Research Article

A New Result for a Blow-up of Solutions to a Logarithmic Flexible Structure with Second Sound

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This paper is concerned with a problem of a logarithmic nonuniform flexible structure with time delay, where the heat flux is given by Cattaneo’s law. We show that the energy of any weak solution blows up infinite time if the initial energy is negative.

1. Introduction

In this work, we consider the vibrations of an inhomogeneous flexible structure system with a constant internal delay and logarithmic nonlinear source term:

\[
\begin{align*}
  m(x)u_{tt} - (p(x)u_t + 2\delta(x)u_x)_t + \mu u_{xx} + \mu u_t(x, t - \tau) = u|u|^{\alpha-1} \ln |u|, \\
  \theta_t + k\theta + \eta u_x = 0, \\
  \tau u_t + \beta q + \lambda \theta = 0
\end{align*}
\]

with boundary conditions

\[
  u(0, t) = u(L, t) = 0; \theta(0, t) = \theta(L, t) = 0, t \geq 0,
\]

and initial conditions

\[
  u(x, 0) = u_0(x), u_t(x, 0) = u_1(x); \theta(x, 0) = \theta_0(x); q(x, 0) = q_0(x), x \in [0, L],
\]

where \( u(x, t) \) is the displacement of a particle at position \( x \in [0, L] \), and the time \( t \geq 0 \), \( \eta > 0 \) is the coupling constant depending on the heating effect, \( p \geq 2, \gamma, \beta, \) and \( k \) are positive constants, and \( \mu \) is a real number. \( \tau > 0 \) is the relaxation time describing the time lag in the response for the temperature, and \( \tau_0 > 0 \) represents the time delay in particular if \( \tau = 0 \) reduces to the viscothermoelastic system with delay, in which the heat flux is given by Fourier’s law instead of Cattaneo’s law, where \( q = q(x, t) \) is the heat flux, and \( m(x), \delta(x) \), and \( p(x) \) are responsible for the inhomogeneous structure of the beam and, respectively, denote mass per unit length of structure, coefficient of internal material damping (viscoelastic property), and a positive function related to the stress acting on the body at a point \( x \). The model of heat condition, originally due to Cattaneo, is of hyperbolic type. We recall the assumptions of \( m(x), \delta(x) \), and \( p(x) \) in [1, 2] such that

\[
m, \delta, p \in W^{1,\infty}(0, L), m(x), \delta(x) \text{ and } p(x) > 0, \forall x \in [0, L].
\]

In these kinds of problems, Gorain [3] in 2013 has established uniform exponential stability of the problem

\[
m(x)u_{tt} - (p(x)u_t + 2\delta(x)u_x)_t = f(x), \text{ on } (0, L) \times \mathbb{R}^+,
\]

which describes the vibrations of an inhomogeneous flexible...
structure with an exterior disturbing force. More recently, Misra et al. [4] showed the exponential stability of the vibrations of a inhomogeneous flexible structure with thermal effect governed by the Fourier law.

$$m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{tx})_x - k\theta_x = f(x),$$
$$\theta_t - \theta_{xx} - ku_{xt} = 0. \tag{6}$$

In addition, we can cite other works in the same form like the system in [5]; Racke studied the exponential stability in linear and nonlinear 1d of thermoelasticity system with second sound given by

$$\begin{cases}
m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{tx})_x - k\theta_x = 0, \text{on}(0, L) \times \mathbb{R}^+ \\
\theta_t + kq_x + \eta u_{tx} = 0, \text{on}(0, L) \times \mathbb{R}^+ \\
\tau q_t + \beta q + k\theta_x = 0, \text{on}(0, L) \times \mathbb{R}^+,
\end{cases} \tag{7}$$

Now for the multidimensional system, Messiah in [6] established a local existence and a blow-up result for a multidimensional nonlinear system of thermoelasticity with second sound (see in this regard Refs. [7–10]); for the same problem above, Alves et al. proved that system (7) is polynomial decay (see [1]), with boundary and initial conditions:

$$u(0, t) = u(L, t) = 0; \theta(0, t) = \theta(L, t) = 0, t \geq 0,$$
$$u(x, 0) = u_0(x), u_t(x, 0) = u_1(x); \quad \theta(x, 0) = \theta_0(x); \quad q(x, 0) = q_0(x), x \in [0, L]. \tag{8}$$

