Guaranteed Phase & Topology Identification in Three Phase Distribution Grids

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Abstract—We present a method for joint phase identification and topology recovery from voltage measurements in unbalanced three phase radial networks. By recovering phases and topology jointly, we make use of all three phase measurements and can handle networks where some buses have only a subset of three phases. Our method is theoretically justified by a novel linearized formulation of unbalanced three phase power flow and makes precisely defined and reasonable assumptions on line impedances and load statistics. We connect our approach to the heuristics of prior work to explain why they often succeed. We validate our method on three IEEE test networks simulated under realistic conditions in OpenDSS, comparing our performance to the state of the art. In addition to providing a new method for phase and topology recovery, our intuitively structured linearized model will provide a foundation for future work in this and other applications.

I. INTRODUCTION

Topology identification determines the connectivity of network nodes from an available set of measurements. It is a vital task in the electric grid, especially in distribution networks [1], which may be switched between multiple operating configurations. Real-time topology awareness is critical to enabling most control and optimization approaches and is essential for detecting unintentional changes that can indicate faulty equipment or cyberattacks. In three phase electrical networks, phase identification, which determines the phase label of each phase at each bus, is one component of topology identification. Correct phase labels are important for several applications, including allocating resources to minimize phase imbalance [2]. Together, topology and phase identification determine the entire network connectivity.

The literature demonstrates several approaches to phase identification—many heuristic—on simulated or real data sets. In [3] and [4] power balance constraints on load measurements are used to identify customer phase connectivity; this approach suffers from multiple feasible solutions. Another class of techniques applies clustering to power or voltage measurements to categorize phases, as in [5] and [6]. Correlations are a popular clustering distance metric for voltage magnitudes [7], [8], [9]. In [10], voltage correlations are used with voltage angle differences to match phases at two buses. Though some of these methods achieve good performance, they don’t provide theoretical guarantees, stymieing understanding of why and how they work, and if and when they fail. For example, clustering algorithms such as k-means [11] can suffer from local minima that may result in incorrect solutions [12]. Further, a theoretical justification enables us to understand how load and network characteristics, such as the radial structure and line impedances, impact the success of an approach. [13] presents a maximum likelihood estimator for phase identification, but it requires knowledge of the grid topology and is non-convex.

Recently, topology identification has been extensively explored in the literature. Several authors use voltage correlations or covariances to recover topology, either through heuristic techniques [14], based on theoretically justified patterns in covariances [15], [16], or conditional independence tests [17], [18]. Another class of approaches estimates the entire network model, including topology and line impedances, by regridding on the relation between nodal voltage and current injections [19], [20], [21], that may be limited to terminal buses in radial grids [22], [23]. All these approaches assume that the system is balanced and can be approximated by a single phase network. This may lead to errors estimating unbalanced distribution networks. [24], [25] present three-phase topology estimation, possibly with incorrect phase labels, using conditional independence or mutual information tests. However the sample requirements of these methods often exceed those of direct/greedy methods such as [1]. Post-topology identification, [24] uses a distance-based metric or mixed-integer program to classify unknown phases. However, this still falls short of a theoretically guaranteed, polynomial-time algorithm for phase recovery.

Contribution: This work presents a provably correct, polynomial-time greedy algorithm for joint phase and topology identification in unbalanced, three phase radial networks using complex nodal voltage measurements. The algorithm is based on a multi-phase power flow model mapping nodal current injections to nodal voltage phasors, and depends on a radial structure and diagonal dominance of line impedance matrices to guarantee correctness. Our method is applicable to real networks, where some buses may have a subset of all three phases. When phases are known, our algorithm reduces to provable, greedy multi-phase topology learning that generalizes prior work [1] for the balanced setting. Similarly, when the topology is known apriori, it reduces to a local approach to phase identification. We validate our algorithm and demonstrate its improvement over prior work in both phase and topology recovery on multiple IEEE test networks, simulated in OpenDSS, an open source distribution system simulator [20]. The code-base of this work will be released publicly.

The paper is organized as follows. Section II presents a linearized model for unbalanced, three-phase networks with missing phases which is the theoretical basis of our work. Section III theoretically justifies a distance metric for phase identification while Section IV justifies one for topology.

