REDUCTION AND DUALITY IN GENERALIZED GEOMETRY

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Abstract. Extending our reduction construction in [15] to the Hamiltonian action of a Poisson Lie group, we show that generalized Kähler reduction exists even when only one generalized complex structure in the pair is preserved by the group action. We show that the constructions in string theory of the (geometrical) T-duality with H-fluxes for principle bundles naturally arise as reductions of factorizable Poisson Lie group actions. In particular, the group may be non-abelian.

1. Introduction

In this article, we propose a candidate of geometric realization of part of the ansatz of T-duality with H-flux in the physics literature, using reductions in generalized Kähler geometry. T-duality has long been intensively studied in physics and has made its marks in mathematics as well, e.g. via mirror symmetry [35]. The context of our reduction construction is the Hamiltonian Poisson action of Poisson Lie group. Classically, such reduction in symplectic category was first discussed in [26] and our construction here should be viewed as the generalization of it to generalized geometry.

Generalized geometry is introduced by Hitchin [13] in the context of generalized Calabi-Yau manifolds. The general theory of generalized complex and Kähler geometries is first developed by Gualtieri in his thesis [12]. Various reduction constructions in the context of generalized geometry are developed by [7,15,24,34,37]. The approach taken here follows the point of view of Hamiltonian symmetries [15].

It is by now well-known that a generalized complex structure induces a canonical Poisson structure, e.g. [1,14,12,15], also §3.9. Let G be a Poisson Lie group with dual group ˆG, then the Hamiltonian Poisson action with moment map as defined in [26] (also see definition 8.12) can be adapted to generalized complex geometry (definition 3.11), as well as generalized Kähler geometry (definition 3.17). We then have the first results on reduction:

Theorem [3.15,3.21]. Suppose (M,J) is an extended complex manifold with Hamiltonian G-action, whose moment map is µ : M → ˆG. Let M0 = µ−1(ˆe), where ˆe ∈ ˆG is the identity element. Suppose that ˆe is a regular value and the geometrical action of G is proper and free on M0. Then there is a natural extended complex structure on the reduced space Q = M0/G.

If furthermore, (M,J1,J2) is an extended Kähler manifold and the G-action is J1-Hamiltonian. Then there is a natural extended Kähler structure on the reduced space Q.

The notion extended (+ structures) is adopted to emphasize that we consider TM as an extension of TM by T∗M, instead of as a direct sum, with an exact Courant algebroid structure (cf. §2.8). When a splitting is chosen, or equivalently, TM is identified with TM with an H-twisted Courant algebroid structure, we will use the notion H-twisted generalized (+ structures). Now, when the action of G preserves a splitting of TM, then the reduced extended tangent bundle in the theorem naturally splits and the twisting form on Q can be explicitly written down (cf. corollary 7.3).

In investigating T-duality, we are guided by the detailed computation in [15] of the example of C2 \ {(0,0)} with non-trivial twisting class, which we recall in §5.10. The following definition is crucial:
Definition 4.1. Let $(\hat{\mathfrak{g}}, \mathfrak{g}, \mathfrak{h})$ be the Manin triple defined by a Poisson Lie group $G$ (cf. theorem 8.6, also 26), with dual group $\hat{G}$. The (infinitesimal) action of $\hat{\mathfrak{g}}$ on $M$ is bi-Hamiltonian if it is induced by a $\mathcal{F}_1$-Hamiltonian action of $G$ together with a $\mathcal{F}_2$-Hamiltonian action of $\hat{G}$.

Suppose that the Manin triple $(\hat{\mathfrak{g}}, \mathfrak{g}, \mathfrak{h})$ is the Lie algebras of the (local) double Lie group $(\hat{G}, G, \mathfrak{g})$ (cf. theorem 8.6). We impose two sets of assumptions, on the group $G$ (assumption 4.2 (0)) and on the action of $\hat{G}$ (the rest of assumption 4.2). Our first result in this direction is the factorizable reduction:

Theorem 4.6. Under assumption 4.2 and suppose that the action of $\hat{G}$ on $M_0$ is proper and free, then the reduced space $\hat{Q} = M_0/G$ of a bi-Hamiltonian action of factorizable Poisson Lie group admits a natural effective Courant algebroid (definition 7.4).

With further restrictions, i.e. the reduction exists with respect to either of the actions of $G$ and $\hat{G}$ as given in theorem 3.2, the factorizable reduction as in theorem 4.6 can be factored in two ways, $M_0 \xrightarrow{\mathcal{F}_G} Q \xrightarrow{\mathcal{F}_G} \hat{Q}$ or $M_0 \xrightarrow{\mathcal{F}_G} \hat{Q} \xrightarrow{\mathcal{F}_G} \hat{Q}$. We then propose

Definition 5.6. The extended Kähler structures on $Q$ and $\hat{Q}$ are Courant dual to each other.

We note that any of the groups $\hat{G}, G$ or $\hat{G}$ could be non-abelian. Thus we have a candidate for the non-abelian duality with background twistings. The more stringent but natural assumption that $G$ and $\hat{G}$ commute in $\hat{G}$ implies that $\hat{G}$ is in fact a torus $\hat{T}$. The choice of terminology in the above is supported by the following theorem when the action of $\hat{T}$ preserves a splitting of $TM$:

Theorem 5.8. After applying a natural $B$-transformation on $M$, which does not change the reduced Courant algebroid on $\hat{Q}$, the twisting forms $h$ and $\hat{h}$ of the structures on $Q$ and $\hat{Q}$ respectively satisfy:

$$\hat{\pi}^* h - \pi^* h = d(\hat{\Theta} \wedge \Theta),$$

where $\pi$ and $\hat{\pi}$ are the quotient maps and $\Theta$ and $\hat{\Theta}$ are connection forms of principle torus bundles.

We point out that the equation above appears as part of the definition of $T$-duality with $H$-flux of principle torus bundles in the literature (also see below). Here, it appears as a geometrical consequence. The notion of $T$-duality group in the literature can be recovered (4.1) with our construction.

We describe the content of the article in the following. It’s helpful to recall the basics of Lu’s construction (see also §5 appendix B). A Poisson Lie group $G$ is a Lie group with a multiplicative Poisson structure, i.e. $m : G \times G \to G$ is a Poisson map. Let $(M, \omega)$ be a symplectic manifold, the action of $G$ on $M$ is called Poisson if the map $G \times M \to M$ defining the action is Poisson, with the product Poisson structure on $G \times M$. In 26, Lu defined momentum mapping for such Poisson actions (see also definition 3.12, theorem 5.13) and went on to show that symplectic reduction can be carried out for Poisson actions with momentum mapping, although in general, the symplectic structure $\omega$ is not invariant under Poisson actions.

The section 3.3 recalls the useful facts concerning the action of the group of generalized symmetries $\hat{\mathfrak{g}} = \text{Diff}(M) \rtimes \Omega^2(M)$, the $H$-twisted Lie bracket on $\mathfrak{X} = \Gamma(TM) \oplus \Omega^1(M)$, Courant algebroid and generalized complex structures and explain in more detail the notion of extended structures. These results are not new and details may be found in, for example 8, 12, 13, 15, and the references therein.

We show, in 4.1, that the momentum mapping as defined in 26 can be extended to the generalized geometry (definition 3.11), and the reduction construction for symplectic manifold can be extended to generalized complex manifold (theorem 3.13), as well as generalized Kähler
manifold (theorem 3.21). Along the way, we obtain lemma 3.4, which can be viewed as an extension of Moser’s argument for symplectic geometry (remark 3.3). We note that similar to the case of symplectic geometry in [24], the generalized complex structure may not be preserved by the group action. In fact, in our construction of generalized Kähler reduction, none of the two generalized complex structures need to be preserved by the group action, as long as certain subbundle of $\mathbb{T}M$ is preserved (remark 3.22). We remark that reduction of Courant algebroid (§ appendix A) as well as reduction of generalized Kähler structure have been discussed in various other works 12 24 25 26.

One of the features of generalized Kähler geometry is that the two generalized complex structures are on the same footing, which is not at all obvious in the classical Kähler geometry. In fact, this is one of the reasons that generalized Kähler geometry could serve as the natural category of discussing duality. Generalized Kähler geometry is relevant also from the result in 12, that it is equivalent to the bi-hermitian geometry, which has been shown to be the string background for $N = (2, 2)$ supersymmetry (11, 1 and references therein). The notion of $T$-duality with $H$-flux in abelian case is proposed in 11 and then has been worked to much more general situations which involve non-commutative 28 and non-associative 1 geometries. The motivation in physics is that the physical theories on $T$-dual spaces are isomorphic and thus provides insights to what the physics is about. Here we concentrate on the more geometrical duality and leave the non-classical cases to future work.

We first describe the construction of $T$-duality with $H$-flux from the existing literature in the following. To simplify matters, we restrict to $T = S^1$, where many complications do not arise. Let $p : E \to M$ be an $S^1$-principal bundle with connection form $\Theta \in \Omega^1(E)$ and curvature form $\Omega \in \Omega^2(M)$. Let $H \in \Omega^3(E)^{S^1}$ be a closed $S^1$-invariant 3-form representing integral class $[H] \in H^3(E, \mathbb{Z})$. By construction, there is a form $h \in \Omega^3(M)$ so that $p^*h = H - \Theta \wedge \Omega$. Let $\hat{\Omega} \in \Omega^2(M)$ be the integration of $H$ along the fiber of $E$, then $[\hat{\Omega}] \in H^2(M, \mathbb{Z})$ and there is a principle $S^1$-bundle $\hat{p} : \hat{E} \to M$ whose first Chern class is $[\hat{\Omega}]$. In particular, we may choose a connection form $\hat{\Theta} \in \Omega^1(\hat{E})$ whose curvature form is $\hat{\Omega}$. Let $\hat{H} = \hat{p}^*h + \hat{\Theta} \wedge \hat{\Omega}$, then $\hat{H} \in \Omega^3(\hat{E})^{S^1}$ is closed and the pair $(\hat{E}, \hat{H})$ is said to be $T$-dual to the pair $(E, H)$. One may also consider the correspondence space $E \times_M \hat{E}$, whose projection to $E$ and $\hat{E}$ is denoted $\pi$ and $\hat{\pi}$ respectively. Then the forms satisfy $\hat{\pi}^*H - \pi^*H = d(\hat{\Theta} \wedge \Theta)$. We may summarize this description with the following diagram:

For higher dimensional torus, it’s argued (see [25], 12 and references therein) that various conditions are needed, on the action and twisting form $H$, in order for the dual space to be classical. Otherwise, it would be one of the non-classical geometries.

The idea of applying generalized geometry in describing $T$-duality is introduced by Gualtieri 12 and Cavalcanti 5, where the first efforts were made. The guiding example for us is the example in § 5.9. By this example, we see that it’s possible for the same function to serve as moment map for Hamiltonian group actions with respect to either generalized complex structure and thus provides a diagram similar to the one above. Another important input is from 5, where Cavalcanti showed that the Courant algebroids defined by invariant sections on $T$-dual $S^1$-principal bundles are isomorphic.
On the physics side, there is vast literature on $T$-duality, both with or without $H$-flux, abelian or non-abelian, for principle bundles or fibration with singular fibers. The approach of realizing dual theories by quotient construction appeared in \cite{12, 13}, where it’s argued that gauging different chiral currents produces dual $\sigma$-models. More recently, there is work of Hull \cite{17}, which discusses $T$-duality in the \textit{doubled formalism}. The formalism is to look at the correspondence space as principle bundle of a doubled torus, consisting of the product of a dual pair of torus with the natural pairing on the Lie algebra. Then the group automorphisms preserving the pairing corresponds to the $T$-duality group. The idea of looking to Poisson Lie group in considering duality goes back to a series of papers by Klimčík and/or Ševera starting with \cite{19, 21, 22}, where Poisson Lie target space duality was proposed as the framework of non-abelian $T$-duality. The papers \cite{20, 17, 21, 28} and references therein contain more recent developments in this direction.

Starting from \cite{16} we discuss $T$-duality with $H$-fluxes in the context of generalized (Kähler) geometry, which includes both abelian and non-abelian groups. In \cite{8} we define the notion of bi-Hamiltonian action (definition \ref{bihamiltonian}) and discuss reduction of bi-Hamiltonian action of factorizable Poisson Lie groups (theorem \ref{bihamiltonian2}). The main point is that the reduced structure is an effective Courant algebroid on the reduced space (definition \ref{bihamiltonian1}). We note that the reduction considered in \cite{8} can be factorized in two ways and in \cite{8} we define the two intermediate stages as being \textit{Courant dual} to each other (definition \ref{bihamiltonian1}). Our construction then provides an isomorphism of Courant algebroids defined by the invariant sections of Courant dual structures (proposition \ref{bihamiltonian2}), extending the result in \cite{8}, with a more geometrical method. The upshot is that in theorem \ref{bihamiltonian3}, we show that $T$-duality, as described above, can arise from a special case of Courant duality. The notion of $T$-duality group is essential in the full picture of $T$-duality with $H$-fluxes and we discuss it in \cite{8}. We note that it’s more desirable that $T$-duality is constructed starting from $(E,H;\Theta)$ instead of from the correspondence space as the approach here. The construction of the correspondence space from one of the reduced space will be discussed in \cite{8}.

To make the paper more self-contained, in \cite{8} appendix A, we present a construction of reduction of extended tangent bundles which is used in this article. In \cite{8} appendix B, we collect various facts on Lie bialgebra, Poisson Lie group and Hamiltonian action.

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\section{Preliminaries}

We recall the preliminaries of generalized geometry and symmetries. As mentioned in the introduction, the results are not new and for details, the readers are referred to the literatures, for example \cite{12, 13, 13, 14}, and the references therein.

