Special functions associated with automorphisms of the space of solutions to special double confluent Heun equation

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Abstract

The family of quads of interrelated functions holomorphic on the universal cover of the complex plane without zero (for brevity, $pqr s$-functions), revealing a number of remarkable properties, is introduced. In particular, under certain conditions the transformations of the argument $z$ of $pqr s$-functions represented by lifts of the replacements $z \leftarrow -1/z$, $z \leftarrow -z$, and $z \leftarrow 1/z$ are equivalent to linear transformations with known coefficients. $Pqr s$-functions arise in a natural way in constructing of certain linear operators acting as automorphisms on the space of solutions to the special double confluent Heun equation (sDCHE). Earlier such symmetries were known to exist only in the case of integer value of one of the constant parameters when the predecessors of $pqr s$-functions appear as polynomials. In the present work, leaning on the generalized notion of $pqr s$-functions, discrete symmetries of the space of solutions to sDCHE are extended to the general case, apart from some natural exceptions.

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1 Definitions and basic properties

Let us consider the following system of linear homogeneous first order ODEs

\begin{align*}
    z^2 p' &= (\mu + (\ell - 1)z)p - q + z^2 r, \\
    q' &= (\lambda - (\ell + 1)z)q + \mu q + s, \\
    z^2 r' &= -(\lambda + \mu^2)p + z(2(\ell - 1) - \mu z)r - s, \\
    z^2 s' &= -(\lambda + \mu^2)q + z^2(\lambda - (\ell + 1)\mu z)r + ((\ell - 1)z - \mu)s.
\end{align*}

Here the symbols $\ell$, $\lambda$, $\mu$ denote some complex constants. The symbols $p, q, r, s$ stand for holomorphic functions of the complex variable $z$. For brevity, we shall refer to them as $pqr s$-functions.

When resolved with respect to the derivatives, i.e. upon division of Eqs (1)-(4) by $z^2$, 1, $z^2$, $z^2$, respectively, the coefficients in their right-hand sides become rational functions holomorphic everywhere except for some of them at zero.
Hence all solutions to the above system are holomorphic in some vicinity of any point \( z_0 \neq 0 \).

One may also regard \( pqr \)-functions as solutions of the Cauchy problem for Eq.s (1)-(4) with arbitrary (but not totally null) initial data specified at arbitrary given \( z = z_0 \neq 0 \). Obviously, such local solution can be analytically continued to any other point of \( \mathbb{C} \) except zero. In particular, all solutions to Eq.s (1)-(4) (i.e. \( pqr \)-functions) are single-valued holomorphic functions on any connected and simply connected subset of \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \).

At the same time it has to be noted that, except for the very special conditions, the natural (inextendible) domain of holomorphicity of \( pqr \)-functions is neither \( \mathbb{C}^* \) nor any its subset but rather the universal cover of \( \mathbb{C}^* \). This is the Riemann surface \( \tilde{\mathbb{C}}^* \) diffeomorphic to \( \mathbb{C} \), the covering projection \( \Pi : \tilde{\mathbb{C}}^* \to \mathbb{C}^* \) being realized by the natural exponential function. However, in what follows, we shall consider, unless otherwise specified, only a part of \( \tilde{\mathbb{C}}^* \) (subdomain) denoting it \( \tilde{\mathbb{C}}^* \). It is representable by the result of the removing from \( \mathbb{C}^* \) of the ray \( \mathbb{R}_- \) of negative reals, \( \{\mathbb{C}^* = \mathbb{C}^* \setminus \mathbb{R}_- \} \). When considered on \( \tilde{\mathbb{C}}^* \), any instance of \( pqr \)-functions combines the four single-valued holomorphic functions uniquely defined by their values (which may be arbitrary but all zero) at any given point \( z_0 \in \tilde{\mathbb{C}}^* \). The two their single-side continuations to \( \mathbb{R}_- \) also exist giving rise to real analytic functions in the common domain \( \mathbb{R}_- \). However, as a rule, these one-side limits do not coincide pointwise.

\( pqr \)-functions reveal a number of noteworthy properties. The first of them is expressed by the following statement.

**Theorem 1** Let \( pqr \)-functions obey at \( z = i \) the constraint

\[
q(i) - pp(i) + r(i) = 0. \tag{5}
\]

Then the following equalities

\[
p(-1/z) = -e^{i\pi z^{2(1-\ell)}} p(z), \tag{6}
\]
\[
q(-1/z) = e^{i\pi z^{-2\ell}} \left( \mu p(z) + z^2 r(z) \right), \tag{7}
\]
\[
r(-1/z) = e^{i\pi z^{2(1-\ell)}} \left( \mu z^2 p(z) + q(z) \right), \tag{8}
\]
\[
s(-1/z) = -e^{i\pi z^{-2\ell}} \left( \mu z^2 p(z) + q(z) \right) + z^2 \left( \mu z^2 r(z) + s(z) \right); \tag{9}
\]

hold true. Conversely, Eq. (5) follows from Eq.s (6)-(9) evaluated at \( z = i \).

**Remark 1** Eq. (5) is obviously implied by Eq. (7) alone. The remaining three equations, when evaluated at \( z = i \), either turn out to be fulfilled identically or follow from Eq. (5) (and, thus, from Eq. (7)).

**Remark 2** The constraint (5) does not affect the value of the function \( s \) at the selected point and, moreover, \( s(\cdot) \) is present only in Eq. (9) which might be considered as decoupled from the preceding ones. However, there is an indirect influence of the selection of \( s \) (via the unrestricted setting of \( s(i) \)) to the other \( pqr \)-functions in view of their “unbreakable interrelation” implied by Eq.s (1)-(4).
Remark 3 The involutive transformation which we shall here refer to as the transformation A, signified in the left-hand sides of Eq.s (6)-(9) by the replacement
\[ z \mapsto -1/z \quad (10) \]
of the argument \( z \) of the functions involved, is here tacitly regarded as the map keeping the particular argument \( z = i \) unchanged. This point is worth mentioning because in the case we deal with, i.e. for functions possessing domains distinct of \( \mathbb{C}^* \), “the reflected imaginary unit” \(-i\) is *not* a fixed point of the implied transformation of the arguments albeit \(-1/(-i) = (-i)\), formally. Moreover, there is another transformation (let us denote it \( \tilde{A} \)) or, one might say, another implementation of the rule (10) recognizing just \(-i\), but not \(+i\), as a fixed point in the domain of a (this time) \( \tilde{A} \)-transformed functions. Accordingly, as long as we consider \(+i\) as the fixed point of the transformation signified by the argument replacement (10), there exist connected and simply connected open sets containing \(+i\) and contained in \( z \mathbb{C}^* \) such that their images through the transformation A also contain \(+i\) and are contained in \( z \mathbb{C}^* \). On them, the asserted relations expressed by Eq.s (6)-(9) are well defined. It is here preferable to consider the subdomain \( z \mathbb{C}^* \) only. The extension of Eq.s (6)-(9) to the whole domain of \( pqrs \)-functions (the universal cover of \( \mathbb{C}^* \)) by means of analytic continuation is obviously feasible although in general it might prove to be not representable by the original formulas.

To clarify some specialties of the above interpretation, we consider the following example. Let \( z \) be continuously moving from \( i \in z \mathbb{C}^* \) towards some \( x \in \mathbb{R}_+ \subset z \mathbb{C}^* \) along a concave curve. Then \(-1/z\), also starting from \(+i\) but further differing from \( z \), is moving around zero in the opposite angular direction, arriving ultimately at \(-x^{-1} \in \mathbb{R}_- \) which does not belong to \( z \mathbb{C}^* \). Thus, when dragging \( z \) farther across \( \mathbb{R}_+ \) inward the half-plane \( \Im z < 0 \), the corresponding (A-transformed argument) \(-1/z\) leaves \( z \mathbb{C}^* \) across ‘the upper edge’ of the cut along the ray \( \mathbb{R}_- \). Notice that we may not consider it entering \( z \mathbb{C}^* \) again through the lower cut edge disconnected from the upper one. This means that in the course of the above process the literal applicability of the formulas (6)-(9) breaks down on the ray of positive reals. Thus, to ensure their meaningfulness, one is compelled to obey the restriction \( \Im z > 0 \). At the same time, it is obvious that such a limitation is only a consequence of certain simplification we had adopted for convenience. It would not arise in case of consideration of \( pqrs \)-functions on their full domain. However, then yet another complication related to certain non-uniqueness of interpretation of Eq.s (6)-(9) would appear. In total, we still prefer here to restrict consideration to the subdomain \( z \mathbb{C}^* \) keeping in mind limitations induced by such a simplification.

Eq. (6) singles out some subset of \( pqrs \)-functions constraining their values (i.e. the initial data for Eq.s (11)-(13)) at \( z = i \). Yet another property leans on their parameterizing by the values at \( z = 1 \). It reads as follows.
Theorem 2  Let $\lambda + \mu^2 \neq 0$ and $pqrs$-functions obey the constraints

\[
\begin{align*}
\mu p(1) + q(1) + r(1) & = 0, \quad (11) \\
\lambda p(1) - \mu q(1) + s(1) & = 0. \quad (12)
\end{align*}
\]

Then the following equalities

\[
\begin{align*}
p(1/z) & = -(\lambda + \mu^2)^{-1} z^{2(1-\ell)} (\mu z^2 r(z) + s(z)), \quad (13) \\
q(1/z) & = -(\lambda + \mu^2)^{-1} z^{-2\ell} (\lambda z^2 r(z) - \mu s(z)), \quad (14) \\
r(1/z) & = -z^{2(1-\ell)} (\mu z^2 p(z) + q(z)), \quad (15) \\
s(1/z) & = -z^{-2\ell} (\lambda z^2 p(z) - \mu q(z)); \quad (16)
\end{align*}
\]

hold true. Conversely, Eqs. (11), (12) follow from Eqs. (13)-(16) evaluated at $z = 1$.

Remark 4  For $z = 1$ the replacement of argument of the functions on the left in Eqs. (13)-(16) (we shall refer to it as the transformation $C$) reveals no effect. Accordingly, there exist open sets containing $+1$ which remain invariant under the action of the transformation $C$. Then it is reasonable to consider first the equalities (13)-(16) on such neighborhoods of the unity and then utilize analytic continuation for their extending to greater domains.

Remark 5  Besides $z = 1$, the point $z = -1$ (excluded, by definition, from $\C^*$) is also unaffected by the replacement $z \mapsto 1/z$ utilized in Eqs. (13)-(16), formally. However, it cannot be considered as a fixed point of the transformation $C$. More precisely, claiming of $z = -1$ to be a fixed point, one must replace the transformation $C$ by “yet another implementation” $\check{C}$ of the above argument replacement. For it, the former fixed point $z = 1$ loses such a property. Besides, for $\check{C}$, the associated (sub-)domain of $pqrs$-functions, playing role of $C^*$, has to contain $\R$ but not $\R_+$. Having thus noted the presence of certain ambiguity in the interpretation of Eqs. (13)-(16), we limit ourselves with the above remark and shall not consider here this issue in greater details.

Combining conditions of the two above theorems we obtain one more relationship in accordance with the following.

