TWO DIMENSIONAL VERONESE GROUPS WITH AN INVARIANT BALL

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Abstract. In this article we characterize the complex hyperbolic groups that leave invariant a copy of the Veronese curve in $\mathbb{P}_2^C$. As a corollary we get that every discrete compact surface group in $\text{PO}^+(2, 1)$ admits a deformation in $\text{PSL}(3, \mathbb{C})$ with a non-empty region of discontinuity which is not conjugate to a complex hyperbolic subgroup. This provides a way to construct new examples of Kleinian groups acting on $\mathbb{P}_2^C$, see [4, 6, 13–15].

Introduction

Back in the 1990s, Seade and Verjovsky began the study of discrete groups acting on projective spaces, see [13–15]. Over the years, new results have been discovered, see [4]. However, it has been hard to construct groups acting on $\mathbb{P}_2^C$ which are neither virtually affine nor complex hyperbolic. In this article we use the irreducible representation $\iota$ of $\text{PSL}(2, \mathbb{C})$ into $\text{PSL}(3, \mathbb{C})$ to produce such groups, more precisely, we show:

**Theorem 0.1.** Let $\Gamma \subset \text{PSL}(2, \mathbb{C})$ be a discrete group of the first kind with non-empty discontinuity region in the Riemann sphere. Then the following claims are equivalent:

1. The group $\Gamma$ is Fuchsian.
2. The group $\iota \Gamma$ is complex hyperbolic.
3. The group $\iota \Gamma$ is $\mathbb{R}$-Fuchsian.

Before we present our next result we should recall the following definition, see [12] page 30. A group $G$ is called a compact surface group, if it is isomorphic to the fundamental group of a compact orientable topological surface $\Sigma_g$ of genus $g \geq 2$.

**Theorem 0.2.** Let $\Sigma_g$ a compact orientable topological surface $\Sigma_g$ of genus $g \geq 2$ and $\rho_0 : \Pi_1(\Sigma_g) \to \text{PO}^+(2, 1)$ be a faithful discrete representation, where $\text{PO}^+(2, 1)$ denotes the projectivization of identity component of $\text{O}(2, 1)$. Then we can find a sequence of discrete faithful representations $\rho_n : \Pi_1(\Sigma_g) \to \text{PSL}(3, \mathbb{C})$ such that:

1. For each $n \in \mathbb{N}$ the group $\rho_n(\Pi_1(\Sigma_g)) = \Gamma_n$ is a complex Kleinian group whose action on $\mathbb{P}_2^C$ is irreducible.
2. For each $n \in \mathbb{N}$ the group $\Gamma_n$ is not conjugate to a subgroup of $\text{PU}(2, 1)$ or $\text{PSL}(3, \mathbb{R})$.
3. The sequence of representations $(\rho_n)$ converge algebraically to $\Gamma_0$, i.e. $\lim_n \rho_n(\gamma) = h$ exists as a projective transformation for all $\gamma \in \Pi_1(\Sigma_g)$ and $\Gamma_0 = \{h : \lim_n \rho_n(\gamma) = h, \gamma \in \Pi_1(\Sigma_g)\}$ compare with the corresponding definition in [10].

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The sequence \((\Gamma_n)\) of compact surface groups converge geometrically to \(\Gamma_0\), i.e. if for every subsequence \((j_n)\) of \((n)\) we get
\[\Gamma_0 = \{g \in \text{PSL}(3, \mathbb{C}) : g = \lim_{j_n} \rho_{j_n}(\gamma_n), \gamma_n \in \Pi_1(\Sigma_g)\},\]
compare with the corresponding definition in [10].

Corollary 0.3. There are complex Kleinian groups acting on \(\mathbb{P}_2^2\) which are neither conjugate to complex hyperbolic groups nor virtually affine groups.

This paper is organized as follows: in Section 1 we review some general facts and introduce the notation used throughout the text. In Section 2 we describe some properties of the Veronese curve which are useful for our purposes. In Section 3 we characterize the complex hyperbolic subgroups that leave invariant a Veronese curve. In Section 4 we depict those real hyperbolic subgroups leaving invariant a Veronese curve. Finally, in Section 5 we show that every discrete compact surface group in \(\text{PO}(2, 1)\) admits a deformation in \(\text{PSL}(3, \mathbb{C})\) which is not conjugate to a complex hyperbolic subgroup and has non-empty Kulkarni region of discontinuity.

1. Preliminaries

1.1. Projective geometry. The complex projective space \(\mathbb{P}_2^2\) is defined as
\[\mathbb{P}_2^2 = (\mathbb{C}^3 \setminus \{0\}) / \mathbb{C}^*,\]
where \(\mathbb{C}^*\) acts by the usual scalar multiplication. This is a compact connected complex 2-dimensional manifold. If \([\cdot] : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}_2^2\) is the quotient map, then a non-empty set \(H \subset \mathbb{P}_2^2\) is said to be a line if there is a \(\mathbb{C}\)-linear subspace \(\tilde{H}\) in \(\mathbb{C}^3\) of dimension 2 such that \([\tilde{H} \setminus \{0\}] = H\). If \(p, q\) are distinct points then \(\overrightarrow{p, q}\) is the unique complex line passing through them. In this article, \(e_1, e_2, e_3\) will denote the standard basis for \(\mathbb{C}^3\).

1.2. Projective transformations. The group of projective automorphisms of \(\mathbb{P}_2^2\) is defined as
\[\text{PSL}(3, \mathbb{C}) := \text{GL}(3, \mathbb{C}) / \mathbb{C}^*,\]
where \(\mathbb{C}^*\) acts by the usual scalar multiplication. Then \(\text{PSL}(3, \mathbb{C})\) is a Lie group acting by biholomorphisms on \(\mathbb{P}_2^2\); its elements are called projective transformations. We denote by \([\cdot] : \text{GL}(3, \mathbb{C}) \to \text{PSL}(3, \mathbb{C})\) the quotient map. Given \(\gamma \in \text{PSL}(3, \mathbb{C})\), we say that \(\tilde{\gamma} \in \text{GL}(3, \mathbb{C})\) is a lift of \(\gamma\) if \([\tilde{\gamma}] = \gamma\).

