Exact and approximate solutions to the minimum of $1 + x + \cdots + x^{2n}$

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Abstract

The polynomial $f_{2n}(x) = 1 + x + \cdots + x^{2n}$ and its minimizer on the real line $x_{2n} = \arg \inf f_{2n}(x)$ for $n \in \mathbb{N}$ are studied. Results show that $x_{2n}$ exists, is unique, corresponds to $\partial_x f_{2n}(x) = 0$, and resides on the interval $[-1, -1/2]$ for all $n$. It is further shown that $\inf f_{2n}(x) = (1 + 2n)/(1 + 2n(1 - x_{2n}))$ and $\inf f_{2n}(x) \in [1/2, 3/4]$ for all $n$ with an exact solution for $x_{2n}$ given in the form of a finite sum of hypergeometric functions of unity argument. Perturbation theory is applied to generate rapidly converging and asymptotically exact approximations to $x_{2n}$. Numerical studies are carried out to show how many terms of the perturbation expansion for $x_{2n}$ are needed to obtain suitably accurate approximations to the exact value.

1 Introduction

The inspiration for this work came from a question posted by Wang [11] on the Mathematics Stack Exchange discussion board March 13, 2021, which sought a solution to the minimum of the polynomial $1 + x + \cdots + x^{2n}$ for $n \in \mathbb{N}$. In the question it was noted that the minimum appeared to correspond to a vanishing derivative and thus could be found by solving for the real roots of $\partial_x (1 + x + \cdots + x^{2n})$. When $n = 1, 2$ these roots are algebraic with their exact forms being recovered using the standard formulae for linear and cubic equations. However for $n \geq 3$, the work of Abel and Galois shows no general algebraic solution exists; hence, motivating the need for more powerful methods [1].

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the broad and pervasive applications of geometric sums in the literature, further
study of this polynomial and its minimum is a worthwhile venture.

2 Preliminaries
Throughout this work we define \( \mathbb{N} = \{1, 2, \ldots\} \), \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), and \( \mathbb{E} = \{2, 4, \ldots\} \) to be the sets of positive integers, nonnegative integers, and positive even integers, respectively. For the sake of brevity we shall denote \( m = 2n \) so that the the polynomial of interest and its minimizer becomes

\[
f_m(x) := 1 + x + \cdots + x^m, \quad m \in \mathbb{E}
\]

and

\[
x_m := \arg \inf_{x \in \mathbb{R}} f_m(x).
\]

The following definitions and relations will also be used.

**Definition 1** (Gamma function).

\[
\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} \, dt, \quad \Re s > 0
\]

**Definition 2** (Factorial power (falling factorial)).

\[
(s)^{(n)} := \frac{\Gamma(s + 1)}{\Gamma(s - n + 1)}
\]

**Definition 3** (Pochhammer symbol (rising factorial)).

\[
(s)_n := \frac{\Gamma(s + n)}{\Gamma(s)}
\]

**Definition 4** (Generalized hypergeometric series).

\[
_\!p\! \!_\! F_q \left( a_1, \ldots, a_p ; b_1, \ldots, b_q ; z \right) := \sum_{k=0}^\infty \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \, \frac{z^k}{k!}
\]

**Definition 5** (k-gamma function and Pochhammer k-symbol [2]). The k-gamma function and Pochhammer k-symbol are given by

\[
\Gamma_k(x) := k^{x-1} \Gamma \left( \frac{x}{k} \right)
\]

and

\[
(x)_{n,k} := \frac{\Gamma_k(x + nk)}{\Gamma_k(x)},
\]

respectively.

**Relation 1.** If \( n \in \mathbb{N}_0 \) then \( (\alpha)_{n,k} = k^n (\alpha/k)_n \) [7, Prop. 3.1].

With these definitions at hand we are ready to begin studying the properties of \( f_m \) and \( x_m \).
3 Properties of \( f_m \) and \( x_m \)

Our first goal is to establish the existence and uniqueness of \( x_m \). To accomplish this it will be helpful to use the closed-form for geometric sums and write \( f_m \) in the form

\[
f_m(x) = \frac{1 - x^{m+1}}{1 - x}.
\]

