Application of topological optimisation methodology to infinitely wide slider bearings operating under compressible flow

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Abstract
It has been over a century since the interest in inventing the optimal topology for bearings arose. A significant achievement was published by Lord Rayleigh, who found the step-bearing geometry which maximise the load-carrying capacity when the classical Reynolds equation is used to model thin film flow of an iso-viscous and incompressible fluid. Since then, new optimisation methods considering some variants of governing equations for finding the best possible bearings have surfaced, one of which will be presented in this paper. Here, two different formulations for compressible flow, i.e. ideal gas and constant bulk modulus compressibility, as well as the classical Reynolds formulation will be used in combination with the method of moving asymptotes for topological optimisation. All three of these problem formulations provide us with unique geometries, which either maximise the load-carrying capacity or minimise friction, for fluids with a wide variety of compressibility.

Keywords
Hydrodynamic lubrication, topological optimisation, Reynolds equation, slider bearing, MMA

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Introduction
The geometry is of highest importance for bearing performance and improvements have the potential to reduce the risk of wear and lower friction, thereby making substantial savings. For this reason, there is great interest in finding the geometry that maximises the load-carrying capacity (LCC) and thus minimises the risk for wear of a hydrodynamically lubricated bearing. Therefore, this work started very long time ago with a publication by Rayleigh, dating back to 1918.¹ Since then, researchers have presented optimal solutions, based on other governing equations than the classical one-dimensional Reynolds equation, and these have resulted in a selection of LCC-maximising geometries, see e.g. literature.²⁻⁸ Yet, there are no LCC-maximising solutions for fluids exhibiting constant bulk modulus type of compressibility, which is of particular interest since it appears in the classical formulations of cavitation models.⁹⁻¹² and later on in literature.¹³⁻¹⁶ In this case, the constitutive relation for the compressibility is defined by

\[
\frac{1}{\beta} = \frac{1}{\rho} \frac{\partial \rho}{\partial p}
\]

where \( \beta \) is a constant referred to as the fluid’s bulk modulus, \( \rho \) is the density of the fluid and \( p \) is the pressure.

The first one who optimised the geometry for minimum coefficient of friction, while applying the

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classical Reynolds formulation to govern the fluid flow, was Rohde. More recently, Rahmani et al. investigated the Rayleigh step bearing for different boundary conditions and presented both LCC-maximising and friction-minimising step geometries. Maday presented a new method for solving Rayleigh’s problem setup of maximising LCC. Shortly thereafter, Maday performed the analysis on an infinitely wide gas slider using the method developed in another study, finding the optimum slider geometry. The resulting bearing geometry turned out to be a converging wedge followed by a step down to the trailing land. In addition, some important findings regarding hydrodynamically lubricated slider bearings are the work of Auloge et al. More precisely, they actually presented the bearing geometry that maximises LCC when a non-Newtonian fluid rheology model is considered and they found that the result was a step bearing different from Rayleigh’s.

Later, Boldyrev analytically solved a much more advanced problem including the Reynolds equation with periodic boundary conditions coupled with the Elrod-Burgdorfer condition. This work resulted in an optimal bearing geometry different from the one obtained when using Dirichlet conditions.

There are also closely related optimisation works for slider bearings, which have advanced into the three-dimensional domain, taking into account the physical behaviour of fluid flow escaping on the sides of the bearing. Fundamental work here are e.g. literature. Rohde and McAllister were early in presenting their very own development of an algorithm for the three-dimensional optimisation problem, which maximises the LCC. About two decades later Boldyrev studied the optimisation problem with the same objective, but with a (compressible) ideal gas model governing the fluid flow, and after additionally two more decades he further perfected the work. With new modern tools, Buscaglia et al. replicated the work by Rohde and McAllister, demonstrating that the same result can be obtained with different techniques, numerically.

There have also been closely related findings for journal bearings. It was Maday who first took Rayleigh’s optimisation of the infinitely wide slider bearing further, using it to find the optimal solution for incompressible flow in a journal bearing application. Later, Buscaglia et al. found an optimal geometry for a journal bearing with three sections using the Burgdorfer slip flow gas model. In addition, they found an optimal solution for three-dimensional slider bearing using the same slip-flow gas model. Both Maday and Buscaglia et al. resulted in converging-gap geometries, similar to the those found by Maday.

Topological optimisation has for a long time been used in the field of structural mechanics, in order to minimise weight and strain energy of various kind of components. One particular type of gradient-based algorithm, for this type of optimisation, is the method of moving asymptotes (MMA) developed by Svanberg, and the globally convergent development thereof (GCMMA). However, in the field of tribology, numerical topological optimisation using these algorithms has not yet been very frequently applied. In fact, to the authors’ knowledge, no one has previously used topological optimisation methods to explore bearing designs for compressible flow that minimises the coefficient of friction, i.e.

$$\mu = \frac{f}{w}$$

where $f$ is the friction and $w$ the normal force (equivalent with the LCC). The objective of this paper is to find out which bearing geometries that either maximise LCC or minimise viscous friction for three classical constitutive relationships of the fluid compressibility. That is, incompressible fluids, ideal gas and compressible fluids that exhibit constant bulk modulus compressibility.

