Fast Algorithms for Diameter-Optimally Augmenting Paths and Trees

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Abstract. We consider the problem of augmenting an n-vertex graph embedded in a metric space, by inserting one additional edge in order to minimize the diameter of the resulting graph. We present exact algorithms for the cases when (i) the input graph is a path, running in $O(n \log^3 n)$ time, and (ii) the input graph is a tree, running in $O(n^2 \log n)$ time. We also present an algorithm that computes a $(1+\varepsilon)$-approximation in $O(n + 1/\varepsilon^3)$ time, for paths in $\mathbb{R}^d$, where $d$ is a constant.

1 Introduction

Let $G = (V, E)$ be a graph in which each edge has a positive weight. The weight (or length) of a path is the sum of the weights of the edges on this path. For any two vertices $x$ and $y$ in $V$, we denote by $\delta_G(x, y)$ their shortest-path distance, i.e., the minimum weight of any path in $G$ between $x$ and $y$. The diameter of $G$ is defined as $\max\{\delta_G(x, y) : x, y \in V\}$.

Assume that we are also given weights for the non-edges of the graph $G$. In the Diameter-Optimal $k$-Augmentation Problem, DOAP($k$), we have to compute a set $F$ of $k$ edges in $(V \times V) \setminus E$ for which the diameter of the graph $(V, E \cup F)$ is minimum.

In this paper, we assume that the given graph is a path or a tree on $n$ vertices that is embedded in a metric space, and the weight of any edge and non-edge is equal to the distance between its vertices. We consider the case when $k = 1$; thus, we want to compute one non-edge which, when added to the graph, results in an augmented graph of minimum diameter. Surprisingly, no non-trivial results were known even for the restricted cases of paths and trees.

Throughout the rest of the paper, we assume that $(V, |.|)$ is a metric space, consisting of a set $V$ of $n$ elements (called points or vertices). The distance between any two points $x$ and $y$ is denoted by $|xy|$. We assume that an oracle

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is available that returns the distance between any pair of points in $O(1)$ time. Our contributions are as follows:

1. If $G$ is a path, we solve problem DOAP(1) in $O(n \log^3 n)$ time.
2. If $G$ is a path and the metric space is $\mathbb{R}^d$, where $d$ is a constant, we compute a $(1 + \varepsilon)$-approximation for DOAP(1) in $O(n + 1/\varepsilon^3)$ time.
3. If $G$ is a tree, we solve problem DOAP(1) in $O(n^2 \log n)$ time.

1.1 Related Work

The Diameter-Optimal $k$-Augmentation Problem for edge-weighted graphs, and many of its variants, have been shown to be NP-hard [17], or even $W[2]$-hard [10, 11]. Because of this, several special classes of graphs have been considered. Chung and Gary [6] and Alon et al. [1] considered paths and cycles with unit edge weights and gave upper and lower bounds on the diameter that can be achieved. Ishii [12] gave a constant factor approximation algorithm (approximating both $k$ and the diameter) for the case when the input graph is outerplanar. Erdős et al. [8] investigated upper and lower bounds for the case when the augmented graph must be triangle-free.

The general problem: The Diameter-Optimal Augmentation Problem can be seen as a bicriteria optimization problem: In addition to the weight, each edge and non-edge has a cost associated with it. Then the two optimization criteria are (1) the total cost of the edges added to the graph and (2) the diameter of the augmented graph. We say that an algorithm is an $(\alpha, \beta)$-approximation algorithm for the DOAP problem, with $\alpha, \beta \geq 1$, if it computes a set $F$ of non-edges of total cost at most $\alpha \cdot B$ such that the diameter of $G' = (V, E \cup F)$ is at most $\beta \cdot D_{\text{opt}}$, where $D_{\text{opt}}$ is the diameter of an optimal solution that augments the graph with edges of total cost at most $B$.

For the restricted version when all costs and all weights are identical [2, 5, 7, 13, 14], Bilò et al. [2] showed that, unless $P=NP$, there does not exist a $(c \log n, \delta < 1 + 1/D_{\text{opt}}^B)$-approximation algorithm for DOAP if $D_{\text{opt}}^B \geq 2$. For the case in which $D_{\text{opt}}^B \geq 6$, they proved that, again unless $P=NP$, there does not exist a $(c \log n, \delta < 5/3 - \frac{7 - (D_{\text{opt}}^B + 1) \mod 3}{3D_{\text{opt}}^B})$-approximation algorithm.

Li et al. [14] showed a $(1, 4 + 2/D_{\text{opt}}^B)$-approximation algorithm. The analysis of the algorithm was later improved by Bilò et al. [2], who showed that it gives a $(1, 2 + 2/D_{\text{opt}}^B)$-approximation. In the same paper they also gave an $(O(\log n), 1)$-approximation algorithm.

For general costs and weights, Dodis and Khanna [7] gave an $O(n \log D_{\text{opt}}^B, 1)$-approximation algorithm. Their result is based on a multi-commodity flow formulation of the problem. Frati et al. [10] recently considered the DOAP problem with arbitrary integer costs and weights. Their main result is a $(1, 4)$-approximation algorithm with running time $O((3^B B^3 + n + \log(Bn))Bn^2)$. 
Geometric graphs: In the geometric setting, when the input is a geometric graph embedded in the Euclidean plane, there are only a few results on graph augmentation in general. Rutter and Wolff [16] proved that the $k$-connectivity and $k$-edge-connectivity augmentation problems are NP-hard on plane geometric graphs, for $k = 2, 3, 4,$ and $5$; the problem is infeasible for $k \geq 6$ because every planar graph has a vertex of degree at most 5. Currently, there are no known approximation algorithms for this problem. Farshi et al. [9] gave approximation algorithms for the problem of adding one edge to a geometric graph while minimizing the dilation. There were several follow-up papers [15, 18], but there is still no non-trivial result known for the case when $k > 1$.