Theorem 3  Let $pqrs$-function obey the conditions of both Theorem 1 and Theorem 2, i.e. they are holomorphic on a connected and simply connected open set containing $+i$ and $+1$ and meet the constraints (5), (11), and (12). Then the equalities

\[
\begin{align*}
\mathcal{M}^{1/2}[p] & = e^{i\ell\pi} (\lambda + \mu^2)^{-1} (\mu z^2 r + s), \quad (17) \\
\mathcal{M}^{1/2}[q] & = -e^{i\ell\pi} (\mu z^2 p + q + \mu(\lambda + \mu^2)^{-1} z^2 (\mu z^2 r + s)), \quad (18) \\
\mathcal{M}^{1/2}[r] & = -e^{i\ell\pi} r, \quad (19) \\
\mathcal{M}^{1/2}[s] & = e^{i\ell\pi} ((\lambda + \mu^2)p + \mu z^2 r), \quad (20)
\end{align*}
\]
hold true, where the arguments $z$ of all the functions coincide and hence are suppressed, and where the operator $\mathcal{M}^{1/2}$ carries out analytic continuation of the function it acts to along the circular arc started at $z$, centered at zero, subtending an angle $\pi$, and oriented counter-clockwise. Moreover, the products of $pqrs$-functions times the power function $z^{-\ell}$ are single-valued and holomorphic on $\mathbb{C}^*$. 

Remark 6 As opposed to transformations of $pqrs$-functions treated by Theorems 1 and 2, the transformation of arguments of functions on the left in Eq.s (17)-(20) (let us call it the transformation $B$) admits no fixed points and is not involutive. Moreover, applying the transformation $B$ twice, the resulting effect turns into the analytic continuation of the function to be transformed along the loop projected to (essentially, coinciding with) the full circle. Such kind of analytic continuation around a singular point (in our case, the center $z = 0$) is commonly named the monodromy transformation. We denote it by the symbol $\mathcal{M}$. We have therefore $\mathcal{M}^{1/2} \circ \mathcal{M}^{1/2} = \mathcal{M}$ by definition. The effect of the operator $\mathcal{M}^{1/2}$ can thus be named semi-monodromy transformation.

In our case $\mathcal{M}$ is the linear operator which sends, in particular, the values of $pqrs$-functions on the “lower” edge of the cut along the ray $\mathbb{R}_-$ to the (generally speaking, distinct) values they assume on its “upper” edge. Since $pqrs$-functions obey on the both edges the same system (1)-(4) of linear homogeneous ODEs (since their coefficients are invariant with respect to $\mathcal{M}$) such a transformation is represented by a constant $4 \times 4$ matrix.

Remark 7 If $z \in \mathbb{C}^*$ and $\Im z < 0$ then $\mathcal{M}^{1/2}z \left( = \mathcal{M}^{1/2}[\text{Id}](z) \right) = e^{i\pi}z = -z \in \mathbb{C}^*$. However, if $\Im z \geq 0$ then an application of $\mathcal{M}^{1/2}$ would yield the argument of $pqrs$-functions in Eq.s (17)-(20) on the left which does not belong to $\mathbb{C}^*$. Evading such a complication, we shall assume $\Im z < 0$ for simplicity unless otherwise specified. Analytic continuation has to be applied for relaxation of the limitation and extending the local form of the equalities (17)-(20) in which $\mathcal{M}^{1/2}$-transformation is regarded as the inversion of sign of the function argument to a greater domain.

Remark 8 In general case, given a prescribed set of constant parameters, simultaneous fulfillment of Eq. (5) and Eq.s (11), (12) for the same instance of $pqrs$-functions should be achievable by means of their appropriate selection. Indeed, the set of all $pqrs$-functions can be indexed by the quad of their values at $z = 1$ fixed up to multiplication by an insignificant (associated with a decoupled degree of freedom) non-zero common factor, i.e. by points of a projective space $\mathbb{CP}^3$. The two linear equations (11), (12) single out the projective line embedded therein. This projective line is conveyed (pushed forward) by the vector flow associated with the equations (1)-(4) into another projective space $\mathbb{CP}^3$ indexing the same set of $pqrs$-functions by their values (also considered up to a common constant factor) at $z = i$. In the latter projective space, the equation (5) singles out certain embedded projective plane. The question equivalent to the issue of consistency of Eq. (5) with Eq.s (11) and (12) reads:
whether the former (conveyed) projective line intersects the latter projective plane or not? This problem remains open yet but numerical computations point in favor of the affirmative upshot, at least, under apparently generic conditions. Thus, most plausibly, inconsistency of Eq. (8) with Eqs. (11) and (12) and the subsequent inanity of Theorem 3, if any, could only occur under the very special conditions (currently unknown). We may state therefore the following.

**Corollary 4** There exists a set of pairwise linearly independent quads of holomorphic functions $p, q, r, s$ parameterized by points of $\mathbb{CP}^2$ such that the equations (6) - (9) are fulfilled.

**Corollary 5** There exists a set of pairwise linearly independent quads of holomorphic functions $p, q, r, s$ parameterized by points of $\mathbb{CP}^1$ such that the equations (13) - (16) are fulfilled.

**Conjecture 1** For almost all values of the constant parameters there exist a discrete set of quads of functions $p, q, r, s$ holomorphic on the universal cover of $\mathbb{C}^*$ such that the equations (6) - (9), (13) - (16), and (17) - (20) are fulfilled.

The last assertion of the Theorem 3 says how the $pqrs$-functions referred to in the above Conjecture are expressed through functions which are single-valued and holomorphic on $\mathbb{C}^*$.

We proceed now with proofs of the three above theorems.

**Proof of Theorem 4.** Let us denote the four differences of the left- and right-hand sides of Eqs. (6), (7), (8), (9), by the symbols $A_p, A_q, A_r, A_s$, respectively, considering them, as they stand, as the functions of $z$. For example, one of such definitions reads $A_p(p(z)) = p(-1/z) + e^{i\pi \mu} z^{2(1-\ell)} p(z)$, etc. As it is shown in Appendix A they obey the following system of linear homogeneous ODEs

\begin{align}
z^2 \frac{d}{dz} A_p &= z(1 - \ell + \mu z)A_p - z^2 A_q + A_r, \\
\end{align}

\begin{align}
z^2 \frac{d}{dz} A_q &= ((\ell - 1) \mu + \lambda z)A_p + \mu z A_q + z A_s, \\
z^2 \frac{d}{dz} A_r &= - (\lambda + \mu^2) z^2 A_p - (\mu + 2(\ell - 1) z) A_r - z^2 A_s, \\
z^3 \frac{d}{dz} A_s &= - (\lambda + \mu^2) z^3 A_q + ((\ell + 1) \mu + \lambda z) A_r - z^2 (\ell - 1 + \mu z) A_s,
\end{align}

provided Eqs. (1)-(4) are fulfilled.

Using the explicit definitions, let us compute the particular values of $A_p(i) = A_p(e^{i\pi})$ for $\Phi \in \{p, q, r, s\}$. Notice that for such a choice of the argument $z$ one has $-1/z = -e^{-i\pi} = i$, $z^{-2\ell} = e^{-i\pi\mu} z^{2(1-\ell)} = -e^{-i\pi}$. Then it follows from Eq. (6) that $A_p(pi) = 0$. The values $A_p(i)$ of the other differences are not automatically zero but one easily finds that in accordance with definitions

$$A_p(i) = \zeta_p \cdot (q(i) - \mu p(i) + r(i))$$

for $\Phi \in \{q, r, s\}$, and $\zeta_q = \zeta_r = 1, \zeta_s = \mu$. 

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Thus if Eq. 5 is fulfilled then \( A_{\Phi}(i) = 0 \) for all ‘the indices’ \( \Phi \in \{p, q, r, s\} \)
This implies the vanishing everywhere of all the functions \( A_{\Phi}(z) \) in view of
uniqueness of solutions of the Cauchy problem for Eq.s (21) with the null initial
data posed at \( z = 1 \).

\[ \text{Proof of Theorem 2.} \]
Building on the notations utilized in the preceding proof, we denote the differences of the left- and right-hand sides of Eq.s (13), (14), (15), (16) by the symbols \( C_{\Delta_p}(z), C_{\Delta_q}(z), C_{\Delta_r}(z), C_{\Delta_s}(z) \), respectively. It is shown in Appendix B that they obey the following system of linear homogeneous ODEs

\[
\begin{align*}
\frac{d^2}{dz^2} C_{\Delta_p} &= - z(\ell + 1 + 1)C_{\Delta_p} + \ell^2 C_{\Delta_q} - C_{\Delta_r}, \\
\frac{d^3}{dz^3} C_{\Delta_q} &= ((\ell + 1)\mu - \lambda z)C_{\Delta_p} - \lambda z C_{\Delta_q} - z C_{\Delta_s}, \\
\frac{d^2}{dz^2} C_{\Delta_r} &= (\lambda + \mu^2)z^2 C_{\Delta_p} + (\mu + 2(1 - \ell)z)C_{\Delta_r} + z^2 C_{\Delta_s}, \\
\frac{d^2}{dz^2} C_{\Delta_s} &= (\lambda + \mu^2)z^3 C_{\Delta_q} + ((\ell + 1)\mu - \lambda z)C_{\Delta_r} + z^2 (1 - \ell + \mu z)C_{\Delta_s},
\end{align*}
\]

provided Eq.s (11)–(14) are fulfilled.

We compute now the values the differences \( C_{\Delta_\Phi}(1) \) \( \Phi \in \{p, q, r, s\} \) acquire after
the plugging \( z = 1 \) in their definitions. The result is as follows:

\[
(\lambda + \mu^2) C_{\Delta_\Phi}(1) = \zeta_\Phi \cdot (s(1) - \mu q(1) + \lambda p(1)) + \sigma_\Phi \cdot (r(1) + q(1) + \mu p(1)),
\]

where \( \zeta_p = 1, \zeta_q = -\mu, \zeta_r = 0, \zeta_s = \lambda + \mu^2; \sigma_p = \mu, \sigma_q = \lambda, \sigma_r = \lambda + \mu^2, \sigma_s = 0. \)

Thus if the constraints (11) and (12) are fulfilled then all the differences \( C_{\Delta_\Phi}(z) \)
vanish at \( z = 1 \). But then they are the identically zero functions, \( C_{\Delta_\Phi}(z) \equiv 0 \), as
a consequence of Eq.s (22). This means exactly that Eq.s (13)–(16) hold true. ☐

\[ \text{Proof of Theorem 3.} \]
As above, let us denote the differences of the left- and right-hand sides of Eq.s (17), (18), (19), (20) by the symbols \( B_{\Delta_p}(z), B_{\Delta_q}(z), B_{\Delta_r}(z), B_{\Delta_s}(z) \), respectively. It is shown in Appendix C that in case of fulfillment of
Eq.s (11)–(14) they obey the following system of linear homogeneous ODEs

\[
\begin{align*}
\frac{d^2}{dz^2} B_{\Delta_p} &= ((\ell - 1)z - \mu)B_{\Delta_p} + B_{\Delta_q} - z^2 B_{\Delta_r}, \\
\frac{d}{dz} B_{\Delta_q} &= - (\lambda + (\ell + 1)z)B_{\Delta_p} - \mu B_{\Delta_q} - B_{\Delta_s}, \\
\frac{d^2}{dz^2} B_{\Delta_r} &= (\lambda + \mu^2)B_{\Delta_p} + (2(\ell - 1) + \mu z)z B_{\Delta_r} + B_{\Delta_s}, \\
\frac{d^2}{dz^2} B_{\Delta_s} &= (\lambda + \mu^2)B_{\Delta_q} - (\lambda + (\ell + 1) \mu z)z^2 B_{\Delta_r} + (\mu + (\ell - 1)z)B_{\Delta_s}.
\end{align*}
\]

The next step should assume computation of the particular values \( B_{\Delta_\Phi}(-1) \),
\( \Phi \in \{p, q, r, s\} \). However, carrying out this by means of the mere substitutions
of the differences of overlapping of sheets of the branching domain $pqrs$-functions live on. To make the computation univocal, we consider first the “deformed” versions $B\Delta_\phi$ of the differences $B\Delta_\phi$, where $\epsilon$ plays role of the deformation parameter. Their distinction is that in case of $B\Delta_\phi$ the factor in argument of the $pqrs$-function on the left is distinct of the one involved in Eqs. (17) - (20), see Remark [7].

