\(\mathcal{PT}\)-symmetric tight-binding chain with gain and loss: A completely solvable model

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We present the analytical solution for the eigenvalues and eigenvectors of a \(\mathcal{PT}\)-symmetric tight-binding chain with gain and loss in a symmetric configuration. We show a simple method to predict the values of the parameters at which exceptional points occur, and we determine the behavior of the eigenvalues and eigenfunctions around these exceptional points perturbatively. The analytical results are used to analyze the transport through the chain. Beyond the exceptional point where the eigenvalues become complex, the ratio of inflow and outflow for the corresponding eigenstates is different from one, and the density in the chain increases or decreases exponentially.

I. INTRODUCTION AND MOTIVATION

\(\mathcal{PT}\)-symmetric Hamiltonians are an important building block of what is commonly known as Non-Hermitian Quantum Mechanics [1]. Such systems were introduced by Bender and Boettcher in 1998 [2] as a generalization of standard Quantum Mechanics [3]. It is known

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that a $\mathcal{PT}$-symmetric Hamiltonian may have real or complex eigenvalues. The first case is denoted as the $\mathcal{PT}$-symmetric unbroken phase and the second as the $\mathcal{PT}$-symmetric broken phase. Equivalently, when the eigenfunctions of a $\mathcal{PT}$-symmetric Hamiltonian are $\mathcal{PT}$-symmetric, the eigenvalues remain real; otherwise the eigenvalues are complex [4]. $\mathcal{PT}$-symmetric Hamiltonians have found various applications, see [5–9] for an overview, essentially because they can be considered as an effective model for systems with gain and loss [10]. Moreover, these systems can show exceptional points, where two (or more) eigenvalues and eigenfunctions coalesce and hence, the Hamiltonian becomes defective [11]. Several remarkable phenomena have been reported recently in the vicinity of these points, for example topological states [12–17], chirality [18–20], unidirectional invisibility [21–23], unidirectional zero sonic reflection [24], enhanced sensing [25] and the possibility to stop light [26].

In this broad context, exactly solvable systems are valuable as they permit a thorough understanding of the phenomena taking place in the system. The analysis of the simplest $2 \times 2$ $\mathcal{PT}$-symmetric matrices has lead to important insights, though certain aspects cannot be captured with such simple models, such as the simultaneous coalescence of more than two eigenvalues [14]. From this perspective, our contribution in this paper has several purposes: The first one is to present an analytical solution for a one-dimensional tight-binding chain, with gain and loss at arbitrary ($\mathcal{PT}$-symmetric) positions along the chain. Previously, studies have focused on either continuous [27], or discrete (i.e. tight-binding) [28], $\mathcal{PT}$-symmetric one-dimensional systems. In particular, the one dimensional tight-binding chain with $\mathcal{PT}$-symmetry has been investigated in the limit of infinite sites [29]. One of the disadvantages of this approach, is that the position of the gain and loss cannot be placed arbitrarily; moreover important finite-size effects, present in experiments, are missed. The solution for $N$ sites in the case where the gain and loss are located at the extreme sites was, to our knowledge, first obtained in Ref. [30] using the Bethe ansatz. However, a general solution for the eigenvalues and eigenvectors for the system with arbitrary ($\mathcal{PT}$-symmetric) gain and loss positions is still missing. Here, we provide that solution.

Our second purpose is to describe, at least approximately, the behavior of the eigenvalues and eigenvectors around the exceptional points. In the system under consideration, the appearance of these points depends on a single real parameter [5, 6, 31], and we can describe the system for the whole range of values in which the parameter varies. A fortunate consequence of describing the system perturbatively is that our scheme provides a method to
determine the parameter values at which exceptional points occur. Finally, the last purpose of this work is to describe transport through the $\mathcal{PT}$-symmetric chain when its corresponding Hamiltonian is considered as an effective model [27]. We derive the corresponding continuity equation and characterize transport before and after the exceptional point.

The paper is organized as follows. In Sec. II we state the problem and the main results; the formalism to obtain the eigenvalues and eigenvectors of a tight-binding chain coupled to a $\mathcal{PT}$-symmetric gain and loss is described in Appendix A. In Sec. III we describe, using perturbation theory, how the eigenvalues and eigenvectors behave around an exceptional point when the gain and loss are at the extreme sites, i.e., in the end-to-end configuration. This methodology is extended, in Sec. IV, to the study of the behavior of the eigenvalues around exceptional points for the case where the gain and loss are at the center of the chain. Next, in Sec. V, we focus on transport along the chain for the end-to-end configuration. Finally in Sec. VI we summarize our results and conclusions.

II. THE MODEL AND ITS SOLUTION

A system is $\mathcal{PT}$-symmetric if the Hamiltonian commutes with the operator $\mathcal{PT}$, where $\mathcal{P}$ and $\mathcal{T}$ are the parity and the time reversal operators, respectively. This is commonly referred to as spacetime-reflection symmetry (see e.g. [31, 32]). Following Bender [4], for a $\mathcal{PT}$-symmetric operator we say that $\mathcal{PT}$ is unbroken if all the eigenfunctions of the operator are also eigenfunctions of $\mathcal{PT}$; otherwise, we say that $\mathcal{PT}$ is broken. Without loss of generality, the time reversal operator $\mathcal{T}$ can be defined simply as the complex conjugation operator [33]

$$\mathcal{T} \equiv *, \quad \mathcal{T}^2 = 1,$$

(1)

where $1$ is the identity. For a matrix $M$, the action of $\mathcal{T}$ is $\mathcal{T}MT = M^*$. Fixing a basis in a finite Hilbert space, the parity operator $\mathcal{P}$ is a matrix such that

$$\mathcal{P} = \mathcal{P}^* \quad \text{and} \quad \mathcal{P}^2 = 1.$$  

(2)

In our case, we choose $\mathcal{P}$ as the matrix $J$ with components

$$J_{ij} = \delta_{i,N-j+1},$$

(3)

which is commonly known as exchange matrix in the mathematical literature [34], sometimes called sip matrix [32].
A tight-binding chain in one dimension with gain and loss in a symmetric configuration (see Fig. 1) can be described by the \( \mathcal{PT} \)-symmetric Hamiltonian
\[
H = t \sum_{i=1}^{N-1} \left( |i\rangle \langle i+1| + |i+1\rangle \langle i| \right) + i\eta \left( |k\rangle \langle N-k+1| - |N-k+1\rangle \langle k| \right),
\]
where \( t \) is the (positive) nearest-neighbor coupling, \( N \) is the length of the chain and corresponds to the dimension of the Hilbert space of the system, \( k \) is the position of the gain and \( N - k + 1 \) is the position of the loss, and \( \eta \) is a real number that describes the strength of the gain and loss. Without loss of generality we fix \( t = 1 \).