\subsection{1. For a smooth manifold $M$, let $\mathbb{T}M = TM \oplus T^*M$ and $\mathcal{G} = \text{Diff}(M) \ltimes \Omega^2(M)$. Let $\lambda, \mu \in \text{Diff}(M)$ and $\alpha, \beta \in \Omega^2(M)$, then the product on $\mathcal{G}$ is given by $(\lambda, \alpha) \cdot (\mu, \beta) = (\lambda\mu, \mu^*\alpha + \beta)$.

Let $X = X + \xi$ with $X \in TM$ and $\xi \in T^*M$, then the (left) action of $\mathcal{G}$ on $\mathbb{T}M$ is given by $(\lambda, \alpha) \circ (X + \xi) = \lambda_\ast X + (\lambda^{-1})^\ast(\xi + \iota_X \alpha)$.

The Lie algebra of $\mathcal{G}$ is $\mathcal{X} = \Gamma(TM) \oplus \Omega^2(M)$ with the following Lie bracket:

$$[X, A] = (X, Y; B)] = ([X, Y], \mathcal{L}_XB - \mathcal{L}_YA).$$
The 1-parameter subgroup generated by \((X, A)\) is given by

\[
e^{t(X,A)} = (\lambda_t, \alpha_t) = \left( e^{tX}, \int_0^t \lambda_s^t \text{Ad}s \right).
\]

Following the above notation, for \(B \in \Omega^2(M)\), we use \(e^B\) to denote the so-called \(B\)-transformation

\[
e^B \circ (X + \xi) = X + \xi + i_X B.
\]

2.2. Let \(H \in \Omega^2_0(M)\), i.e. \(dH = 0\). The \(H\)-twisted Loday bracket on \(TM\) is defined by

\[
(X + \xi) *_H (Y + \eta) = [X, Y] + \mathcal{L}_X \eta - i_Y (d \xi - i_X H).
\]

Let \((X + \xi, Y + \eta) = \frac{1}{2}(\iota_X \eta + \iota_Y \xi)\), then \((\mathcal{T}M, *_H, \langle , \rangle, a)\) defines a structure of Courant algebroid, with \(a : \mathcal{T}M \to \mathcal{T}M\) the natural projection (cf. definition 2.2 below). The Loday bracket is not skew-symmetric, indeed we have

\[
(X + \xi) *_H (Y + \eta) + (Y + \eta) *_H (X + \xi) = d(X + \xi, Y + \eta).
\]

The subgroup \(\mathcal{G} = \text{Diff}(M) \ltimes \Omega^2_0(M)\) is the group of symmetries of the Courant algebroid structure with \(H = 0\). The Lie algebra of \(\mathcal{G}\) is \(\mathfrak{g} = \Gamma(\mathcal{T}M) \oplus \Omega^2_0(M)\) with the induced bracket.

Let \(\mathcal{G}_H \subset \mathcal{G}\) be the symmetry group of the Courant algebroid structure for general \(H\) and \(\mathcal{X}_H\) be its Lie algebra. Consider the linear isomorphism:

\[
\psi_H : \mathcal{X}_H \to \mathcal{X} : (X, A) \mapsto (X, A + i_X H),
\]

and the \(H\)-twisted Lie bracket

\[
[\cdot, \cdot]_H : \mathcal{X}_H \times \mathcal{X}_H \to \mathcal{X}_H : \{(X, A), (Y, B)\}_H = \{(X, Y), \mathcal{L}_X B - \mathcal{L}_Y A + d\iota_Y i_X H\},
\]

then we have

**Proposition 2.3.** [\[L\]] For \(H, H' \in \Omega^2_0(M)\),

\[
[\psi_H(X, A), \psi_H(Y, B)]_{H+H'} = \psi_H[(X, A), (Y, B)]_{H'}
\]

and \(\psi_H : (\mathcal{X}_H, [\cdot, \cdot]_H) \to (\mathcal{X}, [\cdot, \cdot]_H)\) is Lie algebra isomorphism. \(\square\)

Let \(\mathfrak{X} = X + \xi \in \Gamma(\mathcal{T}M)\), then \((X, d\xi) \in \mathcal{X}\) and generates a 1-parameter subgroup in \(\mathcal{G}_H\):

\[
e^{\psi_H^{-1}(X, d\xi)} = (\lambda_t, \alpha_t) = \left( e^{tX}, \int_0^t \lambda_s^t (d \xi - i_X H) ds \right).
\]

The infinitesimal action of \(\mathfrak{X}\) on \(\mathfrak{Y} \in \Gamma(\mathcal{T}M)\) that generates the above subgroup is:

\[
\mathfrak{X} \circ_H \mathfrak{Y} = - \mathfrak{X} *_H \mathfrak{Y}.
\]

2.4. Let \(J : \mathcal{T}M \to \mathcal{T}M\) be a generalized almost complex structure on \(M\), that is, \(J^2 = -\mathbb{I}\) and \(J\) is orthogonal with respect to the pairing \(\langle , \rangle\). Let \(L \subset \mathcal{T}_CM\) be the \(i\)-eigensubbundle of \(J\), then \(L\) is isotropic and \(J\) defines an \(H\)-twisted generalized complex structure if \(L\) is involutive with respect to the \(H\)-twisted Loday bracket \(*_H\). Examples of generalized complex structures include the symplectic and complex structures. Let \(\omega\) (resp. \(J\)) be a symplectic (resp. complex) structure on \(M\), then the corresponding generalized complex structure is defined by the respective isotropic subbundles:

\[
L_\omega = \{ X - i t_X \omega | X \in TM \} \quad \text{and} \quad L_J = \{ X + \xi + i (J(X) - J^*(\xi)) | X \in TM, \xi \in T^*M \}.
\]
2.5. The space of complex valued differential forms \( \Omega^\bullet(M; \mathbb{C}) \) is the spinor space of generalized geometry. Let \( d_H = d - H \wedge \) be the \( H \)-twisted differential on \( \Omega^\bullet(M; \mathbb{C}) \). Each maximally isotropic subbundle \( L \subset T^*_C M \) corresponds to a pure line subbundle \( U \) of \( \wedge^2 T^*_C M \) so that
\[
U = \text{Ann}_C(L) := \{ \rho \in \wedge^2 T^*_C M | \mathcal{X} \cdot \rho = \iota_X \rho + \xi \wedge \rho = 0 \text{ for all } \mathcal{X} = X + \xi \in L \},
\]
where \( \cdot \) stands for the Clifford multiplication. A (nowhere vanishing) local section \( \rho \) of \( U \) is called a pure spinor associated to the subbundle \( L \). The integrability of \( L \) with respect to the \( H \)-twisted Courant bracket is equivalent to the condition \( d_H(\Gamma(U)) \subset \Gamma(U_1) \), where \( U_1 = \Gamma(T^*_C M) \cdot U \) via Clifford multiplication. More explicitly, there is a unique local section \( \mathcal{Y} = Y + \eta \) of \( L \), so that
\[
d_H \rho = d \rho - H \wedge \rho + \mathcal{Y} \cdot \rho = \iota_Y \rho + \eta \wedge \rho,
\]
where we use the convention of \( d_H \) as in \([29]\). For a generalized complex structure \( \mathcal{J} \), the complex line bundle \( U \) is called the canonical bundle of \( \mathcal{J} \).

2.6. Via \([23]\), we have the following definition of a Courant algebroid:

**Definition 2.7.** Let \( E \to M \) be a vector bundle. A Loday bracket \( * \) on \( \Gamma(E) \) is a \( \mathbb{R} \)-bilinear map satisfying the Jacobi identity, i.e. for all \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \Gamma(E) \),
\[
\mathcal{X} \ast (\mathcal{Y} \ast \mathcal{Z}) = (\mathcal{X} \ast \mathcal{Y}) \ast \mathcal{Z} + \mathcal{Y} \ast (\mathcal{X} \ast \mathcal{Z}).
\]

\( E \) is a Courant algebroid if it has a Loday bracket \( * \) and a non-degenerate symmetric pairing \( \langle , \rangle \) on the sections, with an anchor map \( a : E \to TM \) which is a vector bundle homomorphism so that
\[
a(\mathcal{X})(\mathcal{Y}, \mathcal{Z}) = \langle \mathcal{X}, \mathcal{Y} \ast \mathcal{Z} + \mathcal{Z} \ast \mathcal{Y} \rangle
\]
\[
a(\mathcal{X})(\mathcal{Y}, \mathcal{Z}) = \langle \mathcal{X} \ast \mathcal{Y}, \mathcal{Z} \rangle + \langle \mathcal{Y}, \mathcal{X} \ast \mathcal{Z} \rangle.
\]

The skew-symmetrization \([,] \) of \( * \) in the definition is also called the Courant bracket. In particular, the datum \((TM, *, \langle , \rangle, a)\) as given in previous subsections, for \( H \in \Omega^3(M) \), are examples of Courant algebroid, where the corresponding Courant bracket is usually denoted \([,]_H \).

2.8. Let \( TM \) be a Courant algebroid which fits into the following extension:
\[
0 \to T^* M \to TM \xrightarrow{\alpha} TM \to 0,
\]
so that \( \alpha \) is the anchor map. Such Courant algebroid is called exact \([33]\). The set of isotropic splitting \( s : TM \to TM \) is non-empty and is a torsor over \( \Omega^2(M) \). The choice of such \( s \) determines a form \( H \in \Omega^3(M) \) and \( TM \) can then be identified with the datum \((TM, *, \langle , \rangle, a)\) as discussed above. The action of \( B \in \Omega^2(M) \) on the set of splittings translates into \( H \mapsto H + dB \) on the corresponding forms. It follows that \([H] \in H^3(M; \mathbb{R}) \) is well-defined and is the Ševera class of \( TM \). We use the notion extended (+ structures) to emphasize the absence of a splitting while reserve twisted generalized for the situation where a splitting is (or can be explicitly) chosen. For example, an extended complex structure \( \mathcal{J} \) will represent a twisted generalized complex structure \( \mathcal{J} \) on \( TM \) (once a splitting is chosen), which is integrable with respect to a twisted Loday bracket \( *_H \), where \([H] \) gives the Ševera class of the extended tangent bundle \( TM \) defined by the Courant algebroid structure \((TM, *, \langle , \rangle, a)\). Given a different choice of splitting of \( TM \), \( \mathcal{J} \) will represent \( \mathcal{J}_B \), which is \( \mathcal{J} \) transformed by some \( B \in \Omega^2(M) \) and is integrable with respect to \(*_{H+dB} \) on \( TM \). We note that the Courant algebroids are identical (not only isomorphic) in either cases, since the difference is only the choice of a splitting that gives the identification to \( TM \).
3. Poisson Lie actions and reductions

In this section, we describe the reduction of generalized complex and Kähler manifolds via the Hamiltonian action of Poisson Lie groups. This extends the reduction construction of [10] for Hamiltonian action of Lie groups and that of [9] for Poisson Lie action on symplectic manifolds, which we describe in the appendix B ([8]). Again, when we use $\mathbb{T}M$, $\mathbb{J}$ and etc., we assume a choice of splitting of the extended tangent bundle $TM$ and identify the corresponding structures as $H$-twisted generalized structures.

3.1. We first discuss the invariance of $\mathbb{J}$ under generalized actions. Direct computation shows

**Lemma 3.2.** Let $(\lambda, \alpha) \in \mathbb{G}$ and $\rho$ be the pure spinor defining $\mathbb{J}$, then $(\lambda, \alpha) \circ \rho := (\lambda^{-1})^* (\epsilon^{-\alpha} \rho)$ is the pure spinor defining $(\lambda, \alpha) \circ \mathbb{J}$. If $\mathbb{J}$ is $H$-twisted integrable, then $(\lambda, \alpha) \circ \mathbb{J}$ is $(\lambda, \alpha) \circ H$-twisted integrable, where $(\lambda, \alpha) \circ H = (\lambda^{-1})^* (H - d\alpha)$. We have $d((\lambda, \alpha) \circ H, \alpha) \circ \rho = (\lambda, \alpha) \circ dH \rho$. □

**Remark 3.3.** We note that when considering generalized symmetries, we do not have to restrict to real 2-forms to stay with real twisting form, e.g. the group $\text{Diff}(M) \ltimes (\Omega^2(M) \oplus i\Omega^2(M))$ acts on $\mathbb{T}_c M$. The infinitesimal action of $(X,A) \in \mathbb{F}^* \oplus i\mathbb{F}^2(M)$ on the spinors is then given by

$$(X,A) \circ \rho = -\mathcal{L}_X \rho - A \wedge \rho.$$ 

For $\mathbf{X} = X + \xi \in \Gamma(\mathbb{T}_c M)$ so that $X \in \Gamma(TM)$, let $(X,A) = (X, d\xi - \iota_X H)$ and we compute the infinitesimal action on a section $\rho$ of the canonical bundle of $\mathbb{J}$:

$$(3.1) \quad \mathbf{X} \circ_H \rho = (-dH + \mathcal{Y}) \mathbf{X} \cdot \rho - \langle \mathbf{X}, \mathcal{Y} \rangle \rho.$$

We caution that when a generalized complex structure is concerned, such complex actions in general might not preserve the real index.

**Lemma 3.4.** Suppose that $L$ defines the extended complex structure $\mathcal{J}$ and $\mathbf{X}_t = X_t + \xi_t \in \Gamma(L \cap TM \oplus iT^*M)$ is a family of sections parametrized by $\mathbb{R}$. Let $\lambda_t = (\lambda_t, \alpha_t)$ be the family of generalized symmetries generated by $\mathbf{X}_t$. Suppose that for each $p \in M$ there is an open neighbourhood $U_p$ and a compact set $V_p$ so that $\{\lambda_t \circ U_p\} \subseteq V_p$ for all $t$. Then $\lambda_t$ preserves $\mathcal{J}$ for all $t$.

