Doubly Special Relativity versus \( \kappa \)-deformation of relativistic kinematics

by

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Abstract

We argue that recently proposed by Amelino-Camelia et all [1,2] so-called doubly special relativity (DSR), with deformed boost transformations identical with the formulae for \( \kappa \)-deformed kinematics in bicrossproduct basis is a classical special relativity in nonlinear disguise. The choice of symmetric composition law for deformed fourmomenta as advocated in [1, 2] implies that DSR is obtained by considering nonlinear fourmomenta basis of classical Poincaré algebra and it does not lead to noncommutative space-time. We also show how to construct large two classes of doubly special relativity theories - generalizing the choice in [1,2] and the one presented by Magueijo and Smolin [3]. The older version of deformed relativistic kinematics, differing essentially from classical theory in the coalgebra sector and leading to noncommutative \( \kappa \)-deformed Minkowski space is provided by quantum \( \kappa \)-deformation of Poincaré symmetries.

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1 Introduction

The $\kappa$-deformation of Poincaré symmetries in the form of dual pair of Hopf algebras describing respectively quantum deformations of $D = 4$ Poincaré algebra and $D = 4$ Poincaré group [4-11] was proposed in order to introduce modification of relativistic kinematics at extremely high energies. Recently, in the search for phenomenological high energy effects of quantum gravity (see e.g. [12-15]) there were also put forward the generalizations of special relativity theories with two observer-independent parameters – light velocity $c$ and Planck length $L_p$ – under the name of doubly special relativity (DSR) [1,2, 16-17]. In concrete realization of DSR framework the basic formulae coincide with the ones following from the algebraic sector of $\kappa$-deformation of Poincaré algebra in bicrossproduct basis [8-10], with the assumption that the mass-like deformation parameter $\kappa$ is equal to the Planck mass $m_p$. In particular the basic characteristic of DSR – the deformed dispersion relation, described in [1] as ,,key characteristic of DSR1, both conceptually and phenomenologically”

$$m^2 = \left( \frac{\sinh \frac{L_p E}{2}}{\frac{L_p}{2}} \right)^2 - \bar{p}^2 e^{L_p E}$$

is exactly the formula for $\kappa$-deformed mass-shell condition in bicrossproduct basis (assuming $c = 1$); also the differential realizations of the generators of boost transformations in DSR, calculated in [17], can be found in [9].

The aim of this note is however not to intervene in the procedure of referencing to the previous achievements. In fact we are happy that others are applying and developing our ideas, even if this is done without providing proper links with already established results. Here we would like to clarify the notions of DSR from the point of view of Hopf algebras, its coalgebraic structure and its relation with $\kappa$-deformation of Poincaré symmetries [5-12].

A flag property of DSR is the appearance of limiting momentum achieved at infinite energy described by Planck mass, which can not be overpassed if we change reference frame. The main issue is which properties will have an object composed of two constituents having the above described limiting momentum. We can have in principle two possible solutions:

1In [1] there are considered two DSR theories: DSR1 and DSR2 – the second one based on [3]. We consider different versions of DSR in Sect. 2
i) Only single objects (one can call them „Plancktons”) do have the property of limiting momentum. In such a case the Planck mass is just a constant parameter in the theory entering deformed boost transformations, related possibly with some quantum gravity-related quanta.

ii) The property of existence of maximal limiting momentum is valid in the composition process of the momenta. Similarly like in classical special relativity theory one has the relativistic addition law of velocities

\[ v_{12} = v_1 + v_2 \Rightarrow c + c = c \]  

one should have an addition law \( p_{12} = p_1 + p_2 \) providing the existence of the same limiting momenta for \( p_{12} \). It appears that in such a case the deformed symmetries should be described as a Hopf algebra with homomorphic coproducts, what ensures that the boost transformations for the constituents \( p_i \) (\( i = 1, 2 \)) and of total momentum \( p_{12} \) shall have the same form, in particular the same limiting momentum.

