HYBRID DIRAC FIELDS

O.W. Greenberg

Center for Theoretical Physics
Department of Physics
University of Maryland
College Park, MD 20742-4111

Abstract

Hybrid Dirac fields are fields that are general superpositions of the annihilation and creation parts of four Dirac spin 1/2 fields, $\psi^{(\pm)}(x; \pm m)$, whose annihilation and creation parts obey the Dirac equation with mass $m$ and mass $-m$. We discuss a specific case of such fields, which has been called “homeotic.” We show for this case, as is true in general for hybrid Dirac fields (except the ordinary fields whose annihilation and creation parts both obey one or the other Dirac equation), that (1) any interacting theory violates both Lorentz covariance and causality, (2) the discrete transformations $C$, and $CPT$ map the pair $\psi_h(x)$ and $\bar{\psi}_h(x)$ into fields that are not linear combinations of this pair, and (3) the chiral projections of $\psi_h(x)$ are sums of the usual Dirac fields with masses $m$ and $-m$; on these chiral projections $C$, and $CPT$ are defined in the usual way, their interactions do not violate $CPT$, and interactions of chiral projections are Lorentz covariant and causal. In short, the main claims concerning “homeotic” fields are incorrect.

1. Introduction

Since a spin 1/2 field on which parity is defined can obey either of two Dirac equations

$$ (i \not{\partial} - m)\psi(x; m) = 0 $$

or

$$ (i \not{\partial} + m)\psi(x; -m) = 0 $$

1email address, owgreen@physics.umd.edu.
and since the positive (annihilation) and negative (creation) frequencies of a free field can be separated in a Lorentz covariant way, one can consider a family of free “hybrid Dirac fields” which, with suitable normalizations, are linear combinations of the annihilation and creation parts of mass $m$ and mass $-m$ Dirac fields. This type of field, with a specific choice given below, was considered in a recent paper by G. Barenboim and J. Lykken [1] (BL) in connection with a proposed model of CPT violation for neutrinos. They called their field a “homeotic” field. The purpose of this paper is to study the properties of this type of field which, for reasons stated above, I prefer to call a “hybrid Dirac field.” BL proposed their “homeotic” field as a counter-example to my general theorem [2] that interacting fields that violate CPT symmetry necessarily violate Lorentz covariance. I will show below that the BL example does not violate CPT and that their interacting “homeotic” field violates both Lorentz covariance and causality.

The simplest representation of the BL field is

$$\psi_h(x) = \psi^+(x; m) + \psi^-(x; -m),$$

(3)

where to be explicit and to establish notation,

$$\psi^+(x; \pm m) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \sum_{\pm s} b(p, s) u(p, s; \pm m) \exp(-ip \cdot x),$$

(4)

$$\psi^-(x; \pm m) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \sum_{\pm s} d^\dagger(p, s) v(p, s; \pm m) \exp(ip \cdot x),$$

(5)

$$\bar{\psi}^-(x; \pm m) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \sum_{\pm s} b^\dagger(p, s) \bar{u}(p, s; \pm m) \exp(ip \cdot x),$$

(6)

$$\bar{\psi}^+(x; \pm m) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2E_p} \sum_{\pm s} d(p, s) \bar{v}(p, s; \pm m) \exp(-ip \cdot x).$$

(7)

Note the difference between $\bar{\psi}^+(x) = \bar{\psi}^-(x)$ and $\bar{\psi}^+(x)$, etc. To separate $\psi_h(x)$ into $\psi^+(x; m)$ and $\psi^-(x; -m)$, use $[(\pm i \partial_x + m)/2m] \psi_h(x) = \psi^{(\pm)}(x; \pm m); b(p, s)$, etc., can then be calculated in the usual way. The annihilation and creation operators are normalized covariantly,

$$[b(p, s), b^\dagger(q, t)]_+ = 2E_p \delta(p - q) \delta_{st},$$

(8)
etc., and the spinors obey

\[(\not p \mp m) u(p, s; \pm m) = 0, \quad (9)\]

\[(\not p \pm m) v(p, s; \pm m) = 0, \quad (10)\]