1.3. Complex hyperbolic groups. In the rest of this paper, we will be interested in studying those subgroups of \(\text{PSL}(3, \mathbb{C})\) that preserve the unitary complex ball. We start by considering the following Hermitian matrix:
\[H = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.\]
We will set
\[\text{U}(2, 1) = \{g \in \text{GL}(3, \mathbb{C}) : g^* H g^* = H\}\]
\[\text{O}(2, 1) = \{g \in \text{GL}(3, \mathbb{R}) : g^t H g = H\}\]
and $\langle \cdot, \cdot \rangle : \mathbb{C}^3 \to \mathbb{C}$ the Hermitian form induced by $H$. Clearly, $\langle \cdot, \cdot \rangle$ has signature $(2,1)$ and $U(2,1)$ is the group that preserves $\langle \cdot, \cdot \rangle$, see [9]. The projectivization $\text{PU}(2,1)$ preserves the unitary complex ball:

$$\mathbb{H}^2 = \{ [w] \in \mathbb{P}^2_{\mathbb{C}} | \langle w, w \rangle < 0 \}.$$ 

Given a subgroup $\Gamma \subset \text{PU}(2,1)$, we define the following notion of limit set, as in [7].

**Definition 1.1.** Let $\Gamma \subset \text{PU}(2,1)$, then its Chen–Greenberg limit set is $\Lambda_{\text{CG}}(\Gamma) := \bigcup \mathbb{T} \cap \partial \mathbb{H}^2_{\mathbb{C}}$ where the union on the right runs over all points $x \in \mathbb{H}^2_{\mathbb{C}}$.

As in the Fuchsian groups case, $\Lambda_{\text{CG}}(\Gamma)$ has either $1$, $2$ or infinitely many points. A group is said to be non-elementary if $\Lambda_{\text{CG}}(\Gamma)$ has infinitely many points, and in that case it does not depend on the choice of orbit, i.e. $\Lambda_{\text{CG}}(\Gamma) := \bigcup \mathbb{T} \cap \partial \mathbb{H}^2_{\mathbb{C}}$ where $x \in \mathbb{H}^2_{\mathbb{C}}$ is any point.

### 1.4. Pseudo-projective transformations.

The space of linear transformations from $\mathbb{C}^3$ to $\mathbb{C}^3$, denoted by $M(3, \mathbb{C})$, is a complex linear space of dimension $9$, where $\text{GL}(3, \mathbb{C})$ is an open dense set in $M(3, \mathbb{C})$. Then $\text{PSL}(3, \mathbb{C})$ is a open dense set in $QP(3, \mathbb{C}) = (M(3, \mathbb{C}) \setminus \{0\})/\mathbb{C}^*$ called in [5] the space of pseudo-projective maps. Let $\widetilde{M} : \mathbb{C}^3 \to \mathbb{C}^3$ be a non-zero linear transformation. Let $\text{Ker}(\widetilde{M})$ be its kernel and $\text{Ker}(\langle [\tilde{M}] \rangle)$ denote its projectivization. Then $\widetilde{M}$ induces a well defined map $\langle [\tilde{M}] \rangle : \mathbb{P}^2_{\mathbb{C}} \setminus \text{Ker}(\langle [\tilde{M}] \rangle) \to \mathbb{P}^2_{\mathbb{C}}$ by

$$\langle [\tilde{M}] \rangle ([v]) = \widetilde{M}(v).$$

The following result provides a relation between convergence in $QP(3, \mathbb{C})$ viewed as points in a projective space and the convergence viewed as functions.

**Proposition 1.2 (See [5]).** Let $(\gamma_m)_{m \in \mathbb{N}} \subset \text{PSL}(3, \mathbb{C})$ be a sequence of distinct elements. Then:

1. There is a subsequence $(\tau_m)_{m \in \mathbb{N}} \subset (\gamma_m)_{m \in \mathbb{N}}$ and a $\tau_0 \in M(3, \mathbb{C}) \setminus \{0\}$ such that $\tau_m \xrightarrow{m \to \infty} \tau_0$ as points in $QP(3, \mathbb{C})$.

2. If $(\tau_m)_{m \in \mathbb{N}}$ is the sequence given by the previous part of this lemma, then $\tau_m \xrightarrow{m \to \infty} \tau_0$, as functions, uniformly on compact sets of $\mathbb{P}^2_{\mathbb{C}} \setminus \text{Ker}(\tau_0)$.

Moreover, the equicontinuity set of $\{\tau_m|m \in \mathbb{N}\}$ is $\mathbb{P}^n \setminus \text{Ker}(\tau_0)$.

### 1.5. Kulkarni’s limit set.

When we look at the action of a group acting on a general topological space, in general there is no natural notion of limit set. A nice starting point is Kulkarni’s limit set.

**Definition 1.3 (see [11]).** Let $\Gamma \subset \text{PSL}(n + 1, \mathbb{C})$ be a subgroup. We define

1. the set $\Lambda(\Gamma)$ to be the closure of the set of cluster points of $\Gamma z$ as $z$ runs over $\mathbb{P}^n_{\mathbb{C}}$,

2. the set $L_2(\Gamma)$ to be the closure of cluster points of $\Gamma K$ as $K$ runs over all the compact sets in $\mathbb{P}^n_{\mathbb{C}} \setminus \Lambda(\Gamma)$,

3. and Kulkarni’s limit set of $\Gamma$ to be

$$\Lambda_{\text{Kul}}(\Gamma) = \Lambda(\Gamma) \cup L_2(\Gamma),$$

4. Kulkarni’s discontinuity region of $\Gamma$ to be

$$\Omega_{\text{Kul}}(\Gamma) = \mathbb{P}^n_{\mathbb{C}} \setminus \Lambda_{\text{Kul}}(\Gamma).$$
Kulkarni’s limit set has the following properties. For a more detailed discussion of this in the two-dimensional setting, see [3].

Proposition 1.4 (See [1,4,5,11]). Let \( \Gamma \subset \text{PSL}(3, \C) \) be a complex Kleinian group. Then:

1. The sets \( \Lambda_{Kul}(\Gamma), \Lambda(\Gamma), L_2(\Gamma) \) are \( \Gamma \)-invariant closed sets.
2. The group \( \Gamma \) acts properly discontinuously on \( \Omega_{Kul}(\Gamma) \).
3. If \( \Gamma \) does not have any projective invariant subspaces, then

\[ \Omega_{Kul}(\Gamma) = \text{Eq}(\Gamma). \]

Moreover, \( \Omega_{Kul}(\Gamma) \) is complete Kobayashi hyperbolic and is the largest open set on which the group acts properly discontinuously.

