A GEOMETRIC APPROACH TO THE CASCADE APPROXIMATION OPERATOR FOR WAVELETS

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This paper is devoted to an approximation problem for operators in Hilbert space, that appears when one tries to study geometrically the cascade algorithm in wavelet theory. Let $\mathcal{H}$ be a Hilbert space, and let $\pi$ be a representation of $L^\infty(\mathbb{T})$ on $\mathcal{H}$. Let $R$ be a positive operator in $L^\infty(\mathbb{T})$ such that $R(1) = 1$, where $1$ denotes the constant function 1. We study operators $M$ on $\mathcal{H}$ (bounded, but non-contractive) such that

$$\pi(f) M = M \pi(f(z^2)) \quad \text{and} \quad M^* \pi(f) M = \pi(R^* f), \quad f \in L^\infty(\mathbb{T}),$$

where the $*$ refers to Hilbert space adjoint. We give a complete orthogonal expansion of $\mathcal{H}$ which reduces $\pi$ such that $M$ acts as a shift on one part, and the residual part is $\mathcal{H}^{(\infty)} = \bigcap_n [M^n \mathcal{H}]$, where $[M^n \mathcal{H}]$ is the closure of the range of $M^n$. The shift part is present, we show, if and only if $\ker (M^*) \neq \{0\}$. We apply the operator-theoretic results to the refinement operator (or cascade algorithm) from wavelet theory. Using the representation $\pi$, we show that, for this wavelet operator $M$, the components in the decomposition are unitarily, and canonically, equivalent to spaces $L^2(E_n) \subset L^2(\mathbb{R})$, where $E_n \subset \mathbb{R}$, $n = 0, 1, 2, \ldots, \infty$, are measurable subsets which form a tiling of $\mathbb{R}$; i.e., the union is $\mathbb{R}$ up to zero measure, and pairwise intersections of different $E_n$'s have measure zero. We prove two results on the convergence of the cascade algorithm, and identify singular vectors for the starting point of the algorithm.

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**Terminology used in the paper:**
- $\mathbb{T}$: the one-torus
- $\mu$: Haar measure on the torus $\mathbb{T}$
- $Z$: the Zak transform
- $\tilde{X} = ZXZ^{-1}$: transformation of operators
- $\mathcal{H}$: a given Hilbert space
- $\pi$: a representation of $L^\infty(\mathbb{T})$ on $\mathcal{H}$
- $R$: the Ruelle operator on $L^\infty(\mathbb{T})$
- $M$: an operator on $\mathcal{H}$
- $R^*, M^*$: adjoint operators

1. **INTRODUCTION**

In wavelet theory, one is given a *subband filter*, i.e., a function $m_0$ on the unit circle, satisfying (i)–(iii) from below and one wants to construct a scaling function $\varphi$ (relative to $m_0$), i.e., a nonzero function $\varphi$ on $\mathbb{R}$ satisfying the scaling relation $\varphi = M\varphi$ (relation (1.1) in the paper). Here $M = M_{m_0}$ is the so-called cascade operator defined by

$$(Mh)(x) = \sqrt{2} \sum_{n \in \mathbb{Z}} a_n h(2x - n),$$

where $a_n$ are Fourier coefficients of the function $m_0$. The scaling function $\varphi$ is important because its shifts generate (under some analytic conditions) the so-called *multiresolution subspace* $V = V(\varphi)$, which is used to construct the wavelets.

There are several ways of constructing a scaling function $\varphi$. The cascade algorithm is one of the possibilities. In this algorithm one picks some simple function $h$ and considers its iterations $M^nh$. Clearly, if the iterations $M^nh$ converge to a nonzero function $\varphi$, the function $\varphi$ satisfies $\varphi = M\varphi$, so the algorithm gives us a scaling function. We study the problem of convergence of this algorithm in the setting of an abstract Hilbert space. The cascade operator $M$ has some very special structure. The Ruelle operator $R$, which appears naturally in this type of problem, gives us a way to describe this structure. Namely, the operator $M$ is what we shall call a sub-isometry; see Definition 6.1. In fact the concept depends on a pair $(R, \pi)$ where $R$ is a Ruelle operator and $\pi$ is a representation. It turns out that sub-isometries admit an analogue of the Kolmogorov–Wold decomposition for usual isometries. Using this decomposition, we obtain results about convergence of the cascade algorithm in an abstract Hilbert space setting, and then, in the last section, give some applications for wavelet theory.

This paper is technically concerned with approximation problems for operators in Hilbert space, but our initial motivation is the *refinement operator* (alias the *cascade approximation operator*) from wavelet theory. Our starting point is a given function $m_0 \in L^\infty(\mathbb{T})$, which is assumed to satisfy the following three axioms:
(i) $m_0$ is continuous in an open neighborhood of $z = 1$ in $\mathbb{T}$,
(ii) $m_0 = \sqrt{2}$ at $z = 1$,
(iii) $|m_0(z)|^2 + |m_0(-z)|^2 = 2$.

In applications (see [Dau92]), $m_0$ is called a subband filter, and (ii) is called the low pass filter property. That is because of the substitution $z = e^{-i\omega}$, $\omega$ frequency; and (ii) implies that all signals pass at $\omega = 0$, while (i) and (iii) imply that “almost all” pass for small $\omega$, i.e., low frequencies. Finally, axiom (iii) accounts for the name quadrature filter, or “quadrature mirror filter”. They are used in wavelet theory for generating multiresolutions.

Suppose $m_0(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ is the Fourier expansion of $m_0$. A function $\varphi \in L^2(\mathbb{R})$ is called a scaling function (relative to $m_0$) if
\begin{equation}
\varphi(x) = \sqrt{2} \sum_{n \in \mathbb{Z}} a_n \varphi(2x - n),
\end{equation}
and we refer to the operator
\begin{equation}
(Mh)(x) = \sqrt{2} \sum_{n \in \mathbb{Z}} a_n h(2x - n)
\end{equation}
as the cascade operator. The approximation problem is that of finding conditions on the three functions $m_0$, $\varphi$, and $h$ ($h \in L^2(\mathbb{R})$), such that
\begin{equation}
\lim_{n \to \infty} \|\varphi - M^n h\|_{L^2(\mathbb{R})} = 0.
\end{equation}

This is called the cascade approximation, or the cascade problem. Let $R$ be the Ruelle operator (Section 3) corresponding to $m_0$. The basic connection between the two operators $R$ and $M$ will be studied in a geometric Hilbert-space setting (Section 4), where we give a quadratic form $h \mapsto p_2(h)$ on $L^2(\mathbb{R})$ with values in $L^1(\mathbb{T})$. We then introduce the concept of sub-isometries. (The cascade operators $M$ are examples.) We show that $R(p_2(h)) = p_2(Mh)$, so if $M\varphi = \varphi$, then $p_2(\varphi)$ is an eigenfunction for $R$. In addition to $L^2$-convergence questions, there are various classes of pointwise convergence issues. It turns out that, when $m_0$ is given, satisfying (ii)–(iii), and $\varphi$, $h$ satisfy natural criteria, then there is an unexpected obstruction to the approximation (1.3). If $\ker(M^*) \neq \{0\}$ in $L^2(\mathbb{R})$, then the cascade approximation (1.3) is exceedingly “bad”. We show (Proposition 8.2) that $\ker(M^*) = \{0\}$ if and only if $m_0$ does not vanish on a set of positive measure $\subset \mathbb{T}$. So if, for example, $m_0$ is a polynomial, then $\ker(M^*) = \{0\}$.

2. SOME MAIN RESULTS IN THE PAPER

When $\ker(M^*)$ is not an obstruction, we show the following result. We say that $f \in L^2(\mathbb{R})$ is orthogonal if the translates $\{f(\cdot-n) \mid n \in \mathbb{Z}\}$ form an orthonormal family in $L^2(\mathbb{R})$. Let $\varphi$ be a scaling function relative to some $m_0$ satisfying (ii)–(iii), and let $h \in L^2(\mathbb{R})$. Suppose both $\varphi$ and $h$ are orthogonal in this sense. Then we show that the following two conditions are equivalent:

- \begin{enumerate}
  \item $m_0$ is continuous in an open neighborhood of $z = 1$ in $\mathbb{T}$,
  \item $m_0 = \sqrt{2}$ at $z = 1$,
  \item $|m_0(z)|^2 + |m_0(-z)|^2 = 2$.
\end{enumerate}
(a) \( \| \varphi - M^n h \|_2 \rightarrow 0 \), and
(b) the function

\[
p(t) = \sum_{n \in \mathbb{Z}} e^{int} \int_{\mathbb{R}} \varphi(x + n) h(x) \, dx
\]

is continuous near \( t = 0 \), and \( p(0) = 1 \).

The result fails when \( \ker (M^*) \neq \{0\} \); see Section 8.

In studying this, and other related approximation problems in earlier papers [BrJo97, BrJo98], the following general and geometric Hilbert-space framework proved useful. Let \( m_0 \) be given subject to (i)–(iii). (For much of it, only (iii) is needed.) We introduce the Ruelle operator

\[
(R \xi) (z) = \frac{1}{2} \sum_{w^2 = z} |m_0(w)|^2 \xi(w), \quad \xi \in L^\infty(\mathbb{T})
\]

and its dual,

\[
(R^* \xi) (z) = |m_0(z)|^2 \xi(z^2).
\]

Let \( \mathcal{H} \) be an abstract Hilbert space, and let \( \pi \) be a representation of \( L^\infty(\mathbb{T}) \) on \( \mathcal{H} \). An operator \( M \) in \( \mathcal{H} \) is said to be an \((R, \pi)\)-isometry if

\[
M^* \pi(\xi) M = \pi(R^* (\xi)) \quad \text{for all } \xi \in L^\infty(\mathbb{T}).
\]

We also call \( M \) a sub-isometry when \((R, \pi)\) is understood. The cascade operators are special cases of \((R, \pi)\)-isometries (see Section 3), and several of our results for the cascade operator will be derived from a general result about \((R, \pi)\)-isometries; see Section 6. We now summarize briefly our main result for \((R, \pi)\)-isometries. Let \( \mathcal{H} \) be an \((R, \pi)\)-isometry on some Hilbert space \( \mathcal{H} \), and let \( \mathcal{L} := \ker (M^*) \), and \( \mathcal{H}^{(\infty)} := \bigcap_{n=1}^{\infty} [M^n \mathcal{H}] \), where \([\cdot]\) refers to norm-closure in \( \mathcal{H} \). We then establish (Theorem 6.2) the following general orthogonal decomposition,

\[
\mathcal{H} = \bigoplus_{n=0}^{\infty} [M^n \mathcal{L}] \oplus \mathcal{H}^{(\infty)},
\]

and, moreover, we show that each of the (mutually orthogonal) component spaces \([M^n \mathcal{L}]\) and \( \mathcal{H}^{(\infty)} \) is invariant under \( \pi(\xi) \) for all \( \xi \in L^\infty(\mathbb{T}) \). In other words, the spaces in the decomposition reduce the representation \( \pi \); if the restricted representations are denoted \( \pi_n \), we have \( \pi = \pi_0 \oplus \pi_1 \oplus \cdots \oplus \pi_\infty \). (See [Dix96] for the theory of decompositions of representations.) The significance of having the terms \( \mathcal{H}^{(n)} \) in the sum (2.3) invariant under the representation \( \pi \) is that the structure of the subspaces \( \mathcal{H}^{(n)} \) themselves may then be determined from the representation. We show that generically these sequences \( \pi_n = \pi|_{\mathcal{H}^{(n)}} \) is
determined from \( \pi_0 \), and the latter can be computed from the spectral theorem. We also show that every representation of \( L^\infty(T) \) occurs as \( \pi_0 \) in some decomposition (2.3) corresponding to an \((R, \pi)\)-isometry.

While the operator \( M \) plays a central role in the wavelet literature (see, e.g., CoDa96, CoRy95, Dau92, the approach (2.2) (2.3) here is new. It is motivated by the need for a representation-theoretic approach to the classification of wavelets BrJo97, and also by the need for including in the analysis other Hilbert spaces \( \mathcal{H} \) than just \( L^2(\mathbb{R}) \); see Jor98. Even if \( L^2(\mathbb{R}) \) is the final goal, there is a need for understanding the limiting cases when some different Hilbert space \( \mathcal{H} \) (other than \( L^2(\mathbb{R}) \)) is dictated by some more general or different framework and analysis; we refer to Jor98 for more details on this viewpoint.

In Jor98, a framework is adopted which is much more general than the present setting of quadrature mirror filters. (It includes, for example, the orthogonal harmonic analysis of \( \mathcal{H} = L^2(\mu) \) from JoPe98 where \( \mu \) is taken to be a self-affine and singular probability measure on \( \mathbb{R}^d \) of compact support which arises by iteration of a given finite set of affine mappings in \( \mathbb{R}^d \).) Our present approach to multiresolution analysis is inspired by the Lax–Phillips scattering theory LaPh89 for the classical wave equation. This is the continuous case; our aim here is to “discretize” the Lax–Phillips scattering theory, and (2.2) should be viewed in that light. The recent papers by Micchelli et al. MiPr88, Mic96, CDM91 take a somewhat different (but closely related) approach to the discretization problem: Since the functions in a multiresolution subspace \( V(\varphi) \) may be represented by sequences \((\xi_n)_{n \in \mathbb{Z}}\), via \( \sum_n \xi_n \varphi(\cdot - n) \), the operator \( M \) may therefore be studied alternatively as acting on one of the sequence spaces \( \ell^p(\mathbb{Z}) \). In this guise, it takes the form \( M \rightarrow S = \tilde{M} \), where

\[
(S\xi)_n = \sum_{k \in \mathbb{Z}} a_{n-2k} \xi_k.
\]

The motivation for the present paper has several sources. In BEJ97, we showed that certain wavelets, for which the corresponding quadrature mirror filter \( m_0 \) is a polynomial, may be classified by the labels of a corresponding family of irreducible representations of the Cuntz algebra \( \mathcal{O}_2 \). Previously there were known no such clear-cut invariants that classified wavelet families. But the particular wavelets from BEJ97 had in fact been identified earlier (without invariants or classification) in Wel93 and Wic93. Regularity questions for the corresponding scaling functions had been studied in DDL95, GMW94, and MRV96. Our references for quadrature mirror filters are StNg96 and Mal98. Excellent references to wavelets from the operator-theoretic viewpoint are HeWe96 and Hor97. The refinement operators of (2.4) are also called slant Toeplitz operators, and their spectral theory was studied previously in Ho96. A standard reference to the ergodic theory which we use in the present paper is Val82.
3. BACKGROUND, SUMMARY, AND OPERATOR RELATIONS

Let \( T \) denote the one-dimensional torus and let \( \mu \) be its (usual) Haar measure. We will use the form \( T \cong \mathbb{R}/2\pi\mathbb{Z} \) such that functions on \( T \) are identified with \( 2\pi \)-periodic functions on \( \mathbb{R} \). For technical reasons, the identification will be made via \( z = e^{-i\omega}, \omega \in \mathbb{R} \), and functions on \( T \) will be written, alternately, as \( f(z) \) or \( f(\omega) \). Let \( m_0 \in L^\infty(T) \) be given, and suppose \( m_0 = \sqrt{2} \) at \( z = 1 \), and further that

\[
|m_0(z)|^2 + |m_0(-z)|^2 = 2. \tag{3.1}
\]

One approach to wavelets (see [Dau92]) is to first make precise the (formal) infinite product (limit as \( n \to \infty \)):

\[
\prod_{k=1}^{n} \frac{1}{\sqrt{2}} m_0\left(\frac{\omega}{2^k}\right) \chi_{[-2^n\pi,2^n\pi]}(\omega). \tag{3.2}
\]

Suppose this product \((3.2)\), in the limit, represents a function \( F \in L^2(\mathbb{R}) \). Then

\[
\sqrt{2} F(2\omega) = m_0(\omega) F(\omega), \quad \omega \in \mathbb{R}, \tag{3.3}
\]

and we wish to recover the scaling function \( \varphi \) as the inverse Fourier transform of \( F \). The approach is referred to as the Mallat algorithm [Mal89], but it is somewhat indirect. Let

\[
\varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} F(\omega) \, d\omega \tag{3.4}
\]

be the inverse Fourier transform, \( \varphi = F^\vee \), and let

\[
m_0(z) = \sum_{n \in \mathbb{Z}} a_n z^n \tag{3.5}
\]

be the Fourier series expansion of \( m_0 \). (The Fourier basis in \((3.5)\) is \( e_n(z) = z^n \).) Then (again formally),

\[
\varphi(x) = \sqrt{2} \sum_{n \in \mathbb{Z}} a_n \varphi(2x - n). \tag{3.6}
\]

The function \( \varphi \) is called the scaling function, and the closed linear span of the translates

\[
\{ \varphi(\cdot - n) \mid n \in \mathbb{Z} \} \tag{3.7}
\]

generates, under certain analytic conditions [Dau92], a multiresolution subspace \( V = V(\varphi) \) in \( L^2(\mathbb{R}) \). The identity \((3.6)\) allows, for each \( m_0 \), an operator \( M = M_{m_0} \) in \( L^2(\mathbb{R}) \) such that scaling functions \( \varphi \) arise as solutions to a fixed point problem. If we introduce the operator \( M \) by

\[
(Mh)(x) = \sqrt{2} \sum_{n \in \mathbb{Z}} a_n h(2x - n), \quad h \in L^2(\mathbb{R}), \tag{3.8}
\]

then
\[ \varphi = M \varphi, \]

and we should look for choices of \( h \) such that

\[ \lim_{n \to \infty} M^n h = \varphi. \]

We will be interested here primarily in this as an \( L^2 (\mathbb{R}) \)-limit, but other limits (e.g., point-wise) will be considered as well. The issue is both how preassigned properties of the starting function \( h \), and the scaling function \( \varphi \), affect the approximation properties of (3.10). We refer to this as the cascade approximation, and the traditional choice for \( h \) is \( h = \chi_{[0,1]} \), which accidentally is the scaling function for the Haar wavelet. The cascade algorithm is more direct than the Mallat one, as it doesn’t pass via the Fourier transform.

