Duality of a quantum competing system

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Abstract

We have constructed a theory of dual canonical formalism to study the quantum competing systems. In such a system, as the relationship between current and voltage of each, we assumed the duality condition. We considered competing systems of two types. One type is composed of a sandwich structure with a superconductor (SC)/superinsulator (SI)/SC junction, and its dual junction consists of a sandwich structure formed by the SI/SC/SI junction. The other type of system consists of a sandwich structure formed by the SC/FM (ferromagnet)/SC junction, and its dual junction consists of a sandwich structure formed by the FM/SC/FM junction (spin Josephson junctions). We derived the relationship between the phase and the number of particles in a dual system. As an application of the dual competitive systems, we introduce a quantum spin transistor.

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1. Introduction

In superconducting systems, there is a Josephson junction device known as a quantum effect device [1, 2] that operates by means of quantum flux tunneling. The mesoscopic Josephson junction is a quantum effect device that operates by using single Cooper pair tunneling created by a Coulomb blockade [3]. The junction by the superconductor (SC) as the condensation of Cooper pairs and the superinsulator (SI) as the condensation of the quantum flux (vortex), i.e. the junction of SC and SI that are dual to each other [4, 5], forms a competing system. On the other hand, like magnetic duality, spontaneous magnetization and the domain wall are known to have a dual relationship. The junction is formed by the ferromagnet (FM) as the condensation of the spin magnetization and by the SC as the condensation of the domain wall; therefore, the junction of FM and SC is dual. As described above, we considered competing quantum systems of two types, and as an application of these systems, we propose a quantum spin transistor. Our main objective is to build a theory of quantum devices, in which the freedom of a dual particle plays an important role, and we would also like to build upon the duality theory of competitive systems. This paper is organized as follows. In the next section, as the duality system of the charge and magnetic flux, we introduce dual canonical formalism [7] between the SC/SI/SC junction and the SI/SC/SI junction, and by imposing the dual condition to between these, we derived the quantum resistance. In section 3, we derive the partition functions by means of the path integral for a quantum single Josephson junction and its dual model, respectively. In section 4, as the duality of the spin and magnetic domain wall, we study the FM/SC/FM junction and its dual model as an analogy with the Josephson junctions. In section 6, as an application of the junction mentioned in the previous section, we introduce a quantum spin transistor. In section 6, we present a summary and conclusions.

2. Dual canonical formalism between a superconductor/superinsulator/superconductor junction and a superinsulator/superconductor/superinsulator junction

Duality has been known to be a powerful tool in various physical systems such as statistical mechanics [8] and field theory [9, 10]. Furthermore, it has also been recognized in several studies of Josephson junction systems [11, 12]. In this section, we introduce the dual canonical formalism [7] between an SC/SI/SC junction and an SI/SC/SI junction. First, we propose a small SC/SI/SC [7] junction and its equivalent circuit, which are shown in figure 1. It consists of the sandwich structure of the SC/SI/SC. We introduce the order field, which is given by $\psi \equiv \sqrt{N} \exp(i \theta)$. 

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In superconducting systems, for example, \( N = N_1 - N_2 \) represents the relative number operator of a Cooper pair, and \( \theta = \theta_1 - \theta_2 \) represents the relative phase of Cooper pairs. This commutation relation is given by \([\theta, N] = i\). The junction is characterized by its capacitance \( C_\theta \) and the Josephson energy \( E_J = h L_\text{c} / 2e \), where \( L_\text{c} \) is the critical current. Now, when this junction system is considered as the Josephson junctions, it can be described by the Hamiltonian of a quantum single Josephson junction as follows:

\[
H = 4N^2E_c + E_J (1 - \cos \theta). \tag{2.1}
\]

The first term describes the Coulomb energy of the Josephson junction, where \( N \) is the number operator of the Cooper pair, and \( E_c = e^2 / 2C \) is the charging energy per single charge. The second term describes the Josephson coupling energy. From equation (2.1), the Josephson equations are given by

\[
\frac{\partial}{\partial \tau} \theta = \frac{2E_c}{\pi} (1 - \cos \hat{\theta}), \quad \frac{\partial}{\partial \tau} \tilde{N} = 2\tilde{E}_c - E_c \sin \tilde{\theta} / \pi e, \quad \frac{\partial}{\partial \tau} \tilde{V} = \frac{\hbar}{2\pi} E_c \sin \hat{\theta} / \pi e, \quad \frac{\partial}{\partial \tau} \tilde{I} = \frac{\hbar}{2\pi} E_c \sin \tilde{\theta} / \pi e e, \tag{2.2}
\]

