Cooperative Multi-Cell Block Diagonalization with Per-Base-Station Power Constraints

Rui Zhang

Abstract

Block diagonalization (BD) is a practical linear precoding technique that eliminates the inter-user interference in downlink multiuser multiple-input multiple-output (MIMO) systems. In this paper, we apply BD to the downlink transmission in a cooperative multi-cell MIMO system, where the signals from different base stations (BSs) to all the mobile stations (MSs) are jointly designed with the perfect knowledge of the downlink channels and transmit messages. Specifically, we study the optimal BD precoder design to maximize the weighted sum-rate of all the MSs subject to a set of per-BS power constraints. This design problem is formulated in an auxiliary MIMO broadcast channel (BC) with a set of transmit power constraints corresponding to those for individual BSs in the multi-cell system. By applying convex optimization techniques, this paper develops an efficient algorithm to solve this problem, and derives the closed-form expression for the optimal BD precoding matrix. It is revealed that the optimal BD precoding vectors for each MS in the per-BS power constraint case are in general non-orthogonal, which differs from the conventional orthogonal BD precoder design for the MIMO-BC under one single sum-power constraint. Moreover, for the special case of single-antenna BSs and MSs, the proposed solution reduces to the optimal zero-forcing beamforming (ZF-BF) precoder design for the weighted sum-rate maximization in the multiple-input single-output (MISO) BC with per-antenna power constraints. Suboptimal and low-complexity BD/ZF-BF precoding schemes are also presented, and their achievable rates are compared against those with the optimal schemes.

Index Terms

Block diagonalization, convex optimization, cooperative multi-cell system, multi-antenna broadcast channel, network MIMO, per-antenna power constraint, per-base-station power constraint, zero-forcing beamforming.

I. INTRODUCTION

The study of downlink beamforming and power control in cellular systems has been an active area of research for many years. Conventionally, most of the related works have focused on a single-cell setup, where the co-channel interferences experienced by the mobile stations (MSs) in a particular cell caused by the base stations (BSs) of the other cells are treated as additional noises at the receivers. For this setup, the downlink transmission in a single cell with a multi-antenna BS and multiple single-/multi-antenna MSs can be modeled by a multiple-input single-/multiple-output (MISO/MIMO) broadcast channel (BC). It is known that the dirty paper coding (DPC) technique

This paper has been presented in part at IEEE Wireless Communications and Networking Conference (WCNC), Sydney, Australia, April 18-21, 2010.

R. Zhang is with the Institute for Infocomm Research, A*STAR, Singapore (e-mail:rzhang@i2r.a-star.edu.sg) and the Department of Electrical and Computer Engineering, National University of Singapore (e-mail:elezhang@nus.edu.sg).
achieves the capacity region for the Gaussian MISO/MIMO BC, which constitutes all the simultaneously achievable rates for all the MSs [1]. However, DPC requires complicated nonlinear encoding and decoding schemes and is thus difficult to implement in real-time systems. Consequently, linear transmit and receive beamforming schemes for the Gaussian MISO/MIMO BC have drawn a great deal of attention in the literature [2], [3], [4], [5], [6]. In particular, a simple linear precoding scheme for the MIMO BC is known as block diagonalization (BD) [7], [8], [9], [10]. With BD, the transmitted signal from the BS intended for each MS is multiplied by a precoding matrix, which is restricted to be orthogonal to the downlink channels associated with all the other MSs. Thereby, all the inter-user interferences are eliminated and each MS perceives an interference-free MIMO channel. In the special case of MISO BC, BD reduces to the well-known zero-forcing beamforming (ZF-BF) [4]. Although BD is in general inferior in terms of achievable rate as compared to the DPC-based optimal nonlinear precoding scheme or the minimum-mean-squared-error (MMSE)-based optimal linear precoding scheme, it performs very well in the high signal-to-noise-ratio (SNR) regime and achieves the same degrees of freedom (DoF) for the MISO-/MIMO-BC sum-rate as the optimal linear/nonlinear precoding schemes [11]. Moreover, BD can be generalized to incorporate nonlinear DPC processing, which leads to a precoding scheme known as ZF-DPC [11].

Recently, there has been a rapidly growing interest in shifting the design paradigm from the conventional single-cell downlink transmission to the multi-cell cooperative downlink transmission [12], [13], [14], [15], [16], [17]. In these studies, it is assumed that BSs in a cellular network are connected via backhaul links to a central processing unit (e.g., a dedicated control station or a preassigned BS), which has the global knowledge of transmit messages for all the MSs in the network and downlink channels from each BS to all the MSs. Thereby, the central processing unit is able to jointly design the downlink transmissions for all BSs and provide them appropriate signals to transmit. As demonstrated in these works, by utilizing the co-channel interference across different cells in a coherent fashion, the cooperative multi-cell downlink processing leads to enormous throughput gains as compared to the conventional single-cell processing with the co-channel interference treated as noise. Moreover, design of distributed multi-cell downlink beamforming via the use of belief propagation and message passing among BSs has also been recently proposed in [18], without the need of a central controller.

In this work, we focus our study on the BD-based downlink precoding for a fully cooperative multi-cell system equipped with a central processing unit, which is assumed to have the perfect knowledge of all downlink channels and transmit messages in the network. For this setup, the BD precoding design problem can be formulated in an auxiliary MISO/MIMO BC with the number of transmitting antennas equal to the sum of those from all the cooperative BSs. However, instead of adopting the conventional sum-power constraint for the auxiliary
MISO/MIMO BC as in prior works [7], [8], [9], [10], this paper applies a set of transmit power constraints equivalent to those for individual BSs in the multi-cell system. The BD precoder design problem subject to per-BS power constraints is relatively new, and has been studied in, e.g., [19], [20], [21]. In these works, the BD precoders are designed essentially following the same principle as for the conventional sum-power constraint case, i.e., the precoding vectors known for the sum-power constraint case are adopted, and then power allocation is optimization to maximize the sum-rate under per-BS power constraints. However, it remains unclear whether the developed BD precoder solutions therein are indeed optimal for the weighted sum-rate maximization in a cooperative multi-cell system. In this paper, we show that the BD precoder designs following the heuristic of separating the beamforming design and power allocation optimization are indeed suboptimal for rate maximization, while the optimal BD precoder solution requires a new joint optimization approach, as will be proposed in this paper.

It is worth noting that the computation problem for the achievable rate region of the Gaussian MISO/MIMO BC subject to per-antenna power constraints has been studied in [22]. This work has been recently extended in [23], [24] to deal with more general linear transmit power constraints for the MISO/MIMO BC, with the per-antenna power constraint as a special case. The results in [22], [23], [24] can be directly applied for a cooperative multi-cell system to handle the per-BS power constraints, if the DPC-based optimal nonlinear precoder or the MMSE-based optimal linear precoder is used. On the other hand, the ZF-BF precoding design, as a simplified version of BD for the case of MISO BC, has been studied in [19], [24], [25] subject to per-antenna power constraints. In [19], the ZF-BF precoding matrix is taken as the pseudo inverse of the MISO-BC channel matrix and thereby decomposes the MISO BC into parallel interference-free scalar sub-channels for different MSs. The power allocation over the sub-channels is then optimized under per-antenna power constraints. However, it was pointed out in [25] that although the ZF-BF precoding matrix for the MISO BC based on the channel pseudo inverse is optimal for the sum-power constraint case, it is in general suboptimal for the per-antenna power constraint case. Thus, in [25] the authors proposed to apply the principle of generalized matrix inverse to design the ZF-BF precoding with per-antenna power constraints. The scheme in [25] has been improved in terms of computational efficiency and extended to the case of general linear power constraints in [24]. However, these MISO-BC ZF-BF solutions cannot be applied to obtain the optimal BD precoder design for the more general MIMO BC with per-BS power constraints.

The main contributions of this paper are summarized as follows:

- We formulate the MIMO-BC transmit optimization problem with the BD precoding and equivalent per-BS power constraints as a convex optimization problem. By applying convex optimization techniques, we design an efficient algorithm to solve this problem. We also derive the closed-form expression of the optimal BD
precoding matrix to maximize the weighted sum-rate for the MIMO-BC, from which we obtain a lower bound on the number of BSs that should transmit with their maximum power levels. More importantly, we prove that the optimal BD precoding (beamforming) vectors for each MS in the case of per-BS power constraints are in general non-orthogonal, which differs from the conventional orthogonal BD precoder design for the sum-power constraint case. Consequently, the orthogonal BD precoder designs proposed in prior works \cite{19}, \cite{20}, \cite{21} for the per-BS power constraint case are in general suboptimal (for weighted sum-rate maximization).

- For the special case of single-antenna BSs and MSs, we show that the proposed BD precoding design for the MIMO-BC provides the optimal ZF-BF precoder solution to maximize the weighted sum-rate for the MISO BC with per-antenna power constraints. We also compare the proposed solution with existing ones in prior works \cite{24}, \cite{25} based on approaches such as the generalized channel matrix inverse and the semi-definite programming (SDP) with rank-one relaxation.

