Exotic smoothness and quantum gravity

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Abstract
Since the first work on exotic smoothness in physics, it was folklore to assume
a direct influence of exotic smoothness to quantum gravity. Thus, the negative
result of Duston (2009 arXiv:0911.4068) was a surprise. A closer look into
the semi-classical approach uncovered the implicit assumption of a close
connection between geometry and smoothness structure. But both structures,
geometry and smoothness, are independent of each other. In this paper we
calculate the ‘smoothness structure’ part of the path integral in quantum gravity
assuming that the ‘sum over geometries’ is already given. For that purpose
we use the knot surgery of Fintushel and Stern applied to the class $E(n)$ of
elliptic surfaces. We mainly focus our attention to the K3 surfaces $E(2)$. Then
we assume that every exotic smoothness structure of the K3 surface can be
generated by knot or link surgery in the manner of Fintushel and Stern. The
results are applied to the calculation of expectation values. Here we discuss
the two observables, volume and Wilson loop, for the construction of an exotic
4-manifold using the knot 52 and the Whitehead link $Wh$. By using Mostow
rigidity, we obtain a topological contribution to the expectation value of the
volume. Furthermore, we obtain a justification of area quantization.

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1. Introduction

One of the outstanding problems in physics is the unification of the quantum and general
relativity theory. In the case of the quantization of gravity one works on the space of
(pseudo-)Riemannian metrics or better on the space of connections (Levi-Civita or more
general). The diffeomorphism group of the corresponding manifold is the gauge group of this
theory. Knowing more about the structure of this group means knowing more about quantum
gravity. An explicit expression for this correspondence is given by the general relativity theory
(GRT). In this theory we start with a manifold, choose a smooth structure and write down
the field equation. The beginning and the end of this procedure can be motivated by physics but the choice of the smoothness structure is not obvious. The reason for this ambiguity is given by the existence of distinct, i.e. non-diffeomorphic, smoothness structures (exotic smoothness) in dimension 4 (see [2] for an overview). The first examples of exotic 7-spheres were discovered by Milnor in 1956 [3]. Then it was shown that there are only a finite number of smoothness structures in dimensions greater than 4 [4, 5]. In contrast there are countable infinite many (distinct) exotic smoothness structures for some compact 4-manifolds (see the knot surgery below) and uncountable infinite many for some non-compact manifolds (see [6] in the case of $M \times \mathbb{R}$ with $M$ a compact, close 3-manifold, or the overview in [7]) including the exotic $\mathbb{R}^4$ [8–10]. Usually such a variety of structures has a meaning in physics. The first relation between quantum field theory and exotic smoothness was found by Witten [11]. He constructed a topological field theory having the Donaldson polynomials as the ground state. Sładkowski [12–14] discussed the influence of differential structures on the algebra $C(M)$ of functions over the manifold $M$ with methods known as non-commutative geometry. Especially in [13, 14] he stated a remarkable connection between the spectra of differential operators and differential structures. For example, the eta-invariant $\eta(\mathbb{R}P^4, g, \phi)$ (measuring the asymmetry of the spectrum) of a twisted Dirac-operator is different from the eta-invariant of the exotic $\mathbb{R}P^4$ for all metrics $g$ and all spin structures $\phi$ [15]. Thus the most interesting dimension for physics has the richest structure. Now the questions are: What is the physical relevance of the differential structure? How does the observable or its expectation value depend on the smoothness structure? In the case of the exotic $\mathbb{R}^4$ the first question was discussed by Brans and Randall [16] and later by Brans [17, 18] alone to guess that exotic smoothness can be a source of non-standard solutions of Einsteins equation. The author published an article [19] to show the influence of the differential structure to GRT for compact manifolds of simple type. Finally Sładkowski showed in [20] that the exotic $\mathbb{R}^4$ can act as the source of the gravitational field. As shown in [21, 22] the existence of exotic smoothness has a tremendous impact on cosmology, further explored in [23]. Beginning with the work of Krol [24–29] on the relation between categorical constructions and exotic smoothness, the focus in the topic is shifted to show a direct relation between exotic smoothness and quantum gravity [30–32].

In the first section of chapter 10 in [2], the meaning of exotic smoothness for general relativity was discussed on general grounds. Since the first papers about exotic smoothness it was folklore to state an influence of exotic smoothness to the state sum (or path integral) for quantum gravity. Then the negative result of the first paper written by Duston [1] was a surprise. He was able to calculate semi-classical results using exotic smoothness. The reason for the negative result is the loose relation between differential topology and differential geometry for 4-manifolds (in contrast to the strong relation for 3-manifolds known as the geometrization conjecture of Thurston [33]). But the smoothness structure is rather independent of the geometry whereas Duston assumed a coupling between geometry and topology because of its semi-classical approach. Of course, there are some restrictions induced by the smoothness structure like the existence of a Ricci-flat metric but nothing more.

