Correspondences of matter field fluctuations in semiclassical gravity with generalized fluctuations in the hydro approximation

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A correspondence between scalar field fluctuations and generalized fluctuation in a hydrodynamic approximation of fields is obtained. With an intention to explore sub-hydro range mesoscopic physics for the matter in cosmological spacetime, we obtain the results in a closed analytical form. The fluctuations are compared in terms of two point covariances, which can be further used to develop non-equilibrium statistical physics at sub-hydro length scales for the matter fields. The fluid correspondences fall in the classical domain and hence technically make it easier to consider other applications in cosmology than the quantum fields, for which renormalization et al pose difficulties in solutions of the equations. We have obtained a correspondence of imperfect fluid having stochastic effects in the heat flux and anisotropic stresses along with the four-velocity which is the kinetic term. This accounts for thermal fluctuations as well as quantum fluctuations for the fields in the hydro limit. As a special case we have shown the explicit relations for the perfect fluid, and compared with the previous results. The significance of this correspondence is discussed in the concluding section.

I. INTRODUCTION

A correspondence between the stress energy tensor for a field and that of a fluid is well known and widely accepted \cite{1,2}. The field fluid correspondence is of interest in various aspect and is an active area of research \cite{3,4}. In this article which is a sequel of \cite{5,6} we continue to show the correspondence between quantum field fluctuations and the fluctuations in the hydrodynamic approximation of the fields. We carry out our work in terms of the respective stress energy tensors, by defining the fluctuations in terms of a two point noise kernel as has been done for the theory of semiclassical stochastic gravity \cite{8}. In this article we consider the case of a quantum fluid, where the kinetic term, namely the four-velocity also has a random nature along with other fluid variables. This enables one to treat thermal fluctuations and also retain the quantum nature of the fields intact, as opposed to the previous cases where we have considered non-thermal fluctuations and the decoherence limit of quantum fields.

In the theory of stochastic semiclassical gravity \cite{8}, the point separated noise kernel enables one to determine structure formation in the early universe. A similar feature of structure formation in the sub-hydro approximation as we do here, can be applied for a different (later) epoch in the evolution of the universe. This development can also be of significance where the point separated noise kernel with a fluid approximation can enable us to establish non-equilibrium statistical physics for the exotic matter fields that the relativistic stars are composed of. This opens up a new area of investigation related to massive stars. Current efforts towards relativistic fluids for the massive star interiors are being pursued \cite{10,11} and the interiors of relativistic stars are gaining more attention with the achievements in gravitational waves \cite{12,14} detection.

Our framework here is that of building up a theoretical base to address the new scales of interest namely, the sub-hydro mesoscopic scales, in the above two areas of application. These scales are expected to lie a little below the hydro-dynamic(static) scales and substantially above the quantum micro-scales.

For the hydro-limit of fields one has to consider the classical approximation of the stress tensors, as we show later. The spacetime metric $g_{ab}$ and background is considered to be deterministic. In this article the distinct feature is that of considering the kinetic term namely the fluid velocity fluctuations which were considered deterministic for earlier cases, as having a stochastic nature. Therefore the results here are applicable to a wider variety of relativistic imperfect fluids in cosmology and topics in relativistic astrophysics.

In the following sections we intend to formulate and explore the basic principles which probe sub-hydro mesoscopic scales in relativistic fluids, where generalized stochastic variables \cite{5} are applicable. The generalized stochasticity concept takes into consideration "roughness in physical variables" at mesoscopic scales which are yet unexplored in relativistic stars.

II. FLUCTUATIONS OF QUANTUM FIELDS IN CURVED SPACETIME

In this section we revise the semiclassical noise kernel, as has been obtained in the developments of semiclassical stochastic gravity, for fluctuations of quantum scalar fields. Later we will show the relation and correspondences this has with the hydro approximation.

