Reducing the dichromatic number via cycle reversions in infinite digraphs

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Abstract

We prove the following conjecture of S. Thomassé: for every (potentially infinite) digraph $D$ it is possible to iteratively reverse directed cycles in such a way that the dichromatic number of the final reorientation $D^*$ of $D$ is at most two and each edge is flipped only finitely many times. In addition, we guarantee that in every strong component of $D^*$ all the local edge-connectivities are finite and any edge is reversed at most twice.

1 Introduction

Our general motivation is to analyse to what degree a complicated structure can be simplified by using only certain elementary operations. In our case, the structure is a digraph, the operation is reversing a directed cycle and we measure the complexity of the digraphs with the the dichromatic number. The latter, introduced by V. Neumann-Lara in [1], is a directed analogue of the chromatic number.

Definition 1.1. The dichromatic number $\chi(D)$ of a digraph $D$ is the smallest cardinal $\kappa$ such that $V(D)$ can be coloured with $\kappa$ many colours avoiding monochromatic directed cycles.

Various old and fascinating questions related to dichromatic number are still open. For example, Neumann-Lara and Škrekovski conjectured that an orientation of a simple planar graph has always dichromatic number at most two. The best known partial result says that it is true whenever there is no directed cycle of length three (see [2]). In other type of problems the maximal dichromatic number of the possible orientations is in focus. Erdős and Neumann-Lara conjectured the existence of a function $f : \mathbb{N} \to \mathbb{N}$ such that any graph $G$ with chromatic number at least $f(k)$ has an orientation $D$ with dichromatic number $k$ (see [3] and [4]). We should emphasize that even the existence of $f(3)$ is unknown and the analogue question considering infinite chromatic and dichromatic numbers is also open. Partial results were obtained like...
the independence of the statement “Every graph of size and chromatic number $\aleph_1$ admits an orientation with dichromatic number $\aleph_1$” (see [5]).

C. Laflamme, N. Sauer and R. Woodrow drew the attention of the third author to the following conjecture of Thomassé:

**Conjecture 1.2** (S. Thomassé [6]). For every (potentially infinite) digraph $D$, it is possible to reverse the edges of directed cycles iteratively in such a way that each edge is reversed only finitely many times and the dichromatic number of the final reorientation $D^*$ of $D$ is at most two.

Thomassé et al. justified the conjecture for finite tournaments. In fact, they proved a stronger statement:

**Theorem 1.3** (Thomassé et al. [7]). Let $T$ be a tournament on the vertices $v_1, \ldots, v_n$ where for the outdegrees $d^+(v_1) \geq d^+(v_2) \geq \cdots \geq d^+(v_n)$ holds. Then one can reverse directed cycles iteratively in such a way that for the resulting tournament $T^*$: each edge between the vertices $v_1, v_3, v_5, \ldots$ goes forward as well as the edges between the vertices $v_2, v_4, v_6, \ldots$.

Note that $\chi(T^*) \leq 2$ holds in the Theorem above. The restriction of the conjecture to arbitrary finite digraphs was solved by Charbit, a former graduate student of Thomassé.

**Theorem 1.4** (P. Charbit, Theorem 4.5 of [8]). In any finite digraph $D$, it is always possible to reverse directed cycles iteratively in such a way that $\chi(D^*) \leq 2$ holds for the resulting orientation $D^*$.

The Theorem is a consequence of the following interesting characterisation:

**Theorem 1.5** (P. Charbit, Theorem 4.4 of [8]). Let $D$ be a digraph and $k \geq 1$. Then $\chi(D) \leq k$ if and only if there exists a linear order $<$ on $V(D)$ such that for any directed cycle $C$ of $D$ at least $\frac{|C|}{k}$ edges of $C$ are forward edges (with respect to $<$).

Indeed, for the given finite digraph $D$ let us fix an arbitrary linear order $<$ on $V(D)$. If for each directed cycle at least half of the edges goes forward, then $\chi(D) \leq 2$ by Theorem 1.5. Otherwise, we reverse a directed cycle which violates this condition which operation increases the total number of forward edges.

In the infinite case, the condition “each edge is reversed only finitely many times” of Conjecture 1.2 has an important role. Without that we do not have a well-defined orientation after some limit step. Under this “local finiteness” assumption each edge has a stabilized orientation before any limit step of a transfinite sequence of cycle reversions and hence a natural limit orientation can be defined.

Although Theorem 1.5 remains true for infinite digraphs by compactness, the proof of Theorem 1.4 based on it does not seem to adapt for the infinite case. Indeed, after the reversion of a violating cycle the total number of forward edges may remain the same infinite cardinal.

For the infinite case new ideas were needed. The breakthrough was due to the first and third author in [9] for infinite tournaments. They actually proved more than Conjecture 1.2: one can transform any tournament by iteratively reversing directed cycles to a linear order “modulo finite blocks”. More precisely:

**Theorem 1.6** (P. Ellis, D. T. Soukup, [9]). In every tournament $T$, one can iteratively reverse directed cycles (reversing each edge only finitely often) such that in the resulting reorientation $T^*$ each strong component is finite.

By applying Theorem 1.4 to the (already finite) strong components separately, one can reduce the dichromatic number to at most two.
Considering general digraphs, one cannot hope to transform all the strong components to finite. Indeed, take \( \kappa \geq \aleph_0 \) many directed cycles, pick one vertex in each and identify these vertices.\(^1\) The resulting digraph remains the same (up to isomorphism) after any iterative cycle reversion.

Our first main result is somewhat analogous to Theorem 1.6 but for arbitrary digraphs: we can reverse cycles to make the strong components have low connectivity.

**Theorem 1.7.** In every digraph \( D \), one can iteratively reverse directed cycles (reversing each edge only finitely often) such that in the resulting reorientation \( D^* \) in each strong component every local edge-connectivity is finite.

Then, our second main result is the positive answer for the original conjecture in its whole generality:

**Theorem 1.8.** In every digraph \( D \) one can reverse directed cycles iteratively (reversing each edge only finitely often) such that \( \chi(D^*) \leq 2 \) holds for the resulting reorientation \( D^* \).

The paper is organized as follows. We introduce some notation in the following section. Then we develop the necessary tools in sections 3-6 from which we derive our main results in section 7. The next section is devoted to some structural consequences of the main results. We mention some open problems in section 9. In the appendix, we collected the basic facts about elementary submodels that we use.

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### 2 Notation

#### Graphs and digraphs

In graphs we allow multiple edges but not loops. More precisely, an edge \( e \) is an ordered pair \( \{u, v\}, i \) where \( u \neq v \) are its end vertices (the only role of index \( i \) is to represent multiple edges). An undirected graph \( G \) is a set of edges, its vertex set \( V(G) \) is the set of the end vertices of its edges.

The variable \( \overrightarrow{e} \) stands for an orientation of \( e \) from the two possibilities, more precisely \( \overrightarrow{e} \in \{(u, v, i), (v, u, i)\} \). A digraph \( D \) is what we obtain by orienting (the edges of) an undirected graph \( G \). The inverse operation \( \text{Un}(D) \) gives back the underlying undirected graph \( G \). We define \( V(D) \) to be \( V(\text{Un}(D)) \). The set of the outgoing edges of the vertex set \( W \) is denoted by \( \text{out} \( D \)(W) \) and \( \text{in} \( D \)(W) \) stands for the ingoing edges. For \( E \subseteq \text{Un}(D) \), we write \( D(E) \) for the set of the \( D \)-oriented elements of \( E \), i.e., the unique \( D' \subseteq D \) with \( \text{Un}(D') = E \). A digraph \( D^* \) is a reorientation of \( D \) if \( \text{Un}(D^*) = \text{Un}(D) \).

Cycles and paths are always meant to be directed. Formally, a cycle is a digraph of the form \( \{(v_0, v_1, i_0), (v_1, v_2, i_1), \ldots, (v_n, v_0, i_n)\} \) where \( n \geq 1 \) and \( v_i \) are pairwise distinct. Paths

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\(^1\)This was pointed out by Carl Bürger (personal communication).
are defined similarly. The local edge-connectivity from \( u \) to \( v \) in a digraph \( D \) is the maximal size of a system \( \mathcal{P} \) of pairwise edge-disjoint paths from \( u \) to \( v \) in \( D \) and it is denoted by \( \lambda(u, v; D) \). The strong components of \( D \) are the equivalence classes of \( V(D) \) where \( u \sim v \) if \( \lambda(u, v; D), \lambda(v, u; D) > 0 \). We refer to the subdigraphs spanned by the strong components also as strong components but it will not lead to any confusion. A digraph is strongly connected (or strong) if it has only one strong component.

