On Global Flipped $SU(5)$ GUTs in F-theory

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Abstract

We construct an $SU(4)$ spectral divisor and its factorization of types $(3,1)$ and $(2,2)$ based on the construction proposed in [1]. We calculate the chiral spectra of flipped $SU(5)$ GUTs by using the spectral divisor construction. The results agree with those from the analysis of semi-local spectral covers. Our computations provide evidence for the validity of the spectral divisor construction and suggest that the standard heterotic chirality formulae are applicable to the case of F-theory on an elliptically fibered Calabi-Yau fourfold with no heterotic dual.
1 Introduction

F-theory \[2, 3, 4\] is a twelve-dimensional geometric version of string theory. The construction of F-theory is motivated by the \(SL(2, \mathbb{Z})\) symmetry in type IIB string theory. The \(SL(2, \mathbb{Z})\) symmetry becomes the geometrical reparametrization symmetry of the torus when the axio-dilaton in type IIB string theory is identified with the complex modulus of a torus. The ten-dimensional background of type IIB string theory is lifted to a twelve-dimensional manifold which admits an elliptic fibration. Due to the monodromy of \(SL(2, \mathbb{Z})\), F-theory can be regarded as a non-perturbative completion of type IIB string theory \[1\]. In F-theory, it was shown \[6\] that the singularities of elliptic fibers correspond to the gauge groups on the seven-branes. More precisely, the \(A_n, D_n,\) and \(E_n\) singularities of elliptic fibration correspond to \(SU(n + 1), \) \(SO(2n),\) and \(E_n\) gauge groups, respectively. Since F-theory incorporates the exceptional groups, it is believed to be a natural framework for model building. Recently, supersymmetric Grand Unified Theory (GUT) models have been studied extensively in F-theory framework, in particular, the local version of GUT models have been explored in \[7, 32\]. The semi-local and global \(SU(5)\) GUTs in F-theory have been discussed in \[34, 56\]. For the cases of higher rank GUT groups, global \(SO(10)\) GUTs have been studied in \[57\] and semi-local flipped \(SU(5)\) GUTs \[58, 60\] have been constructed in \[61\]. In this paper we mainly focus on flipped \(SU(5)\) GUTs. The purpose of this paper is to promote the semi-local flipped \(SU(5)\) models studied in \[61\] to the global version by using the spectral divisor construction proposed in \[1\].

In F-theory, semi-local GUT models can be constructed by using spectral cover construction \[9, 41\]. In particular, one can use \(SU(4)\) spectral covers to build flipped \(SU(5)\) models \[61\]. We start with an elliptically fibered Calabi-Yau fourfold \(Z_4\) with a base \(B_3\) which contains a divisor \(B_2\) where \(Z_4\) exhibits an \(E_8\) singularity. To avoid full F-theory on a complicated elliptically fibered Calabi-Yau fourfold, we adopt a bottom-up approach to construct models in the decoupling limit, which lead us to consider a contractible complex surface \(B_2\) inside \(B_3\) such that we can reduce full F-theory on \(X_4\) to an effective eight-dimensional supersymmetric gauge theory on \(\mathbb{R}^{3,1} \times B_2\) \[7, 10\]. To achieve the decoupling limit, the surface \(B_2\) has to be a del Pezzo surface \[62, 63\]. To obtain the gauge group \(SU(5) \times U(1)_X\), we unfold

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1See \[5\] for a review.

2See \[33\] for a review.
the $E_8$ singularity into a $D_5$ singularity corresponding to unbroken $SO(10)$. This unfolding can be encoded in an $SU(4)$ spectral cover. It was shown in [9,41] that the spectral cover construction naturally encodes the information of the unfolding $E_8$ singularity and the gauge fluxes. By unfolding an $E_8$ singularity, we can engineer the singularity of types $D_5$, $D_6$, $E_6$, and $E_7$ in the Calabi-Yau fourfold $Z_4$. These operations correspond to the manipulation of the roots of a $SU(4)$ spectral cover. Generally we need to turn on certain fluxes to obtain the chiral spectrum. In F-theory, a natural candidate is the four-form $G$-flux which consists of three-form fluxes and gauge fluxes. In type IIB theory, these three-form fluxes produce back-reaction in the background geometry. It was shown in [30,64] that the three-form fluxes induce noncommutative geometric structures and also modify the texture of the Yukawa couplings. An example of noncommutative geometry is a fuzzy space, which has been studied in the context of F-theory in [65]. In this article we shall turn off these three-form fluxes and focus only on the gauge fluxes. The chirality of the matter fields in the representations of $SO(10)$ is determined by the traceless cover fluxes which are $(1,1)$-forms on the spectral covers. To obtain the gauge group $SU(5) \times U(1)_X$, we turn on a line bundle associated with $U(1)_X$ to break $SO(10)$ down to $SU(5) \times U(1)_X$. The spectrum is then determined by the cover fluxes and $U(1)_X$ fluxes. In this paper we shall focus on the $SU(4)$ spectral cover and also consider the factorizations of the spectral cover to construct realistic flipped $SU(5)$ models. For $U(1)_X$ fluxes breaking $SO(10)$ down to $SU(5) \times U(1)_X$ and numerical models, we refer readers to [61] for the details. A brief review of the semi-local $SU(4)$ spectral cover can be found in section 3.1. The analysis of the chiral spectrum under $(3,1)$ and $(2,2)$ factorizations can be found in section 4.1.

The spectral cover construction discussed above is semi-local. To obtain global flipped $SU(5)$ GUT models, we shall use the spectral divisor construction which has recently been proposed in [1]. This construction is motivated by heterotic/F-theory duality. In the heterotic string framework, one can calculate the chirality of matter fields by specifying a line bundle or its twist $\gamma^{(4)}_H$ on an $SU(4)$ spectral cover. It turns out the net chirality $N_r$ of matter field in the representation $r$ is given by [1,34]

$$N_r = \int_{\Sigma_{r,H}} \gamma^{(4)}_H,$$  \hspace{1cm} (1.1)

\footnote{For another construction from mirror symmetry and the discussion in a global $U(1)$ gauge symmetry arising from global restrictions of the Tate model, see [51] and references therein.}
where $\Sigma_{r,H}$ is the matter curve of representation $r$. It was shown in [70] that the data of the spectral cover can be encoded in a $dP_3$ surface. On the other hand, when the Calabi-Yau fourfold admits a global $K_3$-fibration over $B_2$, the $K_3$ fiber degenerates into two $dP_9$ surfaces glued together along an elliptic curve in the stable degenerate limit. The elliptic fibration over $B_2$ becomes the background Calabi-Yau threefold in the dual heterotic string compactification. Moreover, the spectral cover data of $E_8$ bundles can be encoded in the pair of $dP_9$ surfaces in F-theory geometry. The subbundles of $E_8$ correspond to some singularities in $dP_9$ surfaces. In particular, an unbroken $SO(10)$ gauge group corresponds to a $D_5$ singularity. It turns out that following heterotic/F-theory duality we can define the dual spectral divisor in F-theory framework, which encodes the data of the spectral cover in heterotic theory [1].

In F-theory, the net chirality formula was proposed to be

$$N_r = \int_{\Sigma_r} \gamma_F^{(4)} \circ p_{D_F}^* \gamma_F^{(4)},$$

where $\gamma_F^{(4)}$ is the traceless flux on the $SU(4)$ spectral divisor $D_F^{(4)}$ and $p_{D_F}^* \gamma_F^{(4)}$ is the projective map $p_{D_F}^* \gamma_F^{(4)} : D_F^{(4)} \to B_3$. It was argued [1] that this formula is intrinsic in the sense that it can be applied to the cases of F-theory compactifications without heterotic duals and that spectral divisor construction can be regarded as a global completion of the semi-local spectral cover construction. The case of an $SU(5)$ spectral divisor has been analyzed in [1]. In this article we shall verify this proposal by comparing the computations of chirality from an semi-local $SU(4)$ spectral cover with that from an $SU(4)$ spectral divisor. It turns out that they agree with each other. Our computation provides evidence to support the validity of the spectral divisor construction. The detailed construction of the $SU(4)$ spectral divisor can be found in section 3.2. We also calculate the chirality under $(3, 1)$ and $(2, 2)$ factorizations by using the spectral divisor construction. The results can be found in section 4.2.

The organization of the rest of the paper is as follows: in section 2, we first briefly review the $SU(4)$ spectral cover construction and computation of the chiral spectrum in heterotic string compactification on an elliptically fibered Calabi-Yau threefold. We then turn to the del Pezzo surface construction for $SU(4)$ bundles and stable degenerate limits, which are two important ingredients for heterotic/F-theory duality. We construct an $SU(4)$ spectral divisor in F-theory motivated by heterotic/F-theory duality and calculate the chiral spectrum in the end of section 2. In section 3, we consider the cases of F-theory compactifications without heterotic
duals. We first briefly review the semi-local $SU(4)$ spectral cover construction and then turn to constructing an $SU(4)$ spectral divisor. In section 4, we study $(3, 1)$ and $(2, 2)$ factorizations of the $SU(4)$ spectral cover and $SU(4)$ spectral divisor. We also calculate the chirality induced by traceless fluxes and found agreement between these two constructions. We summarize and conclude in section 5.

2 Preliminaries

In this section we shall briefly review the spectral cover construction in heterotic string. In particular, we shall focus on the case of an $SU(4)$ spectral cover. We then give an introduction to heterotic/F-theory duality. In the end of this section, we construct the dual F-theory spectral divisor [1] motivated by heterotic/F-theory duality.

2.1 $SU(4)$ Cover in Heterotic String

The $\mathcal{N} = 1$ four-dimensional effective theory of heterotic string compactifications is governed by the data $(Z_3, V_1, V_2)$, where $V_1$ and $V_2$ are vector bundles over a six-dimensional manifold $Z_3$. For simplicity, we only focus on one of the vector bundles, denoted by $V$ whose structure group is $G$. Supersymmetry requires that $Z_3$ be a Calabi-Yau threefold and that $V$ admit a connection satisfying the Hermitian Yang-Mills equations [66]

\[ F_{ab} = F_{\bar{a}\bar{b}} = 0, \quad g^{ab}F_{ab} = 0, \]  

(2.1)

where $g$ and $F$ are a metric of $Z_3$ and curvature of the connection, respectively. The unbroken gauge group of the four-dimensional effective theory is then the commutant of $G$ in $E_8$. To obtain an unbroken $SO(10)$ gauge group, we shall focus on the case of $G = SU(4)$. It is an extremely difficult task to construct solutions of the Hermitian Yang-Mills equations Eq. (2.1) for manifolds of dimension greater than one. However, it was proven in [67][69] that there is a one-to-one correspondence between the solutions of the Hermitian Yang-Mills equations and the construction

\footnote{Here we focus on the case of $E_8 \times E_8$ heterotic string compactification with vanishing background three-form flux $H$ and with constant dilaton $\phi$.}
of stable holomorphic vector bundles over the same complex manifold. In other
words, one can either attempt to solve the Hermitian Yang-Mills equations or, simply
construct the associated stable holomorphic vector bundles. It was shown in \cite{70, 71}
that when $Z_3$ admits an elliptic fibration, stable holomorphic bundles with structure
groups $SU(n)$ can be constructed by using spectral covers. In what follows, we
briefly review the spectral cover construction and the computation of net chirality of
the massless matter fields in a four-dimensional effective theory \cite{34, 70, 72}.

Let $Z_3$ be an elliptically fibered Calabi-Yau threefold $\pi_H : Z_3 \rightarrow B_2$ with a
section $\sigma_H : B_2 \rightarrow Z_3$. Due to the presence of the section $\sigma_H$, $Z_3$ can be described
by the Weierstrass model. One can realize $Z_3$ as a hypersurface of $\mathbb{P}^2$-fibration
over $B_2$ given by
\begin{equation}
y^2 = x^3 + f_4 xu^4 + g_6 u^6, \tag{2.2}
\end{equation}
where $x, y, u$ are sections of $\mathcal{O}(2\sigma_H) \otimes K_{B_2}^{-2}$, $\mathcal{O}(3\sigma_H) \otimes K_{B_2}^{-3}$, and $\mathcal{O}(\sigma_H)$, respectively,
while $f_4$ and $g_6$ are sections of $K_{B_2}^{-4}$ and $K_{B_2}^{-6}$, respectively.\footnote{Let $E$ be a holomorphic vector bundle over $Z_3$ and $J_{Z_3}$ be a Kähler form of $Z_3$. The slope $\mu(E)$ is defined by $\mu(E) = \frac{\int_{Z_3} c_1(E) \wedge J_{Z_3} \wedge J_{Z_3}}{\text{rk}(E)}$. The vector bundle $E$ is (semi)stable if for every subbundle or subsheaf $\mathcal{E}$ with $\text{rk}(\mathcal{E}) < \text{rk}(E)$, the inequality $\mu(\mathcal{E}) < (\leq) \mu(E)$ holds. Assume that $E = \oplus_{i=1}^n \mathcal{E}_i$, then $E$ is polystable if each $\mathcal{E}_i$ is a stable bundle with $\mu(\mathcal{E}_i) = \ldots = \mu(\mathcal{E}_k) = \mu(E)$}.\footnote{The globally well-defined $\mathbb{P}^2$-fibration can be realized as the total space of the weighted projective bundle $\mathbb{P}(L^2 \oplus L^3 \otimes \mathcal{O}_{B_2})$. It follows from the condition $c_1(Z_3) = 0$ that $L \cong K_{B_2}^{-1}$, where $K_{B_2}^{-1}$ is the anticanonical bundle of $B_2$. Let $c_1(\mathcal{O}(1)) = \sigma_H$, then the homogeneous coordinates $[x : y : u]$ are sections of $\mathcal{O}(2\sigma_H) \otimes K_{B_2}^{-2}$, $\mathcal{O}(3\sigma_H) \otimes K_{B_2}^{-3}$, and $\mathcal{O}(\sigma_H)$, respectively.} Note that these sections satisfy the following relation:
\begin{equation}
\sigma_H \cdot (\sigma_H + \pi_H^* c_1) = 0, \tag{2.3}
\end{equation}
where $c_1 \equiv c_1(B_2)$. At a generic point $b \in B_2$, the fiber $\mathbb{E}_b$ is an elliptic curve. The
restriction $V|_{\mathbb{E}_b}$ of the bundle $V$ of rank $n$ to the elliptic curve $\mathbb{E}_b$ is split. Namely,$V|_{\mathbb{E}_b}$ can be decomposed as a direct summand of holomorphic line bundles. The
semi-stability of $V$ requires that these line bundles be all of degree zero. Therefore,
we can write $V|_{\mathbb{E}_b} = \oplus_{i=1}^n \mathcal{O}_{\mathbb{E}_b}(q_i - e_0)$, where $q_i \in \mathbb{E}_b$ and $e_0$ is a distinguished point
representing the identity element in the group law on $\mathbb{E}_b$. For $SU(n)$ bundles, it is
required that $c_1(V) = 0$ which leads to the traceless condition $\sum_{i=1}^n (q_i - e_0) = 0$.\footnote{The Donaldson-Uhlenbeck-Yau theorem \cite{67, 69} states that a (split) irreducible holomorphic bundle $E$ admits a hermitian connection satisfying Eq. \cite{24, 1} if and only if $E$ is polystable.}
When the point \( b \) varies along \( B_2, \{q_1, q_2, ..., q_n\} \) spans a \( n \)-fold cover over \( B_2 \), called \( SU(n) \) spectral cover. In particular, the \( SU(4) \) spectral cover is given by

\[
C_H^{(4)} : a_0u^4 + a_2xu^2 + a_3yu + a_4x^2 = 0,
\]

with a projection map \( p_{C_H^{(4)}} : C_H^{(4)} \rightarrow B_2 \). We denote the homological class \([a_0]\) of the section \( a_0 \) by \( \pi_H^*\eta \), where \( \eta \in H_2(B_2, \mathbb{Z}) \) and write the remaining sections as \([a_m]\) = \( \pi_H^*(\eta - mc_1) \), where \( m = 2, 3, 4 \). The sections \( a_0, a_2, a_3 \) and \( a_4 \) encode the information of deformation of \( C_H^{(4)} \) defined by Eq. (2.4) and can be regarded as complex moduli of the spectral cover. On the other hand, the positions of the points \( \{q_1, q_2, q_3, q_4\} \) or the roots of the cover \( C_H^{(4)} \) characterize the deformation of the bundle \( V \). Therefore, \( \{a_0, a_2, a_3, a_4\} \) characterize the deformation\(^7\) of \( V \). It follows from Eq. (2.4) that the homological class of \( C_H^{(4)} \) is given by

\[
[C_H^{(4)}] = 4\sigma_H + \pi_H^*\eta.
\]

