Meshless approach based on MLS with additional constraints for large deformation analysis

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Abstract. In the present work, we are interested to develop a meshless approach, based on the strong form MLS approximation with additional constraints, to solve the nonlinear elastic and elasto-plastic problems for regular and irregular distribution of points. We adopt a plastic behavior law based on the total deformation theory, which is very convenient when the physical nonlinearity is more important than the effect of irreversible process and the loading history. In plasticity, one encounters discontinuities of rigidity where the application of asymptotic developments seems difficult or impossible. To apply the Taylor series expansion, regularization methods have been adapted. The strong form MLS approximation allows us to avoid the inconvenient of the numerical integration, while the asymptotic developments help us to reduce the computation cost observed in the incremental law of plasticity and the iterative methods. For irregular points distribution, we can get an ill-posed least squares problems due to a singular moment matrix of MLS approximation. To avoid this difficulty, we propose a modified MLS approximation by introducing additional constraints which allows to increase the error functional used in the derivation of the shape functions.

1. Introduction

To meet the needs of all industry sectors, in terms of competitiveness and quality, many studies have been devoted to numerical simulations of metal forming processes in order to optimize and ensure the feasibility of some products. The numerical simulations allow to solve the complex problems coupling several nonlinearities (geometric nonlinearity, material nonlinearity and contact nonlinearity). In general, these problems are solved classically by Finite Element Method (FEM) coupled with incremental iterative methods [1-3]. The majority of numerical simulation software use the resolution algorithms based on the Newton-Raphson Method (NRM) to solve nonlinear problems. The Asymptotic Numerical Method (ANM) [4, 5] is an alternative method for solving the nonlinear problems which allows to reduce the computation cost in terms of inversions number of the stiffness matrix observed in the incremental and iterative methods such as the NRM. ANM is characterized by three techniques: perturbation, discretization and continuation. However, this method is based on FEM in most research works in the literature.
the difficulties of the latter reside in the mesh distortion due to the large deformation, the complex geometries and the moving boundary, etc. To overcome these difficulties related to discretization with mesh, meshless methods have been developed since the 90s [5-13].

In the present work, we are interested to develop a meshless approach based on the strong form of the Moving Least Square (MLS) approximation [5, 6, 7, 8] to solve the nonlinear elastic and elasto-plastic problems for regular and irregular distributions of points. In this paper, we adopt a plastic behavior law based on the total deformation theory. This is very convenient when the physical nonlinearity is more important than the effect of irreversible process and the loading history. In plasticity, one encounters discontinuities of behavior laws where the application of asymptotic developments seems difficult or impossible. To apply the Taylor series expansion, regularization methods have been adapted. The strong form MLS approximation allows us to avoid the inconvenient of the numerical integration and the asymptotic developments help us to reduce the computation cost observed in the incremental law of plasticity [8] and the iterative methods. For irregular distribution of points, we can get an ill-posed least squares problems due to a singular moment matrix of MLS approximation. In this paper we propose a modified MLS approximation by introducing an additional constraints which allows to increase the error functional used in the derivation of the shape functions.

The outline of this paper is as follows: Section 2 gives a strong formulation of 2D elasto-plastic contact problems. Section 3 presents the proposed strong form MLS approximation based on a high order approach. While some typical numerical applications for illustrating its performance are presented in Section 4 followed by conclusion in Section 5.

2. Strong formulation of 2D elasto-plastic contact problems

This section presents the equilibrium equations in large deformation elasto-plasticity context.

2.1. Equilibrium equation with contact conditions

The first body is considered deformable which occupies the domain $\Omega$ and the second one is a rigid foundation where $\partial \Omega_C$ is a candidate contact surface between the two bodies (see Fig.1). On the boundaries $\partial \Omega_F$ and $\partial \Omega_d$ a load and displacement are applied respectively on the deformable body.

