N–LAPLACIAN PROBLEMS WITH CRITICAL DOUBLE EXPONENTIAL NONLINEARITIES

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Abstract. In this paper, we prove the existence of a nontrivial solution for the following boundary value problem

\[
\begin{cases}
-\text{div} \left( \omega(x)|\nabla u(x)|^{N-2}\nabla u(x) \right) = f(x,u), & \text{in } B; \\
u = 0, & \text{on } \partial B,
\end{cases}
\]

where $B$ is the unit ball in $\mathbb{R}^N$, $N \geq 2$, the radial positive weight $\omega(x)$ is of logarithmic type, the function $f(x,u)$ is continuous in $B \times \mathbb{R}$ and has critical double exponential growth, which behaves like $\exp\{e^{\alpha|u|^\frac{N}{N-1}}\}$ as $|u| \to \infty$ for some $\alpha > 0$.

1. Introduction. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, and denote with $W^{1,N}_0(\Omega)$ the standard first order Sobolev space given by

\[
W^{1,N}_0(\Omega) = \overline{\{u \in C_0^\infty(\Omega) : \int_\Omega |\nabla u|^N dx < \infty \}} \cap \{u \in W^{1,N}(\Omega) : \|u\|_{W^{1,N}_0(\Omega)} = (\int_\Omega |\nabla u|^N dx)^{\frac{1}{N}} \}.
\]

This space is related to a limiting case for the Sobolev embedding theorem, which yields $W^{1,N}_0(\Omega) \hookrightarrow L^p(\Omega)$ for all $1 \leq p < \infty$, but one knows by easy examples that $W^{1,N}_0(\Omega) \nsubseteq L^\infty(\Omega)$. Hence, one is led to look for a function $g(s) : \mathbb{R} \to \mathbb{R}^+$ with maximal growth such that

\[
\sup_{u \in W^{1,N}_0(\Omega), \|u\|_{W^{1,N}_0(\Omega)} \leq 1} \int_\Omega g(u)dx < \infty.
\]

It was shown that by Trudinger [16] and Moser [13] that the maximal growth is of exponential type. More precisely,

\[
\exp(\alpha|u|^\frac{N}{N-1}) \in L^1(\Omega), \quad \forall \ u \in W^{1,N}_0(\Omega), \quad \forall \ \alpha > 0,
\]

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and
\[ \sup_{\|u\|_{W_0^{1,N}(\Omega)} \leq 1} \int_{\Omega} \exp(\alpha|u|^N) \, dx \leq C(N) \in \mathbb{R}, \quad \text{if and only if} \quad \alpha \leq \alpha_N, \]
where \( \alpha_N = N \omega_{N-1}^{\frac{1}{N}} \) and \( \omega_{N-1} \) is the surface area of the unit ball in \( \mathbb{R}^N \).

Recently, the influence of weights on limiting inequalities of Trudinger-Moser type has been studied, for example, see [3, 9, 4, 5, 14, 15]. If \( \omega \in L^1(\Omega) \) is a non-negative function, we introduce the weighted Sobolev space
\[ W_0^{1,N}(\Omega, \omega) = \text{cl}\left\{ u \in C^\infty_0(\Omega) : \int_\Omega |\nabla u|^N \omega(x) \, dx < \infty \right\}. \tag{1} \]

A general embedding theory for such weighted Sobolev spaces has been developed in Kufner [12]. It turns out that for weighted Sobolev spaces of form (1) logarithmic weights have a particular significance, since they concern limiting situations of such embeddings. However, to obtain interesting results, one needs to restrict attention to radial functions. So let us consider the subspace of radial functions, i.e.
\[ W_0^{1,N}_{0,\text{rad}}(B, \omega) = \text{cl}\left\{ u \in C^\infty_{0,\text{rad}}(B) : \int_B |\nabla u|^N \omega(x) \, dx < \infty \right\}, \]
where \( B = B_1(0) \) be the unit ball in \( \mathbb{R}^N \), the weight
\[ \omega = \left( \log \frac{1}{|x|} \right)^{\beta(N-1)} \quad \text{or} \quad \omega = \left( \log e \frac{|x|}{|x|} \right)^{\beta(N-1)}. \tag{2} \]

Calanchi and Ruf [6] obtained the following results.

**Lemma 1.1.** Let \( \beta \in [0, 1) \) and \( \omega(x) \) be given by (2). Then for all \( u \in W_0^{1,N}_{0,\text{rad}}(B, \omega) \),
(i) \( \int_B e^{\alpha|u|^N} \, dx < +\infty \) if and only if \( \gamma \leq \gamma_{N,\beta} := \frac{N}{(N-1)(1-\beta)} \).
(ii) \( \|u\|_{W_0^{1,N}_{0,\text{rad}}(B, \omega)} \leq 1 \)
\[ \sup_{\|u\|_{W_0^{1,N}_{0,\text{rad}}(B, \omega)} \leq 1} \int_B e^{\alpha|u|^N} \, dx < +\infty \] if and only if \( \alpha \leq \alpha_{N,\beta} = N \left[ \omega_{N-1}^{\frac{1}{N-1}}(1-\beta) \right]^{-\frac{1}{N}}, \]
where \( \omega_{N-1} \) is the area of the unit sphere \( S_{N-1} \) in \( \mathbb{R}^N \).

In the case \( \beta = 1 \), they also showed the embedding of double exponential type: setting \( N' = \frac{N}{N-1} \), then
\[ \int_B e^{e^{\alpha|u|^{N'}}} \, dx < \infty, \quad \forall \ u \in W_0^{1,N}_{0,\text{rad}}(B, \omega), \tag{3} \]
where there and in the follows \( \omega \) is given by
\[ \omega = \left( \log \frac{e}{|x|} \right)^{N-1}, \tag{4} \]
and
\[ \sup_{u \in W_0^{1,N}_{0,\text{rad}}(B, \omega), \|u\|_\omega \leq 1} \int_B e^{\alpha e^{\omega_{N-1}^{\frac{1}{N-1}}|u|^{N'}}} \, dx < \infty \iff \alpha \leq N. \tag{5} \]
In this paper, we deal with the existence of nontrivial solution for the following nonhomogeneous problem

\[
\begin{aligned}
&-\text{div} \left( \omega(x) |\nabla u(x)|^{N-2} \nabla u(x) \right) = f(x,u), & & \text{in } B; \\
&u = 0, & & \text{on } \partial B,
\end{aligned}
\]

where the weight \( \omega(x) \) is defined in (4), the function \( f(x,u) \) is continuous in \( B \times \mathbb{R} \) and has critical double exponential growth, which behaves like \( \exp \{ e^{\alpha |u|^{N \over N-1}} \} \) as \( |u| \to \infty \).

