Digital methods of discrete fourier transform, allowing minimizing the number of algorithmic multiplication operations

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Abstract. The issues related to the digitalization of the spectral analysis of the polyharmonic signal are considered. There are various digital methods of discrete Fourier transform (DFT) of such a signal. The purpose of the study is a comparative analysis of the known methods for transforming a complex signal, which make it possible to minimize the required number of multiplication operations in its transformation algorithms. The research used methods of mathematical and software modeling of digital signal processing algorithms. The methods of discrete Fourier transform based on multiband digital filtering (MDF) with one low-frequency digital filter and with an extended window based on the so-called bank of non-recursive digital filters (BDF) are described. There are the results of a comparative analysis of these methods.

1. Introduction
The gradual development and improvement of the element base of microelectronics and its promising development of nanoelectronics determine the constant demand for computational algorithms for digital signal processing (DSP) [1], [2]. The use of programmable logic device (PLD), due to the peculiarities of their architecture, requires less hardware for the software and hardware implementation of digital algorithms for DFT based on MDF, adaptive to changing the frequency resolution of the Fourier coefficients than the use of digital signal processors. However, until now in PLD, in digital signal processors and in promising nanoprocessors, the equality of the speeds of performing arithmetic operations of multiplication and addition is achieved mainly at the expense of hardware costs. Therefore, further development of the idea of reducing the number of arithmetic multiplication operations in digital DFT algorithms has not yet lost its significance in the hardware and software implementation of DFT methods and is focused on a comprehensive account of reserves for reducing hardware costs required to achieve equality of the speed of performing arithmetic multiplication and addition operations by elementary devices of modern PLD and promising nanoprocessors [3], [4].

Currently, digital DFT methods with a minimum number of arithmetic multiplication operations are widely used: classical DFT methods and their modifications and DFT methods based on MDF [5], [6]. MDF-based DFT methods are digital DFT methods by filtering all bands with one low-pass digital filter (LDF) or by adaptive digital filtering and extended window DFT methods based on BDF.

2. Purpose and research methods
The purpose of the study is a comparative analysis of the known methods for transforming a complex signal, which make it possible to minimize the required number of multiplication operations in its
transformation algorithms. The research used methods of mathematical and software modeling of DSP algorithms.

3. Theoretical basis
The theoretical basis of the study can and should be considered the method of calculating the coefficients of DFT \( \{X(m, \Omega)\}, m=0,1,2...M-1, \Omega=\omega_D/(2M) \) and complex signal \( \{x(n,T)\}, n=0,1,2...N-1, T=2\pi/\omega_D \), according to formula (1) if \( M=N \) [6]:

\[
X(m \cdot \Omega) = \sum_{n=0}^{N-1} x(n \cdot T) \cdot \exp[-i \cdot m \cdot \Omega \cdot n \cdot T], n = 0,1,2...N-1, m = 0,1,2...M-1, M = N
\]

where \( X(m, \Omega) \) is the \( m \)-th instantaneous spectrum component, \( m=0,1,2...M-1, \Omega=\omega_D/(2M), \) of complex signal \( \{x(n,T)\}, n=0,1,2...N-1, T=2\pi/\omega_D; \)

\( M \) is the number of instantaneous spectrum components \( \{X(m, \Omega)\}, m=0,1,2...M-1, \Omega=\omega_D/(2M), \) of complex signal \( \{x(n,T)\}, n=0,1,2...N-1, T=2\pi/\omega_D; \)

\( N \) is the time sample size \( \{x(n,T)\}, n=0,1,2...N-1, T=2\pi/\omega_D; \)

\( T \) is the complex signal sampling period \( \{x(n,T)\}, n=0,1,2...N-1, T=2\pi/\omega_D; \)

\( \Omega \) is the frequency resolution of the instantaneous spectrum \( \{X(m, \Omega)\}, m=0,1,2...M-1, \Omega=\omega_D/(2M); \)

\( \omega_D \) is the complex sampling rate \( \{x(n,T)\}, n=0,1,2...N-1, T=2\pi/\omega_D. \)

About multiplier placement \( 1/N \) before the summation operator, there is no consensus in scientific publications devoted to DSP, and it often appears in the inverse formula DFT [5].

