ON THE 2-ADIC LOGARITHM OF UNITS OF CERTAIN TOTALLY IMAGINARY QUARTIC FIELDS

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Abstract. In this paper, we prove a result on the 2-adic logarithm of the fundamental unit of the field \( \mathbb{Q}(\sqrt{-q}) \), where \( q \equiv 3 \mod 4 \) is a prime. When \( q \equiv 15 \mod 16 \), this result confirms a speculation of Coates-Li and has consequences for certain Iwasawa modules arising in their work.

1. Introduction

Let \( q \) be any prime \( \equiv 3 \mod 4 \), and define
\[ K = \mathbb{Q}(\sqrt{-q}), \quad F = K(\sqrt{-q}). \]
Then there is a unique prime \( \mathfrak{P} \) of \( F \) lying above 2 which is ramified in the extension \( F/\mathbb{Q} \) (see Lemma 3 below), and we write \( \text{ord}_\mathfrak{P} \) for the usual order valuation at \( \mathfrak{P} \). Moreover, \( K \) has odd class number, and it is not difficult to show that \( F \) also has odd class number (see Lemma 3 below). The unit group of \( F \) has rank 1, and we write \( \eta \) for a fundamental unit of \( F \). We have \( \eta \equiv 1 \mod \mathfrak{P} \) when \( q > 3 \), so that the usual logarithmic series \( \log_{\mathfrak{P}}(\eta) \) will converge in the completion \( F_\mathfrak{P} \) of \( F \) at \( \mathfrak{P} \) (see Lemma 4 below, where we also point out how to deal with the slightly exceptional case of \( q = 3 \)). We shall use elementary arguments to prove the following result.

Theorem 1. Let \( q \) be any prime \( \equiv 3 \mod 4 \). Let \( \eta \) be a fundamental unit of \( F \), and let \( \overline{\mathfrak{P}} \) be the unique ramified prime of \( F \) above 2. Then (1) If \( q \equiv 3 \mod 8 \), we have \( \text{ord}_{\overline{\mathfrak{P}}}(\log_{\overline{\mathfrak{P}}}(\eta)) = 0 \); (2) If \( q \equiv 7 \mod 16 \), we have \( \text{ord}_{\overline{\mathfrak{P}}}(\log_{\overline{\mathfrak{P}}}(\eta)) = 2 \); and (3) If \( q \equiv 15 \mod 16 \), we have \( \text{ord}_{\overline{\mathfrak{P}}}(\log_{\overline{\mathfrak{P}}}(\eta)) \geq 4 \).

We first remark that assertions (1) and (2) can be viewed as an exact \( \mathfrak{P} \)-adic form of the Brauer-Siegel theorem as \( q \) varies. Secondly, our motivation for proving the above theorem came from a recent paper of J. Coates and Y. Li [1], which uses 2-adic arguments from Iwasawa theory to prove various non-vanishing theorems for the values at \( s = 1 \) of the complex \( L \)-series of certain elliptic curves with complex multiplication. In fact, the results in [1] are concerned with the field \( F^* = \mathbb{Q}(\sqrt{-\sqrt{-q}}) \), but we note that the fields \( F \) and \( F^* \) are isomorphic extensions of \( \mathbb{Q} \), and so Theorem 1 remains valid with \( F^* \) replacing \( F \). Assume first that \( q \equiv 7 \mod 8 \), so that 2 splits in \( K \), and let \( p \) be the unique prime of \( K \) lying below \( \mathfrak{P} \). By class field theory, there is a unique extension \( K_\infty/K \) with Galois group \( \text{Gal}(K_\infty/K) \cong \mathbb{Z}_2 \), which is unramified outside the prime \( p \). Define \( F_{\infty}^* = F^*K_\infty \), and let \( \Gamma = \text{Gal}(F_{\infty}^*/F) \). Let \( M(F_{\infty}^*) \) (resp. \( M(F^*) \)) denote the maximal abelian 2-extension of \( F_{\infty}^* \) (resp. \( F^* \)) which is unramified outside the primes of \( F_{\infty}^* \) (resp. \( F^* \)) lying above \( p \). Let \( X(F_{\infty}^*) = \text{Gal}(M(F_{\infty}^*)/F_{\infty}^*) \). Now \( M(F_{\infty}^*) \) is clearly a Galois extension of \( F_{\infty}^* \), and hence, as always in Iwasawa theory [2], \( \Gamma \) will act on \( X(F_{\infty}^*) \) by lifting inner automorphisms. Writing \( X(F_{\infty}^*)_\Gamma \) for the \( \Gamma \)-coinvariants of \( X(F_{\infty}^*) \), we see immediately that \( X(F_{\infty}^*)_\Gamma = \text{Gal}(M(F^*)/F_{\infty}^*) \). Moreover we have \( X(F_{\infty}^*)_\Gamma = 0 \) if and only if \( X(F_{\infty}^*)_\Gamma = 0 \). By global class field theory, the Galois group \( \text{Gal}(M(F^*)/F_{\infty}^*) \) is a finite group, and a classical theorem of Coates and Wiles (see [1, Theorem 8.2]) shows that

