Entanglement entropy and entanglement spectrum of the Kitaev model

Hong Yao\textsuperscript{1,2} and Xiao-Liang Qi\textsuperscript{3,4}

\textsuperscript{1}Department of Physics, University of California, Berkeley, CA 94720, USA
\textsuperscript{2}Materials Sciences Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA
\textsuperscript{3}Microsoft Research, Station Q, Elings Hall, University of California, Santa Barbara, CA 93106, USA
\textsuperscript{4}Department of Physics, Stanford University, Stanford, CA 94305, USA

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In this paper, we obtain an exact formula for the entanglement entropy of the ground state and all excited states of the Kitaev model. Remarkably, the entanglement entropy can be expressed in a simple separable form $S = S_G + S_F$, with $S_F$ the entanglement entropy of a free Majorana fermion system and $S_G$ that of a $Z_2$ gauge field. The $Z_2$ gauge field part contributes to the universal “topological entanglement entropy” of the ground state while the fermion part is responsible for the non-local entanglement carried by the $Z_2$ vortices (visons) in the non-Abelian phase. Our result also enables the calculation of the entire entanglement spectrum and the more general Renyi entropy of the Kitaev model. Based on our results we propose a new quantity to characterize topologically ordered states—the capacity of entanglement, which can distinguish the states with and without topologically protected gapless entanglement spectrum.

Exotic phases such as fractional quantum Hall (FQH) states, which are not in the paradigm of conventional symmetry breaking, were termed as topologically ordered \cite{1} since they have robust ground state degeneracy which is protected against all local perturbations, but sensitive to the topology of the system \cite{2}. A topologically ordered state has non-local pattern of quantum entanglement, which is essential for the proposal of topological quantum computation \cite{3–5}.

This paper serves to fill in that gap by providing a simple but exact method to compute the EE for any eigenstate (either ground or excited states) of the Kitaev model \cite{16}, which is one of the most important exact solvable models with non-Abelian anyons. The essence of our method is a rigorous proof that the EE of the Kitaev model is equal to that of two decoupled systems: a sourceless $Z_2$ gauge field and a free Majorana fermion system. Although the EE of the ground state comes only from the $Z_2$ gauge field, the fermionic part is responsible for all nontrivial entanglement properties of the non-Abelian phase. Besides the EE, our method also enables the computation of the whole entanglement spectrum (ES), i.e., the eigenvalue spectrum of the reduced density matrix \cite{17}. We show that the entanglement spectrum is gapless or gapped in the non-Abelian and Abelian phase of the Kitaev model, respectively. We propose a new quantity, the capacity of entanglement, which can be used to distinguish different topological states with gapped and gapless entanglement spectrum.

Kitaev model is a spin-1/2 model originally proposed on the honeycomb lattice \cite{17} with the Hamiltonian

$$H = -\sum_{x\text{-link}} J_x \sigma^x_i \sigma^x_j - \sum_{y\text{-link}} J_y \sigma^y_i \sigma^y_j - \sum_{z\text{-link}} J_z \sigma^z_i \sigma^z_j,$$

where $x, y, z$-link stand for the three types of links. It has a non-Abelian phase when the time-reversal symmetry is broken either explicitly by magnetic field \cite{16} or three-spin couplings \cite{18}, or spontaneously by decorating the honeycomb lattice \cite{19}. For simplicity, hereafter we will present our exact results of EE and entanglement spectrum for the Kitaev model on honeycomb lattice, but our approach can be generalized straightforwardly to a broad class of Hamiltonians, including the Kitaev model on any trivalent lattice and Gamma matrix models \cite{20–24}.

