Finite beta-expansions with negative bases

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Abstract

The finiteness property is an important arithmetical property of beta-expansions. We exhibit classes of Pisot numbers $\beta$ having the negative finiteness property, that is the set of finite $(-\beta)$-expansions is equal to $\mathbb{Z}[\beta^{-1}]$. For a class of numbers including the Tribonacci number, we compute the maximal length of the fractional parts arising in the addition and subtraction of $(-\beta)$-integers. We also give conditions excluding the negative finiteness property.

1 Introduction

Digital expansions in real bases $\beta > 1$ were introduced by Rényi [23]. Of particular interest are bases $\beta$ satisfying the finiteness property, or Property (F), which means that each element of $\mathbb{Z}[\beta^{-1}] \cap [0, \infty)$ has a finite (greedy) $\beta$-expansion. We know from Frougny and Solomyak [13] that each base with Property (F) is a Pisot number, but the converse is not true. Partial characterizations are due to [13] [16] [1]. In [2], Akiyama et al. exhibited an intimate connection to shift radix systems (SRS), following ideas of Hollander [16]. For results on shift radix systems (with the finiteness property), we refer to the survey [18].

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Numeration systems with negative base $-\beta < -1$, or $(-\beta)$-expansions, received considerable attention since the paper [17] of Ito and Sadahiro in 2009. They are given by the $(-\beta)$-transformation

$$T_{-\beta} : [\ell_\beta, \ell_\beta + 1) \rightarrow [\ell_\beta, \ell_\beta + 1), \quad x \mapsto -\beta x - \lfloor -\beta x - \ell_\beta \rfloor,$$

with $\ell_\beta = \frac{-\beta}{\beta + 1}$; see Section 2 for details. Certain arithmetic aspects seem to be analogous to those for positive base systems [12, 20], others are different, e.g., both negative and positive numbers have $(-\beta)$-expansions; for $\beta < \frac{1+\sqrt{5}}{2}$, the only number with finite $(-\beta)$-expansion is 0. We say that $\beta > 1$ has the negative finiteness property, or Property ($-F$), if each element of $\mathbb{Z}[-\beta]$ has a finite $(-\beta)$-expansion. By Dammak and Hbaib [10], we know that $\beta$ must be a Pisot number, as in the positive case. It was shown in [20] that the Pisot roots of $x^2 - mx + n$, with positive integers $m, n, m \geq n + 2$, satisfy the Property ($-F$). This gives a complete characterization for quadratic numbers, as $\beta$ does not possess Property ($-F$) if $\beta$ has a negative Galois conjugate, by [20].

First, we give other simple criteria when $\beta$ does not satisfy Property ($-F$). Surprisingly, this happens when $\ell_\beta$ has a finite $(-\beta)$-expansion, which is somewhat opposite to the positive case, where Property (F) implies that $\beta$ is a simple Parry number.

**Theorem 1.** If $T^k_{-\beta}(\ell_\beta) = 0$ for some $k \geq 1$, or if $\beta$ is the root of a polynomial $p(x) \in \mathbb{Z}[x]$ with $|p(-1)| = 1$, then $\beta$ does not possess Property ($-F$).

The main tool we use is a generalization of shift radix systems. We show that the $(-\beta)$-transformation is conjugated to a certain $\alpha$-SRS. Then we study properties of this dynamical system. We obtain a complete characterization for cubic Pisot units.

**Theorem 2.** Let $\beta > 1$ be a cubic Pisot unit with minimal polynomial $x^3 - ax^2 + bx - c$. Then $\beta$ has Property ($-F$) if and only if $c = 1$ and $-1 \leq b < a$, $|a| + |b| \geq 2$.

Considering Pisot numbers of arbitrary degree, we have the following results.

**Theorem 3.** Let $\beta > 1$ be a root of $x^d - mx^{d-1} - \cdots - mx - m$ for some positive integers $d, m$. Then $\beta$ has Property ($-F$) if and only if $d \in \{1, 3, 5\}$.

**Theorem 4.** Let $\beta > 1$ be a root of $x^d - a_1x^{d-1} + a_2x^{d-2} + \cdots + (-1)^da_d \in \mathbb{Z}[x]$ with $a_i \geq 0$ for $i = 1, \ldots, d$, and $a_1 \geq 2 + \sum_{i=2}^{d} a_i$. Then $\beta$ has Property ($-F$).

These theorems are proved in Section 3. In Section 4 we give a precise bound on the number of fractional digits arising from addition and subtraction of $(-\beta)$-integers in case $\beta > 1$ is a root of $x^3 - mx^2 - mx - m$ for $m \geq 1$. This is based on an extension of shift radix systems. The corresponding numbers for $\beta$-integers have not been calculated yet, although they can be determined in a similar way.
2 $(-\beta)$-expansions

For $\beta > 1$, any $x \in [\ell_\beta, \ell_\beta + 1)$ has an expansion of the form

$$x = \sum_{i=1}^{\infty} \frac{x_i}{(-\beta)^i} \quad \text{with} \quad x_i = [-\beta^i T_{-\beta}^{-1}(x) - \ell_\beta] \quad \text{for all} \quad i \geq 1.$$  

This gives the infinite word $d_{-\beta}(x) = x_1x_2x_3\cdots \in A^\mathbb{N}$ with $A = \{0, 1, \ldots, [\beta]\}$. Since the base is negative, we can represent any $x \in \mathbb{R}$ without the need of a minus sign. Indeed, let $k \in \mathbb{N}$ be minimal such that $\frac{x}{(-\beta)^k} \in (\ell_\beta, \ell_\beta + 1)$ and $d_{-\beta}(\frac{x}{(-\beta)^k}) = x_1x_2x_3\cdots$. Then the $(-\beta)$-expansion of $x$ is defined as

$$\langle x \rangle_{-\beta} = \begin{cases} x_1 \cdots x_{k-1} x_k \cdot x_{k+1} x_{k+2} \cdots & \text{if} \quad k \geq 1, \\ 0 \cdot x_1 x_2 x_3 \cdots & \text{if} \quad k = 0. \end{cases}$$

Similarly to positive base numeration systems, the set of $(-\beta)$-integers can be defined using the notion of $\langle x \rangle_{-\beta}$, by

$$\mathbb{Z}_{-\beta} = \{ x \in \mathbb{R} : \langle x \rangle_{-\beta} = x_1 \cdots x_{k-1} x_k \cdot 0^\omega \} = \bigcup_{k \geq 0} (-\beta)^k T_{-\beta}^{-1}(0),$$

where $0^\omega$ is the infinite repetition of zeros. The set of numbers with finite $(-\beta)$-expansion is

$$\text{Fin}(-\beta) = \{ x \in \mathbb{R} : \langle x \rangle_{-\beta} = x_1 \cdots x_{k-1} x_k \cdot x_{k+1} \cdots x_{k+n} 0^\omega \} = \bigcup_{n \geq 0} \mathbb{Z}_{-\beta}^n.$$  

If $\langle x \rangle_{-\beta} = x_1 \cdots x_{k-1} x_k \cdot x_{k+1} \cdots x_{k+n} 0^\omega$ with $x_{k+n} \neq 0$, then $\text{fr}(x) = n$ denotes the length of the fractional part of $x$; if $x \in \mathbb{Z}_{-\beta}$, then $\text{fr}(x) = 0$.