\[ [M(F^*):F_{\infty}^*] = 2^{(\text{ord}_{\overline{\mathfrak{P}}}(\log_{\overline{\mathfrak{P}}}(\eta)) - 2)/2}, \]

where \( \eta \) now denotes a fundamental unit of the field \( F^* \). Now when \( q \equiv 7 \mod 16 \), Coates and Li show in [1] by a simple Iwasawa theoretic argument based on Nakayama’s lemma that \( X(F_{\infty}^*) = 0 \),
whence it follows from (11) that $\text{ord}_{\mathfrak{p}}(\log_{\mathfrak{p}}(\eta)) = 2$. Based on numerical computations carried out by Zhbihin Liang, they also conjecture in [11] that $\text{ord}_{\mathfrak{p}}(\log_{\mathfrak{p}}(\eta)) \geq 4$ when $q \equiv 15 \mod 16$, but say that they cannot prove this conjecture by the arguments of Iwasawa theory. Thus our theorem above confirms their conjecture, as well as giving a new and simple proof of their result when $q \equiv 7 \mod 16$. In fact, when combined with the arguments from Iwasawa theory given in [11], our result shows that $X(F_{\infty}^*)$ is a free finitely generated $\mathbb{Z}_2$-module of strictly positive rank when $q \equiv 15 \mod 16$. Let $B$ be the abelian variety defined over $K$, which is the restriction of scalars from the Hilbert class field of $K$ to $K$ of the elliptic curve $A$, with complex multiplication by the ring of integers of $K$, which was first defined by Gross (an equation for this elliptic curve is recalled in [11], p. 1). Then in fact, when $q \equiv 15 \mod 16$, our result shows that either $B(F_{\infty}^*)$ contains a point of infinite order, or the Tate-Shafarevich group of $B/F_{\infty}^*$ contains a copy of $\mathbb{Q}_2/\mathbb{Z}_2$. When $q \equiv 3 \mod 8$, none of the above Iwasawa theoretic arguments remain literally valid, because 2 now remains prime in $K$. Nevertheless, we cannot help speculating whether assertion (1) of Theorem [11] for $F^*$ could somehow be used to attack the non-vanishing Conjecture 1.8 of [11]. However, our theorem has the following consequence for primes $q \equiv 3 \mod 8$.

**Corollary 2.** Suppose $q \equiv 3 \mod 8$. Let $F_{\infty}$ be the compositum of all $\mathbb{Z}_2$-extensions of $F$. Let $M(F)$ denote the maximal abelian 2-extension of $F$ which is unramified outside $\mathfrak{p}$. Then $M(F) = F_{\infty}$ and $\text{Gal}(M(F)/F) \cong \mathbb{Z}_2^2$.

We end this Introduction with two unrelated remarks. Firstly, the arguments used to prove Theorem 1 break down completely for primes $q \equiv 1 \mod 4$, because then both $K$ and $F$ have even class numbers. Secondly, the elementary arguments given in the next section hinge on the following simple observations. Firstly, we use repeatedly the identity

$$\eta^2 \pm 1 = \eta(\eta \pm \eta^{-1}).$$

Secondly, since the prime $\mathfrak{p}$ has ramification index 2, we have $\text{ord}_{\mathfrak{p}}(\log_{\mathfrak{p}}(w)) = \text{ord}_{\mathfrak{p}}(w - 1)$ for any element of $w$ of $F$ with $\text{ord}_{\mathfrak{p}}(w - 1) > 2$.

2. Proofs

In this section, we present our elementary proof for Theorem 1. Next we prove Corollary 2 by using a standard result of class field theory. Finally, we give another very simple proof for Theorem [11](3) by the Coates-Wiles formula (11).

