ON THE SPLITTING TYPES OF BUNDLES OF LOGARITHMIC VECTOR FIELDS ALONG PLANE CURVES

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Abstract. We give a formula relating the total Tjurina number and the generic splitting type of the bundle of logarithmic vector fields associated to a reduced plane curve. By using it, we give a characterization of nearly free curves in terms of splitting types. Several applications to free and nearly free arrangements of lines are also given, in particular a proof of a form of Terao’s Conjecture for arrangements having a line with at most 4 intersection points.

1. Introduction

Let \( C : f = 0 \) be a reduced curve of degree \( d \) in \( X = \mathbb{P}^2 \), \( S = \mathbb{C}[x,y,z] \), and \( AR(f) \) the graded \( S \)-module of Jacobian syzygies of \( f \) as in [10], see equation (2.1) below. Note that \( AR(f) \) is isomorphic to the logarithmic derivation module \( D_0(C) \) of the curve \( C \) defined by \( D_0(C) := \{ \theta \in \text{Der} S | \theta(f) = 0 \} \). Let \( mdr(f) := \min \{ k | AR(f)_k \neq (0) \} \). In this paper we assume that \( mdr(f) \geq 1 \), unless otherwise specified. The case \( mdr(f) = 0 \) corresponds to the rather trivial case when \( C \) is a collection of \( d \) lines passing through one common point. Let \( E_C \) be the locally free sheaf on \( X \) corresponding to the graded module \( AR(f) \), and recall that

\[
E_C = T \langle C \rangle (-1),
\]

(1.1)

where \( T \langle C \rangle \) is the sheaf of logarithmic vector fields along \( C \) as considered for instance in [10]. For a line \( L \) in \( X \), the pair of integers \((d_1^L,d_2^L)\), with \( d_1^L \leq d_2^L \), such that \( E_C|_L \cong \mathcal{O}_L(-d_1^L) \oplus \mathcal{O}_L(-d_2^L) \) is called the splitting type of \( E_C \) along \( L \), see for instance [13] [15]. For a generic line \( L_0 \), the corresponding splitting type \((d_1^{L_0},d_2^{L_0})\) is constant. For any line \( L \) in \( X \), we set

\[
I(C,L) = (d-1)^2 - d_1^L d_2^L.
\]

The algebraic structure of the graded \( S \)-module \( AR(f) \) is related to the singularities of \( C \), e.g. the invariants like Milnor numbers and Tjurina numbers. From this viewpoint, when the \( S \)-module \( AR(f) \) is free, which can be considered as the simplest case, then the corresponding curve is called free, a notion going back to K.
Saito [17]. When the minimal resolution of the graded \( S \)-module \( AR(f) \) is slightly more complicated, we get the nearly free curves considered in [11]. See §2 for details.

Recall the definition of the global Tjurina number

\[
\tau(C) = \sum_{p \in C} \tau(C, p)
\]

of the curve \( C \). Also, let \( N(f) = \tilde{J}_f/J_f \), with \( J_f \) the Jacobian ideal of \( f \) in \( S \), spanned by the partial derivatives \( f_x, f_y, f_z \) of \( f \), and \( \tilde{J}_f \) the saturation of the ideal \( J_f \) with respect to the maximal ideal \( \mathfrak{m} = (x, y, z) \) in \( S \). The quotient module \( N(f) = H^0_m(S/J_f) \) plays a key role in this theory. Indeed, let \( \nu(C) = \dim N(f)[T/2] \), where \([ \ ] \) denotes integral part and \( T = 3(d - 2) \). It is known that the curve \( C : f = 0 \) is free (resp. nearly free) if and only if \( \nu(C) = 0 \) (resp. \( \nu(C) = 1 \)), see [7, 9, 11]. The above key invariants of distinct origins are related in the first main result of this paper.

**Theorem 1.1.** With the above notation, for any line \( L \) in \( \mathbb{P}^2 \), and any generic line \( L_0 \) in \( \mathbb{P}^2 \), the following hold.

1. \( \max(mdr(f) - \nu(C), 0) \leq d_1^L \leq d_1^{L_0} \leq \min(mdr(f), [(d - 1)/2]) \).
2. \( I(C, L) \geq I(C, L_0) = \tau(C) + \nu(C) \).

In particular, the reduced curve \( C : f = 0 \) in \( \mathbb{P}^2 \) is free (resp. nearly free) if and only if \( I(C, L_0) = \tau(C) \) (resp. \( I(C, L_0) = \tau(C) + 1 \)).

**Corollary 1.2.** Let \( c_{E_C}(t) = 1 + c_1(E_C)t + c_2(E_C)t^2 \in \mathbb{Z}[t] \) be the Chern polynomial of the vector bundle \( E_C \). Then the curve \( C \) is free (resp. nearly free) if and only if there is a line \( L \subset \mathbb{P}^2 \) such that \( c_2(E_C) - d_1^L d_2^L = 0 \) (resp. \( c_2(E_C) - d_1^{L_0} d_2^{L_0} = 1 \)).

The free case of Corollary [12] is due to Yoshinaga in [25], see Theorem 2.13 below. We give another proof for this case in terms of Tjurina numbers. Indeed, the proof of Theorem 1.1 implies

\[
c_2(E_C) - d_1^{L_0} d_2^{L_0} = (d - 1)^2 - \tau(C) - d_1^{L_0} d_2^{L_0} = \nu(C).
\]

This relation also yields the following analog of a result in [12].

**Corollary 1.3.** For any reduced curve \( C : f = 0 \) in \( \mathbb{P}^2 \) and any line \( L \) in \( \mathbb{P}^2 \) we have

\[
\tau(C) \leq (d - 1)^2 - d_1^L d_2^L.
\]

Moreover, equality holds for a line \( L \) if and only if the curve \( C \) is free, and then it holds for any line \( L \).

The second main result of our paper is the following.

**Theorem 1.4.** Let \( C : f = 0 \) be a reduced curve of degree \( d \) in projective plane \( \mathbb{P}^2 \). Then the following properties are equivalent.
The Chern polynomial $c_{EC}(t) = 1 + c_1(\mathcal{E})t + c_2(\mathcal{E})t^2 \in \mathbb{Z}[t]$ of the vector bundle $\mathcal{E}$ has real roots.

(2) $\tau(C) \geq \frac{3}{4}(d-1)^2$.

(3) $d_{L_0}^1 \leq d_{L_0}^2 - \sqrt{\nu(C)}$ for a generic line $L_0$.

Moreover, these properties imply that $mdr(f)$ coincides with $d_{L_0}^1$ for a generic line $L_0$, and they are implied by either $mdr(f) < \frac{d}{4}$ or

$$mdr(f) \leq \frac{d-1}{2} - \sqrt{\nu(C)}.$$ 

The organization of this paper is as follows. In §2 we recall several definitions and results necessary for the proof of the main results. In §3 we prove Theorem 1.1 and Corollary 1.2. In §4, we prove Theorem 1.4. In §5 and §6 we apply these results to the case of a line arrangement $\mathcal{A} : f = 0$ in $\mathbb{P}^2$. A sample of the results we get in this case is the following special case of Terao’s Conjecture.

**Theorem 1.5.** Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$, such that

(1) there is a line $H \in \mathcal{A}$ containing at most 4 intersection points of $\mathcal{A}$;

(2) this line $H$ does not contain an intersection point of multiplicity $\geq |\mathcal{A}|/2$.

Then the fact that $\nu(\mathcal{A}) \leq 1$, i.e. the fact that $\mathcal{A}$ is either free or nearly free, depends only on the characteristic polynomial $\chi(\mathcal{A}; t)$. In addition, when the characteristic polynomial $\chi(\mathcal{A}; t)$ is not a perfect square, it determines precisely whether $\mathcal{A}$ is free or nearly free.

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2. Preliminaries

First let us recall the definition of free and nearly free curves. In this paper $C$ is a reduced plane curve in $\mathbb{P}^2$, defined by $f = 0$, where $f$ is a degree $d$ homogeneous polynomial. For the coordinate ring $S = \mathbb{C}[x, y, z]$ and a graded $S$-module $M$, let $M_k$ denote the homogeneous degree $k$-part of $M$. For $g \in S$, let $g_x, g_y, g_z$ denote the partial derivative of $g$ by $x, y, z$. Then the graded $S$-module $AR(f) = AR(C) \subset S_{\geq 3}$ of all relations is defined by

$$AR(f)_k := \{(a, b, c) \in S_{k}^{\geq 3} | af_x + bf_y + cf_z = 0\}. \tag{2.1}$$

The module $AR(f)$ is isomorphic to the logarithmic derivations killing $f$, hence we sometimes identify the two types of objects as follows:

$$AR(f) \ni (a, b, c) \mapsto a\partial_x + b\partial_y + c\partial_z \in D_0(C).$$
Its sheafification $E_C := \widehat{AR(f)}$ is a rank two vector bundle on $\mathbb{P}^2$, see [17, 18] for details. In particular we have the following.

