Existence and construction of quasi-stationary distributions for one-dimensional diffusions

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Abstract

In this paper, we study quasi-stationary distributions (QSDs) for one-dimensional diffusions killed at 0, when 0 is a regular boundary and +∞ is a natural boundary. More precisely, we not only give a necessary and sufficient condition for the existence of a QSD, but also we construct all QSDs for one-dimensional diffusions. Moreover, we give a sufficient condition for $R$-positivity of the process killed at the origin. This condition is only based on the drift, which is easy to check.

Keywords: One-dimensional diffusion; Quasi-stationary distribution; $R$-positivity; Natural boundary

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1. Introduction

We are interested in the long-term behavior of an absorbing Markov processes. Conditional stationarity, which we call quasi-stationarity, is one of the most interesting topics in this direction. For quasi-stationary distribution (QSD), we know that the study of QSDs is a long standing problem in several areas of probability theory and a complete understanding of the structure of QSDs seems to be available only in rather special situations such as Markov chains on finite sets or more general processes with compact state space. The main motivation of this work is the existence and construction of QSDs for one-dimensional diffusion $X$ killed at 0, when 0 is a regular boundary and +∞ is a natural boundary. Moreover, we give a sufficient condition in order that the process $X$ killed at 0 is $R$-positive.

To the best of our knowledge, Mandl [12] is the first one to study the existence of a QSD for continuous time diffusion process on the half line. Under Mandl’s conditions are satisfied, the existence of the Yaglom limit and that of a QSD for killed one-dimensional diffusion processes have been proved by various authors (see, e.g., [4, 5, 11, 14, 17]). Under Mandl’s conditions are not satisfied, Cattiaux et al., who study the existence and uniqueness of the QSD for one-dimensional diffusions killed at 0 and whose drift is allowed to go to $-\infty$ at 0 and the process is allowed to have an entrance boundary at $+\infty$, have done a pioneering work (see [1]). In this case, under the most general conditions, Littin C proves the existence of a unique QSD and of the Yaglom limit in [10], which is closely related to [1]. Although in [1, 5, 11, 14, 16, 17] and [1] make the key contributions, the structure of QSDs of killed one-dimensional diffusions has not been completely clarified. This leads us to further study QSDs for one-dimensional diffusions.

Another notion is $R$-positive, which, in general, is not easy to check, is sufficient to facilitate the straightforward calculation of QSDs for a process from the eigenvectors, eigenmeasures and eigenvalues of its transition rate matrix. The classification of killed one-dimensional diffusions has been studied by Martín and San Martín [16]. They give necessary and sufficient conditions, in terms of the bottom eigenvalue function, for $R$-recurrence and $R$-positivity of one-dimensional diffusions killed at the origin.

In this paper, the main novelty is that we not only give a necessary and sufficient condition for the existence of a QSD, but also we construct all QSDs for one-dimensional diffusion $X$ killed at 0, when 0 is a regular boundary and +∞ is a natural boundary. Moreover, compared with [16], we give an explicit criteria for the process $X$ killed at 0 is $R$-positive.

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The remainder of this paper is organized as follows. In Section 2 we present some preliminaries that will be needed in the sequel. In Section 3 we characterise all QSDs for one-dimensional diffusion $X$ killed at 0, when 0 is a regular boundary and $+\infty$ is a natural boundary. In Section 4 we mainly show that the process is $R$-positive. We conclude in Section 5 with some examples.

2. Preliminaries

We consider the generator $Lu = \frac{1}{2} \partial_{xx} u - q \partial_x u$. Denote by $X$ the diffusion whose infinitesimal generator is $L$, or in other words the solution of the SDE

$$dX_t = dB_t - q(X_t) dt, \quad X_0 = x > 0,$$

where $(B_t; t \geq 0)$ is a standard one-dimensional Brownian motion and $q \in C^1([0, \infty))$. Thus, $-q$ is the drift of $X$.

Let $P_x$ and $E_x$ stand for the probability and the expectation, respectively, associated with $X$ when initiated from $x$.

Let $\tau_a = \inf\{t > 0 : X_t = a\}$ be the hitting time of $a$. We are mainly interested in the case $a = 0$ and we denote $\tau = \tau_0$.

As usual $X_\tau$ corresponds to $X$ killed at 0.

