Purely imaginary polar resonances of rapidly-rotating Kerr black holes

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Abstract

We prove the existence of a unique family of non-oscillatory (purely-imaginary) polar quasi-normal resonances of rapidly-rotating Kerr black holes. These purely imaginary resonances can be expressed in the compact form: \( \omega_n = -i2\pi T_{BH}(l+1+n) \), where \( T_{BH} \) is the black-hole temperature, \( l \) is the spheroidal harmonic index of the mode, and \( n = 0, 1, 2, \ldots \) is the resonance parameter. It is shown that our analytical results for the black-hole resonance spectrum agree with new numerical data that recently appeared in the literature.
I. INTRODUCTION

Astrophysically realistic black holes are expected to possess a non-zero spin angular momentum. In fact, recent astrophysical observations [1–3] suggest that rapidly-rotating black holes are ubiquitous in our Universe [4]. It is therefore of astrophysical importance to explore the physical properties of rapidly-spinning black holes.

Rapidly-rotating black holes are also important from the point of view of quantum field theory. In particular, the conjectured relation between the near-horizon quantum states of a near-extremal black hole and the quantum states of a two-dimensional conformal field theory enables one to count the entropy of these black hole [5–7]. Moreover, near-extremal black holes have the interesting property of saturating the recently conjectured universal relaxation bound [8–12].

In the present study we shall explore the resonance spectrum of rapidly-rotating (near-extremal) black holes. Quasinormal (QNM) resonances are the unique ‘sound’ of black holes. (See [13–15] for excellent reviews and detailed lists of references.) Perturbed black holes usually display a characteristic pattern of damped oscillations of the form $e^{-i\omega_R t - \omega_I t}$. These damped oscillations characterize the decay (relaxation) of the perturbation fields in the black-hole spacetime (note that a decaying perturbation is characterized by a complex frequency $\omega = \omega_R - i\omega_I$ with $\omega_I > 0$). The spectrum of quasinormal resonances reflects the physical properties (mass and angular momentum) of the black hole itself [16, 17].

The quasinormal resonances correspond to wave fields propagating in the black-hole spacetime with the physically motivated boundary conditions of purely ingoing waves crossing the black-hole horizon and purely outgoing waves at spatial infinity [18]. For given values of the angular parameters $m$ and $l$ [here $m$ is the azimuthal harmonic index and $l$ is the spheroidal harmonic index of the perturbation mode, see Eq. (5) below] these boundary conditions single out a countable set $\{\omega(n; m, l)\}_{n=0}^{n=\infty}$ of complex black-hole resonances. The fundamental quasinormal resonance, the mode with the smallest imaginary part, determines the characteristic timescale $\tau_{\text{relax}}$ for generic black-hole perturbations to decay: $\tau_{\text{relax}} \sim 1/\omega_I (n = 0)$ [8, 11, 12].

The characteristic spectrum of black-hole resonances is of fundamental importance from both the astrophysical [13–15] and theoretical [19–24] points of view. In particular, these characteristic oscillations are expected to be excited in a variety of astrophysical processes.
involving black holes \cite{13,15}. The excitation of black-hole quasinormal resonances thus provides a promising observational way for identifying the physical parameters (masses and angular momenta) of astrophysical black holes.

It is worth noting that the characteristic resonance spectra of most black-hole spacetimes must be computed numerically \cite{13,15,25}. [See, however, \cite{26,32} for some interesting analytical results in the asymptotic limit $l \gg 1$ (the geometric-optics approximation with $\omega_R \gg 1$), and \cite{19,21,22} for interesting analytical results in the highly-damped asymptotic regime $n \gg 1$ (the $\omega_I \gg 1$ limit)]. A notable exception is provided by the family of rapidly-rotating (near-extremal) black holes with $a \simeq M$. (Here $M$ and $Ma$ are the mass and angular momentum of the black holes, respectively). In particular, it was shown in \cite{8,12} that, for rapidly-rotating black holes, the resonance spectrum of co-rotating ($m > 0$) modes is described by the remarkably simple analytic formula \cite{33}:

$$\omega(n; l, m) = m\Omega - i2\pi T_{\text{BH}}(n + \frac{1}{2} - i\delta),$$

where

$$T_{\text{BH}} \equiv \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)} \quad \text{and} \quad \Omega \equiv \frac{a}{r_+^2 + a^2}$$

are the physical parameters of the black hole: its temperature and angular velocity, respectively. [Here $r_\pm \equiv M \pm \sqrt{M^2 - a^2}$ are the radii of the black-hole horizons]. The parameter $\delta = \delta(l, m)$ is closely related to the angular eigenvalue of the corresponding angular equation, see Eq. (5) below.

It is worth emphasizing that the family (1) of co-rotating $m > 0$ resonances describes perturbations of the rapidly-rotating (near-extremal) black-hole spacetimes with extremely long relaxation times, $\tau_{\text{relax}} \sim 1/\omega_I = O(T_{\text{BH}}^{-1}) \gg M$.

II. RECENTLY PUBLISHED NUMERICAL RESULTS

Recently, Yang et. al. \cite{4} have computed numerically the polar ($m = 0$) quasinormal resonances of near-extremal Kerr black holes. Remarkably, the numerical data presented in \cite{4} (see, in particular, Fig. 7 of \cite{4}) reveals the existence of a previously unknown family of polar resonances which are quite accurately described by the same formula (1) with $m = 0$, namely \cite{34}:

$$\omega_{\text{numerical}}(n; m = 0) \simeq -i2\pi T_{\text{BH}}(n + l + 1).$$
It should be noted that the numerical results of Yang et al. [4] regarding the polar black-hole resonance spectrum (3) are quite surprising for two reasons:

(1) First of all, it should be emphasized that former analytical techniques and approximations which were used in [8, 12, 35–38] in order to derive the analytical formula (1) for the co-rotating \( m > 0 \) resonances [which are characterized by \( M\omega = O(1) \)] are no longer valid (and therefore cannot be applied) in the low frequency \( M\omega \ll 1 \) regime of the polar \( m = 0 \) resonances (3), see Appendix A for details.

(2) To the best of our knowledge, all former numerical studies of the Kerr quasinormal spectrum (see, in particular, the numerical studies of the Kerr spectrum in Refs. [39–42]) have not reported on the existence of these non-oscillatory (purely imaginary, \( \omega_R = 0 \)) polar resonances. In particular, former numerical studies [39–42] have claimed that the fundamental polar resonances are characterized by finite real (\( \omega_R \neq 0 \)) oscillation frequencies. It thus seems that there is some controversy in the literature regarding the nature of the black-hole polar resonances.

It is therefore of physical importance to prove (or disprove) analytically the existence of this unique family of non-oscillatory (purely imaginary) polar quasinormal resonances. The main goal of the present work is to provide such an analytical proof. In particular, as we shall show below, the spectrum of polar black-hole resonances can be studied analytically in the double limit \( M\omega \ll 1 \) with \( MT_{BH} \ll 1 \).

### III. DESCRIPTION OF THE SYSTEM

The physical system we consider consists of a massless spin-\( s \) field linearly coupled to a Kerr black hole of mass \( M \) and angular momentum per unit mass \( a \). In order to facilitate a fully analytical study, we shall assume that the black hole is rapidly-rotating with \( a \simeq M \).

