Bi-Frequency Illumination: A Quantum-Enhanced Protocol

Mateo Casariego,* Yasser Omar, and Mikel Sanz

Quantum-enhanced, idler-free sensing protocol to measure the response of a target object to the frequency of a probe in a noisy and lossy scenario is proposed. In this protocol, a target with frequency-dependent reflectivity \( \eta(\omega) \) embedded in a thermal bath is considered. The aim is to estimate the parameter \( \lambda = \eta(\omega_2) - \eta(\omega_1) \), since it contains relevant information for different problems. For this, a bi-frequency quantum state is employed as the resource, since it is necessary to capture the relevant information about the parameter. Computing the quantum Fisher information \( H \) relative to the parameter \( \lambda \) in an assumed neighborhood of \( \lambda \approx 0 \) for a two-mode squeezed state \((H_Q)\), and a pair of coherent states \((H_C)\), a quantum enhancement is shown in the estimation of \( \lambda \). This quantum enhancement grows with the mean reflectivity of the probed object, and is noise-resilient. Explicit formulas are derived for the optimal observables, and an experimental scheme based on elementary quantum optical transformations is proposed. Furthermore, this work opens the way to applications in both radar and medical imaging, in particular in the microwave domain.

1. Introduction

Quantum information technologies are opening very promising prospects for faster computation, secured communications, and more precise detection and measuring systems, surpassing the capabilities and limits of classical information technologies.[1–5] Namely, in the domain of quantum sensing and metrology,[6] we are currently witnessing a boost of applications to a wide spectrum of physical problems: from gravimetry and geodesy,[7–11] gravitational waves,[12] clock synchronization,[5,13] thermometry[14] and bio-sensors,[15–19] to experimental proposals to seek quantum behavior in macroscopic gravity,[20] to name just a few.

While many of the quantum metrology studies that focus on unlossy and noiseless (unitary) scenarios, the more realistic, lossy case has also been investigated.[21–29] Equivalently, one can talk about quantum metrology with open quantum systems. Understanding what are the precision limits of measurements in the presence of loss is a fundamental endeavor in quantum metrology.[30,31] Certain noise properties have been found to be beneficial in some scenarios,[32,33] and quantum error correction schemes have been proposed to overcome decoherence and restore the quantum-enhancement.[34] Quantum illumination (QI)[35–45] is a particularly interesting example of a lossy and noisy protocol where the use of entanglement proves useful even in an entanglement-breaking scenario. QI shows that the detection of a low-reflectivity object in a noisy thermal environment with a low-intensity signal is enhanced when the signal is entangled to an idler that is kept for a future joint measurement with the reflected state. This makes QI a candidate for a quantum radar,[46] although a more involved protocol is needed.[47,48] The decision problem of whether there is an object or not can be rephrased as a quantum estimation of the object’s reflectivity.
Then, we compute the QFI and show the quantum enhancement. Finally, we compute the optimal observables for both the quantum and the classical probes, and briefly discuss applications.

2. Model and Fundamentals of Quantum Estimation Theory with Gaussian States

2.1. Physics of Gaussian States

When a quantum system has one or more degree of freedom described by operators with a continuous spectrum, we say that the system is a “continuous variable” (CV) system. Within the bosonic CV quantum systems, quantum Gaussian states are defined as the ones arising from Hamiltonians that are at most quadratic in the field operators, which we list in the vector $\mathbf{A} := (a_1, a_2, \ldots, a_N, a_1^\dagger, a_2^\dagger, \ldots, a_N^\dagger)$, where $N$ is the number of modes. This ordering of the creation and annihilation operators is commonly referred to as the “complex basis” or “complex form,” and allows for a compact way of writing down the commutation relations: $[\hat{A}_a, \hat{A}_b] = \delta_{ab} \mathbb{I}$, where $a, b = 1, \ldots, N$, $\mathbb{I}$ is the identity operator, and $\Sigma = \text{diag}(I_N, -I_N)$ is a diagonal matrix, $I_N$ being the $N \times N$ identity matrix.

Instead of having to resort to the infinite-dimensional density operator in order to describe a state, Gaussian systems are fully characterized by an $N$-vector called the displacement vector and an $N \times N$ matrix, the covariance matrix. We can construct the displacement vector

$$d := \text{Tr} \left[ \rho \mathbf{A} \right]$$

and the covariance matrix

$$\Sigma := \text{Tr} \left[ \rho (\Delta \mathbf{A} \Delta \mathbf{A}^\dagger) \right]$$

