On Scheduling Two-Stage Jobs on Multiple Two-Stage Flowshops

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Abstract

Motivated by the current research in data centers and cloud computing, we study the problem of scheduling a set of two-stage jobs on multiple two-stage flowshops. A new formulation for configurations of such scheduling is proposed, which leads directly to improvements to the complexity of scheduling algorithms for the problem. Motivated by the observation that the costs of the two stages can be significantly different, we present deeper study on the structures of the problem that leads to a new approach to designing scheduling algorithms for the problem. With more thorough analysis, we show that the new approach gives very significant improved scheduling algorithms for the problem when the costs of the two stages are different significantly. Improved approximation algorithms for the problem are also presented.

keywords. scheduling, two-stage flowshop, pseudo-polynomial time algorithm, approximation algorithm, cloud computing

1 Introduction

Scheduling is concerned with the problems of optimally allocating available resources to process a given set of jobs. In particular, scheduling jobs on multiple machines has received extensive study in the past four decades, in computer science, operations research, and system sciences [15, 16].

In this paper, we study the scheduling problems for two-stage jobs on multiple two-stage flowshops. A machine \( M \) is a two-stage flowshop (or simply a flowshop) if it consists of an \( R \)-processor \( M_R \) and a \( T \)-processor \( M_T \) that can run in parallel. A job \( J \) is a two-stage job (or simply a job) if it consists of an \( R \)-operation \( R_J \) and a \( T \)-operation \( T_J \) such that the \( T \)-operation \( T_J \) cannot start

*This work is supported in part by the National Natural Science Foundation of China under grants 61232001, 61472449, 61420106009, and 71221061.

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on the $T$-processor $M_T$ of a flowshop $M$ until the $R$-operation $R_J$ has been completed on the $R$-processor $M_R$ of the same flowshop $M$. When a two-stage job $J$ is assigned to a two-stage flowshop $M$, the flowshop $M$ will first use its $R$-processor $M_R$ to process the $R$-operation $R_J$ of $J$, then, at proper time after $M_R$ completes the processing of $R_J$, use its $T$-processor $M_T$ to process the $T$-operation $T_J$ of $J$. Thus, when we consider scheduling two-stage jobs on multiple two-stage flowshops, we need to decide an assignment that assigns each job to a flowshop and, for each flowshop, the execution orders of the $R$- and $T$-operations of the jobs that are assigned to that flowshop.

Thus, the scheduling model studied in the current paper is as follows:

Given a set of $n$ two-stage jobs and a set of $m$ two-stage flowshops, construct a schedule of the jobs on the flowshops that minimizes the makespan, i.e., the total time that elapses from the beginning to the end for completing the execution of all the jobs.

1.1 Motivations

Our scheduling model was motivated by the current research in data centers and cloud computing. A data center is a facility used to house servers, storage systems, and network devices, etc. [1]. Today’s data centers contain hundreds of thousands of servers. Typical cloud computing providers rent infrastructures (IaaS), platforms (PaaS), and softwares (SaaS) as services, while keeping the softwares and data stored in the servers in data centers. Recently, a cloud paradigm called TransCom [23], based on the principle of transparent computing [21], has been proposed. This paradigm considers not only application softwares and data but also traditional system softwares such as operation systems as resources. As a consequence, client devices in such a system can be very light and significantly diversified, as long as they contain a small TransCom kernel and a new-generation input/output system UEFI [24]. Traditional operation systems, application softwares, and data are stored as resources in the cloud. Clients dynamically request these resources selected by users, and the cloud sends the resources to the clients via networks. The infrastructure of such a system is shown in Figure [1]

![Figure 1: The infrastructure of transparent computing](image)

In such a system, a significant amount of resources requested by clients are executable codes of system/application softwares, which in general are large by size and commonly used by many users. Because the main memory of servers is limited, these codes are in general stored in secondary memory such as hard disks that can be accessed by the servers. Therefore, when a server receives a request from a client for a specific code, it will have to first read the code from the secondary
memory into the main memory, then send the code to the client via networks. As a result, a request from a client can be divided into two operations, one is a disk-read operation $R$ that reads the requested code/data from a secondary memory into the main memory, and the other is a network-transmission operation $T$ that sends the code/data via the network to the requesting client. It is also natural to require that the network-transmission operation do not start until the requested code/data has been brought into the main memory. Therefore, in such a system, the data requests become two-stage jobs, consisting of the disk-read and the network-transmission operations, while each server becomes a two-stage flowshop, consisting of the disk-read and network-transmission processors (note that the disk-read and network-transmission can run in parallel in the same server), and scheduling a given set of such requests in a multiple-server center becomes an instance of the scheduling model we have formulated. We should remark that the time for disk-read and the time for network-transmission in a typical server are in general comparable, and, due to the impact of cache systems, they need not to have a linear relation. Therefore, neither can be simply ignored if we want to maintain good performance for the cloud system.

1.2 Previous related work

Multiple machine scheduling and flowshop scheduling have been extensively studied. We first discuss the relationship between our scheduling model and other related scheduling models studied in the literature. Then we review the known results specifically on our scheduling model.

First of all, the classical Makespan problem can be regarded as scheduling one-stage jobs on multiple one-stage machines. On the other hand, scheduling two-stage jobs on a single two-stage flowshop is the classical two-stage flowshop problem.

Other scheduling models that deal with multiple-stage jobs include various “hybrid” shop scheduling problems, such as the hybrid flow shop scheduling problem and the hybrid/flexible job shop scheduling problem. The hybrid shop scheduling problems allow multiple machines for a stage such that the execution of a stage operation of a job can be assigned to any machine for that stage. However, in general there is no specific bonding requirement for the machine that executes an operation for a stage of a job and the machine that executes the operation for the previous stage of the same job. This makes a major difference between this model and our model: our model requires that once a job is assigned to a machine, then the $R$-operation and the $T$-operation of the job must be executed by the $R$-processor and the $T$-processor, respectively, of the same machine.

Indeed, in the hybrid/flexible job shop scheduling model, if each job is given a set of alternative routes, where each route is a sequence of specific flowshops, one for a stage of the job, then our scheduling problem can be formulated as a restricted version of this very general version of the hybrid job shop scheduling problem. However, to the authors’ knowledge, this general version of the hybrid job shop scheduling problem has not been systematically studied. Moreover, since our scheduling problem has a strong constraint that the two stage operations of the same job be bonded to the same flowshop, a general solution to the general version of the hybrid job shop scheduling problem will probably not be efficient and effective enough for our scheduling problem.

Another model that deals with multiple-stage jobs is that of scheduling jobs with setup costs, where a job can also be regarded as a two-stage job in which one stage is the “setup” stage, and the other stage is the “regular” processing stage. However, in the model of scheduling jobs with setup costs assumes one-stage machines — a machine under such a model cannot run the setup stage for one job and the regular processing stage for another job in parallel. On the other

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One may argue that such a disk-read/network-transmission process can be done in pipeline: in this case, we can simply regard each data block of the requested data as a “inseparable” job. Now for each job network-transmission must wait until the disk-read is completed, and the job becomes two-stage. See Section 6 for more discussions.
}
hand, a two-stage flowshop $M$ under our model can have its $R$-processor and $T$-processor run in parallel. Thus, when the $R$-processor of $M$ is processing the $R$-operation for a job, the $T$-processor of $M$ can process the $T$-operation for another job at the same time. Finally, our model is different from that of multi-processor job scheduling problem [5, 6], where a job may require more than one processors and it holds all the requested processors during its execution. On the other hand, a two-stage job under our current model requires the two requested processors to run one after the other, and when one of the processors of a flowshop is running for the job, the other processor of the flowshop may be used for processing other jobs.

Except some research directly related to specific applications, the problem of scheduling two-stage jobs on multiple two-stage flowshops had not been studied thoroughly until very recently. He, Kusiak, and Artiba [11] seem the first group who studied the problem, motivated by applications in glass manufacturing, and proposed a heuristic algorithm. Vairaktarakis and Elhafsi [19] also considered the problem in their study on the hybrid multi-stage flowshop problem. In particular, a pseudo-polynomial time algorithm was proposed in [19] for scheduling two-stage jobs on two two-stage flowshops. Zhang and van de Velde [20] presented constant ratio approximation algorithms for scheduling two-stage jobs on two and three two-stage flowshops. Very recently, following a formulation similar to that of [19], Dong et al. [8] proposed a pseudo-polynomial time algorithm for scheduling two-stage jobs on $m$ two-stage flowshops for a fixed constant $m$, and developed a fully polynomial-time approximation scheme for the problem based on the pseudo-polynomial time algorithm. We also note that approximation algorithms for $k$-stage jobs on multiple $k$-stage flowshops for general $k$ have been studied recently [18].

1.3 Our main results

Our research in the current paper was motivated by our current project on data center and cloud computing, as described in the previous section. Therefore, we are looking for more efficient algorithms for scheduling two-stage jobs on multiple two-stage flowshops, which not only improve previous best theoretical complexity bound, but also run much faster in practice.

First of all, we propose a new formulation to describe configurations for schedules of two-stage jobs on multiple two-stage flowshops. Our formulation is very different from those studied in the literature [19, 8]. We show that dynamic programming based on our formulation directly leads to improvements on the complexity of algorithms for scheduling two-stage jobs on multiple two-stage flowshops, in terms of both theoretical bound and practical performance.

Our further study on the problem was motivated by the observation that in many cases in practice, the execution times for the two stages can differ very significantly. We present deeper study on the structures of the problem that leads to a more carefully designed algorithm. With more thorough analysis, we are able to show that the new approach will give a very significantly improved scheduling algorithm for the problem when the costs of the two stages are significantly different. Improved approximation algorithms for the problem are also presented.