with normalizations \(\bar{u}(p, s; \pm m) u(p, t; \pm m) = \pm 2 m \delta_{st}\) and \(\bar{v}(p, s; \pm m) v(p, t; \pm m) = \mp 2 m \delta_{st}\). I chose these normalizations so that going from \(\psi^{(\pm)}(x; \pm m)\) to \(\psi^{(\pm)}(x; \mp m)\) is accomplished by changing the sign of \(m\). (This would not be the case if I had chosen \(\bar{u}u = \delta_{st}\), for example.) I use the same non-chiral Dirac matrices of [3] and [4] and the same creation and annihilation operators for both the \(m\) and \(-m\) fields. With this choice of gamma matrices, the \(m\) \(u\) spinors have large upper two components and the \(-m\) \(u\) spinors have large lower two components, and vice versa for the \(v\) spinors. The reversal of the signs of the normalizations of the \(u\) and \(v\) spinors for the \(-m\) case may seem strange; however these normalizations are consistent with the anticommutation relation for the \(-m\) field, with the positivity of energy (note that the \(u^\dagger u\) normalization is positive for both the \(m\) and \(-m\) spinors) and with the relation between \(m\) and \(-m\) fields via \(\gamma^5\) given below. If I had started with the \(-m\) field rather than with the \(m\) field, I could have chosen gammas so that the \(-m\) \(u\) spinors would have positive normalization.

(Other linear combinations of the positive and negative frequency parts of Dirac fields of mass \(m\) and mass \(-m\) also fall into the category of hybrid Dirac fields as do the usual Dirac fields, but in this paper I discuss only the example given above). I follow the universal convention, [3], [4] and [5], that the \(p^0 = E_p = \sqrt{m^2 + p^2}\) in spinors is always the positive energy; unfortunately BL violate this convention.

To see that my representation of \(\psi_h(x)\) is identical with the BL field, calculate the anticommutator

\[\left[\psi_{h\alpha}(x), \bar{\psi}_{h\beta}(y)\right]_+ = (i \not \partial_x + m)_{\alpha\beta} \Delta^{(+)}(x - y) - (i \not \partial_x - m)_{\alpha\beta} \Delta^{(-)}(x - y), \quad (11)\]

where

\[\Delta^{(+)}(x) = \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2E_p} \exp(-ip \cdot x), \quad \Delta^{(-)}(x) = \Delta^{(+)}(-x). \quad (12)\]

This agrees with (2.14) of BL, (their \(D(x) = \Delta^{(+)}(x)\)), except that their Dirac indices are transposed; however, this differs from the Dirac result by the sign of the
second term proportional to \( m \), not that of the first term, as stated by BL. It is instructive to rewrite this result as

\[
[\psi_{\alpha}(x), \bar{\psi}_{\beta}(y)]_+ = (i \partial_x)_{\alpha\beta} \Delta(x - y) + m \delta_{\alpha\beta} \Delta^{(1)}(x - y),
\]

where \( i\Delta(x) = \Delta^{(+)} - \Delta^{(-)} \) and \( \Delta^{(1)} = \Delta^{(+)} + \Delta^{(-)} \), since this separates the local and nonlocal terms.

In addition, \( i \partial \psi_{\alpha}(x) = m (\psi^{(+)}(x; m) - \psi^{(-)}(x; -m)) \); it is straightforward to check that

\[
m(\psi^{(+)}(x; m) - \psi^{(-)}(x; -m)) = \frac{-im}{\pi} \mathcal{P} \int dt' \frac{1}{x^0 - t'} \psi_{\alpha}(t', x)
\]

(provided the integral is defined suitably). Thus the present \( \psi_{\alpha} \) obeys the equation of motion (2.7) of BL.

In agreement with my result \([2]\) and with BL, free, or generalized free, hybrid Dirac fields transform covariantly. This is connected with the existence of \( \theta(\pm p^0) \) which exists for timelike momenta, but does not exist for spacelike momenta. I also agree with BL that the propagator of their field \( \psi_{\alpha}(x) \) is not causal; it is also not covariant.

