Network Creation Games with Disconnected Equilibria

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Abstract

In this paper we extend a popular non-cooperative network creation game (NCG) [11] to allow for disconnected equilibrium networks. There are \( n \) players, each is a vertex in a graph, and a strategy is a subset of players to build edges to. For each edge a player must pay a cost \( \alpha \), and the individual cost for a player represents a trade-off between edge costs and shortest path lengths to all other players. We extend the model to a penalized game (PCG), for which we reduce the penalty counted towards the individual cost for a pair of disconnected players to a finite value \( \beta \). Our analysis concentrates on existence, structure, and cost of disconnected Nash and strong equilibria. Although the PCG is not a potential game, pure Nash equilibria always and pure strong equilibria very often exist. We provide tight conditions under which disconnected Nash (strong) equilibria can evolve. Components of these equilibria must be Nash (strong) equilibria of a smaller NCG. However, in contrast to the NCG, for almost all parameter values no tree is a stable component. Finally, we present a detailed characterization of the price of anarchy that reveals cases in which the price of anarchy is \( \Theta(n) \) and thus several orders of magnitude larger than in the NCG. Perhaps surprisingly, the strong price of anarchy increases to at most 4. This indicates that global communication and coordination can be extremely valuable to overcome socially inferior topologies in distributed selfish network design.

1 Introduction

Networks are ubiquitous in modern society. It is therefore not surprising that the study of network creation has attracted much research interest from various disciplines. In recent years, it has been understood that the distributed formation of networks may be subject to economic considerations. In particular, the creation of social, economic, and computational networks was formulated as a game with selfish agents. A general framework for such an approach was proposed by Jackson and Wolinsky [14]. In their games there are \( n \) players and each player is a vertex in a graph. A strategy consists of choosing which incident edges to build. Depending on the network structure there is a payoff for each player, and players adjust their strategy to maximize their payoff. A general finding was that there are games, in which no efficient network is stable for a concept of pairwise stability, which requires bilateral consent to construct a connection. The extensions and adjustments to this model are numerous [13]. In particular, several works extended the model to unilateral link creation and the Nash equilibrium as stability concept [5, 9].

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A particularly interesting variant was proposed in the context of distributed systems and the Internet by Fabrikant et al. [11]. In their network creation game (NCG) the cost of creating an edge is fixed to a parameter $\alpha$. Edge creation is unilateral, and the cost for a player is a trade-off between the costs for created edges and the structural position in the network measured by shortest path distances to all other players. In [11] and consecutive work [2, 8] the inefficiency of Nash equilibria was quantified using the price of anarchy, which captures the deterioration in cost of the worst Nash equilibrium against a social optimal state. The presently known results on the price of anarchy are summarized in Figure 1. Other equilibrium concepts were also studied, e.g. existence and cost of pairwise stable equilibria [7], or of strong equilibria [3]. Extensions to more general edge costs or different player cost trade-offs proved useful in the analysis of mobile peer-to-peer networks [1, 10, 17].

In network analysis [6], the inverse of the sum of shortest path lengths is one of the most commonly used measures of centrality known as closeness [12]. A problem with closeness is that global connectivity is required for the scores to be comparable. This means that in the NCG for moderate to high edge costs the trade-off is distorted by the enforcement of connectivity. Thus, it was not surprising that trees proved to be a prominent equilibrium structure [11].

In this paper, we remedy this problem by replacing the infinite cost of not being connected by a finite penalty $\beta$. This corresponds directly to a variant of closeness centrality proposed by Botafogo et al. [18]. This is also closely related to a measure termed radiality [19], although here $\beta$ depends on the network structure. Such an adjustment to the NCG was also suggested as an open problem in [11]. Our penalized network creation game (PCG) is introduced in Section 2. $\beta$ allows to level off the infinite penalties for disconnectivity and to study the effect of the connection requirement in the NCG on topology and social cost of Nash equilibria. Since the cost of connected equilibria is the same as in the NCG, we will be most interested in existence, structure, and cost of disconnected Nash equilibria. Naturally, if $\beta$ is high, then Nash equilibria of the PCG reveal the same properties as those of the NCG. It is thus not surprising that a number of insights for the NCG can be translated directly to the PCG. If $\beta$ decreases, then properties of Nash equilibria can change. In particular, an interesting insight gained from our structural analysis in Section 3 is presented in Theorem 3.2. It shows that the prevalent tree structures of the NCG are absent in disconnected Nash equilibria whenever $\alpha > 1$ or $\beta > 2$.

Our analysis on the existence of disconnected networks offers relevant insight for the analysis of distributed networks with rational agents. In many scenarios, a priori, a given set of selfish entities has no intrinsic motivation to create a globally connected network. In contrast, our findings indicate a peculiar absence of non-empty disconnected stable networks, which indicate underlying incentives that prohibit their emergence. We failed to identify any non-empty disconnected Nash equilibrium for $\beta > 3$. In addition, structural conditions like constant diameter in all known equilibrium topologies for the NCG led us to conjecture that there is a constant $\beta'$ such that the

| $\alpha$ | $1$ | $\frac{4}{3}$ | $< 4$ | $< 6$ | $\Theta(1)$ | $\alpha(n^\epsilon)$ | $< 1 + \frac{6n \log n}{\alpha}$ |
| --- | --- | --- | --- | --- | --- | --- | --- |
| $\alpha = 0$ | $1$ | $2$ | $\sqrt{\frac{n}{2}}$ | $\sqrt{\frac{n}{2}}$ | $O(n^{1-\epsilon})$ | $12n \log n$ |

Figure 1: Price of anarchy in the NCG
empty network is the only disconnected Nash equilibrium for any PCG with $\beta > \beta'$. This appears somewhat surprising, because the agents in the PCG are not explicitly forced into connection. In addition, it reveals that in terms of topology of Nash equilibria the assumption of infinite penalties in the NCG is not a significant drawback.

In addition, we consider the price of anarchy in Section 4. There are parameter values, for which disconnected Nash equilibria appear but the social optimum is connected, which could lead to an unbounded price of anarchy. However, we show that the price of anarchy in the PCG is always bounded by $O(n)$. In addition, Theorem 4.3 reveals cases with tightness and a matching lower bound of $\Omega(n)$. This bound is strictly larger than any of the known bounds for the NCG. In Section 5 we contrast these findings with the scenario, in which players can play joint coordinated deviations and consider strong equilibria. Unless $\alpha$ and $\beta$ are within a small range, the social optimum is also a strong equilibrium (see Theorem 5.1). In Theorem 5.2 we prove that the price of anarchy for strong equilibria is at most 4. This reveals that in the PCG Nash equilibria can be several orders of magnitude more costly than strong equilibria, a question which is still unsolved for the NCG. More generally, it shows that joint and coordinated actions of selfish agents can drastically reduce inefficiencies in selfish network creation. Finally, Section 6 concludes and presents some problems for further research.

