Gauge theoretical Gromov-Witten invariants and virtual fundamental classes

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Abstract

This article is an expanded version of talks which the authors have given in Oberwolfach, Bochum, and at the Fano Conference in Torino. In these talks we explained the main results of our papers "Gauge theoretical equivariant Gromov-Witten invariants and the full Seiberg-Witten invariants of ruled surfaces" and "Comparing virtual fundamental classes: Gauge theoretical Gromov-Witten invariants for toric varieties".

We have also included new results, e. g. the material concerning flag varieties, Quot spaces over $\mathbb{P}^1$, and the generalized quiver representations.

The common theme is the construction of gauge theoretical Gromov-Witten type invariants of arbitrary genus associated with certain symplectic factorization problems with additional symmetry, and the computation of these invariants in terms of complex geometric objects.

In chapter 1 we introduce the concept of a symplectic factorization problem with additional symmetry (SFPAS), and illustrate it with several important examples: Grassmann manifolds, flag varieties, toric varieties, certain Quot spaces over $\mathbb{P}^1$, and generalized quivers.

Chapter 2 introduces the gauge theoretical problem associated with a SFPAS, i.e. the standard gauge theoretical set up consisting of a configuration space, a partial differential equation (of vortex type), and a gauge group. Our invariants are defined by evaluating canonical cohomology classes on the virtual fundamental class of the moduli space of irreducible solutions to the PDE when this is possible, e.g. when the moduli space is compact.

In chapter 3 we explain the complex geometric interpretation of these moduli spaces in terms of (poly-)stable framed holomorphic objects over a Riemann surface, provided the original SFPAS came from a nice Kahlerian problem. The point is that, in this case, the moment map of the SFPAS together with the Riemannian metric on the base surface gives rise to a

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naturally associated stability concept for these framed holomorphic objects. The "universal Kobayashi-Hitchin correspondence" shows then the existence of an isomorphism \( \iota : \mathcal{M}^* \rightarrow \mathcal{M}^{\text{st}} \) between the gauge theoretic moduli spaces of irreducible solutions of the PDE and the complex geometric moduli space of stable framed holomorphic objects. However, in order to compute the gauge theoretical Gromov-Witten invariants using this explicit description, one also needs to know that the Kobayashi-Hitchin isomorphism \( \iota : \mathcal{M}^* \rightarrow \mathcal{M}^{\text{st}} \) identifies the virtual fundamental classes of these moduli spaces. In general this is a very difficult problem. We show that this is true for the special SFPAS which yields the toric varieties, and we state a conjecture for the general situation. Roughly speaking this conjecture asserts the following: When the gauge theoretic problem is of Fredholm type, and the data for \( \mathcal{M}^{\text{st}} \) are algebraic, then \( \mathcal{M}^{\text{st}} \) admits a canonical perfect obstruction theory in the sense of Behrend-Fantechi, and the Kobayashi-Hitchin isomorphism \( \iota : \mathcal{M}^* \rightarrow \mathcal{M}^{\text{st}} \) identifies the gauge theoretic and the algebraic virtual fundamental classes.

Chapter 4 contains examples, explicit computations of our invariants in an abelian case, as well as some interesting applications to Seiberg-Witten invariants of ruled surfaces. Another nice application concerns an old enumerative problem, namely the counting of maximal sub bundles of a general vector bundle over a Riemann surface.

1 Symplectic factorization problems with additional symmetry

A symplectic factorization problem (SFP) is a system \((F, \alpha, \mu)\), where \(F = (F, \omega)\) is a symplectic manifold, \(\alpha : K \times F \rightarrow F\) is a symplectic action of a compact Lie group \(K\) on \(F\) and \(\mu\) is a moment map for this action. The result of a symplectic factorisation problem \((F, \alpha, \mu)\) is the quotient

\[ F_\mu := \mu^{-1}(0)/K. \]

This quotient becomes a symplectic manifold (respectively orbifold) if \(\mu\) is a submersion in any point of \(\mu^{-1}(0)\) and every point of this set has trivial (respectively finite) stabilizers with respect to the \(K\)-action. In the first case we will say that the SFP \((F, \alpha, \mu)\) is regular and we will denote by \(\omega_\mu\) the induced symplectic form on \(F_\mu\). We agree to call \(F_\mu\) the symplectic quotient of \((F, \alpha, \mu)\) even when this SFP is not regular.

The concept "symplectic factorization problem" is the symplectic analogon of the concept "linearized action of a reductive group" on a polarized algebraic variety in classical Geometric Invariant Theory; the relation between the two concepts is beautifully explained in [Kir].

A compatible almost complex structure of a symplectic factorization problem \((F, \alpha, \mu)\) is an almost complex structure \(J\) on \(F\) which is \(\omega\)-tame (which means that \(\omega(\cdot, J\cdot)\) is a Riemannian metric on \(F\)) and \(K\)-invariant. Such an almost
complex structure defines an $\omega_\mu$-tame complex structure $J_\mu$ on $F_\mu$, if the chosen SFP was regular.

Many remarkable symplectic manifolds (e.g. Grassmann manifolds, flag manifolds, toric varieties, etc) can be regarded in a natural way as symplectic quotients associated with certain SFP’s. In many cases, the input symplectic manifold $F$ has – in a natural way – a larger symmetry than the symmetry used in performing the symplectic factorization. This larger symmetry induces then a symmetry on the resulting symplectic quotient $F_\mu$, and plays an important role in studying the geometry of this quotient.

**Definition 1.1** A symplectic factorization problem with additional symmetry (SFPAS) is a 4-tuple $(F, \alpha, K, \mu)$, where:

1. $F$ is a symplectic manifold,
2. $\alpha : \hat{K} \times F \to F$ is an action of a compact Lie group $\hat{K}$ on $F$,
3. $K$ is a closed normal subgroup of $\hat{K}$, which acts symplectically on $F$ via $\alpha$,
4. $\mu$ is a $\hat{K}$-equivariant moment map for the $K$-action on $F$.

A compatible almost complex structure of a SFPAS $(F, \alpha, K, \mu)$ is an almost complex structure on $F$ which is $\omega$-tame and $\hat{K}$-invariant.

In all interesting examples we know, $\alpha : \hat{K} \times F \to F$ is itself symplectic, and $\mu$ is induced by a moment map for this $\hat{K}$-action via the projection $\hat{k}^\vee \to k^\vee$.

The importance of these concepts comes from the following obvious

**Remark:** If $(F, \alpha, K, \mu)$ is a SFPAS, then the $\hat{K}$-action of $F$ descends to a $K_0 := \hat{K}/K$-action on the symplectic quotient $F_\mu$. This action is symplectic if $\alpha$ was symplectic.

If $J$ is a compatible almost complex structure of the SFPAS $(F, \alpha, K, \mu)$ and the SFP $(F, \alpha|_{K \times F}, \mu)$ is regular, then the induced $\omega_\mu$-tame almost complex structure $J_\mu$ on $F_\mu$ will be $K_0$-invariant.

Therefore a SFPAS (endowed with a compatible almost complex structure) provides a symplectic quotient (respectively an almost Kählerian symplectic quotient) which comes with natural induced symmetry.

Below we give several relevant examples, which motivate the introduction of these concepts and demonstrate their importance:

1. **Grassmann manifolds**

   Consider the manifold $F = \text{Hom}(\mathbb{C}^r, \mathbb{C}^{r_0})$ and the exact sequence of compact Lie groups
   
   $$1 \to U(r) \to U(r) \times U(r_0) \to U(r_0) \to 1$$
The group $\hat{K} := U(r) \times U(r_0)$ acts on $F$ in the obvious way. We denote by $\alpha_{can}$ this action. The $K := U(r)$-action on $F$ has a one parameter family $(\mu_t)_{t \in \mathbb{R}}$ of moment maps

$$\mu_t(f) = \frac{i}{2} f^* \circ f - it \text{id}_{C^r},$$

and the corresponding symplectic quotients are

$$F_{\mu_t} = \begin{cases} \mathbb{G}r_r(C^{r_0}) & \text{if } t > 0 \\ \{\ast\} & \text{if } t = 0 \\ \emptyset & \text{if } t < 0. \end{cases}$$

The moment maps $\mu_t$ are $\hat{K}$-equivariant, therefore the 4-tuples $(\text{Hom}(C^r, C^{r_0}), \alpha_{can}, U(r), \mu_t)$ are SFPAS's. This implies that the $\hat{K}$-action on $F$ descends to a $K_0 := U(r_0)$-action on the symplectic quotients $F_{\mu_t}$. This gives precisely the obvious $U(r_0)$-symmetry of the Grassmannian $\mathbb{G}r_r(C^{r_0})$ of $r$-planes of the complex vector space $C^{r_0}$. This induced symmetry is essential for understanding the geometry of the Grassmann manifolds.

2. Flag manifolds

Let $V_1, \ldots, V_m, V = V_{m+1}$ be Hermitian vector spaces. Put

$$d_i := \dim(V_i), \quad d := \dim(V), \quad F := \bigoplus_{i=1}^{m} \text{Hom}(V_i, V_{i+1}).$$

We consider the exact sequence of compact Lie groups

$$1 \longrightarrow \prod_{i=1}^{m} U(V_i) \longrightarrow \prod_{i=1}^{m+1} U(V_i) \longrightarrow U(V) \longrightarrow 1.$$

and we put $K := \prod_{i=1}^{m} U(V_i), \hat{K} := \prod_{i=1}^{m+1} U(V_i), K_0 := U(V)$. The group $\hat{K}$ acts on $F$ by

$$\alpha_{can}(g_1, \ldots, g_{m+1})(f_1, \ldots, f_m) = (g_2 \circ f_1 \circ g_1^{-1}, \ldots, g_{m+1} \circ f_m \circ g_m^{-1}).$$

The general form of a moment map for the restricted $K$-action on $F$ is

$$\mu_t(f_1, \ldots, f_m) = \frac{i}{2} \begin{pmatrix} f_1^* \circ f_1 \\ f_2^* \circ f_2 - f_1 \circ f_1^* \\ \vdots \\ f_m^* \circ f_m - f_{m-1} \circ f_{m-1}^* \end{pmatrix} - i \begin{pmatrix} t_1 \text{id}_{V_1} \\ t_2 \text{id}_{V_2} \\ \vdots \\ t_m \text{id}_{V_m} \end{pmatrix}.$$
where \( t \in \mathbb{R}^m \). To every \( f = (f_1, \ldots, f_m) \in F \) we associate the subspaces
\[
W_i(f) := (f_m \circ \ldots \circ f_i)(V_i) \subset V, \; 1 \leq i \leq m
\]
One obviously has \( W_i \subset W_{i+1} \) and the map
\[
f \mapsto (W_i(f))_{1 \leq i \leq m}
\]
is constant on orbits.

**Proposition 1.2** Suppose that \( t_i > 0 \), for all \( 1 \leq i \leq m \).

1. Let \( f \in F \). Then the following conditions are equivalent:
   (a) \( f \) is \( \mu_t \)-semistable
   (b) \( f \) is \( \mu_t \)-stable
   (c) all maps \( f_i \) are injective.

