Magnetococonductance signatures of chiral domain-wall bound states in magnetic topological insulators

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(Dated: January 12, 2022)

Recent magnetoconductance measurements performed on magnetic topological insulator candidates have revealed butterfly-shaped hysteresis. This hysteresis has been attributed to the formation of gapless chiral domain-wall bound states during a magnetic field sweep. We treat this phenomenon theoretically, providing a link between microscopic magnetization dynamics and butterfly hysteresis in magnetoconductance. Further, we illustrate how a spatially resolved conductance measurement can probe the most striking feature of the domain-wall bound states: their chirality. This work establishes a regime where a definitive link between butterfly hysteresis in longitudinal magnetoconductance and domain-wall bound states can be made. This analysis provides an important tool for the identification of magnetic topological insulators.

PACS numbers: 75.47.-m, 72.25.-b, 73.90.+f

I. INTRODUCTION

The surface of a strong three-dimensional topological insulator is characterized by an odd number of massless Dirac cones. The addition of static magnetic moments to the surface leads to a local Zeeman-like coupling term. If the moments are ferromagnetically coupled through a direct or indirect [RudermanKittelKasuyaYosida (RKKY)] exchange, they become locked into finite domains of fixed orientation at low temperature. The Zeeman-like term then acts as a local Dirac mass with a sign determined by the orientation of the proximal domain magnetization. Therefore, the electronic spectrum is gapped inside ferromagnetic domains. However, at domain boundaries, where the magnetization (and hence, the Dirac mass) changes sign, the system hosts one-dimensional chiral edge states that follow the domain walls. These domain-wall bound states (DWBS’s) are reminiscent of the chiral edge modes in the quantum-Hall regime. The chirality of these conduction channels is determined by the magnetization of the bounding domains and the spin-orbital structure of the surface states [Fig. 1(c), insets]. If the electronic chemical potential lies within the gap of the surface spectrum, the low-temperature transport properties will be dominated by the chiral and quantized conductance associated with DWBS’s. In this way, electric conductance is determined by the magnetic configuration.

DWBS’s have been discussed in the experimental literature in the context of hysteretic magnetoconductance in magnetic topological insulators. In these experiments, a characteristic butterfly-shaped hysteresis is observed as the applied field is increased, then decreased [see, e.g., Fig. 2(b), below, for an example arising from our model]. Butterfly hysteresis can be attributed to DWBS’s by arguing that the magnetization switches at the coercive field through the creation of domains, resulting in a network of domain walls. The DWBS’s associated with the network of nucleated domains lead to excess conductance. Checkelsky et al. have observed excess conductance as a magnetic field is swept in Mn doped Bi2Te3. Nakajima et al. have observed similar magnetoconductance hysteresis in SmBi2, a candidate topological Kondo insulator. Wang et al. have directly observed ferromagnetic domain formation and growth using magnetic force microscopy in V doped Sb2Te3. The authors of that study observe suppressed (rather than enhanced) longitudinal conductance due to the dominant contribution of bulk carriers to the magnetoconductance. The picture of DWBS transport is compelling, but other mechanisms could explain the observed magnetoconductance hysteresis. Analogous hysteresis curves are seen in distinctly non-topological sys-
tems, e.g., quantum-dot spin valves\textsuperscript{14} similar tunneling magnetoresistance may occur between domains of any conducting ferromagnet. Alternatively, paramagnetic cooling under a cycled magnetic field\textsuperscript{15} in combination with a temperature-dependent conductance could lead to similar hysteretic magnetotransport.

Previous theoretical studies of transport through DWBS’s have focused on current-driven magnetization dynamics. These studies include spin-torque and associated domain-wall motion caused by spin-polarized currents\textsuperscript{16,17} and a proposal for the inverse spin-galvanic effect as a mechanism for magnetization switching\textsuperscript{18}. In contrast, here we focus on the influence of magnetization dynamics on magnetoconductance. In particular, we directly link butterfly hysteresis in magnetoconductance to DWBS dynamics driven by microscopic spin relaxation. Demonstrating that the hysteretic magnetoconductance is effected by chiral transport channels would be definitive evidence of the DWBS hysteresis scenario.