In this paper, we will use the built-in facilities for optimisation in COMSOL Multiphysics to find the geometry of infinitely wide bearings that maximises LCC for constant bulk modulus type of compressible fluids. In addition, we will apply it to find the bearing geometries that minimises the coefficient of friction for incompressible, ideal gas and constant bulk modulus compressible fluids. To verify and ensure the accuracy of the optimisation method applied, we will first use it to replicate Rayleigh’s and Rohde’s solutions for the incompressible case and Maday’s solution for the ideal gas type of compressibility.

### Governing equations and implementation

The Reynolds equation governing thin film flow of compressible fluids under steady conditions reads

$$\frac{d}{dx} \left( \frac{\rho h^3}{12\eta} \frac{dp}{dx} \right) + \frac{u}{2} \frac{d}{dx} \left( \frac{\rho h}{\eta} \right) = 0, \quad 0 < x < L,$$

$$p(0) = p_0 = p(L)$$

where $\rho$ is the density, $\eta$ is the dynamic viscosity, $h$ is the film thickness – including the bearing geometry, $u$ is the speed of the moving surface and $p_0$ is the ambient pressure. In this paper, equation (3) will be employed as the basis for the topological optimisation, i.e., finding the bearing geometry description included in $h$ that either maximises LCC, minimises the coefficient of friction or minimises the friction force for a given applied load. It will be assumed that the lower surface is flat and moves with constant speed $u$ and the optimisation of the geometry will be applied to the upper stationary surface.
The friction force can be calculated by integrating the shear stresses acting on the flat moving surface over the whole length, \( L \), of the bearing, i.e.

\[
f_f = \int_0^L h \frac{\partial p}{\partial x} \, dx + \eta u \frac{\partial h}{\partial x} \tag{4}
\]

In this expression for the friction force, the plus and the minus signs indicates that it is evaluated at the lower moving or the upper stationary surface, respectively. Since the moving surface is commonly also flat in practical applications, we will compute the friction force accordingly.

**Constant bulk modulus compressibility**

From the constitutive relationship between pressure and density in equation (1), we can obtain an explicit relationship between the pressure and the density. Indeed

\[
\rho(p) = \rho_c e^{(p-p_c)\beta}
\]

where \( \rho_c \) is the density at the cavitation pressure \( p_c \) and \( \beta \) is the bulk modulus. During the preliminary studies for the present work, we employed the same cavitation model as in literature,\(^{29,30} \) while performing optimisation for bearing geometry. Since no cavitation was observed – the same bearing geometries as without the cavitation model were generated, and we chose to not include it for the studies presented herein. This implies that \( \rho_c \) can be set to zero, without affecting the resulting optimal geometry. Furthermore, by the application of the transformation \( \theta(x) = \rho(x)/\rho_c \), i.e. \( \theta(x) = e^{(x-x_r)/\beta} \), the resulting linear Reynolds equation can be linearised and non-dimensionalised and, thereafter, expressed as

\[
\frac{d}{dx} \left( \rho_c \frac{d\theta}{dx} \right) = \Lambda \frac{d(\rho h)}{dx}, \quad 0 < x < L/x_r, \quad \theta(0) = 1 = \theta(L/x_r) \tag{6}
\]

where \( \theta = \rho/\rho_c \) and

\[
\Lambda = \frac{6 \eta u x_r}{\rho_c h_r^2} \tag{7}
\]

where \( x_r \) is the reference length and \( h_r \) is the reference film thickness. Obviously, equation (6) can be used to study the transition from compressible to incompressible flow just by increasing the value of the bulk modulus \( \beta \). It will be shown that a \( \beta \)-value larger than 1 GPa will render a solution that will very closely resemble the incompressible result. It will also be shown that as the \( \beta \)-value is lowered, the optimal geometry will differ significantly from both the incompressible and the ideal gas cases.

**Ideal gas compressibility**

An ideal gas exhibits a linear relation between pressure and density. With that in mind, equation (3) can be non-dimensionalised and finally written as

\[
\frac{d}{dx} \left( \rho_i \frac{d\rho_i}{dx} \right) = \Gamma \frac{d(\rho_i h_i)}{dx}, \quad 0 < x < L/x_r, \quad \rho(0) = 1 = \rho(L/x_r) \tag{8}
\]

where

\[
\Gamma = \frac{6 \eta u x_r}{h_r^2 \rho_r} \tag{9}
\]

is referred to as the compressibility number.

**Numerical solution**

Both equations (6) and (8) were implemented inside the *Coefficient Form PDE Physics Interface* in COMSOL Multiphysics.\(^8 \) For the constant bulk modulus model, the initial condition for (6) over the whole length was specified as \( \theta = 1 \), corresponding to an ambient pressure of \( p = 0 \). In all of the numerical solutions, the bearing number was specified as \( \Lambda = 11.1 \times 10^5 \beta \) and \( \beta \) was varied to simulate the transition from compressible to incompressible flow. This setting corresponds to, for instance, \( u = 1 \text{ m/s} \), \( \eta = 2.96 \times 10^{-2} \text{ Pas}, x_r = 0.1 \text{ m}, h_r = 40 \times 10^{-6} \text{ m} \).

