SOME QUESTIONS OF EQUIVARIANT MOVABILITY

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ABSTRACT. In this article some questions of equivariant movability, connected with the substitution of the acting group \( G \) on closed subgroup \( H \) and with transitions to spaces of \( H \)-orbits and \( H \)-fixed points spaces, are investigated. In a special case, the characterization of equivariantly movable \( G \)-spaces is given.

1. Introduction

This paper is devoted to equivariant movability of \( G \)-spaces, i.e., topological spaces endowed with an action of a given compact group \( G \).

More precisely, in §3 we define the notion of equivariant movability or \( G \)-movability and we prove several theorems, including the following ones. If \( X \) is \( p \)-paracompact and \( H \subseteq G \) is a closed subgroup, then \( G \)-movability of \( X \) implies its \( H \)-movability (§3 Theorem 1). \( G \)-movability of \( X \) also implies movability of the space \( X[H] \) of \( H \)-fixed points in \( X \) (§4, Theorem 3). In particular, equivariant movability of a \( G \)-space \( X \) implies ordinary movability of the topological space \( X \) (§3 Corollary 1). We construct a non-trivial example which shows, that the converse, in general, is not true, even if we take for \( G \) the cyclic group \( Z_2 \) of order 2 (§5 Example 1). If \( X \) is a metrizable \( G \)-movable space and \( H \) is a closed normal subgroup of \( G \), then the space \( X|_H \) of its \( H \)-orbits is also \( G \)-movable (§6 Theorem 5). In the case \( H = G \) we obtain that \( G \)-movability of a metrizable \( G \)-space implies ordinary movability of the orbit space \( X|_G \) (§6 Corollary 2). The last assertion, in general, is not invertible (§6 Example 2). However, if \( X \) is metrizable, \( G \) is a compact Lie group and the action of \( G \) on \( X \) is free, then \( X \) is \( G \)-movable if and only if the orbit space \( X|_G \) is movable (§7 Theorem 7).

Examples 2 (§6) and 3 (§8) show that in the last theorem the assumption that the group \( G \) is a Lie group and the assumption that the action is free cannot be omitted.

Some of the above listed results with an outline of proof were given in [9].

Let us denote the category of all topological spaces and continuous maps by \( \text{Top} \), the category of all metrizable spaces and continuous maps by \( \text{M} \) and the category of all \( p \)-paracompact spaces and continuous maps by \( \text{P} \). Always in this article it is assumed that all topological spaces are \( p \)-paracompact spaces and the group \( G \) is compact.

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The reader is referred to the books by K. Borsuk [4] and by S. Mardešić and J. Segal [15] for general information about shape theory and to the book by G. Bredon [5] for introduction to compact transformation groups.

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2. Basic notions and conventions concerning equivariant topology

Let $G$ be a topological group. A topological space $X$ is called a $G$-space if there is a continuous map $\theta : G \times X \to X$ of the direct product $G \times X$ into $X$, $\theta(g, x) = gx$, such that

1) $g(hx) = (gh)x; \quad 2) \ ex = x,$

for all $g, h \in G, \ x \in X$; here $e$ is the unity of $G$. Such a (continuous) map $\theta : G \times X \to X$ is called an (continuous) action of the group $G$ on the topological space $X$. An evident example is the so called trivial action of $G$ on $X$: $gx = x$, for all $g \in G, \ x \in X$. Another example is the action of the group $G$ on itself, defined by $(g, x) \to gx$ for all $g \in G, \ x \in G$.

If $X$ and $Y$ are $G$-spaces, then so is $X \times Y$, where $g(x, y) = (gx, gy), \ g \in G, \ (x, y) \in X \times Y$.

A subset $A$ of a $G$-space $X$ is called invariant provided $g \in G, a \in A$ implies $ga \in A$. It is evident, that an invariant subset of a $G$-space is itself a $G$ space. If $A$ is an invariant subset of a $G$-space $X$, then every neighborhood of $A$ contains an open invariant neighborhood of $A$ (see [17], Proposition 1.1.14).

Let $X$ be any $G$-space and let $H$ be a closed and normal subgroup of the group $G$. The set $Hx = \{hx; \ h \in H\}$ is called the $H$-orbit of the point $x \in X$. Clearly the $H$-orbits of any two points in $X$ are either equal or disjoint, in other words $X$ is partitioned by its $H$-orbits. We denote the set of all $H$-orbits of the $G$-space $X$ by $X|_H$. The set $X|_H$ endowed with the quotient topology is called the $H$-orbit space of $X$. There is a continuous action of the group $G$ on the space $X|_H$ defined by the formula $gHx = Hgx, \ g \in G, x \in X$. So, $X|_H$ is a $G$-space. In case $H = G$ the $G$-orbit of the point $x \in X$ is called the orbit of the point $x$ and the $G$-orbit space is called the orbit space of the $G$-space $X$.