2 The Model and Initial Results

The network connection game (NCG) is a tuple $(V, \alpha)$ and can be described as follows. The set of players $V$ is the set of vertices of a graph. Possible edges $\{i, j\} \in V \times V$ have cost $\alpha$. A strategy $s_i$ of a player $i$ is a subset $s_i \subset V \setminus \{v\}$ and indicates, which edges player $i$ chooses to build. In this way a strategy vector $s$ induces a set of edges between the players. Given a strategy vector $s$ the individual cost for a player $i$ is

$$c_i(s) = \alpha|s_i| + \sum_{j \neq i} \text{dist}_s(i, j),$$

where $\alpha > 0$ and $\text{dist}_s(i, j)$ is the length of a shortest-path in the undirected graph $G_s = (V, E_s)$ induced by the strategy vector $s$. Note that $G_s$ is assumed to be undirected, i.e., each edge can be traversed in any direction, independent of which player pays for it. In the regular connection game $\text{dist}_s(i, j) = \infty$ if players $i$ and $j$ are in different components of $G_s$. In the penalized network creation game (PCG) we are given a penalty value $\beta > 1$, and $\text{dist}_s(i, j) = \beta$ for players $i$ and $j$ in different components. A pure Nash equilibrium (NE) is a state $s$, in which no player can unilaterally decrease his cost $c_i$ by changing his strategy $s_i$. We will restrict our attention to pure equilibria throughout the paper. The social cost $c(s)$ of a state $s$ is simply $c(s) = \sum_{i \in V} c_i(s)$. A social optimum state $s^*$ is a state with minimum social cost. Note that for the cost of a state it does not matter, who builds an edge, and hence we will sometimes consider the graph $G_s$ instead of $s$. States that play an important role in the analysis of the PCG are the empty state $s_0 = (\emptyset, \ldots, \emptyset)$, $s_K$ corresponding to the complete graph, in which each edge $\{i, j\}$ with $i \neq j$ is paid by player $\min\{i, j\}$, and $s_Z$ corresponding to a center-sponsored star, in which one player purchases edges to all other players.

Fabrikant et al. [11] show that there is always a pure NE in the NCG and mention that it might be found by iterative improvement steps. Finding a best-response for a player in a NCG, however, was shown NP-hard [11], and this translates to the PCG for sufficiently large penalty.
cost. In addition, we show that better-response dynamics may cycle, hence the game is no potential
game [16]. As the dynamics involve no disconnectivities, the result follows directly for the PCG.
Nevertheless, in the PCG there is always a pure NE. This serves as a first insight to motivate the
further study of the properties of pure NE in the PCG.

**Theorem 2.1** Every PCG has a pure Nash equilibrium, but neither NCG nor PCG are potential
games.

**Proof.** We first prove the non-existence of a potential function by contradiction. For any \( \alpha > 3 \)
choose an integer \( k \) with \( k < \alpha < \frac{3k}{2} \). Now construct a strategy combination for \( n = 4k \) players as
depicted in Figure 2. The following steps each represent a strict improvement for the players: (1)
player 4 removes edge \( e_1 \), (2) player 2 removes edge \( e_2 \), (3) player 4 builds edges \( e_1 \) and \( e_2 \). The
resulting state is isomorphic to the initial state, in which the roles of players 2 and 4 and edges
\( e_1 \) and \( e_3 \) are switched. In particular, this allows us to construct an infinite improvement path, which
contradicts the existence of a potential function.

For the proof of existence let \( \alpha \geq \beta - 1 \) and consider \( s_\emptyset \). For every player a strategy change
consists of connecting to a number \( t \) of other players. As \( t\alpha + t - t\beta = t(\alpha - (\beta - 1)) \geq 0 \), \( s_\emptyset \)
represents a NE. For \( 1 \leq \alpha < \beta - 1 \) consider the state \( s_Z \) corresponding to a center-sponsored star,
in which the center player pays for all edges. Using the same argument as for \( s_\emptyset \) it is not profitable
for the center player to remove any edges. For a leaf player connecting to additional \( t \) other leaf
players yields a difference of \( t\alpha - t \geq 0 \). Hence, the center-sponsored star represents a NE. Finally,
for \( \alpha < 1 \) and \( \alpha < \beta - 1 \) consider the state \( s_K \). Then, as \( \alpha < 1 \) every edge removal that leaves the
graph connected cannot be profitable. The only possibility to disconnect the graph, however, is for
player 1 to remove all edges. This changes his cost by \( \beta(n - 1) - (\alpha + 1)(n - 1) > 0 \). Hence, for
this case \( s_K \) represents a NE. \( \square \)

### 3 Disconnected Equilibria

In this section we consider existence and structural properties of disconnected NE in the PCG.
First, we clarify the existence of disconnected equilibria.

**Theorem 3.1** For \( \alpha \geq \beta - 1 \) the empty graph is always a disconnected NE. For \( 0 < \alpha < \beta - 1 \)
there is no disconnected NE.
Proof. The first part follows from Theorem 2.1. For the second part consider a player \( v \) in a disconnected NE \( s \). Let \( n_v \) be the size of the component of the graph \( G_s \), in which \( v \) is located. Now suppose \( v \) changes his strategy by connecting to all \( n - n_v \) players in other components. Then the change is
\[
\alpha (n - n_v) + (n - n_v) - \beta (n - n_v) = (n - n_v)(\alpha - (\beta - 1)) < 0.
\]
Hence, under these conditions every player in a disconnected state can decrease his individual cost. \( \square \)

The theorem provides a tight characterization using the empty graph. An interesting issue, however, is to explore whether non-empty disconnected NE are possible, because in many cases the empty graph represents a rather unrealistic prediction for a stable network. Note that a component of \( k \) players in a disconnected NE of a PCG with given \( \alpha \) and \( \beta \) must be a NE in the corresponding NCG with \( \alpha \) and \( k \) players. There are a number of structures that have been identified as components of NE in the NCG, in particular, graphs based on affine planes (including the Petersen graph), cliques, cliques of star graphs, and trees [2]. In the following we consider each of these classes and assume a size of at least 2 vertices to exclude the degenerate case of singleton vertices. The treatment of affine plane graphs is cumbersome but rather straightforward, so we omit details here. These graphs can represent NE for \( \beta < 3 \) and certain restricted values of \( \alpha \). More information is available from the authors upon request.

3.1 Cliques and Cliques of Stars

We refer to a pair as a component consisting of two players linked by a single edge.

Lemma 3.1 For \( \alpha > 1 \) there is no pair in a disconnected NE.

