Inferring Smooth Control: Monte Carlo
Posterior Policy Iteration with Gaussian Processes

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Abstract: Monte Carlo methods have become increasingly relevant for control of non-differentiable systems, approximate dynamics models and learning from data. These methods scale to high-dimensional spaces and are effective at the non-convex optimizations often seen in robot learning. We look at sample-based methods from the perspective of inference-based control, specifically posterior policy iteration. From this perspective, we highlight how Gaussian noise priors produce rough control actions that are unsuitable for physical robot deployment. Considering smoother Gaussian process priors, as used in episodic reinforcement learning and motion planning, we demonstrate how smoother model predictive control can be achieved using online sequential inference. This inference is realized through an efficient factorization of the action distribution and a novel means of optimizing the likelihood temperature to improve importance sampling accuracy. We evaluate this approach on several high-dimensional robot control tasks, matching the sample efficiency of prior heuristic methods while also ensuring smoothness. Simulation results can be seen at monte-carlo-ppi.github.io.

Keywords: approximate inference, policy search, model predictive control

Figure 1: High-dimensional, contact-rich tasks such as manipulation (left) can be performed effectively using sample-based model predictive control. While prior work uses correlated actuator noise to improve sample-efficiency and exploration, these methods do not preserve the smoothness in the downstream actuation $a$, resulting in aggressive control (center). We use smooth Gaussian process priors to infer posterior actions (right), which preserves smoothness while maintaining performance and sample efficiency, as both are using only 32 samples. Rewards $r$ show quartiles over 25 seeds.

1 Introduction

Learning robot control requires optimization to be performed on sampled transitions of the environment [1]. Monte Carlo methods [2] provide a principled means to approach such algorithms, bridging black-box optimization and approximate inference techniques. These methods have been adopted extensively by the community for their impressive simulated [3, 4, 5, 6] and real-world [7, 8, 9, 10, 11, 12, 13, 14, 15] robot learning results. Their appeal includes requiring only function evaluations of the dynamics and objective, so can be applied to complex environments with minimal overhead (Figure 1). Moreover, their stochastic nature also avoids issues with local minima.

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that occur with gradient-based solvers [16, 17]. Finally, while Monte Carlo sampling is expensive, shooting methods can be effectively parallelized across processes and the advent of simulations on GPUs also provides a means of acceleration [18, 13]. However, some aspects of black-box optimization are open to criticism. Sample-based solvers such as the cross-entropy method (CEM) [19] appear ‘wasteful’, ignoring computation by throwing away the majority of samples, while others enforce high-entropy search distributions to avoid premature convergence [18]. Moreover, many design decisions and hyperparameters are heuristic in nature, which is undesirable from both the user- and research perspective when interpreting, tuning or advancing these methods.

In this work, we consider Monte Carlo optimal control through the broader perspective of inference-based control [20, 21, 22, 23, 24, 25, 26, 27], where optimization is achieved through importance sampling [28]. This approach covers settings such as policy search [29], motion planning [8, 30] and model predictive control (MPC) [18]. From this view point, we highlight two key design decisions: the likelihood temperature and the distribution over action sequences. An adaptive temperature scheme is crucial for controlling the optimization behavior across objectives and distributions, but in many methods this aspect is ignored or opaque. Moreover, correlated action sequences are equally crucial for performing effective exploration and control in practical settings. Smoothness, arising from such correlations, is an aspect of human motion [31]. Smooth priors have taken many forms across domains, such as movement primitives [32], smoothed- [11, 12] or coloured noise [4]. We use Gaussian processes [33] as action priors and show how they can be scaled to high-dimensional action spaces through factorization of the covariance. Evaluating on simulated robotic systems, we reproduce prior results on policy search while transferring these ideas to MPC, matching prior performance with respect to sample efficiency while ensuring smooth actuation.

**Contribution.** First, we present a perspective of episodic inference-based control based on Gibbs posteriors. Using this view, we then present novel Monte Carlo variants that incorporate the approximate inference error due to importance sampling, simplifying the hyperparameter while providing regularization. Thirdly, we demonstrate how richer Gaussian process priors can be combined with these regularized Gibbs posteriors for Monte Carlo MPC using online sequential inference, which achieves greater smoothness and sample efficiency than standard white noise priors. We highlight connections between this approach to MPC and effective prior approaches to episodic policy search.

## 2 Monte Carlo Methods for Optimal Control

This section outlines the problem setting and introduces variational optimization and posterior policy iteration methods. We consider the standard (stochastic) optimal control setting in discrete-time, with states \( s \in \mathbb{R}^d_s \), actions \( a \in \mathbb{R}^d_a \). Optimization is framed as maximizing a reward \( r: \mathbb{R}^d_s \times \mathbb{R}^d_a \rightarrow \mathbb{R} \) under dynamics \( p(s_{t+1} \mid s_t, a_t) \) and initial state distribution \( p(s_1) \),

\[
\max_{a_1, \ldots, a_T} \mathbb{E} \left[ \sum_{t=1}^T r(s_t, a_t) \right] \quad \text{s.t.} \quad s_{t+1} \sim p(\cdot \mid s_t, a_t), \quad s_1 \sim p(s_1). \tag{1}
\]

This work focuses on the episodic setting, where optimization is performed after evaluating the current solution over a finite-time horizon \( T \). We frequently use the episodic return \( R \), where \( R(S, A) = \sum_{t=1}^T r(s_t, a_t) \), using upper-case to denote sequences, e.g. \( A := \{a_1, \ldots, a_T\} \).

