Embeddings of 3-Manifolds in $S^4$ from the Point of View of the 11-Tetrahedron Census

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**ABSTRACT**

This is a collection of notes on embedding problems for 3-manifolds. The main question explored is “which 3-manifolds embed smoothly in $S^4$?”. The terrain of exploration is the Burton/Martelli/Matveev/Petronio census of triangulated prime closed 3-manifolds built from 11 or less tetrahedra. There are 13766 manifolds in the census, of which 13400 are orientable. Of the 13400 orientable manifolds, only 149 of them have hyperbolic torsion linking forms and are thus candidates for embedability in $S^4$. The majority of this paper is devoted to the embedding problem for these 149 manifolds. At present 41 are known to embed in $S^4$. Among the remaining manifolds, embeddings into homotopy 4-spheres are constructed for 4. 67 manifolds in the list are known to not embed in $S^4$. This leaves 37 unresolved cases, of which only 3 are geometric manifolds i.e. having a trivial JSJ-decomposition.

**1. Introduction**

Given a smooth manifold $M$, let $ed(M)$ denote the minimum of all integers $n$ such that $M$ admits a smooth embedding into $S^n$. The purpose of these notes is to get a sense for how difficult it is to determine if $ed(M) = 4$, when $M$ is compact, boundaryless 3-dimensional manifold.

Whitney proved that $ed(M) \leq 2n$ for all $n$-manifolds $M$ by a combination of a general position/transversality argument and a double point creation and destruction process now called The Whitney Trick. A basic argument using characteristic classes shows that $ed(RP^{2k}) = 2 \cdot 2^k$ for all $k$, and so Whitney’s result is generally the best one can expect for arbitrary $n$ (see for example Theorem 4.5 and Corollary 11.4 in [40]). For 3-manifolds, C.T.C. Wall improved on Whitney’s result, showing every compact 3-manifold embeds in $S^5$ [53]. Thus, for closed 3-manifolds distinct from $S^3$, the embedding dimension can be one of two possible numbers $ed(M) \in \{4, 5\}$.

Recently Skopenkov has given a complete isotopy classification of embeddings of 3-manifolds into $S^6$ [48]. At the other extreme, the question of which 3-manifolds (with boundary) embed in $S^3$ is quite a difficult problem [1, 13, 38] although there is much known – for example, consider the case of $M$ compact, orientable with boundary a collection of tori. If $M$ embeds in $S^3$, then there is another embedding of $M$ in $S^3$ so that it is the complement of a link [45]. By a Wirtinger presentation, $\pi_1 M$ is generated by the conjugates of $n$ curves on $\partial M$ corresponding to the meridians of the $n$-component link. By the resolution of the Poincaré conjecture, the converse is true – simply fill $M$ along the curves in $\partial M$ to get a homotopy 3-sphere. Although this is an “answer” it is rather difficult to implement in a computationally-effective way [38].

The hope of this paper is that the “intermediate” question of whether or not a 3-manifold embeds in $S^4$ is tractable. This is also problem 3.20 on Kirby’s problem list. The point of view of this paper is that there is no better way to discover than to get one’s hands dirty. The census of prime 3-manifolds which can be triangulated by 11 or less tetrahedra [6] is chosen as a “generic supply” of test cases. Of course, there is good reason to think this problem could be very difficult. There are several significant, closely-related outstanding problems such as the Schönflies problem, and the smooth Poincaré Conjecture in dimension 4 which indicate possible pitfalls. Sometimes in this paper embeddings of 3-manifolds are constructed into homotopy 4-spheres. Likely all the homotopy 4-sphere constructed are the standard $S^4$ but we have not always determined this. There are perhaps simpler obstacles to overcome – at present in the literature there are no known examples of 3-manifolds that embed smoothly in a homology 4-sphere but do not embed in $S^4$. It is rather remarkable that all the obstructions used in this paper are obstructions to embedding into homology 4-spheres, and at least so far they have largely sufficed to determine which 3-manifolds embed in $S^4$.

In Section 2 a brief survey is given of known obstructions to a 3-manifold embedding in $S^4$. Many useful techniques to construct embeddings in $S^4$ are also listed.

We apply the results from Section 2 to the census of 3-manifolds in Sections 3, 4 and 5. To keep the paper a reasonable length, only the manifolds which pass the torsion linking form test (Theorem 2.2) are listed in these sections. Section 3 describes embeddings in $S^4$ for the manifolds in the census which are known to embed in $S^4$.

Section 4 describes embeddings into homotopy 4-spheres of the manifolds in the census that are known to embed in...
homotopy 4-spheres – these homotopy 4-spheres are likely to be diffeomorphic to $S^4$ but this has not been determined.

Section 5 provides obstructions for the manifolds in the census which are known not to embed in $S^4$ (or any homology 4-sphere). Manifolds which fail the torsion linking form test (Theorem 2.2) are not listed as these are too numerous. Manifolds that fail the torsion linking form test are available via the software Regina, see Section 7.

Section 6 lists the manifolds for which it is not yet known if they embed in $S^4$ or homology 4-spheres. Moreover, a list of computed obstructions is provided.

Section 7 provides sketches of some techniques used to compute various invariants of the manifolds from the census. If the reader ever gets lost in the notation used in the tables, usually this section or Section 2 is the appropriate place to look for clarification.

Section 8 contains various observations and comments on the data.

Many of the obstructions and constructions present in this paper were described to the first author by Danny Ruberman. Thanks to Brendan Owens and Saso Strle who kindly let us use their software to compute the $d$-invariant of Seifert fibered rational homology spheres. Thanks also to Jonathan Hillman, Ahmed Issa, Gregor Masbaum, Peter Landweber, Lee Rudolph, Ronald Fintushel, Ronald Stern, Ian Agol, Scott Carter, Nathan Dunfield, Jeff Weeks, Peter Teichner, Tom Goodwillie and Mike Freedman for their suggestions and/or encouragement (whether they remember it or not). A paper such as this requires immense amounts of time for thousands of hand and computer-aided computations. The first author would especially like to thank the Max Planck Institute for Mathematics (Bonn) and the Institut des Hautes Études Scientifiques for giving him the freedom to initiate this open-ended project. Thanks also to IPMU (Tokyo) and KITP (Santa Barbara) for hosting the first author.

2. Obstructions and embedding constructions

There are only a few completely general obstructions to a closed 3-manifold embedding in $S^4$. The first is of course orientability, coming from the generalized Jordan Curve Theorem. There are no other tangent-bundle derived obstructions since the tangent bundle of an orientable 3-manifold is trivial (Stiefel’s Theorem) [30]. A powerful and easy-to-compute obstruction comes from the torsion linking form of a 3-manifold.

Definition 2.1. (Torsion Linking Form) In a compact, boundaryless oriented $n$-manifold $M$ there is a canonical, natural isomorphism (Poincaré duality)

$$H_i(M, Z) \cong H^{n-i}(M, Z) \quad \forall i \in \{0, 1, \ldots, n\}.$$ 

This is a natural short exact sequence (the homology-cohomology Universal Coefficient Theorem)

$$0 \to \text{Ext}_Z(H_{i-1}(M, Z), Z) \to H^i(M, Z) \to \text{Hom}(H_i(M, Z), Z) \to 0 \quad \forall i \in \{0, 1, \ldots, n\}$$

and a canonical isomorphism

$$\text{Ext}_Z(H_i(M, Z), Z) \simeq \text{Hom}_Z(\tau H_i(M, Z), Q/Z) \quad \forall i \in \{0, 1, \ldots, n\}$$

where $\tau H_i(M, Z)$ is the subgroup of torsion elements of $H_i(M, Z)$. This gives two duality pairings on the homology of $M$, the “intersection product” and the “torsion linking form” respectively:

$$f_{H_i}(M, Z) \otimes f_{H_{n-i}}(M, Z) \to \tau H_i(M, Z) \otimes \tau H_{n-i}(M, Z) \to Q/Z$$

where $f_{H_i}(M, Z) = H_i(M, Z)/\tau H_i(M, Z)$ is the “free part” of $H_i(M, Z)$.

See the discussion preceding Figure 1 in Section 7 for details on how one computes the torsion linking form for a triangulated manifold, in practice. In short, given $[x] \in \tau H_i(M)$ and $[y] \in \tau H_{n-i}(M)$, let $k y = \partial Y$, i.e. assume $[y]$ is $k$-torsion, then the linking form is defined as $\langle [x], [y] \rangle = \frac{x \cdot y}{k}$ where $x \cdot y$ indicates the transverse signed intersection number.

Theorem 2.2. [22, 28] (Hantzsche Test) If $M$ is a compact, boundaryless, connected, oriented 3-manifold which embeds in a homology $S^4$ then there is a splitting

$$\tau H_1(M, Z) = A \oplus B,$$

inducing a splitting

$$\text{Hom}_Z(\tau H_1(M, Z), Q/Z) \simeq \text{Hom}_Z(A, Q/Z) \times \text{Hom}_Z(B, Q/Z)$$

which is reversed by Poincaré duality, in the sense that the P.D. isomorphism

$$\tau H_1(M, Z) \to \text{Hom}_Z(\tau H_1(M, Z), Q/Z)$$

restricts to isomorphisms

$$A \to \text{Hom}_Z(B, Q/Z) \quad \text{and} \quad B \to \text{Hom}_Z(A, Q/Z).$$

This uses the convention that

$$\text{Hom}_Z(A, Q/Z) \simeq \text{Hom}_Z(\tau H_1(M, Z), Q/Z)$$

which is zero on $B$, similarly $\text{Hom}_Z(B, Q/Z)$ is the submodule which is zero on $A$.

Proof. $M$ separates the homology 4-sphere into two manifolds, call them $V_1$ and $V_2$, $S^4 = V_1 \cup_M V_2$. Let $A = \tau H_1(V_1, Z)$ and $B = \tau H_1(V_2, Z)$, when we have the isomorphism $A \oplus B \cong H_1(M, Z)$ by the Mayer-Vietoris sequence for $S^4 = V_1 \cup_M V_2$. Let $\{i, j\} = \{1, 2\}$. If a homology class in $\tau H_1(M)$ comes from $\tau H_1(V_i)$ then it must bound a 2-cycle in $V_i$. Thus the torsion linking form is zero on $\tau H_1(V_i) \oplus H_1(V_i)$, giving the result. □

An immediate corollary of Theorem 2.2 is that the only lens space that admits a smooth embedding into $S^4$ is $S^3$. Our convention is that a lens space is a manifold quotient of $S^3$ by a group of isometries, so we exclude $S^1 \times S^2$.

Kawauchi and Kojima call torsion linking forms which have such a splitting “hyperbolic” [38]. Kawauchi and Kojima’s test for hyperbolicity of the torsion linking form has been implemented by the author in the freely-available open-source software package “Regina” [6].

![Figure 1. Dual polyhedral bits inside a tetrahedron $A_1$.](image-url)
As stated in the abstract, there are only 149 manifolds in the census with hyperbolic torsion linking forms, and they are listed in Sections 3, 4, 5 and 6. Since the hyperbolicity computation plays a significant role in this paper, a sketch of the algorithm is given in Section 7.

In general, if a 3-manifold $M$ embeds in a homology 4-sphere $\Sigma^4$, $V_1 \cup_M V_2 = \Sigma^4$. The argument in the proof of Theorem 2.2 gives us (for $\{i,j\} = \{1,2\}$):

$$
\begin{align*}
H_i V_1 &\simeq h^i H_i V_1 \oplus \operatorname{Hom}_Z(\tau H_i V_1, \mathbb{Q}/\mathbb{Z}) \\
H_2 V_1 &\simeq \operatorname{Hom}_Z(h^1 H_i V_1, \mathbb{Z}) \\
H_i V_1 &\simeq *.
\end{align*}
$$

If $M$ is a rational homology sphere, the manifolds $V_1$ and $V_2$ are rational homology balls. If $M$ is a rational homology $S^1 \times S^2$, one of $V_1$, $V_2$ is a rational homology $S^1 \times D^3$, and the other a rational homology $D^2 \times S^2$. If $H_1 M \simeq \mathbb{Z}^2 \oplus \tau H_2 M$ then there are two possibilities: in the first case, one would be a rational genus two 1-handlebody $S^1 \times D^3 \#_p S^1 \times D^3$ and the other a rational genus two 2-handlebody $(D^2 \times S^2) \#_q (D^2 \times S^2)$, in the second case both manifolds would be rational $(S^1 \times D^3) \#_p (D^2 \times S^2)$.

By and large, these complications do not come up much in the census as the majority (13173 of 13766) are rational homology spheres. There are only 201 rational homology $S^1 \times S^2$ manifolds in the census. There are 25 manifolds in the census that have $h^1 V_1 \simeq \mathbb{Z}^2$, and there is only one manifold in the census with $h^1 V_1 \simeq \mathbb{Z}^3$, the manifold $S^1 \times S^1 \times S^1$. There are no manifolds in the census with rank $H_1 M > 3$. Thus, intersection forms such as $H_2 M \otimes H_2 M \to H_1 M$ gives no useful obstruction to census 3-manifolds embedding in $S^4$.

Kawauchi developed an obstruction to a rational homology $S^1 \times S^2$ bounding a rational homology $S^1 \times D^3$, which will be described below.

**Definition 2.3.** (Alexander Polynomial) If $h : H_1(M, \mathbb{Z}) \to \mathbb{Z}$ is an epimorphism, let $M_h \to M$ denote the normal abelian covering space corresponding to $h$, and let $h$ play a double-role as the corresponding generator of the group of covering transformations. Consider $H_1(M_h, \mathbb{Q})$ to be a module over $\Lambda \equiv \mathbb{Q}[\mathbb{Z}]$ (the group ring of the integers $\mathbb{Z}$ with coefficients in the rationals $\mathbb{Q}$), where the action of $\mathbb{Z}$ on $M_h$ is generated by the covering transformation $h$. Notice

$$
\Delta(h) = \prod_{p \in \mathbb{P}} p \in \mathbb{Q}[h^{\pm 1}].
$$

Since it is representing an ideal, it is only well-defined up to multiplication by a unit. We will use the notation $Q(\Lambda)$ for the field of fractions of $\Lambda$.

We use the symbol $\equiv$ to denote either a definition or a canonical identification, while $\simeq$ denotes abstract isomorphism.

When $M$ is compact, orientable and boundaryless, Poincaré duality (of the Blanchfield variety – see for example [23]) and basic linear algebra provides isomorphisms

$$
\tau_A H_1(M_h, Q) \simeq \tau_A H^2(M_h, Q) \simeq \operatorname{Ext}_A(H_1(M_h, Q), \Lambda) \\
\simeq \operatorname{Hom}_A(\tau_A H_1(M_h, Q), Q(\Lambda)/\Lambda)
$$

where cohomology is “cohomology with compact support.”

The inclusion $\Lambda \subset Q(\Lambda)$ is the submodule consisting of elements whose denominator is 1. Given a $\Lambda$-module $A$, $A$ indicates the conjugate $\Lambda$-module – as a $\mathbb{Q}$-vector space it is identical to $A$, but the action of $\mathbb{Z}$ on $A$ is the inverse action.

This statement is the $\Lambda$ analog of the isomorphisms in Definition 2.1. Since $\Lambda$ is a PID, $\tau_A H_1(M_h, Q)$ has a diagonal presentation matrix, thus there is a (not natural) isomorphism between $\tau_A H_1(M_h, Q)$ and $\operatorname{Hom}_A(\tau_A H_1(M_h, Q), Q(\Lambda)/\Lambda)$. Thus, the Alexander polynomial is symmetric $\Delta(h) = \Delta(h^{-1})$.

Notice if $h : H_1(M, \mathbb{Z}) \to \mathbb{Z}$ is an epimorphism, and if $M$ embeds in a homology $S^4$, then one can write $S^4$ as a union $V_1 \cup_M V_2$ and so the homomorphism $h$ factors as a composite

$$
\begin{array}{ccc}
H_1(M, \mathbb{Z}) & \xrightarrow{h} & \mathbb{Z} \\
V_1 & \downarrow{PD} & \\
H_1(V_1, \mathbb{Z}) & \xrightarrow{i} & H_1(V_h, \mathbb{Q}) & \xrightarrow{j} & H_1(M_h, \mathbb{Q}) & \xrightarrow{\mu} & H_2(V_h, \mathbb{Q}) & \xrightarrow{j^*} & H^2(V_h, Q) & \xrightarrow{i^*} & H^3(V_h, M_h, \mathbb{Q}) & \xrightarrow{h^*} & H^3(V_h, M_h, \mathbb{Q}) \xrightarrow{h^*} & H^3(V_h, M_h, \mathbb{Q})
\end{array}
$$

for some $i \in \{1,2\}$.

**Theorem 2.4.** [27] (Kawauchi Test) If $M$ is a rational homology $S^1 \times S^2$ with $h : H_1(M, \mathbb{Z}) \to \mathbb{Z}$ onto, and if $M$ admits an embedding into a homology $S^4$ then $\Delta(h) = f(h)f(h^{-1})$ for some Laurent polynomial $f(h) \in Q[h^{\pm 1}]$.

**Proof.** Let $V$ be the rational homology $S^1 \times D^3$ bounding $M$, as above. Consider the Poincaré Duality long exact sequence of the pair $(V_h, M_h)$:

$$
\begin{array}{ccccccc}
\cdots & \xrightarrow{j^*} & H^2(V_h, M_h, \mathbb{Q}) & \xrightarrow{\partial} & H_1(M_h, \mathbb{Q}) & \xrightarrow{i} & H_1(V_h, \mathbb{Q}) & \xrightarrow{j} & \cdots \\
\cdots & \xrightarrow{j^*} & H^2(V_h, \mathbb{Q}) & \xrightarrow{\partial} & H_1(V_h, \mathbb{Q}) & \xrightarrow{j} & \cdots
\end{array}
$$

that $Q[\mathbb{Z}]$ is isomorphic to a Laurent polynomial ring $Q[h^{\pm 1}]$, which is a principal ideal domain. By the classification of finitely-generated modules over PIDs, $H_1(M_h, \mathbb{Q}) \simeq \Lambda^k \oplus \oplus_{p \in \mathbb{P}} \Lambda/[p]$ for various non-zero polynomials $P$. The order ideal of the $\Lambda$-torsion submodule of $H_1(M_h, \mathbb{Q})$ is called the Alexander polynomial of $h$, and will be denoted

$$
\Delta(h) = \prod_{p \in \mathbb{P}} p \in \mathbb{Q}[h^{\pm 1}].
$$

The next step is to show all six $\Lambda$-modules in the above exact ladder are $\Lambda$-torsion. First consider $H_2(V_h, M_h, \mathbb{Q})$. By the Poincaré Duality isomorphism $H_2(V_h, M_h, \mathbb{Q}) \simeq H^2(V_h, \mathbb{Q})$. The Universal Coefficient Theorem reduces this to showing that $H_2(V_h, \mathbb{Q})$ is a $\Lambda$-torsion module. Consider the long exact sequence

$$
\begin{array}{ccccccc}
\cdots & \xrightarrow{j^*} & H^2(V_h, M_h, \mathbb{Q}) & \xrightarrow{\partial} & H_1(V_h, \mathbb{Q}) & \xrightarrow{j} & \cdots
\end{array}
$$
Where \((t-1)\) indicates multiplication by \((t-1)\), and \(p : V_h \to V\) is the covering projection. \(H_2(V, Q) = 0\) therefore multiplication by \((t-1)\) is onto \(H_2(V, Q)\), thus \(H_2(V, Q)\) is \(\Lambda\)-torsion. Similarly, \(H_1(V, Q)\) is \(\Lambda\)-torsion. \(H_1(M_h, \mathbb{Z})\) is an extension of a quotient of \(H_2(V_h, M_h, \mathbb{Q})\) and a submodule of \(H_1(V_h, Q)\), so it is also torsion.