This paper is organized as follows. In Section II we introduce the model and a set of rate equations governing magnetization dynamics under a magnetic field sweep. In Section III we find an explicit closed-form analytical expression for the trial-averaged magnetoconductance in a controlled limit and predict its dependence on the magnetic-field sweep rate. In Section IV we propose a spatially-resolved measurement capable of explicitly probing the chirality of the DWBS’s. In Section V we discuss the assumptions and limitations of our proposal, concluding with a summary of our work.

II. MODEL

We consider a topological surface state characterized by

\[ H_e = v_F \sum_{\mathbf{k}} c_{\mathbf{k}}^\dagger (k_x \sigma_y - k_y \sigma_x) c_{\mathbf{k}}. \] (1)

Here, $\sigma_i$ are Pauli matrices and $c_{\mathbf{k}} = (c_{\mathbf{k} \uparrow}, c_{\mathbf{k} \downarrow})^T$, where $c_{\mathbf{k}}$ annihilates an electron with 2D surface momentum $\mathbf{k}$ and spin $s$. We assume the electronic system is in contact with a magnetic subsystem at the surface. The magnetic system induces a Zeeman-like coupling giving rise to a local Dirac mass $m(\mathbf{r})$ that is proportional to the local magnetization: $\int \psi^\dagger(\mathbf{r})m(\mathbf{r})\sigma_z\psi(\mathbf{r})d\mathbf{r}$. Here, $\psi(\mathbf{r}) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}c_{\mathbf{k}}/\sqrt{A}$, and $A$ is the area of the 2D surface. For a uniform magnetization (giving $m(\mathbf{r}) = m$), this term gaps the surface spectrum by $2m$. When the magnetization switches (due, e.g., to a change in the ground state during a magnetic-field sweep), it does so through the creation of local domains, each associated with a bounding domain wall. At a domain wall, where $m(\mathbf{r})$ changes sign, the electronic system hosts a chiral DWBS\textsuperscript{19,20}. Each chiral bound state extends a distance $\sim v_F/m$ into the magnetic domains [Fig. I(b)]. Provided the domain walls are separated by more than this distance, there will be no scattering between the bound states and each bound state will support a single quantum of conductance $G_0 = e^2/h$ in a direction determined by the bounding magnetization [Fig. I(b)]. Electronic transport properties for such a system can be probed through source and drain electrodes arranged in a Corbino geometry [Fig. I(a)], providing evidence for the formation and dynamics of chiral DWBS’s. By considering a Corbino geometry, we avoid contributions from sample edges. At sample edges, the projection of the magnetization onto the surface normal may change sign, leading to additional DWBS’s\textsuperscript{21} that complicate the interpretation of magnetotransport measurements.

To model domain-wall dynamics and the resulting conductance during a magnetic-field sweep for the geometry shown in Fig. I(a), we consider a periodic Ising chain of $N$ spins with uniform coupling and a uniform magnetic field, described by Hamiltonian:

\[ H_m = -\sum_{i=1}^{N} \left( \frac{\xi}{2} \sigma_i^z \sigma_{i+1}^z + b \sigma_i^z \right). \] (2)

Here, $\xi$ is the spin-spin coupling, $b$ is the Zeeman energy per spin due to a magnetic field, and $\sigma_i^z$ is the Pauli operator acting on the spin at site $i$. The energy of the magnetic system is then fully characterized by the magnetization $M$ (the difference between the number of spins up and spins down) and by the number of domain walls $w$: $U = \xi w - b M$.

An infinite one-dimensional Ising system with short-range coupling has no ferromagnetically ordered phase at finite temperature, due to the logarithmically diverging entropic advantage in domain-wall formation\textsuperscript{22}. However, a finite chain can order at sufficiently low temperature. For $b < 0$, the ground state has $M = -N$, $w = 0$, while for $b > 0$, the ground state has $M = N$, $w = 0$. As the magnetic field is swept from negative to positive values at low temperature, the new ground state can be reached through the production of transient chiral domain walls, $w \neq 0$. For the geometry shown in Fig. I each pair of domain walls will be associated with one DWBS supporting a single quantum of conductance from source to drain (and another conducting from drain to source), Fig. I(b). This one-dimensional model directly addresses a magnetic topological insulator in which individual magnetic impurities are deposited around the Corbino annulus. To apply this model in the case of a two-dimensional magnetic system, we require that the distance between the source and drain electrodes be small compared to the typical domain size. At the same time, this distance must be large enough that tunneling between the source and drain is suppressed. The relevant distance is set by the decay length of evanescent modes within a gapped region of the magnetic topological insulator surface. This decay length is given by $v_F/m$.

