Robust Generalised Quadratic Discriminant Analysis

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Abstract

Quadratic discriminant analysis (QDA) is a widely used statistical tool to classify observations from different multivariate Normal populations. The generalized quadratic discriminant analysis (GQDA) classification rule/classifier, which generalizes the QDA and the minimum Mahalanobis distance (MMD) classifiers to discriminate between populations with underlying elliptically symmetric distributions competes quite favorably with the QDA classifier when it is optimal and performs much better when QDA fails under non-Normal underlying distributions, e.g. Cauchy distribution. However, the classification rule in GQDA is based on the sample mean vector and the sample dispersion matrix of a training sample, which are extremely non-robust under data contamination. In real world, since it is quite common to face data highly vulnerable to outliers, the lack of robustness of the classical estimators of the mean vector and the dispersion matrix reduces the efficiency of the GQDA classifier significantly, increasing the misclassification errors. The present paper investigates the performance of the GQDA classifier when the classical estimators of the mean vector and the dispersion matrix used therein are replaced by various robust counterparts. Applications to various real data sets as well as simulation studies reveal far better performance of the proposed robust versions of the GQDA classifier. A Comparative study has been made to advocate the appropriate choice of the robust estimators to be used in a specific situation of the degree of contamination of the data sets.

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1 Introduction

Discriminant analysis is a very widely used statistical tool to assign an individual to any of the \(k(\geq 2)\) populations on the basis of a \(p\) dimensional feature vector. Usually it is assumed that the underlying distribution of the feature vectors is multivariate Normal. Under this usual assumption, with equal dispersion matrices of different underlying populations, linear discriminant analysis (LDA) leads to a classification rule based on minimizing the Mahalanobis distance of the new observation from the mean vector of a particular underlying population in question (Zollanvari et al., 2013). This classification rule is also referred to as minimum Mahalanobis distance (MMD) classification rule. When the assumption of equality of the dispersion matrices of the underlying populations is not tenable, quadratic discriminant analysis (QDA) is used, where the classification rule involves the ratio of the determinants of the dispersion matrices, apart from the Mahalanobis distances. More recently, several methodological and computational advancements of LDA and QDA have been developed to generate faster and better performances under specific problems; see, e.g., Hua et al. (2005); Wang et al. (2008); Park and Park (2008); Na et al. (2010); Suzuki and Itoh (2010); Daqi et al. (2014); Ye et al. (2017), among many others.

It is worthwhile to note that, with underlying elliptically symmetric distributions which are not necessarily Normal, the classification rule to discriminate between these populations also involve a similar factor as QDA. Noting this interesting phenomenon and observing that QDA does not perform well in discriminating populations with non-Normal distributions, Bose et al. (2015) generalized the QDA and MMD classifiers. Their proposed method, termed as generalized quadratic discriminant analysis (GQDA), is a simple nonparametric method which is adaptive to any given data set by choosing a threshold value to make the decision on where to classify the new observation. The performance of the classifier under this flexible method matches with that of the QDA classifier when it is optimal and compares quite favorably with the other established complex nonparametric classifiers.

However, like QDA, the proposed GQDA is also based on the mean vectors and the dispersion matrices of the populations, which being unknown, need to be estimated from a part of the data, namely the training set. In practice, quite often the populations get contaminated because of the presence of outlying observations. The susceptibility of the classical sample mean vector and the sample dispersion matrix to outlying observations present in the training set tends to misclassify the new observation, leading to unreliability of the classical LDA and QDA as well as GQDA. As the simple cost effective classifier in GQDA has competitive and sometimes even better performance in a wide range of distributions, not only limited to the Normal distribution, it is worthwhile to make this procedure robust in the face of contamination. This necessity motivates the present authors to make a thorough investigation of the performance of the GQDA classifier in the presence of contamination and undertake a comparative study of different robust versions of GQDA, replacing the classical estimators in the GQDA classifier by their robust counterparts.
The issue of non-robustness of LDA and QDA has been addressed by several researchers replacing the classical estimators in LDA and QDA by their robust counterparts. Randles et al. (1978) proposed to use M estimators and used a rank based rule to estimate the threshold value for discrimination between two populations. Todorov et al. (1990, 1994) worked with minimum covariance determinant (MCD) estimators, while Chork and Rousseeuw (1992) and Kim et al. (2006) used minimum volume ellipsoid (MVE) estimators. Croux and Dehon (2001) advocated S-estimators while Hubert et al. (2012) applied MCD estimates computed by the FAST MCD algorithm. In the present article, we focus on the development of the robust generalized quadratic discriminant analysis (RGQDA) for two class as well as multi-class (more than two classes) classification problems, using such robust estimators of the mean vector and the dispersion matrix along with a detailed (empirical) comparative study. Based on our investigations, suggestions are also made on the specific choice of the robust estimators under situations with different degree of contamination.

The paper is organized as follows. Section 2 gives a brief overview of GQDA and illustrate its unreliability in the presence of outliers through a simulated example of Normal distribution. For ready reference, different robust estimators of the mean vector and the dispersion matrix available in the literature are briefly reviewed in Section 3. Simulation studies related to two class as well as multi class classifications are presented in Section 4 to illustrate the improvement of the performance of our proposed RGQDA over GQDA. The potential of RGQDA is compared with GQDA using some real data sets in Section 5. Finally, Section 6 gives the concluding remarks and future directions of further research.

2 The GQDA classifier and its non-robustness

In this section, we first present a brief overview of GQDA proposed by Bose et al. (2015) for two-class as well as multi-class classification problem.

2.1 Two-class classification

Let us first consider the simpler case where an object is classified into one of the two competing populations or classes using a decision rule formed on the basis of an observation \( x = (x_1, \ldots, x_p)' \) on a \( p \)-variate random feature vector \( X \). The rule is devised using a sample of \( n \) observations on \( X \), called the training set, which represents observations from both the populations. It is assumed that, for the \( j \)th population, \( j = 1, 2 \), the random vector \( X \) has a probability density function \( f_j(x) \), and \( \pi_j \) is the a priori probability for an observation to belong to this population, where \( \pi_1 + \pi_2 = 1 \). Whenever the assumption of equality of prior probabilities \( \pi_1 \) and \( \pi_2 \) holds, the optimal Bayes rule assigns the observation \( x \) to that population \( j \), for which the density function evaluated at \( x \) is larger. Thus the feature space \( \mathcal{X} \) is essentially partitioned into \( R_1 \) and \( R_2 \) such that the object is classified into population 1 or population 2 depending
on whether \( x \in R_1 \) or \( x \in R_2 \), respectively, where

\[
R_1 = \left\{ x : \frac{f_1(x)}{f_2(x)} \geq 1 \right\} = \left\{ x : \log \frac{f_1(x)}{f_2(x)} \geq 0 \right\},
\]

\[
R_2 = \left\{ x : \frac{f_1(x)}{f_2(x)} < 1 \right\} = \left\{ x : \log \frac{f_1(x)}{f_2(x)} < 0 \right\}.
\]

In the case of two underlying populations having multivariate Normal distribution with the mean vectors \( \mu_1, \mu_2 \) and the dispersion matrices \( \Sigma_1 \) and \( \Sigma_2 \), respectively, we get

\[
\log \frac{f_1(x)}{f_2(x)} = \frac{1}{2} \log \left( \frac{\Sigma_2}{\Sigma_1} \right) + \frac{1}{2} \Delta_d^2, \quad \text{say,}
\]

(2.1)

where \( \Delta_d^2 = \left[ (x - \mu_2)' \Sigma_2^{-1} (x - \mu_2) - (x - \mu_1)' \Sigma_1^{-1} (x - \mu_1) \right] \) is nothing but the difference of the squared Mahalanobis distances of \( x \) from the two populations. Thus, for the QDA classification rule, we have

\[
R_1 = \left\{ x : \frac{1}{2} \log \left( \frac{\Sigma_2}{\Sigma_1} \right) + \frac{1}{2} \Delta_d^2 \geq 0 \right\} = \left\{ x : \Delta_d^2 \geq \log \left( \frac{\Sigma_1}{\Sigma_2} \right) \right\},
\]

(2.2)

\[
R_2 = \mathcal{X} - R_1 = \left\{ x : \Delta_d^2 < \log \left( \frac{\Sigma_1}{\Sigma_2} \right) \right\}.
\]

It is to be noted that when the assumption of equality of \( \Sigma_1 \) and \( \Sigma_2 \), holds, simplifying (2.2) we get \( R_1 = \left\{ x : \Delta_d^2 \geq 0 \right\} \), turning the QDA rule identical to the MMD rule. However, in the cases where this assumption of the equality of the dispersion matrices does not hold, the MMD rule fails and the QDA rule turns out to be optimal. On the other hand, it is quite likely that the MMD rule will have a better performance than the QDA rule in terms of reducing the misclassification error, if the underlying population probability densities are not Normal.