Poincaré Duality with the Universal Coefficient Theorem gives us isomorphisms of the three short exact sequences:

\[
\begin{array}{ccccccccc}
0 & \to & \text{img}(\partial) & \to & H_1(M_h, Q) & \to & \text{img}(i_s) & \to & 0 \\
& & \downarrow{PD} & & \downarrow{PD} & & \downarrow{PD} & & \\
0 & \to & \text{img}(i_s) & \to & H^2(M_h, Q) & \to & \text{img}(\delta) & \to & 0 \\
& & \downarrow{UCT} & & \downarrow{UCT} & & \downarrow{UCT} & & \\
0 & \to & \text{img}(\text{Ext}(i_s)) & \to & \text{Ext}(H_1(M_h, Q), \Lambda) & \to & \text{img}(\text{Ext}(\partial)) & \to & 0
\end{array}
\]

where \(\text{Ext}(i_s) : \text{Ext}(H_1(V_h, Q), \Lambda) \to \text{Ext}(H_1(M_h, Q), \Lambda)\) and \(\text{Ext}(\partial) : \text{Ext}(H_1(M_h, Q), \Lambda) \to \text{Ext}(H_2(V_h, M_h, Q), \Lambda)\) are the \(\text{Ext}(-, \Lambda)\)-duals to \(i_s\) and \(\partial\) respectively.

The remainder follows from the following well-known lemma.

\[
0 \to C_s(M_h, \mathbb{Z}) \xrightarrow{t-1} C_s(M_h, \mathbb{Z}) \to C_s(M, \mathbb{Z}) \to 0.
\]

This allows us to compute \(\Delta(h = 1)\) (the Alexander polynomial evaluated at \(h = 1\)) as \(\Delta(h = 1) = \pm |\tau Z H_1(M, \mathbb{Z})|\).

\[\tau Z H_1(M_h, \mathbb{Z}) \cong \text{Hom}_\mathbb{A}(H_1(M_h, \mathbb{Q}), Q(\mathbb{A})/\Lambda)\]

which we think of as a sesquilinear duality pairing

\[\langle \cdot, \cdot \rangle : H_1(M_h, Q) \times H_1(M_h, Q) \to Q(\mathbb{A})/\Lambda.\]

Consider the homology long exact sequence induced from the short exact sequence

\[\tau Z H_1(M_h, \mathbb{Z}) \cong \text{Hom}_\mathbb{A}(H_1(M_h, \mathbb{Q}), Q(\mathbb{A})/\Lambda)\]

where \(\text{Ext}(i_s) : \text{Ext}(H_1(V_h, Q), \Lambda) \to \text{Ext}(H_1(M_h, Q), \Lambda)\) and \(\text{Ext}(\partial) : \text{Ext}(H_1(M_h, Q), \Lambda) \to \text{Ext}(H_2(V_h, M_h, Q), \Lambda)\) are the \(\text{Ext}(-, \Lambda)\)-duals to \(i_s\) and \(\partial\) respectively.

The remainder follows from the following well-known lemma.

\[
0 \to C_s(M_h, \mathbb{Z}) \xrightarrow{t-1} C_s(M_h, \mathbb{Z}) \to C_s(M, \mathbb{Z}) \to 0.
\]

This allows us to compute \(\Delta(h = 1)\) (the Alexander polynomial evaluated at \(h = 1\)) as \(\Delta(h = 1) = \pm |\tau Z H_1(M, \mathbb{Z})|\).

This condition together with the symmetry of the Alexander polynomial provide redundancies that are helpful when doing hand computations of the Alexander polynomial.

There are further obstructions to a rational homology \(S^1 \times S^2\) embedding in a homology \(S^3\), called signature invariants. As we have seen above there is a canonical isomorphism of \(\Lambda\)-modules

\[H_1(M_h, Q) \cong \text{Hom}_\mathbb{A}(H_1(M_h, \mathbb{Q}), Q(\mathbb{A})/\Lambda)\]

\[
\langle [v], [w] \rangle = \frac{1}{A_v} \sum_{z \in \mathbb{Z}} \langle S(v \land h^w)h^z \rangle = \frac{1}{A_w} \sum_{z \in \mathbb{Z}} \langle v \land h^wS(v)h^z \rangle \in Q[h^\pm 1]
\]

\[\equiv \Lambda.
\]

\(\overline{A_w}\) indicates we are taking the conjugate polynomial (conjugation is the automorphism of \(\Lambda \equiv Q[h^\pm 1]\) induced by the non-trivial automorphism of \(\mathbb{Z}\), or equivalently by the operation on polynomials \(h \mapsto h^{-1}\). The symbol \(\land\) indicates we
are taking the oriented intersection number – i.e. one first perturbs the chains to be transverse and then takes the signed intersection number. That the pairing ⟨·,·⟩ is sesqui-linear means that it is Q-linear in both variables and \(h(x,y) = \langle hx,y \rangle = \langle x,h^{-1}y \rangle\) for all \(x,y \in H_1(M_h,Q)\). Moreover, \(\langle x,y \rangle = \langle y,x \rangle\) for all \(x,y \in H_1(M_h,Q)\), where the conjugation is the involution of \(Q(\Lambda)/\Lambda\) induced by conjugation on \(\Lambda\). From the pairing ⟨·,·⟩ we construct an anti-symmetric pairing \([·,·] : H_1(M_h,Q) \times H_1(M_h,Q) \to Q\) by composing with the “Trotter trace” function \(\text{tr} : Q(\Lambda)/\Lambda \to Q\), i.e. \([x,y] = \text{tr}⟨x,y⟩\). See page 182 of [52] for details on the trace function. In brief:

(a) \(\text{tr}\) is a Q-linear function such that \(\text{tr}(x) = -\text{tr}(x)\) for all \(x \in Q(\Lambda)/\Lambda\).

(b) Given \(p,q \in \Lambda\) where \(q\) is not a unit nor divisible by \(1 - h\), and assuming the lowest (resp. highest) degree non-zero coefficient of \(p\) has degree \(\geq\) (resp. \(\leq\)) the lowest (resp. highest) degree non-zero coefficient of \(q\) (say, via the division algorithm), \(\text{tr}(p/q)\) is defined to be the derivative evaluated at 1, \(\text{tr}(p/q) = (p/q)'(1)\).

(c) \(\text{tr}\) is defined on \(Q(\Lambda)/\Lambda\) by extending the definitions (b) and (c) linearly.

(d) An essential property of the Trotter trace is that provided we’re in case (b) and that the highest-order non-zero term of \(p\) is strictly smaller than the highest-order non-zero term for \(q\), then \(\text{tr}((h-1)p/q) = (p/q)(1)\).

From this it follows that composition with the Trotter trace gives an isomorphism

\[
\text{Hom}_\Lambda(H_1(M_h,Q),Q(\Lambda)/\Lambda) \to \text{Hom}_Q(H_1(M_h,Q),Q).
\]

Thus, the pairing \([·,·]\) is non-degenerate, anti-symmetric and multiplication by \(h\) is an isometry \([hx,hy] = [x,y]\). We construct a symmetric bilinear form \(H_1(M_h,Q) \times H_1(M_h,Q) \to Q\) via the formula \([x,y] = [x,ty] + [y,tx]\). Notice that this symmetric form can potentially degenerate: \([x,ty] + [y,tx] = 0\) if and only if \([x,(t^2-1)y] = 0\). Assume \(x \neq 0\) is fixed. Since multiplication by \(-1\) is an isomorphism on \(H_1(M_h,Q)\), \([x,(t^2-1)y] = 0\) for all \(y \in H_1(M_h,Q)\) if and only if \([x,(t+1)y] = 0\) for all \(y\). Therefore if we restrict \([·,·]\) to the maximal \(\Lambda\)-submodule of \(H_1(M_h,Q)\) on which multiplication by \(t+1\) is an isomorphism, we get a non-degenerate symmetric form. Let \(\sigma_p \in \mathbb{Z}\) be the signature of this form. Let \(p\) be any prime symmetric factor of \(\Delta(h)\). By further restricting the above symmetric form to the submodule killed by a power of \(p\), we get further signature invariants \(\sigma_{p,h} \in \mathbb{Z}\), called Milnor signature invariants. These are closely related to Tristram-Levine invariants [23, 35]. The relations among these signature invariants appears in slightly different form in [17, 27, 29].

**Theorem 2.6.** (Signature Test) If \(M\) is a rational homology \(S^1 \times S^2\) and if \(M\) embeds in a homology \(S^4\), then all the above signature invariants are zero.

**Proof.** The proof of Theorem 2.4 gives a commuting ladder

\[
\begin{array}{c}
0 \to \text{im}(\partial) \to H_1(M_h,Q) \to \text{im}(i_\ast) \to 0 \\
0 \to \text{im}(i_\ast^\star) \to \text{Hom}_\Lambda(H_1(M_h,Q),Q(\Lambda)/\Lambda) \to \text{im}(\text{tr}) \to 0
\end{array}
\]

composed with the “Trotter trace” function \(\text{tr} : Q(\Lambda)/\Lambda \to Q\), i.e. \([x,y] = \text{tr}⟨x,y⟩\). The proof of Theorem 2.4 gives a commuting ladder

\[
\begin{array}{c}
\text{im}(\partial) \to H_1(M_h,Q) \to \text{im}(i_\ast) \to 0 \\
\text{im}(i_\ast^\star) \to \text{Hom}_\Lambda(H_1(M_h,Q),Q(\Lambda)/\Lambda) \to \text{im}(\text{tr}) \to 0
\end{array}
\]

where the upper stars indicate \(\text{Hom}_\Lambda(⟨·,·⟩/\Lambda)\)-duals. Thus, the domain of the form \([·,·]\) splits into two subspaces of equal dimension, and the form is zero on one of them. For a non-degenerate form this can happen if and only if the signature is zero.

There are a few obstructions related to particular families of manifolds. For the geometric 3-manifolds among the geometries: \(S^3\), \(S^2\)-fiber, \(E^3\), Sol and Nil, Crisp and Hillman [12] computed precisely which of these manifolds embed in \(S^4\). They do this by a combination of the above obstructions together with a new obstruction derived as a generalization of the Massey-Whitney Theorem on the normal Euler class of 2-manifolds in homology 4-spheres.

Let \(E\) be the total space of a \(D^2\)-bundle \(p : E \to \Sigma\) over a closed surface \(\Sigma\). Let \(q : \partial E \to \Sigma\) be the corresponding \(S^1\)-bundle. The Whitney class \(W_2(q) \in H^2(\Sigma,B) \cong \mathbb{Z}\) is the obstruction to the existence of an everywhere non-zero section of the bundle \(p : E \to \Sigma\). \(W_2(q)\) is an element of the 2nd cohomology group of \(\Sigma\) with coefficients in the bundle of groups \(B = \{(s,\pi^iq^{-1}(s)) : s \in \Sigma\}\).

**Theorem 2.7.** (Whitney-Massey-Crisp-Hillman) [12, 36, 54]

The total space of a disk bundle \(p : E \to \Sigma\) embeds in \(S^4\) (equivalently, a homology \(S^4\)) if and only if

- \(W_2 = 0\), provided \(\Sigma\) is orientable
- \(W_2 \in \{2\chi - 4, 2\chi, 2\chi + 4, \ldots, 4 - 2\chi\}\) if \(\Sigma\) is non-orientable, where \(\chi\) is the Euler characteristic of \(\Sigma\).

A circle bundle over a surface embeds in \(S^4\) (equivalently a homology 4-sphere) if and only if

- \(W_2 \in \{-1, 0, \overline{1}\}\), which is consistent with Ozlik’s unnormalized Seifert notation [41]. A circle bundle over a surface \(\Sigma\) with Euler
number $W_2 = k$ is denoted $SFS[\Sigma : k]$. Whitney constructed all the above embeddings of $D^2$-bundles over surfaces [54] and conjectured it was the complete list of $D^2$-bundles that embed in $S^3$. Massey went on to prove his conjecture [36]. Crisp and Hillman proved the extension for $S^3$-bundles over surfaces [12].

The proof that $W_2 = 0$ when $\Sigma$ is orientable follows from the observation that $W_2$ is the self-intersection number of $\Sigma$ in $S^3$, and that $\Sigma$ can be isotoped off itself in $S^3$. When $\Sigma$ is non-orientable, the same observation tells us that $W_2$ is even. To get the restriction $W_2 \in \{2y_1, 2y_2, 2y_3 + 4, ..., 4 - 2y\}$ Massey employed the $G$-signature Theorem to show that $W_2$ is the signature of a certain form. Precisely, let $X$ be the $\mathbb{Z}_2$-branched cover of $S^4$ branched over $\Sigma$ corresponding to the non-trivial element of $H_1(S^4 \setminus \Sigma, \mathbb{Z}) \cong \mathbb{Z}_2$. The $G$-signature Theorem states that the Euler class of $\Sigma$ in $X$ is the signature of the form $\langle \chi, T, y \rangle$ on $H_2(X, \mathbb{Q})$ where $T : X \to X$ is the covering transformation and $\langle \cdot, \cdot \rangle$ is the intersection product. The result follows from the computations $H_2(X, \mathbb{Q}) \cong \mathbb{Q}^{2-rk[\Sigma]}$, $T_* = -1 + \partial H_2(X, \mathbb{Q})$ and that the Euler class of $\Sigma$ in $S^4$ is twice that of $\Sigma$ in $X$.

In the $S^3$-bundle case, with $\Sigma$ orientable the torsion linking form is the appropriate embedding obstruction. When $\Sigma$ is non-orientable, and $M$ is an $S^3$-bundle over $\Sigma$, the torsion linking form test tells us that $W_2$ must be even. Crisp and Hillman generalized [12] the above argument of Massey’s. Since $W_2$ is even, $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^{2-1} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with one of the $\mathbb{Z}_2$ factors being generated by the fundamental class of the fiber. So if $M$ embeds in $S^4$, we have $S^4 = V_1 \cup M V_2$ and so $H_1 V_1 \oplus H_1 V_2 \simeq H_1 M$, and so the $H_2$-summands corresponding to the fiber inclusion belongs (WLOG) to $H_1 V_1$. Let $W$ be $V_1$ union the $D_2$-bundle over $\Sigma$ with Euler class $W_2$. Let $W'$ be the $\mathbb{Z}_2$-branched cover of $W$ branched over $\Sigma$, and apply the $G$-signature Theorem as in the previous case.

Crisp and Hillman make similar but increasingly complex applications of the $\mathbb{Z}_2$-signature Theorem as formulated in [26] to get further obstructions to the embedding of Seifert-fibered and Sol manifolds. The idea being to use the homology of $M$ to construct 2-sheeted covering spaces $\tilde{M}$ of $M$, and to attach to it the associated covers of $V_1$ or $V_2$, or some associated $\mathbb{Z}_2$-space whose boundary is $\tilde{M}$ and for which the fixed point set is understood. See Proposition 1.2 and Theorem 1.4 of [12].

A link $L \subset S^3$ is said to be slice if there is a manifold $D \subset D^4$ such that $\partial D = L$ and $D$ is diffeomorphic to a disjoint union of discs $D^2$. $D$ is called slice discs for $L$.

**Construction 2.8.** (0-Surgical Embeddings): Let $M$ be a 0-surgery along a link $L \subset S^3$ where $L$ is the union of two links $L = L_1 \cup L_2$ such that $L_i$ is smoothly slice for $i \in \{1, 2\}$. Then $M$ admits a smooth embedding into $S^4$.

**Proof.** The idea of the proof is to consider $S^4$ as the union of two 4-balls, separated via a great 3-sphere. Let $D_1$ be a collection of slice discs in the first semi-sphere whose boundary is $L_1$, and let $D_2$ be a collection of slice discs in the second semi-sphere whose boundary is $L_2$. Then $M$ can be obtained by an embedded surgery on the great 3-sphere along the discs $D_1 \cup D_2$, see Figure 2.

![Figure 2. A 0-surgical embedding.](image)

Some examples of links which are the disjoint union of two slice links are: the Hopf link $M = S^3$, the Whitehead link $(M = S^3 \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) (S^1 \times S^1))$, and the Borromean rings $(M = S^3 \times S^3 \times S^3)$ [12].

**Construction 2.9.** (1-Surgical Embedding): If $M$ is a surgery on a smooth slice link such that the surgery coefficients all belong to the set $\{1, -1\}$, then $M$ admits a smooth embedding into a homotopy 4-sphere.

**Proof.** Write the link $L \subset S^3$ as the union of two disjoint sublinks $L = L_1 \cup L_1$ where the surgery coefficients for the $L_i$ components are $i$ for $i \in \{-1, 1\}$. Let $D \subset D^4$ be the slice discs for $L$, $D = D_1 \cup D_2$ with $\partial D_i = L_i$ for $i \in \{-1, 1\}$. Attach 2-handles to $D^4$ along the components of $L_i$ with framing numbers $i$ appropriately for $i \in \{-1, 1\}$. Let $D'_i$ be the cores of the attaching handles, thus $D'_1 \cup D_1$ is a union of disjointly embedded 2-spheres in $N$ whose normal bundles have Euler number $i$ for $i \in \{-1, 1\}$. Recall that $CP^2$ has this decomposition: it is a $D^2$-bundle over $S^2 (CP^1)$ with Euler number 1, capped-off with a 4-handle. Thus we can replace a tubular neighborhood of $D'_1 \cup D_1$ with a union of 4-handles for $i \in \{-1, 1\}$, giving a manifold $N'$ with $N' = \# N \# CP^2 \# \# \ldots \# CP^2 \# CP^2 \# \ldots \# CP^2$. This is more commonly known as a “blow-down” operation. Thus, $N'$ is contractible, and $\partial N' = M$ so the double of $N'$ is a homotopy $S^4$ containing $M$. See Figure 3.

