Supplementary Materials for

**Enhanced thermal conduction by surface phonon-polaritons**

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Dielectric function measurements
Infrared spectroscopy has been employed to obtain the dielectric function of samples. Reflectance and transmittance spectra have been measured as a function of temperature, in the spectral range going from 400 cm\(^{-1}\) and 8000 cm\(^{-1}\), with a sufficient resolution of 4 cm\(^{-1}\). A Bruker Vertex 80v Fourier Transform Infrared (FTIR) spectrometer coupled to a Bruker Hyperion 3000 microscope has been used to perform the acquisitions. The optical configuration included a Globar source, a Ge/KBr
beam-splitter, a Cassegrain objective (×15, NA=0.4), and liquid nitrogen cooled MCT detector. Samples have been heated up by using a Linkam THSM600 heating stage positioned in the microscope plate, which allows precision X-Y-Z displacement of the sample to properly place it in the focal point of the IR detector (see Fig. 2A). At each temperature, the signal due to the self-thermal emission of the sample has been numerically removed.

Figure 2. | Infrared spectroscopy measurement of SiN membranes. (A), Reflectance and transmittance spectra of a SiN membrane at room temperature. The scheme of a Linkam heating stage along with the optical paths used for the measurements is shown in the inset. (B), dielectric function of SiN membranes with thicknesses of 30 and 100 nm obtained by infrared spectroscopy at room temperature, in comparison with the values reported in the literature [31].

Subsequently, the infrared spectra have been fitted by using the following membrane reflectance $R(\omega)$ and transmittance $T(\omega)$ models, which include multiple coherent reflections within the semi-transparent membrane of thickness $d$ and limited by vacuum at both sides:

$$R(\omega) = \frac{(r(\omega) - r(\omega)e^{2\sqrt{\varepsilon(\omega)kd}})^2}{1 - r^2(\omega)e^{2\sqrt{\varepsilon(\omega)kd}}}, \quad (1a)$$

$$T(\omega) = \frac{4\sqrt{\varepsilon(\omega)}}{(1 + \sqrt{\varepsilon(\omega)})^2(e^{-2i\sqrt{\varepsilon(\omega)kd}} - r(\omega)e^{2\sqrt{\varepsilon(\omega)kd}})}^2, \quad (1b)$$

where $r(\omega) = \frac{1 - \sqrt{\varepsilon(\omega)}}{1 + \sqrt{\varepsilon(\omega)}}$ corresponds to the complex reflectivity coefficient, $\varepsilon(\omega)$ stands for the dielectric function as a function of the angular frequency $\omega$, and the wave vector $k_0 = \omega/c$, with $c$ being the speed of light in vacuum. The dielectric function is then described by the Lorentz dispersion model, as follows:

$$\varepsilon(\omega) = \varepsilon_{\infty} \prod_j \frac{\Omega^2_{jTO} - \omega^2 - i\gamma_{jLO}\omega}{\Omega^2_{jLO} - \omega^2 - i\gamma_{jTO}\omega}, \quad (2)$$

where $\varepsilon_{\infty}$ stands for the high frequency relative dielectric function accounting for the response of the electronic transitions. $\omega_{jTO,LO}$ and $\gamma_{jTO,LO}$ are the angular frequencies and damping of the $j^{th}$ infrared active modes, with TO and LO referring to the transverse and longitudinal optical modes, respectively. We have obtained the optical response of 30 and 100 nm-thick SiN membranes at different temperatures between 300 and 800 K. The predominant terms in Eq. 2 correspond to normal modes with $\Omega_{TO} = 93$ Trad/s, $\Omega_{LO} = 96$ Trad/s and $\gamma_{TO} = 149$ Trad/s, $\gamma_{LO} = 214$ Trad/s at 300 K. The real and imaginary parts of the dielectric function are shown in Fig. 3.
Figure 3. Dielectric function of 30 and 100 nm-thick SiN membranes. (A, B), real and (C, D) imaginary parts of the obtained permittivities.