In [1,2] the authors propose (in linear approximation) the following composition law for energy-momenta in DSR1 theory:

\[ E_1 + E_2 - cL_p\vec{p}_1\vec{p}_2 = E_1' + E_2' - cL_p\vec{p}_1'\vec{p}_2' \] \hspace{1cm} (3)

\[ \vec{p}_1 + \vec{p}_2 - \frac{L_p}{c}(E_1\vec{p}_2 + E_2\vec{p}_1) = \vec{p}_1' + \vec{p}_2' - \frac{L_p}{c}(E_1'\vec{p}_2' + E_2'\vec{p}_1') \] \hspace{1cm} (4)

These composition laws disclose the Hopf algebra structure of the theory proposed in [1,2]: the relations (3, 4) describe the part linear in \( L_p \) of the coproduct obtained by nonlinear transformation of the fourmomenta in classical relativistic theory, i.e. the basic structure remains Einsteinian. Such theories with nonlinear symmetric coproduct for energy were considered [18] just after appearance of \( \kappa \)-deformations of relativistic symmetries in order to use the nonlinear composition law as a tool to describe the dark matter effect, but were rather abandoned in favour of deformed theories providing abelian addition law for energy.

In this note firstly in Sect. 2 we shall show that the deformed boost transformations from [1,2,17] are indeed obtained by nonlinear transformation of momentum generators in classical relativistic theory; besides we shall derive

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2In comparison with [1] we inserted in formula (4) the factor \( \frac{1}{2} \) consistent with the relation \( L_p = \frac{1}{\sqrt{\mu}c} \).
the composition laws (3,4) for the energy and threemomentum by performing the same nonlinear transformation of primitive coproduct for classical fourmomenta (we recall that primitive coproduct describe classical abelian fourmomentum composition law). By generalizing the deformed mass-shell condition (1) we shall describe two large classes of nonlinearly transformed classical theories, generalizing DSR1 type (see [1,2]) and DSR2 type (see [3]) of theories.

In Sect. 3 we shall consider the composition law described by the coproduct of $\kappa$-deformed Poincaré algebra in bicrossproduct basis. In particular following result present in [9] we shall remind why the method of calculating boost formulas, used in [17], is also valid in $\kappa$-deformed quantum-group framework. Further as new result we shall show the consistency of finite boost formulae with $\kappa$-deformed nonabelian addition law for the momenta.

In Sect. 4, we present general discussion. We argue that the reason for selecting the deformed mass-shell condition (1) lies in quantum structure of coproduct – in the presence of symmetric coproduct such a choice is not distinguished at all. We shall show that under quite plausible assumptions there are only two choices of momentum coproduct (addition law of momenta): either one chooses a classical one (symmetric) or the one provided by $\kappa$-deformation. Unfortunately, quantum group structure is quite rigid and can not be cheated.

## 2 Classical relativistic symmetries in nonlinear disguise and DSR theories

Let us consider standard relativistic symmetries, described by the following classical Poincaré Hopf algebra

\[
\begin{align*}
[P^{(0)}_\mu, P^{(0)}_\nu] & = 0 & (5a) \\
[M^{(0)}_{\mu\nu}, P^{(0)}_\rho] & = i(g_{\mu\rho}P^{(0)}_\nu - g_{\nu\rho}P^{(0)}_\mu) & (5b) \\
[M^{(0)}_{\mu\nu}, M^{(0)}_{\rho\tau}] & = i(g_{\mu\rho}M^{(0)}_{\nu\tau} + g_{\nu\tau}M^{(0)}_{\mu\rho} - g_{\mu\tau}M^{(0)}_{\nu\rho} - g_{\nu\rho}M^{(0)}_{\mu\tau}) & (5c)
\end{align*}
\]

with primitive coproduct

\[
\Delta^{(0)}(I_A) = I_A \otimes 1 + 1 \otimes I_A \\
I_A = P^{(0)}_{\mu}, M^{(0)}_{\mu\nu}
\]

\[^3\text{In this paper we consider the hermitean generators of symmetry algebras.}\]
and bilinear mass Casimir representing relativistic dispersion formula 
\( (E^{(0)} = cP_0^{(0)}) \)

\[
C_2(E^{(0)}, \vec{P}^{(0)}) = (E^{(0)})^2 - c^2(\vec{P}^{(0)})^2 = \text{inv} \equiv m_0^2 c^4
\]  

