MORREY SPACES AND CLASSIFICATION OF GLOBAL SOLUTIONS FOR A SUPERCRITICAL SEMILINEAR HEAT EQUATION IN $\mathbb{R}^n$

PHILIPPE SOUPLÉT

Université Paris 13, Sorbonne Paris Cité, CNRS UMR 7539
Laboratoire Analyse Géométrie et Applications
93430 Villetaneuse, France. Email: souplet@math.univ-paris13.fr

Abstract: We prove the boundedness of global classical solutions for the semilinear heat equation $u_t - \Delta u = |u|^{p-1}u$ in the whole space $\mathbb{R}^n$, with $n \geq 3$ and supercritical power $p > (n+2)/(n-2)$. This is proved without any radial symmetry or sign assumptions, unlike in all the previously known results for the Cauchy problem, and under spatial decay assumptions on the initial data that are essentially optimal in view of the known counter-examples. Moreover, we show that any global classical solution has to decay in time faster than $t^{-1/(p-1)}$, which is also optimal and in contrast with the subcritical case.

The proof relies on nontrivial modifications of techniques developed by Chou, Du and Zheng [4] and by Blatt and Struwe [2] for the case of convex bounded domains. They are based on weighted energy estimates of Giga-Kohn type, combined with an analysis of the equation in a suitable Morrey space. We in particular simplify the approach of [2] by establishing and using a result on global existence and decay for small initial data in the critical Morrey space $M^{2,4/(p-1)}(\mathbb{R}^n)$, rather than $\varepsilon$-regularity in a parabolic Morrey space. This method actually works for any convex, bounded or unbounded, smooth domain, but at the same time captures some of the specific behaviors associated with the case of the whole space $\mathbb{R}^n$.

As a consequence we also prove that the set of initial data producing global solutions is open in suitable topologies, and we show that the so-called “borderline” global weak solutions blow up in finite time and then become classical again and decay as $t \to \infty$. All these results put into light the key role played by the Morrey space $M^{2,4/(p-1)}$ in the understanding of the structure of the set of global solutions for $p > p_S$.

Keywords: Semilinear heat equation, supercritical power, boundedness of global solutions, decay, Morrey spaces, weighted energy

1. Introduction.

In this article, we consider the following semilinear heat equation

$$
\begin{cases}
  u_t - \Delta u = |u|^{p-1}u, & x \in \Omega, \ t > 0, \\
  u = 0, & x \in \partial\Omega, \ t > 0, \\
  u(x, 0) = u_0(x), & x \in \Omega.
\end{cases}
$$

Here $p > 1$, $\Omega \subset \mathbb{R}^n$ is a possibly unbounded domain and $u_0 \in L^\infty(\Omega)$. Unless otherwise specified, it is assumed throughout that $\Omega$ is of class $C^{2+\alpha}$. Problem (1.1) admits a
unique, maximal classical solution. Moreover, denoting its maximal existence time by $T_{max} = T_{max}(u_0) \in (0, \infty]$, we have $\lim_{t \to T_{max}} \|u(t)\|_\infty = \infty$ whenever $T_{max} < \infty$. For $1 \leq q \leq \infty$, the norm in $L^q(\Omega)$ will be denoted by $\| \cdot \|_q$ (the domain will be omitted when no confusion arises).

Problem (1.1) is one of the best studied model cases in the theory of semilinear parabolic equations and it has received considerable attention in the last fifty years (see e.g. the monograph [35] for extensive references). In particular, it is well known that (1.1) admits both small data global solutions, and finite time blowup solutions for large initial data $u_0$. It is therefore natural to investigate whether or not it admits other kinds of solutions (namely global unbounded classical solutions).

Denote by $p_S$ the Sobolev critical exponent

$$p_S = (n + 2)/(n - 2)_+.$$

In the subcritical case $p < p_S$ with $\Omega$ bounded, it is known [5] that any global solution is uniformly bounded, i.e.

$$\sup_{t \geq 0} \|u(t)\|_\infty < \infty, \quad (1.2)$$

see also [10, 17, 37, 33, 30, 35] and the references therein for further results such as a priori estimates or universal bounds and for similar properties for nonnegative global solutions of the Cauchy problem. The critical and supercritical cases $p = p_S$ and $p > p_S$ (with $n \geq 3$) are more involved and not as well understood. For a long time, the only available results concerned radially symmetric solutions and were based mainly on intersection-comparison arguments (see [9, 25, 22, 26, 3, 23]). More recently, by introducing some new ideas, the following result about boundedness of global solutions for $p > p_S$ in a nonradial framework was obtained in [4] and [2] (cf. [2, Proposition 6.6] for the general case; the special case $u_0 \geq 0$ follows from the proofs of Proposition 3 and Theorem B in [4]).

**Theorem A.** Assume $p > p_S$, $\Omega$ convex bounded and $u_0 \in L^\infty(\Omega)$. If the solution $u$ of (1.1) is global, then property (1.2) is true. Moreover,

$$\lim_{t \to \infty} \|u(t)\|_\infty = 0.$$

The main goals of the present article are:

(a) to investigate the boundedness and decay of global classical solutions for the Cauchy problem ($\Omega = \mathbb{R}^n$), or more generally in unbounded domains. We will consider also the related question of describing the structure of the set of global solutions, including the so-called “borderline” weak solutions;

(b) to provide an alternative approach, motivated by those in [4] and [2], and in turn give a simpler proof of Theorem A (for bounded domains; as for the case of $\mathbb{R}^n$ or unbounded domains, it involves a wider variety of phenomena, and specific difficulties).

**2. Main results.**

2.1. Boundedness and decay of global classical solutions.

Our main results are the following.
Theorem 1. Let \( p > p_S \), \( \Omega = \mathbb{R}^n \) and \( u_0 \in L^\infty(\mathbb{R}^n) \). Assume that either

\[
\nabla u_0 \in L^q(\mathbb{R}^n) \quad \text{for some} \quad q \in \left[ 2, \frac{n(p-1)}{p+1} \right); \tag{2.1}
\]
or

\[
u_0 \in BC^1(\mathbb{R}^n) \quad \text{and} \quad |\nabla u_0(x)| = o\left(|x|^{-\frac{2}{p-1}}\right) \quad \text{as} \quad |x| \to \infty. \tag{2.2}
\]

If the solution \( u \) of (1.1) is global, then property (1.2) is true. Moreover \( u \) satisfies the decay property

\[
\lim_{t \to \infty} t^{1/(p-1)} \|u(t)\|_\infty = 0. \tag{2.3}
\]

We point out that Theorem 1 extends to the nonradial framework a result of [22] for radial solutions in \( \mathbb{R}^n \) (see [23, Theorem 5.14] and its proof).

We next consider general convex, bounded or unbounded, domains, for which we have the following results. In all this article, we use the following:

**Notation.** The Gaussian heat kernel is denoted by

\[
G_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}, \quad x \in \mathbb{R}^n, \quad t > 0.
\]

Also, for any \( \phi \in L^\infty(\Omega) \), we write \( G_t \ast \phi := G_t \ast \tilde{\phi} \), where \( \tilde{\phi} \in L^\infty(\mathbb{R}^n) \) is the extension of \( \phi \) by 0. Also \( BC^1(\Omega) \) denotes the set of functions of class \( C^1 \) in \( \Omega \) which are bounded along with their first order derivatives.

Theorem 2. Assume \( p > p_S \), \( \Omega \) convex and \( u_0 \in L^\infty(\Omega) \). If \( \Omega \) is unbounded, assume in addition that \( u_0 \in BC^1(\Omega) \) and that

\[
\lim_{t \to \infty} \left( t^{\frac{p+1}{p-1}} \|G_t \ast |\nabla u_0|^2\|_\infty + t^{\frac{2}{p-1}} \|G_t \ast |u_0|^2\|_\infty \right) = 0. \tag{2.4}
\]

If the solution \( u \) of (1.1) is global, then property (1.2) is true. Moreover \( u \) satisfies the decay property (2.3).

As we shall see, condition (2.4) is for instance true under the more familiar assumptions:

\[
|u_0|^{p+1} + |\nabla u_0|^2 \in L^m(\Omega) \quad \text{for some} \quad m \in \left[ 1, \frac{np-1}{2p+1} \right), \tag{2.5}
\]
or \( u_0 \in BC^1(\Omega) \) and

\[
|u_0(x)| + |x| |\nabla u_0(x)| = o\left(|x|^{-\frac{2}{p-1}}\right) \quad \text{as} \quad |x| \to \infty. \tag{2.6}
\]

We thus have:

**Corollary 3.** Assume \( p > p_S \), \( \Omega \) convex and \( u_0 \in L^\infty(\Omega) \). If \( \Omega \) is unbounded, assume in addition that \( u_0 \in BC^1(\Omega) \) and that either (2.5) or (2.6) is satisfied. If the solution \( u \) of (1.1) is global, then property (1.2) is true. Moreover \( u \) satisfies the decay property (2.3).

We stress that the above decay assumptions on \( u_0 \) are not technical. In fact, in the case \( \Omega = \mathbb{R}^n \), the assumption (2.6) on the spatial decay of \( u_0 \), as well as the conclusion (2.3) on the temporal decay of \( u \), are essentially optimal; see Remark 3.1.
2.2. Borderline solutions and structure of the set of global solutions.

A related question is that of “threshold” or “borderline” solutions, introduced in [27]. For \( \Omega \) bounded and a given nontrivial \( \phi \in L^\infty(\Omega) \), consider problem (1.1) with initial data \( u_0 = \lambda \phi \) and let
\[
\lambda^* = \sup\{ \lambda > 0; T_{\text{max}}(\lambda \phi) = \infty \}.
\]
(2.7)

It is well known that if either \( \phi \geq 0 \) or \( \phi \in H^1_0(\Omega) \), then \( \lambda^* \in (0, \infty) \). The behavior of the solution \( u^*(t) = u(t; \lambda^* \phi) \) has been studied in many papers. One of the following three possibilities must occur:

(a) \( u^* \) is global bounded;
(b) \( u^* \) is global unbounded;
(c) \( u^* \) blows up in finite time.

It is known [10, 33] that (a) occurs if \( p < p_S \). If \( p = p_S \), \( \Omega \) is a ball and \( \phi \) is radial nonnegative, then (b) occurs [9]. If \( p > p_S \) and \( \Omega \) is convex then (c) occurs. This is a consequence of the fact that the set
\[
\mathcal{G} = \{ u_0 \in L^\infty(\Omega); T_{\text{max}}(u_0) = \infty \}
\]
(2.8)
of \( u_0 \) in the \( L^\infty \) topology, owing to [2]. Indeed, for any \( u_0 \in \mathcal{G} \), \( \| u(t) \|_\infty \) decays as \( t \to \infty \) by [2, Proposition 6.6]. Property (2.8) then follows easily from the continuous dependence in \( L^\infty \) and the well-known fact that the trivial solution is stable in \( L^\infty \) (with \( \Omega \) bounded). Some results concerning the threshold behavior for \( \Omega = \mathbb{R}^n \) and \( p > p_S \) in the radial case can be found in [25, 26, 23, 24, 34] (see also [35]).

We here give the following extension of property (2.8) to the Cauchy problem (or to convex unbounded domains) in suitable topologies (note that the \( L^\infty \)-topology is no longer relevant, see Remark 3.1(i)). Namely we show that property (2.8) is true with respect to the critical Morrey norm \( M^{2,4/(p-1)} \), as well as to the critical \( L^q \) norm and to the \( L^\infty \) norm weighted by \( |x|^{2/(p-1)} \), corresponding to the borderline decay. All these topologies are quite “natural” in view of the scaling properties of the equation (see Section 6) and the result seems actually new even for convex bounded domains.

**Theorem 4.** Assume \( p > p_S \), \( \Omega \) convex and \( u_0 \in \mathcal{G} \).

(i) If \( \Omega \) is unbounded, assume in addition that \( u_0 \in BC^1(\Omega) \) and that (2.4) holds. Then there exists \( \eta = \eta(u_0) > 0 \) such that
\[
\{ v_0 \in L^\infty(\Omega); \| u_0 - v_0 \|_{M^{2,4/(p-1)}} < \eta \} \subset \mathcal{G},
\]
where the Morrey norm \( \| \cdot \|_{M^{2,4/(p-1)}} \) is defined in (4.1). In particular, there exists \( \eta_1 = \eta_1(u_0) > 0 \) such that
\[
\{ v_0 \in L^\infty(\Omega); \sup_{x \in \Omega} |x|^{2/(p-1)}|u_0(x) - v_0(x)| < \eta_1 \} \subset \mathcal{G}
\]
(2.10)
and
\[
\{ v_0 \in L^\infty(\Omega); \| u_0 - v_0 \|_{n/(p-1)2} < \eta_1 \} \subset \mathcal{G}.
\]
(2.11)
(ii) Let $\phi \in L^\infty(\Omega)$ and assume that $\lambda^* \in (0, \infty)$, where $\lambda^*$ is defined in (2.7). If $\Omega$ is unbounded, assume in addition that $\phi \in BC^1(\Omega)$ and that $\phi$ satisfies (2.4). Then $T_{\text{max}}(\lambda^* \phi) < \infty$.

