Feedback-Induced Flutter Oscillations in a Flexible Tail-Like Appendage for Underwater Propulsion

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Abstract

Oscillating propulsors can provide both propulsive and maneuvering forces to an underwater vehicle. Because the working area is the same, it is possible to provide maneuvering forces that are similar in magnitude to that of the propulsive forces. An oscillating propulsor in the form of a flexible beam is investigated; the flexible beam is appended to an underwater vehicle by a pin joint and actuated by a motor. It is shown that the flexible propulsor can be driven into post-flutter limit cycle oscillations using feedback of the state of the propulsor. The limit cycle oscillations result in traveling waves, which in turn generate propulsive forces. The elastic strain of the propulsor is used to provide feedback; it is shown that changing the location of strain measurement can result in a rich set of stability transitions, each stability transition is associated with a specific mode of flutter-based propulsion with unique thrust and efficiency characteristics.
1 Introduction

Oscillatory modes of propulsion are universally observed in biological systems. Although this has been credited to animals’ “inability to evolve a propeller”, we can more charitably say that natural systems are frugal, and that the ability of these fins, wings, and tails to provide both propulsion and control forces explains their dominance in Nature. Besides reducing the number of subsystems, using the same lifting surface for propulsion and control means that the control forces can have magnitude similar to the propulsive forces, yielding a far more agile platform.

Befitting the large range of aerodynamic and hydrodynamic requirements faced by living creatures, the mechanisms of oscillatory propulsion and control vary widely [1,2]. Among fish swimming with whole-body motions, we observe variation in the degree of forebody motion. Eels’ heads move with an amplitude approximately the same as the tip of the tail, such that an analysis of their propulsion and control necessarily requires knowledge of the entire body’s waveform. In contrast, the fast-swimming scombrids (e.g., tuna, mackerel) limit large amplitudes of motion to the area near the caudal fin. This thunniform motion permits a hydrodynamic analysis which is substantially focused on the caudal fin as a discrete lifting surface.

A number of robotic platforms have been built, which produce propulsion using oscillatory means rather than propellers. Fully-actuated strategies based on the faithful reproduction of biological waveforms have been built [3], for example, as have flexible mechanisms which are substantially underactuated. The latter are particularly interesting, since they allow the storage of elastic energy during the propulsive stroke. Continuous flexible actuators [4], for example, can be driven near their natural frequency to produce a highly efficient drivetrain. It is also possible to drive a compliant system beyond its point of linear stability, so that it produces a limit cycle oscillation, as demonstrated by the fluid-conveying propulsor in [5].

Limit cycle oscillations, originally identified in electrical circuits, appear in diverse nonlinear systems. Hydrodynamic limit cycle oscillations have been used to model vortex-induced vibrations [6] and wake coupling on flapping fins [7]. The production of stable mechanical limit cycle oscillations in a fluid-loaded system is not difficult in itself; flapping flags, aeolian harps, and torsionally fluttering wings will do so when exposed to sufficient flow. Each of the aforementioned are however drag-producing, and extract kinetic energy from the flowing fluid.

Efficient propulsion and control requires that momentum be added to the fluid in the direction of the desired force. There are a number of ways to predict the kinematic conditions under which efficient force production occurs. The Strouhal number [8] and quasi-steady relationships [9] have been used for rolling and pitching fins, and slender-body theory [10,11] is a simple way to predict positive thrust for continuous bodies. Continuous flexible elements require dynamic modeling at the outset of the design, and a significant amount of work has been performed to understand the production and control of traveling waveforms produced by fluid-loaded flexible beams.

Feedback between the state of the continuous body and its terminal end can be used to generate flutter instability [12] and earlier works [5,11] have demonstrated that flutter oscillations in a fluid environment can generate an efficient traveling waveform. The salient features of both [12] and [5,11] are exploited to generate flutter instability through feedback. Similar to [5,11], we consider a flexible beam pinned to a rigid body but a motor at the pinned joint provides the actuation forces instead of internal flow. The motor provides a torque proportional to the strain in
the beam, but unlike the motor is located at the base and not at the terminal end. The physical system is described in section 2 along with a set of assumptions that simplify the mathematical model. The mathematical model is presented in section 3 and the method of solution is discussed in section 4. Flutter investigations are carried out in section 5 along with illustrations of stability transitions between different modes of flutter that occur for a specific set of system parameters. The thrust and efficiency characteristics of the underwater vehicle associated with the different modes of flutter-based propulsion are discussed in section 6. Concluding remarks are provided in section 7.

