Supporting Information of “Pattern phase transitions of self-propelled particles: gases, crystals, liquids, and mills”

1. Proof of Eqs. (3) and (4) of the main body

Let agent $i$’s position be $\vec{x}_i = R[\cos \theta_i, \sin \theta_i]^T$, with $\theta_i$ and $R$ being the angle and the radius of the ring respectively, then $\vec{v}_i = R[-\dot{\theta}_i \sin \theta_i, \dot{\theta}_i \cos \theta_i]^T$. Let agent $i$’s angular velocity be $\omega := d\theta_i/dt$ with $\omega = v_0/R$, one has $\vec{v}_i = R\omega[-\sin \theta_i, \cos \theta_i]^T = v_0[-\sin \theta_i, \cos \theta_i]^T$. Analogously, $\frac{d\vec{v}_i}{dt} = v_0\dot{\theta}_i[-\cos \theta_i, -\sin \theta_i]$. Without loss of generality, let $\theta_i = 0$, then

$$\frac{d\vec{v}_i}{dt} = \frac{v_0^2}{R}[-1, 0]^T. \quad (1)$$

In the steady state, all the agents reach the target speed $||\vec{v}_i|| = v_0$. For a noise-free system, i.e., $\sigma = 0$, Eq. (1) rewrites

$$\frac{d\vec{v}_i}{dt} = b \sum_{j \in N_i} \frac{(d - ||\vec{x}_{ij}||)||\vec{x}_{ij}||^2}{||\vec{x}_{ij}||^3}. \quad (2)$$

Since $N$ agents are evenly distributed along a single ring, the $j$-th agent is located on a circle with radius $R$ and angle $\theta_j = 2\pi j/N$, where $j = 1, 2, ..., N$ and $j \neq i$. Note that the $i$-th agent corresponds to $j = N$ as $\theta_i = 0$. The distance between the $i$-th and the $j$-th agent is

$$||\vec{x}_{ij}|| = 2R \sin(\frac{j\pi}{N}), \quad (3)$$

and the angle of the vector $\vec{x}_{ij} = \vec{x}_i - \vec{x}_j$ is $\theta_{ij} = (\theta_j - \pi)/2$. Therefore,

$$\vec{x}_{ij} = ||\vec{x}_{ij}||[\cos(\theta_{ij}), \sin(\theta_{ij})] = ||\vec{x}_{ij}||[\sin(\frac{j\pi}{N}), -\cos(\frac{j\pi}{N})]^T = 2R \sin(\frac{j\pi}{N})[\sin(\frac{j\pi}{N}), -\cos(\frac{j\pi}{N})]^T. \quad (4)$$

The $i$-th agent contains $N - 1$ neighbors, $N_i = \{j = 1, 2, \ldots, N\}/i$. Substituting Eqs. (1), (3) and (4) to Eq. (2) yields

$$\begin{bmatrix}
\frac{b}{N} \sum_{j=1}^{N-1} \left(\frac{d^2 - 2R \sin(\frac{j\pi}{N}) \sin(\frac{\pi}{N})}{(2R \sin(\frac{j\pi}{N}))^{\beta-1}}\right) \\
\frac{b}{N} \sum_{j=1}^{N-1} \left(-\frac{d - 2R \sin(\frac{j\pi}{N}) \cos(\frac{\pi}{N})}{(2R \sin(\frac{j\pi}{N}))^{\beta-1}}\right)
\end{bmatrix} = \begin{bmatrix}
-\frac{v_0^2}{R} \\
0
\end{bmatrix}. \quad (5)$$

Given $\beta < 1$, one has that the second term $\frac{b}{N} \sum_{j=1}^{N-1} \left(-\frac{d - 2R \sin(\frac{j\pi}{N}) \cos(\frac{\pi}{N})}{(2R \sin(\frac{j\pi}{N}))^{\beta-1}}\right) = 0$ by using the property $g(k) = -g(N-k)$, where $g(k) = \cos(\frac{k\pi}{N}) \left(\sin(\frac{k\pi}{N})\right)^{\tau}$ and $\tau > 0$. Therefore,
the radius $R$ can be estimated by the first term, as

$$\sum_{j=1}^{N-1} \frac{(2R \sin (j\pi/N) - d) \sin(j\pi/N)}{(2R \sin (j\pi/N))^\beta} = \frac{v_0^2}{bR}. \quad (6)$$

Note that $\frac{(2R \sin (j\pi/N) - d) \sin(j\pi/N)}{(2R \sin (j\pi/N))^\beta} = 0$ for $j = N$. Adding this term to the left-hand side of Eq. (6) yields Eq. (3) of the main body.

Analogously, Eq. (4) of the main body can be obtained by given $\|\ddot{\mathbf{v}}_i\| = v_0$ and $\frac{d\mathbf{v}_i}{dt} = [0, 0]^T$. This completes the proof.

2. Effect of parameter $\beta$ in Fig. 2 of the main body

Figs. S1(a)–(d) show the evolution of $V_m$, $V_c$, $D_e$ and $D_r$ along with two parameters $\alpha$ and $\beta$, which correspond to the Fig. 2 of the main body.

![Graphs](image_url)

(a) The migratory motion order $V_m$. (b) The circular motion order $V_c$. (c) The inter-particle distance $D_e$. (d) The inner and outer radius ratio $D_r$ of the mill phase.

Figure S1: The evolution of $V_m$, $V_c$, $D_e$ and $D_r$ with parameters $\alpha$ and $\beta$. 