where $\rho$ is the density operator, $\{,\}$ denotes the anticommutator, and $\Delta \mathbf{A} := \mathbf{A} - d$. It is important to bear in mind that other choices of basis lead to different, but equivalent definitions. In fact, in the following sections we will start by writing down covariance matrices in the so-called “quadrature basis” $(\hat{x}_1, \ldots, \hat{x}_N, \hat{p}_1, \ldots, \hat{p}_N)$ with the canonical position and momentum operators defined by the choice $x_i = 2^{-1/2} \hat{x}_i, y_i = 2^{-1/2} \hat{p}_i$. A key result with important consequences in the context of Gaussian states is the normal mode decomposition, which follows the more general theorem due to Williamson that, from a physical point of view, establishes that any Gaussian Hamiltonian (i.e., quadratic) is equivalent — up to a unitary — to a set of free, non-coupled harmonic oscillators. This apparent simplicity of Gaussian states, however, has a rich structure when it comes to analyzing their Hilbert space properties, as well as information-theoretic quantities such as the quantum Fisher information, entropies, and so on. We can state the result in the following way: any positive-definite Hermitian matrix $\Sigma$ of size $2N \times 2N$ can be diagonalized with a symplectic matrix $S$: $\Sigma = SDS^\dagger$, where $D = \text{diag}(v_1, \ldots, v_N, v_1^\ast, \ldots, v_N^\ast)$ with $v_i$ the symplectic eigenvalues of $\Sigma$, that are the positive eigenvalues of matrix $\Sigma \Sigma^\dagger$. An important result for what follows is that a state is pure if and only if all the symplectic eigenvalues are one: $v_i = 1$ $\forall i$, and $v_i \geq 1$ for any Gaussian state.
2.2. Quantum Estimation

Quantum metrology is so related to quantum estimation that sometimes the two terms are used as synonyms. Incidentally, quantum sensing could be seen as a quantum estimation or metrology problem that deals with a binary question: is the true parameter value localized into one interval or the other? Hence its name: the true parameter value is localized into some interval rather than completely unknown (in this case, the estimation is called global). In the local approach, the QFI matrix emerges as the figure of merit for the quantification of the maximum amount of information one can extract from the system.

While classical parameter estimation deals only with the statistics of measurement outcomes, and answers questions of attainability in the presence of statistical noise (with various properties of measurement outcomes), and answers questions of attainability in the presence of statistical noise (with various properties of measurement outcomes), quantum estimation addresses the problem of what to measure, and imposes additional limits to the precision due to the fundamental probabilistic nature of quantum mechanics. Indeed, the quantum Fisher information (QFI) matrix, can be seen as an optimization of the classical Fisher information—a measure for the amount of information relative to a set of parameters $\lambda$ a system contains—over all possible measurements, or POVMs.

The QFI can be interpreted geometrically by means of a notion of distance in the Hilbert space spanned by density operators. Among the many candidates, the Bures distance

$$D_B^2(\rho_1, \rho_2) := 2 \left( 1 - \sqrt{F(\rho_1, \rho_2)} \right)$$

where $F(\rho_1, \rho_2) := \langle \sqrt{\rho_1} \rho_2 \sqrt{\rho_1} \rangle^2$ is the Uhlmann fidelity between states $\rho_1$ and $\rho_2$, the one correctly linking estimation to geometry. This makes the interpretation of quantum estimation straightforward: it depends upon the distinguishability between states. If $\lambda$ is a vector of parameters that defines a (possibly continuous) family of states $\{\rho_\lambda\}$, then the Bures distance between two infinitesimally close states can be related to a metric tensor, which is no other than the QFI matrix:

$$D_B^2(\rho_\lambda, \rho_{\lambda+\delta\lambda}) = \frac{H(\lambda)}{4}$$

A large QFI translates in a large distinguishability between states. In this paper we will focus on the single parameter case, for which the QFI is a scalar that can be computed using the following basis-dependent formula

$$H(\lambda) = 2 \sum_{m,n} \frac{|\langle \Phi_m | \partial_x \rho_\lambda | \Phi_n \rangle|^2}{\rho_m + \rho_n}$$

where $\{\rho_m | \Phi_n\}$ are the eigensolutions to $\partial_x | \Phi_n\rangle = \rho_m | \Phi_n\rangle$, and $\rho_\lambda$ is the measured, or received state. Moreover, the theory also provides a way of finding an optimal observable, whose outcomes allow us to construct an estimator that

$$\hat{O}_\lambda = \lambda \mathbb{1} + \tilde{L}_\lambda \frac{H(\lambda)}{M}$$

where $\tilde{L}_\lambda$ is a symmetric logarithmic derivative (SLD) that solves the equation $\{ \tilde{L}_\lambda, \rho_\lambda \} = 2 \delta_{\lambda, \rho_\lambda}$, where $\{, \}$ is the anticommutator. When the estimator $\hat{\lambda}$ is constructed using a maximum likelihood method, the so-called quantum Cramér-Rao bound (qCRB)\cite{85,86} is asymptotically achieved, meaning that the observable in Equation (6) has the smallest possible variance:

$$\text{var}(\hat{O}_\lambda) \geq \frac{1}{M H(\lambda)}$$

2.2.1. Gaussian Quantum Estimation

As shown in ref. [60], when we are in the presence of Gaussian states and Gaussian-preserving channels, there is no need to diagonalize the density matrix in Equation (5) in order to find the QFI. For a single parameter, the QFI can be computed using

$$H(\lambda) = \frac{1}{2 \text{det}[A]} \left[ \text{Tr} \left[ (A^{-1} \partial_\lambda A)^2 \right] + \sqrt{\text{det}[I_2 + A^2]} \text{Tr} \left[ (I_2 + A^2)^{-1} \partial_\lambda A^2 \right] \right]$$

$$- 4 \left( v^2 - v^2 \right) \left( \frac{\partial_\lambda v}{v^4 - 1} - 1 \right) + 2 \partial_\lambda d^\dagger \Sigma \partial_\lambda d^\dagger$$

where the dot over $A$ and $\partial_\lambda$ denotes derivative with respect to $\lambda$, and $v$ are the symplectic eigenvalues of $\Sigma$, defined following ref. [61].

