LINEABILITY, SPACEABILITY, AND ADDITIVITY CARDINALS FOR DARBOUX-LIKE FUNCTIONS

KRZYSZTOF CHRIS CIESIELSKI, JOSÉ L. GÁMEZ-MERINO, DANIEL PELLEGRINO, AND JUAN B. SEOANE-SEPÚLVEDA

Abstract. We introduce the concept of maximal lineability cardinal number, $m\mathcal{L}(M)$, of a subset $M$ of a topological vector space and study its relation to the cardinal numbers known as: additivity $A(M)$, homogeneous lineability $\mathcal{H}\mathcal{L}(M)$, and lineability $\mathcal{L}(M)$ of $M$. In particular, we will describe, in terms of $\mathcal{L}$, the lineability and spaceability of the families of the following Darboux-like functions on $\mathbb{R}^n$, $n \geq 1$: extendable, Jones, and almost continuous functions.

1. Preliminaries and background

The work presented here is a contribution to a recent ongoing research concerning the following general question: For an arbitrary subset $M$ of a vector space $W$, how big can be a vector subspace $V$ contained in $M \cup \{0\}$? The current state of knowledge concerning this problem is described in the very recent survey article [4]. So far, the term big in the question was understood as a cardinality of a basis of $V$; however, some other measures of bigness (i.e., in a category sense) can also be considered.

Following [1,23] (see, also, [13]), given a cardinal number $\mu$ we say that $M \subset W$ is $\mu$-lineable if $M \cup \{0\}$ contains a vector subspace $V$ of the dimension $\dim(V) = \mu$. Consider the following lineability cardinal number (see [2]):

$$\mathcal{L}(M) = \min\{\kappa : M \cup \{0\} \text{ contains no vector space of dimension } \kappa\}.$$ 

Notice that $M \subset W$ is $\mu$-lineable if, and only if, $\mu < \mathcal{L}(M)$. In particular, $\mu$ is the maximal dimension of a subspace of $M \cup \{0\}$ if, and only if, $\mathcal{L}(M) = \mu^+$. The number $\mathcal{L}(M)$ need not be a cardinal successor (see, e.g., [1]); thus, the maximal dimension of a subspace of $M \cup \{0\}$ does not necessarily exist.

If $W$ is a vector space over the field $K$ and $M \subset W$, let

$$\text{st}(M) = \{w \in W : (K \setminus \{0\})w \subset M\}.$$ 

Notice that

if $V$ is a subspace of $W$, then $V \subset M \cup \{0\}$ if, and only if, $V \subset \text{st}(M) \cup \{0\}$. \hspace{1cm} (1)

In particular,

$$\mathcal{L}(M) = \mathcal{L}(\text{st}(M)).$$ \hspace{1cm} (2)

Recall also (see, e.g., [15]) that a family $M \subset W$ is said to be star-like provided $\text{st}(M) = M$. Properties (1) and (2) explain why the assumption that $M$ is star-like appears in many results on lineability.

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A simple use of Zorn’s lemma shows that any linear subspace $V_0$ of $M \cup \{0\}$ can be extended to a maximal linear subspace $V$ of $M \cup \{0\}$. Therefore, the following concept is well defined.

**Definition 1.1** (maximal lineability cardinal number). Let $M$ be any arbitrary subset of a vector space $W$. We define

$$m\mathcal{L}(M) = \min\{\dim(V) : V \text{ is a maximal linear subspace of } M \cup \{0\}\}.$$  

Although this notion might seem similar to that of maximal-lineability and maximal-spaceability (introduced by Bernal-González in [3]) they are, in general, not related.

In any case, (1) implies that $m\mathcal{L}(M) = m\mathcal{L}(\text{st}(M))$.  

**Remark 1.2.** It is easy to see that $\mathcal{H}\mathcal{L}(M) = m\mathcal{L}(M) +$, where $\mathcal{H}\mathcal{L}(M)$ is a homogeneous lineability number defined in [2]. (This explains why $\mathcal{H}\mathcal{L}$ is always a successor cardinal, as shown in [2].) Clearly we have

$$\mathcal{H}\mathcal{L}(M) = m\mathcal{L}(M) + \leq \mathcal{L}(M).$$

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$$\mathcal{H}\mathcal{L}(M) = m\mathcal{L}(M) + \leq \mathcal{L}(M).$$

The inequality may be strict, as shown in [2].

For $M \subset W$ we will also consider the following additivity number (compare [2]), which is a generalization of the notion introduced by T. Natkaniec in [20, 21] and thoroughly studied by the first author [7–11] and F.E. Jordan [18] for $V = \mathbb{R}$ (see, also, [10]):

$$A(M, W) = \min\{|F| : F \subset W \& (\forall w \in W)(w + F \not\subset M)\} \cup \{|W| +\},$$

where $|F|$ is the cardinality of $F$ and $w + F = \{w + f : f \in F\}$. Most of the times the space $W$, usually $W = \mathbb{R}$, will be clear by the context. In such cases we will often write $A(M)$ in place of $A(M, W)$.

We are mostly interested in the topological vector spaces $W$. We say that $M \subset W$ is $\mu$-spaceable with respect to a topology $\tau$ on $W$, provided there exists a $\tau$-closed vector space $V \subset M \cup \{0\}$ of dimension $\mu$. In particular, we can consider also the following spaceability cardinal number:

$$\mathcal{L}_\tau(M) = \min\{\kappa : M \cup \{0\} \text{ contains no } \tau\text{-closed subspace of dimension } \kappa\}.$$

Notice that $\mathcal{L}(M) = \mathcal{L}_\tau(M)$ when $\tau$ is the discrete topology.

