Supersymmetric matrix models and branched polymers

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Abstract

We solve a supersymmetric matrix model with a general potential. While matrix models usually describe surfaces, supersymmetry enforces a cancellation of bosonic and fermionic loops and only diagrams corresponding to so-called branched polymers survive. The eigenvalue distribution of the random matrices near the critical point is of a new kind.
1 Introduction

The theory of random matrices has been useful in the analysis of many different physical systems. In particular in condensed-matter physics one encounters situations where physical quantities may not depend on the details of a (partly unknown) complicated Hamiltonian and where the average value of an observable can be calculated by replacing the average over eigenstates of the Hamiltonian with an average over an appropriate ensemble of random matrices. Aspects of vastly different physics such as electron localisation phenomena in disordered conductors and semiconductors [1, 2, 3, 4], disordered quantum wires [5] and integer quantum Hall effects [6] can be described by random matrix theory. Other applications can be found in nuclear physics where properties of highly excited nuclear levels can be described in random matrix language. Indeed, the very idea of random matrix theory started in nuclear physics with Wigner’s seminal work [7] and it was further developed by Dyson [8] and Mehta [9]. For a recent review of the application of random matrix theory to nuclear physics see [10]. More recently random matrices have been used in the study of the low energy chiral properties of QCD [11] and to mention two extremes: the theory of random matrices has been used in the spectral theory of quantum mechanical systems with classically chaotic behaviour [12] and in two-dimensional quantum gravity. In the last case it has even played a derisive role in revealing a whole range of new non-perturbative phenomena (for recent reviews on this vast topic we refer to [13]).

In many of the applications one is interested in computing ensemble averages which involve the density of states and correlation functions of densities of states of the Hamiltonian $H$. The density is conveniently defined in terms of the trace of the resolvent of the Hamiltonian:

$$G(z) = \frac{1}{N} \text{tr} \frac{1}{z - H}$$

where the normalization $N$ indicates that we have already replaced the real Hamiltonian with an $N \times N$ random matrix. The density of states for $H$ is then given by

$$\rho(E) = \frac{i}{2\pi} \left[ G^+(E) - G^-(E) \right],$$

where we have introduced the notation

$$G^\pm(E) = \frac{1}{N} \text{tr} \frac{1}{E - H \pm i\epsilon}.$$  

It is now possible to calculate the average value over a number of microscopic energy levels by taking the average with respect to an appropriate ensemble of random matrices, typically a Gaussian ensemble of $N \times N$ Hermitian or orthogonal or complex matrices. For a general ensemble characterized by a potential $V(H)$, one has for any observable $f(H)$:

$$\langle f(H) \rangle = \frac{1}{Z} \int dH \ f(H) \ \exp \left[ - N \text{tr} V(H) \right],$$
where the partition function $Z$ is defined as

$$Z = \int dH \exp\left[ -N \text{tr} V(H) \right].$$

(1.5)

In [14] it was shown how to calculate $G(z)$ in the large $N$ limit for an arbitrary potential $V(H)$ and the systematic $1/N$ expansion of $G(z)$ was developed in [13]. In addition it is known that the large $N$ limit of the connected density-density correlator, or equivalently

$$G(z_1, z_2) = \left\langle \text{tr} \frac{1}{z_1 - H} \text{tr} \frac{1}{z_2 - H} \right\rangle_{\text{conn}}$$

(1.6)

is universal in the sense that it can be expressed as a function only of the endpoints of the distribution of eigenvalues $\langle \rho(E) \rangle$, independently of the form of the potential $V$. This was first shown in [16] and rediscovered in [17]. Essential aspects of the universality are valid to all orders in the $1/N$ expansion [15].

In many applications one needs both $G^+(E)$ and $G^-(E)$ and in these calculations it has been technically convenient to extend the random matrices to so-called supermatrices, which contain both Fermionic and Bosonic parts [18, 19] (for a review of the supermatrix theory as used in this context, see [20]). Even in the quantum gravity applications [21, 22] as well as in purely combinatorial applications like the Meander problem [23] supersymmetric matrix models have been most useful.

