The noncommutative schemes of generalized Weyl algebras

Robert Won
Wake Forest University

AMS Western Sectional Meeting, Denver, CO
Noncommutative projective schemes

- $k$ algebraically closed, characteristic 0.
- $A$ a noetherian connected graded $k$-algebra

$$A = k \oplus A_1 \oplus A_2 \oplus \cdots$$

- Let $\text{gr-} A$ be the category of finitely generated graded $A$-modules. Consider the quotient category

$$\text{qgr-} A = \text{gr-} A / \text{fdim-} A.$$

**Definition (Artin-Zhang, 1994)**

The noncommutative projective scheme $\text{Proj}_{\text{NC}} A$ is the triple

$$(\text{qgr-} A, A, S).$$
The first Weyl algebra

• Recall the first Weyl algebra:

\[ A_1 = \mathbb{k}\langle x, y \rangle/(xy - yx - 1). \]

• Simple noetherian domain of GK dim 2.
• \( A_1 \) is not connected graded.
• Ring of differential operators on \( \mathbb{k}[t] \)
  • \( x \mapsto t \cdot x \)
  • \( y \mapsto d/dt \)
• \( A_1 \) is \( \mathbb{Z} \)-graded by \( \deg x = 1, \deg y = -1 \)
• \( (A_1)_0 \cong \mathbb{k}[xy] \neq \mathbb{k} \)
Sierra (2009)

• Sue Sierra, *Rings graded equivalent to the Weyl algebra*
• Examined the graded module category $\text{gr-}A_1$:

```
-3 -2 -1  0  1  2  3
```

• For each $\lambda \in \mathbb{k} \setminus \mathbb{Z}$, one simple module $M_\lambda$.
• For each $n \in \mathbb{Z}$, two simple modules, $X\langle n \rangle$ and $Y\langle n \rangle$.
• Classified all rings graded Morita equivalent to $A_1$. 
Sierra (2009)

Computed Pic(gr-A₁), the Picard group of gr-A₁ (autoequivalences modulo natural isomorphism).

**Theorem (Sierra)**
There exist ιₙ, autoequivalences of gr-A₁, permuting X⟨n⟩ and Y⟨n⟩ and fixing all other simple modules.

Also ιᵢιⱼ = ιⱼιᵢ and ιₙ² ∼= Id_{gr-A₁}, so the ιₙ form subgroup isomorphic to

\[
\left(\mathbb{Z}/2\mathbb{Z}\right)^{\mathbb{Z}} \cong \mathbb{Z}_{\text{fin}}.
\]
Smith (2011)

- Paul Smith, *A quotient stack related to the Weyl algebra*
- Constructs a \( \mathbb{Z}_{\text{fin}} \)-graded ring

\[
C := \bigoplus_{J \in \mathbb{Z}_{\text{fin}}} \text{Hom}_{\text{gr-}A}(A, \iota_J A) \cong \frac{\mathbb{L}k[x_n \mid n \in \mathbb{Z}]}{(x_n^2 + n = x_m^2 + m \mid m, n \in \mathbb{Z})} \cong \mathbb{L}k[z][\sqrt{z - n} \mid n \in \mathbb{Z}]
\]

where \( \deg x_n = \{n\} \).
- \( C \) is commutative, integrally closed, non-noetherian PID.

**Theorem (Smith)**

There is an equivalence of categories

\[
\text{Gr-}A_1 \equiv \text{Gr-}(C, \mathbb{Z}_{\text{fin}}).
\]
• $\text{Gr}-A_1 \equiv \text{Gr}-(C, \mathbb{Z}_{\text{fin}}) \equiv \text{Qcoh}(\chi)$.

• Here, $\chi$ is a quotient stack “whose coarse moduli space is the affine line $\text{Spec } \mathbb{k}[z]$, and whose stacky structure consists of stacky points $\mathbb{BZ}_2$ supported at each integer point.”
Z-graded rings

Theorem (Artin-Stafford, 1995)
Let $A$ be a f.g. connected $\mathbb{N}$-graded domain generated in degree 1 with $\text{GKdim}(A) = 2$. Then there exists a projective curve $X$ such that

$$\text{qgr-}A \equiv \text{coh}(X).$$

Theorem (Smith, 2011)
$A_1$ is a f.g. $\mathbb{Z}$-graded domain with $\text{GKdim}(A_1) = 2$. There exists a commutative ring $C$ and quotient stack $\chi$ such that

$$\text{gr-}A_1 \equiv \text{gr}(C, \mathbb{Z}_{\text{fin}}) \equiv \text{coh}(\chi).$$
Rings from autoequivalences

• Smith’s construction:

\[ C := \bigoplus_{J \in \mathbb{Z}_{\text{fin}}} \text{Hom}_{\text{gr-}A}(A, \iota_J A) \]

• \( S \) a \( G \)-graded ring, \( \Gamma \subseteq \text{Pic}(\text{gr-}(S, G)) \) an abelian subgroup.