- We also present a low-complexity, suboptimal scheme for the studied problem, which is obtained by computing the conventional BD precoder design for the sum-power constraint case with an optimal power allocation to meet the per-BS power constraints. This scheme can be considered as an extension of that given in \cite{19} for the MISO BC with the ZF-BF precoding and per-antenna power constraints to the MIMO BC with the BD precoding and per-BS power constraints. We derive an upper bound on the maximum number of BSs transmitting with their full power levels for this scheme, and identify the conditions under which this scheme becomes sum-rate optimal.

The rest of this paper is organized as follows. Section \ref{sec:sigmodel} introduces the signal model for the downlink transmission in a cooperative multi-cell system, and presents the problem formulation for the weighted sum-rate maximization with the BD precoding and per-BS power constraints. Section \ref{sec:opt} derives the optimal solution for this problem, and characterizes the optimal solution for the special case of MISO BC with per-antenna power constraints. Section \ref{sec:subopt} develops a heuristic suboptimal scheme for the studied problem. Section \ref{sec:num} provides numerical examples on the performance of the proposed optimal and suboptimal schemes. Finally, Section \ref{sec:conclusion} concludes the paper.

**Notations:** Scalars are denoted by lower-case letters, vectors denoted by bold-face lower-case letters, and matrices denoted by bold-face upper-case letters. $I$ and $0$ denote an identity matrix and an all-zero matrix, respectively, with appropriate dimensions. For a square matrix $S$, $\text{Tr}(S)$, $|S|$, $S^{-1}$, and $S^{1/2}$ denote the trace, determinant, inverse (if $S$ is full-rank), and square-root of $S$, respectively; and $S \succeq 0$ ($S \preceq 0$) means that $S$ is positive (negative) semi-definite. $\text{Diag}(a)$ denotes a diagonal matrix with the main diagonal given by $a$. For a matrix $M$ of arbitrary size, $M^H$, $M^T$, $\text{Rank}(M)$, and $M^+$ denote the conjugate transpose, transpose, rank, and pseudo inverse of $M$,
respectively. \( \mathbb{E}[\cdot] \) denotes the statistical expectation. The distribution of a circularly symmetric complex Gaussian (CSCG) random vector with mean vector \( x \) and covariance matrix \( \Sigma \) is denoted by \( \mathcal{CN}(x, \Sigma) \); and \( \sim \) stands for “distributed as”, \( \mathbb{C}^{x \times y} \) denotes the space of \( x \times y \) complex matrices. \( \|x\| \) denotes the Euclidean norm of a complex vector \( x \).

II. SYSTEM MODEL AND PROBLEM FORMULATION

We consider a multi-cell system consisting of \( A \) cells, each of which has a BS to coordinate the transmission with \( K_a \) MSs, \( K_a \geq 1 \) and \( a = 1, \cdots, A \). Denote the total number of MSs in the system as \( K = \sum_{a=1}^{A} K_a \). For convenience, we assume that all the BSs are equipped with the same number of antennas, denoted by \( M_B \geq 1 \). Denote the total number of antennas across all the BSs as \( M = M_B A \). We also assume that each of \( K \) MSs is equipped with \( N \) antennas, \( N \geq 1 \). Since we are interested in a fully cooperative multi-cell system, the jointly designed downlink transmission for all the BSs can be conveniently modeled by an auxiliary MIMO BC with \( M \) transmitting antennas and \( K \) MSs each having \( N \) receiving antennas. For convenience, we assign the indices to the transmitting antennas in the auxiliary MIMO BC belonging to different BSs in the multi-cell system according to the BS index, i.e., the \(((a-1)M_B+1)\)-th to \((aM_B)\)-th antennas are taken as the \( M_B \) antennas from the \( a \)th BS, \( a = 1, \cdots, A \). Similarly, the indices of MSs in the auxiliary MIMO BC are assigned according to their cell indices, i.e., the \((\sum_{i=1}^{a-1} K_i+1)\)-th to \((\sum_{i=1}^{a} K_i)\)-th MSs are taken as the \( K_a \) MSs from the \( a \)th cell, \( a = 1, \cdots, A \). Accordingly, the discrete-time baseband signal of the auxiliary MIMO BC is given by

\[
y_k = H_k x_k + \sum_{j \neq k} H_k x_j + z_k, \quad k = 1, \cdots, K
\]

where \( x_k \in \mathbb{C}^{M \times 1} \) and \( y_k \in \mathbb{C}^{N \times 1} \) denote the transmitted and received signals for the \( k \)th MS, respectively; \( H_k \in \mathbb{C}^{N \times M} \) denotes the downlink channel from all the \( M \) base-station antennas to the \( k \)th MS; and \( z_k \in \mathbb{C}^{N \times 1} \) denotes the receiver noise at the \( k \)th MS. For convenience, we assume that \( z_k \sim \mathcal{CN}(0, I), \forall k \).

Without loss of generality, we can further express \( x_k \) as

\[
x_k = T_k s_k, \quad k = 1, \ldots, K
\]

where \( T_k \in \mathbb{C}^{M \times D_k} \) is the precoding matrix (which specifies both the transmit beamforming vectors and allocated power values for different beams) for the \( k \)th MS; \( D_k \) denotes the number of transmitted data streams for the \( k \)th MS due to spatial multiplexing, with \( D_k \leq \min(M,N), \forall k \); and \( s_k \in \mathbb{C}^{D_k \times 1} \) denotes the information-bearing signal for the \( k \)th MS. We assume that \( s_k \)'s are independent over \( k \). It is further assumed that a Gaussian codebook is used for each MS at the transmitter and thus \( s_k \sim \mathcal{CN}(0, I), \forall k \). Denote \( S_k = \mathbb{E}[x_k x_k^H] \) as the transmit covariance matrix for the \( k \)th MS, with \( S_k \in \mathbb{C}^{M \times M} \) and \( S_k \succeq 0 \). It is easy to verify that \( S_k = T_k T_k^H \). The overall
The downlink transmit covariance matrix for the $M$ cooperative transmitting antennas is then $S = \sum_{k=1}^{K} S_k$. Since these transmitting antennas come from more than one BS, they need to satisfy a set of per-BS power constraints expressed as

$$\text{Tr} \left( B_a S \right) \leq P \text{ or } \sum_{k=1}^{K} \text{Tr} \left( B_a S_k \right) \leq P, \quad a = 1, \cdots, A$$

where

$$B_a \triangleq \text{Diag} \left( 0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0 \right)_{(a-1)M_B \rightarrow M_B \rightarrow (A-a)M_B}$$

and $P$ denotes the per-BS power constraint, which is assumed identical for all the BSs. Note that in the special case of single-antenna BSs and MSs, i.e., $M_B = N = 1$, the per-BS power constraints in (3) reduce to the per-antenna power constraints for an equivalent MISO BC.

We assume a quasi-static fading environment and thus the channels of interest in the auxiliary MIMO-BC remain constant for each downlink transmission frame. We consider the BD precoding scheme for each downlink frame transmission in the MIMO BC, which eliminates the inter-user interference, i.e., in (1) we have that for each given $k$, $H_j x_k = 0$ or $H_j^T k = 0, \forall j \neq k$. It is easy to show that the above “ZF constraints” are equivalent to the following constraints

$$H_j S_k H_j^H = 0, \quad \forall j \neq k.$$
cooperative multi-cell system with the BD precoding and per-BS power constraints as follows.

(P1): \[
\begin{align*}
\max_{S_1, \ldots, S_K} & \ \sum_{k=1}^{K} w_k \log |I + H_k S_k H_k^H| \\
\text{s.t.} & \ H_j S_k H_j^H = 0, \ \forall j \neq k \\
& \ \sum_{k=1}^{K} \text{Tr} (B_a S_k) \leq P, \ \forall a \\
& \ S_k \succeq 0, \ \forall k
\end{align*}
\]

where \( w_k \) is the given non-negative rate weight for the \( k \)th MS. For the purpose of exposition, we assume that \( w_k > 0, \forall k \). Note that in (P1), we have used transmit covariance matrices \( S_k \)'s instead of precoding matrices \( T_k \)'s as design variables. This is because with \( S_k \)'s, it is easy to verify that (P1) is a convex optimization problem, since the objective function is concave over \( S_k \)'s and all the constraints specify a convex set over \( S_k \)'s. Thus, (P1) can be solved using standard convex optimization techniques, e.g., the interior-point method [28]. However, such an approach does not reveal the structure of the optimal BD precoding solution. Therefore, in this paper we take a different approach to solve (P1), which is based on the Lagrange duality method [28] for convex optimization problems. As will be shown in Section III, this approach leads to a closed-form expression for the optimal BD precoding matrix, and reveals some interesting properties of the optimal solution.