3 Finiteness

In this section, we discuss the Property (−F) for several classes of Pisot numbers $\beta$. Note that $\text{Fin}(-\beta)$ is a subset of $\mathbb{Z}[\beta^{-1}]$ since $\beta$ is an algebraic integer, hence Property (−F) means that $\text{Fin}(-\beta) = \mathbb{Z}[\beta^{-1}]$, i.e., $\text{Fin}(-\beta)$ is a ring. We start by showing that bases $\beta$ satisfying $d_{-\beta}(\ell_\beta) = d_1d_2\ldots d_k0^\omega$, which can be considered as analogs to simple Parry numbers, do not possess Property (−F). This was conjectured in [19] and supported by the fact that $d_{-\beta}(\ell_\beta) = d_1d_2\ldots d_k0^\omega$ with $d_i \geq d_j + 2$ for all $2 \leq j \leq k$ implies that $d_{-\beta}(\beta-1-d_1) = (d_2+1)(d_3+1)\cdots(d_k+1)1^\omega$. However, the assumption $d_i \geq d_j + 2$ is not necessary for showing that Property (−F) does not hold.

We also prove that a base with Property (−F) cannot be the root of a polynomial of the form $a_0x^d + a_1x^{d-1} + \cdots + a_d$ with $|\sum_{i=0}^{d}(-1)^i a_i| = 1$. 


Proof of Theorem\[\textit{[7]}\]. If $T^k_{-\beta}(\ell_\beta) = 0$, i.e., $d_{-\beta}(\ell_\beta) = d_1d_2\ldots d_k0^\omega$, then we have

$$\frac{-\beta}{\beta+1} = \frac{d_1}{-\beta} + \frac{d_2}{(-\beta)^2} + \cdots + \frac{d_k}{(-\beta)^k}$$

and thus $\frac{1}{\beta+1} \in \mathbb{Z}[\beta^{-1}]$. However, we have $\frac{1}{\beta+1} \notin \text{Fin}(\beta)$ since $T_{-\beta}(\frac{1}{\beta+1}) = \frac{1}{\beta+1}$, i.e., $d_{-\beta}(\frac{1}{\beta+1}) = 1^\omega$. Hence $\beta$ does not possess Property $\text{(-F)}$.

If $p(\beta) = 0$ with $|p(-1)| = 1$, then write

$$p(x - 1) = xf(x) + p(-1),$$

with $f(x) \in \mathbb{Z}[x]$. Then we have $\frac{1}{\beta+1} = |f(\beta+1)| \in \mathbb{Z}[\beta]$ and thus $\frac{(-\beta)^{-j}}{\beta+1} \in \mathbb{Z}[\beta^{-1}]$ for some $j \geq 0$. Now, $d_{-\beta}(\frac{(-\beta)^{-j}}{\beta+1}) = 0^j1^\omega$ implies that $\beta$ does not have the Property $\text{(-F)}$. $\square$

The main tool we will be using in the rest of the paper are $\alpha$-shift radix systems. An $\alpha$-SRS is a dynamical system acting on $\mathbb{Z}^d$ in the following way. For $\alpha \in \mathbb{R}$, $r = (r_0, r_1, \ldots, r_{d-1}) \in \mathbb{R}^d$, and $z = (z_0, z_1, \ldots, z_{d-1}) \in \mathbb{Z}^d$, let $\tau_{r,\alpha}$ be defined as

$$\tau_{r,\alpha}(z_0, z_1, \ldots, z_{d-1}) = (z_1, \ldots, z_{d-1}, z_d),$$

where $z_d$ is the unique integer satisfying

$$0 \leq r_0z_0 + r_1z_1 + \cdots + r_{d-1}z_{d-1} + z_d + \alpha < 1. \quad (1)$$

Alternatively, we can say that

$$\tau_{r,\alpha}(z_0, z_1, \ldots, z_{d-1}) = (z_1, \ldots, z_{d-1}, -\lfloor rz + \alpha \rfloor),$$

where $rz$ stands for the scalar product.

The usefulness of $\alpha$-SRS with $\alpha = 0$ for the study of finiteness of $\beta$-expansions was first shown by Hollander in his thesis \[\textit{[16]}\]. His approach was later formalized in \[\textit{[2]}\] where the case $\alpha = 0$ was extensively studied. The symmetric case with $\alpha = \frac{1}{2}$ was then studied in \[\textit{[4]}\]. Finally, general $\alpha$-SRS were considered by Surer \[\textit{[24]}\].

We say that $\tau_{r,\alpha}$ has the finiteness property if for each $z \in \mathbb{Z}^d$ there exists $k \in \mathbb{N}$ such that $\tau_{r,\alpha}^k(z) = 0$. The finiteness property of $\tau_{r,\alpha}$ is closely related to the Property $\text{(-F)}$, thus it is desirable to study the set

$$\mathcal{D}^0_{d,\alpha} = \{r \in \mathbb{R}^d : \forall z \in \mathbb{Z}^d, \exists k, \tau_{r,\alpha}^k(z) = 0\}.$$

The following proposition shows the link between $(-\beta)$-expansions and $\alpha$-SRS.

**Proposition 5.** Let $\beta > 1$ be an algebraic integer with minimal polynomial $x^d + a_1x^{d-1} + \cdots + a_{d-1}x + a_d$. Set $\alpha = \frac{1}{\beta+1}$ and let $(r_0, r_1, \ldots, r_{d-2}) \in \mathbb{R}^{d-1}$ be such that

$$x^d + (-1)a_1x^{d-1} + \cdots + (-1)^{d}a_d = (x + \beta)(x^{d-1} + r_{d-2}x^{d-2} + \cdots + r_1x + r_0),$$

i.e.,

$$r_i = (-1)^{d-i}\left(\frac{a_{d-i}}{\beta} + \cdots + \frac{a_d}{\beta^{i+1}}\right) \quad \text{for } i = 0, 1, \ldots, d - 2.$$

Then $\beta$ has Property $\text{(-F)}$ if and only if $(r_0, r_1, \ldots, r_{d-2}) \in \mathcal{D}^0_{d-1,\alpha}$.
Proposition 6. Let $\alpha \in [0,1)$ and $r \in \mathbb{R}^d$. Then $r \in \mathcal{D}^0_{d,\alpha}$ if and only if there exists a set of witnesses that does not contain nonzero periodic elements of $\tau_{r,\alpha}$.

Sets of witnesses for several classes of $r \in \mathbb{R}^d$ were derived in [3]. Exploiting their explicit form, several regions of finiteness can be determined; see in particular [3, Theorems 3.3–3.5]. An $\alpha$-SRS analogy of some of those regions was given by Brunotte [7]. Brunotte’s result, however, is unsuitable for our purposes. The next proposition gives several regions of finiteness of $\alpha$-SRS.

Proposition 7. Let $r = (r_0, r_1, \ldots, r_{d-1}) \in \mathbb{R}^d$ and $\alpha \in [0,1)$.

