RIORDAN GROUPS IN HIGHER DIMENSIONS

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Abstract. The classical Riordan groups associated to a given commutative ring are groups of infinite matrices (called Riordan arrays) associated to pairs of formal power series in one variable. The Fundamental Theorem of Riordan Arrays relates matrix multiplication to two group actions on such series, namely formal (convolution) multiplication and formal composition. We define the analogous Riordan groups involving formal power series in several variables, and establish the analogue of the Fundamental Theorem in that context. We discuss related groups of Laurent series and pose some questions.

1. Introduction

The relation between the composition of formal series in one variable and matrix multiplication was indentified at least as early as Eri Jabotinsky’s paper [5] from 1947. It has been much exploited. The original Riordan group was introduced by Shapiro, Getu, Woan and Woodson in 1991 [17] (see also [18, 1]) and named in honour of John F. Riordan. Its elements are infinite matrices with integral entries, and it is an important tool in combinatorics. Riordan arrays and groups with rational, real, complex and other kinds of entries have been studied.

In this paper we explain how to make the corresponding groups in several variables. The several-variable Riordan group and Riordan semigroup defined below are very closely related to structures studied by Luis Verde-Star [19] and by Cheon at al [2]. The main results give a several-variable analogue of the Fundamental Theorem on Riordan Arrays, and we make some remarks about reversibility in these groups. (An element $g$ of a group $G$ is reversible [14] if it is conjugate to its inverse, i.e. there exists $h \in G$ such that $h^{-1}gh = g^{-1}$.) There has been progress on classifying reversibles in one dimension, particularly...
by Luzon, Moron and Prieto-Martinez [7]. Significant and interesting problems remain, even in one dimension.

1.1. **Holomorphs.** Riordan groups are constructed using a special case of a general procedure. The simplest case of this procedure is the construction of the holomorph of a group:

\[ G \mapsto \text{Hol}(G) := G \rtimes \text{Aut}(G), \]

where \( \text{Aut}(G) \) is the group of group automorphisms of \( G \). This semidirect product is the set \( G \times \text{Aut}(G) \), equipped with the multiplication defined by

\[(f, a)(g, b) := (fa(g), a \circ b).\]

More generally, we have this:

For any category \( C \), and object \( A \in C \), we have the group \( \text{Aut}_C(A) \), consisting of the invertible arrows (or \( C \)-morphisms) from \( A \) to \( A \).

**Definition 1.1.** Let \( C \) be a category equipped with a functor \( A \mapsto A^\times \) into the category of groups. For \( A \in C \), the \( C \)-holomorph of \( A \) with respect to \( \times \) is the group

\[ \text{Hol}_C(A, \times) := A^\times \rtimes \text{Aut}_C(A). \]

1.2. **Example.** If \( C \) is the category of rings with identity equipped with the usual functor that associates to the ring \( R \) the group \( R^\times \) of units of \( R \), then \( \text{Hol}_C(R^\times) \) is the subgroup of \( \text{Hol}(R^\times) \) consisting of those \( (f, a) \in R^\times \times \text{Aut}(R^\times) \) such that \( a \) extends to a ring-automorphism of \( R \).

1.3. **Example.** If \( C \) is the category of topological spaces and continuous functions, equipped with the homotopy functor \( X^\times := \pi_1(X) \), then \( \text{Aut}_C(X) \) is the group \( \text{Homeo}(X) \) of bicontinuous bijections of \( X \), and \( \text{Hol}_C(X, \times) \) is the semidirect product

\[ \pi_1(X) \rtimes \text{Homeo}(X). \]

1.4. **Example.** The original holomorph, defined on the category of groups is the special case

\[ \text{Hol}(G) = \text{Hol}_{\text{Group}}(G, \times) \]

when \( \times \) is the identity map on the category, i.e. \( G^\times = G \) for each group \( G \).
1.5. **The Riordan group.** Fix $K$, an integral domain (commutative, with identity), and consider the category $K$-alg of $K$-algebras with $K$-algebra homomorphisms. The formal power series rings $F_d := K[[x_1, \cdots, x_d]]$ in $d$ commuting variables are objects of this category. We refer to [15, Section 1] for basic notation and basic properties of these algebras of power series. We normally write a series $f \in F$ as

$$f = \sum_{m \in S} f_m m,$$

where $S$ is the semigroup of monic monomials in $d$ variables. Multiplication in the algebra $F_d$ is convolution:

$$(fg)_m = \sum_{p|m} f_p g_{m/p},$$

where $p|m$ means that the monomial $p$ is a factor of the monomial $m$. The algebra $F_d$ is a commutative integral domain with identity, and the multiplicative group $F_d^\times$ consists of those series $f$ having ‘constant term’ $f_1 \in K^\times$.

Note that a $K$-algebra automorphism of $F$ preserves the identity series.

**Definition 1.2.** For $d \in \mathbb{N}$, we define the Riordan group in $d$ dimensions over $K$ to be

$$R(K) := R_d(K) := \text{Hol}_{K\text{-alg}}(F_d^\times).$$

In [15] we described the group $G = G_d$ of formal maps of $K^d$ that fix 0, with the operation of formal composition. We denote its identity by $1$.

For $g \in G$, we define the composition map $C_g : F \to F$ by

$$C_g(f) := f \circ g, \ \forall f \in F.$$

The connection to $R_d(K)$ arises from the following well-known fact:

**Theorem 1.** The map $\phi : g \mapsto C_g$ is a group isomorphism from $G_d^{op}$ onto $\text{Aut}_{K\text{-alg}}(F_d)$.

**Proof.** This is straightforward. We have

$$\phi(g \circ h)(f) = f \circ g \circ h = (\phi(h) \circ \phi(g))(f),$$

whenever $g, h \in G$ and $f \in F$, so $\phi$ is a homomorphism from $G_d^{op} \to \text{Aut}_{K\text{-alg}}(F_d)$. If $\phi(g) = \phi(h)$, then for $1 \leq j \leq d$ we have

$$g_j = \phi(g)(x_j) = \phi(h)(x_j) = h_j,$$
so the $j$-th components of the maps $g$ and $h$ coincide. Thus $\phi$ is injective.

Finally, given any $K$-algebra automorphism $\alpha$ of $F$, define
\[
\psi(\alpha) := (\alpha(x_1), \ldots, \alpha(x_d)) \in G.
\]
Then for each $f \in F$, we claim that
\[
\phi(\psi(\alpha))(f) = f \circ \psi(\alpha) = \alpha(f).
\]
This tells us that $\phi$ is surjective. To see the claim, the only point that needs explanation is the second equation. Fix $f \in F$. For $k \in \mathbb{N}$, we can write $f = h + r$, where $h \in K[x_1, \ldots, x_d]$ is a polynomial of degree at most $k$ and $r \in M^{k+1}$, the $(k+1)$-st power of the maximal ideal $M = x_1F + \cdots + x_dF$ of $F$. Since the automorphism $\alpha$ maps $M^{k+1}$ into itself, we have
\[
\alpha(f) - \alpha(h) = \alpha(r) \in M^{k+1}.
\]
Also $\alpha(h) = h \circ \psi(\alpha)$, and
\[
f \circ \psi(\alpha) - h \circ \psi(\alpha) = r \circ \psi(\alpha) \in M^{k+1}.
\]
Thus
\[
f \circ \psi(\alpha) - \alpha(f) \in \bigcap_{k=1}^{\infty} M^{k+1} = (0),
\]
and the claim holds.

In the case $d = 1$, where $F := K[[x]]$ is the algebra of all formal power series in a single indeterminate $x$ having coefficients in $K$, the abelian group $F^\times$ is the set of series of the form
\[
f = f_1 + f_2x + f_2x^2 + \text{HOT}
\]
with $f_1 \in K^\times$. (We use HOT to stand for “higher-order terms”. In this case HOT stands for $\sum_{n=3}^{\infty} f_n x^n$.) For this case, the theorem says that the group $\text{Aut}_{K\text{-alg}}(F)$ is isomorphic to the group $(G, \circ^\text{op})$ of formally-invertible power series of the form
\[
g = g_x x + g_x x^2 + \text{HOT}
\]
with $g_x \in K^\times$, with the action given by
\[
g(f) = f \circ g = f_1 + f_2 g(x) + f_2 g(x)^2 + \text{HOT} = f_1 + (f_2 g_x) x + (f_2 g_x + f_2 g_x^2) x^2 + \text{HOT},
\]
and the group multiplication given by
\[
g \circ^\text{op} g' = g' \circ g.
\]
Corollary 1.1. Let \( d \in \mathbb{N} \) and \( K \) be an integral domain with identity. Then the \( d \)-dimensional Riordan group \( R_d(K) \) is isomorphic to the semidirect product
\[
F^\times_d \ltimes G^\text{op}_d,
\]
with the multiplication
\[
(f, g)(f', g') := (f \cdot (f' \circ g), g' \circ g),
\]
for \( f, f' \in F_d \) and \( g, g' \in G_d \).

