The Eisenbud-Koh-Stillman Conjecture on Linear Syzygies

Mark L. Green*, U.C.L.A.

Although the relationship between minimal free resolutions and Koszul cohomology has been known for a long time, it has been difficult to find a way to fully utilize the “exterior” nature of the Koszul classes. The technique used here seems to be one way to begin to do this. We prove a conjecture of Eisenbud-Koh-Stillman on linear syzygies and in consequence a conjecture of Lazarsfeld and myself on points in projective space. The main novelties in the proof are the use of “exterior minors,” explained below, and showing that certain kinds of linear syzygies in the exterior algebra are impossible. I will work over a field of arbitrary characteristic.

It is a pleasure to acknowledge David Eisenbud for many highly useful conversations regarding this work. In particular, he simplified several arguments in the original version, some of which had only worked in characteristic 0.

**DEFINITION.** Consider two vector spaces $A, B$ of dimensions $a, b$ respectively, and let $V$ be a vector space of dimension $n$. Consider a $b \times a$ matrix of linear forms, which we think of as a linear map $M: A \to B \otimes V$. By a **generalized column** of $M$ we mean, for some non-zero $\alpha \in A$, the map $M(\alpha): B^* \to V$, and by the **rank of a generalized column $\alpha$** we mean the rank of the map $M(\alpha)$; similarly an element $\beta^* \in B^*$ gives a **generalized row** which is a map $M(\beta^*): A \to V$ whose rank is the rank of $M(\beta^*)$. Now $\wedge^k M: \wedge^k A \to \wedge^k (B \otimes V)$. There are natural maps $P_s: \wedge^k (B \otimes V) \to \wedge^k B \otimes S^k V$ and $P_e: \wedge^k (B \otimes V) \to S^k B \otimes \wedge^k V$. The maps $P_s \circ \wedge^k M: \wedge^k A \to \wedge^k B \otimes S^k V$ and $P_e \circ \wedge^k M: \wedge^k A \to S^k B \otimes \wedge^k V$ induce natural maps $\text{Mi}_s^k(M): \wedge^k A \otimes \wedge^k B^* \to S^k V$ and $\text{Mi}_e^k(M): \wedge^k A \otimes S^k B^* \to \wedge^k V$. The image of $\text{Mi}_s^k(M)$ is just the usual ideal $I_s^k(M)$ of $k$ by $k$ minors of $M$; the image of $\text{Mi}_e^k(M)$ we will denote by $I_e^k(M)$ and will call the $k$ by $k$ exterior minors of $M$.

The exterior minors are quite interesting and useful. I do not know of a good reference for their properties, so I will prove what I need.

**PROPOSITION 1.** Let $M$ be a $b \times a$ matrix of linear forms such that every generalized column of $M$ has rank $b$. Then the map $\text{Mi}_s^a(M)$ is injective, and hence there are $(b+1-a)$ linearly independent $a \times a$ exterior minors of $M$.

**PROOF:** For a non-zero $\alpha \in A$, the map $M(\alpha): B^* \to V$ has rank $b$, and thus the set of subspaces $W \subseteq V$ of codimension $a+b-1$ which meet the image of $M(\alpha)$ has codimension $\geq a$ in the Grassmannian. It follows that the set of subspaces $W$ of this dimension meeting the image of some $M(\alpha)$ as $\alpha$ ranges over $P(A)$ has codimension $\geq 1$. If we replace $M$ by the composition $M': A \to B \otimes V \to B \otimes V'$ obtained from a general projection $V \to V'$ to a vector space of dimension $a+b-1$, the hypothesis continues to hold, and the exterior minors of $M'$ are the projection of the exterior minors of $M$ under the map $\wedge^a V \to \wedge^a V'$. It is thus enough to treat the case $\dim(V) = a+b-1$.

We may regard $M$ as an $(a+b-1) \times a$ matrix of linear forms in $B$. The exterior minors of $M$ are the usual minors of this new matrix. The hypothesis on generalized columns of $M$

---

* Research partially supported by the N.S.F.
translates into the hypothesis that the new matrix never drops rank. The Eagon-Northcott complex (cf. Appendix A2 of [6]) now shows that these minors are linearly independent. This completes the proof.