The common exponential multiplier on the right is also modified. Namely, let the factor $e^{i\pi \gamma}$, where $\epsilon \in [0, 1]$ be used instead of $-1 = e^{i\pi}$. For example, one has $B\Delta_\phi(z) = r(e^{i\pi z}) + e^{i\pi \gamma}r(z)$ while $B\Delta_\phi(z) = r(e^{i\pi z}) + e^{i\pi \gamma}r(z)$, etc. Explicit definitions of all $B\Delta_\phi$ are furnished by Eqs. (24).

Now let us notice that in the case $\epsilon = 0$ the functions $pqrs$-functions utilized for computation of $B\Delta_\phi$ coincide with $z$ and no ambiguity in their evaluation can thus arise. Then, starting from these values, we carry out analytic continuation varying $\epsilon$ through the segment $[0, 1]$. We define $B\Delta_\phi(z)$ to be “the final values” the functions $B\Delta_\phi(z)$ arrive at as $\epsilon \to 1$. Such an interpretation leaves no room for ambiguity in the meaning of definitions of $B\Delta_\phi$ and, more generally, the relations Eqs. (17) - (20) represent.

Assuming the above interpretation of $B\Delta_\phi$, it is shown in Appendix [D] that the following equations are fulfilled for arbitrary functions $p, q, r, s$ holomorphic on the circular arc passing through the point $-i, +1$, and $+i$.

\begin{align*}
B\Delta_p(-i) &= e^{i\pi}(\lambda + \mu^2)^{-1}(\mu C\Delta_r(i) - C\Delta_s(i)), \\
B\Delta_q(-i) &= (q(i) - \mu p(i) + r(i)) \\
&\quad + e^{i\pi}(C\Delta_q(i) - \mu C\Delta_p(i) + \mu (\lambda + \mu^2)^{-1}(\mu C\Delta_r(i) - C\Delta_s(i))), \\
B\Delta_r(-i) &= (q(i) - \mu p(i) + r(i)) + e^{i\pi}C\Delta_r(i), \\
B\Delta_s(-i) &= \mu(q(i) - \mu p(i) + r(i)) - e^{i\pi}(\lambda + \mu^2)C\Delta_p(i) + \mu e^{i\pi}C\Delta_r(i).
\end{align*}

The symbols $C\Delta_\phi, \phi \in \{p, q, r, s\}$, were already utilized in the proof of Theorem [2]. They denote the differences of the left- and right-hand sides of Eqs. (13) - (16), considered, as they stand, as the functions of $z$. Every equation from the system (24) can therefore be regarded as the coincidence, upon simplifications, of a pair of certain linear combinations of 4+4 instances of $pqrs$-functions of which some are evaluated at $z = i$ and others at $z = -i$.

On the other hand, the conditions of the theorem to be proven imply, in particular, the fulfillment of the assertion of Theorem [2] which establishes the vanishing of all the four functions $C\Delta_\phi(z)$ irrespectively of the choice of their arguments. Thus all the terms in Eqs. (24) involving those factors may be discarded.

Now, taking into account the fulfillment of Eq. (5), we see that all the expressions on the left in (24), i.e. the functions $B\Delta_\phi(z), \phi \in \{p, q, r, s\}$, evaluated at $z = -i$, actually vanish. Since these functions obey the system of linear homogeneous first order ODEs (see Eqs. (23)) they reduce to identical zero. This means exactly that the equalities (17) - (20) hold true.
Let us consider the second claim of the theorem which establishes, under the restrictions assumed, a simpler domain $\mathbb{C}^*$ for the products of $pqrs$-functions times $z^{-\ell}$ as compared to $pqrs$-functions themselves which are not single-valued on $\mathbb{C}^*$ and hence must be considered on its universal cover. We note first that the four-element vector consisting of the right-hand sides of Eqs (17–20) can be obtained by means of the multiplication of the vector $(p(z), q(z), r(z), s(z))^\top$ by the matrix $M^{1/2}_\ell(z) = e^{i\ell\pi} M^{1/2}(z)$, where

$$M^{1/2}(z) = \begin{pmatrix} 0 & 0 & \mu z^2 (\lambda + \mu^2)^{-1} & (\lambda + \mu^2)^{-1} \\ -\mu z^2 & -1 & -\mu^2 z^4 (\lambda + \mu^2)^{-1} & -\mu z^2 (\lambda + \mu^2)^{-1} \\ 0 & 0 & -1 & 0 \\ \lambda + \mu^2 & 0 & \mu^2 z & 0 \end{pmatrix}.$$ 

Thus under the conditions assumed the action of the operator $M^{1/2}$ to $pqrs$-functions is completely described by the matrix $M^{1/2}_\ell$. Let us examine the effect of this action applied twice. In the language of matrices it is described by the product of the ones associated with the mentioned transformation. However whereas the first operator $M^{1/2}$ of the composition defined as, in a sense, a rotation of the function argument, is associated with $M^{1/2}_\ell(z)$ the second one ‘starts’ with the arguments already rotated acting separately to the matrix $M^{1/2}(z)$ and to the vector of $pqrs$-functions. In other words, the operator composition $M^{1/2} \circ M^{1/2}$ has to be associated with the matrix product $M^{1/2}[M^{1/2}_\ell](z) \cdot M^{1/2}(z) = e^{2i\ell\pi} M^{1/2}[M^{1/2}](z) \cdot M^{1/2}(z)$. We can compute it making use of the following identity

$$M^{1/2}(e^{i\ell\pi} z) \cdot M^{1/2}(z) \equiv I + \frac{(1 - e^{2i\ell\pi}) \mu z^2}{\lambda + \mu^2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (1 - e^{2i\ell\pi}) \mu z^2 \\ 0 & 0 & \lambda + \mu^2 & 0 \\ 0 & 0 & 0 & \lambda + \mu^2 \end{pmatrix},$$

where $I$ is the unit matrix and $\epsilon \in [0, 1]$ is the real parameter. The analytic continuation (“the rotation of the argument”) carried out by the operator $M^{1/2}$ can be represented by the evaluation of the limit as $\epsilon \searrow 1$. Then the factor in parenthesis in front of the second summand on the right goes to zero and we obtain $M^{1/2}[M^{1/2}](z) \cdot M^{1/2}(z) = I$. We see therefore that under the conditions of the theorem the monodromy transformation $M = M^{1/2} \circ M^{1/2}$ of $pqrs$-functions reduces to their multiplication by the constant $e^{2i\ell\pi}$. Accordingly, the products of $pqrs$-functions times $z^{-\ell}$ reveal the trivial (identical) monodromy transformations. Thus they can be continuously extended in both directions across the cut along $\mathbb{R}_{\ell\in\mathbb{N}}$ distinguishing $\mathbb{C}^*$ from $\mathbb{C}^*$.

Since they also obey a system of the first order ODEs (which can easily be derived from Eqs (1–4)) no branching appear showing that they are actually single-valued holomorphic in $\mathbb{C}^*$ itself. The theorem is proved.

$\square$
2 The first integral

It proves sometimes to be useful to take into account the following noteworthy property of all solutions to Eqs. (1)-(4).

Theorem 6 Let $\lambda + \mu^2 \neq 0$. Then the following statements hold true.

0. If holomorphic functions $p, q, r, s$ obey Eqs. (1)-(4) then the value of expression

$$D = z^{2(1-\ell)}(p(z)s(z) - q(z)r(z))$$

does not depend on $z$;

1. if holomorphic functions $p, q, r, s$ obey Eqs. (6)-(9) then

$$Y = \left[ D \right]_{z=1/z},$$

where and in what follows $[D]$ denotes the right-hand side of Eq. (25) considered as a function of $z$;

2. if holomorphic functions $p, q, r, s$ obey Eqs. (13)-(16) then

$$Y = \left[ D \right]_{z=M^{1/2}z},$$

3. if holomorphic functions $p, q, r, s$ obey Eqs. (17)-(20) then

$$Y = \left[ D \right]_{z=M_1^{1/2}z}.$$

It has to be added that the precise meaning of the argument replacements $z \equiv -1/z$, $z \equiv 1/z$, and $z \equiv M^{1/2}z (\approx -z)$ involved in the above formulas is the same as in the corresponding systems of the equations claimed to be fulfilled.

Proof. We shall consider the above assertions one by one.

Assertion 0. Let us expand the expression of the derivative of the right-hand side of Eq. (25) in case of arbitrary holomorphic functions $p, q, r, s$. A straightforward computation establishes the following identity

$$z^{2\ell} \frac{d}{dz} [D] = s \Delta_p - z^2 r \Delta_q - q \Delta_r + p \Delta_s.$$ (29)

Here the symbols $\Delta_\varnothing$, where $\varnothing \in \{p, q, r, s\}$, denote the differences of the left- and right-hand sides of Eqs. (1), (2), (3), (4), respectively, as they stand. Hence if the latter equations are fulfilled then the derivative (29) vanishes and the value of $[D]$ does not depend on $z$.

Assertion 1. Its validity follows from the equality

$$\left[ D \right]_{z=1/z} = e^{-i\pi \ell} \left[ D \right]_{z=M^{1/2}z} + e^{-i\pi \ell} z^{2(1-\ell)} \left[ D \right]_{z=M_1^{1/2}z},$$ (30)
holding true for arbitrary functions $p, q, r, s$, holomorphic at (and in the vicinity of) $z = i$. Here the symbols $\Delta_\mathcal{O}, \mathcal{O} \in \{p, q, r, s\}$, denote the differences of the left- and right-hand sides of Eq.s (6), (7), (8), (9), respectively, considered, as they stand, as the functions of $z$.

Eq. (30) follows, in turn, from the identity (64) given in Appendix E.

Thus if Eq.s (6)-(9) are fulfilled then the equality (26) holds true.

Assertion 2. Let us consider the following identity

$$
\begin{align*}
|\mathcal{D}| & = |\mathcal{D}| + z^{2(\ell-1)}(s(1/z)^{\Delta_p}(z) - r(1/z)^{\Delta_q}(z)) \\
& + (\lambda + \mu^2)^{-1}((\lambda r(z) - \mu z^{-2}s(z))^{\Delta_p}(z) \\
& - (\mu z^2r(z) + s(z))^{\Delta_q}(z))
\end{align*}
$$

(31)

which is verifiable by straightforward computation. Here $^{\Delta_p}(z), \mathcal{O} \in \{p, q, r, s\}$, denote the differences of the left- and right-hand sides of the equations (13), (14), (15), (16), respectively, considered, as they stand, as the functions of $z$. The equality (31) holds true for arbitrary functions $p, q, r, s$ holomorphic at (and in the vicinity of) $z = 1$. It is extended to any other $z \neq 0$ by means of analytic continuation.

