Abstract. The generation and evolution of magnetic fields in the plasma accreting into a rotating black hole is studied in the 3+1 split of the Kerr metric. Attention is focused on effects of the gravitomagnetic potential. The gravitomagnetic force appears as battery term in the generalized Ohm’s law. The gravitomagnetic battery is likely to saturate at higher field strength than the classical Biermann battery.

The coupling of the gravitomagnetic potential with electric fields appears as gravitomagnetic current in Maxwell’s equations. In the magnetohydrodynamic induction equation, this current re-appears as source term for the poloidal magnetic field, which can produce closed magnetic structures around an accreting black hole. In principle, even self-excited axisymmetric dynamo action is possible, which means that Cowling’s anti-dynamo theorem does not hold in the Kerr metric.

Finally, simulations of the $\alpha\Omega$ dynamo in accretion flows into the hole are presented. I assume a simple expression of $\alpha$ in this relativistic context. The modes of the dynamo are oscillating for dynamo numbers which are typical for accretion disks. In a zero angular momentum flow into a Kerr black hole there is still shear, i.e. the angular velocity $\Omega$ of the plasma equals the angular velocity of space, $\omega$, and it has been speculated (Meier 1998) that even then a dynamo could operate. This is shown to be unlikely due to the rapid accretion.

1. Introduction

The influence of the gravitomagnetic field of a rotating compact object on electromagnetic fields has been studied for some 25 years (Wald 1974, Ruffini & Wilson 1975, Blandford & Znajek 1977). The coupling of the gravitomagnetic potential with a magnetic field results in an electromotive force. Currents driven by this electromotive force may extract rotational energy from a black hole. This energy could power relativistic jets by Poynting flux (but see also, e.g., Punsly 1996). Cast in the language of the 3+1 split of the Kerr metric, Maxwell’s equations, together with the ‘ingoing wave boundary condition’ for electromagnetic fields at the horizon, led to the Membrane Paradigm (Thorne et al. 1986). Since a black hole does not carry its own magnetic field, not to mention kGauss fields required for the Blandford-Znajek process to be efficient, strong magnetic fields
must either be accreted into the black hole from the outer accretion disk, or have to be generated and amplified in the plasma surrounding the black hole.

The generation of magnetic fields by a battery operating in the plasma close to a rotating black hole was studied by Khanna (1998b). It was shown that the gravitomagnetic force may play a crucial role in the battery. Khanna & Camenzind (1996a) studied the possibility of an axisymmetric gravitomagnetic dynamo (or \(\omega \Omega\) dynamo), in which the coupling between the gravitomagnetic potential and an electric field is a source for the poloidal magnetic field. This theoretical result invalidates Cowling’s anti-dynamo theorem (see also Núñez 1997), but self-excited growing dynamo modes could not yet be numerically verified for simple kinematics (Khanna & Camenzind 1996b). Egi et al. (1998) have given a criterion for growing modes of the \(\omega \Omega\) dynamo, i.e. that the Poynting flux from close to the horizon (extracted rotational energy of the hole) be positive.

Recently, Meier (1998) speculated that, in a zero angular momentum flow into a Kerr black hole, alternatively to the \(\omega \Omega\) dynamo, an \(\alpha \omega\) dynamo might operate. My simulations of such a scenario show that, due to the relativistic accretion velocity, an \(\alpha \omega\) dynamo is not very likely.

Section 2. gives an introduction to the derivation of MHD in the 3+1 split of the Kerr metric. The generalized Ohm’s law for an electron-ion plasma is presented in Sec. 2.1. The gravitomagnetic battery (along with the relativistic equivalent of Biermann’s battery) are discussed in Sec. 2.2. In Sec. 3. the MHD induction equation in the 3+1 split of the Kerr metric is presented. Applications are the gravitomagnetic dynamo (Sec. 3.1. and 3.2.) and, in Sec. 3.3., the \(\alpha \Omega\) dynamo, or the \(\alpha \omega\) dynamo, respectively. Throughout the paper I set \(G = 1 = c\).