**Lemma 3.** There exists a unique ramified prime ideal $\mathfrak{p}$ of $F$ above 2 which has ramification index 2 in the extension $F/\mathbb{Q}$.

**Proof.** A number field is ramified at a rational prime if and only if its Galois closure is ramified at that prime. It follows that $F/\mathbb{Q}$ is ramified at 2 since its Galois closure $F(\sqrt{-1})$ is clearly ramified at 2. If $q \equiv 3 \mod 8$, then 2 is inert in $K$. Hence $\mathfrak{p} = 2\mathcal{O}_K$ must be ramified in $F/K$, with ramification index 2. Assume next that $q \equiv 7 \mod 8$. Then 2 splits in $K$, say $2\mathcal{O}_K = \mathfrak{p}\mathfrak{p'}$. The prime ideal $\mathfrak{p}$ induces an embedding from $K$ to $\mathbb{Q}_2$. We fix the choice of $\sqrt{-q}$ such that $\sqrt{-q} \equiv 3 \mod 8\mathbb{Z}_2$ when $q \equiv 7 \mod 16$ and that $\sqrt{-q} \equiv 7 \mod 8\mathbb{Z}_2$ when $q \equiv 15 \mod 16$. Then $\mathfrak{p}$ is ramified in $F$. Note that $\mathfrak{p}$ is inert in $F$ when $q \equiv 7 \mod 16$ and that $\mathfrak{p}$ splits in $F$ when $q \equiv 15 \mod 16$. This proves the lemma.

**Lemma 4.** (1) Assume $q > 3$. Then the norm $N(\eta)$ of $\eta$ from $F$ to $K$ is 1 and $\eta$ is congruent to 1 modulo $\mathfrak{p}$.

(2) The class number $h$ of $F$ is odd.

**Proof.** Note that $N(\eta)$ is a unit of $K$ and hence $N(\eta) = \pm 1$. Since $q \equiv 3 \mod 4$, the quadratic Hilbert symbol in the local field $\mathbb{Q}_q(\sqrt{-q})$

$$\left(\frac{-1, \sqrt{-q}}{\mathbb{Q}_q(\sqrt{-q})}\right) = \left(\frac{-1, q}{\mathbb{Q}_q}\right) = -1.$$ 

It follows that $-1 \notin N(F^*)$. In particular, $N(\eta) = 1$. 

If $q \equiv 7 \mod 8$, then $O_F / \mathfrak{P} \cong \mathbb{F}_2$ by the above lemma. Hence $\eta \equiv 1 \mod \mathfrak{P}$ clearly. Suppose next that $q \equiv 3 \mod 8$. Note that the polynomial $(x+1)^2 - \sqrt{-q}$ is Eisenstein in $K_p[x]$ where $K_p = \mathbb{Q}_2(\sqrt[3]{q})$ is the completion of $K$ at $p = 2O_K$. It follows that the ring of integers of $F$ is $O_K[\sqrt{-q}]$. Write $\eta = a + b\sqrt{-q}$ with $a, b \in O_K$. By (1), the conjugate of $\eta$ is $\eta^{-1}$ and hence $\eta + \eta^{-1} = 2a \equiv 0 \mod \mathfrak{P}$. Thus $\eta \equiv 1 \mod \mathfrak{P}$ by the structure of the finite field $O_F / \mathfrak{P} = \mathbb{F}_4$. This proves (1).

For (2), we first note that $K$ has odd class number by genus theory. The ambiguous class number formula \cite[Chapter 13, Lemma 4.1]{1} states that for a cyclic extension $F/K$ of number fields, the order of the Gal($F/K$)-invariant subgroup of the ideal class group $Cl_F$ of $F$ is given by:

$$|Cl_{F}^{Gal(F/K)}| = |Cl_K| \left| \prod_v e_v \frac{|F : K|}{|O_K : O_K \cap N(F^x)|} \right|.$$

Here $Cl_K$ is the ideal class group of $K$, the product runs over all the places of $K$ and $e_v$ are the ramification index of $v$ in $F/K$. In our case, the ramified places are $\sqrt{-q}O_K$ and $p$. Recall that $p$ is the prime of $K$ lying below $\mathfrak{P}$. By (1), we know that $-1 \notin N(F^x)$. Applying the above formula gives $2 \mid |Cl_{F}^{Gal(F/K)}|$. Hence $2 \mid h = |Cl_F|$ by Nakayama’s lemma.

