Properties of Naked Black Holes

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Abstract

We investigate the properties of a class of near-extreme static black hole solutions called naked black holes. These black holes, which occur in string theory, have small curvature invariants but large tidal forces outside their event horizons. We show that these large tidal forces are due to a concentration of dilaton stress-energy near the horizon. We study infalling test strings and find that they are highly excited by the large tidal forces, but remain small. Perhaps most importantly, it turns out that a small amount of infalling matter will cause the curvature invariants to become large outside the horizon. Nevertheless, an exact calculation shows that both the matter trajectories and the classical black hole background are not significantly altered.

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1 Introduction

We recently showed [1] that there are black holes in which the area of the event horizon is large, and all curvature invariants are small near the horizon. Nevertheless, any object which falls in experiences enormous tidal forces outside the horizon. The tidal forces are given by the components of the Riemann tensor in a frame associated with the ingoing geodesics. Thus, for these black holes, the geodesic components are much larger than the invariants constructed from them. This implies that the curvature is nearly null. Although this is not the case for the familiar Reissner-Nordström black hole, we found that it does occur in a wide variety of theories involving scalar fields, including the low energy limit of string theory. The examples were all charged black holes either at or near extremality. These black holes were dubbed naked black holes, as they violate the spirit of cosmic censorship, in that large curvatures are visible outside of the horizon.

In this paper, we investigate the properties of these black holes in more detail. We begin by trying to gain more insight into the nature of the large curvatures seen by geodesic observers. We will see that the size of the components of the Riemann tensor in the geodesic frame is determined by the proper time remaining along the geodesic before the singularity is reached. That is, the relevant length scale for the geodesic curvature is this proper time, just as the relevant length scale for the components in the more familiar static frame is set by the area of the event horizon. For the geodesics we consider, this proper time will only be small for near-extreme black holes. This makes it easier to understand why these black holes must always be near-extreme.

We also show that the large curvature has a matter source. That is, the large contributions to the Riemann tensor come from the Ricci tensor, and hence from the stress-energy of the matter fields. In our examples, we find that it is the dilaton fields’ stress-energy which provides the source. In fact, since the geometry is spherically symmetric, only the dilaton field can have the nearly null stress-energy needed to produce the type of curvature we find. Hence, we would only expect to find black holes of this type in theories with scalar fields.

We then go on to consider the effects on infalling matter. If ordinary matter falls into one of these black holes, it is crushed in the transverse directions. We study the effects on an infalling test string. Considering the portion of the region of large curvatures that an infalling string passes through, we find that it can be approximated by a dilaton plane wave (under the assumption that the string remains small compared to the horizon). This greatly simplifies the calculation, as the propagation of strings through such plane waves has been studied previously [2, 3]. We find that the string becomes very excited before it crosses the horizon. Thus its average mass becomes large. However, the average size remains small, as the modes are all oscillating very quickly near the horizon. In this context, we also consider the effect on outgoing strings which form part of the Hawking radiation.
Perhaps the most important effect of infalling matter is seen when we go beyond the
test particle (string) approximation and include its stress-energy. It then turns out that
a small amount of matter can produce Planck-scale invariants outside the horizon. This
can be viewed as the result of a high energy collision between the infalling matter and
the background dilaton wave. Surprisingly, by carrying out an exact calculation of the
collapse of a charged dust shell, we find that these large invariants do not significantly
change the classical trajectory of the infalling matter. Furthermore, after the shell passes,
the spacetime is simply a charged black hole with slightly larger charge and mass. So
the large curvature invariants have little effect on the classical evolution. In fact we will
see that there is a sense in which certain spacetimes with large curvature invariants are
“close” to spacetimes with small invariants. The main consequence of the Planck scale
curvature invariants is that quantum effects will become important outside the horizon
when matter falls in.

In the next section, we review the results of [1], discussing two examples which will
be used later in this paper. In section 3, we discuss the relation between the tidal forces
and the proper time remaining before the observers reach the singularity. We also show
that the large curvature is associated with a large dilaton stress tensor. In section 4,
we consider infalling matter, and show that the geometry as seen by small infalling
observers is approximately that of a dilaton plane wave. We go on to show that infalling
test strings will be highly excited. In section 5, we consider the stress energy due to
infalling matter, and show that it can lead to large invariants. We then demonstrate
that these do not imply large changes in either the classical solution or the motion of
the matter. A brief discussion is contained in section 6.

2 Review of naked black holes

In [1], we showed that extremal or near-extremal limits of several familiar black hole
solutions arising in general relativity with scalar matter fields or in string theory have
large tidal forces outside large event horizons. We will focus here on four dimensional
black holes with metrics of the form

\[ ds^2 = -\frac{F(r)}{G(r)} dt^2 + \frac{dr^2}{F(r)} + R^2(r) d\Omega. \]  

This metric will have a horizon at \( r = r_+ \) if \( F(r_+) = 0 \). The usual static frame is

\[
\begin{align*}
(e_0)_\mu &= -F^{1/2}(r)G^{-1/2}(r) \partial_\mu t, \\
(e_1)_\mu &= F^{-1/2}(r) \partial_\mu r, \\
(e_2)_\mu &= R(r) \partial_\mu \theta, \\
(e_3)_\mu &= R(r) \sin \theta \partial_\mu \phi.
\end{align*}
\]  

We want to compare the curvature components in this frame with the physical tidal forces
felt by infalling observers. Thus, we consider a frame which is parallelly propagated along
radially infalling geodesics. The geodesics have tangent vector \( u = (\dot{t}, \dot{r}, 0, 0) \), where a dot denotes \( d/d\tau \). There is a conserved energy per unit mass \( E = \dot{t}F(r)/G(r) \), and

\[
\dot{r}^2 = E^2G(r) - F(r).
\]  

(2.3)

We will always assume that \( E \) is of order one, i.e., we will consider geodesics that start at infinity with small velocity. The parallelly propagated orthonormal frame, in which \((e_0')_\mu = u_\mu\), is then related to the static frame by a radial boost,

\[
(e_0')_\mu = u_\mu = -E\partial_\mu t + \frac{\dot{r}}{F(r)}\partial_\mu r
\]

\[
= \cosh \alpha (e_0)_\mu + \sinh \alpha (e_1)_\mu,
\]

(2.4)

and

\[
(e_1')_\mu = \sinh \alpha (e_0)_\mu + \cosh \alpha (e_1)_\mu,
\]

(2.5)

where \( \cosh \alpha = E[G(r)/F(r)]^{1/2} \). Note that since the horizon lies at \( F(r) = 0 \), the boost parameter \( \alpha \) diverges as we approach the horizon. The curvature components \( R_{\theta k\theta' k'} \), \( k = 2, 3 \), can be calculated by using the geodesic deviation equation, with the result

\[
R_{\theta k\theta' k'} = -\frac{\ddot{R}}{R}.
\]

(2.6)

Although the boost parameter diverges at the horizon, these curvature components will generally be finite. However, in certain cases they can be much larger than the curvature components in the static frame.