**Remark 2.1:** It is worth noting that (P1) can be modified to incorporate additional per-antenna power constraints at all the BSs. Let \( P^{(pa)} \) denote the given per-antenna power threshold. Then, a set of \( M \) per-antenna power constraints can be included in (P1) as follows:

\[
\sum_{k=1}^{K} \text{Tr} \left( B_i^{(pa)} S_k \right) \leq P^{(pa)}, \ i = 1, \ldots, M
\]  

where \( B_i^{(pa)} \) is a diagonal matrix with the \( i \)th diagonal element equal to one and all the others equal to zero. Since the resulting optimization problem has similar structure to (P1), it can be solved in a similar way. In this paper, we omit the details for solving this modified version of (P1) for brevity.

**Remark 2.2:** It is also worth noting that (P1) can be modified to solve the weighted sum-rate maximization problem for the cooperative multi-cell downlink transmission with the ZF-DPC precoding [11] subject to the new per-BS/per-antenna power constraints. With ZF-DPC, given a fixed encoding order for the transmitted signals to different MSs (without loss of generality, we assume that the encoding order is given by the MS index), the signal for a later encoded MS is designed with the non-causal knowledge of all the earlier encoded MS signals, of which the associated interferences can be precanceled by DPC. By extending the ZF-DPC scheme in [11] for the MISO BC to the case of MIMO BC, (P1) can be modified to obtain the optimal ZF-DPC precoder design subject to
per-BS power constraints by rewriting the set of ZF constraints in (P1) as
\[ H_j S_k H_j^H = 0, \quad \forall j > k. \] (7)

The resulting problem has similar structure to (P1) and can be solved similarly (the details are omitted for brevity).

III. PROPOSED SOLUTION

In this section, we first present a new algorithm to solve (P1), which reveals the structure of the optimal BD precoding matrix for the general case with arbitrary numbers of antennas at the BS or MS. Then, we investigate the developed solution for the special case of single-antenna BSs and MSs, and compare it with other existing solutions in the literature.

A. General Case

To solve (P1), it is desirable to remove the set of ZF constraints, as follows: Define 
\[ G_k = [H_1^T, \cdots, H_k^T, H_{k+1}^T, \cdots, H_K^T]^T, \quad k = 1, \cdots, K, \]
where 
\[ G_k \in \mathbb{C}^{L \times M} \text{ with } L = N(K - 1). \]
Let the (reduced) singular value decomposition (SVD) of 
\[ G_k = U_k \Sigma_k V_k^H, \]
where 
\[ V_k \in \mathbb{C}^{M \times L} \text{ with } V_k^H V_k = I, \]
and \( \Sigma_k \) is a \( L \times L \) positive diagonal matrix. Note that \( \text{Rank}(G_k) = L < M \) under the previous assumption that \( NK \leq M \). Define the projection matrix \( P_k = (I - V_k V_k^H) \). Without loss of generality, we can express 
\[ P_k = \tilde{V}_k \tilde{V}_k^H, \]
where \( \tilde{V}_k \in \mathbb{C}^{M \times (M-L)} \) satisfies 
\[ V_k^H \tilde{V}_k = 0 \text{ and } \tilde{V}_k^H \tilde{V}_k = I. \]
Note that \( [V_k, \tilde{V}_k] \) forms a \( M \times M \) unitary matrix. Then, we have the following lemma.

**Lemma 3.1:** The optimal solution of (P1) is given by
\[ S_k = \tilde{V}_k Q_k \tilde{V}_k^H, \quad k = 1, \cdots, K \] (8)
where \( Q_k \in \mathbb{C}^{(M-L) \times (M-L)} \) and \( Q_k \succeq 0. \)

**Proof:** Please refer to Appendix A.

**Remark 3.1:** In prior works [7], [8], [9], [10] on the design of BD precoder for the MIMO BC with the sum-power constraint, it has been observed that the columns (precoding vectors) in the BD precoding matrix for the \( k \)th MS, \( T_k \), with 
\[ T_k T_k^H = S_k, \]
should be linear combinations of those in \( \tilde{V}_k \) in order to satisfy the constraints:
\[ H_j T_k = 0, \forall j \neq k. \]Lemma 3.1 extends this result to the case of per-BS power constraints.
With the optimal structures for $S_k$’s given in Lemma \ref{lem3.1}, it can be verified that all the ZF constraints in (P1) are satisfied and thus can be removed. Thus, (P1) reduces to the following equivalent problem

$$(P2) : \max_{Q_1, \ldots, Q_K} \sum_{k=1}^{K} w_k \log \left| I + H_k \bar{V}_k Q_k \bar{V}_k^H H_k^H \right|$$

$$\text{s.t.} \quad \sum_{k=1}^{K} \text{Tr} \left( B_a \bar{V}_k Q_k \bar{V}_k^H \right) \leq P, \ \forall a$$

$$Q_k \succeq 0, \ \forall k.$$ 

Similar to (P1), it can be shown that (P2) is convex. Thus, (P2) is solvable by the Lagrange duality method as follows. By introducing a set of non-negative dual variables, $\mu_a, a = 1, \cdots, A$, associated with the set of per-BS power constraints in (P2), the Lagrangian function of (P2) can be written as

$$L(Q_k, \{\mu_a\}) = \sum_{k=1}^{K} w_k \log \left| I + H_k \bar{V}_k Q_k \bar{V}_k^H H_k^H \right| - \sum_{a=1}^{A} \mu_a \left( \sum_{k=1}^{K} \text{Tr} \left( B_a \bar{V}_k Q_k \bar{V}_k^H \right) - P \right) \tag{9}$$

where $\{Q_k\}$ and $\{\mu_a\}$ denote the set of $Q_k$’s and the set of $\mu_a$’s, respectively. The Lagrange dual function for (P2) is then defined as

$$g(\{\mu_a\}) = \max_{Q_k \succeq 0, \forall k} L(Q_k, \{\mu_a\}). \tag{10}$$

Moreover, the dual problem of (P2) is defined as

$$(P2-D) : \min_{\mu_a \geq 0, \forall a} g(\{\mu_a\}).$$

Since (P2) is convex and satisfies the Slater’s condition \cite{28}, the duality gap between the optimal objective value of (P2) and that of (P2-D) is zero. Thus, (P2) can be solved equivalently by solving (P2-D). Moreover, (P2-D) is convex and can be solved by the subgradient-based method, e.g., the ellipsoid method \cite{29}, given the fact that the subgradient of function $g(\{\mu_a\})$ at a set of fixed $\mu_a$’s is $P - \sum_{k=1}^{K} \text{Tr} \left( B_a \bar{V}_k Q_k \bar{V}_k^H \right)$ for $\mu_a, a = 1, \cdots, A$, where $\{Q_k^*\}$ is the optimal solution for the maximization problem in (10) with the given set of $\mu_a$’s.

Next, we focus on solving for $\{Q_k^*\}$ with a set of fixed $\mu_a$’s. From (9), it is observed that the maximization problem in (10) can be separated into $K$ independent subproblems each involving only one $Q_k$. By discarding the irrelevant terms, the corresponding subproblem, for a given $k$, can be expressed as

$$(P3) : \max_{Q_k \succeq 0} w_k \log \left| I + H_k \bar{V}_k Q_k \bar{V}_k^H H_k^H \right| - \text{Tr} \left( B_\mu \bar{V}_k Q_k \bar{V}_k^H \right)$$

where $B_\mu \triangleq \sum_{a=1}^{A} \mu_a B_a$. Note that $B_\mu$ is a diagonal matrix with the diagonal elements given by different $\mu_a$’s in the order of $a = 1, \cdots, A$. We then have the following lemma.

**Lemma 3.2:** Let $A_\mu$ denote the number of $\mu_a$’s in the main diagonal of $B_\mu$, $a \in \{1, \cdots, A\}$, with $\mu_a > 0$. Then, for (P3) to have a bounded objective value, it holds that $A_\mu \geq \left\lceil \frac{M-N(K-1)}{M_B} \right\rceil$. 

Proof: Please refer to Appendix B.

Remark 3.2: It is noted that by applying the Karash-Kuhn-Tucker (KKT) conditions [28] to (P2), the fact that \( \mu_a > 0 \) for a given \( a \in \{1, \cdots, A\} \) implies that the corresponding power constraint must be tight with the optimal solution for \( \{Q_k\} \). Accordingly, in (P1) the optimal downlink transmit covariance matrices \( S_k \)'s must make the \( a \)th per-BS power constraint tight. Therefore, Lemma 3.2 provides a lower bound on the number of BSs for which the corresponding transmit power constraints must be tight with the optimal \( S_k \)'s for (P1).

