Two-dimensional static black holes with pointlike sources

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Abstract

We study the static black hole solutions of generalized two-dimensional dilaton-gravity theories generated by pointlike mass sources, in the hypothesis that the matter is conformally coupled.

We also discuss the motion of test particles. Due to conformal coupling, these follow the geodesics of a metric obtained by rescaling the canonical metric with the dilaton.

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1 Introduction

Two-dimensional gravity models have been widely investigated in last years as toy models for higher dimensions world [1]. However, they differ in an important respect from higher-dimensional models, because of the necessity of introducing a scalar field $\eta$ (dilaton) in order to obtain nontrivial field equations [2]. Although some authors consider it simply as an auxiliary field, the dilaton plays a role in the interpretation of the theory. For example, the zeroes of the dilaton can be interpreted as singularities of the geometry [3]. Moreover, the dilaton field admits arbitrary kinetic and potential terms, and this opens the possibility for the existence of many inequivalent models of two-dimensional gravity.

Of special interest is of course the study of black hole solutions. These have been obtained for a variety of models, but usually without reference to the specific matter sources. In this paper, we study the solutions generated by static pointlike particles for the models with power-law dilaton potential introduced in ref. [4]. The same problem was previously considered in ref. [5] in the particular case of a linear dilaton potential, but the role of the dilaton was disregarded in that paper. Although the equations for the metric can be solved independently in that special case, the dilaton equations add some constraints on the parameters of the solution. This introduces some difference between our results and those of ref. [5]. Also the thermodynamics is modified if the contribution of the dilaton is taken into account properly.

We also consider the motion of test particles. As we shall see, for conformal coupling of the sources, the newtonian potential grows linearly with the distance, and hence the gravitational force is constant. However, if one assumes for consistency that the test particles are also conformally coupled, they will follow the geodesics of a rescaled metric, and hence the force experienced is modified.

2 Single source

We consider the model of ref. [4] with conformally coupled matter. Conformal coupling appears to be the most natural in this context, since it gives rise to gravitational field equations which relate the geometry to the matter in a fashion similar to their higher-dimensional counterparts.
We start from the action

\[ I = \frac{1}{2\kappa^2} \int d^2x \sqrt{-g} \left( \eta R + \lambda^2 \eta^h + 2\kappa^2 \eta L_M \right) , \]  

(1)

with \( h \) a positive integer, \( \lambda^2 \) a "cosmological" constant, and \( 2\kappa^2 \) the (dimensionless) gravitational constant. Since we are interested in black hole solutions, we only consider the case \( \lambda^2 > 0 \), for which the metric function \( g(x) \) defined below is positive asymptotically. Varying the action (1) yields the field equations

\[ R + \lambda^2 \eta^h - 1 = -2\kappa^2 L_M , \]  

(2)

\[ -(\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2) \eta - \frac{\lambda^2}{2} g_{\mu\nu} \eta^h = \kappa^2 \eta T_{\mu\nu} , \]  

(3)

where, for a free particle of mass \( m \) located at \( x_0 \),

\[ L_M = -m\delta(x - x_0) , \quad T_{\mu\nu} = m\delta(x - x_0)u_\mu u_\nu , \]  

(4)

with \( u^\mu = dx^\mu/ds \) and \( u^\mu u_\mu = -1 \). Note that in two dimensions the sign of \( m \) is not fixed a priori.

We adopt static coordinates, such that

\[ ds^2 = -g(x)dt^2 + g^{-1}(x)dx^2 , \quad \eta = \eta(x) , \]  

(5)

and consider a particle at rest at the origin, for which \( T_{00} = mg^{-1}\delta(x) \), \( T_{11} = 0 \). The field equations can then be put in the form

\[ g'' - \lambda^2 \eta^h - 1 = -2\kappa^2 m \delta(x) , \]  

(6)

\[ g\eta'' = -\kappa^2 m \delta(x) \eta , \]  

(7)

\[ g'\eta' = \lambda^2 \eta^h , \]  

(8)

where a prime denotes a derivative with respect to \( x \).

When \( m = 0 \), the solutions are well known. Of course, the solutions maintain the same form when \( m \neq 0 \), except at the location of the point masses, where the derivative of the metric has a discontinuity proportional to the mass of the source. It is easy to see that the integration of (6) yields a potential for the point particle linear in the distance. This corresponds to a constant gravitational force for minimally coupled test particles.

