Integrable boundary conditions for the Toda lattice

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Abstract

The problem of construction of the boundary conditions for the Toda lattice compatible with its higher symmetries is considered. It is demonstrated that this problem is reduced to finding of the differential constraints consistent with the ZS-AKNS hierarchy. A method of their construction is offered based on the Bäcklund transformations. It is shown that the generalized Toda lattices corresponding to the non-exceptional Lie algebras of finite growth can be obtained by imposing one of the four simplest integrable boundary conditions on the both ends of the lattice. This fact allows, in particular, to solve the problem of reduction of the series $A$ Toda lattices into the series $D$ ones. Deformations of the found boundary conditions are presented which leads to the Painlevé type equations.

Key words: Toda lattice, boundary conditions, integrability, Bäcklund transformation, Lie algebras, Painlevé equations
1 Introduction

In this paper we consider the boundary conditions for the classical completely integrable
differential-difference model — the well known Toda lattice

\[ q_{j,xx} = \exp(q_{j+1} - q_j) - \exp(q_j - q_{j-1}), \quad -\infty < j < +\infty \]  

which are compatible with the integrability property. Imposing of the boundary conditions
of the form

\[ q_M = F(q_{M+1}, \ldots, q_{M+m}), \quad q_N = G(q_{N-1}, \ldots, q_{N-n}), \quad M < N \]  

reduces the infinite-dimensional system (1) to a finite-dimensional one. We require that the
boundary conditions (2) are consistent with integrability at any choice of parameters \( M < N \)
(for fixed \( m, n \)). Thus, the boundary conditions on the left and right ends are assumed
independent from each other, so actually we work with a semi-infinite lattice.

The compatibility of a boundary condition with the integrability property is understood,
in spirit of the symmetry approach, as consistence with the flows determined by higher sym-
metries of the Toda lattice. (The exact definition is given below, in Section 2.) Such in-
terpretation was suggested in the papers [1],[2] and has quite justified itself in the problem
of description of the integrable boundary conditions for such models, as Harry Dym and
Korteweg-de Vries equations, multifield systems of the NLS and Burgers type, Volterra lat-
tice and others (see [3],[4],[5]).

In some cases (for example, for the Burgers equation see [4]) one can prove that a boundary
condition compatible at least with one higher symmetry of the equation is compatible with an
infinite series of its symmetries as well, including a certain one, which can easily be specified
a priori and consequently may be regarded as a test symmetry. Compatibility with the test
symmetry is taken as a basis of the preliminary classification of the integrable boundary
conditions (see Section 3).

The finite-dimensional versions of the lattice (1) called generalized Toda lattices are well
known in literature. Namely, to each simple Lie algebra of the finite growth the generalized
Toda lattice corresponds which is integrable in some sense (see e.g [6],[7]). In this paper
we undertake an attempt to revise the theory of the finite-dimensional integrable systems
of exponential type from the integrable boundary conditions point of view. For example,
we demonstrate that the generalized Toda lattices corresponding to the classical series of
finite-dimensional simple Lie algebras, as well as Kac-Moody algebras, can be represented
as reductions of the infinite lattice \( [1] \) with the boundary conditions \( [2] \) compatible with its
higher symmetries. It should be emphasized that the periodic lattices corresponding to the
series \( \tilde{A} \), as well as the lattices related to the exceptional Lie algebras, remain outside of our
consideration by virtue of the assumption about independence of the boundary conditions on
the left and right ends. On the other hand, the approach developed in the work makes possible
to study also boundary conditions leading beyond the limits of exponential type systems (see,
for example, boundary condition \( [29] \) with a complete set of parameters). The revealing of
the algebraic structures appropriate such boundary conditions is an open problem.

The article is organized as follows. In Section 2 the compatibility criterion of a boundary
condition with a higher symmetry of a lattice is formulated. It involves the dual language of
evolutionary partial differential equations associated to the lattice. In this Section we discuss
also a phenomenon of a degenerate boundary condition, which is characterized by increased
integrability.

In Section 3 we present a method to construct the integrable boundary conditions for the
Toda lattice which is based on this criterion. Symmetries of the Toda lattice, rewritten in the
evolutionary form, coincide with the ZS-AKNS hierarchy and finding of integrable boundary
conditions is equivalent to finding of differential constraints compatible with the odd members
of this hierarchy, in particular, with the system \( [11] \) which plays the role of the test symmetry.
This problem is solved with help of the Bäcklund transformations. It is noticed that by virtue
of the found small order differential constraints the system \( [11] \) is reduced into the integrable
scalar equations KdV, mKdV and Calogero-Degasperis equations.

In Section 4 we demonstrate that imposing of the every possible combinations of the four
simplest boundary conditions found above on the left and right ends of the Toda lattice leads
to the generalized Toda lattices, corresponding to all infinite series of the Lie algebras of the
finite growth (except for the \( \tilde{A} \) series). Moreover, it is proved that the series \( D \) Toda lattices
can be represented as reductions of the series \( A \) lattices. We shall note, that though for the
lattices of the series \( B \) and \( C \) the connection with the series \( A \) is quite obvious and well
known, for the lattices of the series \( D \) this problem remained open.\(^1\) The proposition 4.1 is
proven: let the solution of the Toda lattice corresponding to the Lie algebra \( A_{2n-1} \) satisfies

\(^1\)Authors thank R.I. Yamilov who drew their attention to this problem.
an additional symmetry of the reflection type, then it sets also solution of the Toda lattice corresponding to the Lie algebra $D_n$.