For the ideal gas case, the initial condition for equation (8) over the whole length was specified as \( \bar{\rho} = 1 \), to be able to reproduce Maday’s solution.\(^3 \) An auxiliary parameter \( 0 \leq \epsilon \leq 1 \) was introduced in order to facilitate the convergence of the numerical solution procedure of equation (8), where \( \epsilon = 0 \) represents the incompressible solution and \( \epsilon = 1 \) the one for ideal gas. More precisely, in terms of \( \epsilon \), equation (8) was reformulated as

\[
\frac{d}{dx} \left( (1 - (1 - \bar{\rho})\epsilon) \left( \rho_i^2 \frac{d\rho_i}{dx} - \Gamma \rho_i \right) \right) = 0 \tag{10}
\]

Except for those studies where \( \Gamma \) was varied, the results were obtained using a value of \( \Gamma = 50 \). This setting corresponds to, for instance, \( u = 0.45 \text{ ms}^{-1}, \eta = 18.5 \times 10^{-6} \text{ Pa s}, x_r = 0.1 \text{ m}, h_r = 1 \times 10^{-6} \text{ m} \) and \( p_r = 1 \times 10^5 \text{ Pa} \).

**Optimisation method**

The optimisation problem was setup inside the *Optimization Physics Interface* in COMSOL Multiphysics.\(^8 \) More precisely, it is a gradient-based optimisation solver written by Svanberg specifically designed with topology optimisation in mind. In the literature the method is referred to as the globally convergent method of moving asymptotes (GCMMA) and is available in COMSOL under the name MMA.
In order to perform a topological optimisation, a control variable field and an objective function need to be specified. The control variable for the bearing geometry is chosen as $\tilde{h}$, which can take any value within its lower and upper limits, viz.

$$\tilde{h}_{\text{lower}} \leq \tilde{h} \leq \tilde{h}_{\text{upper}}$$  \hfill (11)

In the present study, the lower limit set to 1 with the exception of the analysis, presented in the section Minimising friction force, comparing friction and LCC optimisation, where a thinner dimensionless fluid film is required for the friction optimisation to obtain the same LCC. The upper limit was set to 100, which is high enough to not restrict the evolution of the bearing geometry during the optimisation procedure. Note that choosing the lower limit of 1 does not restrain generality as both $h$ and LCC scales with $h_0$. There is also a need for initial conditions $\tilde{h}_{\text{init}}$, for the control variable $\tilde{h}$, here specified as

$$\tilde{h}_{\text{init}} = 2 - H(\bar{x} - 1/2)$$  \hfill (12)

where $H(\bar{x} - 1/2)$ is the Heaviside step function.

In the section Minimising friction force, we first specify the objective function as the hydrodynamically generated force by either equation (14) or (15) and present the outcome of the study with the topological optimisation routine, to find the geometry that maximise the LCC of the bearing. Then we take the optimisation one step further by applying the same routine but with the coefficient of friction, i.e. either equation (19) or (20) as objective function.

### Load

To optimise the LCC, the objective function was formulated as the integral of the absolute pressure distribution

$$w = \int_0^L p \, dx$$  \hfill (13)

where for the ideal gas model the absolute pressure can be obtained by subtracting the boundary pressure and in the case of the constant bulk modulus model the boundary pressure is already zero. The objective function can then be made non-dimensional by dividing with the reference length and the reference pressure. Thus, in the case of the ideal gas model the objective function reads

$$\tilde{w} = \frac{w}{\beta \chi_r} = \int_0^{L/x_r} \tilde{p} - 1 \, d\tilde{x}$$  \hfill (14)

For the constant bulk model, the pressure $p$ is substituted with $\ln \theta$ which results in the objective function’s dimensionless form

$$\tilde{w} = \frac{w}{\beta \chi_r} = \int_0^{L/x_r} \ln \theta \, d\tilde{x}$$  \hfill (15)

### Friction

To minimise the coefficient of friction $\mu$, the objective function is specified as fraction between the friction force and the LCC, i.e. equation (2). In practise, we used the scaled coefficient of friction

$$\tilde{\mu} = \frac{h_r \mu}{\chi_r} = \frac{\tilde{f}}{\tilde{w}}$$  \hfill (16)

obtained from the dimensionless forms of the LCC, $\tilde{w}$, from equations (14) and (15) and the corresponding dimensionless friction force for the ideal gas model, $\tilde{f}$

$$\tilde{f} = f \frac{h_r}{h_r p_r} = \int_0^{L/x_r} \tilde{p} \frac{d\tilde{p}}{2 \tilde{d} \tilde{x}} + \frac{\Gamma}{6h} \tilde{d} \tilde{x}$$  \hfill (17)

and for the constant bulk modulus model as

$$\tilde{f} = f \frac{h_r}{h_r p_r} = \int_0^{L/x_r} \tilde{p} \frac{d\ln \theta}{2 \tilde{d} \tilde{x}} + \frac{A}{6h} \tilde{d} \tilde{x}$$  \hfill (18)

This will result in the scaled friction coefficient

$$\tilde{\mu} = \frac{\int_0^{L/x_r} \tilde{p} \frac{d\ln \theta}{2 \tilde{d} \tilde{x}} + \frac{A}{6h} \tilde{d} \tilde{x}}{\int_0^{L/x_r} \tilde{p} - 1 \tilde{d} \tilde{x}}$$  \hfill (19)

and

$$\tilde{\mu} = \frac{\int_0^{L/x_r} \tilde{p} \frac{d\ln \theta}{dx}}{\int_0^{L/x_r} \tilde{p} - 1 \tilde{d} \tilde{x}}$$  \hfill (20)

for the ideal gas and the constant bulk modulus models, respectively.