We denote by $X[H]$ the subspace of fixed points of $H$ on $X$, or the $H$-fixed point subspace of the $G$-space $X$. Let us recall that $X[H] = \{x \in X; \ hx = x, \text{for any} \ h \in H\}$.

The set $G_x = \{g \in G; \ g(x) = x\}$ is a closed subgroup of the group $G$, for every $x \in X$. $G_x$ is called the stationary subgroup (or stabilizer) at the point $x$. The action of the group $G$ on $X$ (or the $G$-space $X$) is called free if the stationary subgroup $G_x$ is trivial, for every $x \in X$. It is clear that $G_{gx} = gG_xg^{-1}$, i.e., the stationary subgroups at any two points of the same orbit are conjugate. The orbits $Gx$ and $Gy$ of points $x$ and $y$, respectively, are said to have the same type if the stationary subgroups $G_x$ and $G_y$ are conjugate.

Let $X, Y$ be $G$-spaces. A (continuous) map $f : X \to Y$ is called a $G$-map, or an equivariant map, if $f(gx) = gf(x)$ for every $g \in G, \ x \in X$. Note that the identity map $i : X \to X$ is equivariant and the composition of equivariant maps is equivariant. Therefore, all $G$-spaces and equivariant maps form a category. Let us denote the category of all topological $G$-spaces and equivariant maps by $Top_G$, the category of all metrizable $G$-spaces and equivariant maps by $M_G$ and the category of all $p$-paracompact $G$-spaces and equivariant maps by $P_G$.

Let $Z$ be a $G$-space and let $Y \subseteq Z$ be an invariant subset. A $G$-retraction of $Z$ to $Y$ is a $G$-map $r : Z \to Y$ such that $r|_Y = 1_Y$.

Let $K_G$ be class of $G$-spaces. A $G$-space $Y$ is called a $G$-absolute neighborhood retract for the class $K_G$ or a $G – \text{ANR}(K_G)$ ($G$-absolute retract for the class $K_G$ or
a $G - AR(K_G)$, provided $Y \subseteq K_G$ and whenever $Y$ is a closed invariant subset of a $G$-space $Z \subseteq K_G$, then there exist an invariant neighborhood $U$ of $Y$ and a $G$-retraction $r : U \to Y$ (there exists a $G$-retraction $r : Z \to Y$).

A $G$-space $Y$ is called a $G$-absolute neighborhood extensor for the class $K_G$ or a $G - ANE(K_G)$ ($G$-absolute extensor for the class $K_G$ or a $G - AE(K_G)$), provided for any $G$-space $X \in K_G$ and any closed invariant subset $A \subseteq X$, every equivariant map $f : A \to Y$ admits an equivariant extension $\tilde{f} : U \to Y$, where $U$ is an invariant neighborhood of $A$ in $X$ ($\tilde{f} : X \to Y$).

3. Movability and equivariant movability

The important shape invariant, called movability, was originally introduced by K. Borsuk [2] for metric compacta. Mardešić and Segal [14] generalized the notion of movability to compacta using the $ANR$-system approach. Kozlowski and Segal in [11] gave a categorical description of this property which applied to arbitrary topological spaces.

Following Mardešić and Segal [14], let us define the notion of equivariant movability or $G$-movability:

**Definition 1.** An inverse $G$-system $\underline{X} = \{X_\alpha, p_{aa'}, A\}$ where each $X_\alpha$, $\alpha \in A$, is a $G$-space and every $p_{aa'} : X_{\alpha'} \to X_\alpha$, $\alpha \leq \alpha'$, is a $G$-homotopy class, is called equivariantly movable or $G$-movable if for every $\alpha \in A$, there exists an $\alpha' \in A$, $\alpha' \geq \alpha$ such that for all $\alpha'' \in A$, $\alpha'' \geq \alpha$ there exists a $G$-homotopy class $r^{\alpha'\alpha''} : X_{\alpha'} \to X_{\alpha''}$ such that

$$p_{aa''} \circ r^{\alpha'\alpha''} = p_{aa'}.$$ 

It is known (see [1], Theorem 2) that every $G$-space $X$ admits a $G - ANR$-expansion in the sense of Mardešić (see [15], I, § 2.1), which is the same as saying that there is an inverse $G - ANR$-system ($G$-system consisting of $G - ANR$'s) $\underline{X} = \{X_\alpha, p_{aa'}, A\}$ associated with $X$ in the sense of Morita [16].

**Definition 2.** A $G$-space $X$ is called equivariantly movable or $G$-movable if there is an equivariantly movable inverse $G - ANR$-system $\underline{X} = \{X_\alpha, p_{aa'}, A\}$ associated with $X$.

