On the Penrose inequality along null hypersurfaces

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Abstract

The null Penrose inequality, i.e. the Penrose inequality in terms of the Bondi energy, is studied by introducing a functional on surfaces and studying its properties along a null hypersurface $\Omega$ extending to past null infinity. We prove a general Penrose-type inequality which involves the limit at infinity of the Hawking energy along a specific class of geodesic foliations called \textit{Geodesic Asymptotically Bondi} (GAB), which are shown to always exist. Whenever this foliation approaches large spheres, this inequality becomes the null Penrose inequality and we recover the results of Ludvigsen–Vickers (1983 \textit{J. Phys. A: Math. Gen.} \textbf{16} 3349–53) and Bergqvist (1997 \textit{Class. Quantum Grav.} \textbf{14} 2577–83). By exploiting further properties of the functional along general geodesic foliations, we introduce an approach to the null Penrose inequality called the \textit{Renormalized Area Method} and find a set of two conditions which imply the validity of the null Penrose inequality. One of the conditions involves a limit at infinity and the other a restriction on the spacetime curvature along the flow. We investigate their range of applicability in two particular but interesting cases, namely the shear-free and vacuum case, where the null Penrose inequality is known to hold from the results by Sauter (2008 \textit{PhD Thesis Zürich ETH}), and the case of null shells propagating in the Minkowski spacetime. Finally, a general inequality bounding the area of the quasi-local black hole in terms of an asymptotic quantity intrinsic of $\Omega$ is derived.

Keywords: Penrose inequality, null hypersurfaces, asymptotic flatness, geometric flows, Hawking energy

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1. Introduction

The Penrose inequality in asymptotically flat spacetimes satisfying the dominant energy condition conjectures that the total energy measured by any observer is bounded below in terms of the area of suitable spacelike surfaces related to quasi-local black holes. This conjecture has received much attention since its formulation in [23] and constitutes an important open problem in gravitation. It has been proved in full generality only in spherical symmetry [12, 16] and for time-symmetric hypersurfaces [5, 13] (extended to space dimensions up to seven in [6]). For a review of the results prior to 2009 the reader is referred to [17].

In recent years, one of the several lines of research that have been pursued involves the Penrose inequality in the null case. Here, the total energy of the spacetime is the Bondi energy $E_B$ measured by an asymptotically inertial observer at a cut $S_\infty$ of past null infinity $\scri^-$ and the quasi-local black hole is a spacelike surface $S_0$ with the two properties of (i) having non-positive future outer null expansion (i.e. it is a weakly outer trapped surface ) and (ii) the outgoing, past directed null hypersurface $\Omega$ starting at $S_0$ extends smoothly all the way to infinity and intersects $\scri^-$ at $S_\infty$. Since, in such a setup, $S_0$ has smaller area than any other surface embedded in $\Omega$ to the past of $S_0$, the Penrose inequality has the form

$$E_B \geq \frac{|S_0|}{16\pi}$$

(1)

and does not require invoking minimal area enclosures of the quasi-local black hole surface $S_0$, as in the general case. This version of the Penrose inequality is often referred to as the null Penrose inequality. It has been proved only in a few special cases, including the case when $\Omega$ is shear-free and vacuum by Sauter [25]. Using a non-linear perturbation argument around a spherically symmetric null hypersurface in the Schwarzschild spacetime (which is indeed shear-free), Alexakis has been able to prove [1] the null Penrose inequality for vacuum spacetimes close enough (in a suitable sense) to the Schwarzschild exterior spacetime. The null Penrose inequality contains, as a particular case, the original formulation due to Penrose involving null shells of dust propagating in the Minkowski spacetime. This problem has also received attention recently, both for shells in Minkowski [18, 27] as well as for a related conjecture in the Schwarzschild spacetime [7].

A proof of the general null Penrose inequality was claimed by Ludvigsen and Vickers [15]. However, a gap was found by Bergqvist [4] who, at the same time, substantially streamlined the argument. Since Ludvigsen–Vickers & Bergqvist’s argument is relevant for this paper, let us describe it in some detail. Their method was based on two facts. The first was the existence of a quasi-local object defined on surfaces which enjoyed monotonicity properties along past directed null geodesic foliations. This functional was introduced by Bergqvist [4] and it has sometimes been called Bergqvist mass in the literature [17, 18]. The second fact was a suitable upper bound for the area of the weakly outer trapped surface $S_0$. Establishing this bound involved that the geodesic null foliation $\{S_i\}$ of $\Omega$ starting at $S_0 = S_0$ (where $r_0 \in \mathbb{R}^+$ and the range of $r$ is $[r_0, \infty)$) dragging $S_0$ to past null infinity satisfied two additional properties. The first was that the future null expansion $\theta_k$ of $S_k$ along the future null generator $k$ tangent to $\Omega$ admits an expansion of the form

$$\theta_k = -\frac{2}{r} + O\left(\frac{1}{r^2}\right),$$

(2)

i.e. with vanishing coefficient in the term $r^{-2}$. The second was that the rescaled metric $r^{-2}\gamma(r)$ (where $\gamma(r)$ is the induced metric of $S_i$) approaches a round metric on the sphere
when \( r \to +\infty \) (one says that \( \{S_r\} \) approaches large spheres). The main result by Ludvigsen–Vickers is that under these circumstances the Penrose inequality (1) follows. Ludvigsen and Vickers took for granted that a geodesic foliation \( \{S_r\} \) satisfying these two properties always exists. Bergqvist noted that under the assumption (2) it was not at all clear that the condition that the metric \( r^{-2\gamma}(r) \) approaches a round sphere needs to be satisfied. This was the gap in the original paper [15]. In [18] we investigated the Penrose inequality for dust null shells in Minkowski and proved its validity for a large class of surfaces. It turns out that the class of surfaces \( S_0 \) for which the Ludvigsen–Vickers method applies is very restrictive [20]. On any given past directed outward null hypersurface \( \Omega \) in the Minkowski spacetime extending smoothly all the way to past null infinity, there was only a one-parameter family of surfaces for which the Ludvigsen–Vickers & Bergqvist argument applies. Our method in [18] was based on geodesic foliations approaching large spheres but did not rely on the condition (2). The arguments however were tailored to the Minkowski spacetime where the null dust shell propagates. It makes sense to try and extend the ideas of [18], which in turn were motivated by Bergqvist’s approach, to find sufficient conditions for the null Penrose inequality in general asymptotically flat spacetimes satisfying the dominant energy condition. This is one of the main objectives of the present paper.

The second main objective is complementary to the previous one. Instead of relaxing condition (2) and keeping the assumption that \( \{S_r\} \) approaches large spheres, it is natural to consider the setup when (2) is kept and we relax the condition of approaching large spheres. Geodesic foliations with this property are named ‘Geodesic Asymptotically Bondi’ in this paper (or GAB for short). A motivation for this name will be given later. GAB foliations turn out to always exist and be (geometrically) unique given any cross-section \( S_0 \) in a past asymptotically flat null hypersurface. Our main result in this setting is a Penrose-type inequality, which relates the area of any weakly outer trapped surface \( S_0 \) and the limit at infinity of the Hawking energy along the GAB foliation associated to \( S_0 \). More precisely (see below for the definitions):

**Theorem 1.** *(A Penrose-type inequality for GAB foliations)*. Let \( \Omega \) be a past asymptotically flat null hypersurface in a spacetime \( (M, g) \) satisfying the dominant energy condition. Let \( S_0 \) be a spacelike cross-section of \( \Omega \). If \( S_0 \) is a weakly outer trapped surface, then

\[
\sqrt{\frac{|S_0|}{16\pi}} \leq \lim_{\lambda \to \infty} m_H(S_0),
\]

where \( m_H(S) \) denotes the Hawking energy of \( S \) and \( \{S_r\} \) is the GAB foliation associated to \( S_0 \).

In combination with a study of the limit of the Hawking energy along general foliations \( \{S_r\} \) of asymptotically flat null hypersurfaces \( \Omega \) carried out in [21], this theorem provides an interesting Penrose-type inequality with potentially useful applications. This theorem immediately extends the Ludvigsen–Vickers & Bergqvist result, because when \( \{S_r\} \) approaches large spheres one automatically has \( m_H(S_0) \to E_B \), where \( E_B \) is the Bondi energy at the cut at \( \mathcal{J}^- \) defined by \( \Omega \) and measured by the observer defined by \( \{S_r\} \).

The key object in this paper is the functional on surfaces

\[
M(S, \ell) = \sqrt{\frac{|S|}{16\pi}} - \frac{1}{16\pi} \int_{S} \theta_r \eta_S.
\]
which has the property that
\[ |S| \leq M(S, \ell) \]
whenever \( S \) is a weakly outer trapped surface. It also has the property that its limit at infinity along foliations approaching large spheres is the Bondi energy. The main objective of this paper is to bound \( M(S, \ell) \) from above by its limit at infinity. For that, the monotonicity properties of \( M(S_0, \ell) \) along suitable foliations will be studied. Although in general, this object is not monotonic, it can be split in two pieces, \( M_0(S_0, \ell) \) and \( D(S_0, \ell) \), where the first is closely related to an object first introduced by Bergqvist in \[4\] and turns out to be monotonically increasing provided the dominant energy condition holds. Thus, discussing under which conditions \( D(S_0, \ell) \) is bounded above by its limit becomes a problem of interest.

We consider various approaches to such an inequality and analyze their range of applicability by applying them to two particular but relevant cases, namely the case when \( \Omega \) is shear-free and vacuum (where, as mentioned, the null Penrose inequality is known to hold by other methods \[25\]) and the case of null shells propagating in the Minkowski spacetime. The latter will allow us in particular to provide a link between the analysis here and the one in \[18\].

This paper is organized as follows. In section 2, after introducing our terminology, we define the functional of surfaces \( M(S, \ell) \) and study its monotonicity properties, as well as its limit at infinity. For the limit, we use the notion of past asymptotically flat null hypersurface, introduced in \[21\] to study the limit of the Hawking energy \( m_{H}(S) \) at infinity. This allows us to relate the limits of \( M(S, \ell) \) and \( m_{H}(S) \) along geodesic foliations. In section 3 we introduce the concept of GAB foliation, and study its existence and uniqueness properties. We split \( M(S, \ell) \) into \( M_0(S_0, \ell) \) and \( D(S, \ell) \) and prove that for GAB foliations \( D(S_0, \ell) \) is bounded above by its limit at infinity, from which theorem 1 follows. The upper bound of \( D(S_0, \ell) \) is obtained by studying the monotonicity properties of yet another functional \( F(S) \). In section 4 we investigate various sufficient conditions implying the property that \( D(S_0, \ell) \) is bounded above by its limit, and hence the Penrose inequality along null hypersurfaces. For this, a slightly stronger notion of asymptotic flatness will be required. We first try to generalize the method valid for GAB foliations to more general settings and discuss the difficulties that arise. We then concentrate on the so-called Renormalized Area Method, where the null Penrose inequality is approached via studying the monotonicity properties of \( D(S_0, \ell) \) itself. The main result here is theorem 4 where two conditions are spelled out from which the null Penrose inequality follows. section 5 is devoted to studying the shear-free vacuum case as an interesting test bed for the previous ideas. After showing that one of the two conditions in theorem 4 fails to hold, we nevertheless find an argument proving the null Penrose inequality using the properties of \( M(S_0, \ell) \). This not only provides an alternative proof of Sauter’s theorem, but yields an explicit formula for the Bondi energy in terms of the geometry of any chosen cross-section \( S_0 \) of the null hypersurface. Section 6 is devoted to studying the renormalized area method in the Minkowski spacetime. This allows us to recover the results in \[18\] in a much more direct and efficient way. Concerning the application of theorem 1 to the Minkowski setting, we derive a general inequality (theorem 5) valid for any closed spacelike surface in Minkowski for which its outer past null cone extends smoothly to past null infinity. In the final section, we quit the method involving \( M(S, \ell) \) and exploit some results derived along the way to show a general inequality bounding the area of a closed spacelike surface embedded in a past asymptotically flat null hypersurface \( \Omega \) in terms of an asymptotic quantity intrinsic to \( \Omega \).
Since the notation of the paper may appear somewhat heavy in places, and following a suggestion by the referee, we have added a list of main symbols in Table 1.

### 2. A functional on two-surfaces

Let \((\mathcal{M}, g)\) be a time-oriented spacetime of dimension four. We denote the scalar product with \(g\) by \(\langle,\rangle\). Given a closed (i.e. compact and without boundary) orientable, spacelike, codimension-two surface \(S\) in \((\mathcal{M}, g)\), its normal bundle \(\mathcal{N}S\) admits a global basis of future directed null vectors \(k\) and \(\ell\). The second fundamental form is \(K(X, Y) = \langle \nabla_X Y, k \rangle\), where \(X\) and \(Y\) are tangent vectors to \(S\), and \(\nabla\) is the covariant derivative in \((\mathcal{M}, g)\). The null curvatures \(Kk(X, Y)\) and \(K\ell(X, Y)\) are defined by \(K(X, Y) = \langle k, K(X, Y) \rangle\) (and similarly for \(\ell\)) and the null-expansions, denoted by \(\theta_k\) and \(\theta_\ell\), are the traces of the null curvatures with respect to the induced metric \(\gamma\). A key object in this paper is the following functional on \(S\)

\[
M(S, \ell) = \sqrt{\frac{[S]}{16\pi}} - \frac{1}{16\pi} \int_S \theta_\ell \eta_S,
\]

where \([S]\) is the area of \(S\) and \(\eta_S\) the metric volume form of \(S\). This quantity has geometric units of length so one may be tempted to assign to it a physical interpretation of quasi-local
mass of $S$. However, $M(S, \ell)$ is not truly a quasi-local quantity on the surface because it depends on the choice of null normal $\ell$, which cannot be uniquely fixed \emph{a priori} in the absence of additional geometric structure. Note, however, that a weakly outer trapped surface $S$ satisfies, by definition, $\theta_\ell \leq 0$ irrespective of the scaling of $\ell$, and hence

$$\sqrt{\frac{|S|}{16\pi}} \leq M(S, \ell).$$

So, if $M(S, \ell)$ enjoyed good monotonicity properties under suitable flows and its value on very large surfaces in an asymptotically flat context could be related to the total mass of the spacetime, this object would be potentially useful to address the Penrose inequality and play perhaps a similar role as the Hawking energy does in the time-symmetric context. In this paper, we restrict ourselves to null flows because in this case $M(S, \ell)$ does satisfy an interesting evolution equation. However, there are many other possibilities that may well be worth exploring.