### Results and discussion

The results have been divided into three sections. In the following section, we verify our approach by first comparing with Rayleigh’s \cite{1} and Rohde’s \cite{17} findings, for incompressible flow. Then we compare our results for the ideal gas case with the findings by Maday. \cite{3} Alongside the convergence studies for LCC optimisation, studies were also conducted to confirm the convergence of the minimisation of the coefficient of friction. The results for both the constant bulk modulus and the ideal gas model are presented in Figures 1 and 2, respectively. Since there are no available results to compare with in this case, the reference geometries were obtained using meshes with 1024 elements. In the section Maximised LCC for constant bulk modulus
fluids, the bearing geometries generated from the topological optimisation for LCC with the constant bulk modulus model will be presented. In the section Minimising the coefficient of friction, the bearing geometries generated for the lowest possible coefficient of friction, for both the constant bulk modulus and the ideal gas models, will be presented.

The geometry that maximises the LCC also generates the thickest fluid film. If one instead optimises for minimum friction force the same LCC comes with a reduced fluid film thickness. In the section Minimising friction force, an investigation of how much the coefficient of friction and film thickness reduces, when optimising for minimum friction force as compared with the solution for maximum LCC.

Validation

In order to investigate the applicability of the present approach implemented in COMSOL Multiphysics’s built-in, MMA based, topological optimisation facility, we first applied it to see if was possible to reproduce Rayleigh’s findings. To do so, topological optimisation for maximum LCC was performed, with the constant bulk modulus model using $\beta = 10$ GPa, which closely resembles an incompressible fluid. To quantify and assure the accuracy of the proposed methodology, a mesh convergence study was conducted. As a measure of the accuracy, an average relative error defined as the average geometrical relative deviation from a reference geometry was employed, i.e.

$$R_{error} = \frac{1}{n} \sum_{i=1}^{n} \left| \frac{h_i - h_{Ref}}{h_{Ref}} \right|$$

The result of the convergence study is shown in Figure 1(a). To be certain that the optimisation for minimum coefficient of friction also converges when the mesh is refined, a similar study was conducted also in this case. The solution presented by Rohde is taken as reference solution to verify the mesh convergence when optimising for minimum coefficient of friction. The result is shown in Figure 1(b).
To leave no uncertainties, similar mesh convergence studies were conducted using the ideal gas model, for both the maximisation of LCC, see Figure 2(a) and the minimisation of the coefficient of friction, see Figure 2(b). For the maximisation of LCC the analytic solution found in Maday,\textsuperscript{3} for $\Gamma = 50$, was used as the reference, and since there is yet no analytic solution for the minimum coefficient of friction, a mesh with 1024 elements was used to obtain a reference. From Figures 1 and 2, it can be seen that the mesh with 256 elements render average relative errors, substantially smaller than 1% in all cases, and it was judged that a mesh with 256 elements is fine enough for carrying out the rest of the optimisation studies presented herein.

The finest mesh in the convergence study for the constant bulk modulus model with $\beta = 10$ GPa has 512 elements. The topologically optimised bearing geometry obtained using this setting is shown in Figure 3, together with the corresponding pressure distribution. Figure 3(a) shows a non-dimensional step-bearing geometry, with a gap of 1.867 at the inlet and a step located at 0.718, which are identical to the values presented in Rayleigh.\textsuperscript{1} At the end of the leading edge land there is small notch, which increases in height when the mesh gets denser. The depth of the notch can be reduced by specifying finer tolerance for the MMA-optimisation procedure; however, this will considerably increase the simulation time. Figure 3(b) shows the classical (piece-wise linear) dimensionless pressure distribution $\tilde{p}$, constructed as $\tilde{p}(\tilde{x}) = \ln(\theta(\tilde{x}))/\Lambda$, with maximum value at approximately $6.86 \times 10^{-2}$ giving rise to a dimensionless LCC of approximately 0.034.

To illustrate the capability of the present methodology, topological optimisation based on the Reynolds type of equation for ideal gas flow, given by equation (10) was next performed. Indeed, the results replicating the LCC-maximising ones that Maday\textsuperscript{3} found were obtained with the setting $\Gamma = 50$ and $\epsilon = 0.99$. The optimal geometry and the corresponding dimensionless pressure distribution are shown in Figure 4. Note that Rayleigh’s solution, but with dimensionless pressure scaled up 50 times to compensate for $\Gamma = 50$, could have been obtained using $\epsilon = 0$.

The resulting optimal bearing geometry and the corresponding dimensionless pressure distribution, which render the lowest coefficients of friction obtained using the ideal gas model with incompressible settings $\Gamma = 1$ and $\epsilon = 0$ are shown in Figure 5(a).

In this case, the solution to equation (10) becomes independent of $\Gamma$, leading to that the optimal bearing geometry is independent of $\Gamma$. The resulting “master” geometry (optimal for all settings) is depicted in Figure 5(a), which also confirms the result of Rohde.\textsuperscript{17} In dimensionless form, the optimal geometry consists of a flat inlet section with the film thickness of $2.00$, and from $\tilde{x} = 0.74$ to 0.82 a tapered step is located which reduces the clearance to the trailing land with a film thickness of $1$. The pressure distribution corresponding to this “master” geometry is shown in Figure 5(b), from which the pressure $p = (6\mu_\infty/r^2)\tilde{p}$, for an arbitrary setting, can be obtained.

