Estimating the finite-time ruin probability of a surplus with a long memory via Malliavin calculus

Shota Nakamura and Yasutaka Shimizu

Abstract

We consider a surplus process of drifted fractional Brownian motion with the Hurst index $H > 1/2$, which appears as a functional limit of drifted compound Poisson risk models with correlated claims. This is a kind of representation of a surplus with a long memory. Our interest is to construct confidence intervals of the ruin probability of the surplus when the volatility parameter is unknown. We will obtain the derivative of the ruin probability w.r.t. the volatility parameter via Malliavin calculus, and apply the delta method to identify the asymptotic distribution of an estimated ruin probability.

Keywords: Finite-time ruin probability; long memory surplus; fractional Brownian motion; Malliavin calculus.

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1 Introduction

In the classical ruin theory initiated by Lundberg [10], the insurance surplus is described by a drifted compound Poisson process such as

$$X_t = x + ct - \sum_{i=1}^{N_t} U_i$$

(1)

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where $x, c > 0$, $N$ is a Poisson process, and $U_i$’s are IID random variables with mean $\mu$, representing claim sizes. One of the direction to extend the model is the following drifted Lévy surplus $(R_t)_{0 < t < T}$:

$$R_t = u + dt + \sigma W_t - V_t,$$

where $u, d, \sigma > 0$, $W$ is a Brownian motion and $V$ is a Lévy subordinator. The model (2) is a natural extension of (1) and considers claim sizes with stationary independent increment. Statistical inference for ruin probability based on the model (2) has been studied by many authors; see, e.g., Asmussen and Albrecher [1], Shimizu [15] and the references therein. However, such an independent assumption is often unrealistic in a certain insurance contract because large claims can be successive once a large claim has occurred. Therefore, it would be better to assume that $(U_i)_{i \in \mathbb{N}}$ are correlated.

Michna [11] assumes that there exists a constant $\alpha \in (0, 1)$ and a slowly varying function $F$:

$$F(\mathbb{N}) \sim F(\mathbb{N}),$$

under which the process $X$ has a long memory: $\sum_{k=1}^{\infty} \text{Cov}(U_1, U_k) = \infty$. Considering a sequence of such a long memory surplus processes $X^n = (X^n_t)_{t \in [0, T]}$ indexed by $n = 1, 2, \ldots$:

$$X^n_t = x_n + c_n t - \sum_{i=0}^{N^n_t} U_i,$$

where $x_n, c_n$ are positive sequences, $(N^n_t)_{t \in [0, T]}$ is a Poisson process with the intensity $\lambda$ and $U_i$’s are correlated random variables as above. Then, according to Theorem 3 in Michna [11], there exists a norming sequence $(\eta_n)_{n \in \mathbb{N}}$ and some constants $u$ and $\theta$ such that the process $X^n/\eta_n$ converges weakly in a functional space $D[0, \infty)$, a space of càdlàg functions with the Skorokhod topology:

$$\frac{X^n_t}{\eta_n} \overset{d}{\to} u + \theta t - W^H_t \text{ in } D[0, \infty) \ (n \to \infty),$$

where $W^H$ is a fractional Brownian motion with the Hurst parameter $H \in (0, 1)$. In such a way, the surplus driven by a fractional Brownian motion naturally appears as a limit of a Poissonian model with a long memory.

Some earlier works model a surplus by fractional Brownian motions. Ji and Robert [8] model the surplus of insurance and reinsurance companies as the two-dimensional fractional Brownian motion and derives asymptotic of the ruin probability when the initial capital tends to infinity. Cai and Xiao [3] consider a drifted mixed fractional Brownian motion as a surplus model and estimate the ruin probability with an unknown drift parameter. In this paper, we are interested in the following drifted fractional Brownian motion as a surplus model:

$$X_t = u + \theta t - \sigma W^H_t,$$
where $\theta > 0$ and $H \in (\frac{1}{2}, 1)$ are known parameters and $\sigma > 0$ is an unknown parameter. Since our model is a normalized limit of a classical type surplus with a known premium rate $c_n$, the drift will be known under a suitable scaling. Therefore we assume $\theta$ is known although the scaling parameter $\sigma$ is unknown.

Our interest is to estimate the finite-time ruin probability: for any $T \in (0, \infty)$,

$$
\Psi_\sigma(u, T) := \mathbb{P}\left( \inf_{0 \leq t \leq T} X_t < 0 \right).
$$

from the past surplus data.

The paper is organized as follows: In Section 2, we prepare some notation and give a brief review of Malliavin calculus. In Section 3, we provide a result on estimating the volatility parameter $\sigma$ and the ruin probability by the delta method. In this procedure, the partial derivative $\frac{\partial}{\partial u} \Psi_\sigma(u, T)$ is required to obtain confidence intervals of the ruin probability, so we derive its explicit form using the integration by parts formula in Malliavin calculus in Section 4.

## 2 Preliminaries

### 2.1 Notation

We use the following notations.

- $A \prec B$ means that there exists a universal constant $c > 0$ such that $A \leq cB$.
- The partial derivative of the function $f$ at the point $x \in \mathbb{R}^d$ with respect to the $i$-th variable is denoted by $\partial_i f(x)$.
- Let $H_n(\cdot)$ denote the $n$-th order Hermite polynomial, which is defined by

$$
H_n(x) = \frac{(-1)^n}{n!} e^{-\frac{x^2}{2}} \frac{d^n}{dx^n} \left( e^{-\frac{x^2}{2}} \right), \quad (n \geq 1).
$$

- $\mathcal{D}(A)$: the Skorokhod space on the set $A \subset \mathbb{R}_+$.
- Let $C_\infty^\omega(\mathbb{R}^n)$ be the set of all infinitely continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $f$ and all of its partial derivatives are of polynomial growth.
- Denote by $C_\infty^\omega_b(\mathbb{R}^n)$ the set of all infinitely continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $f$ and all of its partial derivatives are bounded.
- For any $p > 1$ and $f, g \in L^p(\mathbb{R})$ we define the convolution $f * g$ as

$$
f * g(x) := \int_{\mathbb{R}} f(x - y) g(y) dy.
$$

- We denote the gamma function $\Gamma(\cdot)$ and the beta function $B(\cdot, \cdot)$ by
\[ \Gamma(x) = \int_0^\infty t^{x-1}e^t dt, \quad (x > 0) \]
\[ B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad (x, y > 0). \]

- We denote the left and right-sided fractional integrals \( I_{a^+}^\alpha f(\cdot) \) and derivatives \( D_{a^+}^\alpha f(\cdot) \) by

\[ I_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)dy, \]
\[ I_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y)dy, \]
\[ D_{a^+}^\alpha g(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{g(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{g(x) - g(y)}{(x-y)^{\alpha+1}}dy \right) , \]
\[ D_{b^-}^\alpha g(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{g(x)}{(b-x)^\alpha} + \alpha \int_x^b \frac{g(x) - g(y)}{(y-x)^{\alpha+1}}dy \right) , \]

for any \( 0 < \alpha < 1, x \in (a, b), f \in L^1(a, b) \) and \( g \in I_{a^+}^\alpha (L^p) \) (resp. \( g \in I_{b^-}^\alpha (L^p) \)) where \( p > 1 \) (see Samko et al. [14] for details).