The right-hand side in (3.8) involves a dyadic scaling, and an action by \( L^\infty (\mathbb{T}) \) on \( L^2 (\mathbb{R}) \). We now introduce a representation \( \alpha \mapsto \pi (\alpha) \) of \( L^\infty (\mathbb{T}) \) in the algebra of operators on \( L^2 (\mathbb{R}) \) by

\[ (\pi (\alpha) h) (x) = \sum_{n \in \mathbb{Z}} a_n h (x - n). \]

Then \( M \) can be represented as \( M = D \pi (m_0) \), where \( D \) is the operator of dyadic scaling, \( Dh (x) = \sqrt{2} h (2x) \). Application of the Fourier transform \( \hat{\cdot} \) to both sides in (3.11) then yields

\[ \hat{h} \mapsto \alpha (\omega) \hat{h} (\omega) \]

via the identification

\[ \alpha (\omega) \simeq \alpha (z), \quad z = e^{-i\omega}, \omega \in \mathbb{R}. \]

(In applications, \( \omega \) is a frequency variable.) In either of its forms, (3.11) or (3.12), this representation of \( L^\infty (\mathbb{T}) \) will be denoted \( \pi (\alpha) h \) with \( \pi (\alpha) \) an operator acting on \( L^2 (\mathbb{R}) \), and \( \pi (L^\infty (\mathbb{T})) \) the corresponding algebra of operators.

If \( m_0 \) is given, then \( L^\infty (\mathbb{T}) \) carries two operators \( R \) and \( R^* \), \( R \) defined by

\[ (R \alpha) (z) = \frac{1}{2} \sum_{w \in \mathbb{T}} |m_0 (w)|^2 \alpha (w) \]

and the adjoint operator \( R^* \) by

\[ (R^* \beta) (z) = |m_0 (z)|^2 \beta (z^2),$ \quad \alpha, \beta \in L^\infty (\mathbb{T}). \]

Both of them will also be viewed as \( L^2 (\mathbb{T}) \)-operators, and we have, by a simple calculation,

\[ \int_{\mathbb{T}} (R \alpha) (z) \beta (z) \ d\mu (z) = \int_{\mathbb{T}} \alpha (z) R^* \beta (z) \ d\mu (z), \]

where \( d\mu \) is the usual Haar measure on \( \mathbb{T} \), i.e., \( \frac{1}{2\pi} \int_0^{2\pi} \cdots d\omega \), thus justifying the notation \( R^* \).
The operator $R$ is called the Ruelle operator, or the transfer operator, for reasons we shall go into later; see also [CoRy95].

We further have the usual pairing between $M$ in (3.8), and its $L^2(\mathbb{R})$-adjoint $M^*$, given by

$$\int_{\mathbb{R}} Mh_1(x)h_2(x)\,dx = \int_{\mathbb{R}} h_1(x)M^*h_2(x)\,dx,$$

or equivalently,

$$\langle Mh_1 | h_2 \rangle = \langle h_1 | M^*h_2 \rangle,$$

(3.16)

where $\langle \cdot | \cdot \rangle$ denotes the standard inner product of $L^2(\mathbb{R})$.

Our first result is

**THEOREM 3.1.** Let $m_0 \in L^\infty(\mathbb{T})$ be given, and suppose it satisfies (3.1). Let $M$ be the corresponding cascade operator, and $R$ the Ruelle operator. The respective adjoints are $M^*$ and $R^*$. Finally, let $\pi$ be the representation of $L^\infty(\mathbb{T})$ on $L^2(\mathbb{R})$ given in (3.12). Then we have the following two commutation relations:

(a) $M^*\pi(\alpha)M = \pi(R^*\alpha)$ and
(b) $M\pi(\alpha)M^* = \pi(R\alpha) + \pi(e_1\alpha)T_{1/2}$ for all $\alpha \in L^\infty(\mathbb{T})$, where $e_1(z) = z$ and

$$\left(T_{1/2}h\right)(x) = h(x + 1/2), \quad h \in L^2(\mathbb{R}).$$

The proof will be given in Section 5 below. In this paper, we will study the cascade approximation, and the scaling function $\varphi$, via the abstract algebraic system which is given by the relations (a)-(b) of the theorem. These two operator commutation relations will first be studied abstractly (Section 5) and independently of their origin, and then the results will be specialized, and applied, to the wavelet problems mentioned above. Our proofs will depend on some lemmas of a general nature regarding the Zak transform.

4. THE ZAK TRANSFORM

The Zak transform $Z$ is known [Dau92, p. 109] to be the isometric isomorphism between $L^2(\mathbb{R})$ and $L^2(\mathbb{Z} \times [0,1])$ which is given formally by

$$\left(Zh\right)(z,x) = \sum_{n \in \mathbb{Z}} z^n h(x+n), \quad h \in L^2(\mathbb{R}), \quad x \in \mathbb{R}.\quad (4.1)$$

It is studied in [Dau92, p. 109], [BJR97], and elsewhere. To make it precise, it is convenient to identify its range with functions $H$ on $\mathbb{T} \times \mathbb{R}$ which satisfy the following scaling rule:

$$H(z,x+n) = z^{-n}H(z,x) \quad \text{for all } z \in \mathbb{T}, x \in \mathbb{R}, n \in \mathbb{Z}.\quad (4.2)$$

The norm $\|H\|$, or $\|H\|_2$, will be given by
(4.3) \[ \|H\|^2 = \int_T \int_0^1 |H(z,x)|^2 \, d\mu(z) \, dx, \]
and it can be checked, see [Dau92, p. 109], that
\[ (4.4) \|H\|^2 = \|h\|^2 = \int_R |h(x)|^2 \, dx, \]
and that \( H = Zh \) is an isometric isomorphism of \( L^2(\mathbb{R}) \) onto the Hilbert space \( \mathcal{H}_Z \) of functions which are defined by the scaling formula (4.2), and completion in the norm \( \| \cdot \|_2 \) of (4.3). The simplest wavelet scaling function is \( \varphi_H := \chi_{[0,1]} \) of the Haar wavelet. (The Haar wavelet itself is generated by \( \psi_H = \chi_{[0,\frac{1}{2}]} - \chi_{[\frac{1}{2},1]} \).) The Zak transform \( Z \) has the pleasant feature that \( Z(\varphi_H) \) is the constant function \( \mathbb{1} \) in \( L^2(T \times [0,1]) \).

A second advantage of the \( \mathcal{H}_Z \) formulation (4.2) is that it makes clear a useful direct integral representation
(4.5) \[ \mathcal{H}_Z = \int_T^\oplus \mathcal{H}(z) \, d\mu(z), \]
where each \( \mathcal{H}(z) \) is the Hilbert space \( \{h_z\} \) of functions \( h_z \) on \( \mathbb{R} \) satisfying the \( z \)-scaling rule,
\[ h_z(x + n) = z^{-n}h_z(x), \quad \text{for all } x \in \mathbb{R}, \, n \in \mathbb{Z}. \]
We refer to [Dix69] for details on direct integrals, and note that each \( \mathcal{H}(z) \) is naturally and isometrically isomorphic to \( L^2(0,1) \). Thus \( Z \) is a transform representing \( L^2(\mathbb{R}) \)-functions as direct integrals (over \( T \) with Haar measure) of copies of \( L^2(0,1) \).

The following lemmas about the Zak transform are new. They are needed in the sequel, and are also of independent interest, we feel.

**Lemma 4.1.** Let \( m_0 \in L^\infty(T) \) be given satisfying (3.1), and let \( M \) and \( R \) denote the corresponding cascade operator and Ruelle operator. Let \( h_i, \, i = 1, 2, \) be functions in \( L^2(\mathbb{R}) \) with Zak transforms \( H_i = Zh_i \). Then
(4.6) \[ R(\langle H_1(z) | H_2(z) \rangle)(z) = \langle Z_z Mh_1 | Z_z Mh_2 \rangle, \]
where the inner product \( \langle \cdot | \cdot \rangle \) on both sides of (4.6) is that of \( \mathcal{H}(z) \), i.e.,
(4.7) \[ \langle H_1(z) | H_2(z) \rangle = \int_0^1 H_1(z,x)H_2(z,x) \, dx, \]
and
(4.8) \[ R(\langle H_1(\cdot) | H_2(\cdot) \rangle)(z) = \frac{1}{2} \sum_{w^2 = z} |m_0(w)|^2 \langle H_1(w) | H_2(w) \rangle, \]
and the summation in (4.8) is over the two roots, \( w \in \{ \pm \sqrt{z} \} \subset T. \)
REMARK 4.2. The formula (4.6) is crucial for the use of \( R \) as a transfer operator in the iteration of the cascade algorithm on a given starting vector \( h \in L^2(\mathbb{R}) \), i.e., \( h \to Mh \to M^2h \to \cdots \). Secondly, we will show in Section 6 that (4.6) is equivalent to the first of the two commutation relations, (3), in Theorem 3.1.

\textit{Proof.} Let the functions be as described in the lemma, \( H_i = Zh_i \), \( i = 1, 2 \). We first calculate the term \( Z_z Mh_1 \) from the desired equation (4.6). Keep in mind that all integrals and summations are convergent relative to the respective Hilbert-space norms (due to the isometries which we described before the statement of the lemma). Details are in \cite{Dau92, p. 109} and \cite{BJR97}.

Let

\[
(Uh_1) (x) := \frac{1}{\sqrt{2}} h_1 \left( \frac{x}{2} \right).
\]

Then \( Mh_1 = U^{-1} \pi (m_0) h_1 \), and we now calculate the Zak transform of the two individual operators making up \( M \). First, \( h_1 \mapsto \pi (m_0) h_1 \) transforms into \( H_1 \mapsto m_0 (z) H_1 (z, \cdot) \), since

\[
Z (\pi (m_0) h_1) = \sum_n z^n (\pi (m_0) h_1) (x + n)
\]

\[
= \sum_n \sum_k z^n a_k h_1 (x + n - k)
\]

\[
= \sum_k a_k z^k \sum_n z^{n-k} h_1 (x + n - k)
\]

\[
= \sum_k a_k z^k \sum_n z^n h_1 (x + n)
\]

\[
= m_0 (z) H_1 (z, x)
\]

as claimed. Note that all summation indices range over \( n, k \in \mathbb{Z} \). We have \( \ell^2 (\mathbb{Z}) \)-convergent summations, i.e., relative to the respective \( \ell^2 \)-norms, and the exchange of the summations is justified by the norm-isomorphism property of the Zak transform.

We now turn to the operator \( \bar{U}^{-1} \) given by \( \bar{U}^{-1} = ZU^{-1}Z^* \), where \( Z^* \) is the adjoint of \( Z \). Since \( Z \) is a norm-isomorphism, we have

\[
ZZ^* = \text{id}_{L^2(\mathbb{T} \times [0,1])}, \quad \text{and} \quad Z^* Z = \text{id}_{L^2(\mathbb{R})},
\]

where id refers to the respective identity operators.

We claim that

\[
(Z^* H_1) (x) = \int_H H_1 (z, x) \ d\mu (z).
\]

The proof is the following calculation, for \( h \in L^2 (\mathbb{R}) \):
\begin{align*}
\langle H_1 \mid Zh \rangle &= \int_T \int_0^1 \overline{H_1(z,x)} Zh(z,x) \, d\mu(z) \, dx \\
&= \int_T \int_0^1 \overline{H_1(z,x)} \sum_n z^n h(x+n) \, d\mu(z) \, dx \\
&= \int_T \int_0^1 \sum_n \overline{H_1(z,x+n)} h(x+n) \, dx \, d\mu(z) \\
&= \int_T \int_{-\infty}^\infty \overline{H_1(z,x)} h(x) \, dx \, d\mu(z) \\
&= \int_\mathbb{R} \int_T \overline{H_1(z,x)} h(x) \, dx \, d\mu(z) \\
&= \langle Z^* H_1 \mid h \rangle,
\end{align*}

which proves the stated formula (4.10) for \( Z^* \).

We now calculate

\begin{align*}
\left( \tilde{U}^{-1}H_1 \right) (z,x) &= (ZU^{-1}Z^*) H_1(z,x) \\
&= \sum_n z^n (U^{-1}Z^*H_1)(x+n) \\
&= \sum_n z^n \sqrt{2} (Z^*H_1)(2x+2n) \\
&= \sum_n z^n \sqrt{2} \int_T H_1(\zeta,2x+2n) \, d\mu(\zeta) \\
&= \sqrt{2} \sum_n z^n \int_T \zeta^{-2n} H_1(\zeta,2x) \, d\mu(\zeta) \\
&= \sqrt{2} \sum_n z^n \int_T \zeta^{-n} \sum_{\xi \in T} \frac{1}{2} \sum_{\xi^2 = \zeta} H_1(\xi,2x) \, d\mu(\zeta) \\
&= \frac{1}{\sqrt{2}} \sum_{w^2 = z} H_1(w,2x),
\end{align*}

where the last step simply represents the Fourier series of the final function. The formula

\begin{equation}
\tilde{U}^{-1}H_1(z,x) = \frac{1}{\sqrt{2}} \sum_{w^2 = z} H_1(w,2x)
\end{equation}

is basic in later proofs. Combining the three formulas (4.8), (4.10), and (4.11), we arrive at

\begin{equation}
(Z_z Mh_1)(x) = \frac{1}{\sqrt{2}} \sum_{w^2 = z} m_0(w) H_1(w,2x).
\end{equation}
This is needed in the calculation of the right-hand side in (4.6) from the lemma as follows:

\begin{equation}
\langle Z_{z}Mh_{1} | Z_{z}Mh_{2} \rangle = \int_{0}^{1} \frac{1}{(Z_{z}Mh_{1})(x)} \frac{1}{(Z_{z}Mh_{2})(x)} \, dx \\
= \frac{1}{2} \int_{0}^{1} \sum_{w_{1}^{2}=z} m_{0}(w_{1}) H_{1}(w_{1}, 2x) \sum_{w_{2}^{2}=z} m_{0}(w_{2}) H_{2}(w_{2}, 2x) \, dx \\
= \frac{1}{2} \sum_{w_{1}^{2}=w_{2}^{2}=z} m_{0}(w_{1}) m_{0}(w_{2}) \int_{0}^{1} H_{1}(w_{1}, 2x) H_{2}(w_{2}, 2x) \, dx \\
= \frac{1}{4} \sum_{w_{1}^{2}=w_{2}^{2}=z} m_{0}(w_{1}) m_{0}(w_{2}) \int_{0}^{1} \left[ H_{1}(w_{1}, x) H_{2}(w_{2}, x) + H_{1}(w_{1}, x+1) H_{2}(w_{2}, x+1) \right] \, dx \\
= \frac{1}{4} \sum_{w_{1}^{2}=w_{2}^{2}=z} m_{0}(w_{1}) m_{0}(w_{2}) (1 + w_{1}w_{2}^{-1}) \int_{0}^{1} H_{1}(w_{1}, x) H_{2}(w_{2}, x) \, dx.
\end{equation}

But if \( w_{1} \neq w_{2} \) in the summation, then the factor \( (1 + w_{1}w_{2}^{-1}) = 0 \). If \( w_{1} \neq w_{2} \), then \( w_{2} = -w_{1} \) and \( w_{1}w_{2}^{-1} = -1 \). If \( w_{1} = w_{2} \), then \( (1 + w_{1}w_{2}^{-1}) = 2 \). Continuing the calculation, we get

\begin{equation}
\langle Z_{z}Mh_{1} | Z_{z}Mh_{2} \rangle = \frac{1}{2} \sum_{w_{2}^{2}=z} |m_{0}(w)|^{2} \int_{0}^{1} \frac{H_{1}(w, x) H_{2}(w, x)}{H_{1}(w, x)} \, dx \\
= \frac{1}{2} \sum_{w_{2}^{2}=z} |m_{0}(w)|^{2} \langle H_{1}(w) | H_{2}(w) \rangle \\
= R \left( \langle H_{1}(\cdot) | H_{2}(\cdot) \rangle \right)(z),
\end{equation}

which is the final conclusion of the lemma.

Having formula (4.11) for \( \tilde{U}^{-1} \), we may derive the corresponding formula for \( \tilde{U} = ZUZ^{*} \), but we isolate it here in a separate lemma.

**LEMMA 4.3.** Let \( U \) be the scaling operator (4.3) on \( L^{2}(\mathbb{R}) \), \( Uh(x) = \frac{1}{\sqrt{2}} h \left( \frac{x}{2} \right) \), and let \( \tilde{U} = ZUZ^{*} \) be the corresponding operator on \( \mathcal{H}_{Z} \simeq L^{2}(\mathbb{T} \times [0,1]) \) of the Zak transform. Then

\begin{equation}
(\tilde{U}H)(z, x) = \frac{1}{\sqrt{2}} \left( H \left( z^{2}, \frac{x}{2} \right) + zH \left( z^{2}, \frac{x+1}{2} \right) \right).
\end{equation}
Proof.

\[ \text{ZU} Z^* H (z, x) \]
\[ = \sum_n z^n (U Z^* H) (x + n) \]
\[ = \frac{1}{\sqrt{2}} \sum_n z^n (Z^* H) \left( \frac{x + n}{2} \right) \]
\[ = \frac{1}{\sqrt{2}} \sum_n z^n \int_\mathbb{T} H \left( \zeta, \frac{x + n}{2} \right) d\mu(\zeta) \]
\[ = \frac{1}{\sqrt{2}} \sum_k z^{2k} \int_\mathbb{T} H \left( \zeta, \frac{x}{2} + k \right) d\mu(\zeta) + \frac{1}{\sqrt{2}} \sum_k z^{2k+1} \int_\mathbb{T} H \left( \zeta, \frac{x + 1}{2} + k \right) d\mu(\zeta) \]
\[ = \frac{1}{\sqrt{2}} \left( H \left( z^2, \frac{x}{2} \right) + zH \left( z^2, \frac{x + 1}{2} \right) \right), \]

where, in the last step, we used the standard Fourier series representation of the respective functions.

Our interest in the use of the Zak-transform approach to cascade approximation started in an earlier paper \[BrJo98\] on the special case of compactly supported scaling functions. It is known \[Dau92\] that compact support of \( \varphi \) (as a scaling function in \( L^2(\mathbb{R}) \)) is equivalent to the filter function \( m_0(\cdot) \) being a polynomial, i.e., the sum in (3.5) being finite.

