Hamiltonian, path integral and BRST formulations of the Chern–Simons–Higgs theory in the broken symmetry phase

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Abstract
The Hamiltonian, path integral and BRST formulations of the Chern–Simons–Higgs theory in two-space one-time dimensions are investigated under appropriate gauge-fixing conditions, in the broken (or frozen) symmetry phase, where the phase \( \phi(x^\mu) \) of the complex matter field \( \Phi(x^\mu) \) carries the charge degree of freedom of the complex matter field and is, in fact, akin to the Goldstone boson.

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1. Introduction
The effective theories with excitations, with fractional statistics are supposed to be described by gauge theories with Chern–Simons terms in two plus one dimensions\(^1\) and a study of the gauge field theories and the models of quantum electrodynamics with a Higgs potential, namely, the Abelian Higgs models involving the vector gauge field \( A^\mu(x^\nu) \), and the Chern–Simons–Higgs theories in two-space one-time dimensions has been of wider interest in recent years \([1–3]\). In this work, we study the Hamiltonian, path integral and Becchi–Rouet–Stora and Tyutin (BRST) formulations \([3–6]\) of the Chern–Simons–Higgs theory \([1–3]\), in two-space one-time dimensions, in the broken (or frozen) symmetry phase (BSP) \([2, 3]\), where the phase \( \phi(x^\mu) \) of the complex matter field \( \Phi(x^\mu) \) carries the charge degree of freedom of the complex matter field and is, in fact, akin to the Goldstone boson, in the usual instant-form (IF) of dynamics (on the hyperplanes: \( x^0 = t = \text{constant} \)), under appropriate gauge-fixing conditions. This theory has been quantized canonically in the Schrödinger picture \([2]\), paying particular attention to the understanding of the gauge-invariant wave functionals and operators by Boyanovsky \([2]\). The studies of Boyanovsky \([2]\) were motivated by the twin facts of understanding whether excitations in a broken symmetry state (superfluid) can carry fractional statistics (a question which is relevant for the effective theories in condensed matter) as well as understanding the issue of exotic statistics in gauge-invariant observables \([2]\). For details see \([2]\). In fact, models in the so-called BSP \([2, 3]\), can be thought of as Higgs models where the (Higgs) potential has been set to zero by freezing the scalar field at the degenerate minima of the Higgs potential, and the phase of the complex matter field carries the charge degree of freedom of the complex matter field \([2, 3]\). After very briefly recapitulating some basics of the theory, we study its Hamiltonian and path integral formulations in the next section and its BRST formulation in section 4. The summary and discussion is given in section 5.

2. Chern–Simons–Higgs theory in the BSP

The Chern–Simons–Higgs theory in two-space one-time dimensions is defined by the action (with \( \Phi_0 : \text{real, and } \Phi_0 \neq 0 \)) \([1, 2]\):

\[
S = \int \mathcal{L}_1(\Phi, \Phi^*; A^\mu) \, d^3x,
\]

\[
\mathcal{L}_1 = \left[ \frac{k}{2} \epsilon^{\mu\nu\lambda} \partial_\mu A_\nu A_\lambda + (\bar{D}_\mu \Phi^*) (D^\mu \Phi) - V(|\Phi|^2) \right],
\]

\[
V(|\Phi|^2) = \left[ \alpha_0 + \alpha_2 |\Phi|^2 + \alpha_4 |\Phi|^4 \right],
\]

\[
= \lambda (|\Phi|^2 - \Phi_0^2)^2; \quad \Phi \neq 0
\]

\(^1\) For a comprehensive recent review see e.g. \([1]\).
\[ D_\mu = (\partial_\mu + ieA_\mu); \quad \bar{D}_\mu = (\partial_\mu - ieA_\mu); \quad \kappa = \left( \frac{\theta}{2\pi^2} \right)^2 \]

\[ g^{\mu\nu} := \text{diag}(+1, -1, -1); \quad \mu, \nu = 0, 1, 2; \quad \epsilon^{012} = \epsilon_{012} = +1. \]

Here \( \theta \) is the Chern–Simons parameter. Also, in the following, we keep the Higgs potential rather general, i.e. without making any specific choice for the parameters of the potential except that they are chosen such that the potential remains a double well potential with \( \Phi_0 \neq 0 \). This action thus describes the theory in the so-called symmetry phase \([1, 2]\). In the following we however, study this theory in the so-called BSP \([2, 3]\), of the complex matter field \( \Phi(\equiv \Phi(\chi^\mu)) \). For this purpose, for the complex matter field \( \Phi \) we take \([2, 3]\):

\[ \phi(x^\mu) = \Phi_0(\exp(i\varphi(x^\mu))); \quad \Phi_0 \neq 0. \]

Here \( \varphi(\equiv \varphi(x^\mu)) \) is the phase of the complex matter field \( \Phi(x^\mu) \). The action of the theory in the BSP \([2, 3]\) then becomes:

\[ S = \int \mathcal{L} d^3x; \quad \mathcal{L} := \frac{\kappa}{2} e^{\mu\nu\lambda} \partial_\mu A_\nu A_\lambda + \frac{1}{2}(\partial_\mu \varphi + eA_\mu)(\partial^\mu \varphi + eA^\mu). \]

It is important to notice here that the vector gauge boson \( A^\mu \) becomes massive in the BSP. This mass generation of the vector gauge boson takes place perhaps through a mechanism similar to the Higgs mechanism \([2, 3]\). The phase \( \varphi \) carries the charge degree of freedom of \( \Phi \) and is, in fact, akin to the Goldstone boson and is to be treated as a dynamical field \([2, 3]\). Also, the ground state in the BSP is not rotational invariant. Such studies of the theory in the broken-symmetry (superfluid) state could be relevant for the effective theories in the condensed matter, as the action of the theory describes the low-lying excitations in the BSP \([2, 3]\), as well as for an understanding of the issue of exotic statistics in gauge-invariant observables \([2]\). In the following sections we study the Hamiltonian and BRST formulations of the theory described by the above action \( S \).

