Geometrical quench and dynamical quantum phase transition in the $\alpha - T_3$ lattice

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We investigate quantum quenches and the Loschmidt echo in the two dimensional, three band $\alpha - T_3$ model, a close descendant of the dice lattice. By adding a chemical potential to the central site, the integral of the Berry curvature of the bands in different valleys is continuously tunable by the ratio of the hopping integrals between the sublattices. By investigating one and two filled bands, we find that dynamical quantum phase transition (DQPT), i.e. nonanalytical temporal behaviour in the rate function of the return amplitude, occurs for a certain range of parameters, independent of the band filling. By focusing on the effective low energy description of the model, we find that DQPTs happen not only in the time derivative of the rate function, which is a common feature in two dimensional models, but in the rate function itself. This feature is not related to the change of topological properties of the system during the quench, but rather follows from population inversion for all momenta. This is accompanied by the appearance of dynamical vortices in the time-momentum space of the Pancharatnam geometric phase. The positions of the vortices form an infinite vortex ladder, i.e. a macroscopic phase structure, which allows us to identify the dynamical phases that are separated by the DQPT.

I. INTRODUCTION

The swift progress of measurement techniques, especially ultra-cold atoms in optical lattices,\textsuperscript{12}, have opened the way for the experimental study of real-time dynamics of closed quantum many-body systems. This includes the observation of exotic phenomena such as prethermalization,\textsuperscript{13} many body localization,\textsuperscript{14} particle-antiparticle production in gauge theories.\textsuperscript{15} Motivated by these developments a great amount of theoretical works concerning nonequilibrium phenomena have been conducted in the recent years. One of the more popular method for driving a system out of equilibrium is a sudden quantum quench. In a quantum quench protocol a closed quantum system is prepared in an eigenstate $\ket{\Psi_0}$ of some initial Hamiltonian $H_0$ and then have the system dynamically evolve in time under a different Hamiltonian $H$. An approach to understand dynamics after a quantum quench is to study the behaviour of the return amplitude

$$G(t) = \bra{\Psi_0} \Psi(t) \ket = \bra{\Psi_0} e^{-iHt} \ket{\Psi_0}. \tag{1}$$

There is a formal similarity between Eq. (1) and the partition function $Z = \text{Tr}(e^{-\beta H})$, where $\beta$ is the inverse temperature, which was first pointed out in Ref.\textsuperscript{16}. Based on this formal similarity the theory of dynamical quantum phase transitions (DQPT) have emerged\textsuperscript{17}.

By statistical mechanics the information of a system’s thermodynamical properties is contained within the partition function $Z$. The free energy density of the system is calculated from the partition function as $f = F/N = -(\beta N)^{-1} \ln Z$ with $N$ being the degrees of freedom of the system. A phase transition occurs when $f$ behaves nonanalytically as a function of a control parameter, such as the temperature or external magnetic field. Obviously, in a non-equilibrium situation, the partition function cannot be formulated in the conventional sense. The theory of DQPTs shifts the formal role of the partition function to the complex Loschmidt amplitude $G(t)$. A dynamical quantum phase transition is then defined as the nonanalytic behaviour as a function of time in the dynamical counterpart of the free energy density: $g(t) = -\ln G(t)$, thus, mathematically a DQPT occurs at critical times ($t^*$), whenever $G(t^*) = 0$. Physically, the return amplitude is related to the work done during the quench, which in principle makes DQPTs a measureable phenomenon.

Recently, dynamical quantum phase transitions were observed in various experimental platforms including quantum simulators,\textsuperscript{18} nanomechanical systems,\textsuperscript{19} and single photon resonators.\textsuperscript{20}

The underlying theme of identifying DQPTs is to investigate under when the condition $G(t) = 0$ holds. To this end, the zeros of the return amplitude, referred to as Fisher zeros, are studied in the complex plane: $G(z_n) = \bra{\Psi_0} e^{-Hz_n} \ket{\Psi_0} = 0$, where $z_n \in \mathbb{C}$. In the thermodynamic limit the Fisher zeros coalesce into lines in low dimensions or areas in higher dimensions. The final task is to identify conditions for the Fisher zeros to cross the imaginary axis. DQPTs have been found in both integrable\textsuperscript{21} and nonintegrable\textsuperscript{22} spin systems for quenches across quantum critical points. However, there are also reports of DQPTs that occur when a parameter is not quenched accross a quantum critical point,\textsuperscript{14,15} denying the one-to-one correspondence to conventional phase transitions.\textsuperscript{23}