Teukolsky [44] has shown that the dynamics of a massless spin-\( s \) field \( \Psi \) in the rotating Kerr black-hole spacetime can be described by a single master equation. Decomposing the field \( \Psi \) in the form

\[
s\Psi_{lm}(t, r, \theta, \phi) = e^{im\phi} s\psi_{lm}(r) sS_{lm}(\theta; a\omega) e^{-i\omega t},
\]

one finds [44] that the functions \( s\psi_{lm} \) and \( sS_{lm} \) obey radial and angular equations of the confluent Heun type [44–47] which are coupled by a separation constant \( sA_{lm}(a\omega) \). Here
$(t, r, \theta, \phi)$ are the Boyer-Lindquist coordinates, $\omega$ is the (conserved) frequency of the mode, $m$ is the azimuthal harmonic index of the mode, and $l$ is the spheroidal harmonic index with $l \geq \max\{|m|, |s|\}$. We shall henceforth study the polar $m = 0$ sector of black-hole perturbations.

The angular functions $S_l^m(\theta; a\omega)$ are known as the spin-weighted spheroidal harmonics. These functions satisfy the angular equation \[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial S}{\partial \theta} \right) + \left( a^2 \omega^2 \cos^2 \theta - 2a\omega s \cos \theta - s^2 \cot^2 \theta + s + sA_l - a^2 \omega^2 \right) S = 0 \quad (5) \]
and are required to be regular at the poles $\theta = 0$ and $\theta = \pi$. These boundary conditions pick out a discrete set of eigenvalues $\{sA_l\}$ labeled by the integer $l$.

In the small frequency $a\omega \ll 1$ regime, the regime we are interested in in this study, the angular functions can be approximated by the familiar spin-weighted spherical harmonics. The angular eigenvalues can then be expanded in the form \[ sA_l = l(l + 1) - s(s + 1) + \epsilon(a\omega)^2 + O(a^4\omega^4) , \quad (6) \]
where the expansion coefficient $\epsilon = \epsilon(l, s)$ is given by \[ \epsilon(l, s) = -\frac{3s^4 - 2s^2l(l + 1) - l(l + 1)(l^2 + l - 1)}{l(l + 1)(2l - 1)(2l + 3)} . \quad (7) \]

Thus, in the $a\omega \ll 1$ limit one can write \[ sA_l = l(l + 1) - s(s + 1) , \quad (8) \]
where \[ l \equiv l + \frac{\epsilon}{2l + 1} (a\omega)^2 + O(a^4\omega^4) \quad (9) \]
is nearly an integer with a small correction term of order $O(a^2\omega^2)$. (We shall henceforth omit the indices $s$ and $l$ for brevity.)

The radial function $\psi(r)$ satisfies the differential equation \[ \Delta^{-s} \frac{d}{dr} \left( \Delta^{s+1} \frac{d\psi}{dr} \right) + \left[ (r^2 + a^2)^2 \omega^2 - 2is\omega M(r^2 - a^2) + 2is\omega r - A \right] \psi = 0 \quad (10) \]
where $\Delta \equiv r^2 - 2Mr + a^2$. The scattering process of massless fields in the black-hole spacetime is governed by the radial wave equation (10) supplemented by the physically
motivated boundary conditions of purely ingoing waves crossing the black-hole horizon and a mixture of both ingoing (incident) and outgoing (reflected) waves at spatial infinity \[13–15\]:

\[
\psi \sim \begin{cases} 
  e^{-i\omega y} + R(\omega)e^{i\omega y} & \text{as } r \to \infty \ (y \to \infty) ; \\
  T(\omega)e^{-i\omega y} & \text{as } r \to r_+ \ (y \to -\infty),
\end{cases}
\]

(11)

where the “tortoise” radial coordinate \(y\) is defined by\[dy = \frac{(r^2 + a^2)/\Delta}{dr}.
\]The frequency-dependent functions \(R(\omega)\) and \(T(\omega)\) represent the reflection and transmission amplitudes for a field of conserved frequency \(\omega\) coming from infinity.

