ENTIRE SOLUTIONS OF SUBLINEAR ELLIPTIC EQUATIONS IN ANISOTROPIC MEDIA

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Abstract. We study the nonlinear elliptic problem

$$-\Delta u = \rho(x)f(u) \quad \text{in} \quad \mathbb{R}^N, \quad N \geq 3,$$

where $\ell \geq 0$ is a real number, $\rho(x)$ is a nonnegative potential belonging to a certain Kato class, and $f(u)$ has a sublinear growth. We distinguish the cases $\ell > 0$ and $\ell = 0$ and we prove existence and uniqueness results if the potential $\rho(x)$ decays fast enough at infinity. Our arguments rely on comparison techniques and on a theorem of Brezis and Oswald for sublinear elliptic equations.

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1 Introduction and main results

In their celebrated paper [3], Brezis and Kamin have been concerned with various questions related to the existence of bounded solutions of the sublinear elliptic equation without condition at infinity

$$-\Delta u = \rho(x)u^\alpha \quad \text{in} \quad \mathbb{R}^N, \quad N \geq 3,$$

where $0 < \alpha < 1$, $\rho \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, $\rho \geq 0$ and $\rho \neq 0$. We summarize in what follows the main results obtained in [3]. Brezis and Kamin proved that the nonlinear problem (1) has a bounded solution $u > 0$ if and only if the linear problem

$$-\Delta u = \rho(x) \quad \text{in} \quad \mathbb{R}^N$$

has a bounded solution. In this case, Problem (1) has a minimal positive solution and this solution satisfies $\liminf_{|x| \to \infty} u(x) = 0$. Moreover, the minimal solution is the unique positive solution of (1) which tends to zero at infinity. Brezis and Kamin also showed that if the potential $\rho(x)$ decays fast enough at infinity then Problem (1) has a solution and, moreover, such a solution does not exist if $\rho(x)$ has a slow decay at infinity. For instance, if $\rho(x) = (1 + |x|^p)^{-1}$, then (1) has a bounded solution if and only if $p > 2$. More generally, Brezis and Kamin have proved that Problem (1) has a bounded solution if and only if $\rho(x)$ is potentially bounded, that is, the mapping $x \mapsto \int_{\mathbb{R}^N} \rho(y)|x-y|^{2-N}dy \in L^\infty(\mathbb{R}^N)$. We refer to [4,12] for various results on bounded domains for sublinear elliptic equations with zero Dirichlet boundary condition. Problem (1) in the whole space has been considered in [5,6,8,9,11,13,14] under various assumptions on $\rho$. Sublinear problems (either stationary or evolution ones) appear in the study of population dynamics, of reaction-diffusion processes, of filtration in porous media with absorption, as well as in the study of the scalar curvature of warped products of semi-Riemannian manifolds (see, e.g., [16]).

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Our purpose in this paper is to study the problem

$$\begin{cases}
-\Delta u = \rho(x)f(u) & \text{in } \mathbb{R}^N \\
u > \ell & \text{in } \mathbb{R}^N \\
u(x) \to \ell & \text{as } |x| \to \infty,
\end{cases}$$

(2)

where $N \geq 3$ and $\ell \geq 0$ is a real number.

Throughout the paper we assume that the variable potential $\rho(x)$ satisfies $\rho \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, $\rho \geq 0$ and $\rho \not\equiv 0$.

In our first result we suppose that the growth at infinity of the anisotropic potential $\rho(x)$ is given by

$$(\rho_1) \int_0^{\infty} r\Phi(r)dr < \infty,$$

where $\Phi(r) := \max_{|x|=r} \rho(x)$.

Assumption $(\rho_1)$ has been first introduced in Naito [14].

The nonlinearity $f : (0, \infty) \to (0, \infty)$ satisfies $f \in C^0_{\text{loc}}(0, \infty)$ ($0 < \alpha < 1$) and has a sublinear growth, in the sense that

$(f_1)$ the mapping $u \mapsto -f(u)/u$ is decreasing on $(0, \infty)$ and $\lim_{u \to \infty} f(u)/u = 0$.

