Relaxed Gabor expansion at critical density and a ‘certainty principle’

V.P. Palamodov
Tel Aviv University

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1 Introduction

The functions in $\mathbb{R}^n$ that are the best localized in configuration and in frequency spaces simultaneously are known in quantum field theory as coherent states (shifted modulated gaussian functions). Any coherent state gives exact minimum for the uncertainty relation which means that each coherent state $e_\lambda$ is maximally concentrated at a certain point $\lambda$ of the phase space $\Phi = \mathbb{R}^n \times (\mathbb{R}^n)'$. Another advantage of the family $\{e_\lambda, \lambda \in \Phi\}$ is its large symmetry group: the Weyl-Heisenberg group is acting by shifts and modulation operators and the metaplectic group which is a ‘quantization’ of the group of linear simplectic transformations.

The role of the Weyl-Heisenberg family in the information theory ($n = 1$) was emphasized by Gabor [6]: "each elementary signal conveys exactly one datum, or one quantum of information". Gabor’s idea was to expand an ‘arbitrary’ signal in a series of coherent states $e_\lambda$ for points $\lambda$ in a maximally sparse lattice $\Lambda$ in the phase plane. The property of the Gabor system $\{e_\lambda, \lambda \in \Lambda\}$ depends on the area $a$ of a cell of the lattice $\Lambda$. The system is complete, if and only if $a \leq 1$. Lyubarskii [9] has shown that the Gabor system is a frame, if $a < 1$. We focus on the critical case $a = 1$ when the Gabor system of critical density is still complete, but is not a frame. A function $f \in L_2(\mathbb{R})$ may not have a convergent Gabor series, but there exists always a series that converges in a distribution sense, see Janssen [7]. The book of I.Daubechies [2] contains a survey of the problem, see also Feichtinger, Strohmer [4] for applications of the Gabor analysis.

Lyubarskii and K.Seip [10] has shown that for an arbitrary function $f \in L_2(\mathbb{R})$ that is 1/2-smooth in time and frequency there exists a convergent series over a Gabor system, if a minor deformation of a lattice $\Lambda$ of critical density is made.

We show here that any $\delta$-smooth (in $\Phi$) function for $\delta > 1$ has a unique convergent Gabor expansion with $l_2$-coefficients, provided one more point $\sharp$ is added to
a lattice \( \Lambda \); this point is in the middle of a cell and is called \textit{sharp} point. We call it a \textit{relaxed} Gabor expansion. This approach can save Gabor’s idea. The rate of convergence of the coefficients can be improved, if few more sharp terms are added to the Gabor system. The coefficients of such a series are again uniquely defined for a sufficiently smooth function \( f \).

The relaxed Gabor expansion sheds light to the physical wisdom: \textit{a field supported in certain domain \( D \) in the phase space has about} \( |D| \) \textit{degrees of freedom, where} \( |D| \) \textit{means the simplectic area of the domain} (Nyquist, Wigner, Brillouin, Shannon, Gabor,...). This claim can be called ‘certainty’ principle as opposite of the classical uncertainty one. The Landau-Pollak dimension theorem \[8\] gives an accurate form to this principle in a special situation. Given positive numbers \( T, \Omega \), any function \( f \) ‘concentrated’ in the box \( D \) of the size \( T \times 2\Omega \) can be approximated by a linear combination of \( 2T\Omega + O(\log T\Omega) \) first prolate spheroidal functions. The condition of concentration means that the tail of a unit energy function \( f \) is small in \( \mathbb{R} \setminus [0, T] \) and the tail of Fourier transform \( \hat{f} \) is small in \( \mathbb{R}^* \setminus [-\Omega, \Omega] \). The number \( 2T\Omega \) is equal to the area \( |D| \) of the box measured by the canonical simplectic form in the phase plane. This means the approximation ‘dimension’ of the space of functions localized in the box equals the simplectic area of this up to a logarithmic term.

We formulate and prove here a rigorous form of the certainty principle for domains of arbitrary shape. We use Gabor means \( \langle f | e_\mu \rangle \) to specify the condition of concentration of a function \( f \in L_2 \) in the phase space \( \Phi \). We say that \( f \) is concentrated in a domain \( D \), if the integral over \( \Phi \setminus D \) of the density \( |\langle f | e_\mu \rangle|^2 \) \( d\mu \) is small. We show that for an arbitrary sets \( K \subset D \), where \( D \) is \( r \)-neighborhood of \( K \) for some \( r \geq r_0 \), any function \( f \) concentrated in \( D \) can be written in the form \( g + h + \phi \), where \( g \) is a linear combination of Gabor functions \( e_\lambda \), with \( \lambda \in \Lambda \cap D \), \( h \) belongs to the linear envelope of Gabor functions with sharp \( \lambda \in D \setminus K \) and the norm of \( \phi \) is bounded by the rate of concentration of \( f \) plus an exponentially small term as \( r \to \infty \). The total number \( N \) of such points \( \lambda \) is equal to \( |D| + |D \setminus K| = |D| + O\left(r |D|^{1/2}\right) \).

To summarize briefly, this means that any function \( f \) concentrated in \( D \) is a linear combination of \( N \sim |D| + O\left(r |D|^{1/2}\right) \) coherent states supported in \( D \) up to a exponentially small term.

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2 Gabor transform and localization

Fix a coordinate \( x \) in a line \( \mathbb{R} \); the family of functions in \( \mathbb{R} \)

\[ e_\lambda(x) = 2^{1/4} \exp\left(-\pi (x - p)^2 + 2\pi \theta x\right) \]
are called Gabor functions. Here $\lambda = (p, \theta)$ is a point in the phase space $\Phi \cong \mathbb{R} \times \mathbb{R}'$ ($\mathbb{R}'$ means the dual line). Another term for $e_\lambda$ is **coherent state** at the point $\lambda$. The phase space has the natural Euclidean structure: $|\lambda|^2 = |p|^2 + |\theta|^2$ and Lebesgue measure $d\lambda = dp \, d\theta$. We use the notation $\langle \cdot | \cdot \rangle$ for the scalar product in $L_2 = L_2(\mathbb{R})$ and $\| \cdot \|$ for the norm. For $f \in L_2$ the function $\lambda \mapsto \langle f | e_\lambda \rangle$ defined in $\Phi$ is called **Gabor transform** of $f$. This transform is unitary:

**Proposition 2.1** For an arbitrary function $f \in L_2$ the equation holds

$$\int_{\Phi} |\langle f | e_\lambda \rangle|^2 \, d\lambda = \|f\|^2$$

where $d\lambda = dp \, d\theta$ and

$$f = \int_{\Phi} \langle f | e_\lambda \rangle \, e_\lambda \, d\lambda$$

in weak $L_2$-sense.