Note that the last definition of equivariant movability coincides with the notion of ordinary movability if $G = \{e\}$ is the trivial group.

Let $X$ be an equivariantly movable $G$-space. The evident question arises: does movability of the space $X$ follows from its equivariant movability? The following, more general theorem gives a positive answer (Corollary 1) to the above question.

**Theorem 1.** Let $H$ be a closed subgroup of a group $G$. Every $G$-movable $G$-space is $H$-movable.

To prove this theorem the next result is important.

**Theorem 2.** Let $H$ be a closed subgroup of a group $G$. Every $G - AR(P_G)$ ($G - ANR(P_G)$)-space is an $H - AR(P_H)(H - ANR(P_H))$-space.

**Proof.** According to a theorem of de Vries ([7], Theorem 4.4), it is sufficient to show that if $X$ is a $p$-paracompact $H$-space, then the twisted product $G \times_H X$ is also $p$-paracompact. Indeed, since $X$ is $p$-paracompact and $G$ is compact, $G \times X$ is $p$-paracompact. Therefore, the twisted product $G \times_H X$ is $p$-paracompact. \[\Box\]
Proof of Theorem 1. Let $X$ be any equivariantly movable $G$-space. With respect to the theorem of Smirnov (13, Theorem 1.3), there is a closed and equivariant embedding of the $G$-space $X$ to some $G - AR(P_G)$-space $Y$. Let us consider all open $G$-invariant neighborhoods of type $F_{\rho}$ of the $G$-space $X$ in $Y$. By a result of R. Palais (17, Proposition 1.1.14), these neighborhoods form a cofinal family in the set of all open neighborhoods of $X$ in $Y$, in particular, in the set of all open and $H$-invariant neighborhoods of the $H$-space $X$ in the $H$-space $Y$, which, by Theorem 1 is an $H - AR(P_H)$-space. Hence, from the $G$-movability of the above mentioned family follows its $H$-movability, i.e. from the $G$-movability of the $G$-space $X$ follows the $H$-movability of the $H$-space $X$. $\Box$

From Theorem 1 we obtain the following corollary if we consider the trivial subgroup $H = \{e\}$ of the group $G$.

Corollary 1. Every equivariantly movable $G$-space $X$ is movable.

The converse, in general, is not true, even if one takes for $G$ the cyclic group $Z_2$ of order 2 (see Example 1).

4. MOVABILITY OF THE $H$-FIXED POINT SPACE

Theorem 3. Let $H$ be a closed subgroup of a group $G$. If a $G$-space $X$ is equivariantly movable, then the $H$-fixed point space $X[H]$ is movable.

The proof requires the use of the following theorem.

Theorem 4. Let $H$ be a closed subgroup of a group $G$. Let $X$ be a $G - AR(P_G)(G - ANR(P_G))$-space. Then the $H$-fixed point space $X[H]$ is an $AR(P)(ANR(P))$-space.

Proof. Let $X$ be a $G - AR(P_G)(G - ANR(P_G))$-space. By Theorem 2, it is sufficient to prove the theorem in the case $H = G$. I.e., we must prove that $X[G]$ is $AR(P)$-space. By a theorem of Smirnov (13, Theorem 1.3), we can consider $X$ as a closed $G$-subspace of a $G - AR(P_G)$-space $C(G, V) \times \prod D_{\lambda}$ where $V$ is a normed vector space and thus an $AE(M)$-space, $C(G, V)$ is the space of continuous maps from $G$ to $V$ with the compact-open topology and with the action $(g'f)(g) = f(gg')$, $g, g' \in G, f \in C(G, V)$ of the group $G$ and $D_{\lambda}$ is a closed ball of a finite-dimensional Euclidean space $E_{\lambda}$ with the orthogonal action of the group $G$.

First, let us prove that the set $(C(G, V) \times \prod D_{\lambda})[G]$ of all fixed points of the $G$-space $C(G, V) \times \prod D_{\lambda}$ is an $AR(P)$-space. The spaces $C(G, V)$ and $E_{\lambda}$ are normed spaces. Since the actions of the group $G$ on $C(G, V)$ and $E_{\lambda}$ are linear, the sets $C(G, V)[G]$ and $E_{\lambda}[G]$ will be closed convex sets of locally convex spaces $C(G, V)$ and $E_{\lambda}$, respectively. Therefore, by a well-known theorem of Kuratowski and Dugundji [3], $C(G, V)$ and $E_{\lambda}$ are absolute retracts for metrizable spaces. By a theorem of Lisica [12], they are also absolute retracts for $p$-paracompact spaces. For a closed ball $D_{\lambda} \subset E_{\lambda}$ the last conclusion is true since the set $D_{\lambda}[G] = D_{\lambda} \cap E_{\lambda}[G]$ is closed and convex in $E_{\lambda}$.

