The relation between Hardy’s non-locality and violation of Bell inequality

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(Dated: December 14, 2010)

We give a analytic quantitative relation between Hardy’s non-locality and Bell operator. We find that Hardy’s non-locality is a sufficient condition for violation of Bell inequality, the upper bound of Hardy’s non-locality allowed by information causality just correspond to Tsirelson bound of Bell inequality, and the upper bound of Hardy’s non-locality allowed by the principle of no-signaling just correspond to the algebraic maximum of Bell operator. Then we study the Cabello’s argument of Hardy’s non-locality (a generalization of Hardy’s argument) and find a similar relation between it and violation of Bell inequality. Finally, we give a simple derivation of the bound of Hardy’s non-locality under the constraint of information causality with the aid of above derived relation between Hardy’s non-locality and Bell operator.

PACS numbers: 03.65.Ud, 03.65.Ta

I. INTRODUCTION

Quantum non-locality is always a fundamental problem in physics research. Violation of Bell inequality claims that quantum non-locality cannot be reproduced with the hidden variable local model [1, 2]. In 1992 Hardy proposed his theorem (Hardy’s non-locality) which is a manifestation of quantum non-locality without using inequality [3, 4]. However the amount of quantum non-locality is limited by Tsirelson bound [5], and the whole boundary of quantum non-locality of binary-input and binary-output model has also been studied in [6, 7]. Tsirelson bound (include the boundary of quantum non-locality) doesn’t originate solely from the principle of no-signaling, one class of no-signaling theory, initiated by Popescu and Rohrlich (PR correlation) [8], allow the Bell operator take its algebraic maximum which greatly exceed Tsirelson bound. So identifying the physical principles underlying the limits of quantum non-locality is now a intriguing problem in foundational research of QM. In 2005, W. van Dam [9] has showed that the PR correlation makes communication complexity trivial (communication complexity is not trivial in QM), but which seems highly implausible in nature. Recently, Pawlowski et al. [10] introduced a new physical principle, which is named information causality, this new principle state that communication of m classical bits causes information gain of at most m bits, and they prove that this principle can distinguish physical theories from other no-signaling theories that are endowed with stronger correlations than quantum physics. A new progress has been made in [11, 12] recently, the authors have proved that the quantum non-locality originate from the combination of the principle of no-signaling and local quantum measurement assumption.

In order to investigate quantum correlation, Barret et al. [13] placed them in a general correlation theory—no-signaling theory. They found that no-signaling correlations form a polytope in probabilities space, and this no-signaling polytope contains the quantum correlations as a subset. Their work provide an mathematical framework for research of quantum non-locality. Recently, within this mathematical framework of no-signaling polytope, Ahancj et al. [14] give a bound on Hardy’s non-locality under the constraint of information causality. In the present paper we will study the relation between Hardy’s non-locality and violation of Bell inequality. We find that the Hardy’s non-locality is a sufficient condition for violation of Bell inequality, the upper bound of Hardy’s non-locality limited by information causality just correspond to Tsirelson bound of Bell inequality, and the upper bound of Hardy’s non-locality allowed by the principle of no-signaling just correspond to the algebraic maximum of Bell operator. Then we study the Cabello’s argument of Hardy’s non-locality and find a similar relation between it and violation of Bell inequality. Finally, we give a simple derivation of the bound of Hardy’s non-locality under the constraint of information causality.

II. NO-SIGNALING POLYTOPE AND HARDY’S NON-LOCALITY

In this section we give a brief introduction of no-signaling polytope and Hardy’s non-locality.

Non-signaling polytope Let us still consider the case in which Alice and Bob are each choosing from two input, each of them has two possible outputs. We denote \(X(Y) \in \{0, 1\}\) and \(a(b) \in \{0, 1\}\) as Alice’s (Bob’s) observable and outcome respectively. The joint probabilities \(p_{ab|XY}\) form a table with \(2^4\) entries, although these are not all independent due to the constraints of normalization and the principle of no-signaling. These constraints lead the entire joint probabilities to a convex subset in the form of a polytope in \(2^4\)-dimensional probabilities vector space, one call this polytope as no-signaling polytope [13]. No-signaling polytope is eight dimensional,
have 24 vertices, 16 of which are local vertices and 8 of which are nonlocal vertices (they all correspond to PR correlations).

The local vertices can be expressed as

\[ p_{ab|XY}^{\alpha \beta \gamma} = \begin{cases} 1, & \text{if } a = \alpha X \oplus \beta, \\ 0, & \text{else} \end{cases} \]

where \( \alpha, \beta, \gamma, \delta \in \{0, 1\} \) and \( \oplus \) denotes addition modulo 2.

