Dynamics of global entanglement under decoherence

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We investigate the dynamics of global entanglement, the Meyer-Wallach measure, under decoherence, analytically. We study two important class of multi-partite entangled states, the Greenberger-Horne-Zeilinger and the W state. We obtain exact results for various models of system-environment interactions (decoherence). Our results shows distinctly different scaling behavior for these initially entangled states indicating a relative robustness of the W state, consistent with previous studies.

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I. INTRODUCTION

It is well-known that the notion of quantum entanglement is a key concept in quantum mechanics. It is also responsible for “strange” non-local behavior of quantum systems in marked contrast to classical notions of reality and locality [1,2]. Schrödinger refer to it as the “essence of quantum mechanics” [1]. Besides its fundamental aspects, entanglement constitutes the central part of new modes of information technology, quantum computation and quantum communication [3,4,5] and is therefore a key ingredient of many information processing protocols. Recently, a considerable amount of work has been devoted to characterize, quantify and realize different variety of entangled states [6]. By now, bi-partite entanglement is a relatively well-understood phenomena. However, the situation becomes much more complex when multi-partite systems are considered [7].

On the other hand, entangled states are very fragile when they are exposed to environment. The biggest enemy of entanglement is decoherence which is believed to be the responsible mechanism for emergence of the classical behavior in quantum systems [8]. Since the maintenance and control of entangled states is essential to realization of quantum information processing systems, the study of deteriorating effect of decoherence in entangled states would be of considerable importance from theoretical as well as experimental point of view [9,10,11].

To demonstrate the effects of decoherence on entangled states, an appropriate entanglement measure which could be capable of monitoring the dynamics of entanglement in decoherence processes is needed. However, there are no exact measure of entanglement under general conditions for mixed states. Even for bipartite mixed states, apart from the particular case of two-level systems [12], the exact solution is missing. There is an approximate generalization of concurrence for mixed states which was proposed by Mintert et al [13] and has been used to measure entanglement in multi-qubit systems.

Global entanglement (GE), defined by Meyer-Wallach (MW) entanglement measure of pure-state [14] which is a monotone[15], is a very useful measure of entanglement. As we will show briefly, GE is a measure of total non-local information per particle in a general multi-qubit system. Therefore, GE gives an intuitive meaning to multi-qubit entanglement as well being an experimentally accessible measure [16]. In this paper, we use this measure to monitor the entanglement dynamics of two seminal multi-qubit entangled states, the Greenberger-Horne-Zeilinger (GHZ) state and the W state, under different models of system-environment interaction.

To demonstrate the GE dynamics of multi-qubit entangled state under decoherence, we need to know the generalization of GE to mixed state. Unfortunately, there is no generalization of the primary definition of the MW measure to mixed states, analytically. However, we can monitor the GE dynamics for two important classes of multi-qubit entangled state, the GHZ and W state, by exploiting the relationship between GE and tangles. To elucidate this point, we adopt informational approach which can give an intuitive meaning to GE.

II. GLOBAL ENTANGLEMENT, INFORMATION, AND TANGLES

Finite amount of information can be attributed to N-qubit pure state which is N bit of information according to Brukner-Zeilinger operationally invariant information measure [17]. This information can be distributed in local as well as non-local form, which is associated with entanglement [18]. This information has a complimentary relation:

\[ I_{\text{total}} = I_{\text{local}} + I_{\text{non-local}}. \]  (1)

The total information is conserved unless transferred to environment through decoherence. The amount of information in local form is \( I_{\text{local}} = \sum_{i=1}^{N} I_i \) where, \( I_i = 2Tr\rho_i^2 - 1 \) is the operationally invariant information measure of a qubit [17]. Therefore, according to Eq.(1) \( I_{\text{non-local}} = \sum_{i=1}^{N} 2(1 - Tr\rho_i^2) \) which can be distributed in different forms of quantum correlations, the tangles.
among the system,

\[ I_{\text{non-local}} = 2 \sum_{i_1 < i_2} \tau_{i_1 i_2} + \ldots + N \sum_{i_1 < i_2 < \ldots < i_N} \tau_{i_1 \ldots i_N}, \]

where the first term is referred to as 2-tangle, the next being 3-tangle and the last term the N-tangle of the system. It can easily be seen that the MW measure of GE can be written as:

\[ E_{gl} = \frac{1}{N} [2 \sum_{i_1 < i_2} \tau_{i_1 i_2} + \ldots + N \sum_{i_1 < i_2 < \ldots < i_N} \tau_{i_1 \ldots i_N}]. \]  

