Relativistic Trace Formula for Bound States in Terms of Classical Periodic Orbits

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Abstract

We set up a trace formula for the relativistic density of states in terms of a topological sum of classical periodic orbits. The result is applicable to any relativistic integrable system.
1 Introduction

Gutzwiller’s trace formula of 1971 expresses the density of states \( g(E) \) of a quantum mechanical system approximately as a sum over all periodic classical orbits \([1]\). Later, Balian and Bloch \([2]\) presented a formula which also applies to nonintegrable systems. It arose from a study of sound spectra in cavities with reflecting walls of arbitrary shape in two and more dimensions. Gutzwiller’s formula applies only to systems with isolated orbits. It fails if there exist degenerate families of periodic orbits connected by continuous symmetries \([3]\). The problem arise in the derivation of Gutzwiller’s formula from a stationary phase approximation to the trace integral over the semiclassical Green function at fixed energy. It contains an oscillating exponential of the eikonal function \( S(x, x; E) = \oint p \cdot dx \) of the periodic orbits passing through the point \( x \). A continuous symmetry makes this independent of \( x \) over an entire spatial region swept out by the symmetry operations. Then the second derivatives of the eikonal function vanishes in that region, resulting in a divergence of the stationary-phase integral. Strutinsky and coworkers \([4]\) removed these divergences by going back the convolution integral in the time-dependent propagator and performing exactly as many integrations in that integral and in the trace integral over the Green function, as there are independent parameters describing the degeneracy. Later, Creagh and Littlejohn \([5]\) pursued the same idea in a generalized phase space which also contains room for the continuous symmetry of the system. For integrable systems, their procedure is similar to that of Berry and Tabor \([6]\) who derived a trace formula for integrable systems employing the action-angle variables.

All this development has so far been restricted to the nonrelativistic regime where the particle solves the Schrödinger equation in some external time-independent potential. The purpose of this paper is to begin adapting the methods to relativistic particles described by
the Klein-Gordon equation in the external potential. Our final result will be a relativistic
generalization of Gutzwiller’s trace formula, expressing the density of states as a topological
sum over the relativistic closed classical orbits. The formula is applicable to integrable
relativistic classical systems.

Relativistic quantum mechanics is of course not really a consistent theory. At rel-
avtivistic velocities, particles will be created and absorbed, and the particle number is no
longer conserved, thus violating the current conservation law of the Klein-Gordon equation.
Quantum field theory is certainly the appropriate tool to describe relativistic particles. In
the classical regime, however, the particle number is fixed and these problems are absent,
so that a semiclassical expression for the density of states in terms of relativistic classical
periodic orbits is a consistent approximation expected to render a reliable results for those
systems in which particle creation and annihilation play only a minor role.

2 Relativistic Quantum-Mechanical Trace Formula

Consider a relativistic particle of mass $m$ in an external time-independent potential $V(x)$,
whose quantum mechanics is governed by the Klein-Gordon equation

$$\left\{ [i\hbar \partial_t + mc^2 - V(x)]^2 - c^2 \hbar^2 \left( \partial_x - \frac{e}{c \hbar} A \right)^2 - m^2 c^4 \right\} \phi(x, t) = 0. \quad (1)$$

where $c$ and $\hbar$ are speed of light and Planck’s constant, and $A(x)$ is a magnetic vector
potential. We have shifted the energy origin to the rest energy $mc^2$ in order to have a smooth
limit to nonrelativistic bound-state energies. Since the potentials are time-independent, the
wave functions can be factorized as $\phi(x, t) = e^{-iE_t/\hbar} \Psi(x)$, and (1) takes the Schrödinger-like
form

$$\hat{H}_E \Psi(x) = \varepsilon \Psi(x), \quad (2)$$
where

\[ \varepsilon \equiv \frac{E^2 - m^2 c^4}{2mc^2}, \quad (3) \]

and \( \hat{H}_E \) is the Hamilton operator

\[ \hat{H}_E = \hat{p}^2/2m + \left[2EV(x) - V^2(x)\right]/2mc^2, \quad (4) \]