The Kitaev model can be solved by introducing the Majorana representation of the Pauli matrices \cite{16}: $\sigma^i_\alpha = i\gamma^i_\alpha \eta_i \ (\alpha = x,y,z)$, where $\gamma^i_\alpha$ and $\eta_i$ are Majorana fermion operators. $\gamma^i_\alpha$ and $\eta_i$ on each lattice site define a 4-dimensional Hilbert space, so that the Majorana representation of a spin-1/2 is redundant. The physical Hilbert space is defined by a constraint $D_i = -i\sigma^i_\alpha \sigma^i_\beta \eta_i = 1$. In other words, a state $|\Psi\rangle$ is physical only if $D_i |\Psi\rangle = |\Psi\rangle$ for every $i$. In the Majorana representation we have $\sigma^i_\alpha \sigma^i_\beta = \eta_i \gamma^i_\beta \eta_j = -\hat{u}_{ij} \eta_i \eta_j$, in which the link operators $\hat{u}_{ij} = i\gamma^i_j \gamma^j_i$ mutually com-
The entanglement entropy can be obtained if we can compute $\text{Tr}_A [\rho^n_A]$ for arbitrary positive integer $n$ and then extrapolate the result to $n \in \mathbb{R}$.

To obtain $\rho_A$, we trace out the spin degree of freedom in $B$, which normally can be carried out in terms of fermions and gauge fields. However, the gauge fields on the partition boundary are shared by $A$ and $B$; so we regroup those gauge fields on the boundary links to introduce new $Z_2$ gauge variables which lives in $A$ and $B$ exclusively, as shown in Fig. 1 (see supplementary material [29] for details.) The calculation of $\text{Tr}[\rho^n_A]$ requires some careful treatment of the gauge transformation but is a well-defined mathematical procedure. Thus, we will leave the details involved in obtaining $\rho_A$ and $\text{Tr} \rho^n_A$ to the supplementary material [29] and present only the final result here:

$$\text{Tr}_A [\rho^n_A] = \text{Tr}_{A,G} [\rho^n_{A,G}] \cdot \text{Tr}_{A,F} [\rho^n_{A,F}],$$

for any positive integer $n$. Here $\rho_{A,F} = \text{Tr}_B [\langle \phi(u) \rangle \langle \phi(u) \rangle]$ and $\rho_{A,G} = \text{Tr}_B [\langle G(u) \rangle \langle G(u) \rangle]$ are the reduced density matrices for the free Majorana fermion state $\langle \phi(u) \rangle$, and a pure $Z_2$ gauge field $\mathbf{G}$, respectively, and the ground state of the $Z_2$ gauge field $\langle G(u) \rangle$ is given by a equal weight superposition of all the $2^{N-1}$ gauge field configurations $\hat{u}$ that are gauge equivalent to $u$, i.e., $\langle G(u) \rangle = 2^{-(N-1)/2} \sum_{\hat{u} \sim u} \langle \hat{u} \rangle$. Physically, such a simplification occurs because the effect of the gauge transformation $D_{\hat{g}}$ on the fermion state in region $B$ is canceled out once the trace over gauge field configurations is taken.

Combining Eq. (4) and Eq. (3), it is now obvious that the EE $S$ can be separated into gauge field part $S_G$ and fermion part $S_F$ as follows:

$$S = S_G + S_F.$$

Eq. (4) and (5) are among the central results of this work. By explicit calculation [29] one can obtain $\text{Tr}_{A,G} [\rho^n_{A,G}] = 2^{-L(n-1)}$, so that $S_G = (L-1) \log 2$. As will be shown below, the fermion part has the form $S_F = \alpha L + o(1)$, where $\alpha$ is a positive constant and $o(1)$ represents terms which vanish as $L \to \infty$. In the thermodynamic limit, the total entanglement entropy is given by

$$S = (\alpha + \log 2) L - \log 2,$$

from which we conclude that the TEE is $S_{\text{topo}} = - \log 2$. Our derivation is valid for all phases of Kitaev model, including the Abelian ($Z_2$ gauge theory), non-Abelian (Ising anyon) phases, and also gapless phases. Thus our result directly proves that the TEE for the Abelian and non-Abelian phases are identical, as expected from the total quantum dimensions of their quasiparticles [30].

