CONVERGENCE OF SCALAR CURVATURE OF KÄHLER-RICCI FLOW ON MANIFOLDS OF POSITIVE KODAIRA DIMENSION

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Abstract. In this paper, we consider Kähler-Ricci flow on n-dimensional Kähler manifold with semi-ample canonical line bundle and $0 < m := \text{Kod}(X) < n$. Such manifolds admit a Calabi-Yau fibration over its canonical model. We prove that the scalar curvature of the Kähler metrics along the normalized Kähler-Ricci flow converge to $-m$ outside the singular set of this fibration.

1. Introduction

Let us first recall the set up of Song-Tian [6, 7, 9] where our result will apply. Let $(X^n, \omega_0)$ be a compact Kähler manifold with canonical line bundle $K_X$ being semi-ample and $0 < m := \text{Kod}(X) < n$. Therefore the canonical ring $R(X, K_X)$ is finitely generated, and so the pluricanonical system $|\ell K_X|$ for sufficiently large $\ell \in \mathbb{Z}^+$ induces a holomorphic map

\begin{equation}
\tag{1.1}
f : X \rightarrow B \subset \mathbb{CP}^N := \mathbb{P}H^0(X, K_X^{\otimes \ell}),
\end{equation}

where $B$ is the canonical model of $X$. We have $\text{dim} B = m$.

Let $S'$ be the singular set of $B$ together with the set of critical values of $f$, and we define $S = f^{-1}(S') \subset X$.

Now let $\omega(t)$ be the smooth global solution of the normalized Kähler-Ricci flow

\begin{equation}
\tag{1.2}
\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) - \omega, \quad \omega|_{t=0} = \omega_0.
\end{equation}

It’s well-known [11, 15] that the flow has a global solution on $X \times [0, \infty)$. It’s shown by Song-Tian [6, 7] that $\omega(t)$ collapses nonsingular Calabi-Yau fibers and the flow converges weakly to a generalized Kähler-Einstein metric $\omega_B$ on its canonical model $B$, with $\omega_B$ is smooth and satisfies the generalized Einstein equation on $B \setminus S'$

\begin{equation}
\tag{1.3}
\text{Ric}(\omega_B) = -\omega_B + \omega_{WP},
\end{equation}

where $\omega_{WP}$ is the Weil-Petersson metric induced by the Calabi-Yau fibration $f$. They also proved the $C^0$-convergence on the potential level and in the case when $X$ is an elliptic surface the $C^{1,\alpha}_{\text{loc}}$-convergence of potentials on $X \setminus S$ for any $\alpha < 1$. In [9], Song-Tian showed that the scalar curvature is uniformly bounded on $X \times [0, \infty)$ along the normalized flow. The case
when $X$ is of general type is given by Z. Zhang in [17]. The case for conical Kähler-Ricci flow is given by G. Edwards in [3].

In [2], Fong-Zhang proved the $C^{1,\alpha}$-convergence of potentials when $X$ is a global submersion over $B$ and showed the Gromov-Hausdorff convergence in the special case. In [13] Tosatti-Weinkove-Yang improved the estimate and showed that the metric $\omega(t)$ converges to $f^*\omega_B$ in the $C^0$ local-topology on $X\setminus S$. Moreover, Tosatti-Weinkove-Yang [13] proved that the restricted metric $\omega(t)|_{X_y}$ converges (up to scalings) in the $C^0$-topology to the unique Ricci flat metric on the fibre $X_y$ for any regular value $y$; this result is improved to be smooth convergence by Tosatti-Zhang in [14]. Also see Tosatti’s note [12] for clearer and more unified discussions.

In fact, Tosatti-Weinkove-Yang [13] obtained in their proof that $\|\omega(t) - \tilde{\omega}(t)\|_{\omega(t)} \to 0$ as $t \to \infty$ on $X\setminus S$, where $\tilde{\omega}(t) = e^{-t}\omega_{SRF} + (1 - e^{-t})\omega_B$ (see Section 2 for definition of $\omega_{SRF}$). This enable us to prove that $\|\omega(t)|_{X_y} - m\| + \|\omega_B\|_{\omega(t)}^2 - m \to 0$ as $t \to \infty$ on $X\setminus S$, which then enable us to improve the estimate of scalar curvature on $X\setminus S$, following the argument of Song-Tian [9]. In this paper, we prove that the scalar curvature $R(t)$ converges on the regular part $X\setminus S$.

**Theorem 1.1.** Let $(X, \omega_0)$ be given as above, let $\omega(t)$ be the smooth global solution of the normalized Kähler-Ricci flow (1.2). Then we have

\[
\lim_{t \to \infty} R(t) = -m, \quad \text{on } X\setminus S \times [0, \infty).
\]

In particular, if $S = \emptyset$, then $f$ is a holomorphic submersion and we have

\[
|R(t) + m| \leq Ce^{-\eta t}, \quad \text{on } X \times [0, \infty),
\]

for some constants $\eta, C > 0$ depending on $(X, \omega_0)$.

After rescaling time and space simultaneously, we have the following immediately corollary from Theorem 1.1 of the unnormalized Kähler-Ricci flow.

**Corollary 1.2.** Let $(X, \omega_0)$ be given as above, let $\omega(t)$ be the smooth global solution of the unnormalized Kähler-Ricci flow

\[
\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega), \quad \omega|_{t=0} = \omega_0.
\]

Then we have

\[
\lim_{t \to \infty} (1 + t)R(t) = -m, \quad \text{on } X\setminus S \times [0, \infty).
\]

In particular, if $S = \emptyset$, then $f$ is a holomorphic submersion and we have

\[
|(1 + t)R(t) + m| \leq \frac{C}{(1 + t)^\eta}, \quad \text{on } X \times [0, \infty),
\]

for some constants $\eta, C > 0$ depending on $(X, \omega_0)$.

Note that in Theorem 1.1, the limiting behavior of scalar curvature on the singular set $S$ is unknown. A recent result of the author and two other authors [4] says that: If the canonical bundle $K_X$ is semi-ample, then for any
Kähler class \([\omega]\) on \(X\), there exists \(\delta_{X,[\omega]} > 0\) such that for any \(0 < \delta < \delta_{X,[\omega]}\), there exists a unique cscK metric in the Kähler class \([K_X] + \delta[\omega]\). Hence we can propose the following conjecture.

**Conjecture 1.3.** Let \(X\) be an \(n\)-dimensional Kähler manifold with nef canonical bundle \(K_X\) and positive Kodaira dimension. Then for any initial Kähler metric \(\omega_0\), the solution \(\omega(t)\) of the normalized Kähler-Ricci flow

\[
\frac{\partial}{\partial t} \omega = -\text{Ric}(\omega) - \omega, \quad \omega(0) = \omega_0
\]

converges in Gromov-Hausdorff topology to \(\omega_B\) and the scalar curvature \(R(t)\) converges to \(-\text{Kod}(X)\) in \(C^0(X)\), where \(\text{Kod}(X)\) is the Kodaira dimension of \(X\).

In general, it is natural to ask if the following holds for the maximal solution of the unnormalized Kähler-Ricci flow on \(X \times [0,T]\), where \(X\) is a Kähler manifold and \(T > 0\) is the maximal existence time.

1. If \(T < \infty\), then there exists \(C > 0\) such that

\[-C \leq R(t) \leq C(T-t)^{-1}\]

2. If \(T = \infty\), then there exists \(C > 0\) such that

\[|R(t)| \leq C(1 + t)^{-1}\]

In [5], the answer to the first question is affirmative due to Perelman for the Kähler-Ricci flow on Fano manifolds with finite time extinction. In [13], it is shown that if the Kähler-Ricci flow develops finite time singularity, the scalar curvature blows up at most of rate \((T-t)^{-2}\) if \(X\) is projective and if the initial Kähler class lies in \(H^2(X, \mathbb{Q})\).