We can obtain exactly the eigenvalues and eigenvectors of this Hamiltonian using symbolic calculus \([35, 36]\); see Appendix A for the explicit derivation. The eigenvalues are given by
\[
E_\theta = 2 \cos \theta,
\]
where the values of the “angles” \( \theta \) are those non-trivial solutions (\( \theta \neq m\pi \), with \( m \in \mathbb{Z} \)) that fulfill the equation
\[
\sin (N+1)\theta + \frac{\eta^2}{\sin^2 \theta} \sin[(N-2k+1)\theta] \sin^2(k\theta) = 0.
\]
Writing the eigenvectors in the site basis \( \{|j\rangle, j = 1, 2, \ldots, N\} \), as \( |E_\theta\rangle = \sum_{j=1}^{N} u_j(\theta) |j\rangle \), the \( j \)th-component of the eigenvector is given by
\[
u_j(\theta) = \langle j |E_\theta\rangle = \frac{u_1(\theta)}{\sin \theta} \left[ \frac{\sin j\theta - i\eta \Theta(j - k - 1) \frac{\sin k\theta \sin(j-k)\theta}{\sin \theta}}{\sin \theta} \right. \\
+ \Theta(j - N + k - 2) \frac{\sin(j - N + k - 1)\theta}{\sin \theta} \\
\times \left( i\eta \sin(N - k + 1)\theta - \eta^2 \frac{\sin(N - 2k + 1)\theta \sin k\theta}{\sin \theta} \right]
\]
where \( \Theta(x) \) is the unit step function defined by \( \Theta(x) = 1 \) if \( x \geq 0 \) and \( \Theta(x) = 0 \) otherwise.

In Eq. (7), \( u_1(\theta) \) is the first component of each eigenvector, which can we can set to \( 1/N \), where \( N \) is a normalization constant.

**A. Limiting cases \( \eta = 0 \) and \( \eta \to \infty \)**

Before analyzing specific configurations and addressing the issue of the exceptional points \[6, 11\], we focus first on two limiting cases. When \( \eta = 0 \), Eq. (4) represents a symmetric (real Hermitian) matrix; from Eq. (6) we obtain \( \theta = r\pi/(N+1) \) (\( r = 1, 2, \ldots, N \)), which using Eq. (5) yields the well-known solution for the eigenvalues \( E_{\theta,\eta=0} \) \[35\]. Similarly, the \( j \)th component of the eigenvectors is given by

\[
 u_j(\theta, \eta = 0) = \frac{\sin(j\theta)}{N \sin \theta}. \tag{8} 
\]

In this case, the Hamiltonian in Eq. (4) is “centrosymmetric” and the results in \[34\] hold. The most relevant of these is that the eigenvectors are symmetric or skew-symmetric with respect to the exchange matrix \( J \), i.e. they fulfill

\[
 J |E_{\theta,\eta=0}\rangle = \pm |E_{\theta,\eta=0}\rangle. \tag{9} 
\]

This symmetry is relevant in achieving good transport properties over disordered systems \[37, 39\].

In the opposite limit, when \( \eta \to \infty \), we expect the system to be divided into several sub-systems depending on \( k \): two of them correspond to the uncoupled gain and loss, and the remaining are one, two or three disjoint tight-binding chains depending on the positions \( k \) and \( N - k + 1 \). From Eq. (6), the real parts of \( \theta \) for the disjoint tight-binding chains are given by \( \theta = r\pi/(N - 2k + 1) \) for \( r = 1, 2, \ldots, N - 2k \), and the double-roots \( \theta = r\pi/k \) for \( r = 1, \ldots, k - 1 \). All these \( N - 2 \) eigenvalues have, in the limit \( \eta \to \infty \), an imaginary part which is or tends asymptotically to zero. The asymptotic behavior of the two remaining eigenvalues, which are purely imaginary and are linked to the gain and loss, can be obtained by writing \( \theta = i\phi \); Eq. (6) is transformed to

\[
 \sinh(N + 1)\phi \sinh^2\phi + \eta^2 \sinh(N - 2k + 1)\phi \sinh^2 k\phi = 0, \tag{10} 
\]

which in the limit \( \Re(\phi) \gg 1 \), reduces to

\[
 e^{2\phi} \sim -\eta^2. \tag{11} 
\]
Taking logarithm we obtain

\[ \phi \sim \log \eta + \frac{i\pi}{2} + i\pi m, \ m \in \mathbb{Z}. \]  

(12)

Finally, using Eq. (5), we obtain

\[ E_{\theta,\eta \to \infty} = E_{i\phi,\eta \to \infty} \sim \pm i\left( \eta - \frac{1}{\eta} \right). \]  

(13)

We note that this equation holds independently of \( k \), i.e., for any symmetric configuration of the gain and loss. It is also worth noting that even for small values of the parameters, like \( N = 10, \eta \simeq 3 \), the asymptotic behavior of the eigenvalues is already clear in the numerics; see Figs. 2–4 below.

III. GAIN AND LOSS IN THE END-TO-END CONFIGURATION

Using the results of the previous section, in this section we obtain the eigenvalues and eigenvectors for the case where the gain and loss are in positions 1 and \( N \) respectively. To simplify the description of the behavior close to the exceptional point it is convenient to assume that \( N \) is even.

In this configuration we have \( k = 1 \), and the Hamiltonian reads

\[ H = \sum_{i=1}^{N-1} \left( |i\rangle \langle i+1| + |i+1\rangle \langle i| \right) + i \eta \left( |1\rangle \langle 1| - |N\rangle \langle N| \right). \]  

(14)

From Eqs. (6) and (7) above, we write the corresponding results for this configuration, which we shall use later to obtain expressions for the eigenvalues in the vicinity of the exceptional point. The eigenvalues \( E_\theta \) (see Eq. (5)) depend on \( \theta \), which now satisfies the equation

\[ \sin(N+1)\theta + \eta^2 \sin(N-1)\theta = 0, \]  

(15)

and the components of the eigenvectors are given by (c.f. Eq. (7))

\[ u_j(\theta) = \frac{1}{N \sin \theta} \left( \sin(j\theta) + i\eta \sin(j-1)\theta \right). \]  

(16)

Figure 2 shows the complex eigenvalues \( E_\theta \) as a function of \( \eta \); on the left we show the real part \( \text{Re}(E_\theta) \) of the eigenvalues and on the right their imaginary part \( \text{Im}(E_\theta) \). The figure shows that there is a single exceptional point at \( \eta = 1 \) with two eigenvalues coalescing at \( E_\theta = 0 \), i.e., \( \theta = \pi/2 \).
Figure 2: Eigenvalues $E_\theta$ as a function of the coupling parameter $\eta$ in the end-to-end configuration for $N = 10$. Left: $\text{Re}(E_\theta)$ as a function of $\eta$. Right: $\text{Im}(E_\theta)$ as a function of $\eta$. In this case there is a single exceptional point at $\eta = 1$, all other eigenvalues remain real valued. The inset shows the expansion around $\eta \approx 1$ in more detail.