![Figure 3.](image)

Given $L \subset S^3$ a slice link with slice discs $D \subset D^4$, we say the slice discs $D$ are in ribbon position if the function $d : D^4 \to \mathbb{R}$ given by $d(v) = |v|^2$ when restricted to $D$ ($f : D \to \mathbb{R}$) is a Morse function having no local maxima. If a link $L$ has slice discs $D$ that can be put into ribbon position, $L$ is called a ribbon link. Whether or not every slice knot is ribbon is a long-standing open problem in knot theory, due to Ralph Fox, and is called the slice-ribbon problem.

**Proposition 2.10.** Let $L$ be a ribbon link, then the manifold $N'$ in the proof of Construction 2.9 admits a handle decomposition.
with a single 0-handle, followed by only 1-handle attachments and 2-handle attachments, i.e. there are no 3 or 4-handles.

**Proof.** Let $A$ be the complement of an open tubular neighborhood of $D$ in $D^4$. The distance function $d$ restricts to a Morse function (in the stratified sense) $d_A : A \to \mathbb{R}$ with one local minima, and critical points of index $(-1, -2)$ on $\partial A$ corresponding to the critical points of index $(+1, -1)$ of $f_D : D \to \mathbb{R}$, and critical points of index $(+2, 0)$ corresponding to critical points of index $(+2, 0)$ for $f_D : D \to \mathbb{R}$. So $A$ consists of a 4-ball with 1-handles and 2-handles attached. Let $B$ be $N'$ with an open tubular neighborhood of the spheres $\{D_i \cup D'_i : \forall i\}$ removed. $B$ is $A$ with generalized handles (in the sense of Bott [4]) attached. The generalized handles correspond to the spheres $D_i \cup D'_i$ for each $i$, and are trivial $I$-bundles over their core $S^1 \times S^2$. For each $i$ one can think of this generalized handle as a 2-handle followed by a 3-handle attachment. We construct $N$ by attaching 4-handles to $B$, one for each $i$. The 4-handles cancel the above 3-handle attachments since they satisfy the conditions of Smale’s Handle Cancelation Lemma (see for example [31] VI.7.4) – i.e. the attaching sphere of the 4-handle intersects the belt sphere of the 3-handle transversely in a single point (the belt sphere consists of two points one in $M$ and one not in $M$). Thus $N'$ has a handle decomposition with one 0-handle, and only 1 and 2-handles attached.

Since $N'$ is contractible, the presentation of $\pi_1N'$ coming from Proposition 2.10 must be a presentation of the trivial group, moreover the number of generators and relators is equal, this is called a “balanced presentation.” If the Andrews-Curtis Conjecture were true [21] we could cancel the 1 and 2-handles of $N'$ using handle slides, so $N'$ would be diffeomorphic to the standard $D^4$ and $M$ would embed in $S^4$. The upshot of this observation is that if we use ribbon links in Proposition 2.9 and verify the presentation of $\pi_1N'$ can be trivialized by Andrews-Curtis moves, then we have verified that the manifold $M$ embeds in $S^4$. The presentation of $\pi_1N'$ has the form

$$\pi_1N' = \langle g_1, ..., g_k : r_1, ..., r_p, R_1, ..., R_l \rangle$$

where the generators $g_i$ correspond to the local minima of $d$ on the slice discs, the relators $r_i$ correspond to the saddle points of $d$ on the slice discs, and the relators $R_i$ correspond to the framing curves of the link $L$ – so $k = j + l$. These presentations are readily computed from a ribbon diagram for $L$.

**Constructions 2.9 and 2.8** have relatively simple implementations. For example, given a hyperbolic manifold which satisfies Theorem 2.2, using SnapPea one can drill out selections of curves from $M$ then look for the resulting manifold in previously enumerated tables of hyperbolic link complements. Frequently this technique finds useful surgery presentations. See the beginning of §7 for details.

Notice that it is relatively easy to construct embeddings of many 3-manifolds in homology spheres, for example: A homology 3-sphere embeds in a homology 4-sphere if and only if it is the boundary of a homology 4-ball. The boundary of any homology 4-ball is a homology 3-sphere, thus constructing embeddings of homology 3-spheres in homology 4-spheres is essentially the same problem as constructing homology 4-balls. If $M$ is a homology 3-sphere then $M\#(-M)$ embeds in a homology 4-sphere – simply drill out a tubular neighborhood of $\{\ast \} \times I$ from $M \times I$ to construct a homology 4-ball bounding $M\#(-M)$. If $B$ is an open 3-ball in a homology 3-sphere $M$, the manifold $(M \setminus B) \times S^1 \times S^2 \times D^2$ is another homology 4-sphere containing $M\#(-M)$.

If $M$ has non-trivial homology the situation is a little more subtle. Consider when a 4-manifold $W = V_1 \cup_M V_2$ is a homology 4-sphere. By a simple Mayer-Vietoris argument, this happens if and only if the manifolds $V_1$ and $V_2$ are orientable and the maps $H_iM \to H_iV_1 \oplus H_iV_2$ and $H_2M \to H_2V_1 \oplus H_2V_2$ are isomorphisms. By considering the long exact sequences of the pairs $(V_i,M)$ for $i \in \{1, 2\}$ and a Poincaré Duality argument, this is equivalent to the statement that the horizontal maps in the commutative diagrams below ($\forall \{i,j\} = \{1, 2\}$) are isomorphisms.

$$H_2(V_j, M, \mathbb{Z}) \xrightarrow{\partial} H_1(M, \mathbb{Z}) \xrightarrow{i_*} H_1(V_i, \mathbb{Z})$$

Thus, the problem of constructing an embedding of an arbitrary 3-manifold into a homology $S^4$ can be thought of as a type of “simultaneous cobordism” problem.

**Construction 2.11.** Let $M$ be the result of a surgery along a link $L = L_1 \cup L_2$. Assume that $L_1$ is smooth slice and that the surgery coefficients for $L_1$ are all zero. Further, assume that the matrix of linking numbers $l_{k,i,j}$ where $i$ indexes the component of $L_1$ and $j$ indexes the components of $L_2$ is square and invertible, then $M$ is the boundary of a homology 4-ball. If we weaken this last condition to the matrix $l_{k,i,j}$ is square with non-zero determinant, then $M$ is a rational homology sphere bounding a rational homology ball.

In the case of manifolds that fiber over $S^1$ there is a spinning construction that produces many embeddings.
Construction 2.12. Let $M$ be a closed orientable 3-manifold which fibers over $S^1$. Let $W$ be the fiber of the locally trivial fiber bundle $W \to M \to S^1$ and let $f : W \to W$ be the monodromy, i.e. $M = \mathbb{R} \times_Z W$ where $Z$ acts on $\mathbb{R}$ by translation, and the action on $W$ is generated by $f$. If $W$ admits an embedding into $S^3$ such that $f$ extends to an orientation-preserving diffeomorphism of $S^3$, then $M$ embeds smoothly in $S^4$.

Proof. The diffeomorphism $f : (S^3, W) \to (S^3, W)$ is isotopic to the identity when considered as a diffeomorphism of $S^3$ [9]. Let $F : [0,1] \times S^3 \to S^3$ be such an isotopy: $F(0,x) = x$ and $F(1,x) = f(x)$ for all $x \in S^3$. Let $B$ be an open 3-ball which is disjoint from $W$ and fixed pointwise by $f$. Let $B'$ be the closure of the complement of $B$ in $S^3$, thus $f$ can be assumed to be of the form $f : (B',W) \to (B',W)$, and $f$ restricts to the identity on $\partial B'$. Consider $S^3$ to be the union $S^3 = (D^3 \times S^1) \cup (S^1 \times D^2)$, where we identify $B'$ with $D^2$, then $\{ (F(x,t), e^{2\pi it}) : x \in W, t \in [0,1] \} \subset D^3 \times S^1$ is the embedding of $M$ in $S^4$. \hfill $\blacksquare$

It seems appropriate to call such embeddings “deform-spun” due to the analogy with Litherland’s spinning construction for knots [34]. It has been known since the work of Crisp and Hillman [12] that not all manifolds that fiber over $S^1$ which embed in $S^4$ admit deform-spun embeddings. At present the only examples of this type that are known are 0-surgeries on fibered smooth slice knots (see §3 item 4 for an example).

Embeddings for some special families of 3-manifolds have been worked out in the literature. A class that has received particular attention are the Seifert-fibered homology spheres.

Theorem 2.13. (Casson, Harer [8]) The Brieskorn homology spheres $\Sigma(p,q,r)$ smoothly embed in $S^4$ provided $(p, q, r)$ is of the type:

1. $(p, pa + 1, pa + 2)$ or $(p, pa - 2, pa - 1)$ for $p$ odd.
2. $(p, pa - 1, pa + 1)$ for $p$ even and $a$ odd.
3. $(2, 3, 13)$ or $(2, 5, 7)$
4. $(2, 5, 9)$ or $(3, 4, 7)$

Proof. Casson and Harer prove that these Brieskorn spheres $\Sigma$ bound contractible 4-manifolds $M$ where $M$ has a handle decomposition with a single 0, 1 and 2-handle, and no 3 or 4-handles. Thus the corresponding handle decomposition for $M \times I$ can be trivialized via handle-slides, making $M$ a smooth submanifold of $\partial(M \times I) \simeq S^3$. \hfill $\blacksquare$

The statement of Theorem 2.13 uses the numbering convention of [8] together with the observation that Casson and Harer’s families (3) and (4) are finite. Other useful related references are [3, 19].

Theorem 2.14. (Stern) [49] The Brieskorn spheres $\Sigma(p,q,r)$ bound contractible 4-manifolds provided $(p, q, r)$ is of the form below. Thus, these Brieskorn homology spheres embed in homotopy 4-spheres.

- $(p, pa \pm 1, 2p pa \pm 1)$ or $(p, pa \pm 2, pa \mp 1)$ for $p$ even and a odd.
- $(p, pa \pm 1, 2p pa \pm 1)$ or $(p, pa \pm 2, 2p pa \pm 1)$ for $p$ odd.
- $(p, pa \pm 2, 2p pa \pm 1)$ or $(p, pa \pm 2, 2p pa \pm 1)$ for $p$ odd.

Stern’s contractible 4-manifolds are constructed from a 4-ball by attaching two 1-handles and then two 2-handles.

There is one further construction of embeddings of 3-manifolds in $S^4$ due to Zeeman and Litherland. Let $K$ be a “long knot” i.e. an embedding $K : D^3 \to D^4$ which agrees with the standard inclusion $t \mapsto (t, 0, 0)$ on $\{ t = 1 \} = \partial D^3$. Let $f$ be a diffeomorphism of $D^3$ which fixes pointwise $\partial D^3$ and $\text{img}(K)$. By Cerf’s Theorem [9], there is a smooth 1-parameter family $F : D^3 \times [0,1] \to D^4$ such that $F(x,t) = x$ for all $t \in [0,1]$ and $x \in \partial D^3$, with $F(x,0) = x$ for all $x \in D^3$ and $F(x,1) = f(x)$ for all $x \in D^3$. $F(K(x),t)$ is an isotopy which starts and ends at $K$. Conversely, by the Isotopy Extension Theorem, an isotopy that returns $K$ to itself gives a diffeomorphism of the pair $(D^3, K)$. These two processes are mutually inverse in the sense that there is an isomorphism of the fundamental group of the “space of maps” of type $K$, and the mapping class group of the pair $(D^3, K)$ (see for example [5] for details). Consider $S^4$ to be the union $(D^3 \times S^1) \cup (S^1 \times D^2)$, then the deform spun knot corresponding to $f$ is the embedding $S^2 \equiv (D^1 \times S^1) \cup (S^0 \times D^2) \to (D^3 \times S^1) \cup (S^2 \times D^2) \equiv S^4$

given by

$D^1 \times S^1 \ni (x, e^{2\pi it}) \mapsto (F(K(x),0), e^{2\pi it}) \in D^3 \times S^1$

$S^2 \times D^2 \ni (a,b) \mapsto ((a,0), b) \in S^3 \times D^2$

Theorem 2.15. [34] Let $M : (D^3, K) \to (D^3, K)$ denote the diffeomorphism induced from rotating $K$ by $2\pi$ around the axis $[-1,1] \times \{ 0 \} \subset D^3$, a “meridional Dehn twist.” If $f : (D^3, K) \to (D^3, K)$ preserves a Seifert surface for $K$, then the complement of the deform-spun knot associated to $M^o \circ f$ fibers over $S^3$, provided $n \neq 0$.

Zeeman proved Theorem 2.15 in the case that $f$ was the identity automorphism of $D^3$. He also went on to show that the fiber is the $n$-fold cyclic branch cover of $D^3$ branched over $K$. So for example, if $n = \pm 1$ and $f = \text{Id}$, the associated deform-spun knot is trivial, as it bounds a disk. Litherland identified the fiber in the more general case. Let $\Sigma$ be the preserved Seifert surface. This means that $\Sigma$ is an oriented surface in $D^3$ whose boundary consists of $K$ union a smooth arc in $\partial D^3$ connecting the endpoints of $K$ and that $f(\Sigma) = \Sigma$. Let $C_K$ denote $D^3$ remove an open tubular neighborhood of $K$, and let $X$ denote $C_K$ remove an open tubular neighborhood of $C_K \cap \Sigma$. Denote the two components of the boundary of the tubular neighborhood of $C_K \cap \Sigma$ in $C_K$ by $\Sigma_1$ and $\Sigma_2$ respectively (thought of as the boundary of $\Sigma \times [1,2]$). Litherland shows that the Seifert surface for the deform-spun knot is diffeomorphic to the space $X \times \{ 1, 2, \ldots, n \} / \sim$ where the equivalence relation is defined by $((s, i), i) \sim ((s, i), i + 1)$ for $i \in \{ 1, 2, \ldots, n - 1 \}$ and $((s, 2), n) \sim ((f(s), 1), 1)$, where $(s, i) \in \Sigma$. If one goes on to write $f|_{\Sigma}$ as a product of Dehn twists, this allows the further
Theorem 2.15 gives us a rich source of 3-manifold embeddings in $S^4$, for example, the lens spaces $L_{p,q}$ for $p$ odd are 2-sheeted branched cover over $S^3$ with branch point set the corresponding 2-bridge knot, thus punctured lens spaces with odd order fundamental group embed in $S^3$. Thus the connect sum $L_{p,q} \# - L_{p,-q}$ embed smoothly in $S^4$. Similarly, a punctured Poincaré Dodecahedral Space embeds in $S^4$ by using the 5-fold branch cover of $(D^3, K)$ where $K$ is the trefoil.

If $M_1$ and $M_2$ are lens spaces such that $M_1 \# M_2$ embeds in $S^4$, it follows from Theorem 2.2 that $\pi_1 M_1 \simeq \pi_1 M_2$, and from the torsion linking form that the order of $\pi_1 M_i$ must be odd [28]. Historically the first proof of this is due to Epstein [16], who used different techniques. This led to one of the more interesting conjectures about 3-manifolds embedding in $S^3$, due to Gilmer and Livingston [20] concerning when a connect-sum of two lens spaces embeds in $S^4$. The Gilmer-Livingston conjecture was solved by Fintushel and Stern [19], and recently generalized by Andrew Donald [14] to the case of an arbitrary connect-sum of lens spaces.

Theorem 2.16. [14, 19, 20, 28, 43] A manifold $M$ that is a connect sum of finitely many lens spaces smoothly embeds in $S^4$ if and only if $M$ is a connect sum of finitely many manifolds of the form $L_{p,q} \# L_{p,-q}$ where $p$ is odd. Stated another way, $M$ must be a balanced connect sum of lens spaces and their orientation-reverse, where the lens spaces are required to have fundamental groups of odd order.

So for example $L_{p,1}$ admits an orientation-reversing diffeomorphism, but it does not embed in $S^3$ provided $p \geq 2$. But a connect sum of $k$ copies of $L_{p,1}$ embeds in $S^4$ if and only if $k$ is even and $p \geq 2$ is odd. Fintushel and Stern’s result is the case of the above theorem where there is precisely two prime summands. Donald’s result is a generalization of [33] where Lisca determines when an arbitrary connect-sum of lens spaces bounds a rational homology ball.

Donald makes use of a mixed branch cover/slice disk embedding construction.

Definition 2.17. A link $L \subset S^3$ is doubly-slice if there is an unknotted 2-sphere $M \subset S^4$ such that $M$ intersects $S^3 \times \{0\}$ transversely, and $L \times \{0\} = M \cap (S^3 \times \{0\})$. 

Construction 2.18. [14] If a 3-manifold $M$ is a finite cyclic branched cover of $(S^3, L)$ with $L$ doubly-slice, then $M$ embeds smoothly in $S^4$.

The proof amounts to observing that any finite cyclic branched cover of $S^4$ branched over an unknot is diffeomorphic to $S^4$. Construction 2.18 would not be useful if there wasn’t a large class of doubly-slice links. Donald does so in his Proposition 2.6, constructing a link $L_{a,n} \subset S^3$ such that the double branch cover of $(S^3, L_{a,n})$ is a Seifert fiber space of type $SFS([\frac{1}{2}, \frac{1}{2}, -\frac{1}{a}, -\frac{n}{na+1}])$.

One can algorithmically construct all 3-manifolds that embed smoothly in $S^4$. The algorithm goes like this: Start with any triangulation of $S^4$. Enumerate the vertex-normal 3-manifolds in that triangulation. In particular, find all vertex-normal solutions to the gluing equations, and triangulate them. Barycentrally subdivide the triangulation of $S^4$ and repeat. All 3-manifolds that embed in $S^4$ eventually appear as vertex-normal solutions in any sufficiently-fine triangulation of $S^4$. This is a consequence of Whitehead’s proof that smooth manifolds admit triangulations. This procedure is implemented in Regina (as of version 5.0) and was used to construct several embedding examples in Section 4. This technique also recovers most of the embeddings in Section 3. The downside to this technique is it’s computationally extremely expensive. The upside is it finds embeddings of manifolds that have not been found via any other technique.