Despite its trivial contribution to TEE, the fermion sector $S_F$ is responsible for all the essential differences between the Abelian and non-Abelian phase of Kitaev model in their quantum entanglement properties. The EE of a free fermion system can be computed by the method introduced in Ref. [31]. To obtain an explicit understanding to the fermion EE we consider a torus.
divided by two parallel circles into $A$ and $B$ regions, as shown in Fig. 2 (a). The boundary circle is along the $\hat{y}$ direction. On the torus, the free Majorana fermion Hamiltonian can be block-diagonalized in the basis of $k_y$: $H = \sum_{i,j} \eta_i \eta_j h_{ij} = \sum_{x,\sigma} \frac{1}{2} \sigma \eta(k_y) h_{xx'}(k_y) \eta_{\sigma'}(k_y)$. Thus the system consists of decoupled one-dimensional subsystems of each $k_y$. The EE is given by

$$S_F = -\frac{1}{2} \sum_{n,k_y} \left[ \lambda_n \log \lambda_n + (1 - \lambda_n) \log (1 - \lambda_n) \right] (k_y),$$

where $\lambda_n(k_y)$ are the eigenvalues of the single-particle correlation function $C_{xx'}(k_y) = \langle \eta_i(k_y) \eta_{\sigma'}(k_y) \rangle$ for each $k_y$. $\lambda_n$ plays the role of Fermi-Dirac distribution $1/(e^{\beta \lambda_n} + 1)$ in thermal entropy, so that $\lambda_n = 0 \ (1)$ corresponds to fully occupied (unoccupied) states, respectively. The “entanglement spectrum” (ES) $\lambda_n(k_y)$ has been computed numerically for both non-Abelian and Abelian phases, as shown in Fig. 2 (b), for the Kitaev model with three-spin terms $J'$. The ES is gapped for the Abelian phase, and gapless for the non-Abelian phase, similar to the edge states in the energy spectrum. Similar observation has been made in topological insulators and superconductors and in FQH systems. The two gapless branches in the ES come from the two boundaries between $A$ and $B$. Since $\lambda_n(k_y)$'s are smooth functions of $k_y$, we see from Eq. (7) that in the continuum limit $S_F = \sum_{n} S(k_y) \approx L \int S(k_y) \frac{dk_y}{2\pi}$ satisfies the area law. It is interesting to note that a “gap” always exists between the edge states and other bulk states with $\lambda_n(k_y)$ close to 0 or 1, which is analogous to the “entanglement gap” studied in Ref. \[36\] for FQH system.

The situation becomes more interesting when we consider a cylinder with periodic boundary condition (PBC) and the partition shown in Fig. 2 (a). As shown in Fig. 2 (c), in the non-Abelian phase the numerical calculation gives only one branch of “gapless” states in the entanglement spectrum. Physically, this is because the coupling through the other boundary between $A$ and $B$ is removed by the open boundary condition. However, at $k_y = 0$ there is one isolated additional state with $\lambda = 1/2$, as shown by the blue circle in Fig. 2 (c), which is due to the non-local entanglement between the two Majorana zero modes at the open boundary. Consequently, the entropy $S(k_y)$ is not a smooth function of $k_y$ but has an additional $\log \sqrt{2}$ contributed by $k_y = 0$, as shown in Fig. 2 (d). Compared with the torus case, in the thermodynamic limit we get $S_F = \alpha L + \log \sqrt{2}$, which shows explicitly that in the non-Abelian phase a cylinder with PBC is topologically equivalent to a sphere with two non-Abelian quasiparticles (usually named as $\sigma$ particles), as illustrated in Fig. 2 (a). Each particle carries a $\log \sqrt{2}$ entropy which is solely contributed by the fermion sector.

Besides the EE, more information is contained in our result. The fact that Eq. (11) holds for any positive integer $n$ indicates that the many-body entanglement spectrum—the eigenvalue spectrum of $\rho_A$ is the direct product of the ones of $\rho_{A,G}$ and $\rho_{A,F}$. From $\text{Tr}_{A,G}[\rho_{A,G}^{n}] = 2^{-(L-1)(n-1)}$, one can know that $\rho_{A,G}$ has $2^{L-1}$ non-zero eigenvalues, all of which are degenerate and have the value of $2^{-(L-1)}$. Consequently all non-vanishing eigenvalues of $\rho_A$ are given by those of the Majorana fermion reduced density matrix $\rho_{A,F}$ times $2^{-(L-1)}$. Thus the low “energy” (i.e., close to the maximal eigenvalue of $\rho_A$) feature in the entanglement spectra of $\rho_A$ can be entirely characterized by its fermionic part, which is gapped in the Abelian phase and gapless in the non-Abelian phase, as shown in Fig. 2 (b).