**Proposition 2.1** (Equation (3.1), [10]). For a coherent sheaf $F$ on $\mathbb{P}^2$, consider the graded $S$-module $\Gamma^\ast(F) := \oplus_{k \in \mathbb{Z}} H^0(\mathbb{P}^2, F(k))$. Then $\Gamma^\ast(E_C) = \widehat{AR(f)}$.

**Definition 2.2** ([11]).

1. A curve $C$ is free if the graded $S$-module $\widehat{AR(f)}$ is free, say with a basis $\varphi_1, \varphi_2$. If $\deg \varphi_i = d_i$ ($i = 1, 2$), the multiset of integers $(d_1, d_2)$ is called the exponents of a free curve $C$, and denoted by $\exp(C) = (d_1, d_2)$.

2. A curve $C$ is nearly free if $N(f) \neq 0$ and $\dim N(f)_k \leq 1$ for any $k$.

By [11], the near freeness coincides with the following.

**Proposition 2.3** ([11]). $C$ is nearly free if and only if the graded $S$-module $\widehat{AR(f)}$ has a minimal generator system of syzygies $\theta, \varphi_1, \varphi_2$, such that

$$\deg \theta = d_1 \leq d_2 := \deg \varphi_1 = \deg \varphi_2$$

with a relation

$$h\theta + \beta_1 \varphi_1 + \beta_2 \varphi_2 = 0,$$

for $h \in S$ and linear forms $\beta_1, \beta_2$. The multiset $(d_1, d_2)$ is called the exponents of a nearly free curve $C$, and denoted by $\text{nexp}(C) = (d_1, d_2)$.

Hence in terms of sheaves, for a nearly free curve $C$, the bundle $E_C$ has a minimal resolution of the form

$$0 \to \mathcal{O}(-d_2 - 1) \to \mathcal{O}(-d_1) \oplus \mathcal{O}(-d_2)^{\oplus 2} \to E_C \to 0.$$ 

The following statement is immediate.

**Proposition 2.4.** For a nearly free curve with $\text{nexp}(C) = (d_1, d_2)$, it holds that

$$c_t(E_C) = 1 - (d_1 + d_2 - 1)t + (d_1(d_2 - 1) + 1)t^2.$$ 

Recall also the following characterization of nearly free curves.

**Proposition 2.5** (Proposition 3.8, [11]). $C$ is nearly free if and only if $\nu(C) = 1$.

For the proof of the main results, we need the following. Let $\alpha_L$ be the defining equation of the line $L$. Then one has an exact sequence

$$0 \to \mathcal{O}_X(-1) \overset{\alpha_L}{\to} \mathcal{O}_X \to \mathcal{O}_L \to 0,$$
where the first non-trivial morphism is induced by multiplication by the linear form \( \alpha_L \). Let \( k \) be an integer and tensor the above exact sequence by the vector bundle \( E_C(k) \). We get

\[
0 \to E_C(k-1)^{\alpha_L} \to E_C(k) \to E_C(k)|_L \to 0,
\]

with \( E_C(k)|_L \cong \mathcal{O}_L(k-d_1^L) \oplus \mathcal{O}_L(k-d_2^L) \), since we assume as in the Introduction that \( E_C|_L \cong \mathcal{O}_L(-d_1^L) \oplus \mathcal{O}_L(-d_2^L) \). Then we have the following.

**Proposition 2.6.** The long exact sequence of cohomology groups of the short exact sequence above starts as follows:

\[
(2.2) \quad 0 \to AR(f)_{k-1} \overset{\alpha_L}{\to} AR(f)_k \overset{\pi^*_L}{\to} H^0(L, \mathcal{O}_L(k-d_1^L) \oplus \mathcal{O}_L(k-d_2^L)) \to N(f)_{k+d-2} \overset{\alpha_L}{\to} N(f)_{k+d-1} \to \cdots.
\]

Moreover, for \( k = -1 \), the corresponding morphism \( N(f)_{d-3} \overset{\alpha_L}{\to} N(f)_{d-2} \) is injective and \( d_1^L \geq 0 \) for any line \( L \).

**Proof.** This is exactly as in the proof of [10, Theorem 5.7]. The key point is the identification \( H^1(\mathbb{P}^2, E_C(k)) = N(f)_{k+d-1} \), valid for any integer \( k \), for which we refer to [18, Proposition 2.1]. For the last claim, note that \( N(f)_{d-3} \subset S_{d-3} \) and \( N(f)_{d-2} \subset S_{d-2} \), as the Jacobian ideal is generated in degree \( d-1 \). \( \square \)

The following result is often used to investigate the structure of \( AR(f) \).

**Theorem 2.7** ([25], Theorem 1.45). Let \( F \) be a rank two vector bundle on \( \mathbb{P}^2 \) and \( L \subset \mathbb{P}^2 \) a line. Then

\[
\dim \ker(\pi_L : \Gamma_*(F) \to \Gamma_*(F|_L)) = c_2(F) - d_1^L d_2^L \geq 0,
\]

and the equality holds if and only if \( F \cong \mathcal{O}(-d_1^L) \oplus \mathcal{O}(-d_2^L) \).

For a rank two vector bundle \( E \) on \( \mathbb{P}^2 \), consider the function

\[
(2.3) \quad a_E : (\mathbb{P}^2)^* \to \mathbb{Z}^2, \text{ defined by } a_E(L) := (d_1^L, d_2^L),
\]

where we take \( d_1^L \leq d_2^L \). We say that \( E \) is uniform if the function \( a_E \) is constant. The following classification of the uniform 2-bundles on \( \mathbb{P}^2 \) is often used.

**Theorem 2.8** (e.g., §2.2, Theorem 2.2.2, [15]). A rank two uniform vector bundle on \( \mathbb{P}^2 \) is either (a) a direct sum of line bundles, or (b) isomorphic to \( T_{\mathbb{P}^2}(k) \) for some \( k \in \mathbb{Z} \), where \( T_{\mathbb{P}^2} \) is the tangent bundle of \( \mathbb{P}^2 \).

Next let us introduce some definitions and results on line arrangements in \( \mathbb{P}^2 \), to which we apply our main results. Let \( \mathcal{A} \) be an arrangement of lines in \( \mathbb{P}^2 \), namely, a finite set of lines in \( \mathbb{P}^2 \). It can be naturally identified with a central arrangement \( \overline{\mathcal{A}} \) of planes in \( \mathbb{C}^3 \). Let \( L(\overline{\mathcal{A}}) := \{ \cap_{H \in \mathcal{B}} H \mid \mathcal{B} \subset \overline{\mathcal{A}} \} \) be the intersection lattice of \( \overline{\mathcal{A}} \), with a partial order induced from the reverse inclusion, and let \( \chi(\overline{\mathcal{A}}; t) \) be the corresponding characteristic polynomial, see [16, 8].
Then $\chi(\mathcal{A}; t) = \sum_{i=0}^{3} (-1)^i b_i(\mathcal{A}) t^{3-i}$, where $b_i(\mathcal{A})$ is the $i$-th Betti number of $M(\mathcal{A}) = V \setminus \cup_{H \in \mathcal{A}} H$, see [16] [8]. When $\mathcal{A} \neq \emptyset$, it is known that $\chi(\mathcal{A}; t)$ is divisible by $t-1$. Define $\chi(\mathcal{A}; t) := \chi(\mathcal{A}; t)/(t-1)$ and note that $\chi(\mathcal{A}; t) = t^2 - b_1(\mathcal{A}) t + b_2(\mathcal{A})$, where $b_i(\mathcal{A})$ is the $i$-th Betti number of $M(\mathcal{A}) = \mathbb{P}^2 \setminus \cup_{H \in \mathcal{A}} H$, see [16] [8]. Let us recall the definition of logarithmic vector fields and the freeness of arrangements.

**Definition 2.9.** Let $\alpha_H$ be a defining linear form for $H \in \mathcal{A}$. Then for $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H$, define $AR(\mathcal{A}) := AR(Q(\mathcal{A}))$.

For $H \in \mathcal{A}$, define

$$AR_H(\mathcal{A}) := \{(a, b, c) \in S^{\mathcal{A}} | (a \partial_x + b \partial_y + c \partial_z)(\alpha_L) \in S\alpha_L \ (\forall L \in \mathcal{A} \setminus \{H\}), (a \partial_x + b \partial_y + c \partial_z)(\alpha_H) = 0\}.$$ 

The following is well-known, of which we give a proof for the completeness.

