Generalized Leverage Effects in Asset Returns

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Abstract

We discuss generalizing the correlation between an asset’s return and its volatility—the leverage effect—in the context of stochastic volatility. While it is a long standing consensus that leverage effects exist, empirical evidence paradoxically show that most individual stocks either do not exhibit this correlation or are very weak. This, at least partially, is due to an assumption that the correlation is linear, i.e., the effect of large shocks and small fluctuations changes linearly. We relax this standard assumption of linear correlation to a nonlinear generalization in order to capture the complex nature of this effect using a newly developed Bayesian sequential computation method that is fast, efficient, and on-line. Examining 615 stocks that comprise the S&P500 and Nikkei 225, we find nearly all of the stocks to exhibit leverage effects, of which many would have been lost under the standard linear assumption. We further the analysis by exploring whether there are clear traits that characterize the complexity of the leverage effect and find that different countries and certain sectors to have tendencies for more complex leverage effects.

Keywords: Bayesian Analysis, Particle Learning, State Space Models

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1 Introduction

The estimation, inference, and prediction of volatility is one of the most crucial aspects in analyzing data with variability in order to make informed decisions. In the field of finance and economics, volatility of financial assets has been investigated with great scrutiny to further the understanding of the mechanics and structure of price movement (Taylor 1994; Ghysels et al. 1996; Shephard 1996). One aspect of volatility that has gathered special interest is the correlation between an asset’s return and it’s volatility; coined the leverage effect. It is often claimed that this correlation is negative, implying that a negative (positive) shock to an asset’s return results in an increase (decrease) in volatility. This phenomenon is intuitive, as we can expect– and often observe– that an asset under distress exhibits more variability and uncertainty compared to an asset that is stable or increasing in price. The term leverage refers to an economic interpretation given by Black (1976) and Christie (1982). They state that, when an asset’s price declines, the company’s relative debt increases, making the balance sheet leveraged, resulting in the company being riskier and therefore it’s market value more volatile (see Bekaert and Wu 2000, for example, for different interpretations and comparisons of the leverage effect). Though only a hypothesis, this explanation has held weight in the field and the effect is widely believed to exist with supporting evidence examining major stock indices (Nelson 1991; Glosten et al. 1993; Dumas et al. 1998, for ARCH-type models and Jacquier et al. 1994; West and Harrison 1997; Jacquier et al. 2004; Yu 2005; Omori et al. 2007; Nakajima and Omori 2009, 2012; Takahashi et al. 2013; Shirota et al. 2014, for SV-type models). However, contrary to consensus, the lack of empirical evidence of the effect from individual stocks is paradoxical; with most stocks exhibiting zero or very weak correlation between asset returns and volatility.

We postulate that this is caused by the simple, but common, representation of the correlation: Most volatility models in the literature, basic or advanced, assume that the relationship between an asset’s return and its volatility is linear, even though many advances have been made on other aspects of the model. However, it is counter-intuitive to think that a large shock in return effects volatility with the same linear relationship as small daily fluctuations. This notion has promoted research in considering more com-
plex leverage effects. For example, Hansen et al. (2012) introduced a more general form of the leverage effect by using a leverage function in the form of an Hermite polynomial within a GARCH framework. In the context of stochastic volatility (SV) models, there has been no advances in this direction, to the best of the authors’ knowledge, even though SV models are known to outperform ARCH-type models due to its flexibility in capturing traits seen in asset returns (Geweke 1994; Fridman and Harris 1998; Kim et al. 1998). The advances are hindered, partly, due to the computational complexity SV models entail, as it requires complex Markov chain Monte Carlo (MCMC) methods that are hard to sample and tune.

We respond to this movement by extending the SV model to include the leverage function similar to Hansen et al. (2012) (as well as other forms) to examine the nonlinear dynamics of the correlation between an asset’s return and volatility. To achieve this, we develop an effective Bayesian computation method using sequential Monte Carlo (SMC) by extending the particle learning method of Carvalho et al. (2010) enabling estimation of the parameters of interest in a fast, efficient, and on-line manner. With the new model and algorithm, we are able to examine and analyze the leverage effect over a large number of equity assets and over time, and find strong evidence for the leverage effect where it is unobserved under the simple linear representation. This leads to further insight into the characteristics of the leverage effect for different markets/sectors.

We will define the SV model with nonlinear leverage functions in Section 2 and its estimation method in Section 3. Section 4 will present the empirical study where we apply our model to daily returns of all stocks that compose the S&P500 and Nikkei 225, with additional comments and further discussions in Section 5.

2 Stochastic Volatility Model with Leverage Functions

The basic SV model is given by the following nonlinear dynamic model (West and Harrison 1997; Carvalho and Lopes 2007; Lopes and Polson 2010),

\[
\begin{align*}
    y_t &= \exp \left( \frac{x_t}{2} \right) \epsilon_t, \\
    x_t &= \mu + \beta x_{t-1} + \eta_t,
\end{align*}
\]

\[
\begin{bmatrix}
    \epsilon_t \\
    \eta_t
\end{bmatrix} \sim i.i.d. \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \tau^2 \end{bmatrix} \right),
\]

\( (1) \)
where \( y_t \) is the asset return on day \( t \) and \( x_t \) is the time-varying log volatility. The SV model is a state space model with observation noise \( \epsilon_t \) and state noise \( \eta_t \). Both \( \epsilon_t \) and \( \eta_t \) are mutually and serially independent in this model represented by the off diagonal elements in the covariance matrix being zero. Yu (2005) compares two types of SV models with leverage (addressed as asymmetric SV in the paper) to represent the leverage effect in the model. The widely used of the two assumes the correlation between \( \epsilon_t \) and \( \eta_t \) as the following

\[
\begin{align*}
\epsilon_t &= \exp \left( \frac{x_t}{2} \right) \epsilon_t, \\
\eta_t &= \mu + \beta x_{t-1} + \eta_t, \\
\begin{bmatrix} \epsilon_t \\ \eta_t \end{bmatrix} &\sim i.i.d. \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \tau \\ \rho \tau & \tau^2 \end{bmatrix} \right),
\end{align*}
\]

(2)

where \( \rho \) is the correlation between the observation noise \( \epsilon_t \) and the state noise \( \eta_t \), allowing for a contemporaneous dependence in the variance.