We point out that condition $(f_1)$ does not require that $f$ is smooth at the origin. The standard example of such a nonlinearity is $f(u) = u^p$, where $-\infty < p < 1$. We also observe that we study an equation of the same type as in Brezis and Kamin [3]. The main difference is that we require a certain asymptotic behaviour at infinity of the solution.

Under the above hypotheses $(\rho_1)$ and $(f_1)$, our first result concerns the case $\ell > 0$. We have

**Theorem 1.** Assume that $\ell > 0$. Then Problem (2) has a unique classical solution.

Next, consider the case $\ell = 0$. Instead of $(\rho_1)$ we impose the stronger condition

$$(\rho_2) \int_0^{\infty} r^{N-1}\Phi(r)dr < \infty.$$

We remark that in Edelson [7] it is used the stronger assumption $\int_0^{\infty} r^{N-1+\lambda(N-2)}\Phi(r)dr < \infty$, for some $\lambda \in (0, 1)$.

Additionally, we suppose that

$(f_2)$ $f$ is increasing in $(0, \infty)$ and $\lim_{u \downarrow 0} f(u)/u = +\infty$.

A nonlinearity satisfying both $(f_1)$ and $(f_2)$ is $f(u) = u^p$, where $0 < p < 1$.

Our result in the case $\ell = 0$ is the following.

**Theorem 2.** Assume that $\ell = 0$ and assumptions $(\rho_2)$, $(f_1)$ and $(f_2)$ are fulfilled. Then Problem (2) has a unique classical solution.

We point out that assumptions $(\rho_1)$ and $(\rho_2)$ are related to a celebrated class introduced by Kato, with wide and deep applications in Potential Theory and Brownian Motion. We recall (see [1]) that a real-valued measurable function $\psi$ on $\mathbb{R}^N$ belongs to the Kato class $K$ provided that

$$\lim_{\alpha \to 0} \sup_{x \in \mathbb{R}^N} \int_{|x-y| \leq \alpha} E(y)|\psi(y)|dy = 0,$$

where $E$ denotes the fundamental solution of the Laplace equation. According to this definition and our assumption $(\rho_1)$ (resp., $(\rho_2)$), it follows that $\psi = \psi(|x|) \in K$, where $\psi(|x|) := |x|^{N-3-\Phi(|x|)}$ (resp., $\psi(|x|) := |x|^{-1}\Phi(|x|)$), for all $x \neq 0$. 

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2 Proof of Theorem 1

In order to prove the existence of a solution to Problem (2), we use a result established by Brezis and Oswald (see [4, Theorem 1]) for bounded domains. Consider the problem

\begin{align*}
\begin{cases}
-\Delta u &= g(x, u) \quad \text{in } \Omega \\
u &\geq 0, \quad u \not\equiv 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{align*}

(3)

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with smooth boundary and \( g(x, u) : \Omega \times [0, \infty) \to \mathbb{R} \). Assume that

\begin{align*}
&\text{for a.e. } x \in \Omega \text{ the function } u \mapsto g(x, u) \text{ is continuous on } [0, \infty) \quad \text{(4)} \\
&\text{and the mapping } u \mapsto g(x, u)/u \text{ is decreasing on } (0, \infty); \\
&\text{for each } u \geq 0 \text{ the function } x \mapsto g(x, u) \text{ belongs to } L^\infty(\Omega) \quad \text{(5)} \\
&\exists C > 0 \text{ such that } g(x, u) \leq C(u + 1) \text{ a.e. } x \in \Omega, \quad \forall u \geq 0. \quad \text{(6)}
\end{align*}

Set

\[ a_0(x) = \lim_{u \to 0} g(x, u)/u \quad \text{and} \quad a_\infty(x) = \lim_{u \to \infty} g(x, u)/u, \]

so that \(-\infty < a_0(x) \leq +\infty\) and \(-\infty \leq a_\infty(x) < +\infty\).

