Beyond Schwarzschild–de Sitter spacetimes: I. A new exhaustive class of metrics inspired by Buchdahl for pure $R^2$ gravity in a compact form

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Some sixty years ago Buchdahl pioneered a program in search of static spherically symmetric metrics for pure $R^2$ gravity in vacuo (Nuovo Cimento, Vol 23, No 1, pp 141-157 (1962); https://link.springer.com/article/10.1007/BF02733549 [1]). Surpassing several obstacles, his work culminated in a non-linear second-order ODE which required being solved. However Buchdahl deemed the ODE intractable and abandoned his pursuit for an analytical solution. We have finally managed to overcome this remaining hurdle and bring his program to fruition.

Reformulating Buchdahl’s ODE, we obtain a novel class of metrics (which we shall call the Buchdahl-inspired metrics hereafter) in a compact and transparent expression:

$$ds^2 = e^{k r q(r)} \left\{ p(r) \left[ \frac{-q(r)}{r} dt^2 + \frac{r}{q(r)} dr^2 \right] + r^2 d\Omega^2 \right\}$$

in which the pair \( \{p, q\} \) are two functions of the radial coordinate \( r \) obeying the evolution rules

$$\frac{dp}{dr} = \frac{3 k^2 p}{4 r^2 q^2}$$
$$\frac{dq}{dr} = \left( 1 - \Lambda r^2 \right) p$$

and the Ricci scalar is

$$R(r) = 4 \Lambda e^{-k r} \left[ \frac{q}{r} \right]$$

We are able to verify ex post, via direct inspection, that the metric given above satisfies the $R^2$ vacuo field equation

$$R \left( R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \right) + (g_{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu}) R = 0$$

hence establishing its validity. The compact form above casts the Buchdahl-inspired metric in a parallel resemblance with the classic Schwarzschild–de Sitter (SdS) metric, with the case \( k = 0 \) corresponding to the SdS metric.

We show why the Buchdahl-inspired metric, which exhibits non-constant scalar curvature when \( k \neq 0 \), defeats a “no-go” theorem proved in [2] which posits that pure $R^2$ gravity vacua are restricted to the Einstein spaces, $R_{\mu\nu} = \Lambda g_{\mu\nu}$, and the vanishing Ricci scalar spaces, $R = 0$. The aforementioned “no-go” theorem assumes a rapid asymptotic falloff for the metric as $r \to \infty$. However, we find that the Buchdahl-inspired metric evades that central assumption, which is overly restrictive.

A product of a fourth-derivative gravity, a Buchdahl-inspired metric is specified by 4 parameters: $\Lambda$ measuring the scalar curvature at largest distances, $k$ effecting the variation of the curvature on the manifold, and \( \{p_0, q_0\} \) initiating the “evolution” of \( \{p(r), q(r)\} \) along the radial direction, forming a two-dimensional phase space. The class of Buchdahl-inspired metrics is exhaustive as it covers all “nontrivial” static spherically symmetric metrics admissible for pure $R^2$ gravity in vacuo, with the SdS metric being a special case, $k = 0$. Transparently, the quartet \( \{\Lambda, k, p_0, q_0\} \) spans a topological space with all members in the class of Buchdahl-inspired metrics being smoothly connected to the SdS metrics when $k$ is continuously tuned to 0. In this respect, the Buchdahl-inspired metrics constitute a natural enlargement suitably regarded as a framework “beyond Schwarzschild–de Sitter”.

Our novel solution thereby completes Buchdahl’s six-decades-old program. We also explore the mathematical properties of the Buchdahl-inspired metric in the limit of small $k$ and in the region around the coordinate origin.

I. MOTIVATION

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In a seminal paper entitled “On the Gravitational Field Equations Arising from the Square of the Gaussian Curvature” completed in 1961 [1], Hans A. Buchdahl pio-
neered – yet left unfinished – a program to seek static spherically symmetric metrics for pure $R^2$ gravity in vacuo, a theory that excludes the Einstein-Hilbert term at the outset. Back in his time, Buchdahl was motivated to consider the pure $R^2$ action as an interesting prototype for modified gravity. Recently, the quadratic action has witnessed resurgence $[2][3]$; one attractive feature of the pure $R^2$ action is that it is the only theory that is both ghost-free and scale invariant $[10]$.

Despite making significant progress, unfortunately, Buchdahl discontinued his efforts toward the finish line that was within striking distance. The purpose of our current paper is to bridge the final remaining gap in Buchdahl’s “abandoned” program. The ultimate outcome is a family of static spherically symmetric vacua, expressible in a compact form, for the pure $R^2$ action. We shall focus on the mathematical aspects of these vacua in this paper, while leaving their potential implications in physics for future research.

As Buchdahl indicated therein $[1]$, if one were to adopt the canonical metric using Schwarzschild coordinates

$$ds^2 = -A(r) dt^2 + B(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(1)

then from the $R^2$ vacuo field equation one would confront a coupled system of two equations for $A(r)$ and $B(r)$, one of fourth- and one of third-differential orders. Eliminating one of the two functions would yield a highly non-linear seventh-order ordinary differential equation (ODE).

Nevertheless, Buchdahl was able to devise a judicious choice for the metric alternative to (1), such that the resulting ODE – albeit non-linear – is only of second differential order which remained to be solved. For the reader’s convenience, the original Buchdahl equation (as we shall call it as such, hereafter) is

$$2t \frac{d^2 q}{dt^2} + \left( \frac{1 + t}{1 - t} - \frac{3}{4} \frac{k^2}{q^2} \right) \frac{dq}{dt} = 0$$

(2)

see Eqs. (1.7) and (3.4) in his original paper $[1]$. The metric he chose is then expressible in terms of the function $q(t)$ [N.B.: $t$ is not the time coordinate], with the (Buchdahl) parameter $k$ rendering Eq. (2) non-linear.

The Buchdahl equation (2) is very generic; it captures all “nontrivial” static spherically symmetric vacua admissible for the pure $R^2$ action, besides the Einstein field equation. (viz. $R_{\mu\nu} = \Lambda g_{\mu\nu}$) and the vanishing scalar curvature spaces (viz. $R = 0$). Accordingly, if an analytical solution to the Buchdahl equation can be found, then it would yield a powerful tool to tackle the new physics inherent in pure $R^2$ gravity $[11][13]$. Crucially, as shall be shown in this paper, the new (Buchdahl) parameter $k$ in Eq. (2) would enable the $R^2$ vacua to develop non-constant scalar curvature.

By and large, the Buchdahl equation was an impressive achievement. Yet Buchdahl abandoned his pursuit for an analytical solution as he judged his ODE intractable. 1. This is an unfortunate twist of events as we find that this is not the case. In this paper, we shall advance a number of mathematical maneuvers to reformulate the Buchdahl equation (2) in a more accessible form. From there, we are able to obtain a compact expression for a new class of metrics which we shall call the Buchdahl-inspired metrics, thereby bringing his six-decades-old endeavor to a successful outcome.

