Gaussian bounds of fundamental matrix and maximal $L^1$ regularity for Lamé system with rough coefficients

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Abstract. The purpose of this paper is twofold. First, we use a classical method to establish Gaussian bounds of the fundamental matrix of a generalized parabolic Lamé system with only bounded and measurable coefficients. Second, we derive a maximal $L^1$ regularity result for the abstract Cauchy problem associated with a composite operator. In a concrete example, we also obtain maximal $L^1$ regularity for the Lamé system, from which it follows that the Lipschitz seminorm of the solutions to the Lamé system is globally $L^1$-in-time integrable. As an application, we use a Lagrangian approach to prove a global-in-time well-posedness result for a viscous pressureless flow in a perturbation framework, but with possibly discontinuous densities. The method established in this paper might be a useful tool for studying many issues arising from viscous fluids with truly variable densities.

1. Introduction

This work is motivated by the study of the global well-posedness of the Cauchy problem for a class of hyperbolic–parabolic coupled systems modeling the motion of fluids. Probably, the most famous example is the system of Navier–Stokes equations (see [17, 18]). In a fluid flow, the law of conservation of mass can be formulated mathematically using the continuity equation, given in differential form as

\[ \partial_t \rho + \text{div}(\rho u) = 0, \quad \text{in} \; (0, \infty) \times \mathbb{R}^n, \tag{1.1} \]

where $\rho$ is the density (mass per unit volume) and $u$ is the flow velocity field. The law of conservation applied to momentum gives the momentum equation of the form

\[ \rho (\partial_t u + u \cdot \nabla u) - Au + \nabla P = 0, \quad \text{in} \; (0, \infty) \times \mathbb{R}^n, \tag{1.2} \]

where $P$ is a scalar pressure and $A$ is a dissipative operator. The fluid flow can be either incompressible or compressible. However, in the compressible case, we will only consider pressureless flows, which means we drop the pressure term in (1.2) and write

\[ \rho (\partial_t u + u \cdot \nabla u) - Au = 0, \quad \text{in} \; (0, \infty) \times \mathbb{R}^n. \tag{1.3} \]
Note that (1.1) and (1.3) with \( \mathcal{A} = \Delta \) can be viewed as a viscous regularization for the model of inviscid pressureless gases.

The system of the form (1.1) and (1.3) has been studied by several authors. When \( n = 1 \) and \( \mathcal{A} = \Delta \), Boudin [4] proved the existence of a global smooth solution to (1.1) and (1.3). Perepelitsa [21] considered (1.1) and (1.3) as a simplified model of compressible isentropic Navier–Stokes equations, and he proved the global existence of a small energy weak solution with the density being a nonnegative bounded function throughout the half-space \( \mathbb{R}^3_+ \). Recently, Danchin et al. [8] formally derived the system (1.1) and (1.3), with \( \mathcal{A} \) being the Laplacian or the Lamé operator, as a model of some collective behavior phenomena. They also proved the existence and uniqueness of a global solution with the initial density being only bounded and close to a constant in \( L^\infty \)-norm.

In this work, we are particularly interested in solving (1.1) and (1.2) and (1.1) and (1.3) via the Lagrangian method (see [5,7]). The advantage is that one can convert the hyperbolic–parabolic coupled system into a parabolic system. Then, the uniqueness and stability issues can be tackled in a relatively easy way compared to solving the system in Eulerian coordinates. In this framework, under a scaling-invariant smallness condition on the initial velocity, the heart of the matter is to bound the quantity \( \int_0^\infty \| \nabla u(t) \|_\infty \, dt \); then, this would imply the existence of global-in-time coordinate transformations. Without going into details, we are led to consider the linearized system of the Lagrangian formulation of (1.1) and (1.3) (or, (1.1) and (1.2)) that reads

\[
\rho_0(x) \partial_t u - \mathcal{A} u = f. \tag{1.4}
\]

Note that the coefficient \( \rho_0 \) is now a time-independent function. In the incompressible case, the operator \( \mathcal{A} \) in (1.4) is different from the one in (1.2). Indeed, we need to introduce a so-called Stokes operator to unify the internal force (viscosity and pressure) in (1.2) (see [25]). But \( \mathcal{A} \) is just what it used to be in the compressible case. Now, the main purpose of this paper is to bound \( \int_0^\infty \| \nabla u(t) \|_\infty \, dt \) (by some norms of the initial value and the inhomogeneous term) for solutions \( u \) to (1.4) under the least regularity assumption on the coefficient \( \rho_0 \). To achieve this, we shall study the maximal \( L^1 \)-in-time regularity for solutions to (1.4) in homogeneous type spaces.

If \( \rho_0 \) is close to some constant, (1.4) is essentially a perturbation of a linear system with constant coefficients. In this case, Danchin and Mucha [7] established a maximal \( L^1 \) regularity result for a linear Stokes system with discontinuous coefficients (including piecewise constant densities). In our work, we do not assume any smallness condition on the initial density fluctuation. So (1.4) is no longer a perturbation problem, but we can rewrite it as the following abstract Cauchy problem:

\[
\partial_t u - \rho_0^{-1} \mathcal{A} u = \tilde{f} := \rho_0^{-1} f \tag{1.5}
\]

associated with the composite operator \( \rho_0^{-1} \mathcal{A} \), provided that \( \rho_0 \) has a positive lower bound. What makes the maximal \( L^1 \) regularity for (1.5) possible is the observation that the composite operator \( \rho_0^{-1} \mathcal{A} \) behaves similarly as some operator with constant
coefficients, in the sense that certain Besov-type norms defined via the semigroups generated by both operators are equivalent. This was one of the key observations made in [25] in which the author of the present paper proved the first maximal $L^1$ regularity result concerning viscous incompressible fluids with truly variable densities.

In [25], we were only able to work in the $L^2$ (in space) framework due to the presence of pressure. In this paper, we mainly focus on the pressureless case, and we will work in the general $L^p$ (in space) framework. A practical benefit of doing so is that one can lower the regularity of the density (see [7]). For the analysis in [25] to adapt to the $L^p$ framework, we need to make the extra effort to obtain pointwise bounds for the kernel of $\rho_0^{-1}A$. Let us consider two concrete examples. For $A = \Delta$ (the Laplacian), McIntosh and Nahmod [19] proved that the kernel of the $L^2$ semigroup $e^{t\rho_0^{-1}\Delta}$ generated by $\rho_0^{-1}\Delta$ satisfies Gaussian bounds (see also [14]). This guarantees that the semigroup $e^{t\rho_0^{-1}\Delta}$ extrapolates to a bounded analytic semigroup on $L^p$, $1 < p < \infty$. Note that the kernel of $e^{t\rho_0^{-1}\Delta}$ is essentially a scalar kernel. If $A$ is the Lamé operator, however, (1.4) is a truly coupled system whose fundamental matrix does not necessarily satisfy Gaussian bounds. Nevertheless, we can prove the bounds for the fundamental matrix and its derivatives using a rather classical method if the dimensions of the Euclidean space are no greater than 3. The tricks are due to Davies, one is to use Sobolev inequalities to bound $L^\infty$-norm (see [11]) and the other is a perturbation technique to obtain exponential decay (see [10]). In the spirit of [14,19], once we obtain Gaussian upper bounds of the fundamental matrix (denoted by $K_t(x, y)$), we can easily get the $C^{1,\gamma}$ estimates for the kernel $K_t(x, y)\rho_0^{-1}(y)$.

Before we study the maximal regularity for (1.5), we will establish a maximal $L^1$-in-time regularity result for the abstract Cauchy problem

\begin{equation}
\begin{aligned}
&u'(t) - Su(t) = f(t), \\
&u(0) = x
\end{aligned}
\end{equation}

in homogeneous type spaces. Let us assume that $S$ is an unbounded linear operator on a Banach space $(X, \| \cdot \|)$ that generates a bounded analytic semigroup $e^{tS}$. Given (1.6) with $x = 0$, $S$ is said to have maximal $L^r$-in-time regularity in $X$ for $r \in [1, \infty]$, if for every $f \in L^r((0, \infty); X)$, (1.6) has a unique solution $u$ verifying

\begin{equation}
\|Su\|_{L^r((0,\infty); X)} \leq C\|f\|_{L^r((0,\infty); X)}.
\end{equation}

The maximal $L^r$ regularity issue for $r \in (1, \infty)$ has been extensively studied in the literature. We refer to [9,12,13,16,22], among which [9] also covered the $L^1$ theory, but the global-in-time estimate (1.7) holds only if $0$ belongs to the resolvent set $\rho(S)$ of $S$ (i.e., $S^{-1} \in \mathcal{L}(X)$). It goes without saying that such a condition is demanding in many concrete examples. Recently, Ri and Farwig [23] established maximal $L^1$ regularity for $S$ in inhomogeneous type spaces without assuming $0 \in \rho(S)$. Later, a similar result in the homogeneous space setting was proved by Danchin et al. [6]. The authors in [6] also nicely explained the importance of maximal $L^1$ regularity for parabolic systems in homogeneous spaces. But [6] did not cover maximal regularity
results in homogeneous spaces with negative regularity. For us, working in spaces with negative regularity can weaken the regularity of the density. Our method is motivated by our prior work [25] in which we obtained maximal $L^1$ regularity for a generalized Stokes operator with variable coefficients in homogeneous Besov spaces. It turns out that the strategy of the proof of the concrete result in [25] works equally well for the abstract problem.

**Notations:** Throughout, the letter $C$ denotes a harmless positive constant that may change from line to line, but whose meaning is clear from the context. For two quantities $Q_1, Q_2$, the notation $Q_1 \lesssim Q_2$ means $Q_1 \leq C Q_2$ for some $C$, $Q_1 \simeq Q_2$ means $Q_1 \lesssim Q_2$ and $Q_2 \lesssim Q_1$, $Q_1 \vee Q_2 = \max\{Q_1, Q_2\}$, and $Q_1 \wedge Q_2 = \min\{Q_1, Q_2\}$. We denote by $\| \cdot \|_p$ the Lebesgue $L^p$-norm. For a matrix $A$, $A^\top$ denotes its transpose. For a vector field $v = v(x)$, $\nabla v$ denotes the matrix $(\partial_x v_j)$ and $Dv = (\nabla v)^\top$. For a Banach space $X$, $\mathcal{L}(X)$ denotes the space of all continuous linear operators on $X$. For $q \in [1, \infty]$, we may write $\| \cdot \|_{L^q(X)}$ for the norm of the space $L^q((0, t); X)$, and $\| \cdot \|_{L^q_1(X)}$ for the norm of $L^q(\mathbb{R}_+; X)$, where $\mathbb{R}_+ = (0, \infty)$. We will always denote operators on Banach spaces by “mathcal” letters (e.g., $\mathcal{A}, \mathcal{B}, S$, etc.).

Besides, we adopt the following important notations:

- $\mu, \lambda$ and $\nu$ are constants satisfying $\mu > 0$ and $\nu = \lambda + 2\mu > 0$.
- $\mathcal{L} := \mu \Delta + (\lambda + \mu)\nabla \text{div}$ is the Lamé operator.
- $\rho_0$ is a measurable function defined in $\mathbb{R}^n$ satisfying $m \leq \rho_0(x) \leq \frac{1}{m}$, a.e. $x \in \mathbb{R}^n$, for some $m \in (0, 1]$.
- $b := \rho_0^{-1}$.

