Research Article

Integrability, Variational Principle, Bifurcation, and New Wave Solutions for the Ivancevic Option Pricing Model

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The Ivancevic option pricing model comes as an alternative to the Black-Scholes model and depicts a controlled Brownian motion associated with the nonlinear Schrödinger equation. The applicability and practicality of this model have been studied by many researchers, but the analytical approach has been virtually absent from the literature. This study intends to examine some dynamic features of this model. By using the well-known ARS algorithm, it is demonstrated that this model is not integrable in the Painlevé sense. He’s variational method is utilized to create new abundant solutions, which contain the bright soliton, bright-like soliton, kinky-bright soliton, and periodic solution. The bifurcation theory is applied to investigate the phase portrait and to study some dynamical behavior of this model. Furthermore, we introduce a classification of the wave solutions into periodic, super periodic, kink, and solitary solutions according to the type of the phase plane orbits. Some 3D-graphical representations of some of the obtained solutions are displayed. The influence of the model’s parameters on the obtained wave solutions is discussed and clarified graphically.

1. Introduction

In the modern era, global finance and economics have become one of the most studied fields. Experts and scientists produce more sophisticated products by developing many technological and scientific tools. Experts use these tools to find the best conditions in all kinds of daily situations. To get the most benefit from such devices and to get optimal results, users must know when such devices are in interaction with them. In the past, consumers used their current knowledge to determine the value of products. In this case, they come across many additional problems arriving from customers, sellers, web platforms, banks, and both local and international platforms. Accordingly, such problems were analyzed to elucidate and scrutinize by employing scientific standards. Consequently, such works present more intellectual approaches for the utilizers. Thus, following the financial market is admiringly important. Deeper properties of the modeling of a global financial market produce a global informative system. Specifically, these dynamic systems can be utilized for a serious examination of the productions. Furthermore, transferring productions from production to clients through numerous approaches, such as highway and plane, shipment is another additional significant point in monitoring. The initial step is to accomplish its mathematical models whether they have complex values or real values by employing the wave function. So, several models were built by specialists in obtaining their wave distributions in the present and future trends. Just, in this case, soliton theory considers one of the major utilized theories for the reason that we have exact knowledge such as periodic, singular, dark, bright, complex, and traveling. Such dynamical information gives crucial in insight, expecting, controlling, and upcoming predictions of complex behaviors.
of productions. For these reasons, this branch attracts the attention of scholars. In Reference [1], the authors examined the stochastic differential equations appearing in finance (SDE) and the backward SDE stability has been investigated under small perturbations of both the coefficients and the boundary values [2]. In Reference [3], the authors determined the optimal consumption and portfolio policies in a continuous-time finance model. In recent times, fractional order impulsive stochastic differential equation has been examined in controllability in [4]. The economy splitting schemes have been presented by Samarskii [5]. In Reference [6], the authors planned the robust economic model. They have assumed the concept of the danger factor in the controller design and supplied an algorithm to decide the economic zone to be pursued. Adomian formed and investigated a national economy model, specifically, coupled nonlinear stochastic multidimensional (discrete or continuous) operator equations by making use of the decomposition method to attain the solutions of complex dynamical systems [7]. Due to the advancing and developing in computers, the modeling of large power plants has been considered in [8].

Recently, the study of the deep properties of the mathematical models, which describe finance and economic problems, acquired their significance due to their wide applications. These models are usually nonlinear partial differential equations (NPDEs). NPDEs also appear to govern a wide variety of complex phenomena in biology, chemistry, and physics [9–13]. Closed-form solutions for nonlinear PDEs are crucial for understanding intricate phenomena. In this regard, it is necessary and imperative to find some solutions, especially wave solutions. Researchers are therefore motivated to present new methods and refine existing approaches. Various significant and powerful methods have been introduced such as Darboux transformation [14], Weierstrass elliptic functions methods [15, 16], Bäcklund transformation [17], Lie group [18–21], Hirota’s method [22, 23], bifurcation method [24–33], and for distinct method, as shown in e.g., [34–42]. The analytical and numerical solutions for various types of nonlinear partial differential equations were investigated using traditional Lie symmetry approaches; for instance as shown in [43]. There are several technical methods for solving and investigating several kind of nonlinear partial differential equations, e.g. [44–58].

This traditional Black–Scholes (BS) model is a significant development in financial mathematics. It symbolizes how the cost of economic fairness on the market changes over time, much like stock options [59, 60]. The BS model makes various assumptions under which the volatility, the drift parameter, and the asset price S are assumed to be constants. Additionally, it takes into account a lack of frictionless, ruthless, and arbitrage bazaars [61–63]. Thus, it is necessary to modify certain models (such as the stochastic interest model, the jump-diffusion model, the stochastic volatility model, and models with transaction costs) in order to reduce these assumptions. As a result, IOPM (alternative model of the Black–Scholes equation) has grown more appealing in recent years since it efficiently delivers the values of options. We take into account suppositions: μ is the drift parameter, asset price (or the underlying asset) S following a geometric Brownian motion, and volatility rate σ, which are assumed to be constants. Moreover, we also consider that the arbitrage opportunities are limited (no risk-free profits), while market frictionless are competitive [61–63]. The price function $S = S(t)$, at time $t \in [0, T]$ satisfies the next equation:

$$dS = (\mu dt + \sigma dW_t), \quad S \in [0, \infty),$$  

where $W_t$ refers to the standard Brownian motion. Thus, for the value of the option, $\psi(t, S)$, the Black–Scholes model corresponding to equation (1) admits the following formula:

$$\frac{\partial \psi}{\partial t} + rS \frac{\partial \psi}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 \psi}{\partial S^2} - r\psi = 0,$$  

where $\psi(0, t) = 0$, $\psi(t, S) \to \infty$ as $S \to \infty$, $\psi(t, S) = \max(S - E, 0)$, where $E$ is considered as a constant and

$$S(t) = \psi_0(\psi^{-1}(\min(S/E, 1)) \epsilon dW_t),$$  

$$S_0 = \psi(0).$$

Recently, Vukovic [64] established a connection utilizing the Hamiltonian operator in quantum physics between the Schrodinger equation (SE) and the BS model. As well, it was found that by applying the methods developed in quantum mechanics, the BS equation could be deduced from SE [65]. It is notable that SE is a complex-valued equation, while the BS model is a real-valued for the price function. The BS model (2) may be carried out to one-dimensional option models assigned to $\psi$ and S. As stated in [66], one can utilize the conventional Kolmogorov probability approach to get the probability density function (PDF) derived from the Fokker–Planck equation rather than the value of an option determined by the BS equation.

