THE ELLIPTIC CASIMIR CONNECTION OF A SIMPLE LIE ALGEBRA

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ABSTRACT. We construct a flat connection on the elliptic configuration space associated to any complex semisimple Lie algebra \( g \). This elliptic Casimir connection has logarithmic singularities, and takes values in the deformed double current algebra of \( g \) defined by Guay [19, 20]. It degenerates to the trigonometric Casimir connection of \( g \) constructed by the first author in [38]. By analogy with the rational and trigonometric cases, we conjecture that the monodromy of the elliptic Casimir connection is described by the quantum Weyl group operators of the quantum toroidal algebra of \( g \).

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The first author was supported in part through the NSF grant DMS–1505305.
1. Introduction

1.1. Motivation. Around 1990, Drinfeld proved that the $R$ matrix of quantum groups describes the monodromy of the Knizhnik–Zamolodchikov (KZ) equations [12]. Subsequently, Millson–Toledano Laredo [31, 34], and independently De Concini (unpublished, 1995) and G. Felder et al. [15] introduced another flat connection $\nabla_C$, the Casimir connection of a complex, semisimple Lie algebra $\mathfrak{g}$. The latter connection is distinct from, but dual to the KZ connection, and is described as follows.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\Phi \subset \mathfrak{h}^*$ the corresponding root system, and $\mathfrak{h}_{\text{reg}} \subset \mathfrak{h}$ the complement of the root hyperplanes in $\mathfrak{h}$. For a finite–dimensional $\mathfrak{g}$–module $V$, the Casimir connection $\nabla_C$ is the connection on the holomorphically trivial vector bundle on $\mathfrak{h}_{\text{reg}}$ with fibre $V$ given by

$$\nabla_C = d - \hbar \sum_{\alpha \in \Phi^+} \frac{d\alpha}{\alpha} \kappa_{\alpha},$$

where $\hbar \in \mathbb{C}$ is a deformation parameter, the summation is over a chosen system $\Phi^+$ of positive roots, and $\kappa_{\alpha} \in U_{\mathfrak{s}\mathfrak{l}_2} \subset U\mathfrak{g}$ is the truncated (i.e. Cartan–less) Casimir operator of the sl$_2$–subalgebra of $\mathfrak{g}$ corresponding to the root $\alpha$. This connection is flat and equivariant with respect to the Weyl group $W$, and therefore gives rise to a one–parameter family of monodromy representations of the generalized braid group $B_{\mathfrak{g}} = \pi_1(\mathfrak{h}_{\text{reg}}/W)$ on $V$.

In this setting, a Drinfeld–Kohno theorem was obtained by the first author [34, 36, 37, 39], according to which the monodromy of the Casimir connection of $\mathfrak{g}$ is described by the quantum Weyl group operators of the quantum group $U_{\hbar}\mathfrak{g}$. This result was recently generalised to an arbitrary symmetrisable Kac–Moody algebra by Appel–Toledano Laredo [1, 2, 3, 4].

In related work, the first author constructed a trigonometric version of $\nabla_C$ [38]. Let $G$ be the simply–connected complex Lie group with Lie algebra $\mathfrak{g}$, $H \subset G$ the maximal torus with Lie algebra $\mathfrak{h}$, and $H_{\text{reg}} = H \setminus \bigcup_{\alpha \in \Phi} \{ \alpha \mathfrak{h} = 1 \}$ the complement of the root hypertori in $H$. The trigonometric Casimir connection $\nabla_{\text{trig},C}$ is a connection on the trivial vector bundle $H_{\text{reg}} \times V$, where the fiber $V$ is a finite–dimensional representation of the Yangian $Y_{\hbar}\mathfrak{g}$, which is a deformation of the current algebra $U(\mathfrak{g}[s])$ of $\mathfrak{g}$. Let $\{u_i\}$, and $\{u'\}$ be dual bases of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively. Then, $\nabla_{\text{trig},C}$ is given by

$$\nabla_{\text{trig},C} = d - \hbar \sum_{\alpha \in \Phi^+} \frac{d\alpha}{\alpha} \kappa_{\alpha} - du_i X(u')$$

where $du_i$ is regarded as a translation invariant 1–form on $H$, $X : \mathfrak{h} \to Y_{\hbar}\mathfrak{g}$ is a linear map such that $X(u) \equiv u \otimes s \mod \mathfrak{h}$, and the summation over $i$ is implicit.

The connection $\nabla_{\text{trig},C}$ is flat and $W$–equivariant. Its monodromy yields a one–parameter family of monodromy representations of the affine braid group $B_{\mathfrak{g}}^{\text{aff}} = \pi_1(H_{\text{reg}}/W)$. By analogy with [34], Toledano Laredo conjectured that the monodromy of $\nabla_{\text{trig},C}$ is described by the quantum Weyl group operators of the quantum loop algebra $U_{\hbar}L_{\mathfrak{g}}$ [38]. This conjecture was formulated more precisely in subsequent work of Gautam–Toledano Laredo, and reduced to the case of $\mathfrak{g} = \mathfrak{sl}_2$ [17, 18]. Moreover, for $\mathfrak{sl}_2$, it was proved for the tensor product of evaluation modules in [16]. Work in progress of Bezrukavnikov–Okounkov also proves this conjecture for representations of $Y_{\hbar}\mathfrak{g}$ arising from geometry [7].

The main goal of the present paper is to construct an elliptic analogue of the Casimir connection, and to give a conjectural description of its monodromy.

1.2. The universal KZB connection of a root system. The construction of the elliptic Casimir connection relies on the universal elliptic connection associated to an arbitrary finite (reduced, crystallographic) root system $\Phi$ obtained in [40], which we review below.

Let $Q \subset \mathfrak{h}^*$ and $Q^\vee \subset \mathfrak{h}$ be the root and coroot lattices of $\Phi$ respectively. Let $\tau$ be a point in the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$, and set $\Lambda_{\tau} = \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$. Consider the elliptic curve $E_{\tau} := \mathbb{C}/\Lambda_{\tau}$ with
modular parameter \( \tau \). Set \( T = h/(Q' + \tau Q') \). Any root \( \alpha \in \Phi \) induces a map \( \chi_\alpha : T \to \mathcal{E}_\tau \), with kernel \( T_\alpha \).

We refer to \( T_{\text{reg}} = T \setminus \bigcup_{\alpha \in \Phi} T_\alpha \) as the elliptic configuration space associated to \( \Phi \). The fundamental group \( \pi_1(T_{\text{reg}}/W) \) is the elliptic braid group.

Let \( \theta(z|\tau) \) be the Jacobi theta function, which is a holomorphic function \( \mathbb{C} \times \mathbb{H} \to \mathbb{C} \), whose zero set is \( \{z \mid \theta(z|\tau) = 0\} = \Lambda_\tau \) and such that its residue at \( z = 0 \) is 1 (see Section \$2.2\). Let \( x \) be another complex variable, and set

\[
k(z, x|\tau) := \frac{\theta(z + x|\tau)}{\theta(z|\tau) \theta(x|\tau)} - \frac{1}{x}.
\]

The function \( k(z, x|\tau) \) has only simple poles at \( z \in \Lambda_\tau \), and is regular near \( x = 0 \). It may therefore be regarded as an element of \( 1 + x \text{Hol}(\mathbb{C} - \Lambda_\tau)[[x]] \).

Let \( A \) be an algebra endowed with the following data: a set of elements \( \{t_\alpha\}_{\alpha \in \Phi} \), such that \( t_{-\alpha} = t_\alpha \), and two linear maps \( x : h \to A, y : h \to A \). Consider the following \( A \)-valued meromorphic connection on \( h \).\(^1\)

\[
\nabla_{\text{KZB},\tau} = d - \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}(\frac{x_\alpha}{2})|\tau)(t_\alpha) d\alpha + y(\alpha') du_i \tag{1}
\]

where \( x_\alpha = x(\alpha') \). When \( \Phi \) is the root system of type \( A_\ell \), the connection above coincides with the universal KZB connection introduced by Calaque–Enriquez–Etingof in [8]. In [40], we proved the following.

**Theorem 1.1.** The connection \( \nabla_{\text{KZB},\tau} \) is flat if and only if the following relations hold in \( A \)

1. For any rank 2 root subsystem \( \Psi \) of \( \Phi \), and \( \alpha \in \Psi \),
   \[
   [t_\alpha, \sum_{\beta \in \Psi^+} t_\beta] = 0.
   \]

2. For any \( u, v \in h \)
   \[
   [x(u), x(v)] = 0 = [y(u), y(v)].
   \]

3. For any \( u, v \in h \),
   \[
   [y(u), x(v)] = \sum_{\gamma \in \Phi^+} \langle \gamma, \gamma \rangle(u, y)t_\gamma.
   \]

4. For any \( \alpha \in \Phi \) and \( u \in h \) such that \( \alpha(u) = 0 \),
   \[
   [t_\alpha, x(u)] = 0 = [t_\alpha, y(u)].
   \]

If, moreover, the Weyl group \( W \) of \( \Phi \) acts on \( A \), then \( \nabla_{\text{KZB},\tau} \) is \( W \)-equivariant if and only if

5. For any \( w \in W, \alpha \in \Phi, \) and \( u, v \in h \),
   \[
   w(t_\alpha) = t_{w\alpha}, \quad w(x(u)) = x(wu), \quad \text{and} \quad w(y(v)) = y(wv)
   \]

In what follows, we denote by \( t^\Phi_{\text{tel}} \) the Lie algebra defined by the relations (1)–(4) of Theorem 1.1, endowed with the action of \( W \) given by relation (5). \( t^\Phi_{\text{tel}} \) has an \( \mathbb{N} \)-bigraded given by \( \text{deg}(x(u)) = (1, 0), \text{deg}(y(v)) = (0, 1) \) and \( \text{deg}(t_\alpha) = (1, 1) \).

\(^1\)We are assuming further that \( A \) is a topological algebra such that the infinite sums \( k(\alpha, \text{ad}(\frac{x_\alpha}{2})|\tau)(t_\alpha) \) converge. This is the case for example if \( A \) is complete with respect to a descending filtration, and \( x(u) \) is of positive degree for any \( u \in h \).
1.3. **The deformed double current algebra** $D_\lambda(\mathfrak{g})$. The elliptic Casimir connection constructed in this paper is obtained by specialising the universal connection (1), that is mapping the Lie algebra $\mathfrak{t}_\mathfrak{u}^\Phi$ to an appropriate associative algebra in a $W$–equivariant way. The correct algebra turns out to be the deformed double current algebra $D_\lambda(\mathfrak{g})$ of $\mathfrak{g}$ introduced by Guay [19, 20].

$D_\lambda(\mathfrak{g})$ is an algebra over $\mathbb{C}[\lambda]$, where $\lambda$ is a formal parameter, which deforms the universal central extension of the double current algebra $\mathfrak{g}[u,v]$. It was introduced in [19] for $\mathfrak{g} = \mathfrak{sl}_n$ with $n \geq 4$ and in [20] for an arbitrary simple Lie algebra $\mathfrak{g}$ of rank $\geq 3$, but $\mathfrak{g} \neq \mathfrak{sl}_3$. It is obtained by degenerating the defining relations of the affine Yangian of $\mathfrak{g}$, see [19, Thm. 12.1] and [20, Thm. 5.5]. This degeneration yields a presentation of $D_\lambda(\mathfrak{g})$ which is similar to the Kac–Moody presentation of an affine Lie algebra. A second, double loop, presentation of $D_\lambda(\mathfrak{g})$ is obtained in [20, 22]. It involves two current algebras in one variable which play a symmetric role. Deformed double current algebras for $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ were not defined in [19] due to subtleties arising in small rank. A definition for $\mathfrak{sl}_2, \mathfrak{sl}_3$ will be proposed in [23]. A definition of $D_\lambda(\mathfrak{g})$ in type $B_2$ and $G_2$ is currently unavailable.

In the current paper, we assume that $\text{rank}(\mathfrak{g}) \geq 3$. The following is the second presentation of the deformed double current algebra $D_\lambda(\mathfrak{g})$, when $\mathfrak{g} \neq \mathfrak{sl}_3$. For $\mathfrak{g} = \mathfrak{sl}_3$, more relations need to be imposed in Definition 1.2 so that $D_\lambda(\mathfrak{g})$ is a flat deformation of the universal central extension of $\mathfrak{g}[u,v]$. All other statements in the current paper hold for $\mathfrak{g} = \mathfrak{sl}_3$.

**Definition 1.2.** [22, Def. 2.2] The deformed double current algebra $D_\lambda(\mathfrak{g})$ is generated by elements $X, K(X), Q(X), P(X), X \in \mathfrak{g}$, such that

1. $X, K(X)$ generate a subalgebra which is an image of $\mathfrak{g}[v]$ under the map $X \otimes v \mapsto K(X)$
2. $X, Q(X)$ generate a subalgebra which is an image of $\mathfrak{g}[u]$ under the map $X \otimes u \mapsto Q(X)$
3. $P(X)$ is linear in $X$, and for any $X, X' \in \mathfrak{g}$, $[X, P(X')]=P[X,X']$

and the following relations hold for all root vectors $X_{\beta_1}, X_{\beta_2} \in \mathfrak{g}$ with $\beta_1 \neq -\beta_2$:

$$[K(X_{\beta_1}), Q(X_{\beta_2})] = P([X_{\beta_1}, X_{\beta_2}]) - \lambda \frac{(\beta_1, \beta_2)}{4} S(X_{\beta_1}, X_{\beta_2}) + \frac{1}{2} \sum_{\alpha \in \Phi} S([X_{\beta_1}, X_{\alpha}], [X_{-\alpha}, X_{\beta_2}]),$$

where $S(a_1, a_2) = a_1 a_2 + a_2 a_1 \in U_{\mathfrak{g}}$.

Note that $D_\lambda(\mathfrak{g})$ is $\mathbb{N}$–bigraded by

$$\text{deg}(X) = (0,0), \; \text{deg}(K(X)) = (1,0), \; \text{deg}(Q(X)) = (0,1) \; \text{and} \; \text{deg}(P(X)) = (1,1) = \text{deg}(\lambda)$$

Further, a central element $Z \in D_\lambda(\mathfrak{g})$ is constructed in [22, Prop. 4.1], which is given by

$$Z := \frac{1}{(\beta, \beta)} \left( [K(H_\beta), Q(H_\beta)] - \lambda \frac{1}{4} \sum_{\alpha \in \Phi} S([H_{\beta}, X_{\alpha}], [X_{-\alpha}, H_\beta]) \right)$$

where $\beta$ is any fixed element in $\Phi$ (the formula above is independent of the choice of $\beta$, see Thm. 3.3).

**Remark 1.3.** The deformed double current algebra recently showed up in the work of Costello. In [11], he studies the AdS/CFT correspondence in the reformulation of Koszul duality for algebras. For $M2$ branes in an $\Omega$–background, Costello shows that the algebra of supersymmetric operators on a stack of $k$ branes, when $k \to \infty$, is a deformed double current algebra of type $A$.

1.4. **The elliptic Casimir connection.** Consider the $\mathbb{N}$–grading on $D_\lambda(\mathfrak{g})$ induced by the $\mathbb{N}$–bigrading (2) via the homomorphism $(a, b) \to a$, and let $\tilde{D}_\lambda(\mathfrak{g})$ be the completion of $D_\lambda(\mathfrak{g})$ with respect to this grading. The elliptic Casimir connection is constructed as follows.

**Theorem A** (Theorem 3.6).
is induced from this SL to give rise to a homomorphism.

Theorem B

where $\kappa_\alpha$ is the truncated Casimir operator corresponding to $\alpha$, $h^\vee$ is the dual Coxeter number of $g$, and $Z$ is the central element (3).

(2) The corresponding elliptic Casimir connection valued in $\widehat{D_\lambda(g)}$ is given by

$$\nabla_{\text{Ell}} = d - \frac{1}{2} \sum_{\alpha \in \Phi^+} \left( \frac{\theta(\alpha + \text{ad}(Qg^{-1}))}{\theta(\alpha)} \right) \left( \kappa_\alpha \right) d\alpha - \sum_{\alpha \in \Phi^+} \frac{\theta'(\alpha|\tau)}{\theta(\alpha|\tau)} Z d\alpha + \sum_{i=1}^n K(\alpha^i) du_i$$

(3) $\nabla_{\text{Ell}}$ is flat and $W$-equivariant.

1.5. **Extension to the modular direction.** We proved in [40] that the elliptic connection (1) extends to a flat connection in the modular direction, thus generalising a result of Calaque–Enriquez–Etingof in type A [8]. The extended connection takes values in the semidirect product $\mathfrak{d} \ltimes \ell^0_{\text{Ell}}$, where $\mathfrak{d}$ is an infinite–dimensional Lie algebra introduced in [8]. We prove in this paper that the elliptic Casimir connection also extends in the modular direction by extending the homomorphism $\ell^0_{\text{Ell}} \rightarrow D_\lambda(g)$ given by Theorem A to $\mathfrak{d} \ltimes \ell^0_{\text{Ell}}$.

Let $\mathfrak{d}$ be the Lie algebra with generators $\Delta_0, d, X$, and $\{\delta_{2m}\}_{m \geq 1}$, and relations

$$[d, X] = 2X, \quad [d, \Delta_0] = -2\Delta_0, \quad [X, \Delta_0] = d,$$

$$[\delta_{2m}, X] = 0, \quad [d, \delta_{2m}] = 2m\delta_{2m}, \quad (ad\Delta_0)^{2m+1}(\delta_{2m}) = 0. \quad (4)$$

The generators $X, \Delta_0, d$ span a copy of the Lie algebra $\mathfrak{sl}_2$ inside $\mathfrak{d}$, and the latter decomposes as $\mathfrak{d}_+ \ltimes \mathfrak{sl}_2$, where $\mathfrak{d}_+$ is the subalgebra generated by $\{\delta_{2m}\}_{m \geq 1}$. It is shown in [40, Prop. 7.1] that $\mathfrak{d}$ acts on $\ell^0_{\text{Ell}}$ by derivations (see also Prop. 7.1).

**Theorem B** (Thm. 8.9, Prop. 9.2, Thm. 9.3).

1. There exist elements $\mathbb{E}, \mathbb{F}, \mathbb{H}$, and $\{\mathbb{E}_{2m}\}_{m \geq 1}$ in $D_\lambda(g)$ such that $[\mathbb{E}, \mathbb{F}, \mathbb{H}]$ forms an $\mathfrak{sl}_2$–triple, and which together with $[\mathbb{E}_{2m} \mid m \in \mathbb{N}]$ satisfy the defining relations (4) of $\mathfrak{d}$.

2. The corresponding homomorphism $\mathfrak{d} \rightarrow D_\lambda(g)$, together with the homomorphism $\ell^0_{\text{Ell}} \rightarrow D_\lambda(g)$ given by Theorem A give rise to a homomorphism $\mathfrak{d} \ltimes \ell^0_{\text{Ell}} \rightarrow D_\lambda(g)$.

In Proposition 8.1, we also prove that there is an action of $\text{SL}_2(\mathbb{C})$ on $D_\lambda(g)$ via linear transformations of the two lattices $g[u], g[v]$. The action of the $\mathfrak{sl}_2$-triple $\{\mathbb{E}, \mathbb{F}, \mathbb{H}\}$ in Theorem B is induced from this $\text{SL}_2(\mathbb{C})$-action.

Let $M_{1,1}$ be the moduli space of pointed elliptic curves associated to the root system $\Phi$. More explicitly, let $\bar{\mathcal{H}} \ni \tau$ be the upper half plane. The semidirect product $(Q^\vee \oplus Q^\vee) \ltimes \text{SL}_2(\mathbb{Z})$ acts on $\mathcal{H} \times \bar{\mathcal{H}}$. For $(n, m) \in (Q^\vee \oplus Q^\vee)$ and $(z, \tau) \in \mathcal{H} \times \bar{\mathcal{H}}$, the action is given by translation: $(n, m) \ast (z, \tau) := (z + n + \tau m, \tau)$. For $(a, b \ c \ d) \in \text{SL}_2(\mathbb{Z})$, the action is given by $(a, b \ c \ d) \ast (z, \tau) := \begin{pmatrix} \frac{z + a \tau + b}{c \tau + d} \end{pmatrix} \frac{a \tau + b}{c \tau + d} \ast (z, \tau)$. Let $\alpha(-) : \mathcal{H} \rightarrow \mathcal{H}$ be the map induced by the root $\alpha \in \Phi$. We define $\vec{H}_{\alpha, \tau} \subset \mathcal{H} \times \bar{\mathcal{H}}$ to be

$$\vec{H}_{\alpha, \tau} = \{(z, \tau) \in \mathcal{H} \times \bar{\mathcal{H}} \mid \alpha(z) \in \Lambda_\tau = \mathbb{Z} + \tau \mathbb{Z}\}.$$

The elliptic moduli space $M_{1,1}$ is defined to be the quotient of $\mathcal{H} \times \bar{\mathcal{H}} \setminus \bigcup_{\alpha \in \Phi^+, \tau \in \bar{\mathcal{H}}} \vec{H}_{\alpha, \tau}$ by the action of $(Q^\vee \oplus Q^\vee) \ltimes \text{SL}_2(\mathbb{Z})$.

Let $g(z, x|\tau) := k_a(z, x|\tau)$ be the derivative of function $k(z, x|\tau)$ with respect to variable $x$. Set $a_{2n} := \frac{1}{(2n+1)!} B_{2n+2} \pi^2 |z|^{2n+2}$, where $B_n$ are the Bernoulli numbers and let $E_{2n+2}(\tau)$ be the Eisenstein series. Consider the following function on $\mathcal{H} \times \bar{\mathcal{H}}$

$$\Delta := \Delta(\alpha, \tau) = -\frac{1}{2\pi i} H^\mathbb{E} - \frac{1}{2\pi i} \sum_{n \geq 1} a_{2n} E_{2n+2}(\tau) \mathbb{E}_{2n} + \frac{1}{2\pi i} \sum_{\beta \in \Phi^+} g(\beta, \text{ad} \frac{z \beta}{2}) |\tau|^\frac{1}{2} \kappa_\beta - \frac{Z}{h^\vee}.$$
Let $A$ be an algebra endowed with a set of elements $\{X, α, β, γ, δ\}$, and let $\delta : h → A$ be a simple root. The following relations hold:

\[ \alpha(5) \in \Phi^+ \]

where $δ : h → A$ is the linear map given by $\delta(u) = X(u) - \frac{1}{2} \sum_{α ∈ Φ^+} α(u) t_α$. By [38, Thm. 2.5], $\nabla_{\text{trig}}$ is flat if, and only if, the following relations hold:

1. For any rank 2 root subsystem $Ψ ⊂ Φ$, and $α ∈ Ψ$, $[t_α, \sum_{β ∈ Ψ^+} t_β] = 0$.
2. For any $u, v ∈ h$, $[X(u), X(v)] = 0$.
3. For any $α ∈ Φ^+$, $w ∈ W$ such that $w^{-1}α$ is a simple root and $u ∈ h$, such that $α(u) = 0$,

\[ [t_α, X_w(u)] = 0, \]

where $X_w(u) = X(u) - \sum_{β ∈ Φ^+, r_α(β) = 0} β(u) t_β$.

Modulo (1), the relation (3) is equivalent to the following:

\[ 3' \text{ For } α ∈ Φ^+ \text{ and } v ∈ h \text{ such that } α(v) = 0, [t_α, δ(v)] = 0. \]

Denote by $i_{\text{trig}}$ the Lie algebra defined by the above relations, and note that $i_{\text{trig}}$ is $\mathbb{N}$-graded by $\deg(t_α) = 1 = \deg(X(u)) = \deg(δ(u))$.

1.7.2. Consider now the elliptic connection (1). As $τ → +i∞$, the functions $θ(z|τ)$ and $κ(z, x|τ)$ tend to:

\[ \frac{e^{πiz} - e^{-πiz}}{2πi} \text{ and } 2πi \left( \frac{1}{e^{2πiz} - 1} + \frac{e^{2πix}}{e^{2πix} - 1} - \frac{1}{2πix} \right) \]
respectively. This implies [40, Sect. 4] that $\nabla_{KZB,\tau}$ degenerates to the trigonometric connection
\[
\nabla^\text{deg} = d - \sum_{\alpha \in \Phi^+} 2\pi i d\alpha \left( e^{2\pi i \text{ad}(\frac{\alpha}{2})_{t_\alpha}} - 1 \right) t_\alpha - \sum_{\alpha \in \Phi^+} 2\pi i d\alpha \left( e^{2\pi i \text{ad}(\frac{\alpha}{2})_{y(u')}} - 1 \right) t_\alpha + \sum_i y(u') du_i
\]

Consider the $\mathbb{N}$–grading on $i^\Phi_{\text{trig}}$ induced by the $\mathbb{N}$–bigrading via the homomorphism $(a, b) \to a$, so that $\text{deg}(x(u)) = 1 = \text{deg}(t_\alpha)$ and $\text{deg}(y(v)) = 0$, and let $i^\Phi_{\text{trig}}$ be the completion of $i^\Phi_{\text{trig}}$ with respect to this grading. By universality of $i^\Phi_{\text{trig}}$, the above degeneration gives rise to a map $i^\Phi_{\text{trig}} \to i^\Phi_{\text{trig}}$ given by
\[
t_\alpha \mapsto t_\alpha \quad \text{and} \quad X(u) \mapsto -y(u) + 2\pi i \sum_{\alpha \in \Phi^+} (\alpha, u) \left( e^{2\pi i \text{ad}(\frac{\alpha}{2})_{t_\alpha}} - 1 \right) t_\alpha
\]

1.8. A homomorphism $Y_{h\g} \to D_{\lambda}(\g)$.

1.8.1. Recall that the Yangian $Y_{h\g}$ deforms the current algebra $U(\g[s])$ [Dr1]. It is an associative algebra over $\mathbb{C}[h]$ generated by elements $x, J(x), x \in \g$ subject to the relations in Definition 4.1, where $J(x) \equiv x \otimes s$ mod $\hbar$. In particular, we have $[x, J(y)] = J([x, y])$, for any $x, y \in \g$.

Drinfeld [Dr2] gave another realisation of $Y_{h\g}$, with generators $\{X^+_{i,r}, H_{i,\lambda}\}_{i \in I, \lambda \in \mathbb{N}}$ subject to the relations which are similar to the Kac-Moody presentation of an affine Lie algebra. The Lie subalgebra generated by $\{X_{i,0}, H_{i,0}\}_{i \in I}$ is isomorphic to $\g$, and $X_{i,1} \equiv x \otimes s$ mod $\hbar$. See Theorem 4.8 for a minimal presentation of $Y_{h\g}$ in terms of $\{X^+_{i,r}, H_{i,\lambda}\}_{i \in I, \lambda \in \mathbb{N}}$ given by Guay-Nakajima-Wendlandt. The isomorphism between the two presentations of the Yangian is given by
\[
X^\pm_{i,1} = J(X^\pm_{i}) - \hbar \left( \pm \frac{1}{4} \sum_{\alpha \in \Phi^+} S([X^\pm_{i}, X^\pm_{a}], X^\pm_{a}) - \frac{1}{4} S(X^\pm_{i}, H_i) \right)
\]
\[
H^\pm_{i,1} = J(H^\pm_{i}) - \hbar \left( \frac{1}{4} \sum_{\alpha \in \Phi^+} (\alpha, \alpha) S(X^\pm_{a}, X^\pm_{a}) - \frac{H_i^2}{2} \right)
\]

The trigonometric Casimir connection $\nabla_{\text{trig}, C}$ of $\g$ defined in [38] is the $Y_{h\g}$–valued connection obtained from the universal trigonometric connection (5) via a homomorphism $i^\Phi_{\text{trig}} \to Y_{h\g}$ given by $t_\alpha \mapsto \hbar \kappa_\alpha$ and
\[
\delta(u) \mapsto -2J(u) \quad \text{or equivalently} \quad X(\alpha_i) \mapsto -2\left(H_{i,1} - \frac{\hbar}{2} H_i^2 \right)
\]

1.8.2. Consider now the following diagram
\[
\begin{array}{ccc}
\hat{i}^\Phi_{\text{trig}} & \to & D_{\lambda}(\g) \\
\downarrow & & \downarrow h \\
\hat{i}^\Phi_{\text{trig}} & \to & Y_{h\g}
\end{array}
\]

where the top horizontal arrow is the (completion of) the homomorphism corresponding to the elliptic Casimir connection (Thm. A), the bottom horizontal one corresponds to the trigonometric Casimir connection, and the left vertical arrow is the degeneration homomorphism (6).

We show in this paper that (7) can be completed to a commutative diagram via a homomorphism $\tau : Y_{h\g} \to D_{\lambda}(\g)$. In particular, as $\tau \to \pm i\infty$, the elliptic Casimir connection of $\g$ degenerates to the trigonometric connection of $\g$, viewed as taking values in $D_{\lambda}(\g)$ via the homomorphism $\tau$. Specifically, choose root
vectors $X_\alpha \in \mathfrak{g}_\alpha$ for any $\alpha \in \Phi$ such that $(X_\alpha, X_{-\alpha}) = 1$, and let $\{h_i\}$, $\{h'_i\}$ be dual bases of $\mathfrak{h}$. For any $p, q \geq 0$, let $\Omega_{p,q}$ be the element of $U(\mathfrak{g}[u])$ defined by

$$\Omega_{p,q} = \sum_{\alpha \in \Phi} (X_\alpha \otimes u^p)(X_{-\alpha} \otimes u^q) + \sum_i (h_i \otimes u^p)(h'_i \otimes u^q)$$

(8)

**Theorem D** (Theorem 4.4). There is a unique algebra homomorphism $j : Y_{h\mathfrak{g}} \to \tilde{D}_\lambda(\mathfrak{g})$ such that the diagram (7) commutes. It is given by $h \mapsto \lambda/2$, $X \mapsto X$, $\mathfrak{g} \mapsto \mathfrak{g}$, and

$$J(X) \mapsto \frac{1}{2}K(X) - \frac{j}{4} \left[ Q(X), \sum_{n \geq 0} c_{2n+1} \sum_{p+q=2n} \left( \frac{2n}{p} \right) (-1)^p \Omega_{p,q} \right]$$

where the constants $c_{2n+1}$ are determined by the expansion $\pi i \sum_{n \geq 0} c_{2n+1} x^{2n+1}$.

1.9. **Rational Cherednik algebras.** Recall that a finite–dimensional representation of $\mathfrak{g}$ is small if $2\alpha$ is not a weight for any $\alpha \in \Phi$ [5, 32, 33]. The rational Casimir connection of $\mathfrak{g}$, when taken with values in the zero weight space $V[0]$ of a small representation coincides with the Coxeter KZ connection of the corresponding Weyl group with values in the $W$–module $V[0]$, namely [35, 9]

$$\nabla_{CKZ} = d - \sum_{\alpha \in \Phi^+} \frac{d}{\alpha} k_\alpha s_\alpha$$

where $k_\alpha = h(\alpha, \alpha)$. A similar statement holds for the trigonometric connection $\nabla_{\text{trig},C}$ [38, §7]. Namely, if $V$ is a $Y_{h\mathfrak{g}}$–module whose restriction to $\mathfrak{g}$ is small, the zero weight space $V[0]$ carries a natural action of the degenerate affine Hecke algebra $\mathcal{H}$ of $W$, and the restriction of $\nabla_{\text{trig},C}$ to $V[0]$ coincides with Cherednik’s affine KZ connection for $\mathcal{H}$ [10], up to abelian terms.

In this paper, we establish the elliptic analog of these statements by comparing the elliptic Casimir connection valued in the rational Cherednik algebra $W$ constructed in [40].

Let $H_{h,c}$ be the rational Cherednik algebra of $W$. $H_{h,c}$ is generated by the group algebra $\mathbb{C}[W$, together with a copy of $S\mathfrak{h}$ and $S\mathfrak{h}^*$, and depends on two sets of parameters (see [13], or Definition 5.1 for the defining relations). In [40], we constructed a homomorphism $i_h^\circ : H_{h,c} \to H_{h,c}$. This yields a flat, $W$–equivariant elliptic connection valued in (a completion of) $H_{h,c}$, which is given by

$$\nabla_{H_{h,c}} = d + \sum_{\alpha \in \Phi^+} \frac{2c_\alpha}{(\alpha(\alpha)} \cdot k(\alpha, \alpha) \cdot \frac{\alpha^\vee}{2} (\tau) s_\alpha \, d\alpha - \sum_{\alpha \in \Phi^+} \frac{h}{h'} \theta'(\alpha|\tau) \, d\alpha + \sum_{i=1}^n u^i \, du^i$$

Let now $V$ be a finite–dimensional representation of $D_{d}(\mathfrak{g})$, whose restriction to $\mathfrak{g}$ is small.