We encountered there the following two sesquilinear forms, both indexed by \( \mathbb{T} \):

\[ (4.15) \quad p_1(h_1, h_2) (e^{-i\omega}) = \sum_{n \in \mathbb{Z}} \mathring{h}_1(\omega + 2\pi n) \hat{h}_2(\omega + 2\pi n), \]

where the functions \( h_1, h_2 \) are in \( L^2(\mathbb{R}) \) and of compact support in the \( x \)-variable. This puts the Fourier transform \( \hat{h}_i, i = 1, 2 \), in the Schwartz class so that the sum in (4.15) is well defined. But problem was how to most efficiently remove the compact-support restriction on the functions \( h_i \). (Note the individual terms in the sum on the right-hand side in (4.15) are not periodic. But the sum \( \sum_n \) serves to “periodize” the function \( \omega \mapsto \left( \mathring{h}_1 \hat{h}_2 \right)(\omega), \omega \in \mathbb{R} \).

The other sesquilinear form was

\[ (4.16) \quad p_2(h_1, h_2)(z) = \sum_{k \in \mathbb{Z}} z^k \int_{\mathbb{R}} h_1(x - k) h_2(x) \, dx. \]

With the compact-support restriction, this is even a finite sum. But introducing the Poisson summation formula, or by a direct Fourier series expansion of \( p_1(h_1, h_2) \), we note that, with \( z = e^{-i\omega} \), we have

\[ (4.17) \quad p_1(h_1, h_2)(z) = p_2(h_1, h_2)(z). \]

When \( p_1(h_1, h_2) \) is viewed as a function on \( \mathbb{T} \), its \( k \)’th Fourier series coefficient computes out directly to be \( \int_{\mathbb{R}} \mathring{h}_1(x - k) h_2(x) \, dx \), and (4.17) follows from this.
The following result makes it clear that the compact-support restriction can be removed by use of the Zak-transform approach, and as a bonus, we get some *a priori* estimates that are needed later.

**PROPOSITION 4.4.** Let \( h_i \in L^2(\mathbb{R}) \) be of compact support. Then the two forms \( p_1 \) and \( p_2 \) coincide with \( p_3 \), where

\[
(4.18) \quad p_3 (h_1, h_2) (z) = \langle Z_z h_1 | Z_z h_2 \rangle = \int_0^1 Zh_1 (z,x)Zh_2 (z,x) \, dx.
\]

**Proof.** In the following calculation, convergence is governed by the norm-isometric property of \( Z \), and this also justifies the exchange of summations and integration:

\[
\langle Z_z h_1 | Z_z h_2 \rangle = \int_0^1 \sum_k z^k h_1 (x+k) \sum_l z^l h_2 (x+l) \, dx
\]

\[
= \sum_k \sum_l z^{l-k} \int_0^1 h_1 (x+k)h_2 (x+l) \, dx
\]

\[
= \sum_n z^n \sum_l \int_0^1 h_1 (x+l-n)h_2 (x+l) \, dx
\]

\[
= \sum_n z^n \int_{-\infty}^{\infty} h_1 (x-n)h_2 (x) \, dx
\]

\[
= p_2 (h_1, h_2) (z);
\]

since we already proved the identity \( p_1 = p_2 \) (in (4.17)), the proposition follows. \( \square \)

Having established \( p_1 = p_2 = p_3 \), we will use \( p \) to denote the common form. Since \( p_3 \) is defined for *all* pairs \( h_i \) in \( L^2(\mathbb{R}) \), the compact-support restriction involved in the formulation of \( p_1 \) and \( p_2 \) has been removed.

**COROLLARY 4.5.** Let \( p \) be the form on \( L^2(\mathbb{R}) \times L^2(\mathbb{R}) \) which is defined in (4.18), and taking values in functions on \( \mathbb{T} \). Then in fact \( p \) takes values in \( L^1(\mathbb{T}) \), i.e., the left-hand side of (4.21) below is finite if \( h_1, h_2 \in L^2 \). For restricted pairs \( h_1, h_2 \) of \( L^2(\mathbb{R}) \)-functions, \( p(h_1, h_2) \) can also be checked to take values in \( L^2(\mathbb{T}) \), and with one more restriction in \( L^\infty(\mathbb{T}) \), i.e., the left-hand sides of (4.20) and (4.19), below, are finite with suitable restrictions on \( h_1 \) and \( h_2 \). The restrictions are those which make the right-hand sides of (4.19) and (4.20) finite. The respective bounds (4.19), (4.20), and (4.21) are as follows:

\[
(4.19) \quad \| p(h_1, h_2) \|_{\infty} \leq \text{ess sup}_z \| Z_z h_1 \| \cdot \| Z_z h_2 \|,
\]

\[
(4.20) \quad \| p(h_1, h_2) \|_2 \leq \| p(h_1, h_1) \|_{1/2} \cdot \| h_2 \|_2,
\]

\[
(4.21) \quad \| p(h_1, h_2) \|_1 \leq \| h_1 \|_1 \cdot \| h_2 \|_2.
\]

**Proof.** We begin with (4.21) since it is universal. Before the estimate, we may restrict to \( h_i \) of compact support, and then, after the fact, this restriction is removed by completion.
In the following estimates, we use the Cauchy–Schwarz inequality two times, for the respective Hilbert inner products involved:

\[
\| p(h_1, h_2) \|_1 = \int_T |\langle Z_z h_1 | Z_z h_2 \rangle| \, d\mu(z)
\]

\[
\leq \int_T \| Z_z h_1 \| \cdot \| Z_z h_2 \| \, d\mu(z)
\]

\[
\leq \left( \int_T \| Z_z h_1 \|^2 \, d\mu(z) \cdot \int_T \| Z_w h_2 \|^2 \, d\mu(w) \right)^{\frac{1}{2}}
\]

\[
= \| Z h_1 \|_2 \cdot \| Z h_2 \|_2 = \| h_1 \|_2 \cdot \| h_2 \|_2 ,
\]

proving (4.21).

Formula (4.19) for \( \| p(h_1, h_2) \|_\infty \) is trivial to check, but it is not clear which conditions on the \( h_i \)'s make the right-hand side finite.

Formula (4.20) for \( \| p(h_1, h_2) \|_2 \) is useful later since we will be able to check finiteness of the factor \( \| p(h_1, h_1) \|^{\frac{1}{2}} \). In fact in many cases, we will end up with \( p(h_1, h_1) \) a constant function on \( T \). Checking (4.20) goes as follows: Let \( h_i \in L^2(\mathbb{R}) \), and suppose \( p(h_1, h_1) \in L^\infty(T) \). Then

\[
\| p(h_1, h_2) \|_2^2 = \int_T |\langle Z_z h_1 | Z_z h_2 \rangle|^2 \, d\mu(z)
\]

\[
\leq \int_T \| Z_z h_1 \|^2 \cdot \| Z_z h_2 \|^2 \, d\mu(z)
\]

\[
\leq \text{ess sup}_z |p(h_1, h_1)(z)| \cdot \int_T \| Z_z h_2 \|^2 \, d\mu(z)
\]

\[
= \| p(h_1, h_1) \|_\infty \cdot \| Z h_2 \|_2^2
\]

\[
= \| p(h_1, h_1) \|_\infty \cdot \| h_2 \|_2^2 ,
\]

which is exactly (4.20).

We now use the estimates from Corollary 4.5 to examine boundedness properties of the representation \( \pi \) of \( L^\infty(T) \) on \( L^2(\mathbb{R}) \) which was introduced in (3.11), i.e., \( \pi(\alpha) h = \alpha \ast h \), \( \alpha \in L^\infty(T) \), \( h \in L^2(\mathbb{R}) \). When using \( L^2(\mathbb{R}) \simeq \mathcal{H}_Z \) (via the Zak transform), a unitarily equivalent form of the representation is (also denoted \( \pi \)) the following:

(4.22) \( \pi(\alpha) H(z, x) = \alpha(z) H(z, x) \), \( \text{ for } \alpha \in L^\infty(T), H \in \mathcal{H}_Z \).

The idea from the proof of Corollary 4.5 yields immediately

\[
\| \pi(\alpha) H \|_{\mathcal{H}_Z} \leq \| \alpha \|_\infty \cdot \| H \|_{\mathcal{H}_Z} ,
\]

which is the bound which is required for a representation. In studying scaling vectors, however, we shall also need, for fixed \( H \in \mathcal{H}_Z \), boundedness properties of the map

(4.23) \( C_H : \alpha \mapsto \pi(\alpha) H \)

from \( L^2(T) \) to \( \mathcal{H}_Z \).
PROPOSITION 4.6. The mapping $C_H : \alpha \mapsto \pi(\alpha) H$ in (4.23) is bounded from $L^2(\mathbb{T})$ to $H_Z$ if and only if $p_2(H) = p(H, H)$ is in $L^\infty(\mathbb{T})$, and then the norm of $C_H$ is $\|p_2(H)\|_{L^\infty(Z)}$. Moreover, $C_H$ has a bounded inverse if and only if $p_2(H)$ has an $L^\infty(\mathbb{T})$ inverse, i.e., there is some $\varepsilon \in \mathbb{R}_+$ such that $p_2(H)(z) \geq \varepsilon$ a.e. on $\mathbb{T}$.

Proof. We compute

$$\|C_H(\alpha)\|^2_{H_Z} = \int_T \int_0^1 |\alpha(z) H(z, x)|^2 \, dx \, d\mu(z)$$

$$= \int_T |\alpha(z)|^2 \int_0^1 |H(z, x)|^2 \, dx \, d\mu(z)$$

$$= \int_T |\alpha(z)|^2 p_2(H)(z) \, d\mu(z).$$

If $\alpha \in L^\infty(\mathbb{T})$, then

$$\|C_H(\alpha)\|^2_{H_Z} \leq \|\alpha\|^2_{L^\infty} \int_T p_2(H)(z) \, d\mu(z)$$

$$= \|\alpha\|^2_{L^\infty} \cdot \|H\|^2_{H_Z},$$

and the assertion follows from a standard fact on multiplication operators. The same argument also yields the condition for invertibility of $C_H$. \qed

The significance of the operator $C_H$, $H \in H_Z$, is that if $H = F$ is a scaling function, $\tilde{M}(F) = F$, where $\tilde{M} = ZMZ^{-1}$. Then $C_F$ intertwines $U$ with a special isometry $S_0$ in $L^2(\mathbb{T}) \approx \ell^2(\mathbb{Z})$. Let $m_0$ be a filter satisfying (i)–(iii) in the Introduction, and let $M$ be the corresponding cascade operator. Let $F$ be a scaling function. Let $m_1(z) := zm_0(-z)$ (which is a high-pass filter), and define $(S_j f)(z) = m_j(z) f(z^2)$, $j = 0, 1$, $f \in L^2(\mathbb{T})$, $z \in \mathbb{T}$. Then it is easy to check that

$$S_i^* S_j = \delta_{ij} \text{id}_{L^2(\mathbb{T})}$$

and

$$\sum_{i=0}^1 S_i S_i^* = \text{id}_{L^2(\mathbb{T})}.$$
Table 1. Summary of Theorem 4.7. Embedding of the isometric model into $L^2(\mathbb{R})$

\[
\begin{array}{cccc}
\{0\} & \cdots & V_2(F) & V_1(F) & V_0(F) & \text{finer scales} \\
\cdots & W_2(F) & W_1(F) & W_0(F) & \cdots & \text{rest of } L^2(\mathbb{R}) \\
& \stackrel{C_F}{\leftarrow} U & \stackrel{C_F}{\leftarrow} U & \stackrel{C_F}{\leftarrow} U & L^2(\mathbb{R}) \approx \mathcal{H}_Z \\
& & \uparrow C_F: \alpha \mapsto \pi(\alpha) F & L^2(\mathbb{T}) \\
\{0\} & \cdots & S_0 & S_0 & S_0 & \mathcal{L}_0 = S_1L^2(\mathbb{T}) \\
& \cdots & S_0^2\mathcal{L}_0 & S_0\mathcal{L}_0 & \mathcal{L}_0 & \text{finer scales} \\
& S_0^2L^2(\mathbb{T}) & S_0L^2(\mathbb{T}) & L^2(\mathbb{T}) 
\end{array}
\]

THEOREM 4.7. Let the setting be as above, and consider the cascade problem in $\mathcal{H}_Z \simeq L^2(\mathbb{R})$. Then the following two conditions are equivalent:

(i) $\tilde{M}(F) = F$, and

(ii) $C_F$ intertwines $S_0$ and $\tilde{U}$, i.e., $C_FS_0 = \tilde{U}C_F$.

Let $V_0(F) = [\{\pi(\alpha) F \mid \alpha \in L^2(\mathbb{T})\}]$ where $[\cdot]$ is norm-closure. Let $V_n(F) := \tilde{U}^n(V_0(F))$. If (i) holds, then $V_{n+1}(F) \subset V_n(F)$, and $C_F(S_0^nL^2(\mathbb{T})) = V_n(F)$. Let $W_n(F) := V_n(F) \ominus V_{n+1}(F)$. If further $p_2(F) \equiv 1$, then $C_F(S_0^n\mathcal{L}_0) = W_n(F)$.

The results of the theorem may be summarized as in Table 1 above.

Proof. (i) $\Rightarrow$ (ii). Suppose $\tilde{U}^{-1}\pi(m_0)F = F$; then

\[
\tilde{U}C_F(\alpha) = \tilde{U}\pi(\alpha)F = \pi(\alpha(z^2))\tilde{U}F = \pi(\alpha(z^2))\pi(m_0)F = \pi(m_0(\alpha(z^2)))F = \pi(S_0\alpha)F = C_FS_0(\alpha)
\]

for all $\alpha \in L^2(\mathbb{T})$, which is (ii).

(ii) $\Rightarrow$ (i). Suppose (ii) holds. Then the previous calculation reverses, and shows that $\tilde{M}(F) = F$ where $\tilde{M} = \tilde{U}^{-1}\pi(m_0)$, and we have (i).

Suppose (ii), and let $\alpha \in L^2(\mathbb{T})$; then

\[
C_FS_0^n\alpha = \tilde{U}^nC_F(\alpha) = \tilde{U}^n\pi(\alpha)F \in V_n(F),
\]

and we conclude that $C_F(S_0^nL^2(\mathbb{T})) = V_n(F)$ as claimed.

Now let $\alpha, \beta \in L^2(\mathbb{T})$, and consider the following calculation:

\[
\langle C_FS_1\alpha \mid \tilde{U}\pi(\beta)F \rangle = \langle C_FS_1\alpha \mid C_FS_0\alpha \rangle = \int_{\mathbb{T}} m_1(z)m_0(\alpha(z^2)\beta(z^2)p_2(F)(z) \, d\mu(z).
\]
Table 2. Approximation properties of $m_0^{(n)}$ and $D_n(z)$

| $m_0^{(n)}$ | $D_n(z) = \left| m_0^{(n)}(z) \right|^2$ |
|-------------|----------------------------------|
| $m_0^{(n)} = S^n_0(\mathbb{I})$ | $D_n = R^n(\mathbb{I})$ |
| $\int_{\mathbb{T}} m_0^{(n)} f \, d\mu \xrightarrow{n \to \infty} 0 \quad \forall f \in L^2(\mathbb{T})$ | $\int_{\mathbb{T}} D_n f \, d\mu \xrightarrow{n \to \infty} f(1) \quad \forall f \in C(\mathbb{T})$ if $\{f \mid Rf = f\}$ is one-dimensional |
| $\int_{\mathbb{T}} m_0^{(n)} \, d\mu = (a_0)^n$ if $m_0(z) = \sum_{k=0}^{\infty} a_k z^k$ | $\int_{\mathbb{T}} D_n(z) \, d\mu(z) = 1 \quad \forall n = 1, 2, \ldots$ |

If $p_2(F) \equiv 1$, then this integral is

$$\int_{\mathbb{T}} \frac{1}{2} \sum_{w^2 = z} m_1(w) m_0(w) \overline{\alpha(z)\beta(z)} \, d\mu(z).$$

But the choice of $m_1(z) = zm_0(-z)$ makes it zero:

$$\frac{1}{2} \sum_{w^2 = z} m_1(w) m_0(w) = \frac{1}{2} \sum_{w^2 = z} \bar{w} m_0(-w) m_0(w) = 0,$$

since $\sum_{w^2 = z} w = 0$.

We have proved that if $p_2(F) = \mathbb{I}$, $C_F$ maps $L_0 = S_1(L^2(\mathbb{T}))$ onto $W_0(F) = V_0(F) \ominus UV_0(F)$, and an iteration of the same argument (induction) yields

$$C_F(S^n_0 S_1 L^2(\mathbb{T})) = W_n(F)$$

as claimed.

We also saw that $C_F$ is isometric if and only if $p_2(F) \equiv 1$.

The significance of the spaces $V_0(F)$ and $W_0(F)$ in wavelet theory is that, in the $L^2(\mathbb{R})$-picture, $V_0(F)$, $F = Z \varphi$, is generated by the father function, while $W_1(F)$ is generated by the mother function. It is interesting to summarize the approximation properties of the two function sequences $m_0^{(n)}(z) = m_0(z) m_0(z^2) \cdots m_0(z^{2^{n-1}})$ and $D_n(z) = \left| m_0^{(n)}(z) \right|^2$.

In Table 2, we include results from [BrJo97] in the left-hand column, and results from Meyer and Paiva [MePa93] in the right-hand column. The filter $m_0$ is given as usual, and (i)–(iii) in the Introduction are assumed.

We now show that when $m_0$ is given, and $M$ is the corresponding cascade operator, then the solutions $M(F) = F$ may be identified with a space of intertwining operators. We shall state the details in the Hilbert space $\mathcal{H}_Z$ of the Zak transform.

**COROLLARY 4.8.** An operator $C : L^2(\mathbb{T}) \to \mathcal{H}_Z$ is of the form $C_F$ for some $F \in \mathcal{H}_Z$ satisfying $\tilde{M}(F) = F$ if and only if it satisfies the following two intertwining properties:
(a) $CS_0 = \tilde{UC}$, and
(b) $C(f\alpha) = (\pi(f)C)(\alpha)$, for all $f \in L^\infty(\mathbb{T})$ and $\alpha \in L^2(\mathbb{T})$.