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**2. Preliminary for the Kähler Ricci-flow**

In this section let us recall some known results that we need in our proof. From [11], we have \(f^*O(1) = K_X^{\otimes t}\), hence if we let \(\chi = \frac{1}{t}\omega_{FS}\) on \(\mathbb{P}H^0(X, K_X^{\otimes t})\), we have that \(f^*\chi\) (later, denoted by \(\chi\)) is a smooth semi-positive representative of \(-c_1(X)\). Here, \(\omega_{FS}\) denotes the Fubini-Study metric. Also, we denote by \(\chi\) the restriction of \(\chi\) to \(B \setminus S'\).
Given a Kähler metric $\omega_0$ on $X$, since $X_y := f^{-1}(y)$ are Calabi-Yau for $y \in B \setminus S'$, there exists a unique smooth function $\rho_y$ on $X_y$ with $\int_{X_y} \rho_y \omega_0^{n-m} = 0$, and such that $\omega_0|_{X_y} + \sqrt{-1} \partial \bar{\partial} \rho_y =: \omega_y$ is the unique Ricci-flat Kähler metric on $X_y$. Moreover, $\rho_y$ depends smoothly on $y$, and so define a global smooth function on $X \setminus S$. We define

$$\omega_{SRF} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \rho,$$

which is a closed real $(1,1)$-form on $X \setminus S$, restricts to a Ricci-flat Kähler metric on all fibers $X_y$ of $y \in B \setminus S'$. Let $\Omega$ be the smooth volume form on $X$ with

$$\sqrt{-1} \partial \bar{\partial} \log \Omega = \chi, \quad \int_X \Omega = \left( \frac{n}{m} \right) \int_X \omega_0^{n-m} \wedge \chi^m. \tag{2.1}$$

Define a function $F$ on $X \setminus S$ by

$$F := \frac{\Omega}{\left( \frac{n}{m} \right) \chi^m \wedge \omega_{SRF}^{n-m}}, \tag{2.2}$$

then $F$ is constant along the fiber $X_y$, $y \in B \setminus S'$, so it descends to a smooth function on $B \setminus S'$. Then [7] showed that the Monge-Ampère equation

$$\left( \chi + \sqrt{-1} \partial \bar{\partial} v \right)^m = F \, \chi^m, \tag{2.3}$$

has a unique solution $v \in \text{PSH}(\chi) \cap C^0(B) \cap C^\infty(B \setminus S')$. Define

$$\omega_B = \chi + \sqrt{-1} \partial \bar{\partial} v,$$

which is a smooth Kähler metric on $B \setminus S'$, satisfies the twisted Kähler-Einstein equation

$$\text{Ric}(\omega_B) = -\omega_B + \omega_{WP},$$

where $\omega_{WP}$ is the smooth Weil-Petersson form on $B \setminus S'$.

Now let $\omega = \omega(t)$ be the solution of the normalized Kähler-Ricci flow

$$\frac{\partial}{\partial t} \omega = -\text{Ric}(\omega) - \omega, \quad \omega(0) = \omega_0, \tag{2.4}$$

which exists for all time. Define the reference metrics

$$\hat{\omega}(t) = e^{-t} \omega_0 + (1 - e^{-t}) \chi,$$

which are Kähler for all $t \geq 0$, and we can write $\omega(t) = \hat{\omega}(t) + \sqrt{-1} \partial \bar{\partial} \varphi(t)$, and $\varphi(0) = 0$, then the Kähler-Ricci flow [2.4] is equivalent to the parabolic Monge-Ampère equation

$$\frac{\partial}{\partial t} \varphi = \log e^{(n-m)t} \left( \hat{\omega}(t) + \sqrt{-1} \partial \bar{\partial} \varphi(t) \right)^{n} - \varphi, \quad \varphi(0) = 0. \tag{2.5}$$

From now on, we always set $K = f^{-1}(K')$ where $K' \subset B \setminus S'$ is a compact subset. Then we can choose some open subset $U' \subset B \setminus S'$ such that $K' \subset U'$. Set $U = f^{-1}(U')$, then $K \subset U \subset X \setminus S$. Also, we denote by $h(t)$ some positive decreasing function on $[0, \infty)$ which tends to zero as $t \to \infty$. 

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Now we have the following lemmas. See [13, 12] for unified discussions (and also [2, 7, 9]).

**Lemma 2.1.** There exists some constant \( C = C(K) \) and \( h(t) \) depending on the domain \( K \), such that

1. \( C^{-1} \dot{\omega}(t) \leq \omega(t) \leq C \dot{\omega}(t) \), on \( K \times [0, \infty) \).
2. \( |\varphi - v| + |\dot{\varphi} + \varphi - v| \leq h(t) \), on \( K \times [0, \infty) \).
3. There exists a uniform \( C_0 > 0 \) such that \( |R| \leq C_0 \), on \( X \times [0, \infty) \).
4. \( \text{tr}_{\omega(t)} \omega_B - m \leq h(t) \), on \( K \times [0, \infty) \).
5. Especially, if \( S = \emptyset \), then (1)-(4) hold with \( K \) replaced by \( X \) and \( h(t) \) replaced by \( Ce^{-\eta t} \) for some constants \( \eta, C > 0 \) depending on \( (X, \omega_0) \).

**Lemma 2.2.** Along the normalized Kähler Ricci-flow, we have on \( X \setminus S \times [0, \infty) \)

\[
\left( \frac{\partial}{\partial t} - \Delta \right) (\dot{\varphi} + \varphi - v) = \text{tr}_{\omega(t)} \omega_B - m.
\]

and there exists some \( C = C(K) > 0 \) such that

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \text{tr}_{\omega(t)} \omega_B \leq C, \text{ on } K \times [0, \infty).
\]

Especially, when \( S = \emptyset \), then \((2.6), (2.7)\) holds on \( X \times [0, \infty) \) with \( C \) depending on \( (X, \omega_0) \).

Next we define on \( X \setminus S \) the reference metrics

\[
\tilde{\omega}(t) = e^{-t} \omega_{SRF} + (1 - e^{-t}) \omega_B.
\]

Then we have the following theorem due to [13] (in the proof).

**Theorem 2.3.** There exists \( h(t) \) depending on the domain \( K \) such that

\[
\|\omega(t) - \tilde{\omega}(t)\|_{C^0(K, \omega(t))} \leq h(t).
\]

Especially, when \( S = \emptyset \), then

\[
\|\omega(t) - \tilde{\omega}(t)\|_{C^0(X, \omega(t))} \leq Ce^{-\eta t}.
\]

for some constants \( \eta, C > 0 \) depending on \( (X, \omega_0) \).

We also need the following lemma to choose local coordinates on the regular part, see e.g. Lemma 5.6 of [12].

**Lemma 2.4.** Let \( f : X^n \to Y^m \) be a holomorphic submersion between complex manifolds. Then given any point \( x \in X \) we can find an open set \( U \ni x \) and local holomorphic coordinates \((z_1, \ldots, z_n)\) on \( U \) and \((y_1, \ldots, y_m)\) on \( f(U) \) such that in these coordinates the map \( f \) is given by \((z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_m)\), i.e., \( y_1 = z_1, \ldots, y_m = z_m \).

We can apply Lemma 2.4 to a point \( x \in X \setminus S, y = f(x) \in B \setminus S' \) to choose local coordinates, and we may call such coordinates “local product coordinates”.

3. Convergence of the Trace and Norm of $\omega_B$ Along the Flow

From now on, we denote by $T_0 = \text{tr}_\omega(t)\omega_B$.

In this section, we use Theorem 2.3 to prove $|T_0 - m| + \|\omega_B\|_{\omega(t)}^2 \to 0$ as $t \to \infty$ on $X \setminus S$. As before, we use $h(t), h_1(t), \ldots$ to denote positive decreasing functions on $[0, +\infty)$ which tends to zero as $t \to \infty$.

First, we have the following basic estimate.

**Lemma 3.1.** For any point $x \in U$ with local product coordinates given by Lemma 2.4 around $x$ and $y = f(x)$, say $(z_1, \ldots, z_n)$ around $x$ and $(y_1, \ldots, y_m)$ around $y$. Suppose on such coordinate neighborhood $\omega(t)$ is given by

$$\omega(t) = \sum_{i,j=1}^{n} g(t)_{ij} \, dz_i \wedge d\bar{z}_j,$$

then there exists some constant $C$ depending on the domain such that: for $1 \leq \alpha, \beta \leq m$, $m + 1 \leq i, j \leq n$,

$$|g(t)_{\alpha\beta}| \leq C, \quad \left|g(t)_{\alpha\bar{\beta}}\right| \leq Ce^{-t}, \quad \left|g(t)_{\bar{\alpha}\bar{\beta}}\right| \leq Ce^{-t}.$$  \hspace{1cm} (3.1)

$$\left|g(t)_{\alpha\bar{\beta}}\right| \leq C, \quad \left|g(t)^\alpha_{\bar{\beta}}\right| \leq Ce^{t}, \quad \left|g(t)_{\bar{\beta}}\right| \leq Ce^t.$$  \hspace{1cm} (3.2)

at $x$. In particular, when $S = \emptyset$, (3.1) and (3.2) hold on $X \times [0, \infty)$ with $C$ depending on $(X, \omega_0)$.