And the eight nonlocal vertices are:

\[ p_{ab|XY}^{\alpha \beta \gamma} = \begin{cases} \frac{1}{2}, & \text{if } a \oplus b = XY \oplus \alpha X \oplus \beta Y \oplus \gamma, \\ 0, & \text{else} \end{cases} \]

where \( \alpha, \beta, \gamma \in \{0, 1\} \).

For the case of two inputs and two outputs, there are eight nontrivial facets of the local correlations and they correspond to eight CHSH inequalities as well as eight nonlocal vertices respectively. Let define \( \langle ij \rangle \) as:

\[ \langle ij \rangle = \sum_{a,b=0}^{1} (-1)^{a+b} p_{ab|X=i,Y=j}. \]

Then these eight CHSH inequalities can be express as the following inequalities:

\[ B_{\alpha, \beta, \gamma} = \langle -1 \rangle^0 \langle 00 \rangle + \langle 1 \rangle^\beta \langle 01 \rangle + \langle -1 \rangle^\alpha \langle 10 \rangle + \langle 1 \rangle^{\alpha+\beta+\gamma} \langle 11 \rangle \leq 2, \]

where \( \alpha, \beta, \gamma \in \{0, 1\} \). We can find that the algebraic maximum of Bell operator is \( B_{\alpha, \beta, \gamma} = 4 \), each choice of \( \alpha, \beta, \gamma \) corresponds to one nonlocal vertex of Eq. (2), thus there is a one-to-one correspondence between the nonlocal vertices of no-signaling polytope and the CHSH inequalities. It is easy to check that each nonlocal vertex return a value for the corresponding Bell operator of \( B_{\alpha, \beta, \gamma} = 4 \).

Hardy’s non-locality Consider Alice and Bob share two spin-1/2 particles, denote \( \{ \alpha, A' \} \) as the measurement set of Alice and \( \{ B, B' \} \) as the measurement set of Bob, all outcomes of measurement only take values of \( \pm 1 \). Now consider the following joint probabilities:

\[ P(A = +1, B = +1) = q_1 \]

\[ P(A' = -1, B = -1) = 0 \]

\[ P(A = -1, B' = -1) = 0 \]

\[ P(A' = -1, B' = -1) = q_2 \]

when \( q_1 = 0 \) and \( q_2 > 0 \) above four equations represent Hardy’s argument of Hardy’s non-locality. The general case of \( 0 \leq q_1 < q_2 \) corresponding to the Cabello’s argument which is a generalization of Hardy’s argument. It can be proved that there is no local realistic theory that can reproduce the predictions of Eqs. (5)-(8) [14]. To show this, let us consider that there are some local realistic states for which \( A' = -1 \) and \( B' = -1 \), this correspond to the validity of Eq. (8). For these local realistic states Eqs. (6) and (7) tell that the outcomes of \( A \) and \( B \) must be equal to \( +1 \), so according to local realistic theory \( P(A = +1, B = +1) \) should be at least equal to \( q_2 \) and this contradicts \( q_1 < q_2 \). However quantum entangle state can reproduce Hardy’s non-locality with suitable measurement setting, an example of the most general nonmaximally entangle state which can produce Hardy’s non-locality can be seen in [15]. So Hardy’s non-locality (theorem) manifest the contradiction without inequalities between local realism theory and quantum mechanics.

In [14], the authors have given the expressions of Hardy’s non-locality (Hardy’s argument and Cabello’s argument) in terms of vertices of no-signalling polytope. Using the correspondence as that in [14]: \( (X = 0) \leftrightarrow A, (X = 1) \leftrightarrow A', (Y = 0) \leftrightarrow B, (Y = 1) \leftrightarrow B' \) and \( a, b = 0(1) \leftrightarrow +1(-1) \), one can find that there are only five of the 16 local vertices and one of the 8 nonlocal vertices satisfy Hardy’s argument Eqs. (5)-(8) \((q_1 = 0)\), these vertices are \( p_{ab|XY}^{0000100010001000, 1110000000000000} \) and \( p_{ab|XY}^{00101010, 111101110111} \). So Hardy’s argument of Eq. (5)-(8) can be written as a linear superposition of the above 6 vertices [14]:

\[ p_{ab|XY}^H = c_1 p_{ab|XY}^{0000100010001000} + c_2 p_{ab|XY}^{00101010, 111101110111} + c_3 p_{ab|XY}^{1010, 1100000000000000} \]

where \( \sum_{i=1}^{6} c_i = 1 \).