with \( \hat{p} = -i\hbar \partial_x \). It is useful to view (2) as a special case of a more general eigenvalue equation

\[ \hat{H}_E \Psi(x) = \mathcal{E} \Psi(x), \quad (5) \]

which arises from a Schrödinger-like equation

\[ \hat{H}_E \Psi(x, \tau) = i\hbar \partial_\tau \Psi(x, \tau) \quad (6) \]

by a factorized ansatz \( \Psi(x, \tau) = e^{-i\varepsilon \tau/\hbar} \Psi(x) \). Then the variable \( \tau \) plays the role of a pseudotime, and the Hamilton operator \( \hat{H}_E \) is the pseudotime-evolution operator governing the \( \tau \)-dependence of the system.

Let \( \Psi_n(x) \) be the eigenfunctions of Eq. (5) with eigenvalues \( \mathcal{E}_E(n) \). Then the physical energies \( E_n \) of the particle are given by those values of \( E \) at which the pseudoenergy is equal to \( \varepsilon \):

\[ \mathcal{E}_{En}(n) = \varepsilon. \quad (7) \]

As an example, consider the Coulomb potential \( V(r) = -e^2/r \) of the relativistic hydrogen atom. Equation (4) leads to the radial eigenvalue equation

\[ \frac{d^2 R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} + \left[ \frac{2m}{\hbar^2} \left( \mathcal{E}_E + \frac{Ee^2}{mc^2} \right) - \frac{l(l+1) - \alpha^2}{r^2} \right] R(r) = 0. \quad (8) \]
Its solutions yield the bound state pseudoenergies depending on principal quantum number $n$ and angular momentum $l$, but degenerate in the azimuthal quantum number $m$:

$$E_{E}(n, l, m) = -\frac{E^2/mc^2}{2} \frac{\alpha^2}{\left[(n - l - 1/2)^2 + \sqrt{(l + 1/2)^2 - \alpha^2}\right]^2}, \quad \left\{ \begin{array}{c} n = 1, 2, 3, \ldots \\ l = 0, 1, 2, \ldots \end{array} \right. \quad (9)$$

Inserting these into Eq. (7), we obtain the well-known relativistic bound energies of the Coulomb system:

$$E_{n,l} = \pm mc^2 \left[ 1 + \frac{\alpha^2}{\left[(n - l - 1/2)^2 + \sqrt{(l + 1/2)^2 - \alpha^2}\right]^2} \right]^{-1/2}. \quad (10)$$

The complete information on the spectrum of eigenvalues of the Klein-Gordon equation (1) is contained in the pole terms of the trace of the resolvent $\hat{R}(E) \equiv i(\varepsilon - \hat{H}_E(n) + i\eta)^{-1}$:

$$r(E) \equiv \text{Tr} \hat{R}(E) = i \text{Tr} [\varepsilon - \hat{H}_E(n) + i\eta]^{-1}, \quad (11)$$

where the infinitesimal positive quantity $\eta$ guarantees the causality of the time dependence of the Fourier transform of $r(E)$. The imaginary part of $r(E)$ defines the density of states:

$$g(E) = \frac{1}{\pi} \text{Im} r(E) = \text{Tr} \delta(\varepsilon - \hat{H}_E). \quad (12)$$

In terms of the eigenvalues $E_E(n)$, the density (12) has the spectral representation

$$g(E) = \sum_n \delta(\varepsilon - E_E(n)), \quad (13)$$

where the sum over $n$ covers all quantum numbers. This sum will now be performed in a semiclassical approximation as a sum over periodic classical orbits.