With Lemma 3.2 and \( L = N(K-1) \), we can assume without loss of generality that \( M_B A_{\mu} \geq (M-L) \) since we are only interested in the case where the objective value of (P3) and that of (P1) are both bounded. Accordingly, we have \( \text{Rank}(\hat{V}_k^H B_{\mu} \hat{V}_k) = \min(M_B A_{\mu}, M-L) = M-L \). Thus, \( \hat{V}_k^H B_{\mu} \hat{V}_k \in \mathbb{C}^{(M-L) \times (M-L)} \) is a full-rank matrix and its inverse exists. Moreover, since \( \text{Tr}(XY) = \text{Tr}(YX) \), in (P3) we have

\[
\text{Tr}(B_{\mu} \hat{V}_k Q_k \hat{V}_k^H) = \text{Tr}((\hat{V}_k^H B_{\mu} \hat{V}_k)^{1/2} Q_k (\hat{V}_k^H B_{\mu} \hat{V}_k)^{1/2}).
\]

We thus define

\[
\tilde{Q}_k = (\hat{V}_k^H B_{\mu} \hat{V}_k)^{1/2} Q_k (\hat{V}_k^H B_{\mu} \hat{V}_k)^{1/2}.
\]

Then, (P3) can be reformulated to maximize

\[
w_k \log \left| I + H_k \hat{V}_k (\hat{V}_k^H B_{\mu} \hat{V}_k)^{-1/2} \tilde{Q}_k (\hat{V}_k^H B_{\mu} \hat{V}_k)^{-1/2} \hat{V}_k^H H_k^H \right| - \text{Tr}\left( \hat{Q}_k \right)
\]

subject to \( \hat{Q}_k \succeq 0 \). Note that \( \text{Rank}(H_k \hat{V}_k (\hat{V}_k^H B_{\mu} \hat{V}_k)^{-1/2}) = \min(N, M-L) = N \). Thus, the following (reduced) SVD can be obtained as

\[
H_k \hat{V}_k (\hat{V}_k^H B_{\mu} \hat{V}_k)^{-1/2} = \hat{U}_k \hat{\Sigma}_k \hat{V}_k^H
\]

where \( \hat{U}_k \in \mathbb{C}^{N \times N} \), \( \hat{V}_k \in \mathbb{C}^{(M-L) \times N} \), and \( \hat{\Sigma}_k = \text{Diag}(\hat{\sigma}_{k,1}, \cdots, \hat{\sigma}_{k,N}) \). Substituting the above SVD into (12) and applying the Hadamard’s inequality (see, e.g., [30]) yields the following optimal solution for (12) as

\[
\tilde{Q}_k^* = \hat{V}_k \Lambda_k \hat{V}_k^H,
\]

where \( \Lambda_k = \text{Diag}(\lambda_{k,1}, \cdots, \lambda_{k,N}) \), where \( \lambda_{k,i} \), \( i = 1, \cdots, N \), can be obtained by the standard water-filling algorithm [30] as

\[
\lambda_{k,i} = \left( \frac{w_k - \frac{1}{\hat{\sigma}_{k,i}^2}}{2} \right)^+
\]

where \( (x)^+ = \max(0, x) \). To summarize, the optimal solution of (P3) for a given set of \( \mu_a \)'s can be expressed as

\[
Q_k^* = (\hat{V}_k^H B_{\mu} \hat{V}_k)^{-1/2} \hat{V}_k \Lambda_k \hat{V}_k^H (\hat{V}_k^H B_{\mu} \hat{V}_k)^{-1/2}, \ k = 1, \cdots, K.
\]

Note that when the optimal solution for \( \{\mu_a\} \) in (P2-D) is obtained, the corresponding solution in (15) becomes optimal for (P2). By combining this result with Lemma 3.1 we obtain the following theorem.
Theorem 3.1: The optimal solution of (P1) is given by

\[ S_k^* = \tilde{V}_k (\tilde{V}_k^H B^*_\mu \tilde{V}_k)^{-1/2} \tilde{V}_k \Lambda_k^{1/2}, \quad k = 1, \ldots, K \]  \hspace{1cm} (16)

where \( B^*_\mu = \sum_{a=1}^A \mu^*_a B_a \), with \( \mu^*_a \)'s being the optimal dual solutions of (P2).

The algorithm for solving (P1) is summarized as follows.

**Algorithm (A1):**

- **Initialize** \( \mu_a \geq 0, a = 1, \ldots, A \).
- **Repeat**
  1. Solve \( Q_k^*, k = 1, \ldots, K \) using (15) with the given \( \mu_a \)'s;
  2. Compute the subgradient of \( g(\{\mu_a\}) \) as \( P = -\sum_{k=1}^K \text{Tr} \left( B_a \tilde{V}_k Q_k^* \tilde{V}_k^H \right), a = 1, \ldots, A \), and update \( \mu_a \)'s accordingly based on the ellipsoid method [29];
- **Until** all the \( \mu_a \)'s converge to a prescribed accuracy.
- Set \( S_k^* = \tilde{V}_k Q_k^* \tilde{V}_k^H, k = 1, \ldots, K \).

From Theorem 3.1 and the fact that \( S_k = T_k T_k^H, \forall k \), we obtain the following corollary.

**Corollary 3.1:** The optimal BD precoding matrices to maximize the weighted sum-rate for the MIMO-BC subject to the per-BS power constraints in (3) are given by

\[ T_k^* = \tilde{V}_k (\tilde{V}_k^H B^*_\mu \tilde{V}_k)^{-1/2} \tilde{V}_k \Lambda_k^{1/2}, \quad k = 1, \ldots, K. \]  \hspace{1cm} (17)

In the following remarks, we discuss some interesting observations on the optimal BD precoding matrices given by (17).

**Remark 3.3 (Channel Diagonalization):** One desirable property of linear precoding for a point-to-point MIMO channel is that the precoding matrix, when jointly deployed with a unitary decoding matrix at the receiver, is able to diagonalize the MIMO channel into parallel scalar sub-channels, over which independent encoding and decoding can be applied to simplify the transceiver design. Here, we verify that the optimal \( T_k^* \) given in (17) satisfies this “channel diagonalization” property, as follows:

\[ H_k T_k^* = H_k \tilde{V}_k (\tilde{V}_k^H B^*_\mu \tilde{V}_k)^{-1/2} \tilde{V}_k \Lambda_k^{1/2} \ ]  \hspace{1cm} (18)

\[ = \hat{U}_k \hat{\Sigma}_k \hat{V}_k \Lambda_k^{1/2} \ ]  \hspace{1cm} (19)

\[ = \hat{U}_k \hat{\Sigma}_k \Lambda_k^{1/2} \ ]  \hspace{1cm} (20)

where (19) is due to (13). Therefore, with a unitary decoding matrix \( \hat{U}_k^H \) applied at the \( k \)th MS receiver, the MIMO channel for the \( k \)th MS with BD precoding is diagonalized into \( N \) scalar sub-channels with channel gains
given by the main diagonal of the diagonal matrix $\Sigma_k \Lambda_k^{1/2}$. It is easy to verify that the above linear precoder and decoder processing preserves the single-user MIMO channel capacity.

**Remark 3.4 (Comparison with Conventional Sum-Power Constraint):** It is noted that (P1) can be modified to deal with the case where a single sum-power constraint over all the BSs (instead of a set of per-BS power constraints) is applied. This can be done via replacing the set of per-BS power constraints in (P1) by

$$
\sum_{k=1}^{K} \text{Tr}(S_k) \leq P^{(\text{sum})}
$$

(21)

where $P^{(\text{sum})}$ denotes the given sum-power constraint. Note that (P1) in this case corresponds to the conventional BD precoder design problem for the MIMO BC with a sum-power constraint as studied in [7], [8], [9], [10]. It can be shown that the developed solution for (P1) can be applied to this case, while the corresponding matrix $B^*_\mu$ should be modified as $\mu^* I$ with $\mu^*$ denoting the optimal dual solution associated with the sum-power constraint in (21). From (16), it follows that the optimal solution for this modified version of (P1) is given by

$$
S_k^{**} = \frac{1}{\mu^*} \tilde{V}_k \tilde{V}_k^H \Lambda_k \tilde{V}_k^H \tilde{V}_k^H, \quad k = 1, \cdots, K.
$$

(22)

Moreover, from (13) with $B^*_\mu = \mu^* I$, it follows that $\tilde{V}_k$ is obtained from the SVD: $\frac{1}{\sqrt{\mu}} H_k \tilde{V}_k = \tilde{U}_k \tilde{\Sigma}_k \tilde{V}_k^H$ and is thus independent of $\mu^*$. Accordingly, the optimal precoding matrix in the sum-power constraint case is $T_k^{**} = \frac{1}{\sqrt{\mu}} \tilde{V}_k \Lambda_k^{1/2}$. Comparing $T_k^{**}$ with $T_k^*$ in (17) for the per-BS power constraint case, we see that $T_k^{**}$ consists of orthogonal columns (beamforming vectors) since $\tilde{V}_k \hat{V}_k^H \hat{V}_k \hat{V}_k = I$, while $T_k^*$ in general consists of non-orthogonal columns if $B^*_\mu$ is a non-identity diagonal matrix (i.e., the optimal $\mu_a^*$’s are not all equal). This is the very reason that the BD precoder designs in prior works [19], [20], [21] based on the orthogonal precoder structure $T_k^{**}$ are in general suboptimal for the per-BS power constraint case.

