Analytical and numerical solutions to the problem on a heat wave initiating for the nonlinear heat equation with a source

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Abstract. The article deals with the nonlinear heat (porous medium) equation with a source. This equation has a large number of applications, as well as non-standard non-linear properties. An important class of solutions to the equation is heat waves (waves of filtration), propagating on a zero background with a finite velocity. This paper continues the cycle of articles devoted to the construction and study of heat waves with a closed front. Here we consider a problem with a given boundary condition with a source in cases of plane, circular and spherical symmetry. The solution is constructed in the form of a double power series, the theorem of existence and uniqueness is proved. We also look for some exact solutions of an equation with a power source, the construction of which reduces to the integration of ordinary differential equations (ODE) with a singularity. To solve the obtained ODE, we propose a computational algorithm based on the boundary element method. The results of numerical calculations are presented and discussed.

1. Introduction
We consider a nonlinear parabolic heat [1] (porous medium [2]) equation in the case of power nonlinearity and with a source. This equation has non-standard non-linear properties, for example, disturbances propagate on a zero background with a finite velocity [1], in other words, a heat wave arises. Of the many publications devoted to the study of this equation, we especially note the monographs of L.V. Ovsyannikov [3] and his followers [4], containing some classes of exact solutions. In [5, 6] the authors propose a method for constructing solutions of heat wave type using special series [7]. We also note the book [8], which deals with similar problems in a more general formulation.

The authors of this paper previously completed a series of articles devoted to the construction and study of solutions of such type. Thus, in the articles [9, 10] we consider a one-dimensional problem with a given boundary condition without a source. Papers [11, 12] deal with closed wave fronts. Here we continue these studies and consider the problem with the boundary condition, which is specified on a movable manifold, with a source [13] in the cases of plane, circular and spherical symmetry. In Section 3, we prove a theorem on the existence and uniqueness of an analytical solution, which is constructed in the form of a double power series. Section 4 presents some exact solutions to the equation with a power source, the construction of which reduces to the integration of second-order ordinary differential equations with a singularity. In Section 5,
for solving these ODE, we proposed a computational algorithm based on the boundary element method; the results of a computational experiment are presented.

2. Formulation
We consider nonlinear heat equation

\[ u_t = uu_{\rho \rho} + \nu u_{\rho} + u^2_{\rho}/\sigma + F(u) \]  

(1)

in the cases of plane (\(\nu = 0\)), circular (\(\nu = 1\)) and spherical (\(\nu = 2\)) symmetry. Source function \(F(u)\) obeys \(F(0) = 0, \sigma > 0\) is a constant.

The boundary condition for Eq. (1) has the form

\[ u(t, \rho)\big|_{\rho=R(t)} = f(t), \ R(t) > 0, \ f(0) = 0, \ f'(0) \geq 0, \ [f'(0)]^2 + [R'(0)]^2 \neq 0. \]  

(2)

It generalizes the traditional boundary conditions \(u(t, \rho)\big|_{\rho=R(t)} = 0\) and \(u(t, \rho)\big|_{\rho=R} = f(t)\), \(R = const\). The first condition corresponds to a given heat wave front and the second one means a specified boundary regime. Such conditions are considered in the A.F. Sidorov scientific school [5, 6] for the study of the heat waves propagation on a cold background.

3. Main theorem

Theorem 1. Let \(R(t), f(t)\) and \(F(u)\) be analytical functions in the neighborhoods of \(t = 0\) and \(u = 0\) respectively. Then problem (1), (2) has a unique non-trivial solution, analytical in the neighborhood of \(t = 0, \rho = R(0)\) when choosing the direction of the wave propagation.

The proof contains several steps. First, we construct the solution in the form of a double power series and prove that the coefficients are uniquely determined. Then, using non-degenerate substitutions, we reduce problem (1), (2) to a single equation of a special type. For this equation, which is equivalent to the original problem, we construct a majorant problem. Finally, based on the Cauchy-Kovalevskaya theorem, we prove the existence and uniqueness of an analytical solution of the majorant problem.