**IV. ANALYTIC DERIVATION OF THE QUASINORMAL RESONANCES**

The spectrum of black-hole quasinormal resonances is associated with the poles of the (frequency dependent) transmission and reflection amplitudes \[18, 39\]. As we shall now show, these characteristic resonances can be studied analytically in the double limit

\[ M\omega \ll 1 \quad \text{and} \quad MT_{\text{BH}} \ll 1 \]

(12)

of small frequencies and small black-hole temperatures (that is, rapidly-spinning black holes with \(a \simeq M\)).

To that end, we shall follow the analysis of \[49–51\] in order to calculate the scattering amplitudes in the low frequency regime \(M\omega \ll 1\). It is convenient to define new dimensionless variables

\[
x \equiv \frac{r - r_+}{r_+ - r_-} ; \quad \varpi \equiv \frac{\omega}{2\pi T_{\text{BH}}} ; \quad k \equiv \omega(r_+ - r_-),
\]

(13)

in terms of which the radial wave equation (10) becomes

\[
x^2(x + 1)^2\frac{d^2\psi}{dx^2} + (s + 1)x(x + 1)(2x + 1)\frac{d\psi}{dx} + \left[k^2x^4 + 2iskx^3 - Ax(x + 1) - is\varpi(x + 1/2) + \varpi^2/4\right]\psi = 0.
\]

(14)

The solution of the radial wave equation (14) in the near-horizon regime \(kx \ll 1\) which satisfies the physical boundary condition of purely ingoing waves crossing the black-hole horizon is given by \[49–51\]

\[
\psi = x^{-s - i\varpi/2}(x + 1)^{-s + i\varpi/2} _2F_1(-\ell - s, \ell - s + 1; 1 - s - i\varpi; -x),
\]

(15)

where \(_2F_1(a, b; c; z)\) is the hypergeometric function \[52\].
The solution of the radial wave equation (14) in the far-region \( x \gg |\varpi| + 1 \) is given by

\[
\psi = Ae^{-ikx}x^{\ell-s-1}F_1(\ell - s + 1; 2\ell + 2; 2ikx) + Be^{-ikx}x^{-\ell-s-1}F_1(-\ell - s; -2\ell; 2ikx),
\]

where \( F_1(a; c; z) \) is the confluent hypergeometric function [52].

The coefficients \( A \) and \( B \) can be determined by matching the near-horizon solution (15) with the far-region solution (16) in the overlap region \(|\varpi| + 1 \ll x \ll 1/k\). This matching procedure yields

\[
A = \frac{\Gamma(2\ell + 1)\Gamma(1 - s - i\varpi)}{\Gamma(\ell - s + 1)\Gamma(\ell + 1 - i\varpi)},
\]

and

\[
B = \frac{\Gamma(-2\ell - 1)\Gamma(1 - s - i\varpi)}{\Gamma(-\ell - s)\Gamma(-\ell - i\varpi)}.
\]

Finally, the asymptotic \((x \gg 1)\) form of the confluent hypergeometric functions [52] can be used in order to write the far-region solution (16) in the form given by Eq. (11). After some algebra one finds

\[
|T(\omega)|^2 = \Re \left\{ 4e^{i\pi(s+1/2)} \cos((\ell - s)\pi) \left[ \frac{\Gamma(-2\ell - 1)\Gamma(\ell - s + 1)}{\Gamma(2\ell + 1)\Gamma(-\ell - s)} \right]^2 \frac{\Gamma(\ell + 1 - i\varpi)}{\Gamma(-\ell - i\varpi)} (2k)^{2\ell+1} \right\},
\]

for the transmission probability.

The quasinormal modes represent the scattering resonances of the fields in the black-hole spacetime. These characteristic resonances correspond to the poles of the (frequency dependent) transmission and reflection amplitudes [18, 39]. The well-known pole structure of the Gamma functions [52] implies that the transmission probability (19) has poles at

\[
\ell + 1 - i\varpi = -n,
\]

where \( n = 0, 1, 2, \ldots \) is the resonance parameter. Equation (20) represents the resonance condition for the characteristic quasinormal frequencies of the black-hole spacetime.