If multiplication by $L^\infty(\mathbb{T})$ on $L^2(\mathbb{T})$ is written $\tau(f)\alpha = f\alpha$, then (I) reads:
\[(3)\] $C\tau(f) = \pi(f)C$.

Proof. By Theorem 4.7, it is enough to check that every operator $C: L^2(\mathbb{T}) \to \mathcal{H}_Z$ which satisfies (I) $\simeq$ (III) must be of the form $C = C_F$ for some $F \in \mathcal{H}_Z$. So let $C$ be given, and assume (II). Let $C(\mathbb{I}) := F$. Then, for $\alpha \in L^\infty(\mathbb{T})$, we have $C(\alpha) = C\tau(\alpha)\mathbb{I} = \pi(\alpha)C\mathbb{I} = \pi(\alpha)F = C_F(\alpha)$. We are considering only the case when $C$ is bounded. Since $L^\infty(\mathbb{T})$ is dense in $L^2(\mathbb{T})$, the result follows. If (III) also holds, we saw in Theorem 4.7 that then $\tilde{M}(F) = F$, and the proof is completed. \[\square\]

**REMARK 4.9.** An inner product on the intertwining operators. From (III), we also get the identities
\[C_H^*C_H = \tau(p_2(H)) \quad \text{and} \quad C_H^*C_{H'} = \tau(p(H,H'))\]
for $H,H' \in \mathcal{H}_Z$, where
\[p_2(H)(z) = p(H,H)(z) = \int_0^1 |H(z,x)|^2 \, dx,\]
and $\tau(f)$ denotes multiplication by $f$ on $L^2(\mathbb{T})$. Hence (as noted), $\|C_H\| = \|p_2(H)\|_\infty^{\frac{1}{2}}$, and $C_H$ is bounded if and only if $p_2(H) \in L^\infty(\mathbb{T})$.

**REMARK 4.10.** Proposition 4.8 will be used in Sections 4 & 5 in the study of scaling functions $\varphi \in L^2(\mathbb{R})$, i.e., solutions to $M\varphi = \varphi$ where $M$ is a cascade operator for some given filter $m_0$. If $Z\varphi = F$, then the scaling equation is equivalent to $\tilde{M}(F) = F$. Then, generically, $C_F$ will be bounded, but will not have a bounded inverse.

If $m_0(z) = \frac{1}{\sqrt{2}}(1 + z^3)$, then $\varphi(x) = \frac{1}{3}x\chi_{[0,3]}(x)$, and for $0 \leq x \leq 1$,
\[F(z,x) = Z\varphi(z,x) = \frac{1}{3} \left( x\chi_{[0,3]}(x) + z\chi_{[-1,2]}(x) + z^2\chi_{[-2,1]}(x) \right).\]
It is easy then to check that
\[p_2(H)(z) = p(H,H)(z) = \frac{1}{9} \left( z^{-2} + 2z^{-1} + 3 + 2z + z^2 \right)\]
\[= \frac{1}{9} \left( 3 + 4\cos\omega + 2\cos(2\omega) \right),\]
where $z = e^{-i\omega}$. In this case, $C_F$ is not invertible as $p_2(H)$ vanishes on $\mathbb{T}$. In fact, $p_2(H)(\omega) = \frac{1}{9}(2\cos\omega + 1)^2$ which, of course, vanishes for $\omega = \pm \frac{2\pi}{3}$.

**REMARK 4.11.** Using the usual isomorphism $L^2(\mathbb{T}) \simeq l^2(\mathbb{Z})$ defined by the Plancherel theorem for Fourier series, we note that, if $m(z) = \sum_{n \in \mathbb{Z}} a_n z^n$, then the operator
\[f \mapsto m(z)f(z^2), \quad \text{on} \quad L^2(\mathbb{T}),\]
takes the form

\[(S\xi)_n = \sum_{k \in \mathbb{Z}} a_{n-2k}\xi_k\]  

(4.24)

when realized as an operator on the sequence space $\ell^2(\mathbb{Z})$ via the Fourier series representation

\[f(z) = \sum_{n \in \mathbb{Z}} \xi_n z^n, \quad (\xi_n) \in \ell^2; \quad \sum_{n \in \mathbb{Z}} |\xi_n|^2 = \|f\|_2^2,\]

and this is the connection to the Micchelli operator (2.4) mentioned in the Introduction. We sketch the details of this argument below, and refer to [Mic96] for more details.

Starting with $2\pi$-periodic functions $m$ and $f$, corresponding to the Fourier representation

\[m(z) \sim m(\omega) \sim \sum_{n \in \mathbb{Z}} a_n z^n,\]

and

\[f(z) \sim f(\omega) \sim \sum_{n \in \mathbb{Z}} \xi_n z^n,\]

with $z = e^{-i\omega}$, $\omega \in \mathbb{R}$, as the $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ convention, we have Micchelli’s operator $S$ of (4.24) or (2.4) in the function form

\[(\tilde{S}f)(\omega) = m(\omega) f(2\omega), \quad \omega \in \mathbb{R}.

For each $k$, the iteration $\tilde{S}^k f$ is also $2\pi$-periodic, while $(\tilde{S}^k f)(\frac{\omega}{2^k})$ has period $2^k \cdot (2\pi)$. Introducing $(UF)(x) = 2^{-\frac{x}{2}}F\left(\frac{x}{2}\right)$, $x \in \mathbb{R}$, on functions, or distributions, on $\mathbb{R}$, we arrive at the representation

\[(4.25) \quad U^k \tilde{S}^k f \sim 2^{-\frac{x}{2}}m(\omega) m\left(\frac{\omega}{2}\right) \cdots m\left(\frac{\omega}{2^k}\right) f(\omega).\]

If there is a limit function (or distribution) $F_\xi$, as $k \to \infty$, then the difference

\[(\Delta^k \xi)_j = (S^k \xi)_j - F_\xi\left(\frac{j}{2^k}\right), \quad j \in \mathbb{Z},\]

tends to zero in the limit $k \to \infty$. Note that $\Delta^k$, for each $k$, is acting on sequences, say $\ell^2(\mathbb{Z})$. Hence we get the corresponding scaling function $F_\xi$ from (4.25) at the dyadic rational points $\left\{\frac{j}{2^k} \mid j, k \in \mathbb{Z}\right\} \subset \mathbb{R}$ this way, and we have therefore made the connection to the Micchelli approximation of [Mic96]; see also (2.4) in the Introduction above.

5. PROOF OF THEOREM 3.1

Recall that $\mathcal{H}$ is an isometric isomorphism, viz.:

\[L^2(\mathbb{R}) \simeq \mathcal{H}_Z \simeq L^2(\mathbb{T} \times [0,1]),\]

where the second $\simeq$ amounts to restriction from $\mathbb{R}$ to $[0,1]$ in the $x$-variable: if $H \in \mathcal{H}_Z$ satisfies (4.2), then the restriction $H(z,x)$, $0 \leq x \leq 1$, defines the corresponding element in $L^2(\mathbb{T} \times [0,1])$, and a simple argument shows that this restriction mapping is indeed an
Table 3. Operator correspondence between $L^2 (\mathbb{R})$ and $\mathcal{H}_Z$ (Lemma 5.1)

| $h \in L^2 (\mathbb{R})$ | $H \in \mathcal{H}_Z$ |
|---------------------------|-----------------------|
| $(T_n h) (x) = h (x + n)$ | $\tilde{T}_n H (z, x) = z^{-n} H (z, x)$ |
| $\pi (\alpha) h = \alpha \ast h$ | $\tilde{\pi} (\alpha) H (z, x) = \alpha (z) H (z, x)$ |
| $M = U^{-1} \pi (m_0)$ | $\tilde{M} H (z, x) = \frac{1}{\sqrt{2}} \sum_{w = z} m_0 (w) H (w, 2x)$ |
| $M^* = \pi (\tilde{m}_0) U$ | $\tilde{M}^* H (z, x) = \frac{1}{\sqrt{2}} m_0 (z) \left( H \left( z^2, \frac{x}{2} \right) + z H \left( z^2, \frac{x+1}{2} \right) \right)$ |
| $(E_t h) (x) = e^{itx} h (x)$ | $\tilde{E}_t H (z, x) = e^{itx} H (ze^{it}, x)$ |
| $(\mathcal{F} h) (x) = \int_{\mathbb{R}} e^{-i2\pi yh} (y) dy$ | $\left( \tilde{\mathcal{F}} H \right) (e^{i2\pi \omega}, x) = e^{-i2\pi \omega} H (e^{-i2\pi \omega}, \omega)$ for $\omega, x \in \mathbb{R}$ |

isomorphic isometry of $\mathcal{H}_Z$ onto $L^2 (\mathbb{T} \times [0, 1])$. It follows that operators in one space identify with corresponding operators in the other. If $A$ is a given operator in $L^2 (\mathbb{R})$, then $\tilde{A} := ZAZ^*$ is the corresponding operator in $\mathcal{H}_Z$.

The proof of the following lemma is essentially contained in the previous section: see especially (4.11) and Lemma 4.3.

**Lemma 5.1.** If $A$ is one of the operators in $L^2 (\mathbb{R})$ listed in the first column of Table 3, then $\tilde{A} = ZAZ^*$ in $\mathcal{H}_Z$ is given by the corresponding entry in the second column of Table 3.

**Proof.** In the previous section, we also elaborated on the operators $\pi (\alpha), \alpha \in L^\infty (\mathbb{T})$, $Uh (x) = \frac{1}{\sqrt{2}} h \left( \frac{x}{2} \right)$, and the cascade operator

$$Mh (x) = \sqrt{2} \sum_{n \in \mathbb{Z}} a_n h (2x - n),$$

for $m_0 (z) = \sum_{n \in \mathbb{Z}} a_n z^n$, representing the given low-pass filter. The present proof amounts to a combination of the calculations leading up to Lemma 4.3 and the argument from the proof of that lemma.

**Proof of Theorem 3.1.** With the aid of Lemma 5.1, the proof of the two commutation relations (3)–(4) in Theorem 3.1 now amounts to the following computations. They take place...
in the space $\mathcal{H}_Z$, i.e., the range of the Zak transform, so it is the right-hand column in Table 3 which is used.

Ad (a): Let $\alpha \in L^\infty (T)$ and $H \in \mathcal{H}_Z$. Then

$$\left( \tilde{M}^* \pi (\alpha) \tilde{M} \right) H (z, x) = \frac{1}{\sqrt{2}} m_0 (z) \alpha (z^2) \left( \tilde{M} H \left( z^2, \frac{x}{2} \right) + z \tilde{M} H \left( z^2, \frac{x + 1}{2} \right) \right)$$

$$= \frac{1}{2} m_0 (z) \alpha (z^2) \sum_{w^2 = z^2} m_0 (w) (H (w, x) + z H (w, x + 1))$$

$$= \frac{1}{2} m_0 (z) \alpha (z^2) \sum_{w^2 = z^2} m_0 (w) \left( H (w, x) + zw^{-1} H (w, x) \right)$$

$$= \frac{1}{2} m_0 (z) \alpha (z^2) \sum_{w^2 = z^2} m_0 (w) \left( 1 + zw^{-1} \right) H (w, x).$$

But the summation is over $w \in \{ \pm z \}$, and the term $1 + zw^{-1} = 2$ if $w = z$, and $1 + zw^{-1} = 0$ if $w = -z$. We get

$$\left( \tilde{M}^* \pi (\alpha) \tilde{M} \right) H (z, x) = |m_0 (z)|^2 \alpha (z^2) H (z, x),$$

which is the identity (3) of Theorem 3.1.

Ad (b): As in (a), let $\alpha \in L^\infty (T)$ and $H \in \mathcal{H}_Z$ be given. Then

$$\left( \tilde{M} \pi (\alpha) \tilde{M}^* \right) H (z, x) = \frac{1}{\sqrt{2}} \sum_{w^2 = z} m_0 (w) \alpha (w) \left( \tilde{M}^* H \right) (w, 2x)$$

$$= \frac{1}{2} \sum_{w^2 = z} |m_0 (w)|^2 \cdot \alpha (w) \cdot \left( H (z, x) + w H \left( z, x + \frac{1}{2} \right) \right)$$

$$= R (\alpha) (z) H (z, x) + R (e_1 \alpha) (z) H \left( z, x + \frac{1}{2} \right),$$

which is precisely the second identity (2) of Theorem 3.1. Note that we obtain the identities in $\mathcal{H}_Z$, but since $Z$ is an isomorphism, $Z : L^2 (\mathbb{R}) \rightarrow \mathcal{H}_Z$, we automatically get the same identities in $L^2 (\mathbb{R})$ where the translation to $L^2 (\mathbb{R})$ is made via the dictionary of Lemma 5.1 (Table 3).

REMARK 5.2. The last line in the dictionary of Lemma 5.1 (Table 3) is the correspondence for the Fourier transform $\mathcal{F}$ in $L^2 (\mathbb{R})$, and it shows that the equivalent transform $\tilde{\mathcal{F}}$ in $\mathcal{H}_Z$ is given by a very simple formula: it has a phase factor, and otherwise only involves switching of the two variables $x, \omega$, i.e., time and frequency variables, $z = e^{i2\pi \omega}$. It was included in Table 3 for later use.

6. SUB-ISOMETRIES

Let $m_0 \in L^\infty (T)$ be a low-pass filter, i.e., satisfying conditions (2.1) and (3) in Section 4 and let $R, M$ be the corresponding Ruelle operator and cascade refinement operator; see (2.1) and (3.8) for details. Then $L^\infty (T)$ is represented as an algebra of operators on
$L^2(\mathbb{R})$ via $\pi(\alpha)h = \alpha \ast h$, $\alpha \in L^\infty(\mathbb{T})$, $h \in L^2(\mathbb{R})$, cf. (3.12) or (1.22). Then we get the following two properties for $M$:

(6.1) \[ M^*\pi(\alpha)M = \pi(R^*(\alpha)) \]

and

(6.2) \[ M^*\pi(\alpha) = \pi(\alpha(z^2))M^*, \]

for all $\alpha \in L^\infty(\mathbb{T})$. Taking $\alpha = \mathbb{I}$ in (6.1), we get $M^*M = \pi(|m_0|^2)$. So $M$ is not an isometry unless $|m_0|^2 = \mathbb{I}$. The last condition is inconsistent with the low-pass property (I) of $m_0$, i.e., $m_0 = \sqrt{2}$ at $z = 1$. But we say that $M$ is a sub-isometry. More generally, let $\pi$ be a representation on a Hilbert space $\mathcal{H}$.

**DEFINITION 6.1.** Let $R$ be the Ruelle operator introduced above, i.e., $R: \mathcal{H} \to L^2(\mathbb{R})$, and let $\pi$ be a representation of $L^\infty(\mathbb{T})$ in the algebra of operators on $\mathcal{H}$, such that identities (i), (ii) hold for all $\alpha \in L^\infty(\mathbb{T})$.

We say that an operator $M$ on $\mathcal{H}$ is an $(R, \pi)$-isometry, or a sub-isometry if the data $(R, \pi)$ is understood, if

(i) \[ M^*\pi(\alpha)M = \pi(R^*(\alpha)) \] on $\mathcal{H}$, and

(ii) \[ \pi(\alpha)M = M\pi(\alpha(z^2)) \] on $\mathcal{H}$.

(In the general case, the two conditions (i) and (ii) are independent. A discussion of (ii) and its variant (6.2) will follow.)

If $M$ is an isometry, i.e., $M^*M = \text{id}_\mathcal{H}$, then $M(\mathcal{H})$ is closed, but it may not be so if $M$ is only a sub-isometry. Then we shall denote the closure $[M\mathcal{H}]$. Its orthogonal complement is $\ker M^* = \{h \in \mathcal{H} \mid M^*h = 0\}$. Let $\mathcal{L} := \ker M^*$, and set

(6.3) \[ \mathcal{H}^{(\infty)} := \bigcap_{n=1}^{\infty} [M^n\mathcal{H}]. \]

Again, if $M$ is an isometry, the classical Wold decomposition of $\mathcal{H}$ relative to $M$ states the orthogonal decomposition

(6.4) \[ \mathcal{H} = \sum_{n=0}^{\infty} \oplus M^n\mathcal{L} \oplus \mathcal{H}^{(\infty)}; \]

see [SzFo70] for details. The dimension of $\mathcal{L}$ is then also a complete invariant for the isometry in the pure case, i.e., when $\mathcal{H}^{(\infty)} = \{0\}$.

In this section, we prove an analogue of this result for general $(R, \pi)$-isometries. In that form, the components corresponding to $M^n\mathcal{L}$ in (6.4) will instead be $[M^n\mathcal{L}]$. We will still have orthogonality of the subspaces in the decomposition. The important new element for $(R, \pi)$-isometries is that each of the subspaces in the decomposition is invariant for the representation $\pi$, i.e., $\pi(\alpha)$ maps $[M^n\mathcal{L}]$ into itself for all $\alpha \in L^\infty(\mathbb{T})$, and $n = 0, 1, \ldots$. Similarly $\mathcal{H}^{(\infty)}$ is invariant under the operators $\pi(\alpha)$. The analogy to the classical Wold theorem for isometries raises the question of whether some invariant of the representation $\pi$, when restricted to $\mathcal{L} = \ker (M^*)$, is perhaps a complete invariant in the case when $M$ is
an \((R, \pi)\)-isometry. This question is answered (at least partially) by Theorem 6.2(c) below, while the first two parts of the theorem give a direct analogue of the Wold theorem itself in this new representation-theoretic framework.

**THEOREM 6.2.** Let \(m_0 \in L^\infty(\mathbb{T})\) be a given low-pass filter satisfying the quadratic equation

\[
|m_0(z)|^2 + |m_0(-z)|^2 = 2 \quad \text{a.e. on } \mathbb{T},
\]

and let \(R\) be the corresponding Ruelle operator. Let \(\pi\) be a representation of \(L^\infty(\mathbb{T})\) on a Hilbert space \(\mathcal{H}\), and let \(M\) be an associated \((R, \pi)\)-isometry, i.e., satisfying conditions (i)–(ii) in Definition 6.1.