In Section 5 the found boundary conditions are deformed by transform $q_j \rightarrow q_j + \varepsilon x$. The new boundary conditions are also integrable, but they are compatible with other symmetry set. We demonstrate that closing of the lattice (3) by the boundary conditions with different values of parameter $\varepsilon$ on the right and left ends results in equations of the Painlevé type.

2 Boundary conditions compatible with higher symmetries

Let a lattice of the Toda type

$$q_{j,xx} = f(q_{j-1}, q_j, q_{j+1}), \quad -\infty < j < +\infty$$

(3)

where $\frac{\partial f}{\partial q_{j,x}} \neq 0$ admits a higher symmetry of the form

$$q_{j,t} = g(q_{j-k}, q_{j-k,x}, \ldots, q_{j+k}, q_{j+k,x}).$$

(4)

Using the boundary condition

$$q_0 = F(q_1, q_{1,x}, q_2, q_{2,x}, \ldots, q_m, q_{m,x})$$

(5)

we shall reduce the lattice (3) to a semi-infinite lattice determined on the right half-axis $j > 0$. Everywhere below we assume local analyticity of the functions $f, g, F$. Notice that, generally speaking, for $k > 1$ the boundary condition (3) is not sufficient to close the lattice (4). For this purpose in addition to (3) it is necessary to express the variables $q_{-1}, q_{-1,x}, q_{-2}, q_{-2,x}, \ldots, q_{1-k}, q_{1-k,x}$ through the dynamic variables $q_1, q_{1,x}, q_2, q_{2,x}, \ldots$. Assuming that variables $q_0, q_{-1}, \ldots, q_{1-k}$ also satisfy equation (3), one can easily obtain an algorithm of the boundary condition (3) continuation. First of all, differentiating (3), one finds expression for $q_{0,x}$. Then, solving the equation $f(q_{-1}, q_0, q_1) = D_x^2(F)$ with respect to $q_{-1}$, one obtains

$$q_{-1} = F^{-1}(q_1, q_{1,x}, \ldots, q_{m+1}, q_{m+1,x}).$$

Further on each following step one finds $q_{-j} = F^{-j}(q_1, q_{1,x}, \ldots)$ from equation

$$f(q_{-j}, F^{(1-j)}(q_1, q_{1,x}, \ldots)) = D_x^2(F^{(1-j)}).$$
Definition. The boundary condition (5) is called degenerate if the following identity holds
\[
\frac{\partial f(q_{-1}, q_0, q_1)}{\partial q_{-1}} \bigg|_{q_0=F} = 0.
\]

Obviously, for the degenerate boundary condition the continuation algorithm is not valid.

**Proposition 2.1** Let the boundary condition (5) is degenerate, then \( F \) does not depend on variables \( q_{1,x}, q_2, q_{2,x}, q_3, q_{3,x}, \ldots \) (but can depend on \( q_1 \)).

**Proof.** Assume that variables \( q_1 \) and \( F \) are independent, i.e. \( F \) depends on \( q_m \) or \( q_{m-1,x} \), \( m > 1 \). Then identity \( \frac{\partial f}{\partial q_{-1}}(q_{-1}, F, q_1) \equiv 0 \) yields that the function \( f(q_{j-1}, q_j, q_{j+1}) \) does not depend on variable \( q_{j-1} \) which contradicts the form of the lattice (3). ■

**Consequence.** Let the boundary condition (5) is not degenerate, then the algorithm of continuation given above is correct.

**Proof.** In identity \( \frac{\partial f}{\partial q_{-j}}(q_{-j}, F^{(1-j)}, F^{(2-j)}) \equiv 0 \) the functions \( F^{(2-j)}(q_1, \ldots, q_{m+j-2}) \) and \( F^{(1-j)}(q_1, \ldots, q_{m+j-1}) \) depend on the different sets of dynamic variables and therefore are independent. Hence, from this identity the contradiction follows that the function \( f \) in (3) essentially depends on the third argument. ■

**Proposition 2.2** Let the boundary condition (5) is degenerate, then it completely closes the lattice (4) on the half-axis \( j > 0 \).

**Proof.** From the degeneracy of the boundary condition (5) it follows that variables \( q_{-1}, q_{-1,x}, q_{-2}, q_{-2,x}, \ldots \) can not be expressed through the dynamic variables \( q_1, q_{1,x}, q_2, q_{2,x}, \ldots \) of the problem (3). Therefore they can be considered as independent ones. From the condition of commutation of the flows determined by equations (3), (4) in the point \( j = 1 \) one obtains
\[
D_t(f_1) = D_x^2(g_1)
\]
where the subscript denotes shift on \( j \):
\[
f_j = f(q_{j-1}, q_j, q_{j+1}), \quad g_j = g(q_{j-k}, q_{j-k,x}, \ldots, q_{j+k}, q_{j+k,x}).
\]

