Quantum criticality nearby certain magnetic phase transition beneath the superconducting dome of Ba$_{1-x}$Na$_x$Fe$_2$As$_2$ is attentively studied by virtue of a phenomenological theory in conjunction with renormalization group approach. We report that ordering competition between magnetic and superconducting fluctuations is capable of coaxing incommensurate (IC) magnetic states to experience distinct fates depending upon their spin configurations. C$_2$-symmetry IC magnetic stripe with perpendicular magnetic helix dominates over other C$_2$-symmetry magnetic competitors and hints to a potential candidate for the unknown C$_2$-symmetry magnetic state. Amongst C$_4$-symmetry IC magnetic phases, IC charge spin density wave is substantiated to be the winner shedding light on the significant intertwining of charge and spin degrees of freedom. Meanwhile, ferocious fluctuations render a sharp fall of superfluid density alongside with dip of critical temperature as well as intriguing behavior of London penetration depth.

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**Introduction** - Last dozen years have witnessed considerably intense research devoted to iron pnictides of BaFe$_2$As$_2$ family \[1-13\], whose phase diagrams are ubiquitously born out of both superconducting (SC) and diverse kinds of magnetic orders mediated by quantum phase transitions (QPTs) \[4\]. Notwithstanding magnetism is an antagonistic state versus superconductivity, they compete and collaborate other than coexist with each other \[12, 13, 15\]. This accordingly poses a substantial challenge what is the connection between magnetic and SC states providing a crucial ingredient to substantial challenge what is the connection between magnetic and SC states. In the light of abundant magnetic and SC states, IC charge spin density wave is substantiated to be the winner shedding light on the significant intertwining of charge and spin degrees of freedom. Meanwhile, ferocious fluctuations render a sharp fall of superfluid density alongside with dip of critical temperature as well as intriguing behavior of London penetration depth.

**FIG. 1:** (Color online) Schematic $x - T$ phase diagram of Ba$_{1-x}$Na$_x$Fe$_2$As$_2$ in the vicinity of vital magnetic quantum critical point (QCP) located at $x_c$. PM, SDW, and SC are shortened notations for paramagnetism, spin density wave (SDW), and superconductivity with $T_{2,4}$ and $T_c$ denoting the critical temperatures from PM to C$_2$ SDW and non-SC to SC, respectively. Ordering competition bears out that C$_2$ IC SDW close by the QCP preferable to be IC CSDW, which are manifested and supported by the combination of Table I and Fig. 4. Instead C$_2$ IC SDW can either be C$_2$ IC SDW, DPMH, or MH state as addressed in Supplementary Material (SM) \[21\] (the abbreviations of states hereby are consistent with Table I's).

What is the optimal state characterizing the mystic C$_2$ magnetic state. We are going to make a response taking advantage of a phenomenological theory together with the Wilsonian renormalization group (RG) \[21\]. Answers are of notable help to deeply understand the phase diagram and even offer instructive insights into pairing mechanism. Fig. 4 schematically illustrates our central
results driven by ordering competition.

Effective theory and RG analysis - Fermi surfaces of BaFe$_2$As$_2$ compounds under a three-band model consist of one hole pocket at the center of Brillouin zone $Q_T = (0, 0)$ and two electron pockets centered at two fixed momenta $Q_X = (\pi, 0)$ and $Q_Y = (0, \pi)$ [13, 16–22]. For microscopical consideration, both magnetic and SC states are rooted in interactions among excited quasiparticles from these Fermi pockets [13, 16, 17, 22–25].

Concretely, a magnetic state is composed of two basic magnetic order parameters $M_X$ and $M_Y$, which are designated by $M_j = \sum_k \tilde{c}_j^1 \tilde{\sigma}_{\alpha \beta} c_{j,k+Q_j} \gamma$ with $j = X, Y$ [23–27]. To involve IC magnetic states, ordering vectors are afterwards distributed as $Q_X = (\pi - \delta, 0)$ and $Q_Y = (0, \pi - \delta)$ with $\delta$ being a small correction for generic wavevectors. This indicates that the magnetic order parameters are regarded as a complex quantity $M_{Q_X Y} \neq M_{Q_X Y}^0 \equiv M - Q_{X, Y}$, which is in striking contrast to the commensurate case with $\delta = 0$ and $M_{Q_X Y}^0 = M_{Q_{X, Y}}$.

We begin with the extended Landau-Ginzburg free energy after integrating out the fermionic ingredients [16].

$$L_{\text{eff}} = \left[\frac{1}{2}(\partial_\mu M_X/C) + \alpha M_X + \frac{\beta_X}{2} M_X^2 \right] + \left[\frac{1}{2}(\partial_\nu M_Y/S) + \alpha_Y M_Y^2 + \frac{\beta_Y}{2} M_Y^2 \right] + \left[-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{\alpha_A}{2} A^2 \right]$$

$$+ \left[\frac{1}{2}(\partial_\mu h)^2 + a_h h^2 + \frac{\beta_h}{2} h^2 + \gamma_h h^3 \right] + \alpha_{XY} M_X M_Y + \gamma_{XY} h M_X M_Y h + \gamma_{X^2} h M_X^2 h + \gamma_{Y^2} h M_Y^2 h + \gamma_{A^2} h A^2$$

$$+ \alpha_{XY} M_X^2 M_Y^2 + \lambda_{XY} M_X^2 M_Y^2 h^2 + \lambda_{Y} h M_X M_Y h^2 + \lambda_{XY} h M_X M_Y h^2 + \lambda_{A^2} h A^2,$$

with $C \equiv 1/|n_X \cos \theta|^2$ and $S \equiv 1/|n_Y \sin \theta|^2$. The detailed derivations of this effective theory are presented in SM [21]. $M_{Q_X Y}$ point to magnetic fluctuations and $h, A$ are auxiliary fields to absorb SC fluctuations. We here dub factors in $\alpha_X$ etc. the effective parameters to prevent their being confused with fundamental parameters appearing in Eq. (1). Two series of parameters are bridged by virtue of Eqs. (1)–(19).

To proceed, we compute one-loop corrections to all effective parameters in Eq. (2) and derive the corresponding RG evolutions within Wilsonian RG framework [13, 20, 32] via integrating out the fast fields in the momentum shell $e^{-l} \Lambda < k < \Lambda$ with the running scale $l > 0$. Since the fundamental parameters defined in Eq. (1) dictate the physical properties, it heralds unambiguously that a pillar of task consists in refining their flow equations. To this end, we resort to the strategy in Refs. [18, 32]. Combining RG flows of effective parameters and connections with fundamental parameters [11–19] yields a set of coupled RG equations

\[ \frac{d \chi_i}{dl} = \sum_j F_{ij} \chi_j, \]

with $X_{ij}$ serving as the fundamental parameters [33] and $F_{ij}$ standing for evolution coefficients as a function of $X_{ij}$. It necessitates bearing in mind that the coupled RG evolutions hinge heavily upon the spin configurations of magnetic fluctuations, namely the relationships between $|n_x|^2, |n_y|^2, |n_z|^2$, and $|n_y|^2$, which give rise to seven independent classes of RG evolutions. The details of Eq. (3) are stored completely in SM [21].