To determine analytically the behavior of the eigenvalues around an exceptional point, we use a simple perturbation scheme which provides accurate approximations, in spite of the well known fact that asymptotic series may not be convergent \cite{40}. We write Eq. (15) generically as $F(\theta, \eta^2) = 0$, from which we determine the values of $\theta$ given the strength of the coupling constant $\eta$. In the present case, the equation defining $\theta$ for $\eta = \eta_0 = 1$ can be rewritten as

$$F(\theta, \eta_0^2) = 2 \sin N\theta \cos \theta = 0,$$

(17)

with the solutions $\theta_r = \frac{r\pi}{N}$ ($r = 1, 2, \ldots, N - 1$) and $\theta_0 = \pi/2$. Notice that since $N$ is even, the root $\theta_{N/2}$ is identical to $\theta_0$, so at $\eta = \eta_0 = 1$ these roots coalesce.

If we follow the usual perturbation scheme, we write $\eta^2 = \eta_0^2 + \epsilon$, propose a solution of the form $\theta = \theta_0 + \epsilon \theta^{(1)} + \ldots$ in powers of $\epsilon$, and solve $F(\theta, \eta^2) = 0$, which is also written as a series expansion in $\epsilon$. Each term of that series must be equal to zero, which is used to obtain $\theta^{(1)}$.

However, this procedure breaks down when $\theta_0$ and $\eta_0$ define an exceptional point. Indeed, to first order in $\epsilon$ the expansion reads

$$F(\theta, \eta^2) \approx F(\theta_0, \eta_0^2) + \epsilon \left( \theta^{(1)} \frac{\partial F}{\partial \theta}(\theta_0, \eta_0^2) + \frac{\partial F}{\partial (\eta^2)}(\theta_0, \eta_0^2) \right),$$

(18)
and at the exceptional point we have

$$\frac{\partial F}{\partial \theta}(\theta_0, \eta^2_0) = 0, \quad (19)$$

$$\frac{\partial F}{\partial (\eta^2)}(\theta_0, \eta^2_0) \neq 0, \quad (20)$$

implying that we cannot choose \( \theta^{(1)} \) such that the first order term vanishes. We emphasize that the condition given by Eq. (19) defines the exceptional point.

To overcome the failure of the usual perturbation scheme, we propose to write the solution for \( \theta \) as \( \theta = \theta_0 + \epsilon^{1/2} \theta^{(1)} + \ldots \). Then, to first order in \( \epsilon \) we have

$$F(\theta, \eta) \approx F(\theta_0, \eta^2_0) + \epsilon^{1/2} \theta^{(1)} \frac{\partial F}{\partial \theta}(\theta_0, \eta^2_0) + \epsilon \left( \frac{1}{2} \theta^{(1)} \frac{\partial^2 F}{\partial \theta^2}(\theta_0, \eta^2_0) + \frac{\partial F}{\partial (\eta^2)}(\theta_0, \eta^2_0) \right). \quad (21)$$

The term of order \( \epsilon^{1/2} \) vanishes identically at the exceptional point, due to Eq. (19) above, and from the term proportional to \( \epsilon \) we can obtain \( \theta^{(1)} \). Explicitly, the first order term in \( \epsilon \) leads to

$$-2N\theta^{(1)} \cos \frac{N\pi}{2} + \sin \frac{(N - 1)\pi}{2} = 0, \quad (22)$$

and we have

$$\theta^{(1)} = \pm \frac{i}{\sqrt{2N}}. \quad (23)$$

Consequently, near the exceptional point we have

$$\theta^{\pm} \approx \frac{\pi}{2} \pm \frac{\epsilon^{1/2}}{\sqrt{2N}} = \frac{\pi}{2} \pm i \sqrt{\frac{\eta^2 - 1}{2N}}, \quad (24)$$

in terms of which, the eigenvalues read

$$E_{\theta^{\pm}} = 2 \cos \theta^{\pm} \approx \pm 2i \sinh \sqrt{\frac{\eta^2 - 1}{2N}}. \quad (25)$$

Note that the two solutions of \( \theta^{(1)} \) indicate a coalescence at the exceptional point, and a "complexification" of the eigenvalues in its vicinity. Clearly, if \( \eta < 1 \) the eigenvalues are real, while if \( \eta > 1 \) they become imaginary. The correction term proportional to the square root of \( \eta^2 - 1 \), describes well the results obtained by direct numerical diagonalization of the Hamiltonian matrix in the proximity of the exceptional point, see Fig. 2.

Turning our attention to the eigenvectors: Since \( |E_\theta\rangle \) are eigenvectors of a \( \mathcal{PT} \) invariant Hamiltonian, it is straightforward to see

$$\mathcal{H}\mathcal{PT}|E_\theta\rangle = E_\theta^* \mathcal{PT}|E_\theta\rangle. \quad (26)$$
Figure 3: Eigenvalues $E_\theta$ as a function of the coupling parameter $\eta$ with gain/loss at the center of the chain. Left: Re($E_\theta$) as a function of $\eta$. Right: Im($E_\theta$) as a function of $\eta$. When $\eta = 1$ all the eigenvalues coalesce in pairs. For $\eta > 1$, the imaginary part of only two eigenvalues are proportional to $\pm i\eta$, the rest tend asymptotically to zero.

Thus, as long as the spectrum is real (e.g. as is the case before the exceptional point), the eigenfunctions fulfill

$$\mathcal{PT} |E_\theta\rangle \propto |E_\theta\rangle ,$$

(27)

as can be checked using Eq. (16). More interesting are the symmetries after the exceptional point. Now some eigenvalues $E_\theta$ appearing in Eq. (26) are complex. Thus, complex eigenvalues come in conjugate pairs. Further:

$$\mathcal{PT} |E_\theta\rangle \propto |E_\theta^*\rangle ,$$

(28)

which relates different eigenvectors. These relations will be of use in our discussion regarding transport in the system.

IV. GAIN AND LOSS AT THE CENTER OF THE CHAIN

Now we turn our attention to the case in which the gain and loss are at the center of the chain, that is, at positions $k = N/2$ and $k + 1$ for the gain and the loss, respectively; again we assume for simplicity that $N$ is even. Figure 3 shows the corresponding eigenvalues
for \( N = 10 \) as a function of \( \eta \). On the left we plot the real part of the eigenvalues as a function of \( \eta \); on the right we plot similarly their imaginary parts. In this case, all the eigenvalues coalesce in pairs at \( \eta = \eta_0 = 1 \). For \( \eta > 1 \) the imaginary parts of two of the eigenvalues grow asymptotically as \( \pm i \eta \), c.f. Eq. (13), while the imaginary parts of the rest tend asymptotically to zero.