In the limit of a slow sweep rate, $v = db/dt$ [cf. Eq. 14, below], the magnetization reverses through the creation of only a single counter-polarized domain associated with
a single pair of domain walls. Magnetization dynamics can then be described in the subspace of \( w = 0.2 \). In this subspace, \( U \) is uniquely determined by \( M (M = \pm N) \) correspond to \( w = 0 \), a single domain, while all other \( M \) are reached for \( w = 2 \). We can equivalently describe \( U \) through the number of \('up\) spins, \( n \): \( M = 2n - N \). For each \( n \), the degenerate states of the magnetic system with energy \( E_n \) are labeled by a quantum number \( \alpha \).

Transitions \( n \alpha \rightarrow n' \alpha' \) in the magnetic subsystem (with rate \( \Gamma_{n \alpha \rightarrow n' \alpha'} \)) result through the exchange of energy \( \Delta U_{n' n} = E_{n'} - E_n \) with an environment in thermal equilibrium and therefore obey detailed balance:

\[
\Gamma_{n' \alpha' \rightarrow n \alpha} = \Gamma_{n \alpha \rightarrow n' \alpha'} e^{-\Delta U_{n' n}/k_B T}.
\]

For \( T \rightarrow 0 \), excitation processes (\( \Delta U_{n' n} > 0 \)) are exponentially suppressed, leaving relaxation only. Restricting to single spin flips, the associated golden-rule rates are then differentiated purely through the energy cost, \( \Delta U_{n' n} \), and the number of degenerate and accessible final states, \( g_{n'} \):

\[
\Gamma_{n \rightarrow n'} \equiv \sum_{\alpha} \Gamma_{n \alpha \rightarrow n' \alpha'} = g_{n'} \Gamma(\Delta U_{n' n}).
\] (4)

In Appendix \[A\] we determine \( g_{n'} \) and \( \Gamma(\Delta U_{n' n}) \) in terms of a microscopic system-environment coupling. The form of \( \Gamma(\Delta U) \) depends generally on the coupling and density-of-states of the environment. For single spin flips, \( \Delta M = \pm 2 \) and the change in system energy is \( \Delta U = \xi \Delta w - b \Delta M \). For \( b > \xi \), there are three distinct relaxation rates: domain-wall nucleation, \( \Delta w = +2 \), domain growth, \( \Delta w = 0 \), and domain-wall annihilation, \( \Delta w = -2 \). Thus, with \( b > \xi \), the probability to have \( n \) spins up, \( P_n \), obeys a simple classical (Pauli) master equation:

\[
P_n = \Gamma_{n \rightarrow n-1} P_{n-1} - \Gamma_{n \rightarrow n+1} P_n; \quad n = 0, 1, \ldots, N,
\]

where we set \( \Gamma_{-1 \rightarrow 0} = \Gamma_{N \rightarrow N+1} = 0 \) to apply Eq. (5) for all \( n \). The distinct nonvanishing rates (for nucleation, growth, and annihilation, respectively) are

\[
\Gamma_{0 \rightarrow 1} = N \Gamma(2(\xi - b)); \quad \Gamma_{n \rightarrow n+1} = 2 \Gamma(2b); \quad n = 1, \ldots, N - 2
\]

\[
\Gamma_{N-1 \rightarrow N} = \Gamma(-2(\xi + b)) = \gamma_-. \]

The factor \( g_1 = N \) in the nucleation rate arises from the possibility to flip any of the \( N \) spins and the factor \( g_{n'} = 2 \) (for \( n' = 2, \ldots, N - 1 \)) is due to the possibility to grow a domain by flipping a spin at either end.