It’s easy to check from Eq. (9) that the success probability \( q_2 \) for Hardy’s argument is given by \( q_2 = p_{1111}^H = \frac{c_6}{2} \). We can find that under the no-signaling constraint, the maximum of \( q_2 \) is 1/2 which is achieved when \( c_6 = 1 \) and else \( c_i \)'s = 0. This result has also been derived in [16, 17]. Recently Ahanj et al. [14] gave \( q_{2max} = \frac{\sqrt{2}-1}{2} \) under the constraint of information causality [10].

Cabello’s argument can also be expressed as a linear superposition of the vertices which satisfy Eqs. (5)-(8) in the case of \( 0 \leq q_1 < q_2 \) [14]:

\[ p_{ab|XY}^C = p_{ab|XY}^H + c_7 p_{ab|XY}^{1000100000000000} + c_8 p_{ab|XY}^{1000000000000000} + c_9 p_{ab|XY}^{1000000000000000} \]

where the Hardy’s argument \( p_{ab|XY}^H \) is given in Eq. (9), and the coefficients \( c_i \)'s satisfy the new normalization condition \( \sum_{i=1}^{11} c_i = 1 \). It is easy to calculate the success probability for Cabello’s argument by using Eq. (10)

\[ w = q_2 - q_1 = p_{1111}^H - p_{000000}^C \]

\[ = \frac{c_6 - c_{11}}{2} - c_7 - c_8 - c_9 \]

Recently Ahanj et al. [14] gave \( w_{max} = \frac{\sqrt{2}-1}{2} \) under the constraint of information causality [10].
III. HARDY’S NON-LOCALITY AND BELL INEQUALITY

Now we study the relation between the Hardy’s non-locality and violation of Bell inequality. We first discuss the relation between \(q_2\) and Bell operator in the case of \(q_1 = 0\) (Hardy’s argument). We notice that the expression of Hardy’s argument (Eq. (9)) has only one non-local vertex \(p_{001}^{ab|XY}\), so it’s nature to adopt the Bell inequality corresponding to this non-local vertex:

\[
B_{0,0,1} = -(00) - (01) - (10) + (11) \leq 2. \tag{12}
\]

By using Eq. (3) and Eq. (9), we can get

\[
\begin{align*}
(00) &= -c_1 - c_2 - c_3 - c_4 + c_5 - c_6 + c_7 + c_8 + c_9 + c_{10} + c_{11} \\
(01) &= -c_1 + c_2 - c_3 - c_4 - c_5 + c_6 + c_7 + c_8 - c_9 - c_{10} + c_{11} \\
(10) &= -c_1 - c_2 + c_3 + c_4 - c_5 - c_6 + c_7 + c_8 - c_9 - c_{10} - c_{11} \\
(11) &= -c_1 + c_2 - c_3 + c_4 + c_5 + c_6 + c_7 - c_8 + c_9 + c_{10} - c_{11}
\end{align*}
\]  

Then we obtain the value of Bell operator \(B_{0,0,1}\)

\[
B_{0,0,1} = 2c_1 + 2c_2 + 2c_3 + 2c_4 + 2c_5 + 4c_6 = 2c_6 + 2 = 4q_2 + 2 \tag{14}
\]

In above calculation we used the normalization condition of \(\sum_{i=1}^{6} c_i = 1\) and the relation of \(q_2 = \frac{c_6}{2}\). In 2000, Cereceda derived this result in [17].

From Eq. (14) we find that, if the success probability \(q_2 > 0\) the violation of Bell inequality can be achieved, and it can be also said that the Hardy’s non-locality is a sufficient condition for violation of Bell inequality. We also find that when \(q_2 = \frac{\sqrt{2}}{2}\) the Bell operator \(B_{0,0,1}\) reach Tsirelson bound \(2\sqrt{2}\) and this value \(\left(q_2 = \frac{\sqrt{2}}{2}\right)\) just equal to the upper bound of Hardy’s non-locality under the constraint of information causality [14]. But it must be noticed that in quantum mechanics the maximum probability of success of the Hardy’s non-locality is \(q_2 = 0.09\) [18], the reason is that there are also non-quantum correlations which under Tsirelson bound \(2\sqrt{2}\). Under the no-signaling constraint the maximum of \(q_2\) is \(1/2\), substitute it in Eq. (14) we just get the algebraic maximum of Bell operator: \(B_{0,0,1} = 4\).

Then we discuss the relation between \(w\) and Bell operator in the case of \(0 < q_1 < q_2\) (Cabello’s argument).

