Algebraic identities associated with
KP and AKNS hierarchies

Aristophanes Dimakis
Department of Financial and Management Engineering,
University of the Aegean, 31 Fostini Str., GR-82100 Chios, Greece
dimakis@aegean.gr

Folkert Müller-Hoissen
Max-Planck-Institute for Dynamics and Self-Organization
Bunsenstrasse 10, D-37073 Göttingen, Germany
fmuelle@gwdg.de

Abstract

Explicit KP and AKNS hierarchy equations can be constructed from a certain set of
algebraic identities involving a quasi-shuffle product.

1 Introduction

The equations of the KP hierarchy are well-known to possess multi-soliton solutions. According
to Okhuma and Wadati [1], these solutions can be expressed as formal power series (in some
indeterminate). Substitution into hierarchy equations leads to algebraic sum identities of a
special kind. The structure of these identities has been explored in [2] and abstracted to a
certain algebra which we briefly recall in section 2. Moreover, a map $\Phi$ from the latter algebra
to the algebra of pseudo-differential operators underlying the Gelfand-Dickey formulation [3]
of the KP hierarchy was constructed, which maps a certain set of algebraic identities to KP
hierarchy equations, and the whole KP hierarchy is actually obtained in this way. We briefly
recall this map in section 3. In section 4 we show that, quite surprisingly, the same set of
identities is also related to the AKNS hierarchy in a similar way. In particular, relations
between AKNS and KP emerge from this relation, as will be demonstrated in section 5.

2 The algebra

Let $\mathcal{A} = \bigoplus_{r \geq 1} \mathcal{A}^r$ be a graded linear space over a field $\mathbb{K}$ of characteristic zero, supplied with
two products $\prec : \mathcal{A}^r \times \mathcal{A}^s \to \mathcal{A}^{r+s}$ and $\bullet : \mathcal{A}^r \times \mathcal{A}^s \to \mathcal{A}^{r+s-1}$ which are associative and also

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1These identities are in fact identities of quasi-symmetric functions, see [3] for example.
mixed associative. We assume\(^{2}\) that \(\mathcal{A}\) is generated by \(\mathcal{A}^1\) via the product \(\prec\). Now we define a quasi-shuffle product (see, e.g., \cite{5}) \(\circ\) in \(\mathcal{A}\) by

\[
A \circ B = A \prec B + B \prec A + A \bullet B \tag{1}
\]

\[
A \circ (B \prec \alpha) = A \prec B \prec \alpha + B \prec (A \circ \alpha) + A \bullet B \prec \alpha \tag{2}
\]

\[
(\alpha \prec A) \circ B = A \prec (\alpha \circ B) + B \prec A \prec \alpha + A \bullet B \prec \alpha \tag{3}
\]

\[
(\alpha \prec A) \circ (B \prec \beta) = A \prec (\alpha \circ (B \prec \beta)) + B \prec ((A \prec \alpha) \circ \beta) + (A \bullet B) \prec (\alpha \circ \beta) \tag{4}
\]

for all \(A, B \in \mathcal{A}^1\) and \(\alpha, \beta \in \mathcal{A}\). This is another associative product in \(\mathcal{A}\) which, however, is not mixed associative with the other products. If \((\mathcal{A}^1, \bullet)\) is commutative, then also \((\mathcal{A}, \circ)\) \(^{2}\).

Let \(\mathcal{A}(P)\) be the subalgebra generated by a single element \(P \in \mathcal{A}^1\). We set \(P \bullet n := P \bullet \ldots \bullet P\) (\(n\)-fold product). Introducing the combined associative product \(\succ := \prec + \bullet\), the next formula defines a further associative product in \(\mathcal{A}(P)\),

\[
\alpha \hat{\times} \beta := -\alpha \prec P \succ \beta . \tag{5}
\]

In the following \(\mathcal{I}\) denotes the set of identities in \(\mathcal{A}(P)\) built from the elements \(P \bullet n, n = 1, 2, \ldots\) solely by use of the products \(\circ\) and \(\hat{\times}\).

### 3 From \(\mathcal{A}(P)\) to the KP hierarchy

Let \(\mathcal{R}\) denote the \(\mathbb{K}\)-algebra of formal pseudo-differential operators (ΨDOs) generated by

\[
L = \partial + \sum_{n \geq 1} u_{n+1} \partial^{-n} \tag{6}
\]

with coefficients from an associative algebra \(\mathcal{B}\) (over \(\mathbb{K}\)), together with the projection \((\_)^{<0}\) to that part of a ΨDO containing only negative powers of the partial derivative operator \(\partial\) (with respect to a variable \(x\)). Now

\[
\ell(P) := L , \quad \ell(\alpha \prec P) := -\ell(\alpha)^{<0}L , \quad \ell(\alpha \bullet P) := \ell(\alpha)L \tag{7}
\]

determines iteratively a map \(\ell : \mathcal{A}(P) \to \mathcal{R}\). The map \(\Phi : \mathcal{A}(P) \to \mathcal{B}\) defined by

\[
\Phi(\alpha) := \text{res}(\ell(\alpha)) \tag{8}
\]