**Proof.** By Plancherel Theorem

$$\int_{\mathbb{R}} |\langle f | e_\lambda \rangle|^2 \, d\theta = 2^{1/2} \int \left| \int f(x) \exp(-\pi (x-p)^2) \exp(2\pi i \theta x) \, dx \right|^2 \, d\theta$$

$$= 2^{1/2} \int |F(f_p)(-\theta)|^2 \, d\theta = 2^{1/2} \int |f_p(x)|^2 \, dx,$$

where $f_p(x) = f(x) \exp(-\pi (x-p)^2)$. Next we integrate both sides against $dp$ and changing the variable $x$ to $y = x - p$. This yields

$$\int \int |f_p(x)|^2 \, dx \, dp = \int \int |f(x) \exp(-\pi y^2)|^2 \, dx \, dy$$

$$= \int |f(x)|^2 \, dx \int \exp(-2\pi y^2) \, dy$$

$$= 2^{-1/2} \int |f(x)|^2 \, dx$$

and (1) follows. Now calculate the scalar product of integral (2) with $e_\mu$. We have for arbitrary points $\lambda = (p, \theta), \mu = (q, \eta) \in \Phi$

$$\langle e_\lambda | e_\mu \rangle = \exp(\pi i (p+q)(\theta-\eta) - \pi |\lambda - \mu|^2/2).$$

Therefore

$$\int_{\Phi} \langle f | e_\lambda \rangle \langle e_\lambda | e_\mu \rangle \, d\lambda = \langle f | E_{\lambda,\mu} \rangle,$$

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where

\[ \bar{E}_{\lambda,\mu} = \int_\Phi \langle |e_\lambda\rangle \langle e_\lambda|e_\mu\rangle \, d\lambda \]

\[ = \int_\Phi \langle |e_\lambda\rangle \exp (\pi i (p + q) (\theta - \eta) - \pi |\lambda - \mu|^2 / 2) \, d\lambda \]

\[ = 2^{1/4} \int \exp (-\pi (x - p)^2 - 2\pi i \theta x) \exp (\pi i (p + q) (\theta - \eta) - \pi |\lambda - \mu|^2 / 2) \, d\theta dp \]

Calculate the interior integral by changing \( \theta \) to \( \xi = \theta - \eta \):

\[ E_{\lambda,\mu} (x) = 2^{1/2} \exp (-2\pi \eta x) \int \exp (-\pi (x - p)^2 - \frac{\pi}{2} (p + q - 2x)^2 - \frac{\pi}{2} |p - q|^2) \, dp \]

\[ = 2^{1/2} \exp (-2\pi \eta x) \exp (-\pi (x - q)^2) \int \exp (-2\pi (x - p)^2) \, dp = \bar{e}_\mu (x) \]

This allows to write

\[ \int_\Phi \langle f|e_\lambda\rangle \langle e_\lambda|e_\mu\rangle \, d\lambda = \langle f|e_\mu\rangle \]

This yields for an arbitrary \( g \in L^2 \)

\[ \langle f|g\rangle = \lim_{r \to \infty} \int_{|\lambda| \leq r} \langle f|e_\lambda\rangle \langle e_\lambda|g\rangle \, d\lambda = \int_\Phi \langle f|e_\lambda\rangle \langle e_\lambda|g\rangle \, d\lambda \]

The integral converges, since both factors belong to \( L^2 (\Phi) \).

Now we show that a function \( f \in L^2 \) can be localized in a convex domain \( D \subset \Phi \) in terms of its Gabor transform.

**Corollary 2.2** For an arbitrary \( f \in L^2 \) and any \( q \in \mathbb{R} \) we have

\[ \int_{p \geq q} |\langle f|e_\lambda\rangle|^2 \, d\lambda = \int_{-\infty}^{\infty} I (x - q) |f (x)|^2 \, dx \]

where

\[ I (x) = 2^{1/2} \int_{-\infty}^{x} \exp (-2\pi y^2) \, dy \]

**Remark.** The function \( I (x) \) tends fast to 1 and 0 as \( x \to \infty \), respectively, \( x \to -\infty \) and \( I (0) = 1/2 \).

**Proof** follows from (3).

**Corollary 2.3** Let \( D \) be a domain and \( S \) be a rotation in the phase plane \( \Phi \) and \( M_S \) be the corresponding metaplectic transform. We have

\[ \int |M_S f (x)|^2 I (x - q) \, dx \leq \int_{\Phi \setminus D} |\langle f|e_\lambda\rangle|^2 \, d\lambda \tag{5} \]

where \( q = \sup \{ p; \lambda = (p, \theta) \in S (D) \} \).
Proof. By Proposition 2.2 we have
\[
\int |M_S f(x)|^2 I(x - q) \, dx \leq \int_{\Phi \setminus S(D)} |\langle M_S f | e_{\lambda} \rangle|^2 \, d\lambda.
\]
Because of the transform $M_S$ is unitary, we have by Corollary 11.2 we have
\[
\langle f | e_{\lambda} \rangle = \langle M_S f | M_S e_{\lambda} \rangle = \exp (-i\phi) \langle M_S f | e_{S(\lambda)} \rangle
\]
for some real phase $\phi$. Therefore
\[
\int_{\Phi \setminus S(D)} |\langle M_S f | e_{\lambda} \rangle|^2 \, d\lambda = \int_{\Phi \setminus D} |\langle M_S f | e_{S(\lambda)} \rangle|^2 \, d\lambda = \int_{\Phi \setminus D} |\langle f | e_{\lambda} \rangle|^2 \, d\lambda
\]
which completes the proof. ▶

Suppose that the right-hand side of (5) is small. Taking in account that $I(x) \geq 1/2$ for $x \geq 0$, we conclude that for any rotation $S$ the function $M_S f$ is strongly concentrated in the interval $[r, q] = \pi(S(D))$, where $\pi : \Phi \rightarrow \mathbb{R}$ is the orthogonal projection. The inverse is true if, for instance, $D$ is a convex polygon.

### 3 Gabor series

Choose some elements $\varepsilon \in \mathbb{R}$, $\varepsilon^* \in \mathbb{R}'$ such that $\varepsilon^*(\varepsilon) = 1$ and consider the lattice $\Lambda \subset \Phi = \mathbb{R} \times \mathbb{R}'$ generated by $\varepsilon$ and $\varepsilon^*$:
\[
\Lambda = \{ \lambda = k\varepsilon + j\varepsilon^*, \ k, j \in \mathbb{Z} \}.
\]
The simplectic area of a cell is equal to $\varepsilon^*(\varepsilon) = 1$. Introduce a coordinate $x$ in $\mathbb{R}$ such that $x(\varepsilon) = 1$; then we have $\xi(\varepsilon^*) = 1$ for the dual coordinate $\xi$ in $\mathbb{R}'$. Consider the Gabor system $\{ e_{\lambda}, \ \lambda \in \Lambda \}$.

**Proposition 3.1** For an arbitrary numerical sequence $\{ c_{\lambda} \} \in l^2(\Lambda)$ the series
\[
f = \sum_{\Lambda} c_{\lambda} e_{\lambda}
\]
converges in $L^2$ and the inequality holds
\[
\|f\| \leq \sigma_0 \left( \sum |c_{\lambda}|^2 \right)^{1/2}, \ \sigma_0 = \sum \exp \left( -\frac{\pi}{2} k^2 \right).
\]
**Proof.** By (4) we have for any $\lambda, \mu \in \Lambda$, $\lambda = (p, \theta)$
\[
\langle e_{\lambda} | e_{\mu} \rangle = g_{\lambda - \mu},
\]
\[
g_{\lambda} = \exp \left( \pi ip\theta - \frac{\pi}{2} (p^2 + \theta^2) \right).
\]
Therefore
\[ \|f\|^2 = \sum_{\lambda,\mu} c_\lambda \bar{c}_\mu \langle e_\lambda | e_\mu \rangle = \sum_{\lambda} c_\lambda \sum_{\mu} \bar{c}_\mu g_{\lambda - \mu} \]

The interior sum is a convolution and we can estimate its $l_2$-norm as follows:
\[ \sum_{\lambda} \left| \sum_{\mu} \bar{c}_\mu g_{\lambda - \mu} \right|^2 \leq \left( \sum |g_\lambda| \right)^2 \sum |c_\mu|^2 = \sigma_0^4 \sum |c_\lambda|^2, \]

since $\sum |g_\lambda| = \sigma_0^2$. This yields
\[ \|f\|^4 \leq \sum_{\lambda} |c_\lambda|^2 \sum_{\lambda} \left| \sum_{\mu} \bar{c}_\mu g_{\lambda - \mu} \right|^2 \leq \sigma_0^4 \left( \sum |c_\mu|^2 \right)^2 . \]

\(\forall\)

4 Zak transform

The Zak transform of a function $f \in L^2$ is defined by the series
\[ Zf (y, \xi) = \sum_{q \in \mathbb{Z}} \exp (2\pi iq\xi) f (y + q), \]

which converges almost everywhere in the square $Q = \{0 \leq y, \xi \leq 1\}$. The function $Zf$ fulfills
\[ Zf (y, \xi + 1) = Zf (y, \xi) , \ Zf (y + 1, \xi) = \exp (-2\pi i \xi) Zf (y, \xi). \]

Proposition 4.1 The Zak transform is a unitary operator $L^2 (\mathbb{R}) \rightarrow L^2 (\mathbb{Q})$.