In what follows, we shall focus on spaces $W = \mathbb{R}^X$ of all functions from $X = \mathbb{R}^n$ to $\mathbb{R}$ and consider the topologies $\tau_u$ and $\tau_p$ of uniform and pointwise convergence, respectively. In particular, we write $\mathcal{L}_u(M)$ and $\mathcal{L}_p(M)$ for $\mathcal{L}_{\tau_u}(M)$ and $\mathcal{L}_{\tau_p}(M)$, respectively. Clearly

$$\mathcal{L}_p(M) \leq \mathcal{L}_u(M) \leq \mathcal{L}(M).$$

Recall also a series of definitions that shall be needed throughout the paper.

**Definition 1.3.** For $X \subseteq \mathbb{R}^n$ a function $f : X \to \mathbb{R}$ is said to be

- **Darboux** if $f[K]$ is a connected subset of $\mathbb{R}$ (i.e., an interval) for every connected subset $K$ of $X$;
- **Darboux in the sense of Pawlak** if $f[L]$ is a connected subset of $\mathbb{R}$ for every arc $L$ of $X$ (i.e., $f$ maps path connected sets into connected sets);
- **almost continuous** (in the sense of Stallings) if each open subset of $X \times \mathbb{R}$ containing the graph of $f$ contains also a continuous function from $X$ to $\mathbb{R}$;
• a connectivity function if the graph of $f \upharpoonright Z$ is connected in $Z \times \mathbb{R}$ for any connected subset $Z$ of $X$;
• extendable provided that there exists a connectivity function $F: X \times [0, 1] \to \mathbb{R}$ such that $f(x) = F(x, 0)$ for every $x \in X$;
• peripherally continuous if for every $x \in X$ and for all pairs of open sets $U$ and $V$ containing $x$ and $f(x)$, respectively, there exists an open subset $W$ of $U$ such that $x \in W$ and $f[\text{bd}(W)] \subset V$.

The above classes of functions are denoted by $CIVP$, $SCIVP$, $WCIVP$, and $R$ of extendable functions on additivity and maximal lineability numbers. Sections 3 and 4 focus on the set of extendable functions on $\mathbb{R}$ and $\mathbb{R}^n$, respectively. Surprisingly enough, we shall obtain very different results when moving from $\mathbb{R}$ to $\mathbb{R}^n$. The lineability of some of the above functions have been recently partly studied (see, e.g., [2, 14–16]) but here we shall give definitive answers concerning the lineability and spaceability of several previous studied classes.

The text is organized as follows. In Section 2 we study the relations between additivity and maximal lineability numbers. Sections 3 and 4 focus on the set of extendable functions on $\mathbb{R}$ and $\mathbb{R}^n$, respectively. Notice that all classes defined in the above three definitions are star-like.

Definition 1.4. A function $f: \mathbb{R}^n \to \mathbb{R}$ is called

• everywhere surjective if $f[G] = \mathbb{R}$ for every nonempty open set $G \subset \mathbb{R}^n$;
• strongly everywhere surjective if $f^{-1}(y) \cap G$ has cardinality $\aleph$ for every $y \in \mathbb{R}$ and every nonempty open set $G \subset \mathbb{R}^n$; this class was also studied in [9], under the name of $\aleph$ strongly Darboux functions;
• perfectly everywhere surjective if $f[P] = \mathbb{R}$ for every perfect set $P \subset \mathbb{R}^n$ (i.e., when $f^{-1}(r)$ is a Bernstein set for every $r \in \mathbb{R}$ (compare [6, chap. 7]));
• a Jones function (see [17]) if $f \cap F \neq \emptyset$ for every closed set $F \subset \mathbb{R}^n \times \mathbb{R}$ whose projection on $\mathbb{R}^n$ is uncountable.

The classes of these functions are written as $ES(\mathbb{R}^n)$, $SES(\mathbb{R}^n)$, $PES(\mathbb{R}^n)$, and $J(\mathbb{R}^n)$, respectively. We will drop the domain $\mathbb{R}^n$ if $n = 1$.

Definition 1.5. A function $f: \mathbb{R} \to \mathbb{R}$ has:

• the Cantor intermediate value property if for every $x, y \in \mathbb{R}$ and for each perfect set $K$ between $f(x)$ and $f(y)$ there is a perfect set $C$ between $x$ and $y$ such that $f[C] \subset K$;
• the strong Cantor intermediate value property if for every $x, y \in \mathbb{R}$ and for each perfect set $K$ between $f(x)$ and $f(y)$ there is a perfect set $C$ between $x$ and $y$ such that $f[C] \subset K$ and $f \upharpoonright C$ is continuous;
• the weak Cantor intermediate value property if for every $x, y \in \mathbb{R}$ with $f(x) < f(y)$ there exists a perfect set $C$ between $x$ and $y$ such that $f[C] \subset (f(x), f(y))$;
• perfect roads if for every $x \in \mathbb{R}$ there exists a perfect set $P \subset \mathbb{R}$ having $x$ as a bilateral (i.e., two sided) limit point for which $f \upharpoonright P$ is continuous at $x$.

The above classes of functions shall be denoted by $CIVP$, $SCIVP$, $WCIVP$, and $FR$, respectively.
2. Relation between additivity and lineability numbers

The goal of this section is to examine possible values of numbers \( A(M) \), \( \text{mL}(M) \), and \( \mathcal{L}(M) \) for a subset \( M \) of a linear space \( W \) over an arbitrary field \( K \). We will concentrate on the cases when \( \emptyset \neq M \subset W \), since it is easy for the cases \( M \in \{ \emptyset, W \} \). Indeed, as it can be easily checked, one has \( A(\emptyset) = \mathcal{L}(\emptyset) = 1 \) and \( \text{mL}(\emptyset) = 0 \); \( A(W) = |W|^+, \mathcal{L}(W) = \dim(W)^+, \) and \( \text{mL}(W) = \dim(W) \).

**Proposition 2.1.** Let \( W \) be a vector space over a field \( K \) and let \( \emptyset \neq M \subset W \). Then

(i) \( 2 \leq A(M) \leq |W| \) and \( \text{mL}(M) < \mathcal{L}(M) \leq \dim(W)^+ \);

(ii) if \( A(\text{st}(M)) > |K| \), then \( A(\text{st}(M)) \leq \text{mL}(M) \).