**Set theory**

We use some standard set theoretic notation. We write \( \bigcup \mathcal{X} \) for the union of the elements of the set family \( \mathcal{X} \). Variables \( \alpha, \beta, \xi \) stand for ordinals and \( \kappa \) denotes a cardinal. The set of the natural numbers is \( \omega \). We say a set is countable if it is either finite or countably infinite. The restriction of a sequence \( s \) of length at least \( \alpha \) to \( \alpha \) is \( s \rvert \alpha \). The concatenation of sequences \( s \) and \( z \) is denoted by \( s \circ z \). If a sequence has only one member \( C \), then we abuse the notation and write simply \( C \) for the sequence itself as well. We write \( [X]^\kappa \) and \( [X]^\leq\kappa \) for the set of the \( \kappa \)-sized and smaller than \( \kappa \) subsets of \( X \) respectively.

**Cycle reversions**

Let \( G \) be an undirected graph and let \( \langle D_\xi : \xi < \alpha \rangle \) be a sequence of orientations of \( G \). If the orientation of each \( e \in G \) is stabilizing in the sense that there is some \( \xi_e < \alpha \) such that \( D_\xi(e) = D_\xi(e) \) whenever \( \xi \geq \xi_e \), then we define the limit orientation \( D^* \) of the sequence by letting \( D^*(e) := D_{\xi_e}(e) \) for \( e \in G \). It will be convenient to always have some kind of limit-type object even when the orientations of the edges are not stabilized. The set of generalized limits of \( \langle D_\xi : \xi < \alpha \rangle \) is defined to be the set of those orientations \( D^* \) of \( G \) for which for every finite \( F \subseteq G \) the set

\[
\{ \xi < \alpha : D_\xi(F) = D^*(F) \}
\]

is unbounded in \( \alpha \). One can construct a generalized limit by taking an ultrafilter \( U \) on \( \alpha \) containing all the non-empty terminal segments of \( \alpha \) and orienting an edge \( e \in G \) between \( u \) and \( v \) towards \( v \) iff

\[
\{ \xi < \alpha : D_\xi(e) \text{ points towards } v \} \in U.
\]

For a digraph \( D \), we define the set \( \mathbb{R}(D) \) of reversion sequences acting on \( D \) and their effect by the following recursion. The sequence \( C = \langle C_\xi : \xi < \alpha \rangle \) is in \( \mathbb{R}(D) \) and \( D \cup C = D^* \) if

- either \( \alpha = 0 \) and \( D^* = D \),
- or \( \alpha = \beta + 1, \langle C \upharpoonright \beta \rangle \in \mathbb{R}(D) \), \( C_\beta \) is a directed cycle in \( D \cup \langle C \upharpoonright \beta \rangle \) and we obtain \( D^* \) by reversing the direction of each edge in \( C_\beta \) in \( D \cup \langle C \upharpoonright \beta \rangle \),
- or \( \alpha \) is limit ordinal, \( \langle C \upharpoonright \beta \rangle \in \mathbb{R}(D) \) for \( \beta < \alpha \) and \( \langle D \cup \langle C \upharpoonright \beta \rangle : \beta < \alpha \rangle \) has a limit which is \( D^* \).

Note that if \( C \in \mathbb{R}(D) \), then for every \( e \in \text{ Un}(D) \), there can be only finitely many \( C_\beta \) that contains an orientation of \( e \). We say that a \( C \in \mathbb{R}(D) \) touches (uses) the edge \( e \) \( n \) times if there are exactly \( n \) cycles in the sequence \( C \) that contains an orientation of \( e \). Also, we define \( E(C) \) to be \( \bigcup_{C \in \text{ ran}(C)} \text{ Un}(C) \).

A reorientation \( D^* \) of \( D \) is reachable from \( D \) if \( D^* = D \cup C \) for a suitable \( C \in \mathbb{R}(D) \). We say that \( D^* \) is locally reachable from \( D \) if for all finite \( F \subseteq \text{ Un}(D) \) there is a \( C \in \mathbb{R}(D) \) such that \( (D \cup C)(F) = D^*(F) \).
3 Reversion sequences and local reachability

In this subsection we summarize some structural properties of reversion sequences (some of them were already discovered by the first and third author in [9]).

**Proposition 3.1.** If \((D_\xi : \xi < \alpha)\) is a non-empty sequence of orientations of a graph \(G\) such that for all large enough \(\xi\), \(D_\xi\) is locally reachable from some \(D\), then every generalized limit of the sequence \((D_\xi : \xi < \alpha)\) is locally reachable from \(D\).

**Proof.** Let \(F \subseteq G\) be finite. By the definition of the generalized limit we can pick some \(\xi < \alpha\) such that \(D_\xi(F) = D^*(F)\) and \(D_\xi\) is locally reachable from \(D\). □

**Proposition 3.2.** For every digraph \(D\), there is a reorientation \(D^*\) of \(D\) which is locally reachable from \(D\) and \(\chi(D^*) \leq 2\).

**Proof.** For finite \(D\), we simply apply Theorem 1.4. For an infinite \(D\), one can use standard compactness arguments. Indeed, take for example an ultrafilter \(U\) on the finite subsets \([D]^{<\aleph_0}\) of \(D\) containing the set \(\mathcal{X}_2 = \{H \in [D]^{<\aleph_0} : \vec{\gamma} \in H\}\) for each \(\vec{\gamma} \in D\). For every \(H \in [D]^{<\aleph_0}\), use Theorem 1.4 to fix a \(C_H\) and a two-colouring \(c_H : V(H) \rightarrow 2\) such that \(c\) witnesses \(\chi(H \cup C_H) \leq 2\). It is easy to check that the unique orientation \(D^*\) satisfying

\[\{H \in \mathcal{X}_2 : D^*(e) = (H \cup C_H)[e]\} \in U\]

for every \(e \in \text{Un}(D)\) is locally reachable from \(D\) and \(\chi(D^*) \leq 2\) witnessed by the unique \(c : V(D) \rightarrow 2\) for which for every \(v \in V(D)\)

\[\{H \in [D]^{<\aleph_0} : v \in V(H) \land c_H(v) = c(v)\} \in U.\]

□

**Proposition 3.3.** Let \(D\) be a digraph and let \(C \in \mathcal{RS}(D)\) be finite. Then there is a \(E \in \mathcal{RS}(D)\) which consists of finitely many edge-disjoint cycles of \(D\) in some order such that \(D \cup E = D \cup C\).

**Proof.** We need to show that the set of edges we reverse by applying \(C\) to \(D\), namely \(R := D \setminus (D \cup C)\), can be partitioned into cycles. One can show by induction on the length of \(C\) that for each vertex \(v \in V(D)\), the number of the ingoing and outgoing edges of \(v\) in \(R\) are equal. Since \(R\) is finite, it follows that the desired partition can be constructed “greedily”. □

**Proposition 3.4** (Corollary 8.5 in [9]). Let \(D\) be a digraph, \(C \in \mathcal{RS}(D)\), and let \(F \subseteq \text{Un}(D)\) be finite. Then there is a finite subsequence \(C_F \in \mathcal{RS}(D)\) of \(C\) such that \((D \cup C)(F) = (D \cup C_F)(F)\).

**Corollary 3.5.** The local reachability relation is transitive, i.e., if \(D^*\) is locally reachable from \(D\) and \(D^*\) is locally reachable from \(D^*\), then \(D^*\) is locally reachable from \(D\).

**Proof.** Let \(F \subseteq \text{Un}(D)\) be finite. Pick a \(C_F \in \mathcal{RS}(D)\) for \(D^*\) and \(D^*\) as in Proposition 3.4. Apply Proposition 3.4 again, this time with \(E := E(C_F) \cup F, D\) and \(D^*\) to obtain \(C_E\). Then \((D \cup C_E \cup C_F)(F) = D^{**}(F)\).

Local reachability implies a formally stronger property, namely the countable version of itself.

**Claim 3.6.** If \(D^*\) is locally reachable from \(D\), then for all countable \(E \subseteq \text{Un}(D)\) there is a \(C_E \in \mathcal{RS}(D)\) of length at most \(\omega\) such that \(C_E\) does not use any edge more than twice and \((D \cup C_E)(E) = D^*(E)\). In particular, among countable digraphs local reachability implies reachability.
Proof. If \( E \) is finite, then by Propositions 3.3 and 3.4, we can choose finitely many edge-disjoint cycles in \( D \) in such a way that if \( C_E \) is an enumeration of them then \((D \cup C_E)(E) = D^*(E)\).