2. The map
\[
w : f \mapsto (W_i(f))_{1 \leq i \leq m}
\]
identifies the symplectic quotient \( F_{\mu_t} \) with the flag manifold
\[
\mathbb{F}_{d_1, \ldots, d_m}(V) := \{ (W_1, \ldots, W_m) | W_1 \subset \ldots \subset W_m \subset V, \; \dim(W_i) = d_i \}
\]
Note that \( \mathbb{F}_{d_1, \ldots, d_m}(V) \) is non-empty if and only if \( d_1 \leq \ldots \leq d_m \leq d \) and is interesting when all inequalities are strict (otherwise it can be identified with a flag manifold associated with a smaller number of \( d_i \)-s). We include a short proof for completeness.

**Proof:** 1. We use the standard analytic criterion [B], [MU1], [LT] for testing stability.

   If, in general, \( a \) is a Hermitian endomorphism of a Hermitian space \( W \), and \( \lambda \in \mathbb{R} \), we put
\[
W^\lambda := \ker(a - \lambda \text{id}_W), \; W_\lambda := \bigoplus_{\lambda' \leq \lambda} W^{\lambda'}.
\]
Let
\[
\xi = (\xi_1, \ldots, \xi_m) \in \mathfrak{t} = \bigoplus_{i=1}^m \text{Herm}(V_i),
\]
and let \( \alpha_*(\xi) \in \text{Herm}(F) \) be the induced Hermitian endomorphism. The eigenvalue decomposition of this endomorphism is
\[
\alpha_*(\xi) = \bigoplus_{i=1}^{m-1} \sum_{\eta \in \text{Spec}(\xi_{i+1})} (\eta - \lambda) \text{id}_{\text{Hom}(V^\lambda_i, V^{\eta}_{i+1})} \oplus \sum_{\lambda \in \text{Spec}(\xi_m)} -\lambda \text{id}_{\text{Hom}(V^\lambda_m, V)}.\]
A vector \( f \in F \) is \( \mu_t \)-(semi)stable if and only if for every \( \xi \in \mathfrak{t} \setminus \{0\} \) for which
\[
f \in \bigoplus_{y \leq 0} \text{Eig}(\alpha_*(\xi), y),
\]
one has
\[
\langle t, \xi \rangle = \sum_{i=1}^{m} t_i \text{Tr}(\xi_i) > 0 \ (\geq 0).
\]
The condition \( f \in \bigoplus_{y \leq 0} \text{Eig}(\alpha_*(\xi), y) \) becomes
\[
f_i(V_{i,\lambda}) \subset V_{i+1,\lambda} \ \forall i \in \{1, \ldots, m-1\}, \ \forall \lambda \in \text{Spec}(\xi_i),
\]
\[
f_m(V_{m,0}) = \{0\}.
\]
If all \( f_i \) are injective, then this condition implies \( V_{m,0} = \{0\} \), and by induction, \( V_{i,0} = \{0\} \) for all \( i \in \{1, \ldots, m-1\} \). This means that all \( \xi_i \) have only strictly positive eigen-values, hence indeed \( \langle t, \xi \rangle > 0 \).

Conversely, suppose that \( f_i \) was not injective. Choose \( \xi \) such that \( \xi_j = 0 \) for all \( j \neq i \), and
\[
\xi_i = -\text{pr}_{\text{dker}(f_i)}
\]
Then the condition \( f \in \bigoplus_{y \leq 0} \text{Eig}(\alpha_*(\xi), y) \) is obviously satisfied, but
\[
\langle t, \xi \rangle = -t_i \dim(\ker(f_i)) < 0,
\]
hence \( f \) is not \( \mu_t \)-semistable.

2. The standard theory of Kähler quotients [Kir] gives
\[
F_{\mu_t} = \mu_t^{-1}(0)/K = F_{\mu_t}^{st}/K^c = \{ f \in F | \ker(f_i) = 0 \ \forall i \in \{1, \ldots, m\} / \prod_{i=1}^{m} \text{GL}(V_i) \},
\]
where \( F_{\mu_t}^{st} \) stands for the set of \( \mu_t \)-stable points in \( f \). It easy to see that \( \prod_{i=1}^{m} \text{GL}(V_i) \) acts freely on \( F_{\mu_t}^{st} \), and that the map \( w \) identifies the quotient with \( \mathbb{F}_{d_1, \ldots, d_m}(V) \).

 Again, our input space \( F \) has a larger symmetry than the \( K \)-symmetry used to perform the symplectic factorization, namely the symmetry defined by the \( \hat{K} \)-action. The moment maps \( \mu_i \) are all \( \hat{K} \)-equivariant, so the 4-tuples
\[
(\bigoplus_{i=1}^{m} \text{Hom}(V_i, V_{i+1}), \alpha_{\text{can}}, \prod_{i=1}^{m} \text{U}(V_i), \mu_i)
\]
are SFPAS's.

We get an induced \( K_0 \)-action on the symplectic quotients \( F_{\mu_t} \). In the particular case of Proposition 1.1 (when \( t_i > 0 \)), this is just the tautological \( \text{U}(V) \)-action on \( \mathbb{F}_{d_1, \ldots, d_m}(V) \).
3. Toric varieties

In this case we take $F = \mathbb{C}^r$ and consider an exact sequence of the form

$$1 \rightarrow K_w \rightarrow [S^1]^r \rightarrow [S^1]^m \rightarrow 1,$$

where $w : [S^1]^r \rightarrow [S^1]^m$ is an epimorphism and $K_w := \ker(w)$. Let $v \in \text{Hom}(\mathbb{Z}^r, \mathbb{Z}^m)$ be the morphism defined by the differential $d_e(w)$. The group $\hat{K} := [S^1]^r$ acts on $F$ in a natural way.

The general form of a moment map for the induced $K_w$-action on $\mathbb{C}^r$ is

$$\mu_t(z^1, \ldots, z^r) = -\frac{i}{2} p_v(|z^1|^2, \ldots, |z^r|^2) + it, \quad t \in \text{coker}((v \otimes \text{id}_\mathbb{R})^*),$$

where $(v \otimes \text{id}_\mathbb{R})^* : \mathbb{R}^m \rightarrow \mathbb{R}^r$ is the adjoint of the linear map induced by the morphism $v$, and $p_v$ is the canonical projection

$$p_v : \mathbb{R}^r \rightarrow \text{coker}((v \otimes \text{id}_\mathbb{R})^*) = \mathbb{R}^r/(v \otimes \text{id}_\mathbb{R})^*(\mathbb{R}^m).$$

Here we have used the natural identification

$$\text{Lie}(K_w)^\vee = i\text{coker}[(v \otimes \text{id}_\mathbb{R})^*].$$

The quotient $F_{\mu_t}$ is non-empty if and only if

$$t \in p_v(\mathbb{R}^r_{\geq 0}) = \{t_1, \ldots, t_r \in \mathbb{R}_r : t_i \geq 0 \}.$$

Suppose that the following two conditions are satisfied:

**P1:** For every $j \in \{1, \ldots, r\}$, the column $v_j \in \mathbb{Z}^m$ of the integer matrix $v$ is primitive, i.e. it is a generator of the semigroup $\mathbb{Z}^m \cap \mathbb{R}_{\geq 0}v_j$.

**P2:** $\mathbb{R}^r_{\geq 0} \cap \text{im}(v^*) = \{0\}$ in the dual space $\mathbb{R}^r$ of $\mathbb{R}^r$.

If these conditions are satisfied, then $F_{\mu_t}$ is a projective toric variety with (at most) orbifold singularities, for every $t \in p_v(\mathbb{R}^r_{\geq 0})$ which is a regular value of the map $z \mapsto p_v(|z_1|^2, \ldots, |z_r|^2)$. In order to explain this statement more clearly, let us recall (see [Bat], [Gi]) some simple definitions and results in the theory of toric varieties:

Fix a subset $J \subset \{1, \ldots, r\}$, and let $\Sigma$ be a complete, simplicial fan in $\mathbb{R}^m$ whose 1-skeleton $\Sigma(1)$ is the set of rays $\Sigma(1) = \{\mathbb{R}_{\geq 0}v_j | j \in J\}$. For any $a = (a_1, \ldots, a_r) \in \mathbb{R}_r$ and any strictly convex polyhedral cone $\sigma \in \Sigma$ we define the functional $f^a_\sigma$ on the linear span $\langle \sigma \rangle$ by

$$\langle f^a_\sigma, v_j \rangle = -a_j \quad \text{if } \mathbb{R}_{\geq 0}v_j \text{ is a ray of } \sigma.$$
We introduce the following convex subsets of \( \text{coker}[(v \otimes \text{id}_R)^*] \) (see [OT3] for details):

\[
K(\Sigma) := \{ p_v(a) \mid a_i \geq 0, \langle f^a_\sigma, v_j \rangle \geq -a_j \forall \sigma \in \Sigma, \forall j \in \{1, \ldots, r\} \},
\]

\[
K_0(\Sigma) := \{ p_v(a) \in K(\Sigma) \mid \langle f^a_\sigma, v_j \rangle > -a_j \forall \sigma \in \Sigma, \forall j \in \{1, \ldots, r\} \}
\]

for which \( \mathbb{R}_{\geq 0}v_j \) is not a face of \( \sigma \)

A classical result in the theory of toric varieties states that every complete simplicial fan \( \Sigma \) in \( \mathbb{R}^m \) with \( \Sigma(1) \subset \{ \mathbb{R}_{\geq 0}v_1, \ldots, \mathbb{R}_{\geq 0}v_r \} \) defines an associated compact toric variety \( X_\Sigma \) in the following way: consider first the open set of \( \mathbb{C}^r \)

\[
U(\Sigma) = \{ z \in \mathbb{C}^r \mid \exists \sigma \in \Sigma \text{ such that } z^j \neq 0 \forall j \in \{1, \ldots, r\} \text{ for which } \mathbb{R}_{\geq 0}v_j \text{ is not a face of } \sigma \}.
\]

One proves that there is a geometric quotient

\[
X_\Sigma := U(\Sigma)/K_\Sigma^C
\]

and this quotient is a compact algebraic variety with a natural orbifold structure; it is projective if and only if \( K_0(\Sigma) \neq \emptyset \).

**Theorem 1.3** Let \( \Sigma \) be a complete simplicial fan \( \Sigma \) in \( \mathbb{R}^m \) with

\[
\Sigma(1) \subset \{ \mathbb{R}_{\geq 0}v_1, \ldots, \mathbb{R}_{\geq 0}v_r \}.
\]

For every \( t \in K_0(\Sigma) \), the set of semistable points with respect to the moment map \( \mu_t \) coincides with the corresponding set of stable points, and the symplectic quotient \( U_t^C(0)/K_\Sigma^C \) can be identified as a complex orbifold with the projective toric variety \( X_\Sigma \).

Note also that any regular value of the map \( z \mapsto p_v(|z_1|^2, \ldots, |z_r|^2) \) belongs to a \( K_0(\Sigma) \) for a suitable complete simplicial fan \( \Sigma \) in \( \mathbb{R}^m \), so one gets a complete description of all symplectic quotients of \( \mathbb{C}^r \) by \( K_\Sigma \) which correspond to regular values of the standard moment map \( \mu_0 \).

Concluding, we note that the 4-tuples

\[
(\mathbb{C}^r, \alpha_{\text{can}}, K_\Sigma, \mu_t)
\]

are obviously SFPAS’s. According to our general principle, the \([S^1]^r\)-action \( \alpha_{\text{can}} \) on \( F \) induces an \([S^1]^m\)-action on every symplectic quotient \( F_{\mu_t} \). The complexification of this action has a dense orbit and plays a fundamental role in the study of toric varieties.
4. Strømme’s triples and Quot spaces of trivial sheaves on $P^1$

Consider three Hermitian vector spaces $U$, $V$, $W$ of dimensions $u$, $v := u + r$ and $w$ respectively, where $u$, $w$, $r$ are non-negative integers with $r \leq w$.