(where the residue of a ΨDO is the coefficient of its \(\partial^{-1}\) term) then has the following properties \cite{2},

\[
\Phi(\alpha \hat{\times} \beta) = \Phi(\alpha) \Phi(\beta) , \quad \Phi(P \bullet n \circ \alpha) = \delta_n \Phi(\alpha) \tag{9}
\]

where \(\delta_n\) are derivations with \(\delta_n L := -[(L^n)^{<0}, L]\). They should be regarded as vector fields on the algebra \(\mathcal{R}\). From \cite{2} we recall:

**Theorem.** Writing \(u_2 = \phi_x\) and imposing the flow equations \(\delta_n = \partial_t n\) (where \(t_1 = x\), which imply \(\Phi(P \bullet n) = \phi_{t_n}\), all (combinations of) equations of the KP hierarchy lie in \(\Phi(\mathcal{I})\).  

\(^{2}\)This assumption should have been added in \cite{2}.
We believe that any identity from \( \mathcal{I} \) is mapped to a combination of KP hierarchy equations, so that the correspondence is actually one-to-one. This has not yet been proven, however. The set of algebraic identities specified in the theorem expresses the ‘building rules’ of explicit KP hierarchy equations, which are rather implicitly determined by the sequence of Lax equations \( L_{t_n} = -[(L^n)_<, L] \). For example, the algebraic identity
\[
4 P^{\bullet 3} \circ P - P^{\circ 4} - 6 P \circ (P \times P) = 6 [P^{\bullet 2}, P]_{\times} + 3 P^{\bullet 2} \circ P^{\bullet 2}
\]
is mapped by \( \Phi \) to the (potential) KP equation
\[
(4 \phi_{t_3} - \phi_{xxx} - 6 (\phi_x)^2)_x = 6 [\phi_{t_2}, \phi_x] + 3 \phi_{t_2t_2}.
\]

4 From \( \mathcal{A}(P) \) to the AKNS hierarchy

Let \( \mathcal{B} \) be an associative \( \mathbb{K} \)-algebra with unit \( I \), and \( \mathcal{B}_\lambda \) the algebra of formal series (in an indeterminate \( \lambda \) and its inverse)
\[
X = \sum_{m \leq M} \lambda^m X_m
\]
where \( X_m \in \mathcal{B} \) and \( M \in \mathbb{Z} \). We set
\[
X_{\geq 0} := \sum_{0 \leq m \leq M} \lambda^m X_m, \quad X_{< 0} := X - X_{\geq 0} = \sum_{m < 0} \lambda^m X_m.
\]
Next we choose an element \( V \in \mathcal{B}_\lambda \) of the form
\[
V = v_0 + \lambda^{-1} v_1 + \lambda^{-2} v_2 + \lambda^{-3} v_3 + \ldots, \quad J := v_0
\]
with \( J, v_m \in \mathcal{B} \). Note that \( v_m = V_{-m}, \) \( m = 0, 1, \ldots \). A generalization of the well-known AKNS hierarchy (see also [4]) is then determined by
\[
V_{t_n} = [((\lambda^n V^n)_{\geq 0}, V] = -[(\lambda^n V^n)_{< 0}, V] =: \delta_n V \quad n = 1, 2, \ldots
\]
which requires \( J_{t_n} = 0 \). By a standard argument, the flows commute. The first \( (n = 1) \) hierarchy equation is equivalent to
\[
[J, v_{m+1}] + [v_1, v_m] = v_{m,x} \quad m = 1, 2, \ldots
\]
Next we define two maps \( \ell, \mathfrak{r} : \mathcal{A}(P) \rightarrow \mathcal{B}_\lambda \) via \( \ell(P) = \mathfrak{r}(P) = \lambda V \) and
\[
\ell(\alpha \prec P) = -\ell(\alpha)_{< 0} \lambda V, \quad \ell(\alpha \bullet P) = \ell(\alpha) \lambda V
\]
\[
\mathfrak{r}(P \prec \alpha) = -\lambda V \mathfrak{r}(\alpha)_{\geq 0}, \quad \mathfrak{r}(P \bullet \alpha) = \lambda V \mathfrak{r}(\alpha)
\]
for all \( \alpha \in \mathcal{A}(P) \). As a consequence, we have \( \ell(P^{\bullet m}) = \mathfrak{r}(P^{\bullet m}) = \lambda^m V^m \). The map \( \Phi : \mathcal{A}(P) \rightarrow \mathcal{B} \) defined by
\[
\Phi(\alpha) := (\ell(\alpha)_{< 0} \lambda V)_{\geq 0} = \ell(\alpha)_{-1} J
\]
\[3\]Several properties of \( \ell \) and \( \mathfrak{r} \) in the KP case have been derived in [2]. Most of them in fact remain valid if we replace \( L \) by \( \lambda V \).
then satisfies $\Phi(\alpha) = (r(\alpha)_{<0} \lambda V)_{\geq 0} = r(\alpha)_{-1} J$ and

$$\Phi(P^{\bullet k}) = (\lambda^k V^k)_{-1} J = (V^k)_{-(k+1)} J .$$

(20)

