Confinement induced instability of thin elastic film

BY ANIMANGSU GHATAK ♯ †

† Department of Chemical Engineering, Indian Institute of Technology, Kanpur, UP 208016, India

A confined incompressible elastic film does not deform uniformly when subjected to adhesive interfacial stresses but with undulations which have a characteristic wavelength scaling linearly with the thickness of the film. In the classical peel geometry, undulations appear along the contact line below a critical film thickness or below a critical curvature of the plate. Perturbation analysis of the stress equilibrium equations shows that for a critically confined film the total excess energy indeed attains a minima for a finite amplitude of the perturbations which grow with further increase in the confinement.

Keywords: Confinement, Incompressibility

1. Introduction

Spontaneous surface and interfacial instabilities of thin liquid films have been studied in different contexts, e.g. the classical Saffman-Taylor (Saffman et al 1958) problem in a Hele Shaw cell in which flow driven fingering patterns develop at the moving interface of two viscous or viscoelastic liquids (Homsy et al 1981, Nittmann et al 1985); disjoining pressure induced rupturing and dewetting (Reiter et al 1992, Sharma et al 1998) of ultra thin viscous films; spiral instabilities in viscometric flow of a viscoelastic liquid (Muller et al 1989, McKinley et al 1995); and fingering instability and cavitation during peeling a layer of viscoelastic adhesive (Fields et al 1976, Urhama et al 1989). While most of these viscous and viscoelastic systems have been well characterized experimentally and theoretically, similar surface undulations of confined thin elastic films pose a different kind of problem despite geometric commonalities with many of the liquid systems. The essential difference being that unlike in the liquid system there is no flow of mass and consequent permanent deformation in the elastic body, where the extent of deformation is governed by the equilibrium of the external surface or body forces on the material and the elastic forces developed.

The specific system that will be described in this paper is a thin layer of elastic adhesive confined between a rigid and a flexible plate. While the film remains strongly bonded to the rigid substrate, the flexible plate is detached from it in the classical peel geometry. High aspect ratio of such systems is noteworthy as the lateral length scale far exceeds the thickness of the film resulting in high degree of confinement for the adhesive. As a result, adhesive stresses at the interface does not always result in uniform deformation through out the whole area of contact,

♯ To whom correspondence should be addressed; (aghatak@iitk.ac.in)
rather spatially varying deformations (Gent et al 1958, Gent et al 1969, Ghatak et al 2000, Mönch et al 2001) attain lower energy for the system. Experimentally we see the existence of a critical thickness of the film or a critical curvature of the flexible plate below which the contact line between the film and the flexible plate does not remain straight, but turns undulatory with a characteristic wavelength which increases linearly with the thickness of the film (Ghatak et al 2003). While experimentally this phenomenon has been characterized, there is not much understanding as what drives this instability in a non-flow purely elastic system and how does the curvature of the plate or the thickness of the film result in critical confinement of the film. Here I present a perturbation analysis which addresses these questions highlighting the dual effects of the incompressibility of the elastic film and its confinement.

2. Problem Formulation

![Figure 1](image.png)

Figure 1. (a) Schematic of the experiment in which a model elastic adhesive remains bonded to a rigid substrate and a flexible plate is detached from it with the help of a spacer inserted at the opening of the crack. For a critically confined film, the contact line does not remain smooth but becomes undulatory as shown in video-micrograph (b).

The schematic of our experiment is represented in Figure 1a in which an elastic film of thickness $h$ and shear modulus $\mu$ remains strongly bonded to a rigid substrate while a flexible plate of rigidity $D$ in contact with the film in the form of a curved elastica is supported at one end using a spacer of height $\Delta$. The straight contact line between the film and the plate becomes wavy when the film thickness $h$ decreases below a critical value $h_c$ or the curvature of the plate decreases below a critical value $1/\rho_c$. Figure 1b represents a typical video-micrograph of such undulations which are characterized by two different length scales: the separation distance $\lambda$ between the waves which scales as $\lambda \sim 4h$ and the amplitude $A$ which varies with $D$ and $\mu$ as: $A \sim (D/\mu)^{1/3}$ (Ghatak et al 2003). The figure depicts also the co-ordinate system in which $x$, $y$ and $z$ axes represent respectively the direction of propagation of the contact line, the direction of the wave vector and the thickness coordinate of the film. The $y$ axis is located along the tips of the waves so that the film is completely out of contact with the plate at $0 < x < a$. Assuming the film to be incompressible and purely elastic with no viscous effect, we write the following stress equilibrium
Confinement induced instability of thin elastic film