During a field sweep at low temperature, the magnetic system will initially be in the ‘down’ state (\( P_0 = 1 \)), and domain-wall nucleation may only occur above the coercive field (\( b > \xi \)). For \( b > \xi \), we linearize the nucleation rate:

\[
\Gamma(\Delta U) \approx \Gamma' |\Delta U| \Theta(-\Delta U)
\]

where \( \Gamma' = -d\Gamma(\Delta U = 0^-)/d\Delta U \). This gives \( \gamma_+ \approx \Gamma' 2(b - \xi) \Theta(b - \xi) \) (where \( \Theta(e) \) is the Heaviside step function). While \( \gamma_+ \) vanishes at \( b = \xi \), the growth \( \gamma \) and annihilation \( \gamma_- \) rates remain finite and are taken to be approximately \( b \)-independent over the range of interest. These two rates are related within the range of applicability of the linearization for \( \Gamma(\Delta U) \): \( \gamma_- \approx \Gamma(-4b) = 2\gamma = 2\Gamma(-2b) \). We consider a linear sweep of \( b \) with sweep rate \( \nu \): \( b = \xi + \nu t \), and transform the equation-of-motion: \( dP_n(t)/dt = \nu dP_n(b)/db \). Integrating Eq. (5) then gives (for \( b > \xi \)):

\[
P_0(b) = e^{-\frac{1}{2} \frac{(b-\xi)^2}{\Delta b^2}},
\]

\[
P_N(b_N) = \gamma \left( N - 1, \frac{N_N - b_N}{\Delta b} \right). \quad N - 2!
\]

Here, \( \gamma(s, \nu) = \int_0^\nu \nu s e^{-\nu t} dt \) is an incomplete gamma function, \( P_0(b) \) is the probability that the magnetic system remains in the initial state at field strength \( b \), and \( P_N(b_N) \) is the conditional probability that all \( N \) spins have flipped provided that a nucleation event (initial spin flip) occurred at field strength \( b_N \). We have introduced the scales

\[
\Delta b_+ = \sqrt{\frac{\nu}{2N\gamma^2}},
\]

and

\[
\Delta b = \frac{N\nu}{2\gamma}.
\]

These parameters have natural interpretations: \( \Delta b_+ \) and \( \xi \) determines the typical field at which the nucleation event occurs (Fig. 2(b)), and \( \Delta b \) determines the typical change in field strength during the growth of the domain (Fig. 2(a)).

To guarantee that only a single pair of domain walls is created during magnetization reversal (the single-pair limit), the typical time for domain growth, \( \sim \Delta b/\nu \), should be less than the typical time between nucleation events, \( \sim \Delta b+/\nu \). \( \Delta b \ll \Delta b_+ \), or equivalently:

\[
\nu \ll \frac{2\gamma^2}{N^3\Gamma^3}.
\]

If the linearization of the spin-relaxation rate is valid throughout \( -2\xi \leq \Delta U \leq 0 \), we can relate \( \Gamma' \) and \( \gamma \) through \( \gamma = 2\Gamma' \xi \) and the single-pair limit simplifies to \( \nu \ll \frac{\gamma^2}{\Delta b^3} \). We can substitute reasonable values for these parameters to show that this condition may be realized experimentally. The spin-relaxation rate, \( \gamma \), is not well known for most magnetic topological insulator candidates, but a range of values \( \gamma = 1 - 10^3 \text{s}^{-1} \) is routinely observed for single-electron spins in semiconductor quantum dots at magnetic fields in the range of several Tesla. For example, with \( N = 10, \gamma = 10^3 \text{s}^{-1} \), and a coercive field of 1 T (\( \xi/\mu_B = 0.5 \text{T} \)), where \( g \) is the g-factor, and \( \mu_B \) is the Bohr magneton), the single-pair limit requires \( v/\mu_B \gamma \ll 0.3 \text{T} \text{s}^{-1} \).
The averaged conductance is peaked at \( b \) during a single field sweep. At \( b = b_i > \xi \), a counter-polarized domain nucleates and grows across the sample. At \( b = b_f \), the domain has consumed the sample. The domain has consumed the sample. While the domain is growing, the conductance jumps to \( b_0 \). The trial-averaged conductance is peaked at \( b = \xi + \Delta b_+ \), with maximal value \( \langle G \rangle = G_0(\Delta b/\Delta b_+) \exp(-1/2) \propto \sqrt{\nu} \), where \( v = db/dt \) is the sweep rate.

\[ \langle G (b) \rangle \approx G_0 \frac{\Delta b}{\Delta b_+} \frac{b - \xi}{\sqrt{\nu}} e^{-\frac{1}{2} \left( \frac{b - \xi}{\Delta b_+} \right)^2}. \]  

