Automorphism Groups of Domains that Depend on Fewer Than the Maximal Number of Parameters

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1 Introduction

Let $\Omega \subseteq \mathbb{C}^n$ be a domain, that is a connected, open set. Let $\text{Aut}(\Omega)$ denote the collection of biholomorphic self-mappings of $\Omega$ (see [ISK1] or [GRK] for a survey of this topic). This set forms a group when equipped with the binary operation of composition of mappings. In case $\Omega$ is bounded, then the group is in fact a real (never a complex) Lie group. We call this group the automorphism group of $\Omega$.

It is naturally a matter of considerable interest to describe the automorphism group for a given domain $\Omega$. In the best of all possible circumstances, we would like to give an explicit description of this group. As an instance, in case

$$\Omega = B \equiv \{ (z_1, \cdots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 < 1 \},$$

then the automorphism group of $\Omega$ is generated by (i) the unitary rotations and (ii) the Möbius transformations

$$y(z_1, z_2, \ldots, z_n) \mapsto \left( \frac{z_1 - a}{1 - \bar{a}z_1}, \frac{\sqrt{1 - |a|^2}z_2}{1 - \bar{a}z_1}, \ldots, \frac{\sqrt{1 - |a|^2}z_n}{1 - \bar{a}z_1} \right)$$

for $a \in \mathbb{C}, |a| < 1$. It is worth noting (as this is part of the theme of the present paper) that any description of the automorphisms of $B$ must involve all $n$ variables. Such a statement is already true for the unitary group alone.

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It is not true that the automorphism group depends only on \(n - 1\) variables or \(n - 1\) parameters.\(^2\)

There are other important domains—ones that are currently a focus of considerable study—whose automorphism groups are much simpler. For the Kohn-Nirenberg domain (see [JIS], [KON]), the automorphisms consist of rotations in one variable only. The purpose of the present paper is to produce a geometric criterion which will guarantee that the automorphism group of a given domain \(\Omega \subseteq \mathbb{C}^n\) will depend on fewer than the full number of variables in the ambient space. This result will simplify the example in [JIS], and will also provide further examples for the future. We indicate some of these latter examples at the end of the present paper.\(^3\)

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2 Basic Ideas

Fix a pseudoconvex domain \(\Omega \subseteq \mathbb{C}^2\) with smooth boundary. Because of the discussion in the first section, we may suppose that \(\Omega\) is not (biholomorphically) the ball. In case \(\Omega\) is strongly pseudoconvex, we may then conclude by a theorem of Bun Wong and Rosay (see [KRA1]) that \(\text{Aut}(\Omega)\) is compact (in fact we shall make this a standing hypothesis in the discussion that follows).

Let \((z, w)\) be the coordinates in \(\mathbb{C}^2\). We take it that the origin \(0 = (0, 0)\) lies in \(\partial \Omega\). Now suppose that, near \(0\), \(\Omega\) is defined by the inequality

\[
\rho(z, w) \equiv \text{Re } w + \varphi(z, \overline{z}) + \psi(z, \overline{z}, \text{Im } w) \cdot (\text{Im } w)^2 < 0.
\]

\(^*\)

In what follows we shall always use the letter \(\rho\) to denote a defining function for \(\Omega\). Applying a unitary rotation if necessary, we may arrange that the complex tangent space \(\mathcal{H}_0\) at \(0\) is equal to \(\{(z, 0) : z \in \mathbb{C}\}\).

Let \(\text{Aut}_0(\Omega)\) denote the automorphisms of \(\Omega\) that fix \(0\). We shall assume below that \(\mathcal{H}_0 \cap \Omega\) is an open subset of \(\mathcal{H}_0\). If \(f\) is an automorphism of

\(^2\)Clearly these statements can be formulated in terms of the dimension of the automorphism group. We leave this task as an exercise for the reader. For the purposes here, the formulation in terms of the ambient complex variables is more convenient and more accurate.