### 2.1 Variational Optimization with Gibbs Posteriors

The optimization outlined above is amenable to gradient-based solvers such as stochastic differential dynamic programming [34]. However, to aid optimization through exploration and regularization, we can consider optimizing a parametric belief over action sequences \( q \in Q \). The variational formulation (Equation 3) generalizes Bayes’ rule beyond optimizing likelihoods and resembles many learning algorithms [35, 36]. This work concerns optimizing an open-loop action sequence to maximize an episodic return. Bayesian inference of an action sequence from data, known as input estimation, can be performed using message passing of the appropriate probabilistic graphical model, capturing the sequential structure of the problem and necessary priors [27]. If the measurement log-likelihood is replaced with the control objective, this inference computation can be shown to have precise dualities with dynamic programming-based optimal control [37]. While this switch in objective provides a powerful suite of inference tools for efficient computation, it requires treating the control objective as a Markovian log-likelihood, which is not the case for episodic objectives. The Gibbs likelihood is a general treatment of the objective-as-likelihood (Definition 1) [38, 39].
Definition 1. (Gibbs likelihoods and posteriors) For a loss \( f \) and prior \( p(x) \), the Gibbs posterior \( q_\alpha \) for parameter \( x \) is derived by constructing the Gibbs likelihood \( \exp(-\alpha f(x)) \) from the loss,

\[
q_\alpha(x) = \frac{1}{Z_\alpha} \exp(-\alpha f(x)) p(x), \quad Z_\alpha = \int \exp(-\alpha f(x)) p(x) \, dx, \quad \alpha \geq 0.
\]

This posterior minimizes the following objective

\[
q_\alpha = \arg \min_{q \in \mathcal{Q}} \mathbb{E}_{x \sim q} [f(x)] + \frac{1}{\alpha} \mathbb{D}_{KL}[q(x) \mid\mid p(x)].
\]

This objective appears in PAC-Bayes methods [38], mirror descent methods [40] and Bayesian inference as the evidence lower bound objective when \( f(x) \) is a negative log-likelihood [39].

Augmenting the variational optimization objective with prior regularization (Equation 3), we obtain an expression of the optimal belief in the action sequence (Equation 2). The parameter \( \alpha \) has a range of meanings, depending on context. In PAC-Bayes it is the dataset size, in mirror descent it is an update step size and in risk-sensitive control it is the sensitivity [41, 42]. Example 1 in Appendix A examines a tractable linear-quadratic-Gaussian example of this update, demonstrating its relation to Newton-like optimization and highlighting the effect \( \alpha \) has on the regularized update.

2.2 Posterior Policy Iteration

The optimal control problem (Equation 1) is ambiguous regarding whether the action sequence or state-action trajectory is the optimization variable. Applying the Gibbs posterior to the optimal control setting recovers Rawlik et al.’s posterior policy iteration [41], which can be implemented using the joint distribution or policy. We consider the following joint state-action distribution, that factorizes in the following Markovian fashion \( p(S, A) = p(s_1) \prod_{t=1}^{T-1} p(s_{t+1} \mid s_t, a_t) p(a_t \mid s_t) \). Posterior policy iteration updates the state-action distribution through the policy, constructing a Gibbs likelihood from the reward, as the dynamics and initial state distribution are constant.

Definition 2. (Posterior policy iterations (PPI) [43]) As the initial distribution and dynamics are shared by the prior and posterior joint state-action distribution, the joint Gibbs posterior \( q_\alpha(S, A) \propto \exp(\alpha R(S, A)) p(S, A) \) can be alternatively expressed using the policy posterior update \( q_\alpha(A \mid S) \propto \exp(\alpha R(S, A)) p(A \mid S) \).

Using this update, the key decisions are choosing \( p(A \mid S) \), \( \alpha \) and the inference approximation. If \( p \) and \( q \) are Gaussian, then PPI involves iterative refinement of the distribution. In the Monte Carlo setting, \( q_\alpha \) takes the form of an importance-weighted empirical distribution. To apply iteratively, \( p \) is updated using the M-projection, following the objective (Equation 3), i.e. a weighted maximum likelihood fit of the policy parameters [29]. This approach is a stochastic approximate expectation maximization (SAEM) method [44] and described fully in Algorithm 1 in the Appendix. We argue a key aspect of PPI methods is how to specify the inverse temperature \( \alpha \) during optimization (Section 3), as it has a strong influence on the posterior, which is important when fitting rich distributions such as Gaussian processes (Section 4) from samples. Gaussian process action priors can be applied to several control settings, such as policy search and model predictive control (Section 6).

3 Posterior Policy Constraints for Monte Carlo Optimization

The Gibbs posterior in Definition 2 has been adopted widely in control, albeit from a range of different perspectives, such as Bayesian smoothing [23], solutions to the Feynman-Kac equation [45], maximum entropy [26], mirror descent [46] and entropy-regularized reinforcement learning [47]. An open question is how best to set \( \alpha \) for Monte Carlo optimization? Relative entropy policy search (Definition 3), provides a principled and effective means of deriving \( \alpha \) for stochastic optimization, using the constrained optimization view of entropy-regularized optimal control.

Definition 3. (Episodic relative entropy policy search (eREPS) [29]) Maximize the expected return, subject to a hard KL bound \( \epsilon \) on the policy update,

\[
\max_\theta \mathbb{E}_{s_{t+1} \sim p(\cdot \mid s_t, a_t), a_t \sim q_\alpha(\cdot \mid s_t), s_t \sim p(\cdot)} [R(s_t, a_t)] \quad \text{s.t.} \quad \mathbb{D}_{KL}[q_\alpha(A \mid S) \mid\mid p(A \mid S)] \leq \epsilon.
\]

The posterior policy takes the form \( q_\alpha(A \mid S) \propto \exp(\alpha R) p(A \mid S) \), where \( \alpha \) is derived from Lagrange multiplier calculated by minimizing the empirical dual \( G(\cdot) \) using \( N \) samples,

\[
\min_\alpha G(\alpha) = \frac{\epsilon}{\alpha} + \frac{1}{\alpha} \log \int p(S, A) \exp(\alpha R(S, A)) \, dS \, dA \approx \frac{\epsilon}{\alpha} + \frac{1}{\alpha} \log \frac{1}{N} \sum_{n=1}^{N} \exp(\alpha R_n).
\]
While REPS is a principled approach to stochastic optimization, we posit two weaknesses: The hard KL constraint is difficult to specify, as it depends on the optimization problem, distribution family and dimensionality. Secondly, the Monte Carlo approximation of the dual has no regularization and may poorly adhere to the KL constraint without sufficient samples. Therefore, we desire an alternative approach that resolves these two issues, capturing the Monte Carlo approximation error with a simpler hyperparameter. To tackle this problem, we interpret the REPS update as a pseudo-posterior, where the temperature is calculated using the KL constraint. We make this interpretation concrete by reversing the objective and constraint, switching to an equality constraint for the expectation,

\[
\min_\theta \mathbb{D}_{KL}[q_\theta(A \mid S) \parallel p(A \mid S)] \quad \text{s.t.} \quad \mathbb{E}_{s_{t+1} \sim p(\cdot \mid s_t, a_t), a_{t} \sim q_\theta(\cdot \mid s_t), s_1 \sim p(\cdot)}[\sum_t r(s_t, a_t)] = R^*. 
\]

