Brill Waves

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Abstract
Brill waves are the simplest (non-trivial) solutions to the vacuum constraints of general relativity. They are also rich enough in structure to allow us believe that they capture, at least in part, the generic properties of solutions of the Einstein equations. As such, they deserve the closest attention. This article illustrates this point by showing how Brill waves can be used to investigate the structure of conformal superspace.

1 INTRODUCTION
From time to time I amuse myself by mentally assembling a list of articles I would like to have written. The candidates for this list have to satisfy a number of criteria. Naturally, they have to be both important and interesting to me. Equally, they have to contain results that I can convince myself, however unreasonably, that I could have obtained. Every time I make my list I am struck again by the number of articles by Dieter Brill appearing on it. At first glance, this is explained by the large overlap between our interests. In reality, however, the explanation is to be found by considering the kind of article that Dieter has written over the years and the way in which he manages to convey major insights in a deceptively simple fashion. In this work I wish to return to an article that has a permanent place on my list, the famous Brill waves, and show how some of the the earliest work that Dieter did in general relativity continues to offer valuable insight a third of a century later.

2 BRILL
In his thesis (Brill, 1959) Dieter Brill considered axisymmetric, moment-of-time-symmetry, vacuum initial data for the Einstein equations. The starting point is an axially symmetric three-metric of the following form

\[ g = e^{2Aq}(d\rho^2 + dz^2) + \rho^2d\theta^2, \]  

(1)

where \( A \) is a constant and \( q \) is an (almost) arbitrary function of \( \rho \) and \( z \). We only require that it satisfy \( q = q_\rho = 0 \) along the \( z \)-axis, that it decay fairly rapidly at infinity (faster than \( 1/r \)) and that it be reasonably differentiable.

This metric \( g \) is to be conformally transformed to a metric \( \tilde{g} = \phi^4g \) so that the metric \( \tilde{g} \) has vanishing scalar curvature, so as to satisfy the moment-of-time-symmetry initial
value constraint of the Einstein equations. This is equivalent to seeking a positive solution \( \phi \) to
\[
8\nabla^2_g \phi - R \phi = 0, \quad \phi > 0, \quad \phi \to 1 \text{ at } \infty,
\]
where \( R \) is the scalar curvature of \( g \). It is easy to calculate
\[
R = -2A e^{-2A q} (q_{,\rho\rho} + q_{,zz}).
\]
It is important to notice that \( \sqrt{g} = \rho e^{2A q} \) so that
\[
\int R dv = \int \sqrt{g} R d^3 x = -4A \pi \int \rho (q_{,\rho\rho} + q_{,zz}) d\rho dz
\]
\[
= -4A \pi \int [ (\rho q_{,\rho} - q)_{,\rho} + (\rho q_{,z})_{,z} ] d\rho dz
\]
\[
= 0.
\]
To obtain the last line we have to use both the regularity along the axis and the asymptotic falloff of \( q \). This is a remarkably powerful result, in that it gave Brill the first positivity of energy result in General Relativity.

Let us assume that we obtain a regular solution to (2) (we will return to this issue in Section 3). The total energy, the ADM mass, is contained in the \( 1/r \) part of the physical metric \( \bar{g} \). Since the base-metric \( g \) falls off faster than \( 1/r \) the mass must be contained in the \( 1/r \) part of the conformal factor \( \phi \). More precisely we get
\[
\phi \to 1 + \frac{M}{2r}.
\]
Hence
\[
2\pi M = -\int_{\infty} \nabla^a \phi dS_a = -\int_{\infty} \frac{\nabla^a \phi}{\phi} dS_a
\]
\[
= \int [\nabla^2 \phi \phi + (\nabla \phi)^2 \phi^2] dv = \int [\frac{R}{8} + \frac{(\nabla \phi)^2}{\phi^2}] dv
\]
\[
= \int \frac{(\nabla \phi)^2}{\phi^2} dv > 0.
\]
Notice that we use (4) in going from (7) to (8). The key (as yet) unresolved question is: When can we solve (2)?

