The leptonic Dirac CP-violating phase from sum rules

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Abstract. In the reference 3-neutrino mixing scheme with three light massive neutrinos, CP-violating effects in neutrino oscillations can be caused by the Dirac CP-violating phase $\delta$ present in the unitary neutrino mixing matrix $U$. Using the fact that $U = U_e^\dagger U_\nu$, where $U_e$ and $U_\nu$ are unitary matrices arising from the diagonalisation, respectively, of the charged lepton and neutrino mass matrices, we consider in a systematic way forms of $U_e$ and $U_\nu$ allowing us to express $\delta$ as a function of the neutrino mixing angles present in $U$ and the angles contained in $U_\nu$. After obtaining sum rules for $\cos \delta$, we consider several forms of $U_\nu$ dictated by, or associated with, symmetries, such as tri-bimaximal, bimaximal, etc., for which the angles in $U_\nu$ are fixed. For each of these forms and forms of $U_e$ allowing to reproduce the measured values of the neutrino mixing angles, we construct the likelihood function for $\cos \delta$, using the prospective uncertainties in the determination of the mixing angles. Our results show that the measurement of $\delta$ along with improvement of the precision on the neutrino mixing angles can provide unique information about the possible existence of a new fundamental symmetry in the lepton sector.

1. Introduction

Understanding the origin of the observed pattern of neutrino mixing and establishing the status of the CP symmetry in the lepton sector are among the highest priority goals of the programme of future research in neutrino physics (see, e.g., [1]). One of the major experimental efforts within this programme will be dedicated to the searches for CP-violating (CPV) effects in neutrino oscillations (see, e.g., [2,3]). In the reference 3-neutrino mixing scheme with three light massive neutrinos (see, e.g., [1]), the CPV effects in neutrino oscillations can be caused, as is well known, by the Dirac CPV phase present in the Pontecorvo, Maki, Nakagawa, Sakata (PMNS) neutrino mixing matrix. In the approach we are going to follow to obtain predictions for the Dirac phase, one exploits the fact that the PMNS matrix $U$ has the form [4]:

\begin{equation}
U = U_e^\dagger U_\nu = U_e^\dagger \Psi U_\nu Q_0,
\end{equation}

where $U_e$ and $U_\nu$ are $3 \times 3$ unitary matrices originating from the diagonalisation, respectively, of the charged lepton and neutrino mass matrices. In Eq. (1) $U_e$ and $U_\nu$ are CKM-like $3 \times 3$ unitary matrices, and $\Psi$ and $Q_0$ are diagonal phase matrices each containing in the general case two CPV phases:

\begin{equation}
\Psi = \text{diag}\left(1, e^{-i\psi}, e^{-i\omega}\right), \quad Q_0 = \text{diag}\left(1, e^{i\frac{\phi_1}{2}}, e^{i\frac{\phi_2}{2}}\right).
\end{equation}

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It is further assumed that, up to sub-leading perturbative corrections (and phase matrices), the PMNS matrix $U$ has a specific known form $\tilde{U}_\nu$ that is dictated by continuous and/or discrete symmetries, or by arguments related to symmetries. This assumption seems very natural in view of the observation that the measured values of the three neutrino mixing angles differ from certain possible symmetry values by sub-dominant corrections. Indeed, in terms of angles, the best fit values of the three neutrino mixing parameters, $s^2_{12} \equiv \sin^2 \theta_{12}$, $s^2_{13} \equiv \sin^2 \theta_{13}$ and $s^2_{23} \equiv \sin^2 \theta_{23}$, obtained in the global analysis of neutrino oscillation data performed in [5], imply: $\theta_{12} \approx \pi/5.34$, $\theta_{13} \approx \pi/20$ and $\theta_{23} \approx \pi/4.35$. Thus, $\theta_{12}$ deviates from the possible symmetry value $\pi/4$ by approximately 0.2, $\theta_{13}$ deviates from 0 (or from $\pi/10$) by approximately 0.16, and $\theta_{23}$ deviates from the symmetry value $\pi/4$ by approximately 0.06, where we used $s^2_{23} = 0.437$.

