Lamination links in 3-manifolds

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PRELIMINARY

Abstract

We introduce and define “oriented framed measured lamination links” in a 3-manifold $M$. These generalize oriented framed links in 3-manifolds, and are confined to 2-dimensional improperly embedded subsurfaces of the 3-manifold. Just as some framed links bound Seifert surfaces, so also some framed lamination links bound 2-dimensional measured and oriented “Seifert laminations.” We show that any lamination link which bounds a 2-dimensional Seifert lamination, bounds a “taut” Seifert lamination, i.e. one of maximum Euler characteristic, subject to the condition that the Seifert lamination is carried by an aspherical branched surface. This maximum Euler characteristic function is continuous on certain parametrized families of lamination links carried by a train track neighborhood. Taut Seifert laminations generalize minimal genus Seifert surfaces.

1 Introduction.

Is there a space of oriented links in $S^3$? This would be a space which includes points representing all classical oriented links in $S^3$. The space would have to include additional points interpolating between points representing links. The space would be non-compact. In this paper, we do not answer the question whether a space exists, but we prepare the ground. Our experience of measured lamination spaces in surfaces will serve as a guide, though the end result is very different. “Links” in a surface are curve systems, and we know that the curve complex for a surface can be enlarged to a projective measured lamination space. Thus it is reasonable to try to define knotted and or linked measured laminations, which we will call “framed oriented measured lamination links,” or often just “lamination links.” Perhaps the most distinctive property of oriented knots and links is that they bound Seifert surfaces, so we shall pay special attention to framed measured lamination links which bound 2-dimensional oriented measured “Seifert laminations.” It is best to describe the kind of object we seek using an example:

Example 1.1. Figure[1] shows an oriented branched surface $B$ in $S^3$. We can assign weights to the sectors of the branched surface satisfying the branch equations at branch curves.
Suppose the weights are $x, y, z$ as shown, and the weight vector is $v = (x, y, z)$, satisfying $2z + 2y = 3x$. If we assign integer weights, $B(v)$ is an oriented surface, determined by the weights, whose boundary is an oriented link (with a framing). Even if the weights are rationally related, up to projective equivalence of weight vectors, $\partial B(v)$ represents an oriented link. But there are, of course weight vectors satisfying the branch equations whose entries are not rationally related. Such a weight vector represents a “Seifert lamination” whose boundary is an oriented measured lamination link which is not a classical link.

To picture the lamination $B(v)$, it is best to replace $B$ by a certain fibered neighborhood we call $V(B)$ as shown in Figure 3. There is a projection $\pi : V(B) \to B$, and $\pi^{-1}(\partial B)$ is a 2-dimensional train track neighborhood which we will call $V(\tau)$, where $\tau = \partial B \hookrightarrow S^3$. Observe that $V(\tau)$ is (transversely) oriented, since it lies in $\partial V(B)$, which has an outward orientation. We say $V(\tau)$ with its orientation is a “framing” of the oriented train track $\tau$ in $S^3$, which is determined here by the embedding of $B$, though in general a framing can be chosen at random. The weights on sectors of $B$ give weights on the train track $\tau$ which also satisfy branch or switch equations, and we can imagine the knotted lamination as $V(\tau)$ with the width of different parts of $V(\tau)$ given by the weights of an invariant weight vector $w$ for $\tau$, see Figure 2. The framed train track $\tau$ together with the invariant weight vector $w$ for $\tau$ determines a “prelamination” $V_w(\tau)$. This is a singular foliation of $V(\tau)$, with leaves transverse to fibers of $V(\tau)$ and with a transverse measure with the property that the measure of a fiber corresponding to a point in the interior of a segment of $\tau$ equals the weight $w_k$ assigned by $w$ to the segment. If $s$ is a segment of $\tau$, $\pi^{-1}(\text{int}(s))$ is foliated as a product and $V_w(\tau)$ has a singular foliation (with singularities on $\partial V(\tau)$) as shown in Figure 3. In much the same way the invariant weight vector $v$ for $B$ determines a prelamination representing the Seifert lamination which we denote $B(v)$.

![Figure 1: The branched surface B carrying Seifert laminations for framed lamination links.](image)

**Remarks 1.2.** One might suppose that a more suitable definition of a lamination link would involve 3-dimensional regular neighborhoods of train tracks, with 2-dimensional disk fibers, and with the lamination transverse to the disk fibers. Such a definition is much too permissive; it allows uncontrolled twisting and braiding in the fibered neighborhood. Our lamination links are required to live in a 2-dimensional surface, namely a 2-dimensional train track neighborhood $V(\tau)$. A measured lamination link should be imagined as a closed branched tape with prescribed width on portions without branching.
A fundamental invariant of an oriented knot is the minimal genus of a Seifert surface. Of course the genus of a Seifert surface can also be expressed in terms of the Euler characteristic. The Euler characteristic can be defined for measured 2-laminations (represented by prelaminations), see Section 2. In fact, the Euler characteristic is a linear function on the entries of a weight vector $w$ on $B$. We write the Euler characteristic of a measured lamination $B(w)$ as $\chi(B(w))$. Alternatively, we can denote a measured 2-dimensional lamination $B(v)$ more abstractly as $(\Lambda, \nu)$, where $\nu$ is a transverse measure, assigning a measure to each interval intersecting $\Lambda$ transversely. The abstract lamination $\Lambda$ is obtained by performing an “infinite splitting” on the prelamination.

In order to give a formal definition of “framed oriented measured lamination links,” we must first define certain terms related to prelaminations.

**Definition 1.3.** Suppose $V_w(\tau)$ is a prelamination, either abstract or embedded in a 3-manifold. A leaf of the prelamination is a leaf of the foliation of $V_w(\tau) \setminus \{\text{cusps}\}$ completed by including any cusps at one or both ends. A leaf of the singular foliation on $V(\tau)$ determined by the weight vector $w$ is called a separatrix if it is not contained in $\partial V(\tau)$ and has at least one end at a cusp point on $\partial V(\tau)$. The prelamination $V_w(\tau)$ is irreducible if it is fully carried and there are no compact separatrices homeomorphic to a closed interval (with each end at a cusp point). A measured lamination $V_w(\tau)$ is fully carried by $V(\tau)$ if all the entries of $w$ are strictly positive.

A splitting of the prelamination $V_w(\tau)$ is a prelamination $V_w(\tau')$ obtained by cutting $V(\tau)$ on a compact submanifold (disjoint from $\partial V(\tau) \setminus \{\text{cusps}\}$) of the union of leaves. We also say $\tau'$ is a splitting of $\tau$. A splitting of $V_w(\tau)$ is regular if it can be realized as a finite
sequence of splittings of the following kind: Split on a compact segment of a separatrix with exactly one end at a cusp.

A splitting is *good* if it can be realized as a finite sequence of splittings of the following kind: Split on a compact segment of a separatrix with one or both ends at a cusp. (Splitting on a closed leaf is not allowed in a good splitting, nor is splitting on a compact arc in a leaf with neither end of the arc at a cusp.)

The inverse operation of a (good, regular) splitting is called a *(good, regular) pinching.*

**Definition 1.4.** Suppose $V(\tau) \hookrightarrow M$ is a 2-dimensional fibered train track neighborhood embedded in an oriented 3-manifold $M$. A *complementary digon* for $V(\tau)$ is an embedded disk $D$ such that $D \cap V(\tau) = \partial D$ and $\partial D$ contains exactly two cusp points of $V(\tau)$. (Throughout this paper, the symbol $\hookrightarrow$ means “embedded in.”) We say $V(\tau)$ is adigonal if there are no complementary digons for $V(\tau)$.

A *framed measured oriented lamination link* in an oriented 3-manifold $M$ is a 1-dimensional measured lamination which can be represented as a 2-dimensional adigonal oriented fibered train track neighborhood $V(\tau) \hookrightarrow M$ for an oriented compact train track $\tau \hookrightarrow M$ with an invariant weight vector $w$ on $\tau$ determining the prelamination, which we denote $V_w(\tau)$. We will sometimes refer to a framed measured oriented lamination link simply as a link. We say $V_w(\tau)$ represents the link. Two links represented as $V_{w_1}(\tau_1)$ and $V_{w_2}(\tau_2)$ are equivalent if there is a good splitting (in $M$) $V_{w_1}(\tau_1)$ of $V_{w_2}(\tau_2)$ which, up to isotopy in $M$, is also a good splitting of $V_{w_2}(\tau_2)$. Finally $V_{w_1}(\tau_1)$ is projectively equivalent to $(V_{\lambda}(\tau_1), \lambda w_1)$ for any $0 < \lambda < \infty$. If $L$ is connected, the we say the link $(L, \mu)$ is a knot.

A *trivial leaf* is a closed leaf which bounds an embedded disk whose interior is disjoint from the lamination. An oriented link in the usual sense will be called a *classical link*, and a classical link with a framing will be called a *classical framed link*.

A branched surface $B \hookrightarrow M$ is *aspherical* if it does not carry any sphere.

An oriented 2-dimensional measured lamination $(\Lambda, \nu)$ represented as $V_{\Lambda}(B)$ (usually improperly) embedded in $M$ is called a *Seifert lamination* for the link represented as $V_w(\tau)$ if $V_w(\tau)$ is the oriented boundary of $V_{\Lambda}(B)$. If the link $V_w(\tau)$ has a Seifert lamination, we say the link bounds. For any $V_w(\tau)$ embedded in $M$, not necessarily satisfying the adigonal condition, we define $X(V_w(\tau)) = \sup\{\chi(V_{\Lambda}(B))\}$ where the supremum is over 2-dimensional laminations $V_{\Lambda}(B)$ carried by aspherical branched surfaces $B \hookrightarrow M$ and with $\partial(V_{\Lambda}(B)) = V_w(\tau')$ where $V_w(\tau')$ is equivalent to $V_w(\tau)$. The lamination $V_{\Lambda}(B)$ is *taut* if it achieves the above supremum.

A *peripheral* (framed, measured, oriented) link in $M$ is a link $V_w(\tau)$ with $V(\tau)$ embedded in $\partial M$ or isotopic into $\partial M$. Usually we do not require that a peripheral link be carried by an adigonal $V(\tau)$, but the definition of the function $X$ is otherwise unchanged.

We defined a link as an equivalence class of prelaminations $V_w(\tau)$ embedded in $M$, but we can think of the link as an abstract lamination embedded embedded in $V(\tau)$ obtained by “infinite splitting.” Viewed as a abstract lamination, the link can be written $(L, \mu)$, but we must remember that it lives in a $V(\tau)$.

We remark that if in defining equivalence of representatives $V_w(\tau)$ of a lamination link, we had not insisted on good splittings, then splitting on a compact arc with neither end at a cusp, we would introduce a digon. To eliminate such a digon, we must pinch, and if $M$ is not irreducible, then the digon used to pinch is not unique up to isotopy. Even worse,
splitting on a closed curve introduces a complementary annulus. To reverse this, we can pinch on an annulus, but the annulus may not be unique up to isotopy. On the other hand, we cannot require that there be no complementary annuli, since this would rule out framed links where $V_w(\tau)$ is a union of foliated annuli, so it would rule out framed classical links. These remarks also show that the framed link can be regarded as an abstract lamination, but only if we regard it as an abstract lamination embedded in $V(\tau)$. Thus a framed link is a lamination embedded in a surface embedded in $M$. Our equivalence relation allows only good splittings. If $V_w(\tau)$ is irreducible, a good splitting is also regular, so in this case the underlying surface (with inward cusps on its boundary) containing the lamination does not change. So if $V_w(\tau)$ is irreducible, the lamination link lives in a uniquely determined isotopy class of a surface with inward cusps on its boundary.