This time our input symplectic manifold is

$$F := \text{Hom}(U, V)^{\otimes 2} \oplus \text{Hom}(W, V).$$

hence the space of diagrams of the form

$$\xymatrix{ U \ar[r]^{k} & V \ar[l]_{l} & W.}$$

We consider the exact sequence of compact Lie groups

$$1 \longrightarrow U(U) \times U(V) \longrightarrow U(U) \times U(V) \times U(W) \longrightarrow U(W) \longrightarrow 1$$

and we let the group $\tilde{K} := U(U) \times U(V) \times U(W)$ act on $F$ by

$$\alpha(a, b, c)(k, l, m) = (b \circ k \circ a^{-1}, b \circ l \circ a^{-1}, b \circ m \circ c^{-1}).$$

The general form of a moment map for the induced action of $K := U(U) \times U(V)$ on $F$ is given by

$$\mu_{s, t}(k, l, m) := \frac{i}{2}(k^* \circ k + l^* \circ l, -k \circ k^* - l \circ l^* - m \circ m^*) + i(-t \text{id}_U, s \text{id}_V)$$

for real parameters $s$, $t \in \mathbb{R}$. We refer to [LOT] for the following results

**Theorem 1.4** A triple $(k, l, m)$ is $\mu_1$-(semi)stable if and only if for all subspaces $U_1 \subset U$, $V_1 \subset V$ one has:

1. If

$$(U_1, V_1) \neq (0, 0), \ k(U_1) + l(U_1) \subset V_1$$

then $s \dim(V_1) > t \dim(U_1)$ ($s \dim(V_1) \geq t \dim(U_1)$).

2. If

$$(U_1, V_1) \neq (U, V), \ k(U_1) + l(U_1) \subset V_1, \ \text{im}(m) \subset V_1$$

then $t \dim(U/U_1) > s \dim(V/V_1)$ ($t \dim(U/U_1) \geq s \dim(V/V_1)$).

**Theorem 1.5** Suppose $s = 1 + \varepsilon$, $t = 1$, with $\varepsilon > 0$ sufficiently small.

1. The first (semi)stability condition is equivalent to the condition that the matrix $xk + yl$ has maximal rank $u$ for a general pair $(x, y) \in \mathbb{C}^2$. 

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2. The second (semi)stability condition is equivalent to the condition that the matrix 
\[ [xk + yl|m] \in M_{v,u+w} \] has maximal rank \( v \) for all \((x,y) \in \mathbb{C}^2 \setminus \{(0,0)\} \).

A triple \((k,l,m)\) satisfying the two non-degeneracy conditions of Theorem 1.4 will be called a Strømme triple. The set \( S(u,w,r) \) of Strømme triples is obviously open. The importance of these objects comes from the following construction of Strømme, which provides a simple description of certain Quot spaces on \( \mathbb{P}^1 \). With every Strømme triple \((k,l,m)\) we associate the diagram

\[
\begin{align*}
0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \otimes U & \xrightarrow{(k,l)} \mathcal{O}_{\mathbb{P}^1} \otimes V \to Q \to 0 ,
\end{align*}
\]

where \((k,l)\) is the sheaf morphism induced by \((k,l)\) and the standard isomorphism

\[ H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \simeq \mathbb{C}^2 , \]

\( m \) is the sheaf morphism induced by \( m \), and \( q_{k,l,m} \) is the morphism which makes the diagram commute. The fact that \((k,l,m)\) is a Strømme triple means that \((k,l)\) is a monomorphism and \( q_{k,l,m} \) is an epimorphism.

An important result of Strømme [S] states that

**Theorem 1.6** The group \( \text{GL}(U) \times \text{GL}(V) \) operates freely on the set of Strømme triples and the map \((k,l,m) \mapsto q_{k,l,m}\) descends to an isomorphism

\[ S(u,w,r)/\text{GL}(U) \times \text{GL}(V) \simeq \text{Quot}^{(r,u)}_{\mathcal{O}_{\mathbb{P}^1} \otimes W} , \]

where \( \text{Quot}^{(r,u)}_{\mathcal{O}_{\mathbb{P}^1} \otimes W} \) denotes the Quot space classifying the quotients of \( \mathcal{O}_{\mathbb{P}^1} \otimes W \) which have rank \( r \) and degree \( u \).

Our theorems Theorem 1.3, Theorem 1.4 imply that \( S(u,w,r) \) is just the open set of \( \mu_{1+\varepsilon,t} \) -(semi)stable points of \( F \). Therefore, using the analytic stability criterion, we see that the Quot space \( \text{Quot}^{(r,u)}_{\mathcal{O}_{\mathbb{P}^1} \otimes W} \) can be further identified with a symplectic quotient:

**Corollary 1.7** [LOT] One has a natural isomorphism

\[ F_{\mu_{1+\varepsilon,t}} \simeq \text{Quot}^{(r,u)}_{\mathcal{O}_{\mathbb{P}^1} \otimes W} . \]

Note finally that, in this case too, the input space comes with a larger symmetry than the \( K \)-symmetry used to perform the symplectic factorization, namely with a natural \( \tilde{K} := U(U) \times U(V) \times U(W) \)-symmetry. The moment maps \( \mu_{s,t} \) are all \( \tilde{K} \)-equivariant, so we get the following new examples of SFPAS’s:

\[ (\text{Hom}(U,V)^{\otimes 2} \oplus \text{Hom}(W,V), \alpha_{\text{can}}, U(U) \times U(V), \mu_{s,t}) . \]
Therefore, the symplectic quotients $F_{\mu_s,t}$ have all a natural induced $U(W)$-symmetry. In the special case $(s, t) = (1 + \varepsilon, 1)$, one gets precisely the obvious $U(W)$-action on the Quot space $Quot_{\mathcal{O}_{\mathbb{P}^1} \otimes W}^{(r,u)}$.

5. Quiver Problems

All the examples above are particular cases of the following quiver factorization problem.

Let $Q = (V, A, s, t)$ be a quiver. This means that $V$ and $A$ are finite sets (the set of vertices and the set of arrows) and $s, t$ are maps $s, t : A \to V$ (the source map and the target map).

Let $\hat{K}$ be a compact Lie group. A $Q$-representation of $\hat{K}$ is a system $\rho = (\rho_v)_{v \in V}, \rho_v : \hat{K} \longrightarrow U(W_v)$ of unitary representations of $\hat{K}$.

Take $F := \bigoplus_{a \in A} \text{Hom}(W_s(a), W_t(a))$, and consider the $\hat{K}$-action $\alpha_\rho$ of $\hat{K}$ on $F$ induced by the $Q$-representation $\rho$.

Let $K$ be a closed normal subgroup of $\hat{K}$ endowed with an $\text{ad}_{\hat{K}}$-invariant inner product on its Lie algebra $\mathfrak{k}$. The general form of a moment map for the induced $K$-action is

$$\mu_t((f_a)_{a \in A}) = \frac{i}{2} \rho^*_t \left( \sum_{s(a)=v} f_a^* \circ f_a - \sum_{t(a)=v} f_a \circ f_a^* \right)_{a \in A} - it,$$

where $\rho^*_t$ is the composition

$$it \mapsto it - (id_{\rho_t(u)})_{u \in W} \bigoplus_{v \in V} iu(W_v),$$

$\rho^*_t$ is its adjoint, and $it$ is a central element of $\mathfrak{k}$.

The 4-tuple $((\bigoplus_{a \in A} \text{Hom}(W_s(a), W_t(a)), \alpha_\rho, K, \mu_t)$ is obviously a SFPAS, so one gets an induced $K_0 := \hat{K}/K$-symmetry on the symplectic quotients $F_{\mu_t}$. Such a SFPAS will be called a quiver factorization problem associated with $Q$.

Example: Usually, one takes

$$\hat{K} = \prod_{v \in V} U(W_v),$$
(where $W_v$ is Hermitian vector space) with the canonical representation on $W_v$, and $K$ is chosen to be a closed normal subgroup of this product, for instance a product of factors $U(W_v)$.

In the very special case when $K = \hat{K} = \prod_{v \in V} U(W_v)$, one has a standard quiver factorization problem. This is the case considered in classical GIT [K].

The list below shows that all examples above are just special quiver factorization problems:

\[ \begin{array}{cccc}
\text{SFPAS} & Q & K & K \\
\text{(Hom($\mathbb{C}^r, \mathbb{C}^{r_0}$), $\alpha_{\text{can}}, U(r), \mu_t$)} & \bullet \rightarrow \bullet & U(r) \times U(r_0) & U(r) \\
\left( \bigoplus_{i=1}^m \text{Hom($V_i, V_{i+1}$)}, \alpha_{\text{can}}, \prod_{i=1}^m U(V_i), \mu_t \right) & \bullet \rightarrow \bullet \rightarrow \ldots \rightarrow \bullet & \prod_{i=1}^{m+1} U(V_i) & \prod_{i=1}^m U(V_i) \\
\left( \text{Hom($\mathbb{C}, \mathbb{C}$)$^\oplus r$, $\alpha_{\text{can}}, K_w, \mu_t$)} \right) & \{ \vdots \} & [S^1]^r & K_w \\
\left( \text{Hom($U, V$)$^\oplus 2 \oplus \text{Hom($W, V$), $\alpha, U(U) \times U(V), \mu_{s,t}$)} \right) & \bullet \rightarrow \bullet & U(U) \times U(V) \times U(W) & U(U) \times U(V) \\
\end{array} \]

**Remark:** One can formulate a more general version of the quiver factorization problem explained above in the following way:

Let $\rho = (\rho_v)_{v \in V}$ be a $Q$-representation of $\hat{K}$, and let

\[ r = (r_a)_{a \in A}, \; r_a : \hat{K} \rightarrow U(W_a^0) \]

be a system of representations of $\hat{K}$ indexed by the arrows of $Q$ (the "twisting representations"). Put

\[ F := \bigoplus_{a \in A} \text{Hom($W_{s(a)}, W_{t(a)} \otimes W_a^0$)}, \]

endowed with the $\hat{K}$-action $\alpha_{\rho, r}$ induced by $\rho$ and $r$, and let $K$ be a closed normal subgroup of $\hat{K}$. The general form of the moment map for the induced
The system \((F, \alpha, \rho, r, K, \mu)\) is obviously a SFPAS. Such a SFPAS will be called an \(r\)-twisted quiver factorization problem associated with \(Q\). The particular case when 
\[
\hat{K} := \prod_{v \in V} U(W_v) \times \prod_{a \in A} U(W_0^a), \quad K := \prod_{v \in V} U(W_v)
\]
and \(\rho_v, r_a\) are the canonical representations of \(\hat{K}\) in \(W_v, W_0^a\) was considered in [AlPr1], [AlPr2].

