REMARKS ON THE STEADY PRANDTL BOUNDARY LAYER EXPANSIONS

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Abstract. We continue the study of the validity of the Prandtl boundary layer expansions in [4], where by estimating the stream-function of the remainder, we proved if the Euler flow is perturbation of shear flows when the width of domain is small. In this paper, we obtain a new derivatives estimate of stream-function away from the boundary layer and then prove the validity of expansions for any non-shear Euler flow, provided the width of domain is small.

1. Introduction

We consider the stationary incompressible Navier-Stokes equations

\[
\begin{align*}
    U_\varepsilon U_\varepsilon^X + V_\varepsilon U_\varepsilon^Y - \varepsilon \Delta U_\varepsilon + P_\varepsilon^X &= 0, \\
    U_\varepsilon V_\varepsilon^X + V_\varepsilon V_\varepsilon^Y - \varepsilon \Delta V_\varepsilon + P_\varepsilon^Y &= 0, \\
    U_\varepsilon^X + V_\varepsilon^Y &= 0,
\end{align*}
\]

posed in a two dimensional domain \( \Omega = \{(X,Y) : 0 < X < L, Y > 0\} \). The no-slip boundary conditions are set on the boundary \( Y = 0 \):

\[
U_\varepsilon(X, 0) = 0, \quad V_\varepsilon(X, 0) = 0.
\]

We are concerned with the asymptotic behavior of solution \([U_\varepsilon, V_\varepsilon]\) when \( \varepsilon \) is small. A formal limit \( \varepsilon \to 0^+ \) should lead to the Euler flow \([U^0, V^0]\) inside \( \Omega \):

\[
\begin{align*}
    U^0 U^0_X + V^0 U^0_Y + P^0_X &= 0, \\
    U^0 V^0_X + V^0 V^0_Y + P^0_Y &= 0, \\
    U^0_X + V^0_Y &= 0.
\end{align*}
\]

Naturally, we pose the system \([1,2]\) with no penetration boundary condition on \( Y = 0 \):

\[
V^0(X, 0) = 0.
\]

Generically, there is a mismatch between the tangential velocities of the Euler flow \( U^0(X, 0) \neq 0 \) and the prescribed Navier-Stokes flow \( U_\varepsilon(X, 0) = 0 \) on the boundary, because of the difference of boundary conditions imposed on the two systems.

According to the classical Prandtl boundary layer theory, there exists a thin layer which connect with \( U_\varepsilon(X, 0) \) and \( U_\varepsilon(X, 0) \). Precisely, we take the Prandtl’s variables:

\[
x = X, \quad y = \frac{Y}{\sqrt{\varepsilon}}.
\]

In these variables, we express the solution of the NS equations \([U_\varepsilon, V_\varepsilon]\) via \([u_\varepsilon, v_\varepsilon]\) as

\[
[U_\varepsilon(X, Y), V_\varepsilon(X, Y)] = [u_\varepsilon(x, y), \sqrt{\varepsilon}v_\varepsilon(x, y)]
\]
in which we note that the scaled normal velocity \( v^\varepsilon \) is \( \frac{1}{\sqrt{\varepsilon}} \) of the original velocity \( V^\varepsilon \) to satisfy the divergence-free condition. Similarly, \( P^\varepsilon(X, Y) = p^\varepsilon(x, y) \). In these new variables, the Navier-Stokes equations (1.1) now read

\[
\begin{cases}
    u^\varepsilon u_x^\varepsilon + v^\varepsilon u_y^\varepsilon + p_x^\varepsilon = u_y^\varepsilon + \varepsilon u_{xx}^\varepsilon, \\
    \varepsilon [u^\varepsilon v_x^\varepsilon + v^\varepsilon v_y^\varepsilon] + p_y^\varepsilon = \varepsilon v_y^\varepsilon + \varepsilon v_{xx}^\varepsilon, \\
    u_x^\varepsilon + v_y^\varepsilon = 0.
\end{cases}
\]

(1.4)

Let \( \varepsilon \to 0 \), it leads to the Prandtl equations:

\[
\begin{cases}
    u_0^0 u_{px}^0 + v_0^0 u_{py}^0 - u_0^0 u_{py}^0 + p_{px}^0 = 0, \\
    p_{py}^0 = 0, \\
    u_{px}^0 + v_{py}^0 = 0,
\end{cases}
\]

(1.5)

with \( u_0^0|_{y=0} = 0 \). Prandtl hypothesized that when viscosity \( \varepsilon \) is small the Navier-Stokes flow can be approximately decomposed into two parts:

\[
\begin{align*}
    U^\varepsilon(X, Y) &\approx u^0_e(X, Y) - u^0_e(X, 0) + u_p^0(X, \frac{Y}{\sqrt{\varepsilon}}), \\
    V^\varepsilon(X, Y) &\approx v^0_e(X, Y) + \sqrt{\varepsilon} v_p^0(X, \frac{Y}{\sqrt{\varepsilon}}),
\end{align*}
\]

(1.6)

in which \([u^0_e, v^0_e] = [U^0, V^0]\) denotes the Euler flow.

The verification of the viscosity vanishing limits is a challenging problem in general. There are lots of studies in recent year. For non-stationary case, the problem in the analytic case was proved in [23], [24] and [25]. In 2014, Maekawa [26] proved the convergence under the assumption on the initial vorticity vanishing in the neighbourhood of boundary. Fei, Tao and Zhang [3] generalized this result to 3D case by energy methods. In [27], Gerard-Varet, Maekawa and Masmoudi established the Gevrey stability for Prandtl type shear flows. Chen, Wu and Zhang gave a new proof of Gevrey stability for steady profile by resolvent estimate method in [28]. Later, Gerard-Varet, Maekawa and Masmoudi [29] showed the Prandtl expansion around concave boundary layer in Gevrey space. There are some results of instability in Sobolev space, cf. [8]-[10].

For the steady case, the problem is considered in Sobolev space. In this situation, the Euler flow \([u^0_e, v^0_e]\) in expansion (1.6) is always shear flow \([u^0_e(Y), 0]\). An important progress was made by Guo, Nguyen [30] for Prandtl boundary layer expansions for the steady Navier-Stokes flows over a moving plate. Then Iyer [31] extended this result into a rotating disk. They considered the Euler flow is shear or rotating shear, Prandtl profile is strictly positive and the width of region or the angle of sector is small. Later, Iyer [32] generalized the result in [30] for the perturbation of shear flow. After that, Iyer in [33], [34] and [35] justified the steady Prandtl expansion over a moving plane in \((0, \infty) \times (0, \infty)\) under the assumption of constant shear Euler flow and smallness of the boundary layer profile. In 2018, a significant work by Guo and Iyer [12] showed the convergence result for no-slip boundary conditions in shear Euler flows in the case the width of the region \( L \) is small. Inspired by the methods in this work, we introduced a new quantity, the quotient of stress-function and the approximate solution, in [4]. By estimating that quantity, we justified the validity of expansion (1.6) for the perturbation of shear Euler flow when \( L \) is small. Moreover, we showed when the Euler flow is shear and Prandtl profile is in monotonic class, (1.6) is right even \( L \) is large. Recently, Iyer and Masmoudi in [19] also estimated the quantity introduced in [4] to show the stability of the Prandtl expansion in domain \((0, \infty) \times (0, \infty)\). In that work, the Euler
flow is considered as special shear flow \([1,0]\) and the solution of Prandtl equation is famous Blasius flow in monotonic class. There are also the stability results for Prandtl type shear flow of Navier-Stokes equations with force term in \(X\)-periodic domain cf. \([7]\) and \([2]\).

In this paper, we continue our earlier work in \([1]\). We assume that the outside Euler flow \([u^0_e(X,Y), v^0_e(X,Y)]\) satisfying the following hypothesis:

\[
\begin{align*}
0 < c_0 \leq u^0_e & \leq C_0 < \infty, \\
\|\langle Y \rangle^k \nabla^m [u^0_e, v^0_e]\|_{L^\infty} & < \infty \text{ for } m \geq 1.
\end{align*}
\]  

(1.7)

Here \(\langle Y \rangle = Y + 1\) and \(k\) is a large constant. The special case for Euler flow is shear flow \([u^0_e(0), 0]\) which discussed in \([12]\) and \([4]\).

We consider the Prandtl equations with the positive data.

\[
\begin{align*}
\left\{ 
\begin{array}{l}
u^0_p v^0_{px} + v^0_p v^0_{py} - u^0_p v^0_{py} + p^0_{px} = 0, \\
u^0_p v^0_{pyy} + v^0_p v^0_{py} = 0, \\
u^0_p x = 0 = u^0_p(y), \\
u^0_p y = 0 = v^0_p(y) = 0, \\
u^0_p y = 0 = u^0_e|y = 0.
\end{array}
\right.
\end{align*}
\]

(1.8)

\(U^0_p\) is a prescribing smooth function such that

\[
\begin{align*}
U^0_p > 0 \text{ for } y > 0, \\
\partial_y u^0_p(0) > 0, \\
\partial_y^2 U^0_p - u^0_e(x,0) u^0_e(x,0) \sim y^2 \text{ near } y = 0, \\
\partial_y^m \{U^0_p - u^0_e(x,0)\} \text{ decay fast for any } m \geq 0.
\end{align*}
\]  

(1.9)

By the classical result in \([22]\), under above conditions on \(U^0_p\), if \(L\) is small enough, equations \((1.8)\) admit a classical solution \([u^0_p, v^0_p]\) satisfying:

\[
\begin{align*}
u^0_p & > 0 \text{ for } y > 0, \\
\partial_y u^0_p & > 0, \\
\nabla^m \{u^0_p - u^0_e(0)\} & \text{ decay fast as } y \to \infty \text{ for any } m \geq 0.
\end{align*}
\]  

(1.10)

Now we state our main result.

**Theorem 1.1.** Assume the Euler flow \([u^0_e, v^0_e]\) satisfies \((1.7)\), the Prandtl profile satisfies \((1.10)\), \(L\) is a constant small enough, then there exist \(\varepsilon_0(L) > 0\) depending on \(L\), such that for \(0 < \varepsilon \leq \varepsilon_0\), equations \((1.7)\) admits a solution \([U^\varepsilon, V^\varepsilon]\) \(\in W^{2,2}(\Omega)\), satisfying:

\[
\begin{align*}
\|U^\varepsilon - u^0_e + u^0_e|y = 0 - u^0_p\|_{L^\infty} & \leq C\sqrt{\varepsilon}, \\
\|V^\varepsilon - v^0_e\|_{L^\infty} & \leq C\sqrt{\varepsilon},
\end{align*}
\]  

(1.11)

with the following boundary conditions:

\[
\begin{align*}
[U^\varepsilon, V^\varepsilon]|_{y = 0} & = 0, \\
[U^\varepsilon, V^\varepsilon]|_{x = 0} & = [u^0_e(0,Y) - u^0_e(0,0) + u^0_p(0, \frac{Y}{\sqrt{\varepsilon}}) + \sqrt{\varepsilon} a_0, v^0_e(0,Y) + \sqrt{\varepsilon} b_0], \\
[U^\varepsilon, V^\varepsilon]|_{x = L} & = [u^0_e(L,Y) - u^0_e(L,0) + u^0_p(L, \frac{Y}{\sqrt{\varepsilon}}) + \sqrt{\varepsilon} a_L, v^0_e(L,Y) + \sqrt{\varepsilon} b_L].
\end{align*}
\]  

(1.12)
Here $C$ is a constant independent of $L$ and $\varepsilon$,

$$
a_0(Y) = u^1_e(0, Y) + u^1_b(0, \frac{Y}{\sqrt{\varepsilon}}) + \sqrt{\varepsilon} u^2_e(0, Y) + \sqrt{\varepsilon} u^2_b(0, \frac{Y}{\sqrt{\varepsilon}}),
$$

$$
a_L(Y) = u^1_e(L, Y) + u^1_b(L, \frac{Y}{\sqrt{\varepsilon}}) + \sqrt{\varepsilon} u^2_e(L, Y) + \sqrt{\varepsilon} u^2_b(L, \frac{Y}{\sqrt{\varepsilon}}),
$$

$$
b_0(Y) = v^0_b(0, \frac{Y}{\sqrt{\varepsilon}}) + v^1_e(0, Y) + \sqrt{\varepsilon} v^2_b(0, \frac{Y}{\sqrt{\varepsilon}}) + \sqrt{\varepsilon} v^2_e(0, Y) + \varepsilon v^2_b(0, \frac{Y}{\sqrt{\varepsilon}}),
$$

$$
b_L(Y) = v^0_b(L, \frac{Y}{\sqrt{\varepsilon}}) + v^1_e(L, Y) + \sqrt{\varepsilon} v^2_b(L, \frac{Y}{\sqrt{\varepsilon}}) + \sqrt{\varepsilon} v^2_e(L, Y) + \varepsilon v^2_b(L, \frac{Y}{\sqrt{\varepsilon}}),
$$

are smooth functions constructed in Proposition 4.4.

Unlike the assumptions of shear flows or their perturbation in previous works, this theorem shows the expansions (1.6) for any non-shear Euler flow which $u^0_e$ is strictly positive. This result and the second result in [4] coincide with the classical results of Oleinik and Samokhin [22] for solutions of Prandtl’s equation. They show the local well-posedness ($L$ is small) of Prandtl equation for any $u^0_e(X, 0)$ is strictly positive and the global well-posedness ($L$ is any constant) for $u^0_e(X, 0) \geq 0$. And we show the Prandtl expansions for non-shear Euler flow when $L \ll 1$ in this paper and shear Euler flow when $L$ is any given constant in [4].