2. Main results

The first main result of this work, in the spirit of those in [14, 19], are the Gaussian bounds of the matrix-valued heat kernel of $-b\mathcal{L}$. More precisely, let $K_t(x, y)$ be the kernel of the semigroup $e^{tb\mathcal{L}}$ and denote $S_t(x, y) = K_t(x, y)b(y)$. Then, we have the following:

**Theorem 2.1.** Let $n \in \{2, 3\}$. For any $\gamma \in (0, 1)$ and $t > 0$, each entry of $S_t(x, y)$ is a $C^{1,\gamma}(\mathbb{R}^n \times \mathbb{R}^n)$ function. More precisely, there exist constants $C_1 = C_1(m, \mu, \lambda)$ and $C_2 = C_2(m, \mu, \lambda, \gamma)$ such that for all $t > 0$ and $x, y, h \in \mathbb{R}^n$,

$$|S_t(x, y)| + \sqrt{t}|\nabla_x S_t(x, y)| \leq \frac{C_1}{t^{n/2}} \exp\left\{ -\frac{|x - y|^2}{C_1 t}\right\},$$

$$|\nabla_x S_t(x + h, y) - \nabla_x S_t(x, y)| \leq \left(\frac{|h|}{\sqrt{t}}\right)^\gamma \frac{C_2}{t^{(n+1)/2}} \exp\left\{ -\frac{|x - y|^2}{C_2 t}\right\}, \quad (2.1)$$

and

$$|\nabla_x S_t(x, y + h) - \nabla_x S_t(x, y)| \leq \left(\frac{|h|}{\sqrt{t}}\right)^\gamma \frac{C_2}{t^{(n+1)/2}} \exp\left\{ -\frac{|x - y|^2}{C_2 t}\right\} \tag{2.2}$$

provided $2|h| \leq \sqrt{t}$. 


**Remark 2.1.** In view of Lemma 4.1, we have $S_t(x, y) = S_t^T(y, x)$. So the $y$-derivative also satisfies each of the bounds.

The above theorem is of independent interest on its own, but the pointwise bound for $K_t(x, y)$ is also needed to establish our second main result. Precisely, we need this bound to ensure that $e^{tb\mathcal{L}}$ is a bounded analytic semigroup on $L^p$ $(1 < p < \infty)$ which is asymptotic to zero.

After establishing an abstract $L^1$ theory and some equivalent characterizations of norms in Section 5, we are able to prove the maximal $L^1$ regularity for the following linear system

$$
\begin{align*}
\rho_0 \partial_t u - Au &= f, & \text{in } (0, \infty) \times \mathbb{R}^n, \\
u(0) &= u_0, & \text{on } \mathbb{R}^n.
\end{align*}
$$

**Theorem 2.2.** Let $p \in (1, \infty)$, $s \in (0, 2)$ and $T \in (0, \infty]$. Assume that $n \geq 2$ if $\mathcal{A} = \Delta$, or $n \in [2, 3]$ if $\mathcal{A} = \mathcal{L}$.

(i) Assume that $s \leq \frac{n}{p}$ and $\rho_0, b \in \mathcal{M}(\dot{B}^s_{p,1})$. Then, for $u_0 \in \dot{B}^s_{p,1}$ and $f \in L^1((0, T); \dot{B}^s_{p,1})$, Eq. (2.3) has a unique strong solution $u \in C([0, T); \dot{B}^s_{p,1})$ satisfying

$$
\|u\|_{L_T^\infty(\dot{B}^s_{p,1})} + \|\partial_t u, Au\|_{L_T^1(\dot{B}^s_{p,1})} \leq C\|u_0\|_{\dot{B}^s_{p,1}} + C\|f\|_{L_T^1(\dot{B}^s_{p,1})}
$$

for some constant $C$ depending on $s, m, \mu, \nu$, $\|\rho_0\|_{\mathcal{M}(\dot{B}^s_{p,1})}$, and $\|b\|_{\mathcal{M}(\dot{B}^s_{p,1})}$.

(ii) Assume $\rho_0, b \in \mathcal{M}(\dot{B}^{-s}_{p,1})$. If $u_0 \in \dot{B}^{-s}_{p,1}$ and $f \in L^1((0, T); \dot{B}^{-s}_{p,1})$, then (2.3) has a unique strong solution $u \in C([0, T); \dot{B}^{-s}_{p,1})$ satisfying

$$
\|u\|_{L_T^\infty(\dot{B}^{-s}_{p,1})} + \|\partial_t u, Au\|_{L_T^1(\dot{B}^{-s}_{p,1})} \leq C\|u_0\|_{\dot{B}^{-s}_{p,1}} + C\|f\|_{L_T^1(\dot{B}^{-s}_{p,1})}
$$

for some constant $C$ depending on $s, m, \mu, \nu$, $\|\rho_0\|_{\mathcal{M}(\dot{B}^{-s}_{p,1})}$, and $\|b\|_{\mathcal{M}(\dot{B}^{-s}_{p,1})}$.

The definitions of the Besov spaces $\dot{B}^{\pm s}_{p,1}$ and the multiplier spaces $\mathcal{M}(\dot{B}^{\pm s}_{p,1})$ are given in Section 3 and Section 6, respectively.

As an application of Theorem 2.2, we construct global unique solutions to the viscous pressureless flow

$$
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, & \text{in } (0, \infty) \times \mathbb{R}^n, \\
\rho(\partial_t u + u \cdot \nabla u) - \mathcal{L}u &= 0, & \text{in } (0, \infty) \times \mathbb{R}^n, \\
(\rho, u)|_{t=0} &= (\rho_0, u_0), & \text{on } \mathbb{R}^n.
\end{align*}
$$

Before stating the last main result, let us be clear about what it means by a solution to the system (2.4) with discontinuous density.

**Definition 2.1.** The unknown $(\rho, u)$ is called a global-in-time solution to (2.4) if

$$
\begin{align*}
\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^n) \cap L^\infty(\mathbb{R}_+; \mathcal{M}(\dot{B}^{n/p}_{p,1})), \\
u \in C([0, \infty); \dot{B}^{n/p}_{p,1}), (\partial_t u, \mathcal{L}u) \in \left(L^1(\mathbb{R}_+; \dot{B}^{n/p}_{p,1})\right)^2,
\end{align*}
$$
\( \rho \) is a weak solution to the continuity equation of (2.4) (i.e., \( \rho \) satisfies (1.1) in the sense of distribution), \((\rho, u)\) satisfies the momentum equation of (2.4) for a.e. \( t \in (0, \infty) \), \( u(0) = u_0 \), and \( \rho(t) \rightharpoonup \rho_0 \) in \( L^\infty(\mathbb{R}^n) \) as \( t \to 0^+ \).

**Theorem 2.3.** Let \( n \in \{2, 3\}, p \in (1, 2n) \setminus \{n\}, u_0 \in \dot{B}^{n/p-1}_{p,1} = (\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n))^n \), and \( \rho_0, b \in \mathcal{M}(\dot{B}^{n/p-1}_{p,1}) \). There exists a positive constant \( c \) depending on \( m, p, n, \mu, v, \|\rho_0\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1})}, \|b\|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1})} \) such that if \( \|u_0\|_{\dot{B}^{n/p-1}_{p,1}} \leq c \), then (2.4) has a unique global-in-time solution.

**Remark 2.2.** The above theorem holds without constraint on the dimensions if \( \mathcal{L} \) is replaced by \( \Delta \).

**Remark 2.3.** Note that piecewise-constant initial densities are admissible data provided that \( p \in (n - 1, 2n) \setminus \{n\} \) (see [7, Lemma A.7]).

**Remark 2.4.** Our primary goal is to establish a correct functional framework in which some density-dependent viscous systems are well posed even if the densities are discontinuous. Our framework has wider applications if one allows continuous densities. For an interesting application of our framework to global well-posedness of a model of compressible viscous ideal gases, see [26, Section 4.6].

This paper is organized as follows. Section 3 is a short review of some basics needed in this paper. In Section 4, we prove Theorem 2.1. In Section 5, we follow our prior work [25] closely and derive the maximal \( L^1 \) regularity for the abstract Cauchy problem (1.6) when \( \mathcal{S} \) is a composition of bounded and unbounded operators. Then, in Section 6, we apply the abstract theory together with some equivalent characterizations of Besov norms to prove Theorem 2.2. Section 7 is devoted to the global-in-time well-posedness of the pressureless system (2.4) via Lagrangian coordinates.

### 3. Preliminaries

#### 3.1. Semigroups and abstract Cauchy problem

In this paper, we only consider real vector spaces and no complexification is needed. Let \((X, \|\cdot\|)\) be a real Banach space. We adopt the concept that a \( C_0 \) semigroup \( \{T(t)\}_{t \geq 0} \) on \( X \) is called a bounded \( C_0 \) semigroup if \( \|T(t)\| \leq M < \infty \) for all \( t \geq 0 \), while it is called a contraction semigroup if \( M = 1 \).

**Definition 3.1.** \( \{T(t)\}_{t \geq 0} \) is called a bounded analytic semigroup on \( X \) if it is a bounded \( C_0 \) semigroup with generator \( \mathcal{A} \) such that \( T(t)x \in D(\mathcal{A}) \) for all \( x \in X \) and \( t > 0 \), and

\[
\sup_{t > 0} \|t \mathcal{A} T(t)x\| \leq C\|x\|, \quad \forall x \in X.
\]  

**Remark 3.1.** In applications, one only needs to show (3.1) for \( x \) belonging to a dense subspace of \( X \) since \( \mathcal{A} \) is closed.
Remark 3.2. In fact, (3.1) is also a real characterization of complex analyticity, see, for example, [15, Theorem 4.6], or [2, Theorem 3.7.19].

Let \((H, \langle \cdot, \cdot \rangle)\) be a real Hilbert space. A linear operator \(A : D(A) \subset H \to H\) is called dissipative on \(H\) if

\[
\langle Ax, x \rangle \leq 0, \forall x \in D(A).
\]

We have the following well-known result:

Theorem 3.1. Let \(A\) be a self-adjoint operator on \(H\). Then, \(A\) generates an analytic semigroup of contraction \(\{e^{tA}\}_{t \geq 0}\) if and only if \(A\) is dissipative. Moreover, \(e^{tA}\) is self-adjoint on \(H\) for every \(t \geq 0\).

For the complex version of Theorem 3.1, we refer to [2, Example 3.7.5] and [2, Corollary 3.3.9].

In applications, we will first apply Theorem 3.1 to construct a semigroup on \(L^2\), and then extrapolate it to some Besov spaces. However, it is usually not easy to identify the generator of the new semigroup. In this situation, we wish to identify the generator restricted on a dense subspace of its domain. Recall that a subspace \(Y\) of the domain \(D(A)\) of a linear operator \(A : D(A) \subset X \to X\) is called a core for \(A\) if \(Y\) is dense in \(D(A)\) for the graph norm \(\|x\|_{D(A)} := \|x\| + \|Ax\|\). In other words, \(Y\) is a core for \(A\) if and only if \(A\) is the closure of \(A|_Y\). The next result gives a useful sufficient condition for a subspace to be a core for the generator.

Lemma 3.2. (see [15, p. 53]) Let \(A\) be the infinitesimal generator of a \(C_0\) semigroup \(T(t)\) on \(X\). If \(Y \subset D(A)\) is a dense subspace of \(X\) and invariant under \(T(t)\) (i.e., \(T(t)Y \subset Y\)), then \(Y\) is a core for \(A\).

Next, we recall shortly how to use semigroups to solve abstract Cauchy problems. Suppose that \(A\) is the infinitesimal generator of a \(C_0\) semigroup \(e^{tA}\) on a Banach space \((X, \| \cdot \|)\). We are concerned with the inhomogeneous abstract Cauchy problem

\[
\begin{align*}
\frac{du}{dt}(t) - Au(t) &= f(t), \quad 0 < t \leq T, \\
u(0) &= x.
\end{align*}
\]  

(3.2)

We assume that \(x \in X\) and the inhomogeneous term \(f\) only belongs to \(L^1((0, T); X)\). Then, (3.2) always has a unique mild solution \(u \in C([0, T]; X)\) given by the formula

\[
u(t) = e^{tA}x + \int_0^t e^{(t-\tau)A} f(\tau) \, d\tau.
\]

A continuous function \(u\) is called a strong solution if \(u \in W^{1,1}((0, T); X) \cap L^1((0, T); D(A))\) satisfies (3.2) for a.e. \(t \in (0, T)\). A strong solution is also a mild solution. Conversely, a mild solution with suitable regularity becomes a strong one.