The state noise \( \eta_t \) can then be rewritten as

\[
\eta_t = \rho \epsilon_t + \sqrt{1 - \rho^2} u_t.
\]

(3)

where \( u_t \sim \mathcal{N}(0, 1) \). Hence, eqn. (2) can be rewritten as a nonlinear state space model with uncorrelated observation noise \( u_t \) and state noise \( \epsilon_t \):

\[
\begin{align*}
y_t &= \exp \left( \frac{x_t}{2} \right) \epsilon_t, \\
x_t &= \mu + \beta x_{t-1} + \varphi \epsilon_{t-1} + \omega u_t, \\
\begin{bmatrix} \epsilon_t \\ u_t \end{bmatrix} &\sim i.i.d. \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right),
\end{align*}
\]

(4)

where \( \varphi = \rho \tau \), and \( \omega = \sqrt{1 - \rho^2} \).

The model representation of eqn. (4) assumes linear correlation, as in the term, \( \varphi \epsilon_{t-1} \), is linear, though the assumption is not founded in evidence nor theory but in convenience. We extend the model to include a nonlinear leverage function, \( \ell(\cdot) \):

\[
\begin{align*}
y_t &= \exp \left( \frac{x_t}{2} \right) \epsilon_t, \\
x_t &= \mu + \beta x_{t-1} + \ell(\epsilon_{t-1}) + \omega u_t, \\
\begin{bmatrix} \epsilon_t \\ u_t \end{bmatrix} &\sim i.i.d. \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right),
\end{align*}
\]

(5)

where the nonlinear leverage function \( \ell(\cdot) \) can be any function that transmits an impact
of the previous shock in the asset price $\epsilon_{t-1}$ onto the log volatility $x_t$. If $\ell(\epsilon_{t-1})$ is a linear function, say $\varphi \epsilon_{t-1}$, the SV model with nonlinear leverage in eqn. (5) is reduced to eqn. (4). Although the exact functional form of $\ell(\cdot)$ is unclear, we do not want to impose parametric assumptions upon it. We instead introduce a more flexible polynomial approximation of the leverage function. In this paper, we report the results using Hermite polynomials, following Hansen et al. (2012), to construct the leverage function. We have tested numerous other functional forms (including basic and Laguerre polynomials) but have chosen Hermite polynomials due to its superior predictive properties.

The $k$-th order Hermite polynomial $H_k(z)$ is defined as

$$H_k(z) = (-1)^k \exp \left( \frac{z^2}{2} \right) \frac{d^k}{dz^k} \exp \left( -\frac{z^2}{2} \right),$$

where the first seven Hermite polynomials are

$$H_0(z) = 1, \quad H_1(z) = z, \quad H_2(z) = z^2 - 1, \quad H_3(z) = z^3 - 3z,$$

$$H_4(z) = z^4 - 6z^2, \quad H_5(z) = z^5 - 10z^3 + 15z, \quad H_6(z) = z^6 - 15z^4 + 45z^2 - 15.$$

Hermite polynomials have some desirable properties for the analysis of leverage effects:

1. Zero Expectation:
   When $z$ follows a normal distribution with zero expected value and unit variance, the expected value of the Hermite polynomial is equal to zero for any $k$. In other words,
   $$E(H_k(z)) = \int_{-\infty}^{\infty} H_k(z) \phi(z) dz = 0, \quad z \sim N(0, 1),$$
   where $\phi(z) \propto \exp(-\frac{z^2}{2})$.

2. Orthogonality:
   Hermite polynomials are orthogonal to each other with respect to the weight func-
\[ \phi(z) \propto \exp\left(-\frac{z^2}{2}\right). \] Thus,

\[ E(H_j(z)H_k(z)) = \int_{-\infty}^{\infty} H_j(z)H_k(z)\phi(z)dz = \begin{cases} n! & (k = j); \\ 0 & (k \neq j). \end{cases} \]

Given the two properties above, they form the orthogonal basis of the Hilbert space of leverage functions such that \( \int_{-\infty}^{\infty} |\ell(z)|^2 \phi(z)dz < \infty \).

We now define the approximated leverage function using Hermite polynomials as

\[ \ell^H_k(\epsilon_{t-1}) := \varphi_1 H_1(\epsilon_{t-1}) + \cdots + \varphi_k H_k(\epsilon_{t-1}). \quad (7) \]

Finally, the SV model with the \( k \)-th order nonlinear leverage function is defined as

\[
\begin{cases}
    y_t = \exp\left(\frac{x_t^2}{2}\right) \epsilon_t, \\
    x_t = \mu + \beta x_{t-1} + \ell^H_k(\epsilon_{t-1}) + \omega u_t, \\
    \begin{bmatrix}
        \epsilon_t \\
        u_t
    \end{bmatrix}
    \sim \text{i.i.d. } \mathcal{N}\left(\begin{bmatrix}
        0 \\
        0
    \end{bmatrix}, \begin{bmatrix}
        1 & 0 \\
        0 & 1
    \end{bmatrix}\right).
\end{cases} \quad (8)
\]

This SV model with nonlinear leverage, or SV-NL model in short, is the model that will be used in our empirical study of leverage effects in individual stock price returns (Section 4).