2. The Geometry of the Veronese Curve

Now let us define the Veronese embedding. Set

\[ \psi : \mathbb{P}^1_{\C} \rightarrow \mathbb{P}^2_{\C}, \quad \psi([z, w]) = [z^2, 2zw, w^2]. \]

Let us consider \( \iota : \text{PSL}(2, \C) \rightarrow \text{PSL}(3, \C) \) given by

\[ \iota \left( \frac{az + b}{cz + d} \right) = \left[ \begin{array}{ccc} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & dc & d^2 \end{array} \right]. \]

Trivially, \( \iota \) is well defined. Note that this map is induced by the canonical action of \( \text{SL}(2, \C) \) on the space of homogeneous polynomials of degree two in two complex variables.

Lemma 2.1. The map \( \iota \) is an injective group morphism.

Proof. Let

\[ A = \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right], \quad B = \left[ \begin{array}{cc} e & f \\ g & h \end{array} \right] \in \text{PSL}(2, \C). \]

Then

\[ \iota(AB) = \left[ \begin{array}{ccc} ae + bg & af + bh \\ ce + dg & cf + dh \end{array} \right]. \]

Therefore \( \iota \) is a group morphism. Now suppose \( A = [a_{ij}] \in \text{PSL}(2, \C) \) is such that \( \iota(A) = \text{Id} \). Then

\[ \left[ \begin{array}{ccc} a^2_{11} & a_{11}a_{12} & a^2_{12} \\ 2a_{12}a_{21} & a_{11}a_{22} + a_{12}a_{21} & 2a_{12}a_{21} \\ a^2_{21} & a_{21}a_{22} & a^2_{22} \end{array} \right] = \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right], \]

and so we conclude \( a_{12} = a_{21} = 0 \). Since \( a_{11}a_{22} - a_{12}a_{21} = 1 \), we deduce \( a^2_{11} = a^2_{22} = 1 \), i.e. \( A = \text{Id} \), which concludes the proof.

Proposition 2.2. The morphism \( \iota \) is type preserving. In particular, if \( \Gamma \subset \text{PSL}(2, \C) \) is a discrete subgroup, we must have \( \iota(\Gamma) \) is a discrete group such that each element is strongly loxodromic.
Here, by type preserving, we mean that \( \iota \) carries elliptic elements into elliptic elements, and similarly for loxodromic and parabolic elements.

**Proof.** Consider
\[
A = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \text{PSL}(2, \mathbb{C}).
\]
A straightforward calculation shows
\[
\iota(A) = \begin{bmatrix} a^2 & 0 \\ 0 & a^{-2} \end{bmatrix}, \quad \iota(B) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.
\]
This shows that \( \iota \) is type preserving. Now let
\[
A_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \in \text{PSL}(2, \mathbb{C})
\]
be a sequence such that \( \iota(A_n) \xrightarrow{n \to \infty} \text{Id} \). Then
\[
\begin{bmatrix} a_n^2 & a_nb_n \\ c_n^2 & d_n c_n \end{bmatrix} \xrightarrow{n \to \infty} \text{Id}.
\]
Therefore the \( (a_n^2), (d_n^2) \) are bounded and bounded away from 0, \( b_n \xrightarrow{n \to \infty} 0 \), and \( c_n \xrightarrow{n \to \infty} 0 \), which is a contradiction. □

**Lemma 2.3.** Let \( g \in \text{PSL}(3, \mathbb{C}) \) be such that \( g \) fixes four points in general position. Then \( g = \text{Id} \).

**Proof.** We can assume that the four points in general position fixed by \( g \) are \( \{e_1, e_2, e_3, p\} \). Then
\[
g = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}.
\]
Since \( p, e_1, e_2, e_3 \) are in general position, we conclude \( p = [b_1, b_2, b_3] \) where \( b_1b_2b_3 \neq 0 \). On the other hand, since \( p \) is fixed we deduce
\[
[b_1, b_2, b_3] = [a_1b_1, a_2b_2, a_3b_3],
\]
therefore there is an \( r \in \mathbb{C}^* \) such that \( b_i = ra_i b_i \). In consequence \( a_1 = a_2 = a_3 \), which concludes the proof. □

**Lemma 2.4.** The Veronese curve has four points in general position.

**Proof.** A straightforward calculation shows that \( [1, 0, 0], [0, 0, 1], [1, 2, 1], [1, 2i, -1] \) are points on the Veronese curve. In order to conclude the proof, it is enough to observe
\[
\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2i \\ 0 & 1 & -1 \end{bmatrix} = -2 - 2i, \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 2i \\ 1 & 1 & -1 \end{bmatrix} = -2 + 2i.
\]
□

**Proposition 2.5.** The subgroup of \( \text{PSL}(3, \mathbb{C}) \) leaving \( \psi(P_1^3) \) invariant is \( \iota(\text{PSL}(2, \mathbb{C})) \).
Proof. First, let us prove that $Ver = \psi(\mathbb{P}^1_C)$ is invariant under $\iota(\text{PSL}(2, \mathbb{C}))$. Let $A = [[a_{ij}]] \in \text{PSL}(2, \mathbb{C})$. Then

$$\iota \left( \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) \left( \begin{bmatrix} x \\ 2xy \\ y^2 \end{bmatrix} \right) = \left( \begin{bmatrix} (a_{11}x + a_{12}y)^2 \\ 2(a_{21}x + a_{22}y)(a_{11}x + a_{12}y) \\ (a_{21}x + a_{22}y)^2 \end{bmatrix} \right),$$

and so $Ver$ is invariant under $\iota\text{PSL}(2, \mathbb{C})$ and the following diagram commutes.

\[
\begin{array}{ccc}
\mathbb{P}^1_C & \xrightarrow{\gamma} & \mathbb{P}^1_C \\
\downarrow & & \downarrow \\
Ver & \xrightarrow{\iota \gamma} & Ver \\
\end{array}
\]

Now let $\tau \in \text{PSL}(3, \mathbb{C})$ be an element which leaves $Ver$ invariant. Define

$$\tilde{\tau} : \mathbb{P}^1_C \to \mathbb{P}^1_C, \quad \tilde{\tau}(z) = \psi^{-1}(\tau(\psi(z))).$$

Clearly $\tilde{\tau}$ is well defined and biholomorphic, thus $\tilde{\tau} \in \text{PSL}(2, \mathbb{C})$ and the following diagram commutes.

\[
\begin{array}{ccc}
\mathbb{P}^1_C & \xrightarrow{\tau} & \mathbb{P}^1_C \\
\downarrow & & \downarrow \\
Ver & \xrightarrow{\tau} & Ver \\
\end{array}
\]