**Lemma 1.** The polynomial \( f_m(x) \) is strictly convex on \( x \in \mathbb{R} \) for all \( m \in \mathbb{E} \).

**Proof.** To establish strict convexity it is sufficient to show \( f_m''(x) > 0 \) everywhere on \( x \in \mathbb{R} \). It is trivial to show \( f_m''(x) > 0 \) holds for \( x \geq 0 \) so all that is left to do is to consider the complementary case \( x < 0 \). Equating the second derivative with zero we find \( f_m''(x) = 0 \implies h_m(x) = 0 \), where

\[
h_m(x) = (m - 1)mx^{m+1} - 2(m^2 - 1)x^m + m(m + 1)x^{m-1} - 2.
\]

The signs of the coefficients of \( h_m(-x) \) in order of descending variable exponent yields the sequence \((-1, -1, -1, -1)\), which are all negative. It follows from Descartes’ rule of signs that \( f_m''(x) \) has zero roots on the interval \( x \in (-\infty, 0) \). But, \( f_m''(-1) = \frac{1}{2}m^2 > 0 \); thus, we conclude \( f_m''(x) > 0 \) also holds for all \( x < 0 \). The proof is now complete. \( \square \)

**Theorem 1.** The minimizer \( x_m \) exists, is unique, and resides on the interval \([-1, -1/2]\) for all \( m \in \mathbb{E} \).

**Proof.** We begin by establishing that \( f_m'(x) \) has exactly one real root. It is immediately obvious that \( f_m'(x) > 0 \) for all \( x \geq 0 \). Now assuming \( x < 0 \), we deduce \( f_m'(x) = 0 \iff g_m(x) = 0 \), where \( g_m(x) = mx^{m+1} - (m+1)x^m + 1 \). The signs of the coefficients of \( g_m(-x) \) in order of descending variable exponent gives the sequence \((-1, -1, +1)\); revealing a single variation in sign. Again appealing to Descartes’ rule of signs we conclude \( f_m'(x) \) must have exactly one real root on the interval \( x \in (-\infty, 0) \). However, \( f_m'(-1) = -\frac{1}{2}m \) and \( f_m'(-1/2) = \frac{1}{2}2^{1-m}(2m+1 - 3m - 2) \geq 0 \) with the latter inequality following from induction on \( m \in \mathbb{E} \). Consequently, \( f_m'(x) \) has a single root on the real line contained in the interval \([-1, -1/2]\) for all \( m \in \mathbb{E} \). Furthermore, the strict convexity of \( f_m \) proven in Lemma 1 implies that the solution to \( f_m'(x) = 0 \) also corresponds to the unique global minimum of \( f_m \), which completes the proof. \( \square \)

With the existence and uniqueness of \( x_m \) established, we turn to finding a simple formula for the minimum of \( f_m \) as a function of \( x_m \).

**Lemma 2.** Let \( x_m \in [-1, -1/2] \) denote the unique minimizer of \( f_m \) such that \( f_m(x_m) = \inf_{x \in \mathbb{R}} f_m(x) \). Then,

\[
f_m(x_m) = \frac{1 + m}{1 + m(1 - x_m)}
\]

and \( f_m(x_m) \in [1/2, 3/4] \) for all \( m \in \mathbb{E} \) with \( f_2(x_2) = 3/4 \) and \( \lim_{m \to \infty} f_m(x_m) = 1/2 \).
Proof. From Theorem 1 we know that $x_m$ satisfies $mx_m^{m+1} - (m+1)x_m^m + 1 = 0$, which can be rewritten as $x_m^{m+1} = x_m/(1 + m(1 - x_m))$. Substituting this expression for $x_m^{m+1}$ into (1) yields the desired form for $f_m(x_m)$. The bounds on $f_m(x_m)$ are then found by minimizing and maximizing $f(m, x) = (1 + m)/(1 + m(1 - x))$ on $(m, x) \in \mathbb{E} \times [-1, -1/2]$. We find $\inf f(m, x) = \lim_{m \to \infty} f(m, -1) = 1/2$ and $\sup f(m, x) = f(2, -1/2) = 3/4$, which are equivalent to $\lim_{m \to \infty} f_m(x_m)$ and $f_2(x_2)$, respectively. The proof is now complete.