The way we will make use of exterior minors is the following result, which is essentially the same as what happens in the commutative case:

**PROPOSITION 2.** If $M: A \to B \otimes V$ is a linear map and $f_M: \wedge^m V \otimes A \to \wedge^{m+1} V \otimes B$ is the naturally associated map, then $I^c_\nu(M)$ annihilates ker($f_M$), i.e. if $\phi \in I^c_\nu(M)$ and $\alpha \in \ker(f_M)$, then $\phi \wedge \alpha \in \wedge^{m+a} V \otimes A$ is zero.

**PROOF:** Let $a_1, \ldots, a_a$ be a basis for $A$, and let $\alpha = \sum_{i=1}^a \alpha_i \otimes a_i$ with $\alpha_i \in \wedge^m V$ for all $i$. Let $M(a_i) \in B \otimes V$ be the image of $a_i$ under $M$. The hypothesis is that $\alpha \in \ker(f_M)$ is equivalent to $\sum_i \alpha_i \wedge M(a_i) = 0$, where by $\wedge$ we mean that we multiply elements of $B$ symmetrically and elements of $V$ anti-symmetrically, with the result that $\wedge$ anti-commutes. What we need to show is that for all $i$, $M(a_1) \wedge \cdots \wedge M(a_n) \wedge \alpha_i = 0$. However, we may write the left-hand side as $M(a_1) \wedge \cdots \wedge M(a_i-1) \wedge (\sum_j M(a_j) \wedge \alpha_j) \wedge M(a_{i+1}) \wedge \cdots \wedge M(a_n)$, and this is zero.

**REMARK.** A more elegant approach, suggested by Eisenbud, is to notice that $\text{Mi}_e^{n-1}(M)$ gives a map $B \otimes V \to S^n B \otimes A^* \otimes A \otimes V$. The composition is a map $M': B \otimes V \to S^n B \otimes A^* \otimes A \otimes V \otimes A$ which functions as a “companion matrix” to $M$ because the composition $M'M = \text{Mi}_e(M) \otimes \text{id}_A$. This formula implies Proposition 2.

**PROPOSITION 3.** Let $V$ be a vector space of dimension $n$ and $W \subseteq \wedge^{n-p} V$ a linear subspace of dimension $p > 0$. Then there exists a $0 < k \leq p$ and a $(p-k)$-dimensional subvariety $Z \subseteq G(p-1, W)$ of $(p-1)$-dimensional subspaces $U \subset W$ such that for all $U \in Z$, the image of $U \otimes V \to \wedge^{n-p+1} V$ has codimension $\geq k$ in the image of $W \otimes V \to \wedge^{n-p+1} V$.

**PROOF:** Let $B^* = \text{ker}(W \otimes V \to \wedge^{n-p+1} V)$. Let $M$ be the $b \times p$ matrix of linear relations in $\wedge^p V$ of $W$, which we view as a map $W^* \to B \otimes V$. Let $I_{W,V}$ and $I_{U,V}$ be the images of $W \otimes V$ and $U \otimes V$ respectively in $\wedge^{n-p+1} V$. Let $C^* = \text{ker}(U \otimes V \to \wedge^{n-p+1} V)$. There is then an exact sequence

$$0 \to \frac{B^*}{C^*} \to (W/U) \otimes V \to \frac{I_{W,V}}{I_{U,V}} \to 0.$$ 

We conclude that $\dim(I_{W,V}/I_{U,V}) = n - \dim(B^*/C^*)$. If $w^* \in W^*$ is the annihilator of $U$, then $\dim(B^*/C^*)$ is just the rank of the generalized column of $M$ corresponding to $w^*$. In the $P^{p-1}$ parametrizing generalized columns of $M$, let $Z_r$ be the space of generalized columns of rank $r$, and $d(r) = \dim(Z_r)$. The negation of the conclusion of the proposition is that $Z_r = \phi$ for $r \leq n - p$ and $d(r) < r + p - n$ for all $r > n - p$. We will assume this and derive a contradiction.

