Approximation for Probability Distributions by Wasserstein GAN

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Abstract

In this paper, we show that the approximation for distributions by Wasserstein GAN depends on both the width/depth (capacity) of generators and discriminators, as well as the number of samples in training. A quantified generalization bound is developed for Wasserstein distance between the generated distribution and the target distribution. It implies that with sufficient training samples, for generators and discriminators with proper number of width and depth, the learned Wasserstein GAN can approximate distributions well. We discover that discriminators suffer a lot from the curse of dimensionality, meaning that GANs have higher requirement for the capacity of discriminators than generators, which is consistent with the theory in [4]. More importantly, overly deep (high capacity) generators may cause worse results (after training) than low capacity generators if discriminators are not strong enough. Different from Wasserstein GAN in [3], we adopt GroupSort neural networks [2] in the model for their better approximation to 1-Lipschitz functions. Compared to some existing generalization (convergence) analysis of GANs, we expect our work are more applicable.

1 Introduction

Generative Adversarial Networks (GANs), first proposed by Goodfellow et al. [13], have become really popular in recent years for their remarkable abilities to approximate various kinds of distributions, e.g. common probability distributions in statistics, distributions of pixels in images [13, 27], and even for the natural language generation [6]. Several variants of GANs achieve much better performances not only in generation quality [3, 16, 24, 28, 32] but also in training stability [3, 7, 30] and efficiency [18, 20]. For more details about GANs, please refer to some latest reviews [1, 8, 12, 14].

Although empirical results have shown the success of GANs, theoretical understandings of GANs are still not sufficient. Even if the discriminator is fooled and the generator wins the game in training, we usually cannot answer whether the generated distribution $\mathcal{D}$ is close enough to the target distribution $\mathcal{D}_{targ}$. Arora [4] found that if the discriminator is not strong enough (with fixed architecture and some constraints on parameters), even though the learning has great generalization properties, the generated distribution $\mathcal{D}$ can be far away from the target distribution $\mathcal{D}_{targ}$. Based on the theoretical non-convergence of GANs (with constrained generators and discriminators), Bai et al. [5] designed some special discriminators with restricted approximability to let the trained generator approximate the target distribution in statistical metrics, i.e. Wasserstein distance. However, their
design of discriminators are not applicable enough because it can only be applied to some specific statistical distributions (e.g., Gaussian distributions and exponential families) and invertible as well as injective neural network generators. Liang [19] proposed a kind of oracle inequality to develop the generalization error of GANs’ learning in TV distance. His work shows the effectiveness of GANs in approximating probability distributions to some extent. However, several assumptions are required and KL divergence must be adopted in [19]. It may be inappropriate in generalization (convergence) analysis of GANs because many distributions in natural cases lie on low dimensional manifolds. The generated distribution must coincide well with the support of the target distribution, otherwise, the KL divergence between these two are infinite, which may be not helpful in generalization analysis. After all, error bounds are usually much more strict than empirical results.

Neural networks has been proved to have extraordinary expressive power. Two-layer neural networks with sufficiently large width can approximate any continuous function on compact sets [9, 15, 23]. Deep neural networks with unconstrained number of width and depth are capable of approximating wider ranges of functions [35, 36]. After the wide applications and researches of deep generative model, e.g. VAEs and GANs, researchers began to study the capacity of deep generative networks for approximating distributions. Yang et al. [34] recently proved that neural networks with sufficiently large width and depth can generate a distribution that is arbitrarily close to a high-dimensional target distribution in Wasserstein distances, by inputting only a one-dimensional source distribution (multi-dimensional source distribution has the same conclusion).

Although deep generative networks have capacities to approximate distributions, discriminators play especially important role in GANs. If the discriminator is not strong enough (e.g. ReLU neural networks with limited architectures and constrained parameters), it can easily be fooled by the generator even with imperfect parameters. Both empirical and theoretical results show that GroupSort Neural Networks [2, 31] can approximate any 1-Lipschitz function arbitrary close in a compact set with large enough width and depth. Therefore, in our generalization analysis of Wasserstein GAN, we adopt GroupSort Neural Networks as discriminators.

Motivated by the above mentioned research works, in this paper, we study the approximation for distributions by Wasserstein GAN. Our contributions are given as follows.

1. We connect the capacity of deep generative networks and GroupSort networks and derive a generalization bound for Wasserstein GAN with finite samples. It implies that with sufficient training samples, for generators and discriminators with proper number of width and depth, the Wasserstein GAN can approximate distributions well. To the best of our knowledge, this is the first paper deriving the generalization bound in Wasserstein distance of Wasserstein GAN in terms of the depth/width of the generators (ReLU deep neural networks) and discriminators (GroupSort Neural Networks), as well as the number of both generated and real samples in training.

2. We analyse the importance for both generators and discriminators in GANs. GANs have higher requirement for the capability of discriminators than generators according to our derived generalization bound. We show that for nonparametric discriminators (discriminators are strong enough), the generalization is easier to be obtained as the GroupSort neural networks (as discriminators) suffer a lot in the curse of dimensionality.

3. With the growing capacity (width and depth) of generators, width and depth of discriminators should also be increased, otherwise, the error will be augmented because of the big difference between generators from the same generator class. In other words, overly deep (high
capacity) generators may cause worse results (after training) than low capacity generators if discriminators are not strong enough.

The outline of the paper is given as follows. In Section 2, we give notations and mathematical preliminaries. In Section 3, we study generalization analysis for Wasserstein GAN and show the above mentioned theoretical results. Finally, some concluding remarks are given in Section 4.