In the above situation, it is a natural question whether or not $u^*$ can be continued after $T_{\text{max}}$ as some kind of weak solution. In the case $\phi \geq 0$, then $\lambda \mapsto u(\cdot; \lambda \phi)$ is nondecreasing and a suitable, unique minimal global weak continuation can be constructed by monotone approximation (cf. [27]). The behavior of the minimal global weak continuation for $p > p_S$ was studied in the radial case in [9] and it was later shown [4] that if $\Omega$ is convex and bounded, then $u^*$ becomes classical again for sufficiently large time and eventually decays uniformly to zero. For sign-changing $\phi$, a global weak continuation was also constructed in [4] (see [4, Theorem 1(b))] by using compactness estimates and, as a consequence of the results in [2], the same behavior occurs.

Our next result extends this to the Cauchy problem (or to convex unbounded domains), and also recovers the case of convex bounded domains with simpler proof. Actually, it applies to any pointwise limit of global classical solutions, which in particular covers the weak continuations mentioned above.

**Theorem 5.** Assume $p > p_S$, $\Omega$ convex. Let $u$ be a pointwise limit of global classical solutions of (1.1), namely

$$\lim_{j \to \infty} u_j(x, t) = u(x, t), \quad \text{a.e. in } \Omega \times (0, \infty),$$

where the initial data of $u_j$ are of the form $u_{0,j} = \lambda_j \phi$ with $\phi \in L^\infty(\Omega)$ and $\lambda_j \to \lambda_0 \in (0, \infty)$. If $\Omega$ is unbounded, assume in addition that $\phi \in BC^1(\Omega)$ and that $\phi$ satisfies (2.4). Then there exists $t_0 > 0$ such that $u$ is a bounded classical solution on $\Omega \times (t_0, \infty)$. Moreover $u$ satisfies the decay property (2.3).

The outline of the rest of the article is as follows. Several remarks, especially about the optimality of the results, as well as ideas of proofs, are given in Section 3. The preliminary Section 4 consists of two subsections of reminders, respectively on Morrey spaces and on Giga-Kohn weighted energy. In Sections 5 and 6, we then present our two essential technical results: Proposition 5.1 on the derivation of $M^{2,4/(p-1)}$-Morrey estimates from weighted energy bounds, and Proposition 6.1 on the global existence and decay for small initial data in the critical Morrey space $M^{2,4/(p-1)}(\Omega)$. The main results are then deduced as consequences thereof in Section 7, except for Theorem 4, which is proved in Section 8, as a consequence of a suitable continuous dependence property for problem (1.1) with respect to the critical Morrey norms (Proposition 8.1). Finally, in appendix, for convenience and self-containedness, we provide the proofs of some important known results that we have used, concerning $L^p-L^q$ estimates for the linear heat semigroup in Morrey spaces (Proposition 4.1), basic properties of Giga-Kohn weighted energy (Proposition 4.3), plus some auxiliary lemmas.

---

1 However, weak continuations need not be unique in general (see [7]).
3. Discussion and ideas of proofs.

Remarks 3.1. (i) In the case $\Omega = \mathbb{R}^n$, unlike in the case of bounded domains, boundedness of global solutions cannot hold in general for supercritical $p$ under the mere assumption $u_0 \in L^\infty$. Actually, the spatial decay assumption (2.2) in Theorem 1 is essentially optimal for the boundedness of global solutions. Indeed, for all $p > p_S$, there exist unbounded global solutions for initial data with the borderline decay

$$u_0(x) \sim |x|^{-2/(p-1)} \quad \text{as } |x| \to \infty$$

(cf. [31, 32]). On the other hand, the solution of (1.1) is never global if the spatial decay is slower, namely if

$$\lim_{|x| \to \infty} |x|^{2/(p-1)} u_0(x) = \infty$$

(see [20, 35]).

(ii) The conclusion (2.3) about temporal decay in Theorem 1 is also optimal. Indeed (see e.g. [35, Chapter 20]), for any $p > (n+2)/n$ and $k > 1/(p-1)$, if we take $u_0(x) = \varepsilon(1 + |x|^2)^{-k}$ with $\varepsilon > 0$ small (which satisfies (2.6)), then $u$ is global and has the decay rate $\|u(t)\|_\infty \sim t^{-k}$, which can be made arbitrarily close to $t^{-1/(p-1)}$. Moreover, the conclusion (2.3) is in contrast with the subcritical case $p < p_S$ since in this case, there exist initial data $u_0$ with exponential spatial decay, such that $u$ is global and

$$\|u(t)\|_\infty \sim t^{-1/(p-1)} \quad \text{as } t \to \infty,$$

including forward self-similar solutions (see [14, 18]).

(iii) Under assumption (2.1) in Theorem 1 (or (2.5) in Corollary 3), the solution $u$ actually enjoys better decay properties, depending on the value of $q$. In fact, $u$ behaves like the linear part of the equation, in the sense that there exists a constant $C > 0$ (depending on $u$) such that

$$|u(t)| \leq Ce^{-tA}|u_0| \quad \text{in } \Omega \times (0, \infty),$$

where $e^{-tA}$ is the (Dirichlet) heat semigroup in $\Omega$. In particular, under assumption (2.1) in Theorem 1, we have

$$\|u(t)\|_\infty \leq Ct^{-(n-q)/2q}, \quad t \geq 1,$$

where $(n-q)/2q > 1/(p-1)$; see Remark 7.1.

(iv) For $p > p_S$, $\Omega = \mathbb{R}^n$ and $0 \leq u_0 \in L^\infty$ radially symmetric, the boundedness of global solutions was already known in some cases. It is true if we assume $u_0 \in H^1(\mathbb{R}^n)$ (see [23, Theorem 5.15]). It is also true if $p > p_{JL}$ and $u_0(x) \leq (1-\varepsilon)U_\ast(|x|)$ for $|x|$ large and some $\varepsilon > 0$, where $p_{JL} = 1 + 4\frac{n-4+2\sqrt{n-1}}{(n-2)(n-10)}$ and $U_\ast$ is the singular steady-state (see [26]), or if $p < p_{JL}$ and $u$ satisfies certain finite intersection number properties with $U_\ast$ (see [25, Lemma 2.2]).

(v) For $p = p_S$, the situation is quite different. Indeed if $\Omega$ is a ball, there exist global positive, radial solutions such that $\lim_{t \to \infty} \|u(t)\|_\infty = \infty$ (see [9]), and their growth rates are precisely known [8]. This is conjectured to happen also when $\Omega = \mathbb{R}^n$ if $n \leq 4$ (see [6]).
Remarks 3.2. (i) When \( \Omega \) is bounded, the rate (2.3) can of course be improved. Indeed, in a bounded domain, any decaying solution of (1.1) decays at an exponential rate, since it is well known that the trivial solution is exponentially asymptotically stable. This remains true if \( \Omega \) is unbounded but has finite inradius; see [39].

(ii) In the subcritical case \( p < p_S \), it is well known [10, 33] that global solutions are not only bounded but satisfy an a priori estimate of the form

\[
\sup_{t \geq 0} \|u(t)\|_\infty \leq K(\|u_0\|_\infty),
\]

where the constant \( K \) is bounded on bounded sets. Such an estimate implies in particular that the borderline solutions (cf. Section 2.2) are global and bounded. Consequently, for \( p > p_S \) in convex domains, estimate (3.3) cannot be true, in view of Theorem 4. However, by the proof of Theorem 2 and Corollary 3, the following weaker, delayed a priori estimate of global solutions is true:

\[
\exists t_0 = t_0(\|u_0\|) > 0, \quad \sup_{t \geq t_0} \|u(t)\|_\infty \leq K(\|u_0\|).
\]

Here \( t_0 \) is bounded on bounded sets and we may take \( \|u_0\| = \|u_0\|_\infty \) if \( \Omega \) is convex bounded, and e.g., \( \|u_0\| = \|u_0\|_q + \|\nabla u_0\|_m \) with \( q \in [2, \frac{n(p-1)}{2}) \) and \( m \in [2, \frac{n(p-1)}{p+1}) \) otherwise. This estimate, which remains valid for borderline solutions, is consistent with the fact that any borderline solution blows up in a finite time \( T^* \) (which thus satisfies \( T^* < t_0(\|u_0\|) \)).

(iii) Theorem 1 and Corollary 3 remain true if, instead of (2.1) or (2.5), we assume \( u_0 \in L^q(\Omega) \) for some \( q \in [2, \frac{n(p-1)}{p+1}) \). Indeed, it is easy to show that condition (2.5) is then satisfied at positive time. This avoids any assumption on \( \nabla u_0 \) (which need not exist), but at the expense of a possibly more restrictive condition at infinity on \( u_0 \). On the other hand, it should be possible to extend Theorem 1 (where the integrability or decay assumption is made on \( \nabla u_0 \) only) to general unbounded domains, at the expense of significant additional technicalities. However, for the sake of simplicity, we have refrained ourselves from treating this.

(iv) It is still an open problem whether or not boundedness of global solutions remains true without the convexity assumption on \( \Omega \). If \( p \geq p_S \) and \( \Omega \) is bounded and starshaped (in particular if it is convex), then any bounded global solution of (1.1) has to decay in \( L^\infty \), owing to the existence of a Liapunov functional and to the absence of nontrivial steady states due to Pohozaev’s identity. But the decay of all global solutions cannot be true for general bounded domains, since positive stationary solutions exist for all \( p > 1 \) when \( \Omega \) is for instance an annulus.

The original proofs of Theorem A for bounded domains in [2] and [4] (and nonnegative solutions in the latter) are based on two ingredients:
(a) a monotonicity property, which yields suitably decaying space-time estimates on the solution;

(b) an $\varepsilon$-regularity result in terms of the space-time estimates.

Concerning (a), the authors of [4] use the nondecreasing property of the weighted energy of Giga and Kohn [11, 12] for equation (1.1) rewritten in similarity variables, with rescaling time as a free parameter (see [22] for earlier related arguments in the radial framework). This argument is combined with a Pohozaev-type identity in the original variables. As for [2], a different tool is used, namely a parabolic analogue of an elliptic monotonicity formula obtained and used in [29] to prove partial regularity of weak solutions of the stationary equation $-\Delta u = |u|^{p-1}u$ for $p \geq p_S$. Some of the close relations between these two points of view are noticed in [2]; see [2, Section 3.1 and p. 2348].

As for (b), the $\varepsilon$-regularity result in [4] is derived by a clever use of linear parabolic $L^q$-estimates, leading to local uniform bounds of the solution. A different idea is used in [2], based on a novel analysis of equation (1.1) in a parabolic Morrey space, involving some delicate bootstrap arguments.

We shall here develop a modified approach, which also covers possibly unbounded domains and gives the optimal decay rate in the case of $\mathbb{R}^n$. It actually combines the Giga-Kohn weighted energy framework with an analysis of (1.1) in an elliptic Morrey space of functions of $x$ (instead of a parabolic Morrey space of functions of $x, t$). This new procedure has the advantage to replace the $\varepsilon$-regularity issue with a smoothing and decay property for an initial value problem.

Comparing with (a)(b) above, our strategy can be summarized in the following two steps:

(a') By using properties of the weighted energy of Giga and Kohn, we show that for any global classical solution, the critical Morrey norm $\|u(t)\|_{M^{2,4/(p-1)}(\Omega)}$ decays as $t \to \infty$;\footnote{This is where the condition $p > p_S$ enters and it also requires $u_0$ with suitable spatial decay in the case of unbounded domains}

(b') We prove that any solution starting from an initial data with sufficiently small $M^{2,4/(p-1)}$-norm exists globally and decays in $L^\infty$ as $t \to \infty$.

Property (b') is proved by a semigroup argument, in the line of [43, 36, 38] which, as another important advantage, also yields the correct temporal decay of the solution in $L^\infty$ in the case $\Omega = \mathbb{R}^n$. We here make use of linear smoothing estimates of Kato [16] for the heat semigroup in Morrey spaces.

We note that our proof of Theorem A (bounded domains) is somewhat simpler than those of [2] and [4]. However the tools developed in [2, 4] allow to obtain further delicate properties such as estimates of the Hausdorff dimension of the singular set of certain blowup solutions for $p > p_S$, which we do not consider.
4. Preliminaries.