In view of (31), it is obvious that if Eq.s (13)-(16) are fulfilled then Eq. (27) holds true.

Turning to the assertion 3, let us consider the equation

$$
\begin{align*}
|\mathcal{D}| & = |\mathcal{D}| + e^{-i\ell \pi} z^{2(1-\ell)}(e^{-i\ell \pi} (\mathcal{S}^{\Delta_p} - \mathcal{S}^{\Delta_q}) + (\mu z^2 s + g) \mathcal{B}\Delta_r \\
& + (\lambda + \mu^2)^{-1}(\mu z^2 r + s)(\mu z^2 \mathcal{B}\Delta_r + \mathcal{B}\Delta_s))
\end{align*}
$$

(32)

Here the symbols $^{\Delta_p}$, where $\mathcal{O} \in \{p, q, r, s\}$, denote the differences of the left- and right-hand sides of the equations (17), (18), (19), (20), respectively. ‘The diacritic mark’ $^{\mathcal{S}}$ denotes the transformation of the function argument defined as follows: $^{\mathcal{S}}(z) = \lim_{\epsilon \to 0} \mathcal{O}(e^{i\ell \pi} z)$. Here $\lim_{\epsilon \to 0}$ should be understood as the analytic continuation along the image of the segment $[0, 1] \ni \epsilon$ to the end point corresponding to $\epsilon = 1$. In Theorem 3 such a transformation is associated with the operator $\mathcal{M}^{1/2}$.

Eq. (32) follows from the identity (65) given in Appendix E. In turn, under conditions of the theorem, Eq. (28) is the obvious consequence of Eq. (32).

The constant $\mathcal{O}$ (in fact, the first integral for the system (11)-(14)) may vanish. Indeed, if it is null at some point (this is a quadratic constraint to values of $pqrs$-functions thereat) then it is zero everywhere. Such a case bears many signs of a degeneracy — being nevertheless in no way meaningless. Following here the requirement of genericity, we assume throughout that $\mathcal{O} \neq 0$ without separate mentioning. It is also worth noting that there is another case for which many of the relations discussed here degenerate. Namely, this takes place if $\lambda + \mu^2 = 0$. We evade here clarification of its specialties as well.
3 On applications of \(pqrs\)-functions

The properties of \(pqrs\)-functions established above make them an object of notable interest in itself. However, they arose originally in the context of another important problem, namely, the study of symmetries of the space of solutions to the following ordinary second-order linear homogeneous differential equation

\[
z^2 E'' + ((\ell + 1)z + \mu(1 - z^2)) E' + (- (\ell + 1)\mu z + \lambda) E = 0. \tag{33}
\]

Here \(E = E(z)\) is the unknown holomorphic function, \(\ell, \lambda, \mu\) are the constant parameters which may be identified with the ones involved in Eqs. (1)-(4).

Eq. (33) belongs to the family of double confluent Heun equations (DCHE). They are discussed in Refs [1, 2]. Refs [3, 4] contain some more recent bibliography. Since a generic DCHE is characterized by the four constant parameters, whereas Eq. (33) involves only three ones, Eq. (33) was named a special double confluent Heun equation (sDCHE). This naming is adopted in the present work as well.

It should be noted that Eq. (33) was segregated within the DCHE family because of its intimate relation (in fact, equivalence) to the following nonlinear first-order ODE

\[
\dot{\varphi} + \sin \varphi = B + A \cos \omega t,
\]
in which \(\varphi = \varphi(t)\) is the unknown function, the symbols \(A, B, \omega\) denote real constants, \(t\) is a free real variable, and the overdot denotes deriving with respect to \(t\). The latter equation and its generalizations are, in turn, well known due to their emerging in a number of problems in physics (most notably in the modeling of Josephson junctions) [5, 6, 7], mechanics [8, 9], dynamical systems theory [10], and geometry [11].

In earlier investigations the functions, obeying equations equivalent to Eqs (1)-(4), were utilized for the constructing of linear operators sending the space of solutions to Eq. (33) into itself [12]. It was found that the transformations they determine generate a group which can be regarded as a discrete symmetry of the noted space of solutions. (More precisely, in case of real parameters, one of the three groups arises depending on their values).

The principal limitation of those considerations was, however, the restriction of the parameter \(\ell\) (sometimes called the order of Eq. (33)) to integers only. The simplification following from this assumption (the starting point of derivation of the mentioned symmetry transformations, in fact) is the reducing of the functions equivalent to our \(pqrs\)-functions to polynomials in \(z\) as well as in the parameters \(\lambda, \mu\). Moreover, there exists the recurrent scheme enabling one to compute these polynomials for any given positive integer \(\ell\).

The definition of \(pqrs\)-functions considered in the present work needs no such a restriction that enables us to make a crucial step in revealing of discrete

\[\text{1}\]The basic points of such a relationship are discussed in Ref. [13].

\[\text{2}\]The positive integer order \(\ell \in \mathbb{Z}_+\) determines their degrees linearly increasing with its incrementing. As to the case of negative \(\ell \in \mathbb{Z}_-\), there is a trick enabling one to convert it to the case of the positive order equal to \(|\ell|\).
symmetries of the noted space of solutions in case on non-integer ℓ. We apply the approach closely following the one utilized in the case of integer order although some specific subtleties still have to be taken into account.

To that end, let us define the two families of linear operators, ̂L_A and ̂L_B, depending on the real parameter ε ∈ [−1, 1]. They act to arbitrary functions (denoted E) holomorphic in \( \mathbb{C}^* \) in accordance with the following formulas.

\[
\hat{L}_A[E](z) = e^{\mu(z+1/z)} \left[ z^2 p(z) E'(z) + q(z) E(z) \right],
\]

\[
\hat{L}_B[E](z) = z^{1-\ell} e^{\mu(z+1/z)} \left[ z^2 r(z) E'(z) + s(z) E(z) \right].
\]

The functions p, q, r, s are assumed to be holomorphic in the same domain.

If ε = 0 then the common argument of the functions p, q, E and r, s, E in right-hand sides of (34) and (35) coincide with 1/z and z, respectively. Then the values of ̂L_A[E] and ̂L_B[E] are correctly defined everywhere in \( \mathbb{C}^* \).

We introduce now the particular instances L_A and L_B of the operators ̂L_A and ̂L_B as the results of the analytic continuations, starting from ̂L_A and ̂L_B, along the images of the segment [0, 1] ∈ ε through the corresponding maps \( z \mapsto e^{i\epsilon}/z \) and \( z \mapsto e^{i\epsilon}z \). We may write down these relationships as follows.

\[
L_A = \lim_{\epsilon \to 1} \hat{L}_A, \quad L_B = \lim_{\epsilon \to 1} \hat{L}_B.
\]

**Theorem 7** Let pqrs-functions verify Eq.s (11)-(14) and E verify Eq. (33). Then the functions L_A[E] and L_B[E] also verify Eq. (33).

Proof. Let us introduce the operator \( \mathcal{H} \) associated with Eq. (33), i.e. let

\[
\mathcal{H}[E](z) = z^2 E''(z) + ((\ell + 1)z + \mu(1 - z^2)) E'(z) + (\lambda - \mu(\ell + 1)z) E(z) .
\]

Composing it with the operator L_A, the following expansion of the slightly modified result of their combined action to an arbitrary holomorphic function E can be obtained

\[
\begin{align*}
\left[ e^{-\mu(z+1/z)}(\mathcal{H} \circ L_A)[E] \right] & = z^2 p \mathcal{H}[E] \\
& \quad + (2\mu p - q + 2z^2 r) \mathcal{H}[E] \\
& \quad + z^2 E' \Delta_p + z^2 E' \Delta_q \\
& \quad + ((\lambda - (\ell + 1) \mu)z E + 2z^2 E'' + (\mu(2 - z^2) + 2z) E') \Delta_p \\
& \quad \quad + ((\mu(1 - z^2) - z(\ell - 1 - \mu)z) E + z^2 E'') \Delta_q \\
& \quad \quad + z^2 E' \Delta_r + E \Delta_s,
\end{align*}
\]

provided the functions E, p, q, r, s of the variable z are holomorphic at (and in the vicinity of) z = i. Here the symbols \( \Delta_\Phi \), where \( \Phi \in \{p, q, r, s\} \), denote
the differences of the left- and right-hand sides of Eq.s (1)-(4) considered as the functions of $z$ (they were already used in the proof of Theorem 6), $H' = d/dz \circ H$.

In case of the operator $L_B$ similar expansion looks as follows.

$$z^{\ell-1}e^{-\mu(z+1/z)}(H \circ L_B)[E] = z^2 E^r H'[E] + \left(2(\ell - 1)z E^r + s^r + 2z^2 E^s\right)H'[E] + z^2 E^r \Delta^r_i + E^s \Delta^s_i - (\lambda + \mu^2)(E^r \Delta_p + E^s \Delta_q) - ((\mu - (\ell - 1)z)E^r + (\lambda + (\ell + 1)\mu z)E^s) \Delta_r + (E^r - \mu E^s) \Delta_q.$$  \hfill (39)

The symbols $\Delta^r_i$ have the same meaning as in Eq. \ref{eq:38}. ‘The diacritic mark’ $\hat{\circ}$ denoting the semi-monodromy transformation was also used in the proof of Theorem 6. It is worth reminding that $\hat{\circ} = \hat{\circ}(z) = \lim_{n \to 1} \circ (e^{i\pi z})$ for any holomorphic function $\circ$. If the functions $E, p, q, r, s$ are holomorphic on $\mathbb{C}$ and $\Re z < 0$ then the argument of evaluation of ‘semi-monodromy-transformed’ functions also belongs to $\mathbb{C}$ and all the constituents of Eq. \ref{eq:39} are well defined. Other values of $z$ are to be handled by means of the analytic continuation.

The equalities \ref{eq:38}, \ref{eq:39} follow from the identities \ref{eq:66} and \ref{eq:67}, respectively, given in Appendix F. In turn, the theorem’s assertion follows from Eq.s \ref{eq:38} and \ref{eq:39} since the fulfillment of Eq.s (1)-(4) implies $\Delta^r_i = 0$ and the identical vanishing of $H[E]$ is equivalent to fulfillment of Eq. \ref{eq:33} that had also been assumed.

\textbf{Remark 9} The transformations realized by the operators $L_A$ and $L_B$ carry out the (lifted) replacements $z \mapsto -1/z$ and $z \mapsto -z$ of arguments of the functions involved. There exists the third operator which we denote $L_C$ also sending any solution to Eq. \ref{eq:33} to some its solution and utilizing the missed replacement $z \mapsto 1/z$ of arguments expressing the composition of the preceding ones and constituting in conjunction with them the Klein group of maps naturally acting on $\mathbb{C}$. $L_C$ is not linked to $pqrs$-functions and is well defined for any choice of constant parameters. It can be represented by the following formula.

$$L_C[E](z) = z^{\ell-1} \left[E'(z) - \mu E(z)\right].$$  \hfill (40)

In view of the nontrivial structure of the domain of solutions to Eq. \ref{eq:33} “the implementation” of the rule \ref{eq:40} is not unique. In particular, for one of them (the lift of) $+1$ is the fixed point of the map indicated by the argument replacement $z \mapsto 1/z$ whereas for the other one it is (the lift of) $-1$ which plays a similar role.