We remark that for $q = 3$, multiplying $\eta$ by a third root of unity if needed, we can also assume that $\eta \equiv 1 \mod \mathfrak{P}$.

**Lemma 5.** (1) If $q \equiv 3 \mod 8$, then $\text{ord}_{\mathfrak{P}}(\eta + \eta^{-1}) = \text{ord}_{\mathfrak{P}}(\eta - \eta^{-1}) = 2$; (2) If $q \equiv 7 \mod 16$, then $\text{ord}_{\mathfrak{P}}(\eta + \eta^{-1}) = 4$; (3) If $q \equiv 15 \mod 16$, then $\text{ord}_{\mathfrak{P}}(\eta + \eta^{-1}) \geq 6$.

**Proof of Lemma** \cite{2}. The ideas of the proofs are the same for all cases. We first consider the case $q \equiv 3 \mod 8$ which is slightly easier to handle. If $q = 3$, then $\eta = \frac{\sqrt[3]{q} + 1}{2} - \sqrt{-3}$, and it is readily verified that (1) holds. Assume now that $q > 3$. We have $p = 2O_K = \mathfrak{P}^2$. Then $\mathfrak{P} = \gamma O_F$ for some $\gamma \in O_F$ since the class number $h$ of $F$ is odd. It follows that $\frac{\sqrt{2}}{\gamma}$ is a unit of $O_F$. Thus $\frac{\sqrt{2}}{\gamma} = \pm \eta^k$ for some integer $k$. We claim that $\gamma$ is odd. Indeed, if $k$ is even, we would have that $(\gamma \eta^{-k/2})^2 = \pm 2$, whence $F = K(\sqrt{\pm 2})$, which is a contradiction. This proves the claim. By replacing $\gamma$ by $\gamma^k \frac{\sqrt{2}}{\gamma}$, we may assume that $\frac{\sqrt{2}}{\gamma}$ is the fundamental unit $\eta$. In the proof of part (2) of Lemma \cite{2}, we have shown that $O_F = O_K[\sqrt{-q}]$. Thus we can write $\gamma = a + b\sqrt{-q}$ with $a, b \in O_K$, whence

$$\eta = \frac{a^2 + b^2 \sqrt{-q}}{2} + ab \sqrt{-q} \quad \text{and} \quad N(\gamma) = a^2 - b^2 \sqrt{-q} = \pm 2.$$

In fact, one can show that $N(\gamma) = -2$ by computing the Hilbert symbols of $-2$ and $\sqrt{-q}$, but we will not need this finer result. We need to calculate a mod $2 \in O_K/2O_K \cong \mathbb{F}_4$. It is easy to see that $a \neq 0 \mod 2O_K$. We claim that $a \neq 1 \mod 2O_K$. Note that $\sqrt{-q} \equiv 1 \mod 2O_K$. It follows that $a^2 \equiv b^2 \mod 2O_K$. Suppose $a \equiv 1 \mod 2O_K$. Then $a^2 \equiv b^2 \equiv 1 \mod 4O_K$. This contradicts to the equality $N(\gamma) = \pm 2$ and this proves the claim. Since $a \neq 1 \mod 2O_K$, we have $a^2 + 1 \neq 0 \mod 2O_K$ by the structure of the finite field $\mathbb{F}_4$. Since $N(\gamma) = 1$, the conjugate of $\eta$ is $\eta^{-1}$. We then have $\text{ord}_{\mathfrak{P}}(\eta + \eta^{-1}) = \text{ord}_{\mathfrak{P}}(a^2 + b^2 \sqrt{-q}) = \text{ord}_{\mathfrak{P}}(2(a^2 + 1)) = 2$ and $\text{ord}_{\mathfrak{P}}(\eta - \eta^{-1}) = \text{ord}_{\mathfrak{P}}(2ab \sqrt{-q}) = 2$. This completes the proof for $q \equiv 3 \mod 8$.