In this paper we use a different approach to overcome this problem by dividing the measure in the path integral into two parts: the geometrical part (sum over geometries) and the differential topological part (sum over exotic smoothness). Both parts are more or less independent of each other. If one assumes the existence of the path integral for the geometrical part (state sum $Z_0$ see below) then the state sum is changed by exotic smoothness according to (13). The paper is organized as follows. First we will give a physical motivation for exotic smoothness structures and present the model assumptions. Then we study the construction of the special class of elliptic surfaces and the exotic smoothness structures using the knot surgery. In section 4 we discuss the splitting of the action functional (using the diffeomorphism
invariance of the Einstein–Hilbert action) according to the knot surgery to get relation (5). Then in section 6 we will discuss the calculation of the functional integral to get (13). Then we discuss the expectation value of two observables: the volume and the Wilson loop. The calculation of the examples of the knot \(5_2\) and the Whitehead link \(Wh\) together with some concluding remarks to end the paper.

2. Physical motivation and model assumptions

Einstein’s insight that gravity is the manifestation of geometry leads to a new view on the structure of spacetime. From the mathematical point of view, spacetime is a smooth 4-manifold endowed with a (smooth) metric as a basic variable for general relativity. Later on, the existence question for Lorentz structure and causality problems (see Hawking and Ellis [34]) gave further restrictions on the 4-manifold: causality implies noncompactness, and Lorentz structure needs a codimension-1 foliation. Usually, one starts with a globally foliated, noncompact 4-manifold \(\Sigma \times \mathbb{R}\) fulfilling all restrictions where \(\Sigma\) is a smooth 3-manifold representing the spatial part. But other noncompact 4-manifolds are also possible, i.e. it is enough to assume a noncompact, smooth 4-manifold endowed with a codimension-1 foliation.

All these restrictions on the representation of spacetime by the manifold concept are clearly motivated by physical questions. Among the properties there is one distinguished element: the smoothness. Usually one assumes a smooth, unique atlas of charts covering the manifold where the smoothness is induced by the unique smooth structure on \(\mathbb{R}\). But as discussed in the introduction, that is not the full story. Even in dimension 4, there are an infinity of possible other smoothness structures (i.e. a smooth atlas) non-diffeomorphic to each other.

If two manifolds are homeomorphic but non-diffeomorphic, they are exotic to each other. The smoothness structure is called an exotic smoothness structure. The implications for physics are obvious because we rely on the smooth calculus to formulate field theories. Thus different smoothness structures have to represent different physical situations leading to different measurable results. But it should be stressed that exotic smoothness is not exotic physics! Exotic smoothness is a mathematical possibility which should be further explored to understand its physical relevance.

In this paper we use a special class of the 4-manifold. Currently, there are (uncountable) many examples of exotic, noncompact 4-manifolds which are difficult to describe. Thus we will restrict ourselves to the class of compact 4-manifolds where we have powerful invariants like Seiberg–Witten invariants [35, 36] or Donaldson polynomials [37, 38] to distinguish between different smoothness structures. Secondly, we will not discuss the definition of the path integral and all problems connected with renormalization, definiteness, etc. Thirdly, we will not discuss the Euclidean and/or Lorentzian signature of the metric. Clearly the compactness of the 4-manifold implies the Euclidean metric on the 4-manifold but smoothness question is independent of the metric. Thus, without loss of generality one can remove one point from the compact 4-manifold to get a noncompact 4-manifold with Lorentz signature. Fourth, we use the knot surgery of Fintushel and Stern [39] to construct the exotic smoothness structure. This approach assumes a special class of 4-manifolds (‘complicated-enough’) which contains the class of elliptic surfaces among them the important K3 surface. Thus we summarize the assumptions.

(i) The 4-manifold is the compact and simple-connected elliptic surface \(E(n)\) where \(E(2)\) is the K3 surface.
(ii) All problems with the definition and calculation of the path integral are ignored.
(iii) The signature of the metric is not discussed, i.e. the calculations are the same for Euclidean or Lorentzian metric.
(iv) The exotic smoothness structures of the 4-manifold $E(n)$ are constructed by the Fintushel–Stern knot surgery.

3. Elliptic surfaces and exotic smoothness

In this section we will give some information about elliptic surfaces and its construction. First we will give a short overview of the construction of elliptic surfaces and a special class of elliptic surfaces denoted by $E(n)$ in the literature. Then we present the construction of exotic $E(n)$ by using the knot surgery of Fintushel and Stern.

3.1. Elliptic surfaces

A complex surface $S$ is a two-dimensional complex manifold which is compact and connected. A special complex surface is the elliptic surface, i.e. a complex surfaces $S$ together with a map $\pi : S \rightarrow C$ (C complex curve, i.e. Riemannian surface), so that for nearly every point $p \in S$ the reversed map $F = \pi^{-1}(p)$ is an elliptic curve, i.e. a torus. Now we will construct the special class of elliptic surfaces $E(n)$.

The first step is the construction of $E(1)$ by the unfolding of singularities for two cubic polynomials intersecting each other. The resulting manifold $E(1)$ is the manifold $CP^2 \# 9 CP^2$ but equipped with an elliptic fibration. Then we use the method of fiber sum to produce the surfaces $E(n)$ for every number $n \in \mathbb{N}$. For that purpose we cut out a neighborhood $N(F)$ of one fiber $\pi^{-1}(p)$ of $E(1)$. Now we sew together two copies of $E(1) \backslash N(F)$ along the boundary of $N(F)$ to get $E(2)$, i.e we define the fiber sum $E(2) = E(1) \# E(1)$. Especially we note that $E(2)$ is also known as K3-surface widely used in physics. Thus we get the recursive definition $E(n) = E(n-1) \# E(1)$. The details of the construction can be found in [40].