Proof is straightforward and we omit it. The inversion formula reads
\[ f (x) = \sum_{r \in \mathbb{Z}} \int_0^1 \int_0^1 dyd\xi g (y, \xi) \exp (2\pi i (y - x) (\xi + r)) \]
\[ = \int_0^1 dy \int_\mathbb{R} g (y, \xi) \exp (2\pi i \xi (y - x)) d\xi, \]

where $g = Zf$ is 1-periodic function of $\xi$.

The Weyl-Heisenberg group acts in $L^2$ by the shifts and modulation operators
\[ T_\lambda f (x) = \exp (2\pi i \theta x) f (x - p) , \lambda = (p, \theta) \]

We have for any $\lambda \in \Lambda$ we have
\[ Z (T_\lambda f) (y, \xi) = \exp (2\pi i (p\xi + \theta \xi)) Zf (y, \xi). \]
The sum
\[ \Theta(z) = 2^{1/4} \sum_{q \in \mathbb{Z}} \exp \left(2\pi i q z - \pi q^2 \right) \]
is a Jacobi elliptic function: \( \Theta(z) = 2^{1/2} \theta_3(z; i) \). This function is holomorphic of \( z \) in the whole plane, satisfies the periodicity conditions
\[ \Theta(z + 1) = \Theta(z), \quad \Theta(z + i) = \exp(\pi - 2\pi i z) \Theta(z) \]
has simple zero at the point \( 1/2 + i/2 \) and no other zeros in the closed square \( Q \).

The Zak transform of Gabor function for \( \lambda = (p, \theta) \in \Lambda \) is expressed in terms of \( \Theta \):
\[ Ze_\lambda(y, \xi) = \exp \left(2\pi i (p \xi + \theta y) \right) \exp \left(-\pi y^2 \right) \Theta(\xi + iy). \]  

5 Creation and annihilation operators

The operator
\[ a = \frac{1}{2\pi} \frac{d}{dx} + x, \quad a^+ = -\frac{1}{2\pi} \frac{d}{dx} + x \]
in \( L_2 \) are adjoint one to another. They are called the annihilation and the creation operators. Any Gabor function is an eigenvector of the annihilation operator:
\[ a e_\lambda = \lambda e_\lambda \]  

where \( \lambda = (p, \theta) \) and \( \lambda = p + i \theta \). For any \( \phi \) in the domain of the operator \( a \) we have
\[ Z(a\phi) = AZ\phi, \quad A \doteq \frac{1}{2\pi i} \left( \frac{\partial}{\partial \xi} + i \frac{\partial}{\partial y} \right) + y. \]  

Consider the line bundle \( E \) over the torus \( \mathbb{R}^2/\Lambda \) whose sections \( s \) fulfil the periodicity conditions
\[ s(y, \xi + 1) = s(y, \xi), \quad s(y + 1, \xi) = \exp(-2\pi i \xi) s(y, \xi). \]
in the unit square \( Q \) (which is the fundamental domain of the lattice \( \Lambda \)). Let \( W^\alpha_2(E) \) be the Sobolev space of order \( \alpha \geq 0 \) of sections \( \phi \) of \( E \). By \( \Theta \) the function \( Zf \) is a section of \( E \) of the class \( W^0_2(E) = L_2(Q) \). The differential operator \( A \) acts on smooth sections of \( E \) and defines for any \( \alpha \) a bounded map \( A: W^{\alpha+1}_2(E) \to W^\alpha(E) \).

Proposition 5.1 For any function \( f \in L_2 \) such that
\[ \int \left( |\lambda|^\delta + 1 \right) |\langle f|e_\lambda \rangle|^2 d\lambda < \infty, \quad \delta > 0 \]
we have \( Zf \in W^\delta_2(E) \).
Proof. By (2) we can write
\[ Zf = \int_{\Phi} \langle f | e_{\lambda} \rangle Z e_{\lambda} d\lambda \]
We have for arbitrary \( \lambda = (p, \theta) \in \Phi \)
\[ Ze_{\lambda}(\xi, y) = 2^{1/4} \sum_{q \in \mathbb{Z}} \exp(2\pi i q \xi + \theta (y + q)) \exp(-\pi (y + q - p)^2) \]
Setting \( p = k + r, \theta = j + \eta, k, l \in \mathbb{Z} \) and \( q' = q - k \) yields
\[ Ze_{\lambda}(y, \xi) = 2^{1/4} \sum_{q \in \mathbb{Z}} \exp(2\pi i [k \xi + \theta y + q\eta]) \exp(-\pi (y + q - k)^2) \]
\[ = 2^{1/4} \exp(2\pi i [k (\xi + \eta) + \theta y]) \sum_{q \in \mathbb{Z}} \exp(2\pi i q' (\xi + \eta)) \exp(-\pi (y + q' - r)^2) \]
\[ = \exp(2\pi i [k \xi + jy]) \exp(2\pi i k\eta) Ze_{(r, \eta)}(y, \xi) \tag{17} \]
We can write
\[ Zf(y, \xi) = \sum_{Z \times Z} \exp(2\pi i [k \xi + jy]) \int_{Q} \langle f | e_{\lambda} \rangle E_{\lambda}(y, \xi) dr \, d\eta, \]
The kernel \( E_{\lambda}(y, \xi) = \exp(2\pi i k\eta) Ze_{(r, \eta)}(y, \xi) \) is a real analytic function for \( \lambda, (y, \xi) \in \mathbb{R} \times \mathbb{R} \), is 1-periodic in \( \xi \) and its derivatives in \( y, \xi \) are bounded in any strip \( |y| \leq C \) uniformly for \( \lambda \in \Phi \). Take a test function \( \phi \) in \( \mathbb{R} \times \mathbb{R} \) and apply the Fourier transform \( F = F_{(y, \xi) \rightarrow (s, t)} : \)
\[ F(\phi Zf)(s, t) = \int_{Q} \sum_{k,j} \langle f | e_{\lambda} \rangle F(\phi E_{\lambda})(s - k, t - j) dr \, d\eta \]
The interior sum can be written as convolution on \( \mathbb{Z}^2 \):
\[ G_{r, \eta} = \sum_{k,j \in \mathbb{Z}} \langle f | e_{(k,j)+(r, \eta)} \rangle F(\phi E_{\lambda})(s - k, t - j) \]
We have
\[ |F(\phi E_{\lambda})(s, t)| \leq C_n (|s| + |t| + 1)^{-n} \]
for any natural \( n \), since \( \phi E_{\lambda} \in C^\infty \). Therefore for any fixed \( (r, \eta) \in \mathbb{Q} \)
\[ \int (|s| + |t| + 1)^{\delta} |G_{r, \eta}(s, t)|^2 ds \, dt \leq C \sum_{k,j} (|k| + |j| + 1)^{\delta} \left| \langle f | e_{(k,j)+(r, \eta)} \rangle \right|^2. \]
where the sum in the right-hand side converges for almost all \((r, \eta)\) and the constant \(C\) does not depend on \(f\). Integrating over \(Q\) yields
\[
\int (|s| + |t| + 1)^\delta |F(\phi Z f)(s, t)|^2 \, ds \, dt = \int (|s| + |t| + 1)^\delta \left| \int_Q G_{r, \eta}(s, t) \, dr \, d\eta \right|^2 \, ds \, dt
\]
\[
\leq C \int_Q \sum_{k, j} (|k| + |j| + 1)^\delta \left| \langle f | e_{(k,j)+(r,\eta)} \rangle \right|^2 \, dr \, d\eta
\]
\[
= C \int_\Phi (|k| + |j| + 1)^\delta \left| \langle f | e_{k,j} \rangle \right|^2 \, dp \, d\theta
\]
\[
\leq C' \int_\Phi (|\lambda| + 1)^\delta \left| \langle f | e_{k,j} \rangle \right|^2 \, d\lambda < \infty
\]

This implies that \(\phi Z f\) belongs to \(W^\delta_2\) and our statement follows.