In particular, if \( M \) is star-like, then \( A(M) > |K| \) implies that

(iii) \( A(M) \leq \text{mL}(M) < \mathcal{L}(M) \leq \dim(W)^+ \).

**Proof.**

(i) These inequalities are easy to see.

(ii) This can be proved by an easy transfinite induction. Alternatively, notice that A. Bartoszewicz and S. Głąb proved, in [2, corollary 2.3], that if \( M \subset W \) is star-like and \( A(M) > |K| \), then \( A(M) < \mathcal{H}(M) \). Hence, \( A(\text{st}(M)) > |K| \) implies that \( A(\text{st}(M)) < \mathcal{H}(\text{st}(M)) = \text{mL}(\text{st}(M)) + = \text{mL}(M)^+ \). Therefore, \( A(\text{st}(M)) \leq \text{mL}(M) \).

In what follows, we will restrict our attention to the star-like families, since, by Proposition 2.1, other cases could be reduced to this situation. Our next theorem shows that, for such families and under assumption that \( A(M) > |K| \), the inequalities (3) constitute all that can be said on these numbers.

**Theorem 2.2.** Let \( W \) be an infinite dimensional vector space over an infinite field \( K \) and let \( \alpha, \mu, \) and \( \lambda \) be the cardinal numbers such that \( |K| < \alpha \leq \mu < \lambda \leq \dim(W)^+ \). Then there exists a star-like \( M \subset W \) containing \( 0 \) such that \( A(M) = \alpha \), \( \text{mL}(M) = \mu \), and \( \mathcal{L}(M) = \lambda \).

The proof of this theorem will be based on the following two lemmas. The first of them shows that the theorem holds when \( \alpha = \mu \), while the second shows how such an example can be modified to the general case.

**Lemma 2.3.** Let \( W \) be an infinite dimensional vector space over an infinite field \( K \) and let \( \mu \) and \( \lambda \) be the cardinal numbers such that \( |K| < \mu < \lambda \leq \dim(W)^+ \). Then there exists a star-like \( M \subset W \) containing \( 0 \) such that \( A(M) = \text{mL}(M) = \mu \) and \( \mathcal{L}(M) = \lambda \).

**Proof.** For \( S \subset W \), let \( V(S) \) be the vector subspace of \( W \) spanned by \( S \).

Let \( B \) be a basis for \( W \). For \( w \in W \), let \( \text{supp}(w) \) be the smallest subset \( S \) of \( B \) with \( w \in V(S) \) and let \( c_w : \text{supp}(w) \to K \) be such that \( w = \sum_{b \in \text{supp}(w)} c_w(b)b \). Let \( E \) be the set of all cardinal numbers less than \( \lambda \) and choose a sequence \( \langle B_\eta : \eta \in E \rangle \) of pairwise disjoint subsets of \( B \) such that \( |B_0| = \mu \) and \( |B_\eta| = \eta \) whenever \( \emptyset \neq \eta \in E \). Define

\[
M = A \cup \bigcup_{\eta \in E} V(B_\eta),
\]

where

\[
A = \{ w \in W : c_w(b_0) = c_w(b_1) \text{ for some } b_0 \in \text{supp}(w) \cap B_0, b_1 \in \text{supp}(w) \setminus B_0 \}.
\]
We will show that $M$ is as desired.

Clearly, $M$ is star-like and $0 \in M \subseteq W$. Also, $\mathcal{L}(M) \geq \lambda$, since for any cardinal $\eta < \lambda$ the set $M$ contains a vector subspace $V(B_\eta)$ with $\dim(V(B_\eta)) \geq \eta$.

To see that $A(M) \geq \mu$, choose an $F \subset W$ with $|F| < \mu$. It is enough to show that $|F| < A(M)$, that is, that there exists a $w \in W$ with $w + F \subset A$. As supp$(F) = \bigcup_{v \in F} \text{supp}(v)$ has cardinality at most $|F| + \omega < \mu = |B_0| < \lambda \leq |B \setminus B_0|$, there exist $b_0 \in B_0 \setminus \text{supp}(F)$ and $b_1 \in B \setminus (B_0 \cup \text{supp}(F))$. Let $w = b_0 + b_1$ and notice that $w + F \subset A \subset M$, since for every $v \in F$ we have $b_0 \in \text{supp}(w + v) \cap B_0$, $b_1 \in \text{supp}(w + v) \setminus B_0$, and $c_{w+v}(b_0) = 1 = c_{w+v}(b_1)$.

Next notice that the inequalities $|K| < \mu \leq A(M)$ and Proposition 2.1 imply that $\mu \leq A(M) \leq \text{mL}(M)$. Thus, to finish the proof, it is enough to show that $\text{mL}(M) \leq \mu$ and $\mathcal{L}(M) \leq \lambda$.

To see that $\text{mL}(M) \leq \mu$, it is enough to show that $V(B_0)$ is a maximal vector subspace of $M$. Indeed, if $V$ is a vector subspace of $W$ properly containing $V(B_0)$, then there exists a non-zero $v \in V \cap V(B \setminus B_0)$. Choose a $b_0 \in B_0$ and a non-zero $c \in K \setminus \{c_v(b): b \in \text{supp}(v)\}$. Then $c b_0 + v \in V \setminus M$. So, $V(B_0)$ is a maximal vector subspace of $M$ and indeed $\text{mL}(M) \leq \dim(V(B_0)) = \kappa$.

To see that $\mathcal{L}(M) \leq \lambda$, choose a vector subspace $V$ of $W$ of dimension $\lambda$. It is enough to show that $V \setminus M \neq \emptyset$. To see this, for every ordinal $\eta \leq \lambda$ let us define $B_\eta = \bigcup\{B_z : \zeta \in E \cap \eta\}$. Notice that for every $\eta < \lambda$ there is a non-zero $v \in V$ with supp$(w) \cap B_\eta = \emptyset$.