For the sake of generality, we assume that the particle moves in $D$-dimensions, and assume that the motion has been transformed to $D$ cyclic coordinates whose motion can
easily be quantized (torus quantization). The labels \( n \) will then be integer-valued vectors \( n = (n_1, n_2, \cdots, n_D) \) with non-negative components \( n_i \). For the purpose of deriving a semiclassical approximation to (13), we convert each sum over \( n_i = 0, 1, 2, \ldots \) in Eq. (13) into an integral with the help of the Poisson summation formula [7, 8]

\[
\sum_{n=0}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \int_{0^-}^{\infty} f(n) e^{2\pi i k n} dn. \tag{14}
\]

Here we have assumed that the function \( f(n) \) and its derivatives with respect to \( n \) vanish at infinity, and the lower limit \( 0^- \) on the integral sign indicates that the integration starts on the left-hand side of the origin to include the entire \( \delta \)-function generated by the sum over \( k \). The superscript will be omitted in the sequel. Thus we obtain

\[
g(E) = \sum_{k} \int d^D n \delta (\varepsilon - E_E(n)) e^{2\pi i k n}, \tag{15}
\]

where each component of the integer-valued vector \( k = (k_1, k_2, \cdots, k_D) \) runs from minus to plus infinity, while the now continuous variables \( n_i \) are integrated from \( 0^- \) to infinity.

For integrable systems, the integration variables \( n_i \) in Eq. (15) can be replaced by the values of the action integrals appearing in the relativistic quantum conditions [11]

\[
I_i = \frac{1}{2\pi} \oint_{C_i} p \cdot d\mathbf{x} = \left(n_i + \frac{\mu_i}{4}\right) \hbar, \tag{16}
\]

where \( p \) is the relativistic momentum of the point particle along closed loops \( C_i \) on an invariant torus. The quantum numbers \( n_i \) are the same nonnegative integers as above, while \( \mu_i \) are the numbers of conjugate points along the orbit \( C_i \). Thus we can rewrite Eq. (15) as

\[
g(E) = \frac{1}{\hbar^D} \sum_{k} e^{-ik \cdot \mu \pi/2} \int_{\hbar \mu_1/4}^{\infty} dI_1 \int_{\hbar \mu_2/4}^{\infty} dI_2 \cdots \int_{\hbar \mu_D/4}^{\infty} dI_D \delta (\varepsilon - E_E(I)) e^{2\pi i k \cdot I}, \tag{17}
\]

where we have changed the argument of \( E_E(n) \) to \( E_E(I) \), and introduced vectors \( \mu = (\mu_1, \mu_2, \ldots, \mu_D) \).
Consider now the lowest term with $k = 0$, for which the oscillating exponentials in Eq. (17) are absent. It contributes a smooth density of states

$$\bar{g}(E) = \frac{1}{\hbar^D} \int_0^\infty dI_1 \int_0^\infty dI_2 \cdots \int_0^\infty dI_D \delta (\varepsilon - \mathcal{E}_E(I)), \quad (18)$$

where the lower bounds of the integral has been moved to zero, since the classical orbits for $k = 0$ have zero length, making $\mu$ equal to zero. The multiple integral (18) is just the classical density of states

$$\bar{g}_c(E) = \frac{1}{(2\pi \hbar)^D} \int \int d^Dp d^Dq \delta (\varepsilon - \mathcal{E}_E(p,q)), \quad (19)$$

which in cyclic coordinates reads

$$\bar{g}_c(E) = \frac{1}{(2\pi \hbar)^D} \int_0^\infty dI_1 \int_0^{2\pi} d\varphi_1 \int_0^\infty dI_2 \int_0^{2\pi} d\varphi_2 \cdots \int_0^\infty dI_D \int_0^{2\pi} d\varphi_D \delta (\varepsilon - \mathcal{E}_E(I)), \quad (20)$$

reducing to (18) after integrating out the angular variables. The classical density of states is also referred to as the Thomas-Fermi density [9].

We now turn to the oscillating $k \neq 0$ parts of $g(E)$. With the help of the integral representation for the $\delta$-function

$$\delta (\varepsilon - \mathcal{E}_E(I)) = \frac{1}{2\pi \hbar} \int_{-\infty}^\infty d\tau \ e^{i\tau(\varepsilon - \mathcal{E}_E(I))/\hbar}, \quad (21)$$

we rewrite this as

$$\delta g(E) = \frac{1}{2\pi \hbar} \int_{-\infty}^\infty d\tau \frac{1}{\hbar^D} \sum_k e^{-ik \cdot \mu \pi/2} \int h_{\mu_1/4} \int h_{\mu_2/4} \cdots \int h_{\mu_D/4} dI_1 dI_2 \cdots dI_D e^{i[2\pi k \cdot I + \tau(\varepsilon - \mathcal{E}_E(I))] / \hbar}, \quad (22)$$