2 System Description and Assumptions

We consider a submersible comprised of a rigid body and an appendicular flexible beam immersed in a quiescent fluid - see Fig.1. The center-of-mass of the rigid body is constrained to translate along a channel and the beam is connected to one end of the rigid body by a revolute joint. The submersible is propelled by oscillatory motion of the flexible beam generated by motion of the revolute joint, which is actively controlled. To investigate the dynamics of the submersible, we make several assumptions.

A1. The submersible is neutrally buoyant in a fluid of density $\rho_f$ and its motion is confined to the horizontal plane. We define the reference frame $XY$ in which the $X$ axis is aligned with the channel and the origin is located at the projection of the revolute joint on the $X$ axis; this implies that the $XY$ frame translates along the channel. The revolute joint of the submersible translates with constant velocity $U_e$ along the negative $X$ axis in the quiescent fluid; the $XY$ frame is therefore an inertial reference frame and $U_e$ denotes the external flow relative to this reference frame.

Figure 1: A submersible comprised of a rigid body and an appendicular flexible beam, immersed in a quiescent fluid. The submersible moves with constant velocity $U_e$; therefore $U_e$ is the velocity of the fluid relative to the submersible.
A2. The $x_1y_1$ frame is fixed to the rigid body and is located at its center-of-mass. The orientation of the rigid body ($x_1y_1$ frame) relative to the $XY$ frame is denoted by $\alpha$, where $\alpha$ is measured positive about the vertical axis. The $x_2y_2$ frame is located at the revolute joint and fixed to the beam; the $x_2$ axis is aligned with the slope of the beam at the revolute joint. The orientation of the $x_2y_2$ frame relative to the $x_1y_1$ frame is the angle of rotation of the revolute joint; this angle, denoted by $\delta$, is measured positive about the vertical axis and is assumed to be small. A rotational spring with stiffness $K$ acts with equal and opposite restoring moments $K\delta$ on the rigid body and the flexible beam at the revolute joint.

A3. The rigid body of the submersible is symmetric about the vertical plane that contains the $x_1$ axis. It is assumed to have mass moment of inertia $J$ about its center-of-mass, which is located at a distance $\ell$ from the revolute joint.

A4. The flexible beam satisfies Euler-Bernoulli assumptions. It has length $L$ and a rectangular cross-section with flexural rigidity $EI$, where $E$ and $I$ are the Young’s modulus and area moment of inertia, respectively. The fluid volume associated with the motion of the flexible beam yields an added mass; this fluid is assumed to be non-vortical [13].

A5. The transverse displacement of a point on the beam at a distance $x$ from the revolute joint is denoted by $y(x,t)$ in the $XY$ frame. Therefore, the revolute joint has a transverse displacement of $y_0 \triangleq y(0,t)$. The slope of the beam is denoted by $\theta(x,t) \triangleq \partial y(x,t)/\partial x$. The slope of the beam at $x = 0$, which is denoted by $\theta_0 \triangleq \theta(0,t)$, is small in accordance with Euler-Bernoulli assumptions. It can be verified that $\theta_0 \triangleq (\alpha + \delta)$ and since $\theta_0$ and $\delta$ are both small angles, $\alpha$ is small.

3 Dynamic Model

3.1 Rigid Body Dynamics

The free-body diagrams of the rigid body and the flexible beam are shown in Fig.2. Following Euler-Bernoulli beam conventions, the reaction forces at the hinge are assumed to be $F$ and $S$ along
the $X$ and $Y$ axes; the reaction moment about the vertical axis is assumed to be $\mathbb{M}$. Summing the
forces in the $X$ and $Y$ directions and the moment about the center-of-mass of the rigid body, we
get the following equations:

\[
\begin{align*}
- F + D &= 0 \quad (1a) \\
- S + R &= 0 \quad (1b) \\
\mathbb{M} - \ell (S \cos \alpha - F \sin \alpha) + K\delta &= J\ddot{\alpha} \quad (1c)
\end{align*}
\]

where $D$ is the drag force on the rigid body, which acts in the positive $X$ direction; and $R$ is the
reaction force of the channel on the rigid body, which acts in the positive $Y$ direction. In (1a), the
right hand side is zero since the submersible translates with constant velocity along the $X$
direction. Substituting $\cos \alpha \approx 1$ and ignoring $F \sin \alpha \approx F_0$ for model simplification, we get $F = D$, $S = R$, and

\[
\mathbb{M} - \ell S = J\ddot{\alpha} - K\delta = J\ddot{\alpha} - K(\theta_0 - \alpha) = J\ddot{\alpha} + K\alpha - K\theta_0 \quad (2)
\]

We now make the following assumption in regards to the bending moment $\mathbb{M}$:

**A6.** An actuator located at the revolute joint applies a bending moment that is proportional to
the curvature of the beam at $x = \hat{x}$ - see Fig.1. Therefore,

\[
\mathbb{M} = C \frac{\partial^2 y}{\partial x^2} \bigg|_{x=\hat{x}} \quad (3)
\]

where $C$ is the constant of proportionality.