3 Bayesian Sequential Computation

In this section, we elaborate on our proposed particle learning method for the SV-NL model in eqn. (8). In general, there are two methods for Bayesian computation of SV models: MCMC based (Chib et al. 2002; Omori et al. 2007) and SMC based (Polson et al. 2004; Lopes and Tsay 2011; Creal 2012). While both methods have their pros and cons (see, for example, Gamerman and Lopes 2006), there is merit in using SMC in financial applications because of its on-line nature (i.e. the ability to produce forecasts fast), which is appealing to finance practitioners. However, SMC methods struggled with the problem of parameter learning (Gordon et al. 1993; Kitagawa 1996; Pitt and Shephard 1999) and multiple methods have been proposed (Liu and West 2001; Storvik 2002; Fearnhead 2002; Polson et al. 2008; Johannes and Polson 2008; Johannes et al. 2008). We will
not review the literature of SMC methods, instead we will review and expand on the most recent development by Carvalho et al. (2010), which enables parameter learning in nonlinear, non-Gaussian models under certain conditions.

3.1 Particle learning with auxiliary variables: The algorithm

Parameters of interest in the model are unknown in most applications of finance and econometrics, and for this particular model in eqn. (8), inferring on the parameters in the leverage function is of utmost importance.

When a state space model depends on unknown but static parameters $\theta$, we have the following state space representation,

\[
\begin{align*}
  y_t &\sim p(y_t|x_t, \theta), \\
  x_t &\sim p(x_t|x_{t-1}, \theta),
\end{align*}
\]

where we need to evaluate the posterior distribution of $\theta$ given the observations $y_{1:t}$, $p(\theta|y_{1:t})$, as well as the latent state, $x_{1:T}$. In the framework of SMC methods, $p(\theta|y_{1:t})$ is sequentially updated as new observations arrive in an on-line manner. This is called particle learning, as we learn the parameters using particle filtering. A naïve particle learning algorithm, which is a particle filter for the extended state vector $z_t = (x_t, \theta)$, is defined as follows. Let $\{z_t^{(i)} = (\hat{x}_t^{(i)}, \hat{\theta}_t^{(i)})\}_{i=1}^N$ and $\{\tilde{z}_t^{(i)} = (\tilde{x}_t^{(i)}, \tilde{\theta}_t^{(i)})\}_{i=1}^N$ denote particles, $i = 1 : N$, jointly generated from $p(z_t|y_{1:t-1})$ and $p(z_t|y_{1:t})$, respectively. Then, the particle approximation of the Bayesian learning process is given by

\[
p(z_t|y_{1:t-1}) \simeq \frac{1}{N} \sum_{i=1}^N p(z_t^{(i)}|z_{t-1}^{(i)}),
\]

\[
p(z_t|y_{1:t}) \simeq \sum_{i=1}^N W_t^{(i)} \delta(z_t - \hat{z}_t^{(i)}), \quad W_t^{(i)} = \frac{p(y_t|z_t^{(i)})}{\sum_{i=1}^N p(y_t|z_t^{(i)})}.
\]

Although the above learning algorithm is easy to implement, researchers have been aware that, with the naïve particle learning, particles tend to degenerate in the process of resampling. As a more efficient alternative, Carvalho et al. (2010) propose a particle learning (PL) algorithm that resamples and updates the sufficient statistics of the posterior distributions of the parameters. Let $s_t$ and $S(\cdot)$ denote the vector of the sufficient
where a function $\theta$ sequential Bayes filter (we suppress work. auxiliary particle filter the model in eqn. (8), however, neither is possible. To circumvent these problems, we adopt

$p_{\text{tional form of }} p_{\text{by Pitt and Shephard (1999) within the particle learning frame-}}$

$g$ 

When this is not the case, we introduce an auxiliary variable $g(x_{t-1})$ where a function $g(\cdot)$ is chosen so as to make $p(y_t|g(x_{t-1}))$ resemble $p(y_t|x_{t-1})$ sufficiently

\begin{align}
\text{Algorithm: Particle Learning} \\
\text{Step 0:} \quad & \text{Sample the starting values of } N \text{ particles } \{\tilde{z}_0^{(i)} = (\tilde{x}_0^{(i)}, \tilde{s}_0^{(i)}, \tilde{\theta}_0^{(i)})\}_{i=1}^N \text{ from } p(z_0). \\
\text{Step 1:} \quad & \text{Resample } \{\tilde{z}_t^{(i)} = (\tilde{x}_t^{(i)}, \tilde{s}_t^{(i)}, \tilde{\theta}_t^{(i)})\}_{i=1}^N \text{ from } \{\tilde{z}_{t-1}^{(i)}\}_{i=1}^N \text{ with } W_t^{(i)} \propto p(y_t|\tilde{x}_t^{(i)}, \tilde{\theta}_t^{(i)}). \\
\text{Step 2:} \quad & \text{Propagate } \tilde{x}_t^{(i)} \text{ from } p(x_t|\tilde{x}_{t-1}^{(i)}, y_t, \tilde{\theta}_t^{(i)}), \ i \in \{1, \ldots, N\}. \\
\text{Step 3:} \quad & \text{Update the sufficient statistics } \tilde{s}_t^{(i)} = S(\tilde{s}_{t-1}^{(i)}, \tilde{x}_t^{(i)}, y_t), \ i \in \{1, \ldots, N\}. \\
\text{Step 4:} \quad & \text{Sample } \tilde{\theta}_t^{(i)} \text{ from } p(\theta|\tilde{s}_t^{(i)}), \ i \in \{1, \ldots, N\}. \\
\end{align}

In order to apply the above particle learning algorithm, we need to know the functional form of $p(y_t|x_{t-1}, \theta)$ and be able to draw $x_t$ from $p(x_t|x_{t-1}, y_t, \theta)$. For the SV-NL model in eqn. (8), however, neither is possible. To circumvent these problems, we adopt the auxiliary particle filter by Pitt and Shephard (1999) within the particle learning framework.