![Figure 1. 2D deformable solid in contact with a rigid foundation.](image)

In Fig.1, the initial distance between the body and the rigid foundation and the unit outward normal to boundary $\partial \Omega_C$ are $g$ and $n_C$ respectively. The equilibrium equations of body is given
by:
\[
\text{div} T = 0 \quad \text{in} \quad \Omega_0
\]  
(1)
where \( T \) is the first Piola-Kirchhoff stress tensor which is used in the case of a Lagrangian formulation in large deformations and defined by:
\[
T = ES
\]  
(2)
where \( S \) is the second Piola-Kirchhoff stress tensor or Piola-Lagrange tensor and \( E \) is the transformation gradient tensor.

The boundary conditions can be written as follows:
\[
\begin{align*}
T.N &= \lambda F \text{ on } \partial \Omega_F \\
U &= \lambda U_d \text{ on } \partial \Omega_d
\end{align*}
\]  
(3)
where \( N \) is the unit outward normal to boundary \( \partial \Omega_F \), \( \lambda \) is a control parameter, \( F \) is the force vector applied on the boundary \( \partial \Omega_F \), \( U_d \) is an imposed displacement on the boundary \( \partial \Omega_d \). The contact condition can be written in the form:
\[
T.N = C \text{ on } \partial \Omega_C
\]  
(4)
where \( N \) is the unit outward normal to boundary \( \partial \Omega_C \), \( C \) is the stress vector at a particle of this boundary. In this approach, we consider the unilateral contact without friction which is defined by the following conditions:
\[
\begin{align*}
C_N &= 0 \quad \text{if} \quad \delta \neq 0 \\
C_N &= 0 \quad \text{if} \quad \delta = 0 \\
C_t &= 0
\end{align*}
\]  
(5)
where \( C_N \) and \( C_t \) are the normal and tangential components of stress vector respectively and \( \delta \) is the current clearance.

2.2. Behavior law
The elastoplastic behavior law is based on the deformation theory of plasticity. This theory does not take into account the elastic discharge and it is well suited for problems in which physical nonlinearity is more important than the effect of irreversible process and the loading history. The Ramberg-Osgood law and the hyperbolic law are implemented in this work.

3. The proposed strong form MLS approximation
3.1. Theory and principle of the proposed MLS approximation
Consider an unknown scalar function of a variable field \( u(x) \) defined in a domain \( \Omega \). The MLS approximation \( u^h \), presented by Nayroles et al. in 1992 [13], of \( u(x) \) is defined at a point \( x \in \Omega \) by:
\[
u^h(x) = \sum_{i=1}^{m} p_i(x) a_i(x) = \langle p(x) \rangle \{ a(x) \}
\]  
(6)
where \( \langle p(x) \rangle \) is the basis functions dependent on the spatial coordinates and \( m \) is the number of the basis functions. The basis functions \( \langle p(x) \rangle \) is often constructed using the monomials of Pascal’s triangle. In the equation 6, \( \{ a(x) \} \) is a vector of \( m \) components. Note that the coefficients of the vector \( \{ a(x) \} \) are functions of the spatial variable \( x \). These coefficients can be obtained by minimizing the following weighted discrete form:
\[
J(x) = \sum_{i=1}^{N} W(x - x_i, h)(\langle p(x) \rangle \{ a(x) \} - u_i)^2
\]  
(7)
where $N$ is the number of points in the support domain of coordinate point $x$ for each weight function $W(x - x_i; h) \neq 0$, $u_i$ is the value of $u$ at $x = x_i$ and $W(x - y, h)$ is a weighting function with a compact support $h$. The stationarity of $J(x)$ with respect to $\{a(x)\}$ leads to the following linear system:

$$ [A(x)]\{a(x)\} = \sum_{i=1}^{N} W(x - x_i, h)\{p(x_i)\}u_i $$

where $[A(x)]$ is called the weighted moment matrix. For a given distribution of points, the functional (7) does not include sufficient constraints to ensure an unique solution, so the Eq.(8) show that the vector coefficients $\{a(x)\}$ has multiple solutions if the moment matrix $[A(x)]$ is singular. The idea of enriching this approximation is inspired of the works [14-16]. Based on these works and in the case of quadratic bases we propose to add Additional Constraints (AC) to the quadratic form as follows:

$$ J(x) = \sum_{i=1}^{N} W(x - x_i, h)(< p(x_i) > \{a(x)\} - u_i)^2 + AC $$

where

$$ AC = \mu_xa_x^2 + \mu_ya_y^2 + \mu_za_z^2 $$

with $\mu = m$ positive weights which equal $m - 3$. These constraints ensure that, in the case of the singular moment matrix, the solution is the one whose the coefficients for the higher order monomials in the bases equal to zero. After the minimization of the functional (11), we obtain:

$$ \frac{\partial J(x)}{\partial \{a(x)\} } = ([\tilde{A}(x)]\{a(x)\} - \sum_{i=1}^{N} W(x - x_i, h)\{p(x_i)\}u_i = 0 $$

where $[\tilde{A}(x)] = [A(x)] + [H]$ is the modified moment matrix which is non-singular and $[H] = \begin{bmatrix} O_{3x3} & O_{3x\tilde{m}} \\ O_{\tilde{m}x3} & diag(\mu) \end{bmatrix}$ is a matrix with all elements equal to zero except the last three diagonal entries.

Solving $\{a(x)\}$ from (11) and substituting it into equation (6), the meshless approximation can be defined as:

$$ u^h(x) = \sum_{i \in S_n} \phi_i(x)u_i $$

where $S_n$ is the set of all the points in the support domain, $u_i$ is the point parameter of $u$ at $x = x_i$ and $\phi(x)$ is the modified shape functions defined as

$$ \phi_i(x) = \{p(x)\}^T[\tilde{A}(x)]^{-1}\{p(x_i)\}w(x - x_i, h_i) $$

3.2. A high order approach based on strong form MLS approximation

This approach is based on regularisation technique, Taylor series expansions technique, meshless method and continuation procedure. In this meshless approach, the idea is to draw numerically the branches of solution generating not a sequence of points but a sequence of branches parts. The regularization technique and a quadratic form are required to apply the method of series expansions. In this context, the plasticity conditions of Ramberg-Osgood behavior and
The hyperbolic behavior are replaced by a regularized functions and a quadratic forms. So we can summarize the problem as follows:

\[
\begin{cases}
[\text{div}] \{T\} = 0 & \text{in } \Omega_0 \\
\{T\} = ([III] + [B(G)]) \{S\} \\
[AA(\beta)] \{S\} = (1 + \alpha\eta_1) [D] \{\gamma\} \\
\{\gamma\} = ([II] + \frac{1}{2} [A(G)]) \{G\} \\
\{G\} = [L] \{U\} \\
S_{eqd\beta} = (n - 1)\beta dS_{eq} & \text{(case of Ramberg-Osgood law)} \\
\beta(1 - \chi) = \zeta_E & \text{(case of hyperbolic law (}\eta_1 = 0)\text{)} \\
\chi = \frac{3}{2} \langle S \rangle [M] \{S\} \\
S^2_{eq} = (\chi + \eta_2^2) S^2_y \\
[N] \{T\} = \lambda \{F\} & \text{on } \partial\Omega_F \\
[N] \{T\} = \{C\} & \text{on } \partial\Omega_C \\
\{U\} = \lambda \{U_d\} & \text{on } \partial\Omega_d \\
\end{cases}
\]

Using series expansion technique and identifying term to term according to the powers of an expansion parameter "a", we obtain a sequence of linear problems, which are written at each
order $p$ as follows:

$$
\begin{align*}
\{T_p\}_p &= \{K_M\}_p \{G_p\} + \{T_{nl}\}_p \\
\{S_p\}_p &= (1 + \alpha \eta^p) [BB]^{-1} [D] \{\gamma_p\} + \{S_{nl}\}_p \\
\{\gamma_p\}_p &= ([I] + [A(G_0)]) \{G_p\} + \{\gamma_{nl}\}_p \\
\{G_p\}_p &= [L] \{U_p\} \\
\beta_p &= \frac{(n-1)g_0}{S_{eqp}} S_{eqp} + \beta_{nl} (\text{case of Ramberg-Osgood law}) \\
\beta_p &= \frac{\beta_0}{1 - \chi_0} \chi_p + \beta_{nl} (\text{case of hyperbolic law } (\eta_1 = 0)) \\
\chi_p &= \frac{3}{S_y} \{S\}_0 \{M\} \{S\}_p + \chi_{nl} \\
S_{eqp} &= \frac{1}{2S_{eqp}} \chi_p + S_{nl} \\
[N] \{T_p\}_p &= \lambda_p \{F\} \text{ on } \partial \Omega_F \\
[N] \{T_p\}_p &= \{C_p\} \text{ on } \partial \Omega_C \\
\{U_p\}_p &= \lambda_p \{U_d\} \text{ on } \partial \Omega_d \\
\end{align*}
$$