\(^3\)In the paper [GIK], it is shown that if \(\Omega\) is a hyperbolic domain and the dimension of the automorphism group exceeds \(n^2 + 2\) then the domain must be a ball. The result of the present paper is in a philosophically similar spirit, but the restrictions on the dimension are more severe.
Ω then set \( f(\zeta) = f_1(\zeta, 0) \) for \((\zeta, 0) \in \mathcal{H}_0 \cap \Omega\) and \( f = (f_1, f_2) \). Define \( \mathcal{F}_f(z, w) = (f(z), w) \).

Now we shall list the standing hypotheses that will be in place for the remainder of this paper:

**Standing Hypotheses**

1. The set \( \mathcal{H}_0 \cap \Omega \) is an open subset of \( \mathcal{H}_0 \).
2. The function \( \varphi \) in the definition of \( \rho \) has no harmonic terms.
3. If \( f \in \text{Aut}_0(\Omega) \) then \( \mathcal{F}_f \in \text{Aut}_0(\Omega) \).
4. The automorphism group \( \text{Aut}_0(\Omega) \) is compact.
5. The domain \( \Omega \) is complete hyperbolic.
6. Any automorphism of \( \Omega \) continues analytically to a neighborhood of \( 0 \in \partial \Omega \).

Standing Hypothesis 2 perhaps merits some discussion. In case \( \varphi \) has harmonic terms, we may write

\[
\varphi(z, \overline{z}) = \sum_{k=2}^{\infty} a_k z^k + \sum_{k=2}^{\infty} \overline{a_k \overline{z}^k} + \tilde{\varphi}(z, \overline{z}),
\]

where \( \tilde{\varphi} \) has no harmonic term. Let \( \mu(z) = \sum k a_k z^k \) be holomorphic. Then the holomorphic coordinate change

\[
(\tilde{z}, \tilde{w}) = (z, w + 2 \sum_{k} a_k z^k)
\]

defines a new local defining function

\[
\text{Re} \ \tilde{w} + \tilde{\varphi}(z, \overline{z}) + \psi(z, \overline{z}, \text{Im} \ w) \cdot (\text{Im} \ w)^2 < 0.
\]

Note that the lead term here has no harmonic term.

Now our main result is this:
**Theorem 1** Let $\Omega$ be a domain as described above. Then any automorphism $f(z) = (f^1(z), f^2(z))$ fixing the origin $0$ of $\Omega$ must have the form

$$f(z) = (\varphi(z_1), z_2).$$

In other words, any automorphism of $\Omega$ fixing the origin will depend only on the first variable (and not on the second).

If $f \in \text{Aut}(\Omega)$ then certainly $\rho \circ f$ is also a local defining function for $\Omega$ near $0$. Therefore

$$\rho \circ f(z, 0) = \mu \cdot \rho(z, 0) \quad (***)$$

for some positive function $\mu$ near $0$. As a result, we may conclude (see Standing Hypothesis 2) that the quadratic part of $\mu \cdot \rho(z, 0)$ has no harmonic terms. This observation also applies to the lefthand side of (***) so we see that

$$\text{Re} f_2(z, 0) + \varphi(f_1(z, 0), \overline{f_1(z, 0)}) + \psi(f_1(z, 0), \overline{f_1(z, 0)}, \text{Im} f_2(z, 0)) \cdot (\text{Im} f_2(z, 0))^2$$

has no harmonic terms.

**CLAIM:** We assert that $f_2(z, 0) \equiv 0$.

**Proof of the Claim:** If not, then

$$f_2(z, 0) = \sum_{k \geq 0} a_k z^k.$$

Let $a_{k_0}$ be the nonzero coefficient with least index.

Since $\varphi$ has no harmonic terms and

$$f_1(z, 0) = b_1 z + \sum_{k \geq 2} b_k z^k$$

with $b_1 \neq 0$, we see that

$$\varphi(f_1, \overline{f_1}) = \sum_{k, \ell} b_k \overline{b_\ell} z^k \overline{z^\ell}.$$

Certainly $b_0 = 0$, hence $\varphi(f_1, \overline{f_1})$ has no harmonic terms.