Proof. For convenience, we introduce new variable \(r = \rho - R(t)\), then problem (1), (2) takes the form

\[ u_t - R' u_r = uu_{rr} + \nu u_r/(r + R) + u^2_{r}/\sigma + F(u), \]  

(3)

\[ u(t, r)|_{r=0} = f(t), \]  

(4)

We find the solution to problem (1), (2) as the Taylor series

\[ u(t, r) = \sum_{n,m=0}^{\infty} u_{n,m} t^n/r^m \bigg|_{t=0}. \]  

(5)

In accordance with the assumptions of Theorem 1, \(f(t) = \sum_{n=0}^{\infty} f_n t^n/n!\) and \(R(t) = \sum_{n=0}^{\infty} R_n t^n/n!\). From the boundary condition (4) it follows that \(u_{n,0} = f_n, \text{ and } f_0 = 0, \text{ since } f(0) = 0.\)

Let us assume in Eq. (3) that \(t = r = 0\). Taking into account that the values \(u_{0,0} = 0, u_{1,0} > 0\) have already been found, we obtain the quadratic equation \(f_1 - R_1 u_{0,1} = u_{0,1}^2/\sigma, \) which shows that \(u_{0,1}\) can be found by the formula

\[ u_{0,1} = \frac{1}{2} \left( -\sigma R_1 \pm \sqrt{\sigma^2 R_1^2 + 4\sigma f_1} \right). \]

It can be shown that for any \(f_1 > 0\) and regardless of the values of \(R_1\) coefficient \(u_{0,1} < 0\) if we consider \(-\sqrt{\sigma^2 R_1^2 + 4\sigma f_1}\), otherwise \(u_{0,1} > 0\). This means that the heat wave moves away or approaches the origin of coordinates, respectively.
The remaining coefficients of the series (5) are found by differentiating Eq. (3) with respect to \( t \) and \( r \) and assuming \( t = r = 0 \). Applying operator \( \partial^n[\cdot]/(\partial t^{n-k}\partial r^k)\)|\( t=r=0 \), \( k = 0, 1, ..., n \) to Eq. (3), we obtain the equality

\[
\begin{align*}
u_n + n-k \sum_{i=0}^{n-k} C_{n-k}^i R_{i+1} u_{n-k-i+1} + & \sum_{i=0}^{n-k} \sum_{j=0}^{k} C_{n-k}^i C_{k}^j u_{i,j+1} u_{n-k-i-j+1} + \nu \frac{\partial^n}{\partial t^{n-k}\partial r^k} \left( \frac{u \partial_{rr}}{r + R} \right) \bigg|_{t=r=0} = F_{n-k,k}, \tag{6}
\end{align*}
\]

where \( F_{n-k,k} = \partial^n F(u)/(\partial t^{n-k}\partial r^k)|_{t=r=0} \).

Using formula (6), we find second-order coefficients (the order of the coefficient means the sum of its indices). Let \( n = 1 \) and \( k = 0, 1 \), then we obtain a system of two equations. The solution gives us the coefficients

\[
\begin{align*}
u_{1,1} = (a_1 L_{1,0} - b_1 L_{0,1} - a_1 f_2)/(a_0 a_1 - b_1), & \quad \nu_{0,2} = (a_0 L_{1,0} - L_{1,0} + f_2)/(a_0 a_1 - b_1).
\end{align*}
\]

Here

\[
\begin{align*}
u_0 = -(R_1 + 2u_{0,1}/\sigma), & \quad b_1 = -f_1, \quad a_1 = -[R_1 + (1 + 2/\sigma) u_{0,1}],
\end{align*}
\]

\[
\begin{align*}
u_{1,0} = R_2 u_{0,1} + \nu f_1 u_{0,1}/R_0 + F'(0)f_1, & \quad L_{0,1} = \nu^2 u_{0,1}/R_0 + F'(0)u_{0,1}.
\end{align*}
\]

Note, that \( a_0 a_1 - b_1 \neq 0 \).

Suppose now that all the coefficients up to order \( n \) are found. In (6) we select the coefficients of the highest \((n+1)\)-th order. Then (6) takes the form

\[
\begin{align*}
u_n + n-k \sum_{i=0}^{n-k} C_{n-k}^i R_{i+1} u_{n-k-i+1} + & \sum_{i=0}^{n-k} \sum_{j=0}^{k} C_{n-k}^i C_{k}^j u_{i,j+1} u_{n-k-i-j+1} + \nu \frac{\partial^n}{\partial t^{n-k}\partial r^k} \left( \frac{u \partial_{rr}}{r + R} \right) \bigg|_{t=r=0} = F_{n-k,k}, \tag{7}
\end{align*}
\]

where

\[
\begin{align*}
u_k & = -(R_1 + (k + 2/\sigma) u_{0,1}), \quad b_k = -(n - k)f_1,
\end{align*}
\]

\[
\begin{align*}
u_{n-k,k} = n-k \sum_{i=1}^{n-k} C_{n-k}^i R_{i+1} u_{n-k-i+1} + & \sum_{i=0}^{n-k} \sum_{j=0}^{k} C_{n-k}^i C_{k}^j u_{i,j+1} u_{n-k-i-j+1} + \nu \frac{\partial^n}{\partial t^{n-k}\partial r^k} \left( \frac{u \partial_{rr}}{r + R} \right) \bigg|_{t=r=0} = F_{n-k,k}.
\end{align*}
\]