The stress energy tensor for quantum fields is given by

$$T^{(field)}_{ab}(x) = \phi_{a'} \phi_b - \frac{1}{2} g_{ab} \phi^{c'} \phi_{c'} - \frac{1}{2} g_{ab} m^2 \phi^2 + \xi (g_{ab} \Box - \nabla_a \nabla_b + G_{ab}) \phi^2 \quad (1)$$

Fluctuations in quantum fields are defined by \cite{8} the bitensor which is a two point noise kernel $N_{abcd}(x, x')$
as worked out in the theory of semiclassical stochastic
gravity.

\[ 8N_{abc'd'}(x, x') = \langle \mathcal{T}_{ab}(x), \mathcal{T}_{c'd'}(x') \rangle - 2 \langle \mathcal{T}_{ab}(x) \rangle < \mathcal{T}_{c'd'}(x') > \]

where \(< \ldots >\) denotes the expectation of the quantum
field \(\phi\) on the spacetime background (in this article we
denote the quantum expectation with \(< \ldots >\) and
classical averages or expectation with \(E(\ldots)\). where

Further on we prescribe the noise kernel expressions for the field as \(N_{abc'd'}^{(field)}\) and for fluid as \(N_{abc'd'}^{(fluid)}\). For the scalar
fields then, with non-minimal coupling, the noise kernel is given as

\[
8N_{abc'd'}^{(field)} = (1 - 2\xi)^2(G_{abc}G_{d'b} + G_{abc}G_{d'b}) + 4\xi^2(G_{abc}G_{d'b} + G_{abc}G_{d'b}) - 2\xi(1 - 2\xi)(G_{ab}G_{cd'} + G_{ab}G_{cd'}) + 2\xi(1 - 2\xi)(G_{ab}G_{cd'} + G_{ab}G_{cd'}) - 4\xi^2(G_{ab}G_{cd'} + G_{ab}G_{cd'}) G + 2\xi^2 R_{c'd'} R_{ab}G^2
\]

where \(G \equiv G(x, x')\) are the Wightman functions defined
by \(< \phi(x)\phi(x') >\). Our aim is to show a correspondence
of these fluctuations with the generalized fluctuations in
the sub-hydro limit. The sub-hydro limit that we intend to probe is expected to lie below the classical
hydrodynamic limit and show mesoscopic scale effects in
the fluid approximation. These are different and lie above
the mesoscopic scales of the semiclassical stochastic
gravity, where the quantum fluctuations are considered.
We will take the classical limit of the stress tensor for the
hydro approximation and show that the fluctuations at
the scales give access to a new regime between macro and
micro scales in a straightforward way.

### III. GENERALIZED FLUCTUATIONS IN THE FLUID APPROXIMATION

The quantum fields can be treated as a fluid in the
hydrodynamic approximation such that, the fluid variables
associated with the fields are given by \(\tilde{u}\),

\[
\tilde{u}_a = [\partial_a(\phi)\partial^\phi(\phi)]^{-1/2}\partial_a\phi
\]

\[
e = (1 - \xi\phi^2)^{-1}[\frac{1}{2}\partial_a\phi\partial^\phi(\phi) + V(\phi) + \xi\{\Box(\phi^2) - (\partial^\phi\phi\partial_a\phi)^{-1}\partial^\phi\phi\partial^\phi(\phi^2)\nu_a\nu_b(\phi^2)]
\]

\[
g_a = \xi(1 - \xi\phi^2)^{-1}(\partial^\phi\phi\partial_a\phi)^{-3/2}\partial^\phi\phi\partial^\phi(\phi^2)\partial_a\phi - \nu_a\nu_b(\phi^2)\partial_a\phi
\]