In this paper we will show that a new kind of supersymmetric random matrix theory can be used to described a wide class of so-called branched polymers.

Branched polymers are generalizations of the conventional linear polymers to systems where the polymer chain can branch at each node. The universal scaling behaviour of branched polymers is believed to be important for the understanding of macromolecules in solutions. Contrary to the linear polymers the scaling limit is not described by a conformal field theory [24]. In lattice statistical mechanics they are often modelled as “lattice animals”, i.e. graphs of connected occupied sites on a lattice. Lattice animal exponents in $d$ dimensions have been related to the Lee-Yang edge singularity in $d - 2$ dimensions [25]. One usually introduces contact interactions between occupied neighbour sites which are not linked by a bond and solvent interactions between occupied sites and neighbouring empty sites. In addition one has a chemical potential associated with the growth of the branched polymers, i.e. the occupation of sites of the connected graph. One is interested in temperatures where the branched polymer becomes critical and the number of occupied sites grows to infinity. The detailed critical behaviour depends on the contact and solvent interactions as well as the entropy of the branched polymers. Part of the phase diagram can be approximated by taking the $q \to 1$ limit of an extension of the $q$-states Potts model [26].

The entropy factor is of primary importance for the phase transition. We use the word entropy in a generalized sense since it is intended not only to mean the number of branched polymers of a given size and shape, but also to take into account that a given branching
at a node will occur with a definite probability. In this paper we will concentrate on the analysis of abstract polymers, i.e. we ignore the embedding on a lattice and show how a supersymmetric matrix model of random matrices can be used to determine the universality classes of abstract branched polymers as a function of the weight attributed to branching at the individual nodes. This simplification implies that the partition function for our abstract branched polymer model can be defined as

\[ Z = \sum_{BP} w(BP) e^{-\mu |BP|}, \]  

where \( |BP| \) denotes the number of sites in the branched polymer and \( \mu \) is the chemical potential. \( w(BP) \) is the weight of branching, i.e. it is a product of local weight factors associate with the branching at each site (or vertex):

\[ w(BP) = \prod_s g(s). \]

We assume that \( g(s) \) only depends on the order of the site \( s \), i.e. the number of neighbours connected to \( s \) by links and if \( s \) has order \( k \) we write \( g_k \) for the weight.

This model can be described by the following supersymmetric random matrix model:

\[ Z = \int dW d\bar{W} e^{-\frac{1}{\alpha} N tr V(\bar{W} W)}, \]

\[ V(x) = x - \sum_{k>1} \frac{g_k}{k} x^k, \]

where \( W \) and \( \bar{W} \) are “superfields”

\[ W_a = (B, F), \quad \bar{W}_a = (B^\dagger, \bar{F}), \]

\( a = 1, 2 \) and \( B \) and \( F \) are complex bosonic and fermionic (i.e. Grassmannian) \( N \times N \) matrices, respectively. Supersymmetry means rotation between the \( B \)- and \( F \)-components:

\[ \delta \epsilon B^\dagger = \bar{F} \epsilon, \quad \delta \epsilon F = -\epsilon B, \]  

\[ \delta \bar{\epsilon} B = \bar{F} \epsilon, \quad \delta \bar{\epsilon} F = -B^\dagger \bar{\epsilon}, \]

where \( \epsilon \) and \( \bar{\epsilon} \) are Grassmann valued matrices and it follows trivially that the model is invariant under the supersymmetric transformation (1.12)–(1.13). Note that it is a huge symmetry since \( \epsilon \) is matrix valued.