4.1. Morrey spaces.

Let \( \Omega \) be any domain in \( \mathbb{R}^n \), \( q \in [1, \infty) \) and \( \lambda \in [0, n] \). Recall that the Morrey space \( M^{q,\lambda}(\Omega) \) is defined by

\[
M^{q,\lambda}(\Omega) = \{ f \in L^q_{\text{loc}}(\Omega) : \| f \|_{M^{q,\lambda}} < \infty \},
\]

where

\[
\| f \|_{M^{q,\lambda}}^q = \| f \|_{M^{q,\lambda}(\Omega)}^q = \sup_{a \in \Omega} \sup_{R > 0} r^{n-\lambda} \int_{B_R(a) \cap \Omega} |f|^q \, dx.
\]

Also we set \( M^{\infty,\lambda}(\Omega) = L^\infty(\Omega) \). We note that one can use instead

\[
\sup_{a \in \mathbb{R}^n} \sup_{R > 0} r^{n-\lambda} \int_{B_R(a) \cap \Omega} |f|^q \, dx,
\]

which is easily seen to give an equivalent norm. Observe also that \( M^{q,0}(\Omega) = L^q(\Omega) \), whereas \( M^{q,n}(\Omega) = L^\infty(\Omega) \), owing to the Lebesgue differentiation theorem. For all \( 1 < p \leq r < \infty \), \( m \in (0, r] \) and \( \lambda \in [0, n] \), we have

\[
\| |f|^m \|_{M^{r/m,\lambda}}^r = \| f \|_{M^{r,\lambda}}^m, \quad f \in M^{r,\lambda}(\Omega),
\]

and, as a consequence of Hölder’s inequality,

\[
\| fg \|_{M^{r/p,\lambda}} \leq \| f \|_{M^{r/(p-1),\lambda}} \| g \|_{M^{r,\lambda}}, \quad f \in M^{r/(p-1),\lambda}(\Omega), \ g \in M^{r,\lambda}(\Omega).
\]

Let \( e^{-tA} \) be the Dirichlet heat semigroup on \( L^\infty(\Omega) \). By results in [16], it is known that \( e^{-tA} \) enjoys good smoothing properties in the scale of Morrey spaces.

**Proposition 4.1.** Let \( \Omega \) be an arbitrary domain of \( \mathbb{R}^n \), let \( 1 \leq p \leq q \leq \infty \) and \( 0 \leq \lambda \leq n \). Then there exists a constant \( C = C(n,p,q,\lambda) > 0 \) such that, for all \( f \in L^\infty(\Omega) \cap M^{p,\lambda}(\Omega) \),

\[
\| e^{-tA} f \|_{M^{q,\lambda}} \leq C t^{-\frac{n}{2}(\frac{1}{r} - \frac{1}{q})} \| f \|_{M^{p,\lambda}}, \quad t > 0.
\]

Moreover,

\[
\| e^{-tA} f \|_{M^{p,\lambda}} \leq \| f \|_{M^{p,\lambda}}, \quad t > 0.
\]

A proof is provided in Appendix for convenience.

**Remark 4.1.** Observe that in the case \( \lambda = n \), since \( M^{p,n}(\Omega) = L^p(\Omega) \), this corresponds to the usual \( L^p-L^q \) estimate, and that the case \( \lambda = 0 \) is just \( \| e^{-tA} f \|_\infty \leq C \| f \|_\infty \).

We next note the following elementary properties which will be quite useful, since they relate various “critical” norms associated with problem (1.1). Here \( L^\infty_{2/(p-1)}(\Omega) \) denotes the weighted \( L^\infty \) space with weight \( |x|^{2/(p-1)} \), with norm

\[
\| f \|_{L^\infty_{2/(p-1)}} := \text{ess sup}_{x \in \Omega} |x|^{2/(p-1)} |f(x)|.
\]
Proposition 4.2. Let \( \Omega \) be an arbitrary domain of \( \mathbb{R}^n \) and \( p > 1 + \frac{2}{n} \).

(i) For all \( 1 \leq q < n(p-1)/2 \), we have

\[
L^{n(p-1)/2}(\Omega) \hookrightarrow M^{q,2q/(p-1)}(\Omega) \tag{4.7}
\]

and

\[
L^{\infty}_{2/(p-1)}(\Omega) \hookrightarrow M^{q,2q/(p-1)}(\Omega). \tag{4.8}
\]

(ii) For all \( q \in [1, \infty) \), \( \lambda \in [0, n] \) and \( \phi \in L^\infty(\Omega) \), we have

\[
\|\phi\|_{M^{q,\lambda}} \leq C \sup_{t>0} t^{\lambda/2} \|G_t * |\phi|^q\|_{\infty}. \tag{4.9}
\]

In particular, for \( p > 1 + \frac{4}{n} \), if \( u_0 \in BC^1(\Omega) \) satisfies (2.4), then \( u_0 \in M^{2,4/(p-1)}(\Omega) \).

Remark 4.2. Although we shall not use this property, let us mention that, for \( \lambda > 0 \), the quantity \( \sup_{t>0} t^{\lambda/2} \|G_t * f\|_{\infty} \) is known to be equivalent to the norm of the homogeneous Besov space \( \dot{B}_\infty^{\lambda, \infty} \) (cf. [41, p. 192]) and that condition (2.4) can thus be interpreted in terms of belonging to suitable closed subspaces of Besov spaces.

Proof. Let \( m = n(p-1)/2 \). For all \( a \in \Omega \), \( R > 0 \), by Hölder’s inequality, we have

\[
R^{2q-n/2} \int_{B_R(a) \cap \Omega} |f|^q \, dx \leq CR^{2q-n} R^{n(1-n/2)} \left( \int_{B_R(a) \cap \Omega} |f|^m \, dx \right)^{\frac{n}{m}} \leq C \|f\|^q_m
\]

and (4.7) follows.

To prove (4.8), let us show that

\[
|x|^{-2/(p-1)} \in M^{q,\lambda}(\Omega), \quad \lambda = 2q/(p-1). \tag{4.10}
\]

Setting \( k = 2/(p-1) \), we write

\[
R^{2q-n/2} \int_{B_R(a) \cap \Omega} |x|^{-kq} \, dx \leq R^{kq} \int_{B_1} |a + Ry|^{-kq} \, dy = \int_{B_1} |aR^{-1} + y|^{-kq} \, dy.
\]

If \( aR^{-1} \geq 2 \), then \( \int_{B_1} |aR^{-1} + y|^{-kq} \, dy \leq C := \int_{B_1} |y|^{-kq} \, dy \). If \( aR^{-1} < 2 \), since \( kq < n \), we have

\[
\int_{B_1} |aR^{-1} + y|^{-kq} \, dy = \int_{B_1(aR^{-1})} |y|^{-kq} \, dy \leq C := \int_{B_3} |y|^{-kq} \, dy.
\]

This proves (4.10). For all \( f \in L^\infty_{2/(p-1)}(\Omega) \), since \( |f| \leq \|f\|_{\infty,2/(p-1)} |x|^{-2/(p-1)} \), it follows that \( f \in M^{q,\lambda}(\Omega) \) and \( \|f\|_{M^{q,\lambda}} \leq \||x|^{-2/(p-1)} \|_{M^{q,\lambda}} \|f\|_{\infty,2/(p-1)} \), and (4.8) is proved.

Finally, assertion (4.9) is a consequence of

\[
R^{\lambda-n} \int_{B_R(a) \cap \Omega} |\phi|^q \, dx \leq CR^{\lambda-n} \int_{B_R(a) \cap \Omega} R^n G_{R^2}(x-a) |\phi(x)|^q \, dx \leq CR^\lambda \|G_{R^2} * |\phi|^q\|_{\infty}.
\]
4.2. Similarity variables and weighted energy.

We here recall some fundamental tools from Giga and Kohn [11, 12]. Let \( \Omega \) be a (possibly unbounded) domain of \( \mathbb{R}^n \) of class \( C^{2+\alpha} \). Given \( a \in \Omega \), the backward similarity variables \((y, s)\) with respect to \((a, T)\) are given by
\[
y := \frac{x - a}{\sqrt{T - t}}, \quad s := -\log(T - t).
\] (4.11)

Let \( p > 1 \) and set \( \beta = \frac{1}{p-1} \). If \( u \) is a solution of (1.1) on \((0, T)\), then the corresponding rescaled solution \( w = w_{a,T}(y, s) \) is given by
\[
w(y, s) = w_{a,T}(y, s) := e^{-\beta s}u(a + e^{-s/2}y, T - e^{-s}),
\] (4.12)
and it is defined for all \( s > s_0 := -\log T \) in the rescaled domain
\[
D(s) := \{ y \in \mathbb{R}^n; a + e^{-s/2}y \in \Omega \} = e^{s/2}(\Omega - a).
\] (4.13)

By direct calculation, \( w \) satisfies the equation
\[
\rho w_s - \nabla \cdot (\rho \nabla w) = \rho |w|^{p-1}w - \beta \rho w, \quad \rho(y) = e^{-|y|^2},
\] (4.14)
in \( \bigcup_{s>s_0} D(s) \times \{s\} \). Finally, the weighted energy \( E(s) = E(w_{a,T}(s)) \) is given by
\[
E(w_{a,T}(s)) := \int_{D(s)} \left( \frac{1}{2} |\nabla w|^2 + \frac{\beta}{2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho \, dy.
\] (4.15)

The main properties of the rescaled energy \( E \), in particular its time monotonicity, are described in the following proposition [11, 12].

**Proposition 4.3.** Let \( a \in \Omega \) and assume that \( \Omega \) is starshaped with respect to \( a \). Let \( p > 1 \), \( u_0 \in BC^1(\Omega) \) and assume that the solution \( u \) of (1.1) exists on \((0, T)\). Then, under the notation of the above paragraph, for all \( s > s_0 \), we have
\[
\frac{1}{2} \frac{d}{ds} \int_{\mathbb{R}^n} w^2 \rho \, dy = -2E(w(s)) + \frac{p-1}{p+1} \int_{\mathbb{R}^n} |w|^{p+1} \rho \, dy,
\] (4.16)
\[
\frac{d}{ds} E(w(s)) \leq - \int_{\mathbb{R}^n} w^2 \rho \, dy,
\] (4.17)
\[
E(w(s)) \geq 0
\] (4.18)
and
\[
\int_{\mathbb{R}^n} w^2 \rho \, dy \leq C(n, p)[E(w(s_0))]^{2/(p+1)}.
\] (4.19)

The proof is provided in Appendix for convenience.

5. Estimate of \( u(t) \) in the Morrey space \( M^{2,4/(p-1)}(\Omega) \).

Our first main technical result is the following proposition, which enables one to relate the Morrey norm \( \|u(t)\|_{M^{2,4/(p-1)}(\Omega)} \) to weighted energies for appropriate rescaling times \( T > t \). The latter can be estimated by suitable norms of the initial data, owing to the monotonicity of the weighted energy. This will eventually lead to the decay of \( \|u(t)\|_{M^{2,4/(p-1)}(\Omega)} \).
Proposition 5.1. Let $\Omega \subset \mathbb{R}^n$ be a (possibly unbounded) domain of class $C^{2+\alpha}$, let $u_0 \in BC^1(\Omega)$ and let $u$ be the solution of (1.1).

(i) Let $T > 0$, $a \in \overline{\Omega}$ and assume that $\Omega$ is starshaped with respect to $a$. If $u$ exists on $(0, T)$, then for all $t \in (0, T)$, we have

$$(T - t) \frac{2}{p - 1} - \frac{A}{2} \int_{\Omega} e^{-\frac{|x-a|^2}{4(p-1)}} u^2(x, t) \, dx \leq C(n, p) A \frac{2}{p - 1} (T, u_0, a),$$

where

$$A(T, u_0, a) = \left[ T \frac{p+1}{p-1} G_T * (|\nabla u_0|^2) + T \frac{2}{p-1} G_T * (|u_0|^2) \right] (a).$$

(ii) Assume $\Omega$ convex and set $\mu = \frac{4}{p-1}$. If $u$ is global, then, for all $t_0 > 0$, we have

$$\|u(t_0)\|_{M^{2, \mu}(\Omega)} \leq C(n, p) N \frac{1}{\mu} (u_0, t_0),$$

where

$$N(u_0, t_0) = \sup_{t \geq t_0} \left( t^{\frac{p+1}{p-1}} \|G_t * (|\nabla u_0|^2)\|_{\infty} + t^{\frac{2}{p-1}} \|G_t * (|u_0|^2)\|_{\infty} \right).$$

Proof. In this proof, $C$ denotes a generic positive constant depending only on $n$ and $p$.