3 CANTOR AND BRILL
The next issue to be considered is what choices of \( q \) (or \( Aq \)) allow a regular solution to (2). This question was first seriously discussed by Cantor and Brill (1981). They derived the following
**Theorem I (Cantor and Brill).** There is a \( \bar{g} \) conformally equivalent to a given metric \( g \) such that \( R(\bar{g}) = 0 \) if and only if

\[
\int [8(\nabla f)^2 + R(g) f^2] dv > 0, \tag{9}
\]

for every \( f \) of compact support with \( f \) not identically 0.

Cantor and Brill further show that if \( R(g) \) is small (in a precise sense) then the Sobolev inequality may be used to guarantee inequality (9). The Sobolev inequality states that for any asymptotically flat Riemannian three-manifold there exists a positive constant \( S(g) \) such that

\[
\int (\nabla f)^2 dv > S(g) \left[ \int f^6 dv \right]^{\frac{1}{3}}, \tag{10}
\]

for any \( f \) of compact support. Let us now use the Hölder inequality on the second term in (9) to give

\[
\int R(g) f^2 dv < \left[ \int |R(g)|^2 dv \right]^{\frac{2}{3}} \left[ \int f^6 dv \right]^{\frac{1}{3}}. \tag{11}
\]

Combining (10) and (11) shows that if

\[
\left[ \int |R(g)|^2 dv \right]^{\frac{2}{3}} < 8S(g) \tag{12}
\]

then expression (9) must be positive for any \( f \) and a regular solution to (2) exists.

The Brill waves supply an obvious application of this result. Choose a metric of the form (1) with a fixed \( q \) but allow \( A \) to vary. It is clear from the expression (3) for the scalar curvature that we can always find a small enough \( A \) (which can be either positive or negative) to guarantee that (12) holds.

The next interesting property of the Brill waves is that, for a fixed \( q \), one can always find a large enough value of \( A \) (both positive and negative) such that inequality (9) cannot hold. We know that the scalar curvature integrates to zero. This means that it must have positive and negative regions. It is clear that we can always find an axisymmetric function \( f_+ \) which has support only on the positive regions of the scalar curvature and another function \( f_- \) which has support only on the negative regions.

This choice can be made independent of the value of \( A \). We have that

\[
\int (\nabla f_+)^2 dv = 2\pi \int [(f_+^\rho)^2 + (f_+^z)^2] d\rho dz \tag{13}
\]

entirely independent of \( A \). We also have

\[
\int R(g) f_+^2 dv = 4\pi A \int [-q_{,\rho \rho} + q_{,zz}] f_+^2 d\rho dz \tag{14}
\]
where the integral on the right hand side of (14) is positive (from the choice of $f_+)$ and independent of $A$. Therefore, with this choice of test-function there exists a number $|A_-|$ given by

$$|A_-| = \frac{\int[(f_+)_2 + (f_+)_z]^2 \rho d\rho dz}{2 \int[-(q_{,\rho} + q_{,zz})]f^2_{,\rho} \rho d\rho dz}$$

such that if $A$ is large and negative, i.e., $A < -|A_-|$, inequality (9) cannot hold. A similar bound, using $f_-$, can be derived showing that there exists an $|A_+|$ such that if $A > |A_+|$ (9) again breaks down.

This is not just a mathematical game, conformal transformations are the standard way of constructing solutions to the initial value constraints of General Relativity (Ó Murchadha and York, 1973) and for an interesting set of such data (‘maximal’ initial data) a necessary and sufficient condition is that the metric be conformally transformable to one with zero scalar curvature. What we are doing, therefore, is trying to map out conformal superspace $\tilde{S}$; more precisely, we are trying to delineate the axially symmetric subspace of conformal superspace (a so-called ‘midisuperspace’). The functions $q(\rho, z)$ can be regarded as defining ‘directions’ in conformal superspace with $A$ acting as a ‘distance’ along each given ray. The ‘point’ $A = 0$ is flat space; there is an open interval around the origin on each ray which belongs to conformal superspace and each ray (after a finite ‘distance’) emerges from $\tilde{S}$.