The transformations of the space of solutions to Eq. \ref{eq:33} associated with $pqrs$-functions possess the properties of quasi-involutions similar to ones found earlier in the case of integer order $\ell$, cf Ref. \cite{14}, Eq.s (34), (35).
Theorem 8 Let $\lambda + \mu^2 \neq 0$, the function $E$ obey Eq. (33), and $pqrs$-functions obey Eq.s (1)-(4). If, additionally,

1. Eq.s (16)-(19) hold true then

$$ (L_A \circ L_A)[E] = -e^{i\pi} \mathcal{D} \cdot E; $$

(41)

2. Eq.s (17)-(20) hold true then

$$ (L_B \circ L_B)[E] = -(\lambda + \mu^2)e^{2i\pi} \mathcal{D} \cdot \mathcal{M}[E]. $$

(42)

Proof. The above claims follow from the equalities

$$(L_A \circ L_A)[E](z) + e^{i\pi} \mathcal{D} E(z) = \left((s(z) + \mu z^{-2}q(z))E(z) + (z^2 p(z) + \mu p(z)E(z))^{\Delta q}(z) + (q(z)E(z) + z^2 p(z)E'(z))^{\Delta q}(z) + \mathcal{M}(z)[E](z) \right)$$

$$ e^{i\pi} z^{2(\ell - 1)}((L_B \circ L_B)[E] + \lambda + \mu^2 e^{2i\pi} \mathcal{D} \cdot \mathcal{E}) =$$

$$ = -(\lambda + \mu^2) \tau \cdot \left(z^2 E' \mathcal{B}^{\Delta_p} + E \mathcal{B}^{\Delta_q}\right)$$

$$ = (\mu z^2 \mathcal{B}^{\Delta_q} + \mathcal{B}^{\Delta_p})$$

(43)

involving arbitrary functions $E, p, q, r, s$ and their derivatives. These are, in turn, the consequences of the identities (68) and (69), given in Appendix C. Concerning the notations utilized therein, let us remind that $[\mathcal{D}]$ denote the right-hand side of Eq. (25) considered as a function of $z$. The symbols $\mathcal{A}^{\Delta_p}$ and $\mathcal{B}^{\Delta_q}$, where $\mathcal{O} \in \{p, q, r, s\}$, denote the differences of the left- and right-hand sides for Eq.s (11)-(14), for Eq.s (16)-(19), and for Eq.s (17)-(20), respectively. They are also considered as the functions of $z$.

There are also the two kinds of ‘diacritic marks’ in use. Of them, ‘the accent’ $\accent'$ indicates the transformation of the function argument carrying out its continuous anti-clockwise rotation in the complex plane at an angle $\pi$. It was earlier named the semi-monodromy map. In Theorem 3 such a transformation is associated with the operator $\mathcal{M}^{1/2}$. Evidently, if $\Im z < 0$ then $\mathcal{O}(-z)$. However, if $\Im z \geq 0$ then the semi-monodromy transformation sends such argument out the subdomain $\mathcal{C}^*$ and this can not be expressed by the inversion of the sign. It worth noting here that $\mathcal{M}(z^{-1})$ (see the last but one line in Eq. (43)) is well defined, provided $\Im z > 0$. Indeed, then $\Im z^{-1} < 0$ and the argument of evaluation of the function $p$ when computing $\mathcal{M}(z^{-1}) = \lim_{z^{-1} \to 1} p(e^{i\pi} z^{-1})$ belongs to $\mathcal{C}^*$.

The second ‘accent’ $\accent$ has a similar meaning but “the rotation angle” of a function argument is here twice as much amounting to $2\pi$. Such a transformation looks like a full revolution in $\mathcal{C}^*$ around zero but it does not lead to the
identical map in view of non-trivial structure (distinction of complex plane or any subset of the complex plane) of the domains of the functions we consider. Rather it corresponds to the monodromy transformation.

For some reasons we had agreed above to consider pqr s-functions on their subdomain \( \mathbb{C}^* = \mathbb{C}^* \setminus \mathbb{R}_- \). Here, however, this is not enough and we are forced to introduce for a time a somehow extended one. Indeed, if \( z \in \mathbb{C}^* \) then the point of evaluation of a monodromy-transformed function does not belong to \( \mathbb{C}^* \) due to the cut along the ray of negative reals which the circular path of analytical continuation inevitably meets. “The minimally extended subdomain”, where the monodromy map can still be consistently defined, is constructed, for instance, by means of addition of another copy of \( \mathbb{C}^* \) and the gluing of it to the original one along the opposite edges of their cuts (the two complementary ones remain free). Then if \( z \) belongs to the “lower” (original) sheet of this “double \( \mathbb{C}^* \)” then the point of evaluation of analytic continuation of the function to be monodromy transformed belongs to the upper one and in this way all the constituents of Eq. (44) can be consistently computed (and it is finally fulfilled).

The important circumstance is, however, that under conditions of the theorem the evaluation of many functions and the handling of the associated subtleties it implies is superfluous. Indeed, the fulfillment of certain equations required by the theorem conditions means the vanishing of the expressions \( \mathcal{H}[E], \Delta \Delta, \Delta, \Delta \), and \( \Delta \Delta \) which yield zero independently of the point of their evaluation. They constitute a full collection of the factors in the right-hand sides of Eqs. (43) and (44) such that if all they are zero then all the terms on the right vanish. The only function with transformed argument which ‘survives’ is the monodromy transformed function \( E \) involved in the left-hand side of Eq. (44).

Now, as the right-hand sides of Eqs. (43), (44) are zero, the vanishing of their left-hand sides leads just to Eqs. (41) and (42). □

**Corollary 9** If the conditions of Theorem 8 are fulfilled then the operators \( L_A, L_B \) it concerns determine automorphisms of the space of solutions to Eq. (33).

Indeed, due to (41) and (42) they have no zero eigenvalues and thus their kernels are trivial.

**4 Summary**

We define a family of quads of holomorphic functions (referred to, for brevity, as pqr s-functions) as the non-trivial solutions to the system of linear homogeneous first order ODEs (1)-(4). Each instance of such functions can be constructed as a solution of the Cauchy problem for the initial data specified at any given
point \( z_0 \) except zero. It is shown that, fixing the initial data at \( z_0 = i \) and claiming fulfillment of the linear homogeneous constraint (5), one obtains \( pqrs \)-functions which obey the equalities (6)-(9) (Theorem 1). Similarly, if the initial data are specified at \( z_0 = 1 \) and obey thereat the two linear homogeneous constraints (11), (12) then \( pqrs \)-functions obey the equalities (13)-(16) (Theorem 2). Lastly, if all the three mentioned linear constraints (imposed at two distinct locations) are met then the equalities (17)-(20) involving semi-monodromy map take place as well (Theorem 3). This case is most important since for it the monodromy transformation can also be easily computed. It turns out coinciding with multiplication to a known numerical factor showing that \( pqrs \)-functions are the products of certain power function and functions holomorphic on \( \mathbb{C}^* \) (instead of the universal cover of \( \mathbb{C}^* \)).

\( pqrs \)-functions had found application (if fact, arose) in frameworks of investigation of properties of solutions to special double confluent Heun equation (33). Under conditions here assumed the operators \( L_A, L_B \) defined by the formulas (36), (34), (35) turn out to define the maps of the space of its solutions into itself (Theorem 7). Moreover, they possess quite remarkable composition properties. It particular, the operator \( L_A \) is “almost involutive” (see Theorem 8, Eq. (41)) while \( L_B \), being applied twice, reduces, up to a known constant factor, to the monodromy transformation (see Eq. (42)). Besides, they define automorphisms of the space of solutions to Eq. (33) (Corollary 9).

In the special case of integer values of the constant parameter \( \ell \) the functions almost identical to our \( pqrs \)-functions were originally introduced in Ref. [12]. The distinction of functions with the same notations considered therein against the present ones reduces to different normalizations of the functions \( p \) and \( q \). It is worth mentioning that the variant of \( pqrs \)-functions considered in [12] deals exclusively with polynomials. Moreover, they are polynomial not only in \( z \) but also in the parameters \( \lambda \) and \( \mu \) (while \( \ell \) determines the polynomial degrees). Thus we may claim that in the case of a (positive) integer \( \ell \) Eqs. (1)-(4) admit a polynomial solution.

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A. Identities leading to Eq.s (21)

The equations (21) the proof of Theorem 1 leans on are the straightforward consequences of the four identities displayed below which are, in principle, verifiable by explicit computations. Namely, the following equalities hold true

\[
\begin{aligned}
z^2 \frac{d}{dz} A_{\lambda p}(z) &= -z(\ell + 1 + e^{-i\pi} \mu z) A_{\lambda p}(z) \\
&+ e^{-i\pi} z^2 A_{\Delta q}(z) - e^{i\pi} A_{\Delta r}(z) \\
&- e^{-i\pi} z^2 \Delta_p(e^{i\pi}/z) + e^{i\pi} z^{2(1-\ell)} \Delta_p(z) \\
&(1 + e^{-i\pi}) e^{i\pi} z^{2(1-\ell)} \times \\
&\left( \mu(1 + e^{i\pi})z^2 p(z) - e^{i\pi} q(z) + \mu z^2 r(z) \right), \quad (45)
\end{aligned}
\]

\[
\begin{aligned}
z^3 \frac{d}{dz} A_{\Delta q}(z) &= e^{-i\pi}(\ell + 1) z^2 A_{\lambda p}(z) \\
&- e^{-i\pi} \Delta_q(e^{i\pi}/z) - e^{i\pi} z^{1-2\ell}(\mu \Delta_p(z) + z^2 \Delta_r(z)) \\
&+ (1 + e^{-i\pi}) e^{i\pi} z^{1-2\ell} \times \\
&\left( ((\lambda + \mu^2) z^2 + (1 - e^{i\pi})(\ell + 1) \mu z - \mu^2) p(z) \right) \quad (46)
\end{aligned}
\]

\[
\begin{aligned}
z^2 \frac{d}{dz} A_{\Delta r}(z) &= e^{-i\pi}(\lambda + \mu^2) z^2 A_{\lambda p}(z) + e^{-i\pi} z^2 A_{\Delta q}(z) \\
&+ (e^{i\pi} \mu - 2(\ell - 1) z) A_{\Delta s}(z) \\
&- e^{-i\pi} z^2 \Delta_r(e^{i\pi}/z) - e^{i\pi} z^{2(1-\ell)}(\mu \Delta_p(z) + \Delta_q(z)) \\
&- (1 + e^{-i\pi}) e^{i\pi} z^{2(1-\ell)} \times \\
&\left( (\lambda + (2 - e^{i\pi}) \mu^2) z^2 p(z) + (1 - e^{i\pi}) \mu q(z) \right) \quad (47)
\end{aligned}
\]

\[
\begin{aligned}
z^3 \frac{d}{dz} A_{\Delta s}(z) &= e^{-i\pi}(\lambda + \mu^2) z^3 A_{\Delta q}(z) \\
&- e^{i\pi}(\lambda z - e^{i\pi}(\ell + 1) \mu) A_{\Delta r}(z) \\
&- z^2(\ell + 1 - e^{-i\pi} \mu z) A_{\Lambda s}(z) \\
&- e^{-i\pi} z^3 \Delta_s(e^{i\pi}/z) \\
&+ e^{i\pi} z^{3-2\ell}(\mu^2 \Delta_p(z) + \mu \Delta_q(z) + \mu z^2 \Delta_r(z) + \Delta_s(z)) \\
&+ (1 + e^{-i\pi}) e^{i\pi} z^{2(1-\ell)} \times \\
&\left( \mu z(\lambda + \mu^2 - (e^{i\pi}(\lambda + \mu^2) z^2 \right) \\
&+ \mu e^{i\pi}(e^{i\pi} - 1)(\ell + 1) z) p(z) \right) \quad (48)
\end{aligned}
\]

As a matter of fact, they were handled with help of the computer algebra. Similar remarks concern the majority of formulas in the present paper or, at least, all more or less lengthy ones.
\[-e^{i\pi\ell + (\ell + 1)\mu + \lambda z})q(z) + z^3((\lambda + \mu^2(1 - z^2))r(z) - \mu s(z))\].