Finally, we point out the obvious: The property of tautness for Seifert laminations generalizes the “minimal genus” property of Seifert surfaces.

Peripheral links are easier to deal with, and they give a starting point for studying arbitrary links. Observe that if $\tau \hookrightarrow \partial M$, there is an obvious choice of framing for $\tau$; namely we can take $V(\tau)$ to be a regular neighborhood in $\partial M$ of $\tau$, and we are in a situation where the invariant weight vector on $\tau$ determines the peripheral link, so we can write $\tau(w)$ instead of $V_w(\tau)$ to denote the prelamination determined by the weights.

**Definition 1.5.** The non-zero invariant weight vectors on any train track $\tau$ form a cone $\mathcal{C}(\tau)$. Projectivizing by taking a quotient where $w \in \mathcal{C}(\tau)$ is equivalent to $\lambda w$, we obtain the weight cell, $\mathcal{P}(\tau)$. If $V(\tau)$ is any train track neighborhood of $\tau$, we also write $\mathcal{P}(\tau) = \mathcal{P}(V(\tau))$. Often we use $\mathcal{P}(\tau)$ to denote a particular subspace of $\mathcal{C}(\tau)$, namely $\mathcal{P}(\tau) = \{ w \in \mathcal{C}(\tau) : \sum_i w_i = 1 \}$, where $w_i$ are the entries of $w$.

When $V(\tau) \hookrightarrow M$ is adigonal, the points in the cone $\mathcal{C}(\tau)$ represent links. Some of these bound Seifert laminations, some do not. We show in Section 3 that the set of bounding links carried by $\tau \hookrightarrow \partial M$ forms a subcone $\mathcal{CB}(\tau) \subset \mathcal{C}(\tau)$.

We will prove more detailed results implying:

**Theorem 1.6.** Suppose $M$ is a compact, oriented 3-manifold. Suppose $\tau$ is an embedded oriented train track in $\partial M$. Then
(a) the function $X : \mathcal{CB}(\tau) \rightarrow \mathbb{R}$ defined by $X(w) = X(\tau(w))$ is continuous;
(b) if $\tau(w)$ represents a lamination which bounds a 2-dimensional oriented lamination then there is an aspherical oriented branched surface $B$ and an invariant weight vector $v$, with $\partial V_v(B) = V_w(\tau)$, such that $B(v)$ is taut, i.e. achieves a finite supremum in the definition of $X(V_w(\tau))$.

The method of proof of the above theorem shows that there are some similarities with William Thurston’s norm on homology. We will prove that the function $X$ is piecewise linear, and we will show that there are finitely many oriented branched surfaces $B$ satisfying $\partial B = \tau$ with the property that every lamination carried by $B$ is taut, and for ever $\tau(w)$, there is a branched surface $B$ in the finite collection and an invariant weight vector $v$ for $B$ such that $\partial B(v) = \tau(w)$.

The following theorem clarifies the analogy between the function $X : \mathcal{CB}(\tau) \rightarrow \mathbb{R}$ and Thurston’s norm on homology. The theorem is a simplified version of Theorem 3.12.
Theorem 1.7. Suppose $M$ is a compact orientable 3-manifold. Suppose $\tau \hookrightarrow \partial M$ is an embedded oriented train track in $\partial M$.

The function $X : \mathcal{C}B(\tau) \to \mathbb{R}$ is linear on rays and finite piecewise linear, where $\mathcal{C}B(\tau)$ is the convex subcone of $\mathcal{C}(\tau)$ corresponding to oriented measured laminations $\tau(w)$ bounding oriented 2-dimensional measured laminations in $M$. The function $X$ is concave.

Of course we wish to apply Theorem 1.6 to peripheral lamination links, but we have not required $V(\tau) \subset \partial M$ to be adigonal in $M$. Thus the theorem is stronger than what we need. We remark that allowing lamination links with trivial leaves (which bound disks) does not present a problem for showing that the supremum in the definition of $X$ is finite. A non-trivially measured family of disk leaves in $V(B)$ contributes positive Euler characteristic, but the total positive contribution must be bounded because the family of disks also contributes to the measure on $\tau(w)$. Measured families of spheres in $\Lambda$ could also make positive contributions to $X$, but we have ruled out sphere leaves in $V(B)$ by insisting that $B$ be aspherical. This suggests, but does not prove, that the supremum is finite.

We will show that, in a sense, Theorem 1.6 also applies to non-peripheral links.

Theorem 1.8. Suppose $M$ is a compact, oriented 3-manifold and $(L, \mu) = V(\tau)$ is a lamination link embedded in $M$. Then there is a taut Seifert lamination for $(L, \mu)$. If $V(\tau)$ is irreducible, then there is an aspherical oriented branched surface $B$ and an invariant weight vector $v$, with $\partial B = \tau$, such that $(\Lambda, v) = B(v)$ achieves a finite supremum in the definition of $X((L, \mu))$.

We can also make a statement about the continuity of the function $X$ for non-peripheral links.

Definition 1.9. Suppose $V(\tau) \hookrightarrow M$ is a framed train track. We let $\mathcal{CIB}(V(\tau))$ denote the subspace of invariant weight vectors on $\tau$ which represent irreducible measured laminations fully carried by $V(\tau)$ which bound Seifert laminations.

$$\mathcal{CIB}(V(\tau)) = \{ w \in \mathcal{CB}(\tau) : V_w(\tau) \text{ is irreducible and every entry } w_i > 0 \}$$

Theorem 1.10. Suppose $V(\tau) \hookrightarrow M$ is a framed train track in an oriented compact 3-manifold. Then the function $X : \mathcal{CIB}(V(\tau)) \to \mathbb{R}$ defined by $X(w) = X(V_w(\tau))$ is continuous.

We hope to use this last theorem to describe continuity properties of $X$ on a space of links.

2 Preliminaries

Definitions 2.1. Suppose a surface $V$ can be subdivided into rectangles or charts of the form $I \times J$, where $I = [0, 1]$ and $J = [0, w]$ foliated by leaves $I \times \{u\}$ and by fibers $\{t\} \times J$. We require that $\bigcup I \times \partial J \subset \partial V$, where the union is over charts. Another chart (or the same one) intersects $I \times J$ in sub-intervals of fibers in $\partial I \times J$. Then $V$ with the singular foliation obtained as the union of foliated charts is a dimension 1 prelamination. If adjacent
charts intersect such that the metrics on intersecting fibers coincide on the intersection, the prelamination is *transversely measured*.

The chart foliations yield a foliation of $V \setminus C$, where $C$ is the set of *cusp points* on $\partial V$, which are points that lie on the interior of a fiber of one chart and also at the ends of fibers of two adjacent charts. A *leaf* of the prelamination is a leaf of $V \setminus C$ completed by any cusp points at the ends of the leaf. See Figure 4. A *splitting* of a prelamination, is another prelamination obtained by cutting on a finite union of compact submanifolds of leaves, provided these submanifolds intersect $\partial V(\lambda)$ only at cusps. *Pinching* is the opposite of splitting.

We can make similar definitions in other dimensions. A *dimension $m$ prelamination* is constructed using charts of the form $I \times J$ where $I$ is a disk of dimension $m$ and $J$ is an interval as before; the charts intersect only at their vertical boundaries $\partial I \times J$ and cover an $(m+1)$-manifold $V$ with inward cusps at an $(m-1)$-manifold $C$ in $\partial V$. (One can allow $J$ of higher dimension but we will not use such prelaminations.)

Figure 4: Chart structure for a prelamination.

When a prelamination (of dimension $m$) is embedded in a manifold $M$ of dimension $m+1$, then we often call the embedded submanifold $V$ with its foliation by fibers a *fibered neighborhood of a branched manifold* of dimension $m$. Even if there is no foliation transverse to the fibers, and therefore no prelamination, we refer to the union of charts foliated by fibers as a *fibered neighborhood*. A *branched manifold* is a quotient obtained from a fibered neighborhood by identifying all points on any fiber, where the quotient map is often thought of as a projection. For example, if $m = 1$, the quotient is called a *train track* $\tau$, the fibered neighborhood is denoted $V(\lambda)$ and the quotient map is $\pi : V(\lambda) \to \tau$. For $m = 2$, a fibered neighborhood $V(B)$ projects to a *branched surface* $B$. In any dimension the quotient $B$ is given a smooth structure such that a smooth $m$-disk locally embedded transverse to fibers of projects to a smooth disk in the quotient branched manifold. In any dimension, we let $C$ be the *cusp locus* in $\partial M$. Then $\pi(C)$ is called the *branch locus* (*switch points* for train tracks) and the completions of the components of the complement of the branch locus are called *sectors*. For a train track $\tau$ which contains closed curve components, we choose a vertex in each such component, then we define the *segments* of $\tau$ as the components of the completion of the union of switches and additional vertices. We will use the notation $V(\lambda)$ even when $V(\lambda)$ is embedded in a 3-manifold. Given a prelamination, we say it is *fully carried* by the corresponding branched manifold.

A codimension-1 *lamination* $L \hookrightarrow M$ in an $m+1$-manifold $M$ is a closed foliated subset of $M$.

If a 1-dimensional prelamination is transversely measured, then the chart corresponding
to a segment \(s_i\), which has the form \(I \times [0, w_i]\), is assigned a weight \(w_i > 0\), which yields an “invariant weight vector”, which is characterized by the following property: If \(s_1, s_2,\) and \(s_3\) have a common switch point \(P\), and both \(s_1 \cup s_3\) and \(s_2 \cup s_3\) are locally smooth at \(P\), then \(w_1 + w_2 = w_3\). This last equation is called a switch equation. An invariant weight vector is a weight vector which assigns \(w_i \geq 0\) to each segment \(s_i\) of \(\tau\) and such that the entries of the weight vector satisfy all the switch equations for \(\tau\).

Clearly an invariant weight vector for \(\tau\) determines a measured prelamination fully carried by a sub-train track of \(\tau\).

There is another point of view on laminations and fibered neighborhoods which is sometimes preferable. A lamination can be constructed as an inverse limit of prelaminations obtained by performing “infinite splitting” along leaves intersecting the cusp locus \(C\) of a prelamination. This can be done such that the leaves of the lamination are smooth with respect to a smooth structure on \(M\). A transverse measure on a prelamination then gives a transverse measure on the corresponding lamination. When dealing with laminations rather than prelaminations, one uses a different kind of fibered neighborhood, as in the following definitions.

Definitions 2.2. If \(B \hookrightarrow M\) is a codimension-1 branched manifold of dimension \(m\) embedded in a manifold \(M\), there is a corresponding fibered neighborhood \(N(B) \hookrightarrow M\). Once again, \(N(B)\) is “vertically” foliated by fibers and there is a quotient map \(\pi : N(B) \to B\) which identifies all the points on each fiber and yields the branched manifold \(B\). In Figure 5 we show a typical fibered neighborhood when \(B = \tau\) is a train track and \(M\) is a surface. Instead of a cusp locus there is a vertical boundary \(\partial v N(B)\) contained in \(\partial N(B)\), which is an \(I\)-bundle over an \((m - 1)\)-manifold. For example, if \(B\) is a branched surface in a 3-manifold, the vertical boundary is a union of annuli, fibered by intervals. The horizontal boundary \(\partial h N(B)\) is the closure of \(\partial N(B) \setminus (\partial v N(B) \cup \partial M)\).