Let $\bar{M}$ be a $(n-p+1) \times p$ matrix obtained by choosing $(n-p+1)$ general generalized rows of $M$. For any given generalized column of $M$, let $r$ be its rank. If $r \geq n - p + 1$, which we may assume, then the set of projections of $M$ to a $p \times (n - p + 1)$ matrix for which this generalized column does not have maximal rank has codimension $r + p - n$. 2
Every generalized column of $\tilde{M}$ has maximal rank provided $d(r) < r + p - n$ for all $r$, as then a general projection does not belong to the “bad set” for any generalized column.

Thus a general choice of $\tilde{M}$ has every generalized column of rank $n - p + 1$. By the first proposition, there are at least $\binom{m}{p}$ linearly independent elements of $I^p_p(M) \subseteq \wedge^p V$, and hence $I^p_p(M) = \wedge^p V$. By the second proposition, the elements of $I^p_p(M)$ annihilate $W$, and therefore $W = 0$, which is a contradiction.

As a consequence of the foregoing proposition, we obtain the following result and its corollary, which were conjectured by Eisenbud, Koh, and Stillman (see [1].) We use the notation $K_{p,q}(M,V)$ to denote the Koszul cohomology group $H^p(\wedge^q V \otimes M_{q+p-1}).$

**THEOREM 4.** Let $M = \oplus_{q \geq 0} M_q$ be a finitely generated $S(V)$-module and assume $\text{rank}(M_0) = p > 0$. Let $R \subseteq M_0 \otimes V$ be the module of relations. Then if the (affine) dimension of the rank one relations $R_1$ has dimension $< p$, then the Koszul cohomology group

$$K_{k,0}(M,V) = \ker(\wedge^k V \otimes M_0 \to \wedge^{k-1} V \otimes M_1)$$

vanishes for all $k \geq p$.

**PROOF:** We may proceed by induction on $p$, the case $p = 1$ being obvious. As is well-known, since $M_q = 0$ for $q < 0$, the vanishing of $K_{p,0}(M,V)$ would imply the vanishing of $K_{k,0}(M,V)$ for all $k \geq p$. Let $\alpha \in \wedge^p V \otimes M_0$ be a non-zero element of $K_{p,0}(M,V)$. Under the map $\wedge^p V \otimes M_0 \to \wedge^{p-1} V \otimes V \otimes M_0$, $\alpha$ must map to an element of $\wedge^{p-1} V \otimes R$. Let $W$ be the image of the map $M_0^* \to \wedge^p V$ given by $\alpha$. We may assume that it has dimension $p$, since otherwise we could shrink $M_0$ and $p$. For any $\beta \in \wedge^{p-1} V^*$, the contraction $\langle \beta, \alpha \rangle \in V \otimes M_0$ automatically lies in $R$. Any $m \in M_0$ annihilates a $(p-1)$-dimensional linear subspace of $M_0^*$; let $U_m$ be its image under the map $M_0^* \to \wedge^p V$ determined by $\alpha$. If the image of $U_m \otimes V^* \to \wedge^{p-1} V$ has codimension $k_m$ in the image of $W \otimes V^* \to \wedge^{p-1} V$, then we obtain a $k_m$-dimensional linear space of non-trivial rank one relations in $M_0 \otimes V$ lying in $m \otimes V$.

Now choose a generator $\tau$ for $\wedge^n V^*$, and let $W' = \langle \tau, W \rangle \subseteq \wedge^{n-p} V^*$ and $U'_m = \langle \tau, U_m \rangle$. Then $k_m$ is also the codimension of the image of $U'_m \otimes V^* \to \wedge^{n-p+1} V^*$ in the image of $W' \otimes V^* \to \wedge^{n-p+1} V^*$. We now invoke the preceding proposition to conclude that for some $0 < k \leq p$, there is a variety in $G(p-1, M_0)$ of dimension at least $p - k$ such that for all $m$ in this variety, there exists a $k$-dimensional family of rank one relations in $M$ in $m \otimes V$. This completes the proof.