Assume that \( E = \{e_n\}_{n<\omega} \) and let \( C_0 := \emptyset \). Suppose that \( C_k \in \mathcal{RS}(D) \) is already defined for \( k \leq n \) in such a way that

1. \( C_n \) is finite,
2. \( C_j \subseteq C_k \) if \( j \leq k \leq n \),
3. \( C_n \) touches each edge at most twice,
4. if edge \( e \) is touched by \( C_n \) twice or \( e = e_k \) for some \( k < n \) then \((D \cup C_n)(e) = D^*(e)\).

Observation 3.7. For a finite \( C \in \mathcal{RS}(D) \), there is a (not necessarily unique) \( C^{-1} \in \mathcal{RS}(D \cup C) \) such that \( D \cup C \cup C^{-1} = D \). Indeed, we can flip back the cycles starting from the last one.

By property 1 and Observation 3.7, \( D \) is reachable from \( D \cup C_n \) and hence \( D^* \) is locally reachable from \( D \cup C_n \). Let \( F := E(C_n) \cup \{e_k\}_{k \leq n} \). According to the first paragraph of this proof, there is a finite \( E \in \mathcal{RS}(D \cup C_n) \) consisting of edge-disjoint cycles for which we have \((D \cup C_n \cup E)(F) = D^*(F)\). Then it is easy to check that the extension \( C_{n+1} := C_n \ominus E \) maintains the conditions. Finally let \( C_E := \bigcup_{n<\omega} C_n \).

Remark 3.8. At this point we are able to prove the restriction of Theorem 1.8 to countable digraphs. Indeed, for a countable digraph \( D \) pick a reorientation \( D^* \) with \( \chi(D^*) \leq 2 \) which is locally reachable from \( D \) (see Proposition 3.2) and “reach it” by Claim 3.6. The next remark shows that this proof method cannot be extended to uncountable digraphs.

Remark 3.9. Claim 3.6 is sharp in the sense that it may fail for uncountable \( E \). Indeed, consider the digraph consisting of countably infinite \( u \to v \) edges and \( \aleph_1 \) many \( v \to u \) edges. The reorientation in which each edge points towards \( v \) is locally reachable. But Corollary 4.2 will guarantee that it is not reachable.

We proceed with two further properties of reversion sequences. If \( C' \) is a rearrangement of some \( C \in \mathcal{RS}(D) \) in such a way that \( C' \in \mathcal{RS}(D) \), then necessarily \( D \cup C = D \cup C' \). Indeed, the final direction of an \( e \in \text{Un}(D) \) depends only on \( D(e) \) and on the parity of the number of cycles in the sequence containing some orientation of \( e \). Corollary 8.3 and Theorem 8.4 of [9] ensures that reversion sequences can be always rearranged in such a way that the (new) length is a cardinal and the sequence consists of edge-disjoint \( \omega \)-blocks. More precisely:

**Theorem 3.10.** Let \( D \) be a digraph and let \( C \in \mathcal{RS}(D) \) with \( |C| =: \kappa \geq \aleph_0 \). Then there is a rearrangement \( E \in \mathcal{RS}(D) \) of \( C \) with length \( \kappa \). Moreover, if \( \kappa > \aleph_0 \) then one can choose \( E \) in such a way that for \( \alpha < \kappa \),

\[
E_\alpha := \{C_{\omega\alpha+n} : n < \omega\} \in \mathcal{RS}(D),
\]

and \( E(E_\beta) \cap E(E_\alpha) = \emptyset \) if \( \beta < \alpha < \kappa \) (where \( \omega \alpha \) stands for ordinal multiplication).

By sorting the \( \omega \)-blocks above according if it uses an edge from a given edge set \( E \) or not we obtain the following.

**Corollary 3.11.** Assume that \( D \) is a digraph, \( E \subseteq \text{Un}(D) \) and \( C \in \mathcal{RS}(D) \). Then there is a rearrangement \( E \) of \( C \) such that \( E = E_0 \ominus E_{-0} \) where

- \( E_E, E_{-E} \in \mathcal{RS}(D) \)
- \( E(E_E) \cap E(E_{-E}) = \emptyset \),

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• $E \cap E(\mathcal{E}_{-E}) = \emptyset$,

• the length of $\mathcal{E}_E$ is $|\mathcal{E}_E| \leq |E| + \aleph_0$.

Note that $(D \cup \mathcal{E})(E) = (D \cup \mathcal{E}_E)(E)$ follows.

From Claim 3.6 we obtain (using Theorem 3.10 if the length of $\mathcal{C}$ is more than $\omega$):

**Corollary 3.12.** For every $\mathcal{C} \in \mathbb{R}(D)$, there is a $\mathcal{E} \in \mathbb{R}(D)$ such that:

• $D \cup \mathcal{C} = D \cup \mathcal{E}$,

• $E(\mathcal{E}) \subseteq E(\mathcal{C})$,

• $\mathcal{E}$ touches each edge at most twice.

**Remark 3.13.** One cannot replace “twice” by “once” at the last point of Corollary 3.12. On the one hand, it is easy to see that reversing pairwise edge-disjoint cycles cannot change the local edge-connectivities. On the other hand, a reversion sequence can (see for example in Proposition 4.7).

## 4 Local edge-connectivities and cycle reversions

To prove Theorem 1.8 for arbitrary digraphs, knowing the theorem for countable ones (see Remark 3.8), it seems a natural idea to partition $D$ into digraphs of smaller size by a chain of elementary submodels and solve the problem for these smaller pieces separately. (For Readers not yet familiar with elementary submodels we suggest to look at our short survey at the Appendix 10.) This naive approach does not work in general since we may have cycles not living in just a single member of the partition. One can make this approach work under the (very strong) assumption that in every strong component of $D$ every local edge-connectivity is countable. Indeed, in this case if $M$ is a countable elementary submodel containing $D$, then every cycle $C$ of $D$ that has an edge in $D \cap M$ is entirely in $D \cap M$ (otherwise $C \setminus M$ contains a $u \to v$ path for some vertices $u \neq v \in M$ which ensures $\lambda(u, v; D) > |M|$ by Fact 10.3). Furthermore, each strong component of $D \setminus M$ would contain at most one vertex from $V(D) \setminus M$ because of Fact 10.3. We would then be able to solve the problem for $D \cap M$ and proceed with $D \setminus M$ separately. Hence we shift our focus to investigating the possibility of destroying uncountable edge-connectivities inside strong components by a reversion sequence.

**Claim 4.1.** For every digraph $D$, $\mathcal{C} \in \mathbb{R}(D)$ and $W \subseteq V(D)$, $|\text{out}_D(W)| \geq |\text{out}_{D \cup \mathcal{C}}(W)|$.

**Proof.** Suppose to the contrary that $|\text{out}_D(W)| < |\text{out}_{D \cup \mathcal{C}}(W)|$. If $|\text{out}_D(W)| < \aleph_0$, then by Proposition 3.4, a finite subsequence of $\mathcal{C}$ would already increases the number of outgoing edge of $W$ which is clearly impossible. Let $|\text{out}_D(W)| := \kappa \geq \aleph_0$. By taking a suitable initial segment of $\mathcal{C}$, we may assume $|\text{out}_{D \cup \mathcal{C}}(W)| = \kappa^+$. Without loss of generality we suppose that $\mathcal{C}$ has length $\kappa^+$ (use Corollary 3.11 with $E := \text{Un}(\text{out}_{D \cup \mathcal{C}}(W))$). Note that for any $\alpha < \kappa^+$ for $\mathcal{C}_{\alpha} := \mathcal{C} \restriction \alpha$, we have $|\text{out}_{D \cup \mathcal{C}_{\alpha}}(W)| \leq \kappa$ since $\mathcal{C}_{\alpha}$ reorients at most $\kappa$ many edges. Let $\beta_0 = 0$. If $\beta_n < \kappa^+$ is defined then pick for each $e \in \text{out}_{D \cup \mathcal{C}_{\beta_n}}(W)$ some $\beta_{n+1} > \beta_n$ such that $e \notin E(\mathcal{C}_\beta)$ for $\beta \geq \beta_{n+1}$. Let $\beta_{n+1} := \sup\{\beta_e : e \in E(\text{out}_{D \cup \mathcal{C}_{\beta_n}}(W))\}$ and finally $\alpha := \sup_{n < \omega} \beta_n < \kappa^+$. By construction, none of the edges $\text{out}_{D \cup \mathcal{C}_{\beta}}(W)$ are touched by any $\mathcal{C}_\beta$ with $\beta \geq \alpha$ and therefore none of the edges $\text{in}_{D \cup \mathcal{C}_\alpha}(W)$ as well. Thus $\text{out}_{D \cup \mathcal{C}}(W) = \text{out}_{D \cup \mathcal{C}_{\beta}}(W)$ which is a contradiction.

\[ \square \]
Corollary 4.2. For every digraph \( D \), \( u \neq v \in V(D) \) and \( C \in \mathbb{RS}(D) \), \( \lambda(u, v; D \cup C) \leq \lambda(u, v; D) \).