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contains current injections at non-reference nodes. Let \( I = (1) \) correspond to the impedances of voltages differences to the reference voltage while \( \mathbf{V} \) contains voltages differences to the reference voltage while \( \mathbf{I} \) contains current injections at non-reference nodes. Let \( \mathcal{E}_i \) and \( \mathcal{E}_j \) denote the edge sets on the unique path in the radial system to \( r \) from nodes \( i \) and \( j \) respectively. The value of \( Z_{i,j} \) is given by:

\[
Z_{i,j} = \sum_{k,l \in (\mathcal{E}_i \cap \mathcal{E}_j)} z_{kl}.
\]

Thus, the elements of \( Z \) correspond to the impedances of common paths between node pairs and the reference \( [19], [1] \).

The unbalanced three phase model follows from the single phase one. To clarify definitions, we use ToyNet (Fig. 1b), a simple, unbalanced, three phase radial network, as a running example. We begin with the model for a multiphase line \( ij \), with phases \( \mathcal{M}_{ij} \subseteq \{a,b,c\} \). The voltage across \( ij \) is related to the current along each phase of the line by line impedance matrix \( \mathbf{Y}_{ij} \):

\[
\mathbf{I}_{ij} = \mathbf{Y}_{ij}(\mathbf{V}_i - \mathbf{V}_j)
\]

(2)

\( \mathbf{Y}_{ij} \) is the inverse of the \(|\mathcal{M}_{ij}| \times |\mathcal{M}_{ij}|\) line impedance matrix, \( \mathbf{Z}_{ij} \). Eq. (2) for line 56 in ToyNet is:

\[
\mathbf{Y}_{56} = \begin{bmatrix}
\mathbf{Y}_{56}^{ab} & \mathbf{Y}_{56}^{ac} & \mathbf{Y}_{56}^{bc}
\end{bmatrix}
\]

The node \( i \) current injection, denoted \( \mathbf{I}_i \), is a vector of injections on each phase of \( i \), and is given by the sum of line flows:

\[
\mathbf{I}_i = \sum_{j \in \mathcal{E}} \mathbf{I}_{ij}
\]

Building up from the current-voltage relations across individual lines in (2), the multi-phase voltages and currents injections across the network are related by \( \mathbf{I} = \mathbf{YV} \). Note this model can describe a network with a subset of phases at some nodes. \( \mathbf{Y} \) is the multi-phase system admittance matrix with dimensions \( (\sum_{i \in \mathcal{N}} |\mathcal{M}_{i}|) \times (\sum_{i \in \mathcal{N}} |\mathcal{M}_{i}|) \). The \( i,j \) block of \( \mathbf{Y} \) is

\[
\mathbf{Y}_{ij} = -\mathbf{Y}_{ji}, \quad \mathbf{Y}_{i,j} = \sum_{ij \in \mathcal{E}} \mathbf{Y}_{ij}
\]

Remark 1: \( \mathbf{Y} \) can be factored into an incidence matrix \( \mathbf{A} \), which captures the endpoints of each edge, and \( \mathbf{D} \), a block diagonal matrix of line admittances: \( \mathbf{Y} = \mathbf{A}\mathbf{D}^T \). \( \mathbf{D} \) has

II. UNBALANCED THREE PHASE MODEL

Before introducing our unbalanced three phase network model, we review the single phase balanced power flow model. In the single phase case, each line \( ij \) has an associated scalar admittance \( y_{ij} \) and impedance \( z_{ij} = y_{ij}^{-1} \). The single phase voltage phasors and current injections are related by the \(|\mathcal{N}| \times |\mathcal{N}|\) system admittance matrix \( \mathbf{Y} \) as \( \mathbf{I} = \mathbf{V} \mathbf{Y} \) with the form:

\[
\mathbf{Y}_{ij} = -\mathbf{Y}_{ji}, \quad \mathbf{Y}_{i,j} = \sum_{ij \in \mathcal{E}} \mathbf{Y}_{ij} \Rightarrow \mathbf{Y} = \mathbf{A}\mathbf{D}\mathbf{A}^T
\]