To prove the theorem, we first construct the approximate solutions $U_s = [U_s, V_s]$ of Navier-Stokes equations, which is similar to [4]. The main difficulty is estimating the remainders $U := U^\varepsilon - U_s$, where $U$ satisfies the following linearized Navier-Stokes equations:

$$
-\varepsilon \Delta U + U_s \cdot \nabla U + U \cdot \nabla U_s + \nabla P = F.
$$

But when Euler flows is non-shear, $V_s \approx v^0_e$ is not small. We notice that $v^0_e(X, 0) = 0$, so $v^0_e$ is small near the boundary $\{(X, Y)| Y = 0\}$. It leads us to estimating the stream-function away from the boundary layers. We estimate the derivatives of stream-function in outer area, i.e. $\{Y \geq \delta > 0\}$, and combine this estimate with some estimates we obtained in [4], to show the stream-function can be dominating by $F$, which essentially leads to the proof the theorem.

This paper is organized as follows: In Section 2, we show the main profile of the approximation solution. In Section 3, we estimate the stream-function of remainder. In Section 4, we prove the main theorem. The construction of the high-order approximation solutions is in Appendix.

2. Construction of the approximate solution

The construct of the approximate solutions is similar to [4]. We will need higher order expansions, as compared to (1.6), in order to control the remainder. Actually, the approximate
solutions of the Navier-Stokes equations are as the following form:

\[ U^\varepsilon(X,Y) \approx u^0_\varepsilon(X,Y) + u^0_b(X, Y/\varepsilon) + \sqrt{\varepsilon}[u^1_\varepsilon(X,Y) + u^1_b(X, Y/\varepsilon)] \]

\[ + \varepsilon[u^2_\varepsilon(X,Y) + u^2_b(X, Y/\varepsilon)], \]

\[ V^\varepsilon(X,Y) \approx v^0_\varepsilon(X,Y) + \sqrt{\varepsilon}[v^0_b(X, Y/\varepsilon) + v^1_\varepsilon(X,Y)] + \varepsilon[v^1_b(X, Y/\varepsilon) + v^2_\varepsilon(X,Y)] \]

\[ + \varepsilon^3 v^2_b(X, Y/\varepsilon), \]

\[ P^\varepsilon(X,Y) \approx p^0_\varepsilon(X,Y) + p^0_b(X, Y/\varepsilon) + \sqrt{\varepsilon}[p^1_\varepsilon(X,Y) + p^1_b(X, Y/\varepsilon)] \]

\[ + \varepsilon[p^2_\varepsilon(X,Y) + p^2_b(X, Y/\varepsilon)] + \varepsilon^3 p^3_b(X, Y/\varepsilon), \]

in which \([u^j_\varepsilon, v^j_\varepsilon, p^j_\varepsilon]\) and \([u^j_b, v^j_b, p^j_b]\), with \(j = 0, 1, 2\), denoting the Euler profiles and boundary layer profiles, respectively. Here, we note that these profile solutions also depend on \(\varepsilon\). And the Euler flows are always evaluated at \((X,Y)\), whereas the boundary layer profiles are at \((X,Y)\).

**Notation.** For convenience, we will introduce some notation here. We write

\[ \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2_XY}, \]

\[ \langle \cdot, \cdot \rangle_{Y=0} = \langle \cdot, \cdot \rangle_{L^2_XY(y=0)}; \]

\[ \| \cdot \| = \| \cdot \|_{L^2_XY} \]

and

\[ \| \cdot \|_\infty = \| \cdot \|_{L^\infty_XY} = \| \cdot \|_{L^\infty_{X\varepsilon_Y}}. \]

We denote \(a \lesssim b\) which means there exist a positive constant \(C_0\), s.t. \(a \leq C_0b\), here \(C_0\) is independent on \(\sqrt{\varepsilon}\) and \(L\). And we write \(a = O(b)\) as \(|a| \lesssim b\).

2.1. **The leading order of approximate solution.** Recall the Euler flow \([u^0_\varepsilon, v^0_\varepsilon]\). Let \(\psi\) be the stream-function of \([u^0_\varepsilon, v^0_\varepsilon]\)

\[ \psi(X,Y) := \int_0^Y u^0_\varepsilon(X,Y')dY', \quad \psi_Y = u^0_\varepsilon, \quad \psi_X = -v^0_\varepsilon, \]

then Euler equations (1.2) are equivalent to:

\[ \Delta \psi = F_\varepsilon(\psi). \]

From the assumptions in (1.4), we can know that \(F_\varepsilon\) together with sufficiently many derivatives are bounded and decaying in its argument.

For Prandtl equations, there is a famous result due to Oleinik [22]:

**Proposition 2.1** (Oleinik). Assume boundary data \(U^0_p \in C^\infty\) satisfies (1.4), then for some \(L > 0\), equations (1.8) exists a solution \([u^0_p, v^0_p]\), satisfying, for some \(y_0, m_0 > 0\),

\[ \sup_{(0,L) \times (0,\infty)} \| u^0_p, \partial_y u^0_p, \partial_y u^0_p, \partial_x u^0_p \| \lesssim 1, \]

\[ \inf_{(0,L) \times (0,y_0)} \| \partial_y u^0_p \| > m_0 > 0, \]

\[ u^0_p > 0, \quad \text{for } y > 0. \]
In fact, if \( U^0_p \) satisfies high order parabolic compatibility conditions at the corner \((0,0)\), then \([u^0_p, v^0_p]\) are smooth enough. The compatibility conditions are discussed in our previous works [4]. Following the proof of Oleinik in [22], we have:

**Lemma 2.2.** If \( U^0_p \in C^\infty \) satisfies (1.3) and high order parabolic compatibility conditions, then

\[
\|\langle y \rangle^M \nabla^k (u^0_p(x, y) - u^0_e(x, 0))\|_\infty \lesssim 1, \quad \text{for} \quad 0 \leq k \leq K,
\]

here \( K \) and \( M \) are constants.

After we solved Prandtl’s equation (1.8), we set

\[
u^0_b(X, Y, \sqrt{\varepsilon}) = \int_{\gamma}^{\gamma'} u^0_b(x, y') dy',
\]

and \( p^0_b = 0 \). The construction of high-order approximate solutions \([u^1_e, v^1_e], [u^2_e, v^2_e], [u^1_b, v^1_b] \) and \([u^2_b, v^2_b]\) is in Appendix. We denote the \([U_s, V_s]\) as the following:

\[
U_s(X, Y) = u^0_e(X, Y) + u^0_b(X, \frac{Y}{\sqrt{\varepsilon}}) + \varepsilon [u^1_e(X, Y) + u^1_b(X, \frac{Y}{\sqrt{\varepsilon}})],
\]

\[
V_s(X, Y) = v^0_e(X, Y) + \sqrt{\varepsilon} [v^0_b(X, \frac{Y}{\sqrt{\varepsilon}}) + v^1_b(X, Y)] + \varepsilon [v^1_e(X, \frac{Y}{\sqrt{\varepsilon}}) + v^2_e(X, Y)]
\]

\[
(2.6)
\]

\[
P_s(X, Y) = p^0_e(X, Y) + p^0_b(X, \frac{Y}{\sqrt{\varepsilon}}) + \varepsilon [p^1_e(X, Y) + p^1_b(X, \frac{Y}{\sqrt{\varepsilon}})]
\]

\[
(2.7)
\]

Then the errors

\[
R_1 := U_s U_s X + V_s V_s Y - \varepsilon \Delta U_s + P_{sX},
\]

\[
R_2 := U_s V_s X + V_s V_s Y - \varepsilon \Delta V_s + P_{sY},
\]

satisfy

\[
(2.8) \quad \|R_1\| + \|R_2\| \lesssim \varepsilon^{\frac{3}{4}}.
\]

Next we show the main profile of \([U_s, V_s]\).

### 2.2. The main profile of approximate solution.

In order to obtain the estimates of the remainder, we need to know more information about \([U_s, V_s]\).

\[
U_s(X, Y) = u^0_e(X, Y) + u^0_b(X, \frac{Y}{\sqrt{\varepsilon}}) + O(\sqrt{\varepsilon}),
\]

\[
V_s(X, Y) = v^0_e(X, Y) + \sqrt{\varepsilon} (v^0_b(X, \frac{Y}{\sqrt{\varepsilon}}) + v^1_e(X, Y)) + O(\varepsilon).
\]
Since \( u_0^0(X,Y) \) is strictly positive, \( u_0^0(X, \frac{Y}{\sqrt{\varepsilon}}) > 0 \) for \( Y > 0 \), \( u_0^0(X,0) = 0 \) and \( u_0^0(y_0, X, 0) > 0 \), when \( \frac{Y}{\sqrt{\varepsilon}} \leq 1 \),
\[
    u_e^0(X,Y) + u_b^0(X, \frac{Y}{\sqrt{\varepsilon}}) = u_e^0(X,Y) - u_e^0(X,0) + u_b^0(X, \frac{Y}{\sqrt{\varepsilon}}) \geq -Y + \frac{Y}{\sqrt{\varepsilon}} \geq \frac{Y}{\sqrt{\varepsilon}},
\]
when \( 1 \leq \frac{Y}{\sqrt{\varepsilon}} \leq \varepsilon^{-\frac{1}{4}} \),
\[
    u_e^0(X,Y) + u_b^0(X, \frac{Y}{\sqrt{\varepsilon}}) = u_e^0(X,Y) - u_e^0(X,0) + u_b^0(X, \frac{Y}{\sqrt{\varepsilon}}) \geq -Y + 1 \geq 1,
\]
when \( \frac{Y}{\sqrt{\varepsilon}} \geq \varepsilon^{-\frac{1}{4}} \),
\[
    u_e^0(X,Y) + u_b^0(X, \frac{Y}{\sqrt{\varepsilon}}) \geq 1.
\]

So \( U_s \sim \frac{Y}{\sqrt{\varepsilon}} \), when \( Y \leq \sqrt{\varepsilon} \), and \( U_s \sim 1 \), when \( Y \geq \sqrt{\varepsilon} \).

One can easily see for \( i, j \geq 0 \),
\[
    \| \partial_X^i U_s \| \lesssim 1, \quad \| \partial_X^i V_s \| \lesssim 1,
\]
\[
    \sqrt{\varepsilon}\| Y^j \partial_Y^{i+j+1} U_s \| \lesssim \sqrt{\varepsilon} \| Y^j \partial_Y^{i+j} u_e^0 \| \|
\]
\[
    + \sqrt{\varepsilon} \| \frac{Y^j}{\sqrt{\varepsilon}} \partial_Y^{i+j} u_b^0 \| \|
\]
\[
    \left( \| \partial_Y\| \right) \lesssim 1,
\]
\[
    \sqrt{\varepsilon}\| Y^j \partial_Y^{i+j+1} V_s \| \lesssim \sqrt{\varepsilon} \| Y^j \partial_Y^{i+j} u_e^0 \| \|
\]
\[
    + \sqrt{\varepsilon} \| \frac{Y^j}{\sqrt{\varepsilon}} \partial_Y^{i+j} u_b^0 \| \|
\]
\[
    \left( \| \partial_Y\| \right) \lesssim 1.
\]

Let \( \delta \) be a positive constant satisfying \( \delta \geq \varepsilon^{\frac{1}{4}} \), when \( Y \leq \delta \), we have
\[
    |v_e^0(X,Y)| \lesssim Y = \sqrt{\varepsilon} \frac{Y}{\sqrt{\varepsilon}} \lesssim \sqrt{\varepsilon} U_s \lesssim \delta U_s,
\]
when \( \sqrt{\varepsilon} \leq Y \leq \delta \),
\[
    |v_e^0(X,Y)| \lesssim Y \lesssim Y U_s \lesssim \delta U_s.
\]

When \( Y \geq \delta \), for any \( j \geq 0 \),
\[
    \partial_Y^j U_s(X,Y) = \partial_Y^j u_e^0(X,Y) + O(\sqrt{\varepsilon}),
\]
\[
    \partial_Y^j V_s(X,Y) = \partial_Y^j v_e^0(X,Y) + O(\sqrt{\varepsilon}).
\]

In fact since \( u_b^0(X,y), v_b^0(X,y) \) decay fast when \( y \to \infty \), for some large \( M \),
\[
    |u_b^j(X, \frac{Y}{\sqrt{\varepsilon}})| \lesssim (\frac{Y}{\sqrt{\varepsilon}})^{-M} \lesssim \sqrt{\varepsilon}^M \delta^{-M} \lesssim \varepsilon^M,
\]
\[
    |v_b^j(X, \frac{Y}{\sqrt{\varepsilon}})| \lesssim (\frac{Y}{\sqrt{\varepsilon}})^{-M} \lesssim \sqrt{\varepsilon}^M \delta^{-M} \lesssim \varepsilon^M.
\]
3. Estimates of the remainder

In this section, we show the estimates of the remainder. Let

\[ U^\varepsilon = U_s + U, \quad V^\varepsilon = V_s + V. \]

Then

\[
\begin{aligned}
U_sU_X + U_sXU + V_sU_Y + U_sYV - \varepsilon \Delta U + P_X &= -\{R_1 + UU_X + UU_Y\}, \\
U_sV_X + V_sXU + V_sYV + V_sYV - \varepsilon \Delta V + P_Y &= -\{R_2 + UV_X + UV_Y\}, \\
U_X + V_Y &= 0.
\end{aligned}
\]

We consider the linearized equations:

\[
\begin{aligned}
U_sU_X + U_sXU + V_sU_Y + U_sYV - \varepsilon \Delta U + P_X &= F_1, \\
U_sV_X + V_sXU + V_sYV + V_sYV - \varepsilon \Delta V + P_Y &= F_2, \\
U_X + V_Y &= 0.
\end{aligned}
\]

Our critical estimates is the following proposition.