From geometry, the displacement of the revolute joint can be expressed as

\[
y_0 = \ell \sin \alpha \approx \ell \alpha \quad \Rightarrow \quad \alpha = \frac{1}{\ell} y(0, t) \quad (4)
\]

The above relationship will be useful in simplifying the boundary conditions of the flexible beam.

### 3.2 Fluid-Immersed Beam Dynamics

The transverse vibration of the flexible beam in Fig.[1] is identical to that of a beam in axial flow [14][15]:

\[
EI \frac{\partial^4 y(x, t)}{\partial x^4} + M_c U_e^2 \frac{\partial^2 y(x, t)}{\partial x^2} + 2M_c U_e \frac{\partial^3 y(x, t)}{\partial x \partial t} + (M_c + \rho A) \frac{\partial^2 y(x, t)}{\partial t^2} = 0 \quad (5)
\]

where $E$, $I$, $A$ and $\rho$ are the Young’s modulus of elasticity, area moment of inertia, cross-sectional
area and the mass per unit volume of the beam, and $M_c$ is the mass per unit length (added mass)
of the surrounding fluid [16]. For thin cross sections, such as that of the flexible beam, the added

\[\text{\footnote{Although $\alpha$ is small, $F_0$ is not negligible since $F$ is the thrust produced by the flexible beam.}}\]
mass is equal to the mass of water within the cylinder which circumscribes the beam cross-section. Therefore

$$M_e = \frac{1}{4} \rho_f \pi h^2$$

where $h$ is the height of the beam. The boundary conditions at $x = 0$ are obtained from (2) and (3); using (4), they can be written as

$$EI \frac{\partial^2 y}{\partial x^2}\bigg|_{x=0} = C \frac{\partial^2 y}{\partial x^2}\bigg|_{x=\hat{x}}$$

$$EI \frac{\partial^3 y}{\partial x^3}\bigg|_{x=0} = \frac{C}{\ell} \frac{\partial^2 y}{\partial x^2}\bigg|_{x=\hat{x}} - \frac{1}{\ell} \left[ J \frac{\partial^2 y}{\partial t^2} + Ky - K \frac{\partial y}{\partial x} \right]_{x=0}$$

(6a)

(6b)

The boundary conditions at $x = L$ are

$$EI \frac{\partial^2 y}{\partial x^2}\bigg|_{x=L} = EI \frac{\partial^3 y}{\partial x^3}\bigg|_{x=L} = 0$$

(7)

Using the following change of variables

$$u = \frac{x}{L}, \quad v = \frac{y}{L}, \quad \tau = t \sqrt{\frac{EI}{\rho A L^4}}, \quad u_e = \sqrt{\frac{M_e}{EI}} U_e L, \quad \beta = \frac{M_e}{M_e + \rho A}$$

we get the non-dimensional equation of motion

$$v'''(u, \tau) + u^2_e v''(u, \tau) + 2u^2_e \sqrt{\beta} \ddot{v}'(u, \tau) + \ddot{v}(u, \tau) = 0$$

(8)

where $v'$ and $\dot{v}$ denote the partial derivatives of $v(u, \tau)$ with respect to $u$ and $\tau$, respectively. The non-dimensional boundary conditions can be expressed as

$$v''(0, \tau) = c v''(\gamma, \tau)$$

$$v'''(0, \tau) = c \lambda v''(\gamma, \tau) - \beta_j \ddot{v}(0, \tau) - k\lambda v'(0, \tau) + k\lambda v'(0, \tau)$$

$$v''(1, \tau) = v'''(1, \tau) = 0$$

(9)

where

$$\gamma = \frac{\hat{x}}{L}, \quad \lambda = \frac{L}{\ell}, \quad c = \frac{C}{EI}, \quad k = \frac{KL}{EI}, \quad \beta_j = \frac{J}{(\rho A + M_e)\ell^2}$$

The non-dimensional parameters have the following physical meanings: $\gamma \in [0, 1.0]$ denotes the location of curvature measurement for application of bending moment and $c$ is the feedback gain, $\lambda$ is the ratio of the length of the flexible beam (tail-like appendage) to the length of the rigid body of the submersible, $k$ is the non-dimensional stiffness of the rotational spring at the revolute joint, and $\beta_j$ is the non-dimensional mass moment of inertia of the rigid body.
4 Method of Solution