### 2.2 Malliavin calculus

This section briefly introduces the Malliavin calculus based on Chapters 1 and 5 in Nualart [13]. In the sequel, we denote by \( G \) a real separable Hilbert space.

#### 2.2.1 Malliavin calculus on a real separable Hilbert space

**Definition 1.** We say that a stochastic process \((W_g)_{g \in G}\) is an isonormal Gaussian process associated with the real separable Hilbert space \( G \) if \( W \) is a centered Gaussian family of random variables such that \( \mathbb{E}[W_h W_g] = \langle h, g \rangle_G \) for any \( h, g \in G \).

In the following, we assume that the isonormal gaussian \( W(\cdot) \) is defined on the complete probability space \((\Omega, \mathcal{G}, \mathbb{P})\) where \( \mathcal{G} \) is the \( \sigma \)-algebra generated by \( W \) in this paper.

**Definition 2.** Let \( \sigma \)-algebra \( \mathcal{G} \) be the \( \sigma \)-algebra generated by an isonormal Gaussian \( W \). If a random variable \( F : \Omega \to \mathbb{R} \) satisfies

\[ F = f(W(g_1), \ldots, W(g_n)) \quad (f \in C^\infty_\gamma(\mathbb{R}), g_1, \ldots, g_n \in G), \]  \hspace{1cm} (5)

\( F \) is called a smooth random variable, and the set of all such random variables is denoted by \( S_G \).
Definition 3. The derivative of a smooth random variable $F$ of the form (5) is the $G$-valued random variable given by

$$D^G F = \sum_{i=1}^{n} \partial_i f (W(g_1), \ldots, W(g_n)) g_i.$$ 

Since the operator $D^G$ defined in Definition 3 is a closable operator, we can extend $D^G$ as a closed operator on $\mathbb{D}_G^{1,p} := \overline{S_G}||\cdot||_{1,p}$ where the seminorm $|| \cdot ||_{1,p}$ on $S_G$ is defined by

$$||F||_{1,p} := \left( \mathbb{E}[F^p] + \mathbb{E}[\|D^G F\|_{G,\Omega}^p] \right)^{\frac{1}{p}}$$

for any $p \geq 1$.

The above definitions can be extended to Hilbert-valued random variables. Consider the family $S_G(V)$ of $V$-valued smooth random variables of the form

$$F = \sum_{i=1}^{n} F_i v_i \quad (v_i \in V, F_i \in S_G),$$

Define $D^G F := \sum_{i=1}^{n} D^G F_i \otimes v_i$. Then $D^G$ is a closable operator from $S_G(V)$ into $L^p(\Omega; G \otimes V)$ for any $p \geq 1$. Therefore, $D^G$ is a closed operator on $\mathbb{D}_G^{1,p}(V) = \overline{S_G}||\cdot||_{1,p,V}$ for the seminorm $|| \cdot ||_{1,p}$ determined by

$$||F||_{1,p,V} := \left( \mathbb{E}[\|F\|_V^p] + \mathbb{E}[\|D^G F\|_{G,\Omega,V}^p] \right)^{\frac{1}{p}},$$

on $S_G(V)$. In particular, we define $\mathbb{D}_G^{1,\infty}$ and $\mathbb{D}_G^{1,\infty}(V)$ by

$$\mathbb{D}_G^{1,\infty} := \bigcap_{p=1}^{\infty} \mathbb{D}_G^{1,p}, \quad \mathbb{D}_G^{1,\infty}(V) := \bigcap_{p=1}^{\infty} \mathbb{D}_G^{1,p}(V).$$

The following proposition is the chain rule for $D^G$.

Proposition 1. Suppose that $F = (F^1, \ldots, F^m)$ is a random vector whose components belong to $\mathbb{D}_G^{1,\infty}$. Let $f \in C^{\infty}_p(\mathbb{R}^m)$. Then $f(F) \in \mathbb{D}_G^{1,\infty}$, and we have

$$D^G (f(F)) = \sum_{i=1}^{m} \partial_i f(F) D^G F^i.$$ 

Next, we consider the divergence operator.

Definition 4. The divergence operator $\delta^G$ is an unbounded operator on $L^2(\Omega; G)$ with values in $L^2(\Omega)$ such that:
(1) The domain of $\delta^G$, denoted by $\text{Dom}\, \delta$, is the set of stochastic processes $u \in L^2(\Omega; G)$ such that
\[ \mathbb{E}[\langle D^GF, u \rangle_G] \leq c(u)\|F\|_{L^2(\Omega)}, \]
for any $F \in \mathbb{D}^{1,2}_G$, where $c(u)$ is some constant depending on $u$.

(2) If $u$ belongs to $\text{Dom}\, \delta^G$, then $\delta^G(u)$ is characterized by
\[ \mathbb{E}[F\delta^G(u)] = \mathbb{E}[\langle D^GF, u \rangle_G], \]
for any $F \in \mathbb{D}^{1,2}_G$.

The following proposition allows us to factor out a scalar random variable in a divergence.

**Proposition 2.** Let $F \in \mathbb{D}^{1,2}_G$ and $u \in \text{Dom}\, \delta^G$ such that $Fu \in L^2(\Omega; G)$. Then $Fu \in \text{Dom}\, \delta^G$ and it follows that
\[ \delta^G(Fu) = F\delta^G(u) - \langle D^GF, u \rangle_G. \]

### 2.2.2 Malliavin calculus for the fractional Brownian motion

In this subsection, we introduce the fractional Brownian motion and the Hilbert space associated with the fractional Brownian motion.

**Definition 5.** A centered Gaussian process $(W^H_t)_{t \geq 0}$ is called fractional Brownian motion of Hurst index $H \in (0, 1)$ if it has the covariance function
\[ R_H(t, s) := \mathbb{E}[W^H_t W^H_s] = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}). \]

In this paper we will only use the fractional Brownian motions with Hurst index $H > \frac{1}{2}$. We denote by $\mathcal{E}$ the set of step functions on $[0, T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product
\[ \langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s), \]
which yields that, for any $\phi, \psi \in \mathcal{H}$,
\[ \langle \psi, \phi \rangle_{\mathcal{H}} := H(2H-1) \int_0^T \int_0^T |r-u|^{2H-2} \psi(r)\phi(u) \, dr \, du. \]