where $\{T_{nl}\}_p$, $\{S_{nl}\}_p$, $\{\gamma_{nl}\}_p$, $\gamma_{nl}$, $\chi_{nl}$, $\beta_{nl}$ and $S_{nl}$ are terms that depend on solutions at previous orders and the matrice $[BB] = [AA(\beta_0)] + \frac{3(n-1)g_0}{2S_{eqp}} [D] [M] \{S\}_0 \{S\}_0 [M]$.

3.2.1. Strong form MLS approximation

To apply the meshless approximation to the problem (15), the modified MLS shape functions presented in section 3.1 are used to approximate the displacements $\{U\} = ^T < u, v >$, in the bi-dimensional case, using a set of points in a local support domain of the considered point. By inserting these approximations in the system of equations (15), and after substitution and assembly techniques, the problem (15) is written in the following condensed form:

$$
\begin{align*}
\{K_T\}_p \{U_k\}_p &= \lambda_k \{F\} + \{F_{nl}\}_k \\
\{S_p\}_p &= (1 + \alpha \eta^p) [BB]^{-1} [D] \{\gamma_p\} + \{S_{nl}\}_p (\text{behavior law})
\end{align*}
$$

where $[K_T]$ is the stiffness matrix evaluated at point solution $\{T_0\}$; $\{S_0\}$; $\{\gamma_0\}$; $\{S_{eqp}\}$; $\{G_0\}$; $\{U_0\}$; $\lambda_0$), $\{F_{nl}\}_k$ and $\{S_{nl}\}_p$ are the terms depending on solutions at previous orders, $\{U_k\}_p$ is the unknown vector that collects all nodal displacements and $\eta_1 = 0$ for hyperbolic behavior law. The whole solution of problem is obtained step-by-step using a continuation path following method.

4. Numerical applications

The used mechanical characteristics of the considered structure are: Young’s modulus $E = 200 GPa$, Poisson’s ratio $\nu = 0.3$, elastic yield $S_y = 240 MPa$. In these numerical examples, we
use a quadratic basis functions (number of the basis functions \( m = 6 \)), so the positive weights of the additional constraints \( \tilde{m} = 3 \). For simplicity, all the additional constraints are the same weights \( \mu_x^2 = \mu_{xy} = \mu_y^2 = \mu \) (see Eq.10).

4.1. Stability and accuracy analysis of the proposed approach

We consider the geometrical nonlinearity of a bi-dimensional elastic structure in tension using a 2D plate of geometry (100mm \( \times \) 50mm). This plate is fixed at \( x = 0 \) and submitted to an imposed load \( \lambda F \) at \( x = L \); with \( F = 1 \). In the following we will show the effectiveness of the modified strong form MLS approximation with an irregular distribution of points and \( \mu \neq 0 \). A random perturbation of 20\% compared to the regular distribution is performed. Numerical tests show that the calculation diverges if \( \mu < 0.5 \) as shown in Fig.2-(a) and Fig.2-(b) for two values \( \mu = 0 \) and \( \mu = 0.1 \) respectively. Moreover, the calculation converge if \( \mu = 0.5 \) as shown in Fig.2-(c) which represents initial and deformed configurations of the plate for \( \lambda = 1.3591 \times 10^5 \).

