Many Body Scars as a Group Invariant Sector of Hilbert Space

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We present a class of Hamiltonians $H$ for which a sector of the Hilbert space invariant under a Lie group $G$, which is not a symmetry of $H$, possesses the essential properties of many-body scar states. These include the absence of thermalization and the “revivals” of special initial states in time evolution. Some of the scar states found in earlier work may be viewed as special cases of our construction. A particular class of examples concerns interacting spin-1/2 fermions on a lattice consisting of $N$ sites (it includes deformations of the Fermi-Hubbard model as special cases), and we show that it contains two families of $N+1$ scar states. One of these families, which was found in recent literature, is comprised of the well-known $\eta$-pairing states. We find another family of scar states which is $U(N)$ invariant. Both families and most of the group-invariant scar states produced by our construction in general, give rise to the off-diagonal long range order which survives at high temperatures and is insensitive to the details of the dynamics. Such states could be used for reliable quantum information processing because the information is stored non-locally, and thus cannot be easily erased by local perturbations. In contrast, other scar states we find are product states which could be easily prepared experimentally. The dimension of scar subspace is directly controlled by the choice of group $G$ and can be made exponentially large.

I. INTRODUCTION AND GENERAL SET-UP

The concept of many-body scar states has recently emerged as a novel type of weak ergodicity breaking [1–17]. These states are typically found in the bulk of the spectrum and thus play a role at high temperatures. The scars are special because they have low (area-law) entanglement entropy, do not thermalize, and lead to the exact “revivals” of the initial state of the system initialized with scars. Therefore, the information stored in the system does not dissipate at finite temperature, holding promise for potential applications of such states in quantum information processing.

The current knowledge of the nature of this phenomenon is based on the identification of scars in a variety of systems, such as the AKLT spin chain [1], interacting fermionic models [2, 4, 8, 9], the spin-1 XY model [7], frustrated spin systems [14], and a spin-$\frac{1}{2}$ domain-wall conserving model [15, 16]. In some cases [8, 11, 17], the scar states are related to the well-known $\eta$-pairing states of the Hubbard model, which form a family under the $SU(2)$ symmetry called pseudospin [18–20]. There has been experimental observation of the approximate revivals [21], yet a general understanding of the underlying structures leading to the existence of scars is not yet available.

The Hamiltonians exhibiting scars can be often brought to the form $H = H_0 + H_1$, such that $H_1$ breaks some of the symmetries of $H_0$ and has a special property that it annihilates a subsector of the Hilbert space $S$ consisting of eigenstates of $H_0$. In this paper, we discuss how the symmetry properties of the Hilbert space can be used to construct scars systematically. We analyze a rich class of models where the scar subsector $S$ is invariant under the action of a continuous group $G$, which is bigger than the symmetry of the full Hamiltonian. We will show (see SM A) that the requisite hermitian operator $H_1$ must have the form $H_1 = \sum_j O_jT_j$, where $T_j$ are generators of the symmetry group $G$ and $O_j$ is any operator s.t. the product $O_jT_j$ is Hermitian. For $H_0$, the simplest option is that it has symmetry $G$, i.e. $[H_0, T_j] = 0$, but the most general condition is that $[H_0, C^G_{\alpha}] = W \cdot C^G_{\alpha}$, where $W$ is some operator and $C^G_{\alpha}$ is the quadratic Casimir of the group $G$. Then the states invariant under $G$ are eigenstates of $H_0$.

The dynamics of the scar subssector $S$ is governed by $H_0$ and is decoupled from the rest of the spectrum controlled by $H$. If the ergodic properties of $H_0$ and $H$ are sufficiently different every state in the decoupled sector $S$ will not thermalize with the rest of the system and will thus violate the eigenstate thermalization hypothesis (ETH) [22–24]. Because of the decoupling, the unitary time evolution starting from a state in the invariant sector cannot mix it with the rest of the system. In addition, if the energy gaps between the states from the invariant subssector have a common divisor [25], then the unitary time evolution of a state from the invariant sector will exhibit revivals: the initial state will return to itself after equal time intervals. Therefore the states in $S$ possess all of the defining properties of the many-body scar states. To our knowledge, such general constructions have not been discussed previously, and we present their concrete examples.

The general class of models we study includes the famous Fermi-Hubbard model and its deformations. In this context we show that, in addition to the family of states which transform as spin $N/2$ under the pseudospin symmetry (the $\eta$-pairing states), which were recently shown to be scar states in [8, 11, 17], there is another family of scar states. This second family, whose states may be explicitly written down as (7) or (B2), is invariant under a $U(N)$ symmetry; it forms a multiplet of spin $N/2$ under the $SU(2)$ which is the physical rotational symmetry in the Fermi-Hubbard model.