Since we deal with null flows, we shall work on a null hypersurface $\Omega$. However, we emphasize that $\Omega$ is not our primary object of study. Instead, we deal with null hypersurfaces endowed with a foliation by spacelike surfaces. Thus, we can use all the geometric information coming from $\Omega$ plus the foliation. In fact, $\Omega$ can be endowed [14] with an intrinsic geometric structure independent of any foliation (see also [22] for a framework covering simultaneously hypersurfaces of arbitrary causal character) and this structure could be introduced first in order to distinguish properties intrinsic to $\Omega$ from properties depending on the foliation. However, given the central role that foliations play in this paper, this intermediate step turns out to be unnecessary.

Thus, let $\Omega$ be a smooth, connected null hypersurface embedded in $(\mathcal{M}, g)$ and admitting a global cross-section $S_0$ (i.e. a smooth embedded spacelike surface intersected precisely once by every inextendible curve along the null generators of $\Omega$). We want to investigate the derivative of $M(S_\mu, \ell)$ with respect to $\mu$, where $S_\mu$ is a foliation of $\Omega$ by cross-sections. Such foliations can be uniquely characterized by a choice of future directed null normal $k$ of $\Omega$ as follows. Given $k$, solve the PDE $k(\mu) = -1$ with $\mu = 0$ on $S_0$ and define $S_\mu$ as the level set of the scalar $\mu$. Conversely, given $\{S_\mu\}$ let $k$ be the vector field tangent to the generators of $\Omega$ scaled in such a way that $k(\mu) = -1$. Note that, in order to maintain the generality, we do not make any assumption concerning the null normal $k$ other than being future directed and nowhere zero. Similarly, at this point we do not make any assumption on the null vector field $\ell$ orthogonal to $S_\mu$ (other than being future directed and transverse to $\Omega$). In order to compute the derivative of $M(S_\mu, \ell)$ we need the following well-known identities: let $\gamma_{S_\mu}$ be the induced metric of $S_\mu$ and define $Q_\ell : \Omega \mapsto \mathbb{R}$ by

$$\nabla_k k = Q_\ell k,$$

so that $Q_\ell$ vanishes if and only if the null geodesic generator $k$ is chosen to be affinely parametrized. In this case, the corresponding foliation by level sets will be called a \emph{geodesic foliation}. Given a smooth positive function $\varphi : \Omega \mapsto \mathbb{R}^+$, there is a unique choice of null normal $\ell$ to $S_\mu$ (denoted by $\ell^\varphi$) satisfying

$$(k, \ell^\varphi) = -\varphi.$$

Obviously $\ell^\varphi$ depends on the foliation $S_\mu$, so it is not an intrinsic property of $\Omega$. The choice $\varphi = 2$ will be relevant later and we will denote $\ell^{\varphi=2}$ simply by $\ell$ from now on.
The following evolution equations are standard, see e.g. [9]:

\[ \mathcal{L}_k \gamma = 2 K^k, \]
\[ k(\theta_k) = Q_k \theta_k - \frac{1}{2} \theta_k^2 - \Pi_{AB}^k \Pi^{AB} - \text{Ric}^g(k, k), \]
\[ k(\theta_{\ell^\varphi}) = \left( \frac{1}{\varphi} k(\varphi) - Q_k \right) \theta_{\ell^\varphi} + \text{Ein}^g(k, \ell^\varphi) - \frac{\varphi}{2} (\text{Scal}^S_S - \langle \bar{H}, \bar{H} \rangle) \]
\[ + \varphi (\text{div}_S s_{\ell^\varphi} + |s_{\ell^\varphi}|^2_{S^S}), \]

where \( \gamma \) is the first fundamental form of \( \Omega \), \( \text{Ric}^g \) and \( \text{Ein}^g \) are, respectively, the Ricci and Einstein tensors of \( (\mathcal{M}, g) \), \( D \) is the Levi–Civita covariant derivative of \( (S_\mu, \gamma_S) \), \( \text{Scal}^S \) the corresponding curvature scalar, \( \Pi^k \) is the trace-free part of \( K^k \) \( (K^k = K^k - \frac{1}{2} \theta_k \gamma_S) \), \( \bar{H} \) is the mean curvature vector

\[ \bar{H} = \frac{1}{\varphi} (\theta_k \ell^\varphi + \theta_{\ell^\varphi} k), \]

and the connection of the normal bundle \( s_{\ell^\varphi} \) is the one-form on \( S_\mu \) defined by

\[ s_{\ell^\varphi}(X) = \frac{1}{\varphi} \langle \nabla_X k, \ell^\varphi \rangle, \quad X \in \mathfrak{X}(S_\mu). \]

As before, \( s_{\ell^\varphi} \) is also a foliation-dependent quantity. The evolution of \( M(S_\mu, \ell^\varphi) \) in this general setting is given in the following lemma.

**Lemma 1.** Let \( \Omega \) be a null hypersurface embedded in a spacetime \( (\mathcal{M}, g) \). Assume that \( \Omega \) has topology \( S \times \mathbb{R} \) with the null generator tangent to the \( \mathbb{R} \) factor. Consider a foliation \( \{S_\mu\} \) of \( \Omega \) by spacelike hypersurfaces, all diffeomorphic to \( S \). Let \( k \) be the future null generator satisfying \( k(\mu) = -1 \) and \( \ell^\varphi \) the null normal to \( S_\mu \) satisfying \( (k, \ell^\varphi) = -\varphi \). Then

\[ \frac{dM(S_\mu, \ell^\varphi)}{d\mu} = \frac{1}{\sqrt{64\pi|S_\mu|}} \int_{S_\mu} (-\theta_k) \eta_S + \frac{1}{16\pi} \int_{S_\mu} \left[ \text{Ein}^g(\ell, k) - \frac{\varphi}{2} \text{Scal}^S \right] \eta_S \]
\[ + \varphi (\text{div}_S s_{\ell^\varphi} + |s_{\ell^\varphi}|^2_{S^S}) + \frac{1}{\varphi} k(\varphi) - Q_k \right] \]

\[ \eta_S, \]

where \( Q_k \) is defined by \( \nabla_k k = Q_k k \). If, moreover, \( \varphi \) is constant and \( k \) is geodesic \( (Q_k = 0) \) then

\[ \frac{dM(S_\mu, \ell^\varphi)}{d\mu} = \frac{1}{\sqrt{64\pi|S_\mu|}} \int_{S_\mu} (-\theta_k) \eta_S + \frac{\varphi}{8} \chi(S_\mu) + \frac{1}{16\pi} \int_{S_\mu} \left( \text{Ein}^g(\ell^\varphi, k) + \varphi |s_{\ell^\varphi}|^2_{S^S} \right) \eta_S, \]

where \( \chi(S_\mu) \) is the Euler characteristic of \( S_\mu \).

**Proof.** We drop all reference to \( \mu \) for simplicity. The volume form satisfies

\[ \mathcal{L}_k \eta_S = \theta_k \eta_S. \]
so that the variation along $-k$ of $M(S, \ell^r)$ is, using (6),
\[
\frac{dM(S, \ell^r)}{d\mu} = \frac{1}{\sqrt{64\pi|S|}} \int_S (-\theta_k)\eta_S + \frac{1}{16\pi} \int_S \left[\text{Ein}^k(\ell^r, k) - \frac{\varphi}{2}\text{Scal}^\varsigma\right.
\]
\[
+ \varphi (-\text{div}_S\ell^r + |\ell^r|^2_g) + \left(1 - k(\varphi) - Q_k\right)\theta_{\ell^r}\right] \eta_S,
\]
where we have used $\langle \vec{H}, \vec{H} \rangle = -\frac{2}{\varphi} \theta_k \ell^r$. This is precisely (7). When $\varphi = \text{const}$ and $Q_k = 0$, (8) follows directly from (7) as a consequence of the Gauss–Bonnet theorem $\int_S \text{Scal}^\varsigma \eta_S = 4\pi \chi(S)$.

Our purpose in deriving the general variation formula (7) is to show that indeed $\varphi = \text{const}$ and $Q_k = 0$ are the only clear situations leading to a (nearly) monotonic behaviour. Indeed, the divergence term $\text{div}_S\ell^r$ has no sign which strongly suggests the choice $\varphi = \text{const}$. The term in $\theta_{\ell^r}$, which again has no sign a priori, suggests making the choice $Q_k = 0$ (the seemingly more general condition of making $\varphi$ constant only within the leaves and $Q_k = \varphi^{-1}k(\varphi)$ is simply a reparametrization of the previous one).

Under the dominant energy condition (DEC) on $(M, g)$ (namely, $-\text{Ein}^k(X, \cdot)$ future causal $\forall X$ future causal), this lemma implies that if $S$ is connected and non-spherical, then $M(S_0, \ell^r)$ is monotonically increasing along any geodesic flow for any past expanding (i.e. with $\theta_{\ell^r} \geq 0$) null hypersurface. We will reserve the symbol $\lambda$ for foliations $\{S_0\}$ associated to geodesic generators $k$.

For the Penrose inequality in an asymptotically flat context, the spherical topology is the relevant one. In this setting, $M(S_0, \ell)$ is not always monotonic.

However, under certain circumstances one can relate its value on the initial surface and its asymptotic value at infinity. In fact, obtaining such relations will be the main theme of this paper. We first need to specify our asymptotic conditions. We adopt here the same definitions as in [21], where a detailed analysis of the limit of the Hawking energy along null flows was obtained. For the sake of completeness, we briefly repeat the main definitions. The reader is referred to [21] for further details.

We make the global assumption that $\Omega = \mathbb{S}^2 \times \mathbb{R}$ with the geodesic null generator $k$ tangent along the $\mathbb{R}$-factor. Implicit in this condition is that, fixed a cross-section $S_0$ of $\Omega$ (necessarily of spherical topology), the integral curve of $k$ starting at $p \in S_0$ has maximal domain ($-\infty, \lambda_+(p)$), i.e. the null generators are past complete. After possibly removing portions of $\Omega$ lying to the future of $S_0$ we can assume that $\Omega$ is foliated by the level sets $\{S_t\}$ of the function $\lambda \in C(\Omega)$ defined by $k(\lambda) = -1$, $\lambda|_{S_0} = 0$. Obviously, all such $S_t$ are of spherical topology. The function $\lambda$ is called the level set function of $k$. A null hypersurface $\Omega$ satisfying these properties is called extending to past null infinity. The definition of asymptotic flatness involves so-called Lie constant transversal tensors. A transversal tensor is a covariant tensor on $\Omega$ completely orthogonal to $k$. Any such tensor $T$ is in one-to-one correspondence with a family $T(\lambda)$ of covariant tensors on $S_t$. A transversal tensor is positive definite if each $T(\lambda)$ has this property. A transversal tensor $T$ is Lie constant if $\ell^k T = 0$. Concerning decay at infinity, the symbol $T = o_p(\lambda^q)$ means $\lambda^{i+q}(\ell^i)T = o(1)$ ($i = 0, 1, \ldots, n$), i.e. it has a vanishing limit as $\lambda \to \infty$ in a Lie-propagated basis $\{X_1, \ldots, X_n\}$ of $T\Omega$. $T = o^k_p(\lambda^q)$ means $\lambda^q X_{\alpha_1} \cdots X_{\alpha_i} T = o(1)$ for all $i$-tuple of indices $\{A_1, \ldots, A_i\}$ and $i = 0, 1, \ldots, n$. Similar definitions hold for $O_p(\lambda^q)$ and $Q^k_p(\lambda^q)$.

The definition of asymptotic flatness we use imposes conditions only along $\Omega$ and reads (cf. definition 1 and proposition 3 in [21])
Definition 1. A null hypersurface $\Omega$ in a spacetime $(\mathcal{M}^4, g)$ is past asymptotically flat if it extends to past null infinity and there exists a choice of cross-section $S_0$ and a null geodesic generator $k$ with corresponding level set function $\lambda$ satisfying:

(i) There exist two symmetric 2-covariant transversal and Lie constant tensor fields $\hat{q}$ (positive definite) and $h$ such that $\hat{\gamma} = \gamma - \lambda^2 \hat{q} - \lambda h$ is $\hat{\gamma} = \omega_1(\lambda) \cap \omega_2^0(\lambda)$.

(ii) There exists a transversal, Lie constant one-form $s_1^{(1)}$ such that $\hat{s}_l := s_l - \frac{s_1^{(1)}}{\lambda}$ is $\hat{s}_l = \omega_1(\lambda^{-1})$.

(iii) Denote the Gauss curvature of $\hat{q}$ by $\kappa(q)$. There exists a Lie constant function $\nu^{(1)}$ such that $\frac{\kappa(q)}{\nu^{(1)}(\lambda) \lambda^2} = o(\lambda^{-2})$.

(iv) The function $Riem^a(X_A, X_B, X_C, X_D)$ along $\Omega$ is such that $\lim_{r \to \infty} \frac{1}{r^2} Riem^a(X_A, X_B, X_C, X_D)$ exists while its double trace satisfies $2 \text{Scal} - \frac{1}{2} Riem^a(l, k, l, k) = o(\lambda^{-2})$.

We remark that several parts of this definition involve foliation-dependent quantities. However, the definition is intrinsic to the null hypersurface $\Omega$ because the definition requires the existence of a foliation with the desired properties.