We consider the function

$$\Lambda(x) = \int_0^x e^{Q(y)} dy,$$

where $Q(y) = \int_0^y 2q(x) dx$. Notice that $\Lambda$ is the scale function for $X$. It satisfies $L\Lambda \equiv 0$, $\Lambda(0) = 0$, $\Lambda'(0) = 1$. It will be useful to introduce the following measure defined on $(0, \infty)$:

$$\mu(dy) := e^{-Q(y)} dy.$$

Notice that $\mu$ is the speed measure for $X$.

Let $L^* = \frac{1}{2} \partial_{xx} + \partial_x q'$ be the formal adjoint operator of $L$. We denote by $\varphi_\lambda$ the solution of

$$L^* \varphi_\lambda = -\lambda \varphi_\lambda, \quad \varphi_\lambda(0) = 0, \quad \varphi'_\lambda(0) = 1,$$

and by $\eta_\lambda$ the solution of

$$L \eta_\lambda = -\lambda \eta_\lambda, \quad \eta_\lambda(0) = 0, \quad \eta'_\lambda(0) = 1.$$

A direct computation shows that

$$\varphi_\lambda = e^{-Q} \eta_\lambda.$$  \hspace{1cm} (6)

Let $\lambda_c$ be the smallest point of increase of $\rho(\lambda)$, where $\rho(\lambda)$ denotes the spectral measure of the operator $L^*$. We will assume $\rho(\lambda)$ is left-continuous (see [3], Chapter 9). In ([12], Lemma 2) it was shown that

$$\lambda_c = \sup\{\lambda : \varphi_\lambda(\cdot) \text{ does not change sign}\}.$$

For most of the results in this paper we will use the following hypothesis (H), that is,

**Definition 1.** We say that hypothesis (H) holds if the following explicit conditions on $q$, all together, are satisfied:

(H1) $A(\infty) = \infty$,

(H2) $S = \int_0^\infty e^{Q(y)} \left( \int_y^\infty e^{-Q(z)} dz \right) dy = \infty$.

If (H1) holds, then it is equivalent to $P_x(\tau < \infty) = 1$, for all $x > 0$ (see, e.g., [6], Chapter VI, Theorem 3.2). So that, if (H1) and (H2) are satisfied, then $+\infty$ is a natural boundary according to Feller’s classification (see [6], Chapter 15).

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3. Existence and construction of quasi-stationary distributions

In this section, we study the standard quasi-stationary distributions of a one-dimensional diffusion $X$ killed at 0, when 0 is a regular boundary and $+\infty$ is a natural boundary, a typical problem for absorbing Markov processes. More formally, the following definition captures the main object of interest of this work.

**Definition 2.** We say that a probability measure $\nu$ supported on $(0, +\infty)$ is a quasi-stationary distribution (QSD), if for all $t \geq 0$ and any Borel subset $A$ of $(0, +\infty)$,

$$P_\nu(X_t \in A | \tau > t) = \nu(A).$$  \hfill (7)

It is well known that a basic and useful property is that the time of killing is exponentially distributed when starting from a QSD:

**Proposition 1.** Assume that $\pi$ is a QSD for the process $X$. Then there exists $\lambda > 0$ such that, for all $t > 0$,

$$P_\pi(\tau > t) = e^{-\lambda t}. \hfill (8)$$

The following theorem is one of our main results.

**Theorem 1.** There exists a quasi-stationary distribution for one-dimensional diffusion $X$ satisfying (1) if and only if (H1) is satisfied and the following condition

$$\delta \equiv \sup_{x > 0} \int_0^x \int_x^\infty e^{Q(y)}dy \int_y^\infty 2e^{-Q(y)}dy < \infty \hfill (9)$$

holds. Moreover, if (H2) holds, then for any $0 < \lambda \leq \lambda_c$, $d\nu = 2\lambda \eta_\lambda d\mu$ is a quasi-stationary distribution and all of quasi-stationary distributions for $X$ only have this form, where $\mu$ and $\eta_\lambda$ are defined by (3) and (5) respectively.

A relevant quantity in our study is the exponential decay for the absorption probability $\zeta = -\lim_{t \to \infty} \frac{1}{t} \log P_\pi(\tau > t)$.