Expanding this relation one obtains
\[
= \frac{\partial f_1}{\partial q_0} D_t(F) + \frac{\partial f_1}{\partial q_1} q_1 + \frac{\partial f_1}{\partial q_2} g_2 =
= \frac{\partial g_1}{\partial q_{1-k}} f_{1-k} + \frac{\partial g_1}{\partial q_{1-k,x}} D_x(f_{1-k}) + \ldots + \frac{\partial g_1}{\partial q_{1+k}} f_{1+k} + \frac{\partial g_1}{\partial q_{1+k,x}} D_x(f_{1+k}) + R
\]
where \( R \) contains all terms with the second order partial derivatives of \( g_1 \). It is easy to see that the left hand side of the last equality does not depend on the variables \( q_{-k}, q_{-k,x} \), therefore
the term \( \frac{\partial g}{\partial q_{-k,x}} \frac{\partial f}{\partial q_{-k}} + \frac{\partial g}{\partial q_{-k}} \cdot \frac{\partial f}{\partial q_{-k,x}} \) in the right hand side is equal to 0, under the condition (3). But for \( k > 1 \) it is obvious that \( \frac{\partial f}{\partial q_{-k}} \neq 0 \) \( \frac{\partial g}{\partial q_{-k}} = 0 \). The factor \( \frac{\partial g}{\partial q_{-k}} \) is also equal to 0, for otherwise the first term in the right hand side of equality would essentially depend on variable \( q_{-k} \).

Thus, under condition (3) the function \( g \) does not depend on \( q_{-k}, q_{-k,x} \). Assembling factors at independent variables \( q_{-k}, q_{-k,x} \ldots, q_{-2}, q_{-2,x} \) and repeating the reasoning, one gets that \( \frac{\partial g}{\partial q_{-j}} \bigg|_{(5)} = 0, \frac{\partial g}{\partial q_{-j,x}} \bigg|_{(5)} = 0, j = 1, 2, \ldots, k-1 \), i.e. the right hand side of the equation \( q_{1,t} = g(q_{1-k}, q_{1-k,x}, \ldots, q_{1+k}, q_{1+k,x}) \) under condition (3) actually depends only on dynamic variables \( q_{1}, q_{1,x}, q_{2}, q_{2,x}, \ldots \). The fact that other equations of the lattice (4) for \( n > 0 \) do not depend on variables \( q_{-j}, q_{-j,x} \), if \( q_{0} = F \) is checked similarly.

**Consequence.** The degenerate boundary condition reduces commuting infinite lattices (3),(4) into commuting semi-infinite lattices, given on the half-axis \( j > 0 \).

Notice that a similar fact is true for the lattices \( q_{j,x} = f(q_{j+1}, q_{j}, q_{j-1}) \) of the Volterra type as well.

**Definition.** The boundary problem (3),(5) is called compatible with the higher symmetry (4), if one of the following conditions is fulfilled:

i) Boundary condition (3) is degenerate;

ii) Semi-infinite lattice reduced from (3) by virtue of (3) and its differential consequences obtained by differentiating it with respect to \( x \) by virtue of (3) (see above algorithm of continuation) commutes with the semi-infinite lattice (3),(3).

One can associate with the pair of lattices (3),(4) a system of two partial differential equations. For this purpose one passes from the standard set of dynamical variables \( q_{0}, q_{0,x}, q_{\pm 1}, q_{\pm 1,x}, \ldots \) to the dynamical set consisting of the variables \( q_{0}, q_{1} \) and their derivatives with respect to \( x \). Thus \( q_{-1} \) and \( q_{2} \) are expressed from the conditions \( q_{0,xx} = f(q_{1}, q_{0}, q_{-1}) \) and \( q_{1,xx} = f(q_{2}, q_{1}, q_{0}) \) and so on. Rewriting the lattice (4) in the new variables, one comes to a system of evolutionary equations (cf. (3))

\[
q_{0,t} = g_{+}(q_{0}, q_{1}, q_{0,x}, q_{1,x}, \ldots, q_{0,xx}, q_{1,xx})
\]
\[
q_{1,t} = g_{-}(q_{0}, q_{1}, q_{0,x}, q_{1,x}, \ldots, q_{0,xx}, q_{1,xx})
\]

This transformation maps the boundary condition (3) into the differential constraint of the form

\[
q_{0} = \hat{F}(q_{1}, q_{0,x}, q_{1,x}, \ldots, q_{0,xx}, q_{1,xx})
\]
The following criterion of the boundary condition compatibility with the higher symmetry is a direct consequence of the way the system (6) was constructed.

**Proposition 2.3** In order that the boundary problem (3), (5) be compatible with the symmetry (4), it is necessary and sufficient that the differential constraint (7) be consistent with the dynamics of the equation (6).

### 3 Differential constraints compatible with the ZS-AKNS hierarchy

Let us rewrite the higher symmetries of the Toda lattice

\[ q_{j,t_2} = q_{j,x}^2 + \exp(q_{j+1} - q_j) + \exp(q_j - q_{j-1}) \]  
\[ q_{j,t_3} = q_{j,x}^3 + (2q_{j,x} + q_{j+1,x}) \exp(q_{j+1} - q_j) + (2q_{j,x} + q_{j-1,x}) \exp(q_j - q_{j-1}) \]

as the systems of partial differential equations, introducing new variables \( u_j = \exp q_{j+1}, \ \text{and} \ \ v_j = \exp(-q_j) \) and expressing \( q_k, q_{k,x} \) through derivatives of these variables with respect to \( x \) owing to the Toda lattice (see (8)). By virtue of the lattice (8) each pair of the variables \( u_j, v_j \) satisfies the ZS-AKNS system

\[ u_{t_2} = u_{xx} + 2u^2v, \quad -v_{t_2} = v_{xx} + 2v^2u \]  
\[ u_{t_3} = u_{xxx} + 6uvu_x, \quad v_{t_3} = v_{xxx} + 6uvv_x. \]

Other higher symmetries of the Toda lattice also correspond to the higher symmetries from ZS-AKNS hierarchy. According to the proposition 2.3, finding of a boundary condition of the form (3) compatible with some higher symmetry of the Toda lattice is reduced to finding of differential constraint of the form (7) compatible with dynamics, determined by this symmetry. Rewriting this constraint in variables \( u_0, v_0 \) and omitting the subscript 0 for brevity, we come to the following problem.