$$2 v^2 := \text{Tr}[A^2] \pm \sqrt{\text{Tr}[A^2]^2 - 16 \text{det}[A]}$$

with the matrix $A$ given by $A := i \Omega \Sigma \tau_T$, $\Omega := \text{antidiag}(I_2, -I_2)$, and $\tau_T := \delta_{+4,2} + \delta_{-2,-4} - 1$ is the matrix that changes the basis to the quadrature basis $\{ x_1^\dagger, x_2^\dagger, x_1^\dagger, x_2^\dagger, y_1^\dagger, y_2^\dagger, y_1^\dagger, y_2^\dagger \}$. For a Gaussian
state (Σ, \tilde{d}) written in the complex basis, the symmetric logarithmic derivative in Equation (6) can be obtained as in ref. [60]:

\[ \hat{L}_i = \Delta \hat{A}_i A_i \hat{A} - \text{Tr}[\Sigma A_i] / 2 + 2 \Delta \hat{A} \Sigma A_i / \hat{d}_i \]  

(10)

where \( \Delta \hat{A} := \hat{A} - \hat{d} \), \( \hat{A} \) the complex basis vector of bosonic operators, \( A_i := M^{-1} \hat{d}_i \), where \( M = \Sigma \otimes \Sigma - K \otimes K \), where the bar denotes complex conjugate, and \( K := \text{diag}(1, -1) \). Note that when \( \lambda \rightarrow 0 \) we have \( \hat{O}_{\omega} \equiv \hat{O} = \hat{L}_{\omega=0} / H(\lambda = 0) \), since both limits exist independently. This limit is of our interest because we will work in a neighborhood of \( \lambda \approx 0 \), that is, the measured value of the parameter is expected to be small (i.e. we shall adopt a local estimation strategy).

2.3. Model

The model is synthesized in Figure 1, the target object, modeled as a beam splitter with a frequency-dependent reflectivity is subject to an illumination with a bi-frequency probe. The transmitted signal is lost, and only the reflected part is collected for measurement. For a single frequency, a beam splitter is characterized by a unitary operator

\[ U(\omega) \equiv \exp \left[ \arcsin \left( \sqrt{\eta(\omega)} \right) (\hat{b}^\dagger \hat{b}^\dagger e^{i\varphi} \hat{d}_\omega \hat{b} e^{-i\varphi}) \right] \]  

(11)

where \( \eta(\omega) \) is a frequency-dependent reflectivity, related to transmittivity \( \tau \), we shall adopt an estimator of the parameter \( \lambda \). In this case, \( \lambda \) is expected to be small (i.e. we shall adopt a local estimation approach). The above limit is formulated in Equation (11). The equal thermal photon number is an approximation as long as the frequency difference \( \Delta \omega \equiv \omega_2 - \omega_1 \) is sufficiently small. To make this statement more quantitative, let us assume two different thermal photon densities, \( N_1 \) and \( N_2 \). The Bose–Einstein distribution for photons is \( N_\omega \propto 1 / (e^{\omega/k_B T} - 1) \) where \( \omega \equiv \hbar / k_B T \) is a function of the temperature \( T \). Then,

\[ \frac{N_1}{N_2} = \frac{e^{\omega_1} - 1}{e^{\omega_2} - 1} = \frac{1}{1 + \beta \Delta \omega / \omega_1} \]  

(13)

we see that up to first order in \( \beta \Delta \omega / \omega_1 \), the last expression reduces to \( 1 - \Delta \omega / \omega_1 \). This means that \( N_1 \approx N_2 \), if \( \Delta \omega / \omega_1 \ll 1 \). In particular, for \( T = 300 \) K and \( \omega_1 / 2\pi = 5 \) GHz the expected thermal photon number is roughly 1250. It is straightforward to check that for these frequencies and temperatures, the above approximations are good (that is, \( \approx 4\% \) of relative error) for frequency differences up to 20%.

Because we are working within the local estimation approach and our goal is to find observables that saturate the qCRB, we shall take the true value of \( \lambda \) to be exactly zero. This means that the goal of the protocol is to increase one’s confidence about this initial ansatz of the parameter being zero, and be able to tell when it is close but not exactly zero. Hence, we work in a neighborhood of \( \lambda \approx 0 \)—which can be implemented by taking the limit \( \lambda \rightarrow 0 \) in the derived expressions. Moreover, this relies on a physical assumption, since we are interested in probing regions of \( \eta(\omega) \) that do not change drastically, that are well approximated by a linear function with either no slope or a small one. In this sense, the protocol is a quantum sensing one, since we are interested in an-
svering the question of whether the parameter either vanishes or is small.

It is also worth discussing briefly the effect of absorption loss due to the medium through which the signal travels. These can be accommodated in the model by means of an additional beam splitter. The medium through which the signal travels can be seen as an array of infinitesimal beam splitters, each of which having the same reflectivity, and mixing some incoming signal with the same thermal state. For a travel distance $L$, the flying mode will see a reflectivity

$$\eta_{\text{abs}} = 1 - e^{-\mu L}$$

(14)

where $\mu$ is a parameter characterizing the photon-loss of the medium. A concatenation of beam splitters can be easily put into a single one, as long as they are embedded in the same environment, which is our case. For beam splitters of transmittivities $\tau_1$ and $\tau_2$ their combined resulting transmittivity is simply the product: $\tau = \tau_1 \tau_2$. Thus, accommodating absorption losses into our model is trivially obtained by the transformation $\tau \mapsto e^{-\mu L} \tau$. Since the QFI deals with derivatives with respect to the parameter to be estimated, and ultimately we are interested in QFI ratios between a quantum protocol and its classical counterpart, the above transformation will not affect the overall results, since multiplicative factors will cancel out.