Such a qualitative difference in the entanglement spectrum can be characterized by the Renyi entropy $S_\alpha = \frac{1}{1-\alpha} \log \text{Tr} \rho^\alpha$, which reduces to the EE (or von Neumann entropy) at $\alpha \to 1$. According to Eq. (11) the Renyi entropy of Kitaev model is given by $S_\alpha = S_{FA} + S_{GA}$ for any $\alpha$, with $S_{GA}$ and $S_{FA}$ the contribution from $Z_2$ gauge fields and fermions, respectively. From $\text{Tr}_{A,G}[\rho_{A,G}^{\alpha}] = 2^{-(L-1)(n-1)}$, one can see that $S_{GA} = S_{G} = (L - 1) \log 2$. Thus the TEE in $S_{GA}$ is $\alpha$ independent, which is a generic property of the string-net models. The $\alpha$ dependence of $S_{FA}$ in the Abelian and non-Abelian phases has qualitative difference due to their different entan-

FIG. 2: (Color online) (a) Schematic picture of a torus and a cylinder, each split to two regions $A$ and $B$. The cylinder is equivalent to a sphere with two quasi-particles. (b) The entanglement spectrum $\lambda_n(k_y)$ versus $k_y$ for non-Abelian (red solid lines) and Abelian (blue dotted lines) state on torus. Here and below, we take the parameters $J_x = J_y = J_\perp = 1$ and next-nearest neighbor coupling $J' = 0.2$ for the non-Abelian state, and $J_x = J_y = 1$, $J_\perp = 2.5$, $J' = 0.2$ for the Abelian state. (c) The entanglement spectrum for the non-Abelian state on cylinder. The blue circle marks an additional state with $\lambda = 1/2$ at $k_y = 0$. (d) The entropy $S(k_y)$ versus $k_y$ for non-Abelian (red solid line) and Abelian (blue dotted line) states on the torus and for non-Abelian state on cylinder (black dashed line with circles).
shown in Fig. 3. As expected, in the limit of non-Abelian phase, since the latter has a gapless entanglement spectrum. If we define \( \rho = e^{-\mathcal{H}} \), the quantity \( S_\alpha(1 - 1/\alpha) = -\frac{1}{\alpha} \log \text{Tr} e^{-\alpha \mathcal{H}} \) is the same as the free energy of a thermal system with Hamiltonian \( \mathcal{H} \) and temperature \( t = 1/\alpha \). The behavior of the low energy spectrum of \( \mathcal{H} \) can be obtained from the following quantity:

\[
C_E(t) = -\frac{\partial^2}{\partial t^2} \left[ (1 - t)S_{1/1} \right],
\]

which is termed as “capacity of entanglement” and is the analog of heat capacity \( C_v \) in a thermal system. The explicit expression of \( S_\alpha \) and \( C_E(t) \) is given in the Supplementary material, which leads to the numerical results shown in Fig. 3. As expected, in the limit of \( t \to 0 \), \( C_E(t) \) vanishes exponentially for Abelian phase but linearly for non-Abelian phase, since the latter has a gapless entanglement spectrum with constant density of state. More generally, if the entanglement Hamiltonian \( \mathcal{H} \) describes a \((1 + 1)\) dimensional conformal field theory (CFT) in long wavelength limit \([7, 17]\), the capacity of entanglement is given by \( C_E(t) = (\pi c L/3v) t \) for \( t \to 0 \), with \( L \) the length of the boundary, and \( c \) and \( v \) the central charge and velocity of the CFT, respectively \([40]\). Moreover, if \( \mathcal{H} \) describes a critical theory with dynamical exponent \( z \), from dimensional analysis one can obtain the asymptotic behavior \( C_E(t) \propto Lt^{1/z} \) for \( t \to 0 \). Thus we see that the capacity of entanglement characterizes some important qualitative behavior of the entanglement spectrum in generic systems.