**Definition.** Denote by \(H^\delta\) the space of functions \(f \in L^2\) that satisfies the condition (16) and set
\[
\|f\|_\delta = \left( \int (|\lambda|^\delta + 1) \left| \langle f | e_{\lambda} \rangle \right|^2 \, d\lambda \right)^{1/2}
\]

Denote \(H \equiv \cup_{\delta > 1} H^\delta\).

**Remark 1.** The space \(H^\delta\) coincides with the modulation spaces \(M^w_{2,2}\) of H.Feichtinger, for the weight function \(w(\lambda) = |\lambda|^\delta + 1\).

**Remark 2.** The domain of the harmonic oscillator operator
\[
H \equiv \frac{1}{2} (a^+ a + aa^+) = -\frac{1}{4\pi^2} \frac{d^2}{dx^2} + x^2
\]
is equal to the space \(H^2\).

6 **Relaxed Gabor expansion**

We show now that an arbitrary function \(f \in H\) can be expanded in a Gabor series as above with one more term.

Denote \(\sharp = (1/2, 1/2) \in \Phi\); we call sharp point any element \(\mu \in \Lambda + \sharp\), \(^1\). The

\(^1\)The motivation of this term comes from the interpretation of the lattice \(\Lambda\) as a table for a regular time-frequency notation of music (Brillouin-Wigner). The point \(\sharp\) is then a tune which is shifted inside a rhythm interval (syncope), whose height is shifted by half a tune (sharp= "diese" tune).
sharp Poisson functional is the series

\[ \gamma^\sharp (f) = \frac{1}{i\Theta (0)} \sum_{z} (-1)^q f(q + 1/2) = \frac{Zf (\sharp)}{i\Theta (0)}. \]

We show below that this functional is well defined and continuous in \( H \). In particular, we have \( \gamma^\sharp (e_\sharp) = 1 \), whereas \( \gamma^\sharp (e_\lambda) = 0 \) for arbitrary \( \lambda \in \Lambda \), since \( \Theta \) vanishes at each sharp point. We call the set \( \Lambda^\sharp = \Lambda \cup \{\sharp\} \) the relaxed lattice and consider the relaxed Gabor system \( \{e_\lambda, \lambda \in \Lambda^\sharp\} \).

**Theorem 6.1** There exists a family of continuous functionals \( \gamma^\lambda, \lambda \in \Lambda \) in the space \( H \) such that an arbitrary \( f \in H \) is developed in the series

\[ f = \sum_{\Lambda^\sharp} \gamma^\lambda (f) e_\lambda \] (18)

that converges in \( L_2 (\mathbb{R}) \) and for any \( \delta > 1 \) there exists a constant \( C_\delta \) such that

\[ \sum_{\Lambda^\sharp} |\gamma^\lambda (f)|^2 \leq C_\delta \|f\|_\delta^2. \] (19)

**Lemma 6.2** For an arbitrary \( f \in H \) the Zak transform \( Zf \) is \( \varepsilon \)-Hölder continuous for \( \varepsilon < \delta - 1 \).

**Proof of Lemma.** By Proposition 5.1 \( Zf \in W^{\frac{\delta}{2}}_2 (E) \) for some \( \delta > 1 \). Sobolev’s imbedding theorem implies that \( Zf \) is a continuous section of \( E \) and \( \sup |Zf| \leq C \|f\|_\delta \), [11]. Moreover, \( Zf \) belongs to the Hölder \( \varepsilon \)-class for \( \varepsilon < \delta - 1 \).

**Proof of Theorem.** By Lemma 6.2 the functional \( \gamma^\sharp \) is well defined in \( H \). Set \( f_\sharp = f - \gamma^\sharp (f) e_\sharp \in H \) and define \( F = Zf_\sharp / Ze_0 \). By Lemma 6.3 the function \( F \) is square integrable. It is double periodic with unit periods and can be represented by the double Fourier series

\[ F (y, \xi) = \sum_{\lambda = (p, \theta) \in \Lambda} c_\lambda \exp (2\pi i (py + \theta \xi)), \]

where \( \sum |c_\lambda|^2 = \|F\|^2 \). By Proposition 3.1 the series \( \sum c_\lambda e_\lambda \) converges in \( L_2 (\mathbb{R}) \) to a function \( g \). On the other hand, because of (12),

\[ Zf_\sharp (y, \xi) = Ze_0 \sum c_\lambda \exp (2\pi i (py + \theta \xi)) = \sum c_\lambda Ze_\lambda (y, \xi) \]

This implies \( f_\sharp = g \), since \( Z \) is a unitary operator. Set \( \gamma^\lambda (f) = c_\lambda \) for \( \lambda \in \Lambda \).

**Lemma 6.3** We have \( F \in L_2 (\mathbb{Q}) \).
Proof of Lemma. By the previous Lemma the function $Zf_\sharp$ is $\varepsilon$-Hölder continuous in the interior of $Q$. It vanishes at the sharp point, since $\gamma^\sharp (f_\sharp) = 0$. Therefore $|Zf_\sharp(z)| \leq C_\delta |z - 1/2 - i/2|^\varepsilon$. This implies

$$|F(z)| = \left| \frac{\exp(\pi y^2) Zf_\sharp}{\Theta(z)} \right| \leq C |z - 1/2 - i/2|^\varepsilon - 1,$$

since $\Theta(z)$ has simple zero at each sharp point. This implies the statement. ▶

Remark 1. More strong inequality holds for $f \in H^\delta, \delta > 1$

$$\sum_{\Lambda} (|\lambda| + 1)^{2\varepsilon} |\gamma^\lambda(f)|^2 \leq C_{\delta, \varepsilon} \|f\|_\delta^2, 0 < \varepsilon < \delta - 1$$

Remark 2. We can take an arbitrary sharp point $\mu \in \Lambda + \frac{1}{2}$ instead of $(1/2, 1/2)$ in Theorem 6.1.

7 Gabor coefficients

The coefficients are uniquely defined since of

Proposition 7.1 If $\{c_\lambda\} \in l_2(\Lambda), b \in \mathbb{C}$ and

$$be_\sharp + \sum_{\Lambda} c_\lambda e_\lambda = 0, \quad (20)$$

then $c_\lambda = 0$ for all $\lambda$.

Proof. By Proposition 3.4 the series (20) converges in $L_2$. Apply the Zak transform. By (12) we get

$$-bZe_\sharp = \sum_{\Lambda} c_\lambda Ze_\lambda = Ze_0 \sum_{\Lambda} c_\lambda \exp 2\pi i (p\xi + \theta y)$$

The series in the right-hand side converges to a function $g \in L_2(Q)$ since $Z$ is a unitary operator. It follows $g = -bZe_\sharp/Ze_0$. We have $Ze_\sharp (1/2, 1/2) \neq 0$, whereas the function $Ze_0 (y, \xi) = \exp (-\pi y^2) \Theta (\xi + iy)$ vanishes in the point $1/2 + i/2$. Therefore the function $g$ cannot be square integrable unless $b = 0$. This yields $g = 0$, hence $c_\lambda = 0$ for all $\lambda \in \Lambda$. ▶

Corollary 7.2 We have the following explicit formula for the coefficients:

$$\gamma_\lambda(f) = \int_0^1 \int_0^1 \exp (-2\pi i (py + \theta \xi)) \frac{\exp(\pi y^2) Zf_\sharp(y, \xi)}{\Theta(\xi + iy)} d\xi dy, \quad \lambda \in \Lambda \quad (21)$$

where $f_\sharp = f - \gamma^\sharp (f) e_\sharp$. 