3. Results: Quantum Fisher Information

In this section we compute the QFI for two different probes: an entangled two-mode squeezed (TMS) state, and a pair of coherent beams. The choice of the TMS state over other possible entangled states is motivated by the fact that these are customarily produced in labs, both in optical—for example, with non-linear crystals, and in microwave frequencies—using Josephson parametric amplifiers (JPAs).

3.1. Two-Mode Squeezed Vacuum State

The TMS vacuum (TMSV) state is the continuous-variable equivalent of the Bell state, being the Gaussian state that optimally transforms classical resources (light, or photons) into quantum correlations. The TMSV state is a cornerstone in experiments with quantum microwaves.[87–91] In our case, we are interested in states produced via nondegenerate parametric amplification, in order to have two distinguishable frequencies. The state can be formally written as: $|\psi\rangle_{\text{TMSV}} := (\cosh \eta)^{-1/2} \sum_n (-e^{i\theta} \tanh \eta)^n |n, n\rangle$, where $r \in \mathbb{R}_{>0}$ is the squeezing parameter. For simplicity we take $\phi = 0$. In any realistic application, the TMSV state should be replaced by a TMS thermal state, which can be defined as the one obtained by applying the two-mode squeezing operation to a pair of uncorrelated thermal states $\rho_{\text{in},1}$ and $\rho_{\text{in},2}$ with mean thermal photon numbers $n_1$ and $n_2$, respectively, and hence resulting in a mixed state.[92]

The expected total photon number in these states is given by $N_{\text{TMS}} = (N_1 + N_2) = n_1 + n_2$ and $N_{\text{TMS}} = 2(n_1 + n_2)$, where $N_1 = \sum_i n_i^2$ for $i = 1, 2$. Typically, one has $n_1 = n_2 \equiv n$, which gives us a symmetric TMST state. In this case we define the photon number $N_1$ as the photon number in each of the modes, $N_1 = N_{\text{TMS}}/2 = n(1 + 2N_r) + N_r$, where $N_r = \sinh^2 r/2$.[93,94] In microwaves, a squeezing level $S = -10 \log_{10} [(1 + 2n) \exp(-2r)]$ of $9.1 \text{ dB}$ has been reported [95] for $n = 0.34$ and $r \approx 1.3$, using JPAs operating at roughly $5 \text{ GHz}$ with a filter bandwidth of $430 \text{ kHz}$. This corresponds to $N_r \approx 8$.

The total initial (real) covariance matrix—written in the real basis $(\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2)$—is given by

$$\Sigma = N \begin{pmatrix} N^{-1} \Sigma_{\text{th}} & 0 & 0 & 0 \\ 0 & \Sigma & 0 & \epsilon_x \\ 0 & 0 & N^{-1} \Sigma_{\text{th}} & 0 \\ 0 & \epsilon_x^\dagger & 0 & \Sigma \end{pmatrix}$$

(15)

where $N \equiv 1 + 2n$, $\Sigma_{\text{th}} = (1 + 2N_r) \rho_{\text{th}}(1 + 2N_r)$, is the real covariance matrix of a thermal state, $\Sigma = \cosh(2\lambda) \rho_{\text{th}}$ corresponds to the diagonal part of one of the modes in a TMSV state, and $\epsilon_x = \sinh(2\lambda) \sigma_z$ is the correlation between the two modes, where $\sigma_z$ is the Z Pauli matrix. Note that the covariance matrix of the thermal TMS state is simply $N$ times the one of the TMSV state.

The displacement vector of a TMS state is identically zero $d_{\text{TMS}} = 0$, so the last term of Equation (8) vanishes. Under the assumption that the object does not entangle the two modes, we have that the symplectic transformation is $S(\eta_1, \eta_2) = S_{\text{RS}}(\eta_1) \oplus S_{\text{RS}}(\eta_2)$,[96] where

$$S_{\text{RS}}(x) = \begin{pmatrix} \sqrt{x_1} & \sqrt{1 - x_1} \\ -\sqrt{1 - x_1} & \sqrt{x_1} \end{pmatrix}$$

(16)

is the real symplectic transformation associated with a beam splitter of reflectivity $x$. We define the parameter of interest as $\lambda \equiv \eta_1 - \eta_2$. With this, $S(\eta_1, \eta_2)$ becomes a function of $\lambda$. For simplicity, we define $S_\lambda := S(\eta_1, \eta_1 + \lambda)$. The full state after the signals get mixed with the thermal noise is given by $\Sigma = S_{\text{RS}}(\lambda) \Sigma_{\text{th}} S_{\text{RS}}(\lambda)^\dagger$. In covariance matrix formalism, partial traces are implemented by removing the corresponding rows and columns:[57] in our case the rows and columns 1, 2, 5, and 6. The resulting received covariance matrix reads as follows

$$\Sigma_\lambda = \begin{pmatrix} a & b \sigma_z \\ b \sigma_z & c \end{pmatrix}$$

(17)

with $a \equiv 1 + 2N_{\text{th}} + 2n_1 (2N_r + 4N_r - N_{\text{th}})$, $b \equiv 2(1 + 2n) \sqrt{2N_r} (\eta_1 + \lambda) (2N_r + 1)$, and $c \equiv (1 + 2n) (1 + 4N_r + 4\eta_1 (4N_r - 2N_{\text{th}}) + 2(1 - \lambda) N_{\text{th}})$.