\[
p = (1 - \xi\phi^2)^{-1}[\frac{1}{2}\partial_a\phi\partial^\phi(\phi) - V(\phi) - \xi\{\frac{2}{3}\Box(\phi^2) + \frac{1}{3}(\partial_a\phi\partial^\phi(\phi)^{-1}\partial_a\phi\partial^\phi(\phi^2)\nu_a\nu_b(\phi^2)\partial^\phi(\phi^2)\phi^2\partial^\phi(\phi^2)]
\]
\[ \pi_{ab} = \xi(1 - \xi \phi^2)^{-1}(\partial^f \phi \partial^e \phi)^{-1}\left[ \frac{1}{3}(\partial_a \phi \partial_b \phi - g_{ab} \partial^e \phi \partial^e \phi) \right] \]
\[ + \partial^f \phi \nabla_b \phi \phi^2 \partial_b \phi - \nabla_a \nabla_b \phi \phi^2 \partial_a \phi + \nabla_p \nabla_b \phi \phi^2 \partial_d \phi \partial_a \phi \phi^2 \phi^d \phi \]
\[ - \nabla_p \nabla_b \phi \phi^2 \partial_d \phi + (\partial_c \phi \partial^c \phi)^{-1} \partial^f \phi \nabla_b \phi \phi^2 \partial_a \phi \phi^d \phi \]
\[ \] (12)

for the fluid stress tensor
\[ T_{ab}^{\text{(fluid)}} = u_a u_b(\epsilon + p) - g_{ab} p + q_a u_b + u_a q_b - \pi_{ab} \] (13)

The noise kernel in the hydrodynamic approximation then takes an overall classical form
\[ 8 \tilde{N}_{abc'd'}^{(\text{fluid})}(x,x') = 2(E(T_{ab}^{(\text{fluid})}(x)T_{c'd'}^{(\text{fluid})}(x')) - E(T_{ab}^{(\text{fluid})}(x))E(T_{c'd'}^{(\text{fluid})}(x')) = 2\mathrm{Cov}[T_{ab}^{(\text{fluid})}(x),T_{c'd'}^{(\text{fluid})}(x')] \] (14)

The two point covariance for the hydro-limit stress tensor can be given by \( \mathrm{Cov}[T_{ab}^{(\text{fluid})}(x),T_{c'd'}^{(\text{fluid})}(x')] \) with terms arranged in order, for without the metric coefficient as \( \tilde{N}_{abc'd'}^{(\text{fluid})} \), and with coefficients \( g_{ab}(x), g_{c'd'}(x') \) and \( g_{ab}(x)g_{c'd'}(x') \), which read
\[ \{\mathrm{Cov}[u_a u_b(\epsilon(x) + p(x)), u_c u_d(\epsilon(x') + p(x'))] + \mathrm{Cov}[u_a u_b(\epsilon(x) + p(x)), u_c q_{d'}] + \mathrm{Cov}[u_a u_b(\epsilon(x) + p(x)), q_c u_{d'}] + \mathrm{Cov}[u_{a'} q_b, u_c u_{d'}(\epsilon(x') + p(x'))] + \mathrm{Cov}[u_a q_b, u_c u_{d'}(\epsilon(x') + p(x'))] + \mathrm{Cov}[u_a q_b, u_{c'} u_{d'}] + \mathrm{Cov}[u_a q_b, q_c u_{d'}] + \mathrm{Cov}[\pi_{a'b'}, \pi_{c'd'}] + g_{ab}(x)g_{c'd'}(x') + g_{b'a}(x)g_{c'd'}(x') + \mathrm{Cov}[u_a u_b(\epsilon(x) + p(x)), p(x')] \] (15)