A lamination \(L \hookrightarrow M\) is carried by \(B\) if, after isotopy, it can be embedded in \(N(B)\) transverse to fibers. It is fully carried by \(B\) if it can be isotoped so it is contained in \(N(B)\) and transverse to fibers and intersecting every fiber. If it is fully carried by \(B\), \(L\) can be modified slightly (by replacing any isolated leaf by the boundary of an immersed regular neighborhood of the leaf) so that \(\partial h N(B) \subset L\). When \(\partial h N(B) \subset L\), the completion of \(N(B) \setminus L\) has the structure of an \(I\)-bundle called the interstitial bundle.

If \(B \hookrightarrow M\) is a branched surface, a disk of contact for \(B\) is a disk \(D \hookrightarrow N(B)\) transverse to fibers with \(\partial D \hookrightarrow \text{int}(\partial h N(B))\). Sometimes if \(F\) is a surface fully carried by \(B\), and we
have $\partial_hN(B) \subset F$, a disk of contact $D$ can be isotoped vertically so that $D \subset F$. In this case, we say $D$ is a disk of contact in $F$ although $\partial D \not\subset \text{int}(\partial_hN(B))$.

The notion of splitting a branched manifold is most easily described using the fibered neighborhood $N(B)$.

**Definition 2.3.** If $B \hookrightarrow M$ and $B' \hookrightarrow M$ are codimension-1 branched manifolds, then $B'$ is a splitting of $B$ if $N(B) = N(B') \cup J$, where $J$ is (the total space of) an $I$-bundle $p : J \to V$ and:

1. $\partial_hJ \subset \partial_hN(B')$,
2. $J \cap N(B') \subset \partial J$, $J \cap N(B') \subset \partial N(B')$,
3. $p^{-1}(\partial V) \subset \partial J$ intersects $\partial N(B')$ in finitely many components, each contained in $\partial, N(B')$, with each fiber in $\partial J$ contained in a fiber of $N(B')$.

**Definition 2.4.** If $B \hookrightarrow M$ is a branched surface embedded in a 3-manifold, it has generic branch locus if the projection $V(B) \to B$ maps the cusp locus $C$ to $B$ such that self-intersections of $C$ are transverse and in general position. Then the sectors of $B$ are surfaces with corners, see Figure 3. If $Z$ is such a sector, we assign a geometric Euler characteristic to $Z$, $\chi_g(Z) = \chi(Z) - (1/4)k$, where $k$ is the number of corners of $Z$.

If $L$ is a measured lamination carried by a branched surface $B$ and $L$ induces the weight $v_i$ on the sector $Z_i$, then we define $\chi(L) = \sum_i v_i \chi_g(Z_i)$.

There is some work needed to make sense of the above definitions. For example, one must show that the Euler characteristic of a measured lamination is well-defined, not depending on the branched surface.

**Definition 2.5.** Suppose $v$ is an invariant weight vector on a branched surface $B$ with boundary. (Usually we have $B$ properly embedded in some 3-manifold $M$ and $\partial B$ is a train track embedded in $\partial M$. ) Then we let $\partial v$ denote the invariant weight vector on the train track $\partial B$ obtained by restricting $v$.

Note that each entry of $\partial v$ is an entry of $v$, but a sector $Z$ of $B$ can intersect $\partial B$ in several segments, so an entry of $v$ can appear more than once in $\partial v$.

**Definitions 2.6.** If $\tau$ has $p$ segments, a weight vector $w$ assigns a weight $w_i$ to each segment $s_i$. The weight vector is invariant if it satisfies all the switch equations: For each switch point in $\tau$ there are three segments attached to the switch point, say $s_i$, $s_j$, and $s_k$. If $s_i \cup s_k$ and $s_j \cup s_k$ are smooth, then the switch equation is $w_i + w_j = w_k$. An invariant weight vector $w$ for $\tau$ determines an oriented measured lamination $\tau(w)$ carried by $\tau$.

The weight cone $C(\tau)$ for a train track is the convex cone of invariant weight vectors on $\tau$. More precisely, it is the set $w \in \mathbb{R}^p$ such that $w_i \geq 0$ for all $i$ and all the switch equations are satisfied. The weight cell $\mathcal{PC}(\tau)$ is the intersection of the weight cone with the the hyperplane $\sum w_i = 1$, where the sum is over weights on segments of $\tau$. We similarly define the weight cone and weight cell for branched surfaces.
3 Peripheral links.

Rather than immediately studying arbitrary lamination links in an orientable, compact 3-manifold, it is better to begin by restricting attention to peripheral lamination links in a 3-manifold with boundary.

Definition 3.1. A peripheral oriented measured lamination link (or a peripheral link in an orientable, compact 3-manifold $(M, \partial M)$) is an oriented 1-dimensional measured lamination $(L, \mu)$ in $\partial M$. The link can be represented as $V_w(\tau)$ where $V(\tau)$ is a fibered neighborhood in $\partial M$ of a tangentially oriented train track $\tau$. In this context, when $\tau$ is contained in a surface $\partial M$, $V(\tau) \hookleftarrow \partial M$ is determined by $\tau$ up to isotopy, and we can simply write $V_w(\tau)$ as $\tau(w)$.

We note that for arbitrary links in $M$ we require that $V(\tau)$ be adigonal in $M$. We could impose the same requirement for peripheral links, but the results concerning peripheral links remain true without the condition.

Lemma 3.2. Suppose $M$ is an compact, irreducible, orientable 3-manifold with boundary. Suppose $\tau$ is an embedded oriented train track in $\partial M$. The set of invariant weight vectors $w$ such that $\tau(w)$ is carried by $\tau$ and bounds an oriented 2-dimensional measured lamination $(K, v)$ in $M$ is a convex subcone of $\mathbb{C}(\tau)$.

Proof. We must show that the set of invariant weights $w$ on $\tau$ such that $\tau(w)$ bounds a lamination is closed under addition and multiplication by positive scalars.

First we show that bounding laminations carried by $\tau$ are closed under multiplication by positive scalars. This is easy, since if $\tau(w)$ bounds an 2-lamination, then there is an oriented branched surface $B$ with $\partial B = \tau$, and there exists an invariant weight vector $v$ for $B$ whose restriction to $\tau$ is $w$. This is because the measured lamination $\partial B(v) = \tau(w)$. Then if $\lambda > 0$, $\partial B(\lambda w) = \tau(\lambda w)$, and $\lambda w$ bounds.

Next, we show that $\tau(w_1)$ bounds $B_1(v_1)$ and $\tau(w_2)$ bounds $B_2(v_2)$, then we can construct an oriented branched surface $B$ and an invariant weight vector $v$ for $B$ such that $\tau(w_1 + w_2)$ bounds $B(v)$. We may assume that $B_1$ and $B_2$ are generic, meaning that the branch locus consists of branch curves intersecting transversally at points in the interior of $M$, and $\partial B_1 = \partial B_2 = \tau$. Now we isotope $B_1$ and $B_2$ (rel $\tau$) to a general position in the interior of $M$. This means that $B_1 \cap B_2 = \tau \cup \rho$, where $(\rho, \partial \rho)$ is a train track properly embedded in $(M, \partial M)$. Finally we pinch and identify (respecting orientations) a regular neighborhood of $B_1 \cap B_2$ in $B_1$ with a regular neighborhood of $B_1 \cap B_2$ in $B_2$ to obtain a branched surface $B$ containing $B_1$ and $B_2$ as sub-branched surfaces. The invariant weight vector $v_1$ can be regarded as an invariant weight vector for $B$, with some weights 0, and similarly for $v_2$. Then $\tau(w_1 + w_2)$ bounds the 2-dimensional lamination $B(v_1 + v_2)$. □

Definitions 3.3. With $\tau$ as in the statement of the lemma, the cone of invariant weights on $\tau$ which bound 2-dimensional measured laminations is denoted $\mathbb{C}B(\tau)$. The projectivization of the cone $\mathbb{C}B(\tau)$ is denoted $\mathbb{PC}B(\tau)$.

Suppose $M$ is an irreducible orientable 3-manifold with boundary. An oriented surface $F \hookleftarrow M$ is taut if $\chi(F) = \max\{\chi(G) : \partial G = \partial F$ and $G$ has no closed components$\}$.

When we say “taut surface,” we usually mean the isotopy class of a taut surface.
Another way to phrase the definition of “taut”: If we fix an oriented curve system $C \hookrightarrow \partial M$ which bounds an oriented surface $F$ (with consistent orientations) then $F$ is taut if it has maximal $\chi$ among oriented surfaces with the same boundary and without closed components. The maximum in the definition exists: Fixing a boundary curve system $\tau(w) = C$, if $C$ has $k$ components and $\partial F = C$ is a taut surface, then $\chi(F) \leq k$, because the only components of $F$ which make a positive contribution to $\chi(F)$ are disks.

It is easy to show that a taut surface $F$ is incompressible: Suppose $D$ is a compressing disk for $F$. Surgery on $D$ yields a surface $G$ with the same boundary and larger $\chi$, possibly with closed components, but without sphere components. Discarding closed components of $G$ yields another surface with the same boundary and larger $\chi$, which shows $F$ is not taut.

Note that there could be an integer lattice point $w \in \mathbb{C}B(\tau)$ which represents a curve system which bounds a measured laminations but does not bound a surface. However, the following lemma shows that at least it bounds a weighted surface.

**Lemma 3.4.** If $w \in \mathbb{C}B(\tau)$ is an invariant weight vector with integer or rational entries, then there exists $k \in \mathbb{N}$ such that $\tau(kw)$ bounds a (taut) surface.

**Proof.** Since $\tau(w)$ bounds, we know there exists $B(v)$ with $\partial B(v) = \tau(w)$. We consider the set $A$ of all $v \in \mathbb{C}(B)$ with $\partial v = w$. The cone $\mathbb{C}(B)$ is defined by linear equations and inequalities in entries $v_j$ with integer coefficients. To obtain $A$ we also require one equation of the form $v_j = w_i$ for each entry $w_i$ of $w$, setting $w_i$ equal to one entry of $v$. Thus we impose more equations in entries of $v$ with rational or integer coefficients. Therefore the set $A$ is defined by equations and inequalities with integer coefficients. Since it is non-empty, it must contain a point $u$ with only rational entries. Then there exists $k \in \mathbb{N}$ so $ku$ is an invariant weight vector on $B$ with integer entries, and with $\partial B(ku) = \tau(kw)$. We have shown that $\tau(kw)$ bounds some surface. Then it also bounds a taut surface. $\Box$

**Definition 3.5.** Suppose $\tau \hookrightarrow \partial M$ is an essential oriented train track embedded in $\partial M$. An oriented branched surface $(B, \partial B) \hookrightarrow (M, \partial M)$ is $\tau$-essential in $M$ if

1. $\partial B = \tau$ or a sub-train track of $\tau$,
2. $B$ carries no sphere,
3. $B$ fully carries an oriented surface whose orientation is consistent with the orientation of $B$,
4. $B$ has no disk of contact.

We shall explain the last statement in the following lemma in the course of the proof.