Lemma 3.3. (see [20, Theorem 2.9]) Let \(u \in C([0, T]; X)\) be a mild solution to (3.2). If \(u \in W^{1,1}((0, T), X)\), or \(u \in L^1((0, T), D(A))\), then \(u\) is a strong solution.
3.2. Homogeneous Besov spaces

In most literature on the theory of function spaces, the homogeneous Besov spaces are defined in the ambient space of tempered distributions modulo polynomials (see, e.g., [24]). However, we wish to avoid this type of spaces when solving nonlinear PDEs. In this paper, we adopt the definitions of homogeneous spaces in [3, Section 2.3]. Let $\mathcal{S}'_h(\mathbb{R}^n)$ denote the space of all tempered distributions $u \in \mathcal{S}'_h(\mathbb{R}^n)$ that satisfy

$$u = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j u \text{ in } \mathcal{S}'_h,$$

where $\hat{\Delta}_j$’s are the homogeneous dyadic blocks (see [3, Chapter 2]).

**Definition 3.2.** Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. The homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^n)$ consists of all distributions $u$ in $\mathcal{S}'_h(\mathbb{R}^n)$ such that

$$\|u\|_{\dot{B}^s_{p,r}} := \left( \sum_{j \in \mathbb{Z}} (2^j \|\hat{\Delta}_j u\|_p)^r \right)^{1/r} < \infty.$$

Let us point out a couple of useful facts about the homogeneous Besov spaces. First, as an immediate consequence of the definition, we see that

$$\lim_{N \to \infty} \sum_{|j| < N} \hat{\Delta}_j u = u \text{ in } \dot{B}^s_{p,1}$$

for every $u \in \dot{B}^s_{p,1}$. This implies that $W^{\infty,p}$ is dense in $\dot{B}^s_{p,1}$. Second, for $p \in [1, \infty]$, $\dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$ is continuously embedded in $C_0(\mathbb{R}^n)$, the space of continuous functions that tend to zero at infinity (see [3, Proposition 2.39]). Third, the spaces $\dot{B}^s_{p,r}$ are not always complete. In fact, $\dot{B}^s_{p,r}$ is complete only if $s < \frac{n}{p}$, or $s = \frac{n}{p}$ and $r = 1$ (see [3, Theorem 2.25]). Finally, the product of two distributions in certain Besov spaces can be defined via the so-called paraproduct. In Section 7, we will extensively use the product laws

$$\|uv\|_{\dot{B}^{n/p}_{p,1}} \lesssim \|u\|_{\dot{B}^{n/p}_{p,1}} \|v\|_{\dot{B}^{n/p}_{p,1}}, \quad 1 \leq p < \infty$$

and

$$\|uv\|_{\dot{B}^{n/p-1}_{p,1}} \lesssim \|u\|_{\dot{B}^{n/p}_{p,1}} \|v\|_{\dot{B}^{n/p-1}_{p,1}}, \quad 1 \leq p < 2n.$$

It is sometimes quite useful to associate norms with operators arising from PDEs. For example, we can characterize the homogeneous Besov norms via the heat semigroups.

**Lemma 3.4.** (see [3, Theorem 2.34]) Suppose that $s \in \mathbb{R}$ and $(p, q) \in [1, \infty]^2$. If $k > s/2$ and $k \geq 0$, we have

$$\|u\|_{\dot{B}_{p,q}^{s,k}} := \|t^{-s/2}((t\Delta)^k e^{t\Delta} u\|_{L^q(\mathbb{R}^+, \frac{dt}{t})} \simeq \|u\|_{\dot{B}_{p,q}^s}, \quad \forall u \in \mathcal{S}'_h.$$
Theorem 2.34 in [3] only considers the case \( k = 0 \). But for \( k > 0 \), we can write 
\[
\|u\|_{\dot{B}^{s,\Delta}_{p,q}} = \|\Delta^k u\|_{\dot{B}^{s-2k,\Delta}_{p,q}}.
\]
Then, the equivalence still holds since \( s - 2k < 0 \).

A similar result holds if we replace \( \Delta \) by the Lamé operator \( L \). Let us introduce the Hodge operator \( Q = -\nabla (-\Delta)^{-1} \text{div} \) and let \( \mathcal{P} = I - Q \). The Lamé operator \( L \) and the Laplacian \( \Delta \) can be expressed by each other, namely,
\[
L = (\mu \mathcal{P} + vQ)\Delta = \Delta (\mu \mathcal{P} + vQ) \quad (3.3)
\]
and
\[
\Delta = \left( \frac{1}{\mu} \mathcal{P} + \frac{1}{v} Q \right) L = L \left( \frac{1}{\mu} \mathcal{P} + \frac{1}{v} Q \right). \quad (3.4)
\]
So, for every \( p \in (1, \infty) \) and \( k \in \mathbb{N} \), we have
\[
\|L^k u\|_p \simeq \|\Delta^k u\|_p, \quad u \in W^{2k,p}(\mathbb{R}^n; \mathbb{R}^n). \quad (3.5)
\]

**Lemma 3.5.** Suppose that \( s \in \mathbb{R}, \, p \in (1, \infty) \) and \( q \in [1, \infty] \). If \( k > s/2 \) and \( k \geq 0 \), we have
\[
\|u\|_{\dot{B}^{s,\mathcal{L}}_{p,q}} := \left\| t^{-s/2} \| (t L)^k e^{t\mathcal{L}} u \|_p \right\|_{L_q(\mathbb{R}_+; dt)} \simeq \|u\|_{\dot{B}^{s}_{p,q}}, \quad \forall u \in L^p(\mathbb{R}^n; \mathbb{R}^n).
\]

**Proof.** The lemma is a consequence of Lemma 3.4 along with the identities
\[
e^{t\mathcal{L}} = e^{\mu t\Delta} \mathcal{P} + e^{vt\Delta} Q
\]
and
\[
e^{t\Delta} = \mathcal{P} e^{\mu^{-1} t\mathcal{L}} + Q e^{\nu^{-1} t\mathcal{L}}.
\]

\[\square\]

### 4. Bounds of fundamental matrix

This section is devoted to the proof of Theorem 2.1. For notational convenience, we denote \( L^2 = L^2(\mathbb{R}^n; \mathbb{R}^n), \, H^2 = H^2(\mathbb{R}^n; \mathbb{R}^n) \). Let \( \| \cdot \| \) be the \( L^2 \) norm induced by the standard \( L^2 \) inner product \( \langle \cdot, \cdot \rangle \), and \( \| \cdot \|_{\rho_0} \) the weighted norm induced by the inner product
\[
\langle u, v \rangle_{\rho_0} = \int_{\mathbb{R}^n} u(x) \cdot v(x) \rho_0(x) \, dx.
\]

Roughly, our method is a classical PDE method, and we will study various weighted estimates for the solutions to the parabolic Lamé system
\[
\rho_0(x) \partial_t u - \mathcal{L} u = 0, \quad \text{in } (0, \infty) \times \mathbb{R}^n. \quad (4.1)
\]
Before studying the variable coefficient problem, let us point out a basic fact about the Lamé operator $L$. The assumption on the coefficients $\mu$ and $\lambda$ guarantees the ellipticity of $-L$, and we have
\[
\|(-L)^{1/2}u\|^2 = \langle -Lu, u \rangle = \mu \|\nabla u\|^2 + (\mu + \lambda)\|\text{div } u\|^2 \geq (\mu \wedge \nu)\|\nabla u\|^2 \quad (4.2)
\]
for all vector fields $u \in H^2$.

**Lemma 4.1.** The operator $bL : H^2 \subset L^2 \to L^2$ generates an analytic semigroup of contraction $\{e^{tbL}\}_{t \geq 0}$ on $(L^2, \langle \cdot, \cdot \rangle_{\rho_0})$, and $e^{tbL}b$ is self-adjoint on $(L^2, \langle \cdot, \cdot \rangle_{\rho_0})$ for every $t \geq 0$.

**Proof.** First, it is readily to verify that $bL$ is a self-adjoint operator on $(L^2, \langle \cdot, \cdot \rangle_{\rho_0})$. In view of (4.2), we have $\langle bLu, u \rangle_{\rho_0} \leq 0$ for all $u \in H^2$. So by Theorem 3.1, $bL$ generates an analytic semigroup of contraction $\{e^{tbL}\}_{t \geq 0}$ on $(L^2, \langle \cdot, \cdot \rangle_{\rho_0})$. Since $e^{tbL}$ is self-adjoint on $(L^2, \langle \cdot, \cdot \rangle_{\rho_0})$, we have for all $u, v \in L^2$ that
\[
\langle e^{tbL}bu, v \rangle = \langle e^{tbL}bu, bv \rangle_{\rho_0} = \langle bu, e^{tbL}bv \rangle_{\rho_0} = \langle u, e^{tbL}bv \rangle.
\]
This means that $e^{tbL}b$ is self-adjoint on $(L^2, \langle \cdot, \cdot \rangle)$. \(\square\)

**Lemma 4.2.** Let $n \in \{2, 3\}$. For every $t > 0$, the bounded operator $e^{tbL}$ on $L^2$ admits a Schwartz kernel, denoted by $K_t(x, y)$, which is bounded and satisfies the pointwise bound
\[
|K_t(x, y)| \leq \frac{C}{t^{n/2}}
\]
for some constant $C = C(m, \mu, \lambda)$.

**Proof.** Since $n \in \{2, 3\}$, we get from the Gagliardo–Nirenberg inequality and (3.5) that
\[
\|u\|_{\infty} \leq C\|u\|_{1-n/4}\|Lu\|_{n/4} \quad u \in H^2.
\]
(4.3)

This along with the analyticity of $e^{tbL}$ implies that
\[
\|e^{tbL}u_0\|_{\infty} \leq Ct^{-n/4}\|u_0\|, \quad u_0 \in L^2.
\]
Then, $e^{tbL}$ is also bounded from $L^1$ to $L^2$ due to the self-adjointness of $e^{tbL}b$, and from $L^1$ to $L^{\infty}$ due to the semigroup property. So the Schwartz kernel $K_t(x, y)$ of $e^{tbL}$ is indeed bounded and satisfies the desired bound. This completes the proof. \(\square\)

Next, we adopt the well-known Davies perturbation method (see [10]) to show Gaussian bounds for the kernel $S_t(x, y)$. Let $\mathcal{W}$ denote the set of all bounded real-valued smooth functions $\psi$ on $\mathbb{R}^n$ such that $\|\nabla \psi\|_{\infty} \leq 1$ and $\|\nabla^2 \psi\|_{\infty} \leq 1$. Let $d(x, y) := \sup \{|\psi(x) - \psi(y)| : \psi \in \mathcal{W}\}$. 
Lemma 4.3. (see [11, Lemma 4]) There exists a positive constant $C = C(n)$ such that

$$C^{-1}|x - y| \leq d(x, y) \leq C|x - y|$$

for all $x, y \in \mathbb{R}^n$.

Given $\alpha \in \mathbb{R}$ and $\psi \in H$, define $\psi_\alpha(x) = \psi(\alpha x)$ and $\phi(x) = e^{\psi_\alpha(x)}$. The analysis is based on the key observation that

$$\langle -\phi^{-1}L\phi v, u \rangle = \mu \int (\alpha(\nabla \psi)_\alpha \otimes v + \nabla v) : (-\alpha(\nabla \psi)_\alpha \otimes u + \nabla u) \, dx + (\mu + \lambda) \int (\alpha(\nabla \psi)_\alpha \cdot v + \text{div} \, v)(-\alpha(\nabla \psi)_\alpha \cdot u + \text{div} \, u) \, dx$$

(4.4)

for any smooth vector fields $u$ and $v$. In particular, if $u = v$, we have

$$\langle -\phi^{-1}L\phi v, v \rangle \geq \|(-L)^{1/2}v\|^2 - C\alpha^2 \|v\|^2. \quad (4.5)$$

In what follows, we divide the proof of Theorem 2.1 into three lemmas.