By using Eq. (3) and Eq. (10), we can get

\[
\begin{align*}
(00) &= -c_1 - c_2 - c_3 - c_4 + c_5 - c_6 + c_7 + c_8 + c_9 + c_{10} + c_{11} \\
(01) &= -c_1 + c_2 - c_3 - c_4 - c_5 + c_6 + c_7 + c_8 - c_9 - c_{10} + c_{11} \\
(10) &= -c_1 - c_2 + c_3 + c_4 - c_5 - c_6 + c_7 + c_8 - c_9 - c_{10} - c_{11} \\
(11) &= -c_1 + c_2 - c_3 + c_4 + c_5 + c_6 + c_7 - c_8 + c_9 + c_{10} - c_{11}
\end{align*}
\]  

Here we still adopt the Bell inequality of Eq. (12) which corresponding to non-local vertex \(p_{001}^{ab|XY}\), we obtain the value of Bell operator \(B_{0,0,1}\) for Cabello’s argument of Eq. (10)

\[
B_{0,0,1} = -(00) - (01) - (10) + (11) = 2 + 2c_6 - 2c_{11} - 4(c_7 + c_8 + c_9) = 2 + 4w \tag{16}
\]

In above calculation we used the normalization condition of \(\sum_{i=1}^{11} c_i = 1\) and the relation of Eq. (11).

From Eq. (16) we find a similar relation between \(w\) and violation of Bell inequality as that of \(q_2\’s\). If the success probability for Cabello’s argument \(w = q_2 - q_1 > 0\) the violation of Bell inequality can be achieved. One also can find that when \(w = \frac{\sqrt{2}}{2} - 1\) the Bell operator \(B_{0,0,1}\) reach Tsirelson bound \(2\sqrt{2}\) and this value \((w = \frac{\sqrt{2}}{2} - 1)\) just equal to the upper bound of Cabello’s non-locality under the constraint of information causality [14]. Under the no-signaling constraint, the maximum of \(w\) is \(1/2\) when \(c_6 = 1\) and rest of the \(c_i\’s= 0\), substitute it in Eq. (16) we just get the algebraic maximum of Bell operator of \(B_{0,0,1} = 4\).

IV. THE BOUND OF HARDY’S NON-LOCALITY IN THE LIMITS OF INFORMATION CAUSALITY

In this section, we give a simple derivation of the bound of Hardy’s non-locality under the constraint of information causality with the aid of above derived relation between Hardy’s non-locality and Bell operator.

The general Bell inequality of Eq. (4) can also be written as

\[
B'_{\alpha,\beta,\gamma} = \frac{1}{4} \sum_{X,Y=0}^{1} p(a = b \oplus XY \oplus \alpha X \oplus \beta Y \oplus \gamma | XY) \leq \frac{3}{4}. \tag{17}
\]

where \(\alpha, \beta, \gamma \in \{0, 1\}\). The choice of \(\alpha = 0, \beta = 0, \gamma = 0\) correspond to the standard CHSH inequality [2] and be widely used [11, 12]. The relation between \(B'_{\alpha,\beta,\gamma}\) and \(B_{\alpha,\beta,\gamma}\) is

\[
B'_{\alpha,\beta,\gamma} = \frac{B_{\alpha,\beta,\gamma} + 4}{8} \quad (\text{Hardy}’s \ \text{argument})
\]

\[
= \frac{4q_2 + 6}{8} \quad (\text{Cabello}’s \ \text{argument}). \tag{18}
\]

Tsirelson bound of \(B'_{\alpha,\beta,\gamma}\) is \(\frac{\sqrt{2}+2}{4}\) when \(q_2 = \frac{\sqrt{2}}{2} - 1\) (or \(w = \frac{\sqrt{2}}{2} - 1\)).

The general expression of information causality can written as

\[
A \equiv (2P_1 - 1)^2 + (2P_2 - 1)^2 \leq 1 \tag{19}
\]
where

\[ P_1 = \frac{1}{2} [p(a = b \oplus XY \oplus \alpha X \oplus \beta Y \oplus \gamma |00) + p(a = b \oplus XY \oplus \alpha X \oplus \beta Y \oplus \gamma |10)] \]

\[ P_2 = \frac{1}{2} [p(a = b \oplus XY \oplus \alpha X \oplus \beta Y \oplus \gamma |01) + p(a = b \oplus XY \oplus \alpha X \oplus \beta Y \oplus \gamma |11)]. \]

(20)

the expression of this principle in [10] correspond to the choice of \( \alpha = 0, \beta = 0, \gamma = 0 \).