In [4], Theorem A it was shown that $\zeta$ exists and is independent of $x > 0$. Noticing that, the following three Lemmas (Lemma 1–3) play a key role in this paper, which had been proved in [15].

**Lemma 1.** Assume (H2) holds. The following properties are equivalent:

(i) $\zeta > 0$;

(ii) $\int_0^\infty \varphi_\lambda(y)dy < \infty$ and $\int_0^\infty e^{Q(y)}dy < \infty$;

(iii) $\int_0^\infty \varphi_\lambda(y)dy < \infty$ and $\Lambda(\infty) = \infty$;

(iv) $\lambda_c > 0$ and $\Lambda(\infty) = \infty$;

(v) $\exists \lambda > 0$ such that $\eta_\lambda$ is increasing.

**Lemma 2.** Assume (H) holds. Then $\zeta = \lambda_c$.

**Lemma 3.** Assume (H) holds. The following statements are equivalent for $\lambda \in (0, \lambda_c]$:

(i) $\eta_\lambda$ (or equivalent $\varphi_\lambda$) is positive;

(ii) $\eta_\lambda$ is strictly increasing;

(iii) $\varphi_\lambda$ is strictly positive and integrable.

Moreover, if any of these conditions holds, then

$$\lim_{y \to \infty} \frac{\eta_\lambda(y)}{\Lambda(y)} = 0 \quad \text{and} \quad \int_0^\infty \varphi_\lambda(x)dx = \frac{1}{2\lambda}.$$  \hfill (10)

To yield our result more conveniently, we introduce the following lemma.
Lemma 4. Assume (H) holds. Then $\lambda_c > 0$ if and only if $\delta < \infty$, where $\delta$ is defined by (9).

Proof. Assume (H) holds. If $\lambda_c > 0$, we know from Lemma 1 that the measure $\mu$ is finite. Thanks to Theorem 1.1 in [2], we know that

$$(4\delta)^{-1} \leq \lambda_c \leq (\delta)^{-1}.$$ 

Thus we deduce $\delta < \infty$.

Conversely, we have assumed that 0 is a regular boundary for the process $X$. So if $\delta < \infty$, we can deduce the measure $\mu$ is finite. By suing Theorem 1.1 in [2] again, we deduce $\lambda_c > 0$. □

From (3) and (6), we can define

$$d\nu_\lambda = 2\lambda \varphi_\lambda(y)dy = 2\lambda \eta_\lambda(y)e^{-\lambda y}dy = 2\lambda \eta_\lambda(y)d\mu = 2\lambda \eta_\lambda d\mu.$$ 

We may now study the existence of QSD and construct all QSDs under the condition (H) is satisfied, which is equivalent to the property that $+\infty$ is a natural boundary for $X$ in the sense of Feller. The result is presented in the following proposition.

Proposition 2. Assume both (H1) and (9) hold. Then for any $\lambda \in (0, \lambda_c]$

$$d\nu_\lambda = 2\lambda \eta_\lambda d\mu$$

is a quasi-stationary distribution if and only if the following two conditions are satisfied:

(i) $\int_0^\infty d\nu_\lambda = 1$;

(ii) $L^* \nu_\lambda = -\lambda \nu_\lambda$.

Proof. Thanks to the equality (10), we know that $\nu_\lambda$ is a probability measure, i.e. $\int_0^\infty d\nu_\lambda = \int_0^\infty 2\lambda \eta_\lambda d\mu = 1$. Hence, the condition (i) is satisfied.

Moreover, we know from the equality (3) that

$$L^* \nu_\lambda = L^* 2\lambda \varphi_\lambda = -2\lambda^2 \varphi_\lambda = -\lambda \nu_\lambda.$$ 

Therefore, the condition (ii) is satisfied. Next, we will prove that $\nu_\lambda$ is a quasi-stationary distribution.