(i) Let $(y, s)$ be the backward similarity variables with respect to $(a, T)$, $w = w_{a, T}(y, s)$ the corresponding rescaled solution, $D(s)$ the rescaled domain, and $E(s) = E(w_{a, T}(s))$ the weighted energy (cf. (4.11)–(4.15)). By Proposition 4.3, for all $s \geq s_0 = -\log T$, we have

$$0 \leq E(s) \leq E(s_0), \quad \int_{D(s)} |w|^2 \rho \, dy \leq C[E(s_0)]^{2/(p+1)}.$$

Switching back to the original variables, it follows that

$$(T - t) \frac{2}{p - 1} - \frac{A}{2} \int_{\Omega} e^{-\frac{|x-a|^2}{4(p-1)}} u^2(x, t) \, dx \leq C[E(s_0)]^{2/(p+1)}.$$

To estimate the RHS, we write

$$\int_{D(s_0)} |\nabla w(y, s_0)|^2 \rho \, dy = T \frac{p+1}{p-1} - \frac{A}{2} \int_{\Omega} |\nabla u_0|^2 e^{-\frac{|x-a|^2}{4p-1}} \, dx = CT \frac{p+1}{p-1} (G_T * |\nabla u_0|^2)(a) \quad (5.1)$$

and

$$\int_{D(s_0)} w^2(y, s_0) \rho \, dy = T \frac{2}{p-1} - \frac{A}{2} \int_{\Omega} |u_0|^2 e^{-\frac{|x-a|^2}{4p-1}} \, dx = CT \frac{2}{p-1} (G_T * |u_0|^2)(a), \quad (5.2)$$

Since

$$E(w_{a, T}(s)) \leq \int_{D(s)} \left( \frac{1}{2} |\nabla w|^2 + \frac{\beta}{2} w^2 \right) \rho \, dy,$$
this guarantees the assertion.

(ii) Pick $t_0 > 0$ and $a \in \Omega$. Since $\Omega$ is convex, it is starshaped with respect to $a$. For any $R > 0$, we choose $T = t_0 + R^2$. For each $x \in B(a, R)$, we have $\frac{|x-a|^2}{T-t_0} \leq 1$. It then follows from assertion (i) that

$$
R^{\mu-n} \int_{\Omega \cap B(a, R)} u^2(x, t_0) \, dx \leq C(T-t_0)^{\frac{\mu-n}{2}} \int_{\Omega} e^{-\frac{|x-a|^2}{4(T-t_0)}} u^2(x, t_0) \, dx 
\leq C A^{\frac{2}{q+c}} (T, u_0, a) \leq C N^{\frac{2}{q+c}} (t_0, u_0).
$$

The assertion follows by taking supremum over $a \in \Omega$ and $R > 0$.

**Remark 5.1.** Assume $u_0 \in H^1(\Omega)$ (which is always true in bounded domains, starting from $L^\infty$, after a time shift). Then, since $p > p_S$ is equivalent to $\frac{p+1}{p-1} - \frac{n}{2} < 0$, it is already clear from formulae (5.1)-(5.2) that

$$
E(s_0) = E(w_{a,T}(- \log T)) \to 0, \quad T \to \infty.
$$

This crucial observation is a starting point of the analysis of [2] (although, unlike in our case, it is used there in conjunction with space-time estimates and parabolic Morrey norms).

6. Global existence and decay for small data in critical Morrey spaces.

The remaining task is now to infer the uniform decay of $u(t)$ from its decay in the Morrey space $M^{\frac{2}{q+c}/(p-1)}$. This space – and more generally $M^{q,2q/(p-1)}$ for $p \geq 1 + \frac{2q}{n}$ – turns out (when $\Omega = \mathbb{R}^n$) to be invariant by the scaling of the equation, namely

$$
\lambda \mapsto u_\lambda(x, t) = \lambda^{2/(p-1)} u(\lambda^2 t, \lambda x).
$$

(6.1)

This property is similar to that of the critical $L^q$ space with $q = q_c = n(p-1)/2$ (see, e.g., [35, Chapter 20] and also [15] for further examples). Owing to the linear smoothing estimates of [16], the following small data global existence and decay result for problem (1.1) can be shown in a similar way as for $L^{q_c}$, based on ideas from [43, 38, 36]. In the case of Morrey spaces, results of this type were first obtained in [13, 16] for the Navier-Stokes system. For recent developments concerning other problems such as chemotaxis systems, see e.g. [21, 1]. Related results for semilinear parabolic equations appear in [42, 40, 15], but they do not seem suitable to our needs.

**Proposition 6.1.** Let $\Omega \subset \mathbb{R}^n$ be a (possibly unbounded) domain of class $C^{2+\alpha}$. Let $q \in (1, \infty)$ and $p \geq 1 + \frac{2q}{n}$. There exist $\varepsilon_0 = \varepsilon_0(n, p, q)$ and $C_0 = C_0(n, p, q) > 0$ with the following property. For any $u_0 \in L^\infty(\Omega) \cap M^{q,2q/(p-1)}(\Omega)$, if

$$
\|u_0\|_{M^{q,2q/(p-1)}(\Omega)} \leq \varepsilon_0,
$$

then the corresponding solution $u$ of (1.1) is global and satisfies

$$
\sup_{t > 0} t^{\frac{1}{q+c}} \|u(t)\|_\infty \leq C_0 \|u_0\|_{M^{q,2q/(p-1)}(\Omega)}.
$$
Remarks 6.1. (i) Although this is out of the scope of this paper, we mention that local well-posedness for problem (1.1) is not expected to hold if we only assume \( u_0 \in \mathcal{M}^{q,2q/(p-1)}(\Omega) \) (cf. [16, 40] for related issues concerning the Navier-Stokes system; this is related with the lack of density of smooth functions in Morrey spaces). Actually, local well-posedness should be true under the slightly stronger assumption
\[
u_0 \in \hat{\mathcal{M}}^{q,2q/(p-1)}(\Omega) = \{ u_0 \in \mathcal{M}^{q,2q/(p-1)}(\Omega); \lim_{r \to 0} \sup_{a \in \Omega} r^{\lambda-n} \int_{B_R(a) \cap \Omega} |f|^q \, dx = 0 \}.
\]

(ii) Proposition 5.1 guarantees that any borderline global weak solution \( u^* \) satisfies
\[
\sup_{t>0} \| u^*(t) \|_{\mathcal{M}^{2,4/(p-1)}(\Omega)} < \infty. \tag{6.2}
\]
In spite of this, \( u^* \) blows up in \( L^\infty \) norm at some finite time \( T_{\text{max}}(\lambda^* \phi) \) by Theorem 4. This implies in particular that the (classical) existence time of a solution is not uniform for initial data in bounded sets of \( \mathcal{M}^{2,4/(p-1)} \). Likewise, if \( u_0 \) has the borderline decay (i.e., \( O \) instead of \( o \) in (2.6)), then any global classical solution satisfies (6.2), by Proposition 5.1 and the proof of Corollary 5. However global unbounded solutions do exist for such \( u_0 \) (cf. Remark 3.1(i)).

(iii) The above nonuniformity phenomenon is typical when dealing with critical spaces and it is known to occur for instance for problem (1.1) in the critical Lebesgue space \( L^q(\Omega) \) with \( q = n(p-1)/2 \) (cf. [35, Sections 15-16]). However, \( L^q(\Omega) \) is a strict subspace of \( \mathcal{M}^{2,4/(p-1)}(\Omega) \) with stronger norm (cf. Proposition 4.2) and, as an interesting difference, the critical Morrey norm here does not blow up at \( T_{\text{max}} \), whereas, in all the known examples, the critical \( L^q \) norm always does (although the question is still open in general). Since, on the other hand, problem (1.1) is locally well posed in the critical \( L^q \) space, this difference may be also connected with remark (i). Related blowup results in the critical Lorentz space \( L^{q,\infty} \) can be found in [12] for \( p < p_s \).

(iv) Consider the space \( L^{2/(p-1)}(\Omega) \), defined in (4.6), which is also critical (in the sense that it is invariant by the scaling (6.1)), and is another strict subspace of \( \mathcal{M}^{2,4/(p-1)}(\Omega) \) (cf. Proposition 4.2). It is known [23] that in the radial case, for \( p > p_{JL} \), the \( L^{2/(p-1)} \) norm may or may not blow up at \( T_{\text{max}} \), depending on the type of blowup (type I or II, unfocused or focused at \( x = 0 \)).

Proof of Proposition 6.1. Denoting by \( e^{-tA} \) the Dirichlet heat semigroup on \( L^\infty(\Omega) \) and setting \( u^p = \| u \|^{p-1} u \), we have, for all \( t \in (0,T) \),
\[
u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} u^p(s) \, ds
\]
(in the \( L^\infty \) sense), by the variation-of-constants formula. Set \( \lambda = 2q/(p-1) \in (0,n] \) and denote \( | \cdot |_m = \| \cdot \|_{\mathcal{M}^{m,\lambda}(\Omega)} \) for all \( m \in [1,\infty] \). Fix \( r \in (\max(p,q),pq) \), so that
\[
1 < r/p < q < r
\]
and set
\[ \beta = \frac{\lambda}{2} \left( \frac{1}{q} - \frac{1}{r} \right) = \frac{1}{p-1} \left( \frac{1}{q} - \frac{1}{r} \right) < \frac{1}{p}. \]

In all the proof, \( C_i \) (resp., \( C \)) denote fixed (resp., generic) positive constants depending only on \( n, p, q, r \). By Proposition 4.1, we have
\[ |e^{-tA} \phi|_r \leq C_1 t^{-\beta} |\phi|_q, \quad \phi \in M^{q,\lambda}, \quad t > 0 \] (6.3)

and
\[ |e^{-tA} \phi|_r \leq C_1 t^{-\frac{\lambda(p-1)}{p}} |\phi|_{r/p}, \quad \phi \in M^{r/p,\lambda}, \quad t > 0. \] (6.4)

For any given \( t_0 \in (0, T_{max}) \), set \( M = M(t_0) = \sup_{t \in [0, t_0]} \| u(t) \|_{\infty}^{p-1} < \infty \). On the interval \((0, t_0)\), we get \( |u_t - \Delta u| \leq M |u| \) hence, by the maximum principle,
\[ |u(t)| \leq e^{Mt} e^{-tA} |u_0| \quad \text{in} \ \Omega \times [0, t_0]. \] (6.5)

In particular \( t^\beta |u(t)|_r \leq 2C_1 |u_0|_q \) for \( t > 0 \) small. We may thus define
\[ \tau_1 = \sup \{ t \in (0, T_{max}); s^\beta |u(s)|_r \leq 2C_1 |u_0|_q \text{ for all } s \in (0, t) \} \]

and we have \( \tau_1 \in (0, T_{max}) \). For all \( 0 < s < t < \tau_1 \), by (6.4), (4.2), we obtain
\[ |e^{-(t-s)A} u^p(s)|_r \leq C_1 (t-s)^{-\frac{\lambda(p-1)}{2r}} |u^p(s)|_{r^p} = C_1 (t-s)^{-\frac{1}{r(p-1)}} |u(s)|_r^{p-1} \leq C |u_0|_{q}^p (t-s)^{-\frac{1}{r}} s^{-p}. \]

On the other hand, since \( 1 - \beta (p-1) - q/r = 0 \), we have
\[ t^\beta \int_0^t (t-s)^{-q/r} s^{-\beta p} ds = t^{1-\beta (p-1)-q/r} \int_0^1 (1-\sigma)^{-q/r} \sigma^{-\beta p} d\sigma = C, \quad t > 0, \] (6.6)
where the integrals are finite, due to \( q/r < 1 \) and \( \beta p < 1 \). It then follows that, for all \( t \in (0, \tau_1) \),
\[ t^\beta |u(t)|_r \leq t^\beta |e^{-tA} u_0|_r + t^\beta \int_0^t |e^{-(t-s)A} u^p(s)|_r ds \leq C_1 |u_0|_q + C |u_0|_q^p. \] (6.7)

Assume for contradiction that \( \tau_1 < T_{max} \). By continuity we may take \( t = \tau_1 \) in (6.7) to get \( 2C_1 |u_0|_q \leq C_1 |u_0|_q + C |u_0|_q^p \), which is a contradiction if \( |u_0|_q \leq \varepsilon_0 \) with \( \varepsilon_0 = \varepsilon_0(n, p, q, r) > 0 \) sufficiently small. It follows that \( \tau_1 = T_{max} \) i.e.,
\[ |u(t)|_r \leq 2C_1 |u_0|_q t^{-\beta}, \quad 0 < t < T_{max}. \] (6.8)

Next, by Proposition 4.1, we have
\[ \|e^{-tA} \phi\|_\infty \leq C_2 t^{-\lambda/2r} |\phi|_r, \quad \phi \in M^{r,\lambda}, \quad t > 0. \] (6.9)
Let \( C_0 > 0 \) to be fixed later and set
\[
\tau_2 = \sup \{ t \in (0, T_{\text{max}}); s^{1/(p-1)}\|u(s)\|_{\infty} \leq C_0\|u_0\|_q \text{ for all } s \in (0, t) \} \in (0, T_{\text{max}}].
\]