Widely discussed symmetry forms of $\tilde{U}_\nu$ include: i) tri-bimaximal (TBM) form [7], ii) bimaximal (BM) form, or due to a symmetry corresponding to the conservation of the charged lepton charge $L = L_e - L_\mu - L_\tau$ (LC) [8], [9], iii) golden ratio type A (GRA) form [10], iv) golden ratio type B (GRB) form [11], and v) hexagonal (HG) form [12]. For all these forms the matrix $\tilde{U}_\nu$ represents a product of two orthogonal matrices describing rotations in the 1-2 and 2-3 planes on fixed angles $\theta_{12}$ and $\theta_{23}$:

$$\tilde{U}_\nu = R_{23}(\theta_{23}^\nu) R_{12}(\theta_{12}^\nu),$$

where

$$R_{12}(\theta_{12}^\nu) = \begin{pmatrix} c^\nu_{12} & s^\nu_{12} & 0 \\ -s^\nu_{12} & c^\nu_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_{23}(\theta_{23}^\nu) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c^\nu_{23} & s^\nu_{23} \\ 0 & -s^\nu_{23} & c^\nu_{23} \end{pmatrix},$$

where $c^\nu_{12} \equiv \cos \theta_{12}^\nu$, $s^\nu_{12} \equiv \sin \theta_{12}^\nu$, etc. For all the symmetry forms quoted above one has $\theta_{23}^\nu = -\pi/4$. The forms differ by the value of the angle $\theta_{12}^\nu$, and, correspondingly, of $s^\nu_{12}$: for the TBM, BM (LC), GRA, GRB and HG forms we have, respectively, $s^\nu_{12} = 1/3$, $1/2$, $(2 + r)^{-1} \approx 0.276$, $(3 - r)/4 \approx 0.345$, and $1/4$, $r$ being the golden ratio, $r = (1 + \sqrt{5})/2$.

As is clear from the preceding discussion, the values of the angles in the matrix $\tilde{U}_\nu$, which are fixed by symmetry arguments, typically differ from the values determined experimentally by relatively small perturbative corrections. In the approach we are following, the requisite corrections to the symmetry values are provided by the angles in the matrix $\tilde{U}_\nu$. The matrix $\tilde{U}_\nu$ in the general case depends on three angles and one phase [4]. For certain specific forms of $U_L$ and $U_R$, which allow one to reproduce the measured values of the neutrino mixing angles $\theta_{12}$, $\theta_{23}$ and $\theta_{13}$, the Dirac phase $\delta$ present in the PMNS matrix satisfies sum rules by which it is expressed in terms of the angles $\theta_{12}$, $\theta_{23}$, $\theta_{13}$ and the angles in the matrix $\tilde{U}_\nu$ whose values are fixed (see, e.g., [13–15]).

In the present note we recapitulate the sum rules for cos $\delta$ derived in [14,15] for the following forms of $\tilde{U}_\nu$ and $\tilde{U}_e$:

A. $\tilde{U}_\nu = R_{23}(\theta_{23}^\nu) R_{12}(\theta_{12}^\nu)$ and $\tilde{U}_e = R_{12}^{-1}(\theta_{12}^e)$ (case A1), $\tilde{U}_e = R_{13}^{-1}(\theta_{13}^e)$ (case A2),

$$\tilde{U}_e = R_{13}^{-1}(\theta_{13}^e) R_{12}(\theta_{12}^e) \quad \text{(case A3)}; \quad \tilde{U}_e = R_{23}^{-1}(\theta_{23}^\nu) R_{12}(\theta_{12}^e) \quad \text{(case A4)};$$

B. $\tilde{U}_\nu = R_{23}(\theta_{23}^\nu) R_{13}(\theta_{13}^\nu) R_{12}(\theta_{12}^\nu)$ and $\tilde{U}_e = R_{12}^{-1}(\theta_{12}^e)$ (case B1), $\tilde{U}_e = R_{13}^{-1}(\theta_{13}^e)$ (case B2).

Non-zero values of $\theta_{13}^\nu$ are inspired by certain types of flavour symmetries (see, e.g., [17]). We would like to notice here that in the case of $\tilde{U}_e = R_{23}^{-1}(\theta_{23}^\nu)$ the parameter $s^2_{13} = s^2_{13}^\nu$, i.e., the reactor angle does not get corrected and, in particular, remains zero for the form of $\tilde{U}_\nu$ given in point A. This is why we do not consider this case.

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4 Similar results were obtained in the global analysis of neutrino oscillation data performed in [6].

5 We would like to point out that all the cases considered in this study can be realised when a discrete (lepton) flavour symmetry group is fully broken in the charged lepton sector and is broken to a residual symmetry in the neutrino sector which fixes the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
2. Sum rules and their predictions
We start this section presenting in Table 1 the sum rules of interest, which \(\cos \delta\) satisfies in cases A1 – A4, B1 and B2. It is worth noting that these sum rules are exact within the schemes considered. We give also the expressions for \(s_{23}^2\) for the schemes in which it is fixed by the values of \(\theta_{13}\) and the angles in \(U_{\nu}\).