We start our investigation from the special cases \( h = 0 \) and \( h = 1 \).
2.1 \( h = 0 \)

In this case spacetime is everywhere flat, except at \( x = 0 \). We make the ansatz
\[
\eta = \alpha |x| + \eta_0, \quad g = \beta |x| + \gamma. \tag{9}
\]
Substituting in (6)-(8), we obtain the conditions
\[
2\beta = -2\kappa^2 m, \quad 2\alpha \gamma = -\kappa^2 m \eta_0 \quad \alpha \beta = \lambda^2,
\]
which are easily solved, giving
\[
\beta = -\kappa^2 m, \quad \alpha = -\frac{\lambda^2}{\kappa^2 m}, \quad \gamma = \frac{(\kappa^2 m)^2}{2\lambda^2} \eta_0,
\]
and hence
\[
\eta = -\frac{\lambda^2}{\kappa^2 m} |x| + \eta_0, \quad g = -\kappa^2 m |x| + \frac{(\kappa^2 m)^2}{2\lambda^2} \eta_0. \tag{10}
\]
Regular black hole solutions exist therefore only for \( m < 0 \) and \( \eta_0 < 0 \). The horizons of the black hole are located at \( x_0 = \pm \kappa^2 m \eta_0 / 2\lambda^2 \).

One can calculate the ADM mass of the black hole by means of the Mann formula [7], which for the models (1) reads\(^1\)
\[
M = \frac{1}{2\alpha} \left( \frac{\lambda^2}{h+1} \eta^{h+1} - g \eta^2 \right).
\]
Substituting the solution (10), one obtains \( M = -\frac{1}{4} \kappa^2 m \eta_0 \). The temperature of the horizon can be obtained in the standard way as \( T = |g'(x_0)|/4\pi \) and reads \( T = \frac{\kappa^2|m|}{4\pi} \propto M \). Contrary to the Schwarzschild black hole, it vanishes for \( M \to 0 \). Finally, the entropy can be obtained integrating the thermodynamical relation \( dS = T^{-1}dM \) and results \( S \propto \log M \).

2.2 \( h = 1 \)

This case has also been considered in [5]. The spacetime has constant curvature \(-\lambda^2\), except at the origin. The field equations define \( \eta \) up to a constant \( \eta_0 \), thus the most suitable ansatz for the dilaton and the metric is
\[
\eta = \eta_0 (\alpha |x| + 1), \quad g = \frac{\lambda^2 x^2}{2} + \beta |x| + \gamma. \tag{11}
\]
\(^1\)In the formula reported in [4] a factor of \( h + 1 \) was missing.
Substituting in (6)-(8), one obtains

\[ 2\beta = -2\kappa^2 m, \quad 2\alpha\gamma = -\kappa^2 m, \quad \alpha\beta = \lambda^2, \]

with solution

\[ \beta = -\kappa^2 m, \quad \alpha = -\frac{\lambda^2}{\kappa^2 m}, \quad \gamma = \frac{(\kappa^2 m)^2}{2\lambda^2}. \]

Hence, after a redefinition of \( \eta_0 \),

\[ \eta = \eta_0 \left( \lambda|x| - \frac{\kappa^2 m}{\lambda} \right), \quad g = \frac{\lambda^2 x^2}{2} - \kappa^2 m|x| + \frac{(\kappa^2 m)^2}{2\lambda^2}. \]

(12)

Notice that the dilaton field equations determine the value of \( \gamma \) in terms of the other parameters. This fact was not noticed in [5], since the dilaton equations were disregarded there.

Regular black hole solutions exist if \( m > 0 \), with horizons located at \( x_0 = \pm \kappa^2 m/\lambda^2 \). In this case the metric has a double zero at \( x_0 \) and hence a degenerate horizon is present.