**Lemma 3.6.** Suppose $M$ is a compact, orientable, irreducible 3-manifold with boundary and $\tau \hookrightarrow \partial M$ is an oriented train track in $\partial M$. Then there exists a finite collection $\{B_i : i = 1, 2, \ldots, m\}$ of $\tau$-essential oriented branched surfaces such that every isotopy class of taut surface $F$ with $\partial F$ carried by $\tau$ is represented by a surface fully carried by one of the branched surfaces $B_i$ of the collection.

Further, each taut surface $F$ is carried by some $B_i$ as a least area taut normal surface (among taut surfaces with exactly the same boundary, carried by $\tau$) with respect to a fixed triangulation of $M$. 11
Proof. If $\tau$ does not fully carry a curve system which bounds, without loss of generality we replace $\tau$ by a sub-train track which fully carries some curve system that bounds.

Figure 6: Normal train track.

Choose a triangulation of $M$ such that $\tau$ is a normal train track with respect to the triangulation. This means that the intersection with each 2-simplex in $\partial M$ is one of the train tracks shown in Figure 6 up to isotopy and up to symmetries of the simplex. In particular, we require that the intersection of $\tau$ with each 2-simplex of $\partial M$ be connected. When choosing the triangulation, we also require that each simplex of any dimension intersect $\partial M$ not at all or in a single simplex of dimension $\leq 2$. All of this is possible by choosing a sufficiently fine triangulation of $M$. Suppose we are given an oriented curve system $C$ carried by $\tau$ and suppose $C$ bounds some oriented surface. By our choice of $C$, if $C = \tau(w)$, then $w$ lies in the subcone $\mathcal{CB}(\tau)$ of $\mathcal{C}(\tau)$ representing measured laminations bounding oriented measured laminations. Suppose $F$ is a taut surface bounded by $C$, which means it is a maximal $\chi$ surface without closed components and satisfying $\partial F = C$. We explained earlier that $F$ is incompressible.

The fact that $F$ is incompressible allows us to use a modernized and adapted version of W. Haken’s normal surface theory, [2]. We will put $F$ into normal position with respect to the triangulation of $M$ rel $\partial F = C$. As usual in normal surface theory, we begin by isotoping $F$ (rel $\partial M$) to be in general position with respect to the triangulation, transverse to 1-simplices and 2-simplices and disjoint from vertices. We use the usual “combinatorial area” complexity $\gamma(F)$, equal to the number of intersections of $F$ with the 1-skeleton of the triangulation. By our choice of triangulation and the fact that $\tau$ is oriented and that the orientation of $F$ must be compatible with that of $\tau$, we easily see that if $\sigma$ is a 2-simplex with exactly one edge in $\partial M$, there can be no arc of $F \cap \sigma$ with both ends in $\sigma \cap \partial M$. As in the usual normal surface theory, a closed curve $\alpha$ of $F \cap \sigma$ innermost in $\sigma$ for any 2-simplex $\sigma$ (not contained in $\partial M$) can be eliminated by replacing the disk bounded in $F$ by $\alpha$ by the disk bounded in $\sigma$ by $\alpha$, then isotoping the resulting $F$ slightly to eliminate a closed curve of intersection. If $F \cap \sigma$ contains an arc with both ends in the same 1-simplex $\rho$ not contained in $\partial M$, then isotoping the arc to an arc in $\rho$ and a little beyond, extending the isotopy to $F$, reduces the complexity of $F$. Thus we may assume that $F$ is in normal form with respect to the triangulation. This means that $F$ intersects 3-simplices in disks having the combinatorial type of the normal disk types shown in Figure 7 up to symmetries of the simplex. We now also assume that it has minimal complexity among all normal taut surfaces $F$ with $\partial F = C$.

From a taut surface of minimal area, we construct a branched surface $\hat{B}$ by identifying disks of $F \cap \rho$ for every 3-simplex $\rho$ of the triangulation of $M$ if they belong to the same normal disk type, and extending the identification slightly to regular neighborhoods. Thus $\hat{B}$ is the union of normal disk types that occur in $F$, appropriately joined at 2-simplices to form a branched surface. Evidently, there are just finitely many possibilities (up to isotopy) for $\hat{B}$, provided we extend the locus of identification of disks of the same type in
a standard way. If we replace $F$ by two parallel copies of itself, then we may assume that $F$ is transverse to fibers of $N(\hat{B})$ with $\partial_h N(\hat{B}) \subset F$. This means that there is a well-defined interstitial bundle for $F$ in $N(\hat{B})$.

Our goal now is to modify $\hat{B}$ to ensure that it has no disks of contact. In fact, we want to show that there are finitely many ways to modify $\hat{B}$ such that all the normal minimal complexity surfaces carried by $\hat{B}$ are also carried by one of the finitely many modified branched surfaces. We will modify $\hat{B}$ by removing some interstitial bundle from $N(\hat{B})$ to obtain a new branched surface neighborhood $N(B)$ of a new branched surface which has neither disks of contact nor carries spheres. There is a description of this process in [1]; we will give essentially the same argument here, slightly reorganized.

**Claim 1:** If $\hat{B}$ carries a sphere and fully carries the taut surface $F$, $F$ transverse to fibers of $N(\hat{B})$ with $\partial_h N(\hat{B}) \subset F$, then $\hat{B}$ has a disk of contact $E$ in $F$.

Suppose $\hat{B}$ carries a sphere $S$, transverse to fibers of $N(\hat{B})$ and transverse to the surface $F$ carried by $\hat{B}$. Assume both $S$ and $F$ are transverse to fibers of $N(\hat{B})$ and (for now) disjoint from $\partial_h N(\hat{B})$. We will show that we can replace $S$ with another sphere $S'$ transverse to fibers with fewer curves of intersection with $F$. Given such a sphere $S$, choose a curve $\alpha$ of $F \cap S$ innermost on $S$. Then $\alpha$ bounds a disk $H$ in $S$, so it also bounds a disk $D$ in $F$. Let $H'$ be the closure in $S$ of $S \setminus H$. Then $S' = H \cup D$ is an embedded sphere carried by $\hat{B}$ or $S'' = H' \cup D$ is a sphere transverse to fibers possibly with curves of self-intersection. In the first case, $S'$ can be isotoped to be disjoint from $F$. In the second case, we perform oriented cut-and-paste on all curves of self intersection of $S''$ to get an oriented closed surface $T$, with $\chi(T) = 2$. This means that $T$ has a sphere component $S'''$ having fewer intersection curves with $F$. Repeating, we obtain a sphere transverse to fibers and disjoint from $F$, which we will again call $S$. By irreducibility, $S$ bounds a ball $K$ in $M$, and $F$ does not intersect $K$. We may now assume that both $S$ and $F$ are embedded transverse to fibers in $N(\hat{B})$, with $\partial_h N(\hat{B}) \subset (S \cup F)$.

If there is no interstitial bundle for $S \cup F$ in $K$, then $S \subset \partial_h N(\hat{B})$. Since $F$ is also fully carried and can be isotoped so $\partial_h N(\hat{B}) \subset F$, we conclude $F$ has a sphere component, contrary to assumption.

If there is interstitial bundle for $S \cup F$ in $K$, let $J$ be this interstitial bundle, possibly with many components. Suppose $\{A_1, A_2, \ldots, A_w\}$ are the annuli where $J$ meets $\partial_h N(\hat{B})$. If $A_j$ is one of these annuli, then there are disjoint disks $D_{0j} \leftrightarrow S$ and $D_{1j} \leftrightarrow S$ bounded by the components of $\partial A_j$, such that the topological sphere $A_j \cup D_{0j} \cup D_{1j}$ bounds a ball $K_j \subset K$. We say $A_j$ is outermost if for $s \neq j K_j \not\subset K_s$. Now let $A_j$ be an outermost annulus of the collection. Then there is a component $W$ of $S \cap \partial_h N(\hat{B})$ with $\partial A_j \subset \partial W$. Now, after discarding or ignoring $S$, we can isotope $F$ vertically along fibers such that after the isotopy, once again $\partial_h N(\hat{B}) \subset F$. Then $W \subset F$ and by the incompressibility of $F$, the two
curves $\beta_0$ and $\beta_1$ of $\partial A_j$ bound disks in $F$, say $\beta_0$ bounds $D_{j0}$ and $\beta_1$ bounds $D_{j1}$. Consider $D_{j0}$; either $W \subset D_{j0}$ or $D_{j0}$ is a disk of contact (after a small vertical isotopy). If $W \subset D_{j0}$ then $\beta_1 \subset D_{j0}$ and $D_{j1} \subset D_{j0} \subset F$ yields a disk of contact in the same way.

Now let $J$ denote the interstitial bundle for $F$ in $N(\hat{B})$ (rather than for $F \cup S$). We see that the component $J_0$ of $J$ containing $A_j$ must be a product of the form $P \times I$, where $P$ is planar, with $P \times \{0\} \subset D_{j0}$ and $P \times \{1\} \subset D_{j1}$. There may be other components of $J$ in $K$.

This completes the proof of Claim 1.

**Claim 2:** If $\hat{B}$ has a disk of contact and fully carries the taut minimal complexity surface $F$, $F$ transverse to fibers of $N(\hat{B})$ with $\partial_0 N(\hat{B}) \subset F$, then it has a disk of contact $E \subset F$. Further, if $\partial E \subset A$, where $A$ is an annular component of $\partial_0 N(\hat{B})$, then the component of the interstitial bundle for $F$ in $N(\hat{B})$ containing $A$ has the form $P \times I$, where $P$ is planar.

Suppose $E$ is a disk of contact for $\hat{B}$. This means $E$ is a disk embedded in $N(\hat{B})$ transverse to fibers with $\partial E \subset \text{int}(\partial_0 N(\hat{B}))$. We suppose that $F$ is also embedded in $N(\hat{B})$ transverse to fibers. For now we assume $F$ and $E$ are disjoint from the horizontal boundary. Suppose $E$ is transverse to $F$. Suppose $\alpha$ is a closed curve of intersection innermost in $E$, bounding a disk $H$. Then $H$ is a potential compressing disk for $F$. So since $F$ is incompressible, there is a disk $D$ in $F$ with $\partial D = \partial H$. For one choice of the sense of cut-and-paste on $\alpha$, we obtain a new surface, possibly with self-intersections, and transverse to fibers of $N(\hat{B})$. We perform the cut-and-paste in this sense, and either we get an embedded sphere $S' = H \cup D$ transverse to fibers in $N(\hat{B})$ ($S'$ is carried by $\hat{B}$) or we get a new immersed disk of contact $E' = (E \setminus H) \cup D$. By the previous claim, if $\hat{B}$ carries a sphere, we obtain a disk of contact in $F$. If we get a disk $E'$, and $E'$ happens to be embedded, again after a slight isotopy, there are fewer curves in $E' \cap F$. If $E'$ has self-intersections, then we also have one fewer intersection curve of $E' \cap F$ than $E \cap F$ (after a suitable small homotopy). In this case, perform cut-and-paste on all curves of self-intersection of $E'$ in the appropriate sense to obtain an embedded surface $S''$ with $\partial S'' = \partial E'$ still transverse to fibers. Since $\chi(S'') = \chi(E') = 1$, $S''$ must contain a some component with positive Euler characteristic. Thus $S''$ must contain a new disk of contact $E''$ with fewer curves in $E'' \cap F$ than $E \cap F$, or $S''$ must contain a sphere $S'''$. If we obtain a sphere, we can immediately apply Claim 1 to show there is a disk of contact in $F$. Otherwise, from the new disk of contact, repeating the argument, we obtain a sphere or we get a disk of contact with fewer curves of intersection with $F$. Repeating as often as necessary, we either get a carried embedded sphere, which gives a disk contact disjoint from $F$ isotopic to a disk in $F$ by Claim 1, or we end with a disk of contact (transverse to fibers in $N(\hat{B})$) disjoint from $F$. If $E$ is such a disk of contact, then $\partial E$ is in the interior of an annulus component $A$ of $\partial_0 N(\hat{B})$. By isotoping $F$ we can assume $\partial A \subset F$, since we can isotope $F$ so that $\partial_0 N(\hat{B}) \subset F$. Then we can isotope $F$ back to its previous position, extending the isotopy to $A$, so now we have $\partial A \subset F$, $F \cap F = \emptyset$, and both $E$ and $F$ transverse to fibers. Each component of $\partial A$ must bound a disk in $F$, say the boundary components of $\partial A$ bound disks $D_0$ and $D_1$ in $F$. If isotoping $\partial E$ to $\partial D_0$ (keeping $E$ transverse to fibers) yields a sphere $S_0 = E \cup D_0$ carried by $\hat{B}$ and bounding a ball $K_0$, then up to isotopy of $S_0$, there is a component of $F$ in $K_0$, which is a contradiction. Similarly if isotoping $\partial E$ to $\partial D_1$ (keeping $E$ transverse to fibers) yields a sphere $S_1 = E \cup D_1$ carried by $\hat{B}$ and bounding a ball $K_1$, we obtain a contradiction. It follows that $A \cup D_0 \cup D_1$ bounds a ball $K$ containing a component $J_0$ of interstitial bundle for $F$ in $N(\hat{B})$, with $A \subset J_0$. $J_0$
must have the form $P \times I$, where $P$ is a planar surface, possibly not connected. This says that $D_0$ (or $D_1$) is isotopic to the disk of contact $E$, and we may assume that $E = D_0$. This completes the proof of Claim 2.