2. The MHD description of an electron-ion plasma

The formulation of MHD requires the relativistic definition of a plasma as center-of-mass fluid of its components (Khanna 1998a). The plasma is assumed to be a perfect fluid and is defined by the sum of the ion and electron stress-energy tensors, which contain a collisional coupling term:

\[
(p'_m + p') W^\alpha W^\beta + p' g^{\alpha\beta} \equiv T^{\alpha\beta} = \sum_{x=i,e} (p'^{\alpha}_{mx} + p'^{\beta}_{x}) W^\alpha W^\beta + p'^{\alpha}_{x} g^{\alpha\beta} + T^{\alpha\beta}_{x\text{coll}} .
\] (1)

Subscripts \(i, e\) refer to ion and electron quantities, respectively. Superscripts denote the rest-frame in which the quantity is defined, where ‘\(\cdot\)’ refers to the plasma rest-frame. In the 3+1 split (into hypersurfaces of constant Boyer-Lindquist time \(t\), filled with stationary zero angular momentum fiducial observers) \(T^{\alpha\beta}\) splits into the total density of mass-energy \(\epsilon\) and momentum density \(\vec{S}\):

\[
\epsilon \equiv (\rho'_m + p' v^2) \gamma^2 \approx \rho'_m \gamma^2 \quad \vec{S} \equiv (\rho'_m + p') \gamma^2 \vec{v} \approx \rho'_m \gamma^2 \vec{v} \quad \text{and} \quad \text{the stress-energy tensor of 3-space with metric } \hat{h}
\]

\[
\hat{T} \equiv (\rho'_m + p') \gamma^2 \hat{v} \otimes \vec{v} + p' \hat{h} \approx \rho'_m \gamma^2 \hat{v} \otimes \vec{v} + p' \hat{h} .
\] (3)

The approximate expressions hold for a ‘cold’ plasma. Charge density and current density are given by

\[
\rho_c \equiv \rho_{ci} + \rho_{ce} = Z e n_i \gamma_i - e n_e \gamma_e \quad \vec{j} \equiv \vec{j}_i + \vec{j}_e = Z e n_i \gamma_i \vec{v}_i - e n_e \gamma_e \vec{v}_e .
\] (4)
All quantities resulting from the split are measured locally by FIDOs.

### 2.1. The generalized Ohm’s law in the 3+1 split of the Kerr metric

In the ‘cold’ plasma limit, the local laws of momentum conservation for each species can be re-written as equations of motion, which can then be combined to yield the generalized Ohm’s law for an electron-ion plasma

\[
\frac{j}{\sigma \gamma_e} \approx \vec{E} + \frac{Z n_i \gamma_i \vec{v} \times \vec{B}}{n_e \gamma_e} - \frac{\vec{j} \times \vec{B}}{\sigma \gamma_e} + \nabla (\alpha_g p_e^0) + 4 \pi \gamma_e \rho_c \vec{g} + \frac{\rho_c^i \gamma \vec{v}}{\sigma \gamma_e}
- \frac{4 \pi e (Z n_i \gamma_i^2 - n_e \gamma_e^2)}{\omega_{pe}^2 \gamma_e^2} \left( \frac{d (\gamma_e \vec{v})}{d \tau_p} - \vec{H} \cdot (\gamma_e^2 \vec{v}) \right),
\]

with the conductivity \( \sigma = e^2 n_e / m_e \nu_c \equiv \omega_{pe}^2 / 4 \pi \nu_c \) as measured in the plasma rest frame, the electron plasma frequency \( \omega_{pe} \), the factor of gravitational red-shift \( \alpha_g \) (with \( \vec{g} = -\nabla \ln \alpha_g \)) and the gravitomagnetic tensor field \( \vec{\beta} = \alpha_g^{-1} \vec{H} \), \( \vec{\beta} = \beta^\phi \vec{e}_\phi \equiv -\omega \vec{e}_\phi \) is the gravitomagnetic potential, which drags space into differential rotation with angular velocity \( \omega \). Note that, in the single fluid description, the gravitomagnetic force drives currents, only if the plasma is charged in its rest frame.