To obtain the eigenvalues’ behavior around the value at which each pair coalesces, we use the same method of the previous section. In this case, the angles \( \theta \) satisfy

\[
F(\theta, \eta^2) = \sin(N + 1)\theta \sin \theta + \eta^2 \sin^2 \frac{N\theta}{2} = 0,
\]

which, for \( \eta_0 = 1 \) can be written as

\[
F(\theta, \eta_0^2) = \frac{1}{2} (1 - \cos(N + 2)\theta) = \sin^2 \frac{(N + 2)\theta}{2} = 0.
\]

The nontrivial solutions of this equation are \( \theta_r = 2\pi r/(N + 2) \) with \( r = 1, 2, ..., N + 1 \) and \( r \neq (N + 2)/2 \). We note that at all the values \( \theta_r \), Eq. (19) is satisfied, i.e., \( \partial F/\partial \theta = ((N + 2)/2) \sin(N + 2)\theta \), corresponding to exceptional points. Thus, as observed, at \( \eta = \eta_0 \) the \( N \) eigenvalues coalesce into \( N/2 \) eigenvalues. Indeed, it is easy to see that \( E_{\theta_r} = E_{\theta_{N+2-r}} \). In particular, two eigenvalues coalesce at \( E = 0 \) only when \( N = 2l \) with \( l \) odd, as shown in Fig. 3 for \( N = 10 \). In this case, \( m = (l + 1)/2 \) is an integer, and \( \theta_m = \theta_{N+2-m} = \pi/2 \).

Carrying on the same perturbation scheme described in the previous section, we denote by \( \theta^\pm_r \) the value of the angles near \( \eta_0 \), and finally obtain

\[
\theta^\pm_r = \frac{2\pi r}{N + 2} \pm i\epsilon^{1/2} \frac{\sqrt{2}}{N + 2} \left( 1 - \cos \left( \frac{2\pi Nr}{N + 2} \right) \right)^{1/2},
\]

for \( \epsilon = \eta^2 - 1 \) small, and \( r = 1, 2, ..., N + 1 \) and \( r \neq (N + 2)/2 \).

We consider now the case where \( N = 2l \) with \( l \) even; in Fig. 4 we illustrate it plotting the eigenvalues in terms of \( \eta \) for \( N = 20 \). We first note that there is no coalescence happening at \( E_0 = 0 \) for \( \eta_0 = 1 \), but there is a double coalescence at \( \eta_1 \approx 1.6041339 \). Indeed, at \( \eta = \eta_1 \) the two pairs of complex solutions of \( F(\theta, \eta_1^2) = 0 \), Eq. (29), coalesce into two exceptional points. Writing the angles as \( \theta^\pm_1 = \pi/2 \pm i\delta \), we have obtained numerically \( \delta \approx 0.1787145 \), and therefore the eigenvalues are \( E_{\theta_1^\pm} \approx \mp i0.359335 \). We have verified numerically that \( \partial F/\partial \theta = 0 \) holds at \( (\theta_1, \eta_1^2) \), which implies that they are exceptional points. We observe that, contrary to the common exceptional points, in this case we have \( \text{Re}(E_0) \neq 0 \) and
Figure 4: Eigenvalues $E_\theta$ as a function of the coupling parameter $\eta$ with gain/loss in the center of the chain of length $N = 20$. Notice that although all eigenvalues coalesce at $\eta_0 = 1$, there is another double coalescence for $\eta_1 \sim 1.604 \ldots$ with $\text{Re}(\theta) = \pi/2$.

$\text{Im}(E_\theta) \neq 0$ before the coalescence. Beyond $\eta_1$ the eigenvalues that had coalesced split into four, a pair of branches diverge asymptotically as $\pm i\eta$, while the other pair converges asymptotically to zero. Figure 5 explores this situation.

V. TRANSPORT PROPERTIES

In this section we focus on the transport properties in the $\mathcal{PT}$-symmetric quantum chain, using the analytically calculated eigenvectors. Following [27], our interest is to study one-dimensional transport generated by the effective Hamiltonian Eq. (4). For simplicity, we derive the continuity equation for the case where the contacts are in the end-to-end configuration.

Let $|\Psi(t)\rangle$ be a solution of the Schrödinger equation. Then we have

$$\frac{\partial |\Psi(t)\rangle}{\partial t} = \frac{H}{i} |\Psi(t)\rangle,$$

and its adjoint

$$\frac{\partial \langle \Psi(t)|}{\partial t} = -\frac{1}{i} \langle \Psi(t)| H^*$$
Figure 5: Double appearance of exceptional points in the complex plane for gain/loss at the center of the chain. The black dots are the positions of four eigenvalues at $\eta = 0$. As $\eta > 0$, the eigenvalues move in the complex plane indicated by the arrows. The first pair of exceptional points occur at $\eta = 1$ over the real axis. After this, the four eigenvalues move in the complex plane towards $\text{Re}(\theta) = 0$. The second pair of exceptional points occur at $\eta \approx 1.604$ over the imaginary axis.

where we have set $\hbar = 1$. Using the site basis we have

$$|\Psi(t)\rangle = \sum_{j=1}^{N} c_j(t) |j\rangle,$$  \hspace{1cm} (34)

where the expansion coefficients are given by

$$c_n(t) = \langle n | \Psi(t) \rangle,$$

$$c_n^*(t) = \langle n | \Psi(t) \rangle^* = \langle \Psi(t) | n \rangle.$$  \hspace{1cm} (35)