According to the theory of Dirichlet forms (see [5]), we thus know that the operator $(L, D(L))$ is the generator of a strongly continuous symmetric semigroup of contractions on $L^2(\mu)$ denoted by $(P_t)_{t \geq 0}$, where $D(L) := \{f \in L^2(\mu) : \lim_{t \to 0} \frac{1}{t}(P_tf - f) \text{ exists in } L^2(\mu)\}$. This semigroup is sub-Markovian, that is, $0 \leq P_tf \leq 1$ $\mu$-a.e. if $0 \leq f \leq 1$ (see [5], Theorem 1.4.1). Then $D(L)$ is dense in $L^2(\mu)$ and using the semigroup relation and continuity, we obtain

$$\frac{d}{dt}(P_tf) = P_t Lf = LP_t f, \quad t \geq 0,$$

which implies

$$P_tf - f = \int_0^t P_s Lf ds, \quad t \geq 0.$$ 

(11)

For $\eta_\lambda$, from (5) and (11) we have

$$P_t \eta_\lambda - \eta_\lambda = -\lambda \int_0^t P_s \eta_\lambda ds.$$ 

Straightforward calculations show that

$$P_t \eta_\lambda = e^{-\lambda t} \eta_\lambda.$$ 

Thanks to the symmetry of the semigroup, we have for all $f$ in $L^2(\mu)$,

$$\int P_t f \eta_\lambda d\mu = \int f P_t \eta_\lambda d\mu = e^{-\lambda t} \int f \eta_\lambda d\mu.$$ 

(12)
Since both (H1) and (2) hold, thus we know from Lemma 1–4 that
\[
\int_0^\infty \varphi(y)dy = \int_0^\infty \eta(y)e^{-Qy}dy = \int_0^\infty \eta(y)\mu(dy) = \frac{1}{2l} < \infty.
\]
Then we have \(\eta \in L^1(\mu)\). The equality (12) can extend to all bounded \(f\). In particular, we may use it with \(f = \mathbf{1}_{(0,\infty)}\) and with \(f = \mathbf{1}_A\). Noticing that
\[
\int P_t(\mathbf{1}_{(0,\infty)})2\lambda\eta_1d\mu = \mathbb{P}_\nu(\tau > t)
\]
and
\[
\int P_t\mathbf{1}_A 2\lambda\eta_1d\mu = \mathbb{P}_\nu(X_t \in A, \tau > t),
\]
then
\[
\mathbb{P}_\nu(X_t \in A|\tau > t) = \frac{\mathbb{P}_\nu(X_t \in A, \tau > t)}{\mathbb{P}_\nu(\tau > t)} = \frac{\int P_t(\mathbf{1}_{(0,\infty)})2\lambda\eta_1d\mu}{\int P_t(\mathbf{1}_{(0,\infty)})2\lambda\eta_1d\mu} = \frac{\int \mathbf{1}_A P_t 2\lambda\eta_1d\mu}{\int \mathbf{1}_{(0,\infty)} P_t 2\lambda\eta_1d\mu} = \nu(A).
\]
Thus, we get that \(\nu\) is a QSD.

Conversely, assume that \(\nu\) is a QSD. From the definition of QSD, we know that \(\nu\) is a probability measure, then \(\nu\) satisfies the condition (i).

We only denote \(\nu = \nu_1\) at here. As defined above, a QSD \(\nu\) is a probability measure on \((0, \infty)\) such that for every Borel set \(A\) of \((0, \infty)\),
\[
\nu(A) = \frac{\mathbb{P}_\nu(X_t \in A, \tau > t)}{\mathbb{P}_\nu(\tau > t)} = \frac{\int P_t(\mathbf{1}_A)(x)\nu(dx)}{\int P_t(\mathbf{1}_{(0,\infty)})(x)\nu(dx)} = \frac{P_t^*\nu(\mathbf{1}_A)}{P_t^*\nu(\mathbf{1}_{(0,\infty)})},
\]
where \(P_t^*\nu\) is the measure on \((0, \infty)\) defined for \(f\) measurable and bounded by
\[
P_t^*\nu(f) = \int P_tf(x)\nu(dx).
\]
From the equality (8), we obtain
\[
\int P_t(\mathbf{1}_A)(x)\nu(dx) = P_t^*\nu(\mathbf{1}_A) = e^{-\lambda t}\nu(A).
\]
Thus the probability measure \(\nu\) is an eigenvector for the operator \(P_t^*\) (defined on the signed measure vector space), associated with the eigenvalue \(e^{-\lambda t}\). It is easy to show that
\[
P_t^*\nu = e^{-\lambda t}\nu \Leftrightarrow \nu P_t = e^{-\lambda t}\nu.
\]
Then, it is direct to check that
\[
L^*\nu = -\lambda \nu.
\]
Thus \(\nu\) satisfies the condition (ii). We complete the proof.