For this state, the symplectic eigenvalues $\nu_\lambda$ defined in Equation (9) are strictly larger than one for any value of the parameters $n$, $N_r$, $N_{\text{th}}$, and $\eta_2$, other than $\eta_1 = 1 \land N_{\text{th}} = 0$, so there is no need of any regularization scheme.[61] Indeed, this is due to the mixedness of the received state: regularization is only needed for pure states.

We obtain the function $H_\lambda(\lambda)$ from Equation (8), and compute the two-sided limit $H_\lambda \equiv \lim_{\lambda \rightarrow 0} H_\lambda(\lambda)$ when the parameter
\(\lambda\) goes to zero, finding
\[
H_Q = \kappa [\eta_1 (N_{th}(4nN_r + n + 4N_r - 2N_{th}) - 2\eta_1 (2(n + 1)N_r N_{th}
+ N_{th}(n - N_{th})) + \beta (\eta_1 N_{th}(4nN_r + n + 4N_r - 2N_{th})
- 2\eta_1 (2(n + 1)N_r N_{th} N_{th} + N_{th} (n - N_{th}))) + \beta + 1]
\]
(18)
where
\[
\kappa^{-1} \equiv \eta_1^2 [2\eta_1 (N_{th}N_r (8(n + 1)N_r^2 + 6nN_r + n) - 4N_r)
+ 2N_{th}^2 (4nN_r + n + 6N_r) - 2N_{th}N_r (2(6n + 5)N_r + 2n - 1)
- 2N_{th}^2) - 2\eta_1 (-N_{th}(4n + 1)N_r + n) + N_r ((4n + 2)N_r - 1)
+ N_{th}^2 N_r (2(n + 1)N_r N_{th} + N_r (n - N_{th}))) + 2n(2N_r + 1)N_r N_{th}^2
+ 4N_r^2 (6N_{th}N_r (n + 1) + 1) - 4N_r N_{th}^2 (4N_r + 3) N_{th}^2 + N_{th}^2 N_{th}^2]
\]
(19)
and \(N_{th} \equiv 1 + 2N_{th}\), \(\eta \equiv 1 + 2n\), and \(\beta \equiv nN_{th}^2 + 2N_{th} N_r (n + 1)\).

3.2. Coherent States

Here we use a pair of coherent states as probe: \(|\psi\rangle = |\alpha\rangle \otimes |\alpha\rangle\). The total expected photon number in this state is \(2N_C := 2|\alpha|^2\). For simplicity we take \(\alpha \in \mathbb{R}\). Moreover, since we will compare with the TMST state, we set \(\alpha^2 = n(1 + 2N_r) + N_r\). The initial covariance matrix is simply given by the direct sum of two identity matrices (corresponding to each of the coherent states), and two thermal states. After the interaction and the losses, the measured covariance matrix is
\[
\Sigma = \begin{pmatrix} d & 0 \\ 0 & f \end{pmatrix}
\]
(20)
where \(d = 1 + 2N_r \tau_1\), \(f = 1 + 2N_{th} (\tau_2 - \lambda)\).

The initial displacement vector in the real basis is \(d_0 = (0, 0, \sqrt{2}a, 0, 0, 0, \sqrt{2}a, 0)\) which leads after the interaction and the trace of the losses, to \(d' = \alpha (\sqrt{2}\eta_1, 0, \sqrt{2}N_r, 0)\). The symplectic eigenvalues are also larger than one here. Inserting these in Equation (8), and taking the limit \(\lambda \to 0\), we find that the QFI for the coherent state is
\[
H_C = \frac{4N_{th}^2 (1 + 2N_r \tau_1)^2 + 1}{(1 + 2N_r \tau_1)^3 - 1} + \frac{n(1 + 2N_r) + N_r}{\eta_1 (1 + 2N_r \tau_1)}
\]
(21)
where \(\tau_1 = 1 - \eta_1\) is the transmittivity. Having computed both the quantum and the classical QFIs, in the next section we analyze their ratio \(H_Q / H_C\), a quantifier for the quantum enhancement.