Now

$$\psi(f_1(z, 0), \overline{f_1(z, 0)}, \text{Im} f_2(z, 0)) \cdot (\text{Im} f_2(z, 0))^2.$$
has the term with index $2k_0$ as the first nonvanishing term. Hence the first harmonic term

$$\text{Re } a_{k_0} z^k \equiv 0.$$  

But this implies that $a_{k_0} \equiv 0$. And that is a contradiction. \[\square\]

Now, as a consequence of the claim, we certainly know that

$$f(\mathcal{H}_0 \cap \Omega) \subseteq \mathcal{H}_0.$$  

As a result, $f|_{\mathcal{H}_0 \cap \Omega}$ is an automorphism of $\mathcal{H}_0 \cap \Omega$ that fixes $0$. In our earlier notation, $\tilde{f} \in \text{Aut}(\mathcal{H}_0 \cap \Omega)$.

Referring to Standing Hypothesis 3, we now consider

$$\mathcal{F}_{\tilde{f}^{-1}} \circ f \in \text{Aut}_0(\Omega).$$

We see that

$$\mathcal{F}_{\tilde{f}^{-1}} \circ f(z,0) = \mathcal{F}_{\tilde{f}^{-1}}(\tilde{f}(z),0) = (z,0).$$

Writing $\Phi \equiv \mathcal{F}_{\tilde{f}^{-1}} \circ f$, we may say that $\Phi(z,0) = (z,0)$.

Now we have

$$d\Phi(z,0) = \begin{pmatrix} 1 & a(z) \\ 0 & b(z) \end{pmatrix}$$

for some $a(z), b(z)$ that are holomorphic on $\mathcal{H}_0$. Of course $d\Phi$ takes the real tangent space at $0$ to the real tangent space at $0$ and the complex tangent space at $0$ to the complex tangent space at $0$. Since $\text{Aut}_0(\Omega)$ is compact, we conclude that $\sup |b(z)| = 1$ and $b(0) = 1$ hence (by the maximum principle) $b \equiv 1$. Finally, since $\Omega$ is complete hyperbolic, $a(z) \equiv 0$. We conclude that

$$d\Phi(z,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the identity matrix. By the Cartan Uniqueness Theorem, we conclude that $\Phi(z) = z$ for all $z \in \Omega$.

Therefore, we finally arrived that

$$f(z,w) = (\tilde{f}(z),w).$$

In short, $f$ depends on fewer than the maximal number of parameters.
2.1 Levi Flat Domains

Now we present a variant of our main result.

Standing Hypotheses

1. The local defining function for \( \Omega \) at the origin is \( \Re w = 0 \).
2. Any automorphism of \( \Omega \) continues analytically to a neighborhood of \( 0 \in \partial \Omega \).
3. The automorphism group \( \text{Aut}_0(\Omega) \) is compact.
4. The domain \( \Omega \) is complete hyperbolic.
5. If \( f \in \text{Aut}_0(\Omega) \) then \( \tilde{F} \circ f \in \text{Aut}_0(\Omega) \).

Fix \( f \in \text{Aut}_0(\Omega) \). By the same argument as in the Main Theorem, we have that
\[
\rho \circ f(z,0) = \mu \rho(z,0),
\]
where \( \rho = \Re w \). This implies that \( \Re f_2 \equiv 0 \) where \( f = (f_1, f_2) \). We obtain that \( f \) preseve that \( H_0 \cap \Omega \). By Standing Hypotheses 5, we consider \( \Phi = F_{\tilde{f}} \circ f \). Then \( \Phi(z,0) = (z,0) \).