Now we consider two underlying populations having \( p \)-variate \( t \)-distribution with \( q \) degrees of freedom (d.f.) with densities

\[
f_j(x) = A \left| \Sigma_j \right|^{-\frac{1}{2}} \left[ \left( 1 + \frac{1}{q} (x - \mu_j)' \Sigma_j^{-1} (x - \mu_j) \right)^{-\frac{p+q}{2}} / \Gamma \left( \frac{q}{2} \right) \right], \quad j = 1, 2,
\]

where \( A = \Gamma \left( \frac{p+q}{2} \right) q^{p/2} \pi^{p/2} / \Gamma \left( \frac{q}{2} \right) \). Thus,

\[
\frac{f_1(x)}{f_2(x)} = \frac{\left| \Sigma_2 \right|^{1/2}}{\left| \Sigma_1 \right|^{1/2}} \left[ \left( 1 + \frac{1}{q} (x - \mu_2)' \Sigma_2^{-1} (x - \mu_2) \right)^{-\frac{p+q}{2}} / \Gamma \left( \frac{q}{2} \right) \right],
\]
and it follows that

\[
\log \frac{f_1(x)}{f_2(x)} = \frac{1}{2} \log \frac{|\Sigma_2|}{|\Sigma_1|} + \frac{p + q}{2} \left[ \log \left\{ 1 + \frac{1}{q}(x - \mu_2)'\Sigma_2^{-1}(x - \mu_2) \right\} - \log \left\{ 1 + \frac{1}{q}(x - \mu_1)'\Sigma_1^{-1}(x - \mu_1) \right\} \right]
\]

\[
\cong \frac{1}{2} \log \left( \frac{|\Sigma_2|}{|\Sigma_1|} \right) + \frac{p + q}{2q} \left[ (x - \mu_2)'\Sigma_2^{-1}(x - \mu_2) - (x - \mu_1)'\Sigma_1^{-1}(x - \mu_1) \right]
\]

\[
= \frac{1}{2} \log \left( \frac{|\Sigma_2|}{|\Sigma_1|} \right) + \left( \frac{1}{2} + \frac{p}{2q} \right) \Delta_d^2,
\]

using the Taylor series expansion of terms of the type \(\log(1 + x)\) and neglecting higher-order terms, presuming \(q\) to be sufficiently large. Therefore, we get

\[
R_1 = \left\{ x : \frac{1}{2} \log \left( \frac{|\Sigma_2|}{|\Sigma_1|} \right) + \left( \frac{1}{2} + \frac{p}{2q} \right) \Delta_d^2 \geq 0 \right\}.
\]

\[
= \left\{ x : \Delta_d^2 \geq \frac{q}{p + q} \log \left( \frac{|\Sigma_1|}{|\Sigma_2|} \right) \right\}. \tag{2.3}
\]

It is known that as \(q \to \infty\), the \(t\)-distribution with \(q\) degrees of freedom approaches the Normal distribution, hence the rule (2.3) boils down to the QDA rule in this case, as expected.

In fact it has been shown by Bose et al. (2015) that, for the class of elliptically symmetric distributions with the probability density function having the form

\[
f(x) = \frac{1}{|\Sigma|^{\frac{1}{2}}} g((x - \mu)'\Sigma^{-1}(x - \mu)),
\]

the Bayes rule leads to the partition

\[
R_1 = \left\{ x : \frac{1}{2} \log \left( \frac{|\Sigma_2|}{|\Sigma_1|} \right) + k \Delta_d^2 \geq 0 \right\}, \tag{2.4}
\]

where \(k\) may depend on \(x\).

Therefore, combining (2.2), (2.3), and (2.4) and denoting \(\log \frac{|\Sigma_1|}{|\Sigma_2|}\) by \(\Sigma_d\), a general classification rule/classifier, proposed by Bose et al. (2015), is given by

\[
x \in R_1 \quad \text{if } \Delta_d^2 \geq c\Sigma_d,
\]

\[
x \in R_2 \quad \text{otherwise}, \tag{2.5}
\]

for some constant \(c \geq 0\). Clearly, this classifier boils down to the MMD and the QDA classifiers whenever \(c\) is chosen to be 0 and 1, respectively. In practice, the parameters in the classifier (2.5) are unknown and need to be estimated from the training set.
The simplest and the most popular estimators of the population mean vector and the dispersion matrix are the sample mean vector and the sample dispersion matrix i.e. \( \hat{\mu}_j = \bar{x}_j \) and \( \hat{\Sigma}_j = S_j \), which are obtained from the sample observations from the \( j \)th population in the training set, for \( j = 1, 2 \). Accordingly, for any new observation \( x \), the classification rule referred to as the GQDA classification rule/classifier in Bose et al. (2015) is given by

\[
x \in R_1 \quad \text{if} \quad \hat{\Delta}_d^2 \geq c \hat{\Sigma}_d,
\]

\[
x \in R_2 \quad \text{otherwise},
\]

(2.6)

where

\[
\hat{\Sigma}_d = \log \left( \frac{|S_1|}{|S_2|} \right) \quad \text{and} \quad \hat{\Delta}_d^2 = (x - \bar{x}_2)'S_2^{-1}(x - \bar{x}_2) - (x - \bar{x}_1)'S_1^{-1}(x - \bar{x}_1).
\]

Bose et al. (2015) suggested to choose the threshold \( c \) with a view to maximize the resulting classification accuracy, i.e. to minimize the misclassification error. As noted by them, the constant \( c \) may depend on \( x \) and so a suitable nonparametric approach needs to be adopted to estimate an appropriate value of the constant \( c \) from the training set itself. The major advantage of GQDA is that a proper choice of \( c \) makes the GQDA procedure adaptive to any data set, safeguarding its performance against the possible violation of the normality assumption in the classical QDA. Two methods, namely the minimization of the resubstitution (training set misclassification) error and the cross-validation error, have been proposed by Bose et al. (2015) for selecting the optimal value of \( c \). For a ready reference, the minimization of the resubstitution error based algorithm is described below, which will be used in our proposed RGQDA.

**Algorithm 1. Selection of \( c \) in GQDA for two class classification.**

- **Estimate** \( \Sigma_d \) by \( \hat{\Sigma}_d \) and obtain \( \hat{\Delta}_d^2 \) for each of the observations corresponding to each population in the training set.

- **In the training set**, compute \( \hat{\Delta}_d^2 \) for each observation and denote the ordered \( \frac{\hat{\Delta}_d^2}{\hat{\Sigma}_d} \) values for \( n_j \) observations corresponding to \( j \)th population by \( r_{j(1)}, \ldots, r_{j(n_j)} \), \( j = 1, 2 \).

- **If** \( r_{2(n_2)} < r_{1(1)} \) (i.e. the sets of \( \frac{\hat{\Delta}_d^2}{\hat{\Sigma}_d} \) values for the two populations in the training set are completely disjoint), do:

  - Take \( c \) to be equal to any point in the interval \( [r_{2(n_2)}, r_{1(n_1)}] \) resulting in the resubstitution error to be zero.

- **If** \( r_{2(n_2)} > r_{1(1)} \) (i.e. the sets of \( \frac{\hat{\Delta}_d^2}{\hat{\Sigma}_d} \) values for the two populations in the training set overlap), do:
Find \( s \geq 1 \), such that \( r_2(n_2-s+1),\ldots,r_2(n_2) \) all exceed \( r_1(1) \).

Choose these \( s \) values as candidate values of \( c \).

Compute the resubstitution error for each of these \( s \) values.

Choose that value of \( c \) for which the resubstitution error is the minimum and denote it by \( c^* \).

If \( c^* > 1 \), set \( c^* = 1 \).

### 2.2 Multi-class classification problem

In a general \( g \)-class classification problem where \( g > 2 \), an object is classified into one of the \( g \) populations /classes, say, \( P_1, P_2, \ldots, P_m \) using a decision rule formed on the basis of an observation \( \mathbf{x} = (x_1, \ldots, x_p)' \) on a random feature vector \( \mathbf{X} \). It is assumed that \( \mathbf{X} \) has a probability density function \( f_i(\mathbf{x}) \) in the class \( P_i \), \( i = 1, 2, \ldots, g \). The feature space \( \mathcal{X} \) is essentially partitioned into \( R_1, R_2, \ldots, R_g \) such that the object is classified into \( P_j \) if \( \mathbf{x} \in R_j \). Under the assumption of equality of the prior probabilities \( \pi_1, \pi_2, \ldots, \pi_m \), where \( \sum_{j=1}^m \pi_j = 1 \), the Bayes rule sets

\[
R_i = \{ \mathbf{x} : \frac{f_i(\mathbf{x})}{f_j(\mathbf{x})} \geq 1 \} = \{ \mathbf{x} : \log \frac{f_i(\mathbf{x})}{f_j(\mathbf{x})} \geq 0, \forall j \neq i \} , \quad i = 1, 2, \ldots, g.
\]

If the \( g \) underlying populations are multivariate Normal with the mean vector \( \mu_i \) and the dispersion matrix \( \Sigma_i \) for the \( i \)th population, \( i = 1, 2, \ldots, g \),

\[
\log \frac{f_i(\mathbf{x})}{f_j(\mathbf{x})} = \frac{1}{2} \log \left( \frac{|\Sigma_j|}{|\Sigma_i|} \right) + \frac{1}{2} \Delta_{ij}^2,
\]

where \( \Delta_{ij}^2 = (\mathbf{x} - \mu_j)' \Sigma_j^{-1} (\mathbf{x} - \mu_j) - (\mathbf{x} - \mu_i)' \Sigma_i^{-1} (\mathbf{x} - \mu_i) \). This leads to the partition of the feature space as

\[
R_i = \left\{ \mathbf{x} : \frac{1}{2} \log \left( \frac{|\Sigma_j|}{|\Sigma_i|} \right) + \frac{1}{2} \Delta_{ij}^2 \geq 0, \forall j \neq i \right\}
\]

\[= \left\{ \mathbf{x} : \Delta_{ij}^2 \geq \log \left( \frac{|\Sigma_i|}{|\Sigma_j|} \right), \forall j \neq i \right\}. \tag{2.7}
\]

Bose et al. (2015) proposed an extension of the GQDA classifier (2.6) for the general \( g \)-class classification problem which is described below.

**Algorithm 2. Selection of \( c \) in GQDA for multi \((g > 2)\) class classification.**

- Define \( u_{ij}(\mathbf{x}) = \frac{\Delta_{ij}^2}{\Sigma_{d_{ij}}} \), where \( \Sigma_{d_{ij}} = \log \left( \frac{|\Sigma_i|}{|\Sigma_j|} \right) \), for \( i, j = 1, 2, \ldots, g \).

- Estimate \( \Sigma_{d_{ij}} \) by \( \hat{\Sigma}_{d_{ij}} \) and obtain \( \hat{\Delta}_{ij}^2 \) for each of the observations corresponding to each ordered pair of populations \((P_i, P_j)\), with \( i, j \in \{1, 2, \ldots, g\} \) but \( i \neq j \), by replacing the population mean vectors and the dispersion matrices by the corresponding sample counterparts in the training set.
• Compute and subsequently order \( \hat{\mathbf{u}}_{ij}(\mathbf{x}) = \frac{\Delta^2_{ij}}{\Sigma_{dij}} \) for each training sample \( \mathbf{x} \) from the populations \( P_i \) and \( P_j \), for each ordered pair of populations \( (P_i, P_j) \), with \( i, j \in \{1, 2, \ldots, g\} \) but \( i \neq j \).