Under these hypotheses, Brezis and Oswald proved in [4] that Problem (3) has at most one solution. Moreover, a solution of (3) exists if and only if

\[ \lambda_1(-\Delta - a_0(x)) < 0 \]

(7)

and

\[ \lambda_1(-\Delta - a_\infty(x)) > 0, \]

(8)

where \( \lambda_1(-\Delta - a(x)) \) denotes the first eigenvalue of the operator \(-\Delta - a(x)\) with zero Dirichlet condition. The precise meaning of \( \lambda_1(-\Delta - a(x)) \) is

\[ \lambda_1(-\Delta - a(x)) = \inf_{\varphi \in H_0^1(\Omega), \|\varphi\|_{L^2(\Omega)} = 1} \left( \int |\nabla \varphi|^2 - \int_{[\varphi \neq 0]} a \varphi^2 \right). \]

Note that \( \int_{[\varphi \neq 0]} a \varphi^2 \) makes sense if \( a(x) \) is any measurable function such that either \( a(x) \leq C \) or \( a(x) \geq -C \) a.e. on \( \Omega \).

For any positive integer \( k \) we consider the problem

\begin{align*}
\begin{cases}
-\Delta u_k &= \rho(x) f(u_k) , \quad \text{if } |x| < k \\
u_k &> \ell, \quad \text{if } |x| < k \\
u_k(x) &= \ell, \quad \text{if } |x| = k.
\end{cases}
\end{align*}

(9)
Equivalently, the above boundary value problem can be rewritten
\[
\begin{cases}
-\Delta v_k = \rho(x)f(v_k + \ell), & \text{if } |x| < k \\
v_k(x) = 0, & \text{if } |x| = k.
\end{cases}
\tag{10}
\]

In order to obtain a solution of the problem (10), it is enough to check the hypotheses of the Brezis-Oswald theorem.

- Since \( f \in C(0, \infty) \) and \( \ell > 0 \), it follows that the mapping \( v \mapsto \rho(x)f(v + \ell) \) is continuous in \([0, \infty)\).
- From \( \rho(x)\frac{f(v+\ell)}{v} = \rho(x)\frac{f(v+\ell)}{v+\ell}v \), using positivity of \( \rho \) and (f1) we deduce that the function \( v \mapsto \rho(x)\frac{f(v+\ell)}{v} \) is decreasing on \((0, \infty)\).
- For all \( v \geq 0 \), since \( \rho \in L^\infty_{\text{loc}}(\mathbb{R}^N) \), we obtain that \( \rho \in L^\infty(B(0, k)) \), so the condition (5) is satisfied.
- By \( \lim_{v \to \infty} f(v + \ell)/(v + 1) = 0 \) and \( f \in C(0, \infty) \), there exists \( M > 0 \) such that \( f(v + \ell) \leq M(v + 1) \) for all \( v \geq 0 \). Therefore \( \rho(x)f(v + \ell) \leq ||\rho||_{L^\infty(B(0, k))}M(v + 1) \) for all \( v \geq 0 \).
- We have
  \[
a_0(x) = \lim_{v \to 0} \frac{\rho(x)f(v + \ell)}{v} = +\infty
\]
and
  \[
a_\infty(x) = \lim_{v \to \infty} \frac{\rho(x)f(v + \ell)}{v} = \lim_{v \to \infty} \rho(x)\frac{f(v + \ell)}{v + \ell} \cdot \frac{v + \ell}{v} = 0.
\]

Thus, by Theorem 1 in [4], Problem (10) has a unique solution \( v_k \) which, by the maximum principle, is positive in \( |x| < k \). Then \( u_k = v_k + \ell \) satisfies (3). Define \( u_k = \ell \) for \( |x| > k \). The maximum principle implies that \( \ell \leq u_k \leq u_{k+1} \) in \( \mathbb{R}^N \).