Proof. The local edge-connectivity from \( u \) to \( v \) can be expressed as the smallest size of the \( u \rightarrow v \) cuts (a \( u \rightarrow v \) cut is an edge set such that every \( u \rightarrow v \) path of \( D \) must use an edge of it). Thus by applying the previous claim we obtain the following:

\[
\lambda(u, v; D) = \min \{ \text{out}_{D}(W) : \{ u \} \subseteq W \subseteq V(D) - v \} \geq \min \{ \text{out}_{D \cup C}(W) : \{ u \} \subseteq W \subseteq V(D) - v \} = \lambda(u, v; D \cup C).
\]

\[\square\]

Proposition 4.3. Let \( D \) be a digraph and \( u \neq v \in V(D) \) with \( \lambda(v, u; D) \geq \aleph_0 \). Then for each \( u \rightarrow v \) path \( P \), there is a \( C \in \mathbb{RS}(D) \) of length \( \omega \) such that \( D \setminus (D \cup C) = P \) (i.e., \( C \) flips exactly \( P \)). In addition, \( C \) can be chosen in such a way that it avoids a prescribed edge set of size less than \( \lambda(v, u; D) \) which is disjoint from \( \bigcup P \).

Proof. Let \( \{ Q_n \}_{n < \omega} \) be a set of pairwise edge-disjoint \( v \rightarrow u \) paths avoiding both \( P \) and a prescribed edge set of size less than \( \lambda(v, u; D) \). Then \( P \cup Q_0 \) is the union of finitely many edge-disjoint cycles (see Figure 1), reverse these cycles in an arbitrary order. In the resulting orientation the reverse of \( Q_1 \) together with \( Q_2 \) is the union of finite many cycles thus we can reverse back \( Q_1 \) for the prize of reversing \( Q_2 \). By continuing this recursively we construct the desired \( C \) which flips exactly \( P \).

\[\square\]

Corollary 4.4. Let \( D \) be a digraph and \( u \neq v \in V(D) \) with \( \lambda(v, u; D) \geq \aleph_0 \). Let \( \mathcal{P} \) be a set of edge-disjoint \( u \rightarrow v \) paths of size at most \( \lambda(v, u; D) \). Then there is a \( C \in \mathbb{RS}(D) \) of length at most \( \lambda(v, u; D) \) such that \( D \setminus (D \cup C) = \bigcup \mathcal{P} \). In addition, \( C \) can be chosen in such a way that it avoids a prescribed edge set of size less than \( \lambda(v, u; D) \) which is disjoint from \( \bigcup \mathcal{P} \).

Proof. Take a \( |\mathcal{P}| \)-type enumeration of \( \mathcal{P} \). We use transfinite recursion, in which we apply Proposition 4.3 in each step with the next element of \( \mathcal{P} \) avoiding the prescribed edge set and the edges we touched so far.

\[\square\]

An Erdős-Menger \( u \rightarrow v \) path-system in a digraph \( D \) with \( u \neq v \in V(D) \) is a set \( \mathcal{P} \) of pairwise edge-disjoint \( u \rightarrow v \) paths such that one can choose exactly one edge from each \( P \in \mathcal{P} \) in such a way that the resulting \( T \) is a \( u \rightarrow v \) cut in \( D \). We call such a \( T \) an Erdős-Menger cut orthogonal to \( \mathcal{P} \). Note that \( |\mathcal{P}| = \lambda(u, v; D) \).

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Theorem 4.5 (Infinite Menger’s Theorem, [12]). For every digraph $D$ and $u \neq v \in V(D)$, there is an Erdős-Menger $u \rightarrow v$ path-system.

Remark 4.6. Aharoni and Berger proved originally the vertex-version of Theorem 4.5 but it is known to be equivalent with the edge-version above.

Proposition 4.7. If $\mathcal{P}$ is an Erdős-Menger $u \rightarrow v$ path-system in a digraph $D$ and $D^*$ is the digraph that we obtain from $D$ by reversing the edges $\bigcup \mathcal{P}$, then $D^*$ has no $u \rightarrow v$ path.

Proof. Let $T$ be an Erdős-Menger cut orthogonal to $\mathcal{P}$ and we define $W \subseteq V(D)$ to be the set of those vertices that are reachable from $u$ in $D$ without using any edge from $T$ (see Figure 2). On the one hand, $\text{out}_D(W) = T$. On the other hand, there is no $P \in \mathcal{P}$ which uses an ingoing edge of $W$ because such a $P$ would meet at least two edges from $T$ contradicting the fact that $T$ is orthogonal to $\mathcal{P}$. Thus by reversing $\bigcup \mathcal{P}$ we reverse all the outgoing edges of $W$ and none of the ingoing edges.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The Erdős-Menger cut $T$ and the vertex set $W$.}
\end{figure}

Corollary 4.8. If $u, v$ belong to the same strong component of a digraph $D$ and $\lambda(u, v; D) + \lambda(v, u; D) \geq \aleph_0$, then there is a $C \in \text{RS}(D)$ of length at most $\lambda(u, v; D) + \lambda(v, u; D)$ such that $u$ and $v$ are in different strong component of $D \cup C$.

Proof. By symmetry we may assume that $\lambda(v, u; D) \geq \lambda(u, v; D)$. Take an Erdős-Menger $u \rightarrow v$ path-system $\mathcal{P}$ in $D$ and flip it applying Corollary 4.4. By Proposition 4.7, $u$ and $v$ are no more in the same strong component and hence we are done.

5 Scattered digraphs and elementary submodels

Instead of just destroying all the infinite local edge-connectivities inside every strong component it will be more convenient to acquire a slightly stronger property by a suitable reversion sequence. We call a digraph $D$ \textbf{scattered} if there is no strong component of $D$ that can be subdivided by a suitable reversion sequence. More precisely, whenever $u, v$ are in the same strong component of $D$ they are in the same strong component of $D \cup C$ for every $C \in \text{RS}(D)$. Corollary 4.8 shows that being scattered implies that the local edge-connectivities are finite in every strong component. A digraph $D$ is \textbf{$(W, \kappa)$-scattered}, where $W \subseteq V(D)$ and $\kappa$ is a cardinal, if whenever $u, v \in W$ are in the same strong component of $D$ they are in the same strong component of $D \cup C$ for every $C \in \text{RS}(D)$ of length less than $\kappa$. Note that $(V(D), |D|^+)$-scattered is equivalent with scattered.

Proposition 5.1. Let $D$ be $(W, \kappa)$-scattered where $\kappa > \aleph_0$ and suppose that $u \neq v \in W$ are in the same strong component of $D$. If $\lambda(u, v; D) + \lambda(v, u; D)$ is infinite, then $\lambda(u, v; D), \lambda(v, u; D) \geq \kappa$. 

Proof. By symmetry we may assume that \( \lambda(u, v; D) \leq \lambda(v, u; D) \). Suppose for contradiction that \( \lambda(u, v; D) < \kappa \). Let \( P \) be an Erdős-Menger \( u \rightarrow v \) path-system. By applying Corollary 4.4, there is a \( C \in \text{RS}(D) \) of length \( \lambda(u, v; D) < \kappa \) which reverses exactly \( \bigcup P \) in \( D \). By Proposition 4.7, \( v \) is not reachable from \( u \) in \( D \setminus C \). Since \( u \) and \( v \) were in the same strong component of \( D \), it contradicts the fact that \( D \) is \((W, \kappa)\)-scattered.

Our goal is to prove that for every digraph \( D \) there is a scattered \( D^\ast \) reachable from \( D \) (which is a strengthening of Theorem 1.7). Elementary submodels will play an important role in the proof, and we will need a couple of statements to be able to use them properly.

Proposition 5.2. Suppose that \( M \supseteq |M| \) is an elementary submodel, \( D \in M \) is a digraph and \( D^\ast \) (which may fail to be in \( M \)) is locally reachable from \( D \). Then \( D^\ast \cap M \) is locally reachable from \( D \cap M \).