II. SPECIFIC CONSTRUCTIONS

We will focus on the Hilbert space of $M$ fermionic oscillators

$$\{c_I, c^\dagger_{I'}\} = \delta_{II'}, \quad I, I' = 1, \ldots, M,$$

(1)
that is acted on by $U(M)$ and we can choose $G$ to be any of
its subgroups. The choice of $G$ provides an important handle
on the dimension of the scar subspace: the smaller the group
$G$, the bigger the invariant scar sector $S$. In particular, scar
sectors which are exponentially large in $M$ can be achieved
for sufficiently small groups. We will restrict ourselves to the
groups $G$ whose generators $T_A$ can be expressed as local her-
mitian operators, leading to local $H_1$. If $T_A$ are generators of some algebra $\mathcal{T}$ acting in the Hilbert
space $[T_A, T_B] = f_{AB}^C T_C$ then we construct the Hamiltonian
as follows
\[
H = H_0 + \sum_A O_A T_A, \quad [H_0, \sum_A T_A^2] = W \cdot \sum_A T_A^2
\]
where $O_A$ are arbitrary hermitian operators s.t. $O_A T_A$ is her-
mitian. If the Hilbert space possesses singlets of this algebra
$T_A S_T = 0$, then these states would have energies that do not
depend on $O_A$, while all the other representation of the al-
gebra $\mathcal{T}$ will mix. In all our examples we will use familiar
nearest-neighbour hopping as the generators $T_A$.

To control the scar subsector we will consider the following
integrable fermionic models [26]
\[
H_0 = 2g \left( c_{ab}^\dagger c_{a'b'}^\dagger c_{a'b} c_{a'b'} - c_{ab}^\dagger c_{a'b}^\dagger c_{a'b} c_{a'b'} \right) + 2g(N_2 - N_1)Q + \frac{g}{2} N_1 N_2 (N_2 - N_1),
\]
where summation over repeated indices is implied. We may
interpret the indices of $c_{ab}$ as labelling the sites of a lattice
[27]. Then the model (3) may be viewed as a generalized
Hubbard interaction term which has a continuous symmetry
$O(N_1) \times O(N_2)$, in addition to the usual $U(1)$ symmetry
with conserved charge (particle number) $Q = \frac{1}{2}[c_{ab}^\dagger, c_{ab}]$. It
is a special case, $N_3 = 2$, of the $O(N_1) \times O(N_2) \times O(N_3)$
fermionic tensor model [26, 28]. While in general the tensor
model is not integrable [29], for $N_3 = 2$ it is [26, 30], and all
of the energies are integer in units of $g$. The matrix model (3)
has a ‘t Hooft large $N$ limit where $N_1 = N_2 = N$ is sent to
infinity while keeping $g N$ fixed. Here, for finite $N_1$ and $N_2$,
we will find a scar subspace invariant under $O(N_1) \times O(N_2)$
with dimension exponential in $M = N_1 N_2$.

It is also interesting to study the case where $N_2 = 2$, while
$N_1 = N$ is sent to infinity keeping $g N$ fixed [26, 30]; this
is the well-known vector large $N$ limit and it is natural to inter-
pret only the first index of $c_{ab}$ as labelling lattice sites. For
the vector model and finite $N$, we will exhibit natural scar sub-
sectors which are invariant under $(SU(N) \times SU(N)) \times SU(2)$
and have dimension linear in $M = 2N$.

A. Vector Example

Consider $2N$ fermionic oscillators and two possible choices
for the subgroup of $U(2N)$: $G = SU(N) \times SU(2)$ or $G =
SO(N) \times SU(2)$. We will interpret this Hilbert space as that
of a lattice model with $N$ sites and two fermionic degrees of
freedom per site (they may correspond to the two states of a
spin-1/2 fermion, although the $SU(2)$ symmetry is broken in
our model to $U(1)$). The lattice may be thought of as 1D, as
in fig. 1, but the specific way the $SU(N)$ or $SO(N)$ indices
are mapped to a lattice is not important for the purposes of
finding the scars. In particular, the lattice can be of arbitrary
dimension, frustrated, and can have any boundary conditions.

The Hilbert space we consider is thus identical to that in a
number of models, such as the spin-1/2 Hubbard, Hirsch and
their deformations. The structure of the invariant subspace
$S$ we describe in this section is common to all these spin-1/2
models and does not depend on the details of the Hamiltonian.

Consider the hopping term on this lattice, $T = \sum_{aa',bb'} t_{aa'} c_{ab}^\dagger c_{a'b'}$, where the first index of $c_{ab}$ labels the sites,
the second index the “spin” and $t_{aa'}$ is the hopping strength
hermitian matrix. One can see that, for a general complex
$t_{aa'}$, the hopping $T$ is a generator of $SU(N)$ that acts on the
indices $a$ (see [26] and SM A). For purely imaginary $t_{aa'}$ the
hopping $T$ is a generator of $SO(N)$, and for real $t_{aa'}$ the
situation depends on the parity of $N$ (see SM A).

Combining the strong, $O(N) \times O(2)$-symmetric interaction
described by the vector model Hamiltonian (3) with the
hopping $T$, we obtain a Hubbard-like model $H = H_0 + T$, which
is interesting in its own right. When $T$ is a generator of
$SO(N)$, this $D$-dimensional model is integrable! In
each representation of $O(N)$ the hopping $T$ could be diag-
onalized, and the states would be split with respect to their
Dynkin labels—alogous to the Zeeman splitting for the hy-
drogen atom in a magnetic field. The Hilbert space could be
split into a direct sum of irreducible representations. The
singlet representation $S_O$ is invariant under the action of any
 generator of $SO(N)$ and is therefore annihilated by $T$: $T S_O = 0$.
The singlet states remain at the energies assigned to them by
$H_0$. If $T$ is a generator of $SU(N)$, the system remains inte-
grable (for example, in terms of level statistics), but cannot
be solved easily, because $[H_0, T] \neq 0$. The set of $SU(N)$
singlets, $S_U \subset S_O$, is again annihilated by $T$, will remain
unchanged and at the same integer energies as for $H = H_0$.