This objective is a minimum relative entropy problem [48], which yields the same Gibbs posterior as eREPS (Lemma 1, Appendix A). With exact inference, a suitable prior and oracle knowledge of the maximum return, this program computes the optimal policy in a single step by setting \( R^* \) to the optimal value. However, in this work, the expectation constraint requires self-normalized importance sampling (SNIS) on sampled returns \( R^{(n)} \) using samples from the current policy prior,

\[
\mathbb{E}_{s_{t+1} \sim p(\cdot \mid s_t, a_t), a_{t} \sim q_\theta(\cdot \mid s_t), s_1 \sim p(\cdot)}[\sum_t r(s_t, a_t)] \approx \sum_n w^{(n)} q/p R^{(n)} = \frac{\sum_n R^{(n)} \exp(\alpha R^{(n)})}{\sum_n \exp(\alpha R^{(n)})} = R^*.
\]

Rather than specifying \( R^* \) here, we identify that this estimator is fundamentally limited by inference accuracy. We capture this error by applying an IS-derived concentration inequality to this estimate (Theorem 1) [49]. This lower bound can be used as an objective for optimizing \( \alpha \), balancing policy improvement with approximate inference accuracy.

**Theorem 1.** (Importance sampling estimator concentration inequality (Theorem 2, [49])) Let \( q \) and \( p \) be two probability densities such that \( q \ll p \) and \( d_2[q \mid \mid p] < +\infty \). Let \( x_1, x_2, \ldots, x_N \) i.i.d. random variables sampled from \( p \) and \( f : \mathcal{X} \to \mathbb{R} \) be a bounded function (\( ||f||_\infty < +\infty \)). Then, for any \( 0 < \delta \leq 1 \) and \( N > 0 \) with probability at least \( 1 - \delta \):

\[
\mathbb{E}_{x \sim q(\cdot)}[f(x)] \geq \frac{1}{N} \sum_{i=1}^{N} w_{q/p}(x_i) f(x_i) - ||f||_\infty \sqrt{\frac{(1 - \delta) d_2[q(x) \mid \mid p(x)]}{\delta N}}. \tag{4}
\]

The divergence term \( d_2[q \mid \mid p] \) is the exponentiated Rényi-2 divergence, \( \mathbb{E} \mathbb{D}_2[q \mid \mid p] \). While this is tractable for the multivariate Gaussian, it is otherwise not available in closed form. Fortunately, we can use the effective sample size (ESS) [50] as an approximation, as \( N_\alpha \approx N / d_2[q_\alpha \mid \mid p] \) [49, 51] (Lemma 2, see Section A of the Appendix). Combining Equation 4 with our constraint, instead of setting \( R^* \), we maximize the IS lower bound \( R^*_{\text{LB}} \) to form an objective for the inverse temperature \( \alpha \) which incorporates the inference accuracy due to the sampling given inequality probability 1 - \( \delta \).

\[
\max_\alpha R^*_{\text{LB}}(\alpha, \delta) = \mathbb{E}_{q_\alpha/p}[R] - \mathcal{E}_R(\delta, \hat{N}_\alpha), \quad \mathcal{E}_R(\delta, \hat{N}_\alpha) = ||R||_\infty \sqrt{\frac{(1 - \delta)}{\delta}} \frac{1}{\sqrt{N_\alpha}}. \tag{5}
\]

We refer to this approach as lower-bound policy search (LBPS). This objective combines the expected performance of \( q_\alpha \), based on the IS estimate \( \mathbb{E}_{q_\alpha/p}[\cdot] \), with regularization \( \mathcal{E}_R \) based on the return and inference accuracy. Treating \( p, N, ||R||_\infty \) as task-specific hyperparameters, the only algorithm hyperparameter \( \delta \in (0, 1) \) defines the probability of the bound. In practice, self-normalized importance sampling is used for PPI, as the normalizing constants of the Gibbs likelihoods are not available. While Metelli et al. also derive an SNIS lower bound [49], we found, as they did, that the IS lower bound with SNIS estimates work better in practice due to the conservatism of the SNIS bound. An interpretation of this approach is that the Rényi-2 regularization constrains the Gibbs posterior to be one that can be estimated from the finite samples, as the divergence is used in evaluating IS sample complexity [52, 53]. Moreover, the role of the ESS for regularization is similar to the 'elite' samples in CEM. Connecting these two mechanisms as robust maximum estimators (Section A), we also propose effective sample size policy search (ESSPS), which optimizes \( \alpha \) to achieve a desired ESS \( N^* \), i.e. a Rényi-2 divergence bound, using the objective \( \min_\alpha \mathbb{E}_{q_\alpha} [\hat{N}_\alpha - N^*] \). More details regarding PPI (Section A) and temperature selection methods (Table 1) are in the Appendix.

This section introduces two methods, LBPS and ESSPS, for constraining the Gibbs posteriors for Monte Carlo optimization. These methods provide statistical regularization through soft and hard constraints involving the effective sample size, which avoids the pitfall of fitting high-dimensional distributions to a few effective samples. A popular setting for these methods is MPC, which performs episodic optimization over short planning horizons while adapting each time step to the current state. Moreover, for optimal control, we also need to specify a suitable prior over action sequences. To apply PPI to the MPC setting, we must implement online optimization given this prior over actions.
Smooth Actions, which may be undesirable, though optimal, to fit exactly. Prior methods struggle at providing both effective smooth solutions and action priors, computing the finite-horizon return-based Gibbs likelihood term (Definition 1),

\[ q_0(a_{t:t+H} \mid R_{1:t}) = \int q_0(a_{t:t+H} \mid R_{1:t}) \, da_{1:t-1} \propto \int p(R_{1:t} \mid a_{1:t}) \, p(a_{1:t+H}) \, da_{1:t-1}, \]

where \( \tau \leq t + H \). As an analogy, this is equivalent to combining forecasting with state estimation, i.e. \( p(x_{t:t+H} \mid y_{1:t}) \) for states \( x \) and measurements \( y \). For correlated priors on the action space, this computation is tractable if working with Gaussian processes. In fact, a recurring aspect across several posterior policy iteration-like approaches is the use of Gaussian process policies,

\[
p(A \mid S) = \begin{cases} \prod_t \mathcal{N}(\mu_t, \Sigma_t), & \text{(Independent Gaussian noise, e.g. [18]),} \\
\mathcal{N}(\mu_t, \Sigma_t) \mathcal{N}(\mu_{t+1}, \Sigma_{t+1}) \mathcal{N}(\mu_t, \Sigma_t) & \text{(Bayesian linear regression, e.g. proMP [32]),} \\
\prod_t \mathcal{N}(\mu_t, \Sigma_t), & \text{(time-varying Gaussian e.g. [41, 54, 42]),} \\
\mathcal{GP}(\mu(s), \Sigma(s)) & \text{(non-parametric Gaussian process [55]).} 
\end{cases}
\]