In order to derive the continuity equation, we take the time derivative of the diagonal elements of the density matrix $\rho_{nn}(t) = \langle n | \Psi(t) \rangle \langle \Psi(t) | n \rangle$:

$$\frac{\partial \rho_{nn}}{\partial t} = \frac{1}{i} \left( (\langle n | H | \Psi(t) \rangle \langle \Psi(t) | n \rangle - \langle n | \Psi(t) \rangle \langle \Psi(t) | H^* | n \rangle) \right)$$

$$= \frac{1}{i} \sum_{m} (H_{nm} c_{m}(t) c_{n}^*(t) - H_{nm}^* c_{n}(t) c_{m}^*(t)).$$  \hspace{1cm} (36)
By using the explicit form of the Hamiltonian, Eq. (4), we find

$$\frac{\partial \rho_{nn}}{\partial t} = \begin{cases} c_2(t)c_1^*(t) - c_1(t)c_2^*(t) + 2i|c_1(t)|^2\eta, & n = 1, \\ c_{n-1}(t)c_N^*(t) - c_N(t)c_{n-1}^*(t) - 2i|c_N(t)|^2\eta, & n = N, \\ c_n(t)c_{n+1}^*(t) + c_{n-1}(t)c_n^*(t) - c_n(t)c_{n+1}^*(t) - c_n(t)c_{n-1}^*(t), & \text{otherwise}, \end{cases} (37)$$

which can be written as

$$\frac{\partial \rho_{nn}}{\partial t} + J_{n+1} - J_n = 0 \quad (38)$$

where we have introduced the local fluxes

$$J_n(t) \equiv i(c_n(t)c_{n-1}^*(t) - c_{n-1}(t)c_n^*(t)) = -2\text{Im}(c_n(t)c_{n-1}^*(t)) \quad n \neq 1, N + 1 \quad (39)$$

which represent the density flux from site $n$ to site $n + 1$, and we define $J_1 = 2|c_1(t)|^2\eta$ and $J_{N+1} = 2|c_N(t)|^2\eta$ representing the input and output due to the presence of gain and loss in the chain.

Now consider $|\Psi(t)\rangle$ to be a time dependent eigenstate

$$|E_\theta(t)\rangle \equiv e^{-iE_\theta t}|E_\theta\rangle = e^{-iE_\theta t}\sum_{j=1}^{N} u_j(\theta) |j\rangle, \quad (40)$$

for these states $c_n(t) = e^{-iE_\theta t}u_n(\theta)$. When the eigenvalues are real, products of the form $c_n(t)c_m^*(t) = u_n(\theta)u_m^*(\theta)$, as those appearing in the definition for the flux, are independent of time. Using the explicit coefficients for the eigenfunctions given in Eq. (16), it is straightforward to see that before the occurrence of the exceptional point $J_n = \eta|c_1|^2$ for all $n$. Hence $J_n - J_{n+1} = 0$, indicating that probability is conserved. After the exceptional point, the eigenfunctions are complex, and terms of the form $c_n(t)c_m^*(t) = e^{-2i\text{Im}(E_\theta)t}u_n(\theta)u_m^*(\theta)$ increase or decrease exponentially in time. In view of this, we choose to characterize transport through the coefficient $\xi = |c_N(t)|^2/|c_1(t)|^2$, which corresponds to the ratio of the input to the output in the continuity equation (37). Evaluated on the states corresponding to the eigenfunctions $|E_\theta(t)\rangle$,

$$\xi_E = \frac{|u_N(\theta)|^2}{|u_1(\theta)|^2} \quad (41)$$

becomes independent of time and of the normalization. Further, before the exceptional point, as a direct consequence of Eq. (27), we have $\xi_E = 1$, as can be checked directly using the explicit expression of the coefficients. This indicates that transport is perfectly efficient in the eigenstate corresponding to the (real) eigenvalue $E$. On the other hand, after
the exceptional point, the eigenvalues become complex and $\xi_E$ is no longer equal to one. However, in view of Eq. (28), it is straightforward to see that $\xi_E \xi_E^* = 1$. The actual values of the transport coefficient for the eigenstates near the exceptional point can be evaluated using the perturbation expansion for $\theta$ from Eq. (24):

$$\xi_{E,E^*} \approx \exp(\pm 2N(\eta^2 - 1)^{1/2})$$ (42)

where the exponential form was chosen merely to enforce the fact that the coefficients are positive, and that their product must be equal to one. Having $\xi < 1$ indicates that in this state the flux out of the system exceeds the input and the chain becomes depleted. Conversely, when $\xi > 1$ the density cannot flow out of the system at the rate at which it is injected into the chain. Finally, in the limit $\eta \to \infty$ the transport coefficient is

$$\xi_{E,E^*} \approx [4\eta^{2(N-1)}]^{\pm 1}. \quad (43)$$

Figure 6 shows the transport coefficient $\xi$ as a function of $\eta$ for all the eigenfunctions in the end-to-end configuration with $N = 10$. All the eigenfunctions with eigenvalues that remain real have $\xi = 1.0$. For $\eta \geq 1$, the pair of eigenfunctions with coalescing eigenvalues have a transport coefficient given approximately by Eq. (42) (dashed red line). For large values of $\eta$, they follow Eq. (43) (blue triangles).

VI. SUMMARY AND CONCLUSIONS

In this paper, we have obtained exact expressions for the eigenvalues, Eq. (5)-(6), and eigenvectors, Eq. (7), of a $\mathcal{PT}$-symmetric quantum chain with arbitrary $\mathcal{PT}$-symmetric configuration of the gain and loss. We have focused on two concrete gain and loss configurations: the end-to-end, where the couplings are at the edges of the chain, and center of the chain configuration. In both cases we described how the eigenvalues behave around the exceptional points, and discussed their asymptotic behavior when the coupling $\eta \to \infty$. Using a perturbative approach to study the vicinity of the exceptional points we obtained an equation that defines the exceptional points, Eq. (19). This equation, which makes a naive perturbative scheme fail, is always fulfilled at the exceptional points. Though simple, our method—or straightforward extensions of it—, can also be applied to other types of exceptional points, e.g., when there exists eigenspace condensation [14], or in the case of many indistinguishable photons [11].
Figure 6: Transport coefficient $\xi$ as a function of $\eta$ for the eigenfunctions of the end-to-end configuration ($N = 10$). In this case, the transport coefficients of the eigenfunctions with coalescing eigenvalues are different from 1 after the exceptional point. These transport coefficients indicate the increment ($\xi > 1$) or depletion ($\xi < 1$) of density along the chain.

Furthermore, in the process of solving the finite $\mathcal{PT}$-symmetric tight-binding chain, we have found the solution of the eigenvalues and eigenvectors of a more general finite tridiagonal matrix, Eqs. (A16), (A22) and (A23). Presumably, the same techniques can be applied to more general instances, either mathematically [42], or physically [43] oriented.

In the final part we have analyzed the transport in the end-to-end configuration. We derived an effective continuity equation with which we can identify the flux of density through the system, and the contributions of gain and loss are clearly displayed as input and output terms of the equation. We defined and analyzed a transport coefficient given as the ratio of output to input, and found that for eigenfunctions with complex eigenvalues transport through the system is defective, leading to an exponential increase or depletion of density in the system.
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Appendix A: Solution of the eigenvalue problem

In this section, following Ref. [35], we obtain the eigenvalues and eigenvectors of the general complex tridiagonal matrix defined by

\[
A = b \sum_{j=1}^{N} |j\rangle \langle j| + a \sum_{j=2}^{N} |j-1\rangle \langle j| + c \sum_{j=1}^{N-1} |j\rangle \langle j+1| \\
- \alpha |k\rangle \langle k| - \beta |N-k+1\rangle \langle N-k+1|,
\]

where \(a, b, c, \alpha, \beta \in \mathbb{C}\), \(N\) is the dimension of the Hilbert space which we assume is even for simplicity, and the contacts are in the positions \(k\) and \(N-k+1\). Since the location of the contacts is related by parity we may choose \(k=1, 2, \ldots, N/2\); note that inspite of this, we have not assumed any specific relation among \(\alpha\) and \(\beta\).