The paper is organized as follows. Formal definitions and some preliminary results related to the problem are given in Section 2. The new scheduling formulation for the problem is proposed and improved pseudo-polynomial time exact algorithms based on the new formulation are presented in Section 3. Section 4 is devoted to faster algorithms for the case when the execution times of the two stages are significantly different. An improved approximation algorithm for the problem is given in Section 5 for the problem when the number of flowshops is bounded by a constant. We conclude the paper in Section 6 with remarks and suggested future research.

\[ \text{In fact, our research is independent of [8]: we became aware of the result of [8] only after the current paper had been completed.} \]
2 Single flowshop scheduling and dual scheduling

For \( n \) two-stage jobs \( J_1, \ldots, J_n \) to be processed in a system \( \{ M_1, \ldots, M_m \} \) of \( m \) identical two-stage flowshops, we make the following “standard” assumptions (variations and extensions of this model will be discussed in Section 6):

1. each job consists of an \( R \)-operation and a \( T \)-operation;

2. each flowshop has an \( R \)-processor and a \( T \)-processor that can run in parallel and can process the \( R \)-operations and the \( T \)-operations, respectively, of the jobs;

3. the \( R \)-operation and \( T \)-operation of a job must be executed in the \( R \)-processor and \( T \)-processor, respectively, of the same flowshop, in such a way that the \( T \)-operation cannot start unless the \( R \)-operation is completed;

4. there is no precedence constraints among the jobs; and

5. preemption is not allowed.

Under the model above, each job \( J_i \) can be represented by a pair \((r_i, t_i)\) of integers, where \( r_i \), the \( R \)-time, is the time for processing the \( R \)-operation of \( J_i \) by an \( R \)-processor, and \( t_i \), the \( T \)-time, is the time for processing the \( T \)-operation of \( J_i \) by a \( T \)-processor. A schedule \( S \) of a set of jobs \( \{ J_1, \ldots, J_n \} \) on \( m \) flowshops \( M_1, \ldots, M_m \) consists of an assignment that assigns each job to a flowshop, and, for each flowshop, the execution orders of the \( R \) and \( T \)-operations of the jobs assigned to that flowshop in its corresponding processors. The completion time of a flowshop \( M \) under the schedule \( S \) is the time when \( M \) finishes the execution of the last \( T \)-operation for the jobs assigned to \( M \) (assuming all flowshops are available at the initial time 0). The makespan \( C_{\text{max}} \) of \( S \) is the largest flowshop completion time under the schedule \( S \) over all flowshops. Following the three-field notation \( \alpha|\beta|\gamma \) suggested by Graham et al. [10], this scheduling model can be written as \( P|2\text{FL}|C_{\text{max}} \), or \( P_m|2\text{FL}|C_{\text{max}} \) if the number \( m \) of flowshops is a fixed constant.

2.1 Two-stage job scheduling on a single two-stage flowshop

For \( m = 1 \), the problem \( P_1|2\text{FL}|C_{\text{max}} \) becomes the two-stage flow shop problem. Without loss of generality, a schedule of a set of two-stage jobs on a single two-stage flowshop can be given by an ordered sequence \( \langle J_1, J_2, \ldots, J_t \rangle \) of the jobs such that both the executions of the \( R \)-operations and \( T \)-operations of the jobs, by the \( R \)-processor and \( T \)-processor of the flowshop, respectively, strictly follow the given order [13]. If our interests are in minimizing the makespan of schedules, then we can make the following assumptions.

Lemma 2.1 Let \( S = \langle J_1, J_2, \ldots, J_t \rangle \) be a two-stage job schedule on a single two-stage flowshop, where \( J_i = (r_i, t_i) \), for \( 1 \leq i \leq t \). Let \( \overline{\rho}_h \) and \( \overline{\tau}_h \), respectively, be the times at which the \( R \)-operation and the \( T \)-operation of job \( J_h \) are started. Then for all \( h, 1 \leq h \leq t \), we can assume:

\[
\begin{align*}
(1) & \quad \overline{\rho}_h = \sum_{i=1}^{h-1} r_i; \text{ and} \\
(2) & \quad \overline{\tau}_h = \max\{ \overline{\rho}_h + r_h, \overline{\tau}_{h-1} + t_{h-1} \}.
\end{align*}
\]

Proof. By the assumption, both the executions of the \( R \)-operations and the \( T \)-operations of the jobs follow the given order. Since the \( R \)-operation of the job \( J_h \) cannot start unless the \( R \)-operations of all jobs \( J_1, \ldots, J_{h-1} \) are completed on the \( R \)-processor of the flowshop, we must have \( \overline{\rho}_h \geq \sum_{i=1}^{h-1} r_i \).
If $\hat{h} > \sum_{i=1}^{h-1} r_i$, then we can let the $R$-operation of the job $J_h$ start at time $\hat{h} = \sum_{i=1}^{h-1} r_i$. Note that this change does not delay any other process — in particular, since the $T$-operation of $J_h$ starts at time $\tilde{h}$, which must be at least $\hat{h} + r_i$. Now since the $R$-operation of $J_h$ starts at time $\hat{h} = \sum_{i=1}^{h-1} r_i$ and finishes at time $\hat{h} + r_i < \hat{h} + r_i$, the $T$-operation of $J_h$ can still start at time $\tilde{h} \geq \hat{h} + r_i > \hat{h} + r_i$. For all other jobs, since the starting and finishing times of their $R$-operations and $T$-operations are unchanged, the schedule remains a valid schedule, with no change in the completion time of the flowshop. Applying this process repeatedly, we can fill all “gaps” in the execution of the $R$-processor of the flowshop (i.e., the idle time in the $R$-processor of the flowshop between the finish of the $R$-operation of a job and the start of the $R$-operation of the next job). The result is a valid schedule of the jobs, with no change in the completion time of the flowshop, and satisfies the condition $\hat{h} = \sum_{i=1}^{h-1} r_i$ for all $1 \leq h \leq t$. This proves (1).

The proof of (2) is simple: the $R$-operation of the job $J_h$ is finished at time $\hat{h} + r_h$, and the $T$-operation of the job $J_{h-1}$ is finished at time $\tilde{h}_{h-1} + t_{h-1}$. Therefore, at time $\max\{\hat{h} + r_h, \tilde{h}_{h-1} + t_{h-1}\}$, the $T$-operation of the job $J_h$ can always start, with no reason to further wait if our objective is to minimize the completion time of the flowshop. Moreover, this is the earliest time at which the $T$-operation of $J_h$ can start.

Scheduling two-stage jobs on a single two-stage flowshop, i.e., the two-stage flow shop problem $P_1|\text{2FL}|C_{\text{max}}$, can be solved optimally in time $O(n \log n)$ using the classical Johnson’s algorithm. In terms of our model, Johnson’s algorithm can be described as follows (for more details, see [13]):

**Johnson’s Algorithm [13].**

Given a set of two-stage jobs $(r_i, t_i), 1 \leq i \leq n$, divide the jobs into two disjoint groups $G_1$ and $G_2$, where $G_1$ contains all jobs $(r_h, t_h)$ with $r_h \leq t_h$, and $G_2$ contains all jobs $(r_g, t_g)$ with $r_g > t_g$. Order the jobs in a sequence such that the first part consists of the jobs in $G_1$, sorted in nondecreasing order of $R$-times, and the second part consists of the jobs in $G_2$, sorted in nonincreasing order of $T$-times. The schedule using the order of this sequence minimizes the completion time of the flowshop over all schedules of the jobs on the flowshop.

Johnson’s order of a set of two-stage jobs is to order the jobs into a sequence that satisfies the conditions given by Johnson’s Algorithm above. Therefore, once we determined how the jobs are assigned to the flowshops, Johnson’s order of the jobs assigned to each flowshop will give an optimal execution order for the flowshop. As a result, what that remains unsolved is how we determine the assignment of the jobs to the flowshops. Unfortunately, this task is intractable. In fact, in the special case where the $R$-time of every job is 0, the problem becomes the classical MAKESPAN problem $P||C_{\text{max}}$, where we are asked to optimally schedule a set of (one-stage) jobs on a set of identical (one-stage) machines. The MAKESPAN problem is NP-hard for two machines [4], and is strongly NP-hard for three or more machines [9]. As a consequence, our problem $P_m|\text{2FL}|C_{\text{max}}$ is NP-hard when $m \geq 2$ and NP-hard in the strong sense when $m \geq 3$.

Johnson’s orders of jobs on all flowshops can be constructed by a single sorting process on the input job set, as given by the following lemma, whose proof is straightforward thus is omitted.

**Lemma 2.2** If a job sequence $S$ satisfies Johnson’s order, then every subsequence of $S$ also satisfies Johnson’s order.

Therefore, if we first sort the input job set in Johnson’s order, which can obviously be done in time $O(n \log n)$, then pick the jobs in that order and assign them to flowshops, then every flowshop receives a subset of jobs in their Johnson’s order, which directly gives the optimal execution order.
of the job subset on the flowshop. In the rest of this paper, we will always assume that any sequence of jobs in our consideration is in Johnson’s order, unless we explicitly indicate otherwise.

Lemma 2.1 indicates that in an optimal schedule on a single flowshop based on Johnson’s order, we can simply follow Johnson’s order and let the R-processor of the flowshop consecutively execute the R-operations of the jobs without idle time until all R-operations are completed, and start immediately the T-operation of the job \( J_h \) as soon as the R-operation of \( J_h \) and the T-operation of the job \( J_{h-1} \) are completed. This observation greatly helps us in dealing with two-stage jobs on multiple two-stage flowshops. In particular, for a partial assignment of jobs on a flowshop, its corresponding (optimal) schedule now can be characterized by a pair \((\rho, \tau)\), which gives the finish times of the R-operation and the T-operation of the last job assigned to the flowshop. The pair \((\rho, \tau)\), which will be called the status of the schedule, can be easily updated, based on the formulas given in Lemma 2.1 when a new job is added to the flowshop.

### 2.2 Dual jobs and dual schedules

For a two-stage job \( J_i = (r_i, t_i) \), the dual job of \( J_i \) is \( J_i^d = (t_i, r_i) \) (i.e., the dual job \( J_i^d \) is obtained from the original job \( J_i \) by swapping its R- and T-times). Let \( S = \langle J_1, J_2, \ldots, J_n \rangle \) be a schedule of two-stage jobs on a two-stage flowshop. The dual schedule of \( S \) on the dual jobs of \( S \) is given by \( S^d = \langle J_1^d, J_2^d, \ldots, J_n^d \rangle \), where \( J_i^d \) is the dual job of \( J_i \) for \( 1 \leq i \leq n \). It is interesting to observe and easy to verify that if the schedule \( S \) follows Johnson’s order, then the dual schedule \( S^d \) also follows Johnson’s order. In fact, we have a more general result, as giving in the following theorem.