Furthermore, the appearance of DQPTs in topological band models are also being actively researched, however these are often limited to two band models. In one dimensional topological systems with two bands the number of topologically protected DQPTs are determined by the difference between the initial and final winding num-
In two dimensions the situation is much more complicated. If the modulus of the Chern number of the bands changes under the quench, DQPTs necessarily has to occur, however DQPTs arise even when the Chern numbers are the same. For quantum quenches between gapped phases in a generic multiband system, a robust testing features of this model are that one of the bands is always a zero energy flat band and the integral of the Berry curvature for the other bands of the gapped lattice is independent from the valley index.

The purpose of this paper is to investigate the occurrence of DQPTs in a model that has more than two bands, that can also support different topological phases. To this end we introduce geometrical quench scenarios in the \( \alpha - T_3 \) lattice, depicted in Fig. 1. The main interesting features of this model are that one of the bands is always a zero energy flat band and the integral of the Berry curvature for the other bands of the gapped \( \alpha - T_3 \) lattice depend continuously on the ratio of the sublattice hopping amplitudes. In principle an infinitesimal change in this parameter during the quench results in a different topological phase of the system. By studying different quench scenarios we can deduce how deep the connections between DQPTs and topology grow.

II. THE MODEL

The \( \alpha - T_3 \) lattice is a two dimensional structure whose Bravais lattice is a triangular lattice and the unit cell contains three atoms. The basis includes two rim sites \( (A, B) \) that are connected to a hub site \( (H) \) with hopping amplitudes \( t \) and \( t' \), see Fig. 1. Within the tight binding picture, the \( \alpha - T_3 \) lattice has three bands: a zero energy flat band and two other bands that touch each other at the corners of the hexagonal shaped first Brillouin zone. Similar to graphene, the energy bands are linear near these points and thus the system can be described around the Dirac points with a low-energy effective pseudospin-1 Hamiltonian. By adding a local chemical potential to the central hub site, the low energy Hamiltonian reads as:

\[
H_\xi(q) = \begin{pmatrix}
0 & v_F q_\xi \\
v_F q_\xi & \mu & \alpha v_F q_\xi \\
0 & \alpha v_F q_\xi & 0
\end{pmatrix}.
\]  

Here \( v_F = 3at/2 \) is the Fermi velocity, with \( a \) being the lattice constant, \( \alpha = t'/t \) is the ratio of the hopping amplitudes. The momentum dependence is given in polar coordinates with \( \xi = \pm 1 \) valley index: \( q_\xi = q_x - i q_y = q \exp(-i\vartheta_q) \), where \( q = \sqrt{q_x^2 + q_y^2} \), \( \vartheta_q = \arctan(q_y/q_x) \). Due to the addition of the local chemical potential \( \mu \) to the hub site, the system becomes gapped and develops unique topology. For the Hamiltonian matrix in Eq. 2 the eigenvalues and eigenfunctions are:

\[
E_0(q) = 0, \quad E_{1,2}(q) = \frac{\mu}{2} \pm \sqrt{\frac{\mu^2}{4} + v_F^2(1 + \alpha^2)q^2},
\]

\[
|\psi_0(q)\rangle = \begin{pmatrix}
1 \\
1
\end{pmatrix} \sqrt{1 + \alpha^2} \begin{pmatrix}
-\alpha e^{-i\vartheta_q} \\
0
\end{pmatrix},
\]

\[
|\psi_{1,2}(q)\rangle = \begin{pmatrix}
1 \\
1
\end{pmatrix} \sqrt{E_{1,2}(q)^2 + v_F^2(1 + \alpha^2)q^2} \begin{pmatrix}
v_F q_\xi \\
\alpha v_F q_\xi
\end{pmatrix}.
\]