For the reader’s convenience, we shall briefly present our result in what follows. The Buchdahl-inspired metric is neatly expressible as

$$ds^2 = e^{\frac{k}{2} \frac{dr}{r}} \left\{ p(r) \left[ -\frac{q(r)}{r} \frac{dt^2}{r^2} + \frac{r}{q(r)} dr^2 \right] + r^2 d\Omega^2 \right\}$$

(3)

with the Ricci scalar equal

$$R(r) = 4\Lambda \exp \left( -k \int \frac{dr}{r q(r)} \right)$$

(4)

and the two auxiliary functions $p(r)$ and $q(r)$ evolving along the radial direction $r$ per

$$\frac{dp}{dr} = \frac{3k^2 p}{4r^2 q^2}$$

(5)

$$\frac{dq}{dr} = \left( 1 - \Lambda r^2 \right) p$$

(6)

The deliberate resemblance of Eq. (3) to a Schwarzschild–de Sitter (SdS) metric makes the meaning of terms transparent. The compact form (3)–(6) automatically encompasses the constant-curvature SdS when $k$ equal zero, in which case the non-linear and singular relation in (5) stays silent. A non-zero $k$, however, would trigger an interplay between $p$ and $q$ via (5) and (6), in which case the Buchdahl-inspired metric acquires a non-constant scalar curvature per $[1]$, potentially offering a host of intricate phenomenology and new physics.

Our paper is organized as follows. In Sec. [1] we shall rework Buchdahl’s original paper in a simplified and straightforward approach. Our two-fold aim is to

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1 To quote Buchdahl from his original paper (with notes in square brackets ours). In page 4 of [1]: “Unfortunately the simple appearance of the non-linear second order ODE is deceptive. The best I have been able to achieve is to obtain a solution in the form of a sequence of polynomials of ascending powers of $t$.”

2 No further attempts either by Buchdahl or by others have been made to solve his ODE since its publication.

3 A fact to be shown in Sec. [VI]
derive the results directly from the $R^2$ field equation, and to arrive at an ODE which is more generic than his original ODE. In Sec. [IV] we shall introduce a short-cut towards the (generalized) Buchdahl equation while circumventing his original Hamiltonian-based procedure. In Sec. [V] we shall cast his equation in a more transparent way; then obtain a compact solution describing the new class of Buchdahl-inspired metrics. In Sec. [VI] we shall outline the verification process which confirms the validity of our Buchdahl-inspired metrics. Between Secs. [VI] and [IX] we shall investigate the Buchdahl-inspired metrics in 4 situations: (i) recovering the SdS metric at $k = 0$, (ii) deriving a new metric for the small-$k$ limit; (iii) probing the behavior of the metrics around the coordinate origin; and (iv) uncovering a degeneracy in the overall solution. Section X points out an overly restrictive assumption in a proof proposed in [2] against the existence of non-constant curvature metrics (and the class of Buchdahl-inspired metrics). Sec. [XI] summarizes our work.

II. GENERALIZING THE BUCHDAHL EQUATION: A MORE DIRECT ROUTE

In his original work [I], Buchdahl followed an arduous route. He designed a new Lagrangian, as a “surrogate” to the pure $R^2$ gravity action, then applied the variational principle on it. With the benefits of hindsight, we shall rework Buchdahl’s formulation in a more straightforward manner. We shall start directly from the $R^2$ vacuo field equation, conduct the standard calculations, and reach the generalized Buchdahl equation. We shall try to retain as much as possible Buchdahl’s notation for the reader’s convenience.

Following Buchdahl’s notation, the metric in spherical coordinate is written in the form
\[
d s^2 = -e^{\nu(r)} d t^2 + e^{\lambda(r)} d r^2 + e^{\mu(r)} d \Omega^2
\]
(7)
\[
d \Omega^2 = d \theta^2 + \sin^2 \theta d \phi^2
\]

The vacuo field equation in the pure $R^2$ action is
\[
R \left( R_{\mu \nu} - \frac{1}{4} g_{\mu \nu} R \right) + (g_{\mu \nu} \Box - \nabla_\mu \nabla_\nu) R = 0
\]
(8)
and the “trace” equation in vacuo is
\[
\Box R = 0
\]
(9)
Since $R$ is a function of $r$ only, we have
\[
\nabla_\mu \nabla_\nu R = \partial_\mu \partial_\nu R - \Gamma^r_{\mu \nu} \partial_r R
\]
(10)

The $tt$, $\theta \theta$-, and $rr$- components of the vacuo field equation [8] read
\[
R_{tt} - \frac{1}{4} g_{tt} R = -\Gamma^r_{tt} \frac{R'}{R}
\]
(11)
\[
R_{\theta \theta} - \frac{1}{4} g_{\theta \theta} R = -\Gamma^r_{\theta \theta} \frac{R'}{R}
\]
(12)
\[
R_{rr} - \frac{1}{4} g_{rr} R = -\Gamma^r_{rr} \frac{R'}{R} + \frac{R''}{R}
\]
(13)

The relevant Christoffel symbols and components of the Ricci tensors are
\[
\Gamma^t_{tt} e^{\lambda - \nu} = \frac{\nu'}{2}
\]
(14)
\[
\Gamma^\theta_{\theta \theta} e^{\lambda - \mu} = -\frac{\mu'}{2}
\]
(15)
\[
\Gamma^r_{rr} = \frac{\lambda'}{2}
\]
(16)
and
\[
R_{tt} e^{\lambda - \nu} = \frac{\nu''}{2} + \frac{\nu^2}{4} - \frac{\nu' \lambda'}{4} + \frac{\nu' \mu'}{2}
\]
(17)
\[-R_{\theta \theta} e^{\lambda - \mu} = -e^{\lambda - \mu} + \frac{\mu''}{2} + \frac{\mu^2}{4} + \frac{\nu' \mu'}{2} - \frac{\lambda' \mu'}{4}
\]
(18)
\[-R_{rr} = \frac{\nu''}{2} + \frac{\nu^2}{4} + \mu'' - \frac{\mu^2}{4} - \frac{\nu' \lambda'}{4} - \frac{\lambda' \mu'}{2}
\]
(19)

Furthermore, the Jacobian is
\[
\sqrt{-g} \triangleq \sqrt{-\det g} = e^{\frac{\nu}{2} + \frac{\lambda}{2} + \mu} \sin \theta
\]
(20)
giving
\[
\sqrt{-g} g^{rr} = e^{\frac{\nu}{2} + \frac{\lambda}{2} + \mu} \sin \theta
\]
(21)
The three functions $\nu(r)$, $\lambda(r)$, $\mu(r)$ are subject to an arbitrary coordinate transform. Buchdahl made a judicious choice that
\[
\mu(r) \equiv \frac{1}{2} (\lambda(r) - \nu(r))
\]
(22)
thus making
\[
\sqrt{-g} g^{rr} = \sin \theta
\]
(23)
The “trace” equation [9] \footnote{Recall that for a scalar field $\phi$: $\Box \phi = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu \nu} \partial_{\nu} \phi)$.} is vastly simplified to
\[
R'' = 0
\]
(24)
and
\[
R = \Lambda + k r
\]
(25)

4 Recall that for a scalar field $\phi$: $\nabla_\mu \nabla_\nu \phi = \partial_\mu \partial_\nu \phi - \Gamma^r_{\mu \nu} \partial_r \phi$.  
5 Recall that for a scalar field $\phi$: $\Box \phi = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu \nu} \partial_{\nu} \phi)$. 
6 Recall that for a scalar field $\phi$: $\Box \phi = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu \nu} \partial_{\nu} \phi)$. 


in which \( A \) and \( k \) are 2 constants. If \( k = 0 \) the Ricci scalar is a constant everywhere. For \( k \neq 0 \) the Ricci scalar deviates from constancy.