It is easy to see that the covariance of fractional Brownian motion can be written as
\[
\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T |r-u|^{2H-2} \, dr \, du
\]
\[ = R_H(t, s). \]
Therefore, fractional Brownian motion $W^H$ can be expressed as $W^H_t = W(1_{[0,1]})$ for the isonormal Gaussian $W$ associated with the Hilbert space $\mathcal{H}$. Consider the square integrable kernel

$$K_H(t, s) := c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

where $c_H = \left[ \frac{\Gamma(H+1)}{\Gamma(2H-H-1)} \right]^{\frac{1}{2}}$ and $t > s$. Define the isometric function $K_H^* : \mathcal{E} \to L^2(0, T)$ by

$$(K_H^* \phi)(s) := \int_s^T \phi(t) \frac{\partial K_H}{\partial t}(t, s) dt,$$

then the operator $K_H^*$ is an isometry between $\mathcal{E}$ and $L^2(0, T)$ that can be extended to the Hilbert space $\mathcal{H}$. The operator $K_H^*$ can be expressed in terms of fractional integrals:

$$(K_H^* \phi)(s) = c_H \Gamma(H-\frac{1}{2}) s^{\frac{1}{2}-H} (I_T^{H-\frac{3}{2}}.H-\frac{1}{2} \phi)(s).$$

Finally, we consider the Malliavin calculus on $\mathcal{H}$. For the sake of simplicity, we will use the notation $D^{W^H}, D^{1, p}_{W^H}$, and $\delta^{W^H}$ as the derivative operator, the domain of the derivative, and the divergence operator associated with the Hilbert space $\mathcal{H}$, respectively. In the sequel, we present two results on the derivative operator for fractional Brownian motion.

**Proposition 3.** For any $F \in D^{1, 2}_{W^H} = D^{1, 2}_{L^2(0, T)}$

$$K_H^* D^{W^H} F = D^{L^2([0, T])} F.$$

**Proposition 4.** $\sup_{0 \leq t \leq T} (W^H_t - \theta t)$ belongs to $D^{1, 2}_{W^H}$ and it holds $D^{W^H}_{t, \theta} \sup_{0 \leq t \leq T} (W^H_t - \theta t) = 1_{[0, \tau]}(t)$, for any $t \in [0, T]$, where $\tau$ is the point where the supremum is attained.

**Proof.** For the proof, see Lemma 3.2 in Florit and Nualart [5].

### 3 Statistical problems

Suppose that we have the past surplus data in $[0, T_0]$-interval at discrete time points $\frac{kT_0}{n}$ ($k = 0, 1, \ldots, n$). Our goal is to estimate the finite-time ruin probability for each $T \in (0, \infty]$ from the discrete data $(X_{\frac{kT_0}{n}})_{k \in \{0, 1, \ldots, n\}}$. 
3.1 Estimation of $\sigma$

Fix $0 < H < \frac{1}{2}$. We use the results of Corcuera and Nualart [4] to construct the estimator for the true value of $\sigma$ by using a power variations of the order $p > 0$. We define the power variation of $(X_t)$ as follows:

$$V^p_n(X)_t := \sum_{i=1}^{[nt]} |X_{\frac{i}{n}} - X_{\frac{i-1}{n}}|^p.$$ 

Defining the estimator $\hat{\sigma}_{k,t,n}$ of $\sigma_0$ as

$$\hat{\sigma}_{k,t,n} := \left( \frac{V^p_n(X)_t}{c_p n^{1-pH}} \right)^{\frac{1}{p}},$$

where $c_p = \frac{2^{\frac{p+1}{2}} \Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}$. We obtain the asymptotic normality of $\hat{\sigma}_{k,t,n}$ as in the following theorem.

**Theorem 1.** Let $p > 1$ and $0 < H < \frac{1}{2}$. Then

$$\sqrt{n} \left( (\hat{\sigma}_{k,t,n})^p - \sigma^p \right) \xrightarrow{L} \frac{\nu_1 \sigma^p}{c_p} W_{t^H} \quad (n \to \infty),$$

in law in the space $D([0,T])$ equipped with the Skorohod topology, where

$$\nu_1^2 = \mu_p + 2 \sum_{j \geq 1} \left( \gamma_p (\rho_H(j)) - \gamma_p(0) \right),$$

$$\mu_p = 2^p \left( \frac{1}{\sqrt{\pi}} \Gamma(p + \frac{1}{2}) - \frac{1}{\pi} \Gamma\left( \frac{p+1}{2} \right)^2 \right),$$

$$\gamma_p(x) = (1 - x^2)^{\frac{p+1}{2}} \sum_{k=0}^{\infty} \frac{(2x)^{2k}}{\pi(2k)!} \Gamma\left( \frac{p+1}{2} + k \right)^2,$$

$$\rho_H(n) = \frac{1}{2} \left( (n+1)^{2H} + (n-1)^{2H} - 2n^{2H} \right),$$

and $(W_{t})_{t \in [0,T]}$ is a Brownian motion independent of the fractional Brownian motion $W_{t^H}$.

**Proof.** It it sufficient to prove that

$$n^{-\frac{1}{2} + pH} V^p_n k_p (u + \sigma \theta t) \to 0 \quad (n \to \infty),$$

in probability, uniformly on $[0,T]$, by Corcuera and Nualart [4], p.727. Thus, we show that

...
\( n^{-\frac{1}{2}+pH} V_{k,p}^n (u + \sigma \theta t) = n^{-\frac{1}{2}+pH} \sum_{i=1}^{\lfloor nt \rfloor} \left| \Delta_k (u + \sigma \theta \frac{i-1}{n}) \right|^p \to 0 \quad \text{a.s.,} \quad (8) \)

as \( n \to \infty \), by induction on \( k \). Note that \((1-H)p > \frac{1}{2}\), when \( k = 1 \) we have

\[
\begin{align*}
&n^{-\frac{1}{2}+pH} \sum_{i=1}^{\lfloor nt \rfloor} \left| \Delta_1 (u + \sigma \theta \frac{i-1}{n}) \right|^p = n^{-\frac{1}{2}+pH} \sum_{i=1}^{\lfloor nt \rfloor} \left| \sigma \theta (\frac{i}{n} - \frac{i-1}{n}) \right|^p \\
&= n^{-\frac{1}{2}+pH} |\sigma \theta|^p \frac{\lfloor nt \rfloor}{n^p} \\
&\to 0,
\end{align*}
\]

as \( n \to \infty \). In the same way, we have

\[
\begin{align*}
n^{-\frac{1}{2}+pH} \sum_{i=2}^{\lfloor nt \rfloor+1} \left| \Delta_1 (u + \sigma \theta \frac{i-1}{n}) \right|^p &\to 0 \quad (n \to \infty).
\end{align*}
\]

Assuming

\[
\begin{align*}
&n^{-\frac{1}{2}+pH} \sum_{i=1}^{\lfloor nt \rfloor-k+2} \left| \Delta_{k-1} (u + \sigma \theta \frac{i-1}{n}) \right|^p \to 0 \quad (n \to \infty), \\
&n^{-\frac{1}{2}+pH} \sum_{i=1}^{\lfloor nt \rfloor-k+2} \left| \Delta_{k-1} (u + \sigma \theta \frac{i}{n}) \right|^p \to 0 \quad (n \to \infty),
\end{align*}
\]

holds, we get

\[
\begin{align*}
n^{-\frac{1}{2}+pH} \sum_{i=1}^{\lfloor nt \rfloor-k+1} \left| \Delta_k (u + \sigma \theta \frac{i-1}{n}) \right|^p \\
&\leq n^{-\frac{1}{2}+pH} \sum_{i=1}^{\lfloor nt \rfloor-k+2} \left\{ \left| \Delta_{k-1} (u + \sigma \theta \frac{i}{n}) \right|^p + \left| \Delta_{k-1} (u + \sigma \theta \frac{i-1}{n}) \right|^p \right\} \\
&\to 0,
\end{align*}
\]

as \( n \to \infty \) by \( \Delta_k X_{i-1} = \Delta_{k-1} X_i - \Delta_{k-1} X_{i-1} \). Therefore, since (8) holds for any \( k \in \mathbb{N} \), we get