The quantum-probability construction was executed by Ivancevic [67] for obtaining equal PDF for the value of a stock option for the development of the complex-valued function and proposed a nonlinear model [68]. Thereafter, this model is recalled as the Ivancevic option pricing model (IOPM) which takes the following formula:

$$i \frac{\partial \psi}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 \psi}{\partial S^2} + \beta |\psi|^2 \psi = 0,$$  

where the option price function $\psi(t, S)$ represents the price function at time $t$, the option price PDF is indicated by $|\psi|^2$, $\sigma$ is deemed to be constant, and the adaption potential of the market is represented by the Landau coefficient $\beta$.

The Black–Scholes model proposes that the underlying volatility will remain constant over the life of the derivative and will not be impacted by modifications in the underlying’s price level. Nevertheless, the long-observed features of the surface of implied volatility, such as the smile of volatility and skew, cannot be interpreted by this model, indicating that implied volatility does not tend to differ with respect to the strike price and expiration. However, in this model, it is possible to better correctly simulate derivatives by supposing that the volatility of the underlying price is a stochastic process rather than a constant. The Brownian movement,
just like the Black–Scholes option pricing model, can be employed to derive the Ivancevic option pricing model. Instead of a stochastic structure, we have taken into account the deterministic one in this study.

The Ivancevic option pricing model, which signifies controlled Brownian behavior of financial markets and is a nonlinear wave alternative to the standard Black–Scholes option-pricing model, is formally defined by adaptive nonlinear Schrodinger (NLS) equations that define the option-pricing wave function in terms of the stock price and time. The model contains two parameters: adaptive market potential, which is influenced by the interest rate, and volatility (performing the role of the dispersion frequency coefficient), which can be fixed or stochastic. Some properties of equation (4) are inspected in [69] by applying numerous techniques, for example, tanh expansion method, trial function method, and direct perturbation method. The fractional properties of the this model were examined in [70]. The vector financial wave propagation were yanked in [71]. A nonzero adaptive market potential was analyzed in [72].

In this work, we are interested in studying the model (4) from distinct aspects. The first one is investigating the integrability of equation (4) by applying the Painlevé approach. Although there is a lack of a general definition of completely integrable systems, it is a powerful property for any nonlinear partial differential equation. It is extensively taken that the integrable model must have Lax pair, Hamiltonian and bi-Hamiltonian structure, N-soliton solutions, infinitely many symmetries, bilinear B¨acklund Hamiltonian and bi-Hamiltonian structure, N-soliton that the integrable model must have Lax pair, any nonlinear partial differential equation. It is extensively completely integrable systems, it is a powerful property for approach. Although there is a lack of a general definition of

section 3, Painlev´e is applied to examine the integrability of short description for the methods that will be used. In the model (4).

The present subsection provides a brief description of Painlevé analysis. Painlevé analysis is employed to determine whether the given equation (5) is integrable or not. The equation possesses Painlevé property if its single-valued solutions have no bad singularity other than movable poles about the singular manifold $\phi$, where $\phi(S,t) = 0$. The solution of equation (5) can be written as Laurent series.

$$ u(t,S) = \phi^p \sum_{i=0}^{\infty} u_i(t,S)\phi^i, $$

(6)

where $\phi$ and $u$ are a polynomial and a complex-valued function depending on the two variables $S,t$, respectively.

2.1. ARS-Algorithm. The present subsection provides a brief description of Painlevé analysis. Painlevé analysis is employed to determine whether the given equation (5) is integrable or not. The equation possesses Painlevé property if its single-valued solutions have no bad singularity other than movable poles about the singular manifold $\phi$, where $\phi(S,t) = 0$. The solution of equation (5) can be written as Laurent series.

$$ u(t,S) = \phi^p \sum_{i=0}^{\infty} u_i(t,S)\phi^i, $$

(6)

where $u_0 \neq 0$ and $p$ is a negative integer needed to be found. We follow the ARS algorithm that can be summarized in the following steps [76]:

Step 1: Dominate behavior. We assume the leading order term is assumed to be

$$ u = u_0\phi^p. $$

(7)

Inserting the expression into equation (5) and balancing the dominated terms, we found the value of $p$ and comparing the coefficients of $\phi^p$ in both sides, we obtain an equation determining $u_0$.

Step 2: Resonances. The resonances is defined as the powers at which the arbitrary functions appear in the Laurent series. The resonance can be determined by inserting

$$ u(t,S) = u_0\phi^p + \sum_{i=1}^{\infty} u_i(t,S)\phi^{i+p}, $$

(8)

into equation (5) and comparing different powers of $\phi$. Note that all the resonances are non-negative integers except the resonance $r = -1$, which refers to the arbitrariness of $\phi$. Additionally, the resonance $r = 0$ indicates the coefficient $u_0$ of the leading term is arbitrary. If all the values of resonances are non-negative except $r = -1$, we are going to the next step.