We can now analyze the limit of $M(S, \ell^2)$ at infinity. From item (i) in definition 1, it follows that the volume form $\eta_{S_\lambda}$ of each $S_\lambda$ satisfies

$$\eta_{S_\lambda} = (\lambda^2 + \nu^{(1)}(\lambda) \lambda + o(\lambda)) \eta_{q},$$

where the Lie constant function $\nu^{(1)}$ is defined by the expansion

$$\nu^{(1)} = \frac{-2}{\lambda} + \frac{\nu^{(1)}(\lambda)}{\lambda^2} + o(\lambda^{-2})$$

which is a consequence of definition 1 (see [21]). The expressions become simpler if we introduce the area radius at infinity as

$$R_\hat{q}^2 := \frac{1}{4\pi} \int_S \eta_{\hat{q}},$$

where $\hat{S}$ represents the surface $S$ endowed with the metric $\hat{q}$. The area $S_\lambda$ has the following expansion:

$$[S_\lambda] = \int_{S_\lambda} \eta_{S_\lambda} = \int_{S_\lambda} (\lambda^2 + \nu^{(1)}(\lambda) \lambda + o(\lambda)) \eta_{q} = 4\pi R_\hat{q}^2 \lambda^2 + \left(\int_{S} \nu^{(1)} \eta_{\hat{q}}\right) \lambda + o(\lambda),$$

and therefore

$$\sqrt{|S_\lambda|} = \sqrt{4\pi R_\hat{q}^2 \lambda} + \frac{\sqrt{\int_{S} \nu^{(1)} \eta_{\hat{q}}}}{2\sqrt{4\pi R_\hat{q}^2}} + o(1).$$

We next compute the asymptotic behavior of the second term in $M(S, \ell^2)$. Using item (iii) in definition 1 and noticing that $\nu^{(1)} = \frac{\nu^{(1)}}{\lambda} \nu_l$ (because of the scaling relation $\ell^2 = \frac{\nu}{2} \ell$), it follows (we are assuming $\varphi$ constant here and in what follows)
\[ \int_{S_\lambda} \theta_{\ell^*}(\lambda) \eta_{S_\lambda} = \int_S \left( \varphi K_{\hat{q}} \frac{\hat{q}}{\lambda} + \frac{\varphi \theta_{\ell^*}^{(1)}}{2 \lambda^2} + o(\lambda^2) \right) (\lambda^2 + \theta_{\ell^*}^{(1)} \lambda + o(\lambda)) \eta_\hat{q} \]

\[ = 4\pi \varphi \lambda + \int_S \left( \varphi K_{\hat{q}} \theta_{\ell^*}^{(1)} + \frac{\varphi}{2} \theta_{\ell^*}^{(1)} \right) \eta_\hat{q} + o(1). \]  

(12)

Combining (11) and (12) into (3) gives

\[ M(S, \ell^*) = \left( \frac{R_\hat{q}}{2} - \frac{\varphi}{4} \right) \lambda + \frac{1}{16\pi} \int_S \left( \theta_{\ell^*}^{(1)} \left( \frac{1}{R_\hat{q}} - \varphi K_{\hat{q}} \right) - \frac{\varphi}{2} \theta_{\ell^*}^{(1)} \right) \eta_\hat{q} + o(1). \]

This expression has a finite limit at infinity if and only if the scaling of $\ell^*$ is chosen so that $\varphi = 2R_\hat{q}$. This leads to the following definition:

**Definition 2.** Let $\Omega$ be a past asymptotically flat null hypersurface and $\{S_\lambda\}$ the foliation whose existence is assumed in definition 1. The vector field $\ell^*$ is defined to be null, orthogonal to each leaf $S_\lambda$ and normalized by $\langle k, \ell^* \rangle = -2R_\hat{q}$.

Note that $\ell^*$ is defined by $\varphi = 2R_\hat{q}$ so that its relation to the canonical $\ell$ is $\ell^* = R_\hat{q} \ell$.

With this choice,

\[ \lim_{\lambda \to \infty} M(S_\lambda, \ell^*) = \frac{1}{16\pi} \int_S \left( \theta_{\ell^*}^{(1)} \left( \frac{1}{R_\hat{q}} - 2R_\hat{q} K_{\hat{q}} \right) - R_\hat{q} \theta_{\ell^*}^{(1)} \right) \eta_\hat{q}. \]  

(13)

It is useful to relate this limit to the corresponding limit of the Hawking energy along $\{S_\lambda\}$. For any closed spacelike surface $S$ with spherical topology embedded in a four-dimensional spacetime, the Hawking energy is defined by

\[ m_H(S) = \sqrt{1 - \frac{1}{16\pi} \int_S \langle \mathcal{H}, \mathcal{H} \rangle \eta_S}. \]

The limit of $m_H(S_\lambda)$ was investigated in detail in [21]. In particular, theorem 5 in [21] gives

\[ \lim_{\lambda \to \infty} m_H(S_\lambda) = \frac{-R_\hat{q}}{16\pi} \int_S (K_{\hat{q}} \theta_{\ell^*}^{(1)} + \theta_{\ell^*}^{(1)}) \eta_\hat{q}. \]  

(14)

Combining (13) and (14), the following proposition is proved:

**Proposition 1.** With the choice $\ell^* = R_\hat{q} \ell$, the limits of $M(S_\lambda, \ell^*)$ and $m_H(S_\lambda)$ are related by

\[ \lim_{\lambda \to \infty} M(S_\lambda, \ell^*) = \lim_{\lambda \to \infty} m_H(S_\lambda) + \frac{1}{16\pi} \int_S \theta_{\ell^*}^{(1)} \left( \frac{1}{R_\hat{q}} - R_\hat{q} K_{\hat{q}} \right) \eta_\hat{q}. \]  

(15)

**Remark 1.** There are two interesting cases where the limit of $M(S_\lambda, \ell^*)$ agrees with the limit of the Hawking energy along the foliation. The first occurs when $\hat{q}$ has positive constant curvature, in which case the area radius $R_\hat{q}$ and the Gauss curvature are related by $K_{\hat{q}} = \frac{1}{R_\hat{q}^2}$ and the integrand in the second term of (15) vanishes. Such foliations are called ‘approaching to large spheres’ because the geometry of the leaves tends, after a suitable rescaling, to the round spherical metric. This situation is particularly relevant, because then the limit of the Hawking energy is the Bondi energy measured by the observer defined by the foliation $\{S_\lambda\}$ (see [2, 21, 24, 25] for details).
The other case corresponds to those foliations satisfying $\theta^{(1)}_k = \text{constant}$. In this case, we have
\[
\int_S \theta^{(1)}_k \left( \frac{1}{R_q} - R_q \mathcal{K}_q \right) \eta_q = \theta^{(1)}_k \int_S \left( \frac{1}{R_q} - R_q \mathcal{K}_q \right) \eta_q = \theta^{(1)}_k (4\pi R_q - 4\pi R_q) = 0,
\]
where in the second equality we have used the Gauss–Bonnet theorem. We devote the next section to study in detail geodesic foliations with constant $\theta^{(1)}_k$.

3. GAB foliations and a Penrose-type inequality

As discussed in the introduction, Ludvigsen and Vickers [15] and Bergqvist [4] considered the Penrose inequality for null hypersurfaces. A fundamental ingredient of their work involved geodesic foliations for which $\theta^{(1)}_k$ vanishes identically. As we will see below, such foliations are closely related to geodesic foliations with $\theta^{(1)}_k$ constant. We devote this section to study such foliations. Our main result is a Penrose-type inequality valid in full generality and which reduces to the Penrose inequality when the foliation approaches large spheres. Besides its intrinsic interest, the general Penrose-type inequality helps to put the result of Ludvigsen & Vickers and Bergqvist into a broader perspective and clarifies both its scope and its range of validity.

We first need a lemma showing that, no matter which geodesic foliation is taken, the leading term $\theta^{(1)}_k$ is always strictly positive. This may seem to contradict (2), but this is not the case because $l = 0$ corresponds to a cross-section on $\Omega$, while the corresponding condition for $r$ was not assumed (and in fact does not hold) in (2).

**Lemma 2.** Let $\Omega$ be a past asymptotically flat null hypersurface with a choice of affinely parametrized null generator $k$ and corresponding level set function $\lambda$. Assume that the spacetime satisfies the dominant energy condition, then $\theta^{(1)}_k > 0$.

**Proof.** Let $\{S_0\}$ be the geodesic foliation defined by $\lambda$ and consider the function $\rho(\lambda) = \theta_k |_{S_0} \lambda^2 + 2\lambda$. Using the Raychaudhuri equation (5), which can be written as $\frac{dk}{d\lambda} = \frac{\theta_k^2}{2} + W$ with $W \geq 0$ under DEC, the derivative of $\rho$ satisfies
\[
\rho'(\lambda) = 2\lambda \theta_k + \lambda^2 \left( \frac{1}{2} \theta_k^2 + W \right) + 2 \geq (\lambda \theta_k + 2)^2 \geq 0.
\]
Since $\rho$ vanishes at $\lambda = 0$, it follows that $0 \leq \rho(\lambda) \leq \lim_{\lambda \to -\infty} \rho(\lambda) = \theta^{(1)}_k$ where the last equality follows from the expansion (10). To show the strict inequality $\theta^{(1)}_k > 0$ we argue by contradiction. Assume that there is some null geodesic $\gamma_p$ in $\Omega$ where $\theta^{(1)}_k = 0$. Then $\rho(\lambda)$ necessarily vanishes on this curve and
\[
\theta_k |_{\gamma_p} \lambda^2 + 2\lambda = 0 \implies \theta_k |_{\gamma_p} = \frac{-2}{\lambda} \implies \lim_{\lambda \to 0^+} \theta_k |_{\gamma_p} = -\infty,
\]
which is a contradiction to the smoothness of $\Omega$ at $S_0$. $\square$

The following result deals with the existence of foliations with constant $\theta^{(1)}_k$.

**Lemma 3.** Let $\Omega$ be a past asymptotically flat null hypersurface with a choice of affinely parametrized null generator $k$ and corresponding level set function $\lambda$. There exists a Lie constant positive function $f \in \mathcal{F}(\Omega)$ and a rescaling $k' = f k$ such that the term $\theta^{(1)}_k$ in the asymptotic expansion of $\theta_k'$ is constant.
Proof. Let \( \{ S_\lambda \} \) be the foliation associated to \( k' \). Thus \( \lambda = f \lambda' \) because \( k'(\lambda) = -1 \) and \( \lambda|_{0} = 0 \). It is well known that the null expansion \( \theta_k|_p \) is a property of \( \Omega \) at \( p \in \Omega \), independent of the cross-section passing through \( p \). We can thus transform the expansion (10) under the change of foliation \( \lambda = f \lambda' \) simply as

\[
\theta_k = \frac{-2}{f} \frac{1}{X} + \frac{\theta_k^{(1)}}{f^2} \frac{1}{X^2} + o(X^{-2}).
\]

Using now \( k' = f k \) and the fact that \( \theta_k \) scales as \( k \),

\[
\theta_{k'} = \frac{-2}{X} + \frac{\theta_k^{(1)}}{X^2} + o(X^{-2}). \tag{17}
\]

Since \( \theta_k^{(1)} > 0 \), we can choose \( f = \frac{\theta_k^{(0)}}{c} \) for any given constant \( c > 0 \). The foliation \( \{ S_\lambda \} \) has \( \theta_k^{(1)} = c \), as claimed. \( \square \)

Note that, by construction, the foliation \( \{ S_\lambda \} \) in this lemma is also a geodesic foliation. Once \( \theta_k^{(1)} \) is constant, it can be made zero by a constant shift of \( \lambda \). Indeed, let \( \lambda = \lambda + \lambda_0 \) with \( \lambda_0 \) constant. The null generator \( k \) now remains unchanged and

\[
\theta_k = \frac{-2}{\lambda} + \frac{\theta_k^{(1)}}{\lambda^2} + o(\lambda^{-2}) = \frac{-2}{\lambda'} + \frac{\theta_k^{(1)} + 2\lambda_0}{\lambda'} + o(\lambda^{-2}).
\]

Thus, the coefficient \( \theta_k^{(1)} \) along a geodesic foliation can be made zero by a change of origin if and only if it is constant. As mentioned above, Ludvigsen & Vickers and Bergqvist considered foliations with vanishing \( \theta_k^{(1)} \). Such foliations arise naturally in the context of conformal compactifications of null infinity and are related to the Bondi coordinates near null infinity. This motivates the following definition.

\textbf{Definition 3. (Geodesic Asymptotically Bondi foliation associated to} \( S_0 \)). Consider a past asymptotically flat null hypersurface \( \Omega \) with a choice of cross-section \( S_0 \). A geodesic foliation \( \{ S_\lambda \} \) is called GAB and associated to \( S_0 \) if

(i) \( S_{\lambda=0} = S_0 \)

(ii) \( \theta_k^{(1)} \) is constant.

In the following lemma, we show that two GAB foliations associated to \( S_0 \) are necessarily related by a constant rescaling of parameter, \( \lambda = a \lambda' \) with \( a \in \mathbb{R}^+ \). Thus, the collection of surfaces \( \{ S_\lambda \} \) remain unchanged, and GAB foliations associated to a given \( S_0 \) are geometrically unique. Obviously, when \( S_0 \) changes, the corresponding unique GAB foliation (which exists by lemma 3) also changes.

\textbf{Lemma 4. (Uniqueness of GABs).} Let \( \Omega \) be a past asymptotically flat null hypersurface and \( S_0 \) a cross-section. Two GAB foliations \( \{ S_\lambda \} \) and \( \{ S_\lambda' \} \) associated to \( S_0 \) are related by \( \lambda = a \lambda' \) for some positive constant \( a \).

\textbf{Proof.} Let \( k \) and \( k' \) be the null generators of \( \{ S_\lambda \} \) and \( \{ S_\lambda' \} \). Since both are geodesic, there exists a Lie constant positive function \( f \) such that \( k' = f k \). We have shown in (17) that
By definition of GAB foliation, both $\theta^{(1)}_k$ and $\theta^{(1)\prime}_k$ are constant. Thus $f$ is a positive constant (say $a$) and the affine parameters are related by $\lambda = f X = a X$.

The main result in the work by Ludvigsen & Vickers and Bergqvist can be formulated in terms of GABs as follows.

**Theorem 2.** (Ludvigsen & Vickers [15], Bergqvist [4]) Let $\Omega$ be a past asymptotically flat null hypersurface $\Omega$ in a spacetime satisfying the DEC. Assume that $\Omega$ admits a weakly outer trapped cross-section $S_0$. If $\Omega$ admits a GAB foliation $\{S_t\}$ associated to $S_0$ and approaching large spheres, then the Penrose inequality

$$E_B \geq \sqrt{\frac{|\text{Sol}|}{16\pi}}$$

holds, where $E_B$ is the Bondi energy associated to the observer at infinity defined by the foliation $\{S_t\}$.

As mentioned in the Introduction, the possibility that the foliation can be chosen to approach large spheres was assumed implicitly in the work by Ludvigsen & Vickers. The necessity to add this restriction explicitly was noted by Bergqvist. Since GAB foliations associated to a given $S_0$ are unique, the condition of approaching large spheres is indeed a strong additional assumption, that will only be satisfied in very special circumstances. It makes sense to study GAB foliations in detail, dropping the assumption of approaching large spheres. By doing this we will be able to obtain an interesting Penrose-type inequality relating the area of $S_0$, not to the Bondi energy, but to the limit of the Hawking energy along the foliation. Since the Hawking energy approaches the Bondi energy for asymptotically spherical foliations, our result will automatically include theorem 2 as a corollary. In particular, this will help to clarify the role played by the asymptotically spherical condition in theorem 2.