The singlets in $S$ have several quantum numbers [30] (none
of them are conserved by the full Hamiltonian (5)), which can
be used to distinguish them, such as the charges $Q$ and
\[
Q_2 = -i \left( c_{a1}^\dagger c_{a2} - c_{a2}^\dagger c_{a1} \right) = c_{aa}^\dagger \sigma^z_{\alpha \beta} c_{a\beta}.
\]
To control the energies of the singlets and the period or re-
vivials, we can add these terms to \( H_0 \), i.e. \( \tilde{H}_0 = H_0 + \alpha Q + \beta Q_2 \).

The full Hamiltonian we study reads \([31]\)

\[
H = \tilde{H}_0 + T + 4 \sum_{a=1}^{N} O_a T_a, \quad \text{where} \quad (5)
\]

\[
O_a = \sum_{a, b = (a+2), b'}^2 \left[ g_{a1,2,b,b'}^1 c_{a1,b}^\dagger c_{a2,b'}^\dagger + q_{a1,2,b,b'}^2 c_{a1,b}^\dagger c_{a2,b'}^\dagger + \text{h.c.} \right].
\]

Schematically, the term \( O_a T_a \) is depicted in fig. 1. \( T_a \) induces a hopping from site \( a \) to \( a+1 \) and the operator \( O_a \) acts on sites \( a + 2 \) and \( a + 3 \). Such a structure ensures that each term \( O_a T_a \) is hermitian and local. The coefficients \( \{ d_{a1,2,b,b'}^{1,2} \} \) are random complex numbers and this choice of the operator \( O \) is intended to break the symmetries of \( H_0 \) and to make the bulk of the spectrum ergodic, described by the gaussian unitary ensemble (GUE).

Indeed, most states in the Hilbert space will be mixed because \( T_a \) does not annihilate them, and \( O_a \) mixes all the non-singlet representations, while the effective Hamiltonian for states in \( \mathcal{S} \) remains \( H_0 = \tilde{H}_0 \). Due to this structure, the only remaining symmetry relates the sectors with odd and even \( Q \), both described by GUE (see fig. 2 for the exact numerical results). The time-reversal symmetry is broken by the operator \( c_{ab}^{\dagger} c_{ab} \) in \( O_a \) terms.

For the numerical study, we consider a 1D lattice with \( N = 8 \) and the translation-invariant nearest-neighbour hopping terms \( T \) which are generators of \( G = SU(N) \), with \( t_{a,a+1} = t = 8e^{i\sqrt{2n}} \), periodic boundary conditions and \( \alpha = \beta = 1 \).

The probability distribution \( P(r_k) \) of the level spacings (inset of fig. 2) agrees well with the GUE overlay. It contains information about the correlation functions of close eigenvalues, whereas the spectral form factor, \( g(t, \beta) = \frac{\text{Tr}(e^{-\beta H + iH})}{\text{Tr}(e^{-\beta H})^2} \), also contains information about longer range correlations. The main elements of the spectral form factor (SFF) for a random matrix is a dip ramp plateau structure (for a discussion of their physics, see [32]). The presence of this structure in our system is another evidence of quantum chaos and ergodicity in its bulk spectrum.

A more detailed characterization of ergodicity is provided in the left panel of fig. 3 where we test the eigenstate thermalization hypothesis (ETH), which conjectures that for any measurable local operator \( \mathcal{M} \), its expectation value in an eigenstate must be approximately the same as the window-average over the nearby states at the same energy. We observe that the conjecture holds for most states in the spectrum while it is clearly violated for the eleven \( SU(8) \) singlet states \( \{ |n_{SU} \rangle \} \) that do not thermalize. The situation when the bulk of the spectrum (dimension \( 2^N - N - 3 \)) is ergodic while an exponentially small subset of states is not (there are \( N + 3 \) \( SU(N) \) singlets in our Hilbert space), corresponds by definition to the violation of the strong formulation of ETH (the weak formulation allows for a few “outlier” states) [22–24].

The singlet states violating strong ETH also clearly stand out in the time evolution. Consider two initial states \( \psi_0^n \), made exclusively of singlet eigenstates of \( H \) and \( \psi_0^e \), composed of the same number of generic states. In both cases we can write \( |\psi_0^s/g\rangle = \sum c_n |n\rangle \), where \( |n\rangle \) is an eigenstate of \( H \) with energy \( E_n \). We are interested in the squared projection of the time-evolved state on the initial wavefunction \( f(\tau) = | \langle \psi_0 | e^{-iH\tau} | \psi_0 \rangle |^2 = \sum_{n,m} |c_n c_m|^2 e^{-(E_n - E_m)\tau} \). It should relatively quickly go to zero if the states are generic.
without particular correlations between energies \( E_n \). Exact numerical results confirming this are shown in the right panel of fig. 4. A vanishing overlap with the initial state indicates that the information stored initially has fully dissipated through thermalization. This phenomenon is closely related to the dip seen in the SFF in fig. 2.