\( \mathbf{Y} \) can be factored into \(|\mathcal{N}| \times |\mathcal{E}|\) incidence matrix \( \mathbf{A} \) and diagonal line impedance matrix \( \mathbf{D} \) [24]. Without loss of generality, we choose all edges to be directed away from the network “root”, generally the point of common coupling (PCC) or substation. If edge \( ij \) is oriented from \( i \) to \( j \), the corresponding elements of \( \mathbf{A} \) are

\[
A_{i,ij} = 1, \quad A_{j,ij} = -1, \quad A_{r,ij} = 0 \quad \text{if} \quad r \neq i \neq j
\]

where \( A_{i,ij} \) is the \( i^{th} \) element of the column corresponding to edge \( ij \). By definition, as \( \mathbf{Y}1 = 0 \), \( \mathbf{Y} \) is not invertible. An invertible reduced admittance matrix, \( \hat{\mathbf{Y}} \), is constructed by choosing a reference node \( r \) and removing the corresponding row and column of \( \mathbf{Y} \). Since the system is lossless, its inverse relates voltages and currents as follows:

\[
\hat{\mathbf{Z}} = \mathbf{Y}^{-1}, \quad \hat{\mathbf{V}} = \hat{\mathbf{Z}} \hat{\mathbf{I}}
\]

(1)

\( \mathbf{V} \) contains voltages differences to the reference voltage while \( \hat{\mathbf{I}} \) contains current injections at non-reference nodes. Let \( \hat{\mathcal{E}}_i \) and \( \hat{\mathcal{E}}_j \) denote the edge sets on the unique path in the radial system to \( r \) from nodes \( i \) and \( j \) respectively. The value of \( \hat{Z}_{i,j} \) is given by:

\[
\hat{Z}_{i,j} = \sum_{kl \in (\hat{\mathcal{E}}_i \cap \hat{\mathcal{E}}_j)} \hat{z}_{kl}.
\]

Fig. 1: (a) Notation visualized for two nodes and connecting line. (b) ToyNet: A toy network with three, two, and one phase nodes used as a running example.
dimensions $(\sum_{ij \in \mathcal{E}} |M_{ij}|) \times (\sum_{ij \in \mathcal{E}} |M_{ij}|)$, with line admittance matrices $Y_{ij}$ along the diagonal. $\hat{A}$ is $(\sum_{i \in \mathcal{N}} |M_i|) \times (\sum_{i \in \mathcal{N}} |M_i|)$ dimensional. Its rows correspond to each bus, and columns to phases of each edge. With edges directed toward the root, assume edge $ij \in \mathcal{E}$ is oriented from $i$ to $j$. Then for every $ij \in \mathcal{E}$, with $\phi \in \mathcal{M}_{ij}$: $\hat{A}_{i,ij} = 1$, $\hat{A}_{j,ij} = -1$. All other elements of $\hat{A}$ are zero. $\hat{A}$ for ToyNet is visualized in Fig. 2.

A. Inverting the model

$\hat{Y}$ maps voltages to current injections, but we use the inverse mapping for phase and topology identification. By definition $\hat{Y}$ is singular. Again, a reduction denoted $\hat{Y}$ is invertible. To obtain $\hat{Y}$, we remove the three rows and corresponding to the three phases at reference node $r$. Then, $\hat{Y} = \hat{A} \hat{D} \hat{A}^T$ where $\hat{A}$ is obtained from $A$ by deleting the three rows corresponding to $r$. To derive $\hat{Y}$ inverse, we begin with the right pseudoinverse of $A^T$, which has the following properties.

Lemma 1: Let $\hat{B}$ be the right pseudoinverse of $A^T$, with rows corresponding to nodes and columns to edges. For $i \in \mathcal{N}$ with phases $M_i$, let $\mathcal{E}_i$ be the edge set of the unique path to $r$. Then,

$$B^\phi_{kl} = \begin{cases} -1 & \forall \phi = \psi \in M_i, \forall kl \in \mathcal{E}_i \\ 0 & \text{otherwise} \end{cases}$$

Proof: $B^\phi_{kl}$ is the row of $B$ corresponding to phase $\phi$ at node $i$, while $B^\phi_{ij}$ is the column corresponding to phase $\phi$ of edge $ij$. If $\phi \neq \psi$, $B^\phi_{ij} A^\psi_{kl} = 0$. A column of $A$ has only two nonzero elements, so for $\phi = \psi$, we have

$$B^\phi_{ij} A^\phi_{kl} = -1 \delta(ij \in \mathcal{E}_k) + 1 \delta(ij \in \mathcal{E}_l)$$

For any edge $kl \neq ij$, we will have either $(ij \in \mathcal{E}_k), (ij \in \mathcal{E}_l)$ or $(ij \notin \mathcal{E}_k), (ij \notin \mathcal{E}_l)$. Thus $B^\phi_{ij} A^\phi_{kl} = 0$ for $kl \neq ij$. If $kl = ij$, we have $(ij \notin \mathcal{E}_k), (ij \in \mathcal{E}_l)$ and $B^\phi_{ij} A^\phi_{kl} = 1$. Thus, $B^\phi_{ij} A^\phi_{kl} = 1$ iff $ij = kl, \phi = \psi \Rightarrow B^\phi_{ij} A^\phi_{kl} = 1$.