• For each \( \gamma \in \Gamma \), choose \( \mathcal{F}_\gamma \in \text{Aut}(\text{gr-}(S, G)) \).

• When is

\[ R = \bigoplus_{\gamma \in \Gamma} \text{Hom}_{\text{gr-}(S, G)}(S, \mathcal{F}_\gamma S) \]

an associative ring?
Rings from autoequivalences

- \( \mathcal{F}_\gamma \mathcal{F}_\delta \cong \mathcal{F}_\delta \mathcal{F}_\gamma \cong \mathcal{F}_{\gamma+\delta} \)
- Choose natural isomorphisms \( \eta_{\gamma,\delta} : \mathcal{F}_\gamma \mathcal{F}_\delta \to \mathcal{F}_{\gamma+\delta} \) and let
  \[
  \Theta_{\gamma,\delta} = (\eta_{\gamma,\delta})_S.
  \]
- If
  \[
  \Theta_{\epsilon,\gamma+\delta} \circ \mathcal{F}_\epsilon (\Theta_{\delta,\gamma}) \circ \mathcal{F}_\epsilon \mathcal{F}_\delta (\varphi) = \Theta_{\delta+\epsilon,\gamma} \circ \mathcal{F}_{\delta+\epsilon} (\varphi) \circ \Theta_{\epsilon,\delta}
  \]
  then \( R \) is an associative \( \Gamma \)-graded ring.
- (Morally, this says
  \[
  \text{Aut(gr-(S, G))} \times \text{Aut(gr-(S, G))} \to \text{Aut(gr-(S, G))}
  \]
  \[
  (\mathcal{F}_\gamma, \mathcal{F}_\delta) \mapsto \mathcal{F}_{\gamma+\delta}
  \]
  is associative.)
Rings from autoequivalences

Proposition

Assume setup and notation above. If $P = \bigoplus_{\gamma \in \Gamma} \mathcal{F}_\gamma S$ is a projective generator of $\text{gr-}(S, G)$ then

$$\text{gr-}(R, \Gamma) \equiv \text{gr-}(S, G).$$

- If $G = \mathbb{Z}$, take $\Gamma = \langle S \rangle \cong \mathbb{Z}$.
- Enough autoequivalences in $\Gamma$ so that $P$ generates.
- Few enough autoequivalences in $\Gamma$ so $R$ is nice.
Generalized Weyl algebras (GWAs)

- V. Bavula, *Generalized Weyl algebras and their representations* (1993)
- $D$ a ring; $\sigma \in \text{Aut}(D)$; $a \in \mathbb{Z}(D)$
- The generalized Weyl algebra $D(\sigma, a)$ with base ring $D$

$$D(\sigma, a) = \frac{D\langle x, y \rangle}{\left( \begin{array}{ll}
xy &= a \\
yx &= \sigma(a) \\
dx &= x\sigma(d), d \in D \\
dy &= y\sigma^{-1}(d), d \in D
\end{array} \right)}$$

Theorem (Bell-Rogalski, 2015)

Every simple $\mathbb{Z}$-graded domain of GKdim 2 is graded Morita equivalent to a GWA.
Generalized Weyl algebras

- For us, $D = \mathbb{k}[z]$; $\sigma(z) = z - 1$; $a = f(z)$

$$A(f) \cong \frac{\mathbb{k}\langle x, y, z \rangle}{\begin{pmatrix} xy = f(z) & yx = f(z - 1) \\ xz = (z + 1)x & yz = (z - 1)y \end{pmatrix}}$$

- Two roots $\alpha, \beta$ of $f(z)$ are congruent if $\alpha - \beta \in \mathbb{Z}$

Example (The first Weyl algebra)