It is easy to see, that \( b_0 = 0 \) and \( b_{n-k} \leq 0 \), if \( k = 0, 1, ..., n - 1 \). It can be shown, that if \( f_1 > 0 \), then \( a_k, k = 0, 1, ..., n \) are opposite in sign to \( u_{0,1} \).

Assume in (7) \( k = 0, 1, ..., n \), we obtain the system of equations

\[
\begin{pmatrix}
\begin{array}{cccccccc}
u_0 & b_n & \ldots & 0 & 0 \\
1 & a_1 & \ldots & 0 & 0 \\
& \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & a_{n-1} & b_1 \\
0 & 0 & \ldots & 1 & a_n
\end{array}
\end{pmatrix}
\begin{pmatrix}
u_{n,1} \\
u_{n-1,2} \\
\vdots \\
u_{0,n+1}
\end{pmatrix}
= \begin{pmatrix}L_{n,0} - f_{n+1} \\
L_{n-1,1} \\
\vdots \\
L_{0,n}
\end{pmatrix}
\tag{8}
\]

with a square tridiagonal matrix without diagonal dominance on the left side. By induction on the order of the matrix, we can prove its nondegeneracy [11]. Therefore, system (8) is uniquely solvable. So, by the principle of induction, all coefficients of series (5) are determined uniquely. The solution is constructed, the first stage of the proof is completed.
The following stages of the proof will be briefly discussed (for more details, see [11]). In problem (3), (4) we make several special substitutions. The statement and the physical meaning of the problem imply the presence of a heat wave front, i.e. the line $\rho = \alpha(t)$, on which the desired function vanishes. We make the substitution $s = r - b(t)$, where $b(t) = \alpha(t) - R(t)$. The obtained problem consists of one equation and two boundary conditions $u(t,s)|_{s=-b(t)} = f(t)$ $u(t,s)|_{s=0} = 0$. The substitution $s = s(t,u)$ changes roles of unknown function $u$ and independent variable $s$. We express the function $b$ from the first boundary condition, which is now written in the form $s(t,u)|_{u=f(t)} = -b(t)$, and substitute it into the equation. Next, we make the replacement $v = u - f(t)$, where $v = 0$ becomes the new coordinate axis. Finally, the last substitution $s(u,v) = us_1(v) + u^2S(u,v)$ is a partial decomposition of an unknown function in a Taylor series, taking into account the boundary condition $s|_{u=0} = 0$. The analytical function $s_1 = s_{u|u=0}$ is easily determined from the equation for $u = v = 0$. As a result, we obtain the equation

$$2(1+1/\sigma)S + (4+1/\sigma)uS_u + u^2S_{uu} + \xi_0(v)(S|_{u=0}) + \xi_1(v)u(S_u|_{v=0}) = \eta_0 + w\eta_1 + u^2\eta_2 + u^3\eta_3, \quad (9)$$

where $\xi_i, \ i = 0, 1$ and $\eta_j, j = 0, 1, 2, 3$ are analytical functions of their variables. Note, that $\eta_0 = \eta_0(v),$ and the remaining $\eta_j$ depend on the derivatives of the function $S$ with respect to a variable $u$ of order no more than $j - 1$. The functions $\xi_i(v), \ i = 0, 1$ are positive for $v = 0$. It is easy to verify that all the conditions of Lemma 2 from the paper [11] are satisfied for Eq. (9), and it is solvable in the class of analytical functions.

Now we construct a solution of Eq. (9) in the form of a series $S(u,v) = \sum_{i=0}^{\infty} S_i(v)u^i/i!$. It can be shown that if the majorant estimates $Z_0 >> S_0, Z_1 >> S_1, \ \zeta >> \xi_0 + \xi_1, \ \chi_j >> \eta_j, \ j = 0, 1, 2, 3$ are held, the problem

$$Z_{uu} = (1 + \chi_0 \left( \frac{\partial^2 \chi_0}{\partial u^2} + \frac{\partial \chi_1}{\partial u} + \chi_2 + u\chi_3 \right)), \quad Z(u,v)|_{u=0} = Z_0(v), \quad Z_u(u,v)|_{u=0} = Z_1(v) \quad (10)$$

is the majorant problem for Eq. (9).