Step 3: Compatibility conditions. This step aims to check the existence of a sufficient number of arbitrary functions in the Laurent series (6). This can be performed by inserting the expression

$$ u(t,S) = \phi^p \sum_{i=0}^{r_{\text{max}}} u_i(t,S)\phi^i, $$

(9)

into equation (5) and testing the existence of an arbitrary functions corresponding to the obtained resonances in Step 2, where $r_{\text{max}}$ is the largest value of the resonances. If the compatibility conditions are satisfied then, the given partial differential equation has Painlevé property. Consequently, it is integrable in Painlevé sense or sometimes it is named Painlevé integrable.
2.2. Bifurcation Analysis. The construction of some wave solutions for equation (5) by aiding bifurcation analysis, we follow the next steps [77]:

Step 1: Applying the wave transformation.

\[ u(t, S) = \varphi(\eta)e^{i(k_i t - \omega_i S)}, \quad \eta = k_i t - \omega_i S, \tag{10} \]

where \( k_i, \omega_i, i = 1, 2 \) are arbitrary constants to system (5) and \( \varphi \) is real-valued function, we obtain after some calculations a second order ordinary differential equations of \( \varphi \) which, by substitution, will be rewritten as a one degree of freedom Hamiltonian system, i.e., it takes the following formula:

\[
\begin{align*}
\varphi' &= \frac{\partial H}{\partial y} = y, \\
y' &= -\frac{\partial H}{\partial \xi} = -\frac{\partial V}{\partial \xi},
\end{align*}
\tag{11}
\]

where \( y \) is a new variable introducing to convert the second order differential equation into a dynamical system, \( V(\xi) \) is a real-valued function and \( r \) indicates derivative with respect to \( \eta \).

Step 2: Calculate the Hamiltonian function \( H \) as

\[ H = \frac{1}{2}y^2 + V(\xi) = h, \tag{12} \]

from which we obtain the separable differential equation as

\[ \frac{d\xi}{\sqrt{2(h - V(\xi))}} = \pm d\eta. \tag{13} \]

Step 3: find the equilibrium points for the Hamiltonian system (11) and apply the qualitative theory for the planar dynamical system to specify their nature.

Step 4: find the intervals of real wave propagation which is equivalent to the intervals of real motion of a particle described by the Hamiltonian system (11). Integrating the sides of (12) over certain regions that guarantee real propagation, various kinds of wave solutions are constructed.

3. Painlevé Analysis

The purpose of this section is to investigate the integrability of equation (4) by applying Painlevé singularity analysis which is introduced briefly in section 2. The dependent variable and its complex conjugate are supposed to be \( \psi = \mathbf{I} \) and \( \psi^* = \mathbf{J} \), where * refers to the complex conjugate. Accordingly, equation (4) and its conjugate can be presented as

\[
\begin{align*}
i \mathbf{J}_t + \frac{\sigma^2}{2} \mathbf{J}_{ss} + \beta \mathbf{J}^2 &= 0, \\
i \mathbf{I}_t - \frac{\sigma^2}{2} \mathbf{I}_{ss} + \beta \mathbf{I}^2 &= 0.
\end{align*}
\tag{14}
\]

For both equations, the singularity analysis is accomplished by carrying out a search for the generalized Laurent series for the dependent variables in the neighborhood of the non-characteristic singular manifold \( \mathbf{J}(t, S) \) with non-vanishing derivatives, i.e., \( \mathbf{J} \neq 0, \). Equations of the dependent variable in the neighborhood of the non-characteristic singular manifold \( \mathbf{I}(t, S) \) with non-vanishing derivatives, i.e., \( \mathbf{I} \neq 0, \).

\[
\begin{align*}
I(t, S) &= \varphi^p(t, S) \sum_{i=0}^{\infty} I_i(t, S) \varphi^i(t, S), \\
J(t, S) &= \psi^q(t, S) \sum_{i=0}^{\infty} J_i(t, S) \psi^i(t, S),
\end{align*}
\tag{15}
\]

where \( \mathbf{I}_0 \mathbf{J}_0 \neq 0 \) and \( p, q \) are negative integers required to be found. The leading orders are assumed to be

\[
\begin{align*}
I(t, S) &= I_0 \varphi^p, \\
J(t, S) &= J_0 \psi^q.
\end{align*}
\tag{16}
\]

Inserting expression (16) into equation (14) and balancing the dominant terms, we obtain \( p = q = -1 \) and consequently the coefficient of \( \varphi^{-1} \) becomes

\[ \beta \mathbf{J}_0 + \sigma^2 \mathbf{I}_0^2 = 0. \tag{17} \]

Note that one of the two functions \( \mathbf{I}_0 \) and \( \mathbf{J}_0 \) is an arbitrary function.

We are going to the next step in which we determine the resonances which is defined as the powers at which the arbitrary functions can be entered in Laurent series. The resonances are evaluated by inserting the next two formulas, that is

\[
\begin{align*}
I(t, S) &= I_0 \varphi^{-1} + I_1 + \cdots + I_r \varphi^{-r}, \\
J(t, S) &= J_0 \psi^{-1} + J_1 + \cdots + J_r \psi^{-r},
\end{align*}
\tag{18}
\]

into equation (14) and evaluating the coefficients of \( \varphi^{-r} \), we get a homogeneous linear system presented in a matrix form as

\[
\begin{pmatrix}
\mathbf{\phi}_0^2 (r - 2)(r - 1) + 4\beta \mathbf{I}_0 \mathbf{J}_0 \\
2\beta \mathbf{I}_0^2 \\
-\sigma^2 \mathbf{J}_0^2 (r - 2)(r - 1) - 4\beta \mathbf{I}_0 \mathbf{J}_0 \\
I_r(t, S) \\
J_r(t, S)
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\tag{19}
\]

System (19) has arbitrary solutions if determinant of the last \( 2 \times 2 \) matrix is zero. Hence, taking into account equation (17), we obtain

\[ \sigma^4 \mathbf{J}_0^4 (r + 1)(r - 3)(r - 4) = 0. \tag{20} \]

The zeros of equation (20) are the resonances and they are \( r = 0, r = -1, r = 3, r = 4 \). Note, the resonance \( r = -1 \) refers to the arbitrariness of \( \varphi \) and it is the only agreeable negative resonance. The resonance \( r = 0 \) indicates one of the coefficients for the leading terms is arbitrary and this is clear from equation (17).