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Remark. It follows from a formula of M. Bastiaans \cite{1}, that the coefficients can be written in the form

$$
\gamma^{\lambda}(f) = \langle f^{\sharp} \mid \exp (2\pi i \theta x) \gamma(x - p) \rangle, \quad \lambda = (p, \theta) \in \Lambda,
$$

where

$$
\gamma(x) = 2^{-1/4} \left(\pi / K_0\right)^{3/2} \exp \left(\pi x^2\right) \sum_{n+1/2 \geq x} (-1)^n \exp \left(-\pi (n + 1/2)^2\right)
$$

This function $\gamma$ does not belong to $L_2$, but the integrals $\langle f^{\sharp} \mid \exp (2\pi i \theta x) \gamma(x - p) \rangle$ converge since $f^{\sharp} \in H$ and $\gamma^{\sharp}(f^{\sharp}) = 0$.

8 Improving convergence

To ensure faster convergence of the relaxed Gabor series we impose more sharp conditions on $f$. Fix an integer $m > 0$ and consider the norm

$$
\|f\|_{\delta,m} = \left(\sum_{j=0}^{m} \|a_j f\|_{\delta}^2\right)^{1/2}
$$

Let $H_{\delta,m}$ be the space of functions with finite norm $\|\cdot\|_{\delta,m}$.

Lemma 8.1 For arbitrary different points $\mu_0, \ldots, \mu_m \in \mathbb{C}$ the inverse to the VanderMond matrix $W = \{w^k_j = (\mu_j)^k\}$ is equal to the matrix $V = \{v^k_j\}$ where

$$
v^k_j = \frac{\sigma^{(k)}_{m+1-j}}{p^j(\mu_k)}, \quad p(\lambda) = \Pi_{j=0}^m (\lambda - \mu_j),
$$

$$
\sigma^{(k)}_j = (-1)^j \sigma_j (\mu_0, \ldots, \hat{\mu}_k, \ldots, \mu_m)
$$

and $\sigma_j$ denotes the $j$-th elementary symmetric polynomial.

Proof is by straightforward check.

Lemma 8.2 For an arbitrary natural $m$ and any set of different points $\mu_0, \ldots, \mu_m \in \Lambda + \frac{1}{2}$ there are numbers $\{h^k_j\}$ such that

$$
\gamma^{\sharp}\left(a^k d_{m,j}\right) = \delta^k_j, \quad k, j = 0, \ldots, m
$$

where

$$
d_{m,j} = \sum_{s=0}^{m} h^s_{j} e_{\mu_s}, \quad j = 0, \ldots, m.
$$
PROOF. Calculate the matrix 
\[ g^k_j = \gamma^s (a^k e_{\mu_j}), \quad j, k = 0, \ldots, m. \]
By (17) we have for an arbitrary \( \mu = (q, \eta) \in \Lambda + \sharp \)
\[ Ze_\mu (y, \xi) = \exp \left( -\pi (y - 1/2)^2 + 2\pi i [\eta y + (q - 1/2) (\xi + 1/2)] \right) \]
\[ \times \Theta (\xi + 1/2 + i (y - 1/2)) \]
which yields \( Ze_\mu (\sharp) = \exp (\pi i \eta) \Theta (0). \) By (13)
\[ \gamma^s (a^k e_\mu) = \frac{\mu^k}{i \Theta (0)} Ze_\mu (\sharp) = (-1)^{|\eta|} \mu^k \]
This yields \( g^k_j = (-1)^{|\eta_j|} \mu^k_j. \) The matrix \( \{ g^k_j \} \) is invertible and by Lemma 8.1 the entries of the inverse matrix are
\[ h^k_j = (-1)^{|\eta_j|} \frac{\sigma^{(k)}_{m+1-j}}{p' (\mu_k)}, \quad k, j = 0, \ldots, m. \]
Then (23) implies (22). ▶

Lemma 8.3 We have for any \( \lambda \in \Phi \)
\[ \left\| \sum_{k,j} d_{m,j} \lambda^j \right\|_{\delta,m} \leq (M + 1)^{m+\delta} L^m \] (24)
where
\[ M = \max |\mu_s|, \quad L = \max |\lambda - \mu_s| \]

PROOF. We have for \( k, j = 1, \ldots, m \)
\[ \left\| a^k \sum_j d_{m,j} \lambda^j \right\| = \left\| \sum_{j,s} h^s_j \lambda^j e_{\mu_s} \right\| \leq M^k \left\| \sum_j h^s_j \lambda^j \right\| \]
since \( \|e_{\mu}\| = 1. \) Further
\[ \sum_j h^s_j \lambda^j = \pm \sum_j \frac{\sigma^{(s)}_{m+1-j}}{p' (\mu_s)} \lambda^j = \pm \frac{p_s (\lambda)}{p_s (\mu_s)} \]
where \( p_s (\lambda) = \prod_{j \neq s} (\lambda - \mu_j). \) This yields
\[ \left\| \sum_j h^s_j \lambda^j \right\| = \frac{p_s (\lambda)}{p_s (\mu_s)} \prod_{j \neq s} \left| \frac{\lambda - \mu_j}{\mu_s - \mu_j} \right| \leq L^m \]
and (24) follows. ▶
Theorem 8.4 For an arbitrary natural $m$ and different points $\mu_0, ..., \mu_m \in \Lambda + \frac{1}{2}$ there exists a family of continuous functionals $\gamma^\lambda_m, \lambda \in \Lambda$ in $H^m = \bigcup_{\delta > 1} H^\delta_m$ such that for an arbitrary $f \in H^m$ the equation holds in the space $W^2_k(\mathbb{R})$:

$$f = \sum_{j=0}^{m} \gamma^j \left(a^j f\right) d_{m,j} + \sum_{\lambda} \gamma^\lambda_m \left(f\right) e_\lambda,$$

(25)

where $d_{m,j}$ are as in (23) and there exists for any $\delta > 1$ a constant $C_\delta$ such that

$$\sum_{\lambda} \left(\left|\lambda\right|^2 + 1\right)^m \left|\gamma^\lambda_m \left(f\right)\right|^2 \leq C_\delta \|f\|_{\delta,m}^2.$$

(26)

Proof. Set

$$f_\sharp = f - \sum_{j=0}^{m} \gamma^j \left(a^j f\right) d_j \in H^m$$

By (14) and Lemma 6.2, the function $A^j Z f_\sharp$ is continuous in the interior of $\mathbb{Q}$ for $j \leq m$. We have $A = \bar{\partial}/2 \pi i + y$, where $\bar{\partial} = (\partial_\xi + i \partial_y)$ and

$$A^j Z f_\sharp \left(y, \xi\right) = \Theta \left(\xi + iy\right) A^j \left[\exp \left(-\pi y^2\right) F \left(y, \xi\right)\right] = \Theta \left(\xi + iy\right) \exp \left(-\pi y^2\right) \bar{\partial}^j F \left(y, \xi\right),$$

since the function $\Theta$ is holomorphic and $A \exp \left(-\pi y^2\right) = 0$. Therefore the function $\bar{\partial}^j F$ is Hölder continuous in $\mathbb{Q}$ for $j \leq m$. The property (22) and (14) imply that $\gamma^j \left(a^j Z f_\sharp\right) = 0, j = 0, ..., m$ that is the left-hand side vanishes at the sharp point. By Lemma 6.3, the function

$$\bar{\partial}^j F = \frac{\exp \left(\pi y^2\right) A^j Z f_\sharp \left(y, \xi\right)}{\Theta \left(\xi + iy\right)}$$

is double periodic and belongs to $L_2 \left(\mathbb{Q}\right)$ for $j = 0, ..., m$. Therefore $F \in W^m_2 \left(\mathbb{R}^2 / \Lambda\right)$ and the Fourier coefficients $c_\lambda$ of $F$ satisfy

$$\sum_{\lambda} \left(\left|\lambda\right|^2 + 1\right)^m |c_\lambda|^2 \leq C \|F\|_{W^m_2}^2 \leq C'' \|f_\sharp\|_{\delta,m}^2 \leq C''' \|f\|_{\delta,m}^2.$$