2 Notations and Preliminaries

2.1 Notations

Throughout the paper, we use $\nu$, $\pi$ and $\mu_\theta$ to denote the target distribution, the source distribution (as an input to the generator) and the generated distribution (with $\theta$ as the parameters of the generator) respectively. More concretely, $\mu_\theta$ is the distribution of $g_\theta(Z)$, $Z \sim \pi$, where $g_\theta$ is a generator. The source distribution $\pi$ lies on $\mathbb{R}^r$ and both the target and the generated distribution are defined on $\mathbb{R}^d$, where $r$ and $d$ are the number of dimensions. In section 3, we use $m$ for number of generated samples from $\mu_\theta$ (the same number for samples from the source distribution $\pi$) and $n$ for number of real data from the target distribution $\nu$. The symbol $\lesssim$ is used to hide some unimportant constant factors. For two functions $\ell$, $h$ of a number $n$, $\ell \lesssim h$ means $\lim_{n \to +\infty} \ell/h \leq C$ for a non-negative constant $C$. For two positive real numbers or functions $A$ and $B$, the term $A \lor B$ is equivalent to $\max\{A, B\}$ while $A \land B$ is equivalent to $\min\{A, B\}$.

2.2 Preliminaries

Slightly different from the original model in [13], Generative Adversarial Networks (GANs) can be formulated in a more general form:

$$\min_{g_\theta \in \mathcal{G}} \max_{f \in \mathcal{F}} \mathbb{E}_{Z \sim \pi} f(g_\theta(Z)) - \mathbb{E}_{X \sim \nu} f(X)$$

(1)

where $\mathcal{G}$ is the generator class and $\mathcal{F}$ is the discriminator class. Here we assume that our predefined discriminator class is symmetric, i.e. $f \in \mathcal{F}$ and $-f \in \mathcal{F}$. The symmetry condition is easy to be satisfied, e.g. feed-forward neural networks without activation functions in the last layer. Intuitively, the capacity of generator class $\mathcal{G}$ determines the upper bound for the quality of generated distributions and the discriminator class $\mathcal{F}$ improves learning, pushing the learned parameters close to the optimal one. In practise, we can only access finite samples of target distribution and for lower computation, also finite generated data is considered. Empirically, instead of dealing with (1), we need to solve the following adversarial problem and obtain the estimated (learned) parameters of the generator:

$$\hat{\theta}_{m,n} = \min_{g_\theta \in \mathcal{G}} \max_{f \in \mathcal{F}} \{ \hat{\mathbb{E}}_Z^m f(g_\theta(Z)) - \hat{\mathbb{E}}_X^n f(X) \}$$

(2)

where $\hat{\mathbb{E}}_Z^m$ (and $\hat{\mathbb{E}}_X^n$) denotes the expectation on the empirical distribution of $m$ (and $n$) i.i.d. samples of random variables $Z$ (and $X$) from $\pi$ (and $\nu$). Therefore, the learned generator $g_{\hat{\theta}_{m,n}}$ is obtained from (2).

Suppose that we have the following generator class $\mathcal{G}$:

$$\mathcal{G} = \{ g_\theta = NN(W_g, D_g, \theta) : \|W_i\|_{op} \leq M, \|b_i\| \leq M, 1 \leq i \leq D_g, \theta = \{W, b\} \}$$

(3)
where $NN(W_g, D_g, \theta)$ denotes Neural Networks with $D_g$ layers, $W_g$ neurons in each layer (width), and $\theta$ is the set of parameters. We use $W_i$ to represent the linear transform matrix between $(i-1)$-th and $i$-th layer. Here, we adopt ReLU as the activation function $\sigma$. The architecture of the neural networks is formulated as

$$NN(W_g, D_g, W) = W_{D_g} \circ \sigma(W_{D_{g-1}} \circ \sigma(\cdots \circ \sigma(W_1 x + b_1) + \cdots) + b_{D_{g-1}}) + b_{D_g}$$

For Wasserstein GAN [3], the discriminator class $\mathcal{F}_W$ is

$$\mathcal{F}_W = \{ f : ||f||_{Lip} \leq 1 \}$$

where $||f||_{Lip}$ is the Lipschitz constant of $f$. In section 3.1, we use $\mathcal{F}_W$ as the discriminator class. However, the discriminator class $\mathcal{F}_W$ is empirically impractical. In practice, several strategies are adopted to approximate $\mathcal{F}_W$ in Wasserstein GAN. In this paper, we choose GroupSort neural networks [2][31] as the discriminator class. A GroupSort neural network is defined as

$$GS(W_f, D_f, \alpha) = \mathcal{V}_{D_f} \circ \sigma_k(\mathcal{V}_{D_f-1} \circ \sigma_k(\cdots \circ \sigma_1(\mathcal{V}_1 x + c_1) + \cdots) + c_{D_f-1}) + c_{D_f}$$

where $\alpha = \{ \mathcal{V}, c \}$ denotes parameters of the GroupSort neural network, $W_f$ is the width of each layer, $D_f$ is the depth (number of layers) and $\sigma_k$ is the GroupSort activation function with grouping size $k$. In the paper, $k = 2$. The Discriminator class is defined as

$$\mathcal{F}_{GS} = \{ \hat{f}_{\alpha} = GS(W_f, D_f, \alpha) : GS(W_f, D_f, \alpha) \text{ is a Neural Network of form (5)} \}$$

For more kinds of generator and discriminator classes, please refer to [18][25][26][29].

Several kinds of probability metrics are available to measure the distance between two distributions. Empirical and theoretical results show their great performances [10][11][17][21]. It has been stated above that $KL$ divergence and $JS$ divergence (a special case of $KL$ divergence) are not suitable for generalization (convergence) analysis of GANs due to the low dimensional properties of distributions. Wasserstein distance is a more relaxed and flexible metric and can generalize with some mild conditions (3-Moment of the distribution is finite). In our analysis, we adopt Wasserstein-1 distance ($W_1$) as the evaluation metric between the generated and the target distributions. The Wasserstein-p Distance ($p \geq 1$) between two distributions $\mu$ and $\nu$ is defined as

$$W_p(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu, \nu)} \int ||x - y||^p d(\gamma(x, y))$$

where $\Gamma(\mu, \nu)$ denotes the set of all joint distributions $\gamma(x, y)$ whose marginals are $\mu$ and $\nu$ respectively. When $p = 1$, Kantorovich-Rubinstein duality [33] gives an alternative definition of $W_1$:

$$W_1(\mu, \nu) = \sup_{||f||_{Lip} \leq 1} |E_{X \sim \mu} f(X) - E_{X \sim \nu} f(X)| \leq \frac{1}{2} \inf_{\gamma \in \Gamma(\mu, \nu)} \int ||x - y||^2 d(\gamma(x, y))$$

where $\inf_{\gamma \in \Gamma(\mu, \nu)} \int ||x - y||^2 d(\gamma(x, y))$ is the $W_2$ distance between $\mu$ and $\nu$.