2 The gauge theoretical problem associated with a symplectic factorization problem with additional symmetry

Let \(F = (F, \omega)\) be symplectic manifold and let \(\sigma_\mu = (F, \alpha, K, \mu)\) be a SFPAS. We fix the following topological data:

- a closed, connected, oriented real surface \(Y\),
- a principal \(\hat{K}\)-bundle \(\hat{P}\) on \(Y\),
- a homotopy class \(H \subset \Gamma(Y, E)\) of sections in the associated bundle
  \[
  E := \hat{P} \times_{\hat{K}} F.
  \]

To formulate our gauge theoretical problem, we also need three continuous parameters:

- a compatible almost complex structure \(J\) of the SFPAS \(\sigma_\mu\), i.e. a \(\hat{K}\)-invariant, \(\omega\)-tame almost complex structure \(J\) on the symplectic manifold \((F, \omega)\).
- a Riemannian metric \(g\) on \(Y\),
- a parameter connection \(A_0 \in \mathcal{A}(P_0)\), where \(P_0 := \hat{P}/K\) is the associated \(K_0 := \hat{K}/K\)-bundle of \(\hat{P}\).

We will denote by \(J_g\) the complex structure defined by the Riemannian metric \(g\) and the fixed orientation of \(Y\).

Note that any connection \(\hat{A}\) on \(\hat{P}\) defines an almost complex structure \(J_{\hat{A}}\) on \(E\); this is the unique almost complex structure which agrees with \(J\) on the vertical tangent spaces of \(E\) and with the complex structure \(J_g\) on the \(\hat{A}\)-horizontal
spaces.

With $\sigma_\mu$ and our set of data we associate the following objects:

- a configuration space
  \[ \mathcal{A} := \mathcal{A}_{A_0}(\hat{P}) \times H , \]
  where $\mathcal{A}_{A_0}(\hat{P})$ stands for the affine space of those connections on $\hat{P}$ which induce $A_0$ on $P_0$.

- a differential equation of vortex type for the elements $(\hat{A}, \varphi) \in \mathcal{A}$:
  \[ \begin{cases} 
  \varphi & \text{is } J_{\hat{A}} \text{- holomorphic} \\
  \text{pr}_A \lambda_g F_{\hat{A}} + \mu(\varphi) & = 0 . 
  \end{cases} \]
  \( (V) \)

- a gauge group
  \[ \mathcal{G} := \Gamma(Y, \text{Aut}_{P_0}(\hat{P})) = \Gamma(Y, \hat{P} \times_{A_0} K) , \]
  acting on the configuration space $\mathcal{A}$ and leaving the set of solutions of the equation $(V)$ invariant.

Therefore, once the SFPAS $\sigma_\mu$ is fixed, the equation $(V)$ depends on two systems of data:

- the topological data $\tau = (Y, \hat{P}, H)$,
- the continuous parameters $p := (J, g, A_0)$.

When we want to take into account this dependence, we will write $(V^\tau_p)$, or – when $\tau$ is obvious – $(V^p)$ instead of $(V)$.

We denote by $\mathcal{A}^*$ the open subset of $\mathcal{A}$ consisting of irreducible pairs, i.e. pairs $(\hat{A}, \varphi)$ with trivial stabilizers with respect to the $\mathcal{G}$-action, and by $\mathcal{A}^V$ ($[\mathcal{A}^*]^V$) the closed subspace of $\mathcal{A}$ ($\mathcal{A}^*$) of solutions of $(V)$.

We introduce the quotients
\[ B := \mathcal{A}/\mathcal{G} , \quad B^* := \mathcal{A}^*/\mathcal{G} \]
and the moduli spaces of solutions
\[ \mathcal{M} := \mathcal{A}^V/\mathcal{G} \subset B , \quad \mathcal{M}^* := [\mathcal{A}^*]^V/\mathcal{G} \subset B^* . \]

After suitable Sobolev completions, $B^*$ becomes a Banach manifold, and, by standard elliptic theory, $\mathcal{M}^*$ becomes a finite dimensional subspace of this manifold. We will also use the notations $\mathcal{M}_p$, $\mathcal{M}^*_p$, $\mathcal{M}^*_p(\sigma_\mu)$ when we have to take into account the functoriality of these objects.
As in Donaldson theory or in classical Gromov-Witten theory, in order to introduce invariants associated with these moduli spaces, we have to endow them with canonical cohomology classes and with a virtual fundamental class.

Canonical cohomology classes on $M^*$ can be obtained in the following way:

We regard a section $\varphi \in \Gamma(Y, E)$ as a $\hat{K}$-equivariant map $\hat{P} \to F$. We have a natural evaluation map

$$ev : \mathcal{A} \times \hat{P} \to F,$$

which is $\hat{K}$-equivariant and $\mathcal{G}$-invariant, hence descends to a $\hat{K}$-equivariant map

$$\mathcal{A} \times \hat{P}/\mathcal{G} \to F$$

which restricts to a $\hat{K}$-equivariant map

$$\Phi : \hat{P} := A^* \times \hat{P}/\mathcal{G} \to F.$$

The space $\hat{P}$ can be regarded as $\hat{K}$-bundle over the product $\mathcal{B}^* \times Y$ (the universal $K$-bundle), whereas $\Phi$ can be interpreted as the universal section in the associated bundle $P \times _K F$. The map $\Phi$ induces a morphism

$$\Phi^* : H^*_K(F) \to H^*(\mathcal{B}^* \times Y).$$

For any cohomology class $c \in H^*_K(F, \mathbb{Z})$ and homology class $h \in H_*(Y, \mathbb{Z})$ put

$$\delta^c(h) := \Phi^*(c)/h \in H^*(\mathcal{B}^*, \mathbb{Z}).$$

We will say that the pair $(p, \mu)$ is good if $M^*_p(\sigma_\mu) = M^*_p(\sigma_\mu)^*$, i.e. the corresponding equation $(V)$ has only irreducible solutions.

In many interesting cases (see Theorem 2.2 below) one can show that $M^*$ can be identified with the vanishing locus of a Fredholm section $v$ (induced by the left hand term $v$ of $(V)$) in a Banach vector bundle over the Banach manifold $\mathcal{B}^*$, and that the determinant line bundle of the index of the family of intrinsic differentials of $v$ can be naturally oriented in a neighbourhood of this vanishing locus. In this case, the formalism of Brussee ([Bru], [OT2]) applies and yields a virtual fundamental class

$$[M^*_p]_{\text{vir}} \in H^{cl}_{\text{index}(v)}(M^*, \mathbb{Z}).$$

in the homology with closed supports of $M^*$.

If this situation occurs, and if the moduli space $M$ is compact (see Theorem 2.2 below) one can define gauge theoretical Gromov-Witten invariants of the SFPAS $\sigma_\mu$ with respect to the parameters $(\tau, p)$ by

$$GGW^*_p(\sigma_\mu) \left( \begin{array}{c} c_1 \\ h_1 \\ \vdots \\ c_k \\ h_k \end{array} \right) := \langle \cup_{i=1}^k \delta^{c_i}(h_i), [M^*_p(\sigma_\mu)]_{\text{vir}} \rangle,$$

for every good pair $(p, \mu)$.  

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Remark 2.1 The numbers $GGW^\tau_p(\sigma_\mu)\left(\begin{array}{c} c_1 \\ h_1 \end{array}\right), \ldots, \left(\begin{array}{c} c_k \\ h_k \end{array}\right)$ are not independent, but they satisfy a set of tautological relations. Therefore the map $GGW^\tau_p(\sigma_\mu)$ descends to a graded $\mathbb{Z}$-algebra $\mathbb{A}$ generated by the symbols $\left[c \atop h\right], c \in H^*_K(F; \mathbb{Z})$, $h \in H_*(Y, \mathbb{Z})$ subject to these tautological relations \[OT2\].

For instance, if $\hat{K} = U(r) \times K_\alpha$ and $F$ is contractible, one has

$$\mathbb{A} = \mathbb{Z}[u_1, \ldots, u_r, v_2, \ldots, v_r] \otimes \Lambda^*[\oplus_{l=1}^r H_1(Y, \mathbb{Z})_l],$$

$$\deg(u_i) = 2i, \quad \deg(v_j) = 2j - 2, \quad \deg(H_1(Y)_i) = 2l - 1,$$

$$u_i = \left(\operatorname{chern}_i [s]\right), \quad v_j = \left(\operatorname{chern}_j [Y]\right), \quad H_1(Y, \mathbb{Z})_l = \left\{\left(\begin{array}{c} \operatorname{chern}_h \\ h \end{array}\right) \Big| h \in H_1(Y, \mathbb{Z})\right\}.$$

Here $\operatorname{chern}_h \in H^2_K(F, \mathbb{Z})$ is induced by the isomorphism $H^*_K(F, \mathbb{Z}) \simeq H^*(B\hat{K}, \mathbb{Z})$ and the natural map $H^*(B\hat{K}, \mathbb{Z}) \to H^*(BU(r), \mathbb{Z})$.

Theorem 2.2

1. If the action $\alpha : \hat{K} \times F \to F$ is defined by a unitary representation of $\hat{K}$, then the moduli space $\mathcal{M}^*$ can be regarded as the vanishing locus of a holomorphic Fredholm section $v$ in a complex Banach vector bundle over the Banach manifold $\mathcal{B}^*$.

2. If $\alpha|_{K \times F}$ is defined by a unitary representation $\rho : K \to U(F)$ of $K$ and the standard moment map $\mu_\rho(f) = (d_e \rho)^*\left(\frac{i}{2}f \otimes f^*\right)$ satisfies the properness condition $\mu_\rho^{-1}(0) = \{0\}$, then the moduli space $\mathcal{M}_p^*(\sigma_\mu)$ is always compact for all parameters $(\tau, p)$ and any moment map $\mu$.

Therefore, in the first case Brussee’s formalism \cite{Bru} applies and gives a virtual fundamental class $[\mathcal{M}^*]^{\text{vir}} \in H^\text{cl}_\text{index}(\mathcal{M}^*, \mathbb{Z})$, whereas in the second situation all the moduli spaces $\mathcal{M}_p^*(\sigma_\mu)$ are compact. Combining these results it follows that

Corollary 2.3 If $\alpha$ is defined by a unitary representation $\hat{\rho}$ and $\mu_{\hat{\rho}|_{K_\alpha}}$ satisfies the properness condition above, then the gauge theoretical Gromov-Witten invariants $GGW^\tau_p(\sigma_\mu)$ are well defined for every good pair $(p, \mu)$.

Remark 2.4 One should be able to define $\mathbb{Q}$-valued invariants for almost good pairs $(p, \mu)$, i.e. for pairs $(p, \mu)$ for which the solutions of the corresponding equation $(V)$ have only finite stabilizers. In this case, the moduli space $\mathcal{M}$ must be endowed with an orbifold structure. We refer to \cite{CMS} and \cite{OT3} for details concerning this generalization.
Let us explain two important special cases of our gauge theoretical problem obtained by choosing $K$ trivial or $K_0$ trivial:

1. $K = \{1\}, F$ compact:
   In this case the second equation in the system $(V)$ is identically satisfied, $\tilde{K} = K_0, \tilde{P} = P_0$, and the associated bundle $E$ has a fixed almost complex structure $J_{A_0}$ induced by $A_0$. In this case
   \[ \mathcal{M} = \Gamma_{J_{A_0}}(Y, E) \]
   is the space of $J_{A_0}$-almost holomorphic sections in $E$. Of course, $\mathcal{M}$ is in general non-compact, because bubbling phenomena occur, exactly as in classical Gromov-Witten theory. However, generalizing ideas of Ruan [R] one can define a natural compactification $\overline{\mathcal{M}}$ of such a moduli space and, at least for special fibres $F$, define Gromov-Witten type invariants.
   These Gromov-Witten invariants should be called $K_0$-twisted Gromov-Witten invariants, because they are obtained by replacing the moduli spaces of almost holomorphic $F$-valued maps in Ruan’s Gromov-Witten theory by moduli spaces of sections in $F$-bundles with symmetry group $K_0$ (i.e. moduli spaces of $K_0$-twisted $F$-valued maps).