(7)

Let us introduce by means of the nonlinear relations new fourmomenta \( P_\mu = (E, \vec{P}) \) (see also [19])

\[
E^{(0)} = \kappa^2 c f(E, P^2) \quad \vec{P}^{(0)} = \vec{P} g(E, P^2)
\]

(8)

where the masslike parameter \( \kappa \) describes the measure of nonlinearity 4. One assumes that

\[
f(0, 0) = 0 \quad \frac{\partial f}{\partial x}(0, 0) = g(0, 0) = 1
\]

(9)

what provides in the limit \( \kappa \to \infty \) the relations \( E = E^{(0)}, \vec{P} = \vec{P}^{(0)} \). In new variables \( E, \vec{P} \) the relation (7) looks as follows

\[
\kappa^2 c^2 f^2(E, \vec{P}^2) - \vec{P}^2 g^2(E, \vec{P}^2) = \text{inv} = m_0^2 c^2
\]

(10)

One can introduce two types of nonlinear transformations \( \mathfrak{S} \), leading to different nonlinear pictures of classical symmetries:

i) We assume that both functions \( f \) and \( g \) are analytic, and besides

\[
\lim_{x \to \infty} f(x, y) = \lim_{x \to \infty} g(x, y) = \infty \quad \lim_{x \to \infty} \frac{f^2(x, y)}{g^2(x, y)} = a < \infty
\]

(11)

In such a case one obtains in the limit \( E \to \infty \)

\[
\frac{\vec{P}^2}{\kappa^2 c^2} = \frac{\kappa^2 f^2 - m_0^2}{\kappa^2 g^2} \to \frac{a}{E \to \infty}
\]

(12)

or

\[
\lim_{E \to \infty} \frac{\vec{P}^2}{\kappa^2 c^2} = a \kappa^2 c^2
\]

(13)

Such type of theory is characterized at infinite energy by a maximal value \( \mathfrak{S} \) of three-momenta. An example of such a theory is DSR1 5

\footnote{If we assume for the functions \( f(x, y) \) and \( g(x, y) \) the Taylor expansion at \( x = 0 \), the linear term, proportional to \( 1/\kappa \) describes the leading nonlinear correction for large \( \kappa \).}
of Amelino-Camelia [1,2], which is obtained if for \( a = 1 \) we choose the functions \( f \) and \( g \) relating classical and bicrossproduct basis in \( \kappa \)-deformed theory [19,20]

\[
f(E, \vec{P}^2) = \sinh \frac{E}{\kappa c^2} + \frac{\vec{P}^2}{2\kappa^2 c^2} e^{\frac{E}{\kappa c^2}} \\
g(E, \vec{P}^2) = e^{\frac{E}{\kappa c^2}}
\]

Of course there is an infinite variety of theories having finite limit (13) for \( E \to \infty \).

ii) Other type of theories are obtained if we assume that \( f \) and \( g \) are singular for \( E = \kappa c^2 \), but in the limit \( E \to \kappa c^2 \) the quotient \( f/g \) is finite. Such a theory provides divergent value of \( E^{(0)} \) for \( E = \kappa c^2 \), because it maps the infinite range of energies \( E^{(0)} \) into finite interval \( 0 < E < \kappa c^2 \). A simple example of such nonlinear transformation (see also [21])

\[
f(E) = \frac{E}{\kappa c^2} \left( 1 - \frac{E}{\kappa c^2} \right)^{-1} \\
g(E) = \left( 1 - \frac{E}{\kappa c^2} \right)^{-1}
\]

provides the deformation of Lorentz symmetry proposed recently by Magueijo and Smolin [3].