An \( SU(4) \) bundle can be constructed by specifying a line bundle or its twist \( \gamma_H^{(4)} \) which is \((1, 1)\)-form on \( C_H^{(4)} \). To obtain \( SU(4) \) bundles, it is required that \( \gamma_H^{(4)} \) satisfies the traceless condition \( p_{C_H^{(4)}}^*\gamma_H^{(4)} = 0 \). This can be achieved by setting

\[
\gamma_H^{(4)} = (4 - p_{C_H^{(4)}}^*p_{C_H^{(4)}}^*)([C_H^{(4)}] \cdot \sigma_H).
\]

Turning on an \( SU(4) \) bundle over \( Z_3 \) breaks \( E_8 \) down to \( SO(10) \). Under the breaking pattern \( E_8 \rightarrow SO(10) \times SU(4) \), the adjoint representation of \( E_8 \) is decomposed as

\[
E_8 \rightarrow SO(10) \times SU(4)
\]

\[
248 \rightarrow (1, 15) + (45, 1) + (10, 6) + (16, 4) + (\overline{16}, \overline{4}).
\]

The net chirality of matter fields can be calculated by the Atiyah-Singer index theorem or by intersection numbers of matter curves with \( \gamma_H^{(4)} \).\(^8\) Before computing the net chirality, we need to find the homological classes of matter curves. The

\(^7\)Generically, the spectral cover defined by Eq. (2.4) leads to a semistable bundle.\(^7\) A sufficient condition to obtain a holomorphic stable bundle \( V \) is that \( C_H^{(4)} \) is irreducible, which can be achieved by imposing the following two conditions: (1) The linear system \(|\eta|\) is base-point free in \( B_2 \); (2) \( \eta - mc_1 \) is effective in \( B_2 \).\(^7\)

\(^8\)The moduli space of stable \( SU(4) \) bundles on \( E_8 \) is the projective space \( \mathbb{P}^3 \). Fitting \( \mathbb{P}^3 \)’s together, we obtain the projective bundle \( \mathbb{P}(O_{B_2} \oplus L^{-2} \otimes L^{-3} \oplus L^{-4}) \) over \( B_2 \). In general, the moduli space of stable \( SU(n) \) bundles is the projective bundle \( \mathbb{P}(O_{B_2} \oplus L^{-2} \otimes L^{-3} \oplus ... \otimes L^{-n}) \).\(^7\)
homological class of the matter 16 curve in $Z_3$ is given by the intersection of $C_H^{(4)}$ with the zero section

$$[\Sigma_{16,H}] = [C_H^{(4)}] \cdot \sigma_H. \quad (2.8)$$

The net chirality $N_{16}$ of the matter 16 can be evaluated by

$$N_{16} = \int_{\Sigma_{16,H}} \gamma_H^{(4)} = \gamma_H^{(4)} \cdot [\Sigma_{16,H}] = -\eta \cdot B_2 (\eta - 4c_1). \quad (2.9)$$

To get the net chirality of matter 10, we have to resolve the singularity on the associated cover $C_{\wedge^2 V,H}^{(4)}$ corresponding to the antisymmetric representation 6 in $SU(4)$. It can be done by considering the intersection $C_H^{(4)} \cap \tau C_H^{(4)}$, where $\tau$ is a $\mathbb{Z}_2$ involution acting on the cover $C_H^{(4)}$ by $y \rightarrow -y$ while keeping $x$ and $u$ untouched. More precisely, the intersection $C_H^{(4)} \cap \tau C_H^{(4)}$ is determined by

$$\begin{cases} a_3 y u = 0 \\ a_0 u^4 + a_2 x u^2 + a_4 x^2 = 0. \end{cases} \quad (2.10)$$

The homological class of matter 10 curve in $Z_3$ can be computed as

$$[\Sigma_{10,H}] = [C_H^{(4)}] \cdot \{ [C_H^{(4)}] \cdot [y] \cdot [a_0 u^4] - [u] \cdot [a_4 x^2] \\ = [C_H^{(4)}] \cdot \{ [C_H^{(4)}] - 3(\sigma_H + \pi_H^* c_1) - \sigma_H \}. \quad (2.11)$$

The net chirality $N_{10}$ can be calculated by the intersection number $\gamma_H \cdot [\Sigma_{10,H}]$

$$N_{10} = \gamma_H \cdot [\Sigma_{10,H}] = [C_H^{(4)}] \cdot \{ [4\sigma_H - \pi_H^*(\eta - 4c_1)] \cdot \{ [C_H^{(4)}] - 3(\sigma_H + \pi_H^* c_1) - \sigma_H \} \}
= 0. \quad (2.12)$$

### 2.1.1 Del Pezzo Surface Construction

In the previous section one can see that the information of the bundle $V$ can be encoded in the spectral cover $C_H^{(4)}$ and the twist $\gamma_H^{(4)}$. However, the construction can

\footnote{For the case of $SU(n)$ bundles, $N_{16} = -\eta \cdot B_2 (\eta - nc_1)$ and $N_{10} = -(n - 4)\eta \cdot B_2 (\eta - nc_1)$. The factor $(n - 4)$ in $N_{10}$ can be seen from the fact that $\chi(Z_3, \wedge^2 V) = (n - 4)\chi(Z_3, V)$ where $Z_3$ is a Calabi-Yau threefold and $V$ is a vector bundle of rank $n$ with $c_1(V) = 0.$}
be translated to another form which involves del Pezzo surfaces and is more suitable for the framework of heterotic/F-theory duality. Before introducing the heterotic/F-theory duality, we briefly review the del Pezzo surface construction for \( SU(4) \) bundles. Let \( S \) be a del Pezzo surface \( dP_8 \) which can be obtained by blowing up eight generic points \( p_1, p_2, ..., p_8 \) in \( \mathbb{P}^2 \). The second homology group \( H_2(S, \mathbb{Z}) \) of \( S \) is generated by the basis \( \{ H, E_1, ..., E_8 \} \) with the intersection form given by

\[
H \cdot H = 1, \quad H \cdot E_i = 0, \quad E_i \cdot E_j = -\delta_{ij}, \quad i, j = 1, 2, ..., 8,
\]

where \( H \) is the pullback of the hyperplane divisor in \( \mathbb{P}^2 \) and \( E_i \) are the exceptional divisors from blow-ups. The anticanonical divisor \(-K_S\) of \( S \) is given by

\[
-K_S = 3H - \sum_{i=1}^{8} E_i.
\]

The linear system \(|-K_S|\) has a base point and general elements \( E \) of \(|-K_S|\) are genus one curves. Let us define two subsects of \( H_2(S, \mathbb{Z}) \) as follows \([34, 70]\):

\[
I_8 = \{ l \in H_2(S, \mathbb{Z}) | l \cdot l = -1, \ l \cdot (-K_S) = 1 \}, \quad (2.15)
\]

\[
R_8 = \{ C \in H_2(S, \mathbb{Z}) | C \cdot C = -2, \ C \cdot (-K_S) = 0 \}. \quad (2.16)
\]

Note that \( I_8 \) and \( R_8 \) are in one-to-one correspondence through \( l = C + (-K_S) \) and that the elements in \( R_8 \) are in one-to-one correspondence with roots of \( E_8 \). The generators of \( R_8 \) can be chosen as follows:

\[
C_k = E_k - E_{k+1}, \quad k = 1, 2, ..., 7, \quad C_8 = H - (E_6 + E_7 + E_8).
\]

The intersection matrix of \( R_8 \) is given by \((-C_{E_8})\) where \( C_{E_8} \) is the Cartan matrix of \( E_8 \). Given \( E \in |-K_S| \), a flat bundle on \( E \) is given by

\[
\mathcal{O}_E(C_k|_E) \cong \mathcal{O}_E(q_k - e_0), \quad (2.18)
\]

\(^{10}\)Since \(-K_S\) is ample, \( h^0(S, \mathcal{O}_{B_2}(-K_S)) \neq 0 \). The linear system of \(-K_S\) is defined by \(|-K_S| = \mathbb{P}^0 \mathcal{O}_S(-K_S)\) and the base point locus is defined by \( \bigcap E_8, \ E_8 \in |-K_S| \). For a del Pezzo surface \( dP_8 \), it follows from the Riemann-Roch theorem and Kodaira vanishing theorem that \( \dim(|-K_S|) = \chi(S, \mathcal{O}_S(-K_S)) - 1 = (-K_S)^2 = 1 \). Since \( h^0(S, \mathcal{O}_S(-K_S)) = 2 \), we have two homogeneous polynomials of degree one and the base point is the unique common zero \([0 : 0]\). Moreover, one can show that the linear system \(|-3K_S|\) induces a morphism \( \Phi_{|{-3K_S}|} : S \to W_{\mathbb{P}^3}^{2,3,1,1} \). The image of \( \Phi_{|{-3K_S}|} \) is given by Eq. \([2.19] \ [34]\).
where \( C_k \in R_8 \) and \( q_k \in \mathbb{E} \) given by \( l_k \cdot (-K_S) \). Recall that the spectral cover describes a flat bundle on an elliptic fibration \( \pi_H : Z_3 \to B_2 \) by specifying a set \( \{ q_k \} \) for each fiber \( \mathbb{E}_b \), \( b \in B_2 \). Equivalently, one can describe the bundle by starting with embedding an elliptic curve \( \mathbb{E}_b \) into a fiber of \( dP_8 \)-fibration over \( B_2 \) \( \pi_{W_4} : W_4 \to B_2 \) with \( \pi_{W_4}|_{Z_3} = \pi_H \). Then the local data \( V_{|\mathbb{E}_b} \) of the bundle \( V \) can be described by the cycles \( \{ C_1, C_2, \ldots, C_n \} \) in \( R_8 \) via Eq. \( (2.15) \). On the other hand, one can realize a \( dP_8 \) surface as a divisor in \( W_{\mathbb{P}^3_{2,3,1,1}} \). More precisely, a \( dP_8 \) surface in \( W_{\mathbb{P}^3_{2,3,1,1}} \) can be described by the Weierstrass model as follows:

\[
y^2 = x^3 + \tilde{f}_4(Z_1, Z_2)x + \tilde{g}_6(Z_1, Z_2), \tag{2.19}
\]

where \( [x : y : Z_1 : Z_2] \) are homogeneous coordinates of \( W_{\mathbb{P}^3_{2,3,1,1}} \). \( \tilde{f}_4 \) and \( \tilde{g}_6 \) are homogeneous polynomials of degree four and six, respectively. Through this embedding, one can find that the bundle moduli of a flat bundle on \( \mathbb{E} \) map to the complex structure moduli of the defining equation Eq. \( (2.19) \). For the case of \( G = SU(4) \), one can construct the bundle through the spectral cover construction by specifying points \( \{ q_1, q_2, q_3, q_4 \} \) on \( \mathbb{E}_b \). The bundle moduli are characterized by the coefficients \( \{ a_0, a_2, a_3, a_4 \} \) of the spectral cover defined by Eq. \( (2.1) \). Equivalently, this data can be described by the \((-2)\)-cycles \( \{ C_1, C_2, C_3, C_4 \} \) in \( dP_8 \) and their intersection numbers. The intersection of these cycles form the Cartan matrix of \( SU(4) \). The complement of the extended Dynkin diagram of \( SU(4) \) in \( E_8 \) corresponds to the vanishing cycles which leads to a \( D_5 \) singularity in \( dP_8 \). In other words, the unbroken GUT group \( SO(10) \) corresponds to a \( D_5 \) singularity in \( dP_8 \). Therefore, one can construct an \( SO(10) \) GUT group by engineering a \( D_5 \) singularity in \( dP_8 \). More precisely, one can consider the Weierstrass model

\[
y^2 = x^3 + f_4Z_1^4x + g_6Z_1^6 + Z_2Z_1(b_0Z_1^4 + b_2Z_1^2x + b_4Z_1y + b_4x^2). \tag{2.20}
\]

Note that \( Z_2 = 0 \) is an elliptic curve given by the Weierstrass equation \( y^2 = x^3 + f_4x + g_6 \) and that the parenthesis in Eq. \( (2.20) \) reduces to the spectral cover \( C_H^{(4)} \) given by Eq. \( (2.4) \) when \( Z_1 \to u \) with \( b_m|_{Z_3} = a_m \). It is clear that in this case the bundle moduli \( \{ a_0, a_2, a_3, a_4 \} \) map to the complex moduli of \( dP_8 \) given by Eq. \( (2.20) \). The dual F-theory geometry can be described as a \( dP_9 \)-fibration over \( B_2 \), which is obtained by blowing up the base point. The \( dP_8 \) construction described above for \( SU(n) \) bundles can be realized by a \( dP_9 \) surface whose intersection matrix of \((-2)\)-cycles contains the Cartan matrix of \( E_8 \). It can be seen by taking \([1,34]\)

\[
I_8 = \{ l \in H_2(dP_9, \mathbb{Z})| l \cdot l = -1, l \cdot (-K_{dP_9}) = 1, l \cdot E_9 = 0 \}, \tag{2.21}
\]

\[
R_8 = \{ C \in H_2(dP_9, \mathbb{Z})| C \cdot C = -2, C \cdot (-K_{dP_9}) = 0, C \cdot E_9 = 0 \}. \tag{2.22}
\]
where $E_9$ is an exceptional divisor from the blow-up of the base point. The geometry of a $dP_9$-fibration can be obtained by taking the stable degenerate limit of a $K3$-fibration on $B_2$ in F-theory [9, 34, 70, 73, 74]. Through this degenerate limit, we can embed the data of the bundle $V$ into dual F-theory geometry. We shall describe this degenerate limit in the next section.

### 2.2 Heterotic/F-theory Duality

#### 2.2.1 Stable Degeneration Limit

Let us consider F-theory on an elliptically fibered Calabi-Yau fourfold $\pi: X_4 \to B_3$ with a section $\sigma_{B_3}: B_3 \to X_4$. With the section $\sigma_F$, $X_4$ can be described by the Weierstrass model:

$$y^2 = x^3 + fxu^4 + gu^6.$$  \hfill (2.23)

The Calabi-Yau condition $c_1(X_4) = 0$ requires that $f$ and $g$ are sections of $K_{B_3}^{-4}$ and $K_{B_3}^{-6}$, respectively. The heterotic/F-theory duality requires that $B_3$ admits a $\mathbb{P}^1$-fibration over some surface $B_2$. Let $[Z_1 : Z_2]$ be the homogeneous coordinates of $\mathbb{P}^1$ fiber. Since $f$ and $g$ are the homogeneous polynomials of degree 8 and 12 in terms of $[Z_1 : Z_2]$, respectively, one can expand Eq. (2.23) as

$$y^2 = x^3 + \left( \sum_{i=0}^{8} f_i Z_1^i Z_2^{8-i} \right) x u^4 + \left( \sum_{j=0}^{12} g_j Z_1^j Z_2^{12-j} \right) u^6,$$  \hfill (2.24)

---

11To see this, we can embed $X_4$ as a section of a weighted projective bundle over $B_3$. More precisely, we homogenize Eq. (2.23) to be $y^2 = x^3 + xu^4 + gu^6 \hookrightarrow \mathbb{P}^2_{2,3,1}$, where $f$ and $g$ are sections of line bundles $L^4$ and $L^4$ on $B_3$, respectively. To obtain a globally well-defined fibration, let $\tilde{X}_5$ be the total space of the weighted projective bundle $\mathbb{P}(L^2 \otimes L^3 \otimes O_{B_3})$ over $B_3$ and consider $X_4$ to be a hypersurface in $\tilde{X}_5$. By the adjunction formula [82, 83], we have $c_1(X_4) = \frac{c(B_3)(1+2r+2\pi_X t)(1+3r+3\pi_X t)(1+r)}{(1+6r+6\pi_X t)^3}$, where $r \equiv c_1(O_{\mathbb{P}^1}(1))$ and $t \equiv c_1(L)$. It follows from the condition $c_1(X_4) = 0$ that $L = K_{B_3}^{-1}$.

12Recall that the anticanonical bundle $K_{\mathbb{P}^n}^{-1}$ of $n$-dimensional complex projective space $\mathbb{P}^n$ is $K_{\mathbb{P}^n}^{-1} = (n+1)H \equiv O_{\mathbb{P}^n}(n+1)$. So $K_{\mathbb{P}^1}^{-4} = O_{\mathbb{P}^1}(8)$ and $K_{\mathbb{P}^1}^{-6} = O_{\mathbb{P}^1}(12)$.
where \( f_i \) and \( g_j \) are sections of suitable line bundles over \( B_2 \). When \( Z_1 \to 0 \) and set \( Z_2 = 1 \), Eq. (2.24) becomes

\[
y^2 = x^3 + \left( \sum_{i=0}^{4} f_i z_1^i \right) xu^4 + \left( \sum_{j=0}^{6} g_j z_1^j \right) u^6, \tag{2.25}
\]

where \( z_1 \equiv \frac{Z_1}{Z_2} \). On the other hand, taking \( Z_2 \to 0 \) and set \( Z_1 = 1 \), Eq. (2.24) becomes

\[
y^2 = x^3 + \left( \sum_{m=0}^{4} f_4 z_1^{4-m} \right) xu^4 + \left( \sum_{l=0}^{6} g_6 z_1^{6-l} \right) u^6, \tag{2.26}
\]

where \( z_2 \equiv \frac{Z_2}{Z_1} \). These two limits correspond to two \( dP_9 \) surfaces glued together along an elliptic curve \( E \) with the Weierstrass equation:

\[
y^2 = x^3 + f_4 x u^4 + g_6 u^6. \tag{2.27}
\]

This elliptically fibered Calabi-Yau threefold \( \pi_H : Z_3 \to B_2 \) is the background of heterotic string compactification. Two \( dP_9 \) surfaces, Eq. (2.25) and (2.26), encode the data of bundles \( E_8 \times E_8 \) in the heterotic string. With heterotic/F-theory duality, one can find that constructing a stable \( SU(4) \) bundle on an elliptically fibered \( Z_3 \) with a base \( B_2 \) by using spectral cover construction corresponds to engineering an \( D_5 \) singularity in the geometry of \( dP_9 \)-fibration on \( B_2 \) given by Eq. (2.20).