The auxiliary particle filter is based on the following alternative expression of the sequential Bayes filter (we suppress $\theta$ in the equations for brevity):

\begin{equation}
\begin{aligned}
p(x_t|y_{1:t}) &= \int p(x_t|x_{t-1}, y_{1:t})p(x_{t-1}|y_{1:t})dx_{t-1} \\
&\propto \int p(x_t|x_{t-1}, y_t)p(y_t|x_{t-1})p(x_{t-1}|y_{1:t-1})dx_{t-1}, \\
\end{aligned}
\end{equation}

where we use the fact $p(x_t|x_{t-1}, y_{1:t}) = p(x_t|x_{t-1}, y_t)$ in the state space model. As we mentioned earlier, if we know the functional form of $p(y_t|x_{t-1})$ and can draw $x_t$ from $p(x_t|x_{t-1}, y_t)$, we can apply a resampling-propagation cycle of filtering that the particle learning algorithm adopts. (Note that the original particle filter adopts a propagation-resampling cycle.) When this is not the case, we introduce an auxiliary variable $g(x_{t-1})$ where a function $g(\cdot)$ is chosen so as to make $p(y_t|g(x_{t-1}))$ resemble $p(y_t|x_{t-1})$ sufficiently.
well, and rewrite eqn. (12) as

\[
p(x_t|y_{1:t}) \propto \int \frac{p(y_t|x_t)p(x_t|x_{t-1})}{p(y_t|x_{t-1})} \frac{p(y_t|x_{t-1})}{p(y_t|g(x_{t-1}))} p(x_{t-1}|y_{1:t-1}) \, dx_{t-1}
\]

\[
= \int \frac{p(y_t|x_t)}{p(y_t|g(x_{t-1}))} p(x_t|x_{t-1})p(y_t|g(x_{t-1}))p(x_{t-1}|y_{1:t-1}) \, dx_{t-1}.
\]

(13)

In this paper, we set \( g(x_{t-1}) = \int x_t p(x_t|x_{t-1}) \, dx_t \), which is the conditional expectation of \( x_t \) in the state equation.

Eqn. (13) is the basis for the auxiliary particle filter, which consists of three steps: The first-stage resampling, propagation, and the second-stage resampling. In the first-stage resampling, we resample particles from \( \{ \tilde{x}_t^{(i)} \}_{i=1}^N \), which are drawn from \( p(x_{t-1}|y_{1:t-1}) \) in the previous cycle, with weights proportional to \( p(y_t|g(\tilde{x}_{t-1}^{(i)})) \). This resampling produces the following discretized approximation:

\[
p(\hat{x}_{t-1}|y_{1:t}) \simeq \sum_{i=1}^N \tilde{W}_t^{(i)} \delta(\hat{x}_{t-1} - \tilde{x}_t^{(i)}), \quad \tilde{W}_t^{(i)} = \frac{p(y_t|g(\tilde{x}_{t-1}^{(i)}))}{\sum_{i=1}^N p(y_t|g(\tilde{x}_{t-1}^{(i)}))}.
\]

(14)

Now let \( \{ \hat{x}_t^{(i)} \}_{i=1}^N \) denote the particles sampled in the first-stage resampling. Propagation in the auxiliary particle filter is the same as the original particle filter. Thus, we generate \( \hat{x}_t^{(i)} \) from \( p(x_t|x_{t-1}) \) for \( i \in \{1, \ldots, N\} \). In the second-stage resampling, we resample particles from \( \{ \hat{x}_t^{(i)} \}_{i=1}^N \) with weights proportional to \( p(y_t|x_t^{(i)})/p(y_t|g(\hat{x}_{t-1}^{(i)})) \). Then, the approximated posterior density is given by

\[
p(x_t|y_{1:t}) \simeq \sum_{i=1}^N W_t^{(i)} \delta(x_t - \hat{x}_t^{(i)}), \quad W_t^{(i)} \propto \frac{p(y_t|x_t^{(i)})}{p(y_t|g(\hat{x}_{t-1}^{(i)}))}.
\]

(15)

In summary, the auxiliary particle filter with the auxiliary variable \( g(x_{t-1}) \) is summarized as follows.

**Algorithm: Auxiliary Particle Filter**

**Step 0:** Sample the starting values of \( N \) particles \( \{ \tilde{x}_0^{(i)} \}_{i=1}^N \) from \( p(x_0) \).

**Step 1:** Resample \( \{ \tilde{x}_t^{(i)} \}_{i=1}^N \) from \( \{ \tilde{x}_t^{(i)} \}_{i=1}^N \) with \( \tilde{W}_t^{(i)} \propto p(y_t|g(\tilde{x}_{t-1}^{(i)})) \).

**Step 2:** Propagate \( \hat{x}_t^{(i)} \) from \( p(x_t|\hat{x}_{t-1}^{(i)}) \).
Step 3: Resample $\{\tilde{x}_t^{(i)}\}_{i=1}^N$ from $\{\hat{x}_t^{(i)}\}_{i=1}^N$ with $W_t^{(i)} \propto \frac{p(y_t|x_t^{(i)})}{p(y_t|g(\hat{x}_{t-1}^{(i)}))}$.