**Problem.** Find the differential constraints of the form

\[ F(u, v, u_x, v_x, \ldots, u_{x\ldots x}, v_{x\ldots x}) = 0 \]

(12)
compatible with the $k$-th symmetry of the ZS-AKNS hierarchy, that is those satisfying the identity
\[ D_{t_k}(F) \bigg|_{F=0} \equiv 0. \tag{13} \]

Let us call the order of the differential constraint (12) the maximum order of derivatives with respect to $x$ contained in it. Finding of constraint of the given order compatible with a given flow is reduced to direct, but rather tedious calculations. However, it is easy to prove that because of skew symmetry of the main part, the system (10) and its even order symmetries do not admit any differential constraint, except trivial ones
\[ u = 0 \quad \text{or} \quad v = 0, \tag{14} \]
which correspond to the boundary conditions $\exp(q_1) = 0$ or $\exp(-q_0) = 0$. Thus, the Problem can have nontrivial solution only for the flows, appropriate to the times $t_{2k+1}$. Apparently, if the constraint (12) is compatible with some odd order higher symmetry, then it is compatible with the system (10) as well. Basing on this hypothesis, we shall use the system (11) as a test symmetry when constructing boundary conditions for the Toda lattice. The following statements are proven by direct computations.

**Proposition 3.1** The constraint $u = P(v)$ is compatible with dynamics of the equation (11) in only case if $P$ is a linear function:
\[ u = \alpha v + \beta. \tag{15} \]

**Proposition 3.2** The differential constraint of the first order is compatible with the system (11) if and only if it is of the form $u_x = a(u, v)v_x$, where the factor $a(u, v)$ satisfies the Hopf equation $a_v + aa_u = 0$.

**Remark.** In essence, this constraint is reduced to the linear constraint from the previous proposition. Really, the function $a$ is the first integral of relation $u_x = av_x$; assuming $a = \alpha$ and integrating, one obtains (13).

**Proposition 3.3** The differential constraint of the form $u = P(v, v_x, v_{xx})$ compatible with the system (11) is set by the formula
\[ u(v^2 - c_0) = \frac{v^2 v + c_2(v^2 + c_0) + c_1 v}{v^2 - c_0} - v_{xx}. \tag{16} \]
where $c_0, c_1, c_2$ are arbitrary constants.

Finding of more complex constraints directly along the definition (13) becomes difficult. For further progress we need a method, which would allow us to reproduce new examples from already found ones. For this purpose it is enough to find operations acting on the set of integrable constraints. Two such operations are obvious. Indeed, the system (11) is invariant under the transformations

\begin{align*}
S_c : \quad & u = cu, \quad v = v/c, \quad c = \text{const} \\
R : \quad & u = v, \quad v = u
\end{align*}

hence rewriting the constraint (12) in the new variables yields integrable constraint again. For example, in the formula (16) it is possible to swap places of $u$ and $v$. The less obvious operation is application of the Bäcklund (auto-)transformations. As is well-known (see, e.g [8]) the system (11) admits two essentially different Bäcklund transformations. One of them is given by the explicit formula

\begin{equation}
T : \quad v_1 = 1/u, \quad u_1 = u_{xx} - u_x^2/u + u^2v
\end{equation}

and is equivalent to the shift $q_j \rightarrow q_{j+1}$ in the Toda lattice. Substituting expression (19) into the constraint (12) imposed on the variables $u_1, v_1$ one finds a new constraint on the variables $u, v$. In this case the orders of both constraints differ not more than by 2. For example, from the constraint $u_1 = 1$ one obtains (using in addition the reflection $R$) the constraint (16) with constants $c_0 = c_1 = 0, c_2 = 1$.

The second Bäcklund transformation (more precisely, its ”$x$-part”) is of the form

\begin{equation}
B_\mu : \quad u_x = \ddot{u} + \mu u + u^2\ddot{v}, \quad -v_x = v + \mu \ddot{v} + \ddot{v}^2u
\end{equation}

Differentiation of these relations gives expressions for $v$ and derivatives of $u$ and $v$ of any order through $u$ and derivatives of $\ddot{u}, \ddot{v}$. Substituting these expressions into the constraint (12), one obtains some relation of the form $\hat{F}(u, \ddot{u}, \ddot{v}, \ldots, \ddot{u}_{xx}, \ddot{v}_{xx}) = 0$. Eliminating the variable $u$ from the equations $\hat{F} = 0$ and $D_x(\hat{F}) = 0$ one obtains for the variables $\ddot{u}, \ddot{v}$ a differential constraint

$\hat{F}(\ddot{u}, \ddot{v}, \ldots, \ddot{u}_{xx}, \ddot{v}_{xx}) = 0$.