In order to calculate the coproducts of deformed theories one should provide the formulae inverse to the relations (8) (we use the notation \( P^{(0)} \equiv |\vec{P}^{(0)}| \)):

\[
E = \kappa c^2 F(E^{(0)}, P^{(0)}) \\
\vec{P} = \vec{P}^{(0)} G(E^{(0)}, P^{(0)})
\]

Then we should use the formulae

\[
\Delta(E) = \kappa c^2 F \left( \Delta^{(0)}(E^{(0)}), \Delta^{(0)}(P^{(0)}) \right) \\
\Delta(\vec{P}) = \Delta^{(0)}(\vec{P}^{(0)}) G \left( \Delta^{(0)}(E^{(0)}), \Delta^{(0)}(P^{(0)}) \right)
\]

and reexpress on rhs of (17) the operators \( E^{(0)}, \vec{P}^{(0)} \) in terms of \( E, \vec{P} \) by means of the relations (8).

As an example one can calculate the fourmomentum coproducts (17) for the choice (14), which is the energy-momentum dispersion relation for bicrossproduct basis of \( \kappa \)-Poincaré algebra, further used as basic postulate of DSR1 theory. The inverse formulae look in this case as follows:

\[
F(E^{(0)}, P^{(0)}) = \ln D(E^{(0)}, P^{(0)}) \\
G(E^{(0)}, P^{(0)}) = D^{-1}(E^{(0)}, P^{(0)})
\]
where
\[
D(E^{(0)}, P^{(0)}) = \frac{E^{(0)}}{\kappa c^2} + \sqrt{1 + \left(\frac{E^{(0)}}{\kappa c^2}\right)^2 + \left(\frac{P^{(0)}}{\kappa c^2}\right)^2}
\] (19)

In linear approximation one gets
\[
E = E^{(0)} - \frac{1}{2\kappa} (\vec{P}^{(0)})^2 + O\left(\frac{1}{\kappa^2}\right) \quad \vec{P} = \vec{P}^{(0)} \left(1 - \frac{E^{(0)}}{\kappa c^2}\right) + O\left(\frac{1}{\kappa^2}\right) \] (20)

After the approximation to the formula (8) with the choice (14)
\[
E^{(0)} = E + \frac{\vec{P}^2}{2\kappa} \left(1 + \frac{E}{\kappa c^2}\right) + O\left(\frac{1}{\kappa^2}\right) \quad \vec{P}^{(0)} = \vec{P} \left(1 + \frac{E}{\kappa c^2}\right) + O\left(\frac{1}{\kappa^2}\right) \] (21)
one gets the following deformation of the primitive coproduct
\[
\Delta(E) = E \otimes 1 + 1 \otimes E - \frac{1}{\kappa} \vec{P} \otimes \vec{P} + O\left(\frac{1}{\kappa^2}\right) \quad \Delta(\vec{P}) = \vec{P} \otimes 1 + 1 \otimes \vec{P} - \frac{1}{\kappa c^2} (\vec{P} \otimes E + E \otimes \vec{P}) + O\left(\frac{1}{\kappa^2}\right) \] (22)

Comparing the composition laws (3), (4) for threemomenta and energy in DSR1 theory with (22) we see that they follow from the nonlinear realization of classical Poincaré algebra.