SPhP thermal conductivity modeling

The thermal conductivity $\kappa_{\text{SPhP}}$ of SPhP propagating along a membrane of thickness $d$ and average temperature $T$ is given by [15]

$$\kappa_{\text{SPhP}} = \frac{1}{4\pi d} \int_{\omega_L}^{\omega_H} \omega \beta_R \frac{\partial f_0}{\partial T} d\omega,$$  

(3)

where $2\pi$ is the Planck constant, $\beta_R = \text{Re}(\beta)$ is the real part of the in-plane wave vector $\beta$, $\Lambda = 0.5 \text{Im}(\beta)^{-1}$ is the SPhP mean free path, $f$ is the Bose-Einstein distribution function, and $\omega$ is the SPhP spectral frequency supporting the generation of heat between its minimum $\omega_L$ and maximum $\omega_H$ values. Eq. 3 shows that $\kappa_{\text{SPhP}}$ is determined by the SPhP dispersion relation $\beta = \beta(\omega)$, which, for a film suspended, is given by [16]

$$\epsilon_2 p_1 + \epsilon_1 p_2 \tanh \left( \frac{p_2 d}{2} \right) = 0,$$  

(4)

where $\epsilon_j$ is the relative dielectric function of the membrane ($j=2$) or of its surrounding medium ($j=1$), the transverse wave vectors $p_j$ are related to $\beta$ by $p_j^2 = \beta^2 - \epsilon_j k_0^2$. In order to analyze the possible solutions of Eq. 4 for $\beta$, we rewrite this in-plane wave vector as $\beta = k_0 \sqrt{\epsilon}$, which allows expressing Eq. 4 in a convenient dimensionless form, as follows

$$\epsilon_2 \sqrt{\epsilon} - \epsilon_1 + \epsilon_1 \sqrt{\epsilon} - \epsilon_2 \tanh(\lambda \sqrt{\epsilon - \epsilon_2}) = 0,$$  

(5)
where $\lambda = 0.5k_0d$. Given that the existence of polaritons is determined by their confinement to the membrane surface, they exist provided that the real parts of $p_1 = k_0\sqrt{\varepsilon - \varepsilon_1}$ and $p_2 = k_0\sqrt{\varepsilon - \varepsilon_2}$ are positive. Taking into account that $\varepsilon_1 = 1$ (for the surrounding media), the solution of Eq. 5 for $\varepsilon$ is determined by the normalized thickness $\lambda$ and complex dielectric function $\varepsilon_2 = \varepsilon_R + i\varepsilon_I$ of the suspended membrane. In order to understand the nature of this solution, the following two cases are going to be analyzed:

1. **Lossless membrane ($\varepsilon_I = 0$)**

In this case, $\varepsilon_2(\omega)$ is a real function of $\omega$ and the condition of existence of polaritons establishes that $\varepsilon > \varepsilon_1$ and $\varepsilon_2$, which indicates that the solution of Eq. 5 only exists for $\varepsilon_2(\omega) < 0$ and $\varepsilon > \varepsilon_1$. Interestingly, these conditions can also be derived by using the inequality $\tanh(\lambda\sqrt{\varepsilon - \varepsilon_2}) < 1$ and Eq. 5. The condition $\varepsilon_2(\omega) < 0$ determines the range of frequencies supporting the existence of polaritons and can be even more restrictive depending on the values of $\lambda$. For instance, for an infinitely thick film ($\lambda \to \infty$), Eq. 5 yields $\varepsilon = (\varepsilon_1\varepsilon_2)/(\varepsilon_1 + \varepsilon_2)$, which indicates that the general condition $\varepsilon_2(\omega) < 0$ should be replaced by the well-known constraint $\varepsilon_2(\omega) < -\varepsilon_1$. For a thin membrane ($\lambda \ll 1$), on the other hand, an approximation of Eq. 5 up to $\lambda^2$ takes the form

$$\varepsilon_2\sqrt{\varepsilon - \varepsilon_1} + \lambda\varepsilon_1(\varepsilon - \varepsilon_2) = 0. \quad (6)$$