3. Hamiltonian and path integral formulations

We now study the Hamiltonian and path integral formulations of the theory in the BSP in the usual IF (equal-time) framework, on the hyperplanes \( x^0 = t = \text{constant} \). The canonical momenta obtained from the action \( S \) are:

\[ \pi := \frac{\partial \mathcal{L}}{\partial (\partial_0 \varphi)} = (\partial_0 \varphi + eA_0); \quad \Pi_0 := \frac{\partial \mathcal{L}}{\partial (\partial_0 A_0)} = 0; \]

\[ E_1 := (\Pi^1) := \frac{\partial \mathcal{L}}{\partial (\partial_0 A_1)} = \frac{\kappa}{2} A_2; \]

\[ E_2 := (\Pi^2) := \frac{\partial \mathcal{L}}{\partial (\partial_0 A_2)} = -\frac{\kappa}{2} A_1. \]

Here \( \pi, \Pi_0, E_1 := (\Pi^1) \) and \( E_2 := (\Pi^2) \) are the momenta canonically conjugate respectively to \( \varphi, A_0, A_1 \) and \( A_2 \). The above equations imply that the theory possesses three primary constraints:

\[ \chi_1 = \Pi_0 \approx 0; \quad \chi_2 = \left( E_1 - \frac{\kappa}{2} A_2 \right) \approx 0; \quad \chi_3 = \left( E_2 + \frac{\kappa}{2} A_1 \right) \approx 0. \]

Here the symbol \( \approx \) denotes a weak equality (WE) in the sense of Dirac \([6]\), and it implies that these above constraints hold as strong equalities only on the reduced hypersurface of the constraints and not in the rest of the phase space of the classical theory (and similarly one can consider it as a weak operator equality (WOE) for the corresponding quantum theory) \([6]\). The canonical Hamiltonian density corresponding to \( \mathcal{L} \) (with \( F_{12} := (\partial_1 A_2 - \partial_2 A_1) \)) is:

\[ \mathcal{H}_c := \left[ \pi (\partial_0 \varphi) + \Pi_0 (\partial_0 A_0) + E_1 (\partial_0 A_1) + E_2 (\partial_0 A_2) - \mathcal{L} \right] = \frac{1}{2} \pi^2 - eA_0 \pi - \frac{1}{2} \kappa (A_0 F_{12} + A_1 \delta A_2 - A_2 \delta A_0) + \frac{1}{2} (\partial_1 \varphi)^2 + 2eA_1 \partial_1 \varphi + eA_2 \varphi + eA^2 \varphi_1. \]

After including the primary constraints \( \chi_1, \chi_2 \) and \( \chi_3 \), in the canonical Hamiltonian density \( \mathcal{H}_c \) with the help of Lagrange multiplier fields \( u(x, y, t), v(x, y, t) \) and \( w(x, y, t) \), which are treated as dynamical, the total Hamiltonian density \( \mathcal{H}_T \) could be written as:

\[ \mathcal{H}_T = \mathcal{H}_c + \Pi^0 u + \left( E_1 - \frac{\kappa}{2} A_2 \right) v + \left( E_2 + \frac{\kappa}{2} A_1 \right) w. \]

The Hamilton equations of motion that preserve the constraints of the theory in the course of time could be obtained from the total Hamiltonian; \( \mathcal{H}_T = \int \mathcal{H}_T d^3x \) and are omitted here for the sake of brevity. Now demanding that the primary constraint \( \chi_1 \) be preserved in the course of time, one obtains the secondary Gauss-law constraint of the theory as:

\[ \chi_4 = (e\pi + \kappa F_{12}) \approx 0. \]

The preservation of \( \chi_2 \) and \( \chi_3 \) for all time does not give rise to any further constraints. The theory is thus seen to possess only four constraints \( \chi_1, \chi_2, \chi_3 \) and \( \chi_4 \):

\[ \chi_1 = \Pi_0 \approx 0; \quad \chi_2 = \left( E_1 - \frac{\kappa}{2} A_2 \right) \approx 0; \quad \chi_3 = \left( E_2 + \frac{\kappa}{2} A_1 \right) \approx 0; \quad \chi_4 = (e\pi + \kappa F_{12}) \approx 0. \]

where \( \chi_1, \chi_2 \) and \( \chi_3 \) are primary constraints and \( \chi_4 \) is a secondary constraint. We now calculate the matrix \( M_{\alpha\beta}(x, y) \), where \( x \equiv x(t, x_1, x_2) \), of the Poisson brackets (PBs) among the constraints \( \chi_i \) of the theory. The nonvanishing elements of this matrix are:

\[ M_{23} = -M_{32} = -\kappa \delta(x_1 - y_1) \delta(x_2 - y_2); \]

\[ M_{24} = +M_{42} = +\kappa \delta(x_1 - y_1) \partial_3 \delta(x_2 - y_2); \]

\[ M_{34} = +M_{43} = +\kappa \partial_3 \delta(x_1 - y_1) \delta(x_2 - y_2). \]
It is obvious from the above equations that the constraints $\chi_1$ and $\chi_3$ are first-class while $\chi_2$ and $\chi_3$ are a pair of second-class constraints. Furthermore, the nonvanishing elements of the above matrix correspond to the Gauss anomalies present in the theory. The determinant of the matrix $M_{\alpha\beta}(x, y)$ is zero and therefore the matrix is singular. This implies that the entire set of constraints $\chi_i$ is first-class and that the theory under consideration is gauge-invariant [2–4].