Now consider, $B^\phi_{ij} A^\psi_{kl}$, the inner product of rows. If $\phi \neq \psi$, this is 0. Consider when $\phi = \psi$ and $i \neq j$. If $j$ does not lie along the path from $i$ to the reference, then $jk \notin \mathcal{E}_i$, i.e.

there is no edge connected to $j$ in $\mathcal{E}_i$, and $B^\phi_{ij} A^\phi_{kl} = 0$. In contrast, if $j$ lies along the path from $i$ to the reference, there must be two edges $kj, jl \in \mathcal{E}_i$, oriented to and away from $j$ respectively, as the path passes through $j$. Then, $B^\phi_{ij} A^\phi_{kl} = (-1 \times -1) + (1 \times -1) = 0$. If $i = j$, only edge $il = jl \in \mathcal{E}_i$, and $B^\phi_{ij} A^\phi_{kl} = 1$. Therefore, $B^\phi_{ij} A^\phi_{kl} = 1$ iff $i = j, \phi = \psi$.

Therefore, $\hat{Y}^{-1} = \hat{Z}$ can be written as follows.

Theorem 1: The inverse of $\hat{Y}$ is given by:

$$\hat{Z} = \hat{Y}^{-1} = B\hat{D}^{-1}B^T$$

where $\hat{D}^{-1}$ is block diagonal matrix of line impedance matrices: $D^{-1}_{i,j} = Z_{ij}$. Further, the element of $\hat{Z}$ corresponding to phase $\phi$ at node $i$ and phase $\psi$ at node $j$ is given by:

$$Z_{ij}^\phi = \sum_{kl \in \mathcal{E}_i \cap \mathcal{E}_j} Z_{kl}^{\psi\phi}$$

Proof: Using the structure of $B$ from Lemma 1 we have $\hat{Z} = (B\hat{D}^{-1}B^T)(\hat{A} \hat{D} \hat{A}^T) = I$. Thus $\hat{Z} = \hat{Y}^{-1}$.

Consider the block of $\hat{Z}$ corresponding to nodes $i$ and $j$. Based on Lemma 1

$$Z_{ij}^\phi = \sum_{kl \in \mathcal{E}_i \cap \mathcal{E}_j} B_{i,kl}^{\phi\psi} B_{j,kl}^{\phi\psi} = \sum_{kl \in \mathcal{E}_i \cap \mathcal{E}_j} Z_{kl}^{\psi\phi}$$

Intuitively, (5) says that a change in current injection on phase $\psi$ at node $j$ will affect the voltage at phase $\phi$ at node $i$, proportional to the $(\phi\psi)$ impedance of the shared path $(\mathcal{E}_i \cap \mathcal{E}_j)$ from $i, j$ to $r$. In our definition, $\hat{Y}$ and $\hat{Z}$ are ordered with the phases of each node or edge grouped together. If all phases exist at all nodes, this is equivalent to a permutation of the three-phase model in [25], where entries for one phase across all nodes and edges are grouped together.

Using this theorem, in the unbalanced three phase model, voltages are related to currents as:

$$V = \hat{Z}A$$

Here, $V$ contains nodal voltage differences with the voltage for the matching phase at the reference: $V_\phi = V_\phi - V_{\phi^*}$. $I$ contains all current injections except at the reference. Note that there is no assumption for all nodes having all three phases. In the next section, we will use the model of (6) to determine patterns in voltage statistics to enable phase and topology recovery.

B. Current and Voltage Statistics

We treat voltages as random variables derived from current fluctuations via the model of (6). For our theoretical analysis, we assume the following for current injection statistics:

1) Current injections are uncorrelated across nodes and phases (including at a single node).

$$\text{cov}(i_\phi, i_{\phi'}) = 0 \text{ iff } (i = j) \cap (\phi = \psi)$$

2) Current injections have equal variance at all nodes.

$$\forall i, \phi: \text{var}(i_\phi) = s^2$$
Currents are predominantly driven by load behaviour which is largely uncorrelated over short time intervals on the order of seconds, so we can reasonably model current injections as uncorrelated across nodes and phases when using high resolution measurements such as from PMUs. We assume PMUs report at 120 Hz, but our methods are applicable if the measurement resolution is sufficient for fluctuations in currents (and hence voltages) to be de-trended to remove inter-nodal correlations. Equal variance of all injections is a stronger assumption, but is permissible in reasonably balanced networks. In a later section, we will consider how deviations from these assumptions impact recovery performance.

