On self-dual four circulant codes

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Abstract

Four circulant codes form a special class of 2-generator, index 4, quasi-cyclic codes. Under some conditions on their generator matrices they can be shown to be self-dual. Artin primitive root conjecture shows the existence of an infinite subclass of these codes satisfying a modified Gilbert-Varshamov bound.

Keywords: Quasi-cyclic codes; Self-dual codes; Circulant matrices; Artin primitive root conjecture.

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1 Introduction

A matrix over a finite field $\mathbb{F}_q$ is said to be circulant if its rows are obtained by successive shifts from the first row. If two circulant matrices $A$ and $B$ of order $n$ satisfy $AA^T + BB^T +$
\( I_n = 0 \), where the exponent “\( T \)" denotes transposition, then the code \( C \) generated by

\[
G = \begin{pmatrix}
I_n & 0 & A & B \\
0 & I_n & -B^T & A^T
\end{pmatrix}
\]

is a self-dual code. It is a \([4n, 2n]\) code. This so-called four circulant construction was introduced in [5], revisited in [7], and many self-dual codes with good parameters have been constructed that way.

In the above generator matrix \( G \), \( A \) and \( B \) are circulant matrices and are determined by the polynomials \( a(x), b(x) \in \mathbb{F}_q[x] \) whose \( x \)-expansion are the first row of the matrices \( A \) and \( B \), respectively. A code of length \( 4n \), dimension \( 2n \) is a four circulant code if it is generated by \( G \).

A linear code of length \( N \) over \( \mathbb{F}_q \) is a quasi-cyclic code of index \( l \) for \( l \mid N \) if for each codeword \( c = (c_0, c_1, \ldots, c_{N-1}) \) in \( C \), the vector \( (c_{N-l}, c_{N-l+1}, \ldots, c_{N-1}, c_0, \ldots, c_{N-l-1}) \in C \), where the subscripts are taken modulo \( N \). Hence four circulant codes are quasi-cyclic codes of index 4. Quasi-cyclic codes form an important class of codes, which have been extensively studied [11, 13, 14]. By the Chinese Remainder Theorem (CRT), quasi-cyclic codes can be decomposed into a “CRT product” of shorter codes over larger alphabets [13]. Such a linear code affords a natural structure of module over the auxiliary ring \( R(n, \mathbb{F}_q) = \mathbb{F}_q[x]/(x^n - 1) \). A linear code \( C \) of length 4 over \( R(n, \mathbb{F}_q) \) is an \( R(n, \mathbb{F}_q) \)-submodule of \( R^4(n, \mathbb{F}_q) \). The ring \( R(n, \mathbb{F}_q) \) is never a finite field except if \( n = 1 \). The CRT shows us that, if \( n \) is coprime with the characteristic of \( \mathbb{F}_q \), then the ring is a direct product of finite fields. If this \( R(n, \mathbb{F}_q) \)-submodule is generated by \( r \) generator sets, the code will be called \( r \)-generator code. In view of the generator matrix structures of four circulant codes, we see that these codes are 2-generator.

In the present paper, we study index 4 quasi-cyclic codes of parameters \([4n, 2n]\) inspired by [11, 3, 8]. When \( x^n - 1 \) has only two irreducible factors, we derive an enumeration formula of the self-dual four circulant codes over \( \mathbb{F}_q \) based on that decomposition, and derive an asymptotic lower bound on the minimum distance of these four circulant codes. The asymptotic lower bound is in the spirit of the Gilbert-Varshamov bound. Previous articles [6, 12] also explore this expurgated random coding technique for other families of structured codes.

The material is organized as follows. Section 2 collects the necessary notions and definitions. Section 3 discusses the algebraic structure of four circulant codes. Section 4 presents enumeration formula when \( x^n - 1 \) only has two irreducible factors over \( \mathbb{F}_q \) and establishes the asymptotic of self-dual four circulant codes. Section 5 concludes this article.
2 Definitions and notations

Let \( \mathbb{F}_q \) denote a finite field of characteristic \( p \), and \( R(n, \mathbb{F}_q) = \mathbb{F}_q[x]/(x^n - 1) \). In the following, we consider codes over \( \mathbb{F}_q \) of length \( 4n \) with \( n \) coprime to \( q \). The code \( C \) is generated by a matrix of the form (1).