Here, the first term describes the vortex energy of the dual Josephson junction, where \( E_c = 2\pi^2 E_J = \Phi_0 / 2L_c \) is the vortex energy per single vortex and \( L_c = \Phi_0 / 2\pi I_c \) is the critical inductance. The second term describes the dual Josephson coupling energy. From equation (2.2) dual Josephson equations are given by \( \tilde{V} = (h / \Phi_0) \hat{\theta} / \hat{\theta} = 2\pi I_c \tilde{N}, \quad \tilde{I} = -\Phi_0 \hat{\theta} / \hat{\theta} = 2\tilde{E}_c \sin \hat{\theta} / \pi e, \quad \tilde{V} \) are the voltage of the vortex and the current of the vortex, respectively. Here, we assume the following duality conditions:

\[
\tilde{V} \equiv \tilde{I}, \quad I \equiv \tilde{V}. \tag{2.3}
\]

By imposing these duality conditions, we derived the next two types of relationships. One type is the relationship between the phase of the Cooper pair \( \theta \) and the vortex number \( \tilde{N} \) and the other type is the relationship between the phase of the vortex field \( \hat{\theta} \) and the Cooper pair number \( N \), which is given as follows:

\[
\tilde{N} = -\sin \theta / 2\pi, \quad N = \sin \hat{\theta} / 2\pi. \tag{2.4}
\]

From the Josephson equation, we derived the resistance, given by

\[
R = -\frac{h}{2e^2} \frac{8N E_c}{\pi^2} \frac{2R_Q N E_c}{N E_1}, \tag{2.5}
\]

where \( R_Q = h / (2e^2) \) is the quantum resistance [14, 15]. In the last equality in equation (2.5), we used the relation of equation (2.4). In the same manner, from the dual Josephson equations, we derived the conductance \( R = R^{-1} \). In the case of the condition \( N \gg N_1 \) or \( E_1 \gg E_J \), the junction system becomes an insulator. In particular, in the extreme case \( R \rightarrow \infty \), the junction system becomes an SI. In the reverse case, in the condition of \( N \ll N_1 \) or \( E_1 \ll E_J \), the junction system becomes a conductor. In particular, in the extreme case \( R \rightarrow 0 \), the junction system becomes an SC. As a special case of these conditions, in the case of \( N = N_1 \) and \( E_1 \ll E_J \), in which the resistance \( R \) is equal to the quantum resistance \( R_Q \), which is the state in a self-dual. In the above discussion, we have shown that we can more clearly define the presence of quantum resistance and a quantum critical point by using a dual canonical formalism.

### 3. The partition function by path integral of a quantum single Josephson junction and its dual model

In this section, using the Hamiltonian of the quantum single Josephson junction shown in section 2, we derive the partition function of a quantum single Josephson junction. From the Hamiltonian in equation (2.1), the partition function \( Z(\beta) \) of a quantum single Josephson junction with imaginary time \( \tau \) is as follows:

\[
Z = \int D\theta N \int D\hat{\theta} \exp \left\{ \frac{\hbar}{\beta} \frac{\partial}{\partial \tau} N - 4E_c N^2 - E_J (1 - \cos \theta) \right\}. \tag{3.1}
\]

As a result of integration by \( \theta \), the partition function is as follows:

\[
Z = \int D\theta N \exp \left( \frac{2\hbar}{\beta} \frac{\partial}{\partial \tau} N - 4E_c N^2 + \ln I_{0}(E_1) \right). \tag{3.2}
\]

Here \( I_{0}(E_1) \) are the modified Bessel functions of the zeroth order. We investigated the partition function using the Villain approximation [16] as follows:

\[
I_{0}(E_1) \equiv I_{0}(E_1) \exp \left( -\alpha^2 / 2(E_1) \right), \tag{3.3}
\]

where \( (E_1) \) are Villain’s constants as defined as \( \alpha \equiv -\hbar \tilde{\theta} / \tilde{N} \tilde{\theta} \). We investigated the partition function using the Villain approximation [16] as follows:

\[
Z = \left[ I_{0}(E_1) \right]^{\frac{\hbar \beta}{2E_c} / \sinh \left( \beta \sqrt{2(E_1)} E_c \right)}. \tag{3.4}
\]

![Figure 1. Schematic representation of an SC/SI/SC junction and its equivalent circuit.](image1)