With Buchdahl’s choice (22), the relevant Ricci components become:

\[
\mathcal{R}_{tt} = \frac{\nu''}{2} e^{\nu - \lambda}
\]

\[
\mathcal{R}_{\theta\theta} = 1 + e^{-z} \frac{2}{3} \left( \frac{\nu''}{4} - \frac{\nu'}{4} \right)
\]

\[
\mathcal{R}_{rr} = -\frac{\nu''}{2} + \frac{\nu'}{8} - \frac{3 \nu'^2}{8} + \frac{\nu' \nu}{4}
\]

From (14), (26), (27) the \( tt \)-equation (11) reads:

\[
\frac{\nu''}{2} e^{\nu - \lambda} + \frac{1}{4} e^{\nu} \left( \Lambda + kr \right) = -\frac{\nu'}{2} e^{\nu - \lambda} \frac{k}{\Lambda + kr}
\]

leading to

\[
\nu'' + \frac{k}{\Lambda + kr} \nu' + \frac{1}{2} \left( \Lambda + kr \right) e^\lambda = 0
\]

(31)

From (15), (26), (28) the \( \theta\theta \)-equation (12) reads:

\[
1 + e^{-z} \frac{2}{3} \left( \frac{\nu''}{4} - \frac{\nu'}{4} \right) - \frac{2}{3} \frac{\nu'}{4} (\Lambda + kr) = \left( \frac{\nu'}{4} - \frac{\nu'}{4} \right) e^{-z} \frac{2}{3} \frac{k}{\Lambda + kr}
\]

leading to

\[
\lambda'' - \nu' + \frac{k}{\Lambda + kr} \left( \lambda' - \nu' \right) + (\Lambda + kr) e^\lambda = 4 e^{\frac{z}{2}} + \frac{4}{4}
\]

which, combined with (31), becomes:

\[
\lambda'' + \frac{k}{\Lambda + kr} \lambda' + \frac{3}{2} (\Lambda + kr) e^\lambda = 4 e^{\frac{z}{2}} + \frac{4}{4}
\]

(34)

From (16), (26), (29) the \( rr \)-equation (13) reads:

\[
-\frac{\lambda''}{2} + \frac{\lambda'^2}{8} - \frac{3 \nu'^2}{8} + \nu' \lambda - \frac{1}{4} e^\left( \lambda + kr \right) = -\frac{\lambda'}{2} \frac{k}{\Lambda + kr}
\]

(35)

leading to

\[
\lambda'' - \frac{k}{\Lambda + kr} \lambda' + \frac{\lambda + kr}{2} e^\lambda - \frac{\lambda'^2}{8} + \frac{3 \nu'^2}{8} - \nu' \lambda' = 0
\]

(36)

Now, eliminating \( \lambda'' \) from Eqs. (34) and (36), we get:

\[
2 e^{\frac{z}{2}} + \frac{1}{4} \lambda' - \frac{k}{\Lambda + kr} \lambda' + \frac{\lambda + kr}{2} e^\lambda - \frac{\lambda'^2}{8} + \frac{3 \nu'^2}{8} - \nu' \lambda' = 0
\]

(37)

Next, we make the following coordinate change which is slightly different from Buchdahl in his original paper:

\[
\Lambda + kr = A e^{kz}
\]

(38)

The first and second derivatives acting on \( r \) become:

\[
\frac{d}{dr} = \frac{dz}{dr} \frac{d}{dz} = e^{-kz} \frac{d}{dz}
\]

(39)

\[
\frac{d^2}{dr^2} = \frac{dz}{dr} \frac{d}{dz} \left( \frac{e^{-kz}}{A} \frac{d}{dz} \right)
\]

\[
= \frac{e^{-kz}}{A} \left( -\frac{ke^{-kz}}{A} \frac{d}{dz} + \frac{e^{-kz}}{A} \frac{d^2}{dz^2} \right)
\]

\[
= \frac{e^{-kz}}{A^2} \left( \frac{d^2}{dz^2} - k \frac{d}{dz} \right)
\]

(40)

upon which Eqs. (31), (34), (37), respectively, become:

\[
e^{-2kz} \left( \frac{\nu'' + k}{A^2} \right) + \frac{ke^{-2kz}}{A^2} \frac{\nu''}{2} + \frac{\Lambda}{2} e^{kz} + \lambda = 0
\]

(43)

\[
e^{-2kz} \left( \lambda + kr \right) + \frac{ke^{-2kz}}{A^2} \frac{\nu''}{2} + \frac{3\Lambda}{2} e^{kz} = 4 e^{\frac{z}{2}} + \frac{4}{4}
\]

(44)

\[
\frac{ke^{-2kz}}{A^2} \lambda + \frac{\lambda}{2} e^{kz} + \frac{e^{-2kz}}{8A^2} \nu'' = 8A^2 \nu'' + \nu'' \lambda z = 2 e^{\frac{z}{2}} + \frac{4}{4}
\]

(45)

hence giving

\[
\nu_{zz} + \frac{\Lambda^2}{2} \nu_{zz} = 0
\]

(46)

\[
\lambda_{zz} + \frac{3 \Lambda^2}{2} \nu_{zz} = 4 A^2 e^{kz} + \frac{4}{4}
\]

(47)

\[
\lambda_{zz}^2 - 3 \nu_{zz}^2 + 2 \nu_{zz} \lambda_{zz} + 8 \Lambda \lambda + 4 \Lambda^3 \nu_{zz} = 16 A^2 e^{kz} + \frac{4}{4}
\]

(48)

Further define

\[
\nu = -u + v - k z + \ln 4
\]

\[
\lambda = 3u + v - 3k z + 3 \ln 4
\]

\[
\mu = \frac{\Lambda}{2} - \frac{\nu}{2} = 2u - k z + \ln 4
\]

(49)

(50)

(51)

from which, together with (46)–(48), we obtain

\[
u_{zz} = 16 A^2 e^{u} (1 - \Lambda e^{2u}) e^{v}
\]

(52)

\[
u_{zz} = 16 A^2 e^{u} (1 - 3 \Lambda e^{2u}) e^{v}
\]

(53)

\[
u_{zz} = 16 A^2 e^{u} (1 - 3 \Lambda e^{2u}) e^{v} + 4 k^2
\]

(54)

If \( \Lambda = 1 \), these equations would be equivalent to Eqs. (3.1), (3.3), and (3.4) in Buchdahl’s original paper [1].