\[4\] Explicit expression for $x_m$

In the previous section we showed that the minimizer $x_m$ exists, is unique, and resides on the interval $[-1, -1/2]$ for all $m \in \mathbb{E}$. Furthermore, we were able to establish a very simple expression for $\inf f_m$ as a function of this minimizer so that the problem of evaluating $\inf f_m$ is equivalent to finding $x_m$. For $m = 2, 4$ we may apply the standard equations for roots of linear and cubic equations to derive exact algebraic expressions for $x_m$. Furthermore, as $m \to \infty$ we find for $|x| < 1$: $f_{m}(x) \to (1 - x)^{-1}$, which is strictly increasing on $x \in (-1, -1/2]$. Bringing these observations together we have

\[
\begin{align*}
x_2 &= -\frac{1}{2} \\
x_4 &= -\frac{1}{4} \left(1 + \sqrt[3]{\frac{5}{9}} \left(\sqrt[3]{9 + 4\sqrt{6}} - \sqrt[3]{9 - 4\sqrt{6}}\right)\right) \\
&\vdots \\
x_\infty &= -1.
\end{align*}
\]

While a general algebraic solution for $x_m$ with $m \geq 6$ does not exist, methods for expressing exact solutions to higher-order polynomial roots have been thoroughly studied [10]. For example, the work of Hermite shows that $x_6$ can be solved exactly in terms of nonelementary functions [5]. One way this is accomplished is by reducing the quintic equation $\partial_x f_6(x) = 0$ to its Bring–Jerrard normal form and then using series reversion to express $x_6$ in terms of hypergeometric functions. Using this approach as a clue, Theorem 2 presents an exact and general solution for $x_m$ based on an adaptation of the method used for solving trinomial equations [4, 3, 8].

**Theorem 2.** For all $m \in \mathbb{E}$

\[
x_m = \sum_{k=1}^{m} \frac{(-m)^{k-2}}{(1 + m)x_m^{k}} \frac{\Gamma\left(\frac{m+k}{m} - 1\right)}{(\Gamma\left(\frac{m+k}{m}\right)\Gamma(k))^{m+2}} F_{m+1}^{1} \left(1, \left\{ \frac{k}{m} + \frac{\ell - 1}{m+1} \right\}\frac{m}{m+k} ; \left\{ \frac{k+\ell}{m} \right\}_{\ell=0}^{m} ; 1 \right).
\]

**Proof.** From Theorem 1 we know $x_m$ satisfies $mx_m^{m+1} - (m+1)x_m^m + 1 = 0$, $m \in \mathbb{E}$.
Performing the substitution \( x_m \mapsto -\zeta^{-\frac{m}{m}} \) we obtain the transformed expression

\[
\zeta = 1 + m + m\phi(\zeta),
\]

(2)

with \( \phi(\zeta) = \zeta^{-\frac{m}{m}} \). By Lagrange’s inversion theorem it follows for a function \( F \) analytic in a neighborhood of the root of (2) that

\[
F(\zeta) = F(1 + m) + \sum_{n=1}^{\infty} \frac{m^n}{n!} \left[ \frac{\partial^{n-1} F(w)}{\partial w^{n-1}} \right]_{w = 1+m}.
\]

Choosing \( F(\zeta) = -\zeta^{-\frac{m}{m}} \) we subsequently obtain

\[
x_m = -(1 + m)^{-\frac{1}{m}} + \sum_{n=1}^{\infty} \frac{m^{n-1}}{n!} \left[ \frac{\partial^{n-1} w^{-m}}{\partial w^{n-1}} \right]_{w = 1+m},
\]

which upon further noting that \( \partial^{n-1} w^{-s} = (-1)^{n-1} (s)_{n-1} w^{-s-n+1} \) yields after some algebraic manipulation

\[
x_m = -(1 + m)^{-\frac{1}{m}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{mn+n+1}{m}\right)}{\Gamma\left(\frac{m+n+1}{m}\right)} \frac{(-m(1+m)^{-\frac{m+1}{m}})^n}{n!}.
\]

(3)