1. If $\sum_{i=0}^{d-1} |r_i| \leq \alpha$ and $\sum_{i < 0} r_i > \alpha - 1$, then $r \in \mathcal{D}^0_{d,\alpha}$.

2. If $0 \leq r_0 \leq r_1 \leq \cdots \leq r_{d-1} \leq \alpha$, then $r \in \mathcal{D}^0_{d,\alpha}$.
3. If \( \sum_{i=0}^{d-1} |r_i| \leq \alpha \) and \( r_i < 0 \) for exactly one index \( i = d - k \), then \( \mathbf{r} \in \mathcal{D}^{0}_{d,\alpha} \) if and only if
\[
\sum_{1 \leq j \leq d/k} r_{d-jk} > \alpha - 1.
\] (2)

Proof. 1. The set \( \mathcal{V} = \{-1, 0, 1\}^d \) is closed under \( \tau_{r,0}(\mathbf{z}) \) and \( -\tau_{r,0}(-\mathbf{z}) \), hence it is a set of witnesses. For any \( \mathbf{z} \in \mathcal{V} \) we have \( |\mathbf{r}\mathbf{z}| \leq \alpha \), thus \( |\mathbf{r}\mathbf{z} + \alpha| \in \{0, 1\} \).
Hence any periodic point of \( \tau_{r,\alpha} \) is in \( \{0, -1\}^d \). For \( \mathbf{z} \in \{0, -1\}^d \) we have \( \mathbf{r}\mathbf{z} + \alpha \leq -\sum_{j \leq 0} r_j + \alpha < 1 \). Therefore \( |\mathbf{r}\mathbf{z} + \alpha| = 0 \), so the only period is the trivial one.

2. In this case we take as a set of witnesses the elements of \( \{-1, 0, 1\}^d \) with alternating signs, i.e., \( z_i z_j \leq 0 \) for any pair of indices \( i < j \) such that \( z_k = 0 \) for each \( i < k < j \). For any \( \mathbf{z} \in \mathcal{V} \) we have again \( |\mathbf{r}\mathbf{z}| \leq \alpha \), thus \( |\mathbf{r}\mathbf{z} + \alpha| \in \{0, 1\} \) and \( \tau_{r,\alpha}(\mathbf{z}) \in \mathcal{V} \).
Therefore, we have \( \tau_{r,\alpha}^{-n}(\mathbf{z}) = (-1, 0, \ldots, 0) \) for some \( n \geq 0 \), hence \( \tau_{r,\alpha}^{n+1}(\mathbf{z}) = 0 \).

3. In this case we have \( \mathcal{V} = \{-1, 0, 1\}^d \). As above, all periodic points of \( \tau_{r,\alpha} \) are in \( \{0, -1\}^d \). If \( \mathbf{z} = (z_0, z_1, \ldots, z_{d-1}) \) is a periodic point with \( z_0 = -|\mathbf{r}\mathbf{z} + \alpha| = -1 \), then we must have \( z_{d-k} = -1 \), and consequently \( z_{d-jk} = -1 \) for all \( 1 \leq j \leq d/k \).

Then \( z_{d-k} = -1 \) also implies that \( -\sum_{1 \leq j \leq d/k} r_{d-jk} + \alpha \geq 1 \), i.e., (2) does not hold. On the other hand, if (2) holds, then the vector \((z_0, z_1, \ldots, z_{d-1})\) with \( z_{d-jk} = -1 \) for \( 1 \leq j \leq d/k \), \( z_i = 0 \) otherwise, is a periodic point of \( \tau_{r,\alpha} \).

Next we prove Property (−F) when \( \beta \) is a root of a polynomial with alternating coefficients, where the second highest coefficient is dominant.

Proof of Theorem 4. Let \( \beta > 1 \) be a root of \( p(x) = x^d - a_1 x^{d-1} + a_2 x^{d-2} + \cdots + (-1)^d a_d \in \mathbb{Z}[x] \) with \( a_i \geq 0 \) for \( i = 1, \ldots, d \), and \( a_1 \geq 2 + \sum_{i=0}^{d-2} a_i \). As \( \frac{d}{dx} (p(x) x^{-d}) \geq \frac{a_1}{x} - \frac{a_{i-2}}{x^2} > 0 \) for \( x > 1 \), the polynomial \( p(x) \) has a unique root \( \beta > 1 \), and we have \( \beta > a_1 - 1 \) since \( p(a_1 - 1) \leq -(a_1 - 1)^{d-1} + (a_1 - 2)(a_1 - 1)^{d-2} < 0 \). By Proposition 5 Property (−F) holds if and only if \((r_0, r_1, \ldots, r_{d-2}) \in \mathcal{D}^{0}_{d-1,\alpha} \), with \( r_i = a_{d-i} \beta^{-1} - a_{d-i+1} \beta^{-2} + a_{d-i+2} \beta^{-3} - \cdots + (-1)^{d-i} a_d \beta^{-d+i-1} \). We have
\[
-\sum_{r_i < 0} r_i \leq \frac{a_1 - 2}{\beta^2} + \frac{a_1 - 2}{\beta^4} + \cdots + \frac{a_1 - 2}{\beta^{2(d-2)/2}} \leq \frac{a_1 - 2}{\beta^2 - 1} < \frac{1}{\beta + 1}
\]
Therefore, we assume in the following that $c$ only if Lemma 8.

Proof of Theorem 2. Let $\beta > 1$ be a cubic Pisot unit with minimal polynomial $x^3 - ax^2 + bx - c$. If $c = -1$, then $\beta$ has a negative conjugate, which contradicts Property ($-F$) by [20]. Therefore, we assume in the following that $c = 1$. Then from Lemma 8 we have that $-a - 1 \leq b < a$. By Proposition 5, Property ($-F$) holds if and only if $(r_0, r_1) \in D_{2, a}$, with $(r_0, r_1) = (\frac{1}{\beta}, \frac{b}{\beta} - \frac{1}{\beta^2})$ and $\alpha = \frac{b}{\beta + 1}$. We distinguish five cases for the value of $b$.

1. $b = 0$: If $a \geq 2$, then we have $|r_0| + |r_1| = \frac{1}{\beta} + \frac{1}{\beta^2} < \alpha$ and $r_0 + r_1 > 0 > \alpha - 1$, so we apply item 3 of Proposition 7. If $a = 1$, then we have $T_{-1/\beta}^{-1}(0) = \{0\}$ as $\beta \leq \frac{1 + \sqrt{5}}{2}$, thus $\text{Fin}(\beta) = \{0\}$.

2. $b = -1$: If $a \geq 1$, then $r_0 + r_1 = -\frac{1}{\beta^2} > \frac{1}{\beta^2} = \alpha - 1$. If $a \geq 3$, then we also have $|r_0| + |r_1| < \alpha$ and use item 3 of Proposition 7. If $a = 2$, then $r_0 \approx 0.39, r_1 \approx -0.55, \alpha \approx 0.72, \{-1, 0, 1\}^2$ is a set of witnesses, and Property ($-F$) holds because $\tau_{r, a}$ acts on this set in the following way:

$$\begin{align*}
(-1, 1) \mapsto (1, 1) \mapsto (1, 0) \mapsto (0, -1) \mapsto (-1, -1) \mapsto (-1, 0) \mapsto (0, 0), \\
(0, 1) \mapsto (1, 0), (1, -1) \mapsto (-1, -1).
\end{align*}$$

For $a = 1$, we refer to Theorem 5 which is proved below. If $a = 0$, then $\beta < \frac{1 + \sqrt{5}}{2}$ and thus $\text{Fin}(\beta) = \{0\}$.
3. $1 \leq b \leq a - 2$: For $b \geq 2$, we have $0 < r_0 < r_1 < \alpha$ and thus $(r_0, r_1) \in \mathcal{D}^0_{2, \alpha}$ by item 2 of Proposition 7. If $b = 1$, then we can use item 1 of Proposition 7 because $r_0, r_1 > 0$ and $r_0 + r_1 < \alpha$.

