Yang-Mills and some related algebras

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Dedicated to Jacques Bros

Summary. After a short introduction on the theory of homogeneous algebras we
describe the application of this theory to the analysis of the cubic Yang-Mills al-
gebra, the quadratic self-duality algebras, their “super” versions as well as to some
generalization.

1.1 Introduction

Consider the classical Yang-Mills equations in \((s + 1)\)-dimensional pseudo
Euclidean space \(\mathbb{R}^{s+1}\) with pseudo metric denoted by \(g_{\mu\nu}\) in the canonical
basis of \(\mathbb{R}^{s+1}\) corresponding to coordinates \(x^\lambda\). For the moment the signature
plays no role so \(g_{\mu\nu}\) is simply a real nondegenerate symmetric matrix with
inverse denoted by \(g^{\mu\nu}\). In terms of the covariant derivatives \(\nabla_\mu = \partial_\mu + A_\mu\)
(\(\partial_\mu = \partial/\partial x^\mu\)) the Yang-Mills equations read

\[
g^{\lambda\nu} [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] = 0
\]

for \(\nu \in \{0, \ldots, s\}\). By forgetting the detailed origin of these equations, it
is natural to consider the abstract unital associative algebra \(A\) generated
by \((s + 1)\) elements \(\nabla_\lambda\) with the \((s + 1)\) cubic relations 1.1. This algebra
will be referred to as the Yang-Mills algebra. It is worth noticing here that
Equations 1.1 only involve the product through commutators so that, by its
very definition the Yang-Mills algebra \(A\) is a universal enveloping algebra.

Our aim here is to present the analysis of the Yang-Mills algebra and of
some related algebras based on the recent development of the theory of ho-
mogeneous algebras [2, 3]. This analysis is only partly published in [10].
In the next section we recall some basic concepts and results on homogeneous algebras which will be used in this paper.

Section 3 is devoted to the Yang-Mills algebra. In this section we recall the definitions and the results of [10]. The proofs are omitted since these are in [10] and since very similar proofs are given in Sections 4 and 6. Instead, we describe the structure of the bimodule resolution of the Yang-Mills algebra and the structure of the corresponding small bicomplexes which compute the Hochschild homology.

In Section 4 we define the super Yang-Mills algebra and we prove for this algebra results which are the counterpart of the results of [10] for the Yang-Mills algebra.

In Section 5 we define and study the super self-duality algebra. In particular, we prove for this algebra the analog of the results of [10] for the self-duality algebra and we point out a very surprising connection between the super self-duality algebra and the algebras occurring in our analysis of noncommutative 3-spheres [9], [11].

In Section 6 we describe some deformations of the Yang-Mills algebra and of the super Yang-Mills algebra.

1.2 Homogeneous algebras

Although we shall be concerned in the following with the cubic Yang-Mills algebra \( \mathcal{A} \), the quadratic self-duality algebra \( \mathcal{A}^{(+)} \) [10] and some related algebras, we recall in this section some constructions and some results for general \( N \)-homogeneous algebras [4], [2]. All vector spaces are over a fixed commutative field \( \mathbb{K} \).

A homogeneous algebra of degree \( N \) or \( N \)-homogeneous algebra is an algebra of the form

\[
\mathcal{A} = A(E, R) = T(E)/(R)
\]

where \( E \) is a finite-dimensional vector space, \( R \) is a linear subspace of \( E^{\otimes N} \) and where \((R)\) denotes the two-sided ideal of the tensor algebra \( T(E) \) of \( E \) generated by \( R \). The algebra \( \mathcal{A} \) is naturally a connected graded algebra with graduation induced by the one of \( T(E) \). To \( \mathcal{A} \) is associated another \( N \)-homogeneous algebra, its dual \( \mathcal{A}^! = A(E^*, R^+) \) with \( E^\ast \) denoting the dual vector space of \( E \) and \( R^\perp \subset E^{\otimes N^\ast} = E^*^{\otimes N} \) being the annihilator of \( R \). [4].

The \( N \)-complex \( K(\mathcal{A}) \) of left \( \mathcal{A} \)-modules is then defined to be

\[
\cdots \rightarrow \mathcal{A} \otimes \mathcal{A}_{n+1}^! \rightarrow \mathcal{A} \otimes \mathcal{A}_n^! \rightarrow \cdots \rightarrow \mathcal{A} \rightarrow 0 \quad (1.2)
\]
where $A^*_n$ is the dual vector space of the finite-dimensional vector space $A^i$, and where $d : A \otimes A^*_{n+1} \to A \otimes A^*_{n}$ is induced by the map $a \otimes (e_1 \otimes \cdots \otimes e_{n+1}) \mapsto ae_1 \otimes (e_2 \otimes \cdots \otimes e_{n+1})$ of $A \otimes E^\otimes_{n+1}$ into $A \otimes E^\otimes_{n}$, remembering that $A^*_n \subset E^\otimes_{n}$, (see [3]). In (1.2) the factors $A$ are considered as left $A$-modules. By considering $A$ as right $A$-module and by exchanging the factors one obtains the $N$-complex $\tilde{K}(A)$ of right $A$-modules

$$\ldots \xrightarrow{\tilde{d}} A^*_n \otimes A \xrightarrow{\tilde{d}} A^*_n \otimes A \xrightarrow{\tilde{d}} \ldots \rightarrow A \rightarrow 0$$

(1.3)

where now $\tilde{d}$ is induced by $(e_1 \otimes \cdots \otimes e_{n+1}) \otimes a \mapsto (e_1 \otimes \cdots \otimes e_n) \otimes e_{n+1}a$.

