Dynamical solution of supergravity

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Abstract
We present a class of dynamical solutions for an intersecting D4-D8 brane system in ten-dimensional type IIA supergravity. The dynamical solutions reduce to a static warped AdS6 × S4 geometry in a certain spacetime region. We also consider lower-dimensional effective theories for the warped compactification of general p-brane systems. It is found that an effective p + 1-dimensional description is not possible in general due to the entanglement of the transverse coordinates and the p + 1-dimensional coordinates in the metric components. Then we discuss cosmological solutions. We find a solution that behaves like a Kasner-type cosmological solution at τ → ∞, while it reduces to a warped static solution at τ → 0, where τ is the cosmic time.

1 Introduction
Recently studies on dynamical solutions of supergravity have been a topic of great interest. Conventionally time dependent solutions of higher dimensional supergravity are discussed in the context of lower dimensional effective theories after compactifying the internal space. However, it is unclear how far this effective low-dimensional description is valid. Thus it is much more desirable to discuss the four-dimensional cosmology in terms of the dynamics of the original higher-dimensional theory. This is particularly true in string cosmology in which the behavior of the early universe is to be understood in the light of string theory. Indeed, it was pointed out that the four-dimensional effective theory for warped compactification of ten-dimensional type IIB supergravity allows solutions that cannot be obtained from solutions in the original higher-dimensional theories [1].

In the present work, we consider dynamical solutions for intersecting D4-D8 brane systems in the ten-dimensional type IIA supergravity model [2]. In §2 we first consider p-brane systems in D-dimensions and derive a class of dynamical solutions under a certain metric ansatz. In §3 focusing on intersecting D4-D8 brane systems in the ten-dimensional type IIA supergravity, we extend the metric ansatz used in the previous section to intersecting branes and obtain a class of dynamical solutions. Then further specializing the form of the metric, we consider a cosmological solution. Interestingly, this solution is found to approach a warped static solution as τ → 0 and a Kasner type anisotropic solution as τ → ∞, where τ is the cosmic time. Finally we conclude in §4.

2 Dynamical p-brane solutions
We consider a gravitational theory with the metric gMN, dilaton φ, and an anti-symmetric tensor field of rank (p + 2) in D dimensions. This corresponds to a p-brane system in string theory. The most general action for the p-brane system in the Einstein frame can be written as

\[ S = \frac{1}{2\kappa^2} \int \left( R * 1_D - \frac{1}{2} d\phi \wedge * d\phi - \frac{1}{2} e^{-c\phi} F_{(p+2)} \wedge * F_{(p+2)} \right), \] (2.1)
where \( \kappa^2 \) is the \( D \)-dimensional gravitational constant, \( * \) is the Hodge dual operator in the \( D \)-dimensional spacetime, and \( c \) is a constant given by \( c^2 = 4 - 2(p + 1)(D - p - 3)(D - 2)^{-1} \). The expectation values of fermionic fields are assumed to be zero.

To solve the field equations, we assume the \( D \)-dimensional metric in the form

\[
d s^2 = h^\alpha(x, y)q_{\mu
u}dx^\mu dx^\nu + h^\beta(x, y)u_{ij}dy^i dy^j,
\]

where \( q_{\mu\nu} \) is a \((p + 1)\)-dimensional metric which depends only on the \((p + 1)\)-dimensional coordinates \( x^\mu \), and \( u_{ij} \) is the \((D - p - 1)\)-dimensional metric which depends only on the \((D - p - 1)\)-dimensional coordinates \( y^i \). The parameters \( a \) and \( b \) are given by \( a = -(D - p - 3)(D - 2)^{-1} \), \( b = (p + 1)(D - 2)^{-1} \).

Furthermore, we assume that the scalar field \( \phi \) and the gauge field strength \( F_{(p+2)} \) are given by

\[
e^\phi = h^{-c/2}, \quad F_{(p+2)} = \sqrt{-q}d(h^{-1}) \wedge dx^0 \wedge dx^1 \wedge \cdots \wedge dx^p.
\]