Let us recap: So far, we have obtained the three equations (52), (54) for two unknown functions \( u(z) \) and \( v(z) \). However, the three equations are not independent. Upon taking derivative with respect to \( z \), Eq. (54) yields

\[
u_{zz} + 4k^2 = 16 \left( e^u - 3 \Lambda e^{3u} \right) e^v u_{z} + 16 \left( e^u - \Lambda e^{3u} \right) e^v v_{z}
\]

which is trivially satisfied by Eqs. (52) and (53). Therefore, the system is not over-determined. We shall discard Eq. (54) while keeping Eqs. (52) and (54) from now on.
III. OUR SHORTCUT LEADING TO THE GENERALIZED BUCHDAHL EQUATION

Note that Eq. (65) is of second differential order and Eq. (64) is of first differential order. Eliminating one of the functions \( u \) or \( v \) would in principle produce a third differential order ODE.

To proceed, Buchdahl next exploited some clever analogy of Eqs. (59)–(60) with a Hamiltonian dynamics. However, with the benefit of hindsight, we have found a shortcut to be presented in what follows.

Define \( q \) as a function of \( u \):

\[
q := u_z
\]

which leads to

\[
q_{x} + \frac{2\Lambda x}{1 - \Lambda x^2} q_x = \frac{3k^2}{4x^2} q_x
\]

(69)

At \( \Lambda = 1 \), it duly recovers

\[
xq_{xx} + \left( \frac{2x^2}{1 - x^2} - \frac{3k^2}{4q^2} \right) q_x = 0
\]

(70)

which is precisely Eqs. (4.8) and (3.4) in Buchdahl’s 1962 Nuovo Cimento paper [1].

Remarkably, the resulting ODE is of second (instead of third) differential order. Finally, upon substituting \( x := \sqrt{t} \), Eq. (69) becomes:

\[
2t q_{tt} + \left( \frac{1 + \Lambda t}{1 - \Lambda t} - \frac{3k^2}{4q^2} \right) q_t = 0
\]

(71)

which, at \( \Lambda = 1 \), recovers Eqs. (4.10) and (3.4) in Buchdahl’s paper [1].

We shall call Eq. (70) the generalized Buchdahl equation hereafter. Our next task is to make further progress with this equation.

IV. A NEW CLASS OF BUCHDAHL-INSPIRED METRICS

As we alluded to in the Motivation, Buchdahl deemed that his non-linear ODE (71) – although “deceptively simple” – was insoluble and irreducible to simpler forms. He discontinued his pursuit for an analytical solution and instead sought a power-expansion solution; see Footnote 1 in our current paper for his reasoning.

We find that this is not the case. The task of this section is to reformulate the generalized Buchdahl equation in a more transparent way, via which the final metric can be attained. We shall consider \( \Lambda \in \mathbb{R} \) in general. It turns out that the generalized Buchdahl ODE (69) can be cast in a more convenient form as

\[
\frac{d}{dx} \left( \frac{q_x}{1 - \Lambda x^2} \right) = \frac{3k^2}{4x^2 \sqrt{q_x}} \left( \frac{q_x}{1 - \Lambda x^2} \right)
\]

(72)

Next, let us define a new function \( p(x) \) per

\[
p(x) := \frac{q_x}{1 - \Lambda x^2}
\]

(73)

which, upon combining with (72), produces a set of two coupled non-linear first-order ODEs:

\[
p_x = \frac{3k^2}{4x^2 q^2} \quad p
\]

(74)

\[
q_x = (1 - \Lambda x^2) p
\]

(75)
In terms of $x$, the functions $u$ and $v$ are, using Eqs. (61) and (64)

\[ e^u = x \]
\[ e^v = \frac{q}{16\Lambda^2 (1 - \Lambda x^2)} = \frac{q}{16\Lambda^2} \]  
\text{(76)}
\text{(77)}

and the functions $\nu$, $\lambda$, and $\mu$ are, using Eqs. (49)–(51)

\[ e^\nu = e^{-u+v-kz+ln 4} = \frac{4}{\Lambda^2 e^{kz} 16 x} \]
\[ e^\lambda = e^{3u+v-3kz+3ln 4} = \frac{64}{\Lambda^2 e^{3kz}} \]
\[ e^\mu = e^{2u-kz+ln 4} = \frac{4}{e^{kz} x^2} \]  
\text{(78)}
\text{(79)}
\text{(80)}

From (38) we have

\[ dr = \Lambda e^{kz} dz \]  
\text{(81)}

and since we also know from (56) and (61) that

\[ q = u_z = \frac{du}{dx} \frac{dx}{dz} = \frac{1}{x} \frac{dx}{dz} \]  
\text{(82)}

which leads to

\[ dz = \frac{dx}{x q} \]  
\text{(83)}

we thus have

\[ dr = \Lambda e^{kz} \frac{1}{x q} \frac{dx}{dz} \]  
\text{(84)}

The metric initially expressed in (7) becomes:

\[ ds^2 = -e^v dt^2 + e^\nu dr^2 + e^\mu d\Omega^2 \]
\[ = -\frac{pq}{4\Lambda^2 e^{kz} x} dt^2 + \frac{4pqx^3}{\Lambda^2 e^{3kz}} \left( \frac{\Lambda e^{kz}}{x q} dx \right)^2 + \frac{4x^2}{e^{kz}} d\Omega^2 \]
\[ = \frac{4}{e^{kz}} \left\{ \left( \frac{p}{4} - \frac{q}{4\Lambda x} + \frac{4x}{q} dx \right)^2 + x^2 d\Omega^2 \right\} \]  
\text{(85)}

Finally, using the notation of $r$ in place of $x$, and making the following replacements

\[ \begin{cases} p &\rightarrow 4p \\ q &\rightarrow 4q \\ k &\rightarrow -4k \\ kz &\rightarrow -kz + ln 4 \\ t &\rightarrow \Lambda t \end{cases} \]  
\text{(87)}

we arrive at the family of Buchdahl-inspired metrics presented below.