**Theorem 2.3** For \( 1 \leq i \leq n \), let \( J_i^d \) be the dual job of the two-stage job \( J_i \). On a single two-stage flowshop, the optimal schedule of the job set \( G = \{ J_1, J_2, \ldots, J_n \} \) and the optimal schedule of the dual job set \( G^d = \{ J_1^d, J_2^d, \ldots, J_n^d \} \) have the same completion time. Moreover, if a schedule \( S \) is optimal for the job set \( G \) then its dual schedule \( S^d \) is optimal for the dual job set \( G^d \).

**Proof.** For each \( h \), let \( J_h = (r_h, t_h) \). Thus, the dual job of \( J_h \) is \( J_h^d = (t_h, r_h) \). Let \( S = \langle J_1, J_2, \ldots, J_n \rangle \) be an optimal schedule for the job set \( G \), where, by Lemma 2.1 for each job \( J_h \), the R-operation starts at time \( \tilde{\rho}_h = \sum_{i=1}^{h-1} r_i \) and finishes at time \( \tilde{\rho}_h + r_h \), and the T-operation starts at time \( \tilde{\tau}_h = \max\{\tilde{\rho}_h + r_h, \tilde{\tau}_{h-1} + t_{h-1}\} \) and finishes at time \( \tilde{\tau}_h + t_h \). The completion time of the schedule \( S \) is \( \tau^* = \tilde{\tau}_n + t_n \).

Now consider the schedule \( S_1^d = \langle J_1^d, J_2^d, \ldots, J_n^d \rangle \) for the dual job set \( G^d \), where for each dual job \( J_h^d = (t_h, r_h), 1 \leq h \leq n \), the R-operation starts at time \( \tilde{\rho}_h^d = \tau^* - (\tilde{\tau}_h + t_h) \) and finishes at time \( \tilde{\rho}_h^d = \tau^* - \tilde{\tau}_h \), and the T-operation of \( J_h^d \) starts at time \( \tilde{\tau}_h^d = \tau^* - \sum_{i=1}^{h} r_i \) and finishes at time \( \tilde{\tau}_h^d = \tau^* - \sum_{i=1}^{h-1} r_i \) (see Figure 2 for an illustration).

Note that since \( \tilde{\tau}_{h+1} \geq \tilde{\tau}_h + t_h \), the job \( J_{h+1}^d \) has its R-operation starting at time \( \tilde{\rho}_{h+1}^d = \tau^* - (\tilde{\tau}_h + t_h) \), which is not earlier than the finish time \( \tilde{\rho}_{h+1}^d = \tau^* - \tilde{\tau}_{h+1} \) of the R-operation of the job \( J_{h+1}^d \). Similarly, the job \( J_h^d \) has its T-operation starting at time \( \tilde{\tau}_h^d = \tau^* - \sum_{i=1}^{h} r_i \), which is not earlier than (actually, is equal to) the finish time \( \tilde{\tau}_{h+1}^d = \tau^* - \sum_{i=1}^{h} r_i \) of the T-operation of the job \( J_{h+1}^d \). Finally, since \( \tilde{\tau}_h \geq \tilde{\rho}_h + r_h = \sum_{i=1}^{h} r_i \), the starting time \( \tilde{\tau}_h^d = \tau^* - \sum_{i=1}^{h} r_i \) of the T-operation of the job \( J_h^d \) is not earlier than the finish time \( \tilde{\rho}_h^d = \tau^* - \tilde{\tau}_h \) of the R-operation of the same job \( J_h^d \).

This shows that \( S_1^d \) is a valid schedule for the dual job set \( G^d \). Since the last job \( J_n^d \) in the schedule \( S_n^d \) finishes at time \( \tilde{\tau}_n^d = \tau^* \), the completion time of \( S_n^d \) is \( \tau^* \). Now, by Lemma 2.1 we can convert the schedule \( S_1^d \) into the standard dual schedule \( S^d \) of \( S \), without increasing the completion time, where \( S^d \) is a schedule for the dual job set \( G^d \) and satisfies the conditions in Lemma 2.1.

By our assumption, \( S \) is an optimal schedule for the job set \( G \) and has completion time \( \tau^* \). Thus, the fact that the completion time of the dual schedule \( S^d \) for the dual job set \( G^d \) is not larger
than $\tau^*$ implies that the completion time of an optimal schedule for the job set $G$ is not smaller than that of an optimal schedule for the dual job set $G^d$.

For the other direction, we start with an optimal schedule $S^d$ for the dual job set $G^d$. Using exactly the same procedure, we can show that the schedule $(S^d)^d$ that is dual to $S^d$ for the job set $(G^d)^d$ that is dual to $G^d$ has its completion time not larger than that of $S^d$. Since the job set $(G^d)^d$ that is dual to the dual job set $G^d$ is just the original job set $G$, this shows that the completion time of an optimal schedule for the dual job set $G^d$ is not smaller than that of an optimal schedule for the original job set $G$.

Combining these results, we conclude that the optimal schedule of the job set $G$ and the optimal schedule of the dual job set $G^d$ have the same completion time. This implies that if the completion time of an optimal schedule $S$ for the job set $G$ is $\tau^*$, then the completion time of the dual schedule $S^d$ for the dual job set $G^d$ is also $\tau^*$. Thus, $S^d$ must be optimal for the dual job set $G^d$. \hfill $\Box$

Now consider scheduling two-stage jobs on multiple two-stage flowshops. Let $G = \{J_1, J_2, \ldots, J_n\}$ be a set of two-stage jobs, and let $G^d = \{J^d_1, J^d_2, \ldots, J^d_n\}$ be the dual job set, where for each $h$, $J^d_h$ is the dual job of the job $J_h$.

**Theorem 2.4** On multiple two-stage flowshops, the optimal schedule of the job set $G$ and the optimal schedule of the dual job set $G^d$ have the same makespan. Moreover, an optimal schedule for the job set $G$ can be easily obtained from an optimal schedule for the dual job set $G^d$.

**Proof.** Suppose that $S$ is an optimal schedule of the job set $G$ on $m$ two-stage flowshops, where for each $i$, $1 \leq i \leq m$, $S$ assigns a subset $G_i$ of jobs in $G$ to the $i$-th flowshop. Without loss of generality, we can assume that $S$ optimally schedules the jobs in $G_i$ on the $i$-th flowshop. Now for each $i$, replace the schedule of $G_i$ on the $i$-th flowshop by its dual schedule for the dual job set $G^d_i$. This gives a schedule $S^d$ on the $m$ flowshops for the dual job set $G^d$. By Theorem 2.3, under the schedule $S^d$, the completion time for each flowshop is the same as that under the schedule $S$. Thus, the makespan of the schedule $S^d$ for the dual job set $G^d$ on the $m$ flowshops is the same as that of the schedule $S$ for the job set $G$. This proves that the makespan of an optimal schedule for $G$ on the $m$ flowshops is not smaller than that of an optimal schedule for $G^d$.

Conversely, starting with an optimal schedule of the dual job set $G^d$ on $m$ two-stage flowshops, we can similarly construct a schedule for the original job set $G$ whose makespan is equal to that of the optimal schedule for $G^d$, which implies that the makespan of an optimal schedule for $G^d$ on the $m$ flowshops is not smaller than that of an optimal schedule for $G$.

Combining these results shows that the optimal schedule of the job set $G$ and the optimal schedule of the dual job set $G^d$ have the same makespan. The discussion also explains that starting
with an optimal schedule \( S_d \) for the dual job set \( G^d \), by replacing the schedule on each flowshop with its dual schedule we can obtain an optimal schedule of the job set \( G \).

Theorem 2.4 provides flexibility when we work on scheduling two-stage jobs on multiple two-stage flowshops: sometimes working on the job set that is dual to the given input job set may have certain advantages. In this case, we can simply work on the dual job set, whose optimal solutions can be easily converted into optimal solutions for the original job set. This property will be used in Section 4.

3 Pseudo-polynomial time algorithms for \( P_m|2\text{FL}|C_{\text{max}} \)

In this section, we study the problem \( P_m|2\text{FL}|C_{\text{max}} \), i.e., the problem of scheduling two-stage jobs on \( m \) two-stage identical flowshops, where \( m \) is a fixed constant. Our input is a set of two-stage jobs \( G = \{J_1, J_2, \ldots, J_n\} \), where for each \( i \), \( J_i = (r_i, t_i) \), and we are looking for a schedule of the jobs on \( m \) identical two-stage flowshops \( M_1, \ldots, M_m \), that minimizes the makespan. Let \( R_0 = \sum_{i=1}^n r_i \), \( T_0 = \sum_{i=1}^n t_i \), and for \( 0 \leq k \leq n \), let \( G_k = \{J_1, J_2, \ldots, J_k\} \) be the set of the first \( k \) jobs in \( G \).

With a preprocessing, we can assume that the sequence \( \langle J_1, J_2, \ldots, J_n \rangle \) is in Johnson’s order. If we pick the jobs in this order and assign them to the flowshops, then, by Lemma 2.2, the sequence received by each flowshop \( M_h \) is also in Johnson’s order, which thus gives an optimal schedule of the jobs assigned to the flowshop \( M_h \). Therefore, the status of the flowshop \( M_h \) at any moment can be represented by a pair \((p_h, \tau_h)\) for the corresponding schedule, where \( p_h \) and \( \tau_h \) are the completion times of the \( R \)-processor and the \( T \)-processor, respectively, of the flowshop \( M_h \). By Lemma 2.1, the status \((p_h, \tau_h)\) of the flowshop \( M_h \) can be easily updated when a new job \((r, t)\) is added to the flowshop \( M_h \): the new completion time of the \( R \)-processor will be \( p_h + r \), and the new completion time of the \( T \)-processor will be \( \max\{p_h + r, \tau_h\} + t \). For each schedule \( S \) of the job subset \( G_k \), the tuple \((k; p_1, \tau_1, \ldots, p_m, \tau_m)\) will be called the configuration of \( S \) if under the schedule \( S \) for \( G_k \), the status of the flowshop \( M_h \) is \((p_h, \tau_h)\), for all \( h \).