Here, \( q \) is the determinant of the metric \( q_{\mu\nu} \). Let us first consider the Einstein equations. Using the assumptions (2.2) and (2.3), the Einstein equations are given by

\[
h R_{\mu\nu}(X) - D_\mu D_\nu h - \frac{a}{2}q_{\mu\nu}(\triangle_X h + h^{-1}\triangle_Y h) = 0, \quad R_{ij}(Y) - \frac{b}{2}u_{ij}(\triangle_X h + h^{-1}\triangle_Y h) = 0, \quad \partial_\mu \partial_\nu h = 0,
\]

where \( D_\mu \) is the covariant derivative with respect to the metric \( q_{\mu\nu} \). \( \triangle_X \) and \( \triangle_Y \) are the Laplace operators on the space of \( X \) and the space \( Y \), and \( R_{\mu\nu}(X) \) and \( R_{ij}(Y) \) are the Ricci tensors of the metrics \( q_{\mu\nu} \) and \( u_{ij} \), respectively. From the third equation of (2.3), the warp factor \( h \) must be in the form \( h(x, y) = h_0(x) + h_1(y) \). Let us next consider the gauge field. Under the assumption (2.3), we find \( dF_{(p+2)} = 0 \). Thus, the Bianchi identity is automatically satisfied. Also the equation of motion for the gauge field becomes \( d[e^{-c\phi} F_{(p+2)}] = 0 \). Hence, the gauge field equation is automatically satisfied under the assumption (2.3).

Let us consider the scalar field equation. Substituting the forms of the scalar and the gauge field (2.3), and the warp factor \( h(x, y) = h_0(x) + h_1(y) \) into the equation of motion for the scalar field, we obtain

\[
ch^{-b} (\triangle_X h_0 + h^{-1}\triangle_Y h_1) = 0.
\]

Thus, unless the parameter \( c \) is zero, the warp factor \( h \) should satisfy the equations \( \triangle_X h_0 = 0 \) and \( \triangle_Y h_1 = 0 \). If \( F_{(p+2)} \neq 0 \), the function \( h_1 \) is non-trivial. In this case, the Einstein equations reduce to

\[
R_{\mu\nu}(X) = 0, \quad R_{ij}(Y) = 0, \quad D_\mu D_\nu h_0 = 0.
\]

On the other hand, if \( F_{(p+2)} = 0 \), the function \( h_1 \) becomes trivial. Namely the internal space is no longer warped [1].

Here we mention an important fact about the nature of the dynamical solutions described in the above. In general, we regard the \((p + 1)\)-dimensional spacetime to contain our four-dimensional universe while the remaining space is assumed to be compact and sufficiently small in size. Then one would usually think that an effective \((p + 1)\)-dimensional description of the theory should be possible at low energies. However, solutions of the field equations have the property that they are genuinely \( D \)-dimensional in the sense that one can never neglect the dependence on \( Y \), say of \( h \). This is clear from an inspection of Eqs. (2.3). In particular, the second equation involves the Laplacian of \( h \) with respect to the space \( X \). Hence the equations determining the internal space \( Y \) cannot be determined independently from the geometry of the space \( X \). The origin of this property is due to the existence of a non-trivial gauge field strength which forces the function \( h \) to be a linear combination of a function of \( x^\mu \) and a function of \( y^i \), instead of a product of these two types of functions as conventionally assumed. This fact is in sharp contrast with the case when one is allowed to integrate out the internal space to obtain an effective lower dimensional theory.

Finally we comment on the exceptional case of \( c = 0 \), which happens when \((D, p) = (10, 3), (11, 5), (11, 2) \). The scalar field becomes constant because of the ansatz (2.3), and the scalar field equation is automatically satisfied. Then, the Einstein equations become

\[
R_{\mu\nu}(X) = 0, \quad R_{ij}(Y) = \frac{b}{2}(p + 1)\lambda u_{ij}(Y), \quad D_\mu D_\nu h_0 = \lambda q_{\mu\nu}(X),
\]
where $\lambda$ is a constant. As seen from these equations, the internal space $Y$ is not necessarily Ricci flat, and the function $h_0$ becomes more complicated. For example, when the metric $q_{\mu\nu}$ is Minkowski, $h_0$ is no longer linear in the coordinates $x^\mu$ but quadratic in them [3].