The physical degrees of freedom of the system are governed by the reduced Hamiltonian density of the theory (which is obtained by implementing the constraints of the theory strongly). Also, in the present case, $A_0$ and $\varphi$ play the role of gauge variables and the two pairs $(A_1, E_1)$ and $(A_2, E_2)$ are the pair of inessential eliminable variables and the pair describing the physical degrees of freedom of the system. Accordingly, we choose, in the present case, the first pair namely, $(A_1, E_1)$ as the pair describing the physical degrees of freedom and the other pair as the pair of inessential eliminable variables. So to write the reduced Hamiltonian density of the theory, we choose $\varphi, \pi, A_1$ and $E_1$ as the independent variables and the remaining phase space variables as the dependent variables. The later ones are then expressed in terms of the independent variables as:

$$
\Pi_0 = 0; \quad E_2 = -\frac{\kappa}{2} A_1; \quad A_2 = \frac{\kappa}{2} E_1; \quad \pi = -\frac{\kappa}{e}(\partial_1 A_2 - \partial_2 A_1).
$$

Finally the reduced Hamiltonian density of the theory describing the physical degrees of freedom of the system expressed in terms of the independent variables is then obtained as:

$$
H_R = \left[\frac{1}{2}(\partial_1 \varphi)^2 + \frac{1}{2}(\partial_2 \varphi)^2 + \frac{e^2}{\kappa^2}(A_1)^2 + \frac{2e^2}{\kappa^2}(E_1)^2 + eA_1 \partial_1 \varphi + \frac{4e}{\kappa} E_1 \partial_2 \varphi\right].
$$

Here we remind ourselves that as an alternative to the above, we could have equivalently expressed it in terms of the other pair namely, $(A_2, E_2)$ instead of the pair $(A_1, E_1)$. From the above equation we then obtain the field equations derived from the Heisenberg equations as:

$$
\partial_0 \varphi = -i[\varphi, H_R] = 0,
$$

$$
-\partial_0 E_1 = -i[E_1, H_R] = \left[e^2 A_1 + e\partial_1 \varphi + \frac{2\kappa}{e^2} \partial_2 (\partial_1 E_1 - \frac{\kappa}{2} \partial_2 A_1)\right].
$$

where $H_R = \int H_R dx$ is the reduced Hamiltonian of the theory and it describes the physical degrees of freedom of the system. Furthermore the action of the theory is seen to be invariant under the local vector gauge transformations:

$$
\delta \varphi = e \alpha, \quad \delta A_1 = -\partial_1 \alpha, \quad \delta A_2 = -\partial_2 \alpha, \quad \delta \Pi_0 = \delta \pi = 0,
$$

$$
\delta A_0 = -\partial_0 \alpha, \quad \delta E_1 = -\frac{\kappa}{2} \partial_2 \alpha, \quad \delta E_2 = \frac{\kappa}{2} \partial_1 \alpha, \quad \delta \Pi_0 = \delta \pi = 0,
$$

$$
\delta u = -\partial_0 \delta \alpha, \quad \delta v = -\partial_1 \delta \alpha, \quad \delta w = -\partial_2 \delta \alpha, \quad \delta \Pi_0 = \delta \pi = 0,
$$

where $\alpha \equiv \alpha \left(i, x^1, x^2\right)$ is an arbitrary function of its arguments. The vector gauge current of the theory $J^0 \equiv (J^0, J^1, J^2)$ is:

$$
J^0 = \int j^0 d^2 x = \int d^2 x \left[ e \alpha(\partial_0 \varphi + e A_0) - \frac{\kappa}{2} (A_2 \partial_1 \alpha - A_1 \partial_2 \alpha)\right],
$$

$$
J^1 = \int j^1 d^2 x = \int d^2 x \left[ -e \alpha(\partial_1 \varphi + e A_1) + \frac{\kappa}{2} (A_2 \partial_0 \alpha - A_0 \partial_2 \alpha)\right],
$$

$$
J^2 = \int j^2 d^2 x = \int d^2 x \left[ -e \alpha(\partial_2 \varphi + e A_2) - \frac{\kappa}{2} (A_1 \partial_0 \alpha - A_0 \partial_1 \alpha)\right].
$$

The divergence of the vector gauge current density of the theory could now be easily seen to vanish satisfying the continuity equation: $\partial_0 J^0 = 0$, implying that the theory possesses at the classical level, a local vector-gauge symmetry. Now in order to quantize the theory using Dirac’s procedure, we convert the set of first-class constraints of the theory $\chi_i$ into a set of second-class constraints, by imposing, arbitrarily, some additional constraints on the system called gauge-fixing conditions or the gauge constraints. For this purpose, for the present theory, we could choose, for example, the set of gauge-fixing conditions:

$$
\eta_1 = A_0 \approx 0; \quad \eta_2 = A_1 \approx 0.
$$

It is important to mention here that the gauge $A_0 \approx 0$, is the temporal gauge or the time-axial gauge and the gauge $A_1 \approx 0$, is the coulomb gauge for the electromagnetic field $A^\mu$. These gauge choices are not only acceptable and consistent with our quantization procedures but are also physically more interesting gauge choices representing the time-axial or the temporal gauge and the coulomb gauge respectively. Corresponding to this set of gauge-fixing conditions, the total
Also for later use, for considering the BRST formulation of the theory could be studied becomes:

\[
\psi_1 = \chi_1 = \Pi_0 \approx 0; \quad \psi_2 = \chi_2 = \left( E_1 - \frac{\kappa}{2} A_2 \right) \approx 0, \quad (17a)
\]

\[
\psi_3 = \chi_3 = \left( E_2 + \frac{\kappa}{2} A_1 \right) \approx 0; \quad \psi_4 = \chi_4 = (e\pi + \kappa F_{12}) \approx 0, \quad (17b)
\]

\[
\psi_5 = \eta_1 = A_0 \approx 0; \quad \psi_6 = \eta_2 = A_1 \approx 0. \quad (17c)
\]