We know that the dynamic systems with delay terms have become a significant examination subject in differential equation since the 1970s of the only remaining century. The delay effect that is similar to memory processes is important in the research of applied mathematics such as physics, non-instant transmission phenomena, and biological motivation; model (7) is related to the following problem with delay terms:

$$\begin{cases}
m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{tx})_x + \eta\theta_x + \mu u_t(x, t - \tau_0) = 0 \quad x \in (0, L), t > 0, \\
\theta_t + kq_x + \eta u_{tx} = 0 \quad x \in (0, L), t > 0, \\
\tau q_t + \beta q + k\theta_x = 0 \quad x \in (0, L), t > 0, \\
u(0, t) = u(L, t) = 0; \theta(0, t) = \theta(L, t) = 0, t \geq 0, \\
u(x, 0) = u_0(x), u_t(x, 0) = u_1(x); \quad \theta(x, 0) = \theta_0(x), \quad q(x, 0) = q_0(x), \quad x \in [0, L].
\end{cases} \tag{9}$$

The authors prove that the system (9) is well posed and exponential decay under a small condition on time delay (see [2]). Now in the presence of source term, the system (9) becomes the system studied in this work with a logarithmic source term; this type of problems is encountered in many branches of physics such as nuclear physics, optics, and geophysics. It is well known, from the quantum field theory, that such kind of logarithmic nonlinearity appears naturally in inflation cosmology and in supersymmetric field theories (see [11–13]).

This work is organized as follows: In “Statement of Problem,” we talk briefly about the local existence of the systems (1), (2), and (3), and we define some space and theorem used. In “Blow-up of Solution,” the blow-up result is proved.

2. Statement of Problem

Let us introduce the function

$$z(x, \rho, t) = u_1(x, t - \rho_0), x \in (0, L), \rho \in (0, 1), t > 0. \tag{10}$$

Thus, we have

$$\tau_0 z(x, \rho, t) + z_{\rho}(x, \rho, t) = 0, x \in (0, L), \rho \in (0, 1), t > 0. \tag{11}$$

Then, problems (1)–(3) are equivalent to

$$\begin{cases}
m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{tx})_x + \eta\theta_x + \mu z(x, 1, t) = |u|^p - 2 \ln |u|^4, \quad x \in (0, L), t > 0, \\
\tau_0 z(x, \rho, t) + z_{\rho}(x, \rho, t) = 0 \quad x \in (0, L), \rho \in (0, 1), t > 0, \\
\theta_t + kq_x + \eta u_{tx} = 0 \quad x \in (0, L), t > 0, \\
\tau q_t + \beta q + k\theta_x = 0 \quad x \in (0, L), t > 0,
\end{cases} \tag{12}$$

$$\begin{cases}
u(0, t) = u(L, t) = 0; \theta(0, t) = \theta(L, t) = 0, t \geq 0, \\
u(x, 0) = u_0(x), u_t(x, 0) = u_1(x); \quad \theta(x, 0) = \theta_0(x), x \in [0, L] \\
q(x, 0) = q_0(x), x \in [0, L] \\
z(x, 0, t) = u_1(x, t), x \in (0, L), t > 0, \\
z(x, \rho, 0) = f_0(x, -\rho\tau_0), x \in (0, L), \rho \in (0, 1).
\end{cases} \tag{13}$$

We first state a local existence theorem that can be established by combining the arguments of related works. Let \(v = \Phi_t\) and denote by

$$\Phi = (u, v, \theta, q, z)^T, \quad \Phi(0) = \Phi_0 = (u_0, u_1, \theta_0, q_0, f_0)^T. \tag{14}$$

The state space of \(\Phi\) is the Hilbert space

$$H^1_0(0, L) \times L^2_0(0, L) \times L^2_0(0, L) \times L^2_0((0, 1) \times (0, L)). \tag{15}$$