The averaged conductance is peaked at \( b = \xi + \Delta b_+ \), with maximal value \( \langle G \rangle = G_0(\Delta b/\Delta b_+) \exp(-1/2) \). The trial-averaged conductance from Eq. (11) and magnetization are plotted in Fig. 2(b). A measurement of the correlated change in conductance and magnetization would provide strong evidence for the DWBS picture studied here. Even in the absence of microscopic time-resolved magnetization measurements, the connection between transport and magnetization dynamics can be verified from Eq. (16) through transport alone. In particular, Eq. (16) predicts nontrivial dependences for the averaged conductance peak height \( \langle G \rangle \propto \sqrt{\nu} \) and the maximum position \( \langle \Delta b \rangle \propto \sqrt{\nu} \) as the sweep rate \( v \) is varied. The specific dependences predicted here are non-universal, relying on a linearization of the rates, \( \Gamma(\Delta U) \propto |\Delta U|^\eta \). More generally, for a typical power-law form \( \Gamma(\Delta U) \propto |\Delta U|^\eta \), we find a modified lineshape giving a conductance maximum \( G^* \propto v^{\eta/(\eta+1)} \) and peak width \( \Delta b_+ \propto v^{1/(\eta+1)} \) (see Appendix B for the detailed forms of \( \Delta b_+ \) and \( G^* \) in this general case). Experimental confirmation of these dependencies would be strong evidence for the magnetic origin of the hysteretic conductance and could help to establish the relevant spin-relaxation mechanisms through the exponent \( \eta \).

### III. MAGNETOTRANSPORT HYSTERESIS

In the limit given by Eq. (14), we can use Eqs. (10) and (11) to find closed-form analytical expressions for the trial-averaged magnetoconductance. There will be a single forward-conducting DWBS giving a single conductance quantum \( \langle G \rangle = e^2/\hbar \) for \( 0 < n < N \), and none otherwise. The trial-averaged (denoted \( \langle G \rangle \)) conductance at field strength \( b \) is then

\[ \langle G (b) \rangle = G_0 \int_{b}^{\infty} db_f \int_{\xi}^{b} db_i p_f(b_f \mid b_i) p_i(b_i). \]  

Here, \( p_i(b_i) = -\frac{\partial}{\partial b_i} P_0(b_i) \) is the probability density that a nucleation event occurs at field strength \( b_i \) and \( p_f(b_f \mid b_i) = \frac{\partial}{\partial b_f} P_N(b_f \mid b_i) \) is the conditional probability density that an annihilation event occurs at \( b_f \) given a nucleation event at \( b_i \). As the magnetic field is increased during a single sweep, the conductance should show a jump to \( G \approx G_0 \) starting at a typical field \( \langle b_f \rangle = \xi + \Delta b_+ \). This jump coincides with the nucleation of a pair of domain walls following an initial spin flip. The conductance will then remain at its quantized value as the domain grows through a sequence of spin flips at the domain walls. Finally, the conductance will return to zero when the final spin flips and the magnetization has fully reversed. The final spin flip occurs at a typical field \( \langle b_f \rangle \approx \langle b_i \rangle + \Delta b \) for \( N \gg 1 \) [Fig. 2(a)].

Although each sweep will be associated with random values of \( b_i, b_f \), averaging over many sweeps will result in a robust averaged conductance with reproducible features, given by Eq. [Fig. 2(b)]. The integrals in Eq. (15) can be evaluated approximately to leading order in the slow-field-sweep limit [Eq. (14)], or equivalently \( \Delta b/\Delta b_+ \ll 1 \), giving a simple closed-form expression for the trial-averaged conductance:

\[ \langle G (b) \rangle \approx G_0 \frac{\Delta b}{\Delta b_+} \frac{b - \xi}{\sqrt{\nu}} e^{-\frac{1}{2} \left( \frac{b - \xi}{\Delta b_+} \right)^2}. \]  