**Proof:** Choose a splitting and identify the structures with $H$-twisted structures. Starting from any $p \in M$ and $t_0 \in \mathbb{R}$. Suppose that $\rho_{t_0} = \rho$ is a local section of the canonical bundle $U$ of $\mathbb{J}$ and $\rho_t = (\lambda_t, \alpha_t)^* \rho := (\lambda_t, \alpha_t)^{-1} \circ \rho = e^{\alpha_t} \lambda_t^* \rho$. Then $\rho_t$ is a local section of the canonical bundle $U_t$ of $\mathbb{J}_t = (\lambda_t, \alpha_t)^{-1} \circ \mathbb{J}$. Direct computation shows that

$$\frac{d}{dt} \rho_t \bigg|_{s=0} \rho_{t+s} = (\lambda_t, \alpha_t)^* ((dH - \mathcal{Y}) \mathbf{X}_t \cdot \rho + \langle \mathbf{X}_t, \mathcal{Y} \rangle \rho).$$

Then by assumption $\frac{d}{ds} \rho_t = f_t \rho_t$ for $f_t = \lambda_t^* (\mathbf{X}_t, \mathcal{Y})$. With the initial condition of $\rho_{t_0} = \rho$ we get

$$\rho_t = e^{\int_{t_0}^t f_s ds} \rho.$$

It follows that $U_t = U$ whenever both $\rho$ and $\rho_t$ are defined, e.g. for a neighbourhood of $p$. Since $t_0$ is arbitrary and by compactness assumption, we see that $U_t = U$ for all $t$.

The argument above shows that $L$ is preserved by the family of symmetries generated by $\mathbf{X}_t$, which is independent of the splitting chosen. The proposition then follows. □

**Remark 3.5.** Of course, when $M$ is compact, the condition in the proposition automatically holds. From the proof, we also see that when $dH \rho = 0$, not only the canonical line bundle is preserved, the spinor $\rho$ is preserved as well.
Remark 3.6. The Moser’s argument in symplectic geometry can be seen as a special case of the above lemma. The Moser’s argument goes as following (see [31]). Consider a smooth family of symplectic forms $\omega_t = \omega_0 + d\beta_t$ and $\eta_t = \hat{\beta}_t$. Let $Y_t$ be defined by $i\gamma_t \omega_t + \eta_t = 0$ and $\phi_t$ be the family of diffeomorphisms generated by $Y_t$ via $\frac{d}{dt} \phi_t = \phi_t(Y_t)$, then $\phi_t^* \omega_t = \omega_0$.

In light of lemma 3.5, we consider $\varphi_t = \phi_t^{-1}$, which is generated by the family of vector fields $X_t = -\varphi_t^{-1}(Y_t)$. Then we define $\xi_t = iX_t \omega_0 = -\varphi_t^*(\eta_t)$. It follows that $X_t = X_t - i\xi_t \in \Gamma(L_{\omega_0})$.

The lemma then implies that the following family of symmetries preserves $L_{\omega_0}$:

$$(\varphi_t, \alpha_t) = \left( \varphi_t, -id \int_0^t \varphi_s^* \xi_s ds \right) = \left( \varphi_t, -id \int_0^t \eta_s ds \right).$$

In this case, we have $\rho_0 = e^{i\omega_0}$ and $d\rho_0 = 0$. The proof of the lemma then implies that $\rho_0$ is preserved:

$$e^{i\omega_0} = (\varphi_t, \alpha_t)^* e^{i\omega_0} = e^{-id \int_0^t \eta_s ds} \varphi_t^* (e^{i\omega_0}) = e^{-id \int_0^t (\varphi_t^*)^{-1}(e^{i\omega_0})},$$

which is equivalent to $\phi_t^* \omega_t = \omega_0$ as in Moser’s argument.

3.7. We will use the following conventions:

Convention 3.8. Given a Lie group $G$, the Lie algebra $\mathfrak{g}$ of $G$ is identified as the tangent space $T_eG$ at identity, as well as the space of right invariant vector fields, i.e. $\tau \to X_\tau(g) = (R_g)_\tau \tau$. Then the dual $\hat{\mathfrak{g}} = \mathfrak{g}^*$ of the Lie algebra is identified with the space of right invariant $1$-forms on $G$. Let $\theta^* \in \Omega^1(G)$ denote the right invariant $1$-form on $G$ with $\theta^*_e(e) = \hat{\tau}$ and $\theta^*_\tau$ the left invariant $1$-form on $G$ with $\theta^*_\tau(e) = \hat{\tau}$, for $\hat{\tau} \in \hat{\mathfrak{g}}$. Given a Poisson manifold $P$ with Poisson tensor $\pi_P$, we consider $\pi_P$ also as a map $\pi_P : TP^* \to TP$ defined by $i_{\pi_P(\xi)} = \pi_P(\xi, \eta)$ for $\xi, \eta \in \Omega^1(P)$.

We note that for $\tau \in \mathfrak{g}$, the right invariant vector field $X_\tau$ generates left action on $G$ by the $1$-parameter subgroup $g_t = e^{t\tau}$. Thus the left action of $G$ on $M$ induces homomorphism of Lie algebras $\xi \mapsto X_\xi^M$ where $X_\xi^M$ is the infinitesimal action generated by $\tau$, while the right action of $G$ induces anti-homomorphism of Lie algebras. With this convention, the map $\pi_P$ and the Lie algebra (anti-)homomorphism are opposite to the convention used in [22] and [24]. In the following, we will only consider left actions. We collected the relevant definitions and results on Poisson Lie groups and actions in the appendix (§5).

3.9. The basic setup is the following. Let $(M, \mathcal{J})$ be an extended complex manifold with extended tangent bundle $TM$ and anchor $\alpha : TM \to TM$. Then there is a natural induced Poisson structure $\pi_{\mathcal{J}}$ on $M$ defined by

$$\pi_{\mathcal{J}} : T^*M \to TM \xrightarrow{\mathcal{J}} TM \xrightarrow{\alpha} TM.$$  

Let $(G, \pi_G)$ be a Poisson Lie group with Poisson structure $\pi_G$. Let $\sigma : G \times M \to M$ be a (left) Poisson action with equivariant moment map $\mu : M \to \hat{G}$ (cf. definition §12). If a splitting is chosen, we use $TM$, $\mathcal{J}$ and $H$-twisted when referring to the respective structures. We show

Lemma 3.10. Suppose that $G$ is connected. Let $\mu : M \to \hat{G}$ be an equivariant moment map as in definition §13. Let $\mathcal{J}(\mu^* \hat{\theta}_\tau) = \mathcal{X}_\tau = X_\tau + \xi_\tau$ for $\tau \in \mathfrak{g}$, then

$$i_{X_\tau} \mu^* \hat{\theta}_\tau = i_{X_\tau} \mu^* \xi_\tau = 0 \text{ and } [(X_\tau, d\xi_\tau), (X_\omega, d\xi_\omega)]_H = (X_{[\tau, \omega]}, d\xi_{[\tau, \omega]}).$$

Proof: First $i_{X_\tau} \mu^* \hat{\theta}_\tau = \mu^*(i_{\mathcal{X}_\tau} \mathcal{J}) = \mu^* (\mathcal{J}_G(X_\tau, \hat{X}_\tau)) = 0$, where $\hat{X}_\tau$ is the dressing vector field generated by $\tau \in \mathfrak{g}$ (cf. definition §10). We then compute

$$[\mathcal{J}(\mu^* \hat{\theta}_\tau) + i\mu^* \hat{\theta}_\tau, \mathcal{J}(\mu^* \hat{\theta}_\omega) + i\mu^* \hat{\theta}_\omega]_H$$

$$= [X_\tau, X_\omega] + L_{X_\tau} \xi_\omega - i_{X_\omega} d\xi_\tau + i_{X_\omega} i_{X_\tau} H + i(L_{X_\tau}, \mu^* \hat{\omega}) - i_{X_\omega} \mu^* \hat{\theta}_\tau, \mu^* \hat{\theta}_\tau).$$
and the imaginary part is
\[ \mathcal{L}_X \mu^* \hat{\theta}_\omega - i_{\hat{\mathcal{L}}_X} d\mu^* \hat{\theta}_\tau = \mu^* (\mathcal{L}_{\mu^* X} \hat{\theta}_\omega - i_{\mu^* X} d\hat{\theta}_\tau) \]
\[ = \mu^* (\mathcal{L}_{\hat{X}_\omega} \hat{\theta}_\tau - i_{\hat{X}_\omega} d\hat{\theta}_\tau) = \mu^* \hat{\theta}_{[\omega, \tau]} \]
Thus \( X_{[\tau, \omega]} = [X_{\tau}, X_\omega] \) and \( \xi_{[\tau, \omega]} = \mathcal{L}_{X_{\tau}} \xi_\omega = i_{X_{\tau}} d\xi_\tau + i_{X_\omega} \iota_X H \). The lemma follows. \( \square \)

We note that from above lemma, the symmetry generated by \( X_{[\tau, \omega]} \) coincides with that of \( X_{\tau} \circ H \), which only depends on the Loday bracket.

**Definition 3.11.** The action of a Poisson Lie group \( G \) on an \( H \)-twisted generalized complex manifold \( (M, \mathcal{J}) \) is Hamiltonian with moment map \( \mu : M \to \hat{G} \), if the action is Poisson with respect to \( \pi_\mathcal{J} \), \( \mu \) is an equivariant moment map as in definition \( \ref{def:hamiltonian-action} \) so that the \( G \)-action on \( TM \) is generated by \( \mathcal{J}(\mu^* \hat{\theta}) = X_\mu \), via the Loday bracket \(*\).

**Remark 3.12.** We recall that in the Poisson category, the Poisson action of a Poisson Lie group does not have to preserve the Poisson structure. Thus the action as defined above does not have to preserve the extended complex structure \( \mathcal{J} \). The lemma \( \ref{lem:poisson-action} \) implies that the action on \( TM \) generated by \( \mathcal{J}(\mu^* \hat{\theta}) + i\mu^* \hat{\theta} \) does preserve the structure \( \mathcal{J} \). Thus the non-invariance under the action above can be seen as due to the non-closedness \( \hat{\theta} \). When the Poisson structure on \( G \) is trivial, we have the definition for Hamiltonian actions of Lie groups \( \ref{def:hamiltonian-action} \). By theorem \( \ref{thm:hamiltonian-action} \) \( \mu \) is a Poisson map. Let \( M_0 = \mu^{-1}(\hat{e}) \), then \( \mu_* (\pi_\mathcal{J}|_{M_0}) = \pi_{\mathcal{J}^0}|_{M_0} = 0 \), and \( X_\mu \), i.e. the geometrical action of \( G \), preserves \( M_0 \).

3.13. We may consider reduction by Hamiltonian Poisson Lie group action. Assume that

1. The identity \( \hat{e} \in \hat{G} \) is a regular value of \( \mu \),
2. (the geometrical part of) \( G \) acts freely on \( M_0 \).

**Lemma 3.14.** The restriction of \( (\mu^* \hat{\theta}), \mathcal{J}(\mu^* \hat{\theta}) \) and \( L \oplus (\mu^* \hat{\theta}) \) are \( G \)-equivariant subbundles.

**Proof:** Choose a splitting. It’s enough to show that the infinitesimal actions preserve the subbundles:

\[(X_\omega, d\xi_\omega) \circ H (\mu^* \hat{\theta}_\tau) = \mathcal{L}_{X_\omega} \mu^* \hat{\theta}_\tau = \mu^* (\mathcal{L}_{\hat{X}_\omega} \hat{\theta}_\tau) = \mu^* \hat{\theta}_{[\omega, \tau]},
(X_\omega, d\xi_\omega) \circ H (X_\tau + \xi_\tau) = [X_\omega, X_\tau] + \mathcal{L}_{X_\omega} \xi_\tau - i_{X_\tau} d\xi_\omega + i_{X_\omega} \iota_X H = X_{[\omega, \tau]} + \xi_{[\omega, \tau]},
(X_\omega, d\xi_\omega) \circ H (Y + \eta) = [X_\omega, Y] + \mathcal{L}_{X_\omega} \eta - i_Y d\xi_\omega + i_Y \iota_X H = (X_\omega + \xi_\omega + iY^* \mu^*(d\hat{\theta}_\omega),

for \( Y + \eta \in \Gamma(L) \). We note that \( i_Y \mu^*(d\hat{\theta}_\omega) = -\frac{1}{2} i_Y \mu^*(|[\hat{\theta}, \hat{\theta}]_{\omega} ) \in (\mu^* \hat{\theta}) \). \( \square \)

**Theorem 3.15.** Suppose that \( G \) is compact and assumptions (1) and (2), there is a natural extended complex structure on the quotient \( Q = M_0/G \). When the action of \( G \) preserves a splitting of \( TM \), the reduced structure \( \mathcal{T}Q \) admits a natural splitting up to a choice of connection form of \( Q \to Q \).