For the singlet states, all of the energies \( E_n \) are integer, which means there exists a (greatest) common divisor for all of the energy gaps between singlet states \( E_n - E_m \). After the time \( T = k \frac{2\pi}{\omega} \), \( k \in \mathbb{Z} \) all of the exponents in \( f(\tau) \) are equal to 1. This constructive interference results in “revivals” of the (information stored in the) initial state with period \( T \). The general condition for this effect is that a common divisor exists for the gaps separating the states that dominate the initial state \( \psi_i \). In our numerical example \( \omega = 2 \) and thus we observe the revivals of period \( \pi \) as shown in the left panel of fig. 4. Note that in this calculation 5 percent of generic states were admixed to the initial state but ideal revivals to \( f(\tau) = 1 \) would be observed otherwise. The higher-frequency “revivals” with smaller amplitude are due to the energy differences that are shared only by a subset of the singlet states. The energies of singlets are controlled by the energy differences that are shared only by a subset of the energy gaps between singlet states \( E_n \), which means there exists a (greatest) common divisor for all of the energy gaps between singlet states \( E_n \). The higher-frequency “revivals” with smaller amplitude are due to the energy differences that are shared only by a subset of the singlet states. The energies of singlets are controlled by the energy differences that are shared only by a subset of the energy gaps between singlet states \( E_n \), which means there exists a (greatest) common divisor for all of the energy gaps between singlet states \( E_n \). The higher-frequency “revivals” with smaller amplitude are due to the energy differences that are shared only by a subset of the energy gaps between singlet states \( E_n \), which means there exists a (greatest) common divisor for all of the energy gaps between singlet states \( E_n \).

Having established that the singlet states exhibit all the properties of the many-body scar states, we now examine their structure. The entanglement entropy of these states is noticeably lower compared to the generic states at the same energy as shown in the right panel of fig. 3. Four states are in fact product states. Two of them are vacuum and anti-vacuum with all the orbitals empty/ﬁlled. Two more eigenstates are tensor products of Bell-like states formed on each site:

\[
|S_1\rangle = \bigotimes_a \frac{|0_a1_a2\rangle + i|1_a1_a0\rangle}{\sqrt{2}} = \prod_a c_{a1}^i + ic_{a2}^i |0\rangle \quad (6)
\]

\[
|S_2\rangle = \bigotimes_a \frac{|0_a1_a2\rangle - i|1_a1_a0\rangle}{\sqrt{2}} = \prod_a c_{a1}^i - ic_{a2}^i |0\rangle ,
\]

They are invariant under the \( U(N) \) subgroup of \( U(2N) \). All four states may be easily created in experiment and we provide the corresponding gate sequences in the SM B.

The complete set of \( N + 1 \) states invariant under \( U(N) \) [30] can be constructed by acting repeatedly on the state \( |S_1\rangle \) with the bilinear operator \( \zeta = c_{a1}^i (\sigma^3 - i\sigma^1)_{ba} c_{ab}^i \) (this is a “rotated” version of the zeta-operator in [20]):

\[
|n_{U}\rangle = \frac{\zeta^n}{2^n \sqrt{\frac{N!}{(N-n)!}}} |S_1\rangle , \quad (7)
\]

with \( n = 0, \ldots, N \). Another basis for this family of states is given in (B2). One can see that these states have the maximal possible spin \( N/2 \) with respect to the second index \( b \) which takes values 1 and 2, i.e. they transform as a \( (N + 1) \)-dimensional representation of \( SU(2)_b \), which is the physical spin in the Fermi-Hubbard model. We note that there is only one family which has the maximal spin. Consequently, it is quite robust under the action of any perturbation that preserves this spin. Namely, any spin-preserving term will map this representation to itself which means these states will continue to violate strong ETH while the revivals may disappear as a result of changing their energies.

For a purely imaginary hopping strength, the hopping terms are generators of \( G = SO(N) \), and all the scar states are \( O(N) \) invariant. The Hilbert space may be decomposed according to representations of \( O(N) \times SO(4) \), which is seen easily [30] through writing the \( 4N \) Majorana fermions as \( \psi_{iA} \), where \( i = 1, \ldots, N \) and \( A = 1, \ldots, 4 \) (the relevance of group \( SO(4) \) was noted long ago in the context of Hubbard model [19, 20]). As shown in [30], the \( O(N) \) singlets transform in the \( (N/2, 0) + (0, N/2) \) representation of \( SO(4) \sim SU(2) \times SU(2) \), where we labeled the \( SU(2) \) representations by their spin \( J \). Thus, there are two sets of \( N + 1 \) scar states; each one is invariant under one of the \( SU(2) \) groups and transforms as spin \( N/2 \) under the other. One of these sets is \( |n_{U}\rangle \), for which the \( O(N) \times SU(2) \) symmetry is further enhanced to \( U(N) \). The other set of \( N + 1 \) states is

\[
|n_O\rangle = \frac{\eta^n}{2^n \sqrt{\frac{N!}{(N-n)!}}} |0\rangle , \quad \eta = \sum_{c=1}^N c_1^c \cdot c_2^c , \quad (8)
\]

with \( n = 0, \ldots, N \). They are equivalent to the exact eigenstates of the Hubbard model originally identified using the celebrated \( \eta \)-pairing [18] and recently demonstrated to be many-body scar states [8, 17] (to obtain (8) we need to transform from the real hopping amplitude used in [18] to our imaginary one A). Let us emphasize that the Hamiltonian \( H \) does not respect all the symmetries possessed by the two scar sectors. Thus, the scars appear in the enhanced symmetry sectors of Hilbert space, in accordance with our general arguments.