**B. Special Case: MISO BC with Per-Antenna Power Constraints**

In this subsection, we investigate the developed solution for the special case of $M_B = N = 1$, where the auxiliary MIMO BC with the per-BS power constraints reduces to an equivalent MISO BC with the corresponding per-antenna power constraints, and the BD precoding reduces to the ZF-BF precoding. With $N = 1$, $H_k$ is a row-vector, which we denote by $h_k \in \mathbb{C}^{M \times 1}, k = 1, \cdots, K$. Accordingly, the SVD in (13) is rewritten as

$$
h_k^H \tilde{V}_k (\tilde{V}_k^H B_\mu \tilde{V}_k)^{-1/2} = \tilde{\sigma}_k \hat{v}_k^H
$$

(23)
where \( \hat{\sigma}_k > 0 \) and \( \hat{v}_k \in \mathbb{C}^{(M-L)\times 1} \). From (14), (16), and (23), it follows that the optimal downlink transmit covariance matrix for the \( k \)th MS, \( S_k^\star \), in the case of \( N = 1 \) is expressed as

\[
S_k^\star = \lambda_k \hat{V}_k (\hat{V}_k^H B^*_\mu \hat{V}_k)^{-1/2} \hat{v}_k \hat{v}_k^H (\hat{V}_k^H B^*_\mu \hat{V}_k)^{-1/2} \hat{V}_k^H
\]

(24)

\[
= \lambda_k \hat{\sigma}_k^{-2} \hat{V}_k (\hat{V}_k^H B^*_\mu \hat{V}_k)^{-1} \hat{V}_k^H h_k^H h_k \hat{V}_k (\hat{V}_k^H B^*_\mu \hat{V}_k)^{-1} \hat{V}_k^H
\]

(25)

where \( \lambda_k = (w_k - 1/\hat{\sigma}_k^2)^+ \). It is thus easy to observe that in this case \( \text{Rank}(S_k^\star) \leq 1 \). Thus, the corresponding optimal precoding matrix reduces to a (beamforming) vector denoted by \( t_k^\star \in \mathbb{C}^{M\times 1} \), where \( S_k^\star = t_k^\star (t_k^\star)^H \) and

\[
t_k^\star = \lambda_k^{1/2} \hat{\sigma}_k^{-1} \hat{V}_k (\hat{V}_k^H B^*_\mu \hat{V}_k)^{-1} \hat{V}_k^H h_k.
\]

(26)

Note that (26) holds regardless of \( M_B \), but \( M_B = 1 \) corresponds to the per-antenna power constraint case for the MISO BC. Furthermore, the optimal beamforming vector for the \( k \)th MS in the conventional sum-power constraint case (with \( B^*_\mu = \mu^* I \)) is obtained from (26) as

\[
t_k^\star = \lambda_k^{1/2} \hat{\sigma}_k^{-1} (\mu^*)^{-1} \hat{V}_k^H h_k.
\]

(27)

In the following, we discuss some interesting observations on the optimal ZF-BF precoding design in (26), as compared with other prior results reported in [4], [24], [25].

**Remark 3.5:** Denote \( T = [t_1, \cdots, t_K] \in \mathbb{C}^{M\times K} \) as the precoding matrix for a MISO BC with \( M \) transmitting antennas and \( K \) single-antenna receiving MSs. Then, for the sum-power constraint case with \( t_k = t_k^\star \) given in (27), the corresponding optimal precoding matrix \( T^\star \) becomes the conventional ZF-BF design for the MISO BC based on the channel pseudo inverse [4], i.e., \( T^\star \) can be put in the form \( T^\star = H^\dagger \hat{\Lambda} \), where \( H = [h_1, \cdots, h_K]^H \) and \( \hat{\Lambda} = \text{Diag}(\hat{\lambda}_1, \cdots, \hat{\lambda}_K) \), where \( \hat{\lambda}_k = \lambda_k^{1/2} \hat{\sigma}_k, k = 1, \cdots, K \). However, it is observed that the ZF-BF design based on the channel pseudo inverse is in general suboptimal for the MISO BC with the per-antenna/per-BS power constraints, where the optimal precoding matrix \( T^\star \) is obtained with \( t_k = t_k^\star \) given in (26). Note that \( t_k^\star \) becomes collinear with \( t_k^\star \) regardless of \( \mu_k^\star \)’s when \( M = K \). In this case, \( \hat{V}_k \) becomes a vector, \( \hat{v}_k \in \mathbb{C}^{M\times 1} \), and \( t_k^\star \) and \( t_k^\star \) can both be written in the form \( p_k \hat{v}_k \), with \( p_k \geq 0 \). Furthermore, it can be shown that this result holds regardless of the value of \( M_B \) provided that \( N = 1 \) and \( M = M_B A = K \).

**Remark 3.6:** In [25], the authors proposed a ZF-BF precoding design for the MISO BC with per-antenna power constraints in the form of the generalized inverse of \( H \). The corresponding precoding matrix is expressed as

\[
T = [g_1 a_1, \cdots, g_K a_K] + U^\perp [b_1, \cdots, b_K]
\]

(28)

where \( g_k \) is the normalized (to unit-norm) \( k \)th column in \( H^\dagger \), \( k = 1, \cdots, K \); \( U^\perp \in \mathbb{C}^{M\times (M-K)} \) is a projection matrix onto the orthogonal complement of the space spanned by the row vectors in \( H \), \((U^\perp)^H U^\perp = I \); \( a_k \)’s
and $b_k$’s are design variables, $k = 1, \cdots, K$. In other words, each beamforming vector $t_k$ in $T$ given by (28) is a linear combination of $g_k$ and the columns in $U^\perp$. We see that the beamforming vectors given in (28) are in accordance with the optimal $t^*_k$’s given in (26) due to the fact that for the MISO BC with $N = 1$ and thus $L = M - N(K - 1) = M - K + 1$, the space spanned by the columns in $\tilde{V}_k \in \mathbb{C}^{M \times L}$ is the same as that spanned by $g_k$ and the columns in $U^\perp$. Note that in [24], an algorithm is proposed to obtain the ZF-BF precoding matrix for the MISO BC with per-antenna power constraints by numerically searching over $a_k$’s and $b_k$’s in (28).

In this paper, the optimal ZF-BF precoders are found based on the closed-form expression in (26) and applying a numerical search over the dual variable $\mu_a$’s by the ellipsoid method.

Remark 3.7: It is also worth comparing the proposed method for solving (P1) in the MISO BC case with the method presented in [25]. For the method in [25], a set of transmit beamforming vectors, $t_1, \cdots, t_K$, are used in a MISO BC. Thus, the weighted sum-rate maximization problem with the ZF-BF precoding and per-antenna power constraints can be formulated as

\begin{align*}
(P4) : \quad & \max_{t_1, \cdots, t_K} \sum_{k=1}^{K} w_k \log \left( 1 + \|h^H_k t_k\|^2 \right) \\
& \text{s.t. } h^H_j t_k = 0, \forall j \neq k \\
& \sum_{k=1}^{K} \text{Tr} (B_i t_k t_k^H) \leq P, \forall i
\end{align*}

where $B_i \in \mathbb{C}^{M \times M}$ denotes a diagonal matrix with the $i$th diagonal element equal to one and all the others equal to zero, $i = 1, \cdots, M$; and $P$ refers to the per-antenna power constraint. Note that (P4) is non-convex due to the fact that the objective function is not necessarily concave over $t_k$’s. In [25], it is proposed to convert (P4) into an equivalent problem in terms of $S_k \triangleq t_k t_k^H, k = 1, \cdots, K$, which is expressed as

\begin{align*}
(P5) : \quad & \max_{S_1, \cdots, S_K} \sum_{k=1}^{K} w_k \log \left( 1 + h^H_k S_k h_k \right) \\
& \text{s.t. } h^H_j S_k h_j = 0, \forall j \neq k \\
& \sum_{k=1}^{K} \text{Tr} (B_i S_k) \leq P, \forall i \\
& S_k \succeq 0, \forall k \\
& \text{Rank}(S_k) = 1, \forall k.
\end{align*}

Note that (P5) can be treated as (P1) in the case of $N = 1$ and $M_B = 1$ (thus $M = M_B A = A$), and with an additional set of rank-one constraints for $S_k$’s. However, these rank-one constraints are non-convex and thus render (P5) non-convex in general. As a special form of (P1), (P5) without the rank-one constraints is convex, and thus can be solved efficiently by, e.g., the interior-point method [28]. However, the obtained solution for $S_k$ is not
guaranteed to be rank-one. In [25], it is proved that there always exists a solution that consists of a set of rank-one $S_k$’s for (P5), and a method is provided to construct the rank-one solution from the corresponding solution (with rank greater than one) of (P5) without the rank-one constraints. In contrast, the proposed method in this paper obtains the closed-form solution for (P5), which, as given in [25], is guaranteed to be rank-one.