**Theorem E** (Theorem 5.3).

1. The canonical $W$–action on the zero weight space $V[0]$ together with the assignment

$$x_\alpha \mapsto Q(u), \quad y_\alpha \mapsto K(u'), \quad \text{where } x_\alpha, y_\alpha \in H_{h,c} \text{ for } u \in \mathfrak{h}, u' \in \mathfrak{h}^*$$

yields an action of the rational Cherednik algebra $H_{h,c}$ on $V[0]$.

2. The elliptic Casimir connection with values in $\text{End}(V[0])$ is equal to the sum of the elliptic KZ connection $\nabla_{H_{h,c}}$ and the scalar valued one-form

$$\mathcal{A} = \frac{1}{2} \sum_{\alpha \in \Phi^+} \left( \frac{2h'_\alpha + h''_\alpha}{h'} \right) - (\alpha, \alpha) \frac{\theta'(\alpha|\tau)}{\theta(\alpha|\tau)} \, d\alpha,$$

where $\frac{2h'_\alpha + h''_\alpha}{h'}$ is a constant only depending on the root system $\Phi$. 

1.10. (\(gl_k, gl_n\)) duality for the KZB and elliptic Casimir connections. Let \(M_{k,n}\) be the vector space of \(k \times n\) matrices, and \(\mathbb{C}[M_{k,n}]\) its ring of regular functions. It was discovered in [34] that the commuting actions of \(gl_k\) and \(gl_n\) on \(\mathbb{C}[M_{k,n}]\) give rise to an identification of the rational Casimir connection for \(\mathbb{C}[M_{k,n}]\) with the rational Casimir connection for \(gl_n\) with values in \(\mathbb{C}[M_{k,n}]\) [34].

A similar statement holds in the trigonometric case [16]. Namely, the trigonometric KZ connection for \(gl_n\) with values in \(\mathbb{C}[M_{k,n}]\), which depends upon an additional diagonal matrix \(s = \text{diag}(s_1, \ldots, s_k)\), coincides with the trigonometric Casimir connection of \(gl_n\), when \(\mathbb{C}[M_{k,n}]\) is regarded as the tensor product \(\mathbb{C}[M_{1,n}]^{(r_1)} \otimes \cdots \otimes \mathbb{C}[M_{1,n}]^{(r_k)}\) of \(k\) evaluation modules of the Yangian \(Y_h gl_n\), where each evaluation point \(r_i\) is a function of \(s_i\).

In this paper, we prove that \((gl_k, gl_n)\) duality identifies the (elliptic) KZB connection for \(gl_k\), and the elliptic Casimir connection for \(gl_n\).

Let \(h_k \subset gl_k\) be the Cartan subalgebra of \(gl_k\), and denote by \(h_k^{\text{reg}}\) the set of diagonal matrices in \(h_k\) with distinct eigenvalues. Then, \(\mathbb{C}[h_k^{\text{reg}}] \otimes \mathbb{C}[M_{k,n}]\) is a module over \(B_n = \text{Diff}(h_k^{\text{reg}}) \otimes U(gl_k)^{\otimes n}\). Let \(\mathcal{H}(gl_k, h_k)\) be the Hecke algebra associated to \((gl_k, h_k)\) introduced in [8]. By definition, \(\mathcal{H}(gl_k, h_k)\) is a subquotient of \(B_n = \text{Diff}(h_k^{\text{reg}}) \otimes U(gl_k)^{\otimes n}\) (see [8, Section 6.3], or Definition 6.1). In particular, the zero weight space \(\mathbb{C}[h_k^{\text{reg}}] \otimes \mathbb{C}[M_{k,n}][0]\) is a module over \(\mathcal{H}(gl_k, h_k)\).

Let \(t_{1,n}\) be the Lie algebra with generators \((x_i, y_i)_{i=1}^n\) and \((t_{ij})_{1 \leq i < j \leq n}\) (see [8, Section 6.3], or Sect. 6). \(t_{1,n}\) is a split central extension of \(\mathbb{A}_{n-1}^{\text{el}}\) for the root system \(A_{n-1}\), with kernel spanned by \(\mathcal{T} = \sum_i x_i\) and \(\mathcal{Y} = \sum_i y_i\). In [8, Prop. 41], the authors construct an algebra homomorphism \(t_{1,n} \to \mathcal{H}(gl_k, h_k) \subset B_n / B_n h_k^{\text{diag}}\), which factors through \(t_{1,n}^{\text{A}_{n-1}}\) and is given by

\[
\begin{align*}
x_i &\mapsto \sum_{a=1}^k x_a \otimes E_a^{(i)}, \\
y_i &\mapsto -\sum_{a=1}^k \partial_a \otimes E_a^{(i)} + \sum_{1 \leq a < b \leq k} \frac{1}{x_b - x_a} \otimes E_{ab}^{(i)} E_{ba}^{(j)}, \\
t_{ij} &\mapsto \sum_{1 \leq a, b \leq k} E_{ab}^{(i)} E_{ba}^{(j)}
\end{align*}
\]

This gives rise in particular to an elliptic connection with values in the zero weight space \(\mathbb{C}[h_k^{\text{reg}}] \otimes \mathbb{C}[M_{k,n}][0]\). As pointed out in [8], this connection coincides with the KZB connection for \(gl_k\) on \(n\) points defined in [6, 14].

1.10.2. Consider now the double deformed current algebra \(D_{k,\beta}(sl_n)\) for \(sl_n\) ([19] and 3.3). The latter depends in fact on two parameters \(\lambda, \beta\).

**Theorem F** (Theorem 6.7). The following define an action of the deformed double current algebra \(D_{k,\beta}(sl_n)\) on \(\mathbb{C}[h_k^{\text{reg}}] \otimes \mathbb{C}[M_{1,n}]^{\otimes k}\): for \(1 \leq i \neq j \leq n\), \(E_{ij}\) acts by \(1 \otimes \sum_{a=1}^k (E_{ij})^{(a)}\) and

\[
\begin{align*}
K(E_{ij}) &\text{ acts by } \sum_{a=1}^k x_a \otimes (E_{ij})^{(a)}, \\
Q(E_{ij}) &\text{ acts by } -\sum_{a=1}^k \partial_a \otimes (E_{ij})^{(a)} + \sum_{1 \leq a \neq b \leq k} \frac{1}{x_b - x_a} \otimes \left(\sum_{e=1}^n (E_{ie})^{(a)}(E_{ej})^{(b)} + (E_{ij})^{(a)}\right).
\end{align*}
\]

By composing with the homomorphism \(t_{1,n}^{\text{A}_{n-1}} \to D_{k,\beta}(sl_n)\), Theorem F gives rise to an elliptic Casimir connection for \(sl_n\), with values in \(\text{End}(\mathbb{C}[h_k^{\text{reg}}] \otimes \mathbb{C}[M_{k,n}]\)).
Let $\theta$ be a Euclidean vector space, $\Phi \subset h^*$ a reduced, crystallographic root system. Let $Q \subset h^*$ be the root lattice generated by the roots $\{ \alpha \mid \alpha \in \Phi \}$ and $P \subset h^*$ be the corresponding weight lattice. Let $Q^\vee \subset h$ be the coroot lattice generated by the coroots $\alpha^\vee$, with the inner product $\langle \alpha^\vee, \alpha \rangle = 2$. The coroot lattice is dual to the weight lattice $P$. Let $P^\vee \subset h$ be the dual lattice of $Q$, called the coweight lattice.

**Definition 2.1.** Let $t_{\Phi}^h$ be the Lie algebra generated by a set of elements $\{t_\alpha\}_{\alpha \in \Phi}$, such that $t_\alpha = t_{-\alpha}$, and two linear maps $x : h \to A$, $y : h \to A$. Those generators satisfy the following relations:

1. For any root subsystem $\Psi$ of $\Phi$ (that is, $\langle \Psi \rangle_{\mathbb{Z}} \cap \Phi = \Psi$), we have
   \[ [t_\alpha, \sum_{\beta \in \Psi^+} t_\beta] = 0. \]
2. $[x(u), x(v)] = 0$, $[y(u), y(v)] = 0$, for any $u, v \in h$;
3. $[y(u), x(v)] = \sum_{\gamma \in \Phi^+} \langle v, \gamma \rangle \langle u, \gamma \rangle t_\gamma$.
4. $[t_\alpha, x(u)] = 0$, $[t_\alpha, y(u)] = 0$, if $\langle \alpha, u \rangle = 0$.

The Lie algebra $t_{\Phi}^h$ is bigraded, with grading $\deg(x(u)) = (1, 0), \deg(y(v)) = (0, 1)$, and $\deg(t_\alpha) = (1, 1)$, for any $u, v \in h$ and $\alpha \in \Phi$.

**2.2. Theta functions.** In this subsection, we recall some basic facts about theta functions that will be used in the paper.

Let $\Lambda_\tau := \mathbb{Z} + \mathbb{Z} \tau \subset \mathbb{C}$ and $\mathbb{H}$ be the upper half plane, i.e. $\mathbb{H} := \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$. The following properties of $\theta(z|\tau)$ uniquely characterize the theta function $\theta(z|\tau)$:

1. $\theta(z|\tau)$ is a holomorphic function $\mathbb{C} \times \mathbb{H} \to \mathbb{C}$, such that $\{ z \mid \theta(z|\tau) = 0 \} = \Lambda_\tau$.
2. $\frac{\partial \theta}{\partial z}(0|\tau) = 1$.
3. $\theta(z + 1|\tau) = -\theta(z|\tau)$, and $\theta(z + \tau|\tau) = -e^{-\pi i \tau} e^{-2\pi iz} \theta(z|\tau)$.
4. $\theta(z|\tau + 1) = \theta(z|\tau)$, while $\theta(-z/\tau - 1/\tau) = -(1/\tau) e^{i(\pi/\tau)z^2} \theta(z|\tau)$.
(5) Let \( q := e^{2\pi i r} \) and \( \eta(\tau) := q^{1/24} \prod_{n \geq 1} (1 - q^n) \). If we set \( \theta(z|\tau) := \eta(\tau)^3 \theta(z|\tau) \), then \( \theta(z|\tau) \) satisfies the differential equation

\[
\frac{\partial \theta(z, \tau)}{\partial \tau} = \frac{1}{4\pi i} \frac{\partial^2 \theta(z, \tau)}{\partial z^2}.
\]

We have the product formula of \( \theta(z|\tau) \):

\[
\theta(z|\tau) = u^\frac{1}{2} \prod_{i > 0} (1 - q^i u) \prod_{i \geq 0} (1 - q^i u^{-1}) \frac{1}{2\pi i} \prod_{i > 0} (1 - q^i)^{-2},
\]

where \( u = e^{2\pi i z} \).

2.3. Principal bundles on elliptic configuration space. Consider the elliptic curve \( \mathcal{E}_r := \mathbb{C}/\Lambda_r \) with modular parameter \( \tau \notin \mathbb{R} \). Let \( T := \mathbb{H}/(Q^\vee \otimes \tau Q^\vee) \), which non-canonically isomorphic to \( \mathcal{E}_r^n \), for \( n = \text{rank}(Q^\vee) \).

For any root \( \alpha \in Q \subset \mathfrak{h}^* \), the linear map \( \alpha : \mathfrak{h} = Q^\vee \otimes \mathbb{C} \to \mathbb{C} \) induces a natural map

\[ \chi_\alpha : \mathfrak{h}/(Q^\vee \otimes \tau Q^\vee) \to \mathcal{E}_r. \]

Denote kernel of \( \chi_\alpha \) by \( T_\alpha \), which is a divisor of \( T \). We refer to \( T_{\text{reg}} = T \setminus \bigcup_{\alpha \in \Phi} T_\alpha \) as the elliptic configuration space associated to \( \Phi \). If \( E = \mathbb{R}^n \) with standard coordinates \( \{ e_i \} \), and \( \Phi = \{ e_i - e_j \}_{1 \leq i < j \leq n} \subset E^* \) is the root system of type \( \Lambda_{n-1} \). \( T_{\text{reg}} \) is the configuration space of \( n \) ordered points on the elliptic curve \( \mathcal{E}_r \).

The Lie algebra \( \mathfrak{t}_{\text{nil}}^\Phi \) is positively bi-graded. Let \( \mathfrak{t}_{\text{nil}}^\Phi \) be the (pro–nilpotent) completion of \( \mathfrak{t}_{\text{nil}}^\Phi \) with respect to the grading given by \( \deg(x(u)) = 1 = \deg(y(u)) \), and \( \deg(t_u) = 2 \), and \( \exp(\mathfrak{t}_{\text{nil}}^\Phi) \) is the corresponding pro–unipotent group. We now describe a principal bundle \( \mathcal{P}_{\tau,n} \) on the elliptic configuration space \( T_{\text{reg}} \) with structure group \( \exp(\mathfrak{t}_{\text{nil}}^\Phi) \).

The lattice \( \Lambda_r \otimes Q^\vee \) acts on \( \mathfrak{h} = \mathbb{C} \otimes Q^\vee \) by translations, and \( T = \mathfrak{h}/\Lambda_r \otimes Q^\vee \) is the quotient of \( \mathfrak{h} \) by this action of \( \Lambda_r \otimes Q^\vee \). For any \( g \in \exp(\mathfrak{t}_{\text{nil}}^\Phi) \), and the standard basis \( \{ \alpha_i^\vee \}_{1 \leq i \leq n} \) of \( Q^\vee \), we define an action of \( \Lambda_r \otimes Q^\vee \) on \( \exp(\mathfrak{t}_{\text{nil}}^\Phi) \) by

\[ \alpha_i^\vee (g) = g \text{ and } \tau \alpha_i^\vee (g) = e^{-2\pi i \tau (\alpha_i^\vee)} g. \]

The twisted product \( \widetilde{\mathcal{P}} := \mathfrak{h} \times_{\Lambda_r \otimes Q^\vee} \exp(\mathfrak{t}_{\text{nil}}^\Phi) \) is a principal bundle over \( T \) with structure group \( \exp(\mathfrak{t}_{\text{nil}}^\Phi) \). Denote by \( \mathcal{P}_{\tau,n} \) the restriction of this bundle \( \widetilde{\mathcal{P}} \) on \( T_{\text{reg}} \subset T \).

In other words, let \( \pi : \mathfrak{h} \to \mathfrak{h}/(Q^\vee \otimes \tau Q^\vee) \) be the natural projection. For an open subset \( U \subset T_{\text{reg}} \), the sections of \( \mathcal{P}_{\tau,n} \) on \( U \) are given by

\[ \{ f : \pi^{-1}(U) \to \exp(\mathfrak{t}_{\text{nil}}^\Phi) \mid f(z + \alpha_i^\vee) = f(z), f(z + \tau \alpha_i^\vee) = e^{-2\pi i \tau (\alpha_i^\vee)} f(z) \}. \]

2.4. The universal KZB connection. In this subsection, we recall the universal KZB connection associated to root system \( \Phi \) in [40]. As in [8, 40], we set

\[ k(z,x|\tau) := \frac{\theta(z + x|\tau)}{\theta(z|\tau) \theta(x|\tau)} - \frac{1}{x}, \quad (10) \]

For a fixed \( \tau \), the function \( k(z,x|\tau) \) belongs to \( \text{Hol}(\mathbb{C} - \Lambda_r)[[x]] \), which is holomorphic in \( x \) in the neighborhood of \( x = 0 \). For \( x_u \in \mathfrak{t}_{\text{nil}}^\Phi \), substituting \( x = \text{ad} x_u \) in (10), we get a linear map \( \mathfrak{t}_{\text{nil}}^\Phi \to (\mathfrak{t}_{\text{nil}}^\Phi \otimes \text{Hol}(\mathbb{C} - \Lambda_r))^\vee \), where \( (-)^\vee \) is taking the completion.

We consider the \( \mathfrak{t}_{\text{nil}}^\Phi \)-valued connection on \( T_{\text{reg}} \):

\[ \nabla_{\text{KZB},\tau} = d - \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}(\frac{x\alpha^\vee}{2})|\tau)(t_u) d\alpha + \sum_{i=1}^n y(u') du_i, \quad (11) \]

where \( \Phi^+ \subset \Phi \) is a chosen system of positive roots, \( \{ u_i \} \), and \( \{ u' \} \) are dual basis of \( \mathfrak{h}^* \) and \( \mathfrak{h} \) respectively.
Note that the form (11) is independent of the choice of $\Phi^*$. It follows from the equality $k(z, x|\tau) = -k(-z, -x|\tau)$, which is a direct consequence of the fact that theta function $\theta(z|\tau)$ is an odd function.

Let $W$ be the Weyl group generated by reflections $\{s_\alpha \mid \alpha \in \Phi\}$. Assume there is an action of $W$ on $\mathfrak{t}^\Phi$.

**Theorem 2.2.** [40, Theorem A]

1. The connection $\nabla_{KZB, r}$ (11) is flat if and only if the relations (1)-(4) in $\mathfrak{t}^\Phi$ hold.
2. The connection $\nabla_{KZB, r}$ is $W$-equivariant if and only if the relations hold:

$$
\begin{align*}
s_\alpha(t_\gamma) &= t_{s_\alpha(\gamma)}, \\
s_\alpha(x(u)) &= x(s_\alpha u), \\
s_\alpha(y(v)) &= y(s_\alpha v).
\end{align*}
$$

3. The elliptic Casimir connection

### 3.1. The deformed double current algebras

We assume $\mathfrak{g}$ is a semisimple Lie algebra of rank $\geq 3$. The deformed double current algebras associated to $\mathfrak{g}$ are introduced by Guay in [19] for $\mathfrak{g} = \mathfrak{sl}_n$ with $n \geq 4$ and in [20] for an arbitrary simple Lie algebra $\mathfrak{g}$ of rank $\geq 3$, and $\mathfrak{g} \neq \mathfrak{sl}_3$. It is a deformation of the universal central extension of $U(\mathfrak{g}[u, v])$. The double loop presentations of the deformed double current algebras are established in [19] for $\mathfrak{g} = \mathfrak{sl}_n$, $n \geq 4$ and in [22] for any simple Lie algebra $\mathfrak{g}$, with rank($\mathfrak{g}$) $\geq 3$ and [23] for the $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3$.

We briefly recall the results here. Let $(\cdot, \cdot)$ be the Killing form on $\mathfrak{g}$ and let $X_\beta^+, H_i, 1 \leq i \leq N$ be the Chevalley generators of $\mathfrak{g}$ normalized so that $(X_\beta^+, X_{\beta'}^-) = 1$ and $[X_\beta^+, X_{-\beta}^-] = H_i$. For each positive root $\alpha \in \Phi^+$, we choose generators $X_\alpha^\pm$ of $\mathfrak{g}_\pm\alpha$ such that $(X_\alpha^+, X_\alpha^-) = 1$ and $X_\alpha^\pm = X_\alpha^\mp$. If $\alpha > 0$, set $X_\alpha = X_\alpha^+$; if $\alpha < 0$, set $X_\alpha = X_\alpha^-$. Assume there is an action of $\mathfrak{g}$ on $\mathfrak{g}$.

**Definition 3.1.** [22, Definition 2.2] Assume $\mathfrak{g}$ has rank $\geq 3$, and $\mathfrak{g} \neq \mathfrak{sl}_3$. The deformed double current algebra $D_{\lambda, \beta}(\mathfrak{g})$ is generated by elements $X, K(X), Q(X), P(X), X \in \mathfrak{g}$, such that

1. $K(X), X \in \mathfrak{g}$ generate a subalgebra which is an image of $\mathfrak{g} \otimes \mathbb{C}[v]$ with $X \otimes v \mapsto K(X)$;
2. $Q(X), X \in \mathfrak{g}$ generate a subalgebra which is an image of $\mathfrak{g} \otimes \mathbb{C}[u]$ with $X \otimes u \mapsto Q(X)$;
3. $P(X)$ is linear in $X$, and for any $X, X' \in \mathfrak{g}$, $[P(X), X'] = P[X, X']$.

and the following relations hold for all root vectors $X_{\beta_1}, X_{\beta_2} \in \mathfrak{g}$ with $\beta_1 \neq -\beta_2$:

$$
[K(X_{\beta_1}), Q(X_{\beta_2})] = P([X_{\beta_1}, X_{\beta_2}]) - \lambda \frac{(\beta_1, \beta_2)}{4} S(X_{\beta_1}, X_{\beta_2}) + \lambda \frac{1}{4} \sum_{\alpha \in \Phi} S([X_{\beta_1}, X_\alpha], [X_{-\alpha}, X_{\beta_2}]), \quad (12)
$$

where $S(a_1, a_2) = a_1 a_2 + a_2 a_1 \in U\mathfrak{g}$.

When $\mathfrak{g} = \mathfrak{sl}_n$, the deformed double current algebra $D_{\lambda, \beta}(\mathfrak{sl}_n)$ has two deformation parameters $\lambda, \beta$. Let $\{e_1, \ldots, e_n\}$ be the standard orthogonal basis of $\mathbb{C}^n$. The set of roots of $\mathfrak{sl}_n$ is denoted by $\Phi = \{e_i - e_j \mid 1 \leq i \neq j \leq n\}$, with a choice of positive roots $\Phi^+ = \{e_i - e_j \mid 1 \leq i < j \leq n\}$. The longest positive root $\theta$ equals $e_1 - e_n$. The elementary matrices will be written as $E_{ij} \in \mathfrak{sl}_n$. So $X_\beta^+ = E_{i,i+1}, X_\beta^- = E_{i-1,i}$ and $H_i = E_{ii} - E_{i,i+1}$ for $2 \leq i \leq n - 1$. We set $E_\theta = E_{1n}$.

In the defining relation of $D_{\lambda, \beta}(\mathfrak{sl}_n)$, the relation (12) is modified to the following (see [22, Definition 5.1]).

$$
[K(E_{ab}), Q(E_{cd})] = P([E_{ab}, E_{cd}]) + (\beta - \frac{\lambda}{2})(\delta_{bc}E_{ad} + \delta_{ad}E_{cb}) - \frac{\lambda}{4} (\epsilon_a - \epsilon_b, \epsilon_c - \epsilon_d) S(E_{ab}, E_{cd}) + \frac{\lambda}{4} \sum_{1 \leq i \leq n} S([E_{ab}, E_{ij}], [E_{ji}, E_{cd}]).
$$

When $\beta = \frac{\lambda}{4}$, the relation (13) coincides with the relation (12).

The following equivalent relation of the defining relation (13) of $D_{\lambda, \beta}(\mathfrak{sl}_n)$ is useful in Section 6.2.
Lemma 3.2. In $D_{4,\beta}(sl_n)$, the relation (13) is equivalent to

$$[K(E_{ab}), Q(E_{cd})] = P([E_{ab}, E_{cd}]) + \frac{\lambda}{2} \sum_{1 \leq j \leq n} \delta_{bc} E_{aj} E_{jd} + \frac{\lambda}{2} \sum_{1 \leq j \leq n} \delta_{ad} E_{ci} E_{ib}$$

$$- \lambda E_{ad} E_{cb} + (\beta - \frac{\lambda}{2} - \frac{\lambda}{4} n)(\delta_{bc} E_{ad} + \delta_{ad} E_{cb}).$$

(14)

Proof. It follows from a straightforward computation. For the convenience of the readers, we include a proof here. Assume $1 \leq a \neq b \leq n$, $1 \leq c \neq d \leq n$, and $(a, b) \neq (d, c)$. We expand the right hand side of (13) using the basic equality $[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$ as follows. We have

$$[K(E_{ab}), Q(E_{cd})] = P([E_{ab}, E_{cd}])$$

$$= (\beta - \frac{\lambda}{2})(\delta_{bc} E_{ad} + \delta_{ad} E_{cb}) - \frac{\lambda}{4}(\epsilon_{ab}, \epsilon_{cd}) S(E_{ab}, E_{cd}) + \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S([E_{ab}, E_{ij}], [E_{ji}, E_{cd}])$$

$$= (\beta - \frac{\lambda}{2})(\delta_{bc} E_{ad} + \delta_{ad} E_{cb}) - \frac{\lambda}{4}(\delta_{ac} - \delta_{ad} - \delta_{bc} + \delta_{bd}) S(E_{ab}, E_{cd}) + \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} S(\delta_{hi} E_{aj} - \delta_{aj} E_{ih}, \delta_{ic} E_{jd} - \delta_{jd} E_{ci})$$

$$= (\beta - \frac{\lambda}{2})(\delta_{bc} E_{ad} + \delta_{ad} E_{cb}) - \frac{\lambda}{4}(\delta_{ac} - \delta_{ad} - \delta_{bc} + \delta_{bd}) S(E_{ab}, E_{cd})$$

$$+ \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} \delta_{bc} S(E_{aj}, E_{jd}) - \frac{\lambda}{4}(1 - \delta_{bd}) S(E_{ad}, E_{cb}) - \frac{\lambda}{4}(1 - \delta_{ac}) S(E_{cb}, E_{ad}) + \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} \delta_{ad} S(E_{ib}, E_{ci})$$

$$= (\beta - \frac{\lambda}{2})(\delta_{bc} E_{ad} + \delta_{ad} E_{cb}) + \frac{\lambda}{4} \sum_{1 \leq j \leq n} \delta_{bc} (2 E_{aj} E_{jd} - (E_{ad} - \delta_{ad} E_{jj}))$$

$$+ \frac{\lambda}{4} \sum_{1 \leq i \neq j \leq n} \delta_{ad} (2 E_{ci} E_{ib} - (E_{cb} - \delta_{cb} E_{ii})) - \frac{\lambda}{2} (2 E_{ad} E_{cb})$$

$$= (\beta - \frac{\lambda}{2})(\delta_{bc} E_{ad} + \delta_{ad} E_{cb}) + \frac{\lambda}{2} \sum_{1 \leq i \leq n} \delta_{bc} E_{aj} E_{jd} - \frac{\lambda}{4} n \delta_{bc} E_{ad} + \frac{\lambda}{4} \delta_{bd} \sum_{1 \leq j \leq n} E_{ij}$$

$$+ \frac{\lambda}{2} \sum_{1 \leq i \leq n} \delta_{ad} E_{ci} E_{ib} - \frac{\lambda}{4} n \delta_{ad} E_{cb} + \frac{\lambda}{4} \delta_{ad} \delta_{bc} \sum_{1 \leq i \leq n} E_{ii} - \lambda E_{ad} E_{cb}$$

$$= \frac{\lambda}{2} \sum_{1 \leq i \leq n} \delta_{bc} E_{aj} E_{jd} + \frac{\lambda}{2} \sum_{1 \leq i \leq n} \delta_{ad} E_{ci} E_{ib} - \lambda E_{ad} E_{cb} + (\beta - \frac{\lambda}{2} - \frac{\lambda}{4} n)(\delta_{bc} E_{ad} + \delta_{ad} E_{cb})$$

The last equality follows from the assumption that $\epsilon_a - \epsilon_b \neq \epsilon_d - \epsilon_c$. Therefore, we have $\delta_{ad} \delta_{bc} = 0$. This completes the proof. \qed
For any root $\beta \in \Phi$ of $\mathfrak{g}$, set
\[
Z(\beta) := [K(H_\beta), Q(H_\beta)] - \frac{A}{4} \sum_{\alpha \in \Phi} S([H_\beta, X_\alpha], [X_{-\alpha}, H_\beta]) \in D_1(\mathfrak{g}).
\]

**Theorem 3.3.** [22, Proposition 4.1] In the deformed double current algebra $D_1(\mathfrak{g})$, we have

1. For any two roots $\beta, \gamma \in \Phi$ of $\mathfrak{g}$, we have $Z(\beta) = \frac{Z(\gamma)}{(\beta, \gamma)}$. In particular, $Z := Z(\beta)$ is independent of the choices of $\beta \in \Phi$.
2. The element $Z$ is central in $D_1(\mathfrak{g})$.

In the case of the deformed double current algebra $D_1,\beta(\mathfrak{sl}_n)$ in type $A_{n-1}$ with two parameters. Set
\[
Z_n := \sum_{a=1}^{n} Z_{a,a+1} = \sum_{a=1}^{n} \left( [K(H_a), Q(H_a)] - \frac{A}{4} \sum_{1 \leq i \leq j \leq n} S([H_{a,i}, E_{i,j}],[E_{j,i}, H_{a,j}]) \right),
\]
where $(a, a + 1) = (n, 1)$, when $a = n$.
\[
Z_{ab,cd} := [K(H_{ab}), Q(H_{cd})] - \frac{A}{4} \sum_{1 \leq i \leq j \leq n} S([H_{ab,i}, E_{i,j}],[E_{j,i}, H_{cd,j}]).
\]
\[
W_{ab} := [K(E_{ab}), Q(E_{ba})] - P(H_{ab}) - \frac{A}{4} \sum_{1 \leq i \leq j \leq n} S([E_{ab,i}, E_{i,j}],[E_{j,i}, E_{ba,j}]) - \frac{A}{2} S(E_{ab}, E_{ba})
\]

and denote $Z_{ab,cd}$ by $Z_{ab}$ for short.

**Proposition 3.4.** [22, Proposition 5.1 and Theorem 5.1]

(i) The element $Z_n$ is central in $D_1,\beta(\mathfrak{sl}_n)$.
(ii) For any $1 \leq a \neq b \leq n$, and $1 \leq c \neq b \leq n$, the following relations hold in $D(\mathfrak{sl}_n)$, $n \geq 4$.
\[
Z_{ab,cd} = (\epsilon_c - \epsilon_d, \epsilon_a - \epsilon_b)W_{ab} + (\beta - \frac{A}{2})(\epsilon_a + \epsilon_b, \epsilon_c - \epsilon_d)H_{ab}.
\]
In particular, we have $Z_{ab} = 2W_{ab}$ for any $1 \leq a \neq b \leq n$. When $a, b, c, d$ are distinct, we have $Z_{ab,cd} = 0$.
(iii) For $1 \leq a \neq b \leq n$, and $1 \leq c \neq d \leq n$, in $D_1,\beta(\mathfrak{sl}_n)$, $n \geq 4$, we have:
\[
W_{ab} - W_{cd} = (\beta - \frac{A}{2})(H_{ac} + H_{bd}), \text{ and } Z_{ab} - Z_{cd} = 2(\beta - \frac{A}{2})(H_{ac} + H_{bd})
\]

### 3.2. The elliptic Casimir connection valued in $D_1(\mathfrak{g})$.

We construct the elliptic Casimir connection with values in the deformed double current algebra $D_1(\mathfrak{g})$ by constructing an algebra homomorphism from $t^\Phi_{\mathfrak{sl}}$ to $D_1(\mathfrak{g})$.

**Proposition 3.5.** There is an algebra homomorphism from $t^\Phi_{\mathfrak{sl}} \rightarrow D_1(\mathfrak{g})$, given by, for $h, h' \in \mathfrak{h}$,
\[
x(h) \mapsto Q(h), \quad y(h') \mapsto K(h'), \quad \text{and} \quad t_\alpha \mapsto \lambda \kappa_\alpha + \frac{Z}{h'},
\]
where $\kappa_\alpha := X_\alpha^+X_\alpha^- + X_\alpha^-X_\alpha^+$ is the truncated Casimir operator, $h'$ is the dual Coxeter number, and $Z$ is the central element in Theorem 3.3.

**Proof:** We only prove that the homomorphism preserves the relation $[y(u), x(v)] = \sum_{\alpha \in \Phi^+} \langle v, \gamma \rangle \langle u, \gamma \rangle t_\gamma$ of $t^\Phi_{\mathfrak{sl}}$. All other relations are obviously preserved, proof of which is left as an exercise to the reader.

As a consequence of Theorem 3.3, for any $h, h' \in \mathfrak{h}$, we have:
\[
[K(h), Q(h')] = \frac{A}{4} \sum_{\alpha \in \Phi} S([h, X_\alpha], [X_{-\alpha}, h]) + (h, h') Z = \frac{A}{2} \sum_{\alpha \in \Phi^+} (h, \alpha)(h', \alpha)\kappa_\alpha + (h, h') Z.
\]
The claim now follows from the following equality in [40, Lemma 10.4] that
\[
(h, h') = \frac{1}{h^\vee} \sum_{\alpha \in \Phi^+} (\alpha, h)(\alpha, h'), \text{ for all } h, h' \in \mathfrak{h},
\]
where $h^\vee$ is the dual Coxeter number. Therefore,
\[
[K(h), Q(h')] = \sum_{\alpha \in \Phi^+} (h, \alpha)(h', \alpha) \left( \frac{1}{2} K_\alpha + \frac{Z}{h^\vee} \right).
\]
This completes the proof.

\[\square\]

**Theorem 3.6.** The elliptic Casimir connection valued in the deformed double current algebra $D_A(\mathfrak{g})$

\[
\nabla_{E\text{ll},C} = d - \frac{\alpha}{2} \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}(Q(\alpha^\vee))/2)(\kappa_{\alpha})d\alpha - \sum_{\alpha \in \Phi^+} \theta'(|\alpha|\tau)Z \frac{h^\vee}{h^\vee} d\alpha + \sum_{i=1}^{n} K(u_i)du_i
\]
is flat and $W$-equivariant.