Therefore, the MW measure of GE \( (E_{gl}) \) is a measure of total non-local information per particle (or average of tangles per particles \( \frac{\lambda}{N} \)). This property of \( E_{gl} \) resembles the molar-extensive thermodynamic variables (e.g. heat capacity \( C_p \)). In fact, we expect \( E_{gl} \) to have thermodynamically relevant and experimentally accessible features in multi-qubit (spin) systems[19]. This property of GE in contrast to most other multi-partite entanglement measures. It is therefore of considerable interest to investigate the dynamics of GE under decoherence in the large N limit for the generic, experimentally realizable, entangled states. We note that \( E_{gl} \) does not give detailed knowledge of tangles distribution among the system. For example, \( E_{gl} \) cannot distinguish between entangled states which have equal (\( \tau \)) yet different distribution of tangles, like \( |GHZ\rangle \) and \( |EPR\rangle \). However, \( E_{gl} \) can distinguish between GHZ and W state since their distinctly different type of tangles leads to different values of \( \langle \tau \rangle \).

We consider the two seminal multi-qubit entangled state: \( |GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \) and \( |W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \) as initial states which are known to bear qualitatively different quantum correlation. For example, GHZ state has a different tangle distribution than that of W state. In operational context they cannot be transformed to each other by local operation and classical communication (LOCC) [20]. So it is interesting to investigate the entanglement behavior of this two type of multi-qubit entangled states under different models of system-environment interaction.

In the GHZ state, all the entanglement is contained in the N-tangle form of non-local information which can be computed by N-concurrence measure \( (C_N) \) [21]. \( C_N \) is a generalization of two-qubit measure, concurrence [11], for even number of qubits (N). Therefore, for GHZ state GE is \( E_{gl} = \tau_N = C_N^2 \). This can be easily computed for even N as:

\[ R_N = \rho_N \sigma_y^{\otimes N} \rho_N^* \sigma_y^{\otimes N}, \]

\[ C_N = \max \{ \lambda_1 - \sum_{i=2}^{2N-1} \lambda_i, 0 \}; \]

where \( \sigma_y \) is Pauli matrix in y direction and \( \lambda_i \)s are the eigenvalues of the matrix \( R_N \) with \( \lambda_1 \) being the maximum.

In the W state, only two-qubit entanglement, 2-tangles \( (\tau_2) \), is present with each qubit equally entangled with all other qubits. \( \tau_2 \) can be computed analytically from the above measure. Thus, there are \( \frac{N(N-1)}{2} \) different two-qubit entanglement in the W state. GE is therefore given by:

\[ E_{gl} = (N - 1)\tau_2, \]

for W state. It is important to note that this approach allows us to obtain the exact solution of GE dynamics under decoherence of different environment models. This also shows the great utility of GE as a measure of entanglement.

### III. DECOHERENCE MODEL SYSTEMS

In order to evolve our chosen states under influence of decoherence we use the Lindblad form of master equation [22],

\[ \frac{d\rho}{dt} = \sum_{i=1}^{N} L_i \rho. \]

The Lindblad operators, \( L_i \), describe the local interaction of each qubit with environment independent of other qubit interaction with the environment. We assume \( L_i \) is the same form for all qubit, \( L_i = L \). For markovian process [22]

\[ L_i \rho = \sum_k \frac{\gamma_k}{2} [2J_k \rho J_k^\dagger - \{J_k, J_k^\dagger, \rho\}], \]

where operator \( J_k \) describes the system-environment model of interaction with strength \( \gamma_k \). In this paper we investigate dissipative, dephasing, and noise processes, each with a well-defined \( J_k \). For the two-level systems the operators, \( J_k \), are expressed in terms of Pauli matrices. The solution of Lindblad form of master equation for two-level systems are studied in [23].

For dissipative environment, \( J_1 = \sigma_- \). In this process the system interacts with a thermal bath at zero temperature. This process could be described as spontaneous emission of a two-state atom coupled with the vacuum modes of the ambient electromagnetic field which leads the atom state to the ground state. For dephasing process, \( J_1 = \sigma_+ \). This is a phase-destroying process that does not have a classical counterpart and is therefore intrinsically quantum mechanical. It corresponds to a situation where no energy is exchanged with environment, that is, the population of energy eigenstates of the system do not change with time. Only the phase information which includes quantum correlations is lost. For the noisy environment, \( J_1 = \sigma_- \) and \( J_2 = \sigma_+ \). Noisy dynamics are related to another extreme of thermal bath, i.e. when temperature is extremely high while the system-bath coupling is extremely weak. This process randomizes the state of the system which results in a completely
mixed state eventually. The noise process has a particular interest since its effect is basis independent. That is, the noisy operation is invariant under unitary operation. All these processes could have different effect on the multi-qubit entangled state. But the common feature of them is that under the action of each of these environments any initial entangled state asymptotically evolves to a separable state.