We can even prove more; we can show that each ray passes only once out of $\tilde{S}$. Given $q$, let us assume that we have passed out of $\tilde{S}$, in other words we have an $A_0$ (assumed positive) such that inequality (9) does not hold. This means that there exists a function $f_m$ such that

$$\int[8(\nabla f_m)^2 + R(g)f^2_m]dv \leq 0.$$  

In other words,

$$4\pi \int(4[(f_m,\rho)^2 + (f_m,z)^2] - A_0[(q_{,\rho} + q_{,zz})]f^2_m)\rho d\rho dz \leq 0.$$  

The second term must be negative to counter the positive first term. It is clear that if one increases $A$ while holding $f_m$ and $q$ fixed the inequality must worsen. Therefore, once a ray passes outside $\tilde{S}$, it stays outside. In other words, (the axially symmetric part of) $\tilde{S}$ seems to be simply connected with convex boundary.

4 BEIG AND Ó MURCHADHA

The next question that needs be asked is what happens as one moves along such a ray in $\tilde{S}$ and approaches the critical value of $A$. This question has been addressed
is a number of recent articles (Ó Murchadha, 1987, 1989; Beig and Ó Murchadha, 1991).

There exists an object analogous to the Sobolev constant, $S(g)$, defined in equation (10), the conformal Sobolev or the Yamabe constant. The Yamabe constant is defined by

$$Y(g) = \inf \frac{\int [8(\nabla f)^2 + R(g)f^2]dv}{8(\int f^6dv)^{\frac{1}{3}}},$$

where the infimum is taken over all smooth functions of compact support. This object has a number of interesting properties. It is a conformal invariant; it achieves its maximum value [equalling $3(\frac{\pi}{4})^{2/3}$] at (conformally) flat space (Schoen, 1984). Flat space is its only extremum.

One reason for introducing this concept here is that it allows a different formulation of Theorem I.

**Theorem Ia.** There is an asymptotically flat metric $\bar{g}$ conformally equivalent to a given metric $g$ such that $R(\bar{g}) = 0$ if and only if the Yamabe constant $Y(g)$ of $g$ is positive.

One can immediately see this by comparing expressions (9) and (18).

The function that minimizes the Yamabe functional satisfies the non-linear equation

$$8\nabla^2\theta - R\theta = -8\lambda\theta^5, \quad \theta \rightarrow 0 \text{ at } \infty,$$

where $\lambda$ is a constant that is proportional to the Yamabe constant. It equals the Yamabe constant if $\theta$ is normalized via

$$\int \theta^6dv = 1.$$

I will use this normalization from now on. It can be shown (Schoen, 1984) that $\theta$ exists, is everywhere non-zero, and falls off at infinity like $1/r$.

Let us now combine $\phi$, the solution of eqn.(2), and $\theta$, the solution of eqn.(18), using the Green identity

$$-8 \int_\infty \nabla^a \theta dS_a = 8 \int_\infty (\theta \nabla^a \phi - \phi \nabla^a \theta) dS_a$$

$$= 8 \int (\theta \nabla^2 \phi - \phi \nabla^2 \theta)dv$$

$$= \int (\theta R\phi - \phi (R\theta - Y\theta^5))dv$$

$$= Y \int \theta^5 \phi dv.$$
As $Y$ approaches zero, both $\theta$ itself, and the surface integral of $\theta$ remain well-behaved. In particular, the $1/r$ part of $\theta$ does not vanish in the limit. Therefore the surface integral remains bounded away from zero. This means that the volume integral in (23) must blow up like $1/Y$. Since $\theta$ remains finite, $\phi$ must become unboundedly large. In other words, $\phi$ must blow up like $1/Y$.

It is not enough that $\phi$ blow up like a delta function at one point, it must become large on an extended region so that the integral $\int \theta \phi dv$ blows up like $1/Y$. However, we know that $\phi$ is not a random object, it satisfies a differential equation which forces a broad blow-up on $\phi$. We can make this more precise by returning to eqn.(2). Using the fact that $\sqrt{g} = \rho e^{2Aq}$, we find that $\phi$ must satisfy

$$4\nabla^2_f \phi + A(q_{\rho\rho} + q_{zz})\phi = 0, \quad \phi > 0, \quad \phi \to 1 \text{ at } \infty,$$

(24)

where $\nabla^2_f$ is the flat-space laplacian.