Let \(\lambda\) be an eigenvalue of \(A\) and write its associated (complex) eigenvector in the site basis as \(u = \sum_i u_i |j\rangle\). The eigenvalue problem \(Au = \lambda u\) can be written as the set of linear equations

\[
\begin{align*}
    u_0 &= 0, \\
    au_0 + bu_1 + cu_2 &= \lambda u_1, \\
    &\vdots \\
    au_{k-1} + bu_k + cu_{k+1} &= (\lambda + \alpha) u_k, \\
    &\vdots \\
    au_{N-k} + bu_{N-k+1} + cu_{N-k+2} &= (\lambda + \beta) u_{N-k+1}, \\
    &\vdots \\
    au_{N-1} + bu_N + cu_{N+1} &= \lambda u_N, \\
    u_{N+1} &= 0.
\end{align*}
\]

Note the introduction of two boundary equations, which are used to have a uniform way of writing the linear system. Further, we assume \(ac \neq 0\), otherwise the solution is trivial.
The idea of the approach developed by Yueh [35] is to rewrite this system of equations as an algebraic problem on infinite sequences. Then, one can use the symbolic calculus of Cheng [36] to obtain the required eigenproblem solution. We shall view the components of the eigenvector, \( u_j \), as the \( j \)-th term of the complex sequence \( u = \{u_i\}_{i=0}^\infty \), with \( u_j = 0 \) for \( j = 0 \) and \( j > N \). Notice that \( u_1 \neq 0 \), otherwise if \( u_1 = 0 \) we have that \( u_2 = u_3 = \cdots = 0 \).

Similarly, we define the complex sequence \( f = \{f_j\}_{j=0}^\infty \) with all components identical to zero except for \( f_k = \alpha u_k \) and \( f_{N-k+1} = \beta u_{N-k+1} \), which define the location of the contacts. Then, Eq. (A2) can be expressed as

\[
c\{u_{j+2}\}_{j=0}^\infty + b\{u_{j+1}\}_{j=0}^\infty + a\{u_j\}_{j=0}^\infty = \lambda\{u_{j+1}\}_{j=0}^\infty + \{f_{j+1}\}_{j=0}^\infty.
\]  

(A3)

We now introduce the shift sequence \( S = \{0, 1, 0, \ldots\} \) and the scalar sequence \( \bar{z} = \{z, 0, 0 \ldots\} \), where \( z \in \mathbb{C} \). We take the convolution \( \ast \) of the above equation with \( S^2 \) (see [36] for details and definitions), which yields

\[
c(u - \bar{u}_0 - u_1 S) + (b - \lambda)S(u - \bar{u}_0) + aS^2u = S(f - \bar{f}_0).
\]  

(A4)

Since \( u_0 = f_0 = 0 \), solving for \( u \) we get

\[
(aS^2 + (b - \lambda)S + \bar{c})u = (f + c\bar{u}_1)S.
\]  

(A5)

Since \( c \neq 0 \), the factor \( (aS^2 + (b - \lambda)S + \bar{c}) \) has a multiplicative inverse [36], i.e.

\[
u = \frac{(f + c\bar{u}_1)S}{aS^2 + (b - \lambda)S + \bar{c}}.
\]  

(A6)

The next step is to factorize the denominator, namely

\[
aS^2 + (b - \lambda)S + \bar{c} = a(\gamma_+ - S)(\gamma_- - S),
\]  

(A7)

where

\[
\gamma_\pm = \frac{-(b - \lambda) \pm \sqrt{w}}{2a},
\]  

(A8)

\( w = (b - \lambda)^2 - 4ac \) with \( ac \neq 0 \). Using partial fractions we have

\[
\frac{1}{a(\gamma_- - S)(\gamma_+ - S)} = \frac{1}{\sqrt{w}}\left( \frac{1}{\gamma_- - S} - \frac{1}{\gamma_+ - S} \right).
\]  

(A9)

Since

\[
(\gamma_- - S) \ast \{\gamma_-^{(j+1)}\}_{j=0}^\infty = (\gamma_- - S)\{\gamma_-^{(j+1)}\}_{j=0}^\infty = \{1, 0, \ldots\} = \mathbb{T},
\]  

(A10)
we have \((\gamma_\pm - S)^{-1} = \{\gamma_\pm^{-(j+1)}\}_j=0\). Using this last result in Eq. \((A9)\) we get

\[
\frac{1}{a(\gamma_- - S)(\gamma_+ - S)} = \frac{1}{\sqrt{w}}\{\gamma_-^{-(j+1)} - \gamma_+^{-(j+1)}\}_j=0,
\]
which we insert into Eq. \((A6)\), to obtain

\[
u = \frac{1}{\sqrt{w}}\{\gamma_-^{-(j+1)} - \gamma_+^{-(j+1)}\}_j=0 (f + c\bar{u}) S.
\]

At this point, it is convenient to introduce the following notation. Since \(\gamma_\pm\) are complex numbers, we write

\[
\gamma_\pm = p \pm iq
\]

and

\[
\gamma_\pm = \sqrt{p^2 + q^2} (\cos \theta \pm i \sin \theta) = \frac{e^{\pm i\theta}}{\rho},
\]

where

\[
\rho = \sqrt{a/c}, \cos \theta = \frac{p}{\sqrt{p^2 + q^2}} = \frac{\lambda - b}{2\sqrt{ac}},
\]

with \(p, q, \rho, \theta \in \mathbb{C}\). With these new definitions, Eq. \((A12)\) now reads

\[
u = \frac{1}{\sqrt{w}} \{(\frac{a}{c})^{j+1} (\gamma_+^{j+1} - \gamma_-^{j+1})\} \{f + c\bar{u}\}_j=0 (f + c\bar{u}) S
\]

where in the last equality we have used De Moivre’s theorem.