Since the group $G$ acts on the product $C(G, V) \times \prod D_{\lambda}$ coordinate-wise, $(C(G, V) \times \prod D_{\lambda})[G] = C(G, V)[G] \times \left(\prod D_{\lambda}\right)[G]$.

Hence, $(C(G, V) \times \prod D_{\lambda})[G]$ is an $AR(P)$-space, because it is a product of two $AR(P)$-spaces.
Now let us prove that $X[G]$ is an $AR(P)$-space. Since $X$ is a $G - AR(P_G)$-space, it is a $G$-retract of the product $C(G, V) \times \prod D_\lambda$. Therefore, $X[G]$ is a retract of the $AR(P)$-space $(C(G, V) \times \prod D_\lambda)[G]$, hence, it is an $AR(P)$-space.

The absolute neighborhood retract case is proved similarly. □

Proof of Theorem 3. Let $X$ be a $G$-movable space. By Theorem 1, it is sufficient to prove the theorem in the case $H = G$. So, we must prove movability of the space $X[G]$ of all $G$-fixed points. We consider the $G$-space $X$ as a closed and $G$-invariant space of some $G - AR(P_G)$-space $Y$ ([13], Theorem 1.3). The family of all open, $G$-invariant $F_\sigma$-type neighborhoods $U_\alpha$ of the $G$-space $X$ in $Y$, is cofinal in the set of all open neighborhoods of $X$ in $Y$ ([17], Proposition 1.1.14). It consists of $G - ANR(P_G)$-spaces. The intersections $U_\alpha \cap Y[G] = U_\alpha[G]$ are $ANR(P)$-spaces (Theorem 4). They form a cofinal family of neighborhoods of the space $X[G]$ in $Y[G]$. Indeed, for any neighborhood $U$ of the set $X[G]$ in $Y[G]$ there is a neighborhood $V$ of the set $X[G]$ in $Y$ such that $V \cap Y[G] = U$. Then the set $W = (Y \setminus Y[G]) \cup V$ is a neighborhood of the set $X$ in $Y$, moreover, $W \cap Y[G] = U$. There is an $\alpha$ such that $U_\alpha \subset W$ and therefore $U_\alpha[G] \subset U$. So the family of neighborhoods $U_\alpha[G]$ is cofinal.

Since $X$ is $G$-movable, for every $U_\alpha$ there is a neighborhood $U_{\alpha'} \subset U_\alpha$ such that, for any other neighborhood $U_{\alpha''} \subset U_{\alpha'}$, there exists a $G$-equivariant homotopy $F : U_{\alpha'} \times I \to U_\alpha$ such that $F(y, 0) = y$ and $F(y, 1) \in U_{\alpha''}$, for any $y \in U_{\alpha'}$. It is not difficult to verify that the homotopy $F[G] : U_{\alpha'}[G] \times I \to U_\alpha[G]$, induced by $F$, satisfies the condition of movability of $X[G]$. □

5. EXAMPLE OF A MOVABLE, BUT NOT EQUIVARIANTLY MOVABLE SPACE

Example 1. We will use the idea of S. Mardešić [13]. Let us consider the unit circle $S = \{z \in C; \ |z| = 1\}$. Let us denote $B = [S \times \{1\}] \cup \{(1) \times S\}$. $B$ is the wedge of two copies of the unit circle $S$ with base point $\{1\}$. Let us define a continuous map $f : B \to B$ by the formulas:

$$f(z, 1) = \begin{cases} (z^4, 1), & 0 \leq \arg(z) \leq \frac{\pi}{7} \\ (1, z^4), & \frac{\pi}{2} \leq \arg(z) \leq \pi \\ (z^{-4}, 1), & \pi \leq \arg(z) \leq \frac{3\pi}{2} \\ (1, z^{-4}), & \frac{3\pi}{2} \leq \arg(z) \leq 2\pi \end{cases}$$

$$f(1, t) = \begin{cases} (t^{-4}, 1), & 0 \leq \arg(t) \leq \frac{\pi}{2} \\ (1, t^{-4}), & \frac{\pi}{2} \leq \arg(t) \leq \pi \\ (t^4, 1), & \pi \leq \arg(t) \leq \frac{3\pi}{2} \\ (1, t^4), & \frac{3\pi}{2} \leq \arg(t) \leq 2\pi \end{cases}$$

for every $z$ and $t$ from $S$. Let us consider the $ANR$-sequences

$$B \xleftarrow{t^f} B \xleftarrow{t} B \xleftarrow{f} \ldots$$

and

$$\Sigma B \xleftarrow{\Sigma f} \Sigma B \xleftarrow{\Sigma f} \Sigma B \xleftarrow{\Sigma f} \ldots$$

where $\Sigma$ is the operation of suspension. Let us denote

$$P = \lim \{B, f\}.$$
Then
\[ \Sigma P = \lim_{\rightarrow} \{ \Sigma B, \Sigma f \}. \]
Let us define an action of the group \( G = \{ e, g \} \) on \( \Sigma B \) by the formulas
\[ e[x, t] = [x, t]; \quad g[x, t] = [x, -t]. \]
for every \([x, t] \in \Sigma B\), \(-1 \leq t \leq 1\). It induces an action on \( \Sigma P \).