Taking cognizance of Eq. (13), one can express the polar black-hole resonances in the simple form

\[
\omega_n = -i2\pi T_{BH}(\ell + 1 + n).
\]

It is worth emphasizing again that the resonance spectrum (21) is valid in the small frequency regime \( M\omega \ll 1 \). This implies that the spectrum (21) is valid in the regime of rapidly-rotating (near-extremal) black holes with \( MT_{BH} \ll 1 \) [53]. Note also that the analytically
derived polar spectrum \((21)\) agrees with the recently published numerical data of Yang et. al. \([4]\).

V. SUMMARY

Motivated by the recently published numerical results of Yang et. al. \([4]\), we have studied analytically the polar \((m = 0)\) resonance spectrum of rapidly-rotating (near-extremal) black holes. The numerical results reported by Yang et. al. \([4]\) are quite surprising: while former numerical studies of the Kerr quasinormal spectrum \([39–42]\) have reported on polar quasinormal resonances which are characterized by finite oscillations frequencies \((\omega_R \neq 0)\), the new numerical study of \([4]\) has revealed the existence of a different branch of purely imaginary (non-oscillatory, \(\omega_R = 0\)) polar quasinormal resonances.

This new branch of polar quasinormal resonances seems to have been overlooked in former numerical studies of the Kerr quasinormal spectrum \([39–42]\). Moreover, we have shown (see Appendix A) that former analytical techniques and approximations which were used in \([8, 12, 36, 38]\) in order to study the co-rotating \(m > 0\) resonances of rapidly-rotating black holes are not valid (and therefore cannot be applied) in the low frequency regime of the polar \(m = 0\) quasinormal resonances.

Our main goal in the present study was to prove analytically the existence of this unique family of non-oscillatory (purely-imaginary) black-hole resonances. Using an appropriate small frequency \(M\omega \ll 1\) approximation for the Teukolsky wave equation (instead of the \(\omega - m\Omega \ll \omega\) approximation which was used in \([8, 12, 36, 38]\) in order to study the co-rotating \(m > 0\) resonances), we have established the existence of the new family \((21)\) of purely-imaginary polar \(m = 0\) quasinormal frequencies.

Finally, it is interesting to analyze the complex spectrum of total reflection modes (TRM) which characterizes the black-hole spacetime. These important frequencies are determined by the condition \(T(\omega_{\text{TRM}}) = 0\). Taking cognizance of Eq. \((19)\), one finds that the TRMs are determined by the requirement \(1/\Gamma(-\ell - i\varpi) = 0\) [see Eq. \((19)\) with \(T(\omega) = 0\)]. This condition yields the simple spectrum

\[
\omega_{\text{TRM}}^n = -i2\pi T_{\text{BH}}(-\ell + n)
\]

for the TRMs of the black-hole spacetime.
Taking cognizance of Eqs. (21) and (22), one realizes that the two spectra are almost identical: there are close pairs of QNMs and TRMs. In particular, the difference between a QNM frequency of overtone index $n$ and a nearby TRM frequency of overtone index $n' = n + 2l + 1$ is extremely small, of the order of $O(T_{BH}^3)$ [54]. This observation implies that one must use a numerical scheme of extreme precision in order to compute the QNM frequencies numerically and, in particular, in order to distinguish them numerically from the TRM frequencies. The close proximity between the frequencies (21) of the quasinormal resonances and the frequencies (22) of the total reflection modes may explain the failure of previous numerical studies to observe this unique family of purely imaginary polar resonances [55].

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Appendix A: The range of validity of former analytical studies

In this Appendix we shall discuss the range of validity of the resonance spectrum \[1\]. This resonance spectrum, originally derived in \[12\], stems from the near-extremal resonance condition of Detweiler \[38\], see Eq. (9) of \[38\]. The resonance condition (9) of \[38\] is based on an earlier analysis of Teukolsky and Press, see Appendix A of \[36\]. Thus, in order to find the range of validity of the resonance condition (9) of \[38\], one should carefully examine the range of validity of the analysis presented in Appendix A of \[36\].