where the primes on the summation symbols indicate the omission of $k = 0$. The integrals over $I_i$ and $\tau$ are now evaluated in the stationary phase approximation. Let us abbreviate the action in the exponent by

$$A_k(I, \tau) = 2\pi k \cdot I + \tau [\varepsilon - \mathcal{E}_E(I)]. \quad (23)$$
Its extrema lie at some \( \mathbf{I} = \mathbf{I}, \tau = \bar{\tau} \), where

\[
\frac{\partial A_k}{\partial I_i} \bigg|_{\mathbf{I} = \mathbf{I}, \tau = \bar{\tau}} = 0, \quad \frac{\partial A_k}{\partial \tau} \bigg|_{\mathbf{I} = \mathbf{I}, \tau = \bar{\tau}} = 0. \tag{24}
\]

The first set of equations yields the semiclassical quantization condition

\[
2\pi k_i = \bar{\tau}\omega_i(\bar{\mathbf{I}}), \quad i = 1, 2, \cdots, D, \tag{25}
\]

where

\[
\omega_i(\mathbf{I}) \equiv \frac{\partial E_E(\mathbf{I})}{\partial I_i} \tag{26}
\]

at \( \bar{\mathbf{I}} \) are the angular velocities for the pseudoenergy \( \varepsilon \). The solutions of Eq. (25) yield actions \( \bar{\mathbf{I}} \) as nonlinear functions of \( \mathbf{k} \) and \( \bar{\tau} \):

\[
\bar{\mathbf{I}} = \mathbf{I}(\mathbf{k}, \bar{\tau}). \tag{27}
\]

From Eq. (25) we obtain the important relation for the resonant tori

\[
\frac{k_i}{k_j} = \frac{\omega_i}{\omega_j}, \quad i, j = 1, 2, \cdots, D. \tag{28}
\]

Since \( k_i \) are integer numbers, the orbits on the torus must have commensurate frequencies, so that only closed periodic orbits contribute to the density of states in the saddle point approximation. This establishes the connection between \( \delta g(E) \) and the relativistic periodic orbits of the classical system. If the frequencies are not commensurate, the orbits do not close although the motion is still confined to the torus. Such orbits are called multiply periodic or quasi-periodic.

Each relativistic periodic orbit is specified by \( \mathbf{k} \); it closes after \( k_1 \) turns by \( 2\pi \) of the angle \( \varphi_1 \), \( k_2 \) turns of \( \varphi_2 \), \ldots. Thus \( \mathbf{k} \) plays the role of an index vector characterizing the topology of the periodic orbits. For this reason, the sums in Eq. (17) is also called topological sum. Note that Eq. (28) admits only \( k_i \)-values of the same sign.
The second equation in (24) specifies $\tau$ via

$$\varepsilon - E_E \left( \bar{I}(\bar{\tau}(E)) \right) = 0.$$  \hspace{1cm} (29)

Having determined the saddle points, the semiclassical approximation requires the calculation of the effect of the quadratic fluctuations around these. For this we expand Eq. (23) up to the quadratic terms, and shift the integration variables from $I$ to $I' \equiv I - \bar{I}$. The lower bound of the integrals is then transformed into $\hbar \mu /4 - \bar{I}$. For sufficiently large actions $\bar{I}$, the sharpness of the extrema at small $\hbar$ allows us to move the lower bounds to minus-infinity. This approximation is excellent for highly excited states. We now perform the Gaussian integrals and obtain the oscillating part of the relativistic density of states

$$\delta g^{(2)}(E) = \frac{1}{2\pi} \sqrt{2\pi/\hbar} D^{+1} \sum_k e^{-ik\cdot\mu/2} e^{-i\nu/4} \frac{1}{\bar{\tau}(D-1)/2} |\det M|^{-1/2} e^{i2\pi k \cdot \bar{I}}/\hbar,$$  \hspace{1cm} (30)