The symbols \(\Delta_\Phi(z)\), where \(\Phi \in \{p, q, r, s\}\), stand for the differences of the left- and right-hand sides of the equations (1), (2), (3), (4), respectively. The symbol \(\epsilon\) denotes the real parameter, \(\epsilon \in [-1, 1]\). The definitions of the functions \(A\Delta_\Phi(z)\) read

\[
\begin{align*}
A\Delta_p(z) &= p(e^{i\pi\ell + (\ell + 1)\mu + \lambda z}) + e^{i\ell\pi z}2(1-\ell)p(z), \\
A\Delta_q(z) &= q(e^{i\pi\ell + (\ell + 1)\mu + \lambda z}) - e^{i\ell\pi z} - 2\ell p(z) + z^2 r(z)), \\
A\Delta_r(z) &= r(e^{i\pi\ell + (\ell + 1)\mu + \lambda z}) - e^{i\ell\pi z} - 2\ell p(z) + q(z)), \\
A\Delta_s(z) &= s(e^{i\pi\ell + (\ell + 1)\mu + \lambda z}) + e^{i\ell\pi z} - 2\ell p(z) + q(z)) + z^2 (\mu z^2 p(z) + q(z)) + z^2 (\mu z^2 r(z) + s(z)).
\end{align*}
\]

Hence, as a matter of fact, the equalities (45)-(48) signify the four pairwise coincidences, upon simplification, of certain expressions constructed in two different ways from arbitrary holomorphic functions \(p, q, r, s\) and their first order derivatives.

Let us notice now that \(\lim_{\epsilon \to 1} A\Delta_\Phi(z)\) coincide with the functions \(A\Delta_\Phi(z)\) introduced in the beginning of the proof of Theorem 1 and involved in Eqs. (21). It is worth reminding that they were defined as the differences of the left- and right-hand sides of the equations (4)-(9). They are correctly defined if the functions \(p, q, r, s\) are holomorphic in the vicinity of \(z = i\). Besides, for \(\epsilon = 1\), the last summands in the right-hand sides of the equalities (45)-(48), which are proportional to either \((1 + e^{i\pi\ell})\) or \((1 + e^{-i\pi\ell})\), vanish. Finally, it remains to note that if the functions \(p, q, r, s\) obey Eqs. (1)-(4) then the differences \(\Delta_\Phi(z)\) become identically zero for all \(\Phi \in \{p, q, r, s\}\) and all the summands which contain them can also be dropped out. After such simplifications, comparing the resulting form of Eqs. (45)-(48) with Eqs. (21), one easily finds that they coincide. Thus the equalities (21) hold true.

B Identities leading to Eqs. (22)

Eqs. (22) follow from the identities given below which can be, in principle, verified by straightforward computations. Namely, for any functions \(p, q, r, s\) holomorphic, at least, in the vicinity of \(z = 1\) the following identities hold true

\[
\begin{align*}
\frac{d}{dz} C\Delta_p(z) &= -z(\ell - 1 + \mu z) C\Delta_p(z) + z^2 C\Delta_q(z) - C\Delta_r(z) \\
&\quad + z^2 \Delta_p(1/z) - (\lambda + \mu^2)^{-1}z^2(1-\ell)(\lambda^2 \Delta_r(z) + \Delta_s(z)),
\end{align*}
\]

\[
\begin{align*}
\frac{d}{dz} C\Delta_q(z) &= ((\ell + 1)\mu - \lambda z) C\Delta_p(z) - \mu z C\Delta_q(z) - z C\Delta_s(z) \\
&\quad + z \Delta_q(1/z) - (\lambda + \mu^2)^{-1}z^{1-2\ell}(\lambda z^2 \Delta_r(z) - \mu \Delta_s(z)).
\end{align*}
\]
the differences $\Delta$ identities displayed below which are verifiable by straightforward computations.

Eq. (23), utilized in the proof of Theorem 3, can be obtained from the four identities leading to Eq.s which are therefore the direct consequence of Eq.s (1)-(4).

If the above four functions $p, q, r, s$ their first order derivatives. Obviously, these expressions are correctly defined here the symbols $C(16)$. The symbols $\Delta$ signify the pairwise coincidences, upon simplification, of certain expressions constructed in two different ways from arbitrary holomorphic functions $p, q, r, s$ and their first order derivatives. Obviously, these expressions are correctly defined if the above four functions $p, q, r, s$ are holomorphic in the vicinity of $z = 1$.

Finally, if the functions $p, q, r, s$ are not arbitrary but verify Eq.s (1)-(4) then the differences $\Delta \Phi(z)$ vanish and the identities (50)-(53) convert to Eq.s (22) which are therefore the direct consequence of Eq.s (11)-(14).

C Identities leading to Eq.s (23)

Eq.s (23), utilized in the proof of Theorem 3, can be obtained from the four identities displayed below which are verifiable by straightforward computations. Namely, it can be shown that

$$z^{3} \frac{d}{dz} B_{p}(z) = -(\mu + (\ell - 1)z) B_{\hat{p}}(z) - e^{-ie\pi} B_{\hat{q}}(z) - z^{2} B_{\hat{r}}(z)$$

$$+e^{-ie\pi} B_{\hat{p}}(e^{ie\pi}z) - e^{ie\pi} (\lambda + \mu^{2})^{-1}(\mu z^{2}\Delta_{r}(z) + \Delta_{s}(z)) + (1 + e^{-ie\pi}) (\mu p(e^{ie\pi}z) + e^{ie\pi} z e^{2r(e^{ie\pi}z)} + e^{ie\pi} (q(z) + \mu z^{2} p(z) + \mu (\lambda + \mu^{2})^{-1} z^{2}(s(z) + \mu z^{2} r(z))),$$

$$d \frac{d}{dz} B_{q}(z) = -(\lambda + (\ell + 1)z) B_{\hat{p}}(z) - \mu B_{\hat{q}}(z) - \mu B_{\hat{s}}(z)$$

$$+e^{ie\pi} \Delta_{q}(e^{ie\pi}z) + e^{ie\pi} (\mu \Delta_{p}(z) + \Delta_{q}(z)) + e^{ie\pi} (\lambda + \mu^{2})^{-1}(\mu z^{2}\Delta_{r}(z) + \Delta_{s}(z)) + (1 + e^{ie\pi}) (\lambda (1 - e^{ie\pi}) (\ell + 1)z p(e^{ie\pi}z) + \mu q(e^{ie\pi}z) + s(e^{ie\pi}z)),$$

$$z^{2} \frac{d}{dz} B_{r}(z) = (\lambda + \mu^{2}) B_{\hat{p}}(z) + (2(\ell - 1) + \mu z) z B_{\hat{r}}(z)$$

$$+ \mu \Delta_{q}(e^{ie\pi}z) + e^{ie\pi} \Delta_{r}(z)$$

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\[(1 + e^{-i\pi})(\lambda + \mu^2)p(e^{i\pi}z) + e^{i\pi}\mu z^2 r(e^{i\pi}z) + s(e^{i\pi}z),\]

\[\frac{d}{dz}b^{\Delta_o}_s(z) = (\lambda + \mu^2) b^{\Delta_p}_q(z) - (\lambda + (\ell + 1)\mu z) z^2 b^{\Delta_r}_s(z)
+ (\mu + (\ell - 1)z) b^{\Delta_s}_p(z)
+ e^{-i\pi}\Delta_s(e^{i\pi}z) - e^{i\pi}\Delta_p(z) + \mu z^2 \Delta_r(z)
-(1 + e^{-i\pi})(\lambda + \mu^2)q(e^{i\pi}z) + \mu s(e^{i\pi}z)
- e^{i\pi}z^2(\lambda + (1 - e^{i\pi})(\ell + 1)\mu z)r(e^{i\pi}z).\]  

Here \(\epsilon \in [0, 1]\) is the auxiliary real parameter, the symbols \(p, q, r, s\) stay for arbitrary functions holomorphic in the vicinity of an arc of the circle connecting \(e^{-i\pi/2}\) with \(e^{i\pi/2}\) and passing inbetween them through +1 counter-clockwise. The functions \(b^{\Delta_o}_s(z)\), where \(o \in \{p, q, r, s\}\), are defined as follows.

\[b^{\Delta_p}_p(z) = p(e^{i\pi}z) - e^{i\pi}\lambda + \mu^2(1)(\mu z^2 r(z) + s(z)),\]
\[b^{\Delta_q}_q(z) = q(e^{i\pi}z) + e^{i\pi}(\mu z^2 p(z) + q(z))
+ \mu(\lambda + \mu^2)^{-1}z^2(\mu z^2 r(z) + s(z)),\]
\[b^{\Delta_r}_r(z) = r(e^{i\pi}z) + e^{i\pi}\mu z^2 r(z),\]
\[b^{\Delta_s}_s(z) = s(e^{i\pi}z) - e^{i\pi}(\lambda + \mu^2)p(z) + \mu z^2 r(z).\]

Lastly, the symbols \(\Delta_o(z)\), where \(o \in \{p, q, r, s\}\), stand for the differences of the left- and right-hand sides of the equations (1), (24), (3), (4), respectively. They were also used in Eqs. (15)-(18), (50)-(53).

Thus the equalities (54)-(57) signify the pairwise coincidences, upon simplification, of certain expressions constructed in two different ways from arbitrary holomorphic functions \(p, q, r, s\) and their first order derivatives.

Let us now consider the case \(\epsilon = 1\). The definitions (58) are pertinent if the domain of functions \(p, q, r, s\) covers \(\pm i\) and 1, i.e., in particular, if they are holomorphic on the circular arc passing through \(-i, 1\) and \(+i\). The solutions of the Cauchy problem for Eqs. (11)-(14) with initial data specified at \(z = 1\) possess such a property. One has for them, by definition, \(\Delta_o(z) = 0\). Besides, due to the above choice of \(\epsilon\), the summands in right-hand sides of Eqs. (17)-(20) involving either the factor \((1 + e^{i\pi})\) or the factor \((1 + e^{-i\pi})\) have to be discarded as well.

Taking the above simplifications into account, Eqs. (23) follow since the argument for which the transformed \(pqrs\)-functions on the left in Eqs. (17)-(20) have to be evaluated is exactly the limit of \(e^{i\pi}z\) reached as \(\epsilon \to 1\) (provided \(z\) belongs to the vicinity of \(-i\), at least).