**Claim 3:** If $E$ is a disk of contact in $F$, then $E$ has minimal area among disks $G$ with $\partial G = \partial E$ in general position with respect to the triangulation.

Suppose $E$ does not have minimal area. Let $A$ be the annular component of $\partial_0 N(B)$ containing $\partial E$. We suppose $E \subset F$. Let $G$ be a disk with $\partial G \subset \text{int}(A)$, so $\partial G \cap F = \emptyset$, and $\gamma(G) < \gamma(E)$. We isotope $G$ so it is transverse to $F$. If $G$ is disjoint from $F$, then if $\partial A \subset F$, we can isotope $\partial G$ to $\partial E \subset F$, and we see that $F$ does not have minimal complexity, since we could replace $E$ by $G$. In general, if $G$ intersects $F$, let $\alpha$ be a closed curve of intersection innermost in $G$, bounding a disk $H$ in $G$ and a disk $D$ in $F$. Then $\gamma(D) \leq \gamma(H)$ since $F$ has minimal area. Homotoping $H$ to $D$ and slightly beyond, we obtain a new immersed disk of contact $G_1$ with $\gamma(G_1) \leq \gamma(G)$. Also $G_1$ intersects $F$ in fewer curves than $G$. In order to replace $G_1$ by an embedded disk with the same boundary, we perform certain cut-and-paste operations on selected curves of self-intersection of $G_1$, one by one. Consider a curve $\alpha$ of self-intersection innermost on $G_1$, bounding a disk $E_1$ in $G_1$ whose interior contains no other curves of self-intersection. It is paired with another curve of self-intersection viewed in $G_1$ bounding a disk $E_1'$ in $G_1$. If $E_1$ and $E_1'$ are disjoint in $G_1$, then a suitable cut-and-paste operation on $\alpha$ interchanges $E_1$ and $E_1'$ in $G_1$ and yields a new immersed disk with fewer closed curves of self-intersection and the same area. If $E_1' \supset E_1$, then a suitable cut-and-paste operation on $\alpha$ yields a new immersed disk with fewer curves of self-intersection and area no larger, namely $E_1 \cup (G_1 \setminus E_1')$. Repeating this argument, we obtain an embedded disk $G_2$ without increasing the area. We also have not introduced new curves of intersection with $F$. If $G_2$ is not disjoint from $F$, we now repeat the entire argument, by considering a curve $\alpha$ of $G_2 \cap F$ innermost in $G_2$ and performing cut-and-paste operations as before. Again, we obtain a new embedded disk $G_3$ with fewer curves of intersection with $F$, without increasing area. Ultimately, we obtain some $G_k$ disjoint from $F$, which gives a contradiction.

This completes the proof of Claim 3.

As we explained above, now for each $\hat{B}$ constructed from a normal minimal area taut surface $F$, there is a finite collection of minimal area disks of contact in $F$, such that splitting on all of these disks of contact gives a branched surface $B$ carrying $F$, with $\partial B$ a sub-train track of $\tau$. More precisely, we saw that for every disk of contact $E$ in $\hat{B}$ with boundary in an annulus $A \subset \partial_0 \hat{B}$, there exist discs $D_0$ and $D_1$ in $F$ such that $A \cup D_0 \cup D_1$ is a sphere bounding a ball containing a component of interstitial bundle for $F$ in $N(\hat{B})$ of the form $P \times I$, where $P$ is planar and $A \subset \partial P \times I$. We split to eliminate the disc of contact by removing $P \times I$ and other components of interstitial bundle in $K$ from the interstitial bundle. There are finitely many ways to do this because $D_0$ and $D_1$ are least area disks, as we proved in Claim 3, so there are finitely many possibilities for $D_0$ and $D_1$. We eliminate all disks of contact in this way. So there are finitely many normal branched surfaces $B$ without disks of contact which carry all the minimal area surfaces $F$ carried by $\hat{B}$. We showed that every taut surface $F$ with $\partial F$ carried by $\tau$ is carried by one of finitely many $\hat{B}$’s, so the end result is that we have finitely many branched surfaces $B_i$ which carry all taut surfaces $F$, have no disks of
contact, and do not carry spheres.

The goal now is to show that each \( B_i \) is orientable such that the orientation agrees with the orientations of the taut surfaces. We use ideas from [3]. In the cited paper, oriented surfaces are assumed to have minimal \( \chi \) among oriented surfaces representing the same homology class, which is a little different from our context, but the ideas carry through.

![Figure 8: Replacing \( F \) by \( F' \).](image)

Let \( B \) denote one of the \( B_i \), and we suppose \( F \) is a minimal area taut surface fully carried by \( B \), with \( \partial_h N(B) \subset F \). We let \( J \) denote the interstitial bundle for \( F \). We first observe that no component of \( J \) can have the form \( P \times I \) where \( P \) is planar and \( P \times \{0\} \) is contained in a disk \( D_0 \) in \( F \). In this case \( B \) would have a disk of contact.

Before proceeding, we define an augmented interstitial bundle \( \bar{J} \). The bundle \( \bar{J} \) is obtained from \( J \) by including the ball components of the completion of \( M \setminus N(B) \), which can be given a product structure of the form \( D^2 \times I \) with \( D^2 \times \partial I \) in the horizontal boundary and \( \partial D^2 \times I \) being a product component of the vertical boundary. We have filled in disk \( \times \) interval “bubbles.”

Now we claim that the orientations of \( F \) at opposite ends of fibers of the interstitial bundle for \( F \) in \( N(B) \) must be consistent, to yield an orientation for \( B \). If not, we consider a component \( \bar{J}_0 \) of the augmented interstitial bundle with the property that the orientations of \( F \) at opposite ends of the fiber are inconsistent. A key observation in our situation is that \( \bar{J}_0 \) cannot intersect \( \partial M \), otherwise we have inconsistent orientations of \( C = \partial F \) carried by \( \tau \). (Recall that we chose \( C \) to be carried as an oriented curve system obtaining its orientation from the orientation of \( \tau \).) Clearly \( \bar{J}_0 \) cannot be a product of the form disk \( \times I \), since this would mean \( B \) has a disk of contact. We modify \( F \) as shown in Figure 8 by replacing \( \partial_h \bar{J}_0 \) by \( \partial \bar{J}_0 \). This yields a new oriented surface \( F' \) which has the same boundary exactly, and which can be put in normal position, although it is not normal. If we show that \( F' \) contains no sphere components and \( \chi(F') > \chi(F) \) or \( \chi(F') = \chi(F) \) and the complexity (area) of \( F' \) can be made smaller than that of \( F \) when \( F' \) is isotoped to normal form, then we have a contradiction. First we show that \( F' \) contains no sphere components. Suppose \( F' \) contains a sphere component \( S \), then \( S \) contains at least one annulus of \( \partial_h \bar{J}_0 \). A curve of \( \partial(\partial_h \bar{J}_0) \) innermost on \( S \) yields a disk in \( \partial_h N(B) \), which would have to be in the boundary of a ball of the form \( D^2 \times I \), which we filled in \( \bar{J} \). This is a contradiction. The same argument actually shows that \( F' \) does not contain any “new” disk components intersecting \( \partial_h \bar{J}_0 \). Now it remains to show that either \( \chi(F') > \chi(F) \) or \( F' \) can be isotoped to reduce complexity. If \( \bar{J}_0 \) is a bundle over an annulus, \( \chi(F') = \chi(F) \), but clearly \( F' \) can be isotoped to a normal surface with smaller area than \( F \). Otherwise, \( \chi(\partial_h \bar{J}_0) < 0 \), so \( \chi(F') > \chi(F) \), which contradicts our choice of \( F \).

We conclude that \( F \) is fully carried by one of finitely many oriented normal branched surfaces \( B \), obtained from possible \( \hat{B} \)'s by splitting on finitely many least area disks of contact of each \( \hat{B} \). By construction, no \( B_i \) has a disk of contact. By Claim 1, no \( B_i \) carries a sphere. \( \square \)
Definition 3.7. Suppose \((L, \mu)\) is an oriented measured lamination in \(\partial M\) which bounds some 2-dimensional Seifert lamination. If \(B \hookrightarrow M\) is a properly embedded oriented branched surface in \(M\), we say \(B\) is aspherical if it carries no spheres. We define

\[
X((L, \mu)) := \sup\{\chi(B(v)) : B \text{ aspherical and } \partial B(v) = (L, \mu)\}.
\]

If \(\tau \hookrightarrow \partial M\) is an oriented train track we define a function \(X\) on \(\mathcal{CB}(\tau)\) as follows:

\[
X(w) = X(\tau(w)) := \sup\{\chi(B(v)) : \partial B(v) = \tau(w) \text{ and } B \hookrightarrow M \text{ is aspherical}\}.
\]

A Seifert lamination \((\Lambda, \mu) = B(v)\) for \(\tau(w)\) in \(\partial M\) is taut if \(\chi(B(v)) = X(\tau(w))\) and \(B\) is aspherical.

As we mentioned in the introduction, the property of tautness of a Seifert lamination generalizes the “minimal genus” property of a Seifert surface.

There is a problem with this definition: we do not know whether the supremum is finite. We address this in the following theorem. For weight vectors whose entries are not rationally related, the question which immediately comes to mind is whether the supremum in the definition of \(X(w)\) is achieved by a 2-dimensional measured lamination \(B(v)\). In other words, does a peripheral lamination which bounds a Seifert lamination bound a taut Seifert lamination. We shall prove this in the next theorem. The second question that comes to mind is whether the function \(X\) is continuous on \(\mathcal{CB}(\tau)\). We shall see later that this is also true.