Finally one defines two $N$-differentials $d_L$ and $d_R$ on the sequence of $(A, A)$-bimodules, i.e. of left $A \otimes A^{opp}$-modules, $(A \otimes A^*_n \otimes A)_{n \geq 0}$ by setting $d_L = d \otimes I_A$ and $d_R = I_A \otimes d$ where $I_A$ is the identity mapping of $A$ onto itself. For each of these $N$-differentials $d_L$ and $d_R$ the sequences

$$\ldots \xrightarrow{d_L} A \otimes A^*_n \rightarrow A \xrightarrow{d_R} A \otimes A^*_n \rightarrow \ldots$$

(1.4)

are $N$-complexes of left $A \otimes A^{opp}$-modules and one has

$$d_Ld_R = d_Rd_L$$

(1.5)

which implies that

$$d_L^N - d_R^N = (d_L - d_R) \left( \sum_{p=0}^{N-1} d_L^p d_R^{N-p-1} \right) = \left( \sum_{p=0}^{N-1} d_L^p d_R^{N-p-1} \right) (d_L - d_R) = 0$$

(1.6)

in view of $d_L^N = d_R^N = 0$.

As for any $N$-complex [13] one obtains from $K(A)$ ordinary complexes $C_{p,r}(K(A))$, the contractions of $K(A)$, by putting together alternatively $p$ and $N-p$ arrows of $K(A)$. Explicitly $C_{p,r}(K(A))$ is given by

$$\ldots \xrightarrow{d^{N-p}} A \otimes A^*_n \rightarrow A \otimes A^*_n \xrightarrow{d^{N-p}} A \otimes A^*_n \rightarrow A \otimes A^*_n \xrightarrow{d^{N-p}} \ldots$$

(1.7)

for $0 \leq r < p \leq N - 1$, [14]. These are here chain complexes of free left $A$-modules. As shown in [3] the complex $C_{N-1,0}(K(A))$ coincides with the Koszul complex of [2]; this complex will be denoted by $\mathcal{K}(A, \mathbb{K})$ in the sequel. That is one has

$$\mathcal{K}_{2m}(A, \mathbb{K}) = A \otimes A^*_m, \quad \mathcal{K}_{2m+1}(A, \mathbb{K}) = A \otimes A^*_m$$

(1.8)

for $m \geq 0$, and the differential is $d^{N-1}$ on $\mathcal{K}_{2m}(A, \mathbb{K})$ and $d$ on $\mathcal{K}_{2m+1}(A, \mathbb{K})$. If $\mathcal{K}(A, \mathbb{K})$ is acyclic in positive degrees then $\mathcal{K}$ will be said to be a Koszul algebra. It was shown in [2] and this was confirmed by the analysis of [3] that this is the right generalization for $N$-homogeneous algebra of the usual notion.
of Koszulity for quadratic algebras [17, 16]. One always has $H_0(K(A, K)) \simeq K$ and therefore if $A$ is Koszul, then one has a free resolution $K(A, K) \rightarrow K \rightarrow 0$ of the trivial left $A$-module $K$, that is the exact sequence

$$\cdots \rightarrow A \otimes A^{1*}_{N+1} \xrightarrow{d} A \otimes R \xrightarrow{d} A \otimes E \xrightarrow{d} A \xrightarrow{\varepsilon} K \rightarrow 0$$

(1.9)

of left $A$-modules where $\varepsilon$ is the projection on degree zero. This resolution is a minimal projective resolution of $A$ in the graded category [3].

One defines now the chain complex of free $A \otimes A^{opp}$-modules $K(A, A)$ by setting

$$K_{2m}(A, A) = A \otimes A^{1*}_{Nm} \otimes A, \quad K_{2m+1}(A, A) = A \otimes A^{1*}_{Nm+1} \otimes A$$

(1.10)

for $m \in \mathbb{N}$ with differential $\delta'$ defined by

$$\delta' = d_L - d_R : K_{2m+1}(A, A) \rightarrow K_{2m}(A, A)$$

(1.11)

$$\delta' = \sum_{p=0}^{N-1} d_L^{p} d_{R}^{N-p-1} : K_{2(m+1)}(A, A) \rightarrow K_{2m+1}(A, A)$$

(1.12)

the property $\delta'^2 = 0$ following from (1.6). This complex is acyclic in positive degrees if and only if $A$ is Koszul, that is if and only if $K(A, K)$ is acyclic in positive degrees, [2] and [3]. One always has the obvious exact sequence

$$A \otimes E \otimes A \xrightarrow{\delta'} A \otimes A \xrightarrow{\mu} A \rightarrow 0$$

(1.13)

of left $A \otimes A^{opp}$-modules where $\mu$ denotes the product of $A$. It follows that if $A$ is a Koszul algebra then $K(A, A) \xrightarrow{\mu} A \rightarrow 0$ is a free resolution of the $A \otimes A^{opp}$-module $A$ which will be referred to as the Koszul resolution of $A$. This is a minimal projective resolution of $A \otimes A^{opp}$ in the graded category [3].

Let $A$ be a Koszul algebra and let $M$ be a $(A, A)$-bimodule considered as a right $A \otimes A^{opp}$-module. Then, by interpreting the $M$-valued Hochschild homology $H(A, M)$ as $H_n(A, M) = \text{Tor}_n^{A \otimes A^{opp}}(M, A)$ [3], the complex $M \otimes_{A \otimes A^{opp}} K(A, A)$ computes the $M$-valued Hochschild homology of $A$, (i.e. its homology is the ordinary $M$-valued Hochschild homology of $A$). We shall refer to this complex as the small Hochschild complex of $A$ with coefficients in $M$ and denote it by $S(A, M)$. It reads

$$\cdots \rightarrow M \otimes A^{1*}_{N(m+1)} \xrightarrow{\delta} M \otimes A^{1*}_{Nm+1} \xrightarrow{\delta} M \otimes A^{1*}_{Nm} \xrightarrow{\delta} \cdots$$

(1.14)

where $\delta$ is obtained from $\delta'$ by applying the factors $d_L$ to the right of $M$ and the factors $d_R$ to the left of $M$. 