Then \(\mathcal{H}\) has an orthogonal decomposition:

\[
\mathcal{H} = \sum_{n=0}^{\infty} [M^n \mathcal{L}] \oplus \mathcal{H}^{(\infty)},
\]

where \(\mathcal{L} = \ker (M^*)\), and \(\mathcal{H}^{(\infty)} = \bigcap_{n=1}^{\infty} [M^n \mathcal{H}]\). It has the following three properties:

(a) the individual closed subspaces in the decomposition are mutually orthogonal, i.e., \(M^n \mathcal{L}\) is orthogonal to \([M^k \mathcal{L}]\) if \(n \neq k\), and they are all orthogonal to \(\mathcal{H}^{(\infty)}\);

(b) each of the spaces \([M^n \mathcal{L}]\), for \(n = 0, 1, \ldots\), and \(\mathcal{H}^{(\infty)}\) is invariant under \(\pi(\alpha)\) for all \(\alpha \in L^\infty(\mathbb{T})\); and

(c) every representation \(\pi_0\) of \(L^\infty(\mathbb{T})\) in a Hilbert space \(\mathcal{L}\) arises as the \(n = 0\) term of \((6.6)\) for some \((R, \pi)\)-isometry \(M\).

**Proof.** In the proof, we shall refer to the two properties (i)–(iii) in Definition 6.1. If \(S \subset \mathcal{H}\) is a linear subspace, the orthogonal complement will be denoted

\[
S^\perp = \mathcal{H} \ominus S = \{h \in \mathcal{H} \mid \langle h, s \rangle = 0, \forall s \in S\}.
\]

We clearly have \(\mathcal{L} = (M\mathcal{H})^\perp\).

**Claim 1.** \(\mathcal{L}\) is invariant under \(\pi(\alpha), \alpha \in L^\infty(\mathbb{T})\).

**Proof.** Let \(l \in \mathcal{L}\). To show that \(\pi(\alpha)l \in \mathcal{L}\), we check that

\[
M^* \pi(\alpha)l = \pi(\alpha(z^2))M^*l = 0.
\]

We used (ii) in the calculation, noting that \(M^*l = 0\) by definition. \(\square\)

Our next assertion is this.

**Claim 2.** \(\mathcal{L} \oplus [M\mathcal{L}] = (M^2\mathcal{H})^\perp\).

**Proof.** We prove the claim by showing that a vector \(h_0\) which is orthogonal to all three subspaces \(\mathcal{L}, M\mathcal{L},\) and \(M^2\mathcal{H}\), must be zero. The three subspaces are pairwise mutually orthogonal. This is immediate from the definitions except for the last pair. Let \(l \in \mathcal{L}\) and...
Proof. We first show that, if \( h \in \mathcal{H} \), then 

\[
\langle Ml \mid M^2 h \rangle = \langle M^* Ml \mid h \rangle
\]

\[
= \langle M^* \pi (|m_0|) l \mid h \rangle
\]

\[
= \langle \pi (|m_0|z^2) M^* l \mid h \rangle
\]

\[
= 0,
\]

where we used properties (i) and (ii), in that order. In the last step, we used \( M^* l = 0 \).

The condition that \( h_0 \) is orthogonal to all three subspaces amounts to: \( h_0 \in [M\mathcal{H}], M^* h_0 \in [M\mathcal{H}], \) and \( M^* 2 h_0 = 0 \). Since \( \mathcal{H} = \mathcal{L} \oplus [M\mathcal{H}], \) we have \( [M\mathcal{H}] \subset [M\mathcal{L}] \oplus [M^2 \mathcal{H}], \) so \( h_0 = u + v, u \in [M\mathcal{L}], v \in [M^2 \mathcal{H}]. \) Since \( M^* h_0 \in [M\mathcal{H}], \) we get \( h_0 \in (M\mathcal{L})^\perp. \) Indeed there is a sequence \( h_i \in \mathcal{H} \) such that \( M^* h_0 = \lim_i Mh_i. \) So if \( l \in \mathcal{L}, \) then

\[
\langle Ml \mid h_0 \rangle = \langle l \mid M^* h_0 \rangle = \lim_i \langle l \mid Mh_i \rangle = \lim_i \langle M^* l \mid h_i \rangle = 0,
\]

since \( l \in \ker (M^*). \) Hence \( u = 0 \) in the decomposition of \( h_0: h_0 = 0 + v, v \in [M^2 \mathcal{H}]. \) But \( h_0 \in \ker (M^*^2) = (M^2 \mathcal{H})^\perp, \) and we conclude that \( h_0 = 0. \)

Claim 3. \([M\mathcal{L}] \) is invariant under \( \pi (\alpha), \alpha \in L^\infty (\mathbb{T}). \)

Proof. We first show that, if \( l_0 \in \mathcal{L} \) and \( \alpha \in L^\infty (\mathbb{T}), \) then \( \pi (\alpha) Ml_0 \in \mathcal{L} \oplus [M\mathcal{L}]. \) Using Claim 4, we do this by showing that \( \pi (\alpha) Ml_0 \in \ker (M^*^2). \) But

\[
M^* \pi (\alpha) Ml_0 = \pi (R^* (\alpha)) l_0 \in \mathcal{L},
\]

where we used first (i) and then Claim 3. Then

\[
M^*^2 \pi (\alpha) Ml_0 = \pi (R^* (\alpha) (z^2) ) M^* l_0 = 0,
\]

where we could use (ii) or, alternatively, Claim 3.

This means that \( \pi (\alpha) \mid_{\mathcal{L} \oplus [M\mathcal{L}]} \) has an operator block matrix relative to the orthogonal decomposition \( \mathcal{L} \oplus [M\mathcal{L}] \) of the form

\[
\pi (\alpha) \sim \left( \begin{array}{cc} A & B \\ 0 & D \end{array} \right),
\]

with \( B: [M\mathcal{L}] \to \mathcal{L}, \) and the diagonal operators \( A \) and \( D \) being endomorphisms of the respective spaces \( \mathcal{L} \) and \([M\mathcal{L}]. \) Since \( \pi (\alpha)^* = \pi (\alpha), \) the adjoint \((B^*, 0, 0, D^* )\) must be of the same form, and that forces \( B = 0, \) i.e., \( \pi (\alpha) = (A, 0, 0, D), \) and each of the spaces \( \mathcal{L} \) and \([M\mathcal{L}] \) is then invariant under \( \pi (\alpha). \) In particular, \([M\mathcal{L}] \) is invariant, which is the claim.

Our next assertion merges Claims 3 and 4 into the following induction:

Claim 4. For each \( n = 1, 2, \ldots, \) we have the decomposition

\[
\mathcal{L} \oplus [M\mathcal{L}] \oplus \cdots \oplus [M^{n-1} \mathcal{L}] = (M^n \mathcal{H})^\perp,
\]
where the terms in the decomposition are mutually pairwise orthogonal, and further each of the spaces \([M^k \mathcal{L}]\) is invariant under \(\pi (\alpha )\), \(\alpha \in L^\infty (\mathbb{T})\), \(k = 0, 1, \ldots , n - 1\), where we set \([M^0 \mathcal{L}] = \mathcal{L}\.\)

\[\text{Proof.}\] This is a simple induction which is based on Claims 1–3, and it is left to the reader. Alternatively, we can prove it by using the earlier claims on the operators \(M^2, M^3, \ldots \). \qed

**Claim 5.** We have the decomposition (6.8) of the theorem with the two properties (3)–(4).

\[\text{Proof.}\] For each \(n\), let \(Q_n\) denote the projection onto \([M^n \mathcal{H}]\). Since \([M^{n+1} \mathcal{H}] \subset [M^n \mathcal{H}]\), this is a decreasing sequence of projections in \(\mathcal{H}\). By Hilbert space theory (see, e.g., [SzFo70]), it has a limit \(Q_\infty\), i.e., \(\lim_{n \to \infty} \|Q_n h - Q_\infty h\| = 0\) for all \(h \in \mathcal{H}\), and \(Q_\infty\) is the orthogonal projection onto \(\mathcal{H}^{(\infty)} = \bigcap_{n=1}^\infty Q_n \mathcal{H}\). Recall that \(Q_n \mathcal{H} = [M^n \mathcal{H}]\), by definition! In fact,

\[\|Q_\infty h\| = \inf_n \|Q_n h\|, \quad h \in \mathcal{H}\.\]

But Claim 4 states that \(I - Q_n\) is the projection onto \(\mathcal{L} \oplus [M^1 \mathcal{L}] \oplus \cdots \oplus [M^{n-1} \mathcal{L}]\), where we write \(I\) for the identity operator in \(\mathcal{H}\). Now \(I - Q_n\) is an increasing family of projections, and its limit \(I - Q_\infty\) is the projection onto \(\sum_{n=0}^{\infty} [M^n \mathcal{L}]\). Let \(\mathcal{B}\) denote this space. Vectors \(b\) in \(\mathcal{B}\) are characterized by \((I - Q_\infty) b = b\), or equivalently \(Q_\infty b = 0\), and each \(b\) has the unique representation \(b = \sum_{n=0}^\infty b_n\), \(\|b\|^2 = \sum_{n=0}^\infty \|b_n\|^2\), \(b_n \in [M^n \mathcal{L}]\), \(n = 0, 1, \ldots \)

Since each of the spaces \([M^n \mathcal{L}]\) is invariant under \(\pi (\alpha )\), \(\alpha \in L^\infty (\mathbb{T})\), by Claims 3–4, it follows that \(\mathcal{B}\) is also \(\pi (\alpha )\)-invariant. Since \(\pi (\alpha )^* = \pi (\bar{\alpha})\), it follows that

\[\mathcal{B}^\perp = \mathcal{H}^{(\infty)} = \bigcap_{n=1}^\infty [M^n \mathcal{H}]\]

is also \(\pi (\alpha )\)-invariant. \qed

The proof of Theorem 6.2 will be given after the next three corollaries.

**COROLLARY 6.3.** For each \(n = 1, 2, \ldots \), the space \([M^n \mathcal{H}]\) is invariant under \(\pi (\alpha )\), \(\alpha \in L^\infty (\mathbb{T})\).

\[\text{Proof.}\] We showed in Claim 4 that

\[\left( M^n \mathcal{H} \right)^\perp = \sum_{k=0}^{n-1} [M^k \mathcal{L}]\]

and that the right-hand side has the \(\pi (\alpha )\)-invariance. Since \(\pi (\alpha )^* = \pi (\bar{\alpha})\), we conclude that \(\left( M^n \mathcal{H} \right)^\perp = [M^n \mathcal{H}]\) is also \(\pi (L^\infty (\mathbb{T}))\)-invariant. \qed

Let \((R, \pi, M, \mathcal{H})\) be as in the statement of Theorem 6.2, i.e., \(M : \mathcal{H} \to \mathcal{H}\) is an \((R, \pi )\)-isometry relative to some Ruelle operator \(R\) and representation \(\pi\). We say that a closed subspace \(S \subset \mathcal{H}\) is **double invariant** if \(S\) is invariant under both \(M\) and \(M^*\). It is then immediate from the definition of \(M^*\) that \(S\) is double invariant if and only if both \(S\) and \(S^\perp (= \mathcal{H} \ominus S)\) are invariant under \(M\), i.e., \(M(S) \subset S\), and \(M(S^\perp) \subset S^\perp\.\)
COROLLARY 6.4. Let \((R, \pi, M, \mathcal{H})\) be as described, and let \(\mathcal{B} = (\sum_{n=0}^{\infty} [M^n \mathcal{L}])\) and \(\mathcal{H}^{(\infty)} = \bigcap_{n=1}^{\infty} [M^n \mathcal{H}]\) be as in Theorem 5.2.

Then both \(\mathcal{B}\) and \(\mathcal{H}^{(\infty)}\) are double invariant under \(M\).

Proof. From the comment before the statement of the Corollary, it is enough to show that both \(\mathcal{B}\) and \(\mathcal{H}^{(\infty)}\) are invariant under \(M\). Recall from Theorem 6.2 that \(\mathcal{B} = (\mathcal{H}^{(\infty)})^\perp\), and \(\mathcal{H}^{(\infty)} = \mathcal{B}^\perp\). But it is clear from the definition of \(\mathcal{H}^{(\infty)}\) that \(M (\mathcal{H}^{(\infty)}) \subset \mathcal{H}^{(\infty)}\). Since arbitrary vectors in \(\mathcal{B}\) may be represented as \(b = \sum_{n=0}^{\infty} M^n l_n, l_n \in \mathcal{L}, \|b\|^2 = \sum_{n=0}^{\infty} \|M^n l_n\|^2\), it follows that \(Mb = \sum_{n=0}^{\infty} M^{n+1} l_n\). If \(b\) is encoded with the sequence \((l_0, l_1, l_2, \ldots), l_n \in \mathcal{L}\), then \(Mb \sim (0, l_0, l_1, \ldots), \ldots\), i.e.,

\[
M ((l_0, l_1, l_2, \ldots)) = (0, l_0, l_1, \ldots),
\]

and the \(M\)-invariance for \(\mathcal{B}\) follows from this. By the initial argument we conclude that both \(\mathcal{B}\) and \(\mathcal{H}^{(\infty)}\) are double invariant.

The advantage of the \((l_0, l_1, l_2, \ldots)\) representation of \(\mathcal{B}\) is that \(M^*|_{\mathcal{B}}\) takes an especially simple form:

COROLLARY 6.5. If vectors in \(\mathcal{B}\) are represented in the form \((l_0, l_1, l_2, \ldots)\),

\[
\sum_{n=0}^{\infty} \|l_n\|^2 < \infty, \quad l_n \in \mathcal{L} = \ker (M^*),
\]

then the action of \(M^*\) on \(\mathcal{B}\) is

\[
(l_0, l_1, l_2, \ldots) \mapsto \left( \pi (|m_0|^2) l_1, \pi \left( \left| m_0 (z^2) \right|^2 \right) l_2, \pi \left( \left| m_0 (z^4) \right|^2 \right) l_3, \ldots \right).
\]

Proof. The proof follows from the following calculation: \(M^* l_0 = 0\), and \(M^* M^n l_n = \pi (|m_0|^2) M^{n-1} l_n = M^{n-1} \pi \left( \left| m_0 \left( z^{2n-1} \right) \right|^2 \right) l_n\).

Proof of Theorem 5.2. Let the representation \(\pi_0\) of \(L^\infty (\mathbb{T})\) in \(\mathcal{L}\) be given as in Theorem 5.2. Let \(m_0, h\) be as stated at the outset, i.e., \(R_{m_0} (h) = h\). On the vectors described in (6.9), define the Hilbert-space norm, and corresponding completion, by

\[
\| (l_0)_{\infty} \|^2 := \sum_{n=0}^{\infty} \left\| \pi_0 \left( m_0^{(n)} \right) l_n \right\|^2,
\]

and define \(M\) as in (6.9). A simple computation, using the corresponding inner product

\[
\langle (l_n) | (l'_n) \rangle := \sum_{n=0}^{\infty} \left\langle \pi_0 \left( m_0^{(n)} \right) l_n | \pi_0 \left( m_0^{(n)} \right) l'_n \right\rangle,
\]

and Table 2 then yields an adjoint operator \(M^*\) which turns out to be (6.10). It is now a simple matter to verify that \(M\) is the desired sub-isometry.

REMARK 6.6. It follows from Corollary 5.3 that each projection \(Q_n\) (onto the space \([M^n \mathcal{H}]\)) commutes with \(\pi (\alpha), \alpha \in L^\infty (\mathbb{T})\), i.e., \(Q_n \pi (\alpha) = \pi (\alpha) Q_n\), but it is generally not the case that \(M\) commutes with \(\pi (\alpha)\). However, \(M\) may possibly commute with a special \(\pi (\alpha_0)\) for some \(\alpha_0 \in L^\infty (\mathbb{T})\). We have the following simple result on that.
PROPOSITION 6.7. Let \( m_0 \in L^\infty (\mathbb{T}) \) be as specified in Theorem 6.2, and let \( \pi \) be a faithful representation of \( L^\infty (\mathbb{T}) \) on a Hilbert space \( \mathcal{H} \). Let \( R \) be the Ruelle operator constructed from \( m_0 \), and let \( M \) be a given sub-isometry. Let \( \alpha_0 \in L^\infty (\mathbb{T}) \). If \( M\pi (\alpha_0) = \pi (\alpha_0) M \), then it follows that \( R (\alpha_0) = \alpha_0 \), i.e., \( \alpha_0 \) is an eigenvector for \( R \) with eigenvalue 1.

Proof. By (i) of Definition 6.1,
\[
\pi (|m_0|^2 \alpha_0) = M^* M \pi (\alpha_0) = M^* \pi (\alpha_0) M = \pi (R^* (\alpha_0)).
\]
In the first step, (i) is used on \( 1 \), then commutativity is used, and in the last step, (i) is used on \( \alpha_0 \). Since \( \pi \) is assumed faithful,
\[
|m_0|^2 \alpha_0 = R^* (\alpha_0) = |m_0 (z)|^2 \alpha_0 (z^2),
\]
and
\[
R (\alpha_0) (z) = \frac{1}{2} \sum_{w^2 = z} |m_0 (w)|^2 \alpha_0 (w)
\]
\[
= \frac{1}{2} \sum_{w^2 = z} |m_0 (w)|^2 \alpha_0 (z)
\]
\[
= \alpha_0 (z),
\]
where the quadratic property of \( m_0 \) was used in the last step. Hence \( R (\alpha_0) = \alpha_0 \) as claimed.

REMARK 6.8. The converse implication to the one given in Proposition 6.7 is not true.

Let \( m_0 (z) = \frac{1}{\sqrt{2}} (1 + z^3) \) (see also Section 4), and let \( R \) and \( M \) be the corresponding Ruelle operator and cascade operator. The scaling function \( \varphi \) realized in \( L^2 (\mathbb{R}) \) is \( \varphi = \frac{1}{3} \chi_{[0,3]} \), and a little calculation (see Section 4) shows that
\[
p_2 (\varphi) = p (\varphi, \varphi) = \frac{1}{9} (z^{-2} + 2z^{-1} + 3 + 2z + z^2),
\]
where \( z = e^{-i\omega}, \omega \in \mathbb{R} \). But \( R (p_2 (\varphi)) = p_2 (\varphi) \), so this is an eigenfunction for \( R \). Let \( \alpha_0 := p_2 (\varphi) \). We claim that \( \pi (\alpha_0) \) does not commute with \( M \). In fact, commutativity with \( M \) is equivalent to identity (6.11) from the proof of Proposition 6.7. An inspection shows that (6.11) is not satisfied in this example. In other words,
\[
|m_0 (z)|^2 \alpha_0 (z) \neq |m_0 (z)|^2 \alpha_0 (z^2).
\]
Both sides in (6.12) are polynomials, i.e., in \( \mathbb{C} [z, z^{-1}] \). The right-hand side contains a term \( z^7 \) whereas the left-hand side does not.