In the continuous version of the diameter-optimal augmentation problem, the input graph $G$ is embedded in the plane and the edges to be added to $G$ can have their endpoints anywhere on $G$, i.e., the endpoints can be in the interior of edges of $G$. Moreover, the diameter is considered as the maximum of the shortest-path distances over all points on $G$. Yang [19] considered the continuous version of the problem of adding one edge to a path so as to minimize the continuous diameter. He presented sufficient and necessary conditions for an augmenting edge to be optimal. He also presented an approximation algorithm, having an additive error of $\epsilon$, that runs in $O((n + |P|/\epsilon^2)n)$ time, where $|P|$ denotes the length of the input path $P$ and $\epsilon$ is at most half of the length of a shortest edge in $P$. De Carufel et al. [4] improved the running time to $O(n)$ and also considered the continuous version of the problem for cycles that are embedded in the plane. They showed that adding one edge to any cycle does not decrease the continuous diameter. On the other hand, two edges can always be added that decrease the continuous diameter. De Carufel et al. gave a full characterization of the optimal two edges. If the input cycle is convex, they find the optimal pairs of edges in $O(n)$ time.

2 Augmenting a Path with One Edge

We are given a path $P = (p_1, \ldots, p_n)$ on $n$ vertices in a metric space and assume that it is stored in an array $P[1, \ldots, n]$. To simplify notation, we associate a vertex with its index, that is $p_k = P[k]$ is also referred to as $k$ for $1 \leq k \leq n$. This allows us to extend the total order of the indices to the vertex set of $P$. We denote the start vertex of $P$ by $s$ and the end vertex of $P$ by $e$.

For $1 \leq k < l \leq n$, we denote the subpath $(p_k, \ldots, p_l)$ of $P$ by $P[k,l]$, the cycle we get by adding the edge $e_{P[k,l]}$ to $P[k,l]$ by $C[k,l]$, and the (unicyclic) graph we get by adding the edge $e_{P[k,l]}$ as a shortcut to $P$ by $P[k,l]$; the length of $X \in \{P, P[k,l], C[k,l]\}$ is denoted by $|X|$. We will consider the functions $\overline{d}_{P[k,l]} := \delta_{P[k,l]}$ and $\overline{d}_{C[k,l]} := \delta_{C[k,l]}$, where $\delta_G$ is the length of the shortest path between two vertices in $G$. For $1 \leq k < l \leq n$, we let

$$M(k, l) := \max_{1 \leq x < y \leq n} P_{k,l}(x, y)$$

denote the diameter of the graph $P[k, l]$.
Our goal is to compute a shortcut \( p_{k,l} \) for \( P \) that minimizes the diameter of the resulting unicyclic graph, i.e., we want to compute

\[
m(P) := \min_{1 \leq k < l \leq n} M(k, l).
\]

We will prove the following result:

**Theorem 1.** Given a path \( P \) on \( n \) vertices in a metric space, we can compute \( m(P) \), and a shortcut realizing that diameter, in \( O(n \log^3 n) \) time.

The algorithm consists of two parts. We first describe a sequential algorithm for the decision problem. Given \( P \) and a threshold parameter \( \lambda > 0 \), decide if \( m(P) \leq \lambda \) (see Lemma 1 a) below). In a second step, we argue that the sequential algorithm can be implemented in a parallel fashion (see Lemma 1 b) below), thus enabling us to use the parametric search paradigm of Megiddo.

**Lemma 1.** Given a path \( P \) on \( n \) vertices in a metric space and a real parameter \( \lambda > 0 \), we can decide in

- a) \( O(n \log n) \) time, or in
- b) \( O(\log n) \) parallel time using \( n \) processors

whether \( m(P) \leq \lambda \); the algorithms also produce a feasible shortcut if it exists.

To prove this lemma, observe that

\[
m(P) \leq \lambda \text{ if and only if } \bigvee_{1 \leq k < l \leq n} M(k, l) \leq \lambda.
\]

The algorithm checks, for each \( 1 \leq k < n \), whether there is some \( k < l \leq n \) such that \( M(k, l) \leq \lambda \). If one such index \( k \) is found, we know that \( m(P) \leq \lambda \); otherwise \( m(P) > \lambda \). Clearly this approach also produces a feasible shortcut if it exists.

We decompose the function \( M(k, l) \) into four monotone parts. This will facilitate our search for a feasible shortcut and enable us to do (essentially) binary search: For \( 1 \leq k < l \leq n \), we let

\[
S(k, l) := \max_{k \leq x \leq l} p_{k,l}(s, x), \quad E(k, l) := \max_{k \leq x \leq l} p_{k,l}(x, e),
\]

\[
U(k, l) := p_{k,l}(s, e), \quad O(k, l) := \max_{k \leq x < y \leq l} c_{k,l}(x, y).
\]

Then we have \( M(k, l) = \max\{S(k, l), E(k, l), U(k, l), O(k, l)\} \). The triangle inequality implies that

\[
S(k, l) \leq S(k, l + 1), \quad E(k, l) \geq E(k, l + 1),
\]

\[
U(k, l) \geq U(k, l + 1), \quad O(k, l) \leq O(k, l + 1).
\]

The function \( U \) is easy to evaluate once we have the array \( D[1, \ldots, n] \) of the prefix-sums of the edge lengths: \( D[i] := \sum_{1 \leq j < i} |p_j p_{j+1}| \). These sums can be
computed in $O(n)$ time sequentially or in $O(\log n)$ time using $n$ processors. If in addition to $D$, the vertices $s' = \max \{v \mid \delta_P(s, v) \leq \lambda\}$ and $e' = \min \{v \mid \delta_P(v, e) \leq \lambda\}$ are computed for a fixed $\lambda$ in $O(\log n)$ time (via binary search on $D$), the following decision problems can be answered in constant time:

$$S(k, l) \leq \lambda, \quad E(k, l) \leq \lambda, \quad U(k, l) \leq \lambda.$$ 