From diagram \textbf{(2.1)} we conclude that $\tau \mid_{Ver} = i\tilde{\tau} \mid_{Ver}$. Since the Veronese curve has four points in general position, we conclude $\tau = i\tilde{\tau}$ in $\mathbb{P}^2_C$, which concludes the proof. \hfill $\Box$

**Lemma 2.6.** Given $[1, k] \in \mathbb{P}^1_C$, the tangent line to $Ver$ at $\psi[1, k]$, denoted $T_{\psi[1, k]}Ver$, is given by

$$T_{\psi[1, k]}Ver = \{(x, y, z) \in \mathbb{P}^2_C | z = ky - k^2x\}.$$

Proof. Let us consider the chart $(W_1 = \{(x, y, z) \in \mathbb{P}^2_C | x \neq 0\}, \phi_1 : W_1 \to \mathbb{C}^2)$ of $\mathbb{P}^2_C$ where $\phi_1(x, y, z) = (yx^{-1}, zx^{-1})$ and $(W_2 = \{(x, y) \in \mathbb{P}^1_C | x \neq 0\}, \phi_2 : W_2 \to \mathbb{C}^1)$ of $\mathbb{P}^1_C$ where $\phi_1(x, y) = yx^{-1}$. Let us define

$$\phi : \mathbb{C} \to \mathbb{C}^2, \quad \phi(z) = \phi_1(\psi(\phi^{-1}_2(z))).$$

A straightforward calculation shows that $\phi(z) = (2z, z^2)$, thus the tangent space to the curve $\phi$ at $\phi(k)$ is $\mathbb{C}(1, k) + (2k, k^2)$. Therefore the tangent line to $Ver$ at $[1, 2k, k^2]$ is $[1, 2k, k^2], [1, 2k + 1, k + k^2]$. A simple verification shows

$$T_{\psi[1, k]}Ver = \{(x, y, z) \in \mathbb{P}^1_C | z = ky - k^2x\}. \hfill \Box$$

**Lemma 2.7.** Let $\Gamma \subset \text{PSL}(2, \mathbb{C})$ be a non-elementary subgroup and $x, y, z \in \Lambda(\Gamma)$ be distinct points, then the lines $T_{\psi(x)}Ver, T_{\psi(y)}Ver, T_{\psi(z)}Ver$ are in general position.
Proof. Let us assume that \([1, 0], [0, 1] \notin \Lambda(\Gamma)\). Then there are \(k, r, s \in \mathbb{C}\) such that
\[
\psi(x) = [1, 2k, k^2] \\
\psi(y) = [1, 2r, r^2] \\
\psi(z) = [1, 2s, s^2].
\]
From Lemma 2.6 we know
\[
T_{\psi(x)} \text{Ver} = \{[x, y, z] \in \mathbb{P}_{\mathbb{C}}^1 | z = ky - k^2x\} \\
T_{\psi(y)} \text{Ver} = \{[x, y, z] \in \mathbb{P}_{\mathbb{C}}^1 | z = ry - r^2x\} \\
T_{\psi(z)} \text{Ver} = \{[x, y, z] \in \mathbb{P}_{\mathbb{C}}^1 | z = sy - s^2x\}.
\]
Since
\[
\begin{vmatrix}
  k^2 & -k & 1 \\
  r^2 & -r & 1 \\
  s^2 & -s & 1
\end{vmatrix} = (s - r)(k - s)(k - r) \neq 0
\]
we conclude the proof. □

Lemma 2.8. Let \((\gamma_n) \subset \text{PSL}(2, \mathbb{C})\) be a sequence of distinct elements such that \(\gamma_n \xrightarrow{\infty} x\) uniformly on compact sets of \(\mathbb{P}_{\mathbb{C}}^1 \setminus \{y\}\). Then \(\gamma_n \xrightarrow{\infty} \psi(x)\) uniformly on compact sets of \(\mathbb{P}_{\mathbb{C}}^2 \setminus T_{\psi(y)} \text{Ver}\).

Proof. Let us assume that \(\gamma_n = \begin{bmatrix} a_{ij}(n) \end{bmatrix}\). Note that we can assume \(a_{ij}^{(n)} \xrightarrow{n \to \infty} a_{ij}\) and \(\sum_{i,j=1}^2 |a_{ij}| \neq 0\). Then \(\gamma_n \xrightarrow{n \to \infty} \gamma = [a_{ij}]\) uniformly on compact sets of \(\mathbb{P}_{\mathbb{C}}^1 \setminus \text{Ker}(\gamma)\), thus \(\text{Ker}(\gamma) = \{y\}\) and \(\text{Im}(\gamma) = \{x\}\). Therefore there is a \(k \in \mathbb{C}^*\) such that \(x = [1, k]\), thus \(a_{11} = -ka_{12}\) and \(a_{21} = -ka_{22}\). In consequence
\[
\gamma_n \xrightarrow{n \to \infty} B = \begin{bmatrix}
  k^2a_{12} & -ka_{12} & a_{12}^2 \\
  2k^2a_{12}a_{22} & -2ka_{12}a_{22} & 2a_{12}a_{22} \\
  k^2a_{22} & -ka_{22} & a_{22}^2
\end{bmatrix}.
\]

A simple calculation shows that \(\text{Ker}(B)\) is the line \(\ell = [e_1 - k^2e_3, e_2 + ke_3]\).
Also it is not hard to check that
\[
\ell = \{[x, y, z]|k^2x - ky + z = 0\},
\]
which concludes the proof. □

Proposition 2.9. Let \(\Gamma \subset \text{PSL}(2, \mathbb{C})\) be a non-elementary group. Then \(\iota(\Gamma)\) does not have invariant subspaces in \(\mathbb{P}_{\mathbb{C}}^2\).