relations in the absence of any body forces,

\[ p_x = \mu (u_{xx} + u_{yy} + u_{zz}) \]
\[ p_y = \mu (v_{xx} + v_{yy} + v_{zz}) \]
\[ p_z = \mu (w_{xx} + w_{yy} + w_{zz}) \] (2.1)

where, \( u, v \) and \( w \) are the displacements in the \( x, y \) and \( z \) directions respectively and \( p \) is the pressure. Here and everywhere \( x = \partial s/\partial x \) and \( s_{xx} = \partial^2 s/\partial x^2 \). The incompressibility of the film results in:

\[ u_x + v_y + w_z = 0 \] (2.2)

These equations are solved using the following set of boundary conditions (b.c.):

(a) since the film remains strongly bonded to the substrate we use no slip boundary condition at the interface of the film and the substrate \( (z = 0) \),

\[ u(z = 0) = v(z = 0) = w(z = 0) = 0 \] (2.3)

(b) we assume frictionless contact at the interface of the film and the cover plate, \( z = h \) which results in zero shear stress at the interface,

\[ \sigma_{xz}(x, y, h) = \sigma_{yz}(x, y, h) = 0 \] (2.4)

(c) we assume continuity of normal stress across the interface \( (z = h) \) which implies that the pressure is equal to the bending stress on the plate

\[ p(z = h) = D\nabla^2 \psi(z = h) \text{ at } x < 0 \] (2.5)

here \( \nabla \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 \) is the two dimensional Laplacian, \( D \) is the flexural rigidity of the plate and \( \psi = w|_{x,z=h} \) is its vertical displacement. Since the plate bends only in the direction of \( x \) co-ordinate its vertical displacement \( \psi \) remains uniform along the \( y \) axis; hence, we simplify b.c. 2.5 as

\[ p(z = h) = D\psi_{xxxx}(z = h) \text{ at } x < 0 \] (2.6)

At \( 0 < x < a \) there is no traction either on the film or the plate which yields

\[ \sigma_{xz}|_{z=h} = \sigma_{yz}|_{z=h} = \sigma_{zz}|_{z=h} = 0 \] (2.7)

Equation (2.1)-2.2 can be written in dimensionless form using the following dimensionless quantities,

\[ X = xq, \ Y = y/h, \ Z = z/h, \ U = uq, \ V = v/h, \ W = w/h, \ \Psi = \psi/h, \ P = pe^2/\mu \]

While thickness \( h \) of the film is the characteristic lengths along the \( y \) and \( z \) axes, \( q^{-1} \) is that along \( x \). The length \( q^{-1} \) can be derived as the ratio of the deformability of the plate and the film (Dillard et al 1989, Ghatak et al 2004): \( q^{-1} = (Dh^3/3\mu)^{1/6} \). The quantity \( \epsilon = HQ \) defined as the ratio of the two characteristic lengths is a measure of the confinement of the film such that a lower value of \( \epsilon \) represents a more confined film. Equations 2.1 and 2.2 can then be written in the following dimensionless form

\[ PX = \epsilon^2 U_{XX} - U_{YY} + U_{ZZ} \]
\[ PY = \epsilon^2 V_{XX} + \epsilon^2 (V_{YY} + V_{ZZ}) \]
\[ PZ = \epsilon^2 W_{XX} + \epsilon^2 (W_{YY} + W_{ZZ}) \]
\[ 0 = U_X + V_Y + W_Z \] (2.8)
while the boundary conditions 2.3-2.7 results

\begin{align}
\text{(a)} & \quad U (Z = 0) = V (Z = 0) = 0 \\
\text{(b)} & \quad \sigma_{XZ} (X, Y, Z = 1) = 0 = \sigma_{YZ} (X, Y, Z = 1) \\
\text{(c)} & \quad P (Z = 1) = 3\Psi_{XXX} \quad \text{at} \quad X < 0 \\
\text{(d)} & \quad 0 = \Psi_{XXX} \quad \text{at} \quad 0 < X < aq \quad \tag{2.9}
\end{align}