Lemma 4.4. Let $n \in \{2, 3\}$. There exists a constant $C = C(m, \mu, \lambda)$ such that for all $t > 0$ and $x, y \in \mathbb{R}^n$,

$$|K_t(x, y)| \leq \frac{C}{t^{n/2}} \exp \left\{ -\frac{|x - y|^2}{Ct} \right\}.$$

Proof. Denote $v = \phi^{-1}e^{t\mathcal{L}}(\phi u_0)$, where $u_0 \in L^2$. Then, $v$ is a solution to the system

$$\begin{cases}
\rho_0 \partial_t \nu - \phi^{-1}L\phi \nu = 0, \\
\nu(0) = u_0.
\end{cases} \quad (4.6)$$

We start with the energy estimates for $v$. Taking inner product of (4.6) with $v$, then using (4.5), we get

$$\frac{1}{2} \frac{d}{dt} \|v\|_{\rho_0}^2 + \|(-L)^{1/2}v\|^2 \leq C\alpha^2 \|v\|_{\rho_0}^2.$$

Applying Gronwall’s inequality, we obtain

$$\|v(t)\|^2 + \int_0^t \|(-L)^{1/2}v\|^2 \, d\tau \leq C\|u_0\|^2 e^{C\alpha^2 t}. \quad (4.7)$$

Differentiating (4.6) with respect to $t$, we get by a similar argument that

$$\frac{1}{2} \frac{d}{dt} \|\partial_t \nu\|_{\rho_0}^2 + \|(-L)^{1/2}\partial_t \nu\|^2 \leq C\alpha^2 \|\partial_t \nu\|_{\rho_0}^2.$$
So the function $t \mapsto \|\partial_t v\|_{\rho_0}^2 e^{-Ca^2 t}$ is decreasing. Consequently, we have

$$\|\partial_t v\|_{\rho_0}^2 e^{-Ca^2 t} \leq \frac{2}{t} \int_{t/2}^t \|\partial_t v\|_{\rho_0}^2 e^{-Ca^2 \tau} d\tau.$$  \hfill (4.8)

Next, multiplying (4.6) by $v_t$ and integrating in $x$, then using (4.4) and the Cauchy–Schwarz inequality, we get

$$\|\partial_t v\|_{\rho_0}^2 + \frac{1}{2} \frac{d}{dt} \|(-\mathcal{L})^{1/2} v\|^2 \leq C \left( \alpha^2 \|v\| + |\alpha||\nabla v|\right) \|\partial_t v\|.$$  \hfill (4.9)

This together with (4.7) and (4.2) gives

$$\|\partial_t v\|_{\rho_0}^2 + \frac{d}{dt} \|(-\mathcal{L})^{1/2} v\|^2 \leq C \left( \alpha^4 \|v\|^2 + \alpha^2 \|(-\mathcal{L})^{1/2} v\|^2 \right).$$

So,

$$\|\partial_t v\|_{\rho_0}^2 e^{-Ca^2 t} + \frac{d}{dt} \|(-\mathcal{L})^{1/2} v\|^2 e^{-Ca^2 t} \leq C \alpha^4 \|u_0\|^2.$$  \hfill (4.10)

Combining (4.7) and (4.9), we have

$$\int_{t/2}^t \|\partial_t v\|_{\rho_0}^2 e^{-Ca^2 \tau} d\tau + \|(-\mathcal{L})^{1/2} v(t)\|^2 \leq C \left( \frac{1}{t} + \alpha^4 t \right) \|u_0\|^2 e^{Ca^2 t},$$

which together with (4.8) further implies

$$\|\partial_t v(t)\| \leq C \left( \alpha^2 + \frac{1}{t} \right) \|u_0\| e^{Ca^2 t} \leq \frac{C}{t} \|u_0\| e^{Ca^2 t}.$$  \hfill (4.11)

The above estimate implies the corresponding $L^2$ estimate of $\mathcal{L} v$. To see this, we get by a direct computation that

$$\mathcal{L} v = -\rho_0 \partial_t v + \mu (\alpha^2 |\nabla \psi|^2 v + 2\alpha (\nabla \psi) \cdot \nabla v + \alpha^2 (\Delta \psi) v) \\
+ (\mu + \lambda) (\alpha \nabla v (\nabla \psi) + \alpha \nabla v (\nabla \psi) + \alpha^2 (v \cdot (\nabla \psi)) (\nabla \psi) \\
+ \alpha^2 (\nabla^2 \psi) v).$$  \hfill (4.11)

Then, it is easy to see that

$$\|\mathcal{L} v\| \leq C (\|\partial_t v\| + \alpha^2 \|v\| + |\alpha||\nabla v|).$$

The first-order derivative can be handled by using the interpolation inequality

$$\|\nabla v\| \leq C \|v\|^{1/2} \|\mathcal{L} v\|^{1/2}.$$
So,

\[ \|L v\| \leq C(\|\partial_t v\| + \alpha^2 \|v\|). \]

Substituting for \(\|v\|\) and \(\|\partial_t v\|\) by (4.7) and (4.10), respectively, we have

\[ \|L v(t)\| \leq \frac{C}{t} \|u_0\| e^{C\alpha^2 t}. \]

Now, using the Gagliardo–Nirenberg inequality (4.3), we obtain

\[ \|v(t)\|_\infty \leq \frac{C}{t^{n/4}} \|u_0\|_1 e^{C\alpha^2 t}. \] (4.12)

This means that the operator \(\phi^{-1} e^{tbL} \phi\) is bounded from \(L^2\) to \(L^\infty\). A duality argument gives the bound from \(L^1\) to \(L^2\), that is,

\[ \|v(t)\| \leq \frac{C}{t^{n/4}} \|u_0\|_1 e^{C\alpha^2 t}. \] (4.13)

While this along with the semigroup property of \(\phi^{-1} e^{tbL} \phi\) gives

\[ \|v(t)\|_\infty \leq \frac{C}{t^{n/2}} \|u_0\|_1 e^{C\alpha^2 t}. \] (4.14)

Noticing that the kernel of \(\phi^{-1} e^{tbL} \phi\) is \(K_t(x, y)\), we get

\[ |K_t(x, y)| \leq \frac{C}{t^{n/2}} \exp(C\alpha^2 t + \psi(\alpha x) - \psi(\alpha y)). \]

Replacing \(\psi\) by \(-\psi\), we have

\[ |K_t(x, y)| \leq \frac{C}{t^{n/2}} \exp(C\alpha^2 t - |\psi(\alpha x) - \psi(\alpha y)|). \]

It follows by optimizing with respect to \(\psi \in \mathcal{W}\) and applying Lemma 4.3 that

\[ |K_t(x, y)| \leq \frac{C}{t^{n/2}} \exp(C\alpha^2 t - C^{-1} |x - y|). \]

Finally, minimizing the bound by choosing \(\alpha = \frac{|x - y|}{2Ct}\) completes the proof. \(\square\)

**Lemma 4.5.** Let \(n \in \{2, 3\}\). There exists a constant \(C = C(m, \mu, \lambda)\) such that for all \(t > 0\) and \(x, y \in \mathbb{R}^n\),

\[ |\nabla_x S_t(x, y)| \leq \frac{C}{t^{(n+1)/2}} \exp \left\{ -\frac{|x - y|^2}{Ct} \right\}. \]

**Proof.** Clearly, we only need to show the bound for \(|\nabla_x K_t(x, y)|\). Denote \(u = e^{tbL}(\phi u_0)\) and \(v = \phi^{-1} u\), where \(u_0 \in L^2\). We need to bound the norm \(\|\phi^{-1} \nabla u(t)\|_\infty\). To this end, let us first study the norm \(\|\nabla v(t)\|_\infty\) since

\[ \phi^{-1} \nabla u(t) = \nabla v + \alpha(\nabla \psi)\alpha \otimes v. \] (4.15)
By Eq. (4.11), we see that
\[ \| \mathcal{L}v \|_\infty \leq C(\| \partial_t v \|_\infty + \alpha^2 \| v \|_\infty + |\alpha| \| \nabla v \|_\infty). \] (4.16)

Using Littlewood–Paley and (3.4), one can prove the interpolation inequality
\[ \| \nabla v \|_\infty \leq C \| \frac{1}{2} v(t) \|_\infty \| \frac{1}{2} \mathcal{L}v \|_\infty. \] (4.17)

Plugging (4.17) in (4.16), we easily get
\[ \| \mathcal{L}v \|_\infty \leq C(\| \partial_t v \|_\infty + \alpha^2 \| v \|_\infty). \] (4.18)

Then, combining (4.15), (4.17) and (4.18), we arrive at
\[ \| \phi^{-1} \nabla u(t) \|_\infty \leq C \left( |\alpha| \| v(t) \|_\infty + \| v(t) \|_\infty \| \partial_t v(t) \|_\infty \right). \] (4.19)

Next, in order to bound \( \| \partial_t v(t) \|_\infty \), we observe that
\[ \partial_t v(t) = \phi^{-1} e^{t\frac{i}{2}b\mathcal{L}} \phi[\partial_t v(t/2)]. \]

So, in view of (4.12), (4.10) and (4.13), we get
\[ \| \partial_t v(t) \|_\infty \leq \frac{C}{t^{n/4}} e^{C\alpha^2 t} \| \partial_t v(t/2) \| \leq \frac{C}{t^{1+n/4}} e^{C\alpha^2 t} \| v(t/4) \| \leq \frac{C}{t^{1+n/2}} e^{C\alpha^2 t} \| u_0 \|_1. \] (4.20)

Plugging the above in (4.19) and using (4.14), we have
\[ \| \phi^{-1} \nabla u(t) \|_\infty \leq C \left( \frac{|\alpha|}{t^{n/2}} + \frac{1}{t^{(n+1)/2}} \right) e^{C\alpha^2 t} \| u_0 \|_1 \leq \frac{C}{t^{(n+1)/2}} e^{C\alpha^2 t} \| u_0 \|_1. \] (4.21)

Thus,
\[ |\nabla_x K_t(x, y)| \leq \frac{C}{t^{(n+1)/2}} \exp\{C\alpha^2 t + \psi(\alpha x) - \psi(\alpha y)\}. \]

Again, we finish the proof by optimizing the bound with respect to \( \psi \in \mathcal{W} \) and then \( \alpha \in \mathbb{R} \). \( \square \)

**Remark 4.1.** From (4.20), we also see that the kernel of \( tb\mathcal{L}e^{tb\mathcal{L}} \) has a pointwise Gaussian upper bound. In particular, \( tb\mathcal{L}e^{tb\mathcal{L}} \) extends to a bounded operator on \( L^p \) for every \( t > 0 \).

**Lemma 4.6.** Let \( n \in \{2, 3\} \). For any \( \gamma \in (0, 1) \), there exists a constant \( C = C(m, \mu, \lambda, \gamma) \) such that for all \( t > 0 \) and \( x, y, h \in \mathbb{R}^n \), (2.1) and (2.2) hold whenever \( 2|h| \leq \sqrt{t} \).

**Proof.** Let \( u = e^{tb\mathcal{L}}u_0 \). By Lemmas 4.1 and 4.4, we have
\[ t^{n/4} \| \mathcal{L}u(t) \| + t^{n/2} \| \mathcal{L}u(t) \|_\infty \leq \frac{C}{t} \| u_0 \|_1. \]
For any $\gamma \in (0, 1)$, let $q = \frac{n}{1 - \gamma}$ and $\theta = \frac{2(1 - \gamma)}{n}$. Then, we use the embedding $\dot{W}^{1,q}(\mathbb{R}^n) \hookrightarrow C^\gamma(\mathbb{R}^n)$ to get
\[ \|\nabla u\|_{C^\gamma} \leq C\|\nabla^2 u\|_q \leq C\|\mathcal{L} u\|_{\dot{W}^{1,q}} \leq \frac{C}{t^{(n+1+\gamma)/2}} \|u_0\|_1. \]
Thus, we have for any $h \in \mathbb{R}^n$ that
\[ |\nabla_x K_t(x + h, y) - \nabla_x K_t(x, y)| \leq \frac{C}{t^{(n+1)/2}} \left( \frac{|h|}{\sqrt{t}} \right)^\gamma. \]
The exponential decay factor in (2.1) can be easily obtained by the observation that
\[
|\nabla_x K_t(x + h, y) - \nabla_x K_t(x, y)| \\
\leq (|\nabla_x K_t(x + h, y)| + |\nabla_x K_t(x, y)|)^{1-\beta} |\nabla_x K_t(x + h, y) - \nabla_x K_t(x, y)|^\beta
\]
for any $\beta \in (0, 1)$. This proves (2.1).