4. $1 \leq b = a - 1$: We have $\beta = b + \frac{1}{\beta(\beta - 1)}$. For $b \geq 3$, we have $0 < r_0 < \alpha < r_1 < 1$, the set $\{-1, 0, 1\}^2 \setminus \{(1, 1), (-1, -1)\}$ is a set of witnesses, and $\tau_{r, \alpha}$ acts on this set by

$$(1, 0) \mapsto (0, -1) \mapsto (-1, 1) \mapsto (1, -1) \mapsto (-1, 0) \mapsto (0, 0), \quad (0, 1) \mapsto (1, -1),$$

thus Property $(-F)$ holds. If $b = 2$, then $0 < r_0 < r_1 < \alpha$ and we can use item 2 of Proposition 7. If $b = 1$, then $r_0 \approx 0.57, r_1 \approx 0.25, \alpha \approx 0.64$, thus $\{-1, 0, 1\}^2$ is a set of witnesses, with

$$(-1, -1) \mapsto (-1, 1) \mapsto (1, 0) \mapsto (0, -1) \mapsto (-1, 0) \mapsto (0, 0), \quad (0, 1) \mapsto (1, 0), \quad (1, 1) \mapsto (1, -1) \mapsto (-1, 0),$$

5. $-a - 1 \leq b \leq -2$: We have $-r_0 - r_1 + \alpha = \frac{1}{\beta^2} + \frac{1}{\beta} + \frac{\beta}{\beta + 1} > 1$, thus $\tau_{r, \alpha}(-1, -1) = (-1, 1)$, hence $(r_0, r_1) \notin \mathcal{D}^0_{2, \alpha}$.

Therefore, $\beta$ has Property $(-F)$ if and only if $-1 \leq b < a, |a| + |b| \geq 2$.

Finally, we study generalized $d$-bonacci numbers.

**Proof of Theorem 3** Let $\beta > 1$ be a root of $x^d - m x^{d-1} - \cdots - m x - m$ with $d, m \in \mathbb{N}$.

If $d = 1$ (and $m \geq 2$), then $\beta$ is an integer, and Property $(-F)$ follows from $\mathbb{Z}_{\beta} = \mathbb{Z}$. See e.g. [20].

If $d = 3$, then $\beta = \frac{m}{\beta}$, $0 < r_0 < \alpha < -r_1 < 1$, with $\alpha = \frac{\beta}{\beta + 1}$, and $\tau_{r, \alpha}$ satisfies

$$(0, 1) \mapsto (1, 1) \mapsto (1, 0) \mapsto (0, 1) \mapsto (0, -1) \mapsto (-1, 1) \mapsto (-1, 0) \mapsto (0, 0),$$

with $\{-1, 0, 1\}^2 \setminus \{(1, 1), (-1, -1)\}$ being a set of witnesses.

If $d = 5$, then $\beta = \frac{m}{\beta^2}$, $0 < r_0 < \alpha < -r_1 < r_2 < -r_3 < 1$ and the $\tau_{r, \alpha}$-transitions

$$(0, 1, 0, 0) \mapsto (1, 0, 0, 1) \mapsto (0, 0, 1, 0) \mapsto (0, 1, 0, -1) \mapsto (1, 0, -1, 0) \mapsto (0, -1, 0, 0) \mapsto (-1, 0, 0, 0) \mapsto (1, 0, 0, -1) \mapsto (0, -1, -1, 0) \mapsto (0, 0, 0, 0) \mapsto (0, 0, 0, 0),$$

$$(0, 0, -1, 0) \mapsto (0, -1, 1, 0) \mapsto (-1, 0, 1, 0) \mapsto (0, 1, 0, -1),$$

$$(0, 1, 1, 1) \mapsto (1, 1, 1, 0) \mapsto (1, 1, 0, -1) \mapsto (1, 0, -1, 1) \mapsto (0, -1, 1, 1) \mapsto (0, -1, -1, 1) \mapsto (0, -1, 0, 1),$$

Thus $\tau_{r, \alpha}(-1, -1, 0, 1) = (0, 1, -1, 1) \mapsto (0, 1, 0, 1) \mapsto (1, 1, 1, 0) \mapsto (1, 1, 0, -1) \mapsto (1, 0, -1, -1) \mapsto (0, -1, -1, -1) \mapsto (0, -1, 0, 0)$.
Let $\mathcal{V}$ be the set of these states. We have $\pm \epsilon_i \in \mathcal{V}$, $z \in \mathcal{V}$ if and only if $-z \in \mathcal{V}$ and $\tau_{r,0}(z) \in \mathcal{V}$ for all $z \in \mathcal{V}$, thus $\mathcal{V}$ is a set of witnesses. As $\tau_{r,0}^{11}(z) = (0,0,0,0)$ for all $z \in \mathcal{V}$, $\beta$ has Property ($-F$).

For odd $d \geq 7$, Property ($-F$) does not hold since $T_{-\beta}^{-1}\left(\frac{m}{2^r} + \frac{m}{2^s} + \frac{m}{2^t} - 1\right) = \frac{m}{2^r} + \frac{m}{2^s} + \frac{m}{2^t} - 1$, i.e., $\tau_{r,0}^{2n-1}(-1,0,0,-1,0,0,\ldots,0) = (-1,0,0,-1,0,0,\ldots,0)$. For even $d \geq 2$, we use the second condition of Theorem 11 or that $\tau_{r,0}(-1,\ldots,-1) = (-1,\ldots,-1)$.

Therefore, $\beta$ has Property ($-F$) if and only if $d \in \{1,3,5\}$.

4 Addition and subtraction

In this section, we consider the lengths of fractional parts arising in the addition and subtraction of ($-\beta$)-integers; we prove the following theorem.

**Theorem 9.** Let $\beta > 1$ be a root of $x^3 - m\beta^2 - m\beta - m$, $m \geq 1$. We have

$$\max\{\text{fr}(x \pm y) : x, y \in \mathbb{Z}_{-\beta}\} = 3m + \begin{cases} 3 & \text{if } m = 1 \text{ or } m \text{ is even}, \\ 4 & \text{if } m \geq 3 \text{ is odd}. \end{cases}$$