where \( aq \) is the dimensionless crack length. Equation 2.8 is solved by the regular perturbation technique which assumes that the solutions consist of two components: the base solutions which remain uniform along the \( Y \) co-ordinate and the correction term which incorporates the spatial variation along the \( Y \) axis. Thus the base solutions are of order \( \epsilon^0 \) and the perturbed solutions are of order \( \epsilon^1, \epsilon^2, \ldots \), so that any variable \( T (X, Y, Z) = T_o (X, Z) + \epsilon T_1 (X, Y, Z) + \epsilon^2 T_2 (X, Y, Z) + \ldots \) where \( T = P, U, V \) and \( W \). inserting these definitions in equation 2.8 and separating the base (\( Y \) independent) and the perturbed (\( Y \) dependent) terms yield:

\[
P_{0X} = (\epsilon^2 U_{0XX} + U_{0ZZ}), \quad P_{0Z} = (\epsilon^4 W_{0XX} + \epsilon^2 W_{0ZZ}), \quad 0 = U_{0X} + W_{0Z} \tag{2.10}
\]

and

\[
\begin{align}
\epsilon^2 P_{1X} + \epsilon^4 P_{2X} & = \epsilon^2 (U_{1YY} + U_{1ZZ}) + \epsilon^4 (U_{1XX} + U_{2YY} + U_{2ZZ}) + \epsilon^6 U_{2XX} \\
\epsilon^2 P_{1Y} + \epsilon^4 P_{2Y} & = \epsilon^4 (V_{1YY} + V_{1ZZ}) + \epsilon^6 (V_{1XX} + V_{2YY} + V_{2ZZ}) \\
\epsilon^2 P_{1Z} + \epsilon^4 P_{2Z} & = \epsilon^4 (W_{1YY} + W_{1ZZ}) + \epsilon^6 (W_{1XX} + W_{2YY} + W_{2ZZ}) \\
0 & = U_{1X} + V_{1Y} + W_{1Z} \quad \tag{2.11}
\end{align}
\]

which are solved using b.c. derived from equations 2.9a-d.

**Base Solution:** Since for a thin film, \( \epsilon^2 << 1 \), equation 2.10 can be simplified by neglecting the terms containing \( \epsilon^2 \). Integration of the resulting equations (presented in detail in reference Ghatak et al 2004) finally leads to the following solution for the base components of the displacements in the film and the plate,

\[
\begin{align}
U_0 & = (3Z^2/2 - 3Z) F' \phi_1 (X), \quad W_0 = (3Z^2/2 - Z^3/2) F' \phi_2 (X) \\
\phi_1 & = \epsilon^{X/2} \left( aq \epsilon^{X/2} + (3aq + 4) \sin \left( \sqrt{3}X/2 \right) / \sqrt{3} - aq \cos \left( \sqrt{3}X/2 \right) \right) \\
\phi_2 & = \epsilon^{X/2} \left( aq \epsilon^{X/2} + (3aq + 2) \sin \left( \sqrt{3}X/2 \right) / \sqrt{3} + (aq + 2) \cos \left( \sqrt{3}X/2 \right) \right) \\
F' & = 3\Delta / \left( 6 + 12aq + 9(aq)^2 + 2(aq)^3 \right) \\
\Psi_0 & = F' \phi_2 (X) \quad X < 0 \\
& = F' \left( 2(aq + 1) + (3aq + 2)X + aqX^2 - X^3/3 \right) \quad 0 < X < aq \quad \tag{2.12}
\end{align}
\]

which suggest oscillatory variation along \( X \) with exponentially vanishing amplitude away from the contact line. Such displacements results in the following expression for the dimensionless work of adhesion (Ghatak et al 2004) \( G = W_A / (\mu/q) \):

\[
G = g(aq) \left( 27\Delta^2/2e (aq)^4 \right) \\
g(aq) = 8(aq)^4 \left( 12 + 46(aq) + 72(aq)^2 + 56(aq)^3 + 21(aq)^4 + 3(aq)^5 \right) / \\
3 \left( 6 + 12(aq) + 9(aq)^2 + 2(aq)^3 \right)^3 \quad \tag{2.13}
\]
in which \( g(aq) \) is the correction to the classical result of Obreimoff (Obreimoff et al 1930) for peeling off a rigid substrate.