To prove (2.2), we write
\[
\int [\nabla_x S_t(x, y + h) - \nabla_x S_t(x, y)] u_0(y) dy = \nabla e^{t\mathcal{L}}(b\delta_h u_0)
\]
with $\delta_h u_0(x) = u_0(x - h) - u_0(x)$. Using Lemma 4.5, the right side can be estimated as follows:
\[
\|\nabla e^{t\mathcal{L}}(b\delta_h u_0)\|_\infty \leq \frac{C}{\sqrt{t}} \|e^{\frac{t}{2}\mathcal{L}}(b\delta_h u_0)\|_\infty \leq \frac{C|h|}{t^{1+n/2}} \|u_0\|_1.
\]
The bound in (2.2) can be shown by a similar argument as the first part of the proof. This completes the proof of the lemma.

For completeness, we conclude this section by finishing the proof of Theorem 2.1.

Proof of Theorem 2.1. Lemmas 4.4–4.6 constitute the proof of Theorem 2.1.

5. An abstract $L^1$ theory

In this section, we are concerned with the $L^1$-in-time theory for the abstract Cauchy problem (1.6), where $S$ is a composition of bounded and unbounded operators. We follow our prior work [25] closely, and we do not explicitly use the theory of interpolation spaces.

Let $(X, \| \cdot \|)$ be a Banach space. We temporarily just assume

Assumption 5.1. $S : D(S) \subset X \to X$ is a one-to-one operator that generates a bounded analytic semigroup $e^{tS}$ on $X$. 

Proof of Theorem 2.1. Lemmas 4.4–4.6 constitute the proof of Theorem 2.1. □
Given \( s \in (0, 2) \), we define
\[
\|x\|_{\dot{B}^{s,1}_X} := \|t^{-s/2}\|tS e^{tS} x\|_{L^1(\mathbb{R}^+, \frac{dt}{t})}
\]
and
\[
\|x\|_{\dot{B}^{s,1}_X} := \|t^{s/2}\| e^{tS} x\|_{L^1(\mathbb{R}^+, \frac{dt}{t})},
\]
In view of Lemmas 3.4 and 3.5, the above notations make sense if \( S \) is a second-order elliptic operator. For any \( x \in D(S) \), since \( \|tS e^{tS} x\| \lesssim \|x\| \wedge \|tS x\| \), we easily see that
\[
\|x\|_{\dot{B}^{s,1}_X} \lesssim \|x\|_{D(S)} := \|x\| + \|Sx\|.
\]
While for \( x \in R(S) \), the range of \( S \), we have
\[
\|x\|_{\dot{B}^{-s,1}_X} = \|S^{-1}x\|_{\dot{B}^{2-s,1}_X} \lesssim \|x\|_{R(S)} := \|x\| + \|S^{-1}x\|.
\]

**Definition 5.1.** Let \( s \in (0, 2) \). Define \( \dot{B}^{s,1}_X \) as the completion of \((D(S), \|\cdot\|_{\dot{B}^{s,1}_X})\), and \( \dot{B}^{-s,1}_X \) as the completion of \((R(S), \|\cdot\|_{\dot{B}^{-s,1}_X})\).

The space \( \dot{B}^{s,1}_X \) can also be defined via interpolation (see, e.g., [16, Remark 2.4]), but we do not need this fact in this paper.

For notational convenience, we temporarily denote \( \dot{B}^{s,1}_X \) by \( \dot{B}^{s} \). But we shall not use the abbreviated notations if the norms are associated with different operators.

**Lemma 5.1.** For every \( k \in \mathbb{N} \cup \{0\} \) and \( s \in (0, 2) \), there exists a constant \( C \) depending on \( s \) and \( k \) such that
\[
\sup_{t>0} \|(tS)^k e^{tS} x\|_{\dot{B}^s} \leq C \|x\|_{\dot{B}^s}, \quad \forall x \in D(S),
\]
\[
\sup_{t>0} \|(tS)^k e^{tS} x\|_{\dot{B}^{-s}} \leq C \|x\|_{\dot{B}^{-s}}, \quad \forall x \in R(S),
\]
\[
\left\|(tS)^{k+1} e^{tS} x\right\|_{L^1(\mathbb{R}^+, \frac{dt}{t})} \leq C \|x\|_{\dot{B}^s}, \quad \forall x \in D(S),
\]
and
\[
\left\|(tS)^{k+1} e^{tS} x\right\|_{L^1(\mathbb{R}^+, \frac{dt}{t})} \leq C \|x\|_{\dot{B}^{-s}}, \quad \forall x \in R(S).
\]

**Proof.** The first two inequalities follow immediately from the definitions of the norms and the analyticity of \( e^{tS} \).
The proofs for (5.3) and (5.4) are similar, so let us only prove (5.3). In view of (5.1), we only need to prove (5.3) for \( k = 0 \). Applying Fubini’s theorem, we have

\[
\int_0^\infty \| \tau S e^{\tau S} x \|_{\dot{B}^s} \frac{d\tau}{\tau} = \int_0^\infty \int_0^\infty t^{-s/2} \| S^2 e^{(t+\tau)S} x \| \, dt \, d\tau
\]

\[
= \int_0^\infty \int_0^\infty (t - \tau)^{-s/2} \| S^2 e^{\tau S} x \| \, dt \, d\tau
\]

\[
= \int_0^\infty \| S^2 e^{\tau S} x \| \, dt \int_0^t (t - \tau)^{-s/2} \, d\tau
\]

\[
= \frac{2}{2 - s} \int_0^\infty t^{-s/2} \| t S^2 e^{\tau S} x \| \, dt.
\]

Finally, by the analyticity of \( S \), we end up with

\[
\| (tS) e^{tS} x \|_{\dot{B}^s} \leq C \int_0^\infty t^{-s/2} \| S e^{\tau S} x \| \, dt = C \| x \|_{\dot{B}^s}.
\]

This completes the proof.

The inequality (5.1) (with \( k = 0 \)) guarantees that \( e^{tS} |_{D(S)} \) extends to a bounded operator on \( \dot{B}^s \) with bounds uniform in \( t \). Denote this extension by \( \mathcal{T}_s(t) \). Then, \( \{ \mathcal{T}_s(t) \}_{t \geq 0} \) is a bounded semigroup on \( \dot{B}^s \). Similarly, (5.2) implies that \( e^{tS} \) also extrapolates to a bounded semigroup \( \{ \mathcal{T}_{-s}(t) \}_{t \geq 0} \) on \( \dot{B}^{-s} \). In fact, both semigroups are strongly continuous.

**Lemma 5.2.** \( \{ \mathcal{T}_s(t) \}_{t \geq 0} \) (resp., \( \{ \mathcal{T}_{-s}(t) \}_{t \geq 0} \)) is a bounded \( C_0 \) semigroup on \( \dot{B}^s \) (resp., \( \dot{B}^{-s} \)).

**Proof.** For \( x \in D(S) \), the function \( t \mapsto \mathcal{T}_s(t)x = e^{tS}x \) belongs to \( C([0, \infty); D(S)) \), hence \( C([0, \infty); \dot{B}^s) \) since \( D(S) \hookrightarrow \dot{B}^s \). One can easily get the strong continuity of \( \mathcal{T}_s(t) \) on \( \dot{B}^s \) by a density argument.

The strong continuity of \( \mathcal{T}_{-s}(t) \) on \( \dot{B}^{-s} \) can be proved analogously.

Let us denote by \( \mathcal{G}_s \) and \( \mathcal{G}_{-s} \) the generators of \( \mathcal{T}_s(t) \) and \( \mathcal{T}_{-s}(t) \), respectively. In general, it is not easy to identify the domain of the generator of a semigroup. However, it would be easier to find a core for the generator.

**Lemma 5.3.**

(i) The domain \( D(S^2) \) of \( S^2 \) is a core for \( \mathcal{G}_s \), and it holds that \( \mathcal{G}_s |_{D(S^2)} = S |_{D(S^2)} \), that is, \( \mathcal{G}_s \) is the closure of \( S : D(S^2) \subset \dot{B}^s \rightarrow \dot{B}^s \).

(ii) \( \mathcal{G}_{-s} \) is the closure of \( S : D(S) \cap R(S) \subset \dot{B}^{-s} \rightarrow \dot{B}^{-s} \).

**Proof.** Note that \( D(S^2) \) is dense in \( \dot{B}^s \) since \( D(S^2) \) is dense in \( D(S) \) and \( D(S) \) is dense in \( \dot{B}^s \). For every \( x \in D(S^2) \), we have

\[
\frac{1}{t} (\mathcal{T}_s(t)x - x) = \frac{1}{t} \int_0^t \mathcal{T}_s(\tau)Sx \, d\tau.
\]
Letting $t \rightarrow 0^+$, the right side converges to $Sx$ in $D(S)$, thus, in $\dot{B}^s$. From this, we infer that $D(S^2) \subset D(\mathcal{G}_s)$ and $\mathcal{G}_s|_{D(S^2)} = S|_{D(S^2)}$. Obviously, $D(S^2)$ is invariant under $T_s(t)$. Thus, by Lemma 3.2, $D(S^2)$ is a core for $\mathcal{G}_s$.

We prove the second part along the lines of the above proof. First, $D(S) \cap R(S)$ is dense in $\dot{B}^{-s}$ since $D(S) \cap R(S)$ is dense in $(R(S), \| \cdot \|_{R(S)})$ and $R(S)$ is dense in $\dot{B}^{-s}$. Next, we can show that $D(S) \cap R(S) \subset D(\mathcal{G}_{-s})$ and $\mathcal{G}_{-s}|_{D(S) \cap R(S)} = S|_{D(S) \cap R(S)}$. Moreover, since $D(S) \cap R(S)$ is invariant under $T_{-s}(t)$, so it is a core for $\mathcal{G}_{-s}$. This completes the proof.

**Lemma 5.4.** $\{T_s(t)\}_{t \geq 0}$ (resp., $\{T_{-s}(t)\}_{t \geq 0}$) is a bounded analytic semigroup on $\dot{B}^s$ (resp., $\dot{B}^{-s}$).

**Proof.** We know from Lemma 5.3 that $D(S^2)$ is dense in $\dot{B}^s$, and that $\mathcal{G}_s T_s(t) x = S e^{t\mathcal{G}_s} x$ for $x \in D(S^2)$. It then follows from (5.1) that $\| t \mathcal{G}_s T_s(t) x \|_{\dot{B}^s} \leq C \| x \|_{\dot{B}^s}$ for every $t > 0$. So $T_s(t)$ is a bounded analytic semigroup. An analogous argument gives the analyticity of $T_{-s}(t)$ on $\dot{B}^{-s}$.