The thermodynamical parameters can be computed as in the previous case and are

\[ M = 0, \quad T = 0. \]

2.3 \( h > 1 \)

The case \( h = 1 \) was somehow degenerate, implying vanishing ADM mass and temperature. We pass now to consider the case of \( h \) a generic integer. Substituting the ansatz

\[ \eta = \lambda|x| + \beta, \quad g = \frac{(\lambda|x| + \beta)^{h+1} + \gamma}{h+1}, \]

(13)
in the field equations, we obtain

\[ \beta^h = \left( -\frac{\kappa^2 m}{\lambda} \right), \quad \gamma = \frac{h-1}{2} \beta^{h+1}. \]

Thus the metric function takes the form

\[ g = \frac{1}{h+1} \left[ (\lambda|x| + \beta)^{h+1} + \frac{h-1}{2} \beta^{h+1} \right]. \]

(14)
For $h > 1$ odd, the metric is positive definite, and no horizon is present. For $h > 1$ even, horizons are present if $\lambda > 0$, $m < 0$, with $\beta = -|\kappa^2 m/\lambda|^{1/h}$. The horizons are located at

$$\lambda x_0 = \pm \left| \frac{\kappa^2 m}{\lambda} \right|^{1/h} \left[ \left( \frac{h - 1}{2} \right)^{1/(h+1)} + 1 \right].$$

In this case, the thermodynamical parameters of the black hole are

$$M = \frac{(h-1)}{4(h+1)} \lambda^{-1/h} (\kappa^2 |m|)^{(h+1)/h},$$

$$T \propto \frac{\kappa^2 |m|}{4\pi} \propto M^{h/(h+1)}, \quad S \propto M^{1/(h+1)}.$$

Notice that the ADM mass $M$ is proportional to a power of $m$, and not to $m$ as one could have expected. The reason is the energy due to the dilaton coupling.

The previous solutions can be easily generalized to the case of noninteger $h$, but we shall not discuss in detail this topic.

## 3 Multiple sources

It is interesting to consider the case in which more than one mass is present. Of course, the solutions are always segments of straight lines or arcs of convex parabola, which are connected at the location of the sources with discontinuous derivative. Therefore naked singularities are avoided only if all the point sources lie in the region included between two horizons.

We assume that two point sources of mass $m_1$ and $m_2$ are placed at the points $x_1$ and $x_2$ respectively, and hence $L_M = -m_1 \delta(x-x_1) - m_2 \delta(x-x_2)$, $T_{00} = g^{-1}[m_1 \delta(x-x_1) + m_2 \delta(x-x_2)]$, $T_{11} = 0$.

### 3.1 $h = 0$

Imposing the ansatz

$$\eta = \alpha_1 |x-x_1| + \alpha_2 |x-x_2| + \eta_0,$$

$$g = \beta_1 |x-x_1| + \beta_2 |x-x_2| + \gamma,$$

we obtain the conditions
Figure 1: A typical metric function $g$ with two sources and two horizons in the case $h = 0$.

$$2\beta_1 = -2\kappa^2 m_1, \quad 2\beta_2 = -2\kappa^2 m_2,$$

$$\kappa^2 \Delta(m_1 \alpha_2 - 2m_2 \alpha_1) + 2\gamma \alpha_1 = -\kappa^2 m_1 \eta_0,$$

$$\kappa^2 \Delta(m_2 \alpha_1 - 2m_1 \alpha_2) + 2\gamma \alpha_2 = -\kappa^2 m_2 \eta_0,$$

$$(\alpha_1 + \alpha_2) (\beta_1 + \beta_2) = \lambda^2, \quad (\alpha_1 - \alpha_2) (\beta_1 - \beta_2) = \lambda^2,$$

where $\Delta = |x_1 - x_2|$. Solutions exist for $m_1 \neq m_2$:

$$\beta_1 = -\kappa^2 m_1, \quad \beta_2 = -\kappa^2 m_2, \quad \alpha_1 = -\frac{\lambda^2}{\kappa^2} \frac{m_1}{m_1^2 - m_2^2}, \quad \alpha_2 = \frac{\lambda^2}{\kappa^2} \frac{m_2}{m_1^2 - m_2^2},$$

$$\gamma = \frac{3\kappa^2 \Delta(m_1 + m_2)}{4}, \quad \eta_0 = \frac{3\lambda^2}{2\kappa^2} \frac{\Delta}{m_1 + m_2}.$$

Hence,

$$g = -\kappa^2 m_1 |x - x_1| - \kappa^2 m_2 |x - x_2| + \frac{3\kappa^2 (m_1 + m_2) |x_1 - x_2|}{4}. \quad (16)$$

Notice that now, contrary to the case of a single source, $\eta_0$ is determined by the field equations, and $\gamma$ takes the same value as in the single particle case for such $\eta_0$.

