Admissibility of a posterior predictive decision rule

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Abstract

Recent decades have seen an interest in prediction problems for which Bayesian methodology has been
used ubiquitously. Sampling from or approximating the posterior predictive distribution in a Bayesian
model allows one to make inferential statements about potentially observable random quantities given
observed data. The purpose of this note is to use statistical decision theory as a basis to justify the use
of a posterior predictive distribution for making a point prediction.

1 Introduction and Motivation

As reviewed by [Owhadi and Scovel] the field of statistical decision theory introduced by Wald, building on
a game theoretic foundation developed by von Neumann and Morgenstern, provides links between Bayesian
and frequentist statistical philosophies through the concepts of decision rules, admissibility, and risk functions
amongst others. Moreover, a recent thrust of research motivated by machine learning has put much emphasis
on prediction problems for which Bayesian methodology has been widely used. The purpose of this note
is to demonstrate that classic decision theoretic results can be simply applied to the analysis of prediction
problems. In fact, both [Berger] and [Robert] remark upon the ease of applying statistical decision theory
within the context of prediction however no explicit result is stated in either work; the contribution of this
work, therefore, is to highlight a simple but explicit way in which the results of statistical decision theory
might apply to prediction problems. To the author’s knowledge the most similar lines of thought appear
in work by [Nayak and El-Baz] where the loss function explicitly depends on the underlying parameter in
contrast to what follows; additionally, in [Nayak and El-Baz] a proof of admissibility of a Bayesian prediction
rule is not given albeit alluded to.

In statistical decision theory it is desired that an estimator is admissible, meaning that there exists
no other estimator whose (frequentist) risk is at least as small for all values of the parameter space and
strictly smaller for at least a single value of the parameter space. In this note the notion of the risk of an
estimator is extended to the risk of a prediction rule using statistical decision theory, and furthermore, this
framework is used to evaluate a prediction rule derived by minimizing the Bayes prediction risk analogously
to a decision rule derived by minimizing the Bayes risk. It is shown that such a rule is admissible and can
be derived by minimizing the (Bayes) posterior predictive risk, just as a Bayes rule is admissible and can
be derived by minimizing the posterior risk. These results may motivate the use of a (Bayesian) posterior
predictive distribution to make a prediction, since admissibility is a desirable property and minimizing the
Bayes posterior predictive risk is computationally tractable in many cases, just as minimizing the Bayes risk
is computationally straightforward under commonly used loss functions. For instance the posterior predictive
mean is admissible and minimizes the Bayesian prediction risk under a squared error loss just as the posterior
mean is admissible and minimizes the Bayesian risk under a squared error loss, under weak conditions.

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2 Definitions and Assumptions

Denote the parameter space \( \Theta \subseteq \mathbb{R}^k \). Without essential loss of generality it is assumed all random variables are continuous and have Lebesgue-integrable density functions; the discrete case can be considered using the counting measure. Additionally it is assumed that conditions hold to apply the Fubini theorem (e.g., Theorem 2.8 of [Lehmann and Casella]) and the prior distribution is proper, meaning it integrates to unity and is a valid probability density. Note that the inferential versions of the definitions given below are covered by many sources, including [Berger], [Robert], and [Wasserman].

**Definition(s) 1:** Denote the random variable \( Y_{\text{obs}} \in \mathbb{R}^M \) for data observed, the random variable \( Y_{\text{pred}} \in \mathbb{R}^N \) for data one would like to predict, and \( \theta \) for an unknown that indexes a data generating process, \( f(y_{\text{pred}}, y_{\text{obs}}|\theta) \) a probability density function or probability mass function \( f : \mathbb{R}^N \times \mathbb{R}^M \to [0, \infty) \). In the Bayesian context, one assumes that \( \theta \in \mathbb{R}^k \) is a random variable and has a prior probability distribution \( g(\theta) \), \( g : \mathbb{R}^k \to [0, \infty) \) ascribed to it, such that the joint distribution of \( (Y_{\text{pred}}, Y_{\text{obs}}, \theta) \) is given by \( f(y_{\text{pred}}, y_{\text{obs}}|\theta)g(\theta) \). It is further assumed that \( g(\theta) > 0 \) \( \forall \theta \in \Theta \). The posterior predictive distribution is the conditional distribution of \( Y_{\text{pred}} \) given \( y_{\text{obs}}, p(y_{\text{pred}}|y_{\text{obs}}) \), \( p : \mathbb{R}^N \times \mathbb{R}^M \to [0, \infty) \).