Now we are going to the last step in which we test the presence of the sufficient number of arbitrary functions in the corresponding Laurent series. We insert the two
expressions in equation (18) into the two equation (4) and collect all powers of $\phi$. The coefficient of $\phi^{-3}$ which corresponds to the resonance $r = 0$ gives the same equation (17) which shows one of the two functions $v_0$ or $w_0$ is an arbitrary function.

In the following, we use the Kruskal ansatz $\phi(t, S) = t + \rho(S)$ to simplify the calculations. The coefficient of $\phi^{-3}$ are

\[ 2\sigma^2 \rho^2 w_1 - 2\beta w_0^2 v_1 - 2\sigma^2 \rho^2 w_0 + [5\sigma^2 w_0 \rho^2 + 2i]w_0 = 0, \]

\[ 2\sigma^2 \rho^2 w_1 - \beta w_0^2 v_1 + \sigma \rho^2 w_0' + \left(\frac{\sigma^2}{2} \rho^2 \right)w_0 = 0. \]

(21)

Solving system (21) for $w_1$, we get

\[ v_1 = -\frac{3}{2\beta w_0^3} \left[ 2\sigma^2 \rho^2 w_0 - 3\sigma^2 \rho^3 w_0 - 2iw_0 \right], \]

\[ w_1 = -\frac{2}{\sigma^2 \rho^3} \left[ \sigma^2 \rho^2 w_0 - \sigma^2 \rho^3 w_0 - iw_0 \right]. \]

Similarly, we compute the coefficients of $\phi^{-1}$, we obtain a linear system in the two functions $w_2$ and $v_2$ that has the following solution:

\[ v_2 = \frac{-1}{6\beta \sigma^2 w_0^2 \rho^2} \left[ \sigma^2 \left( 4\sigma^2 \rho^2 w_0 + 24\sigma^2 \rho^2 w_0^2 - 3\sigma^2 \rho^2 w_0^2 \right) - 2i\sigma^2 \left( \rho^2 w_0 - 4\sigma^2 \rho^2 w_0 + 16\sigma^2 \rho^2 w_0 \right) + 4w_0 \right], \]

\[ w_2 = \frac{-1}{12\sigma^2 w_0^2 \rho^2} \left[ \sigma^2 \left( -12\rho^2 w_0 - 48\rho^2 w_0^2 + 4\sigma^2 \rho^2 w_0^2 + 156\rho^2 w_0 \right) - 117\rho^2 w_0 \right] \]

\[ + 4i\sigma^2 w_0 \left( \rho^2 w_0 - 41w_0 \rho^2 - 26\rho w_0 + 26\rho w_0 \right) + 52w_0^2 \].

(23)

4. Bifurcation Analysis and Wave Solutions

As outlined in the previous section, the Ivancevic option pricing model fails to satisfy the Painlevé test for the integrability. Hence, numerical analysis or quasi analytical methods are demanded to construct special solutions for equation (4). This motivates us to search for some wave solutions and furthermore, we aim to investigate the phase space and study the influence of the two parameters $\sigma$ and $\beta$ on the wave solutions. As usual, we apply the following wave transformation as

\[ \psi(t, S) = e^{i(k_1 t - \omega_1 S)} \phi(\eta), \quad \eta = k_2 t - \omega_2 S, \]

where $\phi(\eta)$ is a real-valued function indicating the wave profile, $\omega_1$ is the velocity of the soliton, $k_1$ is the frequency, $\omega_2$ is the number of the soliton, and $k_1$ is an arbitrary real constant. Inserting the transformation (25) into equation (4) and equating the imaginary and real parts in both sides, we obtain

\[ \left( k_2 + \sigma^2 \omega_1 \omega_2 \right) \phi = 0, \]

\[ \sigma^2 \omega_2 \phi'' + 2\beta \phi^3 - \left( 2k_1 + \omega_1 t \sigma^2 \right) \phi = 0. \]

(26)

(27)

It is obvious that equation (26) is satisfied identically if

\[ k_2 = -\sigma^2 \omega_1 \omega_2. \]

Equation (27) can be written as
\[ \dot{\varphi} = y, \]
\[ y = m \varphi - n \varphi^3, \quad (29) \]
where
\[ m = \frac{2k_1 + \omega^2 \sigma^2}{\sigma^2 \omega^2}, \]
\[ n = \frac{2b}{4\sigma^2}. \quad (30) \]

The planar dynamical system (29) is a one-dimensional Hamiltonian system with a Hamiltonian function.

\[ H = \frac{1}{2} y^2 - \frac{m}{2} \varphi^2 + \frac{n}{4} \varphi^4. \quad (31) \]

This Hamiltonian describes the motion of a unit mass particle under the action of two-parameters potential function in the following formula:

\[ V(\varphi) = -\frac{m}{2} \varphi^2 + \frac{n}{4} \varphi^4. \quad (32) \]

It is well known that the Hamiltonian function is a constant of the motion if it does not rely explicitly on the independent variable \( \eta \) that plays the role of the time in Hamiltonian mechanics. Thus, we have

\[ \frac{1}{2} y^2 - \frac{m}{2} \varphi^2 + \frac{n}{4} \varphi^4 = h. \quad (33) \]

where \( h \) is a constant. To describe the phase portrait for the Hamiltonian system (29), we first find the equilibrium points for it. These equilibrium points appear as critical points for the potential function (32), i.e., they are the solution of \( (dV(\varphi)/d\varphi) = -\varphi(m - n\varphi^2) = 0 \). Thus, there is a unique equilibrium point \( F_0 = (0,0) \) if \( mn < 0 \) and there are three equilibrium points \( F_0 = (0,0), F_{1,2} = (\pm \sqrt{mn}/n, 0) \) if \( mn > 0 \). The value of the constant \( h \) at these equilibrium points are

\[ h_0(F_0) = 0, \]
\[ h_1(F_{1,2}) = -\frac{m^2}{4n}. \quad (34) \]

The phase orbits are the energy level curves which are defined as

\[ \mathcal{C}_h = \{(\varphi, y) \in \mathbb{R} \times \mathbb{R}: y^2 = 2[h - V(\varphi)]\}. \quad (35) \]

The type of these equilibrium points can be determined depending on whether they are minimum or maximum points for the potential function (32). It is easy to show \( (d^2V/d\varphi^2)(F_0) = -m \) and \( (d^2V/d\varphi^2)(F_{1,2}) = 2m \). Maple program is employed to depict the phase portrait for system (29).