**Theorem 1.** Assume that

$$2 < p \leq \frac{2n}{n - 2}, \text{if } n \geq 3. \tag{16}$$

Then, for every \(\Phi_0 \in \mathcal{X}\), there exists a unique local solution in the class \(\Phi \in C([0, T]),\).
3. Blow-up of Solution

In this section, we prove that the solutions for the problems (12)–(13) blow up in a finite time when the initial energy is negative. We use the improved method of Salim and Mesmoudi [6]. We define the energy associated with problems (12)–(13) by

\[
E(t) = \frac{1}{2} \left( \|m(x)\|_\infty \|u(t)\|_2^2 + \frac{1}{2} \|p(x)\|_\infty \|u_t\|_2^2 \right) \\
+ \frac{\tau}{2} \|q\|_2^2 + \frac{1}{2} \|\theta\|_2^2 + \frac{\tau_0 |t|}{2} \int_0^1 \|z(x, \rho, t)\|^2 d\rho \\
+ \frac{\gamma}{p^2} \|u\|^p - \frac{1}{p} \int_0^t |u|^p \ln |u|^r dx.
\]

(17)

**Lemma 2.** Suppose that

\[
2 < p \leq \frac{2n}{n-2}, \quad n \geq 3.
\]

Then, there exists a positive constant \( C > 0 \) depending on \([0,L]\) only, such that

\[
\left( \int_0^L |u|^p \ln |u|^r dx \right)^{\frac{1}{r}} \leq C \left( \int_0^L |u|^p \ln |u|^r dx + \|u_x\|_2^2 \right),
\]

(19)

for any \( u \in H^1_0(0,L) \) and \( 2 \leq s \leq p \), provided that \( \int_0^L |u|^p \ln |u|^r dx \geq 0 \).

**Proof.** If \( \int_0^L |u|^p \ln |u|^r dx > 1 \), then

\[
\left( \int_0^L |u|^p \ln |u|^r dx \right)^{\frac{1}{r}} \leq \int_0^L |u|^p \ln |u|^r dx.
\]

(20)

If \( \int_0^L |u|^p \ln |u|^r dx \leq 1 \), then we set

\[
\Gamma_1 = \{ x \in [0,L] | |u| > 1 \},
\]

(21)

and, for any \( \beta \leq 2 \), we have

\[
\left( \int_0^1 |u|^{\beta+1} dx \right)^{\frac{1}{\beta+1}} \leq \int_0^1 |u|^{\beta+1} dx \leq \int_{\Gamma_1} |u|^{\beta+1} dx = \|u\|_{p+1}^{\beta+1}.
\]

(22)

We choose \( \beta = 2p/(p+1) < 2 \) to get

\[
\left( \int_0^L |u|^p \ln |u|^r dx \right)^{\frac{1}{r}} \leq \|u\|_{p+1}^{\beta+1} \leq C \|u_x\|_2^2.
\]

(23)

Combining (20) and (23), the result was obtained.

**Lemma 3.** There exists a positive constant \( C > 0 \) depending on \([0,L]\) only, such that

\[
\|u\|_p^p \leq C \left[ \int_0^L |u|^p \ln |u|^r dx + \|u_x\|_2^2 \right],
\]

(24)

for any \( u \in H^1_0(0,L) \), provided that \( \int_0^L |u|^p \ln |u|^r dx \geq 0 \).

**Proof.** We set

\[
\Gamma_+ = \{ x \in [0,L] | |u| > e \}
\]

and

\[
\Gamma_- = \{ x \in [0,L] | |u| \leq e \},
\]

(25)

thus

\[
\|u\|_p^p \leq \int_{\Gamma_+} |u|^p dx + \int_{\Gamma_-} |u|^p dx \leq \int_{\Gamma_+} |u|^p dx + \int_{\Gamma_-} |u|^p dx + e^p \int_{\Gamma_-} \frac{|u|^p}{e} dx \\
\leq \int_{\Gamma_+} |u|^p dx + e^p \int_{\Gamma_-} \frac{|u|^p}{e} dx \\
\leq \int_0^L |u|^p \ln |u|^r dx + e^p \int_0^L \frac{|u|^p}{e} dx \\
\leq C \left( \int_0^L |u|^p \ln |u|^r dx + \|u_x\|_2^2 \right).
\]

(26)

By using the inequalities \( \|u\|_2^2 \leq C \|u\|_p^2 \leq C(\|u\|_p^{2p}) \), we have the following corollary.