**Proposition 1.** The space \( \Sigma P \) has trivial shape, but it is not \( Z_2 \)-movable.

**Proof.** The triviality of shape of the space \( \Sigma P \) is proved by the method of Mardešić [13]. Let us prove that the space \( \Sigma P \) is not \( Z_2 \)-movable. Consider the set \( \Sigma P[Z_2] \) of all fixed-points of \( Z_2 \)-space \( \Sigma P \). It is obvious that \( \Sigma P[Z_2] = P \). Hence, by Theorem 3, it is sufficient to prove the following proposition. \( \square \)

**Proposition 2.** The space \( P \) is not movable.

**Proof.** Since the movability of an inverse system remains unchanged under the action of a functor, it is sufficient to prove non-movability of the inverse sequence of groups
\[ \pi_1(B) \overset{f_*}{\leftarrow} \pi_1(B) \overset{f_*}{\leftarrow} \pi_1(B) \overset{f_*}{\leftarrow} \cdots, \]
where \( \pi_1(B) \) is the fundamental group of the space \( B \) and \( f_* \) is the homomorphism induced by the mapping \( f : B \to B \).

It is known that for sequences of groups movability implies the following condition of Mittag-Leffler, abbreviated as \( ML \) ([14], p. 166, Corollary 4):

The inverse system \( \{ G_{\alpha'}, p_{\alpha'}, A \} \) of the \( pro-\) \( GROUP \) category is said to be \( ML \) provided for every \( \alpha \in A \), there exist \( \alpha' \in A \), \( \alpha' \geq \alpha \), such that \( p_{\alpha'}(G_{\alpha'}) = p_{\alpha''}(G_{\alpha''}) \), for any \( \alpha'' \in A \), \( \alpha'' \geq \alpha \).

Thus, it sufficient to prove that the sequence (1) does not satisfy condition \( ML \). Let us observe that \( \pi_1(B) \) is a free group with two generators \( a \) and \( b \), and \( f_* \) is the homomorphism defined by the formulas
\[ f_*(a) = aba^{-1}b^{-1}, \quad f_*(b) = a^{-1}b^{-1}ab. \]

\( f_* \) is a monomorphism, because \( f_*(a) \neq f_*(b) \), but not an epimorphism, because, for example, \( f_*(x) \neq a \), for all \( x \in \pi_1(B) \). Hence, for any natural \( m \) and \( n \), \( \text{Im} f_*^m \subseteq \text{Im} f_*^n \) only if \( m > n \). It means that the inverse sequence (1) does not satisfy condition \( ML \). \( \square \)

6. Movability of the orbit space

**Theorem 5.** Let \( X \) be a metrizable \( G \)-space. If \( X \) is \( G \)-movable then for any closed and normal subgroup \( H \) of the group \( G \), the \( H \)-orbit space \( X|_H \) is also \( G \)-movable.

**Proof.** Without losing generality one may suppose that \( X \) is a closed \( G \)-invariant subset of some \( G-AR(M_G) \)-space \( Y \) ([18], Theorem 1.1). \( X|_H \) is a closed \( G \)-invariant subset of \( Y|_H \) ([5], Theorem 3.1).

Let \( \{ X_{\alpha}, \alpha \in A \} \) be the family of all \( G \)-invariant neighborhoods of \( X \) in \( Y \). Let us consider the family \( \{ X_{\alpha}|_H, \alpha \in A \} \), where each \( X_{\alpha}|_H \in G-ANR(M_G) \) and is a \( G \)-invariant neighborhood of \( X|_H \) in \( Y|_H \). Let us prove that the family \( \{ X_{\alpha}|_H, \alpha \in A \} \) is cofinal in the family of all neighborhoods of \( X|_H \) in \( Y|_H \). Let \( U \) be an arbitrary neighborhood of \( X|_H \) in \( Y|_H \). By a theorem of Palais ([17], Proposition 1.1.14), there exists a \( G \)-invariant neighborhood \( V \supseteq X|_H \) laying in \( U \). Let us denote \( \tilde{V} = (pr)^{-1}(V), \)
where \( pr : Y \to Y|_H \) is the \( H \)-orbit projection. It is evident that \( \tilde{V} \) is a \( G \)-invariant neighborhood of the space \( X \) in \( Y \) and \( V = \tilde{V}|_H \). So in any neighborhood of the space \( X|_H \) in \( Y|_H \), there is a neighborhood of type \( X_\alpha|_H \), where \( X_\alpha \) is a \( G \)-invariant neighborhood of \( X \) in \( Y \).