According to Pitt and Shephard (1999), SMC methods provide good estimations of state variables and parameters, where the model is a good approximation of the data, and the conditional density $p(y_t|x_t)$ is reasonably flat in $x_t$. They point out two weaknesses of the algorithm. Firstly, when there are outliers, SMC methods cannot precisely adapt, so that the state variables would be underestimated even when $N$ is large. In this case, the variability of the likelihood, i.e. $p(y_t|x_t)$, would be increased, which reduces the precision of resampling because it is based on the weight proportional to $p(y_t|x_t)$. Secondly, degeneration of particles are a general problem that occurs when the likelihood is similar in each cycle of the particle filter. It causes poor tail representation of the predictive density because the particles placed on the tails are assigned with similarly low weights, increasingly reducing the weights to zero as the algorithm furthers. As a result, the particles may degenerate to a few points. The auxiliary particle filter softens the influence of outliers because the second-stage resampling would be much less variable than the original sampling. Moreover, the first-stage resampling is based on the current observation value $y_t$ so we can expect that the good particles are likely to be propagated forward.

The sufficient statistics $s_t$ and its recursion map $S(\cdot)$ in terms of the parameters in the SV-NL model in eqn. (8) are as follows. The parameters are summarized as $\theta = (\gamma, \omega^2)$ where $\gamma' = [\mu \ \beta \ \varphi_1 \ \ldots \ \varphi_k]$. With the normal-inverse-gamma prior distribution:

$$
\gamma|\omega^2 \sim N_{k+2}(b_0, \omega^2 A_0^{-1}), \quad \omega^2 \sim IG(c_0, d_0), \quad (16)
$$

where $IG(c, d)$ is the inverse gamma distribution with shape parameter $c$ and scale parameter $d$. The conditional posterior distribution of $\theta$ is given by

$$
\gamma|\omega^2 \sim N_{k+2}(b_t, \omega^2 A_t^{-1}), \quad \omega^2 \sim IG(c_t, d_t), \quad (17)
$$

where the sufficient statistics $s_t = (A_t, b_t, c_t, d_t)$ are computed by the following recursion
where $Z'_t = [1 \ x_{t-1} \ H_1(\epsilon_{t-1}) \ ... \ H_k(\epsilon_{t-1})/\sqrt{k!}]$. Finally, the algorithm for particle learning with auxiliary variables (PLAV) is summarized as follows.

**Algorithm: Particle Learning with Auxiliary Variables**

**Step 0:** Sample the starting values of $N$ particles $\{\hat{z}^{(i)}_0\}_{i=1}^N$ from $p(z_0)$.

**Step 1:** Resample $\{\hat{z}^{(i)}_t\}_{i=1}^N$ from $\{\hat{z}^{(i)}_{t-1}\}_{i=1}^N$ with $\hat{W}^{(i)}_t \propto p(y_t|g(\hat{x}^{(i)}_{t-1}, \hat{\theta}^{(i)}_{t-1}))$.

**Step 2:** Propagate $\hat{x}^{(i)}_t$ from $p(x_t|\hat{x}^{(i)}_{t-1}, \hat{\theta}^{(i)}_{t-1})$.

**Step 3:** Resample $\{\hat{x}^{(i)}_t\}_{i=1}^N$ from $\{\hat{x}^{(i)}_t\}_{i=1}^N$ with $\hat{W}^{(i)}_t \propto \frac{p(y_t|\hat{x}^{(i)}_t)}{p(y_t|g(\hat{x}^{(i)}_{t-1}, \hat{\theta}^{(i)}_{t-1}))}$.

**Step 4:** Update the sufficient statistics $\bar{s}^{(i)}_t = S(\bar{s}^{(i)}_{t-1}, \hat{x}^{(i)}_t, y_t)$, $i \in \{1, \ldots, N\}$.

**Step 5:** Sample $\hat{\theta}^{(i)}_t$ from $p(\theta|\bar{s}^{(i)}_t)$, $i \in \{1, \ldots, N\}$.

**Remarks**

(i) The unknown parameters $\theta$ appears only in $p(x_t|x_{t-1}, \theta)$ of the SV-NL model (8).

(ii) Because the auxiliary variable $g(\cdot)$ is the conditional expectation of $x_t$ with respect to $p(x_t|x_{t-1}, \theta)$, it also depends on $\theta$. Thus, it is expressed as $g(x_{t-1}, \theta)$ so as to clarify its dependence on $\theta$.

(iii) The exact expression of $g(x_{t-1}, \theta)$ is

$$g(x_{t-1}, \theta) = Z_{t-1}' \gamma = \mu + \beta x_{t-1} + \varphi_1 H_1(\epsilon_{t-1}) + \cdots + \varphi_k \frac{1}{\sqrt{k!}} H_k(\epsilon_{t-1}).$$

Therefore, technically, $g(x_{t-1}, \theta)$ depends on $y_{t-1}$, since $\epsilon_{t-1} = y_{t-1} \exp(x_{t-1}/2)$. However, we intentionally omit this dependence for notational simplicity.
3.2 Particle learning with auxiliary variables: A simulation example

Before we test the algorithm on actual data, we will compare the results from PLAV with PL and an MCMC alternative (Omori et al. 2007). Since PL cannot estimate the parameters in the linear-leverage SV model in eqn. 4, we will compare the no-leverage SV model (eqn. 1) between the three and linear-leverage SV model (eqn. 4) between PLAV and MCMC. Following Omori et al. (2007), we will generate data from the linear-leverage SV model in eqn. (4) with parameters

\[
\mu = -0.026, \quad \beta = 0.970, \quad \varphi = -0.045, \quad \omega = 0.143
\]

and the no-leverage SV model with parameters

\[
\mu = -0.026, \quad \beta = 0.970, \quad \tau = 0.150.
\]