The original Riordan group corresponds to the case \( d = 1 \) and \( K = \mathbb{Z} \).

Throughout the rest of this paper we identify \( R_d(K) \) and \( F^\times_d \ltimes G^\text{op}_d \), and represent elements of \( R_d(K) \) as pairs \((f, g) \subset F^\times_d \times G_d \).

We define
\[
\mathcal{L} := \{(1, g) : g \in \text{Aut}(F)\}, \quad \mathfrak{A} := \{(f, 1) : f \in F^\times \}.
\]
These are subgroups of \( R \), and in analogy with the case \( d = 1 \) we call \( \mathcal{L} \) the Lagrange subgroup and \( \mathfrak{A} \) the Appell subgroup (cf. [1]). Clearly, \( \mathcal{L} \) is isomorphic to \( G^\text{op} \) and \( \mathfrak{A} \) is isomorphic to \( F^\times \).

Many researchers working on one-dimensional Riordan groups over \( \mathbb{Z} \) or \( \mathbb{C} \) confine attention to \((f, g) \in F^\times_1 \times G_1 \) such that \( f_1 = 1 \) and \( g = x + \text{HOT} \). In this, they follow the lead of Shapiro [18]. So for general \( d \) and \( K \) we define the Shapiro subgroup to be
\[
\mathcal{S} := \{(f, g) \in R : f_1 = 1 \text{ and } g \in \mathcal{T}\},
\]
where
\[
\mathcal{T} := \{g \in G : g = 1 + \text{HOT}\}
\]
is the subgroup of \( G \) consisting of the formal maps tangent to the identity, i.e. maps in the kernel of the homomorphism \( L : G \to \text{GL}(K, d) \) that takes a map \( G \) to the linear term \( L_1 \) in its expansion
\[
g = \sum_{k=1}^{\infty} L_k
\]
in homogeneous polynomials of degree \( k = 1, 2, \ldots \).

2. Indexed matrices

The original description of Riordan arrays, ably described in the undergraduate textbook [1, Chapter 1], is neither in terms of \( \text{Hol}_{K-\text{alg}}(F^\times) \) nor \( F^\times \ltimes G^\text{op} \). The arrays are lower-triangular infinite matrices with integral entries, in which each row is derived from the previous one by an unchanging rule that involves linear combinations. The other descriptions exploit Jabotinski’s idea, compressing the information in the
array into a pair of formal power series. Well-known examples include (1) the array whose rows are those of Pascal’s triangle, corresponding to the series of \( \frac{1}{1-x} \in F_1^\times \) and \( \frac{x}{1-x} \in G_1 \), and (2) the Riordan-group inverse to the array corresponding to the simple finite series \( 1-x \in F \) and \( x-x^2 \in G \), corresponding to \( 1-\sqrt{1-4x^2} \in F \times \) and \( 1-\sqrt{1-4x} \in G \), whose first column lists the Catalan numbers (first named by Riordan).

We proceed to discuss the way to extend the matrix representation of the one-dimensional Riordan groups to dimensions \( d > 1 \).

2.1. Matrix structure. Let \( K \) be an integral domain with identity, and \( n \in \mathbb{N} \). We are used to \( gl(n,K) \), the usual \( K \)-algebra (i.e. ring and \( K \)-module) of \( n \times n \) matrices over \( K \). The typical entry in a matrix \( (a_{ij}) \in gl(n,K) \) is indexed by two natural numbers \( i \) and \( j \) in the set \( \{1,2,\ldots,n\} \). We tend to think of a matrix as an structured object, whose structure is derived from the ordered structure of \( \mathbb{N} \). For some matrix applications the arithmetical structure of \( \mathbb{N} \) is significant; examples are those using Hankel and Toeplitz matrices. For others, the order structure of \( \mathbb{N} \) is significant; examples are those using upper-triangular matrices. But sometimes neither the arithmetical or ordered structure of \( \mathbb{N} \) is important; the natural numbers are just used as labels.

Letting \( S = \{1,\ldots,n\} \), the set \( K^S \) is a \( K \)-module, and the set \( \text{End}_K(K^S) \) of all \( K \)-module endomorphisms of \( K^S \) has a natural \( K \)-algebra structure, where the addition and scalar multiplication are pointwise and the multiplication

\[
\text{End}(K^S) \times \text{End}(K^S) \rightarrow \text{End}(K^S)
\]

is composition. Let \( e_j(i) := 1 \) if \( i = j \) and 0 otherwise. The map that sends the endomorphism \( \phi \in \text{End}(K^S) \) to the matrix \( (a_{ij}) \in gl(n,K) \) defined by

\[
a_{ij} := \phi(e_j)(i), \ \forall i,j \in S
\]

is bijective, and preserves the \( K \)-algebra structures, so this gives another way to view \( gl(n,K) \).

But \( \text{End}(K^S) \) makes sense, and has the same \( K \)-algebra structure for arbitrary sets \( S \), and the same formula (1) defines an object that we can think of as a matrix, i.e. a two-dimensional array, in which the rows and columns are indexed by elements of \( S \). Let us denote the set of all these matrices by \( gl'(S,K) \). If \( S \) is infinite, it is no longer true that the map from endomorphisms to matrices is injective, because the matrix only depends on the values of the endomorphism on the linear
span of the $e_j$, a proper submodule $F$ of $K^S$. When the algebraic operations of pointwise addition and scalar multiplication on $\text{End}(K^S)$ are transferred to $\text{gl}'(S, K)$, the formulas look familiar:

\[(a_{ij}) + (b_{ij}) := (a_{ij} + b_{ij}),\]
\[c \cdot (a_{ij}) := (ca_{ij}),\]

However, since an endomorphism $\phi$ may map an $e_j$ outside the subspace $F$, there is not enough information in the matrix associated to $\phi$ to allow us to calculate the matrix associated to $\psi \circ \phi$, for another endomorphism $\psi$. We can overcome this by restricting attention to $\text{End}_K(F)$, and then composition transfers to the familiar-looking rule:

\[(a_{ij}) \times (b_{ij}) := (c_{ij}),\]

where $c_{ij} = \sum_{k \in S} a_{ik} b_{kj}$.

This rule actually gives a well-defined product on matrices such that, for each $i \in S$, the entry $a_{ij}$ is zero for all but a finite number of $j \in S$, and we denote this set of matrices by $\text{gl}(S, K)$. (Note that this set is somewhat larger than the image of $F$.) With these operations, this set is a $K$-algebra, for arbitrary sets $S$.

This simple idea is useful, because it is sometimes more natural to index the rows and columns of a matrix by elements of a particular finite or infinite set, and it may be artificial (or impossible) to label these using natural numbers. For our immediate purposes, the natural index set that presents itself is the set $S$ of monic monomials in $d$ variables [15], because the semigroup structure of that $S$ is relevant to our purpose.

$\text{gl}(S, K)$ is a semigroup with identity under multiplication; the identity matrix $I$ has $I_{ii} = 1$ and $I_{ij} = 0$ when $i \neq j$. We denote the group of invertible elements by $\text{GL}(S, K)$.

3. Matrix Representation

Let $K$ be an integral domain with identity, $d \in \mathbb{N}$ and $(f, g) \in \mathcal{R} = \mathcal{R}_d(K)$, and let $S$ be the set of monic monomials in $d$ variables. Then we can define an associated matrix $M(f, g) \in \text{GL}(S, K)$ by setting the $(m, n)$ entry equal to the coefficient of the monomial $m$ in $f \cdot (n \circ g)$. This means that you take the composition of the monomial $n \in S$ with the formal map $g \in \mathcal{G}$, getting an element of $\mathcal{F}$, you then multiply this in $\mathcal{F}$ by the formal series $f$, and then you take the coefficient of the monomial $m \in S$. Expanding the product, this gives

\[M(f, g)_{m, n} = \sum_p f_p(p \cdot (n \circ g))_m = \sum_{p|m} f_p(n \circ g)_{m/p}.\]
Theorem 2. The map $M$ is an injective group homomorphism from the Riordan group $R$ into $\text{GL}(S,K)$.

Proof. First, we show that
\begin{equation}
M((f,g)(f',g')) = M(f,g)M(f',g')
\end{equation}
whenever $f, f' \in F^\times$ and $g, g' \in G$.