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To obtain \( \rho_A \), we trace out the spin degrees of freedom in \( B \). For the spins in \( B \) away from the partition boundary, the trace can be carried out in the Hilbert space of fermions and gauge field respectively. However, the spins on the boundary sites need more careful treatment. Suppose the links across the boundary are denoted as \( a_n b_n, n = 1, \cdots, 2L \), as shown in Fig. 1. (Note that we implicitly assumed the boundary length is even here. The odd boundary length case need some extra care but our results below remain valid.) The gauge field \( \tilde{u}_{a_n b_n} = i \gamma^n_\alpha b_n^\alpha \) residing on \( a_n b_n \) has spin degrees of freedom from both \( A \) and \( B \). To trace out \( B \) we introduce new \( Z_2 \) variables \( \tilde{w}_{a,n} = i \gamma_{a_{2n-1}}^\alpha \gamma_{b_{2n}}^\beta \) and \( \tilde{w}_B, n = i \gamma_{b_{2n-1}}^\alpha \gamma_{b_{2n}}^\beta \), which are defined on the dotted links in Fig. 1 and belong to either \( A \) or \( B \) exclusively. The eigenstates of \( \tilde{w}_{A,n} \) and \( \tilde{w}_{B,n} \) are related to those of \( u_{a_{2n-1} b_{2n}} \) by a unitary transformation. Since \(|u\rangle \) is a direct product of gauge fields, we denote \(|u\rangle = |u_A, u_B, u_p\rangle \), where \(|u_A\rangle, |u_B\rangle \), and \(|u_p\rangle \) are gauge fields in \( A \), \( B \), and on the links across the partition boundary respectively. In term of eigenstates of \( \tilde{w}_A \) and \( \tilde{w}_B \) defined above, we obtain

\[
|u_p\rangle = \frac{1}{\sqrt{2^L}} \sum_{w_A = w_B = \{\pm 1\}} |w_A, w_B\rangle ,
\]

(A1)

if all boundary links have eigenvalues \( u_{a_n b_n} = 1 \). For more generic values of \( u_{a_n b_n} \), the only change are the signs of the terms in the right hand side of Eq. (A1), which does not enter the reduced density matrix we are interested in. Here \( w_{A(B)} = (w_{A(B), n}) \) and \( w_A = w_B \) means \( w_{A,n} = w_{B,n} \) for all \( n = 1, \cdots, L \).

In the new basis, the physical eigenstate is given by

\[
|\Psi\rangle = \frac{1}{\sqrt{2^{N+L+1}}} \sum_{g,w} D_g |u_A, w_A; u_B, w_B\rangle |\phi(u)\rangle ,
\]

(A2)

from which one can obtain the reduced density matrix \( \rho_A \) and compute \( \text{Tr} [\rho_A^2] \).

We define \( X_g = i^{(|g|+1)/2} \prod_{j \in g} \gamma^\alpha_j \gamma^\beta_j \) and \( Y_g = i^{(|g|+1)/2} \prod_{j \in g} \gamma^\alpha_j \), where \(|g|\) denotes the number of sites in \( g \) and the ordering of sites in the two products is implicitly taken to be the same such that \( X_g Y_g = D_g \). \( X_g \) and \( Y_g \) are the gauge transformation operators that only act on the gauge fields and Majorana fermions, respectively. We further denote \( X_g = (i)^{|g_A|(|g_B|)} X_{g_B \equiv g \cap B} X_{g_A \equiv g \cap A} \) and \( Y_g = i^{(|g_A|(|g_B|)-1)/2} \prod_{n \in A} \gamma^n_\alpha \gamma^n_\beta \) and \( Y_{g_A} = i^{(|g_A|(|g_B|)-1)/2} \prod_{n \in A} \gamma^n_\alpha \), are the gauge transformation operator acting on \( A \) region, and similar for \( X_{g_B} \) and \( Y_{g_B} \). The ground state (5) is written as

\[
|\Psi\rangle = \frac{1}{\sqrt{2^{N+L+1}}} \sum_{g,w} X_g |u_A, w_A; u_B, w_B\rangle \cdot Y_g |\phi(u)\rangle ,
\]

\[
= \frac{1}{\sqrt{2^{N+L+1}}} \sum_{g,w} X_{g_B} |u_B, w_B\rangle \cdot X_{g_A} |u_A, w_A\rangle \cdot Y_{g_A} Y_{g_B} |\phi(u)\rangle .
\]

(A3)