Let us consider the situation where $q$ has compact support on some ball $B$. The scalar curvature, $R$, has the same compact support. Outside this region the manifold is flat and $\phi$ satisfies $\nabla^2_f \phi = 0$. Hence the min-max principle tells us that the maximum of $\phi$ must be achieved inside the ball $B$. Further, since $\phi$ is positive and satisfies a linear elliptic equation on the compact set $B$ we can use the Harnack inequality (Gilbarg and Trudinger, 1983). This means that there exists a constant $C_B$ (which we can choose independent of $A$) such that

$$\max_B \phi = \max_B \phi \leq C_B \min_B \phi \leq C_B \min_{\delta B} \phi.$$

(25)

In other words, as the maximum of $\phi$ increases inside in $B$, $\phi$ everywhere in $B$ gets dragged up with it. In particular, $\phi$ on the boundary of $B$ becomes large.

As $A$ gets large, as $Y$ gets small, we reach a point where $\max_{\delta B} \phi \geq C_B$ (the Harnack constant). From that point on we have that $\min_{\delta B} \phi \geq 1$. Outside $B$ $\phi$ satisfies $\nabla^2 \phi = 0$. This means that the minimum and maximum of $\phi$ occur on the boundaries of the set. Since $\phi \geq 1$ on $\delta B$ and $\phi = 1$ at infinity, the minimum must be at infinity. Returning to (5), $\phi \to 1 + \frac{M}{2r}$, we can conclude that the constant $M$ must be positive. This is a positive energy proof which is not as strong as the original Brill proof, it only holds in the strong-field region, far away from flat space, as the Yamabe constant becomes small. However, it has the advantage of working for a much larger class of metrics.

By analogy with electrostatics we can regard $\phi$ (or rather $\xi = \phi - 1$) as the potential on and outside a (nonconducting) shell ($\delta B$). The potential on the shell is given and one solves for the potential outside. The ADM mass corresponds to the total charge
that one has to distribute on the shell to give the specified potential distribution. If the potential is everywhere positive on the shell, the charge must be positive.

If the shell were conducting, the potential on it would be constant. Now we can talk about the capacitance $C$, the constant ratio between the ‘charge’ $M$ and the potential on the surface. The capacitance is a kind of Harnack constant because we have

$$C \min_{\delta B} \xi \leq M. \quad (26)$$

$$C \max_{\delta B} \xi \geq M. \quad (27)$$

These are easy to derive. Replacing a non-constant potential on $\delta B$ by its minimum (maximum) value must decrease (increase) the charge while the decreased (increased) charge equals the capacitance multiplied by the potential. Since the capacitance depends only on the size and shape of the surface, not on the charge, we immediately see that as $\min_{\delta B} \xi$ (or $\max_{\delta B} \phi$) increases, so will the ADM mass. If we incorporate (25), (26) and (27) together we can see that the mass grows linearly with $\phi$. More precisely, we have

$$\frac{C}{C_B} \max \phi \leq M \leq C \max \phi, \quad (28)$$

$$C \min_B \phi \leq M \leq C \min_B \phi. \quad (29)$$

I can extract more information from (23). We have

$$Y \int \theta^5 \phi dv \geq Y \int_B \theta \phi dv \geq (Y \min_B \phi) \int \theta^5 dv. \quad (30)$$

The Harnack inequality (25) tells us that if $Y \max \phi$ diverges as $Y$ goes to zero, so also will $Y \min_B \phi$. Eqn. (28) now tells us that $Y \int \theta^5 \phi dv$ must become unboundedly large, which contradicts (23). Hence $Y \max \phi$ remains bounded away from both zero and infinity as $Y$ goes to zero. In turn, this means that both $Y \min_{\delta B} \phi$ and $Y \max_{\delta B} \phi$ remain finite and bounded away from zero as $Y$ goes to zero. Finally, this means that the ADM mass blows up like $1/Y$ as $Y$ goes to zero.