**Perturbation Analysis:** Matching the coefficients for \( \epsilon^i, i = 2, 4 \) in the left and the right hand side of equation 2.11 results in the following set of equations:

\[
\begin{align*}
\epsilon^2: & \quad P_{1X} = U_{1YY} + U_{1ZZ}, \quad P_{1Y} = 0, \quad P_{1Z} = 0 \\
\epsilon^4: & \quad P_{2X} = U_{1XX} + U_{2YY} + U_{2ZZ}, \quad P_{2Y} = V_{1YY} + V_{1ZZ}, \quad P_{2Z} = W_{1YY} + W_{1ZZ} \\
& \quad U_{1X} + V_{1Y} + W_{1Z} = 0
\end{align*}
\]

Equations 2.14 are solved using the following boundary conditions derived from 2.9,

\[
\begin{align*}
\text{(a) at } Z = 0 & \quad \bar{U}_1 = \bar{V}_1 = \bar{W}_1 = \bar{U}_2 = 0 \\
\text{(b) at } Z = 1 & \quad \bar{U}_1 = 0 \quad (\bar{U}_{1Z} + \bar{W}_1) = 0 \quad (\bar{V}_1Z + \bar{W}_1Y) = 0 \\
\text{(c) at } Z = 1, X < 0 & \quad \bar{P}_1 = 3\Psi_{1XXXX}, \quad \text{at } 0 < X < aq \quad 0 = \Psi_{1XXXX}
\end{align*}
\]

where \( \Psi_1 = \bar{W}_1(X, Z = 1) \) is the vertical displacement of the flexible plate at \( Z = 1 \). Assumption of sinusoidal dependence on \( Y \) is a simplification which is apparent from the video micrographs of the contact line as in figure 1b which show that the film and the plate remains in contact whole through the area of the finger implying that the deformation of the film is not perfectly sinusoidal, as it would mean a line contact between the plate and the film. However, here I assume sinusoidal variation to keep the calculations simple. Since, the plate does not bend in the direction of \( Y \), \( \Psi_1 \) and \( \Psi_{1XXX} \) both remain uniform along this axis: consequently in b.c. 2.15c, bending stress on the plate is equated to \( \bar{P}_1 \). Equation 2.14 suggests that \( P_1 \) remains independent of \( Y \) and \( Z \) although \( U_1 \) varies along \( Y \), hence, the only solution for \( P_1 \) that can satisfy equation 2.14 is \( P_1 = 0 \). Other components of the excess displacements are obtained as:

\[
\begin{align*}
U_1 &= 0 \\
V_1 &= \frac{C(X)}{K} \left( \left( \frac{-2K(e^K + e^{-K})}{e^K + e^{-K} + 2Ke^K} \right) \sinh(KZ) + KZ \left( \frac{e^K + e^{-K} - 2Ke^{-K}}{e^K + e^{-K} + 2Ke^K} \right) \right) \cos(KY) \\
W_1 &= \frac{C(X)}{K} \left( \left( \frac{-2e^K + 2e^{-K} + 2K(e^K - e^{-K})}{e^K + e^{-K} + 2Ke^K} \right) \sinh(KZ) + KZ \left( \frac{e^K + e^{-K} - 2Ke^{-K}}{e^K + e^{-K} + 2Ke^K} \right) \right) \sin(KY)
\end{align*}
\]

Here, only the lowest order terms in \( \epsilon \) are computed since the higher order terms enhance accuracy insignificantly. Since for all our experiments, \( \epsilon < 0.3 \), the above solutions imply that the excess deformations in the film occur under very small excess pressure which is of the order \( \epsilon^4 < 0.01 \). This excess pressure, however small, varies along \( Y \) implying that it should depend upon the distance between the plate and the film. Nevertheless, the excess traction which results from the

Article submitted to Royal Society
distance dependent forces (Shenoy et al 2001, Sarkar et al 2004) apply only in the immediate vicinity (< 0.1 μm) of the contact between the film and the plate as the gap between the two increases rather sharply (could be observed in AFM images of the permanent patterns of surface undulations). Hence, it does not contribute any significantly to the overall energetics.