Indeed, if $\pi_\eta : W = V(\hat{B}_\eta) \oplus V(B \setminus \hat{B}_\eta) \to V(\hat{B}_\eta)$ is the natural projection, then there exist distinct $w_1, w_2 \in V$ with $\pi_\eta(w_1) = \pi_\eta(w_2)$ (as $|V(\hat{B}_\eta)| < \lambda = \dim(V)$). Then $w = w_1 - w_2$ is as required.

Now, choose a non-zero $w_1 \in V$ with supp$(w_1) \cap B_0 = \text{supp}(w_1) \cap \hat{B}_1 = \emptyset$. Then, $w_1 \notin A$, and if supp$(w_1) \subset \hat{B}_\lambda = \bigcup_{\eta \in E} B_\eta$, then also $w_1 \notin \bigcup_{\eta \in E} V(B_\eta)$, and we have $w_1 \in V \setminus M$. Therefore, we can assume that supp$(w_1) \subset B_\lambda = \bigcup_{\eta < \lambda} B_\eta$. Let $\eta < \lambda$ be such that supp$(w_1) \subset B_\eta$ and choose a non-zero $w_2 \in V$ with supp$(w_2) \cap B_0 = \emptyset$. Then $w = w_2 - w_1 \in V \setminus M$ (since $w \notin A$, being non-zero with supp$(w) \cap B_0 = \emptyset$, and $w \notin \bigcup_{\eta \in E} V(B_\eta)$, as its support intersects two different $B_\eta$).

**Lemma 2.4.** Let $W$, $W_0$, and $W_1$ be the vector spaces over an infinite field $K$ such that $W = W_0 \oplus W_1$. Let $M \subset W_0$ and

$$\mathcal{F} = M + W_1 = \{m + W : m \in M \text{ and } w \in W_1\}.$$

Then

(i) If $M$ is star-like, then $\mathcal{F}$ is also star-like.

(ii) $A(\mathcal{F}, W) = A(M, W_0)$.

(iii) If $0 \in M$, then $\text{mL}(\mathcal{F}) = \text{mL}(M) + \dim(W_1)$.

(iv) If $0 \in M$ and $\dim(W_1) < \mathcal{L}(M)$, then $\mathcal{L}(\mathcal{F}) = \mathcal{L}(M) + \dim(W_1)$.

**Proof.** In the following, let $\pi_0 : W = W_0 \oplus W_1 \to W_0$ be the canonical projection.

(i) Let $x \in \mathcal{F}$ and $\lambda \in K \setminus \{0\}$. Since $M$ is star-like and $\pi_0(x) \in M$, we have that $\pi_0(\lambda x) = \lambda \pi_0(x) \in M$, and hence $\lambda x \in M + W_1 = \mathcal{F}$.

(ii) Let us see that $A(M, W_0) \leq A(\mathcal{F}, W)$. To this end, let $\kappa < A(M, W_0)$. We need to prove that $\kappa < A(\mathcal{F}, W)$. Indeed, if $F \subset W$ and $|F| = \kappa$, then $|\pi_0(F)| \leq
\[ |F| = \kappa. \] So, there exists a \( w_0 \in W_0 \) such that \( \pi_0[w_0 + F] = w_0 + \pi_0[F] \subset M \), that is, \( w_0 + F \subset M + W_1 = F \). Therefore, \( \kappa < A(F, W) \).

To see that \( A(F, W) \leq A(M, W_0) \) let \( \kappa < A(F, W) \). We need to show that \( \kappa < A(M, W_0) \). Indeed, let \( F \subset W_0 \) be such that \( |F| = \kappa \). Since \( |F| < A(F, W) \), there is a \( w \in W \) with \( w + F \subset F \). Then \( \pi_0(w) \in W_0 \) and \( \pi_0(w) + F = \pi_0[w + F] \subset \pi_0[F] = M \), so indeed \( \kappa < A(M) \).

First notice that it is enough to show that

\[ V \] is a maximal vector subspace of \( F \) if, and only if, \( V = V_0 + W_1 \), where \( V_0 \) is a maximal vector subspace of \( M \).

Indeed, if \( V \) is a maximal vector subspace of \( F \) with \( m\mathcal{L}(F) = \dim(V) \), then, by \( [3] \), \( m\mathcal{L}(F) = \dim(V) = \dim(V_0) + \dim(W_1) \geq m\mathcal{L}(M) + \dim(W_1) \). Conversely, if \( V_0 \) is a maximal vector subspace of \( M \) with \( m\mathcal{L}(M) = \dim(V_0) \), then we have \( m\mathcal{L}(M) + \dim(W_1) = \dim(V_0) + \dim(W_1) = \dim(V_0 + W_1) \geq m\mathcal{L}(F) \).

To see \( [3] \), take a maximal vector subspace \( V \) of \( F \). Notice that \( W_1 \subset V \), since \( V \subset F + W_0 \subset F \) and so, by the maximality, \( V + W_1 = V \). In particular, \( V = V_0 + W_1 \subset F = M + W_1 \), where \( V_0 = \pi_0[V] \). Thus, \( V_0 \) is a vector subspace of \( M \). It must be maximal, since for any its proper extension \( V_0 \subset M \), the vector space \( V_0 + W_1 \subset F \) would be a proper extension of \( V \).

Conversely, if \( V_0 \) is a maximal vector subspace of \( M \), then \( V = V_0 + W_1 \) is a vector subspace of \( F \). If \( V \) cannot have a proper extension \( V \subset F \), then the vector space \( \pi_0[V] \subset M \) would be a proper extension of \( V_0 \).