• Let \( T = \{ \hat{\mathbf{u}}_{ij}(\mathbf{x}) : \hat{\mathbf{u}}_{ij}(\mathbf{x}) \in [0, 1], \ i, j \in \{1, 2, \ldots, g\}, \ i \neq j \} \).

• Take each \( t \in T \) as a candidate value of \( c \) and correctly classify the training samples from the population \( P_i \) by comparing \( \hat{\mathbf{u}}_{ij}(\mathbf{x}) \) with \( c \) for each \( j \neq i \).

• For each \( j \neq i \in \{1, 2, \ldots, g\} \), define the set of correctly classified training samples from \( P_i \) identified through such comparisons as
  \[ R_{ij}(c) = \{ \mathbf{x} \in D_{1i} : \hat{\mathbf{u}}_{ij}(\mathbf{x}) \geq c \}, \]
  where \( c \in T \) and \( D_{1i} \) is the training set for the population \( P_i \).

• Identify the number of misclassified training samples from \( P_i \) as
  \[ MC_i(c) = n_i - | \bigcap_{j \neq i} R_{ij}(c) |, \]
  where \( n_i = |D_{1i}| \) is total number of training samples from the population \( P_i \).

• Repeat the procedure for \( i = 1, 2, \ldots, g \) to obtain the number of misclassified training samples from \( P_i \).

• Compute the total number of instances of misclassification with a particular choice of \( c \) as
  \[ MC(c) = \sum_{i=1}^{g} MC_i(c). \]

• The optimal value of \( c \) is taken as \( c^* = \arg\min_{c \in T} MC(c). \)

2.3 Non-robustness

In reality, quite often the data get contaminated by the presence of outliers. It has been observed by many researchers in recent decades that, under contamination in the data, both the LDA and the QDA procedures result in significantly large misclassification errors failing to provide an appropriate inference on classification, mainly because of the extreme non-robustness nature of the sample mean vector and the sample dispersion matrix used in the process. Since the GQDA classifier also utilizes the sample mean vector and the sample dispersion matrix, it is also non-robust in the presence of outliers in the sample data and, as a result, fails to perform satisfactorily. This issue is illustrated below through a small simulation study pertaining to the two class problem; more numerical illustrations are provided in the later sections of the paper.

Table 1 presents the results from a simulation exercise with 10000 observations, of which 5000 observations are chosen from a tri-variate Normal \( \mathbf{N}_3(-1', \mathbf{I}) \) to form
class 1, and the remaining 5000 observations are chosen from \( \mathbf{N}_3(1', 2I) \) to form class 2. Here, as usual, we denote by \( \mathbf{1} \) and \( \mathbf{I} \), the vector of all ones and the identity matrix, respectively. For each of the two classes so defined, a random selection of 1000 observations form a training set and the remaining 4000 observations form the test set, to be used to check the validity of the classification rule. Under this set-up, the value of the threshold \( c^* \) of the GQDA classifier is obtained following Algorithm 1 on the pooled training set of size 2000 and the % misclassification error (ME%) is calculated on the test data. For comparison, we have also worked solely with the pooled test data set of size 8000 to obtain another threshold, denoted by \( c_{test} \), which will ideally minimize the test set misclassification error and help us to examine how closely it can be approximated by \( c^* \), obtained on the basis of the training set only.

### Table 1: Non-robustness of GQDA under contamination

| Contaminated part of the data | Contamination type and degree | \( c^* \) | ME% with \( c^* \) on the test data | \( c_{test} \) from the test data | ME% with \( c_{test} \) on the test data |
|------------------------------|--------------------------------|---------|-----------------------------------|----------------------------------|----------------------------------|
| Nil                          |                                |         |                                   |                                  |                                  |
| Train mild                    | 5%                             | 0.576   | 11.31                             | 0.997                            | 6.78                             |
|                               | 10%                            | 0.471   | 13.18                             | 0.974                            | 6.793                            |
|                               | 15%                            | 0.413   | 13.969                            | 0.979                            | 6.795                            |
|                               | 20%                            | 0.375   | 13.823                            | 0.981                            | 6.726                            |
| Train hard                    | 5%                             | 0.437   | 25.541                            | 0.996                            | 6.782                            |
|                               | 10%                            | 0.754   | 37.826                            | 1.039                            | 6.759                            |
|                               | 15%                            | 0.838   | 41.254                            | 0.974                            | 6.694                            |
|                               | 20%                            | 0.919   | 43.194                            | 0.941                            | 6.691                            |
| Train and Test mild           | 5%                             | 0.589   | 10.849                            | 0.598                            | 10.444                           |
|                               | 10%                            | 0.491   | 12.462                            | 0.472                            | 12.101                           |
|                               | 15%                            | 0.411   | 12.363                            | 0.419                            | 12.245                           |
|                               | 20%                            | 0.373   | 11.775                            | 0.373                            | 11.511                           |
| Train and Test hard           | 5%                             | 0.437   | 27.061                            | 0.413                            | 26.626                           |
|                               | 10%                            | 0.734   | 38.007                            | 0.741                            | 37.514                           |
|                               | 15%                            | 0.877   | 38.785                            | 0.847                            | 38.287                           |
|                               | 20%                            | 0.936   | 37.775                            | 0.913                            | 37.456                           |

Now to check the performance of the GQDA classifier in the presence of contamination, we serially replace 5%, 10%, 15% and 20% observations of the training set only, without disturbing the test set and also of both the training and the test set separately, with observations generated from two different tri-variate Normal distributions with parameter values set widely apart form those of the original populations. Thus the presence of observations from the outlying populations will contaminate the original data set, the contaminations being termed as mild (hard) depending on whether the mean vector of the outlying population is along the same (opposite) direction of that of the original population. The required percentage of mild contamination for the class 1 and the class 2 are induced with observations generated from \( \mathbf{N}_3(-91', 4I) \) and \( \mathbf{N}_3(91', 16I) \), respectively. Analogously, the required percentage of hard contamination is induced with the observations generated from \( \mathbf{N}_3(91', 4I) \) and \( \mathbf{N}_3(-91', 16I) \) for the class 1 and the class 2, respectively. We apply the GQDA classification rule on the contaminated data sets and obtain the % misclassification error (ME%) corresponding
to $c^*$ and $c_{\text{test}}$ thus emerged. The above procedure is repeated 500 times in the absence as well as in the presence of each type and degree of contamination, and the average % misclassification errors are noted in Table 1.

From Table I, it is seen that, in the absence of contamination, the GQDA classification rule performs quite well as the value of $c^*$ (1.000) and the % misclassification errors (6.938%) are very close to the $c_{\text{test}}$ (1.005) and the corresponding % misclassification error (6.728%). Ideally, no threshold on the test set can perform better than $c_{\text{test}}$ as long as the test set is pure, i.e. devoid of any contamination. Thus, in the pure scenario the justification of the GQDA classification rule is reestablished, as has been observed by Bose et al. (2015).

However, in the presence of contamination, no matter whether it is mild or hard, the performance of the GQDA classification rule starts worsening, suggested by the higher % misclassification errors as the degree of contamination graduates. Whenever only the training set is even mildly contaminated, the % misclassification errors vary from 11.31% to 13.969%, which is double or more, compared to the ones observed in the pure data set. In the case of hard contamination, the same is four fold or more, establishing the fact that the GQDA classification rule is highly unsuitable in the presence of outliers. It is obviously expected that as long as only the training set is contaminated, it will not affect the choice of $c_{\text{test}}$ and the corresponding % misclassification error, which is also clear from our simulation study. In such cases, irrespective of the mild or hard contamination, $c_{\text{test}}$ is 1.00 approximately and the % misclassification errors are close to 6.728%, the one obtained when the GQDA classification rule is applied on the pure test set, as depicted in the first row of Table I.

Whenever both the train and the test sets are contaminated separately with the same type and degree of contamination, we find that the GQDA classification rule behaves miserably with respect to both $c^*$ and $c_{\text{test}}$, failing to capture the nature of the data set. With $c^*$, the % misclassification errors vary from 10.849% to 12.462% (nearly doubled) and from 27.061% to 38.785% (nearly four folds) in cases of mild and hard contamination, respectively, suggesting that the GQDA classification rule is not at all reliable for classification in the presence of outliers, unlike the accuracy which transpires in the case of pure data. This motivates us to re look for a robust classification rule to discriminate between two elliptically symmetric distributions which are not necessarily Normal.

3 RGQDA: robust version of GQDA

In our present work, we take care of the possibility of contamination of populations by outlying observations and propose to replace the classical estimators of the mean vector and the dispersion matrix used in the GQDA classifier (2.6), by different types of robust estimators available in the literature. It has been observed that the % misclassification error reduces drastically using the new version of the classification rule. The proposed robust procedure will be referred to as robust generalized discriminant anal-
ysis (RGQDA) hereafter, and a comparative study on the performances of the GQDA classifier using different types of robust estimators will be presented in the subsequent sections. Before that, for the sake of completeness, we present below a brief summary of the different types of robust estimators of the mean vector and the dispersion matrix considered by us in the determination of the threshold $c$ in the RGQDA classifier.

In the following, we assume that the $p$-dimensional sample observations $x_1, \ldots, x_n$ drawn from a multivariate distribution are independent and identically distributed with a common mean vector $\mu$ and a common dispersion matrix $\Sigma$. Recall that the classical estimators of $\mu$ and $\Sigma$ are the sample mean vector $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ and the sample dispersion matrix $S = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$, respectively. Both of these classical estimators are extremely non-robust having zero breakdown point and unbounded influence function, inspite of being the most efficient under a multivariate Normal model.