IV. RESULTS

Our goal is to obtain the time dependence of the density matrix, $\rho(t) = e^{-Lt} \rho(0)$, of the system in order to determine the time evolution of $E_{gl}(t)$ for the initially prepared multi-qubit entangled states, i.e. W and GHZ. According to the structure of entanglement in W state we can deduce the time dependency of the GE from two-qubit entanglement, $\tau_2$. The two-qubit density matrix, $\rho_{ij}(0) = 2/3 |\psi^+\rangle \langle \psi^+ | + (N-2)/(N(N-1)) |00\rangle \langle 00 |$, is the same for any pair of qubits, $ij$. Therefore, for the initial W state in the dissipative process we have,

$$\rho_{ij}(t) = \begin{pmatrix}
\frac{N-2p}{N} & 0 & 0 & 0 \\
0 & \frac{1}{N} & \frac{p}{N} & 0 \\
0 & \frac{p}{N} & \frac{1}{N} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \tag{8}
$$

where $p = e^{-\gamma t}$ is the decoherence parameter. Similarly, for dephasing process, one obtains,

$$\rho_{ij}(t) = \begin{pmatrix}
\frac{N-2}{N} & 0 & 0 & 0 \\
0 & \frac{1}{N} & \frac{1}{N} & 0 \\
0 & \frac{1}{N} & \frac{1}{N} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \tag{9}
$$

and, therefore, for the initial W state, GE has the simple exact solution,

$$E_{gl}(t) = \frac{4(N-1)}{N^2} e^{-2\gamma t}, \tag{10}
$$

for both dissipative as well as dephasing processes. For noisy process, the density matrix is

$$\rho_{ij}(t) = \frac{2}{N} \begin{pmatrix}
\frac{1-p^2}{4} & 0 & 0 & 0 \\
0 & \frac{1-p^2}{4} & \frac{p^2}{4} & 0 \\
0 & \frac{p^2}{4} & \frac{1-p^2}{4} & 0 \\
0 & 0 & 0 & 1-p^2
\end{pmatrix} + \begin{pmatrix}
\frac{1+p^2}{4} & 0 & 0 & 0 \\
0 & \frac{1+p^2}{4} & \frac{p^2}{4} & 0 \\
0 & \frac{p^2}{4} & \frac{1+p^2}{4} & 0 \\
0 & 0 & 0 & 1-p^2
\end{pmatrix}, \tag{11}
$$

which leads to

$$E_{gl} = \frac{N-1}{4N^2} \left[ max\{4p^2 - (1-p^2)^2 (N^2 - (pN-4p)^2)^2, 0\} \right]^2. \tag{12}
$$

The dynamics of GE in dephasing, dissipation and noisy environment for the initial W state is illustrated in Figs.1 and 2. Eq.(10)(Fig.1) shows that the decay rate($\alpha$), for $E_{gl} \propto \exp(-\alpha t)$ for GE is independent of N for the W state in dissipative and dephasing environment as found previously using numerical solution for a different measure of entanglement [10]. Note, however, that the rate of change of GE decreases with increasing N. For noisy environment, Fig.2, we observe a decay to separable state after a finite time $t_{sep}$ which increases linearly with N, also consistent with previous studies [10].
process is
\[
\rho_N(t) = \frac{1}{2} \left[ (|0\rangle \otimes |0\rangle + p^{\frac{1}{2}} |0\rangle \otimes |1\rangle + p^{\frac{1}{2}} |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \right] + p^{\frac{1}{2}} |0\rangle \otimes |0\rangle + p^{\frac{1}{2}} |1\rangle \otimes |1\rangle )
\]
(13)
which leads simply to \( E_{gl} = e^{-N\gamma t} \). The GHZ density matrix for the dissipative process is
\[
\rho_N(t) = \frac{1}{2} \left( p^{\frac{1}{2}} |0\rangle \otimes |0\rangle + p^{\frac{1}{2}} |1\rangle \otimes |1\rangle \right) + \sum_{q_1,\ldots,q_N=0}^{1} \lambda(Z) |q_1q_2\ldots q_N\rangle \langle q_1q_2\ldots q_N|)
\]
(14)
where
\[
Z = \sum_{i=1}^{N} z_i ; \quad z_i = \frac{1 - \sigma_{z_i}}{2},
\]
and
\[
\langle Z \rangle = \langle q_1q_2\ldots q_N | Z | q_1q_2\ldots q_N \rangle,
\]
and
\[
\lambda(Z) = [p^{\langle Z \rangle} (1-p)^{\langle Z \rangle} + 0_{\langle Z \rangle}].
\]
The GHZ density matrix for the noisy process is
\[
\rho_N(t) = \frac{1}{2} \left( p^{|0\rangle \otimes |0\rangle + p^{|1\rangle \otimes |1\rangle |0\rangle \otimes |0\rangle + p^{|1\rangle \otimes |1\rangle \otimes |1\rangle \rangle + 0_{\langle Z \rangle}].
\]
(15)
where
\[
\lambda(Z) = \frac{1}{N} (1+p)^{\langle Z \rangle} (1-p)^{\langle Z \rangle} + (1-p)^{\langle Z \rangle} (1+p)^{\langle Z \rangle}.
\]
Consequently, our results \( E_{gl} \), for the GHZ state in dephasing, dissipative and noisy environment are shown respectively in Figs. 3, 4, and 5 for various N. As shown in the corresponding insets, the decay rates increases linearly with system size N for all three processes. Also, the rate of change of \( E_{gl} \) increases with system size as well. All these results are consistent with previous studies using different methods than ours[9,10,11]. For the dynamics of GE in GHZ state, although we have the exact solution only for even number of qubit, the behavior of GE under decoherence for the odd number can be inferred from the simplicity and symmetry of our results. For example, our results for \( E_{gl} \) in dephasing process holds for any number of qubit in the initial GHZ state.