**Definition 2:** A prediction rule \( \hat{Y}(.) \) is a function of \( Y_{\text{obs}} \) meant to make a guess at \( Y_{\text{pred}} \), \( \hat{Y} : \mathbb{R}^M \to \mathbb{R}^N \). This is analogous to a decision rule \( \hat{\theta} \) that is meant to be a guess of an unknown \( \theta \).

**Definition 3:** A loss function \( L : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty) \), \( L(\hat{Y}(Y_{\text{obs}}), Y_{\text{pred}}) \), penalizes a discrepancy between \( \hat{Y}(Y_{\text{obs}}) \) and \( Y_{\text{pred}} \). The frequentist notion of prediction risk of the prediction rule can be considered the average of \( L(\hat{Y}, Y_{\text{pred}}) \) holding \( \theta \) fixed, just as frequentist risk of an estimator is the average of \( L(\hat{\theta}, \theta) \) holding \( \theta \) fixed. The Bayesian prediction risk is the average of \( L(\hat{Y}, Y_{\text{pred}}) \) over the joint distribution of \( (Y_{\text{pred}}, Y_{\text{obs}}, \theta) \), just as the Bayes risk is the average of \( L(\hat{\theta}, \theta) \) over the joint distribution of \( (Y_{\text{obs}}, \theta) \). Throughout it is assumed frequentist and Bayesian prediction risk are continuous and well defined \( \forall \theta \in \Theta, \hat{Y} \) and \( Y_{\text{pred}} \).

**Definition 4:** A Bayes prediction rule is one which minimizes the Bayesian prediction risk.

**Definition 5:** A prediction rule \( \hat{Y} \) is admissible if there is no other prediction rule that achieves a frequentist prediction risk at least as small as that of \( \hat{Y} \) \( \forall \theta \in \Theta \) and strictly smaller than that of \( \hat{Y} \) for at least a single value of \( \theta \in \Theta \).

3 Theorems

**Theorem 1:** Under the assumptions of section 2, a Bayes prediction rule can be found by minimizing the average of \( L(\hat{Y}(Y_{\text{obs}}), Y_{\text{pred}}) \) over \( p(y_{\text{pred}}|y_{\text{obs}}) \), referred to as the posterior predictive risk.

**Proof:** The Bayes prediction risk is the average of \( L(\hat{Y}(Y_{\text{obs}}), Y_{\text{pred}}) \) over the joint distribution of \( (Y_{\text{obs}}, Y_{\text{pred}}) \) since \( L(\hat{Y}(Y_{\text{obs}}), Y_{\text{pred}}) \) is not a function of \( \theta \) and so \( \theta \) can be integrated (or summed) out of the joint distribution of \( (Y_{\text{obs}}, Y_{\text{pred}}, \theta) \). Furthermore, the distribution of \( (Y_{\text{obs}}, Y_{\text{pred}}) \) can be written as \( p(y_{\text{pred}}|y_{\text{obs}})p(y_{\text{obs}}) \) (overloading the \( p(.) \) notation for less clutter), so to minimize the Bayes prediction risk it suffices to minimize the average of \( L(\hat{Y}(Y_{\text{obs}}), Y_{\text{pred}}) \) over \( p(y_{\text{pred}}|y_{\text{obs}}) \). In other words, the standard argument for deriving a Bayes rule can be applied by integrating out \( \theta \) as a nuisance parameter.