Now we assume $q \equiv 7 \mod 8$ in the rest of the proof. We have $\mathfrak{P}^h = \gamma O_F$ for some $\gamma \in O_F$. Put $\pi = N(\gamma) \in O_K$. The equalities of ideals $\pi^h O_F = \mathfrak{P}^{2h} = \pi O_F = \gamma^2 O_F$ gives a unit $\frac{\sqrt{2}}{\pi}$ of $F$. We have $\frac{\sqrt{2}}{\pi} = \pm \eta^k$ for some odd integer $k$, for the same reason as in the case $q \equiv 3 \mod 8$. As $\eta \equiv 1 \mod \mathfrak{P}$, we have $\text{ord}_{\mathfrak{P}}(\pm \eta^k) = \text{ord}_{\mathfrak{P}}(\eta + \eta^{-1})$. We may assume that $\frac{\sqrt{2}}{\pi}$ is the fundamental unit $\eta$. Write $\gamma = a + b\sqrt{-q}$ with $a, b \in K$. Then

$$\eta = \frac{a^2 + \sqrt{-q}b^2}{\pi} + \frac{2ab \sqrt{-q}}{\pi} \quad \text{and} \quad a^2 - \sqrt{-q}b^2 = \pi.$$

From now on, we work in $F_{\mathfrak{P}}$, which is a quadratic extension of $K_p = \mathbb{Q}_2$. Recall that as in the proof of Lemma \cite{3}, the embedding induced by $p$ is chosen so that $\sqrt{-q} \equiv 3 \mod 8$ when $q \equiv 7 \mod 16$ and that $\sqrt{-q} \equiv 7 \mod 8$ when $q \equiv 15 \mod 16$. Note that the ring of integers of $F_{\mathfrak{P}}$ is $\mathbb{Z}_2[\sqrt[3]{q}]$. Since $\gamma$ is
integral in \( F_2 \), we have \( a, b \in \mathbb{Z}_2 \). Since \( \text{ord}_{\phi}(\pi) = h \), we can write \( \pi = 2^h u \) with \( u \in \mathbb{Z}_2^\times \). Note that one must have \( \text{ord}_2(a) = \text{ord}_2(b) \). Otherwise, the valuation of \( \pi = N_{F_2/K_2}(a + b\sqrt{-q}) \) at 2 is even which contradicts to the fact that \( h \) is odd. Also note that if \( c, d \in \mathbb{Z}_2^\times \), then \( N_{F_2/K_2}(c+d\sqrt{-q}) \equiv 2 \mod 4\mathbb{Z}_2 \). It follows that \( \text{ord}_2(a) = \text{ord}_2(b) = (h-1)/2 \). Because \( \pi = N_{F_2/K_2}(\gamma) \) is a norm, we conclude the following values of the Hilbert symbols

\[
\left( \frac{2^h u, \sqrt{-q}}{K_p} \right) = \left( \frac{2u, 3}{\mathbb{Q}_2} \right) = 1 \text{ if } q \equiv 7 \mod 16
\]

and

\[
\left( \frac{2^h u, \sqrt{-q}}{K_p} \right) = \left( \frac{2u, 7}{\mathbb{Q}_2} \right) = 1 \text{ if } q \equiv 15 \mod 16.
\]

This implies that \( u \equiv 3 \mod 4 \) if \( q \equiv 7 \mod 16 \) and that \( u \equiv 1 \mod 4 \) if \( q \equiv 15 \mod 16 \). Thus

\[
\frac{\eta + \eta^{-1}}{2} = \frac{a^2 + \sqrt{-q}b^2}{\pi} = \frac{2a^2 - \pi}{\pi} = \left( 2 \frac{a}{2^h u} \right)^2 u^{-1} - 1 = 2 \mod 4 \text{ if } q \equiv 7 \mod 16,
\]

\[
0 \mod 4 \text{ if } q \equiv 15 \mod 16.
\]

This finishes the proof of Lemma 4 by the fact \( \text{ord}_p(2) = 2 \).

**Proof of Theorem 1.** As we mentioned in the end of the introduction, the basic fact that \( \text{ord}_p(\log_{p}(x)) = \text{ord}_p(x-1) \) if \( \text{ord}_p(x-1) > 2 \) will be used. For a proof, see [3, Lemma 5.5]. Assume \( q \equiv 3 \mod 8 \). Then \( \text{ord}_p(\eta^2 + 1) = \text{ord}_p(\eta^2 + \eta^{-1}) = \text{ord}_p(\eta + \eta^{-1}) = 2 \) and \( \text{ord}_p(\eta^2 - 1) = \text{ord}_p(\eta^2 - \eta^{-1}) = \text{ord}_p(\eta - \eta^{-1}) = 2 \). Hence \( \text{ord}_p(\eta^4 - 1) = 4 \). This gives \( \text{ord}_p(\log_{p}(\eta)) = \text{ord}_p(\log_{p}(\eta^4)) = \text{ord}_p(\log_{p}(\eta^8)) = \text{ord}_p(\log_{p}(\eta^{16})) = 0 \). This proves (1).