V. CONCLUSION

In conclusion, in this work we have given an intuitive informational meaning of MW measure of global entanglement. Based on the relationship between global entanglement and tangles, which constitute the non-local form of the information, we identify the exact solution of its dynamics under different system-environment models for the two qualitatively different multi-qubit entangled states, the GHZ and W states. In all the cases considered, we obtain an exponential decay of entanglement as a function of time. For the W state, the results show that the lifetime of the GE is independent of the number of the qubits in dephasing and dissipative processes and the lifetime linearly decreases with N in a noisy process. While for the GHZ state, the lifetime of GE decreases linearly with N. Our results indicate that the quantum correlations in W state are more robust to decoherence effects than that of the GHZ state.

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FIG. 5: $E_{gl}$ vs. $\gamma t$ in an initial GHZ state with noisy process for $N = 2, 4, 6, 8$ qubits. The number of qubits increases from the top curve to bottom. The inset shows the (linear) dependence of decay rate vs. $N$ up to $N = 10$ qubits.

[1] E. Schrödinger, Naturwissenschaften 23, 807 (1935).
[2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. A 47, 777 (1935).
[3] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
[4] C. H. Bennett, G. Brassard, Proceedings of IEEE International Conference on Computers, Systems and Signal Processing 175 (1984); C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. 69, 2881 (1992); A. Ekert, Phys. Rev. Lett. 67, 661 (1991).
[5] R. Raussendorf and H.J. Briegel, Phys. Rev. Lett. 86, 5188 (2001).
[6] For a recent review, see, R. Horodecki et al, quant-ph/0702225 (2007).
[7] L. Amico, R. Fazio, A. Osterloh and V. Vedral, quant-ph/0703044 submitted to Rev. Mod. Phys. (2007).
[8] J. Kofler, C. Brukner, Phys. Rev. Lett. 99, 180403 (2007) ; for example, see, E. Joos, H.D. Zeh, C. Kiefer, D. Giulini, J. Kupsch and I.O. Stamatescu. “Decoherence and the Appearance of a Classical World in Quantum Theory”, 2nd edition, Springer-Verlag (2003).
[9] Dür and H.-J. Briegel, Phys. Rev. Lett. 92, 180403 (2004).
[10] A. R. R. Carvalho, F. Mintert, A. Buchleitner, Phys. Rev. Lett. 93, 230501 (2004).
[11] C. Simon and J. Kempe, Phys. Rev. A 65, 052327 (2002).
[12] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[13] F. Mintert, M. Kus, and A. Buchleitner, Phys. Rev. Lett. 92, 167902 (2004).
[14] D. A. Meyer and N. R. Wallach, J. Math. Phys. 43, 4273 (2002).
[15] G.K. Brennen, Quant. Inf. and Comput. 3, 616 (2003).
[16] Peter J. Love, et al, quant-ph/0602143; G.K. Brennen, Quant. Inf. and Comput. vol. 3, 616 (2003).
[17] Brukner and A. Zeilinger, Phys. Rev. Lett. 83, 3354 (1999).
[18] M. Horodecki and R. Horodecki, Phys.Lett. A 244, 473 (1998); M. Horodecki, K. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), and U. Sen, Phys. Rev. Lett. 90, 100402 (2003); M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), U. Sen, and B. Synak-Radtke, Phys. Rev. A 71, 062307 (2005).
[19] A. Montakhab, and A. Asadian to be submitted.
[20] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000).
[21] A. Wong and N. Christensen, Phys. Rev. A 63, 044301 (2001).
[22] G. Lindblad, Math. Phys. 48, 119 (1976).
[23] H.J. Briegel and B.G. Englert, Phys. Rev. A 47, 3311 (1993).