Up to Eq. \((A15)\) we have followed the same steps as in Yueh \[35\]; the remaining of the derivation is a generalization of Yueh’s. Notice that our eigenvalues are determined by the second equality of Eq. \((A14)\)

\[
\lambda = b + 2\sqrt{ac} \cos \theta,
\]

and also our eigenvectors depend on \(\theta\). Thus, our first task is to obtain an equation for \(\theta\). To do so, we have to calculate the convolutions in Eq. \((A15)\). As noted above, the contacts are at sites \(k\) and \(N-k+1\), and recall that the \(f\) sequence contains this information. Then

\[
(f + c\bar{u}_1) S = \{0, cu_1, 0, \ldots, 0, f_{k+1}, 0, \ldots, 0, f_{N-k+1}, 0, \ldots\},
\]

where we explicitly stated that due to the action of \(S\) over \(f\), \(f_k\) has been switched to position \(k+1\) and similarly for \(f_{N-k+1}\). Next we take the \(j\)-th component of Eq. \((A15)\)

\[
u_j = \frac{2i}{\sqrt{w}} \{(\rho^{j+1} \sin(j+1)\theta) (f + c\bar{u}_1) S\}_j
\]

\[
= \frac{1}{\sqrt{ac} \sin \theta} \left\{(cu_1 \rho^j \sin(j\theta) + \Theta(j-k-1) \alpha u_k \rho^{j-k} \sin[(j-k)\theta] + \Theta(j-N+k-2) \beta u_{N-k+1} \rho^{j-N+k-1} \sin[(j-N+k-1)\theta]\right\},
\]

\[(A18)\]
where $\Theta(x)$ is the unit step function defined by $\Theta(x) = 1$ if $x \geq 0$ and $\Theta(x) = 0$ if $x < 0$, and in the last identity we have used $2i\sqrt{ac}\sin \theta = \sqrt{w}$. In particular, we shall use below the expressions for $u_k$ and $u_{N-k+1}$, which read

$$u_k = \frac{cu_1\rho^k}{\sqrt{ac}\sin \theta} \sin k\theta, \quad (A19)$$

$$u_{N-k+1} = \frac{cu_1\rho^{N-k+1}}{\sqrt{ac}\sin \theta} \left( \sin(N-k+1)\theta + \frac{\alpha \sin k\theta \sin(N-2k+1)\theta}{\sqrt{ac}\sin \theta} \right). \quad (A20)$$

In the last two expressions we have eliminated the terms for which the argument of the step function is negative, and used the simplifying assumption that $N$ is even and $k = 1, 2, \ldots, N/2$.

Using Eq. (A18) for $j = N + 1$ and exploiting the explicit expressions derived for $u_k$ and $u_{N-k+1}$, we obtain

$$u_{N+1} = \frac{cu_1\rho^{N+1}}{\sqrt{ac}\sin \theta} \left[ \sin(N+1)\theta + \frac{\alpha + \beta}{\sqrt{ac}\sin \theta} \sin(N-k+1)\theta \sin k\theta \\
+ \frac{\alpha \beta}{ac\sin^2 \theta} \sin(N-2k+1)\theta \sin^2 k\theta \right], \quad (A21)$$

and from the boundary condition $u_{N+1} = 0$ of the linear system, Eq. (A2), we get

$$\sin(N+1)\theta + \frac{\alpha + \beta}{\sqrt{ac}\sin \theta} \sin(N-k+1)\theta \sin k\theta + \frac{\alpha \beta}{ac\sin^2 \theta} \sin(N-2k+1)\theta \sin^2 k\theta = 0, \quad (A22)$$

Equation (A22) determines $\theta$, with $\theta \neq m\pi$, $m \in \mathbb{Z}$, which excludes all the trivial solutions.

To obtain the components of the eigenvectors $u_j$, $j = 1, 2, \ldots, N$, we proceed similarly with Eq. (A18), where we substitute Eqs. (A19) and (A20), which leads us finally to

$$u_j = \frac{cu_1\rho^j}{\sqrt{ac}\sin \theta} \left[ \sin j\theta + \Theta(j-k-1) \frac{\alpha \sin k\theta \sin(j-k)\theta}{\sqrt{ac}\sin \theta} \\
+ \Theta(j-N+k-2) \frac{\beta}{\sqrt{ac}\sin \theta} \sin(j-N+k-1)\theta \\
\times \left( \sin(N-k+1)\theta + \frac{\alpha}{\sqrt{ac}\sin \theta} \sin(N-2k+1)\theta \sin k\theta \right) \right]. \quad (A23)$$

In the specific case that Eq. (A1) is $\mathcal{PT}$-symmetric, we set $a = c = t$, $\alpha = -i\eta$, $\beta = i\eta$.

Further, we set $b = 0$ since it amounts to a global shift in the energy.

[1] N Moiseyev. *Non-Hermitian Quantum Mechanics*. Cambridge University Press, 2011.
[2] Carl M. Bender and Stefan Boettcher. Real spectra in non-hermitian hamiltonians having $\mathcal{PT}$ symmetry. *Phys. Rev. Lett.*, 80:5243–5246, Jun 1998.

[3] J J Sakurai. *Modern Quantum Mechanics*. Addison-Wesley, rev. ed. edition, 1994.

[4] Carl M Bender. Making sense of non-hermitian hamiltonians. *Reports on Progress in Physics*, 70(6):947–1018, May 2007.

[5] W D Heiss. The physics of exceptional points. *J. Phys. A: Math. Theor.*, 45:44016, 2012.

[6] C M Bender. Pt-symmetric quantum theory. *Journal of Physics: Conference Series*, 631:012002, 2015.

[7] L Feng, R El-Ganainy, and L Ge. Non-hermitian photonics based on parity-time symmetry. *Nature*, 11:752–762, 2017.

[8] R El-Ganainy, K G Makris, M Khajavikhan, Z H Musslimani, S Rotter, and D N Christodoulides. Non-hermitian physics and pt symmetry. *Nature*, 14:11–19, 2018.

[9] Mohammad-Ali Miri and Andrea Alù. Exceptional points in optics and photonics. *Science*, 363(6422):eaar7709, January 2019.

[10] H Schomerus. From scattering theory to complex wave dynamics in non-hermitian $\mathcal{PT}$ - symmetric resonators. *Phil. Trans. R. Soc. A*, 371:20120194, 2013.

[11] T Kato. *Perturbation Theory for Linear Operators*. Springer, 2nd edition, 1995.

[12] Tony E. Lee. Anomalous edge state in a non-hermitian lattice. *Phys. Rev. Lett.*, 116(13):133903, April 2016.

[13] V. M. Martinez Alvarez, J. E. Barrios Vargas, M. Berdakin, and L. E. F. Foa Torres. Topological states of non-hermitian systems. *The European Physical Journal Special Topics*, 227(12):1295–1308, Dec 2018.