\( \tau_p \) is the proper time in the plasma rest frame. The derivation requires the assumption that the species are coupled sufficiently strong that their bulk accelerations

\[
\frac{d (\gamma_e \vec{v})}{d \tau_x} \equiv \left[ \frac{\gamma_e}{\alpha_g} \frac{\partial}{\partial t} + \gamma_e \left( \vec{v} - \frac{\vec{\beta}}{\alpha_g} \right) \cdot \vec{\nabla} \right] (\gamma_e \vec{v})
\]

are synchronized. The same is required for the gravitomagnetic accelerations, i.e. \( | \vec{H} \cdot (\gamma_e^2 \vec{v}) - \vec{H} \cdot (\gamma_e^2 \vec{v}) | \ll | \vec{H} \cdot (\gamma_e^2 \vec{v}) | \). If the MHD-assumption of “synchronized accelerations” is not made, Ohm’s law contains further current acceleration terms, inertial terms and gravitomagnetic terms (Khanna 1998a), which may be important for collisionless reconnection and particle acceleration along magnetic fields. This topic will be discussed elsewhere.

In the limit of quasi-neutral plasma \( (Z n_i \approx n_e) \) and \( \gamma_e \approx \gamma_i \approx \gamma \), Eq. (5) reduces to

\[
\vec{j} \approx \sigma \gamma (\vec{E} + \vec{v} \times \vec{B}) - \frac{\sigma}{e n_e} (\vec{j} \times \vec{B}) + \frac{\sigma}{e n_e \alpha_g} \nabla (\alpha_g p_e^0),
\]

which contains all the terms, familiar from the non-relativistic generalized Ohm’s law, but no gravitomagnetic terms.

### 2.2. The gravitomagnetic battery

The generation of magnetic fields by a plasma battery was originally devised by Biermann (1950) for stars. He showed that, if the centrifugal force acting on a rotating plasma does not possess a potential, the charge separation owing to the electron partial pressure cannot be balanced by an electrostatic field, and thus currents must flow and a magnetic field is generated.
In Khanna (1998b) I have re-formulated Biermann's theory in 3+1 split of the Kerr metric. The base of this battery theory is Ohm's law of eq. (7). Assuming that electrons and ions have non-relativistic bulk velocities in the plasma rest frame, superscripts $i, e$ can be dropped. With $p = p_i + p_e = (n_i + n_e)kT$, the impressed electric field (IEF), $\vec{E}^{(i)} = \nabla (\alpha_g p_e) / en\gamma \alpha_g$, can be re-expressed with the aid of the equation of motion for a 'cold' quasi-neutral plasma to yield

$$\vec{E}^{(i)} = \frac{m_i}{(Z+1)e} \left( \gamma \vec{g} \cdot (\gamma \vec{v}) - \frac{d(\gamma \vec{v})}{d\tau} \right) + \frac{Z \left( \vec{j} \times \vec{B} + (\vec{v} \cdot \vec{j}) \vec{E} \right)}{(Z+1)en\gamma}.$$ (8)

$\tau$ is the proper time in a FIDO frame; i.e. $d/d\tau = \gamma d/d\tau$. The criterium for magnetic field generation is that $\nabla \times \alpha_g \vec{E}^{(i)} \neq 0$. Here I restrict the discussion to the gravitomagnetic IEF $\vec{E}_{gm}^{(i)}$. The function part of $\alpha_g \vec{E}_{gm}^{(i)}$ is

$$\left( \vec{\beta} \cdot \nabla + \vec{v} \cdot \vec{\beta} \right) \cdot (\gamma \vec{v}) = \left( \beta^i (\gamma v^j)_{\mid i} + \gamma \beta^i (v^j)_{\mid i} \right) \vec{e}_j$$

$$= -\gamma v^\phi \omega^2 \vec{\nabla} \omega - \omega \left( (\gamma v^r)_{\mid \phi} \vec{e}_r + (\gamma v^\phi)_{\mid \phi} \vec{e}_\phi \right),$$ (9)

where $\tilde{\omega} = (h_{\phi\phi})^{1/2}$ and $|$ denotes the covariant derivative in 3-space. In axisymmetry $\alpha_g \vec{E}_{gm}^{(i)}$ is clearly rotational, unless some freak $\gamma$ should manage to make $\gamma v^\phi \omega^2$ a function of $\omega$ alone. Thus the gravitomagnetic force drives a poloidal current and generates a toroidal magnetic field. Only if $v^\phi$ is non-axisymmetric, the gravitomagnetic IEF drives toroidal currents. The total IEF $(\alpha_g \vec{E}_{gm}^{(i)} + \alpha_g \vec{E}_{class}^{(i)})$ is likely to rotational in general. This will be quantified for specific velocity fields elsewhere.