In the last line we have denoted \( w_A = w_B \) as \( w \). The reduced density matrix \( \rho_A \) is expressed as

\[
\rho_A = \text{Tr}_B \left[ |\Psi\rangle \langle \Psi| \right]
\]

\[
= \frac{1}{2^{N+L+1}} \sum_{g,w,w'} \langle u_B, w'| X_{g_B}^\dagger X_{g_B} |u_B, w\rangle X_{g_A} |u_A, w\rangle \langle u_A, w'| X_{g_A}^\dagger \cdot \text{Tr}_B \left[ Y_{g_A} Y_{g_B} |\phi(u)\rangle \langle \phi(u)| Y_{g_B}^\dagger Y_{g_A}^\dagger \right] .
\]

(A4)

To make the interproduct \( \langle u_B, w'| X_{g_B}^\dagger X_{g_B} |u_B, w\rangle \) nonzero, the gauge transformation \( X_{g_B} \) and \( X_{g_B}^\dagger \) must be either identical in \( B \) region, or different by a gauge transformation on all sites in the \( B \) region. We define \( X_B \) as \( X_g \) with \( g = B \), and similarly for \( X_A \) and \( Y_{A,B} \). Note that \( X_{g_B}^\dagger = (i)^{|g|} X_{g_B} \). So it is consistent to define \( B - g_B \) such that \( X_{B-g_B} X_{g_B} = X_B \) for any \( g_B \). Consequently, we have

\[
\langle u_B, w'| X_{g_B}^\dagger X_{g_B} |u_B, w\rangle = \delta_{w', w} \left( \delta_{g_B, g_B} + x_B(w) \delta_{g_B, g_B - g_B} \right)
\]

(A5)

with

\[
x_B(w) = \langle u_B, w| X_{B-g_B} X_{g_B} |u_B, w\rangle = \langle u_B, w| X_B |u_B, w\rangle = \prod_{ij \in B} \sum_{n=1}^L u_{ij} w_n .
\]

(A6)
Using this result $\rho_A$ can be simplified to

$$
\rho_A = \frac{1}{2^N+L+1} \sum_{g_A,g_A',w} \sum_{g_B} X_{g_A} |u_A,w\rangle \langle u_A,w| X^\dagger_{g_A'} \cdot Y_{g_A} \text{Tr}_{B,F} \left[ |\phi(u)\rangle \langle \phi(u)| \left( 1 + Y_B \right) \right] Y^\dagger_{g_A'},
$$

$$
= \frac{1}{2^N+L} \sum_{g_A,g_A',w} X_{g_A} |u_A,w\rangle \langle u_A,w| X^\dagger_{g_A'} \cdot Y_{g_A} \text{Tr}_{B,F} \left[ |\phi(u)\rangle \langle \phi(u)| \left( \frac{1 + x_B(w)Y_B}{2} \right) \right] Y^\dagger_{g_A}'.
$$

(A7)

Notice that $Y_B = i^{[B]/[B]-1/2} \prod_{j \in B} \eta_j$ is actually the fermion number parity operator in B region, we see that the operator $\frac{1 + x_B(w)Y_B}{2}$ is actually the projector of the fermion number parity $Y_B = x_B(w)$. For convenience, we define

$$
P^{xB}_{B,F}(w) = \frac{1 + x_B(w)Y_B}{2}
$$

(A8)

and similar for A region. The reduced density matrix

$$
\text{Tr}_{B,F} \left[ |\phi(u)\rangle \langle \phi(u)| P^{xB}_{B,F}(w) \right] = \rho^{xB}_{A,F}
$$

(A9)

is the fermion density matrix with restriction to the fixed fermion number parity $x_B(w)$ in the B region. By using similar interproduct formula (A5) for A region, $\rho_A$ can be computed as

$$
\rho^2_A = \frac{1}{2^N+2L-1} \sum_{g_A,g_A',w} X_{g_A} |u_A,w\rangle \langle u_A,w| X^\dagger_{g_A'} \cdot Y_{g_A} \rho^{xB}_{A,F} P^{xA}_{A,F} \rho^{xB}_{A,F} Y_{g_A'}
$$

(A10)