**Remark 5.1.** By Fatou’s lemma, now (5.3) (resp., (5.4)) actually holds for data in $\dot{B}^s$ (resp., $\dot{B}^{-s}$). In particular, choosing $k = 0$, we have

$$\| \mathcal{G}_s e^{t\mathcal{G}_s} x \|_{L^1(\mathbb{R}_+, \dot{B}^s)} \leq C \| x \|_{\dot{B}^s}, \quad \forall x \in \dot{B}^s$$

and

$$\| \mathcal{G}_{-s} e^{t\mathcal{G}_{-s}} x \|_{L^1(\mathbb{R}_+, \dot{B}^{-s})} \leq C \| x \|_{\dot{B}^{-s}}, \quad \forall x \in \dot{B}^{-s}. \quad (5.6)$$

Next, we take advantage of Lemma 5.4, (5.5) and (5.6) to obtain the maximal $L^1$ regularity for the abstract Cauchy problems

$$u'(t) - \mathcal{G}_s u(t) = f(t), \quad u(0) = x \quad (5.7)$$

and

$$u'(t) - \mathcal{G}_{-s} u(t) = f(t), \quad u(0) = x. \quad (5.8)$$

**Theorem 5.5.** Assume Assumption 5.1. Let $s \in (0, 2)$ and $T \in (0, \infty]$. There exists a constant $C = C(s)$ such that

(i) For any $x \in \dot{B}^s$ and $f \in L^1((0, T); \dot{B}^s)$, Eq. (5.7) has a unique strong solution $u \in C([0, T]; \dot{B}^s)$ satisfying

$$\| u \|_{L^\infty_T(\dot{B}^s)} + \| u' \|_{L^1_T(\dot{B}^s)} \leq C \| x \|_{\dot{B}^s} + C \| f \|_{L^1_T(\dot{B}^s)}.$$

(ii) For any $x \in \dot{B}^{-s}$ and $f \in L^1((0, T); \dot{B}^{-s})$, Eq. (5.8) has a unique strong solution $u \in C([0, T]; \dot{B}^{-s})$ satisfying

$$\| u \|_{L^\infty_T(\dot{B}^{-s})} + \| u' \|_{L^1_T(\dot{B}^{-s})} \leq C \| x \|_{\dot{B}^{-s}} + C \| f \|_{L^1_T(\dot{B}^{-s})},$$
Proof. Let us only give the proof of the first part. The homogeneous part $e^{tG_s}x$ is a classical solution to the homogeneous equation and satisfies the desired estimate by Lemma 5.4 and (5.5). Denote the inhomogeneous part by $I f(t) = \int_0^t e^{(t-\tau)G_s} f(\tau) \, d\tau$. Then, it is easy to see that $\|G_s I f\|_{L^\infty_T(\dot{B}^s)} \lesssim \|f\|_{L^1_T(\dot{B}^s)}$. Using again (5.5) and Fubini’s theorem, we have

$$\|G_s I f\|_{L^1_T(\dot{B}^s)} \leq \int_0^T \int_0^t \|G_s e^{(t-\tau)G_s} f(\tau)\|_{\dot{B}^s} \, d\tau \, dt \lesssim \|f\|_{L^1_T(\dot{B}^s)}.$$ 

So by Lemma 3.3, $u = e^{tG_s}x + I f(t)$ is a strong solution to (5.7). The estimate for $u'$ follows directly by the previous estimates and Eq. (5.7). So the proof is completed.

Now, we consider the abstract Cauchy problem (1.6) associated with a composite operator of the form $S = BA$, where $B$ is a bounded invertible operator and $A$ is an unbounded operator. Our analysis relies on an intriguing equivalent characterization of norms (see [25, Theorem 3.9]). Such a result will be effective for studying density-dependent viscous fluids.

We start with some assumptions.

**Assumption 5.2.** The linear operator $A : D(A) \subset X \to X$ generates a bounded analytic semigroup $e^{tA}$ satisfying $\lim_{t \to \infty} \|e^{tA}x\| = 0$ for every $x \in X$.

**Assumption 5.3.** $B \in \mathcal{L}(X)$ is invertible with an inverse $B^{-1} \in \mathcal{L}(X)$.

**Assumption 5.4.** $S = BA : D(A) \subset X \to X$ generates a bounded analytic semigroup $e^{tS}$ satisfying $\lim_{t \to \infty} \|e^{tS}x\| = 0$ for every $x \in X$.

**Lemma 5.6.** Under Assumptions 5.2–5.4, it holds for any $(s, q) \in (0, 1) \times [1, \infty)$ and $x \in X$ that

$$\left\| t^s \| e^{tS}x \|_{L^q(\mathbb{R}_+, \frac{dt}{t})} \right\| \lesssim \left\| t^s \| e^{tA}B^{-1}x \|_{L^q(\mathbb{R}_+, \frac{dt}{t})} \right\|. \tag{5.9}$$

Consequently, we have for any $x \in D(A)$,

$$\left\| t^{-s} \| tSe^{tS}x \|_{L^q(\mathbb{R}_+, \frac{dt}{t})} \right\| \lesssim \left\| t^{-s} \| tAe^{tA}x \|_{L^q(\mathbb{R}_+, \frac{dt}{t})} \right\|. \tag{5.10}$$

**Proof.** By Assumption 5.2, we have for any $x \in X$ that

$$x = -\lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{1/\varepsilon} \mathcal{A}e^{\tau A}x \, d\tau,$$

where the limit converges in $X$. Replacing $x$ by $B^{-1}x$ gives

$$B^{-1}x = -\int_0^\infty \mathcal{A}e^{\tau A}B^{-1}x \, d\tau.$$
Applying $e^{tS}B$ to both sides of the above identity, we obtain
\[ e^{tS}x = -\int_0^\infty e^{tS}B A e^{\tau A}B^{-1}x \, d\tau. \]

We can bound the integrand in two different ways:
\[ \|e^{tS}B A e^{\tau A}B^{-1}x\| = \|e^{tS}S e^{\tau A}B^{-1}x\| \lesssim \frac{1}{t} \|e^{\tau A}B^{-1}x\| \lesssim \frac{1}{t} \|e^{\frac{\tau}{t}}A B^{-1}x\|, \]
or,
\[ \|e^{tS}B A e^{\tau A}B^{-1}x\| \lesssim \|A e^{\tau A}B^{-1}x\| \lesssim \frac{1}{\tau} \|e^{\tau A}B^{-1}x\|. \]

So we arrive at
\[ \|e^{tS}x\| \lesssim \int_0^\infty \frac{1}{t \sqrt{\tau}} \|e^{\tau A}B^{-1}x\| \, d\tau. \]

Multiplying both sides by $t^s$, we get
\[ t^s \|e^{tS}x\| \lesssim \int_0^\infty \left( \frac{t}{\tau} \right)^s \left( 1 + \frac{\tau}{t} \right) \tau^s \|e^{\tau A}B^{-1}x\| \frac{d\tau}{\tau}. \]

Since $s \in (0, 1)$, it is easy to verify that
\[ \sup_{t>0} \int_0^\infty \left( \frac{t}{\tau} \right)^s \left( \frac{\tau}{t} \wedge 1 \right) \frac{d\tau}{\tau} + \sup_{\tau>0} \int_0^\infty \left( \frac{t}{\tau} \right)^s \left( 1 \wedge 1 \right) \frac{dt}{\tau} \leq C. \]

It then follows from [25, Lemma 3.7] that
\[ \left\| t^s \|e^{tS}x\| \right\|_{L^q(\mathbb{R}^+, \frac{dt}{\tau})} \lesssim \left\| t^s \|e^{\tau A}B^{-1}x\| \right\|_{L^q(\mathbb{R}^+, \frac{dt}{\tau})}. \]

The reverse inequality can be proved in a similar way. By Assumption 5.4, we have for any $x \in X$ that
\[ x = -\int_0^\infty B A e^{\tau S}x \, d\tau. \]

This time we apply $e^{\tau A}B^{-1}$ to both sides of the above identity to get
\[ e^{\tau A}B^{-1}x = -\int_0^\infty e^{\tau A}A e^{\tau S}x \, d\tau. \]

So bounding the integrand in two different ways as before gives rise to
\[ \|e^{\tau A}B^{-1}x\| \lesssim \int_0^\infty \frac{1}{t \sqrt{\tau}} \|e^{\tau S}x\| \, d\tau. \]

This can further imply that
\[ \left\| t^s \|e^{\tau A}B^{-1}x\| \right\|_{L^q(\mathbb{R}^+, \frac{dt}{\tau})} \lesssim \left\| t^s \|e^{tS}x\| \right\|_{L^q(\mathbb{R}^+, \frac{dt}{\tau})}. \]

Thus, we have verified (5.9).
Finally, (5.10) follows by replacing \( x \) by \( Sx \) in (5.9).

We assume additionally that

**Assumption 5.5.** \( A : D(A) \subset X \rightarrow X \) is one-to-one.

So \( S \) satisfies Assumption 5.1. Then, the equivalence of norms implies the equivalence of spaces. More precisely, we get immediately from Lemma 5.6 that

**Corollary 5.1.** Let \( s \in (0, 2) \). Under Assumptions 5.2–5.5, we have

(i) \( \dot{B}^{-s,A}_{X,1} \subset \dot{B}^{-s,S}_{X,1} \) with equivalent norms.

(ii) \( \dot{B}^{-s,A}_{X,1} \) coincides with the completion of \( R(S) \) with respect to the norm \( \| \cdot \|_{\dot{B}^{-s,A}_{X,1}} \),

where the spaces and norms associated with \( A \) are defined in an obvious way.

It turns out that the operator \( B \) acting on \( \dot{B}^{-s,A}_{X,1} \) is meaningful. Indeed, (5.9) implies that \( B|_{R(A)} \) extends to a continuous operator, denoted by \( B \), from \( \dot{B}^{-s,A}_{X,1} \) to \( \dot{B}^{-s,S}_{X,1} \); and that \( B^{-1}|_{R(S)} \) extends to a continuous operator, denoted by \( B^{-1} \), from \( \dot{B}^{-s,S}_{X,1} \) to \( \dot{B}^{-s,A}_{X,1} \). Obviously, \( B \) is invertible and \( B^{-1} = B^{-1} \). These facts can help us identify \( G_{-s} \) in the following

**Lemma 5.7.** Assuming Assumptions 5.2–5.5, then the operator

\[
\hat{A} : D(\hat{A}) \cap R(S) \subset \dot{B}^{-s,S}_{X,1} \rightarrow \dot{B}^{-s,A}_{X,1}
\]

is closable. Moreover, we have \( \hat{G}_{-s} = \overline{B\hat{A}} \), where \( \overline{A} \) is the closure of the above \( A \).

**Proof.** We see from Lemma 5.3 (ii) that \( \hat{G}_{-s} \) is the closure of

\[
\mathcal{B} \hat{A} : D(\mathcal{B}) \cap R(S) \subset \dot{B}^{-s,S}_{X,1} \rightarrow \dot{B}^{-s,S}_{X,1}.
\]

It follows that \( \overline{A} := \overline{B^{-1} \hat{G}_{-s}} \) is the closure of

\[
A : D(A) \cap R(S) \subset \dot{B}^{-s,S}_{X,1} \rightarrow \dot{B}^{-s,A}_{X,1}.
\]

This completes the proof. \( \square \)

We conclude this section with the maximal \( L^1 \) regularity for the Cauchy problem

\[
\overline{B}^{-1} u'(t) - \overline{A} u(t) = f(t), \quad u(0) = x.
\]

**Theorem 5.8.** Let \( s \in (0, 2) \) and \( T \in (0, \infty) \). Assuming Assumptions 5.2–5.5, if

\[
x \in \dot{B}^{-s,S}_{X,1} \quad \text{and} \quad f \in L^1((0, T); \dot{B}^{-s,A}_{X,1}) \quad \text{then (5.11) has a unique strong solution} \ u \quad \text{in the class}
\]

\[
u \in C([0, T); \dot{B}^{-s,S}_{X,1}), \quad u' \in L^1((0, T); \dot{B}^{-s,S}_{X,1}), \quad \overline{A} u \in L^1((0, T); \dot{B}^{-s,A}_{X,1})
\]

Moreover, it holds that

\[
\| \overline{B}^{-1} u \|_{L^\infty_T(\dot{B}^{-s,A}_{X,1})} + \| \overline{B}^{-1} u' \|_{L^1_T(\dot{B}^{-s,A}_{X,1})} \leq C \| \overline{B}^{-1} x \|_{\dot{B}^{-s,A}_{X,1}} + C \| f \|_{L^1_T(\dot{B}^{-s,A}_{X,1})},
\]

where \( C \) depends on \( s, \| \mathcal{B} \|_{\mathcal{L}(X)} \) and \( \| \mathcal{B}^{-1} \|_{\mathcal{L}(X)} \).
Proof. Note that \( \bar{B} f \in L^1((0, T); \dot{B}^{-s}_{X,1} \mathcal{S}) \). Thanks to the continuity of \( \bar{B} \) and \( \bar{B}^{-1} \), and Lemma 5.7, then Theorem 5.8 follows by applying Theorem 5.5 (ii) to the Cauchy problem

\[
  u'(t) - \bar{B}Au(t) = \bar{B}f(t), \quad u(0) = x.
\]

\[\square\]

6. Concrete examples

In this section, we prove Theorem 2.2. We always assume that \( n \geq 2 \) if \( A = \Delta \), or \( n \in \{2, 3\} \) if \( A = \mathcal{L} \).