Using Eq. 3 the corresponding integrals of the Berry curvature for the bands can be calculated as:

\[
C_0 = 0, \quad C_{1,2} = \frac{\xi(\alpha^2 - 1)}{\alpha^2 + 1} (1 - \delta_{\mu 0}).
\]

The topological number for the flat band is zero, while for the dispersive bands, \( C_1 + C_2 = 0 \) always. However, in each valley, the integrals of the Berry curvature continuously depend on the hopping ratio \( \alpha \). This means that around a single valley the electrons occupy bands that can possess non-integer integrals of the Berry curvature, more precisely any real number in the \([-1, 1]\) interval depending on the hopping ratio, although there is a finite gap in the system. By considering both valleys, the non-integer integrals of the Berry curvature for the bands in each valley add up to integer Chern numbers for the bands. As we will see, since the quench dynamics are independent from the valley index \( \xi \), it is satisfactory to focus on a single valley. Henceforward, we are introducing the different quench scenarios in the \( \alpha - T_3 \) model and deduce conditions for the appearance of DQPTs.

Quench protocols. As the quench state we are using the fully filled low energy band \(|\psi_{m}(q)\rangle = |\psi_2(q)\rangle\) with starting parameters \( \mu_0 \) and \( \alpha_0 \). The post quench Hamiltonian that governs the time evolution has parameters
μ and α. In this way the (μ₀, α₀) → (μ, α) quench covers a variety of topological phases, which gives an ideal ground for understanding the connection between topology and DQPTs in this system. The return amplitude is

\[ G(t) = \prod_q G(q, t) \]

with

\[ G(q, t) = \sum_{j=0,1,2} |\langle \psi_j(q) | \psi_{in}(q) \rangle|^2 e^{-iE_j(q)t}. \quad (5) \]

Hereafter, we use \( p_j = |\langle \psi_j(q) | \psi_{in}(q) \rangle|^2 \) for the overlaps between the prequench state and all of the postquench Hamiltonian eigenfunctions, which is the probability of a given single particle state being occupied after the quench. The return amplitude is a product with respect to the momentum q, thus it is zero whenever \( G(q, t) = 0 \). Since \( p_j \geq 0 \) for all q, Eq. (5) can be interpreted as a sum of complex numbers with magnitude \( p_j \), and phase \(-E_j t\). The sum is zero whenever the complex numbers \( p_j e^{-iE_j t} \) form a closed polygon on the complex plane at times \( t^* \). To form a polygon the overlaps must obey the triangle inequality, which combined with the fact that \( \sum_j p_j = 1 \) results in a condition for DQPTs to occur, specifically:

\[ \exists q \neq 0 : p_j(q) \leq \frac{1}{2}, \quad \forall j. \quad (6) \]

For the overlaps \{\( p_j \)\} that satisfy Eq. (6) there exists solutions for the equation \( \sum_j p_j e^{-i\varphi_j} = 0 \) with some \( \varphi_j \) phases. The only remaining question is to whether \( E_j t \) can evolve into these \( \varphi_j \) phases. The answer is affirmative, whenever the energy bands of the Hamiltonian are rationally independent \cite{20,27,28}. In our case if \( \mu \neq 0 \), i.e. the gap is not closed during the quench, this phase condition is automatically satisfied, and as such the only condition for the occurrence of a DQPT is the time independent condition for the overlaps in Eq. (6). The presence of the flat band is invariable in satisfying Eq. (6). As we will see a proportion of the electrons are frozen into the flat band after the quench and the dynamics play out only on the remaining part of the system. This greatly increases the chance for the band populations to satisfy Eq. (6).

Due to rotational invariance of the low energy Hamiltonian in Eq. (2), the overlaps only depend on the magnitude of the momentum \( |q| = q \). The dynamical counterpart of the free energy, the rate function can be calculated, using Eq. (5) as

\[ g(t) = -\lim_{q_F \to \infty} \frac{1}{\sqrt{q_F}} \int_0^{q_F} dq \ln G(q, t), \quad (7) \]

where we introduced a high energy cutoff \( q_F \) in momentum space. The nonanalytic behaviour of the rate function is still accessible within Eq. (7).