Fates of IC magnetic states - With the help of energy-dependent flows of fundamental parameters, we are now in a suitable situation to study the stabilities of IC magnetic states triggered by some magnetic QCP. As to BaFe$_2$As$_2$ compounds, many experimental efforts [4–11] corroborate that magnetism occupies major space of phase diagram in terms of various states with distinguished symmetries and spin configurations. Especially, compound Ba$_{1-x}$Na$_x$Fe$_2$As$_2$ [3, 8, 10] harbors a complicated but fascinating phase diagram sketched in Fig. 1, indicating a string of magnetic states for both $C_2$ and $C_4$ symmetries are allowed with proper variations of temperature and doping. Besides three commensurate states, i.e., stripe spin density wave (SDW),
TABLE I: Collections of low-energy fates for IC magnetic states in Ba$_{1-x}$Na$_x$Fe$_2$As$_2$. The first line enumerates seven distinguished types of IC magnetic states and the second line shows stable constraints as functions of fundamental interaction parameters $\beta$ as well as the third line presents the corresponding low-energy stabilities. Herein, ✔ and ✘ stand for a stable state (i.e., the prevailing candidate by the side of the magnetic QCP) and an unstable state, respectively.

| IC magnetic states | ICS | MH | ICS $\perp$ MH | DPMH | IC CSDW | IC SVC | SWC |
|--------------------|-----|----|----------------|-------|----------|--------|-----|
| Stable constraints | $\beta_1 - \beta_2 < 0$ with $g_2 > |\beta_1 - \beta_2|$, $\beta_1 - \beta_2 > 0$ with $g_2 > |\beta_1 - \beta_2|$ | $\beta_1 - \beta_2 > 0$, $\beta_1 - \beta_2 < 0$, $g_2 < |\beta_1 - \beta_2|$, $g_2 > |\beta_1 - \beta_2|$ | $\beta_1 - \beta_2 > 0$, $\beta_1 - \beta_2 < 0$, $g_2 < |\beta_1 - \beta_2|$, $g_2 > |\beta_1 - \beta_2|$ | $\beta_1 - \beta_2 < 0$, $\beta_1 - \beta_2 > 0$, $g_2 < |\beta_1 - \beta_2|$, $g_2 > |\beta_1 - \beta_2|$ | $\beta_1 - \beta_2 > 0$, $\beta_1 - \beta_2 < 0$, $g_2 < |\beta_1 - \beta_2|$, $g_2 > |\beta_1 - \beta_2|$ |
| Fates of magnetic states | ✘ | ✘ | ✔ | ✘ | ✔ | ✘ | ✘ |

charge spin density wave (CSDW), and spin vortex crystal (SVC) [16, 17, 34, 35]. Christensen et al. [19] recently advocated that potential IC magnetic states are clustered into nine inequivalent breeds. What is more, seven of them can be realized with confined parameters of mean-field free energy in the phase diagram [10], which cover four kinds of C$_2$ IC cases involving C$_2$ IC stripe (ICS), C$_2$ magnetic helix (MH), C$_2$ IC magnetic stripe with perpendicular magnetic helix (ICS $\perp$ MH), and C$_2$ double parallel magnetic helix (DPMH), as well as three distinct C$_4$ IC situations consisting of C$_4$ IC CSDW, C$_4$ IC SVC, and C$_4$ IC spin-whirl crystal (SWC) along with the stable constraints catalogued point-to-point in the second line of Table I.

Despite of an underlying antagonist against SC state, magnetism is assumed to be of intimate relevance to superconductivity as they are closely adjacent to each other or even coexist near the magnetic QPT. To be concrete, we concentrate on an especial point in Fig. 1, namely the QCP at $T = 0$ that separates C$_2$ and C$_4$ IC magnetic states labeled by $x_c$. Generally, the related magnetic fluctuations compete so furiously that are always responsible for physics in the shadow of QPT including quantum critical regime with higher temperatures [14, 23, 24]. Considering individualities of diverse states in spite of hosting common magnetic generalities come up with different consequences, we thereafter contemplate the magnetic states on both sides of this QPT.

As it concerns the issue on intricate relationship between magnetism and superconductivity, a hallmark of fathoming overall phase diagram is tantamount to pinpointing the specific construction of each magnetic state. As a corollary, it is an appropriate pointcut that one investigates how the ordering competition affects the magnetic state on the edge of the QCP by means of RG flows [3] in collaboration with the stable magnetic criteria itemized in the second line of Table I. In order to be baldly relevant with schematic phase diagram, we adopt $T = T_0 e^{-t}$ with $T_0$ the initial temperature to measure the evolution variable $T_e [13, 32]$. Performing numerical analysis not only bears witness to the crucial role of ordering competition but also sheds light on fates of all types of IC magnetic states. Fig. 2 exhibits the temperature (energy) dependence of correlated fundamental parameters, which carry the low-energy characteristics for both C$_2$ ICS $\perp$ MH and C$_4$ IC CSDW. At the outset, we find that stable constraints for C$_2$ ICS $\perp$ MH shown in Fig. 2(a) are well protected with the decrease of temperature. They are sabotaged by extremely strong fluctuations only until the magnetic QCP is sufficiently accessed at the collapsed temperature $T_{col} \sim 10^{-2} T_0$ (taking $T_0 = 100$ K for instance, $T_{col} \sim 10^{-2}$ K). This evidently signals that C$_2$ ICS $\perp$ MH is of particular robustness withstanding ordering competition. In reminiscence of the unknown C$_2$ magnetic state, which locates at a little deviation from the magnetic QCP portrayed in Fig. 1, we are aware that C$_2$ ICS $\perp$ MH is therefore deemed to be a reasonable candidate for this mysterious C$_2$ state that differs substantially from conventional C$_2$ stripe state. In addition, reading off Fig. 2(b) proposes firmly robust temperature-dependent constraints for C$_4$ IC CSDW. We then come to a conclusion that IC CSDW, like its commensurate counterpart [18], behaves dominantly compared to other types of IC C$_4$ magnetic states. This C$_4$ magnetic state is henceforth the most applicable choice in the left side of magnetic QCP $x_c$ in Fig. 1 which compete, coexist, and cooperate with SC state. Furthermore, apart from the two applicable states including C$_2$ ICS $\perp$ MH and C$_4$ IC CSDW, ordering competition surviving by magnetic QCP is not in favour of all other types of IC magnetic...
I clearly exist alone but instead there might be a coexistence of large regions for stable constraints are jeopardized and collapsed. Insets: basic results are insensitive to initial values of parameters. States listed in Table 1. In terminological language, given these states are prone to easily feel plus efficiently receive the fluctuation corrections even far away from a magnetic QCP, they are fairly sensitive and fragile to ordering competition, resulting in undeviating breakdown themselves as temperature is reduced. It broadly suggests that one is unable to solely fix the configuration of $C_2$ IC SDW above $C_4$ IC CSDW and $C_2$ IC $\perp$ MH as displayed in Fig. 1 which may either be $C_2$ ICS, $C_2$ DPMH, or $C_2$ MH. The details of verifying stabilities of IC magnetic states are provided in SM [21]. Last but not the least important, we deliver that, as for the region close enough to the QCP with $T < T_{col}$, ordering competition is so ferociously that any magnetic state cannot exist alone but instead there might be a coexistence of multiple IC magnetic states.