We now justify the existence of a continuous function \( v : \mathbb{R}^N \to \mathbb{R}, v > \ell, \) such that \( u_k \leq v \) in \( \mathbb{R}^N \). We first construct a positive radially symmetric function \( w \) such that \( -\Delta w = \Phi(r) \) (\( r = |x| \)) in \( \mathbb{R}^N \) and \( \lim_{r \to \infty} w(r) = 0 \). A straightforward computation shows that

\[
w(r) = K - \int_0^r \zeta^{1-N} \int_0^\zeta \sigma^{N-1} \Phi(\sigma) d\sigma d\zeta,
\]

where

\[
K = \int_0^\infty \zeta^{1-N} \int_0^\zeta \sigma^{N-1} \Phi(\sigma) d\sigma d\zeta,
\]

provided the integral is finite. An integration by parts yields
\[
\int_0^r \zeta^{1-N} \int_0^\zeta \sigma^{N-1} \Phi(\sigma) d\sigma d\zeta = -\frac{1}{N-2} \int_0^r \frac{d}{dz} \zeta^{2-N} \int_0^\zeta \sigma^{N-1} \Phi(\sigma) d\sigma d\zeta
\]
\[
= \frac{1}{N-2} \left( -r^{2-N} \int_0^r \sigma^{N-1} \Phi(\sigma) d\sigma + \int_0^r \Phi(\zeta) \right)
\]
\[
< \frac{1}{N-2} \int_0^\infty \zeta \Phi(\zeta) < +\infty.
\]
Moreover, if \( w \) is decreasing and satisfies \( 0 < w(r) < K \) for all \( r \geq 0 \). Let \( v > \ell \) be a function such that
\[
w(r) = m^{-1} \int_0^r \frac{t}{f(t+\ell)} dt,
\]
where \( m > 0 \) is chosen such that \( Km \leq \int_0^m \frac{t}{f(t+\ell)} dt \).

Next, by L'Hôpital's rule for the case \( \int \) (see \[15\] Theorem 3, p. 319) we have
\[
\lim_{x \to \infty} \frac{\int_0^x t}{f(t+\ell)} = \lim_{x \to \infty} \frac{x}{f(x+\ell)} = \lim_{x \to \infty} \frac{x + \ell}{f(x+\ell)} \cdot \frac{x}{x + \ell} = +\infty.
\]
This means that there exists \( x_1 > 0 \) such that \( \int_0^x \frac{t}{f(t+\ell)} \geq Kx \) for all \( x \geq x_1 \). It follows that for any \( m \geq x_1 \) we have \( Km \leq \int_0^m \frac{t}{f(t+\ell)} dt \).

Since \( w \) is decreasing, we obtain that \( v \) is a decreasing function, too. Then
\[
\int_0^{v(r)-\ell} \frac{t}{f(t+\ell)} dt \leq \int_0^{v(0)-\ell} \frac{t}{f(t+\ell)} dt = mw(0) = mK \leq \int_0^m \frac{t}{f(t+\ell)} dt.
\]
It follows that \( v(r) \leq m + \ell \) for all \( r > 0 \).

From \( w(r) \to 0 \) as \( r \to \infty \) we deduce that \( v(r) \to \ell \) as \( r \to \infty \).

By the choice of \( v \) we have
\[
\nabla w = \frac{1}{M} \frac{v - \ell}{f(v)} \nabla v \quad \text{and} \quad \Delta w = \frac{1}{m} \frac{v - \ell}{f(v)} \Delta v + \frac{1}{m} \left( \frac{v - \ell}{f(v)} \right)' |\nabla v|^2.
\]
Since the mapping \( u \mapsto f(u)/u \) is decreasing on \((0, \infty)\) we deduce that
\[
\Delta v < \frac{m}{v - \ell} f(v) \Delta w = -\frac{M}{v - \ell} f(v) \Phi(r) \leq -f(v) \Phi(r). \quad (11)
\]

By \( (9) \), \( (11) \) and our hypothesis \( (f1) \), we obtain that \( u_k(x) \leq v(x) \) for each \( |x| \leq k \) and so, for all \( x \in \mathbb{R}^N \).

In conclusion,
\[
u_1 \leq u_2 \leq \ldots \leq u_k \leq u_{k+1} \leq \ldots \leq v,
\]
with \( v(x) \to \ell \) as \( |x| \to \infty \). Thus, there exists a function \( u \leq v \) such that \( u_k \to u \) pointwise in \( \mathbb{R}^N \). In particular, this shows that \( u > \ell \) in \( \mathbb{R}^N \) and \( u(x) \to \ell \) as \( |x| \to \infty \).