**Proof.** We define on such coordinate neighborhood (contained in $U$) the local metrics

$$\omega_E(t) = \omega^{(m)} + e^{-t}\omega^{(n-m)},$$

where $\omega^{(m)}$ and $\omega^{(n-m)}$ denotes the standard Euclidean metrics on the two factors of $\mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^{n-m}$. Thanks to Lemma 2.1, we can find constant $C$ depending on the domain $K$ such that

$$C^{-1}\omega_E(t) \leq \omega(t) \leq C\omega_E(t).$$  \hspace{1cm} (3.4)

Denote

$$\omega_B = \sqrt{-1} \sum_{\alpha, \beta=1}^{m} (g_B)_{\alpha\beta} \, dz_\alpha \wedge d\bar{z}_\beta, \quad \omega_{\text{SRF}} = \sqrt{-1} \sum_{i,j=1}^{n} (g_{\text{SRF}})_{ij} \, dz_i \wedge d\bar{z}_j,$$

Hence using (2.8) of Theorem 2.3 we have

$$C \geq \|\omega(t) - \bar{\omega}(t)\|_{\omega_E(t)}^2,$$

$$\geq \sum_{\alpha, \beta=1}^{m} \left|g(t)_{\alpha\beta} - (1 - e^{-t})(g_B)_{\alpha\beta} - e^{-t}(g_{\text{SRF}})_{\alpha\beta}\right|^2$$

$$+ \sum_{\alpha=1}^{m} \sum_{j=m+1}^{n} e^{t} \left|g(t)_{\alpha\bar{j}} - e^{-t}(g_{\text{SRF}})_{\alpha\bar{j}}\right|^2 + \sum_{i,j=m+1}^{n} e^{2t} \left|g(t)_{i\bar{j}} - e^{-t}(g_{\text{SRF}})_{i\bar{j}}\right|^2,$$
on our coordinate neighborhood. Then we can apply the trivial inequality $|a - b|^2 \geq \frac{1}{2}|a|^2 - |b|^2$ to each term to conclude (3.1). Next, from Lemma 2.1 we have the first and third estimates of (3.2), and the second estimate then follows from Cauchy-Schwarz inequality. □

**Proposition 3.2.** There exists $h(t)$ depending on $K$ such that

(3.5) $|T_0 - m| \leq h(t)$, on $K \times [0, \infty)$.

(3.6) $\|\omega_{B(t)} - m\| \leq h(t)$, on $K \times [0, \infty)$.

In particular, if $S = \emptyset$, then we have

(3.7) $|T_0 - m| + \|\omega_{B(t)} - m\| \leq Ce^{-\eta t}$, on $X \times [0, \infty)$,

where $\eta, C > 0$ are constants depending on $(X, \omega_0)$.

**Proof.** Applying Theorem 2.3 to $U$ there exists some $h_1(t)$ depending on the domain such that

(3.8) $\|\omega(t) - \tilde{\omega}(t)\|_{\mathcal{C}^0(U, \omega(t))} \leq h_1(t)$.

Now given any $x_0 \in K$, we choose local product coordinate like Lemma 3.1, say $U_0 \subset U$ around $x_0$. WLOG, we may assume

$U_0 = B^{(m)}(1) \times B^{(n-m)}(1) \subset \mathbb{C}^n$, $f(U_0) = B^{(m)}(1) \subset \mathbb{C}^m$, $x_0 = (0, 0)$,

where $B^{(m)}(1)$ and $B^{(n-m)}(1)$ denotes Euclidean unit balls in $\mathbb{C}^m$ and $\mathbb{C}^{n-m}$, respectively. The map $f$ is given by $f(z_1, \ldots, z_n) = (z_1, \ldots, z_m)$. Fix a time $t$, we define the transformation

$F_t : U_1 := B^{(m)}(1) \times B^{(n-m)}(1)(e^{-\frac{t}{2}}) \rightarrow U_0$

$F_t(w_1, \ldots, w_n) = (w_1, \ldots, w_m, e^{\frac{t}{2}}w_{m+1}, \ldots, e^{\frac{t}{2}}w_n),$

Then consider on $U_1$ the metrics

$\omega_1(t) = F_t^* \omega(t), \quad \tilde{\omega}_1(t) = F_t^* \tilde{\omega}(t).$

We immediately have from (3.3) that

(3.9) $\|\omega_1(t) - \tilde{\omega}_1(t)\|_{\omega_1(t)}(x_0) = \|\omega(t) - \tilde{\omega}(t)\|_{\omega(t)}(x_0) \leq h_1(t)$.

Denote on $U_1$

$\omega_1(t) = \sqrt{-1} \sum_{i,j=1}^{\infty} h(t)_{ij}dw_i \wedge d\overline{w}_j, \quad \tilde{\omega}_1(t) = \sqrt{-1} \sum_{i,j=1}^{\infty} \tilde{h}(t)_{ij}dw_i \wedge d\overline{w}_j,$
then by definition

\[ \omega_1(t) = F^*_t \left[ \sqrt{-1} \sum_{i,j=1}^n g(t)_{ij} dz_i \wedge d\bar{z}_j \right] \]

\[ = \sqrt{-1} \sum_{\alpha, \beta=1}^m g(t)_{\alpha\beta} dw_\alpha \wedge d\overline{w}_\beta + 2 \text{Re} \left[ \sqrt{-1} \sum_{\alpha=1}^m \sum_{j=m+1}^n e^{\frac{t}{2}} g(t)_{\alpha j} dw_\alpha \wedge d\overline{w}_j \right] \]

\[ + \sqrt{-1} \sum_{i,j=m+1}^n e^{t} g(t)_{ij} dw_i \wedge d\overline{w}_j, \]

dependence on the definition.

hence we obtain: for \( 1 \leq \alpha, \beta \leq m, m+1 \leq i, j \leq n \)

\[ h(t)_{\alpha\beta} = g(t)_{\alpha\beta}, \quad h(t)^{\alpha\beta} = g(t)^{\alpha\beta}, \]

\[ h(t)_{ij} = e^{t} g(t)_{ij}, \quad h(t)^{ij} = e^{-t} g(t)^{ij}. \]

Then we apply Lemma 3.1 to conclude there exists some constant \( C = C(K) > 0 \) such that

\[ \sum_{i,j=1}^n \left( |h(t)_{ij}| + |h(t)^{ij}| \right) (x_0) \leq C. \]

Similarly, we have: for \( 1 \leq \alpha, \beta \leq m, m+1 \leq i, j \leq n \)

\[ \tilde{h}(t)_{\alpha\beta} = (1 - e^{-t})(g_B)_{\alpha\beta} + e^{-t}(g_{SRF})_{\alpha\beta}, \]

\[ \tilde{h}(t)_{ij} = e^{-t} (g_{SRF})_{ij}, \]

\[ \tilde{h}(t)_{\alpha\beta} = (g_{SRF})_{\alpha\beta}. \]

Now at \( x_0 \) we define an \( n \times n \) matrix

\[ A = (a_{ij})_{n \times n} = \begin{pmatrix} (g_B)_{\alpha\beta}(x_0) & 0 \\ 0 & (g_{SRF})_{\alpha\beta}(x_0) \end{pmatrix}_{1 \leq \alpha, \beta \leq m, m+1 \leq i, j \leq n}. \]

We claim that

\[ \|\omega_1(t) - A\|_{\omega_E}^2 (x_0) \leq h_2(t). \]

for some \( h_2(t) \) depending on the domain, where \( \omega_E \) is the standard Euclidean metric on \( U_1 \). To see this, we use (3.9) and (3.11) to obtain

\[ C h_1(t) \geq \|\omega_1(t) - \omega_1(t)\|_{\omega_E}^2 (x_0) \]

\[ = \sum_{i,j=1}^n \left| h(t)_{ij} - \tilde{h}(t)_{ij} \right|^2 (x_0). \]
But at $x_0$

$$\|\omega(t) - A\|_{E}^2 = \sum_{\alpha, \beta = 1}^{m} |h(t)_{\alpha\beta} - a_{\alpha\beta}|^2 + 2 \sum_{\alpha = 1}^{m} \sum_{j = m+1}^{n} |h(t)_{\alpha j}|^2 + \sum_{i, j = m+1}^{n} |h(t)_{i j} - a_{i j}|^2.$$ 

Then we can use (3.12) and (3.14) to estimate three terms on the RHS of the above equality. Indeed, for $1 \leq \alpha, \beta \leq m$, we have at $x_0$

$$Ch_1(t) \geq \left| h(t)_{\alpha\beta} - \tilde{h}(t)_{\alpha\beta} \right|^2 = \left| h(t)_{\alpha\beta} - (1 - e^{-t})(g_B)_{\alpha\beta} - e^{-t}(g_{SRF})_{\alpha\beta} \right|^2 \geq \frac{1}{2} \left| h(t)_{\alpha\beta} - a_{\alpha\beta} \right|^2 - e^{-2t} \left| (g_B)_{\alpha\beta} - (g_{SRF})_{\alpha\beta} \right|^2,$$

which gives

$$\sum_{\alpha, \beta = 1}^{m} \left| h(t)_{\alpha\beta} - a_{\alpha\beta} \right|^2 \leq 2Ch_1(t) + 2e^{-2t} \left| (g_B)_{\alpha\beta} - (g_{SRF})_{\alpha\beta} \right|^2 \leq h_3(t).$$