While equation 2.16 elaborates the variation of excess deformations along \( Y \) and \( Z \), their dependence on \( X \) is incorporated through the coefficient \( C(X) \) which is obtained by solving equations 2.15c using the following boundary conditions

\[
\begin{align*}
(i, ii) \quad & \Psi_1|_{X=0} = C_0 \Phi (K) \\
(iv) \quad & \Psi_{1XX}|_{X=0-} = \Psi_{1XX}|_{X=0+} \\
(vi) \quad & \Psi_1|_{X=aq} = \Psi_{1XX}|_{X=aq} = 0
\end{align*}
\]

Where, \( \xi = \frac{aq}{X} \) is the dimensionless amplitude of the waves and \( C_0 \Phi (K) \) is the excess stretching of the film at \( x = 0 \); \( C_0 \) is a constant and \( \Phi (K) = \frac{W_1}{C(X)}|_{X=0} = (e^{-K} - e^{-2K} + 4Ke^{K}) / K (1 + e^{2K} (1 - 2K)) \). (a) B.C. i, ii, iii and iv occur because the excess displacement and slope of the plate are continuous at \( X = 0 \); (b) at \( X = -\xi \) the displacement of the plate, its slope should vanish, which result in the b.c. v and vi; (c) similarly, at \( X = aq \) excess displacement and curvature of plate is zero, which results in b.c. vii and viii. Incorporating these boundary conditions into the solutions of equations 2.15c yield the following expression for the the excess displacement of the plate:

\[
\begin{align*}
\Psi_1 &= \frac{C_0 \Phi (K)}{aq(3\xi + 4aq)} \left( -3\xi^2 + 6aq\xi + 2(aq)^2 \right) (X/\xi)^3 - 3\left( \xi^2 - 2(aq)^2 \right) (X/\xi) + aq(3\xi + 4aq) \right) \quad \text{at } X < 0 \\
\Psi_1 &= C_0 \Phi (K) \left( 1 + \frac{aq}{\xi(3\xi + 4aq)} \right) (X/aq)^3 - 3(2\xi + 3aq)(X/aq)^2 - 3\left( \xi^2 - 2(aq)^2 \right) (X/(aq)^2) \right) \quad \text{at } 0 < X < aq \quad (2.17)
\end{align*}
\]

**Excess Energy:** Total energy of the system consists of the elastic energy of the film, bending energy of the plate and the interfacial energy:

\[
\Pi = \Pi_e + \Pi_b + \Pi_i
\]

\[
= \frac{\mu}{4} \int_{-\infty}^{0} \int_{0}^{2\pi/k} \int_{0}^{h} \left( (v_z + w_y)^2 + (u_y + v_x)^2 + (u_z + w_x)^2 \right) dz dy dx \\
+ \frac{D}{2} \int_{-\infty}^{a} \int_{0}^{2\pi/k} (\psi_{xx})^2 dy dx + W_A (2\pi a/k + A_{finger}) \quad (2.18)
\]

where \( A_{finger} \) is the interfacial area of contact at \(-a < x < 0\). From equation 2.18 the excess energy of the system is obtained as \( \Pi_{excess} = \Pi - \Pi_0 \) which is written in a dimensionless form using \( \mu/a^3 \) as the characteristic energy and by substituting for variables \( T = T_0 + \varepsilon^2 T_1 + \varepsilon^4 T_2 \), where \( T = U, V, W \) and \( P \).
The expression for excess elastic energy $\Pi_e$ in the film is obtained as,

$$
\Pi_e = \frac{1}{4} \int_{-\xi}^{0} \int_{0}^{2\pi/K} \int_{0}^{1} \left( \epsilon^6 (V_1Z + W_1Y + \epsilon^2 (V_2Z + W_2Y))^2 + \epsilon^8 (U_2Y + V_1X + \epsilon^2 V_2X)^2 + \epsilon^8 ((U_2Z + W_1X) + \epsilon^2 W_2X)^2 + 2\epsilon^4 (U_0Z + \epsilon^2 W_0X) (U_2Z + W_1X + \epsilon^2 W_2X) \right) dZdYdX
$$

where $I$ estimate the excess energy within a distance $-\xi \leq X \leq 0$. Considering only the leading order terms ($\epsilon^4$ and $\epsilon^6$), I can simplify the expression in 2.19 as,

$$
\Pi_e = \epsilon^6 C_0^2 f_2^2 (\xi, aq, K) / 4 = \epsilon^6 C_0^2 \bar{f}_2 (\xi, aq) / 4
$$

Similarly, the dimensionless excess bending energy $\Pi_b$ of the plate:

$$
\Pi_b = (3\pi/K) \int_{-\xi}^{aq} \left( 2\epsilon^2 \Psi_{0XX} \Psi_{1XX} + \epsilon^4 (\Psi_{1XX} + \epsilon^2 \Psi_{2XX})^2 \right) dX
$$