**Winsorized (W) Estimator:**
The first type of the robust estimator we have considered is the simple Winsorized estimators of $\mu$ and $\Sigma$. These were first proposed by [Bickel (1965)] by extending the idea of univariate winsorization in the multivariate context and later discussed by [Zuo and Cui (2004)]. To compute these estimators, a Winsorized sample is formed by trimming a certain percentage of observations from the top and the bottom of the sample, taking into consideration of the shape of the distribution of the data. But, unlike the usual trimming approach which removes the trimmed observations of the sample for subsequent analysis, in Winsorization the trimmed values in the lower and the upper end of the original data are replaced respectively, by the lowest and highest data points of the remaining untrimmed data. We have used the R-function `winsor` from package ‘psych’ to form the Winsorized sample and then use the usual mean vector and the dispersion matrix of the Winsorized sample as robust Winsorized estimators of $\mu$, and $\Sigma$, respectively.

**Minimum Volume Ellipsoid (MVE) Estimator:**
Robust estimators of $\mu$ and $\Sigma$ having high breakdown property, called the minimum volume ellipsoid (MVE) estimators, were proposed by [Rousseeuw (1985)]. These estimators are defined on the basis of a sample of fixed size $h(< n)$, which lies within an ellipsoid of minimum volume, taking the center and the spear of that ellipsoid as the MVE estimators of $\mu$ and $\Sigma$, respectively. For a suitable choice of $h$, one can achieve a very high breakdown point of the MVE estimators [Davies (1987)], which makes these estimators widely popular besides their simple interpretation. However, the MVE estimators are not $\sqrt{n}$-consistent [Davies (1992)] and hence not efficient. We have used the R function ‘CovMve’ from package ‘rrcov’ for computation of the MVE estimators.

**Minimum Covariance Determinant (MCD) Estimator:**
[Rousseeuw (1985)] also proposed another type of robust estimators of $\mu$ and $\Sigma$ by the sample mean vector and the dispersion matrix of $h(< n)$ sample observations which leads to the minimum value of the determinant of the dispersion matrix over
all such samples of fixed size $h$. Hence these estimators are known as the minimum covariance determinant (MCD) estimators. The MCD estimators with $h = \lceil (n + p + 1)/2 \rceil$ achieve the highest possible (finite-sample) breakdown point among the class of affine equivariant scatter estimators [Davies, 1987]. Further, MCD estimators have bounded influence functions as described in [Croux and Haesbroeck, 1999], but they do not have very high efficiency. Unlike the MVE estimators, the MCD estimator of $\mu$ has been shown to be $\sqrt{n}$-consistent and asymptotically Normal by [Butler et al., 1993]. There are several fast algorithms available for computation of this popular estimator. We have used the R function ‘covMcd’ in our implementation that utilizes a fast MCD computation algorithm proposed by [Rousseeuw and van Driessen, 1999].

M-Estimator:

The M-estimators of $\mu$ and $\Sigma$ are defined by [Maronna, 1976] as the respective solutions $l_n \in \mathbb{R}^p$ and $V_n$, a positive definite matrix, of the system of estimating equations

$$
\frac{1}{n} \sum_{i=1}^{n} \psi_1 \left( \sqrt{(x_i - l_n)^T V_n^{-1}(x_i - l_n)} \right) (x_i - l_n) = 0_p,
$$

$$
\frac{1}{n} \sum_{i=1}^{n} \psi_2 \left( (x_i - l_n)^T V_n^{-1}(x_i - l_n) \right) (x_i - l_n)(x_i - l_n)^T = V_n,
$$

for two suitably given weight functions $\psi_1$ and $\psi_2$. [Maronna, 1976] also derived their detailed asymptotic (consistency and normality) and robustness (influence function and breakdown point) properties. In particular, with suitable choice of $\psi_i$s, these M-estimators are highly efficient under Normal model and locally robust having bounded influence function, but they do not have high breakdown point in higher dimensions. Note that the maximum likelihood estimators of $\mu$ and $\Sigma$ under any radically symmetric location-scale model density are also M-estimators for proper choices of weight functions $\psi_i$s. The most commonly suggested weight functions for robust inference are those proposed by Huber (Huber, 1981) as given below

$$
\psi_1(z) = \max(-k, \min(z, k)), \quad \psi_2(z) = \frac{\max(-k^2, \min(z, k^2))}{\mathbb{E}_{X \sim N_0(0_p, I_p)} \left[ \max(-k^2, \min(||X||^2, k^2)) \right]},
$$

for a given tuning parameter $k$. See [Hampel et al., 1986] for further details. In this paper, we have used the R function ‘mvhuberM’ from package ‘SpatialNP’ for the computation of these M-estimators with the above-mentioned Huber’s weight functions.

S-Estimator:

S-estimators of $\mu$ and $\Sigma$ are smoother extensions of the MVE estimators so that the resulting estimators become $\sqrt{n}$-consistent and asymptotically Normal. For a given non-negative, symmetric and continuously differentiable function $\rho$ with $\rho(0) = 0$ and a constant $c < \sup \rho$, the corresponding S-estimators of $\mu$ and $\Sigma$ are defined by [Davies,...]
as the respective solutions $l_n \in \mathbb{R}^p$ and $V_n$, a positive definite matrix, of the constrained optimization problem

$$\min |V_n|, \quad \text{subject to} \quad \frac{1}{n} \sum_{i=1}^n \rho \left( \sqrt{(x_i - l_n)^T V_n^{-1} (x_i - l_n)} \right) \leq c.$$  \hspace{1cm} (3.8)

Further, to achieve robustness one needs to ensure that there exists a constant $b > 0$ such that the function $\rho$ is strictly increasing in $[0, b)$, being constant on $[b, \infty)$. Depending on the choice of the constant $c$ in the above optimization problem (3.8), the S-estimators can have either high efficiency under a Normal model or the maximum possible (finite sample) breakdown point among all affine equivariant estimators, but not the both simultaneously. But S-estimators always have bounded influence function indicating their local robustness and are also closely related to the M-estimators as explored by Lopuhaä (1989). A popular choice of the function $\rho$ is the biweight function proposed by Tukey defined as

$$\rho_{\text{Tukey}}(s) = s \left( 1 - \frac{s^2}{b^2} \right)^2 I(s < b), \quad \text{for } s > 0.$$

A fast deterministic algorithm for the computation of S-estimators, proposed in Hubert et al. (2012), is used in our present work through the R-function ‘CovSest’ from package ‘rrcov’.

Stahel-Donoho (SD) Estimator:
Stahel (1981) and Donoho (1982) proposed other interesting robust estimators of $\mu$ and $\Sigma$, which are affine equivariant as well as have high breakdown point in higher dimensions. They defined a multivariate “outlyingness” measure of a point $x \in \mathbb{R}^p$ with respect to the observed sample (say, $X_n = \{x_1, \ldots, x_n\}$) as

$$O(x, X_n) = \sup_{\{u \in \mathbb{R}^p : ||u|| = 1\}} \frac{|u' x - \mu_n(u' X_n)|}{\sigma_n(u' X_n)},$$

where $\mu_n(u' X_n)$ and $\sigma_n(u' X_n)$ are some given estimators of the univariate mean and the standard deviation of the sample $u' X_n = \{u' x_1, \ldots, u' x_n\}$ for any $u \in \mathbb{R}^p$. Common examples of the univariate estimator of $\mu_n$ are the mean vector or the Median or any general M-estimator of the univariate location (standard deviation or the mean absolute deviation (MAD) or any general M-estimator of the univariate scale parameter) which can be used in $O(x, X_n)$; see Hampel et al. (1986). The Stahel-Donoho (SD) estimators of $\mu$ and $\Sigma$ based on the sample data $X_n$ are then defined respectively, as the weighted mean vector and the dispersion matrix, given by

$$l_{n}^{SD} = \frac{\sum_{i=1}^n w(O(x_i, X_n)) x_i}{\sum_{i=1}^n w(O(x_i, X_n))}, \quad V_{n}^{SD} = \frac{\sum_{i=1}^n w(O(x_i, X_n)) (x_i - l_n^{SD}) (x_i - l_n^{SD})^T}{\sum_{i=1}^n w(O(x_i, X_n))},$$

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where $w$ is a suitable weight function resulting in outlier downweighting. By choosing affine equivariant and high breakdown univariate estimators of $\mu_n$ and $\sigma_n$, one can simultaneously achieve the affine equivariance and high breakdown point for the resulting SD estimators of $\mu$ and $\Sigma$ as well. The $\sqrt{n}$-consistency of these SD estimators was proved by [Maronna and Yohai (1995)], but their asymptotic distributions are yet to be determined. We have used the R-function ‘CovSde’ from the package ‘rrcov’ for the computation of the SD estimators with the extreme 5% observations getting zero weights through their outlyingness measure, computed using the univariate median and the MAD combination.

We have briefly pointed out that all the above robust estimators of $\mu$ and $\Sigma$ have some advantages and disadvantages and none is the uniformly best. In the next section, we will empirically compare the performances of the RGQDA classifiers formed by replacing the classical estimators of the mean vector and the dispersion matrix in the GQDA classifier by the six above-mentioned robust estimators.

4 Empirical illustrations: simulation studies

4.1 Experimental set-ups

In order to investigate the effectiveness of the proposed RGQDA over GQDA, we have conducted several simulation studies. Due to the equivariance of the estimators used, the results obtained are mostly similar and hence, for brevity, we only present the results for one two-class problem and another multi-class problem with four classes, each for three different elliptical symmetric distributions, as specified below.

- **Two-class Problem:** We consider the simulation set-up as described in Section 2.3 with Normal distribution. As mentioned there, 500 repetitions of the simulation study have been undertaken for each case under the pure data as well as under the different types and magnitudes of data contaminations. Here, we also carry out the similar simulation studies for the $t$-distribution with 3 degrees of freedom (d.f.) and the heavy-tailed Cauchy distribution, inducing mild and hard contamination from the outlying population, having the same mean (location) vector and the dispersion (scale) matrix as the case of simulation with Normal distribution.