(a) If \( mn < 0 \), there is a unique equilibrium point \( F_0 \). It is a center point if \( m < 0, n > 0 \) and its phase portrait is described by Figure 1(a). The phase space orbits are defined by \( \mathcal{C}_{h<0} \) and they are bounded periodic orbits around the unique center point \( F_0 \). While the point \( F_0 \) is a saddle point if \( m > 0, n < 0 \) and its phase portrait is outlined by Figure 1(b). All the phase orbits are unbounded for all values of the parameter \( h \). They are \( \mathcal{C}_{h=0} \) in red, \( \mathcal{C}_{h<0} \) in green, and \( \mathcal{C}_{h>0} \) in blue.

(b) If \( mn > 0 \), there are three equilibrium points \( F_{0,1,2} \). If \( m < 0, n < 0 \), the equilibrium point \( F_0 \) is a center while \( F_{1,2} \) are saddle points and the phase portrait for this case is described by Figure 2(a). The Hamiltonian system (29) has unbounded orbits \( \mathcal{C}_{h<0} \) in green, \( \mathcal{C}_{h=0} \) in black, and \( \mathcal{C}_{h>0} \) in brown. It also has bounded heteroclinic orbit in red \( \mathcal{C}_{h=\pm 1} \) which connects the two saddle points \( F_{1,2} \) and its extension represents unbounded orbits. It has also a family of orbits in red \( \mathcal{C}_{h \mid [0,\pm 1]} \) which consists of two type of orbits: one of them is the periodic orbit which lie inside the heteroclinic orbit while the other is unbounded and they appear outside the heteroclinic orbit. On the another side, if \( m > 0, n > 0 \), the equilibrium point \( F_0 \) is a saddle while \( F_{1,2} \) are center points and the phase portrait for this case is described by Figure 2(b). In this case, all the phase orbits for the Hamiltonian system (29) are bounded. There are two families of periodic orbits in green \( \mathcal{C}_{h \mid [0,\pm 1]} \), which appear inside of the homoclinic orbit \( \mathcal{C}_{h=0} \) in red besides the existence of a super periodic orbit \( \mathcal{C}_{h>0} \) in blue.

We construct only the wave solutions for bounded orbits in the following subsection. Additionally, unbounded wave solutions can be computed using the same procedures, but they are not physically meaningful.

### 4.1. Bounded Wave Solutions

Inserting the first equation in equation (29) into equation (33), separating the variable and integrating both sides, we get

\[ \int \frac{d\varphi}{\sqrt{\Phi(\varphi)}} = \int d\eta, \quad (36) \]

where

\[ \Phi(\varphi) = -\frac{n}{2} \left( \varphi^4 - 2m \varphi^2 + 4h \right). \quad (37) \]

The range of the parameters \( m, n, \) and \( h \) are necessary to calculate the integral in the left-hand side of equation (36). By applying the bifurcation theory, we can determine the parameters’ range. As a result, we incorporate the bifurcation constraints on the parameters in order to find wave solutions.

#### 4.1.1. Periodic Wave Solution

We integrate equation (36) along one of the periodic orbits that are determined by the phase portrait analysis.

(a) If \( (m,n,h) \in \mathbb{R}^- \times \mathbb{R}^+ \times \mathbb{R}^+ \), we have a solution in the following formula:
$\phi(\eta) = \varphi_1 \text{cn} \left[ \frac{m(\varphi_1^2 + \varphi_2^2)}{2} (\eta - \eta_0), \frac{\varphi_1}{\sqrt{\varphi_1^2 + \varphi_2^2}} \right]$, \hspace{1cm} \left(38\right)

where \( \pm \varphi_1 \) are the real roots for the quartic polynomial \( (37) \). Hence, we obtain a new solution for equation \( (4) \) in the following formula:

$$\psi(t, S) = \varphi_1 \text{cn} \left[ \frac{m(\varphi_1^2 + \varphi_2^2)}{2} (\eta - \eta_0), \frac{\varphi_1}{\sqrt{\varphi_1^2 + \varphi_2^2}} \right] e^{i (t, u + S)}.$$ \hspace{1cm} \left(39\right)

(b) If \((m, n, h) \in \mathbb{R}^- \times \mathbb{R}^- \times [0, \hbar]_1\), there is a family of periodic orbits in blue as outlined by Figure 2(a).
Integrating equation (36) along one of these orbits, we obtain
\[ \varphi(\eta) = \varphi_1 \frac{1}{\varphi_2} \left( \sqrt{\frac{m}{2}} (\eta - \eta_0) \right) \] (40)

where \( \pm \varphi_{1,2} \) are the roots of the polynomial (37) with \( 0 < \varphi_1 < \varphi_2 \). Hence, equation (4) has a new solution in the following formula:
\[ \psi(t, S) = \varphi_1 \frac{1}{\varphi_2} \left( \sqrt{\frac{m}{2}} (\eta - \eta_0) \right) e^{(k_1 t - \omega_1 S)} \] (41)

(c) If \((m, n, h) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \{0, 1\}, \) there is a periodic family of orbits in green as illustrated by Figure 2(b). Integrating both side of equation (36) along one member of this family, we obtain
\[ \varphi(\eta) = \varphi_1 \text{nd} \left( \sqrt{\frac{m}{2}} (\eta - \eta_0) \right) \] (42)

where \( \pm \varphi_{1,2} \) are real roots for the polynomial (37) with \( 0 < \varphi_1 < \varphi_2 \). Therefore equation (4) has a new wave solution in the following formula:
\[ \psi(t, S) = \varphi_1 \text{nd} \left( \sqrt{\frac{m}{2}} (\eta - \eta_0) \right) e^{(k_1 t - \omega_1 S)} \] (43)