Throughout the section, let $\beta$ be as in Theorem 9 $r = (r_0,r_1) = (\frac{m}{2^r}, -\frac{m}{2^s} - \frac{m}{2^t})$ and $\alpha = \frac{\beta}{\beta + 1}$. Recall that $x, y \in \mathbb{Z}_{-\beta}$ means that $T_{-\beta}^k(x, y) = 0 = T_{-\beta}(\frac{x}{(-\beta)^k}, \frac{y}{(-\beta)^k})$, and fr$(x \pm y) = n$ is the minimal $n \geq 0$ such that $T_{-\beta}^{k+n}(\frac{x+y}{(-\beta)^k}) = 0$, with $k \geq 0$ such that $\frac{x}{(-\beta)^k}, \frac{y}{(-\beta)^k}, \frac{x+y}{(-\beta)^k} \in (\ell_\beta, \ell_\beta + 1)$. To determine fr$(x - y)$, set

$$s_j = T_{-\beta}^j(\frac{x-y}{(-\beta)^k}) + T_{-\beta}^j(\frac{y}{(-\beta)^k}) - T_{-\beta}^j(\frac{x}{(-\beta)^k})$$

for $j \geq 0$. Then we have $s_j = T_{-\beta}^j(\frac{x-y}{(-\beta)^k})$ for $j \geq k$, and, for all $j \geq 0$,

$$s_{j+1} \in -\beta s_j + B \quad \text{with} \quad B = -A - A + A = \{-2m,-2m+1,\ldots,m\},$$

$$s_j \in [l_\beta, l_\beta + 1] + [l_\beta, l_\beta + 1] - [l_\beta, l_\beta + 1] = (l_\beta - 1, l_\beta + 2).$$

As $s_0 = 0$, we have $s_j \in \mathbb{Z}[\beta]$ for $j \geq 0$. Therefore, we extend the bijection $\phi : \mathbb{Z}^2 \to \mathbb{Z}[\beta] \cap [l_\beta, l_\beta + 1]$ to

$$\Phi : \mathbb{Z}^2 \times \{-1,0,1\} \to \mathbb{Z}[\beta] \cap [l_\beta - 1, l_\beta + 2], \quad (z, h) \mapsto rz - \lfloor rz + \alpha \rfloor + h.$$

Note that $\Phi(z,0) = \phi(z)$.

**Lemma 10.** Let $z = (z_0, z_1) \in \mathbb{Z}^2$, $h \in \{-1,0,1\}$ and $b \in B$. Then

$$-\beta \Phi(z,h) + b = \Phi(z_1, h - [rz + \alpha], (z_1 - z_0 - h + [rz + \alpha]) m + [r_0 z_1 + r_1 h - r_1 [rz + \alpha] + \alpha] b).$$
Proof. We have

\[-\beta \Phi(z, h) + b = -z_0m + z_1m + \frac{z_1m}{\beta} + [rz + \alpha] \beta - h\beta + b = r_0z_1 + r_1(h - [rz + \alpha]) + (z_1 - z_0 - h + [rz + \alpha]) m + b.\]

Hence, we have \( s_j \in \Phi(\tilde{\tau}_{r,\alpha}(0, 0)) \), where \( \tilde{\tau}_{r,\alpha} \) extends \( \tau_{r,\alpha} \) to a set-valued function by

\[ \tilde{\tau}_{r,\alpha} : \mathbb{Z}^2 \times \{-1, 0, 1\} \to \mathcal{P}(\mathbb{Z}^2 \times \{-1, 0, 1\}), \quad (z, h) \mapsto \{(z_1, h - [rz + \alpha], h') : h' \in \{-1, 0, 1\} \cap ((z_1 - z_0 - h + [rz + \alpha]) m + [r_0z_1 + r_1h - r_1[rz + \alpha] + \alpha] \cup \mathcal{B})\}. \]

To give a bound for the sets \( \tilde{\tau}_{r,\alpha}(0, 0) \), let

\[ A_k = \{(j, k) : -1 \leq j < k\}, \quad B_k = \{(k, j) : 1 \leq j \leq k\}, \quad C_k = \{(j, j - k) : 1 \leq j \leq k\}, \quad D_k = \{(-j, -k) : 0 \leq j < k\}, \quad E_k = \{(-k, -j) : 2 \leq j \leq k\}, \]

\[ F_k = \{(-j, k - j) : 2 \leq j \leq k + 1\}. \]

Then \( \bigcup_{k \geq 0} \{A_k, B_k, C_k, D_k, E_k, F_k\} \) forms a partition of \( \mathbb{Z}^2 \setminus \{(0, 0), (1, -1)\} \), with the sets \( B_0, C_0, D_0, E_0, F_0 \), and \( E_1 \) being empty; see Figure 4. If \( m \geq 2 \), then let

\[ V = \left( \bigcup_{0 \leq k \leq m} (A_k \cup B_k \cup C_k \cup D_k \cup E_k \cup F_k) \times \{-1, 0, 1\} \right) \setminus \{(m, 1), (0, m, 1)\} \]

\[ \cup \left( (C_{m+1} \setminus \{(m, 1, 0)\}) \times \{1\} \right) \cup \left( (D_{m+1} \times \{1\} \setminus \{(0, m - 1)\}) \times \{0\} \right) \]

\[ \cup \left( (D_{m+1} \setminus \{(0, -m - 1), (1, -m - 1), (-2, -m - 1)\}) \times \{-1\} \right) \]

\[ \cup \left( \{(0, 0), (1, -1), (-1, 1) \cup (E_{m+1} \setminus \{m + 1, m - 1\}) \times \{-1, 0, 1\} \right) \]

\[ \cup \left( (F_{m+1} \setminus \{(-m - 2, -1), (-m - 1, 0)\}) \times \{-1, 0\} \right). \]

If \( m = 1 \), then we add the point \( (-2, 0, -1) \) to this set, i.e.,

\[ V = \left\{ (0, 0), (1, 1), (1, 0), (0, 1), (1, -1), (-1, 0), (-2, -1), (1, -1), (0, 2, -1), (0, -2, 0), (2, 0, 1) \right\} \times \{-1, 0, 1\} \]

\[ \cup \left\{ (1, -1), (0, 1) \right\} \times \{-1, 0\} \cup \left\{ (1, -2), (0, 1) \right\} \cup \left\{ (1, -1, 1), (0, -2, 1), (2, 0, 1) \right\} \]

We call a point \( z \in \mathbb{Z}^2 \) full if \( \{z\} \times \{-1, 0, 1\} \subset V \).

The following result is the key lemma of this section.

Lemma 11. Let \( x, y \in [\ell_\beta, \ell_\beta + 1) \) such that \( x - y \in [\ell_\beta, \ell_\beta + 1) \). Then \( T_{-\beta}(x - y) + T_{-\beta}(y) - T_{-\beta}(x) \in \Phi(V) \) for all \( j \geq 0 \).

To prove Lemma 11, we first determine the value of \( [rz + \alpha] \) for \( (z, h) \in V \).
Lemma 12. Let $z = (z_0, z_1) \in \mathbb{Z}^2$ with $-m-1 \leq z_0 \leq m$, $|z_1| \leq m+1$ and $|z_0-z_1| \leq m+1$. Then

$$\lfloor rz + \alpha \rfloor = z_0 - z_1 + \begin{cases} 0 & \text{if } z_0 \geq 0 \text{ or } z_1 \leq z_0 = -1, \\ 1 & \text{if } z_0 \leq -2 \text{ or } z_1 > z_0 = -1. \end{cases}$$

Proof. We have $z_0 r_0 + z_1 r_1 = z_0 - z_1 - z_0 \frac{m}{\beta^2} + (z_1 - z_0) \frac{m}{\beta^2}$ and

$$\frac{-\beta}{\beta + 1} < -\frac{m^2}{\beta^2} - \frac{(m+1)m}{\beta^3} \leq -\frac{z_0 m}{\beta^2} + \frac{(z_1 - z_0)m}{\beta^3} \leq \frac{(m+1)m}{\beta^2} + \frac{(m+1)m}{\beta^3} < 1 + \frac{1}{\beta + 1},$$

thus $\lfloor rz + \alpha \rfloor \in z_0 - z_1 + \{0, 1\}$.