3 Dynamical solutions for D4-D8 brane system

Now we consider dynamical solutions for the D4-D8 brane system which appears in the ten-dimensional type IIA supergravity. The bosonic action of D4-D8 brane system in the Einstein frame is given by

$$S = \frac{1}{2\kappa^2} \int \left( R * 1 - \frac{1}{2} d\phi \wedge * d\phi - \frac{1}{2} \epsilon^{\phi/2} F_4 \wedge * F_4 - \frac{1}{2} \epsilon^{5\phi/2} m^2 * 1 \right),$$

(3.1)

In the following, we look for a solution whose spacetime metric has the form

$$ds^2 = h^{1/12}(z) \left[ h_4^{-3/8} (x, r, z) q_{\mu\nu} dx^\mu dx^\nu + h_4^{5/8} (x, r, z) \left( dr^2 + r^2 u_{ij} dy^i dy^j + dz^2 \right) \right],$$

(3.2)

where $q_{\mu\nu}$ is the five-dimensional metric depending only on the coordinates $x^\mu$ of $X_5$, and $u_{ij}$ is the three-dimensional metric depending only on the coordinates $y^i$ of $Y_3$. As for the scalar field and the 4-form field strength, we adopt the following assumptions

$$e^\phi = h^{-5/6} p_4^{-1/4}, \quad F_4 = e^{-\phi/2} \left[ \sqrt{-q} d(h_4^{-1}) \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \right].$$

(3.3)

Let us first consider the Einstein equations. Under the assumptions [3.2] and [3.3], the Einstein equations give $h_4(x, y, z) = H_0(x) + H_1(r, z)$. Let us next consider the gauge field $F_4$. Under the assumptions [3.2] and [3.3], the Bianchi identity $dF_4 = 0$ gives

$$\partial_\mu h_4 + (3/r) \partial_r h_4 + \partial_z h_4 + (1/3) \partial_z \ln h \partial_r h_4 = 0, \quad \partial_\mu \partial_r h_4 = 0, \quad \partial_\mu \partial_z h_4 = 0.$$ 

(3.4)

The last two equations are consistent with the result $h_4(x, y, z) = H_0(x) + H_1(r, z)$. Then the first equation [3.2] becomes

$$\partial_\mu H_1 + (3/r) \partial_r H_1 + \partial_z h_4 + (1/3) \partial_z \ln h \partial_r H_1 = 0.$$ 

(3.5)

The gauge field equation $d(e^{\phi/2} F_4) = 0$ is automatically satisfied under the assumption [3.3] and the form of $h_4$ given by $h_4(x, y, z) = H_0(x) + H_1(r, z)$.

Next we consider the scalar field equation. Substituting the assumptions for the metric [3.2], the scalar and gauge fields [3.3], and the form of $h_4(x, y, z) = H_0(x) + H_1(r, z)$ into the scalar field equation, we find

$$\triangle_{X_5} H_0 + (5/4) \left[ \left( 4/9 \right) (\partial_z \ln h)^2 + (2/3) h^{-1} \partial_z^2 h - m^2 h^{-2} \right] = 0,$$ 

(3.6)

where $\triangle_{X_5}$ is the Laplace operator on the space $X_5$, and we used the equation [3.5].

Inserting Eqs. [3.5] and [3.6] into the Einstein equations, we find for non-trivial $H_1$,

$$R_{\mu\nu}(X_5) = 0, \quad R_{ij}(Y_3) = 2 u_{ij}, \quad D_\mu D_\nu H_0 = 0, \quad \triangle_{X_5} H_0 = 0, \quad 4 \left( \partial_z h \right)^2 / 9 - m^2 = 0, \quad \partial_z^2 h = 0,$$ 

(3.7)

where $R_{\mu\nu}(X_5)$, $R_{ij}(Y_3)$ are the Ricci tensors of the metric $q_{\mu\nu}$ and $u_{ij}$, respectively, $D_\mu$ is the covariant derivative with respect to the metric $q_{\mu\nu}$. The last two equations of (3.7) is immediately solved to give $h(z) = 3m(z - z_0)^2/2$, where $z_0$ is an integration constant (corresponding to the position of the D8-brane). Below we set $z_0 = 0$ without loss of generality. Then [3.6] gives the solution $H_1(r, z) = c_1 (r^2 + z^2)^{-5/3} + c_2$, where $c_1$ and $c_2$ are constant parameters.