To evaluate the series (3) we write \( x_m = \sum_{n=0}^{\infty} c_n = \sum_{k=1}^{n} \sum_{r=0}^{\infty} c_{mn+k-1} \) resulting in \( m \) new series containing Pochhammer symbols of the form \((\cdot)_{(m+1)n}\) and \((\cdot)_{mn}\). Then using the identity [7, Eq. 2.13]

\[
(\alpha)_r = r^n \prod_{j=0}^{r-1} \left( \frac{\alpha + j}{r} \right), \quad r \in \mathbb{N}
\]

we arrive at

\[
x_m = \sum_{k=1}^{m} \frac{(-m)^{k-2}}{(1 + m)^{\frac{mk+k}{m} - 1}} \frac{\Gamma\left(\frac{mk+k}{m}\right)}{\Gamma\left(\frac{m+k}{m}\right)} \frac{\Gamma(k)}{\Gamma(\frac{m+k}{m})} \sum_{n=0}^{\infty} \frac{(1)_n \prod_{\ell=0}^{m} \left(\frac{k+\ell}{m}\right)_{n} 1}{n!}
\]

which is the desired result. 

To demonstrate the validity of the closed-form for \( x_m \) given by Theorem 2 we substitute \( m = 2 \) and find

\[
x_2 = \frac{1}{9} \sum_{k=0}^{m} \frac{(-2)^{k-2}}{(1 + 2)^{\frac{mk+k}{2} - 1}} \frac{\Gamma\left(\frac{mk+k}{2}\right)}{\Gamma\left(\frac{m+k}{2}\right)} \frac{\Gamma(k)}{\Gamma(\frac{m+k}{2})} \sum_{n=0}^{\infty} \frac{(1)_n \prod_{\ell=0}^{m} \left(\frac{k+\ell}{2}\right)_{n} 1}{n!}
\]

The \( _3F_2(\cdot) \) term is reduced to \( _2F_1(\cdot) \) by way of [6, Eq. 07.27.03.0120.01]

\[
_3F_2\left(\begin{array}{c}1, 2, \gamma; \\ \beta, \gamma \end{array}; z\right) = \frac{\epsilon - 1}{(\beta - 1)(\gamma - 1)z} _2F_1\left(\begin{array}{c}\beta - 1, \gamma - 1; \\ \epsilon - 1 \end{array}; z\right) - 1.
\]
Gauss’s hypergeometric summation theorem

\[ _2F_1 \left( \alpha, \beta; \gamma, 1 \right) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} \quad \Re(\gamma - \alpha - \beta) > 0 \]

then permits us to write the remaining \( _2F_1(\cdot) \) terms as ratios of gamma functions. After some simplification we find

\[ x_2 = -\frac{1}{2}, \]

which is the exact value of \( x_2 \).

For \( m \geq 4 \), reducing the closed-form for \( x_m \) to more elementary functions in this manner becomes very cumbersome if not impossible. Without the ability to reduce the hypergeometric functions present in \( x_m \), this expression also becomes difficult to implement numerically, especially as \( m \) becomes large. To obtain approximations we could turn to the series given by (3); however, the slow convergence of this series renders it impractical. For example, substituting \( m = 2 \) and adding up the first one-hundred terms of (3) we obtain \( x_2 \approx -0.499885 \), which corresponds to an absolute relative error of \( 2.3 \times 10^{-4} \). Given that numerical root finding methods can achieve more accurate approximations in just a few iterations we find this means of approximation to be less than satisfactory.

5 Perturbation series expansion of \( x_m \)

In the previous section we were able to find an exact expression for \( x_m \) but this expression was not useful for the purpose of computing numerical approximations. Here, we apply the methods of perturbation theory to obtain a faster converging series expansion for this purpose.

We begin by recalling from Theorem 1 that \( x_m \) satisfies

\[ g_m(x_m) = 0, \quad \text{with} \quad g_m(x) = x^m \left( 1 - x + \frac{1}{m} \right) - \frac{1}{m} \]

and \( x_m \to -1 \) as \( m \to \infty \). The fact that \( x_m + 1 \) vanishes as \( m \) becomes large suggests we instead study the perturbed problem

\[ g_{m,\epsilon}(x_m, \epsilon) = 0, \quad \text{with} \quad g_{m,\epsilon}(x) = x^m \left( 2 - (1 + x)\epsilon + \frac{1}{m} \right) - \frac{1}{m}, \]

where

\[ x_m, \epsilon = \sum_{k=0}^{\infty} a_k \epsilon^k. \quad (4) \]