Despite the simplicity of Gaussian action noise, for robotics, more sophisticated noise is often desired for safety and effective exploration [56, 29]. Prior work has proposed first-order smoothing [11, 12]. Using \( v_i^{(n)} \sim \mathcal{N}(0, I), \beta \in [0, 1] \) and \( \Sigma_t = L_tL_t^\top \), actions are sampled using

\[
a_t^{(n)} = \mu_t + L_t n_t^{(n)}, \quad n_t^{(n)} = \beta v_t^{(n)} + (1 - \beta) n_{t-1}^{(n)}, \quad \text{or} \quad n_t^{(n)} = \beta v_t^{(n)} + \sqrt{(1 - \beta^2)} n_{t-1}^{(n)}.
\]

However, in practice it is also implemented as \( a_t^{(n)} = \beta (\mu_t + L_t v_t^{(n)}) + (1 - \beta) a_{t-1}^{(n)} \). While this approach directly smooths the actuation, it also introduces a lag, which may deteriorate performance. Other approaches have used colored noise for sampling the noise \( n \) [4]. Contrast these approaches to Gibbs sampling a multivariate Gaussian joint distribution with 1-step cross-correlations [58], which is \( a_{t-1}^{(n)} = \mu_{t-1} + L_{t-1} v_{t-1}^{(n)} \), where \( \mu_{t-1} = \mu_t + \Sigma_{t-1}^{-1} (a_{t-1} - \mu_{t-1}) \), and \( \Sigma_{t-1} = \Sigma_t - \Sigma_{t-1} \Sigma_{t-1}^{-1} \Sigma_t^\top \). The differences are subtle, but important. The initial proposed sampling scheme essentially adds correlated noise to the mean for exploration, but does not consider the smoothness of the mean itself. The practical implementation incorporates the previous action, but through exponential smoothing, which introduces a fixed lag that potentially degrades the quality of the mean action sequence. Correct sampling of the joint distribution has neither of these issues and naturally extends to correlations over several time steps. We do this in a general form by considering the (continuous time) Gaussian process (see Section G, Appendix), so

\[
p(a_t) = \mathcal{N}(\mu_{t:t}, \Sigma_{t:t}) = \mathcal{GP}(\mu(t), \Sigma(t)) \quad \text{for a discrete-time sequence} \quad t = [t_1, \ldots, t_j].
\]

Proposition 1. Given a Gaussian process prior \( \mathcal{GP}(\mu(t), \Sigma(t)) \) and multivariate normal posterior \( q_0(a_{t:2} \mid R) = \mathcal{N}(\mu_{t:2}, \Sigma_{t:2}) \) for \( t_1 \) to \( t_2 \), the posterior for \( t_2 \) to \( t_4 \) is expressed as

\[
\mu_{t_2:t_4} = \mu_{t_2:t_4} + \Sigma_{t_2:t_4} \Sigma_{t_2:t_2}^{-1} v_{t_2:t_2}, \quad \Sigma_{t_2:t_4} = \Sigma_{t_2:t_2} - \Sigma_{t_2:t_4} \Sigma_{t_2:t_2}^{-1} \Sigma_{t_2:t_4},
\]

where \( v_{t_2:t_2} = \Sigma_{t_2:t_2}^{-1} (\mu_{t_2:t_2} - \mu_{t_2:t_2}) \) and \( \Lambda_{t_2:t_2} = \Sigma_{t_2:t_2}^{-1} (\Sigma_{t_2:t_2} - \Sigma_{t_2:t_2} R) \Sigma_{t_2:t_2}^{-1} \).

This update combines the new sequence prior from \( t_3 \) to \( t_4 \) and the previous likelihood used in the update for \( t_1 \) to \( t_2 \), obtained from the posterior and prior. Note, the cross-covariance \( \Sigma_{t_3:t_4:t_1:t_2} \) is computed using the covariance function of the prior GP. The proof is in Appendix A.

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1See the source code for Nagabandi et al. [11] and MBRL-zib [57].
For a stationary kernel, fixed planning horizon and fixed control frequency, the term $\Sigma_{t:t+H}$ is constant, so can be computed at initialization to avoid repeated inversion. Figure 2 demonstrates how this update lets us combine our prior with previous posterior in a principled fashion. Moreover, its continuous-time construction means that the time resolution can be updated, not just the time window, for planning at different timescales [30].

Compared to the independence assumption, modeling correlations between actions introduces complexity. The full covariance over (flattened) time and action has a complexity $O(T^2d_a^3)$, which is infeasible to work with. Assuming independence between actions, a GP per action has a complexity of $O(T^2d_a)$, requiring $d_a$ GPs to be fit, which is not desirable for online methods such as MPC. To avoid the linear scaling w.r.t. $d_a$, we propose using the matrix Normal distribution (Definition 4) for MPC scalability, as it is parameterized into single $T$ and $d_a$-dimensional covariances.

**Definition 4.** (Matrix Normal Distribution (MaVN) [59]) For a random matrix $X \in \mathbb{R}^{n \times p}$, it follows the distribution $X \sim \text{MN}(\mathbf{M}, \mathbf{K}, \Sigma)$, where $\mathbf{M} \in \mathbb{R}^{n \times p}$, $\mathbf{K} \in \mathbb{S}_p^d$, and $\Sigma \in \mathbb{S}_p^d$, if and only if vec$(X) \sim \mathcal{N}(\text{vec}(\mathbf{M}), \Sigma \otimes \mathbf{K})$, for Kronecker product $\otimes$ and $\mathbb{S}_p^d = \{X \in \mathbb{R}^{k \times k} | X^T = X, X \succeq 0 \}$.

Using the Kronecker-structured covariance provides a useful decomposition of the time-based covariance $\mathbf{K}$, that defines correlations between time steps, and an action covariance $\Sigma$ that captures correlations between actions. Typically we assume actions are independent, but cross-correlations could be learned from experience for richer coordination. While this Kronecker structure does not fully capture the correlations between time and actions, the structure is very useful for MPC on robotic systems, where the actions space could be very high but the planning horizon is sufficiently small for covariance estimation using a reasonable number of Monte Carlo rollouts.