**Corollary 4.** There exists a positive constant \( C > 0 \) depending on \([0,L]\) only, such that

\[
\|u\|_2^2 \leq C \left( \left( \int_0^L |u|^p \ln |u|^r dx \right)^{\frac{1}{r}} + \|u_x\|_2^2 \right).
\]

(27)

provided that \( \int_0^L |u|^p \ln |u|^r dx \geq 0 \).

**Lemma 5.** There exists a positive constant \( C > 0 \) depending on \([0,L]\) only, such that

\[
\|u\|_p^p \leq C \left[ \|u\|_p^p + \|u_x\|_2^2 \right],
\]

(28)

for any \( u \in H^1_0(0,L) \) and \( 2 \leq s \leq p \).

**Proof.** If \( \|u\|_p \geq 1 \), then

\[
\|u\|_p^p \leq \|u\|_p^p.
\]

(29)

If \( \|u\|_p \leq 1 \), then \( \|u\|_p^p \leq \|u\|_p^2 \). Using the Sobolev embedding theorems, we have

\[
\|u\|_p^p \leq \|u\|_p^2 \leq C \|u_x\|_2^2.
\]

(30)

Now we are ready to state and prove our main result. For
this purpose, we define

\begin{align*}
H(t) &= -E(t) = -\frac{1}{2} \left( \|m(x)\|_{\infty} \|u_{t}(t)\|_{2}^{2} \right) - \frac{1}{2} \left( \|p(x)\|_{\infty} \|u_{x}\|_{2}^{2} \right) \\
&\quad - \frac{\gamma}{2} \|u\|_{p}^{2} + \frac{1}{p} \int_{0}^{t} |u|^{p} \ln |u|^{p} dx.
\end{align*}

(31)

Corollary 6. Assume that (18) holds. Then

\begin{align*}
\|u\|_{p}^{2} &\leq C \left\{ \left( 1 - \frac{\gamma}{p} \frac{p}{p(x)\|p(x)\|_{\infty}} \right) \|u\|_{p}^{2} - \frac{2}{\|p(x)\|_{\infty}} H(t) \right. \\
&\quad - \frac{\gamma}{2} \|u_{t}\|_{2}^{2} - \frac{\tau}{2} \|\theta\|_{2}^{2} - \frac{\tau}{2} \|p_{t}\|_{2}^{2} \\
&\quad - \frac{1}{\|p(x)\|_{\infty}} \|\theta\|_{2}^{2} - \frac{\tau}{\|p(x)\|_{\infty}} \int_{0}^{t} |z(x, \rho, t)|^{2} d\rho \\
&\quad + \frac{2}{p} \|p(x)\|_{\infty} \int_{0}^{t} |u|^{p} \ln |u|^{p} dx \left. \right\},
\end{align*}

(32)

for any \(u \in (H_{0}^{4}(0, L))^{n}\) and \(2 \leq s \leq p\).

Theorem 7. Assume that (18) holds. Assume further that

\begin{align*}
E(0) &= \frac{1}{2} \left( \|m(x)\|_{\infty} \|u_{t}(t)\|_{2}^{2} \right) + \frac{1}{2} \left( \|p(x)\|_{\infty} \|\nabla u_{0}\|_{2}^{2} \right) \\
&\quad + \frac{\tau}{2} \|q_{0}\|_{2}^{2} + \frac{1}{2} \|\theta_{0}\|_{2}^{2} + \tau_{0} |p_{0}|^{2} \int_{0}^{t} \int_{0}^{\xi} \left( |z(x, -\rho t)|^{2} \right) d\rho dx \\
&\quad + \frac{\gamma}{2} \|u_{0}\|_{p}^{2} + \frac{1}{p} \int_{0}^{t} |u_{0}|^{p} \ln |u_{0}|^{p} dx < 0.
\end{align*}

(33)

Then, the solution of (12) blows up in finite time.