The last equation holds because $||f||_{Lip} = ||f||_{Lip}$. 

The remaining part studies how well the trained generator (obtained from (2)) is, with respect to its Wasserstein-1 distance to the target distribution $\nu$ (i.e. $W_1(\hat{\mu}_{\theta_{m,n}}, \nu)$) for pre-defined generator class $\mathcal{G}$ and discriminator class $\mathcal{F}$. 

4
3 Generalization Analysis for Wasserstein GAN

3.1 Nonparametric Discriminators

In this part, we let the discriminator class \( \mathcal{F} \) in (2) to be \( \mathcal{F}_W \) (defined in (4)) and the estimated generator \( g_{\theta_{m,n}} \) is obtained from (2). Combining with Lemma 2 the following assumption guarantees that with sufficiently many samples, the empirical distributions \( \hat{\nu} \) and \( \hat{\pi} \) are close (convergent) to \( \nu \) and \( \pi \) in \( W_1 \) distance, respectively. The convergence requirement is essential because they are the base (the source and the target) for GANs, if their empirical distributions are not convergent, let alone distributions from learned generators.

Assumption 1. Both the \( d \) dimensional target distribution \( \nu \) and the \( r \) dimensional source distribution \( \pi \) have finite 3-Moment, i.e. \( M_3(\nu) = \mathbb{E}_{X \sim \nu}|X|^3 < \infty \) and \( M_3(\pi) = \mathbb{E}_{Z \sim \pi}|Z|^3 < \infty \).

Theorem 1 (Generalization of Wasserstein GAN with nonparametric discriminators). Suppose that Assumption 1 holds for the target distribution \( \nu \) (lies in \( \mathbb{R}^d \)) and the source distribution \( \pi \) (lies on \( \mathbb{R}^r \)). For GANs, the generator class \( \mathcal{G} \) is defined in (3) and the discriminator class \( \mathcal{F} = \mathcal{F}_W \) (defined in (4)), i.e. the Wasserstein GAN. Let the learned parameters \( \hat{\theta}_{m,n} \) be obtained from (2), then the following holds,

\[
\mathbb{E} W_1(\mu_{\hat{\theta}_{m,n}}, \nu) \leq \min_{g_\theta \in \mathcal{G}} W_1(\mu_\theta, \nu) + C_1 \cdot M_{D_g} \cdot \begin{cases} m^{-1/2}, & r = 1 \\ m^{-1/2} \log m, & r = 2 \\ m^{-1/r}, & r \geq 3 \end{cases} \cdot \begin{cases} n^{-1/2}, & d = 1 \\ n^{-1/2} \log n, & d = 2 \\ n^{-1/d}, & d \geq 3 \end{cases}
\]

where constants \( C_1 \) and \( C_2 \) are independent of \( M, D_g, W_g, m \) and \( n \).

Lemma 1 (Capacity of deep generative networks to approximate distributions [34]). Let \( p \in [1, \infty) \) and \( \pi \) be an absolutely continuous probability distribution on \( \mathbb{R} \). Assume that the target distribution \( \nu \) is on \( \mathbb{R}^d \) with finite absolute \( q \)-moment \( M_q(\nu) < \infty \) for some \( q > p \). Suppose that \( W \geq 7d + 1 \) and \( D \geq 2 \). Then there exists a neural network \( g_\theta = NN(W, D, \theta) \) such that the generated distribution \( \mu_\theta \) of \( g_\theta(Z) \), \( Z \sim \pi \) has the following property:

\[
W_p(\mu_\theta, \nu) \leq C \left( M_q^*(\nu) + 1 \right)^{1/p} \begin{cases} (W^2D)^{-1/d}, & q > p + p/d \\ (W^2D)^{-1/d} (\log_2 W^2D)^{1/d}, & p < q \leq p + p/d \end{cases}
\]

where \( C \) is a constant depending only on \( p, q \) and \( d \).

Remark 1. Although Lemma 1 holds for 1-dimensional source distribution, the result can be easily generalized to absolutely continuous distributions simply by linear projections.

Based on Lemma 1, we set \( p = 1 \) and \( q = 3 \), the generative net can transform the source distribution \( \pi \) to a distribution that are arbitrary close to the target distribution \( \nu \). That means letting \( W_g, D_g \) and \( M \) be large enough, \( \min_{g_\theta \in \mathcal{G}} W_1(\mu_\theta, \nu) \) can be arbitrarily small, i.e., with sufficiently large \( W_g, D_g \) and \( M \),

\[
\min_{g_\theta \in \mathcal{G}} W_1(\mu_\theta, \nu) \approx 0
\]

With the fixed generator class \( \mathcal{G} \) (\( W_g, D_g \) and \( M \) are fixed), letting \( m \) and \( n \) large enough, by (8) we have

\[
\mathbb{E} W_1(\mu_{\hat{\theta}_{m,n}}, \nu) \approx \min_{g_\theta \in \mathcal{G}} W_1(\mu_\theta, \nu) \approx 0
\]
Because $W_1(\mu_{\theta_{m,n}}, \nu) \geq 0$, we deduce that

$$W_1(\mu_{\hat{\theta}_{m,n}}, \nu) \approx 0,$$

for most of the cases.