Now let us prove the \( G \)-movability of the space \( X|_H \). Let \( X \) be \( G \)-movable. It means that the inverse system \( \{X_\alpha, i_{\alpha\alpha'}, A\} \) is \( G \)-movable. We must prove that the induced inverse system \( \{X_\alpha|_H, i_{\alpha\alpha'}|_H, A\} \) is \( G \)-movable. Let \( \alpha \in A \) be any index. By the \( G \)-movability of the inverse system \( \{X_\alpha, i_{\alpha\alpha'}, A\} \), there is \( \alpha' \in A \), \( \alpha' > \alpha \), such that for any other index \( \alpha'' \in A \), \( \alpha'' > \alpha \), there exists a \( G \)-mapping \( r_{\alpha'\alpha''} : X_{\alpha'} \to X_{\alpha''} \), which makes the following diagram \( G \)-homotopy commutative

\[
\begin{array}{c}
X_\alpha \\
\downarrow \downarrow i_{\alpha\alpha'} \downarrow \downarrow r_{\alpha'\alpha''} \\
X_{\alpha'} \\
\end{array}
\]

Diagram 1.

It turns out that, for given \( \alpha \in A \), the obtained index \( \alpha' \in A \), \( \alpha' > \alpha \), also satisfies the condition of \( G \)-movability of the inverse system \( \{X_\alpha|_H, i_{\alpha\alpha'}|_H, A\} \). This is obvious, because the \( G \)-homotopy commutativity of Diagram 1 implies the \( G \)-homotopy commutativity of the following diagram

\[
\begin{array}{c}
X_\alpha|_H \\
\downarrow \downarrow i_{\alpha\alpha'}|_H \downarrow \downarrow r_{\alpha'\alpha''}|_H \\
X_{\alpha'}|_H \\
\end{array}
\]

Diagram 2.

where \( r_{\alpha'\alpha''}|_H : X_{\alpha'}|_H \to X_{\alpha''}|_H \) is induced by the mapping \( r_{\alpha'\alpha''} \). So, the \( G \)-movability of the space \( X|_H \) is proved. \( \square \)

**Corollary 2.** Let \( X \) be a metrizable \( G \)-space. If \( X \) is \( G \)-movable, then the orbit space \( X|_G \) is movable.

**Proof.** In the case \( H = G \) from the last theorem we obtain that the orbit space \( X|_G \) with the trivial action of the group \( G \) is \( G \)-movable. Therefore, it will be movable by Corollary 1. \( \square \)

Corollary 2 in general is not invertible:

**Example 2.** Let \( \Sigma \) be a solenoid. It is known ([4], Theorem 13.5) that \( \Sigma \) is a non-movable compact metrizable Abelian group. By Corollary 1, the solenoid \( \Sigma \) with the natural group action is not \( \Sigma \)-movable although the orbit space \( \Sigma|_\Sigma \) as a one-point set is movable.

The converse of Corollary 2 is true if the group \( G \) is a Lie group and the action is free (see Theorem 7).
7. Equivariant movability of a free $G$-space

**Theorem 6.** Let $G$ be a compact Lie group and let $Y$ be a metrizable $G$-AR$(M_G)$-space. Suppose that a closed invariant subset $X$ of $Y$ has an invariant neighborhood whose orbits have the same type. If the orbit space $X|_G$ is movable, then $X$ is equivariantly movable.

**Proof.** The orbit space $X|_G$ is closed in $Y|_G$, which is a $G$-AR$(M)$-space. Let $U$ be an arbitrary invariant neighborhood of $X$ in $Y$. By the assumption of the theorem, it follows that there exists a cofinal family of neighborhoods of $X$ in $Y$, whose orbits have the same type. Therefore, one may suppose that all orbits of the neighborhood $U$ have the same type. The orbit set $U|_G$ will be a neighborhood of the space $X|_G$ in $Y|_G$. From the movability of $X|_G$ it follows that, for the neighborhood $U|_G$, there is a neighborhood $\tilde{V}$ of the space $X|_G$ in $Y|_G$, which lies in the neighborhood $U|_G$ and contracts to any preassigned neighborhood of the space $X|_G$.