Proof. we have

\begin{align*}
E(t) &\leq E(0) < 0,
\end{align*}

(34)

and

\begin{align*}
H'(t) &= -E'(t) = 2 \left( \|\delta(x)\|_{\infty} \|u_{t}(t)\|_{2}^{2} \right) + \beta \|q\|_{2}^{2} + |p| \int_{0}^{t} |z(x, 1, t)|^{2} dx.
\end{align*}

(35)

Hence

\begin{align*}
H'(t) &\geq C_{0} \left\{ \left( \|\delta(x)\|_{\infty} \|u_{t}(t)\|_{2}^{2} \right) + |p| \int_{0}^{t} |z(x, 1, t)|^{2} dx \right\} \geq 0; \forall t \in [0, T).
\end{align*}

(36)

Consequently, we get

\begin{align*}
0 < H(0) &\leq H(t) \leq \int_{0}^{t} |u|^{p} \ln |u|^{p} dx; \forall t \in [0, T),
\end{align*}

(37)

by virtue of (17) and (31). We then introduce

\begin{align*}
L(t) &= H^{1-\alpha}(t) + \epsilon \int_{0}^{t} |m(x)u_{t}(t)u(t) + 4\delta(x)|u_{x}|^{2} dx + \epsilon \int_{0}^{t} \frac{\alpha}{L} u dx,
\end{align*}

(38)

where \(\epsilon > 0\) to be specified later and

\begin{align*}
2(p - 2) = \alpha < \frac{p - 2}{2p} < 1.
\end{align*}

(39)

A direct differentiation of \(L(t)\) gives

\begin{align*}
L'(t) &= (1 - \alpha) H^{-\alpha}(t) H'(t) + \epsilon \int_{0}^{t} |m(x)u_{t}|^{2} dx \\
&\quad + \epsilon \frac{\alpha}{L} \int_{0}^{t} \theta u_{x} dx - \epsilon \int_{0}^{t} p(x) u_{x}^{2} dx + 2\epsilon \int_{0}^{t} \theta u_{x} dx \\
&\quad - \epsilon \int_{0}^{t} \mu z(x, 1, t) u dx + \epsilon \int_{0}^{t} |u|^{p} \ln |u|^{p} dx - \epsilon \frac{\alpha}{L} \int_{0}^{t} q u dx,
\end{align*}

(40)

using the inequality of young

\begin{align*}
2\epsilon \int_{0}^{t} \theta u_{x} dx &\geq -\epsilon \|\theta\|_{2}^{2} - \epsilon \|u_{x}\|_{2}^{2},
\end{align*}

(41)

\begin{align*}
-\epsilon \int_{0}^{t} p(x) u_{x}^{2} dx &\geq -\epsilon \|p\|_{\infty} \|u_{x}\|_{2}^{2},
\end{align*}

(42)

\begin{align*}
\epsilon \int_{0}^{t} m(x) |u_{t}|^{2} dx &\geq \epsilon \|m(x)\|_{\infty} \|u_{t}\|_{2}^{2},
\end{align*}

(43)

\begin{align*}
\epsilon \int_{0}^{t} q u_{x} dx &\geq -\epsilon \frac{\alpha}{L} \|u_{x}\|_{2}^{2} - \epsilon \frac{\alpha}{L} \|q\|_{2}^{2},
\end{align*}

(44)

and

\begin{align*}
-\epsilon \int_{0}^{t} \mu z(x, 1, t) u dx &\geq -\epsilon \mu \left\{ \frac{\xi_{1}}{2} \int_{0}^{t} |z(x, 1, t)|^{2} dx + \frac{1}{2k} \|u\|_{2}^{2} \right\}; \forall \xi_{1} > 0,
\end{align*}