The posterior mean and the credible interval of each parameter for the no-leverage SV model and the linear-leverage SV model are given in Tables 1 and 2, respectively. We can see that the estimation results of both SV models, with and without leverage, by three algorithms are comparable and their credible intervals mostly cover the assigned true values (the exception being the MCMC estimation of \(\mu\) for the linear-leverage MCMC model). The execution time for each algorithm is given in Table 3. The results in Table 3 show that MCMC takes much less time for the no-leverage SV model compared to PL and PLAV. However, for the linear-leverage SV model, MCMC takes about nine times as much as PLAV. From this, we can conclude that PLAV is preferred over MCMC in terms of on-line estimation often seen in practice and especially if the model estimated is more complex than the no-leverage SV model.

|        | \(\mu\)     | \(\beta\)     | \(\tau\)     |
|--------|-------------|-------------|-------------|
| True value | -0.026 | 0.970 | 0.150 |
| MCMC   | -0.026 | 0.968 | 0.162 |
|        | [-0.031 -0.021] | [0.951 0.980] | [0.130 0.205] |
| PL     | -0.034 | 0.961 | 0.174 |
|        | [-0.049 -0.023] | [0.947 0.972] | [0.145 0.199] |
| PLAV   | -0.030 | 0.965 | 0.163 |
|        | [-0.042 -0.020] | [0.952 0.975] | [0.146 0.191] |

Table 1: Posterior mean and credible interval (in brackets) of the parameters. Data simulated using the assigned true values in a no-leverage SV model.
Table 2: Posterior mean and credible interval (in brackets) of the parameters. Data simulated using the assigned true values in a linear-leverage SV model.

| Algorithm | No Leverage (min.) | Linear Leverage (min.) |
|-----------|-------------------|------------------------|
| MCMC      | 2.4               | 737.6                  |
| PL        | 66.8              | –                      |
| PLAV      | 64.9              | 81.6                   |

Table 3: Execution time of the three algorithms.

4 Empirical Study

4.1 Data

The object of our empirical study is to estimate the SV-NL model of eqn. (8) for individual stocks that comprise the two major stock indices: The S&P 500 and Nikkei 225. By doing a large scale analysis of this magnitude, we are not only able to explore and compare leverage effects, but also give insight into the characteristics of the volatility structure (leverage function) across markets and sectors. Daily closing prices of each stock from the beginning of 2004 to the end of 2013 are used for the analysis. Since some of the stocks were not listed for the entire time period, they were excluded. A total of 615 stocks were analyzed with 417 stocks from the S&P 500, 198 stocks from the Nikkei 225.

4.2 Model comparison and prior specifications

For each stock return series, we estimate the SV-NL model in eqn. (8) from the 0th-order (no-leverage) up to the 6th-order leverage function. In order to discern the best
order, we compare the one-step predictive marginal likelihood:

\[
p(y_{1:T}|k) = \prod_{t=1}^{T} p(y_t|y_{1:t-1}, k) = \prod_{t=1}^{T} \int p(y_t|x_t)p(x_t|y_{1:t-1}, k)dx_t, \quad k \in \{0, 1, \ldots, 6\},
\]

for each \(t = 1 : T\) and order \(k\); the order of the Hermite polynomial. In the Bayesian model selection procedure, the model with the highest predictive marginal likelihood at time \(t\) is selected as the best model representing the data up to time \(t\). In the framework of particle learning, this can be calculated directly during the filtering procedure with

\[
p(y_{1:T}|k) \simeq \prod_{t=1}^{T} \left\{ \frac{1}{N} \sum_{i=1}^{N} p(y_t|x^{(i)}_{t,k}) \right\},
\]

where \(x^{(i)}_{t,k} \in \{1, \ldots, N\}\) is the particle \(i\) generated from \(p(x_t|\tilde{x}_{t-1}^{(i)}, \tilde{\theta}_{t-1}^{(i)}, k)\), which is the density of the state equation in the SV-NL model in eqn. (8). We set \(N = 10,000\) and evaluate the marginal likelihood with the last eight years of the ten years of data, using the first two years as the learning period. In executing the particle learning algorithm, we set the prior specifications as

\[
A_0 = \text{diag}\{1.0, 0.01, 1.0, \cdots, 1.0\},
\]

\[
b_0' = [0.0, 0.95, 0.0, \cdots 0.0],
\]

\[
c_0 = 5, \quad d_0 = 0.4.
\]

### 4.3 Empirical results

**Comparing against the conventional assumption.** We will first examine whether individual stocks do not, in fact, exhibit leverage effects as expected. To do this, we limit the leverage order to zero and one and examine whether each stock chooses models in eqn. (1) or eqn. (4) under the same specifications. Examining the 615 stocks, we find that the no-leverage model (eqn. (1)) is chosen as the optimal model for nearly half of the stocks and the other half show weak leverage effects (Figure. 1, Figure. 2: left column). Compared to the results reported in other papers (-0.3179 in Yu 2005, -0.3617 in Omori et al. 2007, -0.4825 in Nakajima and Omori 2009), we find that the average
leverage effect to be much smaller (-0.07 if we exclude the zeros and -0.05 if we include the zeros). This confirms that, under the standard linear assumption, most stocks either do not exhibit a leverage effect or are very small.

Relaxing the models to include higher order leverage (Figure. 2: right column), we find that the no-leverage accounts for only 11% under the generalized framework, meaning that around 30% were misspecified, as well as 46% being misspecified as linear leverage, making the total of misspecified models to be around 76%. This reversal strongly indicates that the conventional linear assumption is misleading and biases estimation results towards evidence of no leverage effect.