III. VOLTAGE COVARIANCE FOR PHASE MATCHING

Voltages covariances are informative for phase identification. Under Assumptions (7,8), the covariance of the voltage of phase $\phi$ at node $i$ and phase $\psi$ at node $j$, in the three-phase model (6), is given by:

$$
cov(v_i^\phi, v_j^\psi) = cov(\hat{Z}_i^\phi, \hat{Z}_j^\psi) = s^2 Re((\hat{Z}_i^\phi)^H \hat{Z}_j^\psi) \tag{9}
$$

We are interested in the sum of covariances for a particular phase ordering between nodes $i$ and $j$. Consider the case where the phases at $i$ are a subset of those at $j$ ($M_i \subseteq M_j$). Let $O$ denote the ordering/permutation of phases at $j$, where $O(\phi)$ denotes the specific phase at $j$ matched to the phase $\phi$ at $i$. Then, the covariance sum for matching $O$, denoted by $O_{ij}$, is:

$$
c_{ij}^O = \sum_{\phi \in P_i} cov(v_i^\phi, v_j^{O(\phi)}) \tag{10}
$$

Let $\hat{Z}_j^O$ denote the rows of $\hat{Z}_j$ corresponding to all phases at node $i$, and $\hat{Z}_i^O$ denote the rows corresponding to the phases at $j$ ordered according to $O$. Then $c_{ij}^O$ is:

$$
c_{ij}^O = s^2 Re(\text{vec}(\hat{Z}_i^O)^H \text{vec}(\hat{Z}_j^O)) = s^2 Re(\sum_{k \in N} \text{vec}(\hat{Z}_{ik}^O)^H \text{vec}(\hat{Z}_{jk}^O)) \tag{11}
$$

The last equality follows from (5). The contribution of a node $k$ to $c_{ij}^O$ is the dot product of the common path lengths between $i$, $k$, and $j$. The following result shows how $c_{ij}^O$ enables phase matching.

**Theorem 2:** Consider $c_{ij}^O$ given in (10) for $M_i \subseteq M_j$. If condition (12) holds for each pair of line impedance matrices, then $c_{ij}^O$ is maximized when $O$ corresponds to the correct phase matching between $i$ and $j$.

$$
\forall M \subseteq \{M_1, ..., M_n\}, \forall s, t, k, l \in E : \quad M = \arg \max_O Re(\text{vec}(Z_{st}^M)^H \text{vec}(Z_{kl}^{O(M)})) \tag{12}
$$

where $M$ ranges over every nodal phase set $M_i$. Then, the covariance sum for matching $O$ is:

$$
c_{ij}^O = \sum_{\phi \in P_i} cov(v_i^\phi, v_j^{O(\phi)}) = \sum_{\phi \in P_i} \mathbb{E}[(v_i^\phi - \mathbb{E}(v_i^\phi))^2] \tag{13}
$$

Lemma 2 establishes trends in $c_{ij}$ along one phase.

**Lemma 2:** Given the voltage on phase $\phi$ at node $i$:

$$
\arg \min_j d_{ij}^\phi = \arg \min_j \sum_{\phi \in M_i} \mathbb{E}[(v_i^\phi - v_j^\phi)^2] \in \text{Parent/Child of } i \tag{14}
$$

**Proof:** Expanding the difference, we obtain:

$$
d_{ij}^\phi = \sum_{\phi \in M_i} \mathbb{E}[(v_i^\phi - v_j^\phi)^2] = \sum_{\phi \in M_i} \sum_{n \in N} s_n^2 \|Z_{in}^\phi - Z_{jn}^\phi\|^2 \tag{15}
$$

If the paths from nodes $k$ and $l$ to $r$ merge at node $n$, $\hat{Z}_{kl}^\phi = e_n^{\phi},$ the impedance of the path from $n$ to $r$ along phase coupling $\phi, \psi$:

$$
e_n^{\phi \psi} = \sum_{ij \in E_n} Z_{ij}^{\phi \psi} \tag{16}
$$

To determine the minimizer of (14), consider two cases visualized in Fig. In case A, $j$ is the common ancestor of node $i$, $k$ on the path to the root. In case B, $j$ is an ancestor of $i$, while $k$ is an ancestor of $j$. In both cases, we show that $d_{ij}^\phi < d_{ik}^\phi$ for $d_{ik}^\phi$. Put together, for a given $i$, the minimizer $j$ of $d_{ij}^\phi$ is either the parent or child of $i$.