\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} \left| n^{-\frac{1}{2}+pH} V_{k,p}^n (u + \sigma \theta t) \right| > \varepsilon \right) \leq \frac{1}{\varepsilon^2} \left( n^{-\frac{1}{2}+pH} V_{k,p}^n (u + \sigma \theta T) \right)^2 \to 0.
\]

for any \( \varepsilon > 0 \) and this proof is completed.
3.2 Simulation-based inference for $\Psi_\sigma(u, T)$

Using the estimator $\hat{\Psi}_{k,t,n}$ of $\sigma_0$ given in (6), we can estimate $\Psi_\sigma(u, T)$ by

$$\tilde{\Psi}_{k,t,n}(u, T) := \Psi_{\hat{\sigma}_{k,t,n}}(u, T),$$

and, due to the delta method(cf. van der Vaart [16] P.374), it follows that

$$\sqrt{n} \left( \tilde{\Psi}_{k,t,n}(u, T) - \Psi(u, T) \right) \to \frac{1}{P} \partial_\sigma \Psi_{\sigma_0}(u, T) \frac{1}{p} \sigma_0 \mathcal{W}_t \quad (n \to \infty),$$

in law in the space $\mathcal{D}([0, T])$ equipped with the Skorohod topology, if $\Psi_\sigma(u, T)$ is differentiable at the true volatility parameter $\sigma$. This leads us an $\alpha$-confidence interval for $\Psi_\sigma(u, T)$ such as

$$I_\alpha(\Psi) := \left[ \tilde{\Psi}_{k,T_0,n}(u, T) \pm \frac{z_\alpha/2}{\sqrt{n}} \left| \partial_\sigma \Psi_{\hat{\sigma}_{k,T_0,n}}(u, T) \right| \sqrt{T_0} \right]/p \epsilon_{k,p} \right].$$

(9)

where $[a \pm b]$ stands for the interval $[a - b, a + b]$ for $b > 0$, and $z_\alpha$ stands for the upper $\alpha$-quantile.

Now, the problem is to compute the following quantity;

$$\partial_\sigma^k \Psi_\sigma(u, T) = \left( \frac{\partial}{\partial \sigma} \right)^k \mathbb{P} \left( \inf_{0 \leq t \leq T} X_\sigma^T < 0 \right), \quad k = 0, 1. \quad (10)$$

- For $k = 0$: Since $\Psi_\sigma(u, T)$ does not have a closed expression, we will compute it by the Monte Carlo simulation for a given value of $\sigma$, that is, we generate sample paths of $X_t = u + \sigma \theta t - \sigma W_t^H$ for given $\sigma$, say $(X^{(k)})_{k=1,2,...,m}$ independent each other, observe

$$\tilde{\Psi}(u, T) = \frac{1}{m} \sum_{k=1}^m \mathbf{1}_{\{\tau^{(k)} \leq T\}}, \quad \tau^{(k)} := \inf\{t > 0 | X_t^{(k)} < 0\},$$

which goes to the true $\Psi_\sigma(u, T)$ almost surely as $m \to \infty$ by the strong law of large numbers. However, the event of ruin in $[0, T]$ is often very rare and the most of the indicators of summand will be zero, which will underestimate the true value $\Psi_\sigma(u, T)$. Changing the measure $\mathbb{P}$ into a suitable one, more efficient sampling procedure: importance sampling, will be proposed.

- For $k = 1$: Computing $\partial_\sigma \Psi_\sigma(u, T)$ is not straightforward because the integrand of the righthand side of (10) is not differentiable in $\sigma$, and we cannot differentiate it under the expectation sign $\mathbb{E}$. Moreover, computing numerically, e.g., for small $\epsilon > 0$,

$$\frac{\Psi_{\sigma+\epsilon}(u, T) - \Psi_\sigma(u, T)}{\epsilon} \quad \text{or} \quad \frac{\Psi_{\sigma+\epsilon}(u, T) - \Psi_{\sigma-\epsilon}(u, T)}{2\epsilon}, \quad (11)$$
we have to compute $\Psi_{\sigma+\epsilon}$ and $\Psi_{\sigma-\epsilon}(u,T)$ separately, which usually takes much time. In addition, since the accuracy of the calculation in (11) depends on $\epsilon$, the problem of determining the value of $\epsilon$ also arises. Importance sampling can give the fast convergence with the variance reduction.

4 Differentiability of $\Psi_\sigma$

Fix $H > \frac{1}{2}$. In this section, we discuss the differentiability of $\Psi_\sigma(u,T)$ with respect to $\sigma$ after the ideas of Lanjri and Nualart [9] or Gobet and Kohatsu-Higa [7].

**Theorem 2.** The finite-time ruin probability $\Psi_\sigma(u,T)$ is differentiable with respect to $\sigma$ and we have

$$\frac{\partial \sigma \Psi_\sigma(u,T)}{\partial \sigma} = \mathbb{E}\left[1_{\{\sigma \sup_{0 \leq t \leq T} (W_t^H - \theta_t) < u\}} \frac{u_A(\cdot)}{\sigma} \int_0^T \left(\sup_{0 \leq t \leq T} (W_t^H - \theta_t) - \theta(t)\right) dt \right]$$  \hspace{1cm} (12)

where

$$u_A(t) := \frac{d_H}{c_H \Gamma(H - \frac{1}{2})} t^{1-H} D_T^{H-\frac{1}{2}} \left(\cdot\right)^{2H-1} D_0^{H-\frac{1}{2}} \left(\cdot\right)^{1-H} \psi(Y_t) (t)$$  \hspace{1cm} (13)

$$= \frac{d_H}{c_H B \left(H - \frac{1}{2}, \frac{3}{2} - H\right) \Gamma \left(\frac{3}{2} - H\right)} t^{\frac{1}{2} - H} \cdot$$

$$\times \left\{ \left(1 - \frac{1}{2}\right)^{H-\frac{1}{2}} \left(\psi(Y_t) + \left(\frac{H}{2} - 1\right) t^{2H-1} \int_0^t \left(\frac{1}{2} s^{H-1} \psi(Y_s) - s^{1-H} \psi(Y_s)\right) ds\right) \right\}$$

$$+ \left(\frac{H}{2} - 1\right) t^{2H-1} \int_0^t \left(\frac{1}{2} s^{H-1} \psi(Y_s) - s^{1-H} \psi(Y_s)\right) ds\right\}$$

$$Y_t := 8 \left(4 \int_0^T \int_0^T \frac{|W_s^H - \theta_s - (W_u^H - \theta_u)|^r}{|s - u|^{m+2}} ds du\right)^{\frac{1}{2}} m + \frac{2}{m} t^{rac{1}{2} + \frac{1}{2}}$$  \hspace{1cm} (14)

for any even integers $r, m$ such that $rH > m + 2$ and $\psi \in C_0^m(\mathbb{R}^r)$ satisfies

$$\psi(x) = \begin{cases} 1 & (x \leq \frac{m}{2}) \\ 0 & (x \geq \frac{m}{2}) \end{cases}.$$