We have shown in proposition 1 (cf remark 1) that for GAB foliations, the limit of the functional $M(S, \ell^\varphi)$ is the same as the limit of the Hawking energy at infinity. To obtain a Penrose-type inequality we need to relate the value of $M(S, \ell^\varphi)$ at the initial surface with its asymptotic value. The functional $M(S_0, \ell^\varphi)$ is not monotonic, so this cannot be done straight away. However, the computations in lemma 1 suggest splitting $M(S_0, \ell^\varphi)$ in two terms, one of which will be automatically monotonic. This is useful, because we can then concentrate on studying the non-monotonic term. Define

$$D(S, \ell^\varphi) := \sqrt{\frac{|S|}{16\pi}} - \frac{\varphi}{4} \lambda,$$

$$M_0(S, \ell^\varphi) := \frac{\varphi}{4} \lambda - \frac{1}{16\pi} \int_S \ell^\varphi \eta_S,$$

so that $M(S, \ell^\varphi) = D(S, \ell^\varphi) + M_0(S, \ell^\varphi)$. The computation in lemma 1 implies that for geodesic flows and $\varphi = \text{const}$ (recall that the cross-sections of $\Omega$ are topological spheres, so that $\chi(S) = 2$).
A direct consequence of $M_b(S, \ell^\tau)$ being monotonically increasing is that its initial value is bounded above by its value at infinity. From (12), this limit is given by

$$\lim_{\lambda \to -\infty} M_b(S_\lambda, \ell^\tau) = -\frac{1}{16\pi} \int_S \varphi \left( K_{q^1} \theta_{k^1} + \frac{1}{2} \theta_{j^1}^1 \right) \eta_\lambda,$$

which is finite irrespective of the choice of $\ell^\tau$. On the other hand, $D(S_\lambda, \ell^\tau)$ is not necessarily monotonic and its limit at infinity is finite only for the choice $\ell^\tau = R_q \ell$ and given by (see (11))

$$\lim_{\lambda \to -\infty} D(S_\lambda, \ell^\tau) = \frac{1}{16\pi R_q} \int_S \theta_{k^1} \eta_\lambda.$$

To bound $M(S, \ell^\tau)$ from above we need to find an upper bound for $D(S, \ell^\tau)$. In fact, we shall prove $D(S, \ell^\tau) \leq \lim_{\lambda \to -\infty} D(S_\lambda, \ell^\tau)$ provided the foliation $\{S_\lambda\}$ is GAB. In the following lemma, we introduce a functional that turns out to be monotonic for GAB foliations.

**Lemma 5.** Let $\Omega$ be a past asymptotically flat null hypersurface with a choice of affinely parametrized null generator $k$ and corresponding level set function $\lambda$. Assume that the spacetime satisfies the dominant energy condition. Consider the functional

$$F(S_\lambda) = \frac{|S_\lambda|}{\left(8\pi R^2_0 \lambda + \int_S \theta_{k^1} \eta_\lambda \right)^2}.$$

If $\{S_\lambda\}$ is the GAB foliation associated to $S_0$, then $F(S_\lambda)$ is monotonically increasing.

**Proof.** Writing $F(S_\lambda)$ as

$$F(S_\lambda) = \int_{S_\lambda} \frac{\eta_{S_\lambda}}{\left(8\pi R^2_0 \lambda + \int_S \theta_{k^1} \eta_\lambda \right)^2},$$

and using $\mathcal{L}_k \eta_{S_\lambda} = -\theta_k \eta_{S_\lambda}$, the derivative of $F(S_\lambda)$ is

$$\frac{d}{d\lambda} F(S_\lambda) = \int_{S_\lambda} \left( \frac{-16\pi R^2_0}{\left(8\pi R^2_0 \lambda + \int_S \theta_{k^1} \eta_\lambda \right)^3} + \frac{-\theta_k}{\left(8\pi R^2_0 \lambda + \int_S \theta_{k^1} \eta_\lambda \right)^2} \right) \eta_{S_\lambda}$$

$$= \int_{S_\lambda} \left( \frac{-16\pi R^2_0 \lambda + \int_S \theta_{k^1} \eta_\lambda}{\left(8\pi R^2_0 \lambda + \int_S \theta_{k^1} \eta_\lambda \right)^3} \right) \eta_{S_\lambda}. \quad (19)$$
This derivative is non-negative provided
\[
(\theta_k) \left( 8\pi R_\bar{q}^2 \lambda + \int_{\hat{S}} \theta_k^{(1)} \eta_\bar{q} \right) \geq 16\pi R_\bar{q}^2 \iff \frac{1}{-\theta_k} - \frac{\lambda}{2} \leq \frac{1}{16\pi R_\bar{q}^2} \int_{\hat{S}} \theta_k^{(1)} \eta_\bar{q}. \tag{20}
\]

The Raichaudhuri equation (5) implies that the function \( \frac{1}{-\theta_k} - \frac{\lambda}{2} \) has non-negative derivative (under DEC). Since its limit at infinity is \( \frac{\theta_k^{(0)}}{4} \) it follows
\[
\frac{1}{-\theta_k} - \frac{\lambda}{2} \leq \frac{\theta_k^{(1)}}{4}, \tag{21}
\]
which holds true for any geodesic foliation. For GAB foliations, we have, using
\[
\int_{\hat{S}} \eta_\bar{q} = 4\pi R_\bar{q}^2,
\]
\[
\frac{\theta_k^{(1)}}{4} = \frac{1}{16\pi R_\bar{q}^2} \int_{\hat{S}} \theta_k^{(1)} \eta_\bar{q}
\]
and (21) is exactly (20).

The monotonicity of the functional \( F(S_\lambda) \) is useful to establish an upper bound for \( D(S_\lambda, \ell^*) \), irrespective of whether the foliation is GAB or not.

**Lemma 6.** Let \( \{S_\lambda\} \) be a geodesic foliation with leading term metric \( \hat{q} \). If the functional \( F(S_\lambda) \) is monotonically increasing, then
\[
D(S_\lambda, \ell^*) \leq \lim_{\lambda \to \infty} D(S_\lambda, \ell^*). \tag{22}
\]

**Proof.** The monotonicity of the functional \( F(S_\lambda) \) along \( \{S_\lambda\} \) implies
\[
F(S_\lambda) \leq \lim_{\lambda \to \infty} F(S_\lambda) \tag{23}
\]
To compute this limit we use
\[
|S_\lambda| = \int_{\hat{S}} \eta_\lambda = \int_{\hat{S}} (\lambda^2 + \theta_k^{(1)} \lambda + o(\lambda)) \eta_\lambda = 4\pi R_\bar{q}^2 \lambda^2 + o(\lambda)
\]
which follows from (9) and \( \int_{\hat{S}} \eta_\bar{q} = 4\pi R_\bar{q}^2 \). Hence
\[
\lim_{\lambda \to \infty} F(S_\lambda) = \lim_{\lambda \to \infty} \left( \frac{|S_\lambda|}{8\pi R_\bar{q}^2 \lambda + \int_{\hat{S}} \theta_k^{(1)} \eta_\bar{q}} \right)^2 = \frac{1}{16\pi R_\bar{q}^2}
\]
and (23) yields
\[
\frac{|S_\lambda|}{8\pi R_\bar{q}^2 \lambda + \int_{\hat{S}} \theta_k^{(1)} \eta_\bar{q}}^2 \leq \frac{1}{16\pi R_\bar{q}^2} \iff \frac{|S_\lambda|}{16\pi} \leq \left( \frac{R_\bar{q}^2 \lambda}{2} + \frac{1}{16\pi R_\bar{q}^2} \int_{\hat{S}} \theta_k^{(1)} \eta_\bar{q} \right)^2. \tag{24}
\]
From the definition of \( D(S_\lambda, \ell^*) \) and using \( \varphi = 2R_\bar{q} \) we have
\[
D(S_\lambda, \ell^*) = \sqrt{\frac{|S_\lambda|}{16\pi} - \frac{R_\bar{q}^2 \lambda}{2} - \frac{1}{16\pi R_\bar{q}^2} \int_{\hat{S}} \theta_k^{(1)} \eta_\bar{q}}
\]
after using (24). Since the right-hand side is the limit of \( D(S_\lambda) \) at infinity (18), we conclude (22).
We can now establish our main result concerning GAB foliations.

**Theorem 3** *(A Penrose-type inequality for GAB foliations).* Let $\Omega$ be a past asymptotically flat null hypersurface and $S_0$ a cross-section. Assume that the spacetime satisfies the dominant energy condition. Then, the area $|S_0|$ satisfies the bound

$$\sqrt{\frac{|S_0|}{16\pi}} - \frac{1}{16\pi} \int_{S_0} \theta^{\ell^*} \eta_{\ell^*} \leq \lim_{\lambda \to \infty} m_H(S_\lambda),$$

where the limit is taken along the GAB foliation $\{S_\lambda\}$ associated to $S_0$. In particular, if $S_0$ is a weakly outer trapped cross-section, then

$$\sqrt{\frac{|S_0|}{16\pi}} \leq \lim_{\lambda \to \infty} m_H(S_\lambda).$$

**Proof.** From lemma 5 and 6, $D(S_\lambda, \ell^*)$ is bounded above by its limit at infinity. The monotonicity of $M_\lambda(S_\lambda, \ell^*)$ then implies

$$M(S_\lambda, \ell^*) \leq \lim_{\lambda \to \infty} M(S_\lambda, \ell^*) = \lim_{\lambda \to \infty} m_H(S_\lambda),$$

where the last equality follows from proposition 1, since $\{S_\lambda\}$ is GAB. In particular, for $\lambda = 0$ we have (25). For the last statement, we simply use that $\theta^{\ell^*} \leq 0$ for weakly outer trapped surfaces.

Inequality (26) gives a completely general upper bound for the area of weakly outer trapped surfaces $S_\lambda$ in terms of an energy-type quantity evaluated at infinity along the outward past null hypersurface generated by $S_0$, provided the latter stays regular all the way to infinity. In combination with the general analysis of the limit of the Hawking energy at infinity carried out in [21], this provides a Penrose-type inequality with potentially interesting consequences. Obviously, this inequality will only correspond to the Penrose inequality whenever the limit of the Hawking energy agrees with the Bondi energy of the cut at infinity defined by $\Omega$. As already mentioned, this is known to occur [2, 21, 24, 25] for foliations approaching large spheres. When $\Omega$ admits a GAB foliation approaching large spheres, then the limit of the Hawking energy along this foliation is the Bondi energy $E_B$ associated to this observer at infinity, and the Penrose-type inequality in theorem 3 becomes the standard Penrose inequality, thus recovering the original result by Ludvigsen & Vickers and Bergqvist quoted as theorem 2.

4. On the inequality $D(S_\lambda, \ell^*) \leq \lim_{\lambda \to \infty} D(S_\lambda, \ell^*)$

The key ingredient that allowed us to prove the Penrose-type inequality (26) is $D(S_\lambda, \ell^*) \leq \lim_{\lambda \to \infty} D(S_\lambda, \ell^*)$. In fact, the argument in the proof of theorem 3 combined with proposition 1 shows that any surface $S_0$ satisfying the inequality

$$D(S_0, \ell^*) \leq \lim_{\lambda \to \infty} D(S_\lambda, \ell^*) = \frac{1}{16\pi R^4} \int_{S} \theta_{\ell^*} \eta_{\ell^*},$$

along a geodesic foliation $\{S_\lambda\}$ starting at $S_0$, in a spacetime satisfying the dominant energy condition, automatically satisfies the inequality

$$D(S_0, \ell^*) \leq \lim_{\lambda \to \infty} D(S_\lambda, \ell^*) = \frac{1}{16\pi R^4} \int_{S} \theta_{\ell^*} \eta_{\ell^*},$$

along a geodesic foliation $\{S_\lambda\}$ starting at $S_0$, in a spacetime satisfying the dominant energy condition, automatically satisfies the inequality
This is the Penrose inequality provided \( S_0 \) is a weakly outer trapped surface and the right-hand side is the Bondi energy \( E_B \) along \( \{S_\lambda\} \). For this, it is sufficient that \( \{S_\lambda\} \) approaches large spheres and this will be the case we will be interested in from now on. However, we postpone making the assumption that \( \hat{q} \) is the round metric until Subsection 4.2 because proposition 2 below (which holds for arbitrary geodesic foliations) may be of independent interest.

In the previous section, the validity of (27) followed from the monotonicity of \( F(S_\lambda) \) along GAB foliations. As shown in lemma 6 the monotonicity of \( l_{FS} \) is sufficient to establish (27) for arbitrary geodesic foliations. Since the derivative of (19) is

\[
\frac{d}{d\lambda} F(S_\lambda) = \frac{1}{8\pi R^2 \lambda + \int_{S} \theta^{(1)} k \eta_q} \left( \frac{d}{d\lambda} |S| - \frac{16\pi R^2 |S|}{8\pi R^2 \lambda + \int_{S} \theta^{(1)} k \eta_q} \right)
\]

we have established:

**Proposition 2.** Let \( \Omega \) be a past asymptotically flat null hypersurface in a spacetime satisfying the dominant energy condition and \( \{S_\lambda\} \) a geodesic foliation. If

\[
\frac{d}{d\lambda} |S| \geq \frac{16\pi R^2 |S|}{8\pi R^2 \lambda + \int_{S} \theta^{(1)} k \eta_q}
\]

holds for all \( \lambda \geq 0 \) then the inequality (28) holds. In particular if \( \{S_\lambda\} \) approaches large spheres and (29) is satisfied, then the Penrose inequality \( E_B \geq \sqrt{\frac{|S_0|}{16\pi}} \) holds, where \( E_B \) is the Bondi energy associated to the observer defined by \( \{S_\lambda\} \).

**Remark 2.** Expanding the area as

\[
|S| = 4\pi R^2 \lambda^2 + \left( \int_{S} \theta^{(1)} k \eta_q \right) \lambda + \hat{\Theta},
\]

(29) becomes, after some cancellations,

\[
\left( \int_{S} \theta^{(1)} k \eta_q \right)^2 + \left( 8\pi R^2 \lambda + \int_{S} \theta^{(1)} k \eta_q \right) \frac{d\hat{\Theta}}{d\lambda} \geq 16\pi R^2 \hat{\Theta}.
\]

This alternative form of proposition 2 will be used in section 6 below.