Proof. Let us assume that there is a complex line \(\ell\) invariant under \(\iota(\Gamma)\). By Bzout’s theorem \(\text{Ver} \cap \ell\) has either one or two points. From the following commutative diagram
\[
\begin{array}{c}
\mathbb{P}_{\mathbb{C}}^1 & \xrightarrow{\tau} & \mathbb{P}_{\mathbb{C}}^1 \\
\downarrow{\psi} & & \downarrow{\psi} \\
\text{Ver} & \xrightarrow{\iota \Gamma} & \text{Ver}
\end{array}
\]
where \(\tau \in \Gamma\), we deduce that \(\Gamma\) leaves \(\psi^{-1}(\text{Ver} \cap \ell)\) invariant. Therefore \(\Gamma\) is an elementary group, which is a contradiction, thus \(\iota(\Gamma)\) does not have invariant lines in \(\mathbb{P}_{\mathbb{C}}^2\). Finally, if there is a point \(p \in \mathbb{P}_{\mathbb{C}}^2\) fixed by \(\iota(\Gamma)\), then by Lemmas 2.7, 2.10 and 2.8, there is a sequence of distinct elements \((\gamma_m)_{m \in \mathbb{N}} \subset \Gamma\) and a pseudo-projective transformation \(\gamma \in \text{QP}(3, \mathbb{C})\) such that \(\iota \gamma_m \xrightarrow{m \to \infty} \gamma\) and \(\text{Ker}(\gamma)\) is a complex line.
not containing \( p \). Since \( p \) is invariant and outside \( \text{Ker}(\gamma) \) we conclude \( \{p\} = \text{Im}(\gamma) \). On the other hand, by Lemma 2.8 we deduce \( p \in \text{Ver} \). Therefore \( \Gamma \) is elementary, which is a contradiction.

The following theorem follows easily from the previous discussion.

**Theorem 2.10.** Let \( \Gamma \) be a discrete subgroup of \( \text{PSL}(2, \mathbb{C}) \). Then

\[
\mathbb{P}^2_\mathbb{C} \setminus \text{Eq}(\iota(\Gamma)) = \bigcup_{z \in \Lambda(\Gamma)} T_{\psi(z)}(\mathbb{P}^1_\mathbb{C}).
\]

Moreover \( \Omega_{\text{Kul}}(\iota(\Gamma)) = \text{Eq}(\iota(\Gamma)) \) is Kobayashi hyperbolic, pseudo-convex, and is the largest open set on which \( \Gamma \) acts properly discontinuously.

### 3. Complex Hyperbolic Groups Leaving Ver Invariant

In this section we characterize the subgroups of \( \text{PU}(2,1) \) that leave invariant a projective translation of the Veronese curve \( \text{Ver} \). We need some preliminary lemmas.

**Lemma 3.1.** Let \( B \) be a complex ball. Then

\[
\text{Aut}(BV) = \{ g \in \text{PSL}(3, \mathbb{C}) | g \in \iota(\text{PSL}(2, \mathbb{C})), gB = B \}
\]

is a semi-algebraic group.

**Proof.** Since \( \iota(\text{PSL}(2, \mathbb{C})) \) and \( \text{PU}(2,1) \) are simple Lie groups with trivial centers, we deduce that they are semi-algebraic groups (see [8]). Thus the sets

\[
\{ (g, h, gh) : g, h \in \text{Aut}(BV) \} \quad \text{and} \quad \{ (g, g^{-1}) : g \in \text{Aut}(BV) \}
\]

are semi-algebraic sets. Therefore \( \text{Aut}(BV) \) is a semi-algebraic group. \( \square \)

**Lemma 3.2.** Let \( \Gamma \subset \text{PSL}(2, \mathbb{C}) \) be a discrete non-elementary group such that \( \iota \Gamma \) leaves invariant a complex ball \( B \). Then:

1. The group \( \text{Aut}(BV) \) is a Lie group of positive dimension.
2. We have \( \psi(\Lambda(\Gamma)) \subset \text{Ver} \cap \partial B \).
3. Set \( C = \partial B \cap \text{Ver} \). Then the set \( \psi^{-1}(C) \) is an algebraic curve of degree at most four.
4. The group \( \psi^{-1}(\text{Aut}(BV)) \) can be conjugated to a subgroup of \( \text{Mob}(\mathbb{R}) \), where \( \text{Mob}(\mathbb{R}) = \{ \gamma \in \text{PSL}(2, \mathbb{C}) | \gamma(\mathbb{R} \cup \{\infty\}) = \mathbb{R} \cup \{\infty\} \} \).
5. The set \( \psi^{-1}(C) \) is a circle in the Riemann sphere.
6. The set \( C \) is an \( \mathbb{R} \)-circle, i.e. \( C = \gamma(\partial \mathbb{H}^2_\mathbb{C} \cap \mathbb{P}^2_\mathbb{R}) \), where \( \gamma \in \text{PSL}(3, \mathbb{C}) \) is some element satisfying \( \gamma(\mathbb{H}^2_\mathbb{C}) = B \).
7. The set \( \text{Ver} \cap (\mathbb{P}^2_\mathbb{C} \setminus \overline{B}) \) is non-empty.
8. The set \( \text{Ver} \cap B \) is non-empty.

**Proof.** Let us start by showing (1). Since \( \text{Aut}(BV) \) is semi-algebraic, we deduce that it is a Lie group with a finite number of connected components (see [8]). On the other hand, since \( \text{Aut}(BV) \) contains a discrete subgroup, we conclude \( \text{Aut}(BV) \) has positive dimension.

Now let us prove part (2). Let \( x \in \Lambda(\Gamma) \). Then there is a sequence \( (\gamma_n) \subset \Gamma \) of distinct elements such that \( \gamma_n \xrightarrow{m \to \infty} x \) uniformly on compact sets of \( \mathbb{C} \setminus \{x\} \).
Thus \( \gamma \) is a compact, connected and contains more than two points we conclude that \( \Lambda(1, 2z, z^2) = 0 \) is equivalent to

\[
(3.1) \quad a_{11} + 4Re(a_{12}z) + 2Re(a_{13}z^2) + a_{33}|z|^4 + 4|z|^2Re(a_{23}z) + 4|z|^2a_{22} = 0.
\]

Taking \( z = x + iy \) and \( a_{ij} = b_{ij} + ic_{ij} \), Equation (3.1) can be written as

\[
a_{11} + 4(b_{12}x - c_{12}y) + 2(b_{13}x^2 - y^2) - 2c_{13}xy + a_{33}(x^2 + y^2)^2 + 4(x^2 + y^2)(b_{23}x - c_{23}y) + 4(x^2 + y^2)a_{22} = 0,
\]

which proves the assertion.

Let us prove part (3). Since \( \iota^{-1} \operatorname{Aut}(BV) \) is a Lie group with positive dimension containing a non-elementary discrete subgroup, we deduce that (see [6]) \( \iota^{-1} \operatorname{Aut}(BV) \) can be conjugated either to \( \operatorname{PSL}(2, \mathbb{C}) \) or a subgroup of \( \operatorname{Mob}(\mathbb{R}) \). On the other hand, we know that \( \operatorname{PSL}(2, \mathbb{C}) \) acts transitively on the Riemann sphere, but \( \iota^{-1} \operatorname{Aut}(BV) \) leaves an algebraic curve invariant, plus a point, therefore \( \iota^{-1} \operatorname{Aut}(BV) \) is conjugate to a subgroup of \( \operatorname{Mob}(\mathbb{R}) \), which concludes the proof.