**Proposition 3.1.** Under the assumptions in theorem [1.1], if

\[ [U, V] \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \]

satisfies the equations (3.3), then

\[ \|\sqrt{\varepsilon}U_X, \sqrt{\varepsilon}U_Y, \sqrt{\varepsilon}V_X, \sqrt{\varepsilon}V_Y, U, V\| \leq C(\|F_1\| + \|F_2\|). \]

Let \( \Phi \) be the stream-function of \( U, V \), that is, \( \Phi_X = -V, \Phi_Y = U \). We can solve the stream-function by this way

\[ \Phi(X, Y) = \int_0^Y U(X, Y')dY'. \]

According to the boundary conditions \( [U, V]_{\Omega} = 0 \) and \( U_X + V_Y = 0 \), We have

\[ \Phi|_{X=0} = \Phi|_{X=L} = \Phi|_{Y=0} = 0, \]

\[ \Phi_X|_{X=0} = \Phi_X|_{X=L} = \Phi_Y|_{Y=0} = 0. \]

If \( U, V \in L^2(\Omega) \), then \( \Phi_Y, \Phi_X \in L^2(\Omega) \).

Because \( [U, V] \) satisfy the equations (3.3), We can deduce the equation of stream function

\[
\begin{aligned}
U_s\Phi_X - \Phi_X U_s - \varepsilon \Delta \Phi + V_s\Phi_Y - \Phi_Y V_s = \partial_Y F_1 - \partial_X F_2, \\
\Phi|_{X=0} = \Phi|_{X=L} = \Phi|_{Y=0} = 0 = \Phi_{X=0} = \Phi_{X=L} = \Phi_{Y=0} = 0.
\end{aligned}
\]

It is the fourth-order elliptic equation for \( \Phi \), the boundary conditions are about \( \Phi \) and its derivatives.

Let \( G = \frac{\Phi}{U_s} \), \( G \) and \( \Phi \) satisfy

\[
\begin{aligned}
\partial_{XX}[U_s^2G_X] + \partial_{XY}[U_s^2G_Y] - \varepsilon \Delta^2 \Phi + R[\Phi] &= \partial_Y F_1 - \partial_X F_2, \\
G|_{X=0} = G|_{X=L} = G|_{Y=0} = G|_{X=0} = G|_{X=L} = 0.
\end{aligned}
\]

where \( R[\Phi] = V_s\Delta \Phi_Y - U_sX\Delta \Phi - \Phi_Y \Delta V_s + \Phi \Delta U_sX \). We define two norms of \( G \):

\[ \|G\|_2^2 := \|U_sG_X\|^2 + \|U_sG_Y\|^2, \]

\[ \|G\|_3^2 := \varepsilon\{\|\sqrt{U_s}G_{XX}\|^2 + 2\|\sqrt{U_s}G_{XY}\|^2 + \|\sqrt{U_s}G_{YY}\|^2\}. \]

We start with the following Hardy-type’s inequality in [4].
Lemma 3.1. If \( H \in W^{1,2}(0, \infty) \), \( 0 < \xi \leq 1 \), then
\[
\|H\|_{L^2}^2 \leq C \xi \varepsilon \|\overline{U_s} H Y\|_{L^2}^2 + \frac{C}{\xi} \|U_s H\|_{L^2}^2.
\]

Proof. Let \( \chi : [0, \infty) \to [0, 1] \) be a smooth cut-off function supported in \([0, 2]\), and \( \chi|_{[0,1]} = 1 \),
\[
\int_0^\infty H^2 \sqrt{Y} dY \lesssim \int_0^\infty H^2 \chi(\frac{Y}{\sqrt{\varepsilon}}) dY + \int_0^\infty H^2 (1 - \chi(\frac{Y}{\sqrt{\varepsilon}}))^2 dY.
\]
Recall the leading profile of \( U_s \),
\[
U_s \sim \begin{cases} Y \sqrt{\varepsilon}, & \text{if } Y \leq \sqrt{\varepsilon}, \\ 1, & \text{if } Y > \sqrt{\varepsilon}. \end{cases}
\]
So when \( \frac{Y}{\sqrt{\varepsilon}} \leq 1, \) \( 1 - \chi(\frac{Y}{\sqrt{\varepsilon}}) \lesssim \frac{Y}{\sqrt{\varepsilon}} \), and when \( \frac{Y}{\sqrt{\varepsilon}} \geq 1, \) \( 1 - \chi(\frac{Y}{\sqrt{\varepsilon}}) \lesssim 1 \lesssim \frac{U_s}{\sqrt{\varepsilon}} \). We have
\[
\int_0^\infty H^2 (1 - \chi(\frac{Y}{\sqrt{\varepsilon}}))^2 dY \lesssim \frac{1}{\xi^2} \int_0^\infty U_s^2 H^2 dY.
\]
And
\[
\int_0^\infty H^2 \chi(\frac{Y}{\sqrt{\varepsilon}}) dY = -2 \int_0^\infty Y H Y Y \chi(\frac{Y}{\sqrt{\varepsilon}}) dY - 2 \int_0^\infty \frac{Y}{\sqrt{\varepsilon}} H^2 \chi(\frac{Y}{\sqrt{\varepsilon}}) \chi(\frac{Y}{\sqrt{\varepsilon}}) dY
\lesssim \int_0^\infty Y^2 H^2 \chi(\frac{Y}{\sqrt{\varepsilon}}) dY + \int_0^\infty (\frac{Y}{\sqrt{\varepsilon}})^2 H^2 \chi(\frac{Y}{\sqrt{\varepsilon}}) \chi(\frac{Y}{\sqrt{\varepsilon}}) dY
\lesssim \varepsilon \int_0^\infty U_s H Y^2 dY + \frac{1}{\xi^2} \int_0^\infty U_s^2 H^2 dY.
\]
The proof is complete. \( \square \)

The next lemma is a basic elliptic estimate in \([4]\), it is true even though Euler flows is non-shear.

Lemma 3.2. Let \( G \) be the solution of equation \((3.8)\), \( L > 0 \), then
\[
\|G\|_{L^2}^2 \lesssim \|G\|_{H^3}^2 + |\langle \partial_Y F_1 - \partial_X F_2, G \rangle|.
\]

Proof. Take the inner product of \((3.8)_1\) and \(-G\).
First term
\[
\langle \partial X [U_s^2 G_X], -G \rangle = \langle \partial X [U_s^2 G_X], G_X \rangle = \langle U_s U_s G_X, G_X \rangle
= O(\|G_X\|^2).
\]
Second term
\[
\langle \partial Y [U_s^2 G_Y], -G \rangle = \langle \partial X [U_s^2 G_Y], G_Y \rangle = \langle U_s U_s G_Y, G_Y \rangle
= O(\|G_Y\|^2).
\]
Bi-Laplacian term is
\[
\langle -\varepsilon \Delta^2 \Phi, -G \rangle = \varepsilon \langle \Phi_{XXXX} + 2 \Phi_{XXYY} + \Phi_{YYYY}, G \rangle.
\]
\[
\varepsilon \langle \Phi_{XXXX}, G \rangle = \varepsilon \langle \Phi_{XX}, G_X \rangle = \varepsilon \langle \Phi_{XX}, G_{XX} \rangle
= \varepsilon \langle U_s G_{XX} + 2 U_s G_X + U_s X G, G_{XX} \rangle
= \varepsilon \langle U_s G_{XX}, G_{XX} \rangle - \varepsilon \langle 2 U_s X G + U_s X X G, G_X \rangle,
= \varepsilon \langle U_s G_{XX}, G_{XX} \rangle + O(\varepsilon \|G_X\|^2).
\]
Next

\begin{equation}
\langle -2\varepsilon\Phi_{XYY}, -G \rangle = - \langle 2\varepsilon\Phi_{XY}, G_Y \rangle = \langle 2\varepsilon\Phi_{XY}, G_Y \rangle \\
= 2\varepsilon\langle U_sG_{XY} + U_sXG_Y + U_sYG_X + U_sXYG, G_{XY} \rangle, \\
= 2\varepsilon\langle U_sG_{XY}, G_{XY} \rangle - \varepsilon\langle U_sXXG_Y, G_Y \rangle - \varepsilon \langle U_sYYG_X, G_X \rangle \\
- 2\varepsilon (U_sXYG_Y, G_X) - 2\varepsilon (U_sXYG, G_X) \\
= 2\varepsilon\langle U_sG_{XY}, G_{XY} \rangle + O(\varepsilon\|G_Y\|^2 + \varepsilon\|U_sYY\|\|G_X\|^2 + \varepsilon\|U_sXY\|\|G_X\|\|G\|) \\
= 2\varepsilon\langle U_sG_{XY}, G_{XY} \rangle + O\left(\|G_X\|^2 + \|G_Y\|^2\right).
\end{equation}

\begin{equation}
\langle -\varepsilon\Phi_{YYY}, -G \rangle = - \varepsilon\langle \Phi_{YYY}, G_Y \rangle = \varepsilon\langle \Phi_{YY}, G_Y \rangle |_{Y=0} + \varepsilon \langle \Phi_{YY}, G_{YY} \rangle \\
= \varepsilon\langle U_sG_{YY}, G_Y \rangle + 2U_sYG_Y + U_sYYG, G_Y \rangle |_{Y=0} \\
+ \varepsilon\langle U_sG_{YY}, G_Y \rangle + 2U_sYG_Y + U_sYYG, G_{YY} \rangle \\
= 2\varepsilon\langle U_sG_{YY}, G_Y \rangle |_{Y=0} + \varepsilon\langle U_sG_{YY}, G_{YY} \rangle + \varepsilon\langle 2U_sYG_Y + U_sYYG, G_{YY} \rangle \\
= \varepsilon\langle U_sG_{YY}, G_Y \rangle |_{Y=0} + \varepsilon\langle U_sG_{YY}, G_{YY} \rangle - \varepsilon\langle 2U_sYYG_Y + U_sYYG, G_{YY} \rangle \\
= \varepsilon\langle U_sG_{YY}, G_Y \rangle |_{Y=0} + \varepsilon\langle U_sG_{YY}, G_{YY} \rangle + O\left(\|G_Y\|^2\right),
\end{equation}

where we use the Hardy inequality \(\|G\| \lesssim \|G_Y\|\) and (2.9). Since \(U_s|_{Y=0} = w_{0Y}(X, 0) + \frac{1}{\sqrt{\varepsilon}} v_{0Y}(X, 0) > 0\), the first two terms are positive above. According to (2.9),

\[
\begin{align*}
\|V_s\|_\infty & \lesssim \|\partial_Y (v_0 + \sqrt{\varepsilon} v_0^0)\|_\infty + \sqrt{\varepsilon} \lesssim \|v_0 + v_0^0\|_\infty + \sqrt{\varepsilon} \lesssim 1, \\
\|V_sU_s\|_\infty & \lesssim \|v_0 U_s\|_\infty + 1 \lesssim \|v_0^0\|_\infty \|U_sY\|_\infty + 1 \lesssim 1, \\
\|\Phi_X\| & = \|U_sG_X + U_sXG\| \lesssim \|G_X\|, \\
\|\Phi_Y\| & = \|U_sG_Y + U_sYG\| \lesssim \|G_Y\| + \|U_sY\|\|G\| \lesssim \|G_Y\|.
\end{align*}
\]

The \(R[\Phi]\) term can be estimated as

\begin{equation}
\langle V_s\Phi_{XX}, -G \rangle = \langle V_s\Phi_{XY}, G_X \rangle + \langle V_s\Phi_{XY}, G_Y \rangle \\
= \langle V_s(U_sG_{XY} + U_sXG_Y + U_sYG_X + U_sXYG), G_X \rangle \\
- \langle V_sX\Phi_X, G_Y \rangle - \langle V_sXY\Phi_X, G \rangle \\
= - \frac{1}{2} \langle (V_sU_s)G_X, G_X \rangle + \langle V_s(U_sXG_Y + U_sYG_X + U_sXYG), G_X \rangle \\
- \langle V_sX\Phi_X, G_Y \rangle - \langle V_sXY\Phi_X, G \rangle \\
= O\left(\|V_sU_s\|_\infty \|G_X\|^2 + \|V_sU_sXY\|_\infty \|G\|\|G_X\| + \|G_X\|^2 + \|G_Y\|^2\right) \\
= O\left(\|G_X\|^2 + \|G_Y\|^2\right).
\end{equation}
Collect (3.11)-(3.19), we can obtain the following inequality:
\[ \langle -U_{sX} \Delta \Phi, -G \rangle = - \langle U_{sX} \Phi_X, G_X \rangle - \langle U_{sX} \Phi_X, G \rangle - \langle U_{sX} \Phi_Y, G_Y \rangle - \langle U_{sX} \Phi_Y, G \rangle = O \left( \|G_X\|^2 + \|G_Y\|^2 \right). \]
(3.18)

Collecting (3.11)-(3.19), we can obtain the following inequality:
\[ \|G\|_Y^2 \lesssim \|G_X\|^2 + \|G_Y\|^2 + |\langle \partial_Y F_1 - \partial_X F_2, G \rangle|. \]
(3.19)

By lemma 3.1 for any \(0 < \xi \leq 1\),
\[
\|G_X\|^2 \lesssim \frac{1}{\xi^2} \|U_s G_X\|^2 + \xi \|U_s G_{XY}\|^2 \lesssim \frac{1}{\xi^2} \|G\|_X^2 + \xi \|G\|_Y^2.
\]
Similarly,
\[
\|G_Y\|^2 \lesssim \frac{1}{\xi^2} \|U_s G_Y\|^2 + \xi \|U_s G_{YY}\|^2 \lesssim \frac{1}{\xi^2} \|G\|_X^2 + \xi \|G\|_Y^2.
\]
So we have
\[
\|G\|_Y^2 \lesssim \frac{1}{\xi^2} \|G\|_X^2 + \xi \|G\|_Y^2 + |\langle \partial_Y F_1 - \partial_X F_2, G \rangle|.
\]
By choosing \(\xi\) small enough, we can obtain the inequality (3.10).