In presence of a weak poloidal magnetic field the Biermann battery is limited due to modifications of the rotation law by the Lorentz force, rather than by ohmic dissipation. Then the contribution of the centrifugal force to the IEF becomes irrotational already at weak toroidal fields (Mestel & Roxburgh 1962). The gravitomagnetic battery term, on the other hand, is only linearly dependent on $\vec{v}$. The equilibrium field strength should therefore be higher than for the Biermann battery.

3. The MHD induction equation in the 3+1 split of the Kerr metric

In this section I review the axisymmetric dynamo equations in the 3+1 split of the Kerr metric (Khanna & Camenzind 1996a). Ohm’s law is assumed to be of the standard form for a quasi-neutral plasma; Hall-term and IEF are neglected. Combining Maxwell’s equations (Thorne et al. 1986) with Ohm’s law yields the MHD induction equation

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left( (\alpha_g \vec{v} \times \vec{B}) - \frac{\eta}{\gamma} \left( \nabla \times (\alpha_g \vec{B}) + (\vec{E}_p \cdot \nabla) \tilde{\omega} \vec{e}_\phi \right) \right) + (\vec{B}_p \cdot \nabla) \tilde{\omega} \vec{e}_\phi.$$ (10)

The term standing with the magnetic diffusivity $\eta$ is the current density, which, via Ampère’s law, contains the coupling of the gravitomagnetic field with the
electric field. In axisymmetry this is simply the shear of the poloidal electric field in the differential rotation of space, \( \omega \). Another induction term is the shear of the poloidal magnetic field by \( \omega \). This generates toroidal magnetic field out of poloidal magnetic field even in a zero-angular-momentum flow.

3.1. The gravitomagnetic dynamo

Introducing the flux \( \Psi \) of the poloidal magnetic field and the poloidal current \( T \)

\[
\Psi = \frac{1}{2\pi} \int \vec{B}_p \cdot d\vec{A} = \dot{\omega} A^\phi \quad T = 2 \int \alpha_g \dot{\vec{J}}_p \cdot d\vec{A} = \alpha_g \dot{\omega} B^\phi, \tag{11}
\]

where \( A^\phi \) is the toroidal component of the vector potential, eq. (10) splits into

\[
\frac{\partial \Psi}{\partial t} + \frac{\eta \omega}{\gamma \alpha_g} \left( \vec{V}_p \cdot \vec{\nabla} \right) \Psi - \frac{\eta \omega}{\gamma} \vec{V}_p \cdot \vec{\nabla} \Psi - \frac{\eta \omega^2}{\gamma} \vec{V}_p \cdot \left( \frac{\alpha_g}{\omega^2} \vec{\nabla} \Psi \right) = \frac{\eta \omega}{\gamma \alpha_g} \left( \vec{V}_p \cdot \vec{\nabla} \right) \Psi \tag{12}
\]

\[
\frac{\partial T}{\partial t} + \alpha_g \left( \vec{V}_p \cdot \vec{\nabla} \right) T + \alpha_g \omega^2 \left( \vec{V}_p \cdot \vec{\nabla} \right) T - \alpha_g \omega^2 \vec{V}_p \cdot \left( \frac{\eta \omega^2}{\gamma} \vec{\nabla} T \right) = \alpha_g \omega \left( \vec{V}_p \times \vec{e}_\phi \right) \cdot \vec{\nabla} \Omega. \tag{13}
\]

These equations are the relativistic equivalent of the classical axisymmetric dynamo equations. It is important to note, however, that no mean-field approach was made, but \( \Psi \) has source terms anyway. They result from \( \vec{E}_p \cdot \vec{\nabla} \omega \). Obviously, Cowling’s anti-dynamo theorem does not hold close to a rotating black hole. Growing modes of this gravitomagnetic dynamo were shown to exist for steep gradients of the plasma angular velocity \( \Omega \) (Núñez 1997). According to Egi et al. (1998) growing modes require that the Poynting flux carrying rotational energy extracted from the hole be positive. For simple accretion scenarios, growing modes could not be found in kinematic numerical simulations (Khanna & Camenzind 1996b). If, however, magnetic field is replenished by an outer boundary condition, the gravitomagnetic source terms generate closed loops around the black hole.