We denote the maximum of these three functions by

$$N(k, l) = \max(S(k, l), E(k, l), U(k, l)).$$

Now clearly

$$M(k, l) = \max(N(k, l), O(k, l))$$

and, consequently

$$M(k, l) \leq \lambda \text{ if and only if } N(k, l) \leq \lambda \text{ and } O(k, l) \leq \lambda.$$ 

For fixed $1 \leq k < n$, the algorithm will first check whether there is some $k < l \leq n$ with $N(k, l) \leq \lambda$. If no such $l$ exists, we can conclude that $M(k, l) > \lambda$ for all $k < l \leq n$. The monotonicity of $S$, $E$, and $U$ implies that, for fixed $1 \leq k < n$, the set

$$N_k := \{k < l \leq n \mid N(k, l) \leq \lambda\}$$

is an interval. This interval can be computed (using binary search in $P$ and in $D$ as described above) in $O(\log n)$ time. If $N_k = \emptyset$ we can conclude that for the $1 \leq k < n$ under consideration and for all $k < l \leq n$, we have that $M(k, l) > \lambda$.

If $N_k$ is non-empty, the monotonicity of $O$ implies that it is sufficient to check for $l_k = \min N_k$ (i.e. the starting point of the interval) whether $O(k, l_k) \leq \lambda$:

$$\exists k < l \leq n : O(k, l) \leq \lambda \text{ if and only if } O(k, l_k) \leq \lambda.$$ 

Note that in this case we know that $N(k, l_k) \leq \lambda$. 

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Fig. 1: (a) Illustration of the four distances that define the diameter of a shortcut $p_{kl}$: $U(k, l)$ is the length of the shortest path connecting $s$ and $e$; $O(k, l)$ is the length of the longest shortest path between any two points in $C[k, l]$; $S(k, l)$ ($E(k, l)$) is the length of the longest shortest path from $s$ ($e$) to any vertex in $C(k, l)$. (b) Illustration of the computation of $O(k, l)$. 

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Note that in this case we know that $N(k, l_k) \leq \lambda$. 

Deciding the diameter of small cycles: We now describe how to decide for a
given shortcut $1 \leq k < l \leq n$ if $O(k, l) \leq \lambda$, given that we already know that
$N(k, l) \leq \lambda$. To this end, consider the following sets of vertices from $C[k, l]$: $K := \{k \leq x \leq l \mid \delta_P(k, x) \leq \lambda\}$, $L := \{k \leq x \leq l \mid \delta_P(x, l) \leq \lambda\}$, $M := K \cap L$, $K' := K \setminus L$, $L' := L \setminus K$.

These sets are intervals and can be computed in $O(\log n)$ time by binary
search. Since $N(k, l) \leq \lambda$, we can conclude the following:

- the set of vertices of $C[k, l]$ is $K \cup L$
- $c_{k,l}(x, y) \leq \lambda$ for all $x, y \in K$
- $c_{k,l}(x, y) \leq \lambda$ for all $x, y \in L$
- $c_{k,l}(x, y) \leq \lambda$ for all $x \in M$, $y \in C[k, l]$

Consequently, if $c_{k,l}(x, y) > \lambda$ for $x, y \in C[k, l]$, we can conclude that $x \in K'$
and $y \in L'$. In order to establish that $O(k, l) \leq \lambda$, it therefore suffices to verify that
$$\bigwedge_{x \in K', y \in L'} c_{k,l}(x, y) \leq \lambda.$$ 

Note that on $P$ any vertex $x$ of $K'$ is at least $\lambda$ away from the vertex $l$, i.e.,
$\delta_P(x, l) > \lambda$. Let $x^+$ be point on (a vertex or an edge of) $P$ that is closer (along
$P$) by a distance of $\lambda$ to $l$ than to $x$, i.e., $x^+$ is the unique point on $P$ such that
$$\delta_P(x^+, l) < \delta_P(x, l)$$
and $\delta_P(x, x^+) = \lambda$.

The next (in the direction of $l$) vertex of $P$ will be denoted by $x'$, i.e., $x < x' \leq l$
is the unique vertex of $P$ such that
$$\delta_P(x, x' - 1) \leq \lambda$$
and $\delta_P(x, x') > \lambda$.

Since $x$ is a vertex of $K'$, $x'$ is a vertex of $L'$. For the following discussion we
denote the distance achieved in $C[k, l]$ by using the shortcut by $c_{k,l}^+$ and the
distance achieved by travelling along $P$ only by $c_{k,l}^-$, i.e.,
$$c_{k,l}^-(x, y) := \delta_P(x, y) \quad \text{and} \quad c_{k,l}^+(x, y) := \delta_P(x, k) + |p_{k, l}| + \delta_P(l, y).$$

Clearly
$$c_{k,l}(x, y) = \min(c_{k,l}^+(x, y), c_{k,l}^-(x, y)),$$ 
and $|C[k, l]| = c_{k,l}^+(x, y) + c_{k,l}^-(x, y)$.

For every vertex $y < x'$ on $L'$ we have that $c_{k,l}(x, y) \leq c_{k,l}^-(x, y) \leq \lambda$,
so if there is some vertex $x' \neq y \in L'$ such that $c_{k,l}(x, y) > \lambda$, we know that $x' < y \leq l$; in
that case we have that $c_{k,l}^+(x, y) \leq c_{k,l}^+(x, x')$. Since we assume that $c_{k,l}(x, y) > \lambda$,
we also know that $c_{k,l}^+(x, y) > \lambda$ and we can conclude that $c_{k,l}^+(x, x') > \lambda$, and
consequently that $c_{k,l}(x, x') > \lambda$, i.e., for all $x \in K'$ we have that
$$\bigwedge_{y \in L'} c_{k,l}(x, y) \leq \lambda$$
if and only if $c_{k,l}(x, x') \leq \lambda$.