In particular, we obtain $\Phi(P) = v_2 J$ and

$$\Phi(P^{\bullet 2}) = \{\{J, v_3\} + \{v_1, v_2\}\} J$$

$$\Phi(P^{\bullet 3}) = \left(\frac{1}{2}\{J, J, v_4\} + \{J, v_1, v_3\} + \frac{1}{2}\{J, v_2, v_2\} + \frac{1}{2}\{v_1, v_1, v_2\}\right) J$$

(21)

(22)

where $\{a_1, \ldots, a_k\} := \sum_{\sigma \in S_k} a_{\sigma(1)} \cdots a_{\sigma(k)}$ with the symmetric group $S_k$ of order $k$. It can be shown that $\Phi$ has the following algebra homomorphism property,

$$\Phi(\alpha \times \beta) = (\ell(\alpha)_{<0} \lambda V r(\beta)_{<0})_{-1} J = \Phi(\alpha) \Phi(\beta)$$

(23)

for all $\alpha, \beta \in A(P)$ (cf theorem 6.2 in [2]). Another important formula is$^4$

$$\Phi(\alpha)_{t_n} = \delta_n \Phi(\alpha) = \Phi(P^{\bullet n} \circ \alpha)$$

(24)

where we imposed the flow equations $\delta_n = \partial_{t_n}$ on $B_\lambda$. Applying $\Phi$ to the simple algebraic identities $P^{\bullet k} \circ P^{\bullet n} = P^{\bullet n} \circ P^{\bullet k}$ leads to the relations

$$\Phi(P^{\bullet n})_{t_k} = (\Phi(P^{\bullet k}))_{t_n}$$

(25)

(which in the KP case are trivially satisfied). The identity (11) is mapped by $\Phi$ to

$$\left(4 v_{2,t_3} - v_{2,xxx} - 3 \{\{J, v_3\} + \{v_1, v_2\}\}_{t_2} - 6 (v_2 J v_2)_x\right) J$$

$$+ 6 [v_2 J, \{\{J, v_3\} + \{v_1, v_2\}\}] J = 0 .$$

(26)

### 4.1 $V^2 = V$ reduction

For any polynomial $P$ of $V$ with coefficients in the center of $B$, the constraint $P(V) = 0$, which in particular requires $P(J) = 0$, is preserved by the hierarchy (15). Let us consider the special case $V^2 = V$, which is equivalent to

$$v_m = \sum_{i=0}^{m} v_i v_{m-i} = \{J, v_m\} + \sum_{i=1}^{m-1} v_i v_{m-i} \quad m = 0, 1, 2, \ldots .$$

(27)

We further assume that the first of the hierarchy equations (15), i.e. $V_x = [\lambda J + v_1, V]$ holds, and thus (16). Together with (27), this implies

$$v_{m+1} = -(v_{m,x} + \sum_{i=1}^{m} v_i v_{m+1-i} - [v_1, v_m]) H$$

(28)

where $H := 2J - I$ which satisfies $H^2 = I$. This allows us to express $v_m$, $m > 1$, iteratively in terms of $u := v_1$ and its derivatives with respect to $x$,

$$v_2 = -(u_x + u^2) H, \quad v_3 = u_{xx} - 2 u^3 + [u, u_x]$$

$$v_4 = -(u_{xxx} + [u, u_{xx}]) - 3 \{u^2, u_x\} - (u_x)^2 - 3 u^4) H$$

(29)

(30)