Let $(f,g)$ and $(f',g')$ belong to $R$. Fix $m, n \in S$. Then the $(m,n)$ entry in
\[ M((f,g)(f',g')) = M((f \cdot (f' \circ g) \cdot g' \circ g) \]
is the coefficient of $m$ in
\[ f \cdot (f' \circ g) \cdot n(g' \circ g). \]

We compare this with the $(m,n)$ entry in the matrix product $M((f,g)M((f',g'))$, which is
\[ \sum_{p \in S} (f \cdot p \circ g)_{m} \cdot (f' \cdot n \circ g')_{p}. \]

To begin with we look at three special cases.

Case 1: $(f, \mathbb{I})(f', g')$, i.e. the first factor is in the Appell subgroup.
We have to compare the coefficient of $m$ in $f \cdot f' \cdot (n \circ g')$ with the $(m,n)$ entry of the matrix product $M((f, \mathbb{I})M((f', g'))$. The latter entry is
\[ \sum_{p \mid m} (f \cdot p \circ g)_{m} (f' \cdot (n \circ g'))_{p}, \]
because $(f \cdot p)_{m}$ is obviously zero unless $p$ divides $m$. But $(f \cdot p)_{m} = f_{m/p}$, so the entry is
\[ \sum_{p \mid m} f_{m/p} \cdot (f' \cdot (n \circ g'))_{p}, \]
and this is exactly the coefficient of $m$ in $f \cdot f' \cdot n \circ g'$. So Equation (2) holds in this case.

Case 2: $(f,g)(1, g')$, i.e. the second factor is in the Lagrange subgroup.
We have to compare the coefficient of $m$ in $f \cdot (n \circ g' \circ g)$, which equals
\[ \sum_{p \mid m} f_{m/p} (n \circ g' \circ g)_{p}, \]
with the $(m,n)$ entry of $M((f,g)M((1, g'))$. This entry is
\[ \sum_{p} (f \cdot (p \circ g))_{m} (n \circ g')_{p} \sum_{q \mid m} f_{m/q} (p \circ g)_{q} (n \circ g')_{p}. \]
Interchanging the order of summation, this equals
\[ \sum_{q|m} f_{m/q} \sum_{p} (p \circ g)\_q(n \circ g')\_p. \]

But the inner sum is exactly the coefficient of \( q \) in \((n \circ g') \circ g\), so the entry equals
\[ \sum_{q|m} f_{m/q}(n \circ g' \circ g)\_q. \]

Replacing the dummy variable \( q \) by \( p \), we see that Equation (2) also holds in this case.

Case 3: \((1, g)(f', 1)\), i.e. the first factor is in the Lagrange subgroup and the second in the Appell subgroup.

The \((m, n)\) entry in
\[ M((1, g)(f', 1)) = M(f' \circ g, g) \]
is the coefficient of \( m \) in \((f' \circ g) \cdot (n \circ g)\), which equals
\[ \sum_{p|m} (f' \circ g)\_p \cdot (n \circ g)\_p. \]

Now \( f' \circ g = \sum_q f'_q \cdot (q \circ g)\), so this matrix entry is
\[ \sum_{p|m} \sum_q f'_q \cdot (q \circ g)\_m/p \cdot (n \circ g)\_p = \sum_q f'_q \sum_{p|m} (q \circ g)\_m/p \cdot (n \circ g)\_p. \]

Now \( \sum_{p|m} (q \circ g)\_m/p \cdot (n \circ g)\_p \) is the coefficient of \( m \) in \((q \circ g) \cdot (n \circ g) = (q \cdot n) \circ g\), so the entry is equal to
\[ \sum_q f'_q \cdot ((nq) \circ g)\_m = \sum_{n|m} f'\_n \cdot (p \circ g)\_m. \]

On the other hand, the \((m, n)\) entry in \( M(1, g) M(f', 1)\) is
\[ \sum_{p} (p \circ g)\_m \cdot (f' \cdot n)\_p = \sum_{n|m} (p \circ g)\_m \cdot f'\_n, \]
since \((f' \cdot n)\_p\) is zero unless \( n \) is a factor of \( p \). Thus Equation (2) holds in this case also.

In the general case, for \( f, f' \in \mathcal{F}^\times \) and \( g, g' \in \mathcal{G} \), we can write
\[ (f, g)(f', g') = (f, 1)(1, g)(f', 1)(1, g'), \]
so by Case 1,
\[ M((f, g)(f', g')) = M(f, 1) M((1, g)(f', 1))(1, g'), \]
and by Cases 2 and 3 this is
\[ M(f, 1) M((1, g)(f', 1)) M(1, g')) = M(f, 1) M(1, g) M(f', 1) M(1, g'). \]
Applying Case 1 twice more, this equals \( M(f, g)M(f', g') \), so we have shown that Equation (2) holds for each \( f, f' \in \mathcal{F} \times \) and \( g, g' \in \mathcal{G} \).

It follows that each \( M(f, g) \) is invertible, i.e. lies in \( GL(S, K) \) (and not just in \( gl(S, K) \)), because it is inverted by \( M((f, g)^{-1}) \). Here we use the facts that \( \mathcal{R} \) is a group, that \( M \) maps products in \( \mathcal{R} \) to products in \( gl(S, K) \), and that \( M \) maps the identity \( (1, \mathbb{1}) \) of \( \mathcal{R} \) to the multiplicative identity matrix \( I \) in the ring \( gl(S, K) \).

Finally, we prove that \( M \) is injective: Suppose \( M(f, g) = M(f', g') \).

Let \( m \in S \). The \((m, 1)\) entry of \( M(f, g) \) is the coefficient of \( m \) in \( f \cdot 1 = f \). Thus \( f_m = f'_m \). Since this holds for all \( m \in S \), we have \( f = f' \). Since \( M \) is a group homomorphism, and \((f, g) = (f, \mathbb{1})(1, g)\),

\[
M(1, g) = M(f, \mathbb{1})^{-1}M(f, g) = M(f', \mathbb{1})^{-1}M(f', g') = M(1, g').
\]

For \( 1 \leq j \leq d \), the \((m, x_j)\) entry of \( M(1, g) \) is the coefficient of \( m \) in \( x_j \circ g \), which is just the \( j \)-th component of \( g \). Thus \( g \) and \( g' \) have the same coefficients in each component, hence \( g = g' \). \( \square \)

4. Fundamental Theorem on Riordan Arrays

We continue to assume \( d \in \mathbb{N} \) and to denote by \( S \) the semigroup of all monic monomials in \( d \) variables.

The formula

\[
M(f, g)_{mn} := (f \cdot (n \circ g))_m, \forall m, n \in S
\]

defines a matrix \( M(f, g) \), an element of \( gl(S, K) \), whenever \( f \in \mathcal{F} \) and \( g \in \mathcal{G} \), and not just for \( f \in \mathcal{F} \times \) and \( g \in \mathcal{G} \). The matrix \( M(f, g) \) belongs to \( gl(S, K) \) because its \((m, n)\) entry is zero as soon as the degree of \( n \) exceeds the degree of \( m \).

The right-hand side of the formula

\[
(f, g)(f', g') := (f \cdot f' \circ g, g' \circ g)
\]
does not make sense in this generality, but it does make sense when \( f, f' \in \mathcal{F} \) and \( g, g' \in \mathcal{M}^d \). The right-hand side belongs to the cartesian product \( \mathcal{F} \times \mathcal{M}^d \).

**Definition 4.1.** We define the Riordan semigroup \( \tilde{\mathcal{R}} \) to be the cartesian product \( \mathcal{F} \times \mathcal{M}^d \), equipped with the product defined by Equation (4).

It is straightforward to check that the multiplication is (still) associative, so we do indeed have a semigroup here. The identity \((1, \mathbb{1})\) of \( \mathcal{G} \) is still the identity for \( \tilde{\mathcal{R}} \). The subset \( \mathcal{R} \subset \tilde{\mathcal{R}} \) is precisely the subgroup of invertible elements of \( \tilde{\mathcal{R}} \).
With its usual ring multiplication, the matrix algebra \( \text{gl}(S, K) \) is a semigroup with identity.

**Theorem 3.** The map \( M \) is a semigroup homomorphism from \( \tilde{R} \) into \( \text{gl}(S, K) \).

**Proof.** This amounts to saying that Equation (2) holds for general \((f, g), (f', g') \in \tilde{R}\). Inspection of the proof of Theorem 2 reveals that we did not make any use of the invertibility of \((f, g)\) or \((f', g')\) in proving Equation (2). We did use, at the last step, the factorization
\[
(f, g) = (f, \mathbb{1})(1, g),
\]
but this also holds for general \(f \in F\) and \(g \in M^d\). Thus the present theorem may be viewed as a corollary of the proof of Theorem 2. □

Theorem 3 is an extension to several variables of the so-called Fundamental Theorem on Riordan Arrays (FTRA). The following immediate corollary may be recognised as alternative formulations of FTRA:

**Corollary 3.1.** For \(f, u \in F\) and \(g \in G\), we have
\[
(f \cdot (u \circ g))_m = \sum_{\deg p \leq \deg m} u_p \cdot (f \cdot (p \circ g))_m
\]
whenever \(m \in S\).