### 2.2.2 Dual \( SU(4) \) spectral Divisor in F-theory

Let \( Y_4 \) be a \( dP_9 \)-fibration over a complex surface \( B_2 \) with a projection map \( p : Y_4 \to B_2 \). Since \( dP_9 \) is an elliptic surface, \( Y_4 \) can be regarded as an elliptic fibration over a threefold \( B_3 \) with a section \( \sigma_F : B_3 \to Y_4 \) and \( B_3 \) admits a \( \mathbb{P}^1 \)-fibration over \( B_2 \). The projection map of the elliptic fibration and \( \mathbb{P}^1 \)-fibration are denoted by \( \pi_F : Y_4 \to B_3 \) and \( \varphi : B_3 \to B_2 \), respectively. To describe \( Y_4 \), we embed the elliptic fiber as a

---

\(^{13}\) The hypersurfaces described by Eq. (2.25) and Eq. (2.26) both are homogeneous polynomials of degree six in \( W^{3}_{2,3,1,1} \). They are actually \( dP_8 \) surfaces. It follows from the adjunction formula that \( c_1(S) = x \) and \( c_2(S) = 11x^2 \), where \( x \equiv r + t \). By the Riemann-Roch theorem \( 12\chi(O_S) = c_1^2(S) + c_2(S) \), we obtain \( x^2 = 1 \) and then Euler characteristic \( \chi(S) = 11 \). For \( dP_k \) surfaces, \( \chi(dP_k) = 3 + k \), which implies that \( k = 8 \). One can obtain \( dP_9 \)’s by blowing up the point \( Z_1 = Z_2 = 0 \).

\(^{14}\) The elliptic curve \( E \) is an effective divisor of the linear system \( | - K_S | \). By the adjunction formula, we obtain \( 2g - 2 = \chi(E + K_S) = 0 \), which implies that \( E \) is an elliptic curve.
divisor of $W\mathbb{P}^2_{2,3,1}$ with homogeneous coordinates $[x : y : u]$ and consider the following Weierstrass model:

\[
y^2 = x^3 + f_4(Z_1u)^4x + g_6(Z_1u)^6 + Z_2(Z_1u)^5 - n[b_0(Z_1u)^n + b_2(Z_1u)^{n-2}x + b_3(Z_1u)^{n-3}y + ...],
\]

(2.28)

where the last term in the bracket is $b_nx^{n/2}$ for $n$ even, or $b_nx^{(n-3)/2}y$ for $n$ odd. Note that $x$, $y$, and $u$ are sections of $\mathcal{L}^2$, $\mathcal{L}^3$, and $\mathcal{O}_{B_3}$, respectively, where $\mathcal{L}$ is a line bundle on $B_3$ and will be determined later. To make $W\mathbb{P}^2_{2,3,1}$-fibration globally well-defined, we consider $Y_4$ be a divisor in the weighted projective bundle $W\mathbb{P}(\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}_{B_3})$. We denote the fiber by $\mathcal{O}_F(1)$. Let $c_1(\mathcal{O}_F(1)) = \sigma_F$ and $c_1(\mathcal{L}) = l$. By using the adjunction formula, we obtain

\[
c(Y_4) = c(B_3) \left(1 + 2\sigma_F + 2\pi_F^*l\right) \left(1 + 3\pi_F + 3\pi_F^*l\right) \left(1 + \sigma_F\right) \left(1 + 6\pi_F^*l\right),
\]

(2.29)

where $c$ stand for the total Chern class. It follows from Eq. (2.29) that

\[
c_1(Y_4) = \pi_F^*c_1(B_3) - \pi_F^*l.
\]

(2.30)

Let us turn to the geometry of $B_3$. We take $B_3$ to be a $\mathbb{P}^1$ bundle over $B_2$. To be concrete, let $B_3 = \mathbb{P}(\mathcal{O}_{B_2} \oplus \mathcal{M})$ with $c_1(\mathcal{O}_F(1)) = r$ and $c_1(M) = t$, where $\mathcal{M}$ is a line bundle on $B_2$. By using the adjunction formula, we have

\[
c(B_3) = c(B_2)(1 + r)(1 + r + \varphi^*t),
\]

(2.31)

which implies that

\[
\begin{align*}
    c_1(B_3) &= 2r + \varphi^*(c_1 + t) \\
    c_2(B_3) &= \varphi^*c_2 + \varphi^*c_1 \cdot (\varphi^*t + 2r) \\
    c_3(B_3) &= \varphi^*c_2 \cdot (\varphi^*t + 2r)
\end{align*}
\]

(2.32)

where $c_1 = c_1(B_2)$ and the relation $r \cdot (r + \varphi^*t) = 0$ has been used. On the other hand, it follows from Eq. (2.28) that the heterotic Calabi-Yau threefold $Z_3$ is given by $Z_2 = 0$ which is a submanifold of $Y_4$. By using the adjunction formula, we have

\[
c(Z_3) = \frac{c(Y_4)}{(1 + \pi_F^*r + p^*t)}.
\]

(2.33)

It follows from the Calabi-Yau condition $c_1(Z_3) = 0$, Eq. (2.30), and Eq. (2.32) that

\[
c_1(Y_4) = \pi_F^*r + p^*t, \quad \pi_F^*l = \pi_F^*c_1(B_3) = \pi_F^*r - p^*t = \pi_F^*r + p^*c_1.
\]

(2.34)
Therefore, the homological classes of sections appearing in Eq. (2.40) are as follows:

\[ [x] = 2(\sigma_F + \pi_F^* r + p^* c_1), \quad [y] = 3(\sigma_F + \pi_F^* r + p^* c_1), \quad [u] = \sigma_F, \quad \text{(2.35)} \]

\[ [Z_1] = \pi_F^* r, \quad [Z_2] = \pi_F^* r + p^* t, \quad [b_m] = p^* [(6 - m)c_1 - t], \quad m = 0, 2, 3, 4. \quad \text{(2.36)} \]

Following the proposal in [1], we define the spectral divisor \( \mathcal{D}^{(n)}_F \) of \( Y_4 \) by

\[ \mathcal{D}^{(n)}_F : b_0(Z_1 u)^n + b_2(Z_1 u)^{n-2} x + b_3(Z_1 u)^{n-3} y + ... = 0, \quad \text{(2.37)} \]

where the last term is \( b_n x^{n/2} \) for \( n \) even, or \( b_n x^{(n-3)/2} y \) for \( n \) odd. The projection map is denoted by \( p_{\mathcal{D}^{(n)}_F} : \mathcal{D}^{(n)}_F \to B_3 \). Let \( \gamma^{(n)}_F \) be a \((1,1)\) form on \( \mathcal{D}^{(n)}_F \). It was proposed in [1] that the net chirality formula for matter in the representation \( r \) can be computed as

\[ N_r = [\hat{\Sigma}_r] \cdot \mathcal{G}^{(n)}_F \cdot p_{\mathcal{D}^{(n)}_F}^* B_2, \quad \text{(2.38)} \]

where \( [\hat{\Sigma}_r] \) is the dual matter surface inside \( \mathcal{D}^{(n)}_F \) and \( \mathcal{G}^{(n)}_F \) is defined by \( \gamma^{(n)}_F = [\mathcal{D}^{(n)}_F] \cdot \mathcal{G}^{(n)}_F \) for given \( \gamma^{(n)}_F \). For the case of \( n = 4 \), we have

\[ y^2 = x^3 + f_4(Z_1 u)^4 x + g_6(Z_1 u)^6 + Z_2[b_0(Z_1 u)^5 + b_2(Z_1 u)^3 x + b_3(Z_1 u)^2 y + b_4(Z_1 u)x^2]. \quad \text{(2.39)} \]

Note that when \( Z_2 = 0 \), Eq. (2.39) reduces to \( Z_3 \) defined by Eq. (2.2). In this case the spectral divisor is given by

\[ \mathcal{D}^{(4)}_F : b_0(Z_1 u)^4 + b_2(Z_1 u)^2 x + b_3(Z_1 u)y + b_4 x^2 = 0, \quad \text{(2.40)} \]

with a projection map \( p_{\mathcal{D}^{(4)}_F} : \mathcal{D}^{(4)}_F \to B_3 \). The divisor \( \mathcal{D}^{(4)}_F \) can be realized as the union of four exceptional lines of \( dP_9 \) comprising a fundamental representation of \( SU(4) \) [9331725]. With Eq (2.35) and Eq. (2.36), the homological class of \( \mathcal{D}^{(4)}_F \) is given by

\[ [\mathcal{D}^{(4)}_F] = 4(\sigma_F + \pi_F^* r) + p^*(6c_1 - t). \quad \text{(2.41)} \]

The traceless flux \( \gamma^{(4)}_F \) can be computed as

\[ \gamma^{(4)}_F = (4 - p_{\mathcal{D}^{(4)}_F}^* p_{\mathcal{D}^{(4)}_F})([\mathcal{D}^{(4)}_F] \cdot \sigma_F) = [\mathcal{D}^{(4)}_F] \cdot [4\sigma_F - p^*(2c_1 - t)], \quad \text{(2.42)} \]

where the relation \( \sigma_F \cdot (\sigma_F + \pi_F^* r + p^* c_1) = 0 \) has been used. It follows from Eq. (2.42) that \( \mathcal{G}^{(4)}_F = 4\sigma_F - p^*(2c_1 - t) \). To calculate the chiral spectrum, we need to calculate the homological classes of dual matter surfaces. The dual matter surface \( \hat{\Sigma}_{16} \) sits in
the locus of the intersection \( \{(Z_1 u) = 0\} \cap \{b_4 = 0\} \) and then its homological class is given by

\[
[\hat{\Sigma}_{16}] = (\sigma_F + \pi_{F}^* r) \cdot p^*(2c_1 - t).
\]  

(2.43)

By using the net chirality formula Eq. (2.38), we obtain

\[
N_{16} = [\hat{\Sigma}_{16}] \cdot G_F \cdot \pi_{F}^* r
= -(6c_1 - t) \cdot B_2 \cdot (2c_1 - t).
\]  

(2.44)

On the other hand, the dual matter surface \( \hat{\Sigma}_{10} \) sits in the locus of \( D^{(4)}_F \cap \tau D^{(4)}_F \) where \( \tau \) is a \( \mathbb{Z}_2 \) involution \( y \rightarrow -y \) acting on \( D^{(4)}_F \) while keeping \( x, u, \) and \( Z_1 \) intact. More precisely, the intersection loci of \( D^{(4)}_F \cap \tau D^{(4)}_F \) are determined by

\[
\left\{ \begin{array}{l}
b_3(Z_1 u)y = 0 \\
b_0(Z_1 u)^4 + b_2(Z_1 u)^2x + b_4x^2 = 0.
\end{array} \right.
\]  

(2.45)

It follows from Eq. (2.45) that the homological class of dual matter surface \( \hat{\Sigma}_{10} \) is

\[
[\hat{\Sigma}_{10}] = [D^{(4)}_F] \cdot [D^{(4)}_F] - [Z_1 u] \cdot [b_4] - [y][b_4 x^2] - 2[x][Z_1]
= (\sigma_F + \pi_{F}^* r) \cdot p^*(12c_1 - 4t) + p^*(6c_1 - t) \cdot p^*(3c_1 - t).
\]  

(2.46)

By using Eq. (2.38), the net chirality of matter 10 is

\[
N_{10} = [\hat{\Sigma}_{10}] \cdot G_F^{(4)} \cdot \pi_{F}^* r
= 0.
\]  

(2.47)

These results agree with the computations in the dual heterotic string framework by identifying \( D^{(4)}_F \mid_{Z_3} = G^{(4)}_H \) and \( b_m \mid_{Z_3} = a_m \), which gives rise to the relation \( \eta = 6c_1 - t \).

It was argued in [1] that the chirality formula Eq. (2.38) can be applied to the cases of F-theory compactifications without heterotic duals. In section 3, we shall briefly review semi-local \( SU(4) \) cover construction [61] and its global completion [1].

### 3 Global Completion of \( SU(4) \) Cover

In this section we shall discuss the case of an F-theory compactification on an elliptically fibered Calabi-Yau fourfold without a heterotic dual. We first briefly review the semi-local \( SU(4) \) spectral cover construction studied in [61]. In the second part we construct the \( SU(4) \) spectral divisor following the proposal in [1].


3.1 Semi-local $SU(4)$ Cover

Let us consider an elliptically fibered Calabi-Yau fourfold $\pi: Z_4 \rightarrow B_3$ with a section $\sigma: B_3 \rightarrow Z_4$ and $B_2$ to be a divisor in $B_3$ where $Z_4$ exhibits a $D_5$ singularity. Generically, $Z_4$ can be described by the Tate form as follows:

$$y^2 = x^3 + b_4x^2z + b_3yz^2 + b_2x^2z + b_0z^5. \quad (3.1)$$

Let us define $t \equiv -c_1(N_{B_2/B_3})$ and then the homological classes of the sections $x$, $y$, $z$, and $b_m$ can be expressed as

$$[x] = 3(c_1 - t), \quad [y] = 2(c_1 - t), \quad [z] = -t, \quad [b_m] = (6 - m)c_1 - t = \eta - mc_1. \quad (3.2)$$

Recall that locally $Z_4$ can be described by an ALE fibration over $B_2$. Pick a point $p \in B_2$, the fiber is an ALE space denoted by $ALE_p$. The ALE space can be constructed by resolving an orbifold $\mathbb{C}^2/\Gamma_{ADE}$, where $\Gamma_{ADE}$ is a discrete subgroup of $SU(2)$.

It was shown that the intersection matrix of the exceptional 2-cycles corresponds to the Cartan matrix of $ADE$ type, which can be described by $ADE$ Dynkin diagrams. Let us take $\alpha_i \in H_2(ALE_p, \mathbb{Z})$, $i = 1, 2, ..., 8$ to be the roots of $E_8$ and the extended $E_8$ Dynkin diagram with roots and Dynkin indices to be shown in Fig. 1. Notice that $\alpha_{-\theta}$ is the highest root and satisfies the condition $\alpha_{-\theta} + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 = 0$. To obtain $SO(10)$, we take the volume of the cycles $\{\alpha_4, \alpha_5, ..., \alpha_8\}$ to be vanishing and then $SU(4)$ is generated by $\{\alpha_1, \alpha_2, \alpha_3\}$. An enhancement to $E_6$ happens when $\alpha_3$ or any of its images under the Weyl permutation shrinks to zero size. We define $\{\lambda_1, ..., \lambda_4\}$ to be the periods of these cycles. As described in [10][11],

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15 For more information, see [76][81].
16 By abuse of notation, the corresponding exceptional 2-cycles are also denoted by $\alpha_i$. 

---

Figure 1: The extended $E_8$ Dynkin diagram and indices.
theses \( \lambda_i \) are encoded in the coefficients \( b_m \) as follows:

\[
\begin{aligned}
\sum_i \lambda_i &= \frac{b_1}{b_0} = 0 \\
\sum_{i<j} \lambda_i \lambda_j &= \frac{b_2}{b_0} \\
\sum_{i<j<k} \lambda_i \lambda_j \lambda_k &= \frac{b_3}{b_0} \\
\prod_l \lambda_l &= \frac{b_4}{b_0}.
\end{aligned}
\]  

(3.3)

Equivalently, \( \{\lambda_1, ..., \lambda_4\} \) are the roots of the equation

\[
b_0 \prod_{k=1}^{4}(s + \lambda_k) = b_0 s^4 + b_2 s^2 + b_3 s + b_4 = 0.
\]  

(3.4)

When \( p \in B_2 \) varies along \( B_2 \), Eq. (3.4) defines a fourfold cover \( C^{(4)} \) over \( B_2 \), the semi-local \( SU(4) \) spectral cover. This cover can be described as a section of the canonical bundle \( K_{B_2} \rightarrow B_2 \). When \( \lambda_i \) vanish, \( \prod_i \lambda_i = b_4 = 0 \) and the corresponding gauge group is enhanced to \( E_6 \), which implies that the matter field \( 16 \) is localized at the locus \( \{b_4 = 0\} \). On the other hand, the matter field \( 10 \) corresponds to the anti-symmetric representation \( 6 \) of \( SU(4) \), associated with a sixfold cover \( C^{(6)}_{\wedge^2 V} \) over \( B_2 \). This associated cover \( C^{(6)}_{\wedge^2 V} \) is given by

\[
C^{(6)}_{\wedge^2 V} : b_0^2 \prod_{i<j}(s + \lambda_i + \lambda_j) = b_0^2 s^6 + 2b_0 b_2 s^4 + (b_2^2 - 4b_0 b_4) s^2 - b_3^2 = 0.
\]  

(3.5)

Since matter \( 10 \) corresponds to \( \lambda_i + \lambda_j = 0, \ i \neq j \), it follows from Eq. (3.5) that \( b_3 = 0 \), which means that matter \( 10 \) is localized at the locus \( \{b_3 = 0\} \) as we expected from the \( D_6 \) singularity of Eq. (3.1). From the discussion above, we see that spectral cover indeed encodes the information of singularities and gauge group enhancements.