Note that
\[
\|u(t)\|_{\infty} = e^{-(t/2)A}u(t/2) + \int_{t/2}^t e^{-(t-s)A}u_p(s) \, ds.
\]

Combining (6.9), (6.8) and recalling \( \beta = \frac{1}{p-1} - \frac{1}{2} \), it follows that
\[
t^{\frac{1}{p-1}}\|u(t)\|_{\infty} \leq t^{\frac{1}{p-1}}\|e^{-(t/2)A}u(t/2)\|_{\infty} + t^{\frac{1}{p-1}}\int_{t/2}^t \|u(s)\|^p_\infty \, ds
\]
\[
\leq 2^{\frac{1}{p-1}}C_2t^{\frac{1}{p-1}} - \frac{1}{p} \|u(t/2)\|_r + 2^{\frac{1}{p-1}}C_0^p\|u_0\|^p_q
\]
\[
\leq 2^{\frac{1}{p-1}+\beta+1}C_1C_2\|u_0\|_q + 2^{\frac{1}{p-1}}C_0^p\|u_0\|^p_q.
\]

Choose \( C_0 = 2^{\frac{1}{p-1}+\beta+2}C_1C_2 \). Arguing as for \( \tau_1 \), taking \( \varepsilon_0 \) smaller if necessary, we obtain
\( \tau_2 = T_{\text{max}} \), hence \( T_{\text{max}} = \infty \) and the proposition follows.

\[\square\]

7. Proofs of Theorems 1, 2, 5 and Corollary 3.

Proof of Theorem 2. We may always assume that \( u_0 \in BC^1(\Omega) \). Indeed, if \( \Omega \) is bounded and \( u_0 \in L^\infty \) then this is true after a time shift. Moreover, assumption (2.4) is automatically satisfied if \( \Omega \) is bounded. Indeed, since \( |u_0|^2 \) and \( |\nabla u_0|^2 \) then belong to \( L^1(\mathbb{R}^n) \) (recall that they are extended by 0 outside \( \Omega \)), we have
\[
\|G_t * |\nabla u_0|^2\|_{\infty} + \|G_t * |u_0|^2\|_{\infty} \leq Ct^{-n/2},
\]
so that (2.4) is true since \( \frac{p+1}{p-1} < \frac{n}{2} \) (due to \( p > p_S \)).\(^1\)

Now, by assumption (2.4) and Proposition 5.1, we have
\[
\lim_{t_0 \to \infty} \|u(t_0)\|_{M^{2.4/(p-1)}(\Omega)} = 0.
\]

Pick any \( \varepsilon \in (0, \varepsilon_0) \), where \( \varepsilon_0 \) is given by Proposition 6.1 with \( q = 2 \). Then there exists \( t_0 = t_0(\varepsilon) > 0 \) such that \( \|u(t_0)\|_{M^{2.4/(p-1)}(\Omega)} \leq \varepsilon \), and we deduce from Proposition 6.1 that
\[
\|u(t)\|_{\infty} \leq C_0\varepsilon(t - t_0)^{-1/(p-1)} \leq 2^{1/(p-1)}C_0\varepsilon t^{-1/(p-1)}, \quad t \geq 2t_0.
\]

Theorem 2 follows.\[\square\]

\(^1\) We observe in turn that, since \( \|G_t * \phi\|_{\infty} \geq ct^{-n/2} \) for any nontrivial \( \phi \geq 0 \), assumption (2.4) can be realized by a nontrivial \( u_0 \) only if \( p > p_S \) (for any domain \( \Omega \)).
Proof of Theorem 5. By our assumptions, each \( u_j \) satisfies estimate (7.1) with uniform constants and a common \( t_0 \). Passing to the limit, we see that \( u \) satisfies the same estimate for a.e. \( t \geq 2t_0 \). By parabolic regularity, \( u \) is in particular a classical solution for \( t \geq 2t_0 \). \( \Box \)

Proof of Corollary 3. Let us first assume (2.5). We have \( |u_0|^2 \in L^r(\Omega) \) with \( 1 \leq r < n(p-1)/4 \), hence
\[
t_{\frac{p}{p-r}}^\frac{2}{p-r} \|G_t * |u_0|^2\|_\infty \leq C\| |u_0|^2\|_r \to 0, \ t \to \infty.
\]
Likewise, \( |\nabla u_0|^2 \in L^m(\Omega) \) with \( 1 \leq m < n(p-1)/2(p+1) \), hence
\[
t_{\frac{p+1}{p-1}}^\frac{2}{p-1} \|G_t * |\nabla u_0|^2\|_\infty \leq C\| |\nabla u_0|^2\|_q \to 0, \ t \to \infty.
\]
Let us next assume (2.6). Set \( k = 2/(p-1) \) and choose \( \delta > 0 \) such that \( k + \delta < n/2 \). By (2.6), for any \( \eta > 0 \), there exists \( A > 0 \) such that
\[
|u_0(x)|^2 \leq \eta(1 + |x|^2)^{-k} + A(1 + |x|^2)^{-k-\delta}.
\]
It is well-known that \( \|G_t * (1 + |x|^2)^{-k}\|_\infty \leq C(t+1)^{-k} \) whenever \( k \in (0, n/2) \) (see e.g. [35, Lemma 20.8]). Consequently,
\[
t_{\frac{p}{p-r}}^\frac{2}{p-r} \|G_t * |u_0|^2\|_\infty \leq Ct_{\frac{p}{p-r}}^\frac{2}{p-r} [\eta(t+1)^{-k} + A(t+1)^{-k-\delta}] \leq 2C\eta
\]
for \( t \) large enough, hence \( t_{\frac{p}{p-r}}^\frac{2}{p-r} \|G_t * |u_0|^2\|_\infty \to 0 \) as \( t \to \infty \). Similarly, we show that \( t_{\frac{p+1}{p-1}}^\frac{2}{p-1} \|G_t * |\nabla u_0|^2\|_\infty \to 0 \) as \( t \to \infty \).

In both cases, the conclusion then follows from Theorem 2. \( \Box \)

We shall now derive Theorem 1 as a consequence of Corollary 3. However, unlike in Corollary 3, \( u_0 \) is not assumed to satisfy any integrability or decay hypothesis (only \( \nabla u_0 \) does), nor to belong to \( BC^1 \) in case of assumption (2.1).

To circumvent this, we shall use the following two general lemmas. The first one allows to deduce a suitable integrability or decay property on a function from such an assumption on its gradient. The second one allows to perform a time shift, so as to gain \( BC^1 \) regularity, while preserving assumption (2.1).

**Lemma 7.1.** Let \( n \geq 2 \).

(i) Let \( q \in [1, n) \) and let \( v \in L^\infty(\mathbb{R}^n) \) be such that \( \nabla v \in L^q(\mathbb{R}^n) \). Then there exists a constant \( K \in \mathbb{R} \) such that \( v - K \in L^q(\mathbb{R}^n) \), where \( q^* = nq/(n - q) \).

(ii) Let \( \beta > 0 \) and let \( v \in BC^1(\mathbb{R}^n) \) be such that \( |\nabla v| = o(|x|^{-\beta -1}) \) as \( |x| \to \infty \). Then there exists a constant \( K \in \mathbb{R} \) such that \( v(x) - K = o(|x|^{-\beta}) \) as \( |x| \to \infty \).

**Lemma 7.2.** Let \( \Omega = \mathbb{R}^n \), \( u_0 \in L^\infty(\mathbb{R}^n) \) and let \( u \) be the maximal classical solution of (1.1). There exists \( \tau \in (0, T_{\max}) \) such that \( |\nabla u(t)| \leq 2G_t * |\nabla u_0| \) for all \( t \in (0, \tau] \).
Results related with Lemma 7.1(i) can be found in, e.g., [19, Proposition 3.4] and [28]. Since they do not quite fulfill our needs, we give an elementary proof in Appendix. We stress that in Lemma 7.1(i) we are not assuming \( u \) to belong to the Sobolev space \( W^{1,q}(\mathbb{R}^n) \) (otherwise, the conclusion is immediate by Sobolev embedding and necessarily \( K = 0 \)). The homogeneous space \( W^{1,q}(\mathbb{R}^n) \), defined by completion of \( C_0^\infty(\mathbb{R}^n) \) by the semi-norm \( \| \nabla v \|_q \), is sometimes used in similar situations. Note that addition of a constant cannot be a priori ruled out in this way.

As for Lemma 7.2, which is rather standard, we also postpone its proof to the appendix, in order not to interrupt the main line of argument.

**Proof of Theorem 1.** We first observe that we can always assume \( u_0 \in BC^1(\mathbb{R}^n) \), making a time shift if necessary. Indeed, in case of assumption (2.1), this assumption is still satisfied after replacing \( u_0 \) with \( u(\tau) \in BC^1(\mathbb{R}^n) \) (for some small \( \tau > 0 \)), thanks to Lemma 7.2.

Under assumption (2.1), by Lemma 7.1(i), there exists \( \psi \in L^{q^*} \) and a constant \( K \in \mathbb{R} \) such that \( u_0 = K + \psi \). Since \( q < n(p-1)/(p+1) \implies q^* < n(p-1)/2 \), and \( u_0 \in BC^1 \), it follows that
\[
|u_0 - K|^{p+1} + |\nabla u_0|^2 \in L^m(\mathbb{R}^n) \quad \text{for some } m \in \left[1, \frac{n(p-1)}{2(p+1)}\right].
\]

Under assumption (2.2), by Lemma 7.1(ii), we may write \( u_0 = K + \psi \) with \( \psi = o(|x|^{-2/(p-1)}) \) as \( |x| \to \infty \), hence
\[
|u_0(x) - K| + |x||\nabla u_0(x)| = o(|x|^{-\frac{2}{p-1}}) \quad \text{as } |x| \to \infty.
\]

We claim that \( T_{\max} = \infty \) necessarily imposes \( K = 0 \). To prove this, we use the rescaled solution by similarity variables at \( a = 0, t = T \) and the corresponding weighted energy \( E(s) = E(u_{0,T}(s)) \), cf. Section 4.2 (a more direct comparison argument would apply in the case \( u_0 \geq 0 \)). Assume for contradiction that \( K > 0 \) (without loss of generality). Denoting \( \int = \int_{\mathbb{R}^n} \) and recalling the notation \( G_T(x) = (4\pi T)^{-n/2}e^{-\frac{|x|^2}{4T}} \), we have
\[
T^{-\frac{p+1}{p-1}}E(u_{0,T}(-\log T)) = C_1 \int |\nabla \psi|^2 G_T + \frac{C_2}{T} \int |K + \psi|^2 G_T - C_3 \int |K + \psi|^{p+1} G_T
\leq C_1 \int |\nabla \psi|^2 G_T + \frac{C_2}{T} \left(1 + \int |K + \psi|^{p+1} G_T\right) - C_3 \int |K + \psi|^{p+1} G_T
\leq C_1 \int |\nabla \psi|^2 G_T + \frac{C_2}{T} - \frac{C_3}{2} \int |K + \psi|^{p+1} G_T
\leq C \int (|\nabla \psi|^2 + |\psi|^{p+1}) G_T + \frac{C_2}{T} - C_4 K^{p+1},
\]
for \( T > 0 \) large enough, where we used \( \int G_T = 1 \), and also \( |K + \psi|^{p+1} \geq 2^{-p}K^{p+1} - |\psi|^{p+1} \) in the last inequality. On the other hand, by either (7.2) or (7.3), we have
\[
|\nabla \psi|^2 + |\psi|^{p+1} \in
\]
$L^k(\mathbb{R}^n)$ for some $k \in (1, \infty)$. It follows that the integral term in (7.4) is bounded by $CT^{-n/2k}$, hence converges to 0 as $T \to \infty$. Consequently, $E(w_0, T(-\log T)) < 0$ for $T$ sufficiently large. By Proposition 4.3, this contradicts the global existence of $u$ and we conclude that $K = 0$.

Finally, in view of (7.2) or (7.3) with $K = 0$, Theorem 1 is now a consequence of Corollary 3.

Remark 7.1. Let us justify the assertion in Remark 3.1(iii). From the proof of Corollary 3, under assumption (2.5), we actually get

$$\mathcal{N}(u_0, t) \leq C t^{-n \eta}, \quad t \geq 1,$$

with some $\eta = \eta(n, p, q) > 0$, where $\mathcal{N}(u_0, t)$ is defined in Proposition 5.1. By Proposition 5.1(ii), it follows that $\|u(t)\|_{M^{2,q/(p-1)}} \leq C t^{-(n/2k)}$. For $t \geq t_1$ large enough, we deduce from Proposition 6.1 that

$$\|u(t)\|_\infty \leq C_0 \frac{t}{(2)^{-1/(p-1)}} \|u(t/2)\|_{M^{2,q/(p-1)}} \leq C t^{-\beta}, \quad t \geq t_1,$$

where $\beta = \frac{1}{p-1} + \frac{\eta}{p+1} > \frac{1}{p-1}$. Consequently $h(t) := \int_0^t \|u(s)\|_\infty^{p-1} ds$ satisfies $\sup_{t > 0} h(t) < \infty$ and $z(t) := e^{h(t)} e^{-tA}|u_0|$ solves

$$z_t - \Delta z = h'(t)z = \|u(t)\|_\infty^{p-1} z \geq |u|^{p-1} z.$$

Therefore, $u \leq z$ by the maximum principle, and similarly $u \geq -z$. The claimed property (3.1) follows. On the other hand, by the proof of Theorem 1, under assumption (2.1), we may assume $u_0 \in L^q(\mathbb{R}^n)$ after a time shift. We thus again obtain (3.1) and then (3.2).