**Proof.** Apply Theorem 3 to the \((m, 1)\) components of both sides of Equation (2), and replace \((f, g)\) by \((f, \mathbb{1})\) and \((f', g')\) by \((u, g)\). This gives
\[
(f \cdot (u \circ g))_m = \sum_p (f \cdot (p \circ g))_m \cdot u_p.
\]
Now note that \(p \circ g\) has nonzero \(m\)-coefficient only when the degree of \(p\) is at most the degree of \(m\), and the result follows. □

**4.1.** We remark that the theorem shows that the equation in Corollary 3.1 also holds more generally, for \(g\) belonging to the semigroup \(M^d\):

**Corollary 3.2.** Let \(f, u \in F\) and \(g \in M^d\), \(g \neq 0\), and suppose that a lowest-degree monomial \(n\) with a nonzero \((g_j)_n\) for some \(j \in \{1, \ldots, d\}\) has degree \(k\). Then \(k \geq 1\) and
\[
(f \cdot (u \circ g))_m = \sum_{\deg p \leq (\deg m)/k} u_p \cdot (f \cdot (p \circ g))_m
\]
whenever \(m \in S\).

**Proof.** The point here is that \((p \circ g)_m\) is zero if \(k \cdot \deg p > \deg m\). □
5. Laurent Series and Verde-Star Structures

The rest of this paper was inspired by a lecture of Luis Verde-Star, given in Madrid in 2017, and describing two-way infinite matrices associated with an even larger algebra than $\mathcal{F}$, on which a subgroup of $\mathcal{G}$ acts as a group of automorphisms. The topic was related to his paper [19].

5.1. Monomials. We embed the (commutative, cancellation) semigroup $S$ of monic monomials in the larger group $\hat{S}$ of objects $x^i$, where now we allow any $i \in \mathbb{Z}^d$. As a group, $\hat{S}$ is isomorphic to the additive group $(\mathbb{Z}^d, +)$, the free abelian group on $d$ generators, via the map $x^i \mapsto i$. It has a partial order, defined by

$$x^i \leq x^{i'} \iff i_j \leq i'_j \forall j.$$ (Equivalently, $m \leq n$ means that $n/m \in S$.) It is not hard to see that each subset $A \subset \hat{S}$ that is bounded below has a unique greatest lower bound. We denote this lower bound by $\inf A$.

In case $A \subset S$, $\inf A = \text{hcf} A$ (cf. [15, Subsection 1.1]).

5.2. Laurent series. For an arbitrary formal series $f(x) := \sum_{m \in \hat{S}} f_m m$ in $d$ variables, with coefficients $f_m \in K$, we define the support of $f$ to be the set

$$\text{spt } f = \{m \in \hat{S} : f_m \neq 0\}.$$  

We define the subset of Laurent series by

$$\mathcal{L} := \left\{ f = \sum_{f \in \hat{S}} f_m m : \text{spt } f \text{ is bounded below} \right\}.$$  

The set $\mathcal{L}$ becomes a commutative $K$-algebra with identity when endowed with term-by-term addition and convolution multiplication

$$\left( \sum_m f_mm \right) \left( \sum_m f'_m m \right) := \sum_m \left( \sum_p f_p \cdot f_{m/p} \right) m.$$  

For nonzero $f \in \mathcal{L}$, we define the vertex of $f$ to be

$$v(f) := \inf \text{spt} (f).$$  

This coincides with the definition previously given in [15], in case $f \in \mathcal{F}$.

We refer to $\mathcal{L}$ as the Laurent series algebra associated to the integral domain $K$. It contains $\mathcal{F}$ as a subalgebra.
We refer to the group $L^\times$ as the Laurent series group of $K$. We note that $\hat{S}$ is a subgroup of $L^\times$, i.e. the product of two monomials $mm'$ in $L$ is their product in $\hat{S}$.

For $f \in L$, we always denote the coefficient of the monomial $m$ in the series for $f$ by $f_m$. Notice that $(n \cdot f)_m = f_{m/n}$ whenever $m, n \in \hat{S}$.

5.3. Example. For a set $A \subset \hat{S}$ that is bounded below, let $\chi_A$ denote the series $\sum_{m \in A} m$. Then $\chi_A \in L$. If $B \subset \hat{S}$ is another such set, then the product $h := \chi_A \chi_B$ has coefficients $h_m$ that count the number of ways to factor $m$ as $pq$, with $p \in A$ and $q \in B$.

Thus if $f, f' \in L$, and we let $A := \text{spt}(f)$ and $B := \text{spt}(f')$, then the coefficient $h_m$ measures the complexity of computing $(ff')_m$.

5.4. Other options. There are a good many other ways to construct algebras that can be called Laurent series algebras. The algebra $L$ is just one possibility.

The Laurent series algebra $L$ is a (rather small) subalgebra of the quotient field $\mathcal{F}$ of $F$. Most series $f \in F$ are noninvertible in $L$. For instance, in two dimensions the element $(x_1 + x_2)^{-1} \in \mathcal{F}$ does not belong to $L$ (see below).

It is possible to describe $L$ in terms of fields of fractions of power series rings in one variable, as follows. Let $\Sigma := \Sigma_d$ denote the symmetric group on $\{1, \ldots, d\}$. Then for $\tau \in \Sigma$ we can form, in turn, (1) the (formal) power series ring $K[[x_{\tau(1)}]]$, (2) its field of fractions $K((x_{\tau(1)}))$, (Laurent series in $x_{\tau(1)}$ over $K$), (3) the power series ring $K((x_{\tau(1)}))(x_{\tau(2)})$, (4) its field of fractions $K((x_{\tau(1)}))(x_{\tau(2)})$, (Laurent series in $x_{\tau(2)}$ with coefficients that are Laurent series in $x_{\tau(1)}$), and so on, ending up with the field (2d) of Laurent series

$\mathcal{L}_\tau := K((x_{\tau(1)}))(x_{\tau(2)}) \cdots (x_{\tau(d)})$.

This field $\mathcal{L}_\tau$ is larger than $L$ once $d > 1$, because an element may have nonzero coefficients on monomials involving infinitely many different negative powers of $x_{\tau(1)}$. We have, in fact,

$\mathcal{L} = \bigcap \{\mathcal{L}_\tau : \tau \in \Sigma_d\}$.

The paper of Monforte and Kauers [10] gives a useful historical survey of the various other kinds of Laurent series algebras one might consider, with particular emphasis on those that are fields. It appears
that such algebras are useful in applications to combinatorial problems such as counting lattice-point in polytopes and counting integer partitions. They also arose, as far back as the late 1940’s, in work of A.I. Malcev and B.H. Neumann on embedding group algebras into a division algebra [9, 11], and indeed germs of the subject are found in work of Hans Hahn from forty years before that. There is a substantial literature on Malcev-Neumann fields of power series, related to totally-ordered groups. If we exclude consideration of nonintegral exponents, then the ne plus ultra, when it comes to creating a multiplicatively-closed space of formal series \( f = \sum_{m \in \hat{S}} f_m m \), is to confine attention to \( f \) whose supports lie in a sub-semigroup of \( \hat{S} \). This leads one to consider total orders on \( \mathbb{Z}^d \), and the intersection of \( \mathbb{Z}^d \) with cones in \( \mathbb{R}^d \). There are in fact uncountably many choices, each leading to a different field of “Laurent series”. None of these contains or is contained in the relatively-simpleminded algebra \( \mathcal{L} \) that we consider, which depends on the standard (non-total) lattice partial order on \( \mathbb{Z}^d \), but many of the elementary considerations we meet have, of course, arisen in these previous investigations, and we make no claim to originality for our observations about them, but as far as we know, the two theorems below are new.

Verde-Star considered a family of partial orders on \( \mathbb{Z}^d \) indexed by nonegative integers, with associated cones \( C_s \), and algebras of Laurent series (over \( \mathbb{C} \)) associated to them. Our algebra \( \mathcal{L} \) corresponds to his \( F_s \) with \( s = 0 \), and our \( \mathcal{F} \) corresponds to his \( F_0 \), in that case.

5.5. The Verde-Star-Riordan group.

Definition 5.1. We define the Verde-Star-Riordan group to be
\[
\mathcal{V} = \text{Hol}_{K\text{-alg}}(\mathcal{L}^\times),
\]

Thus
\[
\mathcal{V} = \mathcal{L}^\times \rtimes \text{Aut}_{K\text{-alg}}(\mathcal{L}).
\]

5.6. The group \( \mathcal{L}^\times \).

Proposition 5.1. Let \( f \in \mathcal{L} \). (1) If \( f \neq 0 \), then \( h = v(f)^{-1} \cdot f \in \mathcal{F} \), and \( v(h) = 1 \).
(2) \( \mathcal{L} \) is an integral domain.
(3) \( v(ff') = v(f)v(f) \) whenever \( f, f' \in \mathcal{L} \).
(4) \( f \in \mathcal{L}^\times \) if and only if \( f v(f) \in K^\times \).