In view of inequality (5), we say that \( f \) has subcritical growth at \( +\infty \) if for all \( \alpha > 0 \)

\[
\lim_{s \to \infty} \frac{|f(x,s)|}{\exp \{ e^{\alpha |s|^{N \over N-1}} \}} = 0,
\]

and \( f \) has critical double exponential growth at \( +\infty \) if there exists \( \alpha_0 > 0 \) such that

\[
\begin{aligned}
\lim_{s \to \infty} \frac{|f(x,s)|}{\exp \{ N e^{\alpha |s|^{N \over N-1}} \}} &= 0, \quad \forall \alpha > \alpha_0; \\
\lim_{s \to \infty} \frac{|f(x,s)|}{\exp \{ N e^{\alpha |s|^{N \over N-1}} \}} &= +\infty, \quad \forall \alpha < \alpha_0.
\end{aligned}
\]

We assume the following conditions on the nonlinearity \( f(x,u) \):

\( (F_1) \) \( f : B \times \mathbb{R} \to \mathbb{R} \) is continuous radial in \( x \), and \( f(x,u) = 0 \) for all \( (x,u) \in B \times (-\infty,0] \).

\( (F_2) \) There exist constants \( R_0, M_0 > 0 \) such that for all \( x \in B \) and \( u \geq R_0 \),

\[
F(x,u) \leq M_0 f(x,u).
\]

\( (F_3) \) There exists \( \mu > N \) such that for all \( x \in B \) and \( u > 0 \),

\[
0 < \mu F(x,u) \leq uf(x,u),
\]

where \( F(x,u) = \int_0^u f(x,t)dt \).

\( (F_4) \) \( \limsup_{u \to 0^+} \frac{N F(x,u)}{u^N} < \lambda_1 \), uniformly in \( x \in B \), where \( \lambda_1 > 0 \) is the first eigenvalue associated to the eigenvalue problem

\[
-\text{div}(\omega(x)|\nabla u|^N \nabla u) = \lambda_1 |u|^{N-2} u \quad \text{in } W^{1,N}_{0,\text{rad}}(B,\omega).
\]

\( (F_5) \) There exists a constant \( \beta_0 \) with \( \beta_0 > \frac{1}{\alpha_0} \) such that

\[
\lim_{t \to \infty} \frac{f(x,t)}{\exp \{ N e^{\alpha_0 |t|^{N \over N-1}} \}} \geq \beta_0 \quad \text{uniformly in } x.
\]

Since we are only concerned with nonnegative solution, the condition \( (F_1) \) is natural.

Our result states as follows.

**Theorem 1.2.** Suppose \( f \) has critical double exponential growth at \( +\infty \) with \( \alpha_0 > 0 \) given by (8), and \( (F_1), (F_2), (F_3), (F_4), (F_5) \) hold. Then problem (6) has a nontrivial solution.

**Remark 1.** We give an example of \( f \). Let \( f(t) = F'(t) \), where

\[
F(t) = t^\mu e^{\alpha_0 t^{N \over N-1}},
\]
for \( t \geq 0 \) and \( \mu > N \). Then \( f \) satisfies the hypotheses of Theorem 1.2.

The paper is organized as follows: we give some preliminaries results in Section 2. Section 3 is devoted to study the geometry of the energy functional of problem. We give a more precise information about the minimax level obtained by the Mountain Pass Theorem in Section 4. Section 5 is devoted to prove Theorem 1.2.

2. Preliminaries results. Let us consider the space \( H := W_{0,rad}^{1,N}(B, \omega) \) endowed with the norm

\[
\|u\|_\omega = \left( \int_B |\nabla u|^{N\omega}(x)dx \right)^{1/N}
\]

with \( \omega(x) = \left( \log \frac{e}{|x|} \right)^{N-1}, \forall u \in H, \)

The energy functional \( J : H \rightarrow \mathbb{R} \) is given by

\[
J(u) = \frac{1}{N}\|u\|_\omega^N - \int_B F(x,u)dx.
\]

This functional is of class \( C^1 \), since the hypothesis on the growth of \( f \) ensures the existence of positive constants \( c \) and \( C \) such that

\[
|f(x,t)| \leq C \exp\{e^{ct}\}, \forall x \in B, \forall t \in \mathbb{R}.
\]  

A straightforward calculation shows

\[
\langle J'(u), \phi \rangle = \int_B |\nabla u|^{N-2}\nabla u \nabla \phi \omega(x)dx - \int_B f(x,u)\phi dx,
\]

for all \( \phi \in H \). Hence, a critical point of \( J \) is a weak solution of (6).

Definition 2.1. [17] Let \((X, \| \cdot \|_X)\) be a real Banach space with its dual space \((X^*, \| \cdot \|_{X^*})\) and \(I \in C^1(X, \mathbb{R})\). For \( c \in \mathbb{R} \), we say that \(I\) satisfies the \((PS)_c\) condition if for any sequence \(\{u_k\} \subset X\) with

\[
I(u_k) \rightarrow c, \quad I'(u_k) \rightarrow 0 \quad \text{in} \quad X^*,
\]

there is a subsequence \(\{u_{k_l}\}\) such that \(\{u_{k_l}\}\) converges strongly in \(X\).

Lemma 2.2 (Lions-type lemma). Let \(\{u_k\}_k \subset H\) be such that \(\|u_k\|_\omega = 1\). If \(u_k \rightharpoonup u\) in \(H\), \(\nabla u_k \rightarrow \nabla u\) a.e. in \(B\) and \(u \neq 0\), then

\[
\sup_k \int_B e^{Ne^{\|u\|_\omega^N}} \|u_k\|_{X^*}^{\frac{N}{N-1}} dx < +\infty
\]

for any \(1 < p < P\) where

\[
P := \begin{cases} 
(1 - \|u\|_\omega^N)^{-\frac{1}{N-1}}, & \text{if } \|u\|_\omega < 1; \\
+\infty, & \text{if } \|u\|_\omega = 1.
\end{cases}
\]