**Proposition 2.10.** $AR(\mathcal{A}) \simeq AR_H(\mathcal{A})$ for all $H \in \mathcal{A}$.

**Proof.** Let $\theta_E$ be the Euler derivation and $M := AR(\mathcal{A}) \oplus S\theta_E$. Define a map $\varphi : M \to AR_H(\mathcal{A})$ by

$$M \ni \theta \mapsto \theta - (\theta(\alpha_H)/\alpha_H) \theta_E \in AR_H(\mathcal{A}).$$

The kernel of $\varphi$ is clearly $S \cdot \theta_E$. Also, for any $\theta \in AR_H(\mathcal{A})$, $\theta - (\theta(Q)/(\deg Q)) \theta_E \in AR(\mathcal{A}) \subset M$ is sent to $\theta$ by $\varphi$. Hence

$$AR_H(\mathcal{A}) \simeq M/S \cdot \theta_E \simeq AR(\mathcal{A}).$$

When $\mathcal{A}$ is free, i.e. when $AR(\mathcal{A})$ is a free graded $S$-module, we have the following important result.

**Theorem 2.11** (Terao’s factorization, [20]). Assume that $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, d_2)$. Then $\chi(\mathcal{A}; t) = (t - d_1)(t - d_2)$.

Note that Terao proved Theorem 2.11 in all dimensions, but the above case is enough for our purposes. Here is the nearly free version of this factorization result.

**Theorem 2.12** (Factorization for nearly free arrangements, [11]). Let $\mathcal{A}$ be nearly free with $\exp(\mathcal{A}) = (d_1, d_2)$. Then $d_1 + d_2 = |\mathcal{A}|$ and

$$\chi(\mathcal{A}; t) = (t - d_1)(t - d_2 + 1) + 1.$$ 

However, it is very difficult to determine whether a given arrangement is free or not in general, even for line arrangements. Here we recall a criterion for freeness. For that purpose, we need the following definition.
Definition 2.13. For a central arrangement $\mathcal{A}$ in $\mathbb{C}^3$ and $H \in \mathcal{A}$, define $\mathcal{A}^H := \{H \cap L \mid L \in \mathcal{A} \setminus \{H\}\}$ and $m^H(X) := \{|L \in \mathcal{A} \setminus \{H\} \mid L \cap H = X\|$ for $X \in \mathcal{A}^H$. The pair $(\mathcal{A}^H, m^H)$ is called the Ziegler restriction of $\mathcal{A}$ onto $H$. Also, there is a canonical Ziegler restriction map

$$\pi : \text{AR}_H(\mathcal{A}) \to D(\mathcal{A}^H, m^H),$$

where

$$D(\mathcal{A}^H, m^H) := \{(a, b) \in (S/\alpha_H)^{\oplus 2} \mid (a\partial_x + b\partial_y)(\alpha_X) \in (S/\alpha_H)\alpha_X^{m^H}(X) \quad (\forall X \in \mathcal{A}^H)\}$$

and we choose $\alpha_H = z$ and hence $S/\alpha_H = \mathbb{C}[x, y]$.

Note that the restriction $\mathcal{A}^H$ of a central arrangement $\mathcal{A}$ is also central. Since $D(\mathcal{A}^H, m^H)$ is also reflexive as an $S/\alpha_H$-module, it is free. If its free basis has degrees $(d_1, d_2)$, it is denoted as $\exp(\mathcal{A}^H, m^H) = (d_1, d_2)$. In this article, $d_1 \leq d_2$ unless otherwise specified. Define $b_2(\mathcal{A}^H, m^H) := d_1d_2$.

Theorem 2.14 (\cite{26}). Assume that $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, d_2)$. Then $(\mathcal{A}^H, m^H)$ is also free with $\exp(\mathcal{A}^H, m^H) = (d_1, d_2)$, and the Ziegler restriction map $\text{AR}_H(\mathcal{A}) \to D(\mathcal{A}^H, m^H)$ is surjective for all $H \in \mathcal{A}$.

The following is the arrangement version of Theorem 2.7.

Theorem 2.15 (Yoshinaga’s criterion, \cite{24}). Let $\exp(\mathcal{A}^H, m^H) = (d_1, d_2)$. Then $b_2(\mathcal{A}) - d_1d_2 \geq 0$, which coincides with $\dim \text{coker} \pi$. Moreover, $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, d_2)$ if and only if the equality holds.

In general, the pair $(\mathcal{A}, m)$, where $\mathcal{A}$ is a line arrangement, and $m : \mathcal{A} \to \mathbb{Z}_{>0}$ is called a multiarrangement. To investigate the exponents of the multiarrangement, the following easy lemma is important.

Lemma 2.16 (\cite{3}, Lemma 4.2). Let $(\mathcal{A}, m)$ be a multiarrangement in $\mathbb{C}^2$. For $H \in \mathcal{A}$, let $\delta_H$ be a multiplicity such that $\delta_H(L) = 1$ only when $H = L$, and 0 otherwise. Then there is a homogeneous basis $\theta_1, \theta_2$ for $D(\mathcal{A}, m)$ such that $\alpha_H\theta_1, \theta_2$ form a basis for $D(\mathcal{A}, m + \delta_H)$.

Also, we use the following relation between Betti numbers and Chern classes, regarded as integers under the canonical identification $H^{2i}(\mathbb{P}^2, \mathbb{Z}) = \mathbb{Z}$, for details we refer to \cite[Proposition 5.18]{5} or \cite[Corollary 4.3]{14}.

Proposition 2.17. $b_i(\mathcal{A}) = (-1)^i c_i(E_A)$ for $i = 0, 1, 2$.

3. Proof of Theorem 1.1

The following result is perhaps well-known, but we include a proof for reader’s convenience.

Proposition 3.1. With the notation above, we have the following.
(1) For any line $L$, one has $d_1^L + d_2^L = d - 1$.

(2) If the line $L_0$ is generic and the line $L$ arbitrary, then $d_1^{L_0} \geq d_1^L$. In particular one has $d_1^{L_0} d_2^{L_0} \geq d_1^L d_2^L$.

Proof. For the first claim, note that we have the following values for the first Chern classes, considered now as cohomology classes:

$$c_1(E_C) = c_1(T \langle C \rangle (-1)) = -(d - 1)\alpha,$$

see for instance [10, equation (3.2)] and then obviously

$$c_1(E_C|_L) = c_1(O_L(-d_1^L) \oplus O_L(-d_2^L)) = -(d_1^L + d_2^L)\beta.$$

Here $\alpha$ (resp. $\beta$) are the canonical generators of $H^2(X, \mathbb{Z})$ (resp. $H^2(L, \mathbb{Z})$), with $X = \mathbb{P}^2$. Let $i : L \to X$ denote the inclusion, and note that $i^*(\alpha) = \beta$ and also $i^*(c_1(E_C)) = c_1(i^*(E_C))$, with $i^*(E_C) = E_C|_L$.

For the second claim, see [15, §1, Definition 2.2.3], but note that the ordering of the degrees $(d_1^L, d_2^L)$ in our paper is opposite from the ordering in [15]. Indeed, the generic splitting type as defined in [15, §1, Definition 2.2.3] corresponds to $d_2^L$ being minimal, hence in view of (1), to $d_1^L$ being maximal.

The following result is the key step in proving Theorem 1.1.

Proposition 3.2. Let $L_0$ be a generic line in $\mathbb{P}^2$. With the above notation, we have the following.

(1) For any line $L$ in $\mathbb{P}^2$, one has

$$\max(r - \dim N(f)_{r+d-3}, 0) \leq d_1^L \leq r,$$

where $r = mdr(f)$.

(2) If $d_1^{L_0} < (d-2)/2$, then $mdr(f) = d_1^{L_0}$.

(3) If $d_1^{L_0} \geq (d-2)/2$ and $d = 2m$ is even, then $d_1^{L_0} = m - 1$, $d_2^{L_0} = m$. In particular, this case can occur for a free curve $C$ only if the exponents are $d_1 = m - 1$, $d_2 = m$, while for a nearly free curve $C$ the exponents should

necessarily be either $d_1 = m - 1$, $d_2 = m + 1$, or $d_1 = d_2 = m$.

(4) If $d_1^{L_0} \geq (d-2)/2$ and $d = 2m+1$ is even, then $d_1^{L_0} = d_2^{L_0} = m$. In particular, this case can occur for a free curve $C$ only if the exponents are $d_1 = d_2 = m$, while for a nearly free curve $C$ the exponents should necessarily be $d_1 = m$, $d_2 = m + 1$.