Above lemma shows the second derivatives of \(G\), but it is not good because \(\varepsilon\) is small. Next lemma shows a critical estimate about derivatives of \(G\).

**Lemma 3.3.** Let \(G\) be the solution of equation (3.8), then
\[
\frac{3}{2} \langle U_s^2 G_X, G_X \rangle + \frac{1}{2} \langle U_s^2 G_Y, G_Y \rangle + \langle V_s U_s G_X, G_Y \rangle \lesssim (L + \sqrt{\varepsilon}) (\|G\|_X^2 + \|G\|_Y^2) + |\langle \partial_Y F_1 - \partial_X F_2, G \omega \rangle|.
\]
(3.20)

**Proof.** Take the inner product of (3.8) with \(-G\omega\), where \(\omega = L - x\).
Because $\omega \leq L$, the first term is
\[
\langle \partial_{XX}[U_s^2G_X], -G\omega \rangle = -\langle \partial_X[U_s^2G_X], G \rangle + \langle \partial_X[U_s^2G_X], G_X\omega \rangle
\]
\[
= \frac{3}{2} \langle U_s^2G_X, G_X \rangle + \langle U_s U_sXG_X, G_X\omega \rangle
\]
\[
= \frac{3}{2} \langle U_s^2G_X, G_X \rangle + O(L\|G_X\|^2).
\]

Second term
\[
\langle \partial_{XY}[U_s^2G_Y], -G\omega \rangle = \langle \partial_X[U_s^2G_Y], G_Y\omega \rangle
\]
\[
= \langle U_s^2G_{XY}, G_Y\omega \rangle + \langle 2U_s U_sXG_Y, G_Y\omega \rangle
\]
\[
= \frac{1}{2} \langle U_s^2G_Y, G_Y \rangle + \langle U_s U_sXG_Y, G_Y\omega \rangle
\]
\[
= \frac{1}{2} \langle U_s^2G_Y, G_Y \rangle + O(L\|G_Y\|^2).
\]

Bi-Laplacian term is
\[
\langle -\varepsilon \Delta^2\Phi, -G\omega \rangle = \varepsilon \langle \Phi_{XXXX} + 2\Phi_{XXXY} + \Phi_{YYYY}, G\omega \rangle.
\]
\[
\varepsilon \langle \Phi_{XXXX}, G\omega \rangle = -\varepsilon \langle \Phi_{XXXX}, G\omega \rangle + \varepsilon \langle \Phi_{XXXX}, G \rangle
\]
\[
= \varepsilon \langle U_s G_{XX} + 2U_sXG_X + U_sXXG, G\omega \rangle - 2\varepsilon \langle U_s G_{XX} + 2U_sXG_X + U_sXXG, G \rangle
\]
\[
= \varepsilon \langle U_s G_{XX} - 2U_sXG_X + U_sXXG, G\omega \rangle - \varepsilon \langle 2U_sXG_X + U_sXXG, G \rangle
\]
\[
= \varepsilon \langle U_s G_{XX}, G\omega \rangle + O(\varepsilon\|G_X\|).
\]

Next
\[
2\varepsilon \langle \Phi_{XXXY}, G\omega \rangle = -2\varepsilon \langle \Phi_{XXXY}, G\omega \rangle = 2\varepsilon \langle \Phi_{XY}, G_X\omega \rangle - 2\varepsilon \langle \Phi_{XY}, G \rangle
\]
\[
= 2\varepsilon \langle U_s G_{XY} + U_sXG_Y + U_sYG_X + U_sXYG, G\omega \rangle - 2\varepsilon \langle U_s G_{XY} + U_sXG_Y + U_sYG_X + U_sXYG, G \rangle.
\]

$2\varepsilon \langle U_s G_{XY}, G\omega \rangle$ is good, and
\[
2\varepsilon \langle U_s G_{XY} + U_sYG_X + U_sXGY, G\omega \rangle
\]
\[
= -\varepsilon \langle U_s XG_Y, G\omega \rangle + \varepsilon \langle U_s XG_Y, G \rangle - \varepsilon \langle U_s XGY, G_X\omega \rangle - 2\varepsilon \langle U_s XGY, G_X\omega \rangle - 2\varepsilon \langle U_s XYYG, G\omega \rangle
\]
\[
= O(\varepsilon\|G_Y\|^2 + \|U_s XYY\|\|G_X\|\|\sqrt{\omega}\|^2)
\]
\[
+ \varepsilon\|U_s X\|\|G_X\|\|G\| + \varepsilon\|Y U_s XYY\|\|\frac{G}{Y}\|\|G\|)
\]
\[
= O((L + \sqrt{\varepsilon})(\|G_X\|^2 + \|G_Y\|^2)).
\]
and

\[-2\varepsilon \langle U_s G_{XY} + U_s X G_Y + U_s Y G_X + U_{sXY} G_Y, G_Y \rangle \]
\[= -\varepsilon \langle U_s X G_Y, G_Y \rangle - 2\varepsilon \langle U_s Y G_Y, G_X \rangle - 2\varepsilon \langle U_{sXY} G, G_Y \rangle \]
\[= O(\varepsilon \|G_Y\|^2 + \varepsilon \|U_s Y\|_{\infty} \|G_X\| \varepsilon \|G_Y\| + \varepsilon \|U_{sXY} Y\|_{\infty} \|\frac{G}{Y}\| \varepsilon \|G_Y\|) \]
\[= O((L + \sqrt{\varepsilon})(\|G_X\|^2 + \|G_Y\|^2)). \]

Therefore

\[(3.24) \quad 2\varepsilon \langle \Phi_{XXY}, G_\omega \rangle = 2\varepsilon \langle U_s G_{XY}, G_{XY} \omega \rangle + O((L + \sqrt{\varepsilon})\|\nabla G\|^2). \]

Integrating by parts, we have

\[\varepsilon \langle \Phi_{YY}, G_\omega \rangle = -\varepsilon \langle \Phi_{YY}, G_\omega \rangle|_{Y=0} + \varepsilon \langle \Phi_{YY}, G_\omega \rangle \]
\[= \varepsilon \langle U_s G_{YY} + 2U_y G_Y + U_{sYY} G, G_\omega \rangle|_{Y=0} \]
\[+ \varepsilon \langle U_s G_{YY} + 2U_y G_Y + U_{sYY} G, G_\omega \rangle \]
\[= 2\varepsilon \langle U_s G_{YY} G, G_\omega \rangle|_{Y=0} + \varepsilon \langle U_s G_{YY} + U_{sYY} G, G_Y \omega \rangle + \varepsilon \langle U_{sYY} G + U_{sYY} G, G_\omega \rangle \]
\[= \varepsilon \langle U_s G_{YY} G, G_\omega \rangle|_{Y=0} + \varepsilon \langle U_s G_{YY} + U_{sYY} G, G_Y \omega \rangle - \varepsilon \langle U_{sYY} G + U_{sYY} G, G_\omega \rangle. \]

Because $U_s|_{Y=0} > 0$, the first two terms are positive above, and

\[-\varepsilon \langle 2U_s G_{YY} G, G_\omega \rangle = O(\varepsilon L\|U_s G_{YY}\|_{\infty} \|G_Y\|^2 + \varepsilon L\|U_s Y G_{YY} Y\|_{\infty} \|\frac{G}{Y}\| \varepsilon \|G_Y\|) \]
\[= O(L\|G_Y\|^2), \]

then we obtain

\[(3.25) \quad \varepsilon \langle \Phi_{YY}, G_\omega \rangle = \varepsilon \langle U_s G_{YY}, G_Y \omega \rangle|_{Y=0} + \varepsilon \langle U_s G_{YY}, G_Y \omega \rangle + O(L\|G_Y\|^2). \]

Finally, we deal with $R[\Phi]$ term. Since

\[\langle V_s \Phi_{XY}, -G_\omega \rangle = \langle V_s \Phi_{XY}, G_X \rangle + \langle V_s \Phi_{XY}, G_\omega \rangle - \langle V_s \Phi_{XY}, G \rangle \]
\[= \langle V_s (U_s G_{XY} + U_s X G_Y + U_s Y G_X + U_{sXY} G), G_\omega \rangle \]
\[= \langle V_s \Phi_X G_Y, G_\omega \rangle - \langle V_s \Phi_{XY}, G_\omega \rangle + \langle V_s \Phi_X, G_Y \rangle + \langle V_s \Phi_X, G \rangle \]
\[= -\frac{1}{2} \langle (V_s U_s X) G_X, G_X \rangle + \langle V_s (U_s X G_Y + U_s Y G_X + U_{sXY} G), G_\omega \rangle \]
\[= \langle V_s X \Phi_X, G_\omega \rangle - \langle V_s \Phi_{XY}, G_\omega \rangle + \langle V_s \Phi_X, G_Y \rangle + \langle V_s \Phi_X, G \rangle. \]

Notice that

\[\|V_s U_s Y\|_{\infty} \lesssim \frac{\|V_0^0 y_0^0\|_{\infty}}{\sqrt{\varepsilon}} + 1 \lesssim \frac{\|V_0^0\|_{\infty} y_0^0 y_0^0 \|_{\infty}}{\sqrt{\varepsilon}} + 1 \lesssim 1, \]
\[\|\Phi_X\| = \|U_s G_X + U_s X G\| \lesssim \|G_X\|, \]
\[\|\Phi_Y\| = \|U_s G_Y + U_s Y G\| \lesssim \|G_Y\| + \|U_{sY} Y\|_{\infty} \|\frac{G}{Y}\| \lesssim \|G_Y\|, \]
\[\|G\| \lesssim L\|G_X\|. \]
We obtain

(3.26)

\[
\langle V_s \Phi_{XY}, -G\omega \rangle = \langle V_s \Phi_X, G_Y \rangle + O(L\|V_sU_s\|_\infty \|G_X\|^2 + L\|G_X\|^2 + L\|G_Y\|^2)
\]

\[
= \langle V_sU_sG_X, G_Y \rangle + \langle V_sU_sG, G_Y \rangle + O(L\|G_X\|^2 + \|G_Y\|^2)
\]

\[
= \langle V_sU_sG_X, G_Y \rangle + O(L\|G_X\|^2 + \|G_Y\|^2).
\]

The others are easy. In fact,

(3.27)

\[
\langle V_s \Phi_{YY}, -G\omega \rangle = \langle V_s \Phi_{YY}, G_Y \omega \rangle + \langle V_s \Phi_{YY}, G\omega \rangle
\]

\[
= \langle V_s(U_sG_{YY} + 2U_sG_Y + U_sG_Y), G_Y \omega \rangle
\]

\[
- \langle V_s \Phi_{YY}, G_Y \omega \rangle - \langle V_s \Phi_{YY}, G\omega \rangle
\]

\[
= -\frac{1}{2}(\langle V_sU_s \rangle YG_Y, G_Y \omega \rangle + \langle V_s(2U_sG_Y + U_sG_Y), G_Y \omega \rangle
\]

\[
- \langle V_s \Phi_{YY}, G_Y \omega \rangle - \langle V_s \Phi_{YY}, G\omega \rangle
\]

\[
= O(L\|G_Y\|^2 + L\|V_s\|_\infty \|G_Y\| YV_s \| \|G_Y\| + L\|V_s^2\|_\infty \|G_Y\| YV_s \| \|G\| Y\| Y \| \|\Phi\|)
\]

\[
= O(L\|G\|^2)
\]

(3.28)

\[
\langle -U_s \Delta \Phi, -G\omega \rangle = - \langle U_s \Phi_X, G_X \omega \rangle - \langle U_sX \Phi_X, G\omega \rangle + \langle U_s \Phi_X, G\rangle
\]

\[
- \langle U_s \Phi_Y, G_Y \omega \rangle - \langle U_sX \Phi_Y, G\omega \rangle
\]

\[
= O(L\|G_X\|^2 + \|G_Y\|^2).
\]

(3.29)

\[
\langle -\Phi_Y \Delta V_s + \Phi \Delta U_s, -G\omega \rangle = O(L\|G_X\|^2 + \|G_Y\|^2).
\]

Collect (3.24)-(3.29), we get

\[
\frac{3}{2}(U_s^2G_X, G_X) + \frac{1}{2}(U_s^2G_Y, G_Y) + \langle V_sU_sG_X, G_Y \rangle
\]

\[
\lesssim (L + \sqrt{\delta})\|\nabla G\|^2 + |\langle \partial_Y F_1 - \partial_X F_2, G\omega \rangle|.
\]

By Lemma 3.1 for \(\xi = 1\),

\[
\|\nabla G\|^2 \lesssim \|G\|_X^2 + \|G\|_Y^2.
\]

Finally we have

\[
\frac{3}{2}(U_s^2G_X, G_X) + \frac{1}{2}(U_s^2G_Y, G_Y) + \langle V_sU_sG_X, G_Y \rangle
\]

\[
\lesssim (L + \sqrt{\delta})(\|G\|_X^2 + \|G\|_Y^2) + |\langle \partial_Y F_1 - \partial_X F_2, G\omega \rangle|.
\]

So we finish the proof. \(\square\)

Since \(v_0^0 \neq 0\), we need to deal with \(\langle V_sU_sG_X, G_Y \rangle\) in above lemma. Notice \(V_s \approx v_0^0\) is small near the boundary \(\{(X,Y) | Y = 0\}\), it is sufficient to estimate the stream-function away from the boundary layer. From now on, we write \(\eta_8(Y) := \eta(\frac{Y}{\delta})\). Here \(\eta\) is a smooth non-negative function s.t. \(\eta|_{[0,1]} = 0\) and \(\eta|_{[2,\infty)} = 1\). \(\delta\) is a small constant satisfying \(0 < L\frac{\delta}{4} + \varepsilon \frac{\delta}{4} \leq \delta \ll 1\). Next we show the key estimate of \(\Phi_X + \frac{\sqrt{\delta}}{v_0^0}\Phi_Y\) in the domain \(\{(X,Y) | Y \geq 2\delta\}\).
Lemma 3.4. Let \( G \) be the solution of equation (3.7), \( 0 < L^\frac{1}{2} + \varepsilon^\frac{1}{4} \leq \delta \ll 1 \), then

\[
\langle \Phi_X + \frac{V_s}{U_s} \Phi_Y, (\Phi_X + \frac{V_s}{U_s} \Phi_Y) \eta_\delta \rangle \lesssim (L + \sqrt{\varepsilon})(\|G\|_X^2 + \|G\|_Y^2) + \|F_1\|^2 + \|F_2\|^2
\]

Proof. Take the inner product of (3.7) with \( -\frac{1}{U_s}[\Phi_X + Q_s \Phi_Y] \tilde{\omega}(X) \eta_\delta(Y) \), where \( Q_s = \frac{V_s}{U_s} \), \( \tilde{\omega} = x(L - x) \).