Moreover, we can construct a Higgs bundle to calculate the chiral spectrum for matter \( 16 \) and \( 10 \) by switching on a line bundle on the cover. Let us define \( X \) to be the total space of the canonical bundle \( K_{B_2} \) over \( B_2 \). Note that \( X \) is a local Calabi-Yau threefold, but \( X \) is non-compact. To obtain a compact space, one can compactify \( X \) to the total space \( \bar{X} \) of the projective bundle over \( B_2 \), i.e.

\[
\bar{X} = \mathbb{P}(\mathcal{O}_{B_2} \oplus K_{B_2}),
\]  

(3.6)

with a projection map \( \pi : \bar{X} \rightarrow B_2 \), where \( \mathcal{O}_{B_2} \) is the trivial bundle over \( B_2 \). Notice that \( \bar{X} \) is compact but no longer a Calabi-Yau threefold. Let \( \mathcal{O}_P(1) \) be a hyperplane
section of $\mathbb{P}^1$ fiber and denote its first Chern class by $\sigma_{\infty}$. We define the homogeneous coordinates of the fiber by $[U : W]$. Note that $\{U = 0\}$ and $\{W = 0\}$ are sections of $\mathcal{O}_P(1) \otimes K_S$ and $\mathcal{O}_P(1)$, while the class of $\{U = 0\}$ and $\{W = 0\}$ are $\sigma \equiv \sigma_{\infty} - \pi^* c_1$ and $\sigma_{\infty}$, respectively. By the emptiness of intersection of $\{U = 0\}$ and $\{W = 0\}$, we obtain $\sigma \cdot \sigma = -\sigma \cdot \pi^* c_1$. We define the affine coordinate $s$ by $s = U/W$ and then the $SU(4)$ cover given by Eq. (3.4) can be written as

$$C^{(4)}: b_0 U^4 + b_2 U^2 W^2 + b_3 W^3 + b_4 W^4 = 0$$

with a projection map $p_{C^{(4)}}: C^{(4)} \rightarrow B_2$. It is not difficult to see that the homological class $[C^{(4)}]$ of the cover $C^{(4)}$ is given by $[C^{(4)}] = 4\sigma + \pi^* \eta$. We can calculate the matter curve by intersecting $[C^{(4)}]$ with $\sigma$.

$$[C^{(4)}] \cap \sigma = (4\sigma + \pi^* \eta) \cdot \sigma = \sigma \cdot \pi^*(\eta - 4c_1).$$

On the other hand, it follows from Eq. (3.5) that the homological class of the cover $C^{(6)}_{\Lambda^2 V}$ is given by

$$[C^{(6)}_{\Lambda^2 V}] = 6\sigma + 2\pi^* \eta.$$ (3.9)

However, the cover $C^{(6)}_{\Lambda^2 V}$ is generically singular. To solve this problem, one can consider intersection $C^{(4)} \cap \tau C^{(4)}$ and define [72]

$$[D] = [C^{(4)}] \cap ([C^{(4)}] - 3\sigma_{\infty} - \sigma).$$ (3.10)

where $\tau$ is a $\mathbb{Z}_2$ involution $W \rightarrow -W$ acting on the spectral cover $C^{(4)}$. To obtain chiral spectrum, we turn on a spectral line bundle $L$ on the cover $C^{(4)}$. The corresponding Higgs bundle is given by $p_{C^{(4)}} L$. For $SU(n)$ bundles, it is required that $c_1(p_{C^{(4)}} L) = 0$. It follows that

$$p_{C^{(4)}} c_1(L) = \frac{1}{2} p_{C^{(4)}} r^{(4)} = 0,$$ (3.11)

where $r^{(4)}$ is the ramification divisor given by $r^{(4)} = p_{C^{(4)}} c_1 - c_1(C^{(4)})$. It is convenient to define the cover flux $\gamma^{(4)}$ by

$$c_1(L) = \lambda \gamma^{(4)} + \frac{1}{2} r^{(4)},$$ (3.12)

where $\lambda$ is a rational number used to compensate the non-integral class $\frac{1}{2} r^{(4)}$ such that $c_1(L) \in H_4(\bar{X}, \mathbb{Z})$. The traceless condition $c_1(p_{C^{(4)}}_* L) = 0$ is then equivalent to the condition $p_{C^{(4)}}\gamma^{(4)} = 0$. Up to multiplication of a constant, the only choice of $\gamma^{(4)}$ satisfying the traceless condition is

$$\gamma^{(4)} = (4 - p^{*}_{C^{(4)}} p_{C^{(4)}_*})([C^{(4)}] \cdot \sigma).$$ (3.13)
Since the first Chern class of a line bundle must be integral, it follows that \( \lambda \) and \( \gamma^{(4)} \) have to obey the following quantization condition

\[
\lambda \gamma^{(4)} + \frac{1}{2} [p_{C^{(4)}}^* c_1 - c_1(C^{(4)})] \in H_4(\bar{X}, \mathbb{Z}).
\] (3.14)

With the given cover flux \( \gamma^{(4)} \), the net chirality of matter 16 is calculated by

\[
N_{16} = ([C^{(4)}] \cdot \sigma) \cdot \lambda \gamma^{(4)} = -\lambda \eta \cdot (\eta - 4c_1).
\] (3.15)

On the other hand, the homological class of matter 10 curve is given by Eq. (3.10). It turns out that the net chirality of matter 10 is computed as

\[
N_{10} = [D] \cdot \gamma^{(4)} = 0.
\] (3.16)

One can find that the computations of net chirality agree with those from heterotic spectral cover. Unlike the representation 10 in SU(5) case, the 10 in SO(10) is a real representation. Therefore, it is impossible to engineer a chiral spectrum of 10’s by using a generic SU(4) spectral cover. From Eq. (3.15) and Eq. (3.16), we obtain an SO(10) model with \( -\lambda \eta \cdot (\eta - 4c_1) \) copies of matter on the 16 curve and nothing on the 10 curve. The flux does not have many degrees of freedom to tune and the candidate of 10 Higgs is absent. Therefore, we shall consider factorizations of the SU(4) cover \( C^{(4)} \) to enrich the configuration along the line of the SU(5) cover studied in [39, 40, 44, 45]. Before studying the cove factorizations, we shall construct an SU(4) spectral divisor motivated from heterotic/F-theory duality [1] in section 3.2.

### 3.2 SU(4) Spectral Cover Divisor

Recall that \( Z_4 \) is an elliptically fibered Calabi-Yau fourfold \( \pi : Z_4 \to B_3 \) with a section \( \sigma : B_3 \to Z_4 \). In general, \( Z_4 \) can be described by the Weierstrass model

\[
y^2 = x^3 + f x u^4 + g u^6,
\] (3.17)

where \( f \) and \( g \) are sections of \( K_{B_3}^{-4} \) and \( K_{B_3}^{-6} \), respectively. We now consider the case that \( Z_4 \) exhibits a \( D_5 \) singularity along a divisor \( B_2 \) inside \( B_3 \). We define \( z \) to be a

\[\text{Recall that } Z_4 \text{ can be embedded as a hypersurface of } WP^{3}_{2,3,1} \text{-fibration over } B_3. \text{ It follows from the Calabi-Yau condition } c_1(Z_4) = 0 \text{ that } x, y, \text{ and } u \text{ are sections of } O_{B_3}(2\sigma) \otimes K_{B_3}^{-2}, O_{B_3}(3\sigma) \otimes K_{B_3}^{-3}, \text{ and } O_{B_3}(\sigma), \text{ respectively.}\]
section of the normal bundle $N_{B_2/B_3}$ of $B_2$ in $B_3$. Locally we can expand $f$ and $g$ in the Weierstrass model Eq. \((3.17)\) in terms of $z$. With suitable choice of variables, we obtain

$$y^2 = x^3 + u(zu)[b_0(zu)^4 + b_2(zu)^2x + b_3(zu)y + b_4x^2] + \mathcal{O}(z; u),$$  \(3.18\)

where $\mathcal{O}(z; u)$ stands for the higher order terms of $z$ for each fixed order in $u$. Following the proposal in \([\text{II}]\), we define the $SU(4)$ spectral divisor as

$$\mathcal{D}^{(4)} : b_0(zu)^4 + b_2(zu)^2x + b_3(zu)y + b_4x^2 = 0$$  \(3.19\)

with a projection map $p_{\mathcal{D}^{(4)}} : \mathcal{D}^{(4)} \to B_2$. Note that local behavior of \(\mathcal{D}^{(4)}\) is the same as the union of the exceptional lines described by Eq. \((2.40)\) and that the homological classes of $x, y, u, z$, and $b_m$ in Eq. \((3.19)\) are

$$[x] = 2[\sigma + \pi^*c_1(B_3)], \quad [y] = 3[\sigma + \pi^*c_1(B_3)], \quad [u] = \sigma, \quad [z] = \pi^*B_2$$  \(3.20\)

$$[b_m] = (6 - m)\pi^*c_1(B_3) - (5 - m)\pi^*B_2, \quad m = 0, 2, 3, 4. \quad (3.21)$$

The homological class of the divisor $\mathcal{D}^{(4)}$ is then given by

$$[\mathcal{D}^{(4)}] = 4\sigma + \pi^*[6c_1(B_3) - B_2]. \quad (3.22)$$

In this case the dual matter $16$ surface $\Sigma_{16}$ is determined by the locus of $\{(zu) = 0\} \cap \{b_4 = 0\}$ with homological class

$$[\Sigma_{16}] = (\sigma + \pi^*B_2) \cdot \pi^*[2c_1(B_3) - B_2]. \quad (3.23)$$

On the other hand, the dual matter $10$ surface sits inside the locus of the intersection $\mathcal{D}^{(4)} \cap \tau\mathcal{D}^{(4)}$, where $\tau$ is a $\mathbb{Z}_2$ involution acting on the cover by $y \to -y$ while keeping $x, u, z$ invariant. More precisely, the intersection $\mathcal{D}^{(4)} \cap \tau\mathcal{D}^{(4)}$ is given by

$$\begin{cases} b_3(zu)y = 0 \\ b_0(zu)^4 + b_2(zu)^2x + b_4x^2 = 0. \end{cases} \quad (3.24)$$

We can compute the homological class $[\Sigma_{10}]$ of dual matter $10$ surface as

$$[\Sigma_{10}] = [\mathcal{D}^{(4)}] \cdot [\mathcal{D}^{(4)}] - [zu] \cdot [b_4] - [y] \cdot [b_4x^2] - 2[x] \cdot [z] = \sigma \cdot \pi^*[12c_1(B_3) - 8B_2] + \pi^*[3c_1(B_3) - 2B_2] \cdot \pi^*[6c_1(B_3) - B_2]. \quad (3.25)$$

To obtain chiral spectrum, we turn on a spectral line bundle $\mathcal{N}$ over $\mathcal{D}^{(4)}$. The corresponding Higgs bundle is given by $E = p_{\mathcal{D}^{(4)}*}\mathcal{N}$. For $SU(n)$ bundles, it is required that $c_1(E) = 0$. It follows that

$$c_1(p_{\mathcal{D}^{(4)}*}\mathcal{N}) = p_{\mathcal{D}^{(4)}*}c_1(\mathcal{N}) - \frac{1}{2}p_{\mathcal{D}^{(4)}*}\tau^{(4)} = 0,$$  \(3.26\)
where \( \hat{r}^{(4)} \) is the ramification divisor given by \( \hat{r}^{(4)} = p_{D^{(4)*}}c_1(B_3) - c_1(D^{(4)}) \). It is convenient to define the flux \( \hat{\gamma}^{(4)} \) by

\[
c_1(N) = \lambda \hat{\gamma}^{(4)} + \frac{1}{2} \hat{r}^{(4)},
\]

where \( \lambda \) is a rational number used to compensate the non-integral class \( \frac{1}{2} \hat{r}^{(4)} \) such that \( c_1(N) \in H_2(D^{(4)}, \mathbb{Z}) \). The traceless condition \( c_1(p_{D^{(4)*}}N) = 0 \) is then equivalent to the condition \( p_{D^{(4)*}} \hat{\gamma}^{(4)} = 0 \). Up to multiplication of a constant, the only choice of \( \hat{\gamma}^{(4)} \) satisfying the traceless condition is

\[
\hat{\gamma}^{(4)} = (4 - p_{D^{(4)*}}p_{C^{(4)*}}) ([D^{(4)}] \cdot \sigma).
\]

Since the first Chern class of a line bundle must be integral, it follows that \( \lambda \) and \( \hat{\gamma}^{(4)} \) have to obey the following quantization condition

\[
\lambda \hat{\gamma}^{(4)} + \frac{1}{2} [p_{D^{(4)*}}c_1(B_3) - c_1(D^{(4)})] \in H_2(D^{(4)}, \mathbb{Z}).
\]

In the case of SU(4) spectral divisor, the traceless flux \( \hat{\gamma}^{(4)} \) is given by

\[
\hat{\gamma}^{(4)} = (4 - p_{D^{(4)*}}p_{C^{(4)*}}) ([D^{(4)}] \cdot \sigma) = [D^{(4)}] \cdot \{4\sigma - \pi^*[2c_1(B_3) - B_2]\}.
\]

It follows from Eq. (3.30) and the definition \( \hat{\gamma}^{(4)} = [D^{(4)}] \cdot G^{(4)} \) that \( G^{(4)} = 4\sigma - \pi^*[2c_1(B_3) - B_2] \). With the given cover flux \( \hat{\gamma}^{(4)} \), the net chirality of matter 16 and 10 are respectively given by

\[
N_{16} = [\hat{\Sigma}_{16}] \cdot G^{(4)} \cdot \pi^* B_2 = -(6c_1 - t) \cdot B_2 (2c_1 - t),
\]

\[
N_{10} = [\hat{\Sigma}_{10}] \cdot G^{(4)} \cdot \pi^* B_2 = 0,
\]

where the fact that \( B_2|_{B_2} = -t \) and \( c_1(B_3)|_{B_2} = c_1 - t \) has been used. We found agreement between net chirality from semi-local spectral cover and from spectral divisor construction.

### 4 Chirality

In this section we consider flipped SU(5) GUTs in F-theory. As mentioned in section 1, the construction contains two steps. The first step is to break \( E_8 \) down to \( SO(10) \)
by using $SU(4)$ spectral covers. The second step is to turn on $U(1)_X$ fluxes to break $SO(10)$ down to $SU(5) \times U(1)_X$. In what follows we shall focus on the first step, namely breaking $E_8$ down to $SO(10)$ by using a semi-local $SU(4)$ spectral cover and its global completion, $SU(4)$ spectral divisors. We also analyze the chiral spectra induced by the fluxes. For the analysis of $U(1)_X$ fluxes and numerical models, we refer readers to [61] for the details. We first briefly review (3,1) and (2,2) factorizations of the semi-local $SU(4)$ spectral cover and induced chirality. Then we construct the factorized $SU(4)$ spectral divisor for each factorization and calculate the chirality induced by the fluxes.