Since, according to the definition of the Bäcklund transformation, the new variables $\ddot{u}, \ddot{v}$ also satisfy the system (11), hence the found constraint appears compatible with this system as
well. We call the described procedure dressing of the constraint (12). It is easy to check that dressing also changes the order of constraint not more than by 2.

The indicated transformations generate some group with relations

\[
S_c S_d = S_{cd}, \quad R S_c R = S_d, \quad S_c T = T S_c, \quad S_c B_\mu = B_\mu S_c, \quad R^2 = 1, \quad R T R = T^{-1}, \quad R B_\mu R = S_{-1} B_{\mu}^{-1}, \quad T B_\mu = B_\mu T, \quad B_\mu B_\nu = B_\nu B_\mu.
\]

(21)

Our hypothesis is that all constraints compatible with the odd flows of the ZS-AKNS hierarchy form the orbit of two simplest constraints \( u = 1 \) and \( u = 0 \) under action of this group. Notice that for all constraints obtained in such a way the consistence with the odd flows really takes place. It follows from the facts that these flows are compatible with the constraint \( u = \text{const} \) and that the transformations (17)-(20) act on all hierarchy. The constraints obtained as a result of application of several transformations \( T \) and \( B_\mu \) to the nondegenerate constraint \( u = 1 \) are listed in the table 1. Let us comment the formulae indicated in it. Denoting result of \( m \)-th iteration of the Bäcklund transformation through \( u(m), v(m) \) one obtains from (21) a lattice

\[
u_x(m) = u(m + 1) + \mu(m) u(m) + u^2(m) v(m + 1), \quad (22)
\]

\[-v_x(m + 1) = v(m) + \mu(m) v(m + 1) + v^2(m + 1) u(m).
\]

(23)

It is obvious that by virtue of the constraint \( u(0) = 1 \) the equation (22) at \( m = 0 \) turns into the constraint \( u(1) + v(1) + \mu(0) = 0 \) of the form (13). Notice that variables \( v(0), v(1) \) satisfy the equations KdV and mKdV

\[
\begin{align*}
v_t(0) &= v_{xxx}(0) + 6v(0)v_x(0), \quad v_t(1) = v_{xxx}(1) - 6(v(1) + \mu(0))v(1)v_x(1),
\end{align*}
\]

respectively and the equation (23) at \( m = 0 \) turns into the Miura map

\[
-v(0) = v_x(1) + \mu(0) v(1) + v^2(1)
\]

between them. On the next step the equation (22) at \( m = 1 \) yields

\[
u(1) = -\frac{v_x(2) + \mu(1)v(2) - \mu(0)}{v^2(2) - 1}
\]

(24)

and, substituting in (22) at \( m = 1 \) one obtains, after simple calculations, the constraint (14) with constants

\[
c_0 = 1, \quad c_1 = -\mu^2(0) - \mu^2(1), \quad c_2 = \mu(0)\mu(1)
\]
written for the variables $u(2), v(2)$. Notice that using the transformation (17) one can obtain constraint (16) with arbitrary choice of parameters. Variable $v(2)$ satisfies the equation

$$v_t = v_{xxx} - \frac{6vv_x}{v^2 - 1}\left(v_{xx} - \frac{v v_x^2 + c_2(v^2 + 1) + c_1 v}{v^2 - 1}\right).$$

Equation (24) can be rewritten in the form of differential substitution

$$v(1) = \frac{v_x(2) + \mu(1)v(2) - \mu(0)v^2(2)}{v^2(2) - 1}$$

connecting mKdV and equation (25). Notice, that the point transformation $v = \tanh y$ brings (25) into the so called exponential Calogero-Degasperis equation [9]

$$y_t = y_{xxx} - 2y_x^3 - \frac{3}{8}\left((\mu(1) - \mu(0))^2 e^{4y} + (\mu(1) + \mu(0))^2 e^{-4y} - 2\mu^2(1) - 2\mu^2(0)\right)y_x.$$

Further application of the transformation (20) results in too cumbersome constraint of 4-th order. Appears, however, that the combination $T^{-1}B_{\mu(2)}$ gives constraint of the 2-nd order again. It can conveniently be written in the variables

$$2r = u_{-1}(3) - v_{-1}(3), \quad 2s = u_{-1}(3) + v_{-1}(3).$$

Omitting rather bulky calculations we put only the answer:

$$16r^2(s_x^2 + P) = (2s_{xx} + \dot{P})^2$$

where

$$P = (s - \alpha)(s - \alpha_0)(s - \alpha_1)(s - \alpha_2), \quad 2\alpha = \mu(0) + \mu(1) + \mu(2), \quad \alpha_j = \mu(j) - \alpha.$$

Rewriting the system (11) in the new variables and eliminating $r$ by virtue of the constraint one finds that variable $s$ satisfies the elliptic Calogero-Degasperis equation

$$s_t = s_{xxx} + 6\left(s_x^3 - \frac{2s_{xx} + \dot{P}^2}{16(s_x^2 + P)}\right)s_x$$

introduced in [9] as well. The differential substitution connecting this equation with equation (24) is given by the formula

$$(c_2 + 2\mu(2)s - 2s_x)v^2 + (c_1 + \mu(2)^2 + 4s^2)v + c_2 + 2\mu(2)s + 2s_x = 0.$$
4 Boundary conditions for the Toda lattice