2. $K_0 = \{1\}, F_\mu$ compact:
   In this case $\tilde{K} = K$, so one gets a moduli problem which depends on a SFP $(F, \alpha, \mu)$ rather than a SFPAS. The obtained invariants should be called gauge theoretical (or Hamiltonian) $K$-equivariant Gromov-Witten invariants of $F$. This case was extensively considered and studied by Mundet i Riera [Mu2], Gaio [Ga], Gaio-Salamon [GS] Cieliebak-Gaio-Salamon [CGS], Cieliebak-Gaio-Mundet–Salamon [CGMS]. They state a conjecture – called the adiabatic limit conjecture – asserting that the standard Gromov-Witten invariants of the symplectic quotient $F_\mu$ can be expressed in terms of gauge theoretical Gromov-Witten invariants of the SFP $(F, \alpha, \mu)$, provided this SFP is regular, the quotient $F_\mu$ is compact, and both types of invariants are well defined.
   This conjecture is motivated by the fact that the second equation in $(V)$ tends to
   \[ \mu(\varphi) = 0 \]
   when $g$ is replaced by $tg$ and $t \to \infty$. Progress on this problem was obtained in [GS] for a special class of symplectic manifolds $F$.

Taking into account these discussions, one should probably think of the invariants associated with a SFPAS $(F, \alpha, K, \mu)$ as gauge theoretical $K_0$-twisted, $K$-equivariant Gromov-Witten invariants of $F$.

Of course, the adiabatic limit conjecture should also be true for the general twisted equivariant Gromov-Witten invariants, but one should replace the
usual Gromov-Witten invariants of the symplectic quotient $F_\mu$ by its $K_0$-twisted invariants.

3 The complex geometric interpretation

The natural question at this point is how can one obtain explicit descriptions of moduli spaces $M^*_p(\sigma_\mu)$ (or $M^*_p(\sigma_\mu))$ associated with a given SFPAS $\sigma_\mu$ and parameters $\tau = (Y, \hat{P}, H), \, p = (J, g, A_0)$.

The main observation here is

When the SFPAS $\sigma_\mu$ is Kählerian, i.e. the almost complex structure $J$ on $F$ is integrable and the $\hat{K}$-action on $F$ extends to a holomorphic $\hat{K}^C$-action, then the universal Kobayashi-Hitchin correspondence gives a complex geometric interpretation of these moduli spaces in terms of (poly)stable framed holomorphic pairs.

There are several concepts here which must be explained:

We denote by $G$, $\hat{G}$ and $G_0$ the complexifications of $K$, $\hat{K}$ and $K_0$: these are complex reductive groups. Let $\hat{Q}$ ($Q_0$) be the $\hat{G}$-bundle ($G_0$-bundle) obtained by complexifying $\hat{P}$ ($P_0$). The connection $A_0$ induces a bundle-holomorphic structure $J_0$ on $Q_0$, so one obtains a holomorphic $G_0$-bundle $\hat{Q}_0$ on the Riemann surface $(Y,J_g)$.

**Definition 3.1** A framed holomorphic pair of type $(Q, \alpha, H, Q_0)$ is a triple $(\hat{Q}, \varphi, \lambda)$, where

- $\hat{Q}$ is a holomorphic $\hat{G}$-bundle over $Y$
- $\varphi$ is a holomorphic section in the associated $F$-bundle $\hat{Q} \times \hat{G} F = \hat{P} \times \hat{K} F$ belonging to $H$,
- $\lambda : \hat{Q} \to Q_0$ is $G_0$-framing of $\hat{Q}$, i.e. it is a holomorphic bundle morphism of type $G \to G_0$,

such that $\hat{Q}$ is $C^\infty$-isomorphic to $\hat{Q}$ over $Q_0$, i.e. there exists a commutative diagram

\[
\begin{array}{ccc}
\hat{Q} & \xrightarrow{C^\infty} & \hat{Q} \\
\downarrow\lambda & & \downarrow\lambda \\
\hat{Q}_0 & \xrightarrow{\sim} & \hat{Q}_0
\end{array}
\]

\footnote{We recall that, in general, a bundle-holomorphic structure on a principal $G$-bundle $Q$ is a holomorphic structure on the total space with respect to which the action $G \times Q \to Q$ of the structure group and the projection on the base are both holomorphic.}
where the right hand morphism is just the canonical projection $\hat{Q} \to Q_0$.

An isomorphism between such triples $(\hat{Q}, \varphi, \lambda)$, $(\hat{Q}', \varphi', \lambda')$ is a bundle isomorphism $f : \hat{Q} \to \hat{Q}'$ such that $\lambda' \circ f = \lambda$ and $f_*(\varphi) = \varphi'$.

The classification of framed holomorphic pairs of a given type is a very interesting and important complex geometric problem. Many moduli problems in complex geometry are special cases of this "universal" classification problem. One can develop a complex geometric deformation theory for such objects, i.e. one can introduce in a natural way the notions of holomorphic families of framed holomorphic pairs (of a fixed type) parameterized by a complex space, versal and universal deformation of a fixed framed pair, etc.

As in classical GIT, one cannot construct a moduli space with good properties classifying all framed holomorphic pairs of a given type; one needs a stability condition.

The point is that with every pair $(\mu, g)$ consisting of a $K$-equivariant moment map for the $K$-action on $F$ and a Riemannian metric on $Y$ one can associate a stability condition. The stability condition is open, i.e. if $(\hat{Q}_t, \varphi_t, \lambda_t)_{t \in T}$ is a holomorphic family of framed holomorphic pairs parameterized by a complex space $T$, then the set of parameters $t$ for which $(\hat{Q}_t, \varphi_t, \lambda_t)$ is $(\mu, g)$-stable is open in $T$. Moreover, using complex geometric deformation theory, one gets a moduli space $\mathcal{M}^{(\mu,g)-st}(Q, \alpha, H, Q_0)$ classifying $(\mu, g)$-stable framed pairs of type $(Q, \alpha, H, Q_0)$.

Writing the stability conditions explicitly in the general case is quite technical and requires a long preparation, but is now well understood (see [Mu1] for the case $K_0 = \{1\}$, [LT] for the general case). There exist two important special cases in which it takes a simple form:

**Proposition 3.2** Suppose that $K$ is abelian and that one of the following conditions is satisfied:

1. $F$ is a Hermitian space and $\alpha|_{K \times M} : K \times F \to F$ is induced by a unitary representation, or

2. $F$ is a quasiprojective, and $(\omega, J)$ are induced by a regular embedding $F \to \mathbb{P}^N$, and $\alpha|_{K \times M} : K \times F \to F$ is induced by a unitary representation in $\mathbb{C}^{N+1}$.

Then $(\hat{Q}, \varphi, \lambda)$ is $(\mu, g)$-stable if and only if $\varphi$ is generically $(\mu - 2\pi \mu_{K}(\hat{Q}))$-stable.

Here $\mu_{K}(\hat{Q})$ denotes a topological invariant of $\hat{Q}$ which generalizes the usual slope of a vector bundle [LT].

The universal Kobayashi-Hitchin correspondence states that, with the notations and conventions above, there are natural isomorphisms

$$\mathcal{M}^{\mu}(\sigma_\mu)* \simeq \mathcal{M}^{(\mu,g)-st}(Q, \alpha, H, Q_0). \quad (KH)$$
In particular, this shows that $M^*_\tau(\sigma_\mu)^*$ has a natural complex space structure.

**Remark 3.3**

1. The Kobayashi-Hitchin correspondence can be extended to arbitrary compact complex manifolds $Y$. When $\dim_{\mathbb{C}}(Y) \geq 2$, one must add the integrability condition $F^2_{A_0} = 0$ to the system $(V)$ and require that the parameter connection $A_0$ is integrable.

2. There is a more refined Kobayashi-Hitchin correspondence which identifies the whole moduli space $M^*_\tau(\sigma_\mu)$ with the moduli space of $(\mu, g)$-polystable framed pairs [LT].

The Kobayashi-Hitchin correspondence is a very important tool which can be used to give explicit descriptions of moduli spaces, but it is not quite sufficient for the computation of the invariants. In order to have a complex geometric interpretation of the invariants, one also has to compare the virtual fundamental classes of the moduli spaces involved in the Kobayashi-Hitchin correspondence ($KH$). As we explained in the previous section, a moduli space $M^*$ defined with gauge theoretical methods can naturally be endowed with a virtual fundamental class $[M^*]^{vir}$ when it can be identified with the vanishing locus of a Fredholm section in a vector bundle over a Banach manifold, and the determinant line bundle of the index of the family of intrinsic differentials of this section is oriented in a neighbourhood of this vanishing locus. We agree to call such a gauge theoretical moduli problems of Fredholm type.

On the other hand, many interesting moduli spaces defined in Algebraic Geometry come with a natural perfect obstruction theory in the sense of Behrend-Fantechi [BF], and such a structure allows one to define a virtual fundamental class in the Chow group $A_*$ of the moduli space ([BF], [Kr]).

We state that

**Conjecture:** Let

$$\iota: M^* \rightarrow M^{st}$$

be any Kobayashi-Hitchin type isomorphism. Suppose that

- $M^*$ is associated with a gauge theoretical moduli problem of Fredholm type,
- all the data involved in the definition of $M^{st}$ are algebraic.

Then $M^{st}$ admits a canonical perfect obstruction theory, and $\iota$ maps the virtual fundamental class of $M^*$ in the sense of Brussee to the image of the Behrend-Fantechi virtual fundamental class of $M^{st}$ under the cycle map

$$cl: A_k(M^{st}) \rightarrow H^{2k}_{cl}(M^{st}, \mathbb{Z}).$$

**Remark 3.4**

1. The second assumption can probably be removed, but in order to give a sense
to this more general statement, one needs a complex analytic generalization of the Behrend-Fantechi virtual class theory.

2. The moduli problem introduced in Chapter 2 is always of Fredholm type when the action $\alpha$ is defined by a unitary representation. Note however that the analogous vortex type problems on higher dimensional base manifolds $Y$ are in general not of Fredholm type.

3. The standard vortex moduli problem on Kähler surfaces can be regarded as a moduli problem of Fredholm type [OT2].

The importance of our conjecture is obvious: it provides a universal principle allowing one to identify not only certain moduli spaces defined within the two theories, but also the corresponding numerical invariants.

The conjecture has already been checked in several special cases [OT2], [OT3], and we believe that our method can be used to give a proof of it under the assumption that the base manifold is a Riemann surface.

4 Examples, computations, and applications

We illustrate the universal Kobayashi-Hitchin correspondence in some of the special situations which we considered in the first chapter of this article. In every case we will indicate explicitly the corresponding stability condition and we will state several results concerning the corresponding gauge theoretical invariants, as well as applications of these results.