REMARK 6.9. The eigenvalue problem for \( R \) plays a crucial role for approximation of wavelets; see, e.g., [Str96], [CoDa96], [Vil94].
In the study of refinement operators, the sub-isometries usually have a slightly different formulation, and for the particular cascade operator $M$ in (3.8), the alternative formulation is stated in Lemma 4.1; see (4.12).

We now specialize the general formulation $(\pi, M, \mathcal{H})$ of the present section. Recall that $\pi$ was a representation of $L^\infty (\mathbb{T})$ on a Hilbert space $\mathcal{H}$, and $M: \mathcal{H} \to \mathcal{H}$ was an operator satisfying (3.1)–(3.3) of Definition 3.1 relative to some given Ruelle operator $R = R_{m_0}$. We will specialize as follows: $\mathcal{H} = \mathcal{H}_Z$ (the Hilbert space of Section 3), and

$$
(\pi_Z (\alpha) H (z, x)) = \alpha (z) H (z, x), \quad \alpha \in L^\infty (\mathbb{T}), \, H \in \mathcal{H}_Z.
$$

In this specialized setup, we then have the following.

**PROPOSITION 6.1.** Let the pair $(\pi_Z, \mathcal{H}_Z)$ be as described, and let $M$ be an operator in $\mathcal{H}_Z$. Then the following two conditions are equivalent.

(i) For a.e. $z$ in $\mathbb{T}$, we have the identity

$$
\langle MH_1 (z) | MH_2 (z) \rangle_{L^2(0,1)} = R \left( \langle H_1 (\cdot) | H_2 (\cdot) \rangle_{L^2(0,1)} \right) (z)
$$

for all $H_1, H_2 \in \mathcal{H}_Z$.

(ii) $M^* \pi_Z (\alpha) M = \pi_Z (R^* \alpha)$ for all $\alpha \in L^\infty (\mathbb{T})$.

Here the Ruelle operator is defined from an arbitrary $m_0$ as usual.

**Proof.** (i) $\Rightarrow$ (ii). Let $H_1, H_2 \in \mathcal{H}_Z$. Then

$$
\langle H_1 | (M^* \pi_Z (\alpha) M) H_2 \rangle_{\mathcal{H}_Z} = \int_\mathbb{T} \langle MH_1 (z) | MH_2 (z) \rangle \alpha (z) \, d\mu (z)
$$

$$
= \int_\mathbb{T} \left( \int \langle H_1 (\cdot) | H_2 (\cdot) \rangle \alpha (z) \, d\mu (z) \right) \, d\mu (z)
$$

$$
= \int \langle H_1 (z) | H_2 (z) \rangle |m_0 (z)|^2 \alpha (z^2) \, d\mu (z)
$$

$$
= \langle H_1 | \pi_Z (R^* (\alpha)) H_2 \rangle_{\mathcal{H}_Z},
$$

and this proves (ii).

(ii) $\Rightarrow$ (i). If (ii) holds, the calculation shows that the second and the third terms must agree for all $\alpha \in L^\infty (\mathbb{T})$, and, by duality, this means that (i) must hold a.e. on $\mathbb{T}$. \qed

7. **SINGULAR CASCADE APPROXIMATIONS**

We return in this section to the cascade operator $M$ from Section 4, but it will be convenient to state the results for the Hilbert space $\mathcal{H}_Z$ of Section 3. We will also need the representation $\pi_Z$ of (3.12). From the dictionary in Lemma 5.1 (Table 3) we note that $\mathbb{Z}$-translations in $L^2 (\mathbb{R})$, $h (\cdot) \mapsto h (\cdot + n)$, $n \in \mathbb{Z}$, correspond to $\pi_Z (e_n)$ in $\mathcal{H}_Z$ where $e_n (z) = z^n$. Also we need the fact that the operation $h \mapsto \alpha * h$ on $L^2 (\mathbb{R})$, $h \in L^2 (\mathbb{R})$, $\alpha \in L^\infty (\mathbb{T})$, corresponds to multiplication $\alpha (e^{-i\omega \cdot}) \hat{h} (\omega)$ where $\hat{h} (\omega) = \int_\mathbb{R} e^{-i\omega x} h (x) \, dx$. Hence the result when stated in $\mathcal{H}_Z$ can easily be translated to either $L^2 (\mathbb{R})$ or $\tilde{L}^2 (\mathbb{R})$. 

Let $H_i \in \mathcal{H}_Z$. We shall need the sesquilinear form $p$ from Section 3,
\[ p(H_1, H_2)(z) = \int_0^1 \overline{H_1(z, x)}H_2(z, x) \, dx = \langle H_1(z) \mid H_2(z) \rangle_{L^2(0,1)}. \]
If $H = H_1 = H_2$, we introduce the abbreviation $p_2(H) = p(H, H)$. An important property of a starting vector $h \in L^2(\mathbb{R})$ for the cascade algorithm is orthogonality of the translates \(\{h(\cdot - n) \mid n \in \mathbb{Z}\}\), i.e.,
\[ \int_{-\infty}^{\infty} h(x - n)h(x) \, dx = \delta_{n,0} \|h\|_2^2. \]
(7.1)
The following is immediate from Proposition 4.4 (Section 4):
\begin{lemma}
Let $h \in L^2(\mathbb{R})$, and set $H = Zh$. The following are equivalent:
\begin{enumerate}[(i)]
\item $h$ satisfies the orthogonality condition (7.1); and
\item $p_2(H)(z) \equiv \|H\|_{\mathcal{H}_Z}^2$ a.e. on $T$.
\end{enumerate}
We shall refer to this as the orthogonality condition, meaning orthogonal $\mathbb{Z}$-translates in $L^2(\mathbb{R})$.
\end{lemma}
\begin{remark}
We note similarly that when the Strang–Fix condition
\[ \sum_{n \in \mathbb{Z}} h(x + n) \equiv 1 \]
(7.2)
makes sense for $h \in L^2(\mathbb{R})$, e.g., if $h$ is of compact support, then this corresponds to the following condition on $H = Zh$:
\[ H(z = 1, x) \equiv 1 \quad \text{a.a. } x \in [0, 1]. \]
(7.3)
It is well known that the cascade approximation must start with the orthogonality condition of Lemma 7.1. Suppose $H \in \mathcal{H}_Z$ satisfies (7.3), and
\[ H = H_B + H_\infty \]
(7.4)
is the Wold decomposition from (6.6) in Theorem 6.2, i.e.,
\[ H_B \in \mathcal{B} = \sum_{n=0}^{\infty} [M^n \mathcal{L}], \]
and
\[ H_\infty \in \mathcal{H}^{(\infty)} = \bigcap_{n=1}^{\infty} [M^n \mathcal{H}] \]
relative to a fixed $(R, \pi)$-isometry $M$, where $R = R_{m_0}$, and $m_0$ is given. Then, if $H$ satisfies the orthogonality (7.1), or equivalently (7.3) in Lemma 7.4, then one of the two, $H_B$ or $H_\infty$, can satisfy the same only if the other is zero. This follows from the following formula for the norm in $\mathcal{H} = \mathcal{H}_Z$:
\[ \|H\|_{\mathcal{H}_Z}^2 = \int_T p_2(H)(z) \, d\mu(z). \]
(7.5)
Applied to the decomposition \( H = H_B + H_\infty \) in (7.4), we get
\[
\int_T p_2(H)(z) \, d\mu(z) = \int T p_2(H_B)(z) \, d\mu(z) + \int T p_2(H_\infty)(z) \, d\mu(z),
\]
from which the claim is clear. If \( H \) is orthogonal, then
\[
1 = \|H\|^2 = \int T p_2(H)(z) \, d\mu(z),
\]
so if for example \( p_2(H_B) \equiv 1 \), then
\[
1 = 1 + \int T p_2(H_\infty)(z) \, d\mu(z),
\]
so \( H_\infty = 0 \).

While Remark 7.2 above shows that the \( B \)-decomposition is an obstruction to cascade approximation, it is still the case that solutions \( H \) to \( M(H) = H \) (i.e., \( H = Z\varphi \) for some scaling function \( \varphi \in L^2(\mathbb{R}) \)) yield

**PROPOSITION 7.3.** Let \( m_0 \) be a low-pass filter, and let \( R \) be the corresponding Ruelle operator. Let \( \pi_Z \) be the representation (3.12) or (4.22) of \( L^\infty(T) \) in \( \mathcal{H}_Z \).

(a) Let \( M \) be a sub-isometry. Then
\[
R(p_2(H)) = p_2(MH),
\]
so if \( M(H) = H \), then \( p = p_2(H) \) solves
\[
R(p) = p.
\]
(b) In general, the Fourier series for \( p_2(H) \) is
\[
p_2(H)(z) = \sum_{n \in \mathbb{Z}} z^n \langle \pi(e_n) H \| H \rangle_{\mathcal{H}_Z},
\]
where \( e_n(z) = z^n \), and
\[
\|H\|_{\mathcal{H}_Z}^2 = \sum_{n \in \mathbb{Z}} |\langle \pi(e_n) H \| H \rangle|^2.
\]

**Proof.** Immediate from Lemma 7.1 and Proposition 6.10 in the previous section. The calculation of the expansion (7.9) follows from checking that the Fourier coefficients of \( p_2(H)(z) \) are as stated, i.e., that
\[
\int_T z^{-n} p_2(H)(z) \, d\mu(z) = \int_T p(\pi_Z(e_n) H, H)(z) \, d\mu(z) = \langle \pi_Z(e_n) H \| H \rangle_{\mathcal{H}_Z}.
\]

We show next that \( \lambda = 1 \) is the only point on the unit circle which is an eigenvalue of \( R \), at least when we restrict to continuous eigenfunctions.

**THEOREM 7.4.** Let \( m_0 \) be a continuous low-pass filter, and let \( R \) be the corresponding Ruelle operator in \( L^\infty(T) \). Suppose that the eigenspace \( \{ \xi \in C(T) \mid R(\xi) = \xi \} \) is one-dimensional, i.e., \( \mathbb{C} \xi_i \). Then if \( |\lambda| \geq 1 \), and \( \lambda \neq 1 \), the eigenvalue problem
\[
R(\alpha) = \lambda \alpha
\]
has no nonzero solution in \( C(T) \).
Proof. First note that if \( m_0 \) is assumed continuous, then \( R \) maps \( C(\mathbb{T}) \) into itself. We see this by first approximating \( m_0 \) with finite sums \( \sum_k a_k z^k \). For each such finite sum, the corresponding \( R \) maps \( \mathbb{C}[z, z^{-1}] \) (= the ring of finite Fourier series) into itself. Since the norm of \( R \), as an operator in \( L^\infty(\mathbb{T}) \), is one, i.e., \( \|R\|_{\infty,\infty} = 1 \), we conclude that \( R \) maps the norm closure of \( \mathbb{C}[z, z^{-1}] \) into itself. This norm-closure is \( C(\mathbb{T}) \) by the Stone–Weierstrass theorem.

Since \( R \) leaves \( C(\mathbb{T}) \) invariant, it dualizes to a map on the measures \( M(\mathbb{T}) = C(\mathbb{T})^* \), and we claim that \( R^* (\delta_1) = \delta_1 \) where \( \delta_1 \) denotes the Dirac point measure at 1. Indeed, let \( \xi \in C(\mathbb{T}) \); then

\[
(R\xi) (1) = \frac{1}{2} \sum_{w^2 = 1} |m_0(w)|^2 \xi(w) = \frac{1}{2} (|m_0(1)|^2 \xi(1) + |m_0(-1)|^2 \xi(-1)) = \xi(1)
\]

since \( m_0(1) = \sqrt{2} \) and \( m_0(-1) = 0 \). (This is the low-pass property.) It follows that \( \alpha(1) = 0 \) when \( \alpha \) is the eigenfunction in (7.10).

Suppose we did have a solution \( \alpha \) to (7.10) as stated. Let \( c \in \mathbb{C} \), and set \( \beta_c = 1 + c \alpha \). Then \( \beta_c(1) = 1 \). Pick a scaling function \( H_0 \) for the cascade operator \( M \) corresponding to \( m_0 \), i.e., \( M = M_{m_0} \), and \( M(H_0) = H_0 \). Since \( E_1 = \{ \xi \in C(\mathbb{T}) \mid R\xi = \xi \} \) is one-dimensional, and \( p_2(H_0) \in E_1 \) by Lemma 5.1, we conclude that \( p_2(H_0) = C1 \). Lemma 7.1(i) then implies that \( C = 1 \), so \( p_2(H_0) = 1 \). We then have

\[
(7.11) \quad R^n(\beta_c)(z) = R^n(\beta_c p_2(H_0))(z) = R^n(p(H_0, \pi(\beta_c) H_0))(z)
\]

using the argument from the proof of Proposition 7.3. The left-hand side of (7.11) is \( 1 + c \lambda^n \alpha \), which is not convergent when \( c \neq 0, \lambda \neq 1, || \lambda | \geq 1 \), and \( \alpha \neq 0 \) in \( L^\infty(\mathbb{T}) \).

On the right-hand side in (7.11), some more analysis is needed. Our next claim is that

\[
\lim_{n \to \infty} \left( M^n \pi_Z(\beta) H_0 \right)(z, x) = \beta(1) H_0(z, x)
\]

whenever \( \beta \in C(\mathbb{T}) \), and the limit is in \( \mathcal{H}_Z \), or in \( L^2(\mathbb{R}) \) after a translation of the result via the Zak transform. Using Lemma 5.1, we get

\[
M^n \pi_Z(\beta) H_0(z, \cdot) = 2^{-\frac{n}{2}} \sum_{w^{2^n} = z} m_0(w) m_0(w^2) \cdots m_0(w^{2^{n-1}}) \beta(w) H_0(w, 2^n x).
\]

Let \( m_0^{(n)}(w) := m_0(w) m_0(w^2) \cdots m_0(w^{2^{n-1}}) \). Since \( M(H_0) = H_0 \), we get for the difference (picking \( \beta \) such that \( \beta(1) = 1 \))

\[
H_0 - M^n \pi_Z(\beta) H_0 = 2^{-\frac{n}{2}} \sum_{w^{2^n} = z} m_0^{(n)}(w) \cdot (1 - \beta(w)) H(w, 2^n x).
\]

The \( \mathcal{H}_Z \)-norm is that of \( L^2(\mathbb{T} \times [0, 1]) \); we split up the integral \( \frac{1}{2\pi} \int_{-\pi}^{\pi} \cdots d\omega \) over \( \mathbb{T} \) into two regions, one \( -\delta \leq \omega \leq \delta \), and the other the union of the intervals \( -\pi \leq \omega \leq -\delta \) and \( \delta \leq \omega \leq \pi \). We pick \( \delta \) such that \( |1 - \beta(e^{-i\omega})| \leq \varepsilon \) when \( |\omega| \leq \delta \). After estimating the two
separate contributions, and using Lemma 4.3 and Corollary 4.3, the desired result follows. When it is applied to the right-hand side in (7.11), we get

\[ p(H_0, M^n \pi_Z (\beta_c) H_0)(z) \rightarrow_{n \to \infty} \beta_c(1) p_2(H_0)(z) = p_2(H_0)(z) \]

in the \( L^1(\mathbb{T}) \)-norm. But \( \beta_c(1) = 1 \). Comparing the two results for the limits of the left- and right-hand sides of (7.11), we arrive at the desired contradiction, and conclude that the eigenvalue problem (7.10) does not have eigenvectors as stated.

8. SINGULAR VECTORS

We now return to the subspace \( B \) of the Wold decomposition in Theorem 6.2. But we will specialize to the Hilbert space \( \mathcal{H} = \mathcal{H}_Z \), although the operators on \( \mathcal{H}_Z \) correspond to (unitarily equivalent) versions, on \( L^2(\mathbb{R}) \) and \( \hat{L}^2(\mathbb{R}) \), via the inverse Zak transform \( Z^* \) and the Fourier transform, respectively.

The object is to understand when the singular space \( B \) is present in the decomposition \( \mathcal{H}_Z = B \oplus \mathcal{H}^{(\infty)} \). This is important, as we showed in the previous section that the cascade approximation picks up divergences when \( B \neq 0 \). Since

\[ B = \sum_{n=0}^{\infty} [M^n \mathcal{L}], \]

where \( \mathcal{L} = \ker(M^*) \), we see that the question is decided by the question of when \( \mathcal{L} \neq 0 \). Hence we need to study \( M^* \) more closely, and the second relation (1) from Theorem 3.2 is crucial for that. While there is some connection between the results in this section and those of [DaLa], there are several differences as well: the present approach is general and applies equally well to higher dimensions, matrix dilations, general \( N \)-to-1 maps (onto) in metric spaces, and, for the wavelets, even to the case when the Hilbert space is different from the standard one, i.e., \( L^2(\mathbb{R}) \).