**Proof:** By \( \pi_{\mathcal{J}^0}|_{M_0} = 0 \) we compute on \( M_0 \):

\[ (\mu^* \hat{\theta}_\tau, \mathcal{J}(\mu^* \hat{\theta}_\omega)) = i_{\hat{X}_\omega} \mu^* \hat{\theta}_\tau = \mu^* \iota_{\hat{X}_\omega} \hat{\theta}_\tau = \mu^* \pi_{\mathcal{J}}(\hat{X}_\omega, \hat{X}_\tau) = 0. \]

Then \( \mathcal{K} = \mu^* \hat{\theta}, \mathcal{K}' = \mathcal{J}(\mu^* \hat{\theta}) \) satisfy the conditions of lemma \( \ref{lem:splitting} \) (1). Thus \( T_{\mu} M_0 = \text{Ann}(\mu^* \hat{\theta}, \mathcal{J}(\mu^* \hat{\theta})) \) descends to an extended tangent bundle \( T_{\mu} Q \) on \( Q \). Consider \( (L \oplus (\mu^* \hat{\theta})) \cap \text{Ann}(\mu^* \hat{\theta}, \mathcal{J}(\mu^* \hat{\theta})) \), then it induces a subbundle \( L_0 \) in \( T_{\mu} M_0 \) which coincides with the image of \( L \) under the subquotient. By lemma \( \ref{lem:splitting} \) the bundle \( L_0 \) is \( G \)-equivariant and descends to a subbundle \( L_0 \) of \( T_{\mu} Q \). That \( L_0 \) is maximally isotropic with real index 0 and integrable follows from the same properties of \( L \). Thus \( L_0 \) defines an extended complex structure \( \mathcal{J}_\mu \). The last sentence follows from corollary \( \ref{cor:splitting} \). \( \square \)
3.16. Let \((M, J_1)\) be an extended complex manifold. A second extended complex structure \(J_2\) makes \((M, J_1, J_2)\) into an extended Kähler manifold if they are both defined on the same extended bundle \(TM\) and \(G = -J_1J_2 = -J_2J_1\) defines generalized metric (see [4]) on \(TM\), i.e. \(\langle G, \cdot \rangle\) defines a metric on \(TM\). We show that just as symplectic reduction admits induced Kähler structure when the original manifold is Kähler with \(G\) preserving the complex structure, generalized complex reduction with respect to \(J_1\) would admit extended Kähler structure if \(J_2\) is preserved.

**Definition 3.17.** A Poisson action of Poisson Lie group \(G\) on an extended Kähler manifold \((M, J_1, J_2)\) is \(J_1\)-Hamiltonian if it is Hamiltonian with respect to \(J_1\) and preserves \(J_2\).

3.18. We note that \(\text{Ann}(\mu^*\hat{\theta}, J_1(\mu^*\hat{\theta}))\) is not preserved by \(J_2\):

\[
\text{Ann}(\mu^*\hat{\theta}, J_1(\mu^*\hat{\theta})) \cap J_2(\text{Ann}(\mu^*\hat{\theta}, J_1(\mu^*\hat{\theta}))) = \text{Ann}(\mu^*\hat{\theta}, J_1(\mu^*\hat{\theta}), J_2(\mu^*\hat{\theta}), G(\mu^*\hat{\theta})).
\]

The right hand side of above equation is again \(G\)-equivariant subbundle when restricting to \(M_0\), as the two terms on the left are both so.

**Lemma 3.19.** Let \(\mathbb{V} = \mathbb{V}^* \oplus \mathbb{V}^*\) and \((J_1, J_2; G)\) be a linear generalized Kähler structure. Given subspace \(K \subset \mathbb{V}^*\), let \(U^1_K = \text{Ann}(K, J_1(K)), W_K = \text{Ann}(K, J_1(K), J_2(K), G(K))\) and \((U^1_K)_C, (W_K)_C\) be the respective complexified versions, then \(L_j \cap (W_K)_C = L_j \cap (U^1_K)_C\) for \(j \neq l\). If

1. \(K + J_1(K) \subset U^1_K\),

then the following decomposition holds

\[
U^1_K = W_K \oplus (K + J_1(K)).
\]

The + above becomes \(\oplus\) if we suppose further that

2. \(J_1(1) \cap V^* = \{0\}\).

**Proof:** Obviously \(L_j \cap (W_K)_C \subset L_j \cap (U^1_K)_C\). Let \(X \in L_j \cap (U^1_K)_C\), then \(J_j(X) = iX\) and \(\langle X, K \rangle = \langle X, J_1(K) \rangle = 0\). Then by orthogonality of \(J_j\) we find that \(\langle X, J_j(K) \rangle = \langle X, G(K) \rangle = 0\), i.e. \(X \in L_j \cap (W_K)_C\). For any subspace \(W \subset \mathbb{V}\), let \(W^\perp = \text{Ann}(GW)\), then \(\mathbb{V} = \mathbb{V} \oplus \text{Ann}(GW)\). In particular

\[
\mathbb{V} = W_K \oplus (K + J_1(K) + J_2(K) + G(K)) = W_K \oplus (J_1(K) + J_2(K) + G(K))
\]

\[
= U^1_K \oplus (J_1(K) + G(K)) \quad \text{for} \quad j \neq l,
\]

where \(W_K = \text{Ann}(K, J_1(K), J_2(K))\). With condition (1), by the last expression in (3.2) for \(j = 1\) and \(l = 2\), we see that the decomposition in the statement holds. With the condition (2), we have \(K + J_1(K) = K \oplus J_1(K)\) and it follows that all +’s in (3.2) are ⊕’s. □

**Lemma 3.20.** Continue from lemma 3.12 and let \(N = a \circ J_1(K)\) where \(a : \mathbb{V} \to V\) is the projection, then there is a self-dual exact sequence

\[
0 \to W^\ast_K \xrightarrow{a_K^\ast} \mathbb{W}_K \xrightarrow{a_K} W_K \to 0 \quad \text{where} \quad W_K = \frac{\text{Ann}_V(K)}{N} \quad \text{and} \quad W^\ast_K = \frac{\text{Ann}_V^\ast(N)}{K}.
\]

The restriction \(\langle \cdot, \cdot \rangle_K\) of \(\langle \cdot, \cdot \rangle_K\) on \(\mathbb{W}_K\) is non-degenerate pairing and \((J_1, J_2; G)\) restricts to generalized Kähler structure \((J_1, J_2; G_K)\) on \(\mathbb{W}_K\) with respect to the pairing \(\langle \cdot, \cdot \rangle_K\). The inclusion \(\mathbb{W}_K \hookrightarrow \text{Ann}(K, J_1(K))\) induces natural isomorphism \(\mathbb{W}_K \simeq V_K\) in \([4]\) and the extension sequences correspond.

**Proof:** Note that \(\mathbb{W}_K\) is preserved by \(G\) we see that for any \(X \in \mathbb{W}_K\) such that \(\langle X, \mathbb{W}_K \rangle = 0\), it must satisfy \(\langle X, G(X) \rangle = 0\), i.e. \(X = 0\). It implies that the restriction \(\langle \cdot, \cdot \rangle_K\) is nondegenerate.

Let \(a_K : \mathbb{W}_K \to W_K\) be the map induced from the projection \(a\). The kernel of \(U^1_K \to W_K\) is \(\text{Ann}_V^\ast(N) \oplus J_1(K)\). It follows that the kernel of \(a_K\) is

\[
\ker a_K = (\text{Ann}_V^\ast(N) \oplus J_1(K)) \cap \mathbb{W}_K.
\]
Note that \( \mathcal{U}_k = \mathcal{V}_k \oplus (K \oplus J_1(K)) \) and \( K \oplus J_1(K) \subset \text{Ann}_\nu\cdot(N) \oplus J_1(K) \), we find that \( \text{Ann}_\nu\cdot(N) \oplus J_1(K) = \ker a_K \oplus (K \oplus J_1(K)) \), thus ker \( a_K \simeq \text{Ann}_{\nu_k}(N) \). Now ker \( a_K \) is maximally isotropic with respect to \( \langle \cdot, \cdot \rangle_K \) and the self-duality follows. The last sentence follows from direct checking.

Similar to the classical Kähler case, we have:

**Theorem 3.21.** Let the G action on an extended Kähler manifold \((M, J_1, J_2; G)\) be \( J_1 \)-Hamiltonian with moment map \( \mu : M \to \hat{G} \). Suppose that the assumptions in §3.13 hold, then there is a natural extended Kähler structure on the quotient \( Q = M_0/G \). When the G-action preserves a splitting of \( TM \), the reduced structure splits, up to a choice of connection form on \( M_0 \to Q \).

**Proof:** All the bundles in the proof will be on spaces at \( \mu = \hat{\epsilon} \), either level set or reduced space. Let \( T_{\mu}^\prime M_0 = \text{Ann}(\mu^*\hat{\theta}, J_1(\mu^*\hat{\theta}), J_2(\mu^*\hat{\theta}), \mathcal{G}(\mu^*\hat{\theta})) \) be the subbundle of \( TM|_{M_0} \), then it is a \( G \)-equivariant subbundle. Let \( T_{\mu} M_0 \) be defined as in theorem 8.13 for \( J_1 \). From lemma 3.20, the bundles \( T_{\mu}^\prime M_0 \) and \( T_{\mu} M_0 \) are naturally isomorphic via the inclusion of \( T_{\mu}^\prime M_0 \) in \( \text{Ann}(\mu^*\hat{\theta}, J_1(\mu^*\hat{\theta})) \). We define a new bracket \( \ast_1 \) on \( \Gamma(T_{\mu} M_0)^G \) by the following:

\[
X \ast_1 Y = \pi_1 (X \ast Y), \quad \text{for } X, Y \in \Gamma(T_{\mu}^\prime M_0)^G \subset \Gamma(\text{Ann}(\mu^*\hat{\theta}, J_1(\mu^*\hat{\theta})))^G,
\]

where \( \pi_1 \) is the projection \( \text{Ann}(\mu^*\hat{\theta}, J_1(\mu^*\hat{\theta})) \to T_{\mu}^\prime M_0 \) defined by the first decomposition in lemma 3.19. By definition, \( \Gamma(T_{\mu} M_0)^G \) is closed under \( \ast_1 \). By construction, \( \pi_1 \) coincides with the projection \( \text{Ann}(\mu^*\hat{\theta}, J_1(\mu^*\hat{\theta})) \to T_{\mu} M_0 \) under the natural isomorphism \( T_{\mu}^\prime M_0 \simeq T_{\mu} M_0 \). Thus as in the theorem 3.13, the structure \((T_{\mu} M_0, \langle \cdot, \cdot \rangle_\mu, \pi_1)\) descends to an extended tangent bundle \( TQ \) on \( Q \).

**Remark 3.22.** We notice from the proof that, in order to have extended Kähler reduction, even the extended complex structure \( J_2 \) doesn’t have to be preserved by the \( G \)-action either. The only thing that needs to be preserved is the intersection \( L_2 \cap T_{\mu}^\prime M_0 \). Here, unlike the case in theorem 8.13, where \( L_1 \oplus (\mu^*\hat{\theta}) \) being equivariant provides descending of \( J_1 \), lemma 3.13 implies that such flexibility doesn’t apply to \( J_2 \).

**Remark 3.23.** Generalized Kähler reduction have been constructed by several other works, e.g. [6, 24] and [24], with various generalities. The construction we describe here, which fits our needs for discussing duality, has not appeared in the stated form. In particular, we allow non-trivial \( B \)-field action and we only require the action preserve one of the generalized complex structures.

### 4. Bi-Hamiltonian action and factorizable reduction

We will consider the action of the Manin triple \((\hat{g}, g, \hat{g})\) defined by a Poisson Lie group \( G \) (cf. theorem 3.6 also 20), with dual group \( \hat{G} \). In this context, the notion of Hamiltonian action on an extended Kähler manifold is

**Definition 4.1.** The (infinitesimal) (left) action of \( \hat{g} \) on \( M \) is bi-Hamiltonian if it is induced by a (left) \( J_1 \)-Hamiltonian action of \( G \) together with a (left) \( J_2 \)-Hamiltonian action of \( \hat{G} \).

We will use \( \mu \) and \( \hat{\mu} \) to denote the moment maps of the \( G \) and \( \hat{G} \) actions, respectively. Suppose that \( G \) is a factorizable Poisson Lie group (definition 3.8). Let \( S : \hat{G} \to G \) be the local diffeomorphism defined by \( \hat{\varphi} \) and the exponential maps at the identity elements \( \hat{e} \in \hat{G} \) and \( e \in G \), then \( dS(\hat{e}) = \hat{\varphi} \). We consider the reduction by the bi-Hamiltonian action of \( \hat{G} \).

**Assumption 4.2.** We will need the following conditions:

1. In the following, \( G \) is always a factorizable Poisson Lie group.
2. The identity elements \( e \in G \) and \( \hat{e} \in \hat{G} \) are regular values of \( \mu \) and \( \hat{\mu} \) respectively.
(2) \( \hat{\mu}^{-1}(e) = \mu^{-1}(e) \) and is denoted \( M_0 \).

(3) Restricted over the identity elements, \( d\hat{\mu} = dS \circ d\mu = s \circ d\mu \).

**Remark 4.3.** It follows from (2) that \( M_0 \) is preserved by the \( \hat{G} \) action. By (3), we see that \( \hat{\mu}^* = \mu^* \circ s^* = \mu^* \circ \hat{s} \) when restricted to \( M_0 \), since \( s \) is symmetric. Thus on \( M_0 \) we have

\[
\hat{\mu}^* \theta = \mu^* \circ \hat{s}(\theta) = \mu^* \hat{s}(\theta) \quad \text{for} \quad \hat{s} \in \hat{\mathfrak{g}}.
\]

**Lemma 4.4.** Let \( V = V \oplus V^* \) and \((J_1, J_2; G)\) be a linear generalized Kähler structure. Let \( K \subseteq V^* \) and define \( N_j = a \circ J_j(K) \) for \( j = 1, 2 \) where \( a : V \to V \) is the projection. Assume that for \( j = 1, 2 \)

1. \( K + \mathbb{J}_j(K) \subseteq \text{Ann}(K, \mathbb{J}_j(K)) \),
2. \( \mathbb{J}_j(K) \cap V^* = \{0\} \) and
3. \( N_1 \cap N_2 = \{0\} \).

