the time period \((t_{20} - t_{10}) = 1.38 \times 10^7 t_P\), so evaluating \(\rho\), from eq.(4.32) and comparing with the same from eq.(4.36), it is obtained that

\[ M_f \simeq 54.87 \times 10^{99} \frac{\epsilon^{8/7}}{(r_0 + \epsilon)^{22/7}}. \]  \hspace{1cm} (4.37)

Here also summation is done with the help of the Riemann zeta function defined in the earlier section.

4. Conclusions

Thus, in this part of the series, production of spinless and spin-1/2 particles are discussed in the first two phases of the universe. Contribution of these particles to later development of the universe to third and fourth phases of the universe will be discussed in paper II.

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Appendix A

34
Riccion and Graviton

From the action

\[
S = \int d^4x d^Dy \sqrt{-g_{(4+D)}} \left[ \frac{M^{(2+D)} R_{(4+D)}}{16\pi} + \alpha_{(4+D)} R^2_{(4+D)} + \gamma_{(4+D)} (R^3_{(4+D)} - \frac{6(D+3)}{D-2} \Box_{(D+4)} R^2_{(D+4)}) \right],
\]

(A.1)

the gravitational equations are obtained as

\[
\frac{M^{(2+D)}}{16\pi} (R_{MN} - \frac{1}{2} g_{MN} R_{(4+D)}) + \alpha_{(4+D)} H^{(1)}_{MN} + \gamma_{(4+D)} H^{(2)}_{MN} = 0, \quad (A.2a)
\]

where

\[
H^{(1)}_{MN} = 2R_{;MN} - 2g_{MN} \Box_{(4+D)} R_{(4+D)} - \frac{1}{2} g_{MN} R^2_{(4+D)} + 2R_{(4+D)} R_{MN}, \quad (A.2b)
\]

and

\[
H^{(2)}_{MN} = 3R^2_{;MN} - 3g_{MN} \Box_{(4+D)} R^2_{(4+D)} - \frac{6(D+3)}{(D-2)} \left\{-\frac{1}{2} g_{MN} \Box_{(4+D)} R^2_{(4+D)} \right\}
+ 2 \Box_{(4+D)} R_{(D+4)} R_{MN} + R^2_{ MN} \right\} - \frac{1}{2} g_{MN} R^3_{(4+D)} + 3R^2_{(4+D)} R_{MN}. \quad (A.2c)
\]

Taking \(g_{MN} = \eta_{MN} + h_{MN}\) with \(\eta_{MN}\) being \((4+D)\)-dimensional Minkowskian metric tensor components and \(h_{MN}\) as small fluctuations, the equation for graviton are obtained as

\[
\Box_{(4+D)} h_{MN} = 0 \quad (A.3)
\]

neglecting higher-orders of \(h\).
On compactification of $M^4 \otimes S^D$ to $M^4$, eq.(A.3) reduces to the equation for 4-dimensional graviton as
\[ \Box h_{\mu\nu} + \frac{l(l + D - 1)}{\rho^2} h_{\mu\nu} = 0 \quad (A.4) \]
for the space time
\[ dS^2 = g_{\mu\nu}dx^\mu dx^\nu - \rho^2 [d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \cdots + \sin^2 \theta_1 \cdots \sin^2 \theta_{(D-1)} d\theta_D^2]. \quad (A.5) \]

The 4-dimensional graviton equation (A.4) is like usual 4-dimensional graviton equation (the equation derived from 4-dimensional action) only for $l = 0$. Thus the massless graviton is obtained for $l = 0$ only.

As explained, in section 2, the trace of equations (A.2) leads to the riccion equation
\[ [\Box + \frac{1}{2} \xi R + m_R^2 + \frac{\lambda}{3!} R^2] R + \vartheta = 0, \quad (A.6a) \]
where
\[
\begin{align*}
\xi & = \frac{D}{2(D + 3)} + \eta^2 \lambda R_D \\
m_R^2 & = -\frac{(D + 2)\lambda V_D}{16\pi G_{(4+D)}} + \frac{DR_D}{2(D + 3)} + \frac{1}{2} \eta^2 \lambda R_D^2 \\
\lambda & = \frac{1}{4(D + 3)\alpha}, \\
\vartheta & = -\eta \left[ -\frac{(D + 2)\lambda M^{(2+D)} V_D}{16\pi} + \frac{DR_D^2}{4(D + 3)} + \frac{1}{6} \eta^2 \lambda R_D^3 \right].
\end{align*}
\]
The graviton $h_{\mu\nu}$ has 5 degrees of freedom (2 spin-2 graviton, 2 spin-1 gravi-vector (gravi-photon) and 1 scalar). The scalar mode $f$ satisfies the equation

$$\Box f + \frac{l(l + D - 1)}{\rho^2} f = 0 \quad (A.7)$$

Comparison of eqs.(A.6) and (A.7) show many differences between scalar mode $f$ of graviton and the riccion ($\tilde{R}$) e.g. $\xi, \lambda$ and $\vartheta$, given by eqs.(A.6b,c,d,e), are vanishing for $f$, but non-vanishing for $\tilde{R}$. Eq.(A.7) shows $(mass)^2$ for $f$ as

$$m^2_f = \frac{l(l + D - 1)}{\rho^2} f, \quad (A.8)$$

whereas $(mass)^2$ for $\tilde{R}$, given by eq.(A.6c), depends on $G_{4+D}, V_D$ and $R_D$ (given in section 2). $m^2_f = 0$ for $l = 0$, but $m^2_{\tilde{R}}$ can vanish only when gravity is probed upto $\sim 10^{-33} cm$. As mentioned above, so far, gravity is probed only upto 1cm. Thus $(mass)^2$ of riccion does not vanish.