Considering only the leading order terms: $\epsilon^2$ and $\epsilon^4$, equation 2.22 simplifies to

$$
\Pi_b = \frac{3\pi}{K} \int_{-\xi}^{aq} \left( 2\epsilon^2 \Psi_{0XX} \Psi_{1XX} + \epsilon^4 \Psi_{1XX}^2 \right) dX
$$

Substituting the expressions for $\Psi_0$ and $\Psi_1$ from 2.12 and 2.17, $\Pi_b$ is obtained as,

$$
\Pi_b = \epsilon^2 C_0 \Delta f_3 (\xi, aq, K) + \epsilon^4 C_0^2 f_1 (\xi, aq, K)
$$

The interfacial energy is estimated as,

$$
\Pi_i = 2Ge \int_{0}^{\pi/K} \frac{\xi}{2} \sin (KY) dY = \frac{2G\xi}{K}
$$

Substituting the expression for $G$ from equation 2.13 in equation 2.25 and combining all the three energies, yields the total excess energy as:

$$
\Pi = (\epsilon^6 f_2 (\xi, aq, K) + \epsilon^4 f_1 (\xi, aq, K)) C_0^2 + \epsilon^2 f_3 (\xi, aq, K) \Delta C_0 + \frac{27\Delta^2 \xi g (aq)}{(aq)}
$$

The expressions for $f_1 (\xi, aq, K)$, $f_2 (\xi, aq, K)$ and $f_3 (\xi, aq, K)$ are obtained using Mathematica and are not being presented here since they could not be written in a compact form.

3. Results and Discussion

The expression for excess energy in equation 2.26 accounts for the combined effects of three sets of parameters: $\epsilon$, the confinement parameter, $\xi$ and $K$, the characteristic length scale of the perturbations and $aq$ and $\Delta$, the length scale of the geometry.
of the experiment. In what follows, we look for the solutions of these different sets of parameters which results in negative excess energy associated with the instability.

While it is evident from equation 2.26 that $\xi$ and $K$ are nonlinearly coupled quantities, the physics of the problem is better understood if we study their effects separately. We do that following the observation that the excess displacements in the film but not that of the plate are functions of $Y$, which allows us to assume that the dimensionless wave number $K$ of the perturbations is determined solely by the minima of the excess elastic energy $\Pi_e$ and not the other components of the total excess energy. Although this assumption is not exactly correct as the displacement $\Psi$ of the plate is also a function of $K$, experimental observation that the amplitude remains nearly independent of the wavelength (Ghatak et al 2003) suggests that the above assumption should not insert much inaccuracy into the calculation.

![Graphs](image.png)

Figure 2. (a) Dimensionless elastic energy of the film is plotted against the dimensionless wave number $K$ of the surface undulations. The energy of the film attains a minima at $K = 1.91$ implying that the wavelength varies with thickness of the film as $\lambda = 3.3h$. (b) Amplitude $\xi$ are plotted against the limiting values of $\epsilon$ from equation 3.2 for different values of the dimensionless length $aq$. Curves 1-4 represent $aq = 5, 15, 25$ and 55 respectively. $\epsilon_u = 0.48$ is the upper limit for $\epsilon$ beyond which no real solution for $C_0$ exists.

The plot of excess elastic energy $\Pi_e$ vs. the wave number $K$ in figure 2a then shows that $\Pi_e$ attains a minima when $K = 1.91$. The wavelength of perturbations thus scales with the thickness of the film as $\lambda = 3.3h$ which corroborates with the general observation in a wide range of experimental geometries (Ghatak et al 2000, 2003, Mönch et al 2001) that $\lambda$ remains independent of all the material and the geometric properties of the system except $h$. Furthermore, the proportionality constant matches well with that observed in experiments ($\lambda = 3 - 4h$) with rigid and flexible contacting plates.