By Standing Hypotheses 3, we obtain information on one jet of \( f \):
\[
d\Phi(z,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

By the power series expansion of \( \Phi \),
\[
\Phi_1(z,w) = z + w^2 \sum a_{jk} z^j w^k
\]
\[
\Phi_2(z,w) = w + w^2 \sum b_{jk} z^j w^k.
\]

By Standing Hypotheses 1, there is a positive \( \delta \) such that \( (z,it) \) is in the boundary of \( \Omega \), for all \( |z| < \delta \) and real number \( |t| < \delta \). So, we get \( \Phi(z,it) \) lies in the boundary of \( \Omega \). This implies that
\[
\Re \left( it + (it)^2 \sum b_{jk} z^j (it)^k \right) = 0.
\]
We consider the left hand side as a power series with respect to $t$. For all $j, k \geq 0$,

$$\text{Re} \left(-b_{jk}z^j i^k\right) = 0.$$ 

For each $j, k$, we can choose two different complex numbers $\alpha, \beta$ such that $\alpha^j$ is pure imaginary and $\beta^j$ is real. Note that $j$ and $k$ are fixed. We can assume that $i^k$ is a real number $\pm 1$. We get that

$$\text{Re} \left(-b_{jk}z^j\right) = 0.$$ 

The above equation holds for all sufficient small complex numbers. We can insert $\alpha, \beta$ into it. This implies that $b_{jk}$ is a real and pure imaginary number. Hence, all $b_{jk}$ are zero. We arrive at the conclusion that

$$\Phi_1(z, w) = z + w^2 \sum a_{jk} z^j w^k,$$

$$\Phi_2(z, w) = w.$$ 

We want to prove that $\Phi_1(0, w) = w^2 \sum a_{0k} w^k$ is identically zero. Expecting a contradiction, we assume that $a_{0k}$ is the first nonzero term. We consider the $N$-times composition of $\Phi$ restricted to $(0, w)$. By a calculation,

$$\Phi^N(0, w) = (Na_{0k}w^k + (higher order terms), w).$$

Since $\Phi^N$ is a precompact family, we arrive at a contradiction.

Therefore, $\Phi(0, w) = (0, w)$. Take a derivative at $(0, w)$. Then

$$d\Phi(0, w) = \begin{pmatrix} a(w) & b(w) \\ 0 & 1 \end{pmatrix}.$$ 

By the same technique as in the last section, we have the identity matrix at $(0, w)$. We can apply Cartan’s Uniqueness Theorem. Finally we get that $\Phi$ is the identity map of $\Omega$. This means that $f(z, w) = (\tilde{f}(z), w)$.

### 3 Examples

**EXAMPLE 1** First let us look at the Kohn-Nirenberg domain [KON], which is given (in the complex variables $z, w$) by

$$\text{Re } w + |zw|^2 + |z|^8 + \frac{15}{7} |z|^2 \text{Re } z^6 < 0.$$
It is a simple matter to verify that this domain satisfies the hypothesis about $S$ in our Theorem. Also the domain is of finite type, so every automorphism extends smoothly to the boundary. We may conclude immediately that any automorphism fixing the origin depends on just one variable. Then simple calculations show (see [JIS]) that the only automorphisms are rotations in the $z$ variable.

\[ \square \]

**EXAMPLE 2** Let $\varphi$ be a $C^\infty$ function on $\mathbb{R}$, even, nonnegative, supported in the interval $[-1/10, 1/10]$, constantly equal to 1 on $[-1/100, 1/100]$. Define

$$\Omega = \{ (z_1, z_2) \in \mathbb{C}^2 : (1 - \varphi(|z_2|^2))(1 - \varphi(|1 - z_1|^2)) \cdot [-1 + |z|^2] + \varphi(|z_2|^2)\varphi(|1 - z_1|^2) \cdot [-1 + \epsilon + \text{Re} z_1] < 0 \}.$$

Then $\Omega$ is nothing other than the unit ball in $\mathbb{C}^2$ with a flat bump centered about the spherical boundary point $(1, 0)$. And notice that the boundary point $(1 - \epsilon, 0)$ has a neighborhood in the boundary that lies in the hyperplane $\{\text{Re} z_1 = 1 - \epsilon\}$. It is straightforward to see that the only automorphisms of $\Omega$ are rotations in the $z_2$ variable (see [LER]).