Equation 6 is a quadratic equation for $\sqrt{\varepsilon - \varepsilon_1}$ and its physical solution ($\varepsilon \geq \varepsilon_1$) is given by

$$\varepsilon = \varepsilon_1 + (\lambda\varepsilon_1)^2\left(1 - \frac{\varepsilon_1}{\varepsilon_2}\right)^2, \quad (7)$$

which allows writing the solutions for the in-plane and cross-plane wave vectors, as follows

$$\beta = k_0\sqrt{\varepsilon_1}\left[1 + \frac{\varepsilon_1\lambda^2}{2}\left(1 - \frac{\varepsilon_1}{\varepsilon_2}\right)^2\right], \quad (8a)$$

$$p_1 = k_0\lambda\varepsilon_1\left(1 - \frac{\varepsilon_1}{\varepsilon_2}\right), \quad (8b)$$

$$p_2 = k_0\sqrt{\varepsilon_1 - \varepsilon_2}\left[1 + \frac{\varepsilon_1 - \varepsilon_2}{2}\left(\frac{\lambda\varepsilon_1}{\varepsilon_2}\right)^2\right]. \quad (8c)$$

Equations 8 establish that the polaritons exist ($p_1$ and $p_2 > 0$) and propagate ($\beta > 0$) along the surface of the membrane with a dielectric function $\varepsilon_2(\omega) < 0$. It is therefore clear that the well-known condition $\varepsilon_2 < -\varepsilon_1$ holds for infinitely thick membrane only. A lossless membrane is thus expected to support the propagation of polaritons in a broader range of spectral frequencies than a semi-infinite film. Even though materials may behave like a lossless medium in a certain range of frequencies supporting the propagation of polaritons, they do not absorb energy (for these frequencies) and therefore these polaritons do not generate heat inside them. This fact is confirmed by the heat source $S$ generated by polaritons inside a medium, which is given by $[11] S = 0.5\omega\varepsilon_1|E|^2$, with $\varepsilon_I$ being the imaginary part of the medium $p$ and $|E|$ the magnitude of the polariton electrical field. We thus conclude that the polaritons propagation along a lossless membrane do not contribute to the thermal conductivity in Eq. 3.

2. **Lossy membrane ($\varepsilon_I > 0$)**

In this case, Eqs. 7 and 8 still hold for a thin membrane ($\lambda \ll 1$) with a complex $\varepsilon_2(\omega) = \varepsilon_R + i\varepsilon_I$. By inserting this dielectric function in Eqs. 8, one obtains that the real and imaginary parts of the complex wave vectors $\beta = \beta_R + i\beta_I$ and $p_j = p_{jR} + ip_{jI}$, which are given by [16]
\[
\beta_R = k_0 \sqrt{\varepsilon_1} \left[ 1 + \frac{\varepsilon_1 \lambda^2}{2} \left( 1 - \frac{2 \varepsilon_1 \varepsilon_R}{|\varepsilon_2|^2} + \frac{\varepsilon_1^2}{|\varepsilon_2|^4} (\varepsilon_R^2 - \varepsilon_1^2) \right)^2 \right],
\]
\[
\beta_I = k_0 \sqrt{\varepsilon_1} \left( \frac{\varepsilon_1 \lambda}{|\varepsilon_2|^2} \right)^2 \left( 1 - \frac{\varepsilon_1 \varepsilon_R}{|\varepsilon_2|^2} \right)^2,
\]
\[
p_{1R} = k_0 \varepsilon_1 \lambda \left( 1 - \frac{\varepsilon_1 \varepsilon_R}{|\varepsilon_2|^2} \right),
\]
\[
p_{1I} = k_0 \varepsilon_1 \lambda \left( \frac{\varepsilon_1}{|\varepsilon_2|^2} \right)^2,
\]
\[
p_{2R} = \frac{k_0}{\sqrt{2}} \left[ \chi^+ + \frac{(\varepsilon_1 \lambda)^2}{2|\varepsilon_2|^4} (A\chi^+ - B\chi^-) \right],
\]
\[
p_{2I} = -\frac{k_0}{\sqrt{2}} \left[ \chi^- + \frac{(\varepsilon_1 \lambda)^2}{2|\varepsilon_2|^4} (B\chi^+ + A\chi^-) \right],
\]