We should note that the adaptability of DFT with a minimum number of arithmetic multiplication operations by known methods of \( N \)-point FFT and \( N \)-point DFT by Vinograd's algorithm with Mersenne number-theoretic transformations (NTT) to resizing \( N \) time sampling complex signal \( \{x(n,T)\}, n=0,1,2...N-1, T=2\pi/\omega_D, \) following the value from \( N \) to \( N^* \) is achieved by increasing the number of logical PLD cells with \( \text{SET}_{\text{FFT}}(N,Z) \) to \( \text{SET}_{\text{FFT}}(N^*,Z) \) for a special processor implementation of the FFT method and with \( \text{SET}_{\text{NTT}}(N,Z) \) to \( \text{SET}_{\text{NTT}}(N^*,Z) \) for a similar implementation of the DFT method according to the Vinograd's algorithm with the Mersenne's NTT when the level of the computational error of the FFT changes with \( C_{\text{FFT}}(N,D,Z) \) to \( C_{\text{FFT}}(N^*,D,Z) \), and DFT by Vinograd's algorithm with Mersenne NTT with \( C_{\text{NTT}}(N,D,Z) \) to \( C_{\text{NTT}}(N^*,D,Z) \). Thus, the hardware costs for ensuring the adaptability of the FFT and DFT methods according to the Vinograd's algorithm with the Mersenne TTP, requiring the minimum number of arithmetic multiplication operations for the DFT of a complex signal \( \{x(n,T)\}, n=0,1,2...N-1, T=2\pi/\omega_D, \) with time-varying frequency resolution \( \Omega \) of instant spectrum \( \{X(m, \Omega)\}, m=0,1,2...N-1, \Omega=\omega_D/(2N), \) of this signal in magnitude from \( \Omega \) to \( \Omega^*=\omega_D/(2N^*) \) constitute, respectively, \( \text{SET}_{\text{FFT}}(N,Z)+\text{SET}_{\text{FFT}}(N^*,Z) \) and \( \text{SET}_{\text{NTT}}(N,Z)+\text{SET}_{\text{NTT}}(N^*,Z) \).

Moreover, to calculate only the necessary DFT coefficients using known methods of point FFT and \( N \)-point DFT by Vinograd's algorithm with Mersenne NTT, requiring the execution of the minimum number of arithmetic multiplication operations, it is necessary to calculate all the coefficients DFT \( \{X(m, \Omega)\}, m=0,1,2...N-1, \Omega=\omega_D/(2N), \) of complex signal \( \{x(n,T)\}, n=0,1,2...N-1, T=2\pi/\omega_D. \)

4. Research results
The results of the study confirmed that digital DFT methods, allowing minimizing the number of algorithmic multiplication operations, can and should be considered DFT methods based on MDF.

4.1. DFT Method Based on MDF by one DLF
Methods of frequency selection of signals based on MDF allow, in some cases, to significantly reduce the hardware costs for a special processor implementation of DFT algorithms, since the hardware implementation of the filtering system can be simplified by using a single digital filter in it. The idea of using one arithmetic device for various digital filtering operations is related to the possibility of extending a low-order digital filter to a high-order digital filter [7]. \( M \)-band MDF method corresponds
to the use of a single $K$-th order digital low pass filter (DLF) for the complex signal MDF $\{x(n-T)\}$, $n=0,1,2…N-1$, $T=2\pi/\omega_{OD}$, and the shift in the frequency domain of the spectrum $\{X(m\Omega)\}$, $m=0,1,2…M-1$, $\Omega=\omega_{D}(2M)$, of this signal by multiplying the input MDF by an exponential factor $exp[-i\cdot n\cdot \omega_{M}]$, $n=0,1,2…N-1$, $\omega_{M}=m(\omega_{MAX}/M)$, $m=0,1,2…M-1$, $M=N$, $T=2\pi/\omega_{OD}$.

$M$-band MDF of complex signal $\{x(n-T)\}$, $n=0,1,2…N-1$, $T=2\pi/\omega_{OD}$ with maximum angular frequency $\omega_{MAX}$ is performed by a single DLF of the $K$-th order with an angular cutoff frequency $\omega_{C}=\omega_{MAX}/M$ by shifts of angular frequency $\omega_{m}$. Application to spectrum $\{X(n\Omega)\}$, $n=0,1,2…N-1$, $\Omega=\omega_{D}(2N)$ of complex signal $\{x(n-T)\}$, $n=0,1,2…N-1$, $T=2\pi/\omega_{OD}$, shift of the angular frequency by the amount $\omega_{m}$ allows getting a spectrum $\{X(n\Omega-m\Omega)\}$, $n=0,1,2…N-1$, $m=0,1,2…N-1$, $\Omega=\omega_{D}(2N)$, which corresponds to a complex signal of the form $x(n-T)=x(n-T)\cdot exp[-i\cdot n\cdot \omega_{m\Omega}]$, $n=0,1,2…N-1$, $\omega_{M}=m(\omega_{MAX}/M)$, $m=0,1,2…M-1$, $M=N$, $T=2\pi/\omega_{OD}$. Moreover, all frequency bands of the original spectrum $\{X(n\Omega)\}$, $n=0,1,2…N-1$, $\Omega=\omega_{D}(2N)$ of complex signal $\{x(n-T)\}$, $n=0,1,2…N-1$, $T=2\pi/\omega_{OD}$ will be shifted to the lower frequency band $[0, \omega_{MAX}/M]$. Time counts $\{x(n-T)\}$, $n=0,1,2…N-1$, $T=2\pi/\omega_{OD}$, complexly multiplied only by the coefficients $\{h(k)\}$, $k=0,1,2…K-1$ of one single DLF of the $K$-th order [7], [8].