**Proof.** By Theorem 3.3, $Z$ is in the center of $D_A(\mathfrak{g})$. Therefore, $\text{ad}(Q(\alpha^\vee))/2)^n(Z) = 0$, for all positive $n \in \mathbb{N}$. Recall the function $k(z, x|\tau)$ is given by
\[
k(z, x|\tau) = \frac{\theta(z + x|\tau)}{\theta(z|\tau)\theta(x|\tau)} - \frac{1}{x} \in \text{Hol}(\mathbb{C} - \Lambda_\tau)[[x]],
\]
and it is holomorphic in the neighborhood of $x = 0$. Using the fact that $\theta(z)$ is an odd function, we have $\theta(0) = \theta'(0) = 0$, and $\theta'(0) = 1$. A simple computation shows $\lim_{x \to 0} k(z, x|\tau) = \frac{\theta'(z)}{\theta(z)}$. In other words, viewing $k(z, x|\tau)$ as a function in $x$, the constant term of $k(z, x|\tau)$ with respect to $x$ is then given by $\frac{\theta'(z)}{\theta(z)}$. Hence
\[
k(\alpha, \text{ad}(Q(\alpha^\vee))/2|\tau)(Z) = \frac{\theta'(|\alpha|\tau)}{\theta(|\alpha|\tau)}(Z).
\]
The conclusion now follows from the criterion in Theorem 2.2 and Proposition 3.5.

\[\square\]

### 3.3. The elliptic Casimir connection valued in $D_{A,\beta}(\mathfrak{sl}_n)$

In this section, we consider the type A case with two deformation parameters $\lambda, \beta$. We construct the elliptic Casimir connection with values in $D_{A,\beta}(\mathfrak{sl}_n)$ by constructing an algebra homomorphism from $\mathfrak{t}^\Phi_{e\text{ll}}$ to $D_{A,\beta}(\mathfrak{sl}_n)$.

**Proposition 3.7.** Let $Z_n$ be the central element of $D_{A,\beta}(\mathfrak{sl}_n)$ constructed in Proposition 3.4. There is a morphism of algebras $\mathfrak{t}^\Phi_{e\text{ll}} \to D_{A,\beta}(\mathfrak{sl}_n)$, which is given by:
\[
x(u) \mapsto K(u), \quad y(u) \mapsto Q(u),
\]
\[
t_{ij} \mapsto \frac{\lambda}{2} (E_{ij}E_{ji} + E_{ji}E_{ij}) + \frac{Z_n}{2n^2} + 2(\beta - \frac{\lambda}{2}) \left( \frac{E_{ii} + E_{jj}}{n} - \frac{2}{n} \sum_{e=1}^{n} E_{ee} \right).
\]
As a direct consequence, we have
\textbf{Theorem 3.8.} The elliptic Casimir connection with two parameters $\lambda, \beta$ valued in the deformed double current algebra $D_{\lambda, \beta}(\mathfrak{sl}_n)$
\begin{equation}
\nabla_{\text{EliC}} = d - \frac{\lambda}{2} \sum_{1 \leq i \neq j \leq n} k(z_i - z_j, \text{ad} K(u_i)(\tau)(E_{ij}E_{ji} + E_{ji}E_{ij}))dz_{ij} + \sum_{i=1}^{n} Q(u_i)dz_i \tag{16}
\end{equation}
is flat and $S_n$-equivariant.

For the rest of this subsection, we prove Proposition 3.7.

Using Proposition 3.4, we express $Z_{ab,cd}$ in terms of the central element $Z_n$ and elements in the Cartan subalgebra of $\mathfrak{sl}_n$ as follows.

\textbf{Lemma 3.9.} We have
\[ Z_{ab,cd} = (\epsilon_a - \epsilon_b, \epsilon_c - \epsilon_d) \frac{Z_n}{2n} + 2(\beta - \frac{\lambda}{2})(\delta_{ac} - \delta_{ad})(E_{aa} - \frac{1}{n} \sum_{e=1}^{n} E_{ee}) - 2(\beta - \frac{\lambda}{2})(\delta_{bc} - \delta_{bd})(E_{bb} - \frac{1}{n} \sum_{e=1}^{n} E_{ee}) \]

\textbf{Proof.} By Proposition 3.4(iii), we have $Z_{ab} - Z_{12} = 2(\beta - \frac{4}{3})(H_{a1} + H_{b2})$. This implies
\[ Z_n = \sum_{e=1}^{n} Z_{e,e+1} = \sum_{e=1}^{n} (Z_{12} + 2(\beta - \frac{\lambda}{2})(H_{e1} + H_{e+1,2})) = nZ_{12} + 2(\beta - \frac{\lambda}{2}) \sum_{e=1}^{n} (H_{e1} + H_{e+1,2}) \]

Using Proposition 3.4(iii), we have
\[ W_{ab} = \frac{Z_{12}}{2} + (\beta - \frac{\lambda}{2})(H_{a1} + H_{b2}) = \frac{Z - 2(\beta - \frac{4}{3}) \sum_{e=1}^{n} (H_{e1} + H_{e+1,2})}{2n} + (\beta - \frac{\lambda}{2})(H_{a1} + H_{b2}) \]

Thus, by Proposition 3.4(ii),
\[ Z_{ab,cd} = (\epsilon_a - \epsilon_d, \epsilon_a - \epsilon_b) W_{ab} + (\beta - \frac{\lambda}{2})(\epsilon_a + \epsilon_b, \epsilon_c - \epsilon_d) H_{ab} \]
\[ = (\epsilon_a - \epsilon_d, \epsilon_a - \epsilon_b) \left( \frac{Z_n}{2n} + (\beta - \frac{\lambda}{2})(E_{aa} + E_{bb} - \frac{2}{n} \sum_{e=1}^{n} E_{ee}) \right) + (\beta - \frac{\lambda}{2})(\epsilon_a + \epsilon_b, \epsilon_c - \epsilon_d) H_{ab} \]
\[ = (\epsilon_a - \epsilon_b, \epsilon_c - \epsilon_d) \frac{Z_n}{2n} + 2(\beta - \frac{\lambda}{2})(\delta_{ac} - \delta_{ad})(E_{aa} - \frac{1}{n} \sum_{e=1}^{n} E_{ee}) - 2(\beta - \frac{\lambda}{2})(\delta_{bc} - \delta_{bd})(E_{bb} - \frac{1}{n} \sum_{e=1}^{n} E_{ee}) \]
This completes the proof. \hfill \square

\textbf{Lemma 3.10.} We have the following identity, for any $1 \leq a, b, c, d \leq n$,
\[ \sum_{i \neq j} (\epsilon_{ab}, \epsilon_{ij})(\epsilon_{ij}, \epsilon_{cd}) \left( \frac{E_{ii} + E_{jj}}{n} - \frac{2}{n^2} \sum_{e=1}^{n} E_{ee} \right) = 2(\delta_{ac} - \delta_{ad})(E_{aa} - \frac{1}{n} \sum_{e=1}^{n} E_{ee}) - 2(\delta_{bc} - \delta_{bd})(E_{bb} - \frac{1}{n} \sum_{e=1}^{n} E_{ee}). \]

\textbf{Proof.} The identity follows from a direct computation. For the convenience of the reader, we include a proof here. Set $\epsilon_{ab} = \epsilon_a - \epsilon_b$. We simplify the left hand side using the identity:
\[ (\epsilon_{ab}, \epsilon_{ij})(\epsilon_{ij}, \epsilon_{cd}) = (\delta_{ai} - \delta_{aj} - \delta_{bi} + \delta_{bj})(\delta_{ic} - \delta_{id} - \delta_{jc} + \delta_{jd}). \]
After simplification, we have:

\[
\sum_{i \neq j} (\epsilon_{ab}, \epsilon_{ij})(\epsilon_{ij}, \epsilon_{cd}) \left( \frac{E_{ii} + E_{jj}}{n} - \frac{2}{n^2} \sum_{e=1}^{n} E_{ee} \right) = 2 \sum_{j} (\delta_{ac} - \delta_{ad}) \left( \frac{E_{aa} + E_{jj}}{n} - \frac{2}{n^2} \sum_{e=1}^{n} E_{ee} \right) - 2 \sum_{j} (\delta_{bc} - \delta_{bd}) \left( \frac{E_{bb} + E_{jj}}{n} - \frac{2}{n^2} \sum_{e=1}^{n} E_{ee} \right) = 2(\delta_{ac} - \delta_{ad})(E_{aa} - \frac{1}{n} \sum_{e=1}^{n} E_{ee}) - 2(\delta_{bc} - \delta_{bd})(E_{bb} - \frac{1}{n} \sum_{e=1}^{n} E_{ee}).
\]

Thus, the conclusion follows. \(\Box\)

**Proof of Proposition 3.7.** Using the identity

\[(\epsilon_{ab}, \epsilon_{cd}) = \frac{1}{n} \sum_{i < j} (\epsilon_{ab}, \epsilon_{ij})(\epsilon_{ij}, \epsilon_{cd}),\]

together with Lemma 3.9, Lemma 3.10, we have

\[
Z_{ab,cd} = (\epsilon_{a} - \epsilon_{b}, \epsilon_{c} - \epsilon_{d}) \frac{Z_{n}}{2n} + 2(\beta - \frac{\lambda}{2})(\delta_{ac} - \delta_{ad})(E_{aa} - \frac{1}{n} \sum_{e=1}^{n} E_{ee}) - 2(\beta - \frac{\lambda}{2})(\delta_{bc} - \delta_{bd})(E_{bb} - \frac{1}{n} \sum_{e=1}^{n} E_{ee}) = \sum_{i < j} (\epsilon_{ab}, \epsilon_{ij})(\epsilon_{ij}, \epsilon_{cd}) \frac{Z_{n}}{2n} + \sum_{i < j} (\epsilon_{ab}, \epsilon_{ij})(\epsilon_{ij}, \epsilon_{cd}) 2(\beta - \frac{\lambda}{2})(\frac{E_{ii} + E_{jj}}{n} - \frac{2}{n^2} \sum_{e=1}^{n} E_{ee}).
\]

By the definition of \(Z_{ab,cd}\), we get:

\[
[K(H_{ab}), Q(H_{cd})] = \sum_{i < j} (\epsilon_{ab}, \epsilon_{ij})(\epsilon_{ij}, \epsilon_{cd}) \left( \frac{\lambda}{2}(E_{ij}E_{ji} + E_{ji}E_{ij}) + \frac{Z_{n}}{2n^2} + 2(\beta - \frac{\lambda}{2})(\frac{E_{ii} + E_{jj}}{n} - \frac{2}{n^2} \sum_{e=1}^{n} E_{ee}) \right).
\]

Thus, the map \(t_{\mathfrak{h}_{n}}^{\Phi} \to D_{\lambda,\beta}(\mathfrak{s}_{i_{n}})\) is well-defined. This completes the proof of Proposition 3.7. \(\Box\)

4. **Yangians and the deformed double current algebra**

In this section, we show that as the imaginary part of \(\tau\) approaches \(\infty\), the connection \(\nabla_{\text{Ell,C}}\) degenerates to a trigonometric connection of the form considered in [38]. This gives a map from the trigonometric Lie algebra \(A_{\text{trig}}\) to a completion of \(D_{\lambda}(\mathfrak{g})\). We define the following grading of \(D_{\lambda}(\mathfrak{g})\)

\[\operatorname{deg}(X) = 0, \operatorname{deg}(K(X)) = 0, \operatorname{deg}(Q(X)) = 1, \text{ and } \operatorname{deg}(\lambda) = 1.\]

Let \(\widetilde{D}_{\lambda}(\mathfrak{g})\) be the completion of \(D_{\lambda}(\mathfrak{g})\) with respect to this grading. We expect this map \(A_{\text{trig}} \to \widetilde{D}_{\lambda}(\mathfrak{g})\) extends to a map from the Yangian \(Y_{\mathfrak{h}_{n}}\) to \(\widetilde{D}_{\lambda}(\mathfrak{g})\).

4.1. **Trigonometric Casimir connection.** Toledano Laredo constructed the trigonometric Casimir connection by specializing the universal trigonometric connection \(\nabla_{\text{trig}}\) (5). It is obtained via a Lie algebra homomorphism from the coefficient algebra \(A_{\text{trig}}\) to \(\nabla_{\text{trig}}\) to the Yangian of \(\mathfrak{g}\). We now recall it here.

Let \(H = \operatorname{Hom}_{\mathbb{C}}(Q, \mathbb{C}^{*})\) be the complex algebraic torus with Lie algebra \(\mathfrak{h}\) and coordinate ring given by the group algebra \(\mathbb{C}[Q]\). We denote the function corresponding to \(\lambda \in Q\) by \(e^{\lambda} \in \mathbb{C}[H]\), and set

\[H_{\text{reg}} = H \setminus \bigcup_{\alpha \in \Phi} \{ e^{\alpha} = 1 \}\]

to be the complement of the root hypertori of the maximal torus \(H\).
This trigonometric Casimir connection $\nabla_{\text{trig},C}$ is a connection on the trivial vector bundle $H_\text{reg} \times V$, where the fiber $V$ is a finite-dimensional representation of the Yangian $Y_{\mathfrak{g}}$, which is a deformation of $U(\mathfrak{g}[s])$.

**Definition 4.1.** The Yangian $Y_{\mathfrak{g}}$ is the associative algebra over $\mathbb{C}[h]$ generated by elements $x, J(x), x \in \mathfrak{g}$ subject to the relations:

1. $Ax + \mu y$ (in $Y(\mathfrak{g})$) = $\lambda x + \mu y$ (in $\mathfrak{g}$).
2. $xy - yx = [x, y]$.
3. $J(Ax + \mu y) = \lambda J(x) + \mu J(y)$.
4. $[x, J(y)] = J([x, y])$.
5. $[J(x), J(y)] + [J(z), J([x, y])] + [J(y), J([z, x])] = h^2([x, x_a], [[y, y_b], [z, z_c]])[x^a, x^b, x^c]$.
6. $[J(x), J(y)] + [J(z), J([x, y])] = h^2([x, x_a], [[y, y_b], [z, z_c]])[x^a, x^b, J(x^c)]$,

for any $x, y, z, w \in \mathfrak{g}$ and $\lambda, \mu \in \mathbb{C}$, where $\{x_a\}$ are dual bases of $\mathfrak{g}$ with respect to $\langle, \rangle$ and

$$\sum_{c(1)<c(2)<c(3)} z_{c(1)}z_{c(2)}z_{c(3)} = \frac{1}{24} \sum_{c \in S_3} z_{c(1)}z_{c(2)}z_{c(3)}.$$

**Theorem 4.2 ([38]).** The trigonometric Casimir connection valued in $Y_{\mathfrak{g}}$

$$\nabla_{\text{trig},C} = d - \pi i h \sum_{\alpha \in \Phi^+} \frac{e^{2\pi i \alpha} + 1}{e^{2\pi i \alpha} - 1} d\alpha \kappa_\alpha + 2du(\alpha)$$

is flat and $W$-equivariant.

4.2. In this section, we describe the degeneration of the elliptic Casimir connection $\nabla_{\text{ell},C}$ as the imaginary part of $\tau$ tends to $\infty$.

Let $q = e^{2\pi i \tau}$. As $\Im \tau \to +\infty$, we have $q \to 0$. Using the product formula (9) of theta function, one can show that (see also [40])

$$k(\alpha, \ad(\frac{\alpha}{2})) \rightarrow 2\pi i \left( \frac{1}{e^{2\pi i \alpha} - 1} + \frac{e^{2\pi i \alpha} + 1}{e^{2\pi i \alpha} - 1} \right) - \frac{1}{\ad(\frac{\alpha}{2})}.$$

Therefore, the connection $\nabla_{\text{ell},C}$ degenerates to the following.

$$\nabla_{\text{ell},C} \rightarrow d - \sum_{\alpha \in \Phi^+} \left(2\pi i \left( \frac{1}{e^{2\pi i \alpha} - 1} + \frac{e^{2\pi i \alpha} + 1}{e^{2\pi i \alpha} - 1} \right) - \frac{1}{\ad(\frac{\alpha}{2})}\right) - \frac{1}{\ad(\frac{\alpha}{2})}d\alpha + \sum_{i=1}^{n} K(u^i)du_i$$

$$= d - \sum_{\alpha \in \Phi^+} \left( \frac{\lambda \pi i \kappa_\alpha}{e^{2\pi i \alpha} - 1} + \frac{1}{2} \frac{2\pi e^{2\pi i \alpha} \ad(\frac{\alpha}{2})}{e^{2\pi i \alpha} - 1} - \frac{1}{\ad(\frac{\alpha}{2})}\right) + \sum_{i=1}^{n} K(u^i)du_i$$

$$= d - \sum_{\alpha \in \Phi^+} \left( \frac{\lambda \pi i e^{2\pi i \alpha} + 1}{e^{2\pi i \alpha} - 1} \kappa_\alpha + \frac{1}{2} \left( \frac{2\pi e^{2\pi i \alpha} \ad(\frac{\alpha}{2})}{e^{2\pi i \alpha} - 1} + 1 \right) - \frac{1}{\ad(\frac{\alpha}{2})}\right) + \sum_{i=1}^{n} K(u^i)du_i.$$

Note that the constant term of $\frac{2\pi e^{2\pi i \alpha} \ad(\frac{\alpha}{2})}{e^{2\pi i \alpha} - 1} - \frac{1}{\ad(\frac{\alpha}{2})}$ is $\pi i$. This gives the following.

**Proposition 4.3.** As $\Im \tau \to +\infty$, the elliptic Casimir connection $\nabla_{\text{ell},C}$ degenerates to the following flat connection $\tilde{\nabla}$:

$$\tilde{\nabla} = d - \sum_{\alpha \in \Phi^+} \left( \frac{\lambda \pi i e^{2\pi i \alpha} + 1}{2} \kappa_\alpha + \frac{1}{2} \left( \frac{2\pi e^{2\pi i \alpha} \ad(\frac{\alpha}{2})}{e^{2\pi i \alpha} - 1} + 1 \right) - \frac{1}{\ad(\frac{\alpha}{2})}\right) + \sum_{i=1}^{n} K(u^i)du_i.$$
By the universality of the trigonometric connection $\nabla_{\text{trig}}$ (5), Proposition 4.3 induces a map from the trigonometric Lie algebra $A_{\text{trig}}$ to the completion of $D_{\lambda}(\mathfrak{g})$, given by $t_\alpha \mapsto \kappa_\alpha$, and for $h \in \mathfrak{h}$,
\[
\delta(h) \mapsto -K(h) + \sum_{\alpha \in \Phi^+} (\alpha, h) \left( \frac{\lambda}{2} \left( \frac{\pi i e^{2\pi i \alpha X}}{e^{2\pi i \alpha X} Q(\alpha\tilde{\alpha})} \right) + \frac{1}{\text{ad}(Q(\alpha\tilde{\alpha})/2)} \right) \kappa_\alpha + \left( \frac{e^{2\pi i \alpha X} + 1}{e^{2\pi i \alpha X} - 1} \right) \pi i Z.
\]
Note that the morphism $A_{\text{trig}} \to \widehat{D}_{\lambda}(\mathfrak{g})$ is not unique. For example, the assignment $t_\alpha \mapsto \kappa_\alpha$, and
\[
\delta(h) \mapsto -K(h) + \frac{\lambda}{2} \sum_{\alpha \in \Phi^+} (\alpha, h) \left( \pi i e^{2\pi i \alpha X} \left( \frac{\pi i e^{2\pi i \alpha X}}{e^{2\pi i \alpha X} Q(\alpha\tilde{\alpha})} \right) + \frac{1}{\text{ad}(Q(\alpha\tilde{\alpha})/2)} \right) \kappa_\alpha
\]
also defines a morphism $A_{\text{trig}} \to \widehat{D}_{\lambda}(\mathfrak{g})$. Indeed, the element $Z \in D_{\lambda}(\mathfrak{g})$ is central. The relation
\[
[t_\alpha, \delta(h)] = 0, \text{ if } \alpha(h) = 0
\]
in $A_{\text{trig}}$ is preserved if we shift the image of $\delta(h)$ by the central element $Z$.

4.3. We have the following diagram
\[
\begin{array}{ccc}
A_{\text{trig}} & \longrightarrow & \widehat{D}_{\lambda}(\mathfrak{g}) \\
\downarrow & & \\
Y_{\mathfrak{h}\mathfrak{g}} & & \\
\end{array}
\]
where the horizontal map $A_{\text{trig}} \to \widehat{D}_{\lambda}(\mathfrak{g})$ follows from the degeneration of the elliptic Casimir connection, and the vertical map $A_{\text{trig}} \to Y_{\mathfrak{h}\mathfrak{g}}$ gives the trigonometric Casimir connection in [38]. We show that the Lie algebra homomorphism $A_{\text{trig}} \to \widehat{D}_{\lambda}(\mathfrak{g})$ extends to an algebra homomorphism $Y_{\mathfrak{h}\mathfrak{g}} \to \widehat{D}_{\lambda}(\mathfrak{g})$. In this section, we give an explicit formula for this extension.

**Theorem 4.4.** The following assignment gives an algebra morphism $Y_{\mathfrak{h}\mathfrak{g}} \to \widehat{D}_{\lambda}(\mathfrak{g})$: $\mathfrak{h} \mapsto \frac{1}{2}$, and for any $X \in \mathfrak{g}$,
\[
X \mapsto X, \quad J(X) \mapsto \frac{1}{2} K(X) - \frac{A}{4} \left[ Q(X), \sum_{n \geq 0} c_{2n+1} \sum_{p+q=2n} \left( \frac{2n}{p} \right) (-1)^p \Omega_{p,q} \right],
\]
where
\[
\Omega_{p,q} := \sum_{\alpha \in \Phi} (X_{\alpha} \otimes u^p)(X_{-\alpha} \otimes u^q) + \sum_{i} (h_i \otimes u^i)(h_i^\prime \otimes u^q),
\]
and the constants $c_{2n+1}$ are the coefficients of the expansion $\pi i \sum_{n \geq 0} c_{2n+1} \frac{2n+1}{2^n}$.

We first explain how to obtain the formula in Theorem 4.4. We require that, when restricting on the enveloping algebra $U\mathfrak{g} \subset Y_{\mathfrak{h}\mathfrak{g}}$ of $\mathfrak{g}$, the map is an identity. That is, $X \mapsto X$, for $X \in \mathfrak{g}$. The morphism $A_{\text{trig}} \to \widehat{D}_{\lambda}(\mathfrak{g})$ implies that, for any $h \in \mathfrak{h}$,
\[
2J(h) \mapsto K(h) - \frac{A}{2} \sum_{\alpha \in \Phi^+} (\alpha, h) \left( \frac{\pi i e^{2\pi i \alpha X} Q(\alpha\tilde{\alpha})/2}{e^{2\pi i \alpha X} - 1} \right) \kappa_\alpha.
\]
We use the relation $[X, J(Y)] = J[X, Y]$ of the Yangian $Y_{\mathfrak{h}\mathfrak{g}}$ to deduce the image of $J(X)$ in Theorem 4.4, for $X \in \mathfrak{g}$.

**Lemma 4.5.** For any $n \in \mathbb{N}$, $h \in \mathfrak{h}$, we have
\[
\sum_{\alpha \in \Phi^+} (\alpha, h) \text{ad}(Q(\alpha\tilde{\alpha})/2)^{2n+1} \kappa_\alpha = \sum_{p+q=2n} \left( \frac{2n}{p} \right) (-1)^p [Q(h), \Omega_{p,q}].
\]
Proof. We have, for any \( m \in \mathbb{N} \), for a fixed root \( \alpha \in \Phi 
abla \\
abla \frac{\alpha}{2} \) 

\[
\text{ad}(Q^{\frac{\alpha}{2}})^{m}(\kappa_{\alpha})
= \sum_{p+q=m} \binom{m}{p} \text{ad}(Q^{\frac{\alpha}{2}})^{p}(X_{\alpha}) \text{ad}(Q^{\frac{\alpha}{2}})^{q}(X_{\alpha}) + \sum_{p+q=m} \binom{m}{p} \text{ad}(Q^{\frac{\alpha}{2}})^{p}(X_{\alpha}) \text{ad}(Q^{\frac{\alpha}{2}})^{q}(X_{\alpha})

= \sum_{p+q=m} \binom{m}{p}(-1)^{p}(X_{\alpha} \otimes u^{p})(X_{\alpha} \otimes u^{q}) + \sum_{p+q=m} \binom{m}{p}(-1)^{p}(X_{\alpha} \otimes u^{p})(X_{\alpha} \otimes u^{q}).
\]

Let \( h \in \mathfrak{h} \) be any element in the Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \). We decompose \( h \) as \( h = (h, \alpha)\frac{\alpha}{2} + h' \), where \((\alpha, h') = 0\). Using the fact that \([Q(h'), Q(\frac{\alpha}{2})] = 0\) and \([Q(h'), \kappa_{\alpha}] = 0\), we have

\[
\sum_{\alpha \in \Phi^{+}} (\alpha, h) \text{ad}(Q^{\frac{\alpha}{2}})^{2n+1}(\kappa_{\alpha}) = [Q(h), \sum_{\alpha \in \Phi^{+}} \text{ad}(Q^{\frac{\alpha}{2}})^{2n}(\kappa_{\alpha})]
= [Q(h), \sum_{\alpha \in \Phi^{+}} \sum_{p+q=2n} \binom{2n}{p}(-1)^{p}(X_{\alpha} \otimes u^{p})(X_{\alpha} \otimes u^{q})]
= \sum_{p+q=2n} \binom{2n}{p}(-1)^{p}[Q(h), \Omega_{p,q}].
\]

where the last equity follows from the relation \([Q(h), h' \otimes u^{m}] = 0\), for any \( h' \in \mathfrak{h} \), and \( m \in \mathbb{N} \). This completes the proof. \( \square \)

Lemma 4.6. Let \( X \in \mathfrak{g} \) be any element in \( \mathfrak{g} \) and \( X \not\in \mathfrak{h} \). Choose \( h \in \mathfrak{h} \), such that \([h, X] = X\). We have

\[
J(X) := [J(h), X] = \frac{1}{2} K(X) - \frac{\lambda}{4} [Q(X), \sum_{n \geq 0} c_{2n+1} \sum_{p+q=2n} \binom{2n}{p}(-1)^{p} \Omega_{p,q}].
\]

Proof. Lemma 4.5 together with the identity \([\Omega_{p,q}, X] = 0\) imply for any \( n \in \mathbb{N} \),

\[
\sum_{\alpha \in \Phi^{+}} (\alpha, h) \text{ad}(Q^{\frac{\alpha}{2}})^{2n+1}(\kappa_{\alpha}), \ X = \sum_{p+q=2n} \binom{2n}{p}(-1)^{p}[Q(X), \Omega_{p,q}]. \tag{17}
\]

The function \( (\frac{\pi i^{2n+1}}{e^{2\pi i x} - 1} - \frac{1}{x}) \) is an odd function in \( x \) with non-constant term. Indeed, one can easily check that \( \lim_{x \to 0} (\frac{\pi i^{2n+1}}{e^{2\pi i x} - 1} - \frac{1}{x}) = 0 \). Therefore, we have the expansion in the neighborhood of \( x = 0 \),

\[
(\frac{\pi i^{2n+1}}{e^{2\pi i x} - 1} - \frac{1}{x}) = \sum_{n \geq 0} c_{2n+1} x^{2n+1}, \text{ where } c_{2n+1} \text{ is the efficient.}
\]

By (17), we have

\[
\sum_{\alpha \in \Phi^{+}} (\alpha, h) (\frac{\pi i^{2n} \text{ad}(Q^{\frac{\alpha}{2}})}{e^{2\pi i \text{ad}(Q^{\frac{\alpha}{2}})} - 1} - \frac{1}{\text{ad}(Q^{\frac{\alpha}{2}})}) \kappa_{\alpha}, X = \sum_{n \geq 0} c_{2n+1} \sum_{\alpha \in \Phi^{+}} (\alpha, h) \text{ad}(Q^{\frac{\alpha}{2}})^{2n+1}(\kappa_{\alpha}), X
= \sum_{n \geq 0} c_{2n+1} \sum_{p+q=2n} \binom{2n}{p}(-1)^{p}[Q(X), \Omega_{p,q}].
\]

As a consequence, the image of \( J(X) \) is

\[
J(X) \mapsto \frac{1}{2} K(X) - \frac{\lambda}{4} [Q(X), \sum_{n \geq 0} c_{2n+1} \sum_{p+q=2n} \binom{2n}{p}(-1)^{p} \Omega_{p,q}].
\]
This completes the proof.

Notice that the formula in Lemma 4.6 works for any element $X \in \mathfrak{g}$. The relation $[J(X_\beta), X_{-\beta}] = J(H_\beta)$ in the Yangian forces the image of $J(h)$ to be

$$J(h) \mapsto \frac{1}{2}K(h) - \frac{\lambda}{4} \sum_{\alpha \in \Phi^\ast} (\alpha, h) \left( \frac{\pi i e^{2\pi i \text{ad}(\frac{K(h)}{2})} + 1}{e^{2\pi i \text{ad}(\frac{2\alpha}{2})} - 1} - \frac{1}{\text{ad}(\frac{\lambda}{2})} \right) k_\alpha$$

In other words, there is no shift of the central element $Z$ in the above formula.

**Lemma 4.7.** The assignment in Lemma 4.6 preserves the relation $[J(X), Y] = J([X, Y])$, for any $X, Y \in \mathfrak{g}$.

**Proof.** This follows from the relations $[K(X), Y] = K([X, Y])$, $[Q(X), Y] = Q([X, Y])$, and $[\Omega_{p,q}, X] = 0$, for any $X, Y \in \mathfrak{g}$. □

### 4.4. Proof of Theorem 4.4

We use the following presentation of $Y_{h \mathfrak{g}}$, which was obtained by Guay-Nakajima-Wendlandt in [21] for a symmetrizable Kac-Moody algebra $\mathfrak{g}$ under certain assumption on the Cartan matrix. It eliminates the relation (1.6) in [28, Theorem 1.2] when $\mathfrak{g}$ is finite dimensional. For our purposes, we state the result when $\mathfrak{g}$ is finite dimensional, and $\mathfrak{g} \neq sl_2$.

**Theorem 4.8.** ([28, Thm. 1.2], [21, Thm. 2.12]) The Yangian $Y_{h \mathfrak{g}}$ is isomorphic to the $\mathbb{C}$-algebra generated by elements $X_{i,r}^\pm, H_{i,r}$ for $i \in I$ and $r = 0, 1$ which satisfy the following relations for $i, j \in I$:

$$[H_{i,r}, H_{j,s}] = 0, \quad [H_{i,0}, X_{j,s}^\pm] = \pm(\alpha_i, \alpha_j)X_{j,s}^\pm \quad \forall \ r, s \in \{0, 1\}$$

$$[X_{i,r}^+, X_{j,s}^-] = \delta_{ij}H_{i,r+s} \quad \text{for} \ r + s = 0, 1$$

$$[X_{i,1}^+, X_{j,0}^-] - [X_{i,0}^+, X_{j,1}^-] = \pm \frac{h(\alpha_i, \alpha_j)}{2} S(X_{i,0}^+, X_{j,0}^-)$$

$$[H_{i,1}, X_{j,0}^\pm] - [H_{i,0}, X_{j,1}^\pm] = \pm \frac{h(\alpha_i, \alpha_j)}{2} S(H_{i,0}, X_{j,0}^\pm) \text{ad}(X_{i,0}^\pm)^{1-\alpha_i/(X_{j,0}^\pm)} = 0$$

where $I$ is the set of vertices of the Dynkin diagram of $\mathfrak{g}$, and $a_{ij}$ is the $i$-th entry of the Cartan matrix.

The isomorphism between this presentation of $Y_{h \mathfrak{g}}$ and the one given in Definition 4.1 sends $X_{i,1}^+$ to $J(X_{i}^+) - h\omega_i^+$, and $H_{i,1}^+ = J(H_i) - h\nu_i^+$, where

$$\omega_i^+ = \pm \frac{1}{4} \sum_{\alpha \in \Phi^\ast} S([X_i^+, X_\alpha^\pm], X_\alpha^\pm) - \frac{1}{4} S(X_i^+, H_i) \quad \text{and} \quad \nu_i = [\omega_i^+, X_i^-] = \frac{1}{4} \sum_{\alpha \in \Phi^\ast} (\alpha_i, \alpha) S(X_\alpha^+, X_\alpha^-) - \frac{H_i^2}{2}. \quad (18)$$

**Proof of Theorem 4.4.** To show Theorem 4.4, most of the relations in Theorem 4.8 follow directly from the fact that $[J(X), X'] = J([X, X'])$ for all $X, X' \in \mathfrak{g}$ in Lemma 4.7.