where \(\chi^\pm = \sqrt{|\varepsilon_1 - \varepsilon_2|} \pm (\varepsilon_1 - \varepsilon_R), A = \varepsilon_1 (\varepsilon_R^2 - \varepsilon_1^2) - \varepsilon_R|\varepsilon_2|^2\) and \(B = \varepsilon_1 (2\varepsilon_1 \varepsilon_R - |\varepsilon_2|^2)\). Equations 9b and 9b establish that the propagation and existence of polaritons is determined by the condition \(\varepsilon_R^2 + \varepsilon_1^2 - \varepsilon_1 \varepsilon_R > 0\), which represents the region of the plane \((\varepsilon_R, \varepsilon_1)\) outside of a semicircle with radius \(\varepsilon_1/2\) and centered at \((\varepsilon_1/2, 0)\). This condition indicates that a surrounding medium with lower dielectric function \(\varepsilon_1\) allows the propagation of polaritons in a wider frequency range. The dependence of \(p_{2R}\) and \(p_{2I}\) on \(\varepsilon_1\) can be better seen in the limiting case \(\varepsilon_1 \ll |\varepsilon_1 - \varepsilon_R|\). Under this condition, Eqs. 9e and 9f reduce to

\[
p_{2R} = k_0 \sqrt{\varepsilon_1 - \varepsilon_R},
\]
\[
p_{2I} = -\frac{k_0 \varepsilon_1}{2\sqrt{\varepsilon_1 \varepsilon_R}},
\]

for \(\varepsilon_R < \varepsilon_1\). On the other hand, for \(\varepsilon_R > \varepsilon_1\), Eqs. 9e and 9f take the form

\[
p_{2R} = k_0 \sqrt{\varepsilon_1 - \varepsilon_R},
\]
\[
p_{2I} = -\frac{k_0 \varepsilon_1}{2\sqrt{\varepsilon_1 \varepsilon_R}},
\]

This latter case \((0 < \varepsilon_1 \ll \varepsilon_R - \varepsilon_1)\) usually occurs at low and high enough frequencies and indicates that the lateral confinement \((p_{2R} > 0)\) of polaritons is proportional to \(\varepsilon_1\), as is the case of the heat source \(S\) generated by them. In general, taking into account that this confinement along the membrane and its surrounding medium are determined by \(p_{2R}\) and \(p_{1R}\), respectively; the lateral confinement of polaritons at an interface of a thin membrane is driven by \(p_{1R}p_{2R}\). Furthermore, as polaritons propagate simultaneously along both interfaces of the membrane, the lateral confinement imposed by the whole structure on the propagation of polaritons can be described by the following distribution function

\[
F(\omega) = \left( \frac{p_{1R}(\omega)p_{2R}(\omega)}{p_{1R}(\omega_0)p_{2R}(\omega_0)} \right)^2,
\]

where \(\omega_0\) is the frequency at which the product \(p_{1R}(\omega)p_{2R}(\omega)\) reaches its maximum. For the case of a nanomembrane \((\lambda \ll 1)\) surrounded by air \((\varepsilon_1 = 1)\), Eqs. 9c and 9e establish that \(F\) is nearly independent of the membrane thickness \(d\) and \(\omega_0\) is approximately equal to the frequency at which the minimum of \(\varepsilon_R\) occurs. This is confirmed by Figs. 4a and 4b, which were obtained for a suspended nanomembrane \((d \leq 100 \text{ nm})\) of SiN \((\omega_0 = 175 \text{ Trad/s})\). Considering that the lateral confinement is directly correlated with the absorption of energy driving the polaritons’ heat generation inside the membrane, the integration limits in Eq. 3 are determined by the frequency window where \(F(\omega)\) is significant.

Taking into account that polaritons propagate inside the membrane surrounding medium a lateral distance \((2p_{1R})^{-1}\), whose typical values are in the sub-micrometer scale, they also contribute to the near-field radiation of the membrane. The membrane can thus be considered like a source of thermal radiation, whose radiated energy is driven by the local density of state (LDOS) of polaritons propagating from the membrane to the surrounding medium. This LDOS is the number of electromagnetic-wave modes per unit frequency and volume at a given distance \(L\) from the membrane interface, and it can be determined by combining the Maxwell equations with fluctuational electrodynamics [8]. Given that in non-magnetic media, as is the case considered in
Figure 4. | Real part of the transverse vector, confinement distribution function, and local density of state. (A) Real parts of the transverse wave vectors \( p_1 \) and \( p_2 \) along with their (B) associated confinement distribution function driving the polariton heat transport. (C) Local density of states of polaritons contributing to the near-field radiation in the vicinity of the membrane.