There are several obstructions to embedding rational homology spheres in $S^4$ which utilize Spin-structures and Spin$^c$-structures. We summarize the useful properties of these invariants, but first a quick review of orientation, Spin and Spin$^c$ structures on manifolds. Helpful references for this material are [32, 39, 40, 50].

The group Spin($n$) is the connected 2-sheeted cover of the Lie group $SO_n$, together with the Lie group structure making $Spin(n) \to SO_n$ a homomorphism of Lie groups. Provided $n \geq 3$, Spin($n$) is the universal cover of $SO_n$. The group Spin$^c(n)$ is the twisted-product $Spin(n) \times_{\mathbb{Z}_2} Spin(2)$ where $\mathbb{Z}_2$ acts diagonally as the covering transformation of both factors. Thus, there are Lie group submersions:

$$Z_2 \to Spin(n) \to SO_n \quad Spin(2) \to Spin^c(n) \to SO_n$$

Notice that there is a canonical isomorphism of Lie groups $U_2 \cong Spin^c(3)$, since $SU_2 \subset U_2$ is naturally isomorphic to $Spin(3) = S^3$, and the diagonal matrices in $U_2$ are naturally isomorphic to Spin(2), moreover, $SU_2$ intersects the diagonal matrices at precisely ±1. More generally, $U_n \cong SU_n \times \mathbb{Z}_2 U_1$.

Given an $n$-manifold $N$ let $TN$ denote the tangent bundle of $N$, this the union of all the tangent spaces to $N$. $TN$ is a vector bundle over $N$. The space of all bases to the tangent spaces of $N$ is called the principal $GL_n$-bundle associated to $N$, and will be denoted $GL_n(TN)$. $GL_n(TN)$ is a fiber bundle over $N$ with fiber the Lie group $GL_n$, thus there are fibrations:

$$GL_n \to GL_n(TN) \to N \quad GL_n(TN) \to N \to BGL_n$$

The map $N \to BGL_n$ is called the classifying map for the bundles $TN \to N$ and $GL_n(TN) \to N$ respectively. Since the inclusion $O_n \to GL_n$ is a homotopy-equivalence, a choice of a Riemannian metric on $N$ allows us to replace $GL_n$ by $O_n$ in the discussion above.

An orientation of $N$ is a homotopy class of lifts of the classifying map $N \to BO_n$ to $BSO_n$. For an oriented manifold $N$, a Spin$^c$-structure on $N$ is a homotopy class of lifts of maps $N \to BSO_n$ to $BSpin^c(n)$. Similarly, a Spin$^c$-structure is a homotopy class of lifts of $N \to BSO_n$ to $BSpin(n)$. Essentially by definition, two Spin$^c$-structures on $N$ differ by an element of $[N, B\mathbb{Z}_{2} \times_{\mathbb{Z}_2} H^2(N, \mathbb{Z})]$. Similarly, two Spin$^c$-structures on $N$ differ by an element of $[N, BSpin(2)] \cong H^2(N, \mathbb{Z})$.

Every orientable 3-manifold has a trivial tangent bundle [30], so it has both a Spin(3) and a Spin$^c$(3)-structure. In general, a manifold $N$ has a Spin$^c$ structure if and only if it is orientable and the 2nd Stiefel-Whitney class is zero, $w_2(N) = 0$. Equivalently, if its tangent bundle trivializes...
over the 2-skeleton of \( N \) – moreover, the Spin\((n)\)-structure is taken to be a homotopy class of such a trivialization, once restricted to the 1-skeleton. \( N \) has a Spin\((n)\)-structure if and only if \( \omega_2(N) \) is the reduction of an integral cohomology class. Equivalently, this is if and only if a direct sum with a complex line bundle admits a Spin-structure. Another equivalent definition is that (if \( N \) has odd dimension, stabilize by adding a trivial 1-dimensional vector bundle) a Spin\(^c\)-structure is a homotopy class of almost complex structures over the 2-skeleton such that a representative almost complex structure extends over the 3-skeleton.

**Theorem 2.19.** [30, 42] If \( M \) is a Spin 3-manifold there exists an invariant, called the Rochlin invariant, taking values in \( \mathbb{Q}/2\mathbb{Z} \). The Rochlin invariant of \( M \) is \( \mu(M) = \frac{-\text{det}(W)}{8} \in \mathbb{Q}/2\mathbb{Z} \) where \( W \) is a Spin-manifold such that \( \partial W = M \). \( \text{sign}(W) \) is the signature of the intersection form on \( fH_2(W,\mathbb{Z}) = H_2(W,\mathbb{Z})/\tau H_2(W,\mathbb{Z}) \). When \( M \) is a homology sphere \( \text{sign}(W) \) is divisible by 8, so \( \mu(M) \in \mathbb{Z}/2 \).

The Rochlin invariant has an integral lift for homology spheres, called the \( \bar{\mu} \)-invariant [47]. \( \bar{\mu} \) is a homology cobordism invariant for Seifert fibered homology spheres (see [44] Corollary 7.34).

If \( M \) is a rational homology 3-sphere with a Spin\(^c\)-structure, there is an invariant called the Ozsváth-Szabó \( d \)-invariant or “correction term,” taking values in \( Q \). It is a rational homology Spin\(^c\)-co bordism invariant and additive under connect-sum.

The above theorems explain why we’re interested in Spin and Spin\(^c\) structures – the extra structure given to the tangent bundle allows for more delicate constructions. For our purposes, a Spin structure is the most sensitive tangent bundle structure we’ll ever need. This is because a connected 4-manifold which bounds a non-empty 3-manifold has a trivial tangent bundle if and only if it has admits a Spin structure – to see this, notice such 4-manifolds have the homotopy-type of a 3-complex. The tangent bundle of a 4-manifold with a Spin-structure trivializes over the 2-skeleton, and the obstruction to extending to the 3-skeleton (and thus the entire manifold) lives in a 3-dimensional twisted cohomology group with coefficients \( \pi_2 SO_4 = \pi_2 Spin(4) = \pi_2 (S^3 \times S^3) = 0 \).

**Definition 2.20.** Given a rational homology sphere \( M \), let \( \bar{d}(M) \) be the function whose domain is the Spin-structures on \( M \) and whose values are the Rochlin invariants of \( M \) with the associated Spin-structure. Similarly, let \( \bar{\mu}(M) \) be the function whose domain is the Spin\(^c\) structures on \( M \) and whose values are the associated \( d \)-invariants.

**Corollary 2.21.** (\( \bar{\mu} \) and \( \bar{d} \) tests) Given a rational homology sphere \( M \) which admits a smooth embedding into a homology 4-sphere, \( [H_1(M,\mathbb{Z})] = k \mathbb{Z} \) for some \( k \). Moreover, there are \( 2k-1 \) zeros in \( \bar{d}(M) \). Similarly, \( [H_1(M,\mathbb{Z}_2)] = l \mathbb{Z}_2 \) for some \( l \), and there are \( 2l-1 \) zeros in \( \bar{\mu}(M) \).

**Proof.** Assume \( M \) embeds in \( S^4 \), then \( M \) separates \( S^4 \) into two rational homology balls \( V_1 \) and \( V_2 \). Since \( V_1 \subset S^4 \), \( V_1 \) has a trivial tangent bundle. If we fix a trivialization of \( TV_1 \), the Spin\(^c\) structures on \( V_1 \) correspond to elements of \( [V_1, BSpin(2)] = H^2(V_1,\mathbb{Z}) \).

Consider the problem of determining the Spin\(^c\) structures on \( M \) which restrict from Spin\(^c\) structures on \( V_1 \). If we use the trivialization of \( TM \) coming from considering \( M = \partial V_1 \), this then amounts to determining the image of the restriction map \( [V_1, BSpin(2)] \rightarrow [M, BSpin(2)] \) which by the Brown Representation Theorem is equivalent to the image of the map \( H^2(V_1,\mathbb{Z}) \rightarrow H^2(M,\mathbb{Z}) \). Via Poincaré duality this map is equivalent to \( H_1(V_2,\mathbb{Z}) \rightarrow H_1(M,\mathbb{Z}) \), whose image is the kernel of the map \( H_1(M,\mathbb{Z}) \rightarrow H_1(V_1,\mathbb{Z}) \). In other words, we have the hyperbolic splitting \( H_1(M,\mathbb{Z}) \cong H_1(V_1,\mathbb{Z}) \oplus H_1(V_2,\mathbb{Z}) \), and the Spin\(^c\)-structures on \( M \) that extend to \( V_1 \) correspond to the subgroup \( H_1(V_2,\mathbb{Z}) \). Similarly, the Spin\(^c\)-structures on \( M \) which extend to \( V_2 \) correspond to the subgroup \( H_1(V_1,\mathbb{Z}) \).

Consider the \( \bar{\mu}(M) \)-test. We are considering the image of the map \( [V_1, B\mathbb{Z}_2] \rightarrow [M, B\mathbb{Z}_2] \), which is equivalent to the map \( H^1(V_1,\mathbb{Z}_2) \rightarrow H^1(M,\mathbb{Z}_2) \). The result is analogous, except here we use the splitting \( H^1(M,\mathbb{Z}_2) \cong H^1(V_1,\mathbb{Z}_2) \oplus H^1(V_2,\mathbb{Z}_2) \).

**Corollary 2.21** has a stronger statement, as the zeros in \( \bar{d} \) and \( \bar{\mu} \) have the shape of an “affine \( X \)” in directions specified by the hyperbolic splitting of the torsion linking form.

Perhaps the simplest way to compute the Rochlin vector \( \bar{\mu}(M) \) follows this procedure:

- Find a surgery presentation for \( M \). For hyperbolic 3-manifolds see §7. Graph manifolds in essence have canonical surgery presentations given by their definition, this is also sketched in §7.
- Using inverse “slam-dunk” moves (see figure 5.30 of [21]), find an integral surgery presentation for \( M \).
- Enumerate the Spin-structures on \( M \) via characteristic sublinks (see Proposition 5.7.11 of [21]).
- Use the Kaplan algorithm to find a Spin 4-manifold bounding the Spin 3-manifold specified by a characteristic sublink (Theorem 5.7.14 of [21]).
- From the surgery presentation, the signature is readily computed via basic linear algebra.

The reader will notice that the only obstructions to a 3-manifold embedding in \( S^4 \) that we have mentioned are obstructions to embedding in homology 4-spheres. Theorems 2.16 and 2.7 completely describe, for a very limited class of 3-manifolds, precisely which manifolds from that class admit embeddings in \( S^4 \). Namely, for connect-sums of two lens spaces, and for circle bundles over surfaces there is the curious phenomenon that these 3-manifolds embed in \( S^4 \) if and only if they embed in a homology 4-sphere.

Recently, Issa and McCoy [25] have made progress applying Donaldson’s Theorem to obstructing embeddings of 3-manifolds into \( S^4 \). Specifically, we are referring to the theorem that states that compact, oriented, smooth 4-manifolds have diagonalizable intersection forms, and when the form is definite the diagonalization can be performed over the integers. The basic idea of the argument is that if one has a 3-manifold embedding \( M \rightarrow S^4 \), one replaces the manifold \( V_1 \cup_M V_2 = S^4 \) by \( X_M \cup_M V_2 \), where \( X_M \) is an inspired choice. Issa and McCoy’s techniques work when one can find a 4-manifold where \( X_M \) has a definite intersection pairing. They then study the induced map \( H^2 X_M \rightarrow H^2(X_M \cup_M V_2) \). Donaldson’s theorem
characterizes the geometry of the target, thus if one knows enough about $X_M$ one can obstruct such maps. They take this argument quite far, using both sides ($V_1$ and $V_2$) of the splitting to generate obstructions.

As a warning to the reader, this paper is not exhaustive in its usage of known obstructions to 3-manifolds embedding in $S^4$. Known obstructions to 3-manifolds embedding in homology spheres that have not been employed (yet) include: the Casson-Gordon invariants and their relatives [19], and the w-invariant [44].

3. Manifolds from the census which embed smoothly in $S^4$

In the list below, an attempt was made to give all the manifolds a more-or-less standard name. The Seifert-fibered data is all un-normalized. This means (among other things) that if you sum up all the fiber-data numbers, you get the Euler characteristic of the Seifert bundle over the base orbifold, see Orlik for details [41].

* Spherical manifolds *

1. $S^3, S^3$ is the equator in $S^4$.
2. $SFS\left(\frac{1}{2}\right) SFS\left(\frac{1}{2}\right)$, where $\frac{1}{2}$ is the quaternion group of order 8, i.e. $Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \}$. $S^2/Q_8$ appears as the boundary of a tubular neighborhood of any embedding $\mathbb{RP}^2 \to S^4$ [22]. A standard embedding of $\mathbb{RP}^2$ in $\mathbb{R}^4$ is given by $(x,y,z) \mapsto (xy,xz,y^2-z^2,2yz)$ where we think of $S^2 \subset \mathbb{R}^3$ as the universal cover of $\mathbb{RP}^2$.

* The $\mathbb{R} \times S^2$ manifold *

3. $S^1 \times S^2$. $H_1 = \mathbb{Z}$. Trivial deform-spun embedding (Construction 2.12), also 0-surgery on unknot (Construction 2.8).

* Nil manifolds *

4. $SFS\left[S^1 \times S^1 : 1\right] = \left(S^1 \times S^1\right) \rtimes \left(\begin{array}{c}1 \\ 0 \end{array}\right) \left(S^1 \times S^1\right)$. $H_1 = \mathbb{Z}^2$.

One obtains this manifold as a zero surgery on the link $\langle R : 5\rangle$ [12].

5. $SFS\left[S^1 \times S^1 : 4\right] = \left(S^1 \times S^1\right) \rtimes \left(\begin{array}{c}1 \\ 0 \end{array}\right) \left(S^1 \times S^1\right)$. $H_1 = \mathbb{Z} \oplus \mathbb{Z}^2$.

This manifold is obtained by zero surgery on the link $\langle R : 9\rangle$. Alternatively, it is the unit normal bundle to an embedding of the Klein bottle in $S^4$ [12].

6. $SFS\left[S^1 \times S^1 : -\frac{3}{2}\right] = \mathbb{Z}^2$. This manifold is obtained as the 0-surgery on the $(2,6)$-torus link which is a disjoint union of two unknots [12] (Construction 2.8).

7. $SFS\left[\mathbb{RP}^2 : \frac{1}{2}\right] = \mathbb{Z}^2$. This manifold is obtained as zero surgery on the link $\langle R : 8\rangle$ [12] (Construction 2.8).

* Euclidean manifolds *

8. $S^1 \times S^1 \times S^1$. $H_1 = \mathbb{Z}^3$. Trivial deform-spun embedding (Construction 2.12), also 0-surgery on Borromean rings (Construction 2.8).

9. $(S^1 \times S^1) \times \mathbb{Z} \mathbb{SO}_2$ where $\mathbb{Z} \subset \mathbb{SO}_2$ acts on $S^1 \times S^1$ by $\pi$-rotation on the square torus, so it admits a deform-spun embedding. This manifold is also $SFS\left[(S^1 \times S^1) \times \mathbb{Z} : 0\right]$, so it is the boundary of a tubular neighborhood of an embedding of the Klein bottle in $S^3$. $H_1 = \mathbb{Z} \oplus \mathbb{Z}^2$.

* Sol manifolds *

Crisp and Hillman [12] determined the Sol manifolds that embed in $S^4$. In particular, they showed that none of the Sol manifolds which fiber over $S^3$ embed in $S^4$, and of the remaining Sol manifolds, only three of them embed. Consider the Klein bottle to be $S^1 \times \mathbb{Z}^2$, where $\mathbb{Z}^2 = \{ \pm 1 \}$ acts by $-1.(z_1,z_2) = (z_1,-z_2)$. Given a matrix $A = \left(\begin{array}{cc}a & b \\ c & d\end{array}\right)$ we can describe a Sol-manifold as the union of two orientable $I$-bundles over $S^1 \times \mathbb{Z}^2$. Precisely, if we consider $S^1 \times S^1$ to be the boundary of this $I$-bundle, the gluing map $A_* : S^1 \times S^1 \to S^1 \times S^1$ is given by $A_*(z_1,z_2) = (z_1^a z_2^b, z_1^c z_2^d)$. Alternatively, these manifolds can be described as the union of two manifolds of the form $SFS\left[D^2, \frac{1}{2} \right]$ which identify the boundary with $S^1 \times S^3$ where the first coordinate indicates the fiber direction and the 2nd coordinate the “base” direction, thus such manifolds are specified by a corresponding gluing matrix $B$, which in the notation of Regina would be $B = \left(\begin{array}{cc}d & b \\ c & d-b-a \end{array}\right)$.

10. $SFS\left[D^2, \frac{1}{2} \right]$ U/m $SFS\left[D^2, \frac{1}{2} \right]$ $m = \left(\begin{array}{cc}-1 & 3 \\ 0 & -1\end{array}\right)$. $H_1 = \mathbb{Z}_4$ embeds [12] Crisp-Hillman notation: $(\frac{2}{3} \frac{3}{2})$. 0-surgery on link $\langle R : 9\rangle$ (Construction 2.8).

11. $SFS\left[D^2, \frac{1}{2} \right]$ U/m $SFS\left[D^2, \frac{1}{2} \right]$ $m = \left(\begin{array}{cc}-3 & 5 \\ -2 & 3\end{array}\right)$. $H_1 = \mathbb{Z}_4$ embeds [12] Crisp-Hillman notation: $(\frac{2}{5} \frac{3}{5})$. 0-surgery on $\langle R : 9\rangle$ (Construction 2.8).

12. $SFS\left[D^2, \frac{1}{2} \right]$ U/m $SFS\left[D^2, \frac{1}{2} \right]$ $m = \left(\begin{array}{cc}-9 & 3 \\ -4 & 9\end{array}\right)$. $H_1 = \mathbb{Z}_4$ embeds [12] Crisp-Hillman notation: $(\frac{2}{9} \frac{9}{4})$. 0-surgery on 2-component link (Construction 2.8).

* SL$_2\mathbb{R}$ (Brieskorn) homology spheres *

13. $SFS\left[S^2 \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right] = \Sigma(3,4,5)$ case (1) of Theorem 2.13.

14. $SFS\left[S^2 \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right] = \Sigma(2,5,7)$ case (2) of Theorem 2.13.

15. $SFS\left[S^2 \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right] = \Sigma(3,7,8)$ case (1) of Theorem 2.13.

16. $SFS\left[S^2 \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right] = \Sigma(2,9,11)$ case (2) of Theorem 2.13.