The dilaton black holes discussed in [4, 5, 6] are the simplest examples. These are solutions of a theory with a Maxwell field \( F_{\mu\nu} \) and a scalar field \( \phi \) with the coupling between the Maxwell field and the scalar field governed by an arbitrary constant \( a \). The action is

\[
S = \int d^4x\sqrt{-g} \left[ R - 2(\nabla \phi)^2 - e^{-2a\phi}F_{\mu\nu}F^{\mu\nu} \right],
\]

(2.7)

and the metric for a dilaton black hole is given by

\[
ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + R^2(r)d\Omega,
\]

(2.8)

with

\[
F(r) = \frac{(r - r_+)(r - r_-)}{R^2}
\]

(2.9)

and

\[
R(r) = r \left( 1 - \frac{r_-}{r} \right)^{a^2/(1+a^2)}.
\]

(2.10)
The dilaton is given by

\[ e^{-2\phi} = \left(1 - \frac{r_{-}}{r}\right)^{2a/(1+a^2)} \]  

(2.11)

if the solution carries a magnetic charge. There is a horizon at \( r = r_{+} \) and a singularity at \( r = r_{-} \) for \( a \neq 0 \). For \( a = 0 \), this metric reduces to the Reissner-Nordström metric; \( r = r_{-} \) is an inner horizon, and there is a singularity at \( r = 0 \). The extremal limit in both cases is \( r_{+} = r_{-} \). The ADM mass and charge are

\[ M = \frac{r_{+}}{2} + \left(\frac{1 - a^2}{1 + a^2}\right) \frac{r_{-}}{2}, \]

(2.12)

\[ Q = \left(\frac{r_{+}r_{-}}{1 + a^2}\right)^{1/2}. \]

(2.13)

The Hawking temperature for these black holes is

\[ T = \frac{1}{4\pi} \frac{(r_{+} - r_{-})^{1/2}}{r_{+}^{1+a^2}}. \]

(2.14)

The horizon area will be large and the static curvature will be small (in Planck units) if

\[ R(r_{+}) = r_{+}\epsilon^{2/(1+a^2)} \gg 1, \]

(2.15)

where \( \epsilon \equiv (1 - r_{-}/r_{+}) \). Note that the exponent of \( \epsilon \) is always less than one. The curvature in a freely falling frame is given by

\[ R_{\nu\nu} = -\frac{\ddot{R}}{R} = -\frac{1}{R} \left[ R''(E^2 - F) - \frac{F'R'}{2} \right]. \]

(2.16)

Near the horizon, \( F(r) \) is small, so this will be larger than the Planck scale if

\[ \left| \frac{R''}{R} \right| = \frac{a^2}{(1 + a^2)^2} \frac{(1 - \epsilon)^2}{r_{+}^2 \epsilon^2} > 1. \]

(2.17)

This will be satisfied, for \( a \neq 0 \), if \( r_{+}\epsilon \ll 1 \). Thus we see that there is a range of parameters for which the curvature in the static frame is small, but infalling observers experience large tidal forces near the horizon, namely \( \epsilon \ll 1 \) and \( \epsilon^{-a^2/(1+a^2)} \ll r_{+} \ll \epsilon^{-1} \).

Since \( \epsilon \) is small, these black holes are all close to extremality, and since \( r_{+} \) is large, they have a large mass. For fixed mass, the area of the event horizon goes to zero in the extremal limit. The spacetime develops a null singularity if \( 0 < a \leq 1 \) and a timelike singularity if \( a > 1 \). We are considering a different limit, in which the mass is increased as one approaches extremality, so the horizon area remains large. Note that

\[ (r_{+} - r_{-}) = 4\pi T R(r_{+})^2. \]

(2.18)
Thus, if we take \( r_+ \to r_- \) while keeping the horizon area fixed, the Hawking temperature will go to zero.

In the context of string theory, the black hole solution with electric Neveu-Schwarz charges associated with internal momentum and string winding number has similar behavior. The string metric is 

\[
s^2 = -\Delta^{-1} \left( 1 - \frac{r_0}{r} \right) dt^2 + \left( 1 - \frac{r_0}{r} \right)^{-1} dr^2 + r^2 d\Omega,
\]

where

\[
\Delta = \left( 1 + \frac{r_0 \sinh^2 \gamma_1}{r} \right) \left( 1 + \frac{r_0 \sinh^2 \gamma_p}{r} \right),
\]

and the dilaton is given by

\[
e^{2\phi} = \Delta^{-1/2}.
\]

The ADM mass of these black holes is

\[
M = \frac{r_0 RV}{g^2} (2 + \cosh 2\gamma_1 + \cosh 2\gamma_p),
\]

and the integer normalized charges are

\[
n = \frac{R^2 V}{g^2} r_0 \sinh 2\gamma_p, \quad m = \frac{V}{g^2} r_0 \sinh 2\gamma_1,
\]

where \( R \) is the radius of a compact internal direction, and \((2\pi)^5 V\) is the volume of an internal five-torus. (We are using the same conventions as \[8\] and have set \( \alpha' = 1 \).)

The curvature in the static frame is of order \( 1/r_0^2 \) at the horizon \( r = r_0 \), so we must take \( r_0 \gg 1 \) to keep it small. The curvature in the infalling frame at \( r = r_0 \) is

\[
R_0^{\nu\nu} = -\frac{\dot{R}}{R} = -\frac{R'}{2R} \left( G' E^2 - F' \right) = -\frac{E^2 \Delta'}{2r_0} + \frac{1}{2r_0^2}.
\]

Since

\[
\Delta'(r_0) = -\frac{1}{r_0} (\sinh^2 \gamma_1 \cosh^2 \gamma_p + \cosh^2 \gamma_1 \sinh^2 \gamma_p),
\]

we can make the tidal forces arbitrarily large by increasing \( \gamma_1 \) or \( \gamma_p \). Physically, this just corresponds to increasing the mass and charges.