Although the minima of $\Pi_e$ occurs at $K = 1.91$ irrespective of $C_0$ and $\epsilon$, for both these parameters real values are desired. In fact, equation 2.26, being quadratic w.r.t. $C_0$, suggests that in the limit $\Pi = 0$ the real solutions for $C_0$ exist only when,

\[
(\epsilon^2 f_3 (\xi, aq, K))^2 - 4 \left( \epsilon^6 f_2 (\xi, aq, K) + \epsilon^4 f_1 (\xi, aq, K) \right) (27\xi/K) g (aq) / (aq)^4 \geq 0
\]

resulting in the following inequality for $\epsilon^2$:

\[
\epsilon^2 \leq \frac{f_3^2 (\xi, aq, K) K (aq)^4}{4 f_2 (\xi, aq, K) 27\xi g (aq)} - \frac{f_1 (\xi, aq, K)}{f_2 (\xi, aq, K)} \tag{3.2}
\]
Equation 3.2 sets an upper bound for $\epsilon$ as evident from figure 2b where $\xi$ and $\epsilon$ which satisfy equation 3.2 are plotted for $K = 1.91$ and for different $a$. When $\epsilon$ is smaller than this upper critical limit $\epsilon_u$, two different solutions for $\xi$ exist, the stability of which depends upon whether $\Pi$ attains a minima at these solutions. Hypothesizing that $\Pi$ minimizes when $\partial\Pi/\partial C_0 = 0$, we obtain an expression for $C_0$ which when substituted in 2.26, yields

$$\Pi = \frac{\Delta^2}{4} \frac{f_3(\xi, a, K)}{f_1(\xi, a, K) + \epsilon^2 f_2(K, \xi)} + \frac{27 \Delta^2 \xi}{K} g(a)$$

(3.3)

In figure 3a we plot $\Pi$ from equation 3.3 w.r.t. $\xi$, for $\epsilon = 0.48 - 0.05$. For all these cases, $\Pi$ exhibits a non-monotonic character: with increase in $\xi$, it first increases till it reaches a maxima after which it decreases to attain a minima, there after it increases again. Stability of these systems is determined by the minima of the excess energy which if positive signifies stable base solution for the contact line and unstable solution if negative. For example, for $\epsilon = 0.48$ $\Pi$ remains positive all through implying that the undulation of a straight contact line would increase the total energy of the system so that a straight contact line would remain stable. On the other hand, $\epsilon = 0.365$, presents a limiting case for which minima of the excess energy $\Pi_{\text{min}}$ attains zero and for $\epsilon = 0.25, 0.15, 0.1$ and 0.05, $\Pi_{\text{min}}$ becomes negative, i.e. when the film is more than critically confined i.e. $\epsilon < \epsilon_c = 0.365$, the contact line can become unstable if sufficiently perturbed. The critical value $\epsilon_c = 0.365$ thus obtained for the above set of data corroborates well with 0.3 obtained in experiments of figure 1. Furthermore, a finite energy barrier at $\xi < \xi_i$ suggests that a straight contact line is not unstable for perturbations of all magnitude. Because, for perturbation with amplitude $\xi < \xi_i$ the excess energy remains positive so that these perturbations decay to zero. This result too corroborates with experiment that with increase in confinement of the film, the amplitude of the undulations never
increases from exactly zero, but from a finite value. When $\xi > \xi_i$, $\Pi$ decreases to become negative, so that these perturbations can grow till $\xi$ reaches $\xi_{\text{min}}$, at which $\Pi$ attains the minima $\Pi_{\text{min}}$; $\xi_{\text{min}}$ is then the predicted amplitude of the undulations of the contact line.

Figure 3b depicts the bifurcation diagram where $\xi_i$ (dashed line) and $\xi_{\text{min}}$ (solid line) are plotted with respect to $\epsilon$ for variety of $aq$. The dashed lines signify that the perturbations whose amplitude $\xi < \xi_i$ decay to zero, whereas solid lines mean those with $\xi > \xi_i$ grow to $\xi_{\text{min}}$. The amplitude $\xi_{\text{min}}$ increases with increase in the confinement of the film similar to that observed in experiments, although, the values predicted are somewhat (2-3 times) larger than what is observed. This discrepancy could be due to the underestimation of the excess elastic energy of the film. In figure 4 the combined effects of $\xi$ and $K$ are probed by plotting $\Pi$ w.r.t. $\xi$ and $K$. Here again $\Pi_{\text{min}}$ becomes negative for confinement parameter $\epsilon$ below a critical value $\epsilon_c = 0.365$. However, the wave number $K$ at which the minima occurs does not remain constant, it decreases from 2.12 to 0.5 while $\epsilon$ varies from 0.365 to 0.05. Although this prediction is somewhat different from experiments, in which $\epsilon$ varies between 0.3 to 0.1 at which $K$ is observed to be $1.57 \pm 0.1$, some recent observations (personal communication with Prof. A. Sharma) with very thin elastic films ($\sim 0.5 \mu$m) indeed indicate that $K$ can decrease to 1.0 as $\epsilon$ decreases to 0.07. More experiments are clearly necessary to characterize quantitatively the effect of the coupling of the two length scales with highly confined elastic films.