It remains to show that $[H_{i,1}, H_{j,1}] = 0$ for all $i, j \in I$. This follows from the map $A_{\text{rig}} \to D_{\lambda}(\mathfrak{g})$ in §4.2 obtained from the degeneration of the elliptic Casimir connection. Indeed, we have the following diagram

$$A_{\text{rig}} \xrightarrow{f_1} D_{\lambda}(\mathfrak{g})$$

$$\xrightarrow{f_2} Y_{h}(f_2)$$

(19)
The image of $X(H_i)$ in $\widehat{D}_A(\mathfrak{g})$ is given by
\[
f_1(X(H_i)) = f_1\left(\delta(H_i) + \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha(H_i) t_{\alpha}\right)
= \frac{2}{\hbar} \sum_{\alpha \in \Phi^+} (\alpha, \alpha) \kappa_\alpha
\]
\[
= -2f_2(H_{i,1}) + \hbar H_i^2.
\]
Therefore, the relation $[X(H_i), X(H_j)] = 0$ in $A_{\text{trig}}$, and the obvious relation $[f_2(H_{i,1}), H_i^2] = 0$ in $\widehat{D}_A(\mathfrak{g})$ imply that $[f_2(H_{i,1}), f_2(H_{j,1})] = 0$. This completes the proof. 

5. The elliptic KZ connection

In this section, we show that the rational Cherednik algebra $H_{h,c}$ of $W$ is, very roughly speaking, the "Weyl group" of the deformed double current algebra $D_A(\mathfrak{g})$. This is an analogy relation between the degenerate affine Hecke algebra and the Yangian in [38]. More precisely, we show that if $V$ is a $D_A(\mathfrak{g})$–module whose restriction to $\mathfrak{g}$ is small, the canonical action of $W$ on the zero weight space $V[0]$ extends to one of $H_{h,c}$. Moreover, the elliptic Casimir connection with values in $V[0]$ coincides with elliptic KZ connection with values in this $H_{h,c}$–module.

5.1. The rational Cherednik algebra. In this subsection, we recall the definition of the rational Cherednik algebras. For details, see [13].

Let $W$ be a Weyl group and $\mathfrak{h}$ be its reflection space. For any reflection $s \in W$, fix $\alpha_s \in \mathfrak{h}^*$, such that $s(\alpha_s) = -\alpha_s$. Write $S$ for the collection of linear functions
\[
\{\pm \alpha_s \mid s \text{ is reflections in } W\}.
\]
Let $c : S \to \mathbb{C}, s \mapsto c_s$ be a $W$-invariant function.

**Definition 5.1.** The rational Cherednik algebra $H_{h,c}$ is the quotient of the algebra $\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)[\hat{h}]$ (where $T$ denotes the tensor algebra) by the ideal generated by the relations
\[
[x, x'] = 0, \ [y, y'] = 0, \ [y, x] = \hbar(y, x) - \sum_{s \in S} c_s(\alpha_s, y)(\alpha_s^\vee, x)s,
\]
where $x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}$.

5.2. The elliptic KZ connection. The elliptic KZ connection valued in the rational Cherednik algebra $H_{h,c}$ is constructed by Calaque-Enriquez-Etingof, in [8] for type A and by Toledano Laredo-Yang in [40] for any finite (reduced, crystallographic) root system $\Phi$.

Let $T_{\text{reg}} = T \setminus \cup_{\alpha \in \Phi} T_{\alpha}$ be elliptic configuration space. This elliptic KZ connection $\nabla_{H_{h,c}}$ is a connection on the vector bundle $P_{T,0}^{\infty}$, whose the fiber $V$ is a finite-dimensional representation of the rational Cherednik algebra $H_{h,c}$. Let $\{u_i\}$ and $\{u^i\}$ be the dual basis of $\mathfrak{h}$ and $\mathfrak{h}^*$, the elliptic KZ connection takes the form:
\[
\nabla_{H_{h,c}} = d + \sum_{\alpha \in \Phi^+,} \frac{2c_\alpha}{\alpha(\alpha)}k(\alpha, \text{ad}(\frac{x(\alpha^\vee)}{2})\tau)s_{\alpha}d\alpha - \sum_{\alpha \in \Phi^+} \frac{\hbar}{h^\vee} \frac{\theta'(\alpha|\tau)}{\theta(\alpha|\tau)} \frac{1}{\alpha(\alpha)} d\alpha + \sum_{i=1}^n y(u^i)du_i.
\]
(20)

where
- $x(\alpha^\vee) \in \mathfrak{h}$, $y(u^i) \in \mathfrak{h}^*$ and $s_{\alpha} \in W$ is the reflection associated to the root $\alpha$.
- $h^\vee$ is the dual Coxeter number.

The connection $\nabla_{H_{h,c}}$ is flat and $W$-equivariant and its monodromy yields a one parameter family of monodromy representations of the elliptic braid group $\pi_1(T_{\text{reg}}/W)$. Furthermore, as shown in [8, 40], the monodromy representations factor through the double affine Hecke algebra.
5.3. Small g-modules. Recall that a g–module V is small, if 2α is not a weight of V for any root α [5, 32, 33].

Lemma 5.2. [38, Lemma 7.4] If V is an integrable, small g–module, the following holds on the zero weight space V[0]

\[ \kappa_\alpha = (\alpha, \alpha)(1 - s_\alpha), \]

where the right hand side refers to the action of the reflection \( s_\alpha \in W \) on V[0].

5.4. The elliptic KZ and elliptic Casimir connection. In this subsection, we focus on the category of finite dimensional representations of \( D_4(\mathfrak{g}) \), whose restriction to g is small.

Let V be such module. We have the decomposition \( \Phi^+ = \Phi_1^+ \cup \Phi_s^+ \), where \( \Phi_1^+ \) is the set of positive long roots and \( \Phi_s^+ \) the set of positive short roots. Denote the corresponding dual Coxeter numbers of \( \Phi_l \) and \( \Phi_s \) by \( h_1^\vee \) and \( h_s^\vee \). For ADE case, \( \Phi_s^+ = 0 \), and \( h_s^\vee = 0 \). Assume the central element \( Z \in D_4(\mathfrak{g}) \) acts on V by the scalar \( h - \frac{1}{2}(2h_1^\vee + h_s^\vee) \), and \( \frac{1}{2}(\alpha, \alpha)^2 = 2c_s \).

Theorem 5.3. (i): The canonical \( \mathcal{W} \)–action on the zero weight space V[0] together with the assignment

\[ x_\alpha \mapsto Q(u), \quad y_\alpha \mapsto K(u) \]

yield an action of the rational Cherednik algebra on V[0].

(ii): The elliptic Casimir connection with values in \( \text{End}(V[0]) \) is equal to the sum of the elliptic KZ connection and the scalar valued one-form

\[ \mathcal{A} = \frac{1}{2} \sum_{\alpha, \beta, \gamma} \left( \frac{2h_\gamma^\vee + h_\alpha^\vee}{h_\gamma^\vee} - (\alpha, \alpha) \right) \frac{\theta'(\alpha|\alpha)}{\theta(\alpha|\alpha)} d\alpha \]

For the rest of this subsection, we prove Theorem 5.3. To begin with, we first need two lemmas.

Lemma 5.4. [40, Lemma 10.4] For any root system \( \Phi \), we have \( (u|v) = \frac{1}{\theta_\Phi} \sum_{\gamma \in \Phi_+}(y,u)(\gamma,v) \).

Lemma 5.5. We have \( \sum_{\alpha \in \Phi^+}(u,\alpha)(v,\alpha)(\alpha,\alpha) = (2h_1^\vee + h_s^\vee)(u,v) \).

Proof. For \( \alpha \in \Phi_1^+ \), we have \( (\alpha,\alpha) = 2 \), and for \( \alpha \in \Phi_s^+ \), we have \( (\alpha,\alpha) = 1 \). Then,

\[ \sum_{\alpha \in \Phi^+}(u,\alpha)(v,\alpha)(\alpha,\alpha) = 2 \sum_{\alpha \in \Phi_1^+}(u,\alpha)(v,\alpha) + \sum_{\alpha \in \Phi_s^+}(u,\alpha)(v,\alpha) = (2h_1^\vee + h_s^\vee)(u,v). \]

The last equality follows from Lemma 5.4.

Proof of Theorem 5.3. We rewrite the elliptic Casimir connection based on Lemma 5.2.

\[ \nabla_{\text{Ell,C}} = d - \frac{1}{2} \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}(Q(\alpha^\vee))(\alpha))(\kappa_\alpha) d\alpha - \sum_{\alpha \in \Phi^+} \frac{\theta'(\alpha|\alpha)}{\theta(\alpha|\alpha)} Z \frac{\theta(\alpha|\alpha)}{h_\alpha^\vee} d\alpha + \sum_{i=1}^n K(u^i)du_i \]

\[ = d - \frac{1}{2} \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}(Q(\alpha^\vee))(\alpha))(\alpha)(1 - s_\alpha) d\alpha - \sum_{\alpha \in \Phi^+} \frac{\theta'(\alpha|\alpha)}{\theta(\alpha|\alpha)} Z \frac{\theta(\alpha|\alpha)}{h_\alpha^\vee} d\alpha + \sum_{i=1}^n K(u^i)du_i \]

\[ = d + \frac{1}{2} \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}(Q(\alpha^\vee))(\alpha))s_\alpha d\alpha - \sum_{\alpha \in \Phi^+} \frac{\theta'(\alpha|\alpha)}{\theta(\alpha|\alpha)} \left( \frac{\lambda(\alpha, \alpha)}{2} + Z \right) \frac{\theta(\alpha|\alpha)}{h_\alpha^\vee} d\alpha + \sum_{i=1}^n K(u^i)du_i. \]

Consider the one-form:

\[ \mathcal{A} = \sum_{\alpha \in \Phi^+} \left( \frac{h - Z}{h_\alpha^\vee} - \frac{\lambda(\alpha, \alpha)}{2} \right) \frac{\theta'(\alpha|\alpha)}{\theta(\alpha|\alpha)} d\alpha. \]
By assumption, $Z$ acts on $V[0]$ by the scalar $h - \frac{1}{2}(2h_1^\vee + h_2^\vee)$. The one form $\mathcal{A}$ on $V[0]$ is simplified as follows.

$$\mathcal{A} = \sum_{\alpha \in \Phi^+} \left( \frac{\hbar - Z}{h^\vee} - \frac{\lambda}{2} (\alpha, \alpha) \right) \theta'(\alpha|\tau) d\alpha = \frac{1}{2} \sum_{\alpha \in \Phi^+} \left( \frac{2h_1^\vee + h_2^\vee}{h^\vee} - (\alpha, \alpha) \right) \frac{\theta'(\alpha|\tau)}{\theta(\alpha|\tau)} d\alpha.$$ 

Using the assumption $\frac{1}{2}(\alpha, \alpha) = 2c_1$, we have

$$\nabla_{EIL/C} - \mathcal{A} = d + \sum_{\alpha \in \Phi^+} \frac{2c_1}{(\alpha|\alpha)} \kappa(\alpha, \text{ad}(\frac{Q(\alpha^\vee)}{2})) s_\alpha d\alpha - \sum_{\alpha \in \Phi^+} \frac{h}{h^\vee} \frac{\theta'(\alpha|\tau)}{\theta(\alpha|\tau)} d\alpha + \sum_{i=1}^n K(u_i^\vee) d\lambda_i.$$ 

Note that $\mathcal{A}$ is a scalar valued one-form. Then, the flatness of $\nabla_{EIL/C}$ implies the flatness of $\nabla_{EIL/C} - \mathcal{A}$. The connection $\nabla_{EIL/C} - \mathcal{A}$ is also $W$-equivariant since both $\nabla_{EIL/C}$ and $\mathcal{A}$ are. Define an action of the rational Cherednik algebra on $V[0]$ by:

\[ x_u \mapsto Q(u), \quad y_u \mapsto K(u). \]

We now check this assignment preserves the defining relations of $H_{k,c}$. On $V[0]$, by the relation of $H_{k,c}$, the action of $[y(u), x(v)]$ is the same as the action of

\[ \hbar(u, v) - \sum_{\alpha \in \Phi^+} c_\alpha(\alpha, u)(\alpha^\vee, v)s_\alpha = \hbar(u, v) - \frac{1}{2} \sum_{\alpha \in \Phi^+} (\alpha, \alpha)(\alpha, u)(\alpha, v)s_\alpha, \]

(21)

On the other hand, using the relation of $D_j(g)$, the action of $[K(u), Q(v)]$ is the same as the action of

\[ \frac{1}{2} \sum_{\alpha \in \Phi^+} (u, \alpha)(v, \alpha)s_\alpha + (u, v) Z = \frac{1}{2} \sum_{\alpha \in \Phi^+} (u, \alpha)(v, \alpha)(\alpha, \alpha)(1 - s_\alpha) + (u, v) Z \]

\[ = \frac{1}{2} \sum_{\alpha \in \Phi^+} (u, \alpha)(v, \alpha)(\alpha, \alpha) + (u, v) Z - \frac{1}{2} \sum_{\alpha \in \Phi^+} (u, \alpha)(v, \alpha)(\alpha, s_\alpha). \]

(22)

By Lemma 5.5, and the assumption that $Z$ acts on $V[0]$ by $h - \frac{1}{2}(2h_1^\vee + h_2^\vee)$, we have $\frac{1}{2} \sum_{\alpha \in \Phi^+} (u, \alpha)(v, \alpha)(\alpha, \alpha) + (u, v) Z = \hbar(u, v)$. Therefore, (22) coincides with (21). This completes the proof. \[ \square \]

6. The dual pair $(gl_k, gl_n)$

In this section, we establish a duality between the KZB connection associated to $gl_k$ and the elliptic Casimir connection associated to $gl_n$.

6.1. Reductions. In this section, we recall one realization of the KZB connection from [8, §6.3]. Let $\mathfrak{g} = gl_k$. We have a decomposition $gl_k = \mathfrak{b}_k \oplus \mathfrak{n}_k$, where $\mathfrak{b}_k$ is the set of diagonal matrices, with natural basis $\{ E_{ii} \mid 1 \leq i \leq k \}$, and $\mathfrak{n}_k$ is the set of off-diagonal matrices, with natural basis $\{ E_{ij} \mid 1 \leq i \neq j \leq k \}$.

Identify $\mathfrak{b}_k^\ast$ with $\mathfrak{b}_k$ by the nondegenerate bilinear form $(X, Y) \mapsto \text{tr}(XY)$. Denote by $\mathfrak{b}_k^{\ast\text{reg}} \subset \mathfrak{b}_k^\ast$ the subset of diagonal matrices with distinct eigenvalues, that is $\mathfrak{b}_k^{\ast\text{reg}} = \{ \lambda \in \mathfrak{b}_k^\ast \mid \prod_{1 \leq i < j \leq k} (\lambda_i - \lambda_j) \neq 0 \}$, where $\lambda = \sum_{i=1}^k \lambda_i E_{ii}$.

Let $\text{Diff}(\mathfrak{b}_k)$ be the algebra of algebraic differential operators on $\mathfrak{b}_k$. It has generators $x_i, \partial_i$, for $1 \leq i \leq k$, and relations: $E_{ii} \mapsto x_i, E_{ii} \mapsto \partial_i$ are linear, and $[x_i, x_j] = [\partial_i, \partial_j] = 0$, $[\partial_i, x_j] = \delta_{ij}$.

Set $P := \prod_{1 \leq i < j \leq k} (x_i - x_j) \in S(\mathfrak{b}) \subset \text{Diff}(\mathfrak{b})$. The embedding $\mathfrak{b}_k \subset gl_k$ induces a map

$$\mathfrak{b}_k \to B_n := \text{Diff}(\mathfrak{b}_k)[\frac{1}{P}] \otimes U(gl_k)^{\otimes n}, \quad h \mapsto 1 \otimes \sum_{a=1}^n h^{(a)}.$$ 

Denote by $\mathfrak{b}_k^{\text{diag}}$ its image.
Definition 6.1. [8, Sec. 6.3] The Hecke algebra $\mathcal{H}(\mathfrak{g}_k, \mathfrak{b}_k)$ is defined to be

$$\{ x \in B_n \mid \sum_{i=1}^{n} h^{(i)}(x) \in B_n b_k^{\text{diag}}, \text{ for any } h \in \mathfrak{b}_k \}/B_n b_k^{\text{diag}}.$$ 

Let $V_1, \ldots, V_n$ be $g$–modules, then $S(\mathfrak{b}_k)[1/P] \otimes (\otimes_{i=1}^{n} V_i)$ is a module over $B_n$. Let $(S(\mathfrak{b}_k)[1/P] \otimes (\otimes_{i=1}^{n} V_i))^{b_k} = (S(\mathfrak{b}_k)[1/P] \otimes (\otimes_{i=1}^{n} V_i))[0]$ be the zero weight space. Then, $(S(\mathfrak{b}_k)[1/P] \otimes (\otimes_{i=1}^{n} V_i))^{b_k}$ is a module over $B_n/B_n b_k^{\text{diag}}$, hence a module over $\mathcal{H}(\mathfrak{g}_k, \mathfrak{b}_k)$.

Definition 6.2. [8, Sect. 1.1] Let $t_{1,n}$ be the Lie algebra with generators $x_i, y_i \ (i = 1, \ldots, n)$, and $t_{ij} \ (i \neq j \in \{1, \ldots, n\})$ and relations

$$t_{ij} = t_{ji}, [t_{ij}, t_{kj}] = 0, [t_{ij}, t_{jk}] = 0, [x_i, y_j] = 0, [x_i, y_i] = -\sum_{j \neq i} t_{ij},$$

$$[x_i, t_{jk}] = [y_i, t_{jk}] = 0, [x_i + x_j, t_{ij}] = [y_i + y_j, t_{ij}] = 0$$

$(i, j, k, l$ are distinct). In this Lie algebra $t_{1,n}$, $\sum_i x_i$ and $\sum_i y_i$ are central. We then define

$$\tilde{t}_{1,n} = t_{1,n}/(\sum_i x_i, \sum_i y_i).$$

Proposition 6.3. [8, Prop. 41]

1. There is an algebra homomorphism $t_{1,n} \to \mathcal{H}(\mathfrak{g}_k, \mathfrak{b}_k) \subset B_n/B_n b_k^{\text{diag}}$ given by:

$$x_i \mapsto \sum_{a=1}^{k} x_a \otimes E_{ia}, \quad y_i \mapsto -\sum_{a=1}^{k} \partial_a \otimes E_{ia} + \sum_{j=1}^{n} \sum_{1 \leq a \neq b \leq k} \frac{1}{x_b - x_a} \otimes E_{ab} E_{ib},$$

$$t_{ij} \mapsto \sum_{1 \leq a, b \leq k} E_{ab} E_{ib}, \text{ for } 1 \leq i \neq j \leq n.$$ 

2. The homomorphism factors through the quotient $t_{1,n} \to \tilde{t}_{1,n}$.

In [8, Prop. 41], an algebra homomorphism from $\tilde{t}_{1,n}$ to the Hecke algebra $\mathcal{H}(\mathfrak{g}_k, \mathfrak{b}_k)$ associated to an arbitrary semisimple Lie algebra $g$ is constructed. For the convenience of the reader, we provide a proof in the special case when the Hecke algebra is $\mathcal{H}(\mathfrak{g}_k, \mathfrak{b}_k)$. From the proof, it is clear that the map $t_{1,n} \to B_n/B_n b_k^{\text{diag}}$ does not factor through $B_n$. Indeed, as we will see that, the relations $[x_i, y_j] = 0$, $[x_i, y_i] = 0$ of $t_{1,n}$ are not preserved under the map $t_{1,n} \to B_n$. We now prove the Proposition.

Proof: The relations $[x_a, x_b] = 0$, $t_{ab} = t_{ba}$, $[x_a, t_{bc}] = 0$, where $a, b, c$ are distinct, are obviously preserved. We now check that the relation $[x_a, y_b] = t_{ab}$ for $a \neq b$ is preserved under the map. We have

$$[x_a, y_b] = \sum_{m} x_{E_{mm}} \otimes E_{am}^{(a)} - \sum_{i} \partial_i \otimes E_{ji}^{(b)} + \sum_{c} \sum_{1 \leq j \leq k} \frac{1}{x_j - x_i} \otimes E_{ij}^{(b)} E_{ji}^{(c)}$$

$$= \sum_{i} E_{ii}^{(a)} E_{ii}^{(b)} + \sum_{m \neq i} x_{E_{mm}} \otimes E_{mm}^{(a)} + \sum_{1 \leq j \leq k} \frac{1}{x_j - x_i} \otimes E_{ij}^{(b)} E_{ji}^{(a)}$$

$$= \sum_{i} E_{ii}^{(a)} E_{ii}^{(b)} + \sum_{m \neq i} E_{jm}^{(b)} E_{im}^{(a)} = t_{ab}.$$
We check that the relation \([x_a, y_a] = - \sum_{b \neq a} t_{ab}\) as follows.

\[
[x_a, y_a] = \left[ \sum_{m} x_{Emm} \otimes E^{(a)}_{mm} - \sum_{i} \partial_i \otimes E^{(a)}_{ii} + \sum_{1 \leq i \neq j \leq k} \frac{1}{x_j - x_i} \otimes E^{(a)}_{ij} E^{(c)}_{ji} \right]
\]

\[
= \sum_{i} \left[ \partial_i \otimes E^{(a)}_{ii}, x_i \otimes E^{(a)}_{ii} \right] + \sum_{m,c} \sum_{1 \leq i \neq j \leq k} \frac{x_{Emm}}{x_j - x_i} \otimes \left[ E^{(a)}_{mm}, E^{(a)}_{ij} E^{(c)}_{ji} \right]
\]

\[
= \sum_{i} E^{(a)}_{ii} - \sum_{1 \leq i \neq j \leq k} \frac{1}{x_j - x_i} \otimes \left( (\delta_{mi} E^{(a)}_{mj} - \delta_{mj} E^{(a)}_{im}) E^{(c)}_{ji} + \delta_{ac} E^{(a)}_{ij} (\delta_{mj} E^{(a)}_{mi} - \delta_{mi} E^{(a)}_{jm}) \right)
\]

\[
= \sum_{i} E^{(a)}_{ii} - \sum_{1 \leq i \neq j \leq k} \frac{1}{x_j - x_i} \otimes \left( E^{(a)}_{ij} E^{(c)}_{ji} - \delta_{ac} E^{(a)}_{ij} E^{(a)}_{ji} \right)
\]

\[
= \sum_{i} E^{(a)}_{ii} E^{(a)}_{ii} - \sum_{1 \leq i \neq j \leq k} \frac{1}{x_j - x_i} \otimes \left( E^{(a)}_{ij} E^{(c)}_{ji} + \delta_{ac} E^{(a)}_{ij} E^{(a)}_{ji} \right)
\]

\[
= - \sum_{1 \leq i \leq k} t_{ac} + \sum_{1 \leq i \leq k} E^{(a)}_{ii} E^{(c)}_{ii}
\]

For any \(1 \leq i \leq k\), the element \(\sum_{1 \leq m \leq n} E^{(a)}_{ii} E^{(c)}_{ii} = E^{(a)}_{ii} \cdot (\sum_{1 \leq m \leq n} E^{(c)}_{ii}) \in B_n b^\text{diag}_k\) is zero in \(B_n/B_n b^\text{diag}_k\). Thus, we have the relation

\[
[x_a, y_a] = - \sum_{b \neq a} t_{ab}.
\]

We check that the relation \([y_a, t_{bc}] = 0\) is mapped to zero under the map. We have

\[
[y_a, t_{bc}] = \left[ - \sum_{i} \partial_i \otimes E^{(a)}_{ii} + \sum_{m} \sum_{1 \leq i \neq j \leq k} \frac{1}{x_j - x_i} \otimes E^{(a)}_{ij} E^{(m)}_{ji}, \sum_{ij} E^{(b)}_{ij} E^{(c)}_{ji} \right]
\]

\[
= \left[ \sum_{1 \leq i \neq j \leq k} \frac{1}{x_j - x_i} \otimes E^{(a)}_{ij} (E^{(b)}_{ji} + E^{(c)}_{ji}), \sum_{ij} E^{(b)}_{ij} E^{(c)}_{ji} \right] = 0
\]

We now show the relation \([y_a, y_b] = 0\) is preserved under the map. We will use the following identity: for any \(A, B, C, D\),

\[
[A \otimes B, C \otimes D] = [A, C] \otimes BD + CA \otimes [B, D].
\]

For \(1 \leq a \neq b \leq n\), by definition, we have

\[
[y_a, y_b] = \left[ - \sum_{i=1}^{k} \partial_i \otimes E^{(a)}_{ii}, \sum_{c=1}^{n} \sum_{1 \leq i \neq j \leq k} \frac{1}{x_j - x_i} \otimes E^{(b)}_{ij} E^{(c)}_{ji} \right] \quad \text{(A)}
\]

\[
+ \left[ \sum_{c=1}^{n} \sum_{1 \leq i \neq j \leq k} \frac{1}{x_j - x_i} \otimes E^{(a)}_{ij} E^{(c)}_{ji}, \sum_{i=1}^{k} \partial_i \otimes E^{(b)}_{ii} \right] \quad \text{(B)}
\]

\[
+ \left[ \sum_{c=1}^{n} \sum_{1 \leq i \neq j \leq k} \frac{1}{x_j - x_i} \otimes E^{(a)}_{ij} E^{(c)}_{ji}, \sum_{d=1}^{k} \sum_{1 \leq i \neq j \leq k} \frac{1}{x_j - x_i} \otimes E^{(b)}_{ij} E^{(d)}_{ji} \right] \quad \text{(C)}
\]
We first compute the term (A). We have

$$\begin{align*}
(A) &= - \sum_{i=1}^{k} \sum_{c=1}^{n} \sum_{1 \leq i \neq j \leq k} \left[ \partial_i \frac{1}{x_i - x_j} \right] \otimes E_{ii}^{(a)} E_{si}^{(b)} E_{ts}^{(c)} - \sum_{i=1}^{k} \sum_{c=1}^{n} \sum_{1 \leq i \neq j \leq k} \frac{1}{x_i - x_j} \cdot \partial_i \otimes \left[ E_{ii}^{(a)}, E_{st}^{(b)} E_{is}^{(c)} \right] \\
&= - \sum_{c=1}^{n} \sum_{1 \leq s \neq t \leq k} \frac{1}{(x_t - x_s)^2} \otimes E_{tt}^{(a)} E_{si}^{(b)} E_{ts}^{(c)} - \sum_{c=1}^{n} \sum_{1 \leq s \neq t \leq k} \frac{1}{(x_t - x_s)^2} \otimes E_{ss}^{(a)} E_{st}^{(b)} E_{ts}^{(c)} \\
&\quad - \sum_{1 \leq s \neq t \leq k} \frac{1}{x_t - x_s} \cdot \partial_t \otimes E_{st}^{(b)} E_{ts}^{(a)} + \sum_{1 \leq s \neq t \leq k} \frac{1}{x_t - x_s} \cdot \partial_s \otimes E_{st}^{(a)} E_{ts}^{(b)}
\end{align*}$$

By symmetry of (A) and (B), we get

$$\begin{align*}
(B) &= \sum_{c=1}^{n} \sum_{1 \leq s \neq t \leq k} \frac{1}{(x_t - x_s)^2} \otimes E_{tt}^{(b)} E_{st}^{(a)} E_{ts}^{(c)} + \sum_{c=1}^{n} \sum_{1 \leq s \neq t \leq k} \frac{1}{(x_t - x_s)^2} \otimes E_{ss}^{(b)} E_{st}^{(a)} E_{is}^{(c)} \\
&\quad + \sum_{1 \leq s \neq t \leq k} \frac{1}{x_t - x_s} \cdot \partial_t \otimes E_{st}^{(b)} E_{ts}^{(a)} - \sum_{1 \leq s \neq t \leq k} \frac{1}{x_t - x_s} \cdot \partial_s \otimes E_{st}^{(a)} E_{ts}^{(b)}
\end{align*}$$

Thus, after cancelation, we conclude

$$\begin{align*}
(A) + (B) &= - \sum_{c=1}^{n} \sum_{1 \leq s \neq t \leq k} \frac{1}{(x_t - x_s)^2} \otimes E_{tt}^{(a)} E_{st}^{(b)} E_{ts}^{(c)} - \sum_{c=1}^{n} \sum_{1 \leq s \neq t \leq k} \frac{1}{(x_t - x_s)^2} \otimes E_{ss}^{(a)} E_{st}^{(b)} E_{ts}^{(c)} \\
&\quad + \sum_{c=1}^{n} \sum_{1 \leq s \neq t \leq k} \frac{1}{(x_t - x_s)^2} \otimes E_{tt}^{(b)} E_{st}^{(a)} E_{ts}^{(c)} + \sum_{c=1}^{n} \sum_{1 \leq s \neq t \leq k} \frac{1}{(x_t - x_s)^2} \otimes E_{ss}^{(b)} E_{st}^{(a)} E_{is}^{(c)}
\end{align*}$$

We now compute the term (C). We have

$$\begin{align*}
(C) &= \sum_{c,d=1}^{n} \sum_{1 \leq i \neq j \neq s \leq k} \frac{1}{(x_j - x_i)(x_t - x_s)} \otimes \left[ E_{ij}^{(a)} E_{ji}^{(c)} , E_{st}^{(b)} E_{is}^{(d)} \right] \\
&= \sum_{c=1}^{n} \left( \sum_{1 \neq i \neq j \neq s} \frac{1}{(x_j - x_i)(x_t - x_s)} \otimes E_{ij}^{(a)} E_{ji}^{(c)} - \sum_{i \neq j \neq s} \frac{1}{(x_j - x_i)(x_j - x_s)} \otimes E_{ij}^{(a)} E_{si}^{(b)} E_{ts}^{(c)} \right) \\
&\quad + \sum_{c=1}^{n} \left( \sum_{i \neq j \neq s} \frac{1}{(x_j - x_i)(x_t - x_s)} \otimes E_{ij}^{(a)} E_{st}^{(b)} E_{js}^{(c)} - \sum_{i \neq j \neq s} \frac{1}{(x_j - x_i)(x_t - x_j)} \otimes E_{ij}^{(a)} E_{jt}^{(b)} E_{ti}^{(c)} \right) \\
&\quad + \sum_{c=1}^{n} \left( \sum_{i \neq j \neq s} \frac{1}{(x_j - x_i)(x_t - x_s)} \otimes E_{is}^{(a)} E_{sx}^{(b)} E_{ji}^{(c)} - \sum_{i \neq j \neq t} \frac{1}{(x_j - x_i)(x_t - x_i)} \otimes E_{ij}^{(a)} E_{it}^{(b)} E_{ji}^{(c)} \right)
\end{align*}$$

When the indices $i, j, s$ or $i, j, t$ are distinct, using the identity

$$\frac{1}{(x_j - x_i)(x_t - x_i)} - \frac{1}{(x_j - x_i)(x_t - x_j)} = -\frac{1}{(x_t - x_j)(x_t - x_i)}$$
it is obvious that the corresponding summands in (23) add up to zero. Thus, we simplify the term (C) as follows.

\[
(C) = \sum_{c=1}^{n} \left( \sum_{i \neq j} \frac{1}{(x_j - x_i)^2} \otimes E_{ij}^{(a)} E_{jj}^{(b)} E_{ji}^{(c)} - \sum_{i \neq j} \frac{1}{(x_j - x_i)^2} \otimes E_{ij}^{(a)} E_{ji}^{(b)} E_{jj}^{(c)} \right) \\
+ \sum_{c=1}^{n} \left( \sum_{i \neq j} \frac{-1}{(x_j - x_i)^2} \otimes E_{ij}^{(a)} E_{ji}^{(b)} E_{jj}^{(c)} - \sum_{i \neq j} \frac{-1}{(x_j - x_i)^2} \otimes E_{ij}^{(a)} E_{ji}^{(b)} E_{jj}^{(c)} \right) \\
+ \sum_{c=1}^{n} \left( \sum_{i \neq j} \frac{1}{(x_j - x_i)^2} \otimes E_{ii}^{(a)} E_{ij}^{(b)} E_{ji}^{(c)} - \sum_{i \neq j} \frac{1}{(x_j - x_i)^2} \otimes E_{ij}^{(a)} E_{ji}^{(b)} E_{ii}^{(c)} \right)
\]

To summarise, we have

\[
[y_a, y_b] = (A) + (B) + (C) = \sum_{c=1}^{n} \sum_{i \neq j} \frac{-1}{(x_j - x_i)^2} \otimes E_{ij}^{(a)} E_{ji}^{(b)} E_{jj}^{(c)} - \sum_{i \neq j} \frac{-1}{(x_j - x_i)^2} \otimes E_{ij}^{(a)} E_{ji}^{(b)} E_{jj}^{(c)}
\] (24)

The above formula shows that \([y_a, y_b] \in B_n^{t\text{diag}}\). Therefore, the relation \([y_a, y_b] = 0\) is preserved under the map. It is clear that the map from \(t_{1,2}^{a} B_n B_n^{t\text{diag}}\) factors through the Hecke algebra \(H(gl_k, b_k)\). Indeed the images of the generators \(x_{a, b}\) and \(t_{ab}\) lie in \(H(gl_k, b_k)\). This completes the proof of (1).