We then have \( \hat{\mathbb{V}}_K = \mathbb{V}_K \oplus K \), \( N_1 \oplus N_2 \subseteq \text{Ann}_V(K) \) and the exact sequence

\[
0 \to \frac{\text{Ann}_V(N_1, N_2)}{K} \to \mathbb{V}_K \xrightarrow{\alpha_{K}} \text{Ann}_V(K) \to 0.
\]

**Proof:** Let \( K' = \mathbb{J}_1(K) \oplus \mathbb{J}_2(K) \). It then follows from lemma 4.2. \( \Box \)

**Definition 4.5.** A Courant algebroid \( E \) on \( M \) is an effective Courant algebroid if it fits in the following diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & T^*M & \longrightarrow & E & \overset{p}{\longrightarrow} & E_0 & \longrightarrow & 0 \\
& & \downarrow a & & \downarrow & & \downarrow & & \\
& & TM & & \longrightarrow & & 0 & & \end{array}
\]

where \( a \) is the anchor map and the sequences are all exact.

The usual constructions of \( B \)-transformation for \( B \in \Omega^2(M) \) and twisting of the Courant bracket by \( H \in \Omega^3(M) \) are valid for an effective Courant algebroid \( E \).

**Theorem 4.6.** Given assumption 4.4 and let \((\hat{G}, G, \hat{G})\) be a (local) double Lie group whose Lie algebras form the Manin triple \((\hat{\mathfrak{g}}, \mathfrak{g}, \mathfrak{g})\), where \( \hat{G} \) is connected but not necessarily simply connected (compare to theorem 4.4). Suppose that the action of \( \hat{\mathfrak{g}} \) induces an action of \( \hat{G} \), which is proper and free on \( M_0 \), then there is an effective Courant algebroid \( T_\mu \hat{Q} \) on \( \hat{Q} = \hat{M}_0/\hat{G} \).

**Proof:** Let \( \Gamma(\hat{G}) \) denote the set of \( \hat{G} \)-invariant sections. By lemma 3.14 we see that

\[
(\mu^* \hat{\theta}, J_1(\mu^* \hat{\theta}), J_2(\mu^* \hat{\theta}), \hat{G}(\mu^* \hat{\theta})) = (\mu^* \hat{\theta}, J_1(\mu^* \hat{\theta})) \oplus J_2(\mu^* \hat{\theta}, J_1(\mu^* \hat{\theta}))
\]

is preserved by the \( G \)-action. Similarly, it’s also preserved by \( \hat{G} \) and it follows that it’s preserved by the action of \( \hat{G} \). Analogously, the bundles \( (\mu^* \hat{\theta}, J_1(\mu^* \hat{\theta}), J_2(\mu^* \hat{\theta})) \) and \( (\mu^* \hat{\theta}) \) are preserved by the \( \hat{G} \)-action. Let \( K = (\mu^* \hat{\theta}) \) and \( K' = J_1(\mu^* \hat{\theta}) \oplus J_2(\mu^* \hat{\theta}) \), then the conditions for lemma 4.4 (2) are satisfied. Thus \( T_\mu'' M_0 = \text{Ann}(\mu^* \hat{\theta}, J_1(\mu^* \hat{\theta}), J_2(\mu^* \hat{\theta})) \) descends to a Courant algebroid \( T_\mu \hat{Q} \) on \( \hat{Q} \).

Another way to see the Courant algebroid structure is to follow theorem 3.21. Using the decomposition in lemma 4.4 we define the projection

\[
\pi : \text{Ann}(\mu^* \hat{\theta}, J_1(\mu^* \hat{\theta}), J_2(\mu^* \hat{\theta})) \to T_\mu' M_0
\]

and the bracket \( *_\mu' \):

\[
\mathfrak{X} *_\mu' \mathfrak{Y} = \pi(\mathfrak{X} *_\mu \mathfrak{Y}) \quad \text{for} \quad \mathfrak{X}, \mathfrak{Y} \in \Gamma(T_\mu'' M_0) \quad \text{in} \quad \Gamma(\text{Ann}(\mu^* \hat{\theta}, J_1(\mu^* \hat{\theta}), J_2(\mu^* \hat{\theta}))) \hat{G}.
\]
By definition, $\Gamma(T^*_\mu M_0)\tilde{G}$ is closed under $*_\mu$. Then the inclusion of $T^*_\mu M_0$ into $\text{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta}), \mathcal{J}_2(\mu^*\hat{\theta}))$ induces natural isomorphism to $T^*_\mu M_0$, and the brackets coincide. \hfill \Box

**Corollary 4.7.** With the same assumptions as in theorem 4.4, let $(M, \mathcal{J}_1', \mathcal{J}_2'; \mathcal{G}')$ be the $B_1$-transformed generalized Kähler structure for $B_1 \in \Omega^2(M)\tilde{G}$. Let all other choices be the same. Then the effective Courant algebroid $T_\mu Q'$ induced from $(\mathcal{J}_1', \mathcal{J}_2'; \mathcal{G}')$ is a $b$-transformation of $T_\mu Q$, for some $b \in \Omega^2(\tilde{Q})$.

**Proof:** Choose a connection form $\tilde{\theta}$ of the $\tilde{G}$-principle bundle $M_0 \to \tilde{Q}$ and with respect to a choice of basis of $\mathfrak{g}$ we have $\hat{\theta}_j$ and $\tilde{X}_j$. Consider the form $\hat{b} = \prod_j (1 - \hat{\theta}_j \wedge \iota_{\tilde{X}_j})B_1|_{M_0}$, where the terms in brackets are considered operators on $\Omega^2(M_0)$. Then $\hat{b}$ is horizontal with respect to $\tilde{G}$-action and the transformation $e^\hat{b}$ preserves $\text{Ann}(\mu^*\hat{\theta}, \mathcal{J}_1(\mu^*\hat{\theta}), \mathcal{J}_2(\mu^*\hat{\theta}))$. From which the result follows. \hfill \Box

## 5. Courant and $T$-duality

The **Courant duality** is the following. Consider a bi-Hamiltonian action as in definition 3.21, suppose that reduction of $G$- (resp. $\tilde{G}$-) action at $\hat{e} \in \hat{G}$ (resp. at $e \in G$) as given in theorem 4.4 exists and denote it $(Q, \mathcal{J}_1, \mathcal{J}_2)$ (resp. $(\tilde{Q}, \hat{\mathcal{J}}_1, \hat{\mathcal{J}}_2)$):

**Definition 5.1.** The structures $(Q, \mathcal{J}_1, \mathcal{J}_2)$ and $(\tilde{Q}, \hat{\mathcal{J}}_1, \hat{\mathcal{J}}_2)$ are Hamiltonian dual to each other. When the assumptions of theorem 4.4 holds, the structures are said to be Courant dual to each other.

Geometrically, the Hamiltonian duality as defined above has a significant drawback: \textit{a priori}, the level sets $M_e = \mu^{-1}(\hat{e})$ and $M_{\hat{e}} = \hat{\mu}^{-1}(e)$ might have nothing to do with each other and the relation between the geometry and topology of the quotients $Q$ and $\tilde{Q}$ may not be clear. For Courant duality, the relation of the topology and geometry can be understood much better.

**Proposition 5.2.** Assume the conditions in theorem 4.4 and that $(\hat{G}, G, \tilde{G})$ is a connected double Lie group, we have the following diagram, where the maps are principle bundles of compact Lie groups:

\[
\begin{array}{ccc}
M_0 = Q \times \tilde{Q} & \xrightarrow{\pi} & Q \\
\downarrow p & & \downarrow \pi \\
\tilde{Q} & & \tilde{Q}
\end{array}
\]

**Proof:** Recall that $G \times \hat{G} \to \tilde{G} : (g, \hat{g}) \mapsto gg^{-1}$ as well $\hat{G} \times G \to \tilde{G} : (\hat{g}, g) \mapsto \hat{g}g^{-1}$ are diffeomorphisms for the double Lie group $(\hat{G}, G, \tilde{G})$. The left action of $\hat{G}$ on $Q$ is induced from:

\[
\hat{g} \circ g^{-1}x = \hat{g}g^{-1}x \text{ for } x \in M_0,
\]

while the left action of $G$ on $\tilde{Q}$ is induced from

\[
g \circ \hat{g}^{-1}x = g\hat{g}^{-1}x \text{ for } x \in M_0.
\]

These two actions are both free with the same quotient space $\tilde{Q} = M_0/\tilde{G}$. \hfill \Box

The choice of terminology is justified by the following:

**Proposition 5.3.** With the same conditions as in proposition 5.2, the Courant algebroids on $\tilde{Q}$ formed by the invariant sections of $T\tilde{Q}$ and $\hat{T}\tilde{Q}$ are isomorphic to the one defined by theorem 4.4.
Proof: Note that the invariant sections of $\mathcal{T}Q$ lifts to $M_0$ as the $\tilde{G}$-invariant sections of $\text{Ann}(\mu^*\tilde{\Theta})$, which is isomorphic to $\mathcal{T}_\mu^\dual M_0$ by lemma 3.19. The proposition then follows. □

When the action of $G$ and $\tilde{G}$ commute, we have:

Proposition 5.4. Let $(\hat{g}, g, \tilde{g})$ be the Manin triple defined by a factorizable Lie bialgebra $g$. If $[g, \tilde{g}] = 0$, then $[\hat{g}, g] = 0$, i.e. $\hat{g}$ is abelian, and we write $(\hat{g}, g, \tilde{g}) = (t, t, \hat{t})$.

Proof: By (5.4), we have for $\tau \in g$ and $\hat{\omega} \in \hat{g}$:

$$[(\tau, \tau), (r_\tau(\hat{\omega}), \tilde{r}_\tau(\hat{\omega}))] = [\tau, \tilde{r}_\tau(\hat{\omega})] = 0,$$

which implies that $[\tau, \hat{\omega}] = 0$. Since $\hat{g}$ is invertible, we see that $g$ is abelian. Then $(\hat{g}, g, \tilde{g})$ form a Manin triple implies that $\hat{g}$ and $\tilde{g}$ are abelian as well. □

Because of this, in the following we work under the assumption 4.2 and that $\tilde{G}$ is abelian. The notations $T, T$ and $\tilde{T}$ will mean that the respective groups are compact, i.e. torus.

Lemma 5.5. Both $J_1$ and $J_2$ are preserved by the $\tilde{T}$-action. For any $\tau \in t$ and $\hat{\omega} \in \hat{t}$, we have $d(J_1(\mu^*\tilde{\theta}_\tau), J_2(\tilde{\mu}^*\tilde{\theta}_\omega)) = 0$. We define the pairing

$$P : t \otimes \hat{t} \to R : \tau \otimes \hat{\omega} \mapsto 2\langle J_1(\mu^*\tilde{\theta}_\tau), J_2(\tilde{\mu}^*\tilde{\theta}_\omega) \rangle$$

then $P$ is non-degenerate, i.e. $\tau = 0 \in t \iff P(\tau, \hat{\omega}) = 0$ for all $\hat{\omega} \in \hat{t}$ and vice versa for $\hat{\omega}$.

Proof: By definition, the t-action preserves $J_2$ and t-action preserves $J_1$. Then by the proof of lemma 3.19 and $d\theta = 0$ from abelian-ness, it follows that $J_1$ as well as $\mu^*(\tilde{\theta}_\tau)$ are preserved by $t$. Thus the $\tilde{T}$-action preserves $J_1$. Similarly, $J_2$ and $\tilde{\mu}^*(\tilde{\theta}_\omega)$ are preserved by the $T$-action. We have:

$$J_1(\mu^*\tilde{\theta}_\tau) *_{\tilde{H}} J_2(\tilde{\mu}^*\tilde{\theta}_\omega) = 0 \quad \text{and} \quad J_2(\tilde{\mu}^*\tilde{\theta}_\omega) *_{\tilde{H}} J_1(\mu^*\tilde{\theta}_\tau) = 0.$$

Add the above two equations, we see that along $M_0$,

$$d(J_1(\mu^*\tilde{\theta}_\tau), J_2(\tilde{\mu}^*\tilde{\theta}_\omega)) = 0.$$

That $P$ is non-degenerate follows from non-degenerate-ness of the extended metric $\mathcal{G}$. □

As corollary of theorem 5.21 and theorem 4.6 we note that for bi-Hamiltonian action of $(\tilde{T}, T, \tilde{T})$ where all groups acting properly and freely, the reduction as described in theorem 4.6 factorizes in two ways, via $Q$ or $\tilde{Q}$, thus

Definition 5.6. The structures $\mathcal{T}Q$ and $\tilde{Q} \tilde{Q}$ are said to be (Courant) $T$-dual to each other. □

Assumption 5.7. In the rest of this section, we assume that the $\tilde{T}$-action preserves a splitting of $\mathcal{T}M$.