The averaged conductance is peaked at \( b = \xi + \Delta b_+ \), with maximal value \( \langle G \rangle = G_0(\Delta b/\Delta b_+) \exp(-1/2) \). The trial-averaged conductance from Eq. (11) and magnetization are plotted in Fig. 2(b). A measurement of the correlated change in conductance and magnetization would provide strong evidence for the DWBS picture studied here. Even in the absence of microscopic time-resolved magnetization measurements, the connection between transport and magnetization dynamics can be verified from Eq. (16) through transport alone. In particular, Eq. (16) predicts nontrivial dependences for the averaged conductance peak height \( \langle G \rangle \propto \sqrt{\nu} \) and the maximum position \( \langle \Delta b \rangle \propto \sqrt{\nu} \) as the sweep rate \( v \) is varied. The specific dependences predicted here are non-universal, relying on a linearization of the rates, \( \Gamma(\Delta U) \propto |\Delta U|^\eta \). More generally, for a typical power-law form \( \Gamma(\Delta U) \propto |\Delta U|^\eta \), we find a modified lineshape giving a conductance maximum \( G^* \propto v^{\eta/(\eta+1)} \) and peak width \( \Delta b_+ \propto v^{1/(\eta+1)} \) (see Appendix B for the detailed forms of \( \Delta b_+ \) and \( G^* \) in this general case). Experimental confirmation of these dependencies would be strong evidence for the magnetic origin of the hysteretic conductance and could help to establish the relevant spin-relaxation mechanisms through the exponent \( \eta \).

### IV. SPATIALLY RESOLVED CONDUCTANCE AND BOUND-STATE CHIRALITY

The calculation presented in the previous section could be used to connect microscopic magnetization dynamics to hysteretic conductance. However, it does not directly address the chirality of the associated transport channels. To establish chirality, it may be useful to consider a configuration where the DWBS’s can be spatially resolved by probing conductance as a function of the bias direction and position around the Corbino annulus with segmented electrodes [Fig. 3(a)]. Such a measurement would reveal a conductance peak at one side of a domain for outward bias, and at the opposite side for inward bias. When only a single pair of domain walls is produced, it should be possible to resolve the two domain-wall-induced conductance peaks—they will be separated by a maximum distance comparable to the sample size.

To flesh out the domain-wall dynamics, we now consider an experiment where the ground state is prepared with all spins down at \( b < 0 \), then the magnetic field is rapidly pulsed to a fixed value \( b > \xi \), giving a nonzero nucleation rate and a nonzero spin-flip rate at either end. The integrals in Eq. (15) can be evaluated approximately to leading order in the slow-field-sweep limit [Eq. (14)], or equivalently \( \Delta b/\Delta b_+ \ll 1 \), giving a simple closed-form expression for the trial-averaged conductance:

\[ \langle G (b) \rangle \approx G_0 \frac{\Delta b}{\Delta b_+} \frac{b - \xi}{\sqrt{\nu}} e^{-\frac{1}{2} \left( \frac{b - \xi}{\Delta b_+} \right)^2}. \]  

The averaged conductance is peaked at \( b = \xi + \Delta b_+ \), with maximal value \( \langle G \rangle = G_0(\Delta b/\Delta b_+) \exp(-1/2) \). The trial-averaged conductance from Eq. (11) and magnetization are plotted in Fig. 2(b). A measurement of the correlated change in conductance and magnetization would provide strong evidence for the DWBS picture studied here. Even in the absence of microscopic time-resolved magnetization measurements, the connection between transport and magnetization dynamics can be verified from Eq. (16) through transport alone. In particular, Eq. (16) predicts nontrivial dependences for the averaged conductance peak height \( \langle G \rangle \propto \sqrt{\nu} \) and the maximum position \( \langle \Delta b \rangle \propto \sqrt{\nu} \) as the sweep rate \( v \) is varied. The specific dependences predicted here are non-universal, relying on a linearization of the rates, \( \Gamma(\Delta U) \propto |\Delta U|^\eta \). More generally, for a typical power-law form \( \Gamma(\Delta U) \propto |\Delta U|^\eta \), we find a modified lineshape giving a conductance maximum \( G^* \propto v^{\eta/(\eta+1)} \) and peak width \( \Delta b_+ \propto v^{1/(\eta+1)} \) (see Appendix B for the detailed forms of \( \Delta b_+ \) and \( G^* \) in this general case). Experimental confirmation of these dependencies would be strong evidence for the magnetic origin of the hysteretic conductance and could help to establish the relevant spin-relaxation mechanisms through the exponent \( \eta \).
of a nucleated domain, $\gamma = \Gamma(-2b)$. As before, we can describe the dynamics of magnetization reversal in the single-pair limit. After a nucleation event, the position of a domain wall is described, through Eq. (5), by a directed random walk with Poissonian-distributed step times at rate $\gamma$. The probability that there is a domain wall $x$ sites away from the nucleation site a time $t$ after the nucleation event, $P(x,t)$, is then well-approximated by a gamma distribution for $t \lesssim N/2\gamma$ (before an annihilation event):