It can also be shown (work in progress) that the Fermi-Hubbard Hamiltonian in arbitrary dimension can be written in the form (5); thus, the appearance of many-body scar states in this model is a special case of our construction. As a consequence, the states \( |n_{U}\rangle \) are eigenstates and scar states in the (extended) Hubbard model (and other spin-1/2 models). In table I we summarize the properties of the scar subspace in the \( D \)-dimensional spin-1/2 models of the form (5).

The off-diagonal long range order (ODLRO) has been linked in literature to the high-Tc superconductivity [18, 33] and was shown to be present in the \( |n_{U}\rangle \) scars [18]. Another virtue of ODLRO is that it can be viewed as a spatial distribution of information stored in the scar state, which then protects this information from local perturbations. In the \( |n_{U}\rangle \) scars, the ODLRO is most naturally characterized by the correlator 

\[
G_U = \langle s|c_{i1}^c c_{i2}^c c_{j1}^c |s\rangle ,
\]

which does not depend on the coordinates \( i \) and \( j \) of the sites for any \( |s\rangle \in \mathbb{S}_U \) (see C in SM). In our model (5) we can choose arbitrary operator \( O \), and in our numerical example it includes random coupling leading to quantum chaos. Nevertheless, the ODLRO survives in the scar states corresponding to high temperature for any choice of \( O \) (see SM fig. 6).

Finally, let us note that our construction of \( H_1 \) is similar to that in [11], in that hopping \( T \) is used to annihilate \( \mathbb{S} \). Similarly to [17], [11] has discussed one of the two \( SU(2) \) families
of scar sectors in the context of Hubbard model, although the $O(N)$ invariance of these states was not pointed out explicitly.

### B. Matrix example

Here we consider the Hilbert space spanned by $N_1N_2$ fermion oscillators (1), which is invariant under the action of the $U(N_1N_2)$ group and therefore under its subgroups $U(N_1) \times U(N_2)$ and $O(N_1) \times O(N_2)$. It may be interpreted as a lattice with $N_1N_2$ sites and one fermionic degree of freedom per site. The mapping of the group indices to the spatial lattice sites is again a matter of preference; the simplest choice is that of a rectangular 2D lattice with spinless or spin-polarized fermions. Similarly to the vector case, the scars will be invariant under the group $G = SU(N_1) \times SU(N_2)$ or $G = SO(N_1) \times SO(N_2)$, depending on the choice of the complex or imaginary hopping strength. We will adopt the latter choice where the scar sector is much richer (for the $G = SU(N_1) \times SU(N_2)$ case the invariant subspace consists of only two states: the vacuum and antivacuum).

For imaginary hopping the generators of $G = SO(N_1) \times SO(N_2)$ are linear combinations of the basis generators

$$Q_{ab}^{a'b'} = i(c_{ab}^{\dagger}c_{a'b'} - c_{a'b'}^{\dagger}c_{ab}) \, , \quad Q_{2}^{bb'} = i(c_{ab}^{\dagger}c_{ab'} - c_{ab}^{\dagger}c_{ab'}) \, .$$

Consider hoppings (describing free electrons) in both directions with periodic boundary conditions

$$T_a = tQ_1^{a,a+1}, \quad T_b = tQ_2^{b,b+1}, \quad T = \sum_a T_a + \sum_b T_b \, .$$

The full Hamiltonian reads

$$H = H_0 + T + 4 \sum_{a,b,b'} O_{a,b,b'} T_a + 4 \sum_{a,a',b} O_{a,a',b} T_b,$$

where $H_0$ is given in (3) (like in 1D we add $Q$ to split degenerate singlets), and we choose

$$O_{a,b,b'} = q_{a,b,b'} c_{a+3,b'}^{\dagger}c_{(a+3)b'}, \quad O_{a,a',b} = r_{a,a',b} c_{a+3,b}^{\dagger}c_{(a+3)b},$$

with $q, r$, and complex numbers.

Numerical results for $N_1 = N_2 = 4$ and $t = 8 \sin \sqrt{2}\pi$ can be found in SM D and similarly to the vector case demonstrate that the $G = SO(N_1) \times SO(N_2)$-invariant states become scars. The more complex structure of singlets is reflected in the fact that, unlike in the vector case, only the trivial states (the vacuum $|0\rangle$ and antivacuum $|1\rangle$) have zero entropy and are product states. The following operators may be used (see SM B and [26]) to construct the complete sets of scars in the sense that their linear combinations and products acting on $|0\rangle$ span the full singlet subspace $S$

$$(J_+\epsilon_{a'}) = e_a^{\dagger}e_a^{\dagger}, \quad (K_+\epsilon_{a_1\ldots a_{N_1}}) = \epsilon_{b_1\ldots b_{N_2}} \prod_{i=1}^{N_1/2} e_i^{\dagger}e_i^{\dagger}$$