where $M$ is the stability matrix

$$M = \begin{pmatrix} \bar{\tau} & \partial^2 E_E / \partial I_i \partial I_j & \partial E_E / \partial I_i \\ \partial I_i \partial I_j & \partial^2 E_E / \partial I_i \partial I_j & 0 \end{pmatrix},$$  \hspace{1cm} (31)

whose determinant is, according to formula

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det(D - C^T A^{-1} B),$$  \hspace{1cm} (32)

given by

$$\det M = \det H \omega^T H^{-1} \omega.$$  \hspace{1cm} (33)

where

$$H_{ij} \equiv \frac{\partial^2 E_E}{\partial I_i \partial I_j}.$$  \hspace{1cm} (34)

The Maslov index $\nu$ is equal to $N^+ - N^- - N^0$, where $N^\pm$ denote the numbers of positive and negative eigenvalues of matrix $H_{ij}$, and $N^0$ is unity (zero) if the sign of $\omega^T H^{-1} \omega$ is positive.
(negative). The second factor in (32) has been simplified using the equation of motion for the cyclic variables $\phi$:
$$\frac{d\phi}{d\tau} = \nabla I E E(I) = \omega(I),$$
the right-hand side being also equal to
$$\omega = \frac{2\pi k}{\bar{\tau}}.$$  
(36)

Since for every $k$ there is an equal contribution from $-k$, we may replace the exponential by a cosine and obtain
$$\delta g^{(2)}(E) = \frac{1}{2\pi} \sqrt{\frac{2\pi}{\hbar}} \sum_{k} \frac{1}{\bar{\tau}^{(D-1)/2}} |\det H \omega^T H^{-1} \omega|^{-1/2} \cos \left[2\pi k \cdot \left(\bar{I}/\hbar - \mu/4\right) - \pi \nu/4\right].$$  
(37)

The relativistic trace formula (37) gives us a basis for understanding quantum phenomena at the relativistic level in terms of classical orbits. In general, we just need to evaluate the classical $E_E(\bar{I})$ for integrable systems, and consider some shortest orbits. As in nonrelativistic systems, we expect astonishingly accurate energy spectra from Eq. (37).

3 Three-Dimensional Relativistic Rectangular Billiard

As a first application, consider the motion of a relativistic particle in a three-dimensional rectangular billiard with sides of length $a_1$, $a_2$, and $a_3$ along $q_1$, $q_2$, and $q_3$ axes. The quantum spectrum of Eq. (3) with Dirichlet boundary condition is given by the pseudoenergies
$$E_E(n_1, n_2, n_3) = \frac{\hbar^2 \pi^2}{2m} \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} + \frac{n_3^2}{a_3^2}\right), \quad n_i(i = 1, 2, 3) = 1, 2, 3, \cdots.$$  
(38)

The physical relativistic energy spectrum is obtained from Eq. (7):
$$E_{n_1, n_2, n_3} = \pm \sqrt{\pi^2 \hbar^2 c^2 \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} + \frac{n_3^2}{a_3^2}\right) + m^2 c^4}.$$  
(39)
As in the nonrelativistic case, this result is exactly reproduced by the relativistic quantization according to Eq. (16). The numbers \( \mu_i \) are all equal to 4, since the wave functions have Dirichlet boundary condition. At every encounter with the wall, the action picks up a phase \( \pi \). The relativistic action variables are therefore

\[
I_i = \frac{1}{2\pi} \oint p_i dq_i = n_i \hbar, \quad i = 1, 2, 3; \quad n_i = 1, 2, 3, \ldots.
\]  

The classical Hamiltonian may be expressed as

\[
E_E(I) = \frac{\pi^2}{2m} \left( \frac{I^2_1}{a^2_1} + \frac{I^2_2}{a^2_2} + \frac{I^2_3}{a^2_3} \right),
\]  

and the corresponding angular frequencies are

\[
\omega_i = \frac{\pi^2}{ma^2_i} I_i, \quad i = 1, 2, 3.
\]