The extremal limit for this class of black holes is \( r_0 \to 0, \gamma_1, \gamma_p \to \infty \) with \( n, m \) fixed. It may appear that the large tidal forces are present far from the extremal limit, since we have taken \( r_0 \gg 1 \). However, for fixed charges, the mass above extremality is

\[
\Delta M = M - M_{\text{ext}} \approx \frac{2r_0 RV}{g^2}.
\]
When \( r_0 \gg 1 \), \( \Delta M \) is large, but \( \Delta M/M \) is still small, since \( \gamma_1 \) or \( \gamma_p \) is large.

The Einstein metric is obtained by multiplying (2.19) by \( e^{-2\phi} \). The area of the horizon is thus increased by the factor \( \cosh \gamma_1 \cosh \gamma_p \) and the static frame curvature near the horizon is decreased by the same factor. If one takes \( r_0 \) small and \( \gamma_1, \gamma_p \) sufficiently large, then both the size of the black hole and the tidal forces in the Einstein metric will be large. This example is closely related to the dilaton black hole case. If \( \gamma_1 = \gamma_p \), the Einstein metric is the same as the dilaton black hole with \( a = 1 \). If \( \gamma_1 = 0 \) or \( \gamma_p = 0 \), the metric is the same as the dilaton black hole with \( a = \sqrt{3} \).

3 Source of the large curvature

In this section, we will discuss the origin of the large geodesic curvatures found in naked black holes. One of the first questions one might ask about these curvatures is what sets their size. That is, in the dilaton black hole spacetime, there is a natural large distance scale, namely the radius \( R(r_+) \) of the black hole horizon. This scale determines the size of the curvature in the static frame. Can we identify a short distance scale that determines the size of the curvature components in the geodesic frame? For the simpler examples, the answer is yes. The relevant scale is the proper time remaining along the geodesics before the observers reach the singularity.

Consider the dilaton black hole metrics. Near the horizon, \( F(r) \) is negligibly small, so we can see from (2.3) that \( \dot{r}^2 \approx E^2 \), and hence

\[
\tau \approx \frac{(r - r_-)}{E},
\]

where we have chosen the sign to be positive, so that \( \tau \) corresponds to the ‘time remaining’, and decreases as we follow the geodesic into the black hole. Note that although we have chosen the constant of integration so that the left-hand side vanishes at the singularity, this expression is not valid for \( r \) near \( r_- \) if \( a^2 > 1 \), as in that case \( F(r) \) will eventually diverge as we approach the singularity. Thus, in that case the actual proper time remaining will be somewhat shorter. For \( a^2 \leq 1 \), the expression is valid up to the singularity.

Now consider the function \( R(r) \) in (2.10). At the horizon,

\[
R(r_+) = r_+^{1+a^2} (r_+ - r_-)^{a^2}. \tag{3.2}
\]

Since we are considering the near-extreme case where \( r_+ - r_- \ll 1 \), if we move slightly away from the horizon, the second term will vary much more quickly than the first term.

\footnote{This has also been noticed by B. Hiscock and S. Larsen (private communication). For the Reissner-Nordström metric, timelike geodesics never hit the singularity, but that is not true for the black holes considered here.}
Thus

\[ R(r) \approx r_+^{1+a^2} (r - r_-)^{a^2} \propto \tau^{a^2} \]  

(3.3)

in a neighborhood of the horizon, and hence the curvature is

\[ R_{0'20'2} = -\frac{\ddot{R}}{R} \approx \frac{a^2}{(1 + a^2)^2} \frac{1}{\tau^2}. \]  

(3.4)

Thus, near the horizon, the size of the curvature in the geodesic frame is set by the proper time remaining to the singularity. The fact that the factor in front of $1/\tau^2$ in this expression is independent of the black hole’s mass and charge indicates that this is a useful way to characterise the curvature.

For the Neveu-Schwarz black holes, the situation is somewhat more complicated, as we have one more parameter. If both of the charges are large, we find that $R_{0'k0'k} = \frac{1}{4\tau^2}$, and if one of the charges is zero, $R_{0'k0'k} = \frac{2}{9\tau^2}$. If both charges are non-zero but one is much smaller than the other, we cannot give a simple form for the curvature that is applicable throughout the region of large curvature. However, it is still true that the proper time remaining along the geodesics is short, and hence this may still be the relevant distance scale.

It also worth pointing out that the simple expression obtained above applies only to the large curvatures experienced near the horizon by geodesic observers falling from rest at infinity. In the dilaton black hole, even if we are far from the black hole, we can always find geodesic observers (with sufficiently large $E$) who see large curvatures. However, the analogue of (3.4) for such an observer far from the black hole is $R_{0'k0'k} \propto r^2$, where $r$ is the radial distance to the black hole. So while we can still make the curvature large at any $r$ by taking $\tau$ small enough, $\tau$ doesn’t set its scale.

The fact that the proper time remaining sets the scale of the curvature makes it easier to understand why such large curvatures have only been found for near-extreme black holes. It is only for such black holes that the proper time remaining at the horizon is short for observers who have fallen in from infinity. This makes it seem unlikely that more general black hole solutions will exhibit similar behaviour, even once we include quantum corrections.

Although it is useful to identify the proper time remaining as the relevant length scale, we still need to ask what the source of the large curvatures is. One can decompose the Riemann tensor in terms of the Weyl tensor and the Ricci tensor. For static spherically symmetric black hole solutions, we now show that the Weyl tensor is always invariant under radial boosts. Since the difference between the static frame and the infalling frame is just such a boost, this shows that the difference in curvature must come from the Ricci tensor, and hence the matter fields.

In four spacetime dimensions, one can show that the Weyl tensor is boost invariant by using the fact that every static spherically symmetric spacetime is algebraically special.
of type D \[. The repeated principal null directions are just the radial ingoing and outgoing null vectors. In two component spinor notation, this means that the Weyl spinor can be expressed as
\[
\Psi_{ABCD} = \Psi_{o(AoBt)}^{(A\sigma_1)(B\sigma_2)}.
\]
Since \( o_A \to \lambda o_A \), \( \iota_A \to \lambda^{-1} \iota_A \) under a boost, the Weyl spinor is clearly invariant, and hence so is the Weyl tensor.