The nonvanishing matrix elements of matrix \( R_{\alpha\beta} \) of the PBs among the set of constraints \( \psi_i \) are:

\[
R_{23} = -R_{32} = -\kappa \delta(x_1 - y_1) \delta(x_2 - y_2), \quad (18a)
\]

\[
R_{24} = +R_{42} = +\kappa \delta(x_1 - y_1) \partial_2 \delta(x_2 - y_2), \quad (18b)
\]

\[
R_{34} = +R_{43} = +\kappa \partial_1 \delta(x_1 - y_1) \delta(x_2 - y_2), \quad (18c)
\]

\[
R_{15} = -R_{51} = -\delta(x_1 - y_1) \delta(x_2 - y_2), \quad (18d)
\]

\[
R_{26} = -R_{62} = -\delta(x_1 - y_1) \delta(x_2 - y_2). \quad (18e)
\]

This matrix is seen to be nonsingular with the determinant given by

\[
\sqrt{|\det(R_{\alpha\beta})|} = \left[ \kappa \left[ \partial_1 \delta(x_1 - y_1) | \delta(x_2 - y_2) \right] \right].
\]

Finally, following the standard Dirac quantization procedure, the nonvanishing equal-time commutators of the theory, under the gauge fixing conditions: \( \eta_i \) i.e. under \( A_0 \approx 0 \) and \( A_1 \approx 0 \), are obtained as:

\[
[E_1(t, x_1, x_2), E_2(t, y_1, y_2)] = \left[ \frac{ik}{2} \delta(x_1 - y_1) \delta(x_2 - y_2) \right],
\]

\[
[A_2(t, x_1, x_2), E_2(t, y_1, y_2)] = \left[ 2i \delta(x_1 - y_1) \delta(x_2 - y_2) \right],
\]

\[
[\psi(t, x_1, x_2), E_1(t, y_1, y_2)] = \left[ \frac{ie}{4} \delta(x_1 - y_1) \delta(x_2 - y_2) \right],
\]

\[
[\psi(t, x_1, x_2), A_2(t, y_1, y_2)] = \left[ \frac{ie}{2k} \delta(x_1 - y_1) \delta(x_2 - y_2) \right],
\]

where the step function \( \epsilon(x - y) \) is defined as:

\[
\epsilon(x - y) = \begin{cases} +1, & (x - y) > 0, \\ -1, & (x - y) < 0. \end{cases}
\]

Also for later use, for considering the BRST formulation of the theory we convert the total Hamiltonian density of the theory into the first-order Lagrangian density \( L_{10} \):

\[
L_{10} = \left[ \pi \partial_0 \varphi + \Pi_0 \partial_0 A_0 + E_1 \partial_0 A_1 + E_2 \partial_0 A_2 + \Pi_\alpha \partial_0 u + \Pi_\alpha \partial_0 v + \Pi_\alpha \partial_0 w - H_T \right] = \left[ \pi \partial_0 \varphi + \frac{\kappa}{2} \left( A_2 \partial_0 A_1 - A_1 \partial_0 A_2 \right) \right.
\]

\[
+ \frac{\kappa}{2} (A_0 F_{12} + A_1 \partial_2 A_0 - A_2 \partial_1 A_0) + e A_0 \pi - \frac{1}{2} \pi^2 - \frac{1}{4} \left[ \partial_1 \varphi \right]^2
\]

\[
+ 2e A_1 \partial_1 \varphi + e^2 A_1^2 - \frac{1}{4} \left[ (\partial_2 \varphi)^2 + 2e A_2 \partial_2 \varphi + e^2 A_2^2 \right].
\]

While considering the path integral formulation, the transition to quantum theory is made again by writing the vacuum to vacuum transition amplitude for the theory, called the generating functional \( Z[J_k] \) of the theory, following again the Senjanovic procedure for a theory possessing a set of second-class constraints \([4-6]\), appropriate for our present theory, considered under the gauge-fixing conditions \( \eta_i \) in the presence of the external sources: \( J_k \) as follows \([4-6]\):

\[
Z[J_k] = \int [d\mu] \exp[i \int d^3 x \left( J_k \Phi^k + \pi \partial_0 \varphi + \Pi^0 \partial_0 A_0 
\]

\[
+ E_1 \partial_0 A_1 + E_2 \partial_0 A_2 + \Pi_\alpha \partial_0 u + \Pi_\alpha \partial_0 v + \Pi_\alpha \partial_0 w - H_T \right],
\]

\[
(23)
\]

where the phase space variables of the theory are: \( \Phi^k \equiv (\varphi, A_0, A_1, A_2, u, v, w) \) with the corresponding respective canonical conjugate momenta: \( \Pi_k \equiv (\pi, \Pi_0, E_1, E_2, \Pi_\alpha, \Pi_\alpha, \Pi_\alpha) \). The functional measure \( [d\mu] \) of the generating functional \( Z[J_k] \) under this gauge-fixing is obtained as:

\[
[d\mu] = \left[ \kappa \left[ \partial_1 \delta(x_1 - y_1) \right] \right] \left[ \partial_2 \delta(x_2 - y_2) \right] [d\varphi]
\]

\[
\times [dA_0] [dA_1] [dA_2] [du] [dv] [dw] [d\pi] [d\Pi_0] [dE_1] [dE_2]
\]

\[
\times [dp_\alpha] [dp_\alpha] [dp_\alpha] [dp_\alpha] [dp_\alpha] [dp_\alpha] [dp_\alpha] \Pi_0 \approx 0 \approx [E_1 - \frac{k}{2} A_2] \approx 0
\]

\[
\times \delta \left[ (E_2 + \frac{k}{2} A_1) \approx 0 \right] \delta(\kappa F_{12} \approx 0) \delta[A_0 \approx 0] \delta[A_1 \approx 0].
\]

\[
(24)
\]