where indices of $J_+$, $K_+$ should be contracted with the use of $\delta_{a'a'}$ or the $(J_+\epsilon_{a'})$. For $N_{1,2} = 2$, $K_+$ corresponds to the $\eta$-pairing of the Hubbard model. Because they are singlets, these states again possess ODLRO as we prove in SM C. The structure of the states is much more complex than in the vector case. The long-range order in the language of ref. [20] is described, in part, as a mix of two types: $G_O$, superconducting in one direction while magnetic in the other and $G_U$, magnetic in both directions. In the finite-size system we find numerically (see SM fig. 7) that the following correlator is non-vanishing for all non-trivial scars $|s\rangle$: $G_O(a_1, b_1, 2) = \langle s | c_{a_1}^{\dagger}c_{a_2}^{\dagger}c_{a_2}c_{a_1} | s \rangle$, and is so even at large separations in the $a$ direction, corresponding to the superconducting phase, while in the $b$ direction we would get particle-hole pairs. Further examples of ODLRO that we find numerically lack a simple interpretation in terms of $G_O$ and $G_U$; they are $\langle s | c_{11}^{\dagger}c_{12}^{\dagger}c_{12}c_{11} | s \rangle$ and $\langle s | c_{11}^{\dagger}c_{12}^{\dagger}c_{12}c_{11} | s \rangle$.

In contrast to the vector case, the dimension of the scar subspace grows exponentially with $N_1N_2$ (see ref. [26] and SM D); nevertheless, it spans only an exponentially small fraction of the Hilbert space. It is conceivable that for large $N_1N_2$ the scar states may form continuous energy bands filled with the states possessing ODLRO.

## III. DISCUSSION

In this paper we have identified the many-body scar states with a sector of Hilbert space possessing a greater symmetry than the Hamiltonian. We would like to stress that the presence of the group invariant states $S$ is a property of a Hilbert space and not of a particular Hamiltonian. Once the conditions on the Hamiltonian $H_0 + H_1$ outlined in this paper are satisfied, the states in $S$ have the properties of many-body scar states. This universality explains why the scar states identified to date in different models with the same Hilbert space can be identical.

From the quantum information perspective, the many-body scars made of group-invariant states are appealing because of their non-locality. Indeed, the group-invariance requirement is non-local. As a consequence, the degrees of freedom on all of the sites become entangled which spreads the information over the whole system. This leads to the relative insensitivity of group-invariant states to local perturbations and protection of the quantum information [34]. The invariant scar states form a closed subspace, where non-commuting transformations can act. Furthermore, the scar subspace is decoupled from the rest of the system and does not thermalize.
combination of properties could make the group-invariant scar states an interesting platform for robust quantum information processing.

The gauge/gravity duality [35–37] is a set of correspondences between conventional gauged models without gravity and higher-dimensional gravitational systems. More recently, the gauging of continuous symmetries was advocated in the context of quantum mechanical models of fermionic tensors [28, 38]. In these quantum mechanical models, gauge fields are non-dynamical, so the gauging is equivalent to truncation of the Hilbert space to a group-invariant sector (the counting of such states was performed in [26]). As we have seen, the group invariant states can also play the role of scars. It would be interesting to further explore possible connections between the scars and gauge/gravity duality.

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Supplementary A: Hopping amplitudes and generators

Let us study the Hermitian bilinear operators which preserve charge $Q$,

$$ T_A = \sum_{a,a',b} A_{aa'} c_{ab}^{\dagger} c_{a'b}, \quad A^\dagger = A. \quad (A1) $$

Their commutation relations are

$$ [T_A, T_B] = iT_{[A,B]} . \quad (A2) $$

If we further require $A$ to be traceless, this is isomorphic to the $SU(N)$ algebra.

Each subalgebra of $SU(N)$ corresponds to a subalgebra of (A1) and vice versa. For example, we can take an algebra of antisymmetric matrices $O(N) \subset SU(N)$—they would correspond to the hoppings with purely imaginary amplitudes. It is generated as a Lie algebra by the nearest-neighbor hoppings

$$ T_a = ic_{ab}^{\dagger} c_{(a+1)b} + \text{h.c.}, \quad (A3) $$

which form a maximal subalgebra of $SU(N)$.

Another interesting example of subalgebra arises for the real nearest-neighbor hoppings

$$ \tilde{T}_a = c_{ab}^{\dagger} c_{(a+1)b} + \text{h.c.}. \quad (A4) $$

Let $g \subset su(N)$ be a minimal Lie algebra containing $\tilde{T}_a$. One can see that when $N$ is even we can make a transformation $e_{2a,b} \rightarrow ie_{2a,b}$ that would send $\tilde{T}_a$ to $T_a$. Hence we get $g \approx o(N)$, but it is a different embedding of the original $O(N)$ mentioned before. In this case the family of the invariant states for this group looks like the usual $\eta$-pairing states [33]

$$ |n_{\eta}'\rangle = \frac{\left( \sum_{a} e^{i \pi a_{a}^{\dagger} c_{a1}^{\dagger} c_{a2}} \right)^n}{2^n \sqrt{N^{2n!} / (N-n)!}} \cdot |0\rangle. \quad (A5) $$

For odd $N$ one can show that $\tilde{T}_a$ will comprise the full algebra $su(N)$.

One can consider index $b$ as a lattice index. Therefore we could associate another group $U_b(N)$ with generators

$$ T_b = \sum_{a,b,b'} B_{b,b'} c_{ab}^{\dagger} c_{ab'}. \quad (A6) $$

Then combining these two generators we get that the $U(N_1) \times U(N_2)$ group acts in the Hilbert space, and it could be used as group $G$ in the general construction studied in the main text. However, there are only two $U(N_1) \times U(N_2)$ invariant states [39]. To get richer structure we consider only the hoppings with purely imaginary amplitudes leading to $O(N_1) \times O(N_2)$.