1. The SFPAS which yields the Grassman manifolds

We come back to the SFPAS

$$\sigma_{\mu_t} = (\text{Hom}(\mathbb{C}^r, \mathbb{C}^{r_0}), \alpha_{\text{can}}, U(r), \mu_t),$$

where $\alpha_{\text{can}}$ is the natural action of $\hat{K} := U(r) \times U(r_0)$ on $F = \text{Hom}(\mathbb{C}^r, \mathbb{C}^{r_0})$ endowed with its obvious compatible complex structure $J$, and

$$\mu_t(f) = \frac{i}{2} f^* \circ f - itid_{\mathbb{C}^r}.$$

The data of a principal $\hat{K}$-bundle $\hat{P}$ on a real surface $Y$ is equivalent to the data of a pair $(E, E_0)$ of Hermitian vector bundles of ranks $r, r_0$ over $Y$. Let $d, d_0$ be the degrees of these bundles. We denote by $\tau$ the topological data $(Y, \hat{P}, H)$, where, of course, $H := A^0\text{Hom}(E, E_0)$.

Consider a Hermitian connection $A_0$ on $E_0$, and denote by $\mathcal{E}_0$ the corresponding holomorphic bundle. We also fix a Riemannian metric $g$ on $Y$. So our continuous data is the triple $p = (J, A_0, g)$.

The corresponding gauge theoretical problem becomes: For a given real number $t$, classify pairs $(A, \varphi)$ consisting of a Hermitian connection in $E$ and a
morphism $\varphi \in A^0\text{Hom}(E, E_0)$ such that the following vortex type equation is satisfied:

\[
\begin{cases}
\bar{\partial}_{A, A_0} \varphi = 0 \\
i A F_A - \frac{i}{2} \varphi^* \circ \varphi = -t i d_E.
\end{cases}
\] (V'_{A_0})

Let $\mathcal{M}_t(E, E_0, A_0)$ be the moduli space of solutions of this system and $\mathcal{M}_{+}^*(E, E_0, A_0)$ the open subspace of irreducible solutions.

We explain now the stability condition which corresponds to this gauge theoretical problem [Bra], [HL], [OT2]:

Let $\tau$ be a real constant with $\deg(E)/\text{rk}(E) > -\tau$. A pair $(\mathcal{E}, \varphi)$ consisting of a holomorphic bundle $\mathcal{E}$ of $C^\infty$-type $E$ and a holomorphic morphism $\mathcal{E} \overset{\varphi}{\rightarrow} \mathcal{E}_0$ is $\tau$-(semi)stable if for every nontrival subsheaf $\mathcal{F} \subset \mathcal{E}$ one has

\[
\mu(\mathcal{E}/\mathcal{F}) \geq -\tau \quad \text{if } \text{rk}(\mathcal{F}) < r,
\]

\[
\mu(\mathcal{F}) \leq -\tau \quad \text{if } \mathcal{F} \subset \ker(\varphi).
\]

Let $\mathcal{M}_{+}^{\tau}(E, E_0)$ be the moduli space of $\tau$-stable holomorphic pairs $(\mathcal{E}, \varphi)$ as above. Then the Kobayashi-Hitchin correspondence states:

**Theorem 4.1** [Bra], [OT2] There is a natural real analytic isomorphism

\[
\mathcal{M}_{+}^*(E, E_0, A_0) \cong \mathcal{M}_{+}^{\tau}(E, E_0),
\]

where $\tau = \frac{\text{Vol}_Y}{2\pi} t$.

The complex geometric moduli spaces $\mathcal{M}_{+}^{\tau}(E, E_0)$ are quite complicated in general. However, if either $r = 1$, or $\tau \gg 0$ they have a beautiful algebraic geometric interpretation in terms of Quot spaces. In general, for a $C^\infty$-bundle $F$ and a holomorphic bundle $\mathcal{F}_0$ over $Y$ we denote by $\text{Quot}_{\mathcal{F}_0}^E$ the Quot space of quotients of $\mathcal{F}_0$ with kernels differentiably isomorphic to $F$.

**Proposition 4.2**

i) Suppose $r = 1$. Then

\[
\mathcal{M}_{+}^{\tau}(E, E_0) = \begin{cases}
0 & \text{if } \tau < -\frac{d}{r} \\
\text{Quot}_{\mathcal{F}_0}^E & \text{if } \tau > -\frac{d}{r}.
\end{cases}
\]

ii) There exists a constant $c(E_0, E)$ such that for all $\tau \geq c(E_0, E)$ one has:

1. For every $\tau$-semistable pair $(\mathcal{E}, \varphi)$, $\varphi$ is injective.
2. Every pair $(\mathcal{E}, \varphi)$ with $\varphi$ injective is $\tau$-stable.
3. There is a natural isomorphism $\mathcal{M}_{+}^{\tau}(E, E_0) = \text{Quot}_{\mathcal{F}_0}^E$.

Combining this result with Theorem 4.1, and taking into account the reducible solutions, one obtains
Corollary 4.3
i) In the abelian case $r = 1$ one has
\[ \mathcal{M}_t(E, E_0, A_0) = \mathcal{M}^*_t(E, E_0, A_0) \text{ for } t \neq -\frac{2\pi}{\text{Vol}_g(Y)} \frac{d}{r}, \]
and a real analytic isomorphism
\[ \mathcal{M}_t(E, E_0, A_0) \simeq \begin{cases} \emptyset & \text{if } t < -\frac{2\pi}{\text{Vol}_g(Y)} \frac{d}{2r} \\ \text{Quot}^E_{E_0} & \text{if } t > -\frac{2\pi}{\text{Vol}_g(Y)} \frac{d}{2r}. \end{cases} \]

ii) For sufficiently large $t \in \mathbb{R}$ one has $\mathcal{M}_t(E, E_0, A_0) = \mathcal{M}^*_t(E, E_0, A_0)$ and a natural identification
\[ \mathcal{M}_t(E, E_0, A_0) \simeq \text{Quot}^E_{E_0}. \]

Remark 4.4 All these results can be generalized to the analogous vortex problems on arbitrary Kähler manifolds [OT2].

Using these results and well-known explicit descriptions of the Quot spaces on Riemann surfaces, we are able to compute explicitly all twisted equivariant gauge theoretical Gromov-Witten invariants $GGW^r_p(\sigma_{\mu_1})$ in the abelian case $r = 1$. Using Remark 2.1, put
\[ GGW^r_p(\sigma_{\mu_1})(l) := \sum_{i=0}^{\infty} GGW^r_p(\sigma_{\mu_1})\left( \left( \text{chern}_1 \right)^i \otimes \left( \text{chern}_1 \right) \right), \]
for every $l \in \Lambda^*(H^1(Y, \mathbb{Z}))$. Only one term in this series can be non-zero if $l$ is homogeneous.

Theorem 4.5 [OT2] Suppose $r = 1$, and let $v$ be the expected dimension of the moduli space, i.e. $v := d_0 - r_0d + (r_0 - 1)(1 - g(Y))$. Let $l_{c_1}$ be the generator of $\Lambda^{2g}(H^1(Y, \mathbb{Z}))$ defined by the complex orientation $c_1$ of $H^1(Y, \mathbb{R}) \simeq H^{0,1}(Y)$. Then
\[ GGW^r_p(\sigma_{\mu_1})(l) = \begin{cases} \sum_{i \geq \max(0, v)} \frac{(r_0^2)^i}{i!} \wedge l , l_{c_1} & \text{if } t > -\frac{2\pi}{\text{Vol}_g(Y)} \frac{d}{r} \\ 0 & \text{if } t < -\frac{2\pi}{\text{Vol}_g(Y)} \frac{d}{r}. \end{cases} \]

Applications:

1. If one admits the adiabatic limit conjecture (see section 2) this result would allow the full computation of all "higher genus" twisted Gromov-Witten invariants of $\mathbb{P}^n$.

Note that, according to some experts, the standard (Kontsevich) higher genus Gromov-Witten invariants of $\mathbb{P}^n$ are not completely understood (see for
instance the talk "Relative Gromov-Witten invariants" by Andreas Gathmann (Princeton), Oberwolfach 2002).

2. Let $\text{Quot}_{E_0}^E$ be a Quot space of dimension zero and vanishing expected dimension $v = rd_0 - r_0d + r(r - r_0)(g(Y) - 1)$. A natural question is

How many points has $\text{Quot}_{E_0}^E$?

This question has a beautiful, simple geometric interpretation: equivalently, one can ask:

How many holomorphic subsheaves $E \hookrightarrow E_0$ of rank $r$ and maximal possible degree $d$ exist, when $E_0$ is general?

Using our computation of the gauge theoretical Gromov-Witten invariants, one gets easily the following answer

$\text{Quot}_{E_0}^E$ has $r_0^{g(Y)}$ points, when $r = 1$ and the multiplicities are taken into account.

Note that Lange [La] obtained earlier the inequality

$$\#(Q) \leq 2^{g(Y)}$$

for $r_0 = 2$ using algebraic geometric methods, whereas Oxbury [Oxb] obtained the equality in the smooth case for arbitrary $r_0$.

Results in the non-abelian case $\text{rk}(E) > 1$ were recently announced by Lange-Newstead [LN]:

Suppose $g(Y) = 2$, $r_0 > 2$ and $d$ is odd. Then $\text{Quot}_{E_0}^E$ has $r_0^2(r_0^2 + 2)$ points, if multiplicities are taken into account.

Note finally that, in a recent preprint, Holla [Ho] announced the computation of all twisted equivariant gauge theoretical Gromov-Witten invariants of this SFAPS, which implies a formula for the length of any zero-dimensional Quot space $\text{Quot}_{E_0}^E$ of vanishing expected dimension.

3. We mention finally another interesting result [OT2] which states that

The full Seiberg-Witten invariants [OT1] of ruled surfaces can be naturally identified with certain gauge theoretical Gromov-Witten invariants of the abelian SFAPS $\sigma_{\mu_1} = (\text{Hom}(\mathbb{C}, \mathbb{C}^r), \alpha_{\text{can}}, S^1, \mu_1)$, so they can be deduced from Theorem 4.5.
2. The SFPAS which yields the flag manifolds

We come back to the SFPAS

\[ \bigoplus_{i=1}^{m} \text{Hom}(V_i, V_{i+1}), \alpha_{\text{can}} \prod_{i=1}^{m} U(V_i), \mu_t \]

which yields the flag manifolds with their natural symmetry.

The gauge theoretical problem associated to this SFPAS is the following:

Fix \((m+1)\) Hermitian bundles \(E_1, \ldots, E_m, E = E_{m+1}\) over a Riemannian surface \((Y, g)\), and fix a Hermitian connection \(A_{m+1} \in A(E)\).