Given this result, we can see that the leverage effect is much more persistent than previously expected. A solid majority of stocks analyzed exhibit some order of leverage effect (89%) and we can see that many of them have higher order leverage (76%). We further our analysis by examining the results under the generalized leverage effect model.
Overall results. Figure 3 graphs the number of polynomial orders that are selected within a group using the marginal likelihood. The left panel shows the results for all 615 stocks, while the right panels show the results by market. As seen in the previous section, for all the stocks, we can see that many stocks exhibit 2nd-order leverage (38%) followed by 3rd-order leverage (19%), comprising the majority of stocks. In terms of higher order leverage (leverage in the order of 2-6), they comprise approximately 76% of all stocks. Leverage in the order of 4-6 account for very little of the stocks examined, implying that the leverage is captured fairly simply by lower order functions and is not overfitting.

Similar patterns are observed in the right panels of Figure 3, which indicate that the nonlinear leverage effect is supported and persistent over both the U.S. and Japanese stocks and is not a market dependent phenomenon. However, even though the rough shapes of both histograms are similar, we can see that there is a slight difference between the two markets. For example, the percentage of stocks with no leverage or linear leverage is larger for stocks in the Nikkei 225 compared to the stocks in the S&P 500, while there are more higher order leverage stocks comprising the S&P 500. It is commonly perceived that Japanese companies and investors are more conservative than the U.S., and we conjecture that this difference in risk preference/aversion between the two coun-
tries’ investors may play a role, to some degree, in the volatility structure of the stocks. Nonetheless, higher order leverage functions are supported for the majority of the stocks in both markets, strongly suggesting that the effect exists, but not in a way (i.e. linear) we previously perceived.

**Sector results.** Looking closer at leverage order within sectors (Figure. 4), we observe that, in all sectors, the 2nd-order leverage function is the most selected order—consistent with the overall results—except for Communications, which only has four stocks (all higher order). Though similar, closer examination of the distribution of orders within sector suggests each sector to have some unique characteristics. For example, sectors such as Health Care, Industrials, and Technology have more high order leverage than low order (zero and one) leverage stocks, while Financials, Materials, and Utilities having more low order leverage stocks. These characteristics might be explained by such factors like industry size, investor perceptions, accounting rules, and so on. While examining the exact drivers of these characteristics is beyond the scope of this paper, we will examine whether there are statistical differences between sectors in a later section (Section. 4.4).

**Examining leverage functions.** Table. 4 reports the number of positive/negative highest order coefficients, $\varphi_k$, in the Hermit polynomial of the leverage function and Figure. 5
Figure 4: Number of optimal orders of leverage functions selected using the predictive marginal likelihood for sectors.

exhibits the estimated leverage functions, i.e. news impact curves, for selected stocks that are optimal under that order, as an illustration. Note that, for Figure 5, the stocks are selected arbitrarily and do not represent the whole population, though many stocks exhibit similar traits, as we can infer from Table 4. In terms of the sign of the highest order coefficient, this informs us on the rough shape of the leverage function. For example, the sign for the 1st-order leverage is negative for all stocks, agreeing with the consensus that leverage effects are a negative correlation. For 2nd-order leverage functions, the sign of $\varphi_2$ determines whether the leverage function is concave ($\varphi_2 < 0$) or convex ($\varphi_2 > 0$). As we can see in Table 4, $\varphi_2$ is negative for most of the stocks, meaning that the leverage function is concave, similar to the 2nd-order leverage function in Figure 5. As for the estimated leverage functions in Figure 5, a concave leverage function has a mode somewhere around zero. As a result, a large shock, either positive or negative, will reduce the volatility, which is quite counterintuitive and a stark deviation from the conventional interpretation of the leverage effect.

The two key differences exhibited in the nonlinear leverage compared to the linear
leverage are at zero and at the tail. Firstly, since Hermite polynomials do not necessitate that the function be centered, we see that the assumption that an indifference in price equates in an indifference in volatility is false. This suggests that stocks become more volatile after small, or no, movement in price, in either direction. This may be caused by either investor sentiment to buy/sell stocks that are stagnate, intraday volatility not being accounted for, the model excessively shrinking its volatility estimate, or a combination of all. From an economic perspective, we can infer that, at least for small fluctuations, the leverage effect does not come from financial leverage, but rather reflects investor sentiment and uncertainty.

Secondly, we observe that, although the shapes of the functions are different with order, they all roughly have a negative slope around zero similar to the 1st-order (linear) leverage function with the tail falling off to negative. This result goes against the conventional interpretation of the leverage effect, as it implies that a large shock, either positive or negative, will reduce its volatility. There are multiple possible interpretations as to why this might happen, and why it might better explain price movements in stocks. One possible interpretation is that a sharp rise or fall in a stock price is a first moment phenomenon and not a second moment phenomenon. In other words, a stock price will drastically rise or fall in response to an unexpected change in the expected return of the stock, but not because of its change in volatility. Another possible explanation is that, when a stock price rises or falls, the upward or downward trend tends to continue, i.e., the momentum effect, and this persistency reduces daily volatility. Though these possible

| Order of Leverage | S&P500 Positive | S&P500 Negative | Nikkei225 Positive | Nikkei225 Negative |
|------------------|----------------|----------------|--------------------|-------------------|
| 1                | 0              | 50             | 0                  | 32                |
| 2                | 3              | 160            | 10                 | 60                |
| 3                | 58             | 28             | 18                 | 15                |
| 4                | 15             | 29             | 9                  | 7                 |
| 5                | 9              | 11             | 5                  | 3                 |
| 6                | 12             | 5              | 3                  | 5                 |

Table 4: The number of positive/negative highest order coefficients in the leverage function for each optimal leverage order selected using the predictive marginal likelihood.
Figure 5: Leverage functions of U.S. stocks (gray area: 95% credible interval, vertical lines: 95% interval of the observed returns). The x-axis is the shock to the return and the y-axis is the effect on volatility.