The same calculation can be repeated to obtain

$$
\rho^2_A = \frac{1}{2^{N_A+nL-n+1}} \sum_{g_A,g_A',w} X_{g_A} |u_A,w\rangle \langle u_A,w| X^\dagger_{g_A'} \cdot Y_{g_A} \rho^{xB}_{A,F} \left( P^{xA}_{A,F} \rho^{xB}_{A,F} \right)^{n-1} Y_{g_A'}
$$

(A11)

Finally we take the trace of $\rho^2_A$ by using the same Eq. (A5) to obtain

$$
\text{Tr}_{A} [\rho^2_A] = \frac{1}{2^{n(L-1)}} \sum_{w} \text{Tr}_{A} \left( P^{xA}_{A,F} \rho^{xB}_{A,F} \right)^n
$$

(A12)

For Eq. (A6), we see that flipping the sign of one link $w_i$, for some integer $1 \leq i \leq L$, changes the sign of both $x_A(w)$ and $x_B(w)$. If we define $\prod_{i \in A(B)} u_{ij} = p_{A(B)}$, then $x_{A(B)}(w) = p_{A(B)} \prod_{n=1}^{L} w_n$, and the summation over all $2^L$ configurations of $w = \{ w_i, i = 1, 2, ..., L \}$ leads to

$$
\text{Tr}_{A} [\rho^2_A] = \frac{1}{2^{(n-1)L-1}} \text{Tr}_{A} \left[ \left( P^{PA}_{A,F} \rho^{PB}_{A,F} \right)^n + \left( P^{-PA}_{A,F} \rho^{-PB}_{A,F} \right)^n \right]
$$

(A13)

In the main text we have discussed that the ground state $|u\rangle \langle \phi(u)|$ must satisfy the constraint $D |u\rangle \langle \phi(u)| = |u\rangle \langle \phi(u)|$. Since $D = X_A X_B Y_A Y_B$ and $X_A X_B |u\rangle = p_{APB} |u\rangle$, we obtain $Y_A Y_B |\phi(u)| = p_{APB} |\phi(u)|$. In other words, the total fermion parity is fixed, so that $P^{PA}_{A,F} P^{PB}_{A,F} |\phi(u)\rangle = P^{-PA}_{A,F} P^{-PB}_{A,F} |\phi(u)\rangle = 0$. Thus we have

$$
\rho^{PB}_{A,F} P^{-PA}_{A,F} = \text{Tr}_{B,F} \left[ |\phi(u)\rangle \langle \phi(u)| P^{PB}_{B,F} \right] P^{-PA}_{A,F} = 0
$$

(A14)

so that

$$
\text{Tr}_{A} [\rho^2_A] = \frac{1}{2^{(n-1)L-1}} \text{Tr}_{A} \left[ \left( P^{PA}_{A,F} \rho^{PB}_{A,F} \right)^n + \left( P^{-PA}_{A,F} \rho^{-PB}_{A,F} \right)^n \right],
$$

(A15)

where $\rho_{A,F} = \rho^+_A + \rho^-_A = \text{Tr}_{B} [ |\phi(u)\rangle \langle \phi(u)| ]$ is the free fermion density matrix without fermion number parity constraint in the B region.

To understand this result more intuitively, we note that

$$
\text{Tr}_{A,G} [\rho^2_{A,G}] = \frac{1}{2^{(n-1)L-1}},
$$

(A16)
where $\rho_{A,G} = \text{Tr}_B[(G(u))(G(u))]$ is the reduced density matrix of a pure $Z_2$ gauge field, and the ground state of the $Z_2$ gauge field $|G(u)\rangle$ is given by a equal weight superposition of all the $2^{N-1}$ gauge field configurations $|\tilde{u}\rangle$ that are gauge equivalent to $|u\rangle$, i.e., $|G(u)\rangle = 2^{-(N-1)/2}\sum_{\tilde{u}=u}|\tilde{u}\rangle$. It follows that Eq. (A15) can be written as

$$\text{Tr}_A[\rho^n_A] = \text{Tr}_{A,G}[\rho^n_{A,G}] \cdot \text{Tr}_{A,F}[\rho^n_{A,F}].$$

(A17)

Thus, Eq. (4) in the main text is proved.