Now examine our main theorem. This domain $\Omega$ satisfies the hypotheses of that theorem with the origin replaced by $(1 - \epsilon, 0)$. The automorphism group of this particular $\Omega$ may be determined explicitly (in fact any automorphism of $\Omega$ is just an automorphism of the unit ball—see[LER]), so most of the Standing Hypotheses are automatic. Note particularly that the presence of the bump near $(1 - \epsilon, 0)$ forces standing hypothesis 3 to hold. So this example is an illustration of our main result. The automorphism group only depends on the $z_2$ variable.

### 4 Concluding Remarks

This is the first paper to explore the questions posed here. There is clearly a need for a result in all dimensions, and for results that have more flexible hypotheses.

It would certainly be of interest to have concrete examples to which our results do (or do not) apply. The theorem definitely does not apply to the complex ellipsoids

$$E_{m,n} = \{ (z_1, z_2) : |z_1|^{2m} + |z_2|^{2n} < 1 \}$$
for $m, n \in \mathbb{N}$. And of course they should not. The result also does not apply to the ball or the Siegel upper half space. More, they do not apply to any of the bounded symmetric domains of Cartan (see [GRK] for a discussion of some of these specialized domains).

What would be ideal is to have a theorem that, given $0 < k < n \in \mathbb{N}$, characterizes domains in $\mathbb{C}^n$ whose automorphisms depend only on $k$ variables. This will be the subject of future investigations.
REFERENCES

[BEP] E. Bedford and S. Pinchuk, Domains in $\mathbb{C}^2$ with non-compact holomorphic automorphism group (translated from Russian), *Math. USSR-Sb.* 63(1989), 141–151.

[BER] L. Bers, *Introduction to Several Complex Variables*, New York Univ. Press, New York, 1964.

[JIS] J. Byun and H.R. Cho, Explicit Description for the Automorphism Group of the Kohn-Nirenberg Domain, *Math. Z.*, to appear.

[CHES] S.-C. Chen and M.-C. Shaw, *Partial Differential Equations in Several Complex Variables*, American Mathematical Society, Providence, RI, 2001.

[JPDA] J. P. D’Angelo, *Several Complex Variables and the Geometry of Real Hypersurfaces*, CRC Press, Boca Raton, 1992.

[DIF1] K. Diederich and J. E. Fornæss, Pseudoconvex domains: An example with nontrivial Nebenhülle, *Math. Ann.* 225(1977), 275–292.

[GIK] J. Gifford, A. Isaev, and S. G. Krantz, On the Dimensions of the Automorphism Groups of Hyperbolic Reinhardt Domains, *Illinois Jour. Math.* 44(2000), 602–618.

[GRK] R. E. Greene and S. G. Krantz, Biholomorphic self-maps of domains, *Complex Analysis II* (C. Berenstein, ed.), Springer Lecture Notes, vol. 1276, 1987, 136-207.

[ISK1] A. Isaev and S. G. Krantz, A. Isaev and S. G. Krantz, Domains with non-compact automorphism group: A survey, *Advances in Math.* 146(1999), 1–38.

[JAP] M. Jarnicki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, Walter de Gruyter & Co., Berlin, 1993.

[KOH] J. J. Kohn, Boundary behavior of $\partial$ on weakly pseudoconvex manifolds of dimension two, *J. Diff. Geom.* 6(1972), 523–542.
[KON] J. J. Kohn and L. Nirenberg, A pseudo-convex domain not admitting a holomorphic support function, *Math. Ann.* 201(1973), 265-268.

[KRA1] S. G. Krantz, *Function Theory of Several Complex Variables*, 2nd ed., American Mathematical Society, Providence, RI, 2001.

[KRA2] S. G. Krantz, Determination of a domain in complex space by its automorphism group, *Complex Variables Theory Appl.* 47(2002), 215–223.

[LER] L. Lempert and L. A. Rubel, An independence result in several complex variables. *Proc. Amer. Math. Soc.* 113(1991), 1055–1065.

[NIV] I. Niven, *Irrational Numbers*, The Mathematical Association of America in affiliation with John Wiley & Sons, New York, 1956.

[RUD] W. Rudin, *Function Theory in the Unit Ball of C^n*, Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Springer, Berlin, 1980.