Superfluid density and London penetration depth - As magnetic states steadily compete and coexist with a SC order, it is of great temptation to examine how the superfluid density ($\rho_s$) and London penetration depth ($\lambda_L$) are influenced in the presence of ordering competition, which are of two particular importance implications. In principle, $\rho_s(T)$ can be evaluated as $\rho_s(T) = \rho_s^s(T) - \rho_n(T)$, where $\rho_s^s(T) \propto \alpha_A(T)$ stems from the mass of vector field $A$ that obey RG equations due to Anderson-Higgs mechanism [30] and $\rho_n(T)$ grasps the density of thermally excited normal (non-SC) fermionic quasiparticles (QPs), respectively. Approaching the QCP, ordering competition is dominant and thus the normal QPs effects can be neglected implying $\rho_s(T) \sim \rho_s^s(T)$. Fig. 3 clearly unveils that $\rho_s(T)$ is remarkably suppressed by the ordering competition [18, 31, 37]. Because critical temperature $T_c$ is nominated by $\rho_s(T_c) = \rho_s^s(T_c) - \rho_n(T_c) = 0$, one can infer that it would be intensively reduced in the absence of $\rho_s(T)$. As explicitly delineated in the inset of Fig. 3 it is worth declaring that the drop of $T_c$ caused by the $C_4$ IC CSDW is a little more than its $C_2$ IC $\perp$ MH’s counterpart, which is also apparently exposed in Fig. 1. Albeit a slight splitting, principal tendencies are qualitatively compatible with recent experiments [3, 8, 10]. For qualitative discussions, we single out the s-wave gap symmetry as a toy and tentative substitute. In this respect, the London penetration depth is expressed as $\lambda_L(0)/\lambda_L(T) = \sqrt{\rho_s(T)}$ [38]. As a consequence, $\lambda_L(0)/\lambda_L(T)$ shares the analogous temperature-dependent trajectory with $\rho_s$ under the im-

**FIG. 2:** (Color online) Temperature-dependent stable constraints of (a) $C_2$ ICS $\perp$ MH and (b) $C_4$ IC CSDW under representative starting values of interaction parameters (the basic results are insensitive to initial values of parameters). $T_{col}$ labels the very temperature at which the corresponding stable constrains are jeopardized and collapsed. Insets: enlarged regions for $\beta_1 - \beta_2$ and $g_2/|\beta_1 - \beta_2|$.  

**FIG. 3:** (Color online) Superfluid density and London penetration depth as a function of temperature at $\theta = \pi/6$ affected by $C_2$ ICS $\perp$ MH and $C_4$ IC CSDW neighboring the magnetic QCP. $T$ designates the related critical temperature without ordering competition (the essential features are insensitive to beginning values of interaction parameters). Inset: enlarged regions for $\rho_s$ displaying difference between the two cases.
pact of ordering competition as depicted in Fig. 3. Although BaFe$_2$As$_2$ system possesses a more intricate gap structure, this primitive result might uncover parts of central ingredients that are in charge of $\lambda_L$’s property.

Summary - We study and discern the probable IC magnetic states induced by subtle ordering competition in the proximity of certain QPT below the SC dome of Ba$_{1-x}$Na$_x$Fe$_2$As$_2$. Specifically, we find that $C_2$ ICS $\perp$ MH survives to be a good candidate for the obscure $C_2$ magnetic state and IC $C_4$ CSDW points to the reasonable IC state in the vicinity of the magnetic QPT. In addition, we address that superfluid density in tandem with critical temperature and London penetration depth manifest critical behaviors attesting to ordering competition around the QCP. The conclusions are qualitatively concomitant with recent experiments [6, 8, 10].

We expect our results are profitable to further understand the phase diagram of Ba$_{1-x}$Na$_x$Fe$_2$As$_2$ and explore the correspondence between SC and magnetic states in iron-based superconductors.

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Supplementary Material: “Incommensurate magnetic states induced by ordering competition in Ba$_{1-x}$Na$_x$Fe$_2$As$_2”

DERIVATIONS OF THE EFFECTIVE ACTION

We are going to take into account both magnetic and SC fluctuations in the vicinity of magnetic QCP shown in Fig. 11. To this end, the phenomenological effective action $[16, 23]$ can be casted as

$$S = \int \mathcal{L} = \int \mathcal{L}_{\text{SDW}} + \int \mathcal{L}_{\text{SC}} + \int \mathcal{L}_{\text{SDW-SC}},$$

(4)

where $\mathcal{L}_{\text{SDW}}, \mathcal{L}_{\text{SC}},$ and $\mathcal{L}_{\text{SDW-SC}}$ correspond to SDW, SC orders, and their interplay, respectively.

At first, we go to $\mathcal{L}_{\text{SDW}}$. Designating $M_X \equiv M_X \cos \theta n_X$ and $M_Y \equiv M_Y \sin \theta n_Y$, where $\theta \in (0, \pi/2)$ and $|n_{X,Y}|^2 = 1$ describe the spin configurations of magnetic states, and inserting them into the free energy density $[11]$ with adding the dynamical terms of magnetic order parameters give rise to $[16, 18, 22, 28]
$$
\mathcal{L}_{\text{SDW}} = \left[ n_X \cos \theta \frac{1}{2} (\partial_\mu M_X)^2 + \alpha (|n_X|^2 \cos^2 \theta) M_X^2 \right] + \left[ n_Y \sin \theta \frac{1}{2} (\partial_\mu M_Y)^2 + \alpha (|n_Y|^2 \sin^2 \theta) M_Y^2 \right] + \frac{\beta_1 - \beta_2}{2} (|n_X|^2 \cos^4 \theta M_X^4 + |n_Y|^2 \sin^4 \theta M_Y^4) + \frac{\beta_2}{2} (|n_X|^4 \cos^4 \theta M_X^4 + |n_Y|^4 \sin^4 \theta M_Y^4) + g_1 |n_X|^2 |n_Y|^2 \cos^2 \theta \sin^2 \theta (|n_X| \cdot |n_Y|)^2 + |n_X | \cdot |n_Y|^2) M_X^2 M_Y^2 + g_2 \cos^2 \theta \sin^2 \theta (|n_X| \cdot |n_Y|)^2 M_X^2 M_Y^2. \tag{5}
$$

We next consider $\mathcal{L}_{\text{SC}}$. In order to obtain SC fluctuations in the ordered state, we bring out the the following contribution by employing the condition $\partial_\mu A_\mu = 0$ $[30]
$$
\mathcal{L}_{\text{SC}} = \left[ \partial_\mu \Delta \partial_\nu \Delta + a_s \Delta^2 (k) + \frac{u_s}{2} \Delta^4 (k) \right] + \left[ -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{\alpha_A}{2} A^2 \right] + \lambda_{\Delta A} |\Delta|^2 A^2. \tag{6}
$$

As the system enters into the SC ordered state around the SDW QCP, we need to expand the SC order parameter by introducing two new gapless fields

$$\Delta = V_0 + \frac{1}{\sqrt{2}} (h + i\eta), \quad \langle h \rangle = \langle \eta \rangle = 0, V_0 \equiv \langle \Delta \rangle = \sqrt{-\frac{a_s}{u_s}}, \tag{7}
$$

which help us to extract the potential fluctuation of SC order parameter $[31]$, to make the $A$ massive after absorbing the gapless Goldstone particles.

Combining Eq. (6) and Eq. (7), we get after discarding the constant terms and choosing some transformation to make $\eta = 0$ due to the local gauge invariance $[31],
$$
\mathcal{L}_{\text{SC}} = \left[ \frac{1}{2} (\partial_\mu h)^2 - a_s h^2 + \frac{u_s}{2} h^4 + \frac{\sqrt{-2a_s} u_s}{2} h^3 - \frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{\alpha_A}{2} A^2 + \lambda_{\Delta A} \sqrt{-\frac{2a_s}{u_s}} h A^2 + \frac{\lambda_{\Delta A}}{2} h^2 A^2, \tag{8}
$$

where the “mass” of field $A$ being $\alpha_A \equiv \lambda_{\Delta A} \sqrt{-\frac{2a_s}{u_s}}$.