To see that \( \mathcal{L}(F) \leq \dim(W_1) + \mathcal{L}(M) \), choose a vector space \( V \subset F \). We need to show that \( \dim(V) < \dim(W_1) + \mathcal{L}(M) \). Indeed, \( V = V_1 + W_1 \) is a vector subspace of \( F + W_1 = F \) and \( \dim(V) \leq \dim(V_1) = \dim(W_1) + \dim(\pi_0[V_1]) \), since \( V_1 = W_1 + \pi_0[V_1] \). Therefore, \( \dim(V) \leq \dim(W_1) + \dim(\pi_0[V_1]) < \dim(W_1) + \mathcal{L}(M) \), since \( \dim(W_1) < \mathcal{L}(M) \) and \( \dim(\pi_0[V_1]) < \mathcal{L}(M) \), as \( \pi_0[V_1] \) is a vector subspace of \( M = \pi_0[F] \). So, \( \mathcal{L}(F) \leq \dim(W_1) + \mathcal{L}(M) \).

To see that \( \dim(W_1) + \mathcal{L}(M) \leq \mathcal{L}(F) \), choose a \( \kappa < \dim(W_1) + \mathcal{L}(M) \). We need to show that \( \kappa < \mathcal{L}(F) \), that is, that there exists a vector subspace \( V \subset F \) with \( \dim(V) \geq \kappa \). First, notice that \( \dim(W_1) < \mathcal{L}(M) \) and \( \kappa < \dim(W_1) + \mathcal{L}(M) \) imply that there exists a \( \mu < \mathcal{L}(M) \) such that \( \kappa \leq \dim(W_1) + \mu < \dim(W_1) + \mathcal{L}(M) \). (For finite value of \( \mathcal{L}(M) \), take \( \mu = \max\{\kappa - \dim(W_1), 0\} \); for infinite \( \mathcal{L}(M) \), the number \( \mu = \max\{\kappa, \dim(W_1)\} \) works.) Choose a vector subspace \( V_0 \) of \( M \) with \( \dim(V_0) \geq \mu \). Then the vector subspace \( V = V_0 + W_1 = V_0 + W_1 \subset F \) is as desired, since we have \( \dim(V) = \dim(W_1) + \dim(V_0) \geq \dim(W_1) + \mu \geq \kappa \).

\[ \square \]

**Proof of Theorem** \( 2.2 \). Represent \( W \) as \( W_0 \oplus W_1 \), where \( \dim(W_0) = \lambda \) and \( \dim(W_1) = \mu \). Use Lemma 2.3 to find a star-like \( M \subseteq W_0 \) containing \( 0 \) such that \( A(M, W_0) = m\mathcal{L}(M) = \alpha \) and \( \mathcal{L}(M) = \lambda \). Let \( F = M + W_1 \subset B \). Then, by Lemma 2.4 \( F \ni 0 \) is star-like such that \( A(F) = A(M, W_0) = \alpha \), \( m\mathcal{L}(F) = m\mathcal{L}(M) + \dim(W_1) = \alpha + \mu = \mu \), and \( \mathcal{L}(F) = \mathcal{L}(M) + \dim(W_2) = \lambda + \alpha = \lambda \), as required.

A. Bartoszewicz and S. Głab have asked \[ \text{open question 1} \] whether the inequality \( A(F)^+ \geq \mathcal{H}(L) \) (which is equivalent to \( A(F) \geq m\mathcal{L}(F) \)) holds for any family \( F \subset \mathbb{R}^S \). Of course, for the star-like families \( F \) with \( A(F) > c \), a positive answer to this question would mean that, under these assumptions, we have \( A(F) = m\mathcal{L}(F) \). Notice that Theorem 2.2 gives, in particular, a negative answer to this question.
We do not have a comprehensive example, similar to that provided by Theorem 2.2, for the case when \( A(M) \leq |K| \). However, the machinery built above, together with the results from [2], lead to the following result.

**Theorem 2.5.** Let \( W \) a vector space over an infinite field \( K \) with \( \dim(W) \geq 2^{|K|} \). If \( 2 \leq \kappa \leq |K| \), there exists a star-like family \( F \subseteq W \) containing 0 such that \( A(F) = \kappa \) and \( m\mathcal{L}(F) = \dim(W) \) (so that \( \mathcal{L}(F) = \dim(W) + \)).

**Proof.** Represent \( W \) as \( W = W_0 \oplus W_1 \), where \( \dim(W_0) = 2^{|K|} \) and \( \dim(W_1) = \dim(W) \). If \( 2 \leq \kappa \leq |K| \), then, by [2, Theorem 2.5], there exists a star-like family \( M \subseteq W_0 \) such that \( A(M, W_0) = \kappa \). Notice that the family constructed in that result contains 0. Then, by Lemma 2.4, the family \( F = M + W_1 \) satisfies that \( A(F) = A(M, W_0) = \kappa \) and \( m\mathcal{L}(F) = m\mathcal{L}(M) + \dim(W_1) = \dim(W) \). \( \square \)

### 3. Spaceability of Darboux-like functions on \( \mathbb{R} \)

Recall (see, e.g., [8, chart 1] or [7]) that we have the following strict inclusions, indicated by the arrows, between the Darboux-like functions from \( \mathbb{R} \) to \( \mathbb{R} \). The next theorem, strengthening the results presented in the table from [4, page 14], determines fully the lineability, \( \mathcal{L} \), and spaceability, \( \mathcal{L}_p \), numbers for these classes.

![Figure 1. Relations between the Darboux-like classes of functions from \( \mathbb{R} \) to \( \mathbb{R} \). Arrows indicate strict inclusions.](image)

**Theorem 3.1.** \( \mathcal{L}_p(\text{Ext}) = (2^c)^+ \). In particular, all Darboux-like classes of functions from Figure 1 except C, are \( 2^c \)-spaceable with respect to the topology of pointwise convergence.

**Proof.** In [11, corollary 3.4] it is shown that there exists an \( f \in \text{Ext} \) and an \( F_\sigma \) first category set \( M \subseteq \mathbb{R} \) such that

\[
\text{if } g \in \mathbb{R}^\mathbb{R} \text{ and } g \mid M = f \mid M, \text{ then } g \in \text{Ext}. \quad (4)
\]

It is easy to see that for any real number \( r \neq 0 \) the function \( rf \) satisfies the same property.