Theorem 3.8. Let \((L, \mu)\) be an oriented measured lamination in \(\partial M\) and suppose it is fully carried by an oriented train track \(\tau\) so \((L, \mu) = \tau(w)\). If \((L, \mu)\) bounds a Seifert lamination, i.e. \(w \in \mathcal{CB}(\tau)\), then there exists a taut Seifert lamination \((\Lambda, \mu)\) with \(\partial(\Lambda, \mu) = \tau(w)\) and \(\chi((\Lambda, \mu)) = X(\tau(w)) = \sup\{\chi(B(v)) : \partial B(v) = \tau(w), B \text{ aspherical}\}\). \(X((L, \mu))\) is finite. The lamination \((\Lambda, \mu)\) is carried (but not necessarily fully carried) by one of the finite collection \(\{B_i\}\) of Lemma 3.6.

Proof. Without loss of generality \(w \in \mathcal{PCB}(\tau)\), which means the sum of its entries is 1. There exists a sequence of 2-dimensional laminations \(C_n(v_n)\) with \(\partial C_n(v_n) = (L, \mu)\) and \(C_n\) an aspherical branched surface such that \(\chi(C_n(v_n)) \to X(L, \mu)\). By splitting \(C_n\) sufficiently along leaves of \(C_n(v_n)\), we can assume \(\partial C_n\) is a splitting of \(\tau\), which means any lamination carried by \(\partial C_n\) is carried by \(\tau\). Splitting preserves the asphericity of \(C\), so \(C_n\) is aspherical. We can replace \(v_n\) by invariant weight vectors with only rational entries such that it is still true that \(\chi(C_n(v_n)) \to X(L, \mu)\). We can further require that the sum of the weights induced by \(\partial v_n\) on \(\tau\) is 1. This can be achieved by dividing each entry of \(v_n\) by the sum of the entries induced by \(\partial v_n\) on \(\tau\), which still gives rational entries. Now \(C_n(y_n)\) is a rational weighted surface; in other words, there exist integers \(K_n\) so that \(C_n(K_n y_n) = F_n\) is a surface without sphere components and \(\partial F_n\) is carried by \(\tau\), say \(\partial F_n = \tau(y_n)\). Then \(K_n\) is the sum of the entries of \(y_n\) and the vectors \(w_n = y_n/K_n\) are in \(\mathcal{PCB}(\tau)\) and are the invariant weight vectors induced on \(\tau\) by \(\partial C_n(v_n)\). We know \(\chi(C_n(v_n)) = \chi(F_n)/K_n \to X(w)\). We can replace \(F_n\) by a taut surface \(F'_n\), with \(\partial F'_n = \partial F_n\) and \(\chi(F'_n) \geq \chi(F_n)\), where \(F'_n\) is normal with minimal area with respect to the triangulation chosen in the proof of Lemma 3.6 so it
is fully carried by one of the finite collection \{B_i\} in that lemma. Let us say \( F'_n \) is carried by \( \mathcal{B}_n \). We then also have \( \chi(F'_n) \geq \chi(F_n) = \chi(C_n(K_n, \nu_n)) \).

Suppose \( F'_n = B_i(z_n) \), where \( z_n \) has integer entries. We have constructed \( F'_n \) so \( \partial F'_n = \partial F_n \) are both carried by \( \tau \) and induce the same weights. The sum of the entries of \( \partial z_n \) is \( K_n \). Then \( \partial B_i(z_n/K_n) \to \tau(w) \). We let \( p \) be the number of segments of \( \tau \), after adding a vertex to each closed curve component of \( \tau \). By passing to a subsequence, we can assume \( i_n = i \) for all \( n \), and we can assume \( F'_n = B_i(z_n) \), and \( \partial B_i(z_n/K_n) \to \tau(w) \). (We pass to the corresponding subsequence in \{\( K_n \)\}. Unfortunately, we must normalize \( z_n \) differently, because entries of \( z_n/K_n \) may not be bounded. If we let \( L_n \) be the sum of the entries of \( z_n \), then \( z_n/L_n \in \mathcal{P}(\mathcal{B}_i) \). Passing to a subsequence, \( z_n/L_n \) converges to some \( u \in \mathcal{P}(\mathcal{B}_i) \). Since \( K_n \) is the sum of the the entries of \( \partial z_n \) (some of the entries of \( z_n \), with at most \( p \) repetitions each), we have \( K_n \leq pL_n \). (A sector of \( B_i \) may intersect \( \tau \) in more than one, but not more than \( p \), segments.) Passing to a subsequence again, we arrange that \( K_n/L_n \) converges, \( K_n/L_n \to \lambda \), where \( 0 \leq \lambda \leq p \). Suppose for now that \( \lambda \neq 0 \). Restricting to the boundary, \( \partial z_n/K_n \to w \), so \( \partial z_n/L_n = (\partial z_n/K_n)(K_n/L_n) \to \lambda w \). Then \( \partial z_n/L_n \to \lambda w \), or \( \partial z_n/\lambda L_n \to w \). In this case we can easily conclude that \( \chi(B_i(u/\lambda)) = X(w) \) using the follow sequence of equivalent inequalities:

\[
\begin{align*}
\chi(F'_n) & \geq \chi(F_n) \\
\chi(B_i(z_n)) & \geq \chi(F_n) \\
\chi(B_i(z_n/K_n)) & \geq \chi(F_n)/K_n \\
\chi(B_i) & \left( \frac{z_n}{K_n L_n} \right) \geq \left( \frac{K_n}{L_n} \right) \chi(F_n)/K_n.
\end{align*}
\]

Taking limits as \( n \to \infty \) we get:

\[
\begin{align*}
\chi(B_i(u)) & \geq \lambda X(w) \\
\chi(B_i(u/\lambda)) & \geq X(w)
\end{align*}
\]

Then \( B_i(u/\lambda) \) is a taut Seifert lamination since \( \partial B_i(u/\lambda) = \tau(w) \).

Now we rule out the possibility that \( \lambda = 0 \). If \( \lambda = 0 \), \( \partial z_n/L_n = (\partial z_n/K_n)(K_n/L_n) \to \lambda w = 0 \), which means \( B_i(u) \) is a closed lamination. Eliminating sectors of \( B_i \) assigned 0 entries by \( u \) we obtain an invariant weight vector \( u' \) on a closed sub-branched surface \( B'_i \) of \( B_i \). Since \( C(B'_i) \) is non-empty, \( B'_i \) carries a closed surface \( S = B_i(z) \) for some invariant weight vector \( z \) with integer entries. The weights induced by \( F'_n \) on \( B'_i \) must approach infinity as \( n \to \infty \), otherwise the normalized weights \( z_n/L_n \) on \( \tau \) cannot approach 0. Thus for sufficiently large \( n \) each entry of \( z \) is smaller than the corresponding entry of \( z_n \). In other words, the \( j \)-th entry \( z_j \) of \( z \) is less than the \( j \)-th entry \( z_{nj} \) of \( z_n \) (for the fixed large \( n \)). This means \( B_i(z_n - z) \) is a surface \( G \) with \( \partial F'_n = \partial G \). Since \( B_i \) does not carry spheres \( \chi(S) \leq 0 \). If \( \chi(S) < 0 \), then \( \chi(G) > \chi(F'_n) \) and \( F'_n \) is not a taut surface, a contradiction. If \( \chi(S) = 0 \), then \( \chi(G) = \chi(F'_n) \) but \( \gamma(G) = \gamma(F'_n) - \gamma(S) < \gamma(F'_n) \), a contradiction to our choice of \( F'_n \) as a minimal area normal representative in its isotopy class of a taut surface.

By ruling out the possibility that \( \lambda = 0 \), we have also proved that \( X((L, \mu)) \) is finite. \( \square \)
To show that taut Seifert laminations whose boundaries are carried by an oriented train track \( \tau \) are neatly organized by branched surfaces, we prove the following lemmas.

**Lemma 3.9.** Suppose \( \tau \leftarrow \partial M \) is an oriented train track. Let \( F_1, F_2, \ldots, F_m \) be taut surfaces embedded in \( M \) with \( \partial F_i \) carried by \( \tau \), \( \partial F_i = \tau(y_i) \), and let \( y = \sum i y_i \). Then there exists a taut surface \( F \) with \( \partial F = \tau(y) \), satisfying \( \chi(F) \geq \sum_i \chi(F_i) \).

**Proof.** We use induction. Consider the case \( m = 2 \). We have \( \partial F_1 \) and \( \partial F_2 \) carried by \( \tau \), so we assume that \( \partial F_1 \) and \( \partial F_2 \) are embedded in \( N(\tau) \) transverse to fibers, with \( \partial F_i = \tau(y_i) \), \( i = 1, 2 \). We also assume \( F_1 \) and \( F_2 \) are transverse. If \( F_1 \) and \( F_2 \) intersect on a closed curve which bounds a disk on one of the surfaces, we suppose without loss of generality that \( \alpha \) is such a curve innermost on \( F_1 \), bounding a disk \( E_1 \) in \( F_1 \). By the incompressibility of \( F_2 \), \( \alpha \) also bounds a disk \( E_2 \) in \( F_2 \). We can eliminate at least one curve of intersection between \( F_1 \) and \( F_2 \) by replacing \( E_2 \subset F_2 \) by \( E_1 \) and isotoping a little more to push \( F_2 \) away from \( E_2 \). After removing all closed curves of intersection trivial in \( F_1 \) or \( F_2 \), we perform oriented cut-and-paste on \( F_1 \cup F_2 \) on remaining curves of intersection to obtain \( G \). Note that \( \partial G = \tau(y_1 + y_2) \) and \( \chi(G) = \chi(F_1) + \chi(F_2) \). The cut-and-paste operation cannot introduce spheres by construction; if \( G \) contains other closed surfaces (with \( \chi < 0 \)), we can discard them to obtain an oriented surface \( F \) with the same boundary, and \( \chi(F) \geq \chi(G) \). If \( F \) is not taut, we can replace it with a surface with \( \chi(F) \) even larger.

Finishing the proof by induction is now straightforward. Assume we can replace \( F_1 \cup F_2, \ldots, F_{m-1} \) by a surface \( G \) with \( \chi(G) \geq \sum_{i=1}^{m-1} \chi(F_i) \), and with \( \partial G = \tau(\sum_{i=1}^{m-1} y_i) \). Then we can apply the case \( m = 2 \) to the two surfaces \( G \) and \( F_m \) to prove our lemma for any \( m \). \( \square \)

The branched surfaces \( B_i \) constructed in Lemma 3.6 are oriented aspherical branched surfaces. Therefore any sub- branched surface \( B \) of a \( B_i \) is also an oriented aspherical branched surface, and the following lemma applies to \( B \).

**Lemma 3.10.** Let \( B \leftarrow M \) be a \( \tau \)-essential oriented branched surface, \( \partial B = \tau \leftarrow \partial M \). Suppose \( B(\mathbf{v}) \) is taut and \( \mathbf{v} \) has strictly positive entries. Then any measured lamination carried by \( B \) is taut.