**Feature Approximations.** Despite the matrix Normal factorization, computing the correlations between actions still requires a dense $H \times H$ covariance matrix $\mathbf{K}$ for planning horizon $H$. To sparsify this quantity, we consider kernel approximations, such as the canonical basis functions $\sum_n k(\cdot, \cdot_n)$ and spectral approximations using random features $\sum_n \phi_n(\cdot)$ [60], for a Bayesian linear model $\phi_n W$. Focusing on the squared exponential (SE) kernel, this results in radial basis function (RBF) and random Fourier features (RFF) respectively. Interestingly, RBF features are closely related to probabilistic movement primitives, used extensively in policy search for robotics [32]. For one-dimensional inputs, RFFs are effectively approximated by applying Gauss-Hermite quadrature [61] to the random weights [62]. RBF features and RFFs approximate w.r.t. time and frequency respectively and could be combined [63]. Using these continuous-time features, the optimization is now abstracted from planning horizon and control frequency, providing much greater flexibility. Secondly, due to the features, a factorized weight covariance approximation does not sacrifice smoothness. Moreover, the moment updates described above are not needed, as only $\phi_n$ is updated.

5 Related Work

**Inference-based control.** Posterior policy iteration was proposed by Rawlik et al. [41] and covers prior methods developed from Bayesian smoothing [23, 37], expectation maximization [22, 56], entropy regularization [47, 9] and path integral [64] perspectives. For MPC specifically, the path integral-based MPPI was proposed [18], with alternative formulations based on mirror descent [46] and variational inference [5, 65, 66]. Mukadam et al. [25] models the optimal state-action distribution as a sparse Gaussian process and uses linearization for approximate inference. The same approach is used for Gaussian process motion planning [30], which are also optimized using sampling [8]. Gaussian quadrature is also used for inference-based MPC [27]. Concurrent work uses the ESS for a temperature adjusting heuristic for MPPI [15] and also combines policy search with MPC using PPI techniques [67]. See Section B for a more in-depth discussion on these related works.

**Policy design and regularization.** Smooth actuation is important in robot learning for safety and exploration, having been proposed for Monte Carlo MPC [11, 12, 4] and more broadly incorporated using augmented objectives, parameter sampling and policy design, e.g. [68, 69, 70].

**Stochastic search.** Probabilistic interpretations of black-box optimization algorithms are well established [71, 72, 73], however prior work did not connect the ESS and elite samples. CEM and extensions have also been adopted widely as a solver for MPC [3, 4, 6].

**Gaussian processes for control.** This work adopts GPs for correlated action priors. This is distinct from prior work which uses GPs to approximate dynamics or value functions, e.g. [74, 75, 76, 77].
6 Experimental Results

We assess the Gibbs posterior methods and policy design empirically across various settings. Black-box optimization (Section 6.1) considers standard benchmarks, while policy search (Section 6.2) optimizes action sequences for a robotic task. For MPC (Section 6.3), we evaluate online PPI approaches with white noise and smooth priors on high-dimensional, contact-rich tasks. For the code, see github.com/JoeMWatson/monte-carlo-posterior-policy-iteration.

6.1 Black-box Optimization

To understand the behaviour of the proposed PPI variants, the performance of LBPS and ESSPS on a range of standard black-box optimization functions over a range of hyperparameters, with eREPS and CEM as baselines, are shown in Appendix D.1. Figures 7 – 10 show that ESS is a useful metric for these methods, as each solver exhibits consistent ESS for a given hyperparameter value. However, the uniform weights used by CEM (Figure 7) maintain entropy longer than ESSPS (Figure 8), which can lead to better optima, so the ESS is not sufficient to fully capture the behavior of these solvers.

6.2 Policy Search

As LBPS and ESSPS are closely related to eREPS, we repeat the experiment from prior work performing the ‘ball in a cup’ task using policy search using a Barret WAM [14], which has been shown to transfer to the physical system [78, 14, 79], as a benchmark task. Moreover, we replace ProMPs with Matrix normal RBF and RFF policies. From the kernel perspective, this feature approximation is motivated by the large (T ≃ 1000) task horizon. The results in Appendix D.2 confirm that these solvers are all capable of solving the task, based on success rate, where RBF (Figure 11) and RFF features (Figure 12) perform equally well w.r.t. the convergence of the success rate for each approach.

6.3 Model Predictive Control with Oracle Dynamics

We evaluate online PPI across a range of high-dimensional robotic control tasks in MuJoCo [80], including HumanoidStandup-v2 in Gym [81] and door-v0, hammer-v0 from mj_envs, using the Adroit hand (Figure 1) [82]. To measure smoothness, we adopt the FFT-based score $2Nf_s\sum_{i=1}^N a_i f_i$ [68], with sampling frequency $f_s$ and $N$ resolvable frequencies $f$ with amplitudes $a$. We compute the Euclidean norm of the action sequence over time and apply the smoothness measure to this signal. For the evaluation, we focus on a low computational budget, with 1 or 2 iterations per timestep. To assess the impact of approximate inference, we assess performance over an logarithmic range of sample rollouts, following prior work [4]. Details may be found in Appendix E.2.

White noise priors. Figure 3 shows MPC with white noise priors using LBPS and ESSPS, with MPPI, CEM and P1² baselines (see Table 1). While each solver performs comparably for 1024 rollouts, the low sample regime shows greater performance variance. While MPPI seems particularly effective, Figure 13 shows that its average ESS is particularly low, $\simeq 1$ for many cases. Combined with the fixed variances, this suggests optimization is closer to greedy random search than importance sampling. The poor door-v0 performance of ESSPS is due to slow opening, rather than task failure.
Figure 4: MPC return and smoothness with smooth action priors. Displaying quartiles over 50 seeds. Compared to white noise priors, smooth action priors improve sample efficiency dramatically, but only PPI methods (LBPS, ESSPS) preserve this smoothness in the downstream control.

**Policy design for smooth control.** Figure 4 shows online PPI with action priors. LBPS, ESSPS and MPPI use the SE kernel, with iceM [4] and MPPI with smooth actions and noise as a baseline. The smooth action distributions greatly improves performance across models, tasks and sample sizes, due to effective exploration. As desired, the richer SE kernel provides much greater smoothness, by up to a factor of 2 compared to baselines, with limited impact to performance. It is unsurprising that the smoothness bias reduces performance if optimal behavior is non-smooth, as illustrated in Figure 2. Appendix D.3.2 shows some of the action sequences from Figure 4, where we see GP smoothness varies with increasing rollout samples and also results in significant actuator amplitude reduction.