From Eq. (12), \( \gamma^5 \psi(x; \pm m) \) obeys the Dirac equation for \( \psi(x; \mp m) \), so it is useful to consider the discrete transformation

\[
\mathcal{M}\psi_{\alpha}(x; m)\mathcal{M}^\dagger = \psi_{\alpha}(x; -m) = \gamma^5 \psi_{\alpha}(x; m),
\]

\[
\mathcal{M}\psi_{\alpha}(x; -m)\mathcal{M}^\dagger = \psi_{\alpha}(x; m) = \gamma^5 \psi_{\alpha}(x; -m).
\]

This shows that the \( m \) and \( -m \) spinors are related by \( \gamma^5 \).

The calculation of the BL current is particularly simple for the space components. The result is

\[
[J^i(x), J^j(y)]_- = \bar{\psi}_{\alpha}(x) \gamma^i \partial_x \Delta(x - y) \gamma^j \psi_{\alpha}(y) - \bar{\psi}_{\alpha}(y) \gamma^j \partial_y \Delta(y - x) \gamma^i \psi_{\alpha}(x)

+ m \{-\delta^{ij} [\bar{\psi}_{\alpha}(x) \psi_{\alpha}(y) - \bar{\psi}_{\alpha}(y) \psi_{\alpha}(x)]

- i[\bar{\psi}_{\alpha}(x) \sigma^{ij} \psi_{\alpha}(y) + \bar{\psi}_{\alpha}(y) \sigma^{ij} \psi_{\alpha}(x)] \}\Delta^{(1)}(x - y).
\]

As we expect, the term proportional to the gradient which is the same for \( m \) and for \(-m\) is local, but the term proportional to \( m \) is proportional to \( \Delta^{(1)}(x - y) \) and
is not local. In particular,

\[ \Delta^{(1)}(0, \mathbf{x}) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_p} \cos \mathbf{p} \cdot \mathbf{x} \neq 0, \mathbf{x} \neq 0. \]  

(18)

Note that for the space indices \( i, j \) the BL current is the same as what they call the “Dirac” current. BL correctly point out that causality fails to hold for their “Dirac” current, their Eq.(3.11), but do not notice that this fact directly contradicts their (incorrect) equal time current commutation relations for the space indices, their Eq.(3.3). Thus the statements that BL make concerning the causality condition (their Eq.(3.7))

\[ \left[ \mathcal{H}_I(x), \mathcal{H}_I(y) \right] \sim \delta(x - y), \]  

(19)

the Lorentz covariance of the time-ordering in the Dyson series, and the Lorentz invariance of their \( S \)-matrix are all incorrect and their interacting theory is neither Lorentz covariant nor causal.

In contrast to my disagreement with the assertions concerning equal time commutation relations for the currents and the Hamiltonian density, I agree with BL about the equal time commutation relations for the chiral projections of the hybrid fields. The agreement here is because the chiral projections of the hybrid fields are sums of the usual Dirac fields with masses \( m \) and \(-m\). These chiral projections are not hybrid (or homeotic) fields at all!

I agree with BL that parity and time reversal are realized in the usual way. BL assert that charge conjugation is realized in a different way than usual. This is incorrect. This assertion seems to be based on the tacit assumption that \( C \) and \( CPT \) map the pair \( \psi_h(x) \) and \( \bar{\psi}_h(x) \) onto themselves. Unexpectedly, this is not the case. The discrete transformations \( C \) and \( CPT \) map the terms in \( \psi_h(x) \) into terms that are not present in \( \bar{\psi}_h(x) \) and map the terms in \( \bar{\psi}_h(x) \) into terms that are not present in \( \psi_h(x) \).

Although it is well known, I emphasize that the requirement that charge conjugation changes the sign of the field and the requirement that charge conjugation interchanges particle and antiparticle are equivalent [6, 7, 8]. Here is the standard argument concerning charge conjugation [4]. Thus if \( \psi(x; m) \) obeys

\[ [i \partial_x - e A(x) - m] \psi(x; m) = 0, \]  

(20)
then the charge conjugate field $\psi^c(x; m)$ must obey
\[ [i \partial_x + e A(x) - m]\psi^c(x; m) = 0; \] (21)
and if $\psi(x; -m)$ obeys
\[ [i \partial_x - e A(x) + m]\psi(x; -m) = 0, \] (22)
then the charge conjugate field $\psi^c(x; -m)$ must obey
\[ [i \partial_x + e A(x) + m]\psi^c(x; -m) = 0. \] (23)