Assume that $A$ is a Koszul algebra of finite global dimension $D$. Then the Koszul resolution of $K$ has length $D$, i.e. $D$ is the largest integer such that $K_D(A, K) \neq 0$. By construction, $D$ is also the greatest integer such that $K_D(A, A) \neq 0$. Thus for a Koszul algebra, the global dimension is equal to the Hochschild dimension. Applying then the functor $\text{Hom}_A(\cdot, A)$ to $K(A, K)$ one obtains the cochain complex $L(A, K)$ of free right $A$-modules

$$0 \to L^0(A, K) \to \cdots \to L^D(A, K) \to 0$$

where $L^n(A, K) = \text{Hom}_A(K_n(A, K), A)$. The Koszul algebra $A$ is Gorenstein iff $H^n(L(A, K)) = 0$ for $n < D$ and $H^D(L(A, K)) = K$ (= the trivial right $A$-module). This is clearly a generalisation of the classical Poincaré duality and this implies a precise form of Poincaré duality between Hochschild homology and Hochschild cohomology [5], [20], [21]. In the case of the Yang-Mills algebra and its deformations which are Koszul Gorenstein cubic algebras of global dimension 3, this Poincaré duality gives isomorphisms

$$H_k(A, M) = H^{3-k}(A, M), \quad k \in \{0, 1, 2, 3\}$$

between the Hochschild homology and the Hochschild cohomology with coefficients in a bimodule $M$.

1.3 The Yang-Mills algebra

Let $(g_{\lambda \mu}) \in M_{s+1}(K)$ be an invertible symmetric $(s+1) \times (s+1)$-matrix with inverse $(g^{\lambda \mu})$, i.e. $g_{\lambda \mu}g^{\mu \nu} = \delta^\nu_\lambda$. The Yang-Mills algebra is the cubic algebra $A$ generated by $s+1$ elements $\nabla_\lambda$ ($\lambda \in \{0, \ldots, s\}$) with the $s+1$ relations

$$g^{\lambda \mu}[\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] = 0, \quad \nu \in \{0, \ldots, s\}$$

that is $A = A(E, R)$ with $E = \oplus_\lambda K\nabla_\lambda$ and $R \subset E^{\otimes 3}$ given by

$$R = \sum_\nu K g^{\lambda \mu} [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] \otimes \nabla_\nu$$

$$= \sum_\nu K (g^{\rho \lambda} g^{\mu \nu} + g^{\rho \nu} g^{\lambda \mu} - 2 g^{\rho \mu} g^{\lambda \nu}) \nabla_\lambda \otimes \nabla_\mu \otimes \nabla_\nu$$

In [10] the following theorem was proved.

**Theorem 1.** The cubic Yang-Mills algebra $A$ is Koszul of global dimension 3 and is Gorenstein.

The proof of this theorem relies on the computation of the dual cubic algebra $A^!$ which we now recall.
The dual $A' = A(E^*, R^1)$ of the Yang-Mills algebra is the cubic algebra generated by $s + 1$ elements $\theta^\lambda (\lambda \in \{0, \ldots, s\})$ with relations
\[ \theta^\lambda \theta^\mu \theta^\nu = \frac{1}{s} (g^{\lambda \rho} \theta^\rho + g^{\mu \nu} \theta^\lambda - 2g^{\lambda \nu} \theta^\mu) \]
where $g = g_{\alpha \beta} \theta^\alpha \theta^\beta$. These relations imply that $g \in A'_2$ is central in $A'$ and that one has $A'_0 = \mathbb{K}1 \simeq \mathbb{K}$, $A'_1 = \oplus \lambda \mathbb{K} \theta^\lambda \simeq \mathbb{K} \langle s \rangle$, $A'_2 = \oplus \mu \mathbb{K} \theta^\mu \simeq \mathbb{K} \langle s \rangle$, $A'_3 = \oplus \lambda \mathbb{K} \theta^\lambda g \simeq \mathbb{K} \langle s \rangle$, $A'_4 = \mathbb{K} g^2 \simeq \mathbb{K}$ and $A'_n = 0$ for $n \geq 5$. From this, one obtains the description of $[10]$ of the Koszul complex $K(A, \mathbb{K})$ and the proof of the above theorem. It also follows that the bimodule resolution $K(A, \mathbb{K}) \xrightarrow{\delta} A \rightarrow 0$ of $A$ reads
\[ 0 \rightarrow A \otimes A \xrightarrow{\delta'_1} A \otimes \mathbb{K} \langle s \rangle \xrightarrow{\delta'_2} A \otimes \mathbb{K} \langle s \rangle^2 \xrightarrow{\delta'_3} \cdots \rightarrow 0 \] (1.17)
where the components $\delta'_k$ of $\delta'$ in the different degrees can be computed by using the description of $K(A, \mathbb{K}) = C_{2,0}$ given in Section 3 of $[10]$ and are given by
\[ \begin{align*}
\delta'_1(a \otimes e_\lambda \otimes b) &= a \nabla_\lambda \otimes b - a \otimes \nabla_\lambda b \\
\delta'_2(a \otimes e_\lambda \otimes b) &= (g^{\alpha \beta} \delta'_\lambda + g^{\beta \gamma} g^{\alpha}_\lambda - 2g^{\gamma \alpha} \delta'_\lambda) \times (a \nabla_\alpha \nabla_\beta \otimes e_\gamma \otimes b + a \nabla_\alpha \otimes e_\gamma \otimes \nabla_\beta b + a \otimes e_\gamma \otimes \nabla_\alpha \nabla_\beta b) \\
\delta'_3(a \otimes b) &= g^{\lambda \mu} (a \nabla_\mu \otimes e_\lambda \otimes b - a \otimes e_\lambda \otimes \nabla_\mu b)
\end{align*} \] (1.18)
where $a, b \in A$, $e_\lambda (\lambda = 0, \ldots, s)$ is the canonical basis of $\mathbb{K} \langle s \rangle$ and $\nabla_\lambda$ are the corresponding generators of $A$.