**Proof.** Let \( \mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_k \) be the vertices of the convex polytope \( \mathcal{PC}(B) \). These vertices must have all rational entries. By scaling, we can assume that the sum of the entries of \( \mathbf{v} \) is 1, which means \( \mathbf{v} \in \mathcal{PC}(B) \). Our first goal is to show that \( B(\mathbf{u}_i) \) is taut for each \( i = 0, 1, 2, \ldots, k \). We will show \( B(\mathbf{u}_0) \) is taut by contradiction. Suppose \( B(\mathbf{u}_0) \) is not taut. Then there exists an aspherical branched surface \( B' \) with \( \partial B' \) a sub- train track of \( \tau \) and there exists an invariant weight vector \( \mathbf{u}_i' \) for \( B' \) such that \( \partial B'(\mathbf{u}_i') = \partial B(\mathbf{u}_0) \) and \( \chi(B'(\mathbf{u}_i')) > \chi(B(\mathbf{u}_0)) \). Since \( \chi(B'(\mathbf{u}_i')) \) is linear on \( \mathcal{PC}(B') \) we can assume all entries of \( \mathbf{u}_i' \) are rational. Now \( \mathbf{v} \) can be expressed as a convex combination of the \( \mathbf{u}_i \)'s, \( \mathbf{v} = \sum_{j=0}^k t_j \mathbf{u}_j \). Approximating \( t_j \)'s by rational numbers, we can find sequences \( r_{n,j} \) of rational numbers such that \( \sum_{j=0}^k r_{n,j} \mathbf{u}_j \rightarrow \mathbf{v} \) and \( \sum_{j=0}^k r_{n,j} = 1 \). Of course then the linearity of the Euler characteristic function implies \( \chi(B(\sum_{j=0}^k r_{n,j} \mathbf{u}_j)) \rightarrow \chi(B(\mathbf{v})) \). For a fixed \( n \), scaling all weight vectors by a sufficiently large integer \( K_n \) ensures that \( K_n r_{n,j} \mathbf{u}_j, j = 0, 1, \ldots, k \), and \( K_n r_{n,j} \mathbf{u}_j \) all have only integer entries. This means \( B(\sum_{j=0}^k K_n r_{n,j} \mathbf{u}_j) \) is a surface \( F_n \), while \( B(K_n r_{n,j} \mathbf{u}_j) \) is a surface \( F_{nj} \) and \( B'(K_n r_{n,j} \mathbf{u}_j) \) is a surface \( F'_{nj} \).
Now we have
\[
\chi(B(v)) = \lim_{n \to \infty} \sum_{j=0}^{k} \chi(B(r_{n,j}u_j)) = \lim_{n \to \infty} \sum_{j=0}^{k} \frac{1}{K_n} \chi(B(K_n r_{n,j}u_j)) = \\
= \lim_{n \to \infty} \sum_{j=0}^{k} \frac{1}{K_n} \chi(F_{n,j}) = \lim_{n \to \infty} \frac{1}{K_n} \chi(F_n).
\]
We will arrive at a contradiction by replacing \( F_{n_0} \) in the above by \( F'_{n_0} \) as follows. Since \( \chi(B'(u'_0)) > \chi(B(u_0)) \), say \( \chi(B'(u'_0)) - \chi(B(u_0)) = \epsilon > 0 \), we also have
\[
\chi(F'_{n_0}) - \chi(F_{n_0}) = \chi(B'(K_n r_{n_0}u'_0)) - \chi(B(K_n r_{n_0}u_0)) = \\
= K_n r_{n_0} \chi(B'(u'_0)) - \chi(B(u_0)) = K_n r_{n_0} \epsilon > 0.
\]
We apply Lemma 3.9 to the surfaces \( F'_{n_0}, F_1, F_2, \ldots F_n \). Using the lemma, by suitable isotopy and oriented cut-and-paste of these surfaces (all of whose boundaries are carried by \( \tau \)) we obtain a surface \( F'_n \) with \( \chi(F'_n) \geq \chi(F_{n_0}) + \sum_{i=1}^{k} \chi(F_{n,i}) \). Then \( \chi(F'_n) - \chi(F_n) \geq \left[ \chi(F'_{n_0}) + \sum_{i=0}^{k} \chi(F_{n,i}) \right] - \sum_{i=0}^{k} \chi(F_{n,i}) = \chi(F'_{n_0}) - \chi(F_{n_0}) = K_n r_{n_0} \epsilon \). The surface \( F'_n \) has the same boundary as \( F_n \), and is carried by \( \tau \).

We replace \( F'_n \) by a taut surface, possibly with greater \( \chi \) and still with the same boundary. Then, passing to a subsequence, we can assume \( F'_n \) is normal of minimal area and carried by one of the normal branched surfaces \( B_i \) of Lemma 3.6 with \( \partial F'_n \) unchanged and carried by \( \tau \). So we suppose \( F'_n = B_i(z_n) \). Now we can proceed exactly as in the proof of Theorem 3.8 to take a limit of normalized weighted surfaces \( F'_n/L_n = B_i(z_n/L_n) \) where \( L_n \) is the sum of the entries of \( z_n \), \( z_n/L_n \in \mathcal{PC}(B) \). After passing to a subsequence again, we can arrange that \( z_n/L_n \to u \in \mathcal{PC}(B) \). Now we work backwards to compare \( \chi(B_i(u)) \) and \( \chi(B(v)) \):
\[
\chi(F'_n) - \chi(F_n) \geq K_n r_{n_0} \epsilon \\
\frac{1}{K_n} \chi(F'_n) - \frac{1}{K_n} \chi(F_n) \geq r_{n_0} \epsilon \\
\frac{1}{K_n} \chi(B_i(z'_n)) - \chi(B_i \left( \frac{z'_n K_n}{L_n} \right)) \geq r_{n_0} \epsilon \\
\chi \left( B_i \left( \frac{z'_n K_n}{L_n} \right) \right) - \chi \left( B \left( \frac{K_n}{L_n} \sum_{j=0}^{k} r_{n,j}u_j \right) \right) \geq \frac{K_n}{L_n} r_{n_0} \epsilon.
\]
As in the proof of Theorem 3.8 after passing to a subsequence we can assume \( K_n/L_n \to \lambda \), where \( 0 \leq \lambda \leq p \). As before, we rule out the possibility that \( \lambda = 0 \), showing that otherwise \( F'_n \) is either not taut or is taut but not of minimal area. Taking limits in the last displayed equation above, we get
\[
\chi(B_i(u)) - \lambda \chi(B(v)) \geq \lambda t_0 \epsilon, \quad \chi(B_i(u/\lambda)) - \lambda \chi(B(v)) \geq t_0 \epsilon.
\]
This contradicts the tautness of \( B(v) \) since \( \partial B_i(u/\lambda) = \partial B(v) \).

We can use the same argument to prove that for any \( u_0 \) in \( \mathcal{PC}(B) \) with all rational entries, \( B(u_0) \) is taut. In this case, we assume the extremal points of \( \mathcal{PC}(B) \) are \( u_1, u_2, \ldots, u_k \). Then
we can write $v$ as a convex combination of $u_0, u_1, u_2, \ldots, u_k$ as before, $v = \sum_{i=0}^k t_i u_i$ with $t_0 > 0$, and proceed exactly as before.

Finally, we must show that if $u$ is any point in $P\tilde{C}(B)$, then $B(u)$ is taut. Let $\hat{B}$ be the sub-branched surface of $B$ which fully carries $B(u)$, and let $B(u) = \hat{B}(\hat{u})$ where $\hat{u}$ is the restriction of $u$ to $\hat{B}$. Suppose $B(u)$ is not taut. Then there exists a branched surface $C$ with $\partial C = \partial \hat{B}$ a sub-train track of $\tau$ and an invariant weight vector $y$ for $C$ such that $\chi(C(y)) > \chi(B(u)) = \chi(\hat{B}(\hat{u}))$ and $\partial C(y) = \partial \hat{B}(\hat{u})$. Identifying $\partial C \subset C$ with $\partial \hat{B} \subset \hat{B}$ we obtain a closed branched surface $A$. In fact, we can embed $A$ in the double of the manifold $M$. Also, we have an invariant weight vector $a$ for $A$ whose restriction to $\hat{B}$ and $C$ is $\hat{u}$ and $y$ respectively. We choose a sequence of weight vectors $a_n$, for $A$, each with only rational entries, such that $a_n \to a$. By restricting to $\hat{B}$ and $C$ we get sequences $b_n$ and $c_n$, each with only rational entries, and with $b_n \to \hat{u}$, $c_n \to y$. Since $\lim_{n \to \infty} \chi(C(c_n)) = \chi(C(y)) > \chi(B(u)) = \lim_{n \to \infty} \chi(\hat{B}(b_n))$, we must have $\chi(C(c_n)) > \chi(\hat{B}(b_n))$ for some sufficiently large $N$. On the other hand, since $\partial C(c_N) = \partial \hat{B}(b_N)$, it follows that $\hat{B}(b_N)$ is not taut. This contradicts what we have already proved: that for any rational $u_0$, $B(u_0)$ is taut. □

**Definition 3.11.** Let $B \hookrightarrow M$ an oriented branched surface with $\partial B$ a sub-train track of $\tau$. Then $B$ is $\tau$-taut if every oriented measured lamination carried by $B$ is taut.

Lemma [3.10] tells us that if $B$ is an oriented aspherical branched surface embedded in $M$ and $B$ fully carries a taut lamination, then $B$ is $\tau$-taut.

**Theorem 3.12.** Suppose $M$ is a compact, orientable 3-manifold. Suppose $\tau \hookrightarrow \partial M$ is an oriented train track embedded in $\partial M$.

(a) There exists a finite collection $\{T_1, T_2, \ldots, T_k\}$ of $\tau$-taut branched surfaces such that for every $w \in \tilde{C}(\tau)$, there is a taut $T_j(w)$ such that $\partial T_j(w) = \tau(w)$ (This is not the same collection of branched surfaces as the $\{B_i\}$ in Lemma 3.6)

(b) The function $X$ is linear on rays and finite piecewise linear on the convex subcone $\tilde{C}(\tau)$ of $C(\tau)$ corresponding to oriented measured laminations $\tau(w)$ bounding oriented 2-dimensional measured laminations in $M$. The function $X$ is concave.

**Proof.** We first give the proof for irreducible $M$. To prove statement (a) of the theorem, we will use the collection $\{B_i\}$ of Lemma 3.6. The branched surfaces $T_j$ will be sub-branched surfaces of some of the $B_i$. Suppose $w \in \tilde{C}(\tau)$. By Theorem 3.8, there is a sub-branched surface $B$ of a $B_i$ in the collection from Lemma 3.6 and an invariant weight vector $v$ with strictly positive entries such that $\partial B(v) = \tau(w)$ and $B(v)$ is taut. By Lemma 3.10 $B$ is $\tau$-taut. This $B$ will be one of our branched surfaces $T_j$. As $w$ varies over $\tilde{C}(\tau)$, the possibilities for $B$ are finite, since $B$ is a sub-branched surface of a $B_i$. Thus finitely many taut $T_j$ suffice to carry all laminations whose boundaries have the form $\tau(w)$. Note that there may be branched surfaces in the collection $\{B_i\}$ which are not taut.

We use Proposition 3.13 below, to extend the result to reducible manifolds.

