Forgetting Memories and their Attractiveness

Enzo Marinari
Sapienza Università di Roma, INFN Sezione di Roma 1 and Nanotech-CNR,
UOS di Roma, P.le A. Moro 2, 00185 Roma, Italy
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We study numerically the memory which forgets, introduced in 1986 by Parisi by bounding the synaptic strength, with a mechanism which avoid confusion, allows to remember the pattern learned more recently and has a physiologically very well defined meaning. We analyze a number of features of the learning at finite number of neurons and finite number of patterns. We discuss how the system behaves in the large but finite $N$ limit. We analyze the basin of attraction of the patterns that have been learned, and we show that it is exponentially small in the age of the pattern. This is a clearly non physiological feature of the model.

The model proposed by Hopfield [1] in 1982, implementing in a statistical mechanical setting ideas from Hebb [2] and Eccles [3] has surely been a dramatic breakthrough in the study of neural systems. In the last years a huge amount of work has helped to try and bring these ideas closer to realistic systems (see for example one of [14][15] and of references therein).

Hopfield model can learn a number of patterns that grows linearly in the number of neurons $N$, This is a welcome feature but it goes with a serious drawback. When one tries to have the system learning more than this maximal fraction of patterns the system enters a state of complete confusion, and forgets everything; no patterns can be learned in this situation.

Parisi solved this problem in 1986 with an elegant proposal [14], noticing that bounding the strength of the synapses is enough to allow the memory to forget. Now less patterns are remembered than in the original Hopfield model (but always with a capacity proportional to $N$), but if new patterns are shown to the system it forgets the old patterns and learns the new ones. It is clear that physiologically the synaptic strength cannot grow indefinitely, that gives in this sense to Parisi approach a strong physiological common sense. Parisi model is in principle based on synapses that must be known with high precision: this is a strong physiological weakness, since it is experimentally clear that no more than order of ten levels of synaptic intensity can be relevant (see for example [15] and references therein). Models where a lower synaptic accuracy is needed can be important, but we will not deal with them here. The model proposed by Parisi has been solved analytical, in the limit of $N \to \infty$ in [16]. Also other variations (the so called palimpsestic approaches) have been introduced in [17] (where a threshold is also used) and in [18]. These are basic models that need to be largely elaborated and modified in order to become good potential descriptions of realistic systems. A lot of efforts have been done in this direction (see for example [15][19][20] and references therein).

In this note we will try to clarify some important features of the basic Parisi approach, since this is an important basis of many recent developments. We will analyze numerically detailed features of the learning at finite number of neurons and finite number of patterns, also as a function, for example, of the number of patterns that have been shown to the system. We will show how the system behaves in the large but finite $N$ limit. We will analyze the basin of attraction of the recognizable patterns, and detect what seems to be a clearly non physiological feature of the model.

**Model and Parameters** — In our model we have $N$ neurons $\sigma_i$, $i = 1, ... N$ that can take the values $\pm 1$. We will denote a configuration of the $N$ neurons by $\{\sigma\}$. The model is of the mean field type since in principle each neuron $i$ is connected to all neurons $j$ by a synaptic strength $J_{i,j}$. The synaptic connections are formed by learning from patterns. A learning rule makes the synaptic connections dependent on $M$ patterns (collections of $N$ values $\tau_i = \pm 1$ that we denote by $\{\tau\}$). We call $R = M/N$ the ratio of presented patterns over number of neurons.

The dynamics of the neurons is dictated by the synapses dependent energy function:

$$E[\sigma] = -\frac{1}{2} \sum_{i \neq j} \sigma_i J_{i,j} \sigma_j ,$$

where positive synaptic couplings enhance the affinity among a given couple of neurons, while negative couplings suppress it. When a pattern $\{\tau\}$ is presented to the system the synapses evolve. We write this evolution as:

$$J_{i,j}^{\text{new}} = f \left( J_{i,j}^{\text{old}} + \frac{\tau_i \tau_j}{\sqrt{N}} \right) ,$$

There are $M$ patterns that in the model are presented, one by one, only one time each, to the system. After the learning the system could be able to recognize some of the patterns that have been presented. Finding a pattern is defined here as a situation where after energy minimization the system ends close enough one of the $M$ patterns that have been presented. By recognition we mean starting from one of the pattern presented and used to form the synapses, and after energy minimization ending close enough to this same pattern (by close enough we mean...
that in the final pattern no more than \( \epsilon N \) neurons differ from the ones of the starting pattern: in all this note we take \( \epsilon = 0.02 \). We define the recovery rate \( \rho \) as the number of recognized patterns over the number of neurons \( N \). It is very interesting and telling, as we will see, to compute the basin of attraction of the different stored patterns, by starting, for example, from a random point in phase space (or in a sphere around one of the presented patterns).