**Comparing kernel- and feature-based policies.** To assess feature approximations for smooth MPC, we replace the SE kernel with RBF and RFF features, while keeping the lengthscale fixed. These policies perform worse given fewer samples, but are comparable to the true kernel with sufficient samples (Figure 15). We attribute this to the compounding errors of kernel approximation and fewer effective samples. In contrast to the policy search task, RFFs appear superior to RBF features.

**Learning priors from data.** A benefit of using GP priors is the ability to optimize hyperparameters from expert demonstrations through the likelihood or a divergence. Moreover, the matrix normal distribution is useful for analyzing high-dimensional action sequences, as it decomposes temporal and action correlations into viewable covariance matrices. Section D.3.3 shows the matrix normal distributions of expert demonstrations of the tasks, obtained through human and RL experts. The results show that, surprisingly, the demonstrations are rougher than the smoothness achievable with MPC. We attribute this to control artifacts from demonstration collection and the use of Gaussian noise by RL agents. Applying the same methodology to the demonstrations of the smooth MPC agents proposed here extracts the expected action correlations across tasks. This analysis also raises the question of whether smoothness is an inductive bias we enforce for practicality, or a phenomena we expect to arise from optimality. If the latter, it may be that the simulated environments or objectives considered are lacking components that encourage smoothness, such as energy efficiency.

7 Conclusion

We present a broad perspective on episodic posterior policy iteration method for robotics and new methods for the Monte Carlo setting, based on regularizing the IS approximations. By considering vector-valued Gaussian processes for action priors, we have demonstrated how sample-efficient MPC can be performed as online inference and with greater control over actuator smoothness, connecting Monte Carlo MPC to prior work on policy search. This approach was validated on a set of high dimensional MPC tasks closely matching baseline performance while achieving greater smoothness.

**Limitations.** Much of the prior work is motivated by simplicity, minimizing hyperparameter tuning and numerical procedures such as matrix inversion [64]. In contrast, the contributions of this work introduces complexity, i.e. online temperature optimization and the use of dense covariance matrices in order to perform more sophisticated approximate inference. While this additional complexity has an impact on execution time (Table 2, Appendix), we hope the sample-efficiency when combined with accelerations such as GPU-integration should produce real-time algorithms [13].
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References

[1] R. S. Sutton and A. G. Barto. Reinforcement learning: An introduction. MIT Press, 2018.

[2] A. B. Owen. Monte Carlo theory, methods and examples. 2013.

[3] K. Chua, R. Calandra, R. McAllister, and S. Levine. Deep reinforcement learning in a handful of trials using probabilistic dynamics models. In Advances in Neural Information Processing Systems, 2018.

[4] C. Pinneri, S. Sawant, S. Blaes, J. Achterhold, J. Stueckler, M. Rolinek, and G. Martius. Sample-efficient cross-entropy method for real-time planning. In Conference on Robot Learning, 2020.

[5] M. Okada and T. Taniguchi. Variational inference MPC for Bayesian model-based reinforcement learning. In Conference on Robot Learning, 2019.

[6] M. Lutter, L. Hasenclever, A. Byravan, G. Dulac-Arnold, P. Trochim, N. Heess, J. Merel, and Y. Tassa. Learning dynamics models for model predictive agents. arXiv preprint arXiv:2109.14311, 2021.

[7] F. Stulp, E. Theodorou, J. Buchli, and S. Schaal. Learning to grasp under uncertainty. In 2011 IEEE International Conference on Robotics and Automation, 2011.

[8] M. Kalakrishnan, S. Chitta, E. Theodorou, P. Pastor, and S. Schaal. STOMP: Stochastic trajectory optimization for motion planning. In 2011 IEEE International Conference on Robotics and Automation, 2011.

[9] C. Daniel, G. Neumann, O. Kroemer, J. Peters, et al. Hierarchical relative entropy policy search. Journal of Machine Learning Research, 2016.

[10] G. Williams, P. Drews, B. Goldfain, J. M. Rehg, and E. A. Theodorou. Information-theoretic model predictive control: Theory and applications to autonomous driving. IEEE Transactions on Robotics, 2018.

[11] A. Nagabhandi, K. Konolige, S. Levine, and V. Kumar. Deep dynamics models for learning dexterous manipulation. In Conference on Robot Learning, 2019.

[12] Y. Yang, K. Caluwaert, A. Iscen, T. Zhang, J. Tan, and V. Sindhwani. Data-efficient reinforcement learning for legged robots. In Conference on Robot Learning, 2019.

[13] M. Bhardwaj, B. Sundaralingam, A. Mousavian, N. D. Ratliff, D. Fox, F. Ramos, and B. Boots. STORM: An integrated framework for fast joint-space model predictive control for reactive manipulation. In Conference on Robot Learning, 2021.

[14] M. Lutter, J. Silberbauer, J. Watson, and J. Peters. Differentiable physics models for real-world offline model-based reinforcement learning. In IEEE International Conference on Robotics and Automation, 2021.

[15] J. Carius, R. Ranftl, F. Farshidian, and M. Hutter. Constrained stochastic optimal control with learned importance sampling: A path integral approach. The International Journal of Robotics Research, 2022.

[16] D. Wierstra, T. Schaul, T. Glasmachers, Y. Sun, J. Peters, and J. Schmidhuber. Natural evolution strategies. Journal of Machine Learning Research, 2014.
[17] A. Abdolmaleki, R. Lioutikov, J. R. Peters, N. Lau, L. Pualo Reis, and G. Neumann. Model-based relative entropy stochastic search. In Advances in Neural Information Processing Systems, 2015.

[18] G. Williams, A. Aldrich, and E. A. Theodorou. Model predictive path integral control: From theory to parallel computation. Journal of Guidance, Control, and Dynamics, 2017.

[19] R. Y. Rubinstein and D. P. Kroese. The Cross Entropy Method: A Unified Approach To Combinatorial Optimization, Monte Carlo Simulation. 2004.

[20] P. Dayan and G. E. Hinton. Using expectation-maximization for reinforcement learning. Neural Computation, 1997.

[21] M. Toussaint and A. Storkey. Probabilistic inference for solving discrete and continuous state Markov decision processes. In International Conference of Machine Learning, 2006.