Proof. (1) The fact that \( v(f) = \text{hcf}(\text{spt}(f)) \) implies that for each \( j \in \{1, \ldots, d\} \) there exists \( p \in \text{spt}(f) \) with \( f_p \neq 0 \) and \( p_j = v(f)_j \).
Thus \( q := p/v(f) \) belongs to \( \text{spt}(h) \) and has \( q_j = 1 \) (i.e. \( q \) ‘does not
involve $x_j$). Thus $v(h) = 1$. (cf. [15, proposition 1.1], which has essentially the same proof.)

(2) If $f$ and $f'$ are nonzero elements of $\mathcal{L}$, then $h := f/v(f)$ and $h' := f'/v(f')$ are nonzero elements of $\mathcal{F}$, and if $ff' = 0$, then $hh' = 0$, contradicting the fact that $\mathcal{F}$ is an integral domain. Thus $\mathcal{L}$ is an integral domain.

(3) We have to show that $v(ff')_j = v(f)_jv(f')_j$, for each $j \in \{1, \ldots, d\}$.

We may assume that $j = d$, without loss in generality. By the definition of the vertex, there must be a nonzero coefficients on some term of $f$ of the form $mv(f)_d$, where $m \in \hat{S}$ does not involve $x_d$ at all. In other words, if we write $h := f/v(f)_d$, then

$$k := h(x_1, \ldots, x_{d-1}, 0) \neq 0,$$

and $k \in \mathcal{L}_{d-1}$. Similarly, letting $h' := f'/v(f')_d$,

$$k' := h'(x_1, \ldots, x_{d-1}, 0) \neq 0,$$

and $k' \in \mathcal{L}_{d-1}$. Since $\mathcal{L}_{d-1}$ is an integral domain, we have $kk' \neq 0$, $v(hh')_d = 1$, and $v(ff')_d = v(f)_dv(f')_d$.

(4) Let $h := v(f)^{-1}f$. Then $h \in \mathcal{F}$ and $h_1 = f_v(f)$. Also $v(h) = 1$.

If $f_v(f) \in K^\times$, then $h \in \mathcal{F}^\times$, and

$$f \cdot (v(f)^{-1} \cdot h^{-1}) = (v(f)^{-1} \cdot f) \cdot h^{-1} = h \cdot h^{-1} = 1,$$

so $f \in \mathcal{L}^\times$.

Conversely, if $f$ has an inverse $k \in \mathcal{L}$, then

$$1 = fk = f_v(f)^{-1}v(f)k = hv(f)k,$$

so $h \in \mathcal{L}^\times$. Thus it remains, in view of [15, proposition 1.3] to show that $h^{-1} \in \mathcal{F}$. Suppose $h^{-1} = b$. We have to show that $1 \leq v(b)$.

Suppose, on the contrary, that $1 \not\leq v(b)$. Then some $x_j$ appears with a negative power in $v(b)$, and we may suppose without loss in generality that it is $x_1$, so that $v(b)_1 = x_1^{-k}$ for some $k \in \mathbb{N}$. The series

$$a := \sum_{m \in \hat{S}, m_1 = x_1^{-k}} b_mm$$

is not zero, by definition of $v(b)$.

Similarly, since $v(h) = 1$, the series

$$r := \sum_{m \in \hat{S}, m_1 = 1} h_m m$$

is not zero.
Thus, since $hb = 1$,

$$0 = \sum_{m \in \hat{S}, m_1 = x^{-k}} (hb)_m m = ar,$$

contradicting the fact that $\mathcal{L}$ is an integral domain. Thus $b \in \mathcal{F}$, as required. \hfill \Box

For example, taking $f = x_1 + x_2 \in \mathcal{F}_2$, we have $v(f) = 1$ and $f_1 = 0$, so $f$ is not invertible. This is probably just as well, for its reciprocal seems to have two different formal Laurent series:

$$\frac{1}{x_1 + x_2} \sim \sum_{k=0}^{\infty} x_1^{-k-1} x_2^k \sim \sum_{k=0}^{\infty} x_1^k x_2^{-k-1}.$$

Of course neither series belongs to $\mathcal{L}$.

From the proof, we note:

**Corollary 3.3.**

1. $\mathcal{L}^\times = \hat{S}\mathcal{F}^\times$ and $\hat{S} \cap \mathcal{F}^\times = (1)$, i.e. $\mathcal{L}^\times$ is the inner direct product of $\hat{S}$ and $\mathcal{F}^\times$.

2. The restriction $v|\mathcal{L}^\times$ is a group homomorphism from $\mathcal{L}^\times$ onto $\hat{S}$.

3. $\ker v|\mathcal{L}^\times = \{ f \in \mathcal{L}^\times : v(f) = 1 \} = \mathcal{F}^\times$. \hfill \Box

### 5.7. Composition.

For $g \in \mathcal{G}$, we have seen that the composition map $C_g$ is a $K$-algebra automorphism of $\mathcal{F}$. We cannot define the composition $f \circ g$ for arbitrary $f \in \mathcal{L}$ and $g \in \mathcal{G}$ in a sensible way. For instance, one would expect the composition of $x_1^{-1}$ and the map $(x_1, x_2) \mapsto (x_1 + x_2, x_2)$ to represent $(x_1 + x_2)^{-1}$, but $x_1 + x_2$ is not invertible in $\mathcal{L}$. However, we have the following limited composition:

Given two $d$-tuples $f := (f_1, f_2, \ldots, f_d) \in \mathcal{F}^d$ and $f' := (f'_1, f'_2, \ldots, f'_d) \in \mathcal{F}^d$, we define the coordinatewise product $f * f'$ by setting

$$(f * f')(x) = (f_1 f'_1, \ldots, f_d f'_d).$$

For instance,

$$(x_1, x_2) * (1 + x_1 + x_2^2, 1 - x_2 - x_1^2) = (x_1 + x_1^2 + x_1 x_2^2, x_2 - x_2^2 - x_1 x_2).$$

This product $* \mathcal{F}^d \times \mathcal{F}^d \to \mathcal{F}^d$, and is commutative and associative. If we define addition coordinatewise, then $*$ distributes over addition, and $\mathcal{F}^d$ becomes a $K$-algebra. In fact, it is just the direct

---

1Note that there is room for confusion between $f_1 \in \mathcal{F}$, the first component of an $f \in \mathcal{F}^d$, and $f_1 \in K$, the coefficient of $1_S$ in a series $f \in \mathcal{F}$. In the first case the subscript 1 is $1_Y$, and in the second it is $1_S$. It is necessary to pay attention to the context to avoid this confusion.
product of \( d \) commuting copies of \( \mathcal{F} \). It has the identity \((1, \ldots, 1)\) and its group of units is
\[
(\mathcal{F}^d)^\times = (\mathcal{F}^\times)^d = \{ f \in \mathcal{F}^d : f_j \in \mathcal{F}^\times \ \forall j \}.
\]
\( \mathcal{F}^d \) is not an integral domain when \( d > 1 \), since, for instance,
\[
(1, 0) \ast (0, 1) = (0, 0) = 0_{\mathcal{F}^d}.
\]
In case \( f \) is a formal map, i.e. \( f \in \mathcal{M}^d \), the product \( f \ast f' \) also belongs to \( \mathcal{M}^d \) for each \( f' \in \mathcal{F}^d \). In the particular case \( f = \mathbb{1} \), we use the more suggestive notation
\[
x \ast f' := \mathbb{1} \ast f' = (x_1 f'_1, \ldots, x_d f'_d).
\]
We define the set
\[
\mathcal{K} := \{ x \ast f : f \in (\mathcal{F}^d)^\times \}.
\]
This is a subgroup of \( G \), because the linear part \( L(x \ast f) \) of \( x \ast f \) is represented by the diagonal matrix \( \text{diag}((f_1)_1, (f_2)_1, \ldots, (f_d)_1) \), and this is invertible in \( \text{gl}(d, K) \) if each component \( f_j \in \mathcal{F}^\times \). Note that the group \( L(\mathcal{K}) \leq \text{GL}(K, d) \) of \( K \)-linear maps, isomorphic to \((K^\times, \times)^d\) may be regarded as a subgroup of \( \mathcal{K} \), and then \( \mathcal{K} \) is the semidirect product of
\[
\ker L|\mathcal{K} = \{(x_1(1 + \text{HOT}), \ldots, x_d(1 + \text{HOT}))\}
\]
and \( K^{\times d} \).
Moreover, if \( g = x \ast f \in \mathcal{K} \), then each \( f_j \) belongs to \( \mathcal{F}^\times \), and hence each component \( g_j = x_j f_j \) belongs to \( \mathcal{L}^\times \). This allows us to define \( m \circ g \in \mathcal{L}^\times \), for arbitrary \( m \in \hat{S} \) by writing \( m = x^i \) and defining
\[
m \circ g := \prod_{j=1}^d (x_j f_j)^{i_j}.
\]
Notice that \( m \circ g = m \cdot m(f_1, \ldots, f_d) \in m\mathcal{F}^\times \) when \( m \in \hat{S} \) and \( g = x \ast f \in \mathcal{K} \).