GAB foliations have the property that (29) is always true. It is natural to ask whether the constancy of \( \theta^{(1)} k \) can be relaxed and still obtain sufficiently general conditions under which (29) holds. The issue, however, appears to be difficult. In the next subsection, we study the behaviour of the derivative of \( F(S_\lambda) \) near infinity and show that both cases of \( F(S_\lambda) \) being monotonically increasing or monotonically decreasing near infinity are possible.

**4.1. On the monotonicity of \( F(S_\lambda) \) for large \( \lambda \)**

A necessary condition for (19) to be non-negative for all \( \lambda \) is, of course, that its leading term at infinity is non-negative. To determine the asymptotic behavior at infinity requires one extra term in the expansion of \( \theta_k \) as compared to (10). To make sure this is possible we need a slightly stronger definition of asymptotic flatness.
Definition 4 (Strong past asymptotic flatness). A null hypersurface \( \Omega \) in a spacetime \((\mathcal{M}^4, g)\) is strong past asymptotically flat if it is past asymptotically flat with \((i)\) in definition 1 replaced by the stronger condition

\[(i)' \text{ There exist symmetric 2-covariant transversal and Lie constant tensor fields } \hat{\gamma} \text{ (positive definite), } h \text{ and } \psi_0 \text{ such that } \hat{\gamma} \text{ defined by } \gamma = \lambda^2 \hat{\gamma} + \lambda h + \psi_0 + \hat{\gamma} \text{ is } \hat{\gamma} = o_1(1) \cap o_2^2(1). \]

Remark 3. In strong asymptotically flat null hypersurfaces, all geodesic foliations \( \{S_n\} \) automatically satisfy item \((i)'\) in the definition. Also, there always exist geodesic foliations \( \{S_n\} \) for which the asymptotic metric \( \hat{\gamma} \) is the round metric of unit radius on \( S^2 \) (see [21] for a proof of both facts in the context of asymptotically flat null hypersurfaces, which carries over immediately to the strong asymptotically flat case).

A first consequence of strong asymptotic flatness is that the function \( \Theta \) defined by

\[ \eta_{S_n} = (\lambda^2 + \theta^{(1)} + \Theta) \hat{\eta}_q \]  \hfill (32)

is of the form \( \Theta = \Theta_0 + \hat{\Theta} \), with \( \Theta_0 \) Lie constant and \( \hat{\Theta} = o_1(1) \). A second consequence, which follows from \((4)\), is that \( \partial_k \) admits the expansion

\[ \theta_k = -\frac{2}{\lambda} + \frac{\theta^{(1)}_k + \theta^{(2)}_k}{\lambda^2} + o(\lambda^{-3}) \]  \hfill (33)

with \( \theta^{(2)}_k \) Lie constant. The following proposition relates \( \Theta_0 \) with \( \theta^{(1)}_k \) and \( \theta^{(2)}_k \) and provides a universal bound for \( \theta^{(2)}_k \).

Proposition 3. Let \( \Omega \) be a strong past asymptotically flat null hypersurface and \( \{S_n\} \) a geodesic foliation. Then

\[ \Theta_0 := \lim_{\lambda \to \infty} \Theta = \frac{1}{2}(\theta^{(1)}_k \gamma^2 + \theta^{(1)}_k \gamma^2). \]  \hfill (34)

If, in addition, the spacetime satisfies the dominant energy condition then we will also have

\[ \theta^{(2)}_k \leq -\frac{1}{2} (\theta^{(1)}_k \gamma^2) \leq 0. \]  \hfill (35)

Proof. Inserting \((32)\) and \((33)\) into the evolution equation

\[ \mathcal{L}_{S_k} \eta_{S_n} = -\partial_k \eta_{S_n} \]  \hfill (36)

gives

\[ \left( 2\lambda + \theta^{(1)}_k + \frac{d\hat{\Theta}}{d\lambda} \right) \hat{\eta}_q = -\left( \frac{-2}{\lambda} + \frac{\theta^{(1)}_k}{\lambda^2} + \frac{\theta^{(2)}_k}{\lambda^3} + o(\lambda^{-3}) \right) \times \left( \lambda^2 + \theta^{(1)}_k \lambda + \Theta_0 + o(1) \right) \hat{\eta}_q \]

\[ = \left( 2\lambda + \theta^{(1)}_k + (2\Theta_0 - (\theta^{(1)}_k \gamma^2 - \theta^{(2)}_k)) \frac{1}{\lambda} + o(\lambda^{-1}) \right) \eta_{S_n} \]

Since \( \frac{d\hat{\Theta}}{d\lambda} = o(\lambda^{-1}) \) we conclude \( 2\Theta_0 - (\theta^{(1)}_k \gamma^2 - \theta^{(2)}_k) = 0 \), which proves \((34)\). For the universal bound \((35)\), let us define \( \check{I} (\lambda) = \frac{\eta_{S_n}}{\theta^{(1)}_k \gamma^2 + 2\lambda^3} \). Its Lie derivative is
\[
\mathcal{L}_\lambda \hat{I}(\lambda) = \frac{1}{(\theta^{(1)}_k + 2\lambda)^2} \left( -\theta_k - \frac{4}{\theta^{(1)}_k + 2\lambda} \right) \eta_{S_k} \geq 0,
\]

where in the last inequality we used (21) (here is where the DEC is used). \( \hat{I}(\lambda) \) has limit at infinity \( \frac{1}{2} \eta_q \). In combination with the fact that \( \hat{I} \) is monotonically increasing we conclude

\[
\hat{I}(\lambda) = \frac{\eta_{S_k}}{(\theta^{(1)}_k + 2\lambda)^2} \leq \frac{1}{4} \eta_q.
\]

Inserting (32), a direct computation gives

\[
\left( -\frac{1}{16} (\theta^{(1)}_k)^2 + \frac{1}{4} \Theta_0 \right) \frac{1}{\lambda^2} + o(\lambda^{-2}) \leq 0 \implies \Theta_0 \leq \frac{1}{4} (\theta^{(1)}_k)^2,
\]

which is simply (35) after using the explicit form of \( \Theta_0 \).

Let us now find the asymptotic expansion of the right-hand side of (19). Plugging the asymptotic expansion (33) gives, after a straightforward computation,

\[
\frac{d}{d\lambda} F(S_k) = \frac{1}{(8\pi R_q^2)^2} \left( \left( \int_S \theta^{(1)}_k q \right)^2 - 8\pi R_q^2 \int_S (\theta^{(1)}_k)^2 \eta_q - 8\pi R_q^2 \int_S \theta^{(2)}_k q \right) \frac{1}{\lambda^2} + o(\lambda^{-3}).
\]

The leading coefficient can be rewritten as \( \frac{F_{\infty}}{(8\pi R_q^2)^2} \), where

\[
F_{\infty} = -4\pi R_q^2 \int_S \left( (\theta^{(1)}_k)^2 + 2\theta^{(2)}_k \right) \eta_q + \left[ \left( \int_S \theta^{(1)}_k q \right)^2 - 4\pi R_q^2 \int_S (\theta^{(1)}_k)^2 \eta_q \right]
\]

which is a difference of positive quantities. Indeed, the first term is non-negative because of (35), while the term in brackets is non-positive because

\[
\left( \int_S \theta^{(1)}_k q \right)^2 \leq 4\pi R_q^2 \int_S (\theta^{(1)}_k)^2 \eta_q
\]

by the Hölder inequality. Depending on which term dominates, the functional \( F(S_k) \) will be increasing or decreasing near infinity. Non-negativity of the leading term (37) is obviously a necessary condition for the hypothesis of proposition 2 to hold. However, even when (37) has the right sign, it is not at all obvious how to ensure that \( F(S_k) \) is monotonic for all \( \lambda \) when the foliation is, in addition, assumed to approach large spheres. We have attempted (and failed) to find sufficient condition ensuring \( \frac{d}{d\lambda} F(S_k) \leq 0 \), as this would immediately imply that \( F(S_k) \) is increasing (because \( F'(S_k) \to 0 \) at infinity). Despite the lack of success so far, approaching the null Penrose inequality using the monotonic functional \( F(S_k) \) remains an interesting open problem, particularly in view of the fact that \( F(S_k) \) is always monotonic for GAB foliations.

4.2. Renormalized area method for the Penrose inequality

Monotonicity of \( F(S_k) \) along geodesic foliations approaching large spheres is an interesting sufficient condition for the Penrose inequality along null hypersurfaces. However, as discussed in the previous subsection, it appears to be difficult to find general situations where \( F'(S_k) \geq 0 \) can be guaranteed. In this subsection, we consider an \textit{a priori} different setup which implies the validity of (27) and hence of the Penrose inequality whenever the foliations also satisfy the restriction of approaching large spheres. Let us assume from now on that \( \hat{q} \) is a round metric on the sphere. Without loss of generality we can then assume that \( \hat{q} \) is a round
metric of radius one, which we denote by $q$. Then $R_q = 1$ and $\ell^* = \ell$. We want to investigate the condition
\[
\frac{d}{d\lambda} D(S_\lambda, \ell^*) \geq 0
\] (38)
which indeed implies the validity of (27) and hence the validity of the Penrose inequality. Since $|S_\lambda|$ diverges at infinity like $4\pi L^2$, the functional $D(S_\lambda, \ell)$ can be regarded as a renormalization of the area functional, in order to make it bounded. We thus call the approach to the null Penrose inequality via (38) the renormalized area method. It is interesting that this method is, in fact, a sub-case of the general setup involving monotonicity of $F(S_\lambda)$.

**Proposition 4.** Let $\Omega$ be a strong past asymptotically flat hypersurface and $\{S_\lambda\}$ a geodesic foliation approaching large spheres. Then
\[
\frac{d}{d\lambda} D(S_\lambda, \ell) \geq 0 \implies \frac{d}{d\lambda} F(S_\lambda) \geq 0.
\]

**Proof.** Let $L := \frac{1}{16\pi} \int_{S_\lambda} \theta_k \eta_q > 0$ be the limit of $D(S_\lambda, \ell)$ at infinity. Since $|S_\lambda| = 16\pi \left(D(S_\lambda, \ell) + \frac{\lambda}{2}\right)^2$, we can rewrite $F(S_\lambda)$ as
\[
F(S_\lambda) = |S_\lambda| \left(8\pi \lambda + \int_{S_\lambda} \theta_k \eta_q \right)^2 = \frac{(D_\lambda + \frac{\lambda}{2})^2}{16\pi (L + \frac{\lambda}{2})^2}
\]
where $D_\lambda$ is a short-hand for $D(S_\lambda, \ell)$. Let
\[
f(\lambda) := \sqrt{16\pi} F(S_\lambda) = \frac{D_\lambda + \frac{\lambda}{2}}{L + \frac{\lambda}{2}}
\]
so that
\[
f'(\lambda) \left( L + \frac{\lambda}{2} \right) = \frac{dD_\lambda}{d\lambda} + \frac{1}{2} (1 - f(\lambda)).
\] (39)
If $\frac{dD_\lambda}{d\lambda} \geq 0$ it follows $D_\lambda \leq \lim_{\lambda \to \infty} D_\lambda = L$ so that $f(\lambda) = \frac{D_\lambda + \frac{\lambda}{2}}{L + \frac{\lambda}{2}} \leq 1$ and we conclude from (39) that $f'(\lambda) \geq 0$, which is equivalent to $F'(S_\lambda) \geq 0$. \hfill \Box

The derivative of $D(S_\lambda, \ell)$ is
\[
\frac{d}{d\lambda} D(S_\lambda, \ell) = \frac{1}{2\sqrt{16\pi S_\lambda}} \left( \frac{d}{d\lambda} |S_\lambda| - \sqrt{16\pi S_\lambda} \right) = \frac{1}{2\sqrt{16\pi S_\lambda}} \left( \int_{S_\lambda} (-\theta_k) \eta_k - \sqrt{16\pi S_\lambda} \right).
\]
Given that $\theta_k < 0$, the inequality $\frac{d}{d\lambda} D(S_\lambda, \ell) \geq 0$ can be equivalently written in a slightly more convenient form as $G(\lambda) \geq 0$, where
\[
G(\lambda) := \int_{S_\lambda} (-\theta_k) \eta_k \right)^2 - 16\pi |S_\lambda|.
\]
We start by computing the limit of $G(\lambda)$ at infinity.
Proposition 5. With the same assumptions as in proposition 4,
\[ \lim_{\lambda \to \infty} G(\lambda) = F_\infty = \left( \int_{S^3} \left( \frac{\theta^1_k}{2} \right)^2 - 8\pi \int_{S^3} (\theta^1_k)^2 \eta_k - 8\pi \int_{S^3} \theta^2_k \eta_k \right)^2. \] (40)

Proof. We have shown in proposition 3 that
\[ \eta_{S_k} = \left( \lambda^2 + \theta^1_k \lambda + \frac{1}{2} \left( (\theta^1_k)^2 + \theta^2_k \right) + o(1) \right) \eta_k. \] (41)
From expansion (10), we have \( \theta_k \eta_{S_k} = -2\lambda + \theta^1_k + o(1) \), so that
\[ \left( \int_{S^3} \theta_k \eta_{S_k} \right)^2 = 64\pi^2 \lambda^2 + 16\pi \int_{S^3} \theta^1_k \eta_k + \left( \int_{S^3} \theta^1_k \eta_k \right)^2 + o(1). \]
Also from (41),
\[ |S_k| = 4\pi^2 \lambda^2 + \lambda \int_{S^3} \frac{\theta^1_k}{2} \eta_k + \int_{S^3} \frac{1}{2} \left( (\theta^1_k)^2 + \theta^2_k \right) \eta_k + o(1). \]
Inserting both into \( G(\lambda) \) the divergent terms cancel out and we are left with (40). \( \Box \)

Remark 4. The limit of \( G(\lambda) \) is directly related to the leading term in the asymptotic expansion of \( F(S_k) \) so that the inequality ‘at infinity’ \( F_\infty > 0 \) is necessary for both methods. Thus, for sufficiently large \( \lambda \), the renormalized area method does not only imply \( F'(S_k) > 0 \), but it is in fact equivalent to it (possibly excluding the case \( F_\infty = 0 \) where higher order terms dominate). However, we do not expect this to be true for all \( \lambda \), as it appears that \( \Delta \ell \) should be a proper subset of \( F'(S_k) \).