This yields (26) for $\gamma^\lambda_m \left(f\right) \div c_\lambda, \lambda \in \Lambda$. ▶

A representation like (25) is unique, in spite of the additional terms:

Proposition 8.5 If for some integer $m \geq 0$ the series

$$\sum_{\lambda} \left(\left|\lambda\right|^2 + 1\right)^m |c_\lambda|^2$$

(27)

converges and

$$\sum_{j=0}^{m} b_j d_j + \sum_{\lambda} c_\lambda e_\lambda = 0$$

(28)

for some $b_0, ..., b_m$, then $b_0 = ... = b_m = 0$ and $c_\lambda = 0$ for all $\lambda$. 

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Proof. The series in (28) converges to a function \( g \in W_2^m (\mathbb{R}) \). Moreover \( g \) belongs to the domain of the operators \( a_j \), \( j = 1, \ldots, m \). This follows from convergence of (27) and arguments of Proposition 3.1. By (14) the Zak transform \( Zf \) belongs to the domain of operators \( A_j \), \( j = 1, \ldots, m \). By (12) we have

\[
- \sum_{0}^{m} b_j Zd_j = \sum_{\lambda} c_{\lambda} Ze_{\lambda} = Ze_{0} \sum_{\lambda} c_{\lambda} \exp 2\pi i (p\xi + \theta y)
\]

The series in the right-hand side converges to a function \( g \in L_2 (\mathbb{Q}) \) and \( g = - \sum b_j Zd_j / Ze_{0} \). We have \( Ze_{0} (z_t) = 1 \), whereas the functions \( Zd_1, \ldots, Zd_m \) and \( Ze_{0} (y, \xi) \) vanish at the sharp point. Moreover, we have \( |Ze_{0} (y, \xi)| \leq c |z - 1/2 - i/2| \). Therefore the function \( g \) can not be square integrable unless \( b_0 = 0 \). Similarly, the inclusion \( Ag \in L_2 (\mathbb{Q}) \) implies that \( b_1 = 0 \) and so on. Therefore \( b_0 = \ldots = b_m = 0 \) and \( g = 0 \); which yields \( c_{\lambda} = 0 \) for all \( \lambda \in \Lambda \).

9 Gabor transform of a Gabor series

Let \( G \) be a set in the phase plane \( \Phi \) and \( r > 0 \); for any \( r > 0 \) we denote by \( G (r) \) the \( r \)-neighborhood of \( G \). Consider a convergent series

\[
g = \sum_{\lambda \in G} c_{\lambda} e_{\lambda}
\]

and estimate the Gabor transform of \( g \):

**Proposition 9.1** For any subset \( G \subset \Lambda \) and any \( r > 0 \) we have

\[
\int_{\Phi \setminus G (r)} |\langle g | e_{\mu} \rangle|^2 d\mu \leq \exp (-\pi r^2) \sum_{\lambda} |c_{\lambda}|^2
\]

Proof. By (11) we have for \( \mu \in \Phi \setminus G (r) \)

\[
|\langle g | e_{\mu} \rangle|^2 \leq \left( \sum_{\lambda \in G} |\langle e_{\mu} | e_{\lambda} \rangle| \right)^2 \leq \sum_{G} \exp (-\pi |\lambda - \mu|^2) \sum_{G} |c_{\lambda}|^2.
\]

Integrating the right-hand side yields

\[
\int_{\Phi \setminus G (r)} |\langle g | e_{\mu} \rangle|^2 d\mu \leq \sum_{\lambda \in G} |c_{\lambda}|^2 \int_{\Phi} \exp (-\pi |\lambda - \mu|^2) \phi (\mu) d\mu.
\]

where \( \phi \) is the indicator function of the set \( \Phi \setminus G (r) \). Write the sum over \( \lambda \) as the integral over \( \Phi \) with the singular measure \( \sigma (\lambda) = \sum_{G} |c_{\lambda}|^2 \delta_{\lambda} \). and change the
variable \( \lambda \) by \( \kappa = \lambda - \mu \) in the double integral

\[
\sum_{\lambda \in G} \int_{\Phi} \exp \left( -\pi |\lambda - \mu|^2 \right) h(\mu) \, d\mu \, |c^\lambda|^2 = \int_{\Phi} \int_{\Phi} \exp \left( -\pi |\lambda - \mu|^2 \right) \phi(\mu) \, d\mu \sigma(\lambda)
\]

\[
= \int_{\Phi} \exp \left( -\pi |\kappa|^2 \right) \int_{\Phi} \phi(\mu) \sigma(\kappa + \mu) \, d\mu
\]

The interior integral in the right-hand side vanishes if \( |\kappa| < r \), otherwise it is bounded by the total integral

\[
\int \sigma(\kappa + \mu) \, d\mu = \sum |c^\lambda|^2 \, d\kappa
\]

This yields the estimate for the right-hand side

\[
\sum |c^\lambda|^2 \int_{|\kappa| \geq r} \exp \left( -\pi |\kappa|^2 \right) \, d\kappa = \sum |c^\lambda|^2 \exp \left( -\pi r^2 \right) . \quad \square
\]

10 A ‘certainty’ theorem

Theorem 10.1 Let \( K \subset D \) be arbitrary bounded domains in \( \Phi \), such that \( D \) is \( r \)-neighborhood of \( K \) for some \( r \geq r_0 \). An arbitrary function \( f \in H \) can be written in the form

\[
f = \sum_{\lambda \in \Lambda \cap D} \alpha^\lambda e_\lambda + \sum_{\mu \in (\Lambda + \ell) \cap D \setminus K} \omega^\mu e_\mu + \phi_r
\]

where for \( f \in H^\delta \)

\[
\|
\phi_r\| \leq \left( \int_{\Phi \setminus D} |\langle f | e_\mu \rangle|^2 \, d\mu \right)^{1/2} + C_\delta r^\delta \exp \left( -r/e \right) \| f \|_\delta . \quad (29)
\]

and the constant \( C_\delta \) does not depend on \( K \) and \( D \).

Remark 1. Note that

\[
\alpha^\lambda = \gamma^\lambda (f) + O \left( r^\delta \exp \left( -r/e \right) \right)
\]

for \( \lambda \in K \). The geometry of the two first terms in shown in Fig.1.
Remark 2. The exponential term in the estimate (29) is indispensable, but its form might be made sharper. Indeed, vanishing of Gabor means \( \langle f | e_\mu \rangle \) for \( \mu \in \Phi \setminus D \) does not guarantee that \( f \) can be represented by a Gabor functions supported in \( \Lambda \cap D \) without additional term. This follows from Proposition 9.1 where the exponential factor \( \exp(-\pi r^2) \) is sharp.

Proof of Theorem. Set \( \bar{K}_+ = K(l), \; U = K(r/2), \; D_- = K(r - l) \) where the parameter \( l = O(r^{1/2}) \) will be specified later. We have \( \bar{K} \subset K_+ \subset U \subset D_- \subset D \).