To prove (2), we check the two relations \(\sum_{i=1}^{n} x_i = 0\), and \(\sum_{i=1}^{n} y_i = 0\). We have

\[
\sum_{i=1}^{n} x_i \mapsto \sum_{i=1}^{n} \sum_{a=1}^{k} x_a \otimes E_{aa}^{(i)} = \sum_{a=1}^{k} x_a \otimes \left( \sum_{i=1}^{n} E_{aa}^{(i)} \right),
\]

which is in \(B_n^{t\text{diag}}\), therefore, the map preserves the relation \(\sum_{i=1}^{n} x_i = 0\). To check the map preserves the relation \(\sum_{i=1}^{n} y_i = 0\). We have

\[
\sum_{i=1}^{n} y_i \mapsto -\sum_{i=1}^{n} \sum_{a=1}^{k} \partial_a \otimes E_{aa}^{(i)} + \sum_{1 \leq i, j \leq n, 1 \leq a, b \leq k} \frac{1}{x_b - x_a} \otimes E_{ab}^{(i)} E_{ba}^{(j)}
\]

\[
= -\sum_{i=1}^{n} \sum_{a=1}^{k} \partial_a \otimes E_{aa}^{(i)} + \sum_{1 \leq i \leq n} \sum_{1 \leq a, b \leq k} \frac{1}{x_b - x_a} \otimes E_{ab}^{(i)} E_{ba}^{(j)}
\]

\[
= -\sum_{i=1}^{n} \sum_{a=1}^{k} \partial_a \otimes E_{aa}^{(i)} + \frac{1}{2} \left( \sum_{1 \leq i \leq n} \sum_{1 \leq a, b \leq k} \frac{1}{x_b - x_a} \otimes E_{ab}^{(i)} E_{ba}^{(j)} + \sum_{1 \leq i \leq n} \sum_{1 \leq a, b \leq k} \frac{1}{x_a - x_b} \otimes E_{ba}^{(i)} E_{ab}^{(j)} \right)
\]

\[
= -\sum_{i=1}^{n} \sum_{a=1}^{k} \partial_a \otimes E_{aa}^{(i)} + \frac{1}{2} \sum_{1 \leq i \leq n} \sum_{1 \leq a, b \leq k} \frac{1}{x_b - x_a} \otimes (E_{aa}^{(i)} - E_{bb}^{(i)})
\]

\[
\square
\]

The above term lies in \(B_n^{t\text{diag}}\). This completes the proof.

Let \(C(E, n)\) be the configuration space of \(n\) unordered points on the elliptic curve. The map in Proposition 6.3 gives the following realization of the KZB connection.
Theorem 6.4. [8, Sect. 6.3] The following connection on \( C(E, n) \) valued in the Hecke algebra \( \mathcal{H}(\mathfrak{gl}_k, b_k) \)

\[
\nabla_{\mathcal{H}(\mathfrak{gl}_k, b_k)} = d - \sum_{1 \leq i \leq j \leq n} \sum_{1 \leq a, b \leq k} k(z_i - z_j, \text{ad}(\Sigma_{\lambda=1}^k x_i E_{(i)}^{(j)})(\tau)(E_{ab}^{(i)}E_{ba}^{(j)})dz_i
\]

\[
+ \sum_{1 \leq i, j \leq n} \sum_{1 \leq a \neq b \leq k} \frac{E_{ab}^{(i)}E_{ba}^{(j)}}{x_b - x_a} dz_i - \sum_{i=1}^{n} \sum_{a=1}^{k} \partial_d E_{ad}^{(i)} dz_i,
\]

is flat and \( \mathcal{E}_n \)-equivariant.

Let \( M_{k,n} \) be the vector space of \( k \times n \) matrices and let \( \mathbb{C}[M_{k,n}] \) be the ring of regular functions on \( M_{k,n} \). Thus, \( \mathbb{C}[M_{k,n}] \) is a polynomial ring with \( kn \) variables. The infinite dimensional vector space \( \mathbb{C}[b_k^{\text{reg}}] \otimes \mathbb{C}[M_{k,n}] \) is a module over \( B_n = \text{Diff}(b_k^{\text{reg}}) \otimes U[\mathfrak{gl}_k]^{\otimes n} \). The corresponding zero weight space \( (\mathbb{C}[b_k^{\text{reg}}] \otimes \mathbb{C}[M_{k,n}])[0] \) is a module over the Hecke algebra \( \mathcal{H}(\mathfrak{gl}_k, b_k) \). The connection \( \nabla_{\mathcal{H}(\mathfrak{gl}_k, b_k)} \) in Theorem 6.4 can be valued in the zero weight space \( (\mathbb{C}[b_k^{\text{reg}}] \otimes \mathbb{C}[M_{k,n}])[0] \) which carries an action of \( \mathcal{H}(\mathfrak{gl}_k, b_k) \).

6.2. The \((\mathfrak{gl}_k, \mathfrak{gl}_n)\) duality. In this subsection, we construct an action of \( D_{4,\beta}(\mathfrak{sl}_n) \) on the space \( (\mathbb{C}[b_k^{\text{reg}}] \otimes \mathbb{C}[M_{k,n}]) \), when \( \beta = -\frac{1}{4} \). Using such action, we deduce a \((\mathfrak{gl}_k, \mathfrak{gl}_n)\) duality in the elliptic setting.

The group \( \text{GL}_k \times \text{GL}_n \) acts on \( \mathbb{C}[M_{k,n}] \) by

\[
(g_k, g_n)p(x) = p(g_k^t x g_n), \quad \text{for } (g_k, g_n) \in \text{GL}_k \times \text{GL}_n, \quad \text{and } p(x) \in \mathbb{C}[M_{k,n}].
\]

It induces an action of the dual pair \( (\mathfrak{gl}_k, \mathfrak{gl}_n) \) of the corresponding Lie algebras. To distinguish between the elements of the Lie algebras \( \mathfrak{gl}_n \) and \( \mathfrak{gl}_l \), we denote by \( X^{(p)} \) the elements of \( \mathfrak{gl}_p \). In this notation, we have

\[
(X^{(p)})^{(i)} = 1 \otimes \cdots \otimes X^{(p)} \otimes \cdots \otimes 1 \in U(\mathfrak{gl}_p)^{\otimes n},
\]

where \( X^{(p)} \) lies in the \( i \)-th copy of \( U(\mathfrak{gl}_p) \).

Write \( \mathbb{C}[M_{k,n}] = \mathbb{C}[x_{a,j}]_{1 \leq a \leq k, 1 \leq j \leq n} \). We have the isomorphism

\[
\mathbb{C}[x_{a,1}]_{1 \leq a \leq k} \otimes \cdots \otimes \mathbb{C}[x_{a,n}]_{1 \leq a \leq k} \cong \mathbb{C}[M_{1,k}] \cong \mathbb{C}[M_{k,1}] \cong \mathbb{C}[M_{k,n}] \cong \mathbb{C}[x_{1,1}]_{1 \leq j \leq n} \otimes \cdots \otimes \mathbb{C}[x_{k,1}]_{1 \leq j \leq n}.
\]

The action of \( (\mathfrak{gl}_k, \mathfrak{gl}_n) \) on \( \mathbb{C}[M_{k,n}] \) is given by

\[
(E_{ab}^{(i)})^{(j)} \mapsto x_{ai}\partial_{bi}, \quad (E_{ij}^{(a)})^{(b)} \mapsto x_{ai}\partial_{bj}.
\]

Proposition 6.5. ([16, Sect. 6], [34, Section 3]) The following identities hold on \( \mathbb{C}[M_{k,n}] \).

1. For \( 1 \leq a \leq k, 1 \leq i \leq n \), we have \( (E_{aa}^{(i)})^{(j)} = (E_{ii}^{(a)})^{(j)} \).
2. If \( 1 \leq i \neq j \leq n \) and \( 1 \leq a \neq b \leq k \), then \( (E_{ab}^{(i)})^{(j)}(E_{ba}^{(k)})^{(j)} = (E_{ij}^{(a)})^{(b)}(E_{ji}^{(a)})^{(b)} \).
3. If \( 1 \leq a \neq b \leq k, 1 \leq i \leq n \), then \( (E_{ab}^{(i)})^{(j)}(E_{ba}^{(k)})^{(i)} = (E_{ii}^{(a)})^{(b)}(E_{ii}^{(b)})^{(a)} + (E_{ii}^{(a)})^{(a)} \).

Proof. We only show the last equality. We have

\[
(E_{ab}^{(i)})^{(j)}(E_{ba}^{(k)})^{(i)} \mapsto x_{ai}\partial_{bi}x_{bi}\partial_{ai} = x_{ai}\partial_{ai}x_{bi}\partial_{ai} = x_{ai}\partial_{ai}x_{bi}\partial_{ai} + x_{ai}\partial_{ai},
\]

which is the same as the action of \( (E_{ii}^{(a)})^{(b)}(E_{ii}^{(b)})^{(a)} + (E_{ii}^{(a)})^{(a)} \). \( \square \)

Lemma 6.6. Let \( 1 \leq a \leq k \), the following holds on \( \mathbb{C}[M_{k,n}] \).

\[
(E_{ij}^{(a)})^{(p)}(E_{pq}^{(a)})^{(j)} = (E_{ij}^{(a)})^{(p)}(E_{pq}^{(a)})^{(j)} - \delta_{pq}(E_{ij}^{(a)})^{(a)} + \delta_{jp}(E_{iq}^{(a)})^{(a)},
\]

for \( E_{ij}, E_{pq} \in \mathfrak{gl}_n \).
Proof. We have

\[(E^{(n)}_{ij})^{-1}(E^{(n)}_{pq}) \mapsto x_{ai}\partial_{aj}x_{ap}\partial_{aq} = x_{ai}x_{ap}\partial_{aj}\partial_{aq} + \delta_{jp}x_{ai}\partial_{aq}
\]

This is the same as the action of \((E^{(n)}_{iq})^{-1}(E^{(n)}_{pj})^{-1} - \delta_{pq}(E^{(n)}_{ij})^{-1} + \delta_{jp}(E^{(n)}_{iq})^{-1}\).

\[\square\]

Let \(h_k \subset sl_k\) be the Cartan subalgebra of \(sl_k\).

**Theorem 6.7.** There is an action of the deformed double current algebra \(D_{-1,\frac{\pi}{2}}(sl_n)\) on \(\mathbb{C}[h^\text{reg}_k] \otimes \mathbb{C}[M_{k,n}]\) such that, for \(1 \leq i \neq j \leq n\), \(E_{ij}\) acts by \(1 \otimes \sum_{a=1}^{k} (E^{(n)}_{ij})^{-1}\), and

- \(K(E_{ij})\) acts by \(\sum_{a=1}^{k} x_{a} \otimes (E^{(n)}_{ij})^{-1}\),
- \(Q(E_{ij})\) acts by \(-\sum_{a=1}^{k} \partial_{a} \otimes (E^{(n)}_{ij})^{-1} + \sum_{1 \leq a \neq b \leq k} \frac{1}{x_{b} - x_{a}} \otimes \left(\sum_{e=1}^{n} (E^{(n)}_{ie})^{-1}(E^{(n)}_{ej})^{-1} + (E^{(n)}_{ij})^{-1}\right)\),
- \(P(E_{ij})\) acts by \(-\sum_{a=1}^{k} Y_{a} \otimes (E^{(n)}_{ij})^{-1} + \sum_{1 \leq a \neq b \leq k} \frac{x_{a}}{x_{b} - x_{a}} \otimes (E^{(n)}_{ij})^{-1} + \frac{1}{2} \sum_{1 \leq a \neq b \leq k} \frac{x_{b} + x_{a}}{x_{b} - x_{a}} \otimes \left(\sum_{e=1}^{n} (E^{(n)}_{ie})^{-1}(E^{(n)}_{ej})^{-1}\right)\).

\(Z_{n}\) acts by the scalar \(2(n + 1)\), where \(Y_{a} := \frac{\partial_{a} + x_{a}}{2} \in \text{Diff}(h_k),\) for \(1 \leq a \leq k\).

**Remark 6.8.** We have an isomorphism \(D_{a\lambda_1,\bar{a}\lambda_2}(sl_n) \cong D_{1,\beta}(sl_n)\), for \(a \neq 0\). The isomorphism is given by \(X \mapsto X, \ K(X) \mapsto a_{1}K(X), \ Q(X) \mapsto a_{2}Q(X), \ P(X) \mapsto a_{1}a_{2}P(X)\), where \(a = a_{1}a_{2}\). Therefore, Theorem 6.7 gives an action of \(D_{\lambda,-\frac{\pi}{2},l}(sl_n)\) on \(\mathbb{C}[h^\text{reg}_k] \otimes \mathbb{C}[M_{k,n}]\).

Theorem 6.7 gives the following \((gl_k, gl_n)\) duality in the elliptic case.

**Theorem 6.9.** Under the identification \(\mathbb{C}[M_{k,1}]^\text{reg} \cong \mathbb{C}[M_{k,n}] \cong \mathbb{C}[M_{1,n}]^\otimes_k\),

the KZB connection for \(gl_k\) in Theorem 6.4 with values in \(\mathbb{C}[h^\text{reg}_k] \otimes \mathbb{C}[M_{k,1}]^\otimes_0\) coincides with the sum of

1. the elliptic Casimir connection with two parameters for \(sl_n\) with values in \(\mathbb{C}[h^\text{reg}_k] \otimes \mathbb{C}[M_{1,n}]^\otimes_0\)

2. the closed one-form given by

\[\mathcal{A} = \sum_{1 \leq i \neq j \leq n} \frac{\theta'(z_i - z_j)/\theta(z_i - z_j)}{n} \left(\frac{E_{ii} + E_{jj}}{n} - \frac{1}{n^2}\right)dz_{ij}.\]

**Proof.** Let \(e_1, \cdots, e_n\) be the standard orthonormal basis of \(\mathbb{C}^n\). The root system of \(sl_n\) is given by \(\{e_i - e_j | 1 \leq i \neq j \leq n\}\). There is a natural embedding \(i : \Phi_\text{sl}(sl_n) \hookrightarrow t_{1,n}\) by

\[x(u) \mapsto \sum_{i=1}^{n} (u, e_i)x_i, \ y(u) \mapsto \sum_{i=1}^{n} (u, e_i)y_i, \ t_{e_i - e_j} \mapsto t_{ij}\]
Indeed, we have

\[ [y(u), x(v)] = \sum_{1 \leq i < j \leq n} (u, \epsilon_i) y_j, \sum_{1 \leq i \leq n} (v, \epsilon_i) x_i = \sum_{1 \leq i \leq j \leq n} (u, \epsilon_i)(u, \epsilon_i - \epsilon_j)t_{ij} \]

\[ = \frac{1}{2} \sum_{1 \leq i \neq j \leq n} (u, \epsilon_i - \epsilon_j)(u, \epsilon_i - \epsilon_j)t_{ij} = \sum_{1 \leq i < j \leq n} (u, \epsilon_i - \epsilon_j)(u, \epsilon_i - \epsilon_j)t_{ij} \]

All other relations of \( t^\Phi_{sl}(sl_n) \) are obviously preserved under \( t_1 \).

To summarise, we have two algebra homomorphisms from \( t^\Phi_{sl}(sl_n) \) to \( \text{End}(\mathbb{C}[h^\text{reg}_k] \otimes \mathbb{C}[M_{k,n}])[0] \):

\[
\begin{array}{ccc}
& t_1 & \\
\tau_1 & t_{1,n} & a_1 \\
\tau_2 & a_2 & \text{End}(\mathbb{C}[h^\text{reg}_k] \otimes \mathbb{C}[M_{k,n}])[0]
\end{array}
\]

The horizontal map \( a_{1t_1} \) is obtained by the composition of \( t_1 \) with the map \( a_1 \) in Proposition 3.7. It gives rise to the KZB connection for \( gl_k \) in Theorem 6.4. The other map is the composition \( a_{2t_2} \) in Proposition 6.3, and it gives rise to the elliptic connection for \( sl_n \).

There is a subtlety that the diagram (26) is not commutative. By Proposition 6.3, the action of \( t_{ij} \) using \( a_{1t_1} \) is given by

\[
a_{1t_1}(t_{ij}) = \sum_{1 \leq a, b \leq k} (E_{ab}^{(k)})^{(i)}(E_{ba}^{(k)})^{(j)} = \frac{1}{2} \left( \sum_{1 \leq a, b \leq k} \left( (E^{(n)}_{ij})^{(a)}(E^{(n)}_{ji})^{(b)}(E^{(n)}_{ij})^{(a)} - \sum_{a=1}^{k} (E^{(n)}_{ii})^{(a)} - \sum_{a=1}^{k} (E^{(n)}_{jj})^{(a)} \right) \right) = \frac{1}{2} \left( E_{ij}E_{ji} + E_{ji}E_{ij} - E_{ii} - E_{jj} \right),
\]

where the second equality follows from Proposition 6.5 (see also [34, (3.12)]).

On the other hand, by Proposition 3.7 and Theorem 6.7, the image of \( t_{ij} \) under \( a_{2t_2} \) is given by

\[
a_{2t_2}(t_{ij}) = \frac{\lambda}{2} (E_{ij}E_{ji} + E_{ji}E_{ij}) + \frac{Z_n}{2n^2} + 2(\beta - \frac{\lambda}{2}) (E_{ii} + \frac{E_{jj}}{n} - \frac{2}{n^2} \sum_{1 \leq k \leq n} E_{ee})
\]

\[ = - \frac{1}{2} (E_{ij}E_{ji} + E_{ji}E_{ij}) + \frac{2(n + 1)}{2n^2} + 2n + \frac{1}{2} (E_{ii} + \frac{E_{jj}}{n} - \frac{1}{n^2})
\]

\[ = - \frac{1}{2} (E_{ij}E_{ji} + E_{ji}E_{ij} - E_{ii} - E_{jj}) + \frac{E_{ii} + E_{jj}}{n} - \frac{1}{n^2}
\]

The difference of \( a_{1t_1}(t_{ij}) \) and \( a_{2t_2}(t_{ij}) \) is \( \frac{E_{ii} + E_{jj}}{n} - \frac{1}{n^2} \). For the generators \( x(u), y(u) \in t^\Phi_{sl}(sl_n) \), we have

\[ a_{1t_1}(x(u)) = a_{2t_2}(x(u)), \quad a_{1t_1}(y(u)) = a_{2t_2}(y(u)). \]

As a consequence, the difference of the elliptic Casimir connection in Theorem 3.8 and the KZB connection in Theorem 6.4 is given by the difference of \( a_{1t_1}(t_{ij}) \) and \( a_{2t_2}(t_{ij}) \). Therefore,

\[ \mathcal{A} = \nabla_{\text{Ell},C} - \nabla_{\text{KZB}} = \sum_{1 \leq i < j \leq n} \frac{\theta'(z_i - z_j \tau)}{\theta(z_i - z_j \tau)} \left( \frac{E_{ii} + E_{jj}}{n} - \frac{1}{n^2} \right) dz_{ij}. \]

This completes the proof. \( \square \)
6.3. In this section, we explain how to get the formulas in Theorem 6.7 using Proposition 6.5.

The enveloping algebra $U(\mathfrak{sl}_n)$ is a subalgebra of $D_{\Lambda, \beta}(\mathfrak{sl}_n)$. We require the action of $U(\mathfrak{sl}_n)$ to be the one induced from the natural $GL_n$ action on $\mathbb{C}[M_{k,n}]$. That is, $E_{ij} \mapsto 1 \otimes \sum_{a=1}^k (E_{ij}^{(n)})^{(a)}$.

We now use diagram (26) to deduce the action of $K(E_{ij})$ and $Q(E_{ij})$, for $1 \leq i \neq j \leq n$. The action of $x_i \in \mathfrak{t}_{\mathfrak{sl}}(\mathfrak{sl}_n)$ on $\mathbb{C}[\mathfrak{h}_{k}^{\mathrm{reg}}] \otimes \mathbb{C}[M_{k,n}]$ is given by $\sum_{a=1}^k x_a \otimes (E_{ii}^{(k)})^{(a)}$. By Proposition 6.5, on $\mathbb{C}[\mathfrak{h}_{k}^{\mathrm{reg}}] \otimes \mathbb{C}[M_{k,n}]$, we have the identity

$$\sum_{a=1}^k x_a \otimes (E_{aa}^{(k)})^{(i)} = \sum_{a=1}^k x_a \otimes (E_{ii}^{(n)})^{(a)}.$$ 

Therefore, the action of $K(E_{ij}) \in D_{\Lambda, \beta}(\mathfrak{sl}_n)$, $1 \leq i \neq j \leq n$, is given by:

$$K(E_{ij}) \mapsto \left[ \sum_{a=1}^k x_a \otimes (E_{ii}^{(n)})^{(a)}, E_{ij} \right] = \sum_{a=1}^k x_a \otimes (E_{ij}^{(n)})^{(a)}.$$ 

It is obvious that the assignment preserves the relation $[K(X), Y] = K[X, Y]$, for any $X, Y \in \mathfrak{sl}_n$. Thus, it gives an action of $\mathfrak{sl}_n[v]$ on $\mathbb{C}[\mathfrak{h}_{k}^{\mathrm{reg}}] \otimes \mathbb{C}[M_{k,n}]$.

Similarly, using Proposition 6.5, we rewrite the action of $y_i \in \mathfrak{t}_{\mathfrak{sl}}(\mathfrak{sl}_n)$ on $\mathbb{C}[\mathfrak{h}_{k}^{\mathrm{reg}}] \otimes \mathbb{C}[M_{k,n}]$ by

$$-\sum_{a=1}^k \partial_a \otimes E_{aa}^{(i)} + \sum_{j=1}^n \sum_{1 \leq a \neq b \leq k} \frac{1}{x_b - x_a} \otimes E_{bb}^{(i)} E_{ba}^{(j)} = -\sum_{a=1}^k \partial_a \otimes (E_{ii}^{(n)})^{(a)} + \sum_{j=1}^n \frac{1}{x_b - x_a} \otimes \sum_{1 \leq a \neq b \leq k} \sum_{j=1}^n (E_{ij}^{(n)})^{(a)} (E_{ji}^{(n)})^{(b)} + (E_{ii}^{(n)})^{(a)}.$$ 

Let $i \neq l$, in order to get an action of $Q(E_{il})$, we apply $[ , E_{il}]$ to the above identity. We have

$$[\sum_{a=1}^k \partial_a \otimes (E_{ii}^{(n)})^{(a)}, E_{il}] + \sum_{1 \leq a \neq b \leq k} \frac{1}{x_b - x_a} \otimes \sum_{j=1}^n (E_{ij}^{(n)})^{(a)} (E_{ji}^{(n)})^{(b)} + (E_{ii}^{(n)})^{(a)} E_{il}^{(n)}.$$ 

Therefore, the action of $Q(E_{il})$, $1 \leq i \neq l \leq n$, is given by:

$$Q(E_{il}) \mapsto -\sum_{a=1}^k \partial_a \otimes (E_{il}^{(n)})^{(a)} + \sum_{1 \leq a \neq b \leq k} \frac{1}{x_b - x_a} \otimes \sum_{j=1}^n (E_{ij}^{(n)})^{(a)} (E_{ji}^{(n)})^{(b)} + (E_{ii}^{(n)})^{(a)}.$$ 

It is straightforward to check that the assignment preserves the relation $[Q(X), Y] = Q[X, Y]$, for any $X, Y \in \mathfrak{sl}_n$.

**Proposition 6.10.** There is an action of $\mathfrak{sl}_n[u]$ on $\mathbb{C}[\mathfrak{h}_{k}^{\mathrm{reg}}] \otimes \mathbb{C}[M_{k,n}]$, such that, $E_{ij} \mapsto 1 \otimes \sum_{a=1}^k (E_{ij}^{(a)})^{(a)}$, and the action of $Q(E_{ij})$ is given by (28), for any $1 \leq i \neq j \leq n$.

**Proof.** We have known that $[Q(X), Y] = Q[X, Y]$, for any $X, Y \in \mathfrak{sl}_n$. It is well-known that $Y_{\beta}(\mathfrak{sl}_n)$ is a flat deformation of $\mathfrak{sl}_n[u]$. By Theorem 4.8 (set $\hbar = 0$), we only need to verify the following relation $[Q(H_i), Q(H_j)] = 0$, for any $i, j \in I$.

By (24), we have the following relation in $	ext{Diff}(\mathfrak{h}_{k}^{\mathrm{reg}}) \otimes U(\mathfrak{g}_{k})^{\otimes a}$, for $1 \leq i, j \leq n$,

$$[y_i, y_j] = [Q(E_{ii}), Q(E_{jj})] = \sum_{1 \leq s \neq t \leq k} \frac{1}{(x_t - x_s)^2} \sum_{e=1}^n (-E_{st}^{(i)} E_{ts}^{(j)} E_{ss}^{(e)} + E_{st}^{(j)} E_{ts}^{(i)} E_{ss}^{(e)}).$$
On \(\mathbb{C}[M_{k,n}]\), the operator \(\sum_{e=1}^{n} (E_{it}^{(k)})^{(e)}\) acts by \(\sum_{e=1}^{n} (E_{et}^{(k)})^{(e)} = \sum_{e=1}^{n} (E_{et}^{(n)})^{(e)}\). Therefore,

\[
\sum_{e=1}^{n} \left(-E_{it}^{(i)} E_{is}^{(e)} + E_{st}^{(i)} E_{is}^{(e)}\right)
\]

acts by zero on \(\mathbb{C}[M_{k,n}]\).

This completes the proof. \(\square\)

6.4. **Proof of Theorem 6.7.** In this subsection, we prove Theorem 6.7. We show, when the two parameters are \(\lambda = -1, \beta = \frac{n}{4}\), the formula in Theorem 6.7 gives a well-defined action of \(D_{-1, \frac{n}{4}}(\mathfrak{sl}_n)\) on \(\mathbb{C}[b_k^{\text{reg}}] \otimes \mathbb{C}[M_{k,n}]\).

We have shown that the two subalgebras \(\mathfrak{sl}_n[u]\) and \(\mathfrak{sl}_n[v]\) act on \(\mathbb{C}[b_k^{\text{reg}}] \otimes \mathbb{C}[M_{k,n}]\). It is a straightforward computation to show \([X, P(X')] = P([X, X'])\), for any \(X, X' \in \mathfrak{sl}_n\). It remains to check the main defining relation (14) in Lemma 3.2 on \(\mathbb{C}[b_k^{\text{reg}}] \otimes \mathbb{C}[M_{k,n}]\).

We now compute the action of \([K(E_{ij}), Q(E_{st})]\) on the vector space \(\mathbb{C}[b_k^{\text{reg}}] \otimes \mathbb{C}[M_{k,n}]\). Using the formulas in Theorem 6.7, we have:

\[
[K(E_{ij}), Q(E_{st})] = \left[\sum_a \frac{\partial_a \otimes (E_{st}^{(n)})^{(a)}}{x_a} \right] + \frac{1}{2} \sum_a \frac{\partial_a \otimes (E_{ij}^{(n)})^{(a)}}{x_a} \frac{1}{x_b - x_a} \left(\sum_e (E_{se}^{(n)})^{(e)} (E_{et}^{(n)})^{(b)} + (E_{st}^{(n)})^{(a)}\right)\]  

Set \(Y_a := \frac{\partial_a x_a + x_a \partial_a}{2}\). Using the substitution \(\partial_a x_a = Y_a + \frac{1}{2}\), and \(x_a \partial_a = Y_a - \frac{1}{2}\), we compute the equation (29) as follows.

\[(29) : \left[\sum_a \frac{\partial_a \otimes (E_{ij}^{(n)})^{(a)}}{x_a} \right] + \frac{1}{2} \sum_a \frac{\partial_a \otimes (E_{ij}^{(n)})^{(a)}}{x_a} \frac{1}{x_b - x_a} \left(\sum_e (E_{se}^{(n)})^{(e)} (E_{et}^{(n)})^{(b)} + (E_{st}^{(n)})^{(a)}\right) = \sum_a \frac{Y_a (E_{st}, E_{ij})^{(a)}}{x_a} + \frac{1}{2} \sum_a (E_{st})^{(a)} E_{ij}^{(a)} E_{st}^{(a)}.
\]

We then compute the equation (30). We have

\[(30) = \sum_{c=1}^{k} \sum_{1 \leq a \neq b \leq k} \frac{x_c}{x_b - x_a} \otimes \left(\sum_e (E_{se}^{(n)})^{(e)} (E_{et}^{(n)})^{(b)} + (E_{st}^{(n)})^{(a)}\right)\]

\[= \sum_{1 \leq a \neq b \leq k} \frac{x_a}{x_b - x_a} \otimes \left(\sum_e (E_{ij}, E_{se})^{(a)} (E_{et}^{(n)})^{(b)}\right) + \sum_{1 \leq a \neq b \leq k} \frac{x_b}{x_b - x_a} \otimes \sum_e (E_{se}^{(n)})^{(e)} (E_{ij}, E_{et})^{(b)}\]  

We simplify (31) using the following identities.

\[
\sum_{e=1}^{n} (E_{ij}, E_{se})^{(a)} E_{et}^{(b)} = \sum_{e=1}^{n} (E_{ie} - \delta_{et} E_{sj})^{(a)} E_{et}^{(b)} = \delta_{et} \sum_{e=1}^{n} E_{ie}^{(a)} E_{et}^{(b)} - E_{sj}^{(a)} E_{it}^{(b)}.
\]

\[
\sum_{e=1}^{n} E_{se}^{(a)} (E_{ij}, E_{et})^{(b)} = \sum_{e=1}^{n} E_{se}^{(a)} (E_{it} - \delta_{et} E_{ij})^{(b)} = E_{sj}^{(a)} E_{it}^{(b)} - \delta_{it} \sum_{e=1}^{n} E_{se}^{(a)} E_{ej}^{(b)}.
\]
We have

\[
(31): \sum_{1 \leq a \leq b \leq k} \frac{x_a}{x_b - x_a} \otimes \sum_{e} \left( [E_{ij}, E_{se}]^{(a)}(E_{et}^{(n)})^{(b)} + \sum_{1 \leq a \leq b \leq k} \frac{x_b}{x_b - x_a} \otimes \sum_{e} (E_{se}^{(n)})^{(a)}[E_{ij}, E_{et}]^{(b)} \right)
\]

\[
= \sum_{1 \leq a \leq b \leq k} \frac{x_a}{x_b - x_a} \otimes (\delta_{js} \sum_{e=1}^{n} E_{ie}^{(a)} E_{et}^{(b)} - E_{sj}^{(a)} E_{it}^{(b)}) + \sum_{1 \leq a \leq b \leq k} \frac{x_b}{x_b - x_a} \otimes (E_{sj}^{(a)} E_{it}^{(b)} - \delta_{it} \sum_{e=1}^{n} E_{se}^{(a)} E_{ej}^{(b)})
\]

\[
= \sum_{1 \leq a \leq b \leq k} \frac{x_a}{x_b - x_a} \otimes \left( \delta_{js} \sum_{e=1}^{n} E_{ie}^{(a)} E_{et}^{(b)} - \delta_{it} \sum_{e=1}^{n} E_{se}^{(a)} E_{ej}^{(b)} \right)
\]

\[
= \sum_{1 \leq a \leq b \leq k} \frac{x_a}{x_b - x_a} \otimes \left( \delta_{js} \sum_{e=1}^{n} E_{ie}^{(a)} E_{et}^{(b)} - \delta_{it} \sum_{e=1}^{n} E_{se}^{(a)} E_{ej}^{(b)} \right)
\]