We argue that having two filled bands initially does not differ much from having a single filled band. Preparing initially the ground state with some given chemical potential could mean that the flat band is also occupied. We argue that calculating with the two filled bands scenario would yield identical results to only one filled band.

The two filled bands can be considered as starting from the the fully filled state (all bands are filled) and annihilating electrons from one band for all momenta to reach one empty band. Then, there is one empty hole band, which can be treated mathematically identically to having a single filled electron band. For our specific setting in the \( \alpha = T_3 \), we have checked that all of our results, obtained for the single filled lowest energy band, remain intact for the case of two filled bands, i.e. when the flat band is also occupied, and only the high energy band is empty.

### III. GAPLESS TO GAPPED QUENCH SCENARIO

First, we discuss the case when the prequench Hamiltonian has no gap \( (\mu_0 = 0) \) and during the quench a gap opens \( (\mu \neq 0) \). As mentioned above, in this case the energies of the postquench bands are rationally independent, so we only need to analyze the overlap functions and their properties:

\[ p_0 = \frac{(\alpha_0 - \alpha)^2}{2(1 + \alpha^2)(1 + \alpha_0^2)}, \]

\[ p_{1,2} = \frac{-E_{1,2} \sqrt{1 + \alpha_0^2 + v_F(1 + \alpha_0 q)^2}}{2(1 + \alpha_0^2)(1 + \alpha^2)(1 + \alpha^2)q^2}. \quad (8) \]

In Eq. (8), the flat band overlap is a constant, furthermore \( p_0 < 1/2 \). The \( p_0 \) proportion of the electrons are frozen into the flat band after the quench. Since \( p_1 + p_2 = 1 - p_0 \) is also a constant, their derivatives with respect to \( q \) have opposite signs: \( \partial p_1/\partial q = -\partial p_2/\partial q \). This is important because we notice that \( p_1(q = 0) = 1/2 \) and \( \partial p_1/\partial q < 0, \forall q \), consequently \( p_1 < 1/2, \forall q > 0 \). The determining factor is hence the overlap between the low energy bands of the pre- and postquench Hamiltonians. For \( p_2 \) is an increasing function of \( q \), and to observe a DQPT we need to satisfy the condition

\[ p_2(q \to 0) = \frac{(1 + \alpha \alpha_0)^2}{2(1 + \alpha^2)(1 + \alpha_0^2)} < \frac{1}{2}. \quad (9) \]

Without the presence of the flat band the condition in Eq. (6) is never satisfied and no DQPT would occur. Another observation here is that when \( \alpha = \alpha_0 \) there can be a momenta with magnitude \( q^* \) so that \( p_2(q^*) = 1/2 \) and the condition in Eq. (4) is satisfied in the momentum interval \((0, q^*)\). There is the possibility that \( q^* \) is infinite or \( p_2(q) < 1/2 \) for every \( q \), if that is the case the condition in Eq. (6) is satisfied for every momentum and as we will see, this changes the nature of the DQPT. In order to achieve this the hopping ratios must satisfy:

\[ \frac{1 + \alpha \alpha_0}{\sqrt{(1 + \alpha^2)(1 + \alpha_0^2)}} \leq \frac{1}{\delta_0}, \quad (10) \]
is symmetric to the interchange of $\alpha$ in the same DQPT. For example, if above Eq. (10) originates from the condition $p_2(\infty) \leq 1/2$, and is symmetric to the interchange of $\alpha \leftrightarrow \alpha_0$, raising or lowering the hopping ratio during the quench will result in the same DQPT. For example, if $\alpha_0 = 0$ we must raise $\alpha$ above $\alpha_c = \sqrt{2 + 2\sqrt{2}} \approx 2.2$ and there exists a finite $\alpha$ that satisfy inequality (11) whenever $\alpha_0 < 1/\sqrt{\delta_S - 1} \approx 0.45$.

When Eq. (11) is satisfied, the rate function itself exhibits nonanalytic behaviour, which is visible as kinks on the right hand side of Fig. 2. The rate function develops plateaus, whose value coincides with the fidelity29, namely the overlap between the groundstates of the pre- and postquench Hamiltonians. Its value is calculated from Eq. (10) by replacing $G(q,t)$ with $p_2$ only. This means that during times corresponding to the plateau, the dominant mode is $p_2$, therefore electrons prefer to reside in the ground state of the post quench Hamiltonian.