In \( d \) spacetime dimensions, one can establish the same result as follows. The Weyl tensor can be written as
\[
C_{\mu\nu;\rho\sigma} = R_{\mu\nu;\rho\sigma} - \frac{2}{d-2}(g_{\mu[\rho}R_{\sigma]\nu] - g_{\nu[\rho}R_{\sigma]\mu]) + \frac{2}{(d-1)(d-2)}Rg_{\mu[\rho}g_{\sigma]\nu].
\]
Under a radial boost,
\[
R_{\theta'\theta'\theta'\theta'} = R_{0101}, \quad R_{\theta'\phi'\phi'\phi'} = \cosh \alpha \sinh \alpha (R_{0\theta\theta} + R_{1\phi\phi}),
\]
\[
R_{\theta'\theta'\phi'\phi'} = R_{0\theta\theta} + \sinh^2 \alpha (R_{0\phi\phi} + R_{1\phi\phi}),
\]
\[
R_{\phi'\phi'\phi'\phi'} = R_{1\phi\phi} + \sinh^2 \alpha (R_{0\phi\phi} + R_{1\phi\phi}),
\]
and \( R_{klkl} \) is unchanged, where \( \alpha \) is the boost parameter. It follows that
\[
R_{\theta'\theta'} = R_{\theta'\theta'} + \sum_k R_{\theta'\theta'\phi'\phi'} = R_{00} + \sinh^2 \alpha \sum_k (R_{0\theta\theta} + R_{1\phi\phi})
\]
and
\[
R_{\phi'\phi'} = -R_{\theta'\theta'} + \sum_k R_{\phi'\phi'\phi'\phi'} = R_{11} + \sinh^2 \alpha \sum_k (R_{0\phi\phi} + R_{1\phi\phi})
\]
and that \( R_{kk} \) and \( R_{11} - R_{00} \) are boost invariant. It is then trivial to check that \( C_{0101} \) and \( C_{kklk} \) are boost invariant. We have
\[
C_{\theta'\theta'\phi'\phi'} = R_{\theta'\theta'\phi'\phi'} - \frac{1}{d-2}(-R_{kk} + R_{\theta'\theta'}) - \frac{2}{(d-1)(d-2)}R
\]
\[
= C_{0000} + \sinh^2 \alpha \left[ (R_{0\theta\theta} + R_{1\phi\phi}) - \frac{1}{d-2} \sum_l (R_{0\theta\theta} + R_{1\phi\phi}) \right],
\]
and similar expressions for \( C_{1\phi'\phi'\phi'} \) and \( C_{\phi'\phi'\phi'\phi'} \). Since \( l \) in the summation runs over \( d - 2 \) values, the Weyl tensor will be boost-invariant if all the \( R_{0\theta\theta} + R_{1\phi\phi} \) are the same.

For a spherically symmetric black hole solution, \( l \) just runs over the angular variables, and the spherical symmetry thus implies that all the terms are the same. Thus, for both the dilaton and Neveu-Schwarz examples, the Weyl tensor is boost-invariant. The large value of the Riemann tensor in the geodesic frame is entirely due to the Ricci tensor.

\[3\]This is not the case for the black \( p \)-brane solutions.
The Ricci tensor is determined by the stress-energy tensor of the matter fields. Thus, the large curvatures are not mysterious at all; they are simply the consequence of a large matter source near the horizon. For a radial electric field, \( F \propto (e_0) \wedge (e_1) \), while for a radial magnetic field, \( F \propto \epsilon \), where \( \epsilon \) is the volume element on \( S^2 \), and both of these are unchanged by radial boosts. Any static spherically symmetric electromagnetic field \( F_{\mu\nu} \), and hence the stress-energy tensor associated with it, is therefore always boost-invariant. Thus, the large stress-energy must be associated with the dilaton field. This shows that black holes of this type will only occur in theories with a scalar field.

The curvature is becoming large and nearly null near the horizon. We therefore see that the source of the large geodesic curvatures is a large, nearly null dilaton stress tensor near the horizon. This is closely analogous to the dilaton plane waves studied in \[2\]. The symmetries are also similar, since the timelike Killing field is becoming null near the horizon, and the spherical symmetry implies invariance in the transverse directions. In fact, as we will see in the next section, the region of the geometry explored by small infalling observers is well approximated by such a plane wave.

To conclude this section, consider the effects of large tidal forces on some classical object falling into the black hole. Once the object is in the region where the tidal forces are large, they will dominate over any internal stresses in the body (since such stresses can reasonably be assumed to be very small compared to the Planck scale). Thus, the evolution of the object is well-approximated by regarding it as a cloud of dust, and is characterised by how much this dust cloud is distorted in passing through the region of large tidal forces, up to the horizon. Since the solutions are spherically symmetric, the distortion will be just a uniform shrinking in the transverse directions. In other words, the object is crushed. The amount of shrinking is simply given by the change in \( R(r) \) as we pass through the region of large tidal forces. In our examples, this is always finite, but not necessarily small.

4 Effects on test strings

In this section, we will consider the effects of the large tidal forces on infalling test strings. As we have argued above, the spacetime in a neighborhood of the horizon is closely analogous to one of the plane wave spacetimes studied in \[2\]. We now show explicitly that if we consider the part of the spacetime traversed by an observer of small spatial extent whose center of mass follows a geodesic in from infinity, it can in fact be approximated by one of these plane waves in the region of large curvatures outside the horizon. Since we are interested in the effect on infalling strings, we use the string metric describing a Neveu-Schwarz charged black hole. We will assume that both the winding and momentum charges are large, as this is the simplest case.

First we need to introduce Kruskal-like coordinates for the black hole solution. We
define a tortoise coordinate \( r_* \) such that

\[
dr_* = \Delta^{1/2} \left( 1 - \frac{r_0}{r} \right)^{-1} dr.
\]  

(4.1)

We then define the Kruskal-like coordinates \( U = -e^{-\kappa(t-r_*)}, V = e^{\kappa(t+r_*)} \), where \( \kappa \) is the surface gravity of the black hole. The metric in terms of these new coordinates is

\[
ds^2 = \frac{(1 - \frac{r_0}{r})}{\Delta \kappa^2 UV} dU dV + \frac{1}{r^2} d\Omega,
\]  

(4.2)

where \( r \) is given in terms of \( U \) and \( V \) by

\[
UV \approx -\left( \frac{r - r_0}{r_0} \right).
\]  

(4.6)

For radial geodesics falling into the black hole from infinity, \([2,3]\) and \( \dot{t} = -E\Delta/(1 - r_0/r) \) imply

\[
\dot{r}_* = \Delta^{1/2} \left( \frac{\dot{r}}{1 - \frac{r_0}{r}} \right) = \frac{\Delta^{1/2}}{(1 - \frac{r_0}{r})} \left[ E^2 \Delta - \left( 1 - \frac{r_0}{r} \right) \right]^{1/2} \approx -\dot{t} \left[ 1 - \left( 1 - \frac{r_0}{r} \right) \frac{1}{2E^2 \Delta} \right],
\]  