Lemma 2 provides the upper bound estimation for $\min_{g\in\mathcal{G}} W_1(\mu_\theta, \nu)$, however, for a concrete bound of $\mathbb{E}W_1(\mu_{\hat{\theta}_{m,n}}, \nu)$, we need further assumptions because the generator class in Lemma 1 is not bounded. However, for a fixed target distribution $\nu$ and fixed number of layers ($\|W\| \leq M$), it’s reasonable to assume that Lemma 1 holds for the generator class with bounded parameters and fixed number of layers ($\|W\| \leq M$ for sufficient large $M$ and $D_g$ is fixed but $W_g$ is also bounded by $M$). In the following Assumption 3, we constrain the upper bound of $W_g$ because wider matrix can have larger operation norm.

**Assumption 2.** For fixed target distribution $\nu$ and source distribution $\pi$, generator class $\mathcal{G}$ is defined in (3). Suppose that there exists a fixed depth $D_g$ ($D_g \geq 2$), and let $M$ be large enough that for all $7d + 1 \leq W_g \leq M^{1/2}$, the bound of $W_1(\mu_\theta, \nu)$ in Lemma 1 holds, i.e.

$$\min_{g\in\mathcal{G}} W_1(\mu_\theta, \nu) \leq C \cdot W_g^{-2/d} \quad (9)$$

where $C$ depends on $M_3(\nu)$, $d$ and $D_g$.

**Remark 2.** As we can see that $\|W\|_{op}$ usually grows with increasing width (i.e. $W_g$), that’s why we bound $W_g$ by $M^{1/2}$. The assumption can be revised to other similar forms, as long as $W_g$ is bounded by a monotonically increasing function of $M$. For large enough $m$, we assume that the Assumption 2 holds for some $M \leq m^{1/3r}D_g$.

**Theorem 2** (Generalization bound for Wasserstein GAN with nonparametric discriminators). Suppose that Assumption 1 holds for the target distribution $\nu$ (lies on $\mathbb{R}^d$) and the source distribution $\pi$ (lies on $\mathbb{R}^n$) respectively. For GANs, the generator class $\mathcal{G}$ is defined in (3) and the discriminator class $\mathcal{F} = \mathcal{F}_W$ (defined in (2)), i.e. the Wasserstein GAN. Further assume that Assumption 2 holds for the generators with fixed depth $D_g$ ($D_g \geq 2$), large enough $M \leq m^{1/3r}D_g$, and $7d + 1 \leq W_g \leq M^{1/2}$. Let the parameters $\theta_{m,n}$ be obtained from (2), then the following holds,

$$\mathbb{E}W_1(\mu_{\theta_{m,n}}, \nu) \leq C \cdot W_g^{-2/d} + C_1 \cdot m^{1/3r} \cdot \begin{cases} m^{-1/2}, & r = 1 \\ m^{-1/2} \log m, & r = 2 + C_2 \cdot \begin{cases} n^{-1/2}, & d = 1 \\ n^{-1/2} \log n, & d = 2 \\ n^{-1/d}, & d \geq 3 \end{cases} \end{cases} \quad (10)$$

for large enough $m$ and $n$, where constant $C$ depends on $M_3(\nu)$, $d$ and $D_g$; constant $C_1$ depends on $M_3(\pi)$ and $d$; and constant $C_2$ depends on $M_3(\nu)$ and $d$.

**Proof.** Based on the above Theorem 1 and Assumption 2 (10) can be easily obtained.

### 3.2 Parametric Discriminators (GroupSort Neural Networks)

It’s impractical to use $\mathcal{F}_W$ as the discriminator class $\mathcal{F}$. As one of the approximation strategies, we adopt GroupSort neural networks as discriminator class, i.e. $\mathcal{F} = \mathcal{F}_{GS}$. The following Assumption 3 guarantees that all the GroupSort Neural Networks in $\mathcal{F}_{GS}$ are 1-Lipschitz.

**Assumption 3.** For all $\alpha = (\mathcal{V}_1, \ldots, \mathcal{V}_{D_f}, c_1, \ldots, c_{D_f})$,

$$\|\mathcal{V}_1\|_{2,\infty} \leq 1 \quad \text{and} \quad \max \left(\|\mathcal{V}_2\|_{\infty}, \ldots, \|\mathcal{V}_{D_f}\|_{\infty}\right) \leq 1$$

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Assumption 4 (Decaying condition). Suppose that for a large enough number $\tilde{M} \geq M^{2D_g}$ and $s \geq 3$, there exists a constant $C$ such that
\[
\mathbb{P} \left( \|g_\theta(Z)\| \geq \tilde{M} \right) \leq C \cdot \tilde{M}^{-s} \\
\mathbb{P} \left( \|X\| \geq \tilde{M} \right) \leq C \cdot \tilde{M}^{-s}
\]
hold for all $g_\theta \in \mathcal{G}$.

Remark 3. One of the reasons we bound the parameters of generator class are for the above decaying condition (Assumption 4). For example, if the source distribution $Z \sim \pi$ is standard Gaussian $\mathcal{N}(0, I_r)$ (one of the most accepted source distributions in GANs), then for all $g_\theta \in \mathcal{G}$ (which is $M^{D_g}$ Lipschitz), $g_\theta(Z)$ is sub-Gaussian. To be more concrete, for any $g_\theta \in \mathcal{G}$, we have
\[
\|g_\theta(Z)\| \leq M^{D_g} \|Z\| + D_g \cdot M^{D_g}
\]
therefore,
\[
\mathbb{P} \left( \|g_\theta(Z)\| \geq \tilde{M} \right) \leq \mathbb{P} \left( \|Z\| \geq M^{-D_g} (\tilde{M} - D_g \cdot M^{D_g}) \right) \leq \mathbb{P} \left( \|Z\| \geq M^{-D_g} \tilde{M} \right)
\]

When $\tilde{M} \geq M^{2D_g}$ and $Z$ is from standard Gaussian distribution, then there exists a universal constant $C$ such that
\[
\mathbb{P} \left( \|Z\| \geq M^{-D_g} \tilde{M} \right) \leq C \cdot \tilde{M}^{-s}
\]
because $\mathbb{P} \left( \|Z\| \geq \tilde{M} \right) \leq C' \cdot \exp(-\tilde{M}^2/2)$. Here the depth $D_g$ is always fixed. Any bounded distributions (i.e. values of samples are bounded almost everywhere) apparently satisfy the above assumption. Moreover, with the decaying condition, Assumption 4 for the finite 3-moment of $\nu$ is automatically satisfied.