Let $M$ be a bimodule over $A$. By using the above description of the Koszul resolution of $A$ one easily obtains the one of the small Hochschild complex $S(A, M)$ which reads
\[ 0 \rightarrow M \xrightarrow{\delta_3} M \otimes \mathbb{K} \langle s \rangle \xrightarrow{\delta_2} M \otimes \mathbb{K} \langle s \rangle^2 \xrightarrow{\delta_1} M \otimes \mathbb{K} \langle s \rangle^3 \rightarrow 0 \] (1.19)
with differential $\delta$ given by
\[ \begin{align*}
\delta_1(m^\lambda \otimes e_\lambda) &= m^\lambda \nabla_\lambda - \nabla_\lambda m^\lambda = [m^\lambda, \nabla_\lambda] \\
\delta_2(m^\lambda \otimes e_\lambda) &= (([\nabla_\mu, [\nabla_\mu, m^\lambda]] + [\nabla_\mu, [m^\mu, \nabla_\lambda]] + [m^\mu, [\nabla_\mu, \nabla_\lambda]]) \otimes e_\lambda \\
\delta_3(m) &= g^{\lambda \mu} (m \nabla_\mu - \nabla_\mu m) \otimes e_\lambda = [m, \nabla_\lambda] \otimes e_\lambda
\end{align*} \] (1.20)
with obvious notations. By using (1.20) one easily verifies the duality (1.19). For instance $H_3(A, M) = \text{Ker}(\delta_3)$ which is given by the $m \in M$ such that
\[ \nabla^\lambda m = m \nabla^\lambda \] for \( \lambda = 0, \ldots, s \) that is such that \( am = ma, \forall a \in A \), since \( A \) is generated by the \( \nabla^\lambda \) and it is well known that this coincides with \( H^0(A, M) \).

Similarly \( m^\lambda \otimes e_\lambda \) is in \( \text{Ker}(\delta_2) \) if and only if \( \nabla^\lambda \to D(\nabla^\lambda) = g_{\lambda\mu} m^\mu \) extends as a derivation \( D \) of \( A \) into \( M \) (\( D \in \text{Der}(A, M) \)) while \( m^\lambda \otimes e_\lambda = \delta_3(m) \) means that this derivation is inner \( D = ad(m) \in \text{Int}(A, M) \) from which \( H_2(A, M) \) identifies with \( H^1(A, M) \), and so on.

Assume now that \( M \) is graded in the sense that one has \( M = \oplus_{n \in \mathbb{Z}} M_n \) with \( A_k M_\ell A_m \subset M_{k+\ell+m} \). Then the small Hochschild complex \( S(A, M) \) splits into subcomplexes \( S(A, M) = \oplus_n S^{(n)}(A, M) \) where \( S^{(n)}(A, M) \) is the subcomplex:

\[
0 \to \mathcal{M}_{n-4} \xrightarrow{\delta_1} \mathcal{M}_{n-3} \otimes \mathbb{K}^{s+1} \xrightarrow{\delta_2} \mathcal{M}_{n-2} \otimes \mathbb{K}^{s+1} \xrightarrow{\delta_3} \mathcal{M}_{n-1} \to 0 \quad (1.21)
\]
of (1.19). Assume furthermore that the homogeneous components \( \mathcal{M}_n \) are finite-dimensional vector spaces, i.e. \( \dim(\mathcal{M}_n) \in \mathbb{N} \). Then one has the following Euler-Poincaré formula:

\[
\dim(H_0^{(n)}) - \dim(H_1^{(n)}) + \dim(H_2^{(n)}) - \dim(H_3^{(n)}) = \\
\dim(\mathcal{M}_n) - (s+1)\dim(\mathcal{M}_{n-1}) + (s+1)\dim(\mathcal{M}_{n-2}) - \dim(\mathcal{M}_{n-3})
\]

(1.22) for the homology \( H^{(n)} \) of the chain complex \( S^{(n)}(A, M) \).

In the case where \( M = A \), it follows from the Koszulity of \( A \) that the right hand side of (1.22) vanishes for \( n \neq 0 \). Denoting as usual by \( HH(A) \) the \( A \)-valued Hochschild homology of \( A \) which is here the homology of \( S(A, A) \), we denote by \( HH^{(n)}(A) \) the homology of the subcomplex \( S^{(n)}(A, A) \). Since \( A_n = 0 \) for \( n < 0 \), one has \( HH_0^{(n)}(A) = 0 \) for \( n < 0 \), \( HH_1^{(n)}(A) = 0 \) for \( n \leq 0 \), \( HH_2^{(n)}(A) = 0 \) for \( n \leq 2 \) and \( HH_3^{(n)}(A) = 0 \) for \( n \leq 3 \). Furthermore one has

\[
HH_0^{(0)}(A) = HH_3^{(4)}(A) = \mathbb{K}
\]

(1.23)

\[
HH_0^{(1)}(A) = HH_1^{(1)}(A) = HH_2^{(3)}(A) = \mathbb{K}^{s+1}
\]

(1.24)

\[
HH_0^{(2)}(A) = HH_1^{(2)}(A) = \mathbb{K}^{\frac{(s+1)(s+2)}{2}}
\]

(1.25)

and the Euler Poincaré formula reads here:

\[
\dim(HH_0^{(n)}(A)) + \dim(HH_2^{(n)}(A)) = \dim(HH_1^{(n)}(A)) + \dim(HH_3^{(n)}(A))
\]

(1.26)

for \( n \geq 1 \) which implies

\[
\dim(HH_0^{(3)}(A)) + (s+1) = \dim(HH_1^{(3)}(A))
\]

(1.27)
for \( n = 3 \) while for \( n = 1 \) and \( n = 2 \) it is already contained in [12] and [128].

The complete description of the Hochschild homology and of the cyclic homology of the Yang-Mills algebra will be given in [12].