Let us write down the boundary conditions corresponding to the differential constraints (14)-(16) found above (certainly the constraint (26) gives some boundary condition as well, but we do not need it). Passing to the variables $q_j$ one obtains

$$\exp(q_1) = 0 \quad \text{or} \quad \exp(-q_0) = 0$$  \hspace{1cm} (27)
$$\exp(q_1) = \alpha \exp(-q_0) + \beta$$  \hspace{1cm} (28)
$$c_0 \exp(q_1) = \exp(-q_1) + \frac{c_0 q_{0,x}^2 + c_2 (c_0 \exp(q_0) + \exp(-q_0)) + c_1}{c_0 \exp(q_0) - \exp(-q_0)}$$  \hspace{1cm} (29)

where $\alpha, \beta, c_0, c_1, c_2$ are arbitrary parameters. It should be mentioned that the boundary conditions (27), (28) were found earlier in [11] and [12] respectively.

It is well known (see, e.g. [6], [7], [13]) that each simple Lie algebra of the finite growth corresponds to some integrable generalized Toda lattice. The lattices associated with the Lie algebras $A_{n-1}, B_n, C_n$ are obtained by imposing of the degenerate boundary condition $\exp(-q_0) = 0$ on the left end of the lattice (1) and boundary conditions (27), (28) of particular form on its right end, namely $\exp(q_{n+1}) = 0$, $q_{n+1} = 0$, $q_{n+1} = -q_n$ respectively. Now we shall demonstrate that lattices of the $D$ type correspond to the boundary condition of the form (29). Lie algebra $D_n$ corresponds to the lattice

$$q_{1,xx} = e^{q_2-q_1},$$
$$q_{j,xx} = e^{q_{j+1}-q_j} - e^{q_j-q_{j-1}}, \quad j = 2, \ldots, n-2,$$
$$q_{n-1,xx} = (e^{q_n} + e^{-q_n}) e^{-q_{n-1}} - e^{q_{n-1}-q_{n-2}},$$
$$q_{n,xx} = (e^{-q_n} - e^{q_n}) e^{-q_{n-1}}.$$  \hspace{1cm} (30)

It is clear that the boundary condition on the left end is of the form $e^{-q_0} = 0$, i.e. is degenerate. In order to bring the last but one equation of the system (30) into the standard form we make the change of variables

$$q_j = q_j - \log 2, \quad j = 1, \ldots, n-1, \quad e^{\hat{q}_n} = \cosh q_n.$$  \hspace{1cm} (31)

Thus the last equation of the system accepts the form (we omit the hat over variables $q_j$)

$$q_{n,xx} = (e^{-q_n} - e^{q_n}) e^{-q_{n-1}} + \frac{q_{n,x}^2}{e^{2q_n} - 1}.$$
Assuming \( q_{n,xx} = e^{q_{n+1} - q_n} - e^{q_n - q_{n-1}} \) one obtains a boundary condition for the right end of the lattice

\[
e^{q_{n+1}} = e^{-q_{n-1}} + \frac{q_{n,xx}^2}{2 \sinh q_n}, \tag{32}\]

Obviously, up to the shift \( q_0 \mapsto q_n \), the formula (32) is a particular case of (29) for \( c_0 = 1, \ c_1 = c_2 = 0 \). So, the Toda lattice of the type \( D_n \) is reduced, by use of the change of variables (31), to the following finite-dimensional reduction of the Toda lattice (1):

\[
e^{-q_0} = 0, \quad q_{j,xx} = e^{q_{j+1} - q_j} - e^{q_j - q_{j-1}}, \quad j = 1, \ldots, n \tag{33}\]
\[
e^{q_{n+1}} = e^{-q_{n-1}} + \frac{q_{n,xx}^2}{2 \sinh q_n}.
\]

Now we can represent systems related to the Lie algebras \( A_{n-1}, B_n, C_n, D_n, BD_n, BA_{2n-2}, B_{n-1}, C_{n-1}, CA_{2n-3}, D_{n-1} \) (in other words, only the series \( \tilde{A} \) corresponding to the periodic Toda lattice and exceptional Lie algebras drop out our consideration) in the form of the problem (1)-(2) with boundary conditions of the type (27)-(29). Actually in this matter structure of the beginning and the end of the Lie algebra Coxeter-Dynkin diagram is important only. The left(right) column of the table 2 contains the boundary conditions on the left(right) end of the lattice and corresponding variants of the beginning(end) of the diagram. All possible combinations of the boundary conditions on the left and right ends give us the Toda lattices appropriate to all Lie algebras listed above.

We shall demonstrate now that the series \( D \) lattices can be obtained from the series \( A \) lattices as a result of reflection type reduction combined with the Bäcklund transformation (20). For this purpose we need to rewrite the transformations (17)-(20) in the variables \( q_j \):

\[
S_c : \quad \tilde{q}_j = q_j + \log c, \quad R : \quad \tilde{q}_j = -q_{1-j}, \quad T : \quad \tilde{q}_j = q_{j+1},
\]

\[
B_{\mu} : \quad \begin{cases} 
\tilde{q}_{j,x} = \exp(\tilde{q}_j - q_j) + \exp(q_{j+1} - \tilde{q}_j) + \mu, \\
q_{j,x} = \exp(\tilde{q}_j - q_j) + \exp(q_j - \tilde{q}_{j-1}) + \mu.
\end{cases} \tag{34}\]