**Case A:** We split the sum in (15) into the regions $N_i$ in Fig.
A similar analysis shows $d^\phi_{ik} - d^\psi_{ij} > 0$. 

**Case B.** Now we split (15) over the regions in Fig. 3b. Using (16), we have 

$$d^\phi_{ik} - d^\psi_{ij} = \sum_{n \in N_1, \psi \in M_n} 0 + \sum_{n \in N_2, \psi \in M_n} 0 + \sum_{n \in N_4, \psi \in M_n} 0 + \sum_{n \in N_3, \psi \in M_n} s_n^2 \left( |e_{ij} - e_k|^2 - |e_{ij} - e_k|^2 \right) > 0$$

A similar argument shows $d^\phi_{ik} - d^\psi_{kj} > 0$.  Thus the minimum is given by the parent/child of $i$. 

**Theorem 3:** Given node $i$, the node $j$ which minimizes $d_{ij}$ in (13) is either a parent or child of $i$. If phases at each node are known, Thm. 3 enables correct topology recovery with a greedy algorithm based on $d_{ij}$. Note that Lemma 2 and Thm. 3 hold for all uncorrelated injections even with unequal variances. Thus, Assumption 8 can be relaxed for topology learning. We now have the tools for joint phase and topology recovery, detailed in the next section.

**V. JOINT PHASE & TOPOLOGY IDENTIFICATION**

We propose a greedy algorithm for joint phase and topology identification using the nodal voltages properties derived in Sections III [V]. The pseudo-code is shown in Algorithm 1. We maximize $c^\phi_{ij}$ (9) for phase matching (Theorem 2), and minimize $d_{ij}$ (13) for topology recovery (Theorem 3). The algorithm greedily builds a tree, with node set $T$, from starting node $i_0$. In each iteration, a new node is added to the tree by choosing the node $i \in T$, which minimizes $d_{ij}$ for all $j \in T$, using genNext algorithm. $d_{ij}$ is computed based on the phase matching between $i$ and $j$ that maximizes $c^\phi_{ij}$. The algorithm iterates until all nodes have been added to the tree.

**Algorithm 1** Greedy algorithm for topology and phase matching

**Input:** Multi-phase voltage time series for all nodes in $V$.  
**Output:** Set of edges in network, phase ordering of each node.

1: $N_3, N_2, N_1 \leftarrow$ set of three, two, one phase nodes in $V$. 
2: for all $i \neq j \in N$ do
3: \hspace{1em} $M[i, j] \leftarrow \arg \max c^\phi_{ij}$ in (10) \hspace{1em} \triangleright Initial phase matching 
4: \hspace{1em} $D[i, j] \leftarrow d_{ij}$ in (15) with matched $M[i, j]$ 
5: \hspace{1em} end for
6: \hspace{1em} $T \leftarrow \{i_0\}, N_3 \leftarrow N_3 \setminus i_0, P[i_0] \leftarrow [a, b, c]$ \hspace{1em} \triangleright Add first three phase node to tree $T$ and set phases 
7: while $N_3 \neq \emptyset$ do \hspace{1em} \triangleright Connect 3 phase nodes 
8: \hspace{1em} $i, j \leftarrow \text{genNext}(D, T, N_3)$ 
9: \hspace{1em} $T \leftarrow T \cup i, N_3 \leftarrow N_3 \setminus i, E \leftarrow E \cup e_{ij}, P[j] \leftarrow M[i, j]$ 
10: \hspace{1em} end while
11: while $N_2 \neq \emptyset$ do \hspace{1em} \triangleright Connect 2 phase nodes 
12: \hspace{1em} $i, j \leftarrow \text{genNext}(D, T, N_2)$ 
13: \hspace{1em} $T \leftarrow T \cup i, N_2 \leftarrow N_2 \setminus i, E \leftarrow E \cup e_{ij}, P[j] \leftarrow M[i, j]$ 
14: \hspace{1em} end while
15: while $N_1 \neq \emptyset$ do \hspace{1em} \triangleright Connect 1 phase nodes 
16: \hspace{1em} $i, j \leftarrow \text{genNext}(D, T, N_1)$ 
17: \hspace{1em} $T \leftarrow T \cup i, N_1 \leftarrow N_1 \setminus i, E \leftarrow E \cup e_{ij}, P[j] \leftarrow M[i, j]$ 
18: \hspace{1em} end while