In the proof of (12), it suffices to show that
\[ \begin{align*}
\partial \sigma \mathbb{E} \left[ \phi(\sigma \sup_{0 \leq t \leq T} (W^H_t - \theta t)) \right] &= \mathbb{E} \left[ \phi \left( \sup_{0 \leq t \leq T} (W^H_t - \theta t) \right) \delta_{W^H_t} \left( \frac{\mu_A(\cdot)}{\sigma} \sup_{0 \leq t \leq T} (W^H_t - \theta t) \right) \int_0^T \psi(Y_t) dt \right] \\
&= \mathbb{E} \left[ \phi \left( \sup_{0 \leq t \leq T} (W^H_t - \theta t) \right) \delta_{W^H_t} \left( \frac{\mu_A(\cdot)}{\sigma} \sup_{0 \leq t \leq T} (W^H_t - \theta t) \right) \int_0^T \psi(Y_t) dt \right] \\
&= \mathbb{E} \left[ \phi \left( \sup_{0 \leq t \leq T} (W^H_t - \theta t) \right) \right]
\end{align*} \]

holds for a sufficiently smooth function \( \phi : \mathbb{R}^+ \to \mathbb{R} \) instead of \( 1_{(-\infty,0)}(\cdot) \) in \( \textbf{[12]} \) from the density argument. In the following, we impose the following conditions on \( \phi \):

1. \( \phi \in C^\infty_b(\mathbb{R}^+) \),
2. The function \( \phi \) is constant on \( [0,u] \).

We prove the following properties of \( Y_t \).

**Lemma 1.** For \( Y_t \) defined in \( \textbf{[14]} \), the following (1)–(3) hold.

1. \( |W^H_t - \theta t| \leq Y_t \) \( (t \in [0,T]) \).
2. \( \psi(Y_t) \in D^{1,\infty}_{W^H} (t \in [0,T]) \).
3. There exists a function \( \alpha : \mathbb{R} \to \mathbb{R}^+ \) with \( \lim_{q \to \infty} \alpha(q) = \infty \), such that, for any \( q \geq 1 \), one has: \( \forall t \in [0,T] \quad E[Y^q_t] \leq C_q t^{\alpha(q)} \).

4. \( \left( \int_0^T \psi(Y_t) dt \right)^{-1} \in L^p(\Omega) \quad (p \geq 1) \).

**Proof.** We can show (1) and (2) similarly as in the proof of Lemma 2.1 in Gobet and Kohatsu-Higa \( \textbf{[7]} \). (3) follows immediately from (2). Next, we prove (4). It is sufficient to show that

\[ \mathbb{P} \left( \int_0^T \sigma K_H(\tau, t) \Phi(Y_t) dt < \varepsilon \right) = O(\varepsilon^p) \quad (\varepsilon \downarrow 0), \]

holds from Nualart \( \textbf{[13]} \), p.133, Lemma 2.3.1. Since

\[ \int_0^T \psi(Y_t) dt = \int_0^{\varepsilon} \psi(Y_t) dt + \int_{\varepsilon}^T \psi(Y_t) dt \geq \frac{\varepsilon}{\sigma}, \]

holds on \( \left[ \frac{\mu}{2\sigma} > Y_\varepsilon \right] \), we obtain

\[ \left[ \sigma \int_0^T \psi(Y_t) dt < \varepsilon \right] \subset \left[ \frac{\mu}{2\sigma} \leq Y_\varepsilon \right]. \]

Therefore, for any \( q \geq 1 \) such that \( \alpha(q) \geq p \) we have
The proof is analogous to the proof of Gobet and Kohatsu-Higa [7]. Let
\[ A := [0 \leq \sigma \sup_{0 \leq t \leq T} (W^H_t - \theta t) \leq u]. \]
Since \( \phi'(\sigma \sup_{0 \leq t \leq T} (W^H_t - \theta t)) = 0 \) on \( A \) from the assumption of the function \( \phi \), we get
\[
\left( D^W_H \phi \left( \sigma \sup_{0 \leq t \leq T} (W^H_t - \theta t) \right) \right) \psi(Y_t) = \phi' \left( \sigma \sup_{0 \leq t \leq T} (W^H_t - \theta t) \right) \sigma 1_{\{s \leq \tau\}} \psi(Y_t) = 0 = \phi' \left( \sigma \sup_{0 \leq t \leq T} (W^H_t - \theta t) \right) \sigma \psi(Y_t),
\]
on \( A \). On the other hand, on \( A^c = [\sigma \sup_{0 \leq t \leq T} (W^H_t - \theta t) > u] \), for any \( \omega \in A^c \cap [\phi(Y_t) \neq 0] \) we have
\[
\sigma \sup_{0 \leq t \leq T} (W^H_t(\omega) - \theta t) > u, \quad Y_t(\omega) < \frac{u}{\sigma}.
\]
Therefore, we get
\[
Y_t(\omega) < \frac{u}{\sigma} < \sup_{0 \leq t \leq T} (W^H_t(\omega) - \theta t)) = W^H_t(\omega) - \theta \tau(\omega) \leq Y_\tau(\omega),
\]
so we have \( t \leq \tau \) since \( Y_t \) is a non-decreasing process.

In the sequel, we consider the smoothness of \( u_A \) in the Malliavin sense. Let \( \tilde{K}^*_H \) be the restriction of \( K^*_H \) to \( L^2(0, T) \), and let \( \tilde{K}^{*, adj}_H \) be the adjoint operator of \( \tilde{K}^*_H \) in \( L^2(0, T) \). Then, by Lanjri and Nualart [2], we have
\[
\tilde{K}^{*, adj}_{H}(\psi(Y))(t)
= d_H^{H - \frac{1}{2}} D_0^{H - \frac{1}{2}} t^H \psi(Y_t)
\]
\[ \frac{dH}{\Gamma\left(\frac{3}{2} - H\right)} \left( t^{\frac{1}{2} - H} \psi(Y_t) - \left( H - \frac{1}{2} \right) t^{H - \frac{1}{2}} \int_0^t t^{\frac{1}{2} - H} \psi(Y_s) - s^{\frac{1}{2} - H} \psi(Y_s) \frac{ds}{(t-s)^{H+\frac{1}{2}}} \right), \]

and

\[ \mathcal{K}_H^{-1}(u_t)(t) = \frac{1}{c_H \Gamma\left(\frac{H - \frac{1}{2}}{2}\right)} t^{\frac{1}{2} - H} D_T^{\frac{1}{2} - H} \left( \cdot \right)^{H - \frac{1}{2}} u(\cdot)(t) \]

\[ = \frac{1}{c_H B \left( H - \frac{1}{2}, \frac{3}{2} - H \right)} \left\{ \frac{u_t}{(T-t)^{H - \frac{1}{2}}} + \left( H - \frac{1}{2} \right) t^{\frac{1}{2} - H} \int_t^T s^{H - \frac{1}{2}} u_s - s^{H - \frac{1}{2}} u_t \frac{ds}{(s-t)^{H+\frac{1}{2}}} \right\}, \]

where \( d_H = \left( c_H \Gamma\left(H - \frac{1}{2}\right)\right)^{-1}. \) So we get \( u_A = \mathcal{K}_H^{-1} \circ \mathcal{K}_H^{s, adj-1}(\psi(Y)). \) Since \( \mathcal{K}_H^s : L^2(0,T) \rightarrow K_H^s[H^2(0,T)] \) is the isometric isomorphism, the following lemma holds.