### 1.4 The super Yang-Mills algebra

As pointed out in the introduction, the Yang-Mills algebra is the universal enveloping algebra of a Lie algebra which is graded by giving degree 1 to the generators \( \nabla_\lambda \) (see in [10]). Replacing the Lie bracket by a super Lie bracket, that is replacing in the Yang-Mills equations [11] the commutator by the anticommutator whenever the 2 elements are of odd degrees, one obtains a super version \( \hat{A} \) of the Yang-Mills algebra \( A \). In other words one defines the super Yang-Mills algebra to be the cubic algebra \( \hat{A} \) generated \( s + 1 \) elements \( S_\lambda (\lambda \in \{0, \ldots, s\}) \) with the relations

\[
g^{\lambda\mu}[S_\lambda, \{S_\mu, S_\nu\}] = 0, \quad \nu \in \{0, \ldots, s\} \tag{1.28}
\]

that is \( \hat{A} = A(\hat{E}, \hat{R}) \) with \( \hat{E} = \bigoplus \mathbb{K} S_\lambda \) and \( \hat{R} \subset \hat{E} \otimes^3 \) given by

\[
\hat{R} = \sum_\rho \mathbb{K}(g^{\rho\lambda} g^{\mu\nu} - g^{\rho\nu} g^{\lambda\mu}) S_\lambda \otimes S_\mu \otimes S_\nu \tag{1.29}
\]

Relations (1.28) can be equivalently written as

\[
[g^{\lambda\mu} S_\lambda S_\mu, S_\nu] = 0, \quad \nu \in \{0, \ldots, s\} \tag{1.30}
\]

which mean that \( g^{\lambda\mu} S_\lambda S_\mu \in \hat{A}_2 \) is central in \( \hat{A} \).

It is easy to verify that the dual algebra \( \hat{A}^! = A(\hat{E}^*, \hat{R}^\perp) \) is the cubic algebra generated by \( s + 1 \) elements \( \xi^\lambda (\lambda \in \{0, \ldots, s\}) \) with the relations

\[
\xi^\lambda \xi^\mu \xi^\nu = -\frac{1}{s}(g^{\lambda\mu} \xi^\nu - g^{\mu\nu} \xi^\lambda) g
\]

where \( g = g_{\alpha\beta} \xi^\alpha \xi^\beta \). These relations imply that \( g \xi^\nu + \xi^\nu g = 0 \), i.e.

\[
\{g_{\lambda\mu} \xi^\lambda \xi^\mu, \xi^\nu\} = 0, \quad \nu \in \{0, \ldots, s\} \tag{1.31}
\]

and that one has \( \hat{A}_0^! = \mathbb{K} 1 \simeq \mathbb{K}, \hat{A}_1^! = \bigoplus \mathbb{K} \xi^\lambda \simeq \mathbb{K}^{s+1}, \hat{A}_2^! = \bigoplus_{\mu\nu} \mathbb{K} \xi^\mu \xi^\nu \simeq \mathbb{K}^{(s+1)^2}, \hat{A}_3^! = \bigoplus \mathbb{K} \xi^\lambda g \simeq \mathbb{K}^{s+1}, \hat{A}_4^! = \mathbb{K} g^2 \simeq \mathbb{K} \) and \( \hat{A}_n^! = 0 \) for \( n \geq 5 \).

The Koszul complex \( \mathcal{K}(\hat{A}, \mathbb{K}) \) of \( \hat{A} \) then reads

\[
0 \to \hat{A} \xrightarrow{\xi^\nu} \hat{A}^{s+1} \xrightarrow{N} \hat{A}^{s+1} \xrightarrow{S} \hat{A} \to 0
\]
where $S$ means right multiplication by the column with components $S_{\lambda}$, $S^t$ means right multiplication by the row with components $S_{\lambda}$ and $N$ means right multiplication (matrix product) by the matrix with components $N_{\mu\nu} = (g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta})S_{\alpha}S_{\beta}$.

with $\lambda, \mu, \nu \in \{0, \ldots, s\}$. One has the following result.

**Theorem 2.** The cubic super Yang-Mills algebra $\hat{A}$ is Koszul of global dimension 3 and is Gorenstein.

**Proof.** By the very definition of $\hat{A}$ by generators and relations, the sequence

$$\hat{A}^{s+1} \xrightarrow{N} \hat{A}^{s+1} \xrightarrow{S} \hat{A} \xrightarrow{\varepsilon} K \rightarrow 0$$

is exact. On the other hand it is easy to see that the mapping $\hat{A}^{s+1} \xrightarrow{S^t} \hat{A}^{s+1}$ is injective and that the sequence

$$0 \rightarrow \hat{A} \xrightarrow{S^t} \hat{A}^{s+1} \xrightarrow{N} \hat{A}^{s+1} \xrightarrow{S} \hat{A} \xrightarrow{\varepsilon} K \rightarrow 0$$

is exact which implies that $\hat{A}$ is Koszul of global dimension 3. The Gorenstein property follows from the symmetry by transposition. □

The situation is completely similar to the Yang-Mills case, in particular $\hat{A}$ has Hochschild dimension 3 and, by applying a result of [15], $\hat{A}$ has the same Poincaré series as $A$ i.e. one has the formula

$$\sum_{n \in \mathbb{N}} \dim(\hat{A}_n)t^n = \frac{1}{(1 - t^2)(1 - (s + 1)t + t^2)}$$ (1.32)

which, as will be shown elsewhere, can be interpreted in terms of the quantum group of the bilinear form $(g_{\mu\nu})$ [14] by noting the invariance of Relations (1.30) by this quantum group. For $s = 1$ the Yang-Mills algebra and the super Yang-Mills algebra are particular cubic Artin-Schelter algebras [1] whereas for $s \geq 2$ these algebras have exponential growth as follows from Formula (1.32).