The original generalized Hebb is given by the simple linear relation \( f(x) = x \). Hebb rule is in some sense very successful, since it allows to store and recognize a number of patterns proportional to \( N \). If the number of patterns one wants to store \( M \), is small enough (\( M < \alpha_c N \approx 0.14N \)), this can turn out to be what one needs, but when \( M \) is larger than the critical threshold \( \alpha \) a disaster occurs, since the memory gets completely confused, and everything is forgotten. A very natural and simple mechanism for solving this problem has been proposed in [14] [17], and, in the limit of \( M \to \infty \) and \( N \to \infty \), the Parisi model [14] has been solved analytically in [16]. Here synapses cannot grow more than a constant saturation threshold \( A \). So \( f(x) = x \) if \( -A < x < A \) (adopting in this regime the original generalized Hebb rule), but \( f(x) = -A \) for \( x \leq -A \) and \( f(x) = A \) for \( x \geq A \). Bounding the values of the synapses is physiologically very reasonable (even if more features have to be calibrated in order to get a realistic model, see for example [15] and references therein): the structure of this forgetting model is in this sense both very simple and very reasonable. In his 1986 numerical simulations Parisi estimated that the model does not learn for \( A > A_c \sim 0.7 \) (a very precise estimate since the exact solution gives \( A_c \sim 0.692 \)). On the contrary for small values of \( A \) the model can have a finite storage capability, close to 0.04 (i.e. smaller than the one of the Hopfield model).

The Recovery Rate as a Function of the Threshold \( A \) — As a first step we have measured the recognition rate \( \rho \) as a function of \( A \). Given the couplings that have been learned from patterns we start, one by one, from all patterns that have been shown to the system, minimize the energy function (1) by changing the neurons \( \{ \sigma \} \) in a steepest descent, and check if the arrival point is close enough (with a precision \( \epsilon \), as we have explained before) to the starting point. We use values of \( N \) going from 100 to 3200, values of \( M \) such to have ratios \( R = M/N \) going from 1 to 8. We average over 40 different realizations of the set of patterns but for the case \( N = 3200, M = 3200 \) we have 20 samples and the case \( N = 1600, M = 3200 \) where we have 30 samples. Statistical errors are computed as fluctuations over the independent samples. In the Supplemental Material (SM), available online, we show all the available data and plot them in different ways. We show in Fig. 1(a) the recovery rate as a function of the threshold \( A \) for \( R = 1 \) and for \( R = 2 \) (asymptotically for large \( N \) and \( M \) the system remembers in both case, and independently from \( R \), the last \( N \) patterns shown: see later). The two plots show clear finite \( N \) effects, and a comparison of the two also shows that the value of \( M \) plays a role. In the upper plot one shows one pattern per neuron, and the asymptotic result is closer than in the case where we show to the system two pattern for each neuron: finite size effects increase with \( R \). The large \( A \) region where \( \rho = 0 \) is very clear, and it is moving to lower values of \( A \) for increasing values of \( N \). For large \( N \) \( \rho \) departs linearly from \( A = 0 \), increases linearly in a finite range of \( A \), has a maximum and eventually reaches zero. The value of \( A \) where the recovery rate is maximal, \( A_{\text{max}} \) is well defined already at these values of \( N \).

Fig. 1 makes clear that already for a finite (and reasonably large) number of patterns the behavior of the system is close to the asymptotic behavior (our systems with \( N = 3200 \) neurons can learn more than 120 patterns), and that the details of the learning also depend from the number of patterns presented to the system (for
finite values of $N$ and $M$.