For example, take the adjoint of both sides of Eq. (20) to get
\[ \psi^\dagger(x; m)[-i \partial^\dagger_x - e A^\dagger(x) - m] = 0 \] (24)
where the derivative acts to the left. Next multiply from the right by $\gamma^0$ to get
\[ \bar{\psi}(x; m)[-i \partial_x - e A(x) - m] = 0. \] (25)

Next transpose the equation to get
\[ [-i \partial^T_x - e A(x)^T - m]\bar{\psi}^T(x; m). \] (26)

Finally multiply by the usual $C$ matrix that obeys $C\gamma^\mu T C^\dagger = -\gamma^\mu$ to get
\[ [i \partial_x + e A(x) - m]C\bar{\psi}^T(x; m) = 0. \] (27)

Thus $\psi^c(x; m) = C\bar{\psi}^T(x; m)$. This relation holds separately for the annihilation and creation parts,
\[ C\psi^{(\pm)}(x; m)C^\dagger = \psi^{c(\pm)}(x; m) = C\bar{\psi}^{(\pm)T}(x; m), \] (28)
and analogous relations hold for the relation of $\psi^{(\pm)}(x; -m)$ to $\psi^{c(\pm)}(x; -m)$ up to a minus sign.

Since
\[ \psi_h(x) = \psi^{(+)}(x; m) + \psi^{(-)}(x; -m), \] (29)
charge conjugation takes $\psi_h(x)$ to
\[ \psi_h^c(x) = C\bar{\psi}^{(+)}T(x; m) + C\bar{\psi}^{(-)T}(x; -m). \] (30)
The Pauli adjoint of $\psi_h(x)$ is

$$\bar{\psi}_h(x) = \bar{\psi}^{(+)}(x; -m) + \bar{\psi}^{(-)}(x; m);$$

(31)

thus neither of the terms in $\psi_h^c(x)$ is present in $\bar{\psi}_h(x)$. Another way to look at this is to note that charge conjugation takes, for example, a $b$ annihilation operator to a $d$ annihilation operator; however, the $b$ operator is in $\psi^{(+)}(x; m)$, while the $d$ operator is in $\bar{\psi}^{(+)}(x, m)$, therefore it takes $\psi^{(+)}(x; m)$ to $\bar{\psi}^{(+)}(x; m)$, which does not appear in $\bar{\psi}_h(x)$. A third way to see this is to note that the $b$ annihilation operator in $\psi_h$ is associated with a $u(p, s; m)$ spinor that has large upper components and should be transformed into $Cu^T(p, s; m)$ which has large lower components, while the $d$ annihilation operator in $\bar{\psi}^{(x; -m)}$ is associated with a $\bar{v}(p, s; -m)$ spinor that has large upper components. Thus the discrete transformations $C$, and $CPT$ map the pair $\psi_h(x)$ and $\bar{\psi}_h(x)$ into other fields that are not linear combinations of this pair, and the statements of BL concerning $C$ and $CPT$ are incorrect. (I have suppressed the usual phases that can accompany the definitions of the discrete transformations. The reader who wishes can supply these phases; the conclusions remain unchanged.)

Independent of the discussion just given above, it is important to point out that $CPT$ has a more basic role in relativistic quantum field theory than any of the other discrete transformations $C$, $P$, $T$ or their bilinear products. $CPT$ is the unique discrete symmetry that can be connected to the identity when the proper orthochronous Lorentz group, $L_+^\dagger$, and its associated covering group, $SL(2, C)$, are enlarged to the proper complex Lorentz group, $L_+(C)$, and its covering group, $SL(2, C) \otimes SL(2, C)$. This is not to say that Lorentz invariance alone leads to $CPT$ symmetry. In order for Lorentz invariance to imply $CPT$ symmetry it is necessary and sufficient that a relaxed form of spacelike commutativity (or anticommutativity) called “weak local commutativity” should hold [9]. This last remark shows why a free field with different masses for particle and antiparticle can be Lorentz invariant on-shell and yet not obey $CPT$ symmetry. The reason is that such a field does not obey weak local commutativity. Of course the Green’s functions of such a field will not be Lorentz invariant.