IV. SUBOPTIMAL SOLUTION

In this section, we propose a suboptimal solution for (P1), which can be obtained with less complexity than the optimal solution. First, we define the projected channel of $H_k$ associated with the projection matrix $P_k$ as $H_k^\perp = H_k P_k = H_k \tilde{V}_k \tilde{V}_k^H$, $k = 1, \ldots, K$, where $H_k^\perp \in \mathbb{C}^{N \times M}$, and $\text{Rank}(H_k^\perp) = \min(N, M - L) = N$. Next, define the (reduced) SVD of $H_k^\perp$ as

$$H_k^\perp = U_k^\perp \Sigma_k^\perp (V_k^\perp)^H$$

(29)

where $U_k^\perp \in \mathbb{C}^{N \times N}$, $\Sigma_k^\perp = \text{Diag}(\sigma_{k,1}^\perp, \cdots, \sigma_{k,N}^\perp)$, and $V_k^\perp \in \mathbb{C}^{M \times N}$. Then, the proposed suboptimal solution for (P1) is give by

$$\tilde{S}_k = V_k^\perp \hat{\lambda}_k (V_k^\perp)^H$$

(30)

where $\hat{\lambda}_k = \text{Diag}(\hat{\lambda}_{k,1}, \cdots, \hat{\lambda}_{k,N})$ denotes the power allocation for the $k$th MS. It is worth noting that the above solution for $S_k$ is in general suboptimal for (P1) by comparing it with the optimal solution in (16). Also note that (30) is optimal for the sum-power constraint case as discussed in Section III-A, since it can be shown that in (22) $\tilde{V}_k \hat{V}_k = V_k^\perp$ with $B_k^* = \mu^* I$. With $\tilde{S}_k$’s given in (30), it can be shown that the ZF constraints in (P1) are satisfied and thus can be removed; furthermore, in the objective function of (P1), the following equalities hold:

$$\log |I + H_k \tilde{S}_k H_k^H|$$

(31)

$$= \log |I + (H_k^\perp + H_k V_k V_k^H) \hat{S}_k (H_k^\perp + H_k V_k V_k^H)^H|$$

(32)

$$= \log |I + H_k^\perp \hat{S}_k (H_k^\perp)^H|$$

(33)

$$= \log |I + U_k^\perp \Sigma_k^\perp \hat{\lambda}_k \Sigma_k^\perp (U_k^\perp)^H|$$

(34)

$$= \log |I + (\Sigma_k^\perp)^2 \hat{\lambda}_k|$$

(35)

where (32) is due to the fact that $\tilde{V}_k \hat{V}_k^H + V_k V_k^H = I$; (33) is due to the fact that $\tilde{S}_k V_k = 0$ since $\tilde{V}_k^H V_k = 0$; (34) is due to (29) and (30); and (35) is due to the fact that $\log |I + XY| = \log |I + YX|$. From (35), we see that the MIMO channel for the $k$th MS is diagonalized into $N$ scalar sub-channels with channel gains given by
Accordingly, (P1) is reduced to the following problem

\[
\text{(P6): } \max_{\{\bar{\lambda}_{k,i}\}} \sum_{k=1}^{K} w_k \sum_{i=1}^{N} \log \left(1 + (\sigma_{k,i}^+)^2 \bar{\lambda}_{k,i}\right)
\]

\[
\text{s.t. } \sum_{k=1}^{K} \sum_{i=1}^{N} \|v_k^+ [a, i]\|^2 \bar{\lambda}_{k,i} \leq P, \forall a
\]

\[
\bar{\lambda}_{k,i} \geq 0, \forall k, i
\]

where \(\{\bar{\lambda}_{k,i}\}\) denotes the set of \(\bar{\lambda}_{k,i}\)’s, \(k = 1, \cdots, K\) and \(i = 1, \cdots, N\), while \(v_k^+ [a, i]\) denotes the vector consisting of the elements from the \(i\)th column and the \(((a-1)MB + 1)\)-th to \((aMB)\)-th rows in \(V_k^+, a = 1, \cdots, A\) and \(i = 1, \cdots, N\). It can be verified that (P6) is a convex optimization problem. Thus, similar to (P2), the Lagrange duality method can be applied to solve (P6) by introducing a set of dual variables, \(\mu_a, a = 1, \cdots, A\), associated with the set of per-BS power constraints in (P6). For brevity, we omit here the details for the derivation and present the optimal solution (power allocation) for \(\{\bar{\lambda}_{k,i}\}\) as follows:

\[
\bar{\lambda}_{k,i} = \left(\frac{w_k}{\sum_{a=1}^{A} \mu_a \|v_k^+ [a, i]\|^2} - \frac{1}{(\sigma_{k,i}^+)^2}\right)^+.
\]  

(36)

Similar to (A1), the following algorithm can be used to obtain the proposed suboptimal solution for (P1).

Algorithm (A2):

- **Initialize** \(\mu_a \geq 0, a = 1, \cdots, A\).

- **Compute** the SVDs: \(H_k \tilde{V}_k \tilde{V}_k^H = U_k \Sigma_k \Sigma_k^H V_k^H, k = 1, \cdots, K\).

- **Repeat**

  1. Solve \(\{\bar{\lambda}_{k,i}\}\) using (36) with the given \(\mu_a\)’s;

  2. Compute the subgradient of the dual function for (P6) as

     \[
     P - \sum_{k=1}^{K} \sum_{i=1}^{N} \|v_k^+ [a, i]\|^2 \bar{\lambda}_{k,i}, a = 1, \cdots, A,
     \]

     and update \(\mu_a\)’s accordingly based on the ellipsoid method [29].

- **Until** all the \(\mu_a\)’s converge to a prescribed accuracy.

- **Set** \(S_k = V_k^+ \bar{\Lambda}_k (V_k^+)^H, k = 1, \cdots, K\).

As compared with (A1), (A2) has a lower complexity due to the fact that for each loop in the “Repeat”, only the power allocation computation in (36) is implemented, instead of the precoding matrix computation given in (15). Due to the suboptimal structure of the downlink transmit covariance matrix in (30) for (A2) as compared to the optimal one in (16) for (A1), (A2) in general leads to a suboptimal solution for (P1), except in the special case of \(N = 1\) and \(M = K\) where the transmit covariance structure in (30) is known to be optimal (see Remark 3.5). In this special case, (A2) can be used as an alternative algorithm to (A1) to obtain the optimal solution for (P1).
Remark 4.1: It is worth noting that (A2) can be shown equivalent to the algorithm proposed in [19] for the special case of $M_B = N = 1$, i.e., the MISO BC with the ZF-BF precoding and the per-antenna power constraints. In this case, similar to Remark 3.5 the proposed suboptimal solution in (30) corresponds to a precoding matrix in the form $T = H^\dagger \Theta$, where $H$ denotes the downlink MISO-BC channel, and $\Theta$ is a diagonal matrix with the main diagonal that has been optimized in a similar way as we have used for (P6). According to our previous discussions, this algorithm is indeed suboptimal for (P1) if $M > K$.

At last, as a counterpart of Lemma 3.2, we have the following lemma.

Lemma 4.1: Let $A^*$ denote the number of active per-BS power constraints with the optimal solution for (P6). It then holds that $A^* \leq NK$.

Proof: Please refer to Appendix C.

Lemma 4.1 provides an upper bound on the number of active per-BS power constraints for the suboptimal solution of (P1) obtained by (A2). It follows that in the case of $(A/NK) \gg 1$, most of the BSs in the cooperative multi-cell system cannot transmit with their full power levels with the suboptimal BD precoder design obtained by (A2).

V. Numerical Examples

In this section, we provide numerical examples to illustrate the results in this paper. For the purpose of exposition, we assume that the channel $H_k$’s in (1) are independent over $k$, and all the elements in each channel matrix are independent CSCG random variables with zero mean and unit variance. Moreover, we consider the sum-rate maximization for the cooperative multi-cell downlink transmission, i.e., $w_k$’s are all equal to one in (P1). The obtained numerical results along with related discussions are presented in the following subsections.