Proof. Let \( F \subseteq \text{Un}(D \cap M) \) be finite. By using Propositions 3.4 and 3.3 and the fact that \( D^\ast \) is locally reachable from \( D \), we conclude that there is a finite sequence \( \langle C_i \rangle_{i<n} =: C \in \text{RS}(D) \) of edge-disjoint cycles of \( D \) for which \( (D \cap C)(F) = D^\ast(F) \). For every \( i < n \), the set \( C_i \setminus M \) consists of finitely many pairwise edge-disjoint paths \( P_{i,j} \) where \( P_{i,j} \) goes from some \( u_{i,j} \) to some \( v_{i,j} \) with \( u_{i,j}, v_{i,j} \in M \) (see Figure 3). The idea is to “fix” the cycles by replacing these segments with something that goes inside \( M \). Indeed, the existence of these paths implies (using Fact 10.3) that \( \lambda(u_{i,j}, v_{i,j}; D) > |M| \) and \( \lambda(u_{i,j}, v_{i,j}; D \cap M) = |M| \). Let \( Q_{i,j} \) \((i < n, j < n)\) be pairwise edge-disjoint paths in \( D \cap M \) where \( Q_{i,j} \) is a \( u_{i,j} \rightarrow v_{i,j} \) path and they do not use any edge from \( F \cup E(C) \). For every vertex of the finite subdigraph

\[
\left( \bigcup_{i<n} C_i \setminus \bigcup_{i<n, j<n} P_{i,j} \right) \cup \bigcup_{i<n, j<n} Q_{i,j},
\]

of \( D \cap M \), the indegree is equal to the outdegree therefore it is the union of edge-disjoint cycles. By making a sequence \( E \in \text{RS}(D \cap M) \) from these cycles, we have

\[
((D \cap M) \cap E)(F) = (D \cap C)(F) = (D \cap C)(F) = D^\ast(F) = (D^\ast \cap M)(F).
\]

Thus \( D^\ast \cap M \) is locally reachable from \( D \cap M \).

\[ \text{Figure 3: Replacing the dashed part of cycle } C_0. \]
Now we want to scatter a digraph $D$ as much as possible, reversing edges only inside an elementary submodel $M \supseteq D$ and remaining locally reachable from $D$. In the applications we will have to start with some $L$ which may fail to be in $M$ but which will be locally reachable from $D$ and identical to $D$ outside of $M$.

**Lemma 5.3.** Suppose that $M \supseteq |M|$ is an elementary submodel, $D \in M$ is a digraph and $L$ (which may fail to be in $M$) is locally reachable from $D$ and satisfies $L \setminus M = D \setminus M$. Then there is an $(V(D) \cap M, |M|^\uparrow)$-scattered $D^* = D^*(M, L)$ (does not depend on $D$) with $D^* \setminus M = D \setminus M$ which is locally reachable from $D$.

**Proof.** Let $\kappa := |M|$ and let $\{u_\alpha, v_\alpha\} : \alpha < \kappa^+$ be a sequence with range $[V(D) \cap M]^2$ in which the appearance of every element of $[V(D) \cap M]^2$ is unbounded. We define a sequence $(L_\alpha : \alpha \leq \kappa^+)$ starting with $L_0 := L$. If $L_\beta$ is $(\{u_\beta, v_\beta\}, \kappa^+)$-scattered, then let $L_{\beta+1} := L_\beta$. If it is not and $C_\beta$ is a witness for it, then let $L_\beta' := L_\beta \cap C_\beta$ and we define $L_{\beta+1}$ to be $(L_\beta' \cap M) \cup (L \setminus M)$. If $\alpha$ is a limit ordinal and $L_\beta$ is defined for $\beta < \alpha$, then let $L_\alpha$ be a generalized limit of $(L_\beta : \beta < \alpha)$. Finally we define $D^*$ to be $L_{\kappa^+}$. Note that $L \setminus M = D \setminus M$ by assumption and the recursion preserves this property thus $L_\alpha \setminus M = D \setminus M$ for $\alpha \leq \kappa^+$.

**Proposition 5.4.** $L_\alpha$ is locally reachable from $D$ for every $\alpha \leq \kappa^+$.

**Proof.** For $\alpha=0$ it holds by assumption. For limit steps, it follows from Corollary 3.1. For a successor step, $L_\beta' = L_\beta \cap C_\beta$ is locally reachable from $D$ since $L_\beta$ is locally reachable by induction. By applying Proposition 5.2 with $M, D$ and $L_\beta'$, we obtain that $L_{\beta+1} \cap M$ is locally reachable from $D \cap M$. Since $L_{\beta+1} \setminus M = D \setminus M$, it implies that $L_{\beta+1}$ is locally reachable from $D$.

**Claim 5.5.** If $L_{\beta+1} \neq L_\beta$ for some $\beta < \kappa^+$, then $u_\beta$ and $v_\beta$ are in different strong components of $L_\alpha$ for $\beta < \alpha \leq \kappa^+$.

**Proof.** By construction, $u_\beta, v_\beta$ are in different strong components of $L_\beta'$. By symmetry we may assume that $v_\beta$ is not reachable from $u_\beta$ in $L_\beta'$. First we show that this remains true in $L_{\beta+1}$ as well. Suppose for a contradiction that $P$ is a $u_\beta \rightarrow v_\beta$ path in $L_{\beta+1}$. We show that we must have such a path already in $L_\beta'$. Since $L_{\beta+1} \cap M = L_\beta \cap M$, we have $M \cap P \subseteq L_\beta'$. We prove that the $P \setminus M$ part of $P$ can be replaced in $L_\beta'$. Indeed, $P \setminus M$ consists of edge-disjoint paths $Q_i$ $(i < n)$ where $Q_i$ is a $u_i \rightarrow v_i$ path for some $u_i, v_i \in V(D) \cap M$. Since $L_{\beta+1} \setminus M = D \setminus M$, $Q_i$ lies in $D \setminus M$ and then Fact 10.3 shows that $\lambda(u_i, v_i, D) > |M| = \kappa$. Because of $|D \setminus L_\beta'| \leq |M| + |C_\beta| \cdot |\mathcal{N}| = \kappa$, we may conclude that $\lambda(u_i, v_i, L_\beta') > \kappa$. In particular, $v_i$ is reachable from $u_i$ in $L_\beta'$, but then $v_\beta$ is reachable from $u_\beta$ in $L_\beta'$ which is a contradiction.

Let $T \subseteq L_{\beta+1}$ be a directed cut witnessing the non-existence of a path from $u_\beta$ to $v_\beta$ in $L_{\beta+1}$. Note that the edges $\text{Un}(T \setminus M)$ are oriented in $T$ as in $D$ because $(T \setminus M) \subseteq (L_{\beta+1} \setminus M) = (D \setminus M)$. We show by transfinite induction that $T \subseteq L_\alpha$ whenever $\alpha \geq \beta + 1$. The initial step and the limit steps are obvious. Suppose we know the statement for some $\alpha$. Since the orientation $L_\alpha$ is reachable from $L_\alpha$ by the construction, $T \subseteq L_\alpha$ also holds. We obtain $L_{\alpha+1}$ from $L_\alpha$ by flipping the edges out of $M$ to the direction defined by $D$. Since $D$ and $T$ orients the edges $\text{Un}(T \setminus M)$ the same way, it does not reverse any edge in $T$ and hence $T \subseteq L_{\alpha+1}$.

It remains to show that $D^*$ is $(V(D) \cap M, |M|^\uparrow)$-scattered. For every $u \neq v \in V(D) \cap M$, if there is an ordinal $\beta$ for which $\{u_\beta, v_\beta\} = \{u, v\}$ and $L_\beta$ is not $(\{u_\beta, v_\beta\}, \kappa^+)$-scattered then it must be unique by Claim 5.5 and we define $\beta_{u,v} := \beta + 1$. If there is no such a $\beta$, then $\beta_{u,v} := 0$. The terminal segment of the sequence $(L_\alpha : \alpha < \kappa^+)$ from $\text{sup}(\beta_{u,v} : \{u, v\} \in [V(D) \cap M]^2) < \kappa^+$ is constant. Since the appearance of each $\{u, v\} \in [V(D) \cap M]^2$ is unbounded in $(\{u_\alpha, v_\alpha\} : \alpha < \kappa^+)$, it ensures that $L_{\kappa^+}$ is $(V(D) \cap M, \kappa^+)$-scattered. 