3.2. Knot surgery and exotic elliptic surface

The main technique to construct an exotic elliptic surface was introduced by Fintushel and Stern [39], called the knot surgery. In short, given a simple-connected, compact 4-manifold $M$ with an embedded torus $T^2$ (having special properties, see below), cut out $M \backslash N(T^2)$ a neighborhood $N(T^2) = D^2 \times T^2$ of the torus and glue in $S^1 \times S^3 \backslash N(K)$, with the knot complement $S^3 \backslash N(K)$ (see appendix A). Thus the construction depends on a knot, i.e. an embedding of the circle $S^1$ into $\mathbb{R}^3$ or $S^3$. Then we obtain the new 4-manifold $M_K = (M \backslash N(T)) \cup_{T^3} (S^1 \times (S^3 \backslash N(K)))$ from a given 4-manifold $M$ by gluing $M \backslash N(T^2)$ and $S^1 \times S^3 \backslash N(K)$ along the common boundary, the 3-torus $T^3$. It is the remarkable result of Fintushel and Stern [39] for a non-trivial knot $K$ that $M_K$ is non-diffeomorphic to $M$. We remark that the construction can be easily generalized to links, i.e. the embedding of the disjoint union of circles $S^1 \cup \cdots \cup S^1$ into $\mathbb{R}^3$ or $S^3$. Readers not interested in the details of the construction can now jump to the next section.

The precise definition can be given in the following way for an elliptic surface. Let $\pi : S \rightarrow C$ be an elliptic surface and $\pi^{-1}(t) = F$ a smooth fiber ($t \in C$). As usual, $N(F)$ denotes a neighborhood of the regular fiber $F$ in $S$ (which is diffeomorphic to

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1 We denote this map $\pi$ as elliptic fibration. Thus every complex surface which is equipped with an elliptic fibration is an elliptic surface.
to $D^2 \times T^2$). Delete $N(F)$ from $S$ to get a manifold $S \setminus N(F)$ with the boundary $\partial(S \setminus N(F)) = \partial(D^2 \times F) = S^1 \times F = T^3$, the 3-torus. Then we take the 4-manifold $S^1 \times (S^3 \setminus N(K))$, $K$ being a knot, with the boundary $\partial(S^1 \times (S^3 \setminus N(K))) = T^3$ and reglue it along the common boundary $T^3$. The resulting 4-manifold

$$S_K = (S \setminus N(F)) \cup_{T^3} (S^1 \times (S^3 \setminus N(K)))$$

is obtained from $S$ via knot surgery using the knot $K$. The regular fiber $F$ in the elliptic surface $S$ has two properties which are essential for the whole construction.

(i) In a larger neighborhood $N_c(F)$ of the regular fiber $F$ there is a cups fiber $c$, i.e. an embedded 2-sphere of self-intersection 0 with a single nonlocally flat point whose neighborhood is the cone over the right-hand trefoil knot.

(ii) The complement $S \setminus F$ of the regular fiber is simple-connected $\pi_1(S \setminus F) = 1$.

Then as shown in [39], $S_K$ is not diffeomorphic to $S$. The whole procedure can be generalized to any 4-manifold with an embedded torus of self-intersection 0 in a neighborhood of a cusp.

Before we proceed with the physical interpretation, we will discuss the question when two exotic $S_K$ and $S_K'$ for two knots $K$, $K'$ are diffeomorphic to each other. Currently there are two invariants to distinguish non-diffeomorphic smoothness structures: Donaldson polynomials and Seiberg–Witten invariants. Fintushel and Stern [39] calculated the invariants for $S_K$ and $S_K'$ to show that $S_K$ differs from $S_K'$ if the Alexander polynomials of the two knots differ. Unfortunately the invariants are not complete. Thus we cannot say anything about $S_K$ and $S_K'$ for two knots with the same Alexander polynomial. But Fintushel and Stern [41, 42] constructed counterexamples of two knots $K$, $K'$ with the same Alexander polynomial but with different $S_K$ and $S_K'$. Furthermore Akbulut [43] showed that the knot $K$ and its mirror $\bar{K}$ induce diffeomorphic 4-manifold $S_K = S_{\bar{K}}$.

4. The action functional

In this section we will discuss the Einstein–Hilbert action functional of the exotic 4-manifold. The main part of our argumentation is the additional contribution to the action functional coming from exotic smoothness. Here it is enough to consider the exotic smoothness generated by the knot surgery.

We start with an elliptic surface, the K3 surface $M = E(2)$. In this paper we consider the Einstein–Hilbert action

$$S_{EH}(g) = \int_M R \sqrt{g} \, d^4 x$$

and fix the Ricci-flat metric $g$ as the solution of the vacuum field equations. Now we study the effect to vary the differential structure by a knot surgery. This procedure affected only a submanifold $N(T^2) \subset M$ and we consider a decomposition of the 4-manifold

$$M = (M \setminus N(T^2)) \cup_{T^3} N(T^2)$$

with $N(T^2) = D^2 \times T^2$ leading to a sum in the action

$$S_{EH}(M) = \int_{M \setminus N(T^2)} R \sqrt{g} \, d^4 x + \int_{N(T^2)} R \sqrt{g} \, d^4 x.$$ 