The Hamiltonian and path integral formulations of the theory under the set of gauge-fixing conditions \( A_0 \approx 0 \) and \( A_1 \approx 0 \) are now complete. We now proceed to quantize the present theory under another important set of gauge-fixing conditions namely:

\[
\rho_1 = A_0 \approx 0; \quad \rho_2 = \varphi \approx 0. \quad (25)
\]

This set of gauge-fixing conditions is also perfectly acceptable and consistent with our quantization procedures. Here, the condition \( A_0 \approx 0 \), again represents the time-axial or the temporal gauge and the condition \( \varphi \approx 0 \) is the usual coulomb gauge for the scalar (phase) field \( \varphi \). and, in fact, it could even be a better gauge choice because the scalar (phase) field \( \varphi \) happens to be a gauge variable of the theory. Corresponding to this choice of gauge-fixing conditions, the total set of constraints of the theory under which the quantization of the theory could be studied becomes:

\[
\Psi_1 = \chi_1 = \Pi_0 \approx 0; \quad \Psi_2 = \chi_2 = \left( E_1 - \frac{k}{2} A_2 \right) \approx 0. \quad (26a)
\]
\[ \Psi_3 = \chi_3 = \left( E_2 + \frac{\kappa}{2} A_1 \right) \approx 0; \quad \Psi_4 = \chi_4 = (e\pi + \kappa F_{12}) \approx 0, \]
\[ (26b) \]
\[ \Psi_5 = \rho_1 = A_0 \approx 0; \quad \Psi_6 = \rho_2 = \varphi \approx 0. \]  
\[ (26c) \]
The nonvanishing matrix elements of matrix \( N_{a\beta} \) of the PBs among the set of constraints \( \Psi_i \) are
\[ N_{23} = -N_{32} = -\kappa (x_1 - y_1) \delta(x_2 - y_2), \]
\[ N_{24} = +N_{42} = +\kappa \delta(x_1 - y_1) \delta(x_2 - y_2), \]
\[ N_{34} = +N_{43} = +\kappa \partial_i \delta(x_1 - y_1) \delta(x_2 - y_2), \]
\[ N_{15} = -N_{51} = -\delta(x_1 - y_1) \delta(x_2 - y_2), \]
\[ N_{46} = -N_{64} = -c \delta(x_1 - y_1) \delta(x_2 - y_2). \]
\[ (27a) \]
\[ (27b) \]
\[ (27c) \]
\[ (27d) \]
\[ (27e) \]
This matrix \( N_{a\beta} \) is seen to be nonsingular with the determinant given by
\[ |\text{det}(N_{a\beta})|^{1/2} = |e\kappa [\delta^3(x_1 - y_1)][\delta^3(x_2 - y_2)]| \]
\[ (28) \]
implying that the theory could very well be studied under this set of gauge-fixing conditions also. Finally, the nonvanishing equal-time commutators of the theory as well as the relevant path integral of the theory under the gauge fixing conditions: \( \rho_i \) i.e. under \( A_0 \approx 0 \) and \( \varphi \approx 0 \), could again be obtained after a lengthy but rather straightforward calculation. In the next section, we consider the BRST quantization of the theory.

### 4. BRST formulation

For the BRST formulation of the model, we rewrite the theory as a quantum system that possesses the generalized gauge invariance called BRST symmetry. For this, we first enlarge the Hilbert space of our gauge-invariant theory and replace the notion of gauge-transformation, which shifts operators by \( c \)-number functions, by a BRST transformation, which mixes operators with Bose and Fermi statistics. We then introduce new anti-commuting variables \( c \) and \( \tilde{c} \) (Grassmann numbers on the classical level, operators in the quantized theory) and a commuting variable \( b \) such that [5]:
\[ \slashed{b} = ec, \quad \slashed{\partial} = 0, \quad \slashed{\partial} A_0 = -\partial c, \]
\[ \slashed{\partial} A_1 = -\partial_1 c, \quad \slashed{\partial} A_2 = -\partial_2 c, \]
\[ (29a) \]
\[ \slashed{\partial} E_0 = 0, \quad \slashed{\partial} E_1 = -\frac{\kappa}{2} \partial_2 c, \quad \slashed{\partial} E_2 = \frac{\kappa}{2} \partial_1 c, \]
\[ \slashed{\partial} \Pi_0 = 0, \quad \slashed{\partial} \Pi_1 = \slashed{\partial} \Pi_2 = 0, \]
\[ \slashed{\partial} u = -\partial_0 \partial c, \quad \slashed{\partial} v = -\partial_0 \partial_1 c, \quad \slashed{\partial} w = -\partial_0 \partial_2 c, \]
\[ \slashed{\partial} c = 0, \quad \slashed{\partial} \tilde{c} = b, \quad \slashed{\partial} b = 0 \]
\[ (29b) \]
with the property \( \slashed{\partial}^2 = 0 \). We now define a BRST-invariant function of the dynamical variables to be a function \( f(\tau, \Pi_0, \ldots) \).
so that:

$$\mathcal{H}_{\text{BRST}} = [\pi_0 \partial \phi + \pi_0 \partial A_0 + E_1 \partial_1 A_1 + E_2 \partial_2 A_2 + \pi_4 \partial \mu + \pi_2 \partial \nu + \pi_2 \partial v + \pi_2 (\partial_0 \mu) + \pi_4 \partial \nu] - \mathcal{L}_{\text{BRST}}$$

$$= \left[ \frac{1}{4} \pi^2 + \frac{i}{2} \pi \partial_0 \phi + \frac{1}{2} \kappa A_0 F_{12} - \frac{1}{4} \pi_2 \partial_2 \mu + \pi_2 \partial_2 v + \frac{1}{2} (\partial_0 \phi)^2 + 2e A_1 \partial_1 \nu + e^2 A_1^2 \right]$$

$$+ \frac{1}{2} (\partial_1 \phi \partial_2 \phi - 2e A_2 \partial_2 \phi + e^2 A_2^2).$$  \hfill (37)