Further it can be shown that the \(\kappa\)-deformed boost transformations of DSR1 calculated in [17] can be obtained from classical Lorentz transformations after introducing the nonlinear change (16) of fourmomenta with the functions \(F\) and \(G\) given by (18). We start with the formulae for finite classical Lorentz boosts (we choose \(\vec{n}\) in the direction of relative velocities \(\vec{v} = c\vec{n} \tanh \alpha\) of two Lorentz frames)
\[
E^{(0)}(\alpha) = E^{(0)} \cosh \alpha - c \bar{\vec{n}} \vec{P}^{(0)} \sinh \alpha \\
\vec{P}^{(0)}(\alpha) = \vec{P}^{(0)} + \vec{n} \left[ (\vec{n} \vec{P}^{(0)})(\cosh \alpha - 1) - \frac{E^{(0)}}{c} \sinh \alpha \right] \] (23)

and the relations (16), (18) written in the form
\[
D(E^{(0)}, P^{(0)}) = \exp\left(\frac{E}{\kappa c^2}\right) \quad \vec{P} = D^{-1}(E^{(0)}, P^{(0)}) \vec{P}^{(0)} \] (24)
where \(D\) is given by (19). Because
\[
E(\alpha) = e^{i\alpha(\vec{n} \vec{N})} E^{(0)} e^{-i\alpha(\bar{\vec{n}} \vec{N})} = \kappa c^2 \ln D(E^{(0)}(\alpha), P^{(0)}(\alpha)) \\
\vec{P}(\alpha) = e^{i\alpha(\vec{n} \vec{N})} \vec{P} e^{-i\alpha(\bar{\vec{n}} \vec{N})} = D^{-1} \left(E^{(0)}(\alpha), P^{(0)}(\alpha)\right) \vec{P}^{(0)}(\alpha) \] (25)
and
\[
D(E(0)(\alpha), P(0)(\alpha)) = e^{i\alpha(\vec{n}\vec{N})}D(E(0), P(0))e^{-i\alpha(\vec{n}\vec{N})} = \\
= \frac{E(0)(\alpha)}{\kappa c^2} + \sqrt{1 + \frac{(E(0)(\alpha))^2}{\kappa^2 c^4} - \frac{(P(0)(\alpha))^2}{\kappa^2 c^4}} = \\
= \frac{E(0)(\alpha)}{\kappa c^2} + \sqrt{1 + \frac{(E(0))^2}{\kappa^2 c^4} - \frac{(P(0))^2}{\kappa^2 c^4}} = \\
= \frac{1}{\kappa c^2} \left( E(0)(\cosh \alpha - 1) - c(\vec{n}\vec{P}(0)) \sinh \alpha \right) + D(E(0), P(0))
\]

one can write
\[
W(\alpha) = \exp \left( \frac{E(\alpha) - E}{\kappa c^2} \right) = \\
= 1 + \frac{1}{\kappa c^2}e^{-\frac{E}{\kappa c^2}} \left[ E(0)(\cosh \alpha - 1) - c(\vec{n}\vec{P}(0)) \sinh \alpha \right] = \\
= 1 + B(E)(\cosh \alpha - 1) - \frac{(\vec{n}\vec{P})}{\kappa c} \sinh \alpha
\]

where
\[
B(E) = \frac{1}{2} \left( 1 - e^{-\frac{2E}{\kappa c^2}} + \frac{\vec{P}^2}{\kappa^2 c^4} \right)
\]

From (27) follows the boost transformations for energy
\[
\exp \left( \frac{E(\alpha) - E}{\kappa c^2} \right) = W(\alpha) \Rightarrow E(\alpha) = E + \kappa c^2 \ln W(\alpha)
\]

and for the threemomentum
\[
\vec{P}(\alpha) = \vec{P}(0)e^{-\frac{E(\alpha)}{\kappa c^2}} = W^{-1}(\alpha)D^{-1}(E(0), P(0))\vec{P}(0)(\alpha) = \\
W^{-1}(\alpha)D^{-1}(E(0), P(0)) \left( \vec{P}(0) + \vec{n} \left[ (\vec{n}\vec{P}(0))(\cosh \alpha - 1) - \frac{E(0)}{c} \sinh \alpha \right] \right) = \\
W^{-1}(\alpha) \left( \vec{P} + \vec{n} \left[ (\vec{n}\vec{P})(\cosh \alpha - 1) - \kappa c B(E) \sinh \alpha \right] \right)
\]

The formulae (29(a, b)) in simpler version when \(\vec{n} = (-1, 0, 0)\) were derived by other method in [17] as the finite deformed boost transformations in DSR1 theory.