Proof. Assume there is a pair in a disconnected NE. As one of the players wants to keep the edge, it must be that \( \beta - \alpha - 1 \geq 0 \), and with Theorem 3.1 \( \alpha = \beta - 1 \). Now consider a different player \( v \) that constructs an edge to a player from the pair. The change of the individual cost is
\[
\alpha - 2\beta + 3 = -\alpha + 1 < 0.
\]
Hence, the change is profitable, which proves that a pair cannot appear in a disconnected NE. \( \square \)

Similarly, for larger clique components it must be \( \alpha \leq 1 \), because otherwise a player from the component removes one edge and accepts the increase in distance. Theorem 3.1 yields the following direct corollary.

Corollary 3.1 For \( \beta > 2 \) there is no clique component in a disconnected NE.

A \((k, l)\)-clique of stars is a clique with \( k \) vertices, in which each such node is the center of a star of \( l \) vertices. We consider \( k \geq 3 \), because otherwise the structure is a tree. In [2] it was shown that a \((k, l)\)-clique of stars, in which all edges are created by the players in the clique, is a NE for the NCG with \( \alpha = l \). Here we show that the appearance of such a component in a disconnected NE is quite limited.

Lemma 3.2 For \( \alpha = l \) and \( k \geq 3 \), a \((k, l)\)-clique of stars, in which all edges are created by the players in the clique, can be a component in a disconnected NE if and only if \( \alpha = 1 \) and \( \beta = 2 \).

Proof. Consider a state \( s \), which is a disconnected NE with a component \( C \) of a \((k, l)\)-clique of stars. In addition to the deviations considered in the NCG, for a component in a disconnected NE
we must consider a split of the component and a connection of a vertex outside of the component. For \( k \geq 3 \) we can assign edge costs to the players such that no clique player is able to unilaterally disconnect the clique. As all edges of the stars are also built by players from the clique, we must have \( \alpha + 1 - \beta \leq 0 \), and hence with the general bound from Theorem 3.1 \( \alpha = \beta - 1 \).

Consider a connection from a player outside the component to an arbitrary player \( v \) of the clique. Hence,

\[
\alpha - kl\beta + \sum_{w \in C}(\text{dist}_v(v, w) + 1) = \alpha - kl\beta + (2kl - k - l) + kl \geq 0.
\]

It is \( \alpha = l = \beta - 1 \), so we get as condition

\[
0 \leq l - kl(l + 1) + 3kl - k - l = -k(l^2 - 2l + 1) = -k(l - 1)^2.
\]

Thus, \( l = \alpha = 1 \) and \( \beta = \alpha + 1 = 2 \). It is straightforward to verify that under these conditions the outlined state is a NE.

\[\square\]

### 3.2 Trees

Tree graphs are a structure whose appearance is widespread in the NCG [2, 11]. The following analysis shows that this property is only due to the requirement that a NE must be connected. In the PCG these structures can appear only in very special cases.

**Lemma 3.3** For \( \beta > 2 \) every non-singleton player \( v \) in a disconnected NE has at least one incident edge that was created by a different player \( w \neq v \).

**Proof.** Consider a player \( v \) in a component \( C \) with \( k \) players, who pays for all his \( d_v \) incident edges. As we have a NE, it is not profitable for \( v \) to disconnect from \( C \), i.e., \( \alpha d_v + \sum_{w \in C} \text{dist}(v, w) \leq \beta(k - 1) \). Now consider a different player \( v' \) not in \( C \) that chooses to connect to all neighbors of \( v \). As this must not be profitable, \( \alpha d_v + \sum_{w \in C} \text{dist}(v, w) + 2 \geq 2\beta k \). Adding the two inequalities yields \( \beta \leq 2 \), which proves the lemma. \[\square\]

**Lemma 3.4** Suppose there is a disconnected NE with a component \( C \) of \( k > 1 \) vertices. If \( \alpha > (k - 1)(\beta - 2) + 1 \), then for every player \( v \) there is an incident edge paid by a different player \( w \neq v \).

**Proof.** Suppose there is a player \( v \) that pays for all his \( d_v \geq 1 \) incident edges. As \( v \) does not want to remove all edges, we have \( \alpha d_v + \sum_{w \in C} \text{dist}(v, w) \leq \beta(k - 1) \), and thus

\[
\alpha \leq \frac{1}{d_v} \left( \beta(k - 1) - \sum_{w \in C} \text{dist}(v, w) \right).
\]

For every non-neighbor vertex of \( C \) we have distance at least 2, and thus \( \sum_{w \in C} \text{dist}(v, w) \geq 2(k - 1) - d_v \). Substitution yields \( \alpha \leq (k - 1)(\beta - 2) + 1 \) as desired. \[\square\]

**Theorem 3.2** For \( \beta > 2 \) or \( \alpha > 1 \) no component of a disconnected NE is a tree.

**Proof.** The first bound is a direct consequence of Lemma 3.3 and the fact that for a tree \( |E| = |V| - 1 \). Thus, for disconnected NE with tree components \( \beta \leq 2 \), and the second bound follows with Lemma 3.4. \[\square\]
3.3 Non-empty Equilibria

In the previous paragraphs we have shown that the appearance of known NE topologies from the NCG as components in disconnected NE of the PCG is quite limited. The existence of disconnected NE is guaranteed by the empty network. This raises the question, under which conditions on $\alpha$ and $\beta$ non-empty disconnected NE can evolve. We first present a positive result.

**Lemma 3.5** For $3 \leq \alpha \leq 4$ and $\beta \leq (\alpha + 11)/5$ a cycle $C_5$ of 5 vertices can be a component of a disconnected NE.

**Proof.** Consider a disconnected NE with such a component. Label the players in $C_5$ from 0 to 4 along a Euclidean tour. Each player $i$ pays for edge $(i, i + 1 \mod 5)$. In this case no player can disconnect the component. If a player removes his edge, the increase in distance cost is 4, thus $\alpha \leq 4$. Furthermore, every vertex in the cycle has a sum of distances of 6. As for each vertex $v \notin C_5$ it must not be profitable to connect to a vertex $i \in C_5$, we get $\alpha \geq 5\beta - 11$. Every additional edge yields a cost improvement of at most 3, hence for $\alpha \geq 3$ it is optimal for $v$ to connect to at most one vertex from $C_5$. This means that for $\beta \leq \frac{\alpha + 11}{5} \leq 3$ there can be a component $C_5$ in a disconnected NE. $\square$

Note that a NE with $C_5$ is transient and not strict, i.e., there is a sequence of strategy changes that leaves the individual cost of the changing player identical, but leads into a non-equilibrium state. For instance player 2 can exchange edge (2,3) by edge (2,4) without cost change. Afterwards player 1 can strictly improve by purchasing (1,4) instead of (1,2).