4. Summary

The analysis shows that confinement of an incompressible elastic film leads to favorable energetics for perturbations to grow so that the film can not deform uniformly everywhere when subjected to the tensile stresses at the interface. Furthermore, the nature of the adhesion stress is not important, even the spatial variation of the surface forces play rather an insignificant role. While the theory captures the essential physics of the problem, slight overestimation of the amplitude possibly results from the assumption of sinusoidal variations along the $Y$ axis which is not
perfectly correct. These issues can possibly be resolved by 3D simulation of the force field near the contact line. Nevertheless the results presented in this paper should be important for many other systems in confined geometries.

Acknowledgement: I sincerely thank Prof. M. K. Chaudhury in whose laboratory at Lehigh University and under whose guidance all experiments were carried out. I thank also Prof. L. Mahadevan for suggesting the perturbation analysis for solving the elasticity equations. Many thanks to Prof. Asutosh Sharma and Prof. V. Shankar for many stimulating discussions.

References

Dillard, D. A. 1989 Bending of plates on thin elastomeric foundations. J. Appl. Mech., 56, 382–386.
Fields, R. J. & Ashby, M. F. 1976 Finger-like crack growth in solids and liquids. Philos. Mag. 33, 33–48.
Gent, A. N.; Lindley, P. B. 1958 Internal rupture of bonded rubber cylinders in tension. Proc. R. Soc. London, Ser. A, 195–205.
Gent, A. N. and Tompkins, D. J. 1969 Nucleation and growth of gas bubbles in elastomers. J. App. Phys. 40, 2520–2525.
Ghatak, A.; Shenoy, V.; Chaudhury, M. K.; Sharma, A. 2000 Meniscus instability in thin elastic film. Phys. Rev. Lett. 85, 4329–4332.
Ghatak, A.; Chaudhury, M. K. 2003 Adhesion induced instability patterns in thin confined elastic film. Langmuir 19, 2621–2631.
Ghatak, A.; Mahadevan, L.; Chaudhury, M. K. 2005 Measuring the work of adhesion between a soft confined film and a flexible plate. Langmuir web released on 11th January.
Homsy, G. M. 1981 Viscous fingering in porous media. Annu. Rev. Fluid Mech., 19, 271–311.
McKinley, G. H.; ¨Oztekin, A.; Byars, J.; Brown, R. A. 1985 Self-similar spiral instabilities in elastic flows between a cone and plate. J. Fluid Mech., 285, 123–164.
Mönch, W.; Herminghaus, S. 2001 Elastic instability of rubber films between solid bodies. Europhys. Lett. 53, 525–531.
Müller, S. J.; Larson, R. G.; Shaqfeh, E. S. G. 1989 Rheol. Acta. 28, 499.
Nittmann, J.; Daccord, G.; Stanley, H. E. 1985 Fractal growth of viscous fingers: quantitative characterization of a fluid instability phenomenon. Nature 314, 141–144.
Obreimoff, J. W. 1930 The splitting strength of Mica. Proc. Roy. Soc. Lond. ser. A, 127, 290–297.
Reiter, G. 1992 Dewetting of thin polymer films. Phys. Rev. Lett. 68, 75–78.
Saffman, P. G.; Taylor, G. I. 1958 The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid. Proc. Roy. Soc. London, series A 245, 312–329.
Sarkar, J; Shenoy, V; Sharma, A. 2004 Patterns, forces and metastable pathways in debonding of elastic films. Phys. Rev. Lett. 93, 018302–018305.
Sharma A. and Khanna R. 1998 Pattern formation in unstable thin liquid films. Phys. Rev. Lett. 81, 3463–3466.
Shenoy, V.; Sharma, A. 2001 Pattern formation in a solid film with interactions. Phys. Rev. Lett. 86, 119–122.
Urhama, Y, 1989 Effect of peel load on Stringiness phenomena and peel speed of pressure sensitive Adhesive tape. J. Adhes. 31, 47–58.

Article submitted to Royal Society