A. Convergence Behavior

In Fig. II we show the convergence behavior of Algorithm (A1) for solving (P1). It is assumed that $A = 2$, $M_B = 4$, $K = 4$, and $N = 2$. The transmit power constraint $P$ for each of the two BSs is set equal to 10. The initial values assigned to $\mu_a$’s in (A1) are $\mu_1 = \mu_2 = 0.2$. The achievable sum-rate and the consumed transmit powers by the two BSs are shown for different iterations in (A1), each with a pair of updated values for $\mu_1$ and $\mu_2$. As observed, the plotted rate and power values all converge to fixed values after around 30 iterations. The converged transmit powers for the two BSs are observed both equal to their given constraint value, which is 10. A similar convergence behavior for Algorithm (A2) can be observed and thus omitted here. Generally speaking, the convergence speed of both (A1) and (A2) depends critically on the total number of per-BS power constraints, $A$, which is also the number of dual variable $\mu_a$’s to be searched. With the ellipsoid method, it is known that the
complexity for searching $\mu_a$’s in (A1) or (A2) is $O(A^2)$ for large values of $A$ [29]. Thus, the number of iterations for the algorithm convergence grows asymptotically in the order of the square of the number of BSs in the system.

**B. MISO BC with Per-Antenna Power Constraints**

Next, we consider a special case of the cooperative multi-cell downlink transmission with $M_B = N = 1$, which is equivalent to a MISO BC with the corresponding per-antenna power constraints. The per-BS/per-antenna power constraint is assumed to be $P = 10$. In Fig. 2 we compare the achievable sum-rate with the optimal ZF-BF precoder obtained by (A1) against that with the suboptimal precoder obtained by (A2). The number of MSs is fixed as $K = 2$, while the total number of transmitting antennas $M$ ranges from 2 to 10. It is observed that when $M = K = 2$, the achievable rates for both the optimal and suboptimal precoders are identical, which is in accordance with our discussions in Section IV. It is also observed that when $M > K$, the sum-rate gain of the optimal precoder solution over the suboptimal solution increases with $M$. In order to explain this observation, in Fig. 3 we show the histograms for the number of active per-antenna power constraints with the optimal and suboptimal solutions over 100 random MISO-BC realizations for the case of $M = 8$. It is observed that the number of active per-antenna power constraints with the optimal solution is always no less than $\left\lceil \frac{M-N(K-1)}{M_B} \right\rceil = 7$, while that with the suboptimal solution is always no larger than $NK = 2$, in accordance with Lemmas 3.2 and 4.1 respectively. We thus see that when $M$ becomes much larger than $K$ for the MISO BC, the optimal ZF-BF design can utilize the full transmit powers from at least $(M - K + 1)$ antennas, while the suboptimal design can only have at most $K$ antennas transmitting with their full powers. This explains why in Fig. 2 the rate gap between the optimal and suboptimal ZF-BF designs enlarges as $M$ increases with a fixed $K$.

**C. MIMO BC with Per-Antenna Power Constraints**

Last, we consider the case of multi-antenna MS receivers. For the corresponding auxiliary MIMO BC, we assume that $A = 4, M_B = 1, K = 2$, and $N = 2$. Note that in this case although $M = NK$, i.e., the total number of transmitting antennas are equal to that of receiving antennas, (A2) in general leads to a suboptimal solution for (P1) due to the fact that $N > 1$. In Fig. 4 we show the achievable sum-rates for both the optimal and suboptimal BD precoders vs. the per-BS/per-antenna transmit power constraint $P$. It is observed that although the optimal precoder solution still performs better than the suboptimal one, their rate gap is not as large as that in Fig. 2 when $M > NK$ and $P = 10$. This is due to the fact that in the case of $M = NK$, although the maximum number of antennas transmitting with full powers for the suboptimal solution is still limited by $NK$ according to Lemma 4.1 such a constraint is not useful since $M_B = 1$ and $A = NK$. The practical rule of thumb here is that when
$M_B = 1$ and $A$ is not substantially larger than $NK$, the low-complexity suboptimal BD precoder obtained by (A2) can be applied to achieve the sum-rate performance close to that of the optimal BD precoder obtained by (A1).

VI. CONCLUSION

This paper studies the design of block diagonalization (BD) linear precoder for the fully cooperative multi-cell downlink transmission subject to individual power constraints for the base stations (BSs). By applying convex optimization techniques, this paper derives the closed-form expression for the optimal BD precoding matrix to maximize the weighted sum-rate of all users in the multi-cell system. The optimal BD precoding vectors for each user are shown to be in general non-orthogonal, which differs from the conventional orthogonal precoder design for the sum-power constraint case. A suboptimal heuristic method is also proposed, which combines the conventional orthogonal BD precoder design with an optimized power allocation to meet the per-BS power constraints. Furthermore, this paper shows that the proposed optimal BD precoder solution provides the optimal zero-forcing beamforming (ZF-BF) solution for the special case of MISO BC with per-antenna power constraints. The results in this paper are readily extended to obtain the optimal BD precoders for the MIMO-BC with general linear transmit power constraints, which include per-antenna/per-BS power constraints as special cases.

APPENDIX A
PROOF OF LEMMA 3.1

Let $\{S_1^*, \cdots, S_K^*\}$ denote the optimal solution of (P1). Without loss of generality, for any given $k \in \{1, \cdots, K\}$, we can express $S_k^*$ in the following form

$$S_k^* = [\tilde{V}_k, V_k] \begin{bmatrix} A & B \\ B^H & C \end{bmatrix} [\tilde{V}_k, V_k]^H$$

where $A \in \mathbb{C}^{(M-L) \times (M-L)}$, $B \in \mathbb{C}^{(M-L) \times L}$, and $C \in \mathbb{C}^{L \times L}$. Note that $A = A^H$ and $C = C^H$. Since $S_k^*$ must satisfy the set of ZF constraints in (P1), it follows that

$$V_k^H S_k^* V_k = 0.$$

From (38) and (39), it follows that $C = 0$. Furthermore, from the theory of Schur complement [28], it is known that $S_k^* \succeq 0$ if and only if (iff) the following conditions are satisfied:

$$A \succeq 0$$

$$ (I - AA^\dagger) B = 0$$

$$C - B^H A^\dagger B \succeq 0.$$
Since \( A \succeq 0 \), it follows that \( B^H A^+ B \succeq 0 \). Using this fact together with \( C = 0 \), from (42) it follows that  
\[ B^H A^+ B = 0. \]
Thus, from (41) it follows that \( B = 0 \). With \( B = 0 \) and \( C = 0 \), from (38) it follows that 
\[ S_k^* = \tilde{V}_k A \tilde{V}_k^H. \]
By letting \( A = Q_k \), the proof of Lemma 3.1 thus follows.

**APPENDIX B**

**PROOF OF LEMMA 3.2**

We prove Lemma 3.2 by contradiction. Suppose that there exist a number of strictly positive \( \mu_a \)'s such that 
\[ A_{\mu} < \left[ \frac{M-N(K-1)}{M_B} \right]. \]
Then, it follows that \( A_{\mu} < \frac{M-N(K-1)}{M_B} \). Since \( L = N(K-1) \), it thus follows that \( M_B A_{\mu} < (M - L) \). Let \( S \) denote the set consisting of the indices corresponding to all the non-zero diagonal elements in \( B_{\mu} \), i.e., if \( \mu_a > 0 \) for any \( a \in \{1, \cdots, A\} \), then \( (a-1)M_B + i, i = 1, \cdots, M_B \). Note that the size of \( S \) is denoted by \( |S| = M_B A_{\mu} \). Let \( E_k(S) \) and \( F_k(S^c) \) denote the matrix consisting of the rows in \( \tilde{V}_k \in \mathbb{C}^{M \times (M - L)} \) with the row indices given by the elements in \( S \) and \( S^c \), respectively, where \( S^c \) denotes the complement of \( S \). Note that \( |S| + |S^c| = M \) and \( |S^c| > 0 \) since \( M_B A_{\mu} < (M - L) < M \). From \( E_k(S) \in \mathbb{C}^{M_B A_{\mu} \times(M-L)} \) and \( M_B A_{\mu} < (M - L) \), it follows that \( E_k(S) \) is not full row-rank. Thus, we could find a vector \( q_k \in \mathbb{C}^{(M-L)\times 1} \) with \( \|q_k\| = 1 \) such that \( E_k(S)q_k = 0 \) and \( F_k(S^c)q_k \neq 0 \). Accordingly, we have \( B_{\mu} \tilde{V}_k q_k = 0 \) and \( \tilde{V}_k q_k \neq 0 \). Denote \( w_k = \tilde{V}_k q_k \). Note that the indices of the non-zero elements in \( w_k \) belong to \( S^c \). Suppose that the solution of (P3) is taken as \( Q_k^* = p(q_k q_k^H) \) with \( p \geq 0 \). Substituting this solution into the objective function of (P3) yields 
\[ w_k \log |I + H_k \tilde{V}_k Q_k^* \tilde{V}_k^H H_k^H| - \text{Tr} \left( B_{\mu} \tilde{V}_k Q_k^* \tilde{V}_k^H \right) \]  
(43) 
\[ = w_k \log |I + p H_k w_k w_k^H H_k^H|. \]  
(44)
Let \( R_k = H_k w_k \). Then, (44) can be further expressed as \( w_k \log |I + p R_k R_k^H| \), whose value becomes unbounded as \( p \to \infty \) provided that \( R_k R_k^H \neq 0 \) (which holds with probability one due to independent channel realizations). Therefore, we conclude that the presumption that \( A_{\mu} < \left[ \frac{M-N(K-1)}{M_B} \right] \) cannot be true. Lemma 3.2 thus follows.