Take $f(z) = z$

$$D(\sigma, a) = \frac{\mathbb{k}[z]\langle x, y \rangle}{\begin{pmatrix} xy = z & yx = z - 1 \\ zx = x(z - 1) & zy = y(z + 1) \end{pmatrix}} \cong \frac{\mathbb{k}\langle x, y \rangle}{(xy - yx - 1)} = A_1.$$
Generalized Weyl algebras

Properties of $A(f)$:

- Noetherian domain.
- Krull dimension 1.
- Simple if and only if no congruent roots.
- \[
\text{gl.dim } A(f) = \begin{cases} 
1, & f \text{ has neither multiple nor congruent roots} \\
2, & f \text{ has congruent roots but no multiple roots} \\
\infty, & f \text{ has a multiple root.}
\end{cases}
\]
- We can give $A(f)$ a $\mathbb{Z}$ grading by $\deg x = 1$, $\deg y = -1$, $\deg z = 0$. 
Noncommutative schemes of GWAs

Let $f = z(z + \alpha)$.

If $\alpha \in \mathbb{k} \setminus \mathbb{Z}$ (distinct roots):

If $\alpha = 0$ (double root):

If $\alpha \in \mathbb{N}^+$ (congruent roots):
Noncommutative schemes of GWAs

Theorem(s)

In all cases, there exist numerically trivial autoequivalences $\iota_n$ permuting “$X\langle n\rangle$” and “$Y\langle n\rangle$” and fixing all other simple modules.
Noncommutative schemes of GWAs

Let \( \alpha \in k \setminus \mathbb{Z} \). Define a \( \mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}} \) graded ring \( C \):

\[
C = \bigoplus_{(J,J') \in \mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}}} \text{Hom}_{\text{gr}-A} (A, \iota_{(J,J')A})
\]

\[
\cong \frac{k[a_n, b_n \mid n \in \mathbb{Z}]}{(a_n^2 + n = a_m^2 + m, a_n^2 = b_n^2 + \alpha \mid m, n \in \mathbb{Z})}
\]

with \( \text{deg } a_n = (\{n\}, \emptyset) \) and \( \text{deg } b_n = (\emptyset, \{n\}) \).

Theorem

There is an equivalence of categories \( \text{gr}(C, \mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}}) \cong \text{gr-}A \).
Noncommutative schemes of GWAs

Let $\alpha = 0$. Define a $\mathbb{Z}_{\text{fin}}$-graded ring $B$:

$$B = \bigoplus_{J \in \mathbb{Z}_{\text{fin}}} \text{Hom}_{\text{gr-}A}(A, \nu_J A) \cong \frac{\mathbb{k}[z][b_n \mid n \in \mathbb{Z}]}{(b_n^2 = (z + n)^2 \mid n \in \mathbb{Z})}. $$

**Theorem**

$B$ is a reduced, non-noetherian, non-domain of Kdim 1 with uncountably many prime ideals.

**Theorem**

There is an equivalence of categories $\text{gr-}(B, \mathbb{Z}_{\text{fin}}) \equiv \text{gr-}A$. 
Noncommutative schemes of GWAs

- In congruent root \((\alpha \in \mathbb{N}^+)\), we have finite-dimensional modules.
- Consider the **quotient category**

\[
qgr-A = gr-A/\text{fdim}-A.
\]

**Theorem**

There is an equivalence of categories \(\text{gr-}(B, \mathbb{Z}_{\text{fin}}) \equiv qgr-A\).
Noncommutative schemes of GWAs

To summarize:

**Theorem**
For all quadratic $f$, there exists a commutative ring $B$ such that
\[ \text{gr-}(B, \mathbb{Z}_\text{fin}) \equiv \text{qgr-}A(f). \]
Questions

• \( \mathbb{Z}_{\text{fin}} \)-grading on \( B \) gives an action of \( \text{Spec} \, k\mathbb{Z}_{\text{fin}} \) on \( \text{Spec} \, B \)

\[
\chi = \left[ \frac{\text{Spec} \, B}{\text{Spec} \, k\mathbb{Z}_{\text{fin}}} \right]
\]

• Other \( \mathbb{Z} \)-graded domains of GK dimension 2? Other GWAs?
  • \( U(\mathfrak{sl}(2)) \), Down-up algebras, quantum Weyl algebra, Simple \( \mathbb{Z} \)-graded domains

• Opposite view: \( \mathcal{O} \) a quasicoherent sheaf on \( \chi \):

\[
\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\text{Qcoh}(\chi)}(\mathcal{O}, S^n \mathcal{O})
\]