![Figure 2. Schematic representation of an SI/SC/SI junction and its equivalent circuit.](image2)
where \( \varepsilon \) is a slice unit of imaginary time and \( M \equiv \hbar \beta / \varepsilon \) is the total number of slice units. In addition, we can directly integrate out \( N \) in equation (3.1). Then, the Lagrangian \( L(\hat{\theta}, \theta) \) is expressed as a function of only \( \theta \) as follows:

\[
L = \frac{1}{2} \ln \left( \frac{\hbar^2 \pi}{4E_c} \right) + \frac{-\hbar^2}{16E_c} (\hat{\theta} \theta)^2 - E_J (1 - \cos \theta). \tag{3.3}
\]

On the other hand, we applied the variables transformation of equation (2.4) to the partition function of equation (3.1), and we integrated out \( \hat{\theta} \). Then, the Lagrangian \( \hat{L}(\hat{\theta}, \hat{\theta}) \) is expressed as a function of only \( \hat{\theta} \) as follows:

\[
\hat{L} = \ln \left( \frac{\hbar}{2\pi} \cos \hat{\theta} \right) + \ln [I_\chi (E_\chi)] - \frac{4E_c}{(2\pi)^2} \sin^2 \hat{\theta}, \tag{3.4}
\]

where \( \chi \) are defined as \( 2\pi \chi \equiv -h \cos \theta \hat{\theta} / \partial \tau \). Comparing with equations (3.3) and (3.4), we find that these Lagrangians are a mutually dual representation. Next, using the dual Hamiltonian in equation (2.2), we derive the partition function \( \hat{Z}(\beta) \) of a single dual quantum Josephson junction as follows:

\[
\hat{Z} = \int \! D\hat{h} \hat{N} \int \! D\hat{\theta} \exp \left\{ \int_0^{\beta \hat{\theta}} \! d\tau \left\{ i\hbar \frac{\hat{\theta}}{\partial \tau} \hat{N} - \frac{E_c}{2} \hat{N}^2 - \frac{2E_c}{\pi^2} (1 - \cos \hat{\theta}) \right\} \right\}. \tag{3.5}
\]

As a result of integration by \( \hat{\theta} \), the partition function is as follows:

\[
\hat{Z} = \int \! D\hat{h} \exp \left\{ \int_0^{\beta \hat{\theta}} \! d\tau \left\{ -E_c \hat{N}^2 + \ln [I_\chi (2E_c/\pi^2)] \right\} \right\},
\]

where \( \hat{N} \) is a slice unit of imaginary time and \( \hat{\theta} \) are defined as \( 2\pi \hat{\theta} \equiv -h \theta \hat{\theta} / \partial \tau \). Using the same procedure as in the previous section, we derived the dual partition function as follows:

\[
\hat{Z}(\beta) = \left[ I_\theta (2E_c/\pi^2) \right]^{\theta \hat{\theta} / \beta} \sinh \left( \beta \sqrt{2(2E_c/\pi^2)} \right). \tag{3.6}
\]

If we compare equations (3.2) and (3.6), they are seen to equal under the self-dual conditions of \( E_c = E_J \pi^2 / 2 \). As with the derivation of equation (3.3), we can integrate out \( \hat{N} \) in equation (4.1). Then the Lagrangian \( \hat{L}(\hat{\theta}, \theta) \) is expressed as a function of only theta tilde as follows:

\[
\hat{L} = \frac{1}{2} \ln \left( \frac{\pi \hbar^2}{2\pi^2 E_J} \right) + \frac{-\hbar^2}{8\pi^2 E_J} (\theta \hat{\theta})^2 \right. - \frac{2E_c}{\pi^2} (1 - \cos \hat{\theta}). \tag{3.7}
\]

On the other hand, as with the derivation of equation (3.4), we applied the variables transformation of equation (2.4) to the partition function of equation (3.5), and we integrated out \( \hat{\theta} \). Then the Lagrangian \( \hat{L}(\hat{\theta}, \theta) \) is expressed as a function of only \( \hat{\theta} \) as follows:

\[
\hat{L} = \ln \left( \frac{\hbar}{2\pi} \cos \hat{\theta} \right) + \ln [I_\chi (2E_c/\pi^2)] - \frac{E_J}{2\pi^2} \sin^2 \theta, \tag{3.8}
\]

where \( \chi \) are defined as \( 2\pi \chi \equiv \hbar \cos \theta \hat{\theta}/\partial \tau \). By comparing equations (3.7) and (3.8), we find that these Lagrangians are a mutually dual representation. In addition, at the limit of large \( E_J \) in equation (3.4), if these satisfy the Gaussian approximation of \( \hat{\theta} \), equations (3.4) and (3.7) are the same in the above conditions. Similarly, at the limit of large \( E_c \) in equation (3.8), if these satisfy the Gaussian approximation of \( \theta \), equations (3.8) and (3.3) are the same in the above conditions.