So, even though, $f$ and $\tilde{R}$ are scalars arising from gravity, both are different. Riccion can not arise without higher-derivative curvature terms in the gravitational action, but graviton can be obtained even from Einstein-Hilbert action.

$$\Box = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \right) = \eta^{\mu\nu} \frac{\partial}{\partial X^\mu} \frac{\partial}{\partial X^\nu},$$

where $X^\mu$ are locally inertial co-ordinates and $\eta^{\mu\nu}$ are Minkowskian metric components. It shows that the scalar like operator $\Box$ has the same role on $\tilde{R}$ as it is for other scalar fields $\phi$ due to principle of equivalence.
Appendix B

In eq.(3.6), using the convolution theorem the function $e^{ik.r}[1 - \frac{r_0}{r}]^{-4/7}$ can be written as

$$e^{ik.r}[1 - \frac{r_0}{r}]^{-4/7} = \eta^{-1}e^{ik.r}\int_{r_0+\epsilon}^{\infty} dy e^{-ik.y}[1 - \frac{r_0}{y}]^{-4/7}, \quad (B.1)$$

where $\eta$ has dimension of length, as used in section 2, for dimensional correction. Here in the integral $dy$ introduces a quantity of length dimension which is compensated by $\eta$.

Evaluation of the integral in eq.(B.1), is done as follows.

$$\int_{r_0+\epsilon}^{\infty} dy \frac{e^{-ik.y}}{[1 - \frac{r_0}{y}]^{4/7}} = \int_{r_0+\epsilon}^{\infty} dy \frac{y^{4/7}(y-r_0)^{\omega}}{(y-r_0)^{\omega+4/7}} \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-ik)^n}{n! \Gamma(\omega+4/7)} \int_{r_0+\epsilon}^{\infty} dy y^n y^{4/7} e^{(y-r_0)^{\omega} x}$$

$$\times \left[ \int_0^{\infty} x^\omega e^{-x(y-r_0)} dx \right]$$

$$= \sum_{n=0}^{\infty} \frac{(-ik)^n}{n! \Gamma(\omega+4/7)} \int_{r_0+\epsilon}^{\infty} dx x^{\omega+3/7} e^{x(r_0+y)}$$

$$\times \left[ \int_{r_0+\epsilon}^{\infty} y^{n+4/7} e^{-xy dy} \right]$$

$$\approx \sum_{n=0}^{\infty} \frac{(-ik)^n}{n! \Gamma(\omega+4/7)} \int_{r_0+\epsilon}^{\infty} dx x^{\omega-3/7} e^{-\epsilon x}$$

$$\times (r_0 + \epsilon)^{(n+4/7)} e^{\omega \left[ \frac{1}{x} + \frac{\omega}{\epsilon x^2} + \frac{\omega(\omega-1)}{\epsilon^2 x^3} + \ldots \right]} \quad (B.2)$$

neglecting terms containing $(r_0 + \epsilon)^{-1}$ compared to terms containing $\epsilon^{-1}$.

Performing further integration in eq.(B.2), one obtains that

$$\int_{r_0+\epsilon}^{\infty} dy \frac{e^{-ik.y}}{[1 - \frac{r_0}{y}]^{4/7}} \approx \sum_{n=0}^{\infty} \frac{(-ik)^n}{n! \Gamma(\omega+4/7)} (r_0 + \epsilon)^{(n+4/7)} \times$$

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\[
\begin{align*}
\epsilon^{3/7} \left[ \Gamma(\omega - 3/7) + \omega \Gamma(\omega - 10/7) + \omega(\omega - 1) \Gamma(\omega - 17/7) + \cdots \right] \\
= \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \frac{1}{\Gamma(\omega + 4/7)} (r_0 + \epsilon)^{(n+4/7)} \epsilon^{3/7} \\
= (r_0 + \epsilon)^{4/7} \epsilon^{3/7} \cos k (r_0 + \epsilon),
\end{align*}
\]

when \( \omega \to \infty \). Here, symmetry under \( k \to -k \) caused by symmetry of the model under \( r \to -r \) is used.

The same procedure yields

\[
\int_{r_0+\epsilon}^{\infty} dy e^{-ik \cdot y} y^{-2} \left[ 1 - \frac{r_0}{y} \right]^{-4/7} = (r_0 + \epsilon)^{-10/7} \epsilon^{3/7} \cos k (r_0 + \epsilon),
\]

\( (B.4) \)

\[
\int_{r_0+\epsilon}^{\infty} dy e^{-ik \cdot y} y^{-3/7} (y - r_0)^{-18/7} = (r_0 + \epsilon)^{-3/7} \epsilon^{-11/7} \cos k (r_0 + \epsilon),
\]

\( (B.5) \)

\[
\int_{r_0+\epsilon}^{\infty} dy e^{ik \cdot y} y^{-4/7} (y - r_0)^{4/7} = (r_0 + \epsilon)^{-4/7} \epsilon^{11/7} \cos k (r_0 + \epsilon),
\]

\( (B.6) \)

\[
\int_{r_0+\epsilon}^{\infty} dy e^{-ik \cdot y} y^{-10/7} (y - r_0)^{-18/7} = (r_0 + \epsilon)^{-10/7} \epsilon^{-11/7} \cos k (r_0 + \epsilon).
\]

\( (B.7) \)

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