3.2. The magnetic field structure in the accretion disk close to the hole

In an accretion disk, magnetic fields may be advected into the near-horizon area, where gravitomagnetic effects may become important. This can be simulated by advection/diffusion boundary conditions for \( F = \Psi, T \)

\[
\frac{\partial F}{\partial n} + \frac{\gamma |v^r|}{\eta} F = \frac{\partial F_{\text{out}}}{\partial n} + \frac{\gamma |v^r|}{\eta} F_{\text{out}}, \tag{14}
\]

where \( \partial / \partial n \) is the derivative along the outer boundary normal. Figure shows the stationary final state of a time-dependent simulation (with turbulent magnetic diffusivity), in which \( |B_{p,\text{out}}|/|B_{t,\text{out}}| = 1/50 \). For such a dominantly toroidal magnetic field the gravitomagnetic source terms are strong enough to change the topology of \( \Psi \). This may influence the efficiency of the electromagnetic extraction of rotational energy from the hole.
3.3. The $\alpha\Omega$ dynamo in the Kerr metric

In the innermost region of an accretion disk around a black hole there may also be a turbulent source term of $\alpha$-type. Without any knowledge of the physical source of the term (convection or magnetic shear instability) or its mathematical form in the relativistic context, one can try to assess the physical regime (magnetic diffusivity, accretion velocity, rotation law) in which growing modes of an $\alpha\Omega$ dynamo exist. For a simple mean-field ansatz (Khanna & Camenzind 1996a) the equations of the kinematic $\alpha\Omega$ dynamo are identical to Eqs. (12) and (13) augmented by the $\alpha$-source term, $\alpha T$, for the flux and $\eta$ replaced by $\eta_{\text{turb}}$. In analogy to the expression for $\alpha$ in classical disks, I assume

$$\alpha = (\alpha_g R_0^2 \Omega/H) f(z)/f(H/2) = (3\alpha_g \alpha_{\text{visc}} H \Omega) f(z)/f(H/2),$$

where $H$ is the disk scale height, $R_0$ is the Rossby number and $\alpha_{\text{visc}}$ is the viscosity parameter of standard accretion disk theory. The factor $\alpha_g$ is added in order to suppress the source close to the horizon, where the accretion velocity approaches the speed of light (in properly derived mean-field equations there would probably be a “Rossby number” correlated to the accretion velocity instead). The vertical dependence of $\alpha$ is modelled with $f(z) = \tanh(z) \exp[-(z/H)^2]$. The turbulent diffusivity is described as

$$\eta_{\text{turb}} = \alpha_{\text{visc}} H^2 \Omega \tilde{\omega}/r \sin(\theta),$$

with a vertical scaling $(\exp[-(z/H)^2] + 0.1)/1.1$. The boundary condition at $r = 10 r_g$ is $\partial \Psi / \partial n = 0$ and $T = 0$. Figure 2 shows the first part of a simulation.
Figure 2. A simulation of an $\alpha\Omega$ dynamo with symmetric initial current in a quasi-Keplerian accretion disk. Kerr parameter $a = 0.998M$, $\alpha_{\text{visc}} = 0.065$. Solid contours have positive values, dashed contours have negative values. Kinks at the outer edge are artefacts of transforming data from the spherical grid into a Cartesian plot. Simulation continued in Fig. 3.
Figure 3. The simulation of Fig. 2 has reached a slowly growing, oscillating eigenmode with a period of $\sim 3t_{\text{diff}}$. Deviations from equatorial (anti-)symmetry are probably due to insufficient resolution in $\theta$-direction. Not shown here: The inclusion of non-linear $\alpha$-quenching leads to severe symmetry breaking and chaotic behavior.
Figure 4. Simulation with same setup as above, except that the accretion velocity at $r > 2 r_g$ is 10 times higher.