**Maximised LCC for constant bulk modulus fluids**

With the present methodology for LCC optimisation validated against the results in Rayleigh\textsuperscript{1} and Maday\textsuperscript{3} topological optimisation for maximum LCC was conducted using a 256-element mesh while varying the bulk modulus. The bulk modulus was varied from $10^4$ to $10^9$ Pa with 2/10 decade steps ($10^{4.0}$, $10^{4.2}$, $10^{4.4}$, $10^{4.6}$, $10^{4.8}$), i.e., from highly compressible to nearly incompressible under the specified operating conditions.
conditions. The resulting topologically optimised bearing geometries are separated into two figures to make it easier to differentiate between the results over the whole range of compressibilities. The geometries and the corresponding pressure distributions obtained for $10^4 \leq \beta \leq 10^6$ are shown in Figure 6 and the ones obtained for $10^6 \leq \beta \leq 10^9$ are shown in Figure 7.

It is observed that the corresponding geometry and pressure distribution pairs can be easily identified by locating the point at the $x$-axis where the flat trailing land starts, and where the pressure distribution first reaches its maximum value.

Figure 7(b) does not only show how the pressure distributions vary with $\beta$; it also shows the convergence towards the solution found by Rayleigh, which is virtually indistinguishable from the solution at $\beta = 10^9$ Pa.

Almqvist et al. $^{31}$ in parallel to this work, studied the problem analytically and they found the bearing geometry that maximise the LCC, for fluids exhibiting constant bulk modulus type of compressibility. The results justify the correctness of the numerical findings presented herein, and the mathematical analysis grants even better understanding.

Minimising the coefficient of friction

Next, we will apply the present methodology to deduce the bearing geometries that minimises the coefficient of friction. To this end, the optimisation is performed using the coefficient of friction as the objective function, for a given minimum film thickness. The study was performed both for the ideal gas model, with varying values of $\Gamma$, and for the constant bulk modulus model, with varying values of $\beta$. All numerical simulation results were obtained using a mesh consisting of 256 elements. The resulting optimal bearing geometries that render the lowest
coefficients of friction and the corresponding pressure distributions for the ideal gas model, with varying compressibility numbers $\gamma$, are separated into a lower and a higher range, depicted in Figures 8 and 9. It is observed that LCC-optimised geometries shown in Figure 4(a) and the friction coefficient optimised ones shown in Figure 8(a) exhibit tapered inlet zones and flat lands towards the trailing edge. The major difference is that the LCC-optimised geometry has a vertical step, whereas the corresponding portion of the coefficient of friction optimised geometries has a tapered step. Note that although the tapering may seem linear to the naked eye, they are in fact not. We note that the major difference between the geometries in Figure 8(a) and the ones for $\gamma = 25.5$ and thereafter in Figure 9(a) is that the former are flat at the trailing edge, while the latter exhibit a diverging gap at end of the flat zone.

The resulting optimal bearing geometries that render the lowest coefficients of friction and the corresponding pressure distributions for the constant bulk modulus model with varying $\gamma$-values are also separated into a higher- $(10^6 \leq \gamma \leq 10^9)$ and a lower $(10^4 \leq \gamma \leq 10^6)$ range, as depicted in Figures 10 and 12. From Figure 10(a), it can be seen that the resulting bearing geometry as $\gamma \to 10^9$ is more or less identical to the one for $\epsilon = 0$ depicted in Figure 5(a). Figure 10(b) shows how the pressure distribution changes shape when the geometries gradually change. If the solutions would not have been separated into two groups for different ranges of $\gamma$-values, then it would not have been possible to differentiate between the geometries in the higher range, i.e. the solutions for less compressible flow. Another reason for separating the solutions into these two ranges of $\gamma$-values is that it was observed that the

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Figure 6. The topologically optimised geometries and the corresponding dimensionless pressure distributions, obtained by maximising the LCC for $10^3 \leq \beta \leq 10^6$, using the constant bulk modulus model. (a) Optimised bearing geometries. (b) Corresponding pressure distributions.

Figure 7. The topologically optimised geometries and the corresponding dimensionless pressure distributions, obtained by maximising the LCC for $10^6 < \beta < 10^9$, using the constant bulk modulus model. (a) Optimised bearing geometries. (b) Corresponding pressure distributions.
Figure 8. The topologically optimised geometries and the corresponding dimensionless pressure distributions, obtained by the minimising coefficient of friction for $2^5 < \Gamma < 2^6$, using the ideal gas model. (a) Optimised bearing geometries. (b) Corresponding pressure distributions.

Figure 9. The topologically optimised geometries and the corresponding pressure distributions, obtained by minimising the coefficient of friction for $2^{5.5} < \Gamma < 2^6$, using the ideal gas model. (a) Optimised bearing geometries. (b) Corresponding pressure distribution.