(4.7)

where the approximation is again valid for the region of large curvatures, and an overdot denotes the derivative with respect to the proper time remaining before the singularity is reached. It follows that

\[
\dot{V} = V \kappa (\dot{t} + \dot{r}_*) \approx V \kappa \dot{t} \left( 1 - \frac{r_0}{r} \right) \frac{1}{2E^2 \Delta} = -\frac{V \kappa}{2E},
\]  

(4.8)
so $|\dot{V}| \ll V$, as $\kappa \ll 1$. Thus, infalling observers who follow the geodesics have $V$ nearly constant in the region of large curvatures. Note also that

$$\ddot{U} = -U\kappa(\dot{t} - \dot{r}_*) \approx -U2\kappa \dot{t}. \quad (4.9)$$

We now set

$$V = V_0 \left(1 - \frac{v_1}{2\nu_0^2}\right), \quad (4.10)$$

and ignore the $v_1$ dependence in the metric. We can then write

$$r \approx r_0(1 - UV_0), \quad (4.11)$$

and

$$ds^2 \approx 2(1 - UV_0)V_0dUdv_1 + (1 - UV_0)^2\nu_0^2d\Omega. \quad (4.12)$$

It is convenient to define $u = (1 - UV_0)^2$, so the horizon corresponds to $u = 1$. Since the event horizon is large, and the infalling observer is assumed to be small, we can also approximate the two-sphere metric $r_0^2d\Omega$ by a flat metric. We will use $X_1$ and $X_2$ to denote these flat directions. We can now rewrite the metric as

$$ds^2 \approx -dudv_1 + udX_iX^i. \quad (4.13)$$

where $i = 1, 2$. This is a plane wave metric. To bring it into the form used in [2] we use a change of coordinates discussed in [10],

$$v = v_1 + \frac{1}{2}X_iX^i, \quad (4.14)$$

$$x_i = \sqrt{u}X_i. \quad (4.15)$$

Then the metric is

$$ds^2 \approx -dudv + dx_idx^i + W(u)x_ix^idu^2, \quad (4.16)$$

where

$$W(u) = -\frac{1}{4u^2}. \quad (4.17)$$

Since the horizon corresponds to $u = 1$, it might seem from this form of the metric that the tidal forces are not large there. However, an observer who falls into the black hole with energy $E$ at infinity has

$$P \equiv \dot{u} = -2(1 - UV_0)V_0\dot{U} \approx 4(1 - UV_0)UV_0\kappa \dot{t} \approx 4\frac{r_0^2}{\nu_0^2}\kappa\Delta E \approx 2\frac{\sinh \gamma_1 \sinh \gamma_2}{r_0}E \gg 1. \quad (4.18)$$
Thus, these observers experience large tidal forces, as they are passing through the wave very quickly. This plane wave approximation is valid throughout the region of large curvatures outside the horizon, which is presumably the region in which interesting effects on strings will be produced. One can think of the singularity as being at $u = 0$, although the plane wave approximation is no longer valid there, as we cannot treat the two-sphere as flat once we are sufficiently close to the singularity.

The propagation of first quantized strings through plane waves was discussed in detail in [2], and we will use the same approach and conventions. However, there is one modification. In [2], the main interest was in considering the effects of spacetime singularities on strings, so the spacetime considered consisted of a non-trivial wave with flat spacetime regions before and after it. In our case, we are interested in the behaviour of the string in the region near the horizon, so it is not appropriate to match onto a flat region beyond the wave. However, to interpret the results, we need a static region ‘after’ the wave where we can define positive and negative frequencies. We will therefore consider a spacetime of the form (4.16), with $W$ given by (4.17) up to the horizon $u = 1$, and then match onto $W = -1/4$ in the region $u < 1$. Even though the resulting metric component $W$ is only continuous and not differentiable, there is no induced stress-energy along the matching surface. This is because a metric of the form (4.16) has $R_{\mu\nu} = -W \partial_\mu u \partial_\nu u$.

Since the spacetime is a plane wave, we can choose the light-cone gauge for the string. That is, we choose coordinates $(\sigma, \tau)$ on the worldsheet so that $u = P \tau$. We assume $P \gg 1$ since an unexcited string which falls in from rest at infinity will follow an approximate geodesic until it reaches the region of large tidal forces. The dynamical fields are $x^i(\sigma, \tau)$, $i = 1, 2$. If we decompose the $x^i$ into modes,

$$x^i(\sigma, \tau) = \sum_n x^i_n(\tau)e^{in\sigma},$$

then the worldsheet field equation for $x^i$ becomes

$$\ddot{x}^i_n + n^2 x^i_n - WP^2 x^i_n = 0,$$

where a dot denotes differentiation with respect to $\tau$. Since we have modified the metric at the horizon $u = 1$ ($\tau = 1/P$),

$$WP^2 = -\frac{1}{4\tau^2} \text{ for } \tau > \frac{1}{P},$$

$$WP^2 = -\frac{P^2}{4} \text{ for } \tau < \frac{1}{P}.$$  

Recall that $\tau$ denotes the time remaining to the singularity. So $\tau$ decreases as the string falls in.
The component \( v(\sigma, \tau) \) is determined by

\[
P\dot{v} = (\dot{x}_i)^2 + (x_i')^2 + WP^2x^2, \tag{4.23}
\]

\[
Pv' = 2\dot{x}_i x_i', \tag{4.24}
\]

where a prime denotes differentiation with respect to \( \sigma \).

We can write the solutions of the mode equation (4.20) in terms of a complete set of solutions which are pure positive and negative frequency asymptotically. That is,

\[
x_i^n = i(a_i^n u_n - \bar{a}_i^n \bar{u}_n), \tag{4.25}
\]

where \( u_n \) and \( \bar{u}_n \) are solutions of (4.20) and

\[
u_n \to \frac{1}{2\sqrt{n}}e^{-in\tau}, \quad \bar{u}_n \to \frac{1}{2\sqrt{n}}e^{in\tau} \tag{4.26}
\]

when \( \tau \to \infty \). Similarly, we can write the solution in terms of a set of solutions which are positive and negative frequency in the region \( \tau < 1/P \). That is,

\[
x_i^n = i(b_i^n v_n - \bar{b}_i^n \bar{v}_n), \tag{4.27}
\]

where \( v_n \) and \( \bar{v}_n \) are solutions of (4.20) which are

\[
v_n = \frac{1}{2\sqrt{n}}e^{-in\tau}, \quad \bar{v}_n = \frac{1}{2\sqrt{n}}e^{in\tau} \tag{4.28}
\]

for \( \tau < 1/P \), where

\[
n^2 = n^2 + \frac{P^2}{4} \tag{4.29}
\]

Since both the \( u \)'s and the \( v \)'s are complete sets of states, it must be possible to write them in terms of each other. This implies a linear transformation between the initial and final mode creation and annihilation operators, the Bogoliubov transformation

\[
b_i^n = A_n a_i^n - B_n^* \bar{a}_i^n, \quad \bar{b}_i^n = A_n \bar{a}_i^n - B_n a_i^n. \tag{4.30}
\]

The mass-squared and average size of a string initially in the ground state are related to the Bogoliubov coefficients \( B_n \).