A standard bootstrap argument (see, e.g., \[10\]) shows that \( u \) is a classical solution of the problem \([2]\).

To conclude the proof, it remains to show that the solution found above is unique. Suppose that \( u \) and \( v \) are solutions of \([2]\). It is enough to show that \( u \leq v \) or, equivalently, \( \ln u(x) \leq \ln v(x) \), for any \( x \in \mathbb{R}^N \). Arguing by contradiction, there exists \( \mathbf{x} \in \mathbb{R} \) such that \( u(\mathbf{x}) > v(\mathbf{x}) \). Since \( \lim_{|x| \to \infty} (\ln u(x) - \ln v(x)) = 0 \), we deduce that \( \max_{\mathbb{R}^N} (\ln u(x) - \ln v(x)) \) exists and is positive. At this point, say \( x_0 \), we have
\[
\nabla (\ln u(x_0) - \ln v(x_0)) = 0, \quad (12)
\]
so
\[
\frac{\nabla u(x_0)}{u(x_0)} = \frac{\nabla v(x_0)}{v(x_0)}. \quad (13)
\]

By \( (f1) \) we obtain
\[
\frac{f(u(x_0))}{u(x_0)} < \frac{f(v(x_0))}{v(x_0)}.
\]
So, by (12) and (13),

\[
0 \geq \Delta (\ln u(x_0) - \ln v(x_0))
\]

\[
= \frac{1}{u(x_0)} \cdot \Delta u(x_0) - \frac{1}{v(x_0)} \cdot \Delta v(x_0) - \frac{1}{u^2(x_0)} \cdot |\nabla u(x_0)|^2 + \frac{1}{v^2(x_0)} \cdot |\nabla v(x_0)|^2
\]

\[
= \frac{\Delta u(x_0)}{u(x_0)} - \frac{\Delta v(x_0)}{v(x_0)} = - \rho(x_0) \left( \frac{f(u(x_0))}{u(x_0)} - \frac{f(v(x_0))}{v(x_0)} \right) > 0,
\]

which is a contradiction. Hence \( u \leq v \) and the proof is concluded.

\[\square\]

3 Proof of Theorem 2

3.1 Existence

Since \( f \) is an increasing positive function on \((0, \infty)\), there exists and is finite \( \lim_{u \searrow 0} f(u) \), so \( f \) can be extended by continuity at the origin. Consider the Dirichlet problem

\[
\begin{cases}
-\Delta u_k = \rho(x) f(u_k), & \text{if } |x| < k \\
u_k(x) = 0, & \text{if } |x| = k.
\end{cases}
\]

(14)

Using the same arguments as in case \( \ell > 0 \) we deduce that conditions (4) and (5) are satisfied. In what concerns assumption (6), we use both assumptions (f1) and (f2). Hence \( f(u) \leq f(1) \) if \( u \leq 1 \) and \( f(u)/u \leq f(1) \) if \( u \geq 1 \). Therefore \( f(u) \leq f(1)(u+1) \) for all \( u \geq 0 \), which proves (6). The existence of a solution for (14) follows from (7) and (8). These conditions are direct consequences of our assumptions \( \lim_{u \to \infty} f(u)/u = 0 \) and \( \lim_{u \searrow 0} f(u)/u = +\infty \). Thus, by the Brezis-Oswald theorem, Problem (14) has a unique solution. Define \( u_k(x) = 0 \) for \( |x| > k \). Using the same arguments as in case \( \ell > 0 \), we obtain \( u_k \leq u_{k+1} \) in \( \mathbb{R}^N \).