At last, we introduce $\mathcal{L}_{\text{SC}}$. The interplay between SC and SDW order parameters can be written as $[18],
$$
\mathcal{L}_{\text{SDW-SC}} = \lambda (|M_X|^2 + |M_Y|^2)^2 \Delta^2 + \kappa (|M_X\cdot M_Y| + |M_X\cdot M_Y^*|) \Delta^2. \tag{9}
$$

Based on the information of $\mathcal{L}_{\text{SDW}}$ and $\mathcal{L}_{\text{SC}}$, we are left with
$$
\mathcal{L}_{\text{SDW-SC}} = -\frac{a_s \lambda}{u_s} (|n_X|^2 \cos^2 \theta M_X^2 + |n_Y|^2 \sin^2 \theta M_Y^2) + \lambda \sqrt{-\frac{2a_s}{u_s}} (|n_X|^2 \cos^2 \theta M_X^4 + |n_Y|^2 \sin^2 \theta M_Y^4) h + \frac{\lambda}{2} (|n_X|^2 \cos^2 \theta M_X^2 + |n_Y|^2 \sin^2 \theta M_Y^2) h^2 + \frac{\alpha_A}{2} (|\cos \theta \sin \theta n_X \cdot n_Y| + |\cos \theta \sin \theta n_X \cdot n_Y^*|) M_X M_Y h + \frac{\kappa}{2} (|\cos \theta \sin \theta n_X \cdot n_Y| + |\cos \theta \sin \theta n_X \cdot n_Y^*|) M_X M_Y^* h. \tag{10}
$$
To recapitulate, Eqs. (5), (9), and (10) constitute our effective theory (2). The effective parameters such as $\alpha_X$ etc. are connected with fundamental parameters appearing in Eq. (1) via following relationships

\begin{align}
\alpha_h &\equiv (-a_s), \beta_h \equiv \frac{u_s}{4}, \gamma_h \equiv \sqrt{-\frac{2a_s u_s}{2}}, \alpha_A \equiv -\frac{2\lambda_{\Delta A} a_s}{u_s}, \gamma_{hA} \equiv \lambda_{\Delta A} \sqrt{-\frac{2a_s}{u_s}}, \lambda_{hA} \equiv \frac{\lambda_{\Delta A}}{2}, \tag{11} \\
\alpha_X &\equiv \left( a - \frac{\lambda a_s}{u_s} \right) (|n_X|^2 \cos^2 \theta), \beta_X \equiv \beta_2 \left( |n_X|^2 \cos^4 \theta \right) + (\beta_1 - \beta_2) \left( \frac{1}{2} |n_X|^2 \sin^2 \theta \right), \tag{12} \\
\alpha_Y &\equiv \left( a - \frac{\lambda a_s}{u_s} \right) (|n_Y|^2 \sin^2 \theta), \beta_Y \equiv \beta_2 \left( |n_Y|^2 \sin^4 \theta \right) + (\beta_1 - \beta_2) \left( \frac{1}{2} |n_Y|^2 \sin^2 \theta \right), \tag{13} \\
\alpha_{XY} &\equiv -\frac{a_s \kappa}{u_s} (|\cos \theta \sin \theta n_X \cdot n_Y| + |\cos \theta \sin \theta n_X \cdot n_Y^*|), \tag{14} \\
\gamma_{XYh} &\equiv \sqrt{-\frac{2a_s}{u_s}} (|\cos \theta \sin \theta n_X \cdot n_Y| + |\cos \theta \sin \theta n_X \cdot n_Y^*|), \tag{15} \\
\gamma_{X^2h} &\equiv \sqrt{-\frac{2a_s}{u_s}} (|n_X|^2 \cos^2 \theta), \gamma_{Y^2h} \equiv \sqrt{-\frac{2a_s}{u_s}} (|n_Y|^2 \sin^2 \theta), \tag{16} \\
\lambda_{XY} &\equiv g_1 \cos^2 \theta \sin^2 \theta \left( |n_X|^2 |n_Y|^2 \right) + \frac{g_2}{2} \cos^2 \theta \sin^2 \theta \left( |n_X \cdot n_Y|^2 + |n_X \cdot n_Y^*|^2 \right), \tag{17} \\
\lambda_{Xh} &\equiv \frac{\lambda}{2} \left( |n_X|^2 \cos^2 \theta \right), \lambda_{Yh} \equiv \frac{\lambda}{2} \left( |n_Y|^2 \sin^2 \theta \right), \tag{18} \\
\lambda_{XYh} &\equiv \frac{\lambda}{2} (|\cos \theta \sin \theta n_X \cdot n_Y| + |\cos \theta \sin \theta n_X \cdot n_Y^*|). \tag{19}
\end{align}

We hereby would like to emphasize that $\kappa$ and $\lambda_{\Delta A}$ cannot be represented by original ones $a, a_s, u_s, \lambda, \beta_1, \beta_2, g_1, g_2$ appearing in the free energy (1). This unambiguously substantiates the independence of $\kappa$ and $\lambda_{\Delta A}$ and therefore comes up with two supplementary fundamental parameters.

**COUPLED RG EQUATIONS OF FUNDAMENTAL INTERACTION PARAMETERS**

After performing one-loop analysis of effective theory [18, 20, 32] via integrating out the fields in the momentum shell $e^{-1} \Lambda < k < \Lambda$ with $l > 0$ the running scale, we can derive flows of effective parameters in Eq. (2). Combining these equations and connections (11)-(19) [18, 32], the coupled RG equations for fundamental parameters can be derived.

Before going further, it is necessary to highlight that the fundamental parameters $g_{1,2}$ only appear in Eq. (17). This implies that they do not evolve independently. In this sense, it is hereafter convenient to introduce a parameter

\begin{equation}
\tilde{g} = g_1 \cos^2 \theta \sin^2 \theta (|n_X|^2 |n_Y|^2) + \frac{g_2}{2} \cos^2 \theta \sin^2 \theta (|n_X \cdot n_Y|^2 + |n_X \cdot n_Y^*|^2), \tag{20}
\end{equation}

to describe the information of $g_{1,2}$.

After long but straightforward calculations [18, 32], we eventually obtain the coupled RG equations of all fundamental interaction parameters around the magnetic QCP, which include $\alpha, \beta_{1,2}, \tilde{g}$ and $\kappa$ specifying the characters of spin configurations as well as $a_s, u_s, \lambda_{\Delta A}$ stemming from SC fluctuations. These coupled RG evolutions are closely dependent upon the spin configurations of magnetic fluctuations, namely the relationships between $|n_X|^2$, $|n_X|^4$, $|n_X^2|^2$, $|n_Y|^4$, which are divided into two main sorts of situations.