From an algebraic perspective, we can view such a code \( C \) as an \( R(n, \mathbb{F}_q) \)-submodule in \( R^4(n, \mathbb{F}_q) \) and its two generators are \(((1, 0, a(x), b(x)), (0, 1, -b'(x), a'(x)))\), where \( a'(x) = a(x^{n-1}) \mod (x^n - 1), b'(x) = b(x^{n-1}) \mod (x^n - 1) \).

If \( C(n) \) is a family of codes parameters \([n, k_n, d_n]\), the rate \( \alpha_q(\delta) \) and relative distance \( \delta \) are defined as
\[
\alpha_q(\delta) = \limsup_{n \to \infty} \frac{k_n}{n}
\]
and
\[
\delta = \liminf_{n \to \infty} \frac{d_n}{n}.
\]
Both limits are finite as limits of bounded quantities.

3 Algebraic structure of self-dual four circulant codes

Throughout this paper, we assume \( \gcd(n, q) = 1 \). According to [13], we can cast the factorization of \( x^n - 1 \) into distinct irreducible polynomials over \( \mathbb{F}_q \) in the form
\[
x^n - 1 = \alpha \prod_{i=1}^{s} g_i(x) \prod_{j=1}^{t} h_j(x)h_j^*(x),
\]
where \( \alpha \in \mathbb{F}_q^* \), \( g_i(x) \) is a self-reciprocal polynomial for \( 1 \leq i \leq s \), and \( h_j^*(x) \) is the reciprocal polynomial of \( h_j(x) \) for \( 1 \leq j \leq t \).

By the Chinese Remainder Theorem (CRT), we have
\[
R \simeq \left( \bigoplus_{i=1}^{s} \mathbb{F}_q[x]/\langle g_i(x) \rangle \right) \oplus \left( \bigoplus_{j=1}^{t} \mathbb{F}_q[x]/\langle h_j(x) \rangle \oplus \mathbb{F}_q[x]/\langle h_j^*(x) \rangle \right).
\]
Let \( G_i = \mathbb{F}_q[x]/\langle g_i(x) \rangle, H_j' = \mathbb{F}_q[x]/\langle h_j(x) \rangle, H_j'' = \mathbb{F}_q[x]/\langle h_j^*(x) \rangle \) for simplicity. Note that all of these fields are extensions of \( \mathbb{F}_q \). This decomposition naturally extends to \( R^4 \) as
\[
R^4 \simeq \left( \bigoplus_{i=1}^{s} G_i^4 \right) \oplus \left( \bigoplus_{j=1}^{t} (H_j'^4 \oplus H_j''^4) \right).
\]
In particular, each $R(n,\mathbb{F}_q)$-linear code $C$ of length 4 can be decomposed as the “CRT sum”

$$C \simeq \left( \bigoplus_{i=1}^{s} C_i \right) \oplus \left( \bigoplus_{j=1}^{t} (C'_j \oplus C''_j) \right),$$

where for each $1 \leq i \leq s$, $C_i$ is a linear code over $G_i$ of length 4, and for each $1 \leq j \leq t$, $C'_j$ is a linear code over $H'_j$ of length 4 and $C''_j$ is a linear code over $H''_j$ of length 4, which are called the constituents of $C$.

Note that in terms of constituents of $C$, we have $C$ is self-dual only if $C_i$ is self-dual relative to Hermitian product in $G_i^4$ for $1 \leq i \leq s$, and $C'_j \cap C''_j \neq \{0\}$, $C'_j \cap C''_j \neq \{0\}$ for $1 \leq j \leq t$. Similar to the proof of [8], we can get the following theorem.

**Theorem 3.1.** If $C = ((1,0,a(x),b(x)),(0,1,-b'(x),a'(x))) \subset R^4$ be a four circulant code over $\mathbb{F}_q$, then $C$ is self-dual if and only if $(x^n - 1) | (1 + a(x)a(x^{n-1}) + b(x)b(x^{n-1}))$.

**Remark 3.2.** Under the condition of Theorem 3.1 then $C$ is a linear complementary-dual four circulant code if and only if $\gcd(1 + a(x)a(x^{n-1}) + b(x)b(x^{n-1}),x^n - 1) = 1$.

## 4 Asymptotic of self-dual four circulant codes

### 4.1 Enumeration

In this subsection, we will give enumerative results for self-dual four circulant codes. Recall that the so-called quadratic character $\eta$ of $\mathbb{F}_q$ is defined as $\eta(x) = 1$ if $x \in \mathbb{F}_q^*$ is square and $\eta(x) = -1$ if not.