In order to have asymptotically positive metric, we must take negative values of the masses, as in the single source case. The solution (16) possesses either two or no horizon, depending on the value of $m_1/m_2$. In particular, for $|m_1|/3 < |m_2| < 3|m_1|$, both sources are shielded by a horizon.
3.2 \( h = 1 \)

Consider now the case \( h = 1 \). The appropriate ansatz is

\[
\eta = \eta_0 (\alpha_1 |x - x_1| + \alpha_2 |x - x_2| + 1), \\
g = \frac{\lambda^2 x^2}{2} + \beta_1 |x - x_1| + \beta_2 |x - x_2| + \gamma.
\]

We have imposed the vanishing of a term linear in \( x \) in the metric. This is a choice of gauge and is equivalent to fix the position of the center of mass of the sources. For definiteness, we assume \( x_2 > x_1 \).

As usual, eq. (6) implies \( \beta_1 = -\kappa^2 m_1, \beta_2 = -\kappa^2 m_2 \). Substituting in (8), one obtains three independent equations, that can be cast in the form

\[
x_1 \alpha_1 + x_2 \alpha_2 = 0, \\
\kappa^2 (m_1 \alpha_2 + m_2 \alpha_1) = -\lambda^2 x_1 \alpha_1, \\
\kappa^2 (m_1 \alpha_2 + m_2 \alpha_1) = \lambda^2 (x_1 \alpha_1 - 1),
\]

where the first equation is a consequence of our choice of gauge. From (18) one obtains

\[
\alpha_1 = -\frac{\lambda^2 x_2}{\kappa^2 \Delta (m_1 + m_2)}, \quad \alpha_2 = \frac{\lambda^2 x_1}{\kappa^2 \Delta (m_1 + m_2)},
\]

where \( \Delta = x_2 - x_1 \).

Moreover, eq. (17) gives rise to two conditions:

\[
(\lambda^2 x_1^2 - 2\kappa^2 \Delta m_2 + 2\gamma) \alpha_1 + \kappa^2 \Delta m_1 \alpha_2 + \kappa^2 m_1 = 0, \\
(\lambda^2 x_2^2 - 2\kappa^2 \Delta m_1 + 2\gamma) \alpha_2 + \kappa^2 \Delta m_2 \alpha_1 + \kappa^2 m_2 = 0,
\]

from which one can obtain \( \gamma \) and a further relation between \( x_1, x_2 \) and the other parameters. This condition fixes the distance between \( x_1 \) and \( x_2 \). The final result is

\[
x_1 = -\frac{\kappa^2 \mu_2^2 (\mu_1 + \mu_2)}{\lambda^2}, \quad x_2 = \frac{\kappa^2 \mu_1^2 (\mu_1 + \mu_2)}{\lambda^2},
\]

where \( \mu_{1,2} = (m_{1,2})^{1/3} \). One can now write the parameters of the solution in terms of the mass of the sources only:

\[
\gamma = \frac{\kappa^4 (m_1 + m_2)^2}{2\lambda^2},
\]
Figure 2: The metric function $g$ with $m_1 = m_2$ in the case $h = 1$.

$$\alpha_1 = \frac{-\lambda^2 \mu_1^2}{\kappa^2 (\mu_1^5 + \mu_1^3 \mu_2^2 + \mu_1^2 \mu_2^3 + \mu_2^5)}, \quad \alpha_2 = \frac{-\lambda^2 \mu_2^2}{\kappa^2 (\mu_1^5 + \mu_1^3 \mu_2^2 + \mu_1^2 \mu_2^3 + \mu_2^5)}.$$  

In particular, the metric will take the simple form

$$\frac{\lambda^2 x^2}{2} - \kappa^2 m_1 |x - x_1| - \kappa^2 m_2 |x - x_2| + \frac{\kappa^4 (m_1 + m_2)^2}{2\lambda^2}. \quad (20)$$

The only solutions in which both sources are shielded by horizons are those with $m_1 = m_2 > 0$: in this case the horizons coincide with the location of the sources.

4 Geodesics

If one assumes that test particles experience the same conformal coupling as matter sources, they will not follow the geodesics of the metric $g_{\mu\nu}$, but rather those of the rescaled metric $\hat{g}_{\mu\nu} = \eta^2 g_{\mu\nu}$. This is obvious if one writes the matter action in the form, equivalent to \( \int \)

$$\int d^2x \sqrt{-g} \eta L_M = -m \int \eta ds = -m \int \sqrt{\eta^2 g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} ds, \quad (21)$$

where the dot indicates a derivative with respect to the proper time $s$.