**Algorithm 2** genNext algorithm called in Alg. 1

**Input:** Pairwise distances in $D$.  
**Output:** $i \in N$ to be added, $j \in T$ connected to $i$.

1: $d_{ij} \leftarrow \infty, i \leftarrow \text{None, } j \leftarrow \text{None}$ 
2: for all $b \in T, a \in N$ do 
3: \hspace{1em} if $D[a, b] < d_{ij}$ then 
4: \hspace{2em} $d_{ij} \leftarrow D[a, b], i \leftarrow a, j \leftarrow b$ 
5: \hspace{1em} end if 
6: \hspace{1em} end for

Algorithm 1 adds 3 phase, then 2 phase, then 1 phase nodes to the tree. Therefore, the starting node must be three phase, making the reference an intuitive choice. By adding nodes in order of number of phases, the algorithm implicitly enforces the crucial fact that the number of phases never increases moving from the substation to the network ends; i.e. a single phase node is never the parent of a three phase node. This approach hence avoids issues that can arise when applying a
naive greedy algorithm to a network with a variable number of phases at each node. For example, suppose we are recovering the topology of ToyNet. All nodes have been added to $T$ except 6, 7, and 8. To recover the correct topology, we should connect node 6 to 5 first. Then nodes 7 and 8 will get connected to 6 naturally, as $d_{76} < d_{75}$ and $d_{86} < d_{85}$. However, consider $d_{65}$ and $d_{75}$,

$$d_{65} = \text{var}(v_6^0 - v_5^0) + \text{var}(v_6^0 - v_5^0), \quad d_{75} = \text{var}(v_7^0 - v_5^0)$$

We have no guarantee that $d_{65} < d_{75}$ due to the presence of additional phase variance in $d_{65}$. A naive algorithm that does not consider nodes ordered by decreasing number of phases may result in an incorrectly identified topology, unlike our algorithm.

### A. Alternative Estimation Scenarios

Algorithm 1 recovers both phase and topology from voltage measurements. This is the most general scenario, but our theoretical results also establish estimation algorithms for restricted settings. We discuss them now.

1) **Phase Identification with Topology Information**: If topology is known, phases can be identified by greedily matching adjacent nodes through the network using the distance $c_{kl}^{(9)}$ across edges $ij \in E$. Unlike the algorithms of [5], [6], [7], [8], [9], which cluster all nodal voltage measurements to recover phase, our approach is highly local. In [7], [9], [28] the Pearson correlation coefficient of voltages is the distance for phase matching, which is related to the covariance but not theoretically justified. K-means is a popular clustering algorithm choice [6]. However, even if the correct phase matching is the globally optimal solution of the k-means cost, the optimization is non-convex and may not converge to the global minima. Our greedy approach, however, is guaranteed to result in the correct solution.

2) **Topology Estimation with Phase Information**: If all phase labels are known, the distance $d_{ij}^{(13)}$ can be directly minimized to recover topology. This is a greedy spanning tree learning algorithm that generalizes prior work for the single phase case [1]. Compared to algorithms in [24], [25] that use conditional independence tests and need matrix inversions, our algorithm has improved sample performance, as demonstrated in Section VI.

3) **Estimation using voltage magnitudes only**: While Algorithm 1 is based on nodal voltage phasors, it can also be applied to voltage magnitudes $v_i = |v_i^0|$. To theoretically justify this, we use a linearization of (4) for line $kl$:

$$I_{kl} = Y_{kl} \begin{bmatrix} e^{j\theta^0} (v_k^0 e^{j\theta_k^0} - v_l^0 e^{j\theta_l^0}) \\ e^{j\phi} (v_k^0 e^{j\theta_k^0} - v_l^0 e^{j\theta_l^0}) \\ e^{j\psi} (v_k^0 e^{j\theta_k^0} - v_l^0 e^{j\theta_l^0}) \end{bmatrix}$$

$$\approx Y_{kl} D_r ((V_k - V_l) + j(\theta_k - \theta_l))$$

where $\theta^0$ is the phase $\phi$ reference angle and $\theta_k^0 = \theta_k^0 - \theta^0$. The linearization assumes small magnitude deviations from the reference and small angle difference between neighboring nodes. Properties of voltage magnitudes across the network can then be derived under assumptions on $Y_{kl} D_r, I_{kl}$ to justify Algorithm 1 for phase and topology recovery.