### 1.5 The super self-duality algebra

There are natural quotients $B$ of $A$ and $\tilde{B}$ of $\hat{A}$ which are connected with parastatistics and which have been investigated in [15]. The *parafermionic algebra* $B$ is the cubic algebra generated by elements $\nabla_{\lambda}$ ($\lambda \in \{0, \ldots, s\}$) with relations

$$[\nabla_{\lambda}, [\nabla_{\mu}, \nabla_{\nu}]] = 0$$

for any $\lambda, \mu, \nu \in \{0, \ldots, s\}$, while the *parabosonic algebra* $\tilde{B}$ is the cubic algebra generated by elements $S_{\lambda}$ ($\lambda \in \{0, \ldots, s\}$) with relations
$[S_\lambda, \{S_\mu, S_\nu\}] = 0$

for any $\lambda, \mu, \nu \in \{0, \ldots, s\}$. In contrast to the Yang-Mills and the super Yang-Mills algebras $\mathcal{A}$ and $\tilde{\mathcal{A}}$ which have exponential growth whenever $s \geq 2$, these algebras $\mathcal{B}$ and $\tilde{\mathcal{B}}$ have polynomial growth with Poincaré series given by

$$\sum_n \dim(\mathcal{B}_n)t^n = \sum_n \dim(\tilde{\mathcal{B}}_n)t^n = \left(\frac{1}{1-t}\right)^{s+1} \left(\frac{1}{1-t^2}\right)^{\frac{s(s+1)}{2}}$$

but they are not Koszul for $s \geq 2$, [15].

In a sense, the algebra $\mathcal{B}$ can be considered to be somehow trivial from the point of view of the classical Yang-Mills equations in dimension $s+1 \geq 3$ although the algebras $\mathcal{B}$ and $\tilde{\mathcal{B}}$ are quite interesting for other purposes [15]. It turns out that in dimension $s+1 = 4$ with $g_{\mu\nu} = \delta_{\mu\nu}$ (Euclidean case), the Yang-Mills algebra $\mathcal{A}$ has non trivial quotients $\mathcal{A}^{(\varepsilon)}$ and $\mathcal{A}^{(-)}$ which are quadratic algebras referred to as the self-duality algebra and the anti-self-duality algebra respectively [10]. Let $\varepsilon = \pm$, the algebra $\mathcal{A}^{(\varepsilon)}$ is the quadratic algebra generated by the elements $\nabla_\lambda$ ($\lambda \in \{0, 1, 2, 3\}$) with relations

$$[\nabla_0, \nabla_k] = \varepsilon[\nabla_\ell, \nabla_m]$$

for any cyclic permutation $(k, \ell, m)$ of $(1, 2, 3)$. One passes from $\mathcal{A}^{(-)}$ to $\mathcal{A}^{(\varepsilon)}$ by changing the orientation of $\mathbb{R}^4$ so one can restrict attention to the self-duality algebra $\mathcal{A}^{(+)}$. This algebra has been studied in [10] where it was shown in particular that it is Koszul of global dimension 2. For further details on this algebra and on the Yang-Mills algebra, we refer to [10] and to the forthcoming paper [12]. Our aim now in this section is to define and study the super version of the self-duality algebra.

Let $\varepsilon = +$ or $-$ and define $\tilde{\mathcal{A}}^{(\varepsilon)}$ to be the quadratic algebra generated by the elements $S_0, S_1, S_2, S_3$ with relations

$$i\{S_0, S_k\} = \varepsilon[S_\ell, S_m] \quad (1.33)$$

for any cyclic permutation $(k, \ell, m)$ of $(1, 2, 3)$. One has the following.

**Lemma 1.** Relations (1.33) imply that one has

$$[\sum_{\mu=0}^3 (S_\mu)^2, S_\lambda] = 0$$

for any $\lambda \in \{0, 1, 2, 3\}$. In other words, $\tilde{\mathcal{A}}^{(+)}$ and $\tilde{\mathcal{A}}^{(-)}$ are quotients of the super Yang-Mills algebra $\tilde{\mathcal{A}}$ for $s+1 = 4$ and $g_{\mu\nu} = \delta_{\mu\nu}$.

The proof which is a straightforward verification makes use of the Jacobi identity (see also in [15]). Thus $\tilde{\mathcal{A}}^{(+)}$ and $\tilde{\mathcal{A}}^{(-)}$ play the same role with respect
to $\hat{A}$ as $A^{(+)}$ and $A^{(-)}$ with respect to $A$. Accordingly they will be respectively called the *super self-duality algebra* and the *super anti-self-duality algebra*. Again $\hat{A}^{(+)}$ and $\hat{A}^{(-)}$ are exchanged by changing the orientation of $K^4$ and we shall restrict attention to the super self-duality algebra in the following, i.e. to the quadratic algebra $\hat{A}^{(+)}$ generated by $S_0, S_1, S_2, S_3$ with relations

$$i\{S_0, S_k\} = [S_\ell, S_m]$$

for any cyclic permutation $(k, \ell, m)$ of $(1,2,3)$. One has the following result.

**Theorem 3.** The quadratic super self-duality algebra $\hat{A}^{(+)}$ is a Koszul algebra of global dimension 2.

**Proof.** One verifies that the dual quadratic algebra $\hat{A}^{(+)!}$ is generated by elements $\xi^0, \xi^1, \xi^2, \xi^3$ with relations $(\xi^\lambda)^2 = 0$, for $\lambda = 0, 1, 2, 3$ and $\xi^\ell \xi^m = -\xi^m \xi^\ell = i\xi^0 \xi^\ell = i\xi^\ell \xi^0$ for any cyclic permutation $(k, \ell, m)$ of $(1,2,3)$. So one has $\hat{A}_0^{(+)!} = K I \simeq K$, $\hat{A}_1^{(+)!} = \oplus_\lambda K \xi^\lambda \simeq K^4$, $\hat{A}_2^{(+)!} = \oplus_k K \xi^0 \xi^k \simeq K^3$ and $\hat{A}_n^{(+)!} = 0$ for $n \geq 3$ since the above relations imply $\xi^\lambda \xi^\mu \xi^\nu = 0$ for any $\lambda, \mu, \nu \in \{0,1,2,3\}$. The Koszul complex $K(\hat{A}^{(+)}) = K(\hat{A}^{(+)}, K)$ (quadratic case) then reads

$$0 \rightarrow \hat{A}^{(+)^3} \overset{D}{\rightarrow} \hat{A}^{(+)^4} \overset{S}{\rightarrow} \hat{A}^{(+)} \rightarrow 0$$

where $S$ means right matrix product with the column with components $S_\lambda$ ($\lambda \in \{0,1,2,3\}$) and $D$ means right matrix product with

$$D = \begin{pmatrix}
i S_1 & iS_0 & S_3 & -S_2 \\
i S_2 & -S_3 & iS_0 & S_1 \\
i S_3 & S_2 & -S_1 & iS_0
\end{pmatrix}$$