We can check the consistency of our definitions of the fermionic momenta by looking at the Hamilton equations for the Fermionic variables:

$$\partial_0 \mu = \langle \partial_0 \mathcal{H}_{\text{BRST}}; \partial_0 \mu \rangle; \quad \partial_0 \nu = \langle \partial_0 \mathcal{H}_{\text{BRST}}; \partial_0 \nu \rangle.$$  \hfill (38)

Thus we see that

$$\partial_0 \mu = \langle \partial_0 \mathcal{H}_{\text{BRST}}; \partial_0 \mu \rangle = \partial_0 \nu,$$

$$\partial_0 \nu = \langle \partial_0 \mathcal{H}_{\text{BRST}}; \partial_0 \nu \rangle$$

is in agreement with our definitions of the fermionic momenta. Also, for the operators \( c, \bar{c}, \partial_0 c \) and \( \partial_0 \bar{c}, \) one needs to satisfy the anticommutation relations of \( \partial_0 c \) with \( \bar{c} \) or \( \partial_0 \bar{c} \) with \( c, \) but not of \( c \) with \( \bar{c}. \) In general, \( c \) and \( \bar{c} \) are independent canonical variables and one assumes that \([2, 3]:\)

$$\{\Pi_\mu, \Pi_\nu\} = \{\hat{c}, \bar{c}\} = 0; \quad \partial_0 \{\hat{c}, \bar{c}\} = 0;$$  \hfill (40)

$$\{\partial_0 \hat{c}, \partial_0 \bar{c}\} = (-1)\{\partial_0 c, \partial_0 \bar{c}\},$$

where \{,\} means an anticommutator. We thus see that the anticommutators in the above equation are nontrivial and need to be fixed. In order to fix these, we demand that \( c \) satisfy the Heisenberg equation \([2, 3]:\)

$$[c, \mathcal{H}_{\text{BRST}}] = i\partial_0 c$$  \hfill (41)

and using the property \( c^2 = \bar{c}^2 = 0 \) one obtains

$$[c, \mathcal{H}_{\text{BRST}}] = (\partial_0 \hat{c}, \partial_0 \bar{c}).$$  \hfill (42)

The last three equations then imply:

$$\{\partial_0 \hat{c}, \partial_0 \bar{c}\} = (-1)\{\partial_0 c, \partial_0 \bar{c}\} = i.$$  \hfill (43)

Here the minus sign in the above equation is nontrivial and implies the existence of states with negative norm in the space of state vectors of the theory \([2, 3].\) The BRST charge \( Q \) is the generator of the BRST transformations. It is nilpotent and satisfies \( Q^2 = 0. \) It mixes operators which satisfy Bose and Fermi statistics. According to its conventional definition, its commutators with Bose operators and its anticommutators with Fermi operators for the present theory satisfy:

$$[A_1, Q] = [A_2, Q] = \partial_0 c; \quad [\varphi, Q] = -ec,$$  \hfill (44a)

$$[E_1, Q] = \kappa (\partial_0 c - \frac{1}{2} \partial_0 \bar{c}); \quad [E_2, Q] = -\kappa (\partial_1 c - \frac{1}{2} \partial_1 \bar{c}),$$  \hfill (44b)

$$[\bar{c}, Q] = (-1)(\Pi_0 + E_1 + E_2 + \frac{\kappa}{2} (A_1 - A_2)),$$  \hfill (44c)

$$[\partial_0 \hat{c}, Q] = (-1)(\epsilon \pi + \kappa F_{12}).$$  \hfill (44d)

All other commutators and anticommutators involving \( Q \) vanish. In view of this, the BRST charge operator of the present theory could be written as:

$$Q = \int d^2 x \left[ \kappa (\epsilon \pi + \kappa F_{12}) - i \partial_0 c(\Pi_0 + E_1 + E_2 + \frac{\kappa}{2} (A_1 - A_2)) \right].$$  \hfill (45)

This equation implies that the set of states satisfying the conditions:

$$\Pi^0(\psi) = 0; \quad (E_1 - \frac{\kappa}{2} A_2) |\psi\rangle = 0,$$  \hfill (46a)

$$(E_2 + \frac{\kappa}{2} A_1) |\psi\rangle = 0; \quad (\epsilon \pi + \kappa F_{12}) |\psi\rangle = 0$$  \hfill (46b)

belongs to the dynamically stable subspace of states \(|\psi\rangle\) satisfying \( Q |\psi\rangle = 0, \) i.e. it belongs to the set of BRST-invariant states. In order to understand the condition needed for recovering the physical states of the theory we write the operators \( c \) and \( \bar{c} \) in terms of Fermionic annihilation and creation operators. For this purpose we consider the equation: \( \partial_0 \partial_0 c = -c. \) The solution of this equation gives the Heisenberg operator \( c(t) \) where \( t \equiv x^0 \) is the IF time variable (and correspondingly \( \bar{c}(t) \) as:

$$c(t) = e^{\alpha B + e^{\alpha^2} D}; \quad \bar{c}(t) = e^{-\alpha B^\dagger + e^{\alpha^2} D^\dagger}$$  \hfill (47)

which at the time \( t = 0 \) imply

$$c \equiv c(0) = B + D; \quad \bar{c} \equiv \bar{c}(0) = B^\dagger + D^\dagger.$$  \hfill (48a)

$$\partial_0 c \equiv \partial_0 c(0) = i(B - D); \quad \partial_0 \bar{c} \equiv \partial_0 \bar{c}(0) = -i(B^\dagger - D^\dagger).$$  \hfill (48b)