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**VI. SIMULATION EXPERIMENTS**

We now present simulation results on our algorithm. To quantify the performance, we measure average errors in phase and topology recovery, normalized to the total number in the grid. They are defined as:

$$\text{Topology Error} = \frac{\text{wrong edges} + \text{missing edges}}{\text{total edges}}$$

$$\text{Phase Error} = \frac{\text{wrong nodal phases}}{\text{total nodal phases}}$$

Further, we evaluate the algorithm’s sensitivity to the following parameters.

- **Measurement noise**: We add white noise $n$ to original measurement $v_i$: $\tilde{v}_i = v_i + n$. The noise level is quantified by inverse SNR: $\text{SNR}^{-1}(\tilde{v}_i) = \text{var}(n) / \text{var}(v_i)$. As we use voltage covariances, our method depends on relative precision and not on absolute accuracy. This makes it immune to the stable transducer errors that can afflict distribution PMU data [29].

- **Number of measurement samples**: Assuming 120 Hz measurements from distribution PMUs [30], we record performance using voltages collected over over 1 second to 1
Fig. 5: Reconstructions across all $SNR^{-1}$ and samples in Fig. 4. Opacity indicates the number of times an edge was recovered across trials. Red lines are true edges. Result on voltage phasor measurements $V$ and magnitudes $\|V\|$ are shown.

Fig. 6: Topology recovery error as injections become increasingly correlated. Line is mean, and bands show min-max error range.

- **Load Correlations:** We test our algorithm’s sensitivity to the assumption of uncorrelated injections, by varying the correlations of the loads while maintaining their variance. This is done by changing the covariance matrix $\Sigma = \sigma^2((1 - \epsilon)I + \epsilon 11^T)$. As $\epsilon \rightarrow 1$, injections become more correlated.

Three IEEE distribution test networks, with 13, 34, and 37 buses, are simulated in OpenDSS. The 13 and 34 bus networks have some one and two phases buses, while the 37 bus network has all three phase buses [31]. We modify the models by adding loads at every bus, and by disabling voltage regulators, which invalidate the assumption of voltages driven by injections. We fluctuate the load injections at each phase at each bus, and simulate the network to obtain non-linear voltages.

Fig. 4 shows topology and phase recovery accuracy for three $SNR^{-1}$ levels ranging from 0 (no noise) to 10, with one second to one minute of measurements. PMUs are highly precise; $SNR^{-1}$ would realistically be $\sim 0.001$ [32], [33]. Nevertheless, our method performs well under more noise as measurement samples increase. For all test networks, we achieve perfect topology recovery from voltage magnitudes for $SNR^{-1}$'s 0 and 0.001 across measurement durations. Fig. 5 visualizes the recovered topologies across different trials. Note that the errors are localized to a few nodes, and lower for voltage magnitudes.

Fig. 6 shows topology recovery sensitivity as injections stray from the uncorrelated assumption (7). Error increases rapidly as loads become more correlated. In reality, over short time durations, it is reasonable to assume that injections will be uncorrelated across nodes or can be de-trended [24]. We use at most one minute of data to recovery phase and topology: short enough that the uncorrelated assumption should hold well.

Finally, we compare our method performance to the prior work. We compare topology recovery to that of Liao et al. [24], which uses a mutual information distance metric that is immune to incorrect phase labels. Fig. 7 shows our approach outperforms Liao across test networks, SNRs, and sample durations. We compare phase identification to that of Olivier et al. [7], which uses a normalized correlation distance metric in constrained clustering. Both algorithms achieve zero error on the 34 and 37 bus networks across SNRs and sample durations, but our method performs better on the 13 bus network.

Our polynomial time algorithms are suitable for real time application, taking on the order of seconds to recover phase and topology for the IEEE test cases shared here. On the largest
37 bus test case, the algorithm completes in 15 seconds.

VII. CONCLUSION

We present an algorithm for joint phase and topology identification in unbalanced three phase networks where individual buses can have single, two, or three phase voltages. Our polynomial time approach relies on analytical trends in nodal voltages from three reasonable assumptions on load statistics and line impedances, demonstrating its efficacy on non-linear voltages from three test feeders simulated under realistic conditions. We find our algorithm is robust to measurement non-idealities, and outperforms the prior work in both phase and topology recovery.

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