**DEFINITION.** A relation of rank $\leq r$ is a non-zero element of $\ker(S \otimes V \to M_1)$ for some linear subspace $S \subseteq M_0$ of rank $r$. We will say that such a relation involves the linear subspace $S' \subset M_0$ if $S' \subseteq S$. A corollary of the Theorem is:

**COROLLARY 5.** Let $M = \oplus_{q \geq 0} M_q$ be a finitely generated $S(V)$-module and let $\text{rank}(M_0) = m_0 > 0$. If $K_{p,0}(M,V) \neq 0$, then for a general choice of $(m_0 - p)$-dimensional subspace $S \subseteq M_0$, the affine dimension of the rank $\leq (m_0 - p + 1)$ relations involving $S$ is at least $p$.

**PROOF:** If $S \subseteq M_0$ is a linear subspace, $\tilde{S}$ the submodule of $M$ it generates, and let $\tilde{M} = M/\tilde{S}$. For a general choice of $S$ of dimension $m_0 - p$, our Koszul class $\alpha \in \wedge^p V \otimes M_0$ maps to
choose a subscheme $Z$ at each $P \in \phi H$. Given by evaluation at $P$, there exists a set of rank 1 relations in $\text{im}(V \otimes \lambda)$. Theorem that for any non-zero class $N \in K$, and thus $h$ of property $H$ in which case we are done. Of course, if $h$ in $(P_1)$, then the map $H(V(k)) \to H^1(I_Z(k))$ is given by $l \otimes \sum_i a_i v_i \mapsto \sum_i l(P_i) a_i v_i$. The case $p = 0$ (done with Rob Lazarsfeld at the time of [4]) proceeds as follow—choose a subscheme $Z'$ of $Z$ which violates $N_0$, but such that no proper subscheme of $Z'$ violates $N_0$. This implies that if we write a non-zero element $\phi \in H^1(I_Z(2))\ast$ as $\phi = \sum_{i=1}^{2r+1} \phi_i v_i$, then $\phi_i \neq 0$ for all $i$. Since $h^1(I_Z(1)) \leq r < r+1$, under the multiplication $V \otimes H^1(I_Z(2)) \to H^1(I_Z(1))$, there is some linear form $h \in V$ annihilating $\phi$. Thus $\sum_i h(P_i) \phi_i v_i = 0$. Since the $v_i$ are linearly independent, this implies that $h(P_i) = 0$ for all $i$, and thus $Z'$ lies on the hyperplane $h$. Now either $Z'$ consists of $\leq 2(r-1) + 1$ points, in which case we proceed inductively on the dimension, or it has $\geq 2(r-1) + 2$ points, in which case we are done. Of course, if $h^1(I_Z(2)) = 0$, then $I_Z$ is $3$-regular. Thus if property $N_p$ fails for $p > 0$, it must be that $K_{k,3}(I_Z, V) \neq 0$ for some $k \leq p - 1$, and we may reduce to the case $k = p - 1$. From the Koszul complex $0 \to \wedge^{r+1} V \otimes I_Z(-r-1) \to \cdots \to V \otimes I_Z(-1) \to I_Z \to 0$ twisted by $O_{P^r}(p+3)$, we see that the Koszul group $K_{p-1,3}(I_Z, V) \cong K_{p+1,1}(M, V)$, where $M = \oplus_k M_k = \oplus_k H^1(I_Z(k))$. Since $K_{p,2}(Z, V) \cong K_{p-1,3}(I_Z, V)$, we say that this group being non-zero implies that $K_{r-p,1}(M^*, V) \neq 0$, where $M_k^* = H^1(I_Z(k))$ (in this module, multiplication decreases degree.) We may without loss of generality assume that $Z$ is not contained in a hyperplane. Thus $h^1(I_Z(1)) = r - p$, from which we conclude by the Theorem that for any non-zero class $\lambda \in K_{r-p,1}(M^*, V)$, there are at least an $(r-p)$-dimensional family of rank 1 relations in $\text{im}(\wedge^{r-p-1} V^* \to V \otimes H^1(I_Z(1))^*)$, where the
map is induced by $\lambda$. The $v_i$ are linearly independent, and thus a relation of rank 1 is an element $l \otimes \sum a_i v_i$ such that, for all $i$, either $l(P_i) = 0$ or $a_i = 0$. On any irreducible component of the intersection of the image of $\lambda$ in $V \otimes H^1(I_Z(1))^*$ with the rank one locus, there is a decomposition $Z = Z' + Z''$ of $Z$ into disjoint subsets such that $l(P_i) = 0$ for $P_i \in Z'$ and $a_i = 0$ if $P_i \in Z''$. From the exact sequence