To prove statement (b) of the theorem, we observe that $\chi(T_j(v))$ is a linear function of $v \in \tilde{C}(T_i)$. Because $T_j$ is taut, $\chi(T_j(v)) = X(\partial T_j(v))$ and depends only on $w = \partial v$. This shows $X$ is linear on $\partial \tilde{C}(T_j) = \{w : w = \partial v, \text{ for some } v \in \tilde{C}(T_j)\}$. This proves the piecewise linearity of $X$. To prove the concavity of $X$ we apply Lemma 3.9. The lemma says $X$ is
concave on integer lattice points in $\mathbb{C}B(\tau)$, so it is also concave on rational points in $\mathbb{C}B(\tau)$, so the continuity of $X$ easily proves concavity on all of $\mathbb{C}B(\tau)$. \hfill \Box

We use the following proposition to deal with peripheral links in reducible 3-manifolds.

**Proposition 3.13.** Suppose the compact, orientable 3-manifold $M$ has a prime decomposition $M = M_1 \# M_2 \# \cdots \# M_k$. Suppose for each $i$ that $\tau_i(\omega_i)$ is a possibly empty peripheral lamination link in $\partial M_i$. Let $\tau = \cup_i \tau_i \subset \partial M$ with invariant weight vector $\omega$ assigning the same weights as the $\omega_i$’s. Then $\tau_i(\omega_i)$ bounds a 2-dimensional measured lamination in $M_i$ for each $i$ if and only if $\tau(\omega)$ bounds in $M$. Further, $X(\tau(\omega)) = \sum_i X(\tau_i(\omega_i))$ and there is an oriented aspherical branched surface $B \hookrightarrow M$ and a taut Seifert lamination $B(\nu)$ such that $\partial B(\nu) = \tau(\omega)$. $(X(\tau_i(\omega_i)))$ refers to Seifert laminations in $M_i$.

**Proof.** Let $S = S_1 \cup S_2 \cup \cdots \cup S_m$ be a union of essential spheres in $M$, such that cutting $M$ on $S$ and yields a collection of holed irreducible 3-manifolds, $M'_1, M'_2, \ldots, M'_k$. Capping the boundary spheres yields $M_1, M_2, \ldots, M_k$.

One implication is easy, namely if $\tau_i(\omega_i)$ bounds $(\Lambda_i, \mu_i)$ in $M_i$ for each $i$, then $\tau(\omega)$ bounds $\cup_i \Lambda_i$ in $M$, since we can assume $\Lambda_i$ lies in $M'_i \subset M$. This also shows $X(\tau(\omega)) \leq \sum_i X(\tau_i(\omega_i))$.

For the converse, we will show that if $(\Lambda, \nu)$ is a Seifert lamination in $M$, then we can replace it with a Seifert lamination $(\Lambda', \nu')$ disjoint from $S$. We isotope $\Lambda$ so it is transverse to $S$. Then, because $\Lambda$ is measured, the intersection $\Lambda \cap S_i$ with one of the sphere components of $S$ is a finite collection of measured families of isotopic closed curves. We choose an innermost family on $S_i$, and we surger the lamination just as we would surger a surface. Repeating this process a finite number of times yields a measured lamination disjoint from $S$, and we let $\Lambda_i$ be the component in $M'_i$, which can also be regarded as a lamination in $M_i$.

Next we must show $X(\tau(\omega)) \leq \sum_i X(\tau_i(\omega_i))$. Suppose not. Then $X(\tau(\omega)) > \sum_i X(\tau_i(\omega_i))$. By the continuity of $X$, there exists an invariant weight vector $r$ for $\tau$ with only rational entries such that $X(\tau(r)) > \sum_i X(\tau_i(r_i))$, where $r_i$ is the restriction of $r$ to $\tau_i$. Suppose $X(\tau(r)) = \sum_i X(\tau_i(r_i)) + \epsilon$, where $\epsilon > 0$. Then we can find a Seifert lamination $B(\omega)$ for $\tau(r)$, where $B$ is an aspherical oriented branched surface, $\omega$ has only rational entries, $\partial B(\omega) = \tau(r)$, and $\chi(B(\omega)) > X(\tau(r)) + \epsilon/2$. So $B(\omega)$ is a weighted surface which we will denote $qF$ (abusing notation), where $q$ is a rational weight on $F$. $F$ has no sphere components. Now we make $F$ transverse to $S$. Consider a closed curve of $F \cap S$ innermost on $S$, bounding a disk $H$ in $S$. Surgering $F$ using $H$ yields a new oriented surface $F'$ with fewer intersecting $S$ in fewer curves. If surgery produces a sphere component, we discard it, and $\chi(F)$ is unchanged. Otherwise $\chi(F)$ increases. Repeating surgeries to eliminate intersections with $S$, we end with a new $F$ disjoint from $S$, and with $\chi(F)$ no smaller than for the original $F$. We can write $F = \cup_i F_i$ where $F_i \subset M'_i$. For this $F$, we have $\chi(qF_i) > X(\tau(r_i)) + \epsilon/2$. Hence $\sum_i \chi(qF_i) > X(\tau(r)) + \epsilon/2$. But $\chi(qF_i) \leq X(\tau(r_i))$, because $qF_i$ is a Seifert lamination for $\tau(r_i)$ in $M_i$, carried by an aspherical branched surface (namely $F_i$). This implies $X(\tau(r)) + \epsilon/2 < \sum_i X(qF_i) \leq \sum_i X(\tau(r_i))$, which contradicts our previous statement that $X(\tau(r)) > \sum_i X(\tau(r_i))$. \hfill \Box
4 From peripheral to arbitrary links.

Theorem 1.8 could be regarded as a corollary of Theorem 3.8. It extends the results of the previous section to non-peripheral lamination links, and it also extends the results to compact, reducible, orientable 3-manifolds $M$.

The definition of taut Seifert laminations for non-peripheral links is the same as for peripheral links, and so is the definition of $X$ as a function on the set of links.

**Definition 4.1.** If $(L, \mu)$ is a link in $M$

$$X((L, \mu)) := \sup \{ \chi(B(v)) : B \hookrightarrow M, \ B \text{ aspherical and } \partial B(v) = (L, \mu) \}.$$  

A Seifert lamination $(\Lambda, \mu) = B(v)$ for a link $(L, \mu)$ is taut if $\chi(B(v)) = X((L, \mu))$ and $B$ is aspherical.

The added subtlety in the non-peripheral case comes from the fact that if we represent a link by a fixed $V_w(\tau)$, a (taut) Seifert lamination may not be disjoint from $V_w(\tau)$. This means that we must choose our representation of the link $(L, \mu)$ as $V_w(\tau)$ more carefully. It turns out that one must choose an irreducible representative $V_w(\tau)$.

**Proof of Theorem 1.8.** We represent a lamination link $(L, \mu)$ in a compact, orientable 3-manifold as $V_w(\tau) \hookrightarrow \text{int}(M)$ for some train track neighborhood with no complementary digons. Using a slight modification which suitably separates leaves and possibly replaces isolated leaves by families of parallel leaves, we can embed the lamination in the 2-dimensional fibered neighborhood $N(\tau) \hookrightarrow M$ transverse to fibers with $\partial_0 N(\tau) \subset L$, see Section 2, where $N(\tau)$ corresponds to $V(\tau)$. If the interstitial bundle contains components which are product bundles of the form $I \times I$, we can eliminate these to obtain a splitting of $\tau$ which also carries $L$. Without loss of generality, we therefore assume $V_w(\tau)$ is irreducible. Let $B(v) = V_\tau(B)$ represent any Seifert lamination, so $V_w(\tau) = \partial V_\tau(B)$. Let $R$ be a 3-dimensional regular neighborhood of $\tau$ such that $R \cap V(B) = V(\tau)$. ($R$ is a “fattening” of $V(\tau)$ on one side.) Then $V(\tau)$ is a fibered neighborhood of a train track $\tau$ in $\partial M'$, where $M' = M \setminus \text{int}(R)$.

For now, we will assume that $M'$ is irreducible. Later, we will consider the case that $M'$ is reducible.

We apply Theorem 3.12 with $V(\tau) \hookrightarrow \partial M'$ to show that there is a lamination $(\Lambda, \nu)$ which we denote $B(v)$, whose boundary is $(L, \mu) = V_w(\tau)$ which achieves the supremum in

$$X(V_w(\tau)) = \sup \{ \chi(B(v)) : (B, \partial B) \hookrightarrow (M', \partial M'), \ \partial B(v) = \tau(w), \ B \text{ aspherical} \}.$$  

A priori, it is possible however, that there is a Seifert lamination in $M$ which intersects the interior of $R$ and has larger $\chi$. We can rule this out as follows. If such a Seifert lamination existed, say the Seifert lamination $(\Lambda', \nu') = B'(v')$, $B'$ aspherical, with boundary $V_w(\tau')$ representing the same lamination link, then without loss of generality we can assume $\tau'$ is a regular splitting of $\tau$. Let $J$ be the splitting bundle giving $N(\tau) = N(\tau') \cup J$. Then, by our choices, every component of the splitting bundle is a rectangle $I \times I$ with exactly one end of the rectangle in $\partial_0 N(\tau')$. Consider a component $J_0$ of $J$ with exactly one fiber in $\partial_0 N(\tau')$. The rectangle $J_0$ has 3 sides in $\partial N(\tau')$ with one “free side” which lies in $\partial_0 N(\tau)$. Therefore,
we can simply isotope the intersection of $N(B')$ with int($J_0$) out of $J_0$. Or equivalently, we push the intersections of $\Lambda'$ with the interior of $J_0$ out of $J_0$, see Figure 9.

After performing these isotopies for all components of $J$, we have $\Lambda'$ disjoint from int($R$), if $R$ is chosen sufficiently small. Thus $(\Lambda', \nu')$ is a Seifert lamination for $(L, \mu)$ in $(M', \partial M')$, which is a contradiction to our assumption that $\chi((\Lambda', \nu')) > \chi((\Lambda, \nu))$. We conclude that $(\Lambda, \mu)$ is a taut Seifert lamination.

We use Proposition 3.13 to extend our result to the case where $M'$ is reducible. □

**Proof of Theorem 1.10.** We saw in the proof of Theorem 1.8 that given a lamination link represented as $V_w(\tau)$ which is irreducible and bounds a Seifert lamination, then we can isotope the Seifert lamination into the complement $M'$ of int($R$), where $R$ is a regular neighborhood in $M$ (on one side) of $V_w(\tau)$. Thus the link can be regarded as peripheral in $M'$. Now we take a slightly different point of view. Given a framing $V(\tau)$ of a train track $\tau \hookrightarrow M$, we consider all the invariant weight vectors $w$ such that $V_w(\tau)$ is irreducible, and can therefore be regarded as peripheral in $M'$. This is the subspace $\mathcal{CB}(\tau)$ of $\mathcal{C}(\tau)$. Since now $V(\tau) \hookrightarrow \partial M'$ and $w \in \mathcal{CB}(\tau)$, a Seifert lamination for $V_w(\tau)$ is taut in $M'$ if and only if it is taut in $M$. Now if $M'$ is irreducible by Theorem 3.12, there exist finitely many $\tau$-taut branched surfaces, $\{T_1, T_2, \ldots, T_p\}$ of $\tau$-taut branched surfaces in $M'$ such that for every $w \in \mathcal{CB}(\tau)$, there is a taut $T_i(v)$ such that $\partial T_i(v) = \tau(w)$. This shows that $X$ is continuous (and piecewise linear) on $\mathcal{CB}(\tau)$, where $X$ refers to the function defined for Seifert laminations in $M$. Using Proposition 3.13 we can extend this to the case that $M$ and $M'$ are reducible. □

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