The distance between (the point) $x^+$ and (the vertex) $x'$ on $P$ is called the
defect of $x$ and is denoted by $\Delta(x)$, i.e., $\Delta(x) = \delta_P(x^+, x')$. 
Lemma 2. We have
\[ c_{k,l}(x, x') \leq \lambda \text{ if and only if } |C[k, l]| \leq \Delta(x) + 2\lambda. \]

Proof. Observe that
\[ |C[k,l]| = \delta_P(x, k) + |\mathcal{P}_k| + \delta_P(l, x') + \delta_P(x', x^+) + \delta_P(x^+, x) \]
\[ = \delta_P(x, k) + |\mathcal{P}_k| + \delta_P(l, x') + \Delta(x) + \lambda \]
\[ = c^+_{k,l}(x, x') + \Delta(x) + \lambda. \]
Since \( c^+_{k,l}(x, x') > \lambda \), we have that \( c_{k,l}(x, x') \leq \lambda \) if and only if \( c^+_{k,l}(x, x') \leq \lambda \); the claim follows.

To summarize the above discussion, we have the following chain of equivalences (here \( \Delta_{k,l} := |C[k, l]| - 2\lambda \)):
\[ O(k, l) \leq \lambda \iff \bigwedge_{x \in K'} c_{k,l}(x, x') \leq \lambda \iff \bigwedge_{x \in K'} \Delta_{k,l} \leq \Delta(x) \iff \min_{x \in K'} \Delta(x) \geq \Delta_{k,l}. \]
Since \( K' \) is an interval, the last condition can be tested easily after some preprocessing: To this end we compute a 1d-range tree on \( D \) and associate with each vertex in the tree the minimum \( \Delta \)-value of the corresponding canonical subset. For every vertex \( x \) of \( P \) that is at least \( \lambda \) away from the end vertex of \( P \) we can compute \( \Delta(x) \) in \( O(\log n) \) time by binary search in \( D \). With these values the range tree can be built in \( O(n \log n) \) time. A query for an interval \( K' \) then gives us \( \mu := \min_{x \in K'} \Delta(x) \) in \( O(\log n) \) time and we can check the above condition in \( O(1) \) time.

We describe the algorithm in pseudocode; see Algorithm 1.

The correctness of the algorithm follows from the previous discussion. ComputePrefixSums runs in \( O(n) \) time, ComputeRangeTree runs in \( O(n \log n) \) time, ComputeFeasibleIntervalForN runs in \( O(\log n) \) time, a call to CheckOForShortcut requires \( O(\log n) \) time. The total runtime is therefore \( O(n \log n) \). It is easy to see that with \( n \) processors, the steps ComputePrefixSums and ComputeRangeTree can be realized in \( O(\log n) \) parallel time and that with this number of processors, all calls to CheckOForShortcut can be handled in parallel. Therefore, the entire algorithm can be parallelized and has a parallel runtime of \( O(\log n) \), as stated in Lemma 1 b). This concludes the proof of Lemma 1.

When we plug this result into the parametric search technique of Megiddo, we get the algorithm for the optimization problem as claimed in Theorem 1.

From the above discussion, we note that, since there are only four possible distances to compute to determine the diameter of a path augmented with one shortcut edge, the following corollary follows immediately.

Corollary 1. Given a path \( P \) on \( n \) vertices in a metric space and a shortcut \((u, v)\), the diameter of \( P \cup (u, v) \) can be computed in \( O(n) \) time.
Algorithm 1: Algorithm for deciding if $m(P) \leq \lambda$

\begin{algorithm}
\begin{algorithmic}
\STATE \textbf{DecisionAlgorithm}($P, \lambda$); \hspace{0.5cm} // Decide if $m(P) \leq \lambda$
\begin{algorithmic}[1]
\STATE \textbf{global} $D \leftarrow \text{ComputePrefixSums}(P)$;
\STATE \textbf{global} $s' \leftarrow \max\{v \mid \delta_P(s, v) \leq \lambda\}$;
\STATE \textbf{global} $e' \leftarrow \min\{v \mid \delta_P(v, e) \leq \lambda\}$;
\STATE \textbf{global} $T \leftarrow \text{ComputeRangeTree}(P, \lambda)$;
\FOR{$1 \leq k < n$}
\STATE $N_k \leftarrow \text{ComputeFeasibleIntervalForN}(k, \lambda)$;
\IF{$N_k \neq \emptyset$ \textbf{and} \text{CheckOForShortcut}(k, \min(N_k), \lambda)}
\STATE \textbf{return} TRUE
\ENDIF
\ENDFOR
\STATE \textbf{return} FALSE
\end{algorithmic}
\end{algorithmic}

\begin{algorithm}
\begin{algorithmic}
\STATE \textbf{CheckOForShortcut}(k, l, \lambda); \hspace{0.5cm} // Decide if $O(k, l) \leq \lambda$
\begin{algorithmic}[1]
\STATE $K' \leftarrow \{k \leq x \leq l \mid \delta_P(k, x) \leq \lambda \land \delta_P(x, l) > \lambda\}$; \hspace{0.5cm} // Compute the interval by binary search
\STATE $\mu \leftarrow \min_{x \in K'} \Delta(x)$; \hspace{0.5cm} // Query the range tree $T$
\STATE \textbf{return} ($\mu \geq |C[k, l]| - 2\lambda$)
\end{algorithmic}
\end{algorithmic}
\end{algorithm}

\section{An Approximation Algorithm in Euclidean Space}

In Section 2, we presented an $O(n \log^3 n)$-time algorithm for the problem when the input graph is a path in a metric space. Here we show a simple $(1 + \varepsilon)$-approximation algorithm with running time $O(n + 1/\varepsilon^3)$ for the case when the input graph is a path in $\mathbb{R}^d$, where $d$ is a constant. The algorithm will use two ideas: clustering and the well-separated pair decomposition (WSPD) as introduced by Callahan and Kosaraju [3].