Because of diffeomorphism invariance of the Einstein–Hilbert action, this decomposition do not depend on the concrete realization with respect to any coordinate system. Now we construct a new smoothness structure $M_K$

$$M_K = (M \setminus N(T^2)) \cup_{T^3} (S^1 \times (S^3 \setminus N(K)))$$
by using a knot $K$, i.e. an embedding $K : S^1 \to S^3$. Then the two 4-manifolds $M \setminus N(T^2)$ and $M_K \setminus N(T^2)$ with a boundary of 3-torus $T^3$ are diffeomorphic to each other. Thus we can fix it and its action

$$S_{EH}(M \setminus N(T^2)) = \int_{M \setminus N(T^2)} R \sqrt{g} d^4 x$$

by using a fixed metric $g$ in the interior $\text{int}(M \setminus N(T^2))$. Furthermore we can ignore a possible boundary term because the 3-torus $T^3 = \partial (M \setminus N(T^2))$ is a flat, compact 3-manifold. Thus we obtain

$$S_{EH}(M_K) = S_{EH}(M \setminus N(T^2)) + \int_{S^1 \times (S^3 \setminus N(K))} R_K \sqrt{g_K} d^4 x$$

(2)

with a metric $g_K$ and scalar curvature $R_K$ for the 4-manifold $S^1 \times (S^3 \setminus N(K))$. Now we consider the integral

$$S_{EH}(N(T^2)) = \int_{N(T^2)} R \sqrt{g} d^4 x$$

over $N(T^2) = D^2 \times T^2$ w.r.t. a suitable product metric. The torus $T^2$ is a flat manifold and the disk $D^2$ can be chosen to embed flat in $N(T^2)$. Thus, this integral vanishes $S_{EH}(N(T^2)) = 0$ and

$$S_{EH}(M \setminus N(T^2)) = S_{EH}(M).$$

Using this relation and (2), we obtain the relation

$$S_{EH}(M_K) = S_{EH}(M) + \int_{S^1 \times (S^3 \setminus N(K))} R_K \sqrt{g_K} d^4 x$$

(3)

between the Einstein–Hilbert action on $M$ with and without the knot surgery. Next we will evaluate the integral

$$\int_{S^1 \times (S^3 \setminus N(K))} R_K \sqrt{g_K} d^4 x$$

by using a product metric $g_K$

$$d x^2 = d \theta^2 + h_{ik} d x^i d x^k$$

with the periodic coordinate $\theta$ on $S^1$ and metric $h_{ik}$ on the knot complement $S^3 \setminus N(K)$. We are using the ADM formalism with the lapse $N$ and shift function $N'$ to get a relation between the four-dimensional $R$ and the three-dimensional scalar curvature $R(3)$ (see [44] (21.86) p 520)

$$\sqrt{g_K} R d^4 x = N \sqrt{h} (R(3) + \|n\|^2 ((\text{tr} K)^2 - \text{tr} K^2)) d \theta d^3 x$$

with the normal vector $n$ and the extrinsic curvature $K$. Without loss of generality, using the product metric in $S^1 \times (S^3 \setminus N(K))$ we can embed $S^3 \setminus N(K) \hookrightarrow S^1 \times (S^3 \setminus N(K))$ in such a manner that the extrinsic curvature has a fixed value or vanishes $K = 0$ (parallel transport of the normal vector). Then one obtains

$$\int_{S^1 \times (S^3 \setminus N(K))} R_K \sqrt{g_K} d^4 x = L_{S^1} \cdot \int_{(S^3 \setminus N(K))} R(3) \sqrt{h} N d^3 x$$

with the length $L_{S^1} = \int_{S^1} d \theta$ of the circle $S^1$. Thus the integral on the right side is the three-dimensional Einstein–Hilbert action. As shown by Witten [45–47], this action

$$\int_{(S^3 \setminus N(K))} R(3) \sqrt{h} N d^3 x = L \cdot CS(S^3 \setminus N(K), \Gamma)$$

where $CS(S^3 \setminus N(K), \Gamma)$ is the integral of the Chern–Simons form over the knot complement $S^3 \setminus N(K)$. The three-dimensional action is an integral of the Chern–Simons form over the knot complement $S^3 \setminus N(K)$.
is related to the Chern–Simons action $CS(S^3 \setminus N(K), \Gamma)$ (defined in appendix B)
\[
\int_{S^1 \times (S^3 \setminus N(K))} R_K \sqrt{|g_K|} d^4x = L_{S^1} \cdot L \cdot CS(S^3 \setminus N(K), \Gamma)
\]
with respect to the (Levi-Civita) connection $\Gamma$ and a second length $L$. At least for the class of prime knots, the knot complements $S^3 \setminus N(K)$ have constant curvature and the length has the order of the volume, i.e. $L = \sqrt{\text{Vol}(S^3 \setminus N(K))}$. Finally we have the relation (2)
\[
S_{EH}(M_K) = S_{EH}(M) + L_{S^1} \cdot \sqrt{\text{Vol}(S^3 \setminus N(K))} \cdot CS(S^3 \setminus N(K), \Gamma)
\]
(4) as the correction to the action $S_{EH}(M)$ after the knot surgery. Finally we will write this relation in the usual units
\[
\frac{1}{\hbar} S_{EH}(M_K) = \frac{1}{\hbar} S_{EH}(M) + \frac{L_{S^1} \cdot \sqrt{\text{Vol}(S^3 \setminus N(K))}}{L_p^2} \cdot \pi \cdot CS(S^3 \setminus N(K), \Gamma).
\]
(5)