A string in the static region \( \tau < 1/P \) is similar to one in flat spacetime except that the gravitational field has shifted the frequency of the \( n^{th} \) mode from \( n \) to \( \bar{n} \) (4.29). In particular, the mass of the string is given by

\[
M_s^2 = 8 \sum_{n=1}^{\infty} \bar{n}b_i^n \bar{b}_i^n. \tag{4.31}
\]
This follows by integrating (4.23) over $\sigma$. We have ignored the usual normal ordering constant, since that will be small compared to the mass of the excited string.

We need to calculate the Bogoliubov coefficient $B_n$ between the asymptotically flat region and the region $\tau < 1/P$. Since the $u_n$’s and $v_n$’s are normalized, we can calculate this coefficient by evaluating the inner product between the initial positive-frequency mode $u_n$ and the final negative-frequency mode $\tilde{v}_n$. Fortunately, (4.27) has a simple closed-form solution in the region $\tau > 1/P$ [3]. It is

$$x_n = \sqrt{\tau} [AJ_0(n\tau) + BN_0(n\tau)],$$

(4.32)

where $A$ and $B$ are arbitrary constants, and $J_0$ and $N_0$ are the Bessel functions of the first and second kinds. For $u_n$, the boundary condition (4.26) and the asymptotic expansions of the Bessel functions imply that

$$u_n = \sqrt{\tau/8} e^{-i\pi/4} [J_0(n\tau) - iN_0(n\tau)].$$

(4.33)

We expect that only the modes with $n \ll P$ will be significantly excited by the wave, as it is for these modes that the coefficient of $x_n$ in (4.21) changes significantly as we pass through the wave. We therefore calculate the Bogoliubov coefficients only for $n \ll P$. In this case, we can use the short-distance expansion of the Bessel functions near the matching point at $\tau = 1/P$. That is, for $\tau \sim 1/P$,

$$u_n \approx \sqrt{\pi/8} e^{-i\pi/4} \{1 - 2i[C + \ln(n\tau/2)]\},$$

(4.34)

where $C$ is Euler’s constant. For $\tau < 1/P$, the solution will take the form

$$u_n = T \frac{1}{2\sqrt{n}} e^{-in\tau} + R \frac{1}{2\sqrt{n}} e^{in\tau}.$$

(4.35)

The constants $T$ and $R$, which are determined by matching the two forms at $\tau = 1/P$, are essentially the Bogoliubov coefficients. Performing this matching, we find that $B_n \propto \ln(n/P)$.

Thus if a string is initially in its ground state and falls into the black hole, by the time it reaches the horizon, each mode with $n \ll P$ will be excited to $\langle N_n \rangle \equiv \langle 0|b_n^\dagger b_n|0 \rangle \sim \ln^2(n/P)$. The excitation in modes $n \gg P$ will be highly suppressed. The excitation in modes $n \sim P$ is difficult to calculate, but should be of order one. The string mass (4.31) is then

$$\langle M_s^2 \rangle = 8 \sum_{n=1}^{\infty} n \langle N_n \rangle \sim \sum_{n=1}^{P} n \sim P^2$$

(4.36)

where we have used the fact that for most of the terms, the excitation is of order one. (The first few terms contribute $P \ln^2 P$ which is a subleading contribution.) Thus a
string becomes very massive by the time it crosses the horizon. Notice that this is the invariant rest mass of the string, and not the kinetic energy seen by some observer.

We now check that the mass of the string remains much less than the mass of the black hole. Recall from (2.22) that the mass of the black hole is

\[ M \sim \frac{r_0 RV}{g^2}(\cosh 2\gamma_1 + \cosh 2\gamma_p), \]

Under the reasonable assumption that \( RV \geq 1 \) and \( g < 1 \), (4.18) implies that \( M > r_0^2 P \). In order for the static curvature to be small near the horizon we need \( r_0 \gg 1 \), so indeed \( M \gg M_s \).

The typical size of the string can be estimated as follows [11]. The mean squared radius of the string is roughly

\[ \langle r^2 \rangle \sim \langle \int d\sigma : (x^i(0, \sigma) - x^i_0)^2 : \rangle \sim \sum \frac{\langle N_n \rangle}{n}. \]

Using the above estimate for the excitation yields \( \langle r^2 \rangle \sim O(1) \), so despite the large mass, the string remains small.

We can also consider the process of Hawking radiation of strings. Strings are much more likely to be radiated in their ground state than in a highly excited state. But this refers to the state of the strings near the horizon. By the time they reach infinity, they can be excited. If we model this process by starting the strings in their ground state for \( \tau < 1/P \), the Bogoliubov coefficients are the same as for strings falling in. Thus the excitation of the \( n^{th} \) mode in the asymptotic region will again be \( \langle N_n \rangle \sim \ln^2(n/P) \). The mass of the string in the asymptotic region is given by the usual flat space formula, which again implies \( \langle M_s^2 \rangle \sim P^2 \).

## 5 Back-reaction of infalling matter

So far, we have discussed the behaviour of infalling matter in the test particle (string) approximation. We now ask when this approximation breaks down. For the sake of simplicity, we will just consider infalling dust in this section. Consider the contribution of the dust to the stress tensor. The total stress tensor for the spacetime is

\[ T_{\mu\nu} = \bar{T}_{\mu\nu} + \rho u_\mu u_\nu, \]

where \( \bar{T}_{\mu\nu} \) is the stress tensor for the background spacetime, \( \rho \) is the comoving density of the dust, and \( u_\mu \) is the tangent vector to the flow lines. For the test particle approximation to be valid, the dust should follow the geodesics of the background spacetime, and the backreaction of the dust on the geometry, e.g., the change in the curvature invariants, should be negligible. These conditions are always valid for sufficiently small \( \rho \).
Naively, one might expect that $\rho^2 \ll \bar{T}_{\mu\nu}\bar{T}^{\mu\nu}$ would be sufficient. However if we contract the total stress tensor with itself, we obtain

$$T_{\mu\nu}T^{\mu\nu} = \bar{T}_{\mu\nu}\bar{T}^{\mu\nu} + 2\rho u^\mu u^\nu \bar{T}_{\mu\nu} + \rho^2.$$  \hspace{1cm} (5.2)

We have seen that in the region of large tidal forces the geodesic components of the Einstein tensor become much larger than the curvature invariants. This implies that the middle term on the right in (5.2), $2\rho \bar{T}_{\nu\sigma}^\nu$, can become larger than the first, even if $\rho^2 \ll \bar{T}_{\mu\nu}\bar{T}^{\mu\nu}$. That is, the presence of a small amount of infalling matter can give rise to large curvature invariants in the region of large tidal forces, where previously all the curvature invariants were small. Physically, this is a result of a high energy collision between the infalling matter and the background dilaton wave.