**Proof of Theorem 1.** The theorem follows from Lemma[1–4] and Proposition[2].

**Corollary 1.** If there exists a quasi-stationary distribution for the process \(X\), then \(\mu(0, \infty) < \infty\).
Proof. For any \( x \in (0, \infty) \), we have

\[
\mu(0, \infty) = \int_0^\infty e^{-Q(z)} \, dz = \int_0^x e^{-Q(z)} \, dz + \int_x^\infty e^{-Q(z)} \, dz.
\] (13)

Under the assumption, we know from Theorem \[\text{1}\] that the equality \( (9) \) holds, then for all \( x \in (0, \infty) \), \( \int_0^\infty e^{-Q(z)} \, dz < \infty \). Observe that under condition \( q \in C^1([0, \infty)) \) we have that \( \int_x^\infty e^{-Q(z)} \, dz < \infty \), for all \( x \in (0, \infty) \). Thus \( \mu(0, \infty) < \infty \) follows immediately.

4. \textit{R}-positivity

In this section, we will show that the one-dimensional diffusion \( X \) killed at 0 is \( R \)-positive. This means that the processes \( Y \), whose law is the conditional law of \( X \) to never hit the origin, is positive recurrent.

A direct computation shows that \( \eta_x \) introduced in \[\text{5}\] satisfies:

\[
\eta_x(x) = e^{Q(x)} \left(1 - 2\lambda \int_0^x \eta_x(z) e^{-Q(z)} \, dz\right),
\]

\[
\eta_x(x) = \int_0^x e^{Q(y)} \left(1 - 2\lambda \int_0^y \eta_x(z) e^{-Q(z)} \, dz\right) \, dy.
\] (14)

We know from Theorem \[\text{B}\] in \[\text{4}\] that for \( x > 0 \) fixed, the following limit exists and defines a diffusion \( Y \):

\[
\lim_{t \to \infty} \mathbb{P}_x(X_t \in A | \tau > t) = e^{1+\phi_{x}(1)} \mathbb{P}_x \left(\frac{\eta_x(X_t)}{\eta_x(x)} : X_t \in A, \tau > s\right)
\]

\[
= \mathbb{P}_x(Y_t \in A).
\]

The diffusion \( Y \) satisfies the SDE

\[
dY_t = dB_t - \phi(Y_t) \, dt \quad \text{where} \quad \phi(y) = q(y) - \eta'_x(y),
\] (15)

and it takes values on \((0, \infty)\). In fact, since its drift is of order \(1/\lambda\) for \( x \) near 0, so it never reaches 0.

The connection between the classification of \( Y \) and the \( R \)-classification of the killed diffusion \( X^\tau \) is given in the following definition.

**Definition 3.** If the process \( Y \) is positive recurrent (resp. recurrent, null recurrent, transient), then the process \( X^\tau \) is said to be \( R \)-positive (resp. \( R \)-recurrent, \( R \)-null, \( R \)-transient).

We may now state the following result.

**Theorem 2.** Assume \( (H) \) holds. Then \( X^\tau \) is \( R \)-positive if

\[
\lim_{x \to 0} \mu([x, \infty)) \int_x^\infty e^{Q(y)} \, dy = 0.
\] (16)

Proof. If \( (H) \) is satisfied, since \( \lim_{x \to 0} \mu([x, \infty)) \int_0^\infty e^{Q(y)} \, dy = 0 \) \( \Rightarrow \delta = \sup_{x > 0} \int_0^x e^{Q(y)} \, dy \int_0^\infty 2e^{-Q(y)} \, dy < \infty \), then we know from Lemma \[\text{4}\] that \( \lambda_c > 0 \). Further, from Lemma \[\text{1}\] we obtain \( \mu(0, \infty) < \infty \). Next, we will prove that \( (16) \) is equivalent to

\[
\lim_{n \to \infty} \sup_{r > n} \mu([r, \infty)) \int_r^\infty e^{Q(y)} \, dy = 0.
\] (17)