- **Four-class Problem:** As before, we consider three types of elliptically symmetric distributions, namely multivariate (6-variate) Normal, $t$ with 3 degrees of freedom and Cauchy distribution, with the same size of the train and the test sets. Observations in the four classes are generated from the four postulated distributions having the same dispersion matrix $\Sigma = I_6$ but with different mean vectors, given by $\mu_1 = (1, 1, 1, 1, 1, 1)'$, $\mu_2 = (1, 1, 1, -1, -1, -1)'$, $\mu_3 = (-1, -1, -1, -1, -1, -1)'$ and $\mu_4 = (-1, -1, -1, 1, 1, 1)'$, respectively. The outliers in the $i$-th class for $i = 1, \ldots, 4$, are generated from the same distribution.
as the postulated one for the \( i \)-th class, but with the different mean vectors and the dispersion matrices; the mean vectors being \( \mu_0^i = 9 \times \mu_i \) and \( \mu_0^i = -9 \times \mu_i \) for the mild and hard contaminations, respectively, but the dispersion matrices for both the types of contaminations are taken as \( \Sigma^0 = 4 \Sigma \). Both the types of contaminations are considered for 5%, 10%, 15% and 20% observations from the train set only, without disturbing the test set, as well as from both the train set and the test set separately, as in the case of two-class problem.

In each of the cases, we report the average % misclassification error and the corresponding standard deviation (SD) over 500 replications obtained from the test set using the GQDA and different RGQDA classifier cut-offs, obtained from the respective train set.

4.2 Performances under pure data

The results for the uncontaminated cases i.e. pure data sets are reported in Table 2 for both the two-class and four-class problems and the three types of distributions. One can clearly observe that for the light tailed Normal distribution, the performance of the GQDA classifier is the best but its different robust versions proposed in RGQDA also produce comparable average ME% under the pure data. This is quite natural to expect since the sample estimates of the mean vector and the dispersion matrix are most efficient under a properly specified model with no contamination. However, for the heavy-tailed \( t \) and Cauchy distributions, even under the pure data, several robust versions of the GQDA classifier proposed in RGQDA outperform the GQDA classifier, with greater improvements for the Cauchy distribution, which is due to the presence of potential extreme observations in these heavy tailed distributions. Among the different robust versions, the one with either MCD or SD works the best for \( t \) - and Cauchy distribution.

Table 2: Average ME% (SD) for the test data set using the GQDA classifier and its different robust versions proposed in RGQDA for the pure data

| Method of classification | Normal | Two-class problem | Cauchy | Normal | Four-class problem | Cauchy |
|--------------------------|--------|-------------------|--------|--------|--------------------|--------|
| GQDA                     | 6.938 (0.312) | 12.519 (0.636) | 27.258 (5.131) | 8.653 (0.943) | 18.713 (1.297) | 45.190 (4.359) |
| W                        | 6.986 (0.298) | 12.237 (0.417) | 20.155 (0.589) | 8.887 (0.997) | 16.997 (1.453) | 31.637 (1.596) |
| MVE                      | 6.976 (0.316) | 12.262 (0.424) | 20.216 (0.642) | 8.903 (0.857) | 17.143 (1.450) | 30.257 (1.727) |
| MCD                      | 6.999 (0.317) | 12.292 (0.448) | **20.124 (0.651)** | 8.397 (0.790) | **16.113 (1.350)** | 28.570 (1.808) |
| M                        | 7.070 (0.328) | 12.391 (0.478) | 20.408 (0.850) | 7.313 (0.924) | 17.353 (1.309) | 29.677 (1.791) |
| S                        | 7.077 (0.318) | 12.304 (0.443) | 20.239 (0.588) | 10.263 (1.047) | 17.543 (1.347) | 29.570 (1.920) |
| SD                       | 7.004 (0.308) | **12.230 (0.430)** | 20.226 (0.644) | 8.353 (1.063) | 17.893 (1.539) | **27.860 (1.622)** |

4.3 Performances under contamination: two-class problem

The empirical values of the average ME% and their SDs for simulated experiments with different contaminations for the two-class problems are reported in Table 3 through Ta-
ble for Normal, $t$ and Cauchy distribution, respectively. All these simulation studies emphatically establish that for all the elliptically symmetric distributions considered here, in the case of contaminated data, the use of the robust estimators drastically reduces the % of misclassification error. In general, the improvement increases as the degree of contamination increases from 5% to 20%. As expected, the use of robust estimators exhibits more improvement in terms of reducing the % misclassification error when there is hard contamination compared to the mild contamination.

Table 3: Average ME% (SD) for two-class problems with contaminated Normal distribution, using the GQDA classifier and its different versions proposed in RGQDA

| Contamination degree | Method of classification | Contamination type | Train Mild | Train Hard | Train and Test Mild | Train and Test Hard |
|----------------------|--------------------------|--------------------|------------|------------|--------------------|--------------------|
| 5%                   | GQDA                     | 11.31(0.754)       | 25.541(1.336) | 10.849(0.632) | 27.061(1.269)     |
|                      | W                        | 7.088(0.399)       | 7.13(0.347)   | 9.059(0.389) | 9.371(0.35)       |
|                      | MVE                      | 6.994(0.323)       | 7.02(0.343)   | 9.118(0.288) | 9.312(0.349)      |
|                      | MCD                      | 7.01(0.314)        | 6.947(0.344)  | 9.08(0.287)  | 9.238(0.336)      |
|                      | M                        | 7.109(0.398)       | 7.248(0.286)  | 8.984(0.474) | 9.459(0.23)       |
|                      | S                        | 7.148(0.319)       | 7.124(0.337)  | 9.03(0.482)  | 9.408(0.395)      |
|                      | SD                       | 7.027(0.317)       | 7.019(0.338)  | 9.085(0.389) | 9.306(0.325)      |
| 10%                  | GQDA                     | 13.189(0.769)      | 37.826(0.705) | 12.462(0.841) | 38.007(0.623)     |
|                      | W                        | 7.135(0.346)       | 7.794(0.401)  | 10.473(0.925) | 11.983(0.376)     |
|                      | MVE                      | 7.005(0.339)       | 7.072(0.344)  | 11.255(0.354) | 11.503(0.388)     |
|                      | MCD                      | 6.942(0.341)       | 7.079(0.311)  | 11.207(0.341) | 11.489(0.303)     |
|                      | M                        | 7.561(0.413)       | 10.088(0.582) | 7.415(0.643)  | 14.317(0.597)     |
|                      | S                        | 7.048(0.299)       | 7.037(0.381)  | 11.126(0.648) | 11.851(0.681)     |
|                      | SD                       | 6.998(0.336)       | 6.966(0.319)  | 11.315(0.352) | 11.412(0.286)     |
| 15%                  | GQDA                     | 13.969(0.739)      | 41.254(0.463) | 12.363(0.655) | 38.785(0.526)     |
|                      | W                        | 7.48(0.392)        | 10.299(0.485) | 9.088(1.747)  | 16.273(0.613)     |
|                      | MVE                      | 7.02(0.304)        | 6.987(0.355)  | 13.385(0.365) | 13.755(0.368)     |
|                      | MCD                      | 7.007(0.298)       | 7.04(0.313)   | 13.287(0.794) | 13.758(0.343)     |
|                      | M                        | 9.062(0.493)       | 27.219(1.846) | 7.7(0.425)   | 31.183(1.801)     |
|                      | S                        | 7.066(0.34)        | 7.084(0.314)  | 13.099(1.18)  | 14.209(0.999)     |
|                      | SD                       | 6.982(0.285)       | 7.043(0.348)  | 13.375(0.353) | 13.621(0.298)     |
| 20%                  | GQDA                     | 13.823(0.733)      | 43.194(0.627) | 11.775(0.692) | 37.775(0.411)     |
|                      | W                        | 8.843(0.573)       | 31.401(1.676) | 7.135(0.437)  | 35.175(1.457)     |
|                      | MVE                      | 6.971(0.271)       | 7.021(0.354)  | 15.534(0.334) | 15.926(0.33)      |
|                      | MCD                      | 6.966(0.433)       | 6.973(0.289)  | 15.516(0.363) | 15.984(0.311)     |
|                      | M                        | 10.634(0.526)      | 41.337(0.674) | 8.552(0.418)  | 37.505(0.484)     |
|                      | S                        | 7.064(0.318)       | 7.076(0.367)  | 15.344(0.958) | 16.692(1.303)     |
|                      | SD                       | 6.999(0.295)       | 7.051(0.303)  | 15.403(0.339) | 15.934(0.37)      |