**B: Renyi entropy and entanglement capacitance of Majorana fermions**

As shown in Ref. [31], the free fermion reduced density matrix always have the form

$$\rho = \exp \left[ -\sum_n \epsilon_n \gamma_n^\dagger \gamma_n \right] / \Omega$$

(B1)

with $\Omega = \prod_n (1 + e^{-\epsilon_n})$ the normalization constant and $\gamma_n$ fermion annihilation operators in the diagonal basis of the density matrix. For a system of Majorana fermions $\eta_i$, define the equal time correlation function in the ground state $C_{ij} = \langle \eta_i \eta_j \rangle / 2$ for $i, j$ restricted to the region $A$. The EE between two parts $A$ and $B$ is then given by

$$S_F = -\frac{1}{2} \text{Tr} \left[ C \log C + (1 - C) \log(1 - C) \right]$$

(B2)

with $C$ the matrix with entries of $C_{ij}$. The “single particle density matrix” $C$ plays the role of $(1 + e^{\beta h})^{-1}$ in the thermal entropy, with $h$ the single particle Hamiltonian and the inverse temperature $\beta = 1/T = 1$. The factor 1/2 comes from the fact that a Majorana fermion has half the degree of freedom of a Dirac fermion.

The eigenvalues $\lambda_n$ of the single particle density matrix $C$ in Eq. (B2) and $\epsilon_n$ in Eq. (B1) are related as

$$\lambda_n = \frac{1}{e^{\alpha \epsilon_n} + 1} \Rightarrow e^{-\epsilon_n} = \frac{\lambda_n}{1 - \lambda_n}.$$  

(B3)

Note that for Majorana fermion only $\epsilon_n \geq 0$ states are summed over in Eq. (B1). Using Eq. (B3) we have

$$\text{Tr} \rho^\alpha = \frac{1}{\Omega^\alpha} \prod_{\epsilon_n \geq 0} (1 + e^{-\alpha \epsilon_n})$$

(B4)

so that

$$S_\alpha = -\frac{1}{2} \frac{1}{1 - \alpha} \sum_n \left[ \log (1 + e^{-\alpha \epsilon_n}) - \alpha \log (1 + e^{-\epsilon_n}) \right],$$

$$= \frac{1}{2} \frac{1}{1 - \alpha} \sum_n \log [\lambda_n^\alpha + (1 - \lambda_n)^\alpha].$$

(B5)

The “capacity of entanglement” $C_E$ defined in Eq. (11) can be obtained by

$$C_E = \frac{t}{2} \frac{\partial^2}{\partial t^2} \left[ t \sum_n \log \left[ \lambda_n^{1/t} + (1 - \lambda_n)^{1/t} \right] \right],$$

$$= \frac{1}{2t^2} \sum_n \left( \frac{\epsilon_n}{2 \cosh (\epsilon_n/2t)} \right)^2.$$  

(B6)

It should be noticed that all the results above are for free Majorana fermions. For free complex fermions the only difference is an additional factor of 2.

In the torus case we studied, $k_y$ is a good quantum number, and we have

$$C_E = \frac{L}{2t^2} \sum_n \int \frac{dk_y}{2\pi} \frac{\epsilon_n(k_y)}{2 \cosh (\epsilon_n(k_y)/2t)}.$$  

(B7)
For the Abelian phase $\epsilon_n$ has a gap since $\lambda_n(k_y)$ does not cross $1/2$. If the gap is $E_g = \min(|\epsilon_n(k_y)|)$, the asymptotic behavior of $C_E$ at $t \to 0$ is $C_E \simeq e^{-E_g/t}/t^2$. For the non-Abelian phase, as shown in Fig. 2(b) there is a gapless branch of $\lambda_n(k_y)$ crossing $1/2$, which corresponds to $\epsilon_n(k_y)$ crossing $0$. Near $k_y = 0$ the asymptotic behavior of $|\epsilon_n(k_y)|$ is $|\epsilon_n(k_y)| \simeq v|k_y|$, which gives $C_E \propto t/v$. In the same way as the heat capacity in a thermodynamic system, $C_E/t$ is constant at $t \to 0$ limit and is proportional to the density of state $1/v$ at “maximally entangled limit” $\epsilon_n \to 0$ or $\lambda_n \to 1/2$. 