Proof. To prove the inequality $d_1^L \leq r$ in claim (1), we use increasing induction on $0 \leq k < d_1^L \leq d_2^L$, and prove that $AR(f)_k = 0$ in this range, using the exact sequence (2.2). Note that $AR(f)_0 = 0$ by our assumption $mdr(f) \geq 1$ in Introduction. To
prove the inequality $r - \dim N(f)_{r+d-3} \leq d_1^L$ in claim (1), we use the exact sequence (2.2) for $k = r - 1$, when we get

$$0 \to H^0(L, \mathcal{O}_L(r - 1 - d_1^L)) \oplus H^0(L, \mathcal{O}_L(r - 1 - d_2^L)) \to N(f)_{r+d-3} \to \cdots.$$ 

If $d_1^L < r - \dim N(f)_{r+d-3}$, then $d_1^L \leq r - 1$, and one must have

$$r - d_1^L = \dim H^0(L, \mathcal{O}_L(r - 1 - d_1^L)) \leq \dim N(f)_{r+d-3}.$$ 

This implies $d_1^L \geq r - \dim N(f)_{r+d-3}$, and this is a contradiction. Finally note that $d_1^L \geq 0$, as follows from the last claim in Proposition [2.6]

To prove the claim (2), take $k = d_1^{L_0}$, and note that the condition $d_1^{L_0} < (d - 2)/2$ is equivalent to the condition $k + d - 1 < T/2$, with $T = 3(d - 2)$. Now we use [9, Corollary 4.3] to conclude that the morphism $N(f)_{k+d-2} \to N(f)_{k+d-1}$ in the exact sequence (2.2), which is induced by the multiplication by a generic linear form $\alpha_{L_0}$ defining $L_0$, is injective. Hence $AR(f)_k = H^0(L, \mathcal{O}_L \oplus \mathcal{O}_L(d_1^L - d_2^L)) \neq 0$, which completes the proof for the second claim.

The proof for the claims (3) and (4) goes along the same line, and we leave them to the reader. □

Proof of Theorem 1.1. The claim (1) follows from the claim (1) in Proposition 3.2 (1), using [9, Corollary 4.3]. The inequality in (2) follows from Proposition 3.1. It is known that $c_2(E_C) = (d - 1)^2 - \tau(C)$, see for instance [10, equation (3.2)]. On the other hand, by Theorem 2.7 and the exact sequence (2.2) we get

$$c_2(E_C) - d_1^{L_0}d_2^{L_0} = \dim \ker \{\alpha_{L_0} : N(f) \to N(f)\},$$

for a generic linear form $\alpha_{L_0}$. Using again [9, Corollary 4.3] it is clear that

$$\dim \ker \{\alpha_{L_0} : N(f) \to N(f)\} = \dim N(f)_{[T/2]} = \nu(C).$$

□

By the proof above, we have the following:

Corollary 3.3.

$$c_2(E_C) - d_1^{L_0}d_2^{L_0} = (d - 1)^2 - \tau(C) - d_1^{L_0}d_2^{L_0} = \nu(C).$$

Proof of Corollary 3.3. Combine Corollary 3.3 with Proposition 2.5 and the fact that the freeness is equivalent to $\nu(C) = 0$. □

Using Theorem 1.1 (1) and Proposition 3.1 (1), we get the following result.

Corollary 3.4. For any reduced plane curve $C : f = 0$ of degree $d$, the image $\text{im}(a_{E_C})$ is contained in

$$\{(r_0, d - 1 - r_0), (r_0 - 1, d - r_0), \ldots, (r_0' + 1, d - 2 - r_0'), (r_0', d - 1 - r_0')\},$$
where \( r_0 = \min(\text{mdr}(f), [(d-1)/2]) \) and \( r'_0 = \max(\text{mdr}(f) - \nu(C), 0) \). In particular, this set has at most
\[
  r_0 - r'_0 + 1 \leq \nu(C) + 1
\]
elements. Moreover, if \( C \) is nearly free with \( \text{nexp}(C) = (d_1, d_2) \), \( d_1 \leq d_2 \), and \( L \) is any line, then \( a_{E_C}(L) \) is either \((d_1 - 1, d_2)\) or \((d_1, d_2 - 1)\).

The following result shows in particular that \( E_C \) is a uniform bundle for a nearly free curve \( C \) if and only if the exponents of \( C \) are equal.

**Corollary 3.5.** With the above notation, assume that \( c_2(E_C) = d_1(d_2 - 1) + 1 \), for some integers \( 1 \leq d_1 \leq d_2 \). Then \( C \) is nearly free with \( \text{nexp}(C) = (d_1, d_2) \) if and only if either (1) \( d_1 = d_2 = d' \) and \( a_{E_C} \equiv (d' - 1, d') \), or (2) \( d_1 < d_2 \), \( a_{E_C} \) is not constant and \( \text{Im}(a_{E_C}) = \{(d_1 - 1, d_2), (d_1, d_2 - 1)\} \).

**Proof.** The “if” part follows from Corollary 1.2. Assume that \( C \) is nearly free with \( \text{nexp}(C) = (d_1, d_2) \). Then Corollary 3.4 confirms that the splitting type is either \((d_1 - 1, d_2)\) or \((d_1, d_2 - 1)\). If \( d_1 = d_2 \), then this is clearly the case (1). Assume that \( d_1 < d_2 \), and that \( a_E \) attains only the value \((d_1, d_2 - 1)\). Indeed, this value has to be in the image of \( a_{E_C} \) due to Corollary 1.2. Then Theorem 2.8 says that, combining the fact that \( C \) is nearly free, hence not free, \( E \simeq T_{\mathbb{P}^2}(k) \) for some \( k \in \mathbb{Z} \). Then its splitting type is \((c - 1, c)\), see §2.2, [15], thus \( d_1 + 2 = d_2 \). Then the \( S \)-module \( AR(f) \) is generated by one degree \( d_1 \)-element and two degree \((d_1 + 2)\)-elements. However, by the Euler sequence, the \( S \)-module of global sections of the twisted tangent bundle \( T_{\mathbb{P}^2}(k) \) is generated by three same degree elements, a contradiction. \( \square \)

**4. Proof of Theorem 1.4**

First recall the definition of the local Tjurina number \( \tau(C, p) \), where \( p \in C \). Choose a local system of coordinates \((u, v)\) centered at \( p \), and assume that the analytic germ \((C, p)\) is given by a local equation \( g(u, v) = 0 \). Then one defines
\[
\tau(C, p) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{P}^2, p}}{(g, g_u, g_v)},
\]
where \( g_u, g_v \) are the partial derivatives of \( g \) with respect to \( u \) and \( v \) respectively, and \((g, g_u, g_v)\) is the ideal spanned by these 3 germs in the local ring \( \mathcal{O}_{\mathbb{P}^2, p} \) of analytic function germs at \( p \). Next recall the following basic result relating \( \tau(C) \) and \( r = \text{mdr}(f) \) in [12] Theorem 3.2.

**Proposition 4.1.** For any reduced plane curve of degree \( d \), one has
\[
\tau(d, r)_{\min} \leq \tau(C) \leq \tau(d, r)_{\max},
\]
where \( \tau(d, r)_{\min} = (d - 1)(d - r - 1) \), and \( \tau(d, r)_{\max} = (d - 1)(d - r - 1) + r^2 \) for \( r \leq (d - 1)/2 \) and

\[
\tau(d, r)_{\max} = (d - 1)(d - r - 1) + r^2 - \left( \frac{2r + 2 - d}{2} \right),
\]

for \( (d - 1)/2 < r \leq d - 1 \).

As already noted in [6, Proof of Thm. 1.1], the function

\[ r \mapsto \tau(d, r)_{\max} \]

is strictly decreasing on the interval \([0, d - 1]\).

Now we start the proof of Theorem 1.4. As we have seen in the previous section, one has \( c_1 = -(d - 1) \) and \( c_2 = (d - 1)^2 - \tau(C) \). Using this the equivalence of the claims (1) and (2) is clear, as both are equivalent to \( \Delta = c_1^2 - 4c_2 \geq 0 \). Using the decreasing function \( \tau(d, r)_{\max} \) and the remark that

\[ \tau(d, \frac{d - 1}{2})_{\max} = \frac{3}{4}(d - 1)^2, \]

we see that (1) implies the inequality

\[ r \leq (d - 1)/2. \]  

Suppose now that (1) holds and replace \( c_2 \) by

\[ d_1^{L_0}d_2^{L_0} + \nu(C) = a(d - 1 - a) + \nu(C), \]

where we set \( a = d_1^{L_0} \) for simplicity, and use Theorem 1.1. The condition \( \Delta = c_1^2 - 4c_2 \geq 0 \) now becomes

\[ 4a^2 - 4a(d - 1) + (d - 1)^2 - 4\nu(C) \geq 0. \]  

The associated equation

\[ 4a^2 - 4a(d - 1) + (d - 1)^2 - 4\nu(C) = 0, \]

has roots

\[ a_1 = \frac{d - 1}{2} - \sqrt{\nu(C)} \text{ and } a_2 = \frac{d - 1}{2} + \sqrt{\nu(C)}. \]

In view of the inequality (4.1), it follows that the inequality (4.2) implies \( a \leq a_1 \). Hence we have shown that (1) implies (3). Conversely, if (3) holds, it follows that the inequality (4.2) holds, and hence \( \Delta = c_1^2 - 4c_2 \geq 0 \). Hence (3) implies (1) as well.