At times between the plateaus all the bands contribute to the rate function with nondominant weights and hence a DQPT occurs.

Whenever Eq. (11) is not satisfied the rate function quickly starts to oscillate around the fidelity and approaches it fast with increasing time. The DQPTs in this case only occur in the first derivative of the rate function as kinks which are visible in the left panel of Fig. 2. Previously, in two dimensional systems the nonanalyticities were observed only in the first derivative of the rate function21. Here, we demonstrated that the DQPTs can in fact be observable in the rate function in 2D.

In either case the quenched state is nontopological, i.e. its Chern number is zero. After the quench the available bands can have nontrivial topology. In both cases DQPTs do occur, however the nature of these DQPTs change for values of the parameter $\alpha$ according to (11), which has no connection to the topology of the bands. Why does this change happen? The simplest explanation is that, since the overlaps can be interpreted as occupation probabilities of the postquench bands, a population inversion occurs for the whole system. When there exists a $q^*$ such that $p_2(q^*) = 1/2$ only those states can suffer population inversion whose momentum is in the interval $(0, q^*)$. If there is no such $q^*$, that solves the equation $p_2(q^*) = 1/2$ or it is infinite then every state of the system partakes in the population inversion. This leads to time intervals when every band can participate in the dynamics with nondominant weights and thus DQPTs appear in the rate function itself.

A. Pancharatnam geometric phase and possible dynamical topological order parameter

The topological nature of DQPTs has been theorized20 by identifying a dynamical topological order parameter (DTOP), which has been connected to the Pancharatnam geometric phase of the return amplitude20. In a recent experimental observation of DQPT, the DTOP were identified by the number of dynamical vortices that were created and annihilated in the Brillouin zone at times of the DQPT. Recasting Eq. (10) into the following form:

$$G(q,t) = R_q(t)e^{i\varphi_q(t)},$$

the Pancharatnam geometric phase is available for study. According to rotational symmetry of the system at low energies, this phase only depends on the magnitude of the momentum: $\varphi_q(t) = \varphi(q,t)$. The contourplot of Fig. 3 reveals that in the time-momentum space $(t, q)$ vortices are created at finite times whenever the quench parameters obey inequality (11). We argue that the number of these vortices can be classified as a DTOP. As time passes and reaches the exact moment of a kink before a plateau in the rate function, a countable infinitely many vortices are created. For small momenta the vortices reside close to the middle of the time interval of the plateau. With increasing momentum, however, the vortices appear and disappear closer to the beginning and ending time of the plateau. The exact moment for the DQPT to appear is an accumulation point of the vortices in the momentum-time plane. Every vortex with positive circulation adds +1 to the DTOP, hence it becomes nonzero. By the time we are at the end of the plateau these vortices move to a different momenta and their circulation is shifted to −1. Thus at the exact moment of the cusp at the end of the plateau the sum of the number of vortices with respect to circulation becomes zero again.

As stated above whenever $\alpha_0 < 1/\sqrt{\delta_S - 1}$ there exists a finite $\alpha$ for which inequality (11) holds. With decreasing $\alpha$, it inevitably reaches the point in the $\alpha_0 - \alpha$ parameter space (Fig. 2) when (11) is not satisfied, and the vortices at the times of the different kinks merge together. We can interpret this with the argument that in this case the vortices are created at the infinite far points of $q \to \infty$ and they disappear at the infinite far points of $t \to \infty$. The system has at all times a nonzero unchanged DTOP and the rate function itself does not develop kinks and the DQPTs are only present in the first derivative.
B. Macroscopically different dynamical phases separated by the DQPTs

During a conventional equilibrium phase transition\textsuperscript{18}, the nonanalytic behaviour of the free energy density separates phases of the system, that have different macroscopical properties. The theory of DQPTs rely on the existence of nonanalyticities in the dynamical free energy as a function of time, however these are usually not accompanied by changes in the macroscopical properties.