By imposing the conditions

$$c^2 = \bar{c}^2 = \{\hat{c}, \bar{c}\} = \{\partial_0 \bar{c}, \partial_0 \bar{c}\} = 0,$$  \hfill (49a)

$$\{\partial_0 \bar{c}, \partial_0 \bar{c}\} = i = -\{\partial_0 c, \partial_0 c\}$$  \hfill (49b)

one then obtains

$$B^2 + \{B, D\} + D^2 = B^2 + \{B^\dagger, D^\dagger\} + D^\dagger = 0,$$  \hfill (50a)

$$\{B, B^\dagger\} + \{D, D^\dagger\} + \{B, D^\dagger\} + \{B^\dagger, D\} = 0,$$  \hfill (50b)

$$\{B, B^\dagger\} + \{D, D^\dagger\} - \{B, D^\dagger\} - \{B^\dagger, D\} = 0,$$  \hfill (50c)

$$\{B, B^\dagger\} - \{D, D^\dagger\} - \{B, D^\dagger\} + \{D, B^\dagger\} = -1,$$  \hfill (50d)

$$\{B, B^\dagger\} - \{D, D^\dagger\} + \{B, D^\dagger\} - \{D, B^\dagger\} = -1.$$  \hfill (50e)

with the solution

$$B^2 = D^2 = B^{12} = D^{12} = 0,$$  \hfill (51a)

$$\{B, D\} = \{B^\dagger, D\} = \{B^\dagger, D^\dagger\} = \{B^\dagger, D^\dagger\} = 0,$$  \hfill (51b)

$$\{B^\dagger, B\} = -1/2; \quad \{D^\dagger, D\} = 1/2.$$  \hfill (51c)
We now let \( |0\rangle \) denote the fermionic vacuum for which
\[
B(0) = D(0) = 0.
\] (52)

Now by defining \(|0\rangle\) to have norm one, we have
\[
\langle 0| B B^\dagger |0\rangle = -1/2; \quad \langle 0| D D^\dagger |0\rangle = 1/2
\] (53)
so that
\[
B^\dagger |0\rangle \neq 0; \quad D^\dagger |0\rangle \neq 0.
\] (54)

The theory is thus seen to possess negative norm states in the fermionic sector. The existence of these negative norm states as free states of the fermionic part of \( \mathcal{H}_{\text{BRST}} \) is however, irrelevant to the existence of physical states in the orthogonal subspace of the Hilbert space. In terms of fermionic annihilation and creation operators the quantum Hamiltonian density is:
\[
\mathcal{H}_{\text{BRST}} = \left[ \frac{1}{2} \pi^2 + \frac{1}{e} \pi_0 \varphi + \frac{\kappa}{2} [A_0 (\partial_1 A_2 - \partial_2 A_1)] - \frac{1}{4} \Pi_0^2 + 2 (B^\dagger B + D^\dagger D) + \frac{1}{2} [(\partial_1 \varphi)^2 + 2 e A_1 \partial_1 \varphi + e^2 A_1^2]
\]
\[
+ \frac{1}{2} [(\partial_2 \varphi)^2 + 2 e A_2 \partial_2 \varphi + e^2 A_2^2]
\] (55)
and the BRST charge operator is:
\[
Q = \int d^2 x (i) \left[ B [e \pi + \kappa F_{12} - i(\Pi_0 + E_1 + E_2 + \frac{\kappa}{2} (A_1 - A_2))] + D [e \pi + \kappa F_{12} + i(\Pi_0 + E_1 + E_2 + \frac{\kappa}{2} (A_1 - A_2))] \right].
\] (56)

Now because \( Q |\psi\rangle \geq 0 \), the set of states annihilated by \( Q \) contains not only the set for which the constraints of the theory hold but also additional states for which
\[
B |\psi\rangle = D |\psi\rangle = 0, \quad \Pi^0 |\psi\rangle \neq 0; \quad (E_1 - \frac{\kappa}{2} A_2) |\psi\rangle \neq 0, \quad (E_2 + \frac{\kappa}{2} A_1) |\psi\rangle \neq 0; \quad (e \pi + \kappa F_{12}) |\psi\rangle \neq 0.
\] (57a-c)

The Hamiltonian is also invariant under the anti-BRST transformation given by:
\[
\delta \varphi = -e \bar{c}, \quad \delta \pi = 0, \quad \delta A_0 = \delta_0 \bar{c},
\]
\[
\delta A_1 = \delta_1 \bar{c}, \quad \delta A_2 = \delta_2 \bar{c},
\] (58a)
\[
\delta \Pi_0 = 0, \quad \delta E_1 = \frac{\kappa}{2} \delta_1 \bar{c}, \quad \delta E_2 = -\frac{\kappa}{2} \delta_1 \bar{c},
\]
\[
\delta \Pi_a = \delta P_a = \delta \Pi_0 = 0,
\] (58b)
\[
\delta u = \delta_0 \delta_0 \bar{c}, \quad \delta v = \delta_0 \delta_1 \bar{c}, \quad \delta w = \delta_0 \delta_2 \bar{c},
\]
\[
\delta \bar{c} = 0, \quad \delta \bar{c} = -b, \quad \delta b = 0
\] (58c) with generator or anti-BRST charge
\[
\bar{Q} = \int d^2 x \left[ -i \bar{c} [e \pi + \kappa F_{12}] + i (\delta_0 \bar{c}) ([\Pi_0 + E_1 + E_2 + \frac{\kappa}{2} (A_1 - A_2)]) \right].
\] (59)

which in terms of annihilation and creation operators reads:
\[
\bar{Q} = \int d^2 x (-i) \left[ B^\dagger [e \pi + \kappa F_{12} + i(\Pi_0 + E_1 + E_2 + \frac{\kappa}{2} (A_1 - A_2))] + D^\dagger [e \pi + \kappa F_{12} - i(\Pi_0 + E_1 + E_2 + \frac{\kappa}{2} (A_1 - A_2))] \right].
\] (60)