To make this argument more explicit, mathematical notation is introduced, overloading the \( p(.) \) notation to indicate probability densities. A Bayesian prediction rule is derived by minimizing the Bayesian prediction
risk:
\[
\int \int \int L(\hat{y}(y_{obs}), y_{pred})p(y_{pred}, y_{obs}, \theta) d\theta dy_{pred} dy_{obs} = \int \int \int L(\hat{y}(y_{obs}), y_{pred}) p(y_{pred}, y_{obs}, \theta) d\theta dy_{pred} dy_{obs} = \int \int \int L(\hat{y}(y_{obs}), y_{pred}) p(y_{pred}, y_{obs}) dy_{pred} dy_{obs} = \int \int \int L(\hat{y}(y_{obs}), y_{pred}) p(y_{pred}|y_{obs}) p(y_{obs}) dy_{pred} dy_{obs}
\]

Which can be minimized by minimizing \( \int L(\hat{y}(y_{obs}), y_{pred}) p(y_{pred}|y_{obs}) dy_{pred} \) for arbitrary \( y_{obs} \), i.e., the posterior predictive risk.

**Theorem 2:** Under the assumptions of section 2, Bayesian prediction rules are admissible.

**Proof:** Assume there is a Bayesian prediction rule \( \hat{Y}_{Bayes} \) that is not admissible so that there exists another prediction rule \( \hat{Y}_{new} \) which achieves a strictly smaller frequentist prediction risk for some value \( \theta^* \in \Theta \) and at least as small frequentist prediction risk for all other values in \( \Theta \). Since frequentist risk is assumed to be continuous, there is some measurable subset of \( S \) of \( \mathbb{R}^k \) containing \( \theta^* \) such that the frequentist risk of \( \hat{Y}_{new} \) is strictly smaller than that of \( \hat{Y}_{Bayes} \) on this subset. In other words \( \forall \theta \in S \):
\[
\int \int L(\hat{y}_{new}, y_{pred}) f(y_{pred}, y_{obs}|\theta) dy_{pred} dy_{obs} < \int \int L(\hat{y}_{Bayes}, y_{pred}) f(y_{pred}, y_{obs}|\theta) dy_{pred} dy_{obs}
\]

and \( \forall \theta \in S^c \):
\[
\int \int L(\hat{y}_{new}, y_{pred}) f(y_{pred}, y_{obs}|\theta) dy_{pred} dy_{obs} \leq \int \int L(\hat{y}_{Bayes}, y_{pred}) f(y_{pred}, y_{obs}|\theta) dy_{pred} dy_{obs}
\]

Since the Bayesian prediction risk is the average of the frequentist risk over \( \theta \) and the prior \( g(\theta) \) is strictly non-zero for all points in \( \Theta \) (including those points for which the frequentist risk of \( \hat{Y}_{new} \) is strictly smaller than that of \( \hat{Y}_{Bayes} \)), this yields a prediction rule with strictly smaller Bayesian prediction risk which is a contradiction. In other words, the previous inequalities imply:
\[
\int_S \int L(\hat{y}_{new}, y_{pred}) f(y_{pred}, y_{obs}|\theta) g(\theta) dy_{pred} dy_{obs} d\theta < \int_S \int L(\hat{y}_{Bayes}, y_{pred}) f(y_{pred}, y_{obs}|\theta) g(\theta) dy_{pred} dy_{obs} d\theta
\]

and:
\[
\int_{S^c} \int L(\hat{y}_{new}, y_{pred}) f(y_{pred}, y_{obs}|\theta) g(\theta) dy_{pred} dy_{obs} d\theta \leq \int_{S^c} \int L(\hat{y}_{Bayes}, y_{pred}) f(y_{pred}, y_{obs}|\theta) g(\theta) dy_{pred} dy_{obs} d\theta
\]

so:
\[
\int \int L(\hat{y}_{new}, y_{pred}) f(y_{pred}, y_{obs}|\theta) g(\theta) dy_{pred} dy_{obs} d\theta < \int \int L(\hat{y}_{Bayes}, y_{pred}) f(y_{pred}, y_{obs}|\theta) g(\theta) dy_{pred} dy_{obs} d\theta
\]

which contradicts that \( \hat{Y}_{Bayes} \) is a Bayes rule.
4 Discussion

The contribution of this note has been to show that two results of statistical decision theory can be extended to a prediction setting with natural modifications to standard textbook definitions. Namely, the loss function allows both arguments to be random as opposed to one fixed and one random, and the underlying “true” state of nature $\theta$ is treated as a nuisance parameter when deriving a Bayes rule.

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