The key observation, which can be easily verified, is that for each \( k > 0 \), we have:

**Fact A.** The tuple \((k; p_1, \tau_1, \ldots, p_m, \tau_m)\) is a configuration of a schedule for the job subset \( G_k \) if and only if there is a flowshop \( M_d \) such that the tuple \((k - 1; p_1', \tau_1', \ldots, p_m', \tau_m')\) is a configuration of a schedule for the job subset \( G_{k-1} \), where for \( i \neq d \), \( p_i' = p_i \), \( \tau_i' = \tau_i \), and \( p_d' \) and \( \tau_d' \) satisfy \( p_d = p_d' + r_k \) and \( \tau_d = \max\{\rho_d' + r_k, \tau_d'\} + t_k \), i.e., the schedule given by \((k; p_1, \tau_1, \ldots, p_m, \tau_m)\) is obtained by adding the job \( J_k \) to flowshop \( M_d \) in the schedule given by \((k - 1; p_1', \tau_1', \ldots, p_m', \tau_m')\).

Fact A suggests a dynamic programming algorithm that starts with the tuple \((0; 0, 0, \ldots, 0, 0)\), which corresponds to the unique schedule for the initial empty job subset \( G_0 \), and applies Fact A repeatedly to construct all possible configurations for the schedules for the given job set \( G = G_n \). Moreover, the value \( \max_{\{\{\tau_{\mathbf{h}}\} for a configuration \( (n; p_1, \tau_1, \ldots, p_m, \tau_m) \) gives the makespan of the schedule described by the configuration. Therefore, The configuration \((n; p_1, \tau_1, \ldots, p_m, \tau_m)\) with \( \max_{\{\{\tau_{\mathbf{h}}\} being minimized over all configurations gives a schedule for the job set \( G \) on the \( m \) flowshops whose makespan is the minimum over all schedules of the job set \( G \).

It is easy to see that for all \( 1 \leq h \leq m \), the value \( p_h \) is an integer bounded between 0 and \( R_0 \), and the value \( \tau_h \) is an integer bounded between 0 and \( R_0 + T_0 \). Therefore, a straightforward implementation of the dynamic programming algorithm runs in time \( O(nm^2 R_0^m (R_0 + T_0)^m) \), which will be quite significant when the values of \( R_0 \) and \( T_0 \) are large. In the following, we study how the complexity of the algorithm is improved.
Let \((\rho_h, \tau_h)\) be the status of the flowshop \(M_h\). By definition, we always have \(\rho_h \leq \tau_h\). Moreover, for any job \(J_i\) assigned to the flowshop \(M_h\), by Lemma 2.1 if the \(T\)-operation of \(J_i\) starts no earlier than \(\rho_h\), then it can always start immediately after the \(T\)-operation of the previous job assigned to \(M_h\) is completed, i.e., there is no “gap” in the execution of the \(T\)-processor of \(M_h\) after time \(\rho_h\). This gives \(\tau_h - \rho_h \leq T_0\). This observation suggests that we can use the pair \((\rho_h, \delta_h)\) instead of the pair \((\rho_h, \tau_h)\), where \(\delta_h = \tau_h - \rho_h\), and \(0 \leq \delta_h \leq T_0\). Note that the pair \((\rho_h, \tau_h)\) can be easily obtained from the pair \((\rho_h, \delta_h)\).

Therefore, for a configuration \((k; \rho_1, \delta_1, \ldots, \rho_m, \tau_m)\) for a schedule for the job subset \(G_k\), we will represent it by the tuple \((k; \rho_1, \delta_1, \ldots, \rho_m, \delta_m)\), where for all \(h\), \(\delta_h = \tau_h - \rho_h\) with \(0 \leq \delta_h \leq T_0\), which will be called the \(s\)-configuration of the schedule.

**Remark.** Our configurations and \(s\)-configurations defined above are very different from those proposed in the literature [19, 8], where a configuration is defined based on the makespan of the schedule (see [19, 8] for more details). We will show that based on the formulation of our configurations, much faster algorithms can be developed for the \(P_m|2\text{FL}|C_{\text{max}}\) problem.

Our next improvement is based on reducing the dimension of the \(s\)-configurations. Let \(S_k = (k; \rho_1, \delta_1, \ldots, \rho_m, \delta_m)\) be an \(s\)-configuration for the job subset \(G_k\). Let \(R^0_k = \sum_{i=1}^k r_i\). By Lemma 2.1 there is no “gap” in the execution of the \(R\)-processors of the flowshops. Therefore, \(\sum_{h=1}^m \rho_h = R^0_k\). This gives

**Fact B.** The value \(\rho_1\) can be computed from the values \(\rho_2, \ldots, \rho_m\): \(\rho_1 = R^0_k - \sum_{h=2}^m \rho_h\).

Let \(S_k = (k; \rho_1, \delta_1, \rho_2, \delta_2, \ldots, \rho_m, \delta_m)\) and \(S'_k = (k; \rho_1, \delta'_1, \rho_2, \delta_2, \ldots, \rho_m, \delta_m)\) be \(s\)-configurations for the job subset \(G_k\) that only differ in the completion time of the \(T\)-processor of flowshop \(M_1\), with \(\delta_1 < \delta'_1\). It is easy to see that if we can assign the rest of the jobs \(J_{k+1}, \ldots, J_n\) to \(S'_k\) to build a minimum makespan schedule for the entire job set \(G\), then the same way of assigning the jobs \(J_{k+1}, \ldots, J_n\) to \(S_k\) will also give a minimum makespan schedule of the job set \(G\). Therefore, when all other parameters are identical, we really only have to record the smallest completion time (thus the smallest value \(\delta_1\)) for the \(T\)-processor of the flowshop \(M_1\).

This suggests that we can represent all “useful” \(s\)-configurations for \(G_k\) by a \((2m - 1)\)-dimensional array \(H\) such that

\[
H[k; \rho_2, \delta_2, \ldots, \rho_m, \delta_m] = (\delta_1, d),
\]

if by letting \(\rho_1 = R^0_k - \sum_{h=2}^m \rho_h\), the value \(\delta_1\) is the smallest \(\delta'_1\) such that \((k; \rho_1, \delta'_1, \rho_2, \delta_2, \ldots, \rho_m, \delta_m)\) is a valid \(s\)-configuration for the job subset \(G_k\).

Now we are ready for our algorithm, which is given in Figure 3.

We give some explanations for the algorithm. Steps 3.4-3.7 add the job \(J_{k+1}\) to the \(d\)-th flowshop in the schedule for the job subset \(G_k\) with an \(s\)-configuration \(S = (k; \rho_1, \delta_1, \rho_2, \delta_2, \ldots, \rho_m, \delta_m)\). Thus, before adding the job \(J_{k+1}\), the completion times of the \(R\)-processor and the \(T\)-processor of the \(d\)-th flowshop are \(\rho_d\) and \(\rho_d + \delta_d\), respectively. By Lemma 2.1, after adding the job \(J_{k+1}\), the completion time of the \(R\)-processor becomes \(\rho'_d = \rho_d + r_{k+1}\), and the completion time of the \(T\)-processor is \(\max\{\rho_d + r_{k+1}, \rho_d + \delta_d\} + t_{k+1}\). Therefore, by the definition, after adding the job \(J_{k+1}\), we should have

\[
\delta'_d = (\max\{\rho_d + r_{k+1}, \rho_d + \delta_d\} + t_{k+1}) - \rho'_d = \max\{r_{k+1}, \delta_d\} + \rho_d + t_{k+1} - (\rho_d + r_{k+1}) = \max\{r_{k+1}, \delta_d\} + t_{k+1} - r_{k+1},
\]

as shown in step 3.5 of the algorithm.
Algorithm DynProg-I

INPUT: a set $G = \{J_1, \ldots, J_n\}$ of two-stage jobs, in Johnson's order
OUTPUT: an optimal schedule of $G$ on $m$ two-stage flowshops

1. for all $0 \leq k \leq n$, $0 \leq \rho_k \leq R_0$, $0 \leq \delta_k \leq T_0$, $2 \leq h \leq m$ do
   $H[k; \rho_2, \delta_2, \ldots, \rho_m, \delta_m] = (+\infty, 0)$;
2. $H[0; 0, 0, \ldots, 0] = (0, 0)$;
3. for $k = 0$ to $n - 1$ do
4. for each $H[k, \rho_2, \delta_2, \ldots, \rho_m, \delta_m] = (\delta_1, d_k)$ with $\delta_1 \neq +\infty$ do
5. $\rho_1 = R_0^0 - \sum_{h=2}^m \rho_h$;
6. for $d = 1$ to $m$ do
7. for $(1 \leq h \leq m)$ & $(h \neq d)$ do
8. $\rho_h' = \rho_h; \delta_h' = \delta_h$;
9. $\rho_d' = \rho_d + t_{k+1}; \delta_d' = \max\{r_{k+1}, \delta_d\} + t_{k+1};$
10. if $H[k+1; \rho_2', \delta_2', \ldots, \rho_m', \delta_m'] = (\delta_1, d_{k+1})$ with $\delta_1' < \delta_1$
11. then $H[k+1; \rho_2', \delta_2', \ldots, \rho_m', \delta_m'] = (\delta_1', d)$;
12. return the $H[n; \rho_2, \delta_2, \ldots, \rho_m, \delta_m] = (\delta_1, d_n)$ that minimized the value $\max_{1 \leq h \leq m} \{\rho_h + \delta_h\}$.