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The last tool we will need about elementary submodels is an easy technical lemma which is a strengthening of Fact 10.1. One can show that every uncountable elementary submodel can be written as the union of an increasing continuous chain of smaller sized elementary submodels. We will need a stronger property which is not automatically true. We say that the elementary submodel $M$ has the **chain property** if either $|M| = \aleph_0$ or $|M| = \kappa > \aleph_0$ and there is an increasing continuous chain $\langle M_\alpha : \alpha < \kappa \rangle$ of elementary submodels such that

1. $M = \bigcup_{\alpha < \kappa} M_\alpha$,
2. $\alpha \in M_\alpha$,
3. $|\alpha| \leq |M_\alpha| < |M|$,
4. $\langle M_\xi : \xi \leq \alpha \rangle \in M_{\alpha+1}$,
5. $M_\alpha$ has the chain property.

**Observation 5.6.**
- Chain property is well-defined since to decide if $M$ has it, it is enough to know this for elementary submodels that are elements of $M$.
- For an elementary submodel $M$ with chain property $|M| \subseteq M$ holds.
- If $\langle M_\alpha : \alpha < \kappa \rangle$ witnesses the chain property of $M$, then so does every non-empty terminal segment of it.
- $M_\beta \in M_\alpha$ for $\beta < \alpha$ in the chain above.

**Proposition 5.7.** For every infinite cardinal $\kappa$ and set $X$, there is an elementary submodel $M \ni X$ of size $\kappa$ that has the chain property.

**Proof.** We use induction on $\kappa$. For $\kappa = \aleph_0$, it follows from Fact 10.1. Suppose that $\kappa > \aleph_0$ and the statement is known below $\kappa$. We build the sequence in the definition by transfinite recursion and define $M$ as the union of the chain. Let $M_0$ be an arbitrary countable elementary submodel containing $X$. To build $M_{\alpha+1}$, we use the induction hypothesis with $|\alpha| + \aleph_0$ and $X' := \{\alpha, \langle M_\xi : \xi < \alpha \rangle\}$. We need to check that the defined sequence is really increasing. Fact 10.2 ensures that $\alpha \subseteq M_\alpha$ and hence $M_\beta \in M_\alpha$ for $\beta < \alpha$ since $M_\beta$ is definable from $\beta$ and $\langle M_\xi : \xi < \alpha \rangle$. Then by applying Fact 10.2 again, $M_\beta \subseteq M_\alpha$. 

## 6 The key Lemma

The main difficulty by the elementary submodel approach in this problem is the following. After applying a reversion sequence inside the part of the digraph $D$ covered by some elementary submodel $M$ it is advisable to avoid the edges in $M$ in further cycle reversions (otherwise we may flip eventually some edge infinitely often). How can we guarantee that it is possible? We introduce a lemma that let us overcome this difficulty. Whenever we have a $H \subseteq D$ with some special properties, it guarantees that for any $C \in \mathbb{RS}(D)$ we can find an $E \in \mathbb{RS}(D \setminus H)$ that reverses exactly the same edges among $D \setminus H$ as $C$.

**Lemma 6.1.** Let $D$ be a digraph and let $H \subseteq D$ be infinite. Assume that $\lambda(v, u; D), \lambda(v, u; D) > |H|$ holds for every $u \neq v \in V(H)$ which are in the same strong component of $D$ and satisfy $\lambda(u, v; D \setminus H) > 0$. Then for every $C \in \mathbb{RS}(D)$, there is a $E \in \mathbb{RS}(D \setminus H)$ such that for every $e \in \text{Un}(D \setminus H)$ we have $((D \setminus H) \cup E)(e) = (D \cup C)(e)$.
Proof. We may assume that the length of $\mathcal{C}$ is a (possibly finite) cardinal $\kappa \leq |H| + \aleph_0$ (use Corollary 3.11 with $E := \text{Un}(H)$ and deal only with the part $\mathcal{E}_E$). Let $\mathcal{C} = \langle C_\xi : \xi < \kappa \rangle$ and let us write $C_\alpha$ for $\mathcal{C} \cap \alpha$. We construct by transfinite recursion an increasing continuous sequence $(\mathcal{E}_\alpha : \alpha \leq \kappa)$ where $\mathcal{E}_\alpha \in \text{RS}(D \setminus H)$ such that for each $e \in D \setminus H$, the reversion sequence $\mathcal{E}_\alpha$ touches $e$ either the same number of times as $C_\alpha$ or, for at most $|\alpha| + \aleph_0$ many $e$, exactly two times more. Obviously $\mathcal{E}_0 := \emptyset$ is appropriate. Suppose that $\mathcal{E}_\alpha$ is defined and the conditions hold so far. Note that for $e \in \text{Un}(D \setminus H)$ we have $((D \setminus H) \cap \mathcal{E}_\alpha)(e) = (D \cap C_\alpha)(e)$.

If $C_\alpha \subseteq H$, let $\mathcal{E}_{\alpha+1} := \mathcal{E}_\alpha$. If $C_\alpha \cap H = \emptyset$, then we define $\mathcal{E}_{\alpha+1}$ to be $\mathcal{E}_\alpha \cap C_\alpha$. The remaining case is when $C_\alpha$ has edges both in $H$ and in $D \setminus H$. Then $C_\alpha \setminus H$ consists of finitely many pairwise edge-disjoint paths $P_i$ ($i < n$) where $P_i$ goes from $u_i$ to some $v_i$ for some $u_i, v_i \in V(H)$. The $u_i, v_i$ are in the same strong component in $D \cap C_\alpha$ and hence in $D$ as well (see Corollary 4.2). Path $P_i$ exemplifies $\lambda(u_i, v_i; (D \setminus H) \cap \mathcal{E}_\alpha) > 0$ but then by Corollary 4.2, $\lambda(u_i, v_i; D \setminus H) > 0$. By the assumption about $H$, we have $\lambda(v_i, u_i; D), \lambda(v_i, u_i; D) > |H|$. So for each $P_i$ pick a $v_i \to u_i$ path $Q_i$, such that $Q_i$ has no common edge with: $H, \mathcal{E}_\alpha, Q_j$ ($j \neq i$), $P_j$ ($j < n$). One can partition $\bigcup_{i \leq n} P_i \cup Q_i$ into finitely many edge-disjoint cycles. Extend $\mathcal{E}_\alpha$ with these cycles in an arbitrary order and then apply Proposition 4.3 (combined with Corollary 3.12) to flip back the paths $Q_i$ without flipping any other edges and without using any edge that we touched already. Let $\mathcal{E}_{\alpha+1}$ be the resulting extension. Limit steps preserve the conditions automatically and $\mathcal{E}_\kappa$ is suitable for $\mathcal{E}$.

Corollary 6.2. Assume that $M$ is an elementary submodel, $D \in M$ is a digraph and $L$ (which may fail to be in $M$) is a $(V(D) \cap M, |M|)$-scattered reorientation of $D$ satisfying $L \setminus M = D \setminus M$. Then for every $\mathcal{C} \in \text{RS}(L)$, there is a $\mathcal{E} \in \text{RS}(D \setminus M)$ such that $((L \setminus M) \cap \mathcal{E})(e) = (L \cap \mathcal{C})(e)$ for $e \in \text{Un}(D \setminus M)$.

Proof. It is enough to show that $L$ and $H := L \setminus M$ satisfy the premises of Lemma 6.1. Suppose that $u, v \in V \cap M$ are in the same strong component of $L$ and assume that $\lambda(u, v; L \setminus M) > 0$. Since $L \setminus M = D \setminus M$, it is equivalent with $\lambda(u, v; D \setminus M) > 0$. By assumption $D \in M$, therefore by Fact 10.3, $\lambda(u, v; D) > |M|$ and hence $\lambda(u, v; L) > |M|$ because $D$ and $L$ orient at most $|M|$ edges differently. Since $L$ is $(V(L) \cap M, |M|)$-scattered, Proposition 5.1 ensures that $\lambda(v, u; L) > |M|$ as well.

7 Proof of the main results

Let us restate the main results for convenience.

Theorem 7.1. For every digraph $D$, there is a $\mathcal{C} \in \text{RS}(D)$ such that $D \cap \mathcal{C}$ is scattered and (hence) in every strong component of $D \cap \mathcal{C}$ all the local edge-connectives are finite.

Theorem 7.2. For every digraph $D$, there is a $\mathcal{C} \in \text{RS}(D)$ such that $\chi(D \cap \mathcal{C}) \leq 2$.