Now we show that (3) implies the equality \( mdr(f) = d_1^{L_0} \). If the curve \( C \) is free, the claim is obvious. Otherwise, \( \nu(C) \geq 1 \), and hence the inequality (3) implies \( d_1^{L_0} \leq (d - 3)/2 \). We conclude using Proposition 3.2 (2).
Finally, if \( r = mdr(f) \leq (d - 1)/4 \), it follows that

\[
\tau(C) \geq \tau(d, r)_{\text{min}} \geq \tau(d, \frac{d - 1}{4})_{\text{min}} = \frac{3}{4}(d - 1)^2.
\]

In other words, the inequality \( r = mdr(f) \leq (d - 1)/4 \) implies (2). To show that \( mdr(f) \leq a_1 \) implies (3), we use Proposition \ref{prop:mdr} (1). This ends the proof of Theorem \ref{thm:main}.

Combining Corollary \ref{cor:chern} with Theorem \ref{thm:main}, we have the following.

**Corollary 4.2.** If the Chern polynomial \( c_{EC}(t) = 1 + c_1 t + c_2 t^2 \in \mathbb{Z}[t] \) of \( EC \) has real roots, then

\[
s_C - 1 \leq \nu(C) \leq \left( \frac{d - 1}{2} - mdr(f) \right)^2,
\]

where \( s_C = |\text{im}(a_{EC})| \).

**Example 4.3.** Let \( p_1, \ldots, p_6 \in \mathbb{P}^2 \) be 6 points such that no three are colinear. Let \( \mathcal{A} = \{H_1, \ldots, H_9\} \) be the edges of the corresponding hexagon and three diagonals (more precisely, the lines \( p_1p_2, p_2p_3, p_3p_4, p_4p_5, p_5p_6, p_6p_1, p_1p_4, p_2p_5 \) and \( p_3p_6 \)), such that each line \( H_j \) contains exactly 2 triple points and 4 nodes.

Denote by \( \mathcal{A} : f = 0 \) (resp. by \( \mathcal{A}' : f' = 0 \)) the corresponding line arrangement when the 6 vertices of the hexagon are (resp. are not) on a conic. Then it is known that this pair of Ziegler’s line arrangements has the following properties, see for instance \cite[Remark 8.5]{Ziegler}.

1. Both arrangements have the same intersection lattice. In particular, they have 18 double points and 6 triple points. It follows that \( \tau(\mathcal{A}) = \tau(\mathcal{A}') = 42 < (3/4) \cdot 64 = 48 \). Hence the Chern polynomials

\[
c_{E\mathcal{A}}(t) = c_{E\mathcal{A}'}(t) = 1 - 8t + 22t^2
\]

do not have real roots in view of Theorem \ref{thm:main}.

2. \( r = mdr(f) = 5 \) and \( r' = mdr(f') = 6 \).

3. Since \( d_1^{L_0} \leq (d - 1)/2 = 4 \), it follows that we are in the case (4) of Proposition \ref{prop:mdr}. Hence \( d_1^{L_0} = d_2^{L_0} = 4 \) for both arrangements \( \mathcal{A} \) and \( \mathcal{A}' \). In particular \( d_1^{L_0} \neq mdr(f) \) in these two cases.

Note that one can consider a family \( \mathcal{A}_t : f_t = 0 \) of arrangements as above, where the 6 vertices of the hexagon are not on a conic, for \( t \neq 0 \), degenerating at an arrangement \( \mathcal{A}_0 = \mathcal{A} \), where the 6 vertices of the hexagon are on a conic. Note that under this degeneration the invariant \( mdr(f_t) \) drops by one, while the corresponding splitting invariant \( d_1^{L_0} \) stay constant.
5. Application to line arrangements

In this section let us apply the main results in this paper to the case when $C$ is a finite set of lines in $\mathbb{P}^2$, i.e., an arrangement of lines. For that purpose, let us show the following generalization of Ziegler’s result in [26]. Only in this result $\ell$ is arbitrary.

**Theorem 5.1.** Let $\mathbb{K}$ be an arbitrary field and $\mathcal{A}$ be a central arrangement in $V = \mathbb{K}^\ell$ and $S := \mathbb{K}[x_1, \ldots, x_\ell]$. Let $\pi : AR_H(\mathcal{A}) \to D(\mathcal{A}^H, m^H)$ be the Ziegler restriction map. Let $\theta_1, \ldots, \theta_s \in AR_H(\mathcal{A})$ satisfy that $\pi(\theta_1), \ldots, \pi(\theta_s)$ generate $\text{Im}(\pi)$ as an $(S/\alpha_H)$-module. Then $\theta_1, \ldots, \theta_s$ generate $AR_H(\mathcal{A})$ as an $S$-module.

**Proof.** Let the images of $\theta_1, \ldots, \theta_s$ by $\pi$ generate $M := \text{Im}(\pi)$, $\alpha_H = x_\ell$ and let $0 \neq \theta \in AR_H(\mathcal{A})$ be a minimal degree element. Then we may show that $\pi(\theta) \neq 0$. Assume not. Then $\theta = x_\ell\theta'$ for some $\theta' \in \text{Der} S$. Since $\varphi(\alpha_H) = 0$ for $\varphi \in AR_H(\mathcal{A})$, it follows that $0 \neq \theta' \in AR_H(\mathcal{A})$ with $\deg \theta' < \deg \theta$, a contradiction. Hence $0 \neq \pi(\theta) = \sum_{i=1}^{s} a_i\pi(\theta_i)$ for $a_i \in \mathbb{K}$ such that $a_i = 0$ if $\deg \theta_i > \deg \theta$. Then $\theta - \sum_{i=1}^{s} a_i\theta_i = x_\ell\theta'$. By the same reason, $\theta' \in AR_H(\mathcal{A})$ whose degree is strictly lower than that of $\theta$, a contradiction. Hence $\theta' = 0$, hence the lowest degree derivations in $AR_H(\mathcal{A})$ can be expressed by $\theta_1, \ldots, \theta_s$.

Now assume that the statement holds true for homogeneous derivations in $AR_H(\mathcal{A})$ whose degree is less than $d$. Since $AR_H(\mathcal{A})$ is graded, it suffices to show the statement for homogeneous parts. Let $\theta \in AR_H(\mathcal{A})_k$. If $x_\ell \mid \theta$, then apply the induction hypothesis to $\theta/x_\ell \in AR_H(\mathcal{A})_{k-1}$, which completes the proof. Assume not. Then the same argument as above implies that $\theta - \sum_{i=1}^{s} a_i\theta_i = x_\ell\theta'$ for some $\theta' \in AR_H(\mathcal{A})_{k-1}$. Again the induction hypothesis implies that $\theta' = \sum_{i=1}^{s} g_i\theta_i$, which completes the proof. \qed

**Remark 5.2.** When $\pi$ is surjective, $M = D(\mathcal{A}^H, m^H)$ in terms of Theorem 5.1. Hence the classical result by Ziegler in [26] asserting that $\pi$ is surjective if $\mathcal{A}$ is free can be regarded as a special case of Theorem 5.1.

In general, it is very difficult to investigate the splitting type of vector bundles onto projective lines. Contrary to it, for line arrangements, we can use the technique of multiarrangements to do it. What makes this analysis work well is the following exact sequence which hold true when $\ell = 3$:

$$0 \to \widetilde{AR_H(\mathcal{A})} \to \widetilde{AR_H(\mathcal{A})} \to D(\mathcal{A}^H, m^H) \to 0.$$  

This follows by Proposition 2.6 and the fact that $\ell = 3$. This implies the isomorphism:

$$D(\mathcal{A}^H, m^H) \simeq AR_H(\mathcal{A})|_{H}.$$  

Hence to know $a_E(H)$ for $H \in \mathcal{A}$ is the same as to know $\exp(\mathcal{A}^H, m^H)$. We use this isomorphism frequently in the rest of this article.
Then

\textbf{Theorem 5.7.} Assume that the line arrangement \(\mathcal{A}\) is nearly free with \(nexp(\mathcal{A}) = (d_1, d_2)\). Then \(\exp(\mathcal{A}^H, m^H) = (d_1 - 1, d_2)\) or \((d_1, d_2 - 1)\).