Within the low energy discription of the present model, we can argue that the nonanalytic kinks in the rate function separates macroscopically different phase structures in the momentum-time plane. As stated above, during the time interval of a plateau, the vortices appear and disappear with the exact moment of the DQPT serving as an accumulation point for the positions of these vortices. Since the number of vortices is countable infinity, the developed ladder structure can be considered macroscopic. At the moment of DQPT this macroscopic structure appears or disappears. Hence the nonanalytic kink in the rate function heralds a macroscopical change in one of the properties of the system, making it a true dynamical phase transition.

C. Robustness of DQPTs in the rate function

The nonanalytic kinks that appear in the rate function follow directly from the continuous rotational symmetry of the system, valid at low energies. In this limit, the system behaves as an effective one dimensional system, explaining the kinks in the rate function itself. Moving away from the linearized Hamiltonian in Eq. (2) and including higher order terms of the momentum from the tight binding dispersion would lower the continuous rotational symmetry to discrete 3-fold rotational symmetry. In this case the vortices of Fig. 3 in the momentum-time space would distort and the vortex dynamics would become anisotropic in the $q_x - q_y$ plane as well. This would eventually lead the cusps at the beginning and the end of the plateaus to be rounded. The nonanalytic kinks would appear only in the first derivative of the rate function at times, when the rounding of the cusp begins and ends. Due to the lowering of the symmetry the structure of the Fisher zeroes would also change from being on a line to being on an area in the complex plane. However, if we stay within the realm of the low energy effective theory, such that $\mu$ is much smaller than the bandwidth, the rounding would be very small and any anisotropy in the vortex dynamics would be hardly observable. This is analogous to the effect of trigonal warping on the Fermi surface in graphene and its effect on physical observables such as the optical conductivity\textsuperscript{31,32}.

IV. QUENCH WITHIN THE GAPPED PHASE

The next scenario we consider is when there is a finite gap in the prequench Hamiltonian ($\mu_0 \neq 0$) and it is not closed during the quench ($\mu \neq 0$). As was the case before phase ergodicity holds and we only need to inspect the overlaps:

\[
p_0 = \frac{(\alpha_0 - \alpha)^2 v_F^2 q^2}{(1 + \alpha^2)(E_0^2 + v_F^2(1 + \alpha_0)^2)}, \\
p_{1,2} = \frac{(E_{1,2} E_0 + v_F^2 q^2(1 + \alpha_0)^2)}{(E_0^2 + v_F^2(1 + \alpha_0)^2)(E_{1,2} + v_F^2(1 + \alpha^2)q^2)},
\]

where $E_0 = \mu_0/2 - \sqrt{\mu_0^2/4 + v_F^2(1 + \alpha_0^2)q^2}$. The flat band overlap decreases with the momentum, $\partial p_0/\partial q \leq 0$ and $p_0(0) = 2p_0(\infty)$ with $p_0(\infty) < 1/2$. The high energy band overlap $p_1$ has two zeroes: $p_1(0) = p_1(q') = 0$ and increases with momentum on the $|q'|, \infty$ domain: $\partial p_1/\partial q > 0$, if $q < q'$. Due to $p_1(0) = 0$, we immediately notice that if $\alpha$ and $\alpha_0$ are close $p_2$ will always be above $1/2$, thus the conditions for DQPTs are not met. In the presence of a gap, the prequench state is topological and by changing the hopping ratio the available bands all have different integrals of the Berry curvature. We conclude that no DQPTs occur in spite that the quench reaches a different topological phase. However, there are
certain $\alpha_0$, $\alpha$ parameters for which nonanalytices can occur in the first derivative of the rate function, as well as in the rate function itself. This is shown in Fig. 4. In order to observe DQPTs in the rate function, the condition is $p_2(\alpha) \leq 1/2$, which translates into the same inequality for the hopping ratios as before. For hopping ratios $\alpha, \alpha_0$ that satisfy inequality the low energy band overlap of the pre- and postquench Hamiltonian $(p_2)$ is always below 1/2, and population inversion occurs for all momentum. This time however, there exists a momentum $q^*$, with $p_0(q^*) = 1/2$ and the condition is satisfied for the interval $[q^*, \infty)$. Similarly the Pancharatnam geometric phase in the time-momentum space also develops pairwise vortices with different circulation giving rise to a DTOP. These vortices only appear for momentum larger than $q^*$.