This work, polaritons can only be generated by electromagnetic waves with transverse magnetic (TM) polarization [8], we will only consider the TM component \((\rho_{TM})\) of their LDOS, which is given by [18]

\[
\rho_{TM}(\omega, L) = \frac{k_0^2}{2\pi^2 c} \int_1^\infty \frac{x^3}{x^2 - 1} e^{-2k_0\sqrt{x^2 - 1}} dx, \tag{13}
\]

where the membrane surrounding medium was considered to be air \((\varepsilon_1 = 1)\) and

\[
R_{TM}(\omega, x) = r_{12} \frac{1 - e^{-2k_0\sqrt{x^2 - \varepsilon_2}}}{1 - r_{12}^2 e^{-2k_0\sqrt{x^2 - \varepsilon_2}}}, \tag{14a}
\]

\[
r_{12}(\omega, x) = \frac{\sqrt{x^2 - 1} - \sqrt{\varepsilon_2} \sqrt{\varepsilon_2 x^2 - 1}}{\sqrt{x^2 - 1} + \sqrt{\varepsilon_2} \sqrt{\varepsilon_2 x^2 - 1}}. \tag{14b}
\]

Equation 13 provides the LDOS of polaritons contributing to the near-field radiation in the vicinity of the membrane \((L > 0)\) and therefore the range of frequency where \(\rho_{TM}\) is significant is expected to be different that the one of the confinement distribution function \(F\) (Eq. 12) describing the polaritons contribution to the heat conduction. This is confirmed by Figs. 4b and 4c, which show that the maxima of \(\rho_{TM}\) and \(F\) occur at different frequencies. In fact, the density of states per unit area \(g = \beta R/(2\pi V_g)\) of polaritons contributing to the in-plane thermal conductivity of a membrane is inversely proportional to their group velocity \(V_g\), which cancels out in Eq. 3 [15]. The comparison of this latter expression with Eq. 13, indicates that the heat transport generated by polaritons propagating along the in-plane and out-of-plane directions are quite different due to the different nature of their densities of states.

In Fig. 5, we show the spectral thermal conductivity contributed by SPhP versus frequency. We compare the spectrum at 300 and 800 K and to emphasize the range of meaningful spectral frequencies contributing to the in-plane energy transport of SPhP is independent from temperature.

We have plotted the measured thermal conductivities of the 30, 50, 100 nm-thick membranes of SiN normalized by the corresponding of the 200 nm-thick membrane. Note that this normalized thermal conductivity generally increases with temperature, such that at 800 K, its value for the 100 nm-thick membrane is near unity. Considering that phonons are the dominant heat carriers of the thicker membranes (100 and 200 nm), this comparability between their thermal conductivities indicates that the size effect on the phonon transport is negligible at high temperature. This behavior is consistent with the one shown in Fig. 2B, which clearly displays two aspects: first, the SPhP contribution in the thinner membranes (30 and 50 nm) doubles their thermal conductivities as the temperature rises from 300 to 800 K. Second, the SPhP thermal conductivity of the 200 nm-thick membrane is negligible (see Fig. 3C) in comparison with the predominant phonon counterpart that is inversely proportional to temperature. Figure 6 below provides a complementary way to observe these results.

Heat diffusion model

Let us consider a suspended membrane excited with a laser beam of power \(P\) via a circular aluminum pad of radius \(a\), as shown in Fig. 7. This thin metallic pad converts the laser light into heat that then propagates along the radial and axial directions of
Figure 5. | Spectral thermal conductivity. It is predicted by Eq. (1) for a 30 nm-thick membrane of SiN.

the membrane of thickness $d$. According to the Fourier’s law of heat conduction, the membrane temperature $T(r,z,t)$ at the respective radial and axial positions $r$ and $z$, and a time $t$ is given by

$$\frac{\kappa_r}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \kappa_z \frac{\partial^2 T}{\partial z^2} = \rho c \frac{\partial T}{\partial t},$$

(15)

where $\kappa_r$ and $\kappa_z$ are the in-plane and cross-plane thermal conductivities of the membrane with density $\rho$ and specific heat capacity $c$.