**Lemma 2.** For any stochastic process \( u_t \), we have

\[ u_t \in \mathcal{D}_K^{1,p} \left( K_H^s[H^2(0,T)] \right) \Rightarrow \mathcal{K}_H^{-1}(u_t) \in \mathcal{D}_W^{1,p}(H). \]

**Proof.** When \( u_t = Fv \) holds for \( F \in \mathcal{D}_K^{1,p} \left( K_H^s[H^2(0,T)] \right) \) and \( v \in K_H^s[H^2(0,T)] \) \( \subset \) \( L^2(0,T) \), we have

\[ D_{W,H}^\mathcal{K}_H^{-1}(u_t) = D_{W,H}^\mathcal{K}_H^{-1}(F \mathcal{K}_H^{-1}(v)) \]

\[ = D_{W,H}^\mathcal{K}_H^{-1}(F) \mathcal{K}_H^{-1}(v) \]

\[ = \mathcal{K}_H^{-1}(D_{K_H^s[H^2(0,T)]}^\mathcal{K}_H^{-1}(F)) \mathcal{K}_H^{-1}(v) \]

\[ = \mathcal{K}_H^{-1}(\mathcal{K}_H^s[H^2(0,T)] F) \mathcal{K}_H^{-1}(v) \]

\[ = \mathcal{K}_H^{-1} \mathcal{K}_H^{-1}(D_{K_H^s[H^2(0,T)]}^\mathcal{K}_H^{-1}(F)) \mathcal{K}_H^{-1}(v) \]

\[ = \mathcal{K}_H^{-1} \mathcal{K}_H^{-1}(D_{K_H^s[H^2(0,T)]}^\mathcal{K}_H^{-1}(F)) \mathcal{K}_H^{-1}(v). \]

When \( u_t \in \mathcal{D}_K^{1,p} \left( K_H^s[H^2(0,T)] \right) \), there exists \( u_n \in \mathcal{S}_K^s[H^2(0,T)] \) such that \( u_n \rightarrow u \) \( \text{in} \ \mathcal{D}_K^{1,p} \left( K_H^s[H^2(0,T)] \right) \), and we have

\[ \mathcal{K}_H^{-1} u_n \rightarrow \mathcal{K}_H^{-1} u \ \text{in} \ L^p(\Omega; H), \]

since \( \mathcal{K}_H^s \) is the isometric isomorphism. Thus we have

\[ \left\| D_{W,H}^\mathcal{K}_H^{-1}(u_n) - \mathcal{K}_H^{-1} \mathcal{K}_H^{-1}(D_{W,H}^\mathcal{K}_H^{-1}(u)) \right\|_{L^p(\Omega; H^\otimes \mathcal{H})} \]

\[ = \left\| \mathcal{K}_H^{-1} \mathcal{K}_H^{-1}(D_{K_H^s[H^2(0,T)]}^\mathcal{K}_H^{-1}(u_n)) - D_{K_H^s[H^2(0,T)]}^\mathcal{K}_H^{-1}(u) \right\|_{L^p(\Omega; H^\otimes \mathcal{H})} \]
and so we obtain $K_H^{-1}(u) \in \mathbb{D}_W^{1,p}(\mathcal{H})$.

**Proposition 6.** For $u_A$ given in (13), it holds that

$$u_A \in \mathbb{D}_W^{1,\infty}(\mathcal{H}).$$

**Proof.** It is sufficient to show that $\int_0^1 \frac{t^{\frac{1}{4} - H} \psi(Y_t) - \frac{1}{4} - H \psi(Y_{t+1})}{(t-s)^{H + \frac{1}{2}}} ds \in \mathbb{D}_W^{1,\infty}(\tilde{K}_H (L^2(0,T)))$ from Lemma 2 and $\psi(Y_t) \in \mathbb{D}_W^{1,p}(\mathcal{H})$. Defining $a_n$ and $b_n$ by

$$a_n + b_n := \sum_{i=0}^{n-1} \frac{t^{\frac{1}{4} - H} \psi(Y_i) - \left( \frac{t}{2} + \frac{m}{2n} \right)^{\frac{1}{4} - H} \psi(Y_{i+1})}{(t - \left( \frac{t}{2} + \frac{m}{2n} \right)^{H + \frac{1}{2}}} \frac{t}{2n} + \sum_{i=0}^{n-1} \frac{t^{\frac{1}{4} - H} \psi(Y_i) - \left( \frac{m}{2n} \right)^{\frac{1}{4} - H} \psi(Y_{t+1})}{(t - \left( \frac{m}{2n} \right)^{H + \frac{1}{2}}} \frac{t}{2n},$$

then we can show

$$a_n \to \int_0^1 \frac{t^{\frac{1}{4} - H} \psi(Y_t) - \frac{1}{4} - H \psi(Y_{t+1})}{(t-s)^{H + \frac{1}{2}}} ds \text{ a.s.,}$$

$$b_n \to \int_0^1 \frac{t^{\frac{1}{4} - H} \psi(Y_t) - \left( \frac{m}{2n} \right)^{\frac{1}{4} - H} \psi(Y_{t+1})}{(t-s)^{H + \frac{1}{2}}} ds \text{ a.s.,}$$

as $n \to \infty$. We first show that

$$a_n \xrightarrow{L^p(\Omega)} \int_0^1 \frac{t^{\frac{1}{4} - H} \psi(Y_t) - \frac{1}{4} - H \psi(Y_{t+1})}{(t-s)^{H + \frac{1}{2}}} ds \text{ (n \to \infty),}$$

holds and that $D^{WH} a_n$ converges in $L^p(\Omega)$. Defining

$$A_T := 8 \left( 4 \int_0^T \int_0^T \frac{|W^H \theta s - (W^H - \theta u)|^m}{|s-u|^{m+2}} ds du \right)^{\frac{1}{m}} \frac{m+2}{m},$$

we can evaluate as

$$|a_n| \leq \sum_{i=0}^{n-1} \frac{t^{\frac{1}{4} - H} \psi(Y_i) - \left( \frac{t}{2} + \frac{m}{2n} \right)^{\frac{1}{4} - H} \psi(Y_{i+1})}{(t - \left( \frac{t}{2} + \frac{m}{2n} \right)^{H + \frac{1}{2}}} \frac{t}{2n}.$$
Thus we get