### 3 \(\kappa\)-Deformed relativistic symmetries

The quantum deformation of Poincaré algebra with dimensionfull deformation parameter – the fundamental mass parameter \(\kappa\) – has been introduced in
1991 in so-called standard basis [4-6]; further this deformation was rewritten in bicrossproduct basis [7-11]. In bicrossproduct basis the Lorentz subalgebra remains undeformed

\[
[M_i, M_j] = i\epsilon_{ijk} M_k \\
[M_i, N_j] = i\epsilon_{ijk} N_k \\
[N_i, N_j] = -i\epsilon_{ijk} M_k
\] (30)

where \( \vec{M} = (M_1, M_2, M_3) \) generate space rotations, and \( \vec{N} = (N_1, N_2, N_3) \) the relativistic boosts. In the relations (30a)-(30d) only the relation (30b) is deformed. One gets

\[
[N_i, P_j] = \frac{i}{2}\delta_{ij} \left[ \kappa c \left( 1 - e^{-\frac{E}{\kappa c^2}} \right) \right] - \frac{i}{\kappa c} P_i P_j
\] (31)

All remaining Poincaré algebra relations remain classical.

The Casimir operator for the \( \kappa \)-deformed Poincaré algebra with deformed relations given by (3.2) is given by formula (1).

The Hopf algebra structure is provided by the following coproducts:

\[
\Delta(E) = E \otimes 1 + 1 \otimes E \\
\Delta(\vec{P}) = \vec{P} \otimes 1 + e^{-\frac{E}{\kappa c^2}} \otimes \vec{P} \\
\Delta(\vec{M}) = \vec{M} \otimes 1 + 1 \otimes \vec{M} \\
\Delta(N_i) = N_i \otimes 1 + e^{-\frac{E}{\kappa c^2}} \otimes N_i + \frac{1}{\kappa c} \epsilon_{ijk} P_j \otimes M_k
\] (32a)

(32b)

(32c)

(32d)

The “quantum inverse”, the antipode, is given by the relations

\[
S(E) = -E \\
S(\vec{P}) = -\vec{P} e^{\frac{E}{\kappa c^2}} \\
S(\vec{M}) = -\vec{M} \\
S(N_i) = -e^{\frac{E}{\kappa c^2}} N_i + \frac{1}{\kappa c} \epsilon_{ijk} e^{\frac{E}{\kappa c^2}} P_j M_k
\] (33)

We see that as Hopf algebra only the subalgebra \((\vec{M}, E)\) remains classical.

In order to obtain finite Lorentz transformations of fourmomenta which are consistent with Hopf algebra structure one should introduce the finite
boosts in the pseudo-Euclidean plane $(P_0, (\vec{n} \vec{P}))$ by the following formula (see also [9], Sect.2d.):

$$ P_\mu(\alpha) = \text{ad}_{e^{i\alpha(\vec{n} \vec{N})}} P_\mu $$

(34)

with the quantum adjoint action defined as follows (see e.g. [22])

$$ \text{ad}_Y X = Y^{(1)} X S(Y^{(2)}) $$

(35)

and (in symbolic notation) $\Delta(Y) = Y^{(1)} \otimes Y^{(2)}$. Applying the formula $\text{ad}_{Y_1 Y_2}(X) = \text{ad}_{Y_1}(\text{ad}_{Y_2}(X))$ one gets

$$ P_\mu(\alpha) = \sum_{k=0}^{\infty} \frac{(i\alpha)^k}{k!} \left( \text{ad}_{(\vec{n} \vec{N})} \left( \text{ad}_{(\vec{n} \vec{N})} \ldots (\text{ad}_{(\vec{n} \vec{N})} P_\mu) \ldots \right) \right) $$

(36)