### The Buchdahl-inspired metrics

\[ ds^2 = e^{kz} \frac{dr}{r} \left\{ p(r) \left( -\frac{q(r)}{r} dt^2 + \frac{r}{q(r)} dr^2 \right) + r^2 d\Omega^2 \right\} \]  
\text{(88)}

in which the evolution rules are

\[ \frac{dp}{dr} = \frac{3k^2 p}{4r^2 q^2} \]
\[ \frac{dq}{dr} = (1 - \Lambda r^2) p \]  
\text{(89)}
\text{(90)}

and, using (26), (38), (83) and (87), the Ricci scalar equals to

\[ \mathcal{R}(r) = 4\Lambda e^{-kz} \frac{dr}{r} \]  
\text{(91)}

There are two separate sets of metrics depending on the sign of $\Lambda$:

- **Asymptotically de Sitter**: $\Lambda > 0$ and $r \in [0, \Lambda^{-\frac{1}{2}}]$  
  \[ \mathcal{R}(r) = 4\Lambda \exp \left[ k \int_r^{\Lambda^{-\frac{1}{2}}} \frac{dr'}{r'^2 q(r')} \right] \]  
\text{(92)}

- **Asymptotically anti-de Sitter**: $\Lambda \leq 0$ and $r \in [0, \infty)$  
  \[ \mathcal{R}(r) = 4\Lambda \exp \left[ k \int_r^{\infty} \frac{dr'}{r'^2 q(r')} \right] \]  
\text{(93)}

In either case, the upper bound for the integral in $\mathcal{R}(r)$ is chosen such that, at the largest distance allowable, the Ricci scalar converges to $4\Lambda$.

Compatible with a fourth-derivative action, each metric is specified by four parameters: $\Lambda$ (the large-scale curvature), $k$ (the deviation from constant curvature), $p(r_0)$ and $q(r_0)$ at a reference distance $r_0$.

We shall tentatively call the class of metrics represented in (88)–(93) the Buchdahl-inspired metrics and the coordinate system $(t, r, \theta, \phi)$ used therein the Buchdahl coordinates. The Buchdahl-inspired metrics are complete and exhaustive. All “nontrivial” static spherically symmetric vacuo metrics in pure $\mathcal{R}^2$ gravity fall under the umbrella of the Buchdahl-inspired metrics.

### V. VERIFYING OUR SOLUTION VIA DIRECT INSPECTION

It is desirable to confirm ex post that our solution expressed in (88)–(93) obeys the $\mathcal{R}^2$ vacuo field equation. We shall carry out this due diligence exercise via direct inspection. The task is nontrivial because of the cross dependence between $p(r)$ and $q(r)$. Below is our maneuver.
First, we consider the line element
\[ ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + e^{\mu(r)} d\Omega^2 \] (94)
in which, by virtue of (88)
\[ \nu(r) := \ln \left( \frac{f(r)}{r} \frac{p(r) q(r)}{r} \right) \] (95)
\[ \lambda(r) := \ln \left( \frac{f(r)}{r} \frac{p(r) r}{q(r)} \right) \] (96)
\[ \mu(r) := \ln \left( f(r) r^2 \right) \] (97)
We further equate
\[ f(r) := \exp \left( k \int \frac{dr}{r q(r)} \right) \] (98)
while leaving \( p(r) \) and \( q(r) \) unspecified at the moment.

The relevant Christoffel symbols and Ricci tensor components are given in (14)–(19). We use the symbolic manipulator MAXIMA ONLINE to compute these six components and the Ricci scalar \( R \). They are found to contain \( p(r) \) and \( q(r) \) and their higher-differential order terms up to the fourth order.

Next, we specify
\[ p'(x) = \frac{3k^2}{4r^2} \frac{p(r)}{q^2(r)} \] (99)
\[ q'(x) = (1 - \Lambda r^2) p(r) \] (100)
then use MAXIMA ONLINE to compute \( p''(r), p'''(r), q''(r), q'''(r) \) and express each of them solely in terms of \( p(r) \) and \( q(r) \). We then substitute these quantities into the Christoffel symbols, the Ricci tensor components, and the Ricci scalar obtained above. Despite their cumbersome appearances, after all the dust settles, MAXIMA ONLINE determines that
\[ \mathcal{R} \left( \mathcal{R}_{tt} - \frac{1}{4} g_{tt} \mathcal{R} \right) + \Gamma^r_{tt} \Gamma^r_t \equiv 0 \] (101)
\[ \mathcal{R} \left( \mathcal{R}_{\theta\theta} - \frac{1}{4} g_{\theta\theta} \mathcal{R} \right) + \Gamma^r_{\theta\theta} \Gamma^r_\theta \equiv 0 \] (102)
\[ \mathcal{R} \left( \mathcal{R}_{rr} - \frac{1}{4} g_{rr} \mathcal{R} \right) + \Gamma^r_{rr} \Gamma^r_r - \mathcal{R}'' \equiv 0 \] (103)
identically. In addition, it produces
\[ \mathcal{R}(r) \equiv \frac{4\Lambda}{f(r)} \quad \forall r \] (104)
These outcomes solidly validate that our solution given in (88)–(91) satisfies the \( R^2 \) vacuo field equation.

Our MAXIMA codes used for this section are available in [14]. We must note that another researcher independently and successfully verified our solution using Mathematica; his working notebook is accessible in the public domain [15].

VI. RECOVERING SCHWARZSCHILD–DE SITTER METRIC AS SPECIAL CASE AT \( k = 0 \)

Consider a metric with constant curvature, \( \mathcal{R} \equiv 4\Lambda \; \forall r \). This requires \( k = 0 \) and, from (89)
\[ \frac{dp}{dr} = 0 \] (105)
or \( p = p_0 \equiv 1 \) without loss of generality. Then, from (90), we subsequently have
\[ \frac{dq}{dr} = 1 - \Lambda r^2 \] (106)
\[ q = r - \frac{\Lambda}{3} r^3 - r_s \] (107)
\[ \frac{q}{r} = 1 - \frac{\Lambda}{3} r^2 - r_s \] (108)
with \( r_s \) being a constant of integration. The metric in (88) becomes
\[ ds^2 = - \left( 1 - \frac{\Lambda}{3} r^2 - r_s \right) dt^2 + \frac{dr^2}{1 - \frac{\Lambda}{3} t^2 - \frac{r_s}{r}} + r^2 d\Omega^2 \] (109)
which is nothing but the classic SdS metric. This result also means that the SdS metric is the only vacuo metric with constant curvature available in pure \( R^2 \) gravity.

A Buchdahl-inspired metric can be made arbitrarily close to the SdS metric by tuning the parameter \( k \) to zero. Hence, the quartet \( \{ \Lambda, k, p_0, q_0 \} \) spans a topological space where all members in the space are smoothly connected to the \( k = 0 \) member (namely, the set of SdS metrics).