[22] J. Peters and S. Schaal. Reinforcement learning by reward-weighted regression for operational space control. In International Conference on Machine Learning, 2007.

[23] M. Toussaint. Robot trajectory optimization using approximate inference. In International Conference on Machine Learning, 2009.

[24] H. J. Kappen, V. Gómez, and M. Opper. Optimal control as a graphical model inference problem. In International Conference on Automated Planning and Scheduling, 2013.

[25] M. Mukadam, C. Cheng, X. Yan, and B. Boots. Approximately optimal continuous-time motion planning and control via probabilistic inference. In IEEE International Conference on Robotics and Automation, 2017.

[26] S. Levine. Reinforcement learning and control as probabilistic inference: Tutorial and review. arXiv preprint arXiv:1805.00909, 2018.

[27] J. Watson, H. Abdulsamad, R. Findeisen, and J. Peters. Efficient stochastic optimal control through approximate Bayesian input inference. arXiv preprint arXiv:2105.07693, 2021.

[28] H. J. Kappen and H. C. Ruiz. Adaptive importance sampling for control and inference. Journal of Statistical Physics, 2016.

[29] M. P. Deisenroth, G. Neumann, J. Peters, et al. A survey on policy search for robotics. Foundations and Trends in Robotics, 2013.

[30] M. Mukadam, J. Dong, X. Yan, F. Dellaert, and B. Boots. Continuous-time Gaussian process motion planning via probabilistic inference. The International Journal of Robotics Research, 2018.

[31] T. Flash and N. Hogan. The coordination of arm movements: An experimentally confirmed mathematical model. Journal of Neuroscience, 1985.

[32] A. Paraschos, C. Daniel, J. Peters, G. Neumann, et al. Probabilistic movement primitives. In Advances in Neural Information Processing Systems, 2013.

[33] C. Rasmussen and C. Williams. Gaussian Processes for Machine Learning. MIT Press, 2006.

[34] E. Theodorou, Y. Tassa, and E. Todorov. Stochastic differential dynamic programming. In American Control Conference, 2010.

[35] A. Zellner. Optimal information processing and Bayes’s theorem. The American Statistician, 1988.

[36] M. E. Khan and H. Rue. The Bayesian learning rule. arXiv preprint arXiv:2107.04562, 2021.

[37] J. Watson, H. Abdulsamad, and J. Peters. Stochastic optimal control as approximate input inference. In Conference on Robot Learning, 2019.

[38] B. Guedj. A primer on PAC-Bayesian learning. In Proceedings of the second congress of the French Mathematical Society, 2019.
[39] P. Alquier, J. Ridgway, and N. Chopin. On the properties of variational approximations of Gibbs posteriors. *Journal of Machine Learning Research*, 2016.

[40] B. Dai, N. He, H. Dai, and L. Song. Provable bayesian inference via particle mirror descent. In *International Conference on Artificial Intelligence and Statistics*, 2016.

[41] K. Rawlik, M. Toussaint, and S. Vijayakumar. On stochastic optimal control and reinforcement learning by approximate inference. In *Robotics: Science and Systems*, 2012.

[42] J. Watson and J. Peters. Advancing trajectory optimization with approximate inference: Exploration, covariance control and adaptive risk. In *American Control Conference*, 2021.

[43] K. C. Rawlik. *On probabilistic inference approaches to stochastic optimal control*. PhD thesis, The University of Edinburgh, 2013.

[44] C. Wirth, J. Fürnkranz, and G. Neumann. Model-free preference-based reinforcement learning. In *Conference on Artificial Intelligence*, 2016.

[45] S. Satoh, H. J. Kappen, and M. Saeki. An iterative method for nonlinear stochastic optimal control based on path integrals. *IEEE Transactions on Automatic Control*, 2017.

[46] N. Wagener, C. an Cheng, J. Sacks, and B. Boots. An online learning approach to model predictive control. In *Robotics: Science and Systems*, 2019.

[47] J. Peters, K. Müller, and Y. Altün. Relative entropy policy search. In *Conference on Artificial Intelligence*, 2010.

[48] J. van Campenhout and T. Cover. Maximum entropy and conditional probability. *IEEE Transactions on Information Theory*, 1981.

[49] A. M. Metelli, M. Papini, N. Montali, and M. Restelli. Importance sampling techniques for policy optimization. *Journal of Machine Learning Research*, 2020.

[50] A. Kong. A note on importance sampling using standardized weights. *University of Chicago, Dept. of Statistics, Technical Report*, 1992.

[51] C. Cortes, Y. Mansour, and M. Mohri. Learning bounds for importance weighting. In *Advances in Neural Information Processing Systems*, 2010.

[52] S. Agapiou, O. Papaspiliopoulos, D. Sanz-Alonso, and A. M. Stuart. Importance sampling: Intrinsic dimension and computational cost. *Statistical Science*, 2017.

[53] J. Hernández-González and J. Cerquides. A robust solution to variational importance sampling of minimum variance. *Entropy*, 2020.

[54] V. Gómez, H. J. Kappen, J. Peters, and G. Neumann. Policy search for path integral control. In *Joint European Conference on Machine Learning and Knowledge Discovery in Databases*, 2014.

[55] H. van Hoof, G. Neumann, and J. Peters. Non-parametric policy search with limited information loss. *Journal of Machine Learning Research*, 2017.

[56] J. Kober and J. Peters. Policy search for motor primitives in robotics. In *Advances in Neural Information Processing Systems*, 2009.

[57] L. Pineda, B. Amos, A. Zhang, N. O. Lambert, and R. Calandra. MBRL-Lib: A modular library for model-based reinforcement learning. *arXiv preprint arXiv:2104.10159*, 2021.

[58] A. Doucet. A note on efficient conditional simulation of Gaussian distributions. *Departments of Computer Science and Statistics, University of British Columbia*, 2010.

[59] A. P. Dawid. Some matrix-variate distribution theory: notational considerations and a Bayesian application. *Biometrika*, 1981.

[60] A. Rahimi and B. Recht. Random features for large-scale kernel machines. In *Advances in Neural Information Processing Systems*, 2007.
[61] F. B. Hildebrand. *Introduction to numerical analysis*. Courier Corporation, 1987.

[62] M. Mutny and A. Krause. Efficient high-dimensional Bayesian optimization with additivity and quadrature Fourier features. In *Advances in Neural Information Processing Systems*, 2018.