The rest of this article is organised as follows: in section 2 and 3 we solve the random matrix model (1.9) using Schwinger–Dyson equations and supersymmetry Ward identities and show that the only diagrams which survive the supersymmetry are so-called cactus diagrams. In particular, we derive an equation which determines the possible universality classes. In section 4 we show how the cactus diagrams can be mapped onto branched polymer graphs and we give a purely combinatorial derivation of the important eq. (2.13). Section 5 discusses various generalizations of the model, while section 6 contains a short discussion of the results obtained.
2 Schwinger–Dyson equations and Ward identities

The supersymmetric correlators of the model defined by the partition function (1.3) vanish due to the cancellations between bosonic and fermionic loops (see eq. (2.3)), but the correlators of the bosonic matrices are nontrivial. We shall study the generating function for these correlators:

\[ G(\lambda) = \left\langle \frac{1}{N} \text{tr} \frac{1}{\lambda - B^\dagger B} \right\rangle = \frac{1}{\lambda} + \sum_{n=1}^{\infty} \frac{1}{\lambda^{n+1}} G_n \quad (2.1) \]

with

\[ G_n \equiv \left\langle \frac{1}{N} \text{tr} (B^\dagger B)^n \right\rangle. \quad (2.2) \]

The Schwinger–Dyson equation for \( G(\lambda) \) can be derived from the equality

\[ \int dW d\bar{W} \frac{1}{N^2} \sum_{ij} \frac{\partial}{\partial B_{ij}} \left( e^{-\frac{1}{\alpha N} \text{tr} V(\bar{W} W)} \frac{1}{\lambda - B^\dagger B} \right)_{ij} = 0 \]

and reads

\[ \left\langle \frac{1}{N} \text{tr} V'(\bar{W} W) \frac{B^\dagger B}{\lambda - B^\dagger B} \right\rangle = \alpha \lambda \left\langle \left( \frac{1}{N} \text{tr} \frac{1}{\lambda - B^\dagger B} \right)^2 \right\rangle. \quad (2.3) \]

To close this equation, we use the Ward identity which follows from the invariance of both the action and the measure in (1.9) under the supersymmetry transformations (1.12)–(1.13):

\[ \left\langle \delta_\epsilon \left( F_{\nu} \frac{1}{\nu - \bar{W} W} \frac{1}{\lambda - B^\dagger B} B^\dagger \right) \right\rangle = 0, \]

which, after a proper contraction of matrix indices, can be rewritten as

\[ \left\langle \frac{1}{N} \text{tr} \frac{\bar{W} W}{\nu - \bar{W} W} \frac{1}{\lambda - B^\dagger B} \left( 1 + \frac{1}{N} \text{tr} \frac{B^\dagger B}{\lambda - B^\dagger B} \right) \right\rangle = \left\langle \frac{1}{N} \text{tr} \frac{1}{\nu - \bar{W} W} \frac{B^\dagger B}{\lambda - B^\dagger B} \left( \frac{1}{N} \text{tr} \frac{B^\dagger B}{\lambda - B^\dagger B} \right) \right\rangle. \quad (2.4) \]

This equation can be used to give a formal proof of the fact that the partition function is equal to unity and all supersymmetric correlators vanish. Taking the limit \( \lambda \to \infty \) and retaining \( O(1/\lambda) \) term in eq. (2.4), one finds that

\[ \left\langle \frac{1}{N} \text{tr} \left( \bar{W} W \right)^n \right\rangle = 0 \quad (2.5) \]

for all \( n \geq 1 \). Since such correlators can be obtained by differentiating the partition function with respect to the coupling constants \( g_k \) of the potential (1.11), the partition function is independent of the potential.

On the other hand, the correlator entering the left hand side of eq. (2.3) can be extracted from eq. (2.4) in the large \( N \) limit, when factorisation holds. It can be seen expanding eq. (2.4) in the powers of \( 1/\nu \). It is convenient to introduce the function

\[ Q(\nu, \lambda) = \left\langle \frac{1}{N} \text{tr} \frac{1}{\nu - \bar{W} W} \frac{1}{\lambda - B^\dagger B} \right\rangle. \quad (2.6) \]
and rewrite eqs. (2.3) and (2.4) in the large $N$ limit as

$$\oint \frac{d\omega}{2\pi i} \frac{d\eta}{2\pi i} \frac{\omega V'(\eta)Q(\eta, \omega)}{\lambda - \omega} = \alpha \lambda G^2(\lambda),$$