3.3. Comparison: Quantum Enhancement

We analyze the ratio between the TMST state’s QFI \((H_Q)\) and the coherent pair’s QFI \((H_C)\) for different situations. As a first approximation and to simplify the discussion, we take the limit where \(n \to 0\), which corresponds to a TMSV state input. Finding values of \((\eta_1, N_{th}, N_r)\) such that the ratio \(H_Q / H_C\) is larger than one means that one can extract more information about parameter \(\lambda\) using a TMST state than using a coherent pair, provided an optimal measurement is performed in both cases. In Figure 2 we plot the results for various values of \(\eta_1\). We can immediately see that the ratio gets larger for large values of \(\eta_1\), that is, for highly reflective materials. In particular, we find the high-reflectivity limit the ratio converges even when the individual QFIs do not (since they correspond to a pure state being transmitted):
\[
\lim_{\eta_1 \to 1} \frac{H_Q}{H_C} = \frac{N_{th}^2 (8N_{th} (N_{th} + 1) + 4) + 4N_r^2 N_{th}^2 + N_{th}^2}{N_{th} (N_{th} (4N_{th} + 2) N_{th})}
\]
(22)
which converges to \(1 + 8N_r^2 / (4N_r + 1)\) in the highly noisy scenario \(N_{th} \gg 1\). Using a squeezing of \(r \approx 1.3\) which is experimentally realistic for microwave quantum states, and that corresponds to an expected photon number of \(N_r \approx 2.9\), we expect to find a quantum-enhancement of roughly a factor of six, that is, \(H_Q / H_C \approx 6.4\) in the highly reflective limit.

In the next section we explicitly compute the observables that lead to an optimal extraction of \(\lambda\)’s value for both the classical and the quantum probes.

4. Optimal Observables

Here we address the question of how to extract the maximum information about parameter \(\lambda\) for each of the probes. The theory provides us with explicit ways to compute an optimal POVM, which albeit not unique, provides us with an optimal measurement strategy: upon measuring the outcomes and possibly after some classical data-processing, the results asymptotically tend toward the true value of the parameter to be estimated.

4.1. Optimal Observable for the TMSV State Probe

Computing the SLD in Equation (8) and inserting it in Equation (6) we find
\[
\hat{O}_Q = L_{11} \hat{a}_1^\dagger \hat{a}_1 + L_{22} \hat{a}_2^\dagger \hat{a}_2 + L_{12} (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + L_{0} \hat{1}_{12}
\]
(23)
where the general expressions for the coefficients can be found in Appendix A. The variance of this operator is found to be \(\text{var}(\hat{O}_Q) = 2N_r^2 L_{12} (1 + N_r)\). We can numerically test the validity of the qCR bound for this observable by examining the bound itself for the extreme choice of \(M = 1\). The saturation of the bound produces the following relation:
\[
\text{var}(\hat{O}_Q) H_Q = 1
\]
(24)
Now, as the left hand side a function of \((N_r, N_{th}, \eta_1)\), we can give different values to the reflectivity and find the limiting condition between \(N_r\) and \(N_{th}\), which is depicted in Figure 3. Naturally, the larger \(M\), the better results we can achieve, but \(M = 1\) proves the existence of a choice of parameters for which the bound is saturated.
is implementing photon-counting on the operator \( \hat{\nu} \) in the optimal observable \( \hat{\eta} \equiv \hat{a}^\dagger a + \hat{a}^\dagger \hat{a} \) for the extreme case of just one experimental run \((M = 1)\). As the reflectivity grows, we observe an interesting behavior: the best choice of \( N_{\text{th}} \)—defined as the one that saturates the bound while keeping \( N_2 \) as low as possible—is actually non-vanishing.

Moreover, it is illustrative to study a possible implementation of the noiseless case, since this captures the essence of what is being measured. When \( N_{\text{th}} \rightarrow 0 \) we have that \( \hat{O}^{\text{lim}}_{\text{Q}} = -\mu^2 \hat{a}^\dagger \hat{a}_1 - \hat{a}^\dagger_1 \hat{a}_2 + \mu (\hat{a}^\dagger_1 \hat{a}_1 + \hat{a}_1 \hat{a}_2) - \nu 1_{12} \) where \( \mu^2 \equiv (1 + 1/2N_2) \) and \( \nu \equiv (1 + 1/4N_2) \), and we have taken the limit of vanishing \( N_{\text{th}} \). Notice that we can rewrite this observable as \( \hat{b}_1 \hat{b}_1^\dagger - 1 \), that is, implementing photon-counting on the operator \( b_1 \equiv -i(\hat{a}_1^\dagger - \mu \hat{a}_1) \). This is achieved by means of the transformations captured in Figure 4.

Following that scheme, we have that after the first beam splitter

\[
\hat{a}_1' = \hat{a}_1 \cos \varphi + \hat{a}_2 \sin \varphi \\
\hat{a}_2' = -\hat{a}_1 \sin \varphi + \hat{a}_2 \cos \varphi
\]

(25)

then the Josephson parametric amplifiers (JPA)—ideally squeezing operators—produce \( \hat{a}'' = S(r, \theta) \hat{a}' S(r, \theta) = \hat{a}' \cos r_i - e^{i\theta} \hat{a}'^\dagger \sin r_i \)

\[
\hat{a}'' = S(r, \theta) \hat{a}' S(r, \theta) = \hat{a}' \cos r_i - e^{i\theta} \hat{a}'^\dagger \sin r_i.
\]

(26)

Assuming that the phase shifter \( \phi \) acts as \( \hat{c} \rightarrow e^{-i\phi} \hat{c} \) we find the following output modes

\[
e^{i\phi} \hat{b}_1 = \cos \theta (\hat{a}_1' \cos r_i - e^{i\theta} \hat{a}_1'' \sin r_i) + \sin \theta (\hat{a}_2' \cos r_i - e^{i\theta} \hat{a}_2'' \sin r_i) \\
\hat{b}_2 = -\sin \theta (\hat{a}_1' \cos r_i - e^{i\theta} \hat{a}_1'' \sin r_i) + \cos \theta (\hat{a}_2' \cos r_i - e^{i\theta} \hat{a}_2'' \sin r_i).
\]