Let us prove part (4). We know that \( C \) is \( \operatorname{Aut}(BV) \)-invariant and by part (3) of the present lemma \( \psi^{-1}C \) is an algebraic curve. Thus by Montel’s Lemma we conclude that \( \Lambda_{Gr} \psi^{-1}C \subset \psi^{-1}C \), where \( \Lambda_{Gr} \psi^{-1} \operatorname{Aut}(BV) \) is the Greenberg limit set of \( \iota^{-1} \operatorname{Aut}(BV) \), see [6]. Finally by part (3), we know that \( \iota^{-1} \operatorname{Aut}(BV) \) is conjugate to a subgroup of \( \operatorname{Mob}(\mathbb{R}) \), therefore \( \Lambda_{Gr} \iota^{-1} \operatorname{Aut}(BV) \) is a circle in the Riemann sphere and \( \Lambda_{Gr} \iota^{-1} \operatorname{Aut}(BV) = \psi^{-1}C \).

In order to prove part (5), observe that after a projective change of coordinates we can assume that \( \psi^{-1}C = \mathbb{R} \). Thus \( C = \psi \mathbb{R} = \{[z^2, 2zw, w^2] : z, w \in \mathbb{R}, |a| + |b| \neq 0 \} \). The following claim concludes the proof.

Claim. The sets \( C \) and \( \partial \mathbb{H}^3_R \) are projectively equivalent. Let \( \gamma \in \operatorname{PSL}(3, \mathbb{R}) \) be the projective transformation induced by:

\[
\hat{\gamma} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.
\]

Given \( [p] = [x^2, 2xy, y^2] \in C \), we get \( \gamma(p) = (x^2 - y^2, 2xy, x^2 + y^2) \) and

\[
(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2.
\]

Thus \( \gamma C \subset \partial \mathbb{H}^3_R \). Since \( C \) is a compact, connected and contains more than two points we conclude that \( \gamma \) is a projective equivalence between \( C \) and \( \partial \mathbb{H}^3_R \).
Now we prove part (7). Let \( x \in B \). Then \( x^\perp \) is a complex line in \( \mathbb{P}^2_\mathbb{C} \setminus \hat{B} \); by Bzout’s theorem we know \( \text{Ver} \cap x^\perp \) is non-empty, thus \( \text{Ver} \cap (\mathbb{P}^2_\mathbb{C} \setminus \hat{B}) \neq \emptyset \).

Finally, let us prove part (8). After conjugating by an element in \( \iota \text{PSL}(3, \mathbb{C}) \) we can assume that \( [0, 0, 1] \notin \partial B \). Let \( A = (a_{ij}) \) be the Hermitian matrix introduced in part (3) of the present lemma. Clearly \( a_{33} \neq 0 \). Now let \( F : \mathbb{R}^2 \to \mathbb{R} \) be given by

\[
F(x, y) = a_{11} + 4(b_{12}x - c_{12}y) + 2(b_{13}(x^2 - y^2) - 2c_{13}xy) + a_{23}(x^2 + y^2)^2 + 4(x^2 + y^2)(b_{23}x - c_{23}y + a_{22}).
\]

Thus by part (3) of this lemma we know \( \psi^{-1}C = F^{-1}0 \) is a circle. Moreover

\[
iF^{-1}\mathbb{R}^+ = \text{Ver} \cap \mathbb{P}^2_\mathbb{C} \setminus \hat{B},
\]

\[
iF^{-1}\mathbb{R}^- = \text{Ver} \cap B,
\]

\[
iF^{-1}0 = \text{Ver} \cap \partial B.
\]

If \( \text{Ver} \cap B = \emptyset \), then \( F(x, y) \geq 0 \). A straightforward calculation shows

\[
\triangle F(x, y) = 16(a_{33}(x^2 + y^2) + a_{22} + 2b_{23}x - 2c_{23}y).
\]

Thus \( E = \{(x, y) \in \mathbb{R}^2 : \triangle F(x, y) = 0\} \) is an ellipse.

Claim: We have \( \psi^{-1}C \cap \text{Int}(E) = \emptyset \). On the contrary let us assume that there is an \( x \in \mathbb{C} \cap \text{Int}(E) \neq \emptyset \). Then there is an open neighbourhood \( U \) of \( x \) contained in \( \text{Int}(E) \). Thus \( \triangle F(x, y) \) is negative on \( U \), i.e. \( F \) is super-harmonic on \( U \). However, \( F \) attains its minimum in \( U \), which is a contradiction.

From the previous claim we conclude \( C \) is contained in the closure of \( \text{Ext}(E) \), therefore \( \triangle F(x, y) \leq 0 \) in \( \text{Int}(\psi^{-1}C) \). As a consequence, \( F \) is subharmonic in \( \text{Int}(\psi^{-1}C) \). Let \( c \) be the centre of \( \psi^{-1}C \) and \( r \) its radius. Let \( (r_n) \) be a strictly increasing sequence of positive numbers such that \( r_n \xrightarrow{n \to \infty} r \). Let \( x_n \in B_{r_n}(c) \) be such that

\[
F(x_n) = \max\{F(x) : x \in B_{r_n}(c)\}.
\]

Since \( F \) is subharmonic in \( B_{r_n}(c) \) we conclude \( x_n \in \partial B_{r_n}(c) \) and \( (F(x_n)) \) is a strictly increasing sequence of positive numbers. Since \( \text{Int}(\psi^{-1}C) \cup \psi^{-1}C \) is a compact set, we can assume \( x_n \xrightarrow{n \to \infty} x \), and clearly \( x \in \psi^{-1}C \). On the other hand, since \( F \) is continuous we conclude \( F(x_n) \to F(x) = 0 \), which is a contradiction. \( \square \)

**Corollary 3.3.** There is a \( \gamma_0 \in \text{PSL}(3, \mathbb{R}) \) such that

1. \( \gamma_0\text{PSL}(2, \mathbb{R})\gamma_0^{-1} = \text{PO}^+(2, 1) \), where \( \text{PO}^+(2, 1) \) is the principal connected component of \( \text{PO}(2, 1) \),

2. \( \gamma_0\text{Ver} \cap \mathbb{H}^2_\mathbb{C} \) is non-empty and \( \text{PO}(2, 1)^+ \)-invariant.