To solve (8) for the boundary conditions in (9), we followed the procedure introduced in [14] and used in [15]. In particular, we assume the following separable form for $v(u, \tau)$:

$$v(u, \tau) = f(u) g(\tau); \quad g(\tau) = e^{i\Omega \tau}$$

(10)

where $\Omega$ is the nondimensional frequency of oscillation. Substitution of (10) into (8) and (9) yields

$$f'''(u) + u_e^2 f''(u) + 2u_e^2 \sqrt{\beta} i\Omega f'(u) - \Omega^2 f(u) = 0$$

(11)

$$f''(0) = c f''(\gamma)$$

$$f''(0) = c \lambda f''(\gamma) + \beta_j \Omega^2 f(0) - (k)\lambda^2 f(0) + k\lambda f'(0)$$

$$f''(1) = f'''(1) = 0$$

(12)

Since (11) is an ordinary differential equation with constant coefficients, the solution of $f(u)$ is assumed to be of the form $f(u) = A e^{z_u}$. Accordingly, Eq. (11) results in the characteristic equation

$$z^4 + u_e^2 z^2 + 2u_e^2 \sqrt{\beta} i\Omega z - \Omega^2 = 0$$

(13)

For specific values of $u_e$ and $\beta$, (13) provides four roots of $z_n, n = 1, 2, 3, 4$, which are functions of $\gamma, \lambda, \beta_j, c, k$ and $\Omega$. The solution of $f(u)$ takes the form

$$f(u) = A_1 e^{z_1 u} + A_2 e^{z_2 u} + A_3 e^{z_3 u} + A_4 e^{z_4 u}$$

(14)

Substitution of the boundary conditions in Eq. (12) yields

$$\begin{bmatrix}
\delta_1 \\
\delta_2 \\
\delta_3 \\
\delta_4 \\
\zeta_1 \\
\zeta_2 \\
\zeta_3 \\
\zeta_4 \\
z_1^2 e^{z_1} \\
z_2^2 e^{z_2} \\
z_3^2 e^{z_3} \\
z_4^2 e^{z_4} \\
\end{bmatrix} Z =
\begin{bmatrix}
A_1 \\
A_2 \\
A_3 \\
A_4 \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}$$

(15)

where $\delta_n, \zeta_n, n = 1, 2, 3, 4$, are defined as follows

$$\delta_n \triangleq (1 - ce^{z_n}) z_n^2, \quad \zeta_n \triangleq z_n^3 - k \lambda z_n - c \lambda e^{z_n} + k \lambda^2 - \Omega^2 \beta_j$$

A non-trivial solution of (15) is obtained by solving the transcendental equation $\det\{Z\} = 0$. For specific values of $u_e, \beta, \gamma, \lambda, \beta_j, c$ and $k$, the transcendental equation can be solved numerically to get the complex frequencies $\Omega$, and $z_n, n = 1, 2, 3, 4$, for each $\Omega$. For a particular $\Omega$, the solution of (8) and (9) can be obtained by substituting (14) into (10):

$$v(u, \tau) = \sum_{n=1}^{4} A_n e^{z_n u} e^{i\Omega \tau} = \sum_{n=1}^{4} A_n e^{\Re[z_n] u} e^{i\{\Im[z_n] u + \Re[\Omega] \tau\}} e^{-\Im[\Omega] \tau}$$

(16)
where the coefficients $A_n$, $n = 1, 2, 3, 4$, can be obtained from the null space of $Z$ in (15). It is clear from (16) that $v(u, \tau)$ is a product of three exponential terms where the first term is bounded since $u$ is bounded. The second term is also bounded due to the imaginary exponent and it results in periodic motion. The last term grows unbounded with time if $\text{Im}[\Omega] < 0$; therefore, the point at which $\text{Im}[\Omega]$ changes sign from positive to negative represents the onset of flutter instability.

It should be noted that (16) describes the solution for one nondimensional frequency $\Omega$. At the flutter instability point, one specific value of $\Omega$, $\Omega = \Omega_{cr}$, satisfies $\text{Im}[\Omega] = 0$ whereas all other $\Omega$ values satisfy $\text{Im}[\Omega] > 0$. Since $e^{-\text{Im}[\Omega] \tau} \to 0$ as $\tau \to \infty$ for all $\Omega \neq \Omega_{cr}$, the complete solution at the flutter instability point takes the form

$$v(u, \tau) = \sum_{n=1}^{4} A_n e^{\text{Re}[z_n] u} e^{i\{\text{Im}[z_n] u + \text{Re}[\Omega_{cr}] \tau\}}$$ (17)

Since the imaginary exponent in (17) is a function of both $u$ and $\tau$, the above equation represents a traveling waveform, where the wave travels down the length of the beam with an amplitude increasing along the length.