Finally, putting in (10) $u = 0$, one can find $Z_2(v)$. Differentiating (10) with respect to $u$ and expressing $Z_{uu}$, we obtain a problem of Kovalevskaya type with analytical input data. It follows from the Cauchy-Kovalevskaya theorem that the problem has a unique analytical solution. This proves the theorem.

4. Exact solutions

The proved theorem, like most of its similar assertions, including the classical Cauchy-Kovalevskaya theorem [14], ensures the local analytic solvability of the considered problem without establishing the domain of convergence of series. This reduces the possibilities for their use in the verification of the results of numerical calculations. Therefore, we look for exact (generalized self-similar [15]) solutions of (1), (2), having the form

$$u = \lambda(t) \ w(z), \ z = 1 - \rho/R(t). \quad (11)$$

Here the function $w(z)$ is a solution to the Cauchy problem for the ordinary differential equation with degeneration

$$ww'' + G(z, w, w') = 0, \ w(0) = 0, \ w'(0) = w'_0 \quad (12)$$

on the interval $z \in [0, L]$ or $z \in [-L, 0]$. It depends on the direction of the heat wave moving. Solutions (11), (12) satisfy problem (1), (2) in the case of $f(t) \equiv 0$, i.e. when boundary condition (2) has the form

$$u|_{\rho=R(t)} = 0. \quad (13)$$
Problem (1), (13) is reducible to problem (12) only in some special cases, one of which is given in the following statement.

**Lemma 1** Suppose that \( F = \alpha w^\beta, \beta > 0, \beta \neq 2. \) Then problem (1), (13) allows being reduced to the Cauchy problem (12) and the following equalities hold

\[
G(z, w, w') = \frac{(w')^2}{\sigma} + \left( z - 1 + \frac{\nu w}{z - 1} \right) w' + \frac{2}{\beta - 2} w + \frac{\alpha}{A} w^\beta, w_0' = 0. \tag{14}
\]

\[
R(t) = \left\{ \begin{array}{l}
C_1 e^{C_2 t}, \quad \beta = 1 \\
(C_3 t + C_4) \omega, \quad \beta \neq 1
\end{array} \right., \quad \omega = \frac{\beta - 2}{2 \beta - 2}, \quad A = \left\{ \begin{array}{l}
C_2, \quad \beta = 1 \\
(C_3 \omega)^{2-\beta}, \quad \beta \neq 1
\end{array} \right., \quad \lambda(t) = R(t)R'(t).
\]

The lemma is proved by the direct substitution of (11) into Eq. (1). The equality \( w_0' = \sigma \) is a compatibility condition for (12).

Note that under the conditions of Lemma 1, Theorem 1 ensures the existence and uniqueness of an analytical solution of problem (12) for \( \beta \in \mathbb{N} \). With other values of the parameter \( \beta \), there is no solvability in the class of analytical functions. This follows from the fact that during the differentiation of the function \( F(u) \), a zero appears in the denominator and the procedure for constructing series (5) is terminated.

The question of the solvability of problem (12) with \( \beta > 0, \beta \notin \mathbb{N} \) in a class of twice continuously differentiable functions requires special consideration, which is beyond the scope of this article. Suppose that it holds for \( \beta > 1 \). Taking this hypothesis into account, we carry out a numerical analysis.

### 5. Numerical analysis

To solve problem (12), (14), we apply the iteration algorithm based on the boundary element method (BEM). It was previously used for solving problem (1), (13) on each time step in the articles [10, 12, 13]. Let us write problem (12), (14) as

\[
w'' = -G(z, w, w')/w, \quad z \in [0, L], \quad w(0) = 0, \quad q(0) = -\sigma. \tag{15}
\]

Here \( L < 1, q = \partial w/\partial n \) is a flux, \( n \) is the external normal at the boundary points, \( n(0) = -1, n(L) = 1 \). The iterative solution of problem (15) by the BEM leads to the equality

\[
w_k(\xi) = -\sigma u^*(\xi, 0) + q_2^{(k)} u^*(\xi, L) - w_2^{(k)} q^*(\xi, L) + \int_0^L \frac{G(z, w_{k-1}(z), w_{k-1}^{(k)}(z))}{w_{k-1}(z)} u^*(\xi, z) dz, \tag{16}
\]