Figure 3: An improved algorithm for $P_m|2FL|C_{\text{max}}$

Note that the last row $H[n; *, \ldots, *]$ of the $(2m - 1)$-dimensinal array $H$ includes all possible $s$-configurations of the schedules for the job set $G = G_k$ on the $m$ flowshops. Moreover, the value $\max_{1 \leq h \leq m} \{\rho_h + \delta_h\}$ for an element $H[n; \rho_2, \delta_2, \ldots, \rho_m, \delta_m] = (\delta_1, d)$ (where $\rho_1 = R_0^0 - \sum_{h=2}^m \rho_h$) gives the makespan of the schedule described by $H[n; \rho_2, \delta_2, \ldots, \rho_m, \delta_m] = (\delta_1, d)$. Therefore, the one with $\max_{1 \leq h \leq m} \{\rho_h + \delta_h\}$ being minimized over all $H[n; \rho_2, \delta_2, \ldots, \rho_m, \delta_m] = (\delta_1, d)$ with $\delta_1 \neq +\infty$, as the one returned in step 4 of the algorithm, gives a schedule for the job set $G$ on the $m$ flowshops whose makespan is the minimum over all schedules of the job set $G$. According to the algorithm, the value $H[k; \rho_2, \delta_2, \ldots, \rho_m, \delta_m] = (\delta_1, d)$ also records that the last job $J_k$ in the job subset $G_k$ was added to the flowshop $M_d$ to obtain the $s$-configuration corresponding to $H[k; \rho_2, \delta_2, \ldots, \rho_m, \delta_m] = (\delta_1, d)$. With this information, the actual schedule corresponding to the element $H[k; \rho_2, \delta_2, \ldots, \rho_m, \delta_m] = (\delta_1, d)$ can be re-constructed as follows: (1) if $d \neq 1$, then look through $H[k-1; \rho_2, \delta_2, \ldots, \rho_{d-1}, \delta_{d-1}, \rho_d - r_k, \delta_d', \rho_{d+1}, \delta_{d+1}, \ldots, \rho_m, \delta_m] = (\delta_1, d)$ with $\delta_1 \neq +\infty$ for all $0 \leq \delta_d' \leq T_0$; and (2) if $d = 1$, then look at the element $H[k-1; \rho_2, \delta_2, \ldots, \rho_m, \delta_m] = (\delta_1', d)$, we will find an element $H[k-1; \rho'_2, \delta'_2, \ldots, \rho'_m, \delta'_m]$ for the job subset $G_{k-1}$ that, when $J_k$ is added to the flowshop $M_d$, gives the array element $H[k; \rho_2, \delta_2, \ldots, \rho_m, \delta_m]$. Now with this array element $H[k-1; \rho'_2, \delta'_2, \ldots, \rho'_m, \delta'_m]$ for $G_{k-1}$, we will find where the job $J_{k-1}$ went and what is the corresponding array element for $G_{k-2}$, and so on. Thus, starting from the array element returned in step 4 of the algorithm DynProg-I, we will be able to re-construct an optimal schedule for the job set $G$.

Since we have $0 \leq k \leq n$, and $0 \leq \rho_k \leq R_0$, $0 \leq \delta_k \leq T_0$, for all $2 \leq h \leq m$, the $(2m - 1)$-dimensional array $H$ has a size $O(nR_0^{\text{m}}T_0^{\text{m}-1})$. The algorithm basically goes through the array $H$, element by element, and applies steps 3.2-3.7 on each element, which take time $O(m^2)$. Thus, the algorithm takes time $O(nm^2R_0^{\text{m}}T_0^{\text{m}-1})$ and space $O(nR_0^{\text{m}}T_0^{\text{m}-1})$ (i.e., the space for the array $H$). Note that if we want to re-construct the optimal schedule based on the element returned in step 4 of the algorithm, we can go through the rows $H[k; *, \ldots, *]$ of the array $H$ (i.e., the first index of the array) backwards (i.e., $k$ goes from $n$ to 1), as we described above. This will take additional $O(nmT_0)$ time. Now we are ready to conclude the algorithm with the following theorem.

**Theorem 3.1** An optimal schedule for $n$ two-stage jobs on $m$ two-stage flowshops can be constructed in time $O(nm^2R_0^{\text{m}}T_0^{\text{m}-1})$ and space $O(nR_0^{\text{m}}T_0^{\text{m}-1})$.

We compare Theorem 3.1 with the existing result given in [8], which is the only known result for the $P_m|2FL|C_{\text{max}}$ problem. The algorithm given in [8] is based on a very different definition.
for configurations for schedules of two-stage jobs on \( m \) two-stage flowshops, and has running time \( O(nm^2(R_0 + T_0)^{2m-1}) \) and space \( O(m(R_0 + T_0)^{2m-2}) \). Therefore, in terms of the running time, our algorithm in Theorem 3.1 not only replaces the larger factor \( R_0 + T_0 \) by smaller factors \( R_0 \) and \( T_0 \), but also reduces the exponent from \( 2m - 1 \) to \( 2m - 2 \). In terms of the space complexity, our algorithm seems to use more space because in general \( n > m \). However, a careful examination shows that the algorithm given in [8] seems to only return the value of the makespan of an optimal schedule without giving the actual optimal schedule. In order to also return an actual schedule, the algorithm in [8] seems to have to increase its space complexity to at least \( O(nm(R_0 + T_0)^{2m-2}) \).

On the other hand, if we are only interested in the value of the makespan of an optimal schedule for the given job set, then we can modify our algorithm to run in space \( O(nm) \). This reduces the space complexity by \( 2^{\Theta(m)} \). However, a careful examination shows that the algorithm given in [8] seems to only return the value of the makespan of an optimal schedule without giving the actual optimal schedule. In order to also return an actual schedule, the algorithm in [8] seems to have to increase its space complexity to at least \( O(nm(R_0 + T_0)^{2m-2}) \). In conclusion, our algorithm in Theorem 3.1 improves both time complexity and space complexity of the algorithm given in [8].

4 Dealing with the case when \( R_0 \) and \( T_0 \) differ significantly

In certain cases in practice, the values \( R_0 \) and \( T_0 \) can differ very significantly. Consider the situation in data centers as we described in Section 1. In order to improve the process of data-read/network-transformation, servers in the center may keep certain commonly used software codes in the main memory so that the time-consuming process of data-read can be avoided (see, for example, [22]). Thus, client requests for the code will become two-stage jobs \( J_i = (r_i, t_i) \) with \( r_i = 0 \). As a consequence, the value \( R_0 = \sum_{i=1}^n r_i \) can be significantly smaller than the value \( T_0 = \sum_{i=1}^n t_i \). In certain cases in practice, the values \( R_0 \) and \( T_0 \) can differ very significantly. Consider the situation in data centers as we described in Section 1. In order to improve the process of data-read/network-transformation, servers in the center may keep certain commonly used software codes in the main memory so that the time-consuming process of data-read can be avoided (see, for example, [22]). Thus, client requests for the code will become two-stage jobs \( J_i = (r_i, t_i) \) with \( r_i = 0 \). As a consequence, the value \( R_0 = \sum_{i=1}^n r_i \) can be significantly smaller than the value \( T_0 = \sum_{i=1}^n t_i \). On the other hand, certain data centers may consist of a large number of slow-speed servers (e.g., PC’s) but equipped with high-speed networks [23], which may make \( T_0 \) much smaller than \( R_0 \).

In this section, we will study how to reduce the sizes of the dimensions of the configurations for the schedules in the case where the values \( R_0 \) and \( T_0 \) differ very significantly. This will lead to significant improvements on the complexity of scheduling algorithms. We divide the study into two cases: (1) \( T_0 \) is significantly larger than \( R_0 \) (i.e., \( T_0 \gg R_0 \)), and (2) \( R_0 \) is significantly larger than \( T_0 \) (i.e., \( T_0 \ll R_0 \)). We first consider the case \( T_0 \gg R_0 \).

Since all flowshops are identical, we can arbitrarily re-order the flowshops. In particular, we can order the flowshops so that the completion times of the \( R \)-operations of the flowshops are non-increasing. We call an \( s \)-configuration \((k; \rho_1, \delta_1, \ldots, \rho_m, \delta_m)\) canonical if \( \rho_1 \geq \rho_2 \geq \cdots \geq \rho_m \). Any \( s \)-configuration of a schedule can be converted into a canonical \( s \)-configuration by properly re-ordering the flowshops. Therefore, we only need to consider canonical \( s \)-configurations.

Let \((k; \rho_1, \delta_1, \ldots, \rho_m, \delta_m)\) be a canonical \( s \)-configuration for a schedule \( S_k \) for the job subset \( G_k \). By Lemma 2.1, there is no “gap” in the execution of the \( R \)-processors of the flowshops, so \( \sum_{h=1}^m \rho_i \leq R_0 \). This gives reduced upper bounds for the completion time of the \( R \)-processors of the flowshops:

\[ \text{Fact C.} \text{ In a canonical } s \text{-configuration } (k; \rho_1, \delta_1, \ldots, \rho_m, \delta_m), \rho_h \leq R_0/h, \text{ for all } 1 \leq h \leq m. \]

As we explained in the previous section, if our objective is to minimize the makespan, then when all the values \( k, \rho_1, \rho_2, \delta_2, \ldots, \rho_m, \delta_m \) are given, we only need to record the smallest \( \delta'_1 \) such that \((k; \rho_1, \delta'_1, \rho_2, \delta_2, \ldots, \rho_m, \delta_m)\) corresponds to a valid schedule for the job subset \( G_k \). This
and \( \delta \) reduces the number of dimensions for the s-configurations by 1.

In contrast to Fact C, the values \( \delta_h \) can be very large (recall \( T_0 \gg R_0 \)). We now consider how to deal with the situations when the values \( \delta_h \) are large.

Fix an \( h \), and consider the \( h \)-th flowshop. By Fact C, the completion time of the \( R \)-processor of the flowshop can never be larger than \( R_0/h \leq R_0 \). If the completion time \( \rho_h + \delta_h \) of the \( h \)-th flowshop is larger than or equal to \( R_0 \), then for any further job \( J_p \) assigned to the flowshop, the \( T \)-operation of \( J_p \) can always start immediately when the \( T \)-processor is available. Therefore, all further jobs assigned to the flowshop can have their \( T \)-operations executed consecutively with no execution “gaps” in the \( T \)-processor of the flowshop. Thus, the completion time of the flowshop will only depend on the \( T \)-operations of the further assigned jobs, while is independent of the \( R \)-operations of these jobs. We can use a single value \( \rho_h = R_0/h + 1 \) to record this situation so that the pair \( (R_0/h + 1, \delta_h) \) represents a real status \( (\rho'_h, \delta'_h) \) of the flowshop where \( \rho'_h + \delta'_h \geq R_0 \), and \( \delta_h = \rho'_h + \delta'_h - R_0 \). Note that when a new job \( J_p = (r_p, t_p) \) is added to the flowshop, the corresponding pair of the flowshop is simply changed to \( (R_0/h + 1, \delta_h + t_p) \).