Theorem 3 (Generalization bound for Wasserstein GAN with parametric discriminators). Suppose that Assumption 1 holds for the target distribution $\nu$ (lies on $\mathbb{R}^d$) and the source distribution $\pi$ (lies on $\mathbb{R}^r$) respectively. For GANs, the generator class $\mathcal{G}$ is defined in (4) and the discriminator class $\mathcal{F} = \mathcal{F}_{\text{GS}}$ (defined in (6) and satisfying Assumption 2, i.e. the parametric Wasserstein GAN (GroupSort Neural Networks as discriminators). Further assume that Assumption 2 holds for the generators with fixed depth $D_g$ ($D_g \geq 2$), large enough $M \lesssim m^{1/3rD_g}$ and $7d + 1 \leq W_f \leq M^{1/2}$. Assumption 4 holds for some $s \geq 3$ and $\tilde{M} \geq M^{2D_g}$. Let the learned parameters $\hat{\theta}_{m,n}$ be obtained from (2), then the following holds,
\[
\mathbb{E}W_1(\mu_{\hat{\theta}_{m,n}}, \nu) \lesssim W_g^{-2/d} + \left( m^{-1/6} \log m \lor m^{-2/3r} \right) + \left( n^{-1/2} \log n \lor n^{-1/d} \right) + \left( 2^{-D_f/Cd^2} \lor W_f^{-1/2d^2} \right) + D_f^{-(s-1)} \lor W_f^{-(s-1)/2d^2} \tag{11}
\]
when $m$, $n$, $D_f$ and $W_f$ are sufficiently large and $M^{2D_g} \lesssim \tilde{M} = W_f^{1/2d^2} \lor D_f$. Here, $\lesssim$ hides the parameters w.r.t $d$, $r$, $D_g$ and some universal constants. $C$ is a universal constant.

Remark 4. The first term in the right hand of (11) represents the error from generator class. If we fix the number of layers and let the bound of parameters large enough, the the capacity of generators improves with increasing width (with Assumption 2). The second term is the generalization error of generators with only finite number of samples. As we can see this term is actually related to the
diversity of the generator ($M$ and $D_g$), that’s why we constrain the number of layers and bound the parameters. The third term is generalization error from the target distribution because of finite number of observed data. And last two terms come from the approximation of GroupSort Neural Networks to 1-Lipschitz functions. The error is small if the width and depth of discriminators are large enough (the error decays exponentially and polynomially with depth and width, respectively), which implies the capacity and ability of Neural Networks. Noted that $M^{2D_g} \lesssim \hat{M} = W_f^{1/2d^2} \wedge D_f$ shows severe curse of dimensionality in Wasserstein GAN’s error bound and it also means that GANs require strong discriminators, consistent with some famous empirical and theoretical results. Moreover, $W_g \leq M^{1/2}$ and $M^{2D_g} \lesssim \hat{M} = W_f^{1/2d^2} \wedge D_f$ also indicate strict requirement for discriminators in GANs. It implies that overly deep (high capacity) generators may cause worse results than low capacity generators if discriminators are not strong enough. Based on this, we have reasons to believe that the mode collapse in GANs may attribute to low capacity of discriminator class. The main difference between Theorem 2 and Theorem 3 is the error from the approximation of $\mathcal{F}_{GS}$ to the class of all 1-Lipschitz functions.

Remark 5. Liang [19] derived a generalization bound for the generated distribution and the target distribution in TV distance. The main idea (oracle inequalities) is similar to our method. However, the bound involves KL divergence, which may be infinite if two distributions does not coincide on nonzero measure. Moreover, the pair regularization in his paper requires specially designed architectures of discriminators which may not widely applicable. After all, error bounds are usually much more strict than empirical results. The results from Liang show the effectiveness of GANs in approximating probability distributions to some extent.

4 Conclusion

We present an explicit error bound for the convergence of Wasserstein GAN (with GroupSort neural networks as discriminators) in Wasserstein distance. We show that with sufficient training samples, for generators and discriminators with proper number of width and depth (satisfying Theorem 3), the Wasserstein GAN can approximate distributions well. The error comes from finite number of samples, capacity of generators and the approximation of discriminators to 1-Lipschitz functions. It shows strict requirements for the capacity of discriminators, especially for their width. More importantly, overly deep (high capacity) generators may cause worse results than low capacity generators if discriminators are not strong enough.

We hope our work can help researchers understand properties of GANs. For better results, besides training with more samples, we should simultaneously improve the capacity (width and depth) of generators and discriminators. However, empirical works need to balance between the efficiency and effectiveness. It’s really reasonable as similar results also hold for other deep models.

5 Proof of Main Results in Section 3

5.1 Proof of Theorem 1

For any $\hat{\theta}_{m,n}$ obtained from 2, we have

$$W_1(\mu_{\hat{\theta}_{m,n}}, \nu) = \sup_{||f||_{L_1} \leq 1} \{E_{Z \sim \pi} f(g_{\hat{\theta}_{m,n}}(Z)) - E_{X \sim \nu} f(X)\}$$
\[
\begin{align*}
&\leq \sup_{||f||_{L^p}} \{E_{Z \sim \pi} f(g_{\theta_{m,n}}(Z)) - \hat{E}_m^n f(X) \} + \sup_{||f||_{L^p}} \{E_{X \sim \nu} f(X) - E_{\pi} f(f) \} \\
&\leq \sup_{||f||_{L^p}} \{E_{Z \sim \pi} f(g_{\theta_{m,n}}(Z)) - \hat{E}_m^n f(g_{\theta_{m,n}}(Z)) \} + \sup_{||f||_{L^p}} \{E_{Z \sim \pi} f(g_{\theta}(Z)) - \hat{E}_m^n f(g_{\theta}(Z)) \} \\
&\quad + \sup_{||f||_{L^p}} \{E_{X \sim \nu} f(X) - E_{\pi} f(f) \} \quad \text{it holds for any } g_\theta \in G \text{ due to (2)}.
\end{align*}
\]