The generalized Toda lattice corresponding to the Lie algebra \( A_{m-1} \) is obtained from the infinite lattice by imposing two degenerate boundary conditions:

\[
q_0 = +\infty, \quad q_{j,xx} = e^{q_{j+1} - q_j} - e^{q_j - q_{j-1}}, \quad j = 1, \ldots, m \tag{35}
\]
\[
q_{m+1} = -\infty.
\]
It is easy to check that the formula (34) complemented by the conditions
\[ \tilde{q}_0 = q_0 = +\infty, \quad \tilde{q}_{m+1} = q_{m+1} = -\infty \] (36)
sets the Bäcklund transformation for the system (35). Thus the mapping \( B_\mu : Q \mapsto \tilde{Q} \), where \( Q = (q_1, q_{1,x}, \ldots, q_m, q_{m,x}) \) is given by explicit formulae and is convertible. Indeed, the relations (34), (36) represent a closed system of algebraic equations relatively \( \tilde{q}_{j,x}, \exp(\tilde{q}_j) \), which can easily be solved recurrently, starting from the equation \( q_{1,x} = \exp(\tilde{q}_1 - q_1) + \mu \). Notice, that the given transformation is an example of integrable discrete mapping. It can be interpreted as finite dimensional reduction of the twice discrete lattice induced by transformation (34). About integrable mappings see e.g [14], [15], [16].

The following statement establishes connection between lattices of the series A and D.

**Proposition 4.1** Let \( m = 2n \) and the solution \( Q \) of the system (35) satisfies the reduction condition \( Q = S_{-1}RT^mQ \), that is \( q_j = i\pi - q_{m+1-j} \). Then the new solution \( \tilde{Q} = B_0Q \) of the system (34) possesses the following properties:

1) \( \tilde{Q} = B_0^2RT^m\tilde{Q} \),

2) First half of the vector \( \tilde{Q} \), that is the vector \( (\tilde{q}_1, \tilde{q}_{1,x}, \ldots, \tilde{q}_n, \tilde{q}_{n,x}) \) solves the lattice (34).

**Proof.** The property 1) can easily be proven with the help of relations (21). Further, from reduction conditions on \( q \) it follows that \( q_{n+1} = i\pi - q_n \), that is \( u_n = -v_n \). It follows from the table 1 that dressing of this constraint by transformation \( B_0 \) yields the constraint of the form (16) with constants \( c_0 = 1, c_1 = c_2 = 0 \) connecting the variables \( \tilde{u}_n, \tilde{v}_n \). As we have already seen this constraint is equivalent to the boundary condition (32).

Concluding this section we note that the similar simple connection can be established between the type A lattices and the lattices with the boundary condition (29) of general form on the right end. The proof of the following proposal does not cause difficulties.

**Proposition 4.2** Let \( m = 2n+1 \) and the solution \( Q \) of the system (35) satisfies the reduction condition \( q_j = -q_{m+1-j} \). Then the new solutions of this system \( \tilde{Q} = S_cB_\mu\tilde{Q}, \tilde{\bar{Q}} = B_\nu\tilde{Q} \) where \( c, \mu, \nu \) are arbitrary parameters solve the Toda lattices with the boundary condition \( \exp(-q_0) = 0 \) on the left end and boundary conditions (28), (29) of the general form on the right end respectively.

It follows from the well known fact on the general solution of the type \( A_m \) lattice (see, e.g. [1]) and propositions 4.1 and 4.2 that the general solutions of such lattices can be written explicitly as rational functions on the exponents \( e^{\lambda x} \).
5 Boundary conditions with explicit dependence on $x$.

The boundary conditions found above admit some generalization. Notice that the Toda lattice (1) is invariant under transformation

$$X : \tilde{q}_j = q_j + \varepsilon x,$$

(37)

or, in the variables $u, v$

$$X : \tilde{u} = e^{\varepsilon x} u, \quad \tilde{v} = e^{-\varepsilon x} v.$$

(38)

The difference between this transformation and transformations $S_c, R, T, B_\mu$ considered earlier is that it does not preserve the higher symmetries of the Toda lattice. For example, the system (11) at this change of variables is rewritten as follows

$$u_{ts} = u_{xxx} + 6uvu_x - 3\varepsilon(u_{xx} + 2u^2v) + 3\varepsilon^2u_x - \varepsilon^3u,$$

$$v_{ts} = v_{xxx} + 6uvv_x + 3\varepsilon(v_{xx} + 2v^2u) + 3\varepsilon^2v_x + \varepsilon^3v.$$

In general, it is easy to check that change (38) determines some linear transformation in the space of symmetries of the ZS-AKNS hierarchy. Boundary conditions found in the previous section are consistent with the odd order symmetries subspace. If this subspace were invariant under the transformation $X$ then rewriting of these boundary conditions in variables $\tilde{q}_j$ would give nothing new. The presence of the even order terms in the transformed symmetries results in that the transformed boundary conditions contain explicit dependence on $x$. For example, the formulae (28), (29) are converted into the formulae

$$\exp(q_1) = \alpha \exp(2\varepsilon x - q_0) + \beta \exp(\varepsilon x),$$

(39)

$$c_0 \exp(q_1) = \exp(2\varepsilon x - q_{-1}) + \frac{c_0(q_{0,x} - \varepsilon)^2 + c_2(c_0 \exp(q_0 - \varepsilon x) + \exp(\varepsilon x - q_0)) + c_1}{c_0 \exp(q_0 - 2\varepsilon x) - \exp(-q_0)}$$