We can also generalize $T$ to be not only a nearest-neighbor hopping but to be any hopping in the lattice. It allows us to deform the vector model to be not only a 1D lattice, but an arbitrary dimensional lattice. For example, we can consider a 3D Hubbard model. The fermion operators have 4 index: $c_{x,y,z}^{ab,c}$, where $x, y, z = 1, \ldots, L$ are lattice coordinates. We can rearrange them into a linear index as $i = x + (y-1) * L + (z-1) * L^2$. Then the hoppings from $i$ to $i + 1$ correspond to $x$-directed hoppings, $i$ to $i + N$ is a $y$-directed hopping, and from $i$ to $i + N^2$ is a $z$-directed hopping. Any of these hoppings are the generators of $U(N = L^3)$ and therefore the reasoning presented above would still work for this case.

We end this subsection with a derivation of the structure of the term $H_1$ in our Hamiltonian. Let us take a Hilbert space with the action of some Lie algebra $g$ realized by some hermitian operators. Let $T_i$ be a simple basis of the generators of this algebra $g$, meaning that whole algebra is spanned by $T_i$ and all of their possible commutators. Let $S$ be a space of invariant states $gS = 0$. Then if $K$ is an annihilator of $S$, $i.e. K\dot{S} = 0$; then $K = \sum_j O_j T_j$. The proof is quite simple we can consider an operator $C = \sum_i T_i^2$—it is an annihilator of $S$ since $Cu = 0 \Leftrightarrow T_i v = 0 \Leftrightarrow v \in S$. We can define $C^{-1}$ such that $C^{-1} \cdot C = 1 - P_b$, where $P_b$ is a projector on $S$ and therefore $C^{-1} \cdot C$ is also an annihilator of $S$. Then

$$ K = K(1 - P_b) = K \cdot C^{-1} \cdot C = \sum_i (K \cdot C^{-1} \cdot T_i)T_i = \sum_i O_i T_i, \quad (A7) $$

which was required to prove.

Supplementary B: Construction of group invariant states

As emphasized in the main text, there are 4 states in the vector model with zero entropy. While the states $|0\rangle$ and $|1\rangle$ are the oscillator vacuum and its particle-hole conjugate, the other two states $|S_{1,2}\rangle (6)$ are nontrivial. They may be expressed as a product state with the use of the gates $W_a$ and $\tilde{W}_a$ (see fig. 5):

$$ |S_1\rangle = \prod_{a=1}^{N} W_a |0\rangle, \quad |S_2\rangle = \prod_{a=1}^{N} \tilde{W}_a |0\rangle \quad (B1) $$

Let us also present another, rotated basis for the family of states (7), which are $U(N)$ invariant and transform as spin $N/2$ under the rotational SU(2) symmetry:

$$ |	ilde{n}_{UC}\rangle = \frac{\left( \sum_{a} c_{a1}^{\dagger} c_{a2} \right)^n}{2^n \sqrt{N^{2n!} / (N-n)!}} |\tilde{S}_2\rangle, \quad (B2) $$

![FIG. 5. Left panel: Circuit diagram of $W_a$ for construction of $|S_1\rangle$ states (6). $H$ is the Hadamard gate, $X$ is the Pauli-X gate, the line spanning the two sites represents the CNOT gate from qubit 0 to target qubit 1, and $P$ is the phase gate. Right panel: Circuit diagram of $\tilde{W}_a$ gates needed to construct the singlet state $|S_2\rangle$.](diagram.png)
For the Hamiltonian $H_0$ in (3), the states $\ket{\tilde{n}_U}$ are not eigenstates, while the states $\ket{n_U}$ given in (7) are. However, for the Fermi-Hubbard model, which respects the $SU(2)$ rotational symmetry, both $\ket{\tilde{n}_U}$ and $\ket{n_U}$ are eigenstates.

As shown in the main text, some singlet states in the vector model are related to the $\eta$-pairing states discovered by Yang [18]. We can extend this construction to the matrix model that was discussed in [26]. We start with the vacuum state $\ket{0}$ that is naturally a singlet state, because it is annihilated by any hopping. Then in order to build any other singlet state we can use the products and sums of the operators $A_{a_1} \cdots A_{a_N}$. These operators automatically are singlets under the action of $SO(N)$.

The states with a small number of fermions could be built with the use of only the operator matrix $J_\pm$. We introduce $M_n = \sum_a (J^n)_{aa}$, which is a singlet under the action of $SO(N) \times SO(N)$. For example, acting with $M_n$ and their products we can build singlet states as

$$\ket{s_1} = M_2 \ket{0}, \ket{s_2} = M_3 \ket{0}, \ket{s_3} = M_3M_4M_2^2 \ket{0}, \ldots$$

When the number of fermions is larger than $N$, we can use $K_\pm$ to build singlet states. For example, when $N$ is even we can have

$$\ket{s_N} = (K_+)_{a_1a_2 \cdots a_{N/2}a_{N/2}} \ket{0}.$$ 

For $N = 2$ the operator $(K_+)_{aa}$ is the $\eta$-operator from [18].