Let us denote $V = (pr)^{-1}(\tilde{V})$, where $pr : Y \to Y|_G$ is the orbit projection. It is evident that $V$ is an invariant neighborhood of the space $X$ lying in $U$. Let us prove that $V$ contracts in $U$ to any preassigned invariant neighborhood of $X$. Let $W$ be any invariant neighborhood of $X$ in $Y$. We must prove the existence of an equivariant homotopy $F : V \times I \to U$, which satisfies the condition

$$F(x, 0) = x, \quad F(x, 1) \in W,$$

for any $x \in V$. Since $W|_G$ is a neighborhood of the space $X|_G$ in $Y|_G$, there is a homotopy $\tilde{F} : V|_G \times I \to U|_G$ such that

$$F(\tilde{x}, 0) = \tilde{x}, \quad \tilde{F}(\tilde{x}, 1) \in W|_G,$$

for any $\tilde{x} \in V|_G$. The homotopy $\tilde{F} : V|_G \times I \to U|_G$ preserves the $G$-orbit structure, because $V \subset U$ and all orbits of $U$ have the same types (see Diagram 3).

![Diagram 3](image)

By the covering homotopy theorem of Palais ([17], Theorem 2.4.1), there is an equivariant homotopy $F : V \times I \to U$, which covers the homotopy $\tilde{F}$ and satisfies $F(x, 0) = i(x) = x$. That is, the following diagram is commutative (Diagram 4).

![Diagram 4](image)
Then $F : V \times I \to U$ is the designed equivariant homotopy. It only remains to verify that $F(x, 1) \in W$. But this immediately follows from (2) and the commutativity of Diagram 4.

**Theorem 7.** Let $G$ be a compact Lie group. A metrizable free $G$-space $X$ is equivariantly movable if and only if the orbit space $X|_G$ is movable.

**Proof.** The necessity in a more general case was proved in Corollary 2. Let us prove the sufficiency. Let the orbit space $X|_G$ be movable. One can consider the $G$-space $X$ as a closed and invariant subset of some $G - AR(M_G)$-space $Y$. Let $P \subset X$ be any orbit. From the existence of slices it follows that around $P$ there is such an invariant neighborhood $U(P)$ in $Y$ that $typeQ \geq typeP$, for any orbit $Q$ from $U(P)$ ([5], Corollary 5.5). Since the action of the group $G$ on $X$ is free, $typeQ = typeP = typeG$, for any orbit $Q$ lying in $U(P)$. Let us denote $V = \cup\{U(P); \ P \in X|_G\}$. It is evident that $V$ is an invariant neighborhood of the space $X$ in $Y$ and that all of its orbits have the same type. Then, by Theorem 6, $X$ is equivariantly movable. □

Example 2 shows that the assumption that $G$ is a Lie group is essential in the above theorem. The Example 3 which follows shows that the condition of freeness of the action of the group $G$ is also essential in the above theorem.

8. **Example of a non-free not $Z_2$-movable space with a movable orbit space**

**Example 3.** Let us consider the space $P = \lim\{B, f\}$ constructed in Example 1. Let us define an action of the group $Z_2 = \{e, g\}$ on the space $B$ by the formulas

\[
\begin{align*}
e(z, 1) &= (z, 1) \\
e(1, t) &= (1, t) \\
g(z, 1) &= (1, z^{-1}) \\
g(1, t) &= (t^{-1}, 1),
\end{align*}
\]

for any $z$ and $t$ from $S$. $B$ is a $Z_2 - ANR(M_{Z_2})$ space with the fixed-point $b_0 = (1, 1)$.

**Proposition 3.** The mapping $f : B \to B$, defined by formulas (3), is equivariant.

**Proof.** It is necessary to prove the following two equalities:

\[
\begin{align*}
f(g(z, 1)) &= g(f(z, 1)) \\
f(g(1, t)) &= g(f(1, t)),
\end{align*}
\]

for any $z$ and $t$ from $S$. Let us prove the first one. Consider the following cases:

**Case 1.** $0 \leq argz \leq \frac{\pi}{2} \iff \frac{3\pi}{2} \leq argz^{-1} \leq 2\pi$.

Then $f(g(z, 1)) = f(1, z^{-1}) = (1, z^{-4}) = g(z^4, 1) = g(f(z, 1))$.

**Case 2.** $\frac{\pi}{2} \leq argz \leq \pi \iff \pi \leq argz^{-1} \leq \frac{3\pi}{2}$.

Then $f(g(z, 1)) = f(1, z^{-1}) = (z^{-4}, 1) = g(1, z^4) = f(z, 1)$.

**Case 3.** $\pi \leq argz \leq \frac{3\pi}{2} \iff \frac{\pi}{2} \leq argz^{-1} \leq \pi$.

Then $f(g(z, 1)) = f(1, z^{-1}) = (1, z^4) = g(z^{-4}, 1) = f(z, 1)$.