For type-I case, at which $|n_X^2|^2 \neq |n_X|^4$ and $|n_Y|^2 = |n_Y|^4$ or $|n_X|^2 = |n_X|^4$ and $|n_Y|^2 \neq |n_Y|^4$, both $\beta_1$ and $\beta_2$ flow independently and thus the coupled evolutions are written as

\begin{equation}
\frac{da_s}{dl} = 2a_s - \frac{1}{4 \pi^2} \left\{ \frac{9a_s u_s (1 + 4a_s)}{2} + \frac{2S^2_{\alpha s} a_s \lambda^2}{u_s} \left[ 1 - 4S_1 (a - \frac{\lambda a_s}{u_s}) \right] + \frac{2C D_2 a_s \lambda^2}{u_s} \left[ 1 - 4C_1 (a - \frac{\lambda a_s}{u_s}) \right] \\
+ \frac{32a_s \lambda_{\Delta A}}{3u_s} \left[ 1 + 4\lambda_{\Delta A} a_s \right] + \frac{(S_1 + C D_1) \lambda}{2} + \frac{3u_s (1 + 2a_s)}{4} - (a - \frac{\lambda a_s}{u_s}) (E_7^\alpha S^2 + D_7^\alpha C^2) \lambda \\
+ \lambda_{\Delta A} (1 + 2\lambda_{\Delta A} a_s) + \frac{C S F^2 a_s \kappa^2}{4u_s} \left[ 1 - 2(CD_1 + S_1) (a - \frac{\lambda a_s}{u_s}) \right] \right\}, \tag{21}
\end{equation}
\[
\frac{da}{dl} = 2(a - \frac{\lambda a_s}{u_s}) + \frac{1}{4\pi^2} \left\{ \frac{\lambda}{2} \frac{\tilde{S} \hat{g}}{D_1} + 3C[\beta_2 D_1 + (\beta_1 - \beta_2) D_2] + \left[ \frac{\beta_2}{D_1} - \frac{2S^2 E_1 \hat{g}}{D_1} \right] (a - \frac{\lambda a_s}{u_s}) \right. \\
- 6C^2 (a - \frac{\lambda a_s}{u_s}) [\beta_2 D_2 + (\beta_1 - \beta_2) D_2] + a_\lambda \lambda + \frac{4C^2 D_1 a \lambda^2}{u_s} [1 - 2(CD_1 (a - \frac{\lambda a_s}{u_s}) - a_s)] \\
- \frac{S^2 F^2 \lambda^3}{3D_1} [1 - 2 (S E_1 (a - \frac{\lambda a_s}{u_s}) - a_s)] + \left( \frac{\lambda a_s}{u_s} \frac{da}{dl} + a_\lambda \lambda a_s \frac{dl}{u_s} - \frac{\lambda a_s}{u_s} \frac{du_s}{dl} \right), \tag{22}
\]

\[
\frac{du_s}{dl} = u_s + \frac{1}{2\pi^2} \left\{ -16a_\lambda \lambda u_s (1 + 6a_s) \frac{16C^2 D_1 a \lambda^2}{3u_s} [1 - 6C (a - \frac{\lambda a_s}{u_s}) D_1] - \frac{16S^2 E_1^3 a \lambda^2}{3u_s} [1 - 6(a - \frac{\lambda a_s}{u_s}) E_1] \\
+ 8a_\lambda^2 (S^2 E_1 + C^2 D_1)(a - \frac{\lambda a_s}{u_s}) - 18a_\lambda u_s - \frac{9a_\lambda}{2} - 2C^2 (S^2 E_1^3 + C^2 D_1^3) - \frac{32\lambda a_s}{3} \left( \frac{4\lambda a_s}{u_s} + 1 \right) \\
- \frac{11072a_\lambda \lambda u_s (1 + 6a_s) - \frac{CSF^2 \lambda^3}{D_1} [1 - 2(CD_1 + SE_1) (a - \frac{\lambda a_s}{u_s})] \\
- \frac{C^2 S^2 F^2 a_\lambda \lambda^2 (F^2 \kappa^2 + 2D_1 E_1 \lambda^2}{6u_s} [1 - 4(CD_1 + SE_1) (a - \frac{\lambda a_s}{u_s})] \\
- \frac{2C^2 S^2 F^2 D_1 a \lambda^2 \lambda}{3u_s} [1 - 2(CD_1 + SE_1) (a - \frac{\lambda a_s}{u_s})] \right\}, \tag{23}
\]

\[
\frac{d\lambda}{dl} = \lambda + \frac{1}{2\pi^2} \left\{ \frac{-8S^2 E_1 a_\lambda \lambda \hat{g}}{3D_1 u_s} [1 - 6SE_1 (a - \frac{\lambda a_s}{u_s}) - \frac{8C^2 D_1 a_\lambda^2 \lambda^2 (\beta_2 D_2^1 + (\beta_1 - \beta_2) D_2)}{3u_s} \\
\times [1 - 6C D_1 (a - \frac{\lambda a_s}{u_s}) - 4C D_1 a_\lambda \lambda^2 [2 - (CD_1 (a - \frac{\lambda a_s}{u_s}) - 2a_s)] - \frac{8C^2 D_2 a_\lambda^2 \lambda^3}{3u_s} [1 - 2(2CD_1 (a - \frac{\lambda a_s}{u_s}) - a_s)] \\
+ 3a_\lambda \lambda u_s (1 + 6a_s) \frac{-S^2 F^2 \lambda^3}{D_1} [1 - 2 (S E_1 (a - \frac{\lambda a_s}{u_s}) - a_s)] + \frac{S^2 E_1 \lambda \hat{g}}{D_1} [4SE_1 (a - \frac{\lambda a_s}{u_s}) - 1] \\
- \frac{3a_\lambda \lambda u_s (1 + 4a_s) \lambda}{4} + 3C^2 \lambda [\beta_2 D_2^2 + (\beta_1 - \beta_2) D_2] [4CD_1 (a - \frac{\lambda a_s}{u_s}) - 1] + 4C D_1 \lambda^2 [2(CD_1 (a - \frac{\lambda a_s}{u_s}) - a_s) - 1] \\
- \frac{2S C^2 F^2 a_\lambda \lambda^2 \lambda K_2}{3D_1 u_s} [1 - 2(2SE_1 (a - \frac{\lambda a_s}{u_s}) - a_s)] - \frac{2S C^2 F^2 a_\lambda \lambda^2 \lambda K_2}{3u_s} [1 - 2 (CD_1 + SE_1) (a - \frac{\lambda a_s}{u_s}) + 2a_s] \\
- \frac{C^2 S^2 F^2 a_\lambda \lambda^2 \lambda (\beta_2 D_2^3 + (\beta_1 - \beta_2) D_2)}{2D_1 u_s} [1 - 4(CD_1 (a - \frac{\lambda a_s}{u_s}) + SE_1 (a - \frac{\lambda a_s}{u_s})] \\
- \frac{C^2 S^2 F^2 a_\lambda \lambda^2 \lambda \hat{g}}{6u_s} [1 - 4 (CD_1 + SE_1) (a - \frac{\lambda a_s}{u_s})] \right\}, \tag{24}
\]