4.1.2. Super Periodic Wave Solution. For \((m, n, h) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \{0, 1\}, \) the Hamiltonian system (29) has a super periodic orbits in blue as outlined by Figure 2(b). Integrating both sides of equation (29) along this orbit, we obtain
\[ \varphi(\eta) = \frac{\varphi_1 \varphi_2}{\sqrt{\varphi_1^2 + \varphi_2^2}} \left( \sqrt{\frac{m}{2}} (\eta - \eta_0) \right) \] (44)

where \( \pm \varphi_{1,2} \) are the real roots for the polynomial (37) with \( 0 < \varphi < \varphi_1 \). Hence, equation (4) has a new solution.
\[ \psi(t, S) = \frac{\varphi_1 \varphi_2}{\sqrt{\varphi_1^2 + \varphi_2^2}} \left( \sqrt{\frac{m}{2}} (\eta - \eta_0) \right) e^{(k_1 t - \omega_1 S)} \] (45)

4.1.3. Solitary Wave Solution. The Hamiltonian system (29) has a homoclinic orbit if \((m, n, h) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \{0\} \) in red as clarified by Figure 2(b) and this type of orbits implies to the existence of solitary wave solutions. Integrating both sides of equation (36) along this orbit, we get
\[ \varphi(\eta) = \frac{2m}{n} \text{sech} \left( \sqrt{\frac{m}{n}} (\eta - \eta_0) \right) \] (46)

Consequently, the corresponding solution for equation (4) has the following formula:
\[ \psi(t, S) = \frac{2m}{n} \text{sech} \left( \sqrt{\frac{m}{n}} (\eta - \eta_0) \right) e^{(k_1 t - \omega_1 S)} \] (47)

4.1.4. Kink and Anti-Kink Wave Solutions. For \((m, n, h) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \{1\}, \) the Hamiltonian system (29) owns a heteroclinic orbit which is outline in red by Figure 2(b). This type of orbits refer to the existence of kink or anti-kink wave solution. Integrating both sides of equation (36) along this orbit, we have
\[ \varphi(\eta) = \sqrt{\frac{2m}{n}} \text{tanh} \left( \sqrt{\frac{m}{n}} (\eta - \eta_0) \right) \] (48)

Hence, equation (4) takes a solution in the following formula:
\[ \varphi(\eta) = \sqrt{\frac{2m}{n}} \text{tanh} \left( \sqrt{\frac{m}{n}} (\eta - \eta_0) \right) e^{(k_1 t - \omega_1 S)} \] (49)

5. Variational Principle

Thus, we can find the variational principle of equation (27) by the semi inverse method [78–81], which has been utilized widely to establish the needed variational formulations by introducing an undetermined function as a trial function [82–84].
\[ J = \int \left[ -\sigma \varphi^2 \varphi'' + \frac{\varphi'}{2} - \frac{1}{2} (2k_1 + \omega_1^2 \sigma^3) \varphi^2 \right] \eta \] (50)

5.1. Abundant Solutions

5.1.1. Bright Soliton. He’s variational method, introduced in [85], is a successful method for finding approximated solitary wave solutions. We desire to find the bright soliton for equation (27) in the following formula:
\[ \varphi(\eta) = p \text{sech}(\eta) \] (51)

where \( p \) is a constant. Inserting equation (51) into equation (50), we obtain
\[ J(p) = \int_0^\infty \left[ -\frac{\sigma}{2} \varphi^2 \varphi'' + \frac{\varphi'}{2} - \frac{1}{2} (2k_1 + \omega_1^2 \sigma^3) \varphi^2 \right] \eta \] (52)

\[ = \int_0^\infty \left[ \frac{p^2}{2} (\beta \varphi^2 + \sigma \omega_1^2) \text{sech}(\eta)^4 \right. \] (52)

\[ - \frac{p^2}{2} (\beta \varphi^2 + \sigma \omega_1^2) \text{sech}(\eta)^2 \] (52)

\[ + \frac{p^2}{6} \left[ 2\beta \varphi^2 - 6k_1 - \sigma (\omega_1^2 + 3a_1^2) \right] \] (52)
According to the basis of the Ritz-like method, the stationary condition for equation (52) gives
\[
\frac{\partial J}{\partial p} = \frac{p}{3} \left[ \sigma^2 \left( \omega_2^2 - 3 \omega_1^2 \right) + 4 \beta p^2 - 6 k_1 \right] = 0.
\] (53)

Solving the last equation for \( p \), we get the non-zero solution as
\[
p = \pm \sqrt{\frac{6 k_1 + \sigma^2 (3 \omega_1^2 + \omega_2^2)}{2 \beta}}.
\] (54)

Thus, the bright solution for equation (4) takes the following formula
\[
\psi(t, S) = \pm \sqrt{6 k_1 + \sigma^2 (3 \omega_1^2 + \omega_2^2)} \text{sech}\left(-\sigma^2 \omega_1 \omega_2 t - \omega_2 S\right)e^{i(k_t - \omega_1 S)}. \tag{55}
\]

5.1.2. Bright-Like Soliton. The solution of equation (27) is presumed to have the following formula:
\[
\varphi(\eta) = \frac{p}{1 + \cosh(\eta)}.
\] (56)

Inserting equation (56) into equation (50), we obtain
\[
J(p) = \int_0^{\infty} \left[ -\frac{\sigma \omega_2^2}{2} \varphi^2 + \frac{\beta \varphi^4}{2} - \frac{1}{2} (2 k_1 + \omega_1^2 \sigma^2) \varphi^2 \right] d\eta,
\]
\[
= \int_0^{\infty} \left[ -\frac{\sigma \omega_2^2 p^2}{2} (\cosh(\eta) - 1) \right] \frac{1}{(1 + \cosh(\eta))^2} d\eta,
\]
\[
+ \frac{\beta p^4}{2} - \frac{p^2 (2 k_1 + \sigma^2 \omega_1^2)}{2} \frac{1}{(1 + \cosh(\eta))^2} d\eta.
\] (57)