This observation enables us to represent the status of the \( h \)-th flowshop by a pair \( (\rho_h, \delta_h) \), where either \( 0 \leq \rho_h \leq R_0/h \) and \( 0 \leq \rho_h + \delta_h < R_0 \) (which implies \( 0 \leq \delta_h < R_0 \)), or \( \rho_h = R_0/h + 1 \) and \( 0 \leq \delta_h \leq T_0 \) (which implies that the completion time for the \( T \)-processor of the flowshop is \( R_0 + \delta_h \)). A pair is a valid pair for the \( h \)-th flowshop if it satisfies these conditions. The total number of valid pairs for the \( h \)-th flowshop is bounded by \( (R_0/h + 1)R_0 + (T_0 + 1) = O(R_0^2/h + T_0) \). Note that all valid pairs can be given by a two-dimensional array (i.e., a matrix) with \( R_0/h + 2 \) rows in which each of the first \( R_0/h + 1 \) rows contains \( R_0 \) elements and the last row contains \( T_0 + 1 \) elements (if you like, you can also regard this matrix as an \( (R_0/h + 1) \times R_0 \) matrix plus a one-dimensional array of size \( T_0 + 1 \)).

Summarizing the above discussions, we conclude that all “useful” canonical s-configurations for the job subset \( G_k \), for all \( k \), can be represented by a \((2m)\)-dimensional array \( H' \) whose elements are \((m+1)\)-tuples, such that if

\[
H'[k; \rho_1, \rho_2, \delta_2, \ldots, \rho_m, \delta_m] = (d_k, \delta'_1, \rho'_2, \ldots, \rho'_m),
\]

where \( 0 \leq \rho_1 \leq R_0 \), and for \( 2 \leq h \leq m \), \((\rho_h, \delta_h)\) is a valid pair for the \( h \)-th flowshop, then there is a canonical s-configuration \( (k; \rho'_1, \delta'_1, \rho'_2, \delta'_2, \ldots, \rho'_m, \delta'_m) \) for a valid schedule for the job subset \( G_k \), where \( \rho'_1 = \rho_1 \) and \( \delta'_1 \) is the smallest when all other parameters satisfy their conditions, such that for each \( h, 2 \leq h \leq m \),

1. if \( \rho_h \leq R_0/h \), then \( \rho_h + \delta_h < R_0 \), \( \rho'_h = \rho_h \) and \( \delta'_h = \delta_h \), and
2. if \( \rho_h = R_0/h + 1 \), then \( \rho'_h + \delta'_h \geq R_0 \), and \( \delta_h = \rho'_h + \delta'_h - R_0 \).

Finally the value \( d_k \) in the array element, \( 1 \leq d_k \leq m \), indicates that the last job \( J_k \) in the job subset \( G_k \) is assigned to the \( d_k \)-th flowshop.

Note that in the case \( \rho_h = R_0/h + 1 \), there can be many different values for \( \rho'_h \) that thus correspond to many different canonical s-configurations that satisfy the above conditions. As explained earlier, in this case, different choices of the values \( \rho'_h \) will not affect the makespan of the final schedule of the job set \( G \). Thus, we can pick any valid values (not necessarily the smallest) for these \( \rho'_h \), as long as their sum plus \( \rho_1 \) is equal to \( \sum_{i=1}^{k} r_i \).

\footnote{However, unlike algorithm DynProg-I, we will not be able to remove the dimension for \( \rho_1 \) in s-configurations. This will become clearer in our discussion.}

\footnote{Actually by Fact C, this statement holds true for the \( h \)-th flowshop when \( \rho_h + \delta_h \geq R_0/h \). However, since later we may need to re-order the flowshops to keep the s-configurations canonical, the \( h \)-th flowshop may become the \( h' \)-the flowshop with \( \rho_{h'} + \delta_{h'} < R_0/h' \). Thus, here we pick the looser but more universal bound \( \rho_h + \delta_h \geq R_0 \) that is independent of \( h \) and also simplifies our discussion.}
Since the total number of valid pairs for the \( h \)-th flowshop, for \( 2 \leq h \leq m \), is \( O(R_0^2/h + T_0) \), and \( 0 \leq k \leq n \), we conclude that the number of elements in the array \( H' \) is bounded by

\[
O((n + 1)(R_0 + 1) \prod_{h=2}^{m} (R_0^2/h + T_0)) = O(n(R_0^{2m-1}/m! + R_0 T_0^{m-1})).
\]

Finally, since each element of \( H' \) is an \((m + 1)\)-tuple, we conclude that the array \( H' \) takes space \( O(n m (R_0^{2m-1}/m! + R_0 T_0^{m-1})) \).

We explain how to extend a schedule for the job subset \( G_k \) to a schedule for the job subset \( G_{k+1} \) when the job \( J_{k+1} \) is added. For this, suppose that we have a canonical s-configuration \( S_k = (k; \rho'_1, \delta'_1, \rho'_2, \delta'_2, \ldots, \rho'_m, \delta'_m) \) for \( G_k \) that is given by the element of the array \( H' \):

\[
H'[k; \rho_1, \rho_2, \delta_2, \ldots, \rho_m, \delta_m] = (d_k, \delta'_1, \rho'_2, \ldots, \rho'_m),
\]

as explained above. Note that the s-configuration \( S_k \) can be completely re-constructed when the corresponding element of \( H' \) is given:

1. the status \((\rho'_1, \delta'_1)\) for the first flowshop is \((\rho_1, \delta'_1)\);
2. for \( 2 \leq h \leq m \),
   1. if \( 0 \leq \rho_h \leq R_0/h \), then the status \((\rho'_h, \delta'_h)\) of the \( h \)-th flowshop is \((\rho_h, \delta_h)\); and
   2. if \( \rho_h = R_0/h + 1 \), then the status \((\rho'_h, \delta'_h)\) of the \( h \)-th flowshop is \((\rho'_h, R_0 + \delta_h - \rho'_h)\).

Note that in case (2.2), what matter is that the completion time of the \( T \)-processor is equal to \( \rho'_h + (R_0 + \delta_h - \rho'_h) = R_0 + \delta_h \), while the value \( \rho'_h \) may vary as long as it satisfies \( \rho_1 + \sum_{h=2}^{m} \rho'_h = R_0 \).

Now suppose that we decide to add the job \( J_{k+1} = (r_{k+1}, t_{k+1}) \) to the \( d \)-th flowshop in the s-configuration \( S_k \). Then the resulting configuration for the job subset \( G_{k+1} \) will become

\[
(k + 1; \rho''_1, \delta''_1, \rho''_2, \delta''_2, \ldots, \rho''_m, \delta''_m),
\]

where \( \rho''_d = \rho'_d + r_{k+1} \) and \( \delta''_d = \max\{r_{k+1}, \delta'_d\} + t_{k+1} - r_{k+1} \) (see the explanation given for algorithm DynProg-I in the previous section), and for \( h \neq d \), \( \rho''_h = \rho'_h \) and \( \tau''_h = \tau'_h \). This, after properly sorting the flowshops using the values of \( \rho''_h \), becomes a canonical s-configuration

\[
S_{k+1} = (k; \rho'_1, \delta'_1, \rho'_2, \delta'_2, \ldots, \rho'_m, \delta'_m)
\]

for the job subset \( G_{k+1} \). Assume the \( d \)-th flowshop in \( S_k \) becomes the \( d_{k+1} \)-th flowshop in \( S_{k+1} \). Now let \( \bar{\rho}_1 = \rho'_1 \), and for each \( h, 2 \leq h \leq m \), if \( \rho'_h + \delta'_h < R_0 \) then let \( \bar{\rho}_h = \rho'_h \) and \( \bar{\delta}_h = \delta'_h \), and if \( \rho'_h + \delta'_h \geq R_0 \) then let \( \bar{\rho}_h = \rho_h/R_0 + 1 \) and \( \bar{\delta}_h = \rho'_h + \delta'_h - R_0 \). With these values, look at the array element

\[
H'[k+1; \bar{\rho}_1, \bar{\rho}_2, \bar{\delta}_2, \ldots, \bar{\rho}_m, \bar{\delta}_m].
\]

If the element has not been assigned a value, yet, then assign it the value \((d_{k+1}, \delta'_1, \rho'_2, \ldots, \rho'_m)\). If the element already has a value \((d', \delta'_1, \rho'_2, \ldots, \rho'_m)\) but \( \delta'_1 < \delta''_1 \), then change its value to \((d_{k+1}, \delta'_1, \rho'_2, \ldots, \rho'_m)\). This completes the process of extending the canonical s-configuration given by the array element \( H'[k, \rho_1, \rho_2, \delta_2, \ldots, \rho_m, \delta_m] \), when job \( J_{k+1} \) is added to the \( d \)-th flowshop, to an array element for a canonical s-configuration for the job subset \( G_{k+1} \). It is easy to see that this process takes time \( O(m) \).

Using the above description to replace the steps 3.1-3.7 in the algorithm DynProg-I gives the procedure of extending a canonical s-configuration for \( G_k \) to a canonical s-configuration for \( G_{k+1} \). This, plus certain obvious modifications in other steps, gives a new algorithm DynProg-II for the \( P_m|2\text{FL}|C_{\text{max}} \) problem. Since the number of elements of the array \( H' \) is bounded by
applying the algorithm DynProg-II, we conclude that the time complexity of the algorithm DynProg-II is \(O(nm^2(R_0^{2m-1}/m! + R_0T_0^{m-1}))\). Similarly as we explained for the algorithm DynProg-I, once we apply algorithm DynProg-II and find the array element of \(H'\) that gives a minimum makespan schedule of the job set \(G\) on the \(m\) flowshops, we can use the array to construct the actual schedule by backtracking the array, row by row, in the same amount of time.

Now we describe how to deal with job sets \(G\) when \(T_0 \ll R_0\). Let \(G^d\) be the dual job set of \(G\), and let \(R_0'\) and \(T_0'\) be the sums of the times of the \(R\)-operations and of the \(T\)-operations, respectively, of the jobs in \(G^d\). By the definition, \(R_0' = T_0\) and \(T_0' = R_0\). Therefore, we have \(T_0' \gg R_0'\). Thus, applying the algorithm DynProg-II on the dual job set \(G^d\) will construct an optimal schedule for \(G^d\) in time \(O(mn^2((R_0')^{2m-1}/m! + R_0'(T_0')^{m-1})) = O(nm^2(T_0^{2m-1}/m! + T_0R_0^{m-1}))\). By Theorem 2.4, an optimal schedule for the job set \(G\) can be easily constructed from the optimal schedule for the dual job set \(G^d\) returned by the algorithm DynProg-II.