Figure 10. The topologically optimised geometries and the corresponding dimensionless pressure distributions, obtained by minimising the coefficient of friction for $10^4 < \beta < 10^6$, using the constant bulk modulus model. (a) Optimised bearing geometries. (b) Corresponding pressure distributions.
pressure distributions obtained for $\beta < 10^{6.3}$ are (numerically) identical, representing a “master” pressure distribution. This master pressure distribution and the corresponding optimal geometry are depicted in Figure 11(b). This actually means that the corresponding optimised bearing geometries are just differently scaled versions of a single “master” geometry. We note that, for the incompressible cases for $LCC$ and the coefficient of friction optimisation, the optimal solutions render a single geometry. Furthermore, if the Reynolds equation is non-dimensionalised with $x_r$ as reference length, $h$, as

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**Figure 11.** The topologically optimised geometry and the corresponding $\theta$ solution, obtained by minimising the coefficient of friction for $\beta \approx 10^{6.3}$, using the constant bulk modulus model. (a) Optimised bearing geometry. (b) Corresponding dimensionless density distribution.

**Figure 12.** The topologically optimised geometries and the corresponding dimensionless pressure distributions, obtained by minimising the coefficient of friction for $10^4 \leq \beta \leq 10^6$, using the constant bulk modulus model. (a) Optimised bearing geometries. (b) Corresponding pressure distributions.

**Figure 13.** Bearing geometry selection scheme.
reference film thickness and \( p_r = 6\mu x_c/h_c^2 \) reference pressure, then the pressure distributions, which can be obtained from the dimensionless density result shown in Figure 11(b), would be identical up to the scaling \( 6\mu x_c/h_c^2 \). Moreover, the minimum film thickness, the friction force and the LCC, can be obtained for an arbitrary \( \beta \)-value from \( h_{\text{min}} \approx 0.134\sqrt{6\mu x_c/\beta} \), \( f \approx 300/6\mu x_c \beta \) and \( w \approx 0.6\mu x_c \), respectively.

**Minimising friction force**

The topological optimisation presented in the previous sections was carried out, both for the ideal gas model and the constant bulk modulus model with the objective set to either maximise the LCC or minimise the coefficient of friction, for a given minimum film thickness. For many engineering applications the interest would, however, rather be to minimise the friction force given an applied load. Next, a scheme facilitating the identification and selection of an optimal geometry for a given load and type of compressibility is shown in Figure 13. To this end, some input parameters are required, i.e. \( (w, L, \eta, u, p_u) \), and the type of fluid compressibility needs to be known.

At this stage, the dimensionless LCC should be calculated according to equation (22), i.e.

\[
\frac{w}{Lp_u} = \frac{\bar{w}}{\bar{L}_p} = \tilde{w}
\]

Having calculated the dimensionless LCC, one can identify the corresponding geometry in either Tables 1–3, containing data for the incompressible, ideal gas or constant bulk modulus models, respectively. The friction force, the minimum film thickness and the maximum hydrodynamic pressure can now also be obtained.

For the incompressible flow model, there is only one set of scaling parameters, i.e. the set in Table 1, and this is because of the master geometry depicted in Figure 14 (found by Rohde\(^1\)), which can be rescaled for all LCC.

For the ideal gas model, an interpolation for the value of the dimensionless LCC in Table 2 should be made in order to obtain the bearing geometry from Figure 15 and the dimensionless values for friction force, minimum film thickness and maximum hydrodynamic pressure. Note that, as the LCC decreases, the geometry converges to a geometry closely resembling the one obtained for incompressible flow depicted in Figure 14.

For a constant bulk modulus type of fluid, the procedure is the same as for ideal gas, except for that one should use Table 3 instead of Table 2 and Figure 16 instead of Figure 15.

Lastly, the dimensionless values for the friction force, the minimum film thickness and the maximum hydrodynamic pressure obtained in Tables 1 to 3 will be used in equations (23) to (26) to calculate the performance values of the bearing.

\[
f = \sqrt{6\mu Lp_u\beta^2} = \sqrt{6\mu L\beta f^*}
\]

\[
h_{\text{min}} = \sqrt{\frac{6\mu L}{p_u}} h_{\text{min}}^* = \sqrt{\frac{6\mu L}{\beta}} h_{\text{min}}^*
\]

\[
h = h^* h_{\text{min}}
\]

\[
p_{\text{max}} = Lp_u f_{\text{max}}^* = L\beta \ln(n_{\text{max}})
\]

**Table 1.** Scaling parameters for the incompressible model.

| \( \tilde{f}^* \) | \( h_{\text{min}}^* \) | \( p_{\text{max}}^* \) |
|-------------------|-------------------|-------------------|
| 9.2165 \times 10^{-2} | 1.0000 | 3.1672 \times 10^{-2} |