Proof. From the Brézis-Lieb Lemma, it holds that

\[
\|u_k - u\|_\omega^N = 1 - \|u\|_\omega^N + o_k(1),
\]  

where \(o_k(1) \rightarrow 0\) as \(k \rightarrow \infty\). For every \(x \in B\), it is not difficult to see that

\[
|u_k|^{\frac{N}{N-1}} \leq (1 + \varepsilon)|u_k - u|^{\frac{N}{N-1}} + C|u|^{\frac{N}{N-1}}
\]
for some constant $C$ depending only on $N$ and $\varepsilon$, where $\varepsilon$ is a small positive number to be chosen later. This together with Young inequality, implies that

$$\int_B e^{\frac{N}{q-1}|u_k|^\frac{q}{N-1}} \, dx$$

$$\leq \int_B \exp\{Ne^{\frac{p}{q-1}(1+\varepsilon)|u_k-u|^\frac{N}{N-1}} e^{\frac{1}{q-1}C|u|^\frac{N}{N-1}}\} \, dx$$

$$\leq \int_B \exp \left\{N \left[\frac{1}{q} \exp\frac{1}{q-1}(1+\varepsilon)|u_k-u|^\frac{N}{N-1} + \frac{1}{q} e^{\frac{1}{q-1}C|u|^\frac{N}{N-1}}\right]\right\} \, dx$$

$$\leq \frac{1}{q} \int_B \exp \left\{Ne^{\frac{p}{q-1}(1+\varepsilon)|u_k-u|^\frac{N}{N-1}}\right\} + \frac{1}{q} \int_B \exp \left\{Ne^{\frac{1}{q-1}C|u|^\frac{N}{N-1}}\right\} \, dx,$$

(11)

where $\frac{1}{q} + \frac{1}{q} = 1$. When $\|u\|_\omega < 1$, $p < (1 - \|u\|_\omega)^{-\frac{1}{N-1}}$, from (10) we can choose some $q > 1$ and $\varepsilon > 0$ such that $qp(1+\varepsilon)\|u_k-u\|_\omega^{-\frac{N}{N-1}} \leq 1 - \varepsilon + o_k(1)$. When $\|u\|_\omega = 1$, for any $p > 1$, $q > 1$ and $\varepsilon > 0$, $qp(1+\varepsilon)\|u_k-u\|_\omega^{-\frac{N}{N-1}} = o_k(1)$. The result follows from (5) and (11).

3. Geometry of the function. In this section, we show that the energy functional $J$ satisfies geometric conditions of the mountain pass theorem. Then, we are going to use mountain-pass theorem without a compactness condition such as the one of the $(PS)$ type to prove the existence of the solution. This version of the mountain-pass theorem is a consequence of Ekeland’s variational principle.

Lemma 3.1. Suppose $(F_1)-(F_4)$ hold, $f$ has a critical growth at $+\infty$, then there exist $\delta > 0$ and $\rho > 0$ such that

$$J(u) \geq \delta \quad \forall \quad u \in H, \quad \|u\|_\omega = \rho.$$

Proof. From $(F_4)$, there exist $\tau, \delta_0 > 0$ such that

$$F(x, u) \leq \frac{\lambda_1 - \tau}{N} |u|^N \quad \text{for } |u| \leq \delta_0, \quad x \in B.$$  (12)

On the other hand, from (9), we have that for $q > N$, there exists a constant $C_1 > 0$ such that

$$F(x, u) \leq C_1 |u|^q \exp\{e^{c|u|^\frac{N}{N-1}}\} \quad \forall \quad |u| \geq \delta_0, \quad x \in B.$$  

Thus

$$J(u) \geq \frac{1}{N} \|u\|_\omega^N - \frac{\lambda_1 - \tau}{N} \int_B |u|^N \, dx - C_1 \int_B |u|^q \exp\{e^{c|u|^\frac{N}{N-1}}\} \, dx$$

$$\geq \frac{\tau}{N\lambda_1} \|u\|_\omega^N - C_1 \left(\int_B |u|^{\frac{N}{N-1}} \, dx\right)^{(N-1)/N} \left(\int_B \exp\{Ne^{c|u|^\frac{N}{N-1}}\} \, dx\right)^{1/N}.$$  

Next, we choose $\sigma_0 > 0$ such that $c\sigma_0^\frac{N}{N-1} \leq \omega_\frac{1}{N-1}$ so that for $u \in H$ with $\|u\|_\omega \leq \sigma_0$,

$$\int_B \exp\{Ne^{c|u|^\frac{N}{N-1}}\} \, dx = \int_B \exp\left\{Ne^{c\left(\frac{\sigma_0^N}{\|u\|_\omega^{N-1}}\right)^\frac{N}{N-1}}\right\} \, dx \leq C_2.$$  

Moreover, using the radial lemma [6, Lemma 4(iii)],

$$|u(x)| \leq C_3\|u\|_\omega \left(\log\left(\frac{e}{|x|}\right)\right)^\frac{N-1}{N},$$  

and we get
\[ J(u) \geq \frac{\tau}{N\lambda_1} \|u\|_{H^1}^N - C\|u\|_2^q, \]
for \( \sigma < \sigma_0 \) and \( C \geq C_1 C_2^N C_3^N (\int_B |\log (\log \frac{r}{r_0})|^q dx)^{\frac{N-1}{q}} \).

Now, choose \( \rho > 0 \) as the point where \( g(\sigma) = \frac{\tau}{N\lambda_1} \sigma^N - C\sigma^q \) achieved its maximum on \([0, \sigma_0]\) and \( \delta = g(\rho) \).

**Lemma 3.2.** Suppose \((F_1)\) and \((F_2)\) hold. There exists \( e \in H \) with \( \|e\|_\omega > \rho \) such that \( J(e) < 0 \).

**Proof.** Let \( u_0 \in H \cap L^\infty(B) \) such that \( \|u_0\|_\infty = 1 \). From \((F_1)\) and \((F_2)\), there exists a constant \( C \) such that
\[ F(x, u) \geq C e^{\frac{1}{N}|u|}, \quad \forall |u| \geq R_0, \ x \in B. \]
In particular, for \( p > N \), there exists \( C \) such that
\[ F(x, u) \geq C|u|^p - C, \quad \forall u \in \mathbb{R}, \ x \in B. \]
Since \( p > N \), for \( t > 0 \), we have
\[ J(t u_0) \leq \frac{t^N}{N} \|u_0\|_\omega^N - C t^p \int_B |u_0|^p dx + C \to -\infty \]
as \( t \to \infty \). Setting \( e = t u_0 \) with \( t \) sufficiently large, the conclusion of the lemma follows. \( \square \)

4. **The minimax level.** In order to get a more precise information about the minimax level obtained by the mountain pass theorem, let us consider the function \( \varphi_k = \varphi_k(x) \) defined by means of the identity
\[ \psi_k(t) := \omega_{N-1}^{1/N} \varphi_k(x), \quad \text{with} \ |x| = e^{-t} \]
where \( \{\psi_k\}_k \) is the Moser-type sequence introduced in \([6]\). More precisely
\[ \psi_k(t) = \begin{cases} \frac{\log(1+t)\log(1+k)}{\log(1+k)}, & 0 \leq t \leq k; \\ \frac{\log(1+t)\log(1+k)}{\log(1+k)}, & t \geq k. \end{cases} \]
(14)