This allows us to close this section with the following theorem:

**Theorem 4.1** An optimal schedule for a set of two-stage jobs \(\{J_1, \ldots, J_n\}\) on \(m\) two-stage flowshops, where \(J_k = (r_k, t_k)\), can be constructed in time \(O(nm^2(T_{\operatorname{min}}^{2m-1}/m! + T_{\operatorname{min}}^{m}t_{\operatorname{max}}^{m-1}))\) and space \(O(nm(T_{\operatorname{min}}^{2m-1}/m! + T_{\operatorname{min}}^{m}t_{\operatorname{max}}^{m-1}))\), where \(T_{\operatorname{min}}\) and \(T_{\operatorname{max}}\) are the smaller and the larger, respectively, of the values \(\sum_{k=1}^{n} r_k\) and \(\sum_{k=1}^{n} t_k\).

When \(T_{\operatorname{max}} \gg T_{\operatorname{min}}\), Theorem 4.1 provides significant improvements. For example, if \(T_{\operatorname{max}} = T_{\operatorname{min}}^2\), then, for a fixed constant \(m\), the time complexity of the algorithm given in Theorem 4.1 is of the order \(O(nT_{\operatorname{min}}^{2m-1}) = O(nT_{\operatorname{max}}^{m-1/2})\), which almost matches the best pseudo-polynomial time algorithm for the MAKESPAN problem \(P_m|C_{\max}\) on \(m\) machines [15], which can be regarded as a much simpler version of the \(P_m|\text{2FL}|C_{\max}\) problem in which all jobs are one-stage jobs and all machines are one-stage flowshop. On the other hand, the time complexity of algorithm DynProg-I given in the previous section is of the order \(O(nT_{\operatorname{min}}^{3m-3}) = O(nT_{\operatorname{min}}^{m-1}T_{\operatorname{max}}^{m-1})\).

## 5 Approximation algorithms for \(P_m|\text{2FL}|C_{\max}\)

Based on the well-known techniques in approximation algorithms [12], we can use the pseudo-polynomial time algorithms given in previous sections to develop approximation algorithms for the problem \(P_m|\text{2FL}|C_{\max}\). We present such approximation algorithms in this section.

Since the problem \(P_1|\text{2FL}|C_{\max}\) of scheduling two-stage jobs on a single two-stage flowshop can be solved optimally in polynomial time, we will assume \(m \geq 2\) in our discussion in this section.

Let \(S\) be a schedule for a set \(G = \{J_1, J_2, \ldots, J_n\}\) of two-stage jobs on \(m\) two-stage flowshops. The schedule \(S\) can be described by a partition of the job set \(G\) into \(m\) subsets, which can be given by a job index partition \((P_1, P_2, \ldots, P_m)\), which is a disjoint partition of \(\{1, 2, \ldots, n\}\). Thus, under the schedule \(S\), the job subset \(G_h = \{J_k \mid k \in P_h\}\) of \(G\) is assigned to the \(h\)-th flowshop \(M_h\), for all \(h\). The schedule \(S\) can be easily implemented if for each \(h\), we order the jobs in \(G_h\) in Johnson’s order. For each \(h\), let \(C(S, h)\) be the completion time of the flowshop \(M_h\) under the schedule \(S\). Thus, the makespan \(C_{\max}(S)\) of the schedule \(S\) is equal to \(\max_{1 \leq h \leq m}\{C(S, h)\}\). We say that the schedule \(S\) achieves its makespan on the flowshop \(M_h\) if \(C_{\max}(S) = C(S, h)\).

The following lemma will be useful in the analysis of our approximation algorithms.

**Lemma 5.1** Let \(S_k = \langle J_1, J_2, \ldots, J_k \rangle\) be a schedule on a two-stage flowshop \(M\). If we replace each job \(J_i = (r_i, t_i)\) with the job \(J'_i = (r_i + 1, t_i + 1)\) in the schedule \(S_k\), then the completion time \(\tau'_k\) of the resulting schedule \(S'_k = \langle J_1', J_2', \ldots, J_k' \rangle\) is bounded by \(k + 1\) plus the completion time \(\tau_k\) of \(S_k\).
Proof. We prove the lemma by induction on $k$. For $k = 1$, the lemma holds true since it is easy to see that increasing both the $R$-time and $T$-time of the job $J_1$ increases the completion time of the single-job schedule for $\{J_1\}$ by at most 2.

Now consider the case $k > 1$. Consider the “partial” schedule $S_{k-1} = \langle J_1, J_2, \ldots, J_{k-1} \rangle$ which is obtained by taking off the last job $J_k$ from the schedule $S_k$. Let the completion times of the $R$-processor and the $T$-processor of the flowshop $M$ under the schedule $S_{k-1}$ be $\rho_{k-1}$ and $\tau_{k-1}$, respectively. By Lemma 2.1, $T_k = \max \{ \rho_{k-1} + 1, r_k, \tau_{k-1} \} + t_k$.

Now replace each job $J_i = (r_i, t_i)$ in the schedule $S_{k-1}$ with the job $J'_i = (r_i + 1, t_i + 1)$, for $1 \leq i \leq k - 1$. By the inductive hypothesis, the completion time $\tau'_{k-1}$ of the resulting schedule $S'_{k-1} = \langle J'_1, J'_2, \ldots, J'_{k-1} \rangle$ is bounded by $\tau_{k-1} + (k - 1) + 1 = \tau_{k-1} + k$. Again by Lemma 2.1, the completion time $\rho'_{k-1}$ of the $R$-processor of $M$ on the schedule $S'_{k-1}$ is equal to $\rho_{k-1} + (k - 1)$. Now we can add the job $J'_k = (r_k + 1, t_k + 1)$ to the schedule $S'_{k-1}$ to obtain the schedule $S'_k = \langle J'_1, J'_2, \ldots, J'_k \rangle$. By Lemma 2.1, the completion time $\tau'_k$ of the flowshop $M$ under the schedule $S'_k$ is
\[
\tau'_k = \max \{ \rho'_{k-1} + (r_k + 1), \tau'_{k-1} \} + (t_k + 1) \\
\leq \max \{ \rho_{k-1} + (k - 1) + (r_k + 1), \tau_{k-1} + k \} + (t_k + 1) \\
= \max \{ \rho_{k-1} + r_k, \tau_{k-1} + k \} + (t_k + 1) \\
= (\max \{ \rho_{k-1} + r_k, \tau_{k-1} \} + t_k) + (k + 1) \\
= \tau_k + (k + 1).
\]

This completes the proof of the lemma.

Now we are ready to present the approximation algorithm, which is given in Figure 4.

| Algorithm Approx |
|------------------|
| **INPUT:** a set $G = \{J_1, \ldots, J_n\}$ of two-stage jobs, where $J_k = (r_k, t_k)$ for all $k$, and $\epsilon > 0$. |
| **OUTPUT:** a schedule of $G$ on $m$ identical two-stage flowshops |
| 1. let $T_{\text{max}} = \max \{R_0, T_0\}$ and $K = \epsilon \cdot T_{\text{max}}/(nm)$; |
| 2. for $i = 1$ to $n$ do { $r'_i = \lfloor r_i/K \rfloor$; $t'_i = \lfloor t_i/K \rfloor$ }; |
| 3. let $G' = \{J'_1, \ldots, J'_n\}$, where for each $i$, $J'_i = (r'_i, t'_i)$; |
| 4. apply an algorithm $A$ on $G'$, assuming $A$ returns an optimal schedule $S'$ for $G'$, given by a job index partition $(P_1, \ldots, P_m)$; |
| 5. return the schedule $S$ for $G$ that uses the same job index partition $(P_1, \ldots, P_m)$. |

Figure 4: An approximation algorithm for $P_m|2FL|C_{\text{max}}$

We first study how well the schedule $S$ returned by the algorithm can approximation the optimal schedule for the job set $G$ on the $m$ flowshops.

Both the schedule $S'$ for the job set $G'$ and the schedule $S$ for the job set $G$ use the same job index partition $(P_1, \ldots, P_m)$. Suppose that $S$ achieves its makespan $C_{\text{max}}(S)$ on flowshop $M_h$ and that $S'$ achieves its makespan $C_{\text{max}}(S')$ on flowshop $M_{h'}$. Let $S_0$ be an optimal schedule for the job set $G$ that has a job index partition $(P'_1, \ldots, P'_m)$ and achieves its makespan $C_{\text{max}}(S_0)$ on flowshop $M_d$. Let $S'_0$ be the schedule for the job set $G'$ that also uses the job index partition $(P'_1, \ldots, P'_m)$ and achieves its makespan $C_{\text{max}}(S'_0)$ on flowshop $M_{d'}$.

We need some further notations for our analysis. As we defined, for a schedule $S$ based on the job index partition $(P_1, \ldots, P_m)$ and a flowshop $M_h$, $C(S, h)$ denotes the completion time of the flowshop $M_h$ under the schedule $S$. Let $K$ be the number defined in step 1 of the algorithm.
We will use the notation \( C(S/K, h) \) to denote the completion time of the flowshop \( M_h \) under the schedule \( S \) with each \( J_k = (r_k, t_k) \) of the jobs in the job subset \( \{J_i \mid i \in P_h\} \) replaced by the job \((r_k/K, t_k/K)\), i.e., we shrink each of the jobs by a factor \( K \). Note here the jobs may no longer have integral \( R \)-time and \( T \)-time – this will not affect the complexity of our algorithms and the correctness of our analysis because we will only use this notation and our algorithms will not take advantage of this relaxation. With these notations, we have

\[
C_{\text{max}}(S) = C(S, h) = K \cdot C(S/K, h) \quad (1)
\]

\[
\leq K \cdot C(S', h) + Kn \leq K \cdot C(S', h') + Kn \quad (2)
\]

\[
\leq K \cdot C(S_0', d') + Kn \leq K \cdot C(S_0/K, d') + Kn \quad (3)
\]

\[
= C(S_0, d') + Kn \leq C(S_0, d) + Kn \quad (4)
\]

\[
= \text{Opt}(G) + Kn \quad (5)
\]

We explain the derivations in (1)-(5). The first equality in (1) is because by our assumption, the schedule \( S \) achieves its makespan on flowshop \( M_h \). The second equality in (1) is obvious: if we proportionally shrink the \( R \)-time and the \( T \)-time of each job in flowshop \( M_h \) by a factor \( K \), then the completion time of the flowshop \( M_h \) also shrinks by a factor \( K \).