\textit{Proof.} Apply Corollary \[5.4\] and the exact sequence above. \hfill \square

\textbf{Example 5.4.} Let \(\mathcal{A} = \{xyz(y-z)(x+2y+3z) = 0\}\). Then \(\chi(\mathcal{A}; t) = t^2 - 4t + 5 = (t-2)^2 + 1\). It is easy to check that (e.g., use Theorem \[5.3\] below) \(\mathcal{A}\) is nearly free with \(nexp(\mathcal{A}) = (2, 3)\). Also, \(\exp(\mathcal{A}^H, m^H) = (2, 2)\) and \(\exp(\mathcal{A}^L, m^L) = (1, 3)\) for \(H : z = 0\) and \(L : x + 2y + 3z = 0\). Hence both case in Theorem \[5.3\] can occur in general.

Similarly to Theorem \[2.15\] we may give a sufficient condition for a line arrangement to be nearly free following Theorem \[1.2\]

\textbf{Theorem 5.5} (Near freeness condition). Let \(\chi(\mathcal{A}; t) = (t - d_1)(t - d_2 + 1) + 1\) with \(d_1 \leq d_2\). Then \(\mathcal{A}\) is nearly free if there is \(H \in \mathcal{A}\) such that \(\chi(\mathcal{A}; 0) - b_2(\mathcal{A}^H, m^H) = 1\). Here, \(b_2(\mathcal{A}^H, m^H)\) is the product of exponents of \((\mathcal{A}^H, m^H)\), see \[4\]. In particular, \(nexp(\mathcal{A}) = (d_1, d_2)\) or \((d_1+1, d_2-1)\) if \(d_1 = d_2\) or \(d_1+2 = d_2\), and \(nexp(\mathcal{A}) = (d_1, d_2)\) otherwise.

\textit{Proof.} Apply Theorem \[1.2\] Proposition \[3.2\] and the isomorphism above. \hfill \square

\textbf{Remark 5.6.} We do not know whether the result like Corollary \[3.5\] holds true for \(H \in \mathcal{A}\). In other words, we do not know whether for a nearly free arrangements, there is \(H \in \mathcal{A}\) such that \(\exp(\mathcal{A}^H, m^H) = (d_1, d_2 - 1)\). We do not have any counter example to this statement.

Next let us study the addition-deletion type results for free and nearly free arrangements.

\textbf{Theorem 5.7.} \(1\) Let \(\mathcal{A}\) be free with \(\exp(\mathcal{A}) = (d_1, d_2)\) with \(d_1 \leq d_2\). Let \(L \not\in \mathcal{A}\) be a line. Then \(\mathcal{B} := \mathcal{A} \cup \{L\}\) is nearly free if \(|\mathcal{B}^L| = d_2 + 2\).

\(2\) Let \(\mathcal{A}\) be free with \(\exp(\mathcal{A}) = (d_1, d_2)\) with \(d_1 \leq d_2\). Let \(H \in \mathcal{A}\) be a line. Then \(\mathcal{B} := \mathcal{A} \setminus \{H\}\) is nearly free if \(|\mathcal{A}^H| = d_1\).

\textit{Proof.} \(1\) Assume that \(d_1 = d_2 =: d\). Then \(\exp(\mathcal{B}^H, m^H) = (d, d + 1)\) for any \(H \in \mathcal{B}\) by the argument in \[2\]. Also, \(\chi(\mathcal{B}; 0) = d^2 + d + 1\). Hence their difference is \(1\), so Theorem \[5.5\] completes the proof.

So we may assume that \(d_1 < d_2\). By Theorem \[2.14\] and Lemma \[2.16\] \(\exp(\mathcal{B}^H, m^H)\) are either \((d_1 + 1, d_2)\) or \((d_1, d_2 + 1)\) for \(H \in \mathcal{A}\).

Case 1. Assume that there is \(H \in \mathcal{B}\) such that \(\exp(\mathcal{B}^H, m^H) = (d_1 + 1, d_2)\). Then for \(\pi : AR_H(\mathcal{B}) \to D(\mathcal{B}^H, m^H), \dim coker \pi = d_1d_2 + d_2 + 1 - (d_1 + 1)d_2 = 1\). Hence Theorem \[5.5\] completes the proof.
Case 2. Assume that $\exp(B^H, m^H) = (d_1, d_2 + 1)$ for all $H \in \mathcal{B}$. We may assume that $H \neq L$. Since $b_2(B) = d_1d_2 + d_2 + 1$, it holds that $\dim \ker \pi = d_2 - d_1 + 1$, hence $\mathcal{B}$ is not free by Theorem 2.15. Let $\theta_1, \theta_2$ be a basis for $AR_H(A)$ with $\deg \theta_i = d_i$. Then clearly $\alpha_L \theta_i \in AR_H(B)_{d_i+1}$, and $\alpha_L \theta_1, \alpha_L \theta_2$ are $S$-independent. Let $\eta_1, \eta_2$ be a basis for $D(B^H, m^H)$ with $\deg \eta_1 = d_1, \deg \eta_2 = d_2 + 1$. If $AR_H(B)_{d_i} \neq 0$, then clearly $AR_H(B)$ is free, which is a contradiction. Hence $\eta_1 \notin \Im \pi$. So we may assume that, by putting $\pi = \alpha_L \eta_1 = x \eta_1$. Since $\theta_1$ and $\theta_2$ form a basis for $AR_H(A)$, $\pi(\theta_1)$ and $\pi(\theta_2)$ are $(S/\alpha_L)$-independent. So we may assume that $\pi(\alpha_L \theta_2) = \eta_2$. Now taking $\dim \ker \pi$ into account, there has to be $\theta_3 \in AR_H(B)_{d_2+1}$ such that $\pi(\theta_3) = y^{d_2-d_1+1} \eta_1$. Then by Theorem 5.11 $\alpha_L \theta_1, \alpha_L \theta_2, \theta_3$ satisfy the condition for $\mathcal{B}$ to be nearly free.

(2) Assume that $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, d_2)$. By the deletion-restriction, $\chi(B; t) = (t - d_1)(t - d_2 + 1) + 1$. Let $L \in \mathcal{B}$. Since $\exp(\mathcal{A}^L, m^L) = (d_1, d_2)$ by Theorem 2.14 $\exp(B^L, m^L)$ is either $(d_1 - 1, d_2)$ or $(d_1, d_2 - 1)$. If the latter, then Theorem 5.5 completes the proof. Assume the former for all $L \in \mathcal{B}$. Then

$$b_2(B) - (d_1 - 1)d_2 = d_2 - d_1 + 1.$$ 

By Lemma 2.16, there are a basis $\theta_1, \theta_2$ for $D(B^L, m^L)$ such that $\deg \theta_1 = d_1 - 1$, $\deg \theta_2 = d_2$ and $x \theta_1, \theta_2$ form a basis for $D(A^L, m^L)$, where $\alpha_H := x$. Since $\pi : AR_L(A) \rightarrow D(A^L, m^L)$ is surjective by Theorem 2.14 there are derivations $\varphi_1, \varphi_2 \in AR_L(A)$ such that $\pi(\varphi_1) = x \theta_1$ and $\pi(\varphi_2) = \theta_2$. Since $AR_L(A) \subset AR_L(B)$ and $\dim \ker(\pi_B : AR_L(B) \rightarrow D(B^L, m^L)) = d_2 - d_1 + 1$, there is a derivation $\varphi_3 \in AR_L(B)_{d_2}$ such that $\pi(\varphi_3) = y^{d_2-d_1+1} \theta_1$. Hence

$$\Im \pi_B = \langle x \theta_1, y^{d_2-d_1+1} \theta_1, \theta_2 \rangle_{S/\alpha_L}.$$ 

Since $\deg \varphi_3 = d_2$ and clearly there is a relation among $\varphi_1, \varphi_2, \varphi_3$ at degree $d_2 + 1$, Theorem 5.1 implies that $\mathcal{B}$ is nearly free. □

The following is a nearly free version of the results in [2].

**Theorem 5.8.** Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{P}^2$ with $\chi(\mathcal{A}; t) = t^2 - b_1 t + b_2$, where $b_1 = |\mathcal{A}|-1$. Let $\chi(A; t) = (t-a)(t-b)+1$ with real number $a \leq b$, $a+b = b_1$. Then $\mathcal{A}$ is nearly free if there is $H \in \mathcal{A}$ such that

1. $|\mathcal{A}^H| = b + 1$, or
2. $|\mathcal{A}^H| = a + 1$ and $b \neq a + 2$.

**Proof.** Immediate from Theorem 5.5 and the argument in [2]. □

Now let us apply the results in this paper to show near freeness of some line arrangements.