\textbf{Definition 1} ([3]). Let $s > 0$ be a real number, and let $A$ and $B$ be two finite sets of points in $\mathbb{R}^d$. We say that $A$ and $B$ are well-separated with respect to $s$, if there are two disjoint $d$-dimensional balls $C_A$ and $C_B$, having the same radius, such that (i) $C_A$ contains $A$, (ii) $C_B$ contains $B$, and (iii) the minimum distance between $C_A$ and $C_B$ is at least $s$ times the radius of $C_A$.

The parameter $s$ will be referred to as the separation constant. The next lemma follows easily from Definition 1.

\textbf{Lemma 3} ([3]). Let $A$ and $B$ be two finite sets of points that are well-separated w.r.t. $s$, let $x$ and $p$ be points of $A$, and let $y$ and $q$ be points of $B$. Then (i) $|xy| \leq (1 + 4/s) \cdot |pq|$, and (ii) $|px| \leq (2/s) \cdot |pq|$.

\textbf{Definition 2} ([3]). Let $S$ be a set of $n$ points in $\mathbb{R}^d$, and let $s > 0$ be a real number. A well-separated pair decomposition (WSPD) for $S$ with respect to $s$ is a sequence of pairs of non-empty subsets of $S$, $(A_1, B_1), \ldots, (A_m, B_m)$, such that
1. $A_i \cap B_i = \emptyset$, for all $i = 1, \ldots, m$,
2. for any two distinct points $p$ and $q$ of $S$, there is exactly one pair $(A_i, B_i)$ in the sequence, such that (i) $p \in A_i$ and $q \in B_i$, or (ii) $q \in A_i$ and $p \in B_i$,
3. $A_i$ and $B_i$ are well-separated w.r.t. $s$, for $1 \leq i \leq m$.

The integer $m$ is called the size of the WSPD.

Callahan and Kosaraju showed that a WSPD of size $m = \mathcal{O}(s^d n)$ can be computed in $\mathcal{O}(s^d n + n \log n)$ time.

**Algorithm** We are given a polygonal path $P$ on $n$ vertices in $\mathbb{R}^d$. We assume without loss of generality that the total length of $P$ is 1. Partition $P$ into $m = 1/\varepsilon_1$ subpaths $P_1, \ldots, P_m$, each of length $\varepsilon_1$, for some constant $0 < \varepsilon_1 < 1$ to be defined later. Note that a subpath may have one (or both) endpoint in the interior of an edge. For each subpath $P_i$, $1 \leq i \leq m$, select an arbitrary vertex $r_i$ along $P_i$ as a representative vertex, if it exists. The set of representative vertices is denoted $R_P$; note that the size of this set is at most $m = 1/\varepsilon_1$. Let $P(R)$ be the path consisting of the vertices of $R_P$, in the order in which they appear along the path $P$. We give each edge $(u,v)$ of $P(R)$ a weight equal to $\delta_P(u,v)$.

Imagine that we “straighten” the path $P(R)$, so that it is contained on a line. In this way, the vertices of this path form a point set in $\mathbb{R}^1$; we compute a well-separated pair decomposition $W$ for the one-dimensional set $R_P$, with separation constant $1/\varepsilon_2$, with $0 < \varepsilon_2 < 1/4$ to be defined later. Then, we go through all pairs $(A,B)$ in $W$ and compute the diameter of $P(R) \cup \{(\text{rep}(A), \text{rep}(B))\}$, where $\text{rep}(A)$ and $\text{rep}(B)$ are representative points of $A$ and $B$, respectively, which are arbitrarily chosen from their sets. Note that the number of pairs in $W$ is $O(1/\varepsilon_1 \varepsilon_2)$. Finally the algorithm outputs the best shortcut.

**Analysis** We first discuss the running time and then turn our attention to the approximation factor of the algorithm.

The clustering takes $O(n)$ time, and constructing the WSPD of $R_P$ takes $O(\frac{1}{\varepsilon_1 \varepsilon_2} + \frac{1}{\varepsilon_1} \log \frac{n}{\varepsilon_1})$ time. For each of the $O(1/\varepsilon_1 \varepsilon_2)$ well-separated pairs in $W$, computing the diameter takes, by Corollary 1, time linear in the size of the uni-cyclic graph, that is, $O(\frac{1}{\varepsilon_1 \varepsilon_2})$ time in total.

**Lemma 4.** The running time of the algorithm is $O(n + \frac{1}{\varepsilon_1 \varepsilon_2})$.

Before we consider the approximation bound, we need to define some notation. Consider any vertex $p$ in $P$. Let $r(p)$ denote the representative vertex of the subpath of $P$ containing $p$. For any two vertices $p$ and $q$ in $P$, let $(A,B)$ be the well-separated pair such that $r(p) \in A$ and $r(q) \in B$. The representative points of $A$ and $B$ will be denoted $w(p)$ and $w(q)$, respectively.

**Lemma 5.** For any shortcut $e = (p,q)$ and for any two vertices $x,y \in P$, we have

$$(1 - 4\varepsilon_2) \cdot \delta_G(x,y) - 6\varepsilon_1 \leq \delta_P(w(x), w(y)) \leq \left(\frac{1}{1 - 4\varepsilon_2}\right) \cdot \delta_G(x,y) + 6\varepsilon_1,$$
where $G = P \cup \{(p, q)\}$ and $H = P(R) \cup \{(w(p), w(q))\}$.