In contrast to the restricted interval, for which we can show existence, there is an unbounded region of parameter values, for which the empty network is the only disconnected network - in particular if $\alpha$ or $\beta$ are large compared to $n$.

**Lemma 3.6** In a non-empty disconnected NE let $n_l$ be the minimum size and $diam_l$ the minimum diameter of any non-singleton component. Then

1. $\alpha < 12n_l \log n_l$
2. $\beta \leq 1 + 2 \cdot diam_l$
3. $\beta < 1 + 14\sqrt{n_l \log n_l}$
4. if $n > 6$, then $\beta < n/2$

**Proof.** For the first bound consider $\alpha \geq 12n_l \log n_l$ and a component with $n_l$ players. This component must represent a NE in a NCG with the same $\alpha$ and $n_l$ players, and thus according to [2] must be a tree. This contradicts Theorem 3.2 and the bound follows.

Now consider a non-empty disconnected NE $s$ for $\beta > 2$, and let $C$ be a non-singleton component. As $C$ is no tree, it must contain at least one cycle. Let $U$ be a smallest of all cycles in $C$, and let $v$ be an arbitrary player that constructed some edge $e$ of $U$. Denote by $s'$ the state that evolves if player $v$ removes edge $e$. Note that by this removal no additional pair of players gets disconnected. As $s$ is a NE, we have

$$\alpha \leq \sum_{w \in C} (dist_s'(v, w) - dist_s(v, w)). \quad (1)$$

As we have chosen $U$ to be of minimum size, all shortest distances between vertices of $U$ are given by the paths along the cycle. Thus, there is always a vertex $u$, for which the distances in $s$ and $s'$ are the same. This yields

$$dist_s'(v, w) \leq dist_s'(v, u) + dist_s'(u, w) = dist_s(v, u) + dist_s(u, w)$$
for all \( w \in C \). With \( n_C = |C| \) we can conclude \( \alpha \leq 2n_C \cdot diam(C) - \sum_{w \in C} dist_s(v, w) \). On the other hand, no vertex outside \( C \) must be able to profit from a connection to \( v \), hence \( \alpha + n_C + \sum_{w \in C} dist(v, w) \geq n_C \beta \). The last two inequalities imply that \( \beta \leq 2 \cdot diam(C) + 1 \) and thus deliver the second bound. As each component \( C \) must be a NE of a NCG, we know from [11] that \( diam(C) \leq \sqrt{4\alpha + 1} \). Together with the first bound on \( \alpha \) shown above this implies the third bound \( \beta < 1 + 14\sqrt{n_l \log n_l} \). For the proof of the last bound the inequality (1) and the bound \( \alpha + n_C + \sum_{w \in C} dist(v, w) \geq n_C \beta \) imply

\[
\beta \leq 1 + \frac{1}{n_C} \sum_{w \in C} dist_s(v, w)
\]

The sum of distances for \( v \) is maximal iff \( C \) in \( s \) is a cycle and thus in \( s' \) is a chain with \( v \) being one of its endpoints. In this case

\[
\sum_{w \in C} dist_s(v, w) \leq \frac{(n_C - 1)n_C}{2}.
\]

If \( C \) is not a cycle in \( s \), then the inequality is strict and the previous formulas yield \( \beta < 1 + \frac{n_C - 1}{2} = \frac{n_C + 1}{2} \leq \frac{n}{2} \). If \( C \) is a cycle in \( s \), it is straightforward to show that it must hold \( n_C \leq 5 \) as \( s \) is a NE.

Here we use the assumption that \( n > 6 \) and get \( \beta \leq \frac{n_C + 1}{2} \leq 3 < \frac{n}{2} \). This proves the last bound. \( \square \)

In contrast to these bounds, we have not been able to derive any non-empty disconnected NE for values of \( \beta > 3 \). This led us to formulate the following conjecture.

**Conjecture 3.1 (Constant Penalty Conjecture)** There is a constant \( \beta' \) such that for \( \beta > \beta' \) the only disconnected NE is \( s_\emptyset \).

Note that our bounds imply that if the conjecture is false, then there must be non-tree NE in the NCG with a diameter in \( \omega(1) \). This seems quite unlikely, as all non-tree NE found so far have diameter at most 3.

### 4 Price of Anarchy

In this section we consider the price of anarchy in the PCG. We first present an overview of the social optima of the game in Figure 3.

**Theorem 4.1** The social optimum is \( s^* = s_\emptyset \) for \( 2\beta - 2 < \alpha < 2 \), as well as for \( \alpha \geq \max \{2, \beta n - 2(n - 1)\} \). It holds that \( s^* = s_K \) in the range \( \alpha \leq \min \{2, 2\beta - 2\} \). In the remaining range the social optimum is \( s^* = s_Z \).

**Proof.** First, assume \( \alpha \leq 2 \). In this case \( \alpha + 2 \leq 4 \leq 2dist_s(v, w) \) for every indirectly connected pair of players. Hence, a direct connection can only result in smaller social cost. If in addition \( 2\beta - 2 \leq \alpha \), then disconnectivity is the cheapest alternative and thus \( s_\emptyset \) is optimal. Otherwise, for \( 2\beta - 2 \geq \alpha \) the state \( s_K \) corresponding to the complete graph is optimal.

In the case \( \alpha > 2 \) and \( \beta \leq 2 \) we have \( 2\beta - 2 < \alpha \) and \( 2\beta \leq 2dist_s(v, w) \) for any \( dist_s(v, w) \geq 2 \). Hence, \( s_\emptyset \) represents a social optimum.
For $\alpha \geq \beta n - 2(n - 1)$ and $\beta \geq 2$ consider a state $s$ corresponding to a non-empty graph $G_s$. Suppose $C$ is a non-singleton component of $G_s$ and $n_C = |C| > 1$. By $c(C)$ we consider only the costs introduced by pairs of players in $C$, and by $m_C$ the number of edges within component $C$. It is easy to note $c(C) \geq 2n_C(n_C - 1) + (\alpha - 2)m_C$ (see [11]). If $s$ is a candidate for a social optimum, component $C$ must be a star. However,

$$c(C) \geq 2n_C(n_C - 1) + (\alpha - 2)m_C \geq 2n_C(n_C - 1) + n_C(\beta - 2)(n_C - 1) = n_C(n_C - 1)(\beta - 2). \quad (2)$$

Thus, by removing all edges from non-singleton components we reduce the social cost of the state even further. This implies that $s_\emptyset$ is a social optimum.

Finally, in the case $2 < \alpha < \beta n - 2(n - 1)$ the cheapest connected state is $s_Z$ corresponding to the star. We show here that for every disconnected state there is a strictly cheaper state. Hence, no disconnected state can be a social optimum, and the social optimum remains $s_Z$. This finishes the proof of the theorem.

**Lemma 4.1** If $2 < \alpha < \beta n - 2(n - 1)$, then for each disconnected state $s$ there is a state with strictly less social cost.