(40)

correspondingly. It should be emphasized, that new boundary conditions are not worse than old ones, but they are compatible with other symmetry set. At imposing on both ends of the Toda lattice of boundary conditions relevant to the same value $\varepsilon$ this symmetry set passes into the symmetries of obtained finite-dimensional system and provides its integrability. The situation, however, essentially changes if the boundary conditions imposed on the different ends of the lattice correspond to the different values of parameter $\varepsilon$. It turns out that in small dimensions the equations of the Painlevé type arise (compare with [18], where the Painlevé
equations arise at quasiperiodic closing of integrable lattices). In the examples below the length 1 reductions of the Toda lattice are considered. On the left end we, without loss of generality, impose boundary conditions of the type (28) or (29) and on the right one deformed conditions (39) or (40) at some fixed value of ε. (It is clear that the boundary condition (27) will not give anything new, since the transformation (37) does not change it.)

1) Let us try boundary conditions of the form (28) and (39) at ε = 2:

\[ e^{-q-1} = \alpha e^{q_0} + \beta, \quad e^{q_1} = \gamma e^{4x-q_0} + \delta e^{2x}. \]

On the variable \( q = q_0 \) one obtains the differential equation

\[ q_{xx} = \gamma e^{4x-2q} + \delta e^{2x-q} - \alpha e^{2q} - \beta e^q. \]

It is easy to check that the point transformation \( e^{q(x)} = y(z), \quad e^x = z \) brings it into the 3-rd Painlevé equation

\[ y_{zz} = \left( \frac{1}{2y} + \frac{1}{y-1} \right) y_x^2 - \frac{y_z^2}{y} + \frac{(y-1)^2}{z^2} (Ay + B + Cy^3 + D). \]

with the values of parameters \( A = -\beta, \quad B = \delta, \quad C = -\alpha, \quad D = \gamma. \)

2) Impose the boundary condition of the form (29) with \( c_0 = 1 \) without loosing of generality on the left end and a boundary condition (39) with \( \varepsilon = 1 \) on the right one:

\[ e^{-q-1} = e^{q_1} - \frac{q_{0,x}^2 + c_2(e^{q_0} + e^{-q_0}) + c_1}{e^{q_0} - e^{-q_0}}, \quad e^{q_1} = \alpha e^{2x-q_0} + \beta e^x. \]

On the variable \( q = q_0 \) one obtains the differential equation

\[ q_{xx} = \frac{q_x^2}{1 - e^{-2q}} + \alpha e^{2x}(e^{-2q} - 1) + \beta e^x(e^{-q} - e^q) + \frac{c_2(e^q + e^{-q}) + c_1}{1 - e^{-2q}}. \]

It is easy to check that the point transformation \( e^{q(x)} = y(z)^{-1}, \quad e^x = z \) brings it into the 5-th Painlevé equation

\[ y_{zz} = \left( \frac{1}{2y} + \frac{1}{y-1} \right) y_x^2 - \frac{y_z^2}{y} + \frac{(y-1)^2}{z^2} (Ay + B + Cy + D(y + 1)). \]

with the values of parameters \( 8A = -c_1 - 2c_2, \quad 8B = c_1 - 2c_2, \quad C = 2\beta, \quad D = 2\alpha. \)

3) At last, consider a combination of boundary conditions of the form (29) and (40) (thus, because of small dimension of the system, the left and right ends actually coincide). It is easy to prove that from 7 parameters, contained in these two formulas, 3 can be replaced by
any nonzero constants by use of dilating \( x \), shift \( x \) and shift \( q \). Therefore we may consider boundary conditions

\[
e^{q_1} = e^{-q-1} + \frac{q_0^2}{e^{q_0} - e^{-q_0}} + a(e^{q_0} + e^{-q_0}) + b, \quad e^{q_1} = e^{2x-q-1} + \frac{(q_{0,x} - 1)^2 + c(e^{q_0-x} + e^{x-q_0}) + d}{e^{q_0-2x} - e^{-q_0}}.
\]

Solving these equations relatively \( e^{q_1}, e^{-q-1} \) and substituting found values into the formula \( q_{0,xx} = e^{q_0-q_0} - e^{q_0-q-1} \), one obtains some differential equation of the 2-nd order on the variable \( q = q_0 \). One can check up that change of variables \( e^{q(x)} = \frac{y(z) + \sqrt{z}}{y(z) - \sqrt{z}}, \quad e^{x} = \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \) converts it into the 6-th Painlevé equation

\[
y_{zz} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-z} \right) y_z^2 - \left( \frac{1}{z} + \frac{1}{z-1} + \frac{1}{y-z} \right) y_z + \frac{y(y-1)(y-z)}{z^2(z-1)^2} \left( A + \frac{B}{y^2} + C \frac{z - 1}{(y-1)^2} + D \frac{z(z-1)}{(y-z)^2} \right)
\]

with parameters \( 8A = -b - 2a, \quad 8B = b - 2a, \quad 8C = -d - 2c, \quad 8D = d - 2c + 4 \).

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Figure 1: Dressing of the differential constraints
Figure 2: Boundary conditions corresponding to the ends of Coxeter-Dynkin diagrams