**A REMARK ON THE HARD LEFSCHETZ THEOREM FOR KÄHLER ORBIFOLDS**

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**Abstract.** We give a proof of the Hard Lefschetz Theorem for orbifolds that does not involve intersection homology. We use a foliated version of the Hard Lefschetz Theorem due to El Kacimi.

**Introduction**

Even though orbifolds are not smooth manifolds in general, these mildly singular objects retain a strong flavor of smoothness. For example, for a compact orientable orbifold \( V \), Poincaré duality holds over the reals. If \( V \) is moreover Kähler, it satisfies the Hard Lefschetz Theorem (HLT).

The former was proved by Satake, by adapting the classical proof. However, the only known proof of the HLT for orbifolds uses the far-reaching version of the HLT in intersection homology. Fulton raises the problem of finding a more direct proof (cf. [Ful], p. 105). We propose a solution based on a result of El Kacimi (cf. [EK1]). We emphasize that the methods that we are using here are well known to the specialists of Riemannian foliations (cf. [EK1], [EK2], [GHS], [Mo], [Rum], [Ton]).

1. The resolution of an orbifold as a Seifert bundle

**Definition 1.1.** Let \( |V| \) be a Hausdorff topological space. A Kähler orbifold structure \( V \) on \( |V| \) is given by the following:

(i) An open cover \( \{V_i\}_i \) of \( |V| \).

(ii) For each \( i \in I \), a connected and open \( U_i \subset \mathbb{C}^n \) with a Kähler metric \( h_i \); a finite subgroup \( \Gamma_i \) of holomorphic isometries of \( U_i \); a continuous map \( q_i : U_i \to V_i \), called a local uniformization, inducing a homeomorphism from \( \Gamma_i \setminus U_i \) onto \( V_i \).

(iii) For all \( x_i \in U_i \) and \( x_j \in U_j \) such that \( q_i(x_i) = q_j(x_j) \), there exist \( W_i \subset U_i \) and \( W_j \subset U_j \), open connected neighbourhoods of \( x_i \) and \( x_j \), and a holomorphic isometry \( \phi_{ji} : W_i \to W_j \), called a change of charts, such that \( q_j \circ \phi_{ji} = q_i \) on \( W_i \).

**Remark.** It is true, but not trivial, that \( V \) is Kähler in the above “orbifold” sense if and only if, as a complex space, \( V \) is Kähler in the sense of Grauert (defined in [G]).
Let $V$ be a compact Kähler orbifold. The representation of $V$ as the leaf space of a smooth foliation is due to [GHS], Prop. 4.1. We explain their construction, and check that the foliation is transversely Kähler.

Fix $i$. For any local uniformization $q_i: U_i \to V_i$, we denote by $\pi_i: \tilde{U}_i \to U_i$ the principal bundle of unitary frames. The $\Gamma_i$-action on $U_i$ lifts naturally to $\tilde{U}_i$. Indeed, let $\xi_i = (\xi_i^0, \ldots, \xi_i^n) \in \tilde{U}_i$ be a unitary frame at $x_i$ and let $g_i \in \Gamma_i$. Then $g_i\xi_i := ((g_i)\xi_i^0, \ldots, (g_i)\xi_i^n)$ is a unitary frame at $g_ix_i$. As an isometry is determined by its derivative at one point, this lifted action is free. The quotient $\tilde{V}_i := \Gamma_i \backslash \tilde{U}_i$ is a $C^\infty$-smooth manifold, and $q_i : \tilde{U}_i \to \tilde{V}_i$ is a finite non-ramified covering.

Fix $i$ and $j$. Any change of charts $\phi_{ji}: W_i \subset U_i \to W_j \subset U_j$ lifts to a diffeomorphism $\tilde{\phi}_{ji}: \tilde{W}_i \subset \tilde{U}_i \to \tilde{W}_j \subset \tilde{U}_j$, $\xi_i \mapsto (\phi_{ji}(\xi_i^0), \ldots, (\phi_{ji})(\xi_i^n))$.

**Remark 1.2.** For any two changes of charts $\phi_{ji}$ and $\phi'_{ji}$ defined on the same $W_i \subset U_i$, there exists $g_j \in \Gamma_j$ such that $\phi'_{ji} = g_j \phi_{ji}$ (this follows from the classical case of a non-ramified $q_j$). Accordingly, $\phi'_{ji} = g_j \phi_{ji}$ on $\tilde{W}_i \subset \tilde{U}_i$.