\[
\frac{d\beta_1}{dl} = \left\{ \frac{[D_1^2 - 2D_2 E_1 - (E_1^2 - E_2) D_2]}{(D_1^2 E_2 - E_1^2 D_2)^2} \beta_1 + \frac{[2D_1^2 - D_2]}{2\pi^2 (D_1^2 E_2 - E_1^2 D_2)^2} \left\{ \frac{S^2 \hat{g}^2}{4} [4CD_1 (a - \frac{\lambda a_s}{u_s}) - 1] \\
- \frac{E_1 \lambda^2 (1 + 4a_s)}{4} - 9S^2 [\beta_2 E_1^2 + (\beta_1 - \beta_2) E_2^2] [1 - 4SE_1 (a - \frac{\lambda a_s}{u_s}) - \frac{4S^2 \hat{g}^2 a_\lambda \lambda^2 (\beta_2 E_1^2 + (\beta_1 - \beta_2) E_2^2)}{3u_s} \\
\times [1 - 2 (2SE_1 (a - \frac{\lambda a_s}{u_s}) - a_s)] - \frac{4S C^2 F^2 a_\lambda \lambda^2 \lambda K_2}{3u_s} [1 - 2 (S E_1 (a - \frac{\lambda a_s}{u_s}) - 2a_s)] - \frac{2C^2 F^2 a_\lambda \lambda^2 \hat{g}}{3u_s} \\
\times [1 - 2 (CD_1 (a - \frac{\lambda a_s}{u_s}) - a_s)] \right\} - \frac{2C^2 \hat{g}^2}{2D_1^2 (D_1^2 E_2 - E_1^2 D_2)} [\beta_2 D_2^3 + (\beta_1 - \beta_2) D_2] \\
\times [1 - 2 (2CD_1 (a - \frac{\lambda a_s}{u_s}) - a_s)] + \frac{9C^2 [\beta_2 D_2^3 + (\beta_1 - \beta_2) D_2]}{4CD_1 (a - \frac{\lambda a_s}{u_s}) - \frac{1}{3}] \right\}, \tag{25}
\]

\[
\frac{d\beta_2}{dl} = \left\{ \frac{[E_1 D_1^2 - D_2 E_1^2]}{(D_1^2 E_2 - E_1^2 D_2)^2} \beta_2 + \frac{[2E_1^2 - D_2]}{2\pi^2 (D_1^2 E_2 - E_1^2 D_2)^2} \left\{ \frac{S^2 \hat{g}^2}{4} [4SE_1 (a - \frac{\lambda a_s}{u_s}) - 1] - \frac{D_1^2 \lambda^2 (1 + 4a_s)}{4} \\
+ 9C^2 [\beta_2 E_2^2 + (\beta_1 - \beta_2) D_2^2] [4CD_1 (a - \frac{\lambda a_s}{u_s}) - \frac{1}{3}] - \frac{4C^2 D_1 a \lambda^2 \lambda^2 [\beta_2 D_2^2 + (\beta_1 - \beta_2) D_2]}{3u_s} \\
\times [1 - 2 (2CD_1 (a - \frac{\lambda a_s}{u_s}) - a_s)] - \frac{4S C^2 \hat{g}^2 a_\lambda \lambda^2 \lambda K_2}{3u_s} [1 - 2 (SE_1 (a - \frac{\lambda a_s}{u_s}) - 2a_s)] \\
\times [1 - 2 (2SE_1 (a - \frac{\lambda a_s}{u_s}) - a_s)] - \frac{4S^2 \hat{g}^2 a_\lambda \lambda^2 \lambda K_2}{3u_s} [1 - 2 (SE_1 (a - \frac{\lambda a_s}{u_s}) - 2a_s)] \right\} \right\}, \tag{25}
\]
Here, we would like to stress that

\[ dλ_A \over dt = \lambda_0 + \mp 2 \left( -\lambda_0 + \left[ \lambda_0 + Y + \lambda_2 \right] \right), \]

\[ dκ \over dt = \kappa + \mp 2 \left( -\kappa + \left[ \kappa + \lambda_2 \right] \right), \]

\[ d\hat{g} \over dt = \hat{g} + \mp 2 \left( -\hat{g} + \left[ \hat{g} + \lambda_2 \right] \right), \]

where the variable functions are designated as

\[ D_1 \equiv |n_0|^2 \cos^2 \theta, \quad D_2 \equiv |n_0|^2 \sin^2 \theta, \quad E_1 \equiv |n_Y|^2 \sin^2 \theta, \quad E_2 \equiv |n_Y|^2 \sin^2 \theta, \quad \theta \equiv \frac{1}{|n_Y| \sin \theta} \].

Here, we would like to stress that $\theta \in [0, \pi/2]$, and $\theta = \pi/2$ serve as single magnetic order parameter with $Q_X$ or $Q_Y$, respectively.

For type-II case, at which $|n_i|^2 \neq |n_i|^4$ with $i = X, Y$, only one of $\beta_1$ and $\beta_2$ flows independently. In this circumstance, the flows of $a_s, a, u_s, λ, λ_0, κ$ share the same evolutions with their type-I counterparts. Nevertheless, the parameter $\hat{g}$ evolves under the following way

\[ d\hat{g} \over dt = \hat{g} + \mp 2 \left( -\hat{g} + \left[ \hat{g} + \lambda_2 \right] \right), \]
constant, $\beta_1$ interaction parameters (the qualitative results are insensitive to initial values of parameters): (a) (FIG. 5: (Color online) (a) Temperature-dependent stable constraints of $C_2$-symmetry ICS under representative starting values of interaction parameters (the qualitative results are insensitive to initial values of parameters): (a) $(g - \beta_2 + |\beta_1 - \beta_2|)/|\beta_1 - \beta_2|$ and (b) $\beta_1 - \beta_2$. Insets: sign-change region of $\beta_1 - \beta_2$ (left panel) and (b) behaviors around $t_c$ (right panel).

Further, the RG equations of parameters $\beta_1$ and $\beta_2$ can be broken down into six distinct sorts depending on the concrete conditions.

For type-II case-A with $|n_X|^2 = |n_X|^4$, $|n_Y|^2 = |n_Y|^4$, $|n^Y|^2 = 0$ and $|n^X|^2 \neq 0$, $\beta_1$ evolves but $\beta_2$ is an invariant constant,

$$\frac{d\beta_1}{dt} = \beta_1 + \frac{2}{\beta_2} \left\{ -\frac{4\lambda_2^2}{4 \beta_2^2 \beta_2^2} \left[ 1 - 2(\beta_2^2 - 1) \right] \right\} + \frac{4\beta_2^2}{3 \beta_2} \left[ 1 - 2(\beta_2^2 - 1) \right]$$
β₁ values of interaction parameters (the qualitative results are insensitive to the initial values). Insets: (a) sign-change region of β₁ - β₂ and (b) behaviors around l₁.

\[
- \frac{2S² F² a_s κ² ₂ \dot{g}}{3D₂ u_s} \{1 - 2(2Sₓ₁(a - \frac{λₐ₂}{u_s}) - a_s)\},
\]
\[
\frac{dβ₁}{dl} = 0.
\]

For type-II case-B with |nₓ²|² = |nₓ|⁴, |nᵧ²|² = |nᵧ|⁴, |nₓ²|² = 0 and |nᵧ²|² ≠ 0, β₁ evolves whereas β₂ is an invariant constant,

\[
\frac{dβ₂}{dl} = \frac{2}{2π²} \left\{ \frac{-4S² E₂² a_s λ² β₁}{u_s} \left[1 - 2(2Sₓ₁(a - \frac{λₐ₂}{u_s}) - a_s)\right] - 9S² E₂² β₁² \left[1 - 2(2Sₓ₁(a - \frac{λₐ₂}{u_s}) - 2a_s)\right]
\]
\[+ \frac{C² g²}{E₂} \left[4CD₁(a - \frac{λₐ₂}{u_s}) - 1\right] - \frac{4Sₓ₁ a_s λ²}{3u_s} \left[1 - 2(2Sₓ₁(a - \frac{λₐ₂}{u_s}) - 2a_s)\right] - \frac{λ²(1 + 4a_s)}{4}\]
\[− \frac{2C² F² a_s κ² ₂ \dot{g}}{3E₂ u_s} \{1 - 2(2CD₁(a - \frac{λₐ₂}{u_s}) - a_s)\}\right\}.
\]
\[
\frac{dβ₂}{dl} = 0.
\]