We now determine the saddle points \( \bar{I} \). According to Eq. (25), these are given by

\[
(\bar{I}_1, \bar{I}_2, \bar{I}_3)(\tau) = \left( \frac{2ma^2_1 k_1}{\tau \pi}, \frac{2ma^2_2 k_2}{\tau \pi}, \frac{2ma^2_3 k_3}{\tau \pi} \right),
\]

leading to the pseudoenergies at the saddle point

\[
E_E(\bar{I}(\tau)) = \frac{2m}{\tau^2} \sum_{i=1}^{3} (a_i k_i)^2.
\]

The saddle-point value of \( \tau \) is determined by (29), yielding

\[
\bar{\tau} = \sqrt{\frac{2m}{\varepsilon} \sum_{i=1}^{3} (a_i k_i)^2}.
\]

From these saddle point values, we obtain

\[
\sum_{ij} \omega_i H^{-1}_{ij} \omega_j \bigg|_{I(\bar{\tau})} = 2\varepsilon,
\]
so that the sign of $\omega_i H_{ij}^{-1} \omega_j$ is positive and the number $N^0$ in the Maslov index $\nu = N^+ - N^- - N^0$ vanishes. The determinant of the second-derivative matrix is

$$\det \frac{\partial^2 E}{\partial I_i \partial I_j} = \frac{\pi^6}{m^3 a_1^2 a_2^2 a_3^2}. \tag{47}$$

All eigenvalues of the matrix $\partial^2 E / \partial I_i \partial I_j$ are positive. Thus we identify the indices $N^+ = 3, N^- = 0$. Inserted into Eq. (37), we finally obtain for the oscillating part of the relativistic density of states

$$\delta g^{(2)}(E) = \frac{\pi}{4E_0} \sqrt{\frac{\varepsilon}{E_0}} \frac{a_1 a_2 a_3}{L^3} \sum_{k_1, k_2, k_3 = -\infty}^{\infty} j_0 \left( \frac{S(k)}{\hbar} \right), \tag{48}$$

where $j_0(x)$ is the spherical Bessel function of order zero $j_0(x) = \sin(x)/x$. The symbol $L$ denotes some length scale which may be any average of the three length scales $a_1, a_2, a_3$.

while

$$E_0 \equiv \frac{\pi^2 \hbar^2}{2mL^2} \tag{49}$$

denotes the energy associated with $L$. The quantity $S(k)$ is

$$S(k) = \frac{1}{c} \sqrt{E^2 - m^2 c^4} 2 \sqrt{k_1^2 a_1^2 + k_2^2 a_2^2 + k_3^2 a_3^2} = pL_k. \tag{50}$$

It is precisely the relativistic eikonal $pL_k$ of the classical periodic orbits of momentum $p$ and total length $L_k$. In general, the inclusion of only a few shortest orbits in Eq. (48) yields the correct positions of the quantum energy levels. The three-dimensional relativistic rectangular billiard may serve as a prototype of the relativistic semiclassical treatment for arbitrary billiard systems.

Let us compare the calculation of (48) from our relativistic trace formula (37) with a direct calculation from an inverse Laplace transformation of partition function $Z(\beta)$, i.e,

$$g(E) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} d\beta e^{\beta E} Z(\beta), \tag{51}$$
where the partition function is given by

\[ Z(\beta) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \exp \left\{ -\beta \mathcal{E}_E(n_1, n_2, n_3) \right\} , \]  

(52)

with the pseudoenergies \( \mathcal{E}_E(n_1, n_2, n_3) \) of Eq. (38). The problem is the same as in the calculation of the Casimir energy for the box. Since (52) is a product of three independent sums

\[ Z_i(\beta) = \sum_{n_i=1}^{\infty} \exp \left\{ -\beta \mathcal{E}_E(n_i) \right\} , \quad i = 1, 2, 3 , \]  

(53)

we may process each sum separately. Applying the Poisson formula (14) to the sum over \( n_i \) we find

\[ Z_i(\beta) = \sum_{k_i=-\infty}^{\infty} \int_{-\infty}^{\infty} dn_i e^{-\beta E_0 n_i^2 L^2/a_i^2} e^{2\pi i k_i n_i} - \frac{1}{2} \]  

\[ = \frac{1}{2} a_i \frac{\pi}{L} \int_{-\infty}^{\infty} e^{-(\pi m a_i)^2/\beta E_0 L^2} - \frac{1}{2} \]  