Proof. We prove that for every elementary submodel $M$ that has the chain property (see the definition right before Section 6), for every digraph $D \in M$ and for every reorientation $R$ of $D$ (which may fail to be in $M$) there are $\mathcal{C}_{M, D}, \mathcal{C}_{M, D, R} \in \text{RS}(D \cap M)$, where $\mathcal{C}_{M, D}$ does not depend on $R$ and they satisfy the following properties:

1. $D \cap \mathcal{C}_{M, D}$ is $(V(D) \cap M, |M|)$-scattered,

2. If $R$ is locally reachable from $D \cap \mathcal{C}_{M, D}$, then $(D \cap \mathcal{C}_{M, D, R}) \cap M = R \cap M$.  

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Let us show first that it is enough to justify the main results stated above. Indeed, let a digraph $D$ be given. By Proposition 5.7, we can pick an elementary submodel $M \supseteq D$ of size $|D|$ with chain property. Note that $|D| = |M| \subseteq M$ and hence $D \subseteq M$ by Fact 10.2. Then $D \cap C_{M,D}$ is scattered by property 1 which (together with Corollary 4.8) proves Theorem 7.1. Then by choosing an $R$ with $\chi(R) \leq 2$ which is locally reachable from $D \cap C_{M,D}$ (use Proposition 3.2), $C_{M,D,R}$ satisfies the requirement of Theorem 7.2.

To show that the desired $C_{M,D}, C_{M,D,R} \in \mathbb{R}(D \cap M)$ exist, we apply transfinite induction on $\kappa := |M|$. Suppose that $\kappa = \aleph_0$. First we pick a reorientation $D^*$ of $D$ with $D^* \setminus M = D \setminus M$ which is locally reachable from $D$ and $(V(D) \cap M, \langle N \rangle)$-scattered (use Lemma 5.3 with $L := D$). By Proposition 5.2, $D^* \cap M$ is locally reachable from $D \cap M$. Since $D \cap M$ is countable, local reachability implies reachability by Proposition 3.6. Let $C_{D,M}$ be a witness for this reachability. Suppose that $R$ is locally reachable from $D^*$. By the transitivity of local reachability (see Proposition 3.5), $R$ is locally reachable from $D$. We apply Proposition 5.2 again to conclude that $R \cap M$ is locally reachable from $D \cap M$ and hence by Proposition 3.6 reachable too. We choose $C_{D,M,R}$ to exemplify this reachability.

Let $\kappa > \aleph_0$. Since $M$ has the chain property by assumption, there is an increasing continuous chain $\langle M_\alpha : 1 \leq \alpha < \kappa \rangle$ of elementary submodels such that all the members have the chain property, $\alpha \in M_\alpha$, $|\alpha| \leq |M_\alpha| < \kappa$, $M_\beta = \bigcup_\alpha M_\alpha$ for $\alpha < \kappa$ and $\bigcup_\alpha M_\alpha = M = M_\kappa$.

We point out that Fact 10.2 ensures $\alpha \subseteq M_\alpha$ and $M_\beta \in M_\alpha$ for $\beta < \alpha$. Since $M_\kappa$ contains $D$ by assumption so does some $M_\beta$ where $1 \leq \gamma < \kappa$. By taking a terminal segment of the sequence, we may assume that $\gamma = 1$ (see Observation 5.6). For technical reasons let us define $M_0 := \emptyset$. We construct a sequence $\langle D_\alpha : \alpha \leq \kappa \rangle$ of reorientations of $D$ by transfinite recursion such that for every $\alpha \leq \kappa$:

i. $D_0 = D$,

ii. $D_\beta \in M_\alpha$ for $\beta < \alpha$,

iii. $D_\alpha$ is $(V(D) \cap M_\alpha, |M_\alpha|^\uparrow)$-scattered,

iv. $D_\alpha \setminus M_\alpha = D \setminus M_\alpha$,

v. $D_\alpha$ is locally reachable from $D_\beta$ for $\beta < \alpha$.

Suppose that $D_\xi$ is defined for $\xi < \alpha$ and the conditions i-v hold so far. For $\alpha = 0$, our only choice is $D$ and it clearly preserves the conditions.

Assume that $\alpha = \beta + 1$. Then $M_{\beta+1}$ is an elementary submodel with chain property such that $|M_{\beta+1}| < \kappa$, thus by the induction hypothesis for every digraph $D' \in M_{\beta+1}$ the reversion sequence $C_{M_{\beta+1},D'}$ exists. Then $D_{\beta+1} := D_\beta \cap C_{M_{\beta+1},D_\delta}$ is definable from $M_{\beta+1} \cap D_\beta \in M_{\beta+2}$ and hence $D_{\beta+1} \subseteq M_{\beta+2}$ as demanded by ii. The preservation of the other conditions follows directly from the properties of $C_{M_{\beta+1},D'}$.

Finally let $\alpha$ be a limit ordinal. We take first a generalized limit $L_\alpha$ of $\langle D_\xi : \xi < \alpha \rangle$. Let us remind that generalized limits preserve local reachability (see Proposition 3.1). Then we apply Lemma 5.3 with $D, M_\alpha, L_\alpha$ to obtain $D_\alpha := D^*(M_\alpha, L_\alpha)$. We have $D_\alpha \in M_{\alpha+1}$ since it is definable from $D(M : \beta \leq \alpha) \subseteq M_{\alpha+1}$. Conditions iii and iv follow from Lemma 5.3. Observe that Lemma 5.3 is applicable for the triple $D_\beta, M_\alpha, L_\alpha$ for every $\beta < \alpha$ and gives the same $D_\alpha$. Thus Lemma 5.3 guarantees that $D_\alpha$ is locally reachable from $D_\beta$ for $\beta < \alpha$. The definition of the sequence $\langle D_\alpha : \alpha \leq \kappa \rangle$ is complete.

We will construct $C_{M,D}$ in such a way that $D \cap C_{M,D} = D_\kappa$ holds. To do so, we partition first $D$ and $D_\kappa$ into smaller pieces. Let $D^\alpha := D \cap (M_{\alpha+1} \setminus M_\alpha)$ and $D_\alpha := D_\kappa \cap (M_{\alpha+1} \setminus M_\alpha)$. It is enough to find for $\alpha < \kappa$ a $C_\alpha \in \mathbb{R}(D^\alpha)$ with $D^\alpha \cap E_\alpha = D_\alpha$, since then the concatenation of $E_\alpha(\alpha < \kappa)$ is suitable for $C_{M,D}$. By induction $C_{M_{\alpha+1},D_\alpha, D_\alpha}$ exists for every $\alpha < \kappa$. 

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Proposition 7.3. For every $\alpha < \kappa$, $(D_\alpha \cap C_{M_{\alpha+1},D_\alpha}) \cap M_{\alpha+1} = D_\kappa \cap M_{\alpha+1}$.

Proof. Since $D_\kappa$ is locally reachable from $D_{\alpha+1} = D_\alpha \cap C_{M_{\alpha+1},D_\alpha}$, it follows directly from property 2.

It means that $C_{M_{\alpha+1},D_\alpha}$ transforms simultaneously $D_\beta$ to $D_\kappa$ for every $\beta \leq \alpha$ instead of doing this only for $\alpha$ and not touching $D_\beta$ for $\beta < \alpha$ as required for $E_\alpha$. We fix this by applying Corollary 6.2 with $C_{M_{\alpha+1},D_\alpha}$, $M_\alpha$, $D$, $D_\alpha$. We have just constructed the desired $C_{M,D}$.

Proposition 7.4. For every $\alpha < \kappa$, $(D_\alpha \cap C_{M_{\alpha+1},D_\alpha}) \cap M_{\alpha+1} = R \cap M_{\alpha+1}$

Proof. $R$ is locally reachable from $D_{\alpha+1} = D_\alpha \cap C_{M_{\alpha+1},D_\alpha}$ since we assumed that $R$ is locally reachable from $D_\kappa$ which itself is locally reachable from $D_{\alpha+1}$. In the light of this the Proposition follows from property 2.

Thus $C_{M_{\alpha+1},D_\alpha}$ transforms $D_\beta$ to $R^{\beta} := R \cap (M_{\beta+1} \setminus M_\beta)$ for every $\beta \leq \alpha$. We construct a $E^{*}_{\alpha} \in \mathcal{RS}(D^\alpha)$ by applying Corollary 6.2 with $C_{M_{\alpha+1},D_\alpha}$, $M_\alpha$, $D$, $D_\alpha$ for which $D^{\alpha} \cap E^{*}_{\alpha} = R^{\alpha}$. Let $C_{M,D,R}$ be the concatenation of the sequences $E^{*}_{\alpha}$ ($\alpha < \kappa$).