Next, we prove the existence of a continuous function \( v : \mathbb{R}^N \to \mathbb{R} \) such that \( u_k \leq v \) in \( \mathbb{R}^N \). We first construct a positive radially symmetric function \( w \) satisfying \(-\Delta w = \Phi(r) \ (r = |x|) \) in \( \mathbb{R}^N \) and \( \lim_{r \to \infty} w(r) = 0 \). We obtain

\[
w(r) = K - \int_0^r \zeta^{1-N} \int_0^\zeta \sigma^{N-1} \Phi(\sigma) d\sigma d\zeta,
\]

where

\[
K = \int_0^\infty \zeta^{1-N} \int_0^\zeta \sigma^{N-1} \Phi(\sigma) d\sigma d\zeta,
\]

(15)

provided the integral is finite. By integration by parts we have

\[
\int_0^r \zeta^{1-N} \int_0^\zeta \sigma^{N-1} \Phi(\sigma) d\sigma d\zeta = -\frac{1}{N-2} \int_0^r \frac{d}{d\zeta} \zeta^{2-N} \int_0^\zeta \sigma^{N-1} \Phi(\sigma) d\sigma d\zeta =
\]

\[
\frac{1}{N-2} \left( -r^{2-N} \int_0^r \sigma^{N-1} \Phi(\sigma) d\sigma + \int_0^r \zeta \Phi(\zeta) d\zeta \right) < \frac{1}{N-2} \int_0^\infty \zeta \Phi(\zeta) d\zeta < \infty.
\]

(16)
Therefore
\[ w(r) < \frac{1}{N-2} \cdot \int_0^\infty \zeta \Phi(\zeta) d\zeta, \quad \text{for all } r > 0. \]

Let \( v \) be a positive function such that \( w(r) = c^{-1} \int_0^{v(r)} t/f(t) dt \), where \( c > 0 \) is chosen such that \( Kc \leq \int_0^x t/f(t) dt \). We argue in what follows that we can find \( c > 0 \) with this property. Indeed, by L'Hôpital’s rule,
\[
\lim_{x \to \infty} \frac{\int_0^x \frac{t}{f(t)} dt}{x} = \lim_{x \to \infty} \frac{x}{f(x)} = +\infty.
\]

This means that there exists \( x_1 > 0 \) such that \( \int_0^x t/f(t) dt \geq Kx \) for all \( x \geq x_1 \). It follows that for any \( c \geq x_1 \) we have \( Kc \leq \int_0^x t/f(t) dt \).

On the other hand, since \( w \) is decreasing, we deduce that \( v \) is a decreasing function, too. Hence
\[
\int_0^{v(r)} \frac{t}{f(t)} dt \leq \int_0^{v(0)} \frac{t}{f(t)} dt = c \cdot w(0) = c \cdot K \leq \int_0^c \frac{t}{f(t)} dt.
\]

It follows that \( v(r) \leq c \) for all \( r > 0 \).

From \( w(r) \to 0 \) as \( r \to \infty \) we deduce that \( v(r) \to 0 \) as \( r \to \infty \).

By the choice of \( v \) we have
\[
\nabla w = \frac{1}{c} \cdot \frac{v}{f(v)} \nabla v \quad \text{and} \quad \Delta w = \frac{1}{c} \frac{v}{f(v)} \Delta v + \frac{1}{c} \left( \frac{v}{f(v)} \right)' |\nabla v|^2.
\] (17)

Combining the fact that \( f(u)/u \) is a decreasing function on \((0, \infty)\) with relation (17), we deduce that
\[
\Delta v < c \frac{f(v)}{v} \Delta w = -c \frac{f(v)}{v} \Phi(r) \leq -f(v)\Phi(r).
\] (18)

By (14) and (18) and using our hypothesis (f2), as already done for proving the uniqueness in the case \( \ell > 0 \), we obtain that \( u_k(x) \leq v(x) \) for each \( |x| \leq k \) and so, for all \( x \in \mathbb{R}^N \).

We have obtained a bounded increasing sequence
\[
u_1 \leq u_2 \leq \ldots \leq u_k \leq u_{k+1} \leq \ldots \leq v,
\]
with \( v \) vanishing at infinity. Thus, there exists a function \( u \leq v \) such that \( u_k \to u \) pointwise in \( \mathbb{R}^N \). A standard bootstrap argument implies that \( u \) is a classical solution of the problem (2).