The Optimal Value of the Threshold $A$ — As a first step we want to understand how the maximal recognition rate is reached in the $N \to \infty$ limit at fixed values of $R$. In order to do that we have first interpolated the maxima of the curves of Figs. (1). We have used a quadratic interpolation around the measured maximum value. We plot in Fig. (2) the maximum recognition rate $\rho_{\text{max}}$ and the value of the threshold $A_{\text{max}}$ such that $\rho$ is maximum, versus $1/N$. We fit the data for $\rho_{\text{max}}$ in two different ways.

Both for $R = 1$ and $R = 2$, separately, we use a linear fit and a quadratic fit, in both cases discarding the point with $N = 100$. As it is clear from panel (a) of the figure from all these fits we get compatible results for the value of $\rho_{\text{max}}$ in the $N \to \infty$ limit, where, for example, the discrepancy among the values measured at finite $N$ for $R = 1$ and $R = 2$ shrinks (when using the quadratic fit it becomes of the order of one part per thousand in the $N \to \infty$ limit). The estimated asymptotic value is $0.0431(5)$, and it is slightly larger (of less than 5%) than the theoretical value quoted by [16], maybe because of a pronounced curvature in the $N \to \infty$ limit.

Our estimates of $A_{\text{max}}$ are less precise, so we only use linear fits, and we show them in panel (b) of Fig. (2); also here we discard the $N = 100$ data. Remarkably the finite $N$ corrections have opposite signs in the two cases $R = 1$ and $R = 2$, and the finite $N$ optimal values of the threshold are very different, but in the $N \to \infty$ limit for fixed $R$ the two sets of data converge to very similar values. Averaging the two values we estimate $A_{\text{max}}^{N=\infty} = 0.357(5)$, that has to be compared to the theoretical values $0.354$ [16]. The fact that when different (large) numbers of patterns have been shown to the system the optimal learning threshold can vary is an important feature we are learning here.

The Recovery Rate as a Function of the Pattern Rank — As we have discussed the remarkable feature of our model is that it can forget. When new patterns are learned old patterns are forgotten: the storage capacity is dedicated to the patterns learned more recently. From an intuitive point of view it is not clear if in the $N \to \infty$ limit all this capacity strictly moves to the last patterns shown to the system. Here we analyze this issue from a numerical, finite $N$ point of view. In Fig. (3) we show $\rho$ as a function of the pattern rank $r$ for systems with $R = 1$ and different number of neurons. Here $A = 4$, where the recognition rate is high.

For increasing $N$ the jump from the recognized region
FIG. 4. The fraction of recognized patterns in the transient region $\phi_{\text{transient}}$, as defined in the text, as a function of $N$. Here $R = 1$ and $A = 0.40$. The continuous line is for the function $1.08 x^{-0.5}$.

to the forgotten region becomes sharper: already on our larger systems, here with $N = M = 3200$, there is a very clear jump. In order to make quantitative the visual evidence that the recognition rate $\rho$ is becoming a step function we analyze the region where the recognition rate is smaller than one and larger then zero, and we check if it is shrinking to zero for $N \to \infty$. We look at the ordered patterns (starting with the most recently shown that have smaller rank) and we have the transient region starting when we observe the first pattern with a recognition rate smaller than $1 - \eta$ (we set $\eta = 0.01$). The transient region ends when we meet the first pattern with a recognition rate smaller than $\eta$, and we compute the fraction of patterns $\phi_{\text{transient}}$ in the transient region, that we show in Fig. (4).

$\phi_{\text{transient}}$ goes to zero when $N \to \infty$. We plot in the figure the function $1.08 x^{-0.5}$, that describes perfectly the data. We have shown a clear, unambiguous evidence that the recognition rate as a function of the pattern rank becomes a $\theta$–function in the $N \to \infty$ limit.

It is useful to look at the logarithm of minus the derivative of the recognition rate with respect to the rank

$$
\Lambda \equiv - \log \left( \frac{d\rho}{d(\text{rank})} \right) .
$$

We show it in panel (a) of Fig. (5) for $R = 1$, $A = 0.4$ and different values of $N$ (our statistics for systems with $N = M = 3200$ is not good enough to be useful in this analysis).

The continuous line in Fig. (5) are for a best fit of $\Lambda$ to the logarithm of a Gaussian function, i.e. we use a three parameters fits in $n_c$, $\tilde{r}$ and $\sigma_\Lambda$:

$$
\Lambda(r) \sim c - \frac{(r - \tilde{r})^2}{\sigma_\Lambda^2} .
$$

The best fits are very good, and reconstruct the data with remarkable accuracy. The Gaussian functions become narrow when $N$ increases, and the heights of their maxims increase.