For our purpose, we first give some lemmas without proofs as follows.

**Lemma 4.1.** ([2] Appendix) If $q$ is odd, then the number of solutions $(x,y)$ in $\mathbb{F}_q$ of the equation $x^2 + y^2 = -1$ is $q - \eta(-1)$.

The proof of the next Lemma while given in odd characteristic in [2] is easily seen to hold more generally, for all prime powers $q$.

**Lemma 4.2.** ([2] Appendix) If $n$ is coprime with $q$, then the number of solutions $(a,b)$ in $\mathbb{F}_{q^2}$ of the equation $a^{1+q} + b^{1+q} = -1$ is $(q + 1)(q^2 - q)$.

On the basis of Lemmas 4.1 and 4.2, we are now ready to state and prove the main result of this subsection.

**Theorem 4.3.** Let $n$ be an odd prime, $\gcd(n,q) = 1$ and $q$ be a primitive root modulo $n$. Then, $x^n - 1 = (x - 1)h(x)$ can be factored as a product of two irreducible polynomials over $\mathbb{F}_q$ and the number of self-dual four circulant codes of length $4n$ is $(q - \eta(-1))(q^{n-1} + 1)(q^{n-1} - q^{n-1})$ if $q$ is odd and $q(q^{n-1} + 1)(q^{n-1} - q^{n-1})$ if $q$ is even.
Proof. Since $q$ is a primitive root modulo $n$, the cyclotomic cosets of $q$ modulo $n$ are only two in number, that is $C_0 = \{0\}$ and $C_1 = \{1, q, \ldots, q^{n-2}\}$. Due to $\gcd(n, q) = 1$, the number of monic irreducible factors of $x^n - 1$ over $\mathbb{F}_q$ is equal to the number of cyclotomic cosets of $q$ modulo $n$. Hence $x^n - 1$ can be factored as a product of two irreducible polynomials over $\mathbb{F}_q$. Let $x^n - 1 = (x - 1)h(x)$, where $h(x)$ is an irreducible polynomial and $\deg(h(x)) = n - 1$.

By the CRT approach of [13], $C$ can be decomposed as $C \cong C_1 \oplus C_2$, where $C_1 \subset \mathbb{F}_q$ and $C_2 \subset (\mathbb{F}_q[x]/\langle h(x) \rangle)^4$. In the case $C_1 \subset \mathbb{F}_q^4$, we count self-dual four circulant codes of parameters $[4, 2]$ over $\mathbb{F}_q$. We can obtain the equation $1 + a^2 + b^2 = 0$. When $q$ is odd, it follows from Lemma 4.1 that the number of the solutions of that equation is $q - \eta(-1)$. When $q$ is even, then the equation becomes $a^2 + b^2 = (a + b)^2 = 1$. Hence the number of the solutions of that equation is $q$.

In the case $C_2 \subset (\mathbb{F}_q[x]/\langle h(x) \rangle)^4$, the factor $h(x)$ is a self-reciprocal polynomial of degree $n - 1$, and the number of self-dual four circulant codes of parameters $[4, 2]$ over $\mathbb{F}_q[x]/\langle h(x) \rangle$ is equal to the number of solutions of the equation $1 + a \eta^{n-1} + b \eta^{n-1} = 0$. It follows from Lemma 4.2 that the number of the solutions of that equation is $(q^{n-1} + 1)(q^{n-1} - q^{n-1})$. Hence the number of self-dual four circulant codes of length $4n$ is $(q - \eta(-1))(q^{n-1} + 1)(q^{n-1} - q^{n-1})$ if $q$ is odd and $q(q^{n-1} + 1)(q^{n-1} - q^{n-1})$ if $q$ is even. \qed

4.2 Distance bounds

In number theory, Artin’s conjecture on primitive roots states that a given integer $q$ which is neither a perfect square nor $-1$ is a primitive root modulo infinitely many primes [9]. It was proved conditionally under GRH by Hooley [4]. In this subsection, we study the case when $x^n - 1$ factors as a product of two irreducible polynomials over $\mathbb{F}_q$, i.e. $x^n - 1 = (x - 1)h(x)$, where $h(x)$ is an irreducible polynomial over $\mathbb{F}_q$. We call constant vectors if the codewords of the cyclic code of length $n$ generated by $h(x)$. For example, when $n = 7, q = 3$, then $x^7 - 1 = (x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$. Let $a(x)$ and $b(x)$ denote a polynomial of $\mathbb{F}_q[x]$ coprime with $x^n - 1$, and let $C_{a,b}$ be the self-dual four circulant code with generator $((1, 0, a(x), b(x)), (0, 1, -b'(x), a'(x)))$.