Our moduli space is the space of gauge equivalence classes of systems \(((A_1, \ldots, A_m), (\varphi_1, \ldots, \varphi_m))\), where \(A_i \in A(E_i), \varphi_i \in A^0\text{Hom}(E_i, E_{i+1})\), which solve the equations

\[
\begin{align*}
\partial A_{i+1}(\varphi_i) & = 0 \\
i\Lambda F_{A_i} + \frac{1}{2} (\varphi_{i-1} \circ \varphi_i - \varphi_i \circ \varphi_i) & = -t_i \text{id}_{E_i}, \quad 1 \leq i \leq m .
\end{align*}
\]

Here we put of course \(\varphi_0 := 0\). Two systems \(((A_1, \ldots, A_m), (\varphi_1, \ldots, \varphi_m))\), \(((A'_1, \ldots, A'_m), (\varphi'_1, \ldots, \varphi'_m))\)
are equivalent if they are congruent modulo the gauge group

\[ G = \prod_{i=1}^{m} \text{Aut}(E_i) . \]

The complex geometric classification problem which corresponds to this gauge theoretical problem is the following:

Fix a holomorphic bundle \(\mathcal{E} = \mathcal{E}_{m+1}\) on a complex curve \(Y\). Classify the systems \(((\mathcal{E}_1, \ldots, \mathcal{E}_m), (\varphi_1, \ldots, \varphi_m))\), where \(\mathcal{E}_i\) is a holomorphic bundle of \(\mathcal{C}^\infty\)-type \(E_i\), and \(\varphi_i \in H^0(\mathcal{E}_i^\vee \otimes \mathcal{E}_{i+1})\). Two such systems

\[ ((\mathcal{E}_1, \ldots, \mathcal{E}_m), (\varphi_1, \ldots, \varphi_m)), ((\mathcal{E}_1', \ldots, \mathcal{E}_m'), (\varphi_1', \ldots, \varphi_m')) \]

are equivalent if there exist holomorphic isomorphisms \(f_i : \mathcal{E}_i \to \mathcal{E}_i'\) such that

\[ f_{i+1} \circ \varphi_i = \varphi'_i \circ f_i \quad \text{for} \ i \in \{1, \ldots, m-1\}, \ \varphi_m = \varphi'_m \circ f_m . \]

The \((\mu_t, g)\)-stability condition which corresponds to this complex geometric classification problem follows easily from the universal Kobayashi-Hitchin correspondence for Kählerian SFPAS’s [LT]. This stability condition and applications of the Kobayashi-Hitchin correspondence in this interesting case, will be addressed in a future article.
3. The SFPAS which yields the toric varieties

Let $Y$ be a closed connected oriented real surface. The data of a $[S^1]^r$-bundle $\hat{P}$ on $Y$ is equivalent to the data of a system $L = (L_j)_j$ of $r$ Hermitian line bundles on $Y$.

Fix an integer matrix $v \in M_{m,r}(\mathbb{Z})$ of rank $m$ with the properties $P_1, P_2$ of chapter 1, and let $w : [S^1]^r \to [S^1]^m$ be the associated epimorphism. Put

$$L^0_i := \otimes_{j=1}^r [L^0_j]^{\otimes v^i_j}.$$ 

The gauge theoretical problem associated with the SFAPS $\sigma_{\mu_t} = ((C^r, \alpha_{\text{can}}, K_w, \mu_t), t \in \text{coker}(v \otimes \text{id}_R)^*)$ becomes:

Two such systems are considered equivalent if they are in the same orbit with respect to the natural action of the gauge group

$$\mathcal{G} = \mathcal{C}^\infty(Y, K_w) = \{(f_1, \ldots, f_r) \in \mathcal{C}^\infty(Y, S^1)^r | \prod_{j=1}^r f_j^{v^i_j} = 1 \ \forall i \in \{1, \ldots, m\}\}.$$ 

The complex geometric analoga of these notions are the following

**Definition 4.6** Let $\mathcal{L}^0_i$ be a holomorphic structure on $L^0_i$ for every $i \in \{1, \ldots, m\}$, and put $\mathcal{L}^0 := (\mathcal{L}^0)_{1 \leq i \leq m}$. A holomorphic system of type $(L, v, \mathcal{L}^0)$ is a system

$$((\mathcal{L}_j)_{1 \leq j \leq r}, (\varepsilon_i)_{1 \leq i \leq m}, (\varphi_j)_{1 \leq j \leq r}),$$

where
• $L_j$ is a holomorphic line bundle on $Y$,

• $\varepsilon_i : \bigotimes_{j=1}^r [L_j^i \otimes v'_j] \to L_i^0$ is a holomorphic isomorphism for every $i \in \{1, \ldots, m\}$

and there exist differentiable isomorphisms $L_j \to L_j$ with

$$\varepsilon_i \circ \bigotimes_{j=1}^r [g_j^i \otimes v'_j] = \text{id}_{L_i^0}$$

• $\varphi_j \in H^0(L_j)$.

An isomorphism between two such systems

$$((L_j)_{1 \leq j \leq r}, (\varepsilon_i)_{1 \leq i \leq m}, (\varphi_j)_{1 \leq j \leq r}) \quad Text{and} \quad ((L'_j)_{1 \leq j \leq r}, (\varepsilon'_i)_{1 \leq i \leq m}, (\varphi'_j)_{1 \leq j \leq r})$$

is a system of holomorphic isomorphisms $(u_j)_j, u_j : L_j \to L'_j$ such that $\varphi'_j = u_j(\varphi_j)$ and $\varepsilon'_i \circ [\otimes_j u_j^i \otimes v'_j] = \varepsilon'$.

In order to be able to introduce the corresponding stability condition for our problem, we need some notations: Let $\Sigma$ be a complete simplicial fan with $\Sigma(1)$ vector spaces, we put

$$M := \{\Sigma \to \Sigma \mid \text{is a system of holomorphic isomorphisms}\} \subset \{\text{constant simplicial fans}\}.$$ 

We denote by $M^\text{st}_0(L, v)$ the moduli space of $\Sigma$-stable systems of type $(L, v, L^0)$. With these notations, we can now state the Kobayashi-Hitchin correspondence for our problem:

**Definition 4.7** Let $\Sigma$ be a complete simplicial fan in $\mathbb{R}^m$ such that $\Sigma(1) \subset \{\mathbb{R}_{\geq 0}v^1, \ldots, \mathbb{R}_{\geq 0}v^r\}$.

A system $((L_j)_{1 \leq j \leq r}, (\varepsilon_i)_{1 \leq i \leq m}, (\varphi_j)_{1 \leq j \leq r})$ of type $(L, v, L^0)$ is $\Sigma$-stable if one of the following equivalent conditions is satisfied:

1. There exists a non-empty Zariski open set $Y_0 \subset Y$ such that for every $y \in Y_0$ one has $(\varphi_1(y), \ldots, \varphi_r(y)) \in U(\Sigma, L_y)$.

2. $\varphi \in U(\Sigma, H^0(L))$, where $H^0(L) := (H^0(L_j))_{1 \leq j \leq r}$.

We denote by $\mathcal{M}^\text{st}_0(L, v)$ the moduli space of $\Sigma$-stable systems of type $(L, v, L^0)$. With these notations, we can now state the Kobayashi-Hitchin correspondence for our problem:

**Theorem 4.8** [OT] Let $A^0 = (A^0_i)_i, A^0 \in A(L^0)$ be a fixed system of Hermitian connections and let $L^0$ be the corresponding holomorphic structures. Let $\Sigma$ be a complete simplicial fan with $\Sigma(1) \subset \{\mathbb{R}_{\geq 0}v^1, \ldots, \mathbb{R}_{\geq 0}v^r\}$, and $t \in K_0(\Sigma)$.

Then there is a natural isomorphism of real analytic orbifolds

$$\mathcal{M}_p^L(\sigma_t) \simeq \mathcal{M}^\text{st}_0(L, v).$$
Using this theorem one obtains the following important results [OT3]:

Theorem 4.9 Suppose that \(-t\) is a regular value of the standard moment map \(\mu_0\). Then

1. (complex geometric interpretation) The moduli space \(\mathcal{M}_p^L(\sigma_{\mu_t})\) is a toric fibration over an abelian variety \(P\) of dimension \(g(Y)(r - m)\).

2. (embedding theorem) The moduli space \(\mathcal{M}_p^L(\sigma_{\mu_t})\) can be identified with the vanishing locus of a section \(\sigma\) in a split holomorphic bundle \(E\) over the total space of a locally trivial holomorphic toric fibre bundle \(T\) over \(P\).

In the special case \(Y = \mathbb{P}^1, L_t^0 = \mathcal{O}_{\mathbb{P}^1}\), the complex geometric interpretation was previously obtained and used in [W2], [MP].

Using the ”embedding theorem”, one can endow the moduli space \(\mathcal{M}_p^L(\sigma_{\mu_t})\) with a natural algebraic geometric virtual fundamental class, namely the localized Euler class \([Z(\sigma)]\) of the bundle \(E\) [F]. Roughly speaking, this class – which is an element in the Chow group \(A_{\dim(T) - \rk(E)}(Z(\sigma))\) – is obtained by ”intersecting” the image of \(\sigma\) with the zero section of \(E\) in the (smooth!) total space of this vector bundle.

Theorem 4.10 (comparison theorem) The image of the algebraic geometric virtual fundamental class \([Z(\sigma)]\) of \(Z(\sigma)\) under the cycle map

\[
\text{cl} : A_*(Z(\sigma)) \rightarrow H_*(Z(\sigma))
\]

agrees with the gauge theoretical virtual fundamental class \([\mathcal{M}_p^L(\sigma_{\mu_t})]^\text{vir}\) via the embedding theorem above.

This (rather difficult) result is the first explicit verification of the conjecture stated in section 3.

The computation of the non-twisted gauge theoretical Gromov-Witten invariants in the case \(g(Y) = 0\) was obtained in [CS]. Results in the ”higher genus case” can be found in [Ha], where the comparison Theorem 4.10 plays an important role.

4. The SFPAS of Strømme triples

Now we consider again the SFPAS

\[
\sigma_{\mu_{s,t}} = (\operatorname{Hom}(U, V)^{\oplus 2} \oplus \operatorname{Hom}(W, V), \alpha_{\text{can}}, U(U) \times U(V), \mu_{s,t}),
\]

where

\[
\mu_{s,t}(k, l, m) = \frac{i}{2}(-k^* \circ k + l^* \circ l, -k \circ k^* - l \circ l^* - m \circ m^*) + i(-t\text{id}_U, s\text{id}_V).
\]

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The gauge theoretical problem associated to this SFPAS is the following:

Let $E, F, H_0$ be fixed differentiable Hermitian bundles on $Y$, $A_0$ a fixed Hermitian connection on $H_0$, and $g$ a Riemannian metric on $Y$. We are interested in the moduli space $\mathcal{M}_{A_0,g}(E, F, H_0)$ of equivalence classes of systems $(A, B, (k, l, m))$, where

- $A \in \mathcal{A}(E)$, $B \in \mathcal{A}(F)$,
- $k, l \in A^0\text{Hom}(E, F)$, $m \in A^0\text{Hom}(H, F)$,

such that

- $k, l$ are $\bar{\partial}_{A,B}$-holomorphic, $m$ is $\bar{\partial}_{A_0,B}$-holomorphic,
- the equations
  \[
  \begin{cases}
  i\Lambda F_A - \frac{1}{2}(k^* \circ k + l^* \circ l) = -\text{id}_E \\
  i\Lambda F_B + \frac{1}{2}(k \circ k^* + l \circ l^* + m \circ m^*) = \text{id}_E
  \end{cases}
  \]

are satisfied.

Two such systems are considered equivalent if they are congruent modulo the action of the gauge group

$$G = \text{Aut}(E) \times \text{Aut}(F).$$

The corresponding complex geometric classification problem is the following:

Let $H_0$ be the holomorphic structure on $H_0$ defined by $A_0$. Classify systems $(\mathcal{E}, \mathcal{F}, (k, l, m))$ where

- $\mathcal{E}, \mathcal{F}$ are holomorphic bundles which are differentiably isomorphic with $E$ and $F$ respectively,
- $(k, l) \in H^0(\mathcal{E}^\vee \otimes \mathcal{F})^{\oplus 2}$, $m \in H^0(H_0^\vee \otimes \mathcal{F})$.