Then
\[ \|\varphi_k\|_N = \int_B |\nabla \varphi_k(x)|N |\log \frac{e}{|x|}|^{N-1} dx = \int_0^{\infty} |\psi'_k(t)|N (1+t)^{N-1} dt = 1, \]
and
\[ \int_B \exp \left\{ N e^{\frac{x}{N-1} |\varphi_k|^{N-\frac{1}{N}}} \right\} dx = \omega_{N-1} \int_0^{\infty} \exp \left\{ N e^{\psi_k^{N-\frac{1}{N}}} - Nt \right\} dt. \]

**Lemma 4.1.** We have
\[ \lim_{k \to \infty} \int_0^{\infty} \exp \left\{ N e^{\psi_k^{N-\frac{1}{N}}} - Nt \right\} dt = \frac{N+1}{N} e^N. \]
(15)

**Proof.** By the definition of \( \psi_k \), we have
\[ \int_0^{\infty} \exp \left\{ N e^{\psi_k^{N-\frac{1}{N}}} - Nt \right\} dt = \int_0^k \cdots dt + \int_k^{\infty} \cdots dt := I_1(k) + I_2(k), \]
(16)
where

\[ I_2(k) = \int_k^\infty \exp \left\{ N e^{\frac{\chi_{[1,j]\theta}}{\log(1+j\theta)}} - N t \right\} dt = \int_k^\infty \exp \left\{ Ne^{\log(1+k)} - N t \right\} dt \]

\[ = \int_k^\infty e^{N(1+k) - N t} dt = \frac{e^N}{N}. \tag{17} \]

On the other hand,

\[ I_1(k) = \int_0^k \exp \left\{ N e^{\frac{\chi_{[1,j]\theta}}{\log(1+j\theta)}} - N t \right\} dt \]

\[ = \int_0^k \exp \left\{ Ne^{(\log(1+t)N/(N-1))} - N t \right\} dt \]

\[ = \int_0^k \exp \left\{ N e^{\log(1+t)(\log(1+t))^{1/(N-1)}} - N t \right\} dt \]

\[ = \int_0^k \exp \left\{ N (1 + t)(\log(1+t))^{1/(N-1)} - N t \right\} dt. \]

Setting \( s = 1 + t \) and \( j = k + 1 \), we have

\[ I_1(k) = \int_1^{k+1} \exp \left\{ N s \left( \frac{\log s}{\log(1+s)} \right)^{1/(N-1)} - N(s-1) \right\} ds \]

\[ = e^N \int_1^j \exp \left\{ N s \left( \frac{\log s}{\log(1+s)} \right)^{1/(N-1)} - N s \right\} ds. \]

Writing \( \eta_j(s) = N s \left( \frac{\log s}{\log(1+s)} \right)^{1/(N-1)} \) with \( s \geq 1 \), then

\[ I_1(k) = e^N \int_1^j e^{\eta_j(s)} ds \]

\[ = e^N \left[ \int_1^j e^{\eta_j(s)} ds + \int_j^{j-j\theta} e^{\eta_j(s)} ds + \int_j^{j\theta} e^{\eta_j(s)} ds \right] \]

\[ = e^N \left[ A_1(j) + A_2(j) + A_3(j) \right], \tag{18} \]

where \( \theta \in (0, 1) \). We will estimate each terms as follows.

For \( A_1(j) \), since

\[ \chi_{[1,j]\theta}(s) e^{\eta_j(s)} \leq e^{Ns \frac{\theta}{N}} \cdot N s \in L^1([1, +\infty)), \]

and

\[ \chi_{[1,j]\theta}(s) e^{\eta_j(s)} \to e^{N-s} \text{ for a.e. } s \in [1, +\infty), \text{ as } j \to +\infty. \]