We also have
\[
\delta_0 Q = [Q, H_{\text{BRST}}] = 0; \quad \delta_0 \bar{Q} = [\bar{Q}, H_{\text{BRST}}] = 0
\] (61)
with
\[
H_{\text{BRST}} = \int d^2 x \mathcal{H}_{\text{BRST}}
\] (62)
and we further impose the dual condition that both \( Q \) and \( \bar{Q} \) annihilate physical states, implying that:
\[
Q |\psi\rangle = 0 \quad \text{and} \quad \bar{Q} |\psi\rangle = 0.
\] (63)

The states for which the constraints of the theory hold satisfy both of these conditions and are, in fact, the only states satisfying both of these conditions, since although with (50) and (51),
\[
2 (B^\dagger B + D^\dagger D) = -2 (B B^\dagger + D D^\dagger)
\] (64)
there are no states of this operator with \( B^\dagger |\psi\rangle \neq 0 \) and \( D^\dagger |\psi\rangle \neq 0 \), and hence no free eigenstates of the fermionic part of \( \mathcal{H}_{\text{BRST}} \) that are annihilated by each of \( B, B^\dagger, D, \) and \( D^\dagger \). Thus the only states satisfying \( Q |\psi\rangle = 0 \) and \( \bar{Q} |\psi\rangle = 0 \) are those that satisfy the constraints of the theory. Now because \( Q |\psi\rangle = 0 \), the set of states annihilated by \( Q \) contains not only the set of states for which the constraints of the theory hold but also additional states for which the constraints of the theory do not hold in particular. This situation is, however, easily avoided by additionally imposing on the theory, the dual condition: \( \bar{Q} |\psi\rangle = 0 \) and \( Q |\psi\rangle = 0 \). Thus by imposing both of these conditions on the theory simultaneously, one finds that the states for which the constraints of the theory hold satisfy both of these conditions and, in fact, these are the only states satisfying both of these conditions because in view of the conditions on the fermionic variables \( c \) and \( \bar{c} \) one cannot have simultaneously \( c, \delta_0 \bar{c} \) and \( \bar{c}, \delta_0 c \), applied to \( |\psi\rangle \) to give zero. Thus the only states satisfying \( Q |\psi\rangle = 0 \) and \( \bar{Q} |\psi\rangle = 0 \) are those that satisfy the constraints of the theory and they belong to the set of BRST-invariant as well as to the set of anti-BRST-invariant states.

Alternatively, one can understand the above point in terms of fermionic annihilation and creation operators as follows. The condition \( Q |\psi\rangle = 0 \) implies that the set of states annihilated by \( Q \) contains not only the states for which the constraints of the theory hold but also additional states
for which the constraints do not hold. However, \( \bar{Q} | \psi \rangle = 0 \) guarantees that the set of states annihilated by \( \bar{Q} \) contains only the states for which the constraints hold, simply because \( B^1 | \psi \rangle \neq 0 \) and \( D^1 | \psi \rangle \neq 0 \). Thus, in this alternative way we also see that the states satisfying \( \bar{Q} | \psi \rangle = \bar{Q} | \psi \rangle = 0 \) are only those states that satisfy the constraints of the theory and also that these states belong to the set of BRST-invariant and anti-BRST-invariant states. This completes the BRST formulation of the theory.

5. Summary and discussion

As emphasized in detail in the work of Boyanovsky [2], who has quantized this theory canonically in the Schrödinger picture, paying particular attention to understanding of the gauge-invariant wave functionals and operators, a study of this theory is relevant for understanding the excitations in a broken (or frozen) symmetry state (superfluid) can carry fractional statistics, a question which is relevant for the effective theories in condensed matter, as well as for understanding the issue of exotic statistics in gauge-invariant observables [2]. For further details we refer to the work of [2]. These models in the so-called BSP [2, 3], can, in fact, be thought of as the Higgs models where the (Higgs) potential has been set to zero by freezing the scalar field at the degenerate minima of the Higgs potential, and the phase of the complex matter field carries the charge degree of freedom of the complex matter field [2, 3].

As mentioned previously, it is important to note that the choice of gauge-fixing conditions \( A_0 \approx 0 \) and \( A_1 \approx 0 \) are not only acceptable and consistent with our quantization procedures but are also physically more interesting gauge choices, representing the time-axial or the temporal gauge and the coulomb gauge respectively, for the electromagnetic field \( A^\mu \). This, however, is not a unique gauge choice. Our choice of another set of gauge-fixing conditions namely, \( A_0 \approx 0 \) and \( \varphi \approx 0 \), is also perfectly acceptable and consistent with our quantization procedures and in fact, could even be a better gauge choice because the scalar (phase) field \( \varphi \) happens to be a gauge variable of the theory. In this theory, the condition \( A_0 \approx 0 \), again represents the time-axial or the temporal gauge and the condition \( \varphi \approx 0 \) is the usual coulomb gauge for the scalar (phase) field \( \varphi \).

In the context of the Hamiltonian, path integral and BRST quantizations of this theory, it may be important to emphasize that for the quantization of the present theory which is gauge-invariant, one naturally needs appropriate gauge-fixing conditions. In the Hamiltonian and path integral quantizations, the quantization under gauge-fixing naturally breaks the gauge-invariance of the theory. One of the quantization schemes in which this does not happen is the BRST quantization, where the extended gauge symmetry called the BRST symmetry is maintained even under gauge-fixing. Therefore it is surely desirable to achieve this so-called BRST quantization if possible. This therefore makes a kind of complete quantization of a theory [3–5] as we have done for the present theory in this work.

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