Upon inspection we observe \( g_{m,1}(x) = g_m(x) \) and so it follows that \( x_m \) can be recovered by evaluating the perturbation series (4) at \( \epsilon = 1 \). To determine the coefficients \( a_k \) we first consider the well-known result for integer powers of series to express powers of \( x_{m,\epsilon} \) as

\[ x_{m,\epsilon}^p = \sum_{k=0}^{\infty} c_{k,p} \epsilon^k, \quad p \in \mathbb{N} \]
with
\[ c_{0,p} = a_0^p \]
\[ c_{k,p} = \frac{1}{a_0^k} \sum_{\ell=1}^k ((p + 1)\ell - k)a_{\ell}c_{k-\ell,p}. \]

Using Faà di Bruno’s formula we may also obtain a closed-form for the coefficients \( c_{k,p} \) as
\[ c_{k,p} = \frac{1}{k!} \sum_{\ell=1}^k (p)^{\ell} \frac{B_{k,\ell}(1!\ldots,(k-\ell+1)!)}{a_1\ldots a_\ell}a_{p-\ell}c_{k-\ell+1,p}, \]
where \( B_{n,k}(x_1,\ldots,x_{n-k+1}) \) is the partial Bell-polynomial. Using these results we substitute \( x_{m,\epsilon} \) into \( g_{m,\epsilon} \) and collect terms by powers of \( \epsilon \) yielding
\[ g_{m,\epsilon}(x_{m,\epsilon}) = \left(2 + \frac{1}{m}\right)a_0^m - \frac{1}{m} + \sum_{k=1}^{\infty} \left[ (2 + \frac{1}{m})c_{k,m} - c_{k-1,m} - c_{k-1,m+1}\right] \epsilon^k. \]

Since \( g_{m,\epsilon}(x_{m,\epsilon}) = 0 \) we equate the coefficients of \( \epsilon^k \) to zero to yield an infinite system of equations that recover the coefficients \( a_k \). Setting the constant term equal to zero gives
\[ a_0 = -(1 + 2m)^{-\frac{1}{m}}, \quad (1 + 2m)c_{k,m} - c_{k-1,m} - c_{k-1,m+1} = 0. \]  \( (5) \)
Evaluating the first several coefficients we are able to conjecture a closed-form for \( a_k \), which leads to the following result.

**Theorem 3.** For all \( m \in \mathbb{E} \)
\[ x_m = \sum_{k=0}^{\infty} a_k, \]
where
\[ a_0 = -(1 + 2m)^{-\frac{1}{m}}, \]
\[ a_k = \sum_{\ell=0}^{k} \frac{(\ell + m + 1)_{k-1,m}a_{m\ell+1}^k}{\ell!(k-\ell)!}. \]

**Proof.** We begin by considering the closed-form for \( x_m \) claimed in the statement of Theorem 2, which consists of a sum of \( m \) hypergeometric functions. Denoting \( \{a_{j,k}\}_{j=1}^{m+2} \) as the top parameters and \( \{b_{j,k}\}_{j=1}^{m+1} \) as the bottom parameters of the hypergeometric function in the \( k \)th term we find \( \gamma_k = (b_1 + \cdots + b_{m+1}) - (a_1 + \cdots + a_{m+2}) = \frac{1}{2} \) for all \( k = 1,\ldots,m \). Since \( \gamma_k > 0 \), each of the \( m \)-terms of \( x_m \) can be written as an absolutely convergent series; hence, the entire expression representing \( x_m \) must also be absolutely convergent. Now using the conjectured closed-form for \( a_k \) we write
\[ x_m = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{m^{k-1}}{\ell!(k-\ell)!} \frac{\Gamma(\ell + \frac{\ell+1}{m})\Gamma(k + \frac{\ell+1}{m})}{\ell!(k-\ell)!} a_{m\ell+1}^k. \]
If this expression is equal to that given in the statement of Theorem 2, then it is also absolutely convergent and permits rearrangement of its terms. Inter-
changing the order of summation we find after some simplification
\[
x_m = a_0 \sum_{\ell=0}^{\infty} \frac{\Gamma\left(\frac{m\ell+\ell+1}{m}\right)}{\Gamma\left(\frac{1}{m}\ell+1\right)} \left(\frac{ma_0^{n+1}}{m}\right)^{k} \frac{(ma_0^{n})^{k}}{k!}.
\]
The interior sum over \(k\) can now be evaluated in terms of \(_{1}F_{0}(\alpha; -; z) = (1 - z)^{-\alpha}\). Reintroducing \(a_0\) yields
\[
x_m = -\frac{(1 + m)^{-\frac{1}{m}}}{m} \sum_{\ell=0}^{\infty} \frac{\Gamma\left(\frac{m\ell+\ell+1}{m}\right)}{\Gamma\left(\frac{m+\ell+1}{m}\right)} \left(-m(1 + m)^{-\frac{m+1}{m}}\right)^{\ell},
\]
which is the series expansion for \(x_m\) given in (3). By the uniqueness of Taylor series it follows that the conjectured form for \(a_k\) must be correct.