Considering that the laser beam uniformly illuminates the aluminum pad during a time $t_0$ (pulse length), the solution of Eq. 15 is subjected to the following boundary conditions:

$$T(r \to \infty, z, t) = T_0, \quad T(R, z, t) = T_0, \quad T(r, z, 0) = T_0, \quad (16a)$$

$$\frac{\partial T}{\partial z} |_{z=d} = 0, \quad -\kappa_z \frac{\partial T}{\partial z} |_{z=0} = \frac{P}{\pi a^2} H(t_0-t)H(a-r), \quad (16b)$$

where $R$ is the radius at which the heat conduction vanishes and hence the membrane reaches its initial room temperature $T_0$, and $H$ is the Heaviside function. The solution of Eq. 15 under the boundary condition in Eqs. 16a and 16b can conveniently be obtained by means of the Laplace transform method or the Duhamel’s theorem, both methods yield

$$T(r, z, t) = T_0 + T_c \sum_{n=1}^{\infty} \frac{J_1(x_n)J_0(x_n r/a)}{x_n J_1(\xi_n)} f(x_n, z, t),$$

(17)

where the characteristic temperature $T_c = 2P/(\pi \kappa_d)$, $x_n = a \xi_n/R$, $J_0$ and $J_1$ are the Bessel functions of orders zero and one, respectively, $\xi_n$ are the roots of $J_0(\xi_n) = 0$, such that $\xi_1 < \xi_2 < \ldots$, and

$$f(x_n, z, t) = g(x_n, t) + 2 \sum_{m=1}^{\infty} \frac{(\xi_n r/a)^m}{\beta_{nm}} \cos \left[ m \pi (1 - z/d) \right] g(x_n, \beta_{nm} t),$$

(18a)

$$g(x_n, t) = \psi(x_n, t) - \psi(x_n, t - t_0)H(t - t_0),$$

(18b)

$$\psi(x_n, t) = 1 - \exp(-x_n^2 t/\tau_r),$$

(18c)

$$\beta_{nm} = 1 + \left( m \pi / x_n \right)^2 \tau_r / \tau_z,$$

(18d)

with $\tau_r = a^2 / \alpha_r$, $\tau_z = d^2 / \alpha_z$ being two characteristic times driving the heat conduction along the radial and axial directions of the membrane characterized by the thermal diffusivities $\alpha_r = \kappa_r/(\rho c)$ and $\alpha_z = \kappa_z/(\rho c)$, respectively. Note that $\tau_z$ shows
Figure 6. | Thermal conductivities of 30, 50, 100 nm-thick membranes normalized by that of 200 nm-thick membrane.

up through the β_{nm} parameter only and its impact is weighted by τ_r, such that the temperature profile becomes independent of τ_r and the axial position z, for τ_r ≪ τ_c (d ≪ a). For τ_c ≫ τ_r (d ≫ a), on the other hand, T depends strongly on both τ_r and τ_c. In any case, when the heating radius is much greater than the one of the heating pad (R ≫ a), the difference \(\Delta x = x_{n+1} - x_n = (\xi_{r+1} - \xi_r)a/R\) becomes very small, for any \(n = 1, 2, \ldots\), and therefore the sum in Eq. 17 becomes an integral, as follows

\[
T(r, z, t) = T_0 + \frac{T_c}{2} \int_0^\infty J_1(x) J_0(xr/a)f(x, z, t)\frac{dx}{x^2},
\]

(19)

where we have used the identity \(\xi_{n}(\xi_{n+1} - \xi_n)J_1(\xi_n) = 2\) that can numerically be verified, for any root \(\xi_n\). Equation 19 can also be derived by solving Eq. 15 through the Hankel’s transform, for \(R \to \infty\), as expected. This fact confirms that the application of Eq. 19 is justified for \(R \gg a\) only, otherwise the membrane temperature should be described by Eq. 17. So Eq. 19 establishes that the temperature rise \(\Delta T = T(0, 0, t) - T_0\) recorded by our \(\mu\)TDTR experimental setup at the center of the membrane, is given by

\[
\frac{8\Delta T}{t_c} = N\left(4\frac{t}{\tau_r}\right) - N\left(4\frac{t - t_0}{\tau_r}\right) H(t - t_0) + 8 \sum_{m=1}^{\infty} \left[ M_m\left(\frac{t}{\tau_r}, \frac{\tau_r}{\tau_c}\right) - M_m\left(\frac{t - t_0}{\tau_r}, \frac{\tau_r}{\tau_c}\right) H(t - t_0) \right],
\]

(20)

where

\[
N(u) = u\left(1 - e^{-1/u}\right) + \int_1^{\infty} e^{-x} \frac{dx}{x},
\]