The stationary condition for the last expression is
\[
\frac{\partial J}{\partial p} = \frac{p}{105} \left[ -7 \sigma^2 \left( \omega_2^2 + 5 \omega_1^2 \right) + 12 p^2 \beta - 70 k_1 \right] = 0.
\] (58)

Solving the last expression for \( p \), we get a non-zero solution in the following formula:
\[
p = \pm \sqrt{\frac{7 (10 k_1 + \sigma^2 (4 \omega_1^2 + 5 \omega_2^2))}{12 \beta}}.
\] (59)

Thus, the bright dark solution for equation (4) has the following formula:
\[
\psi(t, S) = \pm \sqrt{7 \sigma^2 (5 \omega_1^2 + \omega_2^2) + 10 k_1}.
\] (60)

5.1.3. Kinky-Bright Soliton. We search for a kinky-bright soliton in the following formula:
\[
\varphi(\eta) = p \text{ sech}(\eta)^2.
\] (61)

Inserting the expression equation (61) into equation (50), we obtain
\[
J(p) = \int_0^{\infty} \left[ -\frac{\sigma \omega_2^2}{2} \varphi^2 + \frac{\beta \varphi^4}{2} - \frac{1}{2} (2 k_1 + \omega_1^2 \sigma^2) \varphi^2 \right] d\eta,
\]
\[
= \int_0^{\infty} \left[ \frac{\beta p^4}{2} \text{sech}(\eta)^8 + 2 \sigma \omega_2^2 p^2 \text{sech}(\eta)^6 \right] d\eta,
\]
\[
- \frac{p^2}{105} \left[ 24 \beta p^2 - 70 k_1 - 7 \sigma^2 \left( 4 \omega_1^2 + 5 \omega_2^2 \right) \right].
\] (62)

Computing the stationary condition, we have
\[
\frac{\partial J}{\partial p} = \frac{2 p}{105} \left[ 48 \beta p^2 - 70 k_1 - 7 \sigma^2 \left( 4 \omega_1^2 + 5 \omega_2^2 \right) \right] = 0.
\] (63)

The solution of the last equation for \( p \) gives
\[
p = \pm \sqrt{\frac{7 (10 k_1 + \sigma^2 (4 \omega_1^2 + 5 \omega_2^2))}{12 \beta}}.
\] (64)

Hence, the kinky-bright soliton for equation (4) is
\[
\psi(t, S) = \pm \sqrt{7 \sigma^2 (5 \omega_1^2 + \omega_2^2) + 10 k_1} \text{sech}^2\left(-\sigma^2 \omega_1 \omega_2 t - \omega_2 S\right)e^{i(k_t - \omega_1 S)}. \tag{65}
\]

5.1.4. Periodic Wave Solution. We try to find periodic wave solution for equation (4) by applying the variational method [85]. To perform our aim, we assume the periodic solution be in the following formula:
\[
\varphi(\eta) = p \cos q \eta, \quad q > 0.
\] (66)

Inserting the expression (66) into equation (50), we obtain
\[ J(p) = \int_0^{T/4} \left\{ -\sigma \omega_1^2 \psi^2 + \frac{\beta}{2} \psi^4 - \frac{1}{2} \left( 2k_1 + \omega_1^2 \right) \psi^2 \right\} d\eta, \]
\[ = \int_0^{T/4} \left[ \frac{\beta}{2} \cos (q\eta)^4 - \frac{p^2}{2} \left( 2k_1 + \sigma^2 \omega_2^2 \right) \right] d\eta, \]
\[ = \int_0^{\pi/2} \frac{\beta}{2} \cos (\gamma)^4 - \frac{p^2}{2} \left( 2k_1 + \sigma^2 \omega_2^2 \right) d\gamma, \]
\[ = \frac{\pi}{32q^2} \left[ 3\beta p^2 - 8k_1 - 4\sigma^2 \left( \omega_1^2 + \omega_2^2 q^2 \right) \right], \quad (67) \]

where \( T \) indicates the period of the nonlinear term of equation (27). Based on the Ritz-like method, the stationary conditions are
\[ \frac{\partial J(p,q)}{\partial p} = -\frac{\pi p}{8q} \left[ 3\beta p^2 - 4k_1 - 2\sigma^2 \left( \omega_1^2 + \omega_2^2 q^2 \right) \right], \]
\[ \frac{\partial J(p,q)}{\partial q} = \frac{\pi p^2}{32q^2} \left[ 3\beta p^2 - 8k_1 + 4\sigma^2 \left( \omega_2^2 q^2 - \omega_1^2 \right) \right]. \quad (68) \]

Solving equation (68) for \( p \) and \( q \), we get the non-zero solution as
\[ p = \pm \sqrt{\frac{8(2k_1 + \sigma^2 \omega_1^2)}{9\beta}}, \]
\[ q = \frac{1}{3\omega_2} \sqrt{3(\sigma^2 \omega_1^2 + 2k_1)}. \quad (69) \]

Hence, the periodic wave solution for equation (4) admits the following formula:
\[ \psi(t,S) = \pm \sqrt{\frac{8(2k_1 + \sigma^2 \omega_1^2)}{9\beta}} \cos \left( \frac{1}{3\sigma \omega_2} \sqrt{3(\sigma^2 \omega_1^2 + 2k_1)} \right) \]
\[ \times \left( -\sigma^2 \omega_1 \omega_2 \right) t - \omega_2 S \right) e^{i(k_1 t - \omega_1 S)}. \quad (70) \]

**6. Physical Interpretations**

This section has two objectives. Firstly, we clarify some of the obtained solutions for certain values of the equation’s parameters. Secondly, we study the influence of the two parameters \( \beta \) and \( \sigma^2 \) on the obtained solution when one of them is fixed and allows the other to vary by solving the dynamical system (29) by using a Fehlberg fourth-fifth-order Runge–Kutta method by aiding the Maple program.