As discrete values of the angular frequency $\omega_{m}$ we use the values corresponding to $\omega_{m}=\omega_{MAX}$, if $\omega_{MAX}=\omega_{OD}/2$, $\omega_{C}=\omega_{MAX}/M$ [7], [8]. In this case, an alternating sign change is applied, and the $n$-th time sample of the complex signal takes the form $x(n-T)=x(n-T)\cdot exp[-i\cdot n\cdot \omega_{m\Omega}]$ $=x(n-T)(-1)^{n}$, $n=0,1,2…N-1$, $T=2\pi/\omega_{OD}$. If $\omega_{OD}=\omega_{MAX}$ and $\omega_{C}=\omega_{MAX}/2$, then LDF of the original complex signal $\{x(n-T)\}$, $n=0,1,2…N-1$, $T=2\pi/\omega_{OD}$, allows getting a complex signal $\{x(n-T)\}$, $n=0,1,2…N-1$, $T=2\pi/\omega_{OD}$, which corresponds to the frequency band $[0, \omega_{MAX}/2]$ of original spectrum $\{X(n\Omega)\}$, $n=0,1,2…N-1$, $\Omega=\omega_{D}(2N)$. LDF of the complex signal $\{x(n\Omega-T)(-1)^{n}\}$, $n=0,1,2…N-1$, $T=2\pi/\omega_{OD}$, gives a complex signal $\{x(n-T)\}$, $n=0,1,2…N-1$, $T=2\pi/\omega_{OD}$, corresponding to the frequency band $[\omega_{MIN}/M, \omega_{MAX}]$ of original spectrum $\{X(n\Omega)\}$, $n=0,1,2…N-1$, $\Omega=\omega_{D}(2N)$, of complex signal $\{x(n-T)\}$, $n=0,1,2…N-1$, $T=2\pi/\omega_{OD}$, with low-frequency shifted $[0, \omega_{MAX}/M]$.

If, using a corner frequency shift equal to $\omega_{MAX}/2=\omega_{OD}/4$, we apply the same processing procedure to signals $\{x(n-T)\}$, $n=0,1,2…N-1$, $T=2\pi/\omega_{OD}$, and $\{x(n-T)\}$, $n=0,1,2…N-1$, $T=2\pi/\omega_{OD}$, and then perform decimation of the samples, considering only one of two consecutive samples, then four bands can be obtained. As a result of sequential execution of $r$ such procedures of the MDF method using the shift algorithm of the complex spectrum of the processed signal and its LDF, the only DLF of the $K$-th order is $2^{r}$ of frequency bands, $q=1,2,3…Q$. Therefore, this method allows the spectrum of this signal to be divided into a rapidly growing number of bands $M=2^{r}$, $q=1,2,3…Q$. [7], [8].

Hardware and software implementation of the $N$-point DFT method of a complex signal $\{x(n-T)\}$, $n=0,1,2…N-1$, $T=2\pi/\omega_{OD}$, based on its $N$-band digital filtering, one DLF of the $K$-th order is provided with a minimum number $MULT_{MDF}(N)$ of $Z$-bit multipliers and the corresponding number $SUMM_{MDF}(N)$ of $Z$-bit adders requiring a total $SET_{MDF}(N, Z)$ of logical cells PLD, calculated by the formula (2):

$$SET_{MDF}(N, Z) = MULT_{MDF}(N) \cdot SET_{Y}(Z) + SUMM_{MDF}(N) \cdot SET_{C}(Z),$$

(2)

where $SET_{MDF}(N, Z)$ is the number of logical cells PLD for $N$-point MDF; $MULT_{MDF}(N)$ is the number of multipliers for $N$-point MDF; $SET_{Y}(Z)$ is the number of logic cells PLD in a $Z$-bit multiplier; $SUMM_{MDF}(N)$ is the number of adders for $N$-point MDF; $SET_{C}(Z)$ is the number of logic cells PLD in $Z$-bit adder.