To summarize, based on the computations of equations (29), (30) and (31), we have

\[
[K(E_{ij}, Q(E_{st}))]
\]

\[
= \sum_{a} Y_{a}[E_{st}, E_{ij}]^{(a)} + \frac{1}{2} \sum_{a} (E_{st}^{(a)} E_{ij}^{(a)} + E_{ij}^{(a)} E_{st}^{(a)}) + \sum_{1 \leq a \leq b \leq k} \frac{x_a}{x_b - x_a} \otimes [E_{ij}, E_{st}]^{(a)}
\]

\[
+ E_{sj} E_{it} - \sum_{e=1}^{n} E_{sj}^{(a)} E_{it}^{(b)} + \sum_{e=1}^{n} \frac{1}{2} \left( \delta_{js} \sum_{e=1}^{n} E_{ie}^{(a)} E_{et}^{(b)} - \delta_{it} \sum_{e=1}^{n} E_{se}^{(a)} E_{ej}^{(b)} \right)
\]

\[
= P(E_{ij}, E_{st}) + \frac{1}{2} \sum_{a} (E_{st}^{(a)} E_{ij}^{(a)} + E_{ij}^{(a)} E_{st}^{(a)}) + E_{sj} E_{it} - \sum_{1 \leq a \leq k} E_{sj}^{(a)} E_{it}^{(a)}
\]

\[
= P(E_{ij}, E_{st}) + \frac{1}{2} \sum_{a} (E_{st}^{(a)} E_{ij}^{(a)} + E_{ij}^{(a)} E_{st}^{(a)}) + \sum_{e=1}^{n} E_{se} E_{ej} + \sum_{e=1}^{n} \left( \delta_{js} \sum_{e=1}^{n} E_{ie}^{(a)} E_{et}^{(a)} + \delta_{it} \sum_{e=1}^{n} E_{se}^{(a)} E_{ej}^{(a)} \right)
\]

\[
= P(E_{ij}, E_{st}) + \frac{1}{2} \sum_{e=1}^{n} E_{se} E_{ej} + \sum_{e=1}^{n} \left( \delta_{js} \sum_{e=1}^{n} E_{ie}^{(a)} E_{et}^{(a)} + \delta_{it} \sum_{e=1}^{n} E_{se}^{(a)} E_{ej}^{(a)} \right)
\]

\[
+ \frac{1}{2} \sum_{1 \leq a \leq k} (E_{st}^{(a)} E_{ij}^{(a)} + E_{ij}^{(a)} E_{st}^{(a)}) - \sum_{1 \leq a \leq k} E_{sj}^{(a)} E_{it}^{(a)} + \frac{1}{2} \sum_{1 \leq a \leq k} \left( \delta_{js} \sum_{e=1}^{n} E_{ie}^{(a)} E_{et}^{(a)} + \delta_{it} \sum_{e=1}^{n} E_{se}^{(a)} E_{ej}^{(a)} \right)
\]

(32)
We compute the action of the equation (32) on the vector space \( \mathbb{C}[M_{k,n}] = (\mathbb{C}[M_{1,n}])^\otimes k \). We use the following identities.

\[
\frac{1}{2} \sum_{1 \leq a \leq k} (E_{st}^{(a)} E_{ij}^{(a)} + E_{ij}^{(a)} E_{st}^{(a)}) - \sum_{1 \leq a \leq k} E_{sj}^{(a)} E_{it}^{(a)} \mapsto \frac{1}{2} \left( x_{as} \partial_{ai} x_{al} \partial_{aj} + x_{al} \partial_{aj} x_{as} \partial_{ai} \right) - x_{as} \partial_{aj} x_{al} \partial_{ai} \\
= \frac{1}{2} \left( x_{as} \partial_{aj} x_{al} \partial_{ai} + \delta_{it} x_{as} \partial_{aj} + x_{al} \partial_{aj} x_{as} \partial_{ai} + \delta_{sij} x_{ai} \partial_{al} \right) - x_{as} \partial_{aj} x_{ai} \partial_{al} \\
= \frac{1}{2} \left( \delta_{it} x_{as} \partial_{aj} + \delta_{sij} x_{ai} \partial_{al} \right) = \frac{1}{2} (\delta_{it} E_{sj}^{(a)} + \delta_{sij} E_{it}^{(a)}).
\]

\[
\sum_{e=1}^n E_{ie}^{(a)} E_{et}^{(a)} \mapsto \sum_{e=1}^n x_{ai} \partial_{ae} x_{ae} \partial_{at} = \sum_{e=1}^n x_{ai} \partial_{ae} \partial_{at} x_{ae} - x_{ai} \partial_{at} \\
= \sum_{e=1}^n x_{ai} \partial_{at} x_{ae} \partial_{ae} + \sum_{e=1}^n x_{ai} \partial_{at} - x_{ai} \partial_{at} = \sum_{e=1}^n E_{it}^{(a)} E_{ee}^{(a)} + \sum_{e=1}^n E_{it}^{(a)} - E_{it}^{(a)} = nE_{it}^{(a)}.
\]

Similarly, \( \sum_{e=1}^n E_{se}^{(a)} E_{ej}^{(a)} \mapsto nE_{sj}^{(a)}. \) Therefore, on \( (\mathbb{C}[M_{1,n}])^\otimes k \), (32) acts by

\[
(32) = \frac{1}{2} \sum_{1 \leq a \leq k} (\delta_{it} E_{sj}^{(a)} + \delta_{sij} E_{it}^{(a)}) + \frac{n}{2} \sum_{1 \leq a \leq k} (\delta_{js} E_{it}^{(a)} + \delta_{it} E_{sj}^{(a)}) = \frac{1}{2} (1 + n)(\delta_{it} E_{sj} + \delta_{sij} E_{it}).
\]

Therefore, on \( \text{End}(\mathbb{C}[b_k^{\text{reg}}] \otimes \mathbb{C}[M_{k,n}]) \) we have the following identity

\[
[K(E_{ij}), Q(E_{st})] = P([E_{ij}, E_{st}]) + E_{sj} E_{it} - \frac{1}{2} (\delta_{js} \sum_{e=1}^n E_{ie} E_{et} + \delta_{it} \sum_{e=1}^n E_{se} E_{ej}) + \frac{1}{2} (1 + n)(\delta_{it} E_{sj} + \delta_{sij} E_{it}).
\]

Comparing the above equality with Lemma 3.2 equation (14), we have

\[
\lambda = -1, \quad \beta = \frac{1}{4} n.
\]

In the rest of this section, we compute the action of the central element \( Z_n \in D_{-1,\frac{4}{n}}(sl_n) \) on \( \mathbb{C}[b_k^{\text{reg}}] \otimes \mathbb{C}[M_{k,n}]. \)

**Lemma 6.11.** The central element \( Z_n \in D_{-1,\frac{4}{n}}(sl_n) \) acts by the scalar \( 2(n + 1) \) on \( \mathbb{C}[b_k^{\text{reg}}] \otimes \mathbb{C}[M_{k,n}]. \)

**Proof.** Recall that, by definition, we have

\[
Z_n = \sum_{a=1}^n Z_{a,a+1} = \sum_{a=1}^n \left( [K(H_a), Q(H_a)] - \frac{4}{4} \sum_{1 \leq i,j \leq n} S([H_a, E_{ij}], [E_{ij}, H_a]) \right).
\]

We first compute the action of

\[
Z_{12} = [K(H_{12}), Q(H_{12})] - \frac{4}{4} \sum_{1 \leq i,j \leq n} S([H_{12}, E_{ij}], [E_{ij}, H_{12}])
\]  

(33)
on $\mathbb{C}^{[\mathbb{b}_k^{reg}]} \otimes \mathbb{C}[M_{k,n}]$ using the formulas in Theorem 6.7. We have

$$[K(H_{12}), Q(H_{12})]$$

$$= \left[ \sum_{a=1}^{k} x_a \otimes H_{12}^{(a)}, \sum_{a=1}^{k} \frac{1}{x_b - x_a} \otimes \left( \sum_{e=1}^{n} E_{1e}^{(a)} E_{e1}^{(b)} - \sum_{e=1}^{n} E_{2e}^{(a)} E_{e2}^{(b)} + H_{12}^{(a)} \right) \right]$$

$$= \sum_{a=1}^{k} H_{12}^{(a)} H_{12}^{(a)} + \sum_{1 \leq a \neq b \leq k} \frac{x_a}{x_b - x_a} [H_{12}^{(a)}, \sum_{e=1}^{n} E_{1e}^{(a)} E_{e1}^{(b)} - \sum_{e=1}^{n} E_{2e}^{(a)} E_{e2}^{(b)} ]$$

$$+ \sum_{1 \leq a \neq b \leq k} \frac{x_b}{x_b - x_a} [H_{12}^{(b)}, \sum_{e=1}^{n} E_{1e}^{(a)} E_{e1}^{(b)} - \sum_{e=1}^{n} E_{2e}^{(a)} E_{e2}^{(b)} ]$$

$$= \sum_{a=1}^{k} H_{12}^{(a)} H_{12}^{(a)} - \sum_{1 \leq a \neq b \leq k} \sum_{e=1}^{n} \left( (\epsilon_1^{(a)}, \epsilon_1^{(b)}) E_{1e}^{(a)} E_{e1}^{(b)} - (\epsilon_1^{(a)}, \epsilon_2^{(b)}) E_{1e}^{(a)} E_{e2}^{(b)} \right)$$

$$= \sum_{a=1}^{k} H_{12}^{(a)} H_{12}^{(a)} - \sum_{1 \leq a \neq b \leq k} \left( \sum_{e=1}^{n} E_{1e}^{(a)} E_{e1}^{(b)} + \sum_{e=1}^{n} E_{2e}^{(a)} E_{e2}^{(b)} - \sum_{e=1}^{n} E_{1e}^{(a)} E_{1e}^{(b)} - \sum_{e=1}^{n} E_{2e}^{(a)} E_{2e}^{(b)} + \sum_{e=1}^{n} E_{1e}^{(a)} E_{e1}^{(b)} + \sum_{e=1}^{n} E_{2e}^{(a)} E_{e2}^{(b)} \right)$$

Therefore, $[K(H_{12}), Q(H_{12})]$ acts on $\mathbb{C}^{[\mathbb{b}_k^{reg}]} \otimes \mathbb{C}[M_{k,n}]$ by

$$- \left( \sum_{e=1}^{n} E_{1e} E_{e1} + \sum_{e=1}^{n} E_{2e} E_{e2} - E_{11} E_{11} - E_{22} E_{22} + E_{12} E_{21} + E_{21} E_{12} \right)$$

$$+ \sum_{1 \leq a \leq k} \left( \sum_{e=1}^{n} E_{1e}^{(a)} E_{e1}^{(a)} + \sum_{e=1}^{n} E_{2e}^{(a)} E_{e2}^{(a)} + \sum_{e=1}^{n} E_{1e}^{(a)} E_{e1}^{(a)} + \sum_{e=1}^{n} E_{2e}^{(a)} E_{e2}^{(a)} - 2 E_{11}^{(a)} E_{22}^{(a)} \right)$$

$$= (n + 1) (E_{11} + E_{22}) - \left( \sum_{e=1}^{n} E_{1e} E_{e1} + \sum_{e=1}^{n} E_{2e} E_{e2} - E_{11} E_{11} - E_{22} E_{22} + E_{12} E_{21} + E_{21} E_{12} \right), \quad (34)$$

where the last equality follows from the following identities

$$\sum_{e=1}^{n} E_{1e}^{(a)} E_{e1}^{(a)} = \sum_{e=1}^{n} x_{a1} \partial_{ae} x_{ae} \partial_{a1} = \sum_{e=1}^{n} x_{a1} \partial_{ae} x_{ae} \partial_{a1} - \sum_{e=1}^{n} x_{a1} \partial_{a1} - x_{1a} \partial_{a1} = n E_{11}^{(a)}$$

$$E_{12}^{(a)} E_{21}^{(a)} + E_{21}^{(a)} E_{12}^{(a)} - 2 E_{11}^{(a)} E_{22}^{(a)} = x_{a1} \partial_{a2} x_{a2} \partial_{a1} + x_{a2} \partial_{a1} x_{a1} \partial_{a2} - 2 x_{a1} \partial_{a1} x_{a2} \partial_{a2} = x_{a1} \partial_{a1} + x_{a2} \partial_{a2} = E_{11}^{(a)} + E_{22}^{(a)}.$$

On the other hand, we have

$$\frac{\lambda}{4} \sum_{1 \leq j \neq f \leq n} S([H_{12}, E_{ij}], [E_{ji}, H_{12}])$$

$$= \frac{\lambda}{2} \left( \sum_{\{j \neq f\}} S(E_{1j}, E_{j1}) + \sum_{\{j \neq f\}} S(E_{2j}, E_{j2}) + 2 S(E_{12}, E_{21}) \right)$$

$$= \frac{\lambda}{2} \left( \sum_{\{j \neq f\}} E_{1j} E_{j1} + \sum_{\{j \neq f\}} E_{2j} E_{j2} + S(E_{12}, E_{21}) \right) + \frac{\lambda}{2} \left( \sum_{\{j \neq f\}} H_{1j} + \sum_{\{j \neq f\}} H_{f2} \right)$$

$$= \frac{\lambda}{2} \left( \sum_{e=1}^{n} E_{1e} E_{e1} + \sum_{e=1}^{n} E_{2e} E_{e2} + S(E_{12}, E_{21}) - E_{11} E_{11} - E_{22} E_{22} \right) + \frac{\lambda}{2} \left( 2 \sum_{e=1}^{n} E_{ee} - n E_{11} - n E_{22} \right) \quad (35)$$
By (33), (34) and (35), when \( \lambda = -1 \), the element \( Z_{12} \) acts on \( \mathbb{C}[b_k]^{\ell \Phi} \otimes \mathbb{C}[M_{t,n}] \) by

\[
Z_{12} \mapsto \left( \frac{n}{2} + 1 \right) (E_{11} + E_{22}) + \sum_{e=1}^{n} E_{ee}.
\]

By symmetry, we know \( Z_{n} \) acts by

\[
Z_{n} = \sum_{a=1}^{n} Z_{a,a+1} \mapsto \sum_{a=1}^{n} \left( \left( \frac{n}{2} + 1 \right) (E_{aa} + E_{a,a+1}) + \sum_{e=1}^{n} E_{ee} \right) = 2(n + 1) \sum_{e=1}^{n} E_{ee}.
\]

This completes the proof of Theorem 6.7.

7. Flat connection on the elliptic moduli space \( M_{t,n} \)

In this section, we collect some results from [40, Sections 7, 8, 9] about the extension of the KZB connection associated to \( \Phi \) to the \( \tau \) direction, which we will need in Section 8 and Section 9.

7.1. Derivation of the Lie algebra \( \mathfrak{t}_{\text{ell}}^\Phi \). Let \( \mathfrak{d} \) be the Lie algebra defined in [8, 40] with generators \( \Delta_0, d, X, \) and \( \delta_{2m}(m \geq 1) \), and relations

\[
[d, X] = 2X, \quad [d, \Delta_0] = -2\Delta_0, \quad [X, \Delta_0] = d,
\]

\[
[\delta_{2m}, X] = 0, \quad [d, \delta_{2m}] = 2m\delta_{2m}, \quad (\text{ad} \Delta_0)^{2m+1}(\delta_{2m}) = 0.
\]

There is a Lie algebra morphism \( \mathfrak{d} \to \text{Der}(\mathfrak{t}_{\text{ell}}^\Phi) \) in [40, Proposition 7.1], denoted by \( \xi \mapsto \tilde{\xi} \). The image \( \tilde{\xi} \) of \( \xi \) acts on \( \mathfrak{t}_{\text{ell}}^\Phi \) by the following formulas.

\[
\tilde{d}(x(u)) = x(u), \quad \tilde{d}(y(u)) = -y(u), \quad \tilde{d}(t_\alpha) = 0,
\]

\[
\tilde{X}(x(u)) = 0, \quad \tilde{X}(y(u)) = x(u), \quad \tilde{X}(t_\alpha) = 0,
\]

\[
\tilde{\Delta}_0(x(u)) = y(u), \quad \tilde{\Delta}_0(y(u)) = 0, \quad \tilde{\Delta}_0(t_\alpha) = 0,
\]

\[
\tilde{\delta}_{2m}(x(u)) = 0, \quad \tilde{\delta}_{2m}(t_\alpha) = [t_\alpha, (\text{ad} \frac{\alpha}{2})^{2m}(t_\alpha)],
\]

and

\[
\tilde{\delta}_{2m}(y(u)) = \frac{1}{2} \sum_{a \in \Phi^+} \alpha(u) \sum_{p+q=2m-1} [(\text{ad} \frac{\alpha}{2})^p(t_\alpha), (\text{ad} -\frac{\alpha}{2})^q(t_\alpha)].
\]

Proposition 7.1. [40, Proposition 7.1] The above map \( \mathfrak{d} \to \text{Der}(\mathfrak{t}_{\text{ell}}^\Phi) \) is a Lie algebra homomorphism.

7.2. A principal bundle. Let \( e, f, h \) be the standard basis of \( \mathfrak{sl}_2 \). There is a Lie algebra morphism \( \mathfrak{d} \to \mathfrak{sl}_2 \) defined by \( \delta_{2m} \to 0, d \to h, X \to e, \Delta_0 \to f \). Let \( \mathfrak{d}_+ \subset \mathfrak{d} \) be the kernel of this homomorphism. Since the morphism has a section, which is given by \( e \mapsto X, f \mapsto \Delta_0 \) and \( h \mapsto d \), we have a semidirect decomposition \( \mathfrak{d} = \mathfrak{d}_+ \rtimes \mathfrak{sl}_2 \). As a consequence, we have the decomposition

\[
\mathfrak{t}_{\text{ell}}^\Phi \rtimes \mathfrak{d} = (\mathfrak{t}_{\text{ell}}^\Phi \rtimes \mathfrak{d}_+) \rtimes \mathfrak{sl}_2.
\]

The Lie algebra \( \mathfrak{t}_{\text{ell}}^\Phi \rtimes \mathfrak{d}_+ \) is positively graded. The \( \mathbb{Z}^2 \)-grading of \( \mathfrak{d} \) and \( \mathfrak{t}_{\text{ell}}^\Phi \) is defined by

\[
\text{deg}(\Delta_0) = (-1, 1), \quad \text{deg}(d) = (0, 0), \quad \text{deg}(X) = (1, -1), \quad \text{deg}(\delta_{2m}) = (2m + 1, 1),
\]

and

\[
\text{deg}(x(u)) = (1, 0), \quad \text{deg}(y(u)) = (0, 1), \quad \text{deg}(t_\alpha) = (1, 1).
\]

We form the following semidirect products

\[
G_n := \exp(\mathfrak{t}_{\text{ell}}^\Phi \rtimes \tilde{\mathfrak{d}}_+) \rtimes \mathfrak{sl}_2(\mathbb{C}),
\]

where \( \mathfrak{t}_{\text{ell}}^\Phi \rtimes \tilde{\mathfrak{d}}_+ \) is the completion of \( \mathfrak{t}_{\text{ell}}^\Phi \rtimes \mathfrak{d}_+ \) with respect to the grading above.
Let $Q^\vee \subset \mathfrak{h}$ be the coroot lattice of $\mathfrak{h}$. The semidirect product $(Q^\vee \oplus Q^\vee) \rtimes \text{SL}_2(\mathbb{Z})$ acts on $\mathfrak{h} \times \mathfrak{S}$ as follows. For $(n, m) \in (Q^\vee \oplus Q^\vee)$ and $(z, \tau) \in \mathfrak{h} \times \mathfrak{S}$, the action is given by translation: $(n, m) \ast (z, \tau) := (z + n + \tau m, \tau)$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, the action is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ast (z, \tau) := \left( \frac{\alpha z + \beta}{\gamma + \delta \tau}, \frac{\alpha \tau + \beta}{\gamma + \delta \tau} \right)$. Let $\alpha(-) : \mathfrak{h} \to \mathbb{C}$ be the map induced by the root $\alpha \in \Phi$. We define $\overline{H}_{a, r} \subset \mathfrak{h} \times \mathfrak{S}$ to be

$$\overline{H}_{a, r} = \{ (z, \tau) \in \mathfrak{h} \times \mathfrak{S} \mid \alpha(z) \in \Lambda_r = \mathbb{Z} + r\mathbb{Z} \}.$$

We define the elliptic moduli space $M_{1, n}$ to be the quotient of $\mathfrak{h} \times \mathfrak{S} \setminus \bigcup_{a \in \Phi^+, \tau \in \mathfrak{S}} \overline{H}_{a, r}$ by the action of $(Q^\vee \oplus Q^\vee) \rtimes \text{SL}_2(\mathbb{Z})$. Let $\pi : \mathfrak{h} \times \mathfrak{S} \setminus \bigcup_{a \in \Phi^+, \tau \in \mathfrak{S}} \overline{H}_{a, r} \to M_{1, n}$ be the natural projection. We define a principal $G_n$–bundle $P_n$ on the elliptic moduli space $M_{1, n}$.

For $u \in \mathbb{C}^*$, $u^d := \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \in \text{SL}_2(\mathbb{C}) \subset G_n$ and for $v \in \mathbb{C}$, $e^vX := \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{C}) \subset G_n$.

**Proposition 7.2.** [40, Proposition 8.3] There exists a unique principal $G_n$–bundle $P_n$ over $M_{1, n}$, such that a section of $U \subset M_{1, n}$ is a function $f : \pi^{-1}(U) \to G_n$, with the properties that

$$f(z + \alpha x \tau | \tau) = f(x \tau)$$

$$f(z | \tau + 1) = f(x | \tau)$$

7.3. **Flat connection on the elliptic moduli space.** In this section, we collect some facts about the universal flat connection constructed in [40, Section 9] on the bundle $P_n$, which is an extension of the universal KZB connection $\nabla_{KZB, \tau}$ to the $\tau$–direction. Recall, in Section 2 (10), we have the function $k(z, x | \tau) = \frac{\theta(z + x \tau)}{\theta(z \tau) \theta(x \tau)} \frac{1}{x} \in \text{Hol}(\mathbb{C} - \Lambda_\tau)\{[x]\}$. Let

$$g(z, x | \tau) := k_1(x, x | \tau) = \frac{\theta(z + x \tau)}{\theta(z \tau) \theta(x \tau)} \left( \theta'(z + x | \tau) - \theta'(x | \tau) \right) + \frac{1}{x^2}$$

be the derivative of function $k(z, x | \tau)$ with respect to variable $x$. We have $g(z, x | \tau) \in \text{Hol}(\mathbb{C} - \Lambda_\tau)\{[x]\}$.

For a power series $\psi(x) = \sum_{n \geq 1} b_{2n} x^{2n} \in \mathbb{C}\{[x]\}$ with positive even degrees, we define two elements in $\overline{t}^\Phi \cong \mathfrak{h}$ by

$$\delta_\psi := \sum_{n \geq 1} b_{2n} \delta_{2n}, \quad \Delta_\psi := \Delta_\psi + \delta_\psi = \Delta_0 + \sum_{n \geq 1} b_{2n} \delta_{2n}.$$ 

As in [8], we consider the following power series

$$\varphi(x) = g(0, 0 | \tau) - g(0, x | \tau) = -\frac{1}{x^2} - \left( \frac{\theta'}{\theta} \right)'(x | \tau) + \left( \frac{1}{x^2} + \left( \frac{\theta'}{\theta} \right)'(x | \tau) \right) |_{x = 0} \in \mathbb{C}\{[x]\}$$

which has positive even degrees. Set $a_{2n} := \frac{-(2n+1)B_{2n+2}(2n+2)!}{(2n+2)!}$, where $B_n$ are the Bernoulli numbers given by the expansion $\frac{1}{e^t - 1} = \sum_{n \geq 0} \frac{B_{2n}}{(2n)!} t^{2n}$. Then, the function $\varphi(x)$ has the expansion $\varphi(x) = \sum_{n \geq 1} a_{2n} E_{2n+2} (\tau) x^{2n}$, for some $E_{2n+2} (\tau)$ only depending on $\tau$. By our convention, we have the following two elements in $\overline{t}^\Phi \cong \mathfrak{h}$:

$$\delta_\varphi = \sum_{n \geq 1} a_{2n} E_{2n+2} (\tau) \delta_{2n}, \quad \Delta_\varphi = \Delta_0 + \delta_\varphi = \Delta_0 + \sum_{n \geq 1} a_{2n} E_{2n+2} (\tau) \delta_{2n}.$$ 

Consider the following function on $\mathfrak{h} \times \mathfrak{S}$:

$$\Delta := \Delta(\mathfrak{h}, \tau) = -\frac{1}{2\pi i} \frac{1}{\Delta_\varphi} + \frac{1}{2\pi i} \sum_{\beta \in \Phi^+} g(\beta, \text{ad} \frac{x\delta_\varphi}{2} | \tau)(t_\beta)$$

$$= -\frac{1}{2\pi i} \Delta_0 - \frac{1}{2\pi i} \sum_{n \geq 1} a_{2n} E_{2n+2} (\tau) \delta_{2n} + \frac{1}{2\pi i} \sum_{\beta \in \Phi^+} g(\beta, \text{ad} \frac{x\delta_\varphi}{2} | \tau)(t_\beta).$$
This is a meromorphic function on \( \mathbb{C}^n \times \mathcal{V} \) valued in \((t^\Phi_{\text{sl}_2} \rtimes \mathfrak{sl}_2) \rtimes \mathfrak{sl}_2 \subset \text{Lie}(G_\mathfrak{h})\), where \( \mathfrak{sl}_2 = \mathbb{C} \Delta_0 \subset \mathfrak{sl}_2 \). It has only poles at \( \bigcup_{a \in \Phi^+} R a_a \).

**Theorem 7.3.** [40, Theorem D] The following \( t^\Phi_{\text{sl}_2} \rtimes \mathfrak{sl}_2 \)-valued KZB connection on \( M_{1,n} \) is flat.

\[
\nabla_{\text{KZB}} = \nabla_{\text{KZB},t} - \Delta \tau = d - \sum_{a \in \Phi^+} k(a, \text{ad} \frac{X^a}{2}(t_a)) d\tau + \sum_{i=1}^n y(u^i) du_i - \Delta \tau.
\]

**8. The \( \mathfrak{sl}_2 \)-triple in the deformed double current algebras**

In this section and Section 9, we prove Theorem C by constructing an algebra homomorphism from \( t^\Phi_{\text{sl}_2} \rtimes \mathfrak{sl}_2 \) to the deformed double current algebra \( D_\lambda(g) \). This shows the elliptic Casimir connection \( \nabla_{\text{Ell,C}} \) extends to a flat connection on \( M_{1,n} \), whose coefficients are in \( D_\lambda(g) \).

To begin with, in this section, we construct an action of \( \mathfrak{sl}_2 \) on the deformed double current algebra \( D_\lambda(g) \). We first construct an action of \( \mathfrak{sl}_2 \) coming from the group \( \text{SL}_2(\mathbb{C}) \) permutation of the two lattices \( \mathfrak{g}[u], \mathfrak{g}[v] \) of \( D_\lambda(g) \). We then show this action is inner. That is, there exists an \( \mathfrak{sl}_2 \)-triple \( \{E, F, H\} \) in \( D_\lambda(g) \), such that the action of \( \mathfrak{sl}_2 \) is given by the commutator \( [X, -] \), for \( X \in \{E, F, H\} \).

**8.1. The action of group \( \text{SL}_2 \).**

**Proposition 8.1.** There is a right action of \( \text{SL}_2(\mathbb{C}) \) on \( D_\lambda(g) \). For \( A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \text{SL}_2(\mathbb{C}) \) and \( z \in \mathfrak{g} \), the action is given by

\[
\begin{align*}
  z &\mapsto z, \\
  K(z) &\mapsto a_{11}K(z) + a_{12}Q(z), \\
  Q(z) &\mapsto a_{22}Q(z) + a_{21}K(z), \\
  \text{and}\ P(z) &\mapsto (a_{11}a_{22} + a_{12}a_{21})P(z) + a_{11}a_{21}[K(y), K(w)] + a_{12}a_{22}[Q(y), Q(w)].
\end{align*}
\]

where \( y, w \) is determined by the equality \( z = [y, w] \).

In particular, we have an order 4 automorphism of \( D_\lambda(g) \) (see [19, Proposition 12.1]):

\[
\begin{align*}
  z &\mapsto z, \\
  K(z) &\mapsto -Q(z), \\
  Q(z) &\mapsto K(z), \\
  P(z) &\mapsto -P(z).
\end{align*}
\]

**Remark 8.2.** The right action in Proposition 8.1 can be made to a left \( \text{SL}_2(\mathbb{C}) \)-action on \( D_\lambda(g) \). We define the left action by \( A \cdot z := zA^T \), where \( A^T \) is the transpose of \( A \).

**Proof of Proposition 8.1.** We first check that the action of \( A \in \text{SL}_2(\mathbb{C}) \) preserves the defining relations of \( D_\lambda(g) \). We only check that the action preserves the relation (12), as the other defining relations are obviously preserved. For simplicity, set

\[
C(X_{\beta_1}, X_{\beta_2}) := -\frac{A}{4}(\beta_1, \beta_2)S(X_{\beta_1}, X_{\beta_2}) + \frac{A}{4} \sum_{a \in \Phi} S([X_{\beta_1}, X_a], [X_{-a}, X_{\beta_2}]).
\]

The equality \( S(a_1, a_2) = S(a_2, a_1) \) implies that \( C(X_{\beta_1}, X_{\beta_2}) = C(X_{\beta_2}, X_{\beta_1}) \). The relation (12) can be rewritten as \( [K(x), Q(y)] = P([x, y]) + C(x, y) \).

Under the action of \( A \in \text{SL}_2(\mathbb{C}) \), it is straightforward to compute the image of \( [K(x), Q(y)] \) under the action. We have

\[
\begin{align*}
  [K(x), Q(y)] &\mapsto [a_{11}K(x) + a_{12}Q(x), a_{22}Q(y) + a_{21}K(y)] \\
  &\mapsto a_{11}a_{22}[K(x), Q(y)] + a_{12}a_{21}[Q(x), K(y)] + a_{11}a_{21}[K(x), K(y)] + a_{12}a_{22}[Q(x), Q(y)] \\
  &\mapsto a_{11}a_{22} + a_{12}a_{21}P([x, y]) + C(x, y) + a_{11}a_{21}[K(x), K(y)] + a_{12}a_{22}[Q(x), Q(y)] \\
  &\mapsto (a_{11}a_{22} + a_{12}a_{21})P([x, y]) + a_{11}a_{21}[K(x), K(y)] + a_{12}a_{22}[Q(x), Q(y)] + C(x, y).
\end{align*}
\]

It is straightforward to see that (37) coincides with \( P([x, y] + C(x, y)) \cdot A \). Thus, the action of \( A \) preserves the relation (12).
We then check that it defines a right action of $SL_2$. It is obvious that $z \Id = z$, for any $z \in D_4(\mathfrak{g})$, where $\Id$ is the identity matrix of $SL_2$. For any $A, B \in SL_2(\mathbb{C})$, and for any $z \in D_4(\mathfrak{g})$, it is a direct calculation to show 
\[ (zA)B = z(AB). \]
For the convenience of the readers, we show the less obvious case when $z = P([x, y])$ as follows.

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ be two elements of $SL_2(\mathbb{C})$, we have
\[ (P([x, y]).A).B = (a_{11}a_{22} + a_{12}a_{21})P([x, y]).B + a_{11}a_{21}[K(x).B, K(y).B] + a_{12}a_{22}[Q(x).B, Q(y).B] \]
\[ = (a_{11}a_{22} + a_{12}a_{21})(b_{11}b_{22} + b_{12}b_{21})P([x, y]) + (a_{11}a_{22} + a_{12}a_{21})(b_{11}b_{21}[K(x), K(y)] + b_{12}b_{22}[Q(x), Q(y)]) \]
\[ + a_{11}a_{21}[b_{11}K(x) + b_{12}Q(x), b_{11}K(y) + b_{12}Q(y)] + a_{12}a_{22}[b_{22}Q(x) + b_{21}K(x), b_{22}Q(y) + b_{21}K(y)] \]
\[ = P([x, y])(a_{11}a_{22} + a_{12}a_{21})(b_{11}b_{22} + b_{12}b_{21}) + 2a_{11}a_{21}b_{11}b_{21}2 + 2a_{12}a_{22}b_{22}b_{22} \]
\[ + [K(x), K(y)](a_{11}a_{22} + a_{12}a_{21})b_{11}b_{21} + a_{11}a_{21}b_{11}^{2} + a_{12}a_{22}b_{21}^{2} \]
\[ + [Q(x), Q(y)](a_{11}a_{22} + a_{12}a_{21})b_{12}b_{22} + a_{11}a_{21}b_{12}^{2} + a_{12}a_{22}b_{22}^{2} \]
\[ + C(x, y)(a_{11}a_{21}b_{12} - a_{11}a_{21}b_{11}b_{12} + a_{12}a_{22}b_{21}b_{22} - a_{12}a_{22}b_{21}b_{22}) \]
\[ = P([x, y])(a_{11}b_{11} + a_{12}b_{12})(a_{12}b_{12} + a_{12}b_{22}) + (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21}) \]
\[ + [K(x), K(y)](a_{11}b_{11} + a_{12}b_{12})(a_{21}b_{12} + a_{22}b_{22}) + [Q(x), Q(y)](a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{12} + a_{22}b_{22}) \]
\[ = P([x, y]).(AB). \]

Therefore, we have a right action of $SL_2(\mathbb{C})$ on $D_4(\mathfrak{g})$. This completes the proof. \hfill \Box

**Corollary 8.3.** The $SL_2(\mathbb{C})$ action in Proposition 8.1 induces a Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ action on $D_4(\mathfrak{g})$.

For $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$, the action is given by

\[ zX = 0, \quad K(z)X = x_{11}K(z) + x_{21}Q(z), \quad Q(z)X = x_{22}Q(z) + x_{12}K(z), \]
\[ P(z)X = x_{21}[K(y), K(w)] + x_{21}[Q(y), Q(w)], \]

where $z \in \mathfrak{sl}_2(\mathbb{C})$ and $y, w$ are determined by the quality $z = [y, w]$. \hfill \Box

In particular, the action of $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2$ is given by $hz = 0$, $hP(z) = 0$, $hK(z) = K(z)$, and $hQ(z) = -Q(z)$.