Let \( h \in L^2(\mathbb{R}) \), and set \( \hat{h}(\omega) = \int_{\mathbb{R}} e^{-i\omega x} h(x) \, dx \). The inverse Fourier transform will be denoted

\[ f^\vee(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega x} f(\omega) \, d\omega. \]

If \( S \subseteq L^2(\mathbb{R}) \) is a closed subspace, we set

\[ S^\vee = \{ f^\vee \mid f \in S \}. \]

If \( E \subseteq \mathbb{R} \) is a measurable subset, consider the projection \( P_E \) given by

\[ (P_E f)(\omega) = \chi_E(\omega) f(\omega), \quad \omega \in \mathbb{R}, \]

and define

\[ L^2(E) := P_E(L^2(\mathbb{R})). \]

We now turn to the closed subspaces in the decomposition (6.4). Considering the \( L^2(\mathbb{R}) \) model, we show that there are measurable subsets \( E_k(m_0) \subseteq \mathbb{R} \) such that

\[ [M^k \mathcal{L}] = L^2(E_k(m_0))^\vee, \quad \text{the } L^2(\mathbb{R}) \text{ picture}, \, k = 0, 1, \ldots, \]
where \(( \cdot )^\vee\) denotes the inverse Fourier transform in \(L^2(\mathbb{R})\). In fact it follows from conclusion (i) of Theorem 6.2, i.e., our Wold-type decomposition theorem (Section 6) that the spaces \([M^kE] \subset L^2(\mathbb{R})\) must have the stated form (8.4), but the object in the present section is to find the sets. To see this, recall the formula

\[
(\pi(\xi)h)\hat{\omega} = \xi(e^{-i\omega})\hat{h}(\omega), \quad \xi \in L^\infty(\mathbb{T}), \ h \in L^2(\mathbb{R}), \ \omega \in \mathbb{R},
\]

for the \(L^\infty(\mathbb{T})\)-representation \(\pi\). Using Theorem 6.2 a second time, we conclude that there is also a measurable \(E^\infty(m_0) \subset \mathbb{R}\) such that \(H^{(\infty)} = (L^2(E^{(\infty)}(m_0)))^\vee\). Using finally the orthogonality part of the conclusion in Theorem 6.2, we note that

\[
\bigcup_{k=0}^\infty E_k(m_0) \cup E^\infty(m_0) = \mathbb{R},
\]

which is to say that they form a tiling of \(\mathbb{R}\). Since the spaces in the decomposition are mutually orthogonal, the sets \(E_k(m_0), k = 0, 1, \ldots, k = \infty\), must be pairwise non-overlapping up to measure zero in \(\mathbb{R}\), i.e.,

\[
E_k(m_0) \cap E_l(m_0)
\]

has Lebesgue measure zero when \(k \neq l\).

When the measurable subset \(E \subset \mathbb{R}\) is given, we use the notation \(2E\) for

\[
2E = \{2\omega \mid \omega \in E\},
\]

and similarly, for \(a \in \mathbb{R}\), set

\[
E + a = \{\omega + a \mid \omega \in E\}.
\]

The mapping \(\omega \mapsto e^{-i\omega}\) is a measurable bijection of \([\pi, \pi]\) onto \(\mathbb{T}\). Let \(c_0\) be its inverse (also measurable). If \(\log\) denotes the corresponding principal branch of the complex logarithm, then

\[
c_0(z) = i\log(z), \quad z \in \mathbb{T}.
\]

Let \(m_0\) be a subband filter satisfying conditions (i)–(iii) in the Introduction, and let

\[
N(m_0) := \{z \in \mathbb{T} \mid m_0(z) = 0\}.
\]

Let \(M\) denote the corresponding cascade operator, i.e.,

\[
(Mh)(x) = \sqrt{2} \sum_{n \in \mathbb{Z}} a_nh(2x - n), \quad h \in L^2(\mathbb{R}),
\]

where \(m_0(z) = \sum_{n \in \mathbb{Z}} a_nz^n\). Finally, let \(E(m_0)\) be given by

\[
E(m_0) = \bigcup_{n \in \mathbb{Z}} 2c_0(N(m_0)) + 4\pi n.
\]

**Lemma 8.1.**

\[
\ker(M^*) = L^2(E(m_0))^\vee = \{f^\vee \mid f \in L^2(E(m_0))\}.
\]
Proof. We have the formula
\[(M^* h)(\omega) = m_0(e^{-i\omega}) \cdot \hat{h}(2\omega), \quad \omega \in \mathbb{R},\]
directly from a Fourier transform of $M^*$. It follows that $\hat{h}(2\omega) = 0$ if $z = e^{-i\omega} \in \mathbb{T \setminus N(m_0)}$. The lemma now follows from (8.12) and (8.13). \hfill \Box

An immediate consequence of this lemma and (8.13) is the following result:

PROPOSITION 8.2. We have the equivalence
\[\ker(M^*) = \{0\} \iff \mu(N(m_0)) = 0,\]
where $\mu$ denotes the Haar measure on $\mathbb{T}$.

We shall need the following second consequence of the lemma:

PROPOSITION 8.3. If $h \in \ker(M^*)$, then $\hat{h} \equiv 0$ on $E(m_0) + 2\pi$.

Proof. We have
\[E(m_0) + 2\pi = \bigcup_{n \in \mathbb{Z}} 2 \cdot (c_0(N(m_0)) + \pi) + 4\pi n = \bigcup_{n \in \mathbb{Z}} 2 \cdot c_0(-N(m_0)) + 4\pi n.\]
But if $z \in N(m_0)$, then $|m_0(-z)| = \sqrt{2}$ (from property (iii) of $m_0$ in the Introduction), so $-z \in \mathbb{T \setminus N(m_0)}$. Formula (8.12) then implies that $\hat{h}(\omega + 2\pi) = 0$ if $\omega \in E(m_0)$, which is the desired conclusion. \hfill \Box

We now turn to $\ker(M^{*k})$, $k > 1$. We find measurable sets $F_k(m_0) \subset \mathbb{R}$ such that
\[\ker(M^{*k}) = (L^2(F_k(m_0)))^\perp.\]
Let $m_0^{(k)}(z) := m_0(z) m_0(z^2) \cdots m_0(z^{2^k-1})$. Then
\[N(m_0^{(k)}) = \left\{ z \in \mathbb{T} \mid m_0^{(k)}(z) = 0 \right\} = N(m_0) \cup \left\{ z \mid z^2 \in N(m_0) \right\} \cup \cdots \cup \left\{ z \mid z^{2^k-1} \in N(m_0) \right\}.\]
By Claim 4 in the proof of Theorem 6.2, we have
\[\ker(M^{*k+1}) \ominus \ker(M^{*k}) = [M^kL],\]
so we will get the components $[M^kL]$ from the following lemma.

LEMMA 8.4. The sets $F_k(m_0)$ from formula (8.14) for $\ker(M^{*k})$ are
\[F_k(m_0) = \bigcup_{n \in \mathbb{Z}} 2^k \cdot c_0(N(m_0^{(k)})) + 2^{k+1} \cdot \pi n.\]

Proof. This is the same argument as the one used in Lemma 8.1 above, and it is based on
\[(M^{*k}h)(\omega) = m_0^{(k)}(e^{-i\omega}) \cdot \hat{h}(2^k \cdot \omega), \quad h \in L^2(\mathbb{R}), \quad \omega \in \mathbb{R}.\]
In using formula (8.14), the following observation on $N\left( m_0^{(k)} \right)$ is useful. Let $\sigma (z) = z^2$ be the square map of $\mathbb{T}$, and let $N \subset \mathbb{T}$ be a subset. Let

$$
\sigma^{-1} (N) = \{ z \in \mathbb{T} \mid \sigma (z) \in N \}.
$$

To understand the dynamical picture of $\sigma$ and its multivalued inverse $\sigma^{-1}$, i.e., their iterations, we have included a graphical illustration in Figures 1 and 2. Then

$$
N \left( m_0^{(n)} \right) = N \left( m_0 \right) \cup \sigma^{-1} (N \left( m_0 \right)) \cup \cdots \cup \sigma^{-(n-1)} (N \left( m_0 \right))
$$

and

$$
N \left( m_0^{(n+1)} \right) \setminus N \left( m_0^{(n)} \right) = \sigma^{-n} (N \left( m_0 \right)) = \{ z \in \mathbb{T} \mid z^{2^n} \in N \left( m_0 \right) \}.
$$

For the example which follows (Example 8.6 below), we will have

$$
N \left( m_0 \right) = \left\{ e^{-i\omega} \mid \frac{\pi}{2} \leq |\omega| \leq \pi \right\},
$$

$$
\sigma^{-1} (N \left( m_0 \right)) = \left\{ e^{-i\omega} \mid \frac{\pi}{4} \leq |\omega| \leq \frac{3\pi}{4} \right\}, \quad \text{and}
$$

$$
\sigma^{-2} (N \left( m_0 \right)) = \left\{ e^{-i\omega} \mid \frac{\pi}{8} \leq |\omega| \leq \frac{3\pi}{8} \right\} \cup \left\{ e^{-i\omega} \mid \frac{5\pi}{8} \leq |\omega| \leq \frac{7\pi}{8} \right\},
$$

the last made up of four arc segments on the unit circle (see Figure 2); and the induction is clear for the general case $\sigma^{-n} (N \left( m_0 \right))$. 

**Figure 1. $\sigma^{-2} (N \left( m_0 \right))$**
When Lemma 8.4 is combined with formula (8.16), we get
\[ [M^k \mathcal{L}] = (L^2 (F_{k+1} (m_0) \setminus F_k (m_0)))^\vee, \]
again with \((\cdot)^\vee\) denoting inverse Fourier transform. We now combine the results above into a proposition.

**PROPOSITION 8.5.** We have
\[ [M^k \mathcal{L}] = L^2 (E_k (m_0))^\vee, \]
where
\[ E_k (m_0) = F_{k+1} (m_0) \setminus F_k (m_0), \]
starting with
\[ E_0 (m_0) = F_1 (m_0) = E (m_0) = \bigcup_{n \in \mathbb{Z}} 2c_0 (N (m_0)) + 4\pi n. \]

*Proof.* Contained in the previous argument. \(\square\)

**EXAMPLE 8.6.** It is known that the function \(\varphi \in L^2 (\mathbb{R})\) given by
\[ \varphi (x) = \chi_{[-\pi,\pi]} (x) = \frac{\sin \pi x}{\pi x} \]
is a scaling function. The subband filter \(m_0\) is
\[ m_0 (e^{-i\omega}) = \sqrt{2} \chi_{[-\pi/2,\pi/2]} (\omega). \]
Hence
\[ c_0 (N (m_0)) = \left[ -\pi, -\frac{\pi}{2} \right) \cup \left[ \frac{\pi}{2}, \pi \right) \]
and
\[ \mathcal{L} = \ker (M^*) = L^2 (E (m_0))^\vee. \]
The corresponding quadratic expression, \( p \) in \( L \), while which in addition to \( \hat{\varphi} \) takes the form

\[
E(m_0) = \bigcup_{n \in \mathbb{Z}} [\pi, 3\pi) + 4\pi n.
\]  

Writing \( m_0 \left( \frac{\omega}{2} + \pi \right) = m_0 \left( e^{-i\pi} \right) \), we see that

\[
m_0 \left( \frac{\omega}{2} + \pi \right) = \sqrt{2} \chi_{E(m_0)}(\omega) \quad \text{and} \quad m_0 \left( \frac{\omega}{2} \right) = \sqrt{2} \chi_{G(m_0)}(\omega),
\]

where \( G(m_0) = \bigcup_{n \in \mathbb{Z}} [-\pi, \pi) + 4\pi n \). We have \( E(m_0) \cup G(m_0) = \mathbb{R} \), and

\[
G(m_0) = E(m_0) + 2\pi,
\]

which yields the following formula for the Ruelle operator:

\[
(R\xi)(\omega) = \chi_{G(m_0)}(\omega) \xi \left( \frac{\omega}{2} \right) + \chi_{E(m_0)}(\omega) \xi \left( \frac{\omega}{2} + \pi \right), \quad \xi \in L^\infty(\mathbb{T}), \ \omega \in \mathbb{R},
\]

where we use \( \mathbb{T} \simeq \mathbb{R} / 2\pi \mathbb{Z} \). It is easy to check directly from (8.22) that

\[
(R\xi)(\omega + 2\pi) = (R\xi)(\omega), \quad \text{for all } \omega \in \mathbb{R}.
\]

Similarly the scaling identity for \( \varphi \), i.e.,

\[
\sqrt{2} \hat{\varphi}(\omega) = m_0 \left( e^{-i\pi/2} \right) \hat{\varphi} \left( \frac{\omega}{2} \right),
\]

takes the form

\[
\hat{\varphi}(\omega) = \chi_{G(m_0)}(\omega) \hat{\varphi} \left( \frac{\omega}{2} \right), \quad \omega \in \mathbb{R},
\]

which in addition to \( \hat{\varphi}_1 = \chi_{[-\pi,\pi]} \) has the solution \( \hat{\varphi}_2 = \chi_{G(m_0)} \), but, of course, \( \chi_{G(m_0)} \) is not in \( L^2(\mathbb{R}) \). Nonetheless, the inverse transform, \( \varphi_2 = \chi_{G(m_0)} \), makes sense as a distribution. The corresponding quadratic expression, \( p_2(\varphi) = p(\varphi, \varphi) \), satisfies

\[
p_2(\varphi_1)(e^{-i\omega}) \equiv 1,
\]

while

\[
p_2(\varphi_2)(e^{-i\omega}) = \sum_{n \in \mathbb{Z}} \chi_{G(m_0)}(\omega + 2\pi n) \equiv \infty.
\]

Since

\[
m_0^{(k)}(\omega) = 2^{k} \chi_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(\omega) \chi_{[-\frac{\pi}{2}, \frac{\pi}{2}])(2\omega) \cdots \chi_{[-\frac{\pi}{2}, \frac{\pi}{2}])(2^{k-1}\omega), \quad \omega \in \mathbb{R},
\]

where \( [\omega] := c_0(e^{-i\omega}) \), we see that the higher cases result from an iteration of an ergodic map on \( \mathbb{T} \) as follows: \( c_0(z) \mapsto c_0(z^2) \), or \( \omega \mapsto 2\omega \mod 2\pi \). Starting with

\[
c_0(N(m_0)) = \left[ -\pi, -\frac{\pi}{2} \right] \cup \left[ \frac{\pi}{2}, \pi \right],
\]

we get

\[
c_0(N(m_0^{(2)})) = \left[ -\pi, -\frac{\pi}{4} \right] \cup \left[ \frac{\pi}{4}, \pi \right],
\]

and by induction

\[
c_0(N(m_0^{(k)})) = \left[ -\pi, -\frac{\pi}{2^k} \right] \cup \left[ \frac{\pi}{2^k}, \pi \right]
\]
(see (8.3) and Figure 2 for the map $c_0$), and

$$F_k \left( m_0^{(k)} \right) = \bigcup_{n \in \mathbb{Z}} \left[ -2^k \pi, -\pi \right] \cup \left[ \pi, 2^k \pi \right] + 2^{k+1} \pi n,$$

$$[M^k \mathcal{L}] = L^2 (E_k (m_0))^\vee,$$

where

$$E_k (m_0) = F_{k+1} (m_0) \setminus F_k (m_0),$$

starting with

$$E_0 (m_0) = E (m_0) = F_1 (m_0) = \bigcup_{n \in \mathbb{Z}} \left[ \pi, 3\pi \right] + 4\pi n$$

and

$$E_1 (m_0) = F_2 (m_0) \setminus F_1 (m_0) = \bigcup_{n \in \mathbb{Z}} \left[ 2\pi, 6\pi \right] + 8\pi n.$$

**Remark 8.7.** Example 8.6 is a special case of a classification from [BrJo97] of filters $m_0$ for which $|m_0|$ takes on only the two values $\sqrt{2}$ and 0. However, the present analysis is focused on convergence questions which are not addressed there.

There is one more feature which sets this class of examples apart from those where $\mathcal{L} = \ker (M^*) = 0$, i.e., those for which the complement of the support of $m_0$ in $T$ has measure zero (see Proposition 8.2). When $\ker (M^*) \neq 0$, then

$$(8.23) \quad \mathcal{E}_1 = \{ \xi \in L^\infty (T) \mid R (\xi) = \xi \}$$

may be infinite-dimensional.

In Example 8.6, recall $\mathcal{H}^{(\infty)} = L^2 (-\pi, \pi)^\vee$, and

$$(8.24) \quad p_2 (f) (e^{-i\omega}) = \chi_{[-\pi,\pi]} (\omega) \left| \hat{f} (\omega) \right|^2$$

$$= P_{[-\pi,\pi]} (|\hat{f}|^2) (\omega), \quad f \in \mathcal{H}^{(\infty)}, \ \omega \in \mathbb{R}.$$

Since $M \varphi = \varphi$, it is clear that $\varphi \in \mathcal{H}^{(\infty)}$, where, in this case,

$$\mathcal{H}^{(\infty)} = \bigcap_{n=1}^{\infty} M^n \left( L^2 (\mathbb{R}) \right) = L^2 (\mathbb{R}) \ominus \left\{ M^k \mathcal{L} \mid k = 0, 1, \ldots \right\},$$

where $\mathcal{L} := \ker M^*$. Since $\mathcal{H}^{(\infty)} = L^2 (E_\infty)^\vee$, it follows that $[-\pi, \pi] \subset E_\infty$, or equivalently $L^2 (-\pi, \pi)^\vee \subset L^2 (E_\infty)^\vee$. But if iteration of (8.13) for $(M^* h)^\wedge$ is combined with the above analysis of the example, we conclude that, in fact, the inclusion is equality, i.e., $[-\pi, \pi] = E_\infty$, up to Lebesgue measure zero in $\mathbb{R}$, and, therefore, $\mathcal{H}^{(\infty)} = L^2 (-\pi, \pi)^\vee$. The main ingredient in this argument is the known ergodicity of the map $z \mapsto z^2$ of $T$; see Figure 2, and [Kea72] for details on $N$-to-1 endomorphisms of measure space.

We now consider the cascade approximation $\lim_{n \to \infty} M^nh$ in $L^2 (\mathbb{R})$. The object is to pick $h$ such that the limit is the scaling function $\varphi$. By Theorem 8.2, the optimal choice dictates $h \in \mathcal{H}^{(\infty)}$, so that excludes (for the example) the standard choice for starting point.
of cascades $h \rightarrow Mh \rightarrow M^2h \rightarrow \cdots$, which is $h = \chi_{(0,1)}$. Clearly $\hat{h}(\omega) = \chi_{(0,1)}(\omega) = e^{-i\frac{\omega}{2}\sin(\frac{\omega}{2})}$ is not in $L^2(-\pi, \pi)$, so $\|M^n\chi_{(0,1)} - \varphi\|_{L^2(\mathbb{R})} \rightarrow 0$ as $n \rightarrow \infty$ is at best a slow approximation.