Let us investigate the geometrical properties of the D4-D8 brane system. As a particular solution to the 3-dimensional metric $u_{ij}$ which satisfies the second equation of (3.7), we take the space $Y_3$ to be a three-dimensional sphere $S^3$. Then if we make a change of coordinates, $z = \tilde{r} \sin \alpha$, $r = \tilde{r} \cos \alpha$ ($0 \leq \alpha \leq \pi/2$), the metric reads

$$ds^2 = h^{1/12} \left[ h_4^{-3/8} q_{\mu\nu} dx^\mu dx^\nu + h_4^{5/8} (d\tilde{r}^2 + \tilde{r}^2 d\Omega_2^2) \right],$$ 

(3.8)
where $d\Omega_2^2 = d\alpha^2 + \cos^2 \alpha d\Omega_3^2$, $h_4(x, \tilde{r}) = H_0(x) + c_1\tilde{r}^{-10/3}$, $h(\tilde{r}, \alpha) = (3m/2)\tilde{r}\sin \alpha$. Here $d\Omega_2^2$ and $d\Omega_3^2$ denote the line elements of the three-dimensional sphere $S^3$ and the four-dimensional sphere $S^4$, respectively.

Now we further define a new coordinate $U$ by $\tilde{r}^2 = U^3$. In the case $q_{\mu\nu}$ is the five-dimensional Minkowski metric $\eta_{\mu\nu}$, the ten-dimensional metric in the limit $U \to 0$ reduced to a warped $AdS_5 \times S^4$ space \[2\].

Let us consider the case $q_{\mu\nu} = \eta_{\mu\nu}$ in more detail. In this case, the solution for the warp factors $h_4$ and $h$ can be obtained explicitly as $h_4(t, \tilde{r}) = \beta t + H_1(\tilde{r})$, $h(\tilde{r}, \alpha) = (3m/2)\tilde{r}\sin \alpha$, where $H_1(\tilde{r}) = c_1\tilde{r}^{-10/3}$, and $\beta$ is a constant parameter.

If we introduce a new time coordinate $\tau$ by $\tau/\tau_0 = (\beta t)^{13/16}$, $\beta \tau_0 = 16/13$, the ten-dimensional metric is given by

\[
ds^2 = h^{1/12} \bigg[ 1 + (\tau/\tau_0)^{-16/13} H_1 \bigg]^{-3/8} \left[ (-d\tau^2 + (\tau/\tau_0)^{-6/13} \delta_{ab} dx^a dx^b) \right. \\
+ \left. \left( 1 + (\tau/\tau_0)^{-16/13} H_1 \right)^{10/13} \left( d\tilde{r}^2 + \tilde{r}^2 d\Omega_2^2 \right) \right]^2.
\]

where the metric $\delta_{ab}$ is the spatial part of the five-dimensional Minkowski metric $\eta_{\mu\nu}$. If we set $H_1 = 0$, the scale factor of the four-dimensional space is proportional to $\tau^{-6/13}$, while that for the remaining five-dimensional space is proportional to $\tau^{10/13}$. Thus in the limit when the terms with $H_1$ are negligible, which is realized in the limit $\tau \to \infty$, we have a cosmological solution. Although this cosmological solution is by no means realistic, it is interesting to note that this cosmological solution is asymptotically static in the past $\tau \to 0$.

### 4 Conclusion

In this work, we investigated dynamical solutions of higher-dimensional supergravity models. We found a class of time-dependent solutions for an intersecting D4-D8 brane system. These solutions were obtained by replacing a constant $A$ in the warp factor $h = A + h_1(y)$ of a supersymmetric solution by a function $h_0(x)$ of the coordinates $x^\mu$ \[3\], where the coordinates $y^i$ would describe the internal space and $x^\mu$ would describe our universe if the spatial dimensions of our universe were four instead of three. In the D4-D8 brane solution, the geometry was found to approach a warped static $AdS_5 \times S^4$ in a certain region of the spacetime.

In particular, we found an interesting solution which is warped and static as $\tau \to 0$ but approaches a Kasner-type solution as $\tau \to \infty$, where $\tau$ is the cosmic time. Although the solution itself is by no means realistic, its interesting behavior suggests a possibility that the universe was originally in a static state of warped compactification and began to evolve toward a universe with a Kaluza-Klein compactified internal space.

Conventionally one would expect an effective theory description in lower dimensions to be valid at low energies. However, as clearly the case of the cosmological solution mentioned above, the solutions we found have the property that they are genuinely $D$-dimensional in the sense that one can never neglect the dependence on $y^i$, say of $h$. Thus our result indicates that we have to be careful when we use a four-dimensional effective theory to analyse the moduli stabilisation problem and the cosmological problems in the framework of warped compactification of supergravity or M-theory.

### References

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