The \( B_{\alpha,\beta,\gamma}' \) can be written as \( \frac{P_1+P_2}{2} \) therefore which can expressed as a function of \( A \):

\[ B_{\alpha,\beta,\gamma}' = \left( \sqrt{A} \sin \theta + \sqrt{A} \cos \theta + 2 \right) \]

\[ \leq \frac{\sqrt{2A} + 2}{4}. \]

(21)

We can find under the constraint of information causality \( A \leq 1 \) the upper bound of \( B_{\alpha,\beta,\gamma}' \) is \( \frac{\sqrt{2}+2}{2} \), therefore from the Eq. (18) we find the upper bound of \( q_2 \) and \( w \) both are \( \frac{\sqrt{2}-1}{2} \) under the constraint of information causality. This upper bound of \( q_2 \) and \( w \) can be achieved in the limits of information causality. For example we can take \( P_1 = P_2 = \frac{\sqrt{2}+2}{4} \), substitute them in Eq. (19) we get \( A = 1 \), so information causality has been observed; at the same time we take them in Eq. (17) and get \( B_{\alpha,\beta,\gamma}' = \frac{1}{2}(P_1 + P_2) = \frac{\sqrt{2}+2}{4} \), so from Eq. (18) we can find the upper bound of \( q_2 \) and \( w \) (\( \frac{\sqrt{2}-1}{2} \)) has been reached. Now we complete the proof of that the bound of Hardy’s non-locality under the constraint of information causality is \( \sqrt{2}-1 \), which is same as the result in [14].

V. CONCLUSION

In this work, we give the quantitative relations between Hardy’s/Cabello’s argument of non-locality and violation of Bell inequality. We obtain the analytic expressions of the relations between the success probabilities of the Hardy’s/Cabello’s argument and the value of Bell operator, and then we find that if and only if the success probabilities of the Hardy’s/Cabello’s argument are greater than zero the violations of Bell inequality can be achieved. The bound values of the success probabilities of the Hardy’s/Cabello’s argument under the constraint of information causality both correspond to Tsirelson bound of Bell operator, and the bound values of these two success probabilities under the no-signaling constraint both correspond to the algebraic maximum of Bell operator. Finally, we give a simple derivation of the bound of Hardy’s non-locality under the constraint of information causality with the aid of above derived relation between Hardy’s non-locality and Bell operator, this bound is the main result of reference [14].

Acknowledgments

The author thank Wei Ren and Shi-Jie Xiong for their help and encouragement. This work was supported by National Foundation of Natural Science in China Grant Nos. 10947142 and 11005031.

[1] J. S. Bell, Physics (Long Island City, N.Y.) 1, 195(1964); J. S. Bell, Speakable and Unspeakable in Quantum Mechanics (Cambridge University Press, Cambridge, England, 1988).
[2] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23, 880(1969); 24, 549(E)(1970).
[3] L. Hardy, Phys. Rev. Lett. 68, 2981(1992).
[4] L. Hardy, Phys. Rev. Lett. 71, 1665(1993).
[5] B. S. Cirel’son, Lett. Math. Phys. 4, 93(1980).
[6] M. Navascués, S. Pironio, and A. Acin, Phys. Rev. Lett. 98, 010401(2007).
[7] M. Navascués, S. Pironio, and A. Acin, New Journal of Physics, 10, 073013(2008).
[8] S. Popescu and D. Rohrlich, Found. Phys. 24, 379(1994).
[9] W. van Dam, e-print arXiv: quant-ph/0501159(2005).
[10] M. Pawłowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Żukowski, Nature 461, 1101(2009).
[11] H. Barnum, S. Beigi, S. Boixo, M. B. Elliott, and S. Wehner, Phys. Rev. Lett. 104, 140401(2010).
[12] A. Acín, R. Augusiak, D. Cavalcanti, C. Hadley, J. K. Korbicz, M. Lewenstein, L. Masanes, and M. Piani, Phys. Rev. Lett. 104, 140404(2010).
[13] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, and D. Roberts, Phys. Rev. A 71, 022101(2005).
[14] A. Ahanj, S. Kunkri, A. Rai, R. Rahaman, and P. S. Joag, Phys. Rev. A 81, 032103(2010).
[15] G. Garbarino, Phys. Rev. A 81, 032106(2010).
[16] S. K. Choudhary, S. Ghosh, G. Kar, S. Kunkri, R. Rahaman, A. Roy, e-print arXiv: quant-ph/0807.4414(2008).
[17] J. L. Cereceda, Found. Phys. Lett. 13, 427(2000).
[18] S. Kunkri, S. K. Choudhary, A. Ahanj, and P. Joag, Phys. Rev. A 73, 022346(2006).
[19] A. Cabello, Phys. Rev. A 65, 032108(2002).