In fact, for any \( r > n \), we have

\[
\mu([r, \infty)) \int_r^\infty e^{Q(y)} \, dy \leq \mu([r, \infty)) \int_0^\infty e^{Q(y)} \, dy,
\]
which implies
\[
\limsup_{n \to \infty} \mu([r, \infty)) \int_0^\infty e^{\xi(x)} dx \leq \limsup_{r \to \infty} \mu([r, \infty)) \int_0^\infty e^{\xi(x)} dx = 0.
\]
Conversely, for any \( n > 0 \), when \( x > n \), we have
\[
\mu([x, \infty)) \int_0^x e^{\xi(y)} dy = \mu([x, \infty)) \int_0^\infty e^{\xi(y)} dy + \mu([x, \infty)) \int_n^\infty e^{\xi(y)} dy
\leq \mu([x, \infty)) \int_0^\infty e^{\xi(y)} dy + \sup_{x > n} \mu([x, \infty)) \int_n^\infty e^{\xi(y)} dy.
\]

By letting \( x \to \infty \) in the above formula, we have
\[
\lim_{n \to \infty} \mu([x, \infty)) \int_0^x e^{\xi(y)} dy \leq \limsup_{n \to \infty} \mu([r, \infty)) \int_r^\infty e^{\xi(x)} dx = 0.
\]

Then we prove the equivalence. We know from Corollary 3.4.14 in [19] that when \( \mu \) is finite then (17) holds if and only if the following super-Poincaré inequality holds, which has been introduced in the work [18]:
\[
\mu(f^2) \leq r E(f, f) + \beta(r) \mu(|f|^2), \quad r > 0,
\]
where \( \beta : (0, \infty) \to (0, \infty) \) is a decreasing function, \( (E, D(E)) \) is a Dirichlet form on \( L^2(\mu) \). At the same time, it has been proved in [18], Theorem 2.1) that (18) is equivalent to \( \sigma_{ess}(L) = 0 \), where \( \sigma_{ess}(L) \) denotes the essential spectrum of \( L \). Then we know that \( -L \) has a purely discrete spectrum \( 0 < \lambda_1 < \lambda_2 < \cdots \), \( \lim_{n \to \infty} \lambda_i = +\infty \) and there exists an orthonormal basis \( \{\eta_i\}_i \) in \( L^2(\mu) \) such that \( -L\eta_i = \lambda_i \eta_i \). Here, we remind the reader that \( \lambda_1 = \lambda_c \).

We denote \( \eta_1 = \eta_c \). For some \( c > 0 \) fixed, we consider the functions
\[
Q^Y(\eta) = \int_c^\infty 2\phi(x) dx = Q(\eta) - Q(c) - 2 \log(\eta(c)/\eta_1(c))
\]
and
\[
\Lambda^Y(\eta) = \int_c^\infty e^{\phi(c)} dz = \eta_1(c)e^{-Q(c)} \int_c^{\eta_1^2(c)} dz \eta_1^2(z)e^{Q(c)} dz.
\]
Because \( \eta_c(x) = x + O(x^2) \) for \( x \) near 0, we first obtain that \( \Lambda^Y(0^+) = -\infty \).

The speed measure \( m \) of \( Y \) is given by
\[
m(dx) = \frac{2dx}{(\Lambda^Y(x))^2}
\]
(see [7], formula (5.51)). So we obtain
\[
m(dx) \leq 2\frac{\eta_1^2(x) e^{Q(c)} dx}{\eta_1^2(c)} = \frac{\eta_1^2(x) e^{Q(c)} dx}{\eta_1^2(c)} < \infty.
\]
We have proved that \( \eta_1 \in L^2(\mu) \), i.e. \( \int_0^\infty \eta_1^2(x) \mu(dx) < \infty \), which implies \( \int_0^\infty \eta_1^2(z) e^{Q(c)} dz < \infty \). Then \( \Lambda^Y(\infty) = \infty \) and from (19) we obtain \( m(0, \infty) < \infty \).

Let \( T_a^Y := \inf \{ t > 0 : Y_t = a \} \) be the hitting time of \( a \) for the process \( Y \). For any \( x, a \in (0, \infty) \), we know that the process \( Y \) is positive recurrent when \( \mathbb{E}_x(T_a^Y) < \infty \). By using the formulas on page 353 in [7], we deduce \( Y \) is positive recurrent. Therefore, \( X^c \) is \( R \)-positive.