(54)

Inserting this into (52), and using the integral formula

\[ \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{d\beta}{\beta^{\mu+1}} e^{\beta \varepsilon} e^{-\kappa/\beta} = \left( \frac{\varepsilon}{\kappa} \right)^{\mu/2} J_\mu \left( 2\sqrt{\kappa \varepsilon} \right) , \]  

(55)

we obtain the exact level density of the relativistic three-dimensional rectangular box

\[ g(E) = g^{(3)}(E) - \frac{1}{2} \left[ g^{(2)}_{12}(E) + g^{(2)}_{23}(E) + g^{(2)}_{31}(E) \right] + \left[ g^{(1)}_1(E) + g^{(1)}_2(E) + g^{(1)}_3(E) \right] - \frac{1}{8} \delta(\varepsilon) . \]  

(56)

The leading term comes from a proper three-fold sum, and is given by

\[ \delta g^{(2)}(E) = \sum_{k_1,k_2,k_3=-\infty}^{\infty} \frac{\pi}{4E_0} \frac{\varepsilon}{E_0} \left( \frac{a_1 a_2 a_3}{L^4} \right) j_0 \left( \frac{S(k)}{\hbar} \right) \]  

(57)

with \( S(k) \) of Eq. (50). This agrees with the semiclassical result (18). The second set of terms gives corrections from the faces of the box:

\[ g^{(2)}_{ij}(E) = \frac{\pi}{4E_0} \frac{a_i a_j}{L^2} \sum_{k_1,k_2=-\infty}^{\infty} j_0 \left( \frac{S_2(k_1,k_2)}{\hbar} \right) , \]  

(58)
where
\[ S_2(k_1, k_2) = \frac{1}{c} \sqrt{E^2 - m^2 c^4} \left( 2 \sqrt{k_1^2 a_1^2 + k_2^2 a_2^2} = p L_{k_1, k_2} \right) \] (59)
are the eikonals of the orbits on the faces. The functions \( g_{ij}^{(2)}(E) \) are the level densities of the planar facial “boxes”.

The third set of terms in (56) stems from the edges of the box, being the level density for these one-dimensional “boxes” of length \( a_i \):

\[ g_i^{(1)}(E) = \frac{a_i}{2L\sqrt{E_0 \varepsilon}} \sum_{k_i = -\infty}^{\infty} \cos \left( \frac{S_1(k_i)}{\hbar} \right), \] (60)
where
\[ S_1(k) = \frac{1}{c} \sqrt{E^2 - m^2 c^4} 2ka_i = p L_{k_1, k_2}. \] (61)

These boundary terms can be obtained also from the general trace formula (14) by calculating higher-order corrections to the semiclassical approximation (37).

The last term in (50) is a delta function at \( \varepsilon = 0 \) which does not contribute to the level density at \( \varepsilon > 0 \).

The classical (Thomas-Fermi) contribution to the density of states is

\[ \bar{g}(E) = \frac{1}{E_0} \left( \frac{\pi}{4} \sqrt{\frac{\varepsilon}{E_0 L^3}} - \frac{\pi}{8} \frac{V_2}{E_0 L^2} + \frac{1}{8} \sqrt{\frac{E_0}{\varepsilon}} \frac{V_1}{L} \right). \] (62)

Here \( V_3 = a_1a_2a_3 \) is the volume of the box, \( V_2 = 2(a_1a_2 + a_2a_3 + a_1a_3) \) the total surface, and \( V_1 = a_1 + a_2 + a_3 \) the sum of the edge lengths.