We can then define the composition \( f \circ g \) for arbitrary \( f \in \mathcal{L} \) and \( g \in \mathcal{K} \) by setting
\[
f \circ g := \sum_{m \in \hat{S}} f_m m \circ g.
\]
this formal Laurent series actually defines an element of \( \mathcal{L} \), because \( m = v(m \circ g) \), so \( v(f) = v(f \circ g) \).

We denote the composition map \( f \mapsto f \circ g \) on \( \mathcal{L} \) by the same symbol \( C_g \) as we already use for its restriction to \( \mathcal{F} \).

**Proposition 5.2.** For each \( g \in \mathcal{K} \), the composition map \( C_g : \mathcal{L} \to \mathcal{L} \) is a \( K \)-algebra automorphism of \( \mathcal{L} \).
Proof. Composition preserves sums, products and scalar multiples. So $C_g$ is a $K$-algebra homomorphism. It is inverted by composition with $g^{-1}$, so it is an isomorphism. \hfill \Box

**Corollary 3.4.** The map $g \mapsto C_g$ is an injective homomorphism of $K^{\text{op}}$ into $\text{Aut}_{K\text{-alg}}(\mathcal{L})$.

*Proof.* The injectivity follows from the fact that $x_j \circ g = g_j$, the $j$-th component of $g$.

The map is a homomorphism because of the associativity of composition. \hfill \Box

5.8. **Permutations.** The permutation group $\Sigma := \Sigma_d$ acts on the symbols $x_j$ by permuting the $j$’s, and this gives another automorphism of $\mathcal{L}$. This is the same as the composition automorphism corresponding the the linear isomorphism of $K^d$ obtained by permuting the entries of the $d$-tuples, so we also regard $\Sigma$ as a subgroup of $G$, and note that it intersects $K$ only in the identity, and acts on $K$ by conjugation. The product set $K\Sigma_d$ is thus a subgroup of $G$, isomorphic to the semidirect product $K \rtimes \Sigma$.

With this convention, we can state:

**Proposition 5.3.** The map $g \mapsto C_g$ is an injective homomorphism of $(K\Sigma)^{\text{op}}$ into $\text{Aut}_{K\text{-alg}}(\mathcal{L})$. \hfill \Box

6. **Automorphisms of $\mathcal{L}$**

We are going to prove:

**Theorem 4.** Assume that the integral domain $K$ has a reciprocal of $n \cdot 1_K$ for some integer $n > 1$. Then the map $g \mapsto C_g$ is an isomorphism $(K\Sigma)^{\text{op}} \simeq \text{Aut}_{K\text{-alg}}(\mathcal{L})$.

The hypothesis on $K$ is satisfied if $K$ has positive characteristic or is a field or contains the ring $\mathbb{Z}_p$ of $p$-adic integers for some prime $p$.

This will immediately give us this description of the Verde-Star-Riordan group:

**Corollary 4.1.** If $n \cdot 1_K$ is invertible in $K$ for some integer $n > 1$, then

$$\text{Hol}_{K\text{-alg}}(\mathcal{L}^\times) \simeq \mathcal{L}^\times \ltimes (K\Sigma)^{\text{op}} = \hat{S}\mathcal{F}^\times \ltimes (K\Sigma)^{\text{op}},$$

with the product given by the usual formula

$$(f,g)(f',g') = (f \cdot (f' \circ g), g' \circ g)$$

for $f, f' \in \mathcal{L}^\times$ and $g, g' \in K$. 
The proof proceeds by a series of lemmas.

**Lemma 6.1.** Each \( x_j \) is prime in \( F \).

**Proof.** Without loss in generality, consider \( j = d \).

The map \( \Phi : f(x_1, \ldots, x_d) \mapsto f(x_1, \ldots, x_{d-1}, 0) \) is a \( K \)-algebra homomorphism \( F_d \to F_{d-1} \). (We make the convention that \( F_0 := K \).) We have

\[
\ker \Phi = \{ f \in F_d : x_d | f \}.
\]

Suppose \( x_d | ff' \). If \( x_d \not| f \) and \( x_d \not| f' \), then

\[
\Phi(ff') = \Phi(f)\Phi(f') \neq 0,
\]

since \( F_{d-1} \) is an integral domain. This is a contradiction. Thus \( x_d | f \) or \( x_d | f' \). \( \square \)

We abbreviate \( \mathcal{A} := \text{Aut}_{K\text{-alg}}(L) \).

**Lemma 6.2.** Suppose \( n \cdot 1_K \) is invertible in \( K \) for some integer \( n > 1 \). Let \( \alpha \in \mathcal{A} \) and \( f \in F^\times \). Then \( \alpha(f) \in F^\times \).

**Proof.** The binomial coefficients \( \binom{1/n}{k} \) are well-defined in \( K \) for each nonnegative integer \( k \) [12]. So if \( h \in M \) the formula

\[
r := \sum_{k=0}^{\infty} \binom{1/n}{k} h^k
\]

defines an element of \( F^\times \) which is an \( n \)-th root of \( 1 + h \): \( r^n = 1 + h \).

If the lemma were false, \( v(\alpha(F^\times)) \subset \hat{S} \) would not be just (1), so there would exist some \( j \) such that the induced homomorphism \( \phi : F^\times \to (\mathbb{Z},+) \) defined by taking \( f \) to the exponent of \( x_j \) in \( v(\alpha(f)) \) is nontrivial, and hence its image would be \( m\mathbb{Z} \) for some nonzero \( m \in \mathbb{Z} \). Choose \( f \in F^\times \) with \( \phi(f) = n \). Replacing \( f \) by \( f/f_1 \), we may assume \( f_1 = 1 \). Then \( f = 1 + h \) with \( h \in M \), and choosing an \( n \)-th root \( r \) as above, we would have \( n\phi(r) = \phi(f) = m \) and \( 0 < \phi(r) < m \), which is impossible. \( \square \)

**Proof of Theorem 4.** Suppose \( \alpha \in \mathcal{A} \). By Lemma 6.2, \( \alpha \) maps \( F^\times \to F^\times \). Since \( \alpha(1) = 1 \), \( \alpha \) maps \( F \to F \). Since the same is true of \( \alpha^{-1} \), the restriction \( \alpha|F \in \text{Aut}_{K\text{-alg}}(F) \). By Theorem 1, \( \alpha \) coincides on \( F \) with composition with the map

\[
g := (\alpha(x_1), \ldots, \alpha(x_d)) \in \mathcal{G}.
\]

Consider \( \alpha(x_j) \). Since \( x_j \in K^\times \), we have \( \alpha(x_j) \in K^\times \), so \( \alpha(x_j) = m_j h_j \) for some monomial \( m_j \in \hat{S} \) and some \( h_j \in F^\times \). Since \( x_j \in F \), we have \( m_j \in F \). Since \( x_j \) is prime in \( F \), so is \( m_j \), so it is equal to \( x_r \) for some \( r = \tau(j) \), say. Since \( L(g) \in \text{GL}(d, K) \), \( \tau \) is a permutation,
Thus \( \tau^{-1} g \in K \), and \( g \in \Sigma K = K \Sigma \). Thus the composition map \( C_g \) is a \( K \)-algebra automorphism of the whole of \( L \).

It remains to show that \( \alpha = C_g \). Fix \( f \in L \). Then \( f = mr \), with \( m = v(f) \in \hat{S} \) and \( r \in F \). Then the automorphism property gives \( \alpha(m) = m \circ g \). Also \( \alpha(r) = r \circ g \), since \( r \in F \), so we get

\[
\alpha(f) = \alpha(m) \alpha(r) = (p \circ g) \cdot (r \circ g) = f \circ g,
\]

as required. \( \square \)

6.1. **Question.** We do not know whether Theorem 4 remains true in characteristic zero without the hypothesis on the existence of a reciprocal for some integer greater than one. In particular, we do not know it for \( K = \mathbb{Z} \). Could there be some more exotic \( \mathbb{Z} \)-algebra automorphism (i.e. ring automorphism) of \( L(\mathbb{Z}) \)?