8. Proof of Theorem 4.

It is based on the following continuous dependence property with respect to the critical Morrey norms.

Proposition 8.1. Let $\Omega \subset \mathbb{R}^n$ be a (possibly unbounded) domain of class $C^{2+\alpha}$. Let $q \in (1, \infty)$ and $p \geq 1 + \frac{2q}{n}$. For any $u_0 \in L^\infty \cap M^{q,2q/(p-1)}(\Omega)$, and any $T_0 \in (0, T_{\max}(u_0))$, there exist $\delta, M > 0$ (depending on $u_0$ and $T_0$) with the following property. If $v_0 \in L^\infty \cap M^{q,2q/(p-1)}(\Omega)$ satisfies

$$\|u_0 - v_0\|_{M^{q,2q/(p-1)}} \leq \delta,$$

then $T_{\max}(v_0) > T_0$ and the corresponding solution $v$ of (1.1) satisfies

$$\|u(t) - v(t)\|_{M^{q,2q/(p-1)}} \leq M \|u_0 - v_0\|_{M^{q,2q/(p-1)}}, \quad 0 \leq t \leq T_0.$$  (8.1)
Proof. It relies on suitable modifications of the proof of Proposition 6.1. We set \( \lambda = 2q/(p-1) \) and denote again \( u^p = |u|^{p-1}u \) and \( |\cdot|_m = \| |\cdot||_{M^{m,\lambda}(\Omega)} \). In all the proof, \( C_i \) (resp., \( C \)) denote fixed (resp., generic) positive constants depending only on \( n, p, q, r \).

**Step 1.** Fix \( r \) such that \( 1 < r/p < q < r \) and let \( \beta = \frac{1}{p-1} - \frac{\lambda}{2r} > 0 \). Let \( C_1 \) be the constant from (6.3). We claim that there exists \( \delta_0 = \delta_0(n, p, q, r) > 0 \) with the following property: for any \( \delta \in (0, \delta_0) \), there exists \( t_1 = t_1(\delta) \in (0, T_0) \) (depending on \( u_0 \)) such that

\[
|u_0 - v_0|_q \leq \delta \quad \Rightarrow \quad \begin{cases} 
\tau^\beta |u(t) - v(t)|_r \leq 2C_1 |u_0 - v_0|_q \\
\text{for all } t \in (0, \min(t_1(\delta), T_{max}(v_0))).
\end{cases}
\]

(8.2)

To prove (8.2), let \( \delta > 0 \) and assume \( |u_0 - v_0|_q \leq \delta \). Since \( u_0 \in M^{q,\lambda} \cap L^\infty \subseteq M^{r,\lambda} \), inequalities (4.5) and (6.5) guarantee the existence of \( t_1 = t_1(\delta) \in (0, T_0) \) such that

\[
s^\beta |u(s)|_r + s^{1/(p-1)} \|u(s)\|_\infty \leq \delta, \quad 0 < s \leq t_1(\delta).
\]

(8.3)

Set \( \tilde{t}_1 = \tilde{t}_1(\delta, v_0) = \min(t_1(\delta), T_{max}(v_0)) \) and let

\[
\tau = \sup \{ t < \tilde{t}_1; s^\beta |u(s) - v(s)|_r \leq 2C_1 |u_0 - v_0|_q \text{ for all } s \in (0, t) \}
\]

(note that \( \tau > 0 \), owing to (6.5) and (4.5) applied to \( u_0, v_0 \in M^{r,\lambda} \) with some \( t_0 < \min(T_{max}(u_0), T_{max}(v_0)) \)). For all \( s \in (0, \tau) \), we have \( s^\beta |v(s)|_r \leq (2C_1 + 1)\delta \) hence, using (4.2), (4.3),

\[
|u^p(s) - v^p(s)|_{r/p} \leq p \left( |u(s)|^{p-1} + |v(s)|^{p-1} \right) |u(s) - v(s)|_r \\
\leq p \left( |u(s)|^{p-1} + |v(s)|^{p-1} \right) |u(s) - v(s)|_r \\
\leq C\delta^{p-1} |u_0 - v_0|_{q^s^{\beta p}}.
\]

Therefore, by (6.3), (6.4) and (6.6), we deduce that, for all \( t \in (0, \tau) \),

\[
t^\beta |u(t) - v(t)|_r \leq t^\beta |e^{-tA}(u_0 - v_0)|_r + t^\beta \int_0^t |e^{-(t-s)A}(u^p(s) - v^p(s))|_r ds \\
\leq C_1 |u_0 - v_0|_q + C_1 t^\beta \int_0^t |e^{-(t-s)A}(u^p(s) - v^p(s))|_{r/p} ds \\
\leq C_1 |u_0 - v_0|_q + C_3 \delta^{p-1} |u_0 - v_0|_q t^\beta \int_0^t |(t-s)^{-q/r} s^{-\beta p} ds \\
\leq (C_1 + C_3 \delta^{p-1}) |u_0 - v_0|_q.
\]

Assuming \( \delta \in (0, \delta_0) \) with \( \delta_0 = \delta_0(n, p, q, r) > 0 \) small enough and arguing as before (6.8), we deduce that \( \tau = \tilde{t}_1 \), hence the claim.

**Step 2.** We claim that there exist \( \delta_1 = \delta_1(n, p, q, r) \in (0, \delta_0) \) and \( C_3, C_4 > 0 \) such that, if \( |u_0 - v_0|_q \leq \delta_1 \), then

\[
T_{max}(v_0) > t_1(\delta_1), \quad t^{1/(p-1)} \|u(t) - v(t)\|_\infty \leq C_3 |u_0 - v_0|_q, \quad t \in (0, t_1(\delta_1)),
\]

(8.5)
where \( t_1(\delta_1) \) is given by Step 1, and

\[
|u(t) - v(t)|_q \leq C_4 |u_0 - v_0|_q, \quad t \in (0, t_1(\delta_1)].
\]

(8.6)

To prove (8.5), for \( C_3 > 0 \) and \( \delta_1 \in (0, \delta_0] \) to be chosen later, we assume \( |u_0 - v_0|_q \leq \delta_1 \). Denote \( \tilde{t}_1 = \tilde{t}_1(\delta_1, v_0) = \min(t_1(\delta_1), T_{\text{max}}(v_0)) \) and let

\[
\tau_1 = \sup \{ t < \tilde{t}_1; \ s^{1/(p-1)}\|u(s) - v(s)\|_\infty \leq C_3 |u_0 - v_0|_q \text{ for all } s \in (0, t) \}
\]

(note that \( \tau_1 > 0 \), since \( \|u(s)\|_\infty \) and \( \|v(s)\|_\infty \) are bounded on \([0, t]\) for some small \( t > 0 \).

In particular, by (8.3), we have \( s^{1/(p-1)}(\|u(s)\|_\infty \vee \|v(s)\|_\infty) \leq (C_3 + 1)\delta_1 \) on \((0, \tau_1)\). For all \( s \in (0, \tau_1) \), we have

\[
\|u^p(s) - v^p(s)\|_\infty \leq p(\|u(s)\|_\infty^{p-1} + \|v(s)\|_\infty^{p-1})\|u(s) - v(s)\|_\infty
\]

\[
\leq 2p(C_3 + 1)^p\delta_1^{p-1}|u_0 - v_0|_q s^{-\frac{p-1}{p}}.
\]

Using (6.10), (6.9), (8.2) and recalling \( \beta = \frac{1}{p-1} - \frac{1}{p} \), we deduce that, for all \( t \in (0, \tau_1) \),

\[
\int_{\frac{t}{2}}^{t} \|u(t) - v(t)\|_\infty \leq \int_{\frac{t}{2}}^{t} e^{-\frac{\beta}{p}(u(\frac{t}{2}) - v(\frac{t}{2}))} ds + \int_{\frac{t}{2}}^{t} \|u^p(s) - v^p(s)\|_\infty ds
\]

\[
\leq 2\int_{\frac{t}{2}}^{t} C_2 t^{\frac{\beta}{p}} \|u(\frac{t}{2}) - v(\frac{t}{2})\|_r + 2\int_{\frac{t}{2}}^{t} p(C_3 + 1)^p\delta_1^{p-1}|u_0 - v_0|_q
\]

\[
\leq \left[ 2\int_{\frac{t}{2}}^{t} C_1 C_2 + 2\int_{\frac{t}{2}}^{t} p(C_3 + 1)^p\delta_1^{p-1} \right]|u_0 - v_0|_q.
\]

Choosing \( C_3 = 2\int_{\frac{t}{2}}^{t} C_1 C_2 \) and \( \delta_1 = \delta_1(n, p, q, r) > 0 \) small enough, we get \( \tau_1 = \tilde{t}_1 \), hence (8.5).

To prove (8.6), we set \( \gamma = \frac{1}{2}(\frac{p}{r} - \frac{1}{q}) > 0 \). Using Proposition 4.1 and (8.4), noting that \( \gamma + \beta p = 1 \), we obtain

\[
|u(t) - v(t)|_q \leq C|e^{-\frac{\gamma A}{r}(u_0 - v_0)}_q + C \int_{0}^{t} (t - s)^{-\gamma}|u^p(s) - v^p(s)|_r ds
\]

\[
\leq C|u_0 - v_0|_q + C\delta_1^{p-1}|u_0 - v_0|_q \int_{0}^{t} (t - s)^{-\gamma}s^{-\beta p} ds \leq C_4 |u_0 - v_0|_q.
\]

**Step 3.** Let \( M_0 = \sup_{t \in [0, T_0]} \|u(t)\|_\infty < \infty \), let \( T_1 := t_1(\delta_1) \) be given by Steps 1 and 2, and assume

\[
|u_0 - v_0|_q < \min(\delta_1, C_3^{-1}T_1^{1/(p-1)}).
\]

Then, by (8.5), we have \( T_2 := \min(T_0, T_{\text{max}}(v_0)) > T_1 \) and

\[
\tau_2 := \sup \{ t \in (T_1, T_2); \|u(s) - v(s)\|_\infty \leq 1 \text{ for all } s \in [T_1, t] \} > T_1.
\]
Next, by the maximum principle, using \( \max(\|u(s)\|_\infty, \|v(s)\|_\infty) \leq M_0 + 1 \) on \([T_1, \tau_2]\), we easily get
\[
|u(t) - v(t)| \leq e^{p(M_0 + 1)^{p-1}(t-T_1)} |e^{-(t-T_1)A}(u(T_1) - v(T_1))| \quad \text{in } \Omega \times [T_1, \tau_2]. \tag{8.7}
\]
Setting \( M_1 = e^{p(M_0 + 1)^{p-1}(T_0-T_1)}, M_2 = M_1 C_3 T_1^{1/(p-1)} \) and using (8.5), we in particular obtain
\[
\|u(t) - v(t)\|_\infty \leq M_1\|u(T_1) - v(T_1)\|_\infty \leq M_2|u_0 - v_0|_q \quad \text{in } \Omega \times [T_1, \tau_2].
\]
Now further assuming \( |u_0 - v_0|_q < \delta := \min(\delta_1, C_3^{-1} T_1^{1/(p-1)}, (2M_2)^{-1}) \), it follows that \( \tau_2 = T_2 \). Consequently, \( T_{\max}(v_0) > T_0 \). Finally, going back to (8.7) and using (4.5) and (6.1), we conclude that
\[
|u(t) - v(t)|_q \leq M_1|e^{-(t-T_1)A}(u(T_1) - v(T_1))|_q \leq M_1\|u(T_1) - v(T_1)\|_q \leq C_4 M_1|u_0 - v_0|_q,
\]
for all \( t \in [T_1, T_0] \). This combined with (8.6) proves (8.1).

**Proof of Theorem 4.** (i) Let \( \varepsilon_0 \) be given by Proposition 6.1 with \( q = 2 \). By assumption (2.4) and Proposition 5.1, since \( u \) is assumed to be global, there exists \( T_0 > 0 \) such that \( \|u(T_0)\|_2 \leq \varepsilon_0/2 \), where \( \| \cdot \|_2 = \| \cdot \|_{M^{2,A/(p-1)}} \).