**Table 1.** Sum rules for \(s_{23}^2\) and \(\cos \delta\) in cases A1 – A4, B1 and B2 (see the Introduction).

| Case | \(s_{23}^2\) | \(\cos \delta\) |
|------|--------------|----------------|
| A1   | \(\frac{s_{23}^2 - s_{13}^2}{1 - s_{13}^2}\) | \(\frac{(c_{13}^2 - c_{23}^2)^{\frac{1}{2}}}{\sin 2\theta_{12} s_{13} c_{23}^2}\) \(\left[ \cos 2\theta_{12} + (s_{12}^2 - c_{12}^2) \frac{s_{23}^2 - (1 + c_{23}^2) c_{13}^2}{c_{13}^2 - c_{23}^2} \right]\) |
| A2   | \(\frac{s_{23}^2}{1 - s_{13}^2}\) | \(- \frac{(c_{23}^2 - s_{23}^2)^{\frac{1}{2}}}{\sin 2\theta_{12} s_{13} s_{23}^2}\) \(\left[ \cos 2\theta_{12} + (s_{12}^2 - c_{12}^2) \frac{c_{23}^2 - (1 + s_{23}^2) s_{13}^2}{c_{13}^2 - s_{23}^2} \right]\) |
| A3   | Not fixed | \(\frac{\tan \theta_{23}}{\sin 2\theta_{12} s_{13}} \left[ \cos 2\theta_{12} + (s_{12}^2 - c_{12}^2) (1 - \cot^2 \theta_{23} s_{13}^2) \right]\) |
| A4   | Not fixed | \(- \frac{\cot \theta_{23}}{\sin 2\theta_{12} s_{13}} \left[ \cos 2\theta_{12} + (s_{12}^2 - c_{12}^2) (1 - \tan^2 \theta_{23} s_{13}^2) \right]\) |
| B1   | \(\frac{c_{13}^2 - c_{13}^2 c_{23}^2}{1 - s_{13}^2}\) | \(\frac{(c_{13}^2 - c_{23}^2 c_{23}^2) s_{13}^2}{\sin 2\theta_{12} s_{13} c_{13}^2 s_{23}^2} \left[ (c_{13}^2 - c_{23}^2 c_{23}^2)^{\frac{1}{2}} + (c_{13}^2 s_{13}^2 - c_{13}^2 c_{23}^2)^{\frac{1}{2}} \right] \) |
| B2   | \(\frac{s_{23}^2 c_{13}^2}{1 - s_{13}^2}\) | \(- \frac{(c_{13}^2 - c_{13}^2 s_{23}^2) s_{13}^2}{\sin 2\theta_{12} s_{13} c_{13}^2 s_{23}^2} \left[ (c_{13}^2 - c_{13}^2 s_{23}^2)^{\frac{1}{2}} \right] \) |

We show in Table 2 the predictions for \(\cos \delta\) for all the schemes considered in the present note using the current best fit values of the neutrino mixing parameters \(s_{12}^2, s_{13}^2\) and \(s_{23}^2\) obtained for neutrino mass spectrum with normal ordering (NO) in [5]. In the case of the BM (LC) symmetry form of \(U_{\nu}\), the corresponding sum rules give unphysical values of \(\cos \delta\), if one uses the current

**Table 2.** The predicted values of \(\cos \delta\) using the current best fit values of \(s_{12}^2, s_{13}^2\) and \(s_{23}^2\) for the NO neutrino mass spectrum [5]. The values in square brackets correspond to \([\theta_{13}, \theta_{12}]\). We have defined \(a = \sin^{-1}(1/3), b = \sin^{-1}(1/\sqrt{2 + r}), c = \sin^{-1}(1/\sqrt{3})\) and \(d = \sin^{-1}(\sqrt{3} - r/2)\).

| Case | TBM | GRA | GRB | HG | BM (LC) |
|------|-----|-----|-----|----|---------|
| A1   | -0.114 | 0.289 | -0.200 | 0.476 | — |
| A2   | 0.114 | -0.289 | 0.200 | -0.476 | — |
| A3   | -0.091 | 0.275 | -0.169 | 0.445 | — |
| A4   | 0.151 | -0.315 | 0.251 | -0.531 | — |
| B1   | -0.222 | 0.760 | 0.911 | -0.775 | -0.562 |
| B2   | -0.866 | 0.222 | -0.760 | -0.911 | -0.791 |
best fit values of $s_{12}^2$, $s_{13}^2$ and $s_{23}^2$ for details see [13, 14]). For the choice of the values of $[\theta_{13}^\prime, \theta_{12}^\prime]$ in cases B1 and B2 in Table 2 see the explanations given in [15].