\[
\sum_{i=0}^{n-1} \frac{\left(\frac{1}{2} + \frac{\mu}{2n}\right)^{\frac{1}{2}} \psi(Y_t) - \left(\frac{1}{2} + \frac{\mu}{2n}\right)^{\frac{1}{2}} \psi(Y_{t+i + \frac{\mu}{2n}})}{(t - \left(\frac{1}{2} + \frac{\mu}{2n}\right))^{H + \frac{1}{2}}} \frac{1}{2n}\]

\[
\leq \sum_{i=0}^{n-1} \frac{\left(\frac{1}{2} \right)^{\frac{1}{2}} \psi(Y_t) - \left(\frac{1}{2} \right)^{\frac{1}{2}} \psi(Y_{t+i + \frac{\mu}{2n}})}{(t - \left(\frac{1}{2} + \frac{\mu}{2n}\right))^{H + \frac{1}{2}}} \frac{1}{2n}\]

\[
\leq \sum_{i=0}^{n-1} \frac{\psi(Y_t) - \psi(Y_{t+i + \frac{\mu}{2n}})}{(t - \left(\frac{1}{2} + \frac{\mu}{2n}\right))^{H + \frac{1}{2}}} \frac{1}{2n}\]

\[
= f_n^1(t) + f_n^2(t) .
\]

Note that \((f_n^1(t))^P \rightarrow L^p(\Omega) \int_{\frac{1}{2}}^{t} \frac{\psi(Y_t)}{(t-s)^{H + \frac{1}{2}}} ds \bigg)^P .\)

Indeed, we have

\[
\left\| f_n^1(t) - \int_{\frac{1}{2}}^{t} \frac{\psi(Y_t)}{(t-s)^{H + \frac{1}{2}}} ds \right\|_{L^p(\Omega)} \leq \|\psi(Y_t)\|_{L^p(\Omega)} \sum_{i=0}^{n-1} \frac{1}{(t - \left(\frac{1}{2} + \frac{\mu}{2n}\right))^{H + \frac{1}{2}}} \frac{1}{2n} - \int_{\frac{1}{2}}^{t} \frac{1}{(t-s)^{H + \frac{1}{2}}} ds \rightarrow 0 \ (n \rightarrow \infty).
\]

Similarly, \((f_n^2(t))^P \rightarrow L^p(\Omega) \int_{\frac{1}{2}}^{t} \frac{\psi(Y_t)}{(t-s)^{H + \frac{1}{2}}} ds \bigg)^P\) holds, so we obtain

\[
a_n \rightarrow \int_{\frac{1}{2}}^{t} \frac{\psi(Y_t) - \psi(Y_s)}{(t-s)^{H + \frac{1}{2}}} ds \quad (n \rightarrow \infty) .
\]

(16)

Also, since

\[
D^{\frac{1}{2}} a_n = \sum_{i=0}^{n-1} D^{\frac{1}{2}} (\frac{1}{2} + \frac{\mu}{2n})^{\frac{1}{2}} \psi(Y_t - \frac{1}{2} + \frac{\mu}{2n}) \left(\frac{1}{2} + \frac{\mu}{2n}\right)^{\frac{1}{2}} \psi(Y_{t+i + \frac{\mu}{2n}}) - \left(\frac{1}{2} + \frac{\mu}{2n}\right)^{\frac{1}{2}} \psi(Y_{t+i + \frac{\mu}{2n}}) \frac{1}{2n}
\]

we have

\[
\bigg| D^{\frac{1}{2}} a_n \bigg| \leq \sum_{i=0}^{n-1} D^{\frac{1}{2}} (\frac{1}{2} + \frac{\mu}{2n})^{\frac{1}{2}} \psi(Y_t) - \left(\frac{1}{2} + \frac{\mu}{2n}\right)^{\frac{1}{2}} \psi(Y_{t+i + \frac{\mu}{2n}}) \frac{1}{2n} \bigg| + \sum_{i=0}^{n-1} D^{\frac{1}{2}} (\frac{1}{2} + \frac{\mu}{2n})^{\frac{1}{2}} \psi(Y_{t+i + \frac{\mu}{2n}}) \frac{1}{2n} \bigg| + \sum_{i=0}^{n-1} \left(\frac{1}{2} + \frac{\mu}{2n}\right)^{\frac{1}{2}} \psi(Y_t) - \left(\frac{1}{2} + \frac{\mu}{2n}\right)^{\frac{1}{2}} \psi(Y_{t+i + \frac{\mu}{2n}}) \frac{1}{2n} \bigg|.
\]

Thus we get
in the same way as in (16). Second, we show that

\[
\begin{align*}
D^W H a_n \xrightarrow{L^p(\Omega)} & \int_0^t \frac{D^W H A_T \left( t^{\frac{\beta+1}{2}} H \psi(Y_t) - s^{\frac{\beta+1}{2}} H \psi(Y_s) \right)}{(t-s)^{H+\frac{1}{2}}} \, ds \quad (n \to \infty)
\end{align*}
\]

holds and that \(D^W H b_n\) converges in \(L^p(\Omega)\). Since it holds that

\[
|b_n| \leq \left( \frac{2}{t} \right)^{H+\frac{1}{2}} \left\{ \sum_{i=0}^{n-1} t^{\frac{\beta+1}{2}} H \psi(Y_{t_{i+1}}) - \left( \int \frac{it}{2n} \frac{\beta+1+H}{\beta+H} \psi(Y_{t_{i+1}}) \right) \right\} \frac{t}{2n}
\]

\[
\leq \left( \frac{2}{t} \right)^{H+\frac{1}{2}} \left\{ \sum_{i=0}^{n-1} \left( \int \frac{it}{2n} \frac{\beta+1+H}{\beta+H} A_T \right) \right\}
\]

\[
= \left( \frac{2}{t} \right)^{H+\frac{1}{2}} f_3(t), \quad (17)
\]

we have

\[
\left\| f_3(t) - \int_0^t \left( t^{\frac{\beta+1}{2}} H \psi(Y_t) - s^{\frac{\beta+1}{2}} H \psi(Y_s) \right) \right\|_{L^p(\Omega)}
\]

\[
\leq \| A_T \|_{L^p(\Omega)} \left\{ \sum_{i=0}^{n-1} \left( \int \frac{it}{2n} \frac{\beta+1+H}{\beta+H} \right) \right\}
\]

\[
\to 0 \quad (n \to \infty).
\]

Thus we have \(b_n \xrightarrow{L^p(\Omega)} \int_0^t \frac{t^{\frac{\beta+1}{2}} H \psi(Y_t) - s^{\frac{\beta+1}{2}} H \psi(Y_s)}{(t-s)^{H+\frac{1}{2}}} \, ds\). From (17), we obtain

\[
D^W H b_n \xrightarrow{L^p(\Omega)} \int_0^t \frac{D^W H A_T \left( t^{\frac{\beta+1}{2}} H \psi(Y_t) - s^{\frac{\beta+1}{2}} H \psi(Y_s) \right)}{(t-s)^{H+\frac{1}{2}}} \, ds \quad (n \to \infty)
\]

in the same way as for \(a_n\). Therefore, since \(D^W H\) is a closable operator, we have

\[
\int_0^t \frac{t^{\frac{\beta+1}{2}} H \psi(Y_t) - s^{\frac{\beta+1}{2}} H \psi(Y_s)}{(t-s)^{H+\frac{1}{2}}} \, ds \in D^{1,\infty}_{W^H}.
\]