4. The ferromagnet/superconductor/ferromagnet junction and its dual model as an analogy with the Josephson junctions

So far, we have dealt with models of a quantum single Josephson junction and its dual model to study the dual canonical formalism. In this section, as the duality of the spin and magnetic domain wall, we propose a single quantum spin device that operates using single quantum spin tunneling. The single spin transistor consists of the sandwich structure of the FM/SC/FM junction. In analogy with a Josephson junction, the FM/SC/FM junction can be thought of as a ferromagnetic junction with a superconducting thin-film barrier, which is shown figure 3. In this case, the superconducting thin film functions as a spin capacitor. As a model for such ferromagnetic junction systems, first we consider the Hamiltonian of the Heisenberg XXZ spin models as follows:

\[
H_{FM} = \sum_{i,j} \left[ -J_{xy}(S_i^x S_j^x + S_i^y S_j^y) + J_{xy}(S_i^z S_j^z) \right],
\]

where \( J_{xy} \) is the exchange coupling constant, and \( S_i^z \) is the spin on site \( i \). The second term describes the Ising spin energy and the first term describes the XY spin energy. We can rewrite the Hamiltonian with the introduction of a spin coherent state and thus derive the Hamiltonian of the quantum single spin junction as follows:

\[
H_{FM} = E_{sc} N_{xy}^2 + E_{xy} (1 - \cos \phi), \tag{4.1}
\]

where \( E_{sc} \equiv (S_0^z)^2 / 2C_s \) is the spin charging energy per single spin, \( C_s \equiv \hbar / 2 \) is the spin quantum unit, \( N_{xy} \equiv S_i^x / S_0^z \) is the relative number operator of XY FM quasi-particles, and \( C_s \equiv 1 / 2J_i \) is the spin capacitance. The \( J_{xy} \) is the critical spin current, and \( \phi \) represents the relative phase of these quasi XY spin particles. Equation (4.1) can be interpreted as the energy band of quasi-spin particles in a periodic potential \( E_{xy} (1 - \cos \phi) \). In particular, in the case of \( E_{sc} \gg E_{xy} \), equation (4.1) is similar to the Bloch wave oscillations in a small Josephson junction. Figure 4 shows the energy band that takes the energy on the vertical axis and spins a magnetic moment on the horizontal axis. Using the Hamiltonian in equation (4.1), spin Josephson equations are given by \( V_i \equiv (h / S_0^z) \hat{\phi} / \partial t = 2N_{xy} E_{sc} / S_0^z \), \( I_\chi \equiv \partial S_i \hat{S}_i / \partial t = -i \sin \phi \), where \( V_i \) and \( I_\chi \) are the spin voltage and spin current of the XY ferromagnetic spin, respectively. The approximate commutation relations

![Figure 3. Schematic representation of an FM/SC/FM junction and its equivalent circuit.](image-url)
between $N_{XY}$ and $\phi$ are given by $[\phi, N_{XY}] \approx i$. The dual Hamiltonian of equation (4.1) is given by

$$H_{dw} = E_{dw} N_{DW}^2 + \tilde{E}_{xy} (1 - \cos \hat{\phi}).$$  \hspace{1cm} (4.2)

In the first term, $E_{dw} \equiv \Phi_0^2 / 2L_z$ describes the domain wall energy, where $L_z \equiv 4/E_{dw}$ is the spin inductance, $\Phi_0$ is the domain wall and $N_{DW} \equiv \Phi_0 / \Phi_0$ is the number operator of the domain wall, where $\Phi_0 \equiv 4\pi$ is the domain wall quanta. In the second term of equation (4.2), $\tilde{E}_{xy} \equiv E_{sc} / 2\pi^2$ is the junction energy of a single domain wall junction, and $N_{DW}$ and $\hat{\phi}$ represent the relative phases of the domain wall fields. Using the Hamiltonian in equation (4.2), dual spin Josephson equations are given by $\dot{V}_s \equiv (h/\Phi_0) \hat{\phi} / \partial t = I_z^2 \pi N_{DW}$. $\dot{I}_z \equiv -\Phi_0 \hat{\phi} / \partial t = E_{sc} \sin \hat{\phi}/\pi S^z$, where $V_s$ and $I_z$ are the dual spin voltage and the dual spin current of the dual $XY$ ferromagnetic spin, respectively. The approximate commutation relations between $N_{DW}$ and $\hat{\phi}$ are given by $[\hat{\phi}, N_{DW}] \approx i$. By imposing similar duality conditions, $V_s \equiv I_z$, $I_z \equiv V_s$ to equation (2.4), we derived the next two of relationships,

