A generalized maximal abelian gauge in SU(3) lattice gauge theory

William W. Tucker and John D. Stack

Department of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois, 68101

We introduce a generalized Maximum Abelian Gauge (MAG). We work with this new gauge on $12^4$ lattices for $\beta = 5.7, 5.8$ and $16^4$ lattices for $\beta = 5.9, 6.0$. We also introduce a form of abelian projection related to the generalized MAG. We measure $U(1) \times U(1)$ wilson loops and single color magnetic current densities.

Abelian dominance is the idea that the confining physics of an $SU(N)$ gauge theory can be explained by the $U(1)^{N-1}$ degrees of freedom. In lattice gauge theory, this is tested by taking link variables with values in the group $SU(N)$ and projecting them onto the $U(1)^{N-1}$ subgroup. In general, some sort of gauge-fixing procedure is used first. The most commonly used gauge is the maximum abelian gauge (MAG). Gauge-fixing to this gauge can be thought of as maximizing the diagonal components of the $SU(N)$ matrix, perhaps squeezing as much physics as possible into the abelian degrees of freedom.

To date, most of the work on the MAG has been done in $SU(2)$ LGT. Here the lattice functional that is maximized can be expressed as

$$G_{mag}^{su2} = \frac{1}{2N_f} \sum_{x, \mu} \text{Tr} \left[ U_{\mu}^\dagger(x) \sigma_3 U_{\mu}(x) \sigma_3 \right],$$

where $N_f$ is the number of lattice links. After sweeping over the lattice performing gauge transformations many times, the gauge-fixed $SU(2)$ link variables are projected onto the $U(1)$ subgroup and calculations of observables are carried out as in $U(1)$ gauge theory. The projected configurations typically give values of the string tension that are about 10% higher than the full $SU(2)$ value at one gauge copy per configuration. These values come down the $SU(2)$ value when Gribov effects are taken into account.

The MAG in $SU(3)$ is constructed similarly to $SU(2)$. We expect the physics to be similar for both of these theories, but the $SU(3)$ structure is more complicated. The simplest MAG functional may now be expressed as

$$G_{mag}^{su3} = \frac{1}{4N_f} \left\{ \sum_{x, \mu} \text{Tr} \left[ U_{\mu}^\dagger(x) \lambda_3 U_{\mu}(x) \lambda_3 \right] + \sum_{x, \mu} \text{Tr} \left[ U_{\mu}^\dagger(x) \lambda_8 U_{\mu}(x) \lambda_8 \right] \right\}.$$  

It has been found that this form of the MAG in $SU(3)$ gives string tensions about 10% lower than the full $SU(2)$ value. This discrepancy increases when Gribov effects are taken into account.

We begin our construction of a new MAG functional by noting that the $SU(2)$ MAG functional can be expressed as the gauge field coupled to an adjoint Higgs field rotated to the $\sigma_3$ direction. An equivalent $SU(3)$ functional would have the Higgs field rotated so as to take values in the Cartan subalgebra of $SU(3)$. The functional from Eq.(2) cannot be duplicated with a single Higgs field, but can be recreated using two or more Higgs fields.

1. Generalized Maximal Abelian Gauge

On the lattice, our new functional will take the form

$$G_{higgs}^{su3} = \frac{1}{N_h N_f} \sum_{l, x, \mu} \text{Tr} \left[ U_{\mu}^\dagger(x) \Phi_l U_{\mu}(x) \Phi_l \right].$$
where the Higgs fields are
\[ \Phi_i = \frac{1}{\sqrt{2}} [\lambda_3 \cos \chi_i + \lambda_8 \sin \chi_i], \tag{4} \]
and \( N_h \) is the number of Higgs fields. In this equation, we could allow the value of \( \chi_i \) to be site-dependent \( \tilde{3} \). Here, we will deal with the case of constant Higgs fields. If we have two fixed Higgs fields, with \( \chi_1 = 0^\circ \) and \( \chi_2 = 90^\circ \) we get back the simple MAG with functional given by Eq.\( \tilde{3} \).

The gauge-fixing procedure consists of maximizing this functional on a site-by-site basis, one \( SU(2) \) subgroup at a time. This can be done analytically. Overrelaxation was used to speed up convergence \( \tilde{3} \). For fixed \( \chi \), an overrelaxation parameter \( \omega = 1.85 \) was used.