In Fig. 1 we present results of a statistical analysis of cases A3 and A4, in which the best fit values of all the three neutrino mixing angles can be reproduced for $\bar{U}_\nu$ having the TBM, GRA, GRB or HG symmetry form. This analysis is performed using the procedure described in [18] (see also [19]), which allows one to get the dependence of the $\chi^2$ function, and hence of the likelihood function $L(\cos \delta) \propto \exp(-\chi^2(\cos \delta)/2)$, on $\cos \delta$. The likelihood function represents the most probable values of $\cos \delta$ for each of the symmetry forms considered. The maxima of $L$ for the different symmetry forms of $U_\nu$ correspond to the values of $\cos \delta$ given in Table 2. The results shown are obtained by marginalising over $s_{13}^2$ and $s_{23}^2$ for a fixed value of $\delta$.

![Figure 1](image-url)

Figure 1. The likelihood function versus $\cos \delta$ for the NO neutrino mass spectrum after marginalising over $s_{13}^2$ and $s_{23}^2$ for the TBM, BM (LC), GRA, GRB and HG symmetry forms of the matrix $\bar{U}_\nu$ in case A3 (left panel) and in case A4 (right panel). The results shown are obtained using the prospective 1σ uncertainties on $s_{12}^2$, $s_{13}^2$ and $s_{23}^2$.

We present $L$ versus $\cos \delta$ within the Gaussian approximation, i.e., constructing $\chi^2$ as a sum: $\chi^2 = \sum_{i=1}^{3} (x_i - \bar{x}_i)^2/\sigma^2_x$, where $x_i = (s_{12}^2, s_{13}^2, s_{23}^2)$. We employ the current best fit values ($\bar{x}_i$) of $s_{12}^2$, $s_{13}^2$, $s_{23}^2$ for the NO spectrum obtained in [5] and the prospective 1σ uncertainties ($\sigma_x$) in the measurement of these mixing parameters, namely, a) 0.7% for $s_{12}^2$, which is the prospective sensitivity of the JUNO experiment [20], b) 5% for $s_{23}^2$, obtained from the prospective uncertainty of 2% [3] on $\sin^2 2}\theta_{23}$ expected to be reached in the NOvA and T2K experiments, and c) 3% for $s_{13}^2$, deduced from the error of 3% on $\sin^2 2}\theta_{13}$ planned to be reached in the Daya Bay experiment [3, 21].

As can be observed from Fig. 1, a rather precise measurement of $\cos \delta$ would allow one to distinguish between the different symmetry forms of $\bar{U}_\nu$ considered by us. The BM case is very sensitive to the best fit values of $s_{12}^2$ and $s_{23}^2$ and is disfavoured at more than 2σ for the current best fit values. This case might turn out to be compatible with the data for larger (smaller) measured values of $s_{12}^2$ ($s_{23}^2$), as demonstrated in [18]. Comparing the left and right panels of Fig. 1, we note first that for a given symmetry form, $\cos \delta$ is predicted to have opposite signs in the two cases considered. In case A4 (right panel) one has $\cos \delta > 0$ for the TBM and GRB symmetry forms, and $\cos \delta < 0$ for the GRA and HG symmetry forms. It is also important to note that due to the fact that the best fit value of $s_{23}^2$ is different from 0.5, the predictions for $\cos \delta$ for each symmetry form, obtained in the two cases, differ not only by sign but also in absolute value. Thus, a precise measurement of $\cos \delta$ would allow one to distinguish not only
between the symmetry forms of $\hat{U}_{\nu}$, but also could provide an indication about the structure of the matrix $\hat{U}_{\nu}$, and thus about the charged lepton mass matrix.

3. Conclusions
In conclusion, we presented the sum rules for the cosine of the Dirac phase $\delta$ in the PMNS mixing matrix, which is assumed to have a particular form dictated by symmetry arguments. These sum rules are exact within the approach employed. The results obtained in [14,15,18] and summarised in this note show that the measurement of the Dirac phase $\delta$ in the PMNS mixing matrix along with improvement of the precision on the mixing angles $\theta_{12}$, $\theta_{13}$ and $\theta_{23}$ can provide unique information as regards the possible existence of a new fundamental (discrete) symmetry in the lepton sector. These measurements could also provide an indication about the charged lepton mass matrix.

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