$$N_{DW} = -\sin \phi / 2\pi, \quad N_{XY} = \sin \hat{\phi} / 2\pi. \hspace{1cm} (4.3)$$

From the spin Josephson equations, we derived the spin resistance $R = V_s / I_z = R_s^2 (N_{XY} / N_{DW}) (E_{sc} / 2\pi^2 E_{xy})$, where $R_s^2 \equiv h/(S^2)$ is the quantum spin resistance. In the same manner, from the dual spin Josephson equation, we derived the spin conductance $R_s = V_s / I_z = R_s^{-1}$. In the case of the condition $N_{XY} \gg N_{DW}$ or $E_{sc} \gg E_{xy}$, the junction system becomes a spin insulator. In particular, in the extreme case of $R_s \to \infty$, the junction system becomes a super-spin insulator. In the reverse case, in the condition of $N_{XY} \ll N_{DW}$ or $E_{sc} \ll E_{xy}$, the junction system becomes a spin conductor. In particular, in the extreme case of $R_s \to 0$, the junction system becomes a super-spin conductor. As a special case of these conditions, in the case of $N_{XY} = N_{DW}$ and $E_{sc} = 2\pi^2 E_{xy}$, the spin resistance $R_s$ is equal to the quantum spin resistance $R_s^* \equiv h/(\pi S^2)$, which is the state in a self-dual.

5. The quantum spin transistor

The single electron transistor [17] is a quantum effect device that operates by using the single electron tunneling created by a Coulomb blockade. Analogously, we propose a single quantum spin transistor that operates by using the single spin tunneling created by a spin blockade. The single quantum spin transistor consists of the FM/SC/FM junction of the sandwich structure. In this section, we consider the mechanisms that lie behind the single quantum spin tunneling, and advance a theoretical analysis to control this device. Now, $V_s$ is the dimension of the frequency $V_s = 2\pi I / e = 2\pi f_s$, where $f_s$ is the frequency of the single electron tunneling oscillations and $I_s$ is the dimension of the energy $I_s = eV / 2\pi$. Here, we used the following assumptions: $\phi \propto \theta$ and $\hat{\phi} \propto -\theta$. As shown in figure 5, as an application of the FM/SC/FM junction, we have devised a spin transistor, where inductance $L$ of the FM/SC/FM junction is defined by $L \equiv 2\pi^2 / J_s e^2$; $L_1$ and $L_2$ are the inductance of junction 1 and junction 2, respectively; $I_s$ is the current of the gate current source and $L_s$ is the inductance of the gate current source. In each of junction 1 and junction 2, the forbidden condition of one quantum spin tunneling is given by the following respective equations: $I_s \equiv \{\pm \pi S^0 + 2\pi S^0 (N_1 - N_2) / eL_1 \}/eL_2$, $I_s \equiv \{\pm \pi S^0 + 2\pi S^0 (N_1 - N_2) / eL_1 \}/eL_2$. For the above conditions, figure 6 shows the operating characteristics of the FM/SC/FM junction. Analogous to the Coulomb diamond in the Coulomb blockade, in the area inside the diamond, the tunneling of spin is blocked. This means that there is a real spin blockade without electron tunneling.

6. Summary and conclusion

The results presented in this paper can be described as follows. As our first result, we showed examples of two types of systems competing with each other. In one type, as the electrical duality, we introduced a duality between the SC and the SI. In the second type, as the magnetic duality, we introduced a duality between the FM and the SC. We applied the duality conditions by dual canonical
formalism to the systems competing with each other, and derived the relationship between the phase and the number of particles in a dual system. We showed that we can more clearly define the presence of quantum resistance and a quantum critical point by a dual canonical formalism. In our second results, we indicated the conditions for a spin blockade in which electron tunneling does not take place, and as its application, we introduced a quantum spin transistor.

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