the above which is the first term of the full expression can more elaborately be written as,
\[ \tilde{N}_{abcd}^{(\text{fluid})} = E(u_a E(u_b) E(u_c) E(u_d) \{\mathrm{Cov}[\epsilon(x), \epsilon(x')] + \mathrm{Cov}[\epsilon(x), p(x')] + \mathrm{Cov}[p(x), \epsilon(x')) + \mathrm{Cov}(p(x), p(x'))\}] + \{E(\epsilon(x)) E(\epsilon(x')) + E(\epsilon(x)) E(p(x')) + E(p(x)) E(\epsilon(x')) + E(p(x)) E(p(x'))\} \{\mathrm{Cov}[u_a, u_c] \mathrm{Cov}[u_b, u_d] + \mathrm{Cov}[u_a, u_d] \mathrm{Cov}[u_b, u_c] + 2E(u_a) E(u_c) \mathrm{Cov}[u_b, u_d] + 2E(u_a) E(u_c) \mathrm{Cov}[u_b, u_c] + 2E(u_a) E(u_c) \mathrm{Cov}[u_b, u_d] + 2E(u_a) E(u_c) \mathrm{Cov}[u_b, u_c] + 2E(u_a) E(u_c) \mathrm{Cov}[u_b, u_d] + 2E(u_a) E(u_c) \mathrm{Cov}[u_b, u_c] \}
\] (16)

Coefficient of \( g_{ab} \)
\[ \mathrm{Cov}[u_a u_b(\epsilon(x) + p(x'))] \] (17)

which can be given by
\[ \tilde{N}_{cd}^{(\text{fluid})} = E(u_c E(u_d) \{\mathrm{Cov}[\epsilon(x), p(x')] \mathrm{Cov}[p(x), \epsilon(x'))\}] \] (18)

coefficient of \( g_{ab} g_{cd} \),
\[ \tilde{N}^{(\text{fluid})} = \mathrm{Cov}[p(x), p(x')] \] (19)

Coefficient of \( g_{c'd'} \)
\[ \mathrm{Cov}[u_a u_b(\epsilon(x) + p(x)), p(x')] \] (20)

which reads
\[ \tilde{N}_{ab}^{(\text{fluid})} = E(u_a) E(u_b) \{\mathrm{Cov}[\epsilon(x), p(x')] + \mathrm{Cov}[p(x), p(x')]\} \] (21)
The full expression for fluid fluctuations reads,

\[ N_{\text{fluid}}^{(abCD')} = E(u_a) E(u_b) E(u_c') E(u_d') \{ \text{Cov}[\epsilon(x), \epsilon(x')] + \text{Cov}[\epsilon(x), p(x')] + \text{Cov}[p(x), \epsilon(x')] + \text{Cov}[p(x), p(x')] \} + \{ E(\epsilon(x)) E(\epsilon(x')) + E(\epsilon(x)) E(p(x')) + E(p(x)) E(\epsilon(x')) + E(p(x)) E(p(x')) \} \{ \text{Cov}[u_{a'}, u_c'] \text{Cov}[u_b, u_d'] + \text{Cov}[u_a, u_d'] \text{Cov}[u_b, u_c'] + 2E(u_a) E(u_c') \text{Cov}[u_b, u_d'] + 2E(u_b) E(u_d') \text{Cov}[u_a, u_c'] + 2E(u_a) E(u_d') \text{Cov}[u_b, u_d'] \}

One can then have correspondence between the fluid and field noise kernels \( N_{\text{fluid}}^{(abCD')} \rightarrow N_{\text{field}}^{(abCD')} \), where term by term we can show that (\( \rightarrow \) is used to denote "term corresponds to")

\[
\begin{align*}
\tilde{N}_{\text{fluid}}^{(abCD')} & \rightarrow \tilde{N}_{\text{field}}^{(abCD')} \quad (23) \\
\tilde{N}_{\text{field}}^{(abCD')} & \rightarrow \tilde{N}_{\text{fluid}}^{(abCD')} \quad (24)
\end{align*}
\]

For the perfect fluid case, where the heat flux \( q_a \) and anisotropic stresses \( \pi_{ab} \) as well as \( \xi \) the non-minimal coupling factor in the stress tensors are vanishing, we have

\[
\begin{align*}
8\tilde{N}_{\text{fluid}}^{(abCD')} & \rightarrow \tilde{N}_{\text{field}}^{(abCD')} \quad (25) \\
\tilde{N}_{\text{field}}^{(abCD')} & \rightarrow \tilde{N}_{\text{fluid}}^{(abCD')} \quad (26)
\end{align*}
\]