### 3.2 Uniqueness

We split the proof into two steps. Assume that \( u_1 \) and \( u_2 \) are solutions of Problem (2). We first prove that if \( u_1 \leq u_2 \) then \( u_1 = u_2 \) in \( \mathbb{R}^N \). In the second step we find a positive solution \( u \leq \min\{u_1, u_2\} \) and thus, using the first step, we deduce that \( u = u_1 \) and \( u = u_2 \), which proves the uniqueness.

**Step I.** We show that \( u_1 \leq u_2 \) in \( \mathbb{R}^N \) implies \( u_1 = u_2 \) in \( \mathbb{R}^N \). Indeed, since
\[
 u_1 \Delta u_2 - u_2 \Delta u_1 = \rho(x) u_1 u_2 \left( \frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1} \right) \geq 0,
\]
it is sufficient to check that
\[ \int_{\mathbb{R}^N} (u_1 \Delta u_2 - u_2 \Delta u_1) = 0 \]  
(19)

Let \( \psi \in C_0^\infty(\mathbb{R}^N) \) be such that \( \psi(x) = 1 \) for \( |x| \leq 1 \) and \( \psi(x) = 0 \) for \( |x| \geq 2 \), and denote \( \psi_n := \psi(x/n) \) for any positive integer \( n \). Set
\[
I_n := \int_{\mathbb{R}^N} (u_1 \Delta u_2 - u_2 \Delta u_1) \psi_n \, dx.
\]
We claim that \( I_n \to 0 \) as \( n \to \infty \). Indeed,
\[
|I_n| \leq \int_{\mathbb{R}^N} |u_1 \Delta u_2| \psi_n \, dx + \int_{\mathbb{R}^N} |u_2 \Delta u_1| \psi_n \, dx.
\]
So, by symmetry, it is enough to prove that \( J_n := \int_{\mathbb{R}^N} |u_1 \Delta u_2| \psi_n \, dx \to 0 \) as \( n \to \infty \). But, from (2),
\[
J_n = \int_{\mathbb{R}^N} |u_1 f(u_2) \rho(x)| \psi_n \, dx = \int_{n}^{2n} \int_{|x|=r} |u_1(x) f(u_2(x)) \rho(x)| \, dx \, dr
\]
\[
\leq \int_{n}^{2n} \Phi(r) \int_{|x|=r} |u_1(x) f(u_2(x))| \, dx \, dr \leq \int_{n}^{2n} \Phi(r) \int_{|x|=r} |u_1(x)| \, M(u_2 + 1) \, dx \, dr.
\]
(20)

Since \( u_1(x), u_2(x) \to 0 \) as \( |x| \to \infty \), we deduce that \( u_1 \) and \( u_2 \) are bounded in \( \mathbb{R}^N \). Returning to (20) we have
\[
J_n \leq M(\|u_2\|_{L^\infty(\mathbb{R}^N)} + 1) \sup_{|x| \geq n} |u_1(x)| \cdot \frac{\omega_N}{N} \int_{n}^{2n} \Phi(r) r^{N-1} \, dr \leq C \int_{0}^{\infty} \Phi(r) r^{N-1} \, dr \cdot \sup_{|x| \geq n} |u_1(x)|.
\]

Since \( u_1(x) \to 0 \) as \( |x| \to \infty \), we have \( \sup_{|x| \geq n} |u_1(x)| \to 0 \) as \( n \to \infty \) which shows that \( J_n \to 0 \). In particular, this implies \( I_n \to 0 \) as \( n \to \infty \).

We recall in what follows the Lebesgue Dominated Convergence Theorem (see [2, Theorem IV.2]).

**Theorem 3.** Let \( f_n : \mathbb{R}^N \to \mathbb{R} \) be a sequence of functions in \( L^1(\mathbb{R}^N) \). We assume that

(i) \( f_n(x) \to f(x) \) a.e. in \( \mathbb{R}^N \),

(ii) there exists \( g \in L^1(\mathbb{R}^N) \) such that, for all \( n \geq 1 \), \( |f_n(x)| \leq g(x) \) a.e. in \( \mathbb{R}^N \).

Then \( f \in L^1(\mathbb{R}^N) \) and \( \|f_n - f\|_{L^1} \to 0 \) as \( n \to \infty \).