Not only does the monopole blow up like $1/Y$, all the other multipoles behave in a similar fashion. This allows us to show that a minimal two-surface must appear in the conformally transformed space if $M$ becomes large enough. Let us consider a surface which has coordinate radius $r$ in the background (flat) space. The area of this surface in the physical space is $A = 4\pi r^2 \phi^4$. $A$, negative is equivalent to $\phi + 2r\phi_r < 0$. The leading terms in $\phi$ are

$$\phi = 1 + \frac{M}{2r} + Q \frac{r^2 - 3z^2}{r^5} + \ldots, \quad (31)$$
where $Q$ is the quadrupole moment. Now we get

$$\phi + 2r \frac{d\phi}{dr} = 1 - \frac{M}{2r} - 5Q \frac{r^2 - 3z^2}{r^5} + \ldots$$

We need to find out if the right hand side of (32) can be negative. We know that $M$ will become large and positive but we have no control over $Q$ except that the ratio $Q/M$ will remain bounded. A similar result holds for all the other multipole moments. We can bound the third term by $10|Q|/r^3$. Let us evaluate (32) at $r = M/4$. This gives

$$\phi + 2r \frac{d\phi}{dr} \leq 1 - 2 + \frac{640|Q|}{M^3} + \ldots$$

It is clear that as $M$ becomes large (as $Y$ becomes small) the third term (and all the higher order terms) becomes small. This implies that the area of the surface in question reduces when it is pushed outwards. (Technically we should evaluate the change in area along the outer normal rather than along the radial vector, however, the difference between the two becomes small as $M$ and $r$ become large.) This means that it is an outer trapped surface in the terminology of Penrose (1965). There is a minimal area surface outside all the trapped surfaces; this is the apparent horizon.

As $M$ increases, the apparent horizon approximates more and more the spherical surface with radius $r = M/2$. The conformal factor on this surface equals 2, so the proper area of the apparent horizon approximates $16\pi M^2$. The location of the apparent horizon moves outward (relative to the flat background) and its area increases as $M$ goes to infinity.

When $M$ reaches infinity, when the horizon reaches infinity, when the Yamabe constant goes to zero, as $\lambda$ approaches the critical value, something catastrophic happens. The solution to (2) blows up and we can no longer construct an asymptotically flat manifold with zero scalar curvature. On the other hand we have a solution to (19) with $\lambda = 0$. This is a conformal factor that transforms the given base manifold into a regular compact manifold with zero scalar curvature. Thus one can construct a closed, vacuum cosmology at moment of time symmetry. If we increase $A$ beyond the critical value, we recover a finite solution to (2), but at the price of allowing $\phi < 0$ in a region. This is John Wheeler’s ‘Bag of Gold’.

5 WHEELER
Let me move back in time. The earliest application I know of the Brill Wave solution was by John Wheeler in 1964. He considered eqn.(24) and brought his understanding of the Shrödinger equation to bear on it. He realised that it was just a scattering problem off a localized potential. The ‘potential’ ($R$) averages to zero, but Wheeler realized that the negative parts of the potential were much more important than
the positive parts. He threw away the positive parts and considered only a negative potential. He even further simplified the problem and considered the problem of scattering off a negative spherical square well potential. He wrote down an analytic solution to this problem.

Wheeler considered a spherically symmetric square well potential in flat space, i.e.,

\[ u = -B, \quad r \leq a \]
\[ u = 0, \quad r > a. \]  \hspace{1cm} (34)

It is easy to solve

\[ \nabla^2 \psi - u \psi = 0 \quad \psi \to 1 \text{ at } \infty. \]  \hspace{1cm} (35)

The solution is

\[
\psi = \frac{1}{\cos \frac{B^{1/2}a}{B^{1/2}r}} \frac{\sin \frac{B^{1/2}r}{B^{1/2}a}}{\cos \frac{B^{1/2}a}{B^{1/2}r}} \quad r \leq a
\]
\[
= 1 + \frac{a}{r} \left( \frac{\tan \frac{B^{1/2}a}{B^{1/2}r}}{\tan \frac{B^{1/2}a}{B^{1/2}r} - 1} \right) \quad r > a.
\]  \hspace{1cm} (36)

So long as \( B^{1/2}a < \pi/2 \), \( \psi \) in (36) is well behaved. As \( B^{1/2}a \) approaches \( \pi/2 \), both \( \psi \) itself, and the coefficient of the \( 1/r \) part (the ADM mass equivalent) become unboundedly large. Further, the surface on which \( \psi = 2 \) (the analogue of the apparent horizon) moves out towards infinity.