5 Flutter Investigations

5.1 System Parameters

To investigate the onset of flutter induced by the bending moment proportional to the curvature of the beam, we consider the rigid body and flexible beam of the submersible to have the following kinematic and dynamic parameters:

$$J = 0.0123 \text{kgm}^2, \quad \ell = 0.2 \text{m}, \quad E = 2.7 \text{GPa}, \quad I = 8.33 \times 10^{-12} \text{m}, \quad \rho = 1420 \text{kg/m}^3$$

The Young’s modulus $E$ and the density $\rho$ of the flexible beam correspond to the commercially available material Cirlex®. The area moment of inertia $I$ was computed by assuming the beam to have a height $h = 0.1 \text{ m}$ and a thickness $t = 0.001 \text{ m}$. These parameter choices resulted in the beam cross-sectional area

$$A = 1.0 \times 10^{-4} \text{m}^2$$

and nondimensional parameters $\beta$ and $\beta_j$ to have the values shown in Table 1. The nondimensional spring stiffness $k$ was chosen to be 5.0 and the nondimensional length of the flexible tail was chosen to be 2.2; the working fluid was assumed to be water, $\rho_f = 1000 \text{ kg/m}^3$.

| Parameters | $\beta$ | $\beta_j$ | $\lambda$ | $k$ |
|------------|---------|-----------|-----------|-----|
| Values     | 0.98    | 0.086     | 2.25      | 5.0 |

From our discussion in section 4, it is clear that the flutter instability point depends on the values of $u_e$, $\beta$, $\gamma$, $\lambda$, $\beta_j$, $c$ and $k$. In the next subsection, we will investigate the value of $c$, i.e., the role of feedback gain, on flutter instability for different values of $u_e$ and $\gamma$. 
5.2 Critical Stability Surface and Modes of Flutter Instability

The critical stability surface is obtained as follows. We start with $c = 0$ and $u_e = 0.01$ for a specific value of $\gamma$ to acquire the first ten natural frequencies, $\Omega_i$, $i = 1, 2, \ldots, 10$, of the flexible beam. These frequencies are then used as a first guess to find the natural frequencies as the magnitude of $c$ is gradually increased. This process is continued till one of the $\Omega$’s, $\Omega_k$, $k \in \{1, 10\}$, satisfies the condition $\text{Im}[\Omega_k] = 0$. This provides the critical value of $c$, $c = c_{cr}$ for $u_e = 0.01$ and the specific value of $\gamma$, and $k$ denotes the mode of flutter instability. The critical stability points are determined on a fine mesh grid for $\gamma \in [0, 0.9]$ and $u_e \in [0, 6.0]$ to obtain the critical stability surface. Since $c$ can be both positive and negative, two separate cases are considered; the critical stability surface for these cases are separately shown in Figs.3(a) and (b) - they appear to have very different topography. For positive $c$, the stability surface in Fig.3(a) can be described by a valley between two peaks for $u_e$ in the approximate range of $[0.0, 5.0]$ and $\gamma$ in the approximate range of $[0.2, 0.7]$. For $u_e > 5.0$, the critical value $c_{cr}$ reduces to $\approx 0$ implying that the flexible beam flutters due to the external flow alone; this instability is similar to that of a flag fluttering in the wind. For $\gamma$ in the approximate range of $[0.7, 0.9]$ and $u_e < 5.0$, the stability surface is comprised of a few narrow ridges and valleys. In contrast to the case of $c_{cr} > 0$, the entire stability surface for $c_{cr} < 0$ is comprised of densely-spaced ridges and valleys. It should be noted that for both cases the ridges and valleys run along the $u_e$ axis; this implies that a change in the point of curvature measurement (value of $\gamma$) has a profound effect on the stability characteristics of the beam.