2. Abelian Projection

The act of projection is another instance in which the \( SU(3) \) case has substantial complications that do not arise in \( SU(2) \). In \( SU(2) \), the diagonal elements are complex conjugates of one another. Taking the phase and using it as a diagonal entry does not give a variable that maximizes the quantity
\[ \Phi_i (x) = \sum_{\mu} \text{Re} \left( \text{Tr} \left[ \Phi_i(x) U' D_{\mu}^\dagger(x) \right] \right), \tag{5} \]
where \( D_{\mu}(x) \) is the diagonal component of the original \( SU(3) \) matrix.

The generalized MAG suggests a generalized form of projection that takes into account the Higgs field we used for gauge-fixing. We now find \( U' \) that maximizes the quantity
\[ \sum_i \text{Re} \left( \text{Tr} \left[ \Phi_i(x) U' \Phi_i(x + \hat{\mu}) D_{\mu}^\dagger(x) \right] \right). \tag{6} \]

We note that for the case of the simple MAG this is the same as the optimal method used previously. As long as the Higgs fields are constant this just amounts to a rescaling of the of the components of \( D_{\mu}(x) \). It also demonstrates the importance of the method of projection used in the variable Higgs case.

For the case of constant Higgs fields, it is useful to talk of the functional
\[ P_\phi = \frac{1}{N_h N_t} \sum_{i,x,\mu} \text{Re} \left( \text{Tr} \left[ \Phi_i^2 U'(x) D_{\mu}^\dagger(x) \right] \right), \tag{7} \]
as an indication of how far we have to project to get to the abelian submanifold.

3. Magnetic Currents

Magnetic currents are extracted for each \( SU(3) \) color by applying the Toussaint DeGrand procedure to the \( U(1) \times U(1) \) links. This produces three magnetic currents. After extracting the magnetic currents, monopole Wilson loops are calculated for each color. This part of the calculation proceeds exactly as in \( U(1) \) or \( SU(2) \) lattice gauge theory \( \tilde{3} \).

4. Results

We gauge-fixed 40 configurations on \( 12^4 \) lattices for each of \( \beta = 5.7, \ 5.8 \). We also gauge-fixed 20 configurations on \( 16^4 \) lattices with \( \beta = 5.9, 6.0 \). A stopping condition similar to that used in \( \tilde{3} \).

The value of the projection functional \( P_\phi \), the \( U(1) \times U(1) \) 3-2 Creutz ratio, and the fractions of links with single color magnetic current are given in Tables 1 and 2. The Creutz ratios may be compared to the full \( SU(3) \) string tensions \( 0.168(1), 0.109(1), 0.073(1), 0.054(1) \) for \( \beta = 5.7, 5.8, 5.9, 6.0 \) respectively \( \tilde{3} \).

Our principle observation is that with two fixed Higgs fields, there are two broad classes of gauges that result. MAG-like gauges result when the Higgs fields are separated by an unbroken \( SU(2) \) symmetry, that is by \( \lambda_8 \)-like Higgs field. An example of an MAG-like gauge would be \( \chi_1 = 15^\circ, \ \chi_2 = 45^\circ \). MAG-like gauges are characterized by a high value for the functional \( P_\phi \) and roughly equal color distribution for magnetic currents. Higgs-like gauges, so named because they include the case of a single Higgs field, occur when the \( \chi \)'s are not so separated. An example of such a gauge would be \( \chi_1 = -15^\circ, \ \chi_2 = 15^\circ \). They are characterized by a range of lower \( P_\phi \) values and an asymmetric distribution of magnetic current. It should be noted that values for the string tension
Table 1
$12^4$ results for $\beta = 5.7$, $\beta = 5.8$.

| $\chi_1$ | $\chi_2$ | $P_\phi$ | $C(3,2)$ | $f_1$ | $f_2$ | $f_3$ |
|----------|----------|----------|----------|-------|-------|-------|
| $0^\circ$ | $90^\circ$ | 0.885 | 0.169(13) | 0.040 | 0.040 | 0.040 |
| $0^\circ$ | $60^\circ$ | 0.886 | 0.173(13) | 0.038 | 0.043 | 0.043 |
| $15^\circ$ | $45^\circ$ | 0.886 | 0.179(15) | 0.040 | 0.047 | 0.047 |
| $-15^\circ$ | $15^\circ$ | 0.883 | 0.189(17) | 0.047 | 0.047 | 0.081 |
| $15^\circ$ | $15^\circ$ | 0.872 | 0.181(15) | 0.043 | 0.101 | 0.134 |
| $0^\circ$ | $0^\circ$ | 0.881 | 0.203(13) | 0.055 | 0.055 | 0.099 |