Lemma 4.4. If $u = (c, d, e, f)$, where $c, d$ is not a constant vector, then there are at most $q^n(q - 1)$ generators $((1, 0, a(x), b(x)), (0, 1, -b'(x), a'(x)))$ such that $u \in C_{a,b}$.

Proof. The condition $u = (c, d, e, f)$ is equivalent to the system of equations

\[
\begin{align*}
    c &\equiv ca - db' \mod (x^n - 1), \\
    f' &\equiv c'b' + da \mod (x^n - 1).
\end{align*}
\]
If $cc' + dd'$ is invertible mod $(x^n - 1)$, then
\[
\begin{align*}
  a &\equiv \frac{ec' + df'}{cc' + dd'} \mod (x^n - 1), \\
  b &\equiv \frac{fc' - de}{cc' + dd'} \mod (x^n - 1).
\end{align*}
\]

In that case $a$ and $b$ are uniquely determined by this system of equations. If $cc' + dd'$ is zero or a zero divisor mod $(x^n - 1)$, we assume there are solutions to the system. So $e \equiv ca - db'$ mod $(x^n - 1)$. Since $\deg a(x) \leq n - 1$, then there are only at most $q^n$ choices for $a$. Given $a$, we will have $(q - 1)$ choices for $b$ by the CRT since $d$ is not a constant vector. In total, there are at most $q^n(q - 1)$ choices for $(a, b)$.

Recall the $q$-ary entropy function defined for $0 \leq t \leq \frac{q - 1}{q}$ by
\[
H_q(t) = \begin{cases} 
0, & \text{if } t = 0, \\
\log_q(q - 1) t - \log_q(1 - t) \log_q(1 - t), & \text{if } 0 < t \leq \frac{q - 1}{q}.
\end{cases}
\]

This quantity is instrumental in the estimation of the volume of high-dimensional Hamming balls when the base field is $\mathbb{F}_q$. The result we are using is that the volume of the Hamming ball of radius $tn$ is, up to subexponential terms, $q^{nH_q(t)}$, when $0 < t < 1$, and $n$ goes to infinity \cite{10} Lemma 2.10.3]. The main result obtained in this paper is as follows.

**Theorem 4.5.** Let $n$ be odd prime, $\gcd(n, q) = 1$ and $q$ be a primitive root modulo $n$, then there are infinite families of self-dual four circulant codes of relative distance $\delta$ satisfying $H_q(\delta) \geq \frac{1}{8}$.

**Proof.** The four circulant codes containing a vector of weight $d \sim 4\delta n$ or less are by standard entropic estimates of \cite{10} Lemma 2.10.3] and Lemma 4.4 bounded above by a quantity of order at most $q^n(q - 1) \times q^{4nH_q(\delta)} = O(q^{n(1 + 4H_q(\delta))})$, up to subexponential terms. This number will be less than the total number of self-dual four circulant codes, which is, by Theorem 4.3 of the order $O(q^{3n/2})$, as long as $H_q(\delta) < \frac{1}{8} + \epsilon$. This ensures the existence of such codes of distance $H_q^{-1}(\frac{1}{8}) - \epsilon$. Letting $\epsilon \to 0$, the result follows.

\[\square\]

**5 Conclusion and Open problems**

In this paper, we have considered self-dual four circulant codes. Inspired by \cite{8}, we have studied four circulant codes from the algebraic perspective and given an enumeration formula of the self-dual subclass in a special case of the factorization of $x^n - 1$. This paper can be considered as a companion paper of \cite{8}. The main difference between the two papers is the nontrivial nature of the existence of a factorization of $x^n - 1$ into two irreducible polynomials, which requires Artin’s conjecture on primitive roots, while the same problem for
$x^n + 1$ can be solved by elementary means. We have only considered enumeration formulae in the case that the factorization of $x^n - 1$ consists of two irreducible polynomials. It would be a worthwhile task to relax this condition by looking at lengths where the factorization of $x^n - 1$ into irreducible polynomials contains more than two elements. In fact extending our enumerative results to a general factorization of $x^n - 1$ seems to be a difficult task, leading to solving complex diagonal equations over finite fields.

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