The ridges and the valleys of the surfaces in Fig.3 correspond to a rich set of stability transitions between different modes of flutter; these transitions can be better appreciated by considering special cases where $\gamma$ or $u_e$ is constant. For positive $c_{cr}$, Figs.4(a) and (b) show the cases where $\gamma = 0.30$ and $u_e = 2.0$, respectively, and for negative $c_{cr}$, Figs.5(a) and (b) show the cases where $\gamma = 0.52$ and $u_e = 2.0$, respectively. For positive $c_{cr}$ and $\gamma = 0.30$, the flexible beam loses stability in the second mode for all values of $u_e$ - see Fig.4(a). As $u_e$ increases, $c_{cr}$ decreases monotonically and becomes $\approx 0$ for $u_e \geq 4.72$; this implies that external flow of $u_e \geq 4.72$ can alone destabilize the flexible beam. In contrast, for negative $c_{cr}$ and $\gamma = 0.52$, the flexible beam loses stability through three different modes - see Fig.5(a). As $u_e$ increases, stability is lost through the seventh mode first, then through the eighth mode, and finally through the second mode. Furthermore, for instability
in the seventh and eighth mode, the value of $c_{cr}$ increases with increase in $u_e$; for instability in the second mode, the value of $c_{cr}$ decreases with increase in $u_e$ and finally becomes $\approx 0$ for $u_e \geq 5.8$.

For a fixed value of $u_e$, $u_e = 2.0$, a much richer set of stability transitions exist - see Fig.4(b) for $c_{cr} > 0$ and Fig.5(b) for $c_{cr} < 0$. For $c_{cr} > 0$, stability is lost through the second mode for a large range of $\gamma \in [0.20, 0.76]$; for $\gamma > 0.76$, the mode of instability switches frequently and randomly. In contrast, for $c_{cr} < 0$, the mode of instability switches frequently and randomly for the entire range

![Figure 4: Transitions of flutter instability modes for the case $c_{cr} > 0$: (a) $\gamma = 0.30$, and (b) $u_e = 2.0$; the value of $k$ denotes the mode of flutter instability.](image)

![Figure 5: Transitions of flutter instability modes for the case $c_{cr} < 0$: (a) $\gamma = 0.52$, and (b) $u_e = 2.0$; the value of $k$ denotes the mode of flutter instability.](image)
Within each band, where the mode of instability remains fixed, the critical stability curve has the shape of a catenary for both positive and negative $c_{cr}$: such catenary shapes have been reported earlier in the literature in the context of critical stability curves [12,17]. Although the catenary shape gives rise to local maxima and minima, the overall trend shows a decrease in the magnitude of $c_{cr}$ as the location of the measurement of curvature changes from the revolute joint end to the distal end of the beam for both positive and negative $c_{cr}$. There are some additional important differences between the positive and negative $c_{cr}$ cases but these differences can only be pointed out in the next section, where we discuss the thrust and efficiency associated with the different modes of flutter.

6 Propulsive Performance

6.1 Thrust

The thrust generated by the flexible propulsor can be estimated using the method proposed by Lighthill [10] for "slender fish". Following the work in [15] where the assumptions in [10] are relaxed to deal with a flexible propulsor rather than a tapered fish, the time-averaged thrust is given by the relation

$$\tau = \frac{1}{2} M_e \left[ \left\{ \left( \frac{\partial y}{\partial t} \right)^2 - U_e^2 \left( \frac{\partial^2 y}{\partial x^2} \right)^2 \right\}_{x=L} - \left\{ \left( \frac{\partial y}{\partial t} \right)^2 - U_e^2 \left( \frac{\partial^2 y}{\partial x^2} \right)^2 \right\}_{x=0} \right]$$  

(18)

where $\langle . \rangle$ refers to a long-term time average of $(.)$. Since the motion of the flexible propulsor is harmonic at the critical stability point, the average can be taken over one time period of oscillation. It is clear that increasing $M_e$ results in higher thrust; the thrust is therefore directly proportional to the square of the height of the flexible beam $h$. The expression for the thrust in (18) can be non-dimensionalized using the change of variables introduced in section 3.2 as follows:

$$\tau^* \triangleq \frac{\tau L^2}{EI} = \frac{\Omega_{cr}}{4\pi} \int_0^{2\pi} \left\{ \left[ \beta \dot{v}(u, \tau)^2 - u_e^2 v'(u, \tau)^2 \right]_{u=1} - \left[ \beta \dot{v}(u, \tau)^2 - u_e^2 v'(u, \tau)^2 \right]_{u=0} \right\} d\tau$$  

(19)

where $\Omega_{cr}$ is the non-dimensional critical frequency.