VII. THE SMALL \( k \) LIMIT

For \( k = 0 \) we already have the solution considered in the preceding section:
\[ p(r) \equiv 1 \quad \text{(without loss of generality)} \] (110)
\[ q(r) = r - \frac{\Lambda}{3} r^3 - r_s \] (111)
Let us consider up to \( \mathcal{O}(k) \)
\[ p(r) = 1 + \mathcal{O}(k) \] (112)
\[ q(r) = r - \frac{\Lambda}{3} r^3 - r_s + \mathcal{O}(k) \] (113)
Plugging them into (89) leads to
\[ \frac{dp}{dr} = \mathcal{O}(k^2) \] (114)
which then means
\[ p(r) = 1 + \mathcal{O}(k^2) \] (115)
Note that this expression is valid up to $O(k^2)$ instead of merely $O(k)$ as in (112). Plugging (115) into (90) yields

$$\frac{dq}{dr} = (1 - \Lambda r^2) + O(k^2)$$  \hspace{1cm} (116)

then

$$q = \left(r - \frac{\Lambda}{3} r^3 - r_s \right) + O(k^2)$$  \hspace{1cm} (117)

Once again, this expression is valid up to $O(k^2)$ instead of merely $O(k)$ as in (113). The conformal factor in the metric is thus

$$e^{k \int \frac{dr}{r(1 - \frac{\Lambda}{3} r^2)}} = e^{k \int \frac{dr}{r^2(1 - \frac{\Lambda}{3} r^2)} + O(k^2)}$$  \hspace{1cm} (118)

The metric in (88) becomes:

$$ds^2 = e^{k \int \frac{dr}{r^2(1 - \frac{\Lambda}{3} r^2)}} \left\{ -\left(1 - \frac{r_s}{r} - \frac{\Lambda}{3} r^2 \right) dt^2 + \frac{dr^2}{1 - \frac{r_s}{r} - \frac{\Lambda}{3} r^2} + r^2 d\Omega^2 \right\} + O(k^2)$$  \hspace{1cm} (119)

and the Ricci scalar is

$$\mathcal{R} = 4\Lambda \left[1 - k \int \frac{dt}{r^2(1 - \frac{\Lambda}{3} r^2)} \right] + O(k^2)$$  \hspace{1cm} (120)

This new metric is valid up to $O(k^2)$, and would be useful for physical situations with small $k$, i.e., with a weak deviation from constant scalar curvature. The metric is determined by three parameters $\Lambda$, $r_s$, and $k$, each representing a length scale.

At $O(k^2)$, the new metric (119) only differs from the SdS metric (109) by the conformal factor $e^{k \int \frac{dr}{r^2(1 - \frac{\Lambda}{3} r^2)}}$. Note that the pure $\mathcal{R}^2$ action is not subject to the conformal symmetry. As a result, the conformal factor is a physical quantity; it explicitly participates in the Ricci scalar rendering the latter non-constant as is evident in (120).

**VIII. BEHAVIOR OF BUCHDHAL-INSPIRED METRIC AROUND THE COORDINATE ORIGIN**

For any metric, the most interesting behavior should be around the origin where singularities might occur. In the limit of $r \to 0$, the “evolution” rules (89) and (90) become

$$\frac{dp}{dr} = \frac{3k^2}{4r} \frac{p}{q^2}$$  \hspace{1cm} (121) \\
$$\frac{dq}{dr} \approx p$$  \hspace{1cm} (122)

The sign of $p$ solely determines the direction of flows for both $p(r)$ and $q(r)$. Figure 1 shows the phase space spanned by $\{p, q\}$ with $q$ the horizontal axis and $p$ the vertical axis. As $r$ moves toward the coordinate origin, Quadrants (I) and (II) correspond to monotonic decreasing $p$ and $q$; Quadrants (III) and (IV) monotonic increasing $p$ and $q$. Figure 1 shows the direction of the flow if we start from a reference distance $r_0 > 0$ and move towards the origin. The SdS flows correspond to $k = 0$ (thus, $p \equiv 1$ and $q(r) = r - r_s + \frac{\Lambda}{3} r^3$ making $\lim \limits_{r \to 0} q(r) = -r_s$) thus their end points belong to Quadrants (II) or (III).

The horizontal axis is an attractor for all quadrants (note: we let $r$ move toward the coordinate origin). This can be shown below.

We shall let $p$ and $q$ converge to $p_s$ and $q_s$ when $r \to 0$ in the following manner

$$p \approx p_s + \tilde{p} r^\eta$$  \hspace{1cm} (123) \\
$$q \approx q_s + \tilde{q} r^\zeta$$  \hspace{1cm} (124)

with $\eta > 0$ and $\zeta > 0$. First, let us assume $p_s \neq 0$; from (121)

$$\frac{dp}{dr} = \frac{3k^2}{4r} \frac{p}{q^2} \approx \frac{3k^2}{4q_s^2} \frac{1}{r}$$  \hspace{1cm} (125)

making

$$p \approx -\frac{3k^2}{4q_s^2} \frac{1}{r^2} + \text{const}$$  \hspace{1cm} (126)

which would diverge as $r \to 0$ in contradiction with the requirement $p \to 0$ as $r \to 0$. Hence, $p_s$ must equal 0. This means that every trajectory must hit the horizontal axis as $r \to 0$ from above. We shall only consider $q_s \neq 0$ to this end.
Since $p_* = 0$, the evolution rules \([121]\) and \([122]\) become:

\[
\eta \bar{p} r^{n-1} \approx \frac{3k^2}{4} \frac{\bar{p}}{r^2} r^{n-1} \quad (127)
\]

\[
\zeta \bar{q} r^{\zeta-1} \approx \bar{p} r^n \quad (128)
\]

giving

\[
\eta = \frac{3k^2}{4q_*^2} > 0 \quad (129)
\]

\[
\zeta = \eta + 1 > 0 \quad (130)
\]

\[
\bar{p} = \eta \bar{q} \quad (131)
\]

Close to the origin, the functions thus are

\[
p(r) \approx (\eta + 1) \bar{q} r^n \quad (132)
\]

\[
q(r) \approx q_* + \bar{q} r^{n+1} \quad (133)
\]

The scalar curvature close to the origin behaves as

\[
R(r) \approx 4\Lambda \exp \left[ -k \int \frac{dr}{r q_*} \right] = 4\Lambda r^{-\frac{\eta}{\zeta}} \quad (134)
\]

As $r \to 0^+$, the Ricci scalar vanishes or diverges depending on the sign of $k/q_*$.