For example, it has been observed that for the Normal distribution, the improvement is in the range of 38%- 50% when training sets are mildly contaminated, whereas the improvement shoots to nearly 72% - 84% for hard contamination in the training sets. Though theoretically the test set is unknown, for the sake of comparison, we also study the improvements by using the RGQDA classifiers when both the training set and the test set are contaminated. The same scenario prevails in this case, showing the improvement for mild (hard) contamination in the range of nearly 17% - 45.5%.
Table 4: Average ME% (SD) for two-class problems with contaminated t distributions with 3 d.f., using the GQDA classifier and its different versions proposed in RGQDA

| Contamination degree | Method of classification | Contamination type          | Train mild          | Train hard          | Train and Test mild | Train and Test hard |
|----------------------|--------------------------|----------------------------|---------------------|---------------------|---------------------|---------------------|
| 5%                   | GQDA                     |                            | 14.411(0.962)       | 22.503(2.528)       | 13.728(1.296)       | 24.426(1.76)        |
|                      | W                        |                            | 12.285(0.406)       | 12.825(0.433)       | 14.039(0.569)       | 14.835(0.487)       |
|                      | MVE                      |                            | 12.335(0.461)       | 12.365(0.492)       | 14.049(0.494)       | 14.281(0.393)       |
|                      | MCD                      |                            | 12.316(0.403)       | 12.202(0.342)       | 14.071(0.402)       | **14.255(0.503)**   |
|                      | M                        |                            | 12.678(0.398)       | 13.276(0.473)       | **13.893(0.671)**   | 15.127(0.552)       |
|                      | S                        |                            | 12.271(0.389)       | 12.304(0.432)       | 14.027(0.447)       | 14.266(0.398)       |
|                      | SD                       |                            | 12.205(0.426)       | 12.261(0.457)       | 14.056(0.466)       | 14.267(0.334)       |
| 10%                  | GQDA                     |                            | 15.098(1.277)       | 40.687(1.145)       | 14.099(1.083)       | 41.666(1.095)       |
|                      | W                        |                            | 12.546(0.487)       | 12.658(0.768)       | 14.629(1.552)       | 18.153(0.6)         |
|                      | MVE                      |                            | 12.262(0.393)       | 12.304(0.482)       | 15.972(0.461)       | 16.312(0.415)       |
|                      | MCD                      |                            | 12.201(0.403)       | 12.171(0.394)       | 16.025(0.62)        | **16.207(0.392)**   |
|                      | M                        |                            | 13.171(0.541)       | 16.865(0.903)       | **13.122(1.082)**   | 20.415(0.616)       |
|                      | S                        |                            | 12.32(0.463)        | 12.185(0.44)        | 15.725(0.834)       | 16.332(0.498)       |
|                      | SD                       |                            | 12.296(0.436)       | 12.243(0.453)       | 15.829(0.722)       | 16.271(0.416)       |
| 15%                  | GQDA                     |                            | 15.181(1.049)       | 43.491(3.959)       | 13.628(1.158)       | 6.982(0.285)        |
|                      | W                        |                            | 12.932(0.568)       | 19.145(1.175)       | 12.386(1.021)       | 23.474(1.015)       |
|                      | MVE                      |                            | 12.249(0.423)       | 12.365(0.407)       | 17.68(0.693)        | 18.401(0.694)       |
|                      | MCD                      |                            | 12.202(0.449)       | 12.255(0.436)       | 17.364(0.771)       | **18.315(0.44)**    |
|                      | M                        |                            | 14.341(0.634)       | 31.945(2.213)       | 12.758(0.592)       | 35.55(1.799)        |
|                      | S                        |                            | 12.261(0.37)        | 12.239(0.444)       | 17.264(1.413)       | 18.718(0.89)        |
|                      | SD                       |                            | 12.339(0.393)       | 12.401(0.442)       | 17.427(1.344)       | 18.382(0.546)       |
| 20%                  | GQDA                     |                            | 14.968(1.053)       | 45.021(5.266)       | 12.893(1.199)       | 39.898(0.703)       |
|                      | W                        |                            | 13.864(0.657)       | 41.288(2.141)       | 11.639(0.671)       | 40.69(0.893)        |
|                      | MVE                      |                            | 12.28(0.442)        | 12.265(0.439)       | 19.185(1.63)        | 20.397(0.794)       |
|                      | MCD                      |                            | 12.292(0.424)       | 12.352(0.467)       | 19.498(0.53)        | **20.386(0.514)**   |
|                      | M                        |                            | 13.381(0.729)       | 44.599(0.651)       | 13.04(0.727)        | 40.531(0.618)       |
|                      | S                        |                            | 12.433(0.448)       | 12.325(0.424)       | 19.263(1.211)       | 20.389(0.566)       |
|                      | SD                       |                            | 12.434(0.427)       | 12.512(0.393)       | 19.307(1.416)       | 20.44(0.563)        |

(58% - 66%). For the t distribution with 3 d.f., the improvement is 15.3% - 19.6% (45.3%-72.7%) in the case of mild (hard) contamination of the training set, whereas the improvement is 6.9% - 9.7% (41.6% - 61.1%) in the case of mild (hard) contamination of both the training and the test set, except for few cases where the GQDA classifier still performs reasonably well. However, for the Cauchy distribution the improvement is rather less compared to the Normal distribution and the t distribution with 3 d.f., as expected, due to its heavy-tail nature, although the absolute amount of decrease in ME% is also quite significant using some RGQDA classifiers based on the estimators having greater robustness. The improvement for mild (hard) contamination of the training set has been noted in the range of 15.9%-19.7% (1.7%-44.8%), while the same for the contamination of both the train and the test set ranges in 17.3% – 30% (24.5% – 41.2%).

In terms of the comparisons among different robust versions of the GQDA classifier proposed in RGQDA, all of them perform similarly well for weaker (5%) contami-
Table 5: Average ME% (SD) for two-class problems with contaminated Cauchy distributions, using the GQDA classifier and its different versions proposed in RGQDA

| Contamination degree | Method of classification | Contamination type | Train mild | Train hard | Train and Test mild | Train and Test hard |
|----------------------|--------------------------|-------------------|------------|------------|--------------------|---------------------|
| 5%                   | GQDA                     | 24.868(4.423)     | 21.582(0.756) | 25.329(4.026) | 36.128(6.321) |
|                      | W                        | 19.951(0.66)     | 21.201(0.73)  | 20.934(0.987) | 22.848(0.747) |
|                      | MVE                      | 20.259(0.753)    | 20.205(0.681) | 21.62(0.804)  | 21.79(0.663)  |
|                      | MCD                      | 20.217(0.68)     | 20.252(0.526) | 21.645(0.835) | 21.831(0.477) |
|                      | M                        | 20.284(0.687)    | 21.548(0.826) | 21.296(1.232) | 23.078(0.695) |
|                      | S                        | 20.096(0.548)    | 20.097(0.564) | 21.47(0.738)  | 21.859(0.571) |
|                      | SD                       | 20.329(0.615)    | 20.302(0.612) | 21.659(0.707) | 21.77(0.601)  |
| 10%                  | GQDA                     | 23.873(3.822)    | 36.462(6.616) | 24.81(3.156)  | 39.747(5.038) |
|                      | W                        | 19.939(0.677)    | 24.069(1.523) | 20.304(1.54)  | 26.462(1.093) |
|                      | MVE                      | 20.244(0.591)    | 20.229(0.684) | 22.803(0.958) | 23.382(0.583) |
|                      | MCD                      | 20.174(0.555)    | 20.177(0.596) | 22.839(0.927) | 23.364(0.47)  |
|                      | M                        | 20.38(0.87)      | 25.596(1.507) | 20.37(1.231)  | 28.032(1.062) |
|                      | S                        | 20.287(0.576)    | 20.319(0.602) | 22.863(0.775) | 23.565(0.571) |
|                      | SD                       | 20.316(0.677)    | 20.267(0.785) | 22.922(1.241) | 23.727(0.541) |
| 15%                  | GQDA                     | 23.523(3.746)    | 36.509(6.699) | 24.265(4.166) | 38.447(5.004) |
|                      | W                        | 19.774(0.669)    | 36.029(4.374) | 18.576(0.817) | 37.523(3.091) |
|                      | MVE                      | 20.414(0.766)    | 20.14(0.607)  | 24.131(1.648) | 25.154(0.825) |
|                      | MCD                      | 20.121(0.658)    | 20.254(0.624) | 24.492(0.984) | 24.989(0.601) |
|                      | M                        | 20.572(0.607)    | 43.169(2.335) | 19.339(0.918) | 43.261(2.04)  |
|                      | S                        | 20.09(0.69)      | 20.148(0.474) | 24.339(0.815) | 25.155(0.552) |
|                      | SD                       | 20.363(0.754)    | 20.427(0.685) | 23.805(2.029) | 25.311(0.724) |
| 20%                  | GQDA                     | 22.867(3.529)    | 30.855(6.096) | 24.544(4.502) | 35.449(5.161) |
|                      | W                        | 19.514(0.589)    | 33.571(7.859) | 17.201(0.641) | 40.142(2.378) |
|                      | MVE                      | 20.391(0.715)    | 20.348(0.758) | 25.36(2.311)  | 26.942(0.988) |
|                      | MCD                      | 20.296(0.691)    | 20.12(0.55)   | 25.546(1.801) | 26.753(0.864) |
|                      | M                        | 20.935(0.751)    | 43.615(4.473) | 18.898(0.786) | 41.615(0.764) |
|                      | S                        | 20.356(0.617)    | 20.393(0.693) | 25.707(1.309) | 26.624(0.825) |
|                      | SD                       | 20.631(0.67)     | 20.61(0.78)   | 25.672(2.068) | 27.133(1.078) |

...and in all the cases considered here. For higher degree of contamination, better performances are observed by the use of strongly robust estimators like MVE, MCD, S or SD. However, interestingly, for the case of *mild* contamination of both the training and the test sets, most of the times the RGQDA classifier based on W or M estimator provides the best performance compared to the other versions.

### 4.4 Performances under contamination: four-class problem

The empirical results from the four-class simulations are reported in Table 6 through Table 8 for the three distributions discussed so far, under different types of contaminations. Besides the general trend of increasing improvement as the degree of contamination increases, the use of different robust statistics lessens the % misclassification error when the contamination is *hard* compared to the case when it is *mild*.