If $z_0 \geq 0$, then we have $-z_0 \frac{m}{\beta^2} + (z_1 - z_0) \frac{m}{\beta^2} \leq \frac{m}{\beta^2} < \frac{1}{\beta + 1}$. If $z_1 \leq z_0 = -1$, then $-z_0 \frac{m}{\beta^2} + (z_1 - z_0) \frac{m}{\beta^2} \leq \frac{m}{\beta^2} < \frac{1}{\beta + 1}$. This shows that $\lfloor rz + \alpha \rfloor = z_0 - z_1$ in these two cases.

If $z_1 > z_0 = -1$, then we have $-z_0 \frac{m}{\beta^2} + (z_1 - z_0) \frac{m}{\beta^2} \leq \frac{m}{\beta^2} > \frac{1}{\beta + 1}$. Finally, if $z_0 \leq -2$, then $-z_0 \frac{m}{\beta^2} + (z_1 - z_0) \frac{m}{\beta^2} \geq 2 \frac{m}{\beta^2} - (m-1) \frac{m}{\beta^2} > \frac{1}{\beta + 1}$, thus $\lfloor rz + \alpha \rfloor = z_0 - z_1 + 1$ in the latter cases. \hfill $\square$

Proof of Lemma 11. We have already seen above that $T^j_{-\beta}(x-y) + T^j_{-\beta}(y) - T^j_{-\beta}(x) \in \Phi(\tilde{\tau}^j_{r,\alpha}(0,0))$. As $(0,0) \in V$, it suffices to show that $\tilde{\tau}^j_{r,\alpha}(V) \subseteq V$.

Let $(z, h) \in V$ and $b \in B$ such that

$$h' = (z_1 - z_0 + \lfloor rz + \alpha \rfloor - h) m + [r_0 z_1 + r_1 h - r_1 \lfloor rz + \alpha \rfloor + \alpha] + b \in \{-1, 0, 1\},$$

Figure 1: The set $V$ for $m = 3$. Full points are represented by disks, points $z$ with $\{z\} \times \{0,1\} \subset V$, $\{z\} \times \{-1,0\} \subset V$ and $\{z\} \times \{1\} \subset V$ by upper half-disks, lower half-disks and circles respectively.
i.e., $(z_1, h - |rz + \alpha|, h') \in \tilde{\tau}_{r, \alpha}(z, h)$. If $(z_1, h - |rz + \alpha|)$ is full, then we clearly have 
$\tilde{\tau}_{r, \alpha}(z, h) \subset V$. Otherwise, we have to consider the possible values of $h'$. We distinguish
seven cases.

1. $z \in \{(0,0), (-1,-1)\}$: We have $|rz + \alpha| = 0$.
   
   If $m \geq 2$, then $(0, h)$ and $(-1, h)$ are full since $(0, -1) \in D_1$, $(0, 1), (-1, 1) \in A_1$, 
   $(−1,0) \in A_0$, and $(0, 0), (−1,−1)$ are also full.  
   If $m = 1$, then $(0,−1), (0,0), (−1,−1), (−1,0)$ are full. For $h = 1$, we have $h' = \neg1 + [r_0z_1 + r_1 + \alpha] + b = b - 2$, thus $h' = 1$. The points $(z_1,1)$ for $z_1 \in \{-1,0\}$ 
   are in $V$.

2. $z = (j,k) \in A_k$: We have $|rz + \alpha| = k - j$ for $j = −1$ and $|rz + \alpha| = j - k$ for 
   $0 \leq j < k$.
   
   If $k = 0$, then $z = (−1,0)$, and $(0,h)$ is full for $m \geq 2$. If $m = 1$, then $(0,0)$ and 
   $(0,−1)$ are full, and $h = 1$ gives that $h' = [r_1 + \alpha] + b = b - 1 \in \{-1, 0\}$, thus 
   $\tilde{\tau}_{r, \alpha}(z,1) \subset V$.
   
   If $1 \leq k < m$, then $(k, h + k), (k, h - j + k)$ lie in $B_k \cup C_k \cup \{(k,k+1)\}$ and are full. 
   If $k = m$, then we have either $h \in \{-1,0\}$, thus $(m, h + m)$ and $(m, h - j + m)$ 
   lie in the set of full points $B_m \cup C_m$, or $h = 1$ and $1 \leq j < m$, in which case 
   $(m, 1 - j + m) \in B_m$ is also full. (Note that $(-1, m, 1), (0, m, 1) \notin V$.)

3. $z = (k, j) \in B_k$: We have $|rz + \alpha| = k - j$, $1 \leq j \leq k$.
   
   For $h \in \{0,1\}$, the point $(j, h + j - k)$ is in $C_k$ and $C_{k-1} \cup B_k$ respectively, hence full.
   The point $(j, j - k - 1) \in C_{k+1}$ is full if $k < m$. Finally, if $k = m$ and $h = 1$, then 
   $h' = m + [r_0j + r_1(j - m - 1) + \alpha] + b = 2m + 1 + b = 1$, and $(j, j - m - 1,1) \in V$.

4. $z = (j, j - k) \in C_k$: We have $|rz + \alpha| = k$, $1 \leq j \leq k$.
   
   The point $(j - k, h - k)$, $h \in \{-1,0,1\}$, is in $D_{k+1}$, $D_k$, and $D_{k-1} \cup E_{k-1} \cup 
   \{(0,0),(−1,−1)\}$ respectively, hence full for all $k < m$, $k \leq m$, and $k \leq m + 1$ respectively. It remains to consider $h = −1, k = m$. For $1 \leq j \leq m - 3$, the point 
   $(j - m, −m - 1)$ is full; we have
   
   $h' = m + [r_0(j - m) − r_1(m + 1) + \alpha] + b = \begin{cases} 2m + 1 + b = 1 & \text{if } j = m, \\
   2m + b \in \{0,1\} & \text{if } j \in \{m - 2, m - 1\}, \end{cases}$
   
   $(0,−m - 1, 1) \in V$, and $\{(j - m, −m - 1)\} \times \{0,1\} \subset V$ for $\max(1, m - 2) \leq j < m$.

5. $z = (−j, k) \in D_k$: We have $|rz + \alpha| = k - j$ if $j \in \{0,1\}$, $|rz + \alpha| = k - j + 1$ if 
   $2 \leq j < k$.
   
   Let first $k = 1$, i.e., $z = (0,−1)$. The point $(-1, h - 1)$ lies in $D_2 \cup \{−1,−1\} \cup A_0$ and 
   is full, except if $m = 1, h = −1$; in the latter case, we have $h' = 1 + [−r_0 - 2r_1 + \alpha] + b = 
   b + 2 \in \{0,1\}$, and $\{((-1),−2)\} \times \{0,1\} \in V$.  