To make this statement quantitative we look at the dependence over $N$ of the maximum of the Gaussian functions and at their width, and we plot them, in log-log scale, in panel (b) of Fig. (5). The straight lines are here for our best power law fits to the values of the peak and of $\sigma_\Lambda^2$, that describe very accurately the data. We find that the maximum of $\Lambda$ grows as $N$ to the power $0.6 \pm 0.1$, while $\sigma_\Lambda^2$ decreases when $N$ increases with a power $-1.0 \pm 0.1$. The evidence for the recognition rate being a $\theta$ function in the $N \to \infty$ limit is clear.

The Basins of Attraction of the Recognized Patterns — We are considering systems with $N$ neurons, where after showing $M$ random, uncorrelated patterns we have been able to “learn” some of these patterns. A very relevant question is about the basin of attraction of these patterns that we are able to recognize (as we have seen, the last that have been shown to the system). We will discuss the answer to a precise, simple question. If we start from one random pattern, extracted with uniform proba-
bility among all the available neuron configurations, how probable it is that we “recognize” one of the patterns that we have stored in our memory? Some of the starting patterns will lead to no recognition, and some will lead instead to “recognize” one of the patterns that have been stored in the system. In a “reasonable” physical systems all stored (random and uncorrelated) patterns should have a similar probability of being retrieved one starting from a random pattern: this probability could mildly decrease with the age of pattern (i.e. with the time that has passed after its presentation to the system), but should not collapse for older patterns. As we will see, and we find this to be a strong inherent weakness of the system we are studying, this is not true here.

We show in Fig. 6 the basin of attraction of recognized patterns (defined as we have just discussed: we start from a random point in phase space, we have the energy decrease by steepest descent, and check if we hit a pattern at distance closer or equal to $\epsilon$, using, as before $\epsilon = 0.02$) as a function of the pattern number. For all $N$ values that we show in Fig. 6 we start from $10^5$ random patterns for each of the 10 different sets of couplings we analyze (samples), and we average over the ten samples. For all $N$ values where we have statistically significant results (these are the ones we show in Fig. 6) there is a clear exponential decrease of the basin of the attraction of the pattern as a function of the age of the pattern: we are able to observe a decrease of the order of $10^5$. Old patterns are only very rarely recognized: they are “stored”, since starting from the exact pattern one falls very close to the pattern itself, but recognizing them when starting not too close is very difficult. The exponential decay of the basins of attraction with the time from presentation is probably related to exponential degradation for increasing noise observed $\text{[15, 20]}$. We are using a here a very simple learning schedule, where patterns are only shown ones: a more complex schedule with repetitions could probably alleviate this problems, but the capacity of the system would decrease, and, crucially, physiologically this does not look like an appropriate mechanism.

The probability that starting from a random pattern the system does not reach any of the stored pattern does not change dramatically with $N$: it is 0.67 for $N = 100$, 0.58 for $N = 200$, 0.65 for $N = 400$ and 0.69 for $N = 800$. This confirms the idea that the basin of convergence to a recognized pattern when starting from a random pattern is a sensible measure of how a recognized pattern is accessible when the systems tries to recognize.

Conclusions — We have analyzed a simple and basic model (proposed by Parisi [4] and solved analytically in the thermodynamical limit by van Hemmen, Keller and Kühn [16]) of a memory that can forget. We have analyzed and unveiled in detail, numerically, its behavior for finite number of neurons $N$ and finite number of patterns shown to the system, $M$, and its dependence over $R = M/N$. We have been able to answer numerically about issues like the concentration of the recognizable pattern on the most recently learned patterns in the limit of $N, M \to \infty$ (where the recognizable pattern are always and only that last that have been learned). We have found that the size of the basin of attraction of the recognized patterns decreases exponentially with the time distance from the moment when the pattern has been presented. This feature is not reasonable from a physiological point of view, and should be cured when trying to describe realistically neural systems: it is possible that modifications of palimpsestic schemes [17, 18] could be useful in this context.

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*enzo.marinari@uniroma1.it*

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