We utilize certain example of equation (4) in order to plot the obtained results. If \( \sigma = 2, \beta = 18 \), equation (4) takes the following formula:
\[ i \frac{\partial \psi}{\partial t} + 2 \frac{\partial^2 \psi}{\partial \sigma^2} \partial S^2 + 18|\psi|^2 \psi = 0. \quad (71) \]

For \( k_1 = 1, \omega_1 = 1, \omega_2 = 2 \), we get bright solution for equation (71) in the following formula:
\[ \psi(t,S) = \frac{34}{36} \text{sech}(8t + 2S)e^{i(t-S)}. \quad (72) \]

Or
\[ \psi(t,S) = -\frac{34}{36} \text{sech}(8t + 2S)e^{i(t-S)}. \quad (73) \]

Figure 3 outlines the plot of the wave forms of the absolute, real, and imaginary parts of the complex bright soliton that is given by equation (72). Also for the same values \( k_1 = 1, \omega_1 = 1, \omega_2 = 2 \), equation (71) has a periodic solution. Based on equation (70), we write the solution as
\[ \psi(t,S) = \frac{16}{3} \cos \left( \frac{\sqrt{2}}{2} (4t + S) \right) e^{i(t-S)}. \quad (74) \]

Or
\[ \psi(t,S) = -\frac{16}{3} \cos \left( \frac{\sqrt{2}}{2} (4t + S) \right) e^{i(t-S)}. \quad (75) \]

Figure 4 clarifies the graph of the wave forms of the absolute, real, and imaginary parts of the complex periodic that is given by equation (74). Figure 4 clarifies the graph of the wave forms of the absolute, real, and imaginary parts of the complex periodic that is given by equation (74).

Now we study the influence of the two parameters \( \beta \) and \( \sigma^2 \) on the obtained solutions in section 4, individually, by solving the dynamical system (29) using a Fehlberg fourth-fifth-order Runge–Kutta method by aiding the Maple program. For more details about this study, as shown in [86].

Figure 5 outlines the changes in the super-wave solution (44) due to the changes of one of the parameters \( \alpha \) and \( \beta \). In the case where \( \sigma^2 \) remains constant and \( \beta \) decreases, the amplitude and width of the solution (44) will both be increased, as shown in Figure 5(a). The amplitude (width) of the solution (44) increase (decrease) if \( \beta \) remains constant while \( \sigma^2 \) decreases, as shown in Figure 6(b).

Figure 6 illustrates the effect of the changes in the two parameters \( \beta \) and \( \sigma^2 \) on the solitary solution (46). Figure 6(a) shows that both the amplitude and the width of the solitary solution (46) increase if \( \sigma^2 \) is still constant and \( \beta \) decreases. Figure 6(b) illustrates the amplitude (width) of the solitary solution (46) decreases (increases) if \( \beta \) takes a fixed constant value while \( \sigma^2 \) decreases.

**Remark 1.** Note that the two Figures 5 and 6 are plotted for the same values of the two parameters \( \beta \) and \( \sigma^2 \), but they are different in initial conditions. Equivalently, Figure 5 is plotted, taking into account certain initial conditions making the value
Figure 3: The behavior of the bright soliton in equation (72). (a) The absolute part, (b) The real part, (c) The imaginary part.

Figure 4: The behavior of the periodic wave solution (74). (a) The absolute part, (b) The real part, (c) The imaginary part.

Figure 5: The behavior of the super periodic wave solution (44). (a) The influence of $\beta$ where $\sigma^2$ is fixed. (b) The influence of $\sigma^2$ where $\beta$ is fixed.
of $h$ positive; while Figure 6 is sketched, regarding with some initial conditions making $h = 0$. Figure 7 clarifies the dependence of the kink solution (48) on the two parameters $\beta$ and $\sigma^2$. As shown in the Figures 7(a) and 7(b), the amplitude of the kink solution (48) increases whether $\sigma^2$ remains fixed and $\beta$ increases or $\beta$ remains unchanged while $\sigma^2$ decreases.

7. Conclusion

This work has been concerned with investigating the Ivancevic option pricing model that comes as an alternative to the Black–Scholes model and depicts a controlled Brownian motion associated with the nonlinear Schrödinger equation from diverse aspects. Our objective here is to examine the integrability of the equation in the Painlevé sense, construct wave solutions and analyze how the parameters $\sigma^2$ and $\beta$ affect the result. Let us display our main results. Applying the well-known ARS algorithm, we have proved equation (4) is not integrable in the Painlevé sense. He's variational method has been utilized to find some new abundant solutions such as the bright soliton, bright-like soliton, kinky-bright soliton, and periodic solution. We have
applied the bifurcation analysis in order to examine the phase portrait and to study some dynamical behavior of this model. Based on the bifurcation analysis, we have introduced a classification of the wave solutions into periodic, super periodic, kink, and solitary solutions according to the type of the phase plane orbit. We have illustrated some of the obtained solution graphically. Moreover, we have studied the influence of the two parameters $\beta$ and $\sigma^2$ on the obtained solutions. If $\sigma^2$ remains constant and $\beta$ decreases, the amplitude and width of the solution (44) will both be increased; see Figure 5(a). The amplitude (width) of the solution (44) increase (decrease) if $\beta$ remains constant while $\sigma^2$ decreases; see Figure 6(b). We have shown the both the amplitude and the width of the solitary solution (46) increase if $\sigma^2$ still constant and $\beta$ decreases. Figure 6(b) illustrates the amplitude (width) of the solitary solution (46) decreases (increases) if $\beta$ takes a fixed constant value while $\sigma^2$ decreases. We have illustrated the amplitude of the kink solution (3.24) increases whether $\sigma^2$ remains fixed and $\beta$ increases or $\beta$ remains unchanged while $\sigma^2$ decreases.

The bifurcation approach can be used to solve higher order NLPDEs, but it gets challenging when the Hamiltonian has more than one dimension. Let us clarify the situation for a 2D-Hamiltonian system as an example. It is integrable if it has two independent first integrals of the motion (one of them is Hamiltonian) which can be solved together to get certain solutions for the given NLPDE. The problem of finding the second first integral beside Hamiltonian is a very hard task and there are no systematic methods generally to find it.

Data Availability
The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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