(2.7)

and

$$\oint \frac{d\eta}{2\pi i} \frac{\eta Q(\eta, \lambda)}{\nu - \eta} \left(1 + \oint \frac{d\omega}{2\pi i} \frac{\omega G(\omega)}{\lambda - \omega}\right) = \oint \frac{d\xi}{2\pi i} \frac{\xi Q(\nu, \xi)}{\lambda - \xi} \oint \frac{d\omega}{2\pi i} \frac{\omega G(\omega)}{\lambda - \omega},$$

(2.8)

where the contours of integration encircle all singularities of $Q$ and $G$, but not $\lambda$, $\nu$ and infinity.

The integrals in eq. (2.8) can be calculated by taking the residues at $\lambda$, $\nu$ and $\infty$. The residue at $\infty$ can be found since the asymptotic behaviour of $Q(\nu, \lambda)$ and $G(\lambda)$ follows from the definitions:

$$Q(\nu, \lambda) = \frac{1}{\nu \lambda} + O \left(\frac{1}{\lambda^2}\right), \quad Q(\nu, \lambda) = \frac{G(\lambda)}{\nu} + O \left(\frac{1}{\nu^2}\right)$$

and

$$G(\lambda) = \frac{1}{\lambda} + O \left(\frac{1}{\lambda^2}\right).$$

After some algebra one finds from (2.8):

$$Q(\nu, \lambda) = \frac{\nu \lambda G^2(\lambda) - \lambda G(\lambda) + 1}{\nu \lambda (\nu G(\lambda) - \lambda G(\lambda) + 1)},$$

(2.9)

This formula is to be substituted in eq. (2.7). Then the integrals over $\eta$ and $\omega$ can be done taking the residues at the poles of $Q(\eta, \omega)$ as a function of $\eta$ and taking the residues at $\omega = \lambda$ and $\omega = \infty$. The result of this calculation reads

$$(\lambda G(\lambda) - 1) V' \left(\lambda - \frac{1}{G(\lambda)}\right) = \alpha \lambda G^2(\lambda).$$

(2.10)

It is now convenient to parametrise $G(\lambda)$ by

$$G(\lambda) = \frac{1}{\lambda - x(\lambda)},$$

(2.11)

and from (2.10) we obtain the following equation for $x(\lambda)$:

$$x V'(x) = \frac{\alpha \lambda}{\lambda - x}.$$

(2.12)

If $V(x)$ is a polynomial of $n$–th order, eq. (2.12) is algebraic of $(n + 1)$–th order.

It follows from (2.11) and (2.12) that

$$x(\infty) = G_1 \left(\equiv \frac{1}{N} \text{tr} B^\dagger B\right)$$
Typical Feynman diagrams for $\langle \frac{1}{N} \text{tr} B^\dagger B \rangle$ in the case where the potential $V(x) = x - \frac{1}{2}g_2x^2 - \frac{1}{3}g_3x^3$.

so that, taking the limit $\lambda \to \infty$ in eq. (2.12), we find the closed equation

$$G_1V'(G_1) = \alpha$$

for $G_1$. The surprising fact that we have obtained a closed equation for the propagator $G_1$ is a consequence of the cancellations between bosonic and fermionic loops. In fig. 1 we have shown some of the diagrams which survive the cancellation. For obvious reasons we name them “cactus diagrams”. Note that the diagrams have an orientation: the cactus loops can only proliferate on the exterior of already existing loops. This is in contradistinction to related cactus diagrams one encounters in the large $N$ limit of purely bosonic or purely fermionic vector models [27, 28].