Consider the reduced structures on $Q = M_0/T$ and $\hat{Q} = M_0/\hat{T}$. By corollary 4.3, the structures are both twisted generalized Kähler structures, whose twisting form can be described with a choice of connection forms. Let $\hat{\Theta}$ be a connection form on $M_0$ as principle $\hat{T}$-bundle. Choose basis $\{\tau_j\}$ and $\{\hat{\tau}_j\}$ of $t$ and $\hat{t}$ respectively, and denote $\theta_j, X_j + \xi_j, \Theta_j$ and $\theta_j, \hat{X}_j + \hat{\xi}_j, \hat{\Theta}_j$ the corresponding components. We define:

$$\hat{B} = B + \hat{B} = \left( \Theta \wedge \xi - \frac{1}{2} \sum_{j,k} \Theta_j \wedge \Theta_k \cdot \tau_{X_k} \xi_j \right) + \left( \hat{\Theta} \wedge \hat{\xi} - \frac{1}{2} \sum_{j,k} \hat{\Theta}_j \wedge \hat{\Theta}_k \cdot \tau_{X_k} \hat{\xi}_j \right).$$

Then $\hat{B}$ is $\hat{T}$-invariant on $M_0$. When the actions generated by $t$ and $\hat{t}$ are proper, the forms $\Theta$ and $\hat{\Theta}$ become connection forms on $M_0$ as respectively $T$ and $\hat{T}$ principle bundles.
Theorem 5.8. Suppose assumption 3.4 holds and let \((M, \mathcal{J}_1, \mathcal{J}_2; G \tilde{B})\) be the \(\tilde{B}\)-transformed structure on \(M\), with \(B\) defined above. Then the induced Courant algebroid on \(\tilde{Q} = M_0 / T\) remains unchanged. Let \(h\) (resp. \(\hat{h}\)) be the twisting form of the corresponding reduced structure on \(Q\) (resp. \(\hat{Q}\)), then
\[
\hat{\pi}^* \hat{h} - \pi^* h = d(\hat{\Theta} \wedge \Theta),
\]
where on the right hand side we use also the pairing \((\hat{\pi}^* h, \pi^* h)\).

Proof: Direct computation shows that the \(T\)-horizontal part of \(\tilde{B}\) is 0. Thus by corollary 4.7, the Courant algebroid structure on \(\tilde{Q}\) remains unchanged under the \(-\tilde{B}\)-transformation.

The \(\tilde{B}\)-transformed structures on \(M\) has twisting form \(\tilde{H} = H + d\tilde{B}\). Let \(\mathcal{J}^B_1 = e^{-\tilde{B}} \mathcal{J}_1 e^B\) and so on. We compute
\[
i_{\mathcal{X}_i} \hat{B} = \xi_i - \Theta \cdot i_{\mathcal{X}_i} \hat{\xi} = \xi_i - \sum_j \hat{\Theta}_j \cdot i_{X_j} \hat{\xi}_j
\]
and it follows that \(\mathcal{J}^B_1(\mu^* \hat{\Theta}_j) = X_i + \xi'_i\) where \(\xi'_i = \sum_j \hat{\Theta}_j \cdot i_{X_j} \hat{\xi}_j\). We note that \(i_{\mathcal{X}_i} \xi'_i = 0\), and the twisting form \(h\) satisfies
\[
\pi^* h = \tilde{H} + d(\Theta \wedge \xi').
\]
Similarly, we have \(\hat{\pi}^* \hat{h} = \hat{H} + d(\hat{\Theta} \wedge \hat{\xi}')\) where \(\hat{\xi}' = \sum_j \hat{\Theta}_j \cdot i_{\mathcal{X}_i} \hat{\xi}_j\). More explicitly, we compute
\[
\hat{\pi}^* \hat{h} - \pi^* h = d \left( \sum_{j,k} \hat{\Theta}_j \wedge \Theta_k \cdot i_{\mathcal{X}_i} \hat{\xi}_k - \sum_{j,k} \Theta_k \wedge \hat{\Theta}_j \cdot i_{\mathcal{X}_k} \hat{\xi}_j \right)
\]
\[
= d \sum_{j,k} \hat{\Theta}_j \wedge \Theta_k \cdot (i_{\mathcal{X}_i} \hat{\xi}_k + i_{\mathcal{X}_k} \hat{\xi}_j)
\]
\[
= d \sum_{j,k} \hat{\Theta}_j \wedge \Theta_k \cdot 2 \langle \mathcal{J}^B_2(\mu^* \hat{\Theta}_j), \mathcal{J}^B_1(\mu^* \hat{\Theta}_k) \rangle
\]
\[
= d(\hat{\Theta} \wedge \Theta),
\]
where the last step we use the pairing \(P\) as given in (5.3).

Remark 5.9. We note that the \(T\)- or \(\hat{T}\)-horizontal part of \(\tilde{B}\) in general do not vanish. Thus the structures on \(Q\) and \(\hat{Q}\) are \(B\)-transformed from their respective original structures. With proposition 5.3 the theorem above states that the Courant algebroid on \(\tilde{Q}\) formed by the set of invariant sections of \(TQ\) or \(\hat{T}Q\) are still isomorphic to the original one. We note also that the equation (5.3) coincides with the equation in the physics literature, where \(M_0\) is to be the correspondence space of the \(T\)-dual bundles \(Q\) and \(\hat{Q}\). It’s shown (e.g. [8]) that the twisted cohomology of \(T\)-dual principle bundles are isomorphic. Since the twisted cohomology only depends on the cohomology class of the twisting, the same is true for the structures on \(Q\) and \(\hat{Q}\) before applying \(B\)-transformation. In [8], proposition 5.3 is shown when \(Q\) and \(\hat{Q}\) are \(T\)-dual \(S^1\)-principle bundles, with twisted generalized complex structures, by directly defining the isomorphism.

5.10 Example. The following example is considered in [14] and we recall the setup and point out its relevance to the current discussion. Let \(M = \mathbb{C}^2 \setminus \{(0,0)\}\) and consider the coordinates \(z = (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) = (x_1, y_1, x_2, y_2)\). Let \(r^2 = |z_1|^2 + |z_2|^2\) and \(J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\).

Consider the following structures:
\[
\mathcal{J}_1 = \begin{pmatrix} 0 & 0 & r^2 J & 0 \\ 0 & -J & 0 & 0 \\ r^{-2} J & 0 & 0 & 0 \\ 0 & 0 & 0 & -J \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} J & 0 & 0 & 0 \\ 0 & 0 & -r^2 J & 0 \\ 0 & 0 & J & 0 \\ 0 & -r^{-2} J & 0 & 0 \end{pmatrix}.
\]
where the labelling on rows are \((Tz_1, Tz_2, T^*z_1, T^*z_2)\). Then \((M, \mathcal{J}_1, \mathcal{J}_2)\) is an \(H\)-twisted generalized Kähler structure where

\[
H = -\sin(2\lambda)d\lambda \wedge d\phi_1 \wedge d\phi_2
\]

in the polar coordinates \((z_1, z_2) = r(e^{i\phi_1} \sin \lambda, e^{i\phi_2} \cos \lambda)\). In particular, \([H] \neq 0 \in H^3(M)\) (cf. [12]).

Let \(\hat{T} = S^1 \times S^1\) and \((e^{i\phi_1}, e^{i\phi_2})\) be the coordinates. It acts on \(M\) via

\[
(e^{i\phi_1}, e^{i\phi_2}) \circ (z_1, z_2) = (e^{i\phi_1}z_1, e^{-i\phi_2}z_2).
\]

Let \(T\) and \(\hat{T}\) be the first and second \(S^1\) respectively, then \((\hat{T}, T, \hat{T})\) is a double Lie group and the action of \(\hat{T}\) is bi-Hamiltonian and satisfies assumption [1,4] with the common moment map \(f = \ln r\). The \(T\) and \(\hat{T}\) actions are generated respectively by

\[
\mathcal{J}_1(df) = -\cos^2 \lambda d\phi_2 \quad \text{and} \quad \mathcal{J}_2(df) = -\cos^2 \lambda d\phi_1.
\]

It follows that \(2(\mathcal{J}_1(df), \mathcal{J}_2(df)) = 1\), i.e. \(T\) and \(\hat{T}\) are dual tori in the standard sense. We note that the actions are not free. Consider the submanifold \(M' = M \setminus \{z_1 = 0 \cup \{z_2 = 0\}\}\), on which the actions are free. The reduced structures of the \(T\) and \(\hat{T}\) action on \(M'\) are respectively the opposite and standard Kähler structures on \(D^2 \setminus \{0\}\). Our results then state that they are \(T\)-dual to each other.

6. \(T\)-duality group

We first consider the linear case and use the notations and assumptions of lemma [1,4]. We note that the natural pairing on \(V\) induces a pairing \(P_K\) on \(\mathcal{J}_1(K) \oplus \mathcal{J}_2(K)\), which can also be seen as induced from the pairing \(\langle \cdot, G(\cdot)\rangle\) defined on \(K\) by \(G\). By the positive definiteness of \(G\) we see that \(P_K\) has signature \((m, m)\) where \(m = \dim K\). Completely parallel to lemma [3,21] we have

**Lemma 6.1.** We use the notations and assumptions of lemma [1,4]. Let \(K' \subset \mathcal{J}_1(K) \oplus \mathcal{J}_2(K)\) be a maximal isotropic subspace with respect to \(P_K\) and \(N' = a(K')\) where \(a : W \rightarrow V\) is the projection, then there is a self-dual exact sequence:

\[
0 \rightarrow W_{K'}^* \xrightarrow{a_{K'}} W_K \xrightarrow{a_K} W_{K'} \rightarrow 0,
\]

with \(W_{K'} = \frac{\text{Ann}_V(K)}{N'}\) and \(W_K = \frac{\text{Ann}_V(N')}{K}\),

where \(W_K = \text{Ann}(K, K', G(K'), G(K)) = \text{Ann}(K, J_1(K), J_2(K), G(K))\).

**Proof:** By the conditions in lemma [1,4], we see that \(K' \cap V^* = \{0\}, K \cap K' \subset \text{Ann}(K, K')\). Let \(a_{K'}\) be the map induced from \(a\). Let \(U_{K'} = \text{Ann}(K, K')\), then the kernel of the induced map \(U_{K'} \rightarrow W_{K'}\) is \(\text{Ann}_{V'}(N') \oplus K'\) and thus the kernel of \(a_{K'}\) is

\[
\ker a_{K'} = (\text{Ann}_{V'}(N') \oplus K') \cap U_{K'}.
\]

By the decomposition \(U_{K'} = \overline{W}_K \oplus (K \cap K')\) and inclusion \(K \cap K' \subset \text{Ann}_{V'}(N') \oplus K'\), we see that \(\ker a_{K'} \simeq \frac{\text{Ann}_{V'}(N')}{K'}\). Since \(\ker a_{K'}\) is maximally isotropic with respect to the induced pairing \(\langle \cdot, K \rangle_{\text{Ann}}\) on \(W_K\), we see that the exact sequence is self-dual.

Using the notations in (the proof of) theorem [3,21] we have

**Proposition 6.2.** Under the condition of theorem [1,8] and let \(T' \subset \hat{T}\) be a maximally isotropic subtorus of \(\hat{T}\) with respect to the pairing \(P\) as in [1,4], i.e., the Lie algebra \(V\) is a Lagrangian subspace of \(T\). Then the reduced space \(Q' = M_0/T'\) has a natural extended Kähler structure.

**Proof:** By the proof of theorem [1,8], the bundles \((\mu^* \hat{\theta}), (\mu^* \hat{\theta}, J_1(\mu^* \hat{\theta}), J_2(\mu^* \hat{\theta}))\) and \(\tau_{\mu} M_0\) are all preserved by the \(\hat{T}\)-action, and thus are preserved by the \(T'\)-action. Let \(K = (\mu^* \hat{\theta})\) and \(K'\) be the subbundle generated by the infinitesimal fields \(\{X_{\tau'} + \xi_{\tau'} \mid \tau' \in K'\}\), then it follows
from the proofs of lemmata \[ \text{(3.1)} \] and \[ \text{(3.3)} \] that \((\mu^*\hat{\theta}, K')\) is preserved by the \(T'\)-action. Since \(T'\) is isotropic, we have \(K \oplus K' \subset \text{Ann}(K, K')\) and lemma \[ \text{(5.3)} \] (1) gives an extended tangent bundle \(\mathcal{T}Q'\) on \(Q'\). It follows from proposition \[ \text{(5.3)} \] that \(J_1\) and \(J_2\) are both invariant with respect to the \(T'\)-action and thus descend to \(J_1'\) and \(J_2'\) on \(\mathcal{T}Q'\), which define an extended Kähler structure. \[ \square \]

The group \(O(m, m; \mathbb{Z})\) is called the \(T\)-duality group in the physics literature \((\cite{29}, \cite{3})\) and the references therein). In the physics literature, for each element of \(O(m, m; \mathbb{Z})\) it is associated a pair of related \(T\)-dual principle bundles with \(H\)-fluxes. The physical theory on such related structures are expected to be the same. In our construction, the following holds.

**Corollary 6.3.** Suppose that the action of \(\hat{T}\) preserves a splitting of \(\mathcal{T}M\). Let \(g \in O(m, m; \mathbb{Z})\) and consider the pair of Lagrangian subgroups \(T_g\) and \(\hat{T}_g\) with Lie algebra \(g(t)\) and \(g(\hat{t})\). Let \(Q_g\) and \(\hat{Q}_g\) be the reduction of \(M_0\) by the groups \(T_g\) and \(\hat{T}_g\) respectively. Then the induced structures on \(Q_g\) and \(\hat{Q}_g\) are twisted generalized Kähler structures and the equation \[ \text{(5.3)} \] holds for this pair after applying certain \(B\)-transformation. The Courant algebroid on \(Q\) defined by the \(\hat{T}_g\)-invariant sections of \(\mathcal{T}Q_g\) is isomorphic to the one given by theorem \[ \text{(5.4)} \].