### 4.1 Semi-local $SU(4)$ Spectral Cover

#### 4.1.1 Constraints

Before computing the chiral spectra, we take a moment to analyze the constraints for the cover fluxes. Let us consider the case of the cover factorization $\mathcal{C}^{(n)} \to \mathcal{C}^{(l)} \times \mathcal{C}^{(m)}$. To obtain well-defined cover fluxes and maintain supersymmetry, we impose the following constraints [40]:

\[
\begin{align*}
    c_1(p_{\mathcal{C}^{(l)}}_* \mathcal{L}^{(l)}) + c_1(p_{\mathcal{C}^{(m)}}_* \mathcal{L}^{(m)}) &= 0, \\ 
    c_1(\mathcal{L}^{(k)}) &\in H_2(\mathcal{C}^{(k)}, \mathbb{Z}), \quad k = l, m, \\ 
    [c_1(p_{\mathcal{C}^{(l)}}_* \mathcal{L}^{(l)}) - c_1(p_{\mathcal{C}^{(m)}}_* \mathcal{L}^{(m)})] \cdot B_2 [\omega] &= 0,
\end{align*}
\]

where $p_{\mathcal{C}^{(k)}}$ denotes the projection map $p_{\mathcal{C}^{(k)}}: \mathcal{C}^{(k)} \to B_2$, $\mathcal{L}^{(k)}$ is a line bundle over $\mathcal{C}^{(k)}$ and $[\omega]$ is an ample divisor dual to a Kähler form of $B_2$. The first constraint Eq. (4.1) is the traceless condition for the induced Higgs bundles\footnote{We may think of Eq. (4.2) as the traceless condition of an $SU(4)$ bundle $V_4$ over $B_2$ split into $V_3 \oplus L$ with $V_3 = p_{a*} \mathcal{L}^{(a)}$ and $L = p_{b*} \mathcal{L}^{(b)}$. Therefore, the traceless condition of $V_4$ can be expressed by $c_1(V_4) = c_1(p_{a*} \mathcal{L}^{(a)}) + c_1(p_{b*} \mathcal{L}^{(b)}) = 0.$} The second constraint Eq. (4.2) requires that the first Chern class of a well-defined line bundle $\mathcal{L}^{(k)}$ must be integral. The third constraint states that the 2-cycle $[c_1(p_{\mathcal{C}^{(l)}}_* \mathcal{L}^{(l)}) - c_1(p_{\mathcal{C}^{(m)}}_* \mathcal{L}^{(m)})]$ in $B_2$ is supersymmetic. Note that Eq. (4.1) can be expressed as

\[
p_{\mathcal{C}^{(l)}} c_1(\mathcal{L}^{(l)}) - \frac{1}{2} p_{\mathcal{C}^{(l)}} c_1 r^{(l)} + p_{\mathcal{C}^{(m)}} c_1(\mathcal{L}^{(m)}) - \frac{1}{2} p_{\mathcal{C}^{(m)}} c_1 r^{(m)} = 0,
\]

where $p_{\mathcal{C}^{(k)}}$ denotes the projection map $p_{\mathcal{C}^{(k)}}: \mathcal{C}^{(k)} \to B_2$, $\mathcal{L}^{(k)}$ is a line bundle over $\mathcal{C}^{(k)}$ and $[\omega]$ is an ample divisor dual to a Kähler form of $B_2$. The first constraint Eq. (4.1) is the traceless condition for the induced Higgs bundles\footnote{We may think of Eq. (4.2) as the traceless condition of an $SU(4)$ bundle $V_4$ over $B_2$ split into $V_3 \oplus L$ with $V_3 = p_{a*} \mathcal{L}^{(a)}$ and $L = p_{b*} \mathcal{L}^{(b)}$. Therefore, the traceless condition of $V_4$ can be expressed by $c_1(V_4) = c_1(p_{a*} \mathcal{L}^{(a)}) + c_1(p_{b*} \mathcal{L}^{(b)}) = 0.$} The second constraint Eq. (4.2) requires that the first Chern class of a well-defined line bundle $\mathcal{L}^{(k)}$ must be integral. The third constraint states that the 2-cycle $[c_1(p_{\mathcal{C}^{(l)}}_* \mathcal{L}^{(l)}) - c_1(p_{\mathcal{C}^{(m)}}_* \mathcal{L}^{(m)})]$ in $B_2$ is supersymmetic. Note that Eq. (4.1) can be expressed as

\[
p_{\mathcal{C}^{(l)}} c_1(\mathcal{L}^{(l)}) - \frac{1}{2} p_{\mathcal{C}^{(l)}} c_1 r^{(l)} + p_{\mathcal{C}^{(m)}} c_1(\mathcal{L}^{(m)}) - \frac{1}{2} p_{\mathcal{C}^{(m)}} c_1 r^{(m)} = 0,
\]

where $p_{\mathcal{C}^{(k)}}$ denotes the projection map $p_{\mathcal{C}^{(k)}}: \mathcal{C}^{(k)} \to B_2$, $\mathcal{L}^{(k)}$ is a line bundle over $\mathcal{C}^{(k)}$ and $[\omega]$ is an ample divisor dual to a Kähler form of $B_2$. The first constraint Eq. (4.1) is the traceless condition for the induced Higgs bundles\footnote{We may think of Eq. (4.2) as the traceless condition of an $SU(4)$ bundle $V_4$ over $B_2$ split into $V_3 \oplus L$ with $V_3 = p_{a*} \mathcal{L}^{(a)}$ and $L = p_{b*} \mathcal{L}^{(b)}$. Therefore, the traceless condition of $V_4$ can be expressed by $c_1(V_4) = c_1(p_{a*} \mathcal{L}^{(a)}) + c_1(p_{b*} \mathcal{L}^{(b)}) = 0.$} The second constraint Eq. (4.2) requires that the first Chern class of a well-defined line bundle $\mathcal{L}^{(k)}$ must be integral. The third constraint states that the 2-cycle $[c_1(p_{\mathcal{C}^{(l)}}_* \mathcal{L}^{(l)}) - c_1(p_{\mathcal{C}^{(m)}}_* \mathcal{L}^{(m)})]$ in $B_2$ is supersusyymmetric. Note that Eq. (4.1) can be expressed as

\[
p_{\mathcal{C}^{(l)}} c_1(\mathcal{L}^{(l)}) - \frac{1}{2} p_{\mathcal{C}^{(l)}} c_1 r^{(l)} + p_{\mathcal{C}^{(m)}} c_1(\mathcal{L}^{(m)}) - \frac{1}{2} p_{\mathcal{C}^{(m)}} c_1 r^{(m)} = 0,
\]
where \( r(l) \) and \( r(m) \) are the ramification divisors for the maps \( p_{C(l)} \) and \( p_{C(m)} \), respectively. Recall that the ramification divisor \( r(k) \) is defined by

\[
r(k) = p_{C(k)}^* c_1 - c_1(C^{(k)}), \quad k = l, m.
\]

(4.5)

It is convenient to define cover fluxes \( \gamma(k) \) as

\[
c_1(L^{(k)}) = \gamma(k) + \frac{1}{2} r(k), \quad k = l, m.
\]

(4.6)

With Eq. (4.6), the traceless condition Eq. (4.1) can be expressed as

\[
p_{C(l)} \gamma(l) + p_{C(m)} \gamma(m) = 0.
\]

(4.7)

We summarize the constraints for the cover fluxes \( \gamma(k) \) as follows:

\[
p_{C(l)} \gamma(l) + p_{C(m)} \gamma(m) = 0,
\gamma(k) + \frac{1}{2} [p_{C(k)}^* c_1 - c_1(C^{(k)})] \in H_2(C^{(k)}, \mathbb{Z}), \quad k = l, m.
p_{C(k)} \gamma(k) \cdot B_2 [\omega] = 0, \quad k = l, m.
\]

(4.8, 4.9)

In the next section, we shall calculate the homological classes of matter curves for \((3,1)\) and \((2,2)\) factorizations. We also compute the chirality induced by the restriction of the fluxes to each matter curve.

### 4.1.2 \((3,1)\) Factorization

We consider the \((3,1)\) factorization, \( C^{(a)} \rightarrow C^{(a)} \times C^{(b)} \) corresponding to the factorization of Eq. (3.7) as follows:

\[
C^{(a)} \times C^{(b)} : \quad (a_0 U^3 + a_1 U^2 W + a_2 U W^2 + a_3 W^3)(d_0 U + d_1 W) = 0.
\]

(4.10)

By comparing with Eq. (3.7), we can obtain the following decomposition:

\[
b_0 = a_0 d_0, \quad b_1 = a_1 d_0 + a_0 d_1 = 0, \quad b_2 = a_2 d_0 + a_1 d_1, \quad b_3 = a_3 d_0 + a_2 d_1, \quad b_4 = a_3 d_1.
\]

(4.11)

We denote the classes \([d_1]\) by \( \pi^* \xi_1 \) and then write

\[
[d_0] = \pi^*(c_1 + \xi_1), \quad [a_k] = \pi^*[\eta - (k+1)c_1 - \xi_1], \quad k = 0, 1, 2, 3.
\]

(4.12)
To solve the traceless condition \( b_1 = 0 \), we use ansatz \( a_0 = \alpha d_0 \) and \( a_1 = -\alpha d_1 \) where 
\[
[a] = \pi^*(\eta - 2c_1 - 2\xi_1).
\]
It is easy to see that the homological classes of \( C^{(a)} \) and \( C^{(b)} \) in \( \tilde{X} \) are
\[
[C^{(a)}] = 3\sigma + \pi^*(\eta - c_1 - \xi_1), \quad [C^{(b)}] = \sigma + \pi^*(c_1 + \xi_1).
\]

To obtain the 10 curves, we follow the method proposed in [39, 40, 44, 72] to calculate the intersection \( C^{(a)} \cap \tau C^{(a)} \), where \( \tau \) is the \( \mathbb{Z}_2 \) involution \( \tau : W \rightarrow -W \) acting on the spectral cover. Since the calculation is straightforward, we omit the detailed calculation here and only summarize the results in Table 1.

| \( \Sigma \) | \( [C^{(b)}] \) | \( 2[C^{(a)}] \) | \( [C^{(a)}] \) |
|---|---|---|---|
| 16 | \( \sigma \cdot \pi^*\xi_1 \) | - | \( \sigma \cdot \pi^*(\eta - 4c_1 - \xi_1) \) |
| 10 | - | \( 2[\sigma + \pi^*(c_1 + \xi_1)] \) | \( [\sigma + \pi^*(\eta - 2c_1 - \xi_1)] \)
| \( \infty \) | \( \sigma_\infty \cdot \pi^*(c_1 + \xi_1) \) | \( 4\sigma_\infty \cdot \pi^*(c_1 + \xi_1) \) | \( \sigma_\infty \cdot \pi^*(\eta - c_1 - \xi_1) + 2\sigma_\infty \cdot \pi^*\xi_1 \)

Table 1: Matter curves for the factorization \( C^{(a)} = C^{(a)} \times C^{(b)} \).

It follows from Table 1 that the homological classes of 16 curves are
\[
\begin{align*}
\Sigma_{16^{(a)}} &= \sigma \cdot \pi^*(\eta - 4c_1 - \xi_1) \quad (4.14) \\
\Sigma_{16^{(b)}} &= \sigma \cdot \pi^*\xi_1 \quad (4.15)
\end{align*}
\]
and that the homological classes of \( \Sigma_{10^{(a)}} \) and \( \Sigma_{10^{(b)}} \) are
\[
\begin{align*}
\Sigma_{10^{(a)}} &= [2\sigma + \pi^*(\eta - 2c_1 - \xi_1)] \cdot \pi^*(\eta - 3c_1 - \xi_1) + 2(\sigma + \pi^*c_1) \cdot \pi^*\xi_1 \quad (4.16) \\
\Sigma_{10^{(b)}} &= [\sigma + \pi^*(c_1 + \xi_1)] \cdot \pi^*(\eta - 3c_1 - \xi_1) + \sigma \cdot \pi^*\xi_1 \quad (4.17)
\end{align*}
\]

19To simplify notations, we denote \( C^{(k)} \cap \tau C^{(l)} \) by \( C^{(k)(l)} \) and notice that \( [C^{(k)(l)}] = [C^{(l)(k)}] \).

20To avoid a singularity of non-Kodaira type, we impose the condition \( \xi_1 \cdot B_2 \cdot (c_1 + \xi_1) = 0 \).

Therefore, \( \Sigma_{10^{(a)(b)}} = \pi^*\xi_1 \cdot \pi^*(c_1 + \xi_1) = 0 \).

21It follows from Eqs. (4.16) and (4.17) that \( \Sigma_{10^{(a)(a)}} \) and \( \Sigma_{10^{(b)(b)}} \) correspond to the same matter curve in \( B_2 \) with homological class \( \eta - 3c_1 \). In other words, \( \Sigma_{10^{(a)(a)}} \) and \( \Sigma_{10^{(b)(b)}} \) both are lifts of the same curve in \( B_2 \). The 10 matter curve inside the cover \( C^{(a)} \) is actually 4-sheeted cover of the corresponding matter curve in \( B_2 \). A nice description of the cover structure for the 10 curve can be found in [24].
For the \((3, 1)\) factorization, the ramification divisors for the spectral covers \(C^{(a)}\) and \(C^{(b)}\) are given by
\[ r^{(a)} = [C^{(a)}] \cdot [\sigma + \pi^*(\eta - 2c_1 - \xi_1)] \]
\[ r^{(b)} = [C^{(b)}] \cdot (\sigma - \pi^*\xi_1), \tag{4.18} \]
respectively. We define traceless fluxes \(\gamma_0^{(a)}\) and \(\gamma_0^{(b)}\) by
\[ \gamma_0^{(a)} = (3 - p_{C^{(a)}P^{(a)}}^*) \gamma^{(a)} = [C^{(a)}] \cdot [3\sigma - \pi^*(\eta - 4c_1 - \xi_1)] \]
\[ \gamma_0^{(b)} = (1 - p_{C^{(b)}P^{(b)}}^*) \gamma^{(b)} = [C^{(b)}] \cdot (\sigma - \pi^*\xi_1), \tag{4.19} \]
where \(\gamma^{(a)}\) and \(\gamma^{(b)}\) are non-traceless fluxes and defined by
\[ \gamma^{(a)} = [C^{(a)}] \cdot \sigma, \quad \gamma^{(b)} = [C^{(b)}] \cdot \sigma. \tag{4.20} \]

Then we can calculate the restriction of fluxes \(\gamma_0^{(a)}\) and \(\gamma_0^{(b)}\) to each matter curve. We omit the calculation here and only summarize the results in Table 2. We also can define additional fluxes \(\delta^{(a)}\) and \(\delta^{(b)}\) by
\[ \delta^{(a)} = (1 - p_{C^{(b)}P^{(a)}}^*) \gamma^{(a)} = [C^{(a)}] \cdot \sigma - [C^{(b)}] \cdot \pi^*(\eta - 4c_1 - \xi_1) \]
\[ \delta^{(b)} = (3 - p_{C^{(a)}P^{(b)}}^*) \gamma^{(b)} = [C^{(b)}] \cdot 3\sigma - [C^{(a)}] \cdot \pi^*\xi_1. \tag{4.21} \]

Another flux we can include is \([40]\)
\[ \rho^{(3,1)} = (3p_{b}^* - p_{a}^*)\rho, \tag{4.22} \]
where \(\rho \in H_2(B_2, \mathbb{R})\). We summarize the restriction of fluxes \(\delta^{(a)}\), \(\delta^{(b)}\) and \(\rho^{(3,1)}\) to each matter curve in Table 3.