$$0 \to H^1(I_{Z'}(1))^* \to H^1(I_Z(1))^* \to H^0(O_{Z''}(1))^*$$

we see that the $\sum_i a_i v_i$ actually belong to $H^1(I_{Z'}(1))^*$ in this circumstance, so that the image of $\lambda$ intersect the rank one locus gives a subvariety $Y \subseteq H^0(I_{Z'}(1))^* \times H^1(I_{Z'}(1))^*$ of dimension $\geq r - p$ consisting of rank 1 relations for $M^*$. This implies that $h^0(I_{Z'}(1))^* + h^1(I_{Z'}(1))^* \geq r - p$. If $Z'$ is $m$ points spanning a $\mathbf{P}^k$, then $h^0(I_{Z'}(1))^* = r - k$ and $h^1(I_{Z'}(1))^* = m - 1 - k$, so that the preceding inequality becomes $m \geq 2k + 1 - p$. Possibly by enlarging $Z'$, we may without loss of generality in this construction assume that for all $p \in Z''$, there exists an $(l, \phi) \in Y$ such that $l(p) \neq 0$.

It remains to show that property $N_p$ fails for $Z'$. Let $\phi_1, \ldots, \phi_m, \psi_1, \ldots, \psi_t$ be a basis for $H^1(I_Z(1))^*$ such that the $\phi_i$ are a basis for $H^1(I_{Z'}(1))^*$. If we write our Koszul class as $\sum_i \alpha_i \otimes \phi_i + \sum j \beta_i \otimes \psi_j$, then if $x_1, \ldots, x_{r+1}$ is a basis for $V$ and $e_1, \ldots, e_{r+1}$ is the dual basis, then the condition that $\lambda$ be a Koszul class is

$$\sum_{i, \nu} < \alpha_i, e_\nu > \otimes x_\nu \phi_i + \sum_{j, \nu} < \beta_j, e_\nu > \otimes x_\nu \psi_j = 0.$$

We read off that the image of $\sum_{j, \nu} < \beta_j, e_\nu > \otimes x_\nu \psi_j = 0$ in $\wedge^{p-1} V \otimes H^0(O_{Z''}(2))^*$. However, the $\psi_j$ are linearly independent there, and thus $\sum_{\nu} < \beta_j, e_\nu > \otimes x_\nu = 0$ on $Z''$ for all $j$. It follows that $< \beta_j, v >/ = 0$ for all $v \in Z''$ and all $j$, where $Z''$ is the linear span of $Z''$. It thus follows that $< \lambda, v >/ = \sum_i < \alpha_i, v >/ \otimes \phi_i$ if $v \in Z''$, and $< \lambda, v >/ > 0$ thus determines an element of $K_{r-p-1,1}(M^*, v^1)$, where $M = \oplus_{q \geq 0} H^1(I_{Z'}(q))^*$. Thus $Z'$ fails to have property $N_p$ when viewed as a subset of $v^1$ unless $< \lambda, v >/ = 0$ in $K_{r-p-1,1}(M, v^1)$ for all $v \in Z''$. Note that $H^1(I_{Z'}(2))^* = 0$ unless $Z'$ fails to have property $N_0$ and hence a fortiori $N_p$. We have arranged things so that the image of $\lambda$ intersected with the rank 1 locus does not lie in $(Z'')^1 \otimes H^1(I_Z(1))^*$. The proof is now done, modulo the following elementary lemma:

**Lemma 7.** Let $M$ be a finitely generated module over $S(V)$. If $\lambda \in K_{p,q}(M, V)$ and $v \in V^*$, then the contraction $< \lambda, v >/ \in K_{p-1,q}(M, v^1)$. If $M_{q-1} = 0$ and the image of the map $\wedge^{p-1} V^* \to V \otimes M_q$ induced by $\lambda$ is not contained in $v^1 \otimes M_q$, then $< \lambda, v >/ \neq 0$ in $K_{p-1,q}(M, v^1)$.

**Proof:** If $m_1, \ldots, m_k$ is a basis for $M_q$, then if $\lambda = \sum \lambda_i \otimes m_i$ with $\lambda_i \in \wedge^p V$, the Koszul condition is $\sum_{i, \nu} < \lambda_i, e_\nu > \otimes x_\nu m_i = 0$; here $x_1, \ldots, x_n$ is a basis for $V$ and $e_1, \ldots, e_n$ the dual basis for $V^*$. We may take $v = e_1$ and $v^1 = \text{span}(x_2, \ldots, x_n)$. If we contract $e_1$ with the equality above, we get $\sum_{i=1}^k \sum_{\nu=1}^n < \lambda_i, e_1 \wedge e_\nu > \otimes x_\nu m_i = 0$. Of course, we may sum over $2 \leq \nu \leq n$ in this equality, and then we have the Koszul condition for $< \lambda, e_1 >/ > 0$ over $v^1$. Further, the image of the map induced by $\lambda$ does not belong to $v^1 \otimes M_q$ if and only
if $\sum_i <\lambda_i, e_1> \otimes m_i \neq 0$, and this is equivalent to $\lambda, e_1 >\neq 0$. If $M_{q-1} = 0$, then this implies that $\lambda, e_1 >$ does not vanish in $K_{p-1,q}(M, v^\perp)$.

**REMARK.** If $Z$ is $2r + 2 - p$ points on a rational normal curve in $\mathbb{P}^r$, then property $N_p$ fails for $Z$.

**REMARK.** A linear map $M: A \rightarrow B \otimes V$ gives maps $\wedge^k A \rightarrow B^\lambda \otimes V^{\lambda^t}$ for $\lambda$ any Young diagram of size $k$, using the Cauchy decomposition

$$\wedge^k (B \otimes V) \cong \oplus_\lambda (B^\lambda \otimes V^{\lambda^t}).$$

One can do similar things using the decomposition of $(B \otimes V)^\lambda$ for other Young diagrams $\lambda$. I call these the **mixed minors** of the matrix $M$ of linear forms. I hope to give some applications of mixed minors in a later paper.

**BIBLIOGRAPHY**

[1] D. Eisenbud and J. Koh, “Some linear syzygy conjectures,” Adv. in Math. 90 (1991), 47-76.

[2] D. Eisenbud and J. Koh, “Remarks on points in projective space,” in *Commutative Algebra*, Conf. Proc. of the 1987 conference in Berkeley (M. Hochster, C. Huneke, and J.D. Sally, Eds.), 157-172, MSRI Publ 15, Springer-Verlag, New York, 1989.

[3] M. Green and R. Lazarsfeld, “On the projective normality of complete linear series on an algebraic curve,” Inv. Math. 83 (1986), 73-90.

[4] M. Green and R. Lazarsfeld, “Some results on the syzygies of finite sets and algebraic curves,” Comp. Math. 67 (1988), 301-314.

[5] R. Lazarsfeld, “A sampling of vector bundle techniques in the study of linear series,” in *Lectures on Riemann Surfaces*, World Scientific, Singapore (1989), 500-559.

[6] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, New York (1995).