(45)

\begin{align*}
-\epsilon \frac{\alpha}{L} \int_{0}^{t} q u dx &\geq \epsilon \frac{\alpha}{L} \left\{ \frac{\xi_{2}}{2} \|q\|_{2}^{2} + \frac{1}{2k} \|u\|_{2}^{2} \right\}; \forall \xi_{2} > 0.
\end{align*}

(46)

Substituting (41), (42), (43), (44), (45), and (46) in (40),
we get

\[ L'(t) \geq (1 - \alpha)H^{-\alpha}(t)H'(t) + \varepsilon \left( \|m(x)\|_{\infty} - \frac{\eta_T}{2k} \right) \|u_t\|_2^2 \]

\[-\varepsilon \left\{ \|p(x)\|_{\infty} + \eta \right\} \|u_x\|_2^2 - \varepsilon \left\{ \|p(x)\|_{\infty} + \eta \right\} \|u_t\|_2^2\]

\[-\varepsilon \left[ \frac{\eta_T}{2k} \right] \|q\|_2^2 + \varepsilon \int_0^L |u|^p \ln |u|^p dx \]

\[-\varepsilon \|u|k \|_2 + \frac{\eta_T}{2k} \|q\|_2^2 \|u_t\|^2.\]  

(47)

We obtain from (35) and (47) the following:

\[ L'(t) \geq \left( 1 - \alpha - \varepsilon C \right)H^{-\alpha}(t)H'(t) + \varepsilon \left( \|m(x)\|_{\infty} - \frac{\eta_T}{2k} \right) \|u_t\|_2^2 \]

\[-\varepsilon \left\{ \|p(x)\|_{\infty} + \eta \right\} \|u_x\|_2^2 - \varepsilon \left\{ \|p(x)\|_{\infty} + \eta \right\} \|u_t\|_2^2\]

\[-\varepsilon \eta_T \|\theta\|_2 - \varepsilon \frac{\eta_T}{2k} \|q\|_2^2 + \varepsilon \int_0^L |u|^p \ln |u|^p dx \]

\[-\varepsilon \left[ \frac{\eta_T}{2k} \right] \|q\|_2^2 \|u_t\|^2.\]  

(48)

We also set \( \xi_1 = \xi_2 = H^{-\alpha}(t) \); hence, (48) gives

\[ L'(t) \geq \left( 1 - \alpha - \varepsilon C \right)H^{-\alpha}(t)H'(t) + \varepsilon \left( \|m(x)\|_{\infty} - \frac{\eta_T}{2k} \right) \|u_t\|_2^2 \]

\[-\varepsilon \left\{ \|p(x)\|_{\infty} + \eta \right\} \|u_x\|_2^2 - \varepsilon \left\{ \|p(x)\|_{\infty} + \eta \right\} \|u_t\|_2^2\]

\[-\varepsilon \left[ \frac{\eta_T}{2k} \right] \|q\|_2^2 + \varepsilon \int_0^L |u|^p \ln |u|^p dx - \varepsilon M \frac{2k}{\alpha} H^{-\alpha}(t)|u_t|^2,\]  

(49)

where \( C \) and \( M \) are strictly positive constants depending only on \( k, \eta, \beta, |\mu| \).

For \( 0 < a < 1 \), we have

\[ L'(t) \geq \left( 1 - \alpha - \varepsilon C \right)H^{-\alpha}(t)H'(t) \]

\[ + \varepsilon \left( \|m(x)\|_{\infty} \left( 1 + \frac{P}{2} (1 - a) \right) - \frac{\eta_T}{2k} \right) \|u_t\|_2^2 \]

\[ + \varepsilon \left[ \|p(x)\|_{\infty} \left( \frac{P}{2} (1 - a) - 1 \right) + \eta \right] \|u_x\|_2^2 \]

\[ + \varepsilon \left\{ \frac{\eta_T}{2k} \right\} \|\theta\|_2^2 - \varepsilon \left\{ \frac{\eta_T}{2k} \right\} \|q\|_2^2 \]

\[ + \varepsilon \int_0^L |u|^p \ln |u|^p dx + \varepsilon M \frac{2k}{\alpha} H^{-\alpha}(t)|u_t|^2,\]  

\[ - \varepsilon M \frac{2k}{\alpha} H^{-\alpha}(t)|u_t|^2 + \varepsilon (1 - a) H(t).\]  