In particular, for the Normal distribution with four classes to classify into, the...
Table 6: Average ME% (SD) for four-class problems with contaminated Normal distributions, using the GQDA classifier and its different versions proposed in RGQDA

| Degree of contamination | Method of classification | Contamination type | Train mild mean (S.D.) | Train hard mean(S.D.) | Train and Test mild mean(S.D.) | Train and Test hard mean (S.D.) |
|-------------------------|-------------------------|-------------------|-----------------------|----------------------|-----------------------------|-------------------------------|
| 5%                      | GQDA                    | mild              | 12.126 (1.824)        | 26.956 (2.788)       | 10.475 (1.403)              | 31.803 (1.969)               |
|                         | W                       | 8.3 (0.920)        | 7.566 (1.040)         | 8.5567 (0.995)       | 12.093 (0.967)              |                               |
|                         | MVE                     | 7.696 (0.766)      | 8.753 (1.076)         | 9.05333 (1.1036)     | 11.26 (0.959)               |                               |
|                         | MCD                     | 7.96 (0.982)       | 13.566 (15.704)       | 8.83167 (1.124)      | 12.726 (0.950)              |                               |
|                         | M                       | 9.023 (0.951)      | 8.943 (0.941)         | 7.97 (0.938)         | 12.516 (1.034)              |                               |
|                         | S                       | 8.396 (1.104)      | 10.9 (1.240)          | 9.608 (1.427)        | 13.9 (1.095)                |                               |
|                         | SD                      | 8.76 (1.147)       | 9.6 (0.994)           | 7.898 (0.973)        | 11.796 (0.933)              |                               |
| 10%                     | GQDA                    | mild              | 13.696 (1.701)        | 52.086 (2.169)       | 11.896 (1.501)              | 53.376 (2.036)               |
|                         | W                       | 8.646 (1.017)      | 9.376 (1.005)         | 7.813 (1.085)        | 17.263 (0.970)              |                               |
|                         | MVE                     | 8.386 (0.879)      | 7.603 (0.884)         | 7.891 (1.168)        | 17.72 (0.962)               |                               |
|                         | MCD                     | 8.78 (0.931)       | 8.676 (0.888)         | 7.831 (1.254)        | 18.393 (0.840)              |                               |
|                         | M                       | 8.526 (1.099)      | 11.283 (1.754)        | 8.108 (1.030)        | 23.18 (1.475)               |                               |
|                         | S                       | 8.303 (0.864)      | 8.03 (0.859)          | 8.353 (1.304)        | 17.9 (0.992)                |                               |
|                         | SD                      | 8.096 (1.055)      | 8.74 (1.040)          | 7.428 (1.066)        | 18.6 (0.186)                |                               |
| 15%                     | GQDA                    | mild              | 13.07 (1.596)         | 44.09 (4.165)        | 13.006 (1.555)              | 47.603 (2.969)               |
|                         | W                       | 8.29 (0.937)       | 10.483 (0.847)        | 7.42 (1.004)         | 22.7 (1.070)                |                               |
|                         | MVE                     | 8.37 (1.039)       | 8.783 (0.954)         | 7.586 (1.535)        | 22.053 (1.012)              |                               |
|                         | MCD                     | 9.333 (1.010)      | 8.953 (0.862)         | 7.673(1.363)         | 21.503 (0.919)              |                               |
|                         | M                       | 9.366 (1.292)      | 49.503 (2.918)        | 9.24 (1.188)         | 49.8 (2.596)                |                               |
|                         | S                       | 10.093 (0.973)     | 9.123 (1.093)         | 10.466 (1.871)       | 22.46 (1.012)               |                               |
|                         | SD                      | 7.61 (0.886)       | 8.713 (0.857)         | 7.523 (0.902)        | 21.55 (0.852)               |                               |
| 20%                     | GQDA                    | mild              | 13.42 (1.381)         | 34.74 (3.061)        | 10.84 (1.607)               | 46.13 (3.573)                |
|                         | W                       | 10.173 (1.171)     | 26.59 (2.469)         | 9.03 (1.394)         | 36.76 (1.902)               |                               |
|                         | MVE                     | 8.813 (1.124)      | 8 (1.067)             | 7.723 (0.945)        | 26.543 (0.885)              |                               |
|                         | MCD                     | 8.073 (1.006)      | 8.986 (1.163)         | 7.613 (1.222)        | 28.07 (1.111)               |                               |
|                         | M                       | 11.226 (1.308)     | 45.83 (4.509)         | 9.85 (1.211)         | 50.56 (3.315)               |                               |
|                         | S                       | 8.04 (0.899)       | 9.31 (0.984)          | 7.776 (1.302)        | 25.91 (0.942)               |                               |
|                         | SD                      | 8.136 (0.994)      | 10.61 (9.342)         | 8.316 (1.605)        | 27.137 (0.899)              |                               |

Improvement is in the range of 36% to 42% with mild contamination and in the range of 72% to 86% in the case of hard contamination in the train set. Similar to the previous comparisons, we study the improvement when both the train and the (theoretically unknown) test set are contaminated. Similar to the two-class problems studied before, an improvement in the range of 24% to 43% has been observed in the case of mild contamination whereas the corresponding range is 45% to 67% for hard contamination.

For the $t$ distribution with 3 d.f. with four classes to classify into, the improvement is in the range of 21% to 29% with mild contamination, and in the range of 50% to 71% in the case of hard contamination in the train set. Similar to the previous comparisons, we study the improvement when both the train and the (theoretically unknown) test set are contaminated. Once again, the use of different RGQDA classifiers results into an improvement in the range of 22% to 32% for mild contamination, and in the range
Table 7: Average ME% (SD) in the test data set for four-class problems with contaminated $t$ distributions with 3 df, using the GQDA classifier and its different robust versions proposed in RGQDA

| Degree of contamination | Method of classification | Contamination type          | Train mild mean (S.D.) | Train hard mean (S.D.) | Train and Test mild mean (S.D.) | Train and Test hard mean (S.D.) |
|-------------------------|--------------------------|----------------------------|------------------------|------------------------|---------------------------------|---------------------------------|
| 5%                      | GQDA                     | Train mild                 | 23.453(2.084)          | 32.573(2.523)          | 23.417(2.214)                   | 35.75(2.616)                   |
|                         | W                        | Train mild                 | 17.323(1.176)          | 16.69(1.184)           | 16.327(0.983)                   | 20.75(1.438)                   |
|                         | MVE                      | Train mild                 | 17.087(1.354)          | 18.706(1.408)          | 19.293(1.363)                   | 21.08(1.242)                   |
|                         | M                        | Train mild                 | 17.91(1.489)           | 16.99(1.247)           | 16.813(1.365)                   | 21.77(1.371)                   |
|                         | M                         | Train and Test mild        | 16.573(1.346)          | 19.063(1.469)          | 16.53(1.451)                    | 22.317(1.31)                   |
|                         | S                         | Train and Test mild        | 17.56(1.321)           | 16.207(1.429)          | 18.21(1.45)                     | 22.387(1.260)                   |
|                         | SD                        | Train and Test mild        | 17.31(1.179)           | 18.61(1.237)           | 17.337(1.277)                   | 20.84(1.404)                   |
| 10%                     | GQDA                     | Train mild                 | 22.76(2.056)           | 58.343(5.557)          | 20.227(1.895)                   | 59.403(3.937)                  |
|                         | W                        | Train mild                 | 17.687(1.345)          | 18.7(1.605)            | 15.573(1.031)                   | 26.23(1.361)                   |
|                         | MVE                      | Train mild                 | 17.907(1.361)          | 17.067(1.314)          | 16.653(1.708)                   | 25.727(1.261)                  |
|                         | M                        | Train mild                 | 17.853(1.392)          | 17.053(1.236)          | 16.613(1.509)                   | 26.4(1.255)                    |
|                         | M                         | Train and Test mild        | 18.073(1.261)          | 22.697(1.772)          | 16.7(1.28)                      | 28.29(1.602)                   |
|                         | S                         | Train and Test mild        | 17.873(1.481)          | 17.927(1.417)          | 18.83333(2.127)                 | 26.227(1.214)                  |
|                         | SD                        | Train and Test mild        | 18.113(1.174)          | 17.48(1.437)           | 16.437(1.75)                    | 26.133(1.351)                  |
| 15%                     | GQDA                     | Train mild                 | 22.233(1.731)          | 51.72(2.810)           | 23.08(1.705)                    | 56.093(2.788)                  |
|                         | W                        | Train mild                 | 18.53(1.398)           | 22.81(1.608)           | 15.657(1.514)                   | 30.7(1.449)                    |
|                         | MVE                      | Train mild                 | 17.093(1.451)          | 17.813(1.410)          | 18.573(3.200)                   | 30.63(1.229)                   |
|                         | M                        | Train mild                 | 17.117(1.346)          | 17.933(1.2738)         | 18.56(2.268)                    | 32.213(8.905)                  |
|                         | M                         | Train and Test mild        | 19.363(1.513)          | 54.277(3.857)          | 17.78667(1.826)                 | 55.843(2.981)                  |
|                         | S                         | Train and Test mild        | 17.787(1.361)          | 16.787(1.368)          | 18.617(2.089)                   | 29.823(1.416)                  |
|                         | SD                        | Train and Test mild        | 18.61(1.472)           | 18.347(1.467)          | 15.583(1.338)                   | 30.203(1.182)                  |
| 20%                     | GQDA                     | Train mild                 | 22.64(2.195)           | 45.877(4.508)          | 20.923(1.900)                   | 52.31(4.467)                   |
|                         | W                        | Train mild                 | 21.977(1.675)          | 51.277(3.472)          | 18.267(1.775)                   | 52.69(3.250)                   |
|                         | MVE                      | Train mild                 | 17.84(5.531)           | 17.717(1.344)          | 17.307(3.126)                   | 31.65(2.226)                   |
|                         | M                        | Train mild                 | 18.153(1.292)          | 16.477(1.72155)        | 19.283(3.393)                   | 33.207(0.996)                  |
|                         | M                         | Train and Test mild        | 21.337(1.783)          | 56.173(6.229)          | 18.323(1.478)                   | 52.743(5.910)                  |
|                         | S                         | Train and Test mild        | 18.863(1.077)          | 15.643(1.469)          | 15.9(2.611)                     | 34.437(1.329)                  |
|                         | SD                        | Train and Test mild        | 18.2(1.305)            | 17.747(1.433)          | 16.317(2.218)                   | 33.423(1.246)                  |

of 39% to 57% for hard contamination. In summary, the justification for the use of different robust versions of the GQDA classifier proposed in RGQDA is emphatically evident from the simulation studies.
Table 8: Average ME% (SD) for four-class problems with contaminated Cauchy distributions, using the GQDA classifier and its different robust versions proposed in RGQDA