Notice also that there exists a family \( \{ h_\xi \in \mathbb{R}^\mathbb{R} : \xi < c \} \) of increasing homeomorphisms such that the sets \( M_\xi = h_\xi[M] \), \( \xi < c \), are pairwise disjoint. (See, e.g., [11, lemma 3.2].) It is easy to see that each function \( f_\xi = f \circ h_\xi^{-1} \) satisfies (4) with the set \( M_\xi \). Increasing one of the sets \( M_\xi \), if necessary, we can also assume that \( \{ M_\xi : \xi < c \} \) is a partition of \( \mathbb{R} \). Let \( \tilde{f} = (f_\xi \mid M_\xi : \xi < c) \) and define

\[
V(\tilde{f}) = \left\{ \bigcup_{\xi < c} t(\xi)(f_\xi \mid M_\xi) : t \in \mathbb{R}^c \right\}. \quad (5)
\]
It is easy to see that $V(\vec{f})$ is 2$^c$-dimensional $\tau_p$-closed linear subspace of Ext.

As the cardinality of the family $\mathcal{B}$ or of Borel functions from $\mathbb{R}$ to $\mathbb{R}$ is $c$, Theorem 3.1 easily implies that Ext \ $\mathcal{B}$ or is 2$^c$-lineable: $\mathcal{L}(\text{Ext} \ \setminus \mathcal{B}$ or) = $(2^c)^+$. Actually, we have an even stronger result:

**Proposition 3.2.** $\mathcal{L}_p(\text{Ext} \cap \text{SES} \ \setminus \mathcal{B}$ or) = $(2^c)^+$.

**Proof.** The function $f \upharpoonright M$ satisfying (4) may also have the property that $M$ is $c$-dense in $\mathbb{R}$ and $f \upharpoonright M$ is SES non-Borel. (6) Indeed, this can be ensured by enlarging $M$ by a $c$-dense first category set $N \subset \mathbb{R} \setminus M$ and redefining $f$ on $N$ so that $f \upharpoonright N$ is non-Borel and SES.

Now, if $f$ satisfies both (4) and (6) and $\vec{f}$ = $\langle f_{\xi} \upharpoonright M_{\xi} \rangle_{\xi < c}$ is defined as in Theorem 3.1, then the space $V(\vec{f})$ given in (5) is as required.

Notice also that Ext $\cap$ PES = PR $\cap$ PES = $\emptyset$. In particular, the space $V$ from Proposition 3.2 is disjoint with PES.

**Remark 3.3.** Clearly, Theorem 3.1 implies that Ext is 2$^c$-lineable. This result has been also independently proved by T. Natkaniec. (See preprint [22].) The technique used in [22] is similar, but different from that used in the proof of Theorem 3.1.

Recall, that it is known that $\mathcal{L}(\text{AC} \ \setminus \text{Ext}) = (2^c)^+$. See [15] or [4, page 14]. However, we do not know what the exact values of the following cardinals are.

**Problem 3.4.** Determine the following numbers:

\[ \mathcal{L}_p(\mathcal{F} \ \setminus \mathcal{G}), \mathcal{L}_u(\mathcal{F} \ \setminus \mathcal{G}), \text{ and } \mathcal{L}_u(\mathcal{F} \ \setminus \mathcal{G}) \]

for $\mathcal{F} \in \{\text{Conn} \ \setminus \text{AC}, \text{D} \ \setminus \text{Conn}, \text{PC} \ \setminus \text{D}\}$ and $\mathcal{G} \in \{\text{SCIVP, CIVP, PR}\}$.

**Problem 3.5.** Is it consistent with the axioms of set theory ZFC that either $A(\mathcal{F}) < m\mathcal{L}(\mathcal{F})$ or $m\mathcal{L}(\mathcal{F})^+ < \mathcal{L}(\mathcal{F})$ for any of the classes $\mathcal{F} \in \{\text{Ext, AC, Conn, D, PC}\}$?

Notice, that the generalized continuum hypothesis GCH implies that $A(\mathcal{F}) = m\mathcal{L}(\mathcal{F})$ and $m\mathcal{L}(\mathcal{F})^+ = \mathcal{L}(\mathcal{F})$ for every $\mathcal{F} \in \{\text{Ext, AC, Conn, D, PC}\}$.

4. **Spaceability of Darboux-like functions on $\mathbb{R}^n$, $n \geq 2$**

Recall (see, e.g., [8, chart 2] or [7]) that we have the following strict inclusions, indicated by the arrows, between the Darboux-like functions from $\mathbb{R}^n$ to $\mathbb{R}$ for $n \geq 2$.

\[
\begin{align*}
\text{Conn}(\mathbb{R}^n) & \quad \text{AC}(\mathbb{R}^n) \\
\text{C}(\mathbb{R}^n) & \quad \text{Ext}(\mathbb{R}^n) \quad \text{AC}(\mathbb{R}^n) \cap \text{D}(\mathbb{R}^n) \\
\text{PC}(\mathbb{R}^n) & \quad \text{D}(\mathbb{R}^n)
\end{align*}
\]

**Figure 2.** Relations between the Darboux-like classes of functions from $\mathbb{R}^n$ to $\mathbb{R}$, $n \geq 2$. Arrows indicate strict inclusions.

The proof of the next theorem will be based on the following result [12, Proposition 2.7]:
Proposition 4.1. Let $n > 0$ and let $f: \mathbb{R}^n \to \mathbb{R}$ be a peripherally continuous function. Then for any $x_0 \in \mathbb{R}^n$ and any open set $W$ in $\mathbb{R}^n$ containing $x_0$, there exists an open set $U \subseteq W$ such that $x_0 \in U$ and the restriction of $f$ to $\text{bd}(U)$ is continuous. Moreover, given any $\varepsilon > 0$, the set $U$ can be chosen so that $|f(x_0) - f(y)| < \varepsilon$ for every $y \in \text{bd}(U)$.