The ring \( L(K) \) becomes a (complete, metrisable) topological ring if we define a neighbourhood of 0 to be a set that contains some power \( M^k \) \((k \in \mathbb{N})\) of the maximal ideal of \( F \). Lemma 6.2 immediately yields the following “automatic continuity” result:

**Corollary 4.2.** If \( n \cdot 1_K \in K^x \) for some integer \( n > 1 \), then each \( K \)-algebra automorphism of \( L \) is continuous. \( \square \)

We also have the following for all \( K \):

**Proposition 6.1.** If \( \alpha \) is a continuous \( K \)-algebra automorphism of \( L \), then \( \alpha \in K \Sigma \).

**Proof.** We claim that \( \alpha \) maps \( F \) into \( F \).

For otherwise there is some \( f \in F \) with \( \alpha(f) \not\in F \), so \( h := f - f_1 \in \mathcal{M} \), and \( \alpha(h) = \alpha(f) - f_1 \not\in F \). For some \( j \in \{1, \ldots, d\} \), \( v(h)_j < 1 \). By continuity, since \( M \) is a neighborhood of 0, there exists some \( N \in \mathbb{N} \) such that \( \alpha(M^N) \subset M \). But \( h^N \in \mathcal{M}^N \), and \( v(\alpha(h^N))_j = v(\alpha(f))^N_j < 1 \), a contradiction.

The rest of the proof proceeds just like that of Theorem 4. \( \square \)

**Corollary 4.3.** For each \( K \) and \( d \), we have

\[
\text{Hol}_C(L, x^x) \simeq L^x \ltimes (K \Sigma)^{\text{op}}
\]

where \( C \) is the category of topological \( K \)-algebras and continuous \( K \)-algebra automorphisms. \( \square \)

Thus \( L^x \ltimes (K \Sigma)^{\text{op}} \) is a holomorph with respect to a different category. We conjecture that there are no exotic automorphisms in any case, and the two holomorphs coincide.
6.2. Distinguished subgroups. By analogy with the terminology used for the Riordan group, we define analogues of $\mathfrak{A}$ and $\mathfrak{L}$, the extended Appell subgroup of $\text{Aut}_{K\text{-alg}}(\mathcal{L})$

$$\widehat{\mathfrak{A}} := \widehat{\mathfrak{A}}_d(K) := \{(f, 1) : f \in \mathcal{L}^\times\},$$

and the restricted Lagrange subgroup

$$\widetilde{\mathfrak{L}} := \widetilde{\mathfrak{L}}_d(K) := \{(1, g) : g \in K\Sigma\}.$$

We need to see what happens when an element of $\widetilde{\mathfrak{L}}$ acts on monomials. First, for $m \in \widehat{S}$ we define the symmetric infimum:

$$\text{sinf}(m) := \inf\{m \circ \tau : \tau \in \Sigma\}.$$

**Proposition 6.2.** Let $m \in \widehat{S}$ and $g \in K\Sigma$. Let $m \circ g = \sum_{p \in \widehat{S}} f_p p$. Then $f_p = 0$ unless $p \geq \text{sinf}(m)$.

**Proof.** We can write $g = k \circ \lambda \circ \tau$ with $\tau \in \Sigma$, $\lambda \in K^{\times d}$ and $k \in \ker L|K = T \cap K$. Then

$$m \circ k = m(x_1(1 + \text{HOT}), \ldots, x_d(1 + \text{HOT})), $$

so $(m \circ k)_p = 0$ unless $m \leq p$. Now $m \circ k \circ \lambda$ is obtained from $m \circ k$ by replacing each $x_j$ by a multiple $\lambda_j x_j$ with $\lambda_j \in K^\times$, so $(m \circ k \circ \lambda)_p = 0$ if and only if $(m \circ k)_p = 0$. Finally, $(m \circ k \circ \lambda \circ \tau)_p$ is nonzero if and only if $m \leq p \circ \tau$, so only if $p \geq \text{sinf}(m)$. □

7. The Verde-Star matrix representation

We define $M : \mathcal{L}^\times \times K\Sigma \to \text{gl}'(\widehat{S}, K)$ by

$$M(f, g)_{mn} := (f \cdot (n \circ g))_m, \forall m, n \in \widehat{S}.$$ 

This is just another example of a matrix indexed by a set, but you may also think of it as a ‘doubly-infinite’ matrix if you like, since the index set $\widehat{S}$ has no lower bound, unlike $S$.

We observe some obvious symmetries of $M(f, g)$:

**Proposition 7.1.** (1) $M(pf, g)_{mn} = M(f, g)_{m/pn}$ if $p \in \widehat{S}$.
(2) $M(f, \lambda \circ g)_{mn} = n(\lambda)M(f, g)_{mn}$ if $\lambda \in K^{\times d}$.
(3) $M(f, \tau \circ g)_{mn} = M(f, g)_{m\circ\tau}$ if $\tau \in \Sigma$. □

If we take $f \in \mathcal{L}^\times$ and $g = x \ast h$ with $h \in (\mathcal{F}^\times)^d$, then for $n, m \in \widehat{S}$ we have

$$(n \circ g) = n \cdot (n \circ h),$$

and if we abbreviate $v = v(f)$, then

$$f \cdot (n \circ g) = v \cdot \left(\frac{f}{v}\right) \cdot n \cdot (n \circ h),$$
so the entry $M(f, g)_{mn}$ is
\[
\left( vn \cdot \left( \frac{f}{v} \right) \cdot (n \circ h) \right) = ((f/v) \cdot (n \circ h))_{m/vn}.
\]
Since $f/v \in \mathcal{F}$ and $n \circ g \in \mathcal{F}$, this entry is zero unless $m/vn \geq 1$. Thus, for fixed $m \in \hat{S}$, the entry is zero unless $n \leq m/v$, and for fixed $n \in \hat{S}$, the entry is zero unless $m \geq vn$. Hence $M$ maps $L^\times \times K$ into the set $\widetilde{\text{gl}}(\hat{S}, K)$ of matrices $M \in \text{gl}^\times(\hat{S}, K)$ such that
\[
\forall m \in \hat{S} \ \exists R \in \hat{S} : n > R \implies M_{mn} = 0, \\
\forall n \in \hat{S} \ \exists T \in \hat{S} : m < T \implies M_{mn} = 0.
\]
If $(f, g)$ is any element of $L^\times \times K \Sigma$, then $g = \tau \circ (x \ast f)$ for some $\tau \in \Sigma$ and $x \ast f \in K$, and then $M(f, g)_{mn} = M(f, x \ast f)_{mno\tau}$ is zero unless $v(f)n \circ \tau \leq m$, so again we see that $M(f, g)$ belongs to $\widetilde{\text{gl}}(\hat{S}, K)$.

The set $\text{gl}(\hat{S}, K)$ is closed under the usual matrix addition and multiplication, and forms a $K$-algebra. We denote its group of invertibles by $\widetilde{\text{GL}}(\hat{S}, K) := \text{gl}(\hat{S}, K)^\times$.

The following is the Laurent series version of the Fundamental Theorem:

**Theorem 5.** $M : L^\times \times K \Sigma \to \widetilde{\text{GL}}(\hat{S}, K)$ is an injective group homomorphism.

**Proof.** This is essentially the same proof as that of Theorem 2, except that we have to be a little careful about the sums involved. The key point is that “intervals” in $\hat{S}$, i.e. sets of the form
\[
\{ m \in \hat{S} : n \leq m \leq p \},
\]
with $n, p \in \hat{S}$ are finite. If nonempty, i.e. if $n \leq p$, then regarded as sets in $\mathbb{Z}^d$ they are rectangular boxes of lattice points.

Fix $(f, g), (f', g') \in L^\times \times K \Sigma$, and $m, n \in \hat{S}$. We need to show that
\[
M((f, g)(f', g'))_{mn} = \sum_{p \in \hat{S}} M(f, g)_{mp}M(f', g')_{pn}.
\]
Once we have this, the remainder of the proof goes as before.

Let $v = v(f)$, $v' = v(f')$.

For $p, r, s \in \hat{S}$, consider the element $a := f_rf'(p \circ g)_{m/r}(n \circ g)_{p/s} \in K$.

This is nonzero only if
\[
r \geq v, s \geq v', \frac{m}{r} \geq p \text{ and } \frac{p}{s} \geq n,
\]
and hence only if
\[ nv' \leq ns \leq p \leq \frac{m}{r} \leq \frac{m}{v}, \]
so only for a finite set of \( p \). Also, for each \( p \), the term \( a \) is nonzero only for \( v \leq r \leq m/p \) and \( v' \leq s \leq p/n \), i.e. for a finite set of pairs \( (r, s) \).