Denote $V_{ji} := V_i \cap V_j$. We want to define a lifting of the obvious gluing map

$$f_{ji} : V_{ji} \subset V_i \to V_{ji} \subset V_j$$

to a gluing diffeomorphism

$$\tilde{f}_{ji} : p_i^{-1}(V_{ji}) \subset \tilde{V}_i \to p_j^{-1}(V_{ji}) \subset \tilde{V}_j.$$ 

Let $z_i \in p_i^{-1}(V_{ji})$. Let $\phi_{ji} : W_i \to W_j$ be a change of charts with $y_i \in \tilde{W}_i$ such that $\tilde{q}_i(y_i) = z_i$. Take $\tilde{f}_{ji}(z_i) := \tilde{q}_i \phi_{ji}(y_i)$. This is well defined: if we choose another point $g_j y_i$ and another change of charts $\phi'_{ji} : g_j W_i \to W'_j$, then both $\phi_{ji}$ and $\phi'_{ji} g_j$ are changes of charts on $W_i$, so, by Remark 1.2, there is a $g_j \in \Gamma_j$ such that $\phi'_{ji} g_j = g_j \phi_{ji}$. Thus $\tilde{q}_j \tilde{\phi}'_{ji} g_j (y_i) = \tilde{q}_j g_j \phi_{ji}(y_i) = \tilde{q}_j \phi_{ji}(y_i)$.

Also from Remark 1.2 for all $i, j, k$, $f_{ki} = f_{kj} f_{ji}$. Therefore, the disjoint collection $\{\tilde{V}_i\}$ glued according to $\{f_{ji}\}_{ij}$ yields a smooth compact manifold that we call $\tilde{V}$. It is Hausdorff as a consequence of the following:

**Proposition 1.3.** For any compact orbifold $V$, the manifold $\tilde{V}$ constructed as above admits a $U(n)$-action, turning $p : \tilde{V} \to V$ into a Seifert principal bundle.
Proposition 1.4. The foliation \( \mathcal{F} \) of \( \tilde{V} \) by \( U(n) \)-orbits is transversely Kähler; i.e., \( \mathcal{F} \) is transversely holomorphic (so the normal bundle admits a complex structure \( J \)), and there exists a closed real 2-form \( \omega \) on \( \tilde{V} \) such that:

(i) the tangent bundle of \( \mathcal{F} \) is the kernel of \( \omega \);

(ii) the quadratic form \( h(-,-) := \omega(J-,-) + \sqrt{-1}\omega(-,-) \) defines a Hermitian metric on the normal bundle of \( \mathcal{F} \).

Proof. For an open subset \( O \subset \tilde{V} \) small enough, we can assume that \( O \subset \tilde{q}_i(\tilde{U}_i) \) for some \( i \) and take a smooth section \( \sigma : O \rightarrow \tilde{V}_i \) of \( \tilde{q}_i \). Then, on \( O \), the foliation \( \mathcal{F} \) is defined by the submersion \( \pi_i \sigma : O \rightarrow U_i \), which induces an isomorphism between the normal bundle of \( \mathcal{F} \) and the tangent bundle of \( U_i \). By hypothesis, we have a \( \Gamma_i \)-invariant Kähler form \( \omega_i \) on \( U_i \). Define \( \omega := \pi_i^* \omega_i \), a well-defined form on \( \tilde{V} \) that satisfies the above properties. \( \square \)

2. A taut and transversely Kähler metric

Let \( g_0 \) be a Riemannian metric on \( \tilde{V} \). Up to averaging it over the action, we can assume that it is \( U(n) \)-invariant. Denote \( m = \dim U(n) \). Fix a global frame on \( U(n) \) given by right-invariant vector fields \( E_1, \ldots, E_m \). Let \( e_1, \ldots, e_m \) be the associated fundamental vector fields on \( \tilde{V} \), and for any \( [\xi] \in \tilde{V} \), let \( M_0[\xi] := \left( g_0(e_k[\xi], e_l[\xi]) \right)_{1 \leq k,l \leq m} \in \mathbb{R}^{m \times m} \), and define a smooth function \( u \) on \( \tilde{V} \) by

\[
u_0[\xi] := \left( \det M_0[\xi] \right)^{-1/2}.
\]

By invariance, \( M_0 \) and \( u_0 \) are constant on each orbit. Now, we define a new Riemannian metric on \( \tilde{V} \) by the conformal change \( g_1 = u_0g_0 \).

Proposition 2.1. With respect to \( g_1 \), the \( U(n) \)-orbits are minimal submanifolds.