5. The functional integral

Now we will discuss the (formal) path integral
\[
Z = \int Dg \exp \left( \frac{i}{\hbar} S[g] \right)
\]
(6)
and its conjectured dependence on the choice of the smoothness structure. In the following we will use frames $e$ instead of the metric $g$. Furthermore we will ignore all problems (ill-definiteness, singularities, etc.) of the path integral approach. Then instead of (6) we have
\[
Z = \int De \exp \left( \frac{i}{\hbar} S_{EH}[e, M] \right)
\]
with the action
\[
S_{EH}[e, M] = \int_M \text{tr}(e \wedge e \wedge R),
\]
where $e$ is a 1-form (coframe), $R$ is the curvature 2-form $R$ and $M$ is the 4-manifold. Next we have to discuss the measure $De$ of the path integral. Currently there is no rigorous definition of this measure and as usual we assume a product measure. Then we have two possible parts which are more or less independent from each other:

1. integration $D_{e_G}$ over geometries
2. integration $D_{e_K}$ over different differential structures parametrized by knots $K$.

We assume that the first integration can be done to obtain formally
\[
Z_0(M) = \int_{\text{Geometries}} D_{e_G} \exp \left( \frac{i}{\hbar} S_{EH}[e, M] \right)
\]
and we are left with the second integration
\[
\int_{\text{Knots}} D_{e_K} \exp \left( \frac{i}{\hbar} S_{EH}[e, M_K] \right)
\]
by varying the differential structure using the knot surgery. But the set of different differential structures on a compact 4-manifold has countable infinite cardinality. Then the integral changes to a sum
\[
\int_{\text{Knots}} D_{e_K} \exp \left( \frac{i}{\hbar} S_{EH}[e, M_K] \right) = \sum_{K} \exp \left( \frac{i}{\hbar} S_{EH}[e, M_K] \right).
\]
Now we made the following main conjecture.
Conjecture 5.1. All distinct exotic smoothness structures of the compact 4-manifold $M = E(2)$ can be generated by knots and links.

As mentioned above, Akbulut [43] showed that the pair of a knot and its mirror knot (same for links) induces diffeomorphic smoothness structures. Thus assuming the conjecture 5.1 and relation (5) for the knot surgery, we obtain for the path integral

$$Z = Z_0 \cdot \sum_K \exp \left( i \frac{L_S^2 \cdot \sqrt{\text{Vol}(S^3 \setminus N(K))}}{L_p^2} \cdot CS(S^3 \setminus N(K), \Gamma) \right),$$

where the connection $\Gamma$ is chosen to be the Levi-Civita connection. Finally exotic smoothness contributes to the state sum of quantum gravity.

6. Observables

Any consideration of quantum gravity is incomplete without considering observables and its expectation values. Here we consider two kind observables:

(i) volume
(ii) holonomy along open and closed path (Wilson loop).

The expectation value for the volume can be calculated via the decomposition of the 4-manifold $M_K$, i.e.

$$M_K = (M \setminus N(T^2)) \cup_T (S^1 \times (S^3 \setminus N(K)))$$

leading to

$$\text{Vol}(M_K) = \text{Vol}(M) + (\text{Vol}(S^1 \times S^3 \setminus N(K)) - \text{Vol}(N(T^2)))$$

Let

$$\langle \text{Vol}(M) \rangle_0 = \frac{\int D e_G \text{Vol}(M, e_G) \exp \left( \frac{i}{\hbar} s_{EH}[e, M] \right)}{\int D e_G \exp \left( \frac{i}{\hbar} s_{EH}[e, M] \right)}$$

be the expectation value of the volume w.r.t. the geometry. Using the linearity of the expectation value, we obtain

$$\langle \text{Vol}(M_K) \rangle_0 = \langle \text{Vol}(M) \rangle_0 + \langle \text{Vol}(S^1 \times S^3 \setminus N(K)) \rangle_0 - \langle \text{Vol}(N(T^2)) \rangle_0$$

and by choosing a unit length scale for the circle in $S^1 \times (S^3 \setminus N(K))$ as well for the torus $T^2$ in $N(T^2)$ we have

$$\langle \text{Vol}(M_K) \rangle_0 = \langle \text{Vol}(M) \rangle_0 + \langle \text{Vol}(S^3 \setminus N(K)) \rangle - 1.$$  (8)

Thus the volume depends on the volume of the knot complement $S^3 \setminus N(K)$ only. This result seems not satisfactory because one may choose the scale of the volume $\text{Vol}(S^3 \setminus N(K))$. Surprisingly, that is not true! As Thurston [48] showed there are two classes of knots: hyperbolic knots and non-hyperbolic knots. The (interior of the) knot complement of a hyperbolic knot admits a homogeneous metric of constant negative curvature (normalized to $-1$). In contrast, the knot complement of non-hyperbolic knots admits no such metric. As Mostow [49] showed, every hyperbolic 3-manifold is rigid, i.e. the volume is a topological invariant. Thus we can scale the knot complement for non-hyperbolic knots in such a manner that

$$\text{Vol}(M_K) \approx \text{Vol}(M) \quad \text{non-hyperbolic knot } K,$$
whereas
\[ \text{Vol}(M_K) = \text{Vol}(M) + \text{Vol}(S^3 \setminus N(K)) - 1 \] for a hyperbolic knot \( K \).