$$P(x,t) \simeq (\gamma t)^{|x|^{-1}} e^{-\gamma t} / (|x| - 1)!.$$  

This equation can be used to directly confirm the presence and dynamics of DWBS’s and their chirality, addressing an important question in the experimental identification of magnetic topological insulators.

**Appendix A: Microscopic spin-flip rate**

In this appendix, we show how the spin-flip rates used in the main text arise from a coupling between the magnetic system and an environment.
Here, \( H_m \) is the magnetic system Hamiltonian given in Eq. (2), \( H_B \) is the environment Hamiltonian and \( V \) is the coupling between system and environment. In the main text, we work in the subspace spanned by eigenstates of the magnetic system Hamiltonian with only zero or two domain walls. We can denote such eigenstates

\[
H_m |n, \alpha\rangle = E_n |n, \alpha\rangle,
\]

where \( n \) is again the number of upward-oriented spins in the system and \( \alpha \) labels the degenerate states with energy \( E_n \). The subspaces with \( n = 0, N \) each contain only a single spin configuration, while those with \( n = 1, \ldots, N - 1 \) each host \( N \) degenerate configurations. We consider a coupling of the form

\[
V = \sum_j \sigma^j_+ B_j + H.c.,
\]

where \( \sigma^j_\pm \) flips the spin at site \( j \), and \( B_j \) is an operator that acts in the environment subspace.

Our goal is to determine the transition rate, \( \Gamma_{n \rightarrow n'} \), from a particular state with \( n \) spins up to any of the \( g_{n'} \) states with \( n' \) spins up. This may be found from \( P_{n \rightarrow n', \alpha' \rightarrow \alpha} \), the probability for a transition from \( |n\alpha\rangle \) to \( |n'\alpha'\rangle \) calculated to second order in time-dependent perturbation theory:

\[
\Gamma_{n \rightarrow n'} = \sum_{\alpha', \alpha} \lim_{t \rightarrow -\infty} \frac{d}{dt} P_{n \rightarrow n', \alpha' \rightarrow \alpha}(t).
\]

As described in the main text, we take an environment initially in thermal equilibrium at temperature \( T \). In the low-\( T \) limit, excitation is exponentially suppressed leaving relaxation rates only (having \( \Delta E_{n', n} = E_{n'} - E_n < 0 \)). We focus on transitions above the coercive field during a magnetic field up-sweep, where relaxation occurs only through spin-flips from down to up. Direct evaluation of Eq. (A4) in this regime gives the relaxation rates:

\[
\Gamma_{n \rightarrow n'} = g_n \Gamma(\Delta U_{n'n}),
\]

where \( g_n = \sum_{\alpha' \alpha} \langle \alpha' \alpha | \sigma^j_+ | n \alpha \rangle \rangle^2 \) gives the number of accessible final states, and where the single-spin relaxation rate is

\[
\Gamma(\Delta U) = \int_{-\infty}^{\infty} dt e^{-i\Delta U t - 0^+ t} \langle B^\dagger(t) B(t) \rangle_0.
\]

Here, \( B^\dagger(t) = e^{iH_B t} B^\dagger e^{-iH_B t} \) describes the interaction picture, \( \langle \cdots \rangle_0 = \langle 0 \cdots | 0 \rangle \) is an average with respect to the ground state of the environment Hamiltonian \( H_B \), and \( 0^+ \) is a positive infinitesimal. In writing Eq. (A6) we have assumed that the environment acting at each site is uncorrelated and equivalent at all sites, i.e.: \( \langle B^\dagger(t) B_j \rangle = \delta_{ij} \langle B^\dagger(t) B \rangle \).