Next, for $1 \leq \alpha \leq m, m + 1 \leq j \leq n$, we have

$$Ch_1(t) \geq \left| h(t)_{\alpha j} - \tilde{h}(t)_{\alpha j} \right|^2 = \left| h(t)_{\alpha j} - e^{-\frac{t}{2}}(g_{SRF})_{\alpha j} \right|^2 \geq \frac{1}{2} \left| h(t)_{\alpha j} \right|^2 - e^{-t} \left| (g_{SRF})_{\alpha j} \right|^2,$$

which gives

$$\sum_{\alpha = 1}^{m} \sum_{j = m+1}^{n} \left| h(t)_{\alpha j} \right|^2 \leq 2Ch_1(t) + 2e^{-t} \left| (g_{SRF})_{\alpha j} \right|^2 \leq h_4(t).$$

Finally, for $m + 1 \leq i, j \leq n$, we have

$$Ch_1(t) \geq \sum_{i, j = m+1}^{n} \left| h(t)_{i j} - (g_{SRF})_{i j} \right|^2 = \sum_{i, j = m+1}^{n} \left| h(t)_{i j} - a_{i j} \right|^2,$$

Combine the above three estimates we obtain (3.13) with

$$h_2(t) = Ch_1(t) + h_3(t) + h_4(t).$$
Now at $x_0$ we can use $(3.11)$ and $(3.13)$ to get

$$\left| \det \omega_1(t) - \det A \right|$$

$$= \left| \sum_{(j_1, \ldots, j_n)} (-1)^{\sigma(j_1, \ldots, j_n)} (h(t)_{j_1} \cdots h(t)_{j_n} - a_{i1} \cdots a_{in}) \right|$$

$$\leq \sum_{(j_1, \ldots, j_n)} \left| h(t)_{j_1} \cdots h(t)_{(n-1)j_{n-1}} (h(t)_{j_n} - a_{nj_n}) \right| + \cdots + \sum_{(j_1, \ldots, j_n)} \left| (h(t)_{1j_1} - a_{1j_1}) \cdots h(t)_{n j_n} \right|$$

$$\leq C \sum_{(j_1, \ldots, j_n)} \left( \left| h(t)_{1j_1} - a_{1j_1} \right| + \cdots + \left| h(t)_{n j_n} - a_{nj_n} \right| \right)$$

$$\leq Ch_2(t)^{\frac{1}{2}}.$$

Hence we obtain

$$(3.15) \quad \left| \det \omega_1(t)(x_0) - \det A \right| \leq h_5(t).$$

But $\det A \in [A_0, A_1]$ for some positive constants $A_0, A_1$ depending on the domain, independent of $t$ (after $t$ is large), so we can choose a large time $T \geq 1$ such that for all $t \geq T$, we have

$$(3.16) \quad \det \omega_1(t)(x_0) \in \left[ \frac{1}{2} A_0, 2A_1 \right],$$

Set $A^{-1} = \left( a^{j}_{i} \right)$. Now we use $(3.11)$, $(3.13)$, $(3.15)$, $(3.16)$ to estimate at $x_0$

$$\left| h(t)^{1T} - a^{1T} \right|$$

$$= \left| \sum_{(j_2, \ldots, j_n)} \left( \frac{(-1)^{\sigma(j_2, \ldots, j_n)} h(t)_{j_2} \cdots h(t)_{j_n}}{\det \omega_1(t)} - \frac{(-1)^{\sigma(j_2, \ldots, j_n)} a_{j_2} \cdots a_{j_n}}{\det A} \right) \right|$$

$$\leq \frac{1}{\det \omega_1(t)} \sum_{(j_2, \ldots, j_n)} \left| h(t)_{j_2} \cdots h(t)_{j_n} - a_{j_2} \cdots a_{j_n} \right|$$

$$+ \sum_{(j_2, \ldots, j_n)} \left| a_{j_2} \cdots a_{j_n} \right| \left| \frac{1}{\det \omega_1(t)} - \frac{1}{\det A} \right|$$

$$\leq \frac{1}{2} C h_2(t)^{\frac{1}{2}} + C \frac{h_5(t)}{\frac{T}{2} A_0 \cdot A_0} \leq h_6(t),$$

Similar argument holds for all $1 \leq i, j \leq n$, hence we obtain

$$(3.17) \quad \sum_{i,j=1}^{n} \left| h(t)^{ij}(x_0) - a^{ij} \right| \leq h_6(t).$$
By the special form of the matrix $A$, we have
\[
\sum_{\alpha, \beta = 1}^{m} a^{\alpha \beta} a_{\alpha \beta} = m, \quad \sum_{i, j, k, l = 1}^{m} a^{i j k l} a_{i j k l} = m,
\]
hence we can apply (3.10), (3.11) and (3.17) to estimate at $x_0$
\[
|T_0 - m| = \left| \sum_{\alpha, \beta = 1}^{m} h(t)^{\alpha \beta} a_{\alpha \beta} - \sum_{\alpha, \beta = 1}^{m} a^{\alpha \beta} a_{\alpha \beta} \right| 
\leq \sum_{\alpha, \beta = 1}^{m} \left| h(t)^{\alpha \beta} - a^{\alpha \beta} \right| \cdot \left| a_{\alpha \beta} \right| \leq h_7(t),
\]
which gives (3.5) at $x_0$, and similarly
\[
\left| \|\omega_B\|^2_{\omega(t)} - m \right| = \left| \sum_{i, j, k, l = 1}^{m} h(t)^{i j k l} a_{i j k l} - \sum_{i, j, k, l = 1}^{m} a^{i j k l} a_{i j k l} \right| 
\leq \sum_{i, j, k, l = 1}^{m} \left( \left| h(t)^{i j} - a^{i j} \right| \left| h(t)^{k l} a_{i j k l} \right| + \left| h(t)^{i j} \right| \left| h(t)^{k l} - a^{k l} \right| \left| a_{i j k l} \right| \right) 
\leq h_8(t),
\]
which gives (3.6) at $x_0$. Since $x_0 \in K$ was arbitrary chosen, we obtain (3.5) and (3.6).

When $S = \emptyset$, the above estimates hold on the whole manifold $X$ with $h(t)$ replaced by $Ce^{-\eta t}$ where $\eta, C > 0$ are constants depending on $(X, \omega_0)$ which may change from line to line. This completes the proof. 

\section{4. The Proof of Theorem 1.1}

In this section we prove Theorem 1.1. All the operators $\nabla, \Delta, \langle, \rangle$ are with respect to the evolving metric $\omega(t)$.

We first need the following basic lemma to improve our decreasing function $h(t)$.

\begin{lemma}
For any $h(t) : [0, \infty) \rightarrow (0, \infty)$, a positive decreasing function which tends to zero as $t \rightarrow \infty$, there exists a smooth positive decreasing function $A(t) : [0, \infty) \rightarrow (0, \infty)$ satisfying that: $h(t) \leq A(t), t \geq 0; A(t) \rightarrow 0$ as $t \rightarrow \infty$; and moreover
\begin{equation}
0 \leq -A'(t) \leq 100A(t), \text{ on } [0, \infty).
\end{equation}
\end{lemma}

\begin{proof}
Choose $0 < \ell_1 < \ell_2 < \cdots$ in the following way: Let $\ell_1 \gg 2$ such that $h(\ell_1) < \frac{1}{2} h(0)$. Then choose $\ell_2 \gg \ell_1 + 2$ such that $h(\ell_2) < \frac{1}{2^2} h(0)$. Repeat this process, for each $k \geq 1$, we choose $\ell_{k+1} \gg \ell_k + 2$ such that $h(\ell_{k+1}) < \frac{1}{2^k} h(0)$. First we define
\[
A(t) \equiv h(0), \quad t \in [0, \ell_1]; \quad A(t) \equiv \frac{1}{2^k} h(0), \quad t \in [\ell_k + 2, \ell_{k+1}], k \geq 1.
\]

\end{proof}
Hence for each $k \geq 1$, we have
\[
\begin{cases}
  h(t) \leq h(\ell_k) < \frac{1}{2^k}h(0), \ t \in [\ell_k, \infty), \\
  A(t) \equiv \frac{1}{2^{k-1}}h(0), \ t \in [\ell_k - 1, \ell_k], \\
  A(t) \equiv \frac{1}{2^k}h(0), \ t \in [\ell_k + 2, \ell_k + 3],
\end{cases}
\]
hence we can define $A(t)$ on $[\ell_k, \ell_k + 2]$ such that $A(t)$ is smooth and decreasing on $[\ell_k - 1, \ell_k + 3]$ (and hence smooth and decreasing on $(0, \infty)$) and moreover
\[
0 \leq -A'(t) \leq 200 \cdot \frac{\frac{1}{2^{k-1}}h(0) - \frac{1}{2^k}h(0)}{(\ell_k + 2) - \ell_k} = \frac{100}{2^k}h(0) \leq 100A(t),
\]
for $t \in [\ell_k, \ell_k + 2]$. Outside such intervals, $A'(t) \equiv 0$, hence (4.1) is verified. Also, on each $[\ell_k, \ell_k + 1]$, we have
\[
h(t) \leq \frac{1}{2^k}h(0) \leq A(t),
\]
hence $h(t) \leq A(t)$ on $[0, \infty)$. Finally, $A(t) \to 0$ as $t \to \infty$ is easy to see. This finishes the proof. \hfill \Box

Next, we need to construct local cutoff function.