With Eqs. \([4.19]\), \([4.21]\), and \([4.22]\), we define the universal cover flux \(\Gamma\) to be \([40]\)
\[ \Gamma = k_a \gamma_0^{(a)} + k_b \gamma_0^{(b)} + m_a \delta^{(a)} + m_b \delta^{(b)} + \rho^{(3,1)} \equiv \Gamma^{(a)} + \Gamma^{(b)}, \tag{4.23} \]
Note that

\[ \Gamma \]

\[ \text{Clearly, then can factorize Eq. (3.7) into the following form:} \]

\[ \text{By comparing the coefficients with Eq. (3.7), we obtain} \]

\[ \delta^{(b)} = \begin{pmatrix} -3c_1 \cdot B_2 \xi_1 \\ -c_1 \cdot B_2 (\eta - 4c_1 - \xi_1) \\ -(\eta - 3c_1 - \xi_1) \cdot B_2 (\eta - 4c_1 - \xi_1) \\ -2 \eta \cdot B_2 (\eta - 3c_1) \end{pmatrix} \]

\[ \delta^{(a)} = \begin{pmatrix} -\xi_1 \cdot B_2 (\eta - 4c_1 - \xi_1) \\ -c_1 \cdot B_2 (\eta - 4c_1 - \xi_1) \\ -(\eta - 3c_1 - \xi_1) \cdot B_2 (\eta - 4c_1 - \xi_1) \\ -2 \eta \cdot B_2 (\eta - 3c_1) \end{pmatrix} \]

\[ \rho^{(3,1)} = \begin{pmatrix} 3 \rho \cdot B_2 \xi_1 \\ -\rho \cdot B_2 (\eta - 4c_1 - \xi_1) \\ 2 \rho \cdot B_2 (\eta - 3c_1) \\ -2 \rho \cdot B_2 (\eta - 3c_1) \end{pmatrix} \]

Table 3: Chirality induced by the fluxes \( \delta^{(a)} \), \( \delta^{(b)} \), and \( \rho^{(3,1)} \).

where \( \Gamma^{(a)} \) and \( \Gamma^{(b)} \) are defined by

\[ \Gamma^{(a)} = [C^{(a)}] \cdot [(3k_a + m_a) \sigma - \pi^*(k_a(\eta - 4c_1 - \xi_1) + m_b \xi_1 + \rho)], \quad (4.24) \]

\[ \Gamma^{(b)} = [C^{(b)}] \cdot [(k_b + 3m_b) \sigma - \pi^*(k_b \xi_1 + m_a(\eta - 4c_1 - \xi_1) - 3\rho)]. \quad (4.25) \]

Note that

\[ p_{C^{(a)} \cdot \Gamma^{(a)}} = -3m_b \xi_1 + m_a(\eta - 4c_1 - \xi_1) - 3\rho, \quad (4.26) \]

\[ p_{C^{(b)} \cdot \Gamma^{(b)}} = 3m_b \xi_1 - m_a(\eta - 4c_1 - \xi_1) + 3\rho. \quad (4.27) \]

Clearly, \( \Gamma^{(a)} \) and \( \Gamma^{(b)} \) obey the traceless condition \( p_{C^{(a)} \cdot \Gamma^{(a)}} + p_{C^{(b)} \cdot \Gamma^{(b)}} = 0 \). Besides, the quantization condition in this case becomes

\[ (3k_a + m_a + \frac{1}{2}) \sigma - \pi^*[k_a(\eta - 4c_1 - \xi_1) + m_b \xi_1 + \rho - \frac{1}{2}(\eta - 2c_1 - \xi_1)] \in H_4(\tilde{X}, \mathbb{Z}), \quad (4.28) \]

\[ (k_b + 3m_b - \frac{1}{2}) \sigma - \pi^*[k_b \xi_1 + m_a(\eta - 4c_1 - \xi_1) - 3\rho - \frac{1}{2}\xi_1] \in H_4(\tilde{X}, \mathbb{Z}). \quad (4.29) \]

The supersymmetry condition is given by

\[ [3m_b \xi_1 - m_a(\eta - 4c_1 - \xi_1) + 3\rho] \cdot B_2 [\omega] = 0. \quad (4.30) \]

4.1.3 (2,2) Factorization

In the case of the (2,2) factorization, the cover is split as \( C^{(4)} \rightarrow C^{(d_1)} \times C^{(d_2)} \). We then can factorize Eq. \( (3.7) \) into the following form:

\[ C^{(d_1)} \times C^{(d_2)} : (e_0 U^2 + e_1 U W + e_2 W^2)(f_0 U^2 + f_1 U W + f_2 W^2) = 0 \quad (4.31) \]

By comparing the coefficients with Eq. \( (3.7) \), we obtain

\[ b_0 = e_0 f_0, \quad b_1 = e_0 f_1 + e_1 f_0 = 0, \quad b_2 = e_0 f_2 + e_1 f_1 + e_2 f_0, \quad b_3 = e_1 f_2 + e_2 f_1, \quad b_4 = e_2 f_2. \quad (4.32) \]
By denoting the homological class of $f_2$ by $\pi^*\xi_2$, the classes of other sections can be written as

$$[f_1] = \pi^*(c_1 + \xi_2), \ [f_0] = \pi^*(2c_1 + \xi_2), \ [e_m] = \pi^*[\eta - (m + 2)c_1 - \xi_2], \ m = 0, 1, 2. \quad (4.33)$$

To solve the traceless condition $b_1 = 0$, we impose the condition $e_0 = \beta f_0$ and $e_1 = -\beta f_1$ where $[\beta] = \pi^*(\eta - 4c_1 - 2\xi_2)$. In this case, the homological classes of $C^{(d_1)}$ and $C^{(d_2)}$ are given by

$$[C^{(d_1)}] = 2\sigma + \pi^*(\eta - 2c_1 - \xi_2), \ [C^{(d_2)}] = 2\sigma + \pi^*(2c_1 + \xi_2). \quad (4.34)$$

To find the 10 curves, we again follow the method proposed in $[39, 40, 44, 72]$ to calculate the intersection $C^{(4)} \cap \pi C^{(4)}$. We omit the detailed calculation here and only summarize the results in Table 4.

| $|C^{(d_2)}(d_2)|$ | $2|C^{(d_1)}(d_2)|$ | $|C^{(d_1)}(d_1)|$ |
|---|---|---|
| 16 | $\sigma \cdot \pi^*\xi_2$ | $\sigma \cdot \pi^*(\eta - 4c_1 - \xi_2)$ |
| 10 | $[2\sigma + \pi^*(2c_1 + \xi_2)] \cdot \pi^*(c_1 + \xi_2)$ | $2[2\sigma + \pi^*(2c_1 + \xi_2)] \cdot \pi^*(\eta - 4c_1 - \xi_2)$ | $\pi^*(\eta - 3c_1 - \xi_2) \cdot \pi^*(\eta - 4c_1 - \xi_2)$ | $+ 2(\sigma + \pi^*c_1) \cdot \pi^*(c_1 + \xi_2)$ |
| $\infty$ | $\sigma_\infty \cdot \pi^*(2c_1 + \xi_2)$ | $4\sigma_\infty \cdot \pi^*(2c_1 + \xi_2)$ | $\sigma_\infty \cdot \pi^*(\eta - 2c_1 - \xi_2)$ | $+ 2\sigma_\infty \cdot \pi^*(\eta - 4c_1 - 2\xi_2)$ |

Table 4: Matter curves for the factorization $C^{(4)} = C^{(d_1)} \times C^{(d_2)}$.

It follows from Table 4 that the homological classes of the factorized 16 curves are

$$[\Sigma_{16^{(d_1)}}] = \sigma \cdot \pi^*(\eta - 4c_1 - \xi_2), \quad (4.35)$$

$$[\Sigma_{16^{(d_2)}}] = \sigma \cdot \pi^*\xi_2, \quad (4.36)$$

and that the homological classes of the factorized 10 curves are

$$[\Sigma_{10^{(d_1)}}] = 2(\sigma + \pi^*c_1) \cdot \pi^*(c_1 + \xi_2) + \pi^*(\eta - 3c_1 - \xi_2) \cdot \pi^*(\eta - 4c_1 - \xi_2), \quad (4.37)$$

$$[\Sigma_{10^{(d_1)}}] = [2\sigma + \pi^*(2c_1 + \xi_2)] \cdot \pi^*(\eta - 4c_1 - \xi_2), \quad (4.38)$$

$$[\Sigma_{10^{(d_2)}}] = [2\sigma + \pi^*(2c_1 + \xi_2)] \cdot \pi^*(c_1 + \xi_2). \quad (4.39)$$

$^{22}$It follows from Eqs. (4.37)-(4.39) that $[\Sigma_{10^{(d_1)}}]$ and $[\Sigma_{10^{(d_2)}}]$ correspond to the same curve with class $c_1 + \xi_2$ in $B_2$, and $[\Sigma_{10^{(d_1)}}]|_\sigma = 2(\eta - 4c_1 - \xi_2)$ in $B_2$. 

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In the (2, 2) factorization, the ramification divisors \( r^{(d_1)} \) and \( r^{(d_2)} \) for the covers \( C^{(d_1)} \) and \( C^{(d_2)} \) are given by
\[
\begin{align*}
  r^{(d_1)} &= [C^{(d_1)}] \cdot \pi^*(\eta - 3c_1 - \xi_2), \\
  r^{(d_2)} &= [C^{(d_2)}] \cdot \pi^*(c_1 + \xi_2),
\end{align*}
\]
respectively. We then define traceless cover fluxes \( \gamma_0^{(d_1)} \) and \( \gamma_0^{(d_2)} \) by
\[
\begin{align*}
  \gamma_0^{(d_1)} &= (2 - p_{C^{(d_1)}pC^{(d_1)\ast}}^*) \gamma^{(d_1)} = [C^{(d_1)}] \cdot [2\sigma - \pi^*(\eta - 4c_1 - \xi_2)], \\
  \gamma_0^{(d_2)} &= (2 - p_{C^{(d_2)}pC^{(d_2)\ast}}^*) \gamma^{(d_2)} = [C^{(d_2)}] \cdot (2\sigma - \pi^*\xi_2),
\end{align*}
\]
where \( \gamma^{(d_1)} \) and \( \gamma^{(d_2)} \) are non-traceless fluxes and defined by
\[
\begin{align*}
  \gamma^{(d_1)} &= [C^{(d_1)}] \cdot \sigma, \\
  \gamma^{(d_2)} &= [C^{(d_2)}] \cdot \sigma.
\end{align*}
\]

We summarize the restriction of the fluxes to each factorized curve in Table 5. We also can define two fluxes
\[
\begin{align*}
  \delta^{(d_1)} &= (2 - p_{C^{(d_2)}pC^{(d_1)\ast}}^*) \gamma^{(d_1)} = [C^{(d_1)}] \cdot 2\sigma - [C^{(d_2)}] \cdot \pi^*(\eta - 4c_1 - \xi_2), \\
  \delta^{(d_2)} &= (2 - p_{C^{(d_1)}pC^{(d_2)\ast}}^*) \gamma^{(d_2)} = [C^{(d_2)}] \cdot 2\sigma - [C^{(d_1)}] \cdot \pi^*\xi_2.
\end{align*}
\]

Another flux we can include is
\[
\rho^{(2,2)} = (p_{C^{(d_2)}pC^{(d_1)\ast}}^* - p_{C^{(d_2)\ast}}^*) \rho,
\]
where \( \rho \in H_2(B_2, \mathbb{R}) \). We summarize the restriction of the fluxes \( \delta^{(d_1)} \), \( \delta^{(d_2)} \), and \( \rho^{(2,2)} \) to each factorized curve in Table 6.

Again we conclude the universal cover flux to be
\[
\Gamma = k_{d_1} \gamma_0^{(d_1)} + k_{d_2} \gamma_0^{(d_2)} + m_{d_1} \delta^{(d_1)} + m_{d_2} \delta^{(d_2)} + \rho^{(2,2)} = \Gamma^{(d_1)} + \Gamma^{(d_2)},
\]
Note that
\[ \Gamma(4) = [C^{(d_1)}] \cdot \{2 (k_{d_1} + m_{d_1}) \sigma - \pi^*[k_{d_1} (\eta - 4c_1 - \xi_2) + m_{d_2} \xi_2 + \rho]\}, \]
\[ \Gamma(4d_2) = [C^{(d_2)}] \cdot \{2 (k_{d_2} + m_{d_2}) \sigma - \pi^*[k_{d_2} \xi_2 + m_{d_1} (\eta - 4c_1 - \xi_2) - \rho]\}. \quad (4.46) \]

Note that
\[ p_{C^{(d_1)*}} \Gamma^{(d_1)} = -2m_{d_2} \xi_2 + 2m_{d_1} (\eta - 4c_1 - \xi_2) - 2\rho, \quad (4.47) \]
\[ p_{C^{(d_2)*}} \Gamma^{(d_2)} = 2m_{d_2} \xi_2 - 2m_{d_1} (\eta - 4c_1 - \xi_2) + 2\rho. \quad (4.48) \]

It is easy to see that \( \Gamma^{(d_1)} \) and \( \Gamma^{(d_2)} \) satisfy the traceless condition \( p_{C^{(d_1)*}} \Gamma^{(d_1)} + p_{C^{(d_2)*}} \Gamma^{(d_2)} = 0 \). In addition, the quantization condition in this case becomes
\[ 2(k_{d_1} + m_{d_1}) \sigma - \pi^*[k_{d_1} (\eta - 4c_1 - \xi_2) + m_{d_2} \xi_2 + \rho - \frac{1}{2}(\eta - 3c_1 - \xi_2)] \in H_4(\bar{X}, \mathbb{Z}), \quad (4.49) \]
\[ 2(k_{d_2} + m_{d_2}) \sigma - \pi^*[k_{d_2} \xi_2 + m_{d_1} (\eta - 4c_1 - \xi_2) - \rho - \frac{1}{2}(c_1 + \xi_2)] \in H_4(\bar{X}, \mathbb{Z}). \quad (4.50) \]

The supersymmetry condition is then given by
\[ [2m_{d_2} \xi_2 - 2m_{d_1} (\eta - 4c_1 - \xi_2) + 2\rho] \cdot_{B_2} [\omega] = 0. \quad (4.51) \]

### 4.2 Global SU(4) Spectral Divisor

#### 4.2.1 Constraints

Similar to the analysis in the last section, we analyze the constraints for the fluxes of the spectral divisors. It was argued in [II] that these constraints could be consistent with that for the semi-local cover fluxes. Let us consider the case of the cover
factorization $D^{(n)} \to D^{(l)} \times D^{(m)}$. To obtain well-defined cover fluxes and maintain supersymmetry, we impose the following constraints [1]:

\[
\begin{align*}
    c_1(p_{D^{(k)}}^* \mathcal{N}^{(l)}) + c_1(p_{D^{(m)}}^* \mathcal{N}^{(m)}) &= 0, \\
    c_1(\mathcal{N}^{(k)}) &\in H_2(D^{(k)}, \mathbb{Z}), \quad k = l, m,
\end{align*}
\]

where $p_{D^{(k)}}$ denotes the projection map $D^{(k)} : D^{(k)} \to B_3$, $\mathcal{N}^{(k)}$ is a line bundle over $C^{(k)}$. The first constraint, Eq. (4.52) is the traceless condition for the induced Higgs bundle. The second constraint, Eq. (4.53) requires that the first Chern class of a well-defined line bundle $\mathcal{N}^{(k)}$ must be integral. Note that Eq. (4.52) can be expressed as

\[
p_{D^{(l)}}^* c_1(\mathcal{N}^{(l)}) - \frac{1}{2} p_{D^{(l)}}^* \mathring{\mathcal{N}}^{(l)} + p_{D^{(m)}}^* c_1(\mathcal{N}^{(m)}) - \frac{1}{2} p_{D^{(m)}}^* \mathring{\mathcal{N}}^{(m)} = 0,
\]

where $\mathring{\mathcal{N}}^{(l)}$ and $\mathring{\mathcal{N}}^{(m)}$ are the ramification divisors for the maps $p_{D^{(l)}}$ and $p_{D^{(m)}}$, respectively. Recall that the ramification divisor $\mathring{\mathcal{N}}^{(k)}$ is defined by

\[
\mathring{\mathcal{N}}^{(k)} = p_{D^{(k)}}^* c_1(B_3) - c_1(D^{(k)}), \quad k = l, m.
\]

It is convenient to define fluxes $\mathring{\gamma}^{(k)}$ as

\[
c_1(\mathcal{N}^{(k)}) = \mathring{\gamma}^{(k)} + \frac{1}{2} \mathring{\mathcal{N}}^{(k)}, \quad k = l, m.
\]

With Eq. (4.56), the traceless condition Eq. (4.52) can be expressed as $p_{D^{(l)}}^* \mathring{\gamma}^{(l)} + p_{D^{(m)}}^* \mathring{\gamma}^{(m)} = 0$. By using Eq. (4.55) and Eq. (4.56), we can recast the quantization condition Eq. (4.53) by $\mathring{\gamma}^{(k)} + \frac{1}{2} [p_{D^{(k)}}^* c_1(B_3) - c_1(D^{(k)})] \in H_2(D^{(k)}, \mathbb{Z}), \quad k = l, m$. We summarize the constraints for the fluxes $\mathring{\gamma}^{(k)}$ as follows:

\[
\begin{align*}
    p_{D^{(l)}}^* \mathring{\gamma}^{(l)} + p_{D^{(m)}}^* \mathring{\gamma}^{(m)} &= 0, \\
    \mathring{\gamma}^{(k)} + \frac{1}{2} [p_{D^{(k)}}^* c_1(B_3) - c_1(D^{(k)})] &\in H_2(D^{(k)}, \mathbb{Z}), \quad k = l, m.
\end{align*}
\]

In the next section, we shall calculate the homological classes of the dual matter surfaces for $(3, 1)$ and $(2, 2)$ factorizations. We also compute the chirality induced by the restriction of the fluxes to each dual matter surface.