Proof. We only prove the second inequality, because the proof of the first inequality is almost identical.

Consider two arbitrary vertices $x, y$ in $P$, and consider a shortest path in $G$ between $x$ and $y$. We have two cases:

**Case 1:** If $\delta_G(x, y) = \delta_P(x, y)$, then $\delta_H(r(x), r(y)) \leq \delta_P(x, y) + 2\varepsilon_1$.

**Case 2:** If $\delta_G(x, y) < \delta_P(x, y)$, then the shortest path in $G$ between $x$ and $y$ must traverse $(p, q).$ Assume that the path is $x \rightarrow p \rightarrow q \rightarrow t$, thus $\delta_G(x, y) = \delta_P(x, p) + |pq| + \delta_P(q, y)$. Consider the following three observations:

1. $|pq| \geq |r(p)r(q)| - 2\varepsilon_1$ and $|w(p)w(q)| \leq (1 + 4\varepsilon_2) \cdot |r(p)r(q)|$. Consequently, $|w(p)w(q)| \leq (1 + 4\varepsilon_2) \cdot (|pq| + 2\varepsilon_1)$.

2. We have
\[
\delta_P(x, p) \geq \delta_P(w(x), w(p)) - \delta_P(w(x), x) - \delta_P(w(p), p)
\geq \delta_P(w(x), w(p)) - (\varepsilon_1 + \delta_P(w(x), r(x))) - (\varepsilon_1 + \delta_P(w(y), r(y)))
\geq \delta_P(w(x), w(p)) - (\varepsilon_1 + 2\varepsilon_2)\delta_P(w(x), w(p)) - (\varepsilon_1 + 2\varepsilon_2)\delta_P(w(x), w(p))
= (1 - 4\varepsilon_2) \cdot \delta_P(w(x), w(p)) - 2\varepsilon_1
\geq (1 - 4\varepsilon_2) \cdot \delta_H(w(x), w(p)) - 2\varepsilon_1.
\]
That is, $\delta_H(w(x), w(p)) \leq \frac{1}{1 - 4\varepsilon_2} \cdot \delta_P(x, p) + 2\varepsilon_1$.

3. We have, $\delta_H(w(y), w(q)) \leq \frac{1}{1 - 4\varepsilon_2} \cdot \delta_P(y, q) + 2\varepsilon_1$, following the same arguments as in (2).

Putting together the three observations we get:
\[
\delta_H(w(x), w(y)) \leq \delta_H(w(x), w(p)) + |w(p)w(q)| + \delta_H(w(q), w(y))
\leq (\frac{1}{1 - 4\varepsilon_2}) \cdot \delta_P(x, p) + 2\varepsilon_1 + (1 + 4\varepsilon_2) \cdot (|pq| + 2\varepsilon_1)
+(\frac{1}{1 - 4\varepsilon_2}) \cdot \delta_P(y, q) + 2\varepsilon_1
< (\frac{1}{1 - 4\varepsilon_2}) \cdot \delta_G(x, y) + 6\varepsilon_1,
\]
where the last inequality follows from the fact that $0 < \varepsilon_2 < 1/4$. This concludes the proof of the lemma. \(\square\)

By setting $\varepsilon_1 = \varepsilon/60$ and $\varepsilon_2 = \varepsilon/32$ and using the fact that the diameter of $H$ is at least $1/2$, we obtain the following theorem that summarizes this section.

**Theorem 2.** Given a path $P$ with $n$ vertices in $\mathbb{R}^d$ and a real number $\varepsilon > 0$, we can compute a shortcut to $P$ in $O(n + 1/\varepsilon^3)$ time such that the resulting uni-cyclic graph has diameter at most $(1 + \varepsilon) \cdot d_{\text{opt}}$, where $d_{\text{opt}}$ is the diameter of an optimal solution.
Fig. 2: Illustrating the input tree \( T \) with \((u, v)\) as an optimal shortcut. The paths in \( T \) between \( x_i \) and \( y_j \), for \( 1 \leq k \leq 4 \) and \( 1 \leq j \leq 3 \), represent all longest paths in \( T \). These paths intersect in the path between \( a \) and \( b \).

4 Augmenting a Tree with One Edge

Next we consider the case when the input graph is a tree \( T = (V, E) \), where \( V \) is a set of \( n \) vertices in a metric space. The aim is to compute an edge \( f \) in \((V \times V) \setminus E\) such that the diameter of the resulting unicyclic graph \((V, E \cup f)\) is minimized.

Let \( P_T \) be the common intersection of all longest paths in \( T \). Observe that \( P_T \) is a non-empty path in \( T \). We denote the endvertices of \( P_T \) by \( a \) and \( b \). Let \( F = T \setminus E(P_T) \) be the forest that results from deleting the edges of \( P_T \) from \( T \).

For any vertex \( u \) of \( T \),
1. let \( \sigma(u) \) be the vertex on \( P_T \) that is in the same tree of \( F \) as \( u \), and
2. let \( \tau(u) \) be the tree of \( F \) that contains \( u \).

Refer to Figure 2 for an illustration.

Consider any augmenting edge \((u, v)\). In the following lemma, we will prove that the augmenting edge \((\sigma(u), \sigma(v))\) is at least as good as \((u, v)\). That is, the diameter of \( T \cup \{(\sigma(u), \sigma(v))\} \) is at most the diameter of \( T \cup \{(u, v)\} \). In case \( \sigma(u) = \sigma(v) \), \( T \cup \{(\sigma(u), \sigma(v))\} \) is equal to \( T \), and the diameter of \( T \cup \{(u, v)\} \) is equal to the diameter of \( T \).

**Lemma 6.** There exists an optimal augmenting edge \( f \) for \( T \) such that both vertices of \( f \) are vertices of \( P_T \).