(50)

Using (27), (37) and Young's inequality, we find

\[ H^{\alpha}(t)|u_t|^2 \leq \left( \int_0^L |u|^p \ln |u|^p dx \right)^{\frac{a}{p}} \|u_t\|^2 \]

\[ \leq C \left( \int_0^L |u|^p \ln |u|^p dx \right)^{\frac{a}{p}} \|u_t\|^2 \]

\[ \leq C \left( \int_0^L |u|^p \ln |u|^p dx \right)^{\frac{a}{p}} + ||u_t||_2^2 + \left( \int_0^L |u|^p \ln |u|^p dx \right)^{\frac{a}{p}}.\]  

(51)

Exploiting (39), we have

\[ 2 < ap + 2 \leq p \text{ and } 2 < \frac{ap^2}{p - 2} \leq p.\]  

(52)

Thus, lemma 1 yields

\[ H^{\alpha}(t)|u_t|^2 \leq C \left( \int_0^L |u|^p \ln |u|^p dx + ||u_t||_2^2 \right).\]  

(53)

Combining (50) and (53), we obtain

\[ L'(t) \geq \left( 1 - \alpha - \varepsilon C \right)H^{-\alpha}(t)H'(t) \]

\[ + \varepsilon \left( \|m(x)\|_{\infty} \left( 1 + \frac{P}{2} (1 - a) \right) - \frac{\eta_T}{2k} \right) \|u_t\|_2^2 \]

\[ + \varepsilon \left[ \|p(x)\|_{\infty} \left( \frac{P}{2} (1 - a) - 1 \right) + \eta \right] \|u_x\|_2^2 \]

\[ + \varepsilon \left\{ \frac{\eta_T}{2k} \right\} \|\theta\|_2^2 - \varepsilon \left\{ \frac{\eta_T}{2k} \right\} \|q\|_2^2 \]

\[ + \varepsilon \int_0^L |u|^p \ln |u|^p dx - \varepsilon M \frac{2k}{\alpha} H^{-\alpha}(t)|u_t|^2,\]

\[ - \eta + \frac{pe(1 - a)}{2} \|\theta\|_2^2 \]

\[ + \varepsilon \left[ \frac{\eta_T}{2k} \right] \|q\|_2^2 + \varepsilon \frac{pe(1 - a)}{2} \|u_t\|^2 \]

\[ + \varepsilon \frac{\eta_T}{2k} \|\theta\|_2^2 + \varepsilon \frac{pe(1 - a)}{2} \|u_t\|^2 \]

\[ + \varepsilon \frac{\eta_T}{2k} \|q\|_2^2 + \varepsilon \frac{pe(1 - a)}{2} \|u_t\|^2 \]

\[ + \varepsilon \frac{\eta_T}{2k} \|\theta\|_2^2 + \varepsilon \frac{pe(1 - a)}{2} \|u_t\|^2.\]  

(54)

At this point, we choose \( a > 0 \) so small that

\[ -\eta + \frac{pe(1 - a)}{2} > 0,\]

\[ \left( \frac{P}{2} (1 - a) - 1 \right) > 0,\]

\[ \frac{\eta_T}{2k} (1 - a) > 0,\]

and \( k \) so large that
\[
\|p(x)\|_x \leq \frac{p}{2} (1 - a) - 1 + r - C \frac{M}{2k} > 0,
\]
\[
a - C \frac{M}{2k} > 0,
\]
\[
\|m(x)\|_x \leq (1 + \frac{p}{2} (1 - a)) - \eta \frac{r}{2k} > 0,
\]
\[
- \frac{\eta \tau}{2k} + \frac{pr(1 - a)}{2} > 0.
\]

Once \(C\) and \(a\) are fixed, we pick \(\varepsilon\) so small so that
\[
(1 - a) - \varepsilon C > 0.
\]