This, by Lebesgue dominated convergence Theorem, yields

\[ \lim_{j \to +\infty} A_1(j) = \lim_{j \to +\infty} \int_1^j e^{\eta_j(s)} ds = \int_1^\infty \chi_{[1,j]\theta}(s) e^{\eta_j(s)} ds = \frac{1}{N}. \]
For $A_2(j)$, we have
\[
\eta_j(j^\theta) = N_j \left( \frac{\log j^\theta}{\log j} \right)^{1/(N-1)} - N_j^\theta = N_j^\theta - \frac{1}{N-1} N_j - N_j^\theta (1 - j^{-\theta(1 - \frac{1}{N-1})}) \\
\leq - \frac{1}{N-1} j^\theta \quad \text{for} \quad j \geq C(N, \theta),
\]
where $C(N, \theta) > 0$ is a constant depending only on $N$ and $\theta$. Moreover,
\[
\eta_j(j - j^\theta) = N(j - j^\theta) \left( \frac{\log(j - j^\theta)}{\log j} \right)^{1/(N-1)} - N(j - j^\theta) \\
= N \exp \left\{ \log(j - j^\theta) \left[ \frac{\log(j - j^\theta)}{\log j} \right]^{1/(N-1)} \right\} - N(j - j^\theta) \\
= N \exp \left\{ \frac{1}{(\log j)^{1/(N-1)}} \left[ \log(j - j^\theta) \right]^{N/(N-1)} \right\} - N(j - j^\theta) \\
= N \exp \left\{ \frac{1}{(\log j)^{1/(N-1)}} \left[ \log \left( 1 - \frac{1}{j^{1-\theta}} \right) \right]^{N/(N-1)} \right\} - N(j - j^\theta).
\]
By a Taylor expansion, we have
\[
\log \left( 1 - \frac{1}{j^{1-\theta}} \right) = - \frac{1}{j^{1-\theta}} + O\left( \frac{1}{j^{2(1-\theta)}} \right), \quad \text{as} \quad j \to \infty,
\]
and
\[
\left[ \log \left( 1 - \frac{1}{j^{1-\theta}} \right) \right]^{N/(N-1)} = \left[ \log j - \frac{1}{j^{1-\theta}} + O\left( \frac{1}{j^{2(1-\theta)}} \right) \right]^{N/(N-1)} \\
= (\log j)^{N/(N-1)} + \left( \log j \right)^{1/(N-1)} \left( - \frac{N}{N - 1} \frac{1}{j^{1-\theta}} + O\left( \frac{1}{j^{2(1-\theta)}} \right) \right).
\]
Then, we have
\[
\frac{1}{(\log j)^{1/(N-1)}} \left[ \log j + \log \left( 1 - \frac{1}{j^{1-\theta}} \right) \right]^{N/(N-1)} = \log j - \frac{N}{N - 1} \frac{1}{j^{1-\theta}} + O\left( \frac{1}{j^{2(1-\theta)}} \right).
\]
Hence
\[
\eta_j(j - j^\theta) = N \exp \left\{ \log j - \frac{N}{N - 1} \frac{1}{j^{1-\theta}} + O\left( \frac{1}{j^{2(1-\theta)}} \right) \right\} - N(j - j^\theta) \\
= N \left[ e^{- \frac{N}{N - 1} \frac{1}{j^{1-\theta}} + O\left( \frac{1}{j^{2(1-\theta)}} \right)} - 1 \right] + N_j^\theta \\
= N \left[ \frac{N}{N - 1} \frac{1}{j^{1-\theta}} + O\left( \frac{1}{j^{2(1-\theta)}} \right) \right] + N_j^\theta \\
= - \frac{N}{N - 1} j^\theta \left[ 1 + O\left( \frac{1}{j^{1-\theta}} \right) \right].
Thus for any $\varepsilon \in (0, 1)$, there exists $j_\varepsilon \geq 1$, such that
\[
\eta_j(j - j^\theta) \leq -\frac{N}{N-1}(1 - \varepsilon)j^\theta \quad \text{for } j \geq j_\varepsilon. \tag{21}
\]
Let us fix $j \geq 1$ and assume $j$ is sufficiently large. A qualitative study of $\eta_j$ on $[1, +\infty)$ shows that $\eta_j \leq 0$ on $[1, j]$ and achieve its unique minimum point at $s_j \in (1, j)$, hence
\[
A_2(j) = \int_{j^\theta}^{j^{j - j^\theta}} e^{\eta_j(s)} ds \leq (j - 2j^\theta)e^{\max\{\eta_j(j^\theta), \eta_j(j - j^\theta)\}}.
\]
From (20) and (21), with $\varepsilon = \frac{N-1}{N}$, we deduce
\[
\max\{\eta_j(j^\theta), \eta_j(j - j^\theta)\} \leq -\frac{1}{N-1}j^\theta
\]
provided $j$ is sufficiently large. Therefore, there exists $j_0 \geq 1$ such that
\[
A_2(j) \leq (j - 2j^\theta)e^{-\frac{1}{N-1}j^\theta}, \quad \forall j \geq j_0.
\]
From (20) and (21), we obtain
\[
\lim_{j \to +\infty} A_2(j) = 0. \tag{22}
\]
For $A_3(j)$. Since $\eta_j$ is convex on $[j - j^\theta, +\infty)$, then for $s \in [j - j^\theta, j]$, we have
\[
\frac{\eta_j(j) - \eta_j(s)}{j - s} \geq \frac{\eta_j(j) - \eta_j(j - j^\theta)}{j^\theta}.
\]
Using the fact $\eta_j(j) = 0$, we find
\[
\eta_j(s) \leq \frac{j - s}{j^\theta} \eta_j(j - j^\theta), \quad \text{for } s \in [j - j^\theta, j].
\]
In view of (21), if $\varepsilon \in (0, 1)$ and $j \geq j_\varepsilon$, we have
\[
\eta_j(s) \leq \frac{N}{N-1}(1 - \varepsilon)(s - j), \quad \text{for } s \in [j - j^\theta, j]. \tag{23}
\]
On the other hand, the convexity of $\eta_j$ on $[j - j^\theta, +\infty)$ and the fact that $\eta'_j(j) = \frac{N}{N-1}$ yields
\[
\eta_j(s) \geq \eta_j(j) + \eta'_j(j)(s - j) = \frac{N}{N-1}(s - j), \quad \text{for } s \in [j - j^\theta, j]. \tag{24}
\]
From (23) and (24), we obtain
\[
\frac{N - 1}{N} \leq \lim_{j \to +\infty} A_3(j) = \lim_{j \to +\infty} \int_{j - j^\theta}^{j} e^{\eta_j(s)} ds \leq \frac{N - 1}{N(1 - \varepsilon)},
\]
and, since $\varepsilon \in (0, 1)$ is arbitrarily fixed, we get
\[
\lim_{j \to +\infty} A_3(j) = \frac{N - 1}{N}. \tag{25}
\]
By (18), (19), (22) and (25), we have
\[
\lim_{k \to +\infty} I_1(k) = e^N. \tag{26}
\]
Thus, (15) follows from (16), (17) and (26). \qed
Lemma 4.2. Suppose that (F1) and (F3) hold. Then there exists $k \in \mathbb{N}$ such that
\[
\max_{t \geq 0} J(t \varphi_k) = \max_{t \geq 0} \left\{ \frac{t^N}{N} - \int_B F(x, t \varphi_k(x))dx \right\} < \frac{1}{N} \left( \frac{\omega_{N-1}^{1/2}}{\alpha_0} \right)^{N-1},
\]
where $\varphi_k(x) = \omega_{N-1}^{-1/N} \psi_k(t)$ with $|x| = e^{-t}$ and $\psi_k$ is defined in (14), $\omega_{N-1}$ is the volume of $(N-1)$-dimensional surface of the unit sphere, and $\alpha_0$ is given in (8).