By Proposition 4.2, we have \( u_0 \in M^{2,A/(p-1)} \cap L^\infty \). By Proposition 8.1, there exists \( \eta > 0 \) such that if \( |u_0 - v_0|_2 \leq \eta \), then \( T_{\max}(v_0) > T_0 \) and \( |u(T_0) - v(T_0)|_2 \leq \varepsilon_0/2 \), hence \( |v(T_0)|_2 \leq \varepsilon_0 \). We then conclude from Proposition 6.1 that \( v \) is global, which proves (2.9). The inclusions (2.10) and (2.11) then follow from Proposition 4.2.

(ii) Assume for contradiction that \( T_{\max}(\lambda^* \phi) = \infty \). Then, since \( \| \phi \|_2 < \infty \) by (2.4) and (4.9), we would have \( T_{\max}(\lambda \phi) = \infty \) for \( \lambda \to \lambda^* \) by assertion (i), contradicting the definition of \( \lambda^* \).

**Appendix.**

In this appendix, for convenience and self-containment, we provide proofs of some important known results that we have used, namely Proposition 4.1 from [16] and Proposition 4.3 from [11, 12]. We also give the proofs of Lemmas 7.1 and 7.2.

**Proof of Proposition 4.1.** By the maximum principle, it suffices to prove the assertion when \( \Omega = \mathbb{R}^n \). Set \( v(\cdot, t) = e^{-tA} f = G_t \ast f \). Let \( p \in [1, \infty) \) (the case \( p = q = \infty \) is obvious due to \( M^{\infty, \lambda} = L^\infty \)). By Jensen’s inequality and \( \|G_t\|_1 = 1 \), we have \( |v(\cdot, t)|^p \leq G_t \ast |f|^p \). By Fubini’s theorem, for any \( R > 0 \), it follows that
\[
\int_{B_R} |v(x, t)|^p \, dx \leq \int_{B_R} \left( \int_{\mathbb{R}^n} G_t(y)|f(x - y)|^p \, dy \right) dx
\]
\[
= \int_{\mathbb{R}^n} G_t(y) \left( \int_{B_R} |f(x - y)|^p \, dx \right) dy = \int_{B_R(x)} |f|^p \, dx \leq R^{n-p} \|f\|_{M^{p, \lambda}}^p.
\]
Consequently, $\|e^{-tA}f\|_{M_{p,\lambda}}^p \leq \|f\|_{M_{p,\lambda}}^p$, hence (4.5).

Next set $\rho(r) = \int_{B_r} |f|^p \, dx$. We claim that, for each $R > 0$ and $\phi \in C^1([0, R])$,
\[
\int_{B_R} \phi(|x|)|f(x)|^p \, dx = \phi(R)\rho(R) - \int_0^R \phi'(r)\rho(r) \, dr.
\] (A.1)

By density of $C(\overline{B}_R)$ in $L^p(B_R)$, it suffices to prove this when $f \in C(\overline{B}_R)$. Set $h(r) = \int_{\partial B_r} |f|^p \, d\sigma_r$, where $d\sigma_r$ is the surface measure on $\partial B_r$. Then $h \in C([0, R])$ and $\rho(r) = \int_0^r h(s) \, ds$. Integrating by parts, we get
\[
\int_{B_R} \phi(|x|)|f(x)|^p \, dx = \int_0^R \phi(r)h(r) \, dr = \int_0^R \phi(r)\rho'(r) \, dr
\]
\[
= \phi(R)\rho(R) - \int_0^R \phi'(r)\rho(r) \, dr.
\]

For fixed $t > 0$, using Hölder’s inequality and then applying (A.1) with $\phi(r) = K_t(r) = (4\pi t)^{-n/2}e^{-r^2/4t}$, we obtain
\[
|v(0, t)|^p \leq \left( \int_{B_R} K_t(|x|)|f(x)| \, dx \right)^p \leq \int_{B_R} K_t(|x|)|f(x)|^p \, dx
\]
\[
\leq K_t(R)\rho(R) - \int_0^R K'_t(r)\rho(r) \, dr
\]
\[
\leq K_t(R)R^{n-\lambda}\|f\|_{M_{p,\lambda}}^p + (4\pi t)^{-n/2}2t^{-1}\|f\|_{M_{p,\lambda}}^p \int_0^R r^{1+n-\lambda}e^{-r^2/4t} \, dr.
\]

Letting $R \to \infty$, we get
\[
|v(0, t)|^p \leq t^{1-(n/2)}\|f\|_{M_{p,\lambda}}^p \int_0^\infty r^{1+n-\lambda}e^{-r^2/4t} \, dr \leq C(n, \lambda)t^{-\lambda/2}\|f\|_{M_{p,\lambda}}^p.
\]

By translation invariance, the same holds at any point $x_0 \in \mathbb{R}^n$ instead of 0. This yields (4.4) for $q = \infty$. The general case follows by interpolating between the cases $q = p$ and $q = \infty$. \hfill \Box

**Proof of Proposition 4.3.** Assume $a = 0$ without loss of generality. For all $s > s_0$, we compute
\[
\frac{d}{ds} \int_{D(s)} |w|^q \rho = \int_{D(s)} \rho \, \partial_s (|w|^q) + e^{s/2} \int_{\partial D(s)} |w|^q \rho \frac{y \cdot \nu}{|y|} \, d\sigma,
\] (A.2)

where $q = 2$ or $p + 1$, and the boundary term vanishes since $w = 0$ on $\partial D(s)$. Here and below, for $s > s_0$, all the procedures are justified owing to the fast decay of the Gaussian weight $\rho$ and parabolic regularity. By integration by parts, we have
\[
\frac{1}{2} \frac{d}{ds} \int_{D(s)} w^2 \rho = \int_{D(s)} w w_s \rho + \int_{D(s)} w \left[ \nabla \cdot (\rho \nabla w) + \rho |w|^{p-1}w - \beta \rho w \right]
\]
\[
= \int_{D(s)} [-\nabla w]^2 - \beta w^2 + |w|^{p+1}] \rho = -2E(w) + \frac{p-1}{p+1} \int_{D(s)} |w|^{p+1} \rho
\]
i.e., (4.16). Next, using
\[ \frac{d}{ds} \int_{D(s)} |\nabla w|^2 \rho = -\frac{d}{ds} \int_{D(s)} w \nabla \cdot (\rho \nabla w) \]
and noting that the variation of the domain again does not produce a boundary term, due to \( w = 0 \) on \( \partial D(s) \), we obtain
\[ \frac{d}{ds} \int_{D(s)} |\nabla w|^2 \rho = -\int_{D(s)} w_s \nabla \cdot (\rho \nabla w) - \int_{D(s)} w \nabla \cdot (\rho \nabla w_s) \]
\[ = -\int_{D(s)} w_s \nabla \cdot (\rho \nabla w) + \int_{D(s)} \rho (\nabla w \cdot \nabla w_s) \]
\[ = -2 \int_{D(s)} w_s \nabla \cdot (\rho \nabla w) + \int_{\partial D(s)} \rho w_s w \nu \, d\sigma. \]

Since \( w_s = -\beta w - \frac{y}{2} \cdot \nabla w + e^{-(\beta+1)s} u_t (a + e^{-s/2} y, T - e^{-s}) \) and since the tangential derivatives of \( w \) vanish on \( \partial D(s) \), we have \( w_s = -\frac{y}{2} \cdot \nabla w \) on \( \partial D(s) \). Therefore, the integrand of the boundary term in (A.3) can be written as \(-\frac{1}{2} \rho (y \cdot \nu) |w_s|^2 \leq 0\), owing to the starshapedness of \( \Omega \) with respect to \( a \). Consequently,
\[ \frac{1}{2} \frac{d}{ds} \int_{D(s)} |\nabla w|^2 \rho \leq -\int_{D(s)} w_s \nabla \cdot (\rho \nabla w). \]

On the other hand, by (A.2), we have
\[ \frac{d}{ds} \int_{D(s)} \left( \beta w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho = \int_{D(s)} (\beta w - |w|^{p-1} w) w_s \rho. \]

Summing the last two formulas and using equation (4.14), we obtain (4.17). Since the continuity of \( E(s) \) at \( s = s_0 \) is guaranteed by the assumption \( u_0 \in BC^1(\Omega) \), we deduce in particular that \( E(s) \leq E(s_0) \) for all \( s > s_0 \).

Next denote \( \psi(s) := \int_{D(s)} w^2(s) \rho \). Then (4.16), Jensen’s inequality and (4.17) imply
\[ \frac{1}{2} \frac{d}{ds} \psi \geq -2E(w(s)) + C(n, p) \psi^{(p+1)/2}(s) \geq -2E(w(s_0)) + C(n, p) \psi^{(p+1)/2}(s). \]

This guarantees (4.18) and (4.19) (otherwise \( \psi \) has to blow up in finite time). \( \square \)

**Proof of Lemma 7.1.** (i) Fix a cut-off function \( \rho \in C_0^\infty([0, \infty)) \), with \( 0 \leq \rho \leq 1 \), such that \( \rho(s) = 1 \) for \( s \leq 1 \) and \( \rho(s) = 0 \) for \( s \geq 2 \). For each integer \( j \geq 1 \), set \( A_j = B_{2j} \setminus \overline{B}_j \) and define
\[ v_j(x) = \rho \left( \frac{|x|}{2} \right) \left( v(x) - \frac{1}{b_j} \right). \]
Since $\rho$ is compactly supported, it is clear that $v_j \in W^{1,q}(\mathbb{R}^n)$. We compute
\[
\nabla v_j(x) - \nabla v(x) = \frac{1}{j} \frac{x}{|x|} \rho'(\frac{|x|}{j}) (v(x) - \int_{A_j} v) + \left( \rho\left(\frac{|x|}{j}\right) - 1 \right) \nabla v(x),
\]
hence
\[
\|\nabla v_j - \nabla v\|_{L^q(\mathbb{R}^n)} \leq C \left( \frac{1}{j} \int_{A_j} v \right)_{L^q(A_j)} + \|\nabla v\|_{L^q(\mathbb{R}^n \setminus B_j)}.
\]

By the Poincaré-Wirtinger inequality, there exists a constant $C_0 = C_0(n,q) > 0$ such that
\[
\|\phi - \int_{A_1} \phi\|_{L^q(A_1)} \leq C_0 \|\nabla \phi\|_{L^q(A_1)} \quad \text{for all } \phi \in W^{1,p}(A_1).
\]

By a simple scaling argument, it follows that
\[
\|\phi - \int_{A_j} \phi\|_{L^q(A_j)} \leq C_0 j \|\nabla \phi\|_{L^q(A_j)} \quad \text{for all } \phi \in W^{1,p}(A_j), \quad j \geq 1.
\]

Therefore,
\[
\|\nabla v_j - \nabla v\|_{L^q(\mathbb{R}^n)} \leq C C_0 \|\nabla v\|_{L^q(A_j)} + \|\nabla v\|_{L^q(\mathbb{R}^n \setminus B_j)} \leq (1 + CC_0) \|\nabla v\|_{L^q(\mathbb{R}^n \setminus B_j)}.
\]

Since the RHS goes to 0 as $j \to \infty$, it follows that $\nabla v_j \to \nabla v$ in $L^q(\mathbb{R}^n)$.

Next, let $K_j := \int_{A_j} v$. Since $|K_j| \leq \|v\|_{L^q}$, we may assume, after extracting a subsequence, that $\lim_{j \to \infty} K_j = K \in \mathbb{R}$. Also, since $v_j \in W^{1,q}(\mathbb{R}^n)$, we have
\[
\|v_j\|_{L^{q^*}(\mathbb{R}^n)} \leq S \|\nabla v_j\|_{L^q(\mathbb{R}^n)}, \quad \text{(A.5)}
\]

where $S = S(n,q) > 0$ is the Sobolev constant. Observe that, by (A.4), for all given $R > 0$, $x \in B_R$ and all $j \geq R$, we have $v_j(x) + K_j = v(x)$, hence
\[
v(x) - K = v_j(x) + (K_j - K), \quad x \in B_R, \quad j \geq R.
\]

Therefore, by (A.5),
\[
\|v - K\|_{L^{q^*}(B_R)} \leq \|v_j\|_{L^{q^*}(B_R)} + |B_R|^{1/q^*} |K_j - K| \leq S \|\nabla v_j\|_{L^q(\mathbb{R}^n)} + |B_R|^{1/q^*} |K_j - K|.
\]

Passing to the limit $j \to \infty$, we obtain
\[
\|v - K\|_{L^{q^*}(B_R)} \leq S \|\nabla v\|_{L^q(\mathbb{R}^n)}.
\]

Uppon letting $R \to \infty$, we conclude that $v - K \in L^{q^*}(\mathbb{R}^n)$. 