At the critical point \( (B^{1/2}a = \pi/2) \), \( \psi \) ceases to exist. We now obtain a solution to a related equation

\[ \nabla^2 \tilde{\psi} - u \tilde{\psi} = 0, \quad \tilde{\psi} \to 0 \text{ at } \infty. \]  \hspace{1cm} (37)

\[
\tilde{\psi} = \frac{\sin \frac{B^{1/2}r}{B^{1/2}a}}{\frac{B^{1/2}r}{B^{1/2}a}} \quad r \leq a
\]
\[
= \frac{1}{B^{1/2}a} \quad r > a.
\]  \hspace{1cm} (38)

This is the equivalent of the Yamabe constant going to zero.

In retrospect, I am amazed how accurately this highly simplified model captures the behaviour of the actual situation. When the well is shallow, we have a regular scattering solution and the energy of the lowest eigenstate is positive. As the well is deepened the energy of the lowest state moves down towards zero. As this happens, we get the phenomenon of resonant scattering, the scattering solution grows bigger and bigger. When the well is just deep enough to give us a zero energy bound state,
the scattering state blows up and ceases to exist. If the well is made even deeper, the scattering solution reappears but now it is no longer everywhere positive, it has a node. This pattern repeats itself as the energy of the next lowest energy state approaches zero, and so on.

The $1/r$ part of the scattering solution also blows up as one approaches the critical point. If one approaches the critical point from the other side, where the scattering solution has a node, the coefficient of the $1/r$ term approaches $-\infty$. This now allows one to give a fairly nonsensical answer to a fairly unreasonable question: What is the energy of a closed universe? My answer is zero, the average of $+\infty$ and $-\infty$!

We now have a fairly precise understanding of what happens as we pump up one of the Brill waves; the Yamabe constant approaches zero for some finite value of the parameter $A$, the mass approaches $+\infty$ and an apparent horizon appears which moves out towards infinity. The horizon, as it moves outwards, becomes more and more spherical. The breakdown of the system coincides with the horizon reaching spacelike infinity. At the critical point we get data which gives us a smooth compact manifold, without boundary, with zero scalar curvature.

I would like to point out here that we have less understanding of what happens as we further increase the factor $A$. Our only real guide is the Wheeler model. If this is valid, the mass should drop to $-\infty$ just on the other side (we would have initial data with a naked singularity), the mass would build up again until it became positive and then blow up to $+\infty$ as the eigenvalue of the first excited state approached zero, and so on. It would be nice to prove that this is (or is not) the correct picture.

The analysis given in Section 4 only deals with the situation when the Yamabe constant is close to zero. Does it hold in general? In particular, if one takes a Brill wave and increase the constant does the mass increase monotonically, does the Yamabe constant decrease monotonically? These are questions that are yet to be answered.

An even more fundamental question: What happens as one approaches the boundary of conformal superspace along a different direction, i.e., not by holding $q$ fixed and increasing $A$, but by changing $q$? I am unable to answer this question definitively, all I can say is that it is currently being investigated by numerical techniques.

6 ABRAHAMS, HEIDERICH, SHAPIRO AND TEUKOLSKY

Brill waves have been used by the numerical relativity community from its earliest days (e.g. Eppley, 1977) and this use has continued right up to the present. The
standard approach (pioneered by Eppley) is to choose a metric of the form (1), with $q$ chosen to be some analytic function. Equation (2) is then solved numerically to find the conformal factor $\phi$. Finally, the ‘physical’ metric $\bar{g} = \phi^4 g$ is constructed. Eppley showed numerically that the overall structure described in the previous sections holds true. With a small scale factor $A$, equation (2) can be solved. As $A$ increases, the conformal factor increases in the middle. Eventually a minimal surface (an apparent horizon) appears in the physical space. Finally, at some finite value of $A$, the solution to (2) blows up.