[63] J. Wilson, V. Borovitskiy, A. Terenin, P. Mostowsky, and M. Deisenroth. Efficiently sampling functions from Gaussian process posteriors. In *International Conference on Machine Learning*, 2020.

[64] E. Theodorou, J. Buchli, and S. Schaal. A generalized path integral control approach to reinforcement learning. *The Journal of Machine Learning Research*, 2010.

[65] A. Lambert, A. Fishman, D. Fox, B. Boots, and F. Ramos. Stein variational model predictive control. In *Conference on Robot Learning*, 2020.

[66] Z. Wang, O. So, J. Gibson, B. Vlahov, M. S. Gandhi, G.-H. Liu, and E. A. Theodorou. Variational inference MPC using Tsallis divergence. In *Robotics: Science and Systems*, 2021.

[67] Y. Song and D. Scaramuzza. Policy search for model predictive control with application to agile drone flight. *IEEE Transactions on Robotics*, 2022.

[68] S. Mysore, B. Mabsout, R. Mancuso, and K. Saenko. Regularizing action policies for smooth control with reinforcement learning. In *IEEE International Conference on Robotics and Automation*, 2021.

[69] A. Raffin, J. Kober, and F. Stulp. Smooth exploration for robotic reinforcement learning. In *Conference on Robot Learning*, 2021.

[70] D. Korenkevych, A. R. Mahmood, G. Vasan, and J. Bergstra. Autoregressive policies for continuous control deep reinforcement learning. In *International Joint Conference on Artificial Intelligence*, 2019.

[71] F. Stulp and O. Sigaud. Path integral policy improvement with covariance matrix adaptation. In *International Conference on Machine Learning*, 2012.

[72] Y. Ollivier, L. Arnold, A. Auger, and N. Hansen. Information-geometric optimization algorithms: A unifying picture via invariance principles. *Journal of Machine Learning Research*, 2017.

[73] A. Abdolmaleki, B. Price, N. Lau, L. P. Reis, and G. Neumann. Deriving and improving CMA-ES with information geometric trust regions. In *Proceedings of the Genetic and Evolutionary Computation Conference*, 2017.

[74] F. Berkenkamp and A. P. Schoellig. Safe and robust learning control with Gaussian processes. In *European Control Conference*, 2015.

[75] J. Kocijan, R. Murray-Smith, C. Rasmussen, and A. Girard. Gaussian process model based predictive control. In *American Control Conference*, 2004.

[76] M. Deisenroth, C. Rasmussen, and J. Peters. Gaussian process dynamic programming. *Neurocomputing*, 2009.

[77] M. Maiworm, D. Limon, J. Maria Manzano, and R. Findeisen. Stability of Gaussian process learning based output feedback model predictive control. In *IFAC Conference on Nonlinear Model Predictive Control*, 2018.

[78] P. Klink, H. Abdulsamad, B. Belousov, C. D’Eramo, J. Peters, and J. Pajarinen. A probabilistic interpretation of self-paced learning with applications to reinforcement learning. *Journal of Machine Learning Research*, 2021.

[79] F. Muratore, C. Eilers, M. Gienger, and J. Peters. Data-efficient domain randomization with Bayesian optimization. *IEEE Robotics and Automation Letters*, 2021.

[80] E. Todorov, T. Erez, and Y. Tassa. MuJoCo: A physics engine for model-based control. In *IEEE International Conference on Intelligent Robots and Systems*, 2012.
[81] G. Brockman, V. Cheung, L. Pettersson, J. Schneider, J. Schulman, J. Tang, and W. Zaremba. OpenAI Gym, 2016.

[82] A. Rajeswaran, V. Kumar, A. Gupta, G. Vezzani, J. Schulman, E. Todorov, and S. Levine. Learning complex dexterous manipulation with deep reinforcement learning and demonstrations. In Robotics: Science and Systems, 2018.

[83] B. D. Anderson and J. B. Moore. Optimal filtering. 2012.

[84] T. D. Barfoot, C. H. Tong, and S. Särkkä. Batch continuous-time trajectory estimation as exactly sparse Gaussian process regression. In Robotics: Science and Systems, 2014.

[85] O. Cappe, S. J. Godsill, and E. Moulines. An overview of existing methods and recent advances in sequential Monte Carlo. Proceedings of the IEEE, 2007.

[86] K. Asadi and M. L. Littman. An alternative softmax operator for reinforcement learning. In International Conference on Machine Learning, 2017.

[87] C. Chu, J. Blanchet, and P. Glynn. Probability functional descent: A unifying perspective on GANs, variational inference, and reinforcement learning. In International Conference on Machine Learning, 2019.

[88] A. Beck and M. Teboulle. Mirror descent and nonlinear projected subgradient methods for convex optimization. Operations Research Letters, 2003.

[89] G. Williams, N. Wagener, B. Goldfain, P. Drews, J. M. Rehg, B. Boots, and E. A. Theodorou. Information-theoretic MPC for model-based reinforcement learning. In IEEE International Conference on Robotics and Automation, 2017.

[90] P. Virtanen, R. Gommers, T. E. Oliphant, M. Haberland, T. Reddy, D. Cournapeau, E. Burovski, P. Peterson, W. Weckesser, J. Bright, S. J. van der Walt, M. Brett, J. Wilson, K. Jarrod Millman, N. Mayorov, A. R. J. Nelson, E. Jones, R. Kern, E. Larson, C. Carey, I. Polat, Y. Feng, E. W. Moore, J. VanderPlas, D. Laxalde, J. Perktold, R. Cimrman, I. Henriksen, E. A. Quintero, C. R. Harris, A. M. Archibald, A. H. Ribeiro, F. Pedregosa, P. van Mulbregt, and . . . Contributors. SciPy 1.0: Fundamental Algorithms for Scientific Computing in Python. Nature Methods, 2020.

[91] P. Dutilleul. The MLE algorithm for the matrix normal distribution. Journal of Statistical Computation and Simulation, 1999.

[92] L. Peng, H. Qian, Z. Shen, C. Zhang, and F. Li. Generative actor-critic: An off-policy algorithm using the push-forward model. arXiv preprint arXiv:2105.03733, 2021.

[93] S. Boyd, S. P. Boyd, and L. Vandenberghe. Convex optimization. Cambridge University Press, 2004.

[94] S. Särkkä and A. Solin. Applied Stochastic Differential Equations. Institute of Mathematical Statistics Textbooks. Cambridge University Press, 2019.