**Theorem 4.2.** For $n \geq 2$, $\mathcal{L}_p(\text{Ext}(\mathbb{R}^n)) = \mathcal{L}_u(\text{Ext}(\mathbb{R}^n)) = \mathcal{L}(\text{Ext}(\mathbb{R}^n)) = \mathcal{C}^+$. In particular, the classes $\mathcal{C}(\mathbb{R}^n)$ and $\text{Ext}(\mathbb{R}^n)$ are $\mathcal{C}$-spaceable with respect to the pointwise convergence topology $\tau_p$ but are not $\mathcal{C}^+$-lineable.

**Proof.** First, notice that $\mathcal{L}_p(\mathcal{C}(\mathbb{R}^n)) = \mathcal{C}^+$ is justified by the space $\mathcal{C}_0$ of all continuous functions linear on the interval $[k, k + 1]$ for every integer $k \in \mathbb{Z}$. Indeed, $\mathcal{C}_0$ is linearly isomorphic to $\mathbb{R}^\mathbb{Z}$.

Now, since $\mathcal{C} = \mathcal{L}_p(\mathcal{C}(\mathbb{R}^n)) \leq \mathcal{L}_p(\text{Ext}(\mathbb{R}^n)) \leq \mathcal{L}_u(\text{Ext}(\mathbb{R}^n)) \leq \mathcal{L}(\text{Ext}(\mathbb{R}^n))$, it is enough to show that $\mathcal{L}(\text{Ext}(\mathbb{R}^n)) \leq \mathcal{C}^+$, that is, that $\text{Ext}(\mathbb{R}^n)$ is not $\mathcal{C}^+$-lineable. To see this, by way of contradiction, assume that there exists a vector space $V \subseteq \text{Ext}(\mathbb{R}^n)$ of cardinality greater than $\mathcal{C}$. Fix a countable dense set $D \subseteq \mathbb{R}^n$ and let $\langle (x_k, \varepsilon_k) : k < \omega \rangle$ be an enumeration of $D \times \{2^{-m} : m < \omega\}$. By Proposition 4.1 for every function $f \in \text{Ext}(\mathbb{R}^n)$ and $k < \omega$ we can choose an open neighborhood $U_k$ of $x_k$ of diameter at most $\varepsilon_k$ such that $f | \text{bd}(U_k)$ is continuous. Consider the mapping $V \ni f \mapsto T_f = \{f | \text{bd}(U_k) : k < \omega\}$. Since its range has cardinality $\mathcal{C}$, there are distinct $f_1, f_2 \in V$ with $T_{f_1} = T_{f_2}$. In particular, $f = f_1 - f_2 \in V$ is equal zero on the set $M = \bigcup_{k < \omega} \text{bd}(U_k)$. Notice that the complement $M^c$ of $M$ is zero-dimensional. We will show that $f$ is not extendable, by showing that it does not satisfy Proposition 4.1.

Indeed, since $f_1 \neq f_2$, there is an $x \in \mathbb{R}^n$ with $f(x) \neq 0$. Let $\varepsilon = |f(x)|$ and let $W$ be any bounded neighborhood of $x$. Then, there is no set $U$ as required by Proposition 4.1.

To see this, notice that for any open set $U \subseteq W$ with $x \in U$, its boundary is of dimension at least 1. In particular, $M \cap \text{bd}(U) \neq \emptyset$ and, for $y \in M \cap \text{bd}(U)$, we have $|f(x) - f(y)| = |f(x)| = \varepsilon$.

Theorem 4.2 determines the values of the numbers $\mathcal{L}_p(\mathcal{F}), \mathcal{L}_u(\mathcal{F})$, and $\mathcal{L}(\mathcal{F})$ for $\mathcal{F} \in \{\mathcal{C}(\mathbb{R}^n), \text{Ext}(\mathbb{R}^n), \text{Conn}(\mathbb{R}^n), \text{PR}(\mathbb{R}^n)\}$ and $n \geq 2$. In the remainder of this section we will examine these cardinal numbers for the remaining classes from the diagram in Figure 2. For this, we will need the following fact, improving a recent result of the second author, see [13] Theorem 2.2.

**Proposition 4.3.** $\mathcal{L}_p(\mathcal{J}(\mathbb{R}^n)) = (2^\mathcal{C})^+$ for every $n \geq 1$. In particular, the families $\mathcal{J}(\mathbb{R}^n)$, $\text{PES}(\mathbb{R}^n)$, $\text{SES}(\mathbb{R}^n)$, and $\text{ES}(\mathbb{R}^n)$ are $2^\mathcal{C}$-spaceable with respect to the topology of pointwise convergence.

**Proof.** Let $\{B_\xi : \xi < \mathcal{C}\}$ be a decomposition of $\mathbb{R}^n$ into pairwise disjoint Bernstein sets. For every $\xi < \mathcal{C}$, let $f_\xi : B_\xi \to \mathbb{R}$ be such that $f_\xi \cap F \neq \emptyset$ for every closed set $F \subseteq \mathbb{R}^n \times \mathbb{R}$ whose projection on $\mathbb{R}^n$ is uncountable. (All of this can be easily constructed by transfinite induction. See, e.g., [1].) Notice that if $g \in \mathbb{R}^R$ and $g | M_\xi = r f_\xi$ for some $\xi < \mathcal{C}$ and $r \neq 0$, then $g \in \mathcal{J}(\mathbb{R}^n)$.

Now, if $f = \langle f_\xi \mid M_\xi : \xi < \mathcal{C} \rangle$ and $V(f)$ is given by [5], then $V(f)$ is $2^\mathcal{C}$-dimensional $\tau_p$-closed linear subspace of $\mathcal{J}(\mathbb{R}^n)$.
Every function in $J(\mathbb{R}^n)$ is surjective. In particular, the above result implies that the class of surjective functions is $2^\mathfrak{c}$-lineable. One could also wonder about the lineability of the family of one-to-one functions from $\mathbb{R}^n$ to $\mathbb{R}$, given below.