**Proof.** Let us prove (1). By Lemma 3.2 we have that \( \iota\text{PSL}(2, \mathbb{R}) \) is a Lie group of dimension three and preserves the quadric in \( \mathbb{P}^2_\mathbb{R} \) given by

\[
\{[w^2, 2wz, z^2] : z, w \in \mathbb{R}\}.
\]

Thus there is a \( \gamma_0 \) in \( \text{PSL}(3, \mathbb{R}) \) such that \( \gamma_0M\delta b(\mathbb{R})\gamma_0^{-1} \) preserves

\[
\{[x, w, z] \in \mathbb{P}^2_\mathbb{R} : |y|^2 + |w|^2 < |x|^2\}.
\]

Hence \( \gamma_0\text{PSL}(2, \mathbb{R})\gamma_0^{-1} = \text{PO}^+(2, 1) \). Part (2) is now trivial. \( \square \)

**Theorem 3.4.** Let \( \Gamma \subset \text{PSL}(2, \mathbb{C}) \) be a discrete non-elementary group. The group \( \iota\Gamma \) is complex hyperbolic if and only if \( \Gamma \) is Fuchsian, i.e. a subgroup of \( \text{PSL}(2, \mathbb{R}) \).
that $\Gamma$ preserves a circle $C$.

Therefore

Assume that $\Gamma$ preserves a complex ball $B$. Then by Lemma 3.2 we deduce that $\Gamma$ preserves a circle $C$ in the Riemann sphere. Let $B^+$ and $B^-$ be the connected components of $\mathbb{P}^1_{\mathbb{C}} \setminus C$ and assume that there is a $\tau \in \Gamma$ such that $\tau(B)^+ = B^-$. Let $x \in \text{Ver} \cap B$ and denote by $Aut^+(BV)$ the principal connected component of $Aut(BV)$ which contains the identity. Then by Lemma 3.2 we deduce

$$Aut^+(BV)x = \psi x^{-1}Aut(BV)\psi^{-1}x = \psi(B^+)$$

and

$$Aut^+(BV)\tau(x) = \psi x^{-1}Aut(BV)\tau\psi^{-1}x = \psi(B^-).$$

Therefore

$$\text{Ver} = Aut^+(BV)x \cup Aut^+(BV)\tau(x) \cup C \subset \mathbb{H}^2,$$

which is a contradiction. Clearly, this concludes the proof.

We arrive at the following theorem:

**Theorem 3.5.** Let $\Gamma \subset \text{PSL}(2, \mathbb{C})$. Then the following claims are equivalent:

1. The group $\Gamma$ is Fuchsian.
2. The group $i\Gamma$ is complex hyperbolic.
3. The group $i\Gamma$ is $\mathbb{R}$-Fuchsian.

4. **Subgroups of PSL(3, $\mathbb{R}$) that Leave Invariant a Veronese Curve**

In this section we characterize those subgroups of $\text{PSL}(3, \mathbb{R})$ which leave invariant a projective copy of $\text{Ver}$.

**Theorem 4.1.** Let $\Gamma \subset \text{PSL}(2, \mathbb{C})$ be a discrete subgroup. Then the following facts are equivalent:

1. The group $\Gamma$ is conjugate to a subgroup of $\text{Mob}(\mathbb{R})$.
2. The group $i\Gamma$ is conjugate to a subgroup of $\text{PSL}(3, \mathbb{R})$.

**Proof.** Let $\Gamma$ be a subgroup of $\text{Mob}(\mathbb{R})$ and $\gamma \in \Gamma$. Then

$$\gamma = \left[ \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right] \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right]$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. A straightforward calculation shows that

$$i\gamma = \left[ \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right] \left[ \begin{array}{ccc} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{array} \right],$$

therefore $i\Gamma \subset \text{PSL}(3, \mathbb{R})$.

Let us assume that there is a real projective space $\mathbb{P}$ which is $i\Gamma$-invariant. Thus, as in Lemma 3.1, we conclude that

$$\text{Aut}(PV) = i\text{PSL}(2, \mathbb{C}) \cap \{ g \in \text{PSL}(3, \mathbb{C}) | g\mathbb{P} = \mathbb{P} \}$$

is a semi-algebraic group. Since $\Gamma \subset i^{-1}\text{Aut}(PV)$, we conclude that $i^{-1}\text{Aut}(PV)$ is a Lie group with positive dimension. From the classification of Lie subgroups of $\text{PSL}(2, \mathbb{C})$ (see [3]), we deduce that $i^{-1}\text{Aut}(PV)$ is either conjugate to $\text{Mob}(\mathbb{C})$ or a subgroup of $\text{Mob}(\mathbb{R})$. In order to conclude the proof, observe that the group $i^{-1}\text{Aut}(PV)$ can not be conjugated to $\text{Mob}(\mathbb{C})$. In fact, assume on the contrary that $i^{-1}\text{Aut}(PV) = \text{Mob}(\mathbb{C})$. Since $\text{Mob}(\mathbb{C})$ acts transitively on $\mathbb{C}$ we deduce that $\text{Aut}(PV)$ acts transitively on $\text{Ver}$. Finally, since $\psi(\Lambda(\Gamma)) \subset \text{Ver} \cap \mathbb{P}$, we deduce $\text{Ver} \subset \mathbb{P}$, which is a contradiction. \qed
5. **Examples of Kleinian Groups with Infinite Lines in General Position**

Let us introduce the following projection, see [9]. For each \( z \in \mathbb{C}^3 \) let \( \eta \) be the function satisfying \( \eta(z)^2 = - \langle z, z \rangle \) and consider the projection \( \Pi : \mathbb{R}^2 \to \mathbb{R}^2 \) given by
\[
\Pi([z_1, z_2, z_3]) = [\eta(z_1, z_2, z_3)(z_1, z_2, z_3) + \eta(z_1, z_2, z_3)(\overline{z_1}, \overline{z_2}, \overline{z_3})].
\]

**Lemma 5.1.** *The projection \( \Pi \) is PO(2, 1)-equivariant.*

**Proof.** Let \( A \in O(2, 1) \) and \( [z] \in \mathbb{R}^2 \). Then
\[
\Pi[Az] = \overline{\eta(Az)Az + \eta(Az)\overline{Az}}
= [\sqrt{-} < Az, Az > Az + \sqrt{-} < Az, Az > Az]
= [\sqrt{-} < z, z > Az + \sqrt{-} < z, z > Az]
= [A] \eta(z)z + \eta(z)\overline{z}
= [A] \Pi[z].
\]

For simplicity in the notation, in the rest of this article we will write \( Ver \) instead of \( \gamma_0(Ver) \), \( \psi \) instead of \( \gamma_0 \circ \psi \), and \( \gamma_0(\cdot) \gamma_0^{-1} \) instead of \( \iota(\cdot) \), where \( \gamma_0 \) is the element given in Corollary 5.3.