This raises the possibility that the behavior of infalling matter in an exact solution will be qualitatively different from the test particle approximation. This would not contradict the known stability of these black holes \cite{12} since this refers only to linearized perturbations. Fortunately, one can check this in a special case involving the collapse of a thin shell. In general relativity, the motion of a thin shell of dust can be determined by matching appropriate metrics on the inside and outside across the shell, using the intrinsic matching conditions developed in \cite{13, 14}. In \cite{15}, Boulware used this technique to describe the collapse of a spherically symmetric charged dust shell to form a Reissner-Nordström black hole. The general technique was extended to theories with gravity coupled to a scalar field in \cite{16, 17}. In the case with a scalar field, there is an additional requirement; the scalar field must be continuous across the shell. This implies that if one tries to match two static solutions across the shell, the position of the shell will be forced to be static unless the scalar field has the same form in the two solutions. Thus, we cannot study the collapse to form a dilaton black hole this way, as the dilaton is constant in flat space, and non-constant in the static black hole solution. Physically, a collapsing spherical charged shell will radiate dilaton waves. The static black hole should be viewed as representing the geometry at late times, long after the collapse has taken place.

However, if we wish to consider a spherical shell falling into an existing black hole, we can use static solutions so long as the shell carries no dilaton charge. This will imply that the dilaton has the same form inside and outside the shell. We will consider the dilaton black hole metric (2.8), as this is the simplest case. The metrics on the inside and the outside of the shell each have the form (2.8), with in general different values of $r_+, r_-$. (We assume that the interior metric has large tidal forces outside the horizon.) The position of the shell in the two metrics will in general also be given by different functions $r(\tau)$, where $\tau$ is the shell’s proper time. However, $R(r)$ measures the proper area of the shell, so it must have the same value at the shell in the two metrics. The dilaton (2.11) must also have the same value at the shell in the two metrics. These two relations imply that if $a \neq 0$, $r_-$ and $r(\tau)$ must both be the same in the two metrics.
The only difference is then in the value of $r_+$. We will use $r_+$ to denote the value in the interior metric, and $r'_+$ to denote the value in the exterior metric. The shell has a proper mass $\mathcal{M}$ and charge $q$. The charge will be equal to the change in the black hole’s charge,

$$q = \left( \frac{r_+}{1 + a^2} \right)^{1/2} \left( r_{n/2}^{r/2} - r_{r/2}^{r/2} \right). \quad (5.3)$$

The black hole’s mass changes by

$$m = \frac{1}{2} (r'_+ - r_+), \quad (5.4)$$

which can be thought of as the total energy carried by the shell. This total energy should be positive, so $r'_+ > r_+$. Note that

$$m = \frac{(r^1/2 + r^{r/2}) \sqrt{1 + a^2}}{r_+^{r/2}} \frac{q > \sqrt{1 + a^2} q > q/\sqrt{1 + a^2}}{= m_{BP S}}, \quad (5.5)$$

so the total energy carried by the shell is greater than the BPS bound. This is not necessarily true of its proper mass $\mathcal{M}$. Note also that the requirement that the shell carries no dilaton charge implies that its charge has the same sign as the black hole’s.

The tangent vector to the shell is

$$u^\mu = (u^0, \dot{r}, 0, 0), \quad (5.6)$$

and the requirement that $u^\mu u_\mu = -1$ implies $u^0 = (F(r) + \dot{r}^2)^{1/2}/F(r)$. The function $F(r)$ is discontinuous across the shell, so the tangent will be different as seen from the inside and the outside. We will use $F_i$ to denote the function that appears in the interior metric, and $F_o$ to denote the function in the exterior metric. The normal is orthogonal to $u^\mu$, and is hence given by

$$n_\mu = (-\dot{r}, u^0, 0, 0). \quad (5.7)$$

The sign choices in the tangent vector and the normal have been made so as to give a future-directed tangent vector and an outward-pointing normal for a shell outside the black hole’s event horizon. If we define the notation $[K]_\pm \equiv K(r + \delta) - K(r - \delta)$ for the discontinuity in some quantity $K$ across the shell, then the intrinsic matching conditions of [13, 14] tell us that the discontinuity in the extrinsic curvature is related to the surface stress-energy of the shell, and hence [15]

$$[u^\mu n_\mu u^\nu + n^\mu n_\mu n_\nu n_\nu]_\pm = -2\mathcal{M}/R^2. \quad (5.8)$$

As in [15], we can use the identity

$$\delta_\sigma^\lambda = \delta_2^\lambda \delta_\sigma^2 + \delta_3^\lambda \delta_\sigma^3 + n^\lambda n_\sigma - u^\lambda u_\sigma, \quad (5.9)$$
and the fact that
\[ \Gamma^2_{a2} = \Gamma^3_{a3} = \delta_a^i R'/R \] (5.10)
to simplify (5.8). The resulting equation is
\[ 2 \frac{R'}{R} [n^1]_\pm = -2 \mathcal{M}/R^2. \] (5.11)

Rearranging and inserting the values of the normals from (5.7), we obtain
\[ \mathcal{M} = RR'[(F_i + \dot{r}^2)^{1/2} - (F_o + \dot{r}^2)^{1/2}] . \] (5.12)

This equation determines the motion of the shells. Rearranging and squaring twice to eliminate the radicals gives
\[ \dot{r}^2 = \left[ \frac{RR'}{2 \mathcal{M}} (F_i - F_o) + \frac{\mathcal{M}}{2 RR'} \right]^2 - F_i = \left[ \frac{m}{\mathcal{M}} \left( 1 - \frac{1}{(1 + a^2)} \frac{r_+}{r} \right) + \frac{\mathcal{M}}{2 RR'} \right]^2 - F_i. \] (5.13)

In the limit in which the mass of the shell goes to zero, (5.13) will reduce to the equation for charged test particles in the interior metric. In this limit, \( \frac{m}{\mathcal{M}} \) will become the energy per unit mass \( E \) of the test particle, and the second term in the brackets vanishes. This second term represents the contribution from the self-gravity of the shell. Note that these test particle trajectories are not geodesics, but describe the motion of charged test particles with charge related to the energy by (5.5). One can easily verify that the tidal forces experienced by these charged test particles also becomes large in the region near the black hole’s event horizon.