**Proof.** Suppose for contradiction that there is a disconnected state $s$ which is the social optimum. Similarly to the last paragraph we see that the components of $s$ must be stars. In addition, $s$ must have at least one non-singleton component, because $s_\emptyset$ is more costly than $s_Z$:

$$n(n - 1)\beta > (n - 1)(\alpha + 2n - 2) = \alpha(n - 1) + 2(n - 1)^2.$$

Suppose $s$ has two (star) components $C_1$ and $C_2$ of $n_1$ and $n_2$ players, respectively, and assume $n_1 > 1$. We will consider two different states, which both must not be cheaper. This delivers a contradiction.

On the one hand, suppose we remove all edges from $C_1$ and instead connect all players of $C_1$ to the center player of $C_2$. Then regarding players $v \notin C_1 \cup C_2$ there is no cost change. We need one additional edge for the center player $v_1$ of $C_1$. Every other player from $C_1$ is now at a distance of 2 to $v_1$, thus there is an increase in distance of $2(n_1 - 1)$. For the newly created distances between players of $C_1$ and $C_2$ we have

$$\sum_{v \in C_1, w \in C_2} dist(v, w) = 2(n_2 - 1)n_1 + n_1.$$
which is counted twice in the social cost. However, we save a penalty of \(2n_1n_2\beta\). By assumption the total changes must not lead to an improved state, hence

\[
\alpha + 2(n_1 - 1) + 4(n_2 - 1)n_1 + 2n_1 - 2n_1n_2\beta \geq 0.
\]

Thus, \(\alpha \geq 2 - 4n_1n_2 + 2n_1n_2\beta\). On the other hand, consider the change in cost when we remove all edges from \(C_1\). Again, by assumption this must not lead to a cheaper state, so

\[
\alpha(n_1 - 1) + 2(n_1 - 1)^2 \leq n_1(n_1 - 1)\beta.
\]

This implies \(\alpha \leq n_1\beta - 2(n_1 - 1)\). Combining the bounds leads to \(n_1(\beta - 2) + 2 \geq 2 - 4n_1n_2 + 2n_1n_2\beta\), which implies \(4n_2 - 2 \geq \beta(2n_2 - 1)\) and thus \(\beta \leq 2\), which is a contradiction to \(2 < \beta n_2 - 2(n - 1)\).

For \(\alpha < \beta - 1\) we have seen in Theorem 3.1 that no disconnected NE exists. In addition, the next theorem shows that in this case a finite penalty for disconnectivity cannot disrupt any NE of the NCG. Hence, for this parameter range the price of anarchy is identical to the NCG.

**Theorem 4.2** For \(\alpha < \beta - 1\) the NE of the PCG are exactly the NE of the NCG, so in this parameter range the prices of anarchy and stability remain the same in the PCG as in the NCG, respectively.

**Proof.** As in this parameter range all NE of the PCG must be connected, every such state must also be a NE in the NCG, because in the PCG the players only consider more potentially profitable deviations. For the converse, suppose for contradiction that there is a NE \(s\) of the NCG, which is not a NE in the PCG. As \(s\) must be connected, there is a player \(v\), who profits from changing her strategy \(s_v\) to a strategy \(s'_v\) that disconnects the resulting graph. Let \(W\) be the set of other players, to which \(v\) is disconnected under \(s' = (s'_v, s_{-v})\). Now suppose \(v\) changes his strategy again to \(s''_v\) by building direct connections to all players from \(W\). As \(\alpha + 1 < \beta\), this is again a profitable deviation. Thus, player \(v\) strictly profits from switching from \(s_v\) to \(s''_v\). Note, however, that in \(s'' = (s''_v, s_{-v})\) the resulting graph is connected, thus \(s''_v\) must be a feasible deviation yielding profit for \(v\) in the NCG. This contradicts the assumption that \(s\) is a NE in the NCG. \(\square\)

In general, however, the price of anarchy for the PCG can be strictly larger than for the NCG. Figure 4 provides an overview of the bounds we obtained. Note that all these bounds are in \(O(n)\) for the respective parameter values. Our proof is divided into ranges, in which different network structures are social optima.

### 4.1 Star Graph

At first, we concentrate on the case \(\max\{2, \beta - 1\} < \alpha < \beta n - 2(n - 1)\), in which disconnected NE can appear and the star is the social optimum.

We start by observing a helpful reduction to the price of anarchy in the NCG. Consider a disconnected NE \(s\) with non-singleton components \(C_1, \ldots, C_r\) and singleton components \(C_{r+1}, \ldots, C_{r+l}\). Let \(n_i = |C_i|\) and \(c(C_i)\) be the cost of \(C_i\) as a NE in a NCG with \(n_i\) players. In particular, \(c(C_i)\) counts only edge and shortest path costs within \(C_i\) but no penalties. In addition, let \(s_{Z_i}\) be a social optimum state for a NCG with \(n_i\) players. For the given parameter range of \(\alpha\) all the \(s_{Z_i}\) are stars.
Lemma 4.2 For the cost of $s$ it holds that

$$\frac{c(s)}{c(s_Z)} \leq \frac{n\beta}{\alpha + 2(n - 1)} + \max_{1 \leq i \leq r} \left\{ \frac{c(C_i)}{c(s_{Z_i})} \right\}.$$  

Proof. Obviously it holds that

$$c(s) = 2\beta \left( \sum_{1 \leq i < j \leq r + l} n_in_j \right) + \sum_{i=1}^{r} c(C_i) \leq \beta n(n - 1) + \sum_{i=1}^{r} c(C_i).$$

With $n = l + \sum_{i=1}^{r} n_i$ we can bound as follows

$$\sum_{i=1}^{r} \frac{c(C_i)}{c(s_{Z_i})} = \sum_{i=1}^{r} \left( \frac{c(C_i) c(s_{Z_i})}{c(s_{Z_i}) c(s_Z)} \right) \leq \max_{1 \leq i \leq r} \left\{ \frac{c(C_i)}{c(s_{Z_i})} \right\} \cdot \sum_{i=1}^{r} \frac{c(s_{Z_i})}{c(s_Z)}$$

$$= \max_{1 \leq i \leq r} \left\{ \frac{c(C_i)}{c(s_{Z_i})} \right\} \cdot \sum_{i=1}^{r} \frac{(n_i - 1)\alpha + 2(n_i - 1)^2}{(n - 1)\alpha + 2(n - 1)^2}$$

$$= \max_{1 \leq i \leq r} \left\{ \frac{c(C_i)}{c(s_{Z_i})} \right\} \cdot \frac{\alpha \sum_{i=1}^{r} (n_i - 1) + 2 \sum_{i=1}^{r} (n_i - 1)^2}{(n - 1)\alpha + 2(n - 1)^2}$$

$$< \max_{1 \leq i \leq r} \left\{ \frac{c(C_i)}{c(s_{Z_i})} \right\},$$

which proves the lemma. \(\square\)

Theorem 4.3 For $2\beta - 2 \leq \alpha \leq n\beta - 2(n - 1)$ the price of anarchy is bounded by

$$\Theta\left(\frac{n\beta}{\alpha}\right) \text{ for } \alpha \geq 12n \log n,$$

$$O\left(5\sqrt{\log n \log n} + \frac{n\beta}{\alpha + n}\right) \text{ for } \alpha < 12n \log n.$$

For $\beta - 1 \leq \alpha \leq 2\beta - 2$ the price of anarchy is $\Theta(\min\{\beta, n\})$. 