with an initial current $T$, which is symmetric with respect to the equatorial plane, and $\Psi = 0$. The parameters are the Kerr parameter $a = 0.998 M$, $\alpha_{\text{visc}} = 0.065$, and the angular momentum of the accreting plasma is 99.999% of the Keplerian value at $r > r_{\text{ms}}$ and constant within, which yields an accretion velocity of $\sim 0.003 c$ at $r = 3 r_g$, increasing to $c$ at the horizon. The dynamo is in a slowly growing quadrupolar mode, oscillating with a period of about three diffusive timescales $t_{\text{diff}} = r_g^2/\eta_0 \approx 2 \times 10^5$ sec $M_9 \left( \frac{\alpha_{\text{visc}}}{0.1} \right)^{-1} \left( \frac{H(r_h)}{0.5 r_g} \right)^{-2}$.

The same setup, but with lower angular momentum in the accretion disk (99.9% of the Keplerian value), which corresponds to a radial velocity of $\sim 0.03 c$ at $r = 3 r_g$, is in a decaying mode, which demonstrates that accretion impedes dynamo action (Fig. 4).

3.4. $\alpha \omega$ dynamo action in a zero-angular momentum flow?

It was mentioned above that the shear of space does also induce a toroidal magnetic field (cf. eq. (10)). In eq. (13) this shear term is obscure, but still there, hidden in $(\nabla \Psi \times e_\phi^0) \cdot \nabla \Omega \propto \vec{B}_p \cdot \nabla \Omega = \vec{B}_p \cdot \nabla (\alpha_g v^\phi + \omega)$. In a zero-angular-momentum flow $v^\phi = 0$ (or, equivalently $\Omega = \omega$) and thus the current $T$ is solely generated by the shear of space. Moreover, $\nabla \omega$ is significantly steeper than $\nabla \Omega_K$, which means that $\vec{B}_p \cdot \nabla \omega$ is a strong source term.

Meier (1998) speculated that, alternatively to the gravitomagnetic dynamo described above, there could be an $\alpha \omega$ dynamo in a zero-angular momentum accretion flow, with $\alpha$ being due to the magnetic shearing instability. Such a flow, however, accretes at relativistic velocities ($\geq 0.1 c$ at $r = 10 r_g$), which should suppress any dynamo action. This conclusion is supported by the simulation shown in Fig. 5. The $\Psi$ and $T$ loops in the corona are transient and depend on the description of $\eta$ and $\alpha$ (here described as in Eq. (15), but not suppressed by $\alpha_g$ in order to have an upper estimate of the source term). Parameters are $\alpha_{\text{visc}} = 0.003$ and $H(r_h) = 0.8 r_g$. A wider parameter study is in progress.
Figure 5. Simulation of an $\alpha\omega$ dynamo. The dynamo generates transient structures in the corona close to the horizon. Within the disk there are no signs whatsoever of dynamo action. The field and current are completely determined by the relativistic advection. $T$ is shown in logarithmic contours.

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References

Biermann, L. 1950, Z. Naturforsch., 5a, 65
Blandford, R.D., Znajek, R.L. 1977, MNRAS, 179, 433
Egi, M., Tomimatsu, A., Takahashi, M. 1998, in: The Central Regions of the Galaxy and Galaxies, IAU Symp. No. 184, Y. Sofue (Ed.), Kluwer, p. 369
Khanna, R. 1998a, MNRAS, 294, 673
Khanna, R. 1998b, MNRAS, 295, L6
Khanna, R., Camenzind, M. 1996a, AA, 307, 665
Khanna, R., Camenzind, M. 1996b, AA, 313, 1028
Núñez, M., 1997, Phys. Rev. Let., 79, 796
Mestel, L., Roxburgh, I.W. 1962, ApJ, 136, 615
Meier, D. 1998, astro-ph/9810352
Punsly, B. 1996, ApJ, 467, 105
Ruffini, R., Wilson, J.R. 1975, Phys. Rev. D, 12, 2959
Thorne, K.S., Price, R.H., Macdonald, D.A., Suen, W.-M., Zhang, X.-H. 1986, in: Black Holes: The Membrane Paradigm, Thorne, K.S., Price, R.H., Macdonald, D.A., (Eds.), Yale Univ. Press, New Haven, pp. 67–120
Wald, R.M., 1974, Phys. Rev. D, 10, 1680