Let $h = h_B + h_\infty$ be the $B \oplus H^{(\infty)}$ decomposition of Theorem 6.2. From Corollary 6.4 we have $M^n h_B \in B$ and $M^n h_\infty \in H^{(\infty)}$, so

$$M^n h - \varphi = M^n h_\infty - \varphi + \overbrace{M^n h_B}^{H^{(\infty)}}$$

and

$$\tag{8.25} \|M^n h - \varphi\|^2 = \|M^n h_\infty - \varphi\|^2 + \|M^n h_B\|^2.$$  

**PROPOSITION 8.8.** Let $m_0(e^{-i\omega}) = \sqrt{2}X_{[-\frac{\pi}{2}, \frac{\pi}{2}]}([\omega])$ as a function on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and let $R, M$ be the respective Ruelle and cascade operators. Let $\varphi = \chi_{[-\pi, \pi]}^\vee$ and $h = \chi_{(0,1)}$. Then $M \varphi = \varphi$ and $h = h_B + h_\infty$, where

$$\hat{h}_B(\omega) = \chi_{R\setminus[-\pi, \pi]}(\omega) e^{-i\frac{\omega}{2}\sin(\frac{\omega}{2})} \quad \text{and} \quad \hat{h}_\infty(\omega) = \chi_{[-\pi, \pi]}(\omega) e^{-i\frac{\omega}{2}\sin(\frac{\omega}{2})},$$

with the approximations:

(i) $\|M^n h_\infty - \varphi\|_2 \xrightarrow{n \to \infty} 0$, and

(ii) $\|M^n h_B\|_2 \xrightarrow{n \to \infty} 0$.

**Proof.** Note that, when (i) and (ii) are combined with (8.25), we get $\|M^n h - \varphi\|_2 \xrightarrow{n \to \infty} 0$, but (i) is a better approximation.

We first prove (ii) by checking that

(a) $\|M^n h_\infty\|_2 \xrightarrow{n \to \infty} 1$, and

(b) $\langle M^n h_\infty | \varphi \rangle \xrightarrow{n \to \infty} 1$.

For the first term (ii) we have

$$\tag{8.26} \|M^n h_\infty\|_2^2 = \int_{\mathbb{T}} R^n (p_2(h_\infty))(z) \, d\mu(z),$$

where

$$p_2(h_\infty)(e^{-i\omega}) = \sum_{n \in \mathbb{Z}} |\hat{h}_\infty(\omega + 2\pi n)|^2.$$  

We take $-\pi \leq \omega < \pi$, and recall the formula $\hat{h}_\infty(\omega) = \chi_{[-\pi, \pi]}(\omega) e^{-i\frac{\omega}{2}\sin(\frac{\omega}{2})}$. Hence $p_2(h_\infty)(e^{-i\omega}) = \left|\sin\left(\frac{\omega}{2}\right)\right|^2$. Since this function is continuous, it follows from a theorem of Meyer and Paiva [MePa93] that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} R^n (p_2(h_\infty))(e^{-i\omega}) \, d\omega \xrightarrow{n \to \infty} p_2(h_\infty)(\omega = 0) = 1.$$  

In view of (8.26), this proves (ii). The application of the Meyer–Paiva theorem (see [MePa93] and line 3 in Table 2 above) requires that $p_2(\varphi) \equiv 1$ on $\mathbb{T}$, which clearly holds for the present $\varphi.$
The argument for (b) is similar:
\[
\langle M^n h_\infty \mid \varphi \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{T}} R^n (p (h_\infty, \varphi)) (z) \, d\mu (z) \xrightarrow{n \to \infty} p (h_\infty, \varphi) (z = 1) = 1.
\]
The last step is based on the formula (|ω| < π)
\[
p (h_\infty, \varphi) (e^{-i\omega}) = \sum_{n \in \mathbb{Z}} \hat{h}_\infty (\omega + 2\pi n) \hat{\varphi} (\omega + 2\pi n)
= (n = 0 \text{ term}) = \hat{\varphi} (\omega) = e^{-i\frac{\omega}{2}} \sin (\frac{\omega}{2}).
\]
It remains to prove (ii). We have
\[
\| M^n h_B \|_2^2 = \int_{\mathbb{T}} R^n (p_2 (h_B)) (z) \, d\mu (z),
\]
and we check that the theorem from [MePa93] applies to that term as well: by the above argument, we have (for |ω| < π)
\[
p_2 (h_B) (e^{-i\omega}) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\sin^2 (\frac{\omega}{2})}{(\frac{\omega}{2} + n\pi)^2}.
\]
By a standard summation formula (see, e.g., [Car95, p. 152]), we have
\[
\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{(\frac{\omega}{2} + n\pi)^2} = \frac{1}{\sin^2 (\frac{\omega}{2})} - \frac{1}{(\frac{\omega}{2})^2},
\]
Hence
\[
p_2 (h_B) (e^{-i\omega}) = 1 - \frac{\sin^2 (\frac{\omega}{2})}{(\frac{\omega}{2})^2},
\]
which is continuous and equals 0 at ω = 0, so (ii) also follows from an application of [MePa93]. □

In the course of the proof, we established the following result.

PROPOSITION 8.9. Let \( \varphi := \chi_{[-\pi, \pi]} \) (inverse Fourier transform), and let \( f \in L^2 (\mathbb{R}) \) be given with \( L^2 \)-Fourier transform \( \hat{f} \). Then we have the following a.e. identity on \( \mathbb{T} \):
\[
p (\varphi, f) (e^{-i\omega}) = \chi_{[-\pi, \pi]} (|\omega|) \hat{f} (\omega),
\]
where \( p \) is defined on \( L^2 (\mathbb{R}) \times L^2 (\mathbb{R}) \) via the Zak transform,
\[
p (\varphi, f) (z) = \int_{0}^{1} (Z \varphi) (z, x) (Z f) (z, x) \, dx
\]
(again for a.a. \( z \in \mathbb{T} \)).
9. APPROXIMATION RESULTS

In this final section we prove a general approximation theorem which is based on some of the same ideas which went into Example 8.6 in the previous section. It uses the approximation kernel

$$D_n(z) := \left| m_0(z) m_0(z^2) \cdots m_0(z^{2^n-1}) \right|^2 = R^{*n}(\mathbb{I})(z).$$

(9.1)

This kernel may be constructed from any filter \(m_0\) subject to conditions (i)–(iii) in the Introduction. If the corresponding scaling function \(\varphi\) with \(\hat{\varphi}(0) = 1\) satisfies \(p_2(\varphi) = 1\), then by a theorem of Meyer and Paiva [MePa93], \(D_n(\cdot)\) is an approximate Dirac delta function \(\delta_1\) on \(C(\mathbb{T})\). In Example 8.6 above, the kernel \(D_n\) computes out as:

$$D_n \left( e^{-i\omega} \right) = 2^n \chi_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(\omega) \chi_{[\frac{2\pi}{2}, \frac{4\pi}{2}]}(\omega) \cdots \chi_{[\frac{2^n\pi}{2}, \frac{2^{n+1}\pi}{2}]}(\omega),$$

where, for \(s \in \mathbb{R}\), we set \([s] = s - k2\pi\) if \(k \in \mathbb{Z}\) is such that \((2k-1)\pi \leq s < (2k+1)\pi\), or equivalently \([s] = c_0(e^{-i\omega})\), \(s \in \mathbb{R}\).

DEFINITION 9.1. Let \(\xi \in L^2(\mathbb{T})\), and \(c \in \mathbb{C}\). While it is not then possible to assign a value to \(\xi(1)\), i.e., \(\xi\) evaluated at \(z = 1\), we will say that \(E \xi(1) = c\) if \(\xi\) is continuous at \(z = 1\) and assumes the value \(c\) there.

THEOREM 9.2. Let \(m_0\) satisfy conditions (i)–(iii) in the Introduction, and let \(R\), \(M\) be the corresponding Ruelle and cascade operators, where we view \(M\) as an operator in \(\mathcal{H}_Z \cong L^2(\mathbb{R})\). Let \(H_0 \in \mathcal{H}_Z\) be a regular scaling function, i.e., \(\|H_0\| = 1\), and \(M(H_0) = H_0\). Let \(F \in \mathcal{H}_Z\) satisfy \(p_2(F) = 1\). Then the following two conditions are equivalent:

(a) \(\|H_0 - M^n F\|_{\mathcal{H}_Z} \xrightarrow{n \to \infty} 0\),

(b) \(E p(H_0, F)(1) = 1\),

where the evaluation at \(z = 1\) in (b) is specified in Definition 9.1.

Proof. Since \(\|H_0\| = 1\), it follows from Lemma 7.1 that \(p_2(H_0) = 1\), i.e., we have both of the families \(\{\pi(e_n)H_0 \mid n \in \mathbb{Z}\}\) and \(\{\pi(e_n)F \mid n \in \mathbb{Z}\}\) orthogonal in \(\mathcal{H}_Z\). For \(F\), we also have

$$\|F\|_{\mathcal{H}_Z}^2 = \int_{\mathbb{T}} p_2(F)(z) \, d\mu(z) = 1.$$

Similarly,

$$\|M^n F\|_{\mathcal{H}_Z}^2 = \int_{\mathbb{T}} p_2(M^n F)(z) \, d\mu(z) = \int_{\mathbb{T}} R^n(p_2(F))(z) \, d\mu(z) = \int_{\mathbb{T}} R^n(\mathbb{I}) \, d\mu = 1,$$
so, for the norm-difference in (8), we have
\[ \|H_0 - M^n(F)\|_{\mathcal{H}_z}^2 = 2 - 2 \Re \langle H_0 | M^n(F) \rangle_{\mathcal{H}_z}, \]
which means that condition (8) is equivalent to
\[ \lim_{n \to \infty} \Re \langle H_0 | M^n(F) \rangle_{\mathcal{H}_z} = 1. \]
This last term computes out as follows: first introduce the sequence
\[ D_n(z) := |m_0(z) m_0(z^2) \cdots m_0(z^{2n-1})|^2 \]
(see (9.1) above), and note that
\[ R^* n(1) = D_n(z). \]
Since \( p_2(H_0) = 1 \), we know from a theorem of Meyer and Paiva [MePa93] that \( D_n(z) \to 0 \)
for \( z \) in the complement of any neighborhood in \( \mathbb{T} \) of \( z = 1 \). But the \( \langle \cdot | \cdot \rangle_{\mathcal{H}_z} \) term in (9.3) is
\[ \langle H_0 | M^nF \rangle_{\mathcal{H}_z} = \langle M^nH_0 | M^nF \rangle_{\mathcal{H}_z} 
\quad \quad = \int_{\mathbb{T}} R^n(p(H_0, F))(z) \, d\mu(z) 
\quad \quad = \int_{\mathbb{T}} R^* n(1)(z) \cdot p(H_0, F)(z) \, d\mu(z) 
\quad \quad = \int_{\mathbb{T}} D_n(z) p(H_0, F)(z) \, d\mu(z). \]
If \( p(H_0, F) \) were continuous, the theorem from [MePa93] would simply give
\[ \int_{\mathbb{T}} D_n p(H_0, F) \, d\mu \longrightarrow p(H_0, F)(1), \]
and the equivalence of (8) and (9) would be clear from this and (9.3). If (9.4) does not hold,
we still have \( p(H_0, F) \in L^2(\mathbb{T}) \) by Corollary 4.5. But
\[ \xi_n(z) := \int_{\mathbb{T}} D_n(z w^{-1}) p(H_0, F)(w) \, d\mu(w) \]
is continuous by the conditions on \( m_0 \), and \( \xi_n \to p(H_0, F) \) in \( L^2(\mathbb{T}) \) by [MePa93] and
Corollary 4.5. Suppose now that (8) is given: the argument from above then shows that (9)
holds in the sense of \( Ep(H_0, F)(1) = 1 \).

For the general approximation problem \( M^n h \to \varphi \), for \( h, \varphi \in \mathcal{H}, M\varphi = \varphi \), we have the following simple result.

**PROPOSITION 9.3.** Consider a system \((M, R, \mathcal{H})\) as in Theorem 5.2 with \( M \) a given sub-isometry and \( \mathcal{H} = \mathcal{B} \oplus \mathcal{H}(\infty) \) the canonical decomposition. Let \( h = h_B + h_\infty \), \( h_B \in \mathcal{B}, h_\infty \in \mathcal{H}(\infty) \) be given.
Then
\[ \| \varphi - M^n h \| \geq \| \varphi - M^n h_\infty \|, \]
so that \( h_\infty \) always gives a better approximation to \( \varphi = M\varphi \).
Proof. From Corollary 6.4, we have $M^n h_B \in B$ and $M^n h_\infty \in H^{(\infty)}$ for all $n = 0, 1, 2, \ldots$, so
\begin{equation}
\varphi - M^n h = \underbrace{\varphi - M^n h_\infty}_{H^{(\infty)}} + M^n h_B,
\end{equation}
and the result is immediate from this. \hfill \square

When Proposition 9.3 is applied to the scaling function $\varphi := \chi_{[-\pi,\pi]}^\vee$ from Example 8.6, we arrive at the following necessary condition on a function $h \in L^2(\mathbb{R})$ for $\|M^n h - \varphi\|_2 \overset{n \to \infty}{\longrightarrow} 0$ to hold.

**COROLLARY 9.4.** Let $\varphi := \chi_{[-\pi,\pi]}^\vee$, and let $h \in L^2(\mathbb{R})$ be given. Assume that the Fourier transform $\hat{h}$ is continuous on $\mathbb{R}$. Then a necessary condition for $\|M^n h - \varphi\|_2 \overset{n \to \infty}{\longrightarrow} 0$ is that
\begin{equation}
\sum_{n \in \mathbb{Z} \setminus \{0\}} |\hat{h}(2\pi n)|^2 = 0,
\end{equation}
i.e., $\hat{h} \equiv 0$ on $\{2\pi n \mid n \in \mathbb{Z} \setminus \{0\}\}$. Specifically, if the sum in (9.7) is $\neq 0$, then the cascade algorithm diverges on $h$.

**Proof.** In the discussion of Example 8.6, we showed that
\begin{equation}
\|M^n h_B\|_2^2 = \int_T R^n (p_2 (h_B))(z) \, d\mu (z),
\end{equation}
and, as $n \to \infty$, the limit of that term is $p_2 (h_B) (z = 1)$. Let $z = e^{-i\omega}$, $-\pi \leq \omega < \pi$. Then we saw that
\begin{equation}
p_2 (h_B) (e^{-i\omega}) = \sum_{n \in \mathbb{Z} \setminus \{0\}} |\hat{h}(\omega + 2\pi n)|^2.
\end{equation}
This follows from the identity
\begin{equation}
\hat{h}_B (\omega) = \chi_{\mathbb{R} \setminus [-\pi,\pi]} (\omega) \hat{h} (\omega), \quad \omega \in \mathbb{R},
\end{equation}
from Example 8.6. Hence
\begin{equation}
\lim_{n \to \infty} \|M^n h_B\|_2^2 = \sum_{l \in \mathbb{Z} \setminus \{0\}} |\hat{h}(2\pi l)|^2,
\end{equation}
and it is clear from (9.6) that $M^n h \not\to \varphi$ in $L^2(\mathbb{R})$ if this term is nonzero. \hfill \square

**REMARK 9.5.** A simple calculation shows that the sum $\sum_{n \neq 0}$ in (9.7) is nonzero if we take $h = h_b = \frac{1}{\sqrt{b}} \chi_{[0,b)}$ when $b \in \mathbb{R}^+, \ b$ irrational. (The argument can be done with the Poisson summation formula.)

Another example where the sum (9.7) is nonzero is a Gaussian, e.g., $h(x) = \pi^{-\frac{1}{2}} e^{-\frac{x^2}{2}}$. The nonzero sum $\sum_{n \in \mathbb{Z} \setminus \{0\}} |\hat{h}(2\pi n)|^2$ can be worked out from Gauss’s lattice sum formula; see, e.g., [DyMc72, p. 140] for details.
We finally note the following application to the cascade operator $M$ in $L^2(\mathbb{R})$ associated with some given filter $m_0$ satisfying (i)–(iii) in the Introduction. For $f, h \in L^2(\mathbb{R})$, recall the form

$$p(f, h)(z) = \sum_{n \in \mathbb{Z}} z^n \int \mathbb{R} f(x-n)h(x) \, dx.$$ 

The results in Section 3 show that $p: L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow L^1(T)$ is well defined and bounded, i.e.,

$$\|p(f, h)\|_1 \leq \|f\|_2 \cdot \|h\|_2$$

for all $f, h \in L^2(\mathbb{R})$.

Returning to the approximation problem

$$M^n h \rightarrow \varphi, \quad h = h_B + h_\infty, \quad M \varphi = \varphi,$$

suppose $z \mapsto p(f, h)(z)$ is continuous when $f, h$ (as a pair) is picked from $\varphi, h_B, h_\infty$. We have the following:

**Proposition 9.6.** If $\varphi$ satisfies $p^2(\varphi) = p(\varphi, \varphi) \equiv 1$, then

$$p(\varphi - h_\infty, h_B)(z = 1) = 0.$$

**Proof.** We have

$$0 = \langle \varphi - M^n h_\infty, M^n h_B \rangle_{L^2(\mathbb{R})}$$

$$= \int_T p(\varphi - M^n h_\infty, M^n h_B)(z) \, d\mu(z)$$

$$= \int_T p(M^n \varphi, M^n h_B) \, d\mu(z) - \int_T p(M^n h_\infty, M^n h_B) \, d\mu(z)$$

$$= \int_T R^n (p(\varphi, h_B))(z) \, d\mu(z) - \int_T R^n (p(h_\infty, h_B))(z) \, d\mu(z)$$

$$\rightarrow n \rightarrow \infty p(\varphi, h_B)(1) - p(h_\infty, h_B)(1),$$

and the claim follows. 

The last step was based on the idea from the proof of Theorem 9.2 above, and the Meyer–Paiva [MePa93] result. The latter states that, if $M$ and $\varphi$ are such that $p^2(\varphi) \equiv 1$, then, for all continuous functions $\xi$ on $T$, we have

$$\lim_{n \rightarrow \infty} \int_T R^n(\xi)(z) \, d\mu(z) = \xi(1).$$

An alternative approach to this limit problem is also given in [Kea72] and [Rue90].

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