17. $SFS\left[S^2 \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right] = \Sigma(2,3,13)$ case (3) of Theorem 2.13.
* $\text{SL}_2 \mathbb{R}$ rational homology spheres *

(18) SFS $[\mathbb{RP}^2/\mathbb{Z}^2; 1, 2, 3, 4]$ $H_1 = \mathbb{Z}_2^2$. Proposition 1.2 from Crisp-Hillman [12].

(19) SFS $[\mathbb{RP}^2/\mathbb{Z}^2; 1, 2, 3]$ $H_1 = \mathbb{Z}_2^2$. Proposition 1.2 from Crisp-Hillman [12].

(20) SFS $[\mathbb{RP}^2/\mathbb{Z}^2; 1, 2, 3]$ $H_1 = \mathbb{Z}_2^4$. Proposition 1.2 from Crisp-Hillman [12]. This is $M_3(1; 2), S = \{(5, 2), (5, 3)\}$ in the notation of Crisp-Hillman. Formerly appeared as §6.1 in this paper.

(21) SFS $[\mathbb{S}^2; 1, 2, 3, 4, 5, 6]$ $H_1 = \mathbb{Z}_2^4$. To construct an embedding of this manifold into $\mathbb{S}^4$ notice that this manifold is obtained by surgery on a regular fiber in the manifold

$$
\text{SFS} \left[ \mathbb{S}^2; \frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right]
$$

which embeds as the unit normal bundle to the “standard” embedding of the Klein bottle in $\mathbb{S}^4$ ($W_2 = 0$). The surgery curve bounds the disk pictured below – thus the surgery can be realized as an embedded surgery.

![Surgery disc](image)

Constructing embedding of SFS $[\mathbb{S}^2; 1, 2, 3, 4, 5, 6]$ SFS $[\mathbb{S}^2; 1, 2, 3, 4, 5, 6]$, $H_1 = \mathbb{Z}_2^4$ 0-surgery on the (2, 8)-torus link, see figure A4 of Crisp and Hillman [12].

(22) SFS $[\mathbb{S}^2; 1, 2, 3, 4, 5, 6]$ $H_1 = \mathbb{Z}_2^4$. 0-surgery on the (2, 8)-torus link, see figure A4 of Crisp and Hillman [12].

(23) SFS $[\mathbb{S}^2; 1, 2, 3, 4, 5, 6]$ $H_1 = \mathbb{Z}_2^4$. $\tilde{d} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 \\ -1 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ Characteristic links $(\{c\}, \{c, e, f\}, \{d, e\}, \{d, f\})$, $\tilde{\mu} = (0, 1, 0, 0)$. Surgery diagram

A. Donald constructs an embedding in Example 2.14 [14].

(24) SFS $[\mathbb{S}^2; 1, 2, 3, 4, 5, 6]$ $H_1 = \mathbb{Z}_2^4$. Characteristic links: $(\{b\}, \{b, c, d\}, \{b, c, e\}, \{b, d, e\}), \tilde{\mu} = (0, 0, 0, 1), \tilde{d} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, surgery diagram:

Embeds via Construction 2.18, manifold is the double branch cover of $(S^3, L_{2, 3})$.

(25) SFS $[\mathbb{S}^2; 1, 2, 3, 4, 5, 6]$ $H_1 = \mathbb{Z}_2^4$. $\tilde{d} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ Characteristic links $(\{c, f\}, \{d, f\}, \{e, f\}, \{c, d, e, f\})$, $\tilde{\mu} = (1, 0, 0, 0)$. Surgery diagram

Embeds via Construction 2.18, manifold is the double branch cover of $(S^3, L_{2, 3})$.

SFS $[\mathbb{S}^2; 1, 2, 3, 4, 5, 6]$ $H_1 = \mathbb{Z}_2^4$. $\tilde{d} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Characteristic links $(\phi, \{c, d\}, \{d, e\}, \{c, e\})$, $\tilde{\mu} = (0, 0, -\frac{1}{2}, 0)$. Surgery diagram

Embeds via Construction 2.18, manifold is the double branch cover of $(S^3, L_{2, 3})$.

(26) SFS $[\mathbb{S}^2; 1, 2, 3, 4, 5, 6]$ $H_1 = \mathbb{Z}_2^4$. $\tilde{d} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Characteristic links $(\phi, \{c, d\}, \{d, e\}, \{c, e\})$, $\tilde{\mu} = (0, 0, -\frac{1}{2}, 0)$. Surgery diagram

Embeds via Construction 2.18, manifold is the double branch cover of $(S^3, L_{2, 3})$.

(27) SFS $[\mathbb{S}^2; 1, 2, 3, 4, 5, 6]$ $H_1 = \mathbb{Z}_2^4$. $\tilde{d} = \begin{pmatrix} 0 \end{pmatrix}$ Characteristic links $(\phi, \{c, d\}, \{d, e\}, \{c, e\})$, $\tilde{\mu} = (0, 0, -\frac{1}{2}, 0)$. Surgery diagram

Embeds via Construction 2.18, manifold is the double branch cover of $(S^3, L_{2, 3})$.

(28) SFS $[\mathbb{S}^2; 1, 2, 3, 4, 5, 6]$ $H_1 = \mathbb{Z}_2^4$. $\tilde{d} = \begin{pmatrix} 0 \end{pmatrix}$ Characteristic links $(\phi, \{c, d\}, \{d, e\}, \{c, e\})$, $\tilde{\mu} = (0, 0, -\frac{1}{2}, 0)$. Surgery diagram

Embeds via Construction 2.18, manifold is the double branch cover of $(S^3, L_{2, 3})$.

(29) SFS $[\mathbb{S}^2; 1, 2, 3, 4, 5, 6]$ $H_1 = \mathbb{Z}_2^4$. $\tilde{d} = \begin{pmatrix} 0 \end{pmatrix}$ Characteristic links $(\phi, \{c, d\}, \{d, e\}, \{c, e\})$, $\tilde{\mu} = (0, 0, -\frac{1}{2}, 0)$. Surgery diagram

Embeds via Construction 2.18, manifold is the double branch cover of $(S^3, L_{2, 3})$.

(30) SFS $[\mathbb{S}^2; 1, 2, 3, 4, 5, 6]$ $H_1 = \mathbb{Z}_2^4$. $\tilde{d} = \begin{pmatrix} 0 \end{pmatrix}$ Characteristic links $(\phi, \{c, d\}, \{d, e\}, \{c, e\})$, $\tilde{\mu} = (0, 0, -\frac{1}{2}, 0)$. Surgery diagram

Embeds via Construction 2.18, manifold is the double branch cover of $(S^3, L_{2, 3})$.

(31) SFS $[\mathbb{S}^2; 1, 2, 3, 4, 5, 6]$ $H_1 = \mathbb{Z}_2^4$. $\tilde{d} = \begin{pmatrix} 0 \end{pmatrix}$ Characteristic links $(\phi, \{c, d\}, \{d, e\}, \{c, e\})$, $\tilde{\mu} = (0, 0, -\frac{1}{2}, 0)$. Surgery diagram

Embeds via Construction 2.18, manifold is the double branch cover of $(S^3, L_{2, 3})$.
via Construction 2.18, manifold is the double branch cover of \((S^1, L_{3,2})\).

\[
\star \text{SL}_2\mathbb{R}\text{-manifolds with infinite } H_1 \star
\]

All three of the manifolds below admit embeddings into \(S^4\) by Lemma 3.2 of Crisp and Hillman [12].

(32) \(\text{SFS } [T : \frac{1}{2}], H_1 = \mathbb{Z}^2\).

(33) \(\text{SFS } [T : \frac{1}{3}], H_1 = \mathbb{Z}^2\).

(34) \(\text{SFS } [T : \frac{1}{4}], H_1 = \mathbb{Z}^2\).

\[
\star \text{ } H^2 \times \mathbb{R} \text{ manifolds } \star
\]

(35) \(\text{SFS } [S^2 : \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{11}{2}], \Sigma_2 \times_{C^2} S^1, H_1 = \mathbb{Z}\). Has “deform-spun” embedding see Construction 2.12. Specifically, the genus 2 surface can be realized as a regular neighborhood of the graph \(G = \{(z_1, z_2) \in C^2 : z_1^2 \in \mathbb{R}, 0 \leq z_1^2 \leq 1, z_2 = \pm \sqrt{1 - |z_2|^2}\}\). The monodromy is given by the order 6 automorphism of \(\Sigma_2^2, (z_1, z_2) \mapsto \left(\frac{z_1^2}{z_1}, e^{i\pi}z_2\right)\).

(36) \(\text{SFS } [S^2 : \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{4}{3}], \Sigma_2 \times_{C^2} S^1, H_1 = \mathbb{Z} + \mathbb{Z}^2\). The surface is the same as the previous case, but the monodromy is given by \((z_1, z_2) \mapsto \left(\frac{z_1^2}{z_1}, e^{i\pi}z_2\right)\) which also allows us to realize the manifold via a deform-spun embedding.

(37) \(\text{SFS } [S^2 : \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{7}{2}], \Sigma_2 \times_{C^2} S^1, H_1 = \mathbb{Z}\). Consider the graph in \(S^3\) given by \(G = \{(z_1, z_2) \in C^2 : z_1^2 \in \mathbb{R}, 0 \leq z_1^2 \leq 1, z_2 = \sqrt{1 - |z_2|^2}\}\). There is a symmetry of \(S^3\) of order 10 preserving this graph \((z_1, z_2) \mapsto \left(\frac{z_1^2}{z_1}, e^{i\pi}z_2\right)\). A surface of genus 4 is the boundary of an equivariant regular neighborhood of \(G\) realizing the monodromy.

\[
\star \text{Hyperbolic manifolds } \star
\]

(38) Hyp 2.10758135 \(H_1 = \mathbb{Z}_3^2\). 0-surgery on the link \(\langle T : 10a_{114} \rangle\). Surgery presentation found via SnapPea.

(39) Hyp 2.25976713, Homology sphere. 0-surgery on \(\langle T : 7a_6 \rangle\). Surgery presentation found via SnapPea.

(40) Hyp 1.39850888, Homology sphere. 1-surgery on \(\langle R : 6i \rangle\), also known as Stevedore’s knot, which is smooth slice.

By Construction 2.9 this manifold embeds in a homotopy \(S^4\) since it bounds a contractible manifold \(N'\). Since \(6i\) a ribbon knot, we can apply Proposition 2.10 and compute the relevant presentation of \(\pi_1N'\). By the nature of the ribbon diagram above, the height function \(d\) has two local minima on the ribbon disk and one saddle point. So we have a presentation of the form \(\langle a, b : r_1, R_1 \rangle\) where \(a, b\) correspond to the local minima of \(d\) on the ribbon disk (which also correspond to the two ribbon singularities of the ribbon disk projected into \(S^3\)), \(r_1\) corresponds to the saddle, which is at the fixed point of the symmetry of the ribbon disk, and \(R_1\) to the surgery framing curve. So \(r_1\) is the relation \(a^{-1}b = b^{-1}a\) and \(R_1\) is the relation \(b = 1\). Since \(\langle a, b, a^{-1}b^{-1}a^{-1}b, b \rangle\) is trivializable by Andrews-Curtis moves, our manifold embeds smoothly in \(S^4\).

Hyp 1.91221025, Homology sphere. \((-1)\)-surgery on \(\langle R : 820 \rangle\) which is smooth slice, so by Construction 2.9 this manifold embeds in a homotopy \(S^4\). As with item 40 we have a ribbon diagram so we can apply Proposition 2.10.

This gives a similar presentation for \(\pi_1 N' = \langle a, b \rangle a^{-1}b^{-1}a^{-1}b, b \rangle\), also trivializable by Andrews-Curtis moves.

4. Manifolds which embed in homotopy 4-spheres

This is a list of manifolds that embed in homotopy 4-spheres. Likely these homotopy 4-spheres are diffeomorphic to \(S^4\) but this has not been determined.

(1) \(\text{SFS } [S^2 : \frac{1}{3}, \frac{1}{3}, -\frac{1}{3}] = \Sigma(2, 3, 25)\). Although Fickle claims [18] that Casson and Harer [8] were the first to show \(\Sigma(2, 3, 25)\) bounds a contractible manifold, his Corollary 3.3 [18] is the earliest written account that I have found.
We use the technique of Casson and Harer [8] to embed this manifold in a homotopy $S^4$. A sketch is given in §7.

5. Manifolds in the census known to not embed in $S^4$

These manifolds fiber over $S^1$ with fiber a hyperbolic surface, and monodromy an automorphism of finite order. Regina stores these manifolds via their Seifert surface, and monodromy an automorphism of finite order. Regina stores these manifolds in $S^4$. A sketch is given in §7.

* Hyperbolic manifolds *

Hyp 1.26370924 $H_1 = \mathbb{Z}_2^2$. $(-5, -5)$-surgery on $(\ell : 5a_1)$ found via SnapPea. $\mu = 0$, computed via the formulae in §4.2.3 of [46].

We use the technique of Casson and Harer [8] to embed this manifold in a homotopy $S^4$. A sketch is given in §7.

* Compound manifolds *

(3) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{2}, \frac{3}{2}]$ $m = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Homology sphere.

Found as a vertex-normal 3-manifold in a 10-pentachoron triangulated homotopy $S^4$-sphere with “isomorphism signature” isoSig(): KLLLQQAQCDHIFJGJGJIII5PAYABAAAPAAaaaadalPCcBCABCYBCB.

(4) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{2}, \frac{3}{2}]$ $m = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $H_1 = \mathbb{Z}_2^2$.

Found as a vertex-normal 3-manifold in a 6-pentachoron triangulated homotopy $S^4$-sphere with “isomorphism signature” isoSig(): gllQQQCDBDEFF4AAYAYGBAAEAAK.

5. Manifolds in the census known to not embed in $S^4$

(1) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}] = \text{SFS}[\mathbb{RP}^2 : 6] = S^3/Q_{23}$. Crisp-Hillman [12].

(2) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}] = S^3/Q_{40}$. $H_1 = \mathbb{Z}_2^2$. Crisp-Hillman [12].

(3) SFS $[S^2 : \frac{1}{2}, \frac{3}{2}, -\frac{4}{2}] = (S^1 \times S^1) \times_{\mathbb{Z}_2} S^1$. Crisp-Hillman [12].

(4) $(S^1 \times S^1) \rtimes A(1, 1) S^1$. $H_1 = \mathbb{Z}$. Crisp-Hillman [12].

(5) SFS $[\mathbb{RP}^2/n2 : \frac{1}{2}, \frac{1}{2}]$. $H_1 = \mathbb{Z}_2^4$. Nil-manifold. Crisp-Hillman [12].

(6) $(S^1 \times S^1) \rtimes A(1, 3) S^1$. $H_1 = \mathbb{Z} \oplus \mathbb{Z}_2^3$. Crisp-Hillman [12].

(7) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{2}, \frac{1}{2}]$ $(-\frac{7}{3}, \frac{5}{3})$ $H_1 = \mathbb{Z}_2^4$

Sol manifold. Crisp-Hillman [12].

* $H^2$-fiber geometry *

These manifolds fiber over $S^1$ with fiber a hyperbolic surface, and monodromy an automorphism of finite order. Regina stores these manifolds via their Seifert data, see the item on computing the monodromy from the Seifert data for details on how we compute the Alexander polynomials of these manifolds in §7.

(8) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, -\frac{7}{3}] = \Sigma_2 \rtimes_{\mathbb{Z}_2^2} S^1$. $H_1 = \mathbb{Z}$. $\Delta = t^2 - t + 1$.

(9) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, -\frac{11}{10}] = \Sigma_2 \rtimes_{\mathbb{Z}_2^2} S^1$. $H_1 = \mathbb{Z}$. $\Delta = t^3 - t^2 + t - 1$.

(10) SFS $[S^2 : \frac{1}{2}, \frac{3}{2}, -\frac{7}{3}] = \Sigma_3 \rtimes_{\mathbb{Z}_2^3} S^1$. $H_1 = \mathbb{Z}$. $\Delta = t^3 - t^2 - 1$.

(11) SFS $[S^2 : \frac{3}{2}, \frac{3}{2}, \frac{11}{13}] = \Sigma_4 \rtimes_{\mathbb{Z}_2^4} S^1$. $H_1 = \mathbb{Z}$. $\Delta = t^4 - t^3 + t - 1$.

In a recent preprint, Jonathan Hillman [24] proves that $H^2 \times \mathbb{R}$ manifolds that fiber over $S^2$ must have an even number of singular fibers, generalizing items 8–11. He also uses the Alexander module as an obstruction.

* Homology spheres with non-zero Rohlin invariant *

These do not embed because they do not satisfy the Rohlin invariant test. See Theorem 2.19. The $\mu$ invariant was computed using formula 2.4.2 in Saveliev’s text [44].

(12) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, -\frac{9}{10}]$. $S^3/P_{120}$ Poincaré Dodecahedral Space. $\mu = -1$.

(13) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, -\frac{9}{10}]$. Brieskorn homology sphere. $\mu = 1$, $d = 0$.

(14) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, -\frac{19}{20}]$. Brieskorn homology sphere. $\mu = -1$, $d = 2$.

(15) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, -\frac{19}{20}]$. Brieskorn homology sphere. $\mu = 1$, $d = 0$.

(16) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, -\frac{19}{20}]$. Brieskorn homology sphere. $\mu = -1$, $d = 2$.

(17) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, -\frac{19}{20}]$. Brieskorn homology sphere. $\mu = 1$, $d = 0$.

(18) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, -\frac{19}{20}]$. Brieskorn homology sphere. $\mu = -1$, $d = 2$.

(19) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, -\frac{19}{20}]$. Brieskorn homology sphere. $\mu = 1$, $d = 0$.

These manifolds fail the $d$-invariant test, see Theorem 2.19.

(20) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}]$, $\mu = 0$, $d = 2$.

(21) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}]$, $\mu = 0$, $d = 2$.

(22) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}]$, $\mu = 0$, $d = 2$.

(23) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}]$, $\mu = 0$, $d = 2$.

(24) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}]$, $\mu = 0$, $d = 2$.

(25) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}]$, $\mu = 0$, $d = 2$.

(26) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}]$, $\mu = -2$, $d = 2$.

* Rational homology spheres which do not satisfy the $d\bar{d}$ test *

(27) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, -\frac{5}{6}]$, $\bar{d} = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 4/3 & 2/3 \end{pmatrix}$.

Corollary 2.21.

(28) SFS $[S^2 : \frac{1}{2}, \frac{1}{2}, -\frac{5}{6}]$, $\bar{d} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 4/3 & 2/3 \end{pmatrix}$.