In terms of the irreducible representations of $L_+^\dagger$

$$\psi(x; m) = \psi_{(1/2, 0)} \oplus \psi_{(0, 1/2)}$$

(32)
\[ \psi(x; -m) = \psi_{(1/2,0)} \Theta \psi_{(0,1/2)}, \] (33)

where \((1/2, 0)\) is the representation with one undotted index and \((0, 1/2)\) is the representation with one dotted index in van der Waerden’s notation [11]. See also [9, 12].

The annihilation and creation parts of these fields each have the corresponding decomposition in irreducibles of \(SL(2, \mathbb{C})\). As stated above one can define \(CPT\) without ever considering the individual discrete symmetries. Pauli [10] showed that \(CPT\) takes each irreducible representation of the homogeneous Lorentz group (without discrete symmetries) into the adjoint of the same irreducible. See also [13]. Thus one can consider \(CPT\) acting on each of the four \(\psi^{(\pm)}(x; \pm m)\) as well as on each of their decompositions into irreducibles separately, regardless of whether or not any of the other ones are added to it to form a hybrid (or homeotic) field.

Using the relation \(\psi(x; -m) = \gamma^5 \psi(x; m)\) we can induce the discrete transformations of the \(-m\) fields from those of the \(m\) fields. Thus

\[ \mathcal{P}\psi(x; -m) = -\gamma^0 \psi(i_x x; -m), \quad i_x x = (x^0, -x^i) \] (34)

\[ \mathcal{C}\psi(x; -m) = i \gamma^2 \psi^T(x; -m), \] (35)

\[ \mathcal{T}\psi(x; -m) = i \gamma^1 \gamma^3 \psi(i_x x; -m), \quad i_x x = (-x^0, x^i) \] (36)

Because \(CPT\) does not map hybrid Dirac fields and their adjoints onto themselves, if the hybrid Dirac field is coupled linearly to a usual Dirac field, the resulting bilinear term violates \(CPT\) and produces an interaction that violates both Lorentz covariance in the sense that its \(T\)-product will not be covariant, and causality in the sense that it fails to commute at spacelike separation. By contrast, the chiral projections of hybrid Dirac fields are sums of the usual Dirac fields with mass \(m\) and \(-m\). For example,

\[
\frac{1 + \gamma^5}{2} \psi^h(x) = \frac{1 + \gamma^5}{2} \left( \psi^+(x; m) + \psi^-(x; m) \right),
\]

\[
= \frac{1 + \gamma^5}{2} \psi(x; m) = \frac{1}{2} (\psi(x; m) + \psi(x; -m))
\] (37)

and

\[
\frac{1 - \gamma^5}{2} \psi^h(x) = \frac{1 - \gamma^5}{2} \left( \psi^+(x; m) - \psi^-(x; m) \right).
\] (38)
(The relative minus sign between $\psi^+(x; m)$ and $\psi^-(x; m)$ in this last equation is not significant since $\psi^+(x; m) \pm \psi^-(x; m)$ both have the same anticommutation relations with their Pauli adjoints and thus have the same contractions, so all their observable expectation values are the same.) $CPT$ acts in the usual way on both $\psi(x; m)$ and $\psi^+(x; m) - \psi^-(x; m)$; thus the bilinear terms that couple the chiral projections of $\psi_h$ to a chiral Dirac field preserve $CPT$ and are Lorentz invariant and causal. This means that the terms in (4.1) of BL do not violate $CPT$ and thus their model fails as an example of $CPT$ violation.

Conclusions: Although free hybrid (or “homeotic”) Dirac fields can be Lorentz covariant on-shell, interacting ones necessarily violate Lorentz covariance in agreement with the theorem in [2]. Such fields also violate causality. Free chiral hybrid Dirac fields are sums of the usual Dirac fields with masses $m$ and $-m$ and because of that they can be local and Lorentz covariant. Further, since they are sums of the usual Dirac fields they must have the usual $CPT$ transformation. This means that their bilinear coupling to a usual chiral Dirac field does not violate $CPT$. The suggestion that “acausal propagation combined with nonlocal interactions yield a causal theory” [11] is incorrect. It seems unlikely that hybrid (or “homeotic”) Dirac fields will be of phenomenological importance.

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