| Degree of contamination | Method of classification | Contamination type | Train mild mean (S.D.) | Train hard mean (S.D.) | Train and Test mild mean(S.D.) | Train and Test hard mean (S.D.) |
|------------------------|--------------------------|---------------------|------------------------|-----------------------|-------------------------------|-------------------------------|
|                        | GQDA                     | 5%                  | 45.193(3.691)          | 49.77(3.960)          | 57.923 (6.113)                | 52.546 (5.897)               |
|                        | W                        | 30.353(1.833)       | 29.626(1.507)          | 32.93(1.874)          | 32.58 (1.490)                 |
|                        | MVE                      | 29.436(1.775)       | 30.31(1.676)           | 33.84(1.612)          | 32.76 (1.898)                 |
|                        | MCD                      | 30.37(1.570)        | 28.55(1.738)           | 33.44(1.448)          | 34.03 (1.623)                 |
|                        | S                        | 30.17(1.610)        | 32.77(1.591)           | 34.15 (1.525)         | 34.95 (1.558)                 |
|                        | SD                       | 29.437(1.574)       | 30.943(1.477)          | 32.65 (6.361)         | 33.05 (2.588)                 |
|                        | 10%                      | GQDA                | 43.85(5.297)           | 75.193(7.158)         | 43.507 (3.640)                | 75.617 (8.686)               |
|                        | W                        | 31.107(1.964)       | 34.82(6.074)           | 28.84(1.706)          | 38.03 (1.575)                 |
|                        | MVE                      | 30.133(1.3496)      | 29.5(1.479)            | 30.06(2.59)           | 36.07 (1.457)                 |
|                        | MCD                      | 29.013(1.780)       | 32.07(1.620)           | 29.65(1.804)          | 37.34 (5.611)                 |
|                        | M                        | 31.54(1.961)        | 35.53(2.177)           | 27.35 (2.112)         | 38.34 (3.868)                 |
|                        | S                        | 31.457(1.294)       | 29.92(1.632)           | 30.14 (2.390)         | 38.52 (5.429)                 |
|                        | SD                       | 31.376(1.702)       | 30.05(1.369)           | 30.2 (2.480)          | 36.53 (1.239)                 |
|                        | 15%                      | GQDA                | 43.307(5.172)          | 74.34 (7.800)         | 40.997 (4.341)                | 72.613 (9.159)               |
|                        | W                        | 32.49(1.798)        | 51.91(5.687)           | 28.32(2.142)          | 47.83 (2.358)                 |
|                        | MVE                      | 31.003(1.469)       | 32.60(1.546)           | 31.41 (2.67)          | 39.99 (1.335)                 |
|                        | MCD                      | 28.987(1.564)       | 31.24(1.960)           | 27.98 (2.268)         | 40.34 (5.140)                 |
|                        | M                        | 30.89(1.820)        | 65.00(7.229)           | 29.18 (1.849)         | 62.56 (4.898)                 |
|                        | S                        | 31.52(1.772)        | 31.90(6.414)           | 30.46 (2.814)         | 46.14 (11.908)                |
|                        | SD                       | 32.09(2.635)        | 30.96(1.508)           | 29.81 (2.363)         | 40.74 (1.782)                 |
|                        | 20%                      | GQDA                | 40.2 (6.182)           | 65.18 (10.238)        | 42.22 (5.169)                 | 75.09 (7.151)               |
|                        | W                        | 30.03(1.821)        | 66.28(7.210)           | 27.73 (1.678)         | 65.97 (7.313)                 |
|                        | MVE                      | 32.05(1.878)        | 30.93(6.622)           | 34.74 (4.19)          | 44.95 (1.958)                 |
|                        | MCD                      | 31.37(4.010)        | 30.31(1.553)           | 35.90 (3.093)         | 41.9 (1.700)                  |
|                        | M                        | 31.38(2.043)        | 64.99(11.06)           | 27.96 (1.915)         | 62.08 (6.732)                 |
|                        | S                        | 30.00(1.981)        | 30.28(6.652)           | 33.43 (3.166)         | 41.88 (1.619)                 |
|                        | SD                       | 31.83(1.323)        | 31.38(1.668)           | 27.16 (2.390)         | 55 (2.659)                    |

5 Real Data Applications

To get a sense of the performances of different RGQDA classifiers in the real data scenario, we have also applied the proposed methods to classify several real data sets obtained from the UCI Machine Learning Repository [Dua and Graff, 2019]. To our expectation, all the findings are seen to support the claim that different RGQDA classifiers outperform the GQDA classifier. However, again for brevity, we present the results for three such data sets from wider application ranges, namely astrophysics, business and biomedical domains.
5.1 Data sets

Let us first present a brief description of each of the three data sets along with the feature lists that we have used in our illustration.

- **Ionosphere Data:** These radar data, collected by a system consisting of a phased array of 16 high-frequency antennas in Goose Bay, Labrador, consist of 351 observations (radar returns) classified into two classes, along with 34 associated features. The two classes are referred to as *Good* and *Bad*, respectively, depending on whether the corresponding signal is returned from the ionosphere indicating some structure there or it is passed through the ionosphere. An autocorrelation function is used to process the received signals with input as the time of a pulse and 17 pulse numbers, which results into 34 covariates corresponding to the real and imaginary parts of the complex electromagnetic signal output from each of the 17 pulse numbers. However, since the variation in the first two variables are observed to be zero, the remaining 32 variables (numbered 3-34) are used to classify the objects into the above-mentioned two classes.

- **Statlog ACA Data:** These data correspond to Australian credit card applications (suitably modified to maintain confidentiality). There are 690 observations belonging to either of the two classes recorded in an (anonymous) class attribute, along with 8 categorical features and 6 numerical features. We consider the 5 continuous variables (except the variable A10) to examine our RGQDA, since the *SD* estimator becomes computationally instable if A10 is included.

- **New Thyroid Data:** This data set, donated by Stefan Aeberhard from Garavan Institute in Sydney, Australia, contains information about 215 patients classified into three classes corresponding to euthyroidism, hypothyroidism or hyperthyroidism thyroid. The purpose is to form a predictive model to classify patient’s thyroid type into the above three classes based on 5 features which are 5 laboratory tests, namely T3-resin uptake test (in percentage), total Serum thyroxin as measured by the isotopic displacement method, total serum triiodothyronine as measured by radioimmuno assay, basal thyroid-stimulating hormone (TSH) as measured by radioimmuno assay, and maximal absolute difference of TSH value after injection of 200 micro grams of thyrotropin-releasing hormone as compared to the basal value. We have used all these five continuous feature variables to perform the proposed RGQDA.

For each of these three datasets, all available observations are randomly partitioned into the training set and the test set of sizes approximately 70% and 30%, respectively, of the entire data set. Further, for robustness illustrations, 10% of the training set has been forcefully misclassified to act like outliers. Then, we have applied different proposed RGQDA classifiers along with the GQDA classifier for the classification of the modified data set. The thresholds obtained from the contaminated training set are
then used to compute the % misclassification errors in the corresponding test set. This process is repeated 500 times with different random partition in each of the replications, and the boxplots of the resulting ME% s are presented for each of the proposed RGQDA classifier and the GQDA classifier in Figure 1 through Figure 3 respectively, for the three above-mentioned datasets.

![Boxplots of ME% using RGQDA classifiers](image)

Figure 1: Boxplots of ME% using the GQDA classifier and different RGQDA classifiers for the Ionosphere data

### 5.2 Results

For all the three above-mentioned data sets (along with several others not reported here for brevity), the advantages of using different RGQDA classifiers over the GQDA classifier is emphatically clear in the presence of contamination. In particular, for the ionosphere data, the box-plots of the ME% s for each of the RGQDA classifiers is way below than the one obtained using the the GQDA classifier. This finding is also consistent with what has been observed in our simulation studies, indicating better performances of the RGQDA classifiers. For the multi class classification in the New Thyroid data as well, it has been resoundingly reinforced that the RGQDA method works much better than the GQDA classifier with significantly lesser spread of the box-plots of the % misclassification errors. Among the different robust versions proposed in RGQDA, the classifiers based on W and S estimators perform the best for the New Thyroid data set, whereas all other classifiers using MCD, M and S estimators exhibit a similar better performance for the Ionosphere data. For the Statlog ACA data, the results are comparatively more sparsed, with the classifier based on W estimator
Figure 2: Boxplots of ME%\text{\textregistered}s using the GQDA classifier and different RGQDA classifiers for the Statlog ACA data

Figure 3: Boxplots of ME%\text{\textregistered}s using the GQDA classifier and different RGQDA classifiers for the New Thyroid data

providing the best performance in RGQDA. Overall, the justification of the use of different RGQDA classifiers for the purpose of classification is profoundly demonstrated in all the three data sets.
6 Conclusion and discussions

In this paper, for the multivariate classification problem, in the likely presence of outliers in the data set, an attempt has been made to study the performance of the GQDA classifier, which is a generalization of the QDA classifier and the MMD classifier, proposed by [Bose et al. (2015)] and suggest a better strategy. Through the simulation studies of some elliptically symmetric distributions as well as working with some real data sets, it has been shown that the GQDA classifier performs miserably when the data set is contaminated. Hence to propose a better alternative, an investigation has been made to replace the estimators used in the GQDA classifier by different robust versions of the classical estimator of the mean vector and the dispersion matrix available in the literature and the new procedure has been termed as RGQDA. An overall commendable performance of different RGQDA classifiers has been transpired so far, in all the examples we tried with, be it simulation or real life data set, varying the degree of improvement across the different robust versions adopted in RGQDA. Thus a comparative study of the performances of different RGQDA classifiers, using different robust versions of the classical estimator in the presence of different degree and the type of contamination suggests the choice of the particular robust estimator to be adopted, to have a reasonably good classification, in terms of reducing the % misclassification error. It is also interesting to note that even when there is no contamination in the data set, the proposed RGQDA classifiers outperform the GQDA classifier in classifying the underlying populations having distributions with heavier tail than the Normal distribution. The performance of different RGQDA classifiers needs to be looked into when we face the dimensionality reduction issue, in case of high-dimensional set-ups with large p-small n problem. Future investigation in this direction will be reported in a separate article.

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