Interpretations are speculative, it is clear that these higher order forms better represent the data compared to the linear leverage effect.
4.4 Further analysis on the characteristics of leverage orders

Using the results in Section 4.3, we are able to infer whether certain markets/sectors have tendencies for lower/higher order leverage and also determine if certain characteristics of a stock contributes to the leverage order. To examine this, we use a basic Bayesian ordered probit model:

\[
y^* = \beta'x + u, \quad u_i \sim i.i.d. N(0,1) \tag{22}
\]

\[
y = \begin{cases} 
0 & (\mu_0 \leq y^* < \mu_1), \\
1 & (\mu_1 \leq y^* < \mu_2), \quad x_i = [1, x_0, \cdots, x_{10}]' \\
2 & (\mu_2 \leq y^* < \mu_3), \quad \beta = [\alpha, \beta_1, \cdots, \beta_{10}]' \\
3, \ldots, 6 & (\mu_3 \leq y^* < \mu_4), 
\end{cases}
\]

where \(\alpha\) is the intercept and \(y^*\) is a latent continuous variable that can take one of four categories \(y\) classified by bins \(\mu_0, \ldots, \mu_4\) where \(\mu_0 = -\infty, \mu_1 = 0, \mu_4 = \infty\). We have grouped leverage orders 3-6 as one group because there were considerably fewer number of higher order leverages as well as them being very similar in shape (as seen in Figure 5). Additionally, our main interest here is to understand which characteristics effect low-high leverage order and not necessarily interested in the exact order number.

In the above, \(\beta\) is a vector of factor loadings and \(x\) is a vector of covariates, which is defined as,

\[x_1 : \text{Volatility (The average volatility of each stock)}\]
\[x_2 : \text{Liquidity (The average trade volume of each stock)}\]
\[x_3 : \text{Country (}x_3 = 1 \text{ if Nikkei225, } x_3 = 0 \text{ if S&P500)}\]
\[x_4 \sim x_{10} : \text{Sector (}x_{i,k-1} = 1 \text{ if } y_i \text{ belongs to sector } k)\]

For the sectors, we have chosen to omit the Finance sector in order to ensure identification and merged the Technologies and Communications sectors because there were only four stocks in the Technologies sector.
For our analysis, we will use the Gibbs sampling method proposed by James and Chib (1993) to sample the posterior distribution of the parameters.

For argument’s sake, let us assume that the leverage effect – and the complexity of it – arises out of investor’s reactions to price change. We can expect, then, that the characteristics of the investor towards a stock is reflected in the complexity of the leverage effect. For example, if the investor is aggressive (conservative) and reacts sharply (dully) to price changes, the leverage effect might tend to be more complex (simple). Thus, if we know that a certain stock or a group of stocks are traded differently than another, the order of the leverage effect might reflect that.

Figure 6 graphs the kernel smoothed posterior distribution for each parameter $\beta_1 \sim \beta_{10}$ and $\alpha$ with its corresponding 90% and 95% credible intervals using the probit model in eqn. (22). Some of the coefficients are notable and gives insight into the characteristics of the leverage effect. For one, we see that quantitative characteristics, such as Volatility and Liquidity have very little to no effect on the leverage order, with Liquidity having a slight but insignificant positive effect. On the other hand, qualitative characteristics show clear signals that effect the leverage order. Looking at the factor Country, we see that there is a significant negative effect, implying that Japanese stocks are less prone to nonlinear leverage compared to U.S. stocks. This aligns with the conventional wisdom that Japanese investors are more conservative than American investors.

Regarding Sector, we can see that a few sectors, such as Technology and Communications and Industrials to have a positive effect on the leverage order. We can infer that the stocks in the above sectors tend to have more complex volatility structures than stocks in the Finance sector (the base sector in this model), which is perceived as one the more conservative sectors.

5 Conclusion

The leverage effect is a perplexing phenomenon. While observations from the market–added with an accompanying sensible economic argument– has brought on consensus among researchers that this effect exists, empirical evidence from individual stocks were often contradictory. In this paper, we have argued that this in not because the leverage effect does not exist, but because of our presumption of the effect, and how it is modeled,
was wrong. Under the stochastic volatility (SV) framework, we generalized the leverage effect, which has been presumed to be linear, to account for nonlinearity.

Applying the new model to stocks comprising the S&P 500 or Nikkei 225, we have found that most stocks exhibit a nonlinear leverage effect. In fact, if we had constrained our analysis to models with linear leverage only, around half of the stocks would have been specified as having zero leverage. Given these results, we can state that the leverage effect puzzle is not a puzzle about the existence of the effect but a matter of model misspecification. Our analysis of the characteristics that drive the nonlinearity gives insight into what causes the leverage effect. We find that the complexity of the leverage effect is, to a certain degree, market/sector specific; suggesting that investors and their traits have an impact. However, our results alone cannot confirm or deny an economic
interpretation for the effect. Further analysis on the leverage order might help in further
our understanding and is an open area for continued research.

Furthermore, this paper discussed new methods for estimating complex non-linear
models using an extension of the particle learning algorithm. With this new algorithm,
we are now able to estimate and infer on models and parameters that were either too
costly or too difficult using conventional methods, allowing researchers to flexibly extend
current models and create new ones. Additionally, the sequential nature of the model
lends itself to real time applications of models that were deemed impractical before.

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