**Proof.** Consider an optimal augmenting edge \((u, v)\). We may assume without loss of generality that \( \sigma(u) \) is on the subpath of \( P_T \) between \( a \) and \( \sigma(v) \). See Figure 2.

Let \( T_{opt} = T \cup \{(u, v)\} \), let \( D_{opt} \) be the diameter of \( T_{opt} \), and let \( T' = T \cup \{(\sigma(u), \sigma(v))\} \). In order to prove the lemma, it suffices to show that the diameter of \( T' \) is at most \( D_{opt} \). If \( D_{opt} \) is equal to the diameter of \( T \), then this obviously holds, because the diameter of \( T' \) is at most the diameter of \( T \). Thus, from now on, we assume that \( D_{opt} \) is less than the diameter of \( T \).

We claim that there exist endvertices \( x \) and \( y \) of some longest path in \( T \) such that
1. $a$ is on the path in $T$ between $x$ and $\sigma(u)$,
2. $b$ is on the path in $T$ between $y$ and $\sigma(v)$,
3. $x$ is not a vertex of $\tau(u) \setminus \{\sigma(u)\}$,
4. $y$ is not a vertex of $\tau(v) \setminus \{\sigma(v)\}$.

To prove this, consider the leaves $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_\ell$ of $T$ such that
1. for each $i$ with $1 \leq i \leq k$, $a$ is on the path in $T$ between $x_i$ and $b$,
2. for each $j$ with $1 \leq j \leq \ell$, $b$ is on the path in $T$ between $y_j$ and $a$,
3. for each $i$ and $j$ with $1 \leq i \leq k$ and $1 \leq j \leq \ell$, the path in $T$ between $x_i$ and $y_j$ is not a vertex of $a$.

Refer to Figure 2. If $k = 1$, then $x_1 = a$ and we take $x = x_1$. Assume that $k \geq 2$.
Consider the maximal subtree of $T$ that contains $a$ and all leaves $x_1, x_2, \ldots, x_k$,
and imagine this subtree to be rooted at $a$. There is a child $a'$ of $a$ such that $u$
is not in the subtree rooted at $a'$. We take $x$ to be any $x_i$ that is in the subtree rooted at $a'$.
By a symmetric argument, we can prove the existence of the vertex $y$.

Recall that we assume that the diameter of $T_{opt}$ (i.e., $D_{opt}$) is less than the diameter of $T$.
This implies that the shortest path in $T_{opt}$ from $x$ to $y$ contains the shortcut $(u, v)$ and, therefore,

$$\delta_T(\sigma(u), u) + |uv| + \delta_T(v, \sigma(v)) < \delta_T(\sigma(u), \sigma(v)). \quad (1)$$

In particular, $\sigma(u) \neq \sigma(v)$.

Now let $s$ and $t$ be any pair of vertices. In the rest of the proof, we will show
that $\delta_T(s, t) \leq D_{opt}$. Up to symmetry, there are three main cases to consider
with respect to the positions of $s$ and $t$:

1. Both vertices are in trees of $F$ that contain the shortcut vertices: $s, t \in \tau(u) \cup \tau(v)$, see Fig 3.
   (a) The vertices are in different trees of $F$: $s \in \tau(u)$ and $t \in \tau(v)$.
   Since
   $$\delta_T(s, \sigma(u)) = \delta_T(s, \sigma(u)) \leq \delta_T(x, \sigma(u)) = \delta_T(x, \sigma(u))$$
   and
   $$\delta_T(\sigma(v), t) = \delta_T(\sigma(v), t) \leq \delta_T(\sigma(v), y) = \delta_T(\sigma(v), y),$$
   we have
   $$\delta_T(s, t) = \delta_T(s, \sigma(u)) + |\sigma(u)\sigma(v)| + \delta_T(\sigma(v), t)$$
   $$\leq \delta_T(x, \sigma(u)) + |\sigma(u)\sigma(v)| + \delta_T(\sigma(v), y)$$
   $$= \delta_T(x, y)$$
   $$\leq \delta_{T_{opt}}(x, y)$$
   $$\leq D_{opt}.$$
Fig. 3: Illustrating (left) case 1(a) and (right) case 1(b).

(b) The vertices are in the same tree of $F$: $s, t \in \tau(u)$.

We will prove that in this case, the shortest paths between $s$ and $t$ in both $T'$ and $T_{\text{opt}}$ do not contain the shortcut, i.e., both these shortest paths are equal to the path in $\tau(u)$ (and, thus, in $T$) between $s$ and $t$.

This will imply that

$$\delta_{T'}(s, t) = \delta_{T_{\text{opt}}}(s, t) \leq D_{\text{opt}}.$$ 

Consider the shortest path $P'(s, t)$ between $s$ and $t$ in $T'$. Observe that shortest paths do not contain repeated vertices. If $P'(s, t)$ contains the shortcut $(\sigma(u), \sigma(v))$, then this path visits the vertex $\sigma(u)$ twice. Thus, $P'(s, t)$ does not contain $(\sigma(u), \sigma(v))$.

Consider the shortest path $P_{\text{opt}}(s, t)$ from $s$ to $t$ in $T_{\text{opt}}$, and assume that this path contains $(u, v)$. We may assume without loss of generality that, starting at $s$, this path traverses $(u, v)$ from $u$ to $v$. (Otherwise, we interchange $s$ and $t$.) Since $P_{\text{opt}}(s, t)$ does not contain repeated vertices, this path contains the subpath in $T$ from $\sigma(v)$ to $\sigma(u)$. This subpath must be the shortest path in $T_{\text{opt}}$ between $\sigma(v)$ and $\sigma(u)$. However, as we have seen in (1), this is not the case. Thus, we conclude that $P_{\text{opt}}(s, t)$ does not contain $(u, v)$.