Thus the sum
\[ \sum_{p, r, s \in \hat{S}} f_{r,s}(p \circ g)_{m/r} (n \circ g')_{p/s} \]
has only a finite number of nonzero terms, and can be summed in any order to the same value. We find that
\[ \sum_{p \in \hat{S}} M(f, g)_{mp} M(f', g')_{pn} = \sum_{p} \sum_{r} \sum_{s} f_{r,s}(p \circ g)_{m/r} (n \circ g')_{p/s}, \]
and rearranging, this equals
\[ \sum_{r} f_{r} \sum_{s} f'_{s} \sum_{p} (p \circ g)_{m/r} (n \circ g')_{p/s}. \]
The inner sum is equal to
\[ (s \cdot n \circ g')_{m/r} = (r \cdot s \circ g \cdot n \circ g' \circ g)_{m} \]
so adding up, the whole is
\[ (f \cdot f' \circ g \cdot n \circ g' \circ g)_{m} = M((f, g)(f', g'))_{mn}, \]
as required. \( \square \)

7.1. Another description. We can view the Verde-Star-Riordan group as a subgroup of the automorphism group of \( L \) in the category \( K \)-mod of \( K \)-modules.

For \( f \in L \), we define the multiplier map \( M_f : L \to L \) by
\[ M_f(f') = f \cdot f', \quad \forall f' \in L. \]
The map \( M : f \mapsto M_f \) is an injection of \( L \) into the \( K \)-algebra of \( K \)-module endomorphisms of \( L \). It maps invertible elements \( f \in L^\times \) to \( K \)-module automorphisms of \( L \).

We have seen that for \( g \in K\Sigma \) the composition map \( C_g \) is a \( K \)-algebra automorphism of \( L \), so, a fortiori, it is a \( K \)-module automorphism.

The topological-\( K \)-algebra holomorph \( L^\times \ltimes (K\Sigma)^{op} \) is mapped injectively into \( \text{Aut}_{K\text{-mod}}(L) \) by the map
\[ (f, g) \mapsto M_f \circ C_g. \]
8. Reversibility

The work of Luzon et al [7] completed the explicit description of the reversible elements of $R_1(\mathbb{C})$. We would like to see similarly explicit descriptions of $R_d(K)$ for all $d \in \mathbb{N}$ and each integral domain with identity.

The following basic result translates the problem into ‘nuts and bolts’:

**Proposition 8.1.** Fix $d \in \mathbb{N}$ and an integral domain $K$ with identity, and abbreviate $R = R_d(K)$. Let $(a,b), (f,g) \in R$. Then $(a,b)$ reverses $(f,g)$ if and only if $b$ is a fixed point of $y \mapsto gyg$ and $a$ is a fixed point of $x \mapsto g(f) \cdot b(x) \cdot b(f)$.

(The dot here represents the multiplication in $F_d$.)

**Proof.** Directly from the definition one deduces that $(a,b)$ reverses $(f,g)$ if and only if $(f,g)(a,b)(f,g) = (a,b)$.

The left hand side evaluates to

$$(g(f) \cdot b(a) \cdot (b^2)(f), gbg).$$

As Polya said, for every unsolved problem there is always an easier problem that you can’t solve, and in this case it may be useful to start by asking which elements of $G$, tangent to the identity, are reversible.
in the whole Riordan group. Also, one could hope that the matrix representation of our main result will help.

9. Other Riordan Group Constructions

9.1. Cheon and collaborators have introduced other higher-dimensional kinds of Riordan groups. A referee suggested we relate them to our work.

The Riordan groups $R_d(K)$ we have just defined in this paper are constructed from structures composed of formal maps and series in several variables, and their elements are best represented as two-dimensional arrays where the rows and columns are indexed by the semigroup of $d$-dimensional monic monomials. It would be possible but not particularly helpful to represent them as multidimensional arrays indexed by integers. The labelling by these semigroup elements preserves information essential to the group structure, and enables simple and transparent formulas.

9.2. The multidimensional Riordan arrays introduced and applied by Cheon and Jin [3] are obtained from a group of formally-invertible formal maps in one variable by a finite number of repeated semidirect products with an abelian group of convolution-product-invertible formal series in one variable, and the elements of their groups are representable as multidimensional arrays (indexed by integers). They work over the ring $K = \mathbb{C}$ of complex numbers, but the construction is valid for any ring. These Cheon-Jin groups relate to the composition action of what we call $G_1(\mathbb{C})$ on the group $(\mathcal{F}_1^\times)^n$, of $n$-tuples of elements of $\mathcal{F}_1^\times$, and also on the space $(\mathcal{F}_1)^\mathbb{N}$ of infinite sequences of power series in one variable. These groups are unlike our $R_d$ for $d > 1$, but the idea could be used to define corresponding extensions $(\mathcal{F}_d^\times)^n \ltimes G_d$.

9.3. In another direction, Cheon, Huang and Kim [2] discuss multivariate Riordan arrays. They work in the context of a power series ring of Krull dimension $d$ over a field $K$. They explain that such a ring can be identified (in various ways) with what we call $\mathcal{F}_d(K)$ by selecting a set of variables $\mathbf{Z}$, i.e. a $d$-tuple of elements that generate the maximal ideal. We observe that the collection of possible ‘sets of variables’ is, in essence and ignoring order, the same as our $G_d(K)$, because composition with a formal map $g \in G_d$ turns one ‘set of variables’ into another. Starting with the idea of a Schauder basis for the ring (a sequence $(b_j)$ such that every element of the ring is an infinite sum $\sum_{j=1}^{\infty} \lambda_j b_j$, converging in the natural topology, with unique coefficients $\lambda_j \in K$) they single out a special kind, which they call a Riordan-Schauder basis.
The key ingredient here is the imposition of some linear order on the set of monomials (our $S_d$), putting them into a sequence. We observe that a general Riordan-Schauder basis for the ring $\mathcal{F}_d$ is in fact what you get when you compose any such sequence of monomials with any $g \in \mathcal{G}_d$. With this setup, they can define a Lagrange group with unit $\mathbb{Z}$ as a collection of ‘sets of variables’, and a Riordan monoid with respect to $\mathbb{Z}$ as a set of products, containing a Riordan group with respect to $\mathbb{Z}$ with an corresponding Appell subgroup. They represent the elements of this group by two-by-two matrices of elements of $K$, using the Riordan-Schauder basis, and they call these matrices Riordan arrays. They give a version of the Fundamental Theorem. They have to be careful about the character of the order imposed on the monomials. Apart from details, it seems that for each choice of units and basis order, the groups of Riordan arrays they construct are in fact isomorphic to our $\mathcal{R}_d(K)$, because each is the semidirect product of groups isomorphic to $\mathcal{F}_d^*$ and $\mathcal{G}_d$. It would be more accurate to say that the Cheon-Huang-Kim groups are isomorphic to $\text{Hol}_C(\mathcal{F}_d,^*)$, where $C$ is the category of topological $K$-algebras and continuous $K$-algebra homomorphisms. These are isomorphic to $\mathcal{R}_d$ because of the automatic continuity of $K$-algebra automorphisms of $\mathcal{F}_d$, when $\mathcal{F}_d$ is equipped with the natural topology.

9.4. We suggest that the focus on the order of the monomials or on the order of the Riordan-Schauder basis is perhaps a distraction from the essentials in the study of the Riordan groups in higher dimensions. The essential thing to focus on is the action on monomials, and the semigroup structure of $S_d$. This is not to say that for a specific application it may not be useful to order the monomials.

9.5. The Fundamental Theorem in Cheon-Huang-Kim is an immediate consequence of the facts that the elements of the Riordan monoid determine continuous linear endomorphims of a topological vector space over $K$ that has a Schauder basis, and the topology is the inductive limit of the discrete topologies on the spans of finite subsets of the basis. Noting this, one remarks that in our context the fundamental equation 2 can also be proven by factoring our map $M$ of Section 3 through the space of continuous $K$-linear endomorphisms of $\mathcal{F}_d$, and using the graded basis of monomials, without worrying about the order of the monomials of each degree.

9.6. We close with a few remarks on functoriality.

$\text{Hol}_C$ is not, in general, a functor from $C$ to the category of groups. However, if we define $C^{\text{iso}}$ to be the category having the same objects as $C$ but only $C$-invertible arrows $A \to B$ as arrows, then
(1) \( C \mapsto C^{\text{iso}} \) is a functor on the category of categories,
(2) the identity map \( C \to C \) is a functor from \( C^{\text{iso}} \) to \( C \), and
(3) the restriction \( \text{Hol}_{C^{\text{iso}}} \) of \( \text{Hol}_C \) to \( C^{\text{iso}} \) is a functor from \( C^{\text{iso}} \) to the category of groups.

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