**Case 4.** $\frac{3\pi}{2} \leq argz \leq 2\pi \iff 0 \leq argz^{-1} \leq \frac{\pi}{2}$.

Then $f(g(z, 1)) = f(1, z^{-1}) = (z^4, 1) = g(1, z^{-4}) = f(z, 1)$.

The second equality of (4) is proved in a similar way. □
Proposition 4. $P$ is a connected, compact, metrizable and equivariantly non-movable $Z_2$-space which is free at all points except at the only fixed point $(b_0, b_0, \ldots)$ and $sh(P|_{Z_2})=0$.

Proof. $P$ is a $Z_2$-space because it is an inverse limit of $Z_2 - ANR(M_{Z_2})$-spaces $B$ and $f$ is an equivariant mapping. The uniqueness of the fixed point is evident. The connectedness, compactness and metrizability follows from the properties of inverse systems ([8], Theorem 6.1.20, Corollary 4.2.5). The non $Z_2$-movability follows from Proposition 2 and Corollary 1.

Let us prove that $sh(P|_{Z_2})=0$ and thus the orbit space $P|_{Z_2}$ is movable.

Let $X = \lim \{B|_{Z_2}, f|_{Z_2}\}$. $X$ is equimorphic to the orbit space $P|_{Z_2}$. Indeed, let us define a mapping $h : X \to P|_{Z_2}$ in the following way:

$$h([(x_1], [x_2], \ldots)) = [(x_1, x_2, \ldots)]$$

where $((x_1], [x_2], \ldots) \in X$, and $x_1, x_2, \ldots$ are selected from the classes $[x_1], [x_2], \ldots$ in such way that $(x_1, x_2, \ldots) \in P$ or what is the same $f(x_{n+1}) = x_n$, for any $n = 1, 2, \ldots$. Let us prove that the mapping $h$ is defined correctly. Let $x_1, \bar{x}_2, \ldots$ be some other representatives of the classes $[x_1], [x_2], \ldots$, respectively, satisfying the conditions $f(g_{x_{n+1}}) = x_n$ for any $n \in N$. Since each class $[x_n]$ has two representatives: $x_n$ and $g_{x_n}$, where $g \in Z_2 = \{e, g\}$, either $\bar{x}_n = g_{x_n}$ or $\bar{x}_n = x_n$. But it is obvious that, if for some $n_0 \in N$, $\bar{x}_{n_0} = g_{x_{n_0}}$, then, for any $n \in N$, $\bar{x}_n = g_{x_n}$, because $f$ is equivariant. Thus, in the case of another choice of the representatives of the classes $[x_1], [x_2], \ldots$, we have

$$h([(x_1], [x_2], \ldots)) = [(\bar{x}_1, \bar{x}_2, \ldots)] = [(g_{x_1}, g_{x_2}, \ldots)] = [g(x_1, x_2, \ldots)] = [(x_1, x_2, \ldots)].$$

However, $h$ is a continuous bijection and thus, it is a homeomorphism ([8], Theorem 3.1.13).

Consequently,

$$P|_{Z_2} = \lim \{B|_{Z_2}, f|_{Z_2}\},$$

where $B|_{Z_2} \cong S$ and the mapping $\bar{f} = f|_{Z_2} : S \to S$ is defined by the formulas:

$$\bar{f}(z) = \begin{cases} z^4, & 0 \leq \arg(z) \leq \frac{\pi}{2} \\ z^{-4}, & \frac{\pi}{2} \leq \arg(z) \leq \frac{3\pi}{2} \\ z^4, & \frac{3\pi}{2} \leq \arg(z) \leq 2\pi \end{cases}$$

for any $z \in S$. Thus, we conclude that the orbit space $P|_{Z_2}$ is a limit of the inverse sequence

$$S \xleftarrow{\bar{f}} S \xleftarrow{\bar{f}} S \xleftarrow{\bar{f}} \ldots$$

By formula (5), the mapping $\bar{f}$ induces a homomorphism $\bar{f}_* : \pi_1(S) \to \pi_1(S)$, which acts as follows:

$$\bar{f}_*(a) = aa^{-1}a^{-1}a,$$

where $a \in \pi_1(S) \cong Z$ is the generator of the group $Z$. From the above formula, it follows that $\bar{f}_*$ is the null-homomorphism and thus, $deg\bar{f} = 0$. For any $k = 1, 2, \ldots$, $\bar{f}_k$ is also a null-homomorphism and thus, $deg\bar{f}_k = 0$. Therefore, by the classical Hopf theorem ([10], Section 2.8, Theorem $H^n$) all $\bar{f}_k : S \to S$ are null-homotopic and $sh(P|_{Z_2}) = 0$. □
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