**Example 5.9.** (1) Let $\mathcal{A}$ be defined by

$$xz(x^2 - y^2)(x^2 - 2y^2)(y - z) = 0.$$
Then it is easy to check (see [16] for example) to show that \( \chi(\mathcal{A}; t) = (t - 3)^2 + (t - 2)(t - 4) + 1 \), but \( \mathcal{A} \) is not free. We can check the non-freeness by several way, here we use Theorem 5.5. It is easy to check that \( \exp(\mathcal{A}^H, m^H) = (2, 4) \) for any \( H \) going through the origin. Hence Theorem 5.5 implies that \( \mathcal{A} \) is nearly free with \( \text{nexp}(\mathcal{A}) = (2, 5) \).

(2) Let \( \mathcal{B} \) be defined by

\[
xyz(x^2 - z^2)(y^2 - z^2)(x - y + z)(x - y - z) = 0.
\]

Then \( \chi(\mathcal{B}; t) = (t - 4)^2 + 1 \). Also, \( |\mathcal{B}^H| = 5 = 4 + 1 \) for \( \ker \alpha_H = x - y \pm z \). Hence Theorem 5.5 implies that \( \mathcal{B} \) is nearly free with \( \text{nexp}(\mathcal{B}) = (4, 5) \).

**Theorem 5.10** (Addition theorem for free and nearly free arrangements). Let \( \mathcal{A} \) be an arrangement in \( \mathbb{P}^2 \), \( H \in \mathcal{A} \) and let \( \mathcal{B} := \mathcal{A} \setminus \{H\} \). Also, let \( d_1 \leq d_2 \) be two non-negative integers. Then the two of the following three implies the third:

1. \( \mathcal{A} \) is nearly free with \( \text{nexp}(\mathcal{A}) = (d_1 + 1, d_2 + 1) \).
2. \( \mathcal{B} \) is free with \( \exp(\mathcal{B}) = (d_1, d_2) \).
3. \( |\mathcal{A}^H| = d_2 + 2 \).

**Proof.** (1) and (2) implies (3) by two factorizations Theorems 2.11 and 2.12 and the deletion-restriction formula. If we assume (2) and (3), then Theorem 5.7 (1) implies (1).

Assume that (1) and (3). Let \( L \in \mathcal{B} \). Then Theorem 5.3 shows that \( \exp(\mathcal{A}^L, m^L) =: \exp(\mathcal{A}'', m) = (d_1, d_2 + 1) \) or \( (d_1 + 1, d_2) \). Hence \( \exp(\mathcal{B}^L, m^L) =: \exp(\mathcal{B}'', k) \) could be one of \( (d_1, d_2) \), \( (d_1 + 1, d_2 - 1) \) or \( (d_1 - 1, d_2 + 1) \). Note that

\[
\chi(\mathcal{B}; 0) = b_2(\mathcal{B}) = d_1 d_2 \geq b_2(\mathcal{B}'', k)
\]

by Theorem 2.15 and the deletion-restriction. Hence the proof is completed if \( \exp(\mathcal{B}'', k) = (d_1, d_2) \). Assume that \( \exp(\mathcal{B}'', k) = (d_1 + 1, d_2 - 1) \). Then by Theorem 2.15

\[
b_2(\mathcal{B}) - b_2(\mathcal{B}'', k) = -d_2 + d_1 + 1 \geq 0.
\]

Hence \( d_2 = d_1 + 1 \) or \( d_2 = d_1 \). For the former, Theorem 2.15 confirms that \( \mathcal{B} \) is free with exponents \( (d_1, d_1 + 1) \). For the latter, \( b_2(\mathcal{B}) - b_2(\mathcal{B}'', k) = 1 \). Since \( b_2(\mathcal{B}) = d_1^2 = (d_1 + 1)(d_1 - 1) + 1 \), Theorem 5.5 shows that \( \mathcal{B} \) is nearly free with \( \text{nexp}(\mathcal{B}) = (d_1 - 1, d_1 + 2) \). Hence \( 0 \neq \alpha_H AR_L(\mathcal{B})_{d_1 - 1} \subset AR_L(\mathcal{A})_{d_1} = 0 \), a contradiction.

Now assume that \( \exp(\mathcal{B}'', k) = (d_1 - 1, d_2 + 1) \). Then \( b_2(\mathcal{B}) - b_2(\mathcal{B}'', k) = d_2 - d_1 + 1 \). This occurs only when \( \exp(\mathcal{A}^L, m^L) = (d_1, d_2 + 1) \). Let \( \alpha_H = y \). Then by Lemma 2.16 we may choose a basis \( \theta_1, \theta_2 \) for \( D(\mathcal{B}'', k) \) with \( \deg \theta_1 = d_1 - 1 \), \( \deg \theta_2 = d_2 + 1 \) such that \( y \theta_1, \theta_2 \) form a basis for \( D(\mathcal{A}^L, m^L) \). Note that \( b_2(\mathcal{A}) - b_2(\mathcal{A}'', m) = d_2 - d_1 + 1 \) too. Since \( \text{nexp}(\mathcal{A}) = (d_1 + 1, d_2 + 1) \), there is \( \varphi \in AR_L(\mathcal{A})_{d_2 + 1} \) such that \( \pi(\varphi) = \theta_2 \), and there are \( \psi_1, \psi_2 \in AR_L(\mathcal{A}) \) such that \( \pi(\psi_1) = xy \theta_1 \) and \( \pi(\psi_2) = y^{d_2 - d_1 + 2} \theta_1 \) for
the Ziegler restriction map \( \pi : AR_L(A) \to D(A^L, m^L) \). Hence \( \text{coker } \pi \) has a basis
\[
y \theta_1, y^2 \theta_1, \ldots, y^{d_2-d_1+1} \theta_1,
\]
whose dimension is surely \( d_2 - d_1 + 1 \). Now consider the basis of \( \text{coker}(\pi' : AR_L(B) \to D(B^n, k)) \). Assume that \( \text{coker } \pi' \ni y \theta_1 \), where this \( i \) is the largest one satisfying this. Then
\[
y^{i+1} \theta_1 \in \text{Im} \pi \quad \text{and} \quad y^i \theta_1 \notin \text{Im} \pi' \quad \iff \quad y^{i+2} \theta_1 \in \text{Im} \pi \quad \text{and} \quad y^{i+1} \theta_1 \notin \text{Im} \pi
\]
\[
\iff \quad 0 \neq y^{i+1} \theta_1 \in \text{coker} \pi'.
\]
Hence \( d_2 - d_1 = i \), and \( \text{coker } \pi' \), whose dimension is \( d_2 - d_1 + 1 \), could contain at most
\[
\theta_1, x \theta_1, \ldots, x^{d_2-d_1+1} \theta_1, y \theta_1, y^2 \theta_1, \ldots, y^{d_2-d_1} \theta_1.
\]
Hence \( x \theta_1 \in D(B^n, k) \) since \( B \) is not free for \( b_2(B) = d_1 d_2 \neq (d_1 - 1)(d_2 + 1) \).

Summarizing, \( AR_L(B) \) has a generator \( \eta_1 \) of degree \( d_1 \), \( \eta_2 \) of degree \( d_2 \) and \( \xi \) of degree \( d_2 + 1 \) such that \( \pi'(\eta_1) = x \theta_1 \), \( \pi'(\eta_2) = y^{d_2-d_1+1} \theta_1 \), \( \pi'(\xi) = \theta_2 \). We show that this cannot occur. By the choice of \( \eta_1, \eta_2 \), there is \( \eta \in AR_L(B) \) such that
\[
y^{d_2-d_1+1} \eta_1 - x \eta_2 = z \eta,
\]
where we set \( z = \alpha L \). Since \( AR_L(B) \leq d_2 \) is generated by \( \{ \eta_1, \eta_2 \} \), there are \( a \in \mathbb{K} \) and \( g \in S_{d_2-d_1} \) such that
\[
\eta = g \eta_1 + a \eta_2 \iff (y^{d_2-d_1+1} - zg) \eta_1 = (x + az) \eta_2.
\]
Since \( y^{d_2-d_1+1} - zg \) and \( x + az \) are coprime, there is \( \eta_0 \in \text{Der}(S)_{d_1-1} \) such that
\[
\eta_0 = \frac{\eta_1}{x + az} = \frac{\eta_2}{y^{d_2-d_1+1} - zg}.
\]
By the former expression of \( \eta_0 \), to show \( \eta_0 \in AR_L(B)_{d_1-1} \), it suffices to show that \( \eta_0(x + az) \in S \cdot (x + az) \) if \( \ker(x + az) \) is \( B \). Since \( (x + az) \nmid (y^{d_2-d_1+1} - zg) \) and \( \eta_2(x + az) \in S \cdot (x + az) \), it follows that
\[
\eta_0(x + az) = \frac{\eta_2(x + az)}{y^{d_2-d_1+1} - zg} \in S(x + az),
\]
which implies \( 0 \neq \eta_0 \in AR_L(B)_{d_1-1} = (0) \), a contradiction. \( \square \)