Because

\[
U_s \Delta \Phi_X + V_s \Delta \Phi_Y = U_s[\Delta \Phi_X + Q_s \Delta \Phi_Y]
\]

\[= U_s[\Delta \Phi_X + Q_s \Phi_Y] - U_s[2 \nabla Q_s \cdot \nabla \Phi_Y + (\Delta Q_s) \Phi_Y],
\]

We have

\[
\langle U_s \Delta \Phi_X + V_s \Delta \Phi_Y, -\frac{1}{U_s}[\Phi_X + Q_s \Phi_Y] \tilde{\omega}(X) \eta_\delta(Y) \rangle
\]

\[= -\langle \Delta [\Phi_X + Q_s \Phi_Y], (\Phi_X + Q_s \Phi_Y) \tilde{\omega} \eta_\delta \rangle
\]

\[+ \langle 2 \nabla Q_s \cdot \nabla \Phi_Y + (\Delta Q_s) \Phi_Y, (\Phi_X + Q_s \Phi_Y) \tilde{\omega} \eta_\delta \rangle.
\]

We calculate the first part in the right hand of above equality:

\[-\langle \Delta [\Phi_X + Q_s \Phi_Y], (\Phi_X + Q_s \Phi_Y) \tilde{\omega} \eta_\delta \rangle = \|\nabla [\Phi_X + Q_s \Phi_Y] \sqrt{\tilde{\omega} \eta_\delta}\|^2
\]

\[+ \langle [\Phi_X + Q_s \Phi_Y]_X, [\Phi_X + Q_s \Phi_Y] \tilde{\omega} X \eta_\delta \rangle
\]

\[+ \langle [\Phi_X + Q_s \Phi_Y]_Y, [\Phi_X + Q_s \Phi_Y] \tilde{\omega} Y \eta_\delta \rangle
\]

\[= \|\nabla [\Phi_X + Q_s \Phi_Y] \sqrt{\tilde{\omega} \eta_\delta}\|^2 + \|\Phi_X + Q_s \Phi_Y \sqrt{\tilde{\omega} \eta_\delta}\|^2
\]

\[+ \frac{1}{2} \langle \Phi_X + Q_s \Phi_Y, \Phi_X + Q_s \Phi_Y \tilde{\omega} \eta_\delta \rangle.
\]

Because \( \tilde{\omega} \lesssim L^2 \), \( |\eta_\delta''| \lesssim \frac{1}{\sqrt{\varepsilon}} \lesssim \frac{1}{\varepsilon} \) and \( Q_s = O(1) \),

\[\langle \Phi_X + Q_s \Phi_Y, \Phi_X + Q_s \Phi_Y \tilde{\omega} \eta_\delta'' \rangle \lesssim L \|\nabla \Phi\|^2.
\]

According to the fact (2.11), when \( y \geq \delta \),

\[\nabla^j U_s = \nabla^j u^0_e + O(\sqrt{\varepsilon}) = O(1),
\]

\[\nabla^j V_s = \nabla^j v^0_e + O(\sqrt{\varepsilon}) = O(1),
\]

\[\nabla^j Q_s = \nabla^j \left[ \frac{\sqrt{\varepsilon}}{u^0_e} \right] + O(\sqrt{\varepsilon}) = O(1),
\]

the second part can be estimated by the following way:

\[\langle 2 \nabla Q_s \cdot \nabla \Phi_Y + (\Delta Q_s) \Phi_Y, (\Phi_X + Q_s \Phi_Y) \tilde{\omega} \eta_\delta \rangle
\]

\[= 2 \langle Q_s X \Phi_X Y, \Phi_X \tilde{\omega} \eta_\delta \rangle + 2 \langle Q_s Y \Phi_X Y, \Phi_X \tilde{\omega} \eta_\delta \rangle
\]

\[+ 2 \langle Q_s X \Phi_X Y, Q_s \Phi_Y \tilde{\omega} \eta_\delta \rangle + 2 \langle Q_s Y \Phi_X Y, Q_s \Phi_Y \tilde{\omega} \eta_\delta \rangle
\]

\[+ \langle (\Delta Q_s) \Phi_Y, (\Phi_X + Q_s \Phi_Y) \tilde{\omega} \eta_\delta \rangle
\]

\[= -\langle [Q_s X \tilde{\omega} \eta_\delta]_Y, \Phi_X^2 \rangle + \langle [Q_s Y \tilde{\omega} \eta_\delta]_X, \Phi_Y^2 \rangle
\]

\[+ 2 \langle [Q_s Y \tilde{\omega} \eta_\delta]_Y, \Phi_Y \Phi_X \rangle - \langle [Q_s X \tilde{\omega} \eta_\delta]_X, \Phi_Y^2 \rangle
\]

\[+ \langle [Q_s Y Q_s \tilde{\omega} \eta_\delta]_Y, \Phi_Y^2 \rangle + \langle (\Delta Q_s) \Phi_Y, (\Phi_X + Q_s \Phi_Y) \tilde{\omega} \eta_\delta \rangle
\]

\[= O(L \|\Phi_X\|^2 + L \|\Phi_Y\|^2).
\]
Here we use the fact $\tilde{\omega} \lesssim L^2$, $|\tilde{\omega}_X| \lesssim L$ and $|\eta_\theta^\delta| \lesssim \delta \lesssim \frac{1}{\sqrt{L}}$. So we conclude that

$$
\langle U_s \Delta \Phi_X + V_s \Delta \Phi_Y, -\frac{1}{U_s} [\Phi_X + Q_s \Phi_Y] \tilde{\omega}(X) \eta_\theta(Y) \rangle
\approx \|\nabla [\Phi_X + Q_s \Phi_Y] \sqrt{\eta_\theta} \|_2^2 + \|\Phi_X + Q_s \Phi_Y \| \sqrt{\eta_\theta} \|_2^2 + O(L \|\nabla \Phi\|_2^2).
$$

(3.32)

Next we deal with the bi-Laplacian term.

$$
\varepsilon \langle \Delta^2 \Phi, \frac{1}{U_s} [\Phi_X + Q_s \Phi_Y] \tilde{\omega} \eta_\theta \rangle = \varepsilon \langle \Phi_{XXX}, 2 \Phi_{XYY} + \Phi_{YY^Y}, \frac{1}{U_s} [\Phi_X + Q_s \Phi_Y] \tilde{\omega} \eta_\theta \rangle.
$$

Integration by Parts is allowed in $X$ direction because $\Phi_X |_{\partial \Omega} = 0$, $\Phi_{\partial \Omega} = 0$ and $\tilde{\omega} \big|_{X=0,Y=L} = 0$, in $Y$ direction because $\eta|_{[0,1]} = 0$. Use the inequality (3.31) again, we can see

$$
\varepsilon \langle \Phi_{XXX}, \Phi_X \frac{Q_s \tilde{\omega} \eta_\theta}{U_s} \rangle = - \varepsilon \langle \Phi_{XXX}, \Phi_X \frac{Q_s \tilde{\omega} \eta_\theta}{U_s} \rangle - \varepsilon \langle \Phi_{XXX}, \Phi_Y \partial_X \frac{Q_s \tilde{\omega} \eta_\theta}{U_s} \rangle
$$

$$
= \varepsilon \langle \Phi_{XXX}, \Phi_X \frac{Q_s \tilde{\omega} \eta_\theta}{U_s} \rangle + 2 \varepsilon \langle \Phi_{XXX}, \Phi_X \partial_X \frac{Q_s \tilde{\omega} \eta_\theta}{U_s} \rangle + \varepsilon \langle \Phi_{XXX}, \Phi_Y \partial_Y \frac{Q_s \tilde{\omega} \eta_\theta}{U_s} \rangle
$$

$$
= O \left( \varepsilon \|\Phi_X\|_2^2 + \varepsilon \|\Phi_Y\|_2^2 \right).
$$
The other terms are easy. We have

\[
\varepsilon \langle \Phi_{XXX} + 2 \Phi_{XYY} + \Phi_{YYY} + \frac{1}{U_s} [\Phi_X + Q_s \Phi_Y] \tilde{\omega} \eta_b \rangle
\]

(3.33)

\[= O \left( \sqrt{\varepsilon} \| \nabla \Phi \| ^2 + (L + \sqrt{\varepsilon}) \varepsilon \| \nabla^2 \Phi \| ^2 \right). \]

The other terms are easy. We have

(3.34)

\[
\langle \Phi_X \Delta U_s + \Phi_Y \Delta V_s, \frac{1}{U_s} [\Phi_X + Q_s \Phi_Y] \tilde{\omega} \eta_b \rangle = O \left( L^2 \| \nabla \Phi \| ^2 \right).
\]

Collect (3.21)–(3.29), we obtain

\[
\| \nabla \Phi_X \| ^2 + \| \nabla \Phi_Y \| ^2 \\
\lesssim (L + \sqrt{\varepsilon}) \left( \| \nabla \Phi \| ^2 + \varepsilon \| \nabla^2 \Phi \| ^2 \right) + \| \partial X F_1 - \partial Y F_2, \frac{1}{U_s} [\Phi_X + Q_s \Phi_Y] \tilde{\omega} \eta_b \| \\
\lesssim (L + \sqrt{\varepsilon}) \left( \| G \| _{L^2}^2 + \| G \| _{L^2}^2 \right) + \| F_1 \| ^2 + \| F_2 \| ^2.
\]

So we end the proof. \( \square \)

**Proof of Proposition 3.1.**

Notice that when \( Y \leq \sqrt{\varepsilon}, V_s \leq \sqrt{\varepsilon} U_s \leq \delta U_s \), and when \( \sqrt{\varepsilon} \leq Y \leq 2 \delta, V_s \leq \delta U_s \). So

\[
\langle V_s U_s G_X, G_Y \rangle = \langle U_s G_X, V_s G_Y \eta_b \rangle + \langle V_s U_s G_X, G_Y (1 - \eta_b) \rangle
\]

\[= - \langle U_s^2 G_X, G_Y \eta_b \rangle + \langle U_s^2 G_X, (G_X + \frac{V_s}{U_s} G_Y) \eta_b \rangle + O \left( \delta \| U_s G_X \| ^2 + \delta \| U_s G_Y \| ^2 \right).
\]

According to Lemma 3.4

\[
\langle U_s^2 G_X, (G_X + \frac{V_s}{U_s} G_Y) \rangle = \langle \Phi_X - U_s X G, (U_s G_X + V_s G_Y) \eta_b \rangle
\]

\[= \langle \Phi_X, (U_s G_X + V_s G_Y) \eta_b \rangle + O \left( \| G \| _{L^2} \| \nabla G \| \right)
\]

\[= \langle \Phi_X, (U_s G_X + \frac{V_s}{U_s} G_Y) \eta_b \rangle - \langle \Phi_X, (U_s X G + \frac{V_s}{U_s} U_s G_Y) \eta_b \rangle + O \left( L \| \nabla G \| ^2 \right)
\]

\[\geq - \frac{1}{4} \| \Phi_X \| ^2 - \| (\Phi_X + \frac{V_s}{U_s} \Phi_Y) \eta_b \| ^2 + O \left( L \| \nabla G \| ^2 \right)
\]

\[\geq - \frac{1}{4} \| U_s G_X \| ^2 + O \left( (L + \sqrt{\varepsilon}) \left( \| G \| _{L^2}^2 + \| G \| _{L^2}^2 \right) + (\| F_1 \| ^2 + \| F_2 \| ^2) \right).
\]

So, we obtain

\[
\langle V_s U_s G_X, G_Y \rangle = \langle U_s G_X, V_s G_Y \eta_b \rangle + \langle V_s U_s G_X, G_Y (1 - \eta_b) \rangle
\]

\[\geq - \frac{5}{4} \langle U_s^2 G_X, G_Y \rangle + O \left( \delta \| U_s G_X \| ^2 + \delta \| U_s G_Y \| ^2 \right)
\]

\[+ O \left( (L + \sqrt{\varepsilon}) \left( \| G \| _{L^2}^2 + \| G \| _{L^2}^2 \right) + (\| F_1 \| ^2 + \| F_2 \| ^2) \right).
\]
We use the method of contraction mapping. Define

\[ B = \{ U \in [U_s, V_s] \mid \|U\| \leq C_0 \varepsilon^{\frac{3}{2}} \}, \]

where \( C_0 \) is chosen later. Next we prove \( \mathcal{T} \) is a contractive mapping in \( B \), if \( \|R\| \leq C_1 \varepsilon^{\frac{3}{2}} \). We write \( F = -R - U \cdot \nabla U \), from Proposition 3.1

\[ \|F\| + \sqrt{\varepsilon} \|\nabla F\| \leq \|F\|. \]