Recently, Abrahams, Heiderich, Shapiro and Teukolsky (1992) showed that this picture of conformal superspace was grossly oversimplified. They chose as $q$ the function

$$q = \rho^2 \exp\left[-\left(\frac{\rho^2}{\lambda^2} + \frac{z^2}{\lambda^2}\right)\right].$$

Now, instead of holding $q$ fixed in the traditional fashion, they varied $q$ by reducing the ‘characteristic wavelength’ $\lambda$ slowly while holding $\lambda$ fixed. Further, they kept changing the scale factor $A$ so that the ADM mass of the physical metric remained at the constant value $M = 1$. They showed that the scale factor $A$ grew rapidly, $A \propto \lambda^{-2}$ as $\lambda \to 0$. It is clear that for all values of $\lambda \neq 0$, the base metric and the physical metric exist and are regular, asymptotically flat, axially symmetric Riemannian metrics with positive Yamabe constant. However, both the base metric and the physical metric become singular when $\lambda = 0$. Most importantly, nowhere along the sequence do apparent horizons appear in the physical metric.

This analysis raises major questions about the simple description of conformal superspace obtained in Section 4 by holding $q$ fixed and changing $A$. AHST are also probing the boundary of conformal superspace and seem to have discovered that the boundary, in addition to smooth metrics with $Y = 0$, also contains singular metrics.

In the spirit of AHST, let me consider the following sequence of metrics

$$g_\lambda = e^{2Cq_\lambda}(d\rho^2 + dz^2) + \rho^2 d\theta^2,$$

with

$$q_\lambda = \frac{\rho^2}{\lambda^2} \exp\left[-\left(\frac{\rho^2}{\lambda^2} + \frac{z^2}{\lambda^2}\right)\right].$$

It is clear that, for some fixed, small $C$, for each $g_\lambda$, as $\lambda \to 0$, one can get a regular solution to Eqn.(2). Further, the ADM mass along the sequence remains (more-or-less) constant and no apparent horizon appears.

Let me now make a (constant) coordinate transformation on each metric $g_\lambda$ replacing $\rho$ with $\lambda\rho$ and $z$ with $\lambda z$. This changes (40) to

$$g'_\lambda = \lambda^2 e^{2Cq'_\lambda}(d\rho^2 + dz^2) + \rho^2 d\theta^2,$$
with
\[ q'_\lambda = \rho^2 \exp[-(\rho^2 + \lambda^2 z^2)]. \tag{43} \]

Now I conformally transform \( g'_\lambda \) by dividing it by \( \lambda^2 \) to finally get
\[ g''_\lambda = e^{2Cq'_\lambda} (d\rho^2 + dz^2) + \rho^2 d\theta^2. \tag{44} \]

This combination of coordinate and conformal transformations cannot change the value of the Yamabe constant. This means that we can still solve (2) for each \( g'_\lambda \). However, the value of the ADM mass is not a conformal invariant; it picks up a factor \( \lambda^{-1} \). Thus the new mass, instead of remaining constant, blows up as \( \lambda \to 0 \).

When one looks at \( q'_\lambda \), eqn.(43), it is clear that the limiting metric is no longer singular. However, it is still unpleasant, as it is no longer asymptotically flat. The ‘z’ dependence in \( q' \) drops out and the metric becomes cylindrically symmetric rather than axially symmetric.

The ADM mass is measured with respect to some ‘unit’ (meters, lightyears, whatever). This combination of coordinate transformation together with a conformal transformation can be regarded as the equivalent of a change of units, and the value of the ADM mass will change appropriately. Thus any relationship between the mass and the Yamabe constant can only be valid in some very restrictive sense, one has to ‘fix the units’.