In general we are able to express the dimension of the singlet subspace as an integral [26]

$$\dim S_O = \frac{4N_1N_2}{V_{N_1}V_{N_2}} \int_{-\pi}^{\pi} dx_1 dy_1 \left( \cos x_1 + \cos y_1 \right)^2 \times \prod_{i \neq j} \left( \cos x_i - \cos y_i \right)^2 \frac{N_1}{\pi} \int_{-\pi}^{\pi} dx_i \times \prod_{i \neq j} \left( \cos x_i - \cos y_i \right)^2,$$

(4)

where $V_{N_1}N_2$ are the dimensions of the $SO(N)$ groups, which are equal to

$$\prod_{i} N_i = \prod_{i} \frac{N_i}{\pi} \int_{-\pi}^{\pi} dx_i \times \prod_{i \neq j} \left( \cos x_i - \cos y_i \right)^2.$$  

(5)

Supplementary C: Off-Diagonal Long Range Order (ODLRO)

Let us show that the singlet states $\ket{s} \in \mathbb{S}$ exhibit the Off-Diagonal Long Range Order (ODLRO) [33]. This means that the correlator

$$G_O = \langle s|c_{11}c_{12}c_{j2}c_{j1}|s\rangle,$$

(6)

does not depend on $i$ and $j$ when they take different values. Indeed, there is an operator $O_{ik} \in O(N)$ which swaps the fermions with indices $i$ and $k$ leaving the others unchanged. For example,

$$O_{ik}c_{i1}O_{ik}^{-1} = c_{k1}, \quad O_{ik}c_{i1}O_{ik}^{-1} = c_{i1}, \quad O_{ik}c_{j1}O_{ik}^{-1} = c_{j1}.$$  

Since $\ket{s}$ is an $O(N)$ invariant state, we have $O_{ik}\ket{s} = \ket{s}$. Using these relations, we see that the correlator $G_O$ does not depend on the positions of $i \neq j$. Indeed,

$$G_O = \langle s|c_{11}c_{12}c_{j2}c_{j1}|s\rangle = \langle s|O_{ik}c_{i1}c_{12}c_{j2}c_{j1}O_{ik}|s\rangle = \langle s|c_{11}c_{12}c_{j2}c_{j1}|s\rangle.$$

(7)

Hence, it is non-vanishing even when the difference between $i$ and $j$ is large. An analogous argument can be applied to the correlator

$$G_U = \langle s|c_{11}c_{21}c_{j2}c_{j1}|s\rangle,$$

(8)

when $\ket{s}$ is a $U(N)$ invariant state, demonstrating ODLRO. We note that $G_O$ is the correlator originally used by Yang [18]. It is related to superconductivity, while $G_U$ is related to the magnetic properties of the system.

For matrix models we also have correlators $G_{O,U}$, where we extend the second index to label spatial coordinates along the other direction. One can notice that $G_O$ now plays a double-role: in one direction it is related to the superconducting properties (since if we separate $G_O$ along the direction $a$, it splits into a product of local cooper pair creation operators) and in the other direction it is related to the magnetic properties (since in the direction $b$ the $G_O$ splits in the product of
Entanglement entropy

FIG. 7. Left panel: Correlator $G_0 = \langle c_{11}^\dagger c_{12}^\dagger c_{22} c_{21}\rangle$ evaluated for every eigenstate of the matrix model with $N = 4$; 7 out of 12 scars have non-vanishing "superconducting" ODLRO. Right panel: Correlator $G_U = \langle c_{11}^\dagger c_{12}^\dagger c_{22}^\dagger c_{21}\rangle$ evaluated for every eigenstate of the matrix model; 8 out of 12 scars have non-vanishing "magnetic" ODLRO. One can show that $G_0$ and $G_U$ are rational numbers which depend linearly on the $SU(N)$ Casimirs of the scar states.

FIG. 8. Matrix model (II B) with $N_1 = N_2 = 4$. Top left panel: Entanglement entropy; Top right panel: ETH plot for the diagonal hopping $M = \langle c_{11}^\dagger c_{22} - c_{22}^\dagger c_{11}\rangle^2$; Bottom: Nearest neighbor eigenvalue spacings (inset) and the spectral form factor for the 442 model

particle-hole creation operators). The correlator $G_U$ is related to the magnetic properties of the states, because in each of the directions it splits into the product of creation and annihilation operators. And again, using the properties of the singlet states, we can interchange indices while the singlet states are left unchanged and we get that the singlet states must have the ODLRO (see fig.7).

Supplementary D: Details of the numerical calculations

1. Time evolution, vector model

Fig. 9 details the composition of the initial state for time evolution calculation for the vector model. Exact revivals to $f^{max}(\tau) = 1$ would occur for an initial state comprised solely of scars. Instead, we admix 5 percent of generic states to analyze the stability of the effect in an (experimental) scenario when the desired initial state can only be created with a finite precision.

Fig. 10 shows the time evolution at late times where in presence of scars, the revivals continue with stabilized amplitude and all information is lost without scars.

2. Matrix model

For matrix models, we can make analogous calculations to the vector model. For example, one can check that the singlet states have a low entanglement entropy in comparison to the other states (see top of fig.8). Also we checked that ETH is violated, for example, the plot of the operator $M = \left(c_{11}^\dagger c_{22} - c_{22}^\dagger c_{11}\right)^2$ as a function of energy $M(E)$ is not smooth for the singlet states. Also we can check that the spacings between the states satisfy the GUE distribution (see bottom of fig. 8).

And again the time evolution of the state consisting mostly of the singlet states would exhibit the revivals.