where \( w_k(\xi) \) is the \( k \)-th iteration of the solution, \( w_0 \equiv 0, u^*(\xi, x) \) and \( q^*(\xi, x) \) are the kernel functions [16], \( w_2^{(k)} = w_k(L) \) and \( q_2^{(k)} = \partial w_k/\partial n|_{\xi=L} \). The values of \( w_2^{(k)} \) and \( q_2^{(k)} \) are found from solving the system of boundary integral equations

\[
\left\{ \begin{array}{l}
w_2^{(k)} = -\sigma L + 2 \int_0^L \frac{G(z, w_{k-1}(z), w_{k-1}^{(k)}(z))}{w_{k-1}(z)} u^*(0, z) dz, \\
w_2^{(k)} - q_2^{(k)} L = -2 \int_0^L \frac{G(z, w_{k-1}(z), w_{k-1}^{(k)}(z))}{w_{k-1}(z)} u^*(L, z) dz.
\end{array} \right. \tag{17}
\]

The iteration process stops when the value of \( w_2^{(k)} \) is close enough to \( w_2^{(k-1)} \). The solution of the initial problem (1), (13), has the form

\[
u = R(t)R'(t)w_k(z), \quad z = 1 - \rho/R(t). \tag{18}
\]
Iteration algorithm (16), (17) allows us to carry out a computational experiment for $\sigma = 3$. Numerical solution (18) of problem (1), (13) when $\nu = 0, \alpha = \beta = 1, R(t) = e^t$ is very close to the exact solution $u(t, \rho) = \sigma e^t(e^t - \rho)$ [13]. This fact demonstrates the effectiveness of the proposed algorithm. Figure 1 shows BEM solutions of problem (15) for $0 < \beta \leq 1$ when $\nu = 0, \alpha = 1, L = 1$. These solutions are nearly linear and defined for $z \in [0,1]$. They make possible to construct the solutions of problem (1), (13) for any point in time $t$ on the segment $\rho \in [R(0), R(t)]$.

The analysis of calculations for $2 < \beta \leq 3$ (see Fig. 2) shows that non-negative solution of problem (15) is defined on $z \in [0, z^*], z^* < 1$, for some values of $\beta$. In these cases, when $R(0) = 0$, the solution of problem (1), (13) at time $t$ is defined on the segment $\rho \in [R^*(t), R(t)]$, $R^*(t) = (1 - z^*)R(t)$, rather than on the segment $\rho \in [R(0), R(t)]$ (see (18)). Therefore, besides the specified zero front $\rho = R(t)$ the solution have the extra zero front $\rho = R^*(t)$ moving in the same direction.

The solution of problem (1), (13) for $1 < \beta < 2$ makes sense only on a finite time interval. This case requires a particular analysis.

Figure 3 shows BEM solutions of problem (15) when $\nu = 1, \alpha = \beta = 1$ for the exponential zero front $R(t) = C_1e^{C_2}t$ with various values of parameter $C_2$. The solutions are constructed on the segment $z \in [0,0.95]$, since for $z \to 1$ they increase indefinitely. To construct the solution of problem (1), (13) on a specified interval $t \in [0,t_\ast]$ one have to solve problem (15)) on the segment $z \in [0,L]$, where $L = 1 - R(0)/R(t_\ast) < 1$.

Figure 4 shows BEM solutions of problem (15) when $\nu = 1, \alpha = 1, \beta = 3, L = 0.95$ for the power zero front $R(t) = (C_3t + C_4)^\omega$ with various values of parameter $C_3$.

Fig. 1. BEM solutions, $\nu = 0, 0 < \beta \leq 1$.

Fig. 2. BEM solutions, $\nu = 0, 2 < \beta \leq 3$.

Fig. 3. BEM solutions, $\nu = 1, \beta = 1$.

Fig. 4. BEM solutions, $\nu = 1, \beta = 3$. 
6. Conclusion
In this paper, we consider the problem of constructing solutions of the nonlinear heat equation with a source having a type of heat wave propagating no a cold (zero) background. The theorem on the existence and uniqueness of heat-wave type analytical solutions is formulated and proved. In the case of a power source, the existence of a generalized self-similar solution is shown. Its construction reduces to the integration of an second -order ordinary differential equation with a singularity before the highest derivative. We propose an algorithm for the numerical solution of the obtained ODE using the boundary element method, which allows us to find the global solution of the initial problem for any finite point in time.

Further research on the subject of the article can be continued in two directions. Firstly, it is an increase in the dimension of the problem. Secondly, it is finding the estimate of the radius of convergence of the constructed series and the proof of the convergence of the numerical method to the exact solution.

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