5. Examples

In this section, we will illustrate our results with the following examples. Moreover, the second example is also given to exhibit the usefulness of the results.
Example 1. The first example we consider the diffusion
\[ dX_t = dB_t - adt, \quad X_0 = x > 0, \]
where \( a > 0 \) constant. In this case, \( q(x) = a, Q(y) = \int_0^y 2adx = 2ay, \Lambda(x) = \int_0^x e^{Q(y)}dy = \frac{1}{2a}(e^{2ax} - 1). \) Then it is direct to check that \( \Lambda(\infty) = \infty, S = \int_0^\infty e^{Q(y)}\left(\int_y^\infty e^{-Q(\xi)}d\xi\right)dy = \int_0^\infty e^{2ay}, \frac{1}{2a}e^{-2ax}dy = \infty. \)

Consider \( \eta_1 \), the solution of
\[ \frac{1}{2}u''(x) - au'(x) = -\lambda u(x), \quad u(0) = 0, \quad u'(0) = 1. \] (20)

If \( a^2 - 2\lambda > 0 \), we have \( \eta_1(x) = \frac{1}{\sqrt{2\alpha^2-2\lambda}}(e^{a\alpha\sqrt{2\lambda-2\alpha^2}} - e^{-a\alpha\sqrt{2\lambda-2\alpha^2}}), \) then in this case, for any \( x > 0, \eta_1(x) > 0. \)
If \( a^2 - 2\lambda = 0 \), we have \( \eta_1(x) = xe^{a\lambda}, \) then in this case, for any \( x > 0, \eta_1(x) > 0. \) If \( a^2 - 2\lambda < 0 \), we have \( \eta_1(x) = \frac{1}{\sqrt{2\lambda-2\alpha^2}}\sin(\sqrt{2\lambda-2\alpha^2}x) \), then in this case, for any \( x > 0, \eta_1(x) \) has to change its sign.

Hence \( \lambda_c = \frac{a^2}{2}. \) By Proposition 2, for any \( 0 < \lambda \leq \lambda_c, \)
\[ dv_1 = 2\lambda\eta_1 dy = 2\lambda\eta_1 e^{-a\lambda}dy \]
is a QSD. In particular, the minimal QSD is \( v_\lambda(dy) = a^2ye^{-\alpha y}dy. \) This result is in keeping with [13].

Example 2. The second example we consider is the Ornstein-Uhlenbeck process
\[ dX_t = dB_t - aX_t dt, \quad X_0 = x > 0, \]
where \( a > 0 \) constant. In this case, \( q(x) = ax, Q(y) = \int_0^y 2axdx = ay^2, \Lambda(x) = \int_0^x e^{Q(y)}dy = \int_0^x e^{y^2}dy. \) From this we have the following behaviors at \( \infty: \)
\[ \int_0^\infty e^{y^2}dy \sim \frac{1}{2ax}e^{ax^2} \quad \text{and} \quad \int_x^\infty e^{-y^2}dy \sim \frac{1}{2ax}e^{-ax^2}. \]

Then, straightforward calculations show that
\[ \Lambda(\infty) = \infty \quad \text{and} \quad S = \int_0^\infty e^{Q(y)}\left(\int_y^\infty e^{-Q(\xi)}d\xi\right)dy = \infty. \]

Consider \( \eta_1 \), the solution of
\[ \frac{1}{2}u''(x) - axu'(x) = -\lambda u(x), \quad u(0) = 0, \quad u'(0) = 1. \] (21)

For (21), we know from Lemma 3.6 in [11] that \( \{\lambda \mid \varphi_\lambda(.) \text{ does not change sign}\} = (-\infty, a] \) and for any \( \lambda \in (0, a], \)
\( \int_0^\infty \varphi_\lambda(x)dx < \infty. \)

Hence \( \lambda_c = a. \) By Proposition 2, for any \( 0 < \lambda \leq \lambda_c, \)
\[ dv_1 = 2\lambda\eta_1 dy = 2\lambda\eta_1 e^{-a\lambda}dy \]
is a QSD. This result is in keeping with [11].

By using the above asymptotic relation, it is direct to check that
\[ \lim_{x \to \infty} \mu(x, \infty) \int_0^\infty e^{Q(y)}dy = \lim_{x \to \infty} \frac{1}{4a^2x^4} = 0. \]

Therefore, by Theorem 2, it follows that the Ornstein-Uhlenbeck process killed at 0 is \( R \)-positive.
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