In addition the Schwinger–Dyson equations for $G_n$ defined by eq. (2.2) allow us to express it in terms of $G_k$ with $k < n$. We give the explicit formulas for the quartic potential $V(x) = x - \frac{1}{2}gx^2$:

$$G_1 = \frac{1 - \sqrt{1 - 4\alpha g}}{2g},$$

$$G_n = \frac{2\alpha G_{n-1} + \alpha \sum_{k+l=n-1}G_kG_l + g \sum_{s=2}^{n+1}(-1)^s \sum_{k_1+\ldots+k_s=n+1,k_i<n}G_{k_1}\cdots G_{k_s}}{1 - 2gG_1}. \quad (2.15)$$

To study the analytical structure of $x(\lambda)$, it is useful to consider the inverse function

$$\lambda = \frac{x^2V'(x)}{xV'(x) - \alpha}, \quad (2.16)$$

which is depicted schematically in fig. 4. It has a pole at $x = G_1$, as follows from eq. (2.13), a maximum at zero and a minimum at some $x_0 > G_1$. Consequently, the function $x(\lambda)$ has two branch points at $\lambda = 0$ and $\lambda = \lambda_0 \equiv \lambda(x_0)$. There may also be other branch points on the unphysical sheets of the Riemann surface of $x(\lambda)$. The branch cut from 0 to $\lambda_0$ is to be identified with the support of the eigenvalue distribution of the positive definite Hermitian matrix $B^\dagger B$, the eigenvalue density being given by the discontinuity of $G(\lambda)$ across this cut.
3 Critical behaviour and scaling limit

Inspection of eq. (2.13) shows that the critical behaviour comes about when the equality

\[ G_c V'(G_c) = \alpha_c \quad (3.1) \]

holds simultaneously with

\[ [G_c V'(G_c)]' = \ldots = [G_c V'(G_c)]^{(m-1)} = 0. \quad (3.2) \]

Near the critical point \( G_1 \) behaves as

\[ G_1 \simeq G_c - \beta^{-1/m} (\alpha_c - \alpha)^{1/m}, \quad (3.3) \]

where \( \beta = (-1)^{m+1} [G_c V'(G_c)]^{(m)}/m! \). In general \( m = 2 \), but the multi-critical points with arbitrarily large \( m \) can be reached by tuning the potential. The susceptibility \( \chi = \frac{\partial G_1}{\partial \alpha} \) at the critical point scales as

\[ \chi \sim (\alpha_c - \alpha)^{-\gamma} \]

with \( \gamma = 1 - \frac{1}{m} \), which coincides with the (multi)critical index of the branched polymers [29, 30]. In the next section we will make the mapping on a class of branched polymers explicit.

Let us turn to the behaviour of the eigenvalue distribution at the critical point. Differentiating eq. (2.14) and expanding around \( \alpha_c \), one finds that \( x_0 \), defined by \( \lambda'(x_0) = 0 \) (fig. 4), scales as \( x_0 \simeq G_c + (\alpha_c - \alpha)^{1/(m-1)} \). Substituting this result into eq. (2.16), we obtain that \( z \simeq G_c \alpha_c (\alpha_c - \alpha)^{-1} \). Thus the eigenvalue distribution exhibits a behaviour which is rather unusual for matrix models — at the critical point it covers the whole
positive real semi-axis. This shows that an appropriate variable in the scaling limit is

\[ p = \frac{\lambda (\alpha_c - \alpha)}{\alpha_c G_c}, \]

and that \( G(\lambda) \) has the following form near the critical point:

\[ G(\lambda) \simeq \frac{1}{\lambda} \left[ 1 + \frac{G_c}{\lambda} - \frac{(\alpha_c - \alpha)^{1 + \frac{1}{m}}}{\alpha_c G_c^{\frac{1}{m}}} f \left( \frac{\lambda (\alpha_c - \alpha)}{\alpha_c G_c} \right) \right]. \quad (3.4) \]

The coefficient in front of the third term in the square brackets is chosen to match with eq. (3.3).