### 8.2. The $\mathfrak{sl}_2$ triple of $D_4(\mathfrak{g})$.

In this subsection, we construct an $\mathfrak{sl}_2$–triple $E, F, H \in D_4(\mathfrak{g})$, such that for any $z \in \mathfrak{g}$, we have

- $[\mathbb{H}, z] = 0$, $[\mathbb{H}, K(z)] = -K(z)$ and $[\mathbb{H}, Q(z)] = Q(z)$.
- $[\mathbb{E}, z] = 0$, $[\mathbb{E}, K(z)] = Q(z)$, and $[\mathbb{E}, Q(z)] = 0$.
- $[\mathbb{F}, z] = 0$, $[\mathbb{F}, K(z)] = 0$, and $[\mathbb{F}, Q(z)] = K(z)$.

Recall we have two subalgebras $\mathfrak{g}[v]$, $\mathfrak{g}[u]$ of $D_4(\mathfrak{g})$. Write $K(z)$ as the element $z \otimes v$ in the subalgebra $\mathfrak{g}[v]$, and $Q(z)$ as $z \otimes u$ in $\mathfrak{g}[u]$. We will need the following higher degree commutation relation of $D_4(\mathfrak{g})$, which is proved in [22, Proposition 6.1] when $\mathfrak{g} = \mathfrak{sl}_4$.

The following result is proved in [22, Prop. 6.1] in type $A$.

**Proposition 8.4.** For any $s \geq 0$ and any $X \in \mathfrak{g}$, there exists in $D_4(\mathfrak{g})$ an element $P_s(X)$ with the property that the assignment $X \mapsto P_s(X)$ is linear, $[P_s(X), X'] = P_s([X, X'])$ for any $X' \in \mathfrak{g}$, and such that, for all root
vectors $X_{\beta_1}, X_{\beta_2} \in \mathfrak{g}$ with $\beta_1 \neq -\beta_2$, the following relation holds:

$$
[K(X_{\beta_1}), X_{\beta_2} \otimes u^s] = P_s([X_{\beta_1}, X_{\beta_2}]) - \frac{\lambda(\beta_1, \beta_2)}{4} \sum_{p+q=s-1} S(X_{\beta_1} \otimes u^p, X_{\beta_2} \otimes u^q)
$$

$$
+ \frac{\lambda}{4} \sum_{a \in \Phi} \sum_{p+q=3-1} S([X_{\beta_1}, X_a] \otimes u^p, [X_{-a}, X_{\beta_2}] \otimes u^q).
$$

**Remark 8.5.** When $s = 0$, $P_0(X) = K(X)$ and the right-hand side equals $K([X_{\beta_1}, X_{\beta_2}])$.

**Remark 8.6.** Write $\sum_{s \geq 1} X_{\beta_s} \otimes u^s = X_{\beta_s} \otimes \frac{u^s}{1-u}$ as a generating series. The relation in Proposition 8.4 is equivalent to the following relation.

$$
[X_{\beta_1} \otimes v, X_{\beta_2} \otimes \frac{u}{1-u}] = \sum_{s \geq 1} P_s([X_{\beta_1}, X_{\beta_2}]) - \frac{\lambda(\beta_1, \beta_2)}{4} S(X_{\beta_1} \otimes \frac{1}{1-u}, X_{\beta_2} \otimes \frac{1}{1-u})
$$

$$
+ \frac{\lambda}{4} \sum_{a \in \Phi} S([X_{\beta_1}, X_a] \otimes \frac{1}{1-u}, [X_{-a}, X_{\beta_2}] \otimes \frac{1}{1-u}).
$$

For any non-zero element $h \in \mathfrak{h}$, let $\overline{E}(h)$ and $\overline{F}(h)$ be the following elements of $D_\lambda(\mathfrak{g})$:

$$
\overline{E}(h) := \frac{1}{(h, h)} \left( [h \otimes v, h \otimes u^s] - \frac{\lambda}{4} \sum_{p+q=2} S([h, X_a] \otimes u^p, [X_{-a}, h] \otimes u^q) \right)
$$

$$
\overline{F}(h) := \frac{1}{(h, h)} \left( [h \otimes u, h \otimes v^s] + \frac{\lambda}{4} \sum_{p+q=2} S([h, X_a] \otimes v^p, [X_{-a}, h] \otimes v^q) \right).
$$

**Lemma 8.7.** The elements $\overline{E}(h), \overline{F}(h)$ are independent of the choice of $h \in \mathfrak{h}$ for $h \neq 0$.

**Proof.** By linearity, it suffices to show for any $\alpha, \beta \in \Phi^+$, we have $\overline{E}(\alpha) = \overline{E}(\beta), \overline{F}(\alpha) = \overline{F}(\beta)$. This follows from the same proof as [22, Proposition 6.2] with $\beta = \frac{4}{4}$ using the higher degree relations in Proposition 8.4. The idea of the proof is essentially in the proof of [22, Proposition 4.1].

Based on Lemma 8.7, we denote $\overline{E}(h)$ by $\overline{E}$, and $\overline{F}(h)$ by $\overline{F}$.

**Corollary 8.8.** For any $h, h' \in \mathfrak{h}$, such that $(h, h') = 0$. We have the following identity in $D_\lambda(\mathfrak{g})$

$$
[h \otimes v, h \otimes u^s] = \frac{\lambda}{4} \sum_{p+q=2} \sum_{\alpha, \beta \in \Phi} (h, \alpha)(h', \alpha) S(X_{\alpha} \otimes u^p, X_{-\alpha} \otimes u^q).
$$

**Proof.** The claim is true for $h = \alpha$, and $h' = \beta$, where $\alpha \in \Phi$ and $\beta \in \Phi$ are two roots such that $(\alpha, \beta) = 0$, which follows from the same proof as [22, Proposition 6.2]. The general statement follows from Lemma 8.7 and linearity of the formula (40) in $h$ and $h'$.

We will show the following elements form an $\mathfrak{sl}_2$-triple of $D_\lambda(\mathfrak{g})$:

$$
\mathcal{E} := \frac{\overline{E}}{C}, \mathcal{F} := \frac{\overline{F}}{C}, \mathcal{H} := [\mathcal{E}, \mathcal{F}],
$$

where the constant $C \in \mathbb{Q}$ depends on the type of the Lie algebra $\mathfrak{g}$, which is given by the following formula.

$$
C = \lambda^2 \sum_{(\alpha, \beta) \in \Phi^+} \frac{1 - (\alpha, \beta)^2}{(\alpha, \beta)(\beta, \beta)} \frac{(\beta, \beta)^2 + (\alpha, \alpha)^2}{16 \dim \mathfrak{h}(\dim \mathfrak{h} - 1)}.
$$

In Appendix A, we compute the constant $C$ explicitly.

We have the following main result in this section.
Theorem 8.9. With notations as above, the following holds.

(1) For any \( z \in \mathfrak{g} \), we have
   (a) \([\mathbb{H}, z] = 0, [\mathbb{E}, K(z)] = -K(z) \) and \([\mathbb{H}, Q(z)] = Q(z)\).
   (b) \([\mathbb{E}, z] = 0, [\mathbb{E}, K(z)] = Q(z), \) and \([\mathbb{E}, Q(z)] = 0\),
   (c) \([\mathbb{F}, z] = 0, [\mathbb{F}, K(z)] = 0, \) and \([\mathbb{F}, Q(z)] = K(z)\).
(2) The elements \( \mathbb{E}, \mathbb{F}, \mathbb{H} \) form an \( \mathfrak{sl}_2 \)-triple.

We prove Theorem 8.9 for the rest of this section.

8.3. Proof of Theorem 8.9(2). In this subsection, we check that the triple \( \mathbb{E}, \mathbb{F}, \mathbb{H} \) form a Lie algebra \( \mathfrak{sl}_2 \) using the part (1) of Theorem 8.9. That is, we check \( [\mathbb{H}, \mathbb{E}] = 2\mathbb{E}, \) and \( [\mathbb{H}, \mathbb{F}] = -2\mathbb{F} \).

By (1) of Theorem 8.9, we claim by induction that

\[
\text{ad}(\mathbb{E})^p(X \otimes v^q) = n!X \otimes u^n, \quad \text{for any } X \in \mathfrak{g}.
\] (42)

Indeed, let \( X = [X_1, X_2] \), we have \( X \otimes v^q = [X_1 \otimes v, X_2 \otimes v^{q-1}] \). Therefore,

\[
\text{ad}(\mathbb{E})^p(X \otimes v^q) = \text{ad}(\mathbb{E})^p[X_1 \otimes v, X_2 \otimes v^{q-1}]
\]

\[
= \sum_{p+q=n} \binom{n}{p} \left[ \text{ad}(\mathbb{E})^p(X_1 \otimes v), \text{ad}(\mathbb{E})^q(X_2 \otimes v^{q-1}) \right]
\]

\[
= n[X_1 \otimes u, (n-1)!X_2 \otimes u^{q-1}] = n!X \otimes u^n.
\]

This shows the claim (42).

To show that \( [\mathbb{H}, \mathbb{E}] = 2\mathbb{E} \), it suffices to show \( \text{ad}(\mathbb{E})^2(\mathbb{F}) = -2\mathbb{F} \). It follows from the following two lemmas and the definitions of \( \mathbb{E} \) (38) and \( \mathbb{F} \) (39).

Lemma 8.10. For any root \( \alpha \in \Phi \), we have

\[
\text{ad}(\mathbb{E})^2[H_\alpha \otimes u, H_\alpha \otimes v^3] = 2[H_\alpha \otimes v, H_\alpha \otimes u^3].
\]

Proof. We have the general identity

\[
\text{ad}(\mathbb{E})^p[A, B] = \sum_{p+q=n} \binom{n}{p} \text{ad}(\mathbb{E})^pA, \text{ad}(\mathbb{E})^qB.
\]

Choose \( A \) to be \( h \otimes v \), \( B \) to be \( h \otimes v^3 \), and apply the operator \( \text{ad}(\mathbb{E})^3 \) to the equality \([A, B] = 0\). This gives the following identity

\[
\text{ad}(\mathbb{E})^3[h \otimes v, h \otimes v^3] = 3[h \otimes u, \text{ad}(\mathbb{E})^2h \otimes v^3] + 6[h \otimes v, h \otimes u^3] = 0.
\]

Rewrite the above identity, we have

\[
[h \otimes u, \text{ad}(\mathbb{E})^2h \otimes v^3] = \text{ad}(\mathbb{E})^2[h \otimes u, h \otimes v^3] = -2[h \otimes v, h \otimes u^3].
\]

This completes the proof. \( \square \)

The following lemma is a directly consequence of the identity

\[
\text{ad}(\mathbb{E})^p(AB) = \sum_{p+q=n} \binom{n}{p} \text{ad}(\mathbb{E})^pA \text{ad}(\mathbb{E})^qB
\]

and Theorem 8.9 (1).

Lemma 8.11. For any \( p, q \in \mathbb{N} \), such that \( p + q = 2 \). We have

\[
\text{ad}(\mathbb{E})^2 \sum_{\alpha \in \Phi} S([h, X_\alpha] \otimes v^p, [X_{-\alpha}, h] \otimes v^q) = 2 \sum_{\alpha \in \Phi} S([h, X_\alpha] \otimes u^p, [X_{-\alpha}, h] \otimes u^q).
\]
Write \( \overline{F} = \overline{F}(H_\alpha) \), by Lemma 8.10 and Lemma 8.11, we have \( \text{ad}(\mathbb{E})^2(\overline{F}) = -2\overline{F} \). Therefore,

\[
[\mathbb{H}, \mathbb{E}] = [[\mathbb{E}, \mathbb{F}], \mathbb{E}] = -\frac{1}{C} \text{ad}(\mathbb{E})^2(\overline{F}) = 2\frac{1}{C} \overline{E} = 2\mathbb{E}.
\]

By symmetry, we have \([\mathbb{H}, \mathbb{F}] = -2\mathbb{F} \). This complete the proof of Theorem 8.9 (2).

8.4. **Proof of Theorem 8.9 (1).** In this subsection, we prove Theorem 8.9 (1). Similar to the proof of [22, Lemma 6.2] with the choice of \( \beta = \frac{1}{7} \), \( s = 3 \), it is straightforward to check that \( \mathbb{E} \) commutes with the subalgebra \( g[u] \subset D_4(g) \). As a consequence, we have \([\mathbb{E}, z] = 0 \) and \([\mathbb{E}, Q(z)] = 0 \). By the symmetry of \( \mathbb{E} \) and \( \mathbb{F} \), it suffices to show \([\mathbb{E}, K(z)] = Q(z) \).

In the reminder of this section, we compute the commutator \([\mathbb{E}, K(z)]\) and identify it with \( CQ(z) \), for \( C \in \mathbb{Q} \) given by the formula (41). This constant \( C \) is simplified further in Appendix A. As \([\mathbb{E}, z] = 0 \), to get a formula of \([\mathbb{E}, K(z)]\) for arbitrary \( z \in g \), we only need to compute \([\mathbb{E}, K(h')]\) for some \( h' \in \mathfrak{h} \).

Recall by Lemma 8.7, the element \( \overline{E}(h) \) (38) is independent of \( h \in \mathfrak{h} \). By convenience of the computation, we choose \( h, h' \in \mathfrak{h} \), such that \((h, h') = 0 \). Under this assumption \((h, h') = 0 \), the computation of \([\mathbb{E}, K(h')]\) is essentially in the proof of [22, Lemma 6.3], which uses the relation in Corollary 8.8. For the convenience of the readers, we include the computation here.

Assume \((h, h') = 0 \). On the one hand, we have

\[
[\hbar \otimes v, h \otimes u^3, h' \otimes v] = -[\hbar \otimes v, [h' \otimes v, h \otimes u^3]] = -\frac{\lambda}{4} \hbar \otimes v, \sum_{p+q=2} \sum_{\alpha \in \Phi} (h, \alpha)(h', \alpha)S(X_\alpha \otimes u^p, X_{-\alpha} \otimes u^q)
\]

\[
= -\frac{\lambda}{4} \sum_{p+q=2} \sum_{\alpha \in \Phi} (h, \alpha)(h', \alpha)S([h \otimes v, X_\alpha \otimes u^p], X_{-\alpha} \otimes u^q)]
\]

\[
- \frac{\lambda}{4} \sum_{\alpha \in \Phi} \sum_{p+q=2} (h, \alpha)(h', \alpha)S(X_\alpha \otimes u^p, [h \otimes v, X_{-\alpha} \otimes u^q])
\]

(43)

On the other hand, we have:

\[
\lfloor S([h, X_\alpha] \otimes u^p, [X_{-\alpha}, h] \otimes u^q), h' \otimes v \rfloor = \lfloor (h, \alpha)^2 S(X_\alpha \otimes u^p, X_{-\alpha} \otimes u^q), h' \otimes v \rfloor
\]

\[
= - (h, \alpha)^2 S((h' \otimes v, X_\alpha \otimes u^p), X_{-\alpha} \otimes u^q) - (h, \alpha)^2 S(X_\alpha \otimes u^p, [h' \otimes v, X_{-\alpha} \otimes u^q])
\]

(44)
Plugging the computations of (43) and (44) into the definition of $\overline{F}$ (38) and arranging the summands, we have:

$$(h,h)[\overline{F}, h' \otimes v] = \left[ h \otimes v, h \otimes u^3 \right] - \frac{A}{4} \sum_{p+q=2 \alpha \in \Phi} S(h, X_\alpha) \otimes u^p, [X_{-\alpha}, h] \otimes u^q) \cdot h' \otimes v$$

$$= -\frac{A}{4} \sum_{p+q=2 \alpha \in \Phi} (h, \alpha)S\left((h', \alpha)h - (h, \alpha)h'\right) \otimes v, X_\alpha \otimes u^q) \cdot X_{-\alpha} \otimes u^q)$$

$$= -\frac{A}{2} \sum_{p+q=2 \alpha \in \Phi} \sum_{\beta \in \Phi} \sum_{s+t=p-1} S\left(S\left((h', \alpha)h - (h, \alpha)h'\right)S(\left[X_\beta, X_\alpha\right] \otimes u^s, [X_{-\beta}, X_\alpha] \otimes u^t) \cdot X_{-\alpha} \otimes u^q$$

$$= -\frac{A^2}{8} \sum_{s+t=p-1 \alpha, \beta \in \Phi} \sum_{\alpha, \beta} \left( (h, \alpha)(h', \alpha)(h, \beta) + (h, \alpha)^2(h', \beta) \right) S\left(X_\alpha \otimes u^s, [X_{-\beta}, X_\alpha] \otimes u^t) \cdot X_{-\alpha} \otimes u^q$$

We use the trick in [22, §6.2] to simplify equation (45). By the formula of [22, §6.2, Page 1357], we have the following identity

$$[\overline{F}, h' \otimes v] = \frac{A^2}{4(h,h)} \sum_{\alpha, \beta \in \Phi} \left( (h, \alpha)(h', \alpha)(h, \beta) + (h, \alpha)^2(h', \beta) \right) \left( [X_\beta, X_\alpha] \otimes u^s, [X_{-\beta}, X_\alpha] \otimes u^t \right) \otimes u$$

where $(h, h') = 0$.

It remains to simply the right hand side of equation (46) under the assumption $(h, h') = 0$. For convenience of the notation, we set

$$F(h, h') := \sum_{\alpha, \beta \in \Phi} \left( (h, \alpha)(h', \alpha)(h, \beta) + (h, \alpha)^2(h', \beta) \right) \left( [X_\beta, X_\alpha] \otimes u^s, [X_{-\beta}, X_\alpha] \otimes u^t \right) \otimes u$$

Note that the element $F(h, h')$ is well-defined for any $h, h' \in \mathfrak{h}$. In general, $F(h, h')$ is some element in $\mathfrak{h}$. For $(h, h') = 0$, equation (46) is the same as $[\overline{F}, h' \otimes v] = \frac{A^2}{(h,h)} F(h, h') \otimes u$. By Lemma 8.7, $\overline{F}$ is independent of the choice of $h \in \mathfrak{h}$. As a consequence, suppose $(h, h') = 0$, then the element $F(h, h')$ is also independent of the choice of $h \in \mathfrak{h}$.

We now list some properties of the element $F(h, h')$.

**Lemma 8.12.** The following holds.

(i) The element $F(h, h')$ is linear in $h'$.

(ii) If $h$ and $h'$ are parallel to each other, we have $F(h, h') = 0$.

(iii) For any $h, h' \in \mathfrak{h}$, write $h' = h'_\parallel + h'_\perp$, where $h'_\parallel$ is parallel to $h$, and $h'_\perp$ is perpendicular to $h$, then we have $F(h, h') = F(h, h'_\perp)$.

(iv) For any $h, h' \in \mathfrak{h}$, the inner product $(F(h, h'), h)$ is zero.

(v) $F(h, h') = (h, h)\tilde{C}h'$, for some constant $\tilde{C} \in \mathbb{C}$.

**Proof.** (i) and (ii) are clear from the definition of $F(h, h')$. (iii) follows easily from (i), (ii).
For (iv), we have
\[(F(h, h'), h) = \sum_{\alpha, \beta \in \Phi} \left( (h, \alpha)(h', \alpha)(h, \beta)^2 - (h, \alpha)^2(h', \beta)(h, \beta) \right) \left( [X_\beta, X_{-\alpha}] | [X_{-\beta}, X_\alpha] \right) \]
\[= \sum_{\alpha, \beta \in \Phi} \left( (h, \beta)(h', \beta)(h, \alpha)^2 - (h, \beta)^2(h', \alpha)(h, \alpha) \right) \left( [X_\alpha, X_{-\beta}] | [X_{-\alpha}, X_\beta] \right) = -(F(h, h'), h).\]

This concludes (iv).

For (v), we fix \( h' \in \mathfrak{h} \) and let \( h \) vary in \( P_{h'} \), where \( P_{h'} \) is the plane perpendicular to \( h' \). We have the fact that if \( (h, h') = 0 \), the element \( \frac{F(h, h')}{(h, h)} \) is independent of the choice of \( h \). By (iv), \( \frac{F(h, h')}{(h, h)} \) is perpendicular to \( P_{h'} \). Therefore, there exists some \( C \), such that \( F(h, h') = (h, h)\overline{C}h' \). This completes the proof. \( \square \)

By Lemma 8.12 (v), we have
\[ [\Xi, h' \otimes v] = \frac{\lambda^2}{4} \overline{C}h' \otimes u, \] therefore, the constant \( C \) is the same as \( C = \frac{\lambda^2}{4} \overline{C} \).

For the rest of this section, we determine the constant \( \overline{C} \), and therefore give an explicit formula of the constant \( C \). Let \( \{h_\alpha\} \) be a basis of \( \mathfrak{h} \), and \( \{h'^\alpha\} \) be the dual basis. On one hand, we have:
\[ \sum_a (F(h, h_\alpha), h'^\alpha) = \sum_{\alpha, \beta \in \Phi} \sum_a ((h, \alpha)(h_\alpha, \alpha)(h, \beta)(h'^\alpha, \beta) - (h, \alpha)^2(h_\alpha, \beta)(h'^\alpha, \beta)) \left( [X_\beta, X_{-\alpha}] | [X_{-\beta}, X_\alpha] \right) \]
\[= \sum_{\alpha, \beta \in \Phi} \left( (h, \alpha)(\alpha, \beta)(h, \beta) - (h, \alpha)^2(\beta, \beta) \right) \left( [X_\beta, X_{-\alpha}] | [X_{-\beta}, X_\alpha] \right). \quad (47) \]

On the other hand, by Lemma 8.12 (iii) and (v), we have \( F(h, h_\alpha) = (h, h)\overline{C} \left( h_\alpha - \frac{h(h, h_\alpha)}{(h, h)} h \right) \). Therefore,
\[ \sum_a (F(h, h_\alpha), h'^\alpha) = \sum_a (h, h)\overline{C} \left( h_\alpha, h'^\alpha \right) - \frac{(h_\alpha, h)(h, h'^\alpha)}{(h, h)} = (h, h)\overline{C} \left( \dim \mathfrak{h} - 1 \right). \quad (48) \]

Combining (47) and (48), we have:
\[ \sum_{\alpha, \beta \in \Phi} \left( (h, \alpha)(\alpha, \beta)(h, \beta) - (h, \alpha)^2(\beta, \beta) \right) \left( [X_\beta, X_{-\alpha}] | [X_{-\beta}, X_\alpha] \right) = (h, h)\overline{C} \left( \dim \mathfrak{h} - 1 \right). \]

Choosing \( h = \gamma \in \Phi^+ \), and taking the sum of \( \gamma \) over \( \Phi^+ \), we have
\[ \sum_{\alpha, \beta \in \Phi} \left( (\gamma, \alpha)(\alpha, \beta)(\gamma, \beta) - (\gamma, \alpha)^2(\beta, \beta) \right) \left( [X_\beta, X_{-\alpha}] | [X_{-\beta}, X_\alpha] \right) = \sum_{\gamma \in \Phi^+} (\gamma, \gamma)\overline{C} \left( \dim \mathfrak{h} - 1 \right). \]

Using the identity \( (\gamma, \gamma) = \sum_{\gamma' \in \Phi^+} (\gamma, \gamma')(\gamma', \gamma) \) in Lemma 5.4, we simplify the above equality as
\[ h^\gamma \sum_{\alpha, \beta \in \Phi} \left( (\beta, \alpha)(\alpha, \beta) - (\alpha, \alpha)(\beta, \beta) \right) \left( [X_\beta, X_{-\alpha}] | [X_{-\beta}, X_\alpha] \right) = \sum_{\gamma \in \Phi^+} (\gamma, \gamma)\overline{C} \left( \dim \mathfrak{h} - 1 \right). \]

Write \( (\gamma, \gamma) = \sum_a (h_\alpha, \gamma)(h'^\alpha, \gamma) \). We have
\[ \sum_{\gamma \in \Phi^+} (\gamma, \gamma) = \sum_{\gamma \in \Phi^+} \sum_a (h_\alpha, \gamma)(h'^\alpha, \gamma) = \sum_a \left( h_\alpha, h'^\alpha \right) = h^\gamma \dim \mathfrak{h}. \]

Therefore, \( h^\gamma \sum_{\alpha, \beta \in \Phi} \left( (\beta, \alpha)(\alpha, \beta) - (\alpha, \alpha)(\beta, \beta) \right) \left( [X_\beta, X_{-\alpha}] | [X_{-\beta}, X_\alpha] \right) = h^\gamma \dim \mathfrak{h} \left( \dim \mathfrak{h} - 1 \right) \overline{C}. \)

This gives an explicit formula of \( \overline{C} \):
\[ \overline{C} = \frac{\sum_{\alpha, \beta \in \Phi} \left( (\alpha, \beta)^2 - (\alpha, \alpha)(\beta, \beta) \right) \left( [X_\beta, X_{-\alpha}] | [X_{-\beta}, X_\alpha] \right)}{\dim \mathfrak{h} \left( \dim \mathfrak{h} - 1 \right)}, \quad (49) \]
and the constant $C$ is given by $C = \frac{1}{4} \bar{C}$. We compute the constant $\bar{C}$ in Appendix A using (49).

### 9. The extension of the elliptic Casimir connection

In this section, we extend the derivation action of $\mathfrak{d}$ on the Lie algebra $\mathfrak{d}_\mathfrak{n}$ in Section 7.1 to an action on the deformed double current algebra $\mathcal{D}_\mathfrak{n}(\mathfrak{g})$. We show furthermore that the action of $\mathfrak{d}$ on $\mathcal{D}_\mathfrak{n}(\mathfrak{g})$ is inner.

#### 9.1. Set

$$
\bar{E}(n) := \frac{1}{(h, h)} \left( [h \otimes v, h \otimes u^n] - \frac{\lambda}{4} \sum_{p+q=n-1} \sum_{\alpha \in \Phi} S([h, X_\alpha] \otimes u^p, [X_{-\alpha}, h] \otimes u^q) \right).
$$

We have the following facts about the element $\bar{E}(n)$:

**Proposition 9.1.**

1. We have the following equality.

$$
\bar{E}(n) = \bar{E}(\alpha, n) := [X_{-\alpha} \otimes v, X_\alpha \otimes u^n] + P_n(h_\alpha) - \frac{\lambda}{4} \sum_{p+q=n-1} S([X_{-\alpha}, v] \otimes u^p, [X_{-\alpha}, h] \otimes u^q)
\quad - \frac{\lambda}{4} \sum_{p+q=n-1} S([X_{-\alpha}, X_\beta] \otimes u^p, [X_{-\beta}, h] \otimes u^q).
$$

2. For any two roots $\alpha, \beta \in \Phi$, we have $\bar{E}(\alpha, n) = \bar{E}(\beta, n)$.

3. $\bar{E}(n)$ commutes with the subalgebra $\mathfrak{g}[u]$ of $\mathcal{D}_\mathfrak{n}(\mathfrak{g})$.

4. If $n = 1$, $\bar{E}(n)$ is a central element of $\mathcal{D}_\mathfrak{n}(\mathfrak{g})$.

5. For any $n \geq 2$, we have:

$$
[\bar{E}(n), z \otimes v] = \frac{C}{3} \binom{n}{2} z \otimes u^{n-2},
$$

where the constant $C \in \mathbb{Q}$ is given as in (41).

**Proof.** The Proposition is a consequence of the higher degree relation in Proposition 8.4. (1) and (2) follow from the same proof as [22, Proposition 6.2]. (3) follows from the same proof as [22, Lemma 6.2]. (4) is proved in [22, Theorem 4.1]. (5) is a similar computation as (46). Indeed, by [22, §6.2], if $(h, h') = 0$, we have

$$
(h, h)[\bar{E}(n), h' \otimes v]
\quad = -\frac{\lambda^2}{8} \sum_{j+i+q=n-2} \sum_{\alpha, \beta, \gamma \in \Phi} ((h, \alpha)(h', \alpha)(h, \beta) - (h, \alpha)^2(h', \beta)) S([X_\beta \otimes u^q, [X_{-\beta}, X_\alpha] \otimes u^q], [X_{-\beta}, X_\alpha])
\quad = \frac{\lambda^2}{4} \binom{n}{2} \sum_{\alpha, \beta, \gamma \in \Phi} ((h, \alpha)(h', \alpha)(h, \beta) - (h, \alpha)^2(h', \beta)) ([X_\beta, X_{-\beta}], [X_{-\beta}, X_\alpha]) \beta \otimes u^{n-2}.
$$

In §8.4, the formula (46) implies $[\bar{E}, h' \otimes v] = C h' \otimes u$, for $C$ given in (41). Comparing the formula (51) with (46), we conclude that $[\bar{E}(n), h' \otimes v] = \frac{C}{3} \binom{n}{2} h' \otimes u^{n-2}$, for the same constant $C$. This completes the proof of (5).
9.2. Let \( \{h_i\}_{1 \leq i \leq n} \) be the basis of \( \mathfrak{h} \), and \( \{h'\}_{1 \leq i \leq n} \) be the corresponding dual basis and write \( h \otimes u := Q(h) \in \mathfrak{g}[u] \). Set

\[
\delta_{2m} = \frac{\lambda}{2} \sum_{p=0}^{2m} (-1)^p \binom{2m}{p} \sum_i (h_i \otimes u^p)(h' \otimes u^{2m-p}) + \frac{3\lambda}{C} \sum_{p=0}^{m-2} (-1)^p \frac{2(2m)!}{(p+2)!(2m-p)!} \tilde{E}(p+2) \tilde{E}(2m-p) + (-1)^{m-1} \frac{(2m)!}{(m+1)!(m+1)!} \tilde{E}(m+1)^2 - \frac{2}{2m+1} \tilde{E}(2m+1) \tilde{E}(1),
\]

(52)

where the constant \( C \) is given in (41).

**Proposition 9.2.** The \( \mathfrak{sl}_2 \)-triple \( \{\mathbb{H}, \mathbb{B}, \mathbb{F}\} \) and \( \delta_{2m} \) satisfy the relations of the derivation algebra \( \mathfrak{d} \).

**Proof.** We need to check that

\[
[\delta_{2m}, \mathbb{B}] = 0, \quad [\mathbb{H}, \delta_{2m}] = 2m\delta_{2m}, \quad \text{and} \quad (\text{ad} \mathbb{F})^{2m+1}(\delta_{2m}) = 0.
\]

Recall that we have, for any \( z \in \mathfrak{g} \),

1. \( [\mathbb{H}, z] = 0, [\mathbb{H}, K(z)] = -K(z) \) and \( [\mathbb{H}, Q(z)] = Q(z) \).
2. \( [\mathbb{B}, z] = 0, [\mathbb{B}, K(z)] = Q(z), [\mathbb{B}, Q(z)] = 0 \).
3. \( [\mathbb{F}, z] = 0, [\mathbb{F}, K(z)] = 0, [\mathbb{F}, Q(z)] = K(z) \).

The fact that \( \mathbb{B} \) commutes with \( \mathfrak{g}[u] \) and \( [\mathbb{B}, K(z)] = Q(z) \) implies \( [\mathbb{B}, \tilde{E}(n)] = 0 \), for any \( n \in \mathbb{N} \). Thus, \( [\delta_{2m}, \mathbb{B}] = 0 \).

By induction, we have \( [\mathbb{H}, z \otimes v^p] = -nz \otimes v^p \), and \( [\mathbb{H}, z \otimes u^p] = nz \otimes u^p \). Thus, \( [\mathbb{H}, \tilde{E}(n)] = (n-1)\tilde{E}(n) \). A direct calculation shows that \( [\mathbb{H}, \delta_{2m}] = 2m\delta_{2m} \).

By induction, \( (\text{ad} \mathbb{F})^k(z \otimes u^p) = 0 \), for \( k < n \), and \( (\text{ad} \mathbb{F})^k(z \otimes u^p) = n!z \otimes v^p \). Thus, \( (\text{ad} \mathbb{F})^k(\tilde{E}(k)) = 0 \), for \( k \leq n \), which implies \( (\text{ad} \mathbb{F})^{2m+1}(\delta_{2m}) = 0 \). This completes the proof. \( \square \)

9.3. We have a subalgebra \( \mathfrak{d} \) of the deformed double current algebra \( D_3(\mathfrak{g}) \). For any element \( X \in \mathfrak{d} \), taking \( [X, \cdot] \) gives a derivation action of \( \mathfrak{d} \) on \( D_3(\mathfrak{g}) \).