By carefully repeating the analysis of \[36\] step-by-step, one realizes that the transition from Eq. (A3) of \[36\] to Eq. (A4) of \[36\] (see Appendix A of \[36\]) is based on the assumption

\[
\omega x \gg \omega - m\Omega ,
\]

(A1)

where

\[
x \equiv (r - r_+)/r_+
\]

(A2)
in the notations of \[36\]. [In particular, when moving from Eq. (A3) to Eq. (A4), the authors of \[36\] keep the term \(4i\omega r_+ x\) in the coefficient of \(dR/dx\) but neglect the terms \(4iM(\omega - m\Omega)\) and \(-(s + 1)\sigma\) in this same coefficient, where \(\sigma \equiv (r_+ - r_-)/r_+\) in the notations of \[36\]. This approximation is valid provided \(\omega x \gg \max(\omega - m\Omega, \sigma/M)\).] In addition, the transition from Eq. (A3) of \[36\] to Eq. (A8) of \[36\] is based on the explicit assumption

\[
x \ll 1 .
\]

(A3)

Thus, the dimensionless variable \(x\) is required to satisfy the two inequalities \((\omega - m\Omega)/\omega \ll x \ll 1\). These two requirements can only be satisfied simultaneously in the regime

\[
\omega - m\Omega \ll \omega .
\]

(A4)
The inequality (A4) is indeed satisfied by the co-rotating \(m > 0\) modes \([1]\) [these modes are characterized by \(M\omega = O(1)\) and \(M(\omega - m\Omega) = O(MT_{\text{BH}}) \ll 1\)]. On the other hand, the inequality (A4) is obviously violated by the family \([3]\) of polar \(m = 0\) quasinormal modes.

One therefore concludes that the near-extremal resonance condition, Eq. (9) of \[38\], is not valid in the low frequency regime of polar modes. Thus, in the present study we shall use a different analytical approach in order to prove the existence of the polar family \([3]\) of quasinormal resonances. This alternative analytical approach is based on a low frequency approximation \(M\omega \ll 1\) to the Teukolsky wave equation (as opposed to the \(\omega - m\Omega \ll \omega\) approximation in \[36\]).
approximation used in Appendix A of [36]). This low frequency approximation is suitable for the analysis of the polar resonance spectrum in the near extremal $MT_{\text{BH}} \ll 1$ regime.

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The parameter $s$ is known as the spin weight of the field. It is given by $s = 0$ for scalar perturbations, $s = \pm \frac{1}{2}$ for massless neutrino perturbations, $s = \pm 1$ for electromagnetic perturbations, and $s = \pm 2$ for gravitational perturbations \cite{44}. 

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\cite{53} It is worth noting that the family of polar resonances, Eq. (21), describes perturbations of the black-hole spacetime which, in the maximally-spinning limit $a \to M$ ($T_{\text{BH}} \to 0$), are characterized by extremely long relaxation times: $\tau = O(T_{\text{BH}}^{-1}) \gg M$.

\cite{54} Specifically, one finds from Eqs. (21) and (22) $\omega_n^{\text{QNM}} - \omega_n^{\text{TRM}} = -i 4\pi T_{\text{BH}}(\ell - l)$. Using the relation $\ell = l + O(a^2\omega^2)$ [see Eq. (19)] with $\omega = O(T_{\text{BH}})$ [see Eq. (21)], one finds $\omega_n^{\text{QNM}} - \omega_n^{\text{TRM}} = O(T_{\text{BH}}^3)$.

\cite{55} As explained above, this previously unknown family of non-oscillatory polar QNMs was identified only recently in the numerical analysis of Yang et. al. \cite{4}.