We choose \( X = L^p = L^p(\mathbb{R}^n; \mathbb{R}^n) \) (\( 1 < p < \infty \)), \( D(A) = W^{2,p} = W^{2,p}(\mathbb{R}^n; \mathbb{R}^n) \), and \( S = bA \). Obviously, Assumptions 5.3 and 5.5 are satisfied. That \( A \) satisfies Assumption 5.2 is a classical result (see, e.g., [2, Example 3.7.6]). That \( bA : W^{2,p} \subset L^p \rightarrow L^p \) satisfies Assumption 5.4 was essentially proved in [14, 19]. Analogously, we can use Lemma 4.1, Lemma 4.4 and Remark 4.1 to show that \( b \mathcal{L} \) satisfies Assumption 5.4 as well.

Let us identify the spaces \( \dot{B}^{s,A}_{X,1} \). Let \( s \in (0, 2) \). We know from Lemmas 3.4 and 3.5 that the \( \dot{B}^{s,A}_{X,1} \)-norm is equivalent to the Besov \( \dot{B}^{s}_{p,1} \)-norm. One can see from (3.3) and (3.4) that \( R(\Delta) = R(\mathcal{L}) \). It is, however, easy to see that \( R(\Delta) \) is dense in \( \dot{B}^{s}_{p,1} \). So \( \dot{B}^{s,A}_{X,1} \) is identified as \( \dot{B}^{s}_{p,1} \) for every \( s \in (0, 2) \). To identify \( \dot{B}^{s,A}_{X,1} \), we assume additionally \( s \leq \frac{n}{p} \) so that \( \dot{B}^{s}_{p,1} \) is complete. Then, applying Corollary 5.1 (i), Lemmas 3.4 and 3.5, and the obvious fact that \( D(A) = W^{2,p} \) is dense in \( \dot{B}^{s}_{p,1} \), we get \( \dot{B}^{s,A}_{X,1} = \dot{B}^{s}_{p,1} \).

We now turn to the central problem of this section, that is, the maximal \( L^1 \) regularity for (2.3). In view of Theorem 5.5 (i) and Lemma 5.3 (i), the smooth solutions to (2.3) should satisfy the \textit{a priori} estimate

\[
  \|u\|_{L^\infty_T(\dot{B}^{s}_{p,1})} + \|u', bAu\|_{L^1_T(\dot{B}^{s}_{p,1})} \lesssim \|u_0\|_{\dot{B}^{s}_{p,1}} + \|bf\|_{L^1_T(\dot{B}^{s}_{p,1})}.
\]

But if \( b \) is merely bounded and measurable, we cannot handle the inhomogeneous term, nor can we obtain the estimate for \( \|Au\|_{L^1_T(\dot{B}^{s}_{p,1})} \). Solving (2.3) in Besov spaces with negative regularity seems to be a more promising way to lower the regularity of the density. In fact, from Theorem 5.8, the \textit{a priori} estimate for smooth solutions becomes

\[
  \|\rho_0 u\|_{L^\infty_T(\dot{B}^{s-1}_{p,1})} + \|\rho_0 u', Au\|_{L^1_T(\dot{B}^{s-1}_{p,1})} \lesssim \|\rho_0 u_0\|_{\dot{B}^{s-1}_{p,1}} + \|f\|_{L^1_T(\dot{B}^{s-1}_{p,1})}.
\]

Unfortunately, the above is not quite true if \( u \) is only a strong solution.

By Corollary 5.1 (ii), the space \( \dot{B}^{s,s}_{X,1} \) agrees with the completion of \( \{bA(W^{2,p}), \|\rho_0 \cdot \|_{\dot{B}^{s-1}_{p,1}}\} \), where \( bA(W^{2,p}) \) is defined as \( \{u = bA v | v \in W^{2,p}\} \). Then, the multiplication by \( \rho_0 \) extends to a bounded operator from \( \dot{B}^{s,s}_{X,1} \) to \( \dot{B}^{s-1}_{p,1} \) with a bounded
inverse that coincides with the extension of the multiplication by \( b \). By Lemma 5.7, the operator
\[ A : W^{2,p} \cap bA(W^{2,p}) \subset \dot{B}^{-s_2}_X \rightarrow \dot{B}^{-s_1}_p \]
is closable, and we denote its closure by \( \overline{A} \). Then, one can directly interpret Theorem 5.8 as follows:

**Corollary 6.1.** Let \( s \in (0, 2) \) and \( T \in (0, \infty) \). If \( u_0 \in \dot{B}^{-s_2}_X \) and \( f \in L^1(0, T) \); \( \dot{B}^{-s_1}_p \), then (2.3) has a unique strong solution \( u \) in the class
\[ u \in C([0, T); \dot{B}_X^{s_2}) \cap L^1(0, T); \dot{B}_X^{s_1}, \ \partial_t u \in L^1(0, T); \dot{B}_X^{s_1}, \ \overline{A} u \in L^1((0, T); \dot{B}^{-s_1}_p). \]

Moreover, there exists some constant \( C = C(s, m, \mu, \nu) \) such that
\[ \| \rho_0 u \|_{L^\infty(B^{-s_1}_p)} + \| \rho_0 u' \|_{L^1(B^{-s_1}_p)} \leq C \| \rho_0 u_0 \|_{B^{-s_1}_p} + C \| f \|_{L^1(B^{-s_1}_p)}. \] (6.1)

Unfortunately, it is not clear whether \( \| \nabla u \|_{\infty} \) can be bounded by \( \| \overline{A} u \|_{B^0/p_{-1}} \) for \( n < p < \infty \). Note that an element in \( \dot{B}_X^{s_1} \) might not even be a distribution. So Theorem 5.5 and Corollary 6.1 may be too abstract to be useful in applications if one insists to work in an \( L^1 \)-in-time framework. For this, we require a little more regularity on the coefficients. So let us recall the definition of multiplier spaces.

**Definition 6.1.** A function \( g \) is called a multiplier for a function space \( (X, \| \cdot \|) \) if \( g \) defines a continuous linear operator on \( X \) by pointwise multiplication. If \( g \) is a multiplier for \( X \), we write \( g \in \mathcal{M}(X) \) and define the multiplier norm by
\[ \| g \|_{\mathcal{M}(X)} := \sup_{\| \phi \| = 1} \| g \phi \|. \]

**Lemma 6.1.** (i) Let \( p \in (1, \infty) \) and \( s \in (0, 2) \cap (0, \frac{n}{p}) \). Assume that \( \rho_0, b \in \mathcal{M}(\dot{B}^{-s}_p) \). Then, \( G_s \) coincides with the operator
\[ bA : \dot{B}^{s_1}_p \subset \dot{B}^{s_2}_p \rightarrow \dot{B}^{s_1}_p. \] (6.2)

(ii) Let \( p \in (1, \infty) \) and \( s \in (0, 2) \). Assume that \( \rho_0, b \in \mathcal{M}(\dot{B}^{-s}_p) \). Then, the space \( \dot{B}^{-s}_p \) coincides with \( \dot{B}^{-s}_p \), and the operator \( \overline{A} \) is given by
\[ A : \dot{B}^{s_1}_p \subset \dot{B}^{s_1}_p \rightarrow \dot{B}^{s_1}_p. \] (6.3)

**Proof.** (i) First, along the same lines of the proof of Lemma 5.3 (i), we can show that \( G_s \) is the closure of
\[ S : \{ u \in D(S) | Su \in \dot{B}^{s_1}_p \} \subset \dot{B}^{s_1}_p \rightarrow \dot{B}^{s_1}_p. \] (6.4)

Since \( \rho_0, b \in \mathcal{M}(\dot{B}^{s_1}_p) \), we can identify \( \{ u \in D(S) | Su \in \dot{B}^{s_1}_p \} = \{ u \in W^{2,p} | bAu \in \dot{B}^{s_1}_p \} \) as the inhomogeneous Besov space \( \dot{B}^{s_1}_p \). On the other hand, it is easy to see that the operator \( bA \) defined in (6.2) is closed and is an extension of the operator \( S \) defined in (6.4). The desired result then follows from the fact that \( \dot{B}^{s_1}_p \) is dense in \( \dot{B}^{s_1}_p \).
(ii) Let us first refine several results in Section 5. Using (5.9) and the fact that $R(A) = \mathcal{A}(W^{2,p})$ is dense in $\hat{B}_{p,1}^{-s}$, we can verify that $\hat{B}_{X,1}^{-s,S}$ agrees with the completion of

$$D_{-s} := \{ u \in L^p | \| \rho_0 u \|_{\hat{B}_{p,1}^{-s}} < \infty \}$$

with respect to the norm $\| \rho_0 \cdot \|_{\hat{B}_{p,1}^{-s}}$. Then, (5.2) holds for every $u \in D_{-s}$, so $\mathcal{T}_{-s}(t)$ is the continuous extension of $e^{tS}|_{D_{-s}}$ to $\hat{B}_{X,1}^{-s,S}$. From this, we can follow the same lines as the proof of Lemma 5.3 (ii) to show that $G_{-s}$ is the closure of $bA : W^{2,p} \cap \hat{B}_{X,1}^{-s,S} \subset \hat{B}_{p,1}^{-s,S}$.

Now, assuming $\rho_0, b \in \mathcal{M}(\hat{B}_{p,1}^{-s})$, it is easy to see that $\hat{B}_{X,1}^{-s,S}$ coincides with $\hat{B}_{p,1}^{-s}$. So $\rho_0G_{-s}$ is the closure of

$$A : W^{2,p} \cap \hat{B}_{p,1}^{-s} \subset \hat{B}_{p,1}^{-s} \to \hat{B}_{p,1}^{-s}.$$

But it is not difficult to see that the closure of the above operator is the one defined by (6.3). This completes the proof. □

Finally, we obtain a concrete version of maximal $L^1$ regularity for (2.3).

Proof of Theorem 2.2. The first part follows from Theorem 5.5 (i), the equivalence between $\hat{B}_{X,1}^{s,S}$ and $\hat{B}_{p,1}^{s}$, and Lemma 6.1 (i). The second part follows from Corollary 6.1 and Lemma 6.1 (ii). □

7. An application to pressureless flows

In this section, we prove Theorem 2.3. The structure of our proof is in the spirit of the one established in [7]. But the substantial progress we make is the removal of the smallness assumption on the fluctuation of the initial density.