$^4$This is the analog of the simplest case expressed by proposition 6.5 in [2].
etc. It is well-known that soliton equations emerge from the hierarchy \([15]\) for \(n > 1\). But now we show how to obtain them from identities in \(I\). Using the above results, the \(\Phi\)-images \([25]\) of algebraic identities for \(n = 1\) and \(k = 2,3\) become

\[
v_{2,y} J = v_{3,x} J, \quad v_{2,t} J = v_{4,x} J
\]

where \(y := t_2\) and \(t := t_3\). Let us choose

\[
J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}
\]

where \(q\) and \(r\) are elements of a (not necessarily commutative) algebra. Then

\[
v_2 = \begin{pmatrix} -qr & q_x \\ -r_x & r_q \end{pmatrix}, \quad v_3 = \begin{pmatrix} qr_x - q_{x}r & q_{xx} - 2qrq \\ r_{xx} - 2rqr & r_{xx} - r_{x}q \end{pmatrix}
\]

\[
v_4 = \begin{pmatrix} -qr_{xx} - q_{xx}r + q_{x}r_{x} + 3qrq \\ -r_{xx} + 3(r_{x}qr + q_{rx}) \\ -r_{xx} + 3(r_{x}qr + q_{rx}) \end{pmatrix}
\]

and equations \([31]\) yield (after an integration)

\[
(q_y - q_{xx} + 2qrq)r = 0, \quad r_y + r_{xx} - 2rqr = 0 \quad (35)
\]

\[
(q_t - q_{xxx} + 3(q_{x}r_{x} + q_{rx})r = 0, \quad r_t - r_{xxx} + 3(r_{x}qr + q_{rx}) = 0 \quad (36)
\]

\([35]\) is a system of coupled nonlinear Schrödinger equations. \([36]\) yields with \(r = 1\) the (noncommutative) KdV equation, and with \(q = r\) the (noncommutative) mKdV equation. Moreover, \([20]\) is satisfied as a consequence of \([35]\) and \([36]\).

### 4.2 \(V^3 = I\) Reduction

In this subsection we sketch another reduction: \(V^3 = I\). Let

\[
J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}, \quad u := v_1 = (1 + 2\zeta) \begin{pmatrix} 0 & r & q \\ q & 0 & r \\ r & q & 0 \end{pmatrix}
\]

where \(\zeta\) is a third root of unity (so that \(\zeta^2 + \zeta + 1 = 0\) and \(q, r \in \mathcal{B}\). \([16]\) determines the non-diagonal part of \(v_m\), the reduction the diagonal part. We obtain

\[
v_2 = \begin{pmatrix} 3qr & \zeta r_x \\ -\zeta q_x & (1 + \zeta) r_q \\ -\zeta/(1 + \zeta) r_x & -3(1 + \zeta) qr & r_x \end{pmatrix}
\]

\[
v_3 = \begin{pmatrix} (\zeta - 1) D/(1 + \zeta) & -R/(2 + \zeta) \\ -Q/(2 + \zeta) & (1 + \zeta) D/(1 + \zeta) \\ \zeta R/(1 + 2\zeta) & (1 + \zeta) Q/(\zeta - 1) \\ \zeta R/(1 + 2\zeta) & (1 + \zeta) Q/(\zeta - 1) \end{pmatrix}
\]

where \(Q := q_{xx} + 9q^2r - 3rr_x, R := r_{xx} + 9qr^2 + 3qq_x\) and \(D := q^3 + r^3 + q_{xx} - q_{xx}r\). Next we compute \([21]\) and evaluate the identity \([25]\) for \(n = 1\) and \(k = 2\). Setting \(t := (1 + 2\zeta)t_2\), this yields the following system of coupled Burgers equations

\[
q_t - q_{xx} + 6rr_x = 0, \quad r_t + r_{xx} + 6qq_x = 0.
\]

\([40]\)
5 From AKNS to KP

The existence of maps $\Phi_{\text{KP}}$ and $\Phi_{\text{AKNS}}$ which map identities in the algebra $\mathcal{A}(P)$ to equations of the KP, respectively AKNS hierarchy suggests a relation between the latter hierarchies. Let us see what happens if we identify their images. The equation $\Phi_{\text{KP}}(P) = \Phi_{\text{AKNS}}(P)$ reads

$$\phi_x = v_2 J$$  \hspace{1cm} (41)

and, more generally, $\Phi_{\text{KP}}(P^n) = \Phi_{\text{AKNS}}(P^n)$ means

$$\phi_t = (\lambda^n V^n)_{-1} J .$$  \hspace{1cm} (42)

With the reduction treated in section 4.1, (41) becomes $\phi_x = -q r$, which indeed is a well-known (symmetry) constraint of the KP equation [6]. As a consequence of it, if $q, r$ satisfy the AKNS equations (35) and (36), then $\phi$ satisfies the potential KP equation (11). (41) generalizes this relation to matrix (potential) KP equations. A thorough analysis of $\Phi_{\text{KP}} = \Phi_{\text{AKNS}}$ has still to be carried out, but we verified with the help of computer algebra in several examples that indeed matrix (potential) KP equations are satisfied by (41) as a consequence of the corresponding AKNS equations. We expect that this relation extends to the whole hierarchies. We plan to report on the relations between the abstract algebra $\mathcal{A}(P)$ and the KP and AKNS hierarchies sketched in this work in more detail in a separate publication.

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