(27)

We insert Equation (25) in the last expression and regroup, finding

\[
e^{i\phi} \hat{b}_1 = \hat{a}_1 (\cos \theta \cos \varphi \cos r_i - \sin \theta \sin \varphi \cos r_i) \\
+ \hat{a}_2 (\cos \theta \sin \varphi \cos r_i + \sin \theta \cos \varphi \cos r_i) \\
+ \hat{a}_1' (e^{-i\theta} \cos \theta \cos \varphi \sin r_i + e^{i\theta} \sin \theta \sin \varphi \sin r_i) \\
+ \hat{a}_2' (e^{-i\theta} \cos \theta \sin \varphi \sin r_i - e^{i\theta} \sin \theta \cos \varphi \sin r_i)
\]

(28)
Because we want to perform photon-counting over the operator $\hat{b}_1 \equiv -i(\hat{a}_2^\dagger - \mu \hat{a}_2)$, we identify:

$$i\mu = \cos \theta \cos \varphi \cosh r_1 - \sin \theta \sin \varphi \cosh r_2$$

$$i = e^{i\varphi} \cos \theta \sinh r_1 + e^{i\varphi} \sin \theta \cosh r_2.$$  

### 4.2. Optimal Observable for the Coherent State Probe

The optimal observable in this case is given by $\hat{O}_C = A1_{(1)} \otimes [(\hat{a}_1^\dagger - \eta_1 \sqrt{\alpha})(\hat{a}_2 - \eta_1 \sqrt{\alpha}) + \frac{1}{2}]$, where $A = 1/\eta_1 (1 - N_1 (\eta_1 - 1))$, and $1_{(1)}$ is the absence of active measurement of mode 1. This expression can then be put as $\hat{O}_C = A1_{(1)} \otimes [(\hat{a}_1^\dagger - \eta_1 \sqrt{\alpha}) \hat{a}_2 - \eta_1 \sqrt{\alpha} + 1/2]$. This operator can be experimentally performed with a displacement $D(-\eta_1 \sqrt{\alpha})$ and photon-counting in the resulting mode. The interpretation is simple: because $\eta_1$ is known (it serves as a reference), there is nothing to be gained by measuring the first mode in the absence of entanglement. Moreover, the observable is separable, as one should expect, and the experimental implementation is straightforward: photon-counting in the—locally displaced—second mode.

We have seen that both quantum and classical observables are non-Gaussian measurements, since they can be related to photon-counting, as expected in order to obtain quantum enhancement. Current photon counters in microwave technologies can resolve up to three photons with an efficiency of 96%. Inefficiencies in the photon-counters can be accounted for with a simple model of an additional beam splitter that mixes the signal with either a vacuum or a low-temperature thermal state. Additionally, the fact that real digital filters are not perfectly sharp should also be accounted for in a full experimental proposal, which we leave for future work.

### 5. Conclusions

We have proposed a novel protocol for achieving a quantum enhancement in the decision problem of whether a target’s reflectivity depends or not on the frequency, using a bi-frequency, entangled probe, in the presence of noise and losses. Crucially, our protocol needs no idler mode, avoiding the necessity of coherently storing a quantum state in a memory. The scaling of the quantum Fisher information (QFI) associated to the estimation problem for the entangled probe is faster than in the case of a coherent signal. This quantum enhancement is more significant in the high reflectivity regime. Moreover, we have derived analytic expressions for the optimal observables, which allow extraction of the maximum available information about the parameter of interest, sketching an implementation with quantum microwaves.

This information can be related to the electromagnetic response of a reflective object to changes in frequency, and, consequently, the protocol can be applied to a wide spectrum of situations. Although the results are general, we suggest two applications within quantum microwave technology: radar physics, motivated by the atmospheric transparency window in the microwaves regime, together with the naturally noisy character of open-air[71,98,100–103] and quantum-enhanced microwave medical contrast-imaging of low penetration depth tissues, motivated not only by the non-ionizing nature of these frequencies, but also because resorting to methods that increase the precision and/or resolution without increasing the intensity of radiation is crucial in order not to heat the sample.

Our work paves the way for extensions of the protocol to accommodate both thermal effects in the input modes, and continuous-variable frequency entanglement[104], where a more realistic model for a beam containing a given distribution of frequencies could be used instead of sharp, ideal bi-frequency states. It also serves as reminder that quantum enhancement provided by entanglement can survive noisy, lossy channels.

### Appendix A: Coefficients for the Optimal Quantum Observable

Here we give the general expressions of the coefficients of the optimal observable for the TMS state:

$$\hat{O}_Q = L_{11} \hat{b}_1^\dagger \hat{b}_1 + L_{22} \hat{b}_2^\dagger \hat{b}_2 + L_{12} \left( \hat{b}_1^\dagger \hat{b}_2^\dagger + \hat{b}_1 \hat{b}_2 \right) + L_0 1_{12}$$  