**Lemma 4.2.** Recall that $K = f^{-1}(K'), U = f^{-1}(U'), K' \subset U' \subset B \setminus S'$. Then there exists a smooth cutoff function $\rho$ with $\text{supp}(\rho) \subset U$, $\rho > 0$ on $U$, $0 \leq \rho \leq 1$, $\rho \equiv 1$ on $K$, satisfying
\[
(4.2) \quad |\nabla \rho|_{\omega(t)}^2 + |\Delta \omega(t)\rho| \leq C,
\]
on $U \times [0, \infty)$ for some constant $C$ depending on the domain $K$.

**Proof.** We first choose cutoff function $\rho_0$ on $B$ such that $\text{supp}(\rho_0) \subset U'$, $\rho_0 > 0$ on $U'$, $0 \leq \rho_0 \leq 1$, $\rho_0 \equiv 1$ on $K'$ and moreover
\[
\sqrt{-1}\partial \rho_0 \wedge \bar{\partial} \rho_0 \leq C\omega_B, \quad -C\omega_B \leq \sqrt{-1}\partial \bar{\partial} \rho_0 \leq C\omega_B,
\]
on $U'$. Then we set $\rho = f^*\rho_0$. Using Lemma 2.1 and Lemma 3.1, we have under local product coordinates
\[
|\nabla \rho|_{\omega(t)}^2 = \sum_{i,j=1}^m g(t)^{ij} \partial_i \rho \partial_j \rho \leq C,
\]
and
\[
|\Delta \omega(t)\rho| = |\text{tr}(\omega(t)\sqrt{-1}\partial \bar{\partial})\rho| \leq C\text{tr}(\omega(t)\omega_B) \leq C,
\]
with some constant $C$ depending on the domain $K$. This finishes the proof. \hfill \Box

In the following, we always use $h(t), A(t), B(t)$ to denote positive decreasing functions which tend to zero as $t \to \infty$, and moreover $A(t), B(t)$ satisfy condition (4.1).
Set $u = \dot{\varphi} + \varphi - v$ on $X \setminus S$. Recall that $T_0 = \text{tr}_B \omega_B$, and
\[
\left(\frac{\partial}{\partial t} - \Delta\right) u = T_0 - m.
\]
Then along the flow we have on $X \setminus S$
\[
\left(\frac{\partial}{\partial t} - \Delta\right) |\nabla u|^2 = |\nabla u|^2 - |\nabla \nabla u|^2 + 2Re(\nabla T_0 \cdot \nabla u).
\]
(4.3)
\[
\left(\frac{\partial}{\partial t} - \Delta\right) \Delta u = \Delta u + \langle \text{Ric}, \sqrt{-1} \partial \bar{\partial} u \rangle + \Delta T_0.
\]
(4.4)

We first estimate $|\nabla u|^2$.

**Proposition 4.3.** There exists $A(t)$ which depends on the domain $K$ such that
\[
|\nabla u|^2 \leq A(t), \text{ on } K \times [0, \infty).
\]
(4.5)

In particular, when $S = \emptyset$, then we have
\[
|\nabla u|^2 \leq Ce^{-\eta t}, \text{ on } X \times [0, \infty).
\]
(4.6)

for some constants $\eta, C > 0$ depending on $(X, \omega_0)$.

**Proof.** Apply Lemma 2.1 and Proposition 3.2, we can find some $h(t)$ depending on $K$ such that
\[
|u| + |T_0 - m| \leq h(t) \text{ on } \mathbb{U} \times [0, \infty).
\]
Then we choose $A(t)$ according to Lemma 4.1, such that $2h(t) \leq A(t)$, $t \geq 0$. Hence
\[
A(t) - u \in \left[ \frac{1}{2}A(t), 2A(t) \right], \quad |T_0 - m| \leq A(t), \text{ on } \mathbb{U} \times [0, \infty).
\]

So we can compute on $U \times [0, \infty)$ (see [9])
\[
\left(\frac{\partial}{\partial t} - \Delta\right) \left(\frac{|\nabla u|^2}{A(t) - u}\right) = \frac{|\nabla u|^2 - (|\nabla \nabla u|^2 + |\nabla \nabla \nabla u|^2) + 2Re(\nabla T_0 \cdot \nabla u)}{A(t) - u} - 2\epsilon Re \left[ \nabla \left(\frac{|\nabla u|^2}{A(t) - u}\right) \cdot \nabla u \right] - \frac{1}{(A(t) - u)^3} \frac{2(1 - \epsilon)}{A(t) - u} Re \left[ \nabla \left(\frac{|\nabla u|^2}{A(t) - u}\right) \cdot \nabla u \right] + (T_0 - m) \frac{|\nabla u|^2}{(A(t) - u)^2} - A'(t) \frac{|\nabla u|^2}{(A(t) - u)^2},
\]
for any $\epsilon \in \mathbb{R}$. Now for some $k \in \mathbb{R}$ to be fixed, we can compute
\[
\left(\frac{\partial}{\partial t} - \Delta\right) \frac{1}{(A(t) - u)^k} = -\frac{k A'(t)}{(A(t) - u)^{1+k}} - \frac{k(k + 1)|\nabla u|^2}{(A(t) - u)^{2+k}} + \frac{k(T_0 - m)}{(A(t) - u)^{1+k}},
\]
and
\[
\frac{2}{(A(t) - u)^{1+k}} Re \left[ \nabla \left( \frac{|\nabla u|^2}{A(t) - u} \right) \cdot \nabla u \right] = \frac{2}{A(t) - u} Re \left[ \nabla \left( \frac{|\nabla u|^2}{(A(t) - u)^{1+k}} \right) \cdot \nabla u \right] - 2k \frac{|\nabla u|^4}{(A(t) - u)^{3+k}},
\]
and hence on \( U \times [0, \infty) \)

(4.7)
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \left( \frac{|\nabla u|^2}{(A(t) - u)^{1+k}} \right) = \frac{2}{(A(t) - u)^{1+k}} \frac{\nabla u^2}{A(t) - u} \left( \frac{\partial}{\partial t} - \Delta \right) \frac{1}{(A(t) - u)^k} - 2Re \left[ \nabla \left( \frac{|\nabla u|^2}{A(t) - u} \right) \cdot \nabla \left( \frac{1}{(A(t) - u)^k} \right) \right] - \frac{2(1 - \epsilon + k)}{A(t) - u} Re \left[ \nabla \left( \frac{|\nabla u|^2}{(A(t) - u)^{1+k}} \right) \cdot \nabla u \right] - (1 + k) A'(t) \frac{|\nabla u|^2}{(A(t) - u)^{2+k}}.
\]

Now we set, in this proof, \( k = -\frac{1}{3}, \epsilon = \frac{2}{3} \), then
\[1 - \epsilon + k = 0, 2\epsilon + k(1 + k) - 2k(1 - \epsilon + k) = \frac{10}{9},\]
hence (4.7) becomes

(4.8)
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \left( \frac{|\nabla u|^2}{(A(t) - u)^{1+k}} \right) = \frac{|\nabla u|^2 - (|\nabla \nabla u|^2 + |\nabla u|^2) + 2Re(\nabla T_0 \cdot \nabla u)}{(A(t) - u)^{1+k}} - \frac{4}{3} Re \frac{|\nabla u|^2 \cdot \nabla u}{(A(t) - u)^{2+k}} - \frac{10}{9} \frac{|\nabla u|^4}{(A(t) - u)^{3+k}} + \frac{2}{3} \frac{(T_0 - m)|\nabla u|^2}{(A(t) - u)^{2+k}} - \frac{2}{3} A'(t) \frac{|\nabla u|^2}{(A(t) - u)^{2+k}}.
\]

We come to estimate each term. First, if we choose normal coordinates around a point in \( U \), then we have

\[|\nabla |\nabla u|^2 \cdot \nabla u| = \left| u_i (u_j u_\gamma) \right| = |u_i u_j u_\gamma + u_i u_\gamma u_\gamma| \leq |\nabla u|^2 \left( |\nabla \nabla u| + |\nabla u| \right),\]
\[ \left| 4 \cdot \text{Re} \left[ \nabla |\nabla u|^2 \cdot \nabla u \right] \right| \leq 2 \left( \frac{2}{3} \cdot \frac{|\nabla u|^2}{(A(t) - u)^{4k+2}} + \frac{|\nabla u|}{(A(t) - u)^{1+k}} + \frac{|\nabla u|}{(A(t) - u)^{1+k}} \right) \]