**D Identities leading to Eqs. (24)**

Let us consider “the deformed differences” \(b^{\Delta_o}_o(z)\), where \(o \in \{p, q, r, s\}\), and \(\epsilon \in [-1, 1]\) is the real parameter, defined by the formulas (58). If \(\epsilon \to 1\) then they
reduce to the differences \( B_\Delta \delta(z) \) of the left- and right-hand sides of Eq.s \((17)-(20)\). We consider their values in the case \( z = -i = e^{-i\pi/2} \). The following four double equalities hold true for arbitrary functions \( p, q, r, s \) holomorphic on the circular arc passing through \(-i, +1, +1\).

\[
B_\Delta p(e^{-i\pi/2}) = p(e^{i\pi/2}) - e^{i\pi}(\lambda + \mu^2)^{-1}\left(\left\lfloor -1 \right\rfloor \mu r(e^{-i\pi/2}) + s(e^{-i\pi/2}) \right) \\
\equiv -\left\lfloor 0 \right\rfloor p(e^{i\pi/2}) - e^{i\pi}(\lambda + \mu^2)^{-1}\left(\left\lfloor -1 \right\rfloor \mu C_\Delta p(e^{i\pi/2}) + C_\Delta s(e^{i\pi/2}) \right),
\]

\[
B_\Delta q(e^{-i\pi/2}) = q(e^{i\pi/2}) + e^{i\pi}\left(\left\lfloor -1 \right\rfloor \mu p(e^{-i\pi/2}) + q(e^{-i\pi/2}) \right) \\
\equiv -\left\lfloor 0 \right\rfloor \left(\mu p(e^{i\pi/2}) - q(e^{i\pi/2}) - r(e^{i\pi/2}) \right)
\]

\[
B_\Delta r(e^{-i\pi/2}) = r(e^{i\pi/2}) + e^{i\pi}\left(\left\lfloor -1 \right\rfloor \mu p(e^{-i\pi/2}) + q(e^{-i\pi/2}) \right) \\
\equiv -\left\lfloor 0 \right\rfloor \left(\mu p(e^{i\pi/2}) - q(e^{i\pi/2}) - r(e^{i\pi/2}) \right)
\]

\[
B_\Delta s(e^{-i\pi/2}) = s(e^{i\pi/2}) - e^{i\pi}\left(\left\lfloor -1 \right\rfloor \mu p(e^{-i\pi/2}) + q(e^{-i\pi/2}) \right) \\
\equiv -\left\lfloor 0 \right\rfloor \left(\mu p(e^{i\pi/2}) - q(e^{i\pi/2}) - r(e^{i\pi/2}) \right)
\]

Here we use the following auxiliary abbreviations:

\[
[-1] = e^{i\pi}, \quad [\left\lfloor -1 \right\rfloor] = e^{-i\pi}, \quad [0] = 1 + e^{i\pi}, \quad [2] = 1 - e^{i\pi}.
\]

The expressions \( C_\Delta \delta \), where \( \delta \in \{p, q, r, s\} \), were introduced in the proof of Theorem 2. They denote the differences of the left- and right-hand sides of Eq.s \((13)-(16)\).

In each of the above four pairs of equalities the first ones are merely the expansions of the corresponding definitions \((58)\) with regard to the particular value of \( z \) picked out above. On the contrary, the second equalities are “the genuine identities” in which the right-hand sides represent some rearrangements of the left-hand ones whose several constituents are aggregated to the expressions \( C_\Delta \delta \). Thus Eq.s \((59)-(62)\) express the coincidences, upon simplification, of some linear combinations of arbitrary fixed functions \( p, q, r, s \) evaluated at \( z = e^{i\pi/2} \) and at \( z = e^{-i\pi/2} \).

If \( \epsilon = 0 \) then the argument of all the \( pqrs \)-functions and the expressions \( B_\Delta \delta \) considered as the functions of \( z \) is +1. Let \( \epsilon \) be further varied through the
particular, ‘the accent’ \(\hat{\circ}\) move along the circular arcs, either clockwise of counter-clockwise. The values the functions assume thereat can be regarded as the result of their analytic continuation from the vicinity of +1. At end points of the noted arcs corresponding to \(\epsilon = 1\) the arguments of the functions become either \(e^{i\pi/2} = i\) or \(e^{-i\pi/2} = -i\) while the expressions \(\hat{\Delta}_\circ\) on the left turn into \(\hat{\Delta}_\circ = \lim_{\epsilon \to 1} \hat{\Delta}_\circ\) evaluated at \(-i\). Besides, it holds \([-1] = \{[-1]\} = -1\), \([0] = 0\), \([2]\) = 2 thereat.

Taking all these simplifications into account, one finds that in the particular case under consideration the equalities of the first and the last expressions in each of the formulas Eq.s (64)-(65) combine to Eq.s (24).

**E Identities utilized in the proof of Theorem 6**

The following two identities verifiable by straightforward computation hold true for arbitrary holomorphic functions \(p, q, r, s\).

\[
e^{-2i\pi} \left[ D \right] = \left[ D \right] + e^{-i\epsilon \pi} \frac{z^{2(1-\ell)}}{s} (1/z) ^{\hat{\Delta}_q} p(z) - r(1/z) ^{\hat{\Delta}_q} p(z)
- (r(z) + \mu z^{-2} p(z)) ^{\hat{\Delta}_r} z - p(z) ^{\hat{\Delta}_r} p(z),
\]

(64)

\[
e^{-2i\pi} \left[ D \right] = \left[ D \right] + e^{-i\epsilon \pi} \frac{z^{2(1-\ell)}}{s} \left( \hat{\Delta}_q p - \hat{\Delta}_q q \right) + \left( \mu z^{2} p + q \right) \hat{\Delta}_r
+ (\lambda + \mu^2)^{-1} \left( \mu z^{2} r + s \right) \left( \mu z^{2} \hat{\Delta}_r + \hat{\Delta}_s \right).
\]

(65)

Here \(\epsilon \in [-1, 1]\) is the auxiliary real parameter. \([D]\) denotes the right-hand side of Eq. (24). Eq.s (64) play role of definitions of the symbols \(\hat{\Delta}_\circ\), where \(\circ \in \{p, q, r, s\}\). Similarly, Eq.s (65) explain the meaning of the symbols \(\hat{\Delta}_\circ\). The renderings of all these abbreviations, as they stand, are considered as the functions of \(z\). Here we also employ in recording some tricks allowing somewhat more compact presentation of formulas than in the preceding Appendices. In particular, ‘the accent’ \(\hat{\circ}\) denotes the transformation of rotation of the function argument at an angle \(\epsilon \pi\), i.e. \(\hat{\circ}(z) = \circ(e^{i\pi} z)\). Note that the arguments of functions are displayed in Eq. (64) (except for \([D]\)) but they are suppressed in Eq. (65) because in the latter case the arguments of all the functions coincide and are equal to \(z\).

To guarantee the meaningfulness of the formulas (64) and (65) one has to ensure the belonging of the values of arguments, for which the functions involved in them are evaluated, to the appropriate domain. These values depend on \(\epsilon\). In particular, if \(\epsilon = 0\) then all the functions are evaluated at either \(z\) or \(1/z\). In such a case one may get any \(z \in \mathbb{C}^*\) for which both formulas (64), (65) prove to be correctly defined — and the equalities they represent hold true. Further, starting with \(\epsilon = 0\), we carry out analytic continuations of all the constituents of Eq.s (64) and (65) varying \(\epsilon \in [0, 1]\) from 0 to 1. In the limit \(\epsilon \to 1\) (i.e. at the end point of the arc of analytic continuation) the expressions denoted \(\hat{\Delta}_\circ\) and \(\hat{\Delta}_\circ\) become identical to the expressions \(\Delta_\circ\) and \(\Delta_\circ\), respectively.

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(see the discussion following Eq. (30) and Eq. (32)), while the transformation indicated by ‘the accent’ ﾀ converts to the semi-monodromy transformation denoted earlier by ‘the accent’ の (indicating application of the operator $\mathcal{M}^{1/2}$).

Inspecting the result of the outlined analytic continuation along the image of the segment $[0,1] \in \epsilon$, one finds that this is nothing else but the equations (31) and (32), provided that $z$ belongs to the vicinity of $+i$ in the former case and $3z < 0$ in the latter one.

F  Identities utilized in the proof of Theorem 7

The following identity, which is verifiable by straightforward computation, holds true for arbitrary holomorphic functions $E, p, q, r, s$.

$$
\left. e^{-\mu(z+1/z)}(\mathcal{H} \circ \hat{L}_A)[E] \right|_{z=e^{i\pi/2}} = z^2p\mathcal{H}'[E] + [\mu([0]2) - 1 + z^2] + [1\mu + 2z)]E')\Delta_p
+ \text{[other terms]} + (1 - [\pm1]z^2)(z^2pE' + qE).
$$

Here $\epsilon \in [-1, 1]$ is the auxiliary real parameter. The abbreviations $[-1], [[-1]], [0], [2]$ are to be expanded in accordance with formulas (68). The operator $\mathcal{H}$ is defined by Eq. (37) (see also Eq. (38)), $\mathcal{H}' = d/dz \circ \mathcal{H}$, the operator $\hat{L}_A$ is defined by Eq. (41). The symbols $\Delta_\bullet(z)$, where $\bullet \in \{p, q, r, s\}$, denote the differences of the left- and right-hand sides of Eqs. (11), (12), (13), (14), respectively, considered as the functions of $z$.

Similar identity describing this time the composition of $\mathcal{H}$ with the operator $\hat{L}_B$ (see Eq. (55)) reads

$$
z^{-1}e^{-\mu(z+1/z)}(\mathcal{H} \circ \hat{L}_B)[E] = z^2\mathcal{H}'[E] + (2(\ell - 1)z\hat{\mathcal{E}}') \mathcal{H}'[E]
+ z^2\hat{\mathcal{E}}'\hat{\Delta}_h + \hat{\mathcal{E}}'\hat{\Delta}_p - (\lambda + \mu^2)(\hat{\mathcal{E}}'\hat{\Delta}_h + \hat{\mathcal{E}}'\hat{\Delta}_p)
+ ((\mu - (\ell - 1)z)\hat{\mathcal{E}}' + (\ell + 1)\mu)\hat{\mathcal{E}}'\hat{\Delta}_p
+ (\hat{\mathcal{E}}' - \mu\hat{\mathcal{E}}')\hat{\Delta}_q
+ [0(W_0\hat{\mathcal{E}} + W_1\hat{\mathcal{E}}' - W_2\hat{\mathcal{E}}'' - W_3\hat{\mathcal{E}}''')].
$$

(67)
where the following abbreviations are employed

\[
W_0 = \left(\ell + 1\right)\left(\lambda + \mu^2\right)\mu z^\gamma
+ z\left(-2\left(\ell - 2\right)\lambda + \left(\ell + 1\right)\mu z\left(4 - 3[0] + |\lambda|\right)\right)^\gamma
+ 2\mu z^\gamma,
\]

\[
W_1 = -\left(\ell - 1\right)\left(\lambda + \mu^2\right)z^\gamma
+ z\left(-2\mu\left(\ell - [3]\right) + \left(\ell - 3\right)\left(4 - 3[0]\right) - |\lambda|z\right)
+ |\lambda|\left|\ell + 1\right|\mu z^2\right)^\gamma
- \left(\ell - 1\right)z + \mu\left(1 - |\lambda|z^2\right)^\gamma
- z^2\left(|\lambda|\left(6 + |\lambda|\right)z + \mu\left(2|\lambda|\right)\right)^\gamma
- z^2\left(|\lambda|\left(6 + |\lambda|\right)z + \mu\left(2|\lambda|\right)\right)^\gamma
- |\lambda|\left|\ell + 1\right|\mu z^2\right)^\gamma.
\]

\[
W_2 = 2|\ell - 1|^2 z^4\gamma
+ z^2\left(|\lambda|\left(10 + |\lambda|\right)\right)^\gamma
+ \mu\left(2|\lambda|\right)^\gamma
- \left(\ell - 1\right)z + |\lambda|\left|\ell + 1\right|\mu z^2\right)^\gamma.
\]

\[
W_3 = |\lambda|\left|\ell + 1\right|^2 z^4\gamma
.\]

The meaning of the \(\Delta\)-symbols and the abbreviations \([-1], [0], [2]\), was explained above. ‘The diacritic mark’ \(\overset{\gamma}{\cdot}\), used also in Eq. \((65)\), denotes the transformation carrying out the rotation of the function argument at an angle \(\epsilon\pi\).