Expanding Eq. (A6) in terms of eigenstates \( (H_B | k \rangle = \epsilon_k | k \rangle; \ \epsilon_k \geq 0 \) gives the standard Fermi’s golden rule result:

\[
\Gamma(\Delta U) = 2\pi \sum_k |B_{0k}|^2 \delta(\Delta U + \epsilon_k),
\]

where \( B_{0k} = \langle 0 | B | k \rangle \). In many cases, the environment coupling can be described (at least approximately) as a unique function of the energy, \( B_{0k} = g(\epsilon_k) \). Taking the continuum limit of Eq. (A7) then gives:

\[
\Gamma(\Delta U) = 2\pi D(\Delta U)|g(\Delta U)|^2,
\]

with environment density of excitations \( D(\epsilon) = \sum_k \delta(\epsilon - \epsilon_k) \).

In general, the energy-dependence of Eq. (A8) will be determined by both the density of excitations in the environment, \( D(\epsilon) \), and the coupling \( g(\epsilon) \). If the low-energy excitations are long-wavelength phonons or magnons, the density of states will vanish at low energy, typically like \( D(\epsilon) \propto e^{\epsilon - 1} \) in \( d = 2, 3 \) dimensions. Provided the coupling \( g(\epsilon) \) does not diverge as \( \epsilon \rightarrow 0 \), this will result in a rate that vanishes typically with some power-law: \( \Gamma(\Delta U) \propto |\Delta U|^\eta \). We consider a general power-law in Appendix B below. In the main text, we considered a rate that is linearizable at low energy. Such a linearizable rate arises naturally, e.g., when spin-flips occur due to cotunneling with a metallic reservoir at low temperature. These second-order tunneling processes give rise to an approximately energy-independent effective coupling \( g(\epsilon) \sim t_c^2/U_c \) with tunnel coupling \( t_c \) and effective charging energy \( U_c \). The environmental excitations are electron-hole pairs created about the Fermi level in an energy window of size \( |\Delta U| \), leading to an overall linear energy dependence at low temperature:

\[
\Gamma(\Delta U) \propto |\Delta U|^\eta
\]

where \( \eta = 2 \). This reflects the way in which the environment will respond to the spin-flip at high rate. The growth and annihilation rates are again taken to be constant about the coercive field. Using this general \( \gamma_+ \) we find

\[
P_0(b) = \exp \left[-\frac{\eta}{\eta + 1} \left(\frac{b - \xi}{\Delta b_+}\right)^{\eta+1}\right].
\]
The conditional probability that all $N$ spins have flipped after a nucleation event, $P_N(b|b_0)$, is unchanged relative to Eq. (11) of the main text since it depends only on $\gamma$ and $\gamma_-$. We have introduced the modified scale
\[
\Delta b_+ = \left( \frac{v \eta}{2 N \Gamma(\eta)} \right)^{1/\eta}.
\]
(B3)

The condition to guarantee that the magnetization reversal occurs through only a single nucleation event is again that the typical time for domain growth, $\sim \Delta b/v$, should be small compared to the typical time between nucleation events (now $\sim \Delta b/\eta v$). The more general single-pair limit is then
\[
v \ll \eta \left( \frac{2 \gamma^{\eta+1}}{N^\eta + 2 \Gamma(\eta)} \right)^{1/\eta}.
\]
(B4)

We evaluate the trial-averaged conductance integral, Eq. (15), using Eq. (B2) in the slow-sweep-limit to find
\[
\langle \langle G(b) \rangle \rangle \simeq G_0 \eta \left( \frac{\Delta b}{\Delta b_+} \right)^\eta \exp \left[ -\frac{\eta}{\eta + 1} \left( \frac{b - \xi}{\Delta b_+} \right)^{\eta + 1} \right].
\]
(B5)

The average conductance is again peaked at $b = \xi + \Delta b_+$, with maximal value
\[
G^* = G_0 \eta \left( \frac{\Delta b}{\Delta b_+} \right) \exp \left[ -\frac{\eta}{\eta + 1} \right].
\]
(B6)

The $v$ scaling presented in the main text is found by examining the $v$-dependence of $\Delta b_+$ and $G^*$. $\Delta b_+$ is proportional to $v \pi^{1/\eta}$. The $v$-dependence in $G^*$ comes from the ratio $\Delta b/\Delta b_+$. $\Delta b$ is proportional to $v$. Substituting the $v$-dependence of $\Delta b_+$, we find $G^* \propto v^{\eta/\eta + 1}$.

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