**Table 2.** Scaling parameters for the ideal gas model.

| \( \tilde{w} \) | Geometry | \( \tilde{f}^* \) | \( h_{\text{min}}^* \) | \( p_{\text{max}}^* \) |
|----------------|----------|-------------------|-------------------|-------------------|
| 1.0000         | 1        | 8.7688 \times 10^{-1} | 9.4209 \times 10^{-2} | 2.9975 |
| 5.6234 \times 10^{-1} | 2        | 6.0630 \times 10^{-1} | 1.6776 \times 10^{-1} | 1.3648 |
| 3.1623 \times 10^{-1} | 3        | 4.2987 \times 10^{-1} | 2.5408 \times 10^{-1} | 6.7846 \times 10^{-1} |
| 1.7783 \times 10^{-1} | 4        | 3.1087 \times 10^{-1} | 3.4113 \times 10^{-1} | 3.5540 \times 10^{-1} |
| 10^{-1}         | 5        | 2.2797 \times 10^{-1} | 4.3761 \times 10^{-1} | 1.9190 \times 10^{-1} |
| 5.6234 \times 10^{-2} | 6        | 1.6870 \times 10^{-1} | 5.7011 \times 10^{-1} | 1.0546 \times 10^{-1} |
| 3.1623 \times 10^{-2} | 7        | 1.2554 \times 10^{-1} | 7.4963 \times 10^{-1} | 5.8494 \times 10^{-2} |
| 1.7783 \times 10^{-2} | 8        | 9.3722 \times 10^{-2} | 9.9234 \times 10^{-1} | 3.2672 \times 10^{-2} |
| 10^{-2}         | 9        | 7.0105 \times 10^{-2} | 1.3176 | 1.8300 \times 10^{-2} |
| 5.6234 \times 10^{-2} | 10       | 5.2496 \times 10^{-2} | 1.7528 | 1.0268 \times 10^{-2} |
| 3.1623 \times 10^{-2} | 11       | 3.9331 \times 10^{-2} | 2.3344 | 5.7654 \times 10^{-3} |
| 1.7783 \times 10^{-2} | 12       | 2.9483 \times 10^{-2} | 3.1098 | 3.2400 \times 10^{-3} |
| 10^{-3}         | 13       | 2.2102 \times 10^{-2} | 4.1473 | 1.8213 \times 10^{-3} |
Figure 15. The topologically optimised geometries and the corresponding dimensionless pressure distributions, obtained by minimizing friction force for $10^{-3} \leq \tilde{w} \leq 10^{2}$, using the ideal gas model. (a) Scaled optimised bearing geometries. (b) Corresponding scaled pressure distributions.

Table 3. Scaling parameters for the constant bulk modulus model.

| $\tilde{w}$   | Geometry | $\tilde{f}^*$ | $\tilde{h}_{\text{min}}$ | $\ln(\theta_{\text{max}})$ |
|--------------|----------|---------------|--------------------------|---------------------------|
| $1$          | 1        | 1.4287        | 7.5613 $\times 10^{-2}$  | 1.6744                    |
| $5.6234 \times 10^{-1}$ | 2        | 7.0111 $\times 10^{-1}$ | 1.3916 $\times 10^{-1}$ | 1.2599                    |
| $3.1623 \times 10^{-1}$ | 3        | 4.4578 $\times 10^{-1}$ | 2.4531 $\times 10^{-1}$ | 7.1086 $\times 10^{-1}$  |
| $1.7783 \times 10^{-1}$ | 4        | 3.1416 $\times 10^{-1}$ | 3.4188 $\times 10^{-1}$ | 3.6086 $\times 10^{-1}$  |
| $10^{-1}$    | 5        | 2.2871 $\times 10^{-1}$ | 4.3806 $\times 10^{-1}$ | 1.9280 $\times 10^{-1}$  |
| $5.6234 \times 10^{-2}$ | 6        | 1.6887 $\times 10^{-1}$ | 5.7039 $\times 10^{-1}$ | 1.0561 $\times 10^{-1}$  |
| $3.1623 \times 10^{-2}$ | 7        | 1.2557 $\times 10^{-1}$ | 7.5019 $\times 10^{-1}$ | 5.8566 $\times 10^{-2}$  |
| $1.7783 \times 10^{-2}$ | 8        | 9.3730 $\times 10^{-2}$ | 9.9268 $\times 10^{-1}$ | 3.2685 $\times 10^{-2}$  |
| $10^{-2}$    | 9        | 7.0106 $\times 10^{-2}$ | 1.3176                   | 1.8302 $\times 10^{-2}$  |
| $5.6234 \times 10^{-3}$ | 10       | 5.2496 $\times 10^{-2}$ | 1.7528                   | 1.0268 $\times 10^{-2}$  |
| $3.1623 \times 10^{-3}$ | 11       | 3.9328 $\times 10^{-2}$ | 2.3344                   | 5.7643 $\times 10^{-3}$  |
| $1.7783 \times 10^{-3}$ | 12       | 2.9482 $\times 10^{-2}$ | 3.1103                   | 3.2401 $\times 10^{-3}$  |
| $10^{-3}$    | 13       | 2.2101 $\times 10^{-2}$ | 4.1469                   | 1.8212 $\times 10^{-3}$  |

Figure 14. The topologically optimised geometry and the corresponding dimensionless pressure distributions, obtained by minimizing the friction force under incompressible flow. (a) Optimised bearing geometries. (b) Corresponding pressure distributions.
To further extend the applicability of the results, the optimisation procedure for maximising the LCC for a given minimum film thickness was conducted for a range of different $\Gamma$-values, for the ideal gas model, and $\beta$-values, for the constant bulk modulus model. Thereafter, optimisation to minimise the friction force was performed without geometrical constraints, but instead with the constraint to retain the same LCC as for the corresponding LCC-optimised geometry. The results describe how much the coefficient of friction may be reduced, in comparison to the case when the LCC was maximised, and also how much the minimum film thickness reduces as a consequence of this. The results for the ideal gas model are shown in Figure 17, where both the film thickness and coefficient of friction reduction, in comparison to the results corresponding to the LCC-maximised bearing geometry are displayed as functions of the compressibility number $\Gamma$.