Varying this action, the geodesic equations can also be written as

$$\frac{D^2 x^\mu}{ds^2} + \frac{\partial \ln \eta}{\partial x^\nu} \left( 2 \frac{Dx^\mu}{ds} \frac{Dx^\nu}{ds} - g^{\mu\nu} \right) = 0$$

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where $D/ds$ denotes the covariant derivative evaluated in the metric $g_{\mu\nu}$.

In terms of the metric function $g(x)$, the previous equations become

\[ \ddot{x} - \frac{g'}{2g} \dot{x}^2 - \frac{1}{2} gg' \dot{t}^2 - \frac{\eta'}{\eta}(g - \dot{x}^2), \]
\[ \ddot{t} + \left( \frac{g'}{g} + 2 \frac{\eta'}{\eta} \right) \dot{t} \dot{t} = 0. \]

These equations can be easily integrated, giving

\[ \dot{t} = \frac{E}{\eta^2 g}, \quad \dot{x} = \frac{1}{\eta^2} \sqrt{E^2 + \epsilon \eta^2 g}, \quad (22) \]

with $\epsilon = 0$ for massless particles, and $\epsilon = 1$ for massive ones, while $E$ is the energy of the test particle.

One can perform a further integration of (22). After simple manipulations one obtains the relations

\[ s = \int \frac{\eta^2}{\sqrt{E^2 + \epsilon \eta^2 g}} \, dx, \quad t = \int \frac{E}{g \sqrt{E^2 + \epsilon \eta^2 g}} \, dx, \quad (23) \]

that permit to compute $x$ as a function of $s$ and $t$, respectively.

These integrals can be easily evaluated for massless particles. For the solution (10), corresponding to $h = 0$, one gets, for positive $x$,

\[ x = \left( \frac{3\kappa^4 m^2 E}{\lambda^4} \right)^{1/3} s^{1/3} + \frac{\kappa^2 m}{\lambda^2} \eta_0 = \frac{\kappa m}{2\lambda^2 \eta_0} \left( 1 - e^{-\kappa^2 mt} \right). \]

It follows that the horizon is reached in a finite proper time $s$, but in an infinite coordinate time $t$. The integrals (23) cannot be evaluated analytically for massive particles, but it is easy to check that the qualitative behavior of the geodesics is the same as for massless particles.

For $h = 1$, the solution (12) yields for massless particles, $x > 0$,

\[ x = \left( \frac{3E}{\lambda^2 \eta_0} \right)^{1/3} s^{1/3} + \frac{\kappa^2 m}{\lambda^2} = -\frac{2}{\lambda^2 t} + \frac{\kappa^2 m}{\lambda^2}. \]

Again, the integrals cannot be explicitly evaluated for massive particles. However, it is easily seen that the behavior of the geodesics is essentially the same as in the previous case.
To compute the force experienced in the Newtonian limit by a test particle in the field generated by a point particle, we assume that the relevant spatial coordinate is that in which the metric $\hat{g}_{\mu\nu}$ takes the Schwarzschild form $ds^2 = -A dt^2 + A^{-1} dr^2$. This is obtained by defining a new coordinate $r = \int \eta^2 dx$. For $h = 0$, for example, from (10), with $m < 0$,

$$r = \frac{\kappa^2|m|}{3\lambda^2} \eta^3,$$

and

$$A = 3\kappa^2|m| \left[ r - \left( \frac{9\kappa^2|m|}{\lambda^2} \right)^{1/3} \eta^0 r^{2/3} \right].$$

The gravitational potential displays a term linear in $r$ and a correction proportional to $\eta^0$. Deriving, one obtains that the force has a constant component proportional to the mass, with a short-range correction diverging at the location of the source.

5 Conclusions

We have studied the complete static solutions of two-dimensional dilaton-gravity theories in the presence of single or multiple sources. The dilaton equations constrain the parameters of the solutions, so that no regular black hole solutions exist for odd $h$, except the case $h = 1$, where they assume a degenerate form.

We also have discussed the action of gravity on test particles. Since these should be conformally coupled, the force exerted by a point mass is not constant, as one would naively expect, except at large distances.

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