Figure 4: Price of anarchy in the PCG
The price of anarchy is bounded by \(\alpha_{\text{connected}}\) connected players.

In every NE every pair of players is either connected or a component of a NE. A directly connected pair yields a social cost \(2\alpha_{\text{connected}}\) social cost for every NE. Using Lemma 4.2 we can bound the price of anarchy for these NE by the sum of the range, for which \(\alpha_{\text{connected}}\) connected pair induces a cost of \(\alpha_{\text{connected}} + 2\leq 2\beta\). Any indirectly connected pair in a NE induces a cost \(2\text{dist}_s(v, w)\leq 2\sqrt{4\alpha + 1}\leq 2\sqrt{8\beta - 7}\leq 2\beta\). Thus, the cost of \(2\beta\) induced by \(s_\emptyset\) is maximal for every pair of players. Therefore the fraction in Equation (3) characterizes the price of anarchy and results in \(\Theta(\min\{\beta, n\})\), which proves the third bound.

For the remaining range with \(\alpha < 12n\log n\) we cannot exclude the possibility that there are worse disconnected NE than \(s_\emptyset\). However, components of these NE must be connected NE of smaller NCGs. Using Lemma 4.2 we can bound the price of anarchy for these NE by the sum of the fraction for \(s_\emptyset\) in Equation (3) plus the maximum factor of any component NE in the corresponding NCG. With the bound of \(5\sqrt{\log n}\log n \in o(n^\epsilon)\) on the price of anarchy for the NCG [8] this proves our second bound \(O\left(5\sqrt{\log n}\log n + \frac{\beta n}{\alpha + n}\right) = O(\max\{5\sqrt{\log n}\log n, \min\{n, \beta\}\})\). In particular, this represents a bound of \(O(n)\) for the price of anarchy.

### 4.2 Complete Graph

In this case we have \(s^* = s_K\), and thus it must hold \(\beta - 1 \leq \alpha \leq \min\{2, 2\beta - 2\}\). The following theorem summarizes the bounds.

**Theorem 4.4** The price of anarchy is bounded by \(4/3\) for \(\alpha < 1\), \(4/3\) for \(1 \leq \alpha \leq 2\) and \(\beta < 2\), and \(3/2\) for \(\alpha < \min\{2, 2\beta - 2\}\) and \(\beta \geq 2\).

**Proof.** Suppose \(s\) is a NE and let \(C\) be a component. For any two players \(v, w\in C\) we have \(\text{dist}_s(v, w)\leq \alpha + 1\), because otherwise building a direct connection is profitable. Thus, for \(\alpha < 1\) there are no indirectly connected players. A directly connected pair of players yields a social cost of \(\alpha + 2 \leq 2\beta\). Hence, the worst NE is \(s_\emptyset\) and the price of anarchy is bounded by

\[
\frac{c(s_\emptyset)}{c(s_K)} = \frac{2\beta}{\alpha + 2} \leq \frac{2\alpha + 2}{\alpha + 2} < \frac{4}{3}.
\]

This proves the first bound. For \(1 \leq \alpha < 2\), the diameter is \(\text{diam}(C)\leq 2\) for every non-singleton component of a NE \(s\). Indirectly connected players yield a social cost of \(2\text{diam}(C)\leq 4\) and directly connected players \(\alpha + 2 < 4\). If \(\beta < 2\), then \(2\beta < 4\) for any disconnected pair of players. Thus, the price of anarchy is bounded by

\[
\frac{4(n(n-1)/2)}{c(s_K)} = \frac{4}{\alpha + 2} \leq \frac{4}{3},
\]

which proves the second bound. In case \(\beta \geq 2\), we get \(2\beta \geq 4 \geq \alpha + 2\), so \(s_\emptyset\) is the worst NE. For the price of anarchy we get the third bound by

\[
\frac{c(s_\emptyset)}{c(s_K)} \leq \frac{2\alpha + 2}{\alpha + 2} < \frac{3}{2}.
\]
4.3 Empty Graph

In this case we have $2\beta - 2 < \alpha < 2$ or $\alpha \geq \max\{2, \beta n - 2(n-1)\}$. The following theorem summarizes the bounds.

**Theorem 4.5** The price of anarchy is bounded by $3/2$ for $2\beta - 2 < \alpha < 1$, 2 for $1 \leq \alpha < 2$ and $\alpha > 2\beta - 2$, and 1 for $\alpha \geq 12n \log n$ and $\alpha > \beta n - 2(n-1)$. It is $O(5^{\log n} \log n \cdot \frac{\alpha + n}{\beta n})$ for the remaining range.

**Proof.** In the range $2\beta - 2 < \alpha < 1$ every component of a NE is a clique. Every directly connected pair of players yields a contribution to the social cost of $\alpha > 2\beta - 2$. For $n > 2$ we can assign the edges of a complete graph to be purchased by the players such that no player can disconnect the graph by removing his edges. Hence, the complete graph represents the worst NE and with

$$\frac{c(s_K)}{c(s_0)} = \frac{\alpha + 2}{2\beta} < \frac{3}{2},$$

the first bound follows.

In the range $1 \leq \alpha < 2$ and $\alpha > 2\beta - 2$ every component has a diameter of at most 2. A connected pair of players yields a social cost of at most $4 > 2\beta$ or $2 + \alpha > 2\beta$. Therefore the worst NE is connected, has diameter at most 2, and as $4 > 2 + \alpha$ as few edges as possible. This means no NE can be more costly than the star graph (which does not represent a NE here). The price of anarchy is bounded by

$$\frac{c(s_Z)}{c(s_0)} = \frac{\alpha(n-1) + 2(n-1)^2}{\beta n(n-1)} < 2,$$

which proves the second bound.

Note that for $\alpha \geq 12n \log n$ Lemma 3.6 shows that NE can only be connected or empty. Lemma 3.4 then shows for $\alpha > n\beta - 2(n-1)$ that $s_0$ is the only NE in this range. Hence, we get a price of anarchy of 1.