**Proof:** Similar to \[ \text{(3.2)} \], we choose basis \(\{\tau^g_j\}\) and \(\{\hat{\tau}^g_j\}\) of \(\mathfrak{t}'\) and \(\mathfrak{t}\) respectively and let \(X^g_j + \xi^g_j, \Theta^g_j\) and \(\hat{X}^g_j + \hat{\xi}^g_j, \hat{\Theta}^g_j\) be the corresponding components, and define

\[
\hat{\mathcal{B}}^g = \mathcal{B}^g + \hat{\mathcal{B}}^g = \left( \Theta^g \wedge \xi^g - \frac{1}{2} \sum_{j,k} \Theta^g_j \wedge \Theta^g_k \cdot t_{X^g_j} \xi^g_k \right) + \left( \hat{\Theta}^g \wedge \hat{\xi}^g - \frac{1}{2} \sum_{j,k} \hat{\Theta}^g_j \wedge \hat{\Theta}^g_k \cdot t_{\hat{X}^g_j} \hat{\xi}^g_k \right).
\]

In particular, the basis \(\{\tau^g_j\}\) and \(\{\hat{\tau}^g_j\}\) can be taken as the transformation of \(\{\tau_j\}\) and \(\{\hat{\tau}_j\}\) by \(g\). The proof of \[ \text{(5.3)} \] is then completely parallel to that of theorem \[ \text{(5.8)} \]. The isomorphism of courant algebroids is straightforward. \[ \square \]

**6.4. Example.** We consider in detail the special case when \(g = e^b\) where \(b : \mathfrak{t} \to \mathfrak{i}\) is skew-symmetric with respect to the pairing \(P\). Then \(g(t) = \text{graph}(b)\) and \(g(\hat{t}) = \hat{\mathfrak{i}}\). Let \(\{\tau_j\}\) and \(\{\hat{\tau}_j\}\) be basis of \(\mathfrak{t}\) and \(\mathfrak{i}\) respectively and \((b_{ij})\) the matrix of \(b\) with respect to these basis. Then \(\{\tau^b_j = \tau_j + \sum_k b_{kj} \hat{\tau}_k\}\) is a basis of \(g(t)\), where \(b(\tau_j) = \sum_k b_{kj} \hat{\tau}_k \in \mathfrak{i}\). Let \(b^b\) to denote the objects for the transformed structures, then

\[
\left\{ \begin{array}{l}
\Theta^b_j = \Theta_j \\
\hat{\Theta}^b_j = \hat{\Theta}_j - \sum_k b_{jk} \Theta_k
\end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l}
\hat{X}^b_j + \hat{\xi}^b_j = \hat{X}_j + \hat{\xi}_j \\
X^b_j + \xi^b_j = X_j + \xi_j + \sum_k b_{kj}(\hat{X}_k + \hat{\xi}_k).
\end{array} \right.
\]

Direct computation gives

\[
(\pi^b)^* h^b = \pi^* h - d \sum_{j,k,l} b_{lj} \Theta_k \wedge \Theta_j \langle X_k + \xi_k, \hat{X}_l + \hat{\xi}_l \rangle
\]

\[
(\hat{\pi}^b)^* \hat{h}^b = \hat{\pi}^* \hat{h} - d \sum_{j,k,l} b_{lj} \Theta_k \wedge \Theta_j \langle X_k + \xi_k, \hat{X}_l + \hat{\xi}_l \rangle
\]

which implies

\[
(\hat{\pi}^b)^* \hat{h}^b - (\pi^b)^* h^b = d \sum_{j,k} \hat{\Theta}^b_j \wedge \Theta^b_k \cdot (t_{\hat{X}^b_j} \xi^b_k + t_{\hat{X}^b_j} \hat{\xi}^b_k),
\]

i.e., the equation \[ \text{(5.3)} \] holds for the pair of reduced structures \(Q^b\) and \(\hat{Q}^b\). Since \(\hat{\mathfrak{i}} = \mathfrak{i}\), we have \(\hat{Q}^b = Q\), while the twisting form \(\hat{h}\) is changed by an exact term. As the situation for \(\mathfrak{t}\) and \(\mathfrak{i}\) is symmetric, we may consider \(e^b\) for skew-symmetric \(\beta : \mathfrak{i} \to \mathfrak{t}\) and obtain similar result.
6.5. Example. The example discussed in §5.10 does not admit interesting T-duality group action, because $O(1,1;\mathbb{Z}) = \{\pm1,\pm\left(\frac{1}{2}\right)\}$. This can be compensated by considering a product of these, e.g. twisted structures on $M^2$, for example, and apply §6.4. Instead, here we consider another situation which is not quite covered by T-duality group. For the example in §5.10 we consider the anti-diagonal action generated by
\[
X_d + \xi_d = \left(\frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2}\right) - (\cos^2 \lambda d\phi_2 + \sin^2 \lambda d\phi_1),
\]
then $\iota_{X_d} \xi_d = -1$. Let $\mathcal{K} = (d\mu)$ and $\mathcal{K}' = (X_d + \xi_d)$, then the condition for lemma 7.2 (1) does not hold. On the other hand, the condition for lemma 7.4 (2) holds and there is an induced effective Courant algebroid on the corresponding reduced space, i.e. $S^2$. A more general result holds:

**Proposition 6.6.** Let $T^+ \subset \tilde{T}$ be a non-degenerate subtorus of $\tilde{T}$ with respect to $P$, i.e. the restriction of $P$ on its Lie algebra $\mathfrak{t}^+$ is non-degenerate, then there is a natural effective Courant algebroid on the reduced space $Q^+ = M_0/T^+$.

\[
\square
\]

7. Appendix A: Reduction of extended tangent bundle

Special case of the reduction of Courant algebroid has been discussed implicitly in our paper \cite{13} in showing that extended complex structure exists as the result of reduction of generalized complex manifold and in general it has been discussed explicitly in the works \cite{8}, \cite{14} and \cite{27}. For the sake of completeness, we prove the reduction of Courant algebroid in the relevant context of our construction in this article, i.e. for extended tangent bundles. We will use the notations in §4.

**Lemma 7.1.** Let $\mathbb{V} = V \oplus V^*$ with the natural pairing $\langle \cdot, \cdot \rangle$, $K \subset V^*$ and $K' \subset \mathbb{V}$ so that $K' \cap V^* = \{0\}$. Define $N' = a \circ K'$ where $a : \mathbb{V} \to V$ is the projection. Assume that

1. $K + K' \subset \text{Ann}(K,K')$ and
2. $K' \cap V^* = \{0\}$.

Let $\mathbb{V}_K = \frac{\text{Ann}(K,K')}{(K,K')}$, then $\langle \cdot, \cdot \rangle$ descends to non-degenerate pairing $\langle \cdot, \cdot \rangle_K$ on $\mathbb{V}_K$ and we have the self-dual exact sequence
\[
0 \to W^*_K \to \mathbb{V}_K \xrightarrow{\alpha_K} W_K \to 0 \quad \text{where} \quad W_K = \frac{\text{Ann}_V(K)}{N'}. \]

**Proof:** See \cite{13}, lemma 4.3. \hfill $\square$

**Lemma 7.2.** Using the same notations as in lemma 7.1 and replacing assumption (1) by one of the following statements that are equivalent to each other:

1. $K \subset \text{Ann}(K,K')$ and $\langle \cdot, \cdot \rangle$ induces a non-degenerate pairing on $\mathbb{V}_K'$ where $\mathbb{V}_K = \frac{\text{Ann}(K,K')}{K}$,
2. $K \subset \text{Ann}(K,K')$ and $\langle \cdot, \cdot \rangle$ restricts to a non-degenerate pairing on $K'$.

then we have the exact sequence:
\[
0 \to W^*_K \to \mathbb{V}_K' \xrightarrow{\alpha_K} \text{Ann}_V(K) \to 0. \]

**Proof:** The surjectivity of $\text{Ann}(K,K') \to \text{Ann}_V(K)$ is easy and everything then follows. \hfill $\square$

**Definition 7.3.** Let $S$ be a subspace of sections in $TM$ which is closed with respect to $\ast$. A closed subspace $S' \subset S$ is a two-sided null ideal if $S \ast S' \subset S'$, $S' \ast S \subset S'$ and $\langle S', S \rangle = 0$.

It follows that when $S'$ is a two-sided null ideal of $S$, the structure $(S, \ast, \langle \cdot, \cdot \rangle)$ induces one such structure on the quotient space $S/S'$, which also satisfies (\cite{23}) and (\cite{24}).
Lemma 7.4. Let $(M, TM)$ be a manifold with an extended tangent bundle $TM$ and $M_0 \subset M$ a submanifold. Let $\mathcal{K} \subset T^*M|_{M_0}$ and $\mathcal{K}' \subset TM|_{M_0}$ be two subbundles of rank $m$ and $m'$ respectively so that $TM_0 = \text{Ann}_{TM}(\mathcal{K})$ and $\mathcal{K}' \cap T^*M = \{0\}$. Suppose that $\mathcal{K}$ is generated by sections $\{\theta_j\}_{j=1}^{m}$ so that $d\theta_j \in \Gamma(\wedge^2 \mathcal{K})$ and $\mathcal{K}'$ is generated by sections $\{X_j\}_{j=1}^{m'}$. Let $\tilde{\sigma}$ be the infinitesimal action generated by $\{X_j\}_{j=1}^{m'}$ via the Loday bracket $\ast$. Suppose that $\text{Ann}(\mathcal{K}, \mathcal{K}')$ is preserved by $\tilde{\sigma}$.

If furthermore, we suppose that the action $\tilde{\sigma}$ on $M_0$ is induced by a morphism $G \to \mathcal{G}_H$ where $G$ is compact of dimension $m'$ and the geometrical action $\sigma$ is free. Let $Q = M_0/G$ then

1. If $(\mathcal{K}, \mathcal{K}') \subset \text{Ann}(\mathcal{K}, \mathcal{K}')$, then $\frac{\text{Ann}(\mathcal{K}, \mathcal{K}')}{\mathcal{K}}$ descends to an extended tangent bundle on $Q$.
2. If $\mathcal{K} \subset \text{Ann}(\mathcal{K}, \mathcal{K}')$ is preserved by $\sigma$ and $\langle , \rangle$ induces a non-degenerate pairing on $\mathcal{K}'$, then $\frac{\text{Ann}(\mathcal{K}, \mathcal{K}')}{\mathcal{K}}$ descends to an effective Courant algebroid on $Q$.

Proof: Let $\alpha : TM \to TM$ be the projection. Let $X, X' \in \Gamma(\text{Ann}(\mathcal{K}, \mathcal{K}'))$, then

$$\langle X_j, X \rangle = 0$$

From the assumptions, $\text{Ann}_{TM}(\mathcal{K})$ is an integrable distribution and $M_0$ is a leaf of this distribution. We also have $\langle X \ast_H X', \theta_j \rangle = \iota_{X,X'} \theta_j = 0$ by assumption on $\theta_j$. It then follows that

$$\langle X \ast_H \theta_j, X' \rangle = X'(X, \theta_j) - \langle X \ast_H X', \theta_j \rangle = 0,$$

i.e. $X \ast_H \theta_j \in \Gamma(\mathcal{K}, \mathcal{K}')$. Similarly we have $\theta_j \ast_H X \in \Gamma(\mathcal{K}, \mathcal{K}')$. Let $N' = a(\mathcal{K}')$, then $N' \subset \text{Ann}_{TM}(\mathcal{K})$. We see that the (geometrical) action of $G$ preserves $M_0$ and the quotient $Q$ is well-defined.

Case (1). Let

$$S_1 = \{ X \in \Gamma(\text{Ann}(\mathcal{K}, \mathcal{K}')) | \forall \mathcal{Y} \in \Gamma(\mathcal{K}), \text{ for all } \mathcal{Y} \in \Gamma(\mathcal{K}') \}.$$

For any $X, X' \in S_1$, $\mathcal{Y} \in \Gamma(\mathcal{K})$ and $\mathcal{Z} \in \Gamma(\text{Ann}(\mathcal{K}, \mathcal{K}'))$ we compute

$$\langle X \ast_H X', \mathcal{Y} \rangle = a(X) \langle X', \mathcal{Y} \rangle - \langle X', X \ast_H \mathcal{Y} \rangle = \langle X', \mathcal{Y} \ast_H X \rangle - a(X) \langle \mathcal{Y}, X \rangle = 0,$$

$$\langle X \ast_H \mathcal{Y}, \mathcal{Z} \rangle = a(\mathcal{Y}) \langle X, \mathcal{Z} \rangle - \langle \mathcal{Y} \ast_H X, \mathcal{Z} \rangle = 0.$$

Thus we have $X \ast_H X' \in \Gamma(\text{Ann}(\mathcal{K}, \mathcal{K}'))$ as well as $X \ast_H \mathcal{W}$ and $\mathcal{W} \ast_H X \in \Gamma(\mathcal{K}, \mathcal{K}')$ for all $\mathcal{W} \in \Gamma(\mathcal{K}', \mathcal{K}')$. It follows that

$$\forall \mathcal{Y} \ast_H (X \ast_H X') = (\mathcal{Y} \ast_H X) \ast_H X' + \mathcal{X} \ast_H (\mathcal{Y} \ast_H X') \Rightarrow X \ast_H X' \in S_1,$$

i.e. $S_1$ is closed under $\ast_H$ and $\Gamma(\mathcal{K}, \mathcal{K}')$ is a two-sided null ideal in $S_1$. Thus the structures $\{\ast_H, \langle , \rangle\}$ descends to $\{\ast^G, \langle, \rangle^G\}$ on $\frac{S_1}{\Gamma(\mathcal{K}, \mathcal{K}')}$.