**Theorem 9.3.** The action of \( \mathfrak{d} \) on \( D_3(\mathfrak{g}) \) is extended from the action of \( \mathfrak{d} \) on \( T^0_{\text{ell}} \). In other words, the following diagram commutes.

\[
\begin{array}{ccc}
\mathfrak{d} \times T^0_{\text{ell}} & \xrightarrow{\text{id}} & T^0_{\text{ell}} \\
\downarrow & & \downarrow \\
\mathfrak{d} \times D_3(\mathfrak{g}) & \xrightarrow{\text{id}} & D_3(\mathfrak{g})
\end{array}
\]

**Proof.** It is clear that the action of the \( \mathfrak{sl}_2 \)-triple \( \{\mathbb{B}, \mathbb{F}, \mathbb{H}\} \) makes the diagram commute.

It remains to show that the action of \( \delta_{2m} \in \mathfrak{d} \) makes the diagram commute. That is, we need to show that:

1. \( [\delta_{2m}, S(X^+_a, X^-_a)] = \frac{4}{5} S(X^+_a, X^-_a), (\text{ad} \frac{Q(a^+)}{2})^{2m} S(X^+_a, X^-_a) \).
2. \( [\delta_{2m}, Q(h)] = 0 \), for any \( h \in \mathfrak{h} \).
3. \( [\delta_{2m}, K(h)] = \frac{4}{5} \sum_{\alpha \in \Phi^+} \alpha(h) \sum_{p+q=2m-1} (\text{ad} \frac{Q(a^+)}{2})^p(\kappa_a), (\text{ad} \frac{Q(a^-)}{2})^q(\kappa_a) \), for any \( h \in \mathfrak{h} \).
For any \( X_\alpha \in g_\alpha \), as \( \tilde{\mathbb{B}}(n), X_\alpha \) = 0, we have

\[
[\delta_{2m}, X_\alpha] = \frac{\lambda^2}{8} \left[ \sum_{\substack{p=0 \to 2m \to \alpha \in \Phi}} \left( \frac{2m}{p} \right) \sum_i (h_i \otimes u^p)(h^\dagger \otimes u^{2m-p}) \right],
\]

Thus,

\[
\delta_{2m}S(X_\alpha^+ \otimes X_\alpha^-) = S(\delta_{2m}(X_\alpha^+ \otimes X_\alpha^-)) + S(X_\alpha^+ \otimes \delta_{2m}(X_\alpha^-))
\]

\[
= \frac{\lambda}{2} S \left[ \left( \frac{Q(\alpha^\vee)}{2} \right)^2 \sum_i (h_i \otimes u^p)(h^\dagger \otimes u^{2m-p}) \right],
\]

The assertion (1) follows.

The assertion (2) is clear. Indeed, \( \tilde{\mathbb{B}}(n) \) commutes with \( g[u] \), and \( \{ h' \otimes u^p, Q(h) \} = 0 \), for any \( h, h' \in \mathfrak{b} \), and \( n \in \mathbb{N} \). This implies \( [\delta_{2m}, Q(h)] = 0 \).

The proof of the assertion (3) is in the next subsection.

9.4. In this subsection, we complete the proof of Theorem 9.3 by checking assertion (3).

For notational reason, we write

\[
\delta_{2m}(K(h)) := \frac{\lambda^2}{8} \sum_{\alpha \in \Phi^+} \alpha(h) \sum_{p+q=2m-1} \left[ (\text{ad} \frac{Q(\alpha^\vee)}{2})^p(\kappa_\alpha), \text{ad} - \frac{Q(\alpha^\vee)}{2} \right]^q(\kappa_\alpha).
\]

The assertion (3) can be rewritten as

\[
[\delta_{2m}, K(h)] = \delta_{2m}(K(h)).
\]

We now compute \( \delta_{2m}(K(h)) \).

**Lemma 9.4.** For any \( h \in \mathfrak{b} \), we have

\[
\delta_{2m}(K(h)) = -\frac{\lambda^2}{4} \sum_{x+t+k=2m-1} (-1)^x \left( \sum_{\beta \in \Phi^+} (\beta, h)S \left( H_\beta \otimes u^k, S(X_\beta^- \otimes u^t, X_\beta^+ \otimes u^k) \right) \right).
\]
Proof. We have

\[
\frac{x^2}{8} \sum_{\beta \in \Phi^+} \beta(h) \sum_{p+q=2m-1} (-1)^q \left[ \frac{O(\beta^s)}{2} p(S(X_\beta^+, X_\beta^-), \frac{O(\beta^s)}{2} p(S(X_\beta^+, X_\beta^-)) \right]
\]

\[
= \frac{x^2}{8} \sum_{\beta \in \Phi^+} \beta(h) \sum_{p+q=2m-1} (-1)^q \left[ \sum_{s+t=p} \sum_{k+j=q} (-1)^j \left( \begin{pmatrix} p \cr s \end{pmatrix} \right) S(X_\beta^+ \otimes u^t, X_\beta^- \otimes u^s) \right]
\]

\[
= \frac{x^2}{8} \sum_{\beta \in \Phi^+} \beta(h) \sum_{p+q=2m-1} (-1)^q \sum_{s+t=p} \sum_{k+j=q} (-1)^{s+j} \left( \begin{pmatrix} p \cr s \end{pmatrix} \right) \left( \begin{pmatrix} q \cr k \end{pmatrix} \right) S(X_\beta^+ \otimes u^t, X_\beta^- \otimes u^s)
\]

\[
= \frac{x^2}{8} \sum_{\beta \in \Phi^+} \beta(h) \sum_{p+q=2m-1} (-1)^q \sum_{s+t=p} \sum_{k+j=q} (-1)^{s+j} \left( \begin{pmatrix} p \cr s \end{pmatrix} \right) \left( \begin{pmatrix} q \cr k \end{pmatrix} \right)
\]

\[
(\langle S(X_\beta^+ \otimes u^k, H_\beta \otimes u^{s+j}), X_\beta^- \otimes u^s \rangle - S(X_\beta^+ \otimes u^s, S(H_\beta \otimes u^{s+k}, X_\beta^- \otimes u^j)))
\]

We compute the summand in (53) as follows. For a fixed positive root \( \beta \in \Phi^+ \), we have

\[
S(X_\beta^+ \otimes u^k, H_\beta \otimes u^{s+j}, X_\beta^- \otimes u^s) - S(X_\beta^+ \otimes u^s, S(H_\beta \otimes u^{s+k}, X_\beta^- \otimes u^j))
\]

\[
= X_\beta^+ \otimes u^k (2(X_\beta^+ \otimes u^k)(H_\beta \otimes u^{s+j}) + (\beta, \beta) X_\beta^+ \otimes u^{k+s+j})
\]

\[
+ (2(H_\beta \otimes u^{s+j})(X_\beta^- \otimes u^k) - (\beta, \beta) X_\beta^- \otimes u^{k+s+j}) X_\beta^- \otimes u^s
\]

\[
- X_\beta^+ \otimes u^s (2(X_\beta^- \otimes u^s)H_\beta \otimes u^{s+k} - (\beta, \beta) X_\beta^- \otimes u^{s+k})
\]

\[
- (2(H_\beta \otimes u^{s+k})(X_\beta^- \otimes u^s) + (\beta, \beta) X_\beta^- \otimes u^{s+k}) X_\beta^- \otimes u^s
\]

\[
= 2(X_\beta^+ \otimes u^s)(X_\beta^- \otimes u^k) H_\beta \otimes u^{s+j} + 2H_\beta \otimes u^{s+j}(X_\beta^- \otimes u^k) (X_\beta^- \otimes u^s)
\]

\[
- 2(X_\beta^+ \otimes u^s)(X_\beta^- \otimes u^s) H_\beta \otimes u^{s+k} - 2H_\beta \otimes u^{s+k}(X_\beta^- \otimes u^s) (X_\beta^- \otimes u^s).
\]

For any \( m \in \mathbb{N} \), we have the equality

\[
\sum_{p+q=2m-1} (-1)^q \sum_{s+t=p} \sum_{k+j=q} (-1)^{s+j} \left( \begin{pmatrix} p \cr s \end{pmatrix} \right) \left( \begin{pmatrix} q \cr k \end{pmatrix} \right) = \sum_{s+t+k+j=2m-1} (-1)^{s+j} \left( \begin{pmatrix} s + t \cr s \end{pmatrix} \right) \left( \begin{pmatrix} k + j \cr k \end{pmatrix} \right)
\]

If we switch the pair \((s, j)\) with the pair \((t, k)\), then the coefficient \((-1)^{k+l} \left( \begin{pmatrix} s+t \cr s \end{pmatrix} \right) \left( \begin{pmatrix} k+j \cr k \end{pmatrix} \right)\) in (53) is changed by a negative sign. Thus, we have two summands of the following form in (53):

\[
2(X_\beta^- \otimes u^s)(X_\beta^- \otimes u^k) H_\beta \otimes u^{s+j} + 2H_\beta \otimes u^{s+j}(X_\beta^- \otimes u^k) (X_\beta^- \otimes u^s)
\]

\[
- 2(X_\beta^- \otimes u^s)(X_\beta^- \otimes u^s) H_\beta \otimes u^{s+k} - 2H_\beta \otimes u^{s+k}(X_\beta^- \otimes u^s) (X_\beta^- \otimes u^s),
\]

and

\[
- 2(X_\beta^- \otimes u^s)(X_\beta^- \otimes u^k) H_\beta \otimes u^{k+s} - 2H_\beta \otimes u^{k+s}(X_\beta^- \otimes u^s) (X_\beta^- \otimes u^s)
\]

\[
+ 2(X_\beta^- \otimes u^s)(X_\beta^- \otimes u^k) H_\beta \otimes u^{s+j} + 2H_\beta \otimes u^{s+j}(X_\beta^- \otimes u^k) (X_\beta^- \otimes u^s).
\]
Combining the above two summands (54) (55) with \((s, t, k, j)\) and \((t, s, j, k)\) (note the coefficient is \((-1)^{k+j}(\binom{s+t}{s}\binom{k+j}{k})\)), we get

\[
\begin{align*}
&\left(2(X_β^+ \otimes u')(X_β^- \otimes u^k) + 2(X_β^+ \otimes u')(X_β^- \otimes u^k)\right)H_β \otimes u^{s+j} \\
&+ H_β \otimes u^{s+j}\left(2(X_β^+ \otimes u')(X_β^- \otimes u') + 2(X_β^+ \otimes u')(X_β^- \otimes u')\right) \\
&- \left(2(X_β^+ \otimes u')(X_β^- \otimes u') + 2(X_β^+ \otimes u')(X_β^- \otimes u')\right)H_β \otimes u^{s+k} \\
&- H_β \otimes u^{s+k}\left(2(X_β^+ \otimes u')(X_β^- \otimes u') + 2(X_β^+ \otimes u')(X_β^- \otimes u')\right) \\
= &(S(X_β^- \otimes u', X_β^+ \otimes u^k) + S(X_β^+ \otimes u', X_β^- \otimes u^k))H_β \otimes u^{s+j} \\
&+ H_β \otimes u^{s+j}\left(S(X_β^+ \otimes u^1, X_β^- \otimes u') + S(X_β^- \otimes u^1, X_β^+ \otimes u')\right) \\
&- \left(S(X_β^+ \otimes u^1, X_β^- \otimes u') + S(X_β^- \otimes u^1, X_β^+ \otimes u')\right)H_β \otimes u^{s+k} \\
&- H_β \otimes u^{s+k}\left(S(X_β^+ \otimes u^1, X_β^- \otimes u') + S(X_β^- \otimes u^1, X_β^+ \otimes u')\right)
\end{align*}
\]

Thus, for a fixed root \(β \in Φ^+\), the summation (53) becomes:

\[
\sum_{s+t+k+j=2m-1} (-1)^{k+j}\left(\binom{s+t}{s}\binom{k+j}{k}\right)(S(X_β^- \otimes u', X_β^+ \otimes u^k), H_β \otimes u^{s+j}) + S(S(X_β^+ \otimes u', X_β^- \otimes u^k), H_β \otimes u^{s+j})
\]

For any integers \(j, k, \) and \(n\) satisfying \(0 \leq j \leq k \leq n\), we have an identity of the binomial coefficients:

\[
\sum_{m=0}^{n}\binom{m}{j}\binom{n-m}{k-j} = \binom{n+1}{k+1}
\]

Now fix the indices \(k, t, \) and fix the sum \(s + j = x\), and we vary the index \(p = 0, \ldots, 2m-1\), thus, the indices \(s, j, q\) are varying according to \(p\). The coefficient of the term \(S(S(X_β^- \otimes u', X_β^+ \otimes u^k), H_β \otimes u^{s+j})\) becomes

\[
(-1)^{k+j}(\binom{s+t}{s}\binom{k+j}{k}) + (-1)^{k+j}(\binom{s+t}{s}\binom{j}{t}) = 2(-1)^{k+j}\sum_{p=0}^{2m-1}(\binom{m}{p})(\binom{2m-1-p}{2m-1-x-t}) = -2(-1)^{k}(\binom{2m}{x}).
\]

We simplify (53) using the above observation, and we get

\[
\tilde{d}_{2m}(K(h)) = -\frac{\lambda^2}{4} \sum_{k=0}^{2m-1} (-1)^k(\binom{2m}{x}) \sum_{β \in Φ^+}(β, h)S(H_β \otimes u^t, \sum_{t+k=2m-1-x} S(X_β^- \otimes u', X_β^+ \otimes u^k)).
\]

This completes the proof.

\[\square\]

**Lemma 9.5.** We have

\[
\tilde{d}_{2m}(K(h)) = \frac{\lambda^2}{2} \sum_{p+q=2m} (-1)^q(\binom{2m}{p})(h \otimes u^{p})(h' \otimes u^{q}), K(h) + \lambda \sum_{p+q=2m} (-1)^q(\binom{2m}{p})(h \otimes u^{p})\tilde{e}(q).
\]
Proof. We compute the commutator $\frac{\lambda}{2} \left[ \frac{\lambda}{2} \sum_{p+q=2m} (-1)^{q} \left( \frac{2m}{p} \right) \left( \lambda_{v} \otimes u^{p} \right) \left( \lambda_{u} \otimes u^{q} \right) \right]$ and compare it with $\delta_{2m}(K(h))$ in Lemma 9.4. We have

$$\frac{\lambda}{2} \left[ \frac{\lambda}{2} \sum_{p+q=2m} (-1)^{q} \left( \frac{2m}{p} \right) \left( \lambda_{v} \otimes u^{p} \right) \left( \lambda_{u} \otimes u^{q} \right) \right]$$

$$= \frac{\lambda}{2} \sum_{p+q=2m} (-1)^{q} \left( \frac{2m}{p} \right) \left( \lambda_{v} \otimes u^{p} \right) \left( \lambda_{u} \otimes u^{q} \right) + \left[ \frac{\lambda}{2} \sum_{p+q=2m} (-1)^{q} \left( \frac{2m}{p} \right) \left( \lambda_{v} \otimes u^{p} \right) \left( \lambda_{u} \otimes u^{q} \right) \right]$$

$$= \frac{\lambda}{2} \sum_{p+q=2m} (-1)^{q} \left( \frac{2m}{p} \right) \left( \lambda_{v} \otimes u^{p} \right) \left( \lambda_{u} \otimes u^{q} \right) + \frac{\lambda^{2}}{8} \sum_{p+q=2m} (-1)^{q} \left( \frac{2m}{p} \right) \sum_{\alpha \in \Phi} (\alpha, h) \left( \sum_{s+i=q-1} S(\lambda_{u} \otimes u^{s}, \lambda_{u} \otimes u^{i}) \right)$$

$$= \frac{\lambda}{2} \sum_{p+q=2m} (-1)^{q} \left( \frac{2m}{p} \right) \left( \lambda_{v} \otimes u^{p} \right) \left( \lambda_{u} \otimes u^{q} \right)$$

$$+ \frac{\lambda^{2}}{4} \sum_{\alpha \in \Phi} (\alpha, h) \sum_{p+s+i=2m-1} (-1)^{q} \left( \frac{2m}{p} \right) S(\lambda_{u} \otimes u^{p}, S(\lambda_{u} \otimes u^{s}, \lambda_{u} \otimes u^{i}))$$

$$= \frac{\lambda}{2} \sum_{p+q=2m} (-1)^{q} \left( \frac{2m}{p} \right) \left( \lambda_{v} \otimes u^{p} \right) \left( \lambda_{u} \otimes u^{q} \right) + \frac{\lambda^{2}}{4} \sum_{\alpha \in \Phi} (\alpha, h) \sum_{p+s+i=2m-1} (-1)^{q} \left( \frac{2m}{p} \right) S(\lambda_{u} \otimes u^{p}, S(\lambda_{u} \otimes u^{s}, \lambda_{u} \otimes u^{i}))$$

where the last equality follows from the definition of $\overline{\mathbb{E}}(n)$ in (50). Using the simple fact that $h = \sum_{i}(h_{i}, h_{i})$, we compute (56) as follows.

$$\frac{\lambda}{2} \sum_{p+q=2m} (-1)^{q} \left( \frac{2m}{p} \right) \left( \lambda_{v} \otimes u^{p} \right) \left( \lambda_{u} \otimes u^{q} \right)$$

$$+ \frac{\lambda^{2}}{4} \sum_{\alpha \in \Phi} (\alpha, h) \sum_{p+s+i=2m-1} (-1)^{q} \left( \frac{2m}{p} \right) S(\lambda_{u} \otimes u^{p}, S(\lambda_{u} \otimes u^{s}, \lambda_{u} \otimes u^{i}))$$

This completes the proof by rearranging the above identity.

\[\square\]

Lemma 9.6. For any $z \in \mathfrak{g}$, we have

$$\lambda \sum_{p+q=2m} (-1)^{q} \left( \frac{2m}{p} \right) \left( \lambda_{z} \otimes u^{p} \right) \overline{\mathbb{E}}(q) = \frac{3 \lambda}{C} \sum_{p=0}^{m-2} (-1)^{p} \frac{2(2m)!}{(q+p+2)(2m-p)!} \overline{\mathbb{E}}(p+2) \overline{\mathbb{E}}(2m-p)$$

$$+ (-1)^{m-1} \frac{2(2m)!}{(m+1)!} \left( \frac{m+1}{2m+1} \right) \left( \frac{m+1}{2m+1} \right) \overline{\mathbb{E}}(m+1) \overline{\mathbb{E}}(1), \quad K(z) \right).$$

Proof. The statement follows from the following calculations. When $q = 0$, we have $\overline{\mathbb{E}}(0) = 0$. So we could assume $q \neq 0$. We separate the cases when $(p, q) = (m-1, m+1)$, and $(p, q) = (2m-1, 1)$ from the set
\{(p, q) \mid p + q = 2m\}. We have
\[
\begin{align*}
&\sum_{p+q=2m} (-1)^p \binom{2m}{p} (z \otimes u^p) \tilde{\mathcal{E}}(q) \\
&= \sum_{p=0}^{m-2} (-1)^p \binom{2m}{p} (z \otimes u^p) \tilde{\mathcal{E}}(2m - p) + \sum_{q=2}^{m} (-1)^q \binom{2m}{q} (z \otimes u^{2m-q}) \tilde{\mathcal{E}}(q) \\
&\quad + (-1)^m \left( 2m \atop m+1 \right) (z \otimes u^{m-1}) \tilde{\mathcal{E}}(m + 1) - 2m(z \otimes u^{2m-1}) \tilde{\mathcal{E}}(1) \\
&= \sum_{p=0}^{m-2} (-1)^p \binom{2m}{p} (z \otimes u^p) \tilde{\mathcal{E}}(2m - p) + (-1)^q \binom{2m}{p+2} (z \otimes u^{2m-p-2}) \tilde{\mathcal{E}}(p + 2) \\
&\quad + (-1)^m \left( 2m \atop m-1 \right) (z \otimes u^{m-1}) \tilde{\mathcal{E}}(m + 1) - 2m(z \otimes u^{2m-1}) \tilde{\mathcal{E}}(1) \\
&= \sum_{p=0}^{m-2} (-1)^p \binom{2m+2}{p+2} (z \otimes u^p) \tilde{\mathcal{E}}(2m - p) + (-1)^q \binom{2m}{p+2} (z \otimes u^{2m-p-2}) \tilde{\mathcal{E}}(p + 2) \\
&\quad + (-1)^m \left( 2m \atop m-1 \right) \tilde{\mathcal{E}}(m + 1) - 2m(z \otimes u^{2m-1}) \tilde{\mathcal{E}}(1) \\
&= \sum_{p=0}^{m-2} \frac{(-1)^p \binom{2m+2}{p+2} (z \otimes u^p) \tilde{\mathcal{E}}(2m - p)}{2m + 1} - \frac{(-1)^m \binom{2m}{m-1} \tilde{\mathcal{E}}(m + 1)}{(m+1)m} - \frac{6}{C(2m+1)} \tilde{\mathcal{E}}(1).
\end{align*}
\]

By Proposition 9.1, \(\tilde{\mathcal{E}}(1)\) is central, and \([	ilde{\mathcal{E}}(n), z \otimes v] = \frac{C}{3} \binom{n}{2} (z \otimes u^{n-2}), \) for \(n \geq 2.\) Furthermore, \(\tilde{\mathcal{E}}(n)\) commutes with the subalgebra \(\mathfrak{g}[u].\) Therefore, we have
\[
\begin{align*}
(57) &= \frac{3}{C} \sum_{p=0}^{m-2} (-1)^p \frac{\binom{2m+2}{p+2}}{(2m + 1)(m+1)} \tilde{\mathcal{E}}(2m - p) + \frac{3}{C} (-1)^{m-1} \frac{\binom{2m}{m-1}}{(m+1)m} \tilde{\mathcal{E}}(m + 1) - \frac{6}{C(2m+1)} \tilde{\mathcal{E}}(1),
\end{align*}
\]
where the constant \(C\) is given by (41). This completes the proof. \(\square\)

It is clear that the assertion (3) follows from Lemma 9.5 and Lemma 9.6, together with the formula (52) of \(\delta_{2m}.\) This completes the proof of Theorem 9.3.

**APPENDIX A.**

In the appendix, we show the constant \(C = \frac{k^2}{4} C\) is given by the formula (41). We furthermore compute (41) explicitly. Recall the following formula of \(\tilde{C}\) in (49):
\[
\tilde{C} = \frac{\sum_{\alpha, \beta \in \Phi} \{ (\alpha, \beta)^2 - (\alpha, \alpha)(\beta, \beta) \} \{(X_\beta, X_{-\alpha}) \} \{ [X_\beta, X_{-\beta}] \}}{\dim \mathfrak{h}(\dim \mathfrak{h} - 1)}.
\]

Let \(\mathfrak{g}\) be a finite dimensional simple Lie algebra. There is a Chevalley basis of \(\mathfrak{g}\) (see [24, \S 25]), which we now recall here. Fix a pair of roots \(\alpha, \beta,\) consider the \(\alpha\)-string through \(\beta:\)
\[
\beta - r\alpha, \ldots, \beta, \ldots, \beta + q\alpha.
\]

**Proposition A.1.** [24, Proposition 25.1] We have

1. \((\beta, \alpha) := \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = r - q.
2. At most two root lengths occur in this string.
3. If \(\alpha + \beta \in \Phi,\) then \(r + 1 = \frac{\alpha + \beta}{\langle \beta, \beta \rangle} \cdot \langle \alpha, \alpha \rangle.\)

As a consequence, for any two roots \(\alpha, \beta,\) if \(\alpha + \beta \in \Phi,\) then we have
\[
q = \frac{(\beta, \beta)}{(\alpha, \alpha)}, \quad r = \frac{(\beta, \beta) + 2(\beta, \alpha)}{(\alpha, \alpha)}.
\]

**Proposition A.2.** [24, Proposition 25.2] It is possible to choose root vectors \(x_\alpha, \alpha \in \Phi\) satisfying
(1) \([x_\alpha, x_{-\alpha}] = h_\alpha\).

(2) If \(\alpha, \beta, \alpha + \beta \in \Phi\), \([x_\alpha, x_\beta] = c_{\alpha\beta} x_{\alpha + \beta}\), then \(c_{\alpha\beta} = -c_{-\alpha, -\beta}\). For any such choice of root vectors, the scalar \(c_{\alpha, \beta}(\alpha, \beta, \alpha + \beta \in \Phi)\) automatically satisfy: \(c_{\alpha, \beta}^2 = q(r + 1)\frac{(\alpha + \beta, \alpha + \beta)}{(\beta, \beta)}, \) where \(\beta - r\alpha, \cdots, \beta, \cdots, \beta + qr \) is the \(\alpha\)-string through \(\beta\).

Therefore, for any two roots \(\alpha, \beta\), if \(\alpha + \beta \in \Phi\), then we have
\[
c_{\alpha, \beta}^2 = q(r + 1)\frac{(\alpha + \beta, \alpha + \beta)}{(\beta, \beta)} = \frac{(\beta, \beta)^2 (\alpha + \beta, \alpha + \beta)^2}{(\alpha, \alpha)^2 (\beta, \beta)^2} = \frac{(\alpha + \beta, \alpha + \beta)^2}{(\alpha, \alpha)^2}.
\]

The killing form of \(\{x_\alpha \mid \alpha \in \Phi\}\) is given by \([24, \text{Page 147}]:
\[
(x_\alpha | x_{-\alpha}) = \frac{2}{(\alpha, \alpha)}.
\]

Choose \(X_\alpha := \sqrt{\frac{(\alpha, \alpha)}{2}} x_\alpha\), for all \(\alpha \in \Phi\). Thus, we have \((X_\alpha | X_{-\alpha}) = 1\). We compute
\[
\sum_{\alpha, \beta \in \Phi} ((\alpha, \beta)^2 - (\alpha, \alpha)(\beta, \beta))(\{X_\beta, X_{-\alpha}\}{X_{-\beta}, X_\alpha})
\]
\[=
\sum_{\alpha, \beta \in \Phi} ((\alpha, \beta)^2 - (\alpha, \alpha)(\beta, \beta))\frac{(\alpha, \alpha)(\beta, \beta)}{2}(x_\alpha, x_\beta)(x_{-\alpha}, x_{-\beta})
\]
\[=
-\sum_{\alpha, \beta \in \Phi} ((\alpha, \beta)^2 - (\alpha, \alpha)(\beta, \beta))\frac{(\alpha, \alpha)(\beta, \beta)}{2} c_{\alpha\beta}^2 \frac{2}{(\alpha + \beta, \alpha + \beta)}
\]
\[=
\sum_{\alpha, \beta \in \Phi} ((\alpha, \alpha)(\beta, \beta) - (\alpha, \beta)^2)\frac{(\beta, \beta)(\alpha + \beta, \alpha + \beta)}{(\alpha, \alpha)}
\]
\[=
\frac{1}{4} \sum_{\alpha, \beta \in \Phi} ((\alpha, \alpha)(\beta, \beta) - (\alpha, \beta)^2)\frac{(\beta, \beta)(\alpha + \beta, \alpha + \beta)}{(\alpha, \alpha)}
\]
\[=
\frac{1}{4} \sum_{\alpha, \beta \in \Phi} (1 - \frac{(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)})(\beta, \beta)^2 + (\alpha, \alpha)^2)(\alpha + \beta, \alpha + \beta)
\]

The above computation gives the following formula of \(\tilde{C}\):
\[
\tilde{C} = \frac{\sum_{\{\alpha, \beta \in \Phi \mid \alpha + \beta \in \Phi\}} (1 - \frac{(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)})(\beta, \beta)^2 + (\alpha, \alpha)^2)(\alpha + \beta, \alpha + \beta)}{4 \dim \mathfrak{h}(\dim \mathfrak{h} - 1)}.
\]

Therefore, \(C = \frac{\tilde{C}}{4}\) is given by the formula (41).

A.1. The simply laced case. When \(\mathfrak{g}\) is simply-laced, we have \(\alpha + \beta \in \Phi\), if and only if \((\alpha, \beta) = -1\). Thus,
\[
\tilde{C} = \frac{3||\{\alpha, \beta \in \Phi \mid \alpha + \beta \in \Phi\}||}{\dim \mathfrak{h}(\dim \mathfrak{h} - 1)} \quad \text{where} \quad \mathfrak{g} \text{ is of type ADE}.
\]

For example, in type \(A_{n-1}\), let \(\mathfrak{g} = \mathfrak{sl}_n\), we have \(\tilde{C} = \frac{3(n-1)2(n-2)}{(n-1)(n-2)} = 6n\).

A.2. Non-simply laced case. Denote by \(\Phi_l\) the set of the long roots in \(\Phi\), and \(\Phi_s\) the set of the short roots. We have the decomposition \(\Phi = \Phi_l \cup \Phi_s\).
A.2.1. Type $B_n$. When $g$ is of type $B_n$, we have

- For any roots $\alpha, \beta \in \Phi_I$ such that $\alpha + \beta$ is a root, then $\alpha + \beta \in \Phi_I$, and $(\alpha, \beta) = -1$.
- For any roots $\alpha, \beta \in \Phi_I$ such that $\alpha + \beta$ is a root, then $\alpha + \beta \in \Phi_I$, and $(\alpha, \beta) = 0$.
- For any roots $\alpha \in \Phi_I, \beta \in \Phi_I$ such that $\alpha + \beta$ is a root, then $\alpha + \beta \in \Phi_I$, and $(\alpha, \beta) = -1$.

Using the above observations, we can simplify (58). We have

$$\sum_{\alpha \in \Phi_I, \beta \in \Phi_I} \left( 1 - \frac{(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} \right) ((\beta, \beta)^2 + (\alpha, \alpha)^2) (\alpha + \beta, \alpha + \beta)$$

Therefore,

$$\overline{C} = \frac{12\#_{\{\alpha \in \Phi_I \mid \alpha + \beta \in \Phi_I\}} + 4\#_{\{\alpha \in \Phi_I \mid \alpha + \beta \in \Phi_I\} + \frac{5}{2}\#_{\{\alpha \in \Phi_I \mid \alpha + \beta \in \Phi_I}\}}}{4n(n - 1)}.$$
Therefore,

\[
\tilde{C} = \frac{12 \#_{(\alpha, \beta \in \Phi_l | \alpha + \beta \in \Phi)} + \frac{5}{2} \#_{(\alpha \in \Phi_l \beta, \beta \in \Phi | \alpha + \beta \in \Phi)} + \frac{3}{2} \#_{(\alpha, \beta \in \Phi_l | \alpha + \beta \in \Phi)} + 4 \#_{(\alpha, \beta \in \Phi_l | \alpha + \beta \in \Phi)}}{48}.
\]

A.3. Type $G_2$. Consider the root system of $G_2$. It is given as in the following picture. (see for example [24, Section 12.1]), where $\{e_1, e_2, e_3\}$ is the standard orthonormal basis of $\mathbb{R}^3$.

\[
\frac{1}{\sqrt{3}}(e_1 + e_2 - 2e_3) \quad \frac{1}{\sqrt{3}}(e_2 - e_3) \quad \frac{1}{\sqrt{3}}(e_1 - e_3) \quad \frac{1}{\sqrt{3}}(2e_1 - e_2 - e_3)
\]

\[
\frac{1}{\sqrt{3}}(e_1 - e_2) \quad \frac{1}{\sqrt{3}}(e_1 - 2e_2 + e_3) \quad \frac{1}{\sqrt{3}}(e_1 - 2e_2 + e_3)
\]

- If $\alpha, \beta \in \Phi_l$, and $\alpha + \beta \in \Phi_l$, then $(\alpha, \beta) = -1$.
- If $\alpha \in \Phi_l, \beta \in \Phi_s$, and $\alpha + \beta \in \Phi_s$, then $(\alpha, \beta) = -1$.
- If both $\alpha, \beta$ are short roots, and $\alpha + \beta \in \Phi_l$, then we have $(\alpha, \beta) = \frac{1}{3}$.
- If both $\alpha, \beta$ are short roots, and $\alpha + \beta \in \Phi_s$, then we have $(\alpha, \beta) = \frac{1}{3}$.

Using the above observations, we can simplify the following.

\[
\sum_{(\alpha, \beta \in \Phi_l | \alpha + \beta \in \Phi)} \left(1 - \frac{(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)}\right)(\beta, \beta)^2 + (\alpha, \alpha)^2)\alpha + \beta, \alpha + \beta) = \sum_{(\alpha, \beta \in \Phi_l | \alpha + \beta \in \Phi)} \frac{12}{\sum_{(\alpha, \beta \in \Phi_l | \alpha + \beta \in \Phi)} \frac{5}{2} \sum_{(\alpha, \beta \in \Phi_l | \alpha + \beta \in \Phi)} \frac{3}{2} \sum_{(\alpha, \beta \in \Phi_l | \alpha + \beta \in \Phi)} \frac{16}{9}.
\]

Therefore,

\[
\tilde{C} = \frac{12 \#_{(\alpha, \beta \in \Phi_l | \alpha + \beta \in \Phi)} + \frac{5}{2} \#_{(\alpha \in \Phi_l \beta, \beta \in \Phi | \alpha + \beta \in \Phi)} + \frac{3}{2} \#_{(\alpha, \beta \in \Phi_l | \alpha + \beta \in \Phi)} + 4 \#_{(\alpha, \beta \in \Phi_l | \alpha + \beta \in \Phi)}}{8}.
\]

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