Surprisingly enough, substituting in (35) the coproduct and antipode from (32b), (33) for arbitrary analytic function $h(P_0, P_1, P_2, P_3)$ one obtains

$$ \text{ad}_{(\vec{n} \vec{N})} h(P_0, P_1, P_2, P_3) = \left[ (\vec{n} \vec{N}), h(P_0, P_1, P_2, P_3) \right] $$

(37)

i.e. one gets the "classical formula" describing finite boost transformations for $\kappa$-deformed Poincaré algebra in bicrossproduct basis:

$$ P_\mu(\alpha) = e^{i\alpha(\vec{n} \vec{N})} P_\mu e^{-i\alpha(\vec{n} \vec{N})} $$

(38)

The differential equations following from the relation (38) (for particular choice $\vec{n} = (-1, 0, 0)$) were used in [17] for the calculation of finite boost transformations (29a, b) in DSR1 theory. We see therefore that the relation (38) can be obtained in two different ways from the formula (36)

i) Using the Hopf algebra structure of $\kappa$-deformed Poincaré algebra, as demonstrated above,

ii) By inserting in (36) the classical coproduct for the boost generators and nonlinear formulae (16), (18) describing threemomentum and energy.

Using the following formula for coproducts [22]

$$ \text{ad}_{\Delta X} \Delta Y = (\text{ad}_{X^{(1)}} Y^{(1)}) \otimes (\text{ad}_{\Delta X^{(2)}} Y^{(2)}) $$

(39)
one can also derive

\[ ad_{\Delta N_k} \Delta P_\mu = [N_k \otimes 1 + 1 \otimes N_k, \Delta P_\mu] \]  

(40)

where we recall that for \( \kappa \)-deformed Poincaré algebra the coproduct for the boosts \( N_i \) and three-momenta \( P_i \) is nonprimitive (see (32b,d)). Using (40) one can show that in general

\[
\vec{P}_\Delta (\alpha) \equiv \text{ad}_{e^{i\alpha(\vec{n} \Delta \vec{N})}} \Delta \vec{P} = e^{i\alpha(\vec{n} \vec{N})} \otimes e^{i\alpha(\vec{n} \vec{N})} \left( \vec{P} \otimes 1 + e^{-\frac{E}{\kappa \kappa c^2}} \otimes \vec{P} \right) e^{-i\alpha(\vec{n} \vec{N})} \otimes e^{-i\alpha(\vec{n} \vec{N})} = \]

(41a)

\[
\vec{E}_\Delta (\alpha) \equiv \text{ad}_{e^{i\alpha(\vec{n} \Delta \vec{N})}} \Delta \vec{E} = e^{i\alpha(\vec{n} \vec{N})} \otimes e^{i\alpha(\vec{n} \vec{N})} (E \otimes 1 + 1 \otimes E) e^{-i\alpha(\vec{n} \vec{N})} \otimes e^{-i\alpha(\vec{n} \vec{N})} = \]

(41b)

i.e. the quantum boost transformations (29a,b) for the coproduct of four-momenta are consistent with the \( \kappa \)-deformed addition law (32a,b).

One can conclude that for the deformed relativistic energy-momentum dispersion relation (1) there are two consistent addition laws for deformed four-momenta, one symmetric (cocommutative) and second noncocommutative.

i) The symmetric addition law is the result of nonlinear transformations of the three-momentum and energy generators in classical framework of relativistic symmetries. One obtains nonabelian symmetric addition law for both energy and three-momenta. Due to this symmetry (cocommutativity of the coproduct) the dual Minkowski space will be a standard one, with standard commutative space-time coordinates.

ii) The nonsymmetric ( noncocommutative ) addition law for deformed three-momenta permits in the framework of \( \kappa \)-deformed quantum group