Case (2). Let

$$S_2 = \{ X \in \Gamma(\text{Ann}(\mathcal{K}, \mathcal{K}')) | X \ast_H X \in \Gamma(\mathcal{K}) \}.$$

Completely parallel to case (1) above, we see that $S_2$ is closed under $\ast_H$. By definition $\Gamma(\mathcal{K})$ is a two-sided null ideal in $S_2$ with respect to $\{\ast_H, \langle , \rangle\}$. By lemma 7.2, $T^*Q \subset \ker a_Q'$.

Let $\Theta$ be a connection form on $\pi : M_0 \to Q$, and $\Theta_j$ the component dual to $X_j$.

Corollary 7.5. In the above lemma, suppose that the action of $G$ on $TM$ preserves a splitting into $TM$ with $H$-twisted structures. Let $X_j = X_j + \xi_j$ under the splitting. Then $TQ$ splits into $TQ$ with $h$-twisted structures, where $\pi^*h = H + dB$ with

$$B = \Theta \wedge \xi_m - \frac{1}{2} \sum_{j,k} \Theta_j \wedge \Theta_k \cdot \iota_{X_j} \xi_k.$$
Proof: The action preserving the splitting implies that $d\xi_j = \iota_X_j H$ and $\mathcal{L}_X_j H = 0$. It follows that $\mathcal{L}_X_j B = 0$ for all $\tau \in \mathfrak{g}$. Let $B' = \prod_j (1 - \Theta_j \wedge \iota_X_j) B$ be the horizontal part of $B$, where $\Theta_j \wedge \iota_X_j$ is interpreted as an operator on $\Omega^2(M)$. Direct computation gives $\iota_X_j B = \xi_j$ and $B' = 0$. Apply $B$-transformation (or choose a different splitting), we have $\text{Ann}(\mathcal{K}, \mathcal{K}') \rightarrow \text{Ann}(\mathcal{K}, \{X_i\})$ and it defines a splitting of $\mathcal{T}Q$. Under the $B$-transformation, the twisting form becomes $H' = H + dB$ and we compute

$$\iota_X_j (H + dB) = \iota_X_j H + \mathcal{L}_X_j B - d\iota_X_j B = \iota_X_j H - d\xi_j = 0.$$  

It follows that there is $h \in \Omega^3_0(Q)$, so that $\pi^* h = H + dB$, which gives the twisting form of the induced splitting of $\mathcal{T}Q$. \hfill \Box

Remark 7.6. We note that $\mathcal{L}_X_j H = 0$ and $d\xi_j = \iota_X_j H$ for all $j$ implies that $d_G(H + \sum_j \xi_j u_j) = 0$, where $d_G$ is the equivariant differential in the equivariant Cartan complex. Then $h$ in the above gives an explicit description of the image of $[H + \sum_j \xi_j u_j]$ under the isomorphism $H_G(M_0) \xrightarrow{\sim} H(Q)$. From here, it again follows that $[h]$ is independent of the choice of the connection form.

8. Appendix B: Poisson Lie group and actions

The material in this subsection is taken from [28, 2] and [1] (the first three chapters). More details can be found there as well as the references therein. We follow the convention [3, 8].

Definition 8.1. A Lie group $G$ is called a Poisson Lie group if it is also a Poisson manifold such that the multiplication map $m : G \times G \rightarrow G$ is a Poisson map, where $G \times G$ is equipped with the product Poisson structure.

Let $\pi$ be a multiplicative Poisson tensor on $G$, then $\pi|_e = 0$ where $e \in G$ is the identity, and the linearization of $\pi$ at $e$ defines on $\mathfrak{g} = \mathfrak{g}^*$ a structure of Lie algebra [1]. From [28].

Theorem 8.2. The right (left) invariant 1-forms on a Poisson Lie group $(G, \pi)$ form a Lie subalgebra of $\Omega^1(G)$ with respect to the bracket

$$(8.1) \quad [\theta_\tau, \theta_\omega]^* = -d\pi(\theta_\tau, \theta_\omega) + \mathcal{L}_{\pi(\theta_\tau)} \theta_\omega - \mathcal{L}_{\pi(\theta_\omega)} \theta_\tau = \mathcal{L}_{\pi(\theta_\tau)} \theta_\omega - \iota_{\pi(\theta_\omega)} d\theta_\tau \quad \text{for } \tau, \omega \in \mathfrak{g}.$$  

The corresponding Lie algebra structure on $\mathfrak{g}$ coincides with the one given by linearizing $\pi$ at the identity $e \in G$. In particular, $\theta_{[\hat{\tau}, \hat{\omega}]} = [\theta_\hat{\tau}, \theta_\hat{\omega}]^*$ for $\hat{\tau}, \hat{\omega} \in \mathfrak{g}$ and $\cdot = l$ or $r$. \hfill \Box

The tangent Lie algebra $\mathfrak{g}$ of a Poisson Lie group $G$ is an example of Lie bialgebra, as defined below.

Definition 8.3. A Lie bialgebra is a vector space $\mathfrak{g}$ with a Lie algebra structure and a Lie coalgebra structure, which are compatible in the following sense: the cocommutator mapping $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ must be a 1-cocycle ($\delta$ acts on $\mathfrak{g} \otimes \mathfrak{g}$ by means of the adjoint representation). A triple of Lie algebras $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ is called Manin triple if $\mathfrak{p}$ has a nondegenerate invariant pairing $\langle , \rangle$ and isotropic Lie subalgebras $\mathfrak{p}_1$ and $\mathfrak{p}_2$ such that as vector space $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$.

The cocommutator $\delta$ induces a Lie bracket on the dual $\mathfrak{g}$ of $\mathfrak{g}$ and $(\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}, \hat{\mathfrak{g}}, \hat{\mathfrak{g}})$ with the natural pairing between $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ form a Manin triple. Conversely, the $\mathfrak{p}_i$ are dual to each other via the nondegenerate pairing $\langle , \rangle$.

Definition 8.4. Let $\hat{G}$ be a Lie group with Lie algebra $(\hat{\mathfrak{g}}, [\hat{\cdot}, \hat{\cdot}])$ with a Poisson Lie structure $\hat{\pi}$ so that the linearization of $\hat{\pi}$ at the $\hat{e} \in \hat{G}$ gives $(\mathfrak{g}, [\cdot, \cdot])$, then $(\hat{G}, \hat{\pi})$ is a dual Poisson Lie group of $G$. When $\hat{G}$ is simply connected, the structure $\hat{\pi}$ always exists and $\hat{G}$ is called the dual group.

Definition 8.5. Three Lie groups $(\hat{G}; G_+, G_-)$ form a double Lie group if $G_{\pm}$ are both closed Lie subgroups of $G$ such that the map $G_+ \times G_- \rightarrow \hat{G} : (g_+, g_-) \mapsto g_+ g_-$ is a diffeomorphism. They form a local double Lie group if there exist Lie subgroups $G'_{\pm}$ of $\hat{G}$ such that $G'_i$ is locally isomorphic to $G_i$ for $i = +,-$ and the map $G'_+ \times G'_- \rightarrow \hat{G} : (g'_+, g'_-) \mapsto g'_+ g'_-$ is a local diffeomorphism near identities.
Theorem 8.6. Let \( G \) be a Poisson Lie group with dual group \( \hat{G} \), then \( \mathfrak{g} \) is naturally a Lie bialgebra. Let \( \hat{G} \) be the connected and simply connected Lie group with Lie algebra \( \hat{\mathfrak{g}} = \mathfrak{g} \oplus \hat{\mathfrak{g}} \) as given above, then \((\hat{G}, G, G)\) form a local double Lie group.

The local double Lie group \((\hat{G}, G, G)\) in the theorem will be called the local double group of \( G \). In general, if the Lie algebras of a (local) double Lie group \((\hat{G}, G, G)\) coincide with the Manin triple defined by the Lie bialgebra \( \mathfrak{g} \), then we say that \((\hat{G}, G, G)\) is a (local) double group of \( G \).

8.7. Let \( r = \sum a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g} \), then it defines a cocommutator \( \delta \) via
\[
\delta : \mathfrak{g} \to \mathfrak{g} \otimes X \mapsto \text{ad}_X r,
\]
which is a 1-cocycle because it is in fact a 1-coboundary. We write \( r = s + a \) where \( s \) (resp. \( a \)) is the symmetric (resp. antisymmetric) part of \( r \), then \( \delta \) as given in (8.2) defines a Lie bialgebra iff
\[
\begin{align*}
(1) & \ s \text{ is ad-invariant and} \\
(2) & \ [r, r] \text{ is ad-invariant, where}
\end{align*}
\]
\[
[r, r] = \sum_{i,j} ([a_i, a_j] \otimes b_i \otimes b_j + a_i \otimes [b_i, a_j] \otimes b_j + a_i \otimes a_j \otimes [b_i, b_j]).
\]

Definition 8.8. The Lie bialgebra defined by \( r \in \mathfrak{g} \otimes \mathfrak{g} \) as above is called a coboundary Lie bialgebra. It is factorizable if \([r, r] = 0\) and \( s \) is invertible. In this case, \( r \) is also called a factorizable r-matrix. A (local) double Lie group \((\hat{G}, G, G)\) is called factorizable if the corresponding Lie bialgebra is factorizable. In this case, we will also call \( G \) a factorizable Poisson Lie group.

For an element \( r \in \mathfrak{g} \otimes \mathfrak{g} \), let \( \tau : \hat{\mathfrak{g}} \to \mathfrak{g} \) be the map defined by \( \tau^* (\omega^*) = (\tau^* \otimes \omega^*)(r) \). Suppose that \( r \) is factorizable and let \((\hat{\mathfrak{g}}, \mathfrak{g}, \hat{\mathfrak{g}})\) be the associated Manin triple, then \( \hat{\mathfrak{g}} \simeq \mathfrak{g} \oplus \hat{\mathfrak{g}} \) as Lie algebra. The isomorphism is given by \( \hat{\mathfrak{g}} \to \mathfrak{g} \oplus \hat{\mathfrak{g}} : \tau \mapsto (\tau, \tau) \)
\[
\hat{\mathfrak{g}} \to \mathfrak{g} \oplus \hat{\mathfrak{g}} : \hat{\omega} \mapsto (r_+ (\hat{\omega}), r_- (\hat{\omega})), \quad \text{with} \quad r_\pm = a \pm s.
\]

8.9. It’s a general fact for Poisson manifolds that \( \pi([\theta_\tau^*, \theta_\omega^*]^*) = [\pi(\theta_\tau^*), \pi(\theta_\omega^*)] \), in the convention 3.8. It follows that the map
\[
\rho : \hat{\mathfrak{g}} \to \Gamma(TG) : \hat{\tau} \mapsto X_{\hat{\tau}}^\rho = \pi(\theta_\tau^*)
\]
is a Lie algebra homomorphism, where \( \cdot \) stands for left or right.

Definition 8.10. For each \( \hat{\tau} \in \hat{\mathfrak{g}} \), the left (resp. right) dressing vector field on \( G \) is
\[
X_{\hat{\tau}}^l = \pi(\theta_{\hat{\tau}}^l) \quad (\text{resp.} \quad X_{\hat{\tau}}^r = -\pi(\theta_{\hat{\tau}}^r)),
\]
and \( \theta_{\hat{\tau}}^l \) is the left or right invariant 1-form on \( G \) determined by \( \hat{\tau} \). Integrating \( X_{\hat{\tau}}^\rho \) gives rise to a local (global if the dressing vector fields are complete) left (or right) dressing action of the dual group \( \hat{G} \) on \( G \), and we say that this left (or right) dressing action consists of left (or right) dressing transformations. The Poisson Lie group \((G, \pi)\) is complete if each left (or equivalently, right) dressing vector field is complete. Analogously, we may define the corresponding concepts on \( G \).

The dressing actions as defined above are the same as those in \([11, 27]\). Following \([29]\):

Definition 8.11. A left action \( \sigma_l : G \times P \to P \) of Poisson Lie group \((G, \pi_G)\) on a Poisson manifold \((P, \pi_P)\) is Poisson if \( \sigma_l \) is a Poisson map, where \( G \times P \) is endowed with the product Poisson structure. Similarly a right action \( \sigma_r : P \times G \to P \) is Poisson when \( \sigma_r \) is Poisson.
Definition 8.12. A $C^\infty$ map $\mu : P \to \hat{G}$ is called a momentum mapping for the left (resp. right) Poisson action $\sigma : G \times P \to P$ if for each $\tau \in g = \hat{g}^*$, the infinitesimal action $X^\tau_l$ (resp. $X^\tau_r$) of $\tau$ is given by
\[
X^\tau_l = \pi_p (\mu^* \bar{\theta}^\tau_l) \quad \text{(resp. } X^\tau_r = -\pi_p (\mu^* \bar{\theta}^\tau_r)\text{)},
\]
where $\bar{\theta}^\tau_l$ (resp. $\bar{\theta}^\tau_r$) is the left (or right) invariant 1-form on $\hat{G}$ determined by $\tau$. The moment map $\mu$ of the Poisson action $\sigma$ is $G$-equivariant if it’s equivariant with respect to the left (or right) dressing action of $G$ on $\hat{G}$.

In particular, when the moment map is equivariant, we have $\mu_*(X^\tau_l) = \hat{X}^\tau_l$, where $\hat{X}^\tau_l$ is the dressing vector field on $\hat{G}$ defined by $\tau \in g$. Then for connected complete Poisson Lie group $G$ (see \[2\] theorem 4.8):

Theorem 8.13. A momentum mapping $\mu : P \to \hat{G}$ for a Poisson action $\sigma$ is $G$-equivariant iff $\mu$ is a Poisson map.

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