Proof. Suppose, by contradiction, that for all $k$ we have
\[
\max_{t \geq 0} \left\{ \frac{t^N}{N} - \int_B F(x, t \varphi_k(x))dx \right\} \geq \frac{1}{N} \left( \frac{\omega_{N-1}^{1/2}}{\alpha_0} \right)^{N-1}.
\]
Then for any $k \geq 1$, there exists $t_k > 0$ satisfying
\[
\frac{1}{N} \left( \frac{\omega_{N-1}^{1/2}}{\alpha_0} \right)^{N-1} \leq \max_{t \geq 0} \left\{ \frac{t^N}{N} - \int_B F(x, t \varphi_k(x))dx \right\} = \frac{t_k^N}{N} - \int_B F(x, t_k \varphi_k(x))dx.
\]
Thus
\[
\frac{t_k^N}{N} - \int_B F(x, t_k \varphi_k(x))dx \geq \frac{1}{N} \left( \frac{\omega_{N-1}^{1/2}}{\alpha_0} \right)^{N-1},
\]
and using the fact that $F(x, u) \geq 0$, we obtain
\[
t_k^N \geq \left( \frac{\omega_{N-1}^{1/2}}{\alpha_0} \right)^{N-1}. \tag{27}
\]
Since at $t = t_k$, we have
\[
\frac{d}{dt} \left( \frac{t^N}{N} - \int_B F(x, t \varphi_k(x))dx \right) = 0,
\]
it follows that
\[
t_k^N = \int_B f(x, t_k \varphi_k \varphi_k dx. \tag{28}
\]
Using hypothesis (F5), given $\tau > 0$, there exists $R_\tau > 0$ such that for all $u \geq R_\tau$, we get
\[
f(x, t) \geq (\beta_0 - \tau) \exp\{N e^{\alpha_1 |t|^{N/2}} \} \quad \forall |t| \geq R_\tau, \text{ uniformly in } x. \tag{29}
\]
By (28) and the definition of $\varphi_k$,
\[
t_k^N = \int_B f(x, t_k \varphi_k) \varphi_k dx \geq \omega_{N-1} \int_k^\infty f \left( e^{-s}, t_k \frac{\omega_{N-1}^{1/2}}{\psi_k} \right) t_k \frac{\psi_k}{\omega_{N-1}^{1/2}} e^{-Ns} ds.
\]
By the definition of $\psi_k$ and (27), for $s \geq k$, we have
\[
t_k \frac{\psi_k}{\omega_{N-1}^{1/2}} = t_k \left( \frac{\log(1 + k)}{\omega_{N-1}^{1/2} (N-1)} \right)^{N-1} \left( \frac{\log(1 + k)}{\alpha_0} \right)^{N-1}.
\]
Therefore, for any $k \geq \bar{k}$ with $\bar{k} = \bar{k}(\tau) \geq 1$ sufficiently large, from (29), we get
\[
t_k^N \geq \omega_{N-1} \int_k^\infty f \left( e^{-s}, t_k \frac{\psi_k}{\omega_{N-1}} \right) t_k \frac{\psi_k}{\omega_{N-1}} e^{-Ns} ds
\]
\[
\geq \omega_{N-1}(\beta_0 - \tau) \int_k^\infty \exp \left\{ N e^{\alpha_0} t_k \frac{\psi_k}{\omega_{N-1}} \right\} e^{-Ns} ds
\]
\[
= \omega_{N-1}(\beta_0 - \tau) \int_k^\infty \exp \left\{ N e^{\alpha_0} t_k \frac{N}{\omega_{N-1}} \log(1+k) - Nk \right\} ds
\]
\[
= \frac{\omega_{N-1}}{N}(\beta_0 - \tau) \exp \left\{ N e^{\alpha_0} (\frac{N}{\omega_{N-1}} |t_k|^N \log(1+k) - Nk \right\}. \tag{30}
\]
So
\[
1 \geq \frac{\omega_{N-1}}{N}(\beta_0 - \tau) \exp \left\{ N e^{\alpha_0} (\frac{N}{\omega_{N-1}} |t_k|^N \log(1+k) - Nk \right\}, \forall k \geq \bar{k},
\]
and thus $\{t_k\}$ is bounded.

Now, if
\[
\lim_{k \to \infty} t_k^N > \left( \frac{\omega_{N-1}}{\alpha_0} \right)^{N-1}, \tag{31}
\]
then (30) would yield a contradiction with the boundedness of $\{t_k\}$. Hence (31) can not hold, it follows that
\[
\lim_{k \to \infty} t_k^N = \left( \frac{\omega_{N-1}}{\alpha_0} \right)^{N-1}. \tag{32}
\]

In order to estimate (28) more precisely, we consider the sets
\[
A_k = \{ x \in B : t_k \varphi_k \geq R_{\tau} \}, \quad C_k = B \setminus A_k
\]
where $R_{\tau} > 0$ is given by (29). By construction,
\[
t_k^N = \int_B f(x, t_k \varphi_k) t_k \varphi_k dx
\]
\[
\geq (\beta_0 - \tau) \int_{A_k} \exp \{ N e^{\alpha_0} |t_k \varphi_k|^\frac{N}{N-1} \} dx + \int_{C_k} f(x, t_k \varphi_k) t_k \varphi_k dx
\]
\[
\geq (\beta_0 - \tau) \int_{A_k} \exp \{ N e^{\alpha_0} |t_k \varphi_k|^\frac{N}{N-1} \} dx - (\beta_0 - \tau) \int_{C_k} \exp \{ N e^{\alpha_0} |t_k \varphi_k|^\frac{N}{N-1} \} dx
\]
\[
+ \int_{C_k} f(x, t_k \varphi_k) t_k \varphi_k dx.
\]
Since $\varphi_k \to 0$ and the characteristic functions $\chi_{C_k} \to 1$ for almost every $x$ in $B$. Therefore, the Lebesgue dominated convergence Theorem implies
\[
\int_{C_k} \exp \{ N e^{\alpha_0} |t_k \varphi_k|^\frac{N}{N-1} \} dx \to \frac{\omega_{N-1}}{N} e^N \quad \text{and} \quad \int_{C_k} f(x, t_k \varphi_k) t_k \varphi_k dx \to 0.
\]
Then, we have
\[
\left( \frac{\omega_{N-1}}{\alpha_0} \right)^{N-1} \geq \left( \beta_0 - \tau \right) \lim_{k \to \infty} t_k^N \int_B \exp \{ Ne^{\alpha_0 |t_k| N} \} dx
\]
\[
= \lim_{k \to \infty} t_k^N \int_B \exp \{ Ne^{\alpha_0 |t_k| N} \} \left( \frac{1}{N} - Nt \right) dx
\]
\[
= \frac{N+1}{N} e^N.
\]
Using (27) and Lemma 15, we have
\[
\lim_{k \to \infty} \int_B \exp \{ Ne^{\alpha_0 |t_k| N} \} dx
\]
\[
= \omega_{N-1} \lim_{k \to \infty} \int_0^\infty \exp \{ Ne^{\alpha_0 |t_k| N} \} \left( \frac{1}{N} - Nt \right) dt
\]
\[
\geq \omega_{N-1} \lim_{k \to \infty} \int_0^\infty \exp \{ Ne^{\alpha_0 |t_k| N} \} - Nt dt
\]
\[
= \frac{N+1}{N} e^N.
\]
Thus, we get
\[
\left( \frac{\omega_{N-1}}{\alpha_0} \right)^{N-1} \geq \left( \beta_0 - \tau \right) \omega_{N-1} e^N,
\]
which implies that
\[
\beta_0 \leq \frac{1}{\omega_{N-1} e^N}
\]
which is a contradiction with (F5), and the proof is complete. \(\square\)

5. **Proof of the existence result.** In this section, we prove the existence result, first we recall the following version of Mountain Pass Theorem.

**Lemma 5.1** ([17, Theorem 2.8]). Let \((X, \| \cdot \|_X)\) be a real Banach space and \(I \in C^1(X, \mathbb{R})\) satisfies \(I(0) = 0\) and
\(I\) has the following properties:
(i) There are constants \(\rho, \alpha > 0\) such that \(I|_{\partial B_\rho} \geq \alpha\).
(ii) There is an \(e \in X \setminus B_\rho\) such that \(I(e) < 0\). Then for the constant
\[
c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)) \geq \alpha
\]
there exists a sequence \(\{u_k\}\) in \(X\) such that
\(I(u_k) \to c, \quad \text{and} \quad I'(u_k) \to 0,\)
where \(\Gamma = \{ \gamma \in C^0([0, 1], X), \gamma(0) = 0, \gamma(1) = e \}\).