(21a)

\[
M_m(u, v) = \int_0^\infty \frac{1 - \exp\left[-(x^2 + (m\pi)^2)v\right]}{x^2 + (m\pi)^2v} J_1(x)dx.
\]

(21b)

The integral term in Eq. 21a is the incomplete gamma function, which is well implemented in common software of numerical analysis (Mathematica, Matlab, Python, etc.) and therefore its evaluation is straightforward. On the other hand, the integral in Eq. 21b indicates that the main contribution to \(M_m\) arises from values of x smaller than and comparable to unity, and hence it reduces to \(M_m(u, v) = v^{-1}(m\pi)^{-2} \exp\left[-uv(m\pi)^2\right]\), for \(m\pi\sqrt{v} \gg 1\). This latter condition is, of course, fulfilled for \(v = \tau_r/\tau_c = (a/d)^2\alpha_r/\alpha_r \gg 1\), that is to say, when the radius of the heating pad is much greater than the membrane thickness \((a \gg d)\). This condition is satisfied by our \(\mu\)TDTR experimental setup operating with \(a = 2.5 \mu m\) and \(d \leq 0.2 \mu m\), and therefore the contribution of the function \(M_m\) in Eq. 20 vanishes. For these pad and sample dimensions, the sum in Eq. 18a can thus be neglected. This fact establishes that the temperature rise \(\Delta T\) recorded by our \(\mu\)TDTR sensor is mainly driven by the in-plane heat conduction and it can thus be used to determine the radial characteristic time \(\tau_r = a^2/\alpha_r\) and hence \(\alpha_r\) of our
membrane samples, by fitting the first two terms of the left-hand side of Eq. 20 to its corresponding $\mu$TDTR experimental data. The in-plane (radial) thermal conductivity is then obtained by means of the relation $\kappa = \alpha_\rho \rho c$, where $\rho$ denotes the density and $c$ the specific heat capacity.

**SPhP thermal conductivity of a SiN membrane on top of an Au nanolayer**

In order to better understand the leakage of fields outside of a SiN membrane, the propagation of surface phonon-polaritons along its interfaces and the thermal energy associated to these energy carriers, we have numerically analyzed their propagation along a single SiN membrane (Fig. 8A), as is the case of our experiments, in comparison with that along a two-layer system consisting of a SiN membrane on top of an Au nanolayer, as shown in Fig. 8B. The dispersion relation of polaritons in the first structure (Fig. 8A) consists of symmetric and antisymmetric modes (with respect to the electric field) given respectively by [15]

\[
\begin{align*}
\frac{p_2}{\varepsilon_2} + \frac{p_1}{\varepsilon_1} \tanh(p_2) \frac{d}{2} &= 0, \\
\frac{p_1}{\varepsilon_1} + \frac{p_2}{\varepsilon_2} \tanh(p_2) \frac{d}{2} &= 0.
\end{align*}
\]

where all parameters are defined within the manuscript. Taking into account that the antisymmetric mode (symmetric with respect to the magnetic field) yields propagation lengths order of magnitude longer than the symmetric one [15], the overwhelming majority of the polariton thermal conductivity is due to this antisymmetric branch. This is the reason why we have only considered this later mode to quantify the polariton thermal conductivity of our SiN membranes, as detailed in the supplementary material and is now shown in the modified Fig. 2A in the revised manuscript. On the other hand, the dispersion relation of polaritons propagating along the structure shown in Fig. 8B is given by [28]

\[
\frac{S_0 + S_1 \tanh(p_0d_0)}{S_1 + S_0 \tanh(p_0d_0)} = \frac{S_0 S_2 + S_1 \tanh(p_2d)}{S_2 S_1 + S_2 \tanh(p_2d)}
\]