Then we obtain on \( U \). Next, let \( k = -\frac{1}{3} \), we have \( 4 + 4k = \frac{8}{3} = 3 + k \), hence

\[ \frac{\text{Re}(\nabla T_0 \cdot \nabla u)}{(A(t) - u)^{1+k}} \leq \frac{2|\nabla T_0|}{(A(t) - u)^{1+k}} \leq |\nabla T_0|^2 + \frac{|\nabla u|^4}{(A(t) - u)^{4k+2}} \]

\[ \leq |\nabla T_0|^2 + \frac{1}{100} \cdot \frac{|\nabla u|^4}{(A(t) - u)^{3k+3}} + 100, \]

The above two terms are the main terms that we need to be careful about, rest three terms are easy to control by using (4.11) and (4.6) (remember that \( k = -\frac{1}{3} \)):

\[ \frac{|\nabla u|^2}{(A(t) - u)^{1+k}} \leq \frac{1}{100} \cdot \frac{|\nabla u|^4}{(A(t) - u)^{3k+3}} + 100(A(t) - u)^{1-k} \]

\[ \leq \frac{1}{100} \cdot \frac{|\nabla u|^4}{(A(t) - u)^{3k+3}} + C, \]

\[ \frac{2}{3} \cdot (T_0 - m)|\nabla u|^2 \leq \frac{A(t)|\nabla u|^2}{(A(t) - u)^{2k+2}} \leq \frac{1}{100} \cdot \frac{|\nabla u|^4}{(A(t) - u)^{3k+3}} + C, \]

\[ \frac{2}{3} \cdot \frac{A'(t)|\nabla u|^2}{(A(t) - u)^{2k+2}} \leq \frac{100A(t)|\nabla u|^2}{(A(t) - u)^{1+k}} \leq \frac{1}{100} \cdot \frac{|\nabla u|^4}{(A(t) - u)^{3k+3}} + C, \]

Hence we obtain on \( U \times [0, \infty) \)

\[ (\frac{\partial}{\partial t} - \Delta) \left( \frac{|\nabla u|^2}{(A(t) - u)^{1+k}} \right) \leq -\frac{1}{10} \cdot \frac{|\nabla u|^4}{(A(t) - u)^{3k+3}} + |\nabla T_0|^2 + C. \]

Next, from Lemma 2.22 we have on \( U \times [0, \infty) \)

\[ (\frac{\partial}{\partial t} - \Delta) T_0^2 = 2T_0 \left( \frac{\partial}{\partial t} - \Delta \right) T_0 - 2|\nabla T_0|^2 \leq C - 2|\nabla T_0|^2. \]

Hence if we set

\[ Q = \frac{|\nabla u|^2}{(A(t) - u)^{1+k}} + T_0^2, \]

then we obtain from (4.9) and (4.10) that on \( U \times [0, \infty) \)

\[ (\frac{\partial}{\partial t} - \Delta) Q \leq -\frac{1}{10} \cdot \frac{|\nabla u|^4}{(A(t) - u)^{3k+3}} + C. \]
Now we choose cutoff function \( \rho \) according to Lemma 4.2. Then we can compute
\[
\left( \frac{\partial}{\partial t} - \Delta \right) (\rho^4 Q) = \rho^4 \left( \frac{\partial}{\partial t} - \Delta \right) Q - Q \Delta \rho^4 - 2 Re \left[ \nabla Q \cdot \nabla \rho^4 \right],
\]

For the second term, we use (4.12) to estimate on \( U \times [0, \infty) \)
\[
- Q \Delta \rho^4 \leq C \rho^2 Q = C \rho^2 \frac{|\nabla u|^2}{(A(t) - u)^{1+k}} + C \rho^2 T_0^2
\leq \frac{1}{100} \cdot \frac{\rho^4 |\nabla u|^4}{(A(t) - u)^{3+k}} + C(A(t) - u)^{1-k} + C \rho^2 T_0^2
\leq \frac{1}{100} \cdot \frac{\rho^4 |\nabla u|^4}{(A(t) - u)^{3+k}} + C,
\]

For the third term, since \( \rho > 0 \) on \( U \), we have on \( U \times [0, \infty) \)
\[
-2 Re \left[ \nabla Q \cdot \nabla \rho^4 \right] = -2 Re \left[ \frac{\nabla (\rho^4 Q) - 4 \rho^3 Q \nabla \rho}{\rho^4} \cdot 4 \rho^3 \nabla \rho \right]
= - \frac{8}{\rho} Re \left[ \nabla (\rho^4 Q) \cdot \nabla \rho \right] + 32 \rho^2 |\nabla \rho|^2 Q
\leq - \frac{8}{\rho} Re \left[ \nabla (\rho^4 Q) \cdot \nabla \rho \right] + \frac{1}{100} \cdot \frac{\rho^4 |\nabla u|^4}{(A(t) - u)^{3+k}} + C,
\]
hence combining (4.11) we obtain on \( U \times [0, \infty) \)
\[
(4.12) \left( \frac{\partial}{\partial t} - \Delta \right) (\rho^4 Q) \leq - \frac{1}{20} \cdot \frac{\rho^4 |\nabla u|^4}{(A(t) - u)^{3+k}} - \frac{8}{\rho} Re \left[ \nabla (\rho^4 Q) \cdot \nabla \rho \right] + C.
\]

Now, assume \( \rho^4 Q \) achieves its maximum at \((x_0, t_0)\) with \( t_0 > 0 \), then \( x_0 \notin \partial U \), hence \( x_0 \in U \) and then \( \rho(x_0) > 0 \) and we have
\[
- \frac{8}{\rho} Re \left[ \nabla (\rho^4 Q) \cdot \nabla \rho \right] (x_0, t_0) = 0,
\]
Then we apply maximum principle to (4.12) to obtain
\[
0 \leq \left( \frac{\partial}{\partial t} - \Delta \right) (\rho^4 Q) (x_0, t_0) \leq - \frac{1}{20} \cdot \frac{\rho^4 |\nabla u|^4}{(A(t) - u)^{3+k}} (x_0, t_0) + C,
\]
which gives at \((x_0, t_0)\)
\[
\frac{\rho^4 |\nabla u|^2}{(A(t) - u)^{1+k}} \leq \frac{\rho^4 |\nabla u|^4}{(A(t) - u)^{3+k}} + \rho^4 (A(t) - u)^{1-k} \leq C.
\]
But \( \rho^4 T_0^2 \leq C \) on \( U \times [0, \infty) \), we conclude that \( \rho^4 Q \leq C \) on \( U \times [0, \infty) \), which gives us that
\[
\frac{|\nabla u|^2}{(A(t) - u)^{\frac{3}{2}}} \leq C, \text{ on } K \times [0, \infty),
\]
where \( C \) is some constant depending on the domain \( K \). Using (4.6) again, we obtain (4.5) with some larger \( A(t) \).
Finally, when $S = \emptyset$, the above arguments are still true on $X \times [0, \infty)$ with all $h(t)$ and $A(t)$ replaced by $Ce^{-\eta t}$ with $\eta, C > 0$ are constants depending on $(X, \omega_0)$ which may change from line to line, since its easy to see that

$$\left| \frac{(Ce^{-\eta t})'}{Ce^{-\eta t}} \right| = \eta \leq 1.$$ (this is the motivation of Lemma 4.1). This completes the proof. \hfill \square

Now we come to estimate $|\Delta u|$ locally. We have the following proposition.

**Proposition 4.4.** There exists $A(t)$ depending on the domain $K$ such that

$$(4.13)$$

$$|\Delta u| \leq A(t), \text{ on } K \times [0, \infty).$$

In particular, when $S = \emptyset$, then we have

$$|\Delta u| \leq Ce^{-\eta t}, \text{ on } X \times [0, \infty).$$

for some constants $\eta, C > 0$ depending on $(X, \omega_0)$.