The main point of this analysis is that the exact equation for the motion of the shell (5.13) only differs from the corresponding test particle equation by the self-gravity term, and hence the motion of the shell is well approximated by test particles so long as this term is negligible. At large distances, this term can only be important if \( m = \mathcal{M} \) and \( \mathcal{M} \) is of order \( r_+ \). This implies that the shell starts at rest and that its mass is comparable to the black hole’s. However, we are more interested in whether this term can become important at smaller radii. Near the black hole, that is, at \( r = r_+(1 + \epsilon \rho) \) (\( \rho \) order one), \( RR' \sim r_+ \epsilon^{(a^2-1)/(1+a^2)} \) where \( \epsilon \equiv (1 - r_-/r_+) \). At moderate distances, that is, \( r = r_+ z \) (\( z \) order one), \( RR' \sim r_+ \). In both these regimes, the other term in the bracket is of order one, so the \( \mathcal{M}/2RR' \) term will only be important if \( \mathcal{M} \) is of the order of the smaller of \( r_+ \) and \( r_+ \epsilon^{(a^2-1)/(1+a^2)} \).

We now wish to compare this condition on the mass of the shell with the condition from the change in the curvature invariant (5.2). In the region of large curvatures, the second term in (5.2) is
\[ 2 \rho \tilde{T}_{0'0'} \sim \frac{\mathcal{M}}{l R^2} \frac{1}{r_+^2 \epsilon^2} \sim \frac{\mathcal{M}}{l r_+^2 \epsilon^{2(1+a^2)/(1+a^2)}}. \] (5.14)
where $l$ is the comoving thickness of the shell, which for simplicity we will now assume to be of order one. Now this term will be of order one (in Planck units) if $\mathcal{M}$ is of order $r^4_+ \epsilon^2 (1+2a^2)/(1+a^2)$. We can choose the black hole’s parameters so that the shell’s self-gravity is still negligible at this value. Thus, the test particle approximation used in computing (5.14) is valid. So the curvature invariant can become as large as the Planck scale (which is certainly much bigger than its background value) without a noticeable departure from the test particle trajectories.

The curvature invariant is only large in the shell, where $\rho \neq 0$. Since we have used a thin shell approximation where the stress energy of the shell is essentially a $\delta$-function, one might wonder if this is responsible for the large invariant. This is certainly not the case. The invariant we compute is small far from the black hole, and was estimated assuming a finite thickness $l$ for the shell. Although we chose $l$ to be of order the Planck length, this was not essential. The point is simply that the curvature invariant (5.14) and the self-gravity term in (5.13) depend on the black hole parameters in different ways, so for any $l$ we can choose the parameters so as to make the curvature invariant as large as we want, while still keeping the self-gravity term negligibly small.

Not only does the large curvature invariant have little effect on the motion of the infalling matter, it also has little effect on the subsequent evolution of the spacetime. After the shell passes, the geometry is again a static charged black hole with mass and charge increased by the shell. The original black hole was near extremal with the mass above extremality of order $\Delta \mathcal{M} \sim r_+ \epsilon$. Remarkably, one can fine-tune the black hole parameters so that the curvature invariant (5.14) is large while keeping $\mathcal{M}/\Delta \mathcal{M}$ small. In other words, one can achieve Planck scale curvature invariants with only a small change in the background metric.

6 Discussion

We have seen that the large tidal forces found for certain near-extreme black hole solutions in [1] have a simple physical interpretation. They result from a large dilaton wave traveling just outside the event horizon. This wave might be viewed as the remnant of the collapse to form the black hole. We saw in the last section that the collapse of a spherical thin charged shell in theories with a dilaton results in the emission of scalar waves. The static black hole solution should be viewed as representing the geometry at late times, long after the collapse has taken place. For near extremal black holes, even after everything has settled down, dilaton waves remain hovering just outside the horizon. The fact that the tidal forces are directly related to the proper time remaining

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5Since the radial tidal forces remain small, $l$ is approximately constant along the trajectory.

6As we have seen, this wave is an approximate description near the horizon. Since the solution is static, it does not carry away any energy.
to the singularity seems to imply that the wave remains significant only for near-extreme black holes.

When a test body falls into a naked black hole, it is crushed by the large tidal forces. The geometry near the horizon seen by an infalling observer can be approximately described by a dilaton plane wave metric. We applied this simplifying approximation to show that when a test string falls in, it becomes highly excited. However, the average size of the string does not grow significantly relative to a fixed length scale as it falls into the black hole.

When one includes the stress energy of the infalling matter, we have seen that the curvature invariants become large near the horizon. However, this does not lead to any significant change in the classical solution. This surprising fact can be understood as follows. Consider the total stress tensor \( T_{\mu\nu} \) in the geodesic frame. Only the \( T_{00} \) component will be different from the background. The background component \( \bar{T}_{00} \) is large and of the same order as the tidal forces, while the correction is just \( \rho \), and hence much smaller than the background. Since there is a frame in which the components of the stress tensor receive only small corrections, the change in the metric must also remain small. The fact that the curvature invariants can still become large is perhaps best illustrated by a simple vector analogy. Consider a large, nearly lightlike vector; that is, one whose time and space components in some orthonormal basis are individually large, say of order \( X \gg 1 \), but whose norm is very small. If we add to it a small vector which only has a time component, of order \( 1/X \), the resulting vector is close to the initial one, but its norm is now of order one.

Since infalling matter can produce Planck scale curvature invariants outside the horizon, quantum effects should become important in this regime. It is even possible that quantum effects are important for naked black holes before infalling matter is added. This cannot take the form of perturbative local corrections since all curvature invariants are small. But nonlocal (or nonperturbative) effects might be important. This issue certainly deserves further investigation. This is especially true in light of the recent ideas in [18], in which it was argued that Hawking radiation from a near extremal Reissner-Nordström black hole may cause the horizon to become unstable. The dilaton wave near the horizon of naked black holes might be viewed as a classical analog of the Hawking radiation in the Reissner-Nordström case.

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