If \(\epsilon = 0\) then the arguments of all the functions involved in Eq.s \((66)\) and \((67)\) are either \(z\) or (somewhere in \((66)\) ) \(1/z\). Accordingly, for any \(z \in \mathbb{C}^\ast\) all the constituents of the both formulas \((66)\), \((67)\) are well defined — and the equalities they signify hold true for arbitrary functions \(E, p, q, r, s\) holomorphic in \(\mathbb{C}^\ast\). Further, we allow \(\epsilon\) to vary through the segment \([0, 1]\) and carry out analytic continuation along the corresponding curves (in fact, the circular arcs) of all the constituents of the formulas \((66), (67)\). Fixing the result of this analytic continuation at the end points corresponding to \(\epsilon = 1\), we obtain the two equalities in which some noted abbreviations acquire the known numerical values as follows: \([-1] = [-1] = -1, [0] = 0, [2] = 2\). Then it is easy to see that we obtain in this way the equations (in fact, identities) \((38)\) and \((39)\).

To prevent egress of the points of evaluation of our functions from \(\mathbb{C}^\ast\), it is enough to pick \(z\) from the vicinity of \(+i\) in the case of Eq. \((38)\) and from the half-plane \(\Im z < 0\) in the case of Eq. \((39)\). The extending to greater domains can be carried out by means of analytic continuation.

**G  Identities utilized in the proof of Theorem 8**

The identities given below represent the appropriately adapted expansions of the iterated linear operators \(L_A\) and \(L_B\) defined by Eq.s \((34), (35)\). Specifically, let \(E, p, q, r, s\) denote arbitrary holomorphic functions and \(\epsilon \in [-1, 1]\) be the real parameter. Then it can be shown by means of straightforward computations...
show that, at first,

\[
(\hat{\mathcal{L}}_A \circ \hat{\mathcal{L}}_A)\{E\}(z) + e^{i\ell\pi} \{\mathcal{D}\} E(z) = \\
(s(z) + \mu z^{-2} q(z))E(z) + (z^2 r(z) + \mu p(z))E'(z)) + \hat{\Delta}_p(z) \\
+ (q(z) + \mu z^{-2} p(z)E''(z)) + \hat{\Delta}_q(z) \\
+ \hat{p}(1/z)(p(z)\mathcal{H}\{E\}(z) + E'(z) \Delta_p(z) + E(z) \Delta_q(z)) \\
+ [0] \hat{p}(1/z)W_1 + (e^{\mu[0]z} + [0]/z - 1)W_2, 
\]

where

\[
W_1 = ([\mathcal{L}]_{1/2} \mu z^{-2} q(z) + q'(z))E(z) + z^2 p(z)E''(z) \\
+ (q(z) + ([\mathcal{L}]_{1/2} + 2z)p(z) + z^2 p'(z))E'(z), 
\]

\[
W_2 = -\hat{p}(1/z)\left(\left(\mu z^{-2}(z^2 - \lvert-1\rvert^2)q(z) + \lvert-1\rvert q'(z)\right)E(z) \\
+ \left((\mu z^{-2} - \lvert-1\rvert^2) + 2\lvert-1\rvert z\right)p(z) \\
+ \lvert-1\rvert q(z) + \lvert-1\rvert z^2 p'(z)\right)E'(z) \\
+ \lvert-1\rvert z^2 p(z)E''(z) + \hat{q}(1/z)(q(z)E(z) + z^2 p(z)E'(z)) \right), 
\]

and, at second,

\[
e^{i\ell\pi} z^{2(\ell - 1)} \left(\left(\hat{L}_B \circ \hat{L}_B\right)\{E\} + e^{2i\ell\pi}(\lambda + \mu)\{\hat{D}\} E(z) \right) = \\
[-1]^{\chi} \hat{z}^2 \hat{E} \cdot \left(\left(\lambda + \mu\right) \hat{z}^2 \hat{E} + \mu z^2 \hat{E} \right) \\
- \hat{E} \cdot \left(\left(\lambda + \mu\right) \hat{E} + \mu z^2 \hat{E} \right) \\
+ [-1]^{\chi} \hat{z}^2 \hat{E} \cdot \left[\mathcal{H}\{E\} + \lvert-1\rvert \hat{E} \cdot \hat{E} \Delta_s \right] \\
- [0] \hat{W}_1 + (e^{\mu[0]z} - 1)\lvert-1\rvert \hat{W}_2. 
\]

where

\[
W_1 = (\mu(1 - \hat{z}^2) \hat{r} - \hat{z} \hat{r})\left([-1]^{\chi} \hat{z}^2 \hat{E} + \hat{z} \hat{E} \right), 
\]

\[
W_2 = \left(\hat{r} \hat{z}^2 \hat{s} + \hat{z} \hat{r} \cdot \left((-\mu(1 - \lvert-1\rvert^2)z^2) + \lvert-1\rvert(\ell - 1)z \hat{z} \hat{E} + \lvert-1\rvert \hat{z}^2 \hat{E} \right) \hat{E} \\
+ [\hat{z}^2 \lvert-1\rvert^3 \left(\hat{z} \hat{r} \hat{E} + \hat{r} \hat{E} \right) \\
+ [-1] \hat{r} \cdot \left((\hat{z}^2 + \lvert-1\rvert^4 z^2 \hat{r}^') \hat{E} \\
+ \left(\hat{z}^2 \hat{E} \right) \cdot \left((-\mu(1 - \lvert-1\rvert^2)z^2) + \lvert-1\rvert(\ell - 3)z \hat{E} + \hat{r} \hat{E} \right) \right). 
\]

Here we employ, in particular, the following abbreviations

\[
[-1] = e^{i\epsilon\pi}, \quad [0] = 1 + e^{i\epsilon\pi}, \quad [0] = 1 - e^{-i\epsilon\pi}, \quad [2] = 1 - e^{i\epsilon\pi}. 
\]

Further abbreviations used for convenience are as follows: \{\mathcal{D}\} stands for the right-hand side of Eq. (26) considered as a function of \(z\). The operator \(\mathcal{H}\) is
defined by Eq. (37). The symbols \( \Delta \), where \( \bullet \in \{ p, q, r, s \} \), denote the differences of the left- and right-hand sides of Eqs. (11, 2, 3, 4), respectively, which are considered as the functions of \( z \). Similarly, the symbols \( \Delta^{\bullet} \) stand for the corresponding ‘\( \epsilon \)-deformed’ differences \( \Delta^{\bullet} \) of the left- and right-hand sides of Eqs. (13)-(19) which are defined by Eqs. (49). 'The diacritic mark' \( \hat{\Delta} \) indicates the transformation carrying out the rotation of the function argument at an angle \( \epsilon \pi \), i.e. \( \hat{\Delta}(z) = \Delta(e^{i\epsilon \pi} z) \). Similar ‘accent’ \( \hat{\Delta} \) carries the concordant meaning: as compared to \( \hat{\Delta} \), the rotation angle for it amounts to \( 2\epsilon \pi \).

Some abuse of notations is related to usage of the symbols \( \hat{\Delta}_{\bullet} \). Similarly to the symbols \( \hat{\Delta}_{\bullet} \), they refer to the ‘\( \epsilon \)-deformed’ differences of the left- and right-hand sides of (this time) Eqs. (17)-(20), which are defined by Eqs. (58). However, now additionally “the \( \hat{\Delta} \)-rotation” of the argument of the \( \epsilon \)-deformation result considered as a function of \( z \) has to be carried out afterwards. Thus in this case one deals, in a sense, with the “\( \epsilon \)-deformed and \( \hat{\Delta} \)-rotated” differences of the left- and right-hand sides of Eqs. (17)-(20).

If \( \epsilon = 0 \) then all the instances of the functions \( E, p, q, r, s \) and their derivatives involved in Eqs. (68), (69) are evaluated either at \( z \) or at \( 1/z \). As a consequence, all the constituents of the these formulas are well defined for arbitrary \( z \in \mathbb{C}^{\ast} \) — and the equalities they represent hold true.

Next, we allow the parameter \( \epsilon \) to vary through the segment \([0, 1]\) and carry out analytic continuation along the corresponding curves (in fact, circular arcs) in the function domains. At their end points corresponding to \( \epsilon = 1 \) the coefficients represented by the abbreviations \([-1], [-1], [0], [2]\) acquire the values \(-1, -1, 0, 2\), respectively, see Eqs. (70). This allows us, in particular, to ignore the last lines in the both formulas (68) and (69). Simultaneously, the effect of the rotation of the function arguments tagged by ‘the accent’ \( \hat{\Delta} \) turns into the action of the semi-monodromy operator \( M^{1/2} \) (see Theorem 3) which we indicate also by ‘the accent’ \( \hat{\Delta} \) over the function symbol, see, e.g., the proof of Theorem 8. As to the exceptional symbols \( \hat{\Delta}_{\bullet} \), \( \bullet \in \{ p, q, r, s \} \), it is easy to see that at the end point of the curve of analytic continuation they become equal to the expressions denoted in Eq. (11) by the symbols \( \hat{\Delta}_{\bullet} \).

A separate note on the effect of ‘the accent’ \( \hat{\Delta} \) is necessary. In the limit as \( \epsilon \rightarrow 1 \) it also ‘rotates’ the argument of the function to be transformed but now the rotation angle amounts to \( 2\pi \) meaning, in a sense, a full revolution. It had been noticed that such transformations are termed monodromy. We denoted the operator carrying out the monodromy transformation by the symbol \( \mathcal{M} \) but in some formulas (e.g. in Eq. (44)) it is also indicated by ‘the diacritic mark’ \( \hat{\Delta} \).

It has also to be noted that in case of monodromy transformation some precaution on structure of the domain of the function to which it acts needs to be taken. This point is briefly discussed in the proof of Theorem 8.

Now, collecting all the modifications of the formulas Eqs. (68) and (69), arising when the analytic continuation corresponding to \( \lim_{\epsilon \rightarrow 1} \) has been carried out, one finds that they finally convert to Eqs. (43) and (44), respectively.
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