For type-II case-C with |nₓ²|² = |nₓ|⁴, |nᵧ²|² ≠ |nᵧ|⁴, |nₓ²|² = 0, and |nᵧ²|² = 0, β₂ evolves but β₁ is an invariant constant,

\[
\frac{dβ₁}{dl} = 0.
\]

\[
\frac{dβ₂}{dl} = \beta₂ + \frac{2}{2π²} \left\{ \frac{-4C² a_s λ² [β₂ E₂² + (β₁ - β₂)D₂]}{u_s} \left[1 - 2(2Sₓ₁(a - \frac{λₐ₂}{u_s}) - a_s)\right] - 9S² β₂ E₂² \left[1 - 4Sₓ₁(a - \frac{λₐ₂}{u_s})\right]
\]
\[+ \frac{C² g²}{E₂} \left[4CD₁(a - \frac{λₐ₂}{u_s}) - 1\right] - \frac{4Sₓ₁ a_s λ²}{3u_s} \left[1 - 2(2Sₓ₁(a - \frac{λₐ₂}{u_s}) - 2a_s)\right] - \frac{λ²(1 + 4a_s)}{4}\]
\[− \frac{2C² F² a_s κ² ₂ \dot{g}}{3E₂ u_s} \{1 - 2(2CD₁(a - \frac{λₐ₂}{u_s}) - a_s)\}\right\}.
\]

For type-II case-D with |nᵧ²|² = |nᵧ|⁴, |nₓ²|² ≠ |nₓ|⁴, |nₓ²|² = 0, and |nᵧ²|² = 0, β₂ evolves but β₁ is an invariant constant,

\[
\frac{dβ₁}{dl} = 0.
\]

\[
\frac{dβ₂}{dl} = \beta₂ + \frac{2}{2π²} \left\{ \frac{-4C² a_s λ² [β₂ D₂² + (β₁ - β₂)D₂]}{u_s} \left[1 - 2(2CD₁(a - \frac{λₐ₂}{u_s}) - a_s)\right] + 9C² β₂ D₂² \left[4CD₁(a - \frac{λₐ₂}{u_s}) - 1\right]
\]
\[+ \frac{S² g²}{D₁²} \left[4Sₓ₁(a - \frac{λₐ₂}{u_s}) - 1\right] - \frac{λ²(1 + 4a_s)}{4}\]
\[− \frac{4CD₁ a_s λ²}{3u_s} \left[1 - 2(2CD₁(a - \frac{λₐ₂}{u_s}) - 2a_s)\right]\right\}.
\]
FIG. 7: (Color online) Temperature-dependent stable constraints of $C_2$-symmetry ICS $\perp$ MH under representative starting values of interaction parameters (the qualitative results are insensitive to the initial values): (a) $\theta = \pi/12$, (b) $\theta = \pi/6$, and (c) $\theta = \pi/3$. Insets: enlarged regions for $\beta_1 - \beta_2$ (left panel) and $g_2/|\beta_1 - \beta_2|$ (right panel).

FIG. 8: (Color online) (a) Temperature-dependent stable constraints of $C_2$-symmetry SVC under representative starting values of interaction parameters (the qualitative results are insensitive to initial values of parameters). Inset: enlarge region for $g_2/|\beta_1 - \beta_2|$ and $(g_1 + |\beta_1 - \beta_2|)/|\beta_1 - \beta_2|$. (b) Sign-change regions at different values of $\theta$.

\[-\frac{2S^2g^2}{3D_1^2u_s}[1 - 2(2SE_1(a - \frac{\lambda a_s}{u_s}) - a_s)].\]  (40)

For type-II case-E with $|n^2_X| \neq |n^2_Y|$, $|n^2_X| \neq |n^2_Y|^4$, $|n^2_Y|^2 = 0$, and $|n^2_X|^2 = 0$, $\beta_2$ evolves but $\beta_1$ is an invariant constant,

$$\frac{d\beta_1}{dl} = 0, \quad \frac{d\beta_2}{dl} = \beta_2 + \frac{2}{2\pi^2}\left\{\frac{-4C^2a_s\lambda^2[\beta_2D_1^2 + (\beta_1 - \beta_2)D_2]}{u_s} [1 - 2(2CD_1(a - \frac{\lambda a_s}{u_s}) - a_s)] + 9C^2\beta_1^2D_1^2[4CD_1(a - \frac{\lambda a_s}{u_s}) - 1] + \frac{S^2g^2}{D_1^2}\{4SE_1(a - \frac{\lambda a_s}{u_s}) - 1\} - \frac{4C^2\lambda a_s^3}{3u_s}[1 - 2(2CD_1(a - \frac{\lambda a_s}{u_s}) - 2a_s)]\right\}. \]  (41)

For type-II case-F with $\mathcal{E}_2D_1^2 - D_2\mathcal{E}_1^2 = 0$, both $\beta_1$ and $\beta_2$ are energy-independent constants,

$$\frac{d\beta_1}{dl} = 0, \quad \frac{d\beta_2}{dl} = 0. \]  (43)
As aforementioned in the maintext, there are seven different types of incommensurate (IC) magnetic states other than three commensurate ones including stripe spin density wave (SDW), charge spin density wave (CSDW), and spin vortex crystal (SVC) \[10, 17, 34, 32\]. To be concrete, these IC magnetic states cover four different \(C_2\) symmetry (\(C_2\)) IC cases consisting of \(C_2\) IC stripe (ICS), \(C_2\) magnetic helix (MH), \(C_2\) IC magnetic stripe with perpendicular magnetic helix (ICS \(\perp\) MH), and \(C_2\) double parallel magnetic helix (DPMH), as well as three distinct \(C_4\)-symmetry (\(C_4\)) IC situations involving \(C_4\) IC CSDW, \(C_4\) IC SVC, and \(C_4\) IC spin-whirl crystal (SWC) \[19\]. In order to examine whether these IC magnetic states are stable against the decrease of energy scales, we within this section lean upon the coupled RG equations \[21\]–\[43\], which are completely encoded with the information of ordering competition, in conjunction with their stable constraints catalogued in Table II of the maintext.

In principle, the energy variable of RG evolution is expressed by \(\Lambda = \Lambda_0 e^{-l}\) with \(l > 0\) denoting the running scale. As our study is concerned with the structure of schematic phase diagram, it is herein of remarkable convenience to associate \(l\) with temperature via designating \(T = T_0 e^{-l}\) with \(T_0\) being the initial temperature to measure the evolution of energy scale \[18, 32\]. On the basis of this transformation and RG equations, we are now in a proper position to judge whether these IC magnets are good candidates residing in the phase diagram of Ba\(_{2-x}\)Na\(_x\)Fe\(_2\)As\(_2\) one by one.