So does the perturbation series for \(x_m\) converge faster than that given by (3)? Substituting \(m = 2\) and adding the first one-hundred terms we find for the absolute relative error \(5.6 \times 10^{-64}\), which is a significant improvement on the absolute relative error of \(2.3 \times 10^{-4}\) obtained from the first one hundred terms of (3).

We conclude this section with an important property of approximations for \(x_m\) obtained via the perturbation series of Theorem 3.

**Corollary 1.** If \(\tilde{x}_{m,n} = \sum_{k=0}^{n} a_k\), then \(x_m \sim \tilde{x}_{m,n}\) as \(m \to \infty\).

**Proof.** Using the expression for \(a_k\) given in Theorem 3 we have \(\lim_{m \to \infty} a_0 = -1\) and \(\lim_{m \to \infty} a_k = 0\) for all \(k \geq 1\); thus, \(\lim_{m \to \infty} \tilde{x}_{m,n} = -1\) for all \(n \in \mathbb{N}_0\). Since \(\lim_{m \to \infty} x_m = -1\) the result follows.

### 6 Numerical results

From Corollary 1 we know \(x_m \sim \tilde{x}_{m,n}\) as \(m \to \infty\) and so we expect the number \(n\) needed to guarantee \(|x_m - \tilde{x}_{m,n}| < \epsilon\) should decrease as \(m\) increases. Since we have closed-forms for \(x_2\) and \(x_4\), which can be computed to arbitrary precision, our first task will be to study the convergence of \(\tilde{x}_{m,n} \to x_m\) as a function of \(n\) for \(m = 2, 4\). Given that we expect less terms will be needed for larger values of \(m\), the results of this exercise should give us a worst case scenario for how large \(n\) must be to obtain the desired accuracy in our approximation.

Using *Mathematica* software, we evaluated \(\tilde{x}_{m,n}\) for \(m = 2, 4\) and \(n = 0, 1, \ldots, 100\). To compare the approximation to the exact values we used the absolute relative error
\[
R_m(n) = \left| \frac{\tilde{x}_{m,n}}{x_m} - 1 \right|
\]
the results of which are plotted in Figure 1. From the figure we see the error decreases exponentially with \(n\) and that \(R_4(n) < R_2(n)\) for each value of \(n\).
Working with the data for $R_4(n)$ we further determined

$$R_4(n) < 5 \times 10^{-(2+0.759n)},$$

which suggests setting $n$ equal to

$$n^* = \max \{0, \left\lceil \frac{\sigma-2}{0.759} \right\rceil \}$$

is sufficient to guarantee $\tilde{x}_{m,n}^*$ agrees with $x_m$ to at least $q$ significant digits for all $m \geq 4$.

To test this hypothesis we first note that $x_m \in (-1, -1/2]$ for all finite $m \in \mathbb{E}$. Since the leading exponent in the decimal expansion of $x_m$ is always negative one it follows that $\tilde{x}_{m,n}$ has $p$ significant digits of $x_m$ if $|x_m - \tilde{x}_{m,n}| \leq 5 \times 10^{-(p+1)}$. Furthermore, we know $x_m$ is the unique real root of $g_m(x) = x^m(1-x+\frac{1}{m}) - \frac{1}{m}$ with $g_m(x_m - \epsilon)$ and $g_m(x_m + \epsilon)$ differing in sign; hence a lower bound on the number of significant digits obtained by $\tilde{x}_{m,n}$ is found by determining the largest nonnegative integer $p$ such that

$$g_m(\tilde{x}_{m,n} - 5 \times 10^{-(p+1)})g_m(\tilde{x}_{m,n} + 5 \times 10^{-(p+1)}) \leq 0.$$