Note that
\[
W_1(\mu_{\theta_{m,n}}, \hat{\mu}_{\theta_{m,n}}) = \sup_{||f||_{L^p}} \{E_{Z \sim \pi} f(g_{\theta_{m,n}}(Z)) - \hat{E}_m^n f(g_{\theta_{m,n}}(Z)) \}
\]
\[
W_1(\mu_\theta, \hat{\mu}_\theta) = \sup_{||f||_{L^p}} \{E_{Z \sim \pi} f(g_\theta(Z)) - E_{Z \sim \pi} f(g_\theta(Z)) \}
\]
\[
W_1(\mu_\theta, \nu) = \sup_{||f||_{L^p}} \{E_{Z \sim \pi} f(g_\theta(Z)) - E_{X \sim \nu} f(X) \}
\]
\[
W_1(\nu, \hat{\nu}) = \sup_{||f||_{L^p}} \{E_{X \sim \nu} f(X) - \hat{E}_m^n f(X) \}
\]

where \(\hat{\mu}_\theta^m\) is an empirical distribution of m i.i.d. sample from \(\mu_\theta\) and \(\hat{\nu}^n\) is an empirical distribution of n iid. sample from \(\nu\).

Furthermore, the inequality holds for any \(g_\theta \in G\), so we take the minimum that
\[
W_1(\mu_{\theta_{m,n}}, \nu) \leq W_1(\mu_{\theta_{m,n}}, \hat{\mu}_{\theta_{m,n}}) + \min_{g_\theta \in G} \{W_1(\mu_\theta, \hat{\mu}_\theta) + W_1(\mu_\theta, \nu) \} + 2W_1(\nu, \hat{\nu}^n)
\]
\[
\leq \max_{g_\theta \in G} W_1(\mu_\theta, \hat{\mu}_\theta) + \min_{g_\theta \in G} \{W_1(\mu_\theta, \nu) + W_1(\mu_\theta, \nu) \} + 2W_1(\nu, \hat{\nu}^n)
\]

Taking the expectation on both m i.i.d. samples of \(Z\) from \(\pi\) and n i.i.d. samples of \(X\) from \(\nu\), we have
\[
\mathbb{E}W_1(\mu_{\theta_{m,n}}, \nu) \leq \mathbb{E} \max_{g_\theta \in G} W_1(\mu_\theta, \hat{\mu}_\theta) + \mathbb{E} \min_{g_\theta \in G} \{W_1(\mu_\theta, \hat{\mu}_\theta) + W_1(\mu_\theta, \nu) \} + 2\mathbb{E}W_1(\nu, \hat{\nu}^n)
\]
\[
\leq \mathbb{E} \max_{g_\theta \in G} W_1(\mu_\theta, \hat{\mu}_\theta) + \min_{g_\theta \in G} \{\mathbb{E}W_1(\mu_\theta, \hat{\mu}_\theta) + W_1(\mu_\theta, \nu) \} + 2\mathbb{E}W_1(\nu, \hat{\nu}^n)
\]
\[
\leq \mathbb{E} \max_{g_\theta \in G} W_1(\mu_\theta, \hat{\mu}_\theta) + \min_{g_\theta \in G} W_1(\mu_\theta, \nu) + \max_{g_\theta \in G} \{\mathbb{E}W_1(\mu_\theta, \hat{\mu}_\theta) \} + 2\mathbb{E}W_1(\nu, \hat{\nu}^n)
\]

The last inequality holds because
\[
\min_{g_\theta \in G} \{\mathbb{E}W_1(\mu_\theta, \hat{\mu}_\theta) + W_1(\mu_\theta, \nu) \} \leq \min_{g_\theta \in G} W_1(\mu_\theta, \nu) + \max_{g_\theta \in G} \{\mathbb{E}W_1(\mu_\theta, \hat{\mu}_\theta) \}
\]
Lemma 2 (Convergence in 1-Wasserstein distance [22]). Consider $F = \{ f : \mathbb{R}^d \to \mathbb{R} : \| f \|_{Lip} \leq 1 \}$. Assume that the distribution $\nu$ satisfies that $M_3 = \mathbb{E}_{X \sim \nu} |X|^3 \leq \infty$. Then there exists a constant $C$ depending on $M_3$ such that

$$\mathbb{E} W_1(\nu, \hat{\nu}^n) \leq C \cdot \begin{cases} n^{-1/2}, & d = 1 \\ n^{-1/2} \log n, & d = 2 \\ n^{-1/d}, & d \geq 3 \end{cases}$$

From the above Lemma 2 and Assumption 1, we have that

$$\mathbb{E} W_1(\nu, \hat{\nu}^n) \leq C_1 \cdot \begin{cases} n^{-1/2}, & d = 1 \\ n^{-1/2} \log n, & d = 2 \\ n^{-1/d}, & d \geq 3 \end{cases}$$

(13)

where constant $C_1$ depends only on 3-Moment of the target distribution $\nu$, irrelevant to the number of samples.

For any $g_\theta \in G$, we have

$$W_1(\mu_\theta, \hat{\mu}_n^\theta) = \sup_{\| f \|_{Lip} \leq 1} \{ \mathbb{E}_{Z \sim \theta} f(g_\theta(Z)) - \hat{\mathbb{E}}_{Z \sim \theta} m f(g_\theta(Z)) \}$$

$$\leq \sup_{\| f \|_{Lip} \leq M_{D_\theta}} \{ \mathbb{E}_{Z \sim \pi} f(Z) - \hat{\mathbb{E}}_{Z \sim \pi} m f(Z) \}$$

(14)

$$= M_{D_\theta} \cdot W_1(\pi, \hat{\nu}^m)$$

The inequality holds because for any $g_\theta \in G$, $\| W_1 \| \leq M$, then

$$\| \nabla g_\theta \| \leq M_{D_\theta}$$

Therefore, $\| g_\theta \|_{Lip} \leq M_{D_\theta}$ and $f \circ g_\theta$ is a $M_{D_\theta}$-Lipschitz function. Similarly, we have

$$\mathbb{E} W_1(\pi, \hat{\nu}^m) \leq C_2 \cdot \begin{cases} m^{-1/2}, & r = 1 \\ m^{-1/2} \log m, & r = 2 \\ m^{-1/r}, & r \geq 3 \end{cases}$$

(15)

Combining (12), (13), (14) and (15), we prove the Theorem 1.