Firstly, we shall convert (2.4) into its Lagrangian formulation. Assume temporarily that $u = u(t, x)$ is a $C^1$ vector field, namely,

$$u \in L^1_{loc}(\mathbb{R}^+; C^1_b(\mathbb{R}^n; \mathbb{R}^n)).$$

By virtue of Cauchy–Lipschitz theorem, the unique trajectory $X(t, \cdot)$ of $u$, defined by the ODE

$$\begin{align*}
\frac{d}{dt}X(t, y) &= u(t, X(t, y)), \\
X(0, y) &= y,
\end{align*}$$

is a $C^1$-diffeomorphism over $\mathbb{R}^n$ for every $t \geq 0$. Let us introduce $A(t, y) = (D_y X(t, y))^{-1}$, $J(t, y) = \det DX(t, y)$, and $\mathcal{A}(t, y) = \text{adj} DX(t, y)$ (the adjugate of $DX$, i.e., $\mathcal{A} = JA$). For any scalar function $\phi = \phi(x)$ and any vector field $v = v(x)$, it is easy to see that

$$(\nabla \phi) \circ X = A^T \nabla (\phi \circ X),$$

(7.2)
and
\[(\text{div } v) \circ X = \text{Tr}[AD(v \circ X)], \tag{7.3}\]
where \(\text{Tr}\) denotes the trace of a square matrix. On the other hand, using an integration by part argument as in the appendix of [7], we also have
\[(\text{div } v) \circ X = J^{-1} \text{div}(\mathcal{A}(v \circ X)). \tag{7.4}\]
Applying (7.2) and (7.3), we see that
\[(\nabla \text{div } v) \circ X = A^T \nabla \text{Tr}(AD(v \circ X)). \tag{7.5}\]
By writing \(\Delta = \text{div } \nabla\), we get from (7.2) and (7.4) that
\[(\Delta v) \circ X = J^{-1} \text{div}(\mathcal{A} A^T \nabla(v \circ X)). \tag{7.6}\]
Now, we introduce new unknowns in Lagrangian coordinates and always denote them by bold letters. So, we define
\[(\rho, u)(t, y) = (\rho, u)(t, X(t, y)). \tag{7.7}\]
The continuity equation in (2.4) has a unique weak solution \(\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^n)\) such that \(J\rho \equiv \rho_0\) (see, e.g., [1, Proposition 2.1]). Using (7.5), (7.6) and the chain rule, one can formally convert the system (2.4) into its Lagrangian formulation that reads
\[\begin{align*}
\rho_0 \partial_t u - \mu \text{div}(\mathcal{A} A^T \nabla u) - (\mu + \lambda) \mathcal{A} A^T \nabla \text{Tr}(A_u D u) &= 0, \\
u \big|_{t=0} &= u_0,
\end{align*} \tag{7.8}\]
where we associate \(\mathcal{A}_u\) and \(A_u\) with the new velocity \(u\), namely,
\[\mathcal{A}_u = \text{adj } DX_u, \quad A_u = (DX_u(t, y))^{-1}\]
with
\[X_u(t, y) = y + \int_0^t u(\tau, y) \, d\tau. \tag{7.9}\]
We shall prove the well-posedness of the fully nonlinear system (7.8) using the contraction mapping theorem. In order to apply the linear theory established in Theorem 2.2, we shall rewrite (7.8) as
\[\rho_0 \partial_t u - \mathcal{L} u = f(u),\]
where
\[f(u) = \mu \text{div}(\mathcal{A}_u A^T u - I) \nabla u) + (\mu + \lambda)(\mathcal{A}_u^T - I) \nabla \text{Tr}(A_u D u) + \nabla \text{Tr}((A_u - I) D u).\]

To bound the nonlinear terms, we need the following
Lemma 7.1. (see [5,7]) Let $v$ be a vector field in $C([0, \infty); \dot{B}^{n/p-1}_{p,1}) \cap L^1(\mathbb{R}_+; \dot{B}^{n/p+1}_{p,1})$ and satisfy
\[ \| \nabla v \|_{L^1(\dot{B}^{n/p}_{p,1})} \leq c_0 \] (7.10)
for some constant $c_0$. It holds that
\[ \| A_v - I \|_{L^\infty(\dot{B}^{n/p}_{p,1})} + \| \mathcal{A}_v - I \|_{L^\infty(\dot{B}^{n/p}_{p,1})} \lesssim \| \nabla v \|_{L^1(\dot{B}^{n/p}_{p,1})}. \] (7.11)

Let $v_1$ and $v_2$ be two vector fields satisfying the same conditions as $v$, and let $\delta v = v_1 - v_2$. Then, we have
\[ \| A_{v_1} - A_{v_2} \|_{L^\infty(\dot{B}^{n/p}_{p,1})} + \| \mathcal{A}_{v_1} - \mathcal{A}_{v_2} \|_{L^\infty(\dot{B}^{n/p}_{p,1})} \lesssim \| \nabla \delta v \|_{L^1(\dot{B}^{n/p}_{p,1})}. \] (7.12)

Now, in view of (7.11) and product laws in Besov spaces, we have
\[ \| f(v) \|_{L^1(\dot{B}^{n/p-1}_{p,1})} \lesssim \| \nabla v \|^{2}_{L^1(\dot{B}^{n/p}_{p,1})} \] (7.13)
whenever $v$ satisfies (7.10).

Again, in view of Theorem 2.2, we shall perform the contraction mapping theorem in the Banach space $E_p$ defined as
\[ E_p := \left\{ u \in C_b([0, \infty); \dot{B}^{n/p-1}_{p,1}) | \partial_t u \in L^1(\mathbb{R}_+; \dot{B}^{n/p-1}_{p,1}), u \in L^1(\mathbb{R}_+; \dot{B}^{n/p+1}_{p,1}) \right\} \]
endowed with the norm
\[ \| u \|_{E_p} := \| u \|_{L^\infty(\dot{B}^{n/p-1}_{p,1})} + \| \partial_t u, \mathcal{L} u \|_{L^1(\dot{B}^{n/p-1}_{p,1})}. \]

Now, we can prove the global-in-time well-posedness for (7.8).

Theorem 7.2. Assume that $n \in \{2, 3\}$, $p \in (1, 2n) \setminus \{n\}$, $u_0 \in \dot{B}^{n/p-1}_{p,1} = (\dot{B}^{n/p-1}_{p,1})^n$, and $\rho_0, b \in \mathcal{M}(\dot{B}^{n/p-1}_{p,1})$. Then, there exists a positive constant $c$ depending on $m, p, n, \mu, \nu, \| \rho_0 \|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1})}$ and $\| \rho_0^{-1} \|_{\mathcal{M}(\dot{B}^{n/p-1}_{p,1})}$ such that if $\| u_0 \|_{\dot{B}^{n/p-1}_{p,1}} \leq c$, then (7.8) has a unique global-in-time strong solution $u \in E_p$ satisfying $\| u \|_{E_p} \lesssim \| u_0 \|_{\dot{B}^{n/p-1}_{p,1}}$.

Proof. For $r > 0$, let $E_p(r)$ denote the closed ball in $E_p$ centered at $u = 0$ with radius $r$. We shall construct a contraction mapping on $E_p(r)$ by solving the linearized system
\[
\begin{cases}
\rho_0 \partial_t u - \mathcal{L} u = f(v), \\
u|_{t=0} = u_0,
\end{cases}
\] (7.14)
where the input $v \in E_p(r)$. To bound the inhomogeneous term, we require $r$ to be small so that
\[ \| \nabla v \|_{L^1(\dot{B}^{n/p}_{p,1})} \leq C\| \nabla v \|_{L^1(\dot{B}^{n/p-1}_{p,1})} \leq C_1 r \leq c_0. \]
This then implies (7.13).

Now, applying Theorem 2.2, we can solve (7.14) for a strong solution \( u \in E_p \) satisfying

\[
\|u\|_{E_p} \leq C \|u_0\|_{\dot{H}^{n/p-1}} + C \|f(v)\|_{L^1(\dot{H}^{n/p-1})} \leq C_2 \|u_0\|_{\dot{H}^{n/p-1}} + C_2 r^2.
\]

To ensure that the mapping \( v \mapsto u \) is a self-map on \( E_p(r) \), we need

\[
r \leq \frac{c_0}{C_1} \wedge \frac{1}{2C_2}
\]

and

\[
\|u_0\|_{\dot{H}^{n/p-1}} \leq \frac{r}{2C_2}.
\]

Next, we need to show the contraction property of the mapping \( v \mapsto u \). Given \( v_1, v_2 \in E_p(r) \), let \( u_1, u_2 \in E_p(r) \) be the corresponding solutions to (7.14). In what follows, for two quantities \( q_1 \) and \( q_2 \), we always denote by \( \delta q \) their difference \( q_1 - q_2 \).

Then, applying Theorem 2.2 to the system satisfied by \( \delta u \), we obtain

\[
\|\delta u\|_{E_p} \leq C \|f(v_1) - f(v_2)\|_{L^1(\dot{H}^{n/p-1})}.
\]

We write

\[
\begin{align*}
(f(v_1) - f(v_2)) &= \mu \text{ div}((\mathcal{A}_1 A_1^T - I)\nabla \delta v) + \mu \text{ div}((\mathcal{A}_2 A_2^T - \mathcal{A}_2)\nabla v_2) \\
&\quad + (\mu + \lambda)(\mathcal{A}_1^T - I)\nabla \text{Tr}(A_1 D \delta v) + (\mu + \lambda)(\mathcal{A}_2^T - I)\nabla \text{Tr}(\delta A D v_2) \\
&\quad + (\mu + \lambda)(\delta \mathcal{A})^T \nabla \text{Tr}(A_2 D v_2) + (\mu + \lambda)\nabla \text{Tr}((A_1 - I) D \delta v) \\
&\quad + (\mu + \lambda)\nabla \text{Tr}(\delta A D v_2),
\end{align*}
\]

where \( \mathcal{A}_i = \mathcal{A}_i \) and \( A_i = A_{\mathcal{A}_i}, i = 1, 2 \). Applying (7.11), (7.12) and product laws in Besov spaces, we arrive at

\[
\|f(v_1) - f(v_2)\|_{L^1(\dot{H}^{n/p-1})} \leq C \|\nabla v_1, \nabla v_2\|_{L^1(\dot{H}^{n/p-1})} \|\nabla \delta v\|_{L^1(\dot{H}^{n/p-1})}.
\]

We thus infer

\[
\|\delta u\|_{E_p} \leq C \|\nabla v_1, \nabla v_2\|_{L^1(\dot{H}^{n/p-1})} \|\nabla \delta v\|_{L^1(\dot{H}^{n/p-1})} \leq C_3 r \|\delta v\|_{E_p},
\]

from which we see that \( \|\delta u\|_{E_p} \leq \frac{1}{2} \|\delta v\|_{E_p} \) if \( r \leq \frac{1}{2C_3} \).

Finally, we choose

\[
r = \frac{c_0}{C_1} \wedge \frac{1}{2C_2} \wedge \frac{1}{2C_3} \quad \text{and} \quad c = \frac{r}{2C_2}.
\]

Then, the mapping \( v \mapsto u \) is a contraction on \( E_p(r) \) and, thus, admits a unique fixed point \( u \in E_p(r) \), which is a solution to (7.8) in \( E_p \). The proof of the uniqueness of strong solutions in \( E_p \) is similar to the proof of the contraction property of the mapping \( v \mapsto u \). This completes the proof of the theorem. \( \square \)
Remark 7.1. For $n < p < 2n$, in view of (6.1), one can prove the global well-posedness under the assumption that $\|\rho_0 u_0\|_{\dot{B}^{n/p-1}_{p,1}} \leq c$ with $c$ only depending on $m, p, n, \mu, \nu$.

Let us conclude this paper by proving Theorem 2.3.

Proof of Theorem 2.3. Let $u$ be the global-in-time solution to (7.8) constructed in Theorem 7.2. Then, (7.9) defines a $C^1$-diffeomorphism $X_{u}(t, \cdot)$ over $\mathbb{R}^n$ for every $t \geq 0$, which enables us to go back to the Eulerian coordinates by introducing

$$\rho(t, x) = \rho_0(X_{u}^{-1}(t, x)) \quad \text{and} \quad u(t, x) = u(t, X_{u}^{-1}(t, x)).$$

Then, $(\rho, u)$ is a solution to (2.4).

Let $(\rho_i, u_i), i = 1, 2,$ be two solutions to (2.4) with the same initial data. Let $X_{u_i}$ be defined via (7.1) and $(\rho_i, u_i)$ via (7.7). Then, $u_1$ and $u_2$ are two solutions to (7.8) with the same initial data. So it follows from the uniqueness part of Theorem 7.2 that $(\rho_1, u_1) = (\rho_2, u_2)$.

We refer the reader to [5,7] for more details.

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