**Theorem 5.11** (Deletion theorem for free and nearly free arrangements). Let \( A \) be an arrangement in \( \mathbb{P}^2 \), \( H \in A \) and let \( B := A \setminus \{H\} \). Also, let \( d_1 \leq d_2 \) be two non-negative integers. Then the two of the following three implies the third:

1. \( A \) is free with \( \exp(A) = (d_1, d_2) \).
2. \( B \) is nearly free with \( \text{nexp}(B) = (d_1, d_2) \).
3. \( |A^H| = d_1 \).
Proof. As in the proof of Theorem \[5.10\], it suffices to show that (2) and (3) imply (1). By Theorem \[2.11\] and the deletion-restriction theorem, \( \chi(\mathcal{A}; t) = (t - d_1)(t - d_2) \) and \( \chi(\mathcal{B}; t) = (t - d_1)(t - d_2 + 1) + 1 \). Since \((a, b, c) = \exp(\mathcal{B}_L, m_L)\) is either \((d_1 - 1, d_2)\) or \((d_1, d_2 - 1)\) by Theorem \[5.3\] for \( L \in \mathcal{B} \), \( \exp(\mathcal{A}_L, m_L) \) is either \((d_1, d_2)\), \((d_1 - 1, d_2 + 1)\) or \((d_1 + 1, d_2 - 1)\) by Lemma \[2.16\]. If it is \((d_1, d_2)\), then Theorem \[2.15\] completes the proof. Assume that \( \exp(\mathcal{A}_L, m_L) = (d_1 + 1, d_2 - 1) \), then
\[
\dim \ker(\pi_L : AR_L(\mathcal{A}) \to D(\mathcal{A}_L, m_L)) = -d_2 + d_1 + 1 \geq 0.
\]
Since \( d_1 \leq d_2 \), we have \( d_1 = d_2 \) or \( d_1 + 1 = d_2 \). Assume that \( d_1 = d_2 \). Then Theorem \[5.5\] shows that \( \mathcal{A} \) is nearly free with \( \exp(\mathcal{A}) = (d_1 - 1, d_1 + 2) \). Hence \((0) \neq AR_L(\mathcal{A})_{d_1-1} \subset AR_L(\mathcal{B})_{d_1-1} = (0) \), a contradiction. If \( d_2 = d_1 + 1 \), then Theorem \[2.15\] says that \( \mathcal{A} \) is free.

Hence we may assume that \( \exp(\mathcal{A}_L, m_L) = (d_1 - 1, d_2 + 1) \) for all \( L \in \mathcal{B} \). If \( d_1 = d_2 \), then the above completes the proof. So we may assume that \( d_1 < d_2 \). This occurs only when \( \exp(\mathcal{B}_L, m_L) = (d_1 - 1, d_2) \) for all \( L \in \mathcal{B} \). Hence putting \( \alpha_H := x \), there is a basis \( \theta_1, \theta_2 \) of degree \( d_1 - 1, d_2 \) for \( D(\mathcal{B}_L, m_L) \) such that \( \theta_1, x\theta_2 \) form a basis for \( D(\mathcal{A}_L, m_L) \). Note that, for \( \pi_B : AR_L(\mathcal{B}) \to D(\mathcal{B}_L, m_L) \),
\[
\dim \ker(\pi_B \pi) = d_2 - d_1 + 1.
\]
This also holds true for \( \pi_A : AR_L(\mathcal{A}) \to D(\mathcal{A}_L, m_L) \) by the assumption of the exponents. Let \( \varphi_1, \varphi_2, \varphi_3 \) be a generator for the nearly free module \( AR_L(\mathcal{B}) \) such that \( \deg \varphi_1 = d_1, \deg \varphi_2 = \deg \varphi_3 = d_2 \) and \( \pi_B(\varphi_1) = \alpha \theta_1, \pi_B(\varphi_2) = \beta d_2 - d_1 + 1 \theta_1 \) and \( \pi_B(\varphi_3) = \theta_2 \) for \( \mathbb{C} \)-independent linear forms \( \alpha, \beta, z \).

Since \( x \varphi_3 \in AR_L(\mathcal{A}) \) and \( \pi_A(x \varphi_3) = x \theta_2 \), \( \dim \ker(\pi_A) \) comes from \( M_A := \text{Im}\pi_A/\theta_1 \subset M_B \), and they have the same codimension. Hence \( \theta_1, \theta_2 \in AR_L(\mathcal{A}) \).

Hence \( \varphi_1, \varphi_2, x \varphi_3 \) generates \( AR_L(\mathcal{A}) \) by Theorem \[5.1\]. Since \( \pi_A(\beta d_2 - d_1 + 1 \varphi_1 - \alpha \varphi_3) = 0 \), the same argument as the above shows that there is \( \varphi \in AR_L(\mathcal{A})_{<d_1} \subset AR_L(\mathcal{B})_{<d_1} \), a contradiction. \( \square \)

6. Proof of Theorem \[1.3\]

Let us prove Theorem \[1.3\]. If \(|\mathcal{A}_H| \leq 2\), there is nothing to show. Assume that \(|\mathcal{A}_H| = 3\). Then by \[2.11\] with the assumption on \( H \), the exponents of the Ziegler restriction onto \( H \) is combinatorial, and it is a generic splitting type. Hence Corollary \[1.2\] completes the proof.

Next assume that \(|\mathcal{A}_H| = 4\). Then by Theorem 1.6 in \[1\] with the assumption on \( H \), \( \exp(\mathcal{A}_H, m_H) = (e_1, e_2) \) is either \((d, d)\), \((d, d + 1)\) or \((d, d + 2)\). For the former two cases, they are generic splitting types. Hence Corollary \[1.2\] and Theorem \[2.15\] says that, if \( \chi(\mathcal{A}; t) \) has integer roots, then they are either free or nearly free if the roots satisfy conditions in Corollary \[1.2\] or Theorem \[2.15\] and not either otherwise. So the rest case is when \(|\mathcal{A}| = 1 + 2d \), \((e_1, e_2) = (d - 1, d + 1)\) and \( b_2(\mathcal{A}) = d^2 \). In this case, Corollary \[1.2\] shows that \( \mathcal{A} \) is nearly free.
Remark 6.1. If $H$ in Theorem 1.5 contains a high multiplicity point, then freeness depends only on $L(A)$, see Proposition 7.4 in [22] or Corollary 1.4 in [3] for example. Also in this case, the following proposition shows the same holds true for near freeness.

Proposition 6.2. Let $A$ be a line arrangement such that $(A^H, m^H)$ is not balanced for some $H \in A$, i.e., there is $X \in A^H$ such that $2m^H(X) \geq |m^H| := \sum_{Y \in A^H} m^H(Y) = |A| - 1$. Then the near freeness of $A$ depends only on $L(A)$.

Proof. Since $(A^H, m^H)$ is not balanced, $\exp(A^H, m^H) = (e_1, e_2) \leq 1$ with $m^H(X) = e_2$ (See [25], p 11 for instance). If $b_2(A) = e_1e_2$, then $A$ is free by Theorem 2.15 hence not nearly free. So assume not, then $A$ is not free. If $b_2(A) - e_1e_2 = 1$, then $A$ is nearly free by Theorem 5.5. By Theorem 2.12 for $A$ to be nearly free, there exist $d_1 \leq d_2$ such that

$$\chi(A; t) = (t - d_1)(t - d_2 + 1).$$

Also, by Corollary 3.5 if $(e_1, e_2)$ is neither $(d_1 - 1, d_2)$ nor $(d_1, d_2 - 1)$, then $A$ is not nearly free. If $(e_1, e_2) = (d_1, d_2 - 1)$, then $A$ is nearly free as shown above. Now assume that $(e_1, e_2) = (d_1 - 1, d_2) = (|A| - 1 - m^H(X), m^H(X))$. Also, we may assume that $d_1 < d_2$. Let $H = \ker z$ and $X = H \cap \{y = 0\}$. Then $\theta_1 := (\prod_{L \in A} a_L)\partial_x \in AR(A)_{d_1 - 1}$. Hence by the Ziegler restriction map $\pi_H : AR_H(A) \to D(A^H, m^H)$, $\theta_1$ goes to $\varphi_1$, where $\varphi_1, \varphi_2$ form a basis for $D(A^H, m^H)$ of degree $e_1, e_2$ respectively. Since $d_1 < d_2$, $(e_1, e_2) = a_{E_4}(H) = (d_1 - 1, d_2)$ is a generic splitting type of $E_A$, which does not coincide with $(d_1, d_2 - 1)$. Hence Corollary 1.2 shows that $A$ is not nearly free. □

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