For $\gamma \in [0.2, 0.9]$ and $u_e \in [0.01, 6.0]$, the regions of positive and negative thrust are shown in Fig 6; the results are shown separately for the two cases where $c_{cr} > 0$ and $c_{cr} < 0$. A comparison of these two cases indicate that $c_{cr} < 0$ results in positive thrust and $c_{cr} > 0$ results in negative thrust in a large fraction of the $u_e-\gamma$ plane. Figure 6(a) provides information that is not revealed in Fig 4. For example, for $\gamma = 0.30$, Fig 4(a) indicates that stability is lost through the second mode for all values of $u_e$; Fig 6(a) reveals that the thrust associated with the second mode of instability is negative. For $u_e = 2.0$, Fig 4(b) indicates that stability is lost through the second mode for $\gamma \in [0.2, 0.76]$ and there are three stability transitions to the ninth, sixth and eighth modes for $\gamma \in [0.76, 0.9]$; Fig 6(a) reveals that the second mode of instability produces negative thrust but positive thrust is produced by the other three modes for $\gamma \in [0.46, 0.9]$. Similar to Fig 6(a), Fig 6(b) provides information that is not revealed in Fig 5. For example, for $\gamma = 0.52$, Fig 6(a) indicates that stability is lost through the seventh mode first, then through the eighth mode, and finally
through the second mode as $u_e$ is increased; Fig.6(b) reveals that positive thrust is generated in the seventh and eighth mode of instabilities and negative thrust is generated in the second mode of instability. For $u_e = 2.0$, Fig.5(b) indicates that there are fourteen stability transitions for $\gamma \in [0, 0.9]$ and Fig.6(b) indicates that all modes of instabilities generate positive thrust. The last example alone makes it abundantly clear that the sign of the thrust does not provide the complete picture; since positive thrust is generated with many different modes of instabilities over the complete range of $\gamma$ investigated, it is important to investigate the magnitude of the thrust and the efficiency associated with different fluttering waveforms generated as a result of using different values of $\gamma$. The procedure for computing the magnitude of the thrust and efficiency of the motion is provided in the next subsection.

### 6.2 Motion Scaling for Computation of Thrust and Efficiency

The dynamics of a submersible operating on the principle of feedback-induced flutter will be nonlinear and limit cycle oscillations will result from using a feedback gain equal to the critical value $c_{cr}$. An increase in the magnitude of the gain, slightly beyond the critical value, will change the nature of the limit cycle oscillation but will not necessarily result in oscillations that grow unbounded; this follows from related prior work on post-flutter behavior - see [18–20], for example. The modeling of the nonlinear behavior is however challenging and its analysis is often intractable; therefore, a linear system model, one that has been studied in the literature [14, 15], is investigated in this paper. A linear system model allows accurate prediction of the onset of instability but the amplitude of oscillations at the critical stability point depends completely on the initial conditions. Since we assume an Euler-Bernoulli beam, the initial conditions should be sufficiently small. Each point on an Euler-Bernoulli beam undergoes a purely transverse displacement; therefore, using trial and error we scale the initial conditions such that the maximum elongation of the beam, irrespective of the mode of flutter, does not exceed 5%. Mathematically, the complex amplitudes $A_n$, $n = 1, 2, 3, 4$, 

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Figure 6: Regions of positive (shaded) and negative (not shaded) thrust in the $u_e$-$\gamma$ plane at the point of critical stability $c = c_{cr}$: (a) $c_{cr} > 0$ and (b) $c_{cr} < 0$. 

---

| $u_e$ | $\gamma$ |
|-------|----------|
| 0.2   | 0.2      |
| 0.4   | 0.4      |
| 0.6   | 0.6      |
| 0.8   | 0.8      |

| $u_e$ | $\gamma$ |
|-------|----------|
| 0.2   | 0.2      |
| 0.4   | 0.4      |
| 0.6   | 0.6      |
| 0.8   | 0.8      |
in (17) are scaled such that

\[
\sup_{\tau \in [0,2\pi/\Omega_{c}]} \int_{0}^{1} \sqrt{1 + (v')^2} \, du = 1.05
\]

(20)
is satisfied. The time-averaged nondimensional thrust computed based on the scaled amplitudes is denoted by \( \tau^*_{\text{scaled}} \). It is laborious to compute \( \tau^*_{\text{scaled}} \) for all points in the \( u_e-\gamma \) plane; therefore, we present results for five select points that are marked in Fig[6] and tabulated below:

Table 2: Computation of scaled thrust and efficiency for five select points

| Case | \( u_e \) | \( \gamma \) | \( c \) | \( c_{cr} \) | \( k \) | \( \tau^*_{\text{scaled}} \) | \( \eta \) |
|------|------|------|------|------|------|------|------|
| (1)  | 2.0  | 0.35 | > 0  | +0.179 | 2    | -0.74 | 0.500 |
| (2)  | 2.0  | 0.35 | < 0  | -0.595 | 5    | 14.47 | 0.508 |
| (3)  | 2.0  | 0.55 | < 0  | -0.300 | 7    | 34.33 | 0.504 |
| (4)  | 2.0  | 0.83 | > 0  | +0.240 | 6    | 23.35 | 0.504 |
| (5)  | 2.0  | 0.83 | < 0  | -0.243 | 5    | 13.42 | 0.506 |

A comparison of cases (1) and (2) indicate that for the same values of \( u_e \) and \( \gamma \) (\( u_e = 2.0, \gamma = 0.35 \)), changing the sign of the proportionality constant \( c \) for the bending moment results in very different behavior: a positive \( c_{cr} \) results in flutter in the second mode and generates negative thrust, and a negative \( c_{cr} \) results in flutter in the fifth mode and generates positive thrust. Interestingly, a change in the location of measurement of curvature from \( \gamma = 0.35 \) to \( \gamma = 0.83 \) generates positive thrust for both positive and negative values of \( c \). A comparison of cases (4) and (5) indicate that for the same values of \( u_e \) and \( \gamma \) (\( u_e = 2.0, \gamma = 0.83 \)), a larger thrust is generated in the sixth mode with a positive \( c_{cr} \) than that generated in the fifth mode with a negative \( c_{cr} \). On the other hand, a comparison of cases (2), (3) and (5) indicate that the value of \( \gamma \) has a significant effect on the thrust generated. For \( u_e = 2.0 \) and \( c < 0 \), these three cases indicate that \( \gamma = 0.55 \) results in the maximum thrust with flutter in the seventh mode; lower thrust is generated in the fifth mode with both \( \gamma = 0.35 \) and \( \gamma = 0.83 \). The overall trend indicates that a higher mode of flutter instability results in a larger magnitude of thrust.

Table 2 also provides the Froude efficiency for the five different cases. The Froude efficiency is determined by computing the power expended in generating the waveform \( y(x,t) \). Using the method outlined in [10, 15, 21], the time-averaged power expended during one oscillation can be expressed as

\[
\bar{P} = U_e M_e \left[ \frac{\partial y}{\partial \tau} \left( \frac{\partial y}{\partial \tau} + U_e \frac{\partial y}{\partial x} \right) \right]_{x=L} - \left[ \frac{\partial y}{\partial \tau} \left( \frac{\partial y}{\partial \tau} + U_e \frac{\partial y}{\partial x} \right) \right]_{x=0}
\]

(21)

Using the change of variables introduced in section 3.2, the nondimensional power can be expressed
as follows:

\[
\mathbf{P}^{*} \triangleq \frac{\mathbf{P} M \frac{1}{2} L^3}{(EI)^{\frac{1}{2}}} = \frac{\Omega_{cr}}{2\pi} \int_{0}^{2\pi} \left\{ \left[ u_{e} \beta \dot{v}(u, \tau)^2 + u_{e}^2 \sqrt{\beta} v'(u, \tau) \dot{v}(u, \tau) \right]_{u=1} - \left[ u_{e} \beta \dot{v}(u, \tau)^2 + u_{e}^2 \sqrt{\beta} v'(u, \tau) \dot{v}(u, \tau) \right]_{u=0} \right\} d\tau
\]

(22)

The Froude efficiency \([10]\) is given by

\[
\eta = \frac{\pi U_{e}}{\mathbf{P}} = \frac{\mathbf{P}^{*} u_{e}}{\mathbf{P}^{*}}
\]

(23)

Since the Froude efficiency is a ratio of \(\mathbf{P}^{*}\) and \(\mathbf{P}^{*}\), its value is unaffected by scaling of the waveform carried out to satisfy the Euler-Bernoulli small displacement assumptions.

7 Conclusion

A new method of propulsion for underwater vehicles is proposed. The propulsor is a flexible tail-like appendage connected to the vehicle by a pin joint and actuated by a motor. The motor is actively controlled using feedback information of the elastic strain on the propulsor at some point along its length. For a proper choice of feedback gain, this form of feedback results in flutter instability. In the post flutter regime, the damping forces in the fluid environment result in stable limit cycle oscillations that are traveling waves. These traveling waves generate propulsive forces; the mode of flutter-based propulsion and the thrust and efficiency characteristics of the waveform depends on the location of strain measurement and the steady state velocity of the vehicle. For a given set of system parameters, numerical simulation results indicate the possibility of switching between high thrust and high efficiency maneuvers. Future work will focus on more realistic computational fluid dynamics simulations and experimental verification.

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