We start out by considering the \(C_2\) IC magnetic states. On one hand, the configurations of spin vectors for \(C_2\) IC magnetic state read \(\mathbf{n}_X = (0, 0, 1)\) and \(\mathbf{n}_Y = (0, 0, 0)\) \[19\], which satisfy the restricted conditions.
of type-II case-A. This indicates the interaction parameters obey the RG evolutions of type-II case-A delineated in Eqs. (21), (24). (27), (28), and (29)-(33). As for $C_2$ ICS, its stable constraints can be either $(\beta_1 - \beta_2) < 0, \ g_2/|\beta_1 - \beta_2| > 0, (g_1 - \beta_2)/|\beta_1 - \beta_2| > -1$ or $(\beta_1 - \beta_2) < 0, \ g_2/|\beta_1 - \beta_2| < 0, (g_1 - \beta_2 - 0.9g_2)/|\beta_1 - \beta_2| > -1$ [18]. Based on these, we perform numerical RG analysis by taking some initial representative values of parameters and obtain the results shown in Fig. 4. On the other, concerning $C_2$ MH and $C_2$ DPMH, the configurations of spin vectors are characterized by $n_X = \frac{1}{\sqrt{2}} (i, 0, 1)/, \ n_Y = (0, 0, 0)$, and $n_Y = \frac{1}{\sqrt{2}} (i, 0, 1)$, respectively [19]. Accordingly, this indicates that the interaction parameters are dictated by the evolutions for type-II case-D provided in Eqs. (21), (24), (27), (28), (29), and (39)-(40). To proceed, we parallel the analogous RG numerical analysis taking advantage of the corresponding constraints [19] $(\beta_1 - \beta_2) > 0, \ g_2/|\beta_1 - \beta_2| > 0, (g_1 - \beta_2)/|\beta_1 - \beta_2| > 0$ or $(\beta_1 - \beta_2) > 0, \ g_2/|\beta_1 - \beta_2| < 0, (g_1 - \beta_2 - 0.9g_2)/|\beta_1 - \beta_2| < -1$ for $C_2$ MH and $(\beta_1 - \beta_2) > 0, \ g_2/|\beta_1 - \beta_2| < 0, (g_1 - \beta_2 - 0.9g_2)/|\beta_1 - \beta_2| < -1$ for $C_2$ DPMH, respectively. The conclusions are underscored in Fig. 5 and Fig. 6 with taking some representative beginning values of parameters. Learning from Fig. 4, Fig. 5, and Fig. 6 we apparently figure out that the sign change of $\beta_1 - \beta_2$ is occurred explicitly once temperature is slightly lowered owing to the effects of ordering competition. As a consequence, we infer that $C_2$ ICS, $C_2$ MH and $C_2$ DPMH are not stable states in the low-energy regime and hence not good candidates for IC magnetic state nearby the QCP in phase diagram of $Ba_{1-x}Na_xFe_2As_2$. However, these three $C_2$ IC states might be suitable states for the quantum critical region with high temperatures, such as $C_2$ IC SDW illustrated in Fig. 11.

In a sharp contrast, with respect to $C_2$ ICS $\perp$ MH, whose the configurations of spin vectors are related to $n_X = (0, 0, 1/\sqrt{2}) (i, 1, 0)$ [19], their interaction parameters are therefore subject to type-I coupled RG equations (21)-(29). Carrying out the similar numerical analysis gives rise to temperature-dependent evolutions depicted in Fig. 7. It manifestly heralds that stable constraints of $C_2$ ICS $\perp$ MH, i.e., $(\beta_1 - \beta_2) > 0, \ (g_1 - \beta_2)/|\beta_1 - \beta_2| < 0, \ g_2/|\beta_1 - \beta_2| > f(g_1 - \beta_2) \approx 2$ for a finite value of $g_1 - \beta_2$ and $\lim_{(g_1 - \beta_2) \to 0} f(g_1 - \beta_2) \to 0$ [19], are remarkably robust against ordering competition as the temperature is decreased. Despite of relative stability, it is very necessary to point out that $C_2$ ICS $\perp$ MH can be destroyed as long as the magnetic QCP is closely accessed, at which the ordering competition becomes so ferocious that any state cannot present solely.

Next, we go to judge $C_4$ IC magnetic states, which include $C_4$ IC SVC, $C_4$ SWC, and $C_4$ IC CSDW. In analogy to $C_2$ IC magnetic states, we inspect low-energy fates of these states by combining their RG equations and stable constraints. For $C_4$ IC SVC with the configurations of spin vectors being $n_X = (0, 0, 1)$ and $n_Y = (0, 1, 0)$ [19], the interaction parameters are governed by the type-II case-A RG equations (21)-(24), (27), (28), (32)-(34) and the stable constraints correspond to $(\beta_1 - \beta_2) < 0, \ g_2/|\beta_1 - \beta_2| > 0$, and $(g_1 - \beta_2)/|\beta_1 - \beta_2| < -1$ [19]. The numerical results presented in Fig. 8 reflect that $C_4$ IC SVC cannot be a well stable state in the phase diagram caused by the influence of ordering competition.

To proceed, we turn to $C_4$ IC SWC, which is well protected by constraints $(\beta_1 - \beta_2) > 0, \ (g_1 - \beta_2)/|\beta_1 - \beta_2| < 0$, and $0 < g_2/|\beta_1 - \beta_2| < 2$ [19]. In addition, the configurations of spin vectors are equivalent to $n_X = (i \cos \phi, 0, \sin \phi)$ and $n_Y = (0, i \cos \phi, \sin \phi)$. Be-
fore going further, it is of particular interest to address that they can be clustered into two sub-situations distinguished by the parameter $\phi$ which characterize the symmetric double-$Q$ noncoplanar SWC with $\phi = \pi/4$ and asymmetric double-$Q$ noncoplanar with $\phi \neq \pi/4$, respectively [19]. As a result, the former interaction parameters are dictated by type-II case-A RG equations (21)-(24), (27), (28), and (32)-(34) but instead the latter ones evolve under type-II case-F RG equations exhibited in Eqs. (21)-(24), (27), (28), (32), and (43). Carrying out analogous RG steps yields to Fig. 9 and Fig. 10 which explicitly signals $C_4$ SWC is not suitable to be present in the phase diagram.

Further, we move to $C_4$ IC CSDW state, at which the configurations of spin vectors are of the form $n_X = (0,0,1)$ and $n_Y = (0,0,1)$ [19], and thus type-II case-A RG equations (21)-(24), (27), (28), and (32)-(34) are in charge of the low-energy fates of interaction parameters. Hereby, it is necessary to highlight that $C_4$-symmetry IC CSDW [19] can be stabilized by either $(\beta_1 - \beta_2) < 0, g_2/|\beta_1 - \beta_2| < 0, (g_1 - \beta_2 - 0.9g_2)/|\beta_1 - \beta_2| < -1$ (case-1) or $(\beta_1 - \beta_2) > 0, g_2/|\beta_1 - \beta_2| < -1, (g_1 - \beta_2 - 0.9g_2)/|\beta_1 - \beta_1| < -1$ (case-2). Fig. 11 and Fig. 12 collect the central results stemming from RG analysis, which manifestly exhibit the temperature (energy) dependence of associated parameters for $C_4$ IC CSDW. In the light of these figures, we are informed that stable constraints for both case-1 and case-2 are considerably robust with the decrease of temperature, which of course can be sabotaged due to sufficiently strong fluctuations so long as the magnetic QCP is closely approached. Consequently, $C_4$ IC CSDW, like its $C_2$ ICS $\perp$ MH counterpart, is of fair robustness against ordering competition and an appropriate candidate for $C_4$ magnetic state in phase diagram of $\text{Ba}_{1-x}\text{Na}_x\text{Fe}_2\text{As}_2$. 

![Temperature-dependent stable constraints](image-url)