Assuming we are in the situation where \( \lim_{\lambda \to \infty} G(\lambda) \geq 0 \), we can ensure \( G(\lambda) \geq 0 \) by the condition \( G'(\lambda) \leq 0 \). This derivative is, from the Raychaudhuri equation (5),
\[ G'(\lambda) = 2 \left( \int_{S^3} \theta_k \eta_{S_k} \right) \left( \frac{d}{d\lambda} \int_{S^3} \theta_k \eta_{S_k} + 8\pi \right) \]
\[ = 2 \left( \int_{S^3} \theta_k \eta_{S_k} \right) \left( \int_{S^3} \left( \mathrm{Ric}^g(k, k) - \frac{1}{2} \theta^2_k + \Pi^k_{AB} \Pi^{AB} \right) \eta_{S_k} + 8\pi \right). \]
Since the first term is always negative, \( G'(\lambda) \leq 0 \) is equivalent to \( H(\lambda) \geq 0 \), where we have defined
\[ H(\lambda) = \int_{S^3} \left( \mathrm{Ric}^g(k, k) - \frac{1}{2} \theta^2_k + \Pi^k_{AB} \Pi^{AB} \right) \eta_{S_k} + 8\pi. \]
We proceed with the computation of the derivative of this function and of its limit at infinity.

Proposition 6. With the same assumptions as in proposition 4, \( \lim_{\lambda \to \infty} H(\lambda) = 0 \) and the derivative of \( H(\lambda) \) is
\[ H'(\lambda) = \int_{S^3} \left( -2\theta \mathrm{Ric}^g(k, k) + 2(\Pi^k_{AB} R_{AB} + \frac{d}{d\lambda} \mathrm{Ric}^g(k, k) \right) \eta_{S_k}, \] (42)
where \( R_{AB} := \mathrm{Riem}^g(X_A, k, X_B, k) \).

Proof. For the limit, we split \( H(\lambda) \) in three terms and show that each tends to zero. We start with \( \int_{S^3} \Pi^k_{AB} \Pi^{AB} \eta_{S_k} \). From equation (4) and the expansion (i) in definition 4 for the metric.
\[
\gamma, \text{ it follows}
\]
\[
K_{AB}^{k} = -\dot{q}_{AB} \lambda - \frac{1}{2} h_{AB} + o(1)
\]
so that its trace-free part is \(\Pi_{AB} = O(1)\). Since \(\gamma(\lambda)_{AB} = \lambda^{2} q_{AB} + o(\lambda)\), its inverse is
\[
\gamma(\lambda)_{AB} = \frac{1}{\lambda^{2}} q_{AB} + o(\lambda^{-2})
\]
and \(\Pi_{AB} = O(\lambda^{-4})\) so that
\[
\int_{S_{\lambda}} (\Pi_{AB})^{k} \eta_{AB} \rightarrow 0
\]
as a consequence of \(\eta_{AB} = \lambda^{2} q_{AB} + O(\lambda)\). Concerning the term in \(\text{Ric}^{k}(k, k)\), we note that inserting the expansion (10) into the Raychaudhuri equation (5) yields \(\Pi_{AB}^{k} + \text{Ric}^{k}(k, k) = O(\lambda^{-4})\) which implies \(\text{Ric}^{k}(k, k) = O(\lambda^{-4})\) and again
\[
\int_{S_{\lambda}} \text{Ric}^{k}(k, k) \eta_{AB} \rightarrow 0.
\]
Finally, \(\theta_{\ell}^{2} \eta_{AB} = (4 + o(1)) q_{AB} + O(\lambda^{-2})\) from which
\[
\int_{S_{\lambda}} \left( -\frac{1}{2} \theta_{\ell}^{2} \right) \eta_{AB} + 8\pi \rightarrow 0.
\]
We next compute the derivative of \(H(\lambda)\). The extrinsic curvature \(K_k\) along a null hypersurface satisfies the Ricatti equation [8]
\[
\frac{d}{d\lambda} (K^{k})_{A} = (K^{k})_{C} (K^{k})_{B} + R_{AB}.
\]
The trace-free part of this equation is
\[
\frac{d}{d\lambda} (\Pi^{k})_{AB} = \theta_{\ell} \Pi^{k}_{AB} + R_{AB} - \frac{1}{2} \text{Ric}^{k}(k, k) \delta^{k}_{AB},
\]
where we have used \(\Pi^{k}_{AB} (\Pi^{k})^{B}_{C} = \frac{1}{2} \text{tr}(\Pi^{k})^{2} \gamma_{AC}\), which is an algebraic property of endomorphisms in two-dimensional vector spaces. Thus
\[
\frac{d}{d\lambda} (\Pi^{k}_{AB}) = 2\theta_{\ell} \text{tr}(\Pi^{k}) + 2R^{AB} \Pi^{k}_{AB}.
\]
Using this together with (36) and the Raychaudhuri equation, the derivative (42) is obtained after a number of cancellations. \(\square\)

We can combine the previous computations to find a set of sufficient conditions under which the renormalized area method applies.

**Theorem 4** (Sufficient conditions for the renormalized area method). Let \(\Omega\) be a strong past asymptotically flat null hypersurface and \(\{S_{\ell}\}\) a geodesic foliation approaching large spheres. Assume that the spacetime satisfies the dominant energy condition. If the two conditions
\[
(i) \left( \int_{S_{\ell}} \theta_{\ell}^{(1)} \eta_{Q} \right)^{2} - 8\pi \int_{S_{\ell}} (\theta_{\ell}^{(1)})^{2} \eta_{Q} - 8\pi \int_{S_{\ell}} \theta_{\ell}^{(2)} \eta_{Q} \geq 0,
\]
\[
(ii) \int_{S_{\ell}} \left( -2\theta_{\ell} \text{Ric}^{k}(k, k) + 2(\Pi^{k})^{AB} R_{AB} + \frac{1}{2\pi} \text{Ric}^{k}(k, k) \right) \eta_{AB} \leq 0, \quad \forall \lambda \geq 0
\]
hold, then
\[
\sqrt{\frac{|S_{\ell}|}{16\pi}} = \frac{1}{16\pi} \int_{S_{\ell}} \theta_{\ell} \eta_{AB} \leq E_{B}
\]
where $E_B$ is the Bondi energy associated to the foliation $\{S_\lambda\}$. In particular, if $S_0$ is a weakly outer trapped surface then the Penrose inequality $E_B \geq \sqrt{\frac{S_0}{16\pi}}$ holds.

**Proof.** From (ii) we have $H'(\lambda) \leq 0$ which implies $H(\lambda) \geq 0$, as this function tends to zero at infinity. Hence $G'(\lambda) \leq 0$. From (i) and proposition 5 we have $\lim_{\lambda \to \infty} G(\lambda) = 0$ and we conclude $G(\lambda) \geq 0$, or equivalently $D'(S_\lambda, \ell) \geq 0$. The theorem follows from (28) using the fact that $\{S_\lambda\}$ approaches large spheres. 

It is remarkable that $H'(\lambda)$ only involves curvature terms. This makes checking the validity of $H'(\lambda) \geq 0$ feasible, at least in some cases. In the next two sections, we explore the validity of conditions (i) and (ii) in two simple, but relevant situations.

5. Shear-free vacuum case

In this section, we consider whether the functional $M(S_\lambda, \ell)$ can be used to prove the Penrose inequality in the case of shear-free null hypersurfaces $\Omega$ (i.e. satisfying $K^i = \frac{1}{\lambda} \partial \gamma$) embedded in a vacuum spacetime. The Penrose inequality in this setup was proved by Sauter [25] in full generality exploiting properties of the Hawking energy. Our interest in analyzing the shear-free case is to gain insight on the range of applicability and limitations of the methods discussed above.

For instance, concerning the renormalized area method in section 4.2, the vacuum and shear-free conditions immediately imply that $H'(\lambda) = 0$, so condition (ii) in theorem 4 is always satisfied. Thus $H(\lambda)$ vanishes identically, which is equivalent to $G(\lambda) = \text{const}$. The method works if and only if this constant is non-negative. It can be computed from its limit at infinity in proposition 5 as

$$G(\lambda) = \lim_{\lambda \to \infty} G(\lambda) = \left( \int_{\Omega^2} \theta^{(1)}_k \eta_q \right)^2 - 8 \pi \int_{\Omega^2} \left( \theta^{(1)}_k \right)^2 \eta_q - 8 \pi \int_{\Omega^2} \theta^{(2)}_k \eta_q. \ (47)$$

In the shear-free vacuum case, the Raychaudhuri equation (5) is simply $\frac{\partial \theta}{\partial \lambda} = -\frac{1}{2} \theta^2$, which integrates to

$$\theta_k = -\frac{2}{\lambda + \alpha},$$

where $\alpha > 0$ (because $\theta_k < 0$ all along $\Omega$) is a Lie constant function. Expanding near infinity

$$\theta_k = -\frac{2}{\lambda} + \frac{2\alpha}{\lambda^2} - \frac{2\alpha^2}{\lambda^3} + O(\lambda^{-4}) \implies \theta_k^{(1)} = 2\alpha, \quad \theta_k^{(2)} = -2\alpha^2,$$

which inserted into (47) yields

$$G(\lambda) = 4 \left( \int_{\Omega^2} \alpha \eta_q \right)^2 - 4 \pi \int_{\Omega^2} \alpha^2 \eta_q.$$
Despite all this, the method involving the functional \( M(S_\beta, \ell) \) is capable of establishing the Penrose inequality in the shear-free vacuum case. However, as we shall see next, the argument is not based on the monotonicity of \( M(S_\beta, \ell) \) (which fails in general, see below) but via an integration of (8), which in turn relies on the fact that all the geometric information along \( \Omega \) can be computed explicitly in the shear-free vacuum case. From the shear-free condition and the expression for \( \theta_\ell \), the metric \( g_\ell \) can be obtained from (4)

\[
\frac{d\gamma_{S_\beta}}{d\lambda} = -2k^\ell = -\theta_\ell \gamma_{S_\beta} = \frac{2}{\lambda + \alpha} \gamma_{S_\beta} \iff \gamma_{S_\beta} = (\lambda + \alpha)^2 \dot{q}
\]

where we used the fact that the foliation \( \{ S_\ell \} \) approaches large spheres. The volume form is \( h_\ell \) = \( (\lambda + \alpha)^2 \eta_\beta \). As shown in lemma 1, the derivative of \( M_0(S_\beta, \ell) \) involves the connection one-form \( s_\ell \). This object satisfies the following well-known evolution equation along an arbitrary foliation defined by a null generator \( k \) (see e.g. [21])

\[
k(s_\ell(X)) = -X(Q_\ell) - s_\ell(X) \theta_\ell + (\text{div}_{S_\beta} K^\ell)(X) - D_\chi \theta_\ell - \text{Ein}^\ell(k, X),
\]

where \( X \) is tangent to \( S_\beta \) and satisfies \([k, X] = 0\). In the vacuum, geodesic and shear-free case this equation becomes

\[
\frac{ds_\ell(X)}{d\lambda} = -\frac{2}{\lambda + \alpha} s_\ell(X) + \frac{1}{(\lambda + \alpha)^2} X(\alpha)
\]

after using the explicit form of \( \theta_\ell \). This equation can be integrated to

\[
s_\ell = \frac{1}{(\lambda + \alpha)^2} (\lambda d\alpha + \omega)
\]

(48)

where \( \omega \) is a Lie constant transversal one-form. In order to investigate the monotonicity of the functional \( M(S_\beta, \ell) \) we need to evaluate (8) and in particular \( |s_\ell|^2_{\gamma_{S_\beta}} \eta_{S_\beta} \). Using (48) and the form of \( \gamma_{S_\beta} \) we have

\[
|s_\ell|^2_{\gamma_{S_\beta}} \eta_{S_\beta} = \frac{1}{(\lambda + \alpha)^2} |\lambda d\alpha + \omega|^2 \eta_\beta
\]

and identity (8) simplifies to

\[
\frac{dM(S_\beta, \ell)}{d\lambda} = \frac{1}{8\pi} \sqrt{\frac{4\pi}{\int_{S^2} (\lambda + \alpha)^2 \eta_\beta}} \int_{S^2} (\lambda + \alpha) \eta_\beta - \frac{1}{2} + \frac{1}{8\pi} \int_{S^2} \frac{1}{(\lambda + \alpha)^2} |\lambda d\alpha + \omega|^2 \eta_\beta.
\]

We want to bound this expression from below. The Lie constant one-form \( \omega \) can be uniquely split into

\[
\omega = -\beta d\alpha + \omega^+, \quad (\omega^+, d\alpha)_\beta = 0
\]

where \( \beta \) is a Lie constant function on \( \Omega \). Thus

\[
\frac{dM(S_\beta, \ell)}{d\lambda} = \frac{1}{2} \left( \frac{\int_{S^2} (\lambda + \alpha) \eta_\beta}{\sqrt{4\pi \int_{S^2} (\lambda + \alpha)^2 \eta_\beta}} - 1 \right) + \frac{1}{8\pi} \int_{S^2} \frac{((\lambda - \beta)^2 |d\alpha|^2_\beta + |\omega^+|^2_\beta)}{(\lambda + \alpha)^4} \eta_\beta.
\]

(49)

The Hölder inequality implies that the term in parenthesis is non-positive and strictly negative unless \( \alpha \) is constant (which corresponds both to the GAB case and to the \( D'(S_\beta, \ell) \geq 0 \) case in the present context). Since \( \beta \) may be positive and constant and \( \omega^+ \) is allowed to be zero, it follows that \( \frac{dM(S_\beta, \ell)}{d\lambda} \) may have either sign. This shows that one cannot expect \( M(S_\beta, \ell) \) to
be a monotonic functional in all cases. Nevertheless, the right-hand side in (49) is an explicit function in \( \lambda \) that can be integrated explicitly

\[
M(S_\lambda, \ell) - M(S_{0}, \ell) = \left[ \frac{1}{2} \left( \int_{\mathbb{S}^2} \frac{(\lambda + \alpha)^2 \eta_q}{4\pi} - \lambda \right) \right]
\]

\[+ \frac{1}{8\pi} \int_{\mathbb{S}^2} \left[ \frac{-\alpha^2}{4} - \frac{1}{3} \left( \beta - \frac{\alpha}{2} \right)^2 + \lambda (\beta - \alpha) - \lambda^2 \right] \left[ d\alpha_q^2 - \frac{1}{3} |\omega_q^1|^2 \right] \eta_q \biggr] \biggr|_{\lambda=0}^h .
\]

Sending \( \lambda \) to infinity, evaluating at \( \lambda = 0 \) and using that the flow approaches large spheres

\[E_{0} = M(S_{0}, \ell)\]

\[= \int_{\mathbb{S}^2} \alpha \eta_q - \sqrt{4\pi} \int_{\mathbb{S}^2} \alpha^2 \eta_q + \int_{\mathbb{S}^2} \left[ \frac{d\alpha_q^2}{4\alpha} + \frac{(\beta - \frac{\alpha}{2})^2 |d\alpha_q^2| + |\omega_q^1|^2}{3\alpha^3} \right] \eta_q \biggr]_{\alpha=0}^{h}.
\]

This identity is valid for any spacelike cross-section \( S_0 \) embedded in a shear-free and vacuum \( \Omega \). We now use a fundamental identity for arbitrary \( C^1 \) functions \( F \) on \( \mathbb{S}^2 \), known as the Beckner inequality [3], which reads

\[\int_{\mathbb{S}^2} \left( F^2 + |dF_q^2| \right) \eta_q \geq \sqrt{4\pi} \int_{\mathbb{S}^2} \nabla^2 F \eta_q \]

with equality only for the constant functions. Writing \( F = \sqrt{\alpha} \) it follows

\[\int_{\mathbb{S}^2} \left( \alpha + \frac{|d\alpha_q^2|}{4\alpha} \right) \eta_q \geq \sqrt{4\pi} \int_{\mathbb{S}^2} \alpha^2 \eta_q \]

and \( I_1 \) is non-negative. The Penrose inequality, in this case, follows because \( I_2 \) is manifestly non-negative and on a weakly outer trapped surface \( \theta \leq 0 \).