From (3.4) one can extract the scaling behaviour of the multi-point correlators of \( B^\dagger B \):

\[ G_n \simeq -\beta^{-\frac{1}{m}} (\alpha_c G_c)^{n-1} (\alpha_c - \alpha)^{-n+1+\frac{1}{m}} f_n + \delta_n G_c; \quad (3.5) \]

and \( f(p) \) is a generating function for \( f_n \). For the quartic potential the scaling behaviour (3.5) can be verified directly by using eq. (2.15). The generating function \( f(p) \) can be found by expansion of eq. (2.10) or eq. (2.12) near the critical point. A simple calculation leads to

\[ f(p) = \frac{1}{p} \left( 1 - \frac{1}{p} \right)^{1/m}. \quad (3.6) \]

4 Mapping on branched polymers

The graphs which survive the supersymmetric cancellation have an interpretation as branched polymers. Given a potential \( V(z) \) with coupling constants \( g_k \) (see eq. (1.10)) we have seen that the calculation of \( \langle B^\dagger B \rangle \) to leading order in \( 1/N \) amounts to a summation over all graphs of the kind shown in fig. 4. The figure can be viewed as a branched polymer in the following way: \( \langle B^\dagger B \rangle \) consists of a closed loop of links with one of the vertices marked. The other vertices are created by expanding the interaction

\[ \exp \left[ -\frac{1}{\alpha} N \text{tr} \left( V(\bar{W}W) - \bar{W}W \right) \right] \]

in powers of \( g_k \) and performing the Gaussian integrals. If the coupling constant is \( g_k \) we attach a “\( k-1 \)-fold blob” with weight factor \( g_k/\alpha \) if the coupling appears as

\[ \frac{1}{\alpha k} \text{tr} \left( \bar{W}W \right)^k. \]

This procedure can now be iterated and we get a complete set of cactus diagrams which is in one-one correspondence with “chiral” branched polymers in the following way: the starting line and the end line of a cactus loop join at a vertex. Cut the loop open such that only the stating line is attached to the vertex. This produces a branched polymer graph which we call “chiral” since branching only occurs at one side of the open line,
corresponding to the fact that the cactus loops only can be attached to the exterior of already existing loops. This is illustrated in fig. 3. We can now write the following combinatorial identity for the chiral branched polymers (see fig. 4):

$$G_1 = \alpha + \alpha^2 \frac{1}{\alpha} (-V'(G_1) + 1) + \alpha^3 \frac{1}{\alpha^2} (-V''(G_1) + 1)^2 + \cdots,$$

from which we get eq. (2.13).

## 5 Ising model on branched polymer

The purpose of the present section is to study a supersymmetric matrix model which describes the Ising model on the branched polymer. It can be constructed by introducing
a pair of the “superfields” $W_1$ and $W_2$ whose components transform under supersymmetry transformations by (1.12), (1.13) with the same parameters $\epsilon$ and $\bar{\epsilon}$ for $W_1$ and $W_2$. We choose the interaction potential in the form

$$N \text{tr} \left( \frac{1}{2} g_+ \bar{W}_1 W_1 W_1 + \frac{1}{2} g_- \bar{W}_2 W_2 W_2 \right),$$

where $g_\pm = g e^{\pm h}$, and attach the spin variables to the vertices of the polymer obtained from the Feynman diagram for $\left( \frac{1}{N} \text{tr} B_1^* B_1 \right)$ as described in sec. [4]. The two types of vertices, the ones corresponding to $g_+$ and $g_-$, are associated with spins of the opposite directions.

The Ising model is recovered if the propagators are defined by

$$\left< B_{\alpha ij}^* B_{\gamma kl} \right>_{\text{Gauss}} = \frac{1}{N} \delta_{ij} \delta_{jk} D_{\alpha \gamma},$$

(5.1)

with

$$D_{\alpha \gamma} = \begin{pmatrix} e^\beta & e^{-\beta} \\ e^{-\beta} & e^\beta \end{pmatrix}. \quad (5.2)$$

The constants $\beta$ and $h$ are to be identified with the inverse temperature and an external magnetic field, respectively.