**APPENDIX C**

**PROOF OF LEMMA 4.1**

We prove Lemma 4.1 by contradiction. Suppose that \( A^* \geq (NK+1) \). Let \( B \) be a subset of \( \{1, \cdots, A\} \) consisting of the indices of the BSs for which the transmit power constraints are tight with the optimal solution for (P6).

Note that \( |B| = A^* \). Let \( \tilde{\lambda}_{k,i}^* \) denote the optimal solution for (P6), \( k = 1, \cdots, K \) and \( i = 1, \cdots, N \). Thus, we have the following equalities from (P6)
\[ \sum_{k=1}^{K} \sum_{i=1}^{N} \| v_k^+ [a, i] \|^2 \tilde{\lambda}_{k,i}^* = P, \quad \forall a \in B. \]  
(45)
Accordingly, $\lambda_{k,i}^*$'s are the solutions for a set of $A^*$ linear independent (which holds with probability one due to independent channel realizations) equations. However, since $A^* \geq (NK + 1)$, we see that the number of equations exceeds that of unknowns, which is equal to $NK$. Thus, given $P > 0$, there exist no feasible solutions for $\lambda_{k,i}^*$'s. We thus conclude that the presumption that $A^* \geq (NK + 1)$ cannot be true. Lemma 4.1 thus follows.

REFERENCES

[1] H. Weingarten, Y. Steinberg, and S. Shamai (Shitz), “The capacity region of the Gaussian multiple-input multiple-output broadcast channel,” IEEE Trans. Inf. Theory, vol. 52, no. 9, pp. 3936-64, Sep. 2006.
[2] F. Rashid-Farrokhi, K. Liu, and L. Tassiulas, “Transmit beamforming and power control for cellular wireless systems,” IEEE J. Sel. Areas Commun., vol. 16, no. 8, pp. 1437-1450, Oct. 1998.
[3] M. Schubert and H. Boche, “Solution of the multiuser downlink beamforming problem with individual SINR constraints,” IEEE Trans. Veh. Technol., vol. 53, no. 1, pp. 18-28, Jan. 2004.
[4] C. B. Peel, B. Hochwald, and A. L. Swindlehurst, “A vector perturbation technique for near capacity multi-antenna multi-user communication – Part I: Channel inversion and regularization,” IEEE Trans. Commun., vol. 53, no. 1, pp. 195-202, Jan. 2005.
[5] A. Wiesel, Y. C. Eldar, and S. Shamai (Shitz), “Linear precoding via conic optimization for fixed MIMO receivers,” IEEE Trans. Sig. Process., vol. 54, no. 1, pp. 161-176, Jan. 2006.
[6] M. Stojnic, H. Vikalo, and B. Hassibi, “Maximizing the sum-rate of multi-antenna broadcast channels using linear preprocessing,” IEEE Trans. Wireless Commun., vol. 5, no. 9, pp. 2338-2342, Sep. 2006.
[7] Q. H. Spencer, A. L. Swindlehurst, and M. Haardt, “Zero-forcing methods for downlink spatial multiplexing in multiuser MIMO channels,” IEEE Trans. Sig. Process., vol. 52, no. 2, pp. 461-471, Feb. 2004.
[8] K. K. Wong, R. D. Murch, and K. B. Letaif, “A joint-channel diagonalization for multiuser MIMO antenna systems,” IEEE Trans. Wireless Commun., vol. 2, no. 4, pp. 773-786, July 2003.
[9] L. U. Choi and R. D. Murch, “A transmit preprocessing technique for multiuser MIMO systems using a decomposition approach,” IEEE Trans. Wireless Commun., vol. 3, no. 1, pp. 20-24, Jan. 2004.
[10] Z. Pan, K. K. Wong, and T. S. Ng, “Generalized multiuser orthogonal space-division multiplexing,” IEEE Trans. Wireless Commun., vol. 3, no. 6, pp. 1969-1973, Nov. 2004.
[11] G. Carie and S. Shamai, “On the achievable throughput of a multiantenna Gaussian broadcast channel,” IEEE Trans. Inf. Theory, vol. 49, no. 7, pp. 1691-1706, July 2003.
[12] S. Shamai (Shitz) and B. M. Zaidel, “Enhancing the cellular downlink capacity via co-processing at the transmitting end,” in Proc. IEEE Veh. Technol. Conf. (VTC), vol. 3, pp. 1745-1749, May 2001.
[13] H. Zhang and H. Dai, “Cochannel interference mitigation and cooperative processing in downlink multicell multiuser MIMO networks,” EURASIP J. Wireless Commun. Netw., no. 2, pp. 222-235, 2004.
[14] M. Karakayali, G. J. Foschini, and R. A. Valenzuela, “Network coordination for spectrally efficient communications in cellular systems,” IEEE Wireless Commun., vol. 13, no. 4, pp. 56-61, Aug. 2006.
[15] O. Somekh, B. Zaidel, and S. Shamai (Shitz), “Sum rate characterization of joint multiple cell-site processing,” IEEE Trans. Inf. Theory, vol. 53, no. 12, pp. 4473-4497, Dec. 2007.
[16] S. Jing, D. Tse, J. Hou, J. Soriaga, J. Sme, and R. Padovani, “Downlink macro-diversity in cellular networks,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), pp. 24-29, June 2007.
[17] S. Kaviani and W. A. Krzymien, “Sum rate maximization of MIMO broadcast channels with coordination of base stations,” in Proc. IEEE Wireless Commun. Net. Conf. (WCNC), Apr. 2008.
[18] B. Ng, J. Evans, S. Hanly, and D. Aktas, “Distributed downlink beamforming with cooperative base stations,” IEEE Trans. Inf. Theory, vol. 54, no. 12, pp. 5491-5499, Dec. 2008.
[19] F. Boccardi and H. Huang, “Optimum power allocation for the MIMO-BC zero-forcing precoder with per-antenna power constraints,” in Proc. Conf. Inf. Sciences Systems (CISS), Mar. 2006.
[20] W. Liu, S. X. Ng, and L. Hanzo, “Multicell cooperation based SVD assisted multi-user MIMO transmission,” in Proc. IEEE Veh. Technol. Conf. (VTC), Apr. 2009.
[21] J. Zhang, R. Chen, J. Andrews, A. Ghosh, and R. W. Heath, “Networked MIMO with clustered linear precoding,” *IEEE Trans. Wireless Commun.*, vol. 8, no. 4, Apr. 2009.

[22] W. Yu and T. Lan, “Transmitter optimization for the multi-antenna downlink with per-antenna power constraints,” *IEEE Trans. Sig. Process.*, vol. 55, no. 6, pp. 2646-2660, June 2007.

[23] L. Zhang, R. Zhang, Y.-C. Liang, Y. Xin, and H. V. Poor, “On the Gaussian MIMO BC-MAC duality with multiple transmit covariance constraints,” in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, June 2009.

[24] H. Huh, H. Papadopoulos, and G. Caire, “MIMO broadcast channel optimization under general linear constraints,” in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, June 2009.

[25] A. Wiesel, Y. Eldar, and S. Shamai, “Zero-forcing precoding and generalized inverses,” *IEEE Trans. Sig. Process.*, vol. 55, no. 9, pp. 4409-4418, Sep. 2008.

[26] T. Yoo and A. Goldsmith, “On the optimality of multi-antenna broadcast scheduling using zero-forcing beamforming,” *IEEE J. Sel. Areas Commun.*, vol. 24, no. 3, pp. 528-541, Mar. 2006.

[27] Z. Shen, R. Chen, J. G. Andrews, R. W. Heath, and B. L. Evans, “Low complexity user selection algorithms for multiuser MIMO systems with block diagonalization,” *IEEE Trans. Sig. Process.*, vol. 54, no. 9, pp. 3658-3663, Sep. 2006.

[28] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.

[29] R. G. Bland, D. Goldfarb, and M. J. Todd, “The ellipsoid method: a survey,” *Operations Research*, vol. 29, no. 6, pp. 1039-1091, 1981.

[30] T. Cover and J. Thomas, *Elements of information theory*, New York: Wiley, 1991.
Fig. 1. Convergence behavior of Algorithm (A1).

Fig. 2. Comparison of the sum-rate in the MISO BC with the ZF-BF precoding and the per-antenna power constraints.
Fig. 3. Comparison of the number of active per-BS power constraints in the MISO BC with the ZF-BF precoding and the per-antenna power constraints.

Fig. 4. Comparison of the sum-rate in the MIMO BC with the BD precoding and the per-antenna/per-BS power constraints.