**Proof of Theorem 1.2.** From Lemmas 3.1 and 3.2, we can apply the Mountain Pass Theorem to obtain a positive level \(c\) and a Palais-Smale sequence \(\{u_k\}\) in \(H\), i.e.
\[
J(u_k) = \frac{1}{N} \|u_k\|^N - \int_B F(x, u_k) dx \to c \quad \text{as} \quad k \to \infty,
\]
\[
|J'(u_k)v| = |\langle u_k, v \rangle| - \int_B f(x, u_k)v dx | \leq \tau_k \|v\|\quad \text{for all} \quad v \in H,
\]
where \(\tau_k \to 0\) as \(k \to \infty\). Using (33), (34) and (F5), we have
\[
\left( \frac{\mu}{N} - 1 \right) \|u_k\|^N \leq C(1 + \|u_k\|),
\]
and hence $\|u_k\|_\omega$ is bounded and thus
\[\int_B f(x, u_k)dx \leq C, \quad \int_B F(x, u_k)dx \leq C.\]
The embedding $H \hookrightarrow L^q(B)$ is compact for all $q \geq N$, by extracting a subsequence, we can assume that $u_k \rightharpoonup u_0$ weakly in $H$ and $u_k \to u_0$ for almost all $x \in B$ as $k \to \infty$. Thanks to Lemma 2.1 in [8], we have
\[f(x, u_k) \to f(x, u_0) \text{ in } L^1(B) \text{ as } k \to \infty. \tag{35}\]

Since
\[0 < F(x, t) \leq M_0 f(x, t) \quad \text{for all } |t| \geq R_0, \quad \text{uniformly in } B.\]

We may apply the Lebesgue dominated convergence Theorem to conclude that
\[F(x, u_k) \to F(x, u_0) \text{ in } L^1(B) \text{ as } k \to \infty.\]

This together with (33) yields,
\[\lim_{k \to +\infty} \|u_k\|_{N'}^N = N\left(c + \int_B F(x, u_0)dx\right). \tag{36}\]

For any $T > 0$, let $S_k^T = \{x \in B | u_k(x) - u_0(x) \geq T\}$. Note that the inequality, for $x \geq 0 Cx \leq N e^x + C(N^{-1}C - 2)$, implies for $x \in B$
\[|u_k(x) - u_0(x)|^{N'} \leq Ne^{-\omega_{N-1}^{-1} \frac{u_k - u_0}{|u_k - u_0|^2}}^{N'} + \omega_{N-1}^{-1} ||u_k - u_0||_{L^2} (N^{-1} \omega_{N-1}^{-1} ||u_k - u_0||_{L^2} - 2).\]

Then for $T > 0$ large enough,
\[\mathcal{L}^N(S_k^T) \leq e^{-T^{N'}} \int_{S_k^T} e^{\|u_k - u_0\|_{L^2}^{N'}} dx \leq e^{-T^{N'}} \int_B \exp\{N e^{-\omega_{N-1}^{-1} \frac{v_k - v_0}{|v_k - v_0|^2}}^{N'} + C'\} dx \leq C(N) e^{-\frac{2}{T^{N'}}},\]

where $\mathcal{L}^N(S_k^T)$ denote the Lebesgue measure of $S_k^T$, $N' = N/(N - 1)$ and the constant $C'$ depends on $N$ and the upper bound of $\|u_k\|_\omega$. It follows
\[\int_{S_k^T} \|\nabla u_k - \nabla u_0\|dx \leq Ce^{-\frac{2}{T^{N'}}} (\int_B \|\nabla u_k - \nabla u_0\|_{L^2}^{2} \omega dx)^{\frac{1}{2}} \to 0 \quad \text{as } T \to \infty. \tag{37}\]

As in [2] the truncation of $v \in H$ at height $T$ denote with $v^T \in H$, that is
\[v^T = \begin{cases} v, & \text{if } |v| \leq T, \\ T v/|v|, & \text{if } |v| > T. \end{cases}\]

Take $v = (u_k - u_0)^T$ in (34), then we have
\[|\int_{B \setminus S_k} (|\nabla u_k|^{N-2} \nabla u_k - |\nabla u_0|^{N-2} \nabla u_0)(\nabla u_k - \nabla u_0)\omega dx| \leq \int_B f(x, u_k)(u_k - u_0)^T dx + o(1) \|u_k - u_0\|_\omega + |\langle u_0, u_k - u_0\rangle_\omega|.\]

By (35) and the Lebesgue dominated convergence Theorem,
\[\int_B f(x, u_k)(u_k - u_0)^T dx \to 0 \quad \text{as } k \to \infty.\]
Since $u_k \to u_0$ weakly in $H$, $|\langle u_0, u_k - u_0 \rangle_\omega| \to 0$ as $k \to \infty$. On the other hand, we recall the elementary inequality that, for $a, b \in \mathbb{R}^N$,

$$|(a)^{N-2}a - (b)^{N-2}b| \cdot (a - b) \geq \begin{cases} |a - b|^2, & N = 2, \\ \frac{1}{N-1}|a - b|^N, & N \geq 3. \end{cases}$$

We get

$$\int_{B \setminus S^2_k} |\nabla u_k - \nabla u_0| dx \leq C \left( \int_{B \setminus S^2_k} |\nabla u_k - \nabla u_0|^N \omega dx \right)^{\frac{1}{N}} = o(1), \quad (38)$$
as $k \to \infty$. We deduce from (37) and (38) that

$$\lim_{k \to \infty} \int_B |\nabla u_k - \nabla u_0| dx = 0,$$

which implies $\nabla u_k \to \nabla u_0$ a.e. in $B$. From (34) we have

$$\int_B |\nabla u_0|^{N-2} \nabla u_0 \nabla v \left( \log \frac{e}{|x|} \right)^{N-1} dx - \int_B f(x, u_0) v dx = 0 \quad \forall v \in C_0^\infty(B),$$

and $u_0$ is a weak solution of (6).