Due to the \( W^{2,2} \) estimate of Stokes equations in convex polygon in [20],

\[ \varepsilon \|\nabla^2 W\| \leq \|F\| + \|\nabla W\| + \frac{1}{\sqrt{\varepsilon}} \|W\| \leq \frac{1}{\sqrt{\varepsilon}} \|F\|. \]

So we get

\[ \|W\| \leq C_2 \|F\|. \]
It’s easy to see
\[ \| U \cdot \nabla U \| \lesssim \| U \|_{L^\infty} \| \nabla U \| \lesssim \| U \|^{\frac{1}{2}} \| \nabla^2 U \|^{\frac{3}{2}} \lesssim \varepsilon^{-\frac{3}{2}} \| U \|_{Z}^2. \]

It implies
\[ \| W \|_Z \leq C_2 \| F \| + C_3 \varepsilon^{-\frac{3}{2}} \| U \|_{Z}^2 \leq (C_1 C_2 + C_3 C_0^2 \varepsilon^\frac{3}{2}) \varepsilon^\frac{3}{2}. \]

Select \( C_0 = C_1 C_2 + 1, \mathcal{T}(B) \subset B \) when \( \varepsilon \) is small enough. And if \( U_1, U_2 \in B, \)
\[ \mathcal{T}(U_1 - U_2)\|_{Z} \leq C_2 \| U_1 \cdot \nabla U_1 - U_2 \cdot \nabla U_2 \| \]
\[ \leq C_2 \| (U_1 - U_2) \cdot \nabla U_1 \| + \| U_2 \cdot \nabla (U_1 - U_2) \| \]
\[ \leq C_3 \varepsilon^{-\frac{3}{2}} (\| U_1 \|_Z + \| U_2 \|_Z) \| U_1 - U_2 \|_Z \]
\[ \leq 2C_0 C_3 \varepsilon^\frac{3}{2} \| U_1 - U_2 \|_Z, \]
so \( \mathcal{T} \) is a contraction mapping on \( B \) when \( \varepsilon \) is small enough, we can conclude equations (4.1) admits a solution and
\[ \| U \|_{L^\infty} \lesssim \varepsilon^{-\frac{3}{2}} \| U \|_Z \lesssim \varepsilon^\frac{3}{2}. \]

So we have
\[ |U^\varepsilon(X,Y) - u_0^0(X,Y) - u_0^0(X, \frac{Y}{\sqrt{\varepsilon}})| \]
\[ = |\sqrt{\varepsilon} u_1^e(X,Y) + \sqrt{\varepsilon} v_1^0(X, \frac{Y}{\sqrt{\varepsilon}}) + \varepsilon^2 u_2^e(X, \frac{Y}{\sqrt{\varepsilon}}) + U(X,Y)| \]
\[ \lesssim \sqrt{\varepsilon}, \]
\[ |V^\varepsilon(X,Y) - v_0^0(X,Y)| \]
\[ = |\sqrt{\varepsilon} v_0^1(X, \frac{Y}{\sqrt{\varepsilon}}) + \sqrt{\varepsilon} v_1^1(X,Y) + \varepsilon v_2^1(X, \frac{Y}{\sqrt{\varepsilon}}) + \varepsilon^\frac{3}{2} v_3^2(X, \frac{Y}{\sqrt{\varepsilon}}) + V(X,Y)| \]
\[ \lesssim \sqrt{\varepsilon}, \]
which ends the proof. \( \square \)

**Appendix**

The high-order approximate solutions are constructed similarly in [4] even though the leading Euler flow is non-shear. By the method of asymptotic matching expansions, we can deduce the equations of \( [u_1^e, v_1^e] \) and \( [v_1^0, v_0^0], j = 1, 2 \). The first order Euler profile \( [u_1^e, v_1^e, p_1^e] \) solves the linearized Euler equations around \( [u_0^e, v_0^e] \):

\[ \begin{cases} 
  u_0^0 u_1^e + u_0^e u_0^1_y + v_0^0 v_1^0 = 0, \\
  u_0^0 v_1^0 + v_0^0 u_0^1 - v_0^1 v_0^0 + v_0^0 v_1^1 + p_0^1 = 0, \\
  \partial_x u_0^1 + \partial_y v_1^1 = 0, \\
  v_1^1 |_{y=0} = -v_0^1 |_{y=0}.
\end{cases} \]

(4.3)
We eliminate the pressure $p^1_e$,
\[
\begin{align*}
\begin{cases}
    v^0_e \Delta u^1_e - u^0_e \Delta v^1_e + v^1_e \Delta u^0_e - u^1_e \Delta v^0_e = 0, \\
    \partial_X u^1_e + \partial_Y v^1_e = 0, \\
    v^1_e|_{Y=0} = -v^0_{0|y=0}.
\end{cases}
\end{align*}
\] (4.4)
We introduce independent variables by
\[
\begin{align*}
\begin{cases}
    \theta(X, Y) = X, \\
    \psi(X, Y) = \int_0^Y u^0_e(X, Y') dY'.
\end{cases}
\end{align*}
\] (4.5)
Let $\psi^1$ be the stream function of $[u^1_e, v^1_e]$,
\[
\psi^1(X, Y) := \int_0^Y u^1_e(X, Y') dY' - \int_0^X v^1_e(X', 0) dX', \quad \psi^1_Y = u^1_e, \quad \psi^1_X = -v^1_e,
\]
then first Euler layer equations (4.3) are equivalent to
\[
\begin{align*}
\begin{cases}
    \Delta \psi^1 - F^e_e(\psi) \psi^1 = 0, \\
    \psi^1|_{X=0} = \psi^1_0(Y), \quad \psi^1|_{X=L} = \psi^1_L(Y), \\
    \psi^1|_{Y=0} = \int_0^X v^0_e(X', 0) dX', \quad \psi^1|_{Y=\infty} = 0.
\end{cases}
\end{align*}
\] (4.6)

It is a standard elliptic problem, we have the following result.

**Lemma 4.1.** If $v^0_0$ is a smooth functions, for any $L > 0$, if $\psi^0_0(Y), \psi^1_L(Y)$ satisfy the compatibility conditions on the corner, then the equations (4.7) admit a unique solution satisfying the following estimate
\[
\|\langle Y \rangle^M \nabla^k \psi^1 \rangle \lesssim 1, \quad \text{for } 1 \leq k \leq K, K \text{ and } M \text{ are large constants}.
\] (4.8)

**Proof.** We homogenize the boundary conditions of system (4.7). Let
\[
\tilde{\psi} = \psi^1 - \frac{L-x}{L} \psi^1_0(Y) - \frac{x}{L} \psi^1_0(Y) - \chi(Y) \int_0^L v^0_e(X', 0) dX' - \chi(Y) \int_0^X v^0_e(X', 0) dX',
\]
here $\chi(Y)$ is a nonnegative smooth cut-off function, $\chi|_{[0, 1]} = 1$ and $\chi|_{[2, \infty]} = 0$. $\tilde{\psi}$ satisfies
\[
\begin{align*}
\begin{cases}
    \Delta \tilde{\psi} - F^e_e(\psi) \tilde{\psi} = \tilde{F}, \\
    \tilde{\psi}|_{\partial \Omega} = 0.
\end{cases}
\end{align*}
\] (4.9)
Notice that
\[
\Delta \psi = F_e(\psi),
\]
so $0 < c_0 \leq u^0_e = \psi_Y \leq C_0 < \infty$ satisfies
\[
\Delta u^0_e = F^e_e(\psi) u^0_e.
\]
Let $w = \frac{\tilde{\psi}}{F_e(\psi)}$, then
\[
u^0_e \Delta w + 2 \nabla u^0_e \cdot \nabla w = \tilde{F}.
\]
Above equation times \( u^0_e \), we show the equation (4.10) is equivalent to

\[
(4.10) \quad \begin{cases} 
\partial_X [(u^0_e)^2 w_X] + \partial_Y [(u^0_e)^2 w_Y] = u^0_e \tilde{F}, \\
w|_{\partial \Omega} = 0.
\end{cases}
\]

We can easily get a prior estimates of equation (4.10). Multiply equation (4.10) by \( w \) and integrate in \( \Omega \),

\[
\|u^0_e w_X\|^2 + \|u^0_e w_Y\|^2 = -(w, u^0_e \tilde{F}) \lesssim \|w\| \|F\| \lesssim \|w_X\| \|F\|.
\]
So we have

\[
(4.11) \quad \|\nabla w\| \lesssim \|\tilde{F}\|.
\]

The equality (4.11) actually shows the existence of solution about equation (4.10). Moreover, if \( \tilde{F} \) is a smooth function decaying fast when \( Y \to \infty \), we can obtain the following estimate by the mathematical induction method:

\[
(4.12) \quad \|\langle Y \rangle^M \nabla^k w\| \lesssim 1, \quad \text{for } 1 \leq k \leq K, K \text{ and } M \text{ are large constants}.
\]

So Lemma 4.1 is right. \( \square \)

**Remark 4.2.** The boundary conditions of \( \psi^1 \) in (4.7) imply the following boundary conditions of \([u^1_e, v^1_e]\)

\[
(4.13) \quad \begin{cases} 
    u^1_e|_{X=0} = \partial_Y \psi^1_0(Y), & u^1_e|_{X=L} = \partial_Y \psi^1_L(Y), \\
    v^1_e|_{Y=0} = -v^0_0(X, 0), & [u^1_e, v^1_e]|_{Y=\infty} = 0.
\end{cases}
\]

So we actually constructed a solution \([u^1_e, v^1_e]\) to equations (4.13) with boundary conditions (4.13).

Next we need to solve the first order boundary layer profile. For simplicity, we introduce some notations.

\[
(4.14) \quad \begin{cases} 
    u^k_p := u^k_p + \sum_{j=0}^{k} \frac{y^j}{j!} \partial^j_X u^k_p|_{Y=0}, & u^k_e := \sum_{j=0}^{k} \frac{y^j}{j!} \partial^j_Y u^k_e|_{Y=0}, \\
    v^k_p := v^k_p - v^0_b|_{y=0} + \sum_{j=0}^{k} \frac{y^{j+1}}{(j+1)!} \partial^{j+1}_Y v^k_e|_{Y=0}, & v^k_e := \sum_{j=0}^{k} \frac{y^{j+1}}{(j+1)!} \partial^{j+1}_Y v^k_e|_{Y=0}.
\end{cases}
\]

And \([u^1_p, v^1_p, p^1_p]\) solves the linearized Prandtl’s equations around \([u^0_p, v^0_p]\):

\[
(4.15) \quad \begin{cases} 
    u^0_p \partial_x u^1_p + u^1_p \partial_x u^0_p + \partial_y u^0_p v^1_p - v^1_p u^1_p|_{y=0} + v^0_p \partial_y u^1_p - \partial_y u^0_p + \partial_x p^1_p = f^{(1)}, \\
    \partial_y p^1_p = 0, \\
    \partial_x u^1_p + \partial_y v^1_p = 0, \\
    u^1_p|_{y=0} = -u^1_e|_{Y=0}, & [u^1_p, v^1_p]|_{y=\infty} = 0,
\end{cases}
\]

where

\[
(4.16) \quad f^{(1)} = -\{u^0_b u^{(1)}_{e, x} + u^0_b u^{(1)}_{e} + v^0_b \partial_y u^{(1)}_e + u^0_b v^{(1)}_e\}.
\]
We see that \( f^{(1)} \) decays fast when \( y \to \infty \) from Lemma 2.2. Since that above equations are linear parabolic type equations, we add the boundary condition on \( u^1_b \rvert_{x=0} \):

\[
\begin{aligned}
\begin{cases}
\partial_x u^1_b &+ \frac{1}{b_1} \partial_x u^1_{b_1} + v^0 u^1_{b_1} + v^0 \partial_y u^1_{b_1} + \left[ v^1_b - v^1_{b_1} \right] \rvert_{y=0} \partial_y u^0_{b_1} - \partial_{yy} u^1_{b_1} + \partial_x p^1_b = f^{(1)}, \\
\partial_y p^1_b &\equiv 0, \\
\partial_x u^1_{b_1} &\equiv \partial_y v^1_{b_1} = 0, \\
u^1_b \rvert_{x=0} &\equiv U^1_B, \quad \left[ u^1_b, v^1_b \right] \rvert_{y=\infty} = 0.
\end{cases}
\end{aligned}
\tag{4.17}
\]

Then notice that \( \vec{u}, \vec{v} \) still using the stream-function of Iyer in [11] proved the well-posedness of above system when \( v^0_t = 0 \). In our case, \( v^0_t \) is different because \( v^0_t \sim yv^0_{eY}(x,0) \) as \( y \) goes to \( \infty \), still we have

**Lemma 4.3.** If \( U^1_B, f^{(1)} \) and their derivatives are bounded and decaying rapidly, they satisfy the parabolic compatibility conditions, then equations \(4.17\) admit a unique solution \([u^1_b, v^1_b]\), and

\[
\begin{aligned}
\| \langle y \rangle^M \nabla^k u^1_b \|_{\infty} + \| \langle y \rangle^M \nabla^k v^1_b \|_{\infty} \lesssim 1 \quad \text{for} \quad 0 \leq k \leq K,
\end{aligned}
\tag{4.18}
\]

where \( K \) and \( M \) are large constants.