As $r \to 0^+$, the metric is approximately

\[
ds^2 \approx r^{\frac{1}{\zeta}} \left\{ (\eta + 1) \bar{q} r^n \left[ -q_* \partial_r^2 + \frac{kr}{q_*} \partial_r \right] + r^2 d\Omega^2 \right\} \quad (135)
\]

which is specified by exactly three parameters \{\(\Lambda, \frac{q_*}{k}, \frac{\bar{q}}{k}\)\} with

\[
\eta = \frac{3}{4} \left( \frac{k}{q_*} \right)^2 \quad (130a)
\]

\[
t : = k t
\]



**IX. A DEGENERACY IN PARAMETER SPACE OF BUCHDAHL-INSPIRED METRIC**

As the limit $k \to 0$ corresponds to the SdS metric, we shall consider only $k \neq 0$ herein. If we make the following substitutions:

\[
q := k \bar{q} \quad (136)
\]

\[
p := k \bar{p} \quad (137)
\]

\[
t := k^{-1} t \quad (138)
\]

then the metric in \([88]\) becomes

\[
ds^2 = e^{\sqrt{\mu v} t} \left\{ \frac{\bar{p}(r)}{r} \left[ -\frac{\bar{q}(r)}{r} \partial_r^2 + \frac{r}{\bar{q}(r)} \partial_r \right] + r^2 d\Omega^2 \right\} \quad (139)
\]

in which

\[
R(r) = 4\Lambda \exp \left[ -\int \frac{dr}{r \bar{q}(r)} \right] \quad (140)
\]

\[
\frac{dp}{dr} = \frac{3}{4r} \frac{\bar{p}}{\bar{q}} \quad (141)
\]

\[
\frac{dq}{dr} = (1 - \Delta r^2) \bar{p} \quad (142)
\]

Accordingly, despite being a product of a fourth-derivative action, a Buchdahl-inspired metric is effectively characterized by only three parameters. This degeneracy helps simplify the classification of Buchdahl-inspired metrics. We shall carry out this task in a companion paper \([10]\).

Note that in Sec. \(\text{VII}\) when treating the weak non-constancy for the Ricci scalar, we made $k$ explicit. Nevertheless, the metric obtained therein was specified by three length scales \(|\Lambda|^{-\frac{1}{2}}, r_*, k\) in perfect agreement with the number of degrees of freedom allowable by the degeneracy uncovered in this section.



**X. HOW DOES BUCHDAHL-INSPIRED METRIC CIRCUMVENT A “PROOF” OF NONEXISTENCE?**

In \([2]\) Kehagias et al sought black hole solutions for the pure quadratic action. Curiously, they omitted the Buchdahl equation and consequently overlooked the new class of Buchdahl-inspired metrics uncovered in our current paper. They considered only the two “automatic” vacuo configurations: (i) the zero-Ricci-scalar spaces, $R = 0$, and (ii) the Einstein spaces, $R_{\mu\nu} = \Lambda g_{\mu\nu}$. Therein, they offered a neat proof that apparently rules out the existence of non-constant curvature metrics (to which Buchdahl-inspired metrics belong). However, the class of Buchdahl-inspired metrics defeat their proof by evading its central assumption. Below is how it happens.

Let us first recap the essence of the proof of Kehagias et al. Their proof is a type of “no-go”, stating that all admissible $R^2$ vacua must have constant scalar curvature. The authors in \([2]\) started with the trace equation of the pure $R^2$ action in vacuo

\[
\Box R = 0 \quad (143)
\]

For the following metric

\[
ds^2 = -\mu(r) dt^2 + \frac{dr^2}{\nu(r)} + r^2 d\Omega^2 \quad (144)
\]

the trace equation takes the form \(6\)

\[
(r^2 \sqrt{\mu v} R')' = 0 \quad (145)
\]

This leads to

\[
(r^2 \sqrt{\mu v} R' R')' = (r^2 \sqrt{\mu v} R' R')' R + r^2 \sqrt{\mu v} (R')^2 \quad (146)
\]

from which one obtains the following identity:

\[
\int_0^\infty dr r^2 \sqrt{\mu v} (R')^2 = \int_0^\infty dr (r^2 \sqrt{\mu v} R' R')' \quad (147)
\]

\(6\) Recall that for a scalar field $\phi$: $\Box \phi = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu\nu} \partial_{\nu} \phi)$.
The right-hand side of (147) can be cast into a three-volume integral which then turns into a two-dimensional surface integral at infinity by virtue of the Gauss-Ostrogradsky divergence theorem: 7

\[
\int_0^{\infty} dr \left( r^2 \sqrt{\mu \nu} \mathcal{R}' \mathcal{R} \right) = \frac{1}{4\pi} \int d\Omega \int_0^{\infty} dr \ r^2 \nabla \cdot \left( \sqrt{\mu \nu} \nabla \left( \nabla \mathcal{R} \right) \right) \tag{148}
\]

\[
\frac{1}{4\pi} \int dV \nabla \cdot \left( \sqrt{\mu \nu} \nabla \mathcal{R} \right) \tag{149}
\]

\[
\frac{1}{4\pi} \int_{S} dS \sqrt{\mu \nu} \nabla \mathcal{R} \tag{150}
\]

\[
\lim_{r \to \infty} r^2 \sqrt{\mu \nu} \mathcal{R} \mathcal{R}' \tag{151}
\]

Now, the authors of [2] posited that if \( \mathcal{R}' \) falls to zero rapidly enough as large distances then the limit in (151) vanishes, making

\[
\int_0^{\infty} dr \ r^2 \sqrt{\mu \nu} \left( \mathcal{R}' \right)^2 = 0 \tag{152}
\]

Because of the non-negativity of the left-hand side of (152), this would force \( \mathcal{R}' = 0 \) everywhere. QED.

However, Buchdahl-inspired metrics invalidate this very assumption: their Ricci scalar decays not as rapidly to warrant (152). As a counterexample, in Sec. VII we obtained a metric with the Ricci scalar behaving at large distances as, per Eq. (120):

\[
\mathcal{R} \approx 4\Lambda - \frac{4k}{r^2} \tag{153}
\]

making

\[
\mathcal{R}' \approx \frac{12k}{r^4} \tag{154}
\]

thence

\[
\lim_{r \to \infty} r^2 \sqrt{\mu \nu} \mathcal{R} \mathcal{R}' = \lim_{r \to \infty} \left| \frac{48\Lambda k}{r^2 \sqrt{\mu \nu}} \right| = 16\Lambda^2 |k| \neq 0 \tag{155}
\]

given that \( \mu \simeq \nu \simeq 1 - \frac{3}{4}r^2 \) as large distances. In general, the growth in \( \mu \) and \( \nu \) balances out the decay in \( \mathcal{R}' \); the proof in [2] overlooked this compensation effect.

The non-zero value in (155) renders the “no-go” proof in [2] inapplicable for the Buchdahl-inspired metric. 8

Before closing this section, we must make two additional comments:

First, the “no-go” proof provided in [2] was previously offered by Nelson for the \( \mathcal{R} + \mathcal{R}^2 + C_{\mu \nu \rho \sigma} C^{\mu \nu \rho \sigma} \) action. Nelson’s proof similarly relied on an overly restrictive assumption on the asymptotic falloff for \( \mathcal{R}' \) as \( r \to \infty \).

Second, in a 2015 paper [4], Liu et al. reported the existence of further black hole solutions (above the Schwarzschild solution) for the Einstein-Weyl gravity, \( \mathcal{R} + C_{\mu \nu \rho \sigma} C^{\mu \nu \rho \sigma} \), viz. with the \( \mathcal{R}^2 \) term being suppressed. These solutions – albeit not in an analytical form – would be in defiance of Nelson’s “no-go” proof. The authors therein [4] identified a (sign) error in Nelson’s proof rendering it inapplicable for the Einstein-Weyl gravity. However, these authors did not refute Nelson’s proof for the pure \( \mathcal{R}^2 \) gravity; they did not point out the problem with the asymptotic falloff assumed in Nelson’s “no-go” proof which would have precluded the existence of Buchdahl-inspired metrics, as we have shown in this section.