Such a system will be called holomorphic systems of type $(E, F, H_0)$. Two holomorphic systems $(\mathcal{E}, \mathcal{F}, (k, l, m))$, $(\mathcal{E}', \mathcal{F}', (k, l, m))$ of type $(E, F, H_0)$ are equivalent if there exist holomorphic isomorphisms $f : \mathcal{E} \to \mathcal{E}'$, $g : \mathcal{F} \to \mathcal{F}'$ such that $k' \circ f = g \circ k$, $l' \circ f = g \circ l$, $g \circ m = m'$.

**Definition 4.11** A holomorphic system $(\mathcal{E}, \mathcal{F}, (k, l, m))$ of type $(E, F, H_0)$ is $(s,t)$-stable if and only if for every pair of subsheaves $\mathcal{E}_1 \subset \mathcal{E}$, $\mathcal{F}_1 \subset \mathcal{F}$ with torsion-free quotients $\mathcal{E}^2 := \mathcal{E}/\mathcal{E}_1$, $\mathcal{F}^2 := \mathcal{F}/\mathcal{F}_1$ one has

1. When $(\mathcal{E}_1, \mathcal{F}_1) \neq (0,0)$ and
   $$k(\mathcal{E}_1) + l(\mathcal{E}_1) \subset \mathcal{F}_1,$$
the following inequality holds
\[
\frac{2\pi}{(n-1)!} \left[ -\deg(\mathcal{E}_1) - \deg(\mathcal{F}_1) \right] + \text{Vol}_g(X) [\text{trk}(\mathcal{F}_1) - \text{trk}(\mathcal{E}_1)] > 0.
\]

2. When \((\mathcal{E}^2, \mathcal{F}^2) \neq (0, 0)\) and
\[
k(\mathcal{E}_1) + l(\mathcal{E}_1) \subset \mathcal{F}_1, \text{ im}(m) \subset \mathcal{F}_1,
\]
the following inequality holds
\[
\frac{2\pi}{(n-1)!} \left[ \deg(\mathcal{E}^2) + \deg(\mathcal{F}^2) \right] + \text{Vol}_g(X) [\text{trk}(\mathcal{E}^2) - \text{srk}(\mathcal{F}^2)] > 0.
\]

There is a moduli space \(\mathcal{M}_{[H_0]}^{(s,t)-st}(E,F)\) classifying stable systems of a fixed type \((E,F,H_0)\). In this case the Kobayashi-Hitchin correspondence for our moduli problem has the following form:

**Theorem 4.12** [LOT] There is a natural isomorphism of real analytic spaces
\[
\mathcal{M}_{A_0,g}(E,F,H_0)^* \simeq \mathcal{M}_{[H_0]}^{(s,t)-st}(E,F).
\]

5. SFPAS’s associated with quiver classification problems

The moduli problem defined by the standard twisted quiver factorization problem (see chapter 1, paragraph 5) and the associated Kobayashi-Hitchin correspondence was studied in [AlPr1], [AlPr2] for arbitrary Kähler manifolds. We recall that this case corresponds to the SFPAS of the form
\[
\left( \bigoplus_{a \in A} \text{Hom}(W_{s(a)}, W_{t(a)} \otimes W_0^a), \alpha_{\rho,r}, \prod_{v \in V} \text{U}(W_v), \mu_t \right),
\]
where \(Q = (V, A, s, t)\) is a quiver, \(W_v\) and \(W_0^a\) are Hermitian vector spaces indexed by \(v \in V\) and \(a \in A\), \(\alpha_{\rho,r}\) is the canonical representation of
\[
\hat{K} := \prod_{v \in V} \text{U}(W_v) \times \prod_{a \in A} \text{U}(W_0^a)
\]
on \(F = \bigoplus_{a \in A} \text{Hom}(W_{s(a)}, W_{t(a)} \otimes W_0^a)\), and \(\mu_t\) is the moment map for the \(K\)-action on \(F\).

The moduli problem corresponding to this SFPAS is the following:

Fix Hermitian bundles \((E_v)_{v \in V}, (E_0^a)_{a \in A}\) on a Riemann surface \((Y, g)\), and fix connections \(A_0^a\) on \(E_0^a\). Classify all systems \(((A_v)_{v \in V}, (\varphi_a)_{a \in A}))\), where

- \(A_v \in \mathcal{A}(E_v)\) is a Hermitian connection,
\[ \varphi_a \in A^0(\text{Hom}(E_{s(a)}, E_{t(a)} \otimes E_0^a)) \]
such that
\[
\begin{cases}
 i\Lambda F_{A_v} + \frac{1}{2} \left( \sum_{t(a)=v} \text{Tr}_{E_0^a}(\varphi_a \circ \varphi_a^*) - \sum_{s(a)=v} (\varphi_a^* \circ \varphi_a) \right) & = 0, \quad \forall a \in A, \\
- \frac{1}{2} t_v \text{id}_{E_v} & = 0 .
\end{cases}
\]

For the corresponding complex geometric classification problem and the Kobayashi-Hitchin correspondence in this case we refer the reader to [AlPr2].

The SFAPS associated with general (twisted) quiver factorization problems is more difficult and leads to interesting moduli problems. The corresponding complex geometric classification problem and the stability condition follows from the universal Kobayashi-Hitchin correspondence for Kählerian SFAPS’s [LT].

References

[AlPr1] Alvarez-Consul, L., Garcia-Prada, O.: Dimensional reduction and quiver bundles, to appear in J. reine angew. Math.

[AlPr2] Alvarez-Consul, L., Garcia-Prada, O.: Hitchin-Kobayashi correspondence, quivers and vortices, to appear in Comm. Math. Phys.

[B] Banfield, D.: Stable pairs and principal bundles, Quart. J. Math. Oxford, 51, 417-436, 2000.

[Bat] Batyrev, V.: Quantum cohomology rings of toric manifolds, Journées de Géométrie Algébrique d’Orsay (Juillet 1992), Astérisque 218, 9-34, Société Mathématique de France, Paris 1993.

[BF] Behrend, K., Fantechi, B.: The intrinsic normal cone, Invent. Math. 128, 1, 45-88, 1997.

[Bra] Bradlow, S., B.: Special metrics and stability for holomorphic bundles with global sections, J. Diff. Geom., 33, 169-214, 1991.

[Bru] Brussee, R.: The canonical class and the $C^\infty$ properties of Kähler surfaces, New York J. Math. 2, 103-146, 1996.

[CMS] Cieliebak, K., Mundet i Riera, I., Salamon, D.: Equivariant moduli problems and the Euler class, to appear in Topology.

[CS] Cieliebak, K., Salamon, D.: Wall for symplectic vortices and quantum cohomology, math.SG/0209170.

[CGMS] Cieliebak, K., Gaio, A. R., Salamon, D., Mundet i Riera, I.: The symplectic vortex equations and invariants of Hamiltonian group actions, math.SG/0111176.
[CGS] Cieliebak, K., Gaio, A. R., Salamon, D.: *J-holomorphic curves, moment maps, and invariants of Hamiltonian group actions*, Internat. Math. Res. Notices no 16, 831-882, 2000.

[DK] Donaldson, S., Kronheimer, P.: *The Geometry of Four-Manifolds*, Oxford Univ. Press, 1990.

[F] Fulton, W.: *Intersection Theory*, Springer Verlag, New York-Berlin-Heidelberg, 1984.

[Ga] Gaio, A. R.: *J-holomorphic curves and moment maps*, Ph. D. Thesis, University of Warwick, November 1999.

[Gi] Givental, A.: *A symplectic fixed point theorem for toric manifolds*, The Floer memorial volume, Progr. Math. 133, Birkhäuser, Basel, 445-481, 1995.

[GS] Gaio, A. R., Salamon, D.: *Gromov-Witten invariants of symplectic quotients and adiabatic limits*, math.SG/0106157.

[Ha] HALIC, M.: *Hamiltonian Gromov-Witten invariants of toric varieties*, in preparation.

[Ho] Holla, Y. I.: *Maximal subbundles and Gromov invariants*, math.AG/0205069.

[HL] Huybrechts, D., Lehn, M.: *Stable pairs on curves and surfaces*, J. Alg. Geom. Vol. 4 , 67-104, 1995.

[K] King, A. D.: *Moduli of representations of finite dimensional algebras*, Quart. J. Math. Oxford, 45, 515-530, 1994.

[Kir] Kirwan, F.: *Cohomology of quotients in symplectic and algebraic geometry*, Princeton Univ. Press, Princeton, NJ, 1984.

[Kr] Kresch, A.: *Cycle groups for Artin stacks*, Invent. Math. 138, no. 3, 495-536, 1999.

[La] Lange, H.: *Höhere Sekantenvarietäten und Vektorbündel auf Kurven*, manuscripta math. Vo. 52, 63-80, 1985.

[LN] Lange, H., Newstead, P. E.: *Maximal subbundles and Gromov-Witten invariants*, math.AG/0204216.

[LOT] Lübke, M., Okonek, Ch., Teleman, A.: *Moduli spaces in gauge theory and algebraic geometry*, in preparation.

[LT] Lübke, M., Teleman, A.: *The universal Kobayashi-Hitchin correspondence for framed pairs*, in preparation.
[MP] Morrison, D., Plesser, M.: *Summing the instantons: quantum cohomology and mirror symmetry in toric varieties*, Nuclear Phys. B, 440, no 1-2, 279-354, 1995.

[Mu1] Mundet i Riera, I.: *A Hitchin–Kobayashi correspondence for Kaehler fibrations*, J. Reine Angew. Math., 528, 41-80, 2000.

[Mu2] Mundet i Riera, I.: *Hamiltonian Gromov–Witten invariants*, math.SG/0002121.

[OT1] Okonek, Ch., Teleman, A.: *Seiberg-Witten invariants for manifolds with \(b_+ = 1\), and the universal wall crossing formula*, Int. J. Math. Vol. 7, No. 6, 811-832, 1996.

[OT2] Okonek, Ch., Teleman, A.: *Gauge theoretical equivariant Gromov-Witten invariants and the full Seiberg-Witten invariants of ruled surfaces*, in Commun. Math. Phys.

[OT3] Okonek, Ch., Teleman, A.: *Comparing virtual fundamental classes. Gauge theoretical Gromov-Witten invariants for toric varieties.*, math.DG/0205137.

[Oxb] Oxbury, W. M.: *Varieties of maximal line subbundles*, Math. Proc. Cambridge Phil. Soc. Vol. 129, no. 1, 9-18, 2000.

[R] Ruan, Y.: *Topological sigma model and Donaldson-type invariants in Gromov theory*, Duke Math. J., Vol. 83, no. 2, 461-500, 1996.

[S] Stroème, S. A.: *On parametrized rational curves in Grassmann varieties*, in ”Space Curves” (Rocca di Papa, 1985), 251-272, Lecture Notes in Math., 1266, Springer Verlag, 1987.

[W1] Witten, E.: *Monopoles and four-manifolds*, Math. Res. Letters 1, 769-796, 1994.

[W2] Witten, E.: *Phases of \(N = 2\) theories in two dimensions*, Nuclear Physics B, 403, no 1-2, 159-222, 1993.

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