For the remaining range of $2 \leq \alpha < 12n \log n$ and $\alpha > \beta n - 2(n-1)$ we use a bounding argument over the components of a disconnected NE. Similar as for the price of anarchy consider a disconnected NE $s$ with non-singleton components $C_1, \ldots, C_r$ and singleton components $C_{r+1}, \ldots, C_{r+t}$. Let $n_i = |C_i|$ and $c(C_i)$ be the cost of $C_i$ as a NE in a NCG with $n_i$ players. In particular, $c(C_i)$ counts only edge and shortest path costs within $C_i$ but no penalties. In addition, let $s_{Z_i}$ be a state for a NCG with $n_i$ players representing a star graph. Then Lemma 4.2 tells us that

$$\sum_{i=1}^{r} \frac{c(C_i)}{c(s_{Z_i})} < \max_{1 \leq i \leq r} \left\{ \frac{c(C_i)}{c(s_{Z_i})} \right\}.$$  

Similarly to Lemma 4.2 we can bound $c(s) \leq \beta n(n-1) + \sum_{i=1}^{r} c(C_i)$, and thus get the third bound

$$\frac{c(s)}{c(s_0)} \leq 1 + \frac{c(s_Z)}{c(s_0)} \sum_{i=1}^{r} \frac{c(C_i)}{c(s_{Z_i})} < 1 + \frac{c(s_Z)}{c(s_0)} \max_{1 \leq i \leq r} \left\{ \frac{c(C_i)}{c(s_{Z_i})} \right\},$$

$$\leq O \left( 5^{\log n} \log n \cdot \frac{\alpha + n}{n^\beta} \right).$$

Note that by restriction to $\alpha < 12n \log n$ this bound is still in $o(n^\epsilon)$. □
5 Strong Equilibria

In this section we assume agents are able to jointly deviate to different strategies. As stability concept we consider the strong equilibrium [4], in which no coalition C of players can decrease the cost for each of its members by taking a joint deviation. More formally, if a state s is a strong equilibrium (SE), then for each coalition of players C and each possible strategy profile s′C for the players in C it holds that if there is a player i ∈ C with ci(s′C, s−C) < ci(s), then there is another player j ∈ C with cj(s′C, s−C) ≥ cj(s). The price of anarchy for SE is a straightforward adaption of the price for NE. It was studied before in [3] for the NCG. The following theorem shows that with the exception of a small range of parameter values strong equilibria always exist in the PCG.

Theorem 5.1 For α < β − 1 the SE of the PCG are exactly the SE of the NCG. For α ≥ β − 1 the social optimum is a SE for all parameter values except β < 3, and βn − 2n + 2 − (β − 1) < α < βn − 2n + 2.

Proof. The first part of the theorem can be proven directly along the lines of Theorem 3.1. Consider a SE s of the NCG and suppose there is a profitable deviation of a coalition C that creates a disconnected graph. Then reconnecting all players across components creates a connected deviation that is cheaper for every player. This is a contradiction to s being a SE.

For the second part, we first note that for the case α = β − 1 we can use the arguments of Theorem 3.1 to show that every SE of the NCG is also a SE of the PCG. Hence, using results from [3] it holds that with the exception of α ∈ (1, 2) (respectively β ∈ (2, 3)) the social optimum is a SE. For the remainder we thus focus on the range α > β − 1.

We will at first concentrate on the case, in which s∅ is a social optimum. In addition, we assume β ≤ min{2, n/2 + 1} holds. Let us consider a deviating coalition C of nC players that builds a connected component, which is disconnected from the remaining n − nC players. In C there must be at least one player v that pays for at least dν/2 (i.e., half of his incident) edges. It requires an easy inductive argument to show that if such a player v does not exist, not all edges of C are being paid for. For such a player we get a cost of

\[ dνα/2 + dν + 2(nC − 1 − dν) + β(n − nC) \]
\[ ≥ βdν + 2(nC − 1 − dν) + β(n − nC) \]
\[ ≥ β(n − nC + dν) + β(nC − 1 − dν) = β(n − 1). \]

Hence, player v ∈ C is not able to strictly decrease his cost.

For the remaining range of β > 2 and α ≥ βn − 2(n − 1) in which s∅ is optimal, we consider a similar argument. Suppose a coalition C of nC players builds a connected network. Then the cost of this network can be lower bounded as in Equation (2). Hence, the new average player cost in C with respect to the coalition is at least β(nC − 1), which is exactly the cost of each player in s∅ with respect to players in C. This proves that whenever s∅ is optimal, it is a SE.

For the case, in which sK is optimal, it is known [3] that there is no connected SE for α ∈ (1, 2) and n ≥ 7. If α ∈ (1, 2) and α/2 + 1 < β ≤ α + 1, then sK is the unique social optimum. This means that for β ∈ (1.5, 3) a social optimum might not be a SE.

For the remainder let us consider the range of β ≥ 3. For α ≤ 2 the game is equivalent to a NCG, so sK is a SE for α ∈ [0, 1] and α = 2. Thus, we concentrate on the case α > 2 and β < α + 1, in which the star sZ is the only social optimum and is not guaranteed to be a SE by previous arguments. Suppose the star is periphery-sponsored, i.e., each leaf vertex pays for the
incident edge. Then the star center will never participate in a deviation: As \( \beta > 2 \), disconnecting a player can only increase the cost for him, and w.r.t. any connected component he can never achieve a better cost. Hence, we focus on the leaf players. If a coalition \( C \) of \( n_C \) leaf players chooses to deviate, it cannot find a profitable deviation that leaves the network connected. This is a result from the fact that the periphery-sponsored star is a SE in the NCG [3, Theorem 4.1]. Hence, let us consider a deviating coalition that builds a connected component \( C \), which is disconnected from the remaining \( n - n_C \) players.

**Case 1:** First, suppose in \( C \) there is a player that pays for an edge and has degree 1. For this player a lower bound on his cost is given by \( \alpha + 1 + 2(n_C - 2) + \beta(n - n_C) \). If the deviation is profitable for the coalition, we must have

\[
\alpha + 2n_C - 2 + \beta(n - n_C) < \alpha + 2n - 3,
\]

because otherwise the player would refuse to join. This gives \( \beta < 2 \), which contradicts \( \beta \geq 3 \).