**Lemma 5.2.** *The map \( \Pi : Ver \cap \mathbb{R}^2 \to \mathbb{R}^2 \) is a homeomorphism.*

**Proof.** Let us prove that the map is onto. Let \( x \in \mathbb{R}^+ \cup \mathbb{R}^- \) be such that \( \psi(x) \in Ver \cap \mathbb{R}^2 \). Then
\[
\mathbb{R}^2 = \text{PSO}^+(2, 1)\Pi(\psi x)
= \Pi(\text{PSO}^+(2, 1)\psi x)
= \Pi(\iota \text{PSL}(2, \mathbb{R}))(\psi(x))
= \Pi(Ver \cap \mathbb{R}^2).
\]

Finally, let us prove that our map is injective. On the contrary, let us assume that there are \( x, y \in Ver \cap \mathbb{R}^2 \) such that \( \Pi(x) = \Pi(y) \). Now define
\[
H_x = \text{Isot}(\text{PSL}(2, \mathbb{R}), \psi^{-1}x),
H_y = \text{Isot}(\text{PSL}(2, \mathbb{R}), \psi^{-1}y).
\]

Clearly \( H_y \) and \( H_x \) are groups where each element is elliptic. On the other hand, observe that
\[
\iota H_x \Pi(x) = \Pi \iota H_x(x) = \Pi(x) \text{ and } \iota H_y \Pi(y) = \Pi \iota H_y(y) = \Pi(y).
\]

Therefore
\[
\iota H_x \cup \iota H_y \subset \text{Isot}(\text{PO}^+(2, 1), \Pi x).
\]

Since \( \Pi(x) \in \mathbb{R}^2 \), we deduce that \( \text{Isot}(\text{PO}^+(2, 1), \Pi x) \) is a Lie group where each element is elliptic. Therefore \( H = \iota^{-1} \text{Isot}(\text{PO}^+(2, 1), \Pi x) \gamma_0 \) is a Lie subgroup of \( \text{PSL}(2, \mathbb{R}) \) where each element is elliptic and \( H_y \cup H_x \subset H \). From the classification of Lie subgroups of \( \text{PSL}(2, \mathbb{C}) \), we deduce that \( H \) is conjugate to a subgroup of \( \text{Rot}_\infty \). Hence \( H_y = H_x \) and so \( x = y \). \( \square \)

**Lemma 5.3.** *Let \( \Gamma \subset \text{PSL}(2, \mathbb{C}) \) be a discrete group. Then \( \Gamma \) is conjugate to a subgroup \( \Sigma \) of \( \text{PSL}(2, \mathbb{R}) \) such that \( \mathbb{H}/\Sigma \) is a compact Riemann surface if and only if \( \iota \Gamma \) is conjugate to a discrete compact surface group of \( \text{PO}^+(2, 1) \).*
Proof. Let $\Gamma \subset PSL(2, \mathbb{R})$ be a subgroup acting properly, discontinuously, freely, and with compact quotient on $\mathbb{H}^+$. Let $R$ be a fundamental region for the action of $\Gamma$ on $\mathbb{H}^+$. We may assume without loss of generality that $\psi(R) \subset \mathbb{H}_\mathbb{C}^2$. Thus $\Pi\psi R$ is a compact subset of $\mathbb{H}_\mathbb{C}^2$ satisfying $\iota\Pi\psi R = \mathbb{H}_\mathbb{C}^2$ which shows that $\iota\Gamma$ is a discrete compact surface group of $PO^+(2, 1)$.

Now, consider the following commutative diagram

\[
\begin{array}{cccccc}
\mathbb{H}_\mathbb{C}^2 & \xrightarrow{\Pi^{-1}} & \text{Ver} \cap \mathbb{H}_\mathbb{C}^2 & \xrightarrow{\psi^{-1}} & \mathbb{H}^+ \\
q_1 \downarrow & & q_2 \downarrow & & q_3 \\
\mathbb{H}_\mathbb{R}^2/\iota\Gamma & \xrightarrow{\tilde{\Pi}} & (\text{Ver} \cap \mathbb{H}_\mathbb{C}^2)/\iota\Gamma & \xrightarrow{\tilde{\psi}} & \mathbb{H}^+ /\Gamma
\end{array}
\]

where $q_1, q_2, q_3$ are the quotient maps, $\tilde{\Pi}(x) = q_2\Pi^{-1}q_1^{-1}x$, and $\tilde{\psi}(x) = q_3\psi^{-1}q_2^{-1}(x)$. By Lemma 5.2 we conclude that $\mathbb{H}_\mathbb{R}^2/\iota\Gamma, (\text{Ver} \cap \mathbb{H}_\mathbb{C}^2)/\iota\Gamma, \mathbb{H}^+ /\Gamma$ are homeomorphic compact surfaces, which concludes the proof. \hfill \Box

Proof of theorem 0.2

Proof. If $\Gamma \subset PO^+(2, 1)$ is a discrete compact surface group, then by Lemma 3.3 we can assume that there is a $\Sigma \subset PSL(2, \mathbb{R})$ such that $\iota\Sigma = \Gamma$. By Theorem 5.3 we know that $\mathbb{H} /\Sigma$ is a compact Riemann surface. From the classic theory of quasi-conformal maps, see [2, 3], it is known that there is a sequence of quasi-conformal maps $(q_n : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}})$ such that $q_n \xrightarrow{n \rightarrow \infty} Id$ and $\Sigma_n = q_n\Sigma q_n^{-1}$ is a quasi-Fuchsian group, which can not be conjugated to a Fuchsian one. In consequence $\Gamma_n = \gamma_0\Sigma_n\gamma_0^{-1}$ is complex Kleinian and neither conjugate to a subgroup of $PU(2, 1)$ nor $PSL(3, \mathbb{R})$, which concludes the proof. \hfill \Box

Now the following result is trivial.

Corollary 5.4. There are complex Kleinian groups acting on $\mathbb{P}_\mathbb{C}^2$ which are not conjugate to either a complex hyperbolic group or a virtually affine group.

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