Proof (Proof of Theorem 2). From Proposition 1 we have

\[
\phi'(\sigma \sup_{0 \leq t \leq T} (W^H_t - \theta t)) \sigma \left( 1_{[0,\tau]} , u_A \right) \mathcal{H}
\]
\[
\begin{align*}
&= \phi'(\sigma \sup_{0 \leq t \leq T} (W_t^H - \theta t))\sigma \left(\mathcal{K}^*_H(1_{[0,\tau]}), \mathcal{K}^*_H (\psi(Y))\right)_{L^2(0,T)} \\
&= \phi'(\sigma \sup_{0 \leq t \leq T} (W_t^H - \theta t))\sigma \left(1_{[0,\tau]}, \psi(Y)\right)_{L^2(0,T)} \\
&= \phi'(\sigma \sup_{0 \leq t \leq T} (W_t^H - \theta t))\sigma \int_0^T \psi(Y_t)dt.
\end{align*}
\]

Thus, since it holds that
\[
\frac{\partial}{\partial \sigma} \mathbb{E}\left[\phi\left(\sigma \sup_{0 \leq t \leq T} (W_t^H - \theta t)\right)\right] = \mathbb{E}\left[\phi'(\sigma \sup_{0 \leq t \leq T} (W_t^H - \theta t)) \sup_{0 \leq t \leq T} (W_t^H - \theta t) \psi(Y_t)Y_t dt\right] \\
= \mathbb{E}\left[\phi'(\sigma \sup_{0 \leq t \leq T} (W_t^H - \theta t)) \psi(Y_t)\right] \int_0^T \psi(Y_t)dt \\
= \mathbb{E}\left[D^{\psi H}_T (\phi(\sigma \sup_{0 \leq t \leq T} (W_t^H - \theta t))), \rho_\psi (\sigma \int_0^T \psi(Y_t)dt)\right]_{\mathcal{H}},
\]

and we have \(u_A(\cdot) \sup_{0 \leq t \leq T} (W_t^H - \theta t) \in \text{Dom}^{\psi H}_T\), we get (12). Next we show that (12) holds. Defining \(g_n : \mathbb{R}_+ \to \mathbb{R}\) by
\[
g_n := 1_{[u+\frac{1}{n}, u+n+\frac{1}{n}]} \ast \rho_\psi (n \geq 2, \rho : \text{molifier}),
\]
then we have
1. \(g_n \in C^0_0(\mathbb{R})\),
2. \(g_n(x) = 0\) on \([0, u]\).

Also we get \(g_n \to 1_{(u, \infty)}\) as \(n \to \infty\) and
\[
f_n(\sigma) := \mathbb{E}[g_n(\sigma \sup_{0 \leq t \leq T} (W_t^H - \theta t))] \\
\to \mathbb{E}[1_{(u, \infty)}(\sigma \sup_{0 \leq t \leq T} (W_t^H - \theta t))] \\
= \mathbb{E}[1_{[u, \infty)}(\sigma \sup_{0 \leq t \leq T} (W_t^H - \theta t))].
\]

Therefore, for any compact set \(K \subset \mathbb{R}_+\), we have
$$\sup_{\sigma \in K} \left( \frac{\partial}{\partial \sigma} f_n(\sigma) - \mathbb{E} \left[ 1_{[u, \infty)}(\sigma \sup_{0 \leq t \leq T} (W_t^H - \theta t)) \delta W^H \left( \frac{u_A(\cdot) \sup_{0 \leq t \leq T} (W_t^H - \theta t)}{\int_0^T \sigma \psi(Y_r) dr} \right) \right] \right)$$

$$\leq \inf_{\sigma \in K} \sup_{\sigma \in K} \mathbb{E} \left[ g_n(\sigma V_r^\ast) \right] \frac{1}{2}$$

$$\leq \sup_{\sigma \in K} \left\{ \mathbb{P} (\sigma V_r^\ast \in [u, u + \frac{2}{n}]) + \mathbb{P} (\sigma V_r^\ast \in [u + n, \infty)) \right\}$$

$$=: \sup_{\sigma \in K} \left\{ h_n^1(\sigma) + h_n^2(\sigma) \right\} .$$

(18)

In considering the compact uniform convergence of $h_n^1(\sigma)$ and $h_n^2(\sigma)$ with respect to $\sigma$, it is sufficient to show only the continuity of $h_n^1(\sigma)$ and $h_n^2(\sigma)$ with respect to $\sigma$ by the Dini theorem. The problem here is that when $\sigma$ changes, the interval $[\frac{u}{\sigma}, \frac{u + \frac{2}{n}}{\sigma}]$ also moves, so the continuity of the measure $\mathbb{P}$ cannot be exploited. Therefore, for any $(\sigma_m) \subset K$ such that $\sigma_m \downarrow \sigma$ as $m \to \infty$, we shall show that for any $m \in \mathbb{N}$ there exists a fixed point that in $[\frac{u}{\sigma_m}, \frac{u + \frac{2}{n}}{\sigma_m}]$. For $(\sigma_m) \in K$ such that $\sigma_m \downarrow \sigma$ as $m \to \infty$, take $\varepsilon > 0$ satisfying $\varepsilon < \frac{2}{2u - \frac{2}{n}}$ and $N \in \mathbb{N}$ large enough to satisfy $\left| \frac{1}{\sigma_N} - \frac{1}{\sigma_m} \right| < \varepsilon$.

Then, it holds that

$$\frac{u}{\sigma_N} < u \left( \frac{1}{\sigma} + \varepsilon \right) < \left( u + \frac{2}{n} \right) \left( \frac{1}{\sigma} - \varepsilon \right) < \frac{u + \frac{2}{n}}{\sigma_N} .$$

from $\left( 2u + \frac{2}{n} \right) \varepsilon < \frac{2}{2u - \frac{2}{n}}$ and $\left| \frac{1}{\sigma_N} - \frac{1}{\sigma_m} \right| < \varepsilon$. Therefore, we can show that $h_n^1(\sigma)$ is right-continuous with respect to $\sigma$ because we can take $a := u \left( \frac{1}{\sigma} + \varepsilon \right)$ independent of $N \in \mathbb{N}$ such that $\frac{u}{\sigma_N} < a < \frac{u + \frac{2}{n}}{\sigma_N}$. In the same way, we can show the case of $\sigma_m \uparrow \sigma$ as $m \to \infty$, so we obtain the continuity of $h_n^1(\sigma)$ and $h_n^2(\sigma)$. Thus, [13] converges to 0 as $n \to \infty$, and this proof is complete.

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