**Remark 4.4.** The family of one-to-one functions from $\mathbb{R}^n$ to $\mathbb{R}$ is 1-lineable but not 2-lineable.

**Proof.** Clearly the family is 1-lineable. To see that it is not 2-lineable, choose two injective linearly independent functions $f$ and $g$ generating a linear space $Z$. Take arbitrary $x \neq y$ in $\mathbb{R}^n$ and consider the function $h = f + \alpha g \in Z \setminus \{0\}$, where $\alpha = (f(x) - f(y))/(g(y) - g(x)) \in \mathbb{R}$. Then, we have $h(x) = h(y)$, so $Z$ contains a function which is not one-to-one. \hfill $\Box$

Other examples of 1-lineable but not 2-lineable sets and, in general, not lineable sets can be found in [4, 5].

**Theorem 4.5.** For $n \geq 2$, $J(\mathbb{R}^n) \subset AC(\mathbb{R}^n) \setminus D(\mathbb{R}^n)$. In particular, the class $AC(\mathbb{R}^n) \setminus D(\mathbb{R}^n)$ is $2^\mathfrak{c}$-spaceable and $\mathcal{L}_p(AC(\mathbb{R}^n) \setminus D(\mathbb{R}^n)) = (2^\mathfrak{c})^+.$

**Proof.** By Proposition 4.3, it is enough to show that $J(\mathbb{R}^n) \subset AC(\mathbb{R}^n) \setminus D(\mathbb{R}^n)$. Clearly, $J(\mathbb{R}^n) \subset AC(\mathbb{R}^n) \cap PES(\mathbb{R}^n)$ for any $n \geq 1$. Thus, it is enough to show that $PES(\mathbb{R}^n) \cap D(\mathbb{R}^n) = \emptyset$ for $n \geq 2$. But this follows immediately from the fact that, under $n \geq 2$, every Bernstein set in $\mathbb{R}^n$ is connected. \hfill $\Box$

**Remark 4.6.** Notice that, since $AC(\mathbb{R}^n) \subset D_p(\mathbb{R}^n)$, then, for $n \geq 2$, we have $\mathcal{L}_p(D_p(\mathbb{R}^n) \setminus D(\mathbb{R}^n)) = (2^\mathfrak{c})^+$. So, $D_p(\mathbb{R}^n) \setminus D(\mathbb{R}^n)$ is also $2^\mathfrak{c}$-spaceable.

**Theorem 4.7.** For $n \geq 2$, $\mathcal{L}_p(D(\mathbb{R}^n) \setminus AC(\mathbb{R}^n)) = (2^\mathfrak{c})^+$. In particular, the class $D(\mathbb{R}^n) \setminus AC(\mathbb{R}^n)$ is $2^\mathfrak{c}$-spaceable.

**Proof.** Let $\pi_1: \mathbb{R}^n \to \mathbb{R}$ the projection of $\mathbb{R}^n$ on its first coordinate. Let $W = V(f) \subset J$ be the vector space of cardinality $2^\mathfrak{c}$ build in Proposition 4.3. Then the vector space

$$V = \{ f \circ \pi_1 : f \in W \}$$

is obviously contained in $D(\mathbb{R}^n)$ and has dimension $2^\mathfrak{c}$. On the other side, if $f \in W$ then $f \circ \pi_1$ cannot be in $AC(\mathbb{R}^n)$, because then $f$ would be continuous. (See [19].) This is not possible, because $J \cap C = \emptyset$. Therefore, $V \subset D(\mathbb{R}^n) \setminus AC(\mathbb{R}^n)$. To finish, let us remark that the space $V$ is also closed by pointwise convergence. \hfill $\Box$

**Remark 4.8.** Notice that, in $\mathbb{R}^n$ (for every $n \in \mathbb{N}$), we have that $AC \setminus Ext$ is $2^\mathfrak{c}$-spaceable (since this class contains the Jones functions). Also, in $\mathbb{R}$, $J \subset AC \setminus SCIVP \subset AC \setminus Ext$ and, since $\mathcal{L}_p(J) = (2^\mathfrak{c})^+$, we have (from the previous results) that

$$\mathcal{L}_p(AC \setminus Ext) = \mathcal{L}_u(AC \setminus Ext) = (2^\mathfrak{c})^+.$$

**Problem 4.9.** For $n \geq 2$, determine the values of the numbers $\mathcal{L}_p(AC(\mathbb{R}^n) \cap D(\mathbb{R}^n))$, $\mathcal{L}_u(AC(\mathbb{R}^n) \cap D(\mathbb{R}^n))$, and $\mathcal{L}_p(AC(\mathbb{R}^n) \cap D(\mathbb{R}^n))$.

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Department of Mathematics,  
West Virginia University, Morgantown,  
WV 26506-6310, USA.  

and  

Department of Radiology, MIPG,  
University of Pennsylvania,  
Blockley Hall - 4th Floor, 423 Guardian Drive,  
Philadelphia, PA 19104-6021, USA.  
E-mail address: KCies@math.wvu.edu

Departamento de Análisis Matemático,  
Facultad de Ciencias Matemáticas,  
Plaza de Ciencias 3,  
Universidad Complutense de Madrid,  
Madrid, 28040, Spain.  
E-mail address: jlgamez@mat.ucm.es

Departamento de Matemática,  
Universidade Federal da Paraíba,  
58.051-900 - João Pessoa, Brazil.  
E-mail address: dpellegrino@gmail.com and pellegrino@pq.cnpq.br

Departamento de Análisis Matemático,  
Facultad de Ciencias Matemáticas,  
Plaza de Ciencias 3,  
Universidad Complutense de Madrid,  
Madrid, 28040, Spain.  
E-mail address: jseoane@mat.ucm.es