It follows from the definition of $\hat{A}^{(+)}$ by generators and relations that the sequence

$$\hat{A}^{(+)^3} \overset{D}{\rightarrow} \hat{A}^{(+)^4} \overset{S}{\rightarrow} \hat{A}^{(+)} \rightarrow K \rightarrow 0$$

is exact. On the other hand one shows easily that the mapping $\hat{A}^{(+)^3} \overset{D}{\rightarrow} \hat{A}^{(+)^4}$ is injective so finally the sequence

$$0 \rightarrow \hat{A}^{(+)^3} \overset{D}{\rightarrow} \hat{A}^{(+)^4} \overset{S}{\rightarrow} \hat{A}^{(+)} \rightarrow K \rightarrow 0$$

is exact which implies the result. □

This theorem implies that the super self-duality algebra $\hat{A}^{(+)}$ has Hochschild dimension 2 and that its Poincaré series is given by

$$P_{\hat{A}^{(+)}}(t) = \frac{1}{(1-t)(1-3t)}$$

in view of the structure of its dual $\hat{A}^{(+)!}$ described in the proof. Thus everything is similar to the case of the self-duality algebra $A^{(+)}$. 
Let us recall that the Sklyanin algebra, in the presentation given by Sklyanin [18], is the quadratic algebra $S(\alpha_1, \alpha_2, \alpha_3)$ generated by 4 elements $S_0, S_1, S_2, S_3$ with relations

$$i\{S_0, S_k\} = [S_\ell, S_m]$$

$$[S_0, S_k] = i\frac{\alpha_\ell - \alpha_m}{\alpha_k}\{S_\ell, S_m\}$$

for any cyclic permutation $(k, \ell, m)$ of $(1, 2, 3)$. One sees that the relations of the super self-duality algebra $\tilde{A}(+)\) are the relations of the Sklyanin algebra which are independent from the parameters $\alpha_k$. Thus one has a sequence of surjective homomorphisms of connected graded algebra

$$\tilde{A} \to \tilde{A}(+) \to S(\alpha_1, \alpha_2, \alpha_3)$$

On the other hand for generic values of the parameters the Sklyanin algebra is Koszul Gorenstein of global dimension 4 [19] with the same Poincaré series as the polynomial algebra $K[X_0, X_1, X_2, X_3]$ and corresponds to the natural ambiant noncommutative 4-dimensional Euclidean space containing the noncommutative 3-spheres described in [9], [11] (their "homogeneisation"). This gives a very surprising connection between the present study and our noncommutative 3-spheres for generic values of the parameters. It is worth noticing here that in the analysis of [11] several bridges between noncommutative differential geometry in the sense of [7], [8] and noncommutative algebraic geometry have been established.

### 1.6 Deformations

The aim of this section is to study deformations of the Yang-Mills algebra and of the super Yang-Mills algebra. We use the notations of Sections 3 and 4.

Let the dimension $s + 1 \geq 2$ and the pseudo metric $g_{\lambda\mu}$ be fixed and let $\zeta \in P_1(K)$ have homogeneous coordinates $\zeta_0, \zeta_1 \in K$. Define $A(\zeta)$ to be the cubic algebra generated by $s + 1$ elements $\nabla_\lambda$ ($\lambda \in \{0, \ldots, s\}$) with relations

$$(\zeta_1(g^{\rho\lambda}g^{\mu\nu} + g^{\rho\mu}g^{\lambda\nu}) - 2\zeta_0 g^{\rho\mu}g^{\lambda\nu})\nabla_\lambda \nabla_\mu \nabla_\nu = 0$$

for $\rho \in \{0, \ldots, s\}$. The Yang-Mills algebra corresponds to the element $\zeta^{YM}$ of $P_1(K)$ with homogeneous coordinates $\zeta_0 = \zeta_1$. Let $\zeta^{sing}$ be the element of $P_1(K)$ with homogeneous coordinates $\zeta_0 = \frac{\zeta_1^2}{2} \zeta_1$; one has the following result.

**Theorem 4.** For $\zeta \neq \zeta^{sing}$ the cubic algebra $A(\zeta)$ is Koszul of global dimension 3 and is Gorenstein.
Proof. The dual algebra \( \mathcal{A}(\zeta)^! \) is the cubic algebra generated by elements \( \theta^\lambda \) with relations
\[
\theta^\lambda \theta^\mu \theta^\nu = \frac{1}{(s+2)\zeta_1 - 2\zeta_0} (\zeta_1 (g^{\lambda \mu} \theta^\nu + g^{\mu \nu} \theta^\lambda) - 2\zeta_0 g^{\lambda \nu} \theta^\mu) \quad (1.36)
\]
for \( \lambda, \mu, \nu \in \{0, \ldots, s\} \) with \( g = g_{\alpha \beta} \theta^\alpha \theta^\beta \). This again implies that \( g \) is in the center and that one has \( A^0 = \mathbb{K} I \cong \mathbb{K} \), \( A^1 = \oplus \lambda \mathbb{K} \theta^\lambda \cong \mathbb{K}^{s+1} \), \( A^2 = \oplus \lambda, \mu \mathbb{K} \theta^\lambda \theta^\mu \cong \mathbb{K}^{(s+1)^2} \), \( A^3 = \oplus \lambda \mathbb{K} \theta^\lambda g \cong \mathbb{K}^{s+1} \), \( A^4 = \mathbb{K}^2 \cong \mathbb{K} \) while \( A^i_n = 0 \) for \( n \geq 5 \), where we have set \( A^i_n = \mathcal{A}(\zeta)^i_n \). Setting \( A = \mathcal{A}(\zeta) \), the Koszul complex \( K(\mathcal{A}(\zeta), \mathbb{K}) \) of \( \mathcal{A}(\zeta) \) reads
\[
0 \to A \xrightarrow{\Sigma^i} A^{s+1} \xrightarrow{M} A^{s+1} \xrightarrow{\Sigma} A \to 0
\]
with the same conventions as before and \( M \) with components
\[
M^{\mu \nu} = \frac{1}{(s+2)\zeta_1 - 2\zeta_0} (\zeta_1 (g^{\mu \nu} g^{\rho \beta} + g^{\mu \alpha} g^{\nu \beta} - 2\zeta_0 g^{\mu \beta} g^{\nu \alpha}) \nabla_\alpha \nabla_\beta
\]
\( \mu, \nu \in \{0, \ldots, s\} \). The theorem follows then by the same arguments as before, using in particular the symmetry by transposition for the Gorenstein property. \( \square \)