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5In particular, \( ad_{N_k} P_i = [N_k, P_i] \Rightarrow \Delta (ad_{N_k} P_i) = [\Delta N_k, \Delta P_i] \neq [N_k \otimes 1 + 1 \otimes N_k, \Delta P_i] = ad_{\Delta N_k} \Delta P_i \). The equality sign holds for the primitive coproduct \( \Delta \vec{N} = \vec{N} \otimes 1 + 1 \otimes \vec{N} \), i.e. for undeformed boosts. In other words, \( \kappa \)-deformed momentum coalgebra is not the left (or right) Lorentz-module coalgebra (see [22,23])
to obtain the abelian addition law for energy. Due to noncocommutativity of the coproduct in the fourmomentum space, one obtains from Hopf algebra duality ($<x_\mu, P_\nu> = i g_{\mu\nu}$) \cite{22} the $\kappa$-deformed space-time \cite{7-11} ($i = 1, 2, 3$)

$$[x_0, x_i] = \frac{i}{\kappa c} x_i \quad [x_i, x_j] = 0 \quad (42)$$

If we are not requiring the classical additivity of energy one can consider any other bases for fourmomenta preserving bicrossproduct property. In particular

- one can describe $\kappa$-deformed framework with classical Poincaré basis \cite{19}
- the nonabelian addition law for energy \cite{20} can be interpreted as introducing geometric 2-particle interactions in standard relativistic theory.
- by performing the nonlinear transformations of fourmomenta it is possible to describe the $\kappa$-deformed Poincaré algebra in the basis which provides deformed energy-momentum dispersion relation given by Magueijo and Smolin \cite{3}, singular for energy $E = \kappa c^2$ (see \cite{21}).

## 4 Conclusions

It follows from our discussion that the introduction of large class of deformed energy-momentum dispersion relations (see (10)) preserved under deformed Lorentz transformations is easy to achieve by a nonlinear change of classical fourmomentum operators (see \cite{8} and \cite{16}). In such a framework all these nonlinearly related symmetry schemes are described by the same Hopf algebra – a mathematical tool for description of given algebraic symmetry. The real distinction between different deformed relativistic symmetry schemes is due to the choice of coproduct, in particular describing the composition law of energy-momentum for multiple states. The nontrivial deformations are obtained if the coproduct is not symmetric. The classification of nontrivial quantum deformations of Poincaré symmetries is known \cite{24} and it is almost complete. From physical reasons there are distinguished these deformations which preserve the commutativity of fourmomentum generators. In such a case it follows from the duality of Hopf algebras describing space-time coordinates and fourmomenta that there are possible only Lie - algebraic deformations of Minkowski space-time. Assuming that we keep classical $O(3)$
invariance only the following two deformations introducing fundamental mass parameter are possible:

i) $\kappa$-deformed Minkowski space, described by the relations (42). Such deformation is physically interesting, because it preserves undeformed nonrelativistic symmetries and the effective deformation parameters $P_0/\kappa c$ and $\vec{P}^2/\kappa^2 c^2$ shows that $\kappa$-deformation is a high energy / high momentum effect.

ii) other possible deformation covariant under classical $O(3)$ rotations is provided by relations:

\[
[\hat{x}_0, \hat{x}_i] = 0 \quad [\hat{x}_i, \hat{x}_j] = \frac{i}{\kappa} \varepsilon_{ijk} \hat{x}_k
\]  

(43)

The second formula (43), used as a description of fuzzy sphere (see e.g. [25,26]) modifies the nonrelativistic sector of classical relativistic symmetries, i.e. one can argue that this second choice for physical applications is less attractive.

We would like to stress that we are aware of difficulties with the adjustment to physical interpretation of nonsymmetric coproducts. It is also true that the complete structure of quantum group sometimes requires more sophisticated mechanisms to detect the deformation effects (see e.g. [27]). We would however to point out that whatever are the efforts to accommodate the nonsymmetric coproducts (32b) into the description of physical processes (see e.g. [28]), at present the alternative for DSR theories is clear: or classical Einsteinian framework or quantum $\kappa$-deformed symmetry, and declaration should be made. Also in discussions of modified relativity principles two formal steps which are sometimes mixed – the nonlinear change of basis and the notion of deformation – should be clearly separated.

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