Finally we can claim

**Proposition 6.1.** The expectation value of the volume for the 4-manifold \( M \) depends on the choice of the smoothness structure if this structure is generated by the knot surgery along a hyperbolic knot or link.

It is known that most knots or links are hyperbolic knots or links among them the figure-8 knot, the Whitehead link and the Borromean rings. The class of knots admitting no hyperbolic structure contains the torus knots and among them the trefoil knot.

Now we discuss the holonomy
\[ \text{hol}(\gamma, \Gamma) = \text{Tr} \left( \exp \left( i \int_{\gamma} \Gamma \right) \right) \]
along a path \( \gamma \) w.r.t. the connection \( \Gamma \), as another possible observable. Then there are two cases:

(i) the path \( \gamma \) lies in \( M \setminus N(T^2) \), i.e. it is not affected by the knot surgery, or
(ii) the path \( \gamma \) lies in \( N(T^2) \) as well as in \( S^1 \times (S^3 \setminus N(K)) \).

Obviously, the expectation value of the holonomy for the first case does not depend on the choice of the differential structure and we will ignore it. In the second case we choose a path in the knot complement \( S^3 \setminus N(K) \). As stated above, the action is identical to the Chern–Simons action. Furthermore it is known that there is an important subclass among all paths, the closed paths. Thus we will concentrate on the closed paths, i.e. we consider the Wilson observable
\[ W(\gamma) = \text{Tr} \left( \exp \left( i \int_{\gamma} \Gamma \right) \right) \]
with the expectation value
\[ \langle W(\gamma) \rangle = \frac{\int D\Gamma \ W(\gamma) \exp \left( i \ell \cdot CS(S^3 \setminus N(K), \Gamma) \right)}{\int D\Gamma \ \exp \left( i \ell \cdot CS(S^3 \setminus N(K), \Gamma) \right)} \] (9)
w.r.t. the setting
\[ L_{S^1} \cdot \sqrt{\text{Vol}(S^3 \setminus N(K))} = L_P^2 \cdot \ell. \]
The Wilson loop depends on the connection only. Thus we switch to the path integral over the connections instead using the frames. As discussed by Witten in the landmark paper [50], the expectation value \( \langle W(\gamma) \rangle \) for a knot \( \gamma \) is a knot invariant depending on \( \ell \), now called the colored Jones polynomial. If \( \gamma \) is the unknot then \( \langle W(\gamma) \rangle \) is an invariant of the 3-manifold in which the unknot embeds. Reshetikhin and Turaev [51] constructed this invariant (the RT-invariant) using quantum groups. Thus,

**Proposition 6.2.** The expectation value for the Wilson loop \( \langle W(\gamma) \rangle \) is a knot invariant of the knot \( \gamma \) in \( S^3 \setminus N(K) \). The knot \( \gamma \) represents a non-contractable loop, i.e. an element in \( \pi_1(S^3 \setminus N(K)) \). If \( \gamma \) is the unknot then \( \langle W(\gamma) \rangle \) is the RT invariant.

But that is not the full story. First consider a closed, contractable curve \( \gamma \) in \( N(T^2) = D^2 \times T^2 = S^1 \times (D^2 \times S^1) \) representing a knot. The knot surgery changed \( N(T^2) \) to \( S^1 \times S^3 \setminus N(K) \) but the knot \( \gamma \) is untouched. Thus, the knot \( \gamma \) in \( S^1 \times S^3 \setminus N(K) \)

\[ 2 \text{ It is known that the path space is fibered over the closed paths by constant paths.} \]
can be approximated by a knot in $S^3$. Then we have the Wilson loop $W_0(\gamma)$ for that special case and obtain [50]
\begin{equation}
\langle W_0(\gamma) \rangle = \text{Jones polynomial } J_\nu(q),
\end{equation}
where we have
\[ q = \exp \left( \frac{2\pi i}{2 + \ell} \right) \]
with the variable $\ell$ defined in (9). For this result we assumed the holonomy group $SU(2)$ or $SL(2, \mathbb{C})$ for the connection, which will be discussed below.

Secondly consider a closed curve, the unknot, in $N(T^2) = D^2 \times T^2 = S^1 \times (D^2 \times S^1)$ going around the non-trivial loop in the solid torus $D^2 \times S^1$. Again, the knot surgery changed $N(T^2)$ to $S^1 \times S^3 \setminus N(K)$. Then the unknot in $N(T^2)$ is the generator of $\pi_1(D^2 \times S^1) = \mathbb{Z}$ which can be mapped to the group $\pi_1(S^3 \setminus N(K))$, the so-called knot group. Now we will concentrate on the semi-classical approach around the classical solution, i.e. the extrema of the Chern–Simons functional $CS(S^3 \setminus N(K), \Gamma)$ in (7). The extrema of the Chern–Simons action are the flat connections (see appendix B). Then the Wilson loop $W(\gamma)$ is a homomorphism
\[ W(\gamma) : \pi_1(S^3 \setminus N(K)) \to G \]
from the fundamental group $\pi_1(S^3 \setminus N(K))$ into the structure group $G$ of the connection (usually called the gauge group). From the physical point of view one can argue that the connection $\Gamma$ is induced by a four-dimensional connection in $S^1 \times S^3 \setminus N(K)$. Thus one has $SO(3, 1)$ or $SO(4)$ (remember we do not discuss the signature). Interestingly this setting is supported by mathematics too. Above we divide the knots $K$ into two classes: hyperbolic knots (=knot complement $S^3 \setminus N(K)$ with hyperbolic geometry) and non-hyperbolic knots. According to Thurston [33] (see [52] for an overview) a hyperbolic geometry is given by the homomorphism
\begin{equation}
\pi_1(S^3 \setminus N(K)) \to SO(3, 1) = PSL(2, \mathbb{C})
\end{equation}
into the Lorentz group isomorphic to the projective, linear group as the isometry group $Isom(\mathbb{H}^3)$ of the three-dimensional hyperbolic space $\mathbb{H}^3$. Thus a natural choice of the structure group for the connection in the knot complement of a hyperbolic knot is the Lorentz group!