The proof by Sauter [25] of this inequality in the vacuum, shear-free case involved computing the Hawking energy for a foliation \( \{ S_\alpha \} \) with the property \( \theta_\alpha(S_\alpha) = \frac{2}{3} \). This is, in general, a different foliation to the one used before (they only agree when \( \alpha \) is constant). A fundamental step in Sauter’s argument was also the Beckner inequality.

Note also, that the Penrose inequality in the shear-free case involves not only the gap given by the Beckner inequality, but a second gap given by \( I_2 \). The stronger Penrose inequality (50) is obviously sharp because if \( S_0 \) is a MOTS (\( \theta = 0 \)) we have equality in (50). It is an interesting question whether one can give a physical interpretation to each of the two positive terms in (50). Note that
that \( \beta \) and \( \omega^\perp \) can be determined in terms of the data on \( S_0 \) and both \( I_1 \) and \( I_2 \) can be written fully in terms of the geometry of the initial surface.

6. Renormalized area method for the shell-Penrose inequality in \( \mathcal{M}^{1,3} \)

The original setup where the Penrose inequality was conjectured [23] involved an incoming null shell of dust matter propagating in the Minkowski spacetime. By exploiting the junction conditions across the shell, the Penrose inequality becomes a geometric inequality for surfaces in the Minkowski spacetime. More precisely, if \( S_0 \) is a closed, connected, spacelike surface embedded in the Minkowski spacetime and satisfying a suitable convexity condition (which corresponds to the condition that its outgoing past null cone extends smoothly to infinity), then

\[
\int_{S_0} \theta_\ell \eta_{S_0} \geq \sqrt{16\pi |S_0|}
\]

where \( \ell \) is the future directed null normal transverse to the \( S_0 \) satisfying \( \langle \ell, k \rangle = -2 \) and \( k \) is future null, tangent to the outgoing past null cone generated by \( S_0 \) and normalized by \( \langle k, \xi \rangle = -1 \), where \( \xi \) is a unit generator of time translations. We call this conjecture the shell-Penrose inequality in Minkowski (it has also been called the Gibbons–Penrose inequality in the literature). Analyzing the validity of this inequality is much simpler than the general null Penrose inequality, but it is still a challenging problem which has received considerable attention in the literature [7, 10, 11, 17–19, 25, 26].

It is a natural question to try and apply the general results concerning the null Penrose inequality discussed above, to the shell-Penrose inequality (51) in the Minkowski spacetime. In this section, we consider the renormalized area method and in section 7 we study the GAB foliation.

The renormalized area method is particularly well suited to the Minkowski spacetime. Indeed, the curvature tensor vanishes identically in this spacetime, so from proposition 6 we have that \( R^A_B = 0 \) makes the Ricatti equation explicitly integrable.

As it is well known (and easy to verify), the solution of the metric evolution equation (4) and the Ricatti equation (45) in Minkowski is given by

\[
(K^A_c)_p = (K^A_0)_C |_{\pi(p)} \left[ (I - \lambda(p)K^B_k |_{\pi(p)})^{-1} \right]_B
\]

(52)

and

\[
(\gamma)_{\lambda} |_{p} = (\gamma)_{\lambda} |_{\pi(p)} \left[ (I - \lambda(p)K^B_k |_{\pi(p)})^{2} \right]_B.
\]

(53)

where \( \pi(p) \) is the (unique) point on \( S_0 \) lying on the null geodesic containing \( p \) and tangent to \( k |_{p} \). Here \( K^k_0 \) denotes the endomorphism with components \( (K^A_0)_B \) and \( K^B_k \) stands to the null second fundamental form of \( S_0 \) along \( k \). Taking the trace of (52) we find \( \theta_k |_{p} = (K^k_0)_c |_{\pi(p)} \left[ (I - \lambda K^A_B |_{\pi(p)})^{-1} \right]_B \), which for the sake of simplicity, we write simply as

\[
\theta_k (\lambda) = tr[K^k_0 o (I - \lambda K^B_k)^{-1}],
\]

dropping all reference to the point \( p \). This expression can be immediately expanded near infinity to give
Thus, \( \theta^{(1)}_k = -\text{tr}((K^k_0)^{-1}) = u, \quad \theta^{(2)}_k = -\text{tr}((K^k_0)^{-2}). \) (54)

Any \( 2 \times 2 \) matrix \( A \) satisfies
\[ \text{tr}(A^2) = \text{tr}(A)^2 - 2 \det(A), \]
which applied to \( K^k_0 \) gives \( \theta^{(2)}_k = 2 \det((K^k_0)^{-1}) - u^2 \). Inserting this into (40) yields
\[ F_\infty = \lim_{\lambda \to \infty} G(\lambda) = \left( \int_{S^2} u \eta_q \right)^2 - 16\pi \int_{S^2} (\det((K^k_0)^{-1})) \eta_q. \] (55)

This expression can be related to the area of \( |S_0| \) as follows. From the definition
\[ q = \lim_{\lambda \to \infty} \gamma(\lambda) = \frac{\gamma(\lambda d - \lambda K^k_0)^2}{\lambda^2} = \gamma(K^k_0)^2, \] (56)
we can relate the volume forms at \( S_0 \) and ‘at infinity’ by
\[ \eta_{S_0} = \det((K^k_0)^{-1}) \eta_q \] (57)
and (55) becomes
\[ F_\infty = \lim_{\lambda \to \infty} G(\lambda) = \left( \int_{S^2} u \eta_q \right)^2 - 16\pi |S_0|. \]

Summarizing, in the Minkowski spacetime \( G(\lambda) = F_\infty \) and \( F_\infty \geq 0 \) implies (cf theorem 4) the validity of (46), which is exactly (51) because the Bondi energy of the Minkowski spacetime vanishes identically. We have thus proved that the shell-Penrose inequality in Minkowski holds provided
\[ \left( \int_{S^2} u \eta_q \right)^2 \geq 16\pi |S_0|. \] (58)

In terms of the support function \( h \) of \( S_0 \) (see [18] for its definition in the present context), this inequality can be rewritten (after some manipulations) in the form
\[ 4\pi \int_{S^2} ((\Delta_q h)^2 + 2 h \Delta_q h) \eta_q \geq 4\pi \int_{S^2} u^2 \eta_q = \left( \int_{S^2} u \eta_q \right)^2, \]
which is precisely the sufficient condition for the shell-Penrose inequality in Minkowski obtained in [18]. This is not surprising since the method in [18] also involved a monotonicity condition for
\[ \frac{\partial}{\partial \lambda} \left( \frac{|S_0|}{\sqrt{16\pi \lambda}} \right) = \frac{1}{2} \lambda. \]
However, the general framework developed here leads to the result in a much more efficient way. In fact, there is an even more direct way of reaching this conclusion as a consequence of proposition 2, or rather of its rewriting in remark 2. Indeed, from (53) and (57),
\[ \eta_{S_0} = \det((K^k_0)^{-1} - \lambda d) \eta_q = (\lambda^2 + \theta^{(1)}_k \lambda + \det((K^k_0)^{-1})) \eta_q \]
\[ \implies |S_0| = 4\pi \lambda^2 + \left( \int_{S^2} \theta^{(1)}_k \eta_q \right) \lambda + \int_{S^2} \det((K^k_0)^{-1}) \eta_q. \] (59)
where in the second equality we used the first expression in (54). Comparing with (30) it follows that \( \Theta \) is Lie constant and takes the value \( \Theta = |S_0| \), so that the necessary condition (31) becomes precisely (58).
The following proposition summarizes the results for the shell-Penrose inequality in Minkowski obtained so far and shows, in addition, that in the Minkowski case monotonicity of $D(S_0)$ is in fact equivalent to the a priori more general conditions (27), or $F'(S_0) \geq 0$.

**Proposition 7** (Equivalence of the monotonicity methods in $\mathcal{M}^{1,3}$). Let $\Omega$ be a past asymptotically flat null hypersurface in $\mathcal{M}^{1,3}$ and $\{S_0\}$ a geodesic foliation approaching large spheres. The following conditions are equivalent:

(i) $\left( \int_{S^2} m_\eta \right)^2 \geq 16\pi |S_0|$

(ii) $\frac{d}{d\lambda} D(S_0) \geq 0$ (renormalized area method),

(iii) $\frac{d}{d\lambda} |S_0| \geq \frac{16\pi}{8\pi \lambda + \int_{S^2} m_\eta} (F'(S_0) \geq 0$ method),

(iv) $D(S_0) \leq \lim_{\lambda \to \infty} D(S_0)$,

where $u = -\text{tr}((K^a_b)^{-1})$. The shell-Penrose inequality for $S_0$ holds if one (and hence any) of these conditions holds.

**Proof.** The implications (ii) $\implies$ (iii) and (ii) $\implies$ (iv) are generally true. The equivalence of (i) and (ii) is a consequence of $G(\lambda) = F_\lambda$ and (55), as discussed above. We have also seen before that (iii) is equivalent to (i) as a consequence of remark 2. It only remains to show that (iv) $\implies$ (iii). Expression (59) for the area $|S_0|$ yields

$$\frac{d}{d\lambda} |S_0| \geq \frac{16\pi |S_0|}{8\pi \lambda + \int_{S^2} \theta^{(1)}_k \eta_\eta} \iff \left( 8\pi \lambda + \int_{S^2} \theta^{(1)}_k \eta_\eta \right)^2 \geq 16\pi |S_0|$$

$$\iff \sqrt{\frac{|S_0|}{16\pi} - \frac{\lambda}{2} \leq \frac{1}{16\pi} \int_{S^2} \theta^{(1)}_k \eta_\eta},$$

which establishes (iv) $\iff$ (iii). $\square$

7. GAB foliations in $\mathcal{M}^{1,3}$: Applications to the shell-Penrose inequality

In the previous section, we studied the renormalized area method for the shell-Penrose inequality in Minkowski. In this section, we investigate in the same setting the consequences of the general Penrose-type inequality obtained in theorem 3. To this end, we need information on the limit of the Hawking energy along GAB foliations. In [21] we have studied the limit of the Hawking energy at infinity for a large class of foliations $\{S_0\}$ along asymptotically flat null hypersurfaces. The results we need from that paper can be summarized as follows:

Let $\{S_0\}$ be a geodesic background foliation approaching large spheres and define $\theta_k^{(1)}$ and $\lambda^{(1)}$ as in definition 1. Consider any other geodesic foliation $\{\tilde{S}_0\}$ starting on the same cross-section $S_0$. The level-set functions $\lambda$ and $\tilde{\lambda}$ are necessarily related by $\lambda = f\tilde{\lambda}$, with $f > 0$ and Lie constant on $\Omega$. Then the limit of the Hawking energy along $\{\tilde{S}_0\}$ is [21]
In order to apply this result in the Minkowski context, we need to compute \( \theta_k^{(1)}, \theta_l^{(1)} \) and \( \theta_q^{(1)} \) for the background foliation, which we fix as follows: choose a Minkowskian coordinate system \((t, x')\) and define the unit Killing \( \xi = \partial_t \). The null generator \( k \) of \( \Omega \) is then uniquely selected by the condition \( (k, \xi) = -1 \) and \( \{S_\lambda\} \) is defined to be the level-set foliation of \( \lambda \in C^\infty(\Omega, \mathbb{R}) \) defined by \( \lambda|_{S_\lambda} = 0 \) and \( k(\lambda) = -1 \). It is immediate to check that \( \{S_\lambda\} \) approaches large spheres.

The time-height function \( \tau_\lambda \) of the level set \( S_\lambda \) is defined to be

\[
\tau_\lambda := t|_{S_\lambda}.
\]

In particular \( \tau_0 = t|_{S_\lambda} \) and, in fact, \( \tau_\lambda|_{\ell} = \tau_{0|_{\ell}} - \lambda \) as a consequence of our choice of normalization for \( k \).