2. Neither vertices are in trees of $F$ that contain the shortcut vertices: $s, t \notin \tau(u) \cup \tau(v)$.

If the shortest path in $T_{\text{opt}}$ from $s$ to $t$ does not contain $(u, v)$, then

$$\delta_{T'}(s, t) \leq \delta_T(s, t) = \delta_{T_{\text{opt}}}(s, t) \leq D_{\text{opt}}.$$ 

Assume that this shortest path contains $(u, v)$. We may assume without loss of generality that this shortest path traverses the edge $(u, v)$ from $u$ to $v$.

We have

$$\delta_{T'}(s, t) \leq \delta_T(s, \sigma(u)) + |\sigma(u)\sigma(v)| + \delta_T(\sigma(v), t)$$

$$\leq \delta_T(s, \sigma(u)) + \delta_T(\sigma(u), u) + |uv| + \delta_T(v, \sigma(v)) + \delta_T(\sigma(v), t)$$

$$= \delta_{T_{\text{opt}}}(s, t)$$

$$\leq D_{\text{opt}}.$$ 

3. One vertex is in a tree of $F$ that contains a shortcut vertex, the other is not: $s \in \tau(u)$ and $t \notin \tau(u) \cup \tau(v)$. 
(a) \( t \) is a vertex in the maximal subtree of \( T \) having \( x \) and \( \sigma(u) \) as leaves, see Fig. 4(left).

As in Case 1(b), it can be shown that the shortest paths between \( s \) and \( t \) (as well as the shortest paths between \( s \) and \( x \)) in both \( T_{opt} \) and in \( T' \) do not contain the shortcut. Thus,

\[
\delta_{T'}(s, t) \leq \delta_{T'}(s, x) = \delta_T(s, x) = \delta_{T_{opt}}(s, x) \leq D_{opt}.
\]

(b) \( t \) is a vertex in the maximal subtree of \( T \) having \( \sigma(u) \) and \( \sigma(v) \) as leaves, see Fig. 4(right).

We first observe that

\[
\delta_{T'}(s, t) = \delta_T(s, \sigma(u)) + \delta_{T'}(\sigma(u), t) \\
\leq \delta_T(x, \sigma(u)) + \delta_{T'}(\sigma(u), t) \\
= \delta_T(x, t).
\]

If the shortest path in \( T_{opt} \) from \( x \) to \( t \) does not contain \( (u, v) \), then

\[
\delta_{T'}(x, t) \leq \delta_T(x, t) = \delta_{T_{opt}}(x, t) \leq D_{opt}.
\]

Assume that the shortest path in \( T_{opt} \) from \( x \) to \( t \) contains \( (u, v) \). Then

\[
\delta_{T_{opt}}(x, t) = \delta_T(x, \sigma(u)) + \delta_{T'}(\sigma(u), u) + |uv| + \delta_T(v, \sigma(v)) + \delta_T(\sigma(v), t).
\]

Observe that

\[
\delta_{T'}(x, t) \leq \delta_T(x, \sigma(u)) + |\sigma(u)\sigma(v)| + \delta_T(\sigma(v), t).
\]

The triangle inequality implies that

\[
\delta_{T'}(x, t) \leq \delta_{T_{opt}}(x, t) \leq D_{opt}.
\]

(c) \( t \) is a vertex in the maximal subtree of \( T \) having \( \sigma(v) \) and \( y \) as leaves.

In this case, we have

\[
\delta_{T'}(s, t) = \delta_T(s, \sigma(u)) + |\sigma(u)\sigma(v)| + \delta_T(\sigma(v), t) \\
\leq \delta_T(x, \sigma(u)) + |\sigma(u)\sigma(v)| + \delta_T(\sigma(v), y) \\
\leq \delta_T(x, \sigma(u)) + \delta_{T_{opt}}(\sigma(u), \sigma(v)) + \delta_T(\sigma(v), y) \\
= \delta_{T_{opt}}(x, y) \\
\leq D_{opt}.
\]

This concludes the proof of the lemma. \( \square \)

As a consequence of Lemma 6, the diameter of a tree cannot be improved by adding a single shortcut, if the intersection of all longest paths is a vertex or a single edge.
4.1 Augmenting a tree

For a tree $T$ with $n$ vertices, let the intersection of all longest paths in $T$ be the path $P_T$. In a preprocessing step, we convert $T$ to a caterpillar tree $T_{cp}$ by replacing every tree $T'$ of $T \setminus E(P_T)$ by a single edge of length $\delta_T(t, v)$, where $v$ is the common vertex of $T'$ and $P_T$, and $t$ is the furthest vertex in $T'$ to $v$, see Figure 5. Note that $T_{cp}$ has a unique longest path.

Recall that for a path, there are only four relevant distances to compute to determine the diameter; the same holds for a tree with a unique longest path. These distances can trivially be computed in $O(n)$ time. Now consider the case when one of the endpoints of the shortcut is fixed at a vertex $v$ and the second endpoint is moving along $P_T$ in $T_{cp}$. As for the path case, the four functions describing the distances are monotonically increasing or decreasing, hence, a simple binary search along $P_T$ for the second endpoint can be used to determine
the optimal placement of the shortcut. As a result, the optimal shortcut, given one fixed endpoint \( v \) of the shortcut, can be computed in \( O(n \log n) \) time. We get:

**Theorem 3.** Given a tree \( T \) on \( n \) vertices in a metric space, we can compute a shortcut that minimizes the diameter of the augmented graph in \( O(n^2 \log n) \) time.

Recall that Lemma 6 states that there exists an optimal shortcut with both its endpoints on \( P_T \). However, our algorithm only requires that one of the endpoints is on \( P_T \). The obvious question is if one can modify the algorithm so that it takes full advantages of the lemma.

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