| $\chi_1$ | $\chi_2$ | $P_\phi$ | $C(3,2)$ | $f_1$ | $f_2$ | $f_3$ |
|----------|----------|----------|----------|-------|-------|-------|
| $0^\circ$ | $90^\circ$ | 0.892 | 0.118(11) | 0.023 | 0.023 | 0.023 |
| $0^\circ$ | $60^\circ$ | 0.892 | 0.120(12) | 0.022 | 0.025 | 0.025 |
| $15^\circ$ | $45^\circ$ | 0.893 | 0.123(13) | 0.023 | 0.027 | 0.027 |
| $-15^\circ$ | $15^\circ$ | 0.890 | 0.136(11) | 0.029 | 0.030 | 0.053 |
| $15^\circ$ | $15^\circ$ | 0.879 | 0.133(12) | 0.026 | 0.080 | 0.102 |
| $0^\circ$ | $0^\circ$ | 0.888 | 0.147(13) | 0.036 | 0.036 | 0.067 |

Table 2
$16^4$ results for $\beta = 5.9$, $\beta = 6.0$.

| $\chi_1$ | $\chi_2$ | $P_\phi$ | $C(3,2)$ | $f_1$ | $f_2$ | $f_3$ |
|----------|----------|----------|----------|-------|-------|-------|
| $0^\circ$ | $90^\circ$ | 0.898 | 0.084(07) | 0.012 | 0.012 | 0.012 |
| $0^\circ$ | $60^\circ$ | 0.898 | 0.087(07) | 0.012 | 0.014 | 0.015 |
| $15^\circ$ | $45^\circ$ | 0.898 | 0.087(07) | 0.013 | 0.016 | 0.015 |
| $-15^\circ$ | $15^\circ$ | 0.896 | 0.101(10) | 0.018 | 0.018 | 0.033 |
| $15^\circ$ | $15^\circ$ | 0.885 | 0.105(08) | 0.016 | 0.064 | 0.079 |
| $0^\circ$ | $0^\circ$ | 0.894 | 0.109(10) | 0.023 | 0.023 | 0.045 |

| $\chi_1$ | $\chi_2$ | $P_\phi$ | $C(3,2)$ | $f_1$ | $f_2$ | $f_3$ |
|----------|----------|----------|----------|-------|-------|-------|
| $0^\circ$ | $90^\circ$ | 0.903 | 0.066(06) | 0.008 | 0.007 | 0.007 |
| $0^\circ$ | $60^\circ$ | 0.903 | 0.067(06) | 0.007 | 0.008 | 0.008 |
| $15^\circ$ | $45^\circ$ | 0.903 | 0.069(06) | 0.008 | 0.009 | 0.009 |
| $-15^\circ$ | $15^\circ$ | 0.901 | 0.079(08) | 0.011 | 0.011 | 0.021 |
| $15^\circ$ | $15^\circ$ | 0.890 | 0.088(08) | 0.010 | 0.052 | 0.062 |
| $0^\circ$ | $0^\circ$ | 0.899 | 0.086(10) | 0.015 | 0.016 | 0.030 |

as determined by Creutz ratios from those Wilson Loops were also unequal.

5. Conclusion

The MAG-type gauges proved to be better than the Higgs-type gauges, at least in terms of how much information was maintained in the projection. All the MAG-type gauges similar Creutz ratios. One note is that there were MAG-type gauges that had slightly higher $P_\phi$ than the original MAG. These gauges also had slightly higher Creutz ratios.

We expect to gather better statistics on all of these cases. There is also a lot to be learned about the variable Higgs case.

REFERENCES

[1] J. D. Stack, W. W. Tucker, and R. J. Wensley, Nucl. Phys. B639 (2002) 203, hep-lat/0205006.
[2] J. Mandula and M. Ogilvie, Phys. Lett. B248 (1990) 156.
[3] J. D. Stack, W. W. Tucker, and R. J. Wensley, hep-lat/0205006.
[4] W. W. Tucker and J. D. Stack, Nucl. Phys. B (Proc. Suppl.) 106, 643, hep-lat/0110165.
[5] J. D. Stack and R. J. Wensley, Nucl. Phys. B371 (1992) 597.
[6] J. D. Stack, S. D. Neiman, and R. J. Wensley, Phys. Rev. D50 (1994) 3399.
[7] K. D. Born et al, Nucl. Phys. B (Proc. Suppl.) 20 (1991), 394.