**Proof.** For convenience, we write \([\vec{u}, \vec{v}] := [u^0_t, v^0_t]\) and we homogenize the system \(4.17\) as the following way:

\[
\begin{aligned}
u(x, y) &= u^1_b(x, y) + u^1_e(x, 0) \eta(y), \\
v(x, y) &= v^1_b(x, y) - v^1_b(x, 0) + u^1_e(x, 0) I_\eta(y), \\
I_\eta(y) &= \int_0^\infty \eta(y') dy'.
\end{aligned}
\tag{4.19}
\]

Here, we select \( \eta \) to be a \( C^\infty \) function satisfying the following:

\[
\eta(0) = 1, \quad \int_0^\infty \eta = 0, \quad \eta \text{ decays fast as } y \to \infty.
\tag{4.20}
\]

Due to \(4.17\), the homogenized unknowns \([u, v]\) satisfy the system

\[
\begin{aligned}
\vec{u} \partial_x u + u \partial_y \vec{u} &+ \vec{v} \partial_y u + v \partial_y \vec{u} - \partial_{yy} u + p_x = f^{(1)} + F =: h, \\
p_y &\equiv 0, \\
\partial_x u &\equiv \partial_y v = 0, \\
u \rvert_{x=0} &= U^1_B + u^1_e(0, 0) \eta \equiv: u_0(y), \quad [u, v] \rvert_{y=0} = 0, \quad u \rvert_{y=\infty} = 0,
\end{aligned}
\tag{4.21}
\]

where

\[
F = \vec{u} u^1_eX(x,0) \eta + \vec{u} u^1_e(x,0) \eta + \vec{v} u^1_e(x,0) \eta' + \vec{u} u^1_eX(x,0) I_\eta - u^1_e(x,0) \eta''.
\tag{4.22}
\]

Notice that \( p \) is independent on \( y \), we evaluate the equation as \( y \to \infty \), we have \( p_x = 0 \). We still using the stream-function of \([u, v]\),

\[
\phi(x, y) := \int_0^y u(x, y') dy', \quad \partial_y \phi = u, \quad \partial_x \phi = -v.
\tag{4.23}
\]

Then \( \phi \) satisfies

\[
\begin{aligned}
\vec{u} \phi_{xy} + \vec{u} \phi_y + \vec{v} \phi_{yy} - \phi_x \vec{u}_y - \phi_{xyy} &= h, \\
\phi \rvert_{x=0} &= \int_0^y u_0(y') dy', \quad \phi \rvert_{y=0} = \phi \rvert_{y=\infty} = 0, \quad \phi \rvert_{y=\infty} = 0.
\end{aligned}
\tag{4.24}
\]
In order to give a priori estimate of \((4.24)\), we denote \(g = \frac{\phi}{u}\). Recall \(\bar{u} \sim y\) when \(y \leq 1\) and \(\bar{u} \sim 1\), when \(y \geq 1\), and \(\phi|_{y=0} = \phi|_{y=0} = 0\), \(g\) is well-defined. And \(g\) satisfies

\[
\begin{aligned}
\partial_x [\bar{u}^2 g_y] - \partial_y [\bar{u} g] + \bar{u} \partial_y^2 [\bar{u} g] - \bar{u} \bar{u}_{yy} g &= h, \\
g|_{x=0} = \bar{u}_0, &\quad g|_{y=0} = 0, &\quad g|_{y \to \infty} = 0.
\end{aligned}
\tag{4.25}
\]

Now we define the norms of \(g\):

\[
\|g\|_{\Xi_0} := \sup_{0 \leq x \leq L \leq \infty} \|\bar{u} g_y\|_{L^2_y(x=x_0)} + \|\sqrt{y} g_{yy}\|_{L^2_y(x=x_0)}.
\]

\[
\|g\|_{\Xi_1} := \sup_{0 \leq x \leq L \leq \infty} \|\bar{u} g_y\|_{L^2_y(x=x_0)} + \|\sqrt{y} g_{yy}\|_{L^2_y(x=x_0)}.
\]

\[
\|g\|_{\Xi_2} := \sup_{0 \leq x \leq L \leq \infty} \|\bar{u} g_y\|_{L^2_y(x=x_0)} + \|\sqrt{y} g_{yy}\|_{L^2_y(x=x_0)}.
\]

here \(\rho = (y)^N\), for \(N\) large constant. Next, let us prove the following priori estimate of \(g\). Suppose \(g\) be a smooth solution of \((4.25)\), \(L > 0\) small enough, then

\[
\|g\|_{\Xi_0} \lesssim \|\bar{u} g_y\|_{L^2_y(x=x_0)} + \|h\rho\|_{L^2_y(x=x_0)}.
\]

\[
\|g\|_{\Xi_1} \lesssim \|\bar{u} g_y\|_{L^2_y(x=x_0)} + \|g\|_{\Xi_0} + \|h\rho\|_{L^2_y(x=x_0)}.
\]

Multiply equation \((4.25)\) by \(g_y \rho^2\) and integrate in \((0, x_0) \times (0, \infty)\).

\[
\int_0^{x_0} \int_0^{\infty} \bar{u}^2 g_y y g_y \rho^2 dxdy = \int_0^{x_0} \int_0^{\infty} \bar{u}^2 g_y y g_y \rho^2 dxdy + \int_0^{x_0} \int_0^{\infty} 2 \bar{u} \bar{u}_{x} g_y \rho^2 dxdy
\]

\[
= \frac{1}{2} \|\bar{u} g_y\|_{L^2_y(x=x_0)}^2 - \frac{1}{2} \|\bar{u} g_y\|_{L^2_y(x=x_0)}^2 + \int_0^{x_0} \int_0^{\infty} \bar{u} \bar{u}_{x} g_y \rho^2 dxdy.
\]

We can dominate \(\|g_y\|\) by \(\|g\|_{\Xi_0}\). Let \(0 < \xi \leq 1\) be a constant being choosing later. \(\chi(y)\) is smooth cut-off function, satisfies \(\chi|_{[0,1]} = 1\), \(\chi|_{[2,\infty]} = 0\). Then,

\[
\|g_y\|_{\Xi_0} \lesssim \|g_y [1 - \chi(\frac{y}{\xi})]\|_{L^2_y, y} + \|g_y \chi(\frac{y}{\xi})\|_{L^2_y, y}.
\]

When \(y \leq 1\), \(1 - \chi(\frac{y}{\xi}) \lesssim \frac{y}{\xi} \lesssim \frac{\bar{u}}{\xi}\), when \(y > 1\), \(1 - \chi(\frac{y}{\xi}) \lesssim \bar{u} \lesssim \frac{\bar{u}}{\xi}\). So

\[
\|g_y [1 - \chi(\frac{y}{\xi})]\|_{\Xi_0} \lesssim \frac{1}{\xi^2} \|\bar{u} g_y\|_{L^2_y, y} \lesssim \frac{L}{\xi^2} \|g\|_{\Xi_0}.
\]

And

\[
\|g_y \chi(\frac{y}{\xi})\|_{L^2_y, y}^2 = \|g_y \chi(\frac{y}{\xi})\|_{L^2_y, y}^2 = -\int_0^{x_0} \int_0^{\infty} 2 y g_y g_{yy} \bar{u}^2 \chi(\frac{y}{\xi}) \rho^2 dxdy - \int_0^{x_0} \int_0^{\infty} 2 \frac{y}{\xi} y g_y^2 \chi(\frac{y}{\xi}) \rho^2 dxdy
\]

\[
\lesssim \|y \chi(\frac{y}{\xi})\|_{L^2_y, y} \lesssim \frac{1}{\xi^2} \|\bar{u} g_y\|_{L^2_y, y} \lesssim \frac{L}{\xi^2} \|g\|_{\Xi_0}.
\]

So we have

\[
\|g_y\|_{L^2_y, y}^2 \lesssim \xi \|\sqrt{y} g_{yy}\|_{L^2_y, y} + \frac{L}{\xi^2} \|g\|_{\Xi_0}.
\]
select \( \xi = L^\frac{1}{3^*} \), then
\[
\|g_y \rho\|^2_{L^2_{x,y}} \lesssim L^\frac{1}{3} \|g\|^2_{\mathbb{L}_0^2}.
\]
So the first term is
\[
\int_0^x \int_0^\infty \left[ \tilde{u}^2 g_y \right] g_y \rho^2 \, dx \, dy = \frac{1}{2} \left\| \tilde{u} g_y \rho \right\|^2_{L^2_{x,y}}(x=x_0) - \frac{1}{2} \left\| \tilde{u} g_y \rho \right\|^2_{L^2_{x,y}}(x=0) + O \left( L^\frac{1}{3} \|g\|^2_{\mathbb{L}_0^2} \right).
\]

So the second term is
\[
\int_0^x \int_0^\infty \partial_y^3 \left[ \tilde{u} g \right] \rho^2 \, dx \, dy = \frac{1}{2} \left\| \tilde{u}^2 g_y \rho \right\|^2_{L^2_{x,y}} - \frac{1}{2} \left\| \tilde{u} g_y \rho \right\|^2_{L^2_{x,y}}(y=0) + O \left( \|g_y \rho\|^2_{L^2_{x,y}} \right).
\]

The second term:
\[
- \int_0^x \int_0^\infty \partial_y^3 \left[ \tilde{u} g \right] \rho^2 \, dx \, dy = \int_0^x \int_0^\infty \partial_y^3 \left[ \tilde{u} g \right] \rho^2 \, dx \, dy + \int_0^x \int_0^\infty \partial_y^3 \left[ \tilde{u} g \right] \rho^2 \, dx \, dy
\]

So the second term is
\[
\int_0^x \int_0^\infty \partial_y^3 \left[ \tilde{u} g \right] \rho^2 \, dx \, dy = \frac{1}{2} \left\| \tilde{u}^2 g_y \rho \right\|^2_{L^2_{x,y}} - \frac{1}{2} \left\| \tilde{u} g_y \rho \right\|^2_{L^2_{x,y}}(y=0) + O \left( \|g_y \rho\|^2_{L^2_{x,y}} \right).
\]

The third term is
\[
\int_0^x \int_0^\infty \tilde{v} \left( \tilde{u} g_y \right) g_y \rho^2 \, dx \, dy = \int_0^x \int_0^\infty \tilde{v} \left( \tilde{u} g_y \right) \rho^2 \, dx \, dy
\]

So the second term is
\[
\int_0^x \int_0^\infty \partial_y^3 \left[ \tilde{u} g \right] \rho^2 \, dx \, dy = \frac{1}{2} \left\| \tilde{u}^2 g_y \rho \right\|^2_{L^2_{x,y}} - \frac{1}{2} \left\| \tilde{u} g_y \rho \right\|^2_{L^2_{x,y}}(y=0) + O \left( \|g_y \rho\|^2_{L^2_{x,y}} \right).
\]
And the last one is

\begin{equation}
\int_{0}^{x_0} \int_{0}^{\infty} \bar{v}_{yy} \bar{u} \bar{g}_y \rho^2 \, dx \, dy = O \left( \left\| g_{yy} \rho \right\|_{L^2}, \left\| \bar{g}_y \rho \right\|_{L^2} \right).
\end{equation}

Collect (4.30), (4.31), (4.32), (4.33), we have

\begin{equation}
\frac{1}{2} \left\| \bar{u} g_y \rho \right\|_{L^2_x(x=x_0)}^2 + \left\| \sqrt{\bar{u}} g_{yy} \rho \right\|_{L^2_x}^2 + \left\| \bar{u}_y g_y \rho \right\|_{L^2_x}^2 = O \left( \left\| g_y \rho \right\|_{L^2_y}^2 \right).
\end{equation}

Take the supremum of 0 \leq x_0 \leq L, notice that L small enough,

\begin{equation}
\sup_{0 \leq x_0 \leq L} \left\| \bar{u} g_y \rho \right\|_{L^2_x(x=x_0)}^2 + \left\| \sqrt{\bar{u}} g_{yy} \rho \right\|_{L^2_x}^2 + \left\| \bar{u}_y g_y \rho \right\|_{L^2_x}^2 \lesssim \left\| \bar{u} g_y \rho \right\|_{L^2_x(x=x_0)}^2 + \left\| h \rho \right\|_{L^2_y}^2.
\end{equation}

The inequality (4.28) is similar to the (4.27). Differential equation (4.25) with respect to \( x \),

\begin{equation}
\partial_x \left[ \bar{u}^2 g_{xy} \right] - \partial_y \left[ \bar{u} \bar{g}_x \right] + \bar{v} \partial_y^2 \left( \bar{u} g_x \right) - \bar{v} \bar{v}_{yy} g_x + \partial_x \left[ 2 \bar{u} \bar{u}_x g_y \right] - \partial_y \left[ u_x g \right] + v_x \partial_y^2 \left( \bar{u}_x g \right) - \bar{v}_x \bar{v}_{yy} g - \bar{v} \bar{v}_{yy} g = h_x.
\end{equation}

Take \( g_{xy} (\bar{y})^2 \) as the test function, like (4.33),

\begin{equation}
\int_{0}^{x_0} \int_{0}^{\infty} \left[ \partial_x \left[ \bar{u}^2 g_{xy} \right] - \partial_y \left[ \bar{u} \bar{g}_x \right] + \bar{v} \partial_y^2 \left( \bar{u} g_x \right) - \bar{v} \bar{v}_{yy} g_x \right] g_{xy} \rho^2 \left( \bar{y} \right)^2 \, dx \, dy \leq \frac{1}{2} \left( \left\| \bar{u}_x g_{xy} \rho \right\|_{L^2_x(x=x_0)}^2 - \left\| \bar{u} g_{xy} \rho \right\|_{L^2_x(x=x_0)}^2 \right) + \left\| \sqrt{\bar{u}} g_{yy} \rho \right\|_{L^2_x}^2 + \left\| \bar{u}_y g_y \rho \right\|_{L^2_x}^2 + O \left( L^\frac{1}{2} \left\| g \right\|_{L^2_{xy}}^2 \right).
\end{equation}