Assume \( q \equiv 7 \mod 16 \). We have \( \text{ord}_p(\eta^2 + 1) = \text{ord}_p(\eta^2 + \eta^{-1}) = \text{ord}_p(\eta + \eta^{-1}) = 2 \) and \( \text{ord}_p(\eta^2 - 1) = \text{ord}_p(\eta^2 - \eta^{-1}) = \text{ord}_p(\eta - \eta^{-1}) = 2 \). Hence \( \text{ord}_p(\log_{p}(\eta)) = 6 \). This proves (2).

Assume \( q \equiv 15 \mod 16 \). Then \( \text{ord}_p(\eta^4 - 1) = \text{ord}_p(\eta^2 + 1) + \text{ord}_p(\eta^2 - 1) \geq 6 + 2 = 8 \). Then \( \text{ord}_p(\log_{p}(\eta^4)) = \text{ord}_p(\eta^4 - 1) \geq 8 \). Thus \( \text{ord}_p(\log_{p}(\eta)) \geq 4 \). This completes the proof of Theorem 1.

Now, we prove Corollary 4, and we begin by recalling a classical result from global class field theory. Let \( L \) be any number field, and \( p \) be a prime number. For a prime ideal \( v \) of \( L \), let \( U_{1,v} \) denote the principal units in the completion \( L_v \) of \( L \), and put \( U_1 = \prod_{v \mid p} U_{1,v} \). Let \( \phi \) be the canonical embedding \( L \hookrightarrow \prod_{v \mid p} L_v \). Denote by \( E_1 \) the group of global units of \( L \) whose images lie in \( U_1 \), and let \( \phi(E_1) \) denote the closure of \( \phi(E_1) \) in \( U_1 \) under the \( p \)-adic topology. Let \( H \) be the \( p \)-Hilbert class field of \( L \). Finally let \( M(L) \) be the maximal abelian \( p \)-extension of \( L \), which is unramified outside the primes of \( L \) lying above \( p \). Then the Artin map induces an isomorphism

\[
U_1/\phi(E_1) \cong \text{Gal}(M(L)/H).
\]

This is a standard consequence of global class field theory (see, for example, [5, Theorem 13.4]). Note that \( U_1 \) is a finitely generated \( \mathbb{Z}_p \)-module of rank \( [L : \mathbb{Q}] \). Moreover, the \( \mathbb{Z}_p \)-module \( \phi(E_1) \) has rank \( \leq r_1 + r_2 - 1 \), and Leopoldt’s conjecture asserts that this rank is always equal to \( r_1 + r_2 - 1 \); here \( r_1 \) and \( r_2 \) are the number of real and complex places of \( L \), respectively.

**Proof of Corollary 4.** We apply the above isomorphism to the field \( F \) with \( q \equiv 3 \mod 8 \) and the prime 2. In this case, \( U_1 = 1 + \mathfrak{P}_F \mathfrak{p} \) has \( Z_2 \)-rank \( [F : \mathbb{Q}] = 4 \), and \( \phi(E_1) = \langle \eta, -1 \rangle \) clearly has \( Z_2 \)-rank 1. Moreover, the 2-Hilbert class field of \( F \) is \( F \) itself since \( F \) has odd class number by Lemma 4. Thus we obtain an isomorphism of \( Z_2 \)-modules

\[
(1 + \mathfrak{P}_F \mathfrak{p})/\langle \eta, -1 \rangle \cong \text{Gal}(M(F)/F).
\]
In order to prove \( M(F) = F_\infty \), it suffices to show that there is no nontrivial torsion element in the group on the left. Consider the commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \to & \{\pm 1\} & \to & \phi(\mathcal{E}_1) & \to \mathbb{Z}_2 \log_\mathfrak{p}(\eta) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mu(1 + \mathfrak{O}_{F_\mathfrak{p}}) & \to & 1 + \mathfrak{O}_{F_\mathfrak{p}} & \to \log_\mathfrak{p}(1 + \mathfrak{O}_{F_\mathfrak{p}}) & \to 0.
\end{array}
\]