![Initial and deformed configurations of the plate for the case of the irregular distribution of points with a random perturbation of 20\%, size of influence domain \( h_i = 2dr \) and different values of \( \mu \)]

\( \text{(a)} : \mu = 0 \quad \text{(b)} : \mu = 0.1 \quad \text{(c)} : \mu = 0.5 \)

**Figure 2.** Initial and deformed configurations of the plate for the case of the irregular distribution of points with a random perturbation of 20\%, size of influence domain \( h_i = 2dr \) and different values of \( \mu \)

In the second test for the case of the irregular distribution of points, we consider a random perturbation of 40\% compared to the regular distribution. Several tests (see Fig.3-(a), Fig.3-(b)) show that the calculation diverges if \( \mu < 1 \). Fig.3-(c) represents initial and deformed configurations of plate for \( \lambda = 1.5220 \times 10^5 \) with \( \mu = 1 \).
Figure 3. Initial and deformed configurations of plate for the case of the irregular distribution of points with a random perturbation of 40%, size of influence domain $h_i = 2dr$ and different values of $\mu$.

In Fig.4, we draw the evolution of load $\lambda$ with respect to displacements $U$ by the proposed approach and the reference approach at node of coordinate $(x = 100, y = 0)$ for irregular distributions of points. This figure indicates a good agreement with the obtained results by the reference approach which the maximum relative error does not exceed 0.05% and 2% for the random perturbation of 20% and 40% respectively.

Figure 4. Evolution of load $\lambda$ with respect to displacements $U$ for irregular distributions of points compared to the reference solution at point of coordinate $(x = 100, y = 0)$

4.2. Analysis of a bi-dimensional elasto-plastic plate in tension

We consider the same bi-dimensional structure with two nonlinearities: geometrical and material. We assume that the plate made of materials described according to Ramberg-Osgood behavior including an elastic term and a plastic term with a hardening exponent $n = 3.5$. We use a total number of points equal 231.

Fig.5-(a) represents the evolution of equivalent stress $S_{eq}$ with respect to the equivalent strain $\gamma_{eq}$ at point of coordinate $(x = 100, y = 0)$. This figure shows the solution obtained by the proposed algorithm with regular distribution of points and $\mu = 0$ and also the solution obtained with irregular distribution of points and $\mu \neq 0$ ($\mu = 0.5$ and $\mu = 1$). This solution requires 27
inversions of the tangent matrix by the proposed approach and 4650 inversions of the tangent matrix by the reference approach based on NRM. We note that the proposed approach gives the similar results to those of reference solution with a reduction of computation time. Fig.5-(a) shows the stress distribution $S_{eq}$ for load $\lambda = 2145.3$ in the deformed configuration of plate.

![Graph](image)

**Figure 5.** (a): Evolution of equivalent stress $S_{eq}$ versus the equivalent strain $\gamma_{eq}$ at point of coordinate (100, 0). (b) Stress distribution in the deformed configuration for load $\lambda = 2079.4$

In the following numerical tests, the results of the proposed approach are controlled by a residue less than or equal to $10^{-5}$. The Fig. 6 shows Ramberg-Osgood stress-strain curve with the same Young’s modulus $E = 200 GPa$ and yield stress $S_y = 240 MPa$ but with different hardening exponents $n = 5, 10, 15$.

![Graph](image)

**Figure 6.** Ramberg-Osgood stress-strain curve with $E = 200 GPa$, $\nu = 0.3$, $S_y = 240 MPa$ and $n = 5, 10, 20$.

In table 1, we note that when the hardening exponent $n$ increases, the number of branches...
decreases for a fixed truncation order $p = 15$. So the proposed algorithm is very efficient and less time consuming for large values of $n$.

Table 1. Influence of hardening exponent $n$ on the number of branches.

| Hardening exponent $n$ | Number of branches |
|------------------------|--------------------|
| 3.5                    | 27                 |
| 5                      | 30                 |
| 10                     | 42                 |
| 15                     | 62                 |

Fig. 7 shows the deformed configuration and the stress distribution for $n = 5$ with $\lambda = 1259$ and $n = 15$ with $\lambda = 527$.

In the second example of this section, we consider a bi-dimensional elasto-plastic plate under traction clamped on the edge $x = 0$ and subjected to an imposed displacement $U = \lambda U_d$ ($U_d$ is a given displacement) on the edge $x = L$. In the other side, the structure is a bi-dimensional elasto-plastic plate made of materials described according to hyperbolic behavior which allows to simulate perfect elasto-plastic material where the hardening is not taken into account. In this law, once the yield stress is reached, the stress becomes constant and only strain continues to increase.