Corollary 2.21.
(29) \( \text{SFS} \left[ S^2 : \frac{1}{3}, \frac{1}{2}, \frac{3}{2} \right] \quad H_1 = \mathbb{Z}_2^2, \quad \tilde{d} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2/3 & 2/3 & 2/3 \\ 0 & 2/3 & 2/3 & 2/3 \end{pmatrix} \) see Corollary 2.21.

(30) \( \text{SFS} \left[ S^2 : \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right] \quad H_1 = \mathbb{Z}_2^3, \quad \tilde{d} = \begin{pmatrix} 2/3 & 2/3 & 2/3 \\ 0 & 2/3 & 2/3 \\ 0 & 0 & 0 \end{pmatrix} \) see Corollary 2.21.

(31) \( \text{SFS} \left[ S^2 : \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right] \quad H_1 = \mathbb{Z}_2^3, \quad \tilde{d} = \begin{pmatrix} 2/3 & 2/3 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) see Corollary 2.21.

(32) \( \text{SFS} \left[ S^2 : \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right] \quad H_1 = \mathbb{Z}_2^3, \quad \tilde{d} = \begin{pmatrix} 2/3 & 2/3 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) see Corollary 2.21.

Rational homology spheres that do not satisfy that \( \tilde{\mu} \)-test

(33) \( \text{SFS} \left[ S^2 : \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right] \quad H_1 = \mathbb{Z}_2^3. \) Characteristic link: \((a, b, c), (a, b, d), (e, c, d, e), \) \( \tilde{\mu} = \left( 0, \frac{1}{2}, -\frac{1}{2}, 0 \right), \)

\( \tilde{d} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \) surgery diagram: \( \begin{array}{ccc} 3a & 0b \\ -1c & 3d & 3e \end{array} \)

(34) \( \text{SFS} \left[ S^2 : \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right] \quad H_1 = \mathbb{Z}_2^3. \) Characteristic link \( (a, d, e), \mu = -\frac{1}{2}, \)

\( \tilde{\mu} = \left( 0, \frac{2}{3}, 0, \frac{2}{3} \right), \) surgery diagram: \( \begin{array}{ccc} 3a & 0b \\ -1c & 5d & 6c & 2d \end{array} \)

(35) \( \text{SFS} \left[ S^2 : \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right] \quad H_1 = \mathbb{Z}_2^3. \) Characteristic link \( (c, e, f), \mu = -\frac{1}{2}, \)

\( \tilde{\mu} = \left( 0, \frac{2}{3}, 0, \frac{2}{3} \right), \) surgery diagram: \( \begin{array}{ccc} 3a & 0b \\ -1c & 5d & 6c & 2d \end{array} \)

(36) \( \text{SFS} \left[ S^2 : \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right] \quad H_1 = \mathbb{Z}_2^3, \) Characteristic link \( \{a, b, f, g\}, \mu = 1, \)

\( \tilde{\mu} = \left( 0, \frac{2}{3}, 0, \frac{2}{3} \right), \) surgery diagram: \( \begin{array}{ccc} 3a & 2b & 0c \\ -2d & -1e & 3f & 3g \end{array} \)

Other rational homology spheres

(37) \( \text{SFS} \left[ \mathbb{R}P^2 / \mathbb{Z}^2 : \frac{1}{2}, \frac{1}{2} \right] \quad H_1 = \mathbb{Z}_2^3, \) Crisp-Hillman Proposition 1.2 [12].

Hyp 1.73198278, Homology sphere, \( \mu = 1. \) +1-surgery on \( \{ a : 4 \} \) (found via SnapPea). \( \mu \) is computed using the surgery formula (Theorem 2.8 of [44]).

\( \frac{1}{3} \)

Manifolds with non-trivial JSJ-decompositions

These manifolds are all of the form \( \text{SFS}[A : \frac{a}{b}] \langle \frac{c}{d} \rangle \) where \( ad - bc = -1. \) These manifolds have \( b_1 = 1 \) if and only if the polynomial \( \beta t^2 + ((d-a)\beta - bx)t + \beta \) does not have a zero at \( t = 1, \) moreover, if \( b_1 = 1, \) this polynomial is the Alexander polynomial of the corresponding covering space. Checking that this polynomial has the form \( rp(t)p(r^{-1}) \) where \( p(t) \) is a rational Laurent polynomial and \( r \) is rational amounts to determining if the number \( ((\tilde{a} - d) + \frac{d}{r^2}) - 4 \) is a rational squared. These five manifolds do not embed since their Alexander polynomials do not satisfy the Kawauchi condition. See Theorem 2.4.

(38) \( \text{SFS} \left[ A : \frac{1}{2} \right] \langle \frac{0}{1} \rangle \quad H_1 = \mathbb{Z}. \)

(39) \( \text{SFS} \left[ A : \frac{1}{2} \right] \langle \frac{0}{1} \rangle \quad H_1 = \mathbb{Z}. \)

(40) \( \text{SFS} \left[ A : \frac{2}{3} \right] \langle \frac{0}{1} \rangle \quad H_1 = \mathbb{Z}. \)

(41) \( \text{SFS} \left[ A : \frac{2}{3} \right] \langle \frac{0}{1} \rangle \quad H_1 = \mathbb{Z}. \)

(42) \( \text{SFS} \left[ A : \frac{2}{3} \right] \langle \frac{0}{1} \rangle \quad H_1 = \mathbb{Z}. \)

(43) \( \text{SFS} \left[ A : \frac{2}{3} \right] \langle \frac{0}{1} \rangle \quad H_1 = \mathbb{Z}. \)

\( \tilde{\mu} \) is computed for the examples below using the splicing additivity formula for \( \tilde{\mu} \), Proposition 2.16 from [44].

(44) \( \text{SFS} \left[ D : \frac{1}{2}, \frac{1}{2} \right] \quad \text{U/m SFS} \left[ D : \frac{1}{2}, \frac{1}{2} \right] \quad m = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \) Homology sphere. \( \Sigma(2, 3, 5) \bowtie \Sigma(2, 5, 7), \) \( \tilde{\mu} = -1. \)

(45) \( \text{SFS} \left[ D : \frac{1}{2}, \frac{1}{2} \right] \quad \text{U/m SFS} \left[ D : \frac{1}{2}, \frac{1}{2} \right] \quad m = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \) Homology sphere. \( \Sigma(2, 3, 5) \bowtie \Sigma(2, 5, 7), \) \( \tilde{\mu} = 1. \)

(46) \( \text{SFS} \left[ D : \frac{1}{2}, \frac{1}{2} \right] \quad \text{U/m SFS} \left[ D : \frac{1}{2}, \frac{1}{2} \right] \quad m = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \) Homology sphere. \( \Sigma(2, 7, 11) \bowtie \Sigma(2, 3, 19), \) IssaMcCoy obstruction [25]. Formed as space \$6.12.

(47) \( \text{SFS} \left[ D : \frac{1}{2}, \frac{1}{2} \right] \quad \text{U/m SFS} \left[ D : \frac{1}{2}, \frac{1}{2} \right] \quad m = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \) Homology sphere. \( \Sigma(3, 5, 7) \bowtie \Sigma(2, 3, 13), \) Issa-McCoy obstruction [25]. Formed as space \$6.13.

(48) \( \text{SFS} \left[ D : \frac{1}{2}, \frac{1}{2} \right] \quad \text{U/m SFS} \left[ D : \frac{1}{2}, \frac{1}{2} \right] \quad m = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \) Homology sphere. \( \Sigma(2, 5, 11) \bowtie \Sigma(3, 4, 11), \) IssaMcCoy obstruction [25]. Formed as space \$6.15.

In the next few examples we need to compute the Alexander polynomials of some graph manifolds. The underlying Seifert-fibered manifolds are all of the type \( \text{SFS} \left[ D : \frac{1}{2}, \frac{1}{2} \right] \). An elementary computation shows that
Lemma 5.1. Consider a manifold $M \cup_T N$ which is the union of two submanifolds $M$ and $N$ along a common boundary torus $T$. Assume $M \cup_T N$ is a rational homology $S^1 \times S^2$, and both $M$ and $N$ are rational homology $S^1 \times D^2$ manifolds.

$$\Delta_{M \cup_T N}(t) = \Delta_M(F) \Delta_N(t^2)(t - 1)^2$$

where $\text{coker}(H_1(M \cup_T N)) = \mathbb{Z}_p$ and $\text{coker}(H_1(N \rightarrow fH_1(M \cup_T N))) = \mathbb{Z}_q$. $p$ and $q$ have a simpler computation since $\text{coker}(H_1 T \rightarrow fH_1 M) = \mathbb{Z}_q$ and $\text{coker}(H_1 T \rightarrow fH_1 N) = \mathbb{Z}_p$. Moreover,

$$\Delta_{SFS}[D : \frac{a}{b'}, \frac{c}{d}] = \frac{(\text{LCM}(b, d) - 1)(t - 1)}{(t^p - 1)(t^q - 1)}$$

with $b' = \frac{b}{\text{GCD}(b, d)}$, $d' = \frac{d}{\text{GCD}(d, b)}$.

The relevant non-embedding result is Theorem 2.4.

(49) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{3}, \frac{1}{3}]$ $m = \left(\begin{array}{c} 1 \\ 3 \\ 4 \end{array}\right)$ $H_1 = \mathbb{Z}$. $\Delta(t) = t^4 - t^2 + 1$.

(50) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{3}, \frac{1}{3}]$ $m = \left(\begin{array}{c} 2 \\ 3 \\ 1 \end{array}\right)$ $H_1 = \mathbb{Z}$. $\Delta(t) = t^4 - t^2 + 1$.

(51) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{3}, \frac{2}{3}]$ $m = \left(\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right)$ $H_1 = \mathbb{Z}$. $\Delta = (t^4 - t^2 + 1)(t^4 - t^3 + t^2 - t + 1)$.

* Fibers over $S^1$ with reducible monodromy *

(52) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{3}, \frac{2}{3}]$ $m = \left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right)$ $\Sigma_4 \times S^1, H_1 = \mathbb{Z}$. The monodromy is reducible with reduction system a union of 5 circles separating $\Sigma_4$ into two 5-punctures spheres. Perhaps the easiest way to describe the monodromy is that it differs from the monodromy of item 37 §3 by a single Dehn twist about a reduction curve. The Alexander polynomial for this manifold is the same as item 37 §3, so it does not provide an obstruction to embedding. Alternatively, the monodromy extends over a handlebody thus this manifolds bounds a genus 4 handlebody bundle over $S^1$ which must be a homology $S^1 \times D^3$. The obstruction to embedding is a variant of the Crisp-Hillman Theorem 2.7. If this manifold embeds in $S^4 = V_1 \cup_m V_2$, then $V_1$ is a homology $S^1 \times D^3$ and $V_2$ is a homology $S^2 \times D^2$. Replace $V_1$ with $V'_1$, the corresponding handlebody bundle over $S^4$, $W = V'_1 \cup_m V_2$ is therefore also a homology $S^4$, but it contains a Klein bottle with normal Euler class $\pm 2$, contradicting Theorem 2.7.

(53) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{3}, \frac{3}{3}]$ $m = \left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right)$ $\Sigma_2 \times S^1, H_1 = \mathbb{Z}$. The monodromy is reducible, the reduction system of 3 curves separates the genus 2 surface into two 3-punctures spheres. The monodromy differs from the monodromy of item 35 §3 by a single Dehn twist about a reduction curve. Again the Alexander polynomial is the same as in item 35 §3 so it is no obstruction to embedding. This does not embed for essentially the same reason as the previous example, only in this case we use the appropriate genus 2 handlebody bundle over $S^1$.

(54) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{2}{3}, \frac{3}{3}]$ $m = \left(\begin{array}{c} -2 \\ -1 \\ 2 \end{array}\right)$ $\Sigma_2 \times S^1, H_1 = \mathbb{Z}$. The monodromy is reducible with a reduction system of 3 curves separating the surface into two pairs of pants. The monodromy differs from the monodromy of item 35 §3 by the cube of a Dehn twist along one of the reduction curves. Thus the manifold bounds a handlebody bundle over $S^1$. Notice this bundle contains a Klein bottle with normal Euler class $W_1 = \pm 6$, which does not embed in a homology $S^4$ by Theorem 2.7. $\Delta(t) = (t^2 - t + 1)^2$.

(55) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{3}, \frac{3}{3}]$ $m = \left(\begin{array}{c} -1 \\ 2 \\ 0 \end{array}\right)$ $\Sigma_2 \times S^1, H_1 = \mathbb{Z} \oplus \mathbb{Z}_3^2$. The monodromy is reducible with reduction system 4 curves separating the surface into two 4-punctured spheres. Like the previous examples, this bundle bounds a handlebody bundle over $S^1$, which in this case contains a Klein bottle with normal Euler class $\pm 2$, and so this 3-manifold does not embed in $S^4$ by Theorem 2.7. $\Delta = (t^2 + 1)^3$.

* Compound rational homology spheres *

These manifolds are primarily the union of two Seifert-fibered manifolds that fiber over a disk, with at most 3 singular fibers. We compute the $\bar{\mu}$-invariant via the Kaplan algorithm (see Theorem 5.7.14 of [21]). We do not compute the $\bar{d}$-invariant as at present there is no simple way to compute $\bar{a}$ for these manifolds. To apply the Kaplan algorithm we need an integral surgery diagram to start with. There is a rather simple way to construct surgery presentations for these manifolds, see §7.

(56) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{3}, \frac{1}{3}]$ $m = \left(\begin{array}{c} 3 \\ 3 \\ 3 \\ 3 \end{array}\right)$ $H_1 = \mathbb{Z}$. We follow the techniques of §7 to construct a surgery presentation for this manifold. If we label the components of the surgery link left-to-right we get

```
1 a 1 b 1 c -2 d -2 e
-3 f -2 g 1 h -1 i
```

as the graph of framing/linking numbers. The four Spin-structures on this manifold correspond to the characteristic links, which are given by
We apply Kaplan’s algorithm to construct surgery presentations of the Spin 4-manifolds bounding each of these Spin 3-manifolds, which will allow us to compute \( \bar{\mu} \). In the order the characteristic links are listed for the above case, \( \bar{\mu} = (\frac{1}{2}, 1, 0, 0) \). So this fails the Rochlin vector test (Corollary 2.21).

(57) SFS [\( D : \frac{1}{2}, \frac{3}{2} \)] U/m SFS [\( D : \frac{1}{2}, \frac{3}{2} \)] \( m = (\frac{3}{2}, \frac{1}{2}) \) \( H_1 = \mathbb{Z}^2 \).

Characteristic links \( \{f, g, h, i\}, \{d, e, f, g, h, i\}, \{a, b, c, d, f, h\}, \{a, b, c, e, f, h\} \), \( \bar{\mu} = (\frac{3}{8}, -\frac{3}{8}, -\frac{7}{8}, -\frac{3}{8}) \). Surgery presentation framing/linking matrix for item 57.

(58) SFS [\( D : \frac{1}{2}, \frac{3}{2} \)] U/m SFS [\( D : \frac{1}{2}, \frac{3}{2} \)] \( m = (\frac{3}{2}, \frac{1}{2}) \) \( H_1 = \mathbb{Z}^2 \).

Characteristic links \( \{a, b, c, d, e\}, \{a, c, d, e\}, \{e, f, g, h\}, \{b, c, e, f, g, h\} \), \( \bar{\mu} = (1, \frac{1}{2}, 1, 1) \).

Surgery diagram:

(59) SFS [\( D : \frac{1}{2}, \frac{3}{2} \)] U/m SFS [\( D : \frac{1}{2}, \frac{3}{2} \)] \( m = (\frac{3}{2}, \frac{1}{2}) \) \( H_1 = \mathbb{Z}^2 \).

Characteristic links \( \{g, h, i\}, \{e, f, g, h, i\}, \{b, c, d, e, g, i\}, \{b, c, d, f, g, i\} \), \( \bar{\mu} = (1, \frac{1}{2}, 1, \frac{1}{2}) \).

Surgery diagram:

(60) SFS [\( D : \frac{1}{2}, \frac{3}{2} \)] U/m SFS [\( D : \frac{1}{2}, \frac{3}{2} \)] \( m = (\frac{3}{2}, \frac{1}{2}) \) \( H_1 = \mathbb{Z}^2 \).

Characteristic links \( \{c, d\}, \{c, e\}, \{a, b, c, d\}, \{a, b, c, e\} \), \( \bar{\mu} = (0, 0, 1, 1) \).

Surgery diagram:

(61) SFS [\( D : \frac{1}{2}, \frac{3}{2} \)] U/m SFS [\( D : \frac{1}{2}, \frac{3}{2} \)] \( m = (\frac{3}{2}, \frac{1}{2}) \) \( H_1 = \mathbb{Z}^2 \).

Characteristic links \( \{d, e, f\}, \{c, e, f\}, \{a, b, d, e, f\}, \{a, b, c, e, f\}\), \( \bar{\mu} = (0, 0, \frac{3}{8}, \frac{3}{8}) \).

Surgery diagram:

(62) SFS [\( D : \frac{1}{2}, \frac{3}{2} \)] U/m SFS [\( D : \frac{1}{2}, \frac{3}{2} \)] \( m = (\frac{3}{2}, \frac{1}{2}) \) \( H_1 = \mathbb{Z}^2 \).

Characteristic links \( \{b, c, d, e, f, g\}, \{a, b, c, e, f, g\}, \{a, c, d, g, h\} \), \( \bar{\mu} = (\frac{1}{2}, 1, 0, 0) \).

Surgery diagram:

6. Manifolds for which embeddability is not known

* \( SL_2 \mathbb{R} \)-manifolds with finite \( H_1 \) *

(1) SFS [\( \mathbb{R}P^2 / n2 : \frac{1}{3}, \frac{2}{3} \)] \( H_1 = \mathbb{Z}_n^2 \), \( \bar{d} \) not computed.

Characteristic links \( \{a, c, d\}, \{b, c, d\}, \{e, f\}, \{a, b, e, f\}\), \( \bar{\mu} = (\frac{1}{2}, 0, 0) \).

Surgery diagram:
Hyperbolic manifolds

These manifolds are uniquely identified in Burton’s census [6] by their volumes. The Rochlin invariant is given from a surgery presentation via Theorem 2.13 [44]. See §7 for notes on how surgery presentations are found. The Rochlin invariant is computed as described in §2. A brief description of the calculation is given below. See Theorem 2.19.

(2) Hyp 1.96273766 $H_1 = \mathbb{Z}^2$. Initial surgery presentation on $\langle R : 8^g_6 \rangle$ found via SnapPea. The first reduction eliminates the unknotted component with framing number $-1/3$ via a Rolfsen twist on that component. A second move creates integral surgery via slam-dunk move on component with framing number $\frac{14}{3}$.