Now consider (2). To simplify the notations without loss of generality, let \( J_1, J_2, \ldots, J_k \) be the schedule on the flowshop \( M_h \) induced from the schedule \( S \). Since there are \( m \geq 2 \) flowshops, we have \( k \leq n - 1 \). The schedule \( S/K \) on the “shrunk” jobs induces a schedule \( \langle J_1/K, J_2/K, \ldots, J_k/K \rangle \) on the flowshop \( M_h \), where \( J_p/K = (r_p/K, t_p/K) \), \( 1 \leq p \leq K \). If we replace each shrunk job \( J_p/K = (r_p/K, t_p/K) \) in the flowshop \( M_h \) by the job \( (r_p/K) + ([r_p/K] + 1) \), with larger \( R \)-time and \( T \)-time, the completion time \( C(S/K)^+, h) \) of the flowshop \( M_h \) under the resulting schedule \((S/K)^+ = \langle (J_1/K)^+, \ldots, (J_k/K)^+ \rangle \) will not be decreased. That is, \( C(S/K, h) \leq C((S/K)^+, h) \).

On the other hand, the schedule \((S/K)^+ \) on the flowshop \( M_h \) can be obtained from the schedule \( S/K \), via the schedule \( S' \) defined in step 4 of the algorithm \textbf{Approx}, as follows: first we replace in \( S/K \) each shrunk job \( J_p/K = (r_p/K, t_p/K) \) in \( M_h \) by the job \( J'_p = ([r_p/K], [t_p/K]) \), as defined in step 3 of the algorithm \textbf{Approx}. Since neither of the \( R \)-time and \( T \)-time of each job is increased, the resulting schedule \( S' \), as given in step 4 of the algorithm that shares the same job index partition with \( S \), has its completion time on the flowshop \( M_h \) not larger than that of \( S/K \). That is, \( C(S', h) \leq C((S/K), h) \). Now the schedule \((S/K)^+ \) on the flowshop \( M_h \) is obtained from the schedule \( S' \) by increasing both \( R \)-time and \( T \)-time of each job in \( M_h \) by 1. By Lemma 5.1, \( C((S/K)^+, h) \leq C(S', h) + (k + 1) \leq C(S', h) + n \) (here we have used the fact \( k \leq n - 1 \)). This, combined with \( C(S/K, h) \leq C((S/K)^+, h) \) proved above, gives immediately \( C(S/K, h) \leq C(S', h) + n \). This proves the first inequality in (2). The second inequality in (2) is because we assume the schedule \( S' \) achieves its makespan on flowshop \( M_{h'} \).

By the algorithm \textbf{Approx}, \( S' \) is an optimal schedule for the job set \( G \) that achieves its makespan on flowshop \( M_{h'} \). By our assumption, \( S_0' \) is also a (not necessarily optimal) schedule for the job set \( G \) that achieves its makespan on flowshop \( M_d \). This explains the first inequality in (3). The second inequality in (3) is based on the observation that if we replace the \( R \)-time and \( T \)-time of each job \( J_p = ([r_p/K], [t_p/K]) \) in flowshop \( M_d \) by not smaller numbers \( r_p/K \) and \( t_p/K \), respectively, the completion time of the flowshop \( M_d \) would not decrease.

The reason for the equality in (4) is the same as that for the second equality in (1). The inequality in (4) is because the schedule \( S_0 \) achieves its makespan on flowshop \( M_d \). Finally, the equality in (5) is because we assume \( S_0 \) is an optimal schedule for the job set \( G \) (here we have used \( \text{Opt}(G) \) for the makespan of an optimal schedule for the job set \( G \)).

According to the derivation in (1)-(5), \( C_{\text{max}}(S) \leq \text{Opt}(G) + Kn \). From \( K = e \cdot T_{\text{max}}/(nm) \), we get \( Kn = e \cdot T_{\text{max}}/m \leq e \cdot \text{Opt}(G) \), where the inequality is based on the obvious fact \( T_{\text{max}}/m \leq \text{Opt}(G) \).
This gives us the following relation:

\[ C_{\text{max}}(S) \leq \text{Opt}(G)(1 + \epsilon) \quad \text{or} \quad C_{\text{max}}(S) / \text{Opt}(G) \leq 1 + \epsilon. \]  

(6)

The time complexity of the algorithm \textbf{Approx} depends on the algorithm \textbf{A} we use in step 4 of the algorithm to construct the optimal schedule \( S' \) for the job set \( G' = \{J_1', J_2', \ldots, J_n'\} \). For example, if we use the algorithm \textbf{DynProg-I} in Figure 3, then by Theorem 3.1, the running time of the algorithm \textbf{DynProg-I}, thus the running time of the algorithm \textbf{Approx} will be \( O(n^2 (R_0')^{m-1} T_0') \), where

\[
R_0' = \sum_{i=1}^{n} r_i' = \sum_{i=1}^{n} \left\lfloor \frac{r_i}{K} \right\rfloor \leq \frac{1}{K} \left( \sum_{i=1}^{n} r_i \right) \leq \frac{nm \cdot T_{\text{max}}}{\epsilon},
\]

here we have used the inequality \( \sum_{i=1}^{n} r_i = R_0 \leq T_{\text{max}} \). Similarly, \( T_0' \leq (nm)/\epsilon \). This shows that the running time of the algorithm \textbf{Approx} is bounded by \( O(n^{2m-1} m^{2m}/\epsilon^{2m-2}) \). This concludes the discussion of this section with the following theorem.

**Theorem 5.2** There is an algorithm for the \( P_m | 2FL| C_{\text{max}} \) problem that on a set \( G \) of \( n \) two-stage jobs and any real number \( \epsilon > 0 \), constructs a schedule for the job set \( G \) on \( m \) two-stage flowshops with a makespan bounded by \( \text{Opt}(G)(1 + \epsilon) \). Moreover, the running time of the algorithm is \( O(n^{2m-1} m^{2m}/\epsilon^{2m-2}) \).

Compared to the approximation algorithm given in [5], which also produces a schedule with makespan bounded by \( \text{Opt}(G)(1 + \epsilon) \) but runs in time \( O(2^{2m-1} n^{2m} m^{2m+1}/\epsilon^{2m-1}) \), our algorithm in Theorem 5.2 gives an obvious improvement on the running time.

When \( m \) is a fixed constant, the algorithm in Theorem 5.2 runs in time polynomial in \( n \) and \( 1/\epsilon \). In the literature of approximation algorithms, such approximation algorithms with a ratio \( 1 + \epsilon \) are called fully polynomial-time approximation schemes (FPTAS) [9]. Thus, Theorem 5.2 claims that the \( P_m | 2FL| C_{\text{max}} \) problem has an FPTAS when \( m \) is a fixed constant.

### 6 Conclusion

Motivated by the current research in data centers and cloud computing, we studied the scheduling problem of two-stage jobs on multiple two-stage flowshops, which in particular addresses the scheduling issues of data transmissions between clients and servers in data centers in the cloud computing framework based on the principle of transparent computing. The problem is NP-hard. Pseudo-polynomial time algorithms for the problem were presented that produce optimal solutions for the problem when the number of flowshops is a fixed constant. In particular, with thorough analysis, we show that for certain cases, much faster pseudo-polynomial time algorithms can be achieved. Approximation algorithms for the problem have also been developed and studied. Our algorithms improve previous known algorithms for the problem.

Needs and considerations in cloud computing practice suggest many further research topics that require the study of variations and extensions of our scheduling model. We list some of them below for future research.

A cloud computing center may have many servers with different powers, ranging from large mainframe computers to small PC’s. The disks connected to the servers and the network bandwidth available for the servers can also differ. Moreover, the disk-read on a server at some moment may even not be needed if the requested data is already in the server’s main memory. This calls for the study of scheduling two-stage jobs on heterogeneous two-stage flowshops. Our scheduling model
can be easily extended to include this situation: suppose that we need to schedule $n$ jobs $J_1, \ldots, J_n$ on $m$ flowshops $M_1, \ldots, M_m$ that may not be identical, then we can represent each two-stage job $J_i$ by $m$ pairs $\{(r_{i,j}, t_{i,j}) \mid 1 \leq j \leq m\}$, where $(r_{i,j}, t_{i,j})$ gives the $R$-time and the $T$-time, respectively, for the job $J_i$ to be processed by the flowshop $M_j$. Of course, constructing optimal schedules and developing good approximation algorithms on this more general model become more challenging. We are currently working on this extended version of the scheduling model.

In many cases, a data request from a client is for a file, which consists of a number of data blocks stored in either secondary or main memory. In a real system, it is possible that the disk-read and network-transmission of the same file are executed in a pipeline manner in units of data blocks. Thus, once a data block of a file is read entirely into the main memory, the block can be transmitted via networks to the client even if some other data blocks for the file have not been read into the main memory, yet. In particular, in such a model, preemption of processing data transmissions from servers to clients becomes possible: after transmitting a few data blocks for a file $F$, a server may switch to processing a different task, and come back later to continue transmitting the remaining data blocks for the file $F$. Under this assumption, each data request from a client can be given by its size, corresponding to the number of data blocks of the data, and the data request can be decomposed into a continuous sequence of two-stage jobs, each corresponds to the disk-read and network-transmission of a data block. Preemptions now are allowed on processing a data request from a client. However, frequent preemptions for processing data requests should be avoided since restarting disk-read for a file will require new disk search, which is significantly more time-consuming compared to reading a data block. Therefore, when we study scheduling on this model, penalty on preemptions should be considered.

Scheduling with job precedences is very common in cloud computing practice. For example, a user who wants to run a Microsoft application on a transparent computing platform may need from the cloud both the code of the application as well as the code of Microsoft Windows software. However, the application cannot be installed until the Windows software is installed on the client device [21]. As a consequence, there is a need to study the scheduling problems under our model in which job precedence is presented.

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