In what follows we shall consider the model without the external field, so that the couplings $g_+$ and $g_-$ are equal to each other. The matrix integral corresponding to this model has the form

$$Z = \int \prod_{\alpha=1}^2 dW^\alpha d\bar{W}_\alpha e^{-N \text{tr} \left( \bar{W} D^{-1} W - \frac{1}{2} g W^+ P^+ W P W^+ W - \frac{1}{2} g W P W P^+ W \right)}, \quad (5.3)$$

where

$$P^\pm = \frac{1 \pm \sigma_3}{2}, \quad (5.4)$$

and the inverse to (5.2) matrix $D^{-1}$ is

$$\left( D^{-1} \right)^\gamma_\alpha = \frac{1}{2 \sinh 2\beta} \begin{pmatrix} e^\beta & -e^{-\beta} \\ -e^{-\beta} & e^\beta \end{pmatrix}. \quad (5.5)$$

so that the quadratic part of the exponent in eq. (5.3) involves

$$\text{tr} \bar{W} D^{-1} W = \frac{1}{2 \sinh 2\beta} \text{tr} \left[ e^\beta \left( \bar{W}_1 W_1 + \bar{W}_2 W_2 \right) - e^{-\beta} \left( \bar{W}_1 W_1^2 + \bar{W}_2 W_2^2 \right) \right]. \quad (5.6)$$

This model is invariant under the supersymmetry transformations, so the bosonic and fermionic loops are mutually cancelled, the partition function is equal to unity, and only the diagrams of the type depicted in fig. [4] survive.

The Schwinger–Dyson equations for the two–point correlator of bosonic matrices can be obtained from the equality

$$\int \prod_{\alpha=1}^2 dW^\alpha d\bar{W}_\alpha \frac{1}{N^2} \sum_{ij} \frac{\partial}{\partial B_{\alpha ij}} \left[ e^{-N \text{tr} \left( \bar{W} D^{-1} W - \frac{1}{2} g W^+ P^+ W P W^+ W - \frac{1}{2} g W P W P^+ W \right)} B_{ij} \right] = 0$$

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and can be rewritten in the form
\[
\langle \frac{1}{N} \text{tr} \left[ (B^\dagger D^{-1})_\alpha B^\gamma \right] \rangle - g \langle \frac{1}{N} \text{tr} \left[ (B^\dagger P^+)_\alpha B^\gamma WP^+W \right] \rangle - g \langle \frac{1}{N} \text{tr} \left[ (B^\dagger P^-)_\alpha B^\gamma WP^-W \right] \rangle = \delta^\alpha_\gamma. \tag{5.7}
\]

Similar to sec. 2, we utilise the Ward identities to close eq. (5.7):
\[
\langle \delta_x \left( F^\beta B^\dagger_\delta B^\gamma B^\dagger_\delta \right) \rangle = 0.
\]

Multiplying this equality by \((P^\pm)_\alpha (P^\pm)_\beta\) we get after some algebra the following equation
\[
\left\langle \frac{1}{N} \text{tr} \left[ (B^\dagger P^\pm)_\alpha B^\gamma WP^\pm W \right] \right\rangle = \left\langle \frac{1}{N} \text{tr} \left[ (B^\dagger P^\pm)_\alpha B^\gamma \right] \frac{1}{N} \text{tr} \left[ B^\dagger P^\pm B \right] \right\rangle. \tag{5.8}
\]

Substituting it into eq. (5.7), one obtains in the large \(N\) limit the closed equation for the \(2 \times 2\) matrix
\[
D^\gamma_\alpha = \left\langle \frac{1}{N} \text{tr} B^\dagger_\alpha B^\gamma \right\rangle. \tag{5.9}
\]

From (5.7) and (5.8) we find
\[
D^{-1}D - g \text{Sp} \left( P^+D \right) P^+D - g \text{Sp} \left( P^-D \right) P^-D = 1. \tag{5.10}
\]

For the symmetry reasons, the dressed coupling constants entering eq. (5.10) should be equal to each other:
\[
g \text{Sp} \left( P^+D \right) = g \text{Sp} \left( P^-D \right) = \frac{g}{2} \text{Sp} D \equiv G. \tag{5.11}
\]