8. On the structure of scattered strong digraphs

In a scattered tournament every strong component is finite (see Theorem 1.6). How simple is the structure of a scattered digraph in general? A simple way to construct arbitrary large scattered strong digraphs is gluing together finite strong digraphs in a tree-like structure where the neighbouring finite digraphs share exactly one vertex. One may hope that under the extra assumption that the underlying undirected graph is 2-vertex-connected a scattered digraph must be finite, however, it is false. It is not too hard to prove that if every edge of a digraph $D$ is in only finitely many directed cycles, then $D$ is scattered. Using this observation, it is easy to draw a counterexample (see Figure 4).

Figure 4: An example for an infinite strong digraph $D$ where the underlying undirected graph is 2-vertex-connected and $D$ is scattered.

It is natural to ask if there are uncountable counterexamples. To show the negative answer, we use only the weaker assumption that the local edge-connectivities are countable instead of being scattered. (Strong and scattered implies that the local edge-connectivities are actually finite, see Corollary 4.8.)

Proposition 8.1. If $D$ is a strongly connected infinite digraph in which every local edge-connectivity is countable and $\text{Un}(D)$ is 2-vertex-connected, then $D$ is countable.

Proof. Pick a countable elementary submodel $M \ni D$. We claim that every strong component of $D \setminus M$ contains exactly one vertex from $V(D) \cap M$. On the one hand, it cannot contain more otherwise we would have an uncountable local edge-connectivity by Fact 10.3, contradicting the assumption. On the other hand, $D$ is strongly connected and we delete only edges with both ends in $V(D) \cap M$ hence it cannot be zero. Let $u, v$ be distinct vertices in $V(D) \cap M$ and let
us denote the corresponding strong components by $K_u$ and $K_v$ respectively. There cannot be edges from $K_u$ to $K_v$ in $D \setminus M$ because it would yield to a $u \to v$ path in $D \setminus M$ from which $\lambda(u, v; D) > \aleph_0$ follows by Fact 10.3. Therefore if an edge leaves $K_u$ or arrives to $K_u$ in $D$ it must be incident with $u$. Since $\text{Un}(D)$ is 2-vertex-connected, it implies $K_u = \{u\}$ and hence $D \subseteq M$.

9 Open problems

Proposition 3.3 ensures that in the finite case (Theorem 1.4) one can actually reverse pairwise edge-disjoint cycles to reduce the dichromatic number to at most two. In our proof of the conjecture we can guarantee by using Corollary 3.12 that for every edge $e$ there are at most two cycles in the sequence that contain some orientation of $e$. It definitely cannot be improved to “at most one” in Theorems 7.1 and 1.6. Indeed, reversing pairwise edge-disjoint cycles cannot change the local edge-connectivities. We do not know if it is always possible the reduce the dichromatic number to at most two by reversing pairwise edge-disjoint cycles.

**Question 9.1.** Is it possible to find in an arbitrary digraph $D$ a family of pairwise edge-disjoint directed cycles in such a way that if we reverse their united edge set then the resulting digraph $D^*$ has dichromatic number at most two?

Yet another interesting line of research would be to allow the use of Z-chains (two-way infinite paths) in both the definition of the dichromatic number and during the cycle reversions. Let us call these notions the generalized dichromatic number and generalized cycle reversion.

**Question 9.2** (R. Diestel). Assume $D$ is an arbitrary digraph. Is there a generalized cycle reversion which lower the generalized dichromatic number of $D$ to 2?

The latter is open even for (countably infinite) tournaments and we also wonder if one can exploit some connections to the theory of ends and tangles.

10 Appendix on elementary submodels

Roughly speaking, an elementary submodel of a larger model $\mathcal{V}$, where in our case $\mathcal{V}$ is always the whole set-theoretic universe, is a set $M \subseteq \mathcal{V}$ such that the structure $(M, \in)$ is a very close approximation of $\mathcal{V}$. Note that $M$ might be countable while $\mathcal{V}$ is a proper class.

By approximation, we mean that whenever a first-order statement holds in the universe $\mathcal{V}$ about some elements of $M$ then it is already true inside the structure $(M, \in)$. The elementary submodel technique is an efficient tool in combinatorial set theory and in several other fields. Let us summarize the basic facts that we use in the paper. For an excellent detailed introduction for elementary submodels we suggest [10] and for more advanced application one can see [11].

First of all, let us make the definition precise. For $k < n$, let $\varphi_k$ be a formula in the language of set theory with $n_k$ free variables and let $\Sigma = \{\varphi_k : k < n\}$. A **$\Sigma$-elementary submodel** is a set $M$ such that

$$\bigwedge_{\varphi \in \Sigma} [(\forall a_1, \ldots, a_{n_k} \in M)(\varphi(a_1, \ldots, a_{n_k}) \iff (M, \in) \models \varphi(a_1, \ldots, a_{n_k})].$$

In plain words, we say that the formulas $\varphi$ are absolute between $\mathcal{V}$ and $M$. In this paper, we need only the following basic facts about elementary submodels.

First, for any infinite cardinal one can construct an elementary submodel containing a given set.
Fact 10.1. For every finite set $\Sigma$ of formulas, infinite cardinal $\kappa$ and set $X$ there is a $\Sigma$-elementary submodel $M$ such that $\kappa \cup \{ X \} \subseteq M$ and $M$ has size $\kappa$.

Second, $M$ has very strong closure properties: if a set $Y$ is uniquely definable by a formula using parameters from an elementary submodel $M$ then $Y$ must be in $M$ already.

Fact 10.2. Let $M$ be a $\Sigma$-elementary submodel and $Y$ is any set.

1. Suppose that there is some $\varphi \in \Sigma$ and $p_1 \in M$ so that $\varphi(p_1, \ldots, p_n, y)$ only holds for $y = Y$. Then $Y \in M$.

2. Suppose that $Y \in M$ and the cardinality $\kappa$ of $Y$ is a subset of $M$. Then $Y \subseteq M$ as well assuming that $\Sigma$ is large enough.\(^2\)

The second statement is somewhat subtle if one encounters elementary submodels for the first time. By the first fact, we can find countable elementary submodels $M$ which contain the real line $\mathbb{R}$ as an element (i.e., $X = \{ \mathbb{R} \}$). However, $\mathbb{R}$ cannot be a subset of $M$ simply by cardinality.

Also, it is common practice when using elementary submodels to omit $\Sigma$ completely from the discussion. In essentially all cases, it is clear (but tedious to write out) for which finitely many formulas the absoluteness arguments are used.

Finally, we make use of the following fact regularly.

Fact 10.3. Let $D$ be a digraph and let $M \supseteq D$ be an elementary submodel with $|M| = \kappa \subseteq M$. If $u, v \in M \cap V(D)$ and $\lambda(u, v; D \setminus M) > 0$ then $\lambda(u, v; D) > \kappa$ and $\lambda(u, v; D \cap M) = \kappa$.

Proof. First, let us prove $\lambda(u, v; D) > \kappa$. Assume on the contrary, that any pairwise edge-disjoint system of paths from $u$ to $v$ has size at most $\kappa$. Take a maximal such family $\mathcal{P}$ which is also an element of $M$ (this can be done by absoluteness). Now, by applying the previous fact twice, all the paths $P \in \mathcal{P}$ must be elements of $M$ and again for these finite paths, we see that all the edges in these paths $P$ are in $M$ too. Now, the assumption $\lambda(u, v; D \setminus M) > 0$ says that there is a path $Q$ from $u$ to $v$ that has no edges in $M$. However, then $\mathcal{P} \cup \{ Q \}$ is a strictly larger edge-disjoint family violating the maximality of $\mathcal{P}$.

Second, since in $D$, there is a family $\mathcal{P}$ of pairwise edge-disjoint paths from $u$ to $v$ of size $\kappa$ (even one of size $\kappa^+$), there must exist such a family $\mathcal{P}'$ in $M$. Now, apply the previous fact twice as before to see that all the paths $P \in \mathcal{P}'$ are in $M$ and all the edges in these paths $P$ are in $M$ too. So, $\lambda(u, v; D \cap M) \geq \kappa$ must hold and hence $\lambda(u, v; D \cap M) = \kappa$ since $M \cap D$ has size $\kappa$.

\[ \square \]

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\(^2\)Contains formulas like “$\kappa$ is the cardinality of $Y$”, “$f$ is a bijection from $\kappa$ to $Y$” etc.