XI. SUMMARY

In this paper, we show that pure \( \mathcal{R}^2 \) gravity admits nontrivial vacuo configurations beyond the vanishing Ricci scalar space (\( \mathcal{R} = 0 \)) and the Einstein space (\( \mathcal{R}_{\mu \nu} = \Lambda \eta_{\mu \nu} \)).

The new solutions are inherent in a program which Hans Buchdahl originated circa 1962. In a seminal – yet obscure – Nuovo Cimento paper [1], Buchdahl set forth to seek static spherically symmetric solutions for the pure \( \mathcal{R}^2 \) action. His work culminated in a non-linear second-order ODE that remained to be solved. If a solution to his ODE can be found, then a complete set of vacua for pure \( \mathcal{R}^2 \) gravity would be readily obtained.

Despite its importance and potential, the Buchdahl equation has largely escaped the attention of the gravitation research community since its inception. Among the mere 40+ publications that cited Buchdahl’s original Nuovo Cimento work, none have attempted to solve his ODE. In this paper, we have finally obtained a novel set of compact solutions to the Buchdahl equation, thereby accomplishing his six-decades-old goal seeking nontrivial vacuo metrics for pure \( \mathcal{R}^2 \) gravity.

Our main result: We reformulated Buchdahl’s original work via a more straightforward route starting directly from the \( \mathcal{R}^2 \) vacuo field equation; we thus departed from Buchdahl’s arduous route that used the variational principle on a “surrogate” Lagrangian. Along the way, we introduced a few shortcuts. We are able to arrive at a generalized Buchdahl equation in the form of a non-linear

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7 Recall that in spherical coordinates, for \( \phi(r) \) and \( \vec{A} = A(r) \hat{r} \):
\[
\nabla \phi = \partial_r \phi(r) \hat{r} \quad \text{and} \quad \nabla \vec{A} = \frac{1}{r} \partial_r \left( r^2 A(r) \hat{r} \right) - \hat{r} \frac{1}{r^2} \frac{d}{dr} \left( r^2 A(r) \right).
\]

8 As an aside comment, the proof in [2] was not water-tight. It should also have handle the intricacy introduced into the 3D divergence theorem by way of the curved space (which in general is not 3D Euclidean).

9 Based on NASA ADS and InspireHEP citation-trackers.
second-order ODE:

\[
\frac{d^2q}{dr^2} + \frac{2\Lambda}{1 - \Lambda r^2} \frac{dq}{dr} = \frac{3k^2}{4r} \frac{dq}{dr}
\]  

(156)

This ODE embodies the four parameters, \(\{\Lambda, k, q(r_0), \frac{dq}{dr} \big|_{r=r_0}\}\), of the fourth-order \(R^2\) theory.

Next, in place of the second-order ODE (156), we are able to recast it in terms of two coupled non-linear first-order ODEs:

\[
\frac{dp}{dr} = \frac{3k^2}{4r} p 
\]  

(157)

\[
\frac{dq}{dr} = \left(1 - \Lambda r^2\right) p 
\]  

(158)

From here, we are able to express the final solution in a neat resemblance to the SdS metric to make the terms transparent and self-explanatory. The Buchdahl-inspired metrics are in a compact representation:

\[
ds^2 = e^{k \int \frac{dr}{r q(r)}} \left\{ p(r) \left[ -\frac{q(r)}{r} dt^2 + \frac{r}{q(r)} dr^2 \right] + r^2 d\Omega^2 \right\}
\]

(159)

with the Ricci scalar equal

\[
R(r) = 4\Lambda \exp \left(-k \frac{dr}{r q(r)}\right)
\]  

(160)

As is generally expected from a fourth-order theory, a Buchdahl-inspired metric is specified by 4 parameters: \(\Lambda\) as the large-distance scalar curvature, the (Buchdahl) parameter \(k\) controlling the deviation of the Ricci scalar from constancy, \(\{p_0, q_0\}\) initiating the “evolution” flow.

Validity of our solution: To allay any doubts, in Sec. VII we verified by direct inspection that the metric given in (157)-(160) obeys the \(R^2\) vacuo field equation

\[
R \left( R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \right) + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) R = 0
\]  

(161)

hence establishing the validity of our solution. The verification process will be detailed in [14]. Note that another researcher also successfully carried out his own verification of our results, with his Mathematica notebook accessible in the public domain [15].

Circumventing a “no-go” theorem: In [2] it was proved that pure \(R^2\) vacua were restricted to the vanishing Ricci scalar spaces, \(R = 0\), and the Einstein spaces, \(R_{\mu\nu} = \Lambda g_{\mu\nu}\). This “no-go” proof, if it were correct, would rule out the existence of vacua with non-constant scalar curvature. Since Buchdahl-inspired metrics project non-constant scalar curvature, as is evident per (160) for \(k \neq 0\), we must identify the cause of the conflict. In Sec. X we found that the “no-go” proof in [2] imposed a rapid asymptotic falloff for the metric at largest distances. Buchdahl-inspired metrics, however, evade this overly restrictive assumption, thereby being able to circumvent the proof.

Recovering the SdS metric at \(k = 0\): The case of \(k = 0\) corresponds to the SdS metric in which \(p(r)\) can be set identically equal to 1 and \(q(r)\) contains a Schwarzschild radius; see Sec. VII.

Properties of the Buchdahl-inspired metrics: We examined the metrics in three situations: (i) the small \(k\) limit; (ii) the region around the coordinate origin; and (iii) a degeneracy in the parameter space of the metrics. These results are shown in Secs. VII, VIII, and IX respectively. A thorough systematic study of the metrics shall be provided in [16].

A framework “beyond Schwarzschild–de Sitter”: The family of Buchdahl-inspired metrics (157)-(160) is exhaustive: it covers all nontrivial static spherically symmetric vacuo configurations admissible in pure \(R^2\) gravity. Its parameters \(\{\Lambda, k, p_0, q_0\}\) form a topological space that encloses the constant-curvature SdS metrics \((k = 0)\) and smoothly connects each non-constant curvature member to an SdS metric when \(k\) is tuned to 0.

The Buchdahl-inspired metrics thus constitute a bona fide enlargement of the SdS metric. It offers a nontrivial example in the context of \(3 + 1\) higher-order gravity that encompasses the SdS metric yet – at the same time – transcends it. Hence the Buchdahl-inspired metrics embody a framework “beyond Schwarzschild–de Sitter”.

In closing, the compact representation (157)-(160) of the Buchdahl-inspired metrics should equip future researchers with a powerful tool to explore new physics in pure \(R^2\) gravity with relative ease.

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