Next we remark that the structure group gauges via conjugation the Wilson loop. Denote by $\text{Hom}(\pi_1(S^3 \setminus N(K)), SO(3, 1))$ the set of homomorphisms (11) and by
\[ \text{Rep}(S^3 \setminus N(K), SO(3, 1)) = \text{Hom}(\pi_1(S^3 \setminus N(K)), SO(3, 1))/SO(3, 1) \]
the representation variety of gauge invariant, flat connections. Usually this variety has singularities but as shown in [53] one can construct a substitute, the character variety, which has the structure of a manifold. This variety (as shown by Gukov [54]) represents the classical solutions. Turaev [55, 56] introduced a deformation quantization procedure to construct a quantum version of $\text{Rep}(S^3 \setminus N(K), SO(3, 1))$. It works in the neighborhood of boundary, i.e. in $\partial(S^3 \setminus N(K)) \times [0, 1]$. Then one arrives at the so-called skein space (see [32] for a discussion). Then the expectation value of Wilson loop observables are given by integrals over some skein space. We will come back to this point in a forthcoming paper.

7. Examples

In this section we will present some concrete calculations based on relation (7), i.e. we consider the relation
\[ \frac{Z(M_K)}{Z_0(M_K)} = \exp \left( \frac{iL_{S^1} \cdot \sqrt{\text{Vol}(S^3 \setminus N(K))}}{L^2} \cdot CS(S^3 \setminus N(K), \Gamma) \right) \]
for a specific exotic 4-manifold $M_K$ (knot surgery of $M$ by knot $K$) with the Levi-Civita connection $\Gamma$, representing the correction from the exotic smoothness on $M$. In the following we will introduce a scaling parameter

$$\frac{L_{S^3} \cdot \sqrt{\text{Vol}(S^3 \setminus N(K))}}{L_p^4} = \ell$$

and the usual units (5) to get

$$\frac{Z(M_K)}{Z_0(M_K)} = \exp \left( i\ell \cdot \pi \cdot CS(S^3 \setminus N(K), \Gamma) \right).$$

(13)

Usually the Chern–Simons invariant has many values depending on the complexity of the fundamental group $\pi_1(S^3 \setminus N(K))$. As argued in appendix B, one has to consider the minimum value of the Chern–Simons invariant to get the Chern–Simons invariant for the Levi-Civita connection.

Here we will calculate the observables for two examples where the value of the volume and of the Chern–Simons invariant is known. The first example is the hyperbolic knot $5_2$ (in Rolfsen notation [57]) whereas the second example is a two-component, hyperbolic link, the Whitehead link $Wh$ (see figure 1). Both complements describe a hyperbolic 3-manifold, i.e. a 3-manifold with a constant negative curvature. The complement $S^3 \setminus N(5_2)$ has the boundary $T^2$ and $S^3 \setminus N(Wh)$ has two boundary components $T^2 \sqcup T^2$. The values of the volume and the Chern–Simon invariants, calculated via the package SnapPea by J Weeks, can be seen in table 1. For the knot $5_2$ we obtain the corrections from (8) and (13) induced by the knot surgery along $5_2$, i.e. for the exotic 4-manifold $M_{5_2}$

$$\langle \text{vol}(M_{5_2}) \rangle = \langle \text{vol}(M) \rangle + 1.828 \cdot L_p^4 \cdot \ell^2$$

(14)

$$Z(M_{5_2}) = Z_0(M_{5_2}) \cdot \exp \left( -i\ell \cdot \frac{3.024 \cdot 128.37}{\pi} \right)$$

(15)

by assuming the length scale $L_p$ for the knot. It is easy to extend the knot surgery to every link. For the two-component Whitehead link $Wh$ we cut out two solid tori $N(T^2 \sqcup T^2)$ and
glue in \( S^1 \times S^3 \setminus N(W_h) \). Then we will produce the exotic 4-manifold \( M_{Wh} \) with

\[
\langle \text{vol}(M_{Wh}) \rangle = \langle \text{vol}(M) \rangle + 2.66386 \cdot L_p^4 \cdot \ell^2
\]

\[
Z(M_{Wh}) = Z_0(M_{Wh}) \cdot \exp \left( i \ell \cdot \frac{2.46742}{\pi} \right).
\]

Thus we obtain non-trivial corrections for the observables coming from exotic smoothness on the 4-manifold \( M \).