Next we show that $u_0$ is nontrivial. In fact, by contradiction, we assume that $u_0 = 0$. From (36), we get

$$\lim_{k \to +\infty} \|u_k\|_\omega = Nc.$$

Thus, given $\varepsilon > 0$, we have

$$\|u_k\|_\omega^N \leq Nc + \varepsilon,$$

for $k$ large enough. From Lemma 4.2, we have that

$$c < \frac{1}{N} \left( \frac{\omega_{N-1}^{-1}}{\alpha_0} \right)^{N-1}.$$

Thus, if we choose $\tau > 0$ sufficiently close to 0 and $\varepsilon$ sufficiently small, we have

$$\alpha_0 (1 + \tau) \|u_k\|_\omega^{\frac{N}{\alpha_0}} < \omega_{N-1}^{\frac{1}{N-1}}. \quad (39)$$

Since $f$ has critical growth, for every $\tau > 0$ and $q > 1$, there exist $R_\tau > 0$ and $C_{\tau, q} > 0$ such that

$$|f(x, t)|^q \leq C_{\tau, q} \exp\{Ne^{\alpha_0(1+\tau)|t|^{\frac{N}{\alpha_0}}}\}, \quad \forall |t| \geq R_\tau \text{ uniformly in } x.$$

Therefore,

$$\int_B |f(x, u_k)|^q dx = \int_{\{|u_k| \leq R_\tau\}} |f(x, u_k)|^q dx + \int_{\{|u_k| \geq R_\tau\}} |f(x, u_k)|^q dx$$

$$\leq \frac{\omega_{N-1}}{N} \max_{B \times [-R_\tau, R_\tau]} |f(x, s)|^q + C_{\tau, q} \int_B \exp\{Ne^{\alpha_0(1+\tau)|u_k|^{\frac{N}{\alpha_0}}}\} dx$$

$$\leq C + C_{\tau, q} \int_B \exp\{Ne^{\alpha_0(1+\tau)|u_k|^{\frac{N}{\alpha_0}}}\} dx.$$
Using (39) and (5), we deduce that
\[
\int_B \exp \{ N e^{\alpha_0 (1 + \tau) |u_k|^{\frac{N}{p-1}}} \} dx
\]
\[
= \int_B \exp \left\{ N e^{\alpha_0 (1 + \tau) \|u\| \|u_k\|^{\frac{N}{\|u_k\|}} \|u_k\| \|u\|} \right\} dx \leq C. \tag{40}
\]
Thus, we get
\[
\int_B |f(x, u_k)|^q dx \leq C.
\]
Then
\[
\left| \int_B f(x, u_k) u_k dx \right| \leq \left( \int_B |f(x, u_k)|^q dx \right)^{1/q} \left( \int_B |u_k|^q \right)^{q'} \to 0,
\]
as \(k \to \infty\), where \(q'\) is the the conjugate exponent of \(q\). Using this and from (34) with \(v = u_k\), we have \(u_k \to 0\) in \(H\). But this is impossible in view if \(\lim_{k \to +\infty} \|u_k\|_\omega = Nc\) and \(c \neq 0\). Consequently \(u_0 \neq 0\).

At last, we shall prove that \(J(u_0) = c > 0\). Note that, from (36)
\[
J(u_0) = \frac{1}{N} \|u_0\|_\omega^N - \int_B F(x, u_0) dx \leq \lim_{k \to \infty} \frac{1}{N} \|u_k\|_\omega^N - \int_B F(x, u_0) dx = c.
\]
Arguing by contradiction, we assume that \(J(u_0) < c\). Set \(v_k = \frac{u_k}{\|u_k\|_\omega}\), then \(\|v_k\|_\omega = 1\) and
\[
v_k \rightharpoonup v_0 = \frac{u_0}{(Nc + \int_B NF(x, u_0) dx)^{1/N}} \neq 0 \quad \text{weakly in } H.
\]
Moreover, \(v_k \to v_0, \nabla v_k \to \nabla v_0\) a.e. in \(B\) and \(\|v_0\|_\omega < 1\). Hence, by Lemma 2.2 that for any \(1 < p < P = (1 - \|v_0\|_\omega^N)\frac{N}{p-1} \leq 1\) we have
\[
\sup_k \int_B \exp \{ N e^{\alpha_{p-1}^p |v_k|^{\frac{N}{N-1}}} \} < +\infty. \tag{41}
\]
By the critical growth of \(f\) at \(+\infty\), for some \(q > 1\) there exist \(\alpha_0 < \alpha < (\alpha_{N-1})^{-\frac{1}{N-1}}\) and \(C > 0\) such that
\[
\int_B |f(x, u_k)|^q dx \leq C + C \int_B \exp \{ N e^{\alpha |u_k|^{\frac{N}{N-1}}} \} dx
\]
\[
\leq C + C \int_B \exp \{ N e^{\alpha_{p-1}^p \frac{1}{p-1} |u_k|^{\frac{N}{N-1}}} \} dx \tag{42}
\]
where \(1 < p < P\), \(\alpha_k = \frac{\alpha_{p-1}^p - \frac{1}{p-1}}{p-1} \|u_k\|_\omega^{\frac{N}{p-1}}\). For sufficiently large \(k\), it holds that
\[
\alpha_k \leq \frac{1}{p} \left( \frac{Nc + \int_B NF(x, u_0) dx}{Nc} \right)^{\frac{1}{p-1}}.
\]
Since \(u_0\) is a nontrivial weak solution to equation (6), it follows
\[
\|u_0\|_\omega^N = \int_B f(u_0) u_0 dx \geq \int_B \mu F(u_0) dx > \int_B NF(u_0) dx.
\]
here the condition \((F_3)\) is used. Thus, then for \(k\) large and \(p\) sufficiently close to \(P\), one has
\[
\alpha_k \leq \frac{P}{p} \left( \frac{Nc + \int_B NF(u_0)dx - \|u_0\|_\omega^N}{Nc + \int_B \mu F(u_0)dx - \|u_0\|_\omega^N} \right)^{\frac{1}{p-1}} < 1.
\]  
Combining with (41), (42), (43) we have
\[
\sup_k \int_B |f(x, u_k)|^q \leq C,
\]
for some \(q > 1\).

From this we see that for \(\varepsilon > 0\), there exists \(\delta > 0\) such that for any measurable set \(\Omega'\) with its Lebesgue measure \(L^N(\Omega') < \delta\), one has
\[
\left| \int_{\Omega'} f(x, u_k)u_k dx \right| \leq \left( \int_{\Omega'} |f(x, u_k)|^q dx \right)^{\frac{1}{q}} \left( \int_{\Omega'} |u_k - u_0|^q dx + \int_{\Omega'} |u_0|^q dx \right)^{\frac{1}{q}} \leq C\varepsilon,
\]
for \(k\) sufficiently large. Since \(u_k \to u_0\) a.e. in \(B\), then by Vitali’s Lemma we have
\[
\|u_k\|_\omega^N - \|u_0\|_\omega^N = \int_B f(x, u_k)u_k dx - \int_B f(x, u_0)u_0 dx + o(1)\|u_k\|_\omega \to 0, \quad \text{as} \quad k \to \infty.
\]
This leads to the contradiction that
\[
c = \lim_{k \to \infty} \frac{1}{N} \|u_k\|_\omega^N - \int_B F(x, u_0)dx = \frac{1}{N} \|u_0\|_\omega^N - \int_B F(x, u_0)dx = J(u_0) < c.
\]
The proof is finished. \(\square\)

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