By Theorem 6.1 we can write \( f = f_U + g \), where

\[
    f_U = \sum_{\Lambda \cap U} \gamma^\lambda(f) e_\lambda, \quad g = \sum_{\Lambda \setminus U} \gamma^\lambda(f) e_\lambda. \tag{30}
\]
and the sharp point $\sharp$ is chosen in $U \setminus K$. Estimate Gabor transform of $g$. By Proposition 9.1 and Theorem 6.1
\[
\int_{K_+} |\langle g|e_\mu \rangle|^2 \, d\mu \leq \exp \left( -\pi \left( \frac{r}{2} - l \right)^2 \right) \sum_{\lambda \in \Lambda \cup U} |\gamma^\Lambda (f)|^2 \leq C \exp \left( -\pi \left( \frac{r}{2} - l \right)^2 \right) \|f\|_\delta^2
\]
If $\mu \in \Phi \setminus D_-$, we can write $\langle g|e_\mu \rangle = \langle f|e_\mu \rangle - \langle f_U|e_\mu \rangle$ since of (30). Apply again Proposition 9.1 and Theorem 6.1 and obtain
\[
\int_{\Phi \setminus D_-} |\langle g|e_\mu \rangle|^2 \, d\mu \leq \int_{\Phi \setminus U_+} |\langle f|e_\mu \rangle|^2 \, d\mu + \exp \left( -\pi \left( \frac{r}{2} - l \right)^2 \right) \|f\|_\delta^2,
\]
since $f_U$ is a sum of functions $e_\lambda$, $\lambda \in U$. By Proposition 2.1 we can write
\[
g = \int_{K_+} \langle g|e_\mu \rangle e_\mu \, d\mu + \int_{\Phi \setminus D_-} \langle g|e_\mu \rangle e_\mu \, d\mu + \int_{D_- \setminus K_+} \langle g|e_\mu \rangle e_\mu \, d\mu
\]
\[
\triangleq g_0 + g_+ + g_-
\]
The norms of the first two terms are bounded as follows
\[
\|g_+\|^2 \leq \exp \left( -\pi \left( \frac{r}{2} - l \right)^2 \right) \|f\|_\delta^2,
\]
\[
\|g_-\|^2 \leq \exp \left( -\pi \left( \frac{r}{2} - l \right)^2 \right) \|f\|_\delta^2 + \int_{\Phi \setminus D} |\langle f|e_\mu \rangle|^2 \, d\mu
\]
We fix a natural $m \leq r - 1$ and transform the integral $g_0$ as follows: for a point $\mu \in D_- \setminus K_+$ we choose a point $\lambda \in \Lambda \cap D \setminus K$ and such that $|\lambda - \mu| \leq 1/\sqrt{2}$ and consider the closed square $Q(\mu)$ centered at $\lambda$ with side $\sqrt{m+1}$, see Fig.2.
The set \( Q(\mu) \) contains \( m+1 \) different points \( \mu_0, \ldots, \mu_m \in \Lambda^+ \) and is contained in the \( l \)-neighborhood of \( \mu \) where \( l = ((m + 1) / 2)^{1/2} + 1 \). It follows that \( Q(\mu) \subset D \setminus K \) and
\[
\max |\mu - \mu_s| \leq l, \quad \max |\lambda - \mu_s| \leq l.
\] (34)

Applying Theorem 8.4 to the function \( \phi = e_{\mu - \lambda} \) and sharp points \( \mu_0 - \lambda, \ldots, \mu_m - \lambda \) yields
\[
e_{\mu - \lambda}(x) = \sum_{j=0}^{m} \gamma_j^\sharp (\alpha^k e_{\mu - \lambda}) d_{m,k} + g_{\mu,\lambda},
\] (35)
\[
g_{\mu,\lambda} = \sum_{\nu} \gamma_{m}^\nu (e_{\mu - \lambda}) e_{\nu},
\]
where \( d_j \) belong to the linear span of \( e_{\mu_s - \lambda}, j = 0, \ldots, m \). We have
\[
\|g_{\mu,\lambda}\|_{\delta,m} \leq \|e_{\mu - \lambda}\|_{\delta,m} + \left\| \sum_j \gamma_j^\sharp (\alpha^k e_{\mu - \lambda}) d_{m,k} \right\|_{\delta,m}.
\]
and \( \alpha^k e_{\mu - \lambda} = (\mu - \lambda)^k e_{\mu - \lambda} \). By Proposition 8.3 and 8.4
\[
\left\| \sum_k \gamma_k^\sharp (\alpha^k e_{\mu - \lambda}) d_{m,k} \right\|_{\delta,m} \leq C (l + 1)^{m+\delta} l^m \leq C (m + 1)^{m+\delta}
\]
\[
\|e_{\mu - \lambda}\|_{\delta,m} \leq C (l + 1)^{m+2} \leq C' (m + 1)^{(m+\delta)/2}
\]
where the constants does not depend on \( m \). This yields
\[
\left\| \sum_j \gamma_j^\sharp (\alpha^k e_{\mu - \lambda}) d_{m,k} \right\|_{\delta,m} \leq C (m + 1)^{m+\delta},
\]
\[
\|g_{\mu,\lambda}\|_{\delta,m} \leq C' (m + 1)^{m+\delta}
\]
By Theorem 8.4
\[
\sum |\nu|^2 + 1 \sum \gamma_{m}^\nu (e_{\mu - \lambda})^2 \leq C (m + 1)^{2(m+\delta)} \|e_{\mu - \lambda}\|_{\delta,m}^2.
\] (36)

Let \( \lambda = (p, \theta) \in \Lambda \); apply the operator \( T_{\lambda} \) as in (9). For \( \mu = (q, \xi), \nu = (r, \eta) \) we obtain
\[
T_{\lambda} e_{\mu - \lambda} = \exp(-2\pi ip (\xi - \theta)) e_{\mu}, \quad T_{\lambda} e_{\nu} = \exp(-2\pi ip\eta) e_{\nu + \lambda}
\]
and
\[
e_{\mu}(x) = \sum_{j=0}^{m} \gamma_j^\sharp (\alpha^k e_{\mu - \lambda}) \exp(2\pi ip (\xi - \theta)) d_{j,m}
\]
\[
+ \sum_{\nu} \gamma_{m}^\nu (e_{\mu - \lambda}) \exp(2\pi ip (\xi - \theta + \eta)) e_{\nu + \lambda}
\] (37)
The function $d_{j,m}$ belongs to the linear envelope of $e_{\kappa}, \kappa \in \Lambda_+$. Change the variable $\nu + \lambda$ by $\nu$ in the second sum and write this equation in the form

$$e_\mu(x) = \sum_{\kappa \in (\Lambda+2) \cap Q(\mu)} \beta^\kappa e_\kappa + \sum_{\lambda \in \Lambda \cap D} \sum_{\nu \in \Lambda \cap D} \varepsilon_\mu^\lambda e_\lambda + \sum_{\nu \in \Lambda \cap D} \omega^\nu e_\nu$$

where $\omega^\nu = \gamma_m^{\nu - \lambda}(e_{\mu - \lambda}) \exp (2\pi i r (\xi - \theta + \eta))$. Estimate the third term by means of (36)

$$\sum_\nu |\omega^\nu|^2 = \sum_{\lambda \in \Lambda \cap D} |\gamma_m^{\nu - \lambda}(e_{\mu - \lambda}) \exp (2\pi i r (\xi - \theta + \eta))|^2$$

$$\leq C (m + 1)^{2m+2} (|\nu - \lambda| + 1)^{-2m} \leq C (m + 1)^{2(m+\delta)} r^{-2m}$$

since $|\nu - \lambda| \geq r^{-2^{1/2}}$. Taking $m = r/e - 1$ we obtain $(m + 1)^{m+\delta} r^{-m} \leq C r^\delta \exp (-r/e)$. The relation $l = O \left( m^{1/2} \right) = O \left( r^{1/2} \right)$ is fulfilled for this choice of $m$. Taking in account that the square $Q(\mu)$ is contained in $D \setminus K$ and integrating (38) on $D \setminus K_+$ against the density $\langle g | e_\mu \rangle d\mu$, yields

$$g_0 = \sum_{\kappa \in (\Lambda+2) \cap D \setminus K} \beta^\kappa e_\kappa + \sum_{\lambda \in \Lambda \cap D} \varepsilon^\lambda e_\lambda + \sum_{\nu \in \Lambda \cap D} \omega^\nu e_\nu,$$

where

$$\beta^\kappa = \int_{D \setminus K_+} \beta^\kappa \langle g | e_\mu \rangle d\mu, \varepsilon^\lambda = ..., \omega^\nu = ...$$