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For $2 \leq k \leq m$, the points $(-k, h+j-k)$, $j \in \{0,1\}$, and $(-k, h+j-k-1)$, $2 \leq j < k$, lie in $\{(−k,−i) : 0 \leq i \leq k+1\}$, and are full, except for $k = m = 2$, $h = −1$, $j = 0$; in the latter case, we have $h′ = 2 + [−2r_0−3r_1+α]+b = b+4 \in \{0,1\}$, and $\{(-2,−3)\} \times \{0,1\} \in V$.

Finally, for $k = m + 1$, we have $j = 0$, $h = 1$, or $1 \leq j \leq \min(m,2)$, $h \in \{0,1\}$, or $3 \leq j \leq m$, $h \in \{-1,0,1\}$, thus the points $(-m−1, h+j−m−1)$, $j \in \{0,1\}$, and $(-m−1, h+j−m−2)$, $2 \leq j \leq m$, lie in $\{(−m−1,−i) : \min(m−1,1) \leq i \leq m\}$ and are full, except for $m = j = h = 1$; in the latter case, we have $h′ = −1 + [−2r_0 + α] + b = b+2 = −1$, and $(-2,0,−1) \in V$.

6. $z = (−k,−j) \in E_k$: We have $|rz + α| = j − k + 1$, $2 \leq j \leq k$.

The point $(-j,h−j+k−1) \in F_{k−2} \cup F_{k−1} \cup F_k \cup \{−k,−2\}$ is full, except for $k = m + 1$, $h = 1$; in the latter case, we have $2 \leq j \leq m$, $h′ = [−r_0j + r_1(m−j+1) + α]+b = b+m \in \{-1,0\}$, and $\{(-j,m−j+1)\} \times \{-1,0\} \subset V$.

7. $z = (−j,k−j) \in F_k$: We have $|rz + α| = 1 − k$, $2 \leq j \leq k + 1$.

If $1 \leq k \leq m$, then the point $(k−j,h+k−1) \in A_{k−2} \cup A_{k−1} \cup A_k \cup \{k−2,k−2\}$ is full, except for $k = m$, $j \in \{m, m+1\}$, $h = 1$; in the latter case, we have $h′ = [r_0(m−j) + r_1m + α]+b = b−m \in \{-1,0\}$, and $\{(m−j,m)\} \times \{-1,0\} \subset V$.

If $k = m + 1$, then $2 \leq j \leq m$, $h \in \{-1,0\}$, or $m = 1$, $j = 2$, $h = −1$, and $(m+1−j,h+m) \in A_{m−1} \cup A_m \cup \{(m−1,m−1)\}$ is full.

Lemma 13. For the following chains of sets, $τ_{r,α}$ maps elements of a set into its successor:

\[ C_k \setminus \{(m+1,0)\} \to D_k \to E_k \to F_{k−1} \quad (3 \leq k \leq m + 1), \]
\[ F_{k+1} \to A_k \to B_k \to C_k \to D_k \quad (1 \leq k \leq m). \]

On the remaining $z = (z_0,z_1) \in \mathbb{Z}^2$ with $−m−1 \leq z_0 \leq m$, $−m−1 \leq z_1 \leq m$ and $|z_0 − z_1| \leq m + 1$, $τ_{r,α}$ acts by

\[ (0,−2) \mapsto (−2,−2) \mapsto (−2,−1) \mapsto (−1,0) \mapsto (0,0), \]
\[ (−1,−2) \mapsto (−2,−1), \quad (0,−1) \mapsto (−1,−1) \mapsto (−1,0). \]

Proof. This is a direct consequence of Lemma 12 except for $(-m−2,−1) \in F_{m+1}$; see also the proof of Lemma 11. As $\frac{1}{\beta+1} < \frac{(m+2)m}{\beta^2} + \frac{(m+1)m}{\beta^3} < 1 + \frac{m}{\beta^2} < 1 + \frac{1}{\beta+1}$, the proof of Lemma 12 shows that $τ_{r,α}(-m−2,−1) = (−1,m) \in A_m$.

Proposition 14. We have

\[ \max\{fr(x−y) : x,y \in \mathbb{Z}_β\} = 3m + \begin{cases} 3 & \text{if } m = 1 \text{ or } m \text{ is even}, \\ 4 & \text{if } m \geq 3 \text{ is odd}. \end{cases} \]
Proof. Let $k \geq 0$ be such that $\frac{x}{(-\beta)^{\frac{x}{\beta}}}, \frac{y}{(-\beta)^{\frac{y}{\beta}}}, \frac{x-y}{(-\beta)^{\frac{x-y}{\beta}}} \in (\ell_\beta, \ell_\beta + 1)$. Then $\text{fr}(x - y) = n$ is the minimal $n \geq 0$ such that $T_{k+1}(\frac{x-y}{(-\beta)^{\frac{x-y}{\beta}}}) = 0$. Let $z \in \mathbb{Z}^2$ be such that $T_{k}(\frac{x-y}{(-\beta)^{\frac{x-y}{\beta}}}) = \phi(z)$. Then $\text{fr}(x - y)$ is the minimal $n \geq 0$ such that $\tau_{k,n}(z) = 0$, and we have $(z, 0) \in V$, i.e.,

$$z \in \{(0, 0), (-1, -1), (0, -1)\} \cup \bigcup_{0 \leq k \leq m} (A_k \cup B_k \cup C_k \cup D_{k+1} \cup E_{k+1} \cup F_{k+1})$$

$$\setminus \{(0, -m - 1), (-m - 1, -m - 1), (-m - 2, -1), (-m - 1, 0)\}.$$

Therefore, $\text{fr}(x - y)$ is bounded by the maximal length of the path from $z$ to $(0, 0)$ given by Lemma 13.

For $1 \leq k \leq m/2$, the sets $F_{2k+1}, A_{2k}, B_{2k},$ and $C_{2k}$ are mapped to $D_2$ in $6k - 2, 6k - 3, 6k - 4,$ and $6k - 5$ steps respectively. For $2 \leq k \leq (m + 1)/2$, the sets $D_{2k}$ and $E_{2k}$ are mapped to $D_2$ in $6k - 6$ and $6k - 7$ steps respectively. The points $(0, -2)$ and $(-1, -2)$ in $D_2$ are mapped to $(0, 0)$ in 4 and 3 steps respectively.

Similarly, for $1 \leq k \leq (m + 1)/2$, the sets $F_{2k}, A_{2k-1}, B_{2k-1},$ and $C_{2k-1}$ are mapped to $D_1 = \{(0, -1)\}$ in $6k - 2, 6k - 3, 6k - 4,$ and $6k - 5$ steps respectively. For $1 \leq k \leq m/2$, the sets $D_{2k+1}$ and $E_{2k+1}$ are mapped to $D_1$ in $6k$ and $6k - 1$ steps respectively. Finally, the point $(0, -1) \in D_1$ is mapped to $(0, 0)$ in 3 steps.

For even $m$, the longest path comes thus from $D_{m+1}$ and has length $3m + 3$. For odd $m \geq 3$, the longest path comes from $F_{m+1}$ and has length $3m + 4$. For $m = 1$, the longest path comes from $A_1$ (since $F_2 \times \{0\} \cap V = \emptyset$ in this case) and has length 6. This proves the upper bound for $\text{fr}(x - y)$.