![Diagram](image)

To which we apply the Kaplan algorithm to get the presentation:

The graph consists of the framing/linking numbers. The characteristic polynomial of the intersection product is $t^4 - 14t^3 - 16t^2 + 49$, thus the signature is zero and $\mu = 0$.

(3) Hyp 2.22671790, Homology sphere. $\mu = 0$. $+\frac{1}{6}$-surgery on $\langle R : 5^g_2 \rangle$ found via SnapPea. $\mu$ computed via Theorem 2.10 in [44].

![Diagram](image)

These manifolds are all of the form $SFS[A : \frac{2}{3}]/\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$ where $ad - bc = -1$. These manifolds have rank$(H_1) = 1$ if and only if the polynomial $bt^2 + ((d - a)\beta - b\alpha)t + \beta$ does not have 1 as a root, moreover this is the Alexander polynomial in this case. Thus the three manifolds below all have Alexander polynomial $\Delta = 2t^2 - 5t + 2$, which satisfies Kawauchi’s Theorem 2.4. Unfortunately, $2t^2 - 5t + 2 = (2t - 1)(t - 2)$ so all signature invariants are zero for these manifolds.

(4) $SFS[A : \frac{1}{6}]/\left(\begin{array}{cc} 0 & 1 \\ 1 & -2 \end{array}\right)$ $H_1 = \mathbb{Z}$.

(5) $SFS[A : \frac{1}{2}]/\left(\begin{array}{cc} 2 & 5 \\ 1 & 2 \end{array}\right)$ $H_1 = \mathbb{Z}$.

(6) $SFS[A : \frac{1}{2}]/\left(\begin{array}{cc} -1 & 2 \\ 1 & -2 \end{array}\right)$ $H_1 = \mathbb{Z}$.

(7) $SFS[A : \frac{1}{2}]/\left(\begin{array}{cc} -1 & \frac{2}{3} \\ -1 & \frac{2}{3} \end{array}\right)$ $H_1 = \mathbb{Z} \oplus \mathbb{Z}_2$. $\Delta = 2t^2 + 5t + 2 = (2t + 1)(t + 2)$ also satisfies Theorem 2.4 and has trivial signature invariants.

(8) $SFS[A : \frac{1}{3}]/\left(\begin{array}{cc} -1 & 3 \\ 1 & -2 \end{array}\right)$ $H_1 = \mathbb{Z}^2$. No tests have been performed for this manifold.

* Compound homology spheres *

Their splicing decomposition is listed and $\bar{\mu}$ is computed using splicing additivity (Proposition 2.16 in [44]).

(9) $SFS[D : \frac{1}{7}, \frac{2}{5}]$ U/m $SFS[D : \frac{1}{7}, \frac{2}{5}]$ $m = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)$ Homology sphere.

(10) $SFS[D : \frac{1}{2}, \frac{2}{5}]$ U/m $SFS[D : \frac{1}{2}, \frac{2}{5}]$ $m = \left(\begin{array}{cc} -2 & 3 \\ -1 & 2 \end{array}\right)$ Homology sphere.

(11) $SFS[D : \frac{1}{2}, \frac{2}{5}]$ U/m $SFS[D : \frac{1}{2}, \frac{2}{5}]$ $m = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)$ Homology sphere.

* Graph manifold with single non-separating torus in [5] *
(12) SFS $[D : \frac{1}{2}, \frac{3}{4}]$ U/m SFS $[A : \frac{1}{2}]$ U/n SFS $[D : \frac{1}{2}, \frac{3}{4}]$

$m = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, n = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ Homology sphere. $\Sigma(2, 3, 11) \cong S^3(L) \cong \Sigma(2, 3, 11)$ where this indicates splicing over the link $L$ in $S^3$ which is the union of two regular fibers in the “(2, 1)-fibering” of $S^3$. $\bar{\mu} = 0$

* Compound rational homology spheres *

See §5 for details on how the Rochlin vector $\bar{\mu}$ is computed for these manifolds.

(13) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{2}, \frac{1}{2}]$ $m = \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}$ $H_1 = \mathbb{Z}_2^2$.

Characteristic links $(\{i,j\}, \{e,f,i,j\}, \{b,c,d,e,i,h\}, \{b, c,d,f,g,i\})$, $\bar{\mu} = (0, \frac{1}{2}, 0, 0)$. This manifold can be thought of as the $(5, 2)$-torus knot complement union the orientable $S^1$-bundle over a Möbius band. There is a natural embedding of such a manifold into $S^4$, since the $(5, 2)$-torus knot bounds a Klein bottle, and the orientable $I$-bundle over the Klein bottle is the $S^1$-bundle over the Möbius band. The above gluing map does not produce this manifold.

Surgery diagram:

(14) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{3}{2}, \frac{1}{2}]$ $m = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$ $H_1 = \mathbb{Z}_2^2$.

Characteristic links $(\{a,b,c\}, \{a,b,c,d,e\}, \{a, c,d,g, h\}, \{a,c,e,g,h\})$, $\bar{\mu} = (0, 0, 0, 0)$.

Surgery diagram:

(15) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{2}, \frac{3}{2}]$ $m = \begin{pmatrix} -2 & 5 \\ -3 & 3 \end{pmatrix}$ $H_1 = \mathbb{Z}_6^2$.

Characteristic links $(\{b, c, d, e, f, g, h\}, \{g, h, i\}, \{g, h, j\}, \{b, c, d, e, f, g, h, i, j\})$, $\bar{\mu} = (0, 0, 0, 0)$.

Surgery diagram:

(16) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{2}, \frac{3}{2}]$ $m = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$ $H_1 = \mathbb{Z}_2^2$.

Characteristic links $(\{b, c, d, e, f, g, h\}, \{b, c, d, e, f, g, h\}, \{a,c,d,h, i\}, \{a,c,e,h,i\})$, $\bar{\mu} = (0, 0, 0, 0)$.

Surgery diagram:

(17) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{2}, \frac{3}{2}]$ $m = \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}$ $H_1 = \mathbb{Z}_2^2$.

Characteristic links $(\{b, c, d, e\}, \{a, c, d, f\}, \{a, i, j\}, \{a, c,d,f, i,j\})$, $\bar{\mu} = (0, 0, 0, 0)$.

Surgery diagram:

(18) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{2}, \frac{3}{2}]$ $m = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $H_1 = \mathbb{Z}_2^2$.

Characteristic links $(\phi, \{a, b, c\}, \{a, b, d\}, \{c, d\})$, $\bar{\mu} = (\frac{1}{2}, 0, 0, 0)$.

Surgery diagram:

(19) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{2}, \frac{3}{2}]$ $m = \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}$ $H_1 = \mathbb{Z}_3^2$.

Characteristic link $(b, c, e), \mu = 0$.

Surgery diagram:

(20) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{2}, \frac{3}{2}]$ $m = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $H_1 = \mathbb{Z}_3^2$.

Characteristic link $(a, c, d), \mu = 0$.

Surgery diagram:

(21) SFS $[D : \frac{1}{2}, \frac{1}{2}]$ U/m SFS $[D : \frac{1}{2}, \frac{3}{2}]$ $m = \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}$ $H_1 = \mathbb{Z}_3^2$.

Characteristic link $(b, c, d, j), \mu = 0$.

Surgery diagram:

(22) SFS $[D : \frac{1}{2}, \frac{3}{2}]$ U/m SFS $[D : \frac{1}{2}, \frac{3}{2}]$ $m = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $H_1 = \mathbb{Z}_3^2$.

Characteristic link $(b), \mu = 0$.

Surgery diagram:

(23) SFS $[D : \frac{1}{2}, \frac{3}{2}]$ U/m SFS $[D : \frac{1}{2}, \frac{3}{2}]$ $m = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$ $H_1 = \mathbb{Z}_3^2$.

Characteristic link $(a, b, d, f, g), \mu = 0$.

Surgery diagram:

(24) SFS $[D : \frac{1}{2}, \frac{3}{2}]$ U/m SFS $[D : \frac{1}{2}, \frac{3}{2}]$ $m = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $H_1 = \mathbb{Z}_3^2$.

Characteristic link $(a, b, c, d), \mu = 0$.

Surgery diagram:

(25) SFS $[D : \frac{1}{2}, \frac{3}{2}]$ U/m SFS $[D : \frac{1}{2}, \frac{3}{2}]$ $m = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ $H_1 = \mathbb{Z}_3^2$.

Characteristic links $(\{a, b, d, e, g, h\}, \{a, c, d, e, g, h\}, \{a, b, d, f, g, h\}, \{a, c, d, f, g, h\}), \bar{\mu} = (0, 0, 0, 0)$.
Characteristic links $\{\{a, c, e, g\}, \{b, c, e, g\}, \{a, c, f, g\}, \{b, c, f, g\}\}$, $\bar{\mu} = (0, 0, 0, 0)$.

Surgery diagram:

\[
\begin{array}{c}
\text{Surgery diagram:} \\
\includegraphics[width=0.5\textwidth]{surgery_diagram}\end{array}
\]

\[
\begin{array}{c}
\text{Surgery diagram:} \\
\includegraphics[width=0.5\textwidth]{surgery_diagram}\end{array}
\]

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\begin{array}{c}
\text{Surgery diagram:} \\
\includegraphics[width=0.5\textwidth]{surgery_diagram}\end{array}
\]

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\begin{array}{c}
\text{Surgery diagram:} \\
\includegraphics[width=0.5\textwidth]{surgery_diagram}\end{array}
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\begin{array}{c}
\text{Surgery diagram:} \\
\includegraphics[width=0.5\textwidth]{surgery_diagram}\end{array}
\]

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\begin{array}{c}
\text{Surgery diagram:} \\
\includegraphics[width=0.5\textwidth]{surgery_diagram}\end{array}
\]

\[
\begin{array}{c}
\text{Surgery diagram:} \\
\includegraphics[width=0.5\textwidth]{surgery_diagram}\end{array}
\]

In order to compute the following Alexander polynomials we need to extend Lemma 5.1 by:

\[
\Delta_{\text{SFS}} \left[ D : \frac{a}{b}, \frac{c}{d}, \frac{e}{f} \right] = \frac{t^{\text{LCM}(b, d, f)} - 1}{(t^{b' - 1} - 1)(t^{d' - 1} - 1)(t^{f' - 1} - 1)}
\]

where $b' = \frac{\text{LCM}(b, d, f)}{a}$, $d' = \frac{\text{LCM}(b, d, f)}{d}$, $f' = \frac{\text{LCM}(b, d, f)}{f}$.

SFS $\left[ D : \frac{a}{b}, \frac{c}{d}, \frac{e}{f} \right]$ U/m SFS $\left[ D : \frac{a}{b}, \frac{c}{d}, \frac{e}{f} \right]$ $n = (0, 0, 0, 0)$, $H_1 = \mathbb{Z}_4$. The homology of the universal $\mathbb{Z}$-cover has presentation $\mathbb{Z}[\mathbb{Z}] / (t^2 + 1) \oplus \mathbb{Z}[\mathbb{Z}] / (t^2 + 1)$. If we represent the generators by $a$ and $b$ then $\langle a, a \rangle = \langle b, b \rangle = 0$ and $\langle a, b \rangle = \frac{1}{t+1}$, which has all signatures equal to zero.

SFS $\left[ D : \frac{a}{b}, \frac{c}{d}, \frac{e}{f} \right]$ U/m SFS $\left[ D : \frac{a}{b}, \frac{c}{d}, \frac{e}{f} \right]$ $m = (0, 0, 0, 0)$, $H_1 = \mathbb{Z}_4$. Exactly as in the previous case, all signatures are zero.

7. Notes on computations and notation

In order to deal with all the manifolds in the census efficiently, extensive use of computers was made while writing this paper.
The census of prime 3-manifolds admitting a triangulation with 11 or less tetrahedra was created independently by Ben Burton, Sergei Matveev [37] and also Bruno Martelli and Carlo Petronio. Burton’s software Regina [6] allows for relatively easy navigation of the census. We use the word triangulated to mean the smooth/PL manifold has been given a compatible unordered delta complex structure. Precisely, denote the n-simplex by $\Delta^n = \{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0 \forall i \text{ and } x_0 + x_1 + \cdots + x_n = 1 \}$. Given $i \in \{0, 1, \ldots, n\}$ the i-th face map of $\Delta^n$ is the map $f_i : \Delta^{n-1} \to \Delta^n$ given by $f_i(x_0, \ldots, x_{n-1}) = (x_0, x_1, \ldots, x_{i-1}, 0, x_i, x_{i+1}, \ldots, x_n)$. Given a permutation $\sigma \in \Sigma(\{0, 1, \ldots, n\})$, the induced automorphism of $\Delta^n$ is given by $\sigma : \Delta^n \to \Delta^n$, $\sigma(x_0, x_1, \ldots, x_n) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$. An unordered delta complex is a CW-complex $X$ such that the domains of the attaching maps are the boundaries of simplices (rather than discs), $\phi : \partial \Delta^n \to X^{(n-1)}$, and for each $i$, the composite satisfies $\phi \circ f_i = \Phi \circ \sigma_i$, where $\Phi : \Delta^{n-1} \to X^{(n-1)}$ is a characteristic map of the $(n - 1)$-skeleton, and $\sigma \in \Sigma(\{0, 1, \ldots, n-1\})$ is some permutation. If $\sigma$ is always the identity permutation, this would be an ordered delta complex.

Surgery presentations for the closed hyperbolic 3-manifolds in the census were created using programs built from SnapPea [7] and Morwen Thistlethwaite’s tables of knots and links. SnapPea allows one to drill a selection of geodesics out of a hyperbolic 3-manifold, computing the canonical polyhedral decomposition on the resulting hyperbolic manifolds. The procedure used to find surgery presentations for closed hyperbolic 3-manifolds is to “randomize” the initial triangulation via a sequence of Pachner moves. SnapPea then drills out an initial curve in the 1-skeleton of the triangulation, resulting in a 1-cusped hyperbolic manifold. If that manifold is in the census of knots, the procedure terminates with a knot surgery diagram. If not, SnapPea is employed to give a list of drillable curves in the dual 1-skeleton of the cusped triangulation. The software then systematically drills out up to two additional geodesics, and then searches for the manifold in Thistlethwaite’s table of hyperbolic link complements. SnapPea’s isometry-checking routines determine the filling slopes if a match is found among the link tables.

Alexander polynomials of knots and smooth 4-ball genus of many knots in the knot tables can be looked up on Cha and Livingston’s web page [10].

Alexander polynomials of knots and smooth 4-ball genus of many knots in the knot tables can be looked up on Cha and Livingston’s web page [10].

The Oszváth-Szabó “d-invariant/correction term” for the Seifert-fibered rational homology spheres in the census were computed using software written by Brendan Owens and Sasо Strle.

The computation of the hyperbolicity of the torsion linking form was implemented by the author in Regina since version 4.4. Details are given below.

Knots and links from tables are referred to via the notation $\langle X : C \rangle$ where $X$ indicates the table name $X = R$ indicates $C$ is taken from the Rolfsen table, $X = T$ indicates $C$ is taken from the Thistlethwaite table. For example, $\langle T : 2a_1 \rangle$ is the Hopf link and $\langle R : 3_1 \rangle$ indicates the trefoil knot. A convenient place to view these tables is the Knot Atlas [51].

NCellularData is a Regina class which implements the computation of the torsion linking form of a 3-manifold and also tests its hyperbolicity via Kawauchi and Kojima’s classification of symmetric bilinear forms on finite abelian groups taking values in $Q/Z$ [28]. Given two elements $[v], [w] \in \pi H_1(M, Z)$, the torsion linking form $\langle [v], [w] \rangle \in Q/Z$ is an intersection number. A multiple of $[v]$ is zero, $n[v] = 0$ for some $n \in Z$, so $nv = \partial S$ for some 2-chain $S$. Perturb $S$ and $w$ to intersect transversely, and let $m \in Z$ be the signed (algebraic) intersection number of $S$ and $w$. The torsion linking form is defined as $\langle [v], [w] \rangle = \frac{m}{n} \in Q/Z$.

The way this is implemented in Regina is to consider $v$ and $w$ as simplicial chains in the simplicial chain-complex of $M$ coming from the triangulation. $M$ has a dual polyhedral-complex where the i-cells of the dual complex correspond to the $(3 - i)$-cells of the triangulation. This is Poincaré’s proof of his duality theorem [46], simplified using CW-complexes. For example, a 2-cell in the dual polyhedral decomposition corresponds to an edge $e$ of the triangulation. Moreover, the 2-cell is an n-gon, and the n-gon is a union of quadrilaterals, one quadrilateral for each time a tetrahedron contains the edge $e$ (e can be contained in a tetrahedron more than once since the triangulation is semi-simplicial). So the 2-cells of the dual polyhedral decomposition intersect the 1-cells of the triangulation transversely. We homotope the identity map on $M$ to be a cellular map from the triangulation to the dual polyhedral decomposition (this is the core of the algorithm). This allows us to express $\nu$ in the simplicial homology of the triangulation of $M$, and $w$ in the cellular homology of the dual polyhedral decomposition. So now $S$ is a simplicial 2-chain and $w$ is a dual 1-chain intersecting transversely, allowing for the computation of the intersection product via $Z$-linear algebra.

The torsion linking form is stored as a square matrix of rational numbers, where the rows and columns are indexed by the invariant factors of $H_1(M, Z)$. The Kawauchi-Kojima classification of torsion linking forms [28] takes as input this matrix and determines hyperbolicity via linear-algebraic manipulations of the matrix.