The results in Figure 17 show that the minimum film thickness has a markedly non-linear dependence of the $\Gamma$-parameter, with the smallest reduction of about 17%, occurring in the region $2^3 < \Gamma < 2^4$. Moreover, under the same conditions the reduction in the friction coefficient is at least 5%, telling us that if 17% smaller gap is still sufficient, then we could reduce power losses by 5% by using the geometry optimised for friction instead of the one optimised for LCC.

It can also be seen that for $\Gamma$-values larger than $2^4$ or so, the friction reduction comes with greater and greater reduction of minimum film thickness. Put in other words, this suggests that in, e.g. a high-speed application – such as turbo machinery or dental drill, the friction force optimised geometry render a quite substantial increased risk of wear.

The result for the constant bulk modulus model is shown in Figure 18, where both the film thickness reduction and the friction force reduction are shown as a function of the value of the bulk modulus $\beta$.

For the constant bulk modulus type of fluid, the friction reduction shows some variation when increasing the $\beta$-value up to about $10^7$ Pa, after which the optimisation leads to more or less the same geometry as the one for an incompressible fluid.

From Figure 18, it can be seen that the minimum film thickness in this case also behaves highly non-linear. It is noteworthy that, as the fluid becomes more and more incompressible, the film thickness reduces and becomes nearly 30% thinner than the corresponding film thickness for the LCC-maximised
geometry. However, the friction force is also smaller, about 7–8% of the one corresponding to the LCC-maximised geometry.

Concluding remarks

The bearing geometry for infinitely wide slider bearings was optimised for maximum LCC and later also for minimum coefficient of friction. The optimal geometries were obtained with the help of the Optimization Physics Interface in the FE-based simulation software COMSOL Multiphysics. The topology optimisation setting that was used is based on the method of moving asymptotes. The resulting bearing geometries were verified against available results for incompressible fluids and ideal gases, as well as analytical results for fluids exhibiting compressibility of the constant bulk modulus type, derived in parallel to the present work.

Mesh convergence studies were performed to verify that the optimisation method can be used to reproduce the LCC-maximising Rayleigh step bearing geometry for incompressible flow, and the bearing geometry minimising the coefficient of friction as presented by Rohde. Moreover, mesh convergence studies for ideal gas compressible flow were conducted to verify Maday’s results for LCC-maximisation and then also for coefficient of friction minimisation.

The LCC-maximised bearing geometries, for the constant bulk modulus type of compressible fluids, consist of a novel contribution and complement the previously available results for incompressible fluids and ideal gases. They exhibit a convergent concave zone at the leading edge, instead the linear wedge that is optimal for ideal gases. For the LCC-optimised geometries, the concavity and the step that follows it moves closer towards the leading edge as the bulk modulus decreases. The step also becomes less pronounced and the bearing exhibits a longer land towards the trailing edge. For the coefficient of friction optimised geometries, a decreasing bulk modulus results in a smoother and smoother geometry, which also exhibits a divergent section towards the trailing edge, and the trailing land seem not to increase in size like it did for the LCC-optimised geometries.

When optimisation is performed for the minimisation of the coefficient of friction, the optimal bearing geometry, for the ideal gas model, is similar to the LCC-maximised one, but exhibit a tapered, instead of a vertical, step. Furthermore, as the Γ-parameter increases, a divergent gap appears at the trailing edge.

Using LCC optimisation as a part of the process in developing of hydrodynamic bearing designs would lead to better film forming capability, resulting in a thicker fluid film and better protection against wear. But also having the tool to optimise for the lowest possible friction force could reduce the power losses with at least 6%, without sacrificing too much of the film forming capability.

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**Appendix**

**Notation**

| Symbol | Description               | Unit   |
|--------|---------------------------|--------|
| $f$    | friction force            | N/m    |
| $f_0$  | dimensionless friction force | –      |
| $h$    | film thickness            | m      |
| $h_r$  | characteristic film thickness | m    |
| $h_d$  | dimensionless film thickness | –    |
| $L$    | bearing length            | m      |
| $p$    | fluid pressure            | Pa     |
| $p_a$  | ambient pressure          | Pa     |

(continued)
| \( \rho_c \) | cavitation pressure | \( \text{Pa} \) |
| \( \rho_r \) | characteristic pressure | \( \text{Pa} \) |
| \( \tilde{p} \) | dimensionless pressure | – |
| \( u \) | speed of moving surface | \( \text{m/s} \) |
| \( w \) | load-carrying capacity | \( \text{N/m} \) |
| \( \tilde{w} \) | dimensionless load carrying capacity | – |
| \( x \) | Cartesian coordinate | \( \text{m} \) |
| \( x_r \) | characteristic length | \( \text{m} \) |
| \( \tilde{x} \) | dimensionless coordinate | – |
| \( \beta \) | lubricant bulk modulus | \( \text{Pa} \) |
| \( \Gamma \) | compressibility number \( 6\nu u x_r / (h^2 \rho_r) \) | – |
| \( \Lambda \) | bearing number \( 6\nu u x_r / (j h^2) \) | – |
| \( \eta \) | lubricant viscosity | \( \text{Pas} \) |
| \( \theta \) | dimensionless density | – |
| \( \mu \) | coefficient of friction | – |
| \( \tilde{\mu} \) | scaled coefficient of friction | – |
| \( \rho \) | lubricant density | \( \text{kg/m}^3 \) |
| \( \rho_c \) | lubricant density at cavitation pressure | \( \text{kg/m}^3 \) |
| \( \epsilon \) | auxiliary parameter | – |