For $m = 1$, this bound is attained by $x = 1 - \beta, y = \beta^4 - \beta^3$, since $\text{fr}(x - y) = \text{fr}(\frac{1}{y} - \frac{1}{y}^3) = 6$. Assume in the following that $m \geq 2$. Then the points $(-m, -m - 1, 0) \in D_{m+1} \times \{0\}$ and $(-2, m - 1, 0) \in F_{m+1} \times \{0\}$ are in $\tau_{k,n}(0,0,0)$ for sufficiently large $j$ because they can be attained from $(0,0,0)$ by transitions

$$(z, h) \xrightarrow{b} (z_1, h - [rz + \alpha], (z_1 - z_0 + [rz + \alpha] - h) m + [r_0 z_1 + r_1 h - r_1 [rz + \alpha] + \alpha] + b)$$

with $b \in B$ by the following paths (cf. the proof of Lemma 11):

$$(0, k, 0) \xrightarrow{1} (k, k, -1) \xrightarrow{m-k+2} (k, -1, -1) \xrightarrow{m-k+2} (-1, -k - 2, -1), \quad (0 \leq k \leq m - 2)$$

$$(-1, -k, -1) \xrightarrow{m} (-k, -k, 1) \xrightarrow{k} (-k, 0, 1) \xrightarrow{k} (0, k, 0), \quad (2 \leq k \leq m)$$

$$(0, m - 1, 0) \xrightarrow{1} (m - 1, m - 1, -1) \xrightarrow{2m} (m - 1, -1, 0) \xrightarrow{m} (-1, -m, -1),$$

$$m \xrightarrow{0} (m, m, 0) \xrightarrow{m} (m, 0, 0) \xrightarrow{m-1} (0, -m, -1),$$

$$(0, -m, -1) \xrightarrow{m-2} (-m, -m - 1, 0) \xrightarrow{1} (-m - 1, -2, 1) \xrightarrow{m} (-2, -m, 0).$$

For even $m \geq 2$, these paths correspond to

$$\text{fr}(000022 000044 \cdots 00000m 0000 \bullet 0^2 - 022000 044000 \cdots 0mm000 0012 \bullet 0^2)$$

$$= \text{fr}((1mm000)^{m/2} 0mm \bullet d_{-\beta}(\phi(-m, -m - 1))) = \text{fr}(\phi(-m, -m - 1)) = 3m + 3;$$
for the second equality, we have used that \((1mm00m00m) \cdot d_{-\beta}(\phi(-m, -m - 1))\) is a \((-\beta)\)-expansion. Indeed, this follows from the lexicographic conditions given in [17] since \(d_{-\beta}(\ell_\beta) = m\ell_m\) and \(d_{-\beta}(\phi(-m, -m - 1))\) starts with 2 (as \(-\beta\phi(-m, -m - 1) = \phi(-m - 1, -2)\)). For odd \(m \geq 3\), we have

\[
\text{fr}(000022000044 \cdots 0000(m-1)(m-1)0000mm0000m \cdot 0^\omega - 02000044000 \cdots 0(m-1)(m-1)0000mm00012000 \cdot 0^\omega) = \text{fr}((1mm00m)0(m+1)/2 \cdot d_{-\beta}(\phi(-2, m - 1))) = \text{fr}(\phi(-2, m - 1)) = 3m + 4.
\]

This concludes the proof of the proposition. 

\[\square\]

**Proposition 15.** We have

\[
\max\{\text{fr}(x + y) : x, y \in \mathbb{Z}_{-\beta}\} \leq \max\{\text{fr}(x - y) : x, y \in \mathbb{Z}_{-\beta}\}.
\]

**Proof.** Let \(\mu = \max\{\text{fr}(x - y) : x, y \in \mathbb{Z}_{-\beta}\}\). For \(x, y \in \mathbb{Z}_{-\beta}\), \(\text{fr}(x + y)\) is the minimal \(n \geq 0\) such that \(T_{-\beta}^k(x + y) = 0\), with \(k \geq 0\) such that \(\frac{x}{\beta}, \frac{y}{\beta}, \frac{x+y}{\beta} \in (\ell_\beta, \ell_\beta + 1)\). By Lemma [11], we have

\[
T_{-\beta}^j(\frac{x}{\beta}) + T_{-\beta}^j(\frac{y}{\beta}) \geq \text{fr}(\phi(0,0,0,0,\ldots,0)) = \text{fr}(\phi(0,0,0,0,\ldots,0)),
\]

for all \(j \geq 0\), thus \(T_{-\beta}^k(\frac{x+y}{\beta}) \in -\Phi(V)\). Therefore, we have \(T_{-\beta}^k(\frac{x+y}{\beta}) = \phi(z) = -\Phi(-z, h)\) for some \(z = (z_0, z_1) \in \mathbb{Z}^2\) and \(h \in \{0, 1\}\) with \((-z, h) \in V\).

If \((z, 0) \in V\), then the proof of Proposition [14] shows that \(\tau_{\epsilon, \alpha}(z) = 0\), thus \(\text{fr}(x+y) \leq \mu\).

Assume now that \((z, 0) \notin V\). Then

\[-z \in D_{m+1} \cup \{(-m - 1, -j) : 1 \leq j \leq m\} \cup \{(m - j + 1) : 1 \leq j \leq m\}.
\]

We can exclude \(-z = (-j, m - j + 1), 1 \leq j \leq m\), because this would imply \(h = 0\) and

\[-\Phi(-z, h) = 1 - \frac{j m}{\beta} - \frac{(m+1)m}{\beta^2} \geq \frac{m^2}{\beta^2} > \frac{1}{\beta + 1}.
\]

This means that \(z \in (A_{m+1} \cup B_{m+1}) \setminus \{(-1, m + 1), (m + 1, m + 1)\}\). With the notation of Lemma [13], we have

\[A_{m+1} \setminus \{(-1, m + 1)\} \to B_{m+1} \setminus \{(m + 1, m + 1)\} \to C_m,
\]

where we have used Lemma [12] and that \(\lfloor r_0(m+1) + r_1j + \alpha \rfloor = m - j\) for \(1 \leq j \leq m\), as

\[
-\frac{\beta}{\beta + 1} < \frac{m^2}{\beta^2} - \frac{m^2}{\beta^2} = \frac{m^2 - m^2}{\beta^2} = \frac{(j - m)m}{\beta^2} = \frac{m^2}{\beta^2} < \frac{1}{\beta + 1}.
\]

Hence, the points in \(A_{m+1} \setminus \{(-1, m + 1)\}\) and \(B_{m+1} \setminus \{(m + 1, m + 1)\}\) are mapped to \((0, 0)\) in the same number of steps as those in \(A_m\) and \(B_m\) respectively, thus \(\text{fr}(x+y) \leq \mu\). 

\[\square\]
Now, Theorem 9 is an immediate consequence of Propositions 14 and 15.

**Remark 16.** It is also possible to determine the exact value of $\max\{\text{fr}(x+y) : x, y \in \mathbb{Z}_{-\beta}\}$ in the same fashion as in the proof of Proposition 14; we have

$$\max\{\text{fr}(x+y) : x, y \in \mathbb{Z}_{-\beta}\} = 3m + \begin{cases} 1 & \text{if } m = 2, \\ 2 & \text{if } m \geq 4 \text{ is even}, \\ 3 & \text{if } m = 1, \\ 4 & \text{if } m \geq 3 \text{ is odd}. \end{cases}$$

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