* Notation – Regina’s naming conventions for 3-manifolds *

For Seifert-fibered manifolds, Regina’s notation is essentially the same as Orlik’s book [41]. Given a surface $\Sigma$ let $M_\Sigma$ denote an orientable $S^1$-bundle over $\Sigma$ with a section. The manifold $SFS \left[ \begin{array}{c} \Sigma, \; \frac{m_1}{k_1}, \ldots, \frac{m_k}{k_k} \end{array} \right]$ is obtained from $M_\Sigma$ by doing surgery on $k$ fibers in $M_\Sigma$, using filling slopes $\frac{m_1}{k_1}, \ldots, \frac{m_k}{k_k}$ (slope zero being the slope of the section). If $\Sigma$ has boundary, the curves in $\partial M_\Sigma$ corresponding to the section will be denoted “a,” and the curves corresponding to the fiber is denoted “f.”
Only a few types of graph manifolds appear in the 11-tetrahedron census. The underlying graphs, if non-trivial, are of the form:

\[
\begin{array}{c}
S^{[1]} \quad S^{[2]} \quad S^{[1]} \quad S^{[1]} \quad S^{[2]} \quad S^{[1]} \quad S^{[1]} \quad S^{[1]} \quad S^{[2]}
\end{array}
\]

Meaning they have at most three vertices: one-vertex graphs have a single edge, and the remaining two graph types are linear. Regina’s convention for naming these manifolds are:

- Manifolds with a single non-separating torus, in this case Regina uses the notation \(M/(c \begin{array}{cc} a & b \\ c & d \end{array})\) where \(M\) is a Seifert fibered manifold with two boundary tori. This indicates that we glue to two boundary tori together so that \(f_2\) is identified with \(af_1 + bo_1\), and \(o_2\) is identified with \(cf_1 + do_1\).
- Manifolds with a single separating torus are denoted \(M_1 U/m M_2, m = (c \begin{array}{cc} a & b \\ c & d \end{array})\) where \(M_1\) and \(M_2\) follow the notation for Seifert-fibered manifolds above. The matrix \(m\) indicates that \(\partial M_2\) is glued to \(\partial M_1\) by a map that identifies \(f_2\) with \(af_1 + bo_1\) and \(o_2\) with \(cf_1 + do_1\).
- The remaining class of manifolds have the form \(M_1 U/m M_2 U/n M_3, m = (a_1 \begin{array}{cc} b_1 & c_1 \\ d_1 & e_1 \end{array}) n = (a_2 \begin{array}{cc} b_2 & c_2 \\ d_2 & e_2 \end{array})\). The matrices \(m\) and \(n\) denote the gluing maps \(m : \partial M_2 \rightarrow \partial M_1\) and \(n : \partial M_3 \rightarrow \partial M_2\), precisely \(m(f_2) = af_1 + bo_1, m(o_2) = cf_1 + do_1\) and \(n(f_1) = af_2 + bo_2, n(o_1) = cf_2 + do_2\).

There are two other classes of manifolds assigned special names by Regina:

- Hyperbolic manifolds are named in a somewhat ad-hoc way. The first part of such a manifold’s name is the initial 8 terms of the decimal expansion of the volume of the manifold, followed by the invariant factor decomposition of its first homology group. If this data does not uniquely identify the manifold in the census, an additional identifier of the shortest geodesic length is given, suitably rounded.
- If the manifold fibers over \(S^1\) with fiber a torus, the manifold is denoted by the notation \(T \times I/(c \begin{array}{cc} a & b \\ c & d \end{array})\) where the matrix describes the monodromy (assuming the tori are parametrized so as to be parallel). In these notes such manifolds are denoted \((S^1 \times S^1) \times_{(c \begin{array}{cc} a & b \\ c & d \end{array})} S^1\).

Surgery presentations of graph manifolds

The technique used to construct surgery presentations is relatively primitive but effective. Lickorish’s proof that 3-manifolds have surgery presentations had a key idea about gluings of manifolds. Let \(M\) and \(N\) be disjoint 3-manifolds and \(f : \partial M \rightarrow \partial N\) a diffeomorphism. Let \(M\cup_f N\) be the manifold obtained by gluing \(\partial M\) to \(\partial N\) along \(f\). Let \(c\) be a curve in \(\partial N\), and let \(D_c : \partial N \rightarrow \partial N\) be the positive Dehn twist about \(c\), then \(M\cup_{D_c} N \simeq M\cup_f N'\) where \(N'\) is the manifold obtained from \(N\) by doing a \(\pm 1\)-Dehn surgery along a curve \(c\) in the interior of \(N\), parallel to \(c\).

For example, consider item 56 of §5. The manifolds SFS \([D : \frac{1}{2}, \frac{1}{2}]\) and SFS \([D : \frac{1}{2}, \frac{3}{2}]\) should be thought of as the “Dehn surgery” for the partially framed links

respectively, in the sense that the unlabeled (green) curves are drilled but not filled. Let \(f_2, o_2\) and \(f_1, o_1\) denote the fiber and base boundary curves for the manifolds SFS \([D : \frac{1}{2}, \frac{1}{2}]\) and SFS \([D : \frac{1}{2}, \frac{3}{2}]\) respectively. Then as a map from the boundary of the first manifold to the boundary of the second our gluing map has the form \((-\frac{3}{4}, -\frac{2}{3})\) (i.e. the transpose of the matrix in item 56) where the bases curves \(\{f_1, o_1\}\) in and in the range \(\{f_2, o_2\}\). Now we “splice” the above two surgery diagrams together using Lickorish’s idea – in particular we compare this union of two solid tori to the genus one Heegaard splitting of \(S^3\). So multiply the gluing map on the left by \(\left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)\) and write this matrix as a product of row/column operations:

\[
\left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} -3 & -2 \\ 4 & 3 \end{array} \right) = \left( \begin{array}{cc} 4 & 3 \\ -3 & -2 \end{array} \right) = \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ -3 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right).
\]

We think of this product as \(\text{D}^{-1} \circ \text{D}^3 \circ \text{D}^1\), i.e. a product of powers of positive Dehn twists about the standard meridians and longitudes in a solid torus in the standard genus 1 Heegaard splitting of \(S^3\). This gives us the “spliced” surgery presentation for the manifold in item 56.
a union of disjoint sub-links $L = L_1 \sqcup L_2$ where $L_1 \subset \mathbb{R}^2 \times \{0\}$ is a collection of nested circles of various radii centered around points in $\{0\} \times \mathbb{R} \times \{0\}$ and $L_2 \subset \{0\} \times \mathbb{R}^2$ is also a collection of nested circles, centered around points in $\{0\} \times \mathbb{R} \times \{0\}$. Such surgery presentations are perhaps most easily represented via a graph, analogous to a plumbing diagram, which represents the framing/linking matrix:

\[
\begin{array}{cccccc}
1 & 1 & 1 & -2 & -2 \\
-3 & -2 & 1 & -1 \\
\end{array}
\]

\* Computing the monodromy from the Seifert data \*

These are the fiber bundles over $S^1$ with fiber a closed surface of genus $g \geq 2$, such that the monodromy is a finite-order diffeomorphism of the surface. Denote such a manifold by $\Sigma \rtimes \mathbb{Z}_n S^1$ where $n$ is the order of the monodromy. Precisely, if $f : \Sigma \rightarrow \Sigma$ denotes the monodromy, $\Sigma \rtimes \mathbb{Z}_n S^1 = (\Sigma \times S^1)/\mathbb{Z}_n$ where $\mathbb{Z}_n$ acts on $\Sigma \times S^1$ by $e^{iz\lambda}(x,z) = (f^k(x), e^{iz}z)$ where we make the identification $\mathbb{Z}_n \equiv \{ e^{iz} : k \in \mathbb{Z}\}$. These manifolds are all Seifert fibered – the fibering being covered by the product fibering of $\Sigma \times S^1$. The fiber $\Sigma$ is the unique horizontal incompressible surface, thus these manifolds all have the form $SFS \left[ S^1 : \frac{a_1}{b_1}, \ldots, \frac{a_k}{b_k} \right]$ where $\sum_{i=1}^{k} \frac{a_i}{b_i} = 0$. Thus $n = LCM\{b_1, b_2, \ldots, b_k\}$. $k$ is the number of non-free orbits of $\mathbb{Z}_n$ acting on $\Sigma$ and $\chi(\Sigma) = n\left(\chi(S^1) + \sum_{i=1}^{k} \left(\frac{1}{b_i} - 1\right)\right)$. The numbers $b_i$ give the cone angles $2\pi/b_i$ for the singular orbits of $\mathbb{Z}_n$ acting on $\Sigma$. For example, items 8 through 11 in §5 all have the form $SFS \left[ S^2 : \frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3} \right]$ where $\text{GCD}(\beta_1, \beta_2, \beta_3) = 1$. The obstruction to show these manifolds do not embed (in any homology sphere) is the Alexander polynomial. For these manifolds an efficient way of computing the Alexander polynomial is by constructing an equivariant CW-decomposition of the fiber – and to consider the Alexander polynomial to be the order ideal of the homology of the fiber as a module $\Lambda$-module, where $\mathbb{Z}$ acts via the monodromy. Since the base space is $S^2$ with three singular points, consider it to be a square with an identification made to the edges. This square lifts to a CW-decomposition of the fiber, and in this case the cell structure reduces to one with $\beta_1 + \beta_2$ 0-cells, $\beta_1 \beta_2$ 1-cells and a single 2-cell. The monodromy has a fixed point which is the center of the 2-cell, and the remaining singular points are the 0-skeleton, allowing a rather direct computation of the Alexander polynomial. Checking that the Alexander polynomial does not have the form $p(t)p(t^{-1})$ can be done readily by using computer algebra software (such as Pari) to compute the roots in $\mathbb{C}$. See Theorem 2.4.

\* A technique of Casson and Harer \*

In their paper Casson and Harer [8] demonstrate a technique to find contractible 4-manifolds bounding 3-manifolds. We show here how this technique allows us to find embeddings of a certain class of 3-manifolds in homotopy
4-spheres. Take for example manifold 40 from the list in §3, this is \((-5, -5)\)-surgery on the Whitehead link.

The above figure starts off with \((-5, -5)\)-surgery on the Whitehead Link, call this manifold \(M\). Think of Step 1 as representing a handle attachment to \(M \times [0,1]\) on the side of \(M \times \{1\}\). Step 2 represents the Kirby "blow down" move. Step 3 is an isotopy. Step 4 a further "fold" isotopy. Step 5 is a further "blow down" equivalence of handle presentations. This leaves us with the manifold \(S^1 \times S^2\) on the boundary, which we attach a 3-handle and then a 4-handle. In summary, we have attached a 2-handle, then a 3-handle and 4-handle to \(M \times [0,1]\) to construct a manifold \(W_1\) bounding \(M \times \{1\}\). By design \(\pi_1 W_1 = \mathbb{Z}_2\) and the inclusion \(H_1(M \times \{1\}) \rightarrow H_1 W_1\) has kernel one of the summands of the hyperbolic splitting \(H_1 M = \mathbb{Z}_2 \oplus \mathbb{Z}_2\). By symmetry of the Whitehead link which switches components, we can repeat the argument on the \(M \times \{0\}\) side of \(M \times [0,1]\), building a manifold \(W_0\) such that the inclusion \(M \times \{0\} \rightarrow W_0\) kills the complementary summand of the hyperbolic splitting. The union of these two bounding manifolds \(W_0 \cup W_1\) is then a homotopy 4-sphere containing \(M\).

\[
\star \text{ A symmetric embedding for } S^3/Q_8 \star
\]

We describe a particularly symmetric embedding of \(S^3/Q_8\) into \(S^4\) that we learned from Rob Kusner.

Let \(A\) be the traceless, symmetric \(3 \times 3\) real matrices,
\[
A = \{ A \in M_{3 \times 3}(\mathbb{R}) : tr(A) = 0, A^t = A \}.
\]

The set \(A\) is a 5-dimensional inner product space with the inner product defined by trace, transpose and the matrix product
\[
\langle A_1, A_2 \rangle = tr(A_1^t A_2).
\]

The group \(SO_3\) acts on \(A\) by conjugation, moreover this action is by isometries. Thus the action restricts to the unit sphere of \(A, S A \equiv S^4\). Symmetric matrices are diagonalizable by orthogonal matrices so the conjugacy class of such a matrix is determined by its three real eigenvalues \(-1 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq 1\). The trace condition \(tr(A) = 0\) is equivalent to the condition \(\lambda_1 + \lambda_2 + \lambda_3 = 0\) and the unit sphere condition is similarly equivalent to the condition \(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1\).

The orbit decomposition of \(SO_3\) acting on \(S A\) consists of two orbits equivalent to \(\mathbb{RP}^2\) (the \(\lambda_1 = \lambda_2\) subspace and the \(\lambda_2 = \lambda_3\) subspace) and a 1-parameter family of orbits equivalent to \(S^3/Q_8\), these are the matrices with distinct eigenvalues. The orbits corresponding to distinct eigenvalues isomorphic to \(SO_3/G\) where \(G\) is the group of rotations by \(\pi\) in the faces of a cube, i.e. the rotations preserving the eigenspaces. This group lifts to \(Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \}\) in \(S^4\), thus these orbits are all embedded manifolds diffeomorphic to \(S^3/Q_8\).

### 8. Observations and questions from the data

A striking feature about the data is that some 3-manifolds from the census are more susceptible to our embedding constructions than others. For example, if the manifold fibers over \(S^1\), we have deform-spun embeddings and surgical embeddings at our disposal. Seifert-fibered spaces have a variety of embedding techniques, largely due to Crisp and Hillman. But when dealing with hyperbolic manifolds, the only technique used is the surgical embedding construction.

**Question 8.1.** Do there exist 3-manifolds \(M\) which embed smoothly in \(S^4\) such that no embedding of \(M\) in \(S^4\) is a surgical embedding in the sense of Constructions 2.8 and 2.9?

For the above question, I know of no relevant obstructions, although presumably the answer is yes. Similarly, if the rank of the 1st homology group of \(M\) is larger than one, it’s not clear if there are any concordance obstructions that one can use. Given \(M\) in \(S^4\), let \(S^4 = V_1 \cup_M V_2\) be the decomposition of \(S^4\) into two 4-manifolds along their common boundary \(M\).

**Question 8.2.** If the rank of \(H_1 M\) is larger than 1, are there any restrictions on the ranks of \(H_1 V_1\) and \(H_1 V_2\)? Similarly, an embedding of \(M\) in \(S^4\) induces a hyperbolic splitting on \(\tau H_1 M\). How many hyperbolic splittings can one generate via embeddings? Are some hyperbolic splittings of \(\tau H_1 M\) impossible to realize via embeddings?

There is at least one such restriction. The inclusion \(M \rightarrow V_i\) induces a restriction map on cohomology \(H^k(V_i) \rightarrow H^k(M)\). So if \(\alpha, \beta \in H^1(V_i)\) restrict to classes in \(H^1(M)\) which have a non-zero cup-product, they must also have a non-zero cup product in \(H^2(V_i)\). This gives a restriction in some cases - for example \(M = (S^3)^3\). Since \(H^1(M)\) has non-trivial cup-products, it is impossible for \(H^2(V_i)\) to have rank three, since \(H^2(V_i)\) would necessarily have rank zero. Thus for any embedding of \((S^3)^3\) in \(S^4\), \(\text{rank}(H_1(V_i)) \geq 1\) for \(i = 1, 2\).

One startling observation from the data in this paper and from the references is that there are as of yet no examples of 3-manifolds that embed in homology 4-spheres which do not embed in \(S^4\). This leads to two questions.

**Question 8.3.** If a 3-manifold \(M\) admits a smooth embedding into a homotopy 4-sphere, does it admit a smooth embedding \(S^4\)? Are there 3-manifolds that embed in homology 4-spheres which do not embed in \(S^4\)?

The earlier question is only interesting if the smooth 4-dimensional Poincaré conjecture is false. But it is perhaps surprising that it’s not immediately clear whether an embedding of a 3-manifold into an exotic \(S^4\) could be pushed into a standard 4-ball. Agol and Freedman [2] have taken a step toward resolving this question, giving an obstruction to a 3-manifold embedding smoothly in \(S^4\) in terms of the handlebody metric on the curve complex. Technically, the Agol-Freedman obstruction obstructs a Heegaard splitting induced by the embedding. It is unclear, at present, if it can be used to obstruct embeddings.

One would think there should be 3-manifolds that embed in homology 4-spheres that do not embed in \(S^4\). I am unaware of any obstructions at present. A reasonable place to look for answers to this question would be homology...
3-spheres. Let $M$ be a homology 3-sphere. As we have observed $M#(-M)$ embeds smoothly in a homology 4-sphere but it is not clear $M#(-M)$ embeds in $S^4$ unless we could realize $M$ as something like a cyclic branched cover on a knot in $S^3$ or some Litherland-style variant on that theme (see Theorem 2.15). This provides a source of 3-manifolds that embed in homology 4-spheres but for which there is no clear embedding in $S^4$.

A reoccurring problem in this paper is that even if a 3-manifold embeds in $S^4$, we have no uniform, standard way of constructing an embedding.

**Question 8.4.** Is there an efficient procedure to determine whether or not a triangulated 3-manifold admits a locally-flat PL-embedding (equivalently, smooth embedding) in $S^4$?

Costantino and Thurston have recently developed an efficient procedure [11] to construct a triangulated 4-manifold that bounds a triangulated 3-manifold. They do this by perturbing a map $M \to \mathbb{R}^2$ associated to the triangulation, and “filling in” the level sets in a natural way. Perhaps one could devise a combinatorial search for embeddings $M \to \mathbb{R}^4$ by considering such an embedding to be a special pair of generic maps $M \to \mathbb{R}^2$?

**Question 8.5.** Is there a computable function $\beta : \mathbb{N} \to \mathbb{N}$ such that for each 3-manifold that embeds in $S^4$ and admits a triangulation with $n$ tetrahedra, $M$ appears as a vertex-normal solution to the gluing equations for a triangulation of $S^4$ with no more than $\beta(n)$ pentachora in the triangulation of $S^4$?

Provided we had such a $\beta$, the problem of determining whether or not a 3-manifold embeds in $S^4$ would be an algorithmically-solvable problem, as there would be a finite list of triangulations of $S^4$ on which to do normal surface enumeration.

On the pessimistic side, Dranishnikov and Repovs [15] have shown there exists a smooth embedding of a 3-manifold $M$ in $S^4$ such that $S^4 = V_1 \cup_M V_2$ with $\pi_1 V_i$ having an unsolvable word problem, for some $i \in \{1, 2\}$. Thus if one attempts to find obstructions to $M$ embedding in $S^4$ based on the fundamental group, one could run into computability problems unless the obstruction is based on a computable invariant of group presentations. Computable invariants of group presentations include things like computable invariants of representation varieties, and the lower central series of the group.

**Question 8.6.** If $M$ admits a smooth embedding into $S^4$, does it admit an embedding where $S^4 = V_1 \cup_M V_2$ with both $\pi_1 V_1$ and $\pi_2 V_2$ having solvable word problems?

**Question 8.7.** (M. Freedman) Given a smooth 3-manifold $M$, if $M#(S^1 \times S^2)$ embeds in $S^4$, does $M$? More generally, does stabilization via connect-sum with copies of $S^1 \times S^2$ make the embedding problem any easier?

The question highlights a technical issue with the kinds of invariants we use to obstruct embedding. All the invariants we use are additive under connect sum.