(A1)
\[ L_{11} = \frac{2N_2(2N_2+1)(2N_2+3)}{A+B+C+D} \]

\[ L_{22} = \frac{4N_1(2N_1-1)N_2(N_2+1) + 2N_1 N_2 (N_2+1)(N_2-1)}{A-B+C-D} \]

\[ L_{12} = \frac{\sqrt{2} \sqrt{N_2(N_2+1)}(\eta_1^2 N_2(N_2+2) - N_{th}^2) + N_2(N_2+1)}{A+B+C-D} \]

\[ L_0 = \frac{\eta_0^2 (N_{th}^2 - 2N_2(2N_2+1))^2 - 2\eta_0^2 (2N_2+5) - 4N_1(N_1 N_2(N_2+1)+2N_2(N_2+1)+3)+1}{E - 4\eta_1(2N_1+1)(-4N_2 N_2 + N_2(2N_2-1) - N_{th}^2) - 8N_2^2(N_2(N_2+1)+1) + 4N_2 N_2(N_2(N_2+3)+1) - 2N_{th}^2 G} \]

\[(A2)\]

where

\[ A \equiv 8(\eta_1 - 1)N_2(N_2+1) \]

\[ B \equiv 4N_2^2 (-\eta_1 + (\eta_1 + 3N_2) \eta_2)^2 - \eta_1 N_2(10N_2+7) + 3N_2(N_2+1) + 1 \]

\[ C \equiv 2N_2 N_2(-\eta_1 + N_2(\eta_1(3\eta_2 - 8) + 4(\eta_1 - 1)(2\eta_1 - 1)N_2 + 3)+1) \]

\[ D \equiv N_{th}^2 (2\eta_1 - 1)N_2(\eta_1 - 1)N_2(N_2 - 1) + 1 \]

\[ E \equiv 4\eta_1^2 (-4N_2 N_2 + N_2(2N_2 - 1) + N_{th}^2) (N_2(4N_2 + 2) - N_{th}^2) \]

\[ F \equiv 2N_2 - N_2 \]

\[ G \equiv 2N_2(N_2 + 1) + 1 \]

In the high reflectivity case \( \eta_1 \rightarrow 1 \) we find:

\[ \lim_{\eta_1 \rightarrow 1} L_{11} = -\frac{2N_2(2N_2+1)(2N_2+3)}{N_2^2(8N_2(N_2+1) + 4) + 4N_2 N_{th}^2 + N_{th}^2} \]

\[ \lim_{\eta_1 \rightarrow 1} L_{22} = -\frac{4N_2(2N_2 N_2 + N_2 + N_2 + N_{th}) + N_{th}^2}{N_2^2(8N_2(N_2+1) + 4) + 4N_2 N_{th}^2 + N_{th}^2} \]

\[ \lim_{\eta_1 \rightarrow 1} L_{12} = 2\sqrt{\eta_2(N_2 + 1)}(N_2(4N_2 + 2) + N_2) \]

\[ \lim_{\eta_1 \rightarrow 1} L_0 = \frac{-2N_2(2N_2(4N_2 + 2) + N_2) + 4N_2 N_{th}^2 + 2N_{th}^2}{8N_2^2(2N_2(N_2+1) + 1) + 8N_2 N_{th}^2 + 2N_{th}^2} \]

Additionally, as shown in the main text, in the noiseless case we get

\[ \mathcal{O}^Q_{\eta_0} \equiv \lim_{N_{th} \rightarrow 0} O_{\eta_0} \equiv -\mu^2 \hat{a}_1 \hat{a}_1 - \hat{a}_1 \hat{a}_2 + \mu (\hat{a}_1 \hat{a}_2 + \hat{a}_1 \hat{a}_2) - \nu \hat{1}_2 \]

\[ (A5) \]

where \( \mu^2 \equiv (1 + 1/2N_2) \) and \( \nu \equiv (1 + 1/4N_2) \) and \( \hat{1}_2 \equiv -i(\hat{a}_2^* - \mu \hat{a}_1) \).

**Conflict of Interest**

The authors declare no conflict of interest.

**Data Availability Statement**

The data that support the findings of this study are available from the corresponding author upon reasonable request.

**Keywords**

entanglement, quantum advantage, quantum enhancement, quantum illumination, quantum microwaves, quantum parameter estimation, quantum sensing

Received: March 30, 2021  
Revised: September 6, 2022  
Published online: October 13, 2022

[1] M. A. Nielsen, I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition*, Cambridge University Press, Cambridge 2010.

[2] F. Arute, K. Arya, R. Babbush, D. Bacon, J. C. Bardin, R. Barends, R. Biswas, S. Boixo, F. G. S. L. Brandao, D. A. Buell, B. Burkett, Y. Chen, Z. Chen, B. Chiaro, R. Collins, W. Courtney, A. Dunsword, E. Farhi, B. Foxen, A. Fowler, C. Gidney, M. Giustina, R. Graff, K. Guerin, S. Haggard, M. P. Harrigan, M. J. Hartmann, A. Ho, M. Hoffmann, T. Huang, et al., *Nature 2019*, 574, 505.

[3] S. L. Braunstein, *Phys. Rev. Lett. 1992*, 69, 3598.
Note that $N_S$ is defined as the expected photon number in a single mode, meaning that the actual photon number in the signal is $2N_S$. In a state language this translates to the total unitary being the tensor product of two beam splitters: $U(\eta_1, \eta_2) = U(\eta_1) \otimes U(\eta_2)$, with $U(\eta_i) \equiv \exp(\arcsin(\sqrt{\eta_i} \omega_i) (\hat{s}_i \hat{b}_i - \hat{s}_i \hat{b}_i^*))$.