It follows that \( \mathcal{A}(\zeta) \) has Hochschild dimension 3 and the same Poincaré series as the Yang-Mills algebra for \( \zeta \neq \zeta^{sing} \).

Remark. One can show that the cubic algebra generated by elements \( \nabla_\lambda \) with relations
\[
(\zeta_1 g^{\rho \lambda} g^{\mu \nu} + \zeta_2 g^{\rho \nu} g^{\mu \lambda} - 2\zeta_0 g^{\rho \mu} g^{\nu \lambda}) \nabla_\lambda \nabla_\mu \nabla_\nu = 0
\]
cannot be Koszul and Gorenstein if \( \zeta_1 \neq \zeta_2 \) and \( \zeta_0 \neq 0 \) or if \( (\zeta_1)^2 \neq (\zeta_2)^2 \).

Let now \( (B_{\lambda \mu}) \in M_{s+1}(\mathbb{K}) \) be an arbitrary invertible \( (s+1) \times (s+1) \)-matrix with inverse \( (B^{\lambda \mu}) \), i.e. \( B_{\lambda \mu} B^{\mu \nu} = \delta^\nu_\lambda \), and let \( \varepsilon = + \) or \(-\). We define \( \mathfrak{A}(B, \varepsilon) \) to be the cubic algebra generated by \( s+1 \) elements \( E_\lambda \) with relations
\[
(B^{\rho \lambda} B^{\mu \nu} + \varepsilon B^{\lambda \mu} B^{\nu \rho}) E_\lambda E_\mu E_\nu = 0 \quad (1.37)
\]
for \( \rho \in \{0, \ldots, s\} \). Notice that \( B \) is not assumed to be symmetric. If \( B_{\lambda \mu} = g_{\lambda \mu} \) and \( \varepsilon = - \) then \( \mathfrak{A}(g, -) \) is the super Yang-Mills algebra \( \mathfrak{A} \) (\( E_\lambda \mapsto S_\lambda \)) while if \( B_{\lambda \mu} = g_{\lambda \mu} \) and \( \varepsilon = + \) then \( \mathfrak{A}(g, +) \) is \( \mathcal{A}(\zeta^0) \) (\( E_\lambda \mapsto \nabla_\lambda \)) where \( \zeta^0 \) has homogeneous coordinates \( \zeta_1 \neq 0 \) and \( \zeta_0 = 0 \). Thus \( \mathfrak{A}(B, +) \) and \( \mathfrak{A}(B, -) \) belong to deformations of the Yang-Mills and of the super Yang-Mills algebra respectively.

Theorem 5. Assume that \( 1 + \varepsilon B^{\rho \lambda} B^{\mu \nu} B_{\lambda \mu} B_{\rho \nu} \neq 0 \), then \( \mathfrak{A}(B, \varepsilon) \) is Koszul of global dimension 3 and is Gorenstein.
Proof. The Koszul complex $K(\mathfrak{A}(B, \varepsilon), \mathbb{K})$ can be put in the form

$$0 \rightarrow \mathfrak{A}^t \rightarrow \mathfrak{A}^{s+1} \xrightarrow{L} \mathfrak{A}^{s+1} \xrightarrow{E} \mathfrak{A} \rightarrow 0$$

where $\mathfrak{A} = \mathfrak{A}(B, \varepsilon)$ and with the previous conventions, the matrix $L$ being given by

$$L^{\mu\nu} = (B^{\mu\alpha} B^{\beta\nu} + \varepsilon B^{\nu\mu} B^{\alpha\beta}) E_\alpha E_\beta$$ (1.38)

for $\mu, \nu \in \{0, \ldots, s\}$. The arrow $\mathfrak{A}^{s+1} \rightarrow \mathfrak{A}^{s+1}$ is always injective and the exactness of $\mathfrak{A}^t \rightarrow \mathfrak{A}^{s+1} \xrightarrow{L} \mathfrak{A}^{s+1}$ follows from the condition $1 + \varepsilon B^{\rho\lambda} B^{\mu\nu} B^{\mu\lambda} B_{\rho\nu} \neq 0$. On the other hand, by definition of $\mathfrak{A}$ by generators and relations, the sequence $\mathfrak{A}^{s+1} \xrightarrow{E} \mathfrak{A}^{s+1} \xrightarrow{\xi} \mathfrak{A} \xrightarrow{\varepsilon} K \rightarrow 0$ is exact. This shows that $\mathfrak{A}$ is Koszul of global dimension 3. The Gorenstein property follows from (see also in [1])

$$B^{\rho\lambda} B^{\mu\nu} + \varepsilon B^{\nu\rho} B^{\lambda\mu} = \varepsilon (B^{\mu\rho} B^{\lambda\mu} + \varepsilon B^{\mu\nu} B^{\rho\lambda})$$

for $\rho, \lambda, \mu, \nu \in \{0, \ldots, s\}$. □

Remark. It is worth noticing here in connection with the analysis of [5] that for all the deformations of the Yang-Mills algebra (resp. the super Yang-Mills algebra) considered here which are cubic Koszul Gorenstein algebras of global dimension 3, the dual cubic algebras are Frobenius algebras with structure automorphism equal to the identity (resp. $(−1)^{\text{degree}} \times$ identity).

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