Let me return to the other feature that AHST observed, the absence of apparent horizons. It does not matter whether one performs this combination of constant coordinate and conformal transformations on the base space or on the physical space. Apparent horizons at moment-of-time-symmetry are equivalent to minimal 2-surfaces. Minimal 2-surfaces are stable under constant scalings. Therefore, if AHST find no apparent horizons neither will I, even though the mass blows up for the sequence I consider. This does not contradict the analysis in Section 4. In that section I assumed the existence of an ‘external’ region where the physical geometry was dominated by the conformal factor. In the sequence of metrics considered here the support of the base scalar curvature spreads ever outwards, preventing us from taking advantage of the increase of \( M \).

It is not even clear what happens to the Yamabe constant in the limit \( \lambda \to 0 \). All I can say is that it will not pass through zero at any finite value of \( \lambda \). There is a hint in Wheeler’s spherically symmetric toy model that the Yamabe constant may remain nonzero, even in the limit.

Let us return to (36), but instead of considering a sequence in which \( B^{1/2}a \) approaches \( \pi/2 \), let us increase \( a \) and simultaneously scale down \( B \) so that \( B^{1/2}a \) remains constant.
(equal to, say, $\pi/4$). Now, the coefficient of the $1/r$ term becomes unboundedly large (it scales with $a$). However, the value of $\psi$ itself never becomes large, we find $\psi \leq \sqrt{2}$ holds true no matter how large $a$ becomes. Hence we never obtain a surface on which $\psi = 2$. Further, the limit $a \to \infty$ does not correspond to the appearance of a zero energy bound state, a solution to (37). Thus, the limit point that AHST investigate almost certainly does not correspond to the Yamabe constant going to zero.

This trick, used by AHST, of scaling the sequence so that the mass remains constant, can also be implemented for the ‘regular’ sequences discussed in Section 4. Wheeler had already realised this in 1964. There are three equivalent choices of scalings:

(i) : Instead of letting the conformal factor $\phi$ go to 1 at infinity, solve (2) with the condition $\max \phi = 1$. This was the choice Wheeler made.

(ii) : Retain $\phi \to 1$ at infinity, but multiply all the coordinates with the Yamabe constant $Y$.

(iii): Retain $\phi \to 1$ at infinity, but divide all the coordinates by the ADM mass.

With each of these constant scalings, the appearance of minimal surfaces is unaffected.

Now, however, as one moves along the sequence, as the minimal surface moves out to infinity, the area of the horizon shrinks to zero. When it pinches off at ‘infinity’, it really is at a finite proper distance from the middle. Further, it does so smoothly, we do not get a conical singularity, instead we get a regular closed manifold. As one moves further along the sequence, into the region of negative Yamabe constant, I expect that this pinched off point moves back from infinity and develops into a conical singularity. Thus we get again an asymptotically flat manifold, but now one with a naked singularity.

7 CONCLUSIONS

What I have been calling conformal superspace, the space of asymptotically flat, Riemannian, 3-manifolds that can be conformally transformed into (asymptotically flat) regular manifolds with positive scalar curvature, is an interesting object. It is the natural space one is left with if one, in a $3 + 1$ analysis of the Einstein equations, factors out the Hamiltonian as well as the momentum constraints. If one does not wish to proceed this far, one can still think of it as an interesting subset of regular superspace, metrics on which we expect to find regular, classical solutions of the constraints.

Conformal superspace, $\tilde{S}$, consists of all asymptotically flat, smooth, Riemannian metrics with positive Yamabe constant. The boundary of this space, the limit points of (smooth) sequences of metrics in $\tilde{S}$ will, presumably, consist of metrics which violate one or other of the defining properties of $\tilde{S}$. In other words we expect to
find metrics which are not asymptotically flat, metrics which are not smooth, metrics which are not uniformly elliptic and metrics for which the Yamabe constant equals zero.

What understanding we have of such issues as the ‘size’ and ‘shape’ of conformal superspace has basically been gained by looking closely at the Brill waves. I am convinced that yet more information can and will be extracted from them. In my introduction I stated that they “offer valuable insight a third of a century later”. I have every expectation that this will continue to hold true for another third of a century.

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