4 Concluding remark

For relativistic integrable systems, we have derived a semiclassical trace formula by transforming the relativistic quantization conditions into the topological sum involving all closed
relativistic classical orbits. Certainly, our final result (37) can also be obtained by an ab initio procedure, starting out from the relativistic path integral for the relativistic fixed-energy amplitude representation [7, 11, 12]

\[ G(x_b, x_a; E) = \frac{\hbar}{2Mc} \int_0^\infty dL \int D\rho \Phi[\rho] \int D^D x e^{iA_E/\hbar}, \]  

(63)

with the action

\[ A_E[x, \dot{x}] = \int_{\tau_a}^{\tau_b} d\tau \left\{ \frac{M}{2\rho(\tau)} \dot{x}^2(\tau) + \frac{e}{c} A \cdot \dot{x}(\tau) + \frac{\rho(\tau)}{2Mc^2} (E - V(x))^2 - \frac{\rho(\tau)}{2} \right\}, \]  

(64)

where \( L \) is defined by

\[ L = \int_{\tau_a}^{\tau_b} d\tau \rho(\tau), \]  

(65)

with \( \rho(\tau) \) being an arbitrary dimensionless fluctuating scale variable, and \( \Phi[\rho] \) is some convenient gauge-fixing functional, such as \( \Phi[\rho] = \delta[\rho - 1] \). The prefactor \( \hbar/Mc \) in (63) is the Compton wavelength of a particle of mass \( M \), the field \( A(x) \) is the vector potential, \( V(x) \) the scalar potential, \( E \) the system energy, and \( x \) the spatial part of the \( D + 1 \)-dimensional vector \( x = (x, i\tau) \). This path integral forms the basis for studying relativistic potential problems. Choosing \( \rho(\tau) \) to be equal to unity, the amplitude (63) becomes

\[ G(x_b, x_a; E) = \frac{\hbar}{2Mc} \int_0^\infty dL \exp \left[ i \frac{i}{\hbar}L \right] \int D^D x \exp \left[ i \frac{i}{\hbar} A_E \right], \]  

(66)

where the fixed-energy action \( A_E \) is given by

\[ A_E = \int_0^L d\tau \left\{ \frac{M}{2} \dot{x}^2(\tau) + \frac{e}{c} A \cdot \dot{x}(\tau) + \frac{1}{2Mc^2} \left[ V^2(x) - 2EV(x) \right] \right\}. \]  

(67)

The semiclassical approximation to the relativistic fixed-energy amplitude (66) is [13]

\[ G_{sc}(x_b, x_a; E) = \frac{\hbar}{2Mc (2\pi\hbar)^{D/2}} \sum_{\text{class.traj.}} \int_0^\infty dL e^{iL/\hbar} \]
\[
\times \det \left[ -\partial_{x_b} \partial_{x_j} A_E(x_b, x_a; L) \right]^{1/2} e^{iA_E(x_b, x_a; L)/\hbar - i\pi \nu/2}. \tag{68}
\]

The associated density of states is obtained from the trace of (68):

\[
\int d^Dx \ G_{sc}(x_b, x_a; E) = \frac{\hbar}{2Mc} \frac{1}{(2\pi \hbar)^{D/2}} \sum_{\text{class.traj.}} \int_0^\infty dL e^{i\varepsilon L/\hbar}
\times \int d^Dx \ det \left[ -\partial_{x_b} \partial_{x_j} A_E(x_b, x_a; L) \right]^{1/2} e^{iA_E(x_b, x_a; L)/\hbar - i\pi \nu/2}. \tag{69}
\]

The trace operation in Eq. (68) is integration over all periodic orbits in the pseudotime “L”. If the relativistic systems is integrable, it can be expressed in terms of action-angle variables as

\[
\int d^Dx \ det \left[ -\partial_{x_b} \partial_{x_j} A_E(x_b, x_a; L) \right]^{1/2} e^{iA_E(x_b, x_a; L)/\hbar - i\pi \nu/2}
= \sum_k \int_0^{2\pi} d^D\varphi L^{-D/2} \ det \left[ \frac{\partial^2 E_E(I)}{\partial I_i \partial I_j} \right]^{-1/2} e^{i(2\pi I \cdot k - E_E(I)L)/\hbar - i\pi \nu/2} \tag{70}
\]

\[
A_E(x_b, x_a; L) = I \cdot (\varphi_b - \varphi_a) - E_E(I)L = 2\pi I \cdot k - E_E(I)L, \tag{71}
\]

thus establishing contact with the earlier treatment in which \(\tau\) plays the role of \(L\).

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