Taking \( f_n := (u_1 \Delta u_2 - u_2 \Delta u_1) \psi_n \) we deduce \( f_n(x) \to u_1(x) \Delta u_2(x) - u_2(x) \Delta u_1(x) \) as \( n \to \infty \). To apply Theorem 3 we need to show that \( u_1 \Delta u_2 - u_2 \Delta u_1 \in L^1(\mathbb{R}^N) \). For this purpose it is sufficient to prove that \( u_1 \Delta u_2 \in L^1(\mathbb{R}^N) \). Indeed,
\[
\int_{\mathbb{R}^N} |u_1 \Delta u_2| \leq \|u_1\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |\Delta u_2| = C \int_{\mathbb{R}^N} |\rho(x) f(u_2)|.
\]

Thus, using \( f(u) \leq f(1)(u + 1) \) and since \( u_2 \) is bounded, the above inequality yields
\[
\int_{\mathbb{R}^N} |u_1 \Delta u_2| \leq C \int_{\mathbb{R}^N} |\rho(x)(u_2 + 1)|
\]
\[
\leq C \int_{0}^{\infty} \int_{|x|=r} \Phi(r) \, dx \, dr \leq C \int_{0}^{\infty} \Phi(r) r^{N-1} < +\infty.
\]
This shows that $u_1 \Delta u_2 \in L^1(\mathbb{R}^N)$ and the proof of Step I is completed.

**Step II.** Let $u_1, u_2$ be arbitrary solutions of Problem (2). For all integer $k \geq 1$, denote $\Omega_k := \{x \in \mathbb{R}^N; |x| < k\}$. The Brezis-Oswald theorem implies that the problem

$$
\begin{cases}
-\Delta v_k = \rho(x)f(v_k) & \text{in } \Omega_k \\
v_k = 0 & \text{on } \partial \Omega_k
\end{cases}
$$

has a unique solution $v_k \geq 0$. Moreover, by the Maximum Principle, $v_k > 0$ in $\Omega_k$. We define $v_k = 0$ for $|x| > k$. Applying again the Maximum Principle we deduce that $v_k \leq v_{k+1}$ in $\mathbb{R}^N$. Now we prove that $v_k \leq u_1$ in $\mathbb{R}^N$, for all $k \geq 1$. Obviously, this happens outside $\Omega_k$. On the other hand

$$
\begin{cases}
-\Delta u_1 = \rho(x)f(u_1) & \text{in } \Omega_k \\
u_1 > 0 & \text{on } \partial \Omega_k
\end{cases}
$$

Arguing by contradiction, we assume that there exists $\overline{\Omega} \in \Omega_k$ such that $v_k(\overline{\Omega}) > u_1(\overline{\Omega})$. Consider the function $h : \Omega_k \to \mathbb{R}, h(x) = \ln v_k(x) - \ln u(x)$. Since $u_1$ is bounded in $\Omega_k$ and $\inf_{\partial \Omega_k} u_1 > 0$ we have $\lim_{|x| \to k} h(x) = -\infty$. We deduce that $\max_{\Omega_k} (\ln v_k(x) - \ln u_1(x))$ exists and is positive. Using the same argument as in the case $\ell > 0$ we deduce that $v_k \leq u_1$ in $\Omega_k$, so in $\mathbb{R}^N$. Similarly we obtain $v_k \leq u_2$ in $\mathbb{R}^N$. Hence $v_k \leq \overline{\Omega} := \min\{u_1, u_2\}$. Therefore $v_k \leq v_{k+1} \leq \ldots \leq \overline{\Omega}$. Thus there exists a function $u$ such that $v_k \to u$ pointwise in $\mathbb{R}^N$. Repeating a previous argument we deduce that $u \leq \overline{\Omega}$ is a classical solution of Problem (2). Moreover, since $u \geq v_k > 0$ in $\Omega_k$ and for all $k \geq 1$, we deduce that $u > 0$ in $\mathbb{R}^N$. This concludes the proof of Step II.

Combining Steps I and II we conclude that $u_1 = u_2$ in $\mathbb{R}^N$. \qed

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