4.2. DFT method based on adaptive digital filtering

Offsetting MDF issues are closely related to sampling rate conversion. Their similarities and differences are most fully considered for the tree-like (multistage) structure of adaptive digital filtering of the pyramidal type [9].

Proposed by professor V.V. Vityazev the method of frequency selection of digital signals based on their adaptive digital filtering makes it possible to construct DFT computational algorithms for a complex signal $\{x(n-T)\}$, $n=0,1,2…N-1$, $T=2\pi/\omega_{OD}$, based on its multicomponent bandpass digital
filtering with a priori specified number $N$ of output channels (frequency bands) $\{X(o_n)\}$, $n=0,1,2,\ldots-N-1$, and uniform frequency step $o_n=o_B/(2N)$ [9], [10]. The implementation of such algorithms is practically reduced to the synthesis of a set of similar bandpass digital filters that differ from each other only in the values of the central frequency of the passband $o_n$, $0\leq o_n \leq o_B/2$. If all bandpass digital filters are implemented independently of each other, then the total cost is proportional to the number of frequency channels $N$. However, the total implementation costs per channel can be significantly reduced due to the multi-stage “preselection” of individual groups of frequency bands of the instantaneous spectrum $\{X(o_n)\}$, $n=0,1,2,\ldots-N-1$, processed signal $\{x(n,T)\}$, $n=0,1,2,\ldots-N-1$, $T=2\pi/o_B$. This approach to the synthesis of the optimal structure of the filtering system is traditionally based on such criteria as minimizing the number of arithmetic multiplication operations per unit time or minimizing the data memory capacity.

The idea of professor V.V. Vityazev to use secondary multistage sampling of digital signals such as $\{x(n-T)\}$, $n=0,1,2,\ldots-N-1$, $T=2\pi/o_B$, for decimation in frequency turned out to be especially fruitful when constructing systems for their frequency selection based on a set of digital band-pass filters with equally spaced frequencies $o_{n+1}+o_B/(2N)=o_n$, $0\leq o_n \leq o_B/2$ [9]. The practical implementation of this idea made it possible to synthesize systems for digital frequency selection of signals based on pyramid structures of elementary comb filters with finite and infinite memory.

Total number of $M_{\text{MDF}}(L,k)$, $M_{\text{MDF}}(L,k)=M_1+M_2+M_3+\ldots+M_L$ digital filters of the $K$-th order in an $L$-step pyramid structure with a base of $B_{\text{MDF}}$, $B_{\text{MDF}}=N$, exceeds the number $M_L$ digital filters in direct parallel form of constructing a set of $N$ digital filters for $M_L=N$. But each individual digital filter from such a set is more difficult to implement than each individual digital filter of a multistage structure [9].

If the spectral structure $\{X(m;\Omega)\}$, $m=0,1,2,\ldots-M-1$, $\Omega=o_B/(2M)$, $0\leq M \leq N$ of input signal $\{x(n-T)\}$, $n=0,1,2,\ldots-N-1$, $T=2\pi/o_B$, is not known, then the pyramid structure of a set of digital filters covering a given operating frequency range $[0, o_B/2]$, adapts to the structure of the input signal $\{x(n-T)\}$, $n=0,1,2,\ldots-N-1$, $T=2\pi/o_B$, by changing the number $L$ of steps of the pyramid structure of elementary filters. Moreover, at each $l$-th step of this structure, $l=1,2,3,\ldots,L$ all frequency bands of the original spectrum $\{X(m;\Omega)\}$, $m=0,1,2,\ldots-M-1$, $\Omega=o_B/(2M)$, $M=B_{\text{MDF}}$, $l=1,2,3,\ldots,L$, $B_{\text{MDF}}=N$, discrete signal $\{x(n-T)\}$, $n=0,1,2,\ldots-N-1$, $T=2\pi/o_B$, are shifted to the lower frequency band $[0, o_B/(2M)]$ and the same coefficients of the only DLF of the $K$-th order are used $\{h(k)\}$, $k=0,1,2,\ldots-K-1$.

The amount of hardware required for a special processor-based PLD implementation of the $N$-point DFT method based on $N$-band adaptive digital filtering by V.V. Vityazev is defined by the number $\text{SET}_{\text{MDF}}(N,Z)$ of logical cells PLD, which depends on the $Z$ value of the bit capacity of the elementary multipliers and adders involved [11], [12].