Here \( \mu(1 + \mathfrak{O}_{F_\mathfrak{p}}) \) is the group of roots of unity in \( 1 + \mathfrak{O}_{F_\mathfrak{p}} \) which equals \( \{\pm 1\} \) as one can check that \( \sqrt{-1} \notin F_\mathfrak{p} \). Thus the logarithm induces an isomorphism

\[
(1 + \mathfrak{O}_{F_\mathfrak{p}})/\langle \eta, -1 \rangle \cong \log_\mathfrak{p}(1 + \mathfrak{O}_{F_\mathfrak{p}})/\mathbb{Z}_2 \log_\mathfrak{p}(\eta).
\]

Since \( \text{ord}_\mathfrak{p}(2) = 2 \), it is clear from the logarithmic series that \( \log_\mathfrak{p}(1 + \mathfrak{O}_{F_\mathfrak{p}}) \subset \mathfrak{O}_{F_\mathfrak{p}} \). We claim that the \( \mathbb{Z}_2 \)-module \( \log_\mathfrak{p}(1 + \mathfrak{O}_{F_\mathfrak{p}})/\mathbb{Z}_2 \log_\mathfrak{p}(\eta) \) is free. Suppose not. Then there exists an element \( a \) in \( \log_\mathfrak{p}(1 + \mathfrak{O}_{F_\mathfrak{p}}) \subset \mathfrak{O}_{F_\mathfrak{p}} \) but not in \( \mathbb{Z}_2 \log_\mathfrak{p}(\eta) \) such that \( 2a \in \mathbb{Z}_2 \log_\mathfrak{p}(\eta) \). Write \( 2a = r \log_\mathfrak{p}(\eta) \) with \( r \in \mathbb{Z}_2 \). Note that \( r \) must be in \( \mathbb{Z}_2^\times \). This would give \( \text{ord}_\mathfrak{p}(\log_\mathfrak{p}(\eta)) = \text{ord}_\mathfrak{p}(2a) > 0 \) which contradicts to Theorem \( \mathbb{A} \). Thus we have that \( \text{Gal}(M(F)/F) \cong \log_\mathfrak{p}(1 + \mathfrak{O}_{F_\mathfrak{p}})/\mathbb{Z}_2 \log_\mathfrak{p}(\eta) \) is a free \( \mathbb{Z}_2 \)-module of rank 3 and hence \( M(F) = F_\infty \). This completes the proof. \( \square \)

We end this paper by noting a second and very simple proof of Theorem \( \mathbb{A} \). Suppose \( q \equiv 7 \mod 8 \), so that \( 2 \) splits in \( K \), and recall that \( \mathfrak{p} \) is the restriction of \( \mathfrak{P} \) to \( K \). As before, let \( M(F) \) be the maximal abelian 2-extension which is unramified outside \( \mathfrak{P} \). By class field theory and the fact that \( F \) has odd class number [2, Theorem 11], we have

\[
(1 + \mathfrak{O}_{F_\mathfrak{p}})/\langle \eta, -1 \rangle \cong \text{Gal}(M(F)/F).
\]

Suppose now \( q \equiv 15 \mod 16 \). The embedding \( K \hookrightarrow K_{\mathfrak{p}} = \mathbb{Q}_2 \) induced by \( \mathfrak{p} \) makes that \( \sqrt{-q} \equiv -1 \mod 8 \) whence \( F_{\mathfrak{p}} = \mathbb{Q}_2(\sqrt{-1}) \). Clearly \( \sqrt{-1} \) is in \( 1 + \mathfrak{O}_{F_\mathfrak{p}} \), but not in \( \langle \eta, -1 \rangle \). Thus \( \text{Gal}(M(F)/F) \) has an element of order 2. Now let \( F_\infty = FK_\infty \), where \( K_\infty \) is the unique \( \mathbb{Z}_2 \)-extension of \( K \) unramified outside \( \mathfrak{p} \). Since \( \text{Gal}(F_\infty/F) \) is a free \( \mathbb{Z}_2 \)-module of rank 1, it follows that \( \text{Gal}(M(F)/F_\infty) \) must contain the element of order 2, and so \( \text{Gal}(M(F)/F_\infty) \neq 0 \). By the formula \( \mathbb{A} \) of Coates-Wiles, it follows that we must have \( \text{ord}_\mathfrak{p}(\log_\mathfrak{p}(\eta)) \geq 4 \), as required.

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