(ii) Let $\varepsilon > 0$. By assumption, there exists $A > 0$ such that

$$|\nabla v(x)| \leq \beta \varepsilon |x|^{-\beta - 1}, \quad |x| \geq A.$$  \hfill (A.6)

For each $\omega \in S^{n-1}$ and $s > r \geq A$, we compute

$$|v(s\omega) - v(r\omega)| = \left| \int_0^1 (s - r)\omega \cdot \nabla v((r + t(s - r))\omega) \, dt \right|$$

$$\leq \int_0^1 (s - r) |\nabla \phi((r + t(s - r))\omega)| \, dt$$

$$\leq \beta \varepsilon \int_0^1 (s - r)(r + t(s - r))^{-\beta - 1} \, dt \leq \varepsilon r^{-\beta}.$$

By the Cauchy property, we deduce that $\ell(\omega) = \lim_{r \to \infty} v(r\omega) \in \mathbb{R}$ exists. Moreover, by letting $s \to \infty$ in the previous inequality, we obtain

$$|v(r\omega) - \ell(\omega)| \leq \varepsilon r^{-\beta}, \quad r \geq A.$$

To prove the assertion, it thus suffices to show that $\ell(\omega)$ is independent of $\omega$. Consider $\omega_1, \omega_2 \in S^{n-1}$ such that $\langle \omega_1, \omega_2 \rangle \geq 0$. We see that, for all $r > 0$ and $t \in [0, 1]$

$$|r\omega_1 + rt(\omega_2 - \omega_1)|^2 = r^2 \left[ 1 + 2t^2(1 - \langle \omega_1, \omega_2 \rangle) + 2t(\langle \omega_1, \omega_2 \rangle - 1) \right]$$

$$= r^2 \left[ 1 - 2t(1 - t) + 2t(1-t)\langle \omega_1, \omega_2 \rangle \right] \geq \frac{1}{2} r^2.$$

For $r \geq A\sqrt{2}$, it follows from (A.6) that

$$|v(r\omega_2) - v(r\omega_1)| = \left| \int_0^1 r(\omega_2 - \omega_1) \cdot \nabla v(r\omega_1 + rt(\omega_2 - \omega_1)) \, dt \right|$$

$$\leq 2r \int_0^1 |\nabla v(r\omega_1 + rt(\omega_2 - \omega_1))| \, dt \leq 2\beta \varepsilon r \left( \frac{r}{\sqrt{2}} \right)^{-\beta - 1} = \beta \varepsilon 2^{\frac{\beta + 3}{2}} r^{-\beta},$$

hence $\ell(\omega_1) = \ell(\omega_2)$ by letting $r \to \infty$. Denoting by $(e_i)_{1 \leq i \leq n}$ the canonical basis of $\mathbb{R}^n$, we have in particular $\ell(e_1) = \ell(e_2) = \ell(-e_1)$. For any $\omega \in S^{n-1}$, since either $\langle \omega, e_1 \rangle \geq 0$ or $\langle \omega, -e_1 \rangle \geq 0$, it follows that $\ell(\omega) = \ell(e_1)$ and the assertion is proved.

\[\square\]

**Proof of Lemma 7.2.** Fix $\tau \in (0, \min(1, T_{max}))$ and let $M = \sup_{t \in (0, \tau)} \|u(t)\|_\infty$. Using

$$\nabla u(t) = \nabla G_t * u_0 + \int_0^t \nabla G_{t-s} * u^p(s) \, ds,$$  \hfill (A.7)

we first obtain, for all $t \in (0, \tau)$,

$$\|\nabla u(t)\|_\infty \leq Ct^{-1/2}\|u_0\|_\infty + CM^p \int_0^t (t-s)^{-1/2} \, ds \leq C_1 t^{-1/2}.$$  \hfill (A.8)
Next rewriting (A.7) as
\[
\nabla u(t) = G_t * \nabla u_0 + \int_0^t G_{t-s} * \nabla u^p(s) \, ds,
\]
we get
\[
|\nabla u(t)| \leq G_t * |\nabla u_0| + N \int_0^t G_{t-s} * |\nabla u(s)| \, ds,
\]
where \(N = pM^{p-1}\). Next set \(z(t) := e^{Nt} G_t * |\nabla u(t)|\). Noting that \(\|\nabla u(t)\|_\infty \in L^1(0, \tau)\) owing to (A.8), we have, by direct computation,
\[
z(t) = G_t * |\nabla u_0| + N \int_0^t G_{t-s} * z(s) \, ds.
\]
Therefore,
\[
\left( |\nabla u(t)| - z(t) \right)_+ \leq N \int_0^t G_{t-s} * \left( |\nabla u(s)| - z(s) \right)_+ \, ds, \quad 0 < t < \tau,
\]
hence
\[
\| \left( |\nabla u(t)| - z(t) \right)_+ \|_\infty \leq N \int_0^t \| \left( |\nabla u(s)| - z(s) \right)_+ \|_\infty \, ds, \quad 0 < t < \tau.
\]
Since \(\| (|\nabla u(t)| - z(t))_+ \|_\infty \in L^1(0, \tau)\), we may apply Gronwall's lemma to conclude that \(\| (|\nabla u(t)| - z(t))_+ \|_\infty = 0\), hence \(|\nabla u(t)| \leq z(t)\) on \((0, \tau)\), which yields the desired conclusion for possibly smaller \(\tau\). \(\square\)

REFERENCES

[1] P. Biler, G. Karch and J. Zienkiewicz, Optimal criteria for blowup of radial and \(n\)-symmetric solutions of chemotaxis systems, Nonlinearity 28 (2015), 4369.
[2] S. Blatt and M. Struwe, An analytic framework for the supercritical Lane-Emden equation and its gradient flow, Int. Math. Res. Notices 2015 (2015), 2342-2385.
[3] X. Chen, M. Fila and J.-S. Guo, Boundedness of global solutions of a supercritical parabolic equation, Nonlinear Anal. 68 (2008), 621-628.
[4] K.-S. Chou, S.-Z. Du, and G.-F. Zheng. On partial regularity of the borderline solution of semilinear parabolic problems, Calc. Var. Partial Differential Equations 30 (2007), 251-275.
[5] M. Fila, Boundedness of global solutions of nonlinear diffusion equations, J. Differential Equations 98 (1992), 226-240.
[6] M. Fila and J.R. King, Grow up and slow decay in the critical Sobolev case, Netw. Heterog. Media 7 (2012), 661-671.
[7] M. Fila and N. Mizoguchi, Multiple continuation beyond blow-up, Diff. Int. Equations 20 (2007), 671-680.
[8] V.A. Galaktionov and J.R. King, Composite structure of global unbounded solutions of nonlinear heat equations with critical Sobolev exponents, *J. Differential Equations* 189 (2003), 199-233.

[9] V.A. Galaktionov and J.L. Vázquez, Continuation of blow-up solutions of nonlinear heat equations in several space dimensions, *Comm. Pure Appl. Math.* 50 (1997), 1-67.

[10] Y. Giga, A bound for global solutions of semilinear heat equations, *Comm. Math. Phys.* 103 (1986), 415-421.

[11] Y. Giga and R.V. Kohn, Asymptotically self-similar blow-up of semilinear heat equations, *Comm. Pure Appl. Math.* 38 (1985), 297-319.

[12] Y. Giga and R.V. Kohn, Characterizing blowup using similarity variables, *Indiana Univ. Math. J.* 36 (1987), 1-40.

[13] Y. Giga and T. Miyakawa, Navier-Stokes flow in \( \mathbb{R}^3 \) with measures as initial vorticity and Morrey spaces, *Comm. Partial Differential Equations* 14 (1989), 577-618.

[14] A. Haraux and F.B. Weissler, Non-uniqueness for a semilinear initial value problem, *Indiana Univ. Math. J.* 31 (1982), 167-189.

[15] G. Karch, Scaling in nonlinear parabolic equations, *J. Math. Anal. Appl.* 234 (1999), 534-558.

[16] T. Kato, Strong Solutions of the Navier-Stokes Equation in Morrey Spaces, *Bol. Soc. Bras. Mat.* 22 (1992), 127-155.

[17] O. Kavian, Remarks on the large time behaviour of a nonlinear diffusion equation, *Ann. Inst. H. Poincaré Analyse Non Linéaire* 4 (1987), 423-452.

[18] T. Kawanago, Asymptotic behavior of solutions of a semilinear heat equation with subcritical nonlinearity, *Ann. Inst. H. Poincaré Anal. non linéaire* 13 (1996), 1-15.

[19] H. Kozono and H. Sohr, New a priori estimates for the Stokes equations in exterior domains, *Indiana Univ. Math. J.* 40 (1991), 1-27.

[20] T. Lee and W.-M. Ni, Global existence, large time behaviour and life span of solutions of a semilinear parabolic Cauchy problem, *Trans. Amer. Math. Soc.* 333 (1992), 365-378.

[21] P.-G. Lemarié-Rieusset, Small data in an optimal Banach space for the parabolic-parabolic and parabolic-elliptic Keller-Segel equations in the whole space, *Adv. Differential Equations* 18 (2013), 1189-1208.

[22] H. Matano and F. Merle, On nonexistence of type II blowup for a supercritical nonlinear heat equation, *Comm. Pure Appl. Math.* 57 (2004), 1494-1541.

[23] H. Matano and F. Merle, Classification of type I and type II behaviors for a supercritical nonlinear heat equation, *J. Funct. Anal.* 256 (2009), 992-1064.

[24] H. Matano and F. Merle, Threshold and generic type I behaviors for a supercritical nonlinear heat equation, *J. Funct. Anal.* 261 (2011) 716-748.

[25] N. Mizoguchi, On the behavior of solutions for a semilinear parabolic equation with supercritical nonlinearity, *Math. Z.* 239 (2002), 215-229.

[26] N. Mizoguchi, Boundedness of global solutions for a supercritical semilinear heat equation and its application, *Indiana Univ. Math. J.* 54 (2005), 1047-1059.

[27] W.-M. Ni, P. Sacks and J. Tavantzis, On the asymptotic behavior of solutions of certain quasilinear parabolic equations, *J. Differential Equations* 54 (1984), 97-120.
[28] C. Ortner and E. Suli, A note on linear elliptic systems on $\mathbb{R}^d$, Preprint ArXiV 1202.3970.
[29] F. Pacard, Partial regularity for weak solutions of a nonlinear elliptic equation, *Manuscripta Math.* 79 (1993), 161-172.
[30] P. Poláčik, P. Quittner and Ph. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems. Part II: parabolic equations, *Indiana Univ. Math. J.* 56 (2007), 879-908.
[31] P. Poláčik and E. Yanagida, On bounded and unbounded global solutions of a supercritical semilinear heat equation, *Math. Ann.* 327 (2003), 745-771.
[32] P. Poláčik and E. Yanagida, Global unbounded solutions of the Fujita equation in the intermediate range, *Math. Ann.* 360 (2014), 255-266.
[33] P. Quittner, A priori bounds for global solutions of a semilinear parabolic problem, *Acta Math. Univ. Comenian. (N.S.)* 68 (1999), 195-203.
[34] P. Quittner, Threshold and strong threshold solutions of a semilinear parabolic equation, Preprint (2016).
[35] P. Quittner, Ph. Souplet, Superlinear parabolic problems. Blow-up, global existence and steady states, Birkhauser Advanced Texts, 2007, 584 p.+xi. ISBN: 978-3-7643-8441-8.
[36] S. Snoussi, S. Tayachi and F.B. Weissler, Asymptotically self-similar global solutions of a general semilinear heat equation, *Math. Ann.* 321 (2001), 131-155.
[37] Ph. Souplet, Sur l’asymptotique des solutions globales pour une équation de la chaleur semi-linéaire dans des domaines non bornés, *C. R. Acad. Sci. Paris Sér. I Math.* 323 (1996), 877-882.
[38] Ph. Souplet, Geometry of unbounded domains, Poincaré inequalities and stability in semilinear parabolic equations, *Comm. Partial Differential Equations* 24 (1999), 951-973.
[39] Ph. Souplet, Decay of heat semigroups in $L^{\infty}$ and applications to nonlinear parabolic problems in unbounded domains, *J. Funct. Anal.* 173 (2000), 343-360.
[40] M.E. Taylor, Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations, *Comm. Partial Differential Equations* 17 (1992), 1407-1456.
[41] H. Triebel, Interpolation theory, function spaces, differential operators, North-Holland Mathematical Library 18, North-Holland, Amsterdam-New York, 1978.
[42] M. Yamazaki and X. Zhou, Semilinear heat equations with distributions in Morrey spaces as initial data, *Hokkaido Math. J.* 30 (2001), 537-571.
[43] F.B. Weissler, Existence and non-existence of global solutions for a semilinear heat equation, *Israel J. Math.* 38 (1981), 29-40.