By Proposition 3.1 and Theorem 6.1 we have

$$(\sum |\omega^\nu|^2)^{1/2} \leq C \exp (-r/e) \left( \int |\langle g | e_\mu \rangle|^2 d\mu \right)^{1/2} \leq C r^\delta \exp (-r/e) \|g\|$$

$$\leq C_{\delta} r^\delta \exp (-r/e) \|f\|$$

Finally we get

$$f = \sum_{\lambda \in \Lambda \cap D} \alpha^\lambda e_\lambda + g_+ + g_- + \sum_{\nu \in (\Lambda+2) \cap D \setminus K} \beta^\nu e_\nu + \sum_{\lambda \in \Lambda \cap D} \omega^\nu e_\nu$$

where $\alpha^\lambda = \gamma^\lambda (f) + \varepsilon^\lambda$ for $\lambda \in \Lambda \cap U$ and $\alpha^\lambda = \varepsilon^\lambda$ for $\lambda \in \Lambda \cap D \setminus U$ and the term $\gamma^\lambda (f) e_\lambda$ is included in the second sum. We arrange this sum as follows

$$f = \sum_{\Lambda \cap D} \alpha^\lambda e_\lambda + \sum_{\kappa \in (\Lambda+2) \cap D \setminus K} \beta^\kappa e_\kappa + \phi_r$$

where and

$$\phi_r = g_+ + g_- + \sum_{\lambda \in \Lambda \cap D} \omega^\lambda e_\lambda.$$
By (32), (33) and (40) we have
\[ \| \phi_r \| \leq \| g_\pm \| + \left\| \sum \omega^r e_\nu \right\| \]
\[ \leq \left( \int_{\Phi \setminus D} |\langle f | e_\mu \rangle|^2 \, d\mu \right)^{1/2} + 2 \exp \left( -\pi (r/2 - l)^2 /2 \right) \| f \|_\delta + C_\delta r^\delta \exp \left( -r/e \right) \| f \|_\delta, \]
which yields (29) for any sufficiently large \( r \).

11 Metaplectic group

Remind that the Weyl-Heisenberg group is the space \( X \times X^* \times \mathbb{R} \) with the group operation
\[ (x, \xi, \tau) \cdot (x', \xi', \tau') = \left( x + x', \xi + \xi', \tau + \tau' + \frac{1}{2} (x' \xi - x \xi') \right). \]

**Definition.** A linear transform \( S \) of \( \Phi \) is called symplectic, if it preserves the canonical bilinear form \( \sigma ([x, \xi], [y, \eta]) = \eta x - \xi y \). We can see that \( \sigma [u, v] = \langle u | J v \rangle \), where
\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]
In particular, \( \lambda = (p, \theta) \mapsto J \lambda = (\theta, -p) \) is a linear symplectic transform. In the case \( X = \mathbb{R} \) a linear transformation \( S \) in \( \mathbb{R} \times \mathbb{R}^* \) is symplectic if the matrix \( (a, b, c, d) \) of \( S \) satisfies \( ad - bc = 1 \). For an arbitrary linear symplectic transformation \( S \) such that \( b \neq 0 \) the integral transform
\[ M_S f (x) = (ib)^{-1/2} \int \exp \left( \pi i \left( \frac{d}{b} x^2 - \frac{2}{b} y x + \frac{a}{b} y^2 \right) \right) f (y) \, dy. \]
is well defined operator in \( L_2 \). If \( n = 0 \), the operator \( M_S \) is defined as composition \( M_T F \), where \( F \) is the Fourier transform and \( T = i^{1/2} S J^{-1} \). The operator \( M_S \) has unitary closure in \( L_2 \). The equation \( M_S M_T = \pm M_{ST} \) holds for any symplectic transformations \( S, T \). It is called metaplectic (two-valued) representation of \( S \). In particular, the metaplectic operator \( M_J \) is the Fourier transform \( F \) times the factor \( i^{1/2} \). This representation is single-valued on the irreducible two-fold covering of the group \( S \). See more information in [5].

Any rotation \( S(x, \xi) = (\cos \varphi \, x - \sin \varphi \, \xi, \sin \varphi \, x + \cos \varphi \, \xi) \) is an orthogonal symplectic transformation.

**Proposition 11.1** If \( S \) is a rotation as above, then
\[ M_S a = \exp (-i \varphi) \, a \, M_S \]
\[ M_S a^+ = \exp (i \varphi) \, a^+ \, M_S \]
where \( a^+, a \) is the creation and the annihilation operator, respectively.
**Proof.** Direct calculation.

**Proposition 11.2** For any rotation $S$ in the phase plane and any point $\lambda$ we have

$$M_S(e_\lambda) = \pm \exp (i\varphi/2) \exp (\pi i (p\theta - q\eta)) e_{S(\lambda)}$$

(41)

where $(q, \eta) = S(p, \theta)$

**Proof** is straightforward.

**Remark.** The equation (41) means that $M_S$ transforms a Gabor function $e_\lambda$ to another Gabor function (up to a phase factor) while the ‘quantum support’ $\lambda$ of a Gabor function moves by action of the corresponding geometric transform $S$. In particular, the Fourier transform belongs to the metaplectic group: $F = M_J$. We have for any $\lambda = (p, \theta)$, $\hat{e}_\lambda(\eta) = \pm \exp (i\varphi/2) \hat{e}_{\lambda}(\eta)$, $\hat{\lambda} = J\lambda = (\theta, -p)$ for a real $\phi$. The metaplectic representation can be thought as a ‘quantization’ of group of simplectic transforms.

**Corollary 11.3** For any $\delta > 1$ and natural $m$ the space $H^{\delta,m}$ is invariant under action of the metaplectic representation of the rotation group.

**Proof.** For any rotation $S$ we have

$$\langle f|e_\lambda \rangle = \langle M_S f|M_S e_\lambda \rangle = \exp (i\varphi/2) \langle M_S f|e_\lambda \rangle$$

which yields $|\langle M_S f|e_\lambda \rangle| = |\langle f|e_\lambda \rangle|$. It follows that $H^{\delta}$ is invariant. The same true for $H^{\delta,m}$ since of (41).

**Corollary 11.4** For an arbitrary orthogonal simplectic transformation $S$ in the phase space, theorem 6.1 holds for functions $f \in H$ and Gabor system $e_\lambda, \lambda \in S(\Lambda^\delta)$.

**Remark.** The operator $M$ relates to the metaplectic representation $\mu$ in the sense of [5] by the equation $M_S = \mu (\tilde{S})$, where the simplectic matrix $\tilde{S}$ is obtained from $S$ by changing sign at $b$ and $c$ and replacing $\theta$ by $-\theta$. This results in the representation $\mu$ which is chosen in [5]. Note that the mapping $S \mapsto \tilde{S}$ is an involution in the simplectic group.

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