**Lemma 7** *(Asymptotic expansion at \( \lambda = +\infty \)).* Let \( \Omega \) be a past asymptotically flat null hypersurface in \( M^{1,3} \) and \( \{S_\lambda\} \) a geodesic foliation associated to a choice of Minkowskian coordinate system \((t, x')\) as described above. Let \( \ell \) be orthogonal to \( \{S_\lambda\} \) and satisfying \( (\ell, k) = -2 \). Then, the following asymptotic expansions hold

\[
\theta_k = \frac{-2}{\lambda} + \frac{u}{\lambda^2} + o(\lambda^{-2}), \quad u = -\text{tr}(K^k_\ell)^{-1}
\]

\[
\theta_l = \frac{2}{\lambda} + \frac{-u + 2\Delta_\ell \tau_0}{\lambda^2} + o(\lambda^{-2}), \quad \tau_0 := t|_{S_\lambda}
\]

\[
s_{\ell A} = -\frac{\nabla_\ell \tau_0}{\lambda} + o(\lambda^{-1}).
\]

**Proof.** In the previous section, we already proved (61). For \( \theta_l \) we exploit the identity

\[
\theta_l + (1 + |D\tau^\ell_\lambda|^2)\theta_k - 2\Delta_\lambda \tau = 0,
\]

valid for any spacelike surface \( S \) in Minkowski whenever \( \tau = t|_{S} \). This identity is a simple consequence of the fact that \( \xi \) is a covariantly constant vector field and it has been used several times in the literature (we refer to [18] for proof). We apply this identity to \( S_\lambda \) and expand for large \( \lambda \) up to order \( \lambda^{-2} \). In particular, we can neglect all terms of order \( O(\lambda^{-3}) \) or higher. Since \( \tau_\lambda = \tau_0 - \lambda \) and \( \gamma_{\lambda 0} \) has the expansion (44), the gradient term is \( |D\tau^\ell_\lambda| = \gamma_{\lambda 0}^{AB} \tau_{0A} \tau_{0B} = O(\lambda^{-2}) \) and the term \( |D\tau^\ell_\lambda|^2 \theta_k \) is \( O(\lambda^{-3}) \) so that it can be ignored. Concerning the Laplacian term, since \( \Delta_\lambda \tau = \Delta_\lambda \tau_0 \) we have, in local coordinates \( \{\lambda, y^A\} \) adapted to the foliation \( \{S_\lambda\} \) (i.e. such that \( k = -\partial_\lambda \))

\[
\Delta_\gamma \tau_\lambda = \frac{1}{\det(g)} \partial_\lambda (\sqrt{|\det(g)|} \gamma^{-1})^{AB} \partial_B \tau_0 \Delta_\gamma \tau_0
\]

\[
= \frac{1}{\det(g)} \partial_\lambda (\sqrt{|\det(g)|} \gamma^{-1})^{AB} \tau_{0A} \tau_{0B} 1 \lambda^2 + O(\lambda^{-3}) = (\Delta_\gamma \tau_0) \frac{1}{\lambda^2} + O(\lambda^{-3})
\]
where we have used $\gamma(\lambda) = \hat{q} \lambda^2 + O(\lambda)$. Inserting $\theta_\eta = \frac{2}{\hat{q}} + \frac{\theta_{\eta0}}{\hat{q}} + o(\lambda^{-2})$ and (61) into (64) and keeping only the terms in $\lambda^{-2}$ we obtain $\theta^{(1)}_\eta + u - 2 \Delta_{\eta} \tau_0 = 0$, which gives (62).

It only remains to compute $s^0_\ell$ in the expansion $s_\ell = s^0_\ell + o(\lambda^{-1})$. To this end, we decompose the Killing vector $\xi$ into normal and tangential components to $S_\lambda$ as

$$\xi = \frac{1}{2} \ell + \left(1 + \frac{|D_{\tau_k}|^2_\lambda}{2}\right) k - \text{grad} \tau_\lambda \quad (65)$$

where grad is the gradient in $S_\lambda$. This decomposition follows directly from the normalization conditions and the definition of $\tau$ (an explicit derivation can be found in [18], cf expression (16)). Solving for $\ell$ in (65) and inserting into the definition of $s_\ell$:

$$s_{\ell A} = \frac{1}{2} \langle \nabla_{\xi} k, \ell \rangle = \langle \nabla_{\xi} k, \xi \rangle - \left(1 + \frac{|D_{\tau_k}|^2_\lambda}{2}\right) k + \text{grad} \tau_\lambda = \langle \nabla_{\xi} k, \xi \rangle + \tau^\alpha_\beta K^\beta_{\alpha A}$$

Now, from $\langle k, \xi \rangle = -1$ we have $\langle \nabla_{\xi} k, \xi \rangle = -\langle k, \nabla_{\xi} \xi \rangle = 0$ because $\xi$ is covariantly constant. We conclude

$$s_{\ell A} = \tau^\alpha_\beta K^\beta_{\alpha A}$$

from which the expansion (63) follows directly after taking into account (44) and (43). □

Lemma 7 allows us to compute the limit of the Hawking energy along very general foliations by exploiting the results in [21]. For geodesic foliations $\lambda = f \lambda$ we simply need to evaluate (60), which becomes

$$\lim_{\lambda \to \infty} m_H(S_\lambda) = \frac{1}{8 \pi \sqrt{16 \pi}} \left( \int_{S^2} f^2 \eta \right) \int_{S^2} \Delta_q (u + 2 \tau_0) \frac{1}{f} \eta$$

In particular, the GAB foliation associated to $S_0$ has rescaling function $f := \frac{\theta_{\eta0}}{\epsilon} = \frac{u}{\epsilon}$, $c > 0$ so that, along this GAB foliation,

$$\lim_{\lambda \to \infty} m_H(S_\lambda) = \frac{1}{8 \pi \sqrt{16 \pi}} \left( \int_{S^2} u^2 \eta \right) \int_{S^2} \Delta_q (u + 2 \tau_0) \frac{1}{u} \eta$$

Thus, the particularization of theorem 3 to the Minkowski setting reads

**Theorem 5.** Let $\Omega$ be a past asymptotically flat null hypersurface in $M^{1,3}$ and $S_0$ a spacelike cross-section of $\Omega$. Then, the following inequality holds:

$$\sqrt{\frac{|S_0|}{16 \pi}} \leq \frac{1}{16 \pi} \int_{S_0} \theta_\xi \eta_{S_0} + \frac{1}{8 \pi \sqrt{16 \pi}} \left( \int_{S^2} u^2 \eta \right) \int_{S^2} \Delta_q (u + 2 \tau_0) \frac{1}{u} \eta$$

where $u = -\text{tr}((K^\beta_\alpha)^{-1})$, $\tau_0 = t|_{S_0}$ with $t$ a Minkowskian time coordinate. The round asymptotic metric $q$ is defined by (56) and $\langle k, \ell \rangle$ are the future directed null normals to $S_0$ with $k$ tangent to $\Omega$ and satisfying $k(t) = 1$ and $\langle k, \ell \rangle = -2$.

**Remark 5.** Whenever $\lim_{\lambda \to \infty} m_H(S_\lambda) \leq 0$, the shell-Penrose inequality in $M^{1,3}$ (51) for $S_0$ follows.

As we discussed in section 5, the Penrose inequality in the shear-free case relies on a highly non-trivial Sobolev-type inequality for functions on the sphere due to Beckner [3].
This inequality plays a core role both in the proof by Sauter [25] and in the proof presented in section 5. The shear-free case in Minkowski corresponds to the case where \( \Omega \) is the past null cone of a point. In fact, the shell-Penrose inequality for cross-section \( S_0 \) on such a past null cone was first proved by Tod [26] using a Sobolev inequality on Euclidean space applied to suitable radially symmetric functions. One might think that Sobolev-type inequalities of some sort should lie behind any method of proving the shell-Penrose inequality for surfaces lying in the past null cone of a point. We find it most remarkable that theorem 3 is capable of proving the shell-Penrose inequality with no reference whatsoever to any Sobolev-type inequality.

**Corollary 1**  (Shell–Penrose inequality in \( \mathcal{M}^3 \) with spherical symmetry). Consider a point \( p \in \mathcal{M}^3 \) and \( \Omega_p \) the past null cone of \( p \). Let \( S_0 \) be a closed spacelike surface embedded in \( \Omega_p \). Then the shell–Penrose inequality for \( S_0 \) holds true as a consequence of theorem 3.

**Proof.** The proof is immediate if we use the relation between \( u \) and the support function \( h \), see [18]. We provide an alternative proof here for the sake of self-consistency.

Consider a Minkowskian time function \( t \) and choose a value \( t_0 < \inf_{\Omega_p} \tau \), where \( \tau = t|_{\Sigma} \). Then, the intersection of \( \Omega_p \) with the hyperplane \( \{ t = t_0 \} \) is a round sphere \( S_t \) of radius \( t(p) - t_0 \) and lying to the past to \( S_0 \). Let \( k \) be the null generator of \( \Omega_p \) satisfying \( k(t) = 1 \). Then, the second fundamental form along \( k \) of \( S_t \) is \( (K^k)^A_B = -\frac{1}{t(p) - t_0} \delta^A_B \). Since \( K^k \) is both a property of \( \Omega \) and of spacelike surfaces embedded in \( \Omega \) we conclude that \( (K^k)^A_B = -\frac{1}{t(p) - t(q)} \delta^A_B \) for all \( q \in \Omega \) and hence \( (K^k)^A_C = -\frac{1}{t(p) - t(q)} \delta^A_B \). Thus \( u = 2(t(p) - t_0) \) which makes the second term in the right-hand side of (67) identically zero.

**Remark 6.** This argument proves, from (66), that the limit of the Hawking energy at infinity on \( \Omega_p \) vanishes for all geodesic foliations. In fact, an explicit computation shows that the Hawking energy is identically zero for any cross-section of \( \Omega_p \).

**8. An upper bound for the area of \( S_\lambda \) along a past asymptotically flat null hypersurface**

We conclude the paper by returning to the general setup of asymptotically flat null hypersurfaces in spacetimes satisfying the dominant energy condition. We also return to geodesic foliations not necessarily approaching large spheres. In this section, we provide a general upper bound for the area \( |S| \) in terms of asymptotic quantities intrinsic to \( \Omega \). We find an inequality which is weaker than the inequality \( D(S_\lambda, t^*) \leq \lim_{\lambda \to -\infty} D(S_\lambda, t^*) \), the difference between both being a Hölder inequality term.

The general idea behind the inequality in the present section is the observation that one possible method to approach the condition \( D(S_\lambda, t^*) \leq \lim_{\lambda \to -\infty} D(S_\lambda, t^*) \) is to obtain an interpolating function \( P(\lambda) \) satisfying \( D(S_\lambda, t^*) \leq P(\lambda) \leq \lim_{\lambda \to -\infty} D(S_\lambda, t^*) \). While this is hard (as finding such a \( P(\lambda) \) would prove the Penrose inequality), we have been able to find a \( P(\lambda) \) satisfying only the first inequality \( D(S_\lambda, t^*) \leq P(\lambda) \), from which a general inequality bounding \( |S| \) from above in terms of asymptotic quantities follows.

**Proposition 8.** Let \( \Omega \) be a past asymptotically flat null hypersurface embedded in a spacetime that satisfies the dominant energy condition, \( S_0 \) a cross-section and \{ \( S_\lambda \) \} a geodesic foliation starting at \( S_0 \). Let \( \vartheta \) be the asymptotic coefficient defined in (10) and \( \hat{q} \) the asymptotic metric associated to \{ \( S_\lambda \) \}. Then,
\[ |S_\lambda| \leq \frac{1}{4} \int_S (\theta_k^{(1)} + 2\lambda)^2 \eta_\phi, \]  
(68)

and in particular \[ |S_\lambda| \leq \frac{1}{4} \int_S (\theta_k^{(1)})^2 \eta_\phi. \]

**Proof.** Let us fix any \( \lambda_0 > 0 \) and consider the volume form on \( S_\lambda (\lambda \geq 0) \) defined by

\[ \hat{\eta}_S := \frac{1}{(\lambda + \lambda_0)^2} \eta_S. \]

Using the evolution equation \( \mathcal{E}_k \eta_S = -\theta_k \eta_S \), the Lie derivative of \( \hat{\eta}_S \) is

\[ \mathcal{E}_k \hat{\eta}_S = -\left( \theta_k + \frac{2}{\lambda + \lambda_0} \right) \hat{\eta}_S. \]

Writing \( \hat{\eta}_S = \hat{f}(\lambda) \eta_\phi \), (69) becomes a differential equation for \( \hat{f} \), which can be integrated as

\[ \hat{f}(\lambda) = \hat{f}(0) e^{-\int_0^\lambda (\theta_k + \frac{2}{\lambda + \lambda_0}) d\lambda}. \]

The initial value \( \hat{f}(0) \) can be computed ‘at infinity’ as a consequence of \( \hat{\eta}_S \rightarrow \eta_\phi \) when \( \lambda \rightarrow \infty \). Thus \( \hat{f}(0) = e^{\int_0^{\infty} (\theta_k + \frac{2}{\lambda + \lambda_0}) d\lambda} \) and therefore

\[ \hat{f}(\lambda) = e^{\int_\lambda^\infty (\theta_k + \frac{2}{\lambda + \lambda_0}) d\lambda}. \]

We aim at finding an upper bound for \( \hat{f}(\lambda) \). We use inequality (21), which implies

\[ \int_\lambda^\infty \left( \theta_k + \frac{2}{s + \lambda_0} \right) ds \leq \int_\lambda^\infty \left( \frac{-4}{2s + \theta_k^{(1)}} + \frac{2}{s + \lambda_0} \right) ds = \log \left( \frac{2\lambda + \theta_k^{(1)}}{2(\lambda + \lambda_0)} \right). \]

Finally,

\[ |S_\lambda| = \int_S \eta_S = \int_S (\lambda + \lambda_0)^2 \hat{f}(\lambda) \eta_\phi = \int_S (\lambda + \lambda_0)^2 e^{\int_\lambda^\infty (\theta_k + \frac{2}{\lambda + \lambda_0}) d\lambda} \eta_\phi \]

\[ \leq (\lambda + \lambda_0)^2 \int_S e^{\log \left( \frac{2\lambda + \theta_k^{(1)}}{2(\lambda + \lambda_0)} \right)^2} \eta_\phi = \frac{1}{4} \int_S (\theta_k^{(1)} + 2\lambda)^2 \eta_\phi. \]

**Remark 7.** The condition \( D(S_\lambda, \ell^*) \leq \lim_{\lambda \rightarrow \infty} D(S_\lambda, \ell^*) \), namely

\[ \sqrt{\frac{|S_\lambda|}{16\pi}} - \frac{R_\ell}{2} \lambda \leq \frac{1}{16\pi R_\ell} \int_S (\theta_k^{(1)}) \eta_\phi, \]

is equivalent to

\[ |S_\lambda| \leq \frac{1}{16\pi R_\ell^2} \left( \int_S (\theta_k^{(1)} + 2\lambda)^2 \eta_\phi \right)^2. \]

As mentioned above, this inequality is stronger than (68), the difference being a Hölder inequality term. Indeed, a direct application of the Hölder inequality yields
\[ |S_{\phi}| \leq \frac{1}{16\pi R_{\tilde{q}}} \left( \int_{S} (\theta_{k}^{(1)} + 2\lambda)\eta_{\tilde{q}} \right)^{2} \leq \frac{1}{4} \int_{S} (\theta_{k}^{(1)} + 2\lambda)^{2}\eta_{\tilde{q}}. \]

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