**Proof.** Applying Lemma 2.1, Proposition 3.2 and Proposition 4.3 we can find $h(t)$ such that

$$\left| T_0 - m \right| + \left| \|\omega_B\|^2_{\omega(t)} - m \right| + |u| + |\nabla u|^2 \leq \frac{1}{2} h(t), \text{ on } \overline{U} \times [0, \infty),$$

Then we apply Lemma 4.1 to find some $B(t)$ which then depends on the domain $K$ such that $h(t) \leq B(t)$ for $t \geq 0$ and $B(t)$ satisfies (4.1). WLOG, we may assume $B(t) \leq 1$ for $t \geq 0$, since otherwise we can consider $t \in [T, \infty)$ for some $T$ large. Now we set $A(t) = B(t)^{\frac{1}{2}} \geq B(t)$, then we have $A(t) \leq 1$ for $t \geq 0$ and moreover

$$(4.14)$$

$$\begin{cases}
|T_0 - m| + \left| \|\omega_B\|^2_{\omega(t)} - m \right| + |u| \leq \frac{1}{2} A(t), \\
|\nabla u|^2 \leq A(t)^2,
\end{cases}$$

on $\overline{U} \times [0, \infty)$. Still, we have

$$0 \leq -A'(t) = -\frac{1}{2} \frac{B'(t)}{B(t)^{\frac{1}{2}}} \leq \frac{1}{2} \frac{100B(t)}{B(t)^{\frac{1}{2}}} \leq 100B(t)^{\frac{1}{2}} = 100A(t),$$

which means that $A(t)$ still satisfies (4.1).

Now we use (4.4) to compute

$$(4.15)$$

$$\left( \frac{\partial}{\partial t} - \Delta \right) \left( \frac{-\Delta u}{A(t) - u} \right) = -\Delta u + |\nabla u|^2 + \langle \sqrt{-1}\partial\overline{\partial}u, \omega_B \rangle - \Delta T_0 \frac{(T_0 - m)\Delta u}{A(t) - u} - \frac{(T_0 - m)\Delta u}{(A(t) - u)^2}$$

$$- \frac{2}{A(t) - u} Re \left[ \nabla \left( \frac{-\Delta u}{A(t) - u} \right) \cdot \nabla u \right] + \frac{A'(t)\Delta u}{(A(t) - u)^2},$$
which is always meaningful on $U \times [0, \infty)$ thanks to (4.14), where we have used the fact that on $X\setminus S \times [0, \infty)$

$$\text{Ric}(\omega) = -\sqrt{-1} \partial \bar{\partial} (\dot{\varphi} + \varphi) - \chi$$

(4.16)

$$= -\sqrt{-1} \partial \bar{\partial} (\dot{\varphi} + \varphi - v) - (\chi + \sqrt{-1} \partial \bar{\partial} v)$$

$$= -\sqrt{-1} \partial \bar{\partial} u - \omega_B.$$

Also, we have

$$\left( \frac{\partial}{\partial t} - \Delta \right) \left( \frac{T_0 - m}{A(t) - u} \right) = \frac{\partial}{\partial t} T_0 - \Delta T_0 \frac{A'(t)(T_0 - m)}{(A(t) - u)^2}$$

$$+ \frac{(T_0 - m)^2}{(A(t) - u)^2} - \frac{2}{A(t) - u} \text{Re} \left[ \nabla \left( \frac{T_0 - m}{A(t) - u} \right) \cdot \nabla u \right].$$

Hence if we set

$$K = \frac{-\Delta u - (T_0 - m)}{A(t) - u},$$

then we have

$$\left( \frac{\partial}{\partial t} - \Delta \right) K = -\Delta u + \frac{\left| \nabla \nabla u \right|^2 + \left\{ \sqrt{-1} \partial \bar{\partial} u, \omega_B \right\}}{A(t) - u} - \frac{2}{A(t) - u} \text{Re} \left[ \nabla K \cdot \nabla u \right]$$

$$- \frac{(T_0 - m)\Delta u}{(A(t) - u)^2} + \frac{A'(t)\Delta u}{(A(t) - u)^2} - \frac{\partial g}{\partial t} T_0 \frac{A(t) - u}{(A(t) - u)^2} + \frac{A'(t)(T_0 - m)}{(A(t) - u)^2} - \frac{(T_0 - m)^2}{(A(t) - u)^2}.$$

But using (4.16) we have on $X\setminus S$

$$\frac{\partial}{\partial t} T_0 = \frac{\partial}{\partial t} \left( g(t) \tilde{\beta} (g_B) \tilde{\beta} \right) = \left( R g_{t} + g(t) g_{t} \right) (g_B)_{t}$$

$$= \left\{ \text{Ric}, \omega_B \right\} + T_0 = - \left\{ \sqrt{-1} \partial \bar{\partial} u, \omega_B \right\} + T_0 - \left\| \omega_B \right\|_{\omega(t)}^2,$$

hence we obtain

$$\left( \frac{\partial}{\partial t} - \Delta \right) K = -\Delta u + \frac{\left| \nabla \nabla u \right|^2 + 2 \left\{ \sqrt{-1} \partial \bar{\partial} u, \omega_B \right\} - T_0 + \left\| \omega_B \right\|_{\omega(t)}^2}{A(t) - u}$$

(4.17)

$$- \frac{2}{A(t) - u} \text{Re} \left[ \nabla K \cdot \nabla u \right] - \frac{(T_0 - m)\Delta u}{(A(t) - u)^2} + \frac{A'(t)\Delta u}{(A(t) - u)^2}$$

$$+ \frac{A'(t)(T_0 - m)}{(A(t) - u)^2} - \frac{(T_0 - m)^2}{(A(t) - u)^2}.$$
Next, set $\epsilon = k = 1$ in (4.7), we have (note that $A(t)$ is changed here)

$$
\left( \frac{\partial}{\partial t} - \Delta \right) \left( \frac{|\nabla u|^2}{(A(t) - u)^2} \right) = \frac{|\nabla u|^2}{(A(t) - u)^2} - (|\nabla \nabla u|^2 + |\nabla \nabla u|^2) + 2\text{Re}(\nabla T_0 \cdot \nabla u)
$$

(4.18)

$$
-2 \text{Re} \frac{\nabla |\nabla u|^2 \cdot \nabla u}{(A(t) - u)^3} - 2 \frac{|\nabla u|^4}{(A(t) - u)^4} + 2\frac{(T_0 - m)|\nabla u|^2}{(A(t) - u)^3} - 2A'(t)|\nabla u|^2
$$

Now set $H = \frac{|\nabla u|^2}{(A(t) - u)^2}$ and then set

$$
Q_\pm = \pm K + 100H + T_0^2,
$$

then combining (4.17) and (4.18) we obtain that on $U \times [0, \infty)$

(4.19)

$$
\left( \frac{\partial}{\partial t} - \Delta \right) Q_\pm = \pm \left\{ -\Delta u + |\nabla \nabla u|^2 + 2(\sqrt{-1} \partial \partial u, \omega_B) - T_0 + ||\omega_B||^2_{\omega_B} + a \right\} \frac{A(t) - u}{(A(t) - u)^2}
$$

$$
- \frac{(T_0 - m)\Delta u}{(A(t) - u)^2} + \frac{A'(t)\Delta u}{(A(t) - u)^2} + \frac{A'(t)(T_0 - m)}{(A(t) - u)^2} - \frac{(T_0 - m)^2}{(A(t) - u)^2}
$$

$$
+ 100 \left\{ \frac{|\nabla u|^2 - (|\nabla u|^2 + |\nabla \nabla u|^2) + 2\text{Re}(\nabla T_0 \cdot \nabla u)}{(A(t) - u)^2} \right\}
$$

$$
- \frac{2\text{Re} \nabla |\nabla u|^2 \cdot \nabla u}{(A(t) - u)^3} - \frac{2|\nabla u|^4}{(A(t) - u)^4} + 2\frac{(T_0 - m)|\nabla u|^2}{(A(t) - u)^3} - \frac{2A'(t)|\nabla u|^2}{(A(t) - u)^3}
$$

$$
- \frac{2}{A(t) - u} \text{Re}(\nabla Q_\pm \cdot \nabla u) + \frac{4T_0}{A(t) - u} \text{Re}(\nabla T_0 \cdot \nabla u)
$$

$$
+ 2T_0 \left( \frac{\partial}{\partial t} - \Delta \right) T_0 - 2|\nabla T_0|^2.
$$