Hence, (54) becomes
\[
L'(t) \geq \{(1 - a) - \varepsilon C\} H^\alpha(t) H(t) + \varepsilon A_2 \|u_t\|^2_2 + \varepsilon A_3 \|\theta\|^2_2 + \varepsilon \left\{a - C \frac{M}{2k}\right\} \int_0^t |u|^p \ln |u|^\gamma dx
\]
\[
+ \frac{\varepsilon r \psi(1 - a)}{2} |u| \int_0^t |z(x, \rho, t)|^2 d\rho dx + \frac{\varepsilon \gamma (1 - a)}{2} \|u\|^2_r + \varepsilon (1 - a) \rho H(t),
\]
where \(A_1 - A_4\) are strictly positive constants depending only on \(p, r, \eta, k, a\).

Thus, for some \(A_0 > 0\), estimate (58) becomes
\[
L'(t) \geq A_0 \left\{H(t) + \|u_t\|^2_2 + \|u_r\|^2_2 + \|u\|^p \|\theta\|^2_2 + \|u\|^p \|q\|^2_2 + |\theta|^2_2 + \int_0^t |u|^p \ln |u|^\gamma dx + \int_0^t |z(x, \rho, t)|^2 d\rho dx, \right\},
\]
and
\[
L(t) \geq L(0) > 0, \forall t \geq 0.
\]

Next, using Hölder’s inequality and the embedding \(\|u\|_2 \leq C \|u\|_p\), we have
\[
\left\| \int_0^t m(x) uu_t dx \right\| \leq \|m(x)\|_x \|u_t\|^2_2 \leq C \|u\|_2 \|u_r\|_2, \quad (61)
\]
and exploiting Young’s inequality, we obtain
\[
\left\| \int_0^t m(x) uu_t dx \right\| \leq C \left\{\|u\|_p^2 + \|u_t\|^2_2\right\}, \quad \text{for } \frac{1}{r} + \frac{1}{r'} = 1.
\]

To be able to use Lemma 5, we take \(r' = 2(1 - a)\) which gives \(r/1 - a = 2/1 - 2a \leq p\).

Therefore, for \(s = 2/1 - 2a\), estimate (62) yields
\[
\left\| \int_0^t m(x) uu_t dx \right\| \leq C \left\{\|u\|_p^2 + \|u_t\|^2_2\right\}, \quad (63)
\]
Hence, Lemma 5 gives
\[
\left\| \int_0^t m(x) uu_t dx \right\| \leq C_1 \left\{\|u\|_p^2 + \|u_t\|^2_2\right\}, \forall C_1 > 0,
\]
and with the same way, we get
\[
\left\| \int_0^t m(x) uu_t dx \right\| \leq C_2 \left\{\|u\|_p^2 + \|q\|^2_2\right\}, \forall C_2 > 0,
\]
From (64), (65), and (66) we obtain
\[
L^{\frac{\alpha}{2}}(t) \leq C \left\{H(t) + \|u\|_p^2 + \|q\|^2_2 + \|u_t\|^2_2 + \|u_r\|^2_2\right\}, \forall t \geq 0, \forall C > 0.
\]
Combining (67) and (59), we arrive at
\[
L'(t) \geq a_0 L^{\frac{\alpha}{2}}(t), \forall t \geq 0,
\]
where \(a_0\) is a positive constant depending only on \(A_0\) and \(C\).

A simple integration of (68) over \((0, t)\) yields
\[
L^{\frac{\alpha}{2}}(t) \geq \frac{1}{L^{-a\alpha}(0) - C_0 t^{1 - a}(1 - a)},
\]
Therefore, \(L(t)\) blows up in time
\[
T^* \leq \frac{1 - a}{C_0 L^{-a\alpha}(0)}.
\]
The proof is completed.

**4. Conclusion**

In this work, we are interested with a problem of a logarithmic nonuniform flexible structure with time delay, where the heat flux is given by Cattaneo’s law. We show that the energy of any weak solution blows up infinite time if the initial energy is negative. The delay effect that is similar to memory processes is important in the research of applied mathematics such as physics, noninstant transmission phenomena, and biological motivation. In the future work, we will try to study the local existence for this problem with respect to some proposal conditions.
Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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