8. Conclusion

In this paper we discuss the influence of exotic smoothness on the observables, the volume and the Wilson loop, in a model for quantum gravity. As one would expect from physics, the path integral (or state sum) as well the expectation values of the observables depend on the choice of the exotic smoothness structure. For the construction of the exotic smoothness structure, we use the knot surgery of Fintushel and Stern [39] for a knot. This procedure can be simply described by substituting a solid (unknotted) torus \( D^2 \times S^1 \) (seen as the complement \( S^3 \setminus N(S^1) \) of the unknot) by a knot complement \( S^3 \setminus N(K) \). Then the result of every calculation depends strongly on this knot complement. Especially we found a strong dependence of the volume from the exotic smoothness for all hyperbolic knots, i.e. knots \( K \) where the (interior of the) complement has a metric of constant, negative curvature (=hyperbolic structure). In that case, Mostow rigidity [49] implies the topological invariance of the volume (see proposition 6.1)! Thus, there is no scaling of the volume so that the contribution to the expectation value of the volume (8) cannot be neglected (see examples (14), (16)). Furthermore, if we consider the scaling parameter \( \ell \) defined by (12) and argue via a consistent quantum field theory based on the Chern–Simons action then Witten [47] showed that this parameter must be quantized. Then we have shown that there is a quantization of the area according to

\[
\frac{L_{S^1} \cdot \sqrt[3]{\text{Vol}(S^3 \setminus N(K))}}{L_p^2} \in \mathbb{N}
\]

agreeing with results in loop quantum gravity [58].

Finally we can support the physically motivated conjecture that quantum gravity depends on exotic smoothness.

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Appendix A. Knot complement

Let \( K : S^1 \to S^3 \) be an embedding of the circle into the 3-sphere, i.e. a knot \( K \). We define by \( N(K) = D^2 \times K \) a thickened knot or a knotted solid torus. The knot complement \( S^3 \setminus N(K) \) results in cutting \( N(K) \) off from the 3-sphere \( S^3 \). Then one obtains a 3-manifold with boundary \( \partial(S^3 \setminus N(K)) = T^2 \). The properties of the knot complement depend strongly on the properties of the knot \( K \). So, the fundamental group \( \pi_1(S^3 \setminus N(K)) \) is also denoted as the knot group. In contrast, the homology group \( H_i(S^3 \setminus N(K)) = \mathbb{Z} \) do not depend on the knot.
Appendix B. Chern–Simons invariant

Let $P$ be a principal $G$ bundle over the 4-manifold $M$ with $\partial M \neq 0$. Furthermore, let $A$ be a connection in $P$ with the curvature

$$ F_A = dA + A \wedge A $$

and Chern class

$$ C_2 = \frac{1}{8\pi^2} \int_M \text{tr}(F_A \wedge F_A) $$

for the classification of the bundle $P$. By using the Stokes theorem, we obtain

$$ \int_M \text{tr}(F_A \wedge F_A) = \int_{\partial M} \text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) $$

with the Chern–Simons invariant

$$ CS(A, \partial M) = \frac{1}{8\pi^2} \int_{\partial M} \text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \tag{B.1} $$

Now we consider the gauge transformation $A \rightarrow g^{-1}Ag + g^{-1}dg$ and obtain

$$ CS(g^{-1}Ag + g^{-1}dg, \partial M) = CS(A, \partial M) + k $$

with the winding number

$$ k = \frac{1}{24\pi^2} \int_{\partial M} (g^{-1}dg)^3 \in \mathbb{Z} $$

of the map $g : M \rightarrow G$. Thus the expression

$$ CS(A, \partial M) \mod 1 $$

is an invariant, the Chern–Simons invariant. Now we will calculate this invariant. For that purpose we consider the functional $(B.1)$ and its first variation vanishes:

$$ \delta CS(A, \partial M) = 0 $$

because of the topological invariance. Then one obtains the equation

$$ dA + A \wedge A = 0, $$

i.e. the extrema of the functional are the connections of vanishing curvature. The set of these connections up to gauge transformations is equal to the set of homomorphisms $\pi_1(\partial M) \rightarrow SU(2)$ up to conjugation. Thus the calculation of the Chern–Simons invariant reduces to the representation theory of the fundamental group into $SU(2)$. In [59] the authors define a further invariant

$$ \tau(\Sigma) = \min \{ CS(\alpha) | \alpha : \pi_1(\Sigma) \rightarrow SU(2) \} $$

for the 3-manifold $\Sigma$. This invariants fulfills the relation

$$ \tau(\Sigma) = \frac{1}{8\pi^2} \int_{\Sigma \times \mathbb{R}} \text{tr}(F_A \wedge F_A) $$

which is the minimum of the Yang–Mills action

$$ \left| \frac{1}{8\pi^2} \int_{\Sigma \times \mathbb{R}} \text{tr}(F_A \wedge F_A) \right| \leq \frac{1}{8\pi^2} \int_{\Sigma \times \mathbb{R}} \text{tr}(F_A \wedge * F_A), $$

i.e. the solutions of the equation $F_A = \pm * F_A$. Thus the invariant $\tau(\Sigma)$ of $\Sigma$ corresponds to the self-dual and anti-self-dual solutions on $\Sigma \times \mathbb{R}$, respectively, or the invariant $\tau(\Sigma)$ is the Chern–Simons invariant for the Levi-Civita connection.
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