[14] V. M. Martinez Alvarez, J. E. Barrios Vargas, and L. E. F. Foa Torres. Non-hermitian robust edge states in one dimension: Anomalous localization and eigenspace condensation at exceptional points. *Phys. Rev. B*, 97:121401, Mar 2018.

[15] Xiang Ni, Daria Smirnova, Alexander Poddubny, Daniel Leykam, Yidong Chong, and Alexander B. Khanikaev. PT phase transitions of edge states at PT symmetric interfaces in non-hermitian topological insulators. *Phys. Rev. B*, 98(16):165129, October 2018.

[16] C. Yuce. Edge states at the interface of non-hermitian systems. *Phys. Rev. A*, 97(4):042118, April 2018.

[17] S. Lin, L. Jin, and Z. Song. Symmetry protected topological phases characterized by isolated
exceptional points. *Phys. Rev. B*, 99(16):165148, apr 2019.

[18] C. Dembowski, H.-D. Gräf, H. L. Harney, A. Heine, W. D. Heiss, H. Rehhfeld, and A. Richter. Experimental observation of the topological structure of exceptional points. *Phys. Rev. Lett.*, 86:787–790, Jan 2001.

[19] Alexei A. Mailybaev, Oleg N. Kirillov, and Alexander P. Seyranian. Geometric phase around exceptional points. *Phys. Rev. A*, 72(1):014104, jul 2005.

[20] Bo Peng, Şahin Kaya ozdemir, Matthias Liertzer, Weijian Chen, Johannes Kramer, Huzeyfe Yılmaz, Jan Wiersig, Stefan Rotter, and Lan Yang. Chiral modes and directional lasing at exceptional points. *PNAS*, 113(25):6845–6850, jun 2016.

[21] Alois Regensburger, Christoph Bersch, Mohammad-Ali Miri, Georgy Onishchukov, Demetrios N. Christodoulides, and Ulf Peschel. Parity–time synthetic photonic lattices. *Nature*, 488(7410):167–171, aug 2012.

[22] Zin Lin, Hamidreza Ramezani, Toni Eichelkraut, Tsampikos Kottos, Hui Cao, and Demetrios N. Christodoulides. Unidirectional invisibility induced by pt-symmetric periodic structures. *Phys. Rev. Lett.*, 106(21):213901, may 2011.

[23] Liang Feng, Ye-Long Xu, William S. Fegadolli, Ming-Hui Lu, José E. B. Oliveira, Wilso R. Almeida, Yan-Feng Chen, and Axel Scherer. Experimental demonstration of a unidirectional reflectionless parity-time metamaterial at optical frequencies. *Nat. Mater.*, 12(2):108–113, nov 2012.

[24] Aurélien Merkel, Vicent Romero-García, Jean-Philippe Groby, Jensen Li, and Johan Christiansen. Unidirectional zero sonic reflection in passive PT-symmetric Willis media. *Phys. Rev. B*, 98(20):201102(R), nov 2018.

[25] Weijian Chen, Şahin Kaya Ozdemir, Guangming Zhao, Jan Wiersig, and Lan Yang. Exceptional points enhance sensing in an optical microcavity. *Nature*, 548(7666):192–196, aug 2017.

[26] Tamar Goldzak, Alexei A. Mailybaev, and Nimrod Moiseyev. Light stops at exceptional points. *Phys. Rev. Lett.*, 120:013901, Jan 2018.

[27] Justin E. Elenewski and Hanning Chen. Real-time transport in open quantum systems from PT-symmetric quantum mechanics. *Phys. Rev. B*, 90:085104, Aug 2014.

[28] Lian-Lian Zhang and Wei-Jiang Gong. Transport properties in a non-hermitian triple-quantum-dot structure. *Phys. Rev. A*, 95:062123, Jun 2017.
[29] Yogesh N. Joglekar, Derek Scott, Mark Babbey, and Avadh Saxena. Robust and fragile $\mathcal{PT}$-symmetric phases in a tight-binding chain. *Phys. Rev. A*, 82:030103, Sep 2010.

[30] L. Jin and Z. Song. Solutions of $\mathcal{PT}$-symmetric tight-binding chain and its equivalent hermitian counterpart. *Phys. Rev. A*, 80:052107, Nov 2009.

[31] C M Bender. Introduction to $\mathcal{PT}$-symmetric quantum theory. *Contemporary Physics*, 46:277–292, 2005.

[32] E M Graefe, U Günther, H J Korsch, and A E Niederle. A non-Hermitian $\mathcal{PT}$ symmetric Bose-Hubbard model: Eigenvalue rings from unfolding higher-order exceptional points. *J. Phys. A: Math. Theor.*, 41:255206, 2008.

[33] Q Wang. 2 $\times$ 2 pt-symmetric matrices and their applications. *Phil. Trans. R. Soc. A*, 371:20120045, 2013.

[34] A Cantoni and P Butler. Eigenvalues and eigenvectors of symmetric centrosymmetric matrices. *Linear Algebra and its Applications*, 13:275–288, 1976.

[35] W Yueh. Eigenvalues of several tridiagonal matrices. *Applied Mathematics E-notes*, pages 5–66, 2005.

[36] S S Cheng. *Partial Difference Equations*. CRC Press, 2003.

[37] Adrian Ortega, Manan Vyas, and Luis Benet. Quantum efficiencies in finite disordered networks connected by many-body interactions. *Annalen der Physik*, 527(9-10):748–756, 2015.

[38] A Ortega, T Stegmann, and L Benet. Efficient quantum transport in disordered interacting many-body networks. *Phys. Rev. E*, 94:042102, Oct 2016.

[39] A Ortega, T Stegmann, and L Benet. Robustness of optimal transport in disordered interacting many-body networks. *Phys. Rev. E*, 98:012141, Jul 2018.

[40] C M Bender and S A Orszag. *Advanced Mathematical Methods for Scientists and Engineers*. McGraw-Hill, 1st edition, 1978.

[41] Mario A. Quiroz-Juárez, Armando Perez-Leija, Konrad Tschernig, Blas M. Rodriguez-Lara, Omar S. Magaña-Loaiza, Kurt Busch, Yogesh N. Joglekar, and Roberto de J. León-Montiel. Exceptional points of any order in a single, lossy, waveguide beamsplitter by photon-number-resolved detection. *arXiv e-prints*, page arXiv:1905.06993, May 2019.

[42] S Kouachi. Eigenvalues and eigenvectors of tridiagonal matrices. *Electronic Journal of Linear Algebra*, 15:115–133, 2006.

[43] Hamidreza Ramezani, Tsampikos Kottos, Vassilios Kovanis, and Demetrios N.
Christodoulides. Exceptional-point dynamics in photonic honeycomb lattices with $\mathcal{PT}$ symmetry. *Phys. Rev. A*, 85:013818, Jan 2012.