A. Perfect fluid case

One can compare this with the expressions for perfect fluid in [8, 9] and observe the difference. As we have considered the four-velocity also as a random variable in the present article, the expressions with the expectation \( E(u_a) \) as well as covariances of four-velocity vectors appear in the results. Therefore these results are relevant for the quantum fluid approximation with the stochastic effects showing up in terms of the kinetic variable
as well as bulk variables of the fluid. Also one can assign thermal fluctuations in the quantum fluid with this prescription. However we emphasise that our scales of interest are with the sub-hydro mesoscopic range physics, but these expressions may also be used for large scale hydrodynamic description, if one considers fluctuations w.r.t time in a conventional stochastic description rather than the generalized stochastic description.

IV. CONCLUDING REMARKS

In this article we have obtained a relation between two point fluctuations of a scalar field and that of its hydrodynamic approximation. These results indicate that, fluctuations of quantum fields can induce or are equivalent to sub-hydro mesoscopic effects in the fluid description of matter. The ”generalized covariances” (or variances) of pressure, energy density, heat flux etc describe these in the background spacetime. These results can be applicable for the perturbative theory in general relativity as the noise or source of perturbations. The significance, also lies in realising their importance for compact astrophysical objects which are coupled to (say) thermal fields and are of interest to collapsing clouds, towards critical phases and end states of collapse where fluctuations can play a critical role. Thus our results can be used to analyse properties of dense compact matter in strong gravity regions, and dynamics at intermediate length scales which become interesting with this formulation. The extended structure and properties given in terms of two point statistical covariances of matter fields in a sub-hydro mesoscopic description is the key feature in this article. Also for the Einstein Langevin equation, it is possible to find solutions applicable to the inflaton field in the hydrodynamic limit, during the evolution of the universe. An interesting direction can be seen to emerge from this work for studying microscopic structure and its connections with kinetic theory in curved spacetime [15]. Such an endeavour can begin by trying to consider these generalized fluctuations (including roughness of physical variables ) of matter fields as fundamental rather than trying to define particles in a curved spacetime. We know that a global definition to particles and to vacuum in a curved spacetime background is not unique. One may then attempt to formulate a kinetic theory using the field fluctuations and its generalization as the basic entity. This approach may find its way through the four-velocity as the kinetic term with its generalized fluctuations as presented in this article . With the framework of two point or higher correlations of fluctuations of matter fields, a tool to study non-local and extended structure of matter in the curved spacetime arises in an interesting way. Thus a kinetic theory of matter in curved spacetime can be based on these fluctuations rather than on the ambiguous localized particles.

[1] M.S.Madsen . Class. Quantum Grav. 5, 627 (1988)
[2] Roberto Mainini. JCAP 07 ,003 (2008)
[3] Mukund Rangamani. CQG , 26 , 224003.
[4] Valerio Faraoni . Phys. Rev D 85 , 024040 (2012).
[5] Ibrahim Semiz. Phys.Rev D 85 , 068501 (2012).
[6] Seema Satin . PRD 100, 044032 (2019).
[7] Seema Satin. CQG, 39,095004 (2022).
[8] Bei Lok Hu, Enric Verdaguer. Living Rev. in Relativity. 7:3
[9] Nicholas G.Phillips, B.L.Hu, Phys.Rev D, 63, 104001 (2001)
[10] Thomas Celora, Nils Andersson, Ian Hawke, Gregory L.Comer. Phys. Rev. D 104, 084090 (2021).
[11] Schmitt and Shternin, Astrophys. Space Sci.Lib 457 (2018)
[12] B.P.Abbot et al. Phys.Rev Lett 116, 061102 (2016).
[13] B.P.Abbot et al. Phys.Rev Lett. 116, 241103 (2016).
[14] Haocun Yu et al., Nature. 583, 43-47 (2020)
[15] Calzetta and Hu . Nonequilibrium Quantum Field Theory. Cambrdige University press (2008).