The Fig. 8-(a) shows the evolution of equivalent stress $S_{eq}$ with respect to the equivalent strain $\gamma_{eq}$ at point of coordinate $(x = 100, y = 0)$ obtained by the proposed approach. To achieve a deformation of $\gamma_{eq} = 0.16$, this solution has required 82 inversions of the tangent matrix by the
proposed approach and 4000 inversions of the tangent matrix by the approach in the research work [8]. We note that the proposed approach based on the hyperbolic plastic law gives better results where it reaches exactly the yield stress value $S_y = 240\, \text{MPa}$. Moreover, the reduction of the computation cost obtained with the proposed approach compared to that of the previous work is very considerable. The Fig.8-(b) shows the deformed configuration in the plate for $\lambda = 16.3$.

**Figure 8.** (a): Evolution of equivalent stress $S_{eq}$ with respect to the equivalent strain $\gamma_{eq}$ at node 21 ($100, 0$). (b): Deformed configuration for load $\lambda = 16.3$

4.3. **Bi-dimensional elasto-plastic plate subjected to an imposed displacement with contact**

In this example, we consider three types of nonlinearities: geometrical nonlinearity, material nonlinearity and contact nonlinearity. Let consider a 2D elasto-plastic plate, made of materials described according to Ramberg-Osgood behavior, with the following geometrical and mechanical characteristics: length $L = 100\, \text{mm}$, height $H = 50\, \text{mm}$, Young’s modulus $E = 200\, \text{GPa}$, Poisson’s ratio $\nu = 0.3$, elastic yield $S_y = 240\, \text{MPa}$ and hardening exponents $n = 3.5$. This plate is simply supported at the edge $x = 0$ and subjected to an imposed displacement $V = \lambda V_d$ at the edge $y = H$; with $V_d$ is a given displacement. In this application, we adopt the mesh ($21 \times 11$) and the regularization parameter of the contact law $\eta_2 = 0.02$. The Fig.9-(a) illustrates the equivalent stress curve $S_{eq}$ versus the equivalent strain $\gamma_{eq}$ at node 21 ($x = 100, y = 0$). The Fig.9-(b) represents the initial and deformed configurations of the plate. This shows the efficiency and the robustness of our approach for calculating the elasto-plasticity contact with large deformation. The proposed approach requires 50 inversions of the tangent matrix. According to a numerical test, we have remarked that when the regularization parameter $\eta_2$ decreases the number of inversions of the tangent matrix increases.

**Figure 9.** (a): Evolution of equivalent stress $S_{eq}$ versus the equivalent strain $\gamma_{eq}$ at node 21 ($100, 0$). (b): Deformed configuration

4.4. **Bending of a bi-dimensional elasto-plastic plate in contact with a curved rigid foundation**

In this example, we consider the contact between a bi-dimensional elasto-plastic plate and a curved rigid foundation. We adopt the same data as in the example 1 (see section 4.1) with the regularization parameter of the contact law $\eta_2 = 0.02$. In this example, we have observed
the same remarks that in previous example for the parameter $\eta_2$ othe regularized contact law. In Fig.10, we present the deformed configuration in red color for $\lambda = 111.61$. The proposed approach requires 350 inversions of the tangent matrix.

![Deformed configuration of the plate for load $\lambda = 111.61$.](image)

5. Conclusion
In this work, we present a contribution for the modeling of nonlinear elastic and elasto-plastic structures with or without contact. This approach allows to overcome the difficulties encountered in the previous work [8]. For example, this approach is well adapted for irregular distributions of points, it allows to take into consideration the effect of hardening using the Ramberg-Osgood behavior law, it also allows to treat the elastic-perfectly plastic problems with good precision. The high flexibility of this mesh-free approach makes it a good alternative for large deformation problems and elasto-plastic analysis of structures with or without contact. An improvement of the developed formulation will also be considered in the future work to consider different type of structures [17, 18, 19].

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