### 4.2.2 $(3, 1)$ Factorization

It will be convenient to define $x = \zeta^2$ and $y = \zeta^3$ where $\zeta$ is a section of $O_{B_3}(\sigma) \otimes K_{B_3}^{-1}$. Then the $SU(4)$ spectral divisor defined by Eq. (3.19) can be written as

\[
D^{(4)} : \quad b_0(zu)^4 + b_2(zu)^2 \zeta^2 + b_3(zu) \zeta^3 + b_4 \zeta^4 = 0.
\]
We now consider the $(3,1)$ factorization $\mathcal{D}^{(4)} \to \mathcal{D}^{(a)} \times \mathcal{D}^{(b)}$ corresponding to the factorization of Eq. (4.59)

$$\mathcal{D}^{(a)} \times \mathcal{D}^{(b)} : \ [\tilde{a}_0(zu)^3 + \tilde{a}_1(zu)^2\zeta + \tilde{a}_2(zu)\zeta^2 + \tilde{a}_3\zeta^3][\tilde{d}_0(zu) + \tilde{d}_1\zeta] = 0,$$  

(4.60)

with projection maps $p_{\mathcal{D}^{(a)}} : \mathcal{D}^{(a)} \to B_3$ and $p_{\mathcal{D}^{(b)}} : \mathcal{D}^{(b)} \to B_3$. By comparing with Eq. (4.59), we can obtain the following relations:

$$b_0 = \tilde{a}_0\tilde{d}_0, \ b_1 = \tilde{a}_1\tilde{d}_0 + \tilde{a}_0\tilde{d}_1, \ b_2 = \tilde{a}_2\tilde{d}_0 + \tilde{a}_1\tilde{d}_1, \ b_3 = \tilde{a}_3\tilde{d}_0 + \tilde{a}_2\tilde{d}_1, \ b_4 = \tilde{a}_3\tilde{d}_1.$$

(4.61)

We denote the homological class of $[\tilde{d}_1]$ by $\pi^*\hat{\xi}_1$ and then write

$$[\tilde{d}_0] = \pi^*[c_1(B_3) - B_2 + \hat{\xi}_1], \ [\tilde{a}_k] = \pi^*[(5-m)c_1(B_3) - (4-m)B_2 - \hat{\xi}_1], \ m = 0, 1, 2, 3.$$  

(4.62)

It is easy to see that the homological classes of $\mathcal{D}^{(a)}$ and $\mathcal{D}^{(b)}$ are given by

$$[\mathcal{D}^{(a)}] = 3\sigma + \pi^*[5c_1(B_3) - B_2 - \hat{\xi}_1], \ [\mathcal{D}^{(b)}] = \sigma + \pi^*[c_1(B_3) + \hat{\xi}_1].$$  

(4.63)

Note that the unfactorized dual matter 16 surface sits inside the locus of $\{(zu) = 0\} \cap \{b_4 = 0\}$. Due to the factorization in Eq. (4.60), the factorized dual matter 16 surfaces sit inside the loci $\{(zu) = 0\} \cap \{\tilde{a}_3 = 0\}$ and $\{(zu) = 0\} \cap \{\tilde{d}_1 = 0\}$. The homological class of dual matter surfaces $\hat{\Sigma}_{16}^{(a)}$ and $\hat{\Sigma}_{16}^{(b)}$ are given by

$$[\hat{\Sigma}_{16}^{(a)}] = (\sigma + \pi^*B_2) \cdot \pi^*[2c_1(B_3) - B_2 - \hat{\xi}_1], \ [\hat{\Sigma}_{16}^{(b)}] = (\sigma + \pi^*B_2) \cdot \pi^*\hat{\xi}_1.$$  

(4.64)

To obtain dual matter surface $\hat{\Sigma}_{10}$’s, we calculate the intersection $\mathcal{D}^{(4)} \cap \tau\mathcal{D}^{(4)}$, where $\tau$ is a $\mathbb{Z}_2$ involution $\zeta \to -\zeta$ acting on $\mathcal{D}^{(4)}$ [39,40,41,72]. Under $(3,1)$ factorization $\mathcal{D}^{(4)} \to \mathcal{D}^{(a)} \times \mathcal{D}^{(b)}$, the intersection $\mathcal{D}^{(4)} \cap \tau\mathcal{D}^{(4)}$ can be decomposed into several components $\mathcal{D}^{(a)} \cap \tau\mathcal{D}^{(a)}$, $\mathcal{D}^{(a)} \cap \tau\mathcal{D}^{(b)}$, and $\mathcal{D}^{(b)} \cap \tau\mathcal{D}^{(b)}$. We first consider the case of $\mathcal{D}^{(a)} \cap \tau\mathcal{D}^{(a)}$. This intersection is determined by

$$\begin{cases} (zu)[\tilde{a}_0(zu)^2 + \tilde{a}_2\zeta^2] = 0 \\ \zeta[\tilde{a}_1(zu)^2 + \tilde{a}_3\zeta^2] = 0. \end{cases}$$  

(4.65)

To solve the constraint $b_1 = \tilde{a}_1\tilde{d}_0 + \tilde{a}_0\tilde{d}_1 = 0$, we use ansatz $\tilde{a}_0 = \tilde{a}\tilde{d}_0$ and $\tilde{a}_1 = -\tilde{a}\tilde{d}_1$ where the homological class of $\tilde{a}$ is $[\tilde{a}] = \pi^*[4c_1(B_3) - 3B_2 - 2\hat{\xi}_1]$. By using the ansatz, we obtain

$$\begin{cases} (zu)[\tilde{a}\tilde{d}_0(zu)^2 + \tilde{a}_2\zeta^2] = 0 \\ \zeta[-\tilde{a}\tilde{d}_1(zu)^2 + \tilde{a}_3\zeta^2] = 0. \end{cases}$$  

(4.66)
It follows from Eq. (4.66) that the homological class of dual matter surface $\tilde{\Sigma}_{10(a)(a)}$ is given by

$$\tilde{\Sigma}_{10(a)(a)} = [D^{(a)}] \cdot [D^{(a)}] - [\zeta] \cdot [\tilde{a}_0] - 9[\zeta] \cdot [zu] + 2[\zeta] \cdot [\tilde{\alpha}]$$

$$\begin{align*}
&= 2\sigma + \pi^* [4c_1(B_3) - B_2 - \tilde{\zeta}_1] \cdot \pi^* [3c_1(B_3) - 2B_2 - \tilde{\zeta}_1] \\
&+ 2[\sigma + \pi^* c_1(B_3)] \cdot \pi^* \tilde{\zeta}_1.
\end{align*}$$

Next we calculate the intersection $D^{(a)} \cap \tau D^{(b)}$ which is given by

$$\begin{align*}
\begin{cases}
\tilde{a}_0(zu)^3 + \tilde{a}_1(zu)^2 \zeta + \tilde{a}_2(zu) \zeta^2 + \tilde{a}_3 \zeta^3 = 0 \\
\tilde{d}_0(zu) - \tilde{d}_1 \zeta = 0.
\end{cases}
\end{align*}$$

By using the ansatz, we can rewrite Eq. (4.68) as

$$\begin{align*}
\begin{cases}
\zeta^2(\tilde{a}_2(zu) + \tilde{a}_3 \zeta) = 0 \\
\tilde{d}_0(zu) - \tilde{d}_1 \zeta = 0.
\end{cases}
\end{align*}$$

It follows from Eq. (4.69) that the homological class of dual matter surface $\tilde{\Sigma}_{10(a)(b)}$ is

$$\tilde{\Sigma}_{10(a)(b)} = [D^{(a)}] \cdot [D^{(b)}] - 3[\zeta] \cdot [\tilde{d}_0] - 2\zeta - 2\zeta^2 - \zeta^3$$

$$\begin{align*}
&= \{\sigma + \pi^* [c_1(B_3) + \tilde{\zeta}_1] \} \cdot \pi^* [3c_1(B_3) - 2B_2 - \tilde{\zeta}_1] \\
&+ (\sigma + \pi^* B_2) \cdot \pi^* \tilde{\zeta}_1.
\end{align*}$$

Let us turn to the case of $D^{(b)} \cap \tau D^{(b)}$ which is determined by

$$\begin{align*}
\begin{cases}
\tilde{d}_0(zu) = 0 \\
\tilde{d}_1 \zeta = 0.
\end{cases}
\end{align*}$$

Then the homological class of dual matter surface $\tilde{\Sigma}_{10(b)(b)}$ is given by

$$\tilde{\Sigma}_{10(b)(b)} = [D^{(b)}] \cdot [D^{(b)}] - [\zeta] \cdot [\tilde{d}_0] - [\zeta] \cdot [\tilde{d}_1] - [zu] \cdot [\zeta] \cdot [zu]$$

$$\begin{align*}
&= \pi^*[c_1(B_3) - B_2 + \tilde{\zeta}_1] \cdot \pi^* \tilde{\zeta}_1.
\end{align*}$$

We summarize the homological classes of dual matter 16 and 10 surfaces in Table 43.

In (3, 1) factorization, the ramification divisors for $D^{(a)}$ and $D^{(b)}$ are given by

$$\tilde{r}^{(a)} = [D^{(a)}] \cdot \{\sigma + \pi^* [4c_1(B_3) - 2B_2 - \tilde{\zeta}_1]\},$$

$$\tilde{r}^{(b)} = [D^{(b)}] \cdot [-\sigma + \pi^*(B_2 - \tilde{\zeta}_1)],$$

\[\text{In the case of } 10^{(b)(b)}, \text{ we impose the condition } \pi^*[c_1(B_3) - B_2 + \tilde{\zeta}_1] \cdot \pi^* \tilde{\zeta}_1 = 0 \text{ to avoid the appearance of a singularity.} \]
By using Eq. (4.76) and following the formula in section 3.2, the net chirality of matter in the representation \( r \) induced by the flux \( \mathcal{G} \) is

\[
N_r = [\hat{\Sigma}_r] \cdot \mathcal{G} \cdot \pi^* B_2,
\]

(4.76)

where \( [\hat{\Sigma}_r] \) is the homological class of dual surface for matter in the representation \( r \).

By using Eq. (4.76) and \( \hat{\xi}_1|_{B_2} = \xi_1 \), we can calculate the restriction of fluxes \( \hat{\gamma}_{0}^{(a)} \) and \( \hat{\gamma}_{0}^{(b)} \) to each dual matter surface. We omit the calculation here and only summarize the results in Table 8.

| Field         | Homological Class                                                                 |
|---------------|-----------------------------------------------------------------------------------|
| \( 16^{(b)} \)| \( (\sigma + \pi^* B_2) \cdot \pi^* \xi_1 \)                                      |
| \( 16^{(a)} \)| \( (\sigma + \pi^* B_2) \cdot \pi^*[2c_1(B_3) - B_2 - \xi_1] \)                |
| \( 10^{(b)(b)} \)| -                                                                                   |
| \( 10^{(a)(b)} \)| \( \{\sigma + \pi^*[c_1(B_3) + \xi_1]\} \cdot \pi^*[3c_1(B_3) - 2B_2 - \xi_1] \) + \( \{\sigma + \pi^* B_2\} \cdot \pi^* \hat{\xi}_1 \) |
| \( 10^{(a)(a)} \)| \( \{2\sigma + \pi^*[4c_1(B_3) - B_2 - \xi_1]\} \cdot \pi^*[3c_1(B_3) - 2B_2 - \xi_1] \) + \( 2[\sigma + \pi^* c_1(B_3)] \cdot \pi^* \hat{\xi}_1 \) |

Table 7: Dual matter surfaces for the factorization \( \mathcal{D}^{(4)} = \mathcal{D}^{(a)} \times \mathcal{D}^{(b)} \).

| Field | \( \hat{\gamma}_{0}^{(b)} \) | \( \hat{\gamma}_{0}^{(a)} \) |
|-------|-----------------------------|-----------------------------|
| \( 16^{(b)} \)| \(-\xi_1 \cdot B_2 (c_1 + \xi_1)\) | 0                           |
| \( 16^{(a)} \)| 0                           | \(-5c_1 - t - \xi_1) \cdot B_2 (2c_1 - t - \xi_1)\) |
| \( 10^{(b)(b)} \)| 0                           | \(-3c_1 - t - 3\xi_1) \cdot B_2 (2c_1 - t - \xi_1)\) |
| \( 10^{(a)(a)} \)| 0                           | \(3c_1 - t - 3\xi_1) \cdot B_2 (2c_1 - t - \xi_1)\) |

Table 8: Chirality induce by the fluxes \( \hat{\gamma}_{0}^{(a)} \) and \( \hat{\gamma}_{0}^{(b)} \).
We also can define additional fluxes \( \tilde{\delta}^{(a)} \) and \( \tilde{\delta}^{(b)} \) by

\[
\tilde{\delta}^{(a)} = (1 - p_{D(a)}^* p_{D(a)_*}) \gamma^{(a)} = [D^{(a)}] \cdot \sigma - [D^{(b)}] \cdot \pi^\dagger [2c_1(B_3) - B_2 - \xi_1],
\]
\[
\tilde{\delta}^{(b)} = (3 - p_{D(a)}^* p_{D(a)_*}) \gamma^{(b)} = [D^{(b)}] \cdot 3\sigma - [D^{(a)}] \cdot \pi^\dagger \xi_1.
\] (4.77)

Another flux we can include is

\[
\tilde{\rho}^{(3,1)} = (3p_{D(a)}^* - p_{D(a)_*}) \tilde{\rho},
\] (4.78)

where \( \tilde{\rho} \in H_2(B_3, \mathbb{R}) \) with \( \tilde{\rho}|_{B_2} = \rho \). We summarize the restriction of fluxes \( \tilde{\delta}^{(a)}, \tilde{\delta}^{(b)} \) and \( \tilde{\rho}^{(3,1)} \) to each matter curve in Table 9.

| \( \delta^{(b)} \) | \( \delta^{(a)} \) | \( \tilde{\rho}^{(3,1)} \) |
|------------------|------------------|------------------|
| 16\(^{(b)}\)     | \(-3c_1 \cdot B_2 \xi_1\) | \(-\xi_1 \cdot B_2 (2c_1 - t - \xi_1)\) | \(3\rho \cdot B_2 \xi_1\) |
| 16\(^{(a)}\)     | \(-\xi_1 \cdot B_2 (2c_1 - t - \xi_1)\) | \(-c_1 \cdot B_2 (2c_1 - t - \xi_1)\) | \(-\rho \cdot B_2 (2c_1 - t - \xi_1)\) |
| 10\(^{(a)}\)\(^(b)\) | \(\xi_1 \cdot B_2 (3c_1 - 2t - 3\xi_1)\) | \(-3(3c_1 - t - \xi_1) \cdot B_2 (2c_1 - t - \xi_1)\) | \(2\rho \cdot B_2 (3c_1 - t)\) |
| 10\(^{(a)}\)\(^{(a)}\) | \(-2\xi_1 \cdot B_2 (3c_1 - t)\) | \((3c_1 - t - \xi_1) \cdot B_2 (2c_1 - t - \xi_1)\) | \(-2\rho \cdot B_2 (3c_1 - t)\) |

Table 9: Chirality induce by the fluxes \( \tilde{\delta}^{(a)}, \tilde{\delta}^{(b)}, \) and \( \tilde{\rho}^{(3,1)} \).

With Eq. (4.74), (4.77), and (4.78), we define the universal flux \( \tilde{\Gamma} \) to be

\[
\tilde{\Gamma} = \tilde{k}_a \tilde{\sigma}^{(a)} + \tilde{k}_b \tilde{\sigma}^{(b)} + \tilde{m}_a \tilde{\delta}^{(a)} + \tilde{m}_b \tilde{\delta}^{(b)} + \tilde{\rho} \equiv \tilde{\Gamma}^{(a)} + \tilde{\Gamma}^{(b)},
\] (4.79)

where \( \tilde{\Gamma}^{(a)} \) and \( \tilde{\Gamma}^{(b)} \) are defined by

\[
\tilde{\Gamma}^{(a)} = [D^{(a)}] \cdot \{ (\tilde{k}_a + \tilde{m}_a)\sigma + \pi^\dagger [2\tilde{k}_a c_1(B_3) - (4\tilde{k}_a + \tilde{m}_a)B_2 + (\tilde{m}_b - \tilde{k}_a)\xi_1 + \tilde{\rho}] \},
\]
\[
\tilde{\Gamma}^{(b)} = [D^{(b)}] \cdot \{ (\tilde{k}_b + 3\tilde{m}_b)\sigma - \pi^\dagger [2\tilde{m}_a c_1(B_3) - (\tilde{k}_b + 4\tilde{m}_b)B_2 + (\tilde{k}_b - \tilde{m}_b)\xi_1 + 3\tilde{\rho}] \}. \]

Note that

\[
p_{D(a)} \tilde{\Gamma}^{(a)} = 2\tilde{m}_a c_1(B_3) - \tilde{m}_a B_2 - (3\tilde{m}_b + \tilde{m}_a)\xi_1 - 3\tilde{\rho},
\]
\[
p_{D(b)} \tilde{\Gamma}^{(b)} = -2\tilde{m}_a c_1(B_3) + \tilde{m}_a B_2 + (3\tilde{m}_b + \tilde{m}_a)\xi_1 + 3\tilde{\rho}.
\] (4.82) (4.83)

Clearly, \( \tilde{\Gamma}^{(a)} \) and \( \tilde{\Gamma}^{(b)} \) obey the traceless condition \( p_{D(a)_*} \tilde{\Gamma}^{(a)} + p_{D(b)_*} \tilde{\Gamma}^{(b)} = 0 \). In this case the quantization conditions are

\[
\{(3\tilde{k}_a + \tilde{m}_a + \frac{1}{2})\sigma + \pi^\dagger [(2\tilde{k}_a - 1)c_1(B_3) - (4\tilde{k}_a + \tilde{m}_a - 1)B_2 + (\tilde{m}_b - \tilde{k}_a + \frac{1}{2})\xi_1 + \tilde{\rho}] \} \in H_4(Z_4, Z),
\]
\[
\{(\tilde{k}_b + 3\tilde{m}_b - \frac{1}{2})\sigma - \pi^\dagger [2\tilde{m}_a c_1(B_3) - (\tilde{k}_b + 4\tilde{m}_b - \frac{1}{2})B_2 + (\tilde{k}_b - \tilde{m}_b - \frac{1}{2})\xi_1 - 3\tilde{\rho}] \} \in H_4(Z_4, Z).
\] (4.84) (4.85)
4.2.3 (2, 2) Factorization