With the help of (4.14), we only need two “good terms”

$$
-100 \frac{|\nabla \nabla u|^2}{(A(t) - u)^2}, \quad -2|\nabla T_0|^2,
$$

to control all other terms except the term which involves $\nabla Q_\pm$. Indeed, we have on $U \times [0, \infty)$

$$
\frac{\Delta u}{A(t) - u} \leq C \frac{|\nabla \nabla u|}{A(t) - u} \leq \frac{|\nabla \nabla u|^2}{(A(t) - u)^2} + C;
$$

$$
\frac{|\nabla \nabla u|^2}{(A(t) - u)^2} \leq \frac{2A(t)|\nabla \nabla u|^2}{(A(t) - u)^2} \leq 2 \frac{|\nabla \nabla u|^2}{(A(t) - u)^2};
$$

$$
$$

$$
$$
we have that on $20 W$ ANGJIAN JIAN
\[
\frac{2 \langle \sqrt{-1} \partial \bar{\partial} u, \omega_B \rangle}{A(t) - u} \leq \frac{2 \| \nabla \nabla u \| \| \omega_B \|}{A(t) - u} \leq \frac{\| \nabla \nabla u \|^2}{(A(t) - u)^2} + C;
\]
\[
\frac{|T_0 - \| \omega_B \|^2/\| \omega(t) \|}{A(t) - u} \leq \frac{|T_0 - m| + \| \omega_B \|^2/\| \omega(t) \|- m|}{A(t) - u} \leq \frac{\| A(t) + \frac{1}{2} A(t) \|}{\frac{1}{2} A(t)} = 2;
\]
\[
\left| \frac{(T_0 - m) \Delta u}{(A(t) - u)^2} \right| \leq \frac{A(t) \cdot C |\nabla \nabla u|}{(A(t) - u)^2} \leq \frac{\| \nabla \nabla u \|^2}{(A(t) - u)^2} + C;
\]
\[
\frac{|A'(t) \Delta u|}{(A(t) - u)^2} \leq \frac{100 A(t) \cdot C |\nabla \nabla u|}{(A(t) - u)^2} \leq \frac{\| \nabla \nabla u \|^2}{(A(t) - u)^2} + C;
\]
\[
\frac{|A'(t)(T_0 - m)|}{(A(t) - u)^2} - \frac{(T_0 - m)^2}{(A(t) - u)^2} \leq C;
\]
\[
100 \frac{|\nabla u|^2}{(A(t) - u)^2} \leq 100 \frac{A(t)^2}{\frac{1}{4} A(t)^2} \leq C;
\]
\[
\frac{2 \Re(\nabla T_0 \cdot \nabla u)}{A(t) - u} \leq \frac{2 |\nabla T_0| \cdot A(t)}{\frac{1}{4} A(t)} \leq |\nabla T_0|^2 + C;
\]
and moreover
\[
100 \left\{ -\frac{2 \Re [\nabla |\nabla u|^2 \cdot \nabla u]}{(A(t) - u)^3} + \frac{2 (T_0 - m)|\nabla u|^2}{(A(t) - u)^3} - \frac{2 A'(t)|\nabla u|^2}{(A(t) - u)^3} \right\}
\]
\[
\leq C \left\{ \frac{A(t)^2 (|\nabla \nabla u| + |\nabla \nabla u|)}{\frac{1}{4} A(t)^2 \cdot (A(t) - u)} + \frac{A(t)^3}{A(t)^3} \right\}
\]
\[
\leq \frac{\| \nabla \nabla u \|^2 + |\nabla \nabla u|^2}{(A(t) - u)^2} + C;
\]
\[
\frac{4 T_0}{A(t) - u} \Re(\nabla T_0 \cdot \nabla u) + 2 T_0 \left( \frac{\partial}{\partial t} - \Delta \right) T_0 \leq |\nabla T_0|^2 + C;
\]
Hence we conclude that on $U \times [0, \infty)$
\[
\left( \frac{\partial}{\partial t} - \Delta \right) Q_{\pm} \leq -5 \frac{|\nabla \nabla u|^2}{(A(t) - u)^2} - \frac{2}{A(t) - u} \Re [\nabla Q_{\pm} \cdot \nabla u] + C.
\]
Now as before, we choose cutoff function $\rho$ according to Lemma 4.2, then we have that on $U \times [0, \infty)$
\[
\left( \frac{\partial}{\partial t} - \Delta \right) (\rho^4 Q_{\pm})
\]
\[
\leq -5 \rho^4 \frac{|\nabla \nabla u|^2}{(A(t) - u)^2} - \frac{2 \rho^4 \Re [\nabla Q_{\pm} \cdot \nabla u]}{A(t) - u} + C - Q_{\pm} \Delta \rho^4 - 2 \Re [\nabla Q_{\pm} \cdot \nabla \rho^4],
\]
For the forth term, we have
\[ -Q_\pm \Delta \rho^4 \leq C \rho^2 |Q_\pm| \]
\[ \leq C \rho^2 \left\{ \frac{|
abla \nabla u|}{A(t) - u} + \frac{|T_0 - m|}{A(t) - u} + 100 \frac{|
abla u|^2}{(A(t) - u)^2} + T_0^2 \right\} \]
\[ \leq \frac{\rho^4 |
abla \nabla u|^2}{(A(t) - u)^2} + C; \]
for the second term, we have
\[ -2 \rho^4 \Re \left[ \nabla Q_\pm \cdot \nabla u \right] A(t) - u \]
\[ = -2 A(t) - u \frac{\rho^4 \Re \left[ \nabla (\rho^4 Q_\pm) \cdot \nabla u \right]}{A(t) - u} + \frac{8 \rho^3 Q_\pm}{A(t) - u} \Re \left[ \nabla \rho \cdot \nabla u \right] \]
\[ \leq -2 A(t) - u \frac{\rho^4 \Re \left[ \nabla (\rho^4 Q_\pm) \cdot \nabla u \right]}{A(t)} \]
\[ \leq -2 A(t) - u \frac{\rho^4 \Re \left[ \nabla (\rho^4 Q_\pm) \cdot \nabla u \right]}{(A(t) - u)^2} + C; \]
and similarly for the last term
\[ -2 \Re \left[ \nabla Q_\pm \cdot \nabla \rho^4 \right] \leq -\frac{8}{\rho} \Re \left[ \nabla (\rho^4 Q_\pm) \cdot \nabla \rho \right] + \frac{\rho^4 |
abla \nabla u|^2}{(A(t) - u)^2} + C. \]

Hence we finally conclude on \( U \times [0, \infty) \)

\[ (4.20) \]
\[ \left( \frac{\partial}{\partial t} - \Delta \right) (\rho^4 Q_\pm) \leq -\frac{\rho^4 |
abla \nabla u|^2}{(A(t) - u)^2} - \frac{2 \Re \left[ \nabla (\rho^4 Q_\pm) \cdot \nabla u \right]}{A(t) - u} - \frac{8}{\rho} \Re \left[ \nabla (\rho^4 Q_\pm) \cdot \nabla \rho \right] + C. \]

Now we assume \( \rho^4 Q_+ \) achieves its maximum at \((x_0, t_0)\) with \( t_0 > 0 \), then if \( x_0 \in \partial U \), we are done. Hence we can assume that \( x_0 \in U \) and then \( \rho(x_0) > 0 \) and hence
\[ -\frac{2 \Re \left[ \nabla (\rho^4 Q_+) \cdot \nabla u \right]}{A(t) - u} (x_0, t_0) - \frac{8}{\rho} \Re \left[ \nabla (\rho^4 Q_+) \cdot \nabla \rho \right] (x_0, t_0) = 0, \]
then we obtain from maximum principle and \((4.20)\) that
\[ 0 \leq \left( \frac{\partial}{\partial t} - \Delta \right) (\rho^4 Q_+)(x_0, t_0) \leq -\frac{\rho^4 |
abla \nabla u|^2}{(A(t) - u)^2}(x_0, t_0) + C, \]
and hence on $U \times [0, \infty)$

$$\rho^4 Q_+ \leq \rho^4 Q_+(x_0, t_0)$$

$$= \rho^4 \left\{ - \Delta u + \frac{(T_0 - m)}{A(t) - u} + 100 \frac{\|\nabla u\|^2}{(A(t) - u)^2} + T_0^2 \right\} (x_0, t_0)$$

$$\leq \frac{\rho^4 \|\nabla u\|}{A(t) - u} (x_0, t_0) + C \leq C,$$

which gives

$$\frac{-\Delta u}{A(t) - u} \leq C, \text{ on } K \times [0, \infty).$$

Similarly, consider $Q_-$ instead gives us

$$\frac{\Delta u}{A(t) - u} \leq C, \text{ on } K \times [0, \infty).$$

and hence we conclude

$$|\Delta u| \leq A(t), \text{ on } K \times [0, \infty),$$

for some larger $A(t)$. Hence we obtain (4.13).

Finally, when $S = \emptyset$, the above arguments are still true on $X \times [0, \infty)$ with all $h(t)$ and $A(t)$ replaced by $Ce^{-\eta t}$ with $\eta, C > 0$ are constants depending on $(X, \omega_0)$ which may change from line to line. This completes the proof. \qed

Now we can prove Theorem 1.1.

**Proof of Theorem 1.1.** From (4.16), we have that on $X \setminus S \times [0, \infty)$

$$R = -T_0 - \Delta u,$$

hence Proposition 3.2 and Proposition 4.4 give us some $h(t)$ depending on the domain $K$ such that

$$|R + m| \leq |T_0 - m| + |\Delta u| \leq h(t), \text{ on } K \times [0, \infty).$$

In particular, if $S = \emptyset$, then

$$|R + m| \leq |T_0 - m| + |\Delta u| \leq Ce^{-\eta t}, \text{ on } X \times [0, \infty),$$

where $\eta, C > 0$ are constants depending on $(X, \omega_0)$. This completes the proof. \qed

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