And

\begin{equation}
\int_{0}^{x_0} \int_{0}^{\infty} \left[ \partial_x \left[ 2 \bar{u} \bar{u}_x g_y \right] + \bar{v} \partial_y^2 \left( \bar{u}_x g \right) - \bar{v}_x \bar{v}_{yy} g - \bar{v} \bar{v}_{yy} g \right] g_{xy} \rho^2 \left( \bar{y} \right)^2 \, dx \, dy = O \left( \left\| g_{yy} \rho \right\|_{L^2_{xy}}^2 + \left\| g_y \rho \right\|_{L^2_{xy}}^2 + \left\| g_{xy} \rho \right\|_{L^2_{xy}}^2 \right) \leq O \left( \left\| g \right\|_{L^2_{xy}}^2 + L^\frac{1}{2} \left\| g \right\|_{L^2_{xy}}^2 \right).
\end{equation}

The difficult term is

\begin{equation}
\int_{0}^{x_0} \int_{0}^{\infty} \partial_y^2 \left[ \bar{u}_x g \right] g_{xy} \rho^2 \left( \bar{y} \right)^2 \, dx \, dy = - \int_{0}^{x_0} \int_{0}^{\infty} \left( \bar{u}_x g_{yy} + 3 \bar{u}_y g_{yy} \right) g_{xy} \rho^2 \left( \bar{y} \right)^2 \, dx \, dy + O \left( \left\| g_y \rho \right\|_{L^2_{xy}}^2 + \left\| g_{xy} \rho \right\|_{L^2_{xy}}^2 \right)
\end{equation}

\begin{equation}
= O \left( \left\| \bar{u} g_{yy} \rho \right\|_{L^2_{xy}}^2 + \left\| g_{yy} \rho \right\|_{L^2_{xy}}^2 \right) \left\| g_{xy} \rho \right\|_{L^2_{xy}}^2 + \left\| g_y \rho \right\|_{L^2_{xy}}^2 + \left\| g_{xy} \rho \right\|_{L^2_{xy}}^2 \right).
\end{equation}

From equation (4.24), we have

\begin{equation}
\left\| \phi_{yy} \rho \right\|_{L^2_{xy}}^2 = O \left( \left\| g \right\|_{L^2_{xy}}^2 + L^\frac{1}{2} \left\| g \right\|_{L^2_{xy}}^2 + \left\| h \rho \right\|_{L^2_{xy}}^2 \right),
\end{equation}
notice that the fact
\[
\| \chi \partial_y^2 \left( \frac{\phi}{y} \right) \|_{L^2_x, y} = O \left( \| \phi_{yy} \|_{L^2_x, y} + \| \phi_y \|_{L^2_x, y} \right),
\]
similarly, we can get
\[
\| \chi g_{yy} \|_{L^2_x, y} = O \left( \| \phi_{yy} \|_{L^2_x, y} + \| \phi_y \|_{L^2_x, y} \right),
\]
so we have
\[
\| g_{yy} \langle y \rangle \|_{L^2_x, y} = \| g_{yy} \chi \|_{L^2_x, y}^2 + \| g_{yy} (1 - \chi) \langle y \rangle \|_{L^2_x, y}^2
\]
\[
= O \left( \| \phi_{yy} \|_{L^2_x, y}^2 + \| \phi_y \|_{L^2_x, y}^2 + \| \sqrt{\| g_{yy} \langle y \rangle \|_{L^2_x, y}^2} \right),
\]
\[
= O \left( \| g \|_{\Xi_0}^2 + L \| g \|_{\Xi_1}^2 + \| h \langle y \rangle \|_{L^2_x, y}^2 \right),
\]
and
\[
\| \bar{u} g_{yy} \langle y \rangle \|_{L^2_x, y}^2 = \| (\phi_{yy} - 3 \bar{u}_y g_{yy} - 3 \bar{u}_{yy} g_y - \bar{u}_{yy} g) \langle y \rangle \|_{L^2_x, y}^2
\]
\[
= O \left( \| g \|_{\Xi_0}^2 + L \| g \|_{\Xi_1}^2 + \| h \langle y \rangle \|_{L^2_x, y}^2 \right).
\]
We conclude (4.39) as
\[
\int_0^\infty \int_0^\infty - \partial_y^2 [\bar{u}_x g_{xy}]^2 \rho \langle y \rangle^2 \, dx \, dy = O \left( \| g \|_{\Xi_0}^2 + L \| g \|_{\Xi_1}^2 + \| h \langle y \rangle \|_{L^2_x, y}^2 \right).
\]
Collect (4.37), (4.38), (4.40), we have
\[
\frac{1}{2} \| \bar{u} g_{xy} \langle y \rangle \|_{L^2_x}^2 (x = x_0) + \| \bar{u} g_{xy} \langle y \rangle \|_{L^2_y}^2 (y = 0)
\]
\[
= \frac{1}{2} \| \bar{u} g_{xy} \langle y \rangle \|_{L^2_x}^2 (x = x_0) + O \left( \| g \|_{\Xi_0}^2 + L \| g \|_{\Xi_1}^2 + \| h \langle y \rangle \|_{L^2_y}^2 \right),
\]
So we get the inequalities (4.27) and (4.28). These inequalities show if \( g \) satisfies the linear parabolic type equation (4.25), then \( \| g \|_{\Xi_0} \) and \( \| g \|_{\Xi_1} \) can be dominated by its initial data and \( h \), we can use the standard method to prove the local existence of solution. Follow this way, we can also get the high order derivatives estimates to show the smoothness of solution. \( \square \)

In fact, the system (4.25) admits a smooth solution \( g \) even if \( L \) is large, since the local well-posedness means the global well-posedness for linear parabolic type equation.

The second order Euler profile \([u_e^2, v_e^2, p_e^2]\) solves the linearized Euler equations around \([u_0^e, v_0^e]\) with the force terms:
\[
\begin{aligned}
&\begin{cases}
  u_e^0 u_e X + u_e^0 u_e Y + u_e Y v_e^0 = F^{(2)}, \\
  u_e^0 v_e X + v_e^0 u_e Y + v_e Y v_e^0 = G^{(2)}, \\
  \partial X u_e^2 + \partial Y v_e^2 = 0,
\end{cases}
\end{aligned}
\]
\[
\begin{aligned}
&\begin{aligned}
  u_e^2 \big| Y = 0 &= - v_e^0 \big| Y = 0, \\
  v_e^0 \big| Y = 0 &= v_e^1 \big| Y = 0, \\
  F^{(2)} &= - ( u_e^1 u_e^1 + v_e^1 u_e^1 Y ), \\
  G^{(2)} &= - ( u_e^1 v_e^1 + v_e^1 v_e^1 Y ),
\end{aligned}
\end{aligned}
\]
We can treat above equations as that of the first order Euler flow.

\( (4.44) \quad \partial_y [\Delta_{XY} \psi^2 - F'_e(\psi) \psi^2 - \frac{F''_e(\psi)}{2}(\psi^1)^2] = \frac{\Delta^2_{XY} \psi}{u^0_e}. \)

Let

\[ H(\theta, \psi) = \int_0^\theta \frac{\Delta^2_{XY} \psi(\theta', Y(\theta', \psi))}{u^0_e(\theta', \psi)} d\theta'. \]

Notice that \( \psi \sim Y \) when \( Y \to \infty \), we have that \( H \) is of fast decay as \( \psi \to \infty \) because of \((1.7)\). We can find a solution of the following equations

\( (4.45) \quad \begin{cases} \Delta_{XY} \psi^2 - F'_e(\psi) \psi^2 - \frac{F''_e(\psi)}{2}(\psi^1)^2 = H(\theta(X, Y), \psi(X, Y)), \\ \psi^2|_{X=0} = \psi^2_0(Y), \quad \psi^2|_{X=L} = \psi^2_L(Y), \\ \psi^2|_{Y=0} = \int_0^X u^1_b(X', 0) dX', \quad \psi^2|_{Y=\infty} = 0, \end{cases} \)

with suitable \( \psi^2_0(Y), \psi^2_L(Y) \), and we have the estimate of the second order Euler flow

\( (4.46) \quad \| \langle y \rangle^M \nabla^k \psi^2 \| \lesssim 1, \quad \text{for} \ 1 \leq k \leq K, \quad K \text{ and } M \text{ large constants} . \)

The second order boundary layer profile \([u^2_b, v^2_b, p^2_b]\) is similar to the first, we need to solve the following equations

\( (4.47) \quad \begin{cases} u^0_b \partial_x u^2_u + u^2_b \partial_x u^0_p + v^0_b \partial_y u^2_p + [v^2_b - v^0_b|_{y=0}] \partial_y u^0_p - \partial_y u^2_b + \partial_x p^2_b = f^{(2)}, \\ \partial_y p^2_b = g^{(2)}, \\ \partial_x u^2_y + \partial_y v^2_b = 0, \\ u^2_b|_{x=0} = U_B, \quad u^2_b|_{y=0} = -u^2_e|_{y=0}, \quad [u^2_b, v^2_b]|_{y=\infty} = 0. \end{cases} \)

Where

\( f^{(2)} = - \{ u^0_b u^2_{ex} + u^0_b u^2_{ey} + v^0_b u^0_{ey} + u^0_b v^0_{ey} \\ + u^1_p u^1_{bx} + u^1_b u^1_{ex} + v^1_b u^1_{ey} + v^1_b v^1_{ey} - u^0_{bx} \}, \)

\( g^{(2)} = - \{ u^0_b v^0_{px} + u^0_b v^0_{py} + v^0_b v^0_{py} + (v^0_e + v^0_e|_{Y=0}) v^0_{by} \\ - v^0_{by} \}. \)

We can see \( f^{(2)} \) and \( g^{(2)} \) decays fast when \( y \to \infty \) from Lemma 2.2 and Lemma 4.3. We can solve \( p^2_b(x, y) = - \int_y^\infty g^{(2)}(x, y') dy' \). By using the same argument of Lemma 4.3 we have

\( (4.49) \quad \| \langle y \rangle^M \nabla^k p^2_b \|_\infty + \| \langle y \rangle^M \nabla^k v^2_b \|_\infty \lesssim 1 \quad \text{for} \quad 0 \leq k \leq K, \)

where \( K \) and \( M \) are large constants.

After that, \( p^3_b \) is solved by

\( (4.50) \quad p^3_b = \int_y^\infty \{ \sum_{j=0}^1 [u^1_{py} v^j + u^{(1-j)}_e v^j_x + v^{1-j}_b v^j_{py} \\ + (v^{(1-j)}_e + v^{2-j}_e|_{Y=0}) v^j_{by}] - v^1_{by} \} dy' \).

We can conclude the following proposition for the approximate profiles.
Proposition 4.4. Under the assumptions of Theorem 1.1 then equations (4.28), (4.49), (4.47) admit smooth solutions \([u_0^i, v_0^i], [u_b^i, v_b^i]\) for \(j = 1, 2\), and the following estimates hold

\[
\|\langle y \rangle^M \nabla^k u_b^i \|_\infty + \|\langle y \rangle^M \nabla^k v_b^i \|_\infty \lesssim 1 \quad \text{for} \quad 0 \leq k \leq K, j = 0, 1, 2,
\]

\[
\|\langle Y \rangle^M \nabla^k u_b^i \|_\infty + \|\langle Y \rangle^M \nabla^k v_b^i \|_\infty \lesssim 1 \quad \text{for} \quad 0 \leq k \leq K, j = 1, 2,
\]

where \(K, M\) sufficiently large constants, \(\langle y \rangle = y + 1\) and \(\langle Y \rangle = Y + 1\).

Notices that \(v_0^i|_{y=0} \neq 0\). We need to match the boundary conditions at \(y = 0\), and also \(v_0^i|_{y \to \infty} = 0\). Then we can modify \([\hat{u}_0^i, \hat{v}_0^i]\) in this way:

\[
\hat{u}_0^i(x, y) := \chi(\sqrt{\varepsilon}y)u_0^i(x, y) - \sqrt{\varepsilon}\chi'(\sqrt{\varepsilon}y)\int_0^y u_0^i(x, y')dy',
\]

\[
\hat{v}_0^i(x, y) := \chi(\sqrt{\varepsilon}y)(v_0^i(x, y) - v_0^i(x, 0)),
\]

where \(\chi\) is a cut-off function satisfying \(\chi|_{[0,1]} = 1\) and \(\chi|_{[2,\infty]} = 0\).

And let \([U_s, V_s, P_s]\) be

\[
U_s(X, Y) = u_0^0(X, Y) + u_0^1(Y - \frac{Y}{\sqrt{\varepsilon}}) + \sqrt{\varepsilon}[u_0^1(X, Y) + u_b^1(Y - \frac{Y}{\sqrt{\varepsilon}})]
\]

\[
+ \varepsilon[u_0^0(X, Y) + \hat{u}_0^0(X, Y)],
\]

\[
V_s(X, Y) = v_0^0(X, Y) + \sqrt{\varepsilon}[v_0^1(X, Y) + v_b^1(X, Y)] + \varepsilon[v_0^1(X, Y) + v_b^1(X, Y)]
\]

\[
+ \varepsilon^2 \hat{v}_0^2(X, Y),
\]

\[
P_s(X, Y) = p_0^0(X, Y) + p_0^1(X, Y) + \sqrt{\varepsilon}[p_0^1(X, Y) + p_b^1(X, Y)]
\]

\[
+ \varepsilon[p_0^1(X, Y) + p_b^1(X, Y)] + \varepsilon^2 p_b^2(X, Y).
\]

Then the errors

\[
R_1 := U_sU_sX + V_sU_sY - \varepsilon \Delta U_s + P_sX,
\]

\[
R_2 := U_sV_sX + V_sV_sY - \varepsilon \Delta V_s + P_sY,
\]

satisfy

\[
\|R_1\| + \|R_2\| \lesssim \varepsilon^{\frac{3}{2}}.
\]

Acknowledgements: L. Zhang is partially supported by NSFC under grant 11471320 and 11631008.

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