5.2 Proof of Theorem 3

Proposition 1 (31). Assume that Assumption 3 is satisfied. Then, for any $k \geq 2$, $F_{GS}$ defined in (2) is a subset of $L$-Lipschitz functions defined on $\mathbb{R}^d$, i.e. $F_{GS} \subseteq F_W$.

For any $f \in F_W = \{ f : \| f \|_{Lip} \leq 1 \}$, suppose that $g_{\theta_m,n}$ is obtained from (2), we have

$$\mathbb{E}_{Z \sim \pi} f(g_{\theta_m,n}(Z)) - \mathbb{E}_{X \sim \nu} f(X)$$

$$= \mathbb{E}_{Z \sim \pi} f(g_{\theta_m,n}(Z)) - \mathbb{E}_{Z \sim \pi} \hat{f}(g_{\theta_m,n}(Z)) + \mathbb{E}_{Z \sim \pi} \hat{f}(g_{\theta_m,n}(Z)) - \hat{\mathbb{E}}_{Z \sim \pi} m f(g_{\theta_m,n}(Z))$$

$$+ \hat{\mathbb{E}}_{Z \sim \pi} m f(g_{\theta_m,n}(Z)) - \hat{\mathbb{E}}_{X \sim \nu} \hat{f}(X) + \hat{\mathbb{E}}_{X \sim \nu} \hat{f}(X) - \mathbb{E}_{X \sim \nu} \hat{f}(X) + \mathbb{E}_{X \sim \nu} f(X)$$

$$\leq \min_{f \in F} \left\{ \mathbb{E}_{Z \sim \pi} f(g_{\theta_m,n}(Z)) - \mathbb{E}_{Z \sim \pi} \hat{f}(g_{\theta_m,n}(Z)) + \mathbb{E}_{X \sim \nu} f(X) - \mathbb{E}_{X \sim \nu} \hat{f}(X) \right\}$$
\[
I_1 = W_1(\mu_{\delta_{m,n}}, \tilde{\mu}^m_{\delta_{m,n}}) \leq \max_{g_\nu \in G} W_1(\mu_\nu, \tilde{\mu}^m_\nu) \\
I_2 = W_1(\nu, \tilde{\nu}^n) \\
I_3 = \max_{f \in F} \left\{ \mathbb{E}_{Z \sim \pi} \tilde{f}(g_{\delta_{m,n}}(Z)) - \mathbb{E}_{X \sim \nu} \tilde{f}(X) \right\} \leq \max_{f \in F} \left\{ \mathbb{E}_{Z \sim \pi} \tilde{f}(g_\theta(Z)) - \mathbb{E}_{X \sim \nu} \tilde{f}(X) \right\} \\
\leq \sup_{||f||_{Lip} \leq 1} \left\{ \mathbb{E}_{Z \sim \pi} \tilde{f}(g_\theta(Z)) - \mathbb{E}_{X \sim \nu} \tilde{f}(X) \right\} \quad \text{it holds for all } g_\theta \in G \\
\leq \sup_{||f||_{Lip} \leq 1} \left\{ \mathbb{E}_{Z \sim \pi} \tilde{f}(g_\theta(Z)) - \mathbb{E}_{Z \sim \pi} f(g_\theta(Z)) + \mathbb{E}_{Z \sim \pi} f(g_\theta(Z)) - \mathbb{E}_{X \sim \nu} f(X) \\
+ \mathbb{E}_{X \sim \nu} f(X) - \mathbb{E}_{X \sim \nu} \tilde{f}(X) \right\} \\
\leq \min_{g_\nu \in G} \sup_{||f||_{Lip} \leq 1} \left\{ \mathbb{E}_{Z \sim \pi} \tilde{f}(g_\theta(Z)) - \mathbb{E}_{Z \sim \pi} f(g_\theta(Z)) + \mathbb{E}_{Z \sim \pi} f(g_\theta(Z)) - \mathbb{E}_{X \sim \nu} f(X) \right\} + W_1(\nu, \tilde{\nu}^n) \\
\leq \min_{g_\nu \in G} W_1(\mu_\nu, \nu) + \max_{g_\nu \in G} W_1(\mu_\nu, \tilde{\mu}^m_\nu) + W_1(\nu, \tilde{\nu}^n) \\
I_4 = \min_{f \in F} \left\{ \mathbb{E}_{Z \sim \pi} \tilde{f}(g_{\delta_{m,n}}(Z)) - \mathbb{E}_{Z \sim \pi} \tilde{f}(g_{\delta_{m,n}}(Z)) + \mathbb{E}_{X \sim \nu} \tilde{f}(X) - \mathbb{E}_{X \sim \nu} f(X) \right\} \\
\leq \min_{f \in F} \left\{ ||f - \tilde{f}||_{L^\infty([-M,M])} \cdot \left( P(||g_{\delta_{m,n}}(Z)|| \leq \tilde{M}) + P(||X|| \leq \tilde{M}) \right) \\
+ \mathbb{E}_{Z \sim \pi} [\tilde{f}(g_{\delta_{m,n}}(Z)) - \tilde{f}(g_{\delta_{m,n}}(Z))] \cdot \mathbb{I}(||g_{\delta_{m,n}}(Z)|| \geq \tilde{M}) + \mathbb{E}_{X \sim \nu} [\tilde{f}(X) - f(X)] \cdot \mathbb{I}(||X|| \leq \tilde{M}) \right\}
\]

where \( \mathbb{I}(\cdot) \) is an indicator function. For a fixed \( f \in \{ f : ||f||_{Lip} \leq 1 \} \), suppose that \( \min_{f \in F} ||f - \tilde{f}||_{L^\infty([-M,M])} \leq 1 \). If \( f(0) = 0 \) and is a 1-Lipschitz function, then
\[
||f(X)|| \leq ||X|| \\
||\tilde{f}(X)|| \leq ||X|| + \tilde{f}(0) \leq ||X|| + 1
\]