For the sake of example we chose $q = 10$ for the number of desired significant digits resulting in $n^* = 11$. Using the above mentioned procedure, the value $p$ was computed for $m = 4, 6, \ldots, 100$ with the results presented in Figure 2. From the figure we observe $p > q$ for each value of $m$ as is expected. Finally, Table 1 presents numerical values for $x_m$ and $f_m(x_m)$ computed using $\tilde{x}_{m,n}$ and the formula in Lemma 2.

$$R_m(n) \text{ versus } n \text{ for } m = 2, 4$$

![Figure 1: Absolute relative error incurred from the approximation $\tilde{x}_{m,n}$ versus $n$ for $m = 2, 4$. Plot produced with matlab2tikz [9].](image-url)
Significant figures obtained by $\tilde{x}_{m,n^{*}}(q)$

Figure 2: Number of significant figures obtained by the approximation $\tilde{x}_{m,n^{*}}(q)$ for $q = 10$ versus $m$.

7 Conclusions

In this note, we were able to establish many useful facts about the polynomial $f_{m}(x) = 1 + x + \cdots + x^{m}$ and its minimum value on the real line. In particular, we were able to show $\arg \inf f_{m}(x) \in [-1, -1/2]$ and $\inf f_{m}(x) \in [1/2, 3/4]$ for all $m \in \mathbb{E}$ as well as provide a very simple formula for the minimum as a function of the minimizer $x_{m}$. Lagrange inversion and perturbation theory were applied to derive two different series expansions for $x_{m}$, which lead to a closed-form in terms of hypergeometric functions. Furthermore, numerical studies were conducted which gave a rule of thumb for how large $n$ must be to achieve a desired accuracy in approximating $x_{m}$ with $\tilde{x}_{m,n}$.

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Table 1: Numerical values for $x_m$ and $f_m(x_m)$.

| $m$  | $x_m$       | $f_m(x_m)$   |
|------|-------------|--------------|
| 2    | -0.5000000000 | 0.7500000000 |
| 4    | -0.6058295862 | 0.6735532235 |
| 6    | -0.670320476  | 0.6350938940 |
| 8    | -0.7145377272 | 0.61566906   |
| 10   | -0.7470540749 | 0.5955429324 |
| 12   | -0.7728416355 | 0.588576922  |
| 14   | -0.7921778546 | 0.5749221276 |
| 16   | -0.8086048979 | 0.5678463037 |
| 18   | -0.823534102  | 0.5532669587 |
| 20   | -0.8340533676 | 0.5498010211 |
| 22   | -0.8441478047 | 0.5458943966 |
| 24   | -0.8529581644 | 0.541823146  |
| 26   | -0.8607238146 | 0.537876878  |
| 28   | -0.8676269763 | 0.533831483  |
| 30   | -0.8738090154 | 0.529928364  |
| 32   | -0.879314184  | 0.526123456  |
| 34   | -0.884333818  | 0.522546897  |
| 36   | -0.889037183  | 0.519137476  |
| 38   | -0.893520563  | 0.515932069  |
| 40   | -0.897127025  | 0.512946888  |
| 42   | -0.9007031162 | 0.510137456  |
| 44   | -0.9040147981 | 0.507519654  |
| 46   | -0.907013919  | 0.505082537  |
| 48   | -0.909995531  | 0.502803624  |
| 50   | -0.912636053  | 0.500674662  |
| 52   | -0.915142114  | 0.498696877  |
| 54   | -0.917487896  | 0.496869237  |
| 56   | -0.919713922  | 0.495192475  |
| 58   | -0.9218020367 | 0.493671387  |
| 60   | -0.9237741513 | 0.492314536  |
| 62   | -0.9256399896 | 0.491024357  |
| 64   | -0.9274082062 | 0.489802435  |
| 66   | -0.929086524  | 0.488638877  |
| 68   | -0.9306818591 | 0.487541835  |
| 70   | -0.9322004214 | 0.486499441  |
| 72   | -0.9336478047 | 0.485499441  |
| 74   | -0.9350290699 | 0.484541835  |
| 76   | -0.9363487901 | 0.483638877  |

$\infty$ -1.0000000000 0.5000000000