![Image](image_url)

FIG. 4. The $\alpha_0 - \alpha$ parameter space, where the colors depict the different regions whether DQPTs cannot occur at all (white), DQPTs occur in the first derivative of the rate function only (blue) and DQPTs occur in the rate function itself (orange). The left panel shows the regions during a gapless to gapped quench scenario $\mu_0 = 0 \to \mu \neq 0$ and the right panel corresponds to quenches within the same gapped phase $\mu_0 = \mu$.

V. CLOSING THE GAP

Finally case we discuss the gapped to gapless quench, i.e. the closing of the gap during the quench: $\mu_0 \neq 0 \to \mu = 0$. In this scenario the postquench bands are rationally dependent: $E_0 + E_1 + E_2 = 0$. The return amplitude is the product with respect to the momentum $q$ of the quantity:

$$G(q, t) = p_0 + (p_1 + p_2) \cos E_1 t + i(p_2 - p_1) \sin E_1 t.$$  

The condition for DQPTs is $G(q, t^*) = 0$, for $q \neq 0$. In this case $p_2 \neq p_1$, $\forall q \neq 0$ and $t^*$ can be exactly calculated:

$$t^* = \left(\frac{2n + 1}{E_1(q^*)}\right),$$  

where $q^*$ is the momentum when $p_0(q^*) = 1/2$ and $p_0$ is given in Eq. (14). With some algebra we determine $q^*$ to be:

$$q^* = \frac{\mu_0}{2\nu_F \sqrt{1 + \alpha_0^2} \sqrt{(\alpha - \alpha_0)^2 - 1}}.$$  

(15)

This momentum has to be real, thus in order to observe a DQPT the hopping ratios must satisfy the following inequality:

$$|\alpha - \alpha_0| - \alpha \alpha_0 > 1.$$  

(16)

We note that the all postquench bands are nontopological and the prequench state has zero Chern number only when $\alpha_0 = 1$. According to (16) DQPTs cannot occur when $\alpha_0 = 1$, thus in this case changing the topology is a necessary but not sufficient condition.

VI. CONCLUSION

The $\alpha - T_3$ lattice model can be realized experimentally with cold fermionic atoms loaded into an optical lattice. Following the proposal for the optical dice lattice ($\alpha = 1$), one simply needs to dephase one of the three pairs of laserbeams to obtain $\alpha \neq 1$. Of late the detection of DQPTs have seen much success with different experimental platforms such as quantum simulators and nanomechanical systems$^{9,11,13}$. Furthermore, the recent experimental success of observing DQPTs using single photon resonators$^{14}$ can also allow the possibility to simulate the different quench scenarios in the Hamilton matrix of Eq. (2).

We investigated different geometrical quench scenarios for the $\alpha - T_3$ lattice model and studied the conditions for the occurrence of dynamical quantum phase transitions. Creating a gap by adding a chemical potential to the central, hub site of the lattice, develops the topology of the bands in a remarkable way. For a single valley, the band Chern numbers can be tuned continuously within the $[-1, 1]$ interval, depending on the hopping ratio $\alpha = t'/t$. Either using the fully filled low energy band or the fully filled low energy and flat band as the prequench state, yield the same identical results.

We showed that in a quench when a gap is opened, DQPTs occur whenever the hopping ratio is also quenched to a different value. Furthermore, the nature of the DQPTs change if the pre- and postquench ratios satisfy inequality. The change is attributed to a population inversion that occurs for every momentum in the system. This reasoning is strengthened by noticing that the same phenomena happens during a quench that maintains the gap. The Pancharatnam geometric phase of the rate function in the time-momentum space also reveals this change. Whenever inequality is satisfied vortices form pairwise and with different circulation, giving rise to kinks in the rate function itself. Interestingly, the positions of the vortices form an infinite vortex ladder, that can be interpreted as a macroscopic structure in the momentum-time plane that appears and disappears.
at a DQPT. For parameters that does not satisfy these vortices are absent at finite times and the DQPTs only occur in the first derivative of the rate function. Finally, closing the gap during a quench can also support DQPTs if the prequench state has nontrivial topology.

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