In the (2,2) factorization $\mathcal{D}^{(4)} \rightarrow \mathcal{D}^{(d_1)} \times \mathcal{D}^{(d_2)}$, the divisor $\mathcal{D}^{(4)}$ splits into two components $\mathcal{D}^{(d_1)}$ and $\mathcal{D}^{(d_2)}$. We then factorize Eq. (4.59) into the following form:

$$\mathcal{D}^{(d_1)} \times \mathcal{D}^{(d_2)} : \ [\tilde{e}_0(zu)^2 + \tilde{e}_1(zu) \zeta + \tilde{e}_2 \zeta^2][\tilde{f}_0(zu)^2 + \tilde{f}_1(zu) \zeta + \tilde{f}_2 \zeta^2] = 0. $$ (4.86)

with projection maps $p_{\mathcal{D}^{(d_1)}} : \mathcal{D}^{(d_1)} \rightarrow B_3$ and $p_{\mathcal{D}^{(d_2)}} : \mathcal{D}^{(d_2)} \rightarrow B_3$. By comparing the coefficients with Eq. (4.59), we obtain the following relations:

$$b_0 = \tilde{e}_0 \tilde{f}_0, \ b_1 = \tilde{e}_0 \tilde{f}_1 + \tilde{e}_1 \tilde{f}_0 = 0, \ b_2 = \tilde{e}_0 \tilde{f}_2 + \tilde{e}_1 \tilde{f}_1 + \tilde{e}_2 \tilde{f}_0, \ b_3 = \tilde{e}_1 \tilde{f}_2 + \tilde{e}_2 \tilde{f}_1, \ b_4 = \tilde{e}_2 \tilde{f}_2. $$ (4.87)

By denoting the homological class of $\tilde{f}_2$ by $\pi^* \hat{\xi}_2$, the homological classes of other sections can be written as

$$[\tilde{f}_k] = \pi^* \{(2 - k)[c_1(B_3) - B_2] + \hat{\xi}_2\}, \ k = 0, 1, $$ (4.88)

$$[\tilde{e}_m] = \pi^* \{(m - 3)B_2 - (m - 4)c_1(B_3) - \hat{\xi}_2\}, \ m = 0, 1, 2. $$ (4.89)

In this case, the homological classes of $\mathcal{D}^{(d_1)}$ and $\mathcal{D}^{(d_2)}$ are given by

$$[\mathcal{D}^{(d_1)}] = 2\sigma + \pi^*[4c_1(B_3) - B_2 - \hat{\xi}_2], \ [\mathcal{D}^{(d_2)}] = 2\sigma + \pi^*[2c_1(B_3) + \hat{\xi}_2]. $$ (4.90)

With Eq. (4.87), the dual matter 16 surfaces sit inside the loci $\{(zu) = 0\} \cap \{\tilde{e}_2 = 0\}$ and $\{(zu) = 0\} \cap \{\tilde{f}_2 = 0\}$. The homological classes of dual matter surfaces $\hat{\Sigma}_{16^{(d_1)}}$ and $\hat{\Sigma}_{16^{(d_2)}}$ are given by

$$[\hat{\Sigma}_{16^{(d_1)}}] = (\sigma + \pi^*B_2) \cdot \pi^*[2c_1(B_3) - B_2 - \hat{\xi}_2], \ [\hat{\Sigma}_{16^{(d_2)}}] = (\sigma + \pi^*B_2) \cdot \pi^*\hat{\xi}_2, $$ (4.91)

respectively. We can obtain the homological classes of dual matter surfaces $\hat{\Sigma}_{10}$’s by calculating the intersection $\mathcal{D}^{(4)} \cap \tau \mathcal{D}^{(4)}$, where $\tau$ is a $\mathbb{Z}_2$ involution $\zeta \rightarrow -\zeta$ [39, 40, 44, 72]. Under (2, 2) factorization $\mathcal{D}^{(4)} \rightarrow \mathcal{D}^{(d_1)} \times \mathcal{D}^{(d_2)}$, $\mathcal{D}^{(4)} \cap \tau \mathcal{D}^{(4)}$ can be decomposed into several components $\mathcal{D}^{(d_1)} \cap \tau \mathcal{D}^{(d_1)}$, $\mathcal{D}^{(d_1)} \cap \tau \mathcal{D}^{(d_2)}$, and $\mathcal{D}^{(d_2)} \cap \tau \mathcal{D}^{(d_2)}$. For the case of $\mathcal{D}^{(d_1)} \cap \tau \mathcal{D}^{(d_1)}$, this intersection is determined by

$$\begin{cases} \tilde{e}_0(zu)^2 + \tilde{e}_2 \zeta^2 = 0 \\
\tilde{e}_1(zu) \zeta = 0. \end{cases} $$ (4.92)

To solve the constraint $b_1 = \tilde{e}_0 \tilde{f}_1 + \tilde{e}_1 \tilde{f}_0 = 0$, we use ansatz $\tilde{e}_0 = \tilde{\beta} \tilde{f}_0$ and $\tilde{e}_1 = -\tilde{\beta} \tilde{f}_1$, where $[\tilde{\beta}] = \pi^*[2c_1(B_3) - B_2 - 2\hat{\xi}_2]$. With the ansatz, Eq. (4.92) can be written as

$$\begin{cases} \tilde{\beta} \tilde{f}_0(zu)^2 + \tilde{e}_2 \zeta^2 = 0 \\
\tilde{\beta} \tilde{f}_1(zu) \zeta = 0. \end{cases} $$ (4.93)
It follows from Eq. (4.93) that the homological class of dual matter surface $\tilde{\Sigma}_{10(d_1)(d_1)}$ can be computed as
\[
[\tilde{\Sigma}_{10(d_1)(d_1)}] = [D^{(d_1)}] \cdot [D^{(d_1)}] - [\zeta] \cdot [\tilde{c}_0] - [zu] \cdot [\tilde{e}_2] - 4[\zeta] \cdot [zu] - 2[\zeta] \cdot [\beta]
\]
Next we calculate the intersection $D^{(d_1)} \cap \tau D^{(d_2)}$ which given by
\[
\begin{cases}
\tilde{c}_0(zu)^2 + \tilde{c}_1(zu)\zeta + \tilde{e}_2\zeta^2 = 0 \\
\tilde{f}_0(zu)^2 - \tilde{f}_1(zu)\zeta + \tilde{f}_2\zeta^2 = 0.
\end{cases}
\]
By using the ansatz, we can recast Eq. (4.95) as
\[
\begin{cases}
\zeta^2[-\beta \tilde{f}_2 + \tilde{e}_2] = 0 \\
\tilde{f}_0(zu)^2 - \tilde{f}_1(zu)\zeta + \tilde{f}_2\zeta^2 = 0.
\end{cases}
\]
Then the homological class of dual matter surface $\tilde{\Sigma}_{10(d_1)(d_2)}$ is given by
\[
[\tilde{\Sigma}_{10(d_1)(d_2)}] = [D^{(d_1)}] \cdot [D^{(d_2)}] - 2[\zeta] \cdot [\tilde{f}_0] - 4[\zeta] \cdot [zu]
\]}
\[
\begin{cases}
\tilde{f}_0(zu)^2 + \tilde{f}_2\zeta^2 = 0 \\
\tilde{f}_1(zu)\zeta = 0
\end{cases}
\]
It follows from Eq. (4.98) that the homological class of dual matter surface $\tilde{\Sigma}_{10(d_2)(d_2)}$ is calculated as
\[
[\tilde{\Sigma}_{10(d_2)(d_2)}] = [D^{(d_2)}] \cdot [D^{(d_2)}] - [\zeta] \cdot [\tilde{f}_0] - [zu] \cdot [\tilde{f}_2] - 4[\zeta] \cdot [zu]
\]}
\[
\begin{cases}
\tilde{f}_0(zu)^2 + \tilde{f}_2\zeta^2 = 0 \\
\tilde{f}_1(zu)\zeta = 0
\end{cases}
\]
We summarize the homological classes of dual matter 16 and 10 surfaces in Table 10.

We can calculate the ramification divisors for the (2, 2) factorization and obtain
\[
\tilde{r}_1^{(d_1)} = [D^{(d_1)}] \cdot \pi^*[3c_1(B_3) - 2B_2 - \hat{\xi}_2],
\]
\[
\tilde{r}_1^{(d_2)} = [D^{(d_2)}] \cdot \pi^*[c_1(B_3) - B_2 + \hat{\xi}_2].
\]
where $\tilde{\gamma}^{(d_1)}$ and $\tilde{\gamma}^{(d_2)}$ are the ramification divisors for the cover $D^{(d_1)}$ and $D^{(d_2)}$, respectively. We then define traceless cover fluxes $\tilde{\gamma}_0^{(d_1)}$ and $\tilde{\gamma}_0^{(d_2)}$ by

$$
\begin{align*}
\tilde{\gamma}_0^{(d_1)} &= (2 - p^{*}_{D^{(d_1)}} p^{*}_{D^{(d_1)*}}) \tilde{\gamma}^{(d_1)} = [D^{(d_1)}] \cdot \{2\sigma - \pi^*[2c_1(B_3) - 3B_2 - \tilde{\xi}_2]\}, \\
\tilde{\gamma}_0^{(d_2)} &= (2 - p^{*}_{D^{(d_2)}} p^{*}_{D^{(d_2)*}}) \tilde{\gamma}^{(d_2)} = [D^{(d_2)}] \cdot \{2\sigma + \pi^*[2B_2 - \tilde{\xi}_2]\},
\end{align*}
$$

(4.101)

where $\tilde{\gamma}^{(d_1)}$ and $\tilde{\gamma}^{(d_2)}$ are non-traceless fluxes and defined by

$$
\begin{align*}
\tilde{\gamma}^{(d_1)} &= [D^{(d_1)}] \cdot \sigma, \\
\tilde{\gamma}^{(d_2)} &= [D^{(d_2)}] \cdot \sigma.
\end{align*}
$$

(4.102)

We summarize the restriction of the fluxes to each factorized curve in Table 11. We also can define two fluxes

$$
\begin{align*}
\widehat{\gamma}^{(d_1)} &= (2 - p^{*}_{D^{(d_2)}} p^{*}_{D^{(d_1)*}}) \tilde{\gamma}^{(d_1)} = [D^{(d_1)}] \cdot \{2\sigma - [C^{(d_2)}] \cdot \pi^*[2c_1(B_3) - B_2 - \tilde{\xi}_2]\}, \\
\widehat{\gamma}^{(d_2)} &= (2 - p^{*}_{D^{(d_1)}} p^{*}_{D^{(d_2)*}}) \tilde{\gamma}^{(d_2)} = [D^{(d_2)}] \cdot \{2\sigma - [C^{(d_1)}] \cdot \pi^*\tilde{\xi}_2\}.
\end{align*}
$$

(4.103)

Another flux we can include is $\tilde{\rho}^{(2,2)}$

$$
\tilde{\rho}^{(2,2)} = (p^{*}_{D^{(d_2)}} - p^{*}_{D^{(d_1)}}) \tilde{\rho},
$$

(4.104)
It is easy to see that \( \tilde{\rho} \in H_2(B_3, \mathbb{R}) \) with \( \tilde{\rho}|_{B_3} = \rho \). We summarize the restriction of the fluxes \( \tilde{\delta}^{(d_1)} \), \( \tilde{\delta}^{(d_2)} \), and \( \tilde{\rho}^{(2,2)} \) to each factorized curve in Table 12.

Again we set the universal flux to be

\[
\tilde{\Gamma} = \tilde{k}_{d_1} \tilde{\gamma}^{(d_1)}_0 + \tilde{k}_{d_2} \tilde{\gamma}^{(d_2)}_0 + \tilde{m}_{d_1} \tilde{\delta}^{(d_1)} + \tilde{m}_{d_2} \tilde{\delta}^{(d_2)} + \tilde{\rho} = \tilde{\Gamma}^{(d_1)} + \tilde{\Gamma}^{(d_2)},
\]

where

\[
\tilde{\Gamma}^{(d_1)} = [D^{(d_1)}] \cdot \left\{ 2(\tilde{k}_{d_1} + \tilde{m}_{d_1})\sigma - \pi^* [2\tilde{k}_{d_1}c_1(B_3) - (3\tilde{k}_{d_1} + 2\tilde{m}_{d_1})B_2 + (\tilde{m}_{d_2} - \tilde{k}_{d_1})\tilde{\xi}_2 + \tilde{\rho}] \right\},
\]

\[
\tilde{\Gamma}^{(d_2)} = [D^{(d_2)}] \cdot \left\{ 2(\tilde{k}_{d_2} + \tilde{m}_{d_2})\sigma - \pi^* [2\tilde{m}_{d_1}c_1(B_3) - (2\tilde{k}_{d_2} + 3\tilde{m}_{d_2})B_2 + (\tilde{k}_{d_2} - \tilde{m}_{d_1})\tilde{\xi}_2 - \tilde{\rho}] \right\}.
\]

Note that

\[
p_{D^{(d_1)}_{\ast}} \tilde{\Gamma}^{(d_1)} = 4\tilde{m}_{d_1}c_1(B_3) - 2\tilde{m}_{d_1}B_2 - 2(\tilde{m}_{d_2} + \tilde{m}_{d_1})\tilde{\xi}_2 - 2\tilde{\rho},
\]

\[
p_{D^{(d_2)}_{\ast}} \tilde{\Gamma}^{(d_2)} = -4\tilde{m}_{d_1}c_1(B_3) + 2\tilde{m}_{d_1}B_2 + 2(\tilde{m}_{d_2} - \tilde{m}_{d_1})\tilde{\xi}_2 + 2\tilde{\rho}.
\]

It is easy to see that \( \tilde{\Gamma}^{(d_1)} \) and \( \tilde{\Gamma}^{(d_2)} \) satisfy the traceless condition \( p_{D^{(d_1)}_{\ast}} \tilde{\Gamma}^{(d_1)} + p_{D^{(d_2)}_{\ast}} \tilde{\Gamma}^{(d_2)} = 0 \). In this case the quantization conditions are given by

\[
m_{d_1}c_1(B_3) - (5\tilde{k}_{d_1} + 4\tilde{m}_{d_1} + 1)B_2 + (\tilde{m}_{d_2} - \tilde{k}_{d_1} + \frac{1}{2})\tilde{\xi}_2 + \tilde{\rho}) \right\} \in H_4(Z_4, \mathbb{Z}),
\]

\[
\{2(\tilde{k}_{d_2} + \tilde{m}_{d_2})\sigma - \pi^* [2\tilde{m}_{d_1}c_1(B_3) - (4\tilde{k}_{d_2} + 5\tilde{m}_{d_2} + 1)B_2 + (\tilde{k}_{d_2} - \tilde{m}_{d_1} + \frac{1}{2})\tilde{\xi}_2 - \tilde{\rho}] \right\} \in H_4(Z_4, \mathbb{Z}).
\]
5 Conclusions

In this paper we construct an $SU(4)$ spectral divisor of F-theory compactified on an elliptically fibered Calabi-Yau fourfold by using heterotic/F-theory duality. We also explicitly calculate the net chirality of matter fields $16$ and $10$ by using the net chirality formula Eq. (1.2). We then found agreement between the computations in F-theory framework and in dual heterotic string. It was argued in [1] that the net chirality formula does not depend on heterotic/F-theory duality and would be intrinsic to F-theory. Therefore, this formula would be applicable to the cases of F-theory compactifications without heterotic duals and the spectral divisors can be regarded as the global completion of semi-local spectral covers. To verify the validity of the net chirality formula, we construct an $SU(4)$ spectral divisor in F-theory geometry with no heterotic dual. By using this spectral divisor and net chirality formula Eq. (1.2), we calculate the net chirality of matter fields $16$ and $10$. It turns out that the computations agree with the analysis of the semi-local $SU(4)$ spectral cover.

To obtain realistic models, we also consider $(3, 1)$ and $(2, 2)$ factorizations of the $SU(4)$ spectral divisor. The explicit computation of chiral spectra shows that the net chirality formula can be applied to the factorized spectral divisors. By comparing with the spectra calculated by using semi-local spectral covers, we again found agreement between the computation in factored spectral divisors and in factored spectral divisors. Our computations provide an example for the validity of the spectral divisor construction and net chirality formula. In heterotic compactifications, the net chirality formula can be recast as an index on a Calabi-Yau threefold. More precisely, it can be expressed as an integral of the third Chern class of a stable holomorphic vector bundle on the Calabi-Yau threefold. It would be interesting to lift the net chirality formula in F-theory framework to an index on a Calabi-Yau fourfold. The structure of the net chirality formula should shed light on the geometry of F-theory compactification and the nature of heterotic/F-theory duality.

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