**Case 2:** Otherwise, suppose each player that pays for an edge in \( C \) has degree at least 2. We again consider a player \( v \), who pays for at least \( d_v/2 \) edges. This player pays for at least one edge and has cost at least \( d_v \alpha/2 + d_v + 2(n_C - 1 - d_v) + \beta(n - n_C) \). The player must be motivated to join the coalition, and hence his cost must decrease:

\[
d_v \alpha/2 + d_v + 2(n_C - 1 - d_v) + \beta(n - n_C) < \alpha + 2n - 3
\]

\[
d_v \alpha - 2d_v + 4n_C + 2\beta(n - n_C) < 2\alpha + 4n - 2
\]

\[
d_v (\alpha - 2) + 2\beta(n - n_C) < 2\alpha + 4(n - n_C) - 2
\]

With \( d_v \geq 2 \) and \( n_C \leq n - 1 \) by assumption we have \( \beta < 2 + \frac{1}{n - n_C} \leq 3 \). This upper bound is again tight. Consider a game with \( n = 5 \) players, in which the four leaf players deviate to a cycle. This deviation can be profitable for any \( \beta < 3 \) and appropriate values of \( \alpha \).

Now consider deviations of a coalition of leaf players to the empty network. This can be profitable if \( \beta(n - 1) < \alpha + 2n - 3 \). Together with the optimality bound of \( s_Z \) this yields \( \beta(n - 1) - 2n + 3 < \alpha < \beta n - 2n + 2 \), the second bound.

At last, consider deviations in which a part of the coalition \( C \) builds disconnected components, while another part of the coalition possibly remains connected to the star of players outside \( C \). For the remaining range, in which the star \( s_Z \) can be a SE, we can use the above arguments to show that there must be a player in a newly created component that is not able to strictly decrease his cost. This shows that \( s_Z \) is indeed a SE in the remaining range.

In combination with results from [3] the theorem shows for the case \( \alpha < \beta - 1 \) that the price of anarchy for SE is strictly larger than 1, but at most 2. The main result in this section is a general constant upper bound on the price of anarchy for SE in the PCG.

**Theorem 5.2** The price of anarchy for SE in the PCG is at most 4.

**Proof.** In case the complete graph is the social optimum, Theorem [4.4] shows that the price of anarchy is at most 1.5. As the SE are a subset of the NE of a game, the theorem follows for this case.
We next show the bound for the empty network as optimum. Suppose $s$ is a non-empty SE, and consider any component $C_i$ of $s$ with $n_i = |C_i| > 1$. Each player that pays for at least one edge in $C_i$ has cost at most $\alpha + n_i - 1$. A joint deviation of this set of players would be to delete all edges, which would result in a cost of $\beta(n_i - 1)$ for each of them w.r.t. the players in $C_i$. Hence, it must be that $\alpha + n_i - 1 \leq \beta(n_i - 1)$.

Note that connection and distance costs within components can be bounded as follows. Each non-empty component $C_i$ has at least one vertex $v_i$ with distance cost at most $\alpha + 2n_i - 3$, because otherwise the whole component could deviate jointly to the star and all decrease their cost w.r.t. vertices from $C_i$. Thus, similarly to [2] and [3, Lemma 4.1] we can bound the distance and edge costs within $C_i$ by

$$(n_i - 1) \left( 2\alpha + n_i - 1 + \sum_{v_j \in C_i, v_j \neq v_i} \text{dist}(v_i, v_j) \right)$$

$$\leq 2\alpha(n_i - 1) + (n_i - 1)^2 + n_i(\alpha + 2n_i - 3)$$

$$\leq 3n_i\beta(n_i - 1) - 2n_i - 2\alpha + 1 \leq 3n_i\beta(n_i - 1)$$

In addition, for each vertex in $C_i$ there are penalties of $\beta \sum_{j \neq i} n_j$. This yields

$$\frac{c(s)}{c(s_0)} \leq \frac{\sum_i 3n_i\beta(n_i - 1) + \beta n_i \sum_{j \neq i} n_j}{\beta n(n-1)} \leq 1 + 3n \max_i \frac{n_i - 1}{n(n-1)} \leq 4,$$

and proves the bound for the empty network as social optimum.

If the star is the social optimum, then for each connected SE the price of anarchy for SE is at most 2. Consider a disconnected SE with $k$ components and number the components such that $n_1 \geq n_2 \geq \ldots \geq n_k$. We can bound the price of anarchy for SE by the maximum factor achieved by any component (see Lemma 4.2), in addition to the costs incurred by the penalties. As each component must represent a SE, we have

$$\frac{c(s)}{c(s_0)} \leq 2 + \frac{\sum_i n_i \beta \sum_{j \neq i} n_j}{\alpha(n-1) + 2(n-1)^2}.$$  

For the remaining part we focus on the penalties. Each player in component $C_i$ has penalty exactly $\beta(n - n_i)$. On the other hand, if all players join and create an additional star, then his new cost for players outside $C_i$ is at most $\alpha + 2(n - n_i)$ when being a leaf. This yields

$$\beta(n - n_1) \leq \beta(n - n_i) \leq \alpha + 2(n - n_i) \leq \alpha + 2(n - 1).$$

Therefore, $\alpha \geq \beta(n - n_1) - 2(n - 1)$, which allows us to bound

$$\frac{\sum_i n_i \beta \sum_{j \neq i} n_j}{\alpha(n-1) + 2(n-1)^2} \leq \frac{\sum_i \beta n_i(n - n_i)}{\beta(n-1)(n-n_1)}$$

Suppose that $r := n - n_1$, then $n_1 = n - r$ and the number of non-connected vertex pairs is $(n - r)r$ for the pairs involving $C_1$ and at most $r^2 - r$ for the remaining components. Hence,

$$\frac{\sum_i n_i \beta \sum_{j \neq i} n_j}{\alpha(n-1) + 2(n-1)^2} \leq \frac{2\beta((n - r)r + r^2 - r)}{\beta(n-1)r} = \frac{2\beta(n-1)}{\beta(n-1)} = 2$$

which proves the theorem for the star network as social optimum. □
6 Conclusions

In this paper we have extended a model for selfish network creation to allow for finite penalty values for disconnectivity. Our analysis of the resulting game and disconnected Nash equilibria brings up a number of interesting insights. All Nash (strong) equilibria of the NCG can be Nash (strong) equilibria for the penalized game under sufficiently large penalty values. Tree structures do almost never appear in disconnected Nash equilibria. There are cases in which the price of anarchy is $\Theta(n)$ and thus strictly higher than in the NCG. In contrast, the strong price of anarchy remains a constant and is at most 4. However, the increase for the price of anarchy is due to the existence of the empty network as a Nash equilibrium. Once we can exclude emptiness of the network, we conjecture that above a constant threshold for the penalty no disconnected non-empty Nash or strong equilibrium exists. This would mean that for all non-empty disconnected Nash equilibria the price of anarchy is similar to the NCG also bounded by $o(n^4)$. Proving or disproving this conjecture remains as an interesting open problem. In addition, it would be interesting to observe similar phenomena in models with different edge costs, e.g. given by hierarchical metrics as in [15]. In general, deriving a deeper understanding of the properties and structural characterizations of Nash and strong equilibria in network creation games like the PCG is an interesting research direction.

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