Multiplying eq. (5.10) by \(D\) from the left, we obtain
\[
D - GD^2D = D. \tag{5.12}
\]

Solving eq. (5.12) for \(D\), we find
\[
D = (1 - GD)^{-1} D = \frac{1}{1 - 2 e^{\beta G} + 2 \sinh 2\beta G^2} \begin{pmatrix} e^\beta - 2 \sinh 2\beta G & e^{-\beta} \\ e^{-\beta} & e^\beta - 2 \sinh 2\beta G \end{pmatrix}. \tag{5.13}
\]

Substitution of the solution (5.13) into the definition of \(G\) (eq. (5.11)) gives the equation from which it should be determined:
\[
2 \sinh 2\beta G^3 - 2 e^{\beta G} + 2 g \sinh 2\beta + 1)G - g e^{\beta} = 0. \tag{5.14}
\]

The third order equation (5.14), together with eq. (5.13), completely determines the two-point correlators of the model. Which one to choose from the three solutions of eq. (5.14) is dictated by matching with perturbation theory.
The critical behaviour comes about when two roots of the equation (5.14) collide. It happens when the following conditions hold simultaneously:

\[ 2 \sinh 2 \beta G_c^3 - 2 e^\beta G_c^2 + (2 g_c \sinh 2 \beta + 1) G_c - g_c e^\beta = 0, \]
\[ 6 \sinh 2 \beta G_c^2 - 4 e^\beta G_c + (2 g_c \sinh 2 \beta + 1) - g_c e^\beta = 0. \]  

The critical curve in the \( \beta - g \) plane is depicted in fig. 5. The critical coupling constant behaves in the low temperature limit, \( \beta \to \infty \), as \( g_c(\beta) \sim \frac{1}{4} e^{-2\beta} \). In this limit the mixing between \( W_1 \) and \( W_2 \) is exponentially small and the above result reproduces the critical point of the solution (2.14), (2.15) of the one–matrix model. The two–point correlators always have the square root singularity along the line of the phase transition; this recovers, in particular, the known result that the Ising model on the branched polymer never becomes critical \[31\].

6 Discussion

We have shown that the simplest non-trivial supersymmetric matrix model produces abstract branched polymers for the general potential (1.10). Certain aspects of these abstract branched polymers have been found earlier by purely combinatorial methods, but the matrix model method allows a detailed study of the spectral properties. We predict that the non-trivial phases of the abstract branched polymers, corresponding the higher critical points with \( m > 2 \), also exist in “real” lattice animals, i.e. branched polymers embedded in regular \( d \)-dimensional lattices. Since these branched polymers look increasing similar to ordinary polymers for large \( m \) (see \[31\]) there might exist a whole sequence of universality classes of lattice animals, interpolating between ordinary polymers with internal (Hausdorff) dimension 1 and “conventional” branched polymers with internal (Hausdorff) dimension 2.
In field theory the use of matrix models has usually been associated with Riemann surfaces. The large \(N\) limit of Hermitian or complex matrix models has a diagrammatic expansion as graphs of spherical topology and an \(1/N\) expansion is an expansion in diagrams with the topology of higher genus surfaces. Adding Supersymmetry to a purely bosonic or purely fermionic theory makes it more rigid and reduce the number of degrees of freedom. Here we observe this in an extreme way: only the tiny subclass of graphs corresponding to branched polymers survives for the model (1.9), (1.10) when we look at observables like \(\text{tr } 1/(z - B^\dagger B)\).

It is an interesting question whether or not supersymmetric matrix models of the type discussed in this paper can be associated with super-Riemann surfaces. As is already mentioned in the previous paragraph, the potential (1.10) is separately invariant under both (1.12) and (1.13). This is the reason only cactus-like diagrams rather than planar graphs survive for the potential (1.10) in the large \(N\) limit. We can reduce the symmetry by modifying the potential in order for a wider class of graphs to survive. Such supersymmetric matrix models deserve further investigations.

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