4.3. BDF-based extended window DFT

Significantly reduce the amount of hardware required for a special processor-based implementation of the computational algorithm for calculating the instantaneous spectrum $\{X(m;\Omega)\}$, $m=0,1,2,\ldots-N-1$, $\Omega=o_B/(2N)$, of complex signal $\{x(n-T)\}$, $n=0,1,2,\ldots-N-1$, $T=2\pi/o_B$, enables $N$-point DFT with an extended window based on BDF that does not require arithmetic multiplication operations [11].

Such DFT is a set of band-pass filters of the same type that cover the entire frequency range under study $[0, o_B/2]$ and is an efficient MDF system for splitting the entire input signal $\{x(n-T)\}$, $n=0,1,2,\ldots-N-1$, $T=2\pi/o_B$ for a given number of $M$ subchannels provided $M\leq N$.

The amount of hardware required for a special processor-based PLD implementation of this DFT method is determined by the number of $\text{SET}_{\text{BDF}}(N,Z)$ of logical cells PLD, which is calculated by the formula (3), and depends on the size of the window $N$ and the value $Z$ of the width of the elementary multipliers and adders involved:

$$\text{SET}_{\text{BDF}}(N,Z) = \text{MULT}_{\text{BDF}}(N) \cdot \text{SET}_{\text{Y}}(Z) + \text{SUMM}_{\text{BDF}}(N) \cdot \text{SET}_{\text{C}}(Z),$$

where $\text{SET}_{\text{BDF}}(N,Z)$ is the number of logical PLD cells for $N$-point DFT with extended window based on BDF;

$\text{MULT}_{\text{BDF}}(N)$ is the number of multipliers for $N$-point DFT with extended window based on BDF;

$\text{SET}_{\text{Y}}(Z)$ is the number of logical cells PLD in the $Z$-bit multiplier;
SUMM_{BDF}(N) is the number of adders for N-point DFT with extended window based on BDF;
SET_{c}(Z) is the number of logical cells PLD in the Z-bit adder.

However, significant savings in hardware using the DFT method with an extended window based on BDF can be achieved only in those cases when it is possible to synthesize such non-recursive digital filters of certain orders of K, under which two conditions are satisfied at once: computational complexity of this DFT method does not exceed the computational complexity of the known FFT methods and (or) DFT according to the Vinograd’s algorithm with Mersenne theoretical numerical transform (TNT), requiring the execution of the minimum number of arithmetic multiplication operations, \( \text{SET}_{BDF}(N,Z) \leq \text{SET}_{FFT}(N,Z) \) and (or) \( \text{SET}_{BDF}(N,Z) \leq \text{SET}_{NTT}(N,Z) \), and the computational error of the DFT method with an extended window based on the BDF does not exceed the computational error of the known FFT and (or) DFT methods according to the Vinograd’s algorithm with Mersenne (TNT), requiring the execution of the minimum number of arithmetic multiplication operations, \( C_{BDF}(N,D,Z) \leq C_{FFT}(N,D,Z) \) and (or) \( C_{BDF}(N,D,Z) \leq C_{NTT}(N,D,Z) \).

Therefore, the research project of D.I. Kaplun and D.V. Minenkov “Synthesis of a new class of non-recursive digital filters without multiplications” [13] was nominated and reached the final at the Business Ideas and Research Development Competition “Young, Daring, Promising” 2010 (http://www.spbinvest.ru/konkurs08.htm).

5. Discussion
The idea of reducing the number of arithmetic multiplication operations in digital DFT algorithms is due to the use of hardware to achieve equality of the speed of performing arithmetic multiplication and addition operations in elementary DSP devices. It has been successfully implemented using MDF-based DFT methods.

However, BDF-based extended window DFT method provides significant hardware savings only when it is possible to synthesize non-recursive digital filters that do not require arithmetic multiplication operations.

Therefore, multiband DFT (MFT) method based on multirate digital filtering can and should be considered more promising. [9]. MFT uses only the difference DLF with trivial integer difference coefficients. Such DLF allows performing DFT without multiplications at all.

6. Conclusion
The results of a comparative analysis of the known DFT methods based on MDF have shown and confirmed that it is possible to minimize the number of algorithmic multiplication operations only in a number of digital algorithms of such DFT.

The minimization of the number of algorithmic multiplication operations by the DFT method with an extended window based on BDF allows reducing the hardware costs for the software and hardware implementation of such DFT algorithms on PLD.

However, the most popular are DFT methods based on MDF with a pyramidal structure, significant simplification of the hardware implementation of which is achieved by using a single digital filter for all DFT spectral bands.

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