Proof (inspired by the proof of [Ton], Prop. 7.6). Since this is a local statement, we can assume that we are on \( \tilde{V}_i \) for some \( i \). Using the covering map \( \tilde{q}_i \), we lift the metric \( g_1 \) to a metric \( h_1 \) on \( \tilde{U}_i \). Let \( X_0 \subset \tilde{U}_i \) be any \( U(n) \)-orbit, and let \( Z \) be an arbitrary vector field that is compactly supported and normal to \( X_0 \). Let \( \{ \varphi_t \}_{t \in \mathbb{R}} \) be the flow associated to \( Z \), and define \( X_t := \varphi_t(X_0) \). By the first variational formula (or [Jos], 3.6.4), it is enough to prove that

\[
\frac{d}{dt} \left( \text{Vol } X_t \right)_{|t=0} = 0.
\]

Because the metric \( h_1 \) is \( U(n) \)-invariant, we can replace \( Z \) with its average over the action. This new \( Z \) is still orthogonal to \( X_0 \), and is \( U(n) \)-invariant, so \( X_t \) is a \( U(n) \)-orbit for every \( t \). On \( \tilde{V} \), the volume form of a \( U(n) \)-orbit is

\[
\left( \det M_1 \right)^{\frac{1}{2}} e_1 \wedge \cdots \wedge e_m,
\]
where \( M_1 = \left( g_1(\epsilon_k, e_l) \right)_{1 \leq k, l \leq m} \) and \( \tilde{e}_1, \ldots, \tilde{e}_m \) is the dual basis. On the other hand, \((\det M_1)^{\frac{1}{2}} = u_{\alpha}^\frac{m}{2}, (\det M_0)^{\frac{1}{2}} = (\det M_0)^{-\frac{1}{2}}(\det M_0)^{\frac{1}{2}} = 1\).

Fix an arbitrary \( t \). Choosing any point on \( M_t \), we get a one-to-one parameterization \( f : U(\alpha) \to X_t \). Then

\[
\text{Vol } X_t = \int_{U(\alpha)} f^* \tilde{q}_t^* (\tilde{e}_1 \wedge \cdots \wedge \tilde{e}_m) = \int_{U(\alpha)} \tilde{E}_1 \wedge \cdots \wedge \tilde{E}_m,
\]

which does not depend on \( t \). \( \square \)

The metric \( g_1 \) determines an embedding \( N \subset T\tilde{V} \) of the normal bundle of \( \tilde{\mathcal{F}} \). Modifying \( g_1 \), we define a metric \( g \) on \( \tilde{V} \) by replacing \( g_{1|N} \) with the metric \( h \) obtained in Prop. \[1.3\]. The proof of Prop. \[2.1\] shows that the leaves of \( \tilde{\mathcal{F}} \) are still minimal submanifolds with respect to \( g \).

### 3. Main result

Let \( M \) be any compact smooth manifold, with a foliation \( \mathcal{F} \). A form \( \alpha \in \Omega^*(M, \mathbb{C}) \) is called basic when for all vectors \( Z \) tangent to the foliation, \( \iota_Z \alpha = 0 = \iota_Z da, \) where \( \iota_Z \) denotes the interior product. The basic forms determine a subcomplex of the usual de Rham complex of \( M \). The associated cohomology groups are called the basic cohomology groups and are denoted by \( H^i_{\mathcal{F}}(\mathcal{F}, \mathbb{C}) \). They can be thought of as a substitute for the de Rham cohomology of the leaf space, which can be very singular and/or not Hausdorff in general. When the leaf space is an orbifold \( V \), the basic cohomology is isomorphic to the singular cohomology of \( V \) over \( \mathbb{C} \) (cf. \[1.3\], 5.3).

Let \( V \) be a compact Kähler orbifold, with \( \tilde{V}, \tilde{\mathcal{F}} \) and \( g \) as described in the previous sections. We want to apply El Kacimi’s Basic Hard Lefschetz Theorem: if \( \tilde{\mathcal{F}} \) is transversely Kähler with respect to \( \omega \) and \( H^j_{\mathcal{F}}(\mathcal{F}, \mathbb{C}) \neq 0 \), then for any \( k, \alpha \mapsto \alpha \wedge \omega^k \) is an isomorphism \( H^{n-k}_{\mathcal{F}}(\mathcal{F}, \mathbb{C}) \to H^{n+k}_{\mathcal{F}}(\mathcal{F}, \mathbb{C}) \), where \( n \) is the complex codimension of \( \tilde{\mathcal{F}} \). We refer to \[1,\] for a more complete statement of the theorem.

By a theorem of Masa (cf. \[Ma\]), the existence of a metric turning the leaves into minimal submanifolds is equivalent to \( H^j_{\mathcal{F}}(\mathcal{F}, \mathbb{C}) \neq 0 \). This completes the proof of the Hard Lefschetz Theorem for \( V \).

**Remark.** In order to avoid the use of \[Ma\], one can easily adapt the proof of \[ELK\] (written when Masa’s theorem was only a conjecture). The assumption \( H^j_{\mathcal{F}}(\mathcal{F}, \mathbb{C}) \neq 0 \) is used in \[ELK\] to prove the classical Hodge-theoretic adjoint formula: \( \delta = (-1)^j * \partial * \). Due to the vanishing of the mean curvature associated to our metric \( g \), this formula follows from \[Ton\], Th. 7.10.

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