where $S_n = p_n/\varepsilon_n$

In order to numerically solve Eq. 23 for the polariton wave vector $p$ and determine the impact of a metallic nanolayer underneath a membrane of SiN, we considered a 30 nm-thick nanolayer of gold, whose dielectric function is shown in Fig. 9 with previous study [29], in comparison with the room temperature one of SiN reported in the supplementary material.
Fig. 10A shows the intrinsic propagation length ($\Lambda$) of polaritons propagating along the Au/SiN structure in comparison with the effective one ($\Lambda_e$) determined by the lateral dimension of our samples. Note that for each SiN thickness $d$, the values of $\Lambda$ and $\Lambda_e$ are significantly smaller than the corresponding ones obtained for a single SiN membrane, as shown in Fig. 10A above. This is reasonable, as the Au nanolayer contributes to increase the absorption of energy of the SiN membrane from the electromagnetic field, which reduces the propagation distance of polaritons along the three interfaces of the two-layer system. Given that the polariton thermal conductivity $k$ is directly determined by the effective propagation length of polaritons (Eq. 22), this reduction of the polariton propagation distance yields $k$ values lower than the corresponding ones obtained for a single SiN membrane, as shown in Fig. 10B. It is thus clear that a single membrane is better than other multi-layered structures (with higher absorption rates) to observe the contribution of polaritons to the heat transport along interfaces. This is the main reason why we have performed our experiments with a single SiN membrane suspended in vacuum.

**Diffuson thermal conductivity in amorphous SiN**

In amorphous materials, there are different types of heat carriers: propagons, diffusons and locons. Propagons and diffusons carry most of the heat in bulk amorphous materials, whereas the flux contribution of locons remains negligible. While propagons do look like phonons, the temperature dependence of propagons might be discarded due to the predominance of surface scattering. The spectrum of propagons involves lower frequencies than the one of diffusons, which in amorphous SiN, have a cutoff frequency at 20 THz, as established by the density of vibrational states [33]. In addition, the spectral diffusivity of diffusons decays quickly as the frequency increases, as shown in Fig. 11B, below. This fact tends to indicate that the contribution of diffusons to the thermal conductivity of SiN decreases as the frequency increases. Diffusons might therefore not contribute significantly to the enhancement of the SiN thermal conductivity in the 300-800 K temperature interval. The temperature dependence of the diffuson thermal conductivity $\kappa_{diff}(T)$ can be written as [33]

$$\kappa_{diff}(T) = \int_{\omega_L}^{\omega_H} \frac{h^2 \omega^2}{k_B T^2} \frac{\frac{k_B T}{e^{\frac{k_B T}{\hbar \omega}} - 1}}{D(\omega) \alpha(\omega)} d\omega$$  \hspace{1cm} (24)$$

where $\omega_L$ and $\omega_H$ are the lower and upper bounds of the diffusons’ frequency spectrum, respectively; $\hbar$ and $k_B$ are the respective reduced Planck and Stefan-Boltzmann constants; while $D(\omega)$ and $\alpha(\omega)$ stand for the density of states and the spectral diffusivity of diffusons, respectively.

The frequency dependences of these two latter parameters are shown in Fig. 11, which allows to integrate Eq. 24 and obtain Fig. 11, for the normalized thermal conductivity $\kappa_{diff}(T)/\kappa_{diff}(300 \text{ K})$. Note that the diffusion thermal conductivity only increases about 6% within the temperature interval of (300-800) K. This weak enhancement is thus negligible in comparison with the 100% increase observed on the measured thermal conductivity, which confirms that diffuson’s contribution cannot explain this latter sizeable enhancement attributed to SPhP.
Figure 9. | Room temperature dielectric functions of SiN and Au.

Figure 10. | Comparison of single SiN membranes and SiN membranes with 30 nm-thick Au nanolayer in terms of propagation length and thermal conductivity. (A) Comparison of the intrinsic (solid lines) and effective (dashed lines) propagation lengths of polaritons propagating along SiN membranes on top of a 30 nm-thick Au nanolayer. (B) Polariton thermal conductivity of our four SiN membranes with and without an Au nanolayer underneath. Calculations were done with Eq. 22 by using the effective propagation length $\Lambda_e$ given by the Mathiessen’s rule $\Lambda_e^{-1} = \Lambda^{-1} + a^{-1}$. This relation establishes that $\Lambda_e \leq a$ and hence allows limiting the propagation distance of polaritons to the real lateral dimension (a) of our samples, as must be.
Figure 11. **Diffusion spectra.** (A) Spectral diffusivity of diffusons and (B) their density of vibrational states in amorphous SiN. Both spectra were plotted with data reported [33].

Figure 12. **Normalized thermal conductivity of diffusons** $\kappa_{diff}(T) / \kappa_{diff}(300 \text{ K})$, as a function of temperature.