Based on Assumption 4 we have
\[
\mathbb{E}_{Z \sim \pi} [\tilde{f}(g_{\delta_{m,n}}(Z)) - \tilde{f}(g_{\delta_{m,n}}(Z))] \cdot \mathbb{I}(||g_{\delta_{m,n}}(Z)|| \geq \tilde{M}) + \mathbb{E}_{X \sim \nu} [\tilde{f}(X) - f(X)] \cdot \mathbb{I}(||X|| \geq \tilde{M}) \leq \tilde{M}^{-s+1}
\]

where \( \lesssim \) hides constant parameters.
Lemma 3 (Theorem 2, [31]). Let \( d \geq 2 \), the grouping size \( k = 2 \). For any \( f \in \text{Lip}_1([-\tilde{M}, \tilde{M}]^d) \), where \( \text{Lip}_1 \) denotes the class of \( 1 \)-Lipschitz functions defined on \([-\tilde{M}, \tilde{M}]^d\), there exists a neural network \( \tilde{f} \) of the form (5) with depth \( D_f \) and width \( W_f \) such that

\[
\|f - \tilde{f}\|_{L^\infty([-\tilde{M}, \tilde{M}]^d)} \leq \tilde{M} \cdot \sqrt{d}(2^{-D_f/Cd^2} \vee W_f^{-1/d^2})
\]

where \( C \) is a constant independent of \( \tilde{M}, D_f \) and \( W_f \).

Proof. By Theorem 2 in [31], for any \( h \in \text{Lip}_1([0,1]^d) \), there exists a GroupSort neural network \( \tilde{h} \) of form (5) such that

\[
\|h - \tilde{h}\|_{L^\infty([0,1]^d)} \leq \sqrt{d} \cdot (2^{-D_f/Cd^2} \vee W_f^{-1/d^2})
\]

For any \( 1 \)-Lipschitz function \( f \) defined on \([-\tilde{M}, \tilde{M}]^d\), we can find a \( h \in \text{Lip}_1([0,1]^d) \) satisfying

\[
f(x) = 2\tilde{M} \cdot h\left(\frac{x}{2\tilde{M}} + \frac{1}{2}\right)
\]

and let

\[
\tilde{f} = 2\tilde{M} \cdot \tilde{h}\left(\frac{x}{2\tilde{M}} + \frac{1}{2}\right) \in \mathcal{F}_{GS}
\]

Therefore,

\[
\|f - \tilde{f}\|_{L^\infty([-\tilde{M}, \tilde{M}]^d)} \leq 2\tilde{M} \cdot \|h - \tilde{h}\|_{L^\infty([0,1]^d)} \leq \tilde{M} \cdot \sqrt{d}(2^{-D_f/Cd^2} \vee W_f^{-1/d^2})
\]

Because the probability is less than or equal to 1, for the above \( f \),

\[
I_4 \lesssim \tilde{M}(2^{-D_f/Cd^2} \vee W_f^{-1/d^2}) + \tilde{M}^{-(s-1)}
\]

(17)

Here, we omit the parameters of \( d \). Letting \( \tilde{M} = W_f^{1/2d^2} \land D_f \), we have

\[
I_4 \lesssim 2^{-D_f/Cd^2} \vee W_f^{-(s-1)/2d^2} + D_f^{-(s-1)/2d^2}
\]

(18)

Then,

\[
W_1(\mu_{\hat{g}_{m,n}}, \nu) = \sup_{\|f\|_{L^\infty} \leq 1} \{\mathbb{E}_{Z \sim \pi} f(g_{\hat{g}_{m,n}}(Z)) - \mathbb{E}_{X \sim \nu} f(X)\} \leq \sup_{\|f\|_{L^\infty} \leq 1} I_1 + I_2 + I_3 + I_4
\]

(19)

\[
\leq \min_{\hat{g}_{m,n}} W_1(\mu_\theta, \nu) + 2 \max_{\hat{g}_{m,n}} W_1(\mu_\theta, \hat{\mu}_\theta^n) + 2W_1(\nu, \hat{\nu}^n) + I_4
\]

To make use of the bound (18) for \( I_4 \), we need to prove that there exists a \( 1 \)-Lipschitz function \( f \) with \( f(0) = 0 \) that maximizes the term \( \mathbb{E}_{Z \sim \pi} f(g_{\hat{g}_{m,n}}(Z)) - \mathbb{E}_{X \sim \nu} f(X) \). Suppose that \( f' \) is one of the maximizers, then \( f' - f'(0) \) also maximize \( \mathbb{E}_{Z \sim \pi} f(g_{\hat{g}_{m,n}}(Z)) - \mathbb{E}_{X \sim \nu} f(X) \) and this finishes the proof.

Taking expectations on (19) and combining it with (12), (13), (14), (15) and (18), we obtain (11).

Remark 6. The above analysis implies that

\[
\sup_{\|f\|_{L^\infty} \leq 1} \mathbb{E}_{Z \sim \pi} f(g_{\theta}(Z)) - \mathbb{E}_{X \sim \nu} f(X) = \sup_{\|f\|_{L^\infty} \leq 1, f(0)=0} \mathbb{E}_{Z \sim \pi} f(g_{\theta}(Z)) - \mathbb{E}_{X \sim \nu} f(X)
\]

If \( f \) is \( 1 \)-Lipschitz and \( f(0) = 0 \), then \( \|f(X)\| \leq \sqrt{d}\tilde{